On the linear evolution of disturbances in plane Poiseuille flow

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June 11, 2020

Abstract

The linear evolution of disturbances due to a ribbon vibrating at frequency $\omega_0$ in plane Poiseuille flow is computed by solving the associated initial boundary value problem in the Fourier-Laplace plane, followed by inversion. A novel algorithm for identifying the temporal modes of the Orr-Sommerfeld equation (OSE) in the complex wavenumber plane, which are required in the inversion, is presented. Unlike in many prior studies, the performance of the Laplace integral first, not only avoids complicated causality arguments and confusion, in locating upstream and downstream modes, that is prevalent in literature but also yields a spatio-temporally uniform solution. It also reveals that the solution consists of a time-periodic part at $\omega_0$, associated with the relevant spatial mode (the Tollmein-Schlichting wave) and a transient wavepacket, associated mainly with the saddle points of the OSE and is computed by the method of steepest descents, which can also include contributions from the spatial pole. Which of these parts dominates depends on the Reynolds number and $\omega_0$. A secondary stability analysis of this dominant part is seen to explain the disturbance growth observed in the seminal experiments of Nishioka, Iida & Ichikawa (J. Fluid Mech., vol.72, 1975, p.731) and Nishioka, Iida & Kanbayashi (NASA TM-75885, 1981). Threshold amplitudes for instability at a subcritical Reynolds number $Re = 5000$ are obtained from the time-averaged three dimensional disturbances, by combining the secondary base states and the growing Floquet modes. The observed minima of the threshold amplitude curves in the experiments are explained in terms of the instabilities of these two base states. Computations, for another subcritical (4000) and a supercritical (6000) Reynolds number, are also validated with the experimental data.

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1 Introduction

It is well-known that shear flows become turbulent much below the critical Reynolds number predicted by linear stability theory (LST), a phenomenon commonly referred to as ‘subcritical transition’. Despite many studies, it remains not completely understood even in simple shear flows like plane Poiseuille flow (pPF), even when subject to controlled conditions like a vibrating ribbon.

Of the handful of studies involving vibrating ribbons in pPF, the pioneering Nishioka et al [1975] (N75, hereafter) is the most exhaustive, when it comes to studying in great detail, the beginning stages of the transition process. Nishioka et al [1981] (N81, hereafter) is a subsequent study aimed at clarifying the role of three-dimensionality in the transition process. Apart from being the first study to observe two-dimensional Tollmein-Schlichting (2D TS) waves in pPF, N75 also sketched the downstream disturbance evolution in the linear regime as in figure 15 (N75F15; the reference to figure x of N75 will appear as N75Fx and that of N81 as N81Fx) and provided, for the first time, a plot of the threshold disturbance amplitude as a function of the ribbon frequency $\omega_0$ for three $Re$ (N75F16). The results presented in N75 and N81 are what concern us predominantly in this work.

Ever since it became evident that LST could not explain real transition scenarios, alternative mechanisms that could produce disturbance growth in subcritical conditions, have been sought. A popular one was the transient growth framework which was used to develop theories based on, for example, optimal disturbances Schmid & Henningson [2001]. However, none of these optimals have been seen, for example, in N75 and there is a gap between how, if at all, the observed features evolve into these optimals. The second one, which actually appeared earlier than the first, is the concept of secondary instability, whose proponents, for example Herbert [1988], have argued that the linearisation about a base state different from the parabolic profile could give rise to growing secondary modes even in subcritical conditions. The discovery of non-linear equilibria and quasi-equilibria in pPF Orszag & Patera [1983], apart from the aforementioned optimals, suggested alternative base states for linearisation. Reddy et al [1998] sketched the supposed route to transition in a controlled disturbance environment-

$$2DTS\,wave \rightarrow 2D\,state \rightarrow 2D\,state\,breakdown \rightarrow transition.$$  

The first component in the above pathway is the least stable spatial Orr-Sommerfeld (OS) mode of finite amplitude, at frequency $\omega_0$. This is supposed
to evolve nonlinearly into a 2D equilibrium state for $Re > 2900$, which being unstable to 3D disturbances, breaks down and leads to transition. It is commonly held (for example, p.270 of Reddy et al. [1998]), that ‘this transition scenario agrees qualitatively and quantitatively with experiments where a two-dimensional TS wave is introduced in the flow via a vibrating ribbon’. However, this description a) assumes that the starting point is always the 2D TS wave and b) begs the question of how the subsequent 2D (nonlinear) state comes about in a vibrating ribbon experiment. We will show later that the starting point may not always be the 2D TS wave. As regards (b), neither have there been any experimental observations of such a state nor has a numerical simulation shown such a state emerge from the 2D TS wave.

How the disturbance input from a vibrating ribbon gets transformed into a disturbance field in the flow, also known as receptivity, is a key step in the evolution of the disturbance. Receptivity study involves solving the linearised NS equations for given initial-boundary conditions, the so-called initial-boundary value problem (IBVP). A variety of analytic solution procedures for the IBVP in parallel shear flows have been advanced over the last 50 years. Most of these procedures have the same two key steps - i) solution of an ODE in the transformed plane and ii) inversion of the transformed solution back to the physical plane; the issues that crop up repeatedly are a) should the problem be treated temporally or spatially or as a combination, b) how to account for contributions of different modes, c) which modes contribute upstream and which downstream and d) how to fix the lines of integration for the inversion.

Gaster [1965] (G65 hereafter) was probably the first to try to simulate the vibrating ribbon problem for the flat plate boundary layer; he was looking to reproduce the spatially growing waves that Schubauer & Skramstad [1948] had observed downstream of a disturbance source in a boundary layer. Till G65, theoreticians mostly used only temporally growing modes and in fact, had reservations about the usage of spatial modes (See Section 47.2 of Drazin & Reid, 1983). Thus, the experimental observations were sought to be explained by calculation of temporal growths which were converted to spatial growths by using the group velocity. The ribbon was approximated by a delta function as has been done here and the possibility of an exponentially growing mode allowed. The inhomogeneous BC was incorporated into the integrand of the inversion integral. G65 fixed the line of integration (LOI) for the $\omega$ inversion in the UHP above all the poles, as per standard Laplace transform theory but then moved it below the real line and claimed that the contribution from the straight line segment vanished in the infinite time limit. This in turn led to the fixing of the LOI in the $\alpha$ plane in one of two ways - a) contour A passing above the pole and b) contour B passing below the pole, dictated by which one would fit the initial unperturbed condition; this was in turn shown to be linked to the sign of $\partial \alpha_r(\omega_0)/\partial \omega_r$ with the LOI being chosen as A if the latter is negative and B if it is positive. For BL flows, the sign of this quantity is shown to be the same sign as that of the group velocity and approximately equal to its reciprocal. After
performing the Fourier inversion, Laplace inversion was done by assuming that the positive frequency axis could be closed by a 'suitable contour' on which the integral vanished. The major shortcoming in G65 is due to non-appreciation of the possibility of non-negligible contributions from this part of the contour. In fact, even in subcritical flows, there is at least one saddle in the \( \omega \) plane which makes the regular Jordan arcs untenable due to intervening hills of the saddle. Thus, parts of the 'suitable contour' have to pass through the saddle, whose contribution then has to be included.

**Ashpis & Reshotko** [1990] (AR90 hereafter) revisit the vibrating ribbon problem for the same flow; their formulation, again in terms of Fourier transforms, is identical to that of G65, though following **Gustavsson** [1979], the continuous spectrum and branch cuts, that G65 had missed, were included. AR90 differs from G65 in including branch cuts associated with the continuous modes which are shown to have an upstream influence. It is similar to G65 in obtaining the time asymptotic solution by lowering the integration contour below the real axis in the \( \omega \) plane. After Fourier inversion, AR90 obtain the same solution as G65 for the discrete part, apart from the fact that the possibility of upstream propagating modes are allowed, if there exist poles in the left half \( \alpha \) plane. They claim that the location of the pole determines the direction of wave propagation with the left half poles (negative phase velocity) contributing to upstream propagating waves and the right half ones (positive phase velocity) to downstream ones. However, the choice of the integration contours, in figure 5, for example, is same as that of Gaster and consequently should have been dictated by considering the sign of the group velocity. While this makes no difference in cases where the phase and group velocities are of the same sign, (ai, bii, ci and dii of figure 5), the sign of the group velocity decides which mode contributes where in the other cases (aii, bi, cii and dii). Similar to G65, their simplistic treatment of the Jordan arc, ignoring the saddle points, leads to a non-accounting of possible downstream-growing wavepackets for supercritical Reynolds numbers.

**Hill** [1995] considers the receptivity of a boundary layer flow to a variety of disturbances including freestream and boundary disturbances. The Lagrange identity and adjoint velocity, pressure and stress fields are used to compute the response for a given OS eigenmode. The response for the vibrating ribbon problem is shown to match that obtained by AR90. The emphasis of **Hill** [1995] is on computing the responses solely from the adjoint field; the physical fields do not seem to be computed and hence issues related to inversion are not discussed.

**Tumin** [1996] considers disturbance evolution due to blowing / suction in a wall slot in pipe Poiseuille flow. Assuming that the response is at the forced frequency \( \omega_0 \) and disturbances decay both upstream and downstream, the solution is written in two alternate ways - as an inverse Fourier transform and as a (countably infinite) sum over all the ‘downstream moving’ spatial OS modes corresponding to \( \omega_0 \). The computed solution is valid only downstream
of the disturbance source. The receptivity coefficients are calculated by using bi-orthogonality between the original disturbance eigenfunctions and a set of adjoint eigenfunctions. However, the procedure has several deficiencies. Apart from the use of the spatial modes, mentioned earlier, it is not clear how the 'downstream moving' modes have been identified. Even though spatial eigenvalues are found in the first, second and fourth quadrants, only the contributions from the first quadrant poles seem to have been considered. Though this simplifies the fixing of the inversion contour (which can be taken as the real line), it is questionable. Another deficiency of this method is to represent the solution as a sum of spatial eigenmodes, whose completeness has not been established till date. Hence, the vibrating ribbon problem necessarily has to be formulated as an IBVP unlike Tumin [1996]. Therefore, receptivity cannot be established by a set of coefficients called receptivity coefficients; it is in fact a spatio-temporal evolution as will be shown in the next section.

Ma et al. [1999] assume Fourier series in the \( \theta \) and \( t \) variables, an eigenmode expansion in \( r \) and derive an ODE system in the streamwise variable \( x \). Adjoint eigenmodes are used in deriving this system. The issue of which modes to include in which part of the domain (\( x > 0, < 0 \)) is settled by appealing to the 'well-known' linear stability of Hagen-Poseuille flow and also by comparison with the DNS results; thus first and second quadrant modes are taken as contributing to the downstream field and the third and fourth quadrant modes contributing to the upstream disturbance field. We will formulate and solve the problem in a way that will obviate the need for making such extraneous assumptions.

Manuilovich [1992] studied time dependent disturbances of pPf, with the disturbance generator being a triangular vibrator of finite length \( l \), mounted on the upper wall, oscillating sinusoidally at frequency \( \omega_0 \). Symmetric and antisymmetric modes are considered separately and subcritical and supercritical cases are distinguished. For the former case, it is shown that, for \( x > l (< -l) \) resp.), the LOI has to be closed in the UHP (LHP resp.) and the solution is written as a sum of discrete spatial modes corresponding to \( \omega_0 \), with UHP poles selected for \( x > 0 \) and LHP poles for \( x < 0 \). As we will see later, this is incorrect.

In summary, all the works cited above (except Hill [1995]) solve the ODE in the transformed plane and then invert the solution back to the physical plane. The method of solution varies and is different from the lifting procedure employed here. Both temporal and spatial inversions are performed in general and a variety of strategies for choice of inversion contours are used; all of these seem to invoke a radiation condition without stating explicitly. Also, all the analyses (apart from G65 who briefly mentions the transients but in a different context) deal with large time asymptotics; these methods are applicable only for \( x/t \approx 0 \). Hence, it is not possible to use these methods at all \( x \) even at large \( t \), in particular for \( x/t \to \infty \). The solutions from these methods are spatially unbounded for supercritical flows. This is a crucial deficiency as Fourier transforms require
spatial boundedness at any given time. This not only makes these procedures mathematically suspect but also render them incapable of producing solutions that can be compared with experimental results.

The use of radiation condition is unavoidable if the disturbance response is assumed to be at the forcing frequency; this assumption results in a boundary value problem. A properly posed problem has to be an IBVP, requiring only boundedness conditions at infinity; the resulting solution is unique and hence there is no need for an additional radiation condition (Schott, 1992). In the present approach to evaluating the integrals (described elsewhere in detail), the global topography of the $\alpha$ and $\omega$ modes is computed and the contributions of the various saddles, branch points and poles are assessed. This leads, in a natural way, without having to make ad hoc assumptions, to the correct spatial decay at infinity.

Such IBVPs have been solved routinely, and in a clear manner, with none of the confusion described earlier, in wave propagation problems in other branches of physics like optics, geophysics and atomic and molecular physics (for e.g. Felsen & Marcuvitz, 1973). A recent exemplary application in fluid mechanics is Gordillo & Perez-Saborid (2002) which describes a procedure for inverting similar integrals, and give numerical results for the forced Ginzburg-Landau equation and the forced Kelvin-Helmholtz problems. By performing the Laplace inversion first, the structure of the solution is laid bare - the solution is seen to consist of a term that gives a response at the signal frequency and a second term which gives the transient and whose evaluation is based on the well-known method of steepest descent. This method has been used in stability studies, but mostly to distinguish between convective and absolute instabilities (Gaster, 1968, Juniper, 2006, Lingwood, 1997). With this procedure, conceptual issues surrounding fixing of inversion contours, choice of upstream / downstream modes and the role of the transient are clarified. We formulate and sketch the solution in \&.

The Fourier inversion involves complex integration in the wavenumber plane and hence, the map of each temporal eigenvalue, as a function of the complex wavenumber, has to be obtained. An understanding of the topography specified by these maps as signified by the knowledge of critical points of the map, like saddles and branch points, is crucial to a correct solution and the absence of a reliable method to sort the eigenmodes correctly has hampered a proper investigation of these problems. Koch (1986) was one of the first to attempt a sort of mode-tracing for pPF in the $\omega$ and $\alpha$ planes as a function of real $\omega$ and $\alpha$ respectively. His procedure was sensitive to mode jumping and his mode indexing (for example in figure 4 of that paper) can only be considered tentative. It can be seen from his figure that what is designated the principal instability mode, in fact is clearly not the least damped mode at all frequencies considered in that figure. The problems with mode tracing and partial solutions are described extensively in Suslov (2006), wherein the author details succes-
sively the problems of sorting eigenvalues based on real parts, imaginary parts or even switching between the two procedures. While the first two fail whenever there are ‘collisions’ of temporal branches, the last one fails for ‘true collisions’ apart from being difficult to implement numerically. He suggested tracking the eigenvalues based on a quantity $\gamma_r$; the eigenvalue with the largest value of $\gamma_r$ corresponds to the dominant mode in a frame with speed $\rightarrow 0$. However, this method can also fail at true collisions. We have used the analytic properties to develop an algorithm to sort eigenmodes corresponding to eigenvalues that have been produced from an eigenvalue solver. It works even at a true collision (for e.g. a double root in the $\omega$ plane) and correctly produces branch points and branch cuts. The algorithm is sketched in Appendix A. Some of the relevant modal maps will be presented in §3. Key aspects of the solution can be deduced by an asymptotic analysis using the method of steepest descent; details are presented in this section.

The spatio-temporal evolution is in general complicated but two idealized secondary states can be identified - a) the TS wave (related to the dominant spatial OS pole at $\omega_0$) and b) a wavepacket (related to the saddle of the phase function of the dominant OS mode). Depending on $\omega_0$ and $Re$, either of these may be dominant; there are also mixed regions where both may be important. Some of these solutions will be presented in §3. The two states have very different decay rates. While the characteristics of TS wave are obtained directly from the OS dispersion equation, those of the wavepacket have to be deduced from the IBVP solution; the decay rate and the propagation velocity of the wavepacket are discussed in detail in this section. The IBVP solutions are used to explain the features of the linear developments presented in N75.

With the solution of the IBVP as a guide, we choose an appropriate secondary base state (either of (a) or (b) above) and perform a Floquet analysis to get the secondary growth rate. Ideally a spatial secondary analysis should be performed at the given drive frequency. However, this is much more complicated than a temporal analysis and also the spatial growth rate can be obtained (approximately) from the temporal one, following Herbert et al(1987). Following most studies in this area, we also do a temporal secondary analysis. The results of the secondary instability analysis are presented in §5.

In the subsequent §6 we present comparisons with experimental results of N75 and N81. We concentrate, in particular, on two figures of these papers N75F16 and N81F15, that have been reproduced in figure 12 and 13, for convenience. Since N75F16 is based on N75F15, we discuss that first.

N75F15 shows growth / decay of a subcritical disturbance at 72 Hz at an $Re = 5000$. It is clear from the figure that the disturbances, below a threshold level, show an initial growth and eventual decay in the streamwise direction; above the threshold, they grow continuously. Kleiser [1982] claims to see similar behavior in the DNS of an initial value problem in pPf; however, instead
of a vibrating ribbon, initial conditions were prescribed. \cite{Trefethen1993} show similar behavior for a $2 \times 2$ nonlinear model. Below some nonlinearity threshold, the curves rise and then decay; above these, they grow and saturate at some amplitude. The former is attributed to non-normal transient growth, the latter to the added nonlinearity, which though not directly contributing to the growth, redistributes the energy such that explosive growth can occur. However, no comparisons with experiments are attempted and the discussion on the relevance of this model to actual flows (p.582) is only at the level of conjecture. To our knowledge, it is yet to be substantiated, say by comparison with an experiment like N75. More importantly, both these studies, and the pathway sketched in \cite{Reddy1998}, involve nonlinearities whereas the lower curves in N75 are in the linear regime, as we will see in detail later. We seek to throw some light on this behavior by a study of the IBVP solution.

N75F16 shows threshold amplitudes $A_t$ vs. $\omega_0$ for three Re (two subcritical Reynolds numbers 4000, 5000 and one mildly supercritical, 6000). We assume that the threshold was deduced by observing an unchanging $u_{\text{max}}/U$ with $x$, much like the (mostly) flat curve (iv) in figure 15.

The threshold curves have the following features -

For fixed $Re$,

a) Two minima $M_i_1, M_i_2$ separated by a maximum $Ma$. ($M_i_2$ is not a true minimum but an endpoint of the interval at which $A_t$ attains a global minimum.)

b) $M_i_1$ is roughly that at which the spatial decay rate is the minimum.

c) $A_t(M_i_2) < A_t(M_i_1)$.

