Abstract

The exact effective field equations of motion, corresponding to the perturbative mixed theory of open and closed (2,2) world-sheet supersymmetric strings, are investigated. It is shown that they are only integrable in the case of an abelian gauge group. The gravitational equations are then stationary with respect to the Born-Infeld-type effective action.
1 Introduction

The closed (2,2) world-sheet supersymmetric string theory is known to have the interpretation of being a theory of self-dual gravity (SDG) \[1\]. Similarly, the open (2,2) string theory can be interpreted as a self-dual Yang-Mills (SDYM) \[2\]. Since open strings can ‘create’ closed strings which, in their turn, can interact with the open ones, there are quantum corrections to the effective field equations of the open (2,2) string theory. Because of the ‘topological’ nature of the (2,2) string theories, only 3-point tree string amplitudes are non-vanishing and local. As a result, quantum perturbative corrections in the mixed theory of open and closed (2,2) strings are still under control. In particular, the SDYM equations receive corrections from diagrams with internal gravitons, so that they become the YM self-duality equations on a Kähler background \[2\]. Therefore, they still respect integrability, as expected. Contrary to the SDYM equations and naive expectations, the effective gravitational equations of motion in the mixed (open/closed) (2,2) string theory get modified in such a way that the resulting ‘spacetime’ is no longer self-dual \[2\]. Accordingly, the integrability property seems to be lost in the mixed (2,2) string theory. I examine the exact effective field equations of motion in the mixed theory, and show that the integrability is nevertheless maintained in the case of an abelian gauge group. The effective field theory is not the Einstein-Maxwell system describing an interaction of the non-linear graviton with photon. Instead, it is of the Born-Infeld-type, and is non-linear with respect to the both fields, gravitational and ‘electromagnetic’.

In sect. 2 the basic facts about closed and open (2,2) strings separately are summarized, while the mixed theory is discussed in sect. 3 along the lines of the Marcus work \[2\]. The mixed effective field equations of motion were also found first by Marcus \[2\]. However, it is the meaning of the gravitational equations that remained mysterious, and their possible integrability properties were not explored. It is the purpose of this Letter to investigate the effective field equations of motion in the mixed (2,2) string theory and give the conditions for their integrability.

2 Basic facts about closed and open (2,2) strings

The (2,2) strings are strings with two world-sheet supersymmetries, both for the left- and right-moving degrees of freedom.\[1\] The critical open and closed (2,2) strings live\[3\] (2,1) and (2,0) heterotic strings can also be defined \[4\]. Since the heterotic strings have to live in a 2 + 1 or 1 + 1 dimensional spacetime where self-duality is lost (or hidden, at least), we do not consider them here (see, however, ref. \[4\]).
in four real dimensions, with the signature $2 + 2$. The physical spectrum consists of a single massless particle, which can be assigned in the adjoint of a gauge group $G$ in the open string case.

The only non-vanishing $(2, 2)$ string tree scattering amplitudes are 3-point trees, while all higher $n$-point functions vanish due to kinematical reasons in $2 + 2$ dimensions. Tree-level calculations of string amplitudes do not require the heavy machinery of BRST quantization [5], or topological methods [6]. The vertex operator for a $(2, 2)$ closed string particle of momentum $k$ reads in $(2, 2)$ world-sheet superspace as

$$V_c = \frac{\kappa}{\pi} \exp \left\{ i \left( k \cdot \bar{Z} + \bar{k} \cdot Z \right) \right\},$$

where $\kappa$ is the $(2, 2)$ closed string coupling constant, and $Z^i(x, \bar{x}, \theta, \bar{\theta})$ are complex $(2, 2)$ chiral superfields.

When using the $(2, 2)$ super-M"obius invariance of the $(2, 2)$ super-Riemann sphere, it is not difficult to calculate the correlation function of three $V_c$. One finds [1]

$$A_{ccc} = \kappa c_{23}^2 , \quad \text{where} \quad c_{23} \equiv \left( k_2 \cdot \bar{k}_3 - \bar{k}_2 \cdot k_3 \right).$$

One can check that the $A_{ccc}$ is totally symmetric on-shell, and it is only invariant under the subgroup $U(1, 1) \cong SL(2, \mathbb{R}) \otimes U(1)$ of the full ‘Lorentz’ group $SO(2, 2) \cong SL(2, \mathbb{R}) \otimes SL(2, \mathbb{R})'$ in $2 + 2$ dimensions.

