On the generation of some Lie-type geometries

This is a pre print version of the following article:

Original:
Cardinali, I., Giuzzi, L., Pasini, A. (2023). On the generation of some Lie-type geometries. JOURNAL OF COMBINATORIAL THEORY. SERIES A, 193 [10.1016/j.jcta.2022.105673].

Availability:
This version is available http://hdl.handle.net/11365/1216736 since 2022-09-28T13:23:17Z

Published:
DOI:10.1016/j.jcta.2022.105673

Terms of use:
Open Access
The terms and conditions for the reuse of this version of the manuscript are specified in the publishing policy. Works made available under a Creative Commons license can be used according to the terms and conditions of said license.
For all terms of use and more information see the publisher's website.

(Article begins on next page)
On the generation of some Lie-type geometries

Ilaria Cardinali\textsuperscript{a}, Luca Giuzzi\textsuperscript{b,*}, Antonio Pasini\textsuperscript{a,1}

\textsuperscript{a}Department of Information Engineering and Mathematics, University of Siena, Via Roma 56, I-53100, Siena, Italy
\textsuperscript{b}DICATAM - Section of Mathematics, University of Brescia, Via Branze 53, I-25123, Brescia, Italy

Abstract

Let $X_n(\mathbb{K})$ be a building of Coxeter type $X_n = A_n$ or $X_n = D_n$ defined over a given division ring $\mathbb{K}$ (a field when $X_n = D_n$). For a non-connected set $J$ of nodes of the diagram $X_n$, let $\Gamma(\mathbb{K}) = \text{Gr}_J(X_n(\mathbb{K}))$ be the $J$-grassmannian of $X_n(\mathbb{K})$. We prove that $\Gamma(\mathbb{K})$ cannot be generated over any proper sub-division ring $\mathbb{K}_0$ of $\mathbb{K}$. As a consequence, the generating rank of $\Gamma(\mathbb{K})$ is infinite when $\mathbb{K}$ is not finitely generated. In particular, if $\mathbb{K}$ is the algebraic closure of a finite field of prime order then the generating rank of $\text{Gr}_{1,n}(A_n(\mathbb{K}))$ is infinite, although its embedding rank is either $(n+1)^2 - 1$ or $(n+1)^2$.

Keywords: Coxeter building, $A_n$, $D_n$, Grassmann geometry, Subgeometry, Generation

2010 MSC: 51E24, 51M30, 51M35, 14M15

1. Introduction

We presume that the reader has some acquaintance with buildings and is familiar with basics of point-line geometry. In case, we refer to Tits [14] for buildings and Shult [13] for point-line geometries. We only recall the notion of generation in point-line geometries. A subspace of a point-line geometry $\Gamma = (P,L)$ is a subset $S$ of the point-set $P$ of $\Gamma$ such that, for every line $\ell \in L$, if $|S \cap \ell| > 1$ then $\ell \subseteq S$. We say that a subset $X$ of a subspace $S$ generates $S$, in symbols $S = \langle X \rangle$, if $S$ is the minimum subspace of $\Gamma$ containing $X$, namely $S$ is the intersection of all subspaces of $\Gamma$ which contain $X$. We also recall that the generating rank of a point-line geometry $\Gamma = (P,L)$ is the number $\text{gr}(\Gamma) := \min\{|X|: X \subseteq P, \langle X \rangle = P\}$.

1.1. Basic definitions and known results

Let $\mathcal{X}$ be a class of buildings such that, for every division ring $\mathbb{K}$, at most one (up to isomorphism) member of $\mathcal{X}$ is defined over $\mathbb{K}$. For instance, $\mathcal{X}$ can be the class of buildings belonging to a given simply laced Coxeter diagram or a given Dynkin diagram, possibly of twisted type. With $\mathcal{X}$ as above, let $\Delta(\mathbb{K})$ be the member of $\mathcal{X}$ defined over $\mathbb{K}$ (provided it exists) and, for a nonempty subset $J$ of the type-set of $\Delta(\mathbb{K})$, let $\text{Gr}_J(\Delta(\mathbb{K}))$ be the $J$-grassmannian of $\Delta(\mathbb{K})$, regarded as a point-line geometry. For a sub-division ring $\mathbb{K}_0$ of $\mathbb{K}$, suppose that $\mathcal{X}$ also contains a member $\Delta(\mathbb{K}_0)$.

*Corresponding author. Tel. +39 030 3715739; Fax. +39 030 3615745

Email addresses: ilaria.cardinali@unisi.it (Ilaria Cardinali), luca.giuzzi@unibs.it (Luca Giuzzi), antonio.pasini@unisi.it (Antonio Pasini)

\textsuperscript{1}(retired)

Preprint submitted to Elsevier July 15, 2022
defined over $\mathbb{K}_0$ and $\text{Gr}_J(\Delta(\mathbb{K}))$ contains $\text{Gr}_J(\Delta(\mathbb{K}_0))$ as a subgeometry (as it is always the case for the geometries to be considered in this paper). We say that $\text{Gr}_J(\Delta(\mathbb{K}))$ is generated over $\mathbb{K}_0$ ($\mathbb{K}_0$-generated for short) if $\text{Gr}_J(\Delta(\mathbb{K}_0))$, as a set of points of the point-line geometry $\text{Gr}_J(\Delta(\mathbb{K}))$, generates $\text{Gr}_J(\Delta(\mathbb{K}))$.

Clearly, if $\text{Gr}_J(\Delta(\mathbb{K}))$ is $\mathbb{K}_0$-generated and $\text{Gr}_J(\Delta(\mathbb{K}_0))$ is $\mathbb{K}_1$-generated for a division ring $\mathbb{K}_1 < \mathbb{K}_0$, then $\text{Gr}_J(\Delta(\mathbb{K}))$ is $\mathbb{K}_1$-generated too. It is also clear that if $\text{Gr}_J(\Delta(\mathbb{K}))$ is $\mathbb{K}_0$-generated then the generating rank of $\text{Gr}_J(\Delta(\mathbb{K}))$ cannot be larger than that of $\text{Gr}_J(\Delta(\mathbb{K}_0))$. On the other hand, suppose that every finite set of points of $\text{Gr}_J(\Delta(\mathbb{K}))$ belongs to a subgeometry of $\text{Gr}_J(\Delta(\mathbb{K}))$ isomorphic to $\text{Gr}_J(\Delta(\mathbb{K}_0))$ for a finitely generated sub-division ring $\mathbb{K}_0$ of $\mathbb{K}$ (as it is often the case). Suppose moreover that $\mathbb{K}$ is not finitely generated as a division-ring and that $\text{Gr}_J(\Delta(\mathbb{K}))$ is not $\mathbb{K}_0$-generated, for any $\mathbb{K}_0 < \mathbb{K}$. Then $\text{Gr}_J(\Delta(\mathbb{K}))$ has infinite generating rank, as we prove in Lemma 1.3.

In short, obvious links exist between the $\mathbb{K}_0$-generation problem and the computation of generating ranks. Less obviously, some relations also seem to exist between $\mathbb{K}_0$-generability and the existence of the absolutely universal embedding. For instance, a number of grassmannians $\text{Gr}_J(\Delta(\mathbb{K}))$ for which the existence of the absolutely universal embedding is still an open problem, cannot be generated over any proper sub-division ring $\mathbb{K}_0$ of $\mathbb{K}$ (see Section 1.3, Remark 1.6).

We shall now briefly summarize what is currently known about $\mathbb{K}_0$-generation. For $X_n$, a simply laced Coxeter diagram of rank $n$ or a Dynkin diagram of rank $n$ (but not of twisted type) and a division ring $\mathbb{K}$ (a field if $X_n \neq A_n$), let $X_n(\mathbb{K})$ be the unique building of type $X_n$ defined over $\mathbb{K}$. In particular, $B_n(\mathbb{K})$ and $C_n(\mathbb{K})$ are the buildings associated to the orthogonal group $O(2n+1,\mathbb{K})$ and the symplectic group $Sp(2n,\mathbb{K})$ respectively.

Suppose firstly that $\text{Gr}_J(X_n(\mathbb{K}))$ is spanned by $\text{Gr}_J(A)$ for an apartment $A$ of $X_n(\mathbb{K})$ (for short, $\text{Gr}_J(X_n(\mathbb{K}))$ is spanned by an apartment). For every sub-division ring $\mathbb{K}_0$ of $\mathbb{K}$, the geometry $\text{Gr}_J(A)$ is contained in a subgeometry of $\text{Gr}_J(X_n(\mathbb{K}))$ isomorphic to $\text{Gr}_J(X_n(\mathbb{K}_0))$. Hence $\text{Gr}_J(X_n(\mathbb{K}))$ is $\mathbb{K}_0$-generated for any $\mathbb{K}_0 \leq \mathbb{K}$. In particular, $\text{Gr}_J(X_n(\mathbb{K}))$ is generated over the prime subfield of $\mathbb{K}$.

It is known (Cooperstein and Shult [8], Blok and Brouwer [1]) that the following grassmannians are generated by apartments, where we take the integers $1, 2, \ldots, n$ as types as usual but when $X_n = D_n$, according to the notation adopted in Section 4.1, we replace $n - 1$ and $n$ with + and −: $\text{Gr}_k(A_n(\mathbb{K}))$ for $1 \leq k \leq n$; $\text{Gr}_1(D_n(\mathbb{K}))$ and $\text{Gr}_+(D_n(\mathbb{K}))$ as well as $\text{Gr}_-(D_n(\mathbb{K}))$; $\text{Gr}_1(C_n(\mathbb{K}))$ and $\text{Gr}_n(B_n(\mathbb{K}))$ but with char($\mathbb{K}$) ≠ 2 in both cases; $\text{Gr}_1(E_6(\mathbb{K}))$, $\text{Gr}_0(E_6(\mathbb{K}))$ and $\text{Gr}_1(E_7(\mathbb{K}))$ (the nodes of the $E_7$-diagram being labeled as in [8]). Therefore, all above mentioned grassmannians are generated over the prime subfield of $\mathbb{K}$. It is easily seen that the same holds for $\text{Gr}_1(B_n(\mathbb{K}))$, even if this geometry is not spanned by any apartment. It is likely that if char($\mathbb{K}$) ≠ 2 then, for every $i \leq n$, the $i$-grassmannian $\text{Gr}_i(C_n(\mathbb{K}))$ is generated over the prime subfield of $\mathbb{K}$, but we are not aware of any explicit proof of this claim.

We now turn to $\text{Gr}_{1,n}(A_n(\mathbb{K}))$. This geometry is interesting in its own right. When $\mathbb{K}$ is a field it is known as the long root geometry for SL($n + 1, \mathbb{K}$). In [2] it is proved that if $n > 2$ then $\text{Gr}_{1,n}(A_n(\mathbb{K}))$ is not $\mathbb{K}_0$-generated, for any proper sub-division ring $\mathbb{K}_0$ of $\mathbb{K}$ (see also [6], Theorem 5.10 for an alternative proof in the special case where $n = 3$ and $\mathbb{K}$ is a field). However, when $\mathbb{K}$ is a field and $\mathbb{K} = \mathbb{K}_0(a_1, \ldots, a_t)$ for suitable elements $a_1, \ldots, a_t \in \mathbb{K} \setminus \mathbb{K}_0$, then $\text{Gr}_{1,n}(A_n(\mathbb{K}_0))$ can be generated by adding at most $t$ elements to $\text{Gr}_{1,n}(A_n(\mathbb{K}_0))$ (Blok and Pasini [2]). In particular, when $\mathbb{K}$ is finite, $(n + 1)^2$ points are enough to generate $\text{Gr}_{1,n}(A_n(\mathbb{K}_0))$. Indeed, in this case $\mathbb{K}$ is a simple extension of its prime subfield $\mathbb{K}_0$ and the generating rank of $\text{Gr}_{1,n}(A_n(\mathbb{K}_0))$ is equal to $(n + 1)^2 - 1$ (Cooperstein [7]).

Not much is known on $\text{Gr}_k(B_n(\mathbb{K}))$ for $1 < k < n$ and $\text{Gr}_k(D_n(\mathbb{K}))$ for $1 < k \leq n - 2$. Probably,
what makes these cases so difficult to investigate is the fact that the special case $\text{Gr}_{1,3}(A_3(\mathbb{K})) \cong \text{Gr}_{+,-}(D_4(\mathbb{K}))$ of $\text{Gr}_{1,n}(A_n(\mathbb{K}))$ somehow enters the game in any attempt to compute the generating rank of $\text{Gr}_k(B_n(\mathbb{K}))$ or $\text{Gr}_k(D_n(\mathbb{K}))$ and, as we have seen above, as far as generation is concerned, $\text{Gr}_{1,n}(A_n(\mathbb{K}))$ can behave wildly. Nevertheless, in [6] we have shown that for $\mathbb{K} = \mathbb{F}_4, \mathbb{F}_8$ or $\mathbb{F}_9$ the grassmannians $\text{Gr}_2(B_n(\mathbb{K}))$ ($n \geq 3$) and $\text{Gr}_2(D_n(\mathbb{K}))$ ($n > 3$) are generated over the corresponding prime subfields $\mathbb{F}_2$ or $\mathbb{F}_3$. The generating ranks of $\text{Gr}_2(B_n(\mathbb{K}_0))$ and $\text{Gr}_2(D_n(\mathbb{K}_0))$, for $\mathbb{K}_0$ a finite field of prime order, are known to be equal to $\binom{2n+1}{2}$ and $\binom{2n}{2}$ respectively (Cooperstein [7]). Hence $\binom{2n+1}{2}$ and $\binom{2n}{2}$ are the generating ranks of $\text{Gr}_2(B_n(\mathbb{K}))$ and $\text{Gr}_2(D_n(\mathbb{K}))$ respectively, with $\mathbb{K}$ as above.

