LOWER ORDER TERMS FOR THE ONE-LEVEL DENSITY OF A SYMPLECTIC FAMILY OF HECKE L-FUNCTIONS

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ABSTRACT. In this paper we apply the $L$-function Ratios Conjecture to compute the one-level density for a symplectic family of $L$-functions attached to Hecke characters of infinite order. When the support of the Fourier transform of the corresponding test function $f$ reaches 1, we observe a transition in the main term, as well as in the lower order term. Assuming GRH, we then directly calculate main and lower order terms for test functions $f$ such that $\text{supp}(\hat{f}) \subset (-1, 1)$, and observe that the result is in agreement with the prediction provided by the Ratios Conjecture. As a corollary, we deduce that, under GRH, at least 75% of these $L$-functions do not vanish at the central point.

1. Introduction

1.1. One-Level Densities and Random Matrices. The statistical distribution of zeros across a family of $L$-functions near the central point has emerged as a popular object of study. This is due to its role in a range of arithmetic applications, such as determining the size of the Mordell-Weil groups of elliptic curves [2, 3], and bounding the size of the class number of imaginary quadratic fields [5]. It is also of purely theoretical interest since such information can be used to determine the symmetry type of the family in question.

Let $\rho_\chi = 1/2 + i\gamma_\chi$ denote a generic non-trivial zero of a given $L$-function $L(s, \chi)$. We may assume that the Generalized Riemann Hypothesis holds for $L(s, \chi)$, so that $\gamma_\chi \in \mathbb{R}$ for all such $\rho_\chi$. Fix $f$ to be an even Schwarz function such that $\hat{f}$ is compactly supported. The (scaled) one-level density is then defined as

$$D_1(\chi; f, R) := \sum_{\rho_\chi} f\left(\frac{\log R}{\pi} \cdot \gamma_\chi\right),$$

where $R$ is a scaling parameter (typically related to the analytic conductor), intended to ensure that the mean spacing between low-lying zeroes is equal to 1.

If $\mathcal{F}(R)$ is a family of $L$-functions with bounded analytic conductor, one
moreover defines the \textbf{averaged (scaled) one-level density} over the family to be

\[ D_1(\mathcal{F}(R); f) := \frac{1}{|\mathcal{F}(R)|} \sum_{\chi \in \mathcal{F}(R)} D_1(\chi; f, R). \]

Much research has been dedicated to studying the one-level density for a variety of families, such as $L$-functions attached to Dirichlet characters, elliptic curves, cuspidal newforms, Maass forms, and many other objects [9, 12, 13, 15]. Usually, one is only able to compute the one-level density for a particular class of test functions, namely those for which $\text{supp}(\widehat{f}) \subseteq (-\alpha, \alpha)$, for a particular $\alpha \in \mathbb{R}$. In all such cases to date, the computation has agreed with that of a random matrix ensemble in the restricted support computed.

Let $G$ denote an $M \times M$ classical matrix group (i.e., the group of unitary, symplectic, orthogonal, or special orthogonal matrices), and set $dA$ to be the normalized Haar measure on $G$. As $A \in G$ is unitary, its eigenvalues $\{e^{i\varphi_1}, e^{i\varphi_2}, \ldots, e^{i\varphi_M}\}$ lie on the unit circle, and we may define the ordered sequence

\[ 0 \leq \varphi_1(A) \leq \varphi_2(A) \leq \cdots \leq \varphi_M(A) < 2\pi. \]

Writing $\varphi_{j+lM}(A) := \varphi_j(A) + 2\pi \cdot l$, we may then extend our definition to the set $\{\varphi_n\}_{n \in \mathbb{Z}}$ of “all” real angles of $A$. Defining $\vartheta(i) := M\varphi(i)/2\pi$, we then consider the \textit{normalized} sequence $\{\vartheta_i\}_{i \in \mathbb{Z}}$ whose average spacing is equal to one.

For $f$ as above, define

\[ W_1(f, A) := \sum_{n \in \mathbb{Z}} f(\vartheta(n)), \]

as well as

\[ W_1(f, G) := \int_G W_1(f, A) dA. \]

The \textit{Katz-Sarnak Density Conjecture} [16, 17] states that in the limit as the conductor tends to infinity, $D_1(\mathcal{F}(R), f)$ is modeled by the one-level scaling density of eigenvalues near 1 of the classical compact group $G$ corresponding to the symmetry type of $\mathcal{F}(R)$. In other words, for an admissible test function $f$, we expect

\[ \lim_{R \to \infty} D_1(\mathcal{F}(R), f) = \lim_{N \to \infty} W_1(f, G) = \int_{\mathbb{R}} f(x) W_{1,G}(x) dx, \]
where

\[
W_{1,G} = \begin{cases} 
  1 & \text{if } G = U \\
  1 - \frac{\sin 2\pi t}{2\pi t} & \text{if } G = USp \\
  1 + \frac{1}{2}\delta_0(t) & \text{if } G = O \\
  1 + \frac{\sin 2\pi t}{2\pi t} & \text{if } G = SO(\text{even}) \\
  1 + \frac{1}{2}\delta_0(t) - \frac{\sin 2\pi t}{2\pi t} & \text{if } G = SO(\text{odd}) 
\end{cases}
\]

is computed explicitly in [16, §AD.12.6]. Here \(\delta_0\) is the Dirac distribution, and \(U, USp, O, SO(\text{even}), SO(\text{odd})\) denote the unitary, symplectic, orthogonal, special even-orthogonal, and special odd-orthogonal matrix groups, respectively.

Random Matrix Theory has proved extremely successful at modelling a variety of statistical quantities related to the zeta function, and across families of \(L\)-functions. Montgomery’s work on the pair correlation of the zeta zeros [23] pointed to a deep connection between statistics attached to zeros of the zeta function and eigenvalues of random matrices, while Rudnick and Sarnak’s work showed that the local spacing distribution between zeros high up on the critical strip of any fixed automorphic \(L\)-function obeys universal behavior in line with Dyson’s computation for the GUE model [25]. The Katz-Sarnak Density Conjecture extends this connection to a relationship between the low-lying zeros across a family of \(L\)-functions near the central point and distribution laws for eigenvalues of random matrices. Here universality breaks down, and, as above, one may use the resulting computation to distinguish between families of different symmetry type.

Motivated by calculations for the characteristic polynomials of matrices averaged over the compact classical groups, Conrey, Farmer, and Zirnbauer [6, 7] suggested a recipe for calculating averages of quotients of products of shifted \(L\)-functions evaluated at certain values on the critical strip. The \(L\)-function Ratios Conjecture can be used to give precise predictions for a variety of statistics across families of \(L\)-functions, including the one-level density [8]. Our goal in this work will be to use the Ratios Conjecture to motivate a precise conjecture for the one-level density of a family of \(L\)-functions attached to Hecke characters of infinite order, and then to demonstrate the accuracy of this conjecture for a restricted class of test functions.

1.2. Main Results. Let \(\alpha \in \mathbb{Z}[i]\) be an element in the ring of Gaussian integers and let \(a = \langle \alpha \rangle \subseteq \mathbb{Z}[i]\) denote the ideal generated by \(\alpha\). For any \(k \in \mathbb{Z}\), the Hecke character \(\Xi_k(a) := (\alpha/\overline{\alpha})^{2k}\) provides a well-defined function on the ideals of \(\mathbb{Z}[i]\). Attached to any such character \(\Xi_k\) is a Hecke \(L\)-function of the form

\[
L_k(s) := \sum_{a \subseteq \mathbb{Z}[i], a \neq 0} \frac{\Xi_k(a)}{N(a)^s} = \prod_{p \text{ prime}} \left( 1 - \frac{\Xi_k(p)}{N(p)^s} \right)^{-1}, \quad \text{Re}(s) > 1,
\]
where $N(a) := \alpha \overline{\alpha}$ is the norm of $a$, and where $p$ runs through the prime ideals in $\mathbb{Z}[i]$. In this work, we study the average one-level density across the family

$$\mathcal{F}(K) := \{L_k(s) : 1 \leq k \leq K\}.$$  

This family of characters arises in several applications connected to the angular distribution of Gaussian primes [19, 20], and, in particular, the zero distribution near the central point across $\mathcal{F}(K)$ affects the variance of Gaussian primes across sectors [4, 24].

Let $\rho_k = 1/2 + i\gamma_k$ denote a generic non-trivial zero of $L_k(s)$. The number of such zeros in a fixed rectangle is

$$\#\{\rho_k : 0 \leq \Im(\rho_k) \leq T_0\} \sim \frac{T_0 \log |k|}{\pi}, \quad k \to \infty, \quad T_0 > 0 \text{ fixed},$$  

so that the density of zeros in $L_k(s)$ near the central point is $\log |k|/\pi$. We thus choose $K$ as our scaling parameter so that the mean value spacing between zeros is 1. The average one-level density across $\mathcal{F}(K)$ is then

$$D_1(\mathcal{F}(K); f) = \frac{1}{K} \sum_{k=1}^{K} \sum_{\gamma_k} f\left(\gamma_k \frac{M}{\pi}\right),$$  

where $M := \log K$.

