OPTIMAL AVERAGE APPROXIMATIONS FOR FUNCTIONS MAPPING IN QUASI-BANACH SPACES

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Abstract. In 1994, M.M. Popov [6] showed that the fundamental theorem of calculus fails, in general, for functions mapping from a compact interval of the real line into the $\ell_p$-spaces for $0 < p < 1$, and the question arose whether such a significant result might hold in some non-locally convex spaces. In this article we completely settle the problem by proving that the fundamental theorem of calculus breaks down in the context of any non-locally convex quasi-Banach space. Our approach introduces the tool of Riemann-integral averages of continuous functions, and uses it to bring out to light the differences in behavior of their approximates in the lack of local convexity. As a by-product of our work we solve a problem raised in [1] on the different types of spaces of differentiable functions with values on a quasi-Banach space.

1. Introduction and background

Continuous maps from a compact interval of the real line into a Banach space $X$ are Riemann-integrable. For each $f : [a, b] \rightarrow X$, the optimal behavior of the averages

$$\frac{1}{t-s} \int_s^t f(u) \, du, \quad a \leq s < t \leq b,$$

in approximating $f$ locally in norm is substantiated by the fact that, thanks to the convexity of the space, for every point $c \in [a, b]$ we have

$$\left\| \frac{1}{t-s} \int_s^t f(u) \, du - f(c) \right\| = \left\| \int_s^t \frac{f(u) - f(c)}{t-s} \, du \right\| \leq \max_{u \in [s,t]} \| f(u) - f(c) \|,$$

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so that
\[
\lim_{(s, t) \to (c, c)} \frac{1}{t - s} \int_{s}^{t} f(u) \, du = f(c).
\]
In other words, the formula
\[
\text{Ave}[f](s, t) = \begin{cases} 
\frac{1}{t - s} \int_{s}^{t} f(u) \, du, & \text{if } a \leq s < t \leq b, \\
f(c), & \text{if } a \leq s = t = c \leq b, \\
\frac{1}{s - t} \int_{t}^{s} f(u) \, du, & \text{if } a \leq t < s \leq b,
\end{cases}
\]
(1.1)
defines a \textit{jointly continuous} function from \([a, b] \times [a, b]\) into \(X\). In particular, \(\text{Ave}[f]\) is \textit{bounded}, i.e.,
\[
\sup_{0 \leq s, t \leq 1} \| \text{Ave}[f](s, t) \| < \infty,
\]
and \textit{separately continuous}, i.e., for fixed \(s_0\) and \(t_0\) in \([a, b]\) we have
\[
\lim_{s \to s_0} \text{Ave}[f](s, t_0) = \lim_{t \to t_0} \text{Ave}[f](s_0, t) = \text{Ave}[f](s_0, t_0).
\]

The study of averages of continuous functions mapping into quasi-Banach spaces faces obstructions from the very beginning. Indeed, if \(X\) is non-locally convex, by an old result of Mazur and Orlicz \([3]\) there exist continuous \(X\)-valued functions failing to be Riemann-integrable. Thus, to extend the problem we suppose that \(f : [a, b] \to X\) is continuous and integrable, and wonder whether, with this extra assumption, the average function defined in (1.1) will retain the optimality it enjoys for Banach spaces. The answer to this question is negative, as the authors showed in [1].

\textbf{Theorem 1.1} ([1, Theorem 1.1]).\textit{ Suppose }\(0 < p < 1\).\textit{ Then there exists a continuous Riemann-integrable function }\(f : [0, 1] \to \ell_p\)\textit{ whose averages are bounded but fail to be optimal, in the sense that}
\[
\lim_{s \to 1^-} \text{Ave}[f](s, 1) \neq \text{Ave}[f](1, 1).
\]

Delving deeper into the subject leads to another less expected “pathology”, namely that the averages \(\text{Ave}[f](s, t)\) need not be bounded even in the case when \(\text{Ave}[f]\) is separately continuous.

\textbf{Theorem 1.2} (cf. [1, Theorem 4.1]).\textit{ Let }\(X\)\textit{ be a non-locally convex quasi-Banach space. Then there exists a continuous Riemann-integrable function }\(f : [0, 1] \to X\)\textit{ whose average function }\(\text{Ave}[f]\)\textit{ is separately continuous and yet }\(\sup_{0 \leq s, t \leq 1} \| \text{Ave}[f](s, t) \| = \infty\).
Hence, in particular this theorem yields the existence of continuous Riemann-integrable functions mapping into non-locally convex spaces whose averages are separately continuous but not jointly continuous.

