Hamiltonian Structures of Multi-component Constrained KP Hierarchy

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ABSTRACT We consider the Hamiltonian theory for the multi-component KP hierarchy. We show that the second Hamiltonian structures constructed by Sidorenko and Strampp [J. Math. Phys. 34, 1429(1993)] are not Hamiltonian. A candidate for the second Hamiltonian Structures is proposed and is proved to lead to hereditary operators.
1 INTRODUCTION

During last few years, the constrained KP hierarchy is studied intensively\textsuperscript{1−3,5}. This hierarchy is the result of generalizing Cao’s\textsuperscript{6} nonlinearization to the 2+1 dimensional case. Very interestingly, as the famous Gelfand-Dikii hierarchy\textsuperscript{7}, the constrained KP hierarchy is both mathematically and physically important. On the one hand, it contains physically applicable models, such as Yajima-Oikawa\textsuperscript{8} model and Melnikov\textsuperscript{9} model. On the other hand, the constrained KP hierarchy is Bi-Hamiltonian\textsuperscript{5}, has Darboux transformation\textsuperscript{10}, can be modified\textsuperscript{11} and is relevant to the theory of the W algebra\textsuperscript{12}. Very recently, the constrained KP is shown to be just a special case of a more general restriction of the KP hierarchy\textsuperscript{13}.

Sidorenko and Strampp\textsuperscript{4} introduced multi-component KP hierarchy, which is a straightforward generalization of the scalar case. This is the hierarchy associated with the following Lax operator

\[
L_n = \partial^n + u_{n-2} \partial^{n-2} + \ldots + u_0 + \sum_{i=1}^{m} q_i \partial^{-1} r_i,
\]

the corresponding flows may be constructed by means of Fractional Power Method\textsuperscript{7}. For \( n = 1 \), one has multi-component AKNS, which includes the important coupled nonlinear Schrödinger equation\textsuperscript{14} as a special case. For the cases \( n = 2 \) and \( n = 3 \), one has the multi-component Yajima-Oikawa hierarchy and Melnikov hierarchy respectively. Sidorenko and Strampp\textsuperscript{4} further constructed recursion operators for the cases \( n = 2 \) and \( n = 3 \) by means of variational calculus developed in the Ref.\textsuperscript{15}. They claimed that their recursion operators have implictic-symplectic factorizations in the sense of Fuchsteiner and Fokas\textsuperscript{17}. That is to say, they claimed that the Bi-Hamiltonian structures are found for the multi-component Yajima-Oikawa hierarchy and Melnikov hierarchy. Unfortunately, they did not prove their statement either directly or indirectly. The partial reason is that their candidates for the Hamiltonian structures are complicated nonlocal matrix operators and direct proof would be too tedious to do by ha

The aim of the paper is to show that Sidorenko and Strampp’s claim is not correct. We will prove that their second Hamiltonian structures are not qualified as Hamiltonian at all. Furthermore, we will present alternative candidates for the second Hamiltonian structures and prove that it leads to hereditary operators. For simplicity, we concentrate on the simplest and non trivial case: two-component case. The generalization to the multi-component case will be commented in the due course.

The paper is arranged as follows. The next section is on the two-component AKNS systems. Sidorenko-Strampp type operator is presented and shown to be not hereditary. A candidate for the second Hamiltonian operators is constructed. Also, we present a hereditary operator for this hierarchy. We do the same thing in the section three for the two-component Yajima-Oikawa hierarchy. Section four contains some comments on generalizations of the results of section two and section three and gives some outlines for the further study.

2 TWO-COMPONENT AKNS HIERARCHY

We consider the two-component AKNS system next. This hierarchy is known for long time. In fact, the so important coupled nonlinear Schrödinger equation\textsuperscript{14} is a reduction of it. However, the explicit form of recursion operator is not written down to the best of my knowledge although it might be known to the specialists. For the motivation of the next section, we present it here.
We first give Sidorenko-Strampp type operator and show it is not hereditary. The Lax operator is

\[ L_1 = \partial - q_1 \partial^{-1} r_1 - q_2 \partial^{-1} r_2, \]  

(2.1)

the flows are

\[ L_{1, k} = [(L_1)^k]_+, L_1, \]  

(2.2)

where subscript + means the projection to the differential part.

