SEMIDISCRETE APPROXIMATION TO BENJAMIN-TYPE EQUATIONS

VASSILIOS A. DOUGALIS AND ANGEL DURAN

Abstract. In this paper a semidiscrete Fourier pseudospectral method for approximating Benjamin-type equations is introduced and analyzed. A study of convergence is presented.

1. Introduction

This paper is concerned with numerical approximations to Benjamin-type equations of the form

\[ u_t - Lu_x + f(u)_x = 0. \] (1.1)

In (1.1) \( u = u(x, t) \) is a real-valued function, \( L \) is the linear, nonlocal, pseudodifferential operator with Fourier symbol

\[ \hat{L}u(\xi) = l(\xi)\hat{u}(\xi) = (\delta|\xi|^{2m} - \gamma|\xi|^{2r})\hat{u}(\xi), \quad \xi \in \mathbb{R}, \] (1.2)

where \( m \geq 1 \) is an integer, \( 0 \leq r < m, \gamma \geq 0, \delta > 0 \) and \( \hat{u}(\xi) \) denotes the Fourier transform of \( u \) at \( \xi \). Finally the nonlinear term \( f \) is of the form

\[ f(u) = \frac{u^{q+1}}{q+1}, \quad q \geq 1. \] (1.3)

The general form (1.1)-(1.3) includes the case of the Benjamin equation, and generalized versions \((m = 1, r = 1/2, q \geq 1)\). Summarized here is some literature on (1.1). This is mainly focused on [21] where Linares and Scialom, based on the theory developed in [17, 18] for the generalized KdV and Benjamin-Ono (BO) equations, respectively, establish local and global well-posedness results for the corresponding initial-value problem for (1.1)-(1.3). More specifically, the generalized Benjamin equation is studied first and for \( u_0 \in H^s(\mathbb{R}), s \geq 1, \) the existence of \( T = T(||u_0||_{H^s}) \) and a unique solution \( u \in C([0, T], H^s(\mathbb{R})) \) with \( u(0) = u_0, ||u||_{L^1_x \cap L^\infty} < \infty \) and

\[ \left( \int_0^T |\partial_x u(x, t)|^2 dt \right)^{1/2} < \infty, \quad x \in \mathbb{R}, \]

are proved. Furthermore, the initial-value problem is globally well-posed in \( H^1(\mathbb{R}) \) for \( q = 2, 3 \) with no restriction on the initial data and for \( q \geq 4 \) when the initial data is small enough. In the general case and for \( u_0 \in H^m(\mathbb{R}), m > r > 0, m > 1, \) the initial-value problem is globally well-posed in the energy space where the energy \( E(u) \) given by

\[ E(u) = \int_{-\infty}^{\infty} (uLudx - 2F(u)) dx, \] (1.4)

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is finite. Here $F'(u) = f(u), F(0) = 0$. Existence of global solutions is obtained without restriction on the initial data when $q < 4m$, while for $q \geq 4m$ the initial condition must be small enough. In addition to (1.4) and for decaying and smooth enough solutions two other quantities

$$I(u) = \int_{-\infty}^{\infty} u^2 dx; \quad C(u) = \int_{-\infty}^{\infty} u dx.$$  \hspace{1cm} \text{(1.5)}$$

are preserved.

Another group of results concerns solitary-wave solutions of (1.1)-(1.3), that is solutions of the form

$$u = \varphi(x - c_s t), \quad c_s > 0,$$

where $\varphi$ and its derivatives go to zero as $X = x - c_s t \to \pm \infty$. Specifically, in [6] existence and asymptotic properties of solitary waves are proved for a range of values of $\gamma$ depending on $r, m$ and the speed $c_s$. Stability (in the orbital sense) of the waves is studied in [2] (see also the references therein).