Of course, by a straightforward scaling argument, Theorem 1.1 and Theorem 1.2 can be proved for any compact interval \([a, b]\) instead of \([0, 1]\). That being said, for simplicity we choose to work on the unit interval of the real line and from now on we will denote \([0, 1]\) by \(I\).

In view of the previous results one may ask whether the optimality of \(\text{Ave}[f]\) in approximating \(f\) will follow when the function \(\text{Ave}[f]\) is simultaneously bounded and separately continuous. Equipped with the machinery we will introduce in Section 2, in Section 3 we provide a negative answer to this question by proving the following theorem:

**Theorem 1.3.** Let \(X\) be a non-locally convex quasi-Banach space. Then there exists \(f : I \to X\) continuous and Riemann-integrable such that \(\text{Ave}[f]\) is bounded and separately continuous but fails to be jointly continuous.

We will also be able to extend Theorem 1.1 by replacing the space \(\ell_p\) for \(p < 1\) with any non-locally convex quasi-Banach space. That is,

**Theorem 1.4.** Let \(X\) be a non-locally convex quasi-Banach space. Then there exists \(f : I \to X\) continuous and Riemann-integrable such that \(\text{Ave}[f]\) is bounded but fails to be separately continuous.

If we put
\[
\mathcal{C}_R(I, X) = \{ f : I \to X : f \text{ is continuous and Riemann-integrable} \};
\]
\[
\mathcal{C}_{jc}(I, X) = \{ f \in \mathcal{C}_R(I, X) : \text{Ave}[f] \text{ is jointly continuous} \};
\]
\[
\mathcal{C}_{sc}(I, X) = \{ f \in \mathcal{C}_R(I, X) : \text{Ave}[f] \text{ is separately continuous} \};
\]
\[
\mathcal{C}_b(I, X) = \{ f \in \mathcal{C}_R(I, X) : \text{Ave}[f] \text{ is bounded} \},
\]

using function-theory terminology the above gives that, on the one hand,
\[
\mathcal{C}_{jc}(I, X) \subsetneq \mathcal{C}_{sc}(I, X) \cap \mathcal{C}_b(I, X) \subsetneq \mathcal{C}_b(I, X)
\]
and, on the other hand,
\[
\mathcal{C}_{sc}(I, X) \cap \mathcal{C}_b(I, X) \subsetneq \mathcal{C}_{sc}(I, X).
\]

In this paper we will complete our study by showing that
\[
\mathcal{C}_b(I, X) \cup \mathcal{C}_{sc}(I, X) \subsetneq \mathcal{C}_R(I, X),
\]
or, equivalently,

**Theorem 1.5.** Let \(X\) be a non-locally convex quasi-Banach space. Then there exists \(f : I \to X\) continuous and Riemann-integrable such that \(\text{Ave}[f]\) is neither bounded nor separately continuous.
It is worth it pointing out that Theorem 1.5 was proved for the particular case $X = \ell^p_{0 < p < 1}$ in [6, Theorem 2.1].

Roughly speaking, the moral of this article will be that to hope for good approximation properties by means of the Riemann integral in the setting of non-locally convex quasi-Banach spaces we need to impose more restrictions to the functions we are working with, a subject that we plan to investigate in a further publication.

From a different perspective, the authors introduced in [1] the following spaces. We will denote by $C^{(1)}(I, X)$ the space of all continuously differentiable functions from a compact interval of the real line into a quasi-Banach space $X$ which vanish at 0. $C^{(1)}_{Lip}(I, X)$ will consist of all $f \in C^{(1)}(I, X)$ which are Lipschitz, equipped with the quasi-norm

$$\|f\|_{Lip} = \sup_{0 \leq s < t \leq 1} \frac{\|f(t) - f(s)\|}{t - s}. \tag{1.2}$$

We will also consider the space $C^{(1)}_{Kal}(I, X)$ of all $f \in C^{(1)}_{Lip}(I, X)$ such that the function $g : I^2 \to X$ given by

$$g(s, t) = \begin{cases} f(s) - f(t) & s \neq t \\ f'(t) & s = t \end{cases}$$

is continuous. Of course, when $X$ is a Banach space, $C^{(1)}_{Kal}(I, X) = C^{(1)}_{Lip}(I, X) = C^{(1)}(I, X)$. However, if $X$ is a non-locally convex quasi-Banach space then $C^{(1)}_{Lip}(I, X) \subsetneq C^{(1)}(I, X)$ as was proved in [1]. As a consequence of Theorem 1.3 we can now complete the picture and state that $C^{(1)}_{Kal}(I, X) \subseteq C^{(1)}_{Lip}(I, X)$.