Set the Fourier transform to be normalized as $f(u) := \int_{\mathbb{R}} \hat{f}(r)e^{2\pi iru}dr$. Based on a Ratios Conjecture model, we suggest the following conjecture:

**Conjecture 1.1.**

$$D_1(\mathcal{F}(K); f) \sim \hat{f}(0) - \frac{1}{2} \int_{\mathbb{R}} \hat{f}(x) \cdot 1_{[-1,1]}(x)dx + \frac{1}{M} \left(c \cdot \hat{f}(0) + d \cdot \hat{f}(1)\right),$$  

where

$$c := 4 \log |\eta(i)| + 3 \log 2 - 1 - \log \pi - 2 \sum_{p \equiv 3(4)} \frac{\log p}{p^2 - 1},$$  

and

$$d := 1 + \log \pi - 2 \log 2 - 4 \log |\eta(i)|.$$  

Here $\eta(s)$ is the Dedekind eta function, and

$$\eta(i) = e^{-\pi} \prod_{n=1}^{\infty} \left(1 - e^{-2\pi n}\right) = \frac{\Gamma\left(\frac{1}{4}\right)}{2\pi^{3/4}}.$$  

Note that by Plancherel’s identity,

$$\int_{\mathbb{R}} f(t) \frac{\sin 2\pi t}{2\pi t} dt = \frac{1}{2} \int_{-1}^{1} \hat{f}(x) \cdot 1_{[-1,1]}(x).$$
It then follows from (1.1) that Conjecture 1.1 agrees with the Density Conjecture and that $\mathcal{F}(K)$ has symplectic monodromy. This agrees with the findings in the function field setting, where the analogue to $\mathcal{F}(K)$ (the family of so-called super-even characters) is also found to have symplectic monodromy [18]. Note, moreover, the emergence of a sharp transition in both the main term and the lower order term of Conjecture 1.1. A similar transition was observed for a symplectic family of Dirichlet $L$-functions by D. Fiorilli, J. Parks, and A. Södergren [10, 11].

To arrive at Conjecture 1.1 we make use of Cauchy’s residue theorem to write $D_1(\mathcal{F}(K); f)$ in terms of two vertical contour integrals, which we then compute by applying the ratios recipe. The resulting expression involves several terms (depending on $K$ and $f$), which we then explicitly compute: i) a term $W_f$ emerging from the infinite place of each $L_k(s)$ ii) a collection of terms $S_\zeta, S_L,$ and $S_{A'}$ emerging from the first piece of the approximate functional equation within the ratios recipe, and iii) a final term $S_T$ emerging from the second piece of the approximate function equation. When supp($\hat{f}$) $\subset (-1, 1)$, $W_f$ and $S_\zeta$ contribute to the main term, while lower order contributions come from $W_f, S_\zeta, S_L,$ and $S_{A'}$. $S_T$ accounts for the sharp transition in both the main and lower order terms, contributing only when supp($\hat{f}$) $\not\subset (-1, 1)$.

Moreover, we prove the following theorem:

**Theorem 1.2.** Assume the Generalized Riemann Hypothesis (GRH), and suppose that supp($\hat{f}$) $\subset (-1, 1)$. Then

$$D_1(\mathcal{F}(K); f) = \hat{f}(0) - \frac{f(0)}{2} + c \cdot \frac{\hat{f}(0)}{M} + O\left(\frac{1}{M^2}\right),$$

where $c$ is as above.

Note that Theorem 1.2 agrees with Conjecture 1.1. In fact, under the assumption of GRH, we show that if supp($\hat{f}$) $\subset (-1, 1)$, then $D_1(\mathcal{F}(K); f)$ agrees with the prediction of the Ratios Conjecture down to an accuracy of size $O(M^{-n})$, for any $n \geq 2$. For conciseness, we only explicitly compute the first such lower order term.

To prove Theorem 1.2, we use the explicit formula to convert the sum over zeros of $L_k(s)$ into a sum over primes. The resulting expression involves a collection of terms: i) $W_f$ (as above) emerging from the infinite place of each $L_k(s)$ ii) a term $S_{\text{inert}}$ emerging from the inert primes iii) a term $S_{\text{ram}}$ emerging from the ramified prime, and iv) a term $S_{\text{split}}$ emerging from the split primes. We then show that

$$S_{\text{inert}} + S_{\text{ram}} = S_\zeta + S_L + S_{A'} + O\left(\frac{1}{K}\right),$$

where $K$ is as above. This agreement also confirms the findings in the function field setting, where the analogue to $\mathcal{F}(K)$ (the family of so-called super-even characters) is also found to have symplectic monodromy [18].
and apply our previous explicit computation of the right side of this equation. When \( \text{supp}(\hat{f}) \subset (-1, 1) \), we then demonstrate that the contribution from \( S_{\text{split}} \) is negligible. The breadth of the support of the test function in Theorem 1.2 is thus limited by our inability to explicitly compute \( S_{\text{split}} \) when \( \text{supp}(\hat{f}) \not\subset (-1, 1) \). Conditional on the Ratios Conjecture, however, we find that \( S_{\text{split}} = S_{\Gamma} \) for any admissible \( f \).

From Theorem 1.2 we moreover obtain the following corollary concerning the non-vanishing of \( L_k(1/2) \).

**Corollary 1.3.** Assume GRH. Then

\[
\lim_{K \to \infty} \frac{1}{|K|} \left| \{ L_k(s) \in \mathcal{F}(K) : L_k(1/2) \neq 0 \} \right| \geq \frac{3}{4} + o(1).
\]

In Section 2 we discuss Hecke characters \( \Xi_k(a) \) and their associated \( L \)-functions. In Section 3 we compute the Ratios Conjecture recipe for the family \( \mathcal{F}(K) \), and then in Section 4 we apply the result to obtain a conjecture for \( D_1(\mathcal{F}(K); f) \) with a power-savings error term. This conjecture is expressed via the collection of terms discussed above, which are then explicitly computed in Section 5, resulting in Conjecture 1.1. Finally, Section 6 provides a proof of Theorem 1.2 and of Corollary 1.3.

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### 2. Hecke Characters

The notation in this section draws from the general set-ups provided in [25] and [14] Ch. 5. Let \( \alpha \in \mathbb{Z}[i] \) be an element in the ring of Gaussian integers and let \( \theta_\alpha \) denote the argument of \( \alpha \). Consider the ideal \( \mathfrak{a} = \langle \alpha \rangle \subseteq \mathbb{Z}[i] \) generated by \( \alpha \). Since \( \mathbb{Z}[i] \) is a principal ideal domain, and the generators of \( \mathfrak{a} \) differ by multiplication by a unit \( \{ \pm 1, \pm i \} \in \mathbb{Z}[i]^\times \), we find that \( \theta_\mathfrak{a} := \theta_\alpha \) is well-defined modulo \( \pi/2 \). We may thus fix \( \theta_\mathfrak{a} \) to lie in \([0, \pi/2)\), which corresponds to choosing a generator \( \alpha \) that lies within the first quadrant of the complex plane.

A rational prime \( p \in \mathbb{Z} \) splits in \( \mathbb{Z}[i] \) if and only if \( p \equiv 1(4) \), in which case we factor \( \langle p \rangle = p_1p_2 \), where \( N(p_1) = N(p_2) = p \). The primes \( p \equiv 3(4) \) are inert, i.e. \( \langle p \rangle \subseteq \mathbb{Z}[i] \) remains prime in \( \mathbb{Z}[i] \), and \( N(\langle p \rangle) = p^2 \). The prime \( 2 = (1 + i)^2 \) is said to ramify in \( \mathbb{Z}[i] \), and \( N((1 + i)) = 2 \).
Consider the Hecke character
\[ \Xi_k(a) = \left(\frac{\alpha}{a}\right)^{2k} = e^{i4k\theta_a}, \quad k \in \mathbb{Z}, \]
and note that to each such character we may associate a degree 2 $L$-function
\[ L_k(s) = \sum_{a \in \mathbb{Z}[i], a \neq 0} \frac{\Xi_k(a)}{N(a)^s} = \prod_{p \text{ prime}} \left(1 - \frac{\Xi_k(p)}{N(p)^s}\right)^{-1}, \quad \text{Re}(s) > 1, \]
and that $L_k(s) = L_{-k}(s)$. Alternatively, we may choose to write $L_k(s)$ as a product over rational primes, namely
\[ L_k(s) = \prod_p L(s, k_p), \]
where
\[ (1 - \Xi_k(p_1)p^{-s})(1 - \Xi_k(p_2)p^{-s}) \quad \text{if} \quad p = p_1p_2 \equiv 1(4), \]
\[ (1 - p^{-s})(1 + p^{-s}) \quad \text{if} \quad p \equiv 3(4), \]
\[ (1 + (-1)^{k+1}2^{-s}) \quad \text{if} \quad p = 2. \]
In terms of the local roots $\alpha_k(p, j)$, this may be expressed as
\[ L(s, k_p)^{-1} = \prod_{j=1}^{2} \left(1 - \frac{\alpha_k(p, j)}{p^s}\right)^{-1}, \]
where at the ramified prime $p = 2$ we understand one of the two local roots to be zero. For Re($s$) > 1, we may then write
\[ \frac{L_k'(s)}{L_k(s)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)a_k(n)}{n^s} = -\sum_{n=1}^{\infty} \frac{c_k(n)}{n^s}, \tag{2.1} \]
where
\[ a_k(p^l) := \sum_{j=1}^{2} \alpha_k(p, j)^l, \]
and
\[ c_k(n) := \Lambda(n)a_k(n). \]
Explicitly,
\[ a_k(p^l) = \begin{cases} (-1)^{lk} & \text{if } p = 2 \\ 0 & \text{if } p \equiv 3(4), l \text{ is odd} \\ 2 & \text{if } p \equiv 3(4), l \text{ is even} \\ \Xi_k(p_1) + \Xi_k(p_2) & \text{if } p = p_1p_2 \equiv 1(4), \end{cases} \]
while
(2.2) \[ c_k(p^j) = \begin{cases} \frac{(-1)^k \Lambda(2^j)}{2} & \text{if } p = 2 \\ 0 & \text{if } p \equiv 3(4), l \text{ is odd} \\ 2 \cdot \Lambda(p^j) & \text{if } p \equiv 3(4), l \text{ is even} \\ (\Xi_k(p_1) + \Xi_k(p_2))\Lambda(p^j) & \text{if } p = p_1p_2 \equiv 1(4). \end{cases} \]

