Grids of white dwarf evolutionary models with masses from $M = 0.1\, M_\odot$ to $M = 1.2\, M_\odot$

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ABSTRACT

We present detailed evolutionary calculations for carbon - oxygen - and helium - core white dwarf models with masses ranging from $M = 0.1$ to $M = 1.2\, M_\odot$ and for metallicities $Z = 0.001$ and $Z = 0$. The sequences cover a wide range of hydrogen envelopes as well. We have taken finite - temperature effects fully into account by means of a detailed white dwarf evolutionary code, in which updated radiative opacities and equations of state for hydrogen and helium plasmas are considered. The energy transport by convection is treated within the formalism of the full - spectrum turbulence theory, as given by the self - consistent model of Canuto, Goldman and Mazzitelli. Convective mixing, crystallization, hydrogen burning and neutrino energy losses are taken into account as well.

The set of models presented here is very detailed and should be valuable, particularly for the interpretation of the observational data on low - mass white dwarfs recently discovered in numerous binary configurations and also for the general problem of determining the theoretical luminosity function for white dwarfs. In this context, we compare our cooling sequences with the observed white dwarf luminosity function recently improved by Leggett, Ruiz & Bergeron and we obtain an age for the Galactic disc of $\approx 8\, Gyr$. Finally, we applied the results of this paper to derive stellar masses of a sample of low - mass white dwarfs.

Key words: evolution - stars: stars: interiors - luminosity function, mass function - pulsars: general - white dwarfs

1 INTRODUCTION

Numerous observations carried out over recent years have presented strong evidence that low - mass, helium white dwarf stars are the product of the evolution of certain close binary systems. Indeed, low - mass white dwarfs have been detected in binary systems containing, for instance, another white dwarf (Marsh 1995; Marsh, Dhillon & Duck 1995; Marsh & Duck 1996; Moran, Marsh & Bragaglia 1997), millisecond pulsar (Lundgren et al. 1996; see also Backer 1998) or a yellow giant (Landsman et al. 1997). In particular, Moran et al. (1997) found the binary system WD 0957 - 666 (consisting of two low - mass white dwarfs) to have an orbital period of only 1.46 h, which is short enough for the binary to merge within only $2.0 \times 10^8$ yr. Very recently, Edmonds et al. (1998) have reported the presence of a candidate helium white dwarf in the globular cluster NGC 6397.

On theoretical grounds, recent population models of close binaries (Iben, Tutukov & Yungelson 1997) suggest a high probability of discovering helium white dwarfs in close binaries.

Detailed evolutionary models of low - mass white dwarfs may provide valuable information not only on the white dwarf itself but also on the companion object and even on the past evolution of the system (see, for instance, Burderi, King & Phinney 1996 and Hansen & Phinney 1998a). In this regard, the analysis carried out, notably by van Kerkwijk, Bergeron & Kulkarni (1996), is worth mentioning. Indeed, from spectroscopic data inferred from its low - mass white dwarf companion combined with a theoretical mass - radius relation for the white dwarf, these authors found the mass of the pulsar PSR J1012 + 5307 to be between 1.5 and $3.2\, M_\odot$. Needless to say, detailed models of helium white dwarfs, together with further observations, are needed in order to achieve a more precise determination of the pulsar mass and hence to constrain the equation of state at the high densities appropriate for neutron stars. In addition,
an independent determination of the age of many millisecond pulsars can be inferred from the study of the cooling of their helium white dwarf companions, which is valuable for understanding the nature and origin of such systems. Another strong motivation for constructing improved white dwarf evolutionary sequences is the fact that, thanks to the *Hubble Space Telescope*, it has been possible to detect the low - luminosity tail of the white dwarf population in globular clusters. Accordingly, white dwarf evolutionary tracks would provide an independent way of measuring the age and distance of such clusters (see, e.g., Richer et al. 1995; Von Hippel, Gilmore & Jones 1995 and Renzini et al. 1996).

In view of these considerations, we present in this paper new grids of white dwarf evolutionary models for different hydrogen envelopes and stellar masses. The emphasis is placed mainly on low - mass, helium white dwarfs, the detailed study of which has recently begun to be undertaken. As a matter of fact, Althaus & Benvenuto (1997a) and Benvenuto & Althaus (1998) carried out an analysis of the structure and evolution of low - mass white dwarfs based on a updated physical description, such as new opacities and equations of state, and the employment of a new convection model more physically sound than the mixing length theory. In a still more recent study, Hansen & Phinney (1998a) presented evolutionary calculations for these objects as well. However, the evolutionary sequences for their more massive models do not converge to the Hamada & Salpeter (1961) predictions for zero - temperature, pure - helium configurations, thus resulting in models with underestimated surface gravities. This can be seen from figure 16 of Hansen & Phinney (1998a). Note that the surface gravity for their more massive models with a hydrogen envelope of $M_H/M_⊙ = 10^{-6}$ is substantially lower than the Hamada - Salpeter values. Such a discrepancy cannot be attributed to the hydrogen layer, since a thin hydrogen envelope introduces a very small correction to the stellar radius of zero - temperature, pure helium models (see Benvenuto & Althaus 1998).

