The universal $\mathcal{C}^*$-algebra generated by $n$ projections has been described. As an immediate corollary one obtains structure theorem for a pair of projections and the solution to an associated index problem. This puts the study of a pair of projections in the proper perspective of noncommutative geometry.

keyword: Index problem, crossed product, Fredholm module, pair of projections.
MSC 46L87.

1 Introduction

There has been considerable amount of work in the operator theory/operator algebra literature on the structure of a pair of projections. Study of this issue goes back to Dixmier ([5]). In [6] Halmos gives a structure theorem for a pair of projections, which has been rederived on various occasions. There is also a survey article by C. Davis ([4]) on this topic. Avron et. al. [1] discusses an index problem in this context. All these approaches use geometric or operator theoretic methods. Our aim is to put these in the proper perspective of noncommutative geometry (NCG). To that end we have described the universal $\mathcal{C}^*$-algebra generated by $n$ projections as a crossed product. An application of the representation of imprimitivity system yields the structure theorem for a pair of projections. We believe, our proof of this fact is the shortest and the first one to connect this issue clearly with imprimitivity. Incidentally it may be remarked that Dixmier’s paper was reviewed by Mackey and it seems that the aforesaid connection went unnoticed. We also show that an index problem, the main result discussed in [1], can be settled easily using general machinery from NCG.
2 Results

We will refer to unitary operators whose square is identity as symmetry.

Theorem 2.1 Let $A_n$ be the universal $C^*$-algebra generated by $n$ projections $P_1, P_2, \cdots, P_n$. Then we have the following isomorphisms,

$$A_n \cong C^*(\mathbb{Z}_2)^n \cong C^*(F_{n-1}) \rtimes \mathbb{Z}_2$$

where $C^*(\mathbb{Z}_2)^n$ is the free product of $n$ copies of the group $C^*$-algebra of $\mathbb{Z}_2$ and $C^*(F_{n-1}) \rtimes \mathbb{Z}_2$ is the crossed product of the group $C^*$-algebra of free group with $n-1$ generators with respect to the $\mathbb{Z}_2$ action that sends each generator to its inverse.

Proof: The first isomorphism follows from the following observations,

a) $C^*(\mathbb{Z}_2)^n$ is the universal $C^*$-algebra generated by $n$ symmetries $U_1, U_2, \cdots, U_n$;

b) $P \mapsto UP = 2P - I$ gives a bijective correspondence between projections and symmetries.

For the second isomorphism note that $C^*(F_{n-1}) \rtimes \mathbb{Z}_2$ is the universal $C^*$-algebra generated by a symmetry $V$ and $n-1$ unitaries $W_1, \cdots, W_{n-1}$ satisfying $VW_i V = W_i^{-1}$ for all $i$. An invertible homomorphism $\pi : C^*(F_{n-1}) \rtimes \mathbb{Z}_2 \to C^*(\mathbb{Z}_2)^n$ is given by