Next, we note that \( L_k(s) \) has conductor \( Q_k = 4 \) and gamma factor

\[ L(s, k, \infty) = \prod_{j=1}^{2} \Gamma(\sigma + \kappa_j), \]

where \( \kappa_1 := |2k| \) and \( \kappa_2 := |2k| + 1 \) are the local parameters of \( L_k(s) \) at infinity, and \( \Gamma(\sigma) := \pi^{-\sigma/2} \Gamma(\sigma/2) \). The completed \( L \)-function is then given by

\[ \Lambda_k(s) := Q_k^{\frac{s}{2}} \prod_{j=1}^{2} \Gamma(\sigma + \kappa_j) \cdot L_k(s) \]

\[ = \left( \frac{2}{\pi} \right)^s \pi^{-\left(\frac{1}{2} + |2k|\right)} \Gamma\left(\frac{s + |2k|}{2}\right) \Gamma\left(\frac{s + |2k| + 1}{2}\right) L_k(s). \]

Hecke showed that if \( k \neq 0 \), these functions have an analytic continuation to the entire complex plane. The root number of \( L_k(s) \) is \( \epsilon(k) = 1 \), which means that it satisfies the functional equation \( \Lambda_k(s) = \Lambda_k(1 - s) \). Moreover, by the Legendre Duplication Formula,

\[ \Gamma(s + |2k|) = \Gamma\left(\frac{s + |2k|}{2}\right) \Gamma\left(\frac{s + |2k| + 1}{2}\right) \pi^{-\frac{1}{2} s + |2k| - 1}, \]

which allows us to represent \( \Lambda_k(s) \) as

\[ \Lambda_k(s) = \Gamma(s + |2k|) L_k(s) 2^{1 - |2k|} \pi^{-s - |2k|}, \]

and yields a functional equation of the form

\[ (2.3) \quad L_k(s) = X_k(s) L_k(1 - s), \]

where

\[ X_k(s) := \pi^{2s - 1} \frac{\Gamma(1 - s + |2k|)}{\Gamma(s + |2k|)}. \]

\( \Lambda_k(s) \) has all its zeros in the critical strip \( 0 < \Re(s) < 1 \) (at the non-trivial zeros of \( L_k(s) \)), and the Generalized Riemann Hypothesis asserts that in fact they all lie on the critical line \( \Re(s) = 1/2 \).
3. Ratios Conjecture

3.1. The Recipe. The Ratios Conjecture described in [6] provides a procedure for calculating the ratio of a product of shifted $L$-functions averaged over a family. Let $L$ be an $L$-function, and $F = \{ f \}$ a family of characters with conductor $c(f)$, as defined in section 3 of [7]. $L(s,f)$ has an approximate functional equation given by

\begin{equation}
L(s,f) = \sum_{n<x} \frac{A_n(f)}{n^s} + X(f,s) \sum_{m<y} \frac{A_m(f)}{m^{1-s}} + \text{remainder}.
\end{equation}

Moreover, one may write

\begin{equation}
\frac{1}{L(s,f)} = \sum_{n=1}^{\infty} \frac{\mu_f(n)}{n^s},
\end{equation}

where the series converges absolutely for $\text{Re}(s) > 1$. To conjecture an asymptotic formula for the average

\[ \sum_{f \in F} L\left(\frac{1}{2} + \alpha, f\right) L\left(\frac{1}{2} + \gamma, f\right), \]

the Ratios Conjecture suggests the following recipe:

- Replace the $L$-function in the numerator with the two sums in its approximate functional equation, ignore the remainder term, and allow both sums to extend to infinity. Replace the $L$-function in the denominator by its series (3.2). Multiply out the resulting expression.
- Replace the $X(f,s)$ factor by its expected value when averaged over the family.
- Replace each summand by its expected value when averaged over the family.
- Call the total $R_F(\alpha, \gamma)$, and let $F = |F|$. Then for

\begin{equation}
-\frac{1}{4} < \text{Re}(\alpha) < \frac{1}{4}, \quad \frac{1}{\log F} \ll \text{Re}(\gamma), \quad \text{Im}(\alpha), \text{Im}(\gamma) \ll \epsilon F^{1-\epsilon},
\end{equation}

the conjecture is that

\[ \sum_{f \in F} \frac{L\left(\frac{1}{2} + \alpha, f\right)}{L\left(\frac{1}{2} + \gamma, f\right)} = \sum_{f \in F} R_F(\alpha, \gamma) \left(1 + O\left(e^{(-\frac{1}{2}+\epsilon)c(f)}\right)\right), \]

for all $\epsilon > 0$.

3.2. Approximate Functional Equation. We seek to apply the above procedure to compute the average

\[ R_K(\alpha, \gamma) = \frac{1}{K} \sum_{k=1}^{K} \frac{L_k\left(\frac{1}{2} + \alpha\right)}{L_k\left(\frac{1}{2} + \gamma\right)}, \]
for specified values of $\alpha$ and $\gamma$. As in [4], the approximate functional equation is given by

$$L_k(s) = \sum_{n<x} \frac{A_k(n)}{n^s} + \pi^{2s-1} \cdot \frac{\Gamma(1-s+|2k|)}{\Gamma(s+|2k|)} \sum_{m<y} \frac{\overline{A_k}(m)}{m^{1-s}} + \text{remainder},$$

where

$$A_k(n) = \sum_{N(a)=n} \Xi_k(a)$$

is a multiplicative function defined explicitly on prime powers as

$$A_k(p^l) = \begin{cases} \sum_{j=-l/2}^{l/2} \Xi_{2j}^k(p) & \text{if } p \equiv 1(4), l \text{ even} \\
\sum_{j=-(l+1)/2}^{(l-1)/2} \Xi_{2j+1}^k(p) & \text{if } p \equiv 1(4), l \text{ odd} \\
0 & \text{if } p \equiv 3(4), l \text{ odd} \\
1 & \text{if } p \equiv 3(4), l \text{ even} \\
(-1)^k & \text{if } p = 2. \end{cases}$$

Here $p|p$, and we note that the above formula is independent of our choice of such $p$. Ignoring the remainder term, and allowing both terms in (3.4) to sum towards infinity, we write

$$L_k(s) \approx \sum_{n=0}^{\infty} \frac{A_k(n)}{n^s} + \pi^{2s-1} \cdot \frac{\Gamma(1-s+|2k|)}{\Gamma(s+|2k|)} \sum_{m=0}^{\infty} \frac{A_k(m)}{m^{1-s}},$$

upon noting that $\overline{A_k}(n) = A_k(n)$ for all $A_k(n)$. Again as in [4], we note that

$$\frac{1}{L_k(s)} = \sum_h \frac{\mu_k(h)}{h^s},$$

where

$$\mu_k(p^h) := \begin{cases} 1 & h = 0 \\
-A_k(p) & h = 1 \\
-1 & h = 2, p \equiv 3(4) \\
1 & h = 2, p \equiv 1(4) \\
0 & \text{otherwise.} \end{cases}$$

Multiplying out the resulting expression yields

$$\prod_p \left( \sum_{n,h} \frac{\mu_k(p^h) A_k(p^n)}{p^{h(\frac{1}{2}+\gamma)+n(\frac{1}{2}+\alpha)}} \right) + \pi^{2\alpha} \cdot \frac{\Gamma(\frac{1}{2}+\gamma+2|k|)}{\Gamma(\frac{1}{2}+\gamma+|2k|)} \prod_p \left( \sum_{m,h} \frac{\mu_k(p^h) A_k(p^m)}{p^{h(\frac{1}{2}+\alpha)+m(\frac{1}{2}-\alpha)}} \right).$$

Next, set

$$k := \frac{1}{2} + |2k|.$$

The algorithm now dictates that we compute an asymptotic for
\[ \left\langle \frac{\Gamma(k-\alpha)}{\Gamma(k+\alpha)} \right\rangle_K \quad \text{and} \quad \left\langle \prod_p \left( \sum_{n,h} \mu_k(p^h) A_k(p^n) \right) \right\rangle_K, \]

where we denote \( \langle \cdot \rangle_K \) to refer to the average over all \( 1 \leq k \leq K \).

