Stagnation Detection in Highly Multimodal Fitness Landscapes

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Abstract

Stagnation detection has been proposed as a mechanism for randomized search heuristics to escape from local optima by automatically increasing the size of the neighborhood to find the so-called gap size, i.e., the distance to the next improvement. Its usefulness has mostly been considered in simple multimodal landscapes with few local optima that could be crossed one after another. In multimodal landscapes with a more complex location of optima of similar gap size, stagnation detection suffers from the fact that the neighborhood size is frequently reset to 1 without using gap sizes that were promising in the past.

In this paper, we investigate a new mechanism called radius memory which can be added to stagnation detection to control the search radius more carefully by giving preference to values that were successful in the past. We implement this idea in an algorithm called SD-RLS\textsuperscript{m} and show compared to previous variants of stagnation detection that it yields speed-ups for linear functions under uniform constraints and the minimum spanning tree problem. Moreover, its running time does not significantly deteriorate on unimodal functions and a generalization of the JUMP benchmark. Finally, we present experimental results carried out to study SD-RLS\textsuperscript{m} and compare it with other algorithms.

1 Introduction

The theory of self-adjusting evolutionary algorithms (EAs) is a research area that has made significant progress in recent years (Doerr and Doerr [2020]). For example, a self-adjusting choice of mutation and crossover probability in the so-called (1+(\(\lambda, \lambda\))) GA allows an expected optimization time of \(O(n)\) on ONE\textsuperscript{MAX}, which is not possible with any static setting (Doerr, Doerr and Ebel [2015] Doerr and Doerr [2018]). Many studies focus on unimodal problems, while self-adjusting EAs for multimodal problems in discrete search spaces, more precisely for pseudo-Boolean optimization, were investigated
only recently from a theoretical runtime perspective. **Stagnation detection** proposed in Rajabi and Witt (2020) addresses a shortcoming of classical self-adjusting EAs, which try to learn promising parameter settings from fitness improvements. Despite the absence of a fitness improvements when the best-so-far solution is at a local optimum, stagnation detection learns from the number of unsuccessful steps and adjusts the mutation rate if this number exceeds a certain threshold. Thanks to this mechanism, the so-called SD-(1+1) EA proposed in Rajabi and Witt (2020) optimizes the classical Jump function with \( n \) bits and gap size \( m \) in expected time \( O(\left(\frac{em}{m}\right)^m) \), which corresponds asymptotically to the best possible time achievable through standard bit mutation, more precisely when each bit is flipped independently with probability \( m/n \). It is worth pointing out that stagnation detection does not have any prior information about the gap size \( m \).

Although leaving a local optimum requires a certain number of bits to be flipped simultaneously, which we call the gap size, the SD-(1+1) EA mentioned above still performs independent bit flips. Therefore, even for the best setting of the mutation rate, only the expected number of flipping bits equals the gap size while the actual number of flipping bits may be different. This has motivated Rajabi and Witt (2021) to consider the \( k \)-bit flip operator flipping a uniform random subset of \( k \) bits as known from randomized local search (RLS) (Doerr and Doerr, 2018) and to adjust \( k \) via stagnation detection. Compared to the SD-(1+1) EA, this allows a speed-up of \( \left(\frac{ne}{m}\right)^m/\left(\binom{n}{m}\right) \) (up to lower-order terms) on functions with gap size \( m \) and a speed-up of up to roughly \( e = 2.718 \ldots \) on unimodal functions while still being able to search globally.

Rajabi and Witt (2021) emphasize that their RLS with self-adjusting \( k \)-bit flip operator resembles variable neighborhood search (Hansen and Mladenovic, 2018) but features less determinism by drawing the \( k \) bits to be flipped uniformly at random instead of searching the neighborhood in a fixed order. The random behavior still maintains many characteristics of the original RLS, including independent stochastic decisions which ease the runtime analysis. If the bit positions to be flipped follow a deterministic scheme as in quasirandom EAs (Doerr, Fouz and Witt, 2010), dependencies complicate the analysis and make it difficult to apply tools like drift analysis. However, a drawback of the randomness is that the independent, uniform choice of the set of bits to be flipped leaves a positive probability of missing an improvement within the given time a specific parameter value \( k \) is tried. Therefore, the first RLS variant with stagnation detection proposed in Rajabi and Witt (2021) and called SD-RLS there has infinite expected runtime in general, but is efficient with high probability, where the success probability of the algorithm is controlled via the threshold value for the number of steps without fitness improvement that trigger a change of \( k \). We remark here that the problem with infinite runtime is not existent with the independent bit flips as long as each bit is flipped with probability in the open interval \((0, 1)\).

In this paper, we denote by SD-RLSP the simple SD-RLS just proposed. To guarantee finite expected optimization time, Rajabi and Witt (2021) introduce a second variant that repeatedly returns to lower so-called (mutation) strengths, i.e., number of bits flipped, while the algorithm is still waiting for an improvement. The largest neighborhood size (i.e., number of bits flipped) is denoted as radius \( r \) and, in essence, the
strengths is decreased in a loop from $r$ to 1 before the radius is increased. Interestingly, the additional time spent at exploring smaller strengths in this loop, with the right choice of phase lengths, contributes only a lower-order term to the typical time that SD-RLS$^p$ has in the absence of errors. The resulting algorithm (Algorithm 2) is called SD-RLS$^r$ in [Rajabi and Witt (2021)], but in this paper referred to by SD-RLS$^r$, where the label $r$ denotes robust.

As already explained in [Rajabi and Witt (2021)], SD-RLS$^r$ (and also the plain SD-RLS$^p$) return to strength 1 after every fitness improvement and try this strength for a sufficiently large time to find an improving one-bit flip with high probability. This behavior can be undesired on highly multimodal landscapes where progress is typically only made via larger strengths. As an example, the minimum spanning tree (MST) problem as originally considered for the (1+1) EA and an RLS variant in [Neumann and Wegener (2007)] requires two-bit flips to make progress in its crucial optimization phase. Both theoretically and experimentally, Rajabi and Witt [Rajabi and Witt (2021)] observed that SD-RLS$^r$ is less efficient than the RLS variant from [Neumann and Wegener (2007)] since low, useless strengths (here 1) are tried for a too long period of time. On the other hand, it can also be risky exclusively to proceed with the strength that was last found to be working if the fitness landscape becomes easier at some point and progress can again be made using smaller strengths.

In this paper, we address this trade-off between exploiting high strengths that were necessary in the past and again trying smaller strengths for a certain amount of time. We propose a mechanism called radius memory that uses the last successful strength value to assign a reduced budget of iterations to smaller strengths. This budget is often much less than the number stereotypically tried in SD-RLS$^r$ after every fitness improvement. However, the budget must be balanced carefully to allow the algorithm to adjust itself to gap sizes becoming smaller over the run of the algorithm. Our choice of budget is based on the number of iterations (which is the same as the number of fitness evaluations) passed to find the latest improvement and assigns the same combined amount of time, divided by $\ln n$, to smaller strengths tried afterwards. This choice, incorporated in our new algorithm SD-RLS$^m$, basically limits the time spent at unsuccessful strengths by less than the waiting time for an improvement with the last successful strength but is still big enough to adjust to smaller strength sufficiently quickly. On the one hand, it (up to lower-order terms) preserves the runtime bounds on general unimodal function classes and jump functions shown for SD-RLS$^r$ in [Rajabi and Witt (2021)]. On the other hand, it significantly reduces the time for the strength to return to larger values on two highly multimodal problems, namely optimization of linear functions under uniform constraints and the MST. Although these ideas are implemented in a simple RLS maintaining one individual only, we implicitly consider stagnation detection as a module that can be added to other algorithms as shown in [Rajabi and Witt (2020)] and very recently in [Doerr and Zheng (2021)] for multi-objective optimization. Concretely, we could also use the stagnation detection with radius memory in population-based algorithms.

This paper is structured as follows. In Section 2, we define the algorithms considered and collect some important technical lemmas. Section 3 presents time bounds for the
new algorithm SD-RLS$^m$ to leave local optima and applies these to obtain bounds on the expected optimization time on unimodal and jump functions. Moreover, it includes in Lemma 4 the crucial analysis of the time for the strength to settle at smaller values when an improvement is missed. Thereafter, these results are used in Section 4 to analyze SD-RLS$^m$ on linear functions under uniform constraints and to show a linear-time speedup compared to the SD-RLS$^r$ algorithm in Rajabi and Witt (2021). Section 5 shows that SD-RLS$^m$ optimizes MST instances on graphs with $m$ edges in expected time at most $(1+o(1))m^2\ln m$, which is by an asymptotic factor of 2 faster than the bound for RLS$^{1,2}$ from Neumann and Wegener (2007) and represents, to the best of our knowledge, the first asymptotically tight analysis of a globally searching (1+1)-type algorithm on the problem. In Section 6, we present an example where the radius memory is detrimental and leads to exponential optimization time with probability $1-o(1)$ while the original SD-RLS$^r$ from Rajabi and Witt (2021) is efficient. Section 7 presents experimental supplements to the analysis of SD-RLS$^r$ and SD-RLS$^m$ and comparisons with other algorithms from the literature before we finish with some conclusions.

2 Preliminaries

2.1 Algorithms

We start by describing a class of classical RLS algorithms and the considered extensions with stagnation detection. Algorithm 1 is a simple hill climber that uses a static strength $s$ and always flips $s$ bits uniformly at random. The special case where $s = 1$, i.e., using one-bit flips, has been investigated thoroughly in the literature Doerr and Doerr (2016) and is mostly just called RLS.

**Algorithm 1** RLS with static strength $s$ for the maximization of $f : \{0,1\}^n \to \mathbb{R}$

Select $x$ uniformly at random from $\{0,1\}^n$

for $t \leftarrow 1, 2, \ldots$ do

Create $y$ by flipping $s$ bit(s) in a copy of $x$.

if $f(y) \geq f(x)$ then $x \leftarrow y$.

The algorithm RLS$^{1,2}$ (which also is just called RLS in Neumann and Wegener (2007)) is an extension of this classical RLS (i.e., Algorithm 1 with strength 1) choosing strength $s \in \{1,2\}$ uniformly before flipping $s$ bits. This extension is crucial for making progress on the MST problem.

