Components of spaces of representations and stable triples

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Abstract

We consider the moduli spaces of representations of the fundamental group of a surface of genus $g \geq 2$ in the Lie groups $SU(2,2)$ and $Sp(4,\mathbb{R})$. It is well known that there is a characteristic number, $d$, of such a representation, satisfying the inequality $|d| \leq 2g - 2$. This allows one to write the moduli space as a union of subspaces indexed by $d$, each of which is a union of connected components. The main result of this paper is that the subspaces corresponding to $d = \pm(2g - 2)$ are connected in the case of representations in $SU(2,2)$, while they break up into $3 \cdot 2^g + 2g - 4$ connected components in the case of representations in $Sp(4,\mathbb{R})$. We obtain our results using the interpretation of the moduli space of representations as a moduli space of Higgs bundles, and an important step is an identification of certain subspaces as moduli spaces of stable triples, as studied by Bradlow and García-Prada.

1 Introduction

Let $\Sigma$ be a closed Riemann surface of genus $g \geq 2$ and let $G$ be a connected Lie group. Consider the space of reductive representations of the fundamental group of $\Sigma$ in $G$ modulo the action of $G$ by conjugation,

$$\mathcal{M}_G = \text{Hom}(\pi_1(\Sigma), G)^+/G,$$

the superscript “+” indicating reductive representations. As is well known, $\mathcal{M}_G$ can also be identified with the moduli space of reductive flat $G$-bundles over $\Sigma$ and it has an algebro-geometric interpretation as a moduli space of Higgs bundles (see Hitchin [13, 16]).

In this paper we study the connected components of $\mathcal{M}_G$ in the cases $G = Sp(4,\mathbb{R})$ and $G = SU(2,2)$. Previous work on this type of problem includes the determination of the number of connected components of $\mathcal{M}_G$ for the groups $PSL(2,\mathbb{R})$ and $PSL(2,\mathbb{C})$ by Goldman [11], for the groups $PSL(n,\mathbb{R})$ ($n \geq 3$) by Hitchin [17], and for the groups $PGL(2,\mathbb{R})$, $PU(2,1)$, and $U(p,1)$ by Xia [26, 23, 27].

The first observation is that there is a characteristic number, $d$, which comes from a characteristic class of the bundle obtained from a reduction of structure group to the maximal compact subgroups $U(2) \subseteq Sp(4,\mathbb{R})$ and $S(U(2) \times U(2)) \subseteq SU(2,2)$, respectively. It is well
known that this satisfies the Milnor-Wood type inequality $|d| \leq 2g - 2$ (cf. Section 3). This allows one to write

$$M_G = \mathcal{M}_{-(2g-2)} \cup \cdots \cup \mathcal{M}_{2g-2},$$

where each $\mathcal{M}_d$ is a union of connected components. We can then state our main result as follows.

**Theorem.** The subspaces $\mathcal{M}_0 \subseteq \mathcal{M}_G$ are connected for $G = \text{Sp}(4, \mathbb{R})$ and $G = \text{SU}(2, 2)$ and the subspaces $\mathcal{M}_{\pm(2g-2)} \subseteq \mathcal{M}_{\text{SU}(2, 2)}$ are connected. The subspaces $\mathcal{M}_{\pm(2g-2)} \subseteq \mathcal{M}_{\text{Sp}(4, \mathbb{R})}$ have $3 \cdot 2^{2g} + 2g - 4$ connected components.

The most remarkable aspect of this result is that $\mathcal{M}_{\pm(2g-2)} \subseteq \text{Sp}(4, \mathbb{R})$ breaks up into a number of different connected components, which are not detected by the first Chern class given by reduction of structure group to $\text{U}(2) \subseteq \text{Sp}(4, \mathbb{R})$. It seems likely that the remaining $\mathcal{M}_d$ are connected and we hope to come back to this question on a later occasion.

The method we use for studying the connected components is via the algebro-geometric interpretation of $\mathcal{M}_G$ as a moduli space of Higgs bundles, due to Hitchin [15, 17]. (A Higgs bundle is a pair $(E, \Phi)$, where $E$ is a holomorphic rank $n$ degree $d$ vector bundle and $\Phi \in H^0(\Sigma; \text{End}(E)_0 \otimes K)$, see Section 2.1 for more details.) From this point of view one can define a Hamiltonian circle action on the moduli space and one uses a moment map for this action as a Morse function, in the sense of Bott, to obtain topological information about the space (cf. Hitchin [15, 17] and Gothen [12]). The central point is then to identify the critical submanifolds of the Morse function and to obtain topological information about them. In particular, to obtain information about connected components, one needs to consider the local minima of the Morse function. It should be remarked that the moduli spaces have singularities and so one cannot directly apply Morse theory, however, in the case of the determination of connected components this difficulty can be circumvented (see Sections 2.4 and 2.5).

In this paper we show that certain critical submanifolds, corresponding to local minima of the Morse function, can be identified with moduli spaces of *stable triples*, or spaces closely related to them, as studied by Bradlow and García-Prada [3, 10]. In the cases $d = 0$ and $|d| = 2g - 2$ the structure of the moduli spaces of triples is particularly simple and this allows us to prove the theorem above. In the case of $G = \text{Sp}(4, \mathbb{R})$ and $|d| = 2g - 2$ we further need to use a spectral curve (see Hitchin [14]) which is an unramified covering of $\Sigma$ and the mod 2 index theorem of Atiyah-Singer to identify the local minima of the Morse function as certain Prym varieties associated to the covering of $\Sigma$.

This paper is organized as follows. In Section 2 we recall the basics of the theory of Higgs bundles and their relation to representations of the fundamental group of a surface in a non-compact Lie group. We also recall the concept of a *$Q$-bundle*, of which a holomorphic triple is a special case, and prove a theorem (Theorem 2.3) which is essential for the identification of the subspace of local minima with a moduli space of holomorphic triples. Finally we describe the Morse theory on the moduli space and, in particular, we describe how to find the Morse indices. It was observed by Hausel [14] that our results on the Morse indices, together with a theorem of his, imply a theorem of Laumon [18] in this context: the nilpotent cone in the moduli space $\mathcal{M}$ of rank $n$ Higgs bundles is a Lagrangian subvariety with respect to
the holomorphic symplectic form on \( \mathcal{M} \). We end this section by briefly describing this. In Section 3 we reprove the known bound \( |d| \leq 2g-2 \), using Higgs bundles; we include the proof because it gives some extra information which is important later on (cf. Proposition 3.2). In Section 4 we analyze the local minima of the Morse function on the space \( \mathcal{M}_G \) for \( G = \text{Sp}(4, \mathbb{R}) \) and \( G = \text{SU}(2, 2) \) in detail. Finally, in Section 5, we finish the proof of our main theorem, using the previous results.

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2 Higgs bundles and the topology of moduli spaces

2.1 Higgs bundles

In this section we review some basic facts about Higgs bundles and set up notation. For details see Hitchin [15] and Simpson [23].

Let \( G_\mathbb{C} \) be a complex semi-simple Lie group with Lie algebra \( g_\mathbb{C} \). Let \( G \subset G_\mathbb{C} \) be a maximal compact subgroup with Lie algebra \( g \). Thus there is a compact real structure \( \tau: g_\mathbb{C} \to g_\mathbb{C} \) whose fixed point set is \( g \). Denoting the \(-1\)-eigenspace of \( \tau \) by \( g^\perp \) we then have \( g_\mathbb{C} = g \oplus g^\perp \).

Non-abelian Hodge theory gives an equivalence between reductive representations of \( \pi_1(\Sigma) \) in \( G_\mathbb{C} \) and Higgs bundles over \( \Sigma \), which we now describe. Let \( \rho: \pi_1(\Sigma) \to G_\mathbb{C} \) be a reductive representation. This data is equivalent to having a principal bundle

\[
P_\mathbb{C} = \tilde{\Sigma} \times_{\rho} G_\mathbb{C}
\]

with a reductive flat connection \( B \in \Omega^1(P_\mathbb{C}; g_\mathbb{C}) \) (here \( \tilde{\Sigma} \) is the universal cover of \( \Sigma \)).

If we have a metric in \( P_\mathbb{C} \), i.e. a reduction of structure group from \( G_\mathbb{C} \) to \( G \), we can write

\[
i^*B = A + \theta
\]

where \( i: P \hookrightarrow P_\mathbb{C} \) is the inclusion of the principal \( G \)-bundle \( P \) given by the reduction of structure group, \( A \) is a connection on \( P \), and \( \theta \in \Omega^1(P; g^\perp) \) is a tensorial form, which can therefore be thought of as an element of \( \Omega^1(\Sigma; P \times_{\text{Ad}} g^\perp) \).

Given a complex representation of \( G \) (e.g. the adjoint representation on \( g_\mathbb{C} \)), we have the usual decomposition of the covariant derivative \( d_A \) in its \((1,0)\)- and \((0,1)\)-parts:

\[
d_A = \partial_A + \bar{\partial}_A.
\]

Similarly, we can write

\[
\theta = \Phi - \tau(\Phi),
\]
for a unique $\Phi \in \Omega^{1,0}(\Sigma; \text{Ad} P_C)$ (by abuse of notation, we denote by $\tau$ the combination of the compact real structure $\tau$ on $g_C$ and conjugation on the form component).

Corlette [6] and Donaldson [8] proved that there exists a harmonic metric in $P$, that is, a metric such that $(A, \Phi)$ obtained via the above procedure satisfy Hitchin’s equations

$$F(A) - [\Phi, \tau(\Phi)] = 0,$$

$$\bar{\partial}_A \Phi = 0.$$  

This, in turn, gives a principal Higgs bundle, i.e. a pair $(P_C, \Phi)$ consisting of a holomorphic principal bundle $P$ (with holomorphic structure defined by $\bar{\partial}_A$) and a Higgs field $\Phi \in \mathcal{H}^0(\Sigma; \text{Ad} P_C \otimes K)$, where $K$ denotes the canonical bundle of $\Sigma$. Given a representation, $V$, of $G_C$ one then obtains a Higgs vector bundle $(E, \Phi)$, where $E = P_C \times_{G_C} V$ and $\Phi \in \mathcal{H}^0(\Sigma; \text{End}(E) \otimes K)$. The two main examples we have in mind are the adjoint representation $V = g_C$ and the fundamental representation of $G_C = \text{SL}(n, \mathbb{C})$. If the original representation $\rho$ of $\pi_1(\Sigma)$ is irreducible then $(E, \Phi)$ is stable, i.e.

$$\mu(F) < \mu(E)$$

for any proper non-trivial $\Phi$-invariant subbundle $F$ of $E$ (here $\mu(E) = \text{deg}(E)/\text{rk}(E)$ is the slope of the holomorphic bundle $E$). Allowing equality in the above inequality gives the notion of a semi-stable Higgs bundle. Finally, if $\rho$ is reductive, then the corresponding Higgs bundle is poly-stable, i.e. it is a direct sum of lower rank Higgs bundles, all of the same slope.

By a theorem of Hitchin [15] and Simpson [22], the above procedure can be reversed, by finding a harmonic metric in the Higgs bundle. This produces a reductive representation of $\pi_1(\Sigma)$ from a poly-stable Higgs bundle. This gives a homeomorphism

$$\mathcal{M}_{G_C} \rightarrow \text{Hom}(\pi_1(\Sigma), G_C)^+/G_C.$$  

where $\mathcal{M}_{G_C}$ is the moduli space of poly-stable principal $G_C$ Higgs bundles.

