Estimation of the Tail Index of Pareto-Type Distributions Using Regularisation

E. Ocran, R. Minkah, G. Kallah-Dagadu, and K. Doku-Amponsah

Department of Statistics and Actuarial Science, School of Physical and Mathematical Sciences, University of Ghana, Accra, Ghana

Correspondence should be addressed to K. Doku-Amponsah; kdoku@ug.edu.gh

Received 2 May 2022; Revised 1 September 2022; Accepted 11 September 2022; Published 27 October 2022

Academic Editor: Ljubisa Kocinac

Copyright © 2022 E. Ocran et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper, we introduce reduced-bias estimators for the estimation of the tail index of Pareto-type distributions. This is achieved through the use of a regularised weighted least squares with an exponential regression model for log-spacings of top-order statistics. The asymptotic properties of the proposed estimators are investigated analytically and found to be asymptotically unbiased, asymptotically consistent, and asymptotically normally distributed. Also, the finite sample behaviour of the estimators are studied through a simulation study. The proposed estimators were found to yield low bias and mean square errors. In addition, the proposed estimators are illustrated through the estimation of the tail index of the underlying distribution of claims from the insurance industry.

1. Introduction

Pareto-type distributions are often encountered in applications in the area of finance [1–3], reinsurance [4–6], risk management [7–9], and telecommunication [10, 11]. This distribution type has tail function

\[ 1 - F(x) = x^{-1/y} \ell_F(x) \text{ as } x \to \infty, \]  

or equivalently upper tail quantile function

\[ U(x) = x^y \ell_u(x) \text{ as } x \to \infty. \]  

The component \( \ell_F \) and \( \ell_u \) are slowly varying functions expressed as

\[ \lim_{t \to \infty} \frac{\ell_F(xt)}{\ell_F(t)} = 1, \text{ for all } x > 0. \]  

The parameter \( y \) is strictly positive for Pareto-type distributions and is also known as the tail index.

Suppose \( X_1, X_2, \ldots, X_n \) denote independent and identically distributed (i.i.d) random variables drawn from a distribution belonging to the maximum domain of attraction of the Pareto family of distributions, then for some auxiliary sequences of constants \( \{a_n > 0; n \geq 1\} \) and \( \{b_n; n \geq 1\} \) [12],

\[ \lim_{n \to \infty} \mathbb{P}\left( \max\{X_1, X_2, \ldots, X_n\} - b_n \leq \frac{x}{a_n} \right) = \exp\left\{ -(1 + yx)^{-1/y} \right\}, 1 + yx \geq 0, \]  

where \( y \geq 0 \). The estimation of \( y \) continues to receive considerable attention in statistics of extremes as all inferences in extreme value analysis depend on the tail index. In practice, we seek estimators with less variance and bias as possible. A parametric or semiparametric approach can be employed to estimate the tail index, [13–16]. However, in
this paper, we employ the semiparametric approach to develop reduced-bias estimators since they result in bias reduction.

Under the semiparametric framework, the tail index estimators are dependent on the $k$ largest observations, with these assumption about $k$:

**Assumption 1.** $k(n) \to \infty$ as $n \to \infty$.

**Assumption 2.** $k = k(n) = O(n)$ as $n \to \infty$.

The most widely used semiparametric tail index estimator is the Hill estimator [17]. The author in [17] approximates the top $k$ order statistics with a Pareto distribution and estimates $y > 0$ using a maximum likelihood estimator (MLE). The Hill estimator has the minimum asymptotic variance among the semiparametric estimators but it is very sensitive to the choice of $k$ [18]. This drawback of the estimator makes its usage challenging in practice, especially in the selection of the tail fraction, $k$. In this paper, we employ the semiparametric approach to reduction.

**2. Estimation Methods**

We let $X_1, X_2, X_3, \ldots, X_n$ denote a sequence of i.i.d random variables drawn from a population with distribution function $F$ and the associated tail quantile function $U$. Let $X_{1,n} \leq X_{2,n} \leq \ldots \leq X_{n,n}$ be the order statistics associated with the sample. Using equation (2), the order statistics can be jointly expressed as

$$\log X_{n-j+1,n}^d - \gamma \log U_{j,n}^{-1} + \log \ell(U_{j,n}^{-1}),$$

where $U_{j,n}^{-1}$, $j = 1, 2, 3, \ldots, n$ represent the order statistics of the standard uniform distribution. Using equation (6), the authors in [29] demonstrated that

$$\log \frac{X_{n-j+1,n}}{X_{n-k,n}} - \gamma \log \frac{U_{k+1,n}}{U_{j,n}} + \log \ell(U_{j,n}^{-1}),$$

where $k \in \{2, 3, \ldots, n-1\}$ and also obtained a more refined expression of equation (7) by imposing a second-order assumption on the rate of convergence to equation (3). This is stated as an assumption as follows:

**Assumption 3.** There exists a real constant $\rho < 0$ and a rate function $b$ that is regularly varying, with index $\rho$, i.e., $b(x) = x^\rho \ell_p(x)$ for some $\ell_p$ satisfying equation (3) for all $v \geq 1$:

$$\lim_{x \to \infty} \left[ \log \frac{\ell_p(xv)}{\ell_p(x)} \right] = b(x)h_p(v),$$

with $h_p(v) = \int_1^v y^{\rho-1} dy$ [30].

Under Assumption 3, the authors in [30] showed that the weighted log-spacings of order statistics

$$T_j = j[\log X_{n-j+1,n} - \log X_{n-j,n}], 1 \leq j \leq k < n,$$

are approximately exponentially distributed. They particularly obtained the expression

$$T_j = \left( \gamma + b_{nk}(\frac{j}{k+1})^{-\rho} \right) f_j, 1 \leq j \leq k,$$

where $b_{nk} = b((n + 1)/(k + 1)) \to 0$ as $k, n \to \infty$, and $f_j$ are i.i.d exponentially distributed with a unit mean and $\rho (\rho < 0)$ is a second-order parameter. The authors employed MLE to the estimate the parameters in equation (10).

Using equation (10) and Assumptions 1 and 2, the authors in [31] demonstrated that $T_j$ can be further approximated as a regression model

$$T_j = T_j(\rho, \varepsilon_j) = \gamma + b_{nk}C_j(\rho) + \varepsilon_j, j \in \{1, 2, \ldots, k\},$$

where $b_{nk} = b(n/k)$ is the slope, $C_j = C_j(\rho) = (j/(k+1))^{-\rho}$ is the covariate, $\gamma$ is the intercept, and $\varepsilon_j$ is the error terms with asymptotic mean, 0, and variance, $\gamma^2$.

