THE ADDITIVE STRUCTURE OF INTEGERS WITH A FLOOR FUNCTION

MOHSEN KHANI, ALI VALIZADEH, AND AFSHIN ZAREI

Abstract. We introduce a complete axiomatization for the structure $\langle \mathbb{Z}, +, 0, 1, f \rangle$ where $f : x \mapsto \lfloor \alpha x \rfloor$ is a unary function with $\alpha$ a fixed transcendental number. This result fits into the more general theme of adding traces of multiplication to integers without losing decidability. When $\alpha$ is either a quadratic number or an irrational number less than one, similar decidability results have been already obtained by applying heavy techniques from the theory of automata. Nevertheless, our approach is based on a clear axiomatization and involves only elementary techniques from model theory.

1. Introduction

The subject of this study is the structure $\langle \mathbb{Z}, +, 0, 1, f \rangle$ which contains the integer addition together with a trace of multiplication, that is the function $\lfloor \alpha x \rfloor$ whose range is the Beatty sequence with modulo $\alpha$.

This structure is definably a part of the more general structure $\langle \mathbb{R}, <, +, 0, \mathbb{Z}, \alpha \mathbb{Z} \rangle$ which has been the subject of various studies in recent years. Most relevant to our work is Hieronymi’s theorem in [H16] which shows that in a special case that $\alpha$ is quadratic, the structure $\langle \mathbb{R}, <, +, 0, \mathbb{Z}, \alpha \mathbb{Z} \rangle$, and as a result $\langle \mathbb{Z}, +, 0, 1, f \rangle$, is decidable. The decidability is proved there by showing that $\langle \mathbb{R}, <, +, 0, \mathbb{Z}, \alpha \mathbb{Z} \rangle$ is $\omega$-automatic, or equivalently by interpreting it in Büchi’s decidable second-order structure $\langle \mathbb{P}(\mathbb{N}), \mathbb{N}, \in, x \mapsto x + 1 \rangle$.

The results on $\langle \mathbb{Z}, +, 0, 1, f \rangle$ have been generalized recently in different directions. In particular it is shown in [H21] that $\langle \mathbb{Z}, +, 0, 1, f \rangle$ is decidable for any irrational number $\alpha < 1$. Again, techniques from the theory of automata are heavily applied in the proof; therein a main feature is designing a Büchi automaton that can perform addition over Ostrowski-numeration systems.

In the present paper, we aim to prove the decidability of $\langle \mathbb{Z}, +, 0, 1, f \rangle$ for a transcendental $\alpha$ using only elementary tools in model theory and number theory. To our knowledge, nothing is known about the decidability of $\langle \mathbb{R}, <, +, 0, \mathbb{Z}, \alpha \mathbb{Z} \rangle$ for the general case of a transcendental $\alpha$; and we believe our result forms a step towards studying this case. This study is indeed along similar lines to a
result by the first and third authors in [KZ21] in which they applied only simple techniques in model theory to provide an alternative proof for the decidability of \( \mathbb{Z}_\alpha \) (for \( \alpha \) the golden ratio).

To have a brief understanding of the model-theory of \( \mathbb{Z}_\alpha \), note that as \( \mathbb{Z}_\alpha \) is in particular a model of Presburger arithmetic, there are definable sets with a "structured" nature. These are typically a congruence class or an arithmetic progression. But as it turns out, the sets that are definable by powers of the function \( f \), have a "random" nature, in that, say, they do not contain an infinite arithmetic progression (see Subsection 5.2 for more details). This randomness, which is partly reflected in Proposition 2.4 is somewhat weaker when \( \alpha \) is quadratic. For example when \( \alpha \) is the golden ratio, then \( f^2(x) \) is equal to \( f(x) + x - 1 \) (see [KZ21]). That is in this case, all equations reduce to a linear form, whereas in the case that \( \alpha \) is transcendental, all different powers of \( f \) take an independent part in creating random behavior.

As it will be expanded in the next sections, the randomness referred to, is, curiously, due to the random distribution of the decimal parts of \( \alpha^i x \), as a consequence of Kronecker’s approximation lemma (Fact 2.1). This lemma says, in particular, that the decimal part of \( \alpha x \), for all natural numbers \( x \), is dense in the interval \( (0, 1) \). More generally by Kronecker’s lemma, if \( \beta_1, \ldots, \beta_n, 1 \) are linearly independent over \( \mathbb{Q} \) and \( (a_i, b_i) \) are arbitrary subintervals of \( (0, 1) \subseteq \mathbb{R} \), then there is \( x \in \mathbb{N} \) such that the decimal part of \( \beta_i x \) lies in the interval \( (a_i, b_i) \).

To capture all mentioned structured and random behavior in the model-theoretic way, we will present a complete axiomatization \( T_\alpha \) for the structure \( \mathbb{Z}_\alpha \) gradually by introducing each axiom-scheme after proving the related property of \( \mathbb{Z}_\alpha \). Each such property and the axiom-scheme that follows it, is obtained as an instance of independence or dependence of some \( \beta_1, \ldots, \beta_n, 1 \) in an application of Kronecker’s lemma. Also axioms in each section ensure some partial model-completeness for the final theory \( T_\alpha \).

By our Main Theorem, we will show that \( T_\alpha \) is model-complete and this suffices for \( \mathbb{Z}_\alpha \) to be decidable.

**Convention.** Let \( \alpha \) be a transcendental number. When there is no mention of a model or a theory, the lemmas and theorems concern the structure \( \mathbb{Z}_\alpha \), and hence the variable and parameters are assumed in \( \mathbb{Z} \).

2. **Primary axioms**

Let \( T_0 \) denote the theory of Presburger arithmetic. In this section we present basic axioms describing the properties of the powers of the function \( f : x \mapsto \lfloor \alpha x \rfloor \), where we let \( f^0(x) = x \).

Although the function \( f \) clearly concerns the integer part of \( \alpha x \), as it turns out, it is the decimal part, which we will denote with \( \lfloor \alpha x \rfloor \), that plays the crucial role. For an example, observe that \( f(a + b) = f(a) + f(b) + \ell \) where \( \ell \in \{0, 1\} \), and \( \ell = 0 \) if and only if \( \lfloor \alpha a \rfloor + \lfloor \alpha b \rfloor < 1 \).
However it is not the decimals themselves that appear in our theory, but their distribution, in the spirits of the following fact (see [HW08, Theorem 442] for a proof).

**Fact 2.1** (Kronecker’s lemma). The set \( \{([\beta_1 x], \ldots, [\beta_n x]) : x \in \mathbb{N} \} \) is dense in \( (0, 1)^n \) whenever \( \beta_1, \ldots, \beta_n, 1 \) are linearly independent over \( \mathbb{Q} \).

Many properties of the function \( f \) in our structures are described as a result of the fact above, and our aim is to exploit the full extent of this fact to axiomatize a complete theory. Let us begin by an immediate consequence.

**Lemma 2.2.** For all \( a, b \in \mathbb{Z} \)

1. \( [\alpha a] < 1 - [\alpha b] \) if and only if \( \mathbb{Z}_\alpha \models f(b + a) = f(b) + f(a) \).
2. \( [\alpha a] < [\alpha b] \) if and only if the following formula holds in \( \mathbb{Z}_\alpha \).
   \[
   \forall z (f(z + a) = f(z) + f(a) + 1 \rightarrow f(z + b) = f(z) + f(b) + 1).
   \]

**Proof.** Part (1) and the forward direction in (2) are obvious. For the backward direction in the second part note that if \( [\alpha a] > [\alpha b] \), then \( 1 - [\alpha a] < 1 - [\alpha b] \). Hence by Fact 2.1 there is an element \( x \) such that \( 1 - [\alpha a] < [\alpha x] < 1 - [\alpha b] \), which contradicts the assumption. \( \square \)

As a result of Kronecker’s lemma, the second part of the lemma above, gives a universal formula that describes the fact that the decimal part of \( \alpha a \) is smaller than the decimal part of \( \alpha b \). This formula, which we denote by \( \varphi(x, y) \), actually witnesses an order property over the integers. We need of course to axiomatize this order, but we do it after introducing a more flexible notation.

**Notation 2.3.**

1. We write \( [\alpha x] < [\alpha y] \) instead of \( \varphi(x, y) \); that is, we regard the former expression as a first-order formula. In a similar vein, we use expressions like \( [\alpha x] \in ([\alpha a], [\alpha b]) \) or \( [\alpha a] < [\alpha x] < [\alpha b] \).
   2. By \( [\alpha x] < 1 - [\alpha a] \) we mean \( f(x + a) = f(x) + f(a) \) and similarly by \( [\alpha x] > 1 - [\alpha a] \) we mean \( f(x + a) = f(x) + f(a) + 1 \).
   3. We use the notation \( 1 - [\alpha x] < 1 - [\alpha y] \) interchangeably with \( [\alpha y] < [\alpha x] \).
   4. For the ease of readability we remove the quotation marks in the formulas introduced in the above items.
   5. We keep using the notation \( [\alpha x] < [\alpha y] \) (and similar notation for other decimal expressions) for its actual meaning in real numbers, and the distinction will be distinguishable from the context.

As one would expect, we need axioms to enforce \( [\alpha x] < [\alpha y] \) to be a linear order. Using Fact 2.1 it is easy to verify that this order is actually a dense linear ordering.
Axiom 1.