We finish by recalling the description of the Zariski tangent space to $\mathcal{M}_{G_C}$ at $(P_C, \Phi)$ given by Biswas and Ramanan [3]. This is the first hyper-cohomology $\mathbb{H}^1(C^\bullet_C)$ of the complex of sheaves

$$C^\bullet_C : \mathcal{O}(\text{Ad} P_C) \xrightarrow{\text{ad}(\Phi)} \mathcal{O}(\text{Ad} P \otimes K).$$  

(2.1)

From this they deduce the long exact sequence

$$0 \rightarrow \mathbb{H}^0(C^\bullet_C) \rightarrow \mathcal{H}^0(\Sigma; \text{Ad} P_C) \rightarrow \mathcal{H}^0(\Sigma; \text{Ad} P \otimes K) \rightarrow T_{(P_C, \Phi)} \mathcal{M}_{G_C} \rightarrow \mathcal{H}^1(\Sigma; \text{Ad} P_C) \rightarrow \mathcal{H}^1(\Sigma; \text{Ad} P \otimes K) \rightarrow \mathbb{H}^2(C^\bullet_C) \rightarrow 0.$$  

(2.2)

$(P_C, \Phi)$ is a smooth point of the moduli space if $\mathbb{H}^0(C^\bullet_C)$ and $\mathbb{H}^2(C^\bullet_C)$ vanish (by Serre duality, it is sufficient to check that $\mathbb{H}^0(C^\bullet_C) = 0$). From this one sees that stable Higgs bundles represent smooth points of the moduli space. The dimension of $\mathcal{M}_{G_C}$ can be calculated using Riemann-Roch to be

$$\dim_{\mathbb{C}} \mathcal{M}_{G_C} = \chi(\mathcal{O}(\text{Ad} P \otimes K)) - \chi(\mathcal{O}(\text{Ad} P_C)) = 2 \dim_{\mathbb{C}} g_C(g - 1)$$
2.2 Real groups

Hitchin [15, 17] showed how to use Higgs bundles to study representations of $\pi_1(\Sigma)$ in real (non-compact) Lie groups. Next we recall the relevant parts of this theory.

Let $G_r \subset G_C$ be a real form, given by a real structure $\sigma$: $g_C \to g_C$. Let $K \subset G_r$ be a maximal compact subgroup and let $\mathfrak{t} \subset \mathfrak{g}_C$ be the corresponding inclusion of Lie algebras. Let $\mathfrak{t}^\perp$ be the orthogonal complement to $\mathfrak{t}$ with respect to the Killing form, then we can write $\mathfrak{g}_r = \mathfrak{t} \oplus \mathfrak{t}^\perp$, where the Killing form is negative definite on $\mathfrak{t}$ and positive definite on $\mathfrak{t}^\perp$.

Define a complex linear involution $\phi$: $g_C \to g_C$ by $\phi|_{\mathfrak{t}} = -1$ and $\phi|_{\mathfrak{t}^\perp} = 1$. Define another real structure on $g_C$ by $\tau = \sigma \phi = \phi \sigma$. It is then easy to see the that the corresponding real subgroup $G \subset G_C$ is a maximal compact subgroup and, clearly, $\mathfrak{t} = \mathfrak{g} \cap \mathfrak{g}_r$.

Now suppose that we have a reductive representation of $\pi_1(\Sigma)$ in $G_r$ and let $B$ be the associated flat connection on the principal $G_r$-bundle $P_{G_r}$. The theorem of Donaldson and Corlette also applies in this case and gives a reduction of structure group to $K \subset G_r$: let $i: P_K \to P_{G_C}$ be the inclusion of principal bundles given by combining the reduction of structure group with the inclusion $G_r \subset G_C$. In the decomposition $i^*B = A + \theta$, $A$ and $\theta$ will be fixed by $\sigma$, while $\tau(A) = A$ and $\tau(\theta) = -\theta$. Thus $\partial_A$ is fixed by $\phi$, and $\Phi$ is in the $-1$-eigenspace of $\phi$. This means that the corresponding Higgs bundle is of the form $(P_{K_C}, \Phi)$, where

- $P_{K_C}$ is a holomorphic principal $K_C$-bundle,
- $\Phi \in H^0(\Sigma; \text{Ad}_{\mathfrak{t}} P_K \otimes K)$ (where we use the notation $\text{Ad}_{\mathfrak{t}} P_K = P_K \times_{\text{Ad}} \mathfrak{t}$).

Conversely, such a Higgs bundle gives a representation of $\pi_1(\Sigma)$ in $G_r$. We then have a homeomorphism

$$\mathcal{M}_{G_r} \to \text{Hom}(\pi_1(\Sigma), G_r)^+/G_r,$$

where $\mathcal{M}_{G_r}$ is the moduli space of poly-stable Higgs bundles of the above type. Alternatively $\mathcal{M}_{G_r}$ can be thought of as the moduli space of solutions $(A, \Phi)$ to Hitchin’s equations modulo $K$-gauge equivalence: then $A$ is a connection on a principal $K$-bundle $P_K$ and $\Phi \in \Omega^{1,0}(\Sigma; \text{Ad}_{\mathfrak{t}} P_K)$.

The analogue to (2.1) in this context is that the Zariski tangent space to $\mathcal{M}_{G_r}$ is the first hyper-cohomology of the complex of sheaves

$$C^*_r: \mathcal{O}(\text{Ad}_{\mathfrak{t}} P_K) \xrightarrow{\text{ad}(\Phi)} \mathcal{O}(\text{Ad}_{\mathfrak{t}} P_K \otimes K),$$

where we use the notation $\text{Ad}_{\mathfrak{t}} P_K = P_K \times_{\text{Ad}} \mathfrak{t}_C = \text{Ad} P_{K_C}$. The analogue to the long exact sequence (2.2) is

$$0 \to \mathbb{H}^0(C^*_r) \to H^0(\Sigma; \text{Ad}_{\mathfrak{t}} P_K) \to H^0(\Sigma; \text{Ad}_{\mathfrak{t}} P_K \otimes K) \to T_{(P_K, \Phi)} \mathcal{M}_{G_r} \to \cdots$$

The smooth points of the moduli space are those for which $\mathbb{H}^0(C^*_r) = \mathbb{H}^2(C^*_r) = 0$ and again the stable Higgs bundles represent smooth points. The dimension of $\mathcal{M}_{G_r}$ can be calculated as before to be $\dim_{\mathbb{C}} \mathcal{M}_{G_r} = \dim_{\mathbb{C}} g_C(g - 1) = \frac{1}{2} \dim_{\mathbb{C}} \mathcal{M}_{G_C}$.
We finish this section by giving two examples of this setup. First consider \( G_r = \text{SU}(n,n) \) which is a real form of \( \text{SL}(2n,\mathbb{C}) \). The Higgs vector bundles \((E, \Phi)\) corresponding to representations of \( \pi_1(\Sigma) \) in \( \text{SU}(n,n) \) are of the form

\[
E = V \oplus V' \quad \text{and} \quad \Phi = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix},
\]

where \( V \) and \( V' \) are rank \( n \) vector bundles with \( \Lambda^nV \otimes \Lambda^nV' \cong \mathcal{O} \), \( b \in H^0(\text{Hom}(V',V) \otimes K) \), and \( c \in H^0(\text{Hom}(V,V') \otimes K) \). Two \( \text{SU}(n,n) \) representations are conjugate if and only if the corresponding Higgs bundles of this form are isomorphic by an isomorphism which is of the form \( \begin{pmatrix} g & 0 \\ 0 & g' \end{pmatrix} \) and of determinant one.

The second example is \( G_r = \text{Sp}(2n,\mathbb{R}) \); this is a split real form of \( \text{Sp}(2n,\mathbb{C}) \). The Higgs vector bundles \((E, \Phi)\) obtained from the standard representation of \( \text{Sp}(2n,\mathbb{C}) \) on \( \mathbb{C}^{2n} \), and corresponding to representations of \( \pi_1(\Sigma) \) in \( \text{Sp}(2n,\mathbb{R}) \) are of the form

\[
E = V \oplus V^* \quad \text{and} \quad \Phi = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix},
\]

where \( V \) is a rank \( n \) vector bundle, \( b \in H^0(S^2V \otimes K) \), and \( c \in H^0(S^2V^* \otimes K) \). Two \( \text{Sp}(2n,\mathbb{R}) \) representations are conjugate if and only if the corresponding Higgs bundles of this form are isomorphic by an isomorphism which is of the form \( \begin{pmatrix} g & 0 \\ 0 & g' \end{pmatrix} \).

### 2.3 \( Q \)-bundles and triples

The special forms (2.5) and (2.6) suggest a different point of view, that of \( Q \)-bundles. This notion, due to Alastair King, provides a general framework for considering a large number of the various kinds of vector bundles with extra structure, which have been studied in recent years. The vortex pairs of Bradlow [4], the triples of García-Prada, introduced in [10] and studied systematically by him and Bradlow in [5] (and also Higgs bundles), are all examples of \( Q \)-bundles.

Let \( Q \) be a quiver, that is, \( Q \) is a directed graph, specified by a set of vertices \( Q_0 \) and a set of arrows \( Q_1 \), together with head and tail maps \( h, t : Q_1 \to Q_0 \).

**Definition 2.1.** A \( Q \)-bundle over a Riemann surface \( \Sigma \) is a collection of holomorphic vector bundles \( \{E_j\}_{j \in Q_0} \) over \( \Sigma \) and a collection of holomorphic maps \( \{\phi_a : E_j(a) \to E_h(a)\}_{a \in Q_1} \). A twisted \( Q \)-bundle is given by in addition specifying a linebundle \( L_a \) for each arrow \( a \). The maps \( \phi_a \) are then required to be holomorphic maps \( \phi_a : E_j(a) \to E_h(a) \otimes L_a \).

We shall only consider \( Q \)-bundles of a particularly simple form: we let \( Q \) be a quiver with 2 vertices and exactly one arrow connecting the vertices in each direction (see fig. [1]).

We denote the the arrows by \( a_{ij} \), where \( a_{ij} \) is the arrow going from \( j \) to \( i \). Also, the maps will be twisted by the canonical bundle \( K \). Thus, from now on, a \( Q \)-bundle is a pair

\[
\mathbf{E} = (E, \Phi),
\]

where \( E = \{E_1, E_2\} \) and \( \Phi = \{\phi_{ij}\} \). Here, each \( E_i \) is a holomorphic vector bundle on \( \Sigma \) and \( \phi_{ij} \) is a holomorphic section of \( \text{Hom}(E_j, E_i \otimes K) \).
A particularly interesting special case occurs when $\phi_{12} = 0$. The data of the above type of $Q$-bundle then comes down to a triple $(E_1, E_2, \phi)$, where $\phi \in H^0(\Sigma; \text{Hom}(E_1, E_2) \otimes K)$. If we define $\tilde{E}_2 = E_2 \otimes K$ then this is equivalent to a holomorphic triple $(E_1, \tilde{E}_2, \phi)$ in the sense of Bradlow and García-Prada [5].

Given a $Q$-bundle $E = (E, \Phi)$, we can define an associated Higgs bundle $(E, \Phi)$ by putting

$$E = E_1 \oplus E_2 \quad \text{and} \quad \Phi = (\phi_{ij}),$$

where $(\phi_{ij})$ is the matrix of $\Phi$ with respect to the above direct sum decomposition of $E$. Note that the Higgs bundles of the form (2.5) or (2.6) arise in this way. Conversely, given a Higgs bundle of the special form (2.5) or (2.6) we get an associated $Q$-bundle.