The authors in [31] proposed the ordinary least squares estimator for the estimation of $\gamma$ in equation (11). Furthermore, based on equation (11), the authors in [13] have introduced the ridge regression estimator for estimating $\gamma$. 
In this paper, we propose the regularised weighted least squares estimators for estimating $y$ in equation (11).

2.1. The Proposed Estimators. In order to estimate $y$, the loss function of the regularised weighted least squares for equation (11) is defined as

$$L_k(y, b_{nk}, \lambda, W) = \frac{1}{k} \sum_{j=1}^{k} W_j \left( T_j - y - b_{nk} C_j \right)^2 + \lambda k b_{nk}^2, \lambda \geq 0.$$  

(12)

Here, $W_j$ is the weight function defined as

$$W_j = W_j(\theta) = \left( 1 - \theta^{j(k)}(k+1) \right), \quad j \in \{1, 2, \ldots, k\},$$  

(13)

where $\theta_j \sim U(0, 1)$. Thus, $W_j \in (0, 1)$ and decreases linearly with respect to $j$. $W_j$ is employed due to its ability to reduce the variability in the estimator. The exponent $\alpha(k) \geq 0$ is chosen such that

$$\Delta = \lim_{k \to \infty} \frac{\alpha(k)}{k} < \infty.$$  

(14)

In this study, we consider $0 < \Delta \leq 1$. Thus, we would define $\alpha(k)$ such that $0 < \alpha(k) \leq k$. Note that, $W_j$ is random through $\theta_j$, and when the exponent is 0, $W_j$ is deterministic. In particular, when $\alpha(k) = 0$, we obtain the weight function $g_j = 1 - j/(k + 1)$, $j \in \{1, 2, \ldots, k\}$, as introduced in [32]. Nevertheless, we can approximate the weight $g_j$ as a limit of the current result by allowing $\Delta$ to approach 0.

We minimize the loss function $L_k$ in equation (12) with respect to $y$ and $b_{nk}$ to obtain jointly estimates $\hat{y}$ and $\hat{b}_{nk}$

$$\hat{b}_{nk}(\lambda, \theta) = \frac{\sum_{j=1}^{k} \hat{W}_j \left( C_j - \sum_{j=1}^{k} \hat{W}_j C_j \right) T_j}{2 \kappa(\alpha(k)) \lambda + \sum_{j=1}^{k} \hat{W}_j C_j^2 - \left( \sum_{j=1}^{k} \hat{W}_j C_j \right)^2},$$  

(15)

and

$$\hat{y}_{RW}(\lambda, \theta) = \sum_{j=1}^{k} \hat{W}_j T_j - \hat{b}_{nk}(\lambda, \theta) \sum_{j=1}^{k} \hat{W}_j C_j.$$  

(16)

while,

$$\hat{W}_j = \frac{W_j}{\sum_{j=1}^{k} W_j}, \quad 1 \leq j \leq k,$$

(17)

$$C_j = \left( \frac{j}{k+1} \right)^{-\rho}, \quad \text{for} \quad j \in \{1, 2, \ldots, k\}, \quad \rho < 0,$$

(18)

and

$$\kappa(\alpha(k)) = \frac{\alpha(k) + 1}{2\alpha(k) + 1}.$$  

(19)

We substitute equation (13) into equation (17) to obtain an explicit expression for equation (17) as

$$\hat{W}_j = \frac{2 \kappa(\alpha(k))}{\lambda \left( 1 - \theta^{j(k)}(k+1) \right)}.$$  

(20)

The main theorem backing the proof of equation (20) is the Kolmogorov’s strong law of large numbers for independent random variables, but for brevity, we present only the results.

The parameter $\rho < 0$ is estimated externally using the minimum variance approach introduced in [13].

In addition, the parameter $\lambda$ in equation (12) is the penalty that regulates the bias coefficient $b_{nk}$. The loss function, $L_k$, minimises the weighted sum of squared residuals and also regulates the size of the bias coefficient $b_{nk}$. The penalty term shrinks the bias term, $b_{nk}$ to 0 as the penalty parameter, $\lambda$, increases. Thus, the larger the value of $\lambda$, the higher the contribution of the penalty term to the loss function and the stronger the regularisation process. To obtain an estimator for the penalty term, $\lambda$, we minimize the asymptotic mean squared error (AMSE) of the proposed estimator, $\hat{y}_{RW}(\lambda)$ (see, example [13]).

Note that, since the weight function depends on the $\theta_j$’s, the estimators in equations (15) and (16) also depend on the $\theta_j$’s through the weight function. Therefore, we would find the AMSE by conditioning on the $\theta_j$’s. From equation (15) and (16), the AMSE for $\hat{y}_{RW}(\lambda)$ is obtained as

$$\text{AMSE} \left( \hat{y}_{RW}(\lambda) | \theta_j = \theta \right) = \gamma^2 \left\{ \frac{4}{3k} \phi(\alpha(k)) + \frac{2S_1(\theta) S_1(\theta)}{2\kappa(\alpha(k)) \lambda + S_2(\theta)} + \frac{S_1^2(\theta) S_1(\theta)}{2\kappa(\alpha(k)) \lambda + S_2(\theta)} \right\} + \left( \frac{2b_{nk}(\lambda) \kappa(\alpha(k)) S_1(\theta) \lambda}{2\kappa(\alpha(k)) \lambda + S_2(\theta)} \right)^2,$$  

(21)
where
\[
S_1(\theta) = \sum_{j=1}^{k} \bar{W}_j C_j, \quad (22)
\]
\[
S_2(\theta) = \sum_{j=1}^{k} \bar{W}_j C_j^2 - \left( \sum_{j=1}^{k} \bar{W}_j C_j \right)^2, \quad (23)
\]
\[
\hat{S}(\theta) = \sum_{j=1}^{k} \bar{W}_j^2 (S_1(\theta) - C_j), (24)
\]

We obtain,
\[
\lambda_k(\theta) = \frac{\gamma^2 S_2(\theta) \hat{S}(\theta) + \gamma^2 S_1(\theta) \hat{S}(\theta)}{2b_n \kappa (a(k)) S_1(\theta) S_2(\theta) - \gamma^2 S_1(\theta) \hat{S}(\theta)} \quad (28)
\]

In order to estimate \( \lambda_k(\theta) \), we assume the slowly varying function \( \ell_\beta \) in (8) is constant \([13] \). Thus, we have
\[
b(x) = \gamma \beta x^2 (1 + o(1)) \text{ as } k \rightarrow \infty,
\]
for some \( \beta \in \mathbb{R} \) and we estimate \( \beta \) via the estimator proposed in [33]. It follows from equation (28) and (29) that
\[
\lambda_k(\theta) = \frac{S_1(\theta) \hat{S}(\theta) + \hat{S}(\theta) S_2(\theta)}{2b_n \kappa (a(k)) S_1(\theta) S_2(\theta) \hat{S}(\theta)^2} \quad (30)
\]