For fixed $\omega_0$,

d) $A_t$ is a decreasing function of $Re$.

Also shown in the figure are the nonlinear calculations of $A_t$ \cite{Itoh1974}. Though N75 claims good agreement with Itoh’s nonlinear calculations, the fact is that only (d), which is qualitative in nature, is reproduced. Not only does \cite{Itoh1974} not produce clear minima at all $Re$, it does not produce $M_i_2$ at all. N75 speculates (p.750) that transition at higher drive frequencies is triggered ‘directly by spot-like fluctuations appearing before the fundamental has grown sufficiently.’ and also express their belief that ‘this may be due to the highly three-dimensional nature of a disturbance with a large $\beta$.’ Not only are we not aware of any subsequent published study, including N81, that clarifies the issue, but also an exhaustive search of the literature did not reveal an explanation for $Ma$ or $M_i_2$. 

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Zhou [1982], Sen & Venkateswarlu [1983] and Suslov & Paolucci [1999] are other nonlinear calculations that tried to reproduce this figure. The first of these does produce threshold amplitude curves that are remarkably similar to the experimental curves, though higher; the difference is attributed to neglect of three-dimensional effects. However, the procedure involves artificially splitting the flow field into a base flow, with real fundamental eigenvalue, and a perturbation and is done to produce solvability conditions, which similar earlier studies could not, with the TS mode. The second of these uses the earlier formulation of Reynolds & Potter [1967] but claims to sum the resulting Stuart-Landau series more accurately by using Shanks method. They also critically discuss Zhou’s results and conclude that there could be convergence issues, without a clear resolution of which, ‘it is perhaps too early to reach any conclusions regarding Zhou’s results.’ The final work solves a cubic Landau equation, based on the Watson model, thus retaining more physical features of the problem, as compared to Zhou [1982]. The threshold amplitude curves in the last two studies do not show the features of the N75 curve; the second one does not show $\text{Mi}_2$ whereas the third shows scattered values with no clear trend. Dhanak [1983] is a nonlinear analysis based on higher order amplitude expansions; the important difference from the other nonlinear studies is that the basic state here is three-dimensional, by considering channel walls in the form of small amplitude spanwise waves. However, no higher order terms are included and the analysis does not capture $\text{Mi}_2$.

We seek to throw light on the local extrema in the threshold amplitude curves by secondary instability analysis. In particular, we provide evidence that the two minima are linked to the secondary instabilities of the TS wave and wavepacket states.

Concluding remarks are presented in §7.

2 The IBVP

2.1 Formulation

The setting for the problem is a plane channel between the walls $y = \pm 1$ with the base flow being the unidirectional, plane Poiseuille flow $U(y) = 1 - y^2$. We consider the problem of creation and evolution of disturbances in such a flow that is subjected to a local unsteady forcing, typically on $y = -1$. The forcing is supposed to simulate the effect of a vibrating ribbon, or a blowing / suction device in a controlled transition experiment.

The linearized equations governing the normal disturbance velocity $v$ and vorticity $\eta$ are the well-known (Schmid & Henningson 2001) OS and Squire equations respectively -
\[ L_{OS}(v) = \left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \left( \nabla_h^2 + D^2 \right) v - D^2 U \frac{\partial v}{\partial x} - \frac{1}{Re} \left( \nabla_h^2 + D^2 \right)^2 v = 0, \tag{2.1a} \]

\[ L_{SQ}(\eta) = \left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} - \frac{1}{Re} \left[ \nabla_h^2 + D^2 \right] \right) \eta = -DU \frac{\partial v}{\partial z}. \tag{2.1b} \]

where \( \nabla_h^2 = \partial_{xx} + \partial_{zz} \) and \( D \) denotes the differential w.r.t \( y \).

The system \((2.1a)\) has to be solved with appropriate initial / boundary conditions. Many facets of the eigenvalue problem, where \((2.1a)\) is solved with homogeneous BCs have been extensively studied, with the literature running to hundreds of papers. In the present case, we are interested in formulating and solving an initial-boundary value problem (IBVP) for \((2.1a)\). In particular, we prescribe the external forcing as a time-dependent boundary condition for the normal disturbance velocity \( v \) on the lower wall \( y = -1 \).

\[ v(x, -1, z, t) = \delta(x) g(z) h(t) \quad \text{and} \quad Dv(x, -1, z, t) = 0. \tag{2.2} \]

\( h(t) \) is usually assumed to be periodic-in-time, starting from \( t = 0 \). For \( t < 0 \), \( h(t) = 0 \). Here, the disturbance source is at \( x = 0 \). The positive and negative values of \( x \) denote streamwise positions downstream and upstream of the disturbance source respectively. Spanwise conditions, expressed through \( g(z) \), usually take a periodic form with \( g(z) = e^{i\beta_0 z} \) with \( \beta_0 \) real, with the understanding that the appropriate part (real or imaginary) of the final solution will be taken.

On the top wall \( y = 1 \), \( v \) and \( Dv \) satisfy homogeneous boundary conditions:

\[ v(x, 1, z, t) = 0 \quad \text{and} \quad Dv(x, 1, z, t) = 0. \tag{2.3} \]

At \( x, z = \pm \infty \), the disturbances and all its derivatives are assumed to decay as \( t \to \infty \). We assume zero initial conditions i.e. \( v(x, y, z, 0) = 0 \). This means that the disturbance generator starts from rest. It is well known that for subcritical and slowly growing supercritical pPf in the linear regime, the least stable disturbances are two-dimensional. We hence restrict the present study to two-dimensional, \( z \)-independent wall forcing; \( g(z) = 1 \).

### 2.2 Solution

The disturbance evolution is governed by equations \((2.1a)\) together with the initial and boundary conditions \((2.2)\) and \((2.3)\). \((2.1a)\) is a homogeneous ODE system for \( v \) with non-homogeneous BC. It turns out to be convenient to transform this to a inhomogeneous ODE system for the auxiliary variable \( v_1 \) satisfying homogeneous BC. This can be achieved by a suitable lifting procedure (p.436, Lanczos 1996) which involves expressing \( v(x, y, t) \) as

\[ v(x, y, t) = v_1(x, y, t) + \delta(x) h(t) f(y) \quad \text{where} \quad f(y) = \frac{2 - 3y + y^3}{4}. \tag{2.4} \]
Substituting (2.4) into (2.1a), the inhomogeneous ODE for $v_1$ is obtained as

$$L\delta(x)h(t)f(y) = -L\delta(x)h(t)f(y),$$  

(2.5)

Assuming $h(0) = 0$, $v_1$ satisfies zero initial condition and homogeneous boundary conditions.

Fourier and Laplace transforming (2.5) and its homogeneous boundary and initial conditions, w.r.t $x$ and $t$ respectively, the well-known Orr-Sommerfeld equation with an inhomogeneous term is obtained -

$$[\mathcal{L} - i\omega\mathcal{M}]\hat{v}_1 = -\hat{h}(\omega)[\mathcal{L} - i\omega\mathcal{M}]f(y),$$  

(2.6)

where

$$\mathcal{L} = i\alpha U(D^2 - \alpha^2) - i\alpha D^2 U - \frac{1}{Re}(D^2 - \alpha^2)^2, \quad \text{and} \quad \mathcal{M} = D^2 - \alpha^2.$$  

The hat symbol denotes the Fourier-Laplace transform of a given function and the overbar denotes Fourier transform w.r.t $x$.

We are interested in sinusoidal forcing, starting from rest i.e. we take $h(t) = \sin\omega_0 t$. For a given $\alpha$ (real) completeness of temporal OS eigenfunctions [Di Prima & Habetler 1969] allows eigenfunction expansion for $\hat{v}_1$ as

$$\hat{v}_1(\alpha, y, \omega) = \sum_{n=1}^{\infty} C_n(\alpha, \omega)\phi_n(\alpha)(y)$$  

(2.7)

Using the bi-orthogonality of $\phi_n^{\alpha}(y)$ and the adjoint eigenfunctions $\xi_n^\alpha(y)$ [Schmid & Henningson 2001],

$$\hat{v}_1(\alpha, y, \omega) = \sum_{n=1}^{\infty} \phi_n^{\alpha}(y)\int_{-1}^{1} [\mathcal{L} - i\omega\mathcal{M}] f\xi_n^{*} dy$$  

where $K_n$ is given by

$$\int_{-1}^{1} \xi_n^{*}(k^2 - D^2)\phi_j dy = K_j\delta_{jk}.$$  

(2.9)

Inverting (2.8) from the $(\alpha, y, \omega)$ to the physical $(x, y, t)$ plane by Laplace and Fourier inversions yields

$$v_1(x, y, z, t) = \frac{1}{4\pi} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \phi_n^{\alpha}(y)\frac{e^{i(\alpha x - \omega_0 t)}I_-(\alpha) - e^{i(\alpha x - \omega_0 t)}I_n^{OS}(\alpha)}{(\omega_n - \omega_0)K_n} d\alpha$$

$$- \frac{1}{4\pi} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \phi_n^{\alpha}(y)\frac{e^{i(\alpha x + \omega_0 t)}I_+(\alpha) - e^{i(\alpha x + \omega_0 t)}I_n^{OS}(\alpha)}{(\omega_n + \omega_0)K_n} d\alpha,$$  

(2.10)
\[ I_{\pm}(\alpha) = \int_{-1}^{1} [\mathcal{L} \pm i\omega_0 M] f_\pm \xi dy, \quad I_n(\alpha) = \int_{-1}^{1} [\mathcal{L} - i\omega_n M] f_\pm \xi dy. \] (2.11)

The explicit forms of the integrals \[I_{\pm}(\alpha)\] and \[I_n(\alpha)\] are given in Appendix C. The second and the fourth integrals are periodic in time with frequency \(\omega_0\). They are similar to the solution obtained by Tumin [1996]; the coefficients in the series are the receptivity coefficients.

The horizontal velocity \(u(x, y, z, t)\) can be obtained, by using the continuity equation. The part of \(u\) corresponding to \(v_1\) can be obtained as [Schmid & Henningson 2001]

\[
u_1(x, y, z, t) = \frac{i}{4\pi} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\alpha} \frac{d\phi_n^{(\alpha)}(y)}{dy} \frac{e^{i(\alpha x - \omega_0 t) I_-(\alpha)} - e^{i(\alpha x - \omega_n t) I_n^{OS}(\alpha)}}{(\omega_n - \omega_0) K_n} \, d\alpha
- \frac{i}{4\pi} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\alpha} \frac{d\phi_n^{(\alpha)}(y)}{dy} \frac{e^{i(\alpha x + \omega_0 t) I_+(\alpha)} - e^{i(\alpha x - \omega_n t) I_n^{OS}(\alpha)}}{(\omega_n + \omega_0) K_n} \, d\alpha.
\] (2.12)

It is not difficult to see that the total velocity \(v(x, y, z, t)\), given by (2.4) is independent of the lifting function \(f(y)\). Since \(u\) is derived from \(v\), it follows that it is independent of \(f\) as well. The parts of \(u\) corresponding to the second term in \(v\) is obtained from continuity equation as

\[
\frac{\partial u_2}{\partial x} = -b(t)f'(y)\delta(x)
\]

Applying the far-field zero conditions, \(u_2 = 0\) for \(x < 0\).

### 2.3 Features of analytic solution

\(v_1\) consists of four integral contributions, the first and third of which are standard integrals that can be evaluated by closing the LOI with a Jordan arc in the UHP (resp. LHP) for \(x > 0\) (resp. \(x < 0\)). For simplicity, we consider the Fourier inversion of only the \(j^{th}\) term of the infinite sums, without the multiplicative factor. Assuming all the spatial modes of the forcing frequency \(\omega_0\) to be distinct, and that \(\omega_j(\alpha) = \omega_0\) at only one point \(\alpha = \alpha_j\) in the complex plane, for \(x > 0\), the first integral becomes

\[
v_{11j} = 2\pi i \frac{e^{i(\alpha_j x - \omega_0 t)} \phi_j^{(\alpha_j)}(y) I_-(\alpha)}{\frac{d\phi_j^{(\alpha_j)}}{d\alpha}|_{\alpha = \alpha_j} K_j}.
\] (2.13)

Since \(\omega_n^{OS}(\alpha) = -\tilde{\omega}_n^{OS}(-\tilde{\alpha})\), we have, the poles for the third integral located at \(\alpha = -\tilde{\alpha}_j\) i.e. in the same half plane as the poles for the first integral. Hence, for \(x > 0\), the third integral becomes

\[
v_{13j} = -2\pi i \frac{e^{-i(\tilde{\alpha}_j x + \omega_0 t)} \phi_j^{(-\tilde{\alpha}_j)}(y) I_-(\tilde{\alpha}_j)}{\frac{d\phi_j^{(-\tilde{\alpha}_j)}}{d\alpha}|_{\alpha = -\tilde{\alpha}_j} K_j}.
\] (2.14)

Thus the poles $\alpha_j$ lying in the UHP ($Im(\alpha_j) > 0$) contribute to the downstream development of the disturbance. A similar analysis for $x < 0$ shows the poles in the LHP contribute to the disturbance development upstream of the source.

We now consider the second and fourth integrals in (2.10). These integrals cannot be evaluated by closures of the contour using Jordan arcs, since $\omega_j(\alpha)$ is not linear in $\alpha$. We now consider each of these in turn. Several approaches for large time asymptotic analysis of similar Fourier integrals exist in the literature; we use the steepest descent method. This involves locating the saddle points of the phase function $p_j(\alpha) = i [\frac{t}{4} \alpha - \omega_j^{OS}(\alpha)]$. $\omega_j^{OS}$ is a complicated function of $\alpha$, possessing in general more than one saddle and a host of branch points whose number increases with increasing $j$, and moreover is only numerically known for pPf; an evaluation of $v_{12j}$ and $v_{14j}$ à la $v_{11j}$ and $v_{13j}$ is thus impossible. We present a first approximation to these integrals here.

The asymptotic solution of an integral of Laplace type as found in (2.10) or (2.12) is given in Appendix D (22). It is assumed in the following asymptotic analysis of these integrals that the phase function has many saddle points and a pole such that the steepest descent path from only one of the saddles passes through the pole; addition of more poles will result in addition of similar terms in the expression for $u$ or $v$. In a later section, the global topography of the phase function will be consulted before applying the formula (22) to (2.10) or (2.12).

We define a saddle path to be the path traced by a saddle point as $v_d$ varies. $v_d$ varies from $-\infty$ to $\infty$. Approximating the second integral by the saddle point contribution at each $v_d$, we get the large-time saddle contribution for $v_{12}(x, y, z, t)$ to be

$$v_{12sj} = -\frac{2\pi}{\sqrt{\gamma}} \bar{\phi}(\alpha_j)(y)I_{\alpha_j}(\alpha_s) e^{-i\gamma/2} e^{p_j(\alpha_s) t}$$  (2.15)

where $\gamma = arg(d^2\omega_j^{OS}/d\alpha^2 |_{\alpha=\alpha_s})$. If the pole arising from $\omega_j^{OS} = \omega_0$ lies between the SDP and the real axis for some interval $X = (v_{d_{min}}, v_{d_{max}})$, its contribution, given by

$$v_{12pj} = -2\pi i e^{i(\alpha_j x - \omega_0 t)} \bar{\phi}(\alpha_j)(y) I_{\alpha_j}$$

$$\frac{d\omega_j}{d\alpha} |_{\alpha=\alpha_j} K_j$$

$$v_d \in X$$

$$= 0 \quad \text{otherwise}$$  (2.16)

has to be included as well, with $v_{12j}$ given by $v_{12j} = v_{12sj} + v_{12pj}$ + a smoothing term. The smoothing term, whose analytic form is presented in Appendix D, arises when the steepest descent path through the saddle point crosses the pole, at $v_{dp} = v_{d_{min}}$ and/or $v_{dp} = v_{d_{max}}$. It smooths the discontinuity arising from the inclusion and exclusion of pole residues. This term can become large when
the saddle point approaches the pole. An increase in the number of relevant saddles can lead to an increase in the number of such smoothing terms as the number of possible descent paths crossing the pole also increases. Let us consider the least stable mode here. For the subcritical case, $v_{d_{min}}$ is positive and $v_{d_{max}} = \infty$; $v_{d_{p}} = v_{d_{min}}$. Similarly, for the supercritical case, $v_{d_{min}}$ is $-\infty$ and $v_{d_{max}}$ is finite and positive; $v_{d_{p}} = v_{d_{max}}$.

For a supercritical Reynolds number, $v_{12pj}$ grows for some $v_{d} \in X$. There may exist a range $Y$ of positive $v_{d}$ over which the real part of the phase function $p_{j}(\alpha)$ can become positive. In this range, $v_{12sj}$ gives rise to a temporally growing wave packet which is convected away from the source. If there is overlap between $X$ and $Y$, for any $v_{d} \in X \cap Y$, both contributions add and a distinct growing TS wave and wavepacket cannot be seen. If the forcing frequency lies outside the neutral stability curve, $v_{12pj}$ decays but $v_{12sj}$ still grows in the interval $Y$, showing a distinct wavepacket. It is possible that the temporal growth rate of $v_{12sj}$ is small, in which case the wavepacket will decay transiently as $t \to \infty$; here too, the wavepacket from $v_{12pj}$ will be prominently seen in the asymptotic limit.

So far, the individual parts (2.13)-(2.15) of the solution (2.10) have been discussed. In the following subsection, various asymptotic limits of $v_{1}$ will be obtained by studying the behavior of their sum. The asymptotic limits of the smoothing terms have not been studied here since it has to lie between the saddle and pole contributions.

### 2.4 Asymptotic limits

We will focus on $v_{11j} + v_{12j}$; similar conclusions can be drawn for $v_{13j} + v_{14j}$.

1. $t \to \infty$. Most of the description will be for downstream locations i.e. $x > 0$. Occasionally, the upstream evolution will also be mentioned.

   (a) $x$ fixed i.e. $v_{d} \to 0$.

   i. Subcritical case. $v_{12pj} = 0$ for $v_{d} < v_{d_{min}}$ where the latter is positive. Since $v_{12sj}$ decays, and $v_{11j} \neq 0$ for all $v_{d} > 0$, the large time asymptotic state is that of a sinusoidal oscillation at the input frequency.

   ii. Supercritical case. $v_{12pj} \neq 0$ as $v_{d} \to 0$ and $v_{11j} = 0$ for $v_{d} > 0$. $v_{12sj}$ decays as $v_{d} \to 0$; otherwise the flow would be absolutely unstable. The asymptotic field is, therefore, a sinusoidal oscillation at the input frequency, which however grows spatially downstream. $v_{12pj}$ would cancel $v_{11j}$ for upstream locations ($v_{d} < 0$)
leaving a decaying wavepacket to propagate upstream.