By contrast, the study of evolution of carbon - oxygen white dwarfs have captured the interest of numerous investigators such as Lamb & Van Horn (1975), Iben & Tutukov (1984), Koester & Schönberner (1986), D’Antona & Mazzitelli (1989), Tassoul, Fontaine & Winget (1990), Wood (1992), Benvenuto & Althaus (1997) and Althaus & Benvenuto (1998). In particular, Iben & Tutukov (1984) were the first in computing evolutionary models of white dwarfs with hydrogen burning, showing that hydrogen burning in cooling white dwarfs could be an important energy source.

With the calculations that we present here we amply extend those presented in Althaus & Benvenuto (1997a) and Benvenuto & Althaus (1998), in which the effects of convection, neutrino losses and different hydrogen envelopes on the structure and evolution of helium white dwarfs were carefully analysed (we should mention that the results shown in Benvenuto & Althaus 1998 correspond to a metallicity of $Z \approx 0$ and not to $Z = 0.001$, as stated in that work). Furthermore, we extend our calculations to the case of more massive carbon - oxygen white dwarfs. Our grid for carbon - oxygen models is likewise very detailed, which may be of relevance in the study of, for instance, the general problem of determining the theoretical white dwarf luminosity function and the assessment of the age of the Galactic disc.

This subject has been recently addressed by Leggett, Ruiz & Bergeron (1998), who have greatly improved the determination of the observed luminosity function for cool white dwarfs. In this regard, we shall derive theoretical luminosity functions from our cooling sequences in order to compare with the Leggett et al. observational data.

The results presented here constitute a very detailed and updated set which will be suitable, for instance, for the interpretation of recent and forthcoming observational data about low - mass white dwarfs in close binary systems. Finally, we applied our evolutionary models with helium cores to derive stellar masses of a sample of low - mass white dwarfs.

2 COMPUTATIONAL DETAILS

The evolutionary sequences have been obtained with the same evolutionary code and input physics we employed in our previous works on white dwarf evolution, and we refer the reader to Althaus & Benvenuto (1997a, 1998) and Benvenuto & Althaus (1998) as well as to the references cited therein for details. In what follows we restrict ourselves to a few brief comments.

The code has been written following the method presented by Kippenhahn, Weigert & Hofmeister (1967) for calculating stellar evolution. In particular, to specify the surface boundary conditions we perform three envelope integrations (at constant luminosity) from photospheric starting values inward to a fitting mass fraction $M_1/M \approx 10^{-16}$, where $M_1$ corresponds to the first Henyey mass shell and $M$ is the total mass of the white dwarf model. In our code the value of $M_1$ is automatically changed over the evolution so as to keep the thickness of the envelope as small as possible. This provides an accurate description of the outer layers of our white dwarf models. The interior integration is treated according to the standard Henyey technique as described by Kippenhahn et al. (1967).

The constitutive physics of our code is as detailed and updated as possible. Briefly, for the low - density regime, we consider the equation of state of Saumon, Chabrier & Van Horn (1995) for hydrogen and helium plasmas. The treatment for the high - density, completely ionized regime appropriate for the white dwarf interior is based on our own equation of state. This includes ionic and photon contributions, coulomb interactions, partially degenerate electrons, quantum corrections for the ions and electron exchange and Thomas - Fermi contributions at finite temperature (see Althaus & Benvenuto 1997a for details). Radiative opacities for the high - temperature regime ($T \geq 6000$ K) are those of OPAL (Iglesias & Rogers 1993), whilst for lower temperatures we use the Alexander & Ferguson (1994) molecular opacities (or the Cox & Stewart 1970 tabulation for pure helium composition). Two extreme values for metallicity have been considered in the envelope: $Z= 0$ and $Z= 0.001$. We should mention that, owing to the lack of reliable low - temperature ($T < 6000$K) opacities for helium composition, our low - luminosity models with helium atmospheres should be regarded with caution, particularly their ages. Conductive opacities for the liquid and crystalline phases and the various mechanisms of neutrino emission relevant to white dwarf interiors are taken from the works of Itoh and collaborators.
Table 1. Available white dwarf evolutionary sequences with hydrogen envelopes

