Large Neutrino Flavor Mixings and Lepton Mass Matrices

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Abstract

Recent atmospheric neutrino data at Super-Kamiokande suggest the near-maximal flavor mixing. Models for the lepton mass matrix, which give the near-maximal flavor mixing, are discussed in this report. Mass matrix models are classified according to the mechanism providing the large flavor mixing, and those are reviewed briefly. “Naturalness” of the mass matrix is also discussed in order to select the neutrino mass matrix. Details of the mass matrix with the $S_3$ flavor symmetry are presented.

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¹To appear in the proceedings of the 1998 "INTERNATIONAL SYMPOSIUM on LEPTON and BARYON NUMBER VIOLATION" at Trento (Italy).
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1 Introduction

The standard model (SM) has still many unexplained features. It is a remarkable property of the quark and the lepton mass spectra that the masses of successive particles increase by large factors. Mixings of the quark sector (CKM matrix) \[1\] seem also to have an hierarchical structure. Those features may provide an important basis for a new physics beyond the SM.

On the other hand, the flavor mixing of the lepton sector is very ambiguous. However, neutrino flavor oscillations provide information of the fundamental property of neutrinos such as masses, flavor mixings and CP violating phase. In these years, there is growing experimental evidences of the neutrino oscillations. The exciting one is the atmospheric neutrino deficit \[2\sim4\] and the solar neutrino deficit \[5\]. Super-Kamiokande \[6\] also presented the large neutrino flavor oscillation in atmospheric neutrinos. Furthermore, a new stage is represented by the long baseline (LBL) neutrino oscillation experiments. The first LBL reactor experiment CHOOZ has already reported a bound of the neutrino oscillation \[8\], which gives a strong constraint of the flavor mixing pattern. The LBL accelerator experiment K2K \[9\] is planned to begin taking data in 1999, whereas the MINOS \[10\] and ICARUS \[11\] experiments will start in the first year of the next century. Those LBL experiments will clarify masses, flavor mixings and CP violation of neutrinos.

The short baseline experiments may be helpful to understand neutrino masses and flavor mixings. The tentative indication has been already given by the LSND experiment \[12\], which is an accelerator experiment for \(\nu_\mu \to \nu_e (\bar{\nu}_\mu \to \bar{\nu}_e)\). The CHORUS and NOMAD experiments \[13\] have reported the new bound for \(\nu_\mu \to \nu_\tau\) oscillation, which has already improved the E531 result \[14\]. The KARMEN experiment \[15\] is also searching for the \(\nu_\mu \to \nu_e (\bar{\nu}_\mu \to \bar{\nu}_e)\) oscillation as well as LSND. The Bugey \[16\] and Krasnoyarsk \[17\] reactor experiments and CDHS \[18\] and CCFR \[19\] accelerator experiments have already given bounds for the neutrino mixing parameters as well as E776 \[20\].

In this report, our starting point as to neutrino mixings is the large \(\nu_\mu \to \nu_\tau\) oscillation of the atmospheric neutrino oscillation with

\[
\Delta m^2_{\text{atm}} = 10^{-3} \sim 10^{-2} \text{eV}^2 , \quad \sin^2 2\theta_{\text{atm}} \geq 0.8 , \tag{1}
\]

which are derived from the recent data of the atmospheric neutrino deficit at Super-Kamiokande \[6\]. Then, questions are raised: Why is there large flavour mixing in the lepton sector in contrast with the quark sector? Are there possible mechanisms providing a large mixing angle from the lepton mass matrices, which are consistent with the quark sector? Answer is ”Yes”. There are many mass matrix models to predict the near-maximal mixing.
2 Origin of Near-Maximal Mixing

In this section, we review lepton mass matrix models, which predict the near-maximal mixing between flavors. We classify mass matrix models according to the mechanism giving the large mixing.

A: See-saw enhancement

The see-saw mechanism of neutrino mass generation gives a very natural and elegant understanding for the smallness of neutrino masses [21]. This mechanism may play another important role, which is to reproduce the large flavor mixing. In the standpoint of the quark-lepton unification, the Dirac mass matrix of neutrinos is similar to the quark mass matrices. Therefore, the neutrino mixings turn out to be typically of the same order of magnitude as the quark mixings. However, the large flavor mixings of neutrinos could be obtained in the see-saw mechanism as a consequence of a certain structure of the right-handed Majorana mass matrix [22][23]. That is the so called see-saw enhancement of the neutrino mixing due to the cooperation between the Dirac and Majorana mass matrices.