(b) $v_d$ fixed.

i. Subcritical case. For $v_d \notin X$, $v_{11j} \neq 0$ while $v_{12j} = v_{12sj}$, which is a decaying wavepacket, so that the asymptotic solution is a decaying TS wave at the signal frequency. For $v_d \in X$, $v_{12pj}$ cancels $v_{11j}$ with the result that only $v_{12sj}$ remains and gives rise to a decaying wavepacket.

ii. Supercritical case. For $v_d \notin X$ and $v_d \in Y$, $v_{11j} = 0$ and $v_{12j} = v_{12sj}$ which is a growing wavepacket over $Y$, which is missing in previous asymptotic solutions published in literature. For $v_d \in X \cap (0, \infty)$ and $v_d \notin Y$, $v_{12j} = v_{12pj} + v_{12sj}$ grows spatially, but remains periodic in time. For $v_d \in X \cap Y$, the wavepacket $v_{12sj}$ grows in time while $v_{12pj}$ is periodic in time.

2. Fixed $t$.

(a) $x \to \infty$. For the subcritical case, the pole contributions from $v_{11j}$ and $v_{12pj}$ cancel, leaving a decaying wavepacket due to $v_{12sj}$. For the supercritical case, $v_{11j} = 0$ for $v_d > 0$ whereas $v_{12pj} = 0$ for $v_d > v_{d_{max}}$. Since $v_{12sj}$ decays outside the finite range $Y$, $v_{12j}$ decays to zero.

(b) $x \to -\infty$. In the subcritical case, the upstream state is again one of a decaying sinusoidal oscillation at the input frequency, whereas in the supercritical case, the asymptotic upstream state is that of a decaying wavepacket.

Unlike the previous studies, the asymptotic solution presented here is spatially bounded at all times for all Reynolds numbers. Earlier similar stability studies invoked Sommerfeld conditions for a periodic signaling problem or Briggs’s criterion for an impulsive disturbance; hence, the disturbances were always studied in the frame $v_d \to 0$. Lingwood [1997], for the first time, obtained the time-asymptotic solutions for impulsive disturbances in rotating disk boundary layer at a non-zero $v_d$; the critical points of the dispersion equation such as the saddle and branch points had to be used. Lingwood [1997], however, considered only one saddle point that varies with $v_d$, thus avoiding the need for computing the global topography of the eigenmodes. The periodic signaling problem, on the other hand, gives rise to a pole in the integrand (as shown above) and hence there is a need for determining the global topography of the Orr-Sommerfeld modes in the application of the steepest descent method.
3 Eigenmodal maps

The Orr-Sommerfeld dispersion equation for pPf has coalescing temporal modes \[ \text{[Jones, 1988]} \]; the corresponding wavenumbers are branch points in the \( \alpha \) plane. Fixing the branch cuts defines the eigenmaps. Here, in order to clearly separate out the temporal eigenmodes, the eigenvalues, \( \omega_j \) are first sorted into modes at the origin \( \alpha = 0 \), based on their imaginary parts. From these, the neighboring values of each temporal mode along the real and imaginary axes are sorted by using discretised Cauchy-Riemann (C-R) equations; the details of the sorting procedure for OS temporal eigenvalues are given in Appendix A. This works in general because each \( \omega_j(\alpha) \) is analytic except at the branch points. Marching in the \( \alpha \) plane is done parallel to the imaginary axis, both into the UHP and the LHP, from the points on the real axis. Even though the C-R equations are not satisfied at a branch point, the algorithm can still deduce the \( \omega \) branches at that point. The vertical marching results in vertical branch cuts away from the real axis in both half planes of \( \alpha \). This vertical mode tracing works as long as there is no double root for any real \( \alpha \) in the wavenumber domain considered. We present here the first two dominant modes of the OS-even, OS-odd families for \( Re = 5000 \) in figure 1.

The understanding of the maps is enhanced by a knowledge of their branch points, saddles and poles. The maps are analytic at almost all points, the only exceptional points being those at which branching occurs. In a given window, these are seen to be finite in number. The double roots appear as half-saddles in the \( \omega \) plane (figures 1 a-d) and as branch points (BPs) in the \( \alpha \) plane. A saddle in the \( \alpha \) plane appears as a cusp in the \( \omega \) plane (figure 1a). Alternatively, one can plot level contours of \( \omega_i \) (or \( \omega_r \)) in the \( \alpha \) plane; the BPs appear with associated branch cuts (BCs). In order to save space, we present details of these points only for the dominant OS even mode. Figure 2 shows the level contours of \( \omega_i \); the bunching of these contours is indicative of BCs (in this case, vertical), emanating from the BPs. Plots can only give a rough indication of the BPs. A procedure to accurately locate them is given in Appendix B.

Saddles of the quantities \( p(\alpha) = \alpha v_d - \omega_n \) are the other entities that are important for computing disturbance wave integrals. The number and type of saddles depends on the mode number and \( v_d \); in particular, saddles can appear and disappear from a modal map as \( v_d \) varies. We track the saddles of the first dominant OS mode for \( 0 < v_d < 1 \); the saddle paths are found by solving \( \partial \omega_i / \partial \alpha_r = 0 \) numerically; the value of \( \partial \omega_r / \partial \alpha_r \) at that point gives the \( v_d \) value for the corresponding saddle. In general, there could be an infinite number of saddles in each mode; however only a few of them make significant contributions to the integral. We present the saddle paths in figure 3. The modal topography is for a fixed value of \( v_d = 0.5 \); the saddle paths for varying \( v_d \) are overlaid on this background. We track saddles only in the RHP, including the imaginary axis; by symmetry the corresponding LHP saddles can be inferred. Mode 1 has three saddles (colored lines in figure 3a) and they move upward with increasing
Figure 1: Temporal maps of the region $R = [-4, 4] \times [-1, 1]$ in the $\alpha$ plane. Dominant modes of a) OS-even Mode 1 b) OS-even Mode 2 c) OS-odd Mode 1 d) OS-odd Mode 2.
Figure 2: Contour plot of $\omega_i$ of mode 1 in the $\alpha$ plane. The branch points and the associated branch cuts in the rectangular domain can be seen. The central pillar in the UHP is actually a cluster of BCs; a close-up view is shown in (b).
Figure 3: a) Movement of three saddle points, A, B and C of the dominant even OS mode in the $\alpha$ plane with varying $v_d$. (b) Heights at saddle point A, B and C and at the branch points BP121 and BP131 (Appendix B) as a function of $v_d$.

$v_d$. Since an eventual goal would be to describe the disturbance evolution in terms of the dynamics of the critical points (BPs, saddles and poles), it is important to study the quantities $q = Im(\omega_{n*} - \alpha_* v_d)$, the real part of the phase function, that govern the growth rate of the disturbance, the asterisk referring to the location of the critical point. Hence we parallelly plot this value at the relevant critical points like the saddles and branch points as a function of $v_d$ in figure 3(b). This will give some indication of which critical points and modes contribute where, as a function of $v_d$. A third important entity is the pole $\alpha_p$ of the wave integral, given by $\omega_n(\alpha_p) - \omega_0 = 0$ where $\omega_0$ is the frequency of the disturbance source. A related quantity which is relevant to the residue calculation is $d\omega_n/d\alpha|_{\alpha_p}$. These are straightforward to calculate by solving the OS equation.

Higher modes are not shown here. However they possess interesting features
worth mentioning here. The movement of a SP along the imaginary axis, as \( v_d \) varies, is a common feature of all modes. These SPs have zero phase (imaginary part) and decay slowly in the neighborhood of \( \alpha = 0 \) and hence, can produce streamwise elongated structures. Collision of saddle points as \( v_d \) varies also occurs frequently in the higher modes. In the second mode, the central saddle increases in height till it collides with another central saddle point. Third mode has two off-axis saddle points which collide at \( v_d = 0.4974 \) forming a monkey saddle point. It will be interesting to study the disturbance velocity patterns corresponding to these saddle points; however, the associated decay rates are much higher than that of the primary mode and hence are not considered in the present study.

### 3.1 Evaluation of Fourier integrals

The results in N75 indicate, and an evaluation of the integrals in §2.2 confirm, that the dominant mode is the two-dimensional first OS even mode. Hence we focus here only on the contributions of this mode to the velocity field.

We evaluate analytically the integral \( v_{121} \), defined in §2.3. From figure 3 of §3, it is seen that one or more of the three saddles could contribute to the integral; we now determine which ones are relevant. The positive real axis ends in a valley of the right-most saddle point (C) and the valleys of the on-axis saddle point (A) connect the left and right half planes. Hence, both the saddle points must contribute to the real axis integral.

Before proceeding to the application of formulae in Appendix D, we show (i) why contribution from the middle saddle B is negligible and (ii) how only the SDP of the off-axis saddle point passes through the pole. Figure 4 shows the steepest descent paths of the saddle points corresponding to various \( v_d \) values shown in the figure; the SDP of the on-axis saddle points are shown in 4(a) and those of the (third) off-axis saddle points are shown in 4(b). The least stable spatial mode corresponding to \( \omega_0 = 0.28 \) is located at \( \alpha = (1.0302, 0.0037) \); it is a pole in the alpha plane with leading contribution to the Fourier integrals. The SDP from the on-axis saddle point A never passes through this pole. It is interesting to note the Stokes phenomenon, when the SDP at \( v_d = 0.1586 \) passes through saddle point B; however, it is not of any consequence as will be shown here. The SDP of the off-axis saddle point C crosses the pole at \( v_d \approx 0.4 \).

In order to obtain asymptotic limits of the \( \alpha \)-integrals for \( v_1 \) and \( u_1 \), we construct an integration path called Olver path, into which the real line is deformed; by definition, an Olver path is the union of descent paths from the saddle points. The topography of the OS first mode shows a three-saddle cluster; such a cluster will always have four hills and four valleys such that all the three saddle points share a hill and a valley. Hence, a set of descent paths from all the three saddles can only be linked through the common valley and hence only one valley of the middle saddle point will be used in the Olver path; whereas, both the valleys of
Figure 4: SDP-Pole crossing; $\omega_0 = 0.28; \alpha_p = (1.032, 0.0037)$
(a) SDP of on-axis saddle point at $v_d = 0.1, 0.1586,$ and $0.17$. (b) SDP of off-axis saddle point $v_d = 0.25, 0.4$ and $0.52$. The SDP for $v_d = 0.4$ crosses the pole.
the saddles A and C are involved. Therefore, the middle saddle point B becomes an ordinary point of the Olver path and hence its contribution amounts to just the error term \cite{Oughtun}. This SP transforms from an open point to an inadmissible SP, when \( \text{Re}[p(\alpha)] \) at this point is greater than the corresponding maximum value on the LOI; if B were admissible, we would obtain a growing mode which is convected downstream, an impossibility in subcritical pPf.

In summary, the Fourier integral has two contributing saddle points and a pole which lies on the SDP of the off-axis saddle point at some \( v_d \); hence the formula presented in Appendix D is readily applicable. The validation of the formula presented in Appendix D is based on the first temporal eigenmode described above. The spatio-temporal solutions of the IBVP (2.10 and 2.11) using the formula are presented in the next section.

4 Results I: Linear disturbance evolution

The streamwise disturbance velocity component \( u_1 \) is computed using Olver method described in Appendix D. It is important to note that the factor \( \alpha \) in the denominator of (2.12) cancels with the same in the numerator and hence is not a pole. This can be verified using the expressions for \( I_\pm(\alpha) \) and \( I_{\text{OS}} \) given in Appendix C.

The asymptotic solutions of \( v_1 \) (and \( u_1 \)) in the previous section show a wavepacket arising from the saddle path along with the TS wave and sometimes distinct from it. In this section, the computed asymptotic spatio-temporal solutions for moderate times will be discussed. The major part of the present study is for \( Re = 5000 \).

The IBVP solutions for various values of \( \omega_0 \) are presented in figure 5. The TS wave and the wavepacket are indistinguishable for \( \omega_0 = 0.3 \), figure 5(a); in this case the disturbance state is a slowly decaying TS wave. At higher frequencies, \( 0.33 < \omega_0 < 0.39 \), the decay rate of TS wave increases more steeply; the wavepacket can be clearly seen as in figure 5 (b and c). However, a large part of the wavepacket is still attached to the TS wave and their amplitudes are comparable over a considerable length of the channel. This state is called a mixed state. At still higher frequencies, \( \omega_0 \geq 0.39 \), as in figure 5(d), the TS wave decay rate is very high and a clear wavepacket is seen. Hence, in this case there are two distinct states of comparable magnitudes emerging from the wall disturbance, a TS wave and a wavepacket.

The two states have very different decay rates and velocities of propagation. The TS wave decays (or grows) temporally in a reference frame which moves with its phase velocity \( c_r \); the phase velocity and decay rate of a TS wave can be deduced from the OS dispersion equation. The wavepacket undergoes
Figure 5: IBVP solution at $t = 268$ (large time) and $Re = 5000$ for four different forcing frequencies. (a) $\omega_0 = 0.3$: TS wave is dominant. (b) $\omega_0 = 0.33$: TS wave and wavepackets are of similar magnitudes and decay rates. (c) $\omega_0 = 0.36$: Clear wavepacket is formed. TS wave and wavepacket are of similar magnitude (d) $\omega_0 = 0.39$ TS wave decays rapidly in the neighborhood of the ribbon; wavepacket travels along the entire channel while decaying slowly.
a slow spatio-temporal elongation. It is almost stationary in a reference frame moving with its group velocity $c_g$; the decay rate $\gamma_{wp}$ of the wavepacket in this reference frame varies slowly with time, unlike the TS wave. Neglecting these small spatio-temporal changes of the wavepacket, the group velocity and growth rate are calculated numerically from two instantaneous solutions.

4.1 Wavepacket characteristics

The speed of the reference frame in which the wavepacket is stationary is computed as follows. It is determined visually by noting the distance traveled by the centre of the wavepacket from the origin of a frame moving with a given velocity. When the centre of the wavepacket is almost stationary in a moving frame, $c$ is chosen as the velocity of that frame. It is also an estimate of the group velocity of the wavepacket; this is a real quantity unlike the complex group velocity of the TS wave.

This procedure is demonstrated in the two movies presented here for $\omega_0 = 0.45$; at this ribbon frequency, there is a distinct wavepacket in the test section. The movement of the wavepacket with respect to a frame moving with $c = 0.395$ is shown in Movie 1. For comparison, the movement of the wavepacket in a frame moving with $c = 0.35$ is shown in Movie 2. In these movies, the moving frame is represented by a box. The wavepacket stays in the box in the first movie while it moves slowly out of the box in the second one.

The temporal decay of the wavepacket is estimated by visually inspecting the temporal constancy of its amplitude when the IBVP solution is multiplied by a suitable exponential time factor. Figure 6(a) shows the IBVP solution at four different instants; the wavepacket decay is shown by the sloped envelope of the wavepacket. Figure 6(b) shows the IBVP solution at the same instants as in the previous figure multiplied by a factor of $e^{0.007t}$. Here, the envelope of the wavepacket is a horizontal line with the inference that the wavepacket decays at the rate of 0.007. The wavepacket arises from a saddle path that is traced as $v_d$ varies. Hence the temporal decay (or growth) of the wavepacket is not purely exponential but also contains an algebraic factor $1/\sqrt{t}$. Here, the exponential decay rate of 0.007 roughly accounts for this algebraic decay too.

Table 1 describes the characteristics of the TS wave and the wavepacket for different Reynolds number-frequency combinations. The third and fourth columns are the phase velocity and the spatial decay (or growth) rate of the TS wave. The spatial decay rate of the wavepacket, $\gamma_{wp}$ is given in the fifth column. It is defined as the spatial decay rate of the wavepacket peak. The group velocity of the wavepacket, $c_g$ is given in the sixth column and the temporal decay rate of a wavepacket is the product of $c_g$ and $\gamma_{wp}$. The wavepacket elongates spatially (but slowly) while moving downstream; the spatio-temporal variation of its group velocity and the decay rate is negligible. The values of $c_g$ and $\gamma_{wp}$ shown here are hence computed from two instantaneous solutions. The decay rates of
Figure 6: IBVP solutions for streamwise disturbance velocity at different instants for $Re = 5000; \omega_0 = 0.45$. 

(a) Temporal decay of the wavepacket, indicated by the enveloping line.

(b) IBVP solutions multiplied by a factor of $e^{0.0071t}$. 

Decay rate of 0.006 is compensated.
the TS wave and the wavepacket are almost equal for $0.3 < \omega_0 < 0.33$ at $Re = 5000$; the respective wavepacket group velocity is only slightly more than the TS phase velocity (Table 1). At higher frequencies, $0.33 < \omega_0 < 0.39$, the decay rate of TS wave increases more steeply compared to that of the wavepacket. At still higher frequencies, $0.39 < \omega_0 < 0.45$, the TS wave decay rate is more than double the decay rate of the wavepacket; we choose this condition for identifying a clear wavepacket state as in figure 5(d). All the parameter combinations for $Re = 6000$ shown in this table satisfy this condition. For $Re = 4000$, the two decay rates are nearly same at $\omega_0 = 0.36$.

### 4.2 Comparison with N75 experiments - Linear stage

We start off with a brief description of the experiment in N75. A pPf was established in a long, quiet (turbulence level $< 0.01\%$) channel, with a demonstration of the parabolic profile to a large degree in N75F3. A sinusoidal disturbance was introduced in this flow through a phosphor bronze ribbon, stretched close to the lower wall and vibrating, at a frequency $\omega_0$, in a direction normal to it. The test section, where the measurements were made, ranged from $\sim 44$ to $\sim 78$ units downstream of the ribbon.