There are equations for preferred special metrics in a $Q$-bundle, the $Q$-vortex equations. Choose a metric compatible with the complex structure on $\Sigma$ and, for convenience, normalize it so that $\text{vol}(\Sigma) = 2\pi$. This of course also gives a Hermitian metric in the canonical bundle $K$. The $Q$-vortex equations are equations for Hermitian metrics in $E_1$ and $E_2$ and in our case they take the form

$$\begin{cases}
  i\Lambda F(A_1) + \phi_{12} \phi_{12}^* - \phi_{21}^* \phi_{21} = \tau_1 \text{Id}_{E_1} \\
  i\Lambda F(A_2) + \phi_{21} \phi_{21}^* - \phi_{12}^* \phi_{12} = \tau_2 \text{Id}_{E_2}
\end{cases}$$

where $F(A_i)$ is the curvature of the metric connection in $E_i$, $\Lambda$ denotes contraction with the Kähler form of $\Sigma$, and $\phi_{ij}^*$ denotes the adjoint taken with respect to the metric obtained from the metrics on $E_i$ and $K$. The parameters $(\tau_1, \tau_2)$ are real, subject to the condition

$$\sum_{i=1}^{2} \left( \deg(E_i) - \tau_i \text{rk}(E_i) \right) = 0,$$

obtained by taking traces in the equations (2.8), summing and integrating over $\Sigma$ (thus there is really only one real parameter involved, which is usually taken to be $\tau = \tau_2$). There is a stability condition for $Q$-bundles, such that any $Q$-bundle which supports a solution to the $Q$-vortex equations is a direct sum of stable $Q$-bundles. In our case the condition is

$$\sum_{i=1}^{2} \left( \deg(F_i) - \tau_i \text{rk}(F_i) \right) < 0$$

for any proper $Q$-subbundle $F$ of $E$. Note that the condition depends on the parameters $(\tau_1, \tau_2)$. Bradlow and García-Prada [5] constructed moduli spaces of stable triples, varying with the parameter $\tau$. 

Figure 1: The quiver $Q$
We shall only need to consider the case $\tau_1 = \tau_2 = \mu(E)$ so we shall assume this from now on. The stability condition (2.9) can then be reformulated as

$$\mu(F) < \mu(E)$$

for any proper $Q$-subbundle $F = (\{F_1, F_2\}, \{\phi_{12}, \phi_{21}\})$ of $E$, and where we write $F = F_1 \oplus F_2$.

But, obviously, $F = F_1 \oplus F_2 \subset E$ is a $\Phi$-invariant subbundle, thus stability of the Higgs bundle $(E, \Phi)$ implies stability of the $Q$-bundle $E$. The following lemma will allow us to conclude that the converse also holds.

**Lemma 2.2.** Let $(E, \Phi)$ be a Higgs bundle of the form $E = E_1 \oplus E_2$ and

$$\Phi = \begin{pmatrix} 0 & \phi_{12} \\ \phi_{21} & 0 \end{pmatrix}.$$ 

Let $E = (\{E_1, E_2\}, \{\phi_{12}, \phi_{21}\})$ be the associated $Q$-bundle. Let $F' \subset E$ be a $\Phi$-invariant subbundle. Then there is a $Q$-subbundle $E' = (\{E'_1, E'_2\}, \{\phi_{12}, \phi_{21}\})$ of $E$ such that

$$\mu(F') \leq \mu(E'),$$

where $E' = E'_1 \oplus E'_2$.

**Proof.** Let $\pi_i: E \to E_i$ be the projection on the $i$th factor. Let $F_i \subset E_i$ and $G_i \subset F'$ be the subbundles which are generated by the image and kernel of $\pi_i$, respectively. Then $F_1$ and $G_2$ are contained in $E_1$, $F_2$ and $G_1$ are contained in $E_2$, and we have sequences of vector bundles

$$0 \to G_i \to F' \to F_i \to 0,$$

which are generically short exact. Hence, $\deg(F') \leq \deg(G_i) + \deg(F_i)$, and putting $F = F_1 \oplus F_2$ and $G = G_2 \oplus G_1$, it follows that

$$2 \deg(F') \leq \deg(F) + \deg(G).$$

Clearly $2 \rk(F') = \rk(F) + \rk(G)$, so that

$$\mu(F') \leq \frac{\rk(F)}{\rk(F) + \rk(G)} \mu(F) + \frac{\rk(G)}{\rk(F) + \rk(G)} \mu(G),$$

and therefore either $\mu(F) \geq \mu(F')$ or $\mu(G) \geq \mu(F')$. Provided that $F$ and $G$ give $Q$-subbundles of $E$ we can then take $E'$ to be the $Q$-bundle associated to either $F$ or $G$.

It thus remains to see show that $F$ and $G$ are $\Phi$-invariant and, therefore, define $Q$-subbundles of $E$. First, let $x_1 \in F_1$. If we write $x_1 = \pi_1(x)$ for some $x = x_1 + x_2$ in $F'$, then

$$\Phi(x) = \Phi(x_1) + \Phi(x_2).$$
By our assumption on the matrix for $\Phi$, it follows that $\Phi(x_1) \in E_2$ and $\Phi(x_2) \in E_1$. Then $\pi_1(\Phi(x)) = \Phi(x_2) \in E_1$ and $\pi_2(\Phi(x)) = \Phi(x_1) \in E_2$. But $\Phi(x) \in F'$ because $F'$ is $\Phi$-invariant, and thus $\Phi(x_2) \in F_1$ and $\Phi(x_1) \in F_2$. Of course, we can repeat the argument with $x_2 \in F_2$ and hence, $F$ is $\Phi$-invariant. The proof that $G$ is $\Phi$-invariant is similar. Let $\Phi(x) \in F'$ because $F'$ is $\Phi$-invariant, and thus $\Phi(x_1) \in F_1$ and $\Phi(x_2) \in F_2$. Of course, we can repeat the argument with $x_2 \in F_2$ and hence, $F$ is $\Phi$-invariant. The proof that $G$ is $\Phi$-invariant is similar. Let $x_1 \in G_2$. By assumption, $\Phi(x_1) \in E_2$. But $G_2 \subseteq F'$, so $\Phi(x_1) \in F'$ as well. It follows that $\Phi(x_1) \in G_1$ and thus, $G$ is $\Phi$-invariant. We have thus seen that $F$ and $G$ define $Q$-subbundles of $E$ and this finishes the proof.

As an immediate consequence we have the following theorem.

**Theorem 2.3.** Let $Q$ be a quiver with two vertices and one arrow connecting the vertices in each direction, and let $E = (\{E_1, E_2\}, \{\phi_{12}, \phi_{21}\})$ be a $Q$-bundle. Let $(E, \Phi)$ be the associated Higgs bundle as above; thus $E = E_1 \oplus E_2$ and

$$\Phi = \begin{pmatrix} 0 & \phi_{12} \\ \phi_{21} & 0 \end{pmatrix}. $$

Then $E$ is stable if and only if $(E, \Phi)$ is. Furthermore, if $(E, \Phi)$ is poly-stable, i.e. the direct sum of lower rank stable Higgs bundles, these lower rank Higgs bundles are $Q$-subbundles of $E$.

**Proof.** The only assertion that requires proof is the final one. Suppose that $(E, \Phi)$ is poly-stable and that $(F', \Phi')$ is a proper stable Higgs subbundle of $(E, \Phi)$ with $\mu(F') = \mu(E)$. By semi-stability of $(E, \Phi)$ the bundle called $G$ in the proof of Lemma 2.2 must then satisfy $\mu(G) = \mu(E) = \mu(F')$ and, since $G \subseteq F'$, it follows by stability of $F'$ that $G = F'$. But $G$ is a $Q$-subbundle so this finishes the proof.

## 2.4 Connected components and Morse theory

We shall use Hitchin’s method \[13, 17\], which we shall now review, for finding the connected components of $M_{G_r}$. The idea is to use discrete invariants of flat bundles for dividing $M_{G_r}$ into subspaces which are unions of connected components. We then show that these subspaces are, in fact, connected. For this consider the moduli space $M_{G_r}$ of Higgs bundles as the space of solutions $(A, \Phi)$ to Hitchin’s equations modulo gauge equivalence. The function

$$f : M \to \mathbb{R}$$

$$(A, \Phi) \mapsto \|\Phi\|^2 = \int_{\Sigma} |\Phi|^2 d\text{vol}$$

is proper. Thus, a subspace $N$ of $M$ is connected if the subspace of local minima of $f$ on $N$ is connected.

Restrict for a moment attention to irreducible solutions to Hitchin’s equations (i.e. stable Higgs bundles); these are smooth points of $M$. In order to identify the subspaces of local minima of $f$ one uses the fact that it is a moment map for the $S^1$ action on $M$, given by $(A, \Phi) \mapsto (A, e^{iu}\Phi)$: this implies that the critical points of $f$ are exactly the fixed points of
the circle action. Now, \((A, \Phi)\) represents a fixed point if and only if there is an infinitesimal gauge transformation \(\psi \in \Omega^0(\Sigma; P_K \times \text{Ad } \mathfrak{k})\) such that

\[
d_A\psi = 0, \quad [\psi, \Phi] = i\Phi. \tag{2.12}
\]

Let \((E, \Phi)\) be a Higgs vector bundle obtained from a complex representation of \(K\), then this can be decomposed in eigenspaces for the covariantly constant gauge transformation \(\psi\). Thus

\[
E = \bigoplus_m U_m, \tag{2.14}
\]

where \(\psi|_{U_m} = im\). Then (2.13) shows that \(\Phi : U_m \to U_{m+1} \otimes K\). The case

\[
E = \text{Ad } P_C = \text{Ad}_{\mathfrak{k}} P_K \oplus \text{Ad}_{\mathfrak{k}} P_K,
\]

is of particular interest. Since the adjoint action of \(\psi \in \mathfrak{k}\) and the involution \(\phi\) on \(\mathfrak{g}_C\) commute, this decomposition of \(\text{Ad } P_C\) is compatible with the decomposition \(\text{Ad } P_C = \bigoplus_m U_m\). From \(\text{ad}(\Phi) : U_m \to U_{m+1} \otimes K\) we conclude that \(\Phi \in H^0(\Sigma; U_1 \otimes K)\) and since \(\Phi \in H^0(\Sigma; \text{Ad}_{\mathfrak{k}} P_K)\) it follows that \(U_1 \subset \text{Ad}_{\mathfrak{k}} P_K\). Furthermore, \(\text{ad}(\Phi)\) interchanges \(\text{Ad}_{\mathfrak{k}} P_K\) and \(\text{Ad}_{\mathfrak{k}} P_K\) and we therefore have

\[
\text{Ad}_{\mathfrak{k}} P_K = \bigoplus_k U_{2k},
\]

\[
\text{Ad}_{\mathfrak{k}} P_K = \bigoplus_k U_{2k+1},
\]

where \(k\) is integer.

Two additional pieces of information will be useful. The first is that there is an isomorphism \(\text{Ad } P_C \xrightarrow{\sim} \text{Ad } P_C\) from the adjoint bundle to the co-adjoint bundle given by the Killing form on \(\mathfrak{g}_C\) and it is trivial to check that under this

\[
U_m \xrightarrow{\sim} U_m^* \tag{2.15}
\]

The second useful piece of information is that one can calculate the value of the Morse function \(f\) at a Higgs bundle of the form (2.14): denoting the component of \(\Phi\) mapping \(U_i\) to \(U_{i+1} \otimes K\) by \(\phi_i\) one shows easily, using Hitchin’s equations, that

\[
\|\phi_i\|^2 = \|\phi_{i-1}\|^2 + (\text{deg}(U_i) - \mu(E) \text{rk}(U_i)). \tag{2.16}
\]

In particular, if \(\mu(U_i) = \mu(E)\) for all \(i\) then \(\Phi\) must vanish.