In Rajabi and Witt (2021), RLS is enhanced by stagnation detection, leading to Algorithm 2. In a nutshell, the algorithm increases its strength after a certain number of unsuccessful steps according to the threshold value $\binom{n}{s} \ln R$ which has been chosen to bound the so-called failure probability at strength $s$, i.e., the probability of not finding an improvement at Hamming distance $s$, by at most $(1 - 1/\binom{n}{s})^{\binom{n}{s}\ln R} \leq 1/R$. It also incorporates logic to return to smaller strengths repeatedly by maintaining the so-called radius value $r$. All variables and parameters will be discussed in detail below when we
come to our extension with radius memory. Algorithm 2 is called SD-RLS∗ in [Rajabi and Witt (2021)] since that paper also discusses a simpler variant called SD-RLS without the logic related to the radius variable. However, that variant is not robust and has infinite expected optimization time in general even on unimodal problems. We call Algorithm 2 SD-RLSr since, as argued in [Rajabi and Witt (2021)], the radius makes the algorithm robust.

Algorithm 2 RLS with robust stagnation detection (SD-RLSr) for the maximization of $f : \{0, 1\}^n \to \mathbb{R}$

Select $x$ uniformly at random from $\{0, 1\}^n$.

$r \leftarrow 1$, $s \leftarrow 1$, $u \leftarrow 0$.

for $t \leftarrow 1, 2, \ldots$ do

Create $y$ by flipping $s$ bits in a copy of $x$ uniformly at random.

$u \leftarrow u + 1$.

if $f(y) > f(x)$ then

$x \leftarrow y$.

$r \leftarrow 1$, $s \leftarrow 1$, $u \leftarrow 0$.

else if $f(y) = f(x)$ and $r = 1$ then

$x \leftarrow y$.

if $u > \binom{n}{s} \ln R$ then

if $s = 1$ then

if $r < n/2$ then $r \leftarrow r + 1$ else $r \leftarrow n$

$s \leftarrow r$.

else

$s \leftarrow s - 1$.

$u \leftarrow 0$.

In the following, we present in Algorithm 3 the new algorithm SD-RLSm using stagnation detection and radius memory. It extends SD-RLSr by adding logic for setting the helper variable $B$ and by minimizing the original threshold $\binom{n}{s} \ln R$ for the number of unsuccessful steps with $\frac{u}{\binom{n}{s} \ln R}$ via the variable. Another minor change is that it increases the strength from 1 to the radius $r$ instead of decreasing it for the sake of simplicity of the new algorithm and the proofs. We describe the algorithm in more detail now.

After a strict improvement with strength $s$ (which becomes the initial radius $r$ for the next search point), the algorithm uses all strengths $s' < r$ for min$\{\frac{u}{\binom{n}{s} \ln R}, \binom{n}{s} \ln R\}$ attempts, where $u$ is the value of the counter at the time that the previous improvement happened. Once the current strength becomes equal to the current radius, the threshold becomes min$\{\infty, \binom{n}{s} \ln R\} = \binom{n}{s} \ln R$ for the rest of iterations with the current search point. Therefore, the cap at $\frac{u}{\binom{n}{s} \ln R}$ is only effective as long as the current radius equals $r$ and the current strengths are smaller than $r$. For technical reasons, the radius increases directly to $n$ when it has passed $n/2$. Moreover, as another technical detail, we accept search points of equal fitness only if the current radius is one (leading to
the same acceptance behavior as in classical RLS, see Algorithm 1, whereas only strict improvements are accepted at larger radii.

The factor $1/\ln n$ appearing in the first argument of the minimum

$$\min\left\{ \frac{u}{(\ln n)(r - 1)}, \left(\frac{n}{s}\right) \ln R \right\}$$

is a parameter choice that has turned out robust and useful in our analyses. The choice $\frac{u}{r - 1}$, i.e., an implicit constant of 1, could seem more natural here since then the algorithm would look at smaller strengths as often as the last successful strength was tried; however, this would make our forthcoming bounds worse by a constant factor.

As mentioned above, stagnation detection has also a parameter $R$ to bound the probability of failing to find an improvement at the “right” strength. We formally prove in Lemma 2 that the probability of not finding an improvement where there is a potential of making progress is at most $\frac{1}{R}$. The recommendation of $R$ for SD-RLS in Rajabi and Witt (2021) is still valid for SD-RLS$^m$, i.e., $R \geq n^{3+\epsilon} \cdot |\text{Im } f|$ for a constant $\epsilon$ (where $\text{Im } f$ is the image set of $f$), resulting in that the probability of ever missing an improvement at the right strength is sufficiently small throughout the run. However, in this paper, by improving some analyses, we recommend a tighter value for $R$, namely $R \geq \max\{S, n^{3+\epsilon}\}$ for an arbitrary constant $\epsilon > 0$ where $S$ is an upper bound on the number of improvements during the run. Obviously, we can always choose $S = |\text{Im } f|$.

The runtime or the optimization time of a search heuristic on a function $f$ is the first point time $t$ where a search point of optimal fitness has been created; often the expected runtime, i.e., the expected value of this time, is analyzed.

### 2.2 Mathematical tools

In the following lemma, which has been taken from Rajabi and Witt (2021), we have some combinatorial inequalities that will be used in the analyses of the algorithms. The part (a) in Lemma 1 seems to be well known and has already been proved in Lugo (2017) and is also a consequence of Lemma 1.10.38 in Doerr (2020). The part (b) follows from elementary manipulations.

**Lemma 1** (Lemma 1 in Rajabi and Witt (2021)). For any integer $m \leq n/2$, we have

(a) $\sum_{i=1}^{m} \binom{n}{i} \leq \frac{n-(m-1)}{n-(2m-1)} \binom{n}{m}$,

(b) $\left(\frac{n}{M}\right)^{M} \leq \binom{n}{m} \left(\frac{n-m}{m}\right)^{M-m}$ for $m < M < n/2$.

### 3 Analysis of the Algorithm SD-RLS$^m$

In this section, we shall show bounds on the optimization time of SD-RLS$^m$ in addition to useful technical lemmas used in different analyses in the paper.
Algorithm 3 RLS with robust stagnation detection and radius memory mechanism (SD-RLS\textsuperscript{m}) for the maximization of \( f : \{0, 1\}^n \rightarrow \mathbb{R} \)

Select \( x \) uniformly at random from \( \{0, 1\}^n \).
\( r \leftarrow 1, \ s \leftarrow 1, \ u \leftarrow 0, \ B \leftarrow \infty \).

\textbf{for} \( t \leftarrow 1, 2, \ldots \) \textbf{do}

Create \( y \) by flipping \( s \) bits in a copy of \( x \) uniformly at random.
\( u \leftarrow u + 1 \).

\textbf{if} \( f(y) > f(x) \) \textbf{then}

\( x \leftarrow y \).
\( r \leftarrow s, \ s \leftarrow 1 \).

\textbf{if} \( r > 1 \) \textbf{then} \( B \leftarrow \frac{u}{\ln(n)(r-1)} \) \textbf{else} \( B \leftarrow \infty \).
\( u \leftarrow 0 \).

\textbf{else if} \( f(y) = f(x) \) and \( r = 1 \) \textbf{then}

\( x \leftarrow y \).

\textbf{if} \( u > \min\{B, (\frac{n}{r}) \ln R\} \) \textbf{then}

\textbf{if} \( s = r \) \textbf{then}

\textbf{if} \( r < n/2 \) \textbf{then} \( r \leftarrow r + 1 \) \textbf{else} \( r \leftarrow n \)
\( s \leftarrow 1 \).

\textbf{else}

\( s \leftarrow s + 1 \).

\textbf{if} \( s = r \) \textbf{then} \( B \leftarrow \infty \)
\( u \leftarrow 0 \).

3.1 Expected Times to Leave a Search Point

In this subsection, we will prove some bounds on the time to leave a search point that has a Hamming distance larger than 1 to all improvements. Let us define by the \textit{epoch} of \( x \) the sequence of iterations where \( x \) is the current search point. In contrast to the previously proposed algorithm SD-RLS\textsuperscript{r} in \cite{Rajabi2021}, the optimization time of SD-RLS\textsuperscript{m} for making progress with the current search point \( x \) is also dependent to the progress in the epoch of the second-to-last search point, i.e., the parent of \( x \).

In detail, the algorithm in epoch \( x \) starts with parameters \( r_0 \) and \( B_0 \), which are set to the strength escaping the parent of \( x \) and the number of fitness calls at that strength divided by \( (r_0 - 1) \ln n \), respectively. Therefore, to analyze the running time, we also need to consider those parameters. Hereinafter, we define \( T_{x,r_0,B_0} \) as the number of steps SD-RLS\textsuperscript{m} takes to find an improvement from the current search point \( x \) with starting radius \( r_0 \) (i.e., at the beginning of the epoch) and budget \( B_0 \) in the current epoch.

We recall the so-called \textit{gap} of the point \( x \in \{0, 1\}^n \) defined in \cite{Rajabi2020} as the minimum Hamming distance to points with the strictly larger fitness function value. Formally, \( \text{gap}(x^*) = \infty \) where \( x^* \) is an optimum, and for the rest of the points, we define

\[
\text{gap}(x) := \min\{H(x, y) : f(y) > f(x), y \in \{0, 1\}^n\}.
\]
It is not possible to make progress by flipping less than \( \text{gap}(x) \) bits of the current search point \( x \), but if the algorithm uses the \( s \)-flip with \( s = \text{gap}(x) \), it can make progress with a positive probability.

In order to estimate the escape time bounds, we consider two cases where \( \text{gap}(x) \geq r_0 \) and where \( \text{gap}(x) < r_0 \). In the first case, it costs \( B_0 \) fitness function calls for the algorithm to increase the radius \( r \) to \( r_0 \). Then, the analysis of the rest of the iterations is the same as Theorem 3 for the algorithm SD-RLS' (the algorithm without radius memory) in Rajabi and Witt (2021). Obviously, the algorithm is asymptotically as efficient as SD-RLS' in this case. However, in the second case where \( \text{gap}(x) < r_0 \), the proposed algorithm can be outperformed by SD-RLS since if it fails to improve at every radius, the algorithm meets larger strengths, which are costly. It means that although the gap size of the current search point is less than its parent, the escape time is the same in the worst case scenario. However, in the rest of the paper, we show that this is not harmful to the optimization time because it is captured by the escape time of the previous epoch, and after a short time, the radius “recovers” and is set to the gap of the current search point, as proved in Lemma 4.