The penalty term, \( \Delta \), is required to be non-negative; therefore, we define \( \lambda_k(\theta) = \max\{\lambda_k(\theta), 0\} \). We then obtain a penalty term and estimators which do not depend on \( \theta_j \)'s, by averaging \( \lambda_k(\theta) \) over the \( \theta_j \)'s, as follows:
\[
\lambda_k(\theta) = E_\theta \left\{ \lambda_k(\theta) \left\{ S_1(\theta) S_2(\theta) \right\} \left\{ \theta_1, \theta_2, \ldots, \theta_j \right\} \right\}, \quad (31)
\]
and we defined the proposed estimator of \( \gamma \) by
\[
\gamma_{RW}(\lambda) = E_\theta (\gamma_{RW}(\lambda, \theta)). \quad (32)
\]

2.2. Asymptotic Properties of the Proposed Estimators. Unbiasedness, consistency, and normality are desirable properties of a good estimator. In this section, we investigate these desirable properties of the proposed estimators.

We shall summarise the asymptotic behaviour of the statistics used to build the AMSE of the proposed estimator in Lemma 1. These properties will be required in the proof of the asymptotic consistency and sampling distribution of the proposed estimator. Henceforth, anytime we use the term \( a.s \) it is with respect to the law of the i.i.d sequence \( \theta_1, \theta_2, \ldots, \theta_k \).

**Lemma 1.** Assume that \( \rho<0 \) is estimated by a consistent estimator \( \hat{\rho} \) and (14) holds, then as \( k \rightarrow \infty \) and \( k/n \rightarrow 0 \):

(i) \( S_1(\theta) = \sum_{j=1}^{k} \bar{W}_j C_j \overset{a.s.}{\rightarrow} 1/(1-\rho) \).
(ii) \( S_2(\theta) = \sum_{j=1}^{k} \bar{W}_j C_j^2 - (\sum_{j=1}^{k} \bar{W}_j C_j)^2 \overset{a.s.}{\rightarrow} \rho^2/(1-2\rho)(1-\rho)^2 \).
(iii) \( \hat{S}(\theta) = \sum_{j=1}^{k} \bar{W}_j^2 (S_1(\theta) - C_j) \overset{a.s.}{\rightarrow} 0 \).
(iv) \( \hat{S}(\theta) = \sum_{j=1}^{k} \bar{W}_j^2 (S_1(\theta) - C_j) \overset{a.s.}{\rightarrow} 0 \).

**Lemma 2.** Suppose \( \rho<0 \) and \( \beta \in \mathbb{R} \) are estimated by their respective consistent estimators \( \hat{\rho} \) and \( \hat{\beta}_k \), then as \( k, n \rightarrow \infty \) and \( k/n \rightarrow 0 \),
\[
\lambda_k(\theta) \overset{a.s.}{\rightarrow} 0. \quad (33)
\]

It follows from Lemma 2 that the regularised weighted least estimator, \( \gamma_{RW}(\lambda) \) is asymptotically unbiased. That is, as \( k \rightarrow \infty \), bias(\( \gamma_{RW}(\lambda) \)) \( \rightarrow 0 \). The bias of the proposed estimator is given by
\[
\text{bias}(\gamma_{RW}(\lambda)) = E_\theta (\gamma_{RW}(\lambda, \theta)) - \gamma = \frac{2b_n \kappa (a(k)) S_1(\theta) \lambda(\theta)}{2 \kappa (a(k)) \lambda(\theta) + S_2(\theta)}, \quad (34)
\]
where
\[
\text{bias}(\hat{\gamma}_{RW}(\lambda)) = \frac{2b_n \kappa (a(k)) S_1(\theta) \lambda(\theta)}{2 \kappa (a(k)) \lambda(\theta) + S_2(\theta)}. \quad (35)
\]

Since the term in the bracket converges to 0 as \( k \rightarrow \infty \) almost surely by Lemma 2, the expectation of the term will converge to 0 as \( k \rightarrow \infty \). Therefore,
\[
\lim_{k \to \infty} \text{bias}(\hat{\theta}(\rho)) = \lim_{k \to \infty} \int_{[0,1]^k} \left\{ 2b_{n,k} \kappa(\alpha(k))S_1(\theta) \lambda(\theta) + S_2(\theta) \right\} d\theta = \int_{[0,1]^k} \left\{ \lim_{k \to \infty} 2b_{n,k} \kappa(\alpha(k))S_1(\theta) \lambda(\theta) + S_2(\theta) \right\} d\theta = 0, \quad (36)
\]

which gives \( \lim_{k \to \infty} \text{bias}(\hat{\theta}(\rho)) = 0 \). Similarly, we can use Lemma 1 and Lemma 2 to show that AMSE(\( \hat{\theta}(\rho) \)) \( \to 0 \) as \( k \to \infty \). We write

\[
\text{AMSE}(\hat{\theta}(\rho), \theta) = \gamma^2 \left[ \frac{4}{3k} \phi(\alpha(k)) + \frac{2S_1(\theta) \lambda(\theta)}{2\kappa(\alpha(k))\lambda + S_2(\theta)} + \frac{S_2(\theta) \lambda(\theta)}{(2\kappa(\alpha(k))\lambda + S_2(\theta))^2} \right] + \left( \text{bias}(\hat{\theta}(\rho), \theta) \right)^2. \quad (37)
\]

Now, observe from the Jensen’s inequality (applied to the expectation taken with respect to the law of \( \theta \)) and the Fubini’s theorem that

\[
\left[ \mathbb{E}_\theta(\text{bias}(\hat{\theta}(\rho), \theta)) \right]^2 \leq \text{AMSE}(\hat{\theta}(\rho), \theta) \leq \mathbb{E}_\theta(\text{bias}(\hat{\theta}(\rho), \theta)) + \mathbb{E}_\theta(\text{bias}(\hat{\theta}(\rho), \theta))^2 = \mathbb{E}_\theta(\text{AMSE}(\hat{\theta}(\rho), \theta, \theta) ), \quad (38)
\]

and therefore, we have

\[
0 \leq \lim_{k \to \infty} \text{AMSE}(\hat{\theta}(\rho), \theta) \leq \lim_{k \to \infty} \int_{[0,1]^k} \left\{ \text{AMSE}(\hat{\theta}(\rho), \theta) \right\} d\theta = \int_{[0,1]^k} \left\{ \lim_{k \to \infty} \text{AMSE}(\hat{\theta}(\rho), \theta) \right\} d\theta = 0.
\]

This implies that the proposed estimator is asymptotically consistent under some conditions.