Mass matrix of light Majorana neutrinos $m_\nu$ has the following form

$$m_\nu \simeq -m_D M_R^{-1} m_D^T ,$$

(2)

where $m_D$ is the neutrino Dirac mass matrix and $M_R$ is the Majorana mass matrix of the right-handed neutrino components. Then, the lepton mixing matrix is [22]

$$V_\ell = S_\ell^T \cdot S_\nu \cdot V_s ,$$

(3)

where $S_\ell$, $S_\nu$ are transformations which diagonalize the Dirac mass matrices of charged leptons and neutrinos, respectively. The $V_s$ specifies the effect of the see-saw mechanism, i.e. the effects of the right-handed Majorana mass matrix. It is determined by

$$V_s^T m_{ss} V_s = \text{diag}(m_1, m_2, m_3) ,$$

(4)

where

$$m_{ss} = -m_D^{diag} M_R^{-1} m_D^{diag} .$$

(5)

Here $m_i (i = 1, 2, 3)$ are the masses of light neutrinos and

$$m_D^{diag} \equiv \text{diag}(m_{1D}, m_{2D}, m_{3D}) ,$$

(6)

is the diagonalized Dirac mass matrix of neutrinos. In the case of two generations, the mixing matrix $V_s$ is easily investigated in terms of one angle $\theta_s$ as follows:

$$\tan 2\theta_s = \frac{\sin 2\theta_M \epsilon_D (1 - \epsilon)}{\epsilon - \epsilon_D^2 + \sin^2 \theta_M (1 - \epsilon) (1 + \epsilon_D^2)} ,$$

(7)

$$\sin^2 \theta_M = \frac{1}{(1 - \epsilon)(1 - \epsilon_D^2)} \left[ \frac{1}{m_3} \left( 1 + \frac{m_2}{m_3} \right) \sqrt{\epsilon_0 \epsilon - \epsilon - \epsilon_D^2} \right] ,$$

(8)
where
\[ \epsilon \equiv \frac{M_2}{M_3}, \quad \epsilon_D \equiv \frac{m_{2D}}{m_{3D}}, \quad \epsilon_0 \equiv \frac{m_{2D}^2 m_3}{m_{3D}^2 m_2}. \] (9)

Thus, the mixing angle is given in terms of the diagonal components in mass matrices. In the range \( 4\epsilon_D/\epsilon_0 \ll \epsilon \ll \epsilon_0 \) with \( m_2 \ll m_3 \), the mixing can be approximately by
\[ \sin^2 \theta_s \simeq \frac{\epsilon_D^2}{\sqrt{\epsilon_0 \epsilon}}. \] (10)

The mixing becomes maximal value \( \sin^2 \theta_s = 1 \) at \( \epsilon = 4\epsilon_D/\epsilon_0 \). That is the enhancement due to the see-saw mechanism. The rich structure of right-handed Majorana mass matrix can lead to the maximal flavor mixing of neutrinos.

In this estimate, mass matrices are assumed to be real. However, the Majorana mass matrix has a non-trivial phase even in the two generation model. The see-saw enhancement condition should be modified including \( CP \) violating phases [23]. It was found that the see-saw enhancement could be obtained due to the phase even if the Majorana mass matrix is proportional to the unit matrix.

Models which satisfy the see-saw enhancement were proposed on the early stage by Harvey, Reiss and Ramond [24], and Babu and Shafi [25] in the framework of \( SO(10) \). Recent works based on the flavor \( U(1) \) symmetry [26] are attractive examples for the see-saw enhancement. The \( U(2) \) symmetry [27] was also studied focusing on the see-saw enhancement [28]. A successful mass matrix is also presented in the phenomenological point of view [29].