3.3. Averaging the \( \Gamma \)-Factors.

Lemma 3.1. Fix \( \text{Re}(\alpha) < 1/2 \). Then
\[
\left\langle \frac{\Gamma(k-\alpha)}{\Gamma(k+\alpha)} \right\rangle_K = \frac{(2K)^{-2\alpha}}{1-2\alpha} + O\left(\frac{1}{K}\right).
\]

Proof. By Stirling’s approximation,
\[
\log \frac{\Gamma(k-\alpha)}{\Gamma(k+\alpha)} = (k-\alpha) \log(k-\alpha) - (k+\alpha) \log(k+\alpha) + 2\alpha + \frac{1}{2} \left( \log(k+\alpha) - \log(k-\alpha) \right) + O\left(\frac{1}{k}\right).
\]

Taylor expanding the logarithm about \( k \) yields
\[
\log(k-\alpha) = \log k - \frac{\alpha}{k} + O\left(\frac{\alpha^2}{k}\right),
\]
so that
\[
\log \frac{\Gamma(k-\alpha)}{\Gamma(k+\alpha)} = (k-\alpha) \left( \log k - \frac{\alpha}{k} \right) - (k+\alpha) \left( \log k + \frac{\alpha}{k} \right) + 2\alpha + \frac{1}{2} \left( \log k + \frac{\alpha}{k} - \log k - \frac{\alpha}{k} \right) + O\left(\frac{1}{k}\right)
= -2\alpha \log k + O\left(\frac{1}{k}\right),
\]
and
\[
\frac{\Gamma(k-\alpha)}{\Gamma(k+\alpha)} = k^{-2\alpha} \left( 1 + O\left(\frac{1}{k}\right) \right).
\]

Setting \( K := \frac{1}{2} + 2K \), it follows that
\[
\frac{1}{K} \sum_{k=1}^{K} \frac{\Gamma(k - \alpha)}{\Gamma(k + \alpha)} = \frac{1}{2K} \int_1^K [k^{2\alpha} + O(k^{-2\alpha - 1})] \, dk + O \left( \frac{1 + K^{-2\alpha}}{K} \right)
\]

\[
= \frac{1}{2K} \left[ \frac{k^{1-2\alpha}}{1 - 2\alpha} + O \left( \frac{k^{-2\alpha}}{-2\alpha} \right) \right]_1^K + O \left( K^{-1} + K^{-2\alpha - 1} \right)
\]

\[
= (2K)^{-2\alpha} \left( \frac{2K + \frac{1}{2}}{2K} \right)^{1-2\alpha} + O \alpha (K^{-1})
\]

\[
= (2K)^{-2\alpha} + O \alpha (K^{-1}),
\]

from which Lemma 3.1 then follows. □

3.4. Averaging the Coefficients. Define

\[\delta_p(h, n) := \lim_{K \to \infty} \langle \mu_k(p^h) A(p^n) \rangle_K,\]

and write

\[
\delta_p(h, n) = \begin{cases} 
\delta_{3(4)}(h, n) & \text{when } p \equiv 3(4) \\
\delta_{1(4)}(h, n) & \text{when } p \equiv 1(4) \\
\delta_2(h, n) & \text{when } p = 2.
\end{cases}
\]

The effect of (3.7) is to limit the choices for \(p^h\) to \(h \in \{0, 1, 2\}\). \(A_k(p^n)\) is a sum of points on the unit circle, and the effect of averaging either \(A_k(p^n)\), \(\mu_k(p) A_k(p^n)\), or \(\mu_k(p^2) A_k(p^n)\), over \(1 \leq k \leq K\) is to eliminate any such terms which are not identically equal to 1. Collecting the significant terms, we then find that

\[
(3.8) \quad \delta_{3(4)}(h, n) = \begin{cases} 
1 & \text{if } h = 0, n \text{ is even} \\
-1 & \text{if } h = 2, n \text{ is even} \\
0 & \text{otherwise},
\end{cases}
\]

\[
(3.9) \quad \delta_{1(4)}(h, n) = \begin{cases} 
1 & \text{if } h = 0, n \text{ is even} \\
-2 & \text{if } h = 1, n \text{ is odd} \\
1 & \text{if } h = 2, n \text{ is even} \\
0 & \text{otherwise},
\end{cases}
\]

and

\[
(3.10) \quad \delta_2(h, n) = \begin{cases} 
1 & \text{if } h = 0, n \text{ is even} \\
-1 & \text{if } h = 1, n \text{ is odd} \\
0 & \text{otherwise}.
\end{cases}
\]

Next, define

\[
G_p(\alpha, \gamma) := \left( \sum_{h,n} \frac{\delta_p(h, n)}{p^{h(\frac{1}{2}+\gamma)+n(\frac{1}{2}+\alpha)}} \right).
\]
If \( p \equiv 3(4) \), we write
\[
G_p(\alpha, \gamma) = \left( \sum_{n \text{ even}} \frac{1}{p^{n(1/2+\alpha)}} - \sum_{n \text{ even}} \frac{1}{p^{1+2\gamma+n(1/2+\alpha)}} \right)
\]
(3.11)
\[
= \sum_{n=0}^{\infty} \left( \frac{1}{p^{n(1+2\alpha)}} - \frac{1}{p^{n(1+2\alpha)+1+2\gamma}} \right)
\]
\[
= \left( 1 - \frac{1}{p^{1+2\gamma}} \right) \left( 1 - \frac{1}{p^{1+2\alpha}} \right)^{-1}
\]
while if \( p \equiv 1(4) \),
\[
G_p(\alpha, \gamma) = \sum_n \left( \frac{\delta_p(0, n)}{p^{n(1/2+\alpha)}} + \frac{\delta_p(1, n)}{p^{n(1/2+\gamma+n(1/2+\alpha))}} + \frac{\delta_p(2, n)}{p^{1+2\gamma+n(1/2+\alpha)}} \right)
\]
(3.12)
\[
= \sum_{n=0}^{\infty} \left( \frac{1}{p^{n(1+2\alpha)}} - \frac{1}{p^{n(1+2\alpha)+2}} + \frac{1}{p^{n(1+2\alpha)+1+2\gamma}} \right)
\]
\[
= \left( 1 - \frac{2}{p^{1+\gamma+\alpha}} + \frac{1}{p^{1+2\gamma}} \right) \left( 1 - \frac{1}{p^{1+2\alpha}} \right)^{-1}.
\]
Lastly, for \( p = 2 \) we obtain
\[
G_2(\alpha, \gamma) = \sum_{n=0}^{\infty} \left( \frac{1}{2^{n(1+2\alpha)}} - \frac{1}{2^{1+\gamma+\alpha}} \frac{1}{2^{n(1+2\alpha)}} \right)
\]
(3.13)
\[
= \left( 1 - \frac{1}{2^{1+\gamma+\alpha}} \right) \left( 1 - \frac{1}{2^{1+2\alpha}} \right)^{-1}.
\]
Combining (3.11), (3.12), and (3.13), we conclude that
\[
G(\alpha, \gamma) := \prod_p G_p(\alpha, \gamma)
\]
\[
= \left( 1 - \frac{1}{2^{1+\gamma+\alpha}} \right) \prod_{p \equiv 1(4)} \left( 1 - \frac{2}{p^{1+\gamma+\alpha}} + \frac{1}{p^{1+2\gamma}} \right)
\]
\[
\times \prod_{p \equiv 3(4)} \left( 1 - \frac{1}{p^{1+2\gamma}} \right) \prod_p \left( 1 - \frac{1}{p^{1+2\alpha}} \right)^{-1}.
\]
Finally, assuming small fixed values of \( \text{Re}(\alpha), \text{Re}(\gamma) \), we factor out all terms which, for fixed \( p \), converge substantially slower than \( 1/p^2 \). We then conclude as follows:

**Conjecture 3.2.** With constraints on \( \alpha \) and \( \gamma \) as described in (3.3),
\[
R_K(\alpha, \gamma) = G(\alpha, \gamma) + \frac{1}{1-2\alpha} \left( \frac{\pi}{2K} \right)^{2\alpha} G(-\alpha, \gamma) + O(K^{-\frac{1}{2}+\epsilon}),
\]
where
\[ G(\alpha, \gamma) = Y(\alpha, \gamma) \times A(\alpha, \gamma) \]
is as above, and
\[ Y(\alpha, \gamma) := L(1 + 2\gamma, \chi_1)\zeta(1 + 2\alpha) / L(1 + \alpha + \gamma, \chi_1)\zeta(1 + \alpha + \gamma). \]
Here \( L(s, \chi_1) \) denotes the Dirichlet \( L \)-function corresponding to the non-principal character \( \chi_1 \in (\mathbb{Z}/4\mathbb{Z})^\times \), and \( A(\alpha, \gamma) := G(\alpha, \gamma) / Y(\alpha, \gamma) \) is an Euler product that converges for small fixed values of \( \text{Re}(\alpha) \) and \( \text{Re}(\gamma) \).

