MODEL THEORETIC REFORMULATION OF THE
BAUM-CONNES AND FARRELL-JONES CONJECTURES

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Abstract. The Isomorphism Conjectures are translated into the language of
homotopical algebra, where they resemble Thomason’s descent theorems.

1. Introduction and statement of the results

In [8], Thomason establishes that algebraic $K$-theory satisfies Zariski and Nis-
nevich descent. This is now considered a profound algebraico-geometric property
of $K$-theory. In [1, 2], we have introduced the sister notion of codescent. Here, we
prove that each one of the so-called Isomorphism Conjectures (see [3, 5]) among

1. the Baum-Connes Conjecture,
2. the real Baum-Connes Conjecture,
3. the Bost Conjecture,
4. the Farrell-Jones Conjecture in $K$-theory,
5. the Farrell-Jones Conjecture in $L$-theory,

is equivalent to the codescent property for a suitable $K$- or $L$-theory functor.

For a (discrete) group $G$, these conjectures aim at computing, in geometrical and
topological terms, the groups $K^*_G(C^*_r G)$, $KO^*_G(C^*_r G)$, $K^*_G(\ell^1 G)$, $K^*_G(\mathbb{R}G)$ and
$L_*^{(-\infty)}(\Lambda G)$ respectively, where $\mathbb{R}$ and $\Lambda$ are associative rings with units, and $\Lambda$ is
equipped with an involution. Davis and Lück [4] express these conjectures as follows
(the equivalence with the original statements is due to Hambleton-Pedersen [6]).

First, fix one of the Conjectures (1)–(5) and denote by $K^*_G(G)$ the corresponding $K$-
or $L$-group among the five listed above (for (4) and (5), $\mathbb{R}$ and $\Lambda$ are understood).
Denote by $C := \text{Or}(G)$ the orbit category of $G$, whose objects are the quotients
$G/H$ with $H$ running among the subgroups of $G$, and the morphisms are the left-
$G$-maps. Let $D := \text{Or}(G, \mathcal{C})$ be the full subcategory of $\text{Or}(G)$ on those objects
$G/H$ for which $H$ is virtually cyclic. We sometimes write $C_G$ and $D_G$ to stress the
dependence on the group $G$. Then, a suitable functor $X_G : \mathcal{C} \to \mathcal{S}$ is constructed,
where $\mathcal{S}$ denotes the usual stable model category of spectra (of compactly generated
Hausdorff spaces), for which the weak equivalences are the stable ones. This functor
$X_G$ has the property that $\pi_*(X_G(G/H))$ is canonically isomorphic to $K_*^G(H)$ for all
$H \leq G$. Then, the fixed Isomorphism Conjecture for $G$ amounts to the statement
that the following composition, called assembly map, is a weak equivalence in $\mathcal{S}$:

\[ \mu^G : \operatorname{hocollim}_D \operatorname{res}_D^C X_G \to \operatorname{hocollim}_C X_G \xrightarrow{\sim} \operatorname{colim}_C X_G \xrightarrow{\sim} X_G(G/G). \]

We turn to homotopical algebra. First, we denote by $\mathcal{U}(\mathcal{C}, \mathcal{D})$ the model category
$\mathcal{S}^C$ of functors $\mathcal{C} \to \mathcal{S}$, where the weak equivalences and fibrations are the $\mathcal{D}$-weak
equivalences and $\mathcal{D}$-fibrations respectively, i.e. they are defined $\mathcal{D}$-objectwise. See

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details in [1, §3], for instance. For a diagram $X \in \mathcal{S}^C$, we let $\xi_X : QX \rightarrow X$ be the cofibrant replacement of $X$ in $\mathcal{U}_S(C, \mathcal{D})$. As in [1, §4], we say that $X$ satisfies $\mathcal{D}$-codescent if the map $\xi_X(c)$ is a weak equivalence in $\mathcal{S}$ for every $c \in C$; if this is only fulfilled at some $c_0 \in C$, we say that $X$ satisfies $\mathcal{D}$-codescent at $c_0$. For a conceptual approach to codescent and a parallel with descent, see [1, §§ 1 and 5]. Let $\mathcal{U}_S(C)$ be the model structure on $\mathcal{S}^C$ with the $C$-weak equivalences and $C$-fibrations; we define $\mathcal{U}_S(\mathcal{D})$ on $\mathcal{S}^D$ similarly. We denote by $\text{Ho}_S(C)$ and $\text{Ho}_S(\mathcal{D})$ the homotopy category of $\mathcal{U}_S(C)$ and $\mathcal{U}_S(\mathcal{D})$ respectively. As in [1, Prop. 13.2], we have the derived adjunction of the Quillen adjunction $\text{ind}^D_\mathcal{D} : \mathcal{U}(\mathcal{D}) \rightleftarrows \mathcal{U}(\mathcal{C}) : \text{res}^C_\mathcal{D}$, namely

$$\text{Lind}^D_\mathcal{D} : \text{Ho}_S(\mathcal{D}) \rightleftarrows \text{Ho}_S(C) : \text{Res}^C_\mathcal{D}.$$

For the sequel, fix a group $G$ and one of the Isomorphism Conjectures (1)–(5); let $X_G \in \mathcal{S}^C$ be the corresponding functor. Keep the other notations as above.