For small disturbance amplitudes, less than 1%, it was established that the disturbance appears in the flow as sinusoidal in time (N75F4) and antisymmetric in the wall normal direction $y$ (N75F5). Similar measurements at various streamwise locations confirmed that the disturbance was indeed a traveling wave, whose wavelength $\lambda$ was estimated. N75F7 shows the spatial evolution of the maximum disturbance value with $x$ for a variety of $Re$ and $\omega_0$. N75F6 and N75F9 present the disturbance wavelengths as a function of $Re$ and $\omega_0$; we will

| $Re$ | $\omega_0$ | $c_r$ | $-Im(\alpha_p)$ | $\gamma_{wp}$ | $c_g$ | State     |
|------|-------------|-------|-----------------|--------------|-------|-----------|
| 5000 | 0.3000      | 0.2776| -0.0046         | -0.0046      | -     | -         |
| 5000 | 0.3100      | 0.2811| -0.0079         | -0.0074      | 0.34  | TS        |
| 5000 | 0.3200      | 0.2889| -0.0139         | -0.0088      | 0.37  | Mixed     |
| 5000 | 0.3300      | 0.2901| -0.0179         | -0.0101      | 0.375 | Mixed     |
| 5000 | 0.3400      | 0.2996| -0.0416         | -0.0138      | 0.39  | TS, WP    |
| 5000 | 0.3500      | 0.3122| -0.1044         | -0.0176      | 0.395 | TS, WP    |
| 6000 | 0.3340      | 0.2815| -0.019          | -0.01        | 0.38  | TS, WP    |
| 6000 | 0.3460      | 0.2866| -0.032          | -0.01        | 0.39  | TS, WP    |
| 6000 | 0.3590      | 0.2927| -0.0595         | -0.0109      | 0.405 | TS, WP    |
| 4000 | 0.3590      | 0.2927| -0.0595         | -0.0109      | 0.405 | TS, WP    |
| 4000 | 0.3640      | 0.3315| -0.01782        | -0.018       | 0.4   | Mixed     |
| 4000 | 0.3690      | 0.3315| -0.01782        | -0.018       | 0.4   | Mixed     |

Table 1: TS wave and wavepacket parameters for $Re = 5000$ over a range of forcing frequencies $\omega_0$. 
consider only N75F6. N75F10 shows the amplification rate $\alpha_i = -h d l n u'_m / dx$ vs. angular frequency. N75F11 shows the experimental stability boundary which is a little different from theory.

The digitized data from N75F6, for $Re = 3000$, $4000$, $5000$ are presented in Table 2, where we have also shown the $\alpha$ of the most dominant spatial mode, obtained by solving the spatial eigenvalue problem. It can be noted that the $\alpha_e$ are higher, in general, than the real part of $\alpha_t$ with a maximum discrepancy of up to 5%.

N75F7 and N75F10 pertain to the damping rate of the disturbance; the former plots the disturbance maximum as a function of downstream distance whereas N75F10 synthesises this information into a single number at each $\omega_0$ and $Re$. The data from N75F7(a), for an $f = 72Hz$, are shown in Table 3, where we have also included the value obtained from a linear stability calculation. For example, for $Re = 5300$, 72Hz corresponds to $\omega_0 = 0.321$ which in turns produces a dominant spatial eigenvalue $\alpha = 1.141 + i0.00831$, the imaginary part of which is used in producing the respective values in the last column in Table 3.

Several things can be noted from the table. For the lowest $Re = 4000$, the amplitude decreases with increasing distance more or less in accordance with linear theory. For $Re = 5300$, there is a slight initial increase in experimental amplitude and later, a precipitous decline, which trends, the linear theory is unable to capture, producing as it does a constantly decreasing amplitude, given that the $Re$ is subcritical. Further, the amplitude curve presented in this figure has a wavy pattern. There are problems for the supercritical $Re = 6400$ as well; the experimental amplitude initially grows faster than what linear theory

| Re  | f(Hz) | $\omega_0$ | $\lambda$(cm) | $\alpha_e$ | $\alpha_t$ |
|-----|-------|------------|---------------|------------|------------|
| 3000| 33    | 0.2597     | 4.919         | 0.9325     | 0.9275 + i 0.031 |
|     | 39    | 0.3069     | 4.208         | 1.09       | 1.0348 + i 0.021 |
|     | 43    | 0.3384     | 3.898         | 1.1767     | 1.1056 + i 0.01915 |
|     | 47    | 0.3699     | 3.697         | 1.2407     | 1.1757 + i 0.0214 |
| 4000| 32.82 | 0.194      | 5.814         | 0.789      | 0.7935 + i 0.0389 |
|     | 38.86 | 0.229      | 4.993         | 0.919      | 0.8802 + i 0.0243 |
|     | 50.34 | 0.297      | 4.117         | 1.114      | 1.0451 + i 0.0103 |
|     | 60.42 | 0.357      | 3.661         | 1.253      | 1.1876 + i 0.017 |
|     | 72    | 0.425      | 3.241         | 1.4152     | 1.3482 + i 0.052 |
| 5000| 38.64 | 0.18       | 5.978         | 0.767      | 0.7732 + i 0.0325 |
|     | 50.41 | 0.238      | 4.628         | 0.99       | 0.9234 + i 0.0097 |
|     | 60.49 | 0.286      | 4.099         | 1.12       | 1.0454 + i 0.00375 |
|     | 72.03 | 0.34       | 3.734         | 1.228      | 1.179 + i 0.014 |

Table 2: Data from N75F6. Theoretical values are in column 6.
predicts and astonishingly, decreases after a certain distance, which the linear theory can never predict.

We turn to the solution of the IBVP for a clue as to what might be producing these behaviors. For \( \omega_0 = 0.425 \) and \( Re = 4000 \), the wavepacket has higher amplitude in the test section due to high decay rate of the TS wave (Table 1) and it arrives much earlier than the 2D TS wave. Figure 7(a) shows the overlap of envelopes of several instantaneous solutions for this case in the time interval \([38, 208]\). The envelope of the wavepacket, shown in red, is above that of the TS wave (black), reflecting that the wavepacket decays at a slower rate. However, the measured decay rate matches closely that of the TS wave, as noted above. The seeming incongruity in the experimental observation of the faster decaying TS wave can be resolved if it is noted (N75) that ‘some distance from the ribbon was required for the disturbances to establish a structure which did not change downstream.’ Thus, in this case, the experimenter can wait for the wavepacket to pass beyond the test section and for the unchanging TS wave to be established. As the ribbon frequency approaches the neutral stability curve (or is in the unstable region) the wavepacket is indistinguishable from the TS wave due to its low group velocity as well as a comparatively higher decay rate than the TS wave. Hence, for such cases, waiting does not amount to any difference in the envelope.

Figure 7(b) shows the overlap of the instantaneous IBVP solutions for the time interval \([198, 208]\). The wavepacket has moved out of the test section before

| Re  | \( x - x_0 \) | \( \frac{u_m'}{u_{m,0}}c \) | \( \frac{u_m'}{u_{m,0}}t \) |
|-----|-------------|----------------|----------------|
| 4000| 6           | 0.751          | 0.732          |
|     | 14          | 0.538          | 0.483          |
|     | 20          | 0.391          | 0.354          |
|     | 27          | 0.269          | 0.246          |
|     | 34          | 0.194          | 0.171          |
| 5300| 6           | 0.962          | 0.951          |
|     | 14          | 0.988          | 0.890          |
|     | 20          | 0.885          | 0.847          |
|     | 27          | 0.732          | 0.799          |
|     | 34          | 0.641          | 0.754          |
| 6400| 6           | 1.086          | 1.015          |
|     | 14          | 1.155          | 1.034          |
|     | 20          | 1.101          | 1.049          |
|     | 27          | 0.973          | 1.067          |
|     | 34          | 0.857          | 1.085          |

Table 3: Data from N75F7. Theoretical values, from linear stability calculations, are in column 4.
Figure 7: Overlap of instantaneous IBVP solution envelopes for $Re = 4000$ and $\omega_0 = 0.425$: The horizontal axis denotes the distance from the ribbon in centimeters. (a) $38 \leq t \leq 208$, (b) $198 \leq t \leq 208$; Filled circles denoted scaled measurements from N75F7a.
(c) Instantaneous IBVP solutions for $Re = 6000$ and $\omega_0 = 0.28$; Dashed line denote the solution envelope. Triangles: Measurements for $Re = 6400$; Squares: Measurements for $Re = 5300$ from N75F7a.

$t = 198$, showing only the TS envelope. The black symbols are the suitably scaled experimental values obtained from N75F7a corresponding to $Re = 4000$. The instantaneous solution for $Re = 6000$ with a ribbon frequency of 72Hz is shown in figure 7(c) in the time interval [38, 208]. In this case, the wavepacket, due to its low group velocity, is not distinguishable from the growing TS wave. The envelope of the instantaneous solutions is shown as dashed line. Clearly, the envelope lies between the measured values for $Re = 6400$ shown in triangles and for $Re = 5300$ (squares); the slight nonlinearity and the apparent disturbance decay farther from the ribbon is also captured well. Thus, the spatial decay in a growing mode ($Re = 6400$) is probably more due to the choice of the time interval than a manifestation of any inherent flow physics.

4.3 Comparison with N75F15

N75F15 records the downstream evolution of the disturbance maximum $u'_m$, being over the channel height. The first recording station is 32 cm ($\approx 44$ units) downstream from the ribbon and the last one, about 57cm ($\approx 78$ units). Experimental points, corresponding to six different initial intensities, are plotted. Six curves are drawn, one through each set of points. Curves (i) - (iii), for initial
Figure 8: Instantaneous IBVP solutions in the test section for Re=5000 and \( \omega_0 = 0.34 \) approximately over two periods of the vibrating ribbon (dotted lines). The three symbols correspond to three lower curves of N75F15 normalized to 1.1 at the peak. Solid line represents the computed maximum amplitudes at the experimental points.

Intensities < 1\%, seem to show that \( u_m' \) increases slightly downstream of the initial station before decreasing continuously. Curve (iv) shows, after the initial rise, a constant disturbance for a considerable downstream distance, before again rising steeply. We will be concerned in this study only with curves (i)-(iii) and the earlier part of (iv); (v) and (vi) depict evolution where the higher initial intensities means nonlinearity plays a role and is beyond the scope of the linear analysis of this paper.

We will attempt to explain the ‘apparent’ spatial growth and subsequent decay of the disturbances (curves i - iii, N75F15) by examining the envelope of the instantaneous solutions. N75’s sampling rate seems to be 10 per time period (for example, N75F4) and we have used the same sampling to produce the envelope in figure 8. This figure shows the instantaneous IBVP solutions for Re = 5000 and \( \omega_0 = 0.34 \) approximately over two ribbon periods \( 158 \leq t \leq 198 \) in steps of two non-dimensional time units, the approximate sampling rate. These times correspond roughly to the residence time of the wavepacket in the test section. The solid line is the envelope formed by marking the maximum amplitude at the experimental points over all these time steps in the interval. The discrete nature of the spatial locations and the time steps results in the irregular shape of the envelope. Also shown in the figure are the lower three curves of N75F15, normalized to a peak value of 1.1 in order to match with the computed peak. The matching between the experimental curves and the theoretical curve is very good; in particular, the initial rise and subsequent decay are demonstrated.
Figure 9: Instantaneous IBVP solutions in the test section for Re=5000 and $\omega_0 = 0.34$ over two periods of the vibrating ribbon (a) $200 \leq t \leq 240$. Solid line represents the computed maximum amplitudes at the experimental points. (b) $38 \leq t \leq 228$. Envelope of the TS wave is shown by bunching of blue lines (inner). Red line shows the envelope of the wavepacket.

If the experimental observations were made at a later time, when the wavepacket moves out of the test section, the envelope is expected to show the TS wave. Figure 9(a) shows the resulting envelope. Note that the irregularity in this case is much less and the envelope follows the rate of decay of the TS wave very well. Hence, the initial rise as well as the different decay rate shown by the lower three curves of N75F15 are due to the combination of the choice of spatial and time steps and the passage of the wavepacket. On the other hand, if the experimental sampling rate was higher and also if more recording stations were located upstream, the envelope will be closer to that shown in figure 9(b); it will show a linear decay with rate different from the decay rate of the TS wave which is shown by the bunched blue lines.

N75 and N81 obtained only instantaneous hot-wire measurements at fixed locations and N75F15 is the maximum disturbance over many such instantaneous measurements at these locations. Hence, these measurements cannot directly show the passage of a wavepacket at an earlier instant. We infer its existence in the experiments from the comparisons of IBVP solution with N75 measurements shown above. Since the wavepacket in the test section is of comparable size, or even bigger, than the TS wave at some ribbon frequencies, it can equally well be considered as a base state for secondary instability.
5 Secondary instability analysis

Secondary instability due to three dimensional background disturbances has long been considered a key mechanism in explaining subcritical transition in wall-bounded shear flows [Herbert et al., 1987]. A variety of base states have been considered for the linearisation - the dominant TS mode with the damping neglected [Herbert 1983; H83 hereafter], nonlinear equilibria and quasi-equilibria [Orszag & Patera, 1983] and streamwise vortices and streaks [Schmid & Henningson, 2001]. Typically, in all such studies, the base state has to be considered in a reference frame moving with an appropriate velocity. This renders the coefficients of the disturbance equations periodic in the frame variable, with the implication that Floquet modes, in that variable, can be sought. The three-dimensional background disturbances are represented by spanwise wavenumbers, $\beta$.

The general strategy is to study temporal secondary instability; a basic traveling wave of wavenumber $\alpha$ is considered and the secondary temporal growth rate determined from the solution of an eigenvalue problem. The traditional secondary instability analyses of H83 and others stop at computing TS threshold amplitude for a neutral Floquet mode at a given $\beta$, which we term neutral threshold amplitude, for easy reference. The neutral threshold amplitude is merely the lowest one for a possible secondary growth and cannot be directly compared with experiments such as N75 or N81, as (a) the 3-D disturbance amplitude is not accounted for and (b) the mild decay of the TS wave is uncompensated for. The experimental studies on flow stability reported in literature, for example, N75 and N81 not only present the wavenumbers of the background three-dimensional disturbances but also the corresponding initial amplitudes $\epsilon_z$. Their threshold amplitude measurements are closely linked to $\epsilon_z$, as is evident from N81. Apart from these experiments, a few other measurements of pPf, (for e.g. Nishioka & Asai 1984, Ramazanov 1984), have also quantified three-dimensionality of the experimental set-up.

We have taken into account both factors in our computation of the threshold amplitude. The amplitude and decay rate of the base state and the magnitude of the three-dimensional background disturbances have been combined into a formula for net growth or decay of the total disturbance over one time-period of the vibrating ribbon; the formula is discussed in the following subsection. This combination is similar to how the primary state is formed by superposition of the TS wave onto pPf.

5.1 Threshold amplitudes

For a small ribbon velocity amplitude $A_R$ the total streamwise velocity is given by

$$u_2(x, y, t) = U(y) + A_R e^{i(\alpha x - \omega t)}u_{TS}(y)$$ (1)
where $u_{TS}$ is the normalized TS eigenfunction. For a given wavenumber $\alpha$, when $\text{Im}(\omega) = \omega_i$ is very small, $u_2$ can be a secondary base state which may be unstable to three dimensional disturbances.

The total disturbance function $\bar{u}(x, y, t)$ is composed of the secondary base state and the corresponding Floquet modes. For simplicity, we define $\bar{u}(x, y, z, t)$ as:

$$
\bar{u}(x, y, z, t) = U(y) + A_R e^{i(\alpha x - \omega_0 t)} \left\{ e^{\omega_i t} u_{TS}(y) + \frac{\epsilon_z}{A_R} \text{Max} \left[ \left( e^{-i\sigma t} - 1, 0 \right) u_f(y) \right] \right\}
$$

(2)

where $\text{Re}(\omega) = \omega_0$ and $\sigma$ is the least stable Floquet mode and $u_f$ is the eigenfunction; for sufficiently small $A_R$, $\sigma_i < 0$. The second term on the R.H.S within the brackets is the three dimensional secondary growth and is modeled such that for either $\epsilon_z = 0$ or $\sigma_i \leq 0$, only the secondary base state remains. The maximum criterion has been used to ensure this happens in the latter case.

The quantities $\beta$ and $\epsilon_z$ are inputs from the measurements. These are presented in N75 and N81 as the wavenumber and amplitude of spanwise variations in the centerline velocity $U_c$. N81F5 and N81F6 show, for some small ribbon amplitudes, that the spanwise percentage variation of TS amplitudes is also roughly the same as that of $U_c$. However, in these cases, the ribbon amplitudes are in the neighborhood of the threshold values and hence secondary growth is already taking place, even though it may not be large enough to compensate the base state decay. The form of three dimensional disturbances for very small ribbon amplitudes is not known. In the absence of this knowledge, the formula shown above is a simple way of incorporating the developing three dimensionality while establishing the two dimensional base state in the absence of secondary growth.

For the Floquet expansion to be valid, it is only necessary that $\epsilon_z \ll u_2$ and hence it can be of the order of $A_R$. Here, we consider only the most unstable fundamental mode $\sigma$ whose real part is often negligibly small. As we are interested only in obtaining the threshold amplitudes, we further assume that there exists a $y = y_1$ such that $u_{TS}(y_1) = u_f(y_1) = 1$ which would maximize $\bar{u}$ across the channel; hence, the threshold amplitudes for $\bar{u}$ for growth under these assumptions will give the minimum threshold amplitude for secondary growth. At the spanwise peaks ($z = 2n\pi/\beta$), the equation given above simplifies to :

$$
\bar{u}(x, y_1, z_{\text{peak}}, t) = U(y) + A_R e^{i(\alpha x - \omega_0 t)} \left\{ e^{\omega_i t} + \frac{\epsilon_z}{A_R} \text{Max} \left[ \left( e^{-i\sigma t} - 1, 0 \right) \right] \right\}
$$

(3)

The expression within the curly brackets models the total growth or decay of the input disturbance. We now describe the two methods of determining the threshold amplitude $A_T$. 

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In amplitude plateauing, which is used in N75, $A_T$ is determined by requiring the average growth/decay of this term over one time-period, $T (= 2\pi/\omega_0)$ to be zero, i.e.

$$\frac{1}{T} \left[ \frac{e^{i\omega_i T} - 1}{\omega_i} + \frac{\epsilon_z}{A_T} \frac{e^{\sigma_i(A_T)T} - 1}{\sigma_i(A_T)} \right] - \frac{\epsilon_z}{A_T} = 1.$$  \hspace{1cm} (4)

In case of peak-valley splitting, we first note that the leading component in the Floquet eigenfunction series is symmetric while the TS eigenfunction is antisymmetric. Hence, their sum, as in the formula for $\bar{u}$, will have sharper peaks and shallow valleys. The time-averaged disturbance amplitude at the valley is given by

$$\bar{u}(z_{\text{valley}}) = \frac{A_T}{T} \left[ \frac{e^{i\omega_i T} - 1}{\omega_i} - \frac{\epsilon_z}{A_T} \frac{e^{\sigma_i(A_T)T} - 1}{\sigma_i(A_T)} \right] + \frac{\epsilon_z}{A_T}$$  \hspace{1cm} (5)

while that at the peak continues to be given by the LHS of (5.4).