| Sequence  | $T_{\text{eff}}$ | Sequence  | $T_{\text{eff}}$ | Sequence  | $T_{\text{eff}}$ | Sequence  | $T_{\text{eff}}$ |
|-----------|----------------|-----------|----------------|-----------|----------------|-----------|----------------|
| He151e3.z3 | 8.12           | He151e4.z3 | 8.97           | He151e5.z3 | 9.53           | He151e6.z3 | 9.97           |
| He152e3.z3 | 7.80           | He152e4.z3 | 8.80           | He152e5.z3 | 7.61           | He152e6.z3 | 8.02           |
| He201e3.z3 | 11.87          | He201e4.z3 | 13.96          | He201e5.z3 | 14.75          | He201e6.z3 | 14.97          |
| He202e3.z3 | 11.93          | He202e4.z3 | 12.71          | He202e5.z3 | 13.38          | He202e6.z3 | 12.71          |
| He251e3.z3 | 17.28          | He251e4.z3 | 19.04          | He251e5.z3 | 20.24          | He251e6.z3 | 21.35          |
| He252e3.z3 | 16.41          | He252e4.z3 | 18.77          | He252e5.z3 | 18.32          | He252e6.z3 | 22.14          |
| He301e3.z3 | 23.92          | He301e4.z3 | 25.76          | He301e5.z3 | 26.15          | He301e6.z3 | 21.73          |
| He302e3.z3 | 23.43          | He302e4.z3 | 22.64          | He302e5.z3 | 23.90          | He302e6.z3 | 24.65          |
| He351e3.z3 | 25.51          | He351e4.z3 | 25.82          | He351e5.z3 | 24.00          | He351e6.z3 | 24.02          |
| He361e3.z3 | 26.14          | He361e4.z3 | 26.57          | He361e5.z3 | 27.66          | He361e6.z3 | 28.78          |
| He401e3.z3 | 22.50          | He401e4.z3 | 27.31          | He401e5.z3 | 28.72          | He401e6.z3 | 30.55          |
| He402e3.z3 | 26.61          | He402e4.z3 | 25.47          | He411e3.z3 | 30.52          | He428e4.z3 | 31.10          |
| He429e3.z3 | 31.28          | He429e4.z3 | 32.00          | He429e5.z3 | 32.84          | He451e4.z3 | 31.11          |
| He451e3.z3 | 29.30          | He451e4.z3 | 30.67          | He452e4.z3 | 29.77          | He454e4.z3 | 27.96          |
| He458e3.z3 | 34.20          | He458e4.z3 | 35.60          | He459e4.z3 | 36.62          | He501e6.z3 | 39.18          |
| He501e3.z3 | 38.22          | He501e4.z3 | 37.07          | He501e5.z3 | 35.85          | He501e6.z3 | 39.18          |
| OC501e4.z3 | 57.91          | OC501e5.z3 | 60.78          | OC501e6.z3 | 61.90          | OC501e7.z3 | 62.30          |
| OC502e4.z3 | 61.42          | OC502e5.z3 | 61.07          | OC502e6.z3 | 70.69          | OC502e7.z3 | 72.04          |
| OC503e4.z3 | 74.69          | OC503e5.z3 | 74.81          | OC503e6.z3 | 73.47          | OC503e7.z3 | 78.56          |
| OC504e4.z3 | 74.33          | OC504e5.z3 | 87.94          | OC504e6.z3 | 88.07          | OC504e7.z3 | 91.60          |
| OC505e4.z3 | 99.48          | OC505e5.z3 | 101.14         | OC505e6.z3 | 108.68         | OC505e7.z3 | 108.83         |
| OC506e4.z3 | 121.03         | OC506e5.z3 | 122.79         | OC506e6.z3 | 120.61         | OC506e7.z3 | 122.81         |
| OC507e4.z3 | 134.59         | OC507e5.z3 | 136.49         | OC507e6.z3 | 136.19         | OC507e7.z3 | 136.37         |
| OC508e4.z3 | 198.47         | OC508e5.z3 | 196.97         | OC508e6.z3 | 195.85         | OC508e7.z3 | 194.00         |
| OC509e4.z3 | 7.70           | He201e4.z0 | 14.05          | He204e4.z0 | 11.37          | He251e3.z0 | 17.01          |
| OC510e4.z3 | 18.93          | He252e3.z0 | 16.38          | He301e3.z0 | 20.55          | He301e4.z0 | 23.63          |
| OC511e4.z3 | 21.70          | He301e4.z0 | 23.50          | He354e4.z0 | 24.83          | He401e3.z0 | 22.69          |
| OC512e4.z0 | 27.14          | He404e4.z0 | 29.49          | OC451e4.z0 | 45.29          | OC501e4.z0 | 58.49          |
| OC513e4.z0 | 43.28          | OC471e4.z0 | 45.29          | OC501e4.z0 | 58.49          | OC521e4.z0 | 60.08          |
| OC514e4.z0 | 53.71          | OC516e4.z0 | 49.94          | OC518e4.z0 | 49.65          | OC600e4.z0 | 58.86          |
| OC515e4.z0 | 57.49          | OC561e4.z0 | 61.79          | OC661e4.z0 | 62.33          | OC681e4.z0 | 67.93          |
| OC516e4.z0 | 77.50          | OC721e4.z0 | 77.90          | OC741e4.z0 | 79.33          | OC761e4.z0 | 83.65          |
| OC517e4.z0 | 85.30          | OC801e4.z0 | 81.06          | OC821e4.z0 | 81.79          | OC841e4.z0 | 89.26          |
| OC518e4.z0 | 106.71         | OC101e4.z0 | 106.95         | OC111e4.z0 | 95.00          | OC121e4.z0 | 96.65          |
| OC519e4.z0 | 97.27          | OC121e4.z0 | 97.27          |              |                |              |                |

