ON REPRESENTATIONS OF RATIONAL CHEREDNIK ALGEBRAS
OF COMPLEX RANK

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Abstract. We study a family of abelian categories $O_{c,\nu}$ depending on complex parameters $c, \nu$ which are interpolations of the $O$-category for the rational Cherednik algebra $H_c(\nu)$ of type $A$, where $\nu$ is a positive integer. We define the notion of a Verma object in such a category (a natural analogue of the notion of Verma module). We give some necessary conditions and some sufficient conditions for the existence of a non-trivial morphism between two such Verma objects. We also compute the character of the irreducible quotient of a Verma object for sufficiently generic values of parameters $c, \nu$, and prove that a Verma object of infinite length exists in $O_{c,\nu}$ only if $c \in \mathbb{Q}_{<0}$. We also show that for every $c \in \mathbb{Q}_{<0}$ there exists $\nu \in \mathbb{Q}_{<0}$ such that there exists a Verma object of infinite length in $O_{c,\nu}$. The latter result is an example of a degeneration phenomenon which can occur in rational values of $\nu$, as was conjectured by P. Etingof.

1. Introduction

The study of representations in complex rank involves defining and studying families of abelian categories depending on a parameter $t$ which are polynomial interpolations of the categories of representations of objects such as finite groups, Lie groups, Lie algebras and more. This was done by P. Deligne in [5] for finite dimensional representations of the general linear group $GL_n$, the orthogonal and symplectic groups $O_{2n}, Sp_{2n}$ and the symmetric group $S_n$. Deligne defined Karoubian tensor categories $Rep(GL_t), Rep(OSp_t), Rep(S_t)$, $t \in \mathbb{C}$, which in points $n = t \in \mathbb{Z}_+$ admit an essentially surjective functor onto the standard categories $Rep(GL_n), Rep(OSp_n), Rep(S_n)$. The category $Rep(S_t)$ was subsequently studied by himself and others (F. Knop in [15], V. Ostrik, J. Comes in [4]).

The resulting categories can be shown to have superexponential growth, and are not equivalent to the category of finite-dimensional representations of any affine algebraic group or supergroup. One can look for rational non-integer values of $t$ where degeneration occurs, which would allow us to understand better how the structure of the classical category of representations is similar for all ranks $n$, and why degenerate phenomena occur at specific values of $n$.

P. Etingof, in [7], suggested using these results to define and study the categories of representations of rational Cherednik algebras, as well as Lie superalgebras, affine Lie algebras and other “non-compact” representation theory objects. A possible approach, as stated in [7], would be using the tensor categories $Rep(GL_t), Rep(OSp_t), Rep(S_t)$ and defining the required category for a “non-compact” object in complex rank as a category of tuples which consist of an ind-object of the corresponding tensor category along with morphisms satisfying some relations.

In this paper, we study the categories which are interpolations of the classical $O$-category of lowest weight modules for the rational Cherednik algebra of type $A$, denoted

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by $H_c(n)$. These categories are BGG-type categories, where one can define Verma objects and study morphisms between these, as well as their irreducible quotients.

The categories $\text{Rep}(H_c(\nu))$ (the parameter $\nu$ replacing the parameter $t$ used above) have been defined in [7] as categories whose objects are ind-objects of Deligne’s category $\text{Rep}(S_t)$ with additional structure; namely, two morphisms in $\text{Rep}(S_t)$, denoted by $x, y$, satisfying some relations. These morphisms represent the actions of the elements $x_1, ..., x_n$ and $y_1, ..., y_n$ of $H_c(n)$ respectively. We proceed to define the full subcategory $O_{c,\nu}$ of $\text{Rep}(H_c(\nu))$, which is an interpolation of the category $O$ of the classical Cherednik algebra $H_c(n)$.

In this category $O_{c,\nu}$ we have objects we will call “Verma objects”, as they are analogues of the Verma modules in the classical $O$ category. They are parameterized by arbitrary partitions $\lambda$, and denoted by $M_{c,\nu}(\lambda)$. As in the $H_c(n)$ case, they have a natural grading as $\text{Rep}(S_t)$ ind-objects, with the lowest grade consisting of an indecomposable $\text{Rep}(S_t)$-object given by the partition $\lambda$.

The Verma objects have about the same properties as the classical Verma modules, and one can ask for which values of parameters $c, \nu$ they are reducible, and for which values of parameters $c, \nu$ there are non-trivial morphisms between different Verma objects. It would also be interesting to compute the character of the unique irreducible quotient of $\text{Rep}(S_t)$-objects.

### 1.1. Summary of the Results

Recall that the category $\text{Rep}(S_t)$ is a Karoubian rigid symmetric monoidal category which is abelian semisimple whenever $\nu \notin \mathbb{Z}_+$; for any $\nu$, its indecomposable objects are parameterized by Young diagrams of arbitrary size. One can treat an indecomposable object $X_\tau$ of $\text{Rep}(S_t)$, with $\tau$ being a Young diagram, as corresponding to the Young diagram obtained from $\tau$ by adding a very long top row (“of size $\nu - |\tau|$”).

Throughout the paper we will assume that $\nu \notin \mathbb{Z}_+$ unless it is explicitly stated otherwise.

The indecomposable object in $\text{Rep}(S_t)$ corresponding to a Young diagram with one box is denoted by $\mathfrak{h}_0$, and has dimension $\nu - 1$ (it corresponds to the reflection representation of $S_n$ when $\nu = n$). One can define an algebra object $\mathfrak{h}_0^*$ in $\text{ind-Rep}(S_t)$ (which would correspond to the ring $\mathbb{C}[x_1, ..., x_n]/\mathbb{C}[x_1, ..., x_n] \langle \sum_i x_i \rangle$ when $\nu = n$).

We define the Verma objects $M_{c,\nu}(\lambda)$ of $O_{c,\nu}$ in Subsection 5.2 (these are sometimes denoted $M(\lambda)$ for short). The underlying ind-object of $\text{Rep}(S_t)$ is just $\mathfrak{h}_0^* \otimes X_\lambda$.

For every non-negative integer $n$, we construct a functor $F_{c,n}$ from $\text{Rep}(H_c(\nu = n))$ (the category $\text{Rep}(H_c(\nu))$ for integer value $n \in \mathbb{Z}_+$ of $\nu$) to $\text{Rep}(H_c(n))$ (the category of representations of $H_c(n)$). Considering the full subcategory of $O_{c,\nu=n}$ whose objects are $M_{c,\nu=n}(\lambda)$ such that $|\lambda| \leq n/2$, we see that the restriction of $F_{c,n}$ to this subcategory is fully faithful; in fact, $F_{c,n}$ takes Verma object $M_{c,\nu=n}(\lambda)$ to Verma module in $O(H_c(n))$ whose lowest weight corresponds to the Young diagram obtained from $\lambda$ by adding a top row of size $n - |\lambda|$.

**Notation 1.1.1.**

- Denote the set of points $(c, \nu)$ on the complex plane where $M_{c,\nu}(\tau)$ is reducible by $\mathcal{B}_\tau$. The image of $\mathcal{B}_\tau \setminus \{c = 0\}$ under the map $c \mapsto 1/c$ will be denoted by $\mathcal{B}_\tau$.
- Denote the set of points $(c, \nu)$ on the complex plane where there is a non-zero morphism $M_{c,\nu}(\mu) \to M_{c,\nu}(\tau)$ by $\mathcal{B}_{\mu,\tau}$. The image of $\mathcal{B}_{\mu,\tau} \setminus \{c = 0\}$ under the map $c \mapsto 1/c$ will be denoted by $\mathcal{B}_{\mu,\tau}$.
As in the classical case, we have: $B_\tau = \bigcup_\mu B_{\mu,\tau}$, the parameter $\mu$ running over all Young diagrams of arbitrary size.

We will show that if $\mu \neq \tau$, $B_{\mu,\tau}$ lies inside a countable (disjoint) union of curves $\bigcup_{m \in \mathbb{Z}_{>0}} L_{\tau,\mu,m}$ for which we give explicit equations. These curves become straight lines, denoted by $L_{\tau,\mu,m}$, when we switch to parameters $(1/c, \nu)$ instead of $(c, \nu)$.

For a fixed positive integer $m$, all points $(c, \nu)$ which satisfy the following condition lie on the curve $L_{\tau,\mu,m}$:

**Condition 1.1.2.** There exists a non-zero map $M_{c,\nu}(\mu) \to M_{c,\nu}(\tau)$ with the lowest grade of $M_{c,\nu}(\mu)$ mapping to grade $m$ of $M_{c,\nu}(\tau)$.

The equations for lines $L_{\tau,\mu,m}$ are given in Section 8.

It is easy to see that the intersection of $B_{\mu,\tau}$ with each of these lines $L_{\tau,\mu,m}$ is either the entire line or a finite number of points. We give a full description of the lines $L_{\tau,\mu,m}$ lying inside $B_{\mu,\tau}$ whenever $|\tau| \neq |\mu|$ in Section 10.

**Theorem 1.** For two distinct Young diagrams $\mu, \tau$ and an integer $m > 0$, the following are equivalent:

1. $|\mu| \neq |\tau|$ and $L_{\tau,\mu,m} \subset B_{\mu,\tau}$,
2. $\mu = \Gamma(\tau, s, \text{sign}(|\mu| - |\tau|))$ for some $s \in C_\tau$ (in particular, $|\mu| \neq |\tau|$), and $|\mu| - |\tau| \equiv m$.

Here $\Gamma(\tau, s, \text{sign}(|\mu| - |\tau|))$ is a construction described explicitly in Section 10 and $C_\tau = \{ |\tau| - 1 + j - \tau_j \mid j \in \mathbb{Z}_{>0} \}$.

Fix $s \in C_\tau$, i.e. $s = |\tau| - 1 + j - \tau_j$ for some $j > 0$. In terms of Young diagrams with “very long top row”, this construction means that

- we start with the Young diagram obtained from $\tau$ by adding a very long top row (“of size $\nu - |\tau|$”),
- take out a hook of size $\nu - s$ with leg $\tau_j + \text{sign}(|\mu| - |\tau|)$ (the Young diagram obtained at this stage is denoted by $\text{core}_\tau(\nu-s)(\tau)$, and is a “normal” Young diagram, i.e. all its rows are of integer size),
- insert a hook of size $\nu - s$ with leg $\tau_j + \text{sign}(|\mu| - |\tau|) + 1$.

The diagram obtained this way would be the Young diagram obtained from $\mu$ by adding a very long top row (“of size $\nu - |\mu|$”), where $\mu = \Gamma(\tau, s, \text{sign}(|\mu| - |\tau|))$.

From the theorem above one can easily see that if $|\mu| \neq |\tau|$, and $L_{\tau,\mu,m} \subset B_{\mu,\tau}$, then for a generic point $(\frac{1}{c}, \nu)$ of $L_{\tau,\mu,m}$, $(\frac{1}{c}, \nu) \notin B_{\mu',\tau}$ for any $\mu' \neq \mu$ (see Proposition 11).

Let $(1/c, \nu)$ be a generic point on a line $L_{\tau,\mu,m} \subset B_{\mu,\tau}$, where $|\tau| \neq |\mu|$. We have proved that in such a case $\mu = \Gamma(\tau, s, \text{sign}(|\mu| - |\tau|))$ for some $s \in C_\tau$. We give a formula for the character of $L_{c,\nu}(\tau)$, and compute explicitly the character of $L_{c,\nu}(\emptyset)$, where $(1/c, \nu)$ is a generic point on a line $L_{\emptyset,\mu,k} = \{(1/c, \nu) \mid \nu = k \}$ (here $k \in \mathbb{Z}_{>0}$ is fixed).

In fact, one can show that in this case there exists a long exact sequence

$$\cdots \to M(\Gamma(\tau, s, \pm l)) \to M(\Gamma(\tau, s, \pm (l-1))) \to \cdots \to M(\Gamma(\tau, s, \pm 1)) \to M(\tau) \to L(\tau) \to 0$$

The sign is $\text{sign}(|\mu| - |\tau|)$, and this is a finite sequence if $|\mu| < |\tau|$.

The character formula for $L_{c,\nu}(\tau)$ is then obtained from Euler’s formula applied to the above long exact sequence.

We conclude the paper by proving the following theorem:

**Theorem 2.** If there exists a Verma object of infinite length in $O_{c,\nu}$, then $c \in \mathbb{Q}_{<0}$. In fact, one can show that for any $c \in \mathbb{Q}_{<0}$, there exists $\nu \in \mathbb{Q}_{<0}$ such that $M_{c,\nu}(\emptyset)$ has infinite length.
Theorem 2 is an example of a degenerate phenomenon which can occur for rational non-integer values of $\nu$. The classical representation theory of Cherednik algebras says that for a non-negative integer $n$, all modules in $O(H_c(n))$ have finite length (cf. [3, Corollary 3.26]), and we see from Theorem 2 that the same is true for Verma objects in $O_{c,\nu}$ for $c \notin \mathbb{Q}$.

It would be interesting to know about other phenomena which can occur in $O_{c,\nu}$ only for $\nu \in \mathbb{Q} \setminus \mathbb{Z}_+$. Such phenomena were conjectured by P. Etingof in [4].

1.2. Structure of the paper. In Section 3 we recall basic definitions and facts about $H_c(n)$, the Cherednik algebra of type $A$. Some further facts about the blocks of the $O$-category for $H_c(n)$ will be given in Section 9.

In Section 4 we recall basic facts about the Deligne category $\text{Rep}(S_\nu)$. We do not give a definition (it can be found in [5]), instead we mention the facts about this category which we will use.

In Sections 5 and 6, we define the category $\text{Rep}(H_c(\nu))$ (interpolation of the category of representations of $H_c(n)$) and the category $O_{c,\nu}$ (interpolation of the $O$-category of $H_c(n)$). We then define the Verma objects $M_{c,\nu}(\tau)$ in $O_{c,\nu}$.

In Section 7, we discuss the functor from $\text{Rep}(H_c(\nu = n))$ (the category $\text{Rep}(H_c(\nu))$ for integer value $n \in \mathbb{Z}_+$ of $\nu$) to $\text{Rep}(H_c(n))$ (the category of representations of $H_c(n)$). In Section 8 we give a necessary condition on $(c, \nu)$ which is required in order for a non-trivial map $M_{c,\nu}(\mu) \rightarrow M_{c,\nu}(\tau)$ to exist. This condition is an equation which defines the lines $L_{\tau,\mu,m}$.

In Section 10 we define constructions for $O_{c,\nu}$ which are necessary in order to describe the triplets $(\tau, \mu, m)$ for which $L_{\tau,\mu,m} \subset B_{\mu,\tau}$ (the latter is done in Section 11) we give a full description in the case $|\tau| \neq |\mu|$, and a partial description when $|\tau| = |\mu|$.

In Section 12 we define the formal character of an object of $O_{c,\nu}$, give a formula for the character of a Verma object $M_{c,\nu}(\tau)$, and compute the character of a simple object $L_{c,\nu}(\tau)$ in the simplest cases. We also give a positive character formula for the Verma objects and look at examples when such a formula can be derived for some simple objects $L_{c,\nu}(\tau)$.

In Section 13 we prove Theorem 2 mentioned above, and in particular give a lower bound on the grade of a given Verma object $M_{c,\nu}(\tau)$ in which a simple $\text{Rep}(S_n)$-object $X_\mu$ can lie.

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2. Notations and conventions

The base field throughout this paper will be $\mathbb{C}$.

2.1.

Notation 2.1.1. For a rational number $q \in \mathbb{Q}$, we write $q = \frac{\text{num}(q)}{\text{den}(q)}$, where $\text{num}(q), \text{den}(q) \in \mathbb{Z}, \text{den}(q) > 0, \gcd(\text{num}(q), \text{den}(q)) = 1$.

Notation 2.1.2 (Symmetric group and Young diagrams).

- $S_n$ will denote the symmetric group ($n \in \mathbb{Z}_+$). We will denote by $S$ the set of reflections $s_{ij}$ in $S_n$. 
• The notation $\lambda$ will stand for a partition (weakly decreasing sequence of non-negative integers), a Young diagram $\lambda$ (considered in the English notation), and the corresponding irreducible representation of $S_{|\lambda|}$. Here $|\lambda|$ is the sum of entries of the partition, or, equivalently, the number of cells in the Young diagram $\lambda$. We will sometimes write $\lambda \vdash n$ to say that $\lambda$ is a partition of $n$.

• When referring to a cell in a Young diagram $\lambda$, $(i,j)$ will be the cell in row $i$ and column $j$, with $i,j \geq 1$.

• The length of the partition $\lambda$, i.e. the number of rows of Young diagram $\lambda$, will be denoted by $l(\lambda)$.

• The $i$-th entry of a partition $\lambda$, as well as the length of the $i$-th row of the corresponding Young diagram, will be denoted by $\lambda_i$ (if $i > l(\lambda)$, then $\lambda_i := 0$). The transpose of the Young diagram $\lambda$ will be denoted by $\lambda^\mathord{\text{\reflectbox{t}}}$ (so the length of the $i$-th column of $\lambda$ is $\lambda^\mathord{\text{\reflectbox{t}}}i$).

Example 2.1.3. Consider the Young diagram $\lambda$ corresponding to the partition $(6,5,4,1)$. The length of $\lambda$ is 4, the size is 16, and the transpose of $\lambda$ is the Young diagram corresponding to partition $(4,3,3,3,3,2,1)$:

\[
\begin{array}{cccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array}
\mapsto
\begin{array}{cccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array}
\]

• $h$ (in context of representations of $S_n$) will denote the permutation representation of $S_n$, i.e. the $n$-dimensional representation $\mathbb{C}^n$ with $S_n$ acting by $g.e_j = e_{g(j)}$ on the standard basis $e_1,..,e_n$ of $\mathbb{C}^n$.

• $h_0$ (in context of representations of $S_n$) will denote the reflection representation of the symmetric group $S_n$: this is the $n – 1$ dimensional irreducible representation of $S_n$ corresponding to the partition $(n – 1,1)$ of $n$. This representation is the subrepresentation of $h$ given by the $S_n$-invariant subspace of all vectors whose sum of coordinates is zero.

We will use the same notation for the corresponding object in $\text{Rep}(S_\nu)$.

• Let $s \in \mathbb{Z}_{\geq -1}, k \in \mathbb{Z}_{\geq 0}$. We will denote by $\tau^s$ a Young diagram consisting of a row with $s + 1$ cells, and by $\pi^k$ a Young diagram consisting of a column with $k$ cells.

2.2. In the symmetric rigid monoidal category $\text{Rep}(S_\nu)$, we will use the following notation:

• $\mathbb{1}$ denotes the unit object.

• For any $U \in \text{Rep}(S_\nu)$, $ev_U : U \otimes U^* \rightarrow \mathbb{1}$ is the evaluation morphism, and $coev_U : \mathbb{1} \rightarrow U^* \otimes U$ is the coevaluation morphism.

• For any $U,U' \in \text{Rep}(S_\nu)$, $\sigma_{U,U'} : U \otimes U' \rightarrow U' \otimes U$ is the symmetry isomorphism in $\text{Rep}(S_\nu)$.

2.3. The following definition of a block in a Karoubian category was given in \cite{[4]}:

Definition 2.3.1 (Block of a Karoubian category). Let $\mathcal{C}$ be a Karoubian category. Consider the weakest equivalence relation on the set of isomorphism classes of indecomposable objects such that any two indecomposable objects with a non-zero morphism between them are equivalent.

An equivalence class in this relation will be called a block.

We will use the same term to refer to a full subcategory in $\mathcal{C}$ whose objects are direct sums of indecomposable objects from a single block (in the above sense).
3. Classical Cherednik algebra of type A

This section follows [9 Chapter 2], [10 Section 2].

3.1. Rational Cherednik algebra of type A. Let \( c \in \mathbb{C} \) (can be considered as a formal parameter).

**Definition 3.1.1.** The rational Cherednik algebra of type A of rank \( n \) with parameter \( c \), denoted by \( \mathcal{H}_c(n) \), is the quotient of the algebra \( \mathbb{C} S_n \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*) \) by the ideal generated by the following relations:

1. \([x, x'] = 0, [y, y'] = 0\) for any \( x, x' \in \mathfrak{h}^*, \ y, y' \in \mathfrak{h} \),
2. \([y, x] = (y, x) - c \sum_{s \in S}(1-s)(y, x)s\), where \( x \in \mathfrak{h}^*, y \in \mathfrak{h}, (,.) \) is the natural pairing between \( \mathfrak{h}, \mathfrak{h}^* \).

If we choose dual bases \( x_1, \ldots, x_n \) of \( \mathfrak{h}^* \), \( y_1, \ldots, y_n \) of \( \mathfrak{h} \) such that \( \forall i, j, (y_j, x_i) = \delta_{ij} \), and \( s_{ij}(y_i) = y_j, s_{ij}(x_i) = x_j \), then the last relation from Definition 3.1.1 can be rewritten as

\[ [y_j, x_i] = cs_{ij} \text{ for any } i \neq j, [y_i, x_i] = 1 - c \sum_{j \neq i} s_{ij} \]

Similarly to the universal enveloping algebra of a Lie algebra, we have:

**Proposition 3.1.2 (PBW-type theorem).** \( \mathcal{H}_c(n) \cong \mathfrak{S} \mathfrak{h}^* \otimes \mathbb{C} S_n \otimes \mathfrak{S} \mathfrak{h} \) as vector spaces.