1. The relation \([\alpha x] < [\alpha y]\) is a dense linear order.

2. For all \(x\) and \(y\) either \([\alpha x] < 1 - [\alpha y]\) or \([\alpha x] > 1 - [\alpha y]\).

Since we are dealing with model-completeness here, congruence relations are not a part of our language. Yet we give the following proposition as its proof grasps our basic intuitions by reflecting the way that we are allowed by Fact 2.1 to “freely control” the decimal part of \(\alpha x\).

Proposition 2.4. The following system of equations has infinitely many solutions in \(\mathbb{Z}\).

\[
\begin{align*}
 x \equiv m \pmod{n} \\
 f(x) \equiv j \pmod{n}
\end{align*}
\]

Proof. It is easy to verify that \(f(x) \equiv j \pmod{n}\) if and only if \([\frac{m}{n} x] \in (\frac{j}{n}, \frac{j+1}{n})\). Hence, to solve the system above, it suffices to find \(y\) such that

\[
\left[\frac{(my + i)\alpha}{n}\right] \in \left(\frac{j}{n}, \frac{j+1}{n}\right).
\]

If \(\frac{j+1}{n} > \left[\frac{j}{n}\alpha\right]\), by Fact 2.1 we choose \(y\) such that

\[
\frac{m\alpha}{n} y < 1 - \left[\frac{i}{n}\alpha\right], \quad \text{and}
\]

\[
\frac{j}{n} - \left[\frac{i}{n}\alpha\right] < \left[\frac{m\alpha}{n} y\right] < \frac{j+1}{n} - \left[\frac{i}{n}\alpha\right].
\]

In the case that \(\frac{j+1}{n} < \left[\frac{j}{n}\alpha\right]\), again by Fact 2.1 we choose \(y\) such that

\[
\left[\frac{m\alpha}{n} y\right] > 1 - \left[\frac{i}{n}\alpha\right], \quad \text{and}
\]

\[
1 + \frac{j}{n} - \left[\frac{i}{n}\alpha\right] < \left[\frac{m\alpha}{n} y\right] < 1 + \frac{j+1}{n} - \left[\frac{i}{n}\alpha\right].
\]

\[\square\]

An interesting, but not relevant, observation in regards to the proposition above, is that each \(n\mathbb{Z}\) is dense in \(\mathbb{Z}\) with respect to the order defined by \(\varphi(x, y)\), and this can be proved similarly. Also, according to the proof above, one can add curious notation like “\([\alpha x] \in (\frac{j}{n}, \frac{j+1}{n})\)”.

We now aim to expand on the idea of the proof above further as follows. By the fact that \(\alpha\) is transcendental, each finite sequence \(1, \alpha, \cdots, \alpha^n\) is independent over \(\mathbb{Q}\) and this provides us with more “control” over the decimals through the following extended version of Fact 2.1.

The following theorem concerns the natural numbers and the word “dense” there has its usual meaning in the reals.
Theorem 2.5. Each set of n-tuples \{([\alpha x], [\alpha f(x)], [\alpha f^2(x)], \ldots, [\alpha f^n(x))]) : x \in \mathbb{N}\} is dense in \((0,1)^n\).

Proof. Assume that \([\alpha x] \in \left(\frac{k}{\alpha^m}, \frac{k+1}{\alpha^m}\right)\) for some \(m, k \in \mathbb{N}\) \((m \geq 2)\). Then
\[
\frac{k}{\alpha^{m-1}} < \alpha [\alpha x] < \frac{k+1}{\alpha^{m-1}}.
\]
Now if \([\alpha^2 x] > \frac{k+1}{\alpha^{m-1}}\), then
\[
[\alpha f(x)] = [\alpha^2 x] - \alpha [\alpha x] \in (0, 1 - \frac{k+1}{\alpha^{m-1}}).
\]
Similarly if \([\alpha^2 x] < \frac{k}{\alpha^{m-1}}\), then
\[
[\alpha f(x)] = [\alpha^2 x] - \alpha [\alpha x] + 1 \in (1 - \frac{k}{\alpha^{m-1}}, 1).
\]
Since \((0, 1) \subseteq (0, 1 - \frac{k+1}{\alpha^{m-1}}) \cup (1 - \frac{k}{\alpha^{m-1}}, 1)\) by applying Kronecker’s lemma to \(\alpha\) and \(\alpha^2\) it is possible to find \(x\) such that \([\alpha x] \in (a, b)\) and \([\alpha f(x)] \in (c, d)\) for any \(a, b, c\) and \(d\).

By induction and similar to the argument above, to control \([\alpha f^n(x)] = [\alpha f(f^{n-1}(x))]\) one needs to control \([\alpha^n f(x)]\). This is also possible, since \([\alpha^n f(x)] = [\alpha^{n+1} x - \alpha^n [\alpha x]]\). Now if \(\alpha^n [\alpha x]\) is in an interval \((\frac{k}{\alpha^m}, \frac{k+1}{\alpha^m})\), one can control \([\alpha^{n+1} x]\) so that \([\alpha^n f(x)]\) belongs to any desired interval. \(\square\)

The theorem above, which we may call “an extended Kronecker’s lemma” is included in our axioms as following.

Axiom scheme 2. For all \(a_1, b_1, \ldots, a_n, b_n\) if \([\alpha b_i] < [\alpha a_i]\) then there exists \(x\) such that
\[
\bigwedge_{i=0}^{n} 1 - [\alpha a_i] < [\alpha f^i(x)] < 1 - [\alpha b_i].
\]

Besides the decimals \([\alpha f^i(x)]\) appearing above, as will be seen in the following sections, there are many other decimals that play role in the theory of \(\mathbb{Z}_\alpha\); in their most general form we are concerned with decimals like \([I(\alpha) x]\) where \(I(\alpha)\) is a rational function with integer coefficients in terms of \(\alpha\).

So according to the decimals involved in our formulas, we arrange our argument towards model-completeness by dividing it in two sections. In Section non-algebraic equations below, we explore results that solely concern the decimals \([\alpha f^i(x)]\); that is, theorems that are proved using the axiom above. Those results whose proof involve more complicated decimals will be treated in Section algebraic equations.

We hope that, by following the corresponding sections, the reader finds our terminology reasonable in naming our equations “algebraic” and “non-algebraic”. To justify this very briefly, note that our algebraic equations are in nature similar to the formula “\(f(x) = a\)" which has
finite many solutions in any model of Th($\mathbb{Z}_\alpha$) (and indeed a unique solution when $\alpha > 1$). Also, a simple corollary of Kronecker’s lemma is that there exist infinitely many elements $x$ satisfying the formula “[$\alpha x] \in (0, 1 - [\alpha a])” which forms a typical example of our non-algebraic equations.

3. Non-algebraic equations

The aim of this section is to provide axiom-schemes which ensure the model-completeness up to the non-algebraic formulas, as in Definition 3.5.

Lemma 3.1. There is a universal formula $\psi_\ell(\bar{a}, \bar{b})$ which expresses the following in $\mathbb{Z}_\alpha$,

$$[\alpha a_1] + \cdots + [\alpha a_n] < [\alpha b_1] + \cdots + [\alpha b_n] + \ell \tag{3.1}$$

for $\ell$ an integer.

Proof. If $\ell \geq n$ (respectively $\ell \leq -n$), then $\psi_\ell$ is equivalent to a logically valid formula (respectively contradiction). Suppose that $0 \leq \ell < n$, then one way to write $\psi_\ell$ is as the conjunction of the following formulas for $\ell < j \leq n$

$$\forall x \left( f(a_1 + \cdots + a_n + x) = f(a_1) + \cdots + f(a_n) + f(x) + j \right) \rightarrow \bigvee_{0<i<n} f(b_1 + \cdots + b_n + x) = f(b_1) + \cdots + f(b_n) + f(x) + (j - \ell) + i \bigg). \tag{3.2}$$

If $\ell < 0$, then $\psi_\ell$ can similarly be obtained as a conjunction of formulas as below

$$\forall x \left( f(b_1 + \cdots + b_n + x) = f(b_1) + \cdots + f(b_n) + f(x) + j \right) \rightarrow \bigvee_{0<i<n} f(a_1 + \cdots + a_n + x) = f(a_1) + \cdots + f(a_n) + f(x) + (j + \ell) + i \bigg). \tag{3.2}$$

Remark 3.2. The proof above shows that the negation of $\psi_\ell$ is also a universal formula. Hence $\psi_\ell$ is equivalent to an existential formula as well.

Lemma 3.3. The following is equivalent to a quantifier-free formula $\varphi_{n,i}(a)$ in $\mathbb{Z}_\alpha$.

$$[\alpha a] = n[\alpha a] - i \tag{3.3}$$

Proof. Simply let $\varphi_{n,i}(a)$ be the formula $f(na) = nf(a) + i$. \qed

Notation 3.4. Instead of $\psi_\ell$ and $\varphi_{n,i}$, we write the equivalent decimal expressions, that is we regard (3.1) and (3.3) as first order formulas. We also use the summation symbol $\sum$ in our formulas with the obvious meaning.
**Definition 3.5.** Call a formula \( \theta(x_1, \ldots, x_n, \bar{a}) \) non-algebraic if it is equivalent to a formula of the following form

\[
\sum_{i=0}^{k} n_i [\alpha f^i(x_1)] + \cdots + \sum_{i=0}^{k} n_i [\alpha f^i(x_n)] < \sum_{i=0}^{k} m_i [\alpha a_i] + \ell
\]

with \( m_i, n_i, \ell \in \mathbb{Z} \). By the non-algebraic type of \( x_1, \ldots, x_n \) over \( A \), we mean the partial type consisting of all non-algebraic formulas \( \theta(\bar{x}, \bar{a}) \) with \( \bar{a} \) in \( A \).