Since all higher correlators are supposed to vanish [1], the local 3-point function (2.2) alone determines the exact effective action [1],

$$S_p = \int d^{2+2} z \left( \frac{1}{2} \eta^{i\bar{j}} \partial_i \phi \partial_{\bar{j}} \phi + \frac{2\kappa}{3} \phi \partial \bar{\phi} \wedge \partial \bar{\phi} \right),$$

which is the Plebański action for self-dual gravity (SDG). Hence, the massless ‘scalar’ of the closed string theory can be identified with a deformation of the Kähler potential $K$ of the self-dual (=Kähler + Ricci-flat) gravity [1], where

$$K = \eta_{i\bar{j}} z^i \bar{z}^\bar{j} + 4\kappa \phi , \quad \eta_{i\bar{j}} = \eta^{i\bar{j}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ (2.4)

The $(2, 2)$ closed string target space metric is therefore given by

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K = \eta_{i\bar{j}} + 4\kappa \partial_i \partial_{\bar{j}} \phi .$$ (2.5)

Similarly, in the open $(2, 2)$ string case, when using the $N = (2, 2)$ superspace vertex

$$V_o = g \exp \left\{ i \left( k \cdot \bar{Z} + \bar{k} \cdot Z \right) \right\},$$

Throughout the paper, complex coordinates $(x, \bar{x})$ are used for a string world-sheet, while $(z^i, \bar{z}^\bar{i})$ denote complex coordinates of the $(2, 2)$ string target space, $i = 1, 2.$
to be assigned to the boundary of the \((2,2)\) supersymmetric upper-half-plane (or \((2,2)\) super-disc) with proper boundary conditions, one finds the three-point function \[ A_{ooo} = -igc_{23}f^{abc}, \] which is essentially a ‘square root’ of \(A_{ccc}\), as it should \((f^{abc}\) are structure constants of \(G\)). The \(A_{ooo}\) can be obtained from the effective action \[ S_{DNS} = \int d^{2+2}z \eta^{ij} \left( \frac{1}{2} \partial_i \varphi^a \partial_j \varphi^a - \frac{ig}{3} f^{abc} \varphi^a \partial_i \varphi^b \partial_j \varphi^c \right) + \ldots . \] Requiring all the higher-point amplitudes to vanish in the field theory (2.8) determines the additional local \(n\)-point interactions, \(n > 3\), which were denoted by dots in eq. (2.8). The full action \(S_{DNS}\) is known as the Donaldson-Nair-Schiff (DNS) action \[ S_{DNS} \] of the SDYM,

\[ \eta^{ij} \partial_j \left( e^{-2ig\varphi} \partial_i e^{2ig\varphi} \right) = 0 \],

where the matrix \(\varphi\) is Lie algebra-valued, \(\varphi = \varphi^a t^a\), and the Lie algebra generators \(t^a\) of \(G\) are taken to be anti-hermitian. The DNS action is known to be dual (in the field theory sense) to the Leznov-Parkes (LP) action \[ \frac{1}{\pi} \delta^{ab} c_{23} \int_{-\infty}^{+\infty} dx \frac{1}{x^2 + 1} = \frac{\kappa}{\pi} \delta^{ab} c_{23} \],

\[ \]
where the integration over the position \( x \) of one of the open string vertices goes along the border of the upper-half-plane (= real line). All higher \( n \)-point mixed amplitudes, \( n \geq 4 \), are believed to vanish, like the purely open or closed string ones. The additional (mixed) term in the \((2,2)\) open string effective field theory action, which has to reproduce the \( A_{\text{ooc}} \), reads as follows \[2\]:

\[
S_M = \int d^{2+2}z \left( 2\kappa \phi \bar{\partial} \partial \phi^a \wedge \partial \bar{\partial} \phi^a \right).
\] (3.3)