Continuing the analogy between the universal enveloping algebra of a Lie algebra and a Cherednik algebra, we have a Cartan-type element:

**Definition 3.1.3.** The element \( \overline{h} \in \mathcal{H}_c(n) \) is defined as

\[ \overline{h} := \sum x_i y_i + \text{dim}(\mathfrak{h})/2 - c \sum_{s \in S} s = \sum x_i y_i + n/2 - c \sum_{s \in S} s \]

This element satisfies: \( g \overline{h} g^{-1} = \overline{h}, [\overline{h}, x_i] = x_i, [\overline{h}, y_i] = -y_i \) for any \( g \in S_n, i, j \in \{1, \ldots, n\} \), and can also be written as \( \overline{h} = \frac{1}{2} \sum (x_i y_i + y_i x_i) \).

We will use the “reduced” Cherednik algebra \( H_c(n) \), which corresponds to the reflection representation \( \mathfrak{h}_0 = \text{span}_\mathbb{C}\{y_i - y_j \mid i, j \in \{1, \ldots, n\}, i \neq j\} \) rather than the permutation representation \( \mathfrak{h} \) of \( S_n \). It is defined in the same way as \( \mathcal{H}_c(n) \), but with \( \mathfrak{h}_0 \) appearing instead of \( \mathfrak{h} \) (see [9], or [13 Section 2]). In fact, we have: \( \mathcal{H}_c(n) = H_c(n) \otimes \mathbb{A}_1 \), \( \mathbb{A}_1 \) being the Weyl algebra. This follows from the fact that the subalgebra of \( \mathcal{H}_c(n) \) generated by \( \sum y_i, \sum x_i \) is isomorphic to \( \mathbb{A}_1 \), and commutes with the subalgebra \( H_c(n) \) of \( \mathcal{H}_c(n) \).

For \( H_c(n) \), \( \overline{h} \) is not an element of \( H_c(n) \), but the element

\[ h := \overline{h} - \frac{(\sum x_i)(\sum y_i) + (\sum y_i)(\sum x_i)}{2n} \]

of \( H_c(n) \) plays the analogous role.

3.2. Category \( O(H_c(n)) \). The category \( O(H_c(n)) \) is defined as the category of all modules over \( H_c(n) \) which are finitely generated over \( \mathfrak{S} \mathfrak{h}_0^* \), and on which \( \mathfrak{h}_0 \) acts locally nilpotently.
3.3. Verma modules. Let $\tau$ be an irreducible representation of $S_n$. Then one can define $M_{c,n}(\tau) := H_c(n) \otimes_{\mathbb{C}S_n} S\mathfrak{h}_0$, where $S\mathfrak{h}_0$ acts on $\tau$ by zero. This is the Verma module corresponding to $\tau$. It is a lowest weight module whose underlying space is isomorphic to $S\mathfrak{h}_0^* \otimes \tau$, where $S\mathfrak{h}$ acts on each subspace $S^m\mathfrak{h}_0^* \otimes \tau$, $\mathfrak{h}_0^*$ acts by multiplication on $S\mathfrak{h}_0^*$, and $\mathfrak{h}_0$ acts according to the commutator relation given in the definition (if $c = 0$, then $\mathfrak{h}_0$ acts by differentiation on $S\mathfrak{h}_0^*$). The element $\mathfrak{h}$ acts locally finitely on modules from $O(H_c(n))$, with finite dimensional generalized eigenspaces. In particular, $\mathfrak{h}$ acts semisimply on $M_{c,n}(\tau)$, with lowest eigenvalue $h_{c,n}(\tau) = \dim(\mathfrak{h}_0)/2 - c \sum_{s \in S} s|_\tau$ (the rest of the eigenvalues on $S^m\mathfrak{h}_0^* \otimes \tau$, $m > 0$ being $h_{c,n}(\tau) + m$).

3.3.1. Sum of transpositions element. Recall that $\mathbb{C}S_n$ has a central element $\Omega := \sum_{s \in S} s \in \mathbb{C}[S_n]$. This element acts on an irreducible representation parameterized by the Young diagram $\tau$ by the scalar $ct(\tau) = \sum_{(i,j) \in \tau} (j - i)$, called “content of $\tau$” (here $(i,j)$ denotes cell in row $i$, column $j$ of the Young diagram $\tau$).

So $h_{c,n}(\tau) = \dim(\mathfrak{h}_0)/2 - c \sum_{s \in S} s|_\tau = \dim(\mathfrak{h}_0)/2 - c\Omega|_\tau = \frac{n-1}{2} - c \cdot ct(\tau)$.

4. Category $Rep(S_\nu)$

4.1. General description. We recall some basic facts about Deligne’s category $Rep(S_\nu)$ (see [5, 7, and 119, Section 2]).

This is a family of Karoubian tensor categories over $\mathbb{C}$ defined for any $\nu \in \mathbb{C}$, and “flat with respect to parameter $\nu$” (one can view $\nu$ as a formal parameter, with $Obj(Rep(S_\nu))$ not depending on $\nu$, and the Hom spaces being modules over $\mathbb{C}[\nu]$).

For any $\nu \in \mathbb{C}$, the category $Rep(S_\nu)$ is generated, as a Karoubian tensor category, by one object, denoted $\mathfrak{h}$. This object is the analogue of the permutation representation of $S_n$, and any object in $Rep(S_\nu)$ is a direct summand in a direct sum of tensor powers of $\mathfrak{h}$.

For $\nu \in \mathbb{Z}_+$, $Rep(S_\nu)$ is a semisimple abelian category.

For $\nu \in \mathbb{Z}_+$, the category $Rep(S_\nu)$ has a tensor ideal $\mathcal{I}_\nu$, called the ideal of negligible morphisms (this is the ideal of morphisms $f : X \rightarrow Y$ such that $tr(fu) = 0$ for any morphism $u : Y \rightarrow X$). In that case, the classical category $Rep(S_n)$ of finite-dimensional representations of the symmetric group for $n := \nu$ is equivalent to $Rep(S_\nu)/\mathcal{I}_\nu$ (equivalent as Karoubian rigid symmetric monoidal categories). Note that $\mathfrak{h}$ is sent to the permutation representation of $S_n$ under this equivalence.

Remark 4.1.1. Although $Rep(S_\nu)$ is not semisimple and not even abelian when $\nu \in \mathbb{Z}_+$, a weaker statement holds (see [5, Proposition 5.1]): consider the full subcategory $Rep(S_\nu)^{\nu/2}$ of $Rep(S_\nu)$ whose objects are direct summands of sums of $\mathfrak{h}^\otimes m$, $0 \leq m \leq \nu$. This subcategory is abelian semisimple.

Notation 4.1.2. We will denote Deligne’s category for integer value $n \geq 0$ of $\nu$ as $Rep(S_{\nu=n})$, to distinguish it from the classical category $Rep(S_n)$ of representations of the symmetric group $S_n$. Similarly for other categories arising in this text.

The indecomposable objects of $Rep(S_\nu)$, regardless of the value of $\nu$, are labeled by all Young diagrams (of arbitrary size). We will denote the indecomposable object in $Rep(S_\nu)$ corresponding to the Young diagram $\tau$ by $X_{\tau}$.

Notation 4.1.3. We will denote by $\lambda(n)$ the Young diagram obtained from $\lambda$ by adding a top row of size $n - |\lambda|$, where $n \geq \lambda_1 + |\lambda|$ is an integer.
Example 4.1.4.

\[ \lambda = \begin{array}{ccccccc} 
\ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\end{array} \quad \mapsto \quad \tilde{\lambda}(31) = \begin{array}{ccccccc} 
\ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\end{array} \]

For non-negative integer \( \nu =: n \), we have: the partitions \( \lambda \) for which \( X_\lambda \) has a non-zero image in the quotient \( \text{Rep}(S_{\nu=n})/I_{\nu=n} \cong \text{Rep}(S_n) \) are exactly the \( \lambda \) for which \( \lambda_1 + |\lambda| \leq n \).

If \( \lambda_1 + |\lambda| \leq n \), then the image of \( X_\lambda \in \text{Rep}(S_{\nu=n}) \) in \( \text{Rep}(S_n) \) would be the irreducible representation of \( S_n \) corresponding to the Young diagram \( \tilde{\lambda}(n) \).

Intuitively, one can treat the indecomposable objects of \( \text{Rep}(S_\nu) \) as if they were parametrized by “Young diagrams with very long top row”. The indecomposable object \( X_\lambda \) would be treated as if it corresponded to \( \tilde{\lambda}(\nu) \), i.e. a Young diagram obtained by adding a very long top row (“of size \( \nu - |\lambda| \)”). This point of view is useful to understand how to extend constructions for \( S_n \) involving Young diagrams to \( \text{Rep}(S_\nu) \).

Example 4.1.5. The indecomposable object \( X_\lambda \), where

\[ \lambda = \begin{array}{ccccccc} 
\ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\end{array} \]

can be thought of as a Young diagram with a “very long top row of length \( \nu - 16 \)”: 

Notation 4.1.6. Let \( h_0 := X_\square \) be the indecomposable object in \( \text{Rep}(S_\nu) \) corresponding to the one-box Young diagram (that would be the analogue of the reflection representation in \( \text{Rep}(S_n) \)).

Its dual in \( \text{Rep}(S_\nu) \), denoted by \( h_0^* \), is isomorphic to \( h_0 \), since all the objects in \( \text{Rep}(S_\nu) \) have a symmetric form defined on them (analogue of the invariant symmetric form on the representations of the symmetric group over a field of characteristic zero).

Remark 4.1.7. \( h \cong h_0 \oplus 1 \) for any \( t \neq 0 \), in particular, for any \( t \notin \mathbb{Z}_+ \). But for \( t = 0 \), we have \( h \cong h_0 \).

4.2. Pieri’s rule. We have an interpolation of Pieri’s rule (see [1, Proposition 5.15], [7], and [11] for Pieri’s rule in the integer \( \nu \) case):

Proposition 4.2.1. [Pieri’s rule] Let \( \nu \notin \mathbb{Z}_+ \) and let \( X_\tau \) be a simple object of \( \text{Rep}(S_\nu) \). As objects of \( \text{Rep}(S_\nu) \),

\[ h_0 \otimes X_\tau \cong \bigoplus_{\mu \in P_\tau^+ \cup P_\tau^- \cup P_\tau^0} X_\mu \oplus \text{cc}(\tau)X_\tau \]

where \( P_\tau^+ \), \( P_\tau^- \), \( P_\tau^0 \) are the sets of all Young diagrams obtained from \( \tau \) by adding, deleting, or moving a corner cell, respectively (note that \( \tau \notin P_\tau^0 \)), and \( \text{cc}(\tau) \) is the number of corner cells of \( \tau \).

Example 4.2.2. \( X_{\pi k} \cong \Lambda^k h_0 \) as objects of \( \text{Rep}(S_\nu) \).
4.3. Analogue of the sum of transpositions element for \( \text{Rep}(S_\nu) \). Consider the following function on Young diagrams \( \mu \):

\[
C(\mu) := \frac{((\nu - |\mu|)(\nu - |\mu| - 1)}{2} - |\mu| + \text{ct}(\mu).
\]

This function interpolates the function \( \text{ct}(\bar{n}(\nu)) \), \( n \in \mathbb{Z}_{\geq |\mu|+\mu_1} \) to \( \nu = n \in \mathbb{C} \).

J. Comes and V. Ostrik defined in \([4]\) an element \( \Omega \) of \( Z(\text{Rep}(S_\nu)) \) for arbitrary \( \nu \) (including non-negative integers) so that for any indecomposable object \( X_\mu \) in \( \text{Rep}(S_\nu) \), \( (\Omega - C(\mu)) \text{Id}_{X_\mu} \) is a nilpotent endomorphism of \( X_\mu \).

For \( X_\mu \in \text{Rep}(S_\nu) \) such that \( \nu \notin \{0, 1, ..., |\mu| + \mu_1 \} \), we have:

\[
\dim \text{End}_{\text{Rep}(S_\nu)}(X_\mu) = 1
\]

and thus

\[
\Omega|_{X_\mu} := C(\mu) \text{Id}_{X_\mu} = \left( \frac{(\nu - |\mu|)(\nu - |\mu| - 1)}{2} - |\mu| + \text{ct}(\mu) \right) \text{Id}_{X_\mu}.
\]

5. Category \( \text{Rep}(H_c(\nu)) \)

Let \( c, \nu \in \mathbb{C} \).

Based on the definition of the category \( \text{Rep}(S_\nu) \) defined by Deligne, P. Etingof defined in \([7]\) the category \( \text{Rep}(H_c(\nu)) \):

**Definition 5.0.1 (Category \( \text{Rep}(H_c(\nu)) \)).** \( \text{Rep}(H_c(\nu)) \) is the category whose objects are triples \((M, x, y)\), where \( M \) is an ind-object of Deligne’s category \( \text{Rep}(S_\nu) \), and

\[
x : h_0^* \otimes M \to M
\]

\[
y : h_0 \otimes M \to M
\]

are morphisms in \( \text{Rep}(S_\nu) \) satisfying the conditions:

1. The morphism

\[
x \circ (\text{Id} \otimes x) \circ ((\text{Id} - \sigma_{b_0^*, h_0^*}) \otimes \text{Id}) : h_0^* \otimes h_0^* \otimes M \to M
\]

is 0.

2. The morphism

\[
y \circ (\text{Id} \otimes y) \circ ((\text{Id} - \sigma_{h_0, h_0}) \otimes \text{Id}) : h_0 \otimes h_0 \otimes M \to M
\]

is 0.

3. The morphism

\[
y \circ (\text{Id} \otimes x) - x \circ (\text{Id} \otimes y) \circ (\sigma_{b_0, h_0^*} \otimes \text{Id}) : h_0 \otimes h_0^* \otimes M \to M
\]

equals

\[
ev_{h_0} \otimes \text{Id} - c(ev_{h_0} \otimes \text{Id}) \circ (\Omega^2 - \Omega^{13})
\]

Here

- \( \sigma \) is the symmetry isomorphism in \( \text{Rep}(S_\nu) \) (i.e. \( \sigma_{U,U'} : U \otimes U' \to U' \otimes U \) is the canonical isomorphism in \( \text{Rep}(S_\nu) \), where \( U, U' \) are objects of \( \text{Rep}(S_\nu) \)),
- \( ev_{h_0} : h_0 \otimes h_0^* \to 1 \) is the evaluation map in category \( \text{Rep}(S_\nu) \),
- \( \Omega \) is the endomorphism of the identity functor of \( \text{Rep}(S_\nu) \) described in \([4, 3]\) and the notation \( \Omega^{i_1 \cdots i_k} \) signifies the endomorphism which acts as \( \Omega \) on the factors \( i_1, ..., i_k \) of the tensor product, and as \( \text{Id} \) on the rest.

Morphisms in \( \text{Rep}(H_c(\nu)) \) are defined to be morphisms of \( M \) in \( \text{ind} - \text{Rep}(S_\nu) \) compatible with maps \( x, y \).
Remark 5.0.2. As in the classical Cherednik algebra, we can replace $\mathfrak{h}_0$ with \mathfrak{h} in the above definition, and get an equivalent definition of the category $\text{Rep}(H_c(\nu))$.

6. Category $O_{c,\nu}$

6.1. Definition of $O_{c,\nu}$. Consider the algebra objects $S\mathfrak{h}_0^*$, $S\mathfrak{h}_0$ in $\text{ind} - \text{Rep}(S_{\nu})$.

Given maps $x, y$ as above, we get maps

$$\overline{x}_M : S\mathfrak{h}_0^* \otimes M \longrightarrow M$$
$$\overline{y}_M : S\mathfrak{h}_0 \otimes M \rightarrow M$$

So we can speak of $M$ as an $S\mathfrak{h}_0^*$-module and an $S\mathfrak{h}_0$-module in $\text{ind} - \text{Rep}(S_{\nu})$.

Definition 6.1.1. The category $O_{c,\nu}$ is defined as the full subcategory of $\text{Rep}(H_c(\nu))$ whose objects are $(M, x, y)$ such that

- $M$ is finitely generated over $S\mathfrak{h}_0^*$ in $\text{Rep}(S_{\nu})$ (i.e. $M$ is a quotient of a “finitely generated free $S\mathfrak{h}_0^*$ module” $S\mathfrak{h}_0^* \otimes V$, where $V$ is an object of $\text{Rep}(S_{\nu})$).
- $M$ is locally nilpotent over $\mathfrak{h}_0$, in the following sense:
  
  for any $\text{Rep}(S_{\nu})$-subobject $T \subset M$, there exists a non-negative integer $m$ (depending on $T$) such that the map $S^m\mathfrak{h}_0 \otimes T \rightarrow M$ (this is the restriction to $T$ of the map $S^m\mathfrak{h}_0 \otimes M \rightarrow M$) equals zero.

Definition 6.1.2. Define $h \in \text{End}(\text{Forget}_{O_{c,\nu} \rightarrow \text{Rep}(S_{\nu})})$ by letting $h_M := \frac{1}{2}(\psi_1 + \psi_2)$ for every $M \in O_{c,\nu}$, where $\psi_1, \psi_2 : M \rightarrow M$,

$$\psi_1 = x \circ (\text{Id} \otimes y) \circ (\text{coev}_{\mathfrak{h}_0^*} \otimes \text{Id}_M) : M \rightarrow \mathfrak{h}^* \otimes \mathfrak{h} \otimes M \rightarrow M$$
$$\psi_2 = y \circ (\text{Id} \otimes x) \circ (\text{coev}_{\mathfrak{h}} \otimes \text{Id}_M) : M \rightarrow \mathfrak{h} \otimes \mathfrak{h}^* \otimes M \rightarrow M$$

Similarly to the integer case, we have:

Lemma 6.1.3.

$$y_M \circ (\text{Id}_{\mathfrak{h}_0} \otimes h) - h \circ y_M = y_M : \mathfrak{h}_0 \otimes M \rightarrow M$$
$$x_M \circ (\text{Id}_{\mathfrak{h}_0^*} \otimes h) - h \circ x_M = -x_M : \mathfrak{h}_0^* \otimes M \rightarrow M$$

Definition 6.1.4 (Indecomposable singular subobject). Let $U \in O_{c,\nu}, V \in \text{Rep}(S_{\nu})$ so that $V \subset U$ and the map $y_U$ restricted to $\mathfrak{h}_0 \otimes V$ is zero. Then $V$ is called “singular in $U$”. If, furthermore, $V$ is an indecomposable (resp. simple) object in $\text{Rep}(S_{\nu})$, then we say that $V$ is an indecomposable (resp. simple) singular $\text{Rep}(S_{\nu})$-subobject of $U$.

Remark 6.1.5. Note that by definition of $O_{c,\nu}$, each object contains an indecomposable singular $\text{Rep}(S_{\nu})$-subobject.

6.2. Verma objects in category $O_{c,\nu}$. One can define Verma objects in $O_{c,\nu}$ category as follows:

Consider the category $\text{Rep}(B_{\nu})$ of pairs $(M, y_M)$, where $M$ is an ind-object of $\text{Rep}(S_{\nu})$,

$$y : \mathfrak{h}_0 \otimes M \rightarrow M$$

is a morphism in $\text{Rep}(S_{\nu})$ satisfying the condition: the morphism

$$y \circ (\text{Id} \otimes y) \circ ((\text{Id} - \sigma_{\mathfrak{h}_0,\mathfrak{h}_0}) \otimes \text{Id}) : \mathfrak{h}_0 \otimes \mathfrak{h}_0 \otimes M \rightarrow M$$

is zero. The morphisms in $\text{Rep}(B_{\nu})$ are morphisms of $\text{Rep}(S_{\nu})$ ind-objects compatible with the $y_M$ maps.

This is the analogue of the category of representations of the “Borel” subalgebra $B_n = \mathbb{C}[S_n] \ltimes S\mathfrak{h}_0$. 

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Then we have the restriction functor \( Res : \text{Rep}(H_c(\nu)) \rightarrow \text{Rep}(B_\nu) \), and it has a left adjoint which is the induction functor \( Ind : \text{Rep}(B_\nu) \rightarrow \text{Rep}(H_c(\nu)) \). The induction functor takes a pair \((M, y_M)\) to a triple \((S_{\nu}^0 \otimes M, y_{S_{\nu}^0 \otimes M}, x_{S_{\nu}^0 \otimes M})\), where the map \(x_{S_{\nu}^0 \otimes M} : h_0^* \otimes S_{\nu}^0 \otimes M \rightarrow S_{\nu}^0 \otimes M\) is the multiplication map of \(h_0^* \otimes S_{\nu}^0 \rightarrow S_{\nu}^0\) tensored with \(\text{Id}_{M}\), and \(y_{S_{\nu}^0 \otimes M} : h_0 \otimes S_{\nu}^0 \otimes M \rightarrow S_{\nu}^0 \otimes M\) is a map satisfying conditions (1,3) of Definition 5.0.1, so that \(y_{S_{\nu}^0 \otimes M} |_{h_0 \otimes S_{\nu}^0 \otimes M} = y_M\).

**Definition 6.2.1** (Verma Object). Consider an indecomposable object \(X_\tau\) in \(\text{Rep}(S_\nu)\), and let the map \(y_{X_\tau} : h_0 \otimes X_\tau \rightarrow X_\tau\) be zero. This makes \((X_\tau, y_{X_\tau})\) an object of \(\text{Rep}(B_\nu)\).