We refer to [1] for additional background and to [4,7] for the needed terminology and notation on quasi-Banach spaces.

2. A technique for customizing functions

Here and subsequently $(X, \| \cdot \|)$ will be a real quasi-Banach space. By the Aoki-Rolewicz theorem we will assume that the quasi-norm on $X$ is $p$-subadditive for some $0 < p \leq 1$, i.e.,

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p, \quad \forall x, y \in X.$$ 

The proof of our theorems in Section 3 relies essentially on the following construction, originally inspired by [6], which we adapted and extended in [1][2].

Let $\lambda = (\lambda_k)_{k=1}^\infty$ be a sequence of positive scalars with $\sum_{k=1}^\infty \lambda_k = 1$. Consider, for $n \in \mathbb{N}$, $t_n = \sum_{k=1}^n \lambda_k$, and $I_n = [t_{n-1}, t_n)$, so that $(I_k)_{k=1}^\infty$
is a sequence of disjoint intervals each one of length $\lambda_k$, whose union is $[0, 1)$.

For each $k \in \mathbb{N}$ let $f_{I_k}$ be the nonnegative piecewise linear function supported on the interval $I_k$ having a node at the midpoint of the interval $c_k = (t_k + t_{k-1})/2$ with $f_{I_k}(c_k) = 2$ and $f(t_{k-1}) = f(t_k) = 0$, i.e.,

$$f_{I_k} = \begin{cases} 
\frac{4}{t_k - t_{k-1}}(t - t_{k-1}) & \text{if } t \in [t_{k-1}, c_k), \\
\frac{4}{t_k - t_{k-1}}(t - t_k) & \text{if } t \in [c_k, t_k), \\
0 & \text{otherwise}.
\end{cases}$$

Let $x = (x_n)_{n=1}^{\infty}$ be a sequence of vectors in $X$. With these ingredients we define the function

$$f = f(\lambda, x) : I \to X$$

as

$$f(t) = \begin{cases} 
f_{I_k}(t)x_k & \text{if } t \in I_k, \\
0 & \text{if } t = 1.
\end{cases} \quad (2.1)$$

Note that for each $s < 1$ the set $f([0, s])$ lies on a finite-dimensional subspace of $X$. Hence, we can consider the corresponding average function $\text{Ave}[f]$ on $[0, 1) \times [0, 1)$ given by

$$\text{Ave}[f](s, t) = \begin{cases} 
\frac{1}{t-s} \int_s^t f(u) \, du, & \text{if } 0 \leq s < t < 1, \\
f(c), & \text{if } 0 \leq s = t = c < 1, \\
\frac{1}{s-t} \int_t^s f(u) \, du, & \text{if } 0 \leq t < s < 1.
\end{cases} \quad (2.2)$$

Clearly, $\text{Ave}[f]$ is jointly continuous on $[0, 1) \times [0, 1)$.

The next auxiliary lemma summarizes several results and ideas contained in [1][2].

**Lemma 2.1.** For a given pair $(\lambda, x)$ we have:

(i) The function $f = f(\lambda, x)$ is continuous on $I$ if and only if $x_k \to 0$.

(ii) Suppose that $X$ is $p$-convex. If $(x_k)_{k=1}^{\infty}$ is bounded and the sequence $(\lambda_k)_{k=1}^{\infty}$ verifies $\sum_{k=1}^{\infty} \lambda_k^p < \infty$, then $\sum_{k=1}^{\infty} \lambda_k x_k$ converges, $f$ is Riemann-integrable on $I$, and $\int_0^1 f(u) \, du = \sum_{k=1}^{\infty} \lambda_k x_k$. 