1.2. Main results

We state in this subsection the main results of this paper. We refer to Section 2 or to [11] and [4] for the notation and further details on the grassmannians we are dealing with. The following, to be proved in Section 3, is our first main result in this paper:

**Theorem 1.1.** For a division ring $\mathbb{K}$, let $\Gamma(\mathbb{K})$ be one of the following: $\text{Gr}_{1,n}(A_n(\mathbb{K}))$ for $n \geq 3$; $\text{Gr}_{+,-}(D_n(\mathbb{K}))$, $n \geq 3$; $\text{Gr}_{1,+,-}(D_n(\mathbb{K}))$ with $n \geq 4$; $\text{Gr}_{1,-}(D_n(\mathbb{K}))$ for $n \geq 4$. Then $\Gamma(\mathbb{K})$ is not $\mathbb{K}_0$-generated for any proper sub-division ring $\mathbb{K}_0$ of $\mathbb{K}$.

As said in Section 1.1 the case of $\text{Gr}_{1,n}(A_n(\mathbb{K}))$ has been already considered in [2], but the proof we shall give in this paper is different and simpler than that of [2]. Theorem 1.1 also contains a proof of a conjecture presented in [6, Remark 5.11].

**Corollary 1.2.** The $(n-1)$-grassmannian $\text{Gr}_{n-1}(B_n^+(\mathbb{K}))$ of the top-thin polar space $B_n^+(\mathbb{K}) = \text{Gr}_1(D_n(\mathbb{K}))$ is not $\mathbb{K}_0$-generated for any proper subfield $\mathbb{K}_0$ of $\mathbb{K}$.

Seemingly, this corollary is an obvious consequence of Theorem 1.1 and the isomorphism $\text{Gr}_{n-1}(B_n^+(\mathbb{K})) \cong \text{Gr}_{+,-}(D_n(\mathbb{K}))$. However its proof is not so trivial as one might believe; we will give it in Section 3.2 As we shall see in Section 3.3 Theorem 1.1 admits the following far reaching generalization:

**Theorem 1.3.** Let $\Gamma(\mathbb{K})$ be either $\text{Gr}_J(A_n(\mathbb{K}))$ or $\text{Gr}_J(D_n(\mathbb{K}))$, with $J$ a non-connected set of nodes of the diagram $A_n$ or $D_n$ respectively. Then $\Gamma(\mathbb{K})$ is not $\mathbb{K}_0$-generated, for any proper sub-division ring $\mathbb{K}_0$ of $\mathbb{K}$.

1.3. Applications to generating ranks and embeddings

Given a point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$, a (full) projective embedding $e : \Gamma \to \text{PG}(V)$ of $\Gamma$ (henceforth often called simply an embedding of $\Gamma$, for short) is an injective map $e : \mathcal{P} \to \text{PG}(V)$ from the point-set $\mathcal{P}$ of $\Gamma$ to the set of points of the projective space $\text{PG}(V)$ of a vector space $V$, such that for every line $\ell \in \mathcal{L}$ of $\Gamma$ the set $e(\ell) := \{e(p) : p \in \ell\}$ is a projective line of $\text{PG}(V)$ and $e(\mathcal{P})$ spans $\text{PG}(V)$. We put $\dim(e) := \dim(V)$, calling $\dim(e)$ the dimension of $e$. If $\mathbb{K}$ is the underlying division ring of $V$, we say that $e$ is defined over $\mathbb{K}$, also that $e$ is a $\mathbb{K}$-embedding. If $\Gamma$ admits a projective embedding we say that $\Gamma$ is projectively embeddable (also embeddable, for short). If $e : \Gamma \to \text{PG}(V)$ and $e' : \Gamma \to \text{PG}(V')$ are two $\mathbb{K}$-embeddings of $\Gamma$ we say that $e$ dominates $e'$ if there is a $\mathbb{K}$-semilinear mapping $\varphi : V \to V'$ such that $e' = \varphi \cdot e$. If $\varphi$ is an isomorphism then we say that $e$ and $e'$ are isomorphic. Following Tits [14], we say that an embedding $e$ is dominant if, modulo isomorphisms, it is not dominated by any embedding other than itself. Every $\mathbb{K}$-embedding $e$ of $\Gamma$ admits a hull $\hat{e}$, uniquely determined up to isomorphism and characterized by the following property: $\hat{e}$ dominates all $\mathbb{K}$-embeddings of $\Gamma$ which dominate $e$ (see Ronan [12]). Accordingly, an
embedding is dominant if and only if it is the hull of at least one embedding; equivalently, if and only if it is its own hull. Finally, an embedding \( \tilde{e} \) of \( \Gamma \) is absolutely universal (henceforth called just universal, for short) if it dominates all embeddings of \( \Gamma \). In other words, \( \Gamma \) admits the universal embedding if and only if all of its embeddings have the same hull, that common hull being the universal embedding of \( \Gamma \). Note that this forces all embeddings of \( \Gamma \) to be defined over the same division ring. Note also that the universal embedding, if it exists, is homogeneous, an embedding \( e \) of \( \Gamma \) being homogeneous if \( eg \cong e \) for every automorphism \( g \) of \( \Gamma \). The embedding rank \( \text{er}(\Gamma) \) of an embeddable geometry \( \Gamma \) is defined as follows:

\[
\text{er}(\Gamma) := \sup \{ \dim(\varepsilon) : \varepsilon \text{ projective embedding of } \Gamma \}.
\]

Obviously, if \( \Gamma \) admits the universal embedding \( \tilde{e} \) then \( \text{er}(\Gamma) = \dim(\tilde{e}) \), but \( \text{er}(\Gamma) \) is defined even if no embedding of \( \Gamma \) is universal. If \( e : \Gamma \to \text{PG}(V) \) is an embedding of \( \Gamma = (P, L) \) then stretching a line in \( \Gamma \) through two collinear points \( p, q \in P \) corresponds to forming the span \( \langle v, w \rangle \subseteq V \) of any two non-zero vectors \( v \in e(p) \) and \( w \in e(q) \). If \( X \subseteq P \) generates \( \Gamma \) then \( P = \bigcup_{n=0}^{\infty} X_n \) where \( X_0 := X \) and \( X_{n+1} := \cup_{p,q \in X_n} \langle p, q \rangle \). Consequently, if we select a non-zero vector \( v_p \in e(p) \) for every point \( p \in X \) then \( \{v_p\}_{p \in X} \) spans \( V \). This makes it clear that \( |X| \geq \dim(e) \). Accordingly,

\[
\dim(e) \leq \text{gr}(\Gamma). \tag{1}
\]

Therefore, if \( \text{gr}(\Gamma) \) is finite and \( \dim(e) = \text{gr}(\Gamma) \) then \( e \) is dominant (hence universal, if \( \Gamma \) admits the universal embedding). In any case, (1) implies the following:

\[
\text{er}(\Gamma) \leq \text{gr}(\Gamma). \tag{2}
\]

In fact the equality \( \text{er}(\Gamma) = \text{gr}(\Gamma) \) holds for many embeddable geometries, but not for all of them. For instance Heiss [9] gives an example where \( \text{gr}(\Gamma) = \text{er}(\Gamma) + 1 \). The example of [9] looks fairly artificial. A more natural example, where \( \text{er}(\Gamma) \) is finite but \( \text{gr}(\Gamma) \) is infinite, is given by Theorem 1.5, to be stated below. That theorem will be obtained in Section 4 with the help of the following lemma. In order to properly state it, we recall that a division ring \( K \) is finitely generated if it is generated as a division ring by a finite subset \( X \subseteq K \), namely no proper sub-division ring of \( K \) contains \( X \). For instance, an algebraic extension of a finite field of prime order \( \mathbb{F}_p \) is finitely generated if and only if it is finite, in which case it is a simple extension of \( \mathbb{F}_p \). On the other hand, no algebraically closed field is finitely generated.

Lemma 1.4. Let \( \Gamma(K) \) be either \( \text{Gr}_J(A_n(K)) \) or \( \text{Gr}_J(D_n(K)) \) for a set of types \( J \) non-connected as a set of nodes of \( A_n \) or \( D_n \). Suppose that \( K \) is not finitely generated. Then the generating rank of \( \Gamma(K) \) is infinite.

Lemma 1.4 will be obtained in Section 4 as a consequence of Theorem 1.3. By exploiting it we will obtain the following:

Theorem 1.5. Let \( \mathbb{F}_p \) be a finite field of prime order and \( \overline{\mathbb{F}}_p \) its algebraic closure. Then, for \( n \geq 3 \), the geometry \( \text{Gr}_{1,n}(A_n(\overline{\mathbb{F}}_p)) \) has infinite generating rank but its embedding rank is equal to either \( (n+1)^2 - 1 \) or \( (n+1)^2 \).

Remark 1.6. It is well known that if \( K \) is a field then \( \text{Gr}_{1,n}(A_n(K)) \) admits an \( (n+1)^2 - 1 \) dimensional embedding, say \( e_{\text{Lie}} \), in (the projective space of) the space of square matrices of order \( n+1 \) with entries in \( K \) and null trace (see e.g. Blok and Pasini [3]; the choice of the symbol \( e_{\text{Lie}} \) for
this embedding is motivated by the fact that it affords the representation of the group $\text{SL}(n+1,\mathbb{K})$ in its action on its own Lie algebra). However $\text{Gr}_{1,n}(A_n(\mathbb{K}))$ does not satisfy the sufficient conditions of Kasikova and Shult \[10\] for the existence of the universal embedding. So, we do not know if it always admits the universal embedding, let alone if $e_{\text{Lie}}$ is universal. A complete answer is known only when $\mathbb{K}$ is a prime field. In this case $e_{\text{Lie}}$ is indeed universal (Blok and Pasini \[3\], Section 3). A bit less is known when $\mathbb{K}$ is a number field or a perfect field of positive characteristic; in this case $e_{\text{Lie}}$ dominates all homogeneous embeddings of $\text{Gr}_{1,n}(A_n(\mathbb{K}))$ (Völklein \[15\]). As for the remaining geometries of Theorem 1.1, namely $\text{Gr}_{+,-}(D_n(\mathbb{K}))$, $\text{Gr}_{1,+,-}(D_n(\mathbb{K}))$ and $\text{Gr}_{1,-}(D_n(\mathbb{K}))$, they too are embeddable (see \[3\]) and, when $\mathbb{K}$ is a prime field, they admit the universal embedding (Blok and Pasini \[3\], Section 4), even if none of them satisfies the conditions of Kasikova and Shult \[10\].

**Remark 1.7.** The geometry $\Delta^+_2$ of $\mathbb{F}$ with $n = 3$ is the same as $\text{Gr}_{1,3}(A_3(\mathbb{F}))$. According to the above, Lemma 4.8 of \[2\], which deals with that geometry and its Weyl embedding $\varepsilon^+_2$ (which is the same as $e_{\text{Lie}}$), might possibly be wrong as stated. It should be corrected as follows: when $n = 3$ and $\mathbb{F}$ is a perfect field of positive characteristic or a number field, then $\varepsilon^+_2$ dominates all homogeneous embeddings of $\Delta^+_2$.

**Remark 1.8.** In our survey of embeddings we have stuck to full projective embeddings, but in the proof of Theorem 1.5 we shall deal also with lax embeddings. *Lax projective embeddings* are defined in the same way as full projective embeddings but for replacing the condition that $e(\ell)$ is a line of $\text{PG}(V)$ with the weaker condition that $e(\ell)$ spans a line of $\text{PG}(V)$, for every line $\ell$ of $\Gamma$. Many authors also require that no two lines of $\Gamma$ span the same line of $\text{PG}(V)$, but in view of our needs in this paper we can safely renounce that requirement. The only fact relevant for us is that inequality (I) holds true even if $e$ is lax, as it is clear from the way we have obtained it.