This table shows available evolutionary sequences for white dwarf models with a hydrogen envelope. We use an abbreviated notation to indicate the core composition, the stellar mass in tenths of solar mass units, the mass fraction of the hydrogen envelope and the envelope metallicity. For instance, He252e3.z0 stands for an evolutionary sequence of 0.25 $M_\odot$, with masses from $M=0.1$ $M_\odot$ to $M=1.2$ $M_\odot$.
The location of the top and the base of the convection zone in terms of the outer mass fraction $q (q = 1 - M_\text{c}/M)$ versus $T_{\text{eff}}$ for a 0.3 $M_\odot$ white dwarf model with a hydrogen envelope of $M_\text{H}/M = 10^{-3}$ according to different theories of convection and metallicities. It is clear that the mixing temperature for models with thin hydrogen envelopes will be dependent upon the assumed theory of convection. At low $T_{\text{eff}}$ values, the depth reached by the base of the convection zone is independent of the treatment of convection. Note also the deeper final extent reached by convection in the case of metallicity $Z = 0$. The convection theory employed. In contrast, the size of the outer convection zone in an intermediate effective temperature ($T_{\text{eff}}$), evolving white dwarf is strongly dependent upon the assumed treatment of convection. Hence, a trustworthy model of stellar convection is to be employed to get reliable $T_{\text{eff}}$ values at which thin hydrogen envelopes mix with the underlying helium (see Benvenuto & Althaus 1998). In this context, our calculations represent an improvement over previous white dwarf studies based on the mixing length theory amply used in white dwarf studies (see, e.g., Tassoul et al. 1990). It is clear that the mixing $T_{\text{eff}}$ for models with thin hydrogen envelopes depends upon the convection theory. To clarify this point better, we show in Figure 1 the behaviour of the evolving outer convection zone in terms of $T_{\text{eff}}$ for our 0.3 $M_\odot$ model with a thick hydrogen envelope. In addition to Camuto et al.’s results, we include in the figure the predictions given by the ML1, ML2 and ML3 versions of the mixing length theory amply used in white dwarf studies (see, e.g., Tassoul et al. 1990). It is clear that the mixing $T_{\text{eff}}$ for models with thin hydrogen envelopes depends upon the convection theory. Another observation that we can make from this figure is that the base of the convection zone (for the case $Z = 0$) ultimately reaches a final extent at an outer mass fraction of $2 \times 10^{-4} (6 \times 10^{-5} M_\odot)$, irrespective of the treatment of convection. This result is in good agreement with the predictions of Hansen & Phinney (1998a) for the same model. Note also the larger final extent of convection zone for lower metallicity, which as we shall see, gives rise to considerable differences in the evolutionary times at low luminosities.

Our white dwarf initial models of different masses and metallicities have been obtained following the artificial evolutionary procedure described by Benvenuto & Althaus (1998). The carbon - oxygen core models all have the same core chemical composition profile shown in Figure 2. This chemical profile was calculated by D’Antona & Mazzitelli (1989) for the progenitor evolution of a 0.55 $M_\odot$ white dwarf. We adopt this profile for all of our models, in spite of the changes that are expected to occur for more massive models as a result of differences in the evolution in progenitor objects. We would need, in order to improve this assumption, detailed calculations of the pre - white dwarf evolution of these objects. To our knowledge, such calculations are not available. Because of the fact that the mass of the hydrogen envelope in white dwarfs is poorly constrained by theoretical calculations of the pre - white dwarf evolution of these objects, particularly in the case of helium white dwarfs where the uncertainties regarding the mass exchange episodes are more severe, we decide to treat the mass of the hydrogen envelope as essentially a free parameter. It is worthwhile to mention that our evolving low - mass white dwarf models should be considered as evolutionary stages that can be asymptotically reached by helium white dwarfs resulting from close binary evolution. In this study we have not computed such binary evolution, and we refer the reader to Iben & Livio (1993) for a review. Needless to say, the starter model choice affects the initial evolution of all of our models, particularly the age (see Althaus & Benvenuto 1997a for details).