(iii) The function \( \text{Ave}[f] \) is bounded on \([0, 1) \times [0, 1)\) if and only if there is \( L > 0 \) so that for all integers \( m, n \) with \( m \leq n \),

\[
\frac{\left\| \sum_{m \leq k \leq n} \lambda_k x_k \right\|}{\sum_{m \leq k \leq n} \lambda_k} \leq L. \tag{2.3}
\]

(iv) Suppose \( x_k \to 0 \). Then \( \text{Ave}[f] \) extends to a separately continuous function on \( I^2 \) mapping the point \((1, 1)\) to \(0 \in X\) if and only if \( \sum_{k=1}^{\infty} \lambda_k x_k \) converges and

\[
\lim_{n \to \infty} \frac{\left\| \sum_{k \geq n} \lambda_k x_k \right\|}{\sum_{k \geq n} \lambda_k} = 0. \tag{2.4}
\]

Moreover, if \( \text{Ave}[f] \) extends to a separately continuous function on \( I^2 \) that maps the point \((1, 1)\) to a vector \( x \in X \), then \( \sum_{k=1}^{\infty} \lambda_k x_k \) converges and

\[
\lim_{n \to \infty} \frac{\sum_{k \geq n} \lambda_k x_k}{\sum_{k \geq n} \lambda_k} = x.
\]

(v) \( \text{Ave}[f] \) extends to a jointly continuous function on \( I^2 \) if and only if

\[
\lim_{m, n \to \infty} \frac{\left\| \sum_{m \leq k \leq n} \lambda_k x_k \right\|}{\sum_{m \leq k \leq n} \lambda_k} = 0. \tag{2.5}
\]

Proving our results requires rigging the technique for tailoring functions exhibited in this lemma by implementing one more layer of complexity in the choice of the sequences \((x_k)_{k=1}^{\infty}\) and \((\lambda_k)_{k=1}^{\infty}\) when \( X \) is non-locally convex.

In what follows, the notation \((\alpha_i) \approx (\beta_i)\) will be used in the regular sense that \( A \alpha_i \leq \beta_i \leq B \alpha_i \) for all \( i \), where \( A \) and \( B \) are some positive constants.
Let \( q \) be a positive integer. For any \((\mu_j)_{j=1}^q \in (0, \infty)\), and \((y_j)_{j=1}^q \in \mathcal{X}\) such that \(\sum_{j=1}^q \mu_j = 1\) and \(\|y_j\| \leq 1\), we have
\[
\left\| \sum_{j=1}^q \mu_j y_j \right\| \leq \left( \sum_{j=1}^q \mu_j^p \right)^{1/p} \leq q^{1/p - 1}.
\]

We set
\[
C_q = \sup \left\{ \left\| \sum_{j=1}^q \mu_j y_j \right\| : \mu_j > 0, \sum_{j=1}^q \mu_j = 1, y_j \in \mathcal{X}, \|y_j\| \leq 1 \right\}.
\]
Clearly \((C_q)_{q=1}^\infty\) is an increasing sequence and, if \(\mathcal{X}\) is not locally convex, \(C_q \to \infty\). Moreover \(C_q \leq q^{1/p - 1}\).

From our choice of \(C_q\), for each \( q \) there exist positive scalars \((\mu_{q,j})_{j=1}^q\) and vectors \((y_{q,j})_{j=1}^q \in \mathcal{X}\) such that \(\sum_{j=1}^q \mu_{q,j} = 1\), \(\|y_{q,j}\| \leq 1\) and
\[
\left\| \sum_{j=1}^q \mu_{q,j} y_{q,j} \right\| \geq \frac{C_q}{2}.
\]

Every natural number \( k \) can be written in a unique way in the form
\[
k = \frac{2q^2 + 1}{2} + \varepsilon \frac{2j - 1}{2}, \tag{2.6}
\]
for some \( q \in \mathbb{N}, \varepsilon \in \{-1, 1\} \) and \(1 \leq j \leq q\). In fact, for a fixed \( q \) we have
- the set \(\left\{ \frac{2q^2 + 1}{2} - \frac{2j - 1}{2} : 1 \leq j \leq q \right\}\) covers all the integers between \(q(q - 1) + 1\) and \(q^2\);
- the set \(\left\{ \frac{2q^2 + 1}{2} + \frac{2j - 1}{2} : 1 \leq j \leq q \right\}\) covers all the integers between \(q^2 + 1\) and \(q(q + 1)\),

so that the numbers \(\left\{ \frac{2q^2 + 1}{2} + \varepsilon \frac{2j - 1}{2} : 1 \leq j \leq q \right\}\) run over all the integers between \(q(q - 1) + 1\) and \(q(q + 1)\).

