An experiment on the shifts of reflected C-lines

W Löffler¹, J F Nye² and J H Hannay²

¹Huygens-Kamerlingh Onnes Laboratory, Leiden University, PO Box 9504, 2300 RA Leiden, The Netherlands
²H H Wills Physics Laboratory, University of Bristol, Tyndall Avenue, Bristol BS8 1TL, UK

E-mail: loeffler@physics.leidenuniv.nl

Received 10 February 2014, revised 5 June 2014
Accepted for publication 10 June 2014
Published 25 July 2014

Abstract

C-lines in vector electromagnetic waves are analogues of the more familiar optical vortices found in the complex scalar waves usually used to describe light. The centroid of a laser beam is shifted on reflection by the well-known Goos–Hänchen and Imbert–Fedorov effects, but if it carries a C-line two separate C-lines appear in the reflected beam, both of which are shifted, their shifts being unrelated to the well-known shift of the beam centroid. An experiment is described that tests the theoretical predictions for the shifts of the C-lines perpendicular to the plane of incidence. It used internal reflection in a glass prism close to the critical angle to enhance the effect. In a simple situation like this, two recently published independent theories of C-line reflection are both applicable and it is shown that their predictions are identical. Our measured differences in the shifts of the two reflected C-lines confirm both theories. Remarkably, the measurable C-line shifts are much larger (hundreds of micrometers) than traditional beam-shift displacements. Theoretically, the difference is infinite at the critical angle itself with a change of sign.

Keywords: C-line, optical vortex, singularity optics, phase singularity, reflection, wave vortex, beam shifts

(Some figures may appear in colour only in the online journal)

1. Introduction

Wave dislocations [1, 2] or optical vortices as they are now called, lines of zero disturbance, are features in the complex scalar fields commonly used to represent light. They are the origin of the speckle pattern seen when a surface is illuminated by a laser beam. The phase circulates around the vortices. They attracted considerable attention when it was discovered [3] that they carried orbital angular momentum (OAM) that could be transferred to small particles. Such phase singularities occur generically, that is, they appear naturally, in any general monochromatic scalar field. However, in the vector electromagnetic field that fully describes light, including its polarization, the end of the electric vector at each point describes a polarization ellipse, which collapses to zero only at special points. Zeros are not a general feature of vector wavefields as they are of scalar fields. A feature in a vector wavefield that is generic and may be thought of as the counterpart of the line vortex in scalar waves is a line where the polarization ellipse is a circle. It is these C-lines that are the concern of this paper. The analogy with wave vorticies arises in the following way. The field at any point may be thought of as the sum of a left-hand and a right-hand circularly polarized component. At a left-hand C-point, for example, the right-hand circular component has zero amplitude and therefore indeterminate phase. Thus, the left-hand C-point is a vortex in the field of the right-hand component.

It is well known that a laser beam incident at an angle on an interface between two different media is not only reflected and transmitted but also shifted in position by the Goos–Hänchen [4] and the Imbert–Fedorov [5] effects. Wolter had proposed in 1950 that the nodal plane arising from the interference of a close pair of plane waves could act as a marker in measuring such shifts (reprint in [6]). Beam-shift studies have hitherto been concerned with shifts of the beam centroid, or a closely related beam feature, even if the beam...
under study carries an optical phase vortex [7, 8]. If the incident beam contains a C-line this is indeed shifted, both longitudinally and laterally, but in a way quite unrelated to the shift of the centroid of the beam. The usual Fresnel coefficients cannot be used directly here because they apply to a simple plane wave. Instead, there are two alternative theoretical schemes that may be applied. The first is described in [9] and the second, which makes use of the relatively new concept of a differentiated plane wave (DPW), in [10, 11]. Each makes a slightly different idealization of the core of the beam and each has its own advantages. The first is especially concerned with C-lines of all orders at the center of the beam while the second is applicable to transmission as well as reflection. Their common ground is reflection of a suitable simple beam, and it is this case that we consider here. The experiment to be described simultaneously tests both theories of the shift caused by reflection. Thus it suffices to present only one of them—we choose the DPW approach and relate it briefly to the other scheme.