For decaying Floquet modes, by the present definition, the peak and valley amplitudes are identical. They start splitting when $\sigma_i = 0$. For nearly two-dimensional disturbances, these amplitudes may not differ noticeably up to a certain $\sigma_i > 0$ and that is why these have not been used in N75; the splitting is more rapid in the case of 3D disturbances as shown in figure 10. The solid and dashed lines refer to the variation of peak and valley amplitudes w.r.t ribbon velocity amplitude respectively; the splitting is clearly seen. The intersection of the peak amplitude with the identity line (dotted) in figure 10 indicates the plateauing amplitude.

The threshold amplitudes in N75 were measured based on amplitude plateauing as shown in N75F15 while peak-valley splitting was chosen in N81. Following the measurements, we have used (average) amplitude plateauing given by (a) for small $\beta$ and the peak-valley splitting amplitude for $\beta = 1.76$ as in N81.

We have considered only the growing fundamental mode, which is (or nearly) always in phase with the TS wave. The decaying Floquet modes are not important as they lead to net decay for subcritical Reynolds numbers. The TS amplitude and the secondary disturbance amplitude are simply added here along with their decay and growth rates respectively; this situation is possible only if the corresponding eigenfunctions have peaks at the same $y$ location. Therefore, the computed threshold values are the lowest possible estimates under the one-ribbon period averaging.

5.2 Background disturbances in N75 and N81

The spanwise distribution of the laminar centreline velocity $U_c$ was found to be wavy for $Re > 3500$ (N75F2), with the authors suggesting that it was due to a slight warping of the upper channel wall. Warping can induce a variety of
spanwise velocity distortions over a range of wavenumbers \( \beta \); the smallest value is zero. The mid-third of the 40 cm wide channel is warped which produces a variation of 1.5\% of the mean channel depth (N75); the velocity on either side of the warped portion is not known. In the absence of velocity data across the entire channel, we assume a spanwise mean flow distortion, of wavelength equal to warping width, \( (\beta = 0.35) \); the corresponding distortion amplitude is assumed to be 1\% \( (\epsilon_z = 0.01) \) based on the given mean channel depth variation of 1.5\%. This set of parameters is a typical one for a mildly three-dimensional background disturbance.

Another set of values for \( (\beta, \epsilon_z) \) can be obtained from the velocity distortions within the warped section as shown in N75F2. Figure 11 shows the Fourier transform of this data. A peak at a wavenumber of \( \beta = 1.5 \) can be seen at all Reynolds numbers. The largest amplitude deviations from the mean centerline velocity are roughly 0.0037 and 0.003 for \( Re = 6000, 5000 \) and 4000 respectively. Another peak of similar amplitude occurs at \( \beta = 2.2 \) for \( Re = 6000 \); however, it is not considered in the present analysis. For comparison with N75, we hence consider two sets of parameters, \( (\beta, \epsilon_z) = (0.35, 0.01) \) and \( (1.5, 0.0037) \), arising from the warping on the top channel wall. These spanwise amplitudes are much smaller than that of the mildly three-dimensional disturbance presented above. However, they are still an order of magnitude higher than the freestream disturbance amplitudes.

The measurements of N81 are for highly three-dimensional disturbances both in terms of the spanwise wavenumbers and the corresponding percentage variation in the mean velocity. A periodic spanwise variation of the base flow was achieved with the help of a damping screen with the wavelength and variation

Figure 10: Peak-Valley splitting. \( Re = 5000; \omega_0 = 0.34 \) (72Hz) and \( \beta = 1.76 \).
in the centerline velocity being roughly 25 mm ($\beta = 1.76$) and 5\% respectively. Unlike N75, the threshold amplitudes in N81 were measured based on peak-valley splitting and are presented in N81F15. Following the experiments, we have computed the threshold amplitudes for $\beta = 1.76$, using a spanwise amplitude of 0.05; the results for peak-valley splitting and amplitude plateauing are shown in Figure 10. Plateauing occurs at a higher amplitude than the peak-valley splitting since, theoretically, peak-valley splitting occurs for any $\sigma_i > 0$, whereas, as shown in subsection 5.1, time-averaged plateauing of the peak amplitude (similar to N75F15), occurs at a positive $\sigma_i$. Even for nearly two dimensional disturbances as in N75, the peak-valley splitting will occur at lower amplitudes compared to N75F16. However, the splitting may not be significant up to some amplitude and hence would not be a convenient criterion for threshold amplitudes in that case.

### 5.3 Floquet analysis of base states

It is clear from Table 1 that distinct TS and wavepacket states and mixed states exist within the ribbon frequency ranges considered in N75 and N81. H83 pioneered the secondary instability analysis with the TS wave as base state; some questions regarding the fundamental and subharmonic instabilities have been reconsidered in [Kidambi & Srinivasan 2018].
5.4 Secondary instability of wavepacket state

It may be noted that only one wavepacket emerges in the solution of the IBVP, while the Floquet framework necessarily implies a periodic system of wavepackets. For this purpose, we construct a periodic wavepacket system based on the IBVP wavepacket, padding with zero on either side so as to control the separations of the packets. The procedure for wavepacket reconstruction using Fourier coefficients is described in Appendix F.

The governing equations for secondary disturbance evolution and their discretized forms are given in Appendix F; the discretised equations have been written for the wavepacket for the first time and reduce to the known form for the TS wave state. The number of Fourier modes in the Floquet expansion depends on the number of significant coefficients, \( N_f \), in the Fourier expansion of the wavepacket; smaller the base \( \alpha \), larger the index \( N_f \), which in turn increases the size of the resulting Floquet matrix. Hence in this analysis, we choose \( N_f \) to be 22 at the maximum.

For the present analysis to have any relevance to the original problem, it is important to know what effect the separation between the wavepackets has on the secondary growth rates. Two different convergence tests have been performed: (i) the wavepacket at different times have been considered and (ii) the number \( N_f \) is varied from 11 up to 20. The convergence of the least stable / most unstable fundamental mode at different amplitudes \( A \) of the wavepacket, for the two times and various \( M \) and \( N_f \) is demonstrated in Table 4 for \( Re = 5000, \omega_0 = 0.45 \) and the spanwise wavenumber \( \beta = 1.84 \). Most of the computations in this paper are done using \( N_f = 11 \) Fourier coefficients.

6 Results II: Comparison with N75F15 and N81F16

We now present the threshold amplitudes for several drive frequencies \( \omega_0 \in (0.25, 0.45) \). From the IBVP solution (for e.g. figure 1), relatively clear base states of TS wave and wavepacket can be established for the lower and upper ends of the frequency range. It is for these ranges that a secondary analysis can be performed and the threshold amplitudes obtained. We have chosen the wavenumber-amplitude combinations, based on the data presented in the introduction, viz. (1) \( \beta = 1.5, \epsilon_z = 0.0037 \) and (2) \( \beta = 0.35, \epsilon_z = 0.01 \), in order to meaningfully compare with N75F16.

We plot the computed threshold amplitudes \( A_T \) for these two sets in figure 12(a), for \( Re = 5000 \) as a function of \( \omega_0 \); experimental data from N75F16 are also shown. The experimental first minimum \( Mi_1 \) occurs at \( \omega_0 = 0.28 \) with an amplitude of 0.0135. The computed \( Mi_1 \) occurs at \( \omega_0 = 0.3 \) and 0.32 for \( (\beta, \epsilon_z) = (0.35, 0.01) \) and \( (1.5, 0.0037) \) respectively; their corresponding amplitudes are 0.01 and 0.013. At higher \( \omega_0 > 0.34 \), the computations for the TS base
Figure 12: Threshold amplitudes as a function of ribbon frequency $\omega_0$, $Re = 5000$. (a) Filled diamond: experimental values from N75; Dash-Dot lines: Itoh’s (1974) nonlinear calculations. Present threshold computations for secondary instability for $\beta = 0.35$; $\epsilon_z = 0.01$ and $\beta = 1.5; \epsilon_z = 0.0037$ are shown. The wavepacket base state computations are indicated. (b) Filled square: experimental values from N81 (strongly three-dimensional) $\beta = 1.76, \epsilon_z = 0.05$.
Table 4: Least stable / most unstable fundamental Floquet eigenvalue for various amplitudes $A$ of the wavepacket. $\omega_0 = 0.45, Re = 5000$.

| $A$  | $N_f$ | $M$  | $\sigma_r$    | $\sigma_i$    |
|------|-------|------|----------------|----------------|
| 0.0022 | 11 | 15 | 0.00227 | -0.00048 |
|       | 17 | 20 | -0.00226 | -0.00042 |
|       | 22 | 25 | 0.00226  | -0.00048 |
| (t = 128) | 22 | 25 | 0.00226  | -0.00077 |
| 0.0024 | 11 | 15 | 0.00235 | 0.00122 |
|       | 17 | 20 | 0.00235  | 0.00127 |
|       | 22 | 25 | 0.00234  | 0.00122 |
| (t = 128) | 22 | 25 | -0.00233 | 0.00093 |
| 0.0026 | 11 | 15 | -0.00243 | 0.00278 |
|       | 17 | 20 | -0.00242 | 0.00282 |
|       | 22 | 25 | 0.00242  | 0.00276 |
| (t = 128) | 22 | 25 | 0.00241  | 0.00247 |
| 0.0028 | 11 | 15 | 0.0025   | 0.00421 |
|       | 17 | 20 | -0.00250 | 0.00425 |
|       | 22 | 25 | -0.00250 | 0.00419 |
| (t = 128) | 22 | 25 | 0.00248  | 0.0039 |

The Floquet analysis for $\omega_0 > 0.39$ is performed on the wavepacket state. The wavepacket threshold amplitude for $(1.5, 0.0037)$ is much lower compared to that of $(0.35, 0.01)$ even though its $\epsilon_z$ is very low. For both sets of $(\beta, \epsilon_z)$, computed threshold amplitudes for $\omega_0 = 0.39$ and 0.45 are almost equal with $A_T = 0.0124$ and 0.0025 respectively; the values at $\omega_0 = 0.45$ are slightly higher. The second minimum $M_{i2}$ of the present calculations, hence occurs at $\omega_0 = 0.39$. The experimental value at $\omega_0 = 0.39$ is roughly 0.012, which is close to the wavepacket threshold amplitude for $(0.35, 0.01)$.

The N81 measurements show three minima at $\omega_0 = 0.2, 0.32,$ and $0.425$. The present computations show a monotonically decreasing threshold amplitude for the TS wave. The base state is in fact a mixed one for $\omega_0 \geq 0.34$ and hence the present threshold computations are not applicable in this range; they are shown in the figure only to indicate what numbers would be obtained...
with such an analysis. The computed threshold amplitude for the wavepacket at 0.39 is roughly 0.0025, which does not vary till $\omega_0 = 0.45$. One difference from the N75 case is that the intermediate peak, demonstrated by experiment in $\omega_0 \in (0.34, 0.39)$ cannot be deduced from the present computations and a separate analysis is required for the mixed object in this range. The computed threshold values at $\omega_0 = 0.34$ and 0.39 are 0.0075 and 0.0025, the first of which is higher than the corresponding experimental minimum of 0.005 at $\omega_0 = 0.32$ whereas the second matches well with the measured minimum. Unsurprisingly, the two-dimensional nonlinear threshold calculations of Itoh are very high compared to both the present computations and the measurements of N81 and neither capture the minima nor their location.

The third minimum $M_{i_L}$ at $\omega_0 = 0.2$ is not shown by the present computations. The IBVP solution is a mixed state at $\omega_0 = 0.2$. For $\omega_0 < 0.2$, a clear wavepacket emerges in the test section, with a decay rate higher than those corresponding to $\omega_0 \geq 0.34$. In addition to this, the least stable OS mode for $\beta = 1.76$ has a decay rate comparable (or even lower) to that of the TS wave. The high initial amplitude $\epsilon_z (=0.05)$ at this $\beta$ will also affect the receptivity of the three-dimensional primary mode for this $\beta$. Hence, in this range of ribbon frequencies, the secondary base state cannot be deduced from the IBVP using the least stable OS mode alone. We have not considered the resulting compound base state in the present study.

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The comparison of the computed threshold amplitudes for $Re=6000$ with the corresponding data of N75F16 is shown in Figure 13(a). Two sets of calculations for the combinations (0.35, 0.01) and (1.5, 0.0037) have been done. N75F16 shows $M_{i_1}$ and $M_{i_2}$ at $\omega_0 = 0.27$ and at $\omega_0 = 0.345$ respectively. The computed minimum $M_{i_1}$ for the TS base state is at $\omega_0 = 0.28$. At $\omega_0 = 0.34$, the threshold amplitudes for TS wave and that of the wavepacket are the same for (0.35, 0.01); on the other hand, for (1.5, 0.0037) the threshold value for the wavepacket base state drops to a very low value of 0.002. For $\omega_0 > 0.34$, the TS base state threshold amplitude continues to grow (not shown here) while the wavepacket threshold amplitudes plateau to 0.0096 and 0.002 respectively for the parameters (0.35, 0.01) and (1.5, 0.0037) respectively. The experimental value at $M_{i_2}$ lies between these two values. The nonlinear threshold amplitudes increase with increasing ribbon frequencies.

The threshold computations for $Re=4000$ at $\beta = 0.35$ is shown in Figure 13(b). The computed values agree very well with the measured values in the frequency range (0.32, 0.34), at which the TS decay rates are the lowest. For all other frequencies, the application of Floquet analysis is more and more in error as the damping rate is no longer negligible. For $\beta = 0.35$, the threshold amplitudes of the wavepacket corresponding to $\omega_0 = 0.4$, 0.425 are less than the minimum threshold of the TS waves. The measured $M_{i_2}$ matches with the computation at $\omega_0 = 0.4$. The wavepacket at $\omega_0 = 0.425$, however, shows a very low threshold amplitude of 0.003 when $\beta = 1.5$. Threshold amplitude compu-
tations were not done for $\beta = 1.5$ and $\omega_o = 0.4$.

Interestingly, the criterion for threshold amplitude presented in subsection 5.1 is not satisfied at all for TS wave at $\beta = 1.5$ and $\epsilon_z = 0.003$, which indicates that a plateauing similar to N75F15 does not occur at all. By increasing $\epsilon_z$ to 0.004, the threshold amplitude condition can be satisfied over a small range of $\omega_0$. This verification, however, is not shown here.

For $Re = 5000$ and 6000, the threshold amplitude at $Mi_1$ is much lower for the $\beta = 0.35$ than for $\beta = 1.5$; this may be due to the higher value of $\epsilon_z$ assumed at $\beta = 0.35$. For the wavepacket state, the growth rate of Floquet modes increases quite rapidly with its amplitude and hence the threshold amplitudes are insensitive to the variation in $\epsilon_z$. The wavepacket thresholds decrease with increasing $\beta$.

6.1 Discussion

As pointed out in the previous section, most of the presented experimental data (for e.g. N75F7 and N75F15) do not represent a constant-rate decay, as required by linear stability analysis; in fact there are regions of spatial growth and plateauing followed by decay even for very small disturbance amplitudes at sub-critical Reynolds numbers. At first glance, these features may seem attributable to transient growth, at least two manifestations of which have been long studied - (a) due to streamwise independent structures [Ellingsen & Palm 1975] and (b) due to the interaction of at least two non-normal modes [Schmid & Henningson 2001]. From the results that have been presented earlier, we argue that both these mechanisms are not in play here. Instead, we have shown that the curves (i) - (iii) of N75F15 are a reflection of the spatio-temporal nature of the interaction between the TS wave and the associated wavepacket. Non-normality of the underlying operator is not directly relevant, as these features are shown by the primary mode itself. As mentioned in the Introduction, Trefethen et al (1993) demonstrated similar behavior with a $2 \times 2$ nonlinear, non-normal model; however, its relevance to an experimental situation like N75 has not been established. The marginal role played by nonlinearity in explaining the behavior of the data considered here is further illustrated by the following facts - (a) Itoh’s (1974) non-linear threshold amplitudes are higher than the experimental values for all Reynolds numbers and (b) Even for a high initial amplitude of 2 %, the initial amplitude and growth of the first harmonic is very small (N75F17). The fact that our results, computed using a secondary instability analysis based on a linear solution, can explain the experimental observations to a large extent, further confirms this.

As is well-known, Floquet analysis allows detuned modes as solutions, the fundamental (resp. subharmonic) being not detuned at all (resp. being the most detuned). The computed thresholds should correspond to whatever detuning produces the lowest values. However, the thresholds presented in figures 12 and
Figure 13: (a) Threshold amplitudes as a function of ribbon frequency $\omega_0$. $Re = 6000$. Filled diamonds represent experimental values from N75. Dash-Dot lines represent Itoh’s (1974) nonlinear calculations. Present threshold computations for secondary instability for $\beta = 0.35; \varepsilon_z = 0.01$ and $\beta = 1.5; \varepsilon_z = 0.0037$ are shown. The wavepacket base state computations are indicated. (b) Threshold amplitudes as a function of ribbon frequency $\omega_0$. $Re = 4000$. Filled diamonds represent experimental values from N75. Dash-Dot lines represent Itoh’s (1974) nonlinear calculations. Present threshold computations for secondary instability for $\beta = 0.35; \varepsilon_z = 0.01$ and $\beta = 1.5; \varepsilon_z = 0.003$ are shown. The wavepacket base state computations are indicated.
13 are based only on the fundamental secondary mode. One reason for this is the lack of experimental observations of signatures of the detuned modes, despite sometimes having lower thresholds, detailed explanations for which have been advanced (for example, in Kim & Moser 1989, Zang & Krist 1989, Kidambi & Srinivasan 2018). Also, the threshold amplitude in the present scenario cannot be read off as the value at which the secondary mode begins to grow but rather has to be computed, taking into account the slight decay of the base state (be it a TS wave or a wavepacket), as detailed in section 5.1. This computation gives unambiguous results for the case of the fundamental mode as it is phase-locked with the primary wave but would have to be further modified to produce sensible results for experimental comparison, if one were to consider detuned modes. In view of the aforementioned lack of experimental observations of such modes, we have considered only the fundamental modes in this study.