Following the idea of Sidorenko and Strampp\(^4\), we easily see that the systems (2.2) have the following presentation

\[ q_{t_k} = B_{ss} \frac{\delta H_{k+1}}{\delta q} = B_{ss1} \frac{\delta H_k}{\delta q}, \]  

(2.3)

where \( q = (q_1, q_2, r_1, r_2)^T \) and

\[
B_{ss} = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{bmatrix},
\]

(2.4)

\[
B_{ss1} = \begin{bmatrix}
2q_1 \partial^{-1} q_1 & 2q_1 \partial^{-1} q_2 & \partial - 2q_1 \partial^{-1} r_1 & -2q_1 \partial^{-1} r_2 \\
2q_2 \partial^{-1} q_1 & 2q_2 \partial^{-1} q_2 & -2q_2 \partial^{-1} r_1 & \partial - 2q_2 \partial^{-1} r_2 \\
\partial - 2r_1 \partial^{-1} q_1 & -2r_1 \partial^{-1} q_2 & 2r_1 \partial^{-1} r_1 & 2r_1 \partial^{-1} r_2 \\
-2r_2 \partial^{-1} q_1 & \partial - 2r_2 \partial^{-1} q_2 & 2r_2 \partial^{-1} r_1 & 2r_2 \partial^{-1} r_2
\end{bmatrix},
\]

(2.5)

and Hamiltonian functionals \( H_k \) may be calculated from the formula:

\[ H_k = \frac{1}{k} \text{Res}(L_1)^k. \]  

(2.6)

The Hamiltonian nature of \( B_{ss} \) is self-evident. However, it is not clear that if \( B_{ss1} \) is or not a Hamiltonian operator although it is skew symmetric. Next, we prove that \( B_{ss1} \) is not Hamiltonian. To show this, let us first simplify \( B_{ss1} \) via coordinate transformations. Motivated by the situation in the scalar case\(^12\), we introduce the following coordinates

\[ S_1 = q_1 r_1, \quad S_2 = q_2 r_2, \quad T_1 = -\frac{r_1 x}{r_1}, \quad T_2 = -\frac{r_2 x}{r_2}, \]  

(2.7)

then, it is ready to see that \( B_{ss} \) and \( B_{ss1} \) are transformed to

\[
\hat{B}_{ss0} = \begin{bmatrix}
0 & 0 & \partial & 0 \\
0 & 0 & 0 & \partial \\
\partial & 0 & 0 & 0 \\
0 & \partial & 0 & 0
\end{bmatrix},
\]

(2.8)

\[
\hat{B}_{ss1} = \begin{bmatrix}
S_1 \partial + \partial S_1 & 0 & \partial^2 + T_1 \partial & 0 \\
0 & S_2 \partial + \partial S_2 & 0 & \partial^2 + T_2 \partial \\
-\partial^2 + \partial T_1 & 0 & -2\partial & -2\partial \\
0 & -\partial^2 + \partial T_2 & -2\partial & -2\partial
\end{bmatrix},
\]

(2.9)

thus, the transformation (2.7) localizes \( B_{ss0} \) and \( B_{ss1} \). With (2.8-9) in hand, we have a recursion operator
Now, if $B_{ss_1}$ would be a Hamiltonian operator, one would have a hereditary operator in the sense of Fuchssteiner\cite{fuchssteiner}. That means the following identity must be hold

$$R'_{ss}[R_{ss}(f)]g - R'_{ss}[R_{ss}(g)]f = R_{ss}(R'_{ss}[f]g - R'_{ss}[g]f),$$

for arbitrary vector function $f$ and $g$. Where $'$ denotes Gateaux derivative.