2. The DPW

To explain the idea of a DPW in the simplest terms let us start with the monochromatic scalar plane wave \( \psi = \exp (i \mathbf{k} \cdot \mathbf{r}) \) and differentiate with respect to the direction of the wave, specified by the components of the wave vector \( \mathbf{k} = (k_x, k_y, k_z) \), rather than with respect to position, given by the vector \( \mathbf{r} = (x, y, z) \). After the differentiation we specialize to the \( z \) direction, as follows:

\[
\left( -i \frac{\partial \psi}{\partial k_x} + i \frac{\partial \psi}{\partial k_z} \right) = (x + iy) \exp (ik_z) \tag{1}
\]

which is an optical vortex, or a pure screw wave dislocation. The differentiation with respect to direction combines an \( x \) variation with a \( y \) variation acting in quadrature, to produce an infinitesimal rotation. This differentiation with respect to direction has produced from a featureless plane wave, a wave with a linear modulation of amplitude across its wavefronts; it is zero on \( x = y = 0 \), which is the dislocation or vortex line. A modulation of amplitude would normally lead to diffraction but with a purely linear modulation the wave propagates unchanged in form. Notice that, because the original plane wave obeys the Helmholtz equation \( \nabla^2 \psi + k^2 \psi = 0 \) the DPW automatically does so too. To construct from this scalar wave a corresponding circularly polarized vector electromagnetic wave we multiply by a complex vector to obtain

\[
E = (1, i, 0) (x + iy) \exp (ik_z) = (1, i, 0) r \exp (i\theta) \exp (ik_z), \tag{2}
\]

where \( r, \theta \) are cylindrical coordinates. A generic C-point is surrounded by a sea of elliptical polarization and has circular polarization only at the point itself. So far as C-lines are concerned this field is therefore very special because it is circularly polarized everywhere. If it were perturbed by a field of finite amplitude and the opposite hand that contained a C-point at a point \( P \) that was not the origin, it would split into two separate C-points, one at the origin and the other at \( P \) [2, p.285]. We shall see shortly that reflection of this unperturbed and degenerate beam removes the degeneracy in a similar way and produces a pair of C-lines of opposite hand.

The incident beam has an amplitude that increases linearly with \( r \) without limit. In practice it arises as the core of a Laguerre–Gauss laser beam with azimuthal index \( \ell = 1 \) and the increase in amplitude must be limited. Formally, this may be represented by multiplying by, for example, a Gaussian envelope of the form \( \exp (-\left( x^2 + y^2 \right)/w^2) \), \( w \) being the waist size. There will now be some small spreading out of the beam by diffraction. But, in contrast to known beam deformation effects happening for focused beams close to the critical angle [12], the C-line shift occurs also for beams with infinite width \( w \). More importantly, the position of a dark C-line can be determined with high accuracy for arbitrary \( w \), in contrast to the beam centroid, which requires detection of the full beam, and is therefore impossible for very large \( w \). We have described a special example of a DPW. A similar, but necessarily not so simple, general treatment is postponed to section 4; it was used to predict the shifts of C-lines on reflection. Although the shifts are analogous to the Goos–Hänchen [4] and Imbert–Fedorov [5] shifts of the centroids of narrow laser beams, they are quite different from them.

Section 3 describes an experiment to test these theoretical predictions. In the theory the incident wave containing the C-lines is infinitely wide. The laser beam used in the experiment had a Gaussian envelope that was comparatively narrow, but it proved to be wide enough for practical purposes.

The shifts are often only a few wavelengths. Various ways of producing an effect large enough to be measured accurately were considered. For example, a Fabry–Perot type slab where the multiple reflections can enhance the effect. But this required a measurement of the thickness accurate to a fraction of a wavelength and so was not readily practicable. Finally, it was decided to work in the vicinity of the critical angle for total internal reflection and this does give a large enough effect—indeed the theory predicts a divergence at the critical angle itself. It should be noted that the theory calculates the spatial shifts that occur at the interface itself and makes no prediction about any purely angular deviation, whereas the experiment measures the shift at a distance from the surface small in comparison with the Rayleigh range. The comparison assumes, it seems plausibly, that any purely angular deviation may be neglected.

3. Experiment

The initial aim was to create in the laboratory a \( \ell = 1 \) OAM [3] beam with homogeneous circular polarization. As shown in figure 1, we used a reflective phase-only spatial light modulator to produce the optical vortex, and a quarter-wave plate to transform linear to circular polarization.