**Remark 3.6.** Note that by Lemma 2.2, \( [\alpha x] < [\alpha a] \) and \( [\alpha x] < 1 - [\alpha a] \) are both examples of non-algebraic formulas. The non-algebraic type of \( x \) over \( a \) in particular determines the numbers \( \ell_i \in \{0, 1\} \) such that

\[
\begin{align*}
f(x + a) &= f(x) + f(a) + \ell_1 \\
f^2(x + a) &= f(f(x) + f(a) + \ell_1) = f^2(x) + f(f(a) + \ell_1) + \ell_2 \\
f^3(x + a) &= f(f^2(x) + f(f(a) + \ell_1) + \ell_2) = f^3(x) + f(f(f(a) + \ell_1) + \ell_2) + \ell_3 \\
&
\end{align*}
\]

By our notation, the above set of equations can be rewritten as follows assuming \( \ell_i = 0 \) (respectively \( \ell_i = 1 \)).

\[
\begin{align*}
[\alpha x] + [\alpha a] &< 1 & (> 1) \\
[\alpha f(x)] + [\alpha (f(a) + \ell_1)] &< 1 & (> 1) \\
[\alpha f^2(x)] + [\alpha (f(f(a) + \ell_1) + \ell_2)] &< 1 & (> 1) \\
&
\end{align*}
\]

More generally, the non-algebraic type of \( x \) over \( A \) determines equalities as

\[
f(k_1 f(\cdots f(k_i x + a_i) \cdots) + a_1) = \sum_i m_i f^i(x) + \sum_i n_i f^i(a_i) + \ell.
\]

**Lemma 3.7.** Let \( \bar{m}, \bar{n}, \bar{\ell} \) be finite tuples. There is a formula \( \chi(\bar{a}) = \chi_{\bar{m}, \bar{n}, \bar{\ell}}(\bar{a}) \) such that \( \mathbb{Z}_a \models \chi(\bar{a}) \) if and only if the following system of equations is solvable in \( \mathbb{Z} \).

\[
\begin{align*}
\sum n_{1j} [\alpha f^j(x)] &< b_1 = \sum m_{1j} [\alpha a_j] + \ell_1 \\
&
\end{align*}
\]

(3.4)

\[
\sum n_{kj} [\alpha f^j(x)] < b_k = \sum m_{kj} [\alpha a_j] + \ell_k
\]
Proof. Note that the solvability of the following system in \( \mathbb{R} \) (with \( x_j \in (0, 1) \))

\[
\sum n_{1j} x_j < b_1 \\
\vdots \\
\sum n_{kj} x_j < b_k
\]

(3.5)

depends on its integer coefficients and comparisons of the numbers \( b_i \). The comparison of \( b_i \) is yet possible in our language using the non-algebraic formulas. \( \square \)

Axiom scheme 3. There is a solution to the system (3.4) if and only if \( \chi_{\bar{m}, \bar{n}, k, \ell}(\bar{a}) \).

Let \( T_{\text{nalg}} \) be \( T_0 \) together with the axiom-schemes 1 to 3. By the next theorem \( T_{\text{nalg}} \) is model-complete up to the non-algebraic formulas. For an interesting connection of \( T_{\text{nalg}} \) to \( o \)-minimality, see Subsection 5.3.

**Theorem 3.8.** Suppose that \( \mathcal{M}_1 \subseteq \mathcal{M}_2 \) are models of \( T_{\text{nalg}} \), then if a system consisting of equations

\[
\sum_{i=1}^{k} n_{1i} [\alpha f^i(x_1)] + \cdots + \sum_{i=1}^{k} n_{pi} [\alpha f^i(x_p)] < \sum_{i=1}^{k} m_i [\alpha a_i] + \ell
\]

with parameters in \( \mathcal{M}_1 \) has a solution in \( \mathcal{M}_2 \), then so does it in \( \mathcal{M}_1 \).

**Proof.** Suppose that the system has a solution \((x_1, \ldots, x_p)\) in \( \mathcal{M}_2 \). Concerning the system above, let \( \Gamma_1(x_1/\bar{a}) \) be the set of all formulas containing the single variable \( x_1 \) and the parameters \( \bar{a} \). Hence \( \Gamma_1(x_1/\bar{a}) \) is a finite subset of the partial non-algebraic type of \( x_1 \) over \( \bar{a} \), so there is an element \( b_1 \in \mathcal{M}_1 \) that satisfies \( \Gamma_1(x_1/\bar{a}) \). Now, replace the variable \( x_1 \) with \( b_1 \) in the system and let \( \Gamma_2(x_2/b_1\bar{a}) \) be the set of all formulas only containing the variable \( x_2 \) and parameters \( b_1\bar{a} \) in the new system. Similarly, one can find \( b_2 \in \mathcal{M}_1 \) that satisfies \( \Gamma_2(x_2/b_1\bar{a}) \) in \( \mathcal{M}_1 \). Continuing this process one reaches a solution \( \bar{b} \in \mathcal{M}_1 \) for the system. \( \square \)

### 4. Algebraic Equations

Our model-completeness theorem so far, Theorem 3.8, is obtained by means of the independency, offered by Kronecker’s lemma, among the decimals \([\alpha f^i(x)]\) for different powers of \( f \), or equivalently the independency of the decimals \([\alpha^i x]\).

However, as mentioned earlier, a thorough understanding of \( Z_\alpha \) is possible only by involving some other decimals as well. These decimals which are typically of the form \([\frac{1}{\alpha} x]\), or more generally \([\frac{1}{F(\alpha)} x]\) for \( F \) a polynomial in terms of \( \alpha \) with integer coefficients, will mainly appear when we are analyzing the range of an algebraic equation. More clarification on this is made at the beginning of Subsection 4.1.
The aim of this section is to provide a number of axiom-schemes meant to support our argument towards model-completeness in the most general case of systems that involve both non-algebraic and algebraic formulas. The main idea would be to omit all algebraic equations in order to reduce to the previously handled case of equations of solely a non-algebraic nature. But to get to this point, some rather untidy calculations will be inevitable.

**Definition 4.1.** We call a formula \( \theta(\bar{x}, A) \) algebraic if it is equivalent to an expression of the following form

\[
\sum f_i^1(m_{i1}x_1) + \ldots + \sum f_i^m(m_{in}x_n) = A.
\]

Systems involving algebraic formulas will be treated separately based on the different number of variables appearing in the system. We found it more pragmatic to explain our reasons for this at the beginning of Subsection 4.3.

### 4.1. One variable.

We begin with the insightful observation that for an element \( A \), being in the range of \( f \) turns out to be an intrinsic property of \( A \) within a model of the full-theory of \( \mathbb{Z}_\alpha \); that is it is preserved in submodels. When \( \alpha > 2 \), this can be proved by observing that \( A \) lies in the range of \( f \) if and only if no \( A - j \) and \( A + j \) does so, for any \( 0 < j < \lfloor \alpha \rfloor \). The latter however is expressible by a universal formula and in turn, follows from the fact that \( f(x + 1) - f(x) = \ell \) for some \( \ell \in \{ \lfloor \alpha \rfloor, \lfloor \alpha \rfloor + 1 \} \). Hence if an equation \( f(x) = A \) is solvable in a model of \( \text{Th}(\mathbb{Z}_\alpha) \) then it is solvable in all of its submodels too.

Nevertheless, we treat this phenomenon rather differently by looking at \( A \in \text{Image}(f) \) in metalanguage. It is straightforward to see that for \( \alpha > 1 \) an element \( A \) lies in the range of \( f \) if and only if \( \lfloor \frac{A}{\alpha} \rfloor > 1 - \frac{1}{\alpha} \). In this case if \( f(x) = A \) then \( x = \lfloor \frac{A}{\alpha} \rfloor + 1 \). The situation becomes easier when \( \alpha \) is less than one, since in that case \( f \) happens to be a surjection.

In solving the algebraic formulas we implement a more general form of this observation with \( \alpha \) replaced by a polynomial \( F(\alpha) \), as described in the following lemma.

**Lemma 4.2.** Let \( H(x) \) denote the term \( n_1x + n_2f(x) + \ldots + n_kf^k(x) \).

1. If solvable, the equation \( H(x) = A \) has at most finitely many solutions scattered in a finite distance to one another. Moreover, the maximum number of solutions as well as their maximum distance is determined in \( \text{Th}(\mathbb{Z}_\alpha) \).

2. There is a natural number \( K \), determinable in \( \text{Th}(\mathbb{Z}_\alpha) \), such that one in each \( K \) successive elements is in the range of \( H(x) \). Moreover, if for some \( j \leq K \) there are elements \( x \) and \( x' \) which are respectively solutions to the equations \( H(x) = A \) and \( H(x') = A + j \), then the distance between \( x \) and \( x' \) cannot exceed \( K \).