Finally, consider the case of local minima of \(f\) which are not represented by stable Higgs bundles. For simplicity we restrict attention to bundles of the form (2.5) or (2.6). These are then direct sums of stable Higgs bundles of lower rank. But from Theorem 2.3 we can conclude that these lower rank Higgs bundles decompose as direct sums of subbundles of the bundles appearing in the direct sum decomposition of the original Higgs bundle.
2.5 Morse indices

Given a poly-stable Higgs bundle \((PK_C, \Phi)\) which represents a critical point of \(f\) it is necessary to decide whether it is a local minimum. This can be done using the observation of Hitchin \([17]\) that if \(\psi\) acts with weight \(m\) on an element of \(\text{Ad}_P K\) then the corresponding eigenvalue of the Hessian of \(f\) is \(-m\), while a weight \(m\) on \((\text{Ad}_P K) \otimes K\) gives the eigenvalue \(1 - m\). It follows that the subspace of \(T_{(PK_C, \Phi)} \mathcal{M}_{Gr}\) on which the Hessian of \(f\) is negative definite is \(H^1(C^\bullet)\), where \(C^\bullet\) is the complex of sheaves

\[
C^\bullet : \mathcal{O}\left( \bigoplus_{k \geq 1} U_{2k} \right) \xrightarrow{\text{ad}(\Phi)} \mathcal{O}\left( \bigoplus_{k \geq 1} U_{2k+1} \otimes K \right).
\]

In other words, the Morse index is \(\dim \mathbb{R} H^1(C^\bullet)\). At a smooth point of \(\mathcal{M}_{Gr}\), where \(H^0(C^\bullet) = H^2(C^\bullet) = 0\), the Morse index can be calculated using the Riemann-Roch theorem:

\[
\dim \mathbb{C} H^1(C^\bullet) = \chi\left( \mathcal{O}\left( \bigoplus_{k \geq 1} U_{2k+1} \otimes K \right) \right) - \chi\left( \mathcal{O}\left( \bigoplus_{k \geq 1} U_{2k} \right) \right)
= (g - 1) \sum_{k \geq 1} (\text{rk}(U_{2k}) + \text{rk}(U_{2k+1})) + \sum_{k \geq 1} (\deg(U_{2k+1}) - \deg(U_{2k})). \tag{2.17}
\]

The stable Higgs bundle \((PK_C, \Phi)\) represents a local minimum of \(f\) if and only if this number is zero.

Finally consider the case of reducible Higgs bundles. It is shown in \([17]\) that if \(H^1(C^\bullet) = 0\) then we continue to have a local minimum and, on the other hand, if there is an element of \(H^1(C^\bullet)\) on which \(\ddot{f}\) is negative and which is tangent to a smooth family of deformations of the Higgs bundle, then this does not represent a local minimum.

2.6 A theorem of Laumon

We conclude this section by a digression to the moduli space of flat \(G_C\) bundles. This space has a holomorphic symplectic form at its smooth points (it is hyper-Kähler). Of course the theory outlined above applies in this case. In particular, if the moduli space is smooth (for example the moduli space of stable Higgs vector bundles with rank and degree co-prime) one can consider the Morse flow on it. At a critical point, we again have the decomposition \(\text{Ad} P_C = \bigoplus_m U_m\). The subspace \(T_{\leq 0}\) of the tangent space to the moduli space on which the Hessian is less than or equal to zero is \(H^1\) of the following complex of sheaves

\[
\mathcal{O}\left( \bigoplus_{m \geq 0} U_m \right) \xrightarrow{\text{ad}(\Phi)} \mathcal{O}\left( \bigoplus_{m \geq 1} U_m \otimes K \right).
\]

The dimension of this is given by Riemann-Roch as

\[
\dim \mathbb{C} T_{\leq 0} = \chi\left( \mathcal{O}\left( \bigoplus_{m \geq 1} U_m \otimes K \right) \right) - \chi\left( \mathcal{O}\left( \bigoplus_{m \geq 0} U_m \right) \right)
= (g - 1) (\text{rk}(U_0) + \sum_{m \geq 1} 2 \text{rk}(U_m)) - \deg(U_0).
\]
But from (2.13) we have $\deg(U_0) = 0$ and $\text{rk}(U_0) + \sum_{m \geq 1} 2 \text{rk}(U_m) = \dim \mathfrak{g}_C$ so that

$$\dim \mathcal{T}_{\leq 0} = \dim \mathfrak{g}_C = \frac{1}{2} \dim \mathcal{M}_{G_C}.$$ 

It was pointed out by Hausel [14] that this fact, together with his theorem that the downwards Morse flow coincides with the nilpotent cone (the pre-image of 0 under the Hitchin map), implies a theorem of Laumon [18, Th. 3.1] in this context: The nilpotent cone in $\mathcal{M}$ is a Lagrangian subvariety with respect to the holomorphic symplectic form on $\mathcal{M}$.

### 3 Milnor-Wood inequalities

For any $G$ there is a locally constant obstruction map

$$o_2: \text{Hom}(\pi_1(\Sigma); G) \to H^2(\Sigma; \pi_1(G)).$$

Note that in both cases, $G = \text{SU}(n,n)$ and $G = \text{Sp}(2n,\mathbb{R})$, $\pi_1(G) = \mathbb{Z}$ so that we have an integer valued function. In the case of representations in $\text{Sp}(2n,\mathbb{R})$, we have $o_2(\rho) = c_1(V)$, where $V$ is the vector bundle appearing in the decomposition (2.6) of the Higgs bundle associated to $\rho$. In the case of $\text{SU}(n,n)$ representations, we have $o_2(\rho) = c_1(V)$, where $V$ is the vector bundle appearing in (2.4). In both cases we thus have an integer valued function $d = \deg(V) = \langle c_1(V), [\Sigma] \rangle$ whose fibres are unions of connected components. There is an outer automorphism of $\mathcal{M}_G$ given by exchanging $V$ with $V^*$ (in the $\text{Sp}(2n,\mathbb{R})$ case), or exchanging $V$ and $V'$ (in the $\text{SU}(n,n)$ case). Thus, in both cases we have an isomorphism between $o_2^{-1}(d)$ and $o_2^{-1}(-d)$, and it therefore suffices to consider the case $d \geq 0$, whenever convenient.

It is well known that there are bounds on the possible values of characteristic numbers of flat bundles, known as Milnor-Wood inequalities and, using Higgs bundles, we shall prove one for flat $\text{SU}(n,n)$- and $\text{Sp}(2n,\mathbb{R})$-bundles. The original inequality proved by J. Milnor [19] concerns $\text{SL}(2,\mathbb{R})$-bundles, while J. Wood [21] considered $\text{SU}(1,1)$-bundles. J. Dupont [4] found a bound for any semi-simple group with finite centre, however, the inequality of Proposition 3.1 below for $G = \text{Sp}(2n,\mathbb{R})$ is sharper than his. Using ideas of Gromov, A. Domic and D. Toledo [7] proved a general result for mappings of a surface into manifolds covered by bounded symmetric domains, and their work implies Proposition 3.1 below. Hitchin obtained a proof in the case of flat reductive $\text{SL}(2,\mathbb{R})$-bundles, using Higgs bundles, in [15], and we obtain our inequality in a similar way. The reason why we include the proof here is, that it gives crucial extra information about the poly-stable Higgs bundles of the form (2.5) and (2.6) (see Proposition 3.2).

**Proposition 3.1.** Let $\rho$ be a reductive representation of $\pi_1(\Sigma)$ in $\text{Sp}(2n,\mathbb{R})$ or $\text{SU}(n,n)$. Then the characteristic number $d = \langle o_2(\rho), [\Sigma] \rangle$ satisfies the inequality

$$|d| \leq n(g - 1).$$

**Proof.** We give the proof in case of $\text{Sp}(2n,\mathbb{R})$-representations, the $\text{SU}(n,n)$ case being completely analogous.
Let \((E, \Phi)\) be the poly-stable Higgs bundle of the form (2.6) corresponding to \(\rho\), as we already noticed \(d = \deg(V)\). Without loss of generality we can assume that \(d > 0\). In this case \(c \neq 0\), as otherwise \(V\) would be \(\Phi\)-invariant, and therefore violate the stability condition. Let \(U\) be the subbundle of \(V^*\), such that \(U \otimes K\) is the vector bundle generated by the image of \(c\). Similarly, let \(U' \subset V\) be the subbundle, which is generated by the kernel of \(c\). Then the bundles \(U'\) and \(V \oplus U\) are both \(\Phi\)-invariant. We therefore get the following inequalities from semi-stability of \((E, \Phi)\):

\[
\begin{align}
\deg(U') &\leq 0 \quad (3.1) \\
\deg(U') + \deg(U) &\leq 0. \quad (3.2)
\end{align}
\]

Note that these inequalities also hold in the case when \(U' = 0\) and \(U = V^*\). Next, we note that \(c\) induces a non-trivial global section of the linebundle

\[
\det(V/U')^{-1} \otimes \det(U \otimes K),
\]

which therefore has positive degree, i.e.

\[
\deg(U') - d + \deg(U) + (2g - 2) \text{rk}(c) \geq 0, \quad (3.3)
\]

where \(\text{rk}(c) = \text{rk}(U)\) is the generic rank of \(c\). Combining this with the inequalities (3.1) and (3.2), we obtain

\[
d \leq (g - 1) \text{rk}(c), \quad (3.4)
\]

so \(d \leq n(g - 1)\) as claimed.

The above proof gives some important additional information: from (3.4) it follows that \(\text{rk}(c) = n\) for \(d > (n - 1)(g - 1)\). In particular, in the extremal case \(d = n(g - 1)\), we have \(\text{rk}(c) = n\), and furthermore equality holds in (3.3). Hence, \(\det(c)\) is a non-zero section of a linebundle of degree 0, and we conclude that \(c\) is an isomorphism. We thus have the following proposition:

**Proposition 3.2.** Let \((E, \Phi)\) be the poly-stable Higgs bundle of the form (2.6) corresponding to a reductive representation of \(\pi_1(\Sigma)\) in \(\text{Sp}(2n, \mathbb{R})\). If \(\deg(V) = n(g - 1)\) then \(c: V \to V^* \otimes K\) is an isomorphism.

Let \((E, \Phi)\) be the poly-stable Higgs bundle of the form (2.3) corresponding to a reductive representation of \(\pi_1(\Sigma)\) in \(\text{SU}(n, n)\). If \(\deg(V) = n(g - 1)\) then \(c: V \to V' \otimes K\) is an isomorphism.

This has as a consequence that there is another discrete invariant on \(\mathcal{M}_{\text{Sp}(2n, \mathbb{R})}\) and we shall come back to this in Section 5.2.

### 4 Minima of \(f\)

In this section we determine the poly-stable Higgs bundles which represent local minima of the function \(f\) on \(\mathcal{M}_{G_r}\) in the cases \(G_r = \text{SU}(2, 2)\) and \(G_r = \text{Sp}(4, \mathbb{R})\). We shall determine
which stable Higgs bundles correspond to critical points of $f$ and then identify those which are local minima using \((2.17)\).

It will be convenient to consider the decomposition \((2.14)\), $E = \bigoplus_m F_m$ of the Higgs bundles $(E, \Phi)$ of the form \((2.5)\) and \((2.6)\), which then gives rise to the decomposition of the adjoint bundle: note that we have $\text{Ad} P_C = \text{End}(V \oplus V')_0$ (the subscript 0 indicating traceless endomorphisms) when $G_r = SU(n, n)$, while $\text{Ad} P_C = \text{End}(V) \oplus S^2V \oplus S^2V^*$ for $G_r = \text{Sp}(2n, \mathbb{R})$.

We begin by finding the minima on $\mathcal{M}_{SU(2,2)}$ which are stable Higgs bundles, leaving the reducible ones for later. As noted in Section 3, we only need to consider Higgs bundles $E = V \oplus V'$ with $\text{deg}(V) \geq 0$.