The beam is internally reflected at the hypotenuse face of a BK7 (\( n = 1.5106 \)) prism, where correction of the angle of
of a C-line.

analyzed using a CCD camera after passing another quarter-wave horizontal polarizer P1 and quarter-wave plate WP1. After internal topological phase of charge one is imprinted using a spatial light modulator (SLM); this vortex beam is then circularly polarized using horizontal polarizer P1 and quarter-wave plate WP1. After internal reflection at the BK7 prism (refractive index 1.5106), the beam is analyzed using a CCD camera after passing another quarter-wave plate WP2 and a linear polarizer P2. The inset shows a typical image of a C-line. Figure 1 Experimental setup. Light from a single mode fiber-coupled superluminescent diode (SLED; λ = 1.8 mm) using a 10x, NA 0.1 microscope objective (M). A topological phase of charge one is imprinted using a spatial light modulator (SLM); this vortex beam is then circularly polarized using horizontal polarizer P1 and quarter-wave plate WP1. After internal reflection at the BK7 prism (refractive index 1.5106), the beam is analyzed using a CCD camera after passing another quarter-wave plate WP2 and a linear polarizer P2. The inset shows a typical image of a C-line.

incidence for refraction at the prism legs is done. After reflection, we use a combination of quarter-wave plate, linear polarizer, and CCD camera (pixel size 4.4 μm) to analyze the reflected field for left and right circular polarization components. We find C-lines in both beams, theoretically (figure 2), and experimentally (figure 3). To determine their position with high accuracy, we use computer analysis: we fit a fourth-order polynomial to the field in the vicinity of the vortex and determine its center position. Since we have no reference, we determine the differential position of the two C-lines by switching the analyzing wave plate WP2 between ±45°. The accuracy of this method is around 10 μm, which we have estimated from repeated measurements. The C-lines are shifted within and perpendicular to the plane of incidence, longitudinally and laterally, respectively. The longitudinal C-line shift is much smaller than the lateral one. More importantly, we find that it is equal for both C-lines; therefore the longitudinal shift difference vanishes even though the individual shifts theoretically diverge at the critical angle. We only discuss lateral C-line shifts in the following.

Figure 3 shows the measured lateral C-line shift difference as a function of the angle of incidence, compared with our theoretical prediction. The apparent systematic deviation is probably due to interface imperfections, as has been observed many times before (e.g., [8, 13]). We see that, as the incident angle passes through the critical angle, the shift difference diverges and changes sign—as theory predicts. A singularity is to be expected because the Fresnel coefficients vary infinitely rapidly in amplitude as the critical angle is approached from below and in phase as it is approached from above. In the formal theory of section 4 it comes from the differentiation of E(k) with respect to k. For a realistic optical beam, which is finite in space and therefore has a nonzero wavevector spread in Fourier space, the divergence at the critical angle disappears because of plane-wave components both below and above critical angle [14]; but for a true infinite DPW infinite in space, the divergence at the critical angle is a real divergence.

4. The C-line shift formulas

The two separate theoretical schemes, Dennis–Götte [9] and Hannay–Nye [10, 11], for describing the core of the beam that carries the C-line are both based on series (Taylor) expansion of the local electric field. Both have the property that although the polarization varies locally, it becomes uniform away from the locality, representing the main polarization of the beam.

The first scheme, which might be called the ‘analytic polynomial wave’ (APW) uses the fact that for Laguerre–Gauss beams the local field can be described by a polynomial function of the single complex variable x + iy where x, y are the transverse coordinates. Higher-order beams are described by higher-order polynomials. The local analyticity feature is preserved in reflection from an interface (but not in transmission). The second scheme, the DPW does not assume analyticity (but does not exclude it). The DPW only deals with the lowest non-trivial level of series expansion: the first order, appropriate to the simplest beam containing a C-line—but that is all that is required here.

The original plane wave obeys Maxwell’s equations and the DPW automatically does so too. For the electric field this means not only satisfying the wave equation for each component separately but also the condition div E = 0. This feature makes the procedure very convenient in dealing with problems of transmission and reflection, for one is then assured of satisfying Maxwell’s equations. In the present problem of reflection if the incident wave is a DPW, so is the reflected wave. This follows because a DPW is a destructive superposition of monochromatic plane waves, all having nearly the same directions and polarizations [11]. Therefore the reflected plane waves will still have nearly the same directions and polarizations and hence make a DPW.