2. Preliminaries

2.1. Setting and notation

We refer to \[11\] Chapter 5 for the definition of the $J$-grassmannian $\text{Gr}_J(\Delta)$ of a geometry $\Delta$. We recall that when $\Delta$ satisfies the so-called Intersection Property (which is always the case when $\Delta$ is a building) then $\text{Gr}_J(\Delta)$ is the same as the $J$-shadow space of $\Delta$ as defined by Tits \[14\] Chapter 12]. According to \[11\] (and \[14\]), the $J$-grassmannian of a geometry $\Delta$ is a geometry with a string-shaped diagram graph and the same rank as $\Delta$, but in this paper, following Buekenhout and Cohen \[4, \S 2.5\], we shall mostly regard it as a point-line geometry, with the $J$-flags of $\Delta$ taken as points, while the lines are the flags of $\Delta$ of type $(J \setminus \{j\}) \cup \text{fr}(j)$ for $j \in J$, where $\text{fr}(j)$ stands for the set of types adjacent to $j$ in the diagram of $\Delta$; a point and a line of $\text{Gr}_J(\Delta)$ are incident precisely when they are incident as flags of $\Delta$. So, the lines of $\text{Gr}_J(\Delta)$ are particular flags of $\Delta$. This setting will indeed be helpful in some respects but it forces to distinguish between a line and its set of points and this distinction often ends in a burden for the exposition; we will often neglect it. This is a harmless abuse. Indeed only grassmannians of buildings are considered in this paper; buildings satisfy the Intersection Property and, if that property holds in a geometry $\Delta$, then no two lines of $\text{Gr}_J(\Delta)$ have the same points (even better: no two lines of $\text{Gr}_J(\Delta)$ have two points in common).

As in Section 1.1, given a division ring $\mathbb{K}$, we denote by $A_n(\mathbb{K})$ the building of type $A_n$ defined over $\mathbb{K}$. Similarly, if the division ring $\mathbb{K}$ is a field (namely, is commutative) then $D_n(\mathbb{K})$ stands for the building of type $D_n$ defined over $\mathbb{K}$. We allow $n = 3$ in $D_n$. So, $D_3 = A_3$. Nevertheless, when writing $D_3(\mathbb{K})$ we always understand that $\mathbb{K}$ is a field, for consistency of notation. Let $X_n$ stand for either $A_n$ or $D_n$. It is well known that the elements of $X_n(\mathbb{K})$ can be identified with suitable vector
subspaces of a vector space $V$ over $\mathbb{K}$ of dimension either $n + 1$ or $2n$ according to whether $X_n = A_n$ or $X_n = D_n$. Similarly, given a proper sub-division ring $\mathbb{K}_0$ of $\mathbb{K}$, the building $X_n(\mathbb{K}_0)$ is realized in a vector space $V_0$ over $\mathbb{K}_0$, of the same dimension as $V$. We can always assume that $V_0$ is the set of $\mathbb{K}_0$-linear combinations of the vectors of a selected basis $E$ of $V$, so that $V$ is obtained from $V_0$ by scalar extension from $\mathbb{K}_0$ to $\mathbb{K}$. Thus, with $E$ suitably selected when $X_n = D_n$, the building $X_n(\mathbb{K}_0)$ is turned into a subgeometry of $X_n(\mathbb{K})$ (see Sections 2.2 and 2.3 for more details). Accordingly, for every subset $J$ of the set of nodes of the diagram $X_n$, the $J$-grassmannian $\text{Gr}_J(X_n(\mathbb{K}_0))$ can be regarded as a subgeometry of $\text{Gr}_J(X_n(\mathbb{K}))$. Our main goal in this paper is to show that, if $J$ consists of extremal nodes of $X_n$ and $|J| > 1$ then $\text{Gr}_J(X_n(\mathbb{K}_0))$ does not generate $\text{Gr}_J(X_n(\mathbb{K}))$.

We firstly consider the $\{1, n\}$-grassmannian $\text{Gr}_{1,n}(A_n(\mathbb{K}))$ of $A_n(\mathbb{K})$; see Fig. 1.

$$\text{Gr}_{1,n}(A_n) : \begin{array}{cccccc}
1 & 2 & 3 & n-1 & n \\
\bullet & - & - & - & -
\end{array}$$

Figure 1: The $\{1, n\}$-grassmannian of $A_n$.

The points of $\text{Gr}_{1,n}(A_n(\mathbb{K}))$ are flags of type $\{1, n\}$ in $A_n(\mathbb{K})$; its lines are flags of type either $\{2, n\}$ or $\{1, n-1\}$; a point $p$ and a line $\ell$ are incident if and only if $p \cup \ell$ is a flag of $A_n(\mathbb{K})$. Turning to $D_n$, we label the nodes of this diagram as in Fig. 2.

$$\begin{array}{ccccccccc}
1 & 2 & 3 & n-3 & n-2 & + & - \\
\bullet & - & - & - & - & - & -
\end{array}$$

Figure 2: Labeling of types for buildings of type $D_n$.

We are interested in the $J$-grassmannians $\text{Gr}_J(D_n(\mathbb{K}))$, where $J = \{+, -\}$ or $J = \{1, +, -\}$ or $J = \{1, -\}$ (we can omit the case $J = \{1, +\}$ since $\text{Gr}_{1,+}(D_n(\mathbb{K})) \cong \text{Gr}_{1,-}(D_n(\mathbb{K}))$); see Fig. 3. Explicitly, the points of $\text{Gr}_{1,+}(D_n(\mathbb{K}))$ are the flags of $D_n(\mathbb{K})$ of type $\{+, -\}$ while the lines are the flags of types $\{n-2, +\}$ and $\{n-2, -\}$ with incidence between a point $p$ and a line $\ell$ given by the condition that $p \cup \ell$ must be a flag of $D_n(\mathbb{K})$. As for $\text{Gr}_{1,-}(D_n(\mathbb{K}))$, its points are the flags of type $\{1, +, -\}$, and the lines are the flags of type $\{2, +, -\}, \{1, n-2, +\}$ or $\{1, n-2, -\}$; incidence is defined as above. Finally, the points of $\text{Gr}_{1,-}(D_n(\mathbb{K}))$ are the flags of type $\{1, -\}$ and the lines are the flags of type either $\{2, -\}$ or $\{1, n-2\}$. Note that when $n = 3$, since $D_3(\mathbb{K}) \cong A_3(\mathbb{K})$, we have $\text{Gr}_{1,3}(A_3(\mathbb{K})) \cong \text{Gr}_{1,-}(D_3(\mathbb{K}))$. In any case, $\text{Gr}_{1,+}(D_n(\mathbb{K})) \cong \text{Gr}_{n-1}(D_n^{++}(\mathbb{K}))$, where $D_n^{++}(\mathbb{K}) := \text{Gr}_1(D_n(\mathbb{K}))$ is the 1-grassmannian of $D_n(\mathbb{K})$ (but regarded as a geometry of rank $n$), namely the top-thin polar space associated to the group $O^{++}(2n, \mathbb{K})$; see Fig. 4.

In the following we shall add more details on the grassmannians introduced before. In particular, we shall better explain in which sense $\text{Gr}_J(A_n(\mathbb{K}))$ and $\text{Gr}_J(D_n(\mathbb{K}))$ contain $\text{Gr}_J(A_n(\mathbb{K}_0))$ and $\text{Gr}_J(D_n(\mathbb{K}_0))$ for a sub-division ring $\mathbb{K}_0$ of $\mathbb{K}$.
Figure 3: Geometries associated to buildings of type $D_n$

Figure 4: Geometry $\text{Gr}_{n-1}(B^+_n(K)) \cong \text{Gr}_{+,-}(D_n(K))$
2.2. The geometry $A_n(\mathbb{K})$ and its grassmannian $\text{Gr}_{1,n}(A(\mathbb{K}))$

Let $A_n(\mathbb{K})$ be a geometry of type $A_n$ defined over a division ring $\mathbb{K}$, with $n \geq 3$. Explicitly, $A_n(\mathbb{K}) \cong \text{PG}(V_{n+1}(\mathbb{K}))$ for a $(n+1)$-dimensional right $\mathbb{K}$-vector space $V_{n+1}(\mathbb{K})$. For $i = 1, 2, \ldots, n$ the elements of $A_n(\mathbb{K})$ of type $i$ are the $i$-dimensional subspaces of $V_{n+1}(\mathbb{K})$, with symmetrized inclusion as the incidence relation. As customary we call the elements of $A_n(\mathbb{K})$ of type $1, 2$ and $n$ points, lines and hyperplanes respectively. The elements of type $n-1$ will be called sub-hyperplanes. Note that, when $n = 3$, lines and sub-hyperplanes are the same objects.

Turning to $\text{Gr}_{1,n}(A_n(\mathbb{K}))$, its points are the point-hyperplane flags $(p, H)$ of $A_n(\mathbb{K})$. Its lines, regarded as sets of points, are of either of the following two types:

$$\ell_{p,S} := \{(p,X) : X \text{ hyperplane}, X \supset S\} \text{ for a (point, sub-hyperplane) flag } (p,S).$$

$$\ell_{L,H} := \{(x,H) : x \text{ a point}, x \subset L\}, \text{ for a line-hyperplane flag } (L,H).$$ (3)

2.3. $D_n(\mathbb{K})$ and $\text{Gr}_J(D_n(\mathbb{K}))$ for $J = \{+, -\}, \{1, -\}$ or $\{1, +, -\}$

Let $\mathbb{K}$ be a field and $V_{2n}(\mathbb{K})$ a vector space of dimension $2n$ over $\mathbb{K}$, with $n \geq 3$. Consider a non-degenerate quadratic form $q$ on $V_{2n}(\mathbb{K})$ of Witt index $n$. As in Section 2.1 let $B_n^+(\mathbb{K})$ be the polar space associated to $q$, namely the (weak) building of rank $n$ whose elements are the vector subspaces of $V_{2n}(\mathbb{K})$ that are totally singular with respect to $q$, with their dimensions taken as types. The elements of $B_n^-(\mathbb{K})$ of dimension 1 are called points and those of dimension 2 lines.

It is well known that we can “unfold” $B_n^+(\mathbb{K})$ as to obtain a building $D_n(\mathbb{K})$ of type $D_n$ (see e.g. Tits [14], Chapter 7). Explicitly, let $\sim$ be the equivalence relation on the set of all $n$-dimensional subspaces of $B_n^+(\mathbb{K})$ defined as follows: $X \sim Y$ if and only if $X \cap Y$ has even codimension in $X$ (equivalently, in $Y$). Let $\mathcal{S}^+$ and $\mathcal{S}^-$ be the two equivalence classes of $\sim$. Take $\{1, 2, \ldots, n-2, +, -\}$ as the set of types. For $1 \leq i \leq n-2$ the $i$-elements of $B_n^+(\mathbb{K})$ are the elements of $D_n(\mathbb{K})$ of type $i$ and the elements of $\mathcal{S}^+$ and $\mathcal{S}^-$ are given types $+$ and $-$ respectively. The $(n-1)$-elements of $B_n^+(\mathbb{K})$ are dropped (but we can recover them as flags of type $\{+,-\}$). Incidence between elements of different types $\{i, j\} \neq \{+, -\}$ is symmetrized inclusion; if $X \in \mathcal{S}^+$ and $Y \in \mathcal{S}^-$ then $X$ is incident with $Y$ if and only if $\dim(X \cap Y) = n-1$.

It is clear from the way $D_n(\mathbb{K})$ is defined that the 1-grassmannian $\text{Gr}_1(D_n(\mathbb{K}))$ of $D_n(\mathbb{K})$, regarded as a geometry of rank $n$, is just the same as $B_n^+(\mathbb{K})$. So, we can go back and forth from $D_n(\mathbb{K})$ to $B_n^+(\mathbb{K})$ as if they were the same object. In the sequel we will sometimes avail ourselves of this opportunity, when profitable.