Concretely, we present the next theorem presenting escape time bounds and prove it by the end of this subsection.

**Theorem 1.** Let \( x \in \{0,1\}^n \) be the new current search point of SD-RLS' with starting radius \( r_0 \), budget \( B_0 \), and \( R \geq n^{3+\epsilon} \) for an arbitrary constant \( \epsilon > 0 \) on a pseudo-Boolean function \( f: \{0,1\}^n \rightarrow \mathbb{R} \). Define \( T_{x,r_0,B_0} \) as the time to create a strict improvement if \( m = \text{gap}(x) \). Then,

(a) if \( m \geq r_0 \),

\[
E(T_{x,r_0,B_0}) \leq \begin{cases} 
\binom{n}{m} \cdot O\left(1 + \frac{m^3}{n-2m} \ln R\right) & \text{if } m < n/2, \\
O(2^n n \ln R) & \text{otherwise},
\end{cases}
\]

(b) or if \( m < r_0 \),

\[
E(T_{x,r_0,B_0}) \leq \begin{cases} 
\binom{n}{r_0} \cdot O\left(1 + \frac{r_0^2}{n-2r_0} \ln R\right) & \text{if } r_0 < n/2, \\
O(2^n \ln R) & \text{otherwise}.
\end{cases}
\]

For proving Theorem 1, we need some definitions and lemmas as follows. Let **phase** \( r \) consist of all points in time where radius \( r \) is used in the algorithm. Let \( E_r \) be the event of **not** finding the optimum within phase \( r \). In Lemma 2, we show that the failure probability is at most \( 1/R \) in phases \( r \) to \( n/2 \) and zero in the last phase (i.e., at radius \( n \)). In the following lemma, we show that in each radius which is at least the gap size, the algorithm makes progress with high probability.
Lemma 2. Let \( x \in \{0, 1\}^n \) be the current search point of SD-RLS\(^m\) on a pseudo-Boolean fitness function \( f : \{0, 1\}^n \to \mathbb{R} \) and let \( m = \text{gap}(x) \) and \( B \geq \binom{n}{m} \ln R \). Then

\[
\Pr(E_r) \leq \begin{cases} 
1 & \text{if } r < m \\
\frac{1}{R} & \text{if } m \leq r < \frac{n}{2} \\
0 & \text{if } r = n.
\end{cases}
\]

Proof. According to the definition of the gap, the algorithm can not make progress where the strength is less than the gap size, so in this case, \( \Pr(E_r) = 1 \). Now, assume \( r \geq m \).

During phase \( r \) (i.e., at radius \( r \)), the algorithm spends \( \min\{B, \binom{n}{m} \ln R\} = \binom{n}{m} \ln R \) steps at strength \( m \) until it changes the strength or radius. Then, the probability of not improving at strength \( s = m \) is at most

\[
\Pr(E_r) = \left(1 - \binom{n}{m}^{-1}\right)^{\binom{n}{m} \ln R} \leq \frac{1}{R}.
\]

During phase \( n \), the algorithm does not change the radius anymore, and it continues to flip \( s \) bits with different \( s \) containing \( m \) until making progress, so the probability of eventually failing to find the improvement in this phase is 0. \( \square \)

Let \( E^j_i \) for \( j > i \) be the event of not finding the optimum during phases \( i \) to \( j \). In other words, \( E^j_i = E_i \cap \cdots \cap E_j \). Obviously, we have \( E^j_j = E_j \) and \( E^j_i = \emptyset \) for \( j < i \).

The next lemma is used to analyze situations where the algorithm does not find an improvement in the first \( k \) phases, where \( k \) is some number depending on the application of the lemma. In the proof of the lemma, we pessimistically do not consider possible improvements at distances larger than the gap size.

Lemma 3. Let \( x \in \{0, 1\}^n \) with \( m = \text{gap}(x) < n/2 \) be the current search point of SD-RLS\(^m\) with \( R \geq n^{3+\epsilon} \) for any arbitrary constant \( \epsilon > 0 \) on a pseudo-Boolean function \( f : \{0, 1\}^n \to \mathbb{R} \). \( T^r_{x,r_0,B_0} \) is defined as the number of steps to find an strict improvement. Then, for \( k \geq \max\{m - 1, r_0\} \), we have

\[
\mathbb{E}\left(T^r_{x,r_0,B_0} \mid E^k_1\right) \leq \begin{cases} 
\binom{n}{m} \cdot O\left(1 + \frac{m^3}{n-2m} \ln R\right) & \text{if } k = m - 1 \\
\binom{n}{k} \cdot O\left(1 + \frac{k^2}{n} \ln R\right) & \text{if } k \geq m
\end{cases},
\]

where \( E^j_i \) is the event of not finding an optimum during phases \( i \) to \( j \) (included).

Proof. According to the definition, we have

\[
\mathbb{E}\left(T^r_{x,r_0,B_0} \mid E^k_1\right) = \mathbb{E}\left(T^r_{x,r_0,B_0} \mid E_1 \cup \cdots \cup E_k\right).
\]

Since the algorithm cannot make progress in a radius smaller than the gap, \( \Pr(E_1) = 1 \) for \( i < m \), so the last term equals

\[
\mathbb{E}\left(T^r_{x,r_0,B_0} \mid E_m \cup \cdots \cup E_k\right),
\]
and using the law of total probability, the term equals
\[
\sum_{i=k+1}^{\lfloor n/2-1 \rfloor} E(T'_x | E_{k+1}^{i-1} \cup E_i) Pr(E_{k+1}^{i-1} \cup E_i) + E(T'_x | E_{k+1}^{\lfloor n/2 \rfloor} \cup E_n) Pr(E_{k+1}^{\lfloor n/2 \rfloor} \cup E_n).
\]

To interpret the last formula, we recall phase \(r\) as all points of time where radius \(r\) is used in the algorithm. In each term in \(S_1\), Variable \(i\) represents the phase in which the algorithm makes progress (i.e., \(E_i\)) and not in smaller phases, i.e., phases from \(k+1\) to \(i-1\) (i.e., \(E_{k+1}^{r-1}\)). Thus, all cases where making improvement happens in one of the phases ranging from \(k+1\) to \(\lfloor n/2-1 \rfloor\) are considered in \(S_1\), and the last case of phase \(n/2\) is computed in \(S_2\). In this manner, we consider all possible cases of success.

In order to estimate \(Pr(E_{k+1}^{i-1} \cup E_i)\), because of the assumption \(k \geq \max\{m-1, r_0\}\), the budget \(B_0\) is not effective in the threshold value so we can use Lemma 2 resulting in
\[
Pr(E_{k+1}^{i-1} \cup E_i) \leq Pr(E_{k+1}^{i-1}) = \prod_{j=k+1}^{i-1} Pr(E_j) < R^{-(i-k-1)}.
\]

Note that we consider only improvements in the Hamming distance of the gap size. Now, For \(S_1\), we compute
\[
\sum_{i=k+1}^{\lfloor n/2-1 \rfloor} E(T'_x | E_{k+1}^{i-1} \cup E_i) Pr(E_{k+1}^{i-1} \cup E_i)
\leq \sum_{i=k+1}^{\lfloor n/2-1 \rfloor} \left( \sum_{r=1}^{i-1} \left( \sum_{s=1}^{r} \left( \frac{n}{s} \right) \ln R + \sum_{r=1}^{m-1} \left( \frac{n}{s} \right) \ln R + \left( \frac{n}{r} \right) \ln R + \left( \frac{n}{m} \right) \right) \right) \cdot R^{-(i-k-1)}.
\]

This is because in the last phase, the success can happen with strength \(m\), so we do not consider the strengths larger than \(m\) in the last phase. Also, in the last phase, the algorithm makes progress in \(\binom{n}{m}\) iterations in expectation.

Now, since \(m \leq i\) resulting in \(\sum_{s=1}^{m-1} \binom{n}{s} \leq \sum_{r=1}^{i-1} \sum_{s=1}^{r} \binom{n}{s}\) so the last term is bounded from above by
\[
\sum_{i=k+1}^{\lfloor n/2-1 \rfloor} \left( \sum_{r=1}^{i-1} 2 \left( \sum_{s=1}^{r} \left( \frac{n}{s} \right) \ln R + \left( \frac{n}{m} \right) \right) \right) \cdot R^{-(i-k-1)}
\leq \sum_{i=k+1}^{\lfloor n/2-1 \rfloor} \left( \sum_{r=1}^{i-1} \left( \frac{n-r+1}{n-2r+1} \right) \left( \frac{n}{r} \right) \ln R + \left( \frac{n}{m} \right) \right) R^{-(i-k-1)}
\leq \sum_{i=k+1}^{\lfloor n/2-1 \rfloor} \left( \left( \frac{n-i}{n-2i+3} \right)^2 \left( \frac{n}{i-1} \right) \ln R + \left( \frac{n}{m} \right) \right) R^{-(i-k-1)}
\leq \sum_{i=k+1}^{\lfloor n/2-1 \rfloor} \left( \left( \frac{1+\frac{i^2}{n-2i}}{n-2i+3} \right) \left( \frac{n}{i-1} \right) \ln R + \left( \frac{n}{m} \right) \right) R^{-(i-k-1)}
\]
In the second and third inequalities, we applied the first inequality in Lemma 1 to eliminate two summations.