In closing, we have included in our calculations the release of latent heat during crystallization (see Benvenuto & Althaus 1997 for details) and convective mixing.
Table 2. Available white dwarf evolutionary sequences without a hydrogen envelope

| Sequence | $T_{\text{eff}}$ | Sequence | $T_{\text{eff}}$ | Sequence | $T_{\text{eff}}$ | Sequence | $T_{\text{eff}}$ | Sequence | $T_{\text{eff}}$ |
|----------|-----------------|----------|-----------------|----------|-----------------|----------|-----------------|----------|-----------------|
| He9990.z3 | 8.62 | He1039.z3 | 7.00 | He1091.z3 | 7.45 | He1146.z3 | 7.93 |
| He1205.z3 | 8.45 | He1263.z3 | 9.02 | He1326.z3 | 9.62 | He1393.z3 | 10.26 |
| He1462.z3 | 10.99 | He1500.z3 | 11.59 | He1575.z3 | 12.17 | He1654.z3 | 12.90 |
| He1736.z3 | 13.51 | He1823.z3 | 14.08 | He1914.z3 | 14.82 | He2010.z3 | 15.85 |
| He2110.z3 | 17.26 | He2216.z3 | 18.27 | He2327.z3 | 20.01 | He2443.z3 | 21.42 |
| He2565.z3 | 22.83 | He2693.z3 | 24.61 | He2829.z3 | 25.29 | He2970.z3 | 26.70 |
| He3118.z3 | 26.44 | He3274.z3 | 26.77 | He3438.z3 | 27.90 | He3601.z3 | 28.07 |
| He3790.z3 | 30.09 | He3979.z3 | 29.63 | He4179.z3 | 30.60 | He4388.z3 | 32.49 |
| He4607.z3 | 34.63 | He4837.z3 | 37.30 | He5079.z3 | 37.69 |
| OC5000.z3 | 57.61 | OC5200.z3 | 47.60 | OC5400.z3 | 51.96 | OC5600.z3 | 54.21 |
| OC5800.z3 | 59.70 | OC6000.z3 | 78.72 | OC7000.z3 | 78.59 | OC8000.z3 | 93.85 |
| OC9000.z3 | 94.82 | OC1000.z3 | 101.57 | OC1100.z3 | 107.97 | OC1200.z3 | 100.01 |

This table shows available evolutionary sequences for white dwarf models without a hydrogen envelope. We use an abbreviated notation to indicate the core composition, the stellar mass in tenths of solar mass units and the envelope metallicity. For instance, He3118.z3 stands for an evolutionary sequence of 0.3118 $M_\odot$ models with a helium core composition and an envelope metallicity of $Z = 0.001$. OC means oxygen - carbon core composition. (Note that OC sequences for model more massive than 1 $M_\odot$ are indicated with the same notation). We also provide a column ($T_{\text{eff}}^i$) with the effective temperature (in thousand K degrees) at which each sequence starts.

3 THE GRIDS OF WHITE DWARF MODELS

In this section we comment on the most important features of the grids. We computed evolutionary sequences with masses ranging from $M = 0.1$ $M_\odot$ to $M = 1.2$ $M_\odot$ and metallicity $Z = 0.001$ and $Z = 0$. For models with $M < 0.5 M_\odot$ we assume a pure helium core and for models with $M > 0.45 M_\odot$ we assume a carbon - oxygen core with the chemical profile of Figure 2. We also vary the mass of the hydrogen envelope $M_H$ in the range $10^{-12} \leq M_H/M_\odot \leq 4 \times 10^{-3}$ and the mass of the helium layer $M_{He}$ (in the case of carbon - oxygen white dwarfs) is taken to be $M_{He}/M_\odot = 10^{-2}$. In Tables 1 and 2 we summarize the main characteristics of all of our available evolutionary sequences. As stated earlier, models have been calculated in the framework of Canuto et al. (1996) theory of convection. We have used OPAL opacity calculations supplemented with the Alexander & Ferguson (1994) molecular opacities (or with the Cox & Stewart 1970 tabulation for helium composition) for low temperatures. In Table 1 and 2 we also give the $T_{\text{eff}}$ values at which each evolutionary sequence starts. In this regard, we emphasise once again that model ages corresponding to the first stages of evolution are meaningless because they are strongly affected by the procedure we use to generate the initial models. At advanced ages, however, this is no longer relevant and age values are meaningful. The sequences have been evolved down to a stellar luminosity log $L/L_\odot = -5$.

We begin by examining the time evolution of our models. From the point of view of an age determination of the disc of our Galaxy from the observed space density of white dwarfs, the evolutionary times of white dwarfs as a function of mass represent obviously an important issue (see Wood 1992 and references cited therein). In this regard, we feel it to be valuable to compare our cooling curves against those published by other authors. We elect those computed by Wood (1995) for models with pure oxygen core on the basis of OPAL radiative opacities. To this end, we have computed additional sequences for models with oxygen cores and with the same outer layer chemical stratification and metallicity as considered by Wood. The comparison is shown in Figure 1 for 0.5 and 0.7 $M_\odot$ models. Note the good agreement between the two set of calculations. At very low luminosities and especially for more massive models than those considered in Figure 1 some divergence appears as a result in part of the different set of low - temperature radiative opacities employed.