Define the Verma object of lowest weight \(X_\tau\) as \(\text{Ind}(X_\tau, y_{X_\tau})\) (so it is an ind-object of \(\text{Rep}(S_\nu)\)). It will be denoted by \(M_{c,\nu}(\tau)\) (or, if \(c, \nu\) are fixed, \(M(\tau)\) for short).

**Remark 6.2.2.** It is easy to see that as a \(\text{Rep}(S_\nu)\) ind-object \(M_{c,\nu}(\tau) = S_{\nu}^0 \otimes X_\tau\), where \(X_\tau\) is an indecomposable object of \(\text{Rep}(S_\nu)\).

The map \(\mathfrak{c}_{M_{c,\nu}(\tau)} : S_{\nu}^0 \otimes S_{\nu}^0 \otimes X_\tau \rightarrow S_{\nu}^0 \otimes X_\tau\) is the multiplication map of \(S_{\nu}^0\) tensored with \(\text{Id}_{X_\tau}\).

The map \(\mathfrak{c}_{M_{c,\nu}(\tau)} : S_{\nu}^0 \otimes S_{\nu}^0 \otimes X_\tau \rightarrow S_{\nu}^0 \otimes X_\tau\) is the map for which condition (3) of Definition 5.0.1 holds, and thus is a deformation of the differentiation map \(S_{\nu}^0 \otimes S_{\nu}^0 \rightarrow S_{\nu}^0\) tensored with \(\text{Id}_{X_\tau}\).

**Notation 6.2.3.** To avoid confusion, the Verma module in \(O(H_c(\nu))\) whose lowest weight corresponds to the Young diagram \(\tilde{\tau}(n)\) is denoted by \(M_{c,\nu}(\tilde{\tau}(n))\), while the Verma object in \(O_{c,\nu=n}\) corresponding to the Young diagram \(\tau\) is denoted by \(M_{c,\nu=n}(\tau)\).

**Definition 6.2.4.** Let

\[
\text{h}_{c,\nu}(\tau) := \frac{\dim(h_0)}{2} - c \cdot \left( (\nu - |\tau|)(\nu - |\tau| - 1) - |\tau| + c t(\tau) \right)
\]

Note that given \(X_\mu \in \text{Rep}(S_\nu)\) such that \(\nu \notin \{0, 1, ..., |\mu| + \mu_1\}\), we have

\[
\text{h}_{c,\nu}(\tau) = \frac{\dim(h_0)}{2} \text{Id} - c \Omega |_{X_\tau}
\]

(here \(\Omega\) is the the endomorphism of the identity functor of \(\text{Rep}(S_\nu)\) described in Section 1.3).

**Example 6.2.5.**

- \(ct(\tau^{s-1}) = \frac{s(s-1)}{2}\), so \(\text{h}_{c,\nu}(\tau^s) = \frac{\dim(h_0)}{2} - c \cdot \left( \frac{s(s-1)}{2} - s \nu + s^2 - s \right)\).
- \(ct(\tau^n) = -\frac{n(n-1)}{2}\), so \(\text{h}_{c,\nu}(\tau^n) = \frac{\dim(h_0)}{2} - c \cdot \left( \frac{n(n-1)}{2} - n \nu \right)\).

**Proposition 6.2.6.** For \(M = M_{c,\nu}(\tau)\), \(h : M_{c,\nu}(\tau) \rightarrow M_{c,\nu}(\tau)\) is an endomorphism of \(M_{c,\nu}(\tau) = \bigoplus_{m \geq 0} S^m h_0^* \otimes X_\tau\) viewed as an ind-object of \(\text{Rep}(S_\nu)\) such that \(h = \bigoplus_{m \geq 0} \phi_m\), where \(\phi_m : S^m h_0^* \otimes X_\tau \rightarrow S^m h_0^* \otimes X_\tau\), \(\phi_0 = \frac{\dim(h_0)}{2} \text{Id} - c \Omega |_{X_\tau}\) and the endomorphism \((\phi_m - (\text{h}_{c,\nu}(\tau) + m) \text{Id})\) of \(S^m h_0^* \otimes X_\tau\) is nilpotent.

**Proof.** Follows immediately from Property [6.3] of the element \(h\). \(\square\)

This proposition means that \(M(\tau)\) (as a \(\text{Rep}(S_\nu)\) ind-object) has a grading by eigenvalues of \(h\) which is the natural \(\mathbb{Z}_+\)-grading on \(M(\tau)\) shifted by \(\text{h}_{c,\nu}(\tau)\).

Now let \(U\) be a subquotient of \(M(\tau)\). Then \(U\) automatically inherits a grading from the grading of \(M(\tau)\) by eigenvalues of \(h\), and the definition of \(\tau\) implies that maps between subquotients of Verma objects preserve this grading.

For simplicity, we will use the natural \(\mathbb{Z}_+\)-grading on \(M(\tau)\) when we refer to the degree in which a \(\text{Rep}(S_\nu)\)-object lies in \(U\).
Proposition 6.2.7. Let $U \in O_{c,\nu}$. Let $X_{\mu}$ be an indecomposable singular $\text{Rep}(S_\nu)$-subobject in $U$. Then there exists a morphism $M_{c,\nu}(\mu) \rightarrow U$ in $O_{c,\nu}$, inducing $\text{Id}_{X_{\mu}}$ on $X_{\mu}$ when regarded as a morphism of $\text{Rep}(S_\nu)$ ind-objects.

Proof. Recall the definition of a Verma object. $X_{\mu}$ being an indecomposable singular $\text{Rep}(S_\nu)$-subobject in $U$ means that $\text{Hom}_{\text{Rep}(B_\nu)}(X_{\mu}, \text{Res}(U)) \neq 0$, where $X_{\mu}$ is considered as a $\text{Rep}(B_\nu)$-object with $y_{x_{\mu}} = 0$, and there exists a non-zero map in $\text{Hom}_{\text{Rep}(B_\nu)}(X_{\mu}, \text{Res}(U))$ whose image is the chosen copy of $X_{\mu}$ in $U$. Then by definition of a Verma object, we have: $\text{Hom}_{\text{Rep}(B_\nu)}(X_{\mu}, \text{Res}(U)) = \text{Hom}_{O_{c,\nu}}(M_{c,\nu}(\mu), U) \neq 0$ and there exists a non-zero map in $\text{Hom}_{O_{c,\nu}}(M_{c,\nu}(\mu), U)$ inducing $\text{Id}_{X_{\mu}}$ on $X_{\mu}$ when regarded as a morphism of $\text{Rep}(S_\nu)$ ind-objects. □

Proposition 6.2.8. Assume $\nu \notin \mathbb{Z}_+$. Then

- Each Verma object $M_{c,\nu}(\tau)$ has a maximal proper $O_{c,\nu}$-subobject $J$, i.e. it has a unique simple quotient $L_{c,\nu}(\tau)$ in $O_{c,\nu}$.
- The simple objects of $O_{c,\nu}$ category are exactly $L_{c,\nu}(\tau)$.

Proof.

- Let $J$ be the sum of all the proper $O_{c,\nu}$-subobjects of $M_{c,\nu}(\tau)$. For any proper $O_{c,\nu}$-subobject $N$ of $M_{c,\nu}(\tau)$, $N$ inherits the grading from $M_{c,\nu}(\tau)$ (as a $\text{Rep}(S_\nu)$ ind-object), with $L$ acting on grade $m$ of $N$ by $(h_{c,\nu}(\tau) + m)\text{Id}$. Since $N$ is a proper subobject, we have $m > 0$. So $J$ can be presented as a direct sum of $\text{Rep}(S_\nu)$-objects on which the restriction of $h$ acts by scalars strictly larger than $h_{c,\nu}(\tau)$. This proves that $J$ is a proper $O_{c,\nu}$-subobject of $M_{c,\nu}(\tau)$.
- Let $L$ be a simple object of $O_{c,\nu}$. Then $L$ has a simple singular $\text{Rep}(S_\nu)$-subobject $X_\tau$. By Proposition 6.2.7, there exists a non-zero map $M(\tau) \rightarrow L$, with induced map on $X_\tau$ being $\text{Id}_\tau$. But $L$ is simple, so $M(\tau) \rightarrow L$ is surjective. □

Remark 6.2.9. Let $M_{c,\nu}(\tau)$ be a Verma object in $O_{c,\nu}$. By the propositions above, to check its reducibility, we only need to check whether there are any non-zero morphisms from other Verma objects to $M_{c,\nu}(\tau)$.

7. Functor $O_{c,\nu} \rightarrow O(H_c(n))$

In this section we construct a functor $F_{c,n} : \text{Rep}(H_c(\nu)) \rightarrow \text{Rep}(H_c(n))$ restricting to $F_{c,n} : O_{c,\nu} \rightarrow O(H_c(n))$ for any $c \in \mathbb{C}, n \in \mathbb{Z}_+$. This functor is analogous to (and based on) the functor $F_n : \text{Rep}(S_\nu) \rightarrow \text{Rep}(S_n)$ which induces the equivalence $\text{Rep}(S_\nu)/\mathcal{J}_\nu \cong \text{Rep}(S_n)$ discussed in Section 4.

The functor $F_{c,n}$ is constructed as follows: let $(M, x, y) \in \text{Rep}(H_c(\nu))$ (see Section 5), and consider $F_n(M) \in \text{ind-Rep}(S_n)$. The maps $F_n(x), F_n(y)$ then make $F_n(M)$ a module over the algebra $CS_n \rtimes (\mathfrak{h} \oplus \mathfrak{h}^*)$, and the conditions on the maps $x, y$ given in Section 5 imply that this action factors through $H_c(n)$, thus making $F_n(M)$ an $H_c(n)$-module.

Recall that a morphism $\phi : (M_1, x_1, y_1) \rightarrow (M_2, x_2, y_2)$ in $\text{Rep}(H_c(\nu))$ is by definition a morphism $\phi : M_1 \rightarrow M_2$ of $\text{Rep}(S_\nu)$ ind-objects such that $x_2 \circ (\text{Id}_\mathfrak{h} \circ \otimes) = \phi \circ x_1$ and $y_2 \circ (\text{Id}_\mathfrak{h} \circ \otimes) = \phi \circ y_1$. So $F_n(\phi)$ is automatically a morphism of $H_c(n)$-modules.

Thus we define

$$F_{c,n}(M, x, y) := F_n(M) \text{ with action of } H_c(n) \text{ given by } F_n(x), F_n(y)$$

$$F_{c,n}(\phi) := F_n(\phi)$$

To see that $F_{c,n}$ restricts to $O_{c,\nu} \rightarrow O(H_c(n))$, note that $F_n(S\mathfrak{h}) = S\mathfrak{h}, F_n(S\mathfrak{h}^*) = S\mathfrak{h}^*$. The definitions of $O_{c,\nu}$, $O(H_c(n))$ then imply that $F_{c,n}(O_{c,\nu}) \subset O(H_c(n))$. In
particular, if $|\lambda| + \lambda_1 \leq n$ then the Verma object of lowest weight $\lambda$ goes to the Verma module of lowest weight $\tilde{\lambda}(n)$.

Finally, a lemma which will be useful to us later on:

**Lemma 7.0.10.** Fix Young diagrams $\mu, \tau$, and let $n \gg 0$ (in fact, it is enough that $n > 2 \max(|\mu|, |\tau|) + 1$). Then

$$\mathcal{F}_{c,n} : \text{Hom}_{O_c,\nu=n}(M_{c,\nu=\mu}(\mu), M_{c,\nu=\mu}(\tau)) \rightarrow \text{Hom}_O(H_c(n))(\tilde{\mu}(n), \tilde{\tau}(n))$$

is a bijection.

**Proof.** Recall that we have:

$$\text{Hom}_{O_c,\nu=n}(M_{c,\nu=\mu}(\mu), M_{c,\nu=\mu}(\tau)) \cong \{ \phi \in \text{Hom}_{\text{Rep}(S_{\nu=\mu})}(X_{\mu}, M_{c,\nu=\mu}(\tau)) : y_{M_{c,\nu=\mu}(\tau)}|_{h_0 \otimes \phi(X_{\mu})} = 0 \}$$

and

$$\text{Hom}_O(H_c(n))(\tilde{\mu}(n), \tilde{\tau}(n)) \cong \{ \psi \in \text{Hom}_{\text{Rep}(S_{\nu=\mu})}(\tilde{\mu}(n), M_{c,\nu=\mu}(\tilde{\tau}(n))) : \mathcal{F}_{c,n}(y_{M_{c,\nu=\mu}(\tau)})|_{h_0 \otimes \psi(\tilde{\mu}(n))} = 0 \}$$

It remains to check that

$$\mathcal{F}_{c,n} : \{ \phi \in \text{Hom}_{\text{Rep}(S_{\nu=\mu})}(X_{\mu}, M_{c,\nu=\mu}(\tau)) : y_{M_{c,\nu=\mu}(\tau)}|_{h_0 \otimes \phi(X_{\mu})} = 0 \} \rightarrow \{ \psi \in \text{Hom}_{\text{Rep}(S_{\nu=\mu})}(\tilde{\mu}(n), M_{c,\nu=\mu}(\tilde{\tau}(n))) : \mathcal{F}_{c,n}(y_{M_{c,\nu=\mu}(\tau)})|_{h_0 \otimes \psi(\tilde{\mu}(n))} = 0 \}$$

is an isomorphism. Let $\psi \in \text{Hom}_{\text{Rep}(S_{\nu=\mu})}(\tilde{\mu}(n), M_{c,\nu=\mu}(\tilde{\tau}(n)))$ such that

$$\mathcal{F}_{c,n}(y_{M_{c,\nu=\mu}(\tau)})|_{h_0 \otimes \psi(\tilde{\mu}(n))} = 0.$$

Since $\mathcal{F}_{c,n}$ is surjective on Hom-spaces, there is a $\phi \in \text{Hom}_{\text{Rep}(S_{\nu=\mu})}(X_{\mu}, M_{c,\nu=\mu}(\tau))$ such that $\mathcal{F}_{c,n}(\phi) = \psi$. We need to check that there exists a unique such $\phi$ so that

$$y_{M_{c,\nu=\mu}(\tau)}|_{h_0 \otimes \phi(X_{\mu})} = 0.$$

Let $m \in \mathbb{Z}_+$ be such that $\psi(\tilde{\mu}(n)) \subset S^m h_0^* \otimes \tilde{\tau}(n) \subset M_{c,\nu=\mu}(\tilde{\tau}(n))$. Recall that

$$\mathcal{F}_{c,n}(y|_{h_0 \otimes \phi(X_{\mu})}) = y_n|_{h_0 \otimes \tilde{\mu}(n)} = 0$$

so $y|_{h_0 \otimes \phi(X_{\mu})} : h_0 \otimes X_{\mu} \rightarrow S^m h_0^* \otimes X_{\tau}$ is a negligible morphism in $\text{Rep}(S_{\nu=\mu})$. But since $n$ is assumed to be very large, $h_0 \otimes \phi(X_{\mu}), S^m h_0^* \otimes X_{\tau} \in \text{Rep}(S_{\nu=\mu})^{(\nu/2)}$, which is a semisimple subcategory of $\text{Rep}(S_{\nu=\mu})$ (see Section 8). So a negligible morphism in this subcategory is zero, and thus $y|_{h_0 \otimes \phi(X_{\mu})} = 0$.

Similarly, the fact that $X_{\mu}, S^m h_0^* \otimes X_{\tau} \in \text{Rep}(S_{\nu=\mu})^{(\nu/2)}$ implies that such a morphism $\phi$ is unique.

\[\square\]

8. **Morphisms between Verma objects: general remarks and necessary conditions**

From now on, we will assume for simplicity that $\nu \notin \mathbb{Z}_+$, unless stated otherwise.

8.1. **Necessary conditions.** The following formula will serve as one of the main tools of this paper.

**Proposition 8.1.1.** Let $\mu$ be a partition and $m > 0$ an integer, and assume there is a non-zero morphism $M_{c,\nu}(\mu) \rightarrow M_{c,\nu}(\tau)$ such that $X_{\mu}$ (image of the lowest weight of $M(\mu)$) sits in degree $m$ of $M_{c,\nu}(\tau)$. Then

$$c \left( \frac{|\tau|^2 - |\mu|^2 - (|\tau| - |\mu|)}{2} + ct(\tau) - ct(\mu) \right) = m + (|\tau| - |\mu|) c \nu$$

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Remark 8.1.8. non-trivial maps $M$.

Proof. Considering the endomorphisms $h$ of objects $M_{c,v}(\mu), M_{c,v}(\tau)$, we get:

If there exists a non-trivial morphism $M_{c,v}(\mu) \longrightarrow M_{c,v}(\tau)$, with the image of the lowest weight $X_\mu$ of $M_{c,v}(\mu)$ lying in degree $m$ of $M_{c,v}(\tau)$, then $h_{c,v}(\tau) + m = h_{c,v}(\mu)$
i.e.

$$\frac{\nu - 1}{2} - c \cdot \left( \frac{(\nu - |\tau|)(\nu - |\tau| - 1)}{2} - |\tau| + ct(\tau) \right) + m =$$

$$= \frac{\nu - 1}{2} - c \cdot \left( \frac{(\nu - |\mu|)(\nu - |\mu| - 1)}{2} - |\mu| + ct(\mu) \right)$$

which can be rewritten as

$$c \cdot \left( \frac{|\tau|^2 - |\mu|^2 - (|\tau| - |\mu|)}{2} - (|\tau| - |\mu|)\nu + ct(\tau) - ct(\mu) \right) = m$$

$\square$

Remark 8.1.2. Note that if $c = 0$, Equation (8.1.1) implies that $m = 0$, $\mu = \tau$ and the morphism $M(\mu) \longrightarrow M(\tau)$ is the identity map. This means that for $c = 0$, all the Verma objects are simple. The category $O(H_0(\nu))$ is a continuation of the categories of modules over $\mathbb{C}S_\nu \times A_n$, which are $O$-coherent $D$-modules and whose Fourier transform has support $\{0\}$. Here $A_n$ is the $n$-th Weyl algebra (the algebra of differential operators on $\mathbb{C}^n$).

From now on, we will assume that $c \neq 0$ and denote:

Notation 8.1.3. $c' := \frac{1}{c}$.

Remark 8.1.4. In this notation, Equation (8.1.1) can be rewritten as

$$\frac{|\tau|^2 - |\mu|^2 - (|\tau| - |\mu|)}{2} + ct(\tau) - ct(\mu) = mc' + (|\tau| - |\mu|)\nu$$

Notation 8.1.5. Let $\tau, \mu$ be Young diagrams, and $m$ be a positive integer. Denote by $L_{\tau,\mu,m}$ the set of points $(c', \nu)$ in $\mathbb{C}^2$ satisfying the Equation (8.1.4).

The above proposition shows (with the notations as in Notation 1.1.1): $B_{\tau,\mu} \subset \bigcup_{m \in \mathbb{Z}_{>0}} L_{\tau,\mu,m}$.

Example 8.1.6. $L_{\tau,\tau,0} = \mathbb{C}^2$ and $L_{\tau,\mu,m} = \emptyset$ for any Young diagram $\tau$ and any $m \in \mathbb{Z}_{>0}$.

Notation 8.1.7. Let $\tau, \mu$ be Young diagrams, and $m$ be a positive integer. Denote: $B_{\tau,\mu,m} := L_{\tau,\mu,m} \cap B_{\tau,\mu}$.

Remark 8.1.8. Note that Proposition 8.1.1 implies the following statement:

Fix $c, \nu$, and consider the lowest weight $X_\mu$ of the Verma object $M_{c,v}(\mu)$. Then for all non-trivial maps $M_{c,v}(\mu) \longrightarrow M_{c,v}(\tau)$ the degree $m$ of $M_{c,v}(\tau)$ in which $X_\mu$ sits is the same, since it is given by Equation (8.1.4).

As a special case of Proposition 8.1.1 we have the following lemma (for its statement and proof for the classical Cherednik algebra, see [10, Lemma 3.5]):

Lemma 8.1.9. Let $X_\mu \subset h_0^* \otimes X_\tau$ in $\text{Rep}(S_\nu)$. We can regard $X_\mu$ as sitting in degree 1 of the $O_{c,v}$-object $M_{c,v}(\tau)$. Then there is a morphism $M_{c,v}(\mu) \longrightarrow M_{c,v}(\tau)$ in $O_{c,v}$, inducing $\text{Id}_{X_\mu}$ on $X_\mu$, if and only if

$$\frac{|\tau|^2 - |\mu|^2 - (|\tau| - |\mu|)}{2} + ct(\tau) - ct(\mu) = c' + (|\tau| - |\mu|)\nu$$
Proof. Consider the morphism \( Y : \mathfrak{h}_0^* \otimes X \tau \longrightarrow \mathfrak{h}_0^* \otimes X \tau \) which is equivalent to \( y \mid_{\mathfrak{h}_0^* \otimes X \tau} : \mathfrak{h}_0 \otimes \mathfrak{h}_0^* \otimes X \tau \longrightarrow X \tau \) under the isomorphism \( \text{End}(\mathfrak{h}_0^* \otimes X \tau) \cong \text{Hom}(\mathfrak{h}_0 \otimes \mathfrak{h}_0^* \otimes X \tau, X \tau) \).