Turning to grassmannians, $\text{Gr}_{1,-}(D_n(\mathbb{K}))$ is the point-line geometry whose points are the flags $(M_1, M_2)$ of $D_n(\mathbb{K})$ of type $\{+, -\}$ and whose lines are of the following two forms:

$$\ell_{U,M_1} := \{(M_1,X) : X \in \mathcal{S}^-, M_1 \cap X \subset U\} \text{ with } (U,M_1) \text{ a flag of type } \{n-2, +\};$$

$$\ell_{U,M_2} := \{(X,M_2) : X \in \mathcal{S}^+, X \cap M_2 \supset U\} \text{ with } (U,M_2) \text{ a flag of type } \{n-2, -\}.\quad (5)$$

Recall that the points of the grassmannian $\text{Gr}_{n-1}(B_n^+(\mathbb{K}))$ of $B_n^+(\mathbb{K})$ are the $(n-1)$-dimensional totally singular subspaces of $V_{2n}(\mathbb{K})$ totally singular for the quadratic form $q$ and the lines are the sets of the form $\ell_{X,M} := \{Y : X \subset Y \subset M\}$ where $\dim(X) = n-2$, $\dim(M) = n$, $X \subset M$ and $M$ is totally singular. Every point $X$ of $\text{Gr}_{n-1}(B_n^+(\mathbb{K}))$ is the intersection $X = M_1 \cap M_2$ of a unique pair $\{M_1, M_2\}$ of $n$-dimensional totally singular subspaces, which necessarily form a $(+, -)$-flag of $D_n(\mathbb{K})$. Conversely, for every $(+, -)$-flag $(M_1, M_2)$ of $D_n(\mathbb{K})$, the intersection $X = M_1 \cap M_2$ is a point of $\text{Gr}_{n-1}(B_n^+(\mathbb{K}))$. A bijective mapping $\iota$ is thus naturally defined from the set of points of $\text{Gr}_{n-1}(B_n^+(\mathbb{K}))$ onto the set of points of $\text{Gr}_{1,-}(D_n(\mathbb{K}))$. The mapping $\iota$ induces a bijection from the set of lines of $\text{Gr}_{n-1}(B_n^+(\mathbb{K}))$.
onto the set of lines of $\text{Gr}_{+,-}(D_n(K))$. In fact, if $\ell_{X,M}$ is a line of $\text{Gr}_{n-1}(B^+_n(K))$ then $\iota(\ell_{X,M})$ is the line of $\text{Gr}_{+,-}(D_n(K))$ denoted by the very same symbol $\ell_{X,M}$ and it has either form \textcircled{5} or \textcircled{6} according to whether $M$ belongs to $\mathfrak{S}^+$ or $\mathfrak{S}^-$. To sum up, $\text{Gr}_{n-1}(B^+_n(K)) \cong \text{Gr}_{+,-}(D_n(K))$.

The grassmannian $\text{Gr}_{1,-}(D_n(K))$ is the point-line geometry where the points are the flags $(p, M)$ of $D_n(K)$ of type $(1, -)$ and the lines are as follows:

$$\ell_{p,U} := \{(p, X) : X \in \mathfrak{S}^-, X \supset U\} \text{ with } (p, U) \text{ a flag of type } (1, n-2);$$

$$\ell_{L,M} := \{(x, M) : \dim(x) = 1, x \subset L\} \text{ with } (L, M) \text{ a flag of type } (2,-).$$

The grassmannian $\text{Gr}_{1,+,-}(D_n(K))$ is the point-line geometry where the points are the flags $(p, M_1, M_2)$ of $D_n(K)$ of type $(1, +, -)$; the lines are as follows:

$$\ell_{L,M_1,M_2} := \{(p, M_1, M_2) : \dim(p) = 1, p \subset L\} \text{ with } (L, M_1, M_2) \text{ a flag of type } (2,+, -);$$

$$\ell_{p,U,M_1} := \{(p, M_1, X) : X \in \mathfrak{S}^-, X \supset U\} \text{ with } (p, U, M_1) \text{ a flag of type } (1, n-2, +);$$

$$\ell_{p,U,M_2} := \{(p, X, M_2) : X \in \mathfrak{S}^+, X \supset U\} \text{ with } (p, U, M_2) \text{ a flag of type } (1, n-2, -).$$

2.4. The subgeometry $\text{Gr}_J(A_n(K_0))$ of $\text{Gr}_J(A_n(K))$ for $K_0 \subseteq K$

Let $E$ be a basis of $V_{n+1}(K)$ and $K_0$ a sub-division ring of $K$. We say that a vector $v \in V_{n+1}(K)$ is $K_0$-rational respect to $E$ (also $K_0$-rational for short, when the basis $E$ is clear from the context) if $v$ is a linear combination of vectors of $E$ with coefficients in $K_0$. The set of $K_0$-rational vectors (with respect to $E$) is a $K_0$-vector space, henceforth denoted $V_{n+1,E}(K_0)$. For a subspace $X$ of $V_{n+1}(K)$, let $X_0 := X \cap V_{n+1,E}(K_0)$. Clearly, $X_0$ is a subspace of $V_{n+1,E}(K_0)$. We say that $X$ is $K_0$-rational with respect to $E$ (also $K_0$-rational for short) if $X_0$ spans $X$ (in $V_{n+1}(K)$); in other words, $X$ is a $K_0$-rational subspace of $V_{n+1}(K)$. Indeed, the rank of a matrix $M$ with entries in $K_0$ does not change if $M$ is regarded as matrix with entries in $K$.

Clearly, the sum of two $K_0$-rational subspaces of $V_{n+1}(K)$ is still $K_0$-rational. Similarly,

Lemma 2.1. The intersection of two $K_0$-rational subspaces is still $K_0$-rational.

Proof. Let $X_0, Y_0$ be two subspaces of $V_{n+1,E}(K_0)$ and $X, Y$ be their spans in $V_{n+1}(K)$. Then $X \cap Y$ contains the span $Z$ of $X_0 \cap Y_0$ in $V_{n+1}(K)$. We must prove that $X \cap Y = Z$. Clearly, $\dim(X) = \dim(X_0)$ and $\dim(Y) = \dim(Y_0)$. Moreover $\dim(X + Y) = \dim(X_0 + Y_0)$. Hence

$$\dim(X \cap Y) = \dim(X) + \dim(Y) - \dim(X + Y) = \dim(X_0) + \dim(Y_0) - \dim(X_0 + Y_0) = \dim(X_0 \cap Y_0) = \dim(Z).$$

Therefore $X \cap Y = Z$.  

The following is now obvious:

Proposition 2.2. The $K_0$-rational elements of $A_n(K)$ form a geometry $A_n,E(K_0) \cong A_n(K_0)$.

In view of Proposition 2.2, we can freely identify $A_n(K_0)$ with $A_n,E(K_0)$, thus regarding $A_n(K_0)$ as a subgeometry of $A_n(K)$. The flags of $A_n(K_0)$ are thus identified with the $K_0$-rational flags of $A_n(K)$, namely the flags of $A_n(K)$ all elements of which are $K_0$-rational (with respect to the selected basis $E$ of $V_{n+1}(K)$). Accordingly, for $\emptyset \neq J \subseteq \{1, 2, \ldots, n\}$ the $J$-grassmannian $\text{Gr}_J(A_n(K_0))$ of
$A_n(\mathbb{K}_0)$ is identified with the subgeometry $\text{Gr}_{J,E}(A_n(\mathbb{K}_0))$ of $\text{Gr}_J(A_n(\mathbb{K}))$ formed by the $\mathbb{K}_0$-rational points and lines of $\text{Gr}_J(A_n(\mathbb{K}))$, namely the points and lines of $\text{Gr}_J(A_n(\mathbb{K}))$ which are $\mathbb{K}_0$-rational as flags of $A_n(\mathbb{K})$.

Henceforth, by a harmless little abuse, we will always regard $\text{Gr}_J(A_n(\mathbb{K}_0))$ as the same as $\text{Gr}_{J,E}(A_n(\mathbb{K}_0))$, thus referring to the span of $\text{Gr}_J(A_n(\mathbb{K}_0))$ in $\text{Gr}_J(A_n(\mathbb{K}))$, as we have done in the Introduction, while in fact we mean the span of $\text{Gr}_{J,E}(A_n(\mathbb{K}_0))$.

The next proposition states that, regarding $\text{Gr}_J(A_n(\mathbb{K}_0))$ as a subgeometry of $\text{Gr}_J(A_n(\mathbb{K}))$, the collinearity graph of $\text{Gr}_J(A_n(\mathbb{K}_0))$ is just the graph induced on its point-set by the collinearity graph of $\text{Gr}_J(A_n(\mathbb{K}))$.

**Proposition 2.3.** A line of $\text{Gr}_J(A_n(\mathbb{K}_0))$ is $\mathbb{K}_0$-rational if and only if at least two of its points are $\mathbb{K}_0$-rational.

**Proof.** The ‘only if’ part of this claim easily follows from the isomorphism of the geometries $\text{Gr}_{J,E}(A_n(\mathbb{K}_0)) \cong \text{Gr}_J(A_n(\mathbb{K}_0))$. Turning to the ‘if’ part, given $j_0 \in J$, let $L$ be a flag of $A_n(\mathbb{K})$ of type $(J \setminus \{j_0\}) \cup \text{fr}(j_0)$ and let $P$ and $P'$ be two distinct $J$-flags of $A_n(\mathbb{K})$ incident with $L$. We must prove that if both $P$ and $P'$ are $\mathbb{K}_0$-rational then $L$ is also $\mathbb{K}_0$-rational. There are three cases to examine: $J$ contains elements $j < j_0$ as well elements $j' > j_0$; $j_0 \leq j$ for every $j \in J$; $j_0 \geq j$ for every $j \in J$. We shall examine only the first case, leaving the remaining two (easier) cases to the reader.

With $j_0$ as in the first case, the flag $L$ has type $(J \setminus \{j_0\}) \cup \{j_0 - 1, j_0 + 1\}$ and contains $Q := P \cap P'$, which is a flag of type $J \setminus \{j_0\}$. Moreover, there are distinct $j_0$-subspaces $S, S'$ of $V_{n+1}(\mathbb{K})$ incident with $L$ such that $P = Q \cup \{S\}$ and $P' = Q \cup \{S'\}$. As $S$ and $S'$ are incident with $L$, the elements of $L$ of type $j_0 - 1$ and $j_0 + 1$ coincide with $S \cap S'$ and $S + S'$ respectively, namely $L = Q \cup \{S \cap S', S + S'\}$. By assumption, $P$ and $P'$ are $\mathbb{K}_0$-rational. Hence $Q = P \cap P'$ as well as $S$ and $S'$ are $\mathbb{K}_0$-rational. If $j_0 - 1 \in J$ then $S \cap S' \in Q$, hence $S \cap S'$ is $\mathbb{K}_0$-rational. Otherwise $S \cap S'$ is $\mathbb{K}_0$-rational by Lemma 2.4. Similarly, $S + S'$ is $\mathbb{K}_0$-rational. Thus, all elements of $L$ are $\mathbb{K}_0$-rational, namely $L$ is $\mathbb{K}_0$-rational.

2.5. The subgeometry $\text{Gr}_J(D_n(\mathbb{K}_0))$ of $\text{Gr}_J(D_n(\mathbb{K}))$ for $\mathbb{K}_0 \leq \mathbb{K}$

Let $\mathbb{K}_0$ be a subfield of $\mathbb{K}$. Let $q : V_{2n}(\mathbb{K}) \rightarrow \mathbb{K}$ be the quadratic form considered in Section 2.4. Without loss of generality we can assume to have chosen the basis $E = (e_1, \ldots, e_{2n})$ of $V_{2n}(\mathbb{K})$ in such a way that $q$ admits the following canonical expression with respect to $E$:

$$q(x_1, \ldots, x_{2n}) = x_1 x_2 + \cdots + x_{2n-1} x_{2n}. \quad (12)$$

As in Section 2.4 we can consider the $\mathbb{K}_0$-vector space $V_{2n,E}(\mathbb{K}_0)$ formed by the $\mathbb{K}_0$-rational vectors (with respect to $E$). The form $q$ induces a quadratic form $q_0$ on $V_{2n,E}(\mathbb{K}_0)$. Clearly, a $\mathbb{K}_0$-rational subspace $X$ of $V_{2n}(\mathbb{K})$ is totally singular for $q$ if and only if $X \cap V_{2n,E}(\mathbb{K}_0)$ is totally singular for $q_0$. Hence the polar space $B^+_n(\mathbb{K}_0)$ associated to $q_0$ can be identified with the subgeometry $B^+_n(\mathbb{K}_0)$ of $B^+_n(\mathbb{K})$ formed by the $\mathbb{K}_0$-rational subspaces of $V_{2n}(\mathbb{K})$ which are totally singular for $q$. Similarly, $D_n(\mathbb{K}_0)$ can be identified with the subgeometry $D_{n,E}(\mathbb{K}_0)$ of $D_n(\mathbb{K})$ formed by the $\mathbb{K}_0$-rational elements of $D_n(\mathbb{K})$.

A flag of $D_n(\mathbb{K})$ is $\mathbb{K}_0$-rational if all of its elements are $\mathbb{K}_0$-rational (with respect to $E$, of course). Given a nonempty subset $J$ of the type-set $\{1, 2, \ldots, n - 2, +, -\}$ of $D_n(\mathbb{K})$, a point or a line of $\text{Gr}_J(D_n(\mathbb{K}))$ are said to be $\mathbb{K}_0$-rational if they are $\mathbb{K}_0$-rational as flags of $D_n(\mathbb{K})$. The $\mathbb{K}_0$-rational points and lines of $\text{Gr}_J(D_n(\mathbb{K}))$ form a subgeometry $\text{Gr}_{J,E}(D_n(\mathbb{K}_0))$ of $\text{Gr}_J(D_n(\mathbb{K}))$ isomorphic to $\text{Gr}_J(D_n(\mathbb{K}_0))$. An analogue of Proposition 2.3 also holds:
Proposition 2.4. A line of \( \text{Gr}_i(D_n(\mathbb{K})) \) is \( \mathbb{K}_0 \)-rational if and only if at least two of its points are \( \mathbb{K}_0 \)-rational.