3.5. Taking the Derivative. Let \( R_K(\alpha, \gamma) \) be as above. Since differentiation operates linearly on finite sums, upon setting \( \gamma = \alpha = r \) we obtain
\[
1 \sum_{k=1}^{K} \frac{L_k'}{L_k} \left( \frac{1}{2} + r \right) = \frac{\partial}{\partial \alpha} R_K(\alpha, \gamma) \bigg|_{\gamma=\alpha=r}.
\]
By Conjecture 3.2
\[
\frac{d}{d\alpha} R_K(\alpha; \gamma) \bigg|_{\alpha=\gamma=r} = \frac{1}{K} \sum_{k=1}^{K} \left( \frac{d}{d\alpha} G(\alpha, \gamma) + \frac{1}{1 - 2\alpha} \left( \frac{\pi}{2K} \right)^{2\alpha} G(-\alpha, \gamma) \right) \bigg|_{\alpha=\gamma=r}
\]
up to an error term, and moreover
\[
\frac{d}{d\alpha} G(\alpha, \gamma) \bigg|_{\alpha=\gamma=r} = \frac{\zeta'}{\zeta}(1 + 2r) - \frac{L'}{L}(1 + 2r, \chi_1) + A'(r, r)
\]
upon noting that \( A(r, r) = 1 \). Similarly,
\[
\frac{d}{d\alpha} \frac{1}{1 - 2\alpha} \left( \frac{\pi}{2K} \right)^{2\alpha} G(-\alpha, \gamma) \bigg|_{\alpha=\gamma=r} = -\frac{1}{1 - 2r} \left( \frac{\pi}{2K} \right)^{2r} \frac{L(1 + 2r, \chi_1)}{L(1, \chi_1)} \zeta(1 - 2r) A(-r, r).
\]
Since the error term remains the same under differentiation we conclude as follows.

**Theorem 3.3.** Assuming Conjecture 3.2, \( 1/\log K < \text{Re}(r) < 1/4 \), and \( \text{Im}(r) \ll K^{-\epsilon} \), we have that
\[
1 \sum_{k=1}^{K} \frac{L_k'}{L_k} \left( \frac{1}{2} + r \right) = \frac{\zeta'}{\zeta}(1 + 2r) + A'(r, r) - \frac{L'}{L}(1 + 2r, \chi_1)
\]
\[
- \frac{1}{1 - 2r} \left( \frac{\pi}{2K} \right)^{2r} \frac{L(1 + 2r, \chi_1)}{L(1, \chi_1)} \zeta(1 - 2r) A(-r, r) + O(K^{-\frac{1}{2}+\epsilon}).
\]

4. One-Level Density Using the Ratios Conjecture

In this section, we use the Ratios Conjecture to compute \( D_1(\mathcal{F}(K); f) \). Let \( f \) be an even Schwarz function that is extended holomorphically to the strip
\(|\text{Im}(z)| < 2\), and recall that \(M = \log K\). By an application of Cauchy’s residue theorem, we find that

\[
D_1(\mathcal{F}(K); f) = \frac{1}{2\pi i} \frac{1}{K} \sum_{k=1}^{K} \left( \int_{(c)} - \int_{(1-c)} \right) \frac{L_k'(s)}{L_k(s)} \cdot f \left( \frac{-iM}{\pi} \left( s - \frac{1}{2} \right) \right) ds,
\]

where \(\text{Re}(s) = c\), and we chose \(1/2 + M^{-1} < c < 3/4\). Here \((c)\) denotes the vertical line from \(c - i\infty\) to \(c + i\infty\).

For the integral along the line \((1-c)\) we apply the change of variables \(s \mapsto 1 - s\), and find that

\[
D_1(\mathcal{F}(K); f) = \frac{1}{2\pi i} \frac{1}{K} \sum_{k=1}^{K} \int_{(c)} f \left( \frac{iM}{\pi} \left( s - \frac{1}{2} \right) \right) \frac{L_k'(s)}{L_k(s)} \cdot \frac{1}{(s - \frac{1}{2})} ds,
\]

upon recalling that \(f\) is even. By (2.3), we may moreover write

\[
D_1(\mathcal{F}(K); f) = \frac{1}{2\pi i} \int_{(c')} f \left( \frac{iMr}{\pi} \right) \left( \langle X_k' \rangle_K \left( \frac{1}{2} + r \right) - \frac{X_k'}{X_k} \left( \frac{1}{2} + r \right) \right) \frac{1}{\zeta(1 + 2r)} dr + O_\varepsilon \left( K^{\varepsilon-1} \right).
\]

Upon applying the change of variables \(\pi i\tau/M = r\), we arrive at the following conjecture.
Conjecture 4.1.

\[ D_1(\mathcal{F}(K); f) = W_f + S_\zeta + S_L + S_A' + S_\Gamma + O\left(K^{-\frac{1}{2} + \epsilon}\right), \]

where

\begin{align*}
W_f & := -\frac{1}{2M} \int_{(C')} f(\tau) \left\langle \frac{X'_k}{X_k} \left(\frac{1}{2} + \frac{\pi i \tau}{M}\right) \right\rangle_{K} d\tau, \\
S_\zeta & := \frac{1}{M} \int_{(C')} f(\tau) \frac{\zeta'}{\zeta} \left(1 + \frac{2\pi i \tau}{M}\right) d\tau, \\
S_L & := -\frac{1}{M} \int_{(C')} f(\tau) \frac{L'}{L} \left(1 + \frac{2\pi i \tau}{M}, \chi_1\right) d\tau, \\
S_A' & := \frac{1}{M} \int_{(C')} f(\tau) A' \left(\frac{\pi i \tau}{M}, \frac{\pi i \tau}{M}\right) d\tau, \\
S_\Gamma & := -\frac{1}{M} \int_{(C')} f(\tau) A \left(\frac{-\pi i \tau}{M}, \frac{-\pi i \tau}{M}\right) \left(\frac{\pi}{2K} \frac{2\pi i \tau}{L(1, \chi_1)} \frac{L(1 + \frac{2\pi i \tau}{M}, \chi_1)}{L(1, \chi_1)} \right) \left(1 - \frac{2\pi i \tau}{M}\right) d\tau,
\end{align*}

and \((C')\) denotes the horizontal line \(\text{Im}(\tau) = -Mc'/\pi\).

Before directly computing \(W_f, S_\zeta, S_A', S_L,\) and \(S_\Gamma,\) we note the following lemma, which will be relevant for our work in Section 6.

**Lemma 4.2.** We have

\begin{align*}
S_\zeta & = -\frac{1}{M} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n} \hat{f} \left(\log \frac{n}{M}\right), \\
S_L & = \frac{1}{M} \sum_{n=1}^{\infty} \frac{\Lambda(n)\chi_1(n)}{n} \hat{f} \left(\log \frac{n}{M}\right),
\end{align*}

and

\[ S_A' = -\frac{2}{M} \sum_{p \equiv 3(4)} \sum_{n=1}^{\infty} \frac{\log p}{p^{2n}} \hat{f} \left(\frac{2n \log p}{M}\right). \]

**Proof.** Write

\[ S_\zeta = -\frac{1}{M} \sum_{n} \frac{\Lambda(n)}{n} \int_{(C')} f(\tau) e^{-2\pi i \tau \frac{\log n}{M}} d\tau, \]

where we note that interchanging the order of integration and summation is justified since the summation inside the integral converges absolutely and uniformly on compact sets. Since the integrand is an entire function, we may shift the contour from \(C'\) to the line \(\text{Im}(\tau) = 0,\) and from which (4.10) then follows.
Equation (4.11) follows from an identical argument, upon noting that for $\text{Re}(z) > 1$, we may write

$$
\frac{L'(z, \chi_1)}{L(z, \chi_1)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n) \chi_1(n)}{n^z}.
$$

Finally, to compute $S_{A'}$, we write

$$
S_{A'} = \frac{2}{M} \sum_{p \equiv 3(4)} \sum_{n=1}^{\infty} \frac{\log p}{p^{2n}} \int_{(C')} e^{-(2\pi i \tau)(r_0 \log p)} f(\tau) d\tau,
$$

upon noting that

$$
A'(r, r) = -2 \sum_{p \equiv 3(4)} \frac{\log p}{p^{2+4r-1}}.
$$

(4.12) then follows upon shifting the contour to the line $\text{Im}(\tau) = 0$.

5. Explicit Computations

In this section, we explicitly compute $W_f$, $S_\zeta$, $S_{A'}$, $S_L$, and $S_\Gamma$, resulting in Conjecture 1.1.