**B:** Type II see-saw model

The conventional see-saw mechanism for neutrino masses is implemented in gauge model such as \( SO(10) \) or the left-right symmetric models. The general form of the see-saw mass matrix is
\[ \left( \begin{array}{cc} f v_L & m_D \\ m_D^T & f M_R \end{array} \right), \] (11)
which gives
\[ m_\nu = f v_L - m_D M_R^{-1} m_D^T \quad \text{with} \quad v_L \simeq \lambda v_{EW}^2/v_R. \] (12)

This is called the type II see-saw formula [30]. Recall that the conventional see-saw formula omits the first term. If due to some symmetry reasons, \( f_{ab} = f_0 \delta_{ab} \), then a degenerate neutrino spectrum emerges. In this model, the flavor symmetry \( S_4 \) or the horizontal \( SU(2)_H \) guarantee the degenerate \( f_{ab} \). For instance, in the range of \( v_R = 10^{13} - 10^{16} \text{GeV} \), the desired value \( v_L \simeq 0.01 \sim 1 \text{eV} \) is quite reasonable. In the minimal SUSY \( SO(10) \) model, the realistic neutrino mixings \( \sin^2 2\theta_\odot \simeq 2.8 \times 10^{-2} \) and \( \sin^2 2\theta_{\text{atm}} \simeq 0.84 \) have been obtained by putting experimental data of masses and CKM matrix elements of the quark sector.

**C:** Exotic fields

The new particle may be essential for the large flavor mixing. The Zee model is a typical model, in which charged gauge singlet Higgs boson plays important role
to give the maximal mixing [31]. Anti-GUT model also need additional some Higgs bosons [32], which lead to the large flavor mixing. The mixing between the ordinary fermions and the exotic ones may be an origin of the large flavor mixing [33][34]. The exogenous mixing based on $SO(10)$ has made possible to investigate the flavor mixing quantitatively [33].

**D:** Large evolution of mixings by RGE’s

There may be another enhancement mechanism of the neutrino flavor mixing. The renormalization group equation (RGE) of the see-saw neutrino mass operators with dimension 5 has been investigated by some authors [35]∼[37]. Babu, Leung and Panteleone pointed out that the neutrino flavor mixing is enhanced by the RGE in the MSSM under the special conditions of the mass matrices. The numerical analyses have been given in ref.[37] focusing on recent experimental data of the atmospheric neutrino deficit.

**E:** Large mixing derived from the charged lepton mass matrix

In the standpoint of the quark-lepton unification, the charged lepton mass matrix is considered to be similar to the down quark one. Then, the mixing following from the charged lepton mass matrix may be considered to be small in the hierarchical base of the quark mass matrix. However, this expectation is not true if the mass matrix is non-Hermitian. In the $SU(5)$ model, the left(right)-handed down quark mixings are related to the right(left)-handed charged lepton mixings because these fermions belong to same representation $5^*$ such as

$$5^* : \psi_L = (d^c_1, d^c_2, d^c_3, e^-, \nu)_L ,$$

and the Yukawa couplings are given by $5^*_i 10_j 5^*_H (i,j=1,2,3)$. This feature was nicely taken into consideration in models of refs.[38] and [39].

It should be noticed that observed quark mass spectra and CKM matrix only constrains the down quark mass matrix as follows [10]:

$$m_{\text{down}} \sim \begin{pmatrix} \lambda^4 & \lambda^3 & \lambda^3 \\ \lambda^2 & \lambda^2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \text{ with } \lambda = 0.22 .$$

The three unknown entries are related to the left-handed lepton mixing in the $SU(5)$ model. Therefore, there is a room in the charged lepton mass matrix to provide a source of the large flavor mixing of the lepton sector.

**F:** Effective neutrino mass matrix

The democratic mass matrix [11] needs the large rotation in order to move to the diagonal base. In the quark sector, this large rotation is canceled each other between down quarks and up quarks. However, the situation of the lepton sector is very different from the quark sector if the effective neutrino mass matrix $m_{\nu LL}^\nu$ is far from the democratic one. Details of the model is discussed in the section 4.
Since the result of the atmospheric neutrino at Super-Kamiokande excites the model building of the lepton mass matrix, new ideas and models will be presented in the near future. Our classification of models with the large flavor mixing will be not enough.

3 Naturalness and Near-Maximal Mixing Angle

Since there are many models, which predict the large flavor mixing, the "naturalness" of the mass matrix is helpful to select models. The idea of the natural mass matrix was proposed in order to restrict severely the arbitrariness in the construction of the quark mass matrices [12].