The IBVP solution shows only one wavepacket downstream of the TS wave. The wavepacket travels downstream with a group velocity, $c_g$, much higher than the phase velocity of the TS wave. Even though the wavepacket evolves spatio-temporally, it is nearly steady in the reference frame moving with the group velocity $c_g$. Given the nearly constant nature of the group velocity and size, secondary instability of the wavepacket is as much a possibility as that of the TS wave. Though wavepackets have been objects of study in the stability community since at least Gaster [1968], which considered three-dimensional wavepacket development in a boundary layer, and the TS wave for even longer, the two have not been considered together in the vibrating ribbon problem. This is possibly (for e.g. Gaster & Davey 1968) because the vibrating ribbon was seen as producing a TS wave and a pulsed point source as producing a wavepacket. The secondary instability of the single wavepacket arising from the IBVP solution is examined by considering a periodic train of wavepackets as the base state; each wavepacket constituting this train is identical to the wavepacket state. The wavepackets are sufficiently separated from each other spatio-temporally; the larger the separation, closer its secondary stability characteristics will be to those of a single wavepacket.

We now discuss the sensitivity of the computed threshold amplitudes to the primary wave characteristics. The computed threshold amplitude $A_T$ depends on the reference frame velocity $c$. Everything else remaining same, an increasing $c$ leads to a decreasing $A_T$, to a certain extent. A representative variation is shown in Table 5 for the fundamental mode at $\alpha = 1.12, \beta = 2, Re = 5000$. $A_T$ attains a minimum for $c \approx 0.8$. It is well-known (for e.g. Croswell 1985) that the transfer of energy from the mean flow to the secondary disturbance is the key instability mechanism and is represented by the term $T_{30} = -c_z^2 \int_{\Omega} u_3 v_3 \frac{dt}{d\Omega} d\Omega$; $u_3$ and $v_3$ are the velocity eigenfunctions in the streamwise and normal directions and $\Omega$ is the channel volume over one wavelength of the TS wave. For the large TS amplitudes considered in H83 and Croswell 1985, the eigenfunctions $u_3$ and $v_3$ (for both fundamental and subharmonic) are peaked in the neigh-
Table 5: Threshold amplitude of the fundamental mode as a function of the reference frame velocity \( c \). \( Re = 5000, \alpha = 1.12 \) and \( \beta = 2 \).

| \( c \) | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.75 | 0.8 | 0.9 | 0.95 |
|---|---|---|---|---|---|---|---|---|---|---|
| \( A_T \) | 0.0362 | 0.0108 | 0.0066 | 0.0052 | 0.0046 | 0.0042 | 0.0041 | 0.004 | 0.0042 | 0.005 |

Figure 14: The dominant component of a) \( u_3 \) and b) \( v_3 \) of the fundamental mode as a function of \( y \). Solid and dashed lines correspond to threshold amplitudes for \( c = 0.3 \) and 0.8 respectively. Dotted line is for an amplitude of 0.0248 and \( c = 0.2818 \). These are the values similar to the ones in the experiment of Nishioka & Asai (1984). Arrows indicate approximate location of the critical layers.

The threshold amplitude \( A_T \), as computed from the formula, also depends on the temporal decay rate of the base state. The measured and computed threshold curves show a parabolic variation with respect to \( \omega_0 \) which follows closely the parabolic curve which the TS spatial decay rate \( \alpha_i \) traces w.r.t \( \omega_0 \).
The first minimum, $M_{i1}$, in N75F16 occurs at the $\omega_0$ corresponding to the least decaying spatial TS wave. In the computations presented in figures 12 and 13, $M_{i1}$ is attained at slightly higher forcing frequencies for all the three Reynolds numbers. The reason for this shift is, the TS decay rates in a small neighborhood of $M_{i1}$ do not vary as rapidly as the phase velocity; as $\omega_0$ increases the phase velocity increases and hence the threshold amplitude decreases. $M_{i1}$ occurs at higher and higher frequency with increasing $\beta$ as can also be seen from the N75 and N81 experiments; the same behavior is shown by the computations presented in figure 12(a) for $Re = 5000$. $M_{i1}$ is not sensitive to $\epsilon_z$ but the corresponding threshold amplitude is mildly sensitive; a larger value of $\epsilon_z$ lowers the minimum threshold amplitude slightly. At higher forcing frequencies where a wavepacket state exists, the threshold amplitude does not show a parabolic variation around $M_{i2}$ but increases mildly beyond $M_{i2}$. For example, for $Re = 5000$, wavepacket states are seen for $\omega_0 > 0.39$. It can be seen from Figure 12(a) that the threshold amplitude for $\omega_0 = 0.45$ is higher than that at 0.39, even though the group velocity $c_g$ for $\omega_0 = 0.45$ is greater than the $c_g$ for $\omega_0 = 0.39$ (Table 1). But, the decay rate of the former is higher which causes the mild increase in the threshold amplitude at $\omega_0 = 0.45$.

Unlike the effect of frame velocity, which comes directly from the Floquet analysis as discussed in the preceding paragraph, the decay rate of the base state is artificially incorporated in the equation (5.4). We have also computed threshold values by using another method (not presented here) where the primary decay rate is compensated by simply adding the computed Floquet growth rate to it; thus, it is a multiplicative compensation as against an additive one for the present method. However, both methods correctly reproduce the parabolic part of the threshold curves, for the TS base state. The two sets of threshold values differ only a little as long as the primary decay rates are small even though the second method does not include $\epsilon_z$ at all. This is not surprising if we recall the weak dependence, on $\epsilon_z$, of threshold amplitudes computed using the formula. The sudden drop in the threshold amplitude when the base state changes is mainly due to the higher frame velocity and hence happens irrespective of how the primary and secondary disturbances are combined. In summary, the two major features of the threshold curves of N75F16 and N81F15, the parabolic nature and the sudden drop in the threshold at higher frequencies, are due to the arrest of primary decay by secondary growth and the change in the base state respectively; in particular, they are not artifacts of the formula that is used for computing the threshold amplitude.

Threshold amplitudes have sometimes been obtained from DNS studies, for a fixed set of wavenumbers ($\alpha, \beta$). We present findings from one such study [Reddy et al. 1998], where different types of disturbances such as TS waves, three dimensional noise (N), oblique waves (OW) and two-dimensional optimal disturbances (2DOP) were considered, alongside the current results and results from N75 in figure 15. It is evident from the present computations that the threshold amplitudes of the two different base states vary differently with
Reynolds numbers. Hence, it is interesting to compare the nature of these variations with the similar results from DNS for other disturbance types such as 2D optimal disturbances and oblique waves etc. The first minima in N75F16 at three Reynolds numbers (4000, 5000 and 6000) are shown by dashed line with diamond symbols. The computed minima at these Reynolds numbers for TS state at $\beta = 0.35$ are shown as solid line with square symbols. The discrepancy between these two curves is the highest at $Re = 6000$; the computations show a smaller threshold amplitude at this supercritical Reynolds number. The amplitudes at $M_{i2}$ of N75F16, which are not necessarily the minimum values, are shown by solid line, triangles. The computed minima for wavepackets at $\beta = 0.35$ are shown by dashed line, triangles. All these threshold amplitudes lie close to the DNS of secondary instability of TS waves but are slightly higher. It has to be noted that the DNS for the TS state was performed for $\alpha = 1$ and $\beta = 1.0$. The computed minima for TS state at $\beta = 1.5$ are much higher than those for $\beta = 0.35$ and hence not shown in this figure. The minimum threshold values for wavepacket at $\beta = 1.5$ are shown by solid line with star symbols. These values are much lower than the group of values for $\beta = 0.35$. These values are comparable to the threshold amplitudes for 2DOPT and random noise. It is evident from figures 12 - 13 that the wavepacket threshold values for $\beta = 2$ are not very different from those of $\beta = 1.5$. The figure also shows that the threshold amplitudes in the 2D vibrating ribbon experiments are closer to the wavepacket thresholds than those of oblique waves and streamwise vortices strengthening our claim that this is indeed the operative mechanism for these parameter values.

7 Concluding remarks

A semi-analytic solution for the IBVP of a vibrating ribbon in pPf has been provided and clearly delineates the distinct states of the TS wave and the wavepacket. The solution is largely made possible by using an algorithm (Appendix A) to sort complex temporal eigenvalues into modal families, identifying salient features like saddles, poles and branch points and then incorporating these features to properly evaluate the relevant disturbance integrals. These states are then used to provide a novel explanation for the incipient stages of subcritical transition in pPf.

This involves not only the well-known secondary instability of the primary TS wave but also a seemingly overlooked secondary instability of a wavepacket that often dominates for higher drive frequencies. To this end, a secondary instability analysis of a wavepacket state has been provided for the first time. This framework can not only explain the behavior of the lower curves in N75F15, but also the reason for the maximum in N75F16, something previous theories based solely on primary linear stability, nonlinearity or transient growth have failed to do. The current model also provides a counterview to the widely accepted route to transition, involving the secondary instability of nonlinear TS states and their
subsequent breakdown, for controlled disturbance environments. N75 claims to have seen spot-like fluctuations directly triggering transition for higher drive frequencies. It is tempting to speculate that these spots are further evolution of the wavepackets that have been observed and documented in this study. In fact, this kind of speculation is quite old, though in a different context; we find, for example, in the Introduction of Gaster & Davey [1968] - ‘Natural transition often occurs through the formation and growth of turbulent spots which are presumably initiated by these linear wavepackets.’ Wavepacket dynamics in the wingtips of turbulent spots has also been investigated (Henningson [1983], Li & Widnall [1989]). A fair amount of print, mostly for boundary layer flows, in the form of DNS studies, has been devoted to evolution of a wavepacket into a spot; a recent example is Cherubini et al [2010]. Post the secondary instability analysis, we have employed heuristic methods, motivated by the experimental ones, to estimate the threshold amplitudes. It may be worthwhile to examine if these methods can be endowed with more rigor. Other interesting and difficult problems would involve an analytic exploration of the nonlinear development of the structures identified in this paper, so as to obtain a better analytic description of the later stages of the transition process.

The authors acknowledge financial support from National Board of Higher Mathematics, Department of Atomic Energy, India through Project No. 2/48(3)/2013/NBHM(RP)/R&DII/685.
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The names of the authors in this translated report are incorrect. The original version is in Japanese published in Proc. 10th Turbulence Symposium, Inst. Space Aeron. Sci., Tokyo Univ., 1978, p. 55-62.
A SPECRE: Sorting Procedure for Eigenvalues based on Cauchy - Riemann Equations

SPEC-RE is based on a well-known result of function theory for polynomials; the roots of such polynomials are then analytic functions of the same parameter with only algebraic singularities [Kato, 1995], whose coefficients are analytic functions of a parameter. Thus, the eigenmode $\omega_j$ is an analytic function of the complex wavenumber $\alpha$ except at isolated branch points. At any such point of analyticity $\alpha = \alpha_p$, the quantity

$$ F = \left| \frac{\partial \text{Re}(\omega_j)}{\partial \alpha_r} - \frac{\partial \text{Im}(\omega_j)}{\partial \alpha_i} \right| + \left| \frac{\partial \text{Re}(\omega_j)}{\partial \alpha_i} + \frac{\partial \text{Im}(\omega_j)}{\partial \alpha_r} \right| $$

has to be negligible since Cauchy-Riemann conditions for analytic functions have to be satisfied. In what follows, this quantity $F(\omega_j; \alpha_p)$ is called the CR residue.

The primary task of SPEC-RE is to sort each eigenvalue of the spectrum from a given initial point in the $\alpha$ plane by minimization of the CR residue at the points of analyticity. The algorithm also makes use of the negation of the CR criterion at the branch points rather than describing a ‘method’ to identify branch points.

The computational domain is a rectangular patch in the $\alpha$ plane with edges parallel to the axes. The grid points are equally spaced along both the axes; however, the grid size in these directions may be different. The sorting algorithm is implemented on a 4-point stencil of this grid (dashed line in figure 16(a)); at any given pivot point $\alpha_{n,m} = (\alpha_r, \alpha_i)$ the stencil consists of the neighbouring points $\alpha_{n-1,m} = (\alpha_r - \delta\alpha_r, \alpha_i), \alpha_{n+1,m} = (\alpha_r + \delta\alpha_r, \alpha_i)$ and $\alpha_{n,m+1} = (\alpha_r, \alpha_i + \delta\alpha_i)$. The inclusion of the upstream point $\alpha_{n-1,m}$ ensures the continuity of slope in the sweeping direction, while the other three points ensure orthogonality (or local harmonicity). Given a particular eigenvalue $\omega_j$ at $\alpha_{n,m}$, the algorithm is designed to pick one (and only one) of the eigenvalues from...
the spectrum at two neighbouring points $\alpha_{n+1,m}$ and $\alpha_{n,m+1}$ such that the CR condition at $\alpha_{n,m}$ is satisfied. Equivalently, the relevant complex derivatives at $\alpha_{n,m}$ must make the CR residue $F$ to be negligible. In the numerical procedure, these derivatives are replaced by the central and forward differences

$$\frac{\partial \omega_j}{\partial \alpha_r} = \frac{\omega_l(\alpha_{n+1,m}) - \omega_j(\alpha_{n-1,m})}{2\delta \alpha_r}, \quad l = 1, 2, 3, ...$$

$$\frac{\partial \omega_j}{\partial \alpha_i} = \frac{\omega_k(\alpha_{n,m+1}) - \omega_j(\alpha_{n,m})}{i\delta \alpha_i}, \quad k = 1, 2, 3, ...$$

The CR residue $F(\omega_j; \alpha_{n,m})$ is defined using these central-forward differences and is actually a set of numbers $F_{lk}(\omega_j; \alpha_{n,m}); l = 1, 2, 3, ... k = 1, 2, 3, ...$. The indices $k_p$ and $l_p$ that correspond to the minimum of these numbers for a given $j$, which is expected to be a negligible quantity, are picked. As the analyticity condition for $\omega_j$ at $\alpha_{n,m}$ is numerically satisfied between $\omega_j, \omega_{l_p}$ and $\omega_{k_p}$, all three $\omega$s belong to the same analytic function. In other words,

$$\omega_j(\alpha_{n+1,m}) = \omega_{l_p}, \quad (a)$$

$$\omega_j(\alpha_{n,m+1}) = \omega_{k_p}, \quad (b)$$

The pivot point can then be moved to one of the two adjacent points either in the horizontal direction or the vertical direction and the sorting procedure can be repeated for the new stencil. Hence, starting from an initial point $\alpha_0$, the sorting procedure picks one and only one value from the spectrum at each grid point and assigns it to the $j$th collection so that an analytic function $\omega_j(\alpha)$ is constructed, on the entire rectangular patch in the $\alpha$ plane.

### A.1 Sweep direction

In a horizontal sweep, the pivot point $\alpha_{n,m}$ moves along the direction of increasing $\Re(\alpha)$, keeping $\Im(\alpha)$ constant. After reaching the right-most point of the grid, the pivot point is moved to $\alpha_{1,m+1}$. Further computations are performed on stencils containing $\alpha_{n,m+1}, \alpha_{n+1,m+1}$ and $\alpha_{n,m+2}$ starting from $n = 1$. It may be noted that the eigenvalues at this level have already been sorted from the computation at the m-th level, as shown in equation (b). Hence, using the eigenvalues at the m+1-st level, either (i) $\omega_j$ at the m + 2-nd level may be sorted, or (ii) re-sorting may be done afresh at the m + 1-st level. Method (ii) will not produce any new arrangement of eigenvalues at the m + 1-st level unless a branch point lies between the m-th and the m + 1-st levels. Eduction of a branch cut along the sweep direction (horizontal) by Method (ii) will be explained in the following subsection. The sweep direction is not rigidly fixed. A vertical sweep, for instance, will produce a different modal map, with vertical branch cuts. One could indeed sweep even along any family of parametric curves; the C-R equations would then have to be satisfied in the appropriate coordinates.
A.2 Mode sorting around a branch point

Assume that there exists a branch point between \( \omega_j \) and \( \omega_{j+k} \) located in the box formed by the \( m \)-th, \( m+1 \)-st, \( n \)-th and \( n+1 \)-st lines as shown in figure 16 (i.e. \( \omega_j(\alpha) \) and \( \omega_{j+k}(\alpha) \) intersect at some \( \alpha_b \)). By design, the sorting algorithm produces an analytic \( \omega_j \) not only up to the \( m \)-th line, but also up to the point \( \alpha_{n+1,m+1} \) on the \( m+1 \)-st line. At the stencil formed by \( \alpha_{n+1,m}, \alpha_{n+2,m} \) and \( \alpha_{n+1,m+1} \), application of CR condition forces analyticity of \( \omega_j \) at both edges of the stencil and hence, does not allow the BC to cut the \( \alpha_{n+1,m} - \alpha_{n+1,m+1} \) edge. The forcing of analyticity on the lower and left edges of the box by the previous stencil leads to the BC cutting the \( \alpha_{n,m+1} - \alpha_{n+1,m+1} \) edge, as shown in figure 16(a). If further computations were to be done using Method (i) to sort eigenvalues at the \( m+2 \)-nd level, application of CR condition for the stencil at \( \alpha_{n,m+1} \) will be erroneous due to the aforementioned non-analyticity at the \( \alpha_{n,m+1} - \alpha_{n+1,m+1} \) edge. By Method (ii), \( \omega_j \) values along that line are rearranged and analytic sorting between \( \omega_j(\alpha_{n,m+1}) \) and \( \omega_j(\alpha_{n+1,m+1}) \) is ensured. Analyticity along this edge forces non-analyticity of \( \omega_j \) along the \( \alpha_{n+1,m} - \alpha_{n+1,m+1} \) edge, which is equivalent to the BC being horizontal in that grid box as shown in figure 16(b). By continuation of the horizontal sweep at the \( m+1 \)-st level, a horizontal BC evolves naturally. A vertical sweep, together with the application of Method (ii) in the vertical direction would produce a vertical BC. It should, in principle, be possible to modify the algorithm to obtain a branch cut along a suitable complex curve from the branch point by allowing non-analyticity at suitable edges of the stencils while sweeping.

B Branch point identification

The coordinates for which \( ||\omega_i - \omega_j|| \) is the smallest will locate a BP between the \( i \)-th and the \( j \)-th modes. This BP location is further verified by mapping a circle, of suitably small radius (typically a value of 0.001 has been used here) and centred around the suspected BP, under the relevant modal maps; it is well-known that the modal maps will generate open curves if a BP is being circled and that the open curves together form a closed curve. For example, the paths
Figure 17: Tracks of $\omega_3$ (blue circles) and $\omega_3$ (red circles) when a circle of radius 0.001 is traced around the 131 BP (0.0226,0.0213) in the $\alpha$ plane. The open tracks indicate the presence of a branch point, which can be inferred to be a square root as the circles of both colours form a closed curve.

traced by modes 1 and 3 when a circle is traced in the $\alpha$ plane around the BP (0.0226,0.0213) are shown in figure 17; each of these paths is an open curve but together they form a closed curve showing that a square root BP is indeed enclosed.