5.1. Computing $W_f$:

**Lemma 5.1.** We have

$$
W_f = \hat{f}(0) + \frac{1}{M} (\log 2 - 1 - \log \pi) \hat{f}(0) + O \left( \frac{1}{K} \right).
$$

*Proof.* Since $X_k'/X_k$ is holomorphic within the horizontal strip $-Mc'/\pi \leq \text{Im}(\tau) \leq 0$, we may shift the line of integration to $\text{Im}(\tau) = 0$, and write

$$
W_f = \frac{1}{2M} \int_{\mathbb{R}} f(\tau) \left( \left\langle \frac{\Gamma'}{\Gamma} \left( k + \frac{\pi i \tau}{M} \right) + \frac{\Gamma'}{\Gamma} \left( k - \frac{\pi i \tau}{M} \right) \right\rangle_{K} - 2 \log \pi \right) d\tau.
$$

By Stirling’s approximation,

$$
\frac{\Gamma'}{\Gamma} \left( k + \alpha \right) + \frac{\Gamma'}{\Gamma} \left( k - \alpha \right) = \log |k + \alpha| + \log |k - \alpha| + O \left( \frac{1}{|k| + |\alpha|} \right)
$$

$$
= 2 \cdot \log |k + \alpha| + O \left( \frac{1}{|k| + |\alpha|} \right).
$$

Thus

$$
W_f = \frac{1}{M} \int_{\mathbb{R}} f(\tau) \left( \left\langle \log \left| k + \frac{\pi i \tau}{M} \right| \right\rangle_{K} - \log \pi \right) d\tau + O \left( \frac{1}{K} \right),
$$

where the error term is calculated upon noting that

$$
O \left( \frac{1}{K} \sum_{k=1}^{K} \frac{1}{|k|} \int_{\mathbb{R}} f(\tau) d\tau \right) = O \left( \frac{M}{K} \right).
$$
Next, write
\[
\int_{\mathbb{R}} f(\tau) \left( \log |k + \frac{\pi i \tau}{M}| \right) d\tau = \int_{\mathbb{R}} f(\tau) \left( \log \left| k \left( 1 + \frac{\pi i \tau}{k \cdot M} \right) \right| \right) d\tau,
\]
\[
= \tilde{f}(0) \cdot \log k + \int_{\mathbb{R}} f(\tau) \left( \log \left( 1 + \frac{\pi i \tau}{k \cdot M} \right) \right) d\tau,
\]
and split
\[
\int_{\mathbb{R}} f(\tau) \log \left( 1 + \frac{\pi i \tau}{k \cdot M} \right) d\tau = \left( \int_{|\tau| \leq \sqrt{M}} + \int_{|\tau| \geq \sqrt{M}} \right) f(\tau) \log \left| 1 + \frac{\pi i \tau}{k \cdot M} \right| d\tau.
\]
Since \(\log |1 + z| = |z| + O(z^2)\), we bound
\[
\int_{|\tau| \leq \sqrt{M}} f(\tau) \log \left| 1 + \frac{\pi i \tau}{k \cdot M} \right| d\tau = O \left( \frac{1}{k \sqrt{M}} \right),
\]
and by the rapid decay of \(f\),
\[
\int_{|\tau| \geq \sqrt{M}} f(\tau) \log \left| 1 + \frac{\pi i \tau}{k \cdot M} \right| d\tau = O \left( \frac{1}{M^{100}} \right).
\]
Thus
\[
\int_{\mathbb{R}} f(\tau) \left( \log |k + \frac{\pi i \tau}{M}| \right) d\tau = \tilde{f}(0) \cdot \log |k| + O \left( \frac{1}{k \cdot \sqrt{M}} \right),
\]
so that
\[
(5.1) \quad W_f = \frac{1}{M} \cdot \hat{f}(0) \left( \frac{1}{K} \sum_{k=1}^{K} \log |k| - \log \pi \right) + O \left( \frac{1}{K} \right).
\]
Finally, since
\[
\sum_{k=1}^{K} \log |2k| \leq \sum_{k=1}^{K} \log |k| \leq \sum_{k=2}^{K+1} \log |2k|,
\]
we have
\[
(5.2) \quad \sum_{k=1}^{K} \log |k| = \sum_{k=1}^{K} \log |k + \log 2| + O(\log K)
\]
\[
= \log K! + (\log 2)K + O(M)
\]
\[
= K \cdot M + (\log 2 - 1)K + O(M).
\]
Lemma 5.1 now follows from (5.1) and (5.2).

5.2. Computing \(S_\zeta\): 

Lemma 5.2. We have
\[
S_\zeta = -\frac{f(0)}{2} + \frac{1}{M} \cdot \gamma_0 \cdot \hat{f}(0) + O(M^{-2}),
\]
where \(\gamma_0\) is the Euler-Mascheroni constant.
Proof. Define
\[ G(x) := \sum_{n<x} \frac{\Lambda(n)}{n}, \]
and note that
\[ \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n} \hat{f} \left( \frac{\log n}{M} \right) = \int_{1}^{\infty} \hat{f} \left( \frac{\log x}{M} \right) dG(x) = -\int_{1}^{\infty} G(x) \left[ \hat{f} \left( \frac{\log x}{M} \right) \right]' dx. \]
By Merten’s theorem,
\[ G(x) = \log x - \gamma_0 + O \left( \frac{1}{\log x} \right), \]
from which it follows that
\[ \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n} \hat{f} \left( \frac{\log n}{M} \right) = -\int_{1}^{\infty} \left( \log x - \gamma_0 \right) \left[ \hat{f} \left( \frac{\log x}{M} \right) \right]' dx + O(M^{-1}). \]
By another application of summation by parts, we then find that
\[ \int_{1}^{\infty} \log x \left[ \hat{f} \left( \frac{\log x}{M} \right) \right]' dx = (\log x) \hat{f} \left( \frac{\log x}{M} \right) \big|_{1}^{\infty} - \int_{1}^{\infty} \hat{f} \left( \frac{\log x}{M} \right) \frac{dx}{x} \]
\[ = -M \int_{0}^{\infty} \hat{f}(u) du \]
\[ = -M \cdot \frac{f(0)}{2}, \]
on upon applying the change of variables \( u = \log x/M \). Moreover,
\[ \int_{1}^{\infty} \left[ \hat{f} \left( \frac{\log x}{M} \right) \right]' dx = \hat{f} \left( \frac{\log x}{M} \right) \big|_{1}^{\infty} = -\hat{f}(0), \]
from which it follows that
\[ S_\zeta = -\frac{1}{M} \left( M \cdot \frac{f(0)}{2} - \gamma_0 \cdot \hat{f}(0) \right) = -\frac{f(0)}{2} + \frac{\gamma_0}{M} \cdot \frac{f(0)}{M} + O(M^{-2}). \]
\[ \square \]

5.3. Computing \( S_L \) and \( S_A \): To compute \( S_L \) we require the following lemma:

Lemma 5.3. We have
\[ \frac{L'}{L}(1, \chi_1) = \left( \gamma_0 - 2 \log 2 - 4 \log |\eta(i)| \right), \]
where
\[ \eta(i) = e^{-\frac{\pi}{12}} \prod_{n=1}^{\infty} \left( 1 - e^{-2\pi n} \right) = \frac{\Gamma \left( \frac{1}{4} \right)}{2\pi^{3/4}}. \]
Proof. Upon inserting \( \tau = i \) into the Kronecker First Limit Formula \[21\], we obtain
\[
\frac{1}{4} \sum_{(m,n) \neq (0,0)} \frac{1}{(m^2 + n^2)^s} = \frac{\pi/4}{s-1} + \frac{\pi}{2} \left( \gamma_0 - \log 2 - 2 \log |\eta(i)| \right) + O(s-1),
\]
where
\[
\eta(\tau) = e^{i\pi \tau} \prod_{n=1}^{\infty} \left( 1 - e^{2\pi i n \tau} \right).
\]
Note that
\[
\frac{1}{4} \sum_{(m,n) \neq (0,0)} \frac{1}{(m^2 + n^2)^s} = \zeta(s) \cdot L(s, \chi_1)
\]
\[
= \left( \frac{\pi/4}{s-1} + \left( \frac{\pi}{4} \cdot \gamma_0 + L'(1, \chi_1) \right) + O(s-1) \right),
\]
since \( L(1, \chi_1) = \pi/4 \). By equating power series coefficients,
\[
L'(1, \chi_1) = \frac{\pi}{2} \left( \frac{\gamma_0}{2} - \log 2 - 2 \log |\eta(i)| \right),
\]
and we conclude that
\[
\frac{L'}{L}(1, \chi_1) = \frac{4}{\pi} \cdot \frac{\pi}{2} \left( \frac{\gamma_0}{2} - \log 2 - 2 \log |\eta(i)| \right)
\]
\[
= \gamma_0 - 2 \log 2 - 4 \log |\eta(i)|.
\]
Equation (5.3) follows from the Chowla-Selberg formula.

\( \square \)

Lemma 5.4. We have
\[
S_L = -\frac{\hat{f}(0)}{M} \left( \gamma_0 - 2 \log 2 - 4 \log |\eta(i)| \right) + O \left( \frac{1}{M^2} \right),
\]
and
\[
S_{\Lambda} = -\frac{2}{M} \sum_{p \equiv 3(4)} \frac{\log p}{p^2 - 1} \hat{f}(0) + O \left( \frac{1}{M^2} \right).
\]

Proof. By Cauchy’s residue theorem, we may shift the contour in (4.7) to the path \( C_0 \cup C_1 \), where
\[
C_0 := \{ \text{Im}(\tau) = 0, |\text{Re}(\tau)| \geq M^\epsilon \} \quad \text{and} \quad C_1 := \{ \text{Im}(\tau) = 0, |\text{Re}(\tau)| \leq M^\epsilon \},
\]
for some \( \epsilon > 0 \). By the rapid decay of \( f \), we then bound
\[
\int_{(C_0)} f(\tau) L' \left( 1 + \frac{2\pi i \tau}{M}, \chi_1 \right) d\tau \ll \frac{1}{M^{100\epsilon}} = O \left( \frac{1}{M^N} \right)
\]
for arbitrarily large \( N > 0 \).