**Proof.** (1) Note that for any \( x \) and \( i \), there exists an integer \( \ell \) such that \( f^i(x) = \lfloor \alpha^i x \rfloor + \ell \). This observation together with elementary properties of the floor function show that for any \( x \) the
equation $H(x) = A$ is equivalent to an equation of the form below
\[ [(n_1 + n_2 \alpha + \ldots n_k \alpha^k)x] = A + j \]
for some $j \in \mathbb{Z}$. The integer interval that $j$ ranges over depends only on the powers of $f$ and the coefficients that appear in $H(x)$.

By setting $F(\alpha) = n_1 + n_2 \alpha + \ldots n_k \alpha^k$, it is easy to verify that the equation above possesses at most $(\lfloor F(\alpha) \rfloor + 1)$-many solutions, and that they are successive. Hence, the maximum number of possible solutions for $H(x) = A$ is bounded by $\lfloor F(\alpha) \rfloor + 1$. But the latter determines the distance of the solutions as well, and it does so uniformly for all $A$ in $\text{Th}(\mathbb{Z}_\alpha)$. We note in passing that for $F(\alpha) > 1$ an equation of the form $[F(\alpha)x] = A$ has the unique solution of $\lfloor A F(\alpha) \rfloor + 1$ whenever it is solvable.

(2) Note that the image of $[F(\alpha)x]$ is unbounded in $\mathbb{Z}_\alpha$, and hence the same holds for $H(x)$. Now, one only needs to note that there is a natural number $K$ such that for all $x$
\[ H(x + 1) - H(x) < K. \]

We need to formulate an axiom-scheme to embrace the content of the lemma above. One needs to carefully observe that the following is a first-order axiom-scheme (which is also recursively enumerable). In fact, we only need to include the second part of Lemma 4.2 for each $H(x) = n_1 x + n_2 f(x) + \ldots + n_k f^k(x)$ as below.

**Axiom scheme 4.** One in each $K$ successive elements, with $K$ as in the lemma above, is in the range of $H(x)$. Also $|H(x) - H(x')| \leq K$ implies $|x - x'| \leq K$ for any $x$ and $x'$.

**Lemma 4.3.** Assume that $\mathcal{M}_1 \subseteq \mathcal{M}_2$ are models of $T_0$ together with the axiom-schemes 1 to 4. Also suppose that $A \in \mathcal{M}_1$ and there is an element $x \in \mathcal{M}_2$ with $n_1 x + n_2 f(x) + \ldots + n_k f^k(x) = A$, then $x$ belongs to $\mathcal{M}_1$.

**Proof.** By axiom-scheme 4, there is an integer $j \leq K$ and an element $x' \in \mathcal{M}_1$ such that $n_1 x' + n_2 f(x') + \ldots + n_k f^k(x') = A + j$. Again, axiom-scheme 4 implies that the distance between $x$ and $x'$ is necessarily an standard integer. This shows that $x$ is actually a member of $\mathcal{M}_1$. \qed

We now proceed by increasing the number of equations and variables.

4.2. **Two variables.**
4.2.1. **One algebraic equation.** Consider the system consisting of the following algebraic equation

\[
\sum_{i=0}^{M} f^i(c_i x) + \sum_{i=0}^{N} f^i(d_i y) = A
\]  

with \(c_i, d_i\) being integers, together with finitely many non-algebraic equations of the form

\[
\sum_{i=0}^{M-1} m_i \lfloor \alpha f^i(x) \rfloor + \sum_{i=0}^{N-1} n_i \lfloor \alpha f^i(y) \rfloor < a
\]

in which as before, \(f^0(x) = x\) and \(m_i, n_i\) are integers.

In the following we will need to get into detailed—yet not very complicated—calculations which lead to an axiom-scheme for \(\mathbb{Z}_\alpha\) to guarantee the solvability of the system above. To make it easier to follow, we first give an overview of how we aim to achieve this goal.

**Overview of the procedure.** We will write Equation (4.1) as

\[
\lfloor F(\alpha)x \rfloor + \lfloor G(\alpha)y \rfloor = A
\]

for some suitable polynomials, \(F\) and \(G\), in terms of \(\alpha\) with integer coefficients. This polynomials are obtained by combining the integer parts at the expense of adding/subtracting small integers. Then letting \(z = \lfloor F(\alpha)x \rfloor\), we take \(x = \lfloor \frac{z}{F(\alpha)} \rfloor + 1\) and \(y = \lfloor \frac{A-z}{G(\alpha)} \rfloor + 1\). We then replace occurrences of \(x\) and \(y\) in \([\alpha f^i(x)]\) and \([\alpha f^i(y)]\) in (4.2) respectively with \(\frac{z}{F(\alpha)}\) and \(\frac{A-z}{G(\alpha)}\). Then we will essentially be dealing with non-algebraic equations for the variables \(\frac{z}{F(\alpha)}\) and \(\frac{A-z}{G(\alpha)}\). We will solve these equations by paying careful attention to the algebraic dependence of the decimal parts incurred by \(F(\alpha)\) and \(G(\alpha)\), and the relaxations that come after Fact 2.1.

**Remark 4.4.** Note that if one of the polynomials mentioned above, say \(F(\alpha)\), happens to be less than one, then the corresponding floor function, \(\lfloor F(\alpha)x \rfloor\), is no longer an injective map and becomes a finite-to-one surjection. Therefore, we must change the corresponding value, \(x = \lfloor \frac{z}{F(\alpha)} \rfloor + 1\), by adding an integer \(\ell \leq \lfloor \frac{1}{F(\alpha)} \rfloor\) whose choice can be constrained by the non-algebraic part of the system. However, this will not affect our argument below and we may continue working without taking in the integer \(\ell\).

**Simplification 1.** Equations in (4.2) can be replaced with equations

\[
\sum_{i=1}^{M} m_i \lfloor \alpha^i x \rfloor + \sum_{i=1}^{N} n_i \lfloor \alpha^i y \rfloor < a
\]

for suitable \(a\). Also (4.1) can be replaced with an equality of the form

\[
\lfloor F(\alpha)x \rfloor + \lfloor G(\alpha)y \rfloor = A
\]

for suitable polynomials \(F, G\) in terms of \(\alpha\) with integer coefficients. This change of format is possible at the expense of adding some new inequalities, to the non-algebraic formulas.
Justification. Note that for any $x$ and $i \leq M$ there exists a natural number $\ell_i$ satisfying

$$f^i(x) = \lfloor \alpha^i x \rfloor - \ell_i.$$ 

In terms of the decimals, this is equivalent to the inequality

$$\ell_i < \lfloor \alpha^i x \rfloor - \alpha \lfloor \alpha^{i-1} x \rfloor - \alpha \ell_{i-1} < \ell_i + 1.$$ 

We add the inequality above to our system of non-algebraic equations. Similarly, for every $y$ and $i \leq N$ there exists a natural number $\ell'_i$ satisfying

$$f^i(y) = \lfloor \alpha^i y \rfloor - \ell'_i.$$ 

Also this is equivalent to the inequality

$$\ell'_i < \lfloor \alpha^i y \rfloor - \alpha \lfloor \alpha^{i-1} y \rfloor - \alpha \ell'_{i-1} < \ell'_i + 1.$$ 

Now our desired equation in the format (4.4) is simply obtained, but we also need to add to our system, non-algebraic equations of the following form.

$$\ell_1 < \sum_{i=1}^{M} m_i \lfloor \alpha^i x \rfloor < \ell_1 + 1,$$

$$\ell_2 < \sum_{i=1}^{N} n_i \lfloor \alpha^i y \rfloor < \ell_2 + 1$$ 

Hence, reusing $A$ and $a$ for the ease of notation, the system we need to solve is the following.

$$A = \lfloor F(\alpha) x \rfloor + \lfloor G(\alpha) y \rfloor$$

$$\sum_{i=1}^{M} m_i \lfloor \alpha^i x \rfloor + \sum_{i=1}^{N} n_i \lfloor \alpha^i y \rfloor < a$$

$$\ell_i < \lfloor \alpha^i x \rfloor - \alpha \lfloor \alpha^{i-1} x \rfloor - \alpha \ell_{i-1} < \ell_i + 1, \quad 1 < i \leq M$$

$$\ell'_i < \lfloor \alpha^i y \rfloor - \alpha \lfloor \alpha^{i-1} y \rfloor - \alpha \ell'_{i-1} < \ell'_i + 1, \quad 1 < i \leq N$$

Note that (4.8) is what we called a non-algebraic equation, as its coefficients are integers. But the inequalities (4.9) and (4.10) have irrational coefficients, and in this regard, we have used and will carefully continue to use the term “inequality”, instead of “non-algebraic equation”, to refer to them. Of course we will later carefully address how we handle the issue of the irrational coefficients.

Now to enter Kronecker’s lemma into the picture, in the rest we need to argue in two cases according to the dependence or independence of $\frac{1}{F(\alpha)}$ and $\frac{1}{G(\alpha)}$, 1.
(a) **The case where \( 1, \frac{1}{F(\alpha)}, \frac{1}{G(\alpha)} \) are independent over \( \mathbb{Q} \).** In this case, it is rather easy to solve Equation (4.7) in \( \mathbb{Z} \). One only needs to find \( z \) such that \( \left\lfloor \frac{z}{F(\alpha)} \right\rfloor \in (1 - \frac{1}{F(\alpha)}, 1) \) and \( \left\lfloor \frac{A-z}{G(\alpha)} \right\rfloor \in (1 - \frac{1}{G(\alpha)}, 1) \), and this is possible by Fact 2.1, and observing that the pairs \((\left\lfloor \frac{z}{F(\alpha)} \right\rfloor, \left\lfloor \frac{A-z}{G(\alpha)} \right\rfloor)\) are dense in \((0, 1)^2\) for \( z \in \mathbb{N} \). The latter results from the fact that indeed the pairs \((\left\lfloor \frac{z}{F(\alpha)} \right\rfloor, \left\lfloor \frac{A-z}{G(\alpha)} \right\rfloor)\) are dense in \((0, 1)^2\).