**Theorem 1.1.** The following statements are equivalent:

(i) $G$ satisfies the considered Isomorphism Conjecture;

(ii) the corresponding functor $X_G \in \mathcal{U}_S(C, \mathcal{D})$ satisfies $\mathcal{D}$-codescent at $G/G \in C$.

**Theorem 1.2.** For subgroups $L \leq H \leq G$, the following statements are equivalent:

(i) $X_H \in \mathcal{U}_S(C_H, \mathcal{D}_H)$ satisfies $\mathcal{D}_H$-codescent at $H/L \in C_H$;

(ii) $X_G \in \mathcal{U}_S(C_G, \mathcal{D}_G)$ satisfies $\mathcal{D}_G$-codescent at $G/L \in C_G$.

In fact, by general results of [1] (without invoking 1.1 above), if $X_G$ satisfies $\mathcal{D}_G$-codescent, then $X_H$ satisfies $\mathcal{D}_H$-codescent for every subgroup $H \leq G$.

**Main Theorem.** The following statements are equivalent:

(i) every subgroup $H$ of $G$ satisfies the considered Isomorphism Conjecture;

(ii) the corresponding functor $X_G \in \mathcal{U}_S(C, \mathcal{D})$ satisfies $\mathcal{D}$-codescent;

(iii) up to isomorphism, the image of $X_G$ in $\text{Ho}_S(C)$ belongs to $\text{Lind}^C_\mathcal{D} (\text{Ho}_S(\mathcal{D}))$.

Note that the usual Baum-Connes and Bost Conjectures are stated with finite subgroups instead of virtually cyclic ones, but this is known to be equivalent. So, in these cases, we could as well set $\mathcal{D}_G := \text{Or}(G, \text{Fin})$ instead of $\text{Or}(G, \mathcal{VC})$.

**Remark 1.3.** Let $X \in \mathcal{S}^C$ be a diagram and let $\zeta_X : QX \rightarrow X$ be an arbitrary cofibrant approximation of $X$ in $\mathcal{U}_S(C, \mathcal{D})$, namely, $\zeta_X$ is merely a $\mathcal{D}$-weak equivalence and $QX$ is cofibrant in $\mathcal{U}_S(C, \mathcal{D})$. Then, $X$ satisfies $\mathcal{D}$-codescent at some object $c \in C$ if and only if $\zeta_X(c)$ is a weak equivalence in $\mathcal{S}$, see [1, Prop. 6.5]. This illustrates the flexibility of the codescent-type reformulation of the Isomorphism Conjectures, namely, every such cofibrant approximation of $X_G$ yields a possibly different assembly map that can be used to test the considered conjecture.

2. The proofs

Let $\mathcal{Gpds}^f$ be the category of groupoids with faithful functors. For the considered conjecture, by [4, 7], there exists a homotopy functor $\mathcal{X} : \mathcal{Gpds}^f \rightarrow \mathcal{S}$, i.e. $\mathcal{X}$ takes equivalences of groupoids to weak equivalences, such that $X_G$ is the composite

$$X_G : \mathcal{C} = \text{Or}(G) \xrightarrow{\iota} \mathcal{Gpds}^f \xrightarrow{\mathcal{X}} \mathcal{S}.$$

The functor $\iota$ takes $G/H$ to its $G$-transport groupoid $G/H^G$ with the set $G/H$ as objects and with $\{ g \in G \mid g g_1 H = g_2 H \}$ as morphisms from $g_1 H$ to $g_2 H$. Moreover, the functor $\mathcal{X}$ takes values in cofibrant spectra, so that $X_G$ is $\mathcal{C}$-objectwise cofibrant.
Let \( \mathcal{C} \) be the category of small categories and \( s\mathcal{S}ets \) that of simplicial sets. Denote by \( \otimes_{\mathcal{D}} : s\mathcal{S}ets^{D^{op}} \times \mathcal{S}^{D} \rightarrow \mathcal{S} \) the tensor product over \( \mathcal{D} \) induced by the simplicial model structure on \( \mathcal{S} \), where \( K \in s\mathcal{S}ets \) “acts” on \( E \in \mathcal{S} \) by \( |K|_+ \wedge E \).