The incident beam in the experiment is the lowest order Laguerre–Gauss beam (beyond a simple Gaussian), having a central zero line of electric field and uniform circular polarization everywhere. Thus its core is very special in that both left and right polarizations are zero along its axis; it contains two coincident C-lines. This core of the incident beam can be described by either scheme. The reflected beam as a whole is no longer a Laguerre–Gauss beam, but its core can again be described by either the DPW or the APW scheme, having a single straight right-handed C-line and a single straight lefthanded one. On reflection the two C-lines have shifted by different amounts and so they are separated. They are both shifted away from the hypothetical straight line that would correspond to specular reflection of the incident double vortex line.

The APW scheme [9] uses expansion of the field close to the beam axis in complex polynomials to calculate the effect of reflection on the position of a vortex in a simple beam, or of the vortices in an incident higher-order vortex beam. It supplies explicitly the formula for the shifts of the two C-lines in reflection of an APW beam. The formula for the C-line
is a vector perpendicular to \( \Delta \) and \( k \). Let this reflected Fourier spectrum of plane waves be denoted \( \mathbf{E}(k) \exp(i \mathbf{k} \cdot \mathbf{r}) \). Its complete form will not be required in our task of finding the local form around the central zero line. This local form is supplied by the DPW for the reflected beam: \( \Delta \mathbf{k} \cdot \frac{\partial}{\partial \mathbf{r}} [\mathbf{E}(k) \exp(i \mathbf{k} \cdot \mathbf{r})] \) evaluated at the central wavevector \( \mathbf{k} \) of the reflected beam. The constant complex vector \( \Delta \mathbf{k} \) is a vector perpendicular to \( \mathbf{k} \) determined by the incident DPW; both \( \mathbf{k} \) and \( \Delta \mathbf{k} \) are mirror reflections in the interface plane of \( \mathbf{k}_{\text{inc}} \) and \( \Delta \mathbf{k}_{\text{inc}} \).

The circular (projected) polarization desired (which may be the same or opposite to that of the incident wave) is selected by taking the dot product of the electric field \( \mathbf{E}(k) \) with a filter polarizer vector. In the case we are considering it is \((1, \pm i, 0)\), with the \( z \) axis now taken in the direction of \( \mathbf{k} \) for the reflected beam, but for generality and notational consistency with [9], it will be denoted by \( \mathbf{F}^* \). Taking the dot product of the reflected beam \( \Delta \mathbf{k} \cdot \frac{\partial}{\partial \mathbf{r}} [\mathbf{E}(k) \exp(i \mathbf{k} \cdot \mathbf{r})] \) with \( \mathbf{F}^* \) gives the scalar field \( \mathbf{F}^* \Delta \mathbf{k} \cdot \mathbf{E}(k) \exp(i \mathbf{k} \cdot \mathbf{r})/\partial \mathbf{r} \). Let this field \( \mathbf{R} = \mathbf{R}_{\text{Re}} + i \mathbf{R}_{\text{Im}} \equiv \partial \log(\mathbf{E} \cdot \mathbf{F}^*)/\partial \mathbf{k} \), \( \Delta \mathbf{k} \equiv \Delta \mathbf{k}_{\text{Re}} + i \Delta \mathbf{k}_{\text{Im}} \). Then \( \mathbf{R} \cdot (\Delta \mathbf{k}_{\text{Re}} + \Delta \mathbf{k}_{\text{Im}}) = 0 \) whose real and imaginary equations are simultaneous linear equations for \( \mathbf{r} \cdot \Delta \mathbf{k}_{\text{Re}} \) and \( \mathbf{r} \cdot \Delta \mathbf{k}_{\text{Im}} \). Now if we add any complex multiple of \( \mathbf{k} \) to \( \mathbf{r} \), it makes no difference to the two equations, and it follows, therefore, that the solution for \( \mathbf{r} \) describes a straight line parallel to the vector \( \mathbf{k} \). After some reduction we obtain an equation involving \( \mathbf{r} \) on the left-hand side in terms of an outer product matrix on the right,

\[
(\mathbf{r} + \mathbf{R}_{\text{Im}}) \times \mathbf{k} = \mathbf{k}^2 \left[ \Delta \mathbf{k}_{\text{Re}} \otimes \Delta \mathbf{k}_{\text{Re}} + \Delta \mathbf{k}_{\text{Im}} \otimes \Delta \mathbf{k}_{\text{Im}} \right] \mathbf{R}_{\text{Re}} \\
\left( \Delta \mathbf{k}_{\text{Re}} \times \Delta \mathbf{k}_{\text{Im}} \right) \cdot \mathbf{k}.
\]