Putting \( t = \frac{z}{F(\alpha)} \) and \( t' = \frac{A-z}{G(\alpha)} \), we consider all solutions of Equation (4.7) of the form

\[ x = \lfloor t \rfloor + 1, \quad y = \lfloor t' \rfloor + 1 \]

and we add the following inequalities:

\begin{align*}
(4.11) & \quad F(\alpha) - 1 < F(\alpha) \lfloor t \rfloor < F(\alpha) \\
(4.12) & \quad G(\alpha) - 1 < G(\alpha) \lfloor t' \rfloor < G(\alpha)
\end{align*}

**Simplification 2.** One can adapt the system so that all occurrences of \( \lfloor \alpha^i x \rfloor \) for \( i \leq M \) are replaced with \( \lfloor \alpha^i t \rfloor \) (and similarly \( \lfloor \alpha^i y \rfloor \) are replaced with \( \lfloor \alpha^i t' \rfloor \)).

**Justification.** For each \( i \leq M \),

\begin{align*}
(4.13) & \quad \lfloor \alpha^i x \rfloor = \lfloor \alpha^i t \rfloor + \alpha^i - \alpha^i \lfloor t \rfloor - u_i, \\
& \text{provided that} \\
(4.14) & \quad u_i < \lfloor \alpha^i t \rfloor + \alpha^i - \alpha^i \lfloor t \rfloor < u_i + 1.
\end{align*}

Similarly, for \( i \leq N \),

\begin{align*}
(4.15) & \quad \lfloor \alpha^i y \rfloor = \lfloor \alpha^i t' \rfloor + \alpha^i - \alpha^i \lfloor t' \rfloor - v_i, \\
& \text{provided that} \\
(4.16) & \quad v_i < \lfloor \alpha^i t' \rfloor + \alpha^i - \alpha^i \lfloor t' \rfloor < v_i + 1.
\end{align*}

So, Equation (4.8) can be rewritten as

\begin{align*}
(4.17) & \quad \sum_{i=1}^{M} m_i \lfloor \alpha^i t \rfloor - \lfloor t \rfloor \sum_{i=1}^{M} m_i \alpha^i + \sum_{i=1}^{N} n_i \lfloor \alpha^i t' \rfloor - \lfloor t' \rfloor \sum_{i=1}^{N} n_i \alpha^i < a - s \\
& \text{with } s = \sum_{i=1}^{M} m_i (\alpha^i - u_i) + \sum_{i=1}^{N} n_i (\alpha^i - v_i). \quad \text{Also, Equations (4.9) and (4.10) are rewritten as} \\
(4.18) & \quad \ell_i < \lfloor \alpha^i t \rfloor - \alpha \lfloor \alpha^{i-1} t \rfloor + u_{i-1} - u_i < \ell_i + 1, \quad 1 < i \leq M \\
(4.19) & \quad \ell_i' < \lfloor \alpha^i t' \rfloor - \alpha \lfloor \alpha^{i-1} t' \rfloor + v_{i-1} - v_i < \ell_i' + 1, \quad 1 < i \leq N.
\end{align*}

□ end of justification.
Applying the dependencies. Note that \([\alpha^M t] \) is dependent, in the way explained in the following, to the rest of \([\alpha^i t] \) for \( i < M \). (Similarly, \([\alpha^N t'] \) is dependent to the rest of \([\alpha^i t'] \)). Write \( F(\alpha) = c_0 + c_1\alpha + \ldots + c_M\alpha^M \) with \( c_M \neq 0 \). Then

\[
\frac{\alpha^n}{F(\alpha)} = \frac{1}{c_M} - \left( \frac{r_0}{F(\alpha)} + \frac{r_1\alpha}{F(\alpha)} + \ldots + \frac{r_{M-1}\alpha^{M-1}}{F(\alpha)} \right),
\]

where \( r_i = \frac{c_i}{c_M} \) for each \( i \in \{0, \ldots, M-1\} \). Multiplying the above by \( z \) and taking the decimal parts,

\[
[a^M t] = -\sum_{i=0}^{M-1} r_i[a^i t] + q
\]

with \( q \) a rational number. This necessitates us to add the following non-algebraic equation.

\[
q < -\sum_{i=0}^{M-1} r_i[a^i t] < q + 1.
\]

Similarly

\[
[a^N t'] = -\sum_{i=0}^{N-1} r'_i[a^i t'] + q'
\]

and the following non-algebraic equation is added.

\[
q' < -\sum_{i=0}^{N-1} r'_i[a^i t'] < q' + 1.
\]

By replacing Equations (4.21) and (4.23) in (4.17), we obtain

\[
\sum_{i=1}^{M-1} (m_i - m_M r_i) [a^i t] - [t] (m_M r_0 + \sum_{i=1}^{M} m_i\alpha^i) + m_M q
\]

\[
+ \sum_{i=1}^{N-1} (n_i - n_N r'_i) [a^i t'] - [t'] (n_N r'_0 + \sum_{i=1}^{N} n_i\alpha^i) + n_N q' < a - s
\]

Also, by replacing Equations (4.21) in (4.18) for \( i = M \) and (4.23) in (4.19) for \( i = N \), we obtain

\[
\ell_M < -\sum_{i=0}^{M-1} r_i[a^i t] - \alpha [a^{M-1} t] + u_{M-1}\alpha - u_M + q < \ell_M + 1,
\]

\[
\ell'_N < -\sum_{i=0}^{N-1} r'_i[a^i t'] - \alpha [a^{N-1} t'] + v_{N-1}\alpha - v_N + q' < \ell'_N + 1.
\]

Treating \([\alpha^i t] \) and \([\alpha^j t'] \) as independent variables. For all \( 0 \leq i < M \) and \( 0 \leq j < N \), \([\alpha^i t] \) and \([\alpha^j t'] \) can be treated as independent variables, hence one can respectively replace \([\alpha^i t] \) and \([\alpha^j t'] \) with
variables $z_i$ and $w_j$. So the system is converted to the following form

\[(4.28)\quad \sum_{i=1}^{M-1} m'_i z_i + \sum_{j=1}^{N-1} n'_j w_j - z_0 d - w_0 d' < a' \]

\[(4.29)\quad b_i < z_i + \alpha z_{i-1} < b_i + 1, \quad 1 < i \leq M - 1 \]

\[(4.30)\quad b'_i < w_i + \alpha w_{i-1} < b'_i + 1, \quad 1 < i \leq N - 1 \]

\[(4.31)\quad b_M < -\sum_{i=0}^{m-1} r_i z_i - \alpha z_{m-1} < b_M + 1 \]

\[(4.32)\quad b'_N < -\sum_{i=0}^{N-1} r'_i w_i - \alpha w_{N-1} < b'_N + 1. \]

with $m'_i = m_i - m_M r_i$ for all $i < M$ and $n'_j = n_j - n_N r'_j$ for all $j < N$, and $d = m_M r_0 + \sum_{i=1}^{M} m_i \alpha^i$, and $d' = n_N r'_0 + \sum_{j=1}^{N} n_j \alpha^j$.

Any system consisting of inequalities of the forms (4.29), (4.30), (4.31), and (4.32), always has a solution, since it deals with hyperplanes which mutually intersect each other—and the hyperplane in (4.28). Hence, the system above has a solution if and only if all equations of the form (4.28) are simultaneously solvable. We now get to the point to give a concise account of all calculations above.

**Lemma 4.5.** Let $\Omega$ be a system consisting of an equation of the form

\[ \sum_{i=0}^{M} f^i(c_i x) + \sum_{i=0}^{N} f^i(d_i y) = A \]

with $c_i, d_i$ integers, together with finitely many non-algebraic equations of the form

\[ \sum_{i=0}^{M-1} m_i [\alpha f^i(x)] + \sum_{i=0}^{N-1} n_i [\alpha f^i(y)] < a \]

with integer coefficients.

1. $\Omega$ has a solution in $\mathbb{Z}$ if the system $\Omega'$ consisting only of modified equations of the form (4.28) has a solution in $\mathbb{R}$, forcing that the variables are in the interval $(0, 1)$.

2. There is a formula $\varepsilon_\Omega$ (in terms of the parameters in the system) such that $\mathcal{Z}_{\alpha} \models \varepsilon_\Omega$ if and only if $\Omega$ has a solution in $\mathbb{R}$.

**Proof.** We only need to prove the second item. Note that each equation of the form (4.28) is actually the equation of a hyperplane in $\mathbb{R}$ in variables $z_i, w_i$. In order to solve such a system, one needs to analyze these hyperplanes regarding if they are parallel or intersecting each other. But two such hyperplanes are parallel if and only if the ratio of the corresponding coefficients is equal, we denote this ratio by $\lambda$. But it is easy to verify if $\lambda$ is a rational (note that the only way $\lambda$ could be irrational would be when all $m'_i, n'_j$ were zero, but an easy calculation reveals that also in this case the ratio in question is rational).