The complete non-abelian effective action can be determined by demanding all higher-point amplitudes to vanish in the field theory describing the mixed \((2,2)\) strings, order by order in \( n \). Rescaling \( \phi \) by a factor of \( 4\kappa \), and \( \varphi \) by a factor of \( g \), one finds \[2\]

\[
S_{\text{tot}} = \frac{1}{16\kappa^2} \int d^{2+2}z \left[ \frac{1}{2} \eta^{ij} \partial_i \phi \bar{\partial}_j \phi + \frac{1}{6} \phi \bar{\partial} \partial \phi \wedge \partial \bar{\partial} \phi \right] + \frac{1}{g^2} \int d^{2+2}z \eta^{\bar{ij}} \times
\]

\[
\times \left[ -\frac{1}{2} \text{Tr} \left( \partial_i \varphi \bar{\partial}_j \varphi \right) - \frac{2i}{3!} \text{Tr} \left( \bar{\partial}_j \varphi \left[ \partial_i \varphi, \varphi \right] \right) + \frac{2^2}{4!} \text{Tr} \left( \bar{\partial}_j \varphi \left[ \left[ \partial_i \varphi, \varphi \right], \varphi \right] \right) + \ldots \right]
\]

\[+ \frac{1}{g^2} \int d^{2+2}z \left[ \frac{1}{2} \partial \bar{\partial} \phi \wedge \text{Tr} \left( \varphi \partial \bar{\partial} \varphi \right) + \frac{2i}{3!} \partial \bar{\partial} \phi \wedge \text{Tr} \left( \varphi \left[ \partial \varphi, \bar{\partial} \varphi \right] \right) + \ldots \right].
\] (3.4)

Despite of its rather complicated form, the equations of motion resulting from that action can be written down in simple geometrical terms \[2\], namely

\[
g^{\bar{i}j}(\phi) \bar{\partial_j} \left( e^{-2i\varphi} \partial_i e^{2i\varphi} \right) = 0,
\] (3.5)

and

\[- \det g_{ij} = +1 + \frac{2\kappa^2}{g^2} \text{Tr} \left( F_{ij} F^{ij} \right),
\] (3.6)

where \( F_{ij} \) is the YM field strength of the YM gauge fields

\[
A \equiv e^{-i\varphi} \partial e^{i\varphi}, \quad \bar{A} \equiv e^{i\varphi} \bar{\partial} e^{-i\varphi},
\] (3.7)

\( g_{ij} = \eta_{ij} + \partial_i \bar{\partial}_j \phi \) is a Kähler metric, \( g^{\bar{i}j} \) is its inverse, and the indices \((i, \bar{j})\) are raised and lowered by using the totally antisymmetric Levi-Civita symbols \( \varepsilon^{ij}, \varepsilon^{\bar{i}j}, \) and \( \varepsilon_{ij}, \varepsilon_{\bar{i}j} \) \((\varepsilon_{12} = \varepsilon^{12} = 1)\).

Eq. (3.5) is just the Yang equation (of motion) of the DNS action describing the SDYM on a curved Kähler background, as it should have been expected. It is a meaning of the gravitational equation (3.6) that is of our interest.

### 4 Effective field theory of mixed \((2,2)\) strings as an integrable deformation of self-duality

Associated with the Kähler metric

\[
ds^2 = 2g_{ij} dz^i dz^{\bar{j}} \equiv 2K_{ij} dz^i dz^{\bar{j}},
\] (4.1)
is the fundamental (Kähler) closed two-form
\[ \Omega = g_{ij}dz^i \wedge d\bar{z}^j \equiv K_{ij}dz^i \wedge d\bar{z}^j , \]