**Proposition 4.1.** The stable Higgs bundles of the form \((2.5)\) with $\text{deg}(V) \geq 0$, which correspond to a local minimum of $f$ on $\mathcal{M}_{SU(2,2)}$ are the ones which have $b = 0$, $c \neq 0$, and $\text{deg}(V) > 0$.

**Proof.** Let $(E, \Phi)$ be a Higgs bundle of the form \((2.5)\) which represents a critical point of $f$. $E$ comes from the standard representation of $S(U(2) \times U(2))$ on $\mathbb{C}^2 \oplus \mathbb{C}^2$ and the infinitesimal gauge transformation $\psi$ which produces the decomposition $E = \bigoplus_m F_m$ is fibrewise in $\mathfrak{su}(2) \times \mathfrak{su}(2))$. Hence each of the bundles $F_m$ is of the form $F_m = F_{m1} \oplus F_{m2}$ where $F_{mi} = V_i \cap F_m \subseteq V_i$ for $i = 1, 2$. We claim that either $F_{m1}$ or $F_{m2}$ must be zero (unless $E = U_0$). To see this, let $m_0$ be the smallest $m$ such that $F_{mi} = V_i \cap F_m \subseteq V_i$ for $i = 1, 2$. We claim that either $F_{m1}$ or $F_{m2}$ must be zero (unless $E = U_0$). To see this, let $m_0$ be the smallest $m$ such that $F_{m1}$ and $F_{m2}$ are both non-zero. Then $\Phi(U_{m_0-1})$ is contained in either $V$ or $V'$, since the same is true for $U_{m_0-1}$ and $\Phi$ interchanges $V$ and $V'$. Without loss of generality we may suppose that $\Phi(U_{m_0-1}) \subseteq V$.

Then each of the bundles

$$
\bigoplus_{m < m_0} F_m \oplus \left(V \cap \bigoplus_{m \geq m_0} F_m\right)
$$

and

$$
V' \cap \bigoplus_{m \geq m_0} F_m
$$

is $\Phi$-invariant, and so we have a decomposition of $(E, \Phi)$ as a direct sum of lower rank Higgs bundles. This is impossible because $(E, \Phi)$ is stable.

Let $r = (\text{rk}(F_m))$ be the rank vector whose entries are the ranks of the bundles $F_m$. We analyze the possibilities for $r$ case by case.

1st case: $r = (1, 1, 1, 1)$. Note that $0 = \text{tr}(\psi) = i \sum m \text{rk}(F_m)$. In this case we therefore have $E = F_{-3/2} \oplus F_{-1/2} \oplus F_{1/2} \oplus F_{3/2}$, where each $F$ is a linebundle. Hence the decomposition \((2.14)\) of $\text{Ad} P_C$ is of the form

$$
\text{Ad} P_C = U_{-3} \oplus \cdots \oplus U_3,
$$

where

$$
U_2 = \text{Hom}(F_{-3/2}, F_{1/2}) \oplus \text{Hom}(F_{-1/2}, F_{3/2}),
$$

$$
U_3 = \text{Hom}(F_{-3/2}, F_{3/2}).
$$
The formula (2.17) for the Morse index then takes the form

\[ \dim \mathbb{H}^1(C^\ast) = 3(g-1) + \deg(F_{3/2}) - \deg(F_{-3/2}) - (\deg(F_{1/2}) - \deg(F_{-3/2}) + \deg(F_{3/2}) - \deg(F_{-1/2})) \]
\[ = 3(g-1) + \deg(F_{-1/2}) - \deg(F_{1/2}). \]

Now we note that \( F_{1/2} \oplus F_{3/2} \) is a \( \Phi \)-invariant subbundle of \( E \) and thus, by stability, \( \deg(F_{1/2}) + \deg(F_{3/2}) < 0 \). Combining this with the above result we get

\[ \dim \mathbb{H}^1(C^\ast) > 3(g-1) + \deg(F_{-1/2}) + \deg(F_{3/2}). \]

But since \( \Phi \) interchanges \( V \) and \( V' \) we must have \( V = F_{-3/2} \oplus F_{1/2} \) and \( V' = F_{3/2} \oplus F_{-1/2} \) or vice-versa. Therefore \( |d| = |\deg(V)| = |\deg(V')| = |\deg(F_{-1/2}) + \deg(F_{3/2})| \). Combining this with the above inequality we get

\[ \dim \mathbb{H}^1(C^\ast) > 3(g-1) - |d| > 0, \]

where the last inequality comes from the Milnor-Wood inequality \( |d| \leq 2(g-1) \) of Proposition 3.1. We conclude that a critical point of this type always has strictly positive Morse index and hence it cannot be a local minimum of \( f \).

2nd case: \( r = (1, 2, 1) \). Again using \( \sum m \text{rk}(F_m) = 0 \) we see that in this case \( E = F_{-1} \oplus F_0 \oplus F_1 \). We then have \( \text{Ad} P_{C} = U_{-2} \oplus \cdots \oplus U_2 \) and

\[ U_2 = \text{Hom}(F_{-1}, F_1), \]

and so, from (2.17), we get

\[ \dim \mathbb{H}^1(C^\ast) = g - 1 - (\deg(F_1) - \deg(F_{-1})) \]
\[ = g - 1 - (2\deg(F_1) + \deg(F_0)), \]

where the second equality is due to the fact that \( \deg(E) = 0 \). Since \( F_0 \oplus F_1 \) and \( F_1 \) are \( \Phi \)-invariant we get from stability that \( \deg(F_0) + \deg(F_1) < 0 \) and \( \deg(F_1) < 0 \). Hence \( 2\deg(F_1) + \deg(F_0) < 0 \), which shows that \( \dim \mathbb{H}^1(C^\ast) > 0 \). Therefore a critical point of this type cannot be a minimum of \( f \) either.

3rd case: \( r = (1, 1, 2) \) (or \( r = (2, 1, 1) \)). In this case \( E = F_{m_1} \oplus F_{m_2} \oplus F_{m_3} \) where \( V = F_{m_1} \oplus F_{m_2} \) and \( V' = F_{m_3} \) (or vice-versa). Since \( \Phi \) interchanges \( V \) and \( V' \), it follows that \( \Phi|_{F_{m_1}} = 0 \) and so, \( (E, \Phi) \) is reducible. Thus this case cannot occur. The case \( r = (2, 1, 1) \) is analogous.

4th case: \( r = (2, 2) \). In this case \( E = F_{-1/2} \oplus F_{1/2} \). Then \( U_m = 0 \) for \( m \geq 2 \) and hence we see from (2.17) that these critical points are local minima of \( f \). Clearly \( F_{-1/2} = V \) and \( F_{1/2} = V' \), or vice-versa. If \( V = F_{1/2} \) it would be \( \Phi \)-invariant and so \( \deg(V) < 0 \) which is absurd. Thus, in fact, \( V = F_{-1/2} \) and \( V' = F_{1/2} \). In the notation of (2.5) this means that \( c = \Phi|_V \) and \( b = \Phi|_{V'} = 0 \). This gives the minima with \( \deg(V) > 0 \). Finally, note that if \( \deg(V) = \deg(V') = 0 \) then either \( V \) or \( V' \) is a \( \Phi \)-invariant subbundle which violates stability. Thus there are no stable Higgs bundles with \( \deg(V) = 0 \) which are local minima of \( f \). 

\[ \square \]
The fact that there is another discrete invariant for flat $\text{Sp}(2n, \mathbb{R})$-bundles (cf. Section 5.2) is reflected in the difference between the previous and the following result.

**Proposition 4.2.** The stable Higgs bundles of the form (2.6) with $\deg(V) \geq 0$, which correspond to a local minimum of $f$ on $\mathcal{M}_{\text{Sp}(4, \mathbb{R})}$ are the ones which have

1. $b = 0$, $c \neq 0$, and $\deg(V) > 0$.
2. $\deg(V) = 2g - 2$, $V = L_1 \oplus L_2$, and $\Phi$ of the form

$$
\begin{pmatrix}
0 & 0 & 0 & \tilde{c} \\
0 & 0 & \tilde{c} & 0 \\
0 & 0 & 0 & 0 \\
0 & \tilde{b} & 0 & 0
\end{pmatrix}
$$

with respect to the decomposition $E = V \oplus V^* = L_1 \oplus L_2 \oplus L_1^{-1} \oplus L_2^{-1}$.

**Proof.** This is analogous to the proof of Proposition 4.1. However, in this case the infinitesimal gauge transformation $\psi$ which produces the decomposition $E = \bigoplus F_m$ of the Higgs bundle of the form (2.6) belongs to $\Omega^0(\Sigma; \text{Ad} P_K)$, that is, it is fibrewise in $u(2)$. Thus there are only two possibilities: either $V = F_{-1/2}$ and $V^* = F_{1/2}$ with $\Phi: V \to V^* \otimes K$, that is, $b = 0$ (here we are using that $\deg(V) \geq 0$). These Higgs bundles are seen to be minima as before. The other possibility is that $V = F_{m_1} \oplus F_{m_2}$ and $V^* = F_{-m_1} \oplus F_{-m_2}$, where $F_{-m} = F_{m}^*$. Note that either $(m_1, m_2) = (-3/2, 1/2)$ or $(m_1, m_2) = (-1/2, 3/2)$. In this case the decomposition (2.14) has the form

$$
\text{Ad} P_C = U_{-3} \oplus \cdots \oplus U_3,
$$

where

$$
U_2 = \text{Hom}(F_{-3/2}, F_{1/2}) \cong \text{Hom}(F_{-1/2}, F_{3/2}), \\
U_3 = \text{Hom}(F_{-3/2}, F_{3/2}).
$$

From (2.17) we therefore get the Morse index

$$
\dim_{\mathbb{C}} \mathbb{H}^1(C^\bullet) = 2(g - 1) + \deg(F_{3/2}) - \deg(F_{-3/2}) - (\deg(F_{1/2}) - \deg(F_{-3/2})) \\
= 2(g - 1) - (\deg(F_{-3/2}) + \deg(F_{1/2})) \\
= 2(g - 1) \pm \deg(V).
$$

Thus we cannot have a minimum unless $\deg(V) = 2g - 2$, and in this case, from Proposition 3.2, we have $V = F_{-3/2} \oplus F_{1/2}$ since otherwise $c$ would not be of rank 2. This gives the second case of the proposition.

It remains to identify the local minima of $f$ which are not stable Higgs bundles.
Proposition 4.3. The reducible Higgs bundles of the form (2.3) with deg(V) \geq 0 which correspond to a local minimum of \( f \) on \( \mathcal{M}_{SU(2,2)} \) either have \( \Phi = 0 \) and deg(V) = deg(V') = 0, or, if \( \Phi \neq 0 \), they are direct sums of rank 2 Higgs bundles \( (E_1, \Phi_1) \) and \( (E_2, \Phi_2) \), where \( E_i = L_i \oplus L_i' \), \( L_i \) a line-bundle with deg(\( L_i \)) \geq 0 and \( \Phi_i: L_i \rightarrow L_i' \otimes K \). If \( \Phi_i \neq 0 \) then deg(\( L_i \)) > 0.

Proof. Let \( (E, \Phi) \) be a reducible Higgs bundle of the form (2.3) which is a local minimum of \( f \). Consider \( \mathcal{M}_{SU(2,2)} \) as the space of solutions \( (A, \Phi) \) to Hitchin’s equations modulo \( S(U(2) \times U(2)) \) gauge equivalence.

First consider the case \( \Phi = 0 \). Then, by poly-stability, deg(V) = deg(V') = 0, and V and V' are poly-stable vector bundles. On the other hand, it is clear that such Higgs bundles are, in fact, reducible (absolute) minima of \( f \). This gives the first case of the proposition.

Suppose now that \( \Phi \neq 0 \). The possible reductions of structure group are the following.