As we mentioned, white dwarf evolutionary times represent a powerful tool for constraining the age of the disc of our Galaxy. Indeed, the existence of an abrupt falloff in the observed white dwarf luminosity function (see Liebert, Dahn & Monet 1988 and earlier references cited therein) has been interpreted in terms of a finite age of the disc of the Galaxy (D’Antona & Mazzitelli 1978). By fitting the observations with theoretical white dwarf luminosity functions, this interpretation was quantitatively explored by numerous investigators such as Winget et al. (1987), Iben & Laughlin (1989) and Wood (1992). Recently, Leggett et al. (1998) have substantially improved the determination of the observed luminosity function for cool white dwarfs. To compare with observations, we constructed integrated luminosity functions from our evolutionary sequences. To this end, we follow the treatment presented in Iben & Laughlin (1989). Specifically, the space density of white dwarfs per unit of $\ell$, $\ell \equiv \log(L/L_\odot)$, is calculated from

$$\frac{dn}{d\ell} = -\psi_0 \int_{m_i}^{m_o} \phi(m) \left(\frac{dL_{\text{cool}}}{d\ell}\right)_M dm.$$  

Here, $\phi(m)$ is the Salpeter initial mass function of white dwarf progenitors with stellar mass $m$ (which predicts that the created star distribution is proportional to $1/m^{1.35}$) and $L_{\text{cool}}$ is the white dwarf cooling time at a given $\ell$, which is a function of the white dwarf mass $M$, $m_i$ and $m_o$ denote respectively the minimum and the maximum mass of the main sequence stars which contribute to the white dwarf space density at $\ell$. We take $m_i \approx 8 M_\odot$ (Wood 1992) and $m_o$ is obtained by solving the equation $t_{MS}(m) + t_{\text{cool}}(\ell, M) = t_d$, where $t_d$ is the assumed disc age. The pre - white dwarf evolutionary times $t_{MS}(m)$ are those of Iben & Laughlin (1989). As far as the initial($m$) - final($M$) mass relation is concerned,
Figure 3. Age - surface luminosity relation for 0.5 (upper panel) and 0.7 (lower panel) $M_{\odot}$ white dwarf models with pure oxygen cores and hydrogen envelope mass of $M_{H}/M = 10^{-4}$. We compare our results (solid lines) to those given by the Wood (1995) models (open circles) having the same stellar mass and chemical stratification as ours. The calculations are for a metallicity of $Z = 0$. At high luminosities, the discrepancy between the two sets of calculations are due to the different procedure employed to generate the initial models.

Figure 4. Theoretical white dwarf luminosity functions (dotted lines) corresponding to our carbon - oxygen core, white dwarf models with a hydrogen envelope mass of $M_{H}/M = 10^{-4}$ and metallicity $Z = 0$. The curves, which correspond to assumed disc ages of 6 - 10 Gyr, are compared to the observational data of Leggett et al. (1998) and they have been normalized to the observed white dwarf space density of 0.00339 stars per cubic parsec. Note that the best fit to the dimmest white dwarfs observed corresponds to a disc age of approximately 8 Gyr.

We use an exponential model: $M = 0.40 e^{0.125 \ell}$ (Wood 1992). In deriving Equation (1), the star formation rate $\psi_0$ has been assumed to be constant. Finally, for each of the selected luminosity values, we calculate $\partial t_{\text{cool}}/\partial \ell$ at a given $M$ by using linear interpolation between the $\partial t_{\text{cool}}/\partial \ell$ values of the sequences that bracket $M$. The resulting luminosity functions for assumed disc ages of 6 - 10 Gyr are shown in Figure 4 (in Figure 4, the luminosity functions have been converted into intervals of bolometric magnitude $M_{bol}$). It is worth mentioning that all of our theoretical curves have been normalized to the observed space density of 0.00339 white dwarfs per cubic parsec (Leggett et al. 1998). The best fit to the coolest white dwarfs observed is obtained for assumed disc ages of $\approx 8$ Gyr, which is in agreement with the ages quoted by Leggett et al. on the basis of the Wood (1995) cooling sequences.