Now, via the second inequality in Lemma 1, and then excluding the first term from the summation, we bound the last term from above by

\[
2 \left(1 + \frac{(k + 1)^2}{n - 2k - 2}\right) \frac{k + 1}{n - k - 1} \left(\frac{n}{k + 1}\right) \ln R + \left(\frac{n}{m}\right)
+ 4 \sum_{i=k+2}^{\lfloor n/2 - 1 \rfloor} \left(1 + \frac{i^2}{n - 2i}\right) \left(\frac{n - k - 1}{k + 1}\right)^i \ln R \cdot R^{-(i-k-1)}
\]

\[
= 2 \left(1 + \frac{(k + 1)^2}{n - 2k - 2}\right) \frac{k + 1}{n - k - 1} \left(\frac{n}{k + 1}\right) \ln R + \left(\frac{n}{m}\right)
+ 4 \ln R \left(\frac{n}{k + 1}\right) \sum_{i=k+2}^{\lfloor n/2 - 1 \rfloor} \left(1 + \frac{i^2}{n - 2i}\right) \left(\frac{n}{Rk}\right)^{i-k-2}.
\]

Using the fact that \(R \geq n^{3+\epsilon}\) and \(i > k\), the last expression is bounded from above by

\[
O\left(\frac{\ln R}{R} n^2 \left(\frac{n}{k+1}\right)\right) = O\left(\frac{\ln R}{R} n^2 \left(\frac{n}{k}\right)\right) = o\left(\frac{n}{k}\right).\]

If \(k \geq m\), the last inequality is bounded by

\[
2 \left(1 + \frac{(k + 1)^2}{n - 2k - 2}\right) \left(\frac{n}{k}\right) \ln R + o\left(\left(\frac{n}{k}\right)\right) \leq \left(\frac{n}{k}\right) \cdot O\left(1 + \frac{k^2}{n} \ln R\right).
\]

However, if \(k = m - 1\), the inequality is bounded by

\[
2 \left(1 + \frac{m^2}{n - 2m}\right) \frac{m}{n - m + 1} \left(\frac{n}{m}\right) \ln R + \left(\frac{n}{m}\right) + o\left(\left(\frac{n}{m}\right)\right)
\leq \left(\frac{n}{m}\right) O\left(1 + \frac{m^3}{n - 2m} \ln R\right).
\]

Regarding \(S_2\), when radius \(r\) is increased to \(n\), the algorithm mutates \(s\) bits of the the current search point for all possible strengths of 1 to \(n\) periodically. In each cycle through different strengths, according to lemma 2, the algorithm escapes from the local optimum with probability \(1 - 1/R\) so there are \(R/(R - 1)\) cycles in expectation via geometric distribution. Besides, each cycle of radius \(n\) costs \(\sum_{s=1}^{n} \binom{n}{s} \ln R\). Overall, we have \(\frac{R}{R - 1} \sum_{s=1}^{n} \binom{n}{s} \ln R\) extra fitness function calls if the algorithm fails to find the optimum in the first \(\lfloor n/2 - 1 \rfloor\) phases happened with the probability of \(R^{-(\lfloor n/2 \rfloor - k - 1)}\). Thus, we have
Altogether, we finally have $E_{T'_{x,r_0,B_0} | E_{k+1}^{[n/2]} \cup \overline{E_n}} = S_1 + S_2$, resulting in the statement. \qed

Now, by using Lemma 3, we prove Theorem 1.

Proof of Theorem 1. Regarding Part (a), the starting radius $r_0$ is less than or equal to the gap of the current search point, i.e., $r_0 \leq m = \text{gap}(x)$. For $m < n/2$, using Lemma 3 with $k = m - 1$, we have $E(T_{x,r_0,B_0}) \leq \frac{n}{m} \cdot O\left(1 + \frac{m^3}{n-2m} \ln R\right)$.

For $m \geq n/2$, the algorithm is not able to make an improvement for radius $r$ less than $n/2$. It can be pessimistically assumed that $B_0 = \infty$. However, as radius $r$ is increased to $n$, the algorithm mutates $m$ bits of the current search point for all possible strengths of 1 to $n$ periodically. Thus, according to Lemma 2, the algorithm escapes from the local optimum with probability at least $1 - \frac{1}{R}$ so there are at most $R/(R-1)$ cycles in expectation in this phase (i.e., at radius $n$) by the geometric distribution. Finally, we compute

$$E(T_{x,r_0,B_0} | E_{m-1}^{[n/2]}) < \sum_{i=1}^{[n/2]-1} \sum_{j=1}^{i} \left(\frac{n}{j}\right) \ln R + \frac{R}{R-1} \sum_{i=1}^{n} \left(\frac{n}{i}\right) \ln R$$

$$\leq O(2^n n \ln R).$$

Regarding Part (b), the algorithm starts with a radius which is larger than the gap size. Thus, in phase $r_0$, if $B_0 \geq \left(\frac{n}{m}\right) \ln R$, then the probability of failure in this phase would be $Pr(E_{r_0}) \leq 1/R$ via Lemma 2. Otherwise, the probability of failure can be greater than $1/R$.

We pessimistically assume that $Pr(E_{r_0}) = 1$, which means that the algorithm does not make progress in this phase. Therefore, using Lemma 3 with $k = r_0$ the time for leaving the point is $\left(\frac{n}{r_0}\right) \cdot O\left(1 + k^2/n \ln R\right)$.
For $r_0 \geq n/2$, we pessimistically assume that the algorithm is not able to make an improvement for the strengths less than $r_0$, costing $(1/\ln n) \cdot (r_0 - 1) \cdot B_0$ iterations. Then the algorithm tries strength $r_0$. If no improvement is found, the radius is increased to $n$, and the algorithm mutates $m$ bits of the current search point for all possible strengths $m$ from 1 to $n$ periodically. Thus, like part (a), according to Lemma 2 the algorithm escapes from the local optimum with probability at least $1 - 1/R$, so there are at most $R/(R - 1)$ cycles in expectation through the geometric distribution in this phase. Finally, we compute

$$E(T_{x,r_0,B_0} \mid E_{x}^{m-1}) < \frac{(r_0 - 1)B_0}{\ln n} + \binom{n}{r_0} \ln R + \frac{R}{R - 1} \sum_{i=1}^{n} \binom{n}{i} \ln R = O(2^n \ln R).$$

\[\square\]

### 3.2 Expected Optimization Times

In this subsection, we will prove a crucial technical lemma on recover times and use it to obtain bounds on the expected optimization time on unimodal and jump functions.

**Recover times for strengths.** In the previous subsection, we analyzed the time of SD-RLS for leaving only a single search point. We observed that the duration of epochs depended on the starting radius denoted as $r_0$ set from the previous epoch. This can be inconvenient to estimate an upper bound on the running time on an arbitrary function. Therefore, in the following lemma, we show that if the algorithm uses larger strengths than the gap size to make progress, after a relatively small number of iterations the algorithm chooses the gap size of a current search point as the strength.

**Lemma 4.** Let $x \in \{0, 1\}^n$ with $m = \text{gap}(x) < n/2$ be the current search point of SD-RLS with $R \geq n^{3+\epsilon}$ for an arbitrary constant $\epsilon > 0$ on a pseudo-Boolean function $f : \{0, 1\}^n \rightarrow \mathbb{R}$. Assume that the radius is $k > m$. Define $S_x$ as the number of iterations spent from that point in time on until the algorithm sets the radius to at most the gap size of the current search point. Assume that $B_0$ is the value of the variable $B$ in the beginning. Then, for $B_0 \geq \left(\frac{n}{m}\right) \ln R$, we have

$$E(S_x) \leq o\left(\binom{n}{k - 1}\right),$$

and for $B_0 < \left(\frac{n}{m}\right) \ln n$, we have

$$E(S_x) \leq o\left(R\binom{n}{k - 1}\right).$$

The idea of the proof is that in the case of making progress with larger strengths than $m$ or failing to improve and increasing strength and radius even further, the algorithm also tries all smaller strengths often enough, more precisely, as often as the threshold value $(\text{gap}(x)) \ln R$ that would hold if the current search point was $x$, $s = \text{gap}(x)$ and
\( B = \infty \) in a phase of Algorithm 3. Thus, the algorithm can make progress when the strength equals the current gap with good probability.

**Proof of Lemma 3** We recall the epoch of \( x \) as the sequence of iterations where \( x \) is the current search point. We assume that the gap size of the current search point does not become equal to or larger than the current radius value in all epochs. Otherwise, \( S_x \) is no bigger than the following estimation.

Assume \( x' \) is the current search point and \( r' \) is a radius value larger than \( g_{x'} := \text{gap}(x') \). Note that \( g_{x'} \) equals \( m \) in the beginning (in the first epoch), but it may be different when a strict improvement is made.

We now claim that for each at most \( \ln n \cdot r'(\binom{n}{g_{x'}}) \ln R \) iterations with strength \( r' \) in phase \( r' \) (i.e., at radius \( r' \)) (even in different epochs), the algorithm uses smaller strengths including strength \( g_{x'} \) for \( \binom{n}{g_{x'}} \ln R \) iterations. After proving the claim, we can show that with probability \( 1 - 1/R \) the algorithm makes progress with the strength which equals the gap size of the current search point via Lemma 2.

To prove this claim, we consider two cases. First, if the algorithm does not make progress with strength \( r' \), then in the next phase the algorithm uses strength smaller than \( r' \) including \( g_{x'} \) in \( \binom{n}{g_{x'}} \ln R \) iterations. Thus, \( \binom{n}{g_{x'}} \ln R \) iterations with strength \( r' \) are enough for satisfying the claim in this case.

In the second case, assume that the algorithm makes an improvement with strength \( r' \) in its \( u_{th} \) attempt. For the next epoch, the algorithm tries strength \( g_{x'} \) for \( (1/\ln n) \cdot u/(r' - 1) \) times (according to the variable \( B \) in Algorithm 3). Assume that \( u_1, \ldots, u_{\ell} \) are the counter values where the algorithm makes progress with strength \( r' \). Thus, after at most \( \sum_{i=1}^{\ell} u_i \leq (\ln n) \cdot (r' - 1) \binom{n}{g_{x'}} \ln R \) improvements with strength \( r' \), the number of iterations with strength \( g_{x'} \) is at least \( \sum_{i=1}^{\ell} (1/\ln n) \cdot u_i/(r' - 1) \geq \binom{n}{g_{x'}} \ln R - 1 \). In the next epoch, there would be at most \( \binom{n}{g_{x'}} \ln R \) iterations with strength \( r' \) for having the last required iteration.