However, a long calculation shows that it is not the case here. In fact, letting $f = (f_1, f_2, f_3, f_4)^T$ and $g = (g_1, g_2, g_3, g_4)^T$, we have

$$R'_{ss}[R_{ss}(f)]g - R'_{ss}[R_{ss}(g)]f - R_{ss}(R'_{ss}[f]g - R'_{ss}[g]f) = 2(-f_2g_1 + g_2f_1),$$

which is not identical zero. This means that $R_{ss}$ is not hereditary. So we conclude that $B_{ss_1}$ is not Hamiltonian.

In the remain part of the section, we construct a hereditary operator for the hierarchy(2.2). To do this, let us rewrite the corresponding spectral problem as matrix form

$$\begin{bmatrix}
\phi \\
\phi_1 \\
\phi_2
\end{bmatrix} =
\begin{bmatrix}
\lambda & q_1 & q_2 \\
r_1 & 0 & 0 \\
r_2 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\phi \\
\phi_1 \\
\phi_2
\end{bmatrix} \equiv U\Phi,
$$

as usual, we adjoin (2.13) with time evolution of wave function $\Phi$: $\Phi_t = V\Phi$. Then, calculating zero-curvature equation $U_t - V_x + [U, V] = 0$ leads us to

$$q_{k} = B_0 \frac{\delta H_{k+1}}{\delta q} = B_1 \frac{\delta H_{k}}{\delta q},$$

where

$$B_0 = B_{ss_0},$$

$$B_1 =
\begin{bmatrix}
2q_1\partial^{-1}q_1 & q_1\partial^{-1}q_2 + q_2\partial^{-1}q_1 & R_1 & -q_1\partial^{-1}r_2 \\
q_1\partial^{-1}q_2 + q_2\partial^{-1}q_1 & 2q_2\partial^{-1}q_2 & -q_2\partial^{-1}r_1 & R_2 \\
-(R_1) & -(R_2) & r_1\partial^{-1}r_2 + r_2\partial^{-1}r_1 & 2r_2\partial^{-1}r_2 \\
-q_1\partial^{-1}r_1 & -q_1\partial^{-1}r_2 & R_1 & -q_1\partial^{-1}r_2 + q_2\partial^{-1}r_1
\end{bmatrix},$$

$$R_1 = \partial - 2q_1\partial^{-1}r_1 - q_2\partial^{-1}r_2, \quad R_2 = \partial - q_1\partial^{-1}r_1 - 2q_2\partial^{-1}r_2.$$
\[ R = \begin{bmatrix}
R_1 & -q_1 \partial^{-1} r_2 & -2q_1 \partial^{-1} q_1 & -q_1 \partial^{-1} q_2 - q_2 \partial^{-1} q_1 \\
-q_2 \partial^{-1} r_1 & R_2 & -q_1 \partial^{-1} q_2 - q_2 \partial^{-1} q_1 & -2q_2 \partial^{-1} q_2 \\
2r_1 \partial^{-1} r_1 & r_1 \partial^{-1} r_2 + r_2 \partial^{-1} r_1 & (R_1)^* & r_1 \partial^{-1} q_2 \\
r_1 \partial^{-1} r_2 + r_2 \partial^{-1} r_1 & 2r_2 \partial^{-1} r_2 & r_2 \partial^{-1} q_1 & (R_2)^*
\end{bmatrix}, \tag{2, 18}
\]

where \( R_1 \) and \( R_2 \) are defined by (2.17).

Then, straightforward but cumbersome calculation shows that \( R \) is indeed hereditary. This also supports our conjecture: \( B_1 \) is Hamiltonian.

### 3 COUPLED YAJIMA-OIKAWA HIERARCHY

As promised in the Introduction, we prove that Sidorenko-Strampp’s operator (see Ref. 4) is not Hamiltonian. After this, we give a candidate for the second Hamiltonian operator.