Let us consider the $2 \times 2$ Hermitian quark mass matrix $M_i (i = u, d)$. Assume for simplicity that it can be diagonalized by some orthogonal matrices $O_i (i = u, d)$ as follows:

$$O_i^T M_i O_i = M_i^{diag} \equiv \begin{pmatrix} m_{i1} & 0 \\ 0 & m_{i2} \end{pmatrix}, \quad (i = u, d), \quad (15)$$

with

$$O_i = \begin{pmatrix} \cos \theta_i & \sin \theta_i \\ -\sin \theta_i & \cos \theta_i \end{pmatrix}, \quad V_q = O_u^T O_d = \begin{pmatrix} \cos \theta_c & \sin \theta_c \\ -\sin \theta_c & \cos \theta_c \end{pmatrix}, \quad (16)$$

where $\theta_c = \theta_d - \theta_u$. Then the mass matrices can be written as

$$M_i = \begin{pmatrix} c_i^2 m_{i1} + s_i^2 m_{i2} & c_i s_i (m_{i2} - m_{i1}) \\ c_i s_i (m_{i2} - m_{i1}) & s_i^2 m_{i1} + c_i^2 m_{i2} \end{pmatrix}, \quad (i = u, d), \quad (17)$$

where $c_i \equiv \cos \theta_i$ and $s_i \equiv \sin \theta_i$. The mixing matrix $V_q$ is invariant under the changes

$$O_d \rightarrow O O_d, \quad O_u \rightarrow O O_u, \quad (18)$$

where $O$ is some arbitrary orthogonal matrix. Thus, using the fact that $\sin \theta_c \ll 1$, we can assume both $\theta_d \ll 1$ and $\theta_u \ll 1$ without loss of generality. Taking into account the quark mass hierarchy, we set

$$m_u = a \lambda^4 m_c, \quad m_d = b \lambda^2 m_s, \quad (19)$$

where $a$ and $b$ are $O(1)$ coefficients, and $\lambda \equiv \sin \theta_c \simeq 0.22$. Thus the mass matrices are expressed in terms of $\theta_u$ and $\theta_d$ as following:

$$M_u \simeq \begin{pmatrix} a \lambda^4 + \sin^2 \theta_u & \sin \theta_u \\ \sin \theta_u & 1 \end{pmatrix} m_c, \quad M_d \simeq \begin{pmatrix} b \lambda^2 + \sin^2 \theta_d & \sin \theta_d \\ \sin \theta_d & 1 \end{pmatrix} m_s. \quad (20)$$

There are three different options now for the angles $\theta_u$ and $\theta_d$:

1. $\sin \theta_d \sim \lambda$, $\sin \theta_u \sim \lambda$,
2. $\sin \theta_d \sim \lambda$, $\sin \theta_u \leq \lambda^2$,
3. $\sin \theta_d \leq \lambda^2$, $\sin \theta_u \sim \lambda$. 

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The options 1 and 3 are unnatural. Indeed, one would require a severe fine-tuning of the matrix element \([M_u]_{11}\) forcing \([M_u]_{11} \simeq ([M_u]_{12})^2\) to arrive at the large \(m_u/m_c \sim \lambda^4\) hierarchy. On the other hand, option 2 gives a natural quark mass matrix without fine-tuning. Here the fine-tuning means a tuning of \(O(\lambda^2)\), which comes from the following inspection. There is a well known phenomenological relation between the CKM matrix element and the quark mass ratio as

\[ |V_{us}| \simeq \sqrt{\frac{m_d}{m_s}} \simeq \lambda . \]  

The down quark mass ratio dominates \(V_{us}\), while the contribution of the up quark mass ratio is at most \(\sqrt{m_u/m_c} \simeq O(\lambda^2)\). This relation is consistent with the option 2. Thus, the criterion of \(O(\lambda^2)\) tuning is very useful in order to select the option 2 by excluding options 1 and 3. The extension of the natural mass matrices to the three family case in quark sector is also given in Ref.[42].