$$\pi(V) = U_1, \quad \pi(VW_i) = U_{i+1} \text{ for } i = 1, \cdots, n-1.$$

$\square$

Remark 2.2 We can identify representations of $A_2$ with that of $C(S^1)$ on some Hilbert space $\mathcal{H}$ along with a symmetry $V$ implementing the conjugation action of $\mathbb{Z}_2$ on $S^1$. Clearly $\mathcal{H}$ must look like $K_+ \oplus K_- \oplus L^2(S_+, \mu_+) \oplus L^2(S_-, \mu_-)$ where $S_+ = \{z \in S^1 : Imz > 0\}$, $S_- = \{z \in S^1 : Imz < 0\}$ and the measures $\mu_\pm$ satisfy $\mu_+ = \mu_- \circ \text{Conjugation}$. The symmetry $V$ switches $L^2(S_+, \mu_\pm)$ and respects $K_\pm$. The representation $\pi : C(S^1) \to \mathcal{H}$ acts as $\pi(f) = \text{multiplication by } f \text{ on } L^2(S_\pm, \mu_\pm)$ and $\pi(f)|_{K_\pm} = f(\pm 1)I_{K_\pm}$. If we denote by $\widetilde{P}_1, \widetilde{P}_2$ the images of the generators $P_1, P_2$ of $A_2$, then they respect $K_+ \oplus K_-$ and commute there. Using the correspondence between projections and symmetries stated in the proof of theorem 2.1 we get that on $L^2(S_+, \mu_+) \oplus L^2(S_-, \mu_-)$, $\widetilde{P}_1 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, and $\widetilde{P}_2 = \frac{1}{2} \begin{pmatrix} 1 & \pi(z) \\ \pi(z) & 1 \end{pmatrix}$. This is the structure theorem for a pair of projections. Note that in this part of the space $\widetilde{P}_1 - \widetilde{P}_2$ is an offdiagonal operator matrix and on $K_- \oplus K_+$ has eigenvalues $\pm 1, 0$. Therefore under the assumption that $(\widetilde{P}_1 - \widetilde{P}_2)^{2k+1}$ is trace class for some $k$, we have the following equality, $tr_\mathcal{H}(\widetilde{P}_1 - \widetilde{P}_2)^{2k+1+2m} = tr_\mathcal{H}(\widetilde{P}_1 - \widetilde{P}_2)^{2k+1}$, $\forall m \geq 0$.

Now we intend to put the index problem tackled by Avron et. al. ([1]) in the framework of NCG. There it has been remarked that the problem has “relations to the work of Connes
We show that the main result of [1] can be deduced as a simple application from results in [2].

**Theorem 2.3** Let $\tilde{P}, \tilde{Q}$ be projections on $\mathcal{H}$ such that $\tilde{P} - \tilde{Q} \in \mathcal{B}_{2k+1}$, the Schatten ideal. Then $\tilde{Q}\tilde{P} : \tilde{P}\mathcal{H} \rightarrow \tilde{Q}\mathcal{H}$ is Fredholm with $\text{Index} (\tilde{Q}\tilde{P}) = \text{tr} (\tilde{P} - \tilde{Q})^{2k+1}$.

**Proof:** Let $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$, $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $\pi : A_2 \rightarrow \mathcal{B}(\mathcal{K})$ be the representation given on the generators $P_1, P_2$ by the prescription $\pi(P_1) = \begin{pmatrix} \tilde{P} & 0 \\ 0 & \tilde{Q} \end{pmatrix}$, $\pi(P_2) = \begin{pmatrix} \tilde{Q} & 0 \\ 0 & \tilde{P} \end{pmatrix}$. Then $(A_2, \pi, \mathcal{K}, \gamma, F)$ is an even Fredholm module. Hence by proposition 2, page 289, and proposition 4, page 296, [3] we get $\tilde{Q}\tilde{P} : \tilde{P}\mathcal{H} \rightarrow \tilde{Q}\mathcal{H}$ is Fredholm with $\text{Index} (\tilde{Q}\tilde{P}) = (-1)^{k+1} \text{tr} \gamma \pi(P_1)[F, \pi(P_1)]^{2k+2} = \text{tr} (\tilde{P} - \tilde{Q})^{2k+3}$. Now remark 2.2 completes the proof. □

**References**

[1] Avron, J.; Seiler, R.; Simon, B.: The index of a pair of projections. *J. Funct. Anal.*, **120**, 1994, no. 1, 220–237.

[2] Connes, A.: Noncommutative differential geometry. *Inst. Hautes tudes Sci. Publ. Math.*, No. 62, 1985, 257–360.

[3] Connes, A.: *Noncommutative Geometry*, Academic Press, 1994.

[4] Davis, Chandler; Kahan, W. M.: The rotation of eigenvectors by a perturbation. III. *SIAM J. Numer. Anal.*, **7**, 1970, 1–46.

[5] Dixmier, J: Position relative de deux varits linaires fermes dans un espace de Hilbert. (French) *Revue Sci.*, **86**, 1948, 387–399.

[6] Halmos, P. R.: Two subspaces *Trans. Amer. Math. Soc.*, **144**, 1969, 381–389.