For ease of reference, we label the branch points with 3 digit numbers - the first two digits are the mode numbers sharing the branch point and the last is its serial number in the list of branch points between those modes. The branch points are listed in the order they occur from top to bottom in the complex plane. Thus BP342 is the second of the branch points between modes 3 and 4. Though we refer primarily to BPs in the RHP, it is understood that the images of these in the left half plane are also BPs and the number, in general, refers to both. Also, BPxyn and BPxyx refer to the same BP.

C Explicit forms of integrals in (2.11)

We give explicit forms for the integrals in (2.11) that involve even and odd adjoint OS and Squire eigenfunctions. We recall that these eigenfunctions are first computed on the half-domain $[0,1]$ and reflected in the appropriate manner. The function and second derivative are required to vanish at $y = 0$ for an odd eigenfunction whereas vanishing of the first and third derivatives lead to an even eigenfunction.

We now consider one part of the integral $I_+$ viz.

$$I_L = \int_{-1}^{1} \mathcal{L} f \xi_n^\ast dy.$$
Splitting $Lf$ into an even and odd part, we have

$$Lf = f_{Le} + f_{Lo}$$  \hspace{1cm} (6)$$

For an even $\xi_n$, $I_L$ can be written as

$$I_L \equiv I_{Le}^n = 2 \int_0^1 f_{Le} \xi_n(y) dy$$  \hspace{1cm} (7)$$

Since $\hat{\xi_n}(\hat{y}), \hat{y} \in [-1,1]$ is what is actually computed as a Chebyshev expansion, we need to express (C 2) in terms of $\hat{y}$. The relation between the hatted and unhatted $y$ is given by $y = (1 + \hat{y})/2$. Dropping the hat, with the understanding that the integrand is a function of $\hat{y}$, we have

$$I_{Le}^n = \int_{-1}^1 f_{Le} \xi_n d\hat{y}. \hspace{1cm} (8)$$

A similar expression obtains for the integral $I_{Lo}$ involving the odd eigenfunctions, with $f_{Lo}$ in place of $f_{Le}$. After some algebra, $f_{Le}$ and $f_{Lo}$ can be shown to be

$$f_{Le} = (G_0 + G_2/2)T_0 + G_1T_1 + G_2T_2 \hspace{1cm} (9)$$

where $G_0 = \frac{1}{2} \left( 2i\alpha - \frac{3}{4}i\alpha^3 - \frac{\alpha^4}{Re} \right)$, $G_1 = \frac{i\alpha^3}{4}$ and $G_2 = \frac{i\alpha^3}{8}$.

The $T_n(\hat{y})$ are Chebyshev polynomials of the first kind.

$$f_{Lo} = H_0T_0 + H_1T_1 + H_2T_2 + H_3T_3 + H_4T_4 + H_5T_5 \hspace{1cm} (10)$$

where $H_0 = a_0 + \frac{a_2}{2} + \frac{3}{8}a_4$, $H_1 = a_1 + \frac{3}{4}a_3 + \frac{5}{8}a_5$, $H_2 = \frac{1}{2}(a_2 + a_4)$,

$H_3 = \frac{a_3}{4} + \frac{a_5}{2}$, $H_4 = \frac{a_4}{8}$, and $H_5 = \frac{a_5}{16}$ with, $a_0 = \frac{33}{128}i\alpha^3 - \frac{3}{8}i\alpha + \frac{3}{2}a_2^2 + \frac{11}{32}i\alpha^4$,

$H_3 = \frac{3}{8}i\alpha + \frac{3}{2}a_2^2$ and $H_5 = \frac{3}{32}Re$,

$H_2 = \frac{1}{2}(a_2 + a_4)$,

$H_3 = \frac{5}{128}f_4 + \frac{9}{8}a^2 + \frac{9}{2}a_0^2$, $a_1 = -\frac{19}{64}i\alpha^3 + \frac{3}{8}i\alpha + \frac{3}{32}a_2^4$,

$H_2 = \frac{5}{128}i\alpha^3 + \frac{1}{8}a_3^4$, $a_3 = \frac{5}{128}i\alpha^3$, $a_5 = \frac{i\alpha^3}{128}$.

Similarly we have

$$Mf = f_{Me} + f_{Mo} \hspace{1cm} (11)$$

with

$$f_{Me} = -\frac{\alpha^2}{2}T_0, \quad \text{and} \quad f_{Mo} = (F_0 + \frac{F_2}{2})T_0 + (F_1 + \frac{F_3}{4})T_1 + \frac{F_2}{2}T_2 + \frac{F_3}{4}T_3, \hspace{1cm} (12)$$

with $F_0 = \frac{3}{4} + \frac{11}{32}a_2$, $F_1 = \frac{3}{4} + \frac{9}{32}a_2$, $F_2 = \frac{3}{32}a_2^2$, $F_3 = -\frac{\alpha^2}{32}$.
Using the fact that \( \xi_n^*(\hat{y}) \) is given by the Chebyshev series

\[
\xi_n^* = \sum_{i=0}^{\infty} b_{ni}^e T_i(\hat{y})
\]

and the fact that

\[
\int_{-1}^{1} T_m(x) T_n(x) dx = \begin{cases} 
\frac{1}{2} & \text{if } m + n \text{ is even} \\
0 & \text{if } m + n \text{ is odd}
\end{cases}
\]

(C 3) becomes

\[
I_{Le}^n = 2 \sum_{k=0}^{\infty} \left[ \left( \frac{a_0}{1 - 4k^2} + \frac{4k^2 + 3}{(9 - 4k^2)(4k^2 - 1)} a_2 \right) b_{n,2k}^e + \frac{1}{(1 - 2k)(2k + 3)} a_1 b_{n,2k+1}^e \right].
\] (13)

In this expression,

\[
a_0 = G_0 + \frac{G_2}{2}, \quad a_1 = G_1, \quad a_2 = \frac{G_2}{2},
\]

and \( b_{ni}^e \) are the Chebyshev coefficients determining the even eigenfunction. Similarly,

\[
I_{Lo}^n = 2 \sum_{k=0}^{\infty} \left[ \left( \frac{H_0}{1 - 4k^2} + \frac{4k^2 + 3}{(9 - 4k^2)(4k^2 - 1)} H_2 + \frac{4k^2 + 15}{(25 - 4k^2)(4k^2 - 9)} H_4 \right) b_{n,2k}^o \\
+ \frac{1}{(1 - 2k)(2k + 3)} H_1 + \frac{10k + 9}{(9 - 4k^2)(2k + 1)(2k + 5)} H_3 + \frac{4k^2 + 4k + 25}{(25 - 4k^2)(2k - 3)(2k + 7)} H_5 \right) b_{n,2k+1}^o \right].
\] (14)

The \( b_{ni}^o \) are the Chebyshev coefficients determining the odd eigenfunction.

As for the second part of the integral, we have

\[
I_{Me}^n = -2 \sum_{k=0}^{\infty} \frac{\alpha^2}{1 - 4k^2} b_{n,2k}^e,
\] (15)

\[
I_{Mo}^n = 2 \sum_{k=0}^{\infty} \left[ \left( \frac{a_0}{1 - 4k^2} + \frac{4k^2 + 3}{(9 - 4k^2)(4k^2 - 1)} a_2 \right) b_{n,2k}^e \right. \\
\left. + \frac{1}{(1 - 2k)(2k + 3)} a_1 + \frac{10k + 9}{(9 - 4k^2)(2k + 1)(2k + 5)} a_3 \right] b_{n,2k+1}^e \right]. \] (16)

In the above expression,

\[
a_0 = F_0 + \frac{F_2}{2}, \quad a_1 = F_1 + \frac{3}{4} F_3, \quad a_2 = \frac{F_2}{2}, \quad a_3 = \frac{F_3}{4}
\]
D Integral asymptotics by Olver method

We collect here asymptotic expansions for the case of interacting saddle and pole. Most of the material is sourced from Oughstun (2009) which has the original references.

The formulae are presented for the case of \( N \) isolated saddles \( \alpha_{si}, i = 1, \ldots, N \) interacting with a pole \( \alpha_{p1} \). It is also assumed that the steepest descent path from only one of the saddles crosses the pole with varying \( \nu_d \), a real parameter. For the case considered in the text, \( N = 2 \).

\[
I_{sp}(t; \nu_d) \approx \sum_{i=1}^{2} q(\alpha_{si}) \left( -\frac{2\pi}{tp'(\alpha_{si})} \right)^{1/2} e^{tp(\alpha_{si})}
\]

\[
+ \gamma_1 \left[ \pm i\pi erfc(\mp i\Delta_1 \sqrt{t}) e^{tp(\alpha_{p1})} + \sqrt{\frac{\pi}{t}} \frac{e^{tp(\alpha_{si})}}{\Delta_1} \right],
\]

(17)

where

\[
\Delta_1 = |p(\alpha_{s1}) - p(\alpha_{p1})|^{1/2},
\]

\[
\gamma_1 = \lim_{\alpha \to \alpha_{p1}} [(\alpha - \alpha_{p1})] q(\alpha),
\]

and

\[
erfc(z) = 1 - \frac{2}{\sqrt{\pi}} \int_{0}^{z} e^{-\xi^2} \, d\xi.
\]

A crucial point in the computation is that the correct branch be chosen for \( \Delta_1 \). For this, we make use of the following -

\[
\sqrt{\Delta_1^2} = \begin{cases} 
\Delta_1 & \text{if } -\pi < \theta_1 \leq \frac{\pi}{2} \\
-\Delta_1 & \text{if } -\frac{\pi}{2} < \theta_1 \leq \pi \text{ or } \frac{\pi}{2} < \theta_1 < \pi,
\end{cases}
\]

where \( \theta_1 = arg(\Delta_1) \), in turn is computed by eq.(10.91) of Oughstun (2009) as

\[
\lim_{\alpha_{p1} \to \alpha_{s1}} \theta_1 = \theta_c + arg\{[-p''(\alpha_{s1})]^{1/2}\} + 2n\pi
\]

with \( \theta_c \) being the angle made by the vector from \( \alpha_{s1} \) to \( \alpha_{p1} \) and \( n \) an integer chosen such that \( \theta_1 \) lies in the principal range \((-\pi, \pi)\). The upper (lower) signs are to be used when \( Im(\Delta_1) > (\leq) 0 \). When \( Im(\Delta_1) = 0 \) but \( \Delta_1 \neq 0 \), we have

\[
I_{sp}(t; \theta) \approx \sum_{i=1}^{2} q(\alpha_{si}) \left( -\frac{2\pi}{tp'(\alpha_{si})} \right)^{1/2} e^{tp(\alpha_{si})}
\]

\[
+ \gamma_1 \left[ \mp i\pi erfc(-i\Delta_1 \sqrt{t}) e^{tp(\alpha_{p1})} + \sqrt{\frac{\pi}{t}} \frac{e^{tp(\alpha_{si})}}{\Delta_1} \right],
\]

(18)
For $\Delta_1 = 0$, we have

$$I_{sp}(t; v_d) \approx \sum_{i=1}^{2} q(\alpha_{si}) \left( - \frac{2\pi}{tp'(\alpha_{si})} \right)^{1/2} e^{tp(\alpha_{si})} - \gamma_{1} \left( - \frac{2\pi}{tp'(\alpha_{s1})} \right)^{1/2} e^{tp(\alpha_{s1})} \left[ \frac{1}{\alpha_{s1} - \alpha_{p1}} + \frac{p''(\alpha_{s1})}{6p'(\alpha_{s1})} \right].$$

(19)

The total integral $I$ is then given as below. The assumption is that the pole is fixed and the saddle moves upward with increasing $v_d$. There are two cases to consider. If, for small $v_d$, the LOI is in between the SDP and the pole, $I$ is given by

$$I = \begin{cases} 
I_{sp} & \text{if } v_d < (v_d)_p \\
I_{sp} + i\pi\gamma_{1} e^{tp(\alpha_{p1})} & \text{if } v_d = (v_d)_p \\
I_{sp} + 2i\pi\gamma_{1} e^{tp(\alpha_{p1})} & \text{if } v_d > (v_d)_p.
\end{cases}$$

In this case, the pole is encircled in an anticlockwise manner when the LOI is deformed into the SDP. $(v_d)_p$ is the value of $v_d$ at which the pole and saddle collide.

On the other hand, if, for small $v_d$, the pole is between the LOI and the SDP, then we have

$$I = \begin{cases} 
I_{sp} & \text{if } v_d > v_{d_p} \\
I_{sp} - i\pi\gamma_{1} e^{tp(\alpha_{p1})} & \text{if } v_d = v_{d_p} \\
I_{sp} - 2i\pi\gamma_{1} e^{tp(\alpha_{p1})} & \text{if } v_d < (v_{d_p}).
\end{cases}$$

In this case, the pole is encircled in a clockwise manner when the LOI is deformed into the SDP.

We now consider the numerical implementation of formulae 17-19. If $\Delta_1$ is bounded away from zero, the implementation is straightforward. This is the most likely scenario when an off-axis pole and an off-axis saddle interact; though $Im(\Delta_1)$ passes through zero when the SDP passes through the pole, $Re(\Delta_1)$ and hence $\Delta_1$ itself remain non-zero, in general. However, when $\alpha_{s1}$ and $\alpha_{p1}$ lie on the imaginary axis, it is inevitable that $\Delta_1 = 0$ for some $v_d$; this happens when the pole and saddle collide. For other $v_d$, $\Delta_1$ is pure imaginary. Formulae 17 and 19 are to be used in this case. From 17 it appears that $I_{sp} \to \infty$ as $\Delta_1 \to 0$. However, the first term also tends to infinity and indeed the resultant cancellations result in 19 which is valid for $\Delta = 0$. For $\Delta \neq 0$ but small, large errors can result if 17 is used as is. For numerical purposes, we adopt the following procedure -
Choose an \( \epsilon > 0 \). For \( 0 < \text{Im}(\Delta_1) < \epsilon \), we write

\[
I_{sp}(t; v_d) \approx \sum_{i=1}^{2} q(\alpha_{si}) \left( -\frac{2\pi}{tp(\alpha_{si})} \right)^{1/2} e^{tp(\alpha_{si})} \nonumber \\
- \gamma_1 \left( -\frac{2\pi}{tp(\alpha_{si})} \right)^{1/2} e^{tp(\alpha_{si})} \left[ \frac{1}{\alpha_{s1} - \alpha_{p1}} + \frac{p''(\alpha_{s1})}{6p''(\alpha_{s1})} \right] \nonumber \\
- \gamma_1 i\pi(1 - e^{-i\Delta_1\sqrt{t}}) e^{tp(\alpha_{p1})}, \tag{20}
\]

and for \( -\epsilon < \text{Im}(\Delta_1) < 0 \), we write

\[
I_{sp}(t; v_d) \approx \sum_{i=1}^{2} q(\alpha_{si}) \left( -\frac{2\pi}{tp(\alpha_{si})} \right)^{1/2} e^{tp(\alpha_{si})} \nonumber \\
- \gamma_1 \left( -\frac{2\pi}{tp(\alpha_{si})} \right)^{1/2} e^{tp(\alpha_{si})} \left[ \frac{1}{\alpha_{s1} - \alpha_{p1}} + \frac{p''(\alpha_{s1})}{6p''(\alpha_{s1})} \right] \nonumber \\
- \gamma_1 i\pi e^{-i\Delta_1\sqrt{t}} e^{tp(\alpha_{p1})}. \tag{21}
\]

Typically, we take \( \epsilon = 0.01 \). We now check that \( 19, 21 \) result in \( I \) being a continuous function of \( v_d \), even though the constituents of \( I \) viz. \( I_{sp} \) and \( I_p \) are discontinuous. To avoid clutter, we denote the sum in \( 20 \) as \( S \) and the next term as \( T \). Note that \( T \) is the limit of the second term in the square bracket of \( 17 \) (which we denote by \( U \)) as the pole approaches the saddle. Assume the pole does not contribute when \( \text{Im}(\Delta_1) > 0 \). Then, we have,

\[
I = \begin{cases} 
S + U + i\pi\gamma_1 e^{-i\Delta_1\sqrt{t}} & \text{if } \text{Im}(\Delta_1) > \epsilon \\
S + T + \gamma_1 i\pi e^{-i\Delta_1\sqrt{t}} & \text{if } 0 < \text{Im}(\Delta_1) < \epsilon \\
S + T - \gamma_1 i\pi e^{i\Delta_1\sqrt{t}} + 2\pi\gamma_1 e^{tp(\alpha_{p1})} & \text{if } -\epsilon < \text{Im}(\Delta_1) < 0 \\
S + U - \gamma_1 i\pi e^{i\Delta_1\sqrt{t}} + 2\pi\gamma_1 e^{tp(\alpha_{p1})} & \text{if } \text{Im}(\Delta_1) < -\epsilon.
\end{cases} \tag{22}
\]

The first line is just \( 17 \) the second, \( 20 \) plus the pole contribution, the third, \( 21 \) plus the pole contribution and the fourth again \( 17 \) with the appropriate sign. It can be checked that \( I \) is continuous at a) \( \text{Im}(\Delta_1) = \epsilon \) as \( T \) approaches \( U \) as \( \text{Im}(\Delta_1) \rightarrow \epsilon \), b) \( \Delta_1 = 0 \), as the discontinuity in \( I_{sp} \) exactly cancels the discontinuity in the pole contribution and finally at c) \( \text{Im}(\Delta_1) = -\epsilon \) for the same reason as in (a).

Similar expressions can be obtained for the second case, when the pole is encircled in a clockwise manner; the signs of the residue and error function terms will be different.