Typically the dynamics of these solitary waves is the main motivation to study numerically nonlinear dispersive wave equations like (1.1), by constructing schemes approximating to corresponding initial- and periodic boundary-value problems on sufficiently large intervals. With this aim the present paper is focused on the numerical analysis. Consider, for simplicity, the initial- and periodic boundary-value problem (ipbvp) for (1.1)-(1.3) on $[-\pi, \pi]$

$$u_t - Lu_x + f(u)_x = 0, \quad x \in [-\pi, \pi], t > 0,$$

$$u(x, 0) = u_0(x), \quad x \in [-\pi, \pi],$$  \hspace{1cm} \text{(1.6)}$$

where $u_0$ is $2\pi$-periodic. In this case, to our knowledge, the literature on the problem (1.6) can be summarized as follows. In [20], using the methods by Bourgain, [9], and Kenig et al., [17], for the study of the well-posedness of the ipbvp for the KdV equation, global well-posedness for the Benjamin equation is established for data in $L^2$. Recently, Shi and Li, [23], proved local well-posedness for small initial data in $H^\mu(T), \mu \geq -1/2$, by using a different technique. Finally Cascaval, [12], studies the local and global well-posedness of the initial value problem (ivp) and the ipbvp for the class of nonlinear dispersive equations

$$u_t - Mu_x + G(u)_x = 0,$$

where $\hat{M}u(\xi) = |\xi|^{2\beta} \hat{u}(\xi), \beta \geq 1/2$ and $G$ is sufficiently smooth and satisfies

$$\lim_{|r| \to \infty} \frac{G'(r)}{|r|^p} < \infty,$$

for some $p < 4\beta$. (This includes the cases $G(u)_x = u^p u_x, p \geq 1$.) The author of [12] obtains global well-posedness in $H^\mu(T)$ with $\mu = \max\{2\beta, 3/2 + \epsilon\}$ for some $\epsilon > 0$ and in the cases:

- $\beta = 1, \mu = 2$.
- $\beta = 1/2, G'(u) = u, \mu \in (3/2, 2]$.
- $\beta > 3/2, \mu = 2\beta$.

Furthermore, for $1/2 < \beta \leq 3/2$ the problem is locally well-posed in $H^\mu(T)$ with $\mu = 2\beta$ for $\beta > 3/4$ and $\mu \in (3/2, 2\beta + 1]$ for $1/2 < \beta \leq 3/4$. In particular, global well-posedness of the periodic gKdV equation (for $p < 4$) in $H^2(T)$ is obtained.
On the other hand, in \[13\] well-posedness of the ivp of some models of the form (1.6) in relatively smooth, periodic function spaces is assumed. These spaces must have at least finite energy in the sense that the solutions

\[ u(x, t) = \sum_{n=-\infty}^{\infty} u_n(t)e^{inx}, \]

decomposed into its Fourier series has the property

\[ \sum_{n=-\infty}^{\infty} |l(n)||u_n(t)|^2 < \infty, \quad t > 0, \]

where \(l(\xi)\) is defined in (1.2) as the Fourier symbol of \(L\). Also, for smooth solutions, periodic conditions ensure the preservation of the corresponding versions of the functionals (1.4), (1.5) with the integrals on the interval \((-\pi, \pi)\), that is

\[ C_\pi(u) = \int_{-\pi}^{\pi} u dx; \quad I_\pi(u) = \int_{-\pi}^{\pi} u^2 dx, \quad E_\pi(u) = \int_{-\pi}^{\pi} (uLudx - 2F(u)) dx, \quad (1.7) \]

where \(F'(u) = f(u), F(0) = 0\).