Overall, it costs less than \( (\ln n) \cdot r'(\binom{n}{g_{x'}}) \ln R + \binom{n}{r} \ln R < (\ln n) \cdot r'(\binom{n}{g_{x'}}) \) iterations to observe \( \binom{n}{g_{x'}} \ln R \) iterations with strength \( g_{x'} \), and consecutively with a probability of at least \( 1 - 1/R \), the algorithm makes progress with strength \( g_{x'} \) via Lemma 2.

If we pessimistically assume that after each \( \binom{n}{g_{x'}} \ln R \) iterations with strength \( g_{x'} \), the radius is increased by one, by the law of total probability and Lemma 1 the expected number of steps at larger strengths than the gap of the current search point is at most

\[
\sum_{r=k}^{[n/2-1]} \frac{n - (r - 1)}{n - (2r - 1)} r \ln n \binom{n}{r} \ln R \cdot R^{-(r-k-1)}
\]

\[
< R \cdot \sum_{r=k}^{[n/2-1]} n^2 \ln n \left( \frac{n - k + 1}{k - 1} \right)^{r-k+1} \binom{n}{k-1} \ln R \cdot R^{-(r-k)}
\]

for the phases ranging from \( k \) to \([n/2 - 1]\) and at most
\[
\frac{R}{R-1} \cdot 2n \ln n \binom{n}{(n/2)} \ln R \cdot R^{-(r-m-1)}
\]

\[
< R \cdot 4n \ln n \left(\frac{n-k+1}{k-1}\right)^{\lceil n/2 \rceil - k + 1} \binom{n}{k-1} \ln R \cdot R^{-(\lceil n/2 \rceil - k - 2)}
\]

for the last phase (i.e., at radius \(n\)). Since \(R > n^{3+\epsilon}\), both are bounded from above by 
\(o\left(R \left(\frac{n}{k-1}\right)\right)\).

However, in phase \(k\), before reaching the strength \(k\), the algorithm uses strength \(m\)
\(B_0\) times. If \(B_0 \geq \binom{n}{m} \ln R\), then with probability at least \(1 - \frac{1}{R^2}\), the algorithm finds
an improvement at the Hamming distance corresponding to the gap size, resulting in

\[
E(S_x) \leq \frac{1}{R} \cdot o\left(R \left(\frac{n}{k-1}\right)\right) = o\left(\binom{n}{k-1}\right).
\]

**Analysis on unimodal functions**  On unimodal functions, the gap of all points in the
search space (except for global optima) is one, so the algorithm can make progress with
strength 1. In the following theorem, we show how SD-RLS\(m\) behaves on unimodal func-
tions compared RLS using an upper bounds based on the fitness-level method (Wegener,
2001). The proof is similar to the proof of Lemma 4 in Rajabi and Witt (2021).

**Theorem 2.** Let \(f : \{0,1\}^n \to \mathbb{R}\) be a unimodal function and consider SD-RLS\(m\) with
\(R \geq \max\{S, n^{3+\epsilon}\}\) for an arbitrary constant \(\epsilon > 0\) where \(S\) is an upper bound on the
number of strict improvements during the run, e.g., \(S = |\text{Im } f|\). Then, with probability
at least \(1 - S/R^2\), SD-RLS\(m\) never uses strengths larger than 1 and behaves stochastically
like RLS before finding an optimum of \(f\).

Denote by \(T\) the runtime of SD-RLS\(m\) on \(f\). Let \(f_i\) be the \(i\)-th fitness value of an
increasing order of all fitness values in \(f\) and \(s_i\) be a lower bound for the probabilities
that RLS finds an improvement from search points with fitness value \(f_i\), then
\(E(T) \leq \sum_{i=1}^{||\text{Im } f||} 1/s_i + o(n)\).

**Proof.** The algorithm SD-RLS\(m\) uses strength 1 for \(\binom{n}{m} \ln R\) times when the radius is 1
and \(\binom{n}{m} \ln R\) times when the radius is 2 but the strength is still 1. (Only considering
the first case would not be sufficient for the result of this lemma.) Overall, the algorithm
tries \(2\binom{n}{m} \ln R\) steps with strength 1 before setting the strength to 2.

As on unimodal functions, the gap of all points is 1, the probability of not finding
and improvement is

\[
\left(1 - \binom{n}{m}^{-1}\right)^{2\binom{n}{m} \ln R} \leq \frac{1}{R^2}.
\]
This argumentation holds for each improvement that has to be found. Since at most $|\text{Im } f|$ improving steps happen before finding the optimum, by a union bound the probability of SD-RLS$^m$ ever increasing the strength beyond 1 is at most $S \frac{1}{R^m}$, which proves the lemma.

To prove the second claim, we consider all fitness levels $A_1, \ldots, A_{|\text{Im } f|}$ such that $A_i$ contains search points with fitness value $f_i$ and sum up upper bounds on the expected times to leave each of these fitness levels. Under the condition that the strength is not increased before leaving a fitness level, the worst-case time to leave fitness level $A_i$ is $1/s_i$ similarly to RLS. Hence, we bound the expected optimization time of SD-RLS$^m$ from above by adding the waiting times on all fitness levels for RLS, which is given by $\sum_{i=1}^{|\text{Im } f|} 1/s_i$.

We let the random set $W$ contain the search points from which SD-RLS$^m$ does not find an improvement within phase 1 (i.e., while $r = 1$) so the radius is increased. Assume $T_x$ is the number of iterations spent where the radius is larger than 1 and increasing the radius happening where $x$ is the current search point; formally,

$$E(T) \leq \sum_{i=1}^{|\text{Im } f|} \frac{1}{s_i} + \sum_{x \in W} E(T_x).$$

Each search point selected by the algorithm contributes with probability $\Pr(E_1)$ to $W$. Hence, as $S$ is an upper bound on the number of improvements, $E(|W|) \leq S \cdot \Pr(E_1)$. As on unimodal functions, the gap of all points is 1, by Lemma 4 we compute

$$\sum_{x \in W} E(T_x) \leq S \cdot \Pr(E_1) \cdot E(T_x \mid \text{gap}(x) = 1) \leq S \cdot R^{-1} o\left(\binom{n}{1}\right) = o(n).$$

Thus, we finally have

$$E(T) \leq \sum_{i=1}^{|\text{Im } f|} \frac{1}{s_i} + o(n),$$
as suggested. \hfill \Box

**Analysis on JumpOff** We use the results developed so far to prove a bound on a newly designed function called JUMPOff with two parameters $m$ and $c$. Formally,

$$\text{JUMPOff}_{m,c} = \begin{cases} 
  m + |x|_1 & \text{if } |x|_1 \leq n - m - c \text{ or } |x|_1 \geq n - c, \\
  n - |x|_1 - c & \text{otherwise}.
\end{cases}$$

This function can be considered as a variant of well-known jump benchmark. In this function, we move the location of the jump with size $m$ to an earlier point than it is in
the function JUMP such that after the jump there is a unimodal sub-problem behaving like ONE\textsc{Max} of length $c$. The JUMP function is a special case of JUMP\textsc{Off} with $c = 0$, i.e. JUMP\textsc{Off},$_{m,0} = \text{JUMP}_m$.

The following theorem shows that SD-RLS$^m$ optimizes JUMP\textsc{Off} in a time that is essentially determined by the time to overcome the gap only. The proof idea is that the algorithm can quickly re-adapt its radius value to the gap size of the current search point after escaping the local optimum.

**Theorem 3.** Let $n \in \mathbb{N}$. For all $2 \leq m < O(\ln n)$ and $0 \leq c < O(\ln n)$, the expected runtime $E(T)$ of SD-RLS$^m$ with $R \geq n^{3+\epsilon}$ for an arbitrary constant $\epsilon > 0$ on JUMP\textsc{Off},$_{m,c}$ satisfies $E(T) = O\left(\binom{n}{m}\right)$, conditioned on an event that happens with probability $1 - o(1)$.

**Proof.** Before reaching the plateau consisting of all points of $n - m - c$ one-bits, JUMP\textsc{Off},$_{m,c}$ is equivalent to ONE\textsc{Max}; hence, according to Lemma 2 the expected time SD-RLS$^m$ takes to reach the plateau is at most $O(n \ln n)$. Note that this bound was obtained via the fitness level method with $s_i = (n - i)/n$ as minimum probability for leaving the set of search points with $i$ one-bits.

Every plateau point $x$ with $n - m - c$ one-bits satisfies $\text{gap}(x) = m$ according to the definition of JUMP\textsc{Off},$_{m,c}$. Thus, using Theorem 1 the algorithm finds one of the $\binom{m+c}{m}$ improvements within expected time at most $\binom{n}{m}$.

According to Lemma 2 this success happens with strength $m$ (and not larger) with probability at least $1 - 1/R$.

Now, we compute the probability that $B_0 < n \ln R$, resulting from making progress within less steps than $\ln n(m - 1)\binom{n}{1} \ln R$ in the previous epoch.

$$
\Pr(u < (m - 1)n \ln n \ln R) \leq 1 - \left(1 - \frac{(m+c)}{\binom{m}{m}}\right)^{(m-1)n \ln n \ln R} 
\leq (m - 1)n \ln n \ln R \frac{(m+c)\binom{n}{m}}{\binom{n}{m}} \leq (m - 1)n \ln n \ln R \frac{(m+c)\binom{n}{m}}{n^m}
$$

According to the assumption on $m$ and $c$, the last term is bounded from above by

$$(m - 1)n \ln n \ln R \frac{(e \ln n)^m}{n^m} \leq \frac{m(e \ln n)^{2m+2}}{n^{m-1}}$$

This means that with probability at least $1 - o(1)$, the variable $B$ is not effective in the beginning of the next epoch with strength 1.

After making progress over the jump, the starting radius $r_0$ is at least $m$, although an improvement can be found at the Hamming distance of 1, i.e., $\text{gap}(x) = 1$. Via Lemma 4 with $k = m$ and $B_0 \geq \binom{n}{1} \ln R$, within $o(\binom{n-1}{m-1})$ steps in expectation, the algorithm sets the radius to 1, if it pessimistically does not find the optimum point. Now, the algorithm needs to optimize a ONE\textsc{Max} of length $c$, so its expected time again can be obtained from Lemma 2 which is at most $O(n \ln n)$.