The left-hand side does not give \( \mathbf{r} \) directly (the solution is a line not a point). To obtain a formula for \( \mathbf{r} \) explicitly, pre-multiply, with a cross product, both sides of equation (3) by

![Figure 2](image-url) Calculated intensity maps (angle of incidence 41.5°, \( \lambda = 825 \text{ nm}, n = 1.5106 \)) of a reflected C-line field, for the two circular polarized components (a), (b) and the total intensity (c). Of course, the component of the same handedness as the incident field (b) is orders of magnitude stronger than the opposite component (a), which is evident from the total intensity plot (c). The crosses indicate the exact C-line positions, the arrow their lateral shift difference. The longitudinal shifts of the C-lines are much smaller and equal, and although they both diverge at the critical angle the difference between them remains zero.

![Figure 3](image-url) Predicted (full line) and measured (circles) lateral C-line shift difference around the critical angle for total internal reflection. We see experimentally the onset of the divergence of the shift, and that it changes sign at critical incidence (vertical line). The observed shift difference is very large in comparison with conventional beam shifts of the beam centroid.

Shifts in a DPW was described in brief in [10, 11] but not carried through, so an explicit general derivation is given now. Where an APW is appropriate the result for a DWP is the same as that for an APW.

The incident wave (2) can be considered as arising from the differentiation of a bundle of plane waves with ‘constant’ polarization (circular in the case of (2)). Thus one starts with a form \( \Delta \mathbf{k}_{\text{inc}} \cdot \frac{\partial}{\partial \mathbf{r}} [\mathbf{E}_{\text{inc}}(\mathbf{k}_{\text{inc}}) \exp(i \mathbf{k}_{\text{inc}} \cdot \mathbf{r})] \). Differentiating the product, one term is eliminated by choosing \( \Delta \mathbf{k}_{\text{inc}} \cdot \frac{\partial}{\partial \mathbf{r}} \mathbf{E}_{\text{inc}}(\mathbf{k}_{\text{inc}}) = 0 \), representing the ‘constant’ polarization condition of the bundle. This leaves \( \Delta \mathbf{k}_{\text{inc}} \cdot i \mathbf{E}_{\text{inc}}(\mathbf{k}_{\text{inc}}) \). With a temporary \( z \) axis defined to be along \( \mathbf{k}_{\text{inc}} \) one takes \( \mathbf{E}_{\text{inc}} \equiv (-i, 1, 0) \) and \( \Delta \mathbf{k}_{\text{inc}} = (1, i, 0) \). This gives the wave specified by (2).

The reflected wave is generated from the incident Fourier spectrum of plane waves, each modified by multiplication by its own Fresnel reflection coefficient depending on its direction of propagation or wavevector \( \mathbf{k} \).
\( \left( \frac{x}{k^2} \right) \) to give
\[
r = \frac{k \times [\Delta k_{\text{Re}} \otimes \Delta k_{\text{Re}} + \Delta k_{\text{Im}} \otimes \Delta k_{\text{Im}}] \mathbf{R}_{\text{Re}}}{(\Delta k_{\text{Re}} \times \Delta k_{\text{Im}}) \cdot \mathbf{k}} - R_{\text{Im}} + \mu \mathbf{k},
\]
which is in a rather different notation. To relate the notions note that the cross product in our equation (4) is represented by the right-angle rotation operator \( \mathbf{r}_{\sigma_z} \). On this common ground of reflection of a simple beam the two formulas agree.

5. Discussion

We have demonstrated experimentally how C-lines, points of pure apparent circular polarization, are displaced upon reflection at planar interfaces. We have also shown that the observed displacement of a vortex follows the prediction of both the theoretical schemes referred to in the Introduction.