For each \( k \in \mathbb{N} \), let \( q = q(k), j = j(k), \) and \( \varepsilon = \varepsilon(k) \) uniquely determined by the representation \((2.6)\).

Let \( a = (A_q)_{q=1}^\infty \) and \( b = (\beta_q)_{q=1}^\infty \) be two sequences of positive scalars with \(\sum_{q=1}^\infty \beta_q = 1/2\). We consider \( x = (x_k)_{k=1}^\infty \) in \(\mathcal{X}\) given by
\[
x_k = \varepsilon A_q y_{q,j}, \tag{2.7}
\]
and \( \lambda = (\lambda_k)_{k=1}^\infty \) given by
\[
\lambda_k = \beta_q \mu_{q,j}. \tag{2.8}
\]
Note that
\[ \sum_{k=1}^{\infty} \sum_{q=1}^{\infty} \sum_{j=1}^{q} \mu_{q,j} \beta_q = 2 \sum_{q=1}^{\infty} \sum_{j=1}^{q} \mu_{q,j} \beta_q = 2 \sum_{q=1}^{\infty} \beta_q = 1, \]
so that we can define maps \( f : I \to X \) and \( \text{Ave}[f] : [0, 1) \times [0, 1) \to X \) as told in \( (2.1) \) and \( (2.2) \), respectively.

Note that now the maps \( f \) and \( \text{Ave}[f] \) depend also on the choice of the scalars \( \mu_{q,j} \) and the vectors \( y_{q,j} \). In order to avoid cumbersome notations and to emphasize the role of the sequences \( a \) and \( b \) in the construction of the specific functions that will best suit our needs in Section 3, in the next lemma we will refer to them as \( f_{a,b} \) and \( \text{Ave}[f_{a,b}] \).

**Lemma 2.2.** For a given pair \( (a, b) \) with \( a = (A_q)_{q=1}^{\infty} \) and \( b = (\beta_q)_{q=1}^{\infty} \), we have:

(i) If \( A_q \to 0 \), then \( f_{a,b} \) is continuous on \( I \).

(ii) Suppose that \( X \) is \( p \)-convex for some \( 0 < p \leq 1 \). If \( (A_q)_{q=1}^{\infty} \) is bounded and \( \sum_{q=1}^{\infty} q^{1-p} \beta_q^p < \infty \), then \( f_{a,b} \) is Riemann-integrable on \( I \) and \( \int_0^1 f_{a,b}(u) \, du = 0 \).

(iii) The average function \( \text{Ave}[f_{a,b}] \) is bounded on \( [0, 1) \times [0, 1) \) if and only if the sequence \( (A_q C_q)_{q=1}^{\infty} \) is bounded.

(iv) Suppose \( A_q \to 0 \). Then \( \text{Ave}[f_{a,b}] \) extends to a separately continuous function on \( I^2 \) if and only if
\[ \lim_{q \to \infty} \frac{A_q C_q \beta_q}{\sum_{r \geq q} \beta_r} = 0. \]
Moreover, in the positive case, the extended function maps the point \((1, 1)\) to \(0 \in X\).

(v) \( \text{Ave}[f_{a,b}] \) extends to a jointly continuous function on \( I^2 \) if and only if \( A_q C_q \to 0 \).

**Proof.** Let \( (\lambda_k)_{k=1}^{\infty} \) and \( (x_k)_{k=1}^{\infty} \) as defined in \( (2.8) \) and \( (2.7) \) respectively.

(i) is clear from Lemma 2.1(i).

(ii) follows from Lemma 2.1(ii) since
\[ \sum_{k=1}^{\infty} \lambda_k = \sum_{k=1}^{\infty} q \sum_{j=1}^{q} \sum_{\varepsilon = \pm 1} \mu_{q,j} \beta_q^p = 2 \sum_{k=1}^{\infty} q \sum_{j=1}^{q} \mu_{q,j} \beta_q^p = 2 \sum_{q=1}^{\infty} q^{1-p} \beta_q^p < \infty. \]