To our knowledge, the numerical approximation to (1.6) has only been considered for the particular cases of the KdV equation and generalized KdV equation (\(m = 1, \gamma = 0\)) and for the Benjamin equation and generalized Benjamin equation (\(m = 1, r = 1/2\)). Focused on this last case, we first mention the method used in [1, 8] based on pseudospectral collocation in space and a second-order time-stepping code. On the other hand, the method considered in [14] has the same type of spatial discretization with a third-order singly diagonally implicit Runge-Kutta scheme, combined with a projection technique to preserve invariant quantities, [16], as time integrator. In [15], a hybrid spectral-finite element scheme along with a 2-stage Gauss-Legendre implicit Runge-Kutta method is constructed while structure preserving integrators are proposed in [19]. As mentioned above, the application of all of them was, to a greater or lesser extent, related to solitary wave dynamics.

The purpose of the present paper is the introduction and analysis of a semidiscrete numerical method to approximate (1.6), based on a Fourier pseudospectral discretization. As observed before, this is, to our knowledge, the first proposal in the literature to approximate the periodic-initial value problem (1.6), (1.2), (1.3) in its full generality. The possible nonlocal character of the linear operator \(L\) in (1.2) justifies the use of a Fourier-Galerkin discretization in space. The semidiscrete scheme is studied in Section 2. We first establish the existence of a unique, local in time solution of the semidiscrete problem. The existence of a global in time solution is then derived from the preservation of the quantities (1.7) by the semidiscretization. The convergence of the pseudospectral discretization is then established in Theorem 2.1. The proof is based on the introduction of a linear, intermediate problem, [3, 22], whose solution approximates both those of the continuous and semidiscrete problems, with estimates that depend on the regularity of the solution of the first one.
For real $\mu \geq 0$ and $1 \leq p \leq \infty$ we denote by $W^\mu_p = W^\mu_p(-\pi, \pi)$ the real Sobolev space on $(-\pi, \pi)$ with norm $|| \cdot ||_{\mu,p}$. Let $H^\mu := W^\mu_2$ and for $g \in H^\mu$ put

$$
||g||_\mu = ||g||_{\mu,2} = \left( \sum_{k \in \mathbb{Z}} (1 + k^2)^\mu |\hat{g}(k)|^2 \right)^{1/2},
$$

$$
\hat{g}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} g(x) dx
$$

$| \cdot |_\infty$ will stand for the norm in $L^\infty(-\pi, \pi)$. Finally, the inner product in $H^0 = L^2(-\pi, \pi)$ is denoted by $(\cdot, \cdot)$, defined by

$$(u, v) = \int_{-\pi}^{\pi} u(x)\overline{v(x)}dx,$$

with $|| \cdot ||$ standing for the corresponding norm. According to previous comments and the form of $L$ in (1.2), we will assume that problem (1.6), (1.2), (1.3) is well-posed on $H^{\mu,m} = H^{\mu,m}(-\pi, \pi)$.

2. Analysis of convergence of the semidiscrete scheme

2.1. A Fourier pseudospectral approximation of the periodic-initial value problem. For $N \geq 1$ integer we consider

$$S_N = \text{span}\{e^{ikx}, k \in \mathbb{Z}, -N \leq k \leq N\}.$$ 

The semidiscrete Fourier-Galerkin approximation to (1.6), (1.2), (1.3) is defined as a real-valued map $u^N : [0, \infty) \to S_N$ such that, for all $\chi \in S_N$,

$$(u^N, \chi) + ((-Lu^N + f(u^N))_x, \chi) = 0, \quad t > 0,
$$

$$
u^N(x, 0) = P_N u_0(x),
$$

where $P_N$ is the $L^2$-projection of $L^2$ onto $S_N$. For $v \in L^2$ we have

$$P_N v = \sum_{|k| \leq N} \hat{v}_k e^{ikx},$$

with $\hat{v}_k$ the $k$-th Fourier coefficient of $v$. Some properties of $P_N$ will be used throughout the paper. First, it is well known that $P_N$ commutes with the differential operator $\partial_x$. Moreover, cf. [11], given integers $0 \leq j \leq \mu$ there exists a constant $C$ independent of $N$ such that for any $v \in H^\mu$,

$$||v - P_N v||_j \leq CN^{1-j} ||v||_{\mu}, \quad \mu \geq 0,$$

$$||v - P_N v||_\infty \leq CN^{1/2-\mu} ||v||_{\mu}, \quad \mu \geq 1.$$ 

(2.2)