**Theorem 3.** Suppose equations (2), (8), and (14) are satisfied. Assume also that \( \rho \) is estimated by a consistent estimator \( \hat{\rho} \) with

\[
\mathbb{E} \left[ \lim_{k \to \infty} S_2(\hat{\rho}) \right] = \mathbb{E} \left[ \frac{(1 - 2\hat{\rho})(1 - \hat{\rho})^2}{\hat{\rho}^2} \right] < \infty. \quad (40)
\]

Then, if assumptions A.1 and A.2 holds, and \( \sqrt{k} b_{n,k} \to 0 \), then, we have

\[
\sqrt{k} \left( \mathbb{E}_\theta(\hat{\theta}(\rho), \theta) - \gamma \right) \to_d N\left( 0, \gamma^2 \right). \quad (41)
\]

Theorem 3 discusses the asymptotic normality of \( \hat{\theta}(\rho) \) defined in equation (16). To prove Theorem 3, we require the following properties in addition.

We write

\[
\mathfrak{M}_k(\theta) := \frac{\hat{W}}{2\kappa(\alpha(k))\lambda + \sum_{j=1}^k \hat{W}^2 C_j^2 - \left( \sum_{j=1}^k \hat{W} C_j \right)^2} \quad (42)
\]

**Lemma 4.** Let \( C_j = (j/k + 1)^{-p}, \ W_j = W_j / \sum_{j=1}^k W_j, j \in \{1, 2, \ldots, k\} \) and \( \rho < 0 \). Then, as \( k \to \infty \),

\[
\mathfrak{M}_k(\theta) \to O\left( \frac{1}{k^{1+\omega}} \right), \quad 0 < \omega \leq 0.1. \quad (43)
\]

Lemma 4 is required in the Proof Proof 5.

**Lemma 5.** Let \( T_1, T_2, T_3, \ldots \) be independent random variables from an exponential distribution with mean \( \mu < \infty \), for all \( i \). Then, for any \( \epsilon > 0 \),

\[
\lim_{k \to \infty} P\left( \sqrt{k} \mathfrak{M}_k \geq \epsilon \right) = 0, \quad (44)
\]

where \( \mathfrak{M}_k \) is defined by equation (15).

**Remark 1.** Lemma 5 shows the statistics \( \sqrt{k} \mathfrak{M}_k \to 0 \) as \( k, n \to \infty \) and \( k/n \to 0 \).

The next lemma is about the satisfaction of the Lyapunov’s version of the central limit theorem. The Lyapunov’s variant of the central limit theorem assumes the existence of a finite moment of an order higher than two.

**Lemma 7.** Suppose that \( Z_1, Z_2, \ldots \) are independent random variables such that \( \mathbb{E}(Z_k) = \bar{Z}_k \) and \( \text{Var}(Z_k) = \sigma_k^2 < \infty \), then, there exists \( \delta > 0 \) such that

\[
\lim_{k \to \infty} \frac{1}{s_{n,k}} \sum_{j=1}^k \mathbb{E}\left( \left| Y_j - \bar{Z}_j \right|^{2+\delta} \right) = 0, \quad (45)
\]

where \( Z_k = \mathbb{E}_\theta(\hat{W}_k(\theta_k) | T_k) \).
Remark 2. Setting the penalty term to 0 reduces the regularised weighted least squares estimator to a weighted least squares estimator. The difference between this weighted least squares estimator and the one introduced by [32] is that, this weighted least squares estimator has smaller asymptotic variance and this is due to the introduction of randomness into the weight function. The resulting weighted least squares estimator is also asymptotically unbiased, asymptotically consistent, and asymptotically normally distributed with mean 0 and variance $\gamma^2$.

3. Simulation Study

In the previous section, we proposed the regularised weighted least squares estimators under the semiparametric setting to estimate the tail index of the underlying distribution of a given data from the Pareto-type of distributions. In this section, we perform a simulation study to compare the performance of our proposed estimators to other existing semiparametric tail index estimators. Particularly, the regularised weighted least squares, RWLS, the reduced-bias weighted least squares with modified weight function, WLS, the ridge regression, RR [13], the least squares, LS [31], the Hill estimator, HILL [17], and the bias-corrected Hill, BCHILL [29] in the case of Pareto-type distributions are compared.

3.1. Simulation Design. We consider the Fréchet and Burr XII from the Pareto-type distributions as shown in Table 1. For each distribution $F$, we generate 1000 repetitions of samples of size $n = 50, 500,$ and 2000. For the Fréchet distributions, we consider $\alpha = 10, 2$, and 1.0; and for the Burr XII we consider the mixtures.

(i) $\xi = \sqrt{10}$ & $\tau = \sqrt{10}$,
(ii) $\xi = \sqrt{2}$ & $\tau = \sqrt{2}$ and
(iii) $\xi = 2$ & $\tau = 1/2$

To obtain the tail index values $\gamma = 0.1, 0.5$ and 1.0, respectively. We consider the finite sample behaviour of the proposed estimators, RWLS and WLS, and also compared these estimators with RR, LS, BCHILL and HILL. The mean square errors (MSE) and bias are plotted as a function of the number of top-order statistics, $k$, to investigate the estimators’ sample path behaviour.

In the case of the weight function, the $\theta_j$’s will be replaced with their point estimate, in this case, the mean of a standard uniform distribution. In the case of $\alpha (k)$, we select $\alpha (k)$ such that $\Delta \rightarrow 0$, as $k \rightarrow \infty$. This choice of $\alpha (k)$ is made because in practice we have observed that it yields much more stable estimates compared to when $\alpha (k)$ is selected such that $\Delta \rightarrow c > 0$ in application.

3.2. Discussion of Simulation Results. In this section, we discuss the behaviour of RWLS and WLS relative to RR, LS, HILL, and BCHILL. The MSE and bias are the performance measures in the simulation studies. The simulation results for the Burr distribution with different tail indexes are shown in Figures 1–3. Also, Figures 4–6 present the simulation results for the Fréchet distribution with varying tail indexes.

From these figures, the plots of WLS and RWLS follow the same sample path for $k \leq 0.4n$, i.e., their performance are relatively the same on that interval. WLS and LS are very close to each other, though generally, WLS slightly outperforms LS in terms of MSE and bias. Thus, generally the WLS can be considered the most appropriate estimator of the tail index among the regression-based estimators (i.e., RR, LS, WLS, and RWLS) since it mostly has smallest bias and MSE across all samples.