Proof. This statement can be proved in the same way as Proposition 2.3 but for a couple of cases in the proof of the ‘only if’ part, which we shall now discuss.

1. Suppose that \( J \) contains at least one of the types \(+\) and \( -\), say \( + \in J \). Suppose moreover that \( n - 2 \notin J \). Let \( L \) be a flag of \( D_n(\mathbb{K}) \) of type \( (J \setminus \{+\}) \cup \text{fr}(n-2) = (J \setminus \{+\}) \cup \{n-2\} \) and let \( P, P' \) be distinct \( \mathbb{K}_0 \)-rational flags of type \( J \), both incident with \( L \). Then \( Q = P \cap P' \) is a \( \mathbb{K}_0 \)-rational flag, \( P = Q \cup \{M\} \) and \( P' = Q \cup \{M'\} \) for distinct \( \mathbb{K}_0 \)-rational element \( M, M' \in \mathcal{S}^+ \). Also, \( L = Q \cup S \) for an \( (n-2) \)-element \( S \) incident with \( Q \). We have \( S \subseteq M \cap M' \) since \( P \) and \( P' \) are incident with \( L \). However, \( \dim(M \cap M') \) has even codimension in \( M \) and \( M' \), since \( M \) and \( M' \) belong to the same family of \( n \)-elements of \( B_r^+(\mathbb{K}) \), namely \( \mathcal{S}^+ \). Therefore \( S = M \cap M' \). Hence \( S \) is \( \mathbb{K}_0 \)-rational by Lemma 2.1. Thus, \( L \) is \( \mathbb{K}_0 \)-rational.

2. The set \( J \) contains none of the types \(+\) or \( -\) but it contains \( n-2 \). To fix ideas, suppose that \( n > 3 \). Let \( L \) be a flag of \( D_n(\mathbb{K}) \) of type \( (J \setminus \{n-2\}) \cup \text{fr}(n-2) = (J \setminus \{n-2\}) \cup \{n-3, +, -\} \) and let \( P, P' \) be distinct \( \mathbb{K}_0 \)-rational flags of type \( J \), both incident with \( L \). Then \( Q = P \cap P' \) is a \( \mathbb{K}_0 \)-rational flag, \( P = Q \cup \{S\} \) and \( P' = Q \cup \{S'\} \) for distinct \( \mathbb{K}_0 \)-rational \( (n-2) \)-elements \( S, S' \) of \( D_n(\mathbb{K}) \) and \( L = Q \cup \{R, M_1, M_2\} \) for an \( (n-3, +, -) \)-flag \( (R, M_1, M_2) \) incident with \( Q \). As both \( P \) and \( P' \) are incident with \( L \), the sum \( S + S' \) is contained in \( M \cap M' \). However \( \dim(M \cap M') = n - 1 \) while \( \dim(S + S') \geq n - 1 \) since \( S \neq S' \). Consequently, \( M \cap M' = S + S' \). On the other hand, \( S + S' \) is a \( \mathbb{K}_0 \)-rational subspace of \( V_{2n}(\mathbb{K}) \), since both \( S \) and \( S' \) are \( \mathbb{K}_0 \)-rational. Hence \( M \cap M' \) is \( \mathbb{K}_0 \)-rational. Therefore \( M \cap M' \) is an \((n-1)\)-element of \( B_{r,E}^+(\mathbb{K}_0) = \text{Gr}_1(D_{n,E}(\mathbb{K}_0)) \). Accordingly, \( M \cap M' = M_0 \cap M_0' \) for a \((+,-)\)-flag \((M_0, M_0')\) of \( D_{n,E}(\mathbb{K}_0) \). On the other hand, all \((+,-)\)-flags of \( D_{n,E}(\mathbb{K}_0) \) are \((+,-)\)-flags of \( D_n(\mathbb{K}) \) too and two \((+,-)\)-flags \((M, M')\) and \((M_0, M_0')\) of \( D_{n}(\mathbb{K}) \) coincide if \( M \cap M' = M_0 \cap M_0' \). It follows that \( M = M_0 \) and \( M' = M_0 \), namely both \( M \) and \( M' \) are \( \mathbb{K}_0 \)-rational. It remains to prove that \( R \) too is \( \mathbb{K}_0 \)-rational. If \( n-3 \in J \) then \( R \in Q \) and there is nothing to prove. Otherwise \( R = S \cap S' \). Hence \( R \) is \( \mathbb{K}_0 \)-rational by Lemma 2.1. Therefore \( L \) is \( \mathbb{K}_0 \)-rational.

We have assumed that \( n > 3 \). When \( n = 3 \) we have \( J = \{n-2\} \) and \( L = (M_1, M_2) \), of type \((+,-)\); we get the conclusion as above, but now with no \( R \) to take care of. \( \square \)

All we have said for \( D_n(\mathbb{K}_0) \) and \( \text{Gr}_j(D_n(\mathbb{K}_0)) \) in this section holds for \( B_r^+(\mathbb{K}_0) \) and \( \text{Gr}_j(B_r^+(\mathbb{K}_0)) \) as well.

3. Proof of Theorems 1.1 and 1.3

For \( X_n \) equal to \( A_n \) or \( D_n \) and a nonempty set of types \( J \), let \( \Gamma(\mathbb{K}) := \text{Gr}_j(X_n(\mathbb{K})) \) and \( \Gamma(\mathbb{K}_0) := \text{Gr}_j(X_n(\mathbb{K}_0)) \) be its \( \mathbb{K}_0 \)-rational subgeometry for a proper sub-division ring \( \mathbb{K}_0 \) of \( \mathbb{K} \) (Sections 2.1 and 2.3).

Definition 3.1. We say that a node \( t \) of \( X_n \) splits \( J \) if \( t \notin J \) and \( J \) is not contained in one single connected component of \( X_n \setminus \{t\} \). In other words, \( t \) separates at least two of the types of \( J \).

Definition 3.2. We say that a \( J \)-flag \( F \) (point of \( \Gamma(\mathbb{K}) \)) is nearly \( \mathbb{K}_0 \)-rational if either at least one of its elements is \( \mathbb{K}_0 \)-rational or there exists a \( \mathbb{K}_0 \)-rational element of \( X_n(\mathbb{K}) \) incident with \( F \) and such that its type splits \( J \). We denote by \( \Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K})) \) the set of all nearly \( \mathbb{K}_0 \)-rational points of \( \Gamma(\mathbb{K}) \).
Obviously, $\Gamma(\mathbb{K}_0) \subseteq \Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$. We shall prove the following:

**Theorem 3.3.** If $\Gamma(\mathbb{K})$ is $\mathbb{Gr}_{1,n}(A_n(\mathbb{K}))$, $\mathbb{Gr}_{1,-}(D_n(\mathbb{K}))$, $\mathbb{Gr}_{1,+}(D_n(\mathbb{K}))$ or $\mathbb{Gr}_{+,-}(D_n(\mathbb{K}))$ then $\Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$ is a proper subspace of $\Gamma(\mathbb{K})$.

Theorem 3.1 then immediately follows from Theorem 3.3 and the inclusion $\Gamma(\mathbb{K}_0) \subseteq \Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$.

### 3.1. Proof of Theorem 3.3

We need a preliminary result from multi-linear algebra, to be exploited later, when discussing the case $\Gamma(\mathbb{K}) = \mathbb{Gr}_{+,-}(D_n(\mathbb{K}))$.

**Lemma 3.4.** Suppose that $\mathbb{K}$ is a field and let $V := V_4(\mathbb{K})$. Given a basis $E = (e_1, e_2, e_3, e_4)$ of $V$, let $E \wedge V = (e_i \wedge e_j), i < j$ be the corresponding basis of the second exterior power $V \wedge V$ of $V$. Then all the following hold:

1. The span $\langle v, w \rangle$ of two independent vectors $v, w \in V$ is $\mathbb{K}_0$-rational with respect to $E$ if and only if $v \wedge w$ is proportional to a vector of $V \wedge V$ which is $\mathbb{K}_0$-rational with respect to $E \wedge E$.
2. A non-zero vector $v \in V$ is proportional to a $\mathbb{K}_0$-rational vector if and only if the subspace $S_v := \langle v \wedge x \rangle_{x \in V}$ of $V \wedge V$ is $\mathbb{K}_0$-rational with respect to $E \wedge E$.
3. The span $\langle u, v, w \rangle$ of three independent vectors $u, v, w \in V$ is $\mathbb{K}_0$-rational (with respect to $E$) if and only if $\langle u \wedge v, u \wedge w, v \wedge w \rangle$ is $\mathbb{K}_0$-rational with respect to $E \wedge E$.

**Proof.** (1) Without loss of generality, we can assume $v = e_1 + e_2 a_3 + e_4 a_4$ and $w = e_2 + e_3 b_3 + e_4 b_4$ for $a_3, a_4, b_3, b_4 \in \mathbb{K}$. Hence $v \wedge w = e_{1,2} + e_{1,3} b_3 + e_{1,4} b_4 - e_{2,3} a_3 - e_{2,4} a_4 + e_{3,4}(a_3 b_4 - a_4 b_3)$, where we write $e_{i,j}$ for $e_i \wedge e_j$. Both parts of (1) are equivalent to the following single claim: $a_3, a_4, b_3, b_4 \in \mathbb{K}_0$. Hence they are mutually equivalent.

(2) Without loss of generality, we can assume that $v = e_1 + e_2 a_2 + e_3 a_3 + e_4 a_4$. Hence $S_v := \langle v \wedge e_2, v \wedge e_3, v \wedge e_4 \rangle$. We have

$v \wedge e_2 = e_{1,2} - e_{2,3} a_3 - e_{2,4} a_4$, $v \wedge e_3 = e_{1,3} + e_{2,3} a_2 - e_{3,4} a_4$, $v \wedge e_4 = e_{1,4} + e_{2,4} a_2 + e_{3,4} a_3,$

with $e_{i,j} := e_i \wedge e_j$, as above. Both parts of (2) are thus equivalent to this: $a_2, a_3, a_4 \in \mathbb{K}_0$. So Claim (2) is proved.

(3) Without loss of generality, we can assume that $u = e_1 + e_4 a$, $v = e_2 + e_3 b$ and $w = e_3 + e_4 c$. Hence $u \wedge v = e_{1,2} + e_{1,4} b - e_{2,4} a$, $u \wedge w = e_{1,3} - e_{1,4} c - e_{3,4} a$ and $v \wedge w = e_{2,3} + e_{2,4} c - e_{3,4} b$. Both parts of (3) are equivalent to this: $a, b, c \in \mathbb{K}_0$. Claim (3) follows.

**Lemma 3.5.** If $\Gamma(\mathbb{K})$ is as in the hypotheses of Theorem 3.3 then the set $\Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$ is a subspace of $\Gamma(\mathbb{K})$.

**Proof.** We must show that, for any two nearly $\mathbb{K}_0$-rational collinear points $F, F'$ of $\Gamma(\mathbb{K})$, the line $(F, F')_{\Gamma(\mathbb{K})}$ is fully contained in $\Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$. There are several cases to consider:

1. $\Gamma(\mathbb{K}) = \mathbb{Gr}_{1,n}(A_n(\mathbb{K}))$. Let $F = (p, H)$ and $F' = (p', H')$ be two distinct collinear points of $\Gamma(\mathbb{K})$, namely two point-hyperplane flags with either $p \neq p'$ and $H = H'$ or $p = p'$ but $H \neq H'$. Suppose moreover that $F$ and $F'$ are nearly $\mathbb{K}_0$-rational.
3) \( \Gamma(K) = \emptyset \) if \( a < 1 \). We have \( \ell_{L, H} = \ell_{L, H} \) is the line corresponding to the line \( L, M \), which now plays the role of \( H \) and \( L \), has dimension \( n - 2 \) instead of \( n - 1 \). If \( M = M' \) then an argument similar to that used for \( (b) \) of \( \Gamma(K) \) yields the conclusion. We leave the details to the reader.

2) \( \Gamma(K) = \emptyset \) if \( a < 1 \). Let \( F = (p, M_1, M_2) \) be two collinear points of \( \Gamma(K) \) and suppose they both are nearly \( \mathbb{K}_0 \)-rational. Two subcases can occur:

(a) \( M_i = M_i' \) for \( i = 1, 2 \). If at least one of the \( n \)-spaces \( M_1 \) and \( M_2 \) is \( \mathbb{K}_0 \)-rational, there is nothing to prove. Suppose that neither of them is \( \mathbb{K}_0 \)-rational. Then, since \( F \) and \( F' \) are nearly \( \mathbb{K}_0 \)-rational by assumption, there are \( \mathbb{K}_0 \)-rational subspaces \( U_0, U_0' \) with \( p \subseteq U_0 \subseteq M_1 \cap M_2 \) and \( p' \subseteq U_0' \subseteq M_1 \cap M_2 \). We have \( \langle F, F' \rangle_{\Gamma(K)} = \ell_{L, M_1, M_2} \) as in \( (1) \) with \( L = p + p' \). The sum \( U_0 + U_0' \) is a \( \mathbb{K}_0 \)-rational subspace of \( V_{2n}(K) \) and contains \( L \).