When $\chi = e^{ikx}, |k| = 0, \ldots, N$ then (2.1) becomes the initial value problem for the Fourier coefficients of $u^N$,

$$\widehat{u^N_t}(k, t) = (ik)((\delta |k|^2 - \gamma |k|^2)\widehat{u^N}(k, t) - \widehat{f(u^N)}(k, t)), \quad t > 0,
$$

$$\widehat{u^N}(k, 0) = \widehat{u_0}(k),$$

(2.3)

Remark 2.1. In the sequel the following properties of $f$ and $f'$ will be used:

(i) $f(u) - f(v) = (u - v)g(u, v, q)$ with

$$g(u, v, q) = \frac{1}{q+1} \sum_{j=0}^{q} u^j v^{q-j},$$

where $j$ is a non-negative integer.
and therefore

\[ ||f(u) - f(v)|| \leq ||u - v|||g|\infty, \quad |g|\infty \leq (\max\{|u|\infty,|v|\infty\})^{q}. \]

(ii) \( f'(u) - f'(v) = (u - v)h(u,v,q) \) with

\[ h(u,v,q) = \sum_{j=0}^{q-1} u^j v^{q-1-j}, \]

and therefore

\[ ||f'(u) - f'(v)|| \leq ||u - v|||h|\infty, \quad |h|\infty \leq q(\max\{|u|\infty,|v|\infty\})^{q-1}. \]

2.2. Existence and uniqueness of solutions of the semidiscrete problem.
In particular, (i) implies that the right hand side of (2.3) is at least locally Lipschitz
continuous with respect to the \( L^2 \) norm in \( S_N \). Then, using standard theory of
ordinary differential equations, we obtain the existence of a unique, local in time
solution of (2.3). Also, standard arguments prove the existence of a global in time
solution if the semidiscretization preserves the \( L^2 \) norm. In our case, we have the
following result.

**Lemma 2.1.** The solution \( u^N \) of (2.1) satisfies, for \( t > 0 \),

\[ \frac{d}{dt} C_\pi(u^N) = \frac{d}{dt} I_\pi(u^N) = \frac{d}{dt} E_\pi(u^N) = 0. \]

where \( C_\pi, I_\pi \) and \( E_\pi \) are given by (1.7).

**Proof.** The preservation of \( C_\pi \) is obtained directly by taking \( \chi = 1 \) in (2.1) while
if we take \( \chi = u^N \) we have

\[ (f(u^N)_x, u^N) = F(u^N(\pi,t)) - F(u^N(-\pi,t)) = 0, \]

(where \( F \) is a primitive of \( f \)). Moreover, if \( v \) is a real-valued element of \( S_N \) it
follows, since \( \hat{v}(k) = \hat{v}(-k) \), that

\[ (Lv_x, v) = 0. \]

Therefore \( (Lu^N_x, u^N) = 0 \), which implies the preservation of \( I_\pi \). Finally, with
\( \chi = P_N f(u^N) - Lu^N \) we have

\[ (u^N_t, \chi) = \int_{-\pi}^{\pi} u^N_t(f(u^N) - Lu^N) dx \]

\[ = \frac{1}{2} \frac{d}{dt} \int_{-\pi}^{\pi} \left( u^N Lu^N - \frac{2}{(q + 1)(q + 2)}(u^N)^{q+2} \right) dx \]

\[ = \frac{1}{2} \frac{d}{dt} E_\pi(u^N(\cdot,t)), \]

and because of periodicity and properties of \( P_N \), we have

\[ (-Lu^N_x + f(u^N)_x, \chi) = (-Lu^N_x + f(u^N)_x, P_N f(u^N) - Lu^N) \]