The Lax operator in this case is
\[ L_2 = \partial^2 - u - q_1 \partial^{-1} r_1 - q_2 \partial^{-1} r_2, \tag{3, 1} \]

two Hamiltonian operators given by Sidorenko and Strampp\(^4\) for the hierarchy associated with (3.1) are
\[
B_{ss_0} = \begin{bmatrix}
-2\partial & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0
\end{bmatrix}, \tag{3, 2}
\]

\[
B_{ss_1} = \begin{bmatrix}
-\frac{1}{3} \partial^3 + u \partial + \partial u & q_1 \partial + \frac{1}{2} \partial q_1 & q_2 \partial + \frac{1}{2} \partial q_2 & r_1 \partial + \frac{1}{2} \partial r_1 & r_2 \partial + \frac{1}{2} \partial r_2 \\
\partial q_1 + \frac{1}{2} q_1 \partial & \frac{3}{4} q_1 \partial^{-1} q_1 & \frac{3}{4} q_1 \partial^{-1} q_2 & J(q_1, r_1) & -\frac{3}{4} q_1 \partial^{-1} r_2 \\
\partial q_2 + \frac{1}{2} q_2 \partial & \frac{3}{4} q_2 \partial^{-1} q_1 & \frac{3}{4} q_2 \partial^{-1} q_2 & J(q_1, r_2) & -\frac{3}{4} q_2 \partial^{-1} r_2 \\
\partial r_1 + \frac{1}{2} r_1 \partial & -(J(q_1, r_1))^* & -\frac{3}{4} r_1 \partial^{-1} q_2 & \frac{3}{4} r_1 \partial^{-1} r_1 & \frac{3}{4} r_1 \partial^{-1} r_2 \\
\partial r_2 + \frac{1}{2} r_2 \partial & -(J(q_2, r_1))^* & \frac{3}{4} r_2 \partial^{-1} q_1 & \frac{3}{4} r_2 \partial^{-1} r_1 & \frac{3}{4} r_2 \partial^{-1} r_2
\end{bmatrix}, \tag{3, 3}
\]

with \( J(q, r) \equiv \partial^2 - u - \frac{3}{2} q \partial^{-1} r \)

We notice that \( B_{ss_1} \) is exactly the one given in the Ref.4 apart from a scaling. As before, we introduce new coordinates
\[ u = u, \quad S_1 = q_1 r_1, \quad S_2 = q_2 r_2, \quad T_1 = -\frac{r_1}{r_2}, \quad T_2 = -\frac{r_2}{r_1}, \tag{3, 4} \]
then, \( B_{ss_1} \) take the following forms
\[
\tilde{B}_{ss_0} = \begin{bmatrix}
-2\partial & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \partial & 0 \\
0 & 0 & 0 & 0 & \partial \\
0 & \partial & 0 & 0 & 0 \\
0 & 0 & \partial & 0 & 0
\end{bmatrix}, \tag{3, 5}
\]

5
The following representation of the hierarchy
may be generalized along two directions: generic multi-component case and higher order constrained KP case. While both generalizations are straightforward, the calculations will be extremely involved. Here, we just comment that the results presented in the Appendix of the Ref.4 are not correct.

We considered the Hamiltonian theory for the multi-component constrained KP hierarchy. It is
proved, in the simplest non trivial case, that the Hamiltonian operators calculated by Sidorenko and Strampp are by no means Hamiltonian. Alternative Hamiltonian structures are proposed and they are shown to lead to hereditary operators.

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4 CONCLUSIONS

We considered the Hamiltonian theory for the multi-component constrained KP hierarchy. It is
proved, in the simplest non trivial case, that the Hamiltonian operators calculated by Sidorenko and Strampp are by no means Hamiltonian. Alternative Hamiltonian structures are proposed and they are shown to lead to hereditary operators.

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There are several points which deserves further consideration. We list them here: (1). Hamiltonian nature of the second structures(2,16) and(3,8). As pointed above, a direct verification is too cumbersome to do by bare hand. This may be completed by a symbolic program, such as Maple or Mathematica. Another way to do this is to use a Miura map, which should simplify the
second structure considerably if it exists. Finding a Miura map is interesting in its own right; (2). Modifications of the hierarchies presented here. In the scalar case, this problem is solved in the Ref.11. The generalization to the multi-component case is important; (3). Associated classical W algebras. Once again, this problem in the scalar case is solved12. Here we just remark that the structure(3.6) is closely related to so-called bosonic analogy of the Knizhnik-Bershadsky superconformal algebra18. Some of these problems are under investigation.

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