Additionally, the MSE plots of the proposed estimators are low and near constant over the central part of $k$, except in the case of Burr XII with $\gamma = 1.0$. With the exception of the HILL estimator (which globally has the highest MSE), the MSE curves of the estimators are mostly close to each other in the central $k$ region, especially in the case of the Fréchet distribution. This implies that the proposed estimators are competitive with the existing estimators. However, the proposed estimators, WLS and RWLS, generally attain the lowest bias for small samples, i.e., $n = 50$. Furthermore, for medium to large values of $k$, the sample paths of RWLS in the MSE and bias plots are between HILL and RR. Even though the BCHILL estimator mostly has the smallest MSE and bias, the proposed estimators (RWLS and WLS) outperform it for large values of $k$.

Hence, from the simulation results, WLS and RWLS are appropriate alternatives for the estimation of the tail index of the Pareto-type distributions in terms of MSE and bias.

In addition, no single tail index estimator under investigation was found to be universally the best in terms of the MSE and bias. Finally, the R codes for the simulation and the application studies are available at https://github.com/kikiocran/RegularisedTail.

4. Applications

In this section, we consider the estimation of the tail index of the underlying distribution of two datasets from the insurance industry. First, the SOA Group Medical Insurance dataset which consists of over 170,000 claims recorded from 1991 to 1992. In this study, we consider the 1991 dataset, which comprised 75,789 claims and have been studied widely in the extreme value context (see, for example, [4, 18]). Considering the large size of this dataset, we focus on the extreme tail of the data and hence consider the top 10% data points, (i.e, 0 < $k \leq 0.1n$). The SOA dataset is available at https://lstat.kuleuven.be/Wiley/Data/soa.txt.

Second, an automobile insurance data from Ghana which consists of 452 claims from July 7, 2020, to May 11, 2021, and can be found at https://github.com/kikiocran/RegularisedTail. We will refer to this dataset as the GH claims in this study. To the best of our knowledge, this dataset has never been used in the extreme value theory literature.

The scatter plots of the SOA, and the GH claims are shown in Figure 7. We observe that two claims and one claim in the SOA and GH claims, respectively, appear to be far
Table 1: Heavy-tailed distributions from the Pareto-type distribution.

| Distribution     | $1 - F(x)$                  | $\ell_F(x)$                | $\gamma$ |
|------------------|-----------------------------|----------------------------|----------|
| Burr type XII    | $(1 + x^\tau)^{-\xi}$      | $(1 + x^{-\tau})^{-\xi}$  | $1/\tau\xi$ |
| Fréchet          | $1 - F(x) = 1 - \exp(-x^{-\alpha})$ | $1 - x^{-\alpha}/2 + O(x^{-\alpha})$ | $1/\alpha$ |

Figure 1: Results for Burr type XII distribution with $\gamma = 0.1$: Bias (top row) and MSE (bottom row). First column: $n = 50$; second column: $n = 500$; and third column: $n = 2000$.

Figure 2: Results for Burr type XII distribution with $\gamma = 0.5$: Bias (top row) and MSE (bottom row). First column: $n = 50$; second column: $n = 500$; and third column: $n = 2000$. 
detached from the bulk of the data. These observations can also be seen to deviate from linearity and far removed from the bulk of the points, respectively, in the Pareto and exponential Q-Q plots (Figure 8) of the two datasets. Such large observations are suspected outliers and may significantly influence the tail index estimates (see, for example,
The convex curvature of the exponential Q-Q plots and the near linearity of the Pareto Q-Q plots of the datasets indicate the datasets suggest they belong to the Pareto-type distributions.

Figure 5 shows the sample paths of the tail index estimators for the underlying distributions of the two datasets. The plot of HILL diverges as $k$ increases, i.e., it is very sensitive to the changes in $k$. Hence, it is not an appropriate

Figure 6: Results for Fréchet distribution with $\gamma = 0.5$: Bias (top row) and MSE (bottom row). First column: $n = 50$; second column: $n = 500$; and third column: $n = 2000$.

Figure 9 shows the sample paths of the tail index estimators for the underlying distributions of the two datasets. The plot of HILL diverges as $k$ increases, i.e., it is very sensitive to the changes in $k$. Hence, it is not an appropriate
estimator for estimating the tail index. The other estimators exhibit some form of stability; however, the sample paths of the proposed estimators (i.e., RWLS and WLS) are smooth, that is, these estimators are less sensitive to changes in $k$. All the tail index estimators considered are very unstable for small values of $k$ due to the small number of exceedances. A
specific tail index estimate can be obtained from the plots of WLS and RWLS for both datasets.

5. Conclusion

In this paper, we proposed tail index estimators for the Pareto-type of distributions using the regression model. In addition to the ordinary least squares and the ridge regression estimators, we proposed the regularised weighted least squares and the weighted least squares estimators as alternative regression-based reduced-bias estimators. The tail index estimates by the proposed estimators are generally stable and smooth across a broader path of \( k \). The characteristics of the proposed estimators are as follows:

(i) They are asymptotically consistent, asymptotically unbiased, and asymptotically normally distributed with mean 0 and variance \( \gamma^2 \).

(ii) The MSE curves are low and flat over the central part of \( k \).

(iii) The plots of their tail index estimates are more stable, smooth, and near horizontal than the Hill, ordinary least squares, and the bias-corrected Hill estimators.

In conclusion, comparatively, the proposed estimators are competitive to the existing estimators and can be considered as appropriate estimators of the tail index in terms of MSE, bias, and in real-life application.

6. Proofs

Proof of Lemma 1.

(i) From equation (20) and (22), we have

\[
S_1(\theta) = \sum_{j=1}^{k} \hat{W}_j C_j
\]

\[
= 2 \left( \frac{\alpha(k) + 1}{2\alpha(k) + 1} \right) \int_{0}^{1} \left( 1 - E(\theta^{\alpha(k)}) u \right) u^{-\rho} du + o(1)
\]

\[
= \frac{2(2(\alpha(k)/k) - \rho(\alpha(k)/k + 1/k))}{(2(\alpha(k)/k) + (1/k))(1 - \rho)(2 - \rho)} + o(1).
\] (46)

\[
\lim_{k \to \infty} \left[ \frac{2(2(\alpha(k)/k) - \rho(\alpha(k)/k + 1/k))}{(2(\alpha(k)/k) + (1/k))(1 - \rho)(2 - \rho)} + o(1) \right] = \frac{2(2\Delta - \rho\Delta)}{(2\Delta)(1 - \rho)(2 - \rho)} = \frac{1}{(1 - \rho)}. \] (47)
where \( \Delta = \lim_{k \to \infty} \alpha(k)/k \) and hence, as \( k \to \infty \), we have \( S_1(\theta) \to 1/(1 - \rho) \).