In \( O(H_c(n)) \), we have:

\[
Y = 1 - c \sum_{s \in S} 1 \otimes s + c \sum_{s \in S} s \otimes s
\]

so in our case, we would have:

\[
Y = 1 - c\Omega^2 + c\Omega^{1,2} = 1 + h_{c,\nu}(\tau) - \frac{\nu - 1}{2} + c\Omega^{1,2}
\]

(\( \Omega \) as in Definition 5.0.1). Thus \( Y \) acts on a simple object \( X_\mu \subset \mathfrak{h}_0 \otimes X \tau \) by

\[
1 + h_{c,\nu}(\tau) - \frac{\nu - 1}{2} - \left( h_{c,\nu}(\mu) - \frac{\nu - 1}{2} \right) = 1 + h_{c,\nu}(\tau) - h_{c,\nu}(\mu)
\]

This proves that \( X_\mu \) is a simple singular \( \text{Rep}(S_c) \)-subobject of \( M(\tau) \) iff \( 1 + h_{c,\nu}(\tau) - h_{c,\nu}(\mu) = 0 \), which is equivalent to

\[
\frac{|\tau|^2 - |\mu|^2 - (|\tau| - |\mu|)}{2} + ct(\tau) - ct(\mu) = c' + (|\tau| - |\mu|)\nu
\]

\( \square \)

An additional condition for \( X_\mu \) to sit in degree \( m \) of \( M_{c,\nu}(\tau) \) arises from Pieri’s rule (cf. Proposition 1.2.1).

In terms of Notations 8.1.5 8.1.7 8.1.9, Lemma 8.1.9 means that \( \mathcal{B}_{\tau,\mu,1} = \mathcal{L}_{\tau,\mu,1} \) whenever \( \mu \) is obtained from \( \tau \) as in Pieri’s rule, i.e., by adding/moving/deleting a cell, or if \( \mu = \tau \).

### 8.2. Remarks on Equation (8.1.4)

For a Young diagram \( \mu \), denote:

\[
f(\mu) := \frac{|\mu|^2 - |\mu|}{2} + ct(\mu)
\]

Then Equation (8.1.4) can be rewritten as

\[
(1) \quad f(\tau) - f(\mu) = c'm + (|\tau| - |\mu|)\nu
\]

**Lemma 8.2.1.** Let \( \mu \) be a Young diagram. Then \( f(\mu) \) is a non-negative integer less or equal to \( |\mu|^2 - |\mu| \), and it is equal to zero if \( \mu \) is a column diagram (i.e. \( l(\mu) \leq 1 \)), and to \( |\mu|^2 - |\mu| \) if \( \mu \) is a row diagram (i.e. \( l(\mu) \leq 1 \)).

**Proof.** This is equivalent to saying that \( |ct(\mu)| \leq \frac{|\mu|^2 - |\mu|}{2} \). The latter can be proved by induction on the number of columns of \( \mu \):

Base: Assume the number of columns of \( \mu \) is zero, i.e. \( \mu = \emptyset \). Then \( |ct(\mu)| = 0 = \frac{|\mu|^2 - |\mu|}{2} \).

Also, if \( \mu \) is a column diagram, then \( ct(\mu) = -\frac{|\mu|^2 - |\mu|}{2} \).

Step: Denote by \( k \) the number of columns of \( \mu \) (\( k > 1 \)), by \( \mu' \) the diagram \( \mu \) without the last column, and by \( l \) the number of boxes in the last column of \( \mu \). By induction assumption, \( |ct(\mu')| \leq \frac{|\mu'|^2 - |\mu'|}{2} \). Next, \( ct(\mu) = ct(\mu') + (k - 1) \cdot l - l(l - 1)/2 \), and so \( \frac{|\mu|^2 - |\mu|}{2} = \frac{|\mu'|^2 - |\mu'|}{2} + |\mu'| l + \frac{l^2 - l}{2} \geq |ct(\mu')| + \frac{l^2 - l}{2} + |\mu'| l \geq |ct(\mu')| + (k - 1) \cdot l - l(l - 1)/2 = |ct(\mu)| \) (for the last inequality, note that by definition, \( |\mu'| \geq k - 1 \), with equality if and only if \( \mu \) is a row diagram). For a row diagram \( \mu \) (with \( k \) cells), there is an equality \( ct(\mu) = \frac{|\mu|^2 - |\mu|}{2} \), and in general, \( |ct(\mu)| < \frac{|\mu|^2 - |\mu|}{2} \) for \( \mu \) having \( k > 1 \) columns and not a row diagram.

\( \square \)
For the right hand side of Equation (1), note that we have, by Pieri’s rule (Proposition 12.1), \( m \geq ||\mu| - |\tau||. \)

9. Blocks of category \( O(H_c(n)) \) (classical case)

9.1. KZ functor and connection to the representations of the Hecke algebra.

A powerful tool in studying the \( O \) category for Cherednik algebra is the KZ functor (see e.g. [14, 2.8.2], [9, Section 6], [2, Section 3.3]). The KZ functor is a functor \( O(H_c(n)) \to \text{Rep}(\mathcal{H}_q(n)) \), \( \text{Rep}(\mathcal{H}_q(n)) \) being the category of representations of the Hecke algebra of type A, where \( q = \exp(2\pi ic) \).

It turns out that this functor is essentially surjective on objects, surjective on Homs and exact (see [12]). This functor induces an equivalence of categories \( \mathcal{KZ} : O/\mathcal{O}_\text{tor} \to \text{Rep}(\mathcal{H}_q(n)) \), where \( \mathcal{O}_\text{tor} \) is a full subcategory of the \( O \) category whose objects are modules which, when considered as \( \mathbb{C}[h_0] \)-modules, have Krull dimension less than \( n - 1 \) (see [9, Section 6.3]). The non-zero objects of \( O/\mathcal{O}_\text{tor} \) would thus be \( H_c(n) \)-modules which are supported on the whole \( \mathbb{C}^n \).

Moreover, by the result mentioned in [2, Proposition 3.4], the KZ functor is faithful on the full subcategory of Verma modules in \( O(H_c(n)) \).

We will use the fact that the KZ functor takes \( M(\lambda) \) to \( S_{\lambda}^\natural \) (the dual of the Specht module) if \( c \geq 0 \), and to \( S_{\lambda} \) (Specht module) if \( c < 0 \). For \( c < 0 \), the module \( L(\lambda) \) goes to \( D_{\lambda} \) (which is either the unique simple quotient of the Specht module \( S_{\lambda} \), if it exists, or zero).

9.2. Representations of Hecke algebra of type A. We will use the following facts about the representations of Hecke algebra of type A (see [18], [2, Section 3]):

Some definitions:

Notation 9.2.1. Denote \( e := \text{ord}_C(q) \) (i.e. \( e \) is the order of \( q \) if \( q \) is a root of unity and \( \infty \) otherwise), and denote \( s := n - e. \)

First of all, we have:

Theorem 9.2.2. If \( e > n \) (in particular, if \( e = \infty \)), then the category \( \text{Rep}(\mathcal{H}_q(n)) \) is semisimple.

From now on, we will assume \( e \leq n \), and thus \( 0 \leq s \leq n \).

Definition 9.2.3 (\( e \)-hook).

- Hooks are parameterized by cells \((i, j)\) in the diagram; a hook corresponding to \((i, j)\) consists of all the cells \((i', j')\) such that either \( i' = i, j' \geq j \), or \( i' \geq i, j' = j \). The cell \((i, j)\) is the “vertex” of the hook corresponding to it. Cells \((i, j'), j' > j\) are called the “arm” of the hook, while cells \((i', j), i' > i\) are called the “leg” of the hook.

- An \( e \)-hook of a Young diagram is a hook of length \( e \) (i.e. contains exactly \( e \) cells).

Example 9.2.4. Let \( \lambda = (5, 4, 2, 2) \). The hook corresponding to \((1, 2)\) is a 7-hook with arm 3 and leg 3:

\begin{center}
\begin{tabular}{|c|c|c|c|}
\hline
\( \circ \) & \( \circ \) & \( \circ \) & \( \circ \) \\
\hline
\( \circ \) & \( \circ \) & \( \circ \) & \( \circ \) \\
\hline
\end{tabular}
\end{center}

Lemma 9.2.5. For \( n \) large relatively to \( s \) (namely, \( n > 2s, n > 1 \)), a Young diagram \( \tau \) with \( |\tau| = n \) can have at most one \( e \)-hook.
Proof. Indeed, given a Young diagram \( \tau \) with \( |\tau| = n \), if there are two distinct \( e \)-hooks in \( \tau \), then their intersection contains at most one cell, and this gives us the inequality \( 2e - 1 \leq |\tau| = n \), and thus \( n \leq 2s + 1 \). Note that, in fact, a Young diagram cannot consist of two hooks of equal size intersecting each other in one cell unless this diagram contains just one cell. So if \( n \geq 2 \), then \( 2e - 1 < |\tau| = n \), i.e. \( n < 2s + 1 \). So for \( n > 2s, n > 1 \), there is at most one \( e \)-hook in \( \tau \). \( \square 

Definition 9.2.6 (Core). Let \( s \) be a non-negative integer. The \((n - s)\)-core of a Young diagram \( \tau \) is the Young diagram obtained by performing the following procedure on \( \lambda := \tau \):

1. Take the Young diagram \( \lambda \).
2. If \( \lambda \) has no \((n - s)\)-hook, stop. The diagram \( \lambda \) is then the \((n - s)\)-core of \( \tau \).
3. Otherwise, remove an \((n - s)\)-hook from \( \lambda \) and move the boxes underneath this \((n - s)\)-hook one position up and one position to the left. Denote the obtained Young diagram by \( \lambda^{(1)} \).
4. Repeat the procedure for \( \lambda := \lambda^{(1)} \).

So for \( n > \max(2s, 1) \), either \( \text{core}_{(n-s)}(\tau) = \tau \), or \( \text{core}_{(n-s)}(\tau) \) has \( s \) boxes.

Example 9.2.7. Let \( \lambda = (5, 4, 2, 2) \). Its 7-core is obtained by removing the hook corresponding to (1, 2) and moving the boxes below it one space up and left, i.e. \( \text{core}_7(\lambda) = (3, 1, 1, 1) \).

Now, assume \( n > \max(2s, 1) \). Classification of partitions of size \( n \) with given core \( \lambda \) such that \( |\lambda| = s \) is given by the following proposition:

Proposition 9.2.8. Let \( \text{hook}(l, (n - s)) \) be the \((n - s)\)-hook with leg \( l \): \((n - s) - l, 1, 1, 1, ..., 1 \). Then there is a unique Young diagram \( \text{rec}(l, \lambda) \) of size \( n \) such that \( \lambda \) is obtained by deleting the hook \( \text{hook}(l, n - s) \) from \( \text{rec}(l, \lambda) \).

Proof. Basically, \( \text{rec}(l, \lambda) \) is obtained by “inserting” a hook \( \text{hook}(l, (n - s)) \) to \( \lambda \) in a certain not completely trivial way. To insert a hook \( \text{hook}(l, (n - s)) \) into a diagram \( \lambda \), one should put its vertex in the position \((i, j)\) such that the following equations are satisfied:

\[
\begin{align*}
\lambda_i + 1 &\leq j - 1 + n - s - l \\
&\leq \lambda_{i-1} \\
\lambda_j + 1 &\leq i + l \leq \lambda_{j-1}
\end{align*}
\]

\((\lambda_{0}, \lambda_{0} := \infty)\).

We now explain how to find such \( i, j \).

First, we show that if the inequalities \( (2) \) hold, then \( i = 1 \) or \( j = 1 \). Indeed, adding the two inequalities \( (2) \), we get:

\[
\begin{align*}
j - 1 + n - s - l + i + l = i + j - 1 + n - s &\leq \lambda_{i-1} + \lambda_{j-1}
\end{align*}
\]

But \( n - s > s = |\lambda| \), so \( i + j - 1 + n - s \geq i + j + |\lambda| \geq |\lambda| + 2 \). Thus we get:

\[
|\lambda| + 2 \leq \lambda_{i-1} + \lambda_{j-1}.
\]

But this is clearly possible only when \( i = 1 \) or \( j = 1 \).

Next, find \( j \) such that

\[
\lambda_j + 1 \leq l + 1 \leq \lambda_{j-1}
\]

(which is always possible). We now have two cases:

1. If we get \( j \geq 2 \), then we put \( i := 1 \) and we only need to check that \( \lambda_1 + 1 \leq j - 1 + n - s - l \). But we assumed that \( n > 2s \), so \( n - s > |\lambda| = s \), and thus

\[
\begin{align*}
j - 1 + n - s - l &\geq j + |\lambda| - l \geq 2 + |\lambda| - \lambda_{j-1} \geq \lambda_1 + 1
\end{align*}
\]

so we found indices \((i, j)\) satisfying the inequalities \( (2) \).
(2) If we get \( j = 1 \), then \( \lambda_j \leq l \). We now find \( i \) such that
\[
\lambda_i + 1 \leq n - s - l \leq \lambda_{i-1}
\]
(again, this is always possible). To show that we found indices \((i, j)\) satisfying the inequalities \([2]\), we only need to check that \( \lambda_i + 1 \leq i + l \), which is true since we have \( i \geq 1 \) and \( \lambda_j \leq l \).

This shows that there exists exactly one way to insert the hook \( hook(l, (n-s)) \) into \( \lambda \).

\[\square\]

**Example 9.2.9.** Let \( \lambda = (5, 4, 2, 2) \).

1. Adding the 7-hook \((6, 1)\) to its 7-core \( core_7(\lambda) = (3, 1, 1, 1) \), we obtain the Young diagram \( rec(1, core_7(\lambda)) = (7, 4, 1, 1) \):
\[
\lambda = \begin{array}{cccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \\
\bullet & \bullet & \bullet & \bullet & \\
\bullet & \bullet & \\
\end{array} \quad \leftrightarrow \quad core_7(\lambda) = \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array} \quad \leftrightarrow \quad rec(1, \lambda) = \begin{array}{ccc}
\bullet & \bullet & \\
\bullet & \\
\bullet & \\
\end{array}
\]

2. Adding the 7-hook \((2, 1, 1, 1, 1)\) to \( core_7(\lambda) \), we obtain the Young diagram \( rec(5, core_7(\lambda)) = (3, 2, 2, 2, 1, 1) \):
\[
\lambda = \begin{array}{ccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \\
\bullet & \bullet & \bullet & \\
\bullet & \bullet & \\
\end{array} \quad \leftrightarrow \quad core_7(\lambda) = \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array} \quad \leftrightarrow \quad rec(5, \lambda) = \begin{array}{cccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \\
\bullet & \bullet & \bullet & \\
\bullet & \bullet & \\
\bullet & \\
\end{array}
\]

**Theorem 9.2.10** (cf. [18]). Let \( \tau, \mu \) be partitions of \( n \).

1. All the simple modules of \( H_q(n) \) are \( D_{\lambda} \), where \( \lambda \) is \( e \)-regular (i.e. the multiplicity of each part of \( \lambda \) is smaller than \( e \)). These are exactly the partitions for which \( D_{\lambda} \neq 0 \).
2. The \( H_q(n) \)-modules \( S_{\tau} \) and \( S_{\tau'} \) have the same composition factors.
3. Two modules \( S_{\tau}, S_{\mu} \) belong to the same block iff \( \tau, \mu \) have the same \((n-s)\)-core.
4. If \( \tau \) has no \((n-s)\)-hook, then \( S_{\tau} \) (and hence \( S_{\tau'} \)) is irreducible.

2. Assume \( n > 2s \). Then we have:
   - Let \( \tau = rec(l, \beta), 0 \leq l \leq (n-s-2) \) and \( \mu \neq \tau \). Then there is a non-trivial morphism \( S_{\mu} \rightarrow S_{\tau} \) iff \( \mu = rec(l+1, \beta) \).
   - If \( \tau = rec(l, \beta), \mu = rec(l+1, \beta) \), then the composition factors of \( S_{\tau} \) are \( D_{\tau}, D_{\mu} \), with multiplicity 1 (only one of them if the other is zero).

### 9.3. Blocks of category \( O(H_c(n)) \).

We now give the relevant results for the category \( O(H_c(n)) \):

Let \( n \) be a non-negative integer.

#### 9.3.1. Equivalences.

First of all, we have a useful theorem proved by Rouquier and expanded by Losev (see [20] Theorem 5.12, [16]):

**Theorem 9.3.1** (Rouquier). Let \( n > 1, r, a > 0, b \neq 0 \) be integers, \( gcd(r,a) = 1 \). Then the categories \( O(H_{c_1=b/a}(n)) \), \( O(H_{c_2=br/a}(n)) \) are equivalent if \( c_1 \not\in \{ \frac{2k+1}{r}, k \in \mathbb{Z} \} \), with a correspondence:
- morphism \( M_{c_1,n}(\mu) \rightarrow M_{c_1,n}(\tau) \) (\(|\tau| = |\mu| = n\)) such that \( \mu \) sits in degree \( m \) of \( M_{c_1,n}(\tau) \) corresponds to
- morphism \( M_{c_2,n}(\mu) \rightarrow M_{c_2,n}(\tau) \) (\(|\tau| = |\mu| = n\)) such that \( \mu \) sits in degree \( rm \) of \( M_{c_2,n}(\tau) \).

Next, we have the following simple equivalence of categories: (see [21] 3.1.4):
Observation 9.3.2. The rational Cherednik algebras $H_{-c}(n), H_{c}(n)$ are isomorphic:

$$H_{c}(n) \rightarrow H_{(-c)}(n), x \mapsto x, y \mapsto y, \sigma \in S_{n} \mapsto (\text{sign}(\sigma) \cdot \sigma) \in \mathbb{C}[S_{n}]$$

This means that the categories of representations of these algebras are equivalent, with equivalence given by

$$O(H_{c}(n)) \rightarrow O(H_{(-c)}(n)), V \mapsto \text{sign} \otimes V,$$

where $\text{sign}$ is the sign representation of $S_{n}$. Note that $\text{sign} \otimes \mu \cong \bar{\mu}$ for representation $\mu$ of $S_{n}$.

We also have the following statement (see [12, Section 6.2]):

**Proposition 9.3.3.** If $c \not\in \frac{1}{2} + \mathbb{Z}$, then

$$\text{Hom}_{H_{c}(n)}(M_{c,n}(\mu), M_{c,n}(\tau)) \cong \text{Hom}_{H_{-c}(n)}(M_{(-c),n}(\tau), M_{(-c),n}(\mu))$$

9.3.2. Results on $O(H_{c}(n))$ obtained from the theory of representations of the Hecke algebra $H_{c}(n)$. The correspondence between the lowest weight representations of $H_{c}(n)$ and the finite dimensional representations of the Hecke algebra $H_{q}(n)$ gives us the following theory (see [5]):

- First of all, we have (see [11, 6.13]):

**Proposition 9.3.4.** For $c \leq 0$, $KZ(M_{c,n}(\tau)) = S_{\tau}, KZ(L_{c,n}(\tau)) = D_{\tau}$, and thus $KZ(L_{c,n}(\tau)) \neq 0$ iff $\tau$ is $(n-s)$-regular (i.e. the multiplicity of each part of $\tau$ is smaller than $n-s$).

- For $c > 0$, $KZ(L_{c,n}(\tau)) \neq 0$ iff $\tau$ is $(n-s)$-restricted (i.e. $\tau_{i} - \tau_{i+1} < n-s$ for all $i$).

We next have the following theorem:

**Theorem 9.3.5** (Dipper, James; cf. [5]). If $c'$ satisfies one of the following conditions:

- $c' \not\in \mathbb{Q},$
- $c' \in \mathbb{Q}$, and $c' = \frac{d}{a}, \gcd(a,d) = 1, d > n$,

then the category $O(H_{c}(n))$ is semisimple (in particular, all Verma modules are simple).

**Proof.** This is a direct consequence of Theorem 9.2.2, the standard property of Verma modules (Proposition 6.2.7), and the fact that the KZ functor is faithful on the full subcategory of Verma modules in $O(H_{c}(n))$. □

This means that we remain with only one interesting case:

$$c' = \frac{d}{a}, \gcd(a,d) = 1, a > 0, 0 \leq d \leq n$$

For $d \leq n/2$, the theory is more complicated, but for $d > n/2$, it is rather simple and explained below. Note that Theorem 9.3.1 allows us to assume that $a = 1$ and study just the case $c' = d \in \{0,1,...,n\}$.

Fix integer $s \geq 0$, and from now on, consider $n > 2s$, $c' = d = n-s$, and a Young diagram $\tau$ with $|\tau| = n$.

**Corollary 9.3.6.** $M(\tau)$ is simple if and only if $\tau$ has no $(n-s)$-hook, or its $(n-s)$-hook is a vertical strip.

**Proof.** This is a consequence of Theorem 9.2.10 and of the standard property of Verma modules (Proposition 6.2.7). □

Furthermore, we have (see [2, Sections 3.4, 3.5]):
Theorem 9.3.7. Let $\beta$ be an arbitrary Young diagram of size $s$. Let $\tau = \text{rec}(l, \beta)$, with $0 \leq l \leq (n - s - 2)$ and $\mu \neq \tau$. Then there is a non-trivial morphism $\psi_l : M(\mu) \rightarrow M(\tau)$ iff $\mu = \text{rec}(l + 1, \beta)$.