Naturalness is also expanded to the lepton sector [43]. If neutrino sector has the hierarchical mass structure, the naturalness argument is exactly the same as in the example just described for the quark sector. However, in the neutrino sector the inverse mass hierarchy for the flavor is still allowed by the constraints obtained in the disappearance experiments of the neutrino oscillations in Bugey [16], Krasnoyarsk [17], CDHS [18] and CCFR [19]. Since the neutrino mixing is chosen to be \(\sin \theta_\nu \sim 1\) in this case, one cannot always guarantee to have both \(\theta_\nu \ll 1\) and \(\theta_E \ll 1\) (charged lepton mixing) by the change in eq.(18). Therefore, we should reconsider the naturalness of the quark mass matrices in the lepton sector.

In the case of \(s \simeq 1(c \ll 1)\) with \(m_1 \leq m_2\) in eq.(17), the mass matrix is expressed approximately:

\[ M_\nu \simeq \begin{pmatrix} m_2 & cm_2 \\ cm_2 & m_1 + c^2m_2 \end{pmatrix}. \]  

The natural mass matrix without the fine-tuning requires

\[ m_1 \geq c^2 m_2 . \]  

For example, if we take \(c \simeq a'\lambda, \ m_1 \simeq a\lambda m_2\), which satisfies the condition(23), then matrix

\[ M_\nu \simeq \begin{pmatrix} 1 & a'\lambda \\ a'\lambda & a\lambda \end{pmatrix} m_2, \]  

gives us the masses \(m_2\) and \(a\lambda m_2\), correspondingly. Thus, the naturalness condition follows from the (2,2) entry in the case of the inverse mass hierarchy of the flavor, while in the quark sector this condition follows from the (1,1) entry.

It is necessary to clarify the concepts of naturalness for the large mixing angle. Furthermore, the recent experiments [2]~[6] also indicate the large flavor mixing in the neutrinos. Let us consider the \(2 \times 2\) symmetric matrix \(M_\nu\). Again, assume for simplicity that it can be diagonalized by some orthogonal matrix \(O_\nu\):

\[ O_\nu^T M_\nu O_\nu = M_\nu^{diag} \equiv \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}. \]
Here the orthogonal matrix $O_\nu$ is

$$O_\nu = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}, \quad c \equiv \cos \theta_\nu \simeq \frac{1}{\sqrt{2}}, \quad s \equiv \sin \theta_\nu \simeq \frac{1}{\sqrt{2}}. \quad (26)$$

Then $M_\nu$ is expressed in terms of the mass eigenvalues and mixing angle as follows:

$$M_\nu = O_\nu M_\nu^{\text{diag}} O_\nu^T = \begin{pmatrix} m_1 c^2 + m_2 s^2 & cs(m_2 - m_1) \\ cs(m_2 - m_1) & m_1 s^2 + m_2 c^2 \end{pmatrix}. \quad (27)$$

If the neutrino masses have the hierarchy such as $m_1/m_2 = \epsilon \ll 1$ with the large mixing angle, the mass matrix $M_\nu$ is

$$M_\nu \simeq \begin{pmatrix} \epsilon c^2 + s^2 & cs \\ cs & c^2 + \epsilon s^2 \end{pmatrix} m_2. \quad (28)$$

In the (1,1) entry, $\epsilon c^2$ should be fine tuned against $s^2$ in order to arrive at $m_1/m_2 \sim \epsilon$. For example, taking $c \simeq s \simeq 1/\sqrt{2}$, the matrix

$$M_\nu \simeq \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} m_2, \quad (29)$$

gives $m_1 \simeq 0$ and $m_2$. Further if we assume that $\epsilon \simeq a\lambda^2$, the matrix $M_\nu$ in (29) should be replaced by the following matrix

$$M_\nu \simeq \begin{pmatrix} \frac{1+a\lambda^2}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1+a\lambda^2}{2} \end{pmatrix} m_2. \quad (30)$$

The latter gives the mass $m_1 = O(\lambda^2)m_2$ provided that the order of $O(\lambda^2)$ is fine-tuned. Thus, the hierarchical neutrino masses are unnatural in the case of the large mixing angle.