To show the other statements we need to estimate \( \| \sum_{m \leq k \leq n} \lambda_k x_k \| \).
First, we consider the case \( q(m) = q(n) = q \).
Suppose that \( m = (2q^2 + 1)/2 - (2j_0 - 1)/2 \) and \( n = (2q^2 + 1)/2 + (2j_1 - 1)/2 \). Then,

\[
\sum_{k=m}^{n} \lambda_k x_k = \min_{\{j_0, j_1\}} \sum_{i=1}^{\min\{j_0, j_1\}} \varepsilon A_q \beta_q \mu_{q, j} y_{q, j} + \max_{\{j_0, j_1\}} \sum_{i=1+\min\{j_0, j_1\}}^{\max\{j_0, j_1\}} A_q \beta_q \mu_{q, j} y_{q, j}
\]

\[
= \pm A_q \beta_q \sum_{i=1+\min\{j_0, j_1\}}^{\max\{j_0, j_1\}} \mu_{q, j} y_{q, j}.
\]

In particular, if \( j_0 = j_1 \), \( \sum_{k=m}^{n} \lambda_k x_k = 0 \). Therefore,

\[
\sum_{k=q(q+1)}^{q(q+1)+1} \lambda_k x_k = 0.
\]

Also,

\[
\left\| \sum_{k=q^2+1}^{q(q+1)} \lambda_k x_k \right\| \geq A_q D_q \beta_q. \tag{2.10}
\]

Suppose that \( m = (2q^2 + 1)/2 - (2j_0 - 1)/2 \), \( n = (2q^2 + 1)/2 - (2j_0 - 1)/2 \) (or \( m = (2q^2 + 1)/2 + (2j_0 - 1)/2 \), \( n = (2q^2 + 1)/2 + (2j_1 - 1)/2 \), with \( j_0 \leq j_1 \). Then,

\[
\sum_{k=m}^{n} \lambda_k x_k = \pm A_q \beta_q \sum_{i=j_0}^{j_1} \mu_{q, j} y_{q, j}.
\]

Since for any \( 1 \leq i_0 \leq i_1 \leq q \) we have

\[
\left\| \sum_{i=i_0}^{i_1} \mu_{q, j} y_{q, j} \right\| \leq C_{i_1 - i_0 + 1} \sum_{i=i_0}^{i_1} \mu_{q, j} \leq C_q \sum_{i=i_0}^{i_1} \mu_{q, j},
\]

we get that, in any case, for \( m \) and \( n \) such that \( q(m) = q(n) \),

\[
\left\| \sum_{k=m}^{n} \lambda_k x_k \right\| \leq A_q C_q \sum_{k=m}^{n} \lambda_k, \tag{2.11}
\]

and

\[
\left\| \sum_{k=m}^{n} \lambda_k x_k \right\| \leq A_q C_q \beta_q. \tag{2.12}
\]

Now we consider the case \( q_0 = q(m) < q(n) = q_1 \). Using (2.9), the \( p \)-subadditivity of the quasi-norm, and (2.11),

\[
\left\| \sum_{k=m}^{n} \lambda_k x_k \right\|^p = \left\| \sum_{k=m}^{q_0(q_0+1)} \lambda_k x_k + \sum_{k=q_1(q_1-1)+1}^{n} \lambda_k x_k \right\|^p.
\]
Using (2.3) with \( q_0(q_0 + 1) \) and \( \lambda_k x_k \)
From this estimate we obtain
Notice that both inequalities are true even if \( q \leq 1 \)
Combining (2.9) and (2.12), and taking into account that the functions
\[ \sum_{k=m}^{n} \lambda_k x_k \]
\[ \leq 2^{1/p - 1} \max \{ A_{q_0} C_{q_0}, A_{q_1} C_{q_1} \} \sum_{k=m}^{n} \lambda_k, \quad (2.13) \]
and
\[ \left\| \sum_{k=m}^{n} \lambda_k x_k \right\| \leq 2^{1/p} (A_{q_0} C_{q_0} \beta_{q_0} + A_{q_1} C_{q_1} \beta_{q_1}). \quad (2.14) \]
Notice that both inequalities are true even if \( q(n) = q(m) \).
Now we are ready to tackle (iii). Suppose that \( \operatorname{Ave}[f_{a,b}] \) is bounded. Using (2.3) with \( m = q^2 + 1, n = q(q + 1) \) in combination with (2.10) we obtain that \( (A_q C_q)_{q=1}^{\infty} \) is bounded.
Conversely, assume that there is a positive constant \( M \) such that \( A_q C_q \leq M \) for all \( q \in \mathbb{N} \). Let \( n, m \in \mathbb{N} \) with \( m \leq n \). Denote \( q_0 = q(m), q_1 = q(n) \). Inequality (2.13) yields
\[ \left\| \sum_{k=m}^{n} \lambda_k x_k \right\| \leq 2^{1/p - 1} M \sum_{k=m}^{n} \lambda_k. \]
To show (iv), assume that \( \operatorname{Ave}[f_{a,b}] \) extends to a separately continuous function from \( I^2 \) into \( X \). By (2.3) we have that \( \sum_{r=q(q-1)+1}^{\infty} \lambda_k x_k = 0 \). Hence, the extension must send the point \( (1,1) \) to 0 and so (2.4) holds. Putting \( n = q^2 + 1 \) and appealing to (2.9) and (2.10) we obtain
\[ \lim_{q \rightarrow q^2 + 1} \frac{A_q C_q \beta_q}{2(\beta_q + 2 \sum_{r>q} \beta_r)} = 0, \]
which yields the desired conclusion.
Let us show the converse. To see that \( \sum_{k=1}^{\infty} \lambda_k x_k \) is a Cauchy series, note that \( A_q C_q \beta_q \rightarrow 0 \) and use (2.14). Let \( n \in \mathbb{N} \) and consider \( q = q(n) \). Combining (2.9) and (2.12), and taking into account that the functions
of the form $t \mapsto \frac{t}{t+a}$ ($a > 0$) are increasing on $(0, \infty)$,