In that case, we have:
- $\dim \text{Hom}(M(\mu), M(\tau)) = 1$.
- There is a short exact sequence
  
  $$0 \longrightarrow L(\mu) \longrightarrow M(\tau) \longrightarrow L(\tau) \longrightarrow 0.$$ 

Remark 9.3.8. In [2] Section 3, this is proved for the case $s = 0$, i.e. $e = n$. The only thing which we need to understand to generalize this proof is the following: the method used to prove that there is such a short exact sequence uses the fact that for $c' = n/r, r > 0, \gcd(n, r) = 1$, $L(\lambda)$ is thin (i.e. doesn’t have full support, and goes to zero under $KZ$) iff $\lambda = \emptyset$. But for $c' = (n - s)/r$, we have other simple thin modules (actually, those are exactly $L(\lambda)$ such that $\lambda$ is not $(n - s)$-restricted), which can appear as composition factors in $M(\tau)$.

Recall that the Young diagrams $\lambda$ of size $n$ come in two types (we still assume that $n > 2s$):

- Young diagrams $\lambda$ which do not have an $(n - s)$-hook, and thus $\text{core}_{(n-s)}(\lambda) = \lambda$. In this case $M(\lambda)$ is simple and lies in a semisimple block (see Corollary 9.3.6), so $L(\lambda)$ cannot appear as a composition factor in $M(\tau)$ unless $\lambda = \tau$.
- For a fixed Young diagram $\beta$ of size $s$, consider all the Young diagrams $\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}, \ldots$ with $(n - s)$-core $\beta$, arranged so that $\lambda^{(0)}$ has maximal content among all $\lambda^{(i)}$ (that is, $\lambda^{(0)}$ is the Young diagram obtained from $\beta$ by adding $(n - s)$ cells to first row). One immediately sees that $\lambda^{(i)}$ are $(n - s)$-restricted for $i > 0$, and $\lambda^{(0)}$ is not. Thus an argument similar to the one in [2] Section 3] would work here as well.

Example 9.3.9. The Verma module $M_{c,n}(\tau^{n-1})$ (whose lowest weight is the trivial representation $\tau^{n-1} = \emptyset(n)$ of $S_n$) is reducible if and only if the following condition on $c' = \frac{n - s}{r}, n$ holds:

$$c' = \frac{n - s}{r}, \quad s \geq 0, \quad r \geq 1$$

In this case, $M_{c,n}(\tau^{n-1})$ contains a lowest weight $H_c(n)$-submodule, whose lowest weight $\mu$ lies in degree $r(s + 1)$ of $M_{c,n}(\tau^{n-1})$. The Young diagram of $\mu$ is a two row diagram, the lower row containing $s + 1$ boxes.

Theorem 9.3.11 gives the following corollary:

Corollary 9.3.10. Let $n > 3, s \geq 0, r \geq 1$ be integers, $n > 2s, \gcd(r, n - s) = 1$. Then the blocks of the category $O(H_{\infty,n}(n))$ correspond to $(n - s)$-cores, and we have two types of blocks: blocks containing only one Verma module (up to isomorphism), corresponding to a diagram which doesn’t have an $(n - s)$-hook, and blocks which correspond to a Young diagrams of size $s$; in such a block $\text{Block}_\beta$ ($|\beta| = s$) lie the Verma modules whose $(n - s)$-core is $\beta$, and these Verma modules form a long exact sequence:

$$0 \longrightarrow M(\text{rec}((n - s - 1), \beta)) \longrightarrow M(\text{rec}((n - s - 2), \beta)) \longrightarrow M(\text{rec}((n - s - 3), \beta)) \longrightarrow \cdots$$

(Note that $\text{rec}((n - s - 1), \beta)$ is the diagram $\beta$ with a vertical $(n - s)$-hook added, and $\text{rec}(0, \beta)$ is the diagram $\beta$ with a horizontal $(n - s)$-hook added).
Remark 9.3.11. For $s \geq n/2$, the situation is more complicated, but one can still say the following (see \[3\]).

Let $c' = n - s$. If $M_{c,n}(\mu), M_{c,n}(\tau)$ belong to the same block, then they have the same $(n - s)$-core. In particular, if $\tau$ has no $(n - s)$-hooks, then $M_{c,n}(\tau)$ is a simple module lying in a semisimple block.

10. Constructions for $O_{c,\nu}$

We now define constructions analogous to those described in Section \[9\] for the category $O_{c,\nu}$, where we consider generic values of $\nu$. We use the idea described in Section \[4\] of treating a simple object $X_{\nu}$ of $\text{Rep}(S_{\nu})$ as a Young diagram obtained by adding a very long top row to $\tau$ ("the top row having length $(\nu - |\tau|)$"). This idea is only meant to provide some intuition, but it is true that for $n \in \mathbb{Z}, n >> 0$, the image of $X_{\nu}$ under the functor $\text{Rep}(S_{\nu}) \rightarrow \text{Rep}(S_n)$ is isomorphic to $\tilde{\tau}(n)$, which is exactly a Young diagram obtained by adding a top row of length $(n - |\tau|)$ to $\tau$.

So instead of adding and removing $(n - s)$-hooks as we did in Section \[9\], we would be adding and removing "$(\nu - s)$-hooks" from diagrams with "very long top row of size $(\nu - |\tau|)$". Below is a formal description of the relevant constructions.

Let $\tau$ be a Young diagram, and $s \geq 0$ be an integer.

Definition 10.0.12.

- Define $C_{\tau} := \{|\tau| - 1 + j - \tau_j | j \geq 1\}$.
- For $s \in C_{\tau}, s = |\tau| - 1 + j_s - \tau_{j_s}$ for some $j_s \geq 1$, define the $(\nu - s)$-core of $\tau$ as $\text{core}_{(\nu-s)}(\tau)_j = \tau_j + 1$ if $1 \leq j < j_s$, and $\text{core}_{(\nu-s)}(\tau)_j = \tau_{j+1}$ if $j \geq j_s$.

That is, $\text{core}_{(\nu-s)}(\tau)$ is the Young diagram obtained by taking out column $j_s$ of $\tau$, moving columns $1, \ldots, j_s - 1$ down and adding a row of length $j_s - 1$ on top of them, and moving left the columns $j_s + 1, \ldots$.

Example 10.0.13. Let $\tau = (8, 5, 4, 3, 3, 2)$. Then $|\tau| = 25$, $C_{\tau} = \{19, 20, 22, 25, 27, 29, 30, 31\} \cup \mathbb{Z}_{\geq 33}$. Let $s = 22$. Then $s \in C_{\tau}, s = |\tau| - 1 + 3 - \tau_3$. So $j_s = 3$, and $\text{core}_{(\nu-s)}(\tau) = (7, 4, 3, 2, 2, 2)$.

Let $s = 34$. Then $s \in C_{\tau}, s = |\tau| - 1 + 10 - \tau_{10}$. So $j_s = 10$, $\text{core}_{(\nu-s)}(\tau) = (9, 8, 5, 4, 3, 3, 2)$.

Remark 10.0.14. Deligne defined $\text{core}_{(\nu-s)}(\lambda)$, which he denoted by $\{\lambda\}^+_s$, with $n := s$, in \[5\, 7.5\].

The process of "reconstruction" (corresponding to inserting a hook into a Young diagram) is defined as follows:

Construction 10.0.15. Given any Young diagram $\eta$ and an integer $l \geq 0$, we define $\tau := \text{rec}(l, \eta)$ as the Young diagram obtained through the following steps:

- Find the index $k \geq 1$ such that $\eta_{k-1} \geq l + 1 > \eta_k$ (here $\eta_0 := \infty$).
Define \( \tau = \text{rec}(l, \eta) \) by \( \tau_j := \eta_j - 1 \) for \( j < k \), \( \tau_k := l \) and \( \tau_j := \eta_j - 1 \) for \( j > k \).

That is, we divide \( \eta \) into two parts: part 1 consisting of columns 1, ..., \( k - 1 \) and part 2 consisting of columns \( k, k + 1, \ldots \). Then we delete the top row of part 1, add a \( k \)-th column of length \( l \), and add part 2 as columns \( k + 1, k + 2, \ldots \).

**Example 10.0.16.**
- For \( l > \eta_1 \), \( \text{rec}(l, \eta) \) is the Young diagram obtained from \( \eta \) by adding a column of length \( l \) to \( \eta \) (this will become the first column).
- For \( l = 0 \), \( \text{rec}(l, \eta) \) is the Young diagram obtained from \( \eta \) by removing its top row.

**Example 10.0.17.** \( \text{rec}(\tau_j, \text{core}_{(\nu - s)}(\tau)) = \tau \).

**Example 10.0.18.** Let \( \tau = (10, 8, 6, 5, 4, 1) \), \( s = 42 \) (so \( j_s = 6 \)). Then \( \text{rec}(7, \text{core}_{(\nu - s)}(\tau)) = (10, 8, 6, 6, 6, 5) \).

**Notation 10.0.19.** Let \( \tau \) be a Young diagram, \( s \in C_\tau, l \in \mathbb{Z}_{\geq -\tau_j} \) \( (j_s \text{ given by } s = |\tau| - 1 + j_s - \tau_j) \). We will denote the Young diagram \( \text{rec}(\tau_j + l, \text{core}_{(\nu - s)}(\tau)) \) by \( \Gamma(\tau, s, l) \).

**Example 10.0.20.** Consider \( \tau = \emptyset \). Given \( s \geq 0 \), the \((\nu - s)\)-core of \( \emptyset \) is a row of length \( s \). Then \( \Gamma(\emptyset, s, l) = \text{rec}(l, \text{core}_{(\nu - s)}(\emptyset)) \) is a hook with arm length \( s \) and leg length \( l - 1 \).

**Lemma 10.0.21.** Let \( \tau \) be a Young diagram, \( l > 0 \) an integer, \( s := |\tau| - 1 + j_s - \tau_j \) for some \( j_s \geq 1 \), \( \mu = \Gamma(\tau, s, l) \). Then \( |\mu| - |\tau| > 0 \), and \( s = \frac{|f(\mu) - f(\tau)|}{|\mu| - |\tau|} \) (\( f \) is defined in Subsection 8.2).

**Proof.** We first describe \( \mu \) in terms of \( \tau \). By Constructions [10.0.12, 10.0.15] described above, \( \mu \) is obtained from \( \tau \) by taking out column \( j_s \) of \( \tau \), inserting a column of length \( \tau_j + l \) after column \( k_{s,l} - 1 \) for some \( k_{s,l} \leq j_s \) \( (k_{s,l} \text{ is uniquely determined by Construction [10.0.15])} \), and adding a cell to each of the columns \( k_{s,l} + 1, \ldots, j_s \) of the newly constructed diagram. Thus \( \mu_j = \tau_j \) for \( j < k_{s,l} \) and \( j \geq j_s \), \( \mu_j = \tau_j + l \), and \( \mu_j = \tau_j + 1 \) for \( j = k_{s,l} + 1, \ldots, j_s \). This means that we have:

\[
|\mu| - |\tau| = j_s - k_{s,l} + l
\]

Note that since \( k_{s,l} \leq j_s, l > 0 \), we have: \( |\mu| - |\tau| > 0 \).

From this description of \( \mu \), we see that
Assume \( (\tau_j,\nu) \). Let \( \lambda \) be as above.

Let \( \lambda_0 := \text{rec}(\lambda) \). Then \( \lambda_0(\nu) = \text{rec}(\lambda_0(\nu)) \).

Proof. One can easily see that the procedure for constructing \( \text{rec}(\lambda) \) coincides with the procedure for constructing \( \text{rec}(\lambda_0(\nu)) \) and then removing the top row.

\( \square \)

11. Blocks in category \( O_{c,\nu} \)

11.1. Morphisms between two Verma objects. We now give some necessary and some sufficient conditions for the existence of a non-trivial morphism between Verma objects. Fix Young diagrams \( \tau, \mu \).

Proposition 11.1.1. Let \( r \in \mathbb{Z} \setminus \{0\} \), \( s \in C_{\tau} \) (i.e. \( s = |\tau| - 1 + j_s - \tau_j \) for some \( j_s \geq 1 \)). Assume \( \tau_{j_s} + \text{sign}(r) \geq 0 \).

Let \( \mu = \Gamma(\tau, s, \text{sign}(r)) \).

If \( \nu = \frac{\nu - s}{r} \), then there exists a non-trivial morphism \( M_{c,\nu}(\mu) \to M_{c,\nu}(\tau) \).

Proof. Assume \( (\nu, \mu) \) and \( s \) as above.

Let \( n >> 0 \) be an integer, such that \( \text{gcd}(n - s, r) = 1 \) (there are infinitely many such positive integers).
By Corollary 11.1.10 Proposition 9.3.2 and Proposition 10.1.22 we have a non-trivial morphism $M_{c,n}(\tilde{\mu}(n)) \to M_{c,n}(\tilde{\tau}(n))$. Moreover, $s = \frac{f(\mu) - f(\tau)}{|\mu| - |\tau|}$ (f defined in Subsection 8.2) by Lemma 10.0.21 and the image of $\tilde{\mu}(n)$ sits in degree $m := r(|\mu| - |\tau|)$ of $M_{c,n}(\tilde{\tau}(n))$ (this is a direct consequence of Equation (11)).

Then by Lemma 7.0.10 there is a non-trivial morphism $M_{c,\nu} = M_{c,\nu} = M_{c,\nu} = M_{c,\nu}$ with the image of $X_\mu$ sitting in degree $m = r(|\mu| - |\tau|)$ of $M_{c,\nu} = M_{c,\nu} = M_{c,\nu} = M_{c,\nu}$, and $s = \frac{f(\mu) - f(\tau)}{|\mu| - |\tau|}$.

Consider the space $V_\nu = \text{Hom}_{Rep(S)}(X_\mu, S^{m-1} \otimes X_\tau)$ for any $\nu \in \mathbb{C}$. By the construction of Deligne’s categories, these spaces are isomorphic for almost every $\nu \in \mathbb{C}$ (in particular, for all $\nu \notin \mathbb{Z}_+$). We will denote the locus of such “generic” points $\nu \in \mathbb{C}$ by $A$ (so $|\mathbb{C} \setminus A| < \infty, \mathbb{C} \setminus \mathbb{Z}_+ \subset A$), and write just $V$ instead of $V_\nu$ whenever $\nu \in A$.

The map $y_{M,\nu}(\tau)$ from the definition of $\text{Rep}(H_c(\nu))$ defines a polynomial family of maps $Y_{c,\nu} : V \to \text{Hom}_{Rep(S)}(\mathfrak{h}_0 \otimes X_\mu, S^{m-1} \mathfrak{h}_0 \otimes X_\tau)$

$$\phi \mapsto y_{M,\nu}(\tau) \circ (\text{Id}_{\mathfrak{h}_0} \otimes \phi)$$

It is easy to see that the locus of points $(c, \nu) \in \mathbb{C} \times A$ such that $Y_{c,\nu}$ is not injective is a Zariski closed subset of $\mathbb{C} \times A$. Denote it by $\mathcal{C}$.

By definition, $\mathcal{C}$ is exactly the locus of points $(c, \nu) \in \mathbb{C} \times A$ such that $y_{M,\nu}(\tau) |_{\mathfrak{h}_0} = 0$ for some $\phi_0 \in V$ (the latter condition being equivalent to the existence of a non-trivial morphism $\Theta_\nu : M_{c,\nu}(\mu) \to M_{c,\nu}(\tau)$ such that $\Theta_\nu(X_\mu)$ sits in degree $m$ of $M_{c,\nu}(\tau)$).

The morphisms $M_{c,\nu} = M_{c,\nu} = M_{c,\nu}$ are injective whenever $n \in \mathbb{Z}_{>0}, c' = \frac{1}{c} = \frac{n - s}{r}$, which means that $(c, \nu = n) \in \mathcal{C}$ for any $c, n$ satisfying $n > 0, \frac{1}{c} = \frac{n - s}{r}$.

Thus $(c, \nu) \in \mathbb{C}^2 | c' = \frac{1}{c} = \frac{n - s}{r} \subset \mathcal{C}$, and we get the required statement.

\[\square\]

**Remark 11.1.2.** In the proof of this proposition, we established that $s = \frac{f(\mu) - f(\tau)}{|\mu| - |\tau|}$, so by Equation (11), the image of $X_\mu$ sits in degree $m$ of $M_{c,\nu}(\tau)$ for all $(c', \nu)$ such that $c' = \frac{n - s}{r}$.

In general, for each pair $(c', \nu)$, Equation (11) allows us to compute the unique degree $m$ in which the image of $X_\mu$ could sit in $M_{c,\nu}(\tau)$.

We continue with Young diagrams $\tau, \mu$ fixed.

Consider the line $\mathcal{L}_{\tau,\mu,m} \subset \mathbb{C}^2$ (defined in Notation 8.1.5). We would like to check whether $\mathcal{L}_{\tau,\mu,m}$ satisfies the following condition:

**Condition 11.1.3.** For all $\nu$ there exist non-trivial morphisms $\Theta_\nu : M_{c,\nu}(\mu) \to M_{c,\nu}(\tau)$ such that $(c', \nu) \in \mathcal{L}_{\tau,\mu,m}$ and $\Theta_\nu(X_\mu)$ sits in degree $m$ of $M_{c,\nu}(\tau)$.

Assume this is indeed the case.

Then by Lemma 7.0.10 for integer $n > 0$ there must exist non-trivial morphisms $\tilde{\Theta}_n : M_{c,n}(\tilde{\mu}(n)) \to M_{c,n}(\tilde{\tau}(n))$ (corresponding to $\Theta_\nu$) with the image of $\tilde{\mu}(n)$ sitting in degree $m$ of $M_{c,n}(\tilde{\tau}(n))$.

Recall that for $n \in \mathbb{Z}_+, (c', \nu) \in \mathcal{L}_{\tau,\mu,m}$ is equivalent to $c' = \frac{(|\mu| - |\tau|)n - (f(\mu) - f(\tau))}{m}$. Denote: $a := \frac{|\mu| - |\tau|}{m}, b := f(\mu) - f(\tau), d_n := \text{gcd}(a, b, m)$. Then $c' = \frac{an - b}{m} = \frac{(an - b)/d_n}{m/d_n}$, where $(an - b)/d_n, m/d_n \in \mathbb{Z}, m/d_n > 0, \text{gcd}((an - b)/d_n, m/d_n) = 1$.

By Theorem 9.3.1 if $(an - b)/d_n \neq 2$, we can pass to the case $(c' = (an - b)/d_n, n)$, with the image of $\tilde{\mu}(n)$ sitting in degree $d_n$ of $M_{c,n}(\tilde{\tau}(n))$.

Note that if $a \neq 0$, then, since $0 < d_n \leq m$, we have $(an - b)/d_n \neq 2$ for $n > 0$.

Now we have the following cases:
Case $a > 0$.

**Lemma 11.1.4.** For $n >> 0$, there are no non-trivial morphisms $\tilde{\Theta}_n$ unless $a | b, m$, i.e. $d_n = a$.

**Proof.** We assume these morphisms are not trivial. Then we have, by Theorem 9.3.6, $(an - b)/d_n < n$ for $n >> 0$, i.e. $a \leq d_n$. Also, putting $d := \gcd(a, b), a' := a/d, b' := b/d$, we have: $d_n = \gcd(d(a'n - b'), m) \geq a = da' \geq 1$ for $n >> 0$. This implies, for $n >> 0$: either $d_n \mid d$ (possible only if $d_n = d = a$, and then we have $a \mid b, m$), or $\gcd(a'n - b', m) \neq 1$.

So we only need to check that there is no $n_0 \in \mathbb{Z}$ such that $\forall n > n_0, \gcd(a'n - b', m) \neq 1$. Indeed, $b', \gcd(a', m)$ are relatively prime (since $a', b'$ are), and so $
\gcd(a'n - b', m) = \gcd(\frac{a'}{\gcd(a', m)}n - b', \frac{m}{\gcd(a', m)})$. Now

$$\gcd(\frac{a'}{\gcd(a', m)}n - b', \frac{m}{\gcd(a', m)}) = 1$$

so

$$\gcd(a'n - b', m) = \gcd(\frac{a'}{\gcd(a', m)}n - b', \frac{m}{\gcd(a', m)}) = 1$$

for infinitely many integer values of $n$. $\Box$

Thus $d_n = a$, and $(an - b)/d_n \neq 2$ for $n >> 0$. So to understand what happens for $(c', n) \in \mathcal{L}_{\tau, \mu, m}$, it is enough to check what happens the case $(c' = n - (b/a), n)$ (recall: $b/a = -\left(\frac{\mu(a) - 1}{|\mu|} \right) \in \mathbb{Z}$). Still assuming the existence of non-trivial morphisms $\tilde{\Theta}_n$, Proposition 10.0.22 and Corollary 9.3.10 give:

For $n >> 0$ and $c' = n - b/a$, put $s := b/a$, and we have

$$\mu = \Gamma(\tau, s, 1)$$

for $j_s \geq 1$, such that $s = |\tau| - 1 + j_s - \tau j_s$.

Note that a priori, $j_s$ could depend on $n$, but the equality $s = |\tau| - 1 + j_s - \tau j_s$ defines $j_s$ uniquely for each $s$, so since $s$ doesn’t depend on $n$, neither does $j_s$.

**Conclusion 11.1.5.** For $|\mu| > |\tau|$, $\mathcal{L}_{\tau, \mu, m}$ satisfies Condition 11.1.3 if and only if the following hold:

1. $s := \frac{\Gamma(\mu) - \Gamma(\tau)}{|\mu| - |\tau|} \in C_\tau \subset \mathbb{Z}$,
2. $\mu = \Gamma(\tau, s, 1)$,
3. $(|\mu| - |\tau|) \mid m$.