It is worthy of comment that the age of helium white dwarf models depends on the mass of the hydrogen envelope. This is particularly true for models with very thick hydrogen envelopes, for which hydrogen burning contributes substantially to the total luminosity, thus leading to a delay in cooling even down to very low $T_{\text{eff}}$. Unfortunately, the maximum mass of the hydrogen envelope resulting from binary evolution is still an open question. Evidence favouring "thin" envelopes was presented by Iben & Tutukov (1986) from self - consistent binary evolutionary calculations. Indeed, these authors found that the hydrogen envelope remaining at the top of their 0.3 $M_{\odot}$ remnant after shell flash episodes is too small ($M_{H} \approx 1.4 \times 10^{-4} M_{\odot}$) to sustain any further nuclear burning. Needless to say, a larger hydrogen remnant would lead to longer evolutionary times. In the context of age determinations of millisecond pulsars with helium white dwarf companions, this fact is a clearly important one to be taken into account. To place this assertion on a more quantitative basis, we consider the pulsar PSR J1012+5307. The surface gravity and $T_{\text{eff}}$ for its low - mass helium white dwarf companion have been determined to be $\log g = 6.75 \pm 0.07$ and $T_{\text{eff}} = 8550 \pm 25 K$ (van Kerkwijk et al. 1996). At $T_{\text{eff}} \approx 8500 K$, we find that our 0.21 $M_{\odot}$ helium white dwarf model with $M_{H}/M = 2 \times 10^{-4}$ has $\log g = 6.82$ and age 0.44 Gyr, in good agreement with the Hansen & Phinney (1998b) predictions. The same fit to the $T_{\text{eff}}$ and gravity values would be achieved with a 0.213 $M_{\odot}$ model with $M_{H}/M \approx 6 \times 10^{-5}$. However, in this case hydrogen burning supplies 70 per cent of the surface luminosity and the model age becomes as high as 0.9 Gyr.

Our main motivation for the publication of a detailed set of low - mass white dwarf models like the one presented...
Figure 5 shows the surface gravity \( g \) models we computed here by employing the full scheme of mass white dwarfs. We think that the detailed low-mass treatment of the recent and future observational data about low-mass white dwarfs. These features may turn out to be very important for the interpretation of the recent and future observational data about low-mass white dwarfs. We think that the detailed low-mass models we computed here by employing the full scheme of stellar evolution theory may help such an endeavour.

In the context of the foregoing paragraph we show in Figure 5 the surface gravity \( g \) (in cgs units) in terms of \( T_{\text{eff}} \) for some selected models. The effects of finite temperature are clearly noticeable, particularly for less massive models. As is well known, at a given \( T_{\text{eff}} \) more massive models are characterized by smaller radii. As \( T_{\text{eff}} \) decreases the model radius (gravity) gradually becomes smaller (larger), ultimately reaching an almost constant value as expected for a strongly degenerate configuration, in which the mechanical structure is determined mainly by degenerate electron pressure. As a result, stellar parameters asymptotically reach constant values corresponding to zero temperature configurations. Note also the changes in the \( g \) values brought about by rather thick hydrogen envelopes. Another observation we can make from this figure is that, at low \( T_{\text{eff}} \) convective mixing between hydrogen and helium layers increases the \( g \) values of models with thin hydrogen envelopes. In fact, convective mixing changes the outer layer composition from a hydrogen-dominated to a helium-dominated one, thus giving rise to denser outer layers. From then on, their subsequent evolution resembles that of a white dwarf model without a hydrogen envelope, as can be noted from Figure 5 (see also Figure 5), particularly for less massive models.

To clarify better the role played by hydrogen envelopes in the \( g \) values of low-mass models, we show in Figure 5 the reduction in the \( g \) values of pure helium configurations resulting from adding hydrogen envelopes of different thickness. More precisely, we plot in terms of \( T_{\text{eff}} \) the quantity \( \Delta g/g_0 \equiv (g_0 - g_{\text{H}})/g_0 \) for various stellar masses and hydrogen envelopes \( (g_0 \) and \( g_{\text{H}} \) stand for the surface gravity of a helium-core configuration of a given stellar mass without and with a hydrogen envelope, respectively). It is clear that thick hydrogen envelopes appreciably reduce the \( g \) values of pure helium models. At the low \( T_{\text{eff}} \) of 15,000K for instance, the \( g \) values of the 0.35 and 0.25 \( M_\odot \) pure helium models are reduced, respectively, by 20 and 30 per cent if a hydrogen envelope of \( M_\text{H}/M_\odot = 10^{-3} \) is added. Such values increase considerably at higher \( T_{\text{eff}} \). At \( T_{\text{eff}} \approx 17,000 \) K, there is a