Since \( C_1 \) is compact, we may switch the order of integration and summation.
and compute the integral along $C_1$ using the Taylor expansion of $L'/L$. This allows us to write
\[ \int_{(C_1)} f(\tau) \frac{L'}{L} \left(1 + i \frac{2\pi \tau}{M}, \chi_1 \right) d\tau = \frac{L'}{L}(1, \chi_1) \int_{(C_1)} f(\tau) d\tau + O \left( \frac{1}{M} \right). \]
By the rapid decay of $f$,
\[ \int_{(C_1)} f(\tau) d\tau = \int_{\mathbb{R}} f(\tau) d\tau + O \left( \frac{1}{M^{100}} \right), \]
so that
\[ \int_{(C_1)} f(\tau) \frac{L'}{L} \left(1 + i \frac{2\pi \tau}{M}, \chi_1 \right) d\tau = \tilde{f}(0) \frac{L'}{L}(1, \chi_1) + O \left( \frac{1}{M} \right). \]
Equation (5.4) then follows from Lemma 5.3. Equation (5.5) follows from an identical argument, upon noting that
\[ \frac{\partial}{\partial \alpha} A(\alpha, \alpha) \bigg|_{\alpha = 0} = -2 \sum_{p \equiv 3(4)} \log p p^2 - 1. \]

5.4. Computing $S_\Gamma$:

Lemma 5.5. We have the asymptotic formula
\[ S_\Gamma = \frac{f(0)}{2} - \frac{1}{2} \int_{\mathbb{R}} \tilde{f}(x) \cdot 1_{[-1,1]}(x) dx + \frac{d \cdot \tilde{f}(1)}{M} + O \left( \frac{1}{M^2} \right), \]
where $d$ is as in (1.4). In particular, for $\text{supp}(\tilde{f}) \subseteq (-1, 1)$, we have that $S_\Gamma \ll M^{-n}$ for any $n > 0$.

Proof. Set
\[ h(\tau) := \frac{A \left( \frac{\pi i \tau}{M}, \frac{\pi i \tau}{M} \right)}{1 - \frac{2\pi i \tau}{M}} \left( \frac{\pi}{2K} \right)^{\frac{2\pi i \tau}{M}} \frac{L \left(1 + \frac{2\pi i \tau}{M}, \chi_1 \right)}{L(1, \chi_1)} \zeta \left(1 - \frac{2\pi i \tau}{M} \right), \]
so that we may write
\[ S_\Gamma = -\frac{1}{M} \int_{(C')} f(\tau) h(\tau) d\tau. \]
We shift the contour of integration in $S_\Gamma$ to the path $C := C_0 \cup C_1 \cup C_\eta$, where
\[ C_0 := \{ \text{Im}(\tau) = 0, \text{Re}(\tau) \geq M^\delta \}, \quad C_1 := \{ \text{Im}(\tau) = 0, \eta \leq \text{Re}(\tau) \leq M^\delta \}, \]
and
\[ C_\eta := \{ \tau = \eta e^{i\theta}, \theta \in [-\pi, 0] \}. \]
For the part of the integral over $C_0$, we use the rapid decay of $f$, as well as upper bounds on the components of the integrand within the critical strip, to bound
\[ \int_{(C_0)} f(\tau) h(\tau) d\tau \ll \frac{1}{M^{107}}. \]

Since the integrand is uniformly continuous on the compact set $\mathcal{C}_1 \cup \mathcal{C}_\eta$, we then Taylor expand each component of the integrand, i.e. write

$$\frac{1}{1 - \frac{2\pi i \tau}{M}} \left( \frac{\pi}{2K} \right)^{\frac{2\pi i \tau}{M}} = e^{-2\pi i \tau} + e^{2\pi i \tau} \left( \log \frac{\pi}{2} + 1 \right) + O \left( \frac{\tau^2}{M^2} \right),$$

(5.6)

$$\zeta \left( 1 - \frac{2\pi i \tau}{M} \right) = \frac{M}{2\pi i} i + \gamma_0 + O \left( \frac{\tau}{M} \right),$$

(5.7)

$$L \left( \frac{1 + \frac{2\pi i \tau}{M}}{\log (1, \chi_1)} \right) = 1 + \frac{L'}{L} (1, \chi_1) \cdot \frac{2\pi i \tau}{M} + O \left( \frac{\tau^2}{M^2} \right),$$

(5.8)

and

$$A \left( -\frac{\pi i \tau}{M}, \frac{\pi i \tau}{M} \right) = 1 + \log 4 \cdot \frac{\pi i \tau}{M} + O \left( \frac{\tau^2}{M^2} \right),$$

(5.9)

where in (5.9) we note that

$$\left. \frac{\partial}{\partial \alpha} A(-\alpha, \alpha) \right|_{\alpha=0} = \log 4.$$

Combining equations (5.6) - (5.9), we find that

$$S_\Gamma = -\frac{1}{M} \int_{(\mathcal{C}_1 \cup \mathcal{C}_\eta)} f(\tau) h(\tau) d\tau + \left( \frac{1}{M^{100}} \right)$$

(5.10)

$$= -\frac{1}{M} \int_{(\mathcal{C}_1 \cup \mathcal{C}_\eta)} f(\tau) \left( e^{-2\pi i \tau} \left( 1 + \left( \log \frac{\pi}{2} + 1 \right) \frac{2\pi i \tau}{M} \right) \left( 1 + \frac{L'}{L} (1, \chi_1) \cdot \frac{2\pi i \tau}{M} \right) \right) \left( \frac{M}{2\pi i} i + \gamma_0 \right) \left( 1 + \log 2 \cdot \frac{2\pi i \tau}{M} \right) + O \left( \frac{\tau^2}{M^2} \right) d\tau + \left( \frac{1}{M^{100}} \right)$$

$$= \int_{(\mathcal{C}_1 \cup \mathcal{C}_\eta)} f(\tau) e^{-2\pi i \tau} \cdot \frac{d}{2\pi i \tau} d\tau + \frac{d}{M} \cdot \int_{(\mathcal{C}_1 \cup \mathcal{C}_\eta)} f(\tau) \cdot e^{-2\pi i \tau} d\tau + O \left( \frac{1}{M^2} \right)$$

$$= \int_{(\mathcal{C}_1 \cup \mathcal{C}_\eta)} f(\tau) e^{-2\pi i \tau} \cdot \frac{d}{2\pi i \tau} d\tau + \frac{d}{M} \cdot \hat{f}(1) + O \left( \frac{1}{M^2} \right),$$

where $d$ is as in (1.4), and where we note that by the holomorphy of the integrand, and the rapid decay of $f$,

$$\int_{(\mathcal{C}_1 \cup \mathcal{C}_\eta)} f(\tau) \cdot e^{-2\pi i \tau} d\tau = \hat{\xi}(1) + O \left( \frac{1}{M^{100}} \right).$$

Moreover, we write

$$\int_{(\mathcal{C}_1 \cup \mathcal{C}_\eta)} f(\tau) e^{-2\pi i \tau} \cdot \frac{d}{2\pi i \tau} d\tau = \frac{1}{2\pi i} \int_{(c)} f(\tau) e^{-2\pi i \tau} d\tau + O \left( \frac{1}{M^{100}} \right)$$

(5.11)

$$= J_1 + J_2 + O \left( \frac{1}{M^{100}} \right),$$
where
\[ J_1 := \frac{1}{2\pi i} \int_{(C)} \cos(2\pi \tau) \frac{f(\tau)}{\tau} d\tau, \]
and
\[ J_2 := -\int_{(C)} \frac{\sin(2\pi \tau)}{2\pi \tau} f(\tau) d\tau, \]
both of which are well-defined by the rapid decay of \( f \).

**Computing \( J_1 \):** Since the integrand in \( J_1 \) is odd, integration over the path \( C_0 \cup C_1 \) is zero. Hence
\[ J_1 = \frac{1}{2\pi i} \int_{C_0} \cos(2\pi \tau) \frac{f(\tau)}{\tau} d\tau = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(2\pi \eta e^{i\theta}) f(\eta e^{i\theta}) d\theta, \]
where we apply the change of variables \( \tau = \eta e^{i\theta} \). Since
\[ \cos(2\pi \eta e^{i\theta}) = 1 + O(\eta^2), \]
we find that in the limit as \( \eta \to 0 \),
\[ J_1 = \lim_{\eta \to 0} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\eta e^{i\theta}) d\theta = \frac{1}{2\pi} \cdot \pi f(0) = \frac{f(0)}{2}. \quad (5.12) \]

**Computing \( J_2 \):** Note that since \( \sin(2\pi \tau)/2\pi \tau \) is entire except for a removable singularity at \( \tau = 0 \), we may shift the line of integration to obtain
\[ J_2 = -\int_{\mathbb{R}} \frac{\sin(2\pi \tau)}{2\pi \tau} f(\tau) d\tau = -\frac{1}{2} \int_{-1}^{1} \tilde{f}(\tau) d\tau, \quad (5.13) \]
as in (1.5). This coincides with the second term in the Katz-Sarnak prediction. Since all the error terms are independent of \( \eta \), Lemma 5.5 then follows from (5.10), (5.11), (5.12), and (5.13).