In that case, points $(c', \nu) \in \mathcal{L}_{\tau, \mu, m}$ satisfy the condition: $c' = \frac{\nu - s}{r}$ for $r = \frac{m}{|\mu| - |\tau|}$.

Case $a < 0$. By Observation 9.3.2, the existence of a non-trivial morphism $\tilde{\Theta}_n$ for $n >> 0$ is equivalent to the existence of a non-trivial morphism $\tilde{\Upsilon}_n : M_{-c', n}(\tilde{\tau}(n)) \rightarrow M_{-c', n}(\tilde{\mu}(n))$. The necessary and sufficient conditions for the existence of a non-trivial morphism $\tilde{\Upsilon}_n$ for all $n >> 0$ are given in Conclusion 11.1.5 with $\mu$ and $\tau$ switching roles.

**Conclusion 11.1.6.** For $|\mu| < |\tau|$, $\mathcal{L}_{\tau, \mu, m}$ satisfies Condition 11.1.3 if and only if the following hold:

1. $s := \frac{\Gamma(\mu) - \Gamma(\tau)}{|\mu| - |\tau|} \in C_\mu \subset \mathbb{Z}$,
2. $\mu = \Gamma(\tau, s, -1)$,
3. $(|\tau| - |\mu|) \mid m$.

In that case, points $(c', \nu) \in \mathcal{L}_{\tau, \mu, m}$ satisfy the condition: $c' = -\frac{\nu - s}{r}$ for $r = \frac{m}{|\tau| - |\mu|}$.
Case $a = 0$. In this case, we are considering lines $c' = \frac{f(\tau) - f(\mu)}{m} = \frac{ct(\tau) - ct(\mu)}{m}$, i.e. $c'$ is constant. Put $d := \gcd(ct(\tau) - ct(\mu), m)$ (previously denoted $d_m$; now doesn't depend on $n$). By definition, $0 < d \leq m$. If $d \neq \frac{ct(\tau) - ct(\mu)}{m}$, Theorem 9.3.1 allows us to pass to the case $c' = (ct(\tau) - ct(\mu))/d \in \mathbb{Z}$ (i.e. we will now consider the line $L_{\tau,\mu,m/d} = \{(c' = (ct(\tau) - ct(\mu))/d, \nu) \mid \nu \in \mathbb{C}\}$).

From now on, we will assume $d \neq \frac{ct(\tau) - ct(\mu)}{m}$.

It would be difficult to determine whether morphisms $\tilde{\Theta}_n$ exist for $(c',n) \in L_{\tau,\mu,m/d}, n \in \mathbb{Z}, n >> 0$, but we can rule out some cases using the tools we have so far. First of all, if there exists an integer $n > 2|\mu| = 2|\tau|$ such that $c' = (ct(\tau) - ct(\mu))/d > n$ (i.e. if $(ct(\tau) - ct(\mu))/d > 2|\tau| + 1$), then by Theorem 9.3.5 there are no non-trivial morphisms $\tilde{\Theta}_n$, and we are done. Theorem 9.3.5 also tells us that the same argument applies to the case $c' = (ct(\tau) - ct(\mu))/d < -(2|\tau| + 1)$, so we ruled out the possibilities $|c'| = |(ct(\tau) - ct(\mu))/d| > 2|\tau| + 1$.

We will show that something can be said if $|\tau| < |(ct(\tau) - ct(\mu))/d| \leq 2|\tau| + 1$.

For this, we should consider the sign of $(ct(\tau) - ct(\mu))$.

Case 1. Let $(ct(\tau) - ct(\mu)) > 0, |\tau| < |(ct(\tau) - ct(\mu))/d| \leq 2|\tau| + 1$. Let $n := 2|\tau| + 1$. Assume we have a non-trivial morphism $\tilde{\Theta}_n$. Then, putting $s := n - (ct(\tau) - ct(\mu))/d$, we have: $n > 2s$, so by Corollary 9.3.10 and Proposition 10.0.22

$$\mu = \Gamma(\tau, s, 1)$$

for $j_s \geq 1$, such that $s = |\tau| - 1 + j_s - \tau j_s$.

But then we would get, by Lemma 10.0.21 $|\mu| > |\tau|$. Contradiction.

Case 2. Let $(ct(\tau) - ct(\mu)) < 0, |\tau| < |(ct(\tau) - ct(\mu))/d| \leq 2|\tau| + 1$. Let $n := 2|\tau| + 1$. By Proposition 9.3.2, we can pass to the previous case (Case 1.1) by switching the roles of $\mu$ and $\tau$, so we conclude there are no non-trivial morphisms $\tilde{\Theta}_n$ in that case.

**Conclusion 11.1.7.** For $|\tau| = |\mu|$, if $\frac{ct(\tau) - ct(\mu)}{\gcd(ct(\tau) - ct(\mu), m)} > |\tau|$ then $L_{\tau,\mu,m}$ does not satisfy Condition 11.1.3.

The results of this subsection prove the following theorem:

**Theorem 11.1.8.** For two distinct Young diagrams $\mu, \tau$ and an integer $m > 0$, the following are equivalent:

1. $|\mu| \neq |\tau|$ and $L_{\tau,\mu,m} \subset B_{\mu,\tau}$.
2. $\mu = \Gamma(\tau, s, \text{sign}(|\mu| - |\tau|))$ for some $s \in C_\tau$ (in particular, $|\mu| \neq |\tau|$), and $(|\mu| - |\tau|) \mid m$.

**Remark 11.1.9.** We know that for every $c \in \mathbb{C}, n \in \mathbb{Z}_{>0}$, $\text{dim Hom}(M_{c,n}(\mu(n)), M_{c,n}(\tau(n))) \leq 1$ for any $\mu, \tau$.

Similarly to the arguments in this subsection, Lemma 7.0.10 implies that that $\text{dim Hom}(M_{c,n}(\mu(n)), M_{c,n}(\tau(n))) \leq 1$ for $n \in \mathbb{Z}_{+}, n >> 0$, so we have: $\{(c, \nu) \mid \text{dim Hom}(M_{c,n}(\mu), M_{c,n}(\tau)) \geq 1\}$ is a Zariski closed set in $\mathbb{C}^2$ of dimension at most 1, and the set $\{(c, \nu) \mid \text{dim Hom}(M_{c,n}(\mu), M_{c,n}(\tau)) \geq 2\}$ is a finite set of points in $\mathbb{C}^2$.

**Remark 11.1.10.** The results in this subsection agree with the statement of Lemma 8.1.9 when $m = 1$, but they imply the statement of Lemma 8.1.9 only if $|\mu| \neq |\tau|$. 26
Lemma 11.2.1. Let \( m_1, m_2 \in \mathbb{Z}_+ \), and let \( \tau, \mu, \mu^{(1)}, \mu^{(2)} \) be Young diagrams. Then
- \( \mathcal{L}_{\tau, \mu, m_1} \cap \mathcal{L}_{\tau, \mu, m_2} \neq \emptyset \iff m_1 = m_2 \).
- If \((c', \nu) \in \mathcal{L}_{\tau, \mu^{(1)}, m_1} \cap \mathcal{L}_{\tau, \mu^{(2)}, m_2}\), and the lines \( \mathcal{L}_{\tau, \mu^{(1)}, m_1}, \mathcal{L}_{\tau, \mu^{(2)}, m_2} \) do not coincide, then \( c', \nu \in \mathbb{Q} \) (and there is only one point \((c', \nu)\) like this).
- Assume the lines \( \mathcal{L}_{\tau, \mu^{(1)}, m_1}, \mathcal{L}_{\tau, \mu^{(2)}, m_2} \) coincide, \( |\mu^{(1)}| \neq |\tau| \), and \( \mathcal{L}_{\tau, \mu^{(1)}, m_1} \subset B_{\mu^{(1)}, \tau} \). Then \( \mathcal{L}_{\tau, \mu^{(2)}, m_2} \subset B_{\mu^{(2)}, \tau} \) iff \( \mu^{(1)} = \mu^{(2)} \).

Proof.
- Follows immediately from the definition of \( \mathcal{L}_{\tau, \mu, m} \) (see Notation 8.1.5).
- Follows immediately from the definition of \( \mathcal{L}_{\tau, \mu, m} \) by a linear equation with rational coefficients.
- First, \( |\mu^{(1)}| \neq |\tau| \), and \( \mathcal{L}_{\tau, \mu^{(1)}, m_1} \subset B_{\mu^{(1)}, \tau} \), so by Theorem 11.1.8 putting
  \[
  s := \frac{f(\mu^{(1)}) - f(\tau)}{|\mu^{(1)}| - |\tau|}
  \]
  we get
  \[
  \mu^{(1)} = \Gamma(\tau, s, \text{sign}(|\mu^{(1)}| - |\tau|))
  \]
  with \( j_s \) given by \( s = |\tau| - 1 + j_s - \tau j_s \).
  Next, since the lines \( \mathcal{L}_{\tau, \mu^{(1)}, m_1}, \mathcal{L}_{\tau, \mu^{(2)}, m_2} \) coincide and \( |\mu^{(1)}| \neq |\tau| \), we get:
  \[
  [m_1 : (|\mu^{(1)}| - |\tau|) : f(\mu^{(1)}) - f(\tau)] = [m_2 : (|\mu^{(2)}| - |\tau|) : f(\mu^{(2)}) - f(\tau)]
  \]
  which implies
  \[
  s = \frac{f(\mu^{(1)}) - f(\tau)}{|\mu^{(1)}| - |\tau|} = \frac{f(\mu^{(2)}) - f(\tau)}{|\mu^{(2)}| - |\tau|}.
  \]
  We also get: \( |\mu^{(2)}| \neq |\tau| \) and \( \text{sign}(|\mu^{(1)}| - |\tau|) = \text{sign}(|\mu^{(2)}| - |\tau|) \).
  Now assume that \( \mathcal{L}_{\tau, \mu^{(2)}, m_2} \subset B_{\mu^{(2)}, \tau} \). Since
  \[
  |\mu^{(2)}| \neq |\tau|, \quad s = \frac{f(\mu^{(1)}) - f(\tau)}{|\mu^{(1)}| - |\tau|} = \frac{f(\mu^{(2)}) - f(\tau)}{|\mu^{(2)}| - |\tau|}
  \]
  and
  \[
  \text{sign}(|\mu^{(1)}| - |\tau|) = \text{sign}(|\mu^{(2)}| - |\tau|)\]
  we get, from 11.1.8
  \[
  \mu^{(2)} = \Gamma(\tau, s, \text{sign}(|\mu^{(2)}| - |\tau|)) = \mu^{(1)}
  \]

From the third part of the above lemma, we immediately get:

Corollary 11.2.2. Let \( |\mu| \neq |\tau| \), and \( \mathcal{L}_{\tau, \mu, m} \subset B_{\mu, \tau} \). Then for a generic point \((1/c, \nu) \in \mathcal{L}_{\tau, \mu, m}, (1/c, \nu) \notin B_{\mu', \tau} \) for any \( \mu' \neq \mu \).

12. Characters of simple objects in \( \mathcal{O}_{c, \nu} \)

We continue with \( \nu \notin \mathbb{Z}_+, c \neq 0 \). We also continue to identify \( \mathfrak{h}_0 \) with \( \mathfrak{h}_0^* \), \( \mathfrak{h} \) with \( \mathfrak{h}^* \).
12.1. Definitions.

Definition 12.1.1. The character of a graded vector space $V = \bigoplus_{k \in \mathbb{Z}} V_k$ will be defined as $ch_q(V) := \sum_k q^k \dim V_k$.

Definition 12.1.2 (Character of a graded ind-object of $Rep(S_\nu)$). The character of an ind-object $V$ of $Rep(S_\nu)$ with the grading $V = \bigoplus_{j \in \mathbb{Z}_+} V_j$, $V_j \in Rep(S_\nu)$ is defined to be a formal power series in $t$:

$$ch_t V = \sum_{\text{Young diagrams } \mu \text{ of arbitrary size}} X_\mu ch_t (\text{Hom}_{S_\nu}(X_\mu, V))$$

Definition 12.1.3 (Character of a object of $O_{c,\nu}$). Let $M$ be an object of the category $O_{c,\nu}$. The character of $M$ is defined to be a formal power series in $t$:

$$ch_t M := \sum_{\text{Young diagrams } \mu \text{ of arbitrary size}} X_\mu tr \left( t^h \big|_{\text{Hom}_{S_\nu}(X_\mu, M)} \right)$$

with complex degrees, and coefficients in $K_0(Rep(S_\nu))$, which is the Grothendieck ring of $Rep(S_\nu)$. We will usually write $chM$ instead of $ch_t M$ for short.

Remark 12.1.4. The trace of the operator $t^h$ is defined since $h$ acts locally finitely on objects in $O_{c,\nu}$.

12.2. Characters of Verma objects. Recall that $M(\lambda) \cong S_{\lambda_0} \otimes X_\lambda$ as ind-objects of $Rep(S_\nu)$. So it is enough to compute the character of $S_{\lambda_0}$, and then use the formula

$$ch_t M(\lambda) = ch_t (S_{\lambda_0}) X_\lambda^{h_{c,\nu}(\lambda)}$$

We now give a formula for computing the character of the ind-object $S_{\lambda_0}$ of $Rep(S_\nu)$, which comes from the character formula in [9, 3.7].

Proposition 12.2.1. We have the following formula for the character of $S_{\lambda_0}$:

$$ch_t (S_{\lambda_0}) = \frac{1}{\sum_{n \geq 0} (-1)^n \Lambda^n \lambda_0 t^n}$$

Proof. Consider the category $Schur$ of all Schur functors: the objects are Schur functors

$$S_R : (FinVect) \to (FinVect), S_R(\cdot) = \bigoplus_{n \geq 0} R_n \otimes_{\mathbb{C}[S_n]} (\cdot)^\otimes_n$$

where $R = (R_n)_{n \geq 0}$, each $R_n$ being a finite-dimensional representation of $S_n$, and only finitely many of these representations being nonzero; the morphisms are natural transformations between such functors. This category is equivalent to the category $\bigoplus_{n \in \mathbb{Z}} Rep(S_n)$, and thus semisimple.

One can see that $Fun(Schur, Rep(S_\nu)) \cong Rep(S_\nu)$ (this is true for any symmetric tensor category), where $Fun(Schur, Rep(S_\nu))$ are functors between additive linear tensor categories.

Fix the object $V$ in $Schur$ corresponding to the identity functor (which is, of course, a Schur functor as well). In $Schur$, we have a commutative algebra $SV$.

We then have an exact Koszul-type complex in $Schur \cong \bigoplus_{n \in \mathbb{Z}} Rep(S_n)$:

$$\ldots \to SV \otimes \Lambda^m V \to SV \otimes \Lambda^{m-1} V \to \ldots \to SV \otimes V \to SV \to \mathbb{C} \to 0$$

This gives us an exact complex of ind-objects of $Rep(S_\nu)$:

$$\ldots \to S_{\lambda_0} \otimes \Lambda^m \lambda_0 \to S_{\lambda_0} \otimes \Lambda^{m-1} \lambda_0 \to \ldots \to S_{\lambda_0} \otimes \lambda_0 \to S_{\lambda_0} \to 1 \to 0$$
Now the character formula follows directly from Euler's formula applied to this complex.

12.3. The graded space $\text{Hom}_{\text{Rep}(S_{\nu})}(X_{\mu}, S_{\theta_{0}} \otimes X_{T})$. In this subsection we describe explicitly the decomposition of $S_{\theta} \otimes X_{T}$ and $S_{\theta_{0}} \otimes X_{T}$ as $\text{Rep}(S_{\nu})$ ind-objects into graded sums of simple objects.

**Lemma 12.3.1.** For $n \gg 0$ (in fact, for $n > 2(|\mu| + |\tau|)$, we have:

$$ch_{q}(\text{Hom}_{S_{\nu}}(\bar{\mu}(n), \mathbb{C}[x_{1}, \ldots, x_{n}] \otimes \bar{T}(n))) = (s_{\bar{\mu}(n)} * s_{\bar{\mu}(n)})(1, q, q^{2}, \ldots)$$

The latter expression can be rewritten as

$$ch_{q}(\text{Hom}_{S_{\nu}}(\bar{\mu}(n), \mathbb{C}[x_{1}, \ldots, x_{n}] \otimes \bar{T}(n))) = \sum_{\lambda, \rho, \chi \lambda \leq n} \gamma_{\bar{\lambda}(n)}^{\bar{\lambda}(n)} s_{\lambda}(n)(1, q, q^{2}, \ldots) =$$

$$= \sum_{\lambda} \pi_{\lambda \rho}^{\lambda} s_{\lambda}(n)(1, q, q^{2}, \ldots)$$

Here

- $s_{\lambda}(n)$ is the Schur symmetric function corresponding to the partition $\lambda(n)$ of $n$ (see [17, Chapter I, Par. 3, p.41]),
- $s_{\bar{\mu}(n)} * s_{\bar{\mu}(n)}$ is the internal product of Schur symmetric functions as defined in [17, Chapter I, Par. 7, p.116],
- $\gamma_{\alpha' \alpha''}^{\alpha'} := \frac{1}{n!} \sum_{w \in S_{n}} \chi_{\alpha'}(w) \chi_{\alpha''}(w)$ is the Kronecker coefficient of partitions $\alpha', \alpha''$ of $n$ (here $\chi_{\alpha'}(w)$ is the value at $w \in S_{n}$ of the character of the irreducible representation of $S_{n}$ corresponding to the Young diagram $\alpha'$).
- $\pi_{\lambda \rho}^{\lambda}$ is the reduced Kronecker coefficient (equals $\gamma_{\lambda(n)}^{\lambda(n)}$ for $n \gg 0$). A good reference for standard and reduced Kronecker coefficients is [13].

**Proof.** We define the map $\mathcal{F}$ (denoted by $ch$ in [17, Chapter I, Par. 7]) from conjugation-invariant functions on $S_{n}$ to symmetric functions in countably many variables by putting

$$\mathcal{F}(f) := \frac{1}{n!} \sum_{w \in S_{n}} f(w) \prod_{j \geq 1} \left( \sum_{i} x_{i}^{\rho_{j}(w)} \right)$$

where $\rho(w) = (\rho_{1}, \rho_{2}, \ldots)$ is the cycle-type of $w$.

Denoting

$$p_{\rho} := \prod_{j \geq 1} \left( \sum_{i} x_{i}^{\rho_{j}(w)} \right), m(\rho) := \text{number of parts of size } i \text{ in } \rho, z_{\rho} := \prod_{i \geq 1} i^{m(\rho)_{i}} m(\rho)_{i}!$$

we get:

$$\mathcal{F}(f) := \sum_{\rho \geq 0} \frac{1}{z_{\rho}} f(\rho) p_{\rho}$$

(here $f(\rho)$ is the value of $f$ on the conjugacy class of $S_{n}$ consisting of permutations of cycle-type $\rho$).

Let $V$ be a representation of $S_{n}$. Denote by $\chi^{V}$ the character of $V$, and by abuse of notation, $\mathcal{F}(V) := \mathcal{F}(\chi^{V})$. If $V = \bigoplus_{j} V_{j}$ is a $\mathbb{Z}$-graded representation of $S_{n}$, then put $\mathcal{F}_{g}(V) := \bigoplus_{j \in \mathbb{Z}} \mathcal{F}(\chi^{V_{j}}) q^{j}$.

We have: $\mathcal{F}(\chi^{\alpha}) = s_{\alpha}$ for partition $\alpha$ of $n$, and $\mathcal{F}(\chi^{\alpha'} \chi^{\alpha''}) := \mathcal{F}(V') * \mathcal{F}(V'')$ ($\chi^{\alpha'} \chi^{\alpha''}$ is the character of the representation $\alpha' \otimes \alpha''$ of $S_{n}$).

Denote by $g_{w}$ the action of $w \in S_{n}$ on the $n$-dimensional complex vector space $\mathbb{C}^{n}$, $g_{w}$ given by the permutation matrix corresponding to $w$. 