| White Dwarf | Source | \( T_{\text{eff}} \) (K) | log \( g \) | \( M/M_\odot \) (1) | \( M/M_\odot \) (2) | \( M/M_\odot \) (3) |
|------------|--------|-----------------|----------|----------------|----------------|----------------|
| 0316 + 345 | BSL    | 14800           | 7.61     | 0.40           | 0.414          | 0.452          |
| 0339 + 523 | BSL    | 13350           | 7.47     | 0.34           | 0.354          | 0.395          |
| 0710 + 741 | BSL    | 18930           | 7.45     | 0.35           | 0.377          | 0.418          |
| 0957 - 666 | BRB    | 27047           | 7.285    | 0.335          | 0.381          | 0.417          |
| 1022 + 050 | BRB    | 14481           | 7.483    | 0.351          | 0.364          | 0.405          |
| 1101 + 364 | BSL    | 13610           | 7.38     | 0.31           | 0.325          | 0.369          |
| 1241 + 010 | BSL    | 24010           | 7.22     | 0.31           | 0.342          | 0.390          |
| 1317 + 453 | BSL    | 14000           | 7.43     | 0.33           | 0.343          | 0.385          |
| 1353 + 409 | BSL    | 23580           | 7.54     | 0.40           | 0.427          | 0.470          |
| 1614 + 136 | BSL    | 22430           | 7.34     | 0.33           | 0.361          | 0.409          |
| 1713 + 332 | BSL    | 22030           | 7.40     | 0.35           | 0.376          | 0.420          |
| 1824 + 040 | BRB    | 14795           | 7.608    | 0.394          | 0.411          | 0.451          |
| 2032 + 188 | BSL    | 18540           | 7.48     | 0.36           | 0.385          | 0.425          |
| 2331 + 290 | BSL    | 27830           | 7.50     | 0.39           | 0.431          | 0.479          |
| 2337 - 760 | BRB    | 14295           | 7.507    | 0.354          | 0.372          | 0.410          |

\( T_{\text{eff}} \), the surface gravity \( g \) and the stellar mass \( M \) of the objects were taken from Bergeron, Saffer & Liebert (BSL) (1992) and Bragaglia, Renzini & Bergeron (BRB) (1995). The next column (2) gives the stellar mass according to our helium-core models without a hydrogen envelope and the last column lists the stellar mass according to our helium-core models with a hydrogen envelope of \( M_\text{H}/M_\odot = 10^{-3} \).
change in the slope of the curves stemming from the decrease in the radiative opacity values after helium recombination. This causes pure helium models to become denser and hence to have larger $g$ values.

Non-negligible differences in the structure and cooling of white dwarfs may also arise from the employment of different metallicities in the envelope, particularly at low luminosities where central temperature of models becomes strongly tied to the details of the outer layer chemical stratification (see Tassoul et al. 1990). This expectation is born out by Figure 7, in which we compare the cooling times of helium white dwarf models for two extreme metallicities assumed in the envelope. When convection reaches the domain of degeneracy, the central temperature drops substantially and the star has initially an excess of internal energy to be radiated, thus giving rise to a lengthening of evolutionary times during that stage of evolution. Because models with lower metallicities are characterized by deeper convection zones (see Figure 2), this effect occurs at higher luminosities in such models and this explains their greater ages as compared with high-metallicity models. Eventually, at very low luminosities, more transparent models evolve more rapidly, as expected. We have also analysed the effect of metallicity on surface gravity for helium models (see Figure 8) and we found that surface gravity is almost insensitive to a specific choice of metallicity in the envelope.

Lastly, we have applied our evolutionary models with helium cores to derive stellar masses of some selected low-mass white dwarfs. To this end, we have picked out low-surface gravity white dwarfs from the sample of white dwarfs analysed by Bergeron, Saffer & Liebert (1992) and Bragaglia, Renzini & Bergeron (1995), and we list the results in Table 3. The above cited authors estimated the white dwarf masses from evolutionary models with pure carbon-core composition. However, using our evolutionary models with helium cores and no hydrogen envelope we find the mass values to be appreciably underestimated, particularly at high temperatures. It is worth mentioning that the objects listed in Table 3 are white dwarfs that most likely have a hydrogen envelope. This being the case, the stellar mass should be estimated from evolutionary models with hydrogen envelopes. As shown in Table 3, there is an appreciable difference in the white dwarf mass when stellar masses are derived from models with thick hydrogen envelopes ($M_H/M \approx 10^3$).

In view of the preceding considerations, we conclude that detailed models of low-mass white dwarfs such as presented in this study should be carefully taken into account.
Grids of white dwarf evolutionary models with masses from $M = 0.1 M_\odot$ to $M = 1.2 M_\odot$

Figure 8. Surface gravities versus $T_{\text{eff}}$ for (from top to bottom) 0.30, 0.25 and 0.20 $M_\odot$ helium white dwarf models with $M_H/M = 10^{-4}$ and for metallicities $Z = 0$ (solid lines) and $Z = 0.001$ (dashed lines).

should the mass of a white dwarf be measured by applying the surface gravity - $T_{\text{eff}}$ relation. This is particularly true regarding the possibility of constraining the equation of state at neutron star densities as inferred from observations of low - mass white dwarf companions to millisecond pulsars, such as those studied by van Kerkwijk et al. (1996).

Complete tables containing the results of our calculations are available at the World Wide Web site [http://www.fcaglp.unlp.edu.ar/~althaus/]. Additional evolutionary sequences are obtained upon request to the authors at their e-mail address. Features such as surface luminosity, $T_{\text{eff}}$, central density and temperature, surface gravity, stellar radius, age and hydrogen surface abundance are listed in the tables.

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