Altogether, $E(T) \leq O(n \ln n) + \binom{n}{m} + \binom{n}{m-1} + O(n \ln n)$, conditioned on the mentioned events of having enough iterations with the strength passing the jump part and escaping from the local optimum with strength $m$, happening with probability $1 - o(1)$. \qed
4 Speed-ups By Using Radius Memory

In this section, we consider the problem of minimizing a linear function under a uniform constraint as analyzed in Neumann, Pourhassan and Witt (2019): given a linear pseudo-Boolean function \( f(x_1, \ldots, x_n) = \sum_{i=1}^{n} w_i x_i \), the aim is to find a search point \( x \) minimizing \( f \) under the constraint \( |x|_1 \geq B \) for some \( B \in \{1, \ldots, n\} \). W.l.o.g., \( w_1 \leq \cdots \leq w_n \).

Neumann, Pourhassan and Witt (2019) obtain a tight worst-case runtime bound \( \Theta(n^2) \) for RLS\(^{1,2}\) and a bound for the (1+1) EA which is \( O(n^2 \log B) \) and therefore tight up to logarithmic factors. We will see in Theorem 4 that with high probability, SD-RLS\(^m\) achieves the same bound \( O(n^2) \) despite being able to search globally like the (1+1) EA. Afterwards, we will identify a scenario where SD-RLS\(^r\) is by a factor of \( \Omega(n) \) slower.

We start with the general result on the worst-case expected optimization time, assuming the set-up of Neumann, Pourhassan and Witt (2019).

**Theorem 4.** Starting with an arbitrary initial solution, the expected optimization time of SD-RLS\(^m\) with \( R \geq n^{3+\epsilon} \), where \( \epsilon > 0 \) is an arbitrary constant, on a linear function with a uniform constraint is \( O(n^2) \) conditioned on an event that happens with probability \( 1 - O(1/n) \).

**Proof.** We follow closely the proof of Theorem 4.2 in Neumann, Pourhassan and Witt (2019) who analyze RLS\(^{1,2}\). The first phase of optimization (covered in Lemma 4.1 of the paper) deals with the time to reach a feasible search point and proves this to be \( O(n \log n) \) in expectation. Since the proof uses multiplicative drift, it is easily seen that the time is \( O(n \log n) \) with probability at least \( 1 - O(1/n) \) thanks to the tail bounds for multiplicative drift (Lengler, 2020). The second phase deals with the time to reach a tight search point (i.e., containing \( B \) one-bits, which is \( O(n \log n) \) with probability at least \( 1 - O(1/n) \) by the very same type of arguments. The analyses so far rely exclusively on one-bit flips so that the bounds also hold for SD-RLS\(^m\) thanks to Lemma 2 up to a failure event of \( O(S/R^2) = o(1/n) \) since it holds for the number of improvements \( S \) that \( S \leq n \). By definition of the fitness function, only tight search points will be accepted in the following.

The third phase in the analysis from Neumann, Pourhassan and Witt (2019) considers the potential function \( \phi(x) = |\{x_i = 1 : i > B\}| \) denoting the number of one-bits outside the \( B \) least significant positions. At the same time, \( \phi(x) \) describes the number of zero-bits at the \( B \) optimal positions. Given \( \phi(x) = i > 0 \) for the current search point, the probability of improving the potential is \( \Theta(i^2/n^2) \) since there are \( i^2 \) improving two-bit flips (\( i \) choices for a one-bit to be flipped to 0 at the non-optimal positions and \( i \) choices for a zero-bit to be flipped to 1 at the other positions). This results in an expected optimization time of at most \( \sum_{i=1}^{\infty} O(n^2/i^2) = O(n^2) \) for RLS\(^{1,2}\).

SD-RLS\(^m\) can achieve the same time bound since after every two-bit flip, the radius memory only allocates the time for the last improvement (via two-bit flips) to iterations trying strength 2. Hence, as long as no strengths larger than 2 are chosen, the expected optimization time of SD-RLS\(^m\) is \( O(n^2) \).
To estimate the failure probability, we need a bound on the number of strict improvements of $f$, which may be larger than the number of improvements of the $\phi$-value since steps that flip a zero-bit and a one-bit both located in the prefix or suffix may be strictly improving without changing $\phi$. Let us assign a value to each zero-bit, representing the number of one-bits to its left (at higher indices). In other words, each of these values shows the number of one-bits that can be flipped with the respective zero-bit to make a strict improvement. Let us define by $S$ the sum of these values. Clearly, $S \leq n(n-1)/2 = O(n^2)$. Now, we claim that each strict improvement decreases $S$ by at least one, resulting in bounding the number of strict improvements from above by $O(n^2)$. Assume that in a strict improvement, the algorithm flips one one-bit at position $i$ and one zero-bit at position $j$. Obviously, $j < i$. The corresponding value for the zero-bit at the new position $i$ is less than the corresponding number for the zero-bit at position $j$ before flipping because there is at least one one-bit less for the new zero-bit, which was the one-bit at the position of $i$. Altogether, the number of strict improvements at strength 2 is at most $O(n^2)$.

The proof is completed by noting that the strength never exceeds 2 with probability at least $1 - O(n^2)/R = 1 - O(1/n)$ thanks to Lemma 2 and a union bound. \qed

With a little more effort, including an application of Lemma 4, the bound from Theorem 4 can be turned into a bound on the expected optimization time. We will see an example of such arguments later in the proof of Theorem 5 and in the proof of Theorem 6.

We now illustrate why the original SD-RLS$^r$ is less efficient on linear functions under uniform constraints than SD-RLS$^m$. To this end, we study the following instance: the weights of the objective function are $n$ pairwise different natural numbers (sorted increasingly), and the constraint bound is $B = n/2$, i.e., only search points having at least $n/2$ one-bits are valid. Writing search points in big-endian as $x = (x_n, \ldots, x_1)$, we assume the point $1^{n/2}0^{n/2}$ as starting point of our search heuristic. The optimum is then $0^{n/2}1^{n/2}$ since the $n/2$ one-bits are at the least significant positions. We call the latter positions the suffix and the other the prefix.

Considering the potential $\phi(x)$ defined as the number of one-bits in the prefix, we note that the expected time for RLS$^{1,2}$ and for SD-RLS$^m$ (up to a failure event) to reduce the potential from its initial value $n/4$ to $n/8$ is $O(n)$ since during this period there is always an improvement probability of at least $\Omega((n/8)^2/n^2) = \Omega(1)$. We claim that SD-RLS$^r$ needs time $\Omega(n^2 \log n)$ for this since after each improvement the strength and radius are reset to 1, resulting in $\Omega(n)$ phases where the algorithm is forced to iterate unsuccessfully with strength 1 until the threshold $\binom{n}{1} \ln R$ is reached. By contrast, SD-RLS$^m$ will spend $O(n \ln n)$ steps to set the strength and radius to 2. Afterwards, during the $n/4$ improvements it will only spend an expected number of $O(1)$ steps at strength 1 before it returns to strength 2 and finds an improvement in expected time $O(1)$. The total time is $O(n \log n + n) = O(n \log n)$. Based on these ideas, we formulate the following theorem.
Theorem 5. Consider a linear function with \( n \) pairwise different weights under uniform constraint with \( B = n/2 \) and let \( S_b = \{ \phi(y) \leq b \mid y \in \{0, 1\}^n \} \). Starting with a feasible solution \( x \) such that \( x \) contains \( B \) one-bits and \( \phi(x) = a \), the expected time to find a search point in \( S_b \) for SD-RLS\(^m\) with \( R \geq n^{3+\epsilon} \) for an arbitrary constant \( \epsilon > 0 \) is at most

\[
O \left( n \ln R + n^2 \sum_{i=b}^{a} \frac{1}{i^2} \right).
\]

For SD-RLS\(^r\) with \( R \geq n^{3+\epsilon} \) for an arbitrary constant \( \epsilon > 0 \) it is at least

\[
\Omega \left( P \cdot n \ln R + n^2 \sum_{i=b}^{a} \frac{1}{i^2} \right),
\]

where \( P \) is the number of improvements with strengths larger than 1 and \( P > a - b \) with probability \( 1 - o(1/n) \).

Proof. We first find an upper bound for SD-RLS\(^m\). First, SD-RLS\(^m\) spends \( O(n \ln R) \) steps to set the strength and radius to \( 2 \). Afterwards, the number of iterations with strength \( 1 \) is \( 1/\ln n \) times of the number of iterations with strength \( 2 \), conditioned on not exceeding the threshold when \( r = 2 \), which will be studied later.

When the strength equals \( 2 \), the probability of improving the potential is \( \Theta(i^2/n^2) \), resulting in the expected time of \( \Theta(n^2/i^2) \) for each improvement. Thus, in expectation, there are \( n^2 \sum_{i=b}^{a} 1/i^2 \) iterations to find a search point in \( S_b \). In the case that the counter exceeds the threshold, happening with probability \( 1/R \) for each improvement using Lemma 2, it costs \( o\left( \binom{n}{2} \right) \) iterations with different strengths in expectation, according to Lemma 3 to set the radius to \( 2 \) again. Since the number of fitness improvements at strength \( 2 \) is at most \( n^2 \) (see the proof of Th. 5), we obtain a failure probability of at most \( n^2 \cdot 1/R = o(1/n) \) for the event of exceeding the threshold. Hence, the expected number of iterations with different strengths is \( o(\frac{1}{n}) \binom{n}{2} = o(n) \) and therefore a lower-order term of the claimed bound on the expected time to reach \( S_b \).

In case of a failure, we repeat the argumentation. Hence, after an expected number of \( 1/(1 - o(1/n)) = 1 + o(1) \) repetitions no failure occurs and \( S_b \) is reached.

Overall, the expected time for SD-RLS\(^m\) to find a search point in \( S_b \) is at most

\[
(1 + o(1)) \left( n \ln R + (1 + 1/\ln n) \cdot \sum_{i=b}^{a} \frac{n^2}{i^2} \right) = O \left( n \ln R + n^2 \sum_{i=b}^{a} \frac{1}{i^2} \right).
\]

In order to compute a lower bound on the optimization time of SD-RLS\(^r\), we claim that the number of potential improvements is at least \( a - b \) with probability \( 1 - o(1/n) \), i.e., each improvement decreases the potential function roughly by at most one in expectation.