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By MacMahon’s Master theorem (see [9] Lemma 3.28); the proof relies on an argument similar to the one used in [12,22], we have:

\[
 \mathcal{F}_q(\mathbb{C}[x_1, \ldots, x_n]) = \frac{1}{n!} \sum_{w \in S_n} tr(Sym^k(g_w))q^k \rho(w) = \frac{1}{n!} \sum_{w \in S_n} \frac{1}{det(1-qg_w)} \rho(w) = \frac{1}{n!} \sum_{w \in S_n} \prod_{1 \leq k \leq l(p(w))} \frac{1}{1 - q^k} \rho(w) = \frac{1}{n!} \sum_{w \in S_n} \prod_{1 \leq k \leq l(p(w))} \frac{1}{1 - q^k} \rho(w)
\]

Similarly,

\[
 \mathcal{F}_q(\mathbb{C}[x_1, \ldots, x_n] \otimes \hat{\tau}(n))) = \frac{1}{z_\rho} \prod_{1 \leq k \leq l(\rho)} \frac{1}{1 - q^k} \rho_\tau(\hat{\tau}(n)) = \sum_{\alpha,\rho \in S \tau} \frac{1}{z_\rho} \prod_{1 \leq k \leq l(\rho)} \frac{1}{1 - q^k} \rho_\tau(\hat{\tau}(n)) = (s_\tau(n) \ast s_\tau(n))(1, q, q^2, \ldots)
\]

Thus

\[
 ch_q(\text{Hom}_{S_n}(\hat{\mu}(n), \mathbb{C}[x_1, \ldots, x_n] \otimes \hat{\tau}(n))) = \text{coefficient of } s_\mu(n) \text{ in } \mathcal{F}_q(\mathbb{C}[x_1, \ldots, x_n] \otimes \hat{\tau}(n))) = \sum_{\rho \in S \tau} \frac{1}{z_\rho} \prod_{1 \leq k \leq l(\rho)} \frac{1}{1 - q^k} \rho_\tau(\hat{\tau}(n)) = (s_\tau(n) \ast s_\tau(n))(1, q, q^2, \ldots)
\]

The last equality holds by [17] Chapter 6, par. 8., p.363]. □

**Corollary 12.3.2.** Taking \( h_0 \) to be the reflection representation of \( S_n \), we get:

\[
 ch_q(\text{Hom}_{S_n}(\hat{\mu}(n), S h_0 \otimes \hat{\tau}(n))) = (1 - q)(s_\tau(n) \ast s_\tau(n))(1, q, q^2, \ldots) = (1 - q) \sum_{\lambda} \overline{\gamma}_{\tau,\mu}^\lambda s_{\lambda(n)}(1, q, q^2, \ldots)
\]

**Proof.** This follows directly from the fact that \( \mathbb{C}[x_1, \ldots, x_n] = S h_0 \), where \( h = h_0 \oplus \mathbb{C} \) is the permutation representation of \( S_n \), and so \( S^n h = \bigoplus_{0 \leq j \leq m} S j h_0 \). □

**Corollary 12.3.3.**

\[
 ch_q(\text{Hom}_{S_n}(X_{\mu}, S h_0 \otimes X_{\tau})) = (1 - q) \sum_{\lambda} \overline{\gamma}_{\tau,\mu}^\lambda s_{\lambda}(1, q, q^2, \ldots)
\]

where \( \overline{\gamma}_{\tau,\mu}^\lambda \) is the reduced Kronecker coefficient, and

\[
 s_{\lambda}(1, q, q^2, \ldots) := \frac{q^{[\lambda]}s_{\lambda}(1, q, q^2, \ldots)}{\prod_{j \geq 1} (1 - q^j)}
\]

**Proof.** By the structure of Deligne’s category described in Section 4, if there exist integers \( k, N \) such that for any \( n \geq N \), \( \dim \text{Hom}_{S_n}(\hat{\mu}(n), S^n h_0) = k \) (here for each \( n \), \( h_0 \) is the reflection representation of \( S_n \)), then \( \dim \text{Hom}_{S_n}(X_{\mu}, S^n h_0) = k \) in Deligne’s category \( \text{Rep}(S_n) \) as well.

Thus the power series in \( ch_q(\text{Hom}_{S_n}(X_{\mu}, S h_0 \otimes X_{\tau})) \) is the \( q \)-adic limit, as \( n \) tends to infinity, of \( ch_q(\text{Hom}_{S_n}(\hat{\mu}(n), S h_0 \otimes \hat{\tau}(n))) \); recall that, by definition, a sequence \( \{P^{(n)}(q)\}_n \) is \( q \)-adic convergent, in the \( q \)-adic metric, to the formal power series \( P(q) \) if for any \( k \geq 0 \) there exists \( N_k \in \mathbb{Z}_{>0} \) such that for any \( n > N_k \), \( q^{k+1} \mid (P(q) - P^{(n)}(q)) \).

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By [17, Chapter I, Par. 5, Example 1],
\[
s_\lambda(1, q, q^2, \ldots) = \frac{q^{\sum_{i \geq 1} i \lambda_i - |\lambda|}}{\prod_{x \in \lambda}(1 - q^{h(x)})}
\]
where \(h(x)\) is the size of the hook in \(\lambda\) with vertex \(x\), so
\[
s_{\lambda(n)}(1, q, q^2, \ldots) = \frac{q^{\sum_{i \leq n} i \lambda_i}}{\prod_{x \in \lambda}(1 - q^{h(x)})} \cdot \prod_{1 \leq j \leq n - |\lambda|} \frac{1}{1 - q^{n-|\lambda|-j+1+\lambda_j}}
\]
The \(q\)-adic limit of \(s_{\lambda(n)}(1, q, q^2, \ldots)\), as \(n\) tends to infinity, is then
\[
\overline{s}_\lambda(1, q, q^2, \ldots) := \frac{q^{\sum_{i \geq 1} i \lambda_i}}{\prod_{x \in \lambda}(1 - q^{h(x)})} \cdot \prod_{j \geq 1} \frac{1}{1 - q^{j}} = \frac{q^{|\lambda|} s_\lambda(1, q, q^2, \ldots)}{\prod_{j \geq 1}(1 - q^j)}
\]
This completes the proof of the statement. \(\square\)

**Example 12.3.4.** For \(\tau = \emptyset\), we get:
\[
ch_q(\text{Hom}_{S_\nu}(X_\mu, S\mathfrak{h}_0)) = (1 - q)\overline{s}_\mu(1, q, q^2, \ldots) = \frac{q^{|\mu|} s_\mu(1, q, q^2, \ldots)}{\prod_{j \geq 1}(1 - q^j)} = \frac{q^{\sum_{i \geq 1} \mu_i} \overline{s}_\mu(1, q, q^2, \ldots)}{\prod_{j \geq 1}(1 - q^j)}
\]

### 12.4. Characters of simple objects: generic cases.

#### 12.4.1. Fix a Young diagram \(\tau\). Recall that for each Young diagram \(\mu \neq \tau\) and \(m \in \mathbb{Z}_+\), there is a set of points \((c', \nu)\) for which \(M_{c',\nu}(\mu)\) maps to \(M_{c',\nu}(\tau)\) in degree \(m\). These sets are either straight lines or finite sets.

The union of these sets is the reducibility locus of \(M_{c',\nu}(\tau)\), denoted by \(B_\tau\). Outside this reducibility locus \(L_{c',\nu}(\tau) = M_{c',\nu}(\tau)\), and the character of this object is given by the formula
\[
chL(\tau) = chM(\tau) = \frac{t^{he_{c',\nu}(\tau)}X_{\tau}}{\sum_{n \geq 0}(-1)^n \Lambda^n \mathfrak{h}_0 t^n}
\]
This is the most generic case.

#### 12.4.2. The next most generic case is a generic point on a line \(L_{c,\nu,m} \subset B_{\mu,\tau}\).

Fix a pair \(s, r\) such that \(s, r \in \mathbb{Z}, r \neq 0, s \geq 0\), and consider a generic point \((c', \nu)\) on the line \(c' = \frac{\nu - s}{r}\).

Let \(\tau\) be a Young diagram such that \(s \in C_\tau\) (this exactly means that \(M(\tau)\) is actually reducible at this point \((c, \nu)\)), and denote \(\mu := \Gamma(\tau, s, \pm 1)\), where the sign equals \(\text{sign}(\tau)\).

Then there are only two distinct Verma objects which map non-trivially into \(M(\tau): M(\mu)\) and \(M(\tau)\) itself (see Subsection 11.2). Both map uniquely (up to scalar multiple) into \(M(\tau)\).

**Lemma 12.4.1.** The image of \(M(\mu)\) in \(M(\tau)\) is a simple \(O_{c',\nu}\)-object.

**Proof.** Assume \(X_\lambda\) is a simple singular \(\text{Rep}(S_\nu)\)-subobject in \(\text{Im}(M(\mu))\), and \(X_\lambda \subset M(\tau)\) lies in degree \(m'\). Then the action of \(h\) on \(M(\tau)\) and \(J(\tau)/\text{Im}(M(\mu))\) gives (see Proposition 8.1.1).
Lemma 12.4.2. \( \text{Assume} \)

Proof. \( \text{Assume} \) \( X_\lambda \) is a simple singular \( \text{Rep}(S_\nu) \)-subobject in \( \text{Im}(M(\mu)) \), this means that \( X_\lambda \) coincides with the lowest weight in \( \text{Im}(M(\mu)) \).

Thus \( \text{Im}(M(\mu)) \) is simple. \( \Box \)

We now want to show that there is a short exact sequence:

\[ 0 \rightarrow L(\mu) \rightarrow M(\tau) \rightarrow L(\tau) \rightarrow 0 \]

This is equivalent to saying that \( \text{Im}(M(\mu)) = J(\tau) \).

Lemma 12.4.2. \( M(\tau)/\text{Im}(M(\mu)) \) is a simple \( O_{c,\nu} \)-object.

Proof. \( \text{Assume} X_\lambda \) is a simple singular \( \text{Rep}(S_\nu) \)-subobject in \( M(\tau)/\text{Im}(M(\mu)) \), and its preimage \( X_\lambda \subset M(\tau) \) lies in degree \( m' \).

Then the action of \( h \) on \( M(\tau) \) and \( J(\tau)/\text{Im}(M(\mu)) \) gives (see Proposition 8.1.1):

\[
c' = \frac{\nu - s}{r} + \frac{|\tau|^2 - |\lambda|^2 - (|\tau| - |\lambda|)}{2} + c(\tau) - c(\lambda) = c'm' + (|\tau| - |\lambda|)\nu
\]

We now need to consider separately the case when \( \lambda = \mu \). Indeed, in that case the above equations mean that \( m' = m \), and thus we have \( X := X_\mu \subset M(\tau) \) (not a singular subobject) which lies in degree \( m' \) and whose image in \( M(\tau)/\text{Im}(M(\mu)) \) is not zero and singular.

The former implies that \( y_{M(\tau)}(h_0 \otimes X) \neq 0 \) is a direct sum of simple \( \text{Rep}(S_\nu) \)-objects lying in degree \( m - 1 \) of \( M(\tau) \).

The fact that image of \( X \) in \( M(\tau)/\text{Im}(M(\mu)) \) is singular would mean that \( y_{M(\tau)}(h_0 \otimes X) \subset \text{Im}(M(\mu)) \)

But this leads to a contradiction, since \( \text{Im}(M(\mu)) \) lies in degrees strictly higher than \( m - 1 \) of \( M(\tau) \).

So \( \lambda \neq \mu \), and this contradicts our assumption. \( \Box \)

Thus for generic \((c', \nu)\) such that \( c' = \frac{\nu - s}{r} \), we have a short exact sequence:

\[ 0 \rightarrow L(\mu) \rightarrow M(\tau) \rightarrow L(\tau) \rightarrow 0 \]

and a long exact sequence

\[ \ldots \rightarrow M(\Gamma(\tau, s, \pm l)) \rightarrow M(\Gamma(\tau, s, \pm (l - 1))) \rightarrow \ldots \rightarrow M(\Gamma(\tau, s, \pm 1)) \rightarrow M(\tau) \rightarrow L(\tau) \rightarrow 0 \]

As before, the sign corresponds to the sign of \( r \), and this sequence ends (on the left) with \( M(\text{rec}(0, \text{core}_{(\nu-s)}(\tau))) = M(\Gamma(\tau, s, -\tau_\nu)) \) if \( r < 0 \).

This allows us to compute the character of \( L(\tau) \) by Euler formula:

\[
chL(\tau) = \sum_{l \in \mathbb{Z}_+, \text{and } l \leq \tau_\nu, \text{if } r < 0} (-1)^l chM(\Gamma(\tau, s, \pm l))
\]
Now, 
\[ h_{c,v}(\Gamma(\tau, s, \pm l)) - h_{c,v}(\tau) = \]
\[ = \sum_{1 \leq l' \leq l} h_{c,v}(\Gamma(\tau, s, \pm l')) - h_{c,v}(\Gamma(\tau, s, \pm (l' - 1))) = \]
\[ = \sum_{1 \leq l' \leq l} m_{l'} = \sum_{1 \leq l' \leq l} c(\nu - s) (|\Gamma(\tau, s, \pm l')| - |\Gamma(\tau, s, \pm (l' - 1))|) = \]
\[ = c(\nu - s) ([\Gamma(\tau, s, \pm l]) - |\tau|) = c(\nu - s) (j_s - k_{s,l} \pm l) \]

Here we denote by \( m_{l'} \) the degree of \( M(\Gamma(\tau, s, \pm (l' - 1))) \) in which lies the image of the lowest weight of \( M(\Gamma(\tau, s, \pm l')) \), and use Lemma 10.0.21.

Using Subsection 12.2 and the fact that \( \text{chL} \) (see Lemma 8.2.1) has only one non-trivial singular \( \text{Rep}(\lambda) \)-subobject, then Equation (1) and Pieri’s rule (Proposition 4.2.1) implies that

\[ h_{c,v}(\Gamma(\tau, s, \pm l)) - h_{c,v}(\tau) = c(\nu - s) (j_s - k_{s,l} \pm l) \]
we get:

\[ \text{chL}(\tau) = \frac{t^{hc,v}(\tau)}{\sum_{l \in \mathbb{Z}_+} \text{chL}(\lambda^t)} \left( \sum_{l \in \mathbb{Z}_+} (-1)^l X_{n+l} (\nu-s) (j_s-k_{s,l} \pm l) \right) \]

As before, the sign corresponds to the sign of \( r \).

**Example 12.4.3.** Let \( cv = 1, Re(c) > 0 \).

If \( X_\lambda \) lies in degree \( m' \) of \( M(\pi^n) \) as a singular \( \text{Rep}(S_\nu)-\text{subobject} \) (\( \pi^n \) is a column diagram with \( n \) cells), then Equation (11) gives us:

\[ m = (|\lambda| - n)c\nu - f(\lambda)c = (|\lambda| - n) - f(\lambda)c \]
and Pieri’s rule (Proposition 4.2.1) implies that \( m \geq ||\lambda| - n| \). Since \( m, |\lambda|, f(\lambda) \in \mathbb{Z}_+ \), we get that \( c \in \mathbb{Q}_{>0} \).

We also have: \( f(\lambda) \geq 0 \), with equality iff \( \lambda \) is a column diagram (see Lemma 8.2.1). So

\[ ||\lambda| - n| \leq m = (|\lambda| - n) - f(\lambda)c \leq |\lambda| - n \]
which means that \( f(\lambda) = 0, |\lambda| \geq n, i.e. \lambda = \pi^k \) for some \( k \geq n \) (in fact, \( k > n \)), and \( X_{\pi^k} \) lies in degree \( m = k - n \) of \( M(\pi^n) \).

But by Pieri’s rule (Proposition 4.2.1), for \( k > 1 \), \( T^k \mathfrak{h}_0 \) (k-th tensor power of \( \mathfrak{h}_0 \)) contains only one copy of \( X_{\pi^k} \cong \Lambda^k(\mathfrak{h}_0) \), and it does not lie inside \( \text{Sym}^k(\mathfrak{h}_0) \). So \( \text{Sym}^k(\mathfrak{h}_0) \) doesn’t contain subobjects isomorphic to \( X_{\pi^k} \).

So \( M(\pi^n) \) has only one non-trivial singular \( \text{Rep}(S_\nu)-\text{subobject}: X_{\pi^{n+1}} \), which lies in degree 1 of \( M(\pi^n) \).

In this case, we can compute the character of \( L(\pi^n) \) as in Equation (3). We assumed that \( cv = 1 \), so \( h_{c,v}(\pi^n) = \frac{(\nu-1)(1-c\nu)}{2} + n = n \), hence

\[ \text{chL}(\pi^n) = \frac{\sum_{l \in \mathbb{Z}_+} (-1)^l X_{\pi^{n+l}}}{\sum_{l \in \mathbb{Z}_+} (-1)^l \Lambda^l \mathfrak{h}_0^l} \]

But \( \Lambda^l \mathfrak{h}_0 \cong X_{\pi^{n+l}} \) in \( \text{Rep}(S_\nu) \), so

\[ \text{chL}(\pi^n) = \frac{\sum_{l \in \mathbb{Z}_+} (-1)^l \Lambda^l \mathfrak{h}_0^l}{\sum_{l \in \mathbb{Z}_+} (-1)^l \Lambda^l \mathfrak{h}_0^l} \]

Note that \( L(\emptyset) \cong X_\emptyset = 1 \) as \( \text{Rep}(S_\nu) \) (ind)-objects, with maps \( x_{L(\emptyset)}, y_{L(\emptyset)} = 0 \).
Example 12.4.4. Similarly, for a generic point \((c, \nu)\) on the line \(c\nu = k\), \(k \in \mathbb{Z}_{>0}\), we have: \(s = 0, r = k\) and so
\[
\Gamma(\pi^n, 0, \pm t) = X_{\pi^n \pm t}
\]
\[
chL(\pi^n) = t^{kn} \cdot \frac{\sum_{l \in \mathbb{Z}_+} (-1)^l \Lambda_{l} h_0 t^{kl}}{\sum_{l \in \mathbb{Z}_+} (-1)^l \Lambda_{l} h_0 t^{l}}
\]
and in particular
\[
chL(\emptyset) = t^{-\frac{(r-1)(k-1)}{2}} \cdot \frac{\sum_{l \in \mathbb{Z}_+} (-1)^l \Lambda_{l} h_0 t^{kl}}{\sum_{l \in \mathbb{Z}_+} (-1)^l \Lambda_{l} h_0 t^{l}} = t^{-\frac{(r-1)(k-1)}{2}} \cdot \frac{ch_1 S h_0}{ch_{l_k} S h_0}
\]
(see [2] or [9, Corollary 3.50] for the corresponding result for \(O(H_c(n))\)).

As in the case when \(k = 1\), we can also compute the character of \(L_{c,\nu}(\emptyset)\) explicitly:

**Proposition 12.4.5.** Let \((c, \nu)\) be a generic point on the line \(c\nu = k\), where \(k \in \mathbb{Z}_{>0}\) is fixed. Then
\[
ch_q \text{Hom}_{S_{\nu}}(X_{\mu}, L_{c,\nu}(\emptyset)) = q^{\mu | s_{\mu}(1, q, q^2, \ldots, q^{k-2})} \prod_{2 \leq j \leq k} (1 - q^j)
\]

**Proof.** From the exact sequence
\[
\ldots \to M(\pi^t) \to M(\pi^{t-1}) \to \ldots \to M(\pi^1) \to M(\emptyset) \to L(\emptyset) \to 0
\]
and the character formula Corollary [12.3.3] for \(\text{Hom}_{\text{Rep}(S_{\nu})}(X_{\mu}, S h_0 \otimes X_{\pi^t})\), we have:
\[
ch_q \text{Hom}_{S_{\nu}}(X_{\mu}, L(\emptyset)) = (1 - q) \sum_{l \geq 0} \sum_{\lambda \vdash n} (-1)^l q^{\lambda | s_{\mu}(1, q, q^2, \ldots)} \frac{q^{\lambda | \rho_{\mu} \lambda}}{\prod_{j \geq 1} (1 - q^j)}
\]

One immediately sees that this is the \(q\)-adic limit, as \(n\) tends to infinity, of the sequence of formal power series
\[
P_n(q) := (1 - q) \sum_{l \geq 0} \sum_{\lambda \vdash n} (-1)^l q^{\lambda | \rho_{\mu} \lambda} s_{\lambda}(1, q, q^2, \ldots)
\]
where \(h_0\) is the reflection representation of \(S_n\).

We now give a formula for \(P_n(q)\). First, recall that (see [17, Chapter I, Par. 7])
\[
\gamma_{\lambda h_0, \lambda}(n) = \frac{1}{n!} \sum_{w \in S_n} \chi_{\lambda h_0}(w) \chi_{\lambda}(w) \chi_{\lambda}(w)
\]
(here \(\chi_{\beta}^\lambda(w) = tr |_{\beta} (w)\) is the value at \(w\) of the character of \(S_n\) corresponding to the irreducible representation \(\beta\)).

Now, from the exact sequence
\[
0 \to S h_0 \otimes \Lambda^{n-1} h_0 \to \ldots \to S h_0 \otimes \Lambda^{n} h_0 \to \ldots \to S h_0 \otimes h_0 \to S h_0 \to \mathbb{C} \to 0
\]
we have:
\[
\sum_{l \geq 0} (-1)^l \frac{q^{kl} \chi_{\lambda h_0}(w)}{\sum_{l \geq 0} q^{kl} tr |_{S h_0} (w)} = \ldots
\]
\[
\ldots = \frac{1}{\sum_{l \geq 0} q^{kl} (1 - q^k) tr |_{S h_0} (w)} = \frac{1}{1 - q^k} \prod_{1 \leq j \leq l(\rho(w))} (1 - q^{k \rho(w)_j})
\]
(here \(\rho(w)\) is the cycle type of \(w\)).
On the other hand, we have (see e.g. [17], Par. 1, (7.7), (7.8), p. 114):

\[ \sum_{\lambda \vdash n} \chi^\lambda(w)s_\lambda(1,q,q^2,...) = \prod_{1 \leq j \leq \ell(\rho(w))} \frac{1}{1-q^{\rho(w)}}, \]

Thus we obtain:

\[ P_n(q) = (1-q)\sum_{l \geq 0} \sum_{\lambda \vdash n} (-1)^l q^{kl} \chi^\lambda(\Lambda_{h_0, \mu(n)})s_\lambda(1,q,q^2,...) = \]

\[ = \frac{1-q}{1-q^k} \frac{1}{k!} \sum_{\mu \in S_n} \chi^{\mu(n)}(w) \prod_{1 \leq j \leq \ell(\rho(w))} \frac{1}{1-q^{\rho(w)}}, \]

\[ = \frac{1-q}{1-q^k} \frac{1}{k!} \sum_{w \in S_n} \chi^{\mu(n)}(w) \prod_{1 \leq j \leq \ell(\rho(w))} (1+q^{\rho(w)}+q^{2\rho(w)}+...+q^{k\rho(w)}) = \frac{1-q}{1-q^k} s_{\mu(n)}(1,q,...,q^{k-1}) \]

Taking the \( q \)-adic limit when \( n \to \infty \), we obtain: the \( q \)-adic limit of \( s_{\mu(n)}(1,q,...,q^{k-1}) \) is \( \frac{q^{\mu(1,0,0,0,0,0)}(1,q^2,...,q^{k-2})}{\prod_{1 \leq j \leq k}(1-q^j)} \), and thus

\[ ch_q \text{Hom}_{S_n}(X_\mu, L(\emptyset)) = \frac{q^{\mu(1,0,0,0,0,0)}(1,q^2,...,q^{k-2})}{\prod_{2 \leq j \leq k}(1-q^j)} \]

\[ \Box \]

**Remark 12.4.6.** Note that \( s_{\mu}(1,q,q^2,...,q^{k-2}) = 0 \) if \( l(\mu) \geq k \), so \( \text{Hom}_{S_n}(X_\mu, L(\emptyset)) = 0 \) whenever \( l(\mu) \geq k \).