where \( K \) is the (locally defined) Kähler potential, and all subscripts after a comma denote partial differentiations. We regard the complex coordinates \((z^i, \bar{z}^\bar{i})\) as independent variables, so that our complexified ‘spacetime’ \( \mathcal{M} \) is locally a direct product of two 2-dimensional complex manifolds \( \mathcal{M} \cong \mathcal{M}_2 \otimes \bar{\mathcal{M}}_2 \), where both \( \mathcal{M}_2 \) and \( \bar{\mathcal{M}}_2 \) are endowed with complex structures, i.e. possess closed non-degenerate two-forms \( \omega \) and \( \bar{\omega} \), respectively. Hence, the effective equations of motion (3.5) and (3.6) in the mixed (2,2) string theory can be rewritten to the even more geometrical form as

\[ \Omega \wedge F = 0 , \]

and

\[ \Omega \wedge \Omega + \frac{4\kappa^2}{g^2} \text{Tr}(F \wedge F) = 2\omega \wedge \bar{\omega} , \]

where \( F \) is the YM Lie algebra-valued field strength two-form satisfying

\[ \omega \wedge F = \bar{\omega} \wedge F = 0 . \]

Eqs. (4.3) and (4.5) are just the self-dual Yang-Mills equations in the Kähler ‘spacetime’. They are therefore integrable and their solutions describe Yang-Mills instantons (see e.g., the recent paper and references therein for some explicit constructions of the solutions). In particular, one can always locally change the flat SDYM equations of motion into the SDYM equations on a curved Kähler background by a diffeomorphism transformation compatible with the Kähler structure.

The integrability condition for the gravitational equations of motion in the complexified ‘spacetime’ is known to be precisely equivalent to the (anti)self-duality of the Weyl curvature tensor \([13]\). The famous twistor construction of Penrose \([13]\) transforms the problem of solving the non-linear partial differential equations of conformally self-dual gravity into the standard Riemann-Hilbert problem of patching together certain holomorphic data.

As far as the Kähler spaces are concerned, the self-duality of the Weyl tensor is precisely equivalent to the vanishing Ricci scalar curvature \([14, 15]\), while the Ricci tensor itself is known to be simply related to the Kähler metric as

\[ R_{ij} = \partial_i \partial_j \log \det(g_{kk}) . \]

\(^6\)The normalization of the holomorphic two-forms \( \omega \) and \( \bar{\omega} \) is fixed by the flat ‘spacetime’ limit where \( \omega = dz^1 \wedge d\bar{z}^2 \) and \( \bar{\omega} = d\bar{z}^1 \wedge d\bar{z}^2 \).
Eq. (3.6) or (4.4) therefore yields
\[
R_{ij} = \partial_i \bar{\partial}_j \log \left[ 1 + \frac{2\kappa^2}{g^2} \text{Tr}(F_{ij} F^{ij}) \right], \tag{4.7}
\]
and, hence,
\[
R = g^{m\bar{n}} \partial_m \bar{\partial}_n \log \left[ 1 + \frac{2\kappa^2}{g^2} \text{Tr}(F_{ij} F^{ij}) \right], \tag{4.8}
\]
while both do not vanish on shell. It is also obvious that the ‘matter’ stress-energy tensor to be equal to the Einstein tensor in accordance with eqs. (4.7) and (4.8), does not vanish too. It is to be compared to the standard gravitational equations of motion in the case of the Einstein-Yang-Mills coupled system to be described by the standard action given by a sum of the Einstein-Hilbert and the Yang-Mills terms. There, the YM stress-energy tensor is quadratic with respect to the YM field strength, and it vanishes under the SDYM condition. In our case, the YM stress-energy tensor is not even polynomial in the YM field strength, and it has to correspond to a non-polynomial (in $F$) effective action.