$$\left\| \sum_{k \geq n} \lambda_k x_k \right\| = \left\| \sum_{k \geq n} \lambda_k x_k \right\| \leq \frac{A_q C_q \sum_{k \geq n} \lambda_k^{(q+1)}}{q(q+1) \sum_{k \geq n} \lambda_k}$$

$$= \frac{A_q C_q \sum_{k \geq n} \lambda_k}{q(q+1) \sum_{k \geq n} \lambda_k + 2 \sum_{r > q} \beta_r} \leq \frac{A_q C_q \beta_q}{\sum_{r \geq q} \beta_r} \to 0.$$

(v) Suppose that Ave$[f_{a,b}]$ extends to a jointly continuous function on $I^2$. Using (2.5) with $m = q^2 + 1$ and $n = q(q + 1)$ and appealing to (2.10) we obtain $A_q C_q \to 0$. The converse follows from (2.13). □

3. COMPLETION OF THE PROOFS OF THE MAIN THEOREMS

Throughout this section $X$ will be a non-locally convex quasi-Banach space whose quasi-norm is assumed to be $p$-subadditive for some $p < 1$.

Proof of Theorem 1.3. Pick $b > 2(1-p)/p$ and let $a = (A_q)_{q=1}^\infty$ and $b = (\beta_q)_{q=1}^\infty$ be scalar sequences given by

$$A_q = \frac{1}{C_q}, \quad \beta_q = \frac{1}{2} \left( \frac{1}{q^b} - \frac{1}{(1+q)^b} \right), \quad q = 1, 2, \ldots,$$

where the sequence $(C_q)_{q=1}^\infty$ is defined in the forerunners of Lemma 2.2. Recall that $C_q \to \infty$ because $X$ is non-locally convex.

Notice that

$$\sum_{q=1}^\infty \beta_q = \frac{1}{2} \left( 1 - \lim_q \frac{1}{(1+q)^b} \right) = \frac{1}{2},$$

so that we can construct maps $f_{a,b}$ and Ave$[f_{a,b}]$ as in Section 2.

Since $A_q \to 0$, $f_{a,b}$ is continuous on $I$. That $f_{a,b}$ is Riemann-integrable on $I$ follows from Lemma 2.2(ii). Indeed, taking into account
that $\beta_q \approx q^{-b-1}$,

$$\sum_{q=1}^{\infty} q^{1-p} \beta_q^p \approx \sum_{q=1}^{\infty} q^{1-p} q^{-(b+1)p} = \sum_{q=1}^{\infty} \frac{1}{q^{bp+2p-1}},$$

and this last series converges because $bp + 2p - 1 > 2(1 - p) + 2p - 1 = 1$.

Now, we can use formula (1.1) to extend $\text{Ave}[f_{a,b}]$ to a function $F_{a,b}$ on the whole square $I^2$.