Reduction to \( S((U(1) \times U(1)) \times (U(1) \times U(1))) \). In this case we have \( V = L_1 \oplus L_2 \) for linebundles \( L_1 \) and \( L_2 \), while \( V' = L_1' \oplus L_2' \). Thus \( (E, \Phi) \) is the direct sum of two Higgs bundles \( (E_1, \Phi_1) \) and \( (E_2, \Phi_2) \), where \( E_i = L_i \oplus L_i' \), \( L_1L_1' \otimes L_2L_2' = \mathcal{O} \), and the Higgs field \( \Phi_i \) has zeros along the diagonal. Note also that deg(\( E_i \)) = 0 by poly-stability of \( (E, \Phi) \). Each of the bundles \( (E_i, \Phi_i) \) is a minimum on the moduli space of rank 2 Higgs bundles of this form and hence of the form (2.14), in other words all components of \( \Phi_i \) are zero, except one off-diagonal entry. (cf. Hitchin [15], Section 10).

There are now two cases to consider. The first case is when \( \Phi \) is zero on one of the bundles \( V \) and \( V' \); since deg(V) \geq 0 we must have \( \Phi: V \rightarrow V' \otimes K \). In other words, \( \Phi \) is of the form

\[
\begin{pmatrix}
0 & 0 \\
c & 0
\end{pmatrix}
\]

with respect to the decomposition \( E = V \oplus V' \). Thus \( (E, \Phi) \) is of the form considered in Proposition [14]. As in the proof of that proposition one sees that there is no subspace of the Zariski tangent space with negative weights and, therefore, these Higgs bundles represent local minima of \( f \). This case includes the case of one of the \( \Phi_i \) being equal to zero. Note that, if deg(\( L_i \)) = 0 then it follows from (2.16) and the remark following it that \( \Phi_i = 0 \). This case gives the remaining local minima of the statement of the proposition.

The other case is when \( \Phi \) is non-zero on both \( V \) and \( V' \), say that \( \Phi_1: L_1 \rightarrow L_1' \otimes K \) and \( \Phi_2: L_2' \rightarrow L_2 \otimes K \). By stability, and since \( \Phi \neq 0 \), we then have deg(L_1') < 0 and deg(L_2) < 0, and so deg(L_1) > 0 and deg(L_2') > 0. We shall show that in this case \( (E, \Phi) \) is not a local minimum of \( f \). Let the infinitesimal gauge transformation \( \psi \) which produces the decomposition \( E_i = L_i \oplus L_i' \) of (2.14) have weights \( m_i \) on \( L_i \), and weights \( m_i' \) on \( L_i' \). We then have the following equations relating these numbers:

\[
m_1' = m_1 + 1, \\
m_2 = m_2' + 1,
\]

From these equations it follows that \((m_2 - m_1) + (m_1' - m_2') = 2\) and hence, either \( m_2 - m_1 \geq 1 \) or \( m_1' - m_2' \geq 1 \). For definiteness suppose that \( m_2 - m_1 \geq 1 \) (the other case is entirely
similar). This means that
\[ \text{Hom}(L_1, L_2) \subseteq \text{Ad}_{\mathfrak{c}} P_K = (\text{End}(V) \oplus \text{End}(V'))_0 \]
has weight \( \geq 1 \) and that this is a subspace of the highest weight space of \( \psi \). Note that \( \text{ad}(\Phi) \) is zero restricted to the highest weight space and so \( H^1(\Sigma; \text{Hom}(L_1, L_2)) \) gives a subspace of \( \mathbb{H}^1(C^*) \) on which \( \tilde{f} \) is negative. But since \( \text{deg}(L_1) \geq 0 \) and \( \text{deg}(L_2) < 0 \) we have \( H^0(\Sigma; \text{Hom}(L_1, L_2)) = 0 \) and therefore, from Riemann-Roch, \( H^1(\Sigma; \text{Hom}(L_1, L_2)) \neq 0 \).
It only remains to find a smooth family of Higgs bundles in \( \mathcal{M}_{\text{SU}(2,2)} \) to which an element in \( H^1(\Sigma; \text{Hom}(L_1, L_2)) \) is tangent (cf. Section 2.5). By hypothesis \( (E, \Phi) \) is the direct sum of the stable Higgs bundles \( (E_1, \Phi_1) \) and \( (E_2, \Phi_2) \). All extensions
\[ 0 \rightarrow E_2 \rightarrow E \rightarrow E_1 \rightarrow 0 \]
are parametrized by \( H^1(\Sigma; \text{Hom}(E_1, E_2)) \) so, in particular, \( t \in H^1(\Sigma; \text{Hom}(L_1, L_2)) \) defines an extension
\[ 0 \rightarrow E_2 \rightarrow E_t \rightarrow E_1 \rightarrow 0 \]
which is non-trivial if \( t \neq 0 \). Note that \( E = V_t \oplus V' \), where \( V_t \) is the non-trivial extension
\[ 0 \rightarrow L_2 \rightarrow V_t \rightarrow L_1 \rightarrow 0 \]
defined by \( t \). We define a Higgs field \( \Phi = \begin{pmatrix} 0 & b_t \\ 0 & 0 \end{pmatrix} \) on \( E_t \) of the appropriate form in the following way. To define \( b_t: V' \rightarrow V_t \otimes K \) we use the composition
\[ V' \xrightarrow{b} L_2 \otimes K \rightarrow V_t \otimes K, \]
while to define \( c_t: V_t \rightarrow V' \otimes K \) we use the composition
\[ V_t \rightarrow L_1 \xrightarrow{c} V' \otimes K. \]
For \( t \neq 0 \) the Higgs bundle \( (E_t, \Phi_t) \) is a non-trivial extension of stable Higgs bundles and therefore stable. For \( \alpha \in \mathbb{C} \), the family \( (E_{\alpha t}, \Phi_{\alpha t}) \) is thus a smooth family of Higgs bundles in \( \mathcal{M}_{\text{SU}(2,2)} \) to which \( t \in H^1(\Sigma; \text{Hom}(L_1, L_2)) \) is tangent.

Reduction to \( S((U(1) \times U(1)) \times U(2)) \). In this case we have a decomposition of \( (E, \Phi) \) as a direct sum of Higgs bundles \( (E_1, \Phi_1) \) and \( (E_2, \Phi_2) \), where \( E_1 = V \oplus L_1 \) and \( E_2 = L_2 \) with \( L_1 \) and \( L_2 \) linebundles. Again \( (E_1, \Phi_1) \) and \( (E_2, \Phi_2) \) represent local minima on lower rank moduli spaces and so, \( \Phi_2 = 0 \). If \( (E_1, \Phi_1) \) is reducible we are back in one of the previous cases so we may assume that \( (E_1, \Phi_1) \) is stable. Since we are at a minimum it must be of the form \( \mathbb{Z} \mathbb{I} \mathbb{A} \) and again there are several possibilities. If \( \Phi_1 \) is zero on either \( V \) or \( L_1 \) then it is zero on either \( V \) or \( V' \) and \( (E, \Phi) \) is a minimum as above. Thus the only case that remains is when \( V = F_{-1} \oplus F_1 \) and \( \Phi: F_{-1} \rightarrow L_1 \otimes K \), and \( \Phi: L_1 \rightarrow F_1 \otimes K \). The weights of the infinitesimal gauge transformation producing this decomposition are \(-1, 0, \) and \( 1 \) on \( F_{-1}, L_1, \) and \( F_1 \), respectively. As above one sees that \( H^1(\Sigma; \text{Hom}(F_{-1}, F_1)) \) gives a subspace of \( \mathbb{H}^1(C^*) \) on which \( \tilde{f} \) is negative and that \( t \in H^1(\Sigma; \text{Hom}(F_{-1}, F_1)) \) is tangent to a smooth family of Higgs bundles, so that \( (E, \Phi) \) does not represent a local minimum. We omit the details.

Reduction to \( S(U(2) \times (U(1) \times U(1))) \). This case is analogous to the previous one. \( \square \)
In an analogous manner one can prove the following proposition.

**Proposition 4.4.** The reducible Higgs bundles of the form (2.6) with \( \text{deg}(V) \geq 0 \) which correspond to a local minimum of \( f \) on \( \mathcal{M}_{\text{Sp}(2,\mathbb{R})} \) either have \( \Phi = 0 \) and \( \text{deg}(V) = 0 \), or, if \( \Phi \neq 0 \), they are direct sums of rank 2 Higgs bundles \((E_1, \Phi_1)\) and \((E_2, \Phi_2)\), where \( E_i = L_i \oplus L_i^{-1} \), \( L_i \) a line-bundle with \( \text{deg}(L_i) \geq 0 \) and \( \Phi_i : L_i \to L_i^{-1} \otimes K \). If \( \Phi_i \neq 0 \) then \( \text{deg}(L_i) > 0 \).

\[ \square \]

## 5 Connected Components

### 5.1 Components of \( \mathcal{M}_{\text{SU}(2,2)} \)

In this section we consider the connected components of \( \mathcal{M}_{\text{SU}(2,2)} \). Using Proposition 3.1 we can write

\[ \mathcal{M}_{\text{SU}(2,2)} = \mathcal{M}_{-(2g-2)} \cup \cdots \cup \mathcal{M}_{2g-2}, \]

where \( \mathcal{M}_d \), the subspace of Higgs bundles of the form 2.5 with \( \text{deg}(V) = d \), is a union of connected components. Denote the subspace of local minima of \( f \) on \( \mathcal{M}_d \) by \( \mathcal{N}_d \). Note that, since \( f \) is proper, connectedness of \( \mathcal{N}_d \) implies connectedness of \( \mathcal{M}_d \). As noted in Section 3 we can without loss of generality assume that \( d \geq 0 \). The results of the previous section then give the following identification of \( \mathcal{N}_d \).

**Proposition 5.1.** The subspace of local minima of \( f \) on \( \mathcal{M}_d \), \( \mathcal{N}_d \), is the space of poly-stable Higgs bundles of the form (2.5) with \( b = 0 \).

**Proof.** Immediate from Propositions 4.1 and 4.3.  

We can use this to identify \( \mathcal{N}_d \) with a moduli space of triples, as studied by Bradlow and García-Prada [5, 10]. Denote the moduli space of stable triples \((V, \tilde{V}', \phi)\) (where \( \phi \in H^0(\Sigma; \text{Hom}(V, \tilde{V}')) \), \( \text{deg}(V) = d \), \( \text{deg}(\tilde{V}') = 2g - 2 - d \), and \( \text{rk}(V) = \text{rk}(\tilde{V}') = 2 \)) by \( \mathcal{M}_d^{\text{triples}} \) (cf. Section 2.3). To each such triple we associate a Higgs bundle \((V \oplus V', (\phi_0, 0))\), where \( V' = \tilde{V}' \otimes K^{-1} \). The following theorem is then an immediate consequence of Theorem 2.3.

**Theorem 5.2.** \( \mathcal{N}_d \) is isomorphic to the fibre, over the trivial bundle \( O \), of the map

\[ \mathcal{M}_d^{\text{triples}} \to \text{Jac}(\Sigma) \]

\[ (V, \tilde{V}', \phi) \mapsto \Lambda^2(V) \otimes \Lambda^2(\tilde{V}') \otimes K^{-2}. \]

\[ \square \]

Thus information about connectedness of moduli spaces of stable triples would give information about connectedness of the \( \mathcal{M}_d \) and we hope to come back to this on a later occasion. At present we can prove the following theorem.