If \( \dim(U_0 + U_0') < n - 1 \) then \( U_0 + U_0' \) is a \( \mathbb{K}_0 \)-rational element of \( D_n(K) \) incident with the flag \( (L, M_1, M_2) \), which corresponds to the line \( \ell_{L, M_1, M_2} \). As in (b) of \( (1) \), it follows that all points of \( \ell_{L, M_1, M_2} \) are nearly \( \mathbb{K}_0 \)-rational.

If \( \dim(U_0 + U_0') > n - 2 \) then necessarily \( U_0 + U_0' = M_1 \cap M_2 \). In this case \( U_0 + U_0' \) is not an element of \( D_n(K) \), but it is a \( \mathbb{K}_0 \)-rational \((n - 1)\)-element of \( B_n^+(K) \), hence an \((n - 1)\)-element of the subgeometry \( B_n^+(K) \) of \( B_n^+(K) \). As such, \( U_0 + U_0' \) is contained in just two \( n \)-elements \( N_1 \) and \( N_2 \) of \( B_n^+(K) \). However \( N_1 \) and \( N_2 \) also belong to \( B_n^+(K) \). In fact, they are the unique two \( n \)-elements of \( B_n^+(K) \) which contain \( U_0 + U_0' \). On the other hand, \( U_0 + U_0' \) is contained in \( M_1 \) and \( M_2 \). Therefore \( \{M_1, M_2\} = \{N_1, N_2\} \). However \( N_1 \) and \( N_2 \) are \( \mathbb{K}_0 \)-rational. Hence \( M_1 \) and \( M_2 \) are \( \mathbb{K}_0 \)-rational, contrary to our assumptions. We have reached a contradiction. The proof is complete, as far as the present subcase is concerned.

(b) Let \( p = p', M_i = M_i' \) but \( M_j \neq M_j' \), for \( \{i, j\} = \{1, 2\} \). To fix ideas, assume that \( M_1 = M_1' \) and \( M_2 = M_2' \). If \( M_1 \) or \( p \) are \( \mathbb{K}_0 \)-rational, then there is nothing to prove. Suppose that neither \( M_1 \) nor \( p \) are \( \mathbb{K}_0 \)-rational. Recalling that \( F \) and \( F' \) are nearly \( \mathbb{K}_0 \)-rational, one of the following occurs:

(b1) There are \( \mathbb{K}_0 \)-rational subspaces \( U_0, U_0' \) of dimension at most \( n - 2 \) such that \( p \subseteq U_0 \subseteq M_1 \cap M_2 \) and \( p \subseteq U_0' \subseteq M_1 \cap M_2' \).
(b2) Just one of $M_2$ and $M'_2$ is $\mathbb{K}_0$-rational. To fix ideas, let $M'_2$ be the $\mathbb{K}_0$-rational one. Then there exists a $\mathbb{K}_0$-rational subspace $U_0$ of dimension $\dim(U_0) \leq n - 2$ such that $p \subseteq U_0 \subset M_1 \cap M_2$. 

(b3) Both $M_2$ and $M'_2$ are $\mathbb{K}_0$-rational. 

In subcases (b1) and (b2) we can consider the element $U_0 \cap U'_0$ or $U_0 \cap M'_2$ respectively. This element contains $p$ and is $\mathbb{K}_0$-rational by Lemma 2.4. So, we get the conclusion as in (a) of 1. In subcase (b3), the intersection $U_0 = M_2 \cap M'_2$ is $\mathbb{K}_0$-rational by Lemma 2.1 and has dimension $\dim(U_0) = n - 2k$ for a positive integer $k < n/2$, since $M_2$ and $M'_2$ belong to the same class $\mathcal{S}^-$. Hence $\dim(U_0) \leq n - 2$. Moreover, $U_0 \subset M_1$, since both $M_2$ and $M'_2$ are incident with $M_1$ in $D_n(\mathbb{K})$. Clearly, $p \subseteq U_0$. Again, the conclusion follows as in (a) of 1.

4) $\Gamma(\mathbb{K}) = \text{Gr}_{+, -}(D_n(\mathbb{K}))$. Assume firstly that $n = 3$. We have discussed this case in [6, Theorem 5.10] but we turn back to it here, using an argument different from that of [6].

By the Klein correspondence, $V_{2n}(\mathbb{K}) = V_6(\mathbb{K})$ can be regarded as the exterior square of $V_4(\mathbb{K})$, with the basis $E = (e_1, \ldots, e_6)$ of $V_6(\mathbb{K})$, to be chosen as in Section 2.5 realized as the exterior square $E = E' \wedge E'$ of a suitable basis $E'$ of $V_4(\mathbb{K})$. The elements of $D_6(\mathbb{K})$ of type $+ \,$ or $\,$ $-$ correspond to 1- and 3-dimensional subspaces of $V_6(\mathbb{K})$ and the 1-elements of $D_6(\mathbb{K})$ correspond to 2-subspaces of $V_4(\mathbb{K})$. By Lemma 3.4, an element of $D_6(\mathbb{K})$ is $\mathbb{K}_0$-rational with respect to $E$ if and only if the subspace corresponding to it in $V_6(\mathbb{K})$ is $\mathbb{K}_0$-rational with respect to $E'$. Accordingly, a $(+, -)$-flag of $D_6(\mathbb{K})$ is nearly $\mathbb{K}_0$-rational if and only if the corresponding $(1, 3)$-flag of $A_4(\mathbb{K})$ is nearly $\mathbb{K}_0$-rational. Thus, we are driven back to the special case $\text{Gr}_{1, 3}(A_3(\mathbb{K}))$ of $\text{Gr}_{1, n}(A_n(\mathbb{K}))$, already discussed in 11 of this proof. It follows that $\Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$ is a subspace of $\Gamma(\mathbb{K})$, as claimed.

Consider now $n > 3$. Let $F = (M_1, M_2)$ and $F' = (M'_1, M'_2)$ be two distinct nearly $\mathbb{K}_0$-rational collinear points of $\Gamma(\mathbb{K})$. As $F$ and $F'$ are collinear, either $M_1 = M'_1$ or $M_2 = M'_2$. To fix ideas, let $M_2 = M'_2$. Hence

$$\ell_{U, M_2} = \{(M, M_2) : M \in \mathcal{S}^+, M \cap M_2 \supset U\}$$

is the line of $\Gamma(\mathbb{K})$ through $F$ and $F'$, where $U = M_1 \cap M'_1 \subset M_2$, $\dim(U) = n - 2$ (see 11). If $M_2$ is $\mathbb{K}_0$-rational, there is nothing to prove. Assuming that $M_2$ is not $\mathbb{K}_0$-rational, there are still a number of subcases to examine:

(a) Both $M_1$ and $M'_1$ are $\mathbb{K}_0$-rational. Hence $U = M_1 \cap M'_1$ is $\mathbb{K}_0$-rational. Accordingly, every $(+, -)$-flag $(M, M_2) \in \ell_{U, M_2}$ is nearly $\mathbb{K}_0$-rational.

(b) Neither $M_1$ nor $M'_1$ are $\mathbb{K}_0$-rational. Hence there exist $\mathbb{K}_0$-rational $\dim(U_0) = n - 2$-elements $U_0$ and $U'_0$ such that $U_0 \subset M_1 \cap M_2$ and $U'_0 \subset M'_1 \cap M'_2$. If $U_0 = U'_0$ then $U_0 = M_1 \cap M'_1$. However $M_1 \cap M'_1 = U$. Hence $U = U_0$ is $\mathbb{K}_0$-rational. In this case we are done: all $(+, -)$-flags incident to $U$ are nearly $\mathbb{K}_0$-rational. On the other hand, suppose $U_0 \neq U'_0$. Then $U_0 + U'_0$ is a $\mathbb{K}_0$-rational $(n - 1)$-dimensional subspace of $M_2$. Being $\mathbb{K}_0$-rational, $U_0 + U'_0$ is an $(n - 1)$-element of $B_n^+(\mathbb{K}_0)$. As such, it is contained in just two $n$-elements of $B_n^+(\mathbb{K}_0)$. In other words, both $n$-elements of $B_n^+(\mathbb{K}_0)$ containing $U_0 + U'_0$ are $\mathbb{K}_0$-rational. However $M_2$ is indeed one of those two elements. Therefore $M_2$ is $\mathbb{K}_0$-rational. This contradicts the assumptions made on $M_2$. Consequently, this case we have now been considering cannot occur.
(c) Just one of $M_1$ and $M'_1$ is $K_0$-rational. To fix ideas, let $M_1$ be the $K_0$-rational one. As $(M'_1, M_2)$ is nearly $K_0$-rational by assumption, but neither $M'_1$ nor $M_2$ are $K_0$-rational, there exists a $K_0$-rational $(n-2)$-element $U' \subset M'_1 \cap M_2$. If $U' = U$ then $U$ is $K_0$-rational and we are done.

Suppose that $U' \neq U$. Therefore $U' \not\subset M_1$, otherwise $U' = M_1 \cap M'_1 = U$. As both $M_1$ and $U'$ are $K_0$-rational, their intersection $W := M_1 \cap U'$ is $K_0$-rational. Note that $\dim(W) = n-3$, as one can see by noticing that $M_1 \cap U' = (M_1 \cap U') \cap U' = (M_1 \cap M_2) \cap U'$ and recalling that $M_1 \cap M_2$ is a hyperplane of $M_2$.

Consider the orthogonal $W^\perp$ of $W$ with respect to the form $q$ of Section 2. Taking equation (12) into account and recalling that $W$ is $K_0$-rational vector subspace of $V_{2n}(K)$. In fact $W^\perp$ is the span in $V_{2n}(K)$ of the orthogonal $W_0^\perp \subset V_{2n, E}(K_0)$ of $W_0 := W \cap V_{2n, E}(K_0)$ with respect to the form $q_0$ induced by $q$ on $V_{2n, E}(K_0)$. All of the spaces $M_1, M'_1, U, U'$ and $M_2$ contain $W$ and are totally singular, hence they are contained in $W^\perp$. Moreover, $M_1$ and $U'$ are $K_0$-rational.

As $W^\perp$ is $K_0$-rational, we can choose a basis $B = (w_1, \ldots, w_{n-3})$ of $W^\perp$ formed by $K_0$-rational vectors. We can also assume that $w_1, \ldots, w_{n-3}$ span $W$. As $B$ consists of $K_0$-rational vectors, a vector subspace of $W^\perp$ is $K_0$-rational with respect to $B$ if and only if it is $K_0$-rational with respect to $E$. Accordingly, $M_1$ and $U'$ are $K_0$-rational with respect to $B$ while $M'_1, U$ and $M_2$ are not.

We now switch to the quotient $W^\perp/W$, taking the cosets $\bar{w}_i := w_{n-3+i} + W$ for $i = 1, 2, \ldots, 6$ to form a basis $\bar{B}$ of $W^\perp/W$. Since $W$ is totally singular, the form $q$ induces a quadratic form $\bar{q}$ on $W^\perp/W$. Let $\bar{\Delta}$ be the $D_3$-building associated to $\bar{q}$ and let $\bar{\Gamma}$ be its $(+, -, -)$-grassmannian. So, $(M_1/W, M_2/W)$ and $(M'_1/W, M_2/W)$ are points of the line $\ell_{U/W,M_2/W}$ of $\bar{\Gamma}$ with $U/W \neq U'/W$ and $U'/W \subset M'_1/W \cap M_2/W$. By the above, $M_1/W$ and $U'/W$ are $K_0$-rational while $M'_1/W$, $M_2/W$ and $U/W$ are not. We now switch from the $D_3$-building $\bar{\Delta}$ to the corresponding $A_3$-geometry, with elements of type $+$ and $-$ realized as points and planes of $\text{PG}(V_4(K))$. In this new perspective, the above situation looks as follows: we have two distinct points $p$ and $p'$ (corresponding to $M_1/W$ and $M'_1/W$), two distinct lines $L$ and $L'$ (corresponding to $U/W$ and $U'/W$) and a plane $S$ (corresponding to $M_2/W$). Both $p$ and $p'$ belong to $L$, $p' \not\in L'$. Moreover, $S$ contains both $L$ and $L'$. Hence $S$ is spanned by $p$ and $L'$. However, $M_1/W$ and $U'/W$ are $K_0$-rational. Therefore, in view of Lemma 3.4 both $p$ and $L'$ are $K_0$-rational with respect to a suitable basis $B'$ of $V_4(K)$. Hence $S$ is $K_0$-rational with respect to $B'$, since it is spanned by $p$ and $L'$. By exploiting Lemma 3.3 once again, we obtain that $M_2/W$ is $K_0$-rational with respect to $B$. We have reached a final contradiction, which shows that the case we have been considering cannot occur.

The proof is complete.