It is striking to see that our experiment with realistic, finite optical beams agrees very well with theory, despite the fact that the theory is based on infinitely extended fields, i.e., DPWs. This confirms that only the field in the vicinity of the optical vortex determines the C-line shift, and that the Gaussian envelope is less relevant. We note that, in our experiment, we did not have to use a very sensitive position detection method such as a split detector in combination with a lock-in amplifier as is common in beam-shift experiments [15], because the vortex core position can be determined with very high accuracy. Further, we were able to observe experimentally lateral shifts differences ten times larger than the Goos–Hänchen shift, which has been observed experimentally to be less than 20 \( \mu \text{m} \) (see, e.g., [15]). Our results confirm Wolter’s suggestion made more than 60 years ago, that the nodal line of a pair of plane waves [16] should be easier to detect than the centroid of an optical beam.

Our work connects the field of polarization singularities [17], which is based on exact solutions of the 3D Maxwell equations, to optical beam shifts [18]; in particular to shifts of beams with OAM [7, 8, 19, 20], or the appearance of ‘OAM sidebands’ [21].

Acknowledgements

We thank J B Götte and M R Dennis for fruitful discussions on the properties of C-lines.

References

[1] Nye J F and Berry M V 1974 Dislocations in wave trains. Proc. R. Soc. A 336 165
[2] Nye J F 1999 Natural Focusing And Fine Structure Of Light: Caustics And Wave Dislocations (Bristol: IOP Publishing)
[3] Allen L, Beijersbergen M W, Spreeuw R J C and Woerdman J P 1992 Orbital angular momentum of light and the transformation of Laguerre–Gaussian laser modes Phys. Rev. A 45 8185
[4] Goos F and Hänchen H 1947 Ein neuer und fundamentaler Versuch zur totalreflexion Ann. Phys. 436 333–46
[5] Imbert C 1972 Calculation and experimental proof of the transverse shift induced by total internal reflection of a circularly polarized light beam Phys. Rev. D 5 787
[6] Wolter H 2009 Concerning the path of light upon total reflection J. Opt. A 11 090401
[7] Fedoseyev V G 2001 Spin-independent transverse shift of the centre of gravity of a reflected and of a refracted light beam Opt. Commun. 193 9
[8] Merano M, Hermosa N, Woerdman J P and Aiello A 2010 How orbital angular momentum affects beam shifts in optical reflection Phys. Rev. A 82 023817
[9] Dennis M R and Götte J B 2012 Topological aberration of optical vortex beams: determining dielectric interfaces by optical singularity shifts Phys. Rev. Lett. 109 183903
[10] Hannay J H and Nye J F 2013 Refraction of C-line vortices J. Opt. 15 014008
[11] Hannay J H and Nye J F 2013 A differentiated plane wave: its passage through a slab J. Opt. 15 044025
[12] Okuda H and Sasada H 2006 Huge transverse deformation in nonspecular reflection of a light beam possessing orbital angular momentum near critical incidence Opt. Express 14 8393
[13] Löfler W, Hermosa N, Aiello A and Woerdman J P 2013 Total internal reflection of orbital angular momentum beams J. Opt. 15 014012
[14] Lai H, Cheng F and Tang W 1986 Goos–Hänchen effect around and off the critical angle J. Opt. Soc. Am. A 3 550
[15] Gilles H, Girard S and Hamel J 2002 Simple technique for measuring the Goos–Hänchen effect with polarization modulation and a position-sensitive detector Opt. Lett. 27 1421
[16] Dennis M R and Götte J B 2013 Beam shifts for pairs of plane waves J. Opt. 15 014015
[17] Nye J F 1983 Lines of circular polarization in electromagnetic wave fields Proc. R. Soc. A 389 279
[18] Bliokh K Y and Aiello A 2013 Goos–Hänchen and Imbert–Fedorov beam shifts: an overview J. Opt. 15 014001
[19] Götte J B and Dennis M R 2012 Generalized shifts and weak values for polarization components of reflected light beams New J. Phys. 14 073016
[20] Aiello A 2012 Goos–Hänchen and Imbert–Fedorov shifts: a novel perspective New J. Phys. 14 013058
[21] Löfler W, Aiello A and Woerdman J P 2012 Observation of orbital angular momentum sidebands due to optical reflection Phys. Rev. Lett. 109 113602