In the case when neutrino masses are approximately degenerate such as $m_1 \simeq m_2$ and $(m_2 - m_1)/m_2 = \epsilon \ll 1$, the mass matrix $M_\nu$ is

$$M_\nu \simeq \begin{pmatrix} 1 & cse \\ cse & 1 \end{pmatrix} m_2. \quad (31)$$

Hence, no fine tuning is required for any entry. We can explore the same argument when the mass eigenvalues are $-m_1$ and $m_2$, in which case

$$M_\nu \simeq \begin{pmatrix} c^2 \epsilon & 2cs \\ 2cs & s^2 \epsilon \end{pmatrix} m_2. \quad (32)$$

Therefore in the case of the large mixing angle we call eqs. (31,32) the natural mass matrix.

In the sequel the naturalness of the mass matrix for the lepton sector is investigated for the three family model. The conclusion in the case of three families as follows: the neutrino mass matrices with quasi degenerate masses and maximal mixing are the natural ones. Quasi degenerate masses means $m_3 \simeq m_2 \gg m_1$ or $m_3 \simeq m_2 \simeq m_1$.

In the next section, we present a texture with the $S_3$ symmetry, which is a typical natural mass matrix.
4 Neutrino Mass Matrix with $S_3$ Symmetry

One of the most attractive description of the quark sector in the phenomenological mass matrix approach starts with an $S_3(R) \times S_3(L)$ symmetric mass term (often called “democratic” mass matrix) [11]:

$$M_q = \frac{c_q}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

(33)

where $q = \text{up and down}$, and quarks belong to $3=2\oplus 1$ of $S_3(L)$ or $S_3(R)$. The same form is still available for the charged lepton sector. However, the neutrino mass matrix is different if they are Majorana particles. The $S_3(L)$ symmetric mass term is given as follows:

$$M_\nu = c_\nu \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + c_\nu r \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

(34)

where $r$ is an arbitrary parameter. The eigenvalues of this matrix are given as $c_\nu (1 + 2r, 1 - r, 1 - r)$, which means that there are at least two degenerate masses in the $S_3(L)$ symmetric Majorana mass matrix [14][15].

If three degenerate light neutrinos are required, the parameter $r$ should be taken as $r = 0$ or $r = -2$. The first case was discussed in ref. [14] and the second case was discussed in ref. [45]. The difference of $r$ leads to the difference in the CP property of neutrinos. If $r = -1/2$, one finds two massive neutrinos and one massless neutrino. So the $S_3(L)$ symmetry could be reconciled with the LSND data [12] by including the symmetry breaking terms.

Alternative representation of the $S_3(L)$ symmetric mass matrix is given as

$$M_\nu = c_\nu \begin{pmatrix} e^{i\alpha} & 1 & 1 \\ 1 & e^{i\alpha} & 1 \\ 1 & 1 & e^{i\alpha} \end{pmatrix},$$

(35)

which is based on the universal strength for Yukawa couplings (USY) hypothesis [16]. If $\alpha = 2\pi/3$, three neutrino masses are degenerate.

In order to reproduce the atmospheric neutrino deficit by the large neutrino oscillation, the symmetry breaking terms are required. Since results are almost same, we show the numerical analyses in ref. [14], where the LSND data is disregarded.

Let us start with discussing the following charged lepton mass matrix:

$$M_\ell = \frac{c_\ell}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} \delta_1^\ell & 0 & 0 \\ 0 & \delta_2^\ell & 0 \\ 0 & 0 & \delta_3^\ell \end{pmatrix}.$$  

(36)

The first term is a unique representation of the $S_3(R) \times S_3(L)$ symmetric matrix and the second one is a symmetry breaking matrix given by Koide [17]. This matrix is diagonalised as

$$U_\ell^\dagger M_\ell U_\ell = \text{diag}(m_1^\ell, m_2^\ell, m_3^\ell),$$

(37)
where
\[ m_1^\ell = (\delta_1^\ell + \delta_2^\ell + \delta_3^\ell)/3 - \xi^\ell/6, \]
\[ m_2^\ell = (\delta_1^\ell + \delta_2^\ell + \delta_3^\ell)/3 + \xi^\ell/6, \]
\[ m_3^\ell = c_\ell + (\delta_1^\ell + \delta_2^\ell + \delta_3^\ell)/3, \]
with
\[ \xi^\ell = [(2\delta_3^\ell - \delta_2^\ell - \delta_1^\ell)^2 + 3(\delta_2^\ell - \delta_1^\ell)^2]^{1/2}. \]