As long as the strength does not become greater than \( 2 \), the number of one-bits remains \( B \), so when the strength is at most \( 2 \), the algorithm can only improve the potential by 1 since it cannot make progress by flipping at least two zero-bits in \( B \).
least significant positions. The radius becomes 3 with probability at most $1/R$ for each improvement at strength 2. Since there are at most $n^2$ improvements, the probability of not increasing the current radius to 3 during the run is at least $1 - n^2/R = 1 - o(1/n)$.

Now, since the algorithm spends $n \ln R$ steps for each improvement, the number of steps with strength 1 is $\Omega((a-b)n \ln R)$ with probability $1 - o(1/n)$. Also, the number of steps with strength 2 is $\Theta(n^2 \sum_{i=1}^b 1/i^2)$ in expectation. Note that we ignore the number of iterations with strengths larger than 2 for the lower bound.

Overall, with probability at least $1 - o(1/n)$ the time of SD-RLS is at least

$$\Omega\left((a-b)n \ln R + n^2 \sum_{i=b}^a \frac{1}{i^2}\right).$$

In the following corollary, it can be seen that SD-RLS is faster than SD-RLS in the middle of the run by a factor of roughly $n$.

**Corollary 1.** The relative speed-up of SD-RLS with $R \geq n^{3+\epsilon}$ for an arbitrary constant $\epsilon > 0$ compared to SD-RLS with $R \geq n^{3+\epsilon'}$ for an arbitrary constant $\epsilon' > 0$ to find a search point in $S_b$ with $b = n/8$ for a starting search point $x$ with $\phi(x) = n/4$ is $\Omega(n)$ with probability at least $1 - o(1/n)$.

**Proof.** Assume that $T_r$ and $T_m$ are the considered hitting times of SD-RLS and SD-RLS, respectively, with $R \geq n^{3+\epsilon}$ for an arbitrary constant $\epsilon > 0$. Using Theorem 5 with $a = n/4$ and $b = n/8$, we have

$$\frac{E(T_r)}{E(T_m)} \geq \frac{\Omega(n^2 \ln R)}{O(n \ln R)} \geq \Omega(n),$$

with probability at least $1 - o(1/n)$.

### 5 Minimum Spanning Trees

In Theorem 6 in Rajabi and Witt (2021), the authors studied SD-RLS on the MST problem as formulated in Neumann and Wegener (2007) using the fitness function $f$ that returns the total weight of the spanning tree and penalizes unconnected graphs as well as graphs containing cycles. They showed that SD-RLS with $R = m^4$ can find an MST starting with an arbitrary spanning tree in $(1 + o(1))(m^2 \ln m + (4m \ln m)E(S))$ fitness calls where $E(S)$ is the expected number of strict improvements. The reason behind $E(S)$ is that for each improvement with strength 2, the algorithm resets the radius to one for the next epoch and explores this radius more or less completely in $m \ln R$ iterations. This can be costly for the graphs requiring many improvements.

However, with SD-RLS, we do not need to consider the number of improvements for estimating the number of iterations with strength 1 since with the radius memory mechanism, the number of iterations with strength 1 is asymptotically in the order of $1/\ln n$ times of the number of successes. This leads to the following simple bound.
Theorem 6. The expected optimization time of SD-RLS\(^m\) with \(R = m^4\) on the MST problem starting with an arbitrary spanning tree is at most

\[
(1 + o(1))(m^2/2)(1 + \ln(r_1 + \cdots + r_m)) \leq (1 + o(1))(m^2 \ln m),
\]

where \(r_i\) is the rank of the \(i\)th edge in the sequence sorted by increasing edge weights.

The proof is similar to the proof of Theorem 6 in [Rajabi and Witt (2021)] by using drift multiplicative analysis [Lengler (2020)]. However, we show that the radius memory mechanism controls the number of iterations with strength 1, and we apply Lemma 4 to show that if the algorithm uses strengths larger than 2, the algorithm shortly after makes an improvement with strength 2 again.

Proof of Theorem 6 We aim at using multiplicative drift analysis using \(g(x) = \sum_{i=1}^m x_ir_i\) as potential function. Since the algorithm has different states we do not have the same lower bound on the drift towards the optimum. However, at strength 1 no mutation is accepted since the fitness function from Neumann and Wegener (2007) gives a huge penalty to non-trees. Hence, our plan is to conduct the drift analysis conditioned on the strength is at most 2 and account for the steps spent at strength 1 separately. Cases where the strength exceeds 2 will be handled by an error analysis and a restart argument.

Let \(X(t) := g(x(t)) - g(x_{\text{opt}})\) for the current search point \(x(t)\) and an optimal search point \(x_{\text{opt}}\). Since the algorithm behaves stochastically the same on the original fitness function \(f\) and the potential function \(g\), we obtain that \(E(X(t) - X(t+1) \mid X(t)) \geq X(t)/m^2 \geq 2X(t)/m^2\) since the \(g\)-value can be decreased by altogether \(g(x(t)) - g(x_{\text{opt}})\) via a sequence of at most \((m^2)/2\) disjoint two-bit flips; see also the proof of Theorem 15 in Doerr, Johannsen and Winzen (2012) for the underlying combinatorial argument.

Let \(T\) denote the number of steps at strength 2 until \(g\) is minimized, assuming no larger strength to occur. Using the multiplicative drift theorem, we have \(E(T) \leq (m^2/2)(1 + \ln(r_1 + \cdots + r_m)) \leq (m^2/2)(1 + \ln(m^2))\) and by the tail bounds for multiplicative drift (e.g., Lengler (2020)) it holds that \(\Pr(T > (m^2/2)(\ln(m^2) + \ln(m^2))) \leq e^{-\ln(m^2)} = 1/m^2\). Note that this bound on \(T\) is below the threshold for strength 2 since \((m^2)/2 \ln R = (m^2 - m) \ln(m^4) \geq (m^2/2)(4\ln m)\) for \(m\) large enough. Hence, with probability at most \(1/m^2\) the algorithm fails to find the optimum before the strength can change from 2 to a different value due to the threshold being exceeded.

We next bound the expected number of steps spent at larger strengths. By Lemma 4 if the algorithm fails to find an improvement with the right radius, i.e., when the radius becomes \(r = 3 > \text{gap}(x) = 2\), then, in at most \(o((m^2))\) iterations in expectation, the radius is set to the gap size of the current search point at that time. Thus, by using Lemma 4 an increase of radius above 2 costs at most \(o((m^2))\) iterations to make improvements with strength 2 again and set the radius to two. This time is only a lower-order term of the runtime bound claimed in the theorem. If the strength exceeds 2, we wait for it to become 2 again and restart the previous drift analysis, which is conditional on strength at most 2. Since the probability of a failure is at most \(1/m^2\), this gives an expected number of at most \(1/(1 - m^{-2})\) restarts, which is \(1 + o(1)\) as well.
It remains to bound the number of steps at strength 1. For each strict improvement, $B$ is set to $\frac{1}{\ln n \cdot u}$ where $u$ is the counter value, the counter is reset, and radius $r$ is set to 2. Thereafter, $B$ steps pass before the strength becomes 2 again. Hence, if the strength does not exceed 2 before the optimum is reached, this contributes a term of $(1 + \frac{1}{\ln n})$ to the number of iterations with strength 2 in the previous epoch. Also, in the beginning of the algorithm, there is a complete phase with strength 1 costing $m \ln R$, which only contributes a lower-order term.

Theorem 6 is interesting since it is asymptotically tight and does not suffer from the additional $\log(w_{\text{max}})$ factor known from the analysis of the classical $(1+1)$ EA (Neumann and Wegener 2007). In fact, this seems to be the first asymptotically tight analysis of a globally searching $(1+1)$-type algorithm on the MST. So far a tight analysis of evolutionary algorithms on the MST was only known for $\text{RLS}^{1,2}$ with one- and two-bit flip mutations (Raidl, Koller and Julstrom 2006). Our bound in Theorem 6 is by a factor of roughly 2 better since it avoids an expected waiting time of 2 for a two-bit flip. On the technical side, it is interesting that we could apply drift analysis in its proof despite the algorithm being able to switch between different mutation strengths influencing the current drift.

6 Radius Memory can be Detrimental

After a high mutation strength has been selected, e.g., to overcome a local optimum, the radius memory decreases the threshold values for phase lengths related to lower strengths. As we have seen in Lemma 4, $\text{SD-RLS}^m$ can often return to a smaller strength quickly. However, we can also point out situations where using the smaller strength with their original threshold values as used in the original $\text{SD-RLS}^r$ from Rajabi and Witt (2021) is crucial.

Our example is based on a general construction principle that can be traced back to Witt (2003) and was picked up in Rajabi and Witt (2020) to show situations where stagnation in the context of the $(1+1)$ EA is detrimental; see that paper for a detailed account of the construction principle. In Rajabi and Witt (2021), the idea was used to demonstrate situations where bit-flip mutations outperforms $\text{SD-RLS}^r$. Roughly speaking, the functions combine two gradients one of which is easier to exploit for an algorithm $A$ while the other is easier to exploit for another algorithm $B$. By appropriately defining local and global optima close to the end of the search space in direction of the gradients, either Algorithm $A$ significantly outperforms Algorithm $B$ or the other way round.

We will now define a function on which $\text{SD-RLS}^m$ is exponentially slower than $\text{SD-RLS}^r$. In the following, we will imagine any bit string $x$ of length $n$ as being split into a prefix $a := a(x)$ of length $n - m$ and a suffix $b := b(x)$ of length $m$, where $m$ is defined below. Hence, $x = a(x) \circ b(x)$, where $\circ$ denotes the concatenation. The prefix $a(x)$ is called valid if it is of the form $1^i0^{n-m-i}$, i.e., $i$ leading ones and $n - m - i$ trailing zeros. The prefix fitness $\text{pre}(x)$ of a string $x \in \{0, 1\}^n$ with valid prefix $a(x) = 1^i0^{n-m-i}$
equals $i$, the number of leading ones. The suffix consists of $[n^{1/8}]$ consecutive blocks of $[n^{3/4}]$ bits each, altogether $m = O(n^{7/9})$ bits. Such a block is called valid if it contains either 0 or 2 one-bits; moreover, it is called active if it contains 2 and inactive if it contains 0 one-bits. A suffix where all blocks are valid and where all blocks following first inactive block are also inactive is called valid itself, and the suffix fitness $\text{suff}(x)$ of a string $x$ with valid suffix $b(x)$ is the number of leading active blocks before the first inactive one. Finally, we call $x \in \{0,1\}^n$ valid if both its prefix and suffix are valid.