(ii) From equation (23),

\[
S_2(\theta) = \sum_{j=1}^k \bar{W}_j C_j^2 - \left( \sum_{j=1}^k \bar{W}_j C_j \right)^2 = \sum_{j=1}^k \bar{W}_j C_j^2 - \left( \frac{1}{1 - \rho} \right)^2. 
\] (48)

\[
\sum_{j=1}^k \bar{W}_j C_j^2 = 2 \left( \frac{\alpha(k) + 1}{2\alpha(k) + 1} \right) \int_0^1 \left( 1 - \mathbb{E}(\bar{\theta}^{(k)}) u \right)^2 \text{d}u + o(1)
\]

\[
= \frac{(2(\alpha(k)/k) - 2\rho(\alpha(k)/k) + (1/k))}{(2(\alpha(k)/k) + (1/k))(1 - \rho)(1 - 2\rho)} + o(1).
\] (49)

It follows that

\[
\lim_{k \to \infty} \left[ \frac{(2(\alpha(k)/k) - 2\rho(\alpha(k)/k) + (1/k))}{(2(\alpha(k)/k) + (1/k))(1 - \rho)(1 - 2\rho)} + o(1) \right] = \frac{2\Delta - 2\rho \Delta}{2\Delta (1 - \rho)(1 - 2\rho)} = \frac{1}{(1 - 2\rho)}. \] (50)

where \( \Delta = \lim_{k \to \infty} \alpha(k)/k \). Therefore,

\[
S_2(\theta) \to \frac{1}{1 - 2\rho} \left( \frac{1}{1 - \rho} \right)^2 = \frac{\rho^2}{(1 - 2\rho)(1 - \rho)^2}. \] (51)

(iii) The expression \( \hat{S}(\theta) = \sum_{j=1}^k \bar{W}_j^2 (S_1(\theta) - C_j) \) can also be written as

\[
\hat{S}(\theta) = \frac{4}{k} \left( \frac{\alpha(k) + 1}{2\alpha(k) + 1} \right) \left( \frac{1}{k} \sum_{j=1}^k \left( 1 - \bar{\theta}^{(k)} j \right) \left( \frac{j}{k + 1} \right)^{-\rho} \left( S_1(\theta) - \frac{j}{k + 1} \right)^{-\rho} \right)
\]

\[
= \frac{4}{k} \left( \frac{\alpha(k)/k + (1/k)}{2(\alpha(k)/k) + (1/k)} \right) \int_0^1 \left( 1 - \mathbb{E}(\bar{\theta}^{(k)}) u \right)^2 \left( \frac{1}{1 - \rho} - u^{-\rho} \right) \text{d}u + o(1).
\] (52)

Therefore, as \( k \to \infty \), \( \hat{S}(\theta) \xrightarrow{as} 0 \).

(iv) \( \hat{S}(\theta) = \sum_{j=1}^k \bar{W}_j^2 (S_1(\theta) - C_j)^2 \) can also be expressed as

\[
\hat{S}(\theta) = \frac{4}{k} \left( \frac{\alpha(k) + 1}{2\alpha(k) + 1} \right) \left( \frac{1}{k} \sum_{j=1}^k \left( 1 - \bar{\theta}^{(k)} j \right) \left( \frac{j}{k + 1} \right)^{-\rho} \left( S_1(\theta) - \frac{j}{k + 1} \right)^{-\rho} \right)
\]

\[
= \frac{4}{k} \left( \frac{\alpha(k)/k + (1/k)}{2(\alpha(k)/k) + (1/k)} \right) \int_0^1 \left( 1 - \mathbb{E}(\bar{\theta}^{(k)}) u \right)^2 \left( \frac{1}{1 - \rho} - u^{-\rho} \right) \text{d}u + o(1).
\] (53)

It also follows that, as \( k \to \infty \), \( \hat{S}(\theta) \xrightarrow{as} 0 \). □

**Proof of Lemma 3.** We observe that

\[
\hat{S}(\theta) = \frac{k^\rho}{k^{1 + \alpha + \rho}} \leq \mathcal{M}_1(\theta, C_j) \leq \frac{k^\rho C_j^2}{k^{1 + \alpha + \rho} + n^\rho}.
\] (54)

**Proof of Lemma 2.** The proof easily follows by using Lemma 1.
Therefore, we have \( M_1(\theta, C_j) \rightarrow o(1/k^{1+w}) \), 0 < \( \omega \leq 0.1 \), as \( k \rightarrow \infty \), which completes the Proof of Proof 3.

Proof of Lemma 4. The proof requires the use of large deviation principles (LDP). From equations (15) and (33), \( \tilde{b}_{n,k}(\theta, \bar{\rho}) = \sum_{j=1}^{k} M_1(\theta, C_j)T_j \). Given \( \{\bar{\rho} = \rho\} \), \( T_j \) is exponentially distributed with mean

\[
\mu_j(\theta, \rho) = \mathbb{E}[T_j | \bar{\rho} = \rho, \theta] = y + \tilde{b}_{n,k}(\rho, \theta)C_j(\bar{\rho}).
\]

Therefore, we have

\[
M_{T_j, \bar{\rho}, \theta}(t) = \mathbb{E}[e^{tT_j} | \bar{\rho} = \rho, \theta] = \left( \frac{1}{1 - \mu_j(\theta, \bar{\rho})t} \right).
\]

Using equation (36) and similar calculations as in [32], the moment generating function of \( \tilde{b}_{n,k} \) given the law of \( \{\bar{\rho} = \rho\} \) is

\[
M_{\tilde{b}_{n,k}(\rho)}(t) = \left\{ \prod_{j=1}^{k} \left( \frac{1}{1 - \mu_j(\theta, \bar{\rho})M_1(\theta, t)} \right) \right\}.
\]

It follows that

\[
M_{\tilde{b}_{n,k}}(kt) = \left\{ \prod_{j=1}^{k} \left( \frac{1}{1 - k\mu_j(\theta, \bar{\rho})M_1(\theta, C_j)} \right) \right\}.
\]

Now using the bound on \( M_1(\theta, C_j) \), see Lemma 4 and the Squeeze Theorem, we obtain

\[
\frac{1}{k} \log M_{\tilde{b}_{n,k}}(kt) = 0.
\]