The matrix that diagonalises \( U_\ell = A B_\ell \) reads
\[ A = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}, \]
\[ B_\ell \simeq \begin{pmatrix} \cos \theta^\ell & -\sin \theta^\ell & \lambda^\ell \sin 2\theta^\ell \\ \sin \theta^\ell & \cos \theta^\ell & \lambda^\ell \cos 2\theta^\ell \\ -\lambda^\ell \sin 3\theta^\ell & \lambda^\ell \cos 3\theta^\ell & 1 \end{pmatrix}, \]
with
\[ \tan 2\theta^\ell \simeq -\sqrt{3} \frac{\delta_2^\ell - \delta_1^\ell}{2\delta_3^\ell - \delta_2^\ell - \delta_1^\ell}, \quad \lambda_\ell = \frac{1}{\sqrt{2} 3c_\ell} \xi^\ell. \]

It has been shown that all quark masses and mixing angles are successfully given by taking \( \delta_1 = -\epsilon, \delta_2 = \epsilon \) and \( \delta_3 = \delta \). Analogous to the quark sector, \( \delta_1^\ell = -\epsilon_\ell, \delta_2^\ell = \epsilon_\ell \) and \( \delta_3^\ell = \delta_\ell \) are taken. The three mass eigenvalues are then
\[ m_1^\ell \simeq -\epsilon_\ell^2/2\delta_\ell, \quad m_2^\ell \simeq 2\delta_\ell/3 + \epsilon_\ell^2/2\delta_\ell, \quad m_3^\ell \simeq c_\ell + \delta_\ell/3, \]
and the angle \( \theta^\ell \) is
\[ \sin \theta^\ell \simeq -\sqrt{\frac{m_1^\ell}{m_2^\ell}}. \]

Let us turn to the neutrino mass matrix:
\[ M_\nu = c_\nu \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \epsilon_\nu & 0 \\ \epsilon_\nu & 0 & 0 \\ 0 & 0 & \delta_\nu \end{pmatrix}, \]
where the symmetry braking is given by a small term with two adjustable parameters. An alternative natural choice to lift the mass degeneracy may be \( \text{diag}(-\epsilon_\nu, \epsilon_\nu, \delta_\nu) \), which we shall also discuss later. The mass eigenvalues are \( c_\nu \pm \epsilon_\nu \), and \( c_\nu + \delta_\nu \), and the matrix that diagonalises \( M_\nu (U_\nu^T M_\nu U = \text{diagonal}) \) is
\[ U_\nu = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]
That is, our $M_\nu$ represents three degenerate neutrinos, with the degeneracy lifted by a small parameter.

The lepton mixing angle as defined by $V_\ell = (U_\ell)^\dagger U_\nu = (AB_\ell)^\dagger U_\nu$ is thus given by

$$V_\ell \simeq \begin{pmatrix}
 1 & -\left(\frac{1}{\sqrt{3}}\right)\sqrt{\frac{m_e}{m_\mu}} & \left(\frac{2}{\sqrt{6}}\right)\sqrt{\frac{m_e}{m_\mu}} \\
\sqrt{\frac{m_e}{m_\mu}} & 1/\sqrt{3} & -2/\sqrt{6} \\
0 & 2/\sqrt{6} & 1/\sqrt{3}
\end{pmatrix}, \tag{47}
$$

where the neutrino mass parameters do not appear. The parameters $\epsilon_\nu c_\nu$ and $\delta_\nu c_\nu$ are fixed by the neutrino mass difference explored by the oscillation effect. The normalisation $c_\nu$ is not fixed unless one of the neutrino masses is known, but it is not important for this argument, since the lepton mixing matrix is almost independent of the details of these parameters except for the $m_e/m_\mu$ ratio. If we retain all small terms, the lepton mixing angle is predicted to be

$$V_\ell = \begin{pmatrix}
0.998 & -0.045 & 0.05 \\
0.066 & 0.613 & -0.787 \\
0.005 & 0.789 & 0.614
\end{pmatrix}, \tag{48}
$$