The final fitness function is a weighted combination of $\text{pre}(x)$ and $\text{suff}(x)$. We define for $x \in \{0,1\}^n$, where $x = a \circ b$ with the above-introduced $a$ and $b$,

$$\text{PreferOneBitFlip}(x) := \begin{cases} n - m - \text{pre}(x) + \text{suff}(x) & \text{if } \text{suff}(x) \leq n^{1/9} \land x \text{ valid} \\ n^2 \text{pre}(x) + \text{suff}(x) & \text{if } n^{1/9} < \text{suff}(x) \leq n^{1/8}/2 \land x \text{ valid} \\ n^2(n-m) + \text{suff}(x) - n - 1 & \text{if } \text{suff}(x) > n^{1/8}/2 \land x \text{ valid} \\ -\text{OneMax}(x) & \text{otherwise}. \end{cases}$$

We note that all search points in the third case have a fitness of at least $n^2(n-m) - n - 1$, which is bigger than $n^2(n-m-1) + n$, an upper bound on the fitness of search points that fall into the second case without having $m$ leading active blocks in the suffix. Hence, search points $x$ where $\text{pre}(x) = n - m$ and $\text{suff}(x) = \lceil n^{1/8} \rceil$ represent local optima of second-best overall fitness. The set of global optima equals the points where $\text{suff}(x) = \lceil n^{1/8}/2 \rceil$ and $\text{pre}(x) = n - m$, which implies that at least $n^{1/8}$ bits (two from each block) have to be flipped simultaneously to escape from the local toward the global optimum. The first case is special in that the function is decreasing in the $\text{pre}$-value as long as $\text{suff}(x) \leq n^{1/9}$. Typically, the first valid search point falls into the first case. Then two-bit flips are essential to make progress and the radius memory of SD-RLS$^m$ will be used when waiting for the next improvement. After leaving the first case, since two-bit flips happen quickly enough, the memory will make progress via one-bit flips unlikely, leading to the local optimum.

We note that function $\text{PreferOneBitFlip}$ shares some features with the function $\text{NeedHighMut}$ from [Rajabi and Witt (2020)] and the function $\text{NeedGlobalMut}$ from [Rajabi and Witt (2021)]. However, it contains an extra case for small suffix values, uses different block sizes and block count for the suffix, and inverts roles of prefix and suffix by leading to a local optimum when the suffix is optimized first.

In the following, we make the above ideas precise and show that SD-RLS$^m$ outperforms SD-RLS$^m$ on $\text{PreferOneBitFlip}$.

**Theorem 7.** With probability at least $1 - 1/n^{1/8}$, SD-RLS$^m$ with $R \geq n^{3+\epsilon}$ for an arbitrary constant $\epsilon > 0$ needs time $2^{\Omega(n^{1/8})}$ to optimize $\text{PreferOneBitFlip}$. With probability at least $1 - 1/n$, SD-RLS$^m$ with $R \geq n^{3+\epsilon}$ optimizes the function in time $O(n^2)$.

**Proof.** As in the proof of Theorem 4.1 in [Rajabi and Witt (2020)], we have that the first valid search point (i.e., search point of non-negative fitness) of both SD-RLS$^m$ and SD-
RLS\textsuperscript{m} has both \textsc{pre}- and \textsc{suff}-value value of at most $n^{1/9}/2$ with probability $2^{-\Omega(n^{1/9})}$. In the following, we tacitly assume that we have reached a valid search point of the described maximum \textsc{pre}- and \textsc{suff}-value and note that this changes the required number of improvements to reach local or global maximum only by a $1 - o(1)$ factor. For readability this factor will not be spelled out any more.

Given the situation with a valid search point $x$ where \textsc{suff}(x) < $n^{1/9}/2$, fitness improvements only possible by increasing the \textsc{suff}-value. Since the probability of a \textsc{suff}-improving steps is at least $(n^{3/2})/n^2 = \Omega(n^{-1/2})$ at strength 2, the time for both algorithms to reach the second case of the definition of \textsc{PreferOneBitFlip} is $O(n^{1/9}n^{1/2}) = O(n^{11/18})$ according to Lemma 3, and by repeating independent phases and Markov’s inequality, the time is $O(n)$ with probability exponentially close to 1. Afterwards, one-bit flips increasing the \textsc{pre}-value are strictly improving and happen while the strength is 1 with probability at least $1 - 1/R$ in SD-RLS\textsuperscript{p}, which does not have radius memory. Therefore, by a union bound over all $O(n)$ improvements, with probability at least $1 - 1/n$, SD-RLS\textsuperscript{p} increases the \textsc{pre}-value to its maximum before the \textsc{suff}-value becomes greater than $\lceil n^{1/9} \rceil$. The time for this is $O(n^2)$ even with probability exponentially close to 1 by Chernoff bounds. Afterwards, by reusing the above analysis to leave the first case, with probability at least $1 - 1/n$ a number of $o(n^2)$ steps is sufficient for SD-RLS\textsuperscript{p} to find the global optimum. This proves the second statement of the theorem.

To prove the first statement, i.e., the claim for SD-RLS\textsuperscript{m}, we first note that the probability of improving the \textsc{pre}-value at strength 2 is $O(n^2)$ since two specific bits would have to flip. Hence, such steps never happen in $\Theta(n)$ steps with probability $1 - O(1/n)$. By contrast, a \textsc{suff}-improving step at strength 2, which has probability $\Omega(n^{-1/2})$, happens within $O(n^{3/4})$ steps with probability at least $1 - 1/n^{1/4}$ according to Markov’s inequality. In this case, the radius memory of SD-RLS\textsuperscript{m} will set a threshold of $B = n^{3/4}$ for the subsequent iterations at strength 1. The probability of improving the \textsc{pre}-value within this time is $O(1/n^{1/4})$ by a union bound, noting the success probability of at most $1/n$. Hence, the probability of having at least $n^{1/8}$ improvements of the \textsc{suff}-value within $n^{3/4}$ steps each before an improvement of the \textsc{pre}-value (at strength 1) happens, is at least $1 - 1/n^{1/8}$ by a union bound. If all this happens, the algorithm has to flip at least $n^{1/8}$ bits simultaneously, which requires $2^{\Omega(n^{1/8})}$ steps already to reach the required strength. The total failure probability is $O(1/n^{1/8})$. \hfill $\Box$

## 7 Experiments

We ran an implementation of five algorithms SD-RLS\textsuperscript{m}, SD-RLS\textsuperscript{p}, (1+1) FEA\textsubscript{$\beta$} with $\beta = 1.5$ from Doerr et al. (2017), the standard (1+1) EA and RLS\textsuperscript{1,2} on the MST problem with the fitness function from Neumann and Wegener (2007) for three types of graphs called TG, Erdős–Rényi with $p = (2 \ln n)/n$, and $K_n$. We carried out a similar experiment to Rajabi and Witt (2021) with the more additional class of complete graphs $K_n$ to illustrate the performance of the new algorithm and compare with the other algorithms.

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The graph TG with \( n \) vertices and \( m = 3n/4 + \binom{n/2}{2} \) edges contains a sequence of \( p = n/4 \) triangles which are connected to each other, and the last triangle is connected to a complete graph of size \( q = n/2 \). Regarding the weights, the edges of the complete graph have the weight 1, and we set the weights of edges in triangle to 2\( a \) and 3\( a \) for the side edges and the main edge, respectively. In this paper, we consider \( a = n^2 \). The graph TG is used for estimating lower bounds on the expected runtime of the (1+1) EA and RLS in the literature (Neumann and Wegener, 2007). As can be seen in Figure 1a, (1+1) EA with heavy-tailed mutation, (1+1) FEA\(_\beta\), with \( \beta = 1.5 \) outperformed the rest of the algorithms. However, SD-RLS\(_r\) and SD-RLS\(_m\) also outperformed the standard (1+1) EA and RLS\(_{1,2}\).

Regarding the graphs Erdős–Rényi, we produced some random Erdős–Rényi graphs with \( p = (2 \ln n)/n \) and assigned each edge an integer weight in the range \([1, n^2]\) uniformly at random. We also checked that the graphs are certainly connected. Then, we ran the implementation to find the MST of these graphs. The obtained results can be seen in Figure 1b. As discussed in Section 6 in Rajabi and Witt (2021), SD-RLS\(_r\) does not outperform the (1+1) EA and RLS\(_{1,2}\) on MST with graphs when the number of strict improvements in SD-RLS\(_r\) is large. However, the proposed algorithm in this paper, SD-RLS\(_m\) outperformed the rest of the algorithms, although there can be a relatively large number of improvements on such graphs. We can also see this superiority in Figure 1c for the complete graphs \( K_n \) with random edge weights in the range \([1, n^2]\).

For statistical tests, we ran the algorithms on the graphs TG and Erdős–Rényi 200 times, and all p-values obtained from a Mann-Whitney U-test between the algorithms, with respect to the null hypothesis of identical behavior, are less than \( 10^{-2} \) except for the results regarding the smallest size in each set of graphs.

**Conclusions**

We have investigated stagnation detection with the s-bit flip operator as known from randomized local search and introduced a mechanism called *radius memory* that allows...
continued exploitation of large $s$ values that were useful in the past. Improving earlier work from Rajabi and Witt (2021), this leads to tight bounds on complex multimodal problems like linear functions with uniform constraints and the minimum spanning tree problem, while still optimizing unimodal and jump functions as efficiently as in earlier work. The bound for the MST is the first tight runtime bound for a global search heuristics and improves upon the runtime of classical RLS algorithms by a factor of roughly 2. We have also pointed out situations where the radius memory is detrimental for the optimization process. In the future, we would like to investigate the concept of stagnation detection with radius memory in population-based algorithms and plan analyses on further combinatorial optimization problems.

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