Conjecture 1.1 now follows from Conjecture 4.1 and Lemmas 5.1, 5.2, 5.4, and 5.5. In particular, when \( \text{supp}(\tilde{f}) \subseteq (-1, 1) \), the conjecture implies that
\[ D_1(\mathcal{F}(K); f) = \tilde{f}(0) - \frac{f(0)}{2} + c \cdot \frac{\tilde{f}(0)}{M} + O\left(\frac{1}{M^2}\right), \]
in agreement with Theorem 1.2.

6. Proof of Theorem 1.2 and Corollary 1.3

6.1. Explicit Formula. Recall from (4.4) that
\[ D_1(\mathcal{F}(K); f) = \frac{1}{2\pi i} \int_{(c')} \left( 2 \left\langle \frac{L'_k}{L_k} \left(\frac{1}{2} + r\right) \right\rangle_K - \left\langle X'_{k_1}/X_k \left(\frac{1}{2} + r\right) \right\rangle_K \right) f\left(\frac{iMr}{\pi}\right) \, dr. \]
By (2.1), and an identical argument to that found in Lemma 4.2.
\[ \frac{1}{\pi i} \int_{(c')} L_k \left( \frac{1}{2} + r \right) f \left( \frac{iMr}{\pi} \right) \, dr = -\frac{1}{M} \sum_{n=1}^{\infty} \frac{c_k(n)}{\sqrt{n}} \hat{f} \left( \log \frac{n}{2M} \right). \]

It follows that the averaged scaled one-level density is given by

\[ D_1(F(K); f) = W_f + S_{\text{split}} + S_{\text{inert}} + S_{\text{ram}}, \quad (6.1) \]

where

\[ S_{\text{inert}} := -\frac{2}{M} \sum_{p \text{ prime}} \frac{\Lambda(p^l)}{p^{l/2}} \hat{f} \left( \frac{l \log p}{2M} \right), \]

\[ S_{\text{ram}} := -\frac{1}{M} \frac{1}{K} \sum_{k=1}^{K} \sum_{l=1}^{\infty} \frac{(-1)^{l-k} \Lambda(2^l)}{2^{l/2}} \hat{f} \left( \frac{l \log 2}{2M} \right), \]

and

\[ S_{\text{split}} := -\frac{1}{M} \sum_{p \text{ prime}} \frac{\Lambda(p^l)}{p^{l/2}} \hat{f} \left( \frac{l \log p}{2M} \right) \left( \langle \Xi_k(p^l_1) \rangle_K + \langle \Xi_k(p^l_2) \rangle_K \right). \]

We now proceed to compute \( S_{\text{inert}}, S_{\text{ram}}, \) and \( S_{\text{split}}. \)

**6.2. Ramified and Inert Primes.** We compute

\[ S_{\text{ram}} = -\frac{1}{M} \frac{1}{K} \sum_{k=1}^{K} \sum_{l=2 \text{ even}}^{\infty} \frac{(-1)^{l-k} \Lambda(2^l)}{2^{l/2}} \hat{f} \left( \frac{l \log 2}{2M} \right) \]

\[ = \frac{-1}{M} \frac{1}{K} \left( \sum_{k=1}^{K} \sum_{l=2 \text{ even}}^{\infty} \log 2 \hat{f} \left( \frac{l \log 2}{2M} \right) + \sum_{k=1}^{K} \sum_{l=1 \text{ odd}}^{\infty} \log 2 \hat{f} \left( \frac{l \log 2}{2M} \right) \right) \]

\[ = -\frac{\log 2}{M} \sum_{n=1}^{\infty} \frac{1}{2^n} \hat{f} \left( \frac{n \log 2}{M} \right) + O \left( \frac{1}{K} \right). \]

Upon noting that

\[ \chi_1(n) - 1 = \begin{cases} 
-2 & \text{if } n \equiv 3(4) \\
0 & \text{if } n \equiv 1(4) \\
-1 & \text{otherwise,}
\end{cases} \]
it follows from Lemma 4.2 that
\[ S_\zeta + S_L = \frac{1}{M} \sum_{n=1}^{\infty} \frac{\Lambda(n)(\chi_1(n) - 1)}{n} \hat{f} \left( \frac{\log n}{M} \right) \]
\[ = -\frac{2}{M} \sum_{n=3(4)}^{\infty} \frac{\Lambda(n)}{n} \hat{f} \left( \frac{\log n}{M} \right) - \frac{1}{M} \sum_{n=1}^{\infty} \log 2 \hat{f} \left( \frac{n \log 2}{M} \right) \]
\[ = -\frac{2}{M} \sum_{p=3(4)}^{\infty} \sum_{n=1}^{\infty} \frac{\log p}{p^n} \hat{f} \left( \frac{n \log p}{M} \right) + S_{\text{ram}} + O \left( \frac{1}{K} \right) \]
\[ = S_{\text{inert}} - S_{\mathcal{A}}' + S_{\text{ram}} + O \left( \frac{1}{K} \right), \]
yielding (1.7).

6.3. Split Primes. By (1.7) and (6.1) we obtain the following corollary:

**Corollary 6.1.** If Conjecture 4.1 is true, then

\[ S_{\text{split}} = S_\Gamma + O \left( e^{-\frac{M}{2}} \right) \]
\[ = \frac{f(0)}{2} - \frac{1}{2} \int_{\mathbb{R}} \hat{f}(x) \cdot 1_{[-1,1]}(x) \, dx + \frac{d}{M} \cdot \hat{f}(1) + O \left( \frac{1}{M^2} \right). \]

We now compute \( S_{\text{split}} \) when \( \text{supp}(\hat{f}) \subset (-1,1) \). First, bound

\[ S_{\text{split}} \ll \frac{1}{M} \sum_{a \in \mathbb{Z}[i]} \frac{\Lambda(N(a))}{N(a)^{1/2}} \hat{f} \left( \frac{\log N(a)}{2M} \right) \langle \Xi_k(a) \rangle_K. \]

Since \( \theta_a \notin \pi \mathbb{Z} \), we have that

\[ \langle \Xi_k(a) \rangle_K = \frac{1}{K} \sum_{k=1}^{K} e^{4ki\theta_a} = \frac{1}{K} \cdot \frac{\Xi_1(a) - \Xi_K(a)}{1 - \Xi_1(a)} \ll \frac{1}{K \theta_a}, \]
where we note that

\[ |1 - \Xi_1(a)| = 2| \sin(2\theta_a)| \gg \theta_a. \]

Moreover, by Lemma 2.1 in [24], we have that

\[ \theta_a \gg \frac{1}{\sqrt{N(a)}}, \]
i.e. the smallest possible angle attached to an ideal of norm $x$ is $\gg x^{-1/2}$.
Since $\text{supp}(\hat{f}) \subset (-\alpha, \alpha)$,

\[
S_{\text{split}} \ll \frac{1}{K} \frac{1}{M} \int_{1}^{\infty} \frac{\log x}{\sqrt{x}} \hat{f} \left( \frac{\log x}{2M} \right) \left( \int_{x^{-1/2}}^{x^{1/2}} \frac{d\theta}{\theta} \right) dx \\
\ll \frac{1}{K} \frac{1}{M} \int_{1}^{\infty} \frac{(\log x)^2}{\sqrt{x}} \hat{f} \left( \frac{\log x}{2M} \right) dx
\]

for any $\epsilon > 0$. In particular, we conclude that when $\text{supp}(\hat{f}) \subset (-1, 1)$,

(6.2) \quad \quad S_{\text{split}} = O \left( K^{-\delta} \right)

for some $\delta > 0$. Theorem 1.2 now follows upon collecting the results in 1.7, Lemma 5.1, Lemma 5.2, Lemma 5.4, (6.1), and (6.2).

6.3.1. Proof of Corollary 1.3

Proof. Let

\[
p_m(K) = \frac{1}{K} \# \{ L_k(s) \in \mathcal{F}(K) : \text{ord}_{s=\frac{1}{2}} L_k(1/2) = m \},
\]

and choose an admissible test function $f(x) \geq 0$ such that $f(0) = 1$. Then under the assumption of GRH,

\[
\sum_{m=1}^{\infty} m p_m(K) \leq D_1(\mathcal{F}(K); f).
\]

Moreover, by the symmetry of the functional equation, we find that the order of vanishing at the central point must be even, i.e. $p_{2m+1}(K) = 0$. If $\text{supp}(\hat{f}) \subset (-1, 1)$, we then obtain

\[
p_0(K) = 1 - \sum_{m=1}^{\infty} p_m(K) \geq 1 - \frac{1}{2} D_1(\mathcal{F}(K); f) = \frac{5}{4} - \frac{\hat{f}(0)}{2} + o(1),
\]

by Theorem 1.2 Corollary 1.3 then follows upon choosing the Fourier pair

\[
f(x) = \left( \frac{\sin \pi x}{\pi x} \right)^2, \quad \hat{f}(y) = (1 - |y|) \text{ if } |y| \leq 1.
\]

\[\square\]

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