**Proof of Theorem 1.1.** *A priori*, to test whether \( X_G \) satisfies \( \mathcal{D} \)-codescent at some \( c \in \mathcal{C} \) requires a thorough understanding of the usually mysterious cofibrant replacement of \( X_G \). A key point here is the freedom to use *any* cofibrant approximation instead, see Remark 1.3. We provide in [2, §6] a general construction of cofibrant approximations in \( \mathcal{U}_{\mathcal{S}}(\mathcal{C}, \mathcal{D}) \), one of which is exactly suited for our present purposes [2, Cor. 6.9]. Evaluated at the terminal object \( G/G \in \mathcal{C} \), this cofibrant approximation \( \zeta_{X_G} : QX_G \rightarrow X_G \) is a certain map (described at the end of the proof)

\[
\zeta_{X_G}(G/G) : \quad QX_G(G/G) = B(? \setminus_{\mathcal{D}})^{op} \otimes_{\mathcal{D}} \text{res}_{\mathcal{D}}^{G} X_G(?) \rightarrow X_G(G/G).
\]

Indeed, using the notations of [2, Not. 6.1], this follows from the canonical identification \( (? \setminus_{\mathcal{D}} \otimes_{\mathcal{D}} G/G)^{op} = (? \setminus_{\mathcal{D}})^{op} \) of diagrams in \( \mathcal{Cat}^{D^{op}} \) and from the fact that \( X_G \) is \( \mathcal{C} \)-objectwise cofibrant. By definition of the homotopy colimit, we have

\[
B(? \setminus_{\mathcal{D}})^{op} \otimes_{\mathcal{D}} \text{res}_{\mathcal{D}}^{G} X_G(?) = \text{hocolim}_{\mathcal{D}} \text{res}_{\mathcal{D}}^{G} X_G.
\]

So, it suffices to show that \( \zeta_{X_G}(G/G) \) coincides with the assembly map \( \mu^G \). In the notations of [2, Not. 5.1], we have \( \text{mor}_{\mathcal{D}, \mathcal{C}}(?, G/G) = * \otimes_{\mathcal{D}} \text{res}_{\mathcal{D}}^{G} X_G \) (the constant diagram with value the point). By [2, Lem. 5.3], the spectrum \( * \otimes_{\mathcal{D}} \text{res}_{\mathcal{D}}^{G} X_G \) identifies with \( \text{ind}_{\mathcal{D}} \text{res}_{\mathcal{D}}^{G} X_G(G/G) \). Letting \( \epsilon \) denote the counit of the adjunction \( (\text{ind}_{\mathcal{D}}, \text{res}_{\mathcal{D}}) \), it is routine to verify that there is a canonical commutative diagram

\[
\begin{array}{ccc}
\text{hocolim}_{\mathcal{D}} \text{res}_{\mathcal{D}}^{G} X_G & \longrightarrow & B(? \setminus_{\mathcal{D}})^{op} \otimes_{\mathcal{D}} \text{res}_{\mathcal{D}}^{G} X_G(?) \\
\downarrow & & \downarrow \\
\text{hocolim}_{\mathcal{D}} \text{res}_{\mathcal{D}}^{G} X_G & \longrightarrow & \text{colim}_{\mathcal{D}} \text{res}_{\mathcal{D}}^{G} X_G \\
\downarrow & & \downarrow ^{\approx} \\
\text{hocolim}_{\mathcal{C}} X_G & \sim & \text{colim}_{\mathcal{C}} X_G
\end{array}
\]

\[
\text{colim}_{\mathcal{C}} X_G \sim X_G(G/G)
\]

The composition of the first column followed by the last row is the assembly map \( \mu^G \). The composition in the last column is \( \zeta_{X_G}(G/G) \), see [2, Cor. 6.9].

More generally, one can prove that the “\((X, \mathcal{F}, G)\)-Isomorphism Conjecture” of [4, Def. 5.1] is equivalent to \( X \) satisfying \( \text{Or}(G, \mathcal{F}) \)-codescent at \( G/G \), for any objectwise cofibrant diagram \( X \in S^{\text{Or}(G)} \) and any family \( \mathcal{F} \) of subgroups of \( G \).

For \( g \in G \) and \( H \trianglelefteq G \), we write \( gH := gHg^{-1} \). In the orbit category \( \text{Or}(G) = \mathcal{C}_G \), for an element \( g \in G \) such that \( gH \leq K \) for some subgroups \( H \) and \( K \) of \( G \), we designate by the right coset \( Kg \) the morphism \( G/H \rightarrow G/K \) taking \( gH \) to \( ggy^{-1}K \).