Does it also imply that eq. (4.4) is not integrable? First, let us rewrite eq. (3.6) to the form:
\[
\det(g_{ij}) + \frac{2\kappa^2}{g^2} \text{Tr} \det(F_{ij}) = -1 , \tag{4.9}
\]
where both determinants are two-dimensional. Given an abelian field strength $F$ satisfying the self-duality condition (4.3) or, equivalently, $g_{11}F_{22} + g_{22}F_{11} - g_{12}F_{21} - g_{21}F_{12} = 0$, there is a remarkable identity
\[
\det(g) + \frac{2\kappa^2}{g^2} \det(F) = \det \left( g + \frac{\kappa \sqrt{2}}{g} F \right) . \tag{4.10}
\]
In addition, eq. (3.7) in the abelian case implies $A = i\partial \varphi$, $\bar{A} = -i\bar{\partial} \varphi$, and, hence,
\[
F = 2i\partial \bar{\partial} \varphi . \tag{4.11}
\]
Taken together, they allow us to represent eq. (4.9) as the Plebański heavenly equation
\[
\det \left( \partial \bar{\partial} K \right) = -1 , \tag{4.12}
\]
with the complex potential
\[
K \equiv K + \frac{2\sqrt{2} \kappa}{g} \varphi , \tag{4.13}
\]
whose imaginary part is a harmonic function (because of the self-duality of $F$), and of order $\hbar^{1/2} g$ since eq. (3.1). Eq. (4.12) is the consistency condition for the linear system
\[
L_1 \psi \equiv \left[ \partial_1 + i\lambda B_1 \right] \psi \equiv \left[ \bar{\partial}_1 + i\lambda \left( \mathcal{K}_{21} \partial_1 - \mathcal{K}_{11} \partial_2 \right) \right] \psi = 0 , \\
L_2 \psi \equiv \left[ \partial_2 + i\lambda B_2 \right] \psi \equiv \left[ \bar{\partial}_2 + i\lambda \left( \mathcal{K}_{22} \partial_1 - \mathcal{K}_{12} \partial_2 \right) \right] \psi = 0 , \tag{4.14}
\]
where $\lambda$ is a complex spectral parameter. The linear equations (4.14) describe a fibering of the associated twistor space in the sense of Penrose [13].

Hence, one still has the Frobenius integrability, just like that in the usual case of the Plebański heavenly equation with a real Kähler potential. The generalized ‘metric’ to be defined with respect to the complex Kähler potential is not real, and its only use is to make the integrability apparent, while the true metric in eq. (4.1) is real of course.

When trying to generalize that results to the non-abelian situation, one arrives at an obstruction, since the crucial relation (4.10) is no longer valid. If, nevertheless, one wants to impose such a relation, one finds that

$$F^a \wedge F^b = 0, \quad \text{when} \quad a \neq b .$$  \hspace{1cm} (4.15)

Eq. (4.15) implies that different directions in the YM group space do not ‘see’ each other. Hence, insisting on integrability seems to send us back to the abelian case.

It should be noticed that the solutions to the gravitational equations of motion (4.12) are all stationary with respect to the Born-Infeld-type effective action

$$S = \int d^{2+2}z \sqrt{-\det \left( g_{i\bar{j}} + \frac{\kappa \sqrt{2}}{g} F_{i\bar{j}} \right)} .$$  \hspace{1cm} (4.16)

The action $S$ is not the standard Born-Infeld action [16] since the determinant in eq. (4.16) is two-dimensional, not four-dimensional.

5 Conclusion

Our result is given by the title. To conclude, it is worth mentioning that an infinite hierarchy of conservation laws and the infinite number of symmetries [17] exist as the consequences of Penrose’s twistor construction when it is formally applied to our ‘almost self-dual’ gravity with a complex Kähler potential. The underlying symmetry is known to be a loop group $S^1 \to SDiff(2)$ of the area-preserving (holomorphic) diffeomorphisms (of a 2-plane), which can be considered as a ‘large N limit’ ($W_\infty$) of the $W_N$ symmetries in two-dimensional conformal field theory [18, 19]. The area-preserving holomorphic diffeomorphisms,

$$\partial_i \bar{\partial}_j K(z, \bar{z}) \to \partial_i \xi^k(z) \partial_k \bar{\partial}_j K(\xi, \bar{\xi}) \bar{\partial}_j \xi^k(\bar{z}) ,$$  \hspace{1cm} (5.1)

leave eqs. (4.12) and (4.16) invariant, since

$$\left| \det(\partial_i \xi^k) \right| = 1$$  \hspace{1cm} (5.2)

by their definition.
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