Hence, by the G "rtner Ellis Theorem, conditional on \( \{\theta, \rho\} \), the statistics \( \tilde{b}_{n,k}(\theta, \rho) \) follows a Large Deviation Principle (LDP) with speed \( k \) and a rate function \( I(x) \) defined as

\[
I(x) = \sup_{\eta \in \mathbb{R}} \{ \eta x - \delta(\eta) \} = \begin{cases} 0 & \text{if } x = 0, \\ \infty & \text{if } x \neq 0 \end{cases}
\]

where \( x \in [0, \infty) \). This implies for every \( \varepsilon > 0 \), we have

\[
P(\sqrt{k} \tilde{b}_{n,k} > \varepsilon) = \mathbb{P}(\tilde{b}_{n,k} > \varepsilon) = \mathbb{P}(\tilde{b}_{n,k} \geq \varepsilon) \leq e^{-\inf_{x \in \mathbb{R}} I(x) + o(\varepsilon)}.
\]

where \( \Gamma_k = (e/k, k) \). The typical behaviour of the rate function is when \( x \neq 0 \), therefore

\[
\lim_{k \rightarrow \infty} \mathbb{P}(\sqrt{k} \tilde{b}_{n,k} > \varepsilon) \leq 0.
\]

Thus, \( \sqrt{k} \tilde{b}_{n,k} \rightarrow \rho \) as \( k \rightarrow \infty \) and this ends the proof.

Proof of Lemma 5. We observe that \( T_1, T_2, T_3, \ldots \) are independent but not identical distributed random variables.

(1)
(ii) Let \( f \) be the probability density function of \( T_j \) given \((\theta, \hat{p})\) and we observe that
\[
\mathbb{E}\left(|Z_j - \mu_j|^{2+\delta} | \theta, \hat{p} = \rho \right) = \int_0^\infty |Z_j - \mu_j|^{2+\delta} f(z_j)dz_j
\]
\[
= - \int_0^{\rho_j} (Z_j - \mu_j)^{2+\delta} f(z_j)dz_j + \int_{\rho_j}^\infty (Z_j - \mu_j)^{2+\delta} f(z_j)dz_j
\]
\[
eq e^{-1} \left( \frac{(-\mu_j)^{3+\delta}(7 + 2\delta)}{(3 + \delta)(4 + \delta)} + \mu_j^{3+\delta}(2 + \delta)! \right) \{1 + o(1) \}
\]
\[
= \mu_j^{2+\delta} \eta(\delta) \{1 + o(1) \},
\]
(65)

\[
\mathbb{E}\left(|Z_j - \mu_j|^{2+\delta} \right) = \mathbb{E}\left[ \mathbb{E}\left(|T_j - \mu_j|^{2+\delta} | \theta, \hat{p} \right) \right] \leq \eta(\delta) \{1 + o(1) \} \mu_j^{2+\delta}.
\]
(66)

Now define \( Z_j = \mathbb{E}_{\theta}[\hat{W}_j(\theta)]T_j - \mu_j, 1 \leq j \leq k \), and note that
\[
S_n^2 = \text{Var} \left( \sum_{j=1}^k \mathbb{E}_{\theta} Z_j \right) = \sum_{j=1}^k \text{Var} (\mathbb{E}_{\theta} \hat{W}_j(\theta)T_j) = \sum_{j=1}^k \left[ \mathbb{E}_{\theta} \hat{W}_j(\theta) \right]^2 \text{Var}(T_j) = \frac{k}{\gamma} \sum_{j=1}^k \left[ \mathbb{E}_{\theta} \hat{W}_j(\theta) \right]^2.
\]
(67)

Hence,

\[
\lim_{k \to \infty} \frac{1}{S_n^{2+\delta}} \sum_{j=1}^k \mathbb{E}\left(|Z_j - \mu_j|^{2+\delta} \right) \leq \lim_{k \to \infty} \left[ \frac{\eta(\delta) \{1 + o(1) \} \{ \gamma + o(1/k^n) \} + o(1) \mu_j^{2+\delta} \sum_{j=1}^k \left[ \mathbb{E}_{\theta} \hat{W}_j(\theta) \right]^{2+\delta}} {\gamma^2 \sum_{j=1}^k \left[ \mathbb{E}_{\theta} \hat{W}_j(\theta) \right]^2} \right]
\]
(68)

\[
= \eta(\delta) \lim_{k \to \infty} \left[ \frac{\sum_{j=1}^k \mathbb{E}_{\theta} \hat{W}_j(\theta)^{2+\delta}} {\left( \sum_{j=1}^k \left[ \mathbb{E}_{\theta} \hat{W}_j(\theta) \right]^2 \right)^{(1+\delta)/2}} \right] \leq \eta(\delta) \lim_{k \to \infty} \left[ \frac{2^{(1+\delta)/2} k^{-\delta/2} \kappa(\alpha(k))^{(1+\delta)/2} (1 + o(k))}{(1 - o(k))^{(1+\delta)/2}} \right] = 0.
\]

Proof of Theorem 1. Using Lemma 5 and Lemma 7, we can prove Theorem 3. It has been established in Lemma 5 that \( \sqrt{k} \tilde{b}_{nk} \to 0 \) as \( k \to \infty \). Lemma 7 also establishes that the Lyapunov’s condition holds for Central Limit Theorem; hence, by the Lyapunov’s Central Limit Theorem,
\[
\sqrt{k} ( \mathbb{E}_{\theta} \tilde{Y}_n(\theta) - \gamma ) \to \mathcal{N}(\mu, \sigma^2), \text{ as } k \to \infty.
\]
(69)

Therefore, all we need to complete the Proof Proof 6, is to specify the parameters of the normal distribution.

Recall from (16) that
\[
\tilde{Y}_k(\lambda) = \mathbb{E}_{\theta} \left[ \tilde{Y}_{RW}(\lambda, \theta) \right] = \mathbb{E}_{\theta} \left[ \sum_{j=1}^k \hat{w}_j T_j - \tilde{b}_{nk} \sum_{j=1}^k \hat{w}_j C_j \right].
\]
(70)

Also, from Lemma 5, asymptotically, the second term on the right hand side vanishes, so we would concentrate on the first term of the expression only. Let
\[ S_k = \sqrt{k} (\bar{\gamma}_k (\lambda) - \gamma) = \sqrt{k} \left\{ \mathbb{E}_\theta \left[ \sum_{j=1}^k \bar{W}_j T_j - \tilde{b}_{nk} \sum_{j=1}^k \bar{W}_j C_j \right] - \gamma \right\}. \] (71)