**Proof of Theorem 1.2.** Consider the functor \( \Phi : \mathcal{C}_H \rightarrow \mathcal{C}_G \) taking a coset \( H/L \in \mathcal{C}_H \) to \( G/L \). For any \( L \leq H \), we have canonical equivalences of groupoids in \( G\mathcal{P}ls^f \)

\[
\frac{H/L}{G/L} \rightleftarrows \frac{L}{G/L},
\]

where \( \underline{L} \) is \( L \) viewed as a one-object groupoid. Since \( \mathcal{X} \) is a homotopy functor, one checks that there is a canonical zig-zag of two \( \mathcal{C}_H \)-weak equivalences between \( X_H \) and \( \Phi^*X_G = X_G \circ \Phi \) in \( \mathcal{U}_{\mathcal{S}}(\mathcal{C}_H) \). By weak invariance of codescent [1, Prop. 6.10], \( X_H \) and \( \Phi^*X_G \) satisfy \( \mathcal{D}_H \)-codescent at exactly the same objects \( H/L \) of \( \mathcal{C}_H \).
Fix an object $H/K \in \mathcal{D}_H$. Let $E_{H/K} \subset G$ be a set of representatives for the quotient $H \backslash \{ g \in G \mid gK \leq H \}$. Let $M\gamma : \Phi(H/K) = G/K \to G/M = \Phi(H/M)$ be a morphism in $\mathcal{C}_G$ with $M \leq H$ (and $\gamma \in G$). It is straightforward that there is a unique pair $(g, Mh)$ with $g \in E_{H/K}$ and $Mh \in \text{mor}_{\mathcal{C}_G}(H/\gamma K, H/M)$ (namely characterized by $Hg = H\gamma$ and $Mh = M\gamma g^{-1}$) such that $M\gamma$ decomposes in $\mathcal{C}_G$ as

$$
\begin{array}{ccc}
G/K & \xrightarrow{Kg} & G/\gamma K \\
\downarrow \quad & & \downarrow \quad \\
G/M & \xrightarrow{Mh} & G/M
\end{array}
$$

Since $\Phi(\mathcal{D}_H) \subset \mathcal{D}_G$, this precisely says that $\Phi$ is a left glossy morphism of pairs of small categories in the sense of [1, Defs. 7.3 and 8.1]. By left glossy invariance of codescent [1, Thm. 9.14], $\Phi^* X_G$ satisfies $\mathcal{D}_H$-codescent at some $H/L \in \mathcal{C}_H$ if and only if $X_G$ satisfies $\mathcal{D}_1$-codescent at $G/L \in \mathcal{C}_G$, where $\mathcal{D}_1 := \Phi(\mathcal{D}_H)$. Set $\mathcal{D}_2 := \mathcal{D}_G$ and fix $H/L \in \mathcal{C}_H$. For $i = 1, 2$, consider the full subcategory $\mathcal{E}_i$ of $\mathcal{D}_i$ given by

$$
\mathcal{E}_i := \{ G/K \in \mathcal{D}_i \mid \text{mor}_{\mathcal{C}_G}(G/K, G/L) \neq \emptyset \}.
$$

By the Pruning Lemma [1, Thm. 11.5], $X_G$ satisfies $\mathcal{D}_1$-codescent at $G/L$ if and only if it satisfies $\mathcal{E}_1$-codescent at $G/L$. Since $L \leq H$, every object of $\mathcal{E}_1$ is isomorphic, inside $\mathcal{C}_G$, to some object of $\mathcal{E}_2$ and conversely; in other words, $\mathcal{E}_1$ and $\mathcal{E}_2$ are essentially equivalent in $\mathcal{C}_G$, in the sense of [1, Def. 3.12]. So, by [1, Prop. 10.1], $X_G$ satisfies $\mathcal{E}_1$-codescent at $G/L$ if and only if it satisfies $\mathcal{E}_2$-codescent at $G/L$. By the Pruning Lemma again, $X_G$ satisfies $\mathcal{E}_2$-codescent at $G/L$ if and only if it satisfies $\mathcal{D}_2$-codescent at $G/L$, i.e. $\mathcal{D}_G$-codescent at $G/L$.

In total, we have proven that $X_H$ satisfies $\mathcal{D}_H$-codescent at an object $H/L \in \mathcal{C}_H$ if and only if $X_G$ satisfies $\mathcal{D}_G$-codescent at $G/L$, as was to be shown.

\begin{proof}[Proof of the Main Theorem] The equivalence between (i) and (ii) follows from Theorems 1.1 and 1.2; (ii) and (iii) are equivalent by [1, Thm. 13.5]. \end{proof}

\begin{thebibliography}{99}

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