Now when the hyperplanes are parallel, to check the solvability of the system one needs to compare the constant terms. But, by the previous section this comparison is expressible by first order formulas in our language.

Looking at the right-hand-sides of the equations above, the reader needs also to makes sure that expressions like $[\alpha x] < F(\alpha)$ are also allowed by first order formulas. Indeed instead of referring to
$F(\alpha)$ one can refer to $[F(\alpha) \cdot 1]$, as the integer part of $F(\alpha) \cdot 1$ is a finite number determined by the coefficients of the system. But, as in simplifications above, each $[\alpha^i \cdot 1]$ is expressible and hence so is $[F(\alpha) \cdot 1]$. Finally, this brings us to the following axiom-scheme.

**Axiom scheme 5.** $\Omega$ has a solution if and only if $\varepsilon_\alpha$.

(b) The case where $1, \frac{1}{F(\alpha)}$ and $\frac{1}{G(\alpha)}$ are dependent over $\mathbb{Q}$. In this case we need to combine the previous methods of solving algebraic equations with one and two variables (respectively appeared in Subsection 4.2.1 and Case (b) above). This time we do not get into as much details as in the previous case since the idea is essentially the same.

The only way $\frac{1}{F(\alpha)}, \frac{1}{G(\alpha)}, 1$ are dependent over $\mathbb{Q}$, is that $G(\alpha) = rF(\alpha)$ for some rational number $r$. Hence our main Equation (4.11) becomes of the form

\[
[F(\alpha)(x + ry)] = A,
\]

for a suitable $A$, and for this to be solvable one first needs $A$ to be in the range of the function $[F(\alpha)z]$. That is, the existence of an element $A'$ is needed such that

\[
x + ry = A' = \left[\frac{A}{F(\alpha)}\right] + 1.
\]

Now we replace $x$ with $A' - ry$ in all non-algebraic equations (and inequalities) and we note that if $\deg F = M$ then $[\alpha^M y]$ is dependent to the rest of $[\alpha^i y]$ for $i < M$.

But our axiom-schemes so far, are enough to guarantee the following. If $A$ equals to $[F(\alpha)z]$ for some $z$ then the algebraic equation is solvable. Now in order for the algebraic equation to be consistent with the non-algebraic equations and inequalities, one needs to ensure that the conditions on $[\alpha^i x]$ remain consistent with the rest when replacing $x$ with $A' - ry$. Hence we essentially have non-algebraic equations and inequalities in terms of $y$.

**Corollary 4.6.** If the system $\Omega$ has a solution in a model $\mathcal{M}$ of $T_0$ together with the axiom-schemes 1 to 5, then it is also solvable in any submodel of $\mathcal{M}$.

**Proof.** In addition to what already established, one needs to notice that if there exists an element $A' \in \mathcal{M}$ satisfying $[F(\alpha)A'] = A$ with $A$ in a submodel of $\mathcal{M}$, then $A'$ does belong to the same submodel as well. $\square$

4.2.2. **Two algebraic equations.** Now consider a system consisting of two algebraic equations

\[
H_1(x, y) = A_1
\]

\[
H_2(x, y) = A_2,
\]

each in the form of (4.1), together with finitely many non-algebraic equations of the form (4.2). As in Simplification 1 in Subsection 4.2.1 above, by enriching the non-algebraic part of the system we can transform it into a system of equations of the form

\[
[F_1(\alpha)x] + [G_1(\alpha)y] = A_1
\]

\[
[F_2(\alpha)x] + [G_2(\alpha)y] = A_2.
\]

Based on Equation (4.36) and similar to our arguments in Subsection 4.2.1 we let $x = \left[\frac{A_1}{F_1(\alpha)}\right] + 1$ and $y = \left[\frac{A_2}{G_1(\alpha)}\right] + 1$. In particular, we recall Remark 4.4 to justify that our argument below also works for the cases that some of the polynomials involved are less than one.

Using these values for $x$ and $y$, we can recalculate Equation (4.37) to obtain an equation of the form

\[
[I(\alpha)z] = B + j
\]
where \( I(\alpha) \) is a rational function with integer coefficients in terms of \( \alpha \), \( j \) is an integer, and the value of \( B \) is determined by \( A_2 \) and \( \left[ \frac{S_2(\alpha)}{S_1(\alpha)} \right] A_1 \).

We now proceed as in the proof of Lemma 1.2 to obtain a natural number \( K \) such that one in each successive elements lies in the range of \([I(\alpha)]\), and such that \([I(\alpha)] - [I(\alpha)]z'\) \( < K \) implies \(|z - z'| < K\). However, there is no easy way to include this fact about \( I(\alpha) \) in our axioms.

Instead, similar to the first part of Lemma 1.2, we note that for a solution \( z \) of the Equation (4.38) there are at most finitely many pairs of integers \( (x, y) \) satisfying \( x = \left\lfloor \frac{z + 1}{z} \right\rfloor + 1 \) and \( y = \left\lfloor \frac{z^2 + 1}{z} \right\rfloor + 1 \). That is, given \( A_1 \) and \( A_2 \), there are at most finitely many pairs of solutions \( (x, y) \) satisfying the system (4.35), and in addition each of the components in \((x, y)\) varies only in a finite range that is determined by the coefficients appearing in the system.

To conclude, we establish the axiom-scheme below for any \( H_1(x, y) \) and \( H_2(x, y) \).

**Axiom scheme 6.** There is a natural number \( K \) such that for any \( A_1, A_2 \) there exist some \( 0 \leq j \leq K \) and a solution \( (x, y) \) to the system obtained from (4.35) by replacing \( A_1 \) with \( A_1 + j \). In addition, for any \((x, y)\) and \((x', y')\) we have
\[
\max \left\{ \left| x - x' \right|, \left| y - y' \right| \right\} < K
\]
whenever
\[
\max \left\{ \left| H_1(x, y) - H_1(x', y') \right|, \left| H_2(x, y) - H_2(x', y') \right| \right\} < K.
\]

The following corollary can be proved similar to Lemma 4.3.

**Corollary 4.7.** Assume that \( T \) is the theory consisting of \( T_0 \) together with the axioms-schemes 1 to 6. If \( M_1 \) and \( M_2 \) are two models of \( T \) with \( M_1 \subseteq M_2 \), then any system consisting of two algebraic and finitely many non-algebraic equations with two variables, is solvable in \( M_1 \) whenever it is solvable in \( M_2 \).

**Remark 4.8.** In general, when the number of algebraic formulas appearing in a system is not less than the number of variables, a similar argument as above shows that the same constraints are forced to hold over the solutions of that system. Consequently, as in Lemma 4.3, solutions of such systems are transferred in a model to each of its submodels.

### 4.3. More than two variables

As we have seen in Subsection 1.2 in solving the systems with exactly two variables there can occur two different types of subtleties which are dictated by the algebraic part of the system.

On the one hand, for each of the variables, say \( x \) for example, there exists a dependency among the decimals \( \lfloor \alpha^i x \rfloor \) for different powers of \( \alpha \). If \( F(\alpha) \), as a polynomial in terms of \( \alpha \), is of degree \( M \) then the decimal part of \( \frac{\alpha^M}{F(\alpha)} x \) does depend on the decimals \( \lfloor \frac{\alpha^i}{F(\alpha)} x \rfloor \) that contain smaller powers of \( \alpha \). This type of dependency reveals itself via the Equation (1.20).

On the other hand, there might be a dependency that relates the two variables \( x \) and \( y \) to one another. This type of dependency can occur in the cases that \( 1, \frac{1}{E}, \) and \( \frac{1}{C} \) are linearly dependent over \( Q \); this happens simply because the latter condition entails the existence of a rational number \( r \) with \( F(\alpha) = rG(\alpha) \). As can be seen in Case (b) of Subsection 1.2 this situation necessitates us to have a more careful treatment regarding the two-fold subtleties involved in the system.

The equations with more number of variables are approached in essentially a similar manner, but with somewhat less subtleties. As will be seen below, when the number of variables increase either the system reduces to a system with a less number of variables, or the decimals involving the last variable can be controlled independently of the rest, as desired by Kronecker’s lemma; we will elaborate on this in Case 3 below.
We initially consider systems involving only a single algebraic equation, and afterwards we explain how we can handle the general case. Let the following be an algebraic equation:

\[ \sum_{i=0}^{N_1} f^i(c_1;x_1) + \cdots + \sum_{i=0}^{N_k} f^i(c_{ki};x_k) = A \]

with \( k > 2 \). Similar to Simplification 1 in Subsection 4.2.1, by adding extra non-algebraic equations, we can rewrite this equation in the form of

\[ \lfloor F_1(\alpha)x_1 \rfloor + \cdots + \lfloor F_k(\alpha)x_k \rfloor = A \]

for some polynomials \( F_i(\alpha) \) in terms of \( \alpha \) with integer coefficients.

To have a clearer presentation we consider three different cases below, and for a better readability we drop \( \alpha \) from our notation and will simply write \( F_i \) instead of \( F_i(\alpha) \).

**Case 1.** There exists a polynomial that generates the others by rational multiples. Assume that \( F_1 \) is such a polynomial and \( r_2, \ldots, r_k \) are rational multiples with \( F_i = r_i F_1 \) for any \( i = 2, \ldots, k \).