**Theorem 5.3.** The subspaces \( \mathcal{M}_d \) of \( \mathcal{M}_{\text{SU}(2,2)} \) are connected for \( d = 0 \) and \( d = \pm(2g-2) \).
Proof. First consider the case \( d = 0 \). To see that \( \mathcal{N}_0 \) is connected, consider the continuous map

\[
\mathcal{N}_0 \times \mathcal{N}_0 \times \text{Jac}(\Sigma) \to \mathcal{N}_0
\]

\[
(E, E', L) \mapsto ((E \otimes L) \oplus (E' \otimes L^{-1}), 0),
\]

where \( \mathcal{N}_0 \) denotes the moduli space of rank 2 poly-stable vector bundles with fixed trivial determinant bundle. From Proposition \[5.1\] we see that this is surjective and, since \( \mathcal{N}_0 \) and \( \text{Jac}(\Sigma) \) are connected, that \( \mathcal{N}_0 \) is connected.

Next consider the case \( d = 2g - 2 \) (as already noticed, this also takes care of the case \( d = -(2g - 2) \)). From Propositions \[3.2\] and \[5.1\] we see that \( \mathcal{N}_{2g-2} \) is isomorphic to the moduli space of rank 2, degree \( 2g - 2 \) vector bundles with fixed determinant, which is known to be connected.

5.2 Components of \( \mathcal{M}_{\text{Sp}(4, \mathbb{R})} \)

In this section we consider the connected components of \( \mathcal{M}_{\text{Sp}(4, \mathbb{R})} \). Again using Proposition \[3.1\] we can write

\[
\mathcal{M}_{\text{Sp}(4, \mathbb{R})} = \mathcal{M}_{-2g-2} \cup \cdots \cup \mathcal{M}_{2g-2},
\]

where \( \mathcal{M}_d \), the subspace of Higgs bundles of the form \[2.6\] with \( \deg(V) = d \), is a union of connected components. Again we denote the subspace of local minima of \( f \) on \( \mathcal{M}_d \) by \( \mathcal{N}_d \), and connectedness of \( \mathcal{N}_d \) implies connectedness of \( \mathcal{M}_d \).

We can also identify \( \mathcal{N}_d \) with a moduli space of triples, using Theorem \[2.3\], as follows.

**Theorem 5.4.** For \( -(2g - 2) \leq d \leq 2g - 2 \), \( \mathcal{N}_d \) is the isomorphic to the fixed point set of the involution on the moduli space \( \mathcal{M}_d^{\text{triples}} \) of poly-stable triples (as defined in the previous section), defined by

\[
\mathcal{M}_d^{\text{triples}} \to \mathcal{M}_d^{\text{triples}}
\]

\[
(V, \tilde{V}', \phi) \mapsto \Lambda^2(V) \otimes \Lambda^2(\tilde{V}') \otimes K^{-2}.
\]

With regard to connectedness, we consider the cases \(|d| < 2g - 2 \) and \(|d| = 2g - 2 \) separately.

**The case \(|d| < 2g - 2 \).** In this case everything is completely analogous to the case of \( \text{SU}(2, 2) \)-bundles. To begin with, we have the following result.

**Proposition 5.5.** For \(|d| < 2g - 2 \), the subspace of local minima of \( f \) on \( \mathcal{M}_d \), \( \mathcal{N}_d \), is the space of poly-stable Higgs bundles of the form \[2.7\] with \( b = 0 \).

**Proof.** Follows from Propositions \[4.2\] and \[4.4\].

We have the following result about connectedness of \( \mathcal{M}_0 \).
Theorem 5.6. The subspace $\mathcal{M}_0$ of $\mathcal{M}_{\text{Sp}(4,\mathbb{R})}$ is connected.

Proof. From Proposition 5.5 it follows in particular that $\mathcal{N}_0$ is isomorphic to the moduli space of rank 2, degree 0 poly-stable vector bundles. Since this space is connected, the result is proved. □

The case $|d| = 2g - 2$. In this case the results are entirely different from those of $\text{SU}(2,2)$-bundles, due to Proposition 4.2.

Let $(E, \Phi)$ be a Higgs bundle of the form (2.6) with $d = n(g - 1)$. Choosing a square root $L_0$ of the canonical bundle on $\Sigma$, we can define a rank $n$ vector bundle $W$ by

$$W = V \otimes L_0^{-1},$$

and we can define $C \in H^0(\Sigma; S^2W^*)$ and $\phi \in H^0(\Sigma; \text{End}(W) \otimes K^2)$ by

$$C = c \otimes 1_{L_0^{-1}},$$

and

$$\phi = (b \otimes 1_{L_0}) \circ (c \otimes 1_{L_0^{-1}}).$$

Note that $\phi$ is symmetric with respect to the quadratic form $C$.

From Proposition 3.2 we know that $c$ is an isomorphism when $(E, \Phi)$ is poly-stable, and thus we can recover $(E, \Phi)$ from this data. Therefore the set of isomorphism classes of Higgs bundles of the form (2.6) is equal to the set of isomorphism classes of Higgs bundles

$$(W, C, \phi), \quad (5.1)$$

where $W$ has a non-degenerate quadratic form $C$, and the Higgs field $\Phi$ is twisted by $K^2$ and symmetric with respect to $C$. There is an obvious stability condition for $(W, C, \phi)$, namely that

$$\mu(U) < \mu(W) \quad (5.2)$$

for all $\phi$-invariant subbundles $U$ of $W$. Next, we shall prove that $(W, C, \phi)$ is stable, if and only if $(E, \Phi)$ is.

Theorem 5.7. The subspace $\mathcal{M}_{n(g-1)} \subset \mathcal{M}_{\text{Sp}(2n,\mathbb{R})}$ of Higgs bundles of the form (2.6), with $d = n(g - 1)$ is isomorphic to the moduli space of poly-stable Higgs bundles of the form (5.1).

Proof. We have to prove that $(E, \Phi)$ is stable if and only if $(W, C, \phi)$ is. From Theorem 2.3 we know that stability of $(E, \Phi)$ is equivalent to stability of the $Q$-bundle $E = (E, \Phi)$. Thus, all we need to prove is that $E$ is stable if and only if $(W, C, \phi)$ is. Because stability is unaffected by tensoring with a line bundle, we can equally well prove that $(V, b \circ c)$ is stable. Note, that $\mu(V) = g - 1$.

Assume $E$ is a stable $Q$-bundle. Let $U \subset V$ be a $\phi$-invariant subbundle. Let $U' \subset V^*$ be the subbundle such that $U' \otimes K$ is generically the image of $U$ under $c$. Then $b$ maps $U'$
to $U$, because of the $\phi$-invariance of $U$. Hence, $F = (\{U, U'\}, \{b, c\})$ defines a $Q$-subbundle of $E$, and it follows that

$$\mu(F) < \mu(E).$$

But, as $c$ is an isomorphism

$$\mu(E) = \mu(E) = \mu(V \oplus V \otimes K^{-1}) = \mu(V) - (g-1),$$

and similarly $\mu(F) = \mu(U) - (g - 1)$. Therefore $\mu(U) < \mu(V)$ and so, $(W, C, \phi)$ is stable.

Conversely, assume that $(W, C, \phi)$ is stable. Let $F = (\{U, U'\}, \{b, c\})$ be a $Q$-subbundle of $E$. Let $\tilde{U} \subset V^*$ be the subbundle which is generically the image of $U' \otimes K$ under $c^{-1}$. Both $U$ and $\tilde{U}$ are $\phi$-invariant subbundles of $V$, because $F$ is a $Q$-subbundle. Hence, $\mu(U) < \mu(V)$ and $\mu(U') < \mu(V)$, by stability of $(W, C, \phi)$. Recalling that $\mu(V) = g - 1$ and $\mu(U) = \mu(U') - (2g - 2)$, we get

$$\mu(U) < g - 1, \quad (5.4)$$

and

$$\mu(U') < -(g - 1). \quad (5.5)$$

Note also that

$$\text{rk}(U') \geq \text{rk}(U), \quad (5.6)$$

because $c$ is an isomorphism, and the image of $U$ under $c$ is contained in $U' \otimes K$, by the assumption that $F$ is a $Q$-subbundle. Combining (5.4), (5.5), and (5.6), we get:

$$\mu(F) = \mu(U \oplus U')$$

$$= \frac{\text{rk}(U)}{\text{rk}(U \oplus U')} \mu(U) + \frac{\text{rk}(U')}{\text{rk}(U \oplus U')} \mu(U')$$

$$< \frac{\text{rk}(U) - \text{rk}(U')}{\text{rk}(U \oplus U')} (g - 1)$$

$$\leq 0.$$

Of course, $\mu(E) = 0$ and hence the proof is finished.

The existence of the quadratic form $C$ on $W$ means that the structure group is $O(n, \mathbb{C})$. The maximal compact subgroup of $O(n, \mathbb{C})$ is $O(n)$ and, therefore, we have the Stiefel-Whitney classes $w_1$ and $w_2$ as topological invariants. We now specialize to the case $n = 2$. The first Stiefel-Whitney class can then be seen in holomorphic terms as follows: the quadratic form $C$ gives an isomorphism $(\Lambda^2 W)^2 \cong \mathcal{O}$; hence, $\Lambda^2 W$ gives an element of $H^1(\Sigma; \mathbb{Z}/2)$, and it is easy to see that this element is $w_1(W)$. It follows that $\Lambda^2 W = \mathcal{O}$
if and only if \( w_1(W) = 0 \). This, in turn, is equivalent to the existence of a reduction of structure group to \( \text{SO}(2, \mathbb{C}) \subset \text{O}(2, \mathbb{C}) \). Using the identification \( \mathbb{C}^* \cong \text{SO}(2, \mathbb{C}) \) via

\[
\lambda \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},
\]

we see that this happens exactly when \( W \) decomposes as a direct sum

\[
W = L \oplus L^{-1},
\]

and \( C \) is of the form

\[
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

with respect to this decomposition. Now it is clear that, in this case, \( w_2(W) \) is given by

\[
w_2 = c_1(L) \mod 2.
\]

By interchanging \( L \) with its dual if necessary, we may assume that \( \deg(L) \geq 0 \). Furthermore, when \( \deg(L) > 0 \), the Higgs field \( \phi \) must induce a non-zero holomorphic map

\[
L \to L^{-1}K^2,
\]

because otherwise \( L \subset W \) would violate stability. Hence, we have

\[
\deg(L) \leq 2g - 2.
\]

We, therefore, have a decomposition of \( \mathcal{M}_{2g-2} \) into subspaces, each of which is a union of connected components, as follows:

\[
\mathcal{M}_{2g-2} = \left( \bigcup_{u, v} M_u^v \right) \cup \left( \bigcup_{l=0}^{2g-2} M_0^l \right),
\]

where \( M_u^v \) is the moduli space of poly-stable Higgs bundles \( (W, C, \phi) \) with \( w_1(W) = u \in H^1(\Sigma; \mathbb{Z}/2) - \{0\} \) and \( w_2(W) = v \in H^2(\Sigma; \mathbb{Z}/2) \), and where \( M_0^l \) is the moduli space of poly-stable Higgs bundles \( (W, C, \phi) \) with \( w_1(W) = 0 \) and \( \deg(L) = l \). We shall prove that each of these subspaces is connected, except \( M_0^{2g-2} \): in this case the Higgs field \( \phi \) induces a non-zero section of the degree 0 linebundle \( L^{-2}K^2 \) and thus \( L^2 = K^2 \). We therefore have a further decomposition of of \( M_0^{2g-2} \) into subspaces \( M_{0,L}^{2g-2} \), indexed by the \( 2g \) square roots \( L \in \text{Jac}^{2g-2}(\Sigma) \) of \( K^2 \) (note the analogy with the breaking up of \( \mathcal{M}_{2g-2} \) into several connected components).

We can now state our main result, to be proved in the remaining part of this section.