Notice that

$$\frac{A_q C_q \beta_q}{\sum_{r=q}^{\infty} \beta_r} = \frac{q^{-b} - (1 + q)^{-b}}{(q + 1)^{-b}} = \left(\frac{q + 1}{q}\right)^b - 1 \to 0.$$

By Lemma 2.2 (iv), $\text{Ave}[f_{a,b}]$ has a separately continuous extension to $I^2$ that maps the point $(1,1)$ to 0. This extension must be $F_{a,b}$.

That $F_{a,b}$ is bounded and discontinuous on $I^2$ follows from Lemma 2.2 (iii) and (v), respectively, since $A_q C_q = 1$.

**Proof of Theorem 1.4.** Let $a = (A_q)_{q=1}^{\infty}$ and $b = (\beta_q)_{q=1}^{\infty}$ be given by

$$A_q = \frac{1}{C_q}, \quad \beta_q = 2^{-q-1}, \quad q = 1, 2, \ldots$$

Notice that $\sum_{q=1}^{\infty} \beta_q = 1/2$ so that we can construct the corresponding maps $f_{a,b}$ and $\text{Ave}[f_{a,b}]$.

Since $A_q \to 0$, $f_{a,b}$ is continuous on $I$. Moreover,

$$\sum_{q=1}^{\infty} q^{1-p} \beta_q^p = 2^{-p} \sum_{q=1}^{\infty} q^{1-p} (2^{-p})^q < \infty,$$

and so $f_{a,b}$ is Riemann-integrable on $I$. Now, we can use (1.1) to define a map $F_{a,b}$ that extends $\text{Ave}[f_{a,b}]$ to $I^2$.

To prove that $F_{a,b}$ is bounded it suffices to see that $\text{Ave}[f_{a,b}]$ is bounded. But this follows from Lemma 2.2 (iii) since $A_q C_q = 1$.

We get easily

$$\frac{A_q C_q \beta_q}{\sum_{r=q}^{\infty} \beta_r} = \frac{2^{-q-1}}{2-q} = \frac{1}{2}.$$

By Lemma 2.2 (iv), $\text{Ave}[f_{a,b}]$ does not have a separately continuous extension to $I^2$. Hence $F_{a,b}$ is not separately continuous. □
Proof of Theorem 1.5. Let $a = (A_q)_{q=1}^\infty$ and $b = (\beta_q)_{q=1}^\infty$ be given by

$$A_q = \frac{1}{\sqrt{C_q}}, \quad \beta_q = 2^{-q-1}, \quad q = 1, 2, \ldots,$$

and construct the maps $f_{a,b}$ and $\text{Ave}[f_{a,b}]$. As in the proof of Theorem 1.4 the function $f_{a,b}$ is continuous and Riemann-integrable on $I$, so we are able to define $F_{a,b}$, which extends $\text{Ave}[f_{a,b}]$ to $I^2$.

In order to prove that $F_{a,b}$ is not bounded it suffices to see that $\text{Ave}[f_{a,b}]$ is not bounded. This follows from Lemma 2.2 (iii) since $A_q C_q = \sqrt{C_q} \to \infty$.

Finally, since

$$\sum_{r=q}^{\infty} \beta_r = \frac{2^{-q-1} \sqrt{C_q}}{2-q} = \frac{\sqrt{C_q}}{2} \to \infty,$$

Lemma 2.2 (iv) yields that $\text{Ave}[f_{a,b}]$ does not have a separately continuous extension to $I^2$. Hence $F_{a,b}$ is not separately continuous. □

Remark 3.1. Note that both Theorem 1.4 and Theorem 1.5 yield that given a non-locally convex quasi-Banach space $X$, there exists a continuous function $f : I \to X$ whose integral function $t \mapsto \int_0^t f(u) \, du$ fails to have a left derivative at 1. That is, the fundamental theorem of calculus breaks down, not only for $\ell_p$ when $p < 1$ as was showed by Popov in the aforementioned [6, Theorem 2.1], but for any non-locally convex quasi-Banach space!

Remark 3.2. The alert reader might wonder whether it is possible to define an integral for quasi-Banach spaces that interacts well with differentiation, in the sense that the fundamental theorem of calculus remains true. Vogt introduced in 1967 a concept of integrability quite different from that of Riemann specifically designed for $p$-Banach spaces with $p < 1$. We refer to [3] for details. As it happens, it has been recently shown in [3] that the Lebesgue differentiation theorem does hold for functions mapping in quasi-Banach spaces that are integrable in the sense of Vogt.

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