Similar to Case (b) in Subsection 4.2.1, by enriching the non-algebraic part of the system, the Equation (4.40) can be transformed into an equation of the form

\[ \lfloor F_1(x_1 + r_2 x_2 + \cdots + r_k x_k) \rfloor = A. \]

But, as also described there, such an equation is essentially a non-algebraic system and is already handled by our axioms.

**Case 2.** There are two polynomials \( F_1 \) and \( F_2 \) that are not a rational multiple of each other, but each of the remaining polynomials are a rational multiple of either \( F_1 \) or \( F_2 \). That is, by renaming the variables, the Equation (4.40) eventually turns into an equation of the form

\[ \lfloor F_1(x_1 + r_2 x_2 + \cdots + r_k x_k) \rfloor + \lfloor F_2(y_1 + r'_2 y_2 + \cdots + r'_k y_k) \rfloor = A. \]

Such equations were dealt with in Case (a) of Subsection 4.2.1.

**Case 3.** There are at least three polynomials none of which being a rational multiple of the others. Consider the following equation including three polynomials \( F_1, F_2 \) and \( F_3 \) possessing the described property:

\[ \lfloor F_1 x_1 \rfloor + \lfloor F_2 x_2 \rfloor + \lfloor F_3 x_3 \rfloor = A. \]

Let \( z_1 = \lfloor F_1 x_1 \rfloor \) and \( z_2 = \lfloor F_2 x_2 \rfloor \). Now, finding a solution to the equation above is equivalent to finding the integers \( z_1 \) and \( z_2 \) satisfying

\[ 1 - \frac{1}{F_1} < \left[ \frac{z_1}{F_1} \right] < 1, \]

\[ 1 - \frac{1}{F_2} \leq \left[ \frac{z_2}{F_2} \right] < 1, \text{ and} \]

\[ 1 - \frac{1}{F_3} < \left[ \frac{A - (z_1 + z_2)}{F_3} \right] < 1, \]

where the latter is equivalent to the condition that \( A - (z_1 + z_2) \) lies in the range of the function \( \lfloor F_3 \rfloor \).

Recall from Case (b) in Subsection 4.2.1 that the set of numbers \( \{1, \frac{1}{F_1}, \frac{1}{F_2} \} \) is dependent over \( \mathbb{Q} \) if and only if \( F_2 \) is a rational multiple of \( F_1 \). But such a situation is already excluded in Case 3. That is, \( z_1 \) and \( z_2 \) can freely be chosen to satisfy the conditions (4.41) and (4.42) without any correlation in place.
However, comparing to the systems of two variables there is still rather a significant advantage here in fulfilling the desired condition of (4.43). In fact, even if the set of numbers \( \{1, \frac{1}{F_1}, \frac{1}{F_2}, \frac{1}{F_3}\} \) is linearly dependent over \( \mathbb{Q} \), it is not very hard to verify that the existence of two completely unrelated factors, \( z_1 F_1 \) and \( z_2 F_2 \), here enables us to control them so that the decimal \( \left[ \frac{A-(z_1+z_2)}{F_3} \right] \) is forced to lie in the desired interval. The situation becomes even easier when there are more than three variables available in the system.

Now, suppose that there is more than one algebraic equation in a system containing \( k \)-many variables. Using the same procedure as above, we can reduce the first algebraic equation to a system of non-algebraic equations containing \( (k-1) \)-many variables; that is by setting \( z_i := \lfloor F_i x_i \rfloor \), for any \( i \in \{1, \ldots, k-1\} \), and \( z_k := A - \sum_{i=1}^{k-1} z_i \). By Remark 4.3 without loss of generality we can assume each \( F_i \) to be greater than one. Hence in the rest of the algebraic equations and for any \( i \in \{1, \ldots, k-1\} \) we can replace each \( x_i \) by \( \lfloor z_i F_i \rfloor + 1 \). Also we replace \( x_k \) with

\[
\left\lfloor \frac{A - \sum_{i=1}^{k-1} z_i}{F_k} \right\rfloor + 1
\]

to transform each of the remaining equations to an algebraic equation of \( (k-1) \)-many variables whose coefficients are rational functions in terms of \( \alpha \) (as in Subsection 4.2.2).

We continue the same procedure to employ each of the algebraic equations in reducing to a system containing a fewer number of variables, which has already been handled. That is, a system consisting of some algebraic equations of the form (4.39) and finitely many non-algebraic equations of the form

\[
\sum_{i=0}^{N_1} m_{1i} \left[ \alpha f^i(x_1) \right] + \cdots + \sum_{i=0}^{N_k} m_{ki} \left[ \alpha f^i(x_k) \right] < a,
\]

(4.44)

will turn, as in Subsection 4.2.1 to a system of inequalities involving a set of hyperplanes in \( \mathbb{R} \).

In the case that the number of algebraic equations is greater than or equal to the number of variables, the procedure will terminate by reducing to the case studied in Subsection 4.2.2 (see Remark 4.8).

Hence, in each of the mentioned cases the formula \( \xi_{k,\ell}(\bar{a}) \) needed in the following axiom exists.

**Axiom scheme 7.** A system consisting of \( k \)-many algebraic equations and \( \ell \)-many non-algebraic equations with coefficients \( \bar{a} \) has a solution if and only if \( \xi_{k,\ell}(\bar{a}) \).

**Notation 4.9.** We denote by \( \mathcal{T}_\alpha \) the theory consisting of the axiom-schemes 1 to 7 together with \( T_0 \).

Finally we can conclude that our axioms, collected in \( \mathcal{T}_\alpha \), are model-complete and fully axiomatize the theory \( \text{Th}(\mathbb{Z}_\alpha) \). Completeness and hence decidability of \( \mathcal{T}_\alpha \) follows from the fact that \( \mathbb{Z}_\alpha \) is a prime model of \( \mathcal{T}_\alpha \).

**Main Theorem 4.10.** \( \mathcal{T}_\alpha \) is model-complete, complete, and decidable.

5. Concluding remarks

5.1. **The case of an algebraic \( \alpha \).** We expect that, with enough rigour, our proof would work even when \( \alpha \) is an algebraic number. In this case, a certain dependency is constantly imposed on each of the systems of equations of the theory. In fact, when \( \alpha \) satisfies an equation of a minimal degree \( n \) as

\[
\alpha^n + k_{n-1}\alpha^{n-1} + \ldots + k_0 = 0,
\]

(5.1)

with integer coefficients, any system in the theory turns into a system of equations constrained by a dependency over the decimal parts of variables.
To be more precise, consider a system of two variables as in Subsection 4.2. Then for each of the variables, say $x$, Equation (5.1) forces the decimal $[\alpha^n x]$ to depend on the decimals involving the smaller powers of $\alpha$. Also any already-existing dependency in the system that involves larger powers of $\alpha$ will turn, using Equation (5.1), into a dependency on the decimals $[\alpha^i x]$ containing powers of $\alpha$ less than $n$. That is, we finally end in a system whose dependencies solely involve the decimals $[\alpha^i x]$ with a power $i$ strictly less than $n$.

Hence, as long as the polynomials $F(\alpha)$ and $G(\alpha)$, that are literally part of the system, are so that the numbers $1, \frac{1}{F(\alpha)}$ and $\frac{1}{G(\alpha)}$ are independent over $\mathbb{Q}$, the proof of Case (a) in Subsection 4.2 would smoothly be carried out.

But an extra subtlety occurs when $1, \frac{1}{F(\alpha)}$ and $\frac{1}{G(\alpha)}$ are dependent over $\mathbb{Q}$. Despite what we have done in Case (b) of Subsection 4.2, for an algebraic $\alpha$ we cannot necessarily conclude in this case that $F(\alpha)$ is a rational multiple of $G(\alpha)$, and hence the argument in that case cannot directly be applied.

However, the mentioned dependency means that there are rational numbers $r$ and $r'$ with $\frac{1}{F(\alpha)} = rF(\alpha) + r'F(\alpha)G(\alpha)$. Hence, as in Simplification 1 in Subsection 4.2, by adding some extra non-algebraic equations we can transform an algebraic equation
\[
[F(\alpha)x] + [G(\alpha)y] = A \tag{5.2}
\]
into an equation of the form
\[
[F(\alpha)(x + (r + r'G(\alpha))y)] = A.
\]

Note that the solvability of the equation above in a model $M$ is equivalent to existence of an element $A'$ such that
\[
[F(\alpha)A'] = A \tag{5.3}
\]
and moreover that
\[
A' = x + (r + r'G(\alpha))y
\]
for some $x, y \in M$. By adding new non-algebraic equations, we can turn the latter equation into
\[
A' = x + [(r + r'G(\alpha))]y
\]
which, by setting $G'(\alpha) = r + r'G(\alpha)$, turns into
\[
A' = x + [G'(\alpha)y]. \tag{5.4}
\]

Hence, at the expense of adding new non-algebraic equations, we can reduce Equation (5.2) to finding an $A'$ that satisfies Equation (5.3) and for which Equation (5.4) is solvable.

Using the same idea as in Case (b) of Subsection 4.2 we can conclude that for a submodel $M' \subseteq M$ with $A \in M'$ solvability of Equation (5.3) in $M$ results in having $A'$ as a member of $M'$. Now, with $A' \in M'$ solving an equation of the form (5.4) in $M$, as in Case (a) of Subsection 4.2 can be transformed to its solvability in $M'$ since Equation (5.4) does not contain a dependency over $\mathbb{Q}$ anymore.