**Theorem 5.8.**

i) The spaces \( M_{0,L}^{2g-2} \) are connected.

ii) The spaces \( M_0^l \) are connected for \( 0 \leq l < 2g - 2 \).
Remark 5.9. Hitchin showed in \[17\] that for any split real form $G_r$ of a complex simple Lie group the moduli space of reductive representations of $\pi_1(\Sigma)$ in $G_r$ contains a connected component which is homeomorphic to an Euclidean space of dimension $(2g-2)\dim G_r$. This component is called the Teichmüller component. The group $\text{Sp}(4, \mathbb{R})$ is a split real form of $\text{Sp}(4, \mathbb{C})$ so there is a Teichmüller component in this case. As a matter of fact, each of the subspaces $M_{2g-2}^{2g-2}$ is isomorphic to a vector space: note that $W = L \oplus L^{-1}$ is completely determined by $L$ and that any $(W, C, \phi)$ is stable. Hence $M_{2g-2}^{2g-2}$ is isomorphic to the vector space

$$H^0(\Sigma; K^2) \oplus H^0(\Sigma; K^2) \oplus H^0(\Sigma; K^4).$$

Note that this proves $i)$ of the theorem.

Remark 5.10. One can see (see \[13\] for details), that the subspaces $M_d^l$, $M_d^v$, $M_{l0}^l$, and $M_{2g-2}^{2g-2}$ are non-empty. Therefore, Theorem 5.8 shows that $\mathcal{M}_{\text{Sp}(4, \mathbb{R})}$ has at least $3 \cdot 2^{2g} + 8g - 13$ connected components.

Proof that the subspaces $M_0^l \subset M_{2g-2}$ are connected. Recall that any $(W, C, \phi)$ in $M_0^l$ is of the form

$$W = L \oplus L^{-1},$$

with $l = \deg(L)$ and $C$ of the form $(0 1\ 1 0)$. First, we consider the case of $l > 0$. In this case, the Higgs field $\phi$ must be non-zero, as otherwise the subbundle $L \subset W$ would violate stability. But any critical point of the type described in $i)$ of Proposition 4.2 has $\phi = 0$ so, it follows that all the critical points in $M_0^l$ for $l > 0$ are of the type described in $ii)$ of Proposition 4.2. We therefore see that the critical points correspond to Higgs bundles $(W, C, \phi)$, which are of the form described above and where, furthermore, $\phi$ is of the form

$$\phi = \begin{pmatrix} 0 & 0 \\ \tilde{\phi} & 0 \end{pmatrix},$$

with $\tilde{\phi} \in H^0(\Sigma; L^{-2}K^2)$. Using this, it is now easy to give an explicit description of the subspace of local minima of $f$ on $M_0^l$.

Proposition 5.11. The subspace of local minima $N_0^l \subset M_0^l$ fits into a pull-back diagram

$$
\begin{array}{ccc}
N_0^l & \longrightarrow & \text{Jac}^l(\Sigma) \\
\downarrow \pi & & \downarrow L \rightarrow L^{-2}K^2 \\
S^{4g-4-2l}\Sigma & \xrightarrow{D-\epsilon[D]} & \text{Jac}^{4g-4-2l}(\Sigma),
\end{array}
$$

where $\pi(W, C, \phi) = (\phi)$. 

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Proof. The only thing there is to remark is that any \((W,C,\phi)\), of the form given above, is stable. But, \(L^{-1} \subset W\) is the only \(\phi\)-invariant subbundle so, this is obvious.

From this proposition, it is clear that \(N^l_0\) is connected so, from the properness of \(f\), it follows that \(M^l_0\) is connected for \(l > 0\).

In the case \(l = 0\), we have the following result.

**Proposition 5.12.** Any local minimum of \(f\) on \(M^0_0\) has \(\phi = 0\) and is, therefore, of the type described in \(i)\) of Proposition 4.2.

**Proof.** Suppose we have a critical point of the type described in \(ii)\) of Proposition 4.2, with \(\phi \neq 0\). Then, \(L^{-1} \subset W\) is \(\phi\)-invariant and therefore, \((W,C,\phi)\) is semi-stable, but not stable. Since we are considering the moduli space of poly-stable Higgs bundles, \((W,C,\phi)\) decomposes as a direct sum of rank 1 Higgs bundles of degree 0. The only subbundles of \(W\) of rank 1 and degree 0 are \(L\) and \(L^{-1}\), and \(L\) is not \(\phi\)-invariant so, we conclude that this situation cannot occur.

Consequently, we have the following description of the subspace of local minima of \(f\) on \(M^0_0\).

**Proposition 5.13.** The subspace \(N^0_0 \subset M^0_0\) of local minima of \(f\) is isomorphic to the moduli space of poly-stable \((W,C)\), where \(W\) is of the form

\[
W = L \oplus L^{-1},
\]

for a linebundle \(L\) of degree 0, and \(C\) is of the form \((0 1 1 0)\), with respect to this decomposition.

Note that the pair \((W,C)\) decomposes into a direct sum of Higgs linebundles exactly when \(L^2 = \mathcal{O}\), and it is then poly-stable, but not stable. All other \((W,C)\) are stable. It follows that there is a surjective continuous map

\[
\text{Jac}^0(\Sigma) \rightarrow N^0_0,
\]

given by taking \(L\) to \((W,C)\) of the form given above. Therefore, \(N^0_0\) is connected, finishing the proof that the subspaces \(M^l_0\) are connected.

**Proof that the subspaces** \(M^v_u \subset M_{2g-2}\) **are connected.** We begin by noting that the \((W,C,\phi)\) corresponding to a critical point of the type described in \(ii)\) of Proposition 4.2 has \(w_1(W) = 0\), thus we see that the subspaces of local minima \(N^v_u \subset M^v_u\) consist of critical points of the type described in \(i)\) of Proposition 4.2. Recall that for these \(b = 0\); in terms of the Higgs bundle \((W,C,\phi)\), this means that \(\phi = 0\). Thus, \(N^v_u\) is the moduli space of stable pairs \((W,C)\) with the given characteristic classes. From \(\Lambda^2W \neq \mathcal{O}\), one sees easily that any such pair is stable.

There is a connected double cover \(\tilde{\Sigma} \xrightarrow{\pi} \Sigma\) given by

\[
w_1(W) \in H^1(\Sigma; \mathbb{Z}/2) = \text{Hom}(\pi_1\Sigma, \mathbb{Z}/2).
\]
Clearly, the pull-back of $W$ to $\tilde{\Sigma}$ is of the form $\pi^* W = M \oplus M^{-1}$ with $\pi^* C = \left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)$ and $M$ a linebundle. Let $\tau: \tilde{\Sigma} \to \tilde{\Sigma}$ be the involution interchanging the sheets of the covering, then, clearly,

$$\tau^* M = M^{-1}.$$ 

Conversely, if $M$ is a linebundle on $\tilde{\Sigma}$ which satisfies this condition, then $W = \pi_* M$ is a rank 2 vector bundle with a non-degenerate quadratic form $C$. In fact $\tilde{\Sigma}$ is the spectral curve associated to $(W, C)$ (see Hitchin [1] and Beauville, Narasimhan and Ramanan [2]). Hence $N_u^0 \cup N_u^1$ can be identified with the kernel of the map

$$1 + \tau^*: \text{Jac}(\tilde{\Sigma}) \to \text{Jac}(\tilde{\Sigma}),$$

where $\tilde{\Sigma}$ is the unramified double cover of $\Sigma$ given by $u \in H^1(\Sigma; \mathbb{Z}/2)$. It remains to distinguish between $w_2$ being equal to 0 or 1. When the cover is unramified, the kernel of $1 + \tau^*$ splits into two components,

$$\ker(1 + \tau^*) = P^+ \cup P^-,$$

each of them a translate of the Prym variety of the covering. It is a classical theorem of Wirtinger, that the function $\delta: P^+ \cup P^- \to \mathbb{Z}/2$, defined by

$$\delta(M) = \dim \mathbb{C} H^0(\tilde{\Sigma}; M \otimes \pi^* L_0) \mod 2$$

$$= \dim \mathbb{C} H^0(\tilde{\Sigma}; \pi_* M \otimes L_0) \mod 2,$$

is constant on each of $P^+$ and $P^-$ and takes different values on them. For proofs of these facts, see Mumford [20] or [21].

Now, let $F \to \Sigma$ be a real vector bundle. Choosing a metric on $F$, the complexification $F^c = F \otimes_{\mathbb{R}} \mathbb{C}$ acquires a holomorphic structure and therefore, there is a $\overline{\partial}$-operator

$$\overline{\partial}_{L_0}(F): \Omega^0(\Sigma; L_0 \otimes F^c) \to \Omega^{0,1}(\Sigma; L_0 \otimes F^c).$$

Atiyah [1] shows that the function

$$\delta_{L_0}(F) = \dim \mathbb{C} \ker(\overline{\partial}_{L_0}(F)) \mod 2$$

is independent of the choice of the metric, and that it extends to give a group homomorphism

$$\delta_{L_0}: KO(\Sigma) \to \mathbb{Z}/2.$$ 

Define $\gamma \in KO(S^2)$ to be the pull-back of the generator of $KO(S^2)$ under a map $\tilde{\Sigma} \to S^2$ of degree 1. Atiyah [1, Lemma (2.3)] shows that

$$\delta_{L_0}(\gamma) = 1.$$ 

Furthermore, the total Stiefel-Whitney class gives an isomorphism

$$w: KO(\Sigma) \to \{1\} \oplus H^1(\Sigma; \mathbb{Z}/2) \oplus H^2(\Sigma; \mathbb{Z}/2).$$
of the additive group $\widetilde{KO}(\Sigma)$ onto the multiplicative group of the cohomology ring $H^{*}(\Sigma; \mathbb{Z}/2)$ (see \[1\], Remark, p. 54). Clearly,

$$w(\gamma) = (1, 0, 1),$$

where we identify $H^{2}(\Sigma; \mathbb{Z}/2) = \mathbb{Z}/2$. We may, therefore, think of $\delta_{L_{0}}$ as a homomorphism of the multiplicative group of $H^{*}(\Sigma; \mathbb{Z}/2)$ to $\mathbb{Z}/2$, which takes the value 1 on the element $(1, 0, 1)$. Let $u \in H^{1}(\Sigma; \mathbb{Z}/2)$; then,

$$(1, u, 0) = (1, u, 1) \cdot (1, 0, 1)$$

in $H^{*}(\Sigma; \mathbb{Z}/2)$. Therefore,

$$\delta_{L_{0}}(1, u, 0) = \delta_{L_{0}}(1, u, 1) + 1. \quad (5.7)$$

Returning to $(W, C)$ with $W = \pi_{*}M$ for $M \in \ker(1 + \tau^{*})$, we see that

$$\delta(M) = \delta_{L_{0}}(W^{r}),$$

where $W^{r}$ is a real rank two bundle, whose complexification is $W$. It follows from (5.7), that $\delta$ takes different values for different values of $w_{2}(W)$ and hence, that $w_{2}(W)$ determines whether $M$ lies in $P^{+}$ or $P^{-}$.

From this discussion, we obtain the following explicit description of the subvariety $N_{u}^{v} \subset M_{u}^{w}$ of local minima of $f$.

**Proposition 5.14.** Let $u \in H^{1}(\Sigma; \mathbb{Z}/2) - \{0\}$, let $v \in H^{2}(\Sigma; \mathbb{Z}/2) = \mathbb{Z}/2$ and let $P^{+}$ and $P^{-}$ be the Abelian varieties associated to the double cover of $\Sigma$, given by $u$ as above. Then, the subvariety $N_{u}^{v} \subset M_{u}^{w}$ of local minima of $f$ is equal to $P^{+}$ and $P^{-}$, respectively, for the two values of $v$. \[ \square \]

Consequently, $N_{u}^{v}$ is connected and, from the properness of $f$, it follows that $M_{u}^{w}$ is connected, finally finishing the proof of Theorem 5.8.

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