\[ = (f(u^N)_x, f(u^N)) - (-Lu^N_x, f(u^N)) \]

\[ - (f(u^N)_x, Lu^N) + (Lu^N, Lu^N) = 0. \quad \square \]
2.3. **Analysis of convergence.** In order to study the convergence of the semidiscrete scheme, we consider the intermediate problem of searching for \( w^N \in S_N \) such that for all \( \chi \in S_N \) and any \( t^* > 0 \),

\[
(w^N_t, \chi) + (-\mathcal{L}w^N_x + f'(u)w^N_x, \chi) = 0, \quad 0 \leq t \leq t^*,
\]

\[
w^N(0) = P_N u_0,
\]

where \( u \) is a solution of (1.6).

**Lemma 2.2.** Let \( u \), the solution of (1.6), belong to \( H^\mu \) for \( \mu > 3/2 \). Then the ivp (2.5) has a unique solution \( w^N \) which satisfies

\[
\max_{0 \leq t \leq t^*} |w^N|_{\infty} \leq C,
\]

and

\[
\max_{0 \leq t \leq t^*} ||u - w^N|| \leq CN^{1-\mu},
\]

for some constant \( C = C(u, t^*) \). In addition, if \( \mu \geq 5/2 \) then

\[
\max_{0 \leq t \leq t^*} ||w^N||_{1, \infty} \leq C,
\]

where again \( C \) is a constant depending on \( u \) and \( t^* \) only.

**Proof.** As before, standard theory ensures existence and uniqueness of a local solution for (2.5). Furthermore, while \( w^N \) exists, putting \( \chi = w^N \) in (2.5) gives, in view of (2.4), that

\[
\frac{d}{dt} ||w^N||^2 = \frac{1}{2} (\partial_x f'(u), (w^N)^2) \leq \frac{1}{2} |\partial_x (f'(u))|_{\infty} ||w^N||^2.
\]

Now

\[
|\partial_x (f'(u))|_{\infty} = |u^{q-1} u_x|_{\infty},
\]

which is bounded for all \( t > 0 \) if \( \mu > 3/2 \). Then by Gronwall’s lemma, there is a constant \( C \) such that

\[
||w^N|| \leq Ce^{Ct},
\]

and we can extend the local solution to a solution on \([0, t^*]\).

Now we define \( \rho^N = P_N u - w^N \) and estimate the difference

\[
u - w^N = u - P_N u + \rho^N.
\]

In view of (2.5), \( \rho^N \) satisfies, for all \( \chi \in S_N \),

\[
(\rho^N_t, \chi) + (-\mathcal{L}\rho^N_x + f'(u)\rho^N_x, \chi) = 0, \quad 0 \leq t \leq t^*,
\]

\[
\rho^N(0) = 0,
\]

Note first that for \( \chi \in S_N \)

\[
(f(u)_x - f'(u)\rho^N_x), \chi) = (f'(u)(u - P_N(u))x + \rho^N_x, \chi).
\]

Therefore, if we take \( \chi = \rho^N \) in (2.9) we have

\[
\frac{1}{2} \frac{d}{dt} ||\rho^N||^2 + (f'(u)(u - P_N(u))x, \rho^N) + (f'(u)\rho^N, \rho^N) = 0,
\]
and by periodicity
\[ (f'(u)\rho_x^N, \rho^N) = -\frac{1}{2}(\partial_x f'(u), (\rho^N)^2). \]

Hence
\[ \frac{1}{2} \frac{d}{dt} ||\rho^N||^2 = -(f'(u)(u - P_N(u))_x, \rho^N) + \frac{1}{2}(\partial_x f'(u), (\rho^N)^2) \]
\[ \leq |f'(u)|_{\infty}||(u - P_N(u))_x|| ||\rho^N|| + \frac{1}{2} |\partial_x f'(u)|_{\infty} ||\rho^N||^2 \]
\[ \leq C \frac{1}{N^{\mu - \tau}} ||\rho^N|| + C ||\rho^N||^2 \]
\[ \leq \left(\frac{C}{N^{\mu - \tau}}\right)^2 + ||\rho^N||^2. \]

Finally, the initial condition and Gronwall’s lemma lead to
\[ \max_{0 \leq t \leq t^*} ||\rho^N|| \leq \frac{C}{N^{\mu - \tau}}, \quad (2.10) \]
and, consequently, to (2.7).