13. **Length of Verma objects**

In this section, we will discuss the set of points \((c,\nu)\) such that the Verma object \( M_{c,\nu}(\tau) \) is of finite length. We will prove the following theorem:

**Theorem 13.0.7.** For any \( c \notin \mathbb{Q}_{<0} \) and any Young diagram \( \tau \), the Verma object \( M_{c,\nu}(\tau) \) is of finite length.

By Subsection 11.1, \( M_{c,\nu}(\tau) \) has infinite length whenever \((c,\nu)\) is the intersection point of infinitely many curves of form \( \frac{1}{c} = \frac{\nu - s}{r} \) where \( s \in C_\tau, r \in \mathbb{Z} \setminus \{0\} \).

For instance, when \( \tau = \emptyset \), we have:

**Lemma 13.0.8.** For any \( c \in \mathbb{Q}_{<0} \) there exists \( \nu \in \mathbb{Q} \) such that the Verma object \( M_{c,\nu}(\emptyset) \) has infinite length.

**Proof.** Indeed, for any \( c \in \mathbb{Q}_{<0} \) and any \( \nu \in \mathbb{Q} \) such that \( \text{den}(\nu) \mid \text{num}(\nu), \text{den}(\nu) \mid (\text{num}(\nu)+\text{den}(\nu)) \) (i.e. \( \text{num}(\nu)\text{den}(\nu)^{-1} \equiv -1 \ mod \ \text{den}(\nu) \)), we have:

\[ \{s \in \mathbb{Z}_+ \mid c(\nu-s) \in \mathbb{Z}_{>0}\} = \{s \in \mathbb{Z}_+ \mid s \equiv -1 \ mod \ \text{den}(\nu), s > \nu\} \]

Now, recall that \( C_\emptyset = \mathbb{Z}_+ \). So any \( s \) such that \( s \equiv -1 \ mod \ \text{den}(\nu), s > \nu \) gives us a non-trivial morphism \( M_{c,\nu}(\tau^s) \rightarrow M_{c,\nu}(\emptyset) \). The image of each of these morphisms has a simple object \( L_{c,\nu}(\tau^s) \) as a quotient, and since all the weights \( \tau^s \) are different, each of these simple objects \( L_{c,\nu}(\tau^s) \) contributes to the length of \( M_{c,\nu}(\emptyset) \). This proves that \( M_{c,\nu}(\emptyset) \) has infinite length.

The proof consists of two parts: proving that whenever \( c \in \mathbb{R}_{>0} \), the Verma object \( M_{c,\nu}(\tau) \) is of finite length (we will prove this statement in Subsection 13.2), and proving that whenever \( c \notin \mathbb{Q} \), the Verma object \( M_{c,\nu}(\tau) \) is of finite length, too (we will prove that in Subsection 13.3).
Remark 13.0.9. In the classical case, all modules in \(O(H_c(n))\) have finite length. See [3] Corollary 3.26.

13.1. Bounds on the graded space \(\text{Hom}_{\text{Rep}(S_\nu)}(X_\mu, S_\theta \otimes X_\tau)\). In this subsection, we give a bound on the least degree in which a simple \(\text{Rep}(S_\nu)\) object \(X_\mu\) can lie in the \(\text{Rep}(S_\nu)\) ind-object \(S_\theta \otimes X_\tau\).

Recall that we have the following corollary of Lemma 12.3.1.

Corollary 13.1.1. For \(\tau = \emptyset\), we have:

\[
ch_q(\text{Hom}_{S_n}(\bar{\mu}(n), \mathbb{C}[x_1, \ldots, x_n])) = s_{\bar{\mu}(n)}(1, q, q^2, \ldots)
\]

By the definition of a Schur symmetric function, \(s_{\bar{\mu}(n)}(1, q, q^2, \ldots)\) is divisible by \(q^{\sum_k \mu_k k}\). So for any \(n >> 0\), the minimal degree of \(S_\tau = \mathbb{C}[x_1, \ldots, x_n]\) in which \(\bar{\mu}(n)\) appears is greater than or equal \(\sum_k \mu_k k\).

Corollary 13.1.2. For any \(\nu\), we have, in \(\text{Rep}(S_\nu)\): the minimal degree of \(S_\tau\) (and thus \(S_\theta\) as well) in which \(X_\mu\) appears is greater than or equal \(\sum_k \mu_k k\).

We now generalize this result to the following:

Proposition 13.1.3. Consider the \(\mathbb{Z}_+\)-graded complex vector space \(\text{Hom}_{\text{Rep}(S_\nu)}(X_\mu, S_\theta \otimes X_\tau)\) (the grading inherited from \(S_\theta\)). Let \(m\) be a grade of this space containing a non-zero morphism. Then

\[
m \geq \sum_{k \geq 1} \mu_k k - l(\mu) |\tau| - \frac{3}{2} |\tau|^2 - \frac{|\tau|}{2}
\]

Proof. Since any object of \(\text{Rep}(S_\nu)\) is isomorphic to its dual, \(\text{Hom}_{\text{Rep}(S_\nu)}(X_\mu, S_\theta \otimes X_\tau) \cong \text{Hom}_{\text{Rep}(S_\nu)}(X_\mu \otimes X_\tau, S_\theta)\) as \(\mathbb{Z}_+\)-graded vector spaces.

By the definition of \(m\), this means that \(X_\mu \otimes X_\tau\) and \(S^m S_\theta\) have a common composition factor. Now, \(X_\tau \subset S_\theta \otimes |\tau|\), so \(X_\mu \otimes S_\theta \otimes |\tau|\) and \(S^m S_\theta\) have a common composition factor.

A simple subobject \(X_\lambda\) of \(X_\mu \otimes S_\theta \otimes |\tau|\) satisfies the condition arising from Pieri’s rule (Proposition 4.2.1):

Condition 13.1.4. \(\lambda\) can be obtained from \(\mu\) by performing at most \(|\tau|\) steps, each consisting of either adding a cell, deleting a cell or moving a cell (i.e. deleting a cell and then adding a cell).

We know that for some \(\lambda\) as above, \(X_\lambda \subset S^m S_\theta\), so by Corollary 13.1.2

\[
m \geq \sum_k \lambda_k k
\]

We are left with the problem of giving a lower bound for \(\sum_{k \geq 1} \lambda_k k - \sum_{k \geq 1} \mu_k k\) for two Young diagrams satisfying Condition 13.1.4.

By the Condition 13.1.4 \(\sum_{k \geq 1} \lambda_k k\) is maximal when \(\lambda\) is obtained from \(\mu\) by adding a column of \(|\tau|\) cells to \(\mu\), i.e.

\[
\lambda_i := \begin{cases} 
\mu_i & \text{if } 1 \leq i \leq l(\mu) \\
1 & \text{if } l(\mu) + 1 \leq i \leq l(\mu) + |\tau| 
\end{cases}
\]
So

\[ \sum_{k \geq 1} \mu_k k \leq \sum_{k \geq 1} \lambda_k k + l(\lambda) |\tau| + \frac{|\tau|(|\tau| + 1)}{2} \leq \sum_{k \geq 1} \lambda_k k + l(\mu) |\tau| + \frac{|\tau|(|\tau| + 1)}{2} \]

Thus

\[ \sum_{k \geq 1} \lambda_k k \geq \sum_{k \geq 1} \mu_k k - l(\mu) |\tau| - |\tau|^2 - \frac{|\tau|(|\tau| + 1)}{2} \]

and, finally, we get:

\[ m \geq \sum_{k} \lambda_k k \geq \sum_{k \geq 1} \mu_k k - l(\mu) |\tau| - |\tau|^2 - \frac{|\tau|(|\tau| + 1)}{2} = \sum_{k \geq 1} \mu_k k - l(\mu) |\tau| - \frac{3}{2} |\tau|^2 - \frac{|\tau|}{2} \]

\[ \square \]

13.2. Length of Verma objects for \( c \in \mathbb{R}_{>0} \).

**Proposition 13.2.1.** For any \( c \in \mathbb{R}_{>0} \), the Verma object \( M_{c,\nu}(\tau) \) is of finite length.

**Proof.** Fix \( c, \nu \) such that \( c \in \mathbb{R}_{>0} \). We need to prove that \( M(\tau) \) has only finitely many composition factors. Let \( \{L(\mu^j)\}_{j \in J} \) be the composition factors of \( M(\tau) \).

Since \( L(\mu^j) \) is a composition factor of \( M(\tau) \), there exists a positive integer \( m^j \) such that \( h_{c,\nu}(\mu^j) = h_{c,\nu}(\tau) + m^j \). By Proposition 8.1.1 to prove that \( J \) is a finite set, it is enough to prove that there exist only finitely many Young diagrams \( \mu \) such that \( X_\mu \) lies in degree \( m \) of \( M(\tau) \) and

\[ m = c\nu(|\mu| - |\tau|) + c \left( \frac{|\tau|^2 - |\mu|^2 - (|\tau| - |\mu|)}{2} + ct(\tau) - ct(\mu) \right) \]

By Proposition 13.1.3, it is, in fact, enough to prove that

\[ c\nu(|\mu| - |\tau|) + c \left( \frac{|\tau|^2 - |\mu|^2 - (|\tau| - |\mu|)}{2} + ct(\tau) - ct(\mu) \right) < \sum_{k \geq 1} \mu_k k - l(\mu) |\tau| - \frac{3}{2} |\tau|^2 - \frac{|\tau|}{2} \]

for all but finitely many Young diagrams \( \mu \).

Recall that \( ct(\mu) = \sum_{k \geq 1} \mu_k k - \sum_{k \geq 1} \mu_k k \), so subtracting LHS from RHS in the above expression, we get

\[ -c\nu|\mu| + c\nu|\tau| - cf(\tau) + c \frac{|\mu|^2 - |\mu|}{2} + c \sum_{k \geq 1} \mu_k k - c \sum_{k \geq 1} \mu_k k + \]

\[ + \sum_{k \geq 1} \mu_k k - l(\mu) |\tau| - \frac{3}{2} |\tau|^2 - \frac{|\tau|}{2} = c \frac{|\mu|^2 - |\mu|}{2} - \]

\[ - c\nu|\mu| + c \sum_{k \geq 1} \mu_k k - (c - 1) \sum_{k \geq 1} \mu_k k - l(\mu) |\tau| - \frac{3}{2} |\tau|^2 - \frac{|\tau|}{2} + c\nu|\tau| - cf(\tau) \]

where \( f(\tau) := \frac{|\tau|^2 - |\tau|}{2} + ct(\tau) \).
So we need to show that for all but finitely many Young diagrams $\mu$,
\[
\frac{c}{2} |\mu|^2 - c |\mu| - cv |\mu| + c \sum_{k \geq 1} \bar{\mu}_k k - (c - 1) \sum_{k \geq 1} \mu_k k - l(\mu) |\tau| - \frac{3}{2} |\tau|^2 - \frac{|\tau|}{2} + cv |\tau| - cf(\tau) > 0
\]

Since $l(\mu) \leq |\mu|$, it is in fact enough to check that all but finitely many Young diagrams $\mu$ satisfy the following condition:

**Condition 13.2.2.**

\[
\frac{c}{2} |\mu|^2 - \frac{c}{2} |\mu| - cv |\mu| - |\mu| |\tau| + c \sum_{k \geq 1} \bar{\mu}_k k - (c - 1) \sum_{k \geq 1} \mu_k k - \frac{3}{2} |\tau|^2 - \frac{|\tau|}{2} + cv |\tau| - cf(\tau) > 0
\]

We now have to consider two cases separately:

- If $0 < c < 1$, then $c \sum_{k \geq 1} \bar{\mu}_k k - (c - 1) \sum_{k \geq 1} \mu_k k > 0$, and

\[
\frac{c}{2} |\mu|^2 - \frac{c}{2} |\mu| - cv |\mu| - |\mu| |\tau| + c \sum_{k \geq 1} \bar{\mu}_k k - (c - 1) \sum_{k \geq 1} \mu_k k \geq 0
\]

for $\mu$ such that $|\mu|$ is large enough.

- If $c \geq 1$, then using the facts that $\sum_{k \geq 1} \bar{\mu}_k k \geq |\mu|$ and $\sum_{k \geq 1} \mu_k k \leq \frac{|\mu|(|\mu|+1)}{2}$, we get:

\[
\frac{c}{2} |\mu|^2 - \frac{c}{2} |\mu| - cv |\mu| - |\mu| |\tau| + c \sum_{k \geq 1} \bar{\mu}_k k - (c - 1) \sum_{k \geq 1} \mu_k k \geq \frac{c}{2} |\mu|^2 - \frac{c}{2} |\mu| - cv |\mu| - |\mu| |\tau| + c |\mu| - (c - 1) |\mu| (|\mu| + 1) = \frac{1}{2} |\mu|^2 + \left( \frac{1}{2} - cv - |\tau| \right) |\mu|
\]

So the RHS of Equation (13.2.2) is at least

\[
\frac{1}{2} |\mu|^2 + \left( \frac{1}{2} - cv - |\tau| \right) |\mu| - \frac{3}{2} |\tau|^2 - \frac{|\tau|}{2} + cv |\tau| - cf(\tau)
\]

which is obviously positive whenever $|\mu|$ is large enough, i.e. for all but finitely many Young diagrams $\mu$.

\[\square\]

13.3. Length of Verma objects for $c \notin \mathbb{Q}$.

**Proposition 13.3.1.** For any $c \notin \mathbb{Q}$, the Verma object $M_{c,\nu}(\tau)$ is of finite length.

**Proof.** Assume that for some $c \notin \mathbb{Q}, \nu$, a Verma object $M_{c,\nu}(\tau)$ has infinite length.

Let $\{L(\mu^j)\}_{j \in I}$ be the composition factors of $M(\tau)$.

Since $L(\mu^j)$ is a composition factor of $M(\tau)$, there exists a positive integer $m^j$ such that $h_{c,\nu}(\mu^j) = h_{c,\nu}(\tau) + m^j$, and $X_{\mu} \subset S^{m^j} \mathbb{H}_0 \otimes \tau$. Similarly to the proof of Proposition 8.1.1 for any $j$ we have:

\[
\frac{1}{c} (1/c, \nu) \in L_{\tau,\mu^j, m^j}
\]

Since $(1/c, \nu) \in L_{\tau,\mu^j, m^j}$ for any $j$, and $c \notin \mathbb{Q}$, Subsection 11.2 implies that for all $j$ the lines $L_{\tau,\mu^j, m^j}$ coincide; that is, the equations

\[
\frac{1}{c} m^j + \nu(|\tau| - |\mu^j|) = \frac{1}{2} \left( |\tau|^2 - |\mu^j|^2 - (|\tau| - |\mu^j|) \right) + ct(\tau) - ct(\mu^j)
\]

define the same line for all $j$. 38
This means that there exist constants $C, C' \in \mathbb{Q}$ such that for any $j$,

$$\frac{1}{|\mu^j| - |\tau|} \left( \frac{|\tau|^2 - |\mu^j|^2 - (|\tau| - |\mu^j|)}{2} + ct(\tau) - ct(\mu^j) \right) = C'$$

and $\frac{|\mu^j| - |\tau|}{m^j} = C$.

Since $X_{\mu^j} \subset S_{m^j}^{\mu^j} h_0 \otimes \tau$, Pieri’s rule (Proposition 4.2.1) implies that $|C| \leq 1$, and Proposition 13.1.3 implies that for any $j$,

$$m^j \geq \sum_{k \geq 1} (\mu^j)_k k - l(\mu^j) |\tau| - \frac{3}{2} |\tau|^2 - \frac{|\tau|}{2}$$

So there exist constants $C, C'$ such that for any $j$, the following conditions hold for $\mu := \mu^j, m := m^j$:

**Condition 13.3.2.**

$$\frac{1}{|\mu| - |\tau|} \left( \frac{|\mu|^2 - |\mu|}{2} + ct(\mu) - f(\tau) \right) = C'$$

$$\frac{|\mu| - |\tau|}{m} = C$$

$$m^j \geq \sum_{k \geq 1} (\mu)_k k - l(\mu) |\tau| - \frac{3}{2} |\tau|^2 - \frac{|\tau|}{2}$$

with $f(\tau) = \frac{|\tau|^2 - |\tau|}{2} + ct(\tau)$ as in Subsection 8.2.

We will show that for given $C, C'$ as above, Condition 13.3.2 can only hold for a finite number of Young diagrams $\mu$.

First, notice that we can assume that $C > 0$ (otherwise $\mu \leq |\tau|$ and we are done).

The second and third parts of Condition 13.3.2 give:

$$C^{-1}(|\mu| - |\tau|) \geq \sum_{k \geq 1} \mu_k k - l(\mu) |\tau| - \frac{3}{2} |\tau|^2 - \frac{|\tau|}{2}$$

i.e.

$$\sum_{k \geq 1} \mu_k k \leq \frac{3}{2} |\tau|^2 + \frac{|\tau|}{2} - C^{-1} |\tau| + l(\mu) |\tau| + C^{-1} |\mu|$$

The first part of Condition 13.3.2 gives:

$$\frac{|\mu|^2 - |\mu|}{2} + ct(\mu) - C' |\mu| = f(\tau) - C' |\tau|$$

Now, $ct(\mu) = \sum_{k \geq 1} \mu_k k - \sum_{k \geq 1} \mu_k k$, so

$$\frac{|\mu|^2 - |\mu|}{2} - C' |\mu| = f(\tau) - C' |\tau| - ct(\mu) = f(\tau) - C' |\tau| + k \sum_{k \geq 1} \mu_k k - \sum_{k \geq 1} \mu_k k \leq$$

$$\leq f(\tau) - C' |\tau| + \frac{3}{2} |\tau|^2 + \frac{|\tau|}{2} - C^{-1} |\tau| + l(\mu) |\tau| + C^{-1} |\mu| - \sum_{k \geq 1} \mu_k k$$

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Which means that
\[
\frac{|\mu|^2}{2} - \left( \frac{1}{2} + C' + C^{-1} \right) |\mu| - l(\mu) |\tau| + \sum_{k \geq 1} \mu_k k \leq f(\tau) - C' |\tau| + \frac{3}{2} |\tau|^2 + \frac{|\tau|}{2} - C^{-1} |\tau|
\]

Now consider the expression \( \sum_{k \geq 1} \mu_k k \). We can rewrite it as
\[
\sum_{k \geq 1} \mu_k k = \sum_{1 \leq k \leq l(\mu)} (\mu_k - 1)(k - 1) + \frac{l(\mu)^2}{2} - \frac{l(\mu)}{2} + |\mu|
\]

The first summand in this expression, \( \sum_{1 \leq k \leq l(\mu)} (\mu_k - 1)(k - 1) \) is clearly non-negative, so
\[
\sum_{k \geq 1} \mu_k k \geq \frac{l(\mu)^2}{2} - \frac{l(\mu)}{2} + |\mu|
\]

Back to our previous result, we had:
\[
\frac{|\mu|^2}{2} - \left( \frac{1}{2} + C' + C^{-1} \right) |\mu| - l(\mu) |\tau| + \sum_{k \geq 1} \mu_k k \leq f(\tau) - C' |\tau| + \frac{3}{2} |\tau|^2 + \frac{|\tau|}{2} - C^{-1} |\tau|
\]

Applying the inequality \( \sum_{k \geq 1} \mu_k k \geq \frac{l(\mu)^2}{2} - \frac{l(\mu)}{2} + |\mu| \), we now know that
\[
\frac{|\mu|^2}{2} + \left( \frac{1}{2} - C' - C^{-1} \right) |\mu| + \frac{l(\mu)^2}{2} - \left( \frac{1}{2} + |\tau| \right) l(\mu) \leq f(\tau) - C' |\tau| + \frac{3}{2} |\tau|^2 + \frac{|\tau|}{2} - C^{-1} |\tau|
\]

The inequality we obtained clearly means that \(|\mu|\) is bounded (by a function of \(\tau\)), which proves the proposition.

\[\square\]

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