5.2. On definable sets. Based on the terminology used in [T08 Section 3.1], there appear three fundamental types of sets in various areas of mathematics: The “structured” sets, the “random” sets, and sets of a “hybrid” nature. Below, we make a concise clarification on this phenomena concerning the definable sets in $\mathbb{Z}_\alpha$.

If a power of $f$ does not appear in an existential formula $\varphi(x)$ with a single free variable $x$, then the quantifier elimination available in Presburger arithmetic shows that $\varphi(x)$ is actually describing a
congruence class to which \( x \) belongs. So in this case, \( \varphi(x) \) defines an infinite arithmetic progression which is a typical example of a “structured” set by having a completely predictable behaviour.

On the contrary, Connell proved in \( [C60] \) Theorem 2] that no Beatty sequence with an irrational modulo can contain an infinite arithmetic progression. That is the set of solutions of a formula like \( \exists y (x = f^n(y)) \), that forms a typical example of an existential formula containing a power of \( f \), cannot contain an infinite arithmetic progression. It is not clear to us if the same fact holds for formulas of the form \( \exists y (x = f(y) + f^2(y)) \) that contain an addition of terms; the latter question might be of interest from the perspective of additive combinatorics.

However in proposition below, and using Theorem \ref{thm:2.1} we prove that in the range of any term of one variable we can find finite arithmetic progressions of arbitrary large length. This slightly generalizes a similar result by Connell in \( [C60] \) Theorem 2].

**Proposition 5.1.** Let \( h(x) = \sum_{i=0}^{k} m_i f^i(x) \). For any natural number \( n \) there exists an arithmetic progression of length \( n \) in the range of \( h \) in \( \mathbb{Z}_\alpha \).

**Proof.** To have an arithmetic progression of length \( n \), it suffices to find \( x \) and \( y \) such that for any \( \ell \leq n \) we have that \( h(x + \ell y) = h(x) + \ell h(y) \). And the latter holds whenever \( \mathbb{Z}_\alpha \) satisfies the following non-algebraic formula for any \( 1 \leq i \leq k \)

\[
f^i(x + ny) = f^i(x) + nf^i(y),
\]

or equivalently whenever we have the following for any \( 0 \leq i \leq k - 1 \)

\[
\mathbb{Z}_\alpha \models 0 < \lfloor \alpha f^i(x) \rfloor + n \lfloor \alpha f^i(y) \rfloor < 1.
\]

But, Theorem \ref{thm:2.1} allows us to find \( x \) and \( y \) with the desired properties. \( \square \)

A similar argument as in the proof of proposition above shows that for an existential formula \( \varphi(x) \) of more than one existential variable, the set of solutions \( \varphi(\mathbb{Z}_\alpha) \) contains arithmetic progressions of arbitrary finite lengths.

The latter observation shows that the formulas containing a power of \( f \) behaves more-or-less similar to prime numbers in that they do not contain infinite arithmetic progression whereas they do contain arbitrary long finite arithmetic progressions \( (\mathbb{G}108) \). However, Proposition \ref{prop:2.4} shows that such definable sets may differ from the primes in intersecting each congruence class at infinitely many points. But it seems reasonable to consider them as hybrid sets in \( \mathbb{Z}_\alpha \) just like as we do for primes in \( \mathbb{Z} \).

To sum up, the structured definable sets in \( \mathbb{Z}_\alpha \) are disjoint-by-finite from the hybrid sets, and still another interesting phenomena occurs in \( \mathbb{Z}_\alpha \) when we consider two mentioned types of sets from the perspective of the order topology available in \( \mathbb{Z}_\alpha \) by the formula \( [\alpha x] < [\alpha y] \). In fact, both the structured and hybrid sets find a uniform description in this topology by being simultaneously dense and co-dense there.

### 5.3. A connection to \( \omega \)-minimality.

We show that the non-algebraic part of \( T_\alpha \), denoted by \( T_{\text{nalg}} \) in Section \ref{section:5} gives rise to an \( \omega \)-minimal theory that consists of, and is actually determined by, the main features of \( T_{\text{nalg}} \).

For a model \( M \models T_{\text{nalg}} \) we associate a structure \( A_M \) in a language \( L^* \) that contains a set of predicates meant to capture the non-algebraic content of \( M \). We make use of an auxiliary intermediate theory \( T^* \) and will return to \( A_M \) in the end. Actually, the symbols of \( L^* \) find their complete intended meaning only when they are considered in a model of \( T^* \) that is already an \( A_M \) associated to a model \( M \) of \( T_{\text{nalg}} \). Also, as becomes clear below, the \( \omega \)-minimal component referred to earlier arises only in the latter case; that is, when we are considering those models of \( T^* \) that are associated to a model of \( T_{\text{nalg}} \).

Let \( L^* = \{ <, P_{\bar{m}, \bar{n}, \ell} \}_{\bar{m}, \bar{n}, \ell \in \mathbb{Z}} \) where each \( P_{\bar{m}, \bar{n}, \ell} \) accepts tuples of arity \( |\bar{m}| + |\bar{n}| \).
Fix some $\mathcal{M} \models \mathcal{T}_{\text{nal}g}$ and let $A_{\mathcal{M}}$ be the subset of non-standard reals defined as

$$A_{\mathcal{M}} := \{ [aa] \mid a \in M \}. $$

For $[aa], \ldots, [ab], \ldots \in A_{\mathcal{M}}$, we let $P_{m, \bar{n}, \ell}([[aa], \ldots, [ab], \ldots])$ hold in $A_{\mathcal{M}}$ if and only if

$$\mathcal{M} \models \sum m_i [aa_i] < \sum n_i [ab_i] + \ell. $$

Note in particular that the predicate $P_{1,1,0}([aa], [ab])$ holds in $A_{\mathcal{M}}$ if and only if

$$\mathcal{M} \models [aa] < [ab]. $$

That is, $P_{1,1,0}$ coincides with the relation $<$ in $A_{\mathcal{M}}$. Hence by Axiom 1 this predicate defines a dense linear ordering on $A_{\mathcal{M}}$.

Towards introducing $T^*$, we keep using the notation $[ax]$ for elements of an arbitrary $L^*$-structure $A$. Also, for simplicity and particularly in axiom-schemes (2) and (3) below, we keep thinking of predicates $P_{m, \bar{n}, \ell}$ as if they are reflecting the content of the inequality appeared in (5.5), while we carefully have this reservation in mind that an expression like $\sum m_i [aa_i]$ is, by itself, just meaningless in $T^*$ and does not refer to an actual point.

Let $T^*$ be the theory that describes the following:

1. The relation $<$ is a dense linear order.
2. The predicates $P_{m, \bar{n}, \ell}$ are consistent with the usual addition and ordering of real numbers. That is $T^*$ describes how elements can be moved from each side of (5.5) to the other. For example if $P_{2,1,0}(a, c)$ holds, then we have that $P_{1,1,0}(a, c, a)$. This example reflects the content of the fact that $2[aa] < [ac]$ implies $[aa] < [ac] - [aa]$ in real numbers.
3. If $\sum m_i [aa_i] < \sum n_i [ab_i] + \ell < 1$ then there is $[ax]$ such that

$$\sum m_i [aa_i] < [ax] < \sum n_i [ab_i] + \ell. $$

Because of the density enforced on the predicates of $L^*$ by the axioms (1) and (3) above, it is easy to verify the following proposition.

**Proposition 5.2.** $T^*$ admits quantifier elimination in $L^*$.

Now, for some model $\mathcal{A}_M \models \mathcal{T}_{\text{nal}g}$ it is easy to see that the associated $\mathcal{A}_M$ is a model of $T^*$. On the other hand, $T^*$ is similar to an $o$-minimal theory in the sense that any set defined by a formula $\varphi(x, \bar{a}b)$ is a finite union of intervals of the form below

$$\{ x : \sum m_i [aa_i] < m [ax] < \sum n_i [ab_i] + \ell \}. $$

But, as mentioned earlier, the endpoints of this interval are not some actual points in an arbitrary model of $T^*$. However, in each of the structures $A_M$ these endpoints turn out to be elements of the form $[aa]$. Moreover, at the expense of adding/subtracting an integer value to/from $\ell$, we can write $m [ax]$ as $[max]$ or equivalently as $[az]$ for some $z$ in $M$. That is, each $L^*$-formula $\varphi(x, \bar{a}b)$ becomes equivalent to a finite disjunction of formulas of the form below in $A_M$:

$$[aa] < [ax] < [ab]. $$

Hence for any models $\mathcal{M}, \mathcal{N} \models \mathcal{T}_{\text{nal}g}$ the two associated structures $A_M$ and $A_N$ are elementary equivalent since we are able to form a back-and-forth system between them. In other words, there exists a completion of $T^*$ that is $o$-minimal and is determined by $T_{\text{nal}g}$.
A note on overlapping results. Short before submitting this paper, we learnt of a similar independent work by Günaydin and Özsahakyan uploaded in the arxiv [GO21]. The main difference between the two papers, is that they let the Beatty sequence as a predicate in the language, where we put the function \( f = \lfloor \alpha x \rfloor \), which is not definable in their structure. For the same reason, we have to deal with decimals that concern the powers of the function \( f \) where they need only to treat decimals of the linear combinations.

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