Note now that if \(0 \leq t \leq t^*\)
\[ |w^N(t)|_{\infty} \leq |u(t)|_{\infty} + |w^N(t) - P_N u(t)|_{\infty} + |P_N u(t) - u(t)|_{\infty} \]
\[ = |u(t)|_{\infty} + |\rho^N(t)|_{\infty} + |P_N u(t) - u(t)|_{\infty} \]
Recall now that the following inverse inequalities hold in \(S_N\). Given \(0 \leq s \leq r\), there exists a constant \(C_0\) such that
\[ ||\psi||_r \leq C_0 N^{-\frac{s}{r}} ||\psi||_s, \quad ||\psi||_r,\infty \leq C_0 N^{1/2 + r - s} ||\psi||_s, \quad (2.11) \]
for all \(\psi \in S_N\). Then (2.2) and (2.10) give
\[ |w^N(t)|_{\infty} \leq C + \frac{C}{N^{\mu - 3/2}} + \frac{C}{N^{\mu - 1/2}} \leq \tilde{C}, \]
for some constant \(C\) and (2.6) follows. If now \(\mu \geq 5/2\) we have
\[ |w_x^N(t)|_{\infty} \leq |u_x(t)|_{\infty} + |\rho_x^N(t)|_{\infty} + |\partial_x (P_N u(t) - u(t))|_{\infty} \]
\[ \leq C + \frac{C}{N^{\mu - 5/2}} + \frac{C}{N^{\mu - 3/2}} \leq \tilde{C}, \]
Hence (2.8) follows. \(\square\)

We now proceed to study the convergence of the semidiscrete scheme (2.1).

**Theorem 2.1.** Let \(u^N\) be the solution of (2.2) and suppose that \(u\), the solution of (1.3), belongs to \(H^\mu, \mu \geq 5/2\). Then given \(0 \leq t^* < \infty\) there exists a constant \(C = C(u, t^*)\) such that
\[ \max_{0 \leq t \leq t^*} ||u - u^N|| \leq \frac{C}{N^{\mu - 1}}. \quad (2.12) \]