For validation, we consider the integral \( I \) defined by

\[
I = \int_{-\infty}^{\infty} \frac{\alpha(1 - \frac{\gamma}{\omega_0} \alpha^2) \text{Exp} [i(\alpha x - \omega t)]}{\omega - \omega_0} \, d\alpha.
\]

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where $\omega$ is the first Orr-Sommerfeld mode. We validate the analytic computation by integrating $I$ numerically. For the numerical integration, although the line of integration is the real line, it is more convenient, due to the slow decay rate of the exponent along the real axis, to integrate along a path such as shown in figure 18 where we have also shown the contour plot of $\omega_i - \alpha_i \frac{\pi}{2}$ to convey an impression of how fast the integrand decays. As can be seen, we have chosen the numerical integration path to follow the steepest descent path beyond $\|\alpha_r\| = 2$ (shown as dotted lines in figure 18). The contour levels at two points on the descent paths are indicated in the figure. The descent paths reach a level of $\approx -0.275$ within a rectangular domain $[-4, 4] \times [-4, 4]$. Hence, for $t = 40$, for example, the magnitude of the integrand along the descent path decays to nearly zero within this domain. This assures that the length of the integration path is sufficient to obtain a converged value. The grid independence of the numerical integral has been checked by doubling the number of integration points along the integration path. For $t < 40$, numerical integration requires much longer descent paths. We show the real and imaginary parts of $I$, as a function of $v_d$ in figures 6 for $t = 100$: the drive frequency $\omega_0 = 0.3$.

It can be discerned from the analytic computation that the initial transient is almost entirely due to the saddle family C (green line in figure 3a) while the final oscillatory part is due to the pole contribution, with the wavelength and damping related to the real and imaginary parts of the pole coordinates. Though the central saddle (red line in figure 3a) is at a similar height as the right saddle for a range of $v_d$, (figure 3b) its contribution still turns out to be negligible because of the factor $\alpha$ in the integrand. Figures 19 (a) and (b)
Figure 19: a) Real and b) Imaginary parts of $I_{1}^{OSE}$. Solid and dashed lines represent numerical and analytic computations. $\omega_{0} = 0.3, t = 100$.

| $n$ | 0   | 1      | 2 | 3 | 4   |
|-----|-----|--------|---|---|-----|
| $a_n$ | -0.0065 | -0.0056 + 0.0037 i | -0.0016 + 0.0066 i | 0.0128 + 0.0092 i | 0.0657 + 0.0416 i |

| $n$ | 5 | 6 | 7 | 8 | 9 |
|-----|---|---|---|---|---|
| $a_n$ | 0.0312 + 0.257 i | -0.4041 + 0.0926 i | -0.0586 - 0.2329 i | 0.0344 - 0.048 i | 0.0079 - 0.0127 i |

| $n$ | 10 | 11 |
|-----|----|----|
| $a_n$ | 0.0009 - 0.0057 i | -0.0008 - 0.0028 i |

Table 6: Fourier coefficients of the wavepacket at $t = 108$ for $\omega_{0} = 0.45$. $N = 11$.

show the comparison of the numerical (solid lines) and analytical (dashed lines) values of $\text{Real}(I)$ and $\text{Im}(I)$ respectively. The agreement between the numerical integration and the analytical values is very good. Validations were also done for many values of $\omega_{0}$, though not shown in this paper.

E Wavepacket reconstruction

It may be noted that only one wavepacket emerges in the solution of the IBVP, while the Floquet framework necessarily implies a periodic system of wavepackets. For the present analysis to have any relevance to the original problem, it is important to know what effect the separation between the wavepackets has on the secondary growth rates. For this purpose, we construct a periodic wavepacket system based on the IBVP wavepacket, with zero padding on either side so as to control the separations of the packets. Assuming the length of the padded wavepacket to be the wavelength $\lambda_{b}$, the Fourier coefficients of the periodic wavepacket system are obtained. With $I$ denoting the index of the maximum of these fourier coefficients, $I\alpha$ gives the central wavenumber $\alpha_{c}$. The smaller the $\alpha$, the larger the number of fourier modes $N$. Too small an
Table 7: Fourier coefficients of the wavepacket at \( t = 108 \) for \( \omega_0 = 0.45 \).\( N = 17. \)

| \( n \) | \( \alpha_n \) |
|-------|-------------|
| 0     | -0.0033     |
|       | 0.0031 - 0.0009 i |
| 1     | -0.0028 + 0.0018 i |
| 2     | 0.0021 - 0.0026 i |
| 3     | -0.0008 + 0.0034 i |
| 4     | -0.0015 - 0.0039 i |
| 5     | 0.0064 + 0.0046 i |
| 6     | -0.016 - 0.0076 i |
| 7     | 0.0328 + 0.0208 i |
| 8     | -0.0462 - 0.0604 i |
| 9     | 0.0156 + 0.1285 i |
| 10    | 0.0982 - 0.1554 i |
| 11    | -0.2021 + 0.0463 i |
| 12    | 0.1509 + 0.0991 i |
| 13    | -0.0129 - 0.1164 i |
| 14    | 0.0197 + 0.0597 i |
| 15    | 0.0172 - 0.0241 i |
| 16    | -0.0088 + 0.0106 i |
| 17    |                     |

\( \alpha \) does lead to an increased separation of the wavepackets but will also entail large \( N \) and consequently large computational times. In the procedure adopted here, we first find \( \alpha_c \) accurately by choosing a very small base wavenumber \( \alpha \). The base wavenumber is then redefined to be an integral fraction of \( \alpha_c \) and the fourier coefficients are recomputed. For the wavepacket at \( t = 108 \), \( \alpha_c \) is found to be \( \approx 1.0802 \) and we have calculated using \( \alpha = \alpha_c/6, \alpha_c/12 \) and \( \alpha_c/16 \) (0.18004, 0.09002 and 0.06751 respectively). The fourier coefficients for \( \alpha_c/6 \) \( \alpha_c/12 \) are tabulated in Tables 6 and 7. It may be noted that the seventh and the thirteenth coefficient are the largest in these two representations respectively.

Figures 20 and 21 illustrate the periodic wavepacket systems constructed with these \( \alpha \), the effect of which on the separation length can be clearly seen. The larger the separation between the wavepackets, the closer they may be assumed to approximate the actual situation of a single wavepacket. In these two figures, the solid line denotes the localized wavepacket from the IBVP solution; the dashed line, indistinguishable from the solid one, is the periodic reconstruction of the wavepacket. The magnitudes of the fourier coefficients corresponding to \( \alpha = \alpha_c/6 \) and \( \alpha_c/12 \) are shown in figure 22.
Figure 21: Periodic wavepacket system for $Re = 5000, \omega_0 = 0.45$ at $t = 108$. $\alpha = \alpha_c/12$.

Figure 22: Fourier coefficients of the periodic wavepacket systems corresponding to $\alpha = \alpha_c/6$, and $\alpha_c/12$ for $Re = 5000; \omega_0 = 0.45$. 

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F Floquet equations

The secondary base state velocity is given by

\[ \vec{v}_2(\xi, y) = U(y) + A \vec{v}(\xi, y); \quad \xi = x - ct \]  (23)

where \( \vec{v}(\xi, y) \) is the normalized TS wave or the wavepacket from the IBVP solution. The secondary disturbances \( \vec{v}_3(\xi, y, z, t) \) are defined as perturbations in the total velocity:

\[ \vec{v}_T(\xi, y, z, t) = \vec{v}_2(\xi, y) + \epsilon \vec{v}_3(\xi, y, z, t) \]  (24)

Substituting (F 2) into the N-S equations, linearising in \( \epsilon \), eliminating the pressure and the spanwise velocity component \( w_3 \) by taking curl and using continuity respectively, a coupled PDE system for the streamwise and normal velocity components \( u_3 \) and \( v_3 \) is obtained as (Herbert et al 1987) -

\[
\begin{align*}
\frac{1}{Re} \nabla^2 - (U - c) \frac{\partial}{\partial x} - \frac{\partial}{\partial t} + \frac{\partial^2 v_3}{\partial z^2} + \zeta \frac{\partial^2 u_3}{\partial y \partial z} \\
+ A \left[ -\frac{\partial^2 \psi_1}{\partial x \partial y} \frac{\partial}{\partial x} + \frac{\partial^2 \psi_1}{\partial x^2} \frac{\partial}{\partial y^2} - \frac{\partial^2 \psi_1}{\partial x \partial y} \frac{\partial^2 v_3}{\partial z^2} \right] = 0,
\end{align*}
\]  (25)

\[
\begin{align*}
\frac{1}{Re} \nabla^2 - (U - c) \frac{\partial}{\partial x} - \frac{\partial}{\partial t} \nabla^2 v_3 - \frac{d\zeta_0}{dy} \frac{\partial v_3}{\partial x} \\
+ A \left[ -\frac{\partial^2 \psi_1}{\partial x \partial y} \frac{\partial}{\partial x} + \frac{\partial^2 \psi_1}{\partial x^2} \frac{\partial}{\partial y^2} \right] \nabla^2 v_3 + \frac{\partial^2 \psi_1}{\partial x^2} \left( \frac{\partial \xi_3}{\partial y} + \frac{\partial \eta_3}{\partial y} \right) \\
- \frac{\partial^2 \psi_1}{\partial x \partial y} \left( \frac{\partial \xi_3}{\partial z} + \frac{\partial \eta_3}{\partial z} \right) - \frac{\partial^2 \psi_1}{\partial x^2} \left( \frac{\partial v_3}{\partial y} + \frac{\partial v_3}{\partial y} \right) \\
- \frac{\partial \xi_3}{\partial y} \frac{\partial v_3}{\partial x} - \left( u_3 \frac{\partial}{\partial x} + v_3 \frac{\partial}{\partial y} \right) \frac{\partial \xi_3}{\partial x} \right] = 0.
\]  (26)

The boundary conditions are \( u_3 = v_3 = \partial v_3/\partial y = 0 \) at \( y = \pm 1 \). Standard Floquet theory (for e.g. Nayfeh & Mook 1995) indicates that the disturbance equations, with periodic coefficients of period \( \lambda = 2\pi/\alpha \) admit solutions of the form

\[
\begin{bmatrix} u_3(\xi, y, z, t) \\ v_3(\xi, y, z, t) \end{bmatrix} = e^{\gamma \xi} e^{\sigma t} e^{i\beta z} \sum_{m=-\infty}^{\infty} \begin{bmatrix} u_m(y) \\ v_m(y) \end{bmatrix} e^{ima\xi}.
\]  (27)
Substituting (F 5) into (F 3) and (F 4), and collecting coefficients of $e^{i\alpha \xi}$, we have, for each $m$, the following sets of equations:

\[
\frac{\left(\delta_m^2 - \beta^2\right)^2}{Re} + (U - c)\delta_m(\beta^2 - \delta_m^2) - A\delta_m(\delta_m^2 - \beta^2)(a_0\phi' + \bar{u}_0\phi') u_m + \frac{\delta_m^2 - \beta^2}{Re} \frac{d^2 u_m}{dy^2} \\
+ \beta^2 \left[U' + A(a_0\phi'' + b_0\phi'')\right] v_m + \left[\delta_m(\delta_m^2 - \beta^2) - (U - c)\delta_m^2 - A\delta_m^2 (a_0\phi' + \bar{u}_0\phi')\right] \frac{dv_m}{dy} \\
+ \frac{\delta_m}{Re} \frac{d^3 v_m}{dy^3} - A\phi' \sum_{n=1}^{N} a_n(\delta_{m-n}^2 - \beta^2)(\delta_{m-n} + i\alpha n)u_{m-n} \\
+ A\phi \alpha \sum_{n=1}^{N} \left[i(\delta_{m-n}^2 - \beta^2) - \alpha n^2 \delta_{m-n}\right] a_n \frac{du_{m-n}}{dy} \\
- A\phi \sum_{n=1}^{N} \bar{u}_n(\delta_{m+n}^2 - \beta^2)(\delta_{m+n} + i\alpha n)u_{m+n} \\
- A\phi \alpha \sum_{n=1}^{N} \left[i(\delta_{m+n}^2 - \beta^2) + \alpha n^2 \delta_{m+n}\right] \bar{u}_n \frac{du_{m+n}}{dy} + A\beta^2 \phi' \sum_{n=1}^{N} a_n v_{m-n} \\
- A\phi \sum_{n=1}^{N} \delta_{m-n}(\delta_{m-n} + i\alpha n)a_n \frac{dv_{m-n}}{dy} + A\alpha \phi \sum_{n=1}^{N} (i\delta_{m-n} - \alpha n)a_n \frac{d^2 v_{m-n}}{dy^2} \\
+ A\beta^2 \bar{\phi}' \sum_{n=1}^{N} \bar{u}_n v_{m+n} - A\phi' \sum_{n=1}^{N} \delta_{m+n}(\delta_{m+n} - i\alpha n) \bar{u}_n \frac{dv_{m+n}}{dy} \\
- A\alpha \phi \sum_{n=1}^{N} (i\delta_{m+n} + \alpha n)m \bar{u}_n \frac{d^2 v_{m+n}}{dy^2} = \sigma \left[(\delta_m^2 - \beta^2)u_m + \delta_m \frac{dv_m}{dy}\right], \quad (28)
\]

\[
\frac{\left(\delta_m^2 - \beta^2\right)^2}{Re} - (U - c)\delta_m(\delta_m^2 - \beta^2) - 2\delta_m \\
+ A\delta_m \left(a_0\phi'' + \bar{u}_0\phi'' - (\delta_m^2 - \beta^2) \ast (a_0\phi' + \bar{u}_0\phi')\right) v_m \\
+ \left[\frac{2}{Re}(\delta_m^2 - \beta^2) - (U - c)\delta_m\right] \frac{d^2 v_m}{dy^2} + \frac{1}{Re} \frac{d^4 v_m}{dy^4} \\
+ A \sum_{n=1}^{N} \left[(n^2 \alpha^2 \phi - \phi')(\alpha n - 2i\delta_{m-n}) - \alpha n(\delta_{m-n}^2 - \beta^2)\right] n a_n u_{m-n} \\
+ 2i\alpha A\phi \sum_{n=1}^{N} n a_n \delta_{m-n} \frac{du_{m-n}}{dy} + A\alpha^2 \phi \sum_{n=1}^{N} n^2 a_n \frac{d^2 u_{m-n}}{dy^2} \\
+ A \sum_{n=1}^{N} \left[(n^2 \alpha^2 \phi - \phi')(\alpha n + 2i\delta_{m+n}) - \alpha n(\delta_{m+n}^2 - \beta^2)\right] n a_n u_{m+n} 
\]
\[-2i\alpha A\phi \sum_{n=1}^{N} n a_n \delta_{m+n} \frac{du_{m+n}}{dy} + A\alpha^2 \phi \sum_{n=1}^{N} n^2 a_n \frac{d^2 u_{m+n}}{dy^2} - A \]

\[\sum_{n=1}^{N} \left[ \phi' \left( \delta_{m-n} (\delta_{m-n}^2 - \beta^2) + i\alpha n (\delta_{m-n}^2 + \beta^2) \right) + (n^2 \alpha^2 \phi' - \phi''') (i\alpha n + \delta_{m-n}) \right] v_{m-n} \]

\[+ A \sum_{n=1}^{N} \left[ i\alpha n (\delta_{m-n}^2 - \beta^2) - 2\alpha^2 n^2 \phi \delta_{m-n} - i\alpha n (n^2 \alpha^2 \phi - \phi''') \right] a_n \frac{dv_{m-n}}{dy} \]

\[+ A\phi' \sum_{n=1}^{N} (i\alpha n - \delta_{m+n}) a_n \frac{d^2 v_{m-n}}{dy^2} + A\alpha \phi \sum_{n=1}^{N} n a_n \frac{d^3 v_{m-n}}{dy^3} - A \]

\[\sum_{n=1}^{N} \left[ \phi (\delta_{m+n} (\delta_{m+n}^2 - \beta^2) + i\alpha n (\delta_{m+n}^2 + \beta^2) \right) + (n^2 \alpha^2 \phi - \phi''') (-i\alpha n + \delta_{m+n}) \right] v_{m+n} \]

\[- A \sum_{n=1}^{N} \left[ i\alpha n (\delta_{m+n}^2 - \beta^2) - 2\alpha^2 n^2 \phi \delta_{m+n} - i\alpha n (n^2 \alpha^2 \phi - \phi''') \right] a_n \frac{dv_{m+n}}{dy} \]

\[- A\phi' \sum_{n=1}^{N} (i\alpha n + \delta_{m+n}) a_n \frac{d^2 v_{m+n}}{dy^2} + A\alpha \phi \sum_{n=1}^{N} n a_n \frac{d^3 v_{m+n}}{dy^3} = \sigma \left( \delta_{m}^2 - \beta^2 \right) v_{m} + \frac{d^2 v_{m}}{dy^2} \right].

(29)

In the above, \( \delta_m = \gamma + i\alpha \). As is evident, the equation for the \( m \)th Fourier coefficient involves Fourier coefficients from the \((m - N)\)th to the \((m + N)\)th levels. Even though \( m \) ranges over the real line, for numerical purposes, we truncate to a maximum of \( m = M \) where \( M \geq N \). Chebyshev collocation at \( K + 1 \) points in \( y \) renders this ODE system into a matrix eigenvalue problem

\[SV = \sigma TV\]

where \( V \) is the \( 2(K + 1)(2M + 1) \) dimensional vector with components

\[\left( u_{-M,0}, \ldots, u_{-M,K}, v_{-M,0}, \ldots, v_{-M,K}, \ldots, u_{M,0}, \ldots, u_{M,K}, v_{M,0}, \ldots, v_{M,K} \right)\]

and \( S \) and \( T \) are square matrices of the same dimension. \( u_{i,j} \) is the value of the Fourier coefficient \( u_i \) at the \( j \)th collocation point. \( S \) and \( T \) are both banded matrices, the former with varying bandwidth and the latter with a fixed width,
as shown below -

\[
S = \begin{pmatrix}
s_{1,1} & \cdots & \cdots & s_{1,j_1} & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots \\
s_{i_1,1} & \cdots & \cdots & \cdots & s_{i_1,i_2} \\
\vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & s_{i_2,j_2} & \cdots & s_{i_2,i_2}
\end{pmatrix}, \quad (30)
\]

\[
T = \begin{pmatrix}
t_{1,1} & \cdots & t_{1,j_3} & 0 & 0 & 0 \\
0 & \ddots & \ddots & 0 & 0 & \vdots \\
\vdots & \ddots & t_{i_1,i_1} & \cdots & t_{i_1,j_4} & \vdots \\
0 & 0 & \ddots & \ddots & \vdots & 0 \\
0 & 0 & 0 & t_{i_2,j_5} & \cdots & t_{i_2,i_2}
\end{pmatrix}. \quad (31)
\]

In the above,

\[i_1 = 2M(K + 1) + 1, \quad i_2 = 2(2M + 1)(K + 1),\]
\[j_1 = 2(N + 1)(K + 1), \quad j_2 = 2(2M - N)(K + 1) + 1,\]
\[j_3 = 2K + 2, \quad j_4 = 2(M + 1)(K + 1), \quad j_5 = 4M(K + 1) + 1.\]