Lemma 3.6. Let $V$ be a vector space over a division ring $K$ and $E = (e_1, \ldots, e_n)$ a basis of $V$. Let $K_0$ be a proper sub-division ring of $K$ and take $\eta \in K \setminus K_0$. Suppose $S$ is a subspace of $V$ containing $e_1 + e_2\eta$. If $S$ is $K_0$-rational (with respect to $E$) then $e_1, e_2 \in S$.

Proof. Following our conventions, we assume that $V$ is a right vector space. Let $V_0$ be the $K_0$-vector space of the $K_0$-rational vectors of $V$ (with respect to $E$). In order to avoid any confusion, we denote spans in $V$ by the symbol $\langle \ldots \rangle_V$ and spans in $V_0$ by the symbol $\langle \ldots \rangle_{V_0}$.
Assuming that $S$ is $\mathbb{K}_0$-rational, let $(v_1, \ldots, v_k)$ be a basis of $S$ consisting of $\mathbb{K}_0$-rational vectors and suppose that $c_1 + e_2 \eta \in S$. Then $\dim(S \cap \langle e_1, e_2 \rangle) \geq 1$. Note that the vector space $S_0 := (v_1, \ldots, v_k)_V = S \cap V_0$ has the same dimension as $S$. Thus, since $\dim(S \cap \langle e_1, e_2 \rangle) \geq 1$, we also have $\dim(S_0 \cap \langle e_1, e_2 \rangle V_0) \geq 1$ by the well known Grassmann dimension formula. It follows that there exists a non-zero vector $w \in S_0$ which is a linear combination $w = e_1 c_1 + e_2 c_2$ with $c_1, c_2 \in \mathbb{K}_0$ and $(c_1, c_2) \neq (0, 0)$. If either $c_1 = 0$ or $c_2 = 0$, then we are done. So, we can assume that $c_1 \neq 0 \neq c_2$.

Without loss of generality, we can put $c_2 = 1$, so that $w = e_1 + e_2 c_2$ with $c_2 \in \mathbb{K}_0$. Now we claim that there exists $j_0 \in \{1, \ldots, k\}$ such that $v_{j_0} = e_1 a_{1,j_0} + e_2 a_{2,j_0} + \cdots + e_n a_{n,j_0}$ with $c_2 a_{1,j_0} \neq a_{2,j_0}$.

By way of contradiction, suppose that for all $v_j \in \{v_1, \ldots, v_k\}$ we have

$$v_j = e_1 a_{1,j} + e_2 a_{2,j} + e_3 a_{3,j} + \cdots + e_n a_{n,j}$$

with $c_2 a_{1,j} = a_{2,j}$, i.e. $(a_{1,j}, a_{2,j}) = (1, c_2) d_j$ for some $d_j \in \mathbb{K}$. This implies that for all vectors $v \in S$ we have

$$v = v_1 \lambda_1 + v_2 \lambda_2 + \cdots + v_k \lambda_k = e_1 \left( \sum_{i=1}^{k} \lambda_i + \sum_{i=1}^{k} d_i \right) + e_1 \sum_{i=1}^{k} \lambda_i + \sum_{i=1}^{k} d_i + u$$

with $u \in (e_3, \ldots, e_n)$, $\lambda_i, d_i \in \mathbb{K}$ and $c_2 \in \mathbb{K}_0$. In particular, taking $v = e_1 + e_2 \eta \in S$ we have $\sum_{i=1}^{k} \lambda_i + \sum_{i=1}^{k} d_i = 1$ and $e_1 (\sum_{i=1}^{k} \lambda_i + \sum_{i=1}^{k} d_i) = \eta$, forcing $\eta = c_2 \in \mathbb{K}_0$ which is a contradiction. The claim is proved.

Consider the ordered pair $(w, v_{j_0})$. As $w, v_{j_0} \in S_0$, we can complete this pair to an ordered basis $B$ of $S_0$ by choosing $k - 2$ suitable vectors from the $k - 1$ vectors in $\{v_1, \ldots, v_k\} \setminus \{v_{j_0}\}$. Without getting out of $V_0$, we can now apply a full Gaussian reduction to the sequence of vectors of $B$ to obtain another basis $(v'_1, \ldots, v'_k)$ of $S_0$ such that the $(n \times k)$-matrix $M$ of the coefficients of the vectors $v'_1, \ldots, v'_k$ with respect to $e_1, \ldots, e_n$ is in Column Reduced Echelon Form. (Note that, according to our convention to deal with right vector spaces, vectors should be represented as columns.) By construction, the matrix $M$ contains the identity matrix $I_k$ as a minor. Up to a permutation of the vectors $e_1, \ldots, e_n$ we can suppose that this minor encompasses the first $k$ rows of the matrix $M$. The remaining $n - k$ rows form an $((n - k) \times k)$-matrix

$$N = (b_{k+i,j})_{i=1}^{n-k,k}$$

with entries $b_{k+i,j} \in \mathbb{K}_0$. However $e_1 + e_2 \eta \in S = \langle S_0 \rangle_V = \langle v'_1, v'_2, \ldots, v'_k \rangle_V$. Hence there exist $\alpha_1, \ldots, \alpha_k \in \mathbb{K}$ such that $e_1 + e_2 \eta = v'_1 \alpha_1 + v'_2 \alpha_2 + \cdots + v'_k \alpha_k$. For every $i = 1, \ldots, k$ we have $v'_i = e_i + \sum_{j=k+1}^{n} e_j b_{j,i}$. Therefore

$$e_1 + e_2 \eta = \sum_{i=1}^{k} e_i \alpha_i + e_{k+1} (\sum_{j=1}^{k} b_{k+1,j} \alpha_j) + e_{k+2} (\sum_{j=1}^{k} b_{k+2,j} \alpha_j) + \cdots + e_n (\sum_{j=1}^{k} b_{n,j} \alpha_j),$$

which implies $\alpha_1 = 1, \alpha_2 = \eta, \alpha_3 = \alpha_4 = \cdots = \alpha_k = 0$ and

$$\sum_{j=1}^{k} b_{k+1,j} \alpha_j = \sum_{j=1}^{k} b_{k+2,j} \alpha_j = \cdots = \sum_{j=1}^{k} b_{n,j} \alpha_j = 0.$$

It follows that $e_1 + e_2 \eta = v'_1 + v'_2 \eta$, whence $(e_1 - v'_1) = (v'_2 - e_2) \eta$. However,

$$(e_1 - v'_1) = \sum_{i=k+1}^{n} e_i (b_{i,1}) \eta, \quad (v'_2 - e_2) = \sum_{i=k+1}^{n} e_i (b_{i,2}) \eta,$$
whence $-b_{i+1} = b_{i+2} \eta$ for all $i \geq k + 1$. Since $b_{i+1} \in \mathbb{K}_0$ for all $i, j$ and the elements $1, \eta \in \mathbb{K}$ are linearly independent over $\mathbb{K}_0$, it follows that $b_{i+1} = b_{i+2} = 0$ for all $i \geq k + 1$. So, $v'_1 = e_1$ and $v'_2 = e_2$. Since $v'_1, v'_2 \in S$, we obtain $e_1, e_2 \in S$, which proves the lemma.

**Lemma 3.7.** If $\Gamma(\mathbb{K})$ is as in the hypotheses of Theorem 3.3, then not all points of $\Gamma(\mathbb{K})$ belong to $\Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$.

**Proof.** We first consider the case $\Gamma(\mathbb{K}) = \text{Gr}_{1,n}(A_n(\mathbb{K}))$. Pick $\eta \in \mathbb{K} \setminus \mathbb{K}_0$. With $E = (e_1, \ldots, e_{n+1})$ as in Section 2.4, put $p = \langle e_1 + e_2 \eta \rangle$ and $H = \langle e_1 + e_2 \eta, e_3, \ldots, e_n, e_{n+1} \rangle$. (Needless to say, the symbol ($\ldots$) refers to spans in $V_{n+1}(\mathbb{K})$.) The flag $(p, H)$ is a point of $\text{Gr}_{1,n}(A_n(\mathbb{K}))$. Let $S$ be a subspace of $V_{n+1}(\mathbb{K})$ such that $p \subseteq S \subseteq H$. Any such subspace contains the vector $e_1 + e_2 \eta$ but neither $e_1$ nor $e_2$. Hence $S$ cannot be $\mathbb{K}_0$-rational, by Lemma 3.4. Consequently, $(p, H) \notin \Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$.

The case $\Gamma(\mathbb{K}) = \text{Gr}_{1,+,\ldots,\ldots,1,\ldots,\ldots,1}(D_n(\mathbb{K}))$ is entirely analogous. With $E = (e_1, e_2, \ldots, e_{2n})$ as in Section 2.5 and $\eta$ as above, put $p = \langle e_1 + e_3 \eta \rangle$, $M_1 = \langle e_1 + e_3 \eta, e_2 \eta - e_4, e_5, e_7, \ldots, e_{2n-1} \rangle$, and $M_2 = \langle e_1 + e_3 \eta, e_2 \eta - e_4, e_5, e_7, \ldots, e_{2n-1} \rangle$. Taking equation (12) into account, it is straightforward to see that $p, M_1, M_2$ belong to $D_n(\mathbb{K})$. It is also easy to see that they form a $(1, +, \ldots, -, \ldots, 1)$-flag of $D_n(\mathbb{K})$, namely a point of $\Gamma(\mathbb{K})$. Clearly, none of $p, M_1$ or $M_2$ is $\mathbb{K}_0$-rational and Lemma 3.6 implies that none of the subspaces contained in $M_1 \cap M_2$ and containing $p$ can be $\mathbb{K}_0$-rational. Hence $(p, M_1, M_2) \notin \Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$.

When $\Gamma(\mathbb{K}) = \text{Gr}_{1,+,\ldots,\ldots,1,\ldots,\ldots,1}(D_n(\mathbb{K}))$ we can consider the flag $(p, M)$ where $p = \langle e_1 + e_3 \eta \rangle$ and $M = \langle e_1 + e_3 \eta, e_2 \eta - e_4, e_5, e_7, \ldots, e_{2n-1} \rangle$. The subspace $M$ is $n$-dimensional and totally singular for $\eta$. We can also assume to have chosen the signs $+$ and $-$ in such a way that $\mathfrak{S}^-$ is indeed the class which $M$ belongs to. So, $(p, M)$ is a point of $\Gamma(\mathbb{K})$. Once again, by Lemma 3.6 we see that $(p, M) \notin \Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$.

Finally, let $\Gamma(\mathbb{K}) = \text{Gr}_{1,+,\ldots,\ldots,1,\ldots,\ldots,1}(D_n(\mathbb{K}))$. In view of Lemma 3.4 if $n = 3$ we are back to $A_3$. So, assume $n > 3$. With $\eta \in \mathbb{K} \setminus \mathbb{K}_0$ and $E = (e_1, \ldots, e_{2n})$ as in Section 2.5, put

$$
M_1 := \langle e_1 + e_3, e_2 - e_4, e_5 + e_7 \eta, e_6 \eta - e_8, e_{10}, e_{12}, \ldots, e_{2n} \rangle,
$$

$$
M_2 := \langle e_1 + e_4, e_2 - e_3, e_5 + e_7 \eta, e_6 \eta - e_8, e_{10}, e_{12}, \ldots, e_{2n} \rangle.
$$

Then $M_1$ and $M_2$ are $n$-dimensional totally singular subspaces of $D_n(\mathbb{K})$ but neither of them is $\mathbb{K}_0$-rational. Moreover $M_1 \cap M_2 = \langle e_1 - e_2 + e_3 + e_4, e_5 + e_7 \eta, e_6 \eta - e_8, e_{10}, e_{12}, \ldots, e_{2n} \rangle$. Hence $\{M_1, M_2\}$ is a $\{+, -\}$-flag of $D_n(\mathbb{K})$, necessarily not $\mathbb{K}_0$-rational, since neither $M_1$ nor $M_2$ is $\mathbb{K}_0$-rational. Accordingly, $M_1 \cap M_2$ is not $\mathbb{K}_0$-rational. In fact all $\mathbb{K}_0$-rational subspaces of $M_1 \cap M_2$ are contained in $\langle e_1 - e_2 + e_3 + e_4, e_{10}, e_{12}, \ldots, e_{2n} \rangle$, which is $(n-3)$-dimensional. Their dimensions are too small for them to split $(+, -)$. Therefore $(M_1, M_2) \notin \Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$.

Lemmas 3.6 and 3.7 yield Theorem 3.3.