**Proof.** We have already established the existence of \(u^N\), the solution of the initial-value problem (2.4), on any temporal interval \([0, t^*]\). Let \(w^N\) be the solution of the intermediate problem (2.5) and \(e^N = w^N - u^N\). Since \(u - u^N = u - w^N + e^N\), in view of (2.7) we only need to estimate \(e^N\). This satisfies, for all \(\chi \in S_N\)
\[ (e^N_x, \chi) + (-L e^N_x - (f(u^N)_x - f'(u)w^N_x), \chi) = 0, \quad 0 \leq t \leq t^*, \quad (2.13) \]
\[ e^N(0) = 0, \]
Since \( f(u_N)^x - f'(u)w_N^x = (f(u_N)^x - f(w_N)^x) + (f(w_N)^x - f'(u)w_N^x) \), taking \( \chi = e^N \) in (2.13) gives, in view of (2.4), that for \( 0 \leq t \leq t^* \)
\[
(e_1^N, e_1^N) = (f(u_N)^x - f(w_N)^x, e^N) + ((f'(w_N)^x - f'(u))w_N^x, e^N).
\] (2.14)
In order to estimate the second term in the right hand side of (2.14), using the inclusion of Remark 2.1(ii) we observe that
\[
||f'(w_N) - f'(u)|| \leq q||w_N - u|| (\max(||w_N||, ||u||))^{q-1}.
\] Therefore, by (2.7), (2.8)
\[
||(f'(w_N) - f'(u))w_N^x, e^N)|| \leq C ||f'(w_N) - f'(u)|| ||w_N|| ||e^N|| \leq \frac{C}{N^{\mu-1}} ||e^N||,
\] (2.15)
for some constant \( C = C(u, t^*) \).
In order to estimate the first term in the right hand side of (2.14), we make use of formulas (3.10)-(3.13) of [7] to write
\[
f(u_N)^x - f(w_N)^x = f(u_N^x - e^N)^x - f(w_N)^x = f(-e^N)^x + R(w_N, -e^N),
\]
where
\[
R(v, w) = \frac{1}{q + 1} \partial_x \left( \sum_{j=1}^q \binom{q + 1}{j} v^{q+1-j}w^j \right).
\]
Then, by periodicity
\[
(f(u_N)^x - f(w_N)^x, e^N) = ((R(w_N, -e^N), e^N).
\] (2.16)
Hence, using the estimate (3.13) of [7] we have
\[
|(R(w_N, -e^N), e^N)| \leq C_{q, \max} \int_{1 \leq m \leq q} ||w_N||^m_\infty \sum_{j=1}^q \int_{-\pi}^{\pi} |e^N|^{j+1} dx,
\] (2.17)
and we observe that
\[
\sum_{j=1}^q \int_{-\pi}^{\pi} |e^N|^{j+1} dx \leq \max_{1 \leq j \leq q} |e^N|^{j-1}_\infty ||e^N||^2.
\]
Since \( e^N(0) = 0 \), there exists by continuity a maximal temporal value \( t_N > 0 \) (assume without loss of generality that \( t_N < t^* \)) such that
\[
|e^N||_\infty \leq 1, \quad 0 \leq t \leq t_N.
\] (2.18)
For \( t \leq t_N \) therefore, (2.10), (2.17), (2.8) and (2.18) yield
\[
|(f(u_N)^x - f(w_N)^x, e^N)| \leq C ||e^N||^2,
\] (2.19)
for some constant \( C = C(u, t^*) \). Hence by (2.14), (2.15), and (2.19) we have for \( t \in [0, t_N] \)
\[
\frac{1}{2} \frac{d}{dt} ||e^N||^2 \leq C ||e^N||^2 + \frac{C}{N^{\mu-1}} ||e^N|| \leq C ||e^N||^2 + \left( \frac{C}{N^{\mu-1}} \right)^2.
\]
Since \( e^N(0) = 0 \), Gronwall’s lemma implies
\[
\max_{0 \leq t \leq t_N} ||e^N|| \leq \frac{C}{N^{\mu-1}}.
\] (2.20)
for some constant $C = C(u, t^*)$. Hence, by the inverse properties (2.11) of $S_N$ we obtain that
\[
\max_{0 \leq t \leq t_N} |e^N|_\infty \leq C N^{3/2-\mu},
\]
which contradicts the maximality of $t_N$ in (2.15) if $N$ is taken sufficiently large. We conclude that $t_N = t^*$ and (2.12) follows from (2.20). □

\textbf{Remark 2.2.} Under the hypothesis of Theorem 2.1 we have for $0 \leq t \leq t^*$, using (2.8), inverse properties of $S_N$, (2.7) and (2.12)
\[
||u^N||_{1,\infty} \leq ||u^N - w^N||_{1,\infty} + ||w^N||_{1,\infty}
\leq C N^{3/2} ||u^N - w^N|| + C
\]
\[
\leq C N^{3/2} (||u^N - u|| + ||u - w^N||) + C \leq C N^{5/2-\mu} + C,
\]
i.e.
\[
||u^N||_{1,\infty} \leq B,
\]
for some constant $B = B(u, t^*)$.

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