The expected value of \( S_k \) is given as

\[
\mathbb{E}(S_k) = \mathbb{E}_\theta \left\{ \sqrt{k} \sum_{j=1}^k \bar{W}_j \mathbb{E}(T_j) - \sqrt{k} \tilde{b}_{nk} \sum_{j=1}^k \bar{W}_j C_j - \sqrt{k} \gamma \right\}
= \mathbb{E}_\theta \mathbb{E}_\rho \left\{ \sqrt{k} \sum_{j=1}^k \bar{W}_j (\gamma + \tilde{b}_{nk}(\rho) C_j(\rho)) - \sqrt{k} \tilde{b}_{nk}(\rho) \sum_{j=1}^k \bar{W}_j C_j(\rho) - \sqrt{k} \gamma \right\} = \mathbb{E}_\theta \mathbb{E}_\rho [0] = 0. \quad (72)

Hence, \( \mathbb{E}(S_k) \to \mu = 0 \) as \( k \to \infty \). Recall that \( k \to \infty, \sqrt{k} \tilde{b}_{nk} \to p_0 \), \( 2\kappa(\alpha(k)) \to 1 \) and by assumption \( \theta_j \) and \( T_j \) are independent, therefore, the variance of \( S_k \) is given by

\[
\text{Var}(S_k) = k \text{Var} \left( \mathbb{E}_\theta \left[ \sum_{j=1}^k \bar{W}_j T_j \right] \right)
= \frac{4}{k} \kappa^2(\alpha(k)) \sum_{j=1}^k \text{Var} \left\{ \mathbb{E}_\theta \left( 1 - \theta^{(k)}_j \frac{1}{k+1} \right) T_j \right\}
= \frac{4}{k} \kappa^2(\alpha(k)) \sum_{j=1}^k \left\{ \mathbb{E}_\theta \left( 1 - \frac{1}{\alpha(k) + 1} \frac{j}{k+1} \right) \right\}^2 \text{Var}(T_j)
= \frac{4}{k} \kappa^2(\alpha(k)) \sum_{j=1}^k \left\{ \left( 1 - \frac{1}{\alpha(k) + 1} \frac{j}{k+1} \right) \right\}^2 \text{Var}(T_j)
= (2\kappa(\alpha(k)))^2 \gamma^2 \left\{ \left( 1 - \frac{1}{\alpha(k) + 1} \frac{o(k)}{k} \right) \right\}
= \gamma^2 \left\{ \left( 1 - \frac{1}{\alpha(k) + 1} \frac{o(k)}{k} \right) \right\} + o(1). \quad (73)

Using the assumption \( \alpha(k)/k \to \Delta < \infty \), as \( k \to \infty \), we have \( \text{Var}(S_k) \to \sigma^2 = \gamma^2 \), which completes the Proof of Theorem 1.

Acknowledgments

Ocran, E. would like to thank the University of Ghana Building a New Generation of Academics in Africa (BANGA-Africa) Project (funded by Carnegie Corporation of New York) for providing financial support for this Ph.D. research work.

References

[1] F. Longin, Extreme Events in Finance: A Handbook of Extreme Value Theory and its Applications, John Wiley & Sons, New York, NY, USA, 2016.
[2] M. M. Kithinji, P. N. Mwita, and A. O. Kube, “Adjusted extreme conditional quantile autoregression with application to risk measurement,” Journal of Probability and Statistics, vol. 2021, Article ID 6697120, 10 pages, 2021.
[3] K. Gkillas and P. Katsiampa, “An application of extreme value theory to cryptocurrencies,” Economics Letters, vol. 164, pp. 109–111, 2018.
[4] R. Minkah, T. de Wet, and A. Ghosh, "Robust estimation of pareto-type tail index through an exponential regression model," *Communications in Statistics—Theory and Methods*, pp. 1–19, 2021.

[5] R. Minkah, "Tail index estimation of the generalised pareto distribution using a pivot from a transformed pareto distribution," *Science and Development Journal*, vol. 4, no. 1, pp. 1–19, 2020.

[6] C. Rohrbeck, E. F. Eastoe, A. Frigessi, and J. A. Tawn, "Extreme value modelling of water-related insurance claims," *Annals of Applied Statistics*, vol. 12, no. 1, pp. 246–282, 2018.

[7] A. J. McNeil, "Estimating the tailsoflossseveritydistributions usingextremetheory," *ASTIN Bulletin: The Journal of the IA*, vol. 27, no. 1, pp. 117–137, 1997.

[8] G. Magnou, "An application of extreme value theory for measuring financial risk in the uruguayan pension fund," *Compendium: Cuadernos de Economía y Administración*, vol. 4, no. 7, pp. 1–19, 2017.

[9] E. Afuecheta, C. Utazi, E. Ranganai, and C. Nnanatu, "An application of extreme value theory for measuring financial risk in brics economies," *Annals of Data Science*, pp. 1–40, 2020.

[10] N. Mehrnia and S. Coleri, "Wireless channel modeling based on extreme value theory for ultra-reliable communications," *IEEE Transactions on Wireless Communications*, vol. 21, no. 2, pp. 1064–1076, 2022.

[11] B. Finkenstadt and H. Rootzén, *Extreme Values in Finance, Telecommunications, and the Environment*, CRC Press, Boca Raton, FL, USA, 2003.

[12] M. I. Gomes and M. J. Martins, "Asymptotically unbiased estimators for parameters of a pareto distribution with a restricted scale," *Statistical Methodology*, vol. 5, no. 1, pp. 5–31, 2008.

[13] S. Tripathi, S. Kumar, and C. Petropoulos, "Improved estimators for parameters of a pareto distribution with a restricted scale," *Statistical Methodology*, vol. 18, pp. 1–13, 2014.

[14] M. B. Hill, "A simple general approach to inference about the tail of a distribution," *Annals of Statistics*, vol. 3, no. 5, pp. 1163–1174, 1975.

[15] J. Beirlant, Y. Goengebeur, J. Teugels, and J. Segers, *Statistics of Extremes: Theory and Applications*, John Wiley & Sons, New York, NY, USA, 2004.

[16] S. Csorgo, P. Deheuvels, and D. Mason, "Kernel estimates of the tail index of a distribution," *The Annals of Statistics*, vol. 13, no. 3, pp. 1050–1077, 1985.

[17] J. Danielsson, D. W. Jansen, and C. G. De vries, "The method of moments ratio estimator for the tail shape parameter," *Communications in Statistics - Theory and Methods*, vol. 25, no. 4, pp. 711–720, 1996.

[18] J. Beirlant, P. Vynckier, and J. L. Teugels, "Excess functions and estimation of the extreme-value index," *Bernoulli*, vol. 2, pp. 293–318, 1996.