3.2. Proof of Corollary 3.6

As already remarked in Section 2.3, the function $\iota$ that maps every $(n-1)$-element of $B^n_+ (\mathbb{K})$ onto the pair of $n$-elements containing it is an isomorphism from $\text{Gr}_{n-1}(B^n_+ (\mathbb{K}))$ to $\text{Gr}_{n,-}(D_n(\mathbb{K}))$. We know from Theorem 1.1 that if $\mathbb{K}_0 < \mathbb{K}$ then $\text{Gr}_{n,-}(D_n(\mathbb{K}_0))$ spans a proper subspace of $\text{Gr}_{n,-}(D_n(\mathbb{K}))$. In order to show that the same holds for $\text{Gr}_{n-1}(B^n_+ (\mathbb{K}_0))$ and $\text{Gr}_{n-1}(B^n_+ (\mathbb{K}))$, as claimed in Corollary 1.2, we only need to prove the following:

**Proposition 3.8.** The isomorphism $\iota$ maps the subgeometry $\text{Gr}_{n-1}(B^n_+ (\mathbb{K}_0))$ of $\text{Gr}_{n-1}(B^n_+ (\mathbb{K}_0))$ onto the subgeometry $\text{Gr}_{n,-}(D_n(\mathbb{K}_0))$ of $\text{Gr}_{n,-}(D_n(\mathbb{K}))$.
Proof. It goes without saying that both \( \text{Gr}_{n-1}(B^+_n(\mathbb{K}_0)) = \text{Gr}_{n-1,E}(B^+_n(\mathbb{K}_0)) \) and \( \text{Gr}_{+,E}(D_n(\mathbb{K}_0)) = \text{Gr}_{+,E}(D_n(\mathbb{K}_0)) \) for the same basis \( E \) of \( V_{2n}(\mathbb{K}) \), chosen as in Section 2.5.

Let \( U = M_1 \cap M_2 \) for a \((+,-)\)-flag \((M_1,M_2)\) of \( D_n(\mathbb{K}) \). If both \( M_1 \) and \( M_2 \) are \( \mathbb{K}_0 \)-rational then \( U \) is \( \mathbb{K}_0 \)-rational, by Lemma 2.1. Conversely, let \( U \) be \( \mathbb{K}_0 \)-rational. Let \( M'_1 \) and \( M'_2 \) be the two \( n \)-elements of \( B^+_{n,E}(\mathbb{K}_0) \) containing \( U \). Then \( M'_1 \) and \( M'_2 \) are \( \mathbb{K}_0 \)-rational, as they belong to \( B^+_{n,E}(\mathbb{K}_0) \). However, they are the only two \( n \)-elements on \( U \). Hence \( \{M'_1, M'_2\} = \{M_1, M_2\} \). Accordingly, \( M_1 \) and \( M_2 \) are \( \mathbb{K}_0 \)-rational.

3.3. Proof of Theorem 1.3

We are not going to give a detailed proof of this theorem. We will only offer a sketch of it, leaving the details to reader.

As stated since the beginning of this section, \( \mathbb{K}_0 \) is a proper sub-division ring of \( \mathbb{K} \) and \( \Gamma(\mathbb{K}) = \text{Gr}_J(X_n(\mathbb{K})) \), where \( X_n \) stands for \( A_n \) or \( D_n \). According to the hypotheses of Theorem 1.3 we assume that \( J \) is not connected.

Suppose firstly that \( J \) contains two types \( j_1 \) and \( j_2 \), with \( j_1, j_2 \leq n - 2 \) when \( X_n = D_n \), such that \( j_1 + 1 < j_2 \) and \( i \not\in J \) for every type \( i \in \{j_1 + 1, j_1 + 2, \ldots, j_2 - 1\} \). We say that a \( J \)-flag \( F \) of \( X_n(\mathbb{K}) \) (point of \( \Gamma(\mathbb{K}) \)) is nearly \( \mathbb{K}_0 \)-rational at \((j_1, j_2)\) if there exists a \( \mathbb{K}_0 \)-rational element \( X \) of \( X_n(\mathbb{K}) \) incident to \( F \) and such that \( j_1 \leq \dim(X) \leq j_2 \). Let \( \Omega_{\mathbb{K}_0,j_1,j_2}(\Gamma(\mathbb{K})) \) be the set of \( J \)-flags which are nearly \( \mathbb{K}_0 \)-rational at \((j_1, j_2)\). Using the same argument as in 1) in the proof of Lemma 3.5, with the roles of 1 and \( n \) respectively taken by \( j_1 \) and \( j_2 \) we see that \( \Omega_{\mathbb{K}_0,j_1,j_2}(\Gamma(\mathbb{K})) \) is a subspace of \( \Gamma(\mathbb{K}) \). Next, by an argument similar to that used for \( \text{Gr}_1,2(\mathbb{K}_n(\mathbb{K})) \) in the proof of Lemma 3.7, we obtain that \( \Omega_{\mathbb{K}_0,j_1,j_2}(\Gamma(\mathbb{K})) \neq \Gamma(\mathbb{K}) \), namely \( \Omega_{\mathbb{K}_0,j_1,j_2}(\Gamma(\mathbb{K})) \) is a proper subspace of \( \Gamma(\mathbb{K}) \). However \( \Gamma(\mathbb{K}_0) := \text{Gr}_J(X_n(\mathbb{K}_0)) \) is contained in \( \Omega_{\mathbb{K}_0,j_1,j_2}(\Gamma(\mathbb{K})) \). Hence \( \Gamma(\mathbb{K}_0) \) spans a proper subspace of \( \Gamma(\mathbb{K}) \), as stated in Theorem 1.3.

Two more possibilities remain to examine, which are not considered in Theorem 1.1, namely \( X_n(\mathbb{K}) = D_n(\mathbb{K}) \) and \( J \) as follows:

1. \( J = \{j, j + 1, \ldots, j + k\} \cup \{+, -\} \) for \( j \geq 1 \), \( j + k < n - 2 \) and either \( j > 1 \) or \( k > 0 \). In this case we can use the same arguments as for \( J = \{1, +, -\} \) in the proof of Theorem 1.1 with \( j + k \) playing the role of 1.

2. \( J = \{j, j + 1, \ldots, j + \} \cup \{-\} \) or \( J = \{j, j + 1, \ldots, j + k\} \cup \{+, -\} \) for \( j \geq 1 \), \( j + k < n - 2 \) and either \( j > 1 \) or \( k > 0 \). The arguments used for \( J = \{1, -\} \) work for this case as well, with 1 replaced by \( j + k \).

4. Proof of Lemma 1.4 and Theorem 1.5

4.1. Proof of Lemma 1.4

Assume that \( J \) is non-connected and \( \mathbb{K} \) is not finitely generated. Let \( S \) be a finite set of points of \( \Gamma(\mathbb{K}) = \text{Gr}_J(X_n(\mathbb{K})) \), where \( X_n \) stands for \( A_n \) or \( D_n \). Each element \( F \) of \( S \) is a \( J \)-flag \( F = \{U_1, U_2, \ldots, U_t\} \) of vector subspaces \( U_i \) of \( V_N(\mathbb{K}) \), where \( t := |J| \) and \( N = n + 1 \) or \( 2n \) according as \( X_n \) is \( A_n \) or \( D_n \). Fix a basis \( B_{i,F} \) for each of the vector subspaces \( U_i \in F \) and each \( F \in S \) and let \( C(S) \) be the set of all the coordinates of the vectors of \( \cup_{F \in S} \cup_{i=1}^t B_{i,F} \) with respect to a given basis of \( V_N(\mathbb{K}) \) (chosen as in Section 2.5) when \( X_n = D_n \).

Since \( S \) is finite, \( C(S) \) is finite as well; in fact \( |C(S)| \leq t \cdot N \cdot |S| \). Therefore, and since \( \mathbb{K} \) is not finitely generated, \( C(S) \) generates a proper sub-division ring \( \mathbb{K}_0 \) of \( \mathbb{K} \). Then \( \Gamma(\mathbb{K}_0) := \text{Gr}_J(X_n(\mathbb{K}_0)) \)
spans a proper subspace of \( \Gamma(K) \), by Theorem 1.4. Obviously, \( S \) is contained in \( \Gamma(K_0) \). Hence \( S \) spans a proper subspace of \( \Gamma(K) \). Thus we have proved that no finite subset of \( \Gamma(K) \) generates \( \Gamma(K) \), as claimed in Lemma 1.4.

### 4.2. Proof of Theorem 1.5

Put \( \Gamma \equiv \text{Gr}_{1,n}(A_n(F_p)) \). We have \( \text{gr}(\Gamma) = \infty \) by Lemma 1.4, since \( F_p \) is not finitely generated. The geometry \( \Gamma \) admits a (full) projective embedding of dimension \( (n + 1)^2 - 1 \), namely the embedding \( e_{\text{Lie}} \) mentioned in Remark 1.6. Therefore \( \text{er}(\Gamma) \geq (n + 1)^2 - 1 \).

By way of contradiction, suppose that \( \text{er}(\Gamma) > (n + 1)^2 \). Then \( \Gamma \) admits a (full) projective embedding \( e : \Gamma \rightarrow \text{PG}(V) \) of dimension \( \dim(e) \geq (n + 1)^2 + 1 \). Consequently, there exists a set \( S \) of \( (n + 1)^2 + 1 \) points of \( \Gamma \) such that \( \bigcup_{x \in S} e(x) \subset V \) spans a subspace \( V_S \) of \( V \) of dimension \( \dim(V_S) = (n + 1)^2 + 1 \).

Every point \( x \in S \) is a point-hyperplane flag \( (p_x, H_x) \) of \( A_n(F_p) \). For every \( x \in S \) we choose a non-zero vector \( v_x \in p_x \) and a basis \( B_x \) of \( H_x \). Chosen a basis \( E \) of \( V_{n+1}(F_p) \), let \( C(S) \) be the set of all elements of \( F_p \) which occur as coordinates (with respect to \( E \)) of either \( v_x \) or a vector of \( B_x \), for \( x \in S \). The set \( C(S) \) is finite. Hence it generates a finite subfield \( L \) of \( F_p \). Every point \( x \in S \) is obviously \( L \)-rational. Therefore \( S \subset \Gamma_L \equiv \text{Gr}_{1,n}(A_n(L)) \subset \Gamma \).

Let \( V_L \) be the subspace of \( V \) corresponding to the span of \( e(\Gamma_L) \). Clearly \( V_L \supseteq V_S \). Hence \( \dim(V_L) \geq \dim(V_S) = (n + 1)^2 + 1 \). The restriction \( e_L \) of \( e \) to \( \Gamma_L \) is a lax embedding of \( \Gamma_L \) in \( \text{PG}(V_L) \). As noticed in Remark 1.8, inequality (1) holds for lax embeddings too. Therefore \( \Gamma_L \) has generating rank \( \text{gr}(\Gamma_L) \geq \dim(e_L) = \dim(V_L) > (n + 1)^2 \).

On the other hand, the field \( L \) is a simple extension of the prime field \( F_p \) and \( \text{Gr}_{1,n}(A_n(F_p)) \) has generating rank equal to \( (n + 1)^2 - 1 \), by Cooperstein [7]. Therefore \( \text{gr}(\Gamma_L) \leq (n + 1)^2 \). By Blok and Pasini [2, Corollary 4.8]. We have reached a contradiction. Consequently, \( \text{er}(\Gamma) \leq (n + 1)^2 \). The proof of Theorem 1.5 is complete.

### References

**References**

[1] R. J. Blok and A. E. Brouwer, Spanning point-line geometries in buildings of spherical type, J. Geom. 62 (1998), 26–35.

[2] R. J. Blok and A. Pasini, Point-line geometries with a generating set that depends on the underlying field, in: Finite Geometries (eds. A. Blokhuys et al.), Kluwer, Dordrecht (2001), 1–25.

[3] R. J. Blok and A. Pasini, On absolutely universal embeddings, Discr. Math. 267 (2003), 45–62.

[4] F. Buekenhout and A. M. Cohen, Diagram Geometry, Springer Verlag, Berlin, 2013.

[5] I. Cardinali and A. Pasini, Grassmann and Weyl embeddings of orthogonal Grassmannians, J. Alg. Combin. 38 (2013), 863–888.

[6] I. Cardinali, L. Giuzzi and A. Pasini, The generating rank of a polar Grassmannian, Adv. Geom. 21 (2021), 515–539.
[7] B. N. Cooperstein, Generating long root subgroup geometries of classical groups over finite prime fields, Bull. Belg. Math. Soc. 5 (1998), 531–548.

[8] B. N. Cooperstein and E.E. Shult, Frames and bases of lie incidence geometries, J. Geom 60 (1997), 17-46.

[9] S. Heiss, A note on embeddable $\mathbb{F}_2$-geometries, J. Combin. Theory Series A 92 (2000), 83–87.

[10] A. Kasikova and E. E. Shult, Absolute embeddings of point-line geometries, J. Algebra 238 (2001), 265–291.

[11] A. Pasini, Diagram geometries, Oxford University Press, Oxford 1994.

[12] M. A. Ronan, Embeddings and hyperplanes of discrete geometries European J. Combin. 8 (1987), 179–185.

[13] E. E. Shult, Points and Lines, Springer-Verlag, Berlin 2011.

[14] J. Tits, Buildings of Spherical Type and Finite BN pairs, Springer L. N. in Maths. 386, Springer-Verlag, Berlin 1974.

[15] H. Völllein, On the geometry of the adjoint representation of a Chevalley group, J. Algebra 127 (1989), 139–154.