which leads the large $\nu_\mu - \nu_\tau$ oscillation $\sin^2 2\theta_{\text{atm}} \simeq 8/9$. For the $\nu_e - \nu_\mu$ oscillation $\sin^2 2\theta_\odot \simeq 8 \times 10^{-3}$, which also agrees with the neutrino mixing corresponding to the small angle solution of the MSW scenario [18] for the solar neutrino problem [19]. It is remarked that predicted $V_{\ell e3} \simeq 0.05$ is stable against the symmetry breaking parameters. In the future, this prediction will be tested in the following long baseline experiments $\nu_\mu \rightarrow \nu_e$ and $\nu_e \rightarrow \nu_\tau$:

$$P(\nu_\mu \rightarrow \nu_e) \simeq 4V_{\mu e}^2 V_{e3}^2 \sin^2 \frac{\Delta m_{31}^2 L}{4E},
$$

$$P(\nu_e \rightarrow \nu_\tau) \simeq 4V_{e\tau}^2 V_{e3}^2 \sin^2 \frac{\Delta m_{31}^2 L}{4E}. \tag{49}
$$

A very important constraint comes from neutrinoless double beta decay experiments. The latest result on the lifetime of $^{76}\text{Ge} \rightarrow ^{76}\text{Se}$, $\tau_{1/2} > 1.1 \times 10^{25}$ yr [50] yields an upper limit on the Majorana neutrino mass $0.4$ eV [51] to $1.1$ eV [52] depending on which nuclear model is adopted for nuclear matrix elements. We are then left with quite a narrow window for the neutrino mass $0.1\text{eV} \leq m_{\nu_e} \simeq m_{\nu_\mu} \simeq m_{\nu_\tau} \leq 1\text{eV}$ for the present scenario to be viable. It will be most interesting to push down the lower limit of neutrinoless double beta decay lifetime. If the limit on neutrino mass is lowered by one order of magnitude the our degenerate neutrino mass scenario will be ruled out.

The argument we have made above is of course by no means unique, and a different assumption on the matrix leads to a different mass-mixing relation. Let us briefly discuss the consequence of the other matrices we have encountered in the line of our argument above. If we adopt the symmetry breaking term alternative to eq.(45),

$$\begin{pmatrix}
-\epsilon_\nu & 0 & 0 \\
0 & \epsilon_\nu & 0 \\
0 & 0 & \delta_\nu
\end{pmatrix}, \tag{50}
$$
in parallel to the charged lepton and quark sectors, we obtain the lepton mixing matrix to be

\[ V_\ell \simeq \begin{pmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}
\end{pmatrix}. \] (51)

This is identical to the matrix presented by Fritzsch and Xing [53]. For this case one gets

\[ \sin^2 2\theta_\odot \simeq 1, \quad \sin^2 2\theta_{\text{atm}} \simeq 8/9. \] (52)

This case can accommodate the "just-so" scenario for the solar neutrino problem due to neutrino oscillation in vacuum [54]. This matrix has been investigated in detail [55] focusing on recent data at Super-Kamiokande [56]. The "just so solution" leads to the bi-maximal flavor mixing, which may be interesting for the theoretical origin [50].

5 Summary

Atmospheric neutrino deficit at Super-Kamiokande suggests the near-maximal flavor mixing, which have excited the model building of the lepton mass matrices. Mass matrix models are classified according to the mechanism providing the large(maximal) flavor mixing. However, more quantitative studies are needed in order to understand the origin of the large flavor mixings deeply. The studies of the lepton mass matrices will give clues of new symmetry such as the flavor symmetry and will indicate the particular directions for the unification of matter. Furthermore, the structure of the neutrino mass matrix will give strong impact on other fields such as the leptogenesis [62]. The $CP$ violating phase structure in the neutrino mass matrix is also a very attractive subject as well as the $CP$ violation of the quark sector.

Acknowledgements

I thank M. Fukugita and T. Yanagida for collaboration on the lepton mass matrix model with the $S_3$ symmetry, and M. Matsuda for the collaboration on the naturalness of the lepton mass matrices. This research is supported by the Grant-in-Aid for Science Research, Ministry of Education, Science and Culture, Japan(No.10140218, No.10640274).

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