Contact analysis of a planetary gear train using linear complementarity

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Abstract. The current computation models for gear contact analysis and wear prediction are mostly based on finite element principles which consumes a lot of computation time and effort. In this paper, an alternate approach of linear complementarity is used for gear contact analysis. This approach was successfully applied for a pair of rigid spur gears in our previous paper. In this paper, we are extending this to a planetary gear train. The linear complementarity problem formulation is revised to take bilateral constraints into consideration. A linear complementarity solver computes the contact forces between meshing teeth of gears in gear train. From the contact forces, sliding wear in gear teeth is predicted. Archard’s wear model is used for the wear prediction.

1. Introduction

Gear trains are one of the effective and popular means for power and motion transmission between shafts. Contact analysis is an essential part to evaluate life expectancy, safety parameters and integrity of gear trains. Currently, two types of analysis models are being followed: 1) Lumped Parameter Model [1] 2) Finite Element Model [2]. Gear train is a multi-body system with multiple contact configurations that cannot be solved using equations of motion alone. Hence, we have used linear complementarity [3] for doing multiple contact analysis in gear trains. This approach was successfully applied to a pair of rigid spur gears in our previous work and the results were compared with those from finite element model [4]. The contribution of current work is twofold. First, we have extended our linear complementarity analysis of a pair of meshing spur gears to a more complicated case of planetary gear train. Second, we have incorporated the reaction forces at bearings as bilateral constraints in linear complementarity problem (LCP) formulation. From linear complementarity analysis, we have calculated the contact forces at meshing teeth in gear train. From the contact forces, sliding wear in gear teeth is predicted using Archard’s wear model [5].

2. Method

The first step in contact analysis is to create kinematic model for determining the contact points along path of contact of meshing gears and second step is the linear complementarity analysis to get the contact forces at the contact points.
2.1. Kinematic Model
For a pair of mating spur gears, contact points always lie on the line of action. This line of action is fixed for a given gear pair owing to the involute profile of gear tooth. Sometimes there is only one tooth in contact and other times there are two teeth in contact. Thus, the contact ratio is in between 1 and 2.

![Figure 1. Kinematic model of planetary gear train](image1)

![Figure 2. Input and output torques for linear complementarity model](image2)

The kinematic model for planetary gear train is shown in Fig. 1. The input motion is given to sun gear. The carrier arm is fixed and the output is ring gear. The equations for the involute profile of spur gears and internal gear are used for the generation of exact gear tooth geometry [6].

2.2. Linear Complementarity Model
LCP consists of a set of non-negative variables that are pairwise complementary (i.e. either one variable is positive and the other is zero or vice versa) and are related by a set of linear equations. According to Anitescu-Potra’s model [7], for every contact point \( j \), the normal contact velocity \( v_j \) and the normal contact force \( F_j \) form a complementary pair, i.e., \( F_j \geq 0, \ v_j \geq 0, \) and \( F_j v_j = 0 \). This statement is also represented as \( 0 \leq F_j \perp v_j \geq 0 \).

Since \( v_j \) and \( F_j \) are linearly related, \( \sum_{j=1}^{n} F_j v_j = F^T v = F^T(AF + b) = 0 \), where \( n \) is the number of contact points, \( A \in \mathbb{R}^{n \times n} \) is a positive semidefinite matrix having information on body mass and inertia and \( b \in \mathbb{R}^n \) is a vector consisting of all external forces and body forces.

If friction need to be considered into the model, a tangential component of force and velocity at every contact point also need to be considered. The magnitude of the friction force at contact point \( j \) is linearly related to the normal force by Coulomb’s law.

Hence the polyhedral approximation of the friction cone is considered. For two dimensional case, the tangent space is a line separating two bodies at contact point [7], [8].

The Anitescu-Potra’s model [7] gives a complementarity formulation for rigid body contact problems with dynamic friction that can be solved using Lemke’s algorithm [3]. In Anitescu-Potra’s model, at each contact point \( j \), the complementary constraints are:

\[
0 \leq v_{nj} \perp p_{nj} \geq 0 \quad (1)
\]

\[
0 \leq \rho_j = \lambda_j e_j + W^T f_j v, \perp p_{fj} \geq 0 \quad (2)
\]

\[
0 \leq \zeta_j = \mu p_{nj} - e_j^T p_{fj}, \perp \lambda_j \geq 0 \quad (3)
\]
where $\perp$ indicates the complementary variables, $v_{nj} (= c_{nj}^T v)$ is the normal velocity between the two bodies at contact point $j$, $p_{nj}$ is the normal component of the contact impulse, $p_{fj}$ is the friction component of the contact impulse, $e_j \in \mathbb{R}^D$ is a vector of ones, $D$ is the number of edges of the friction pyramid, $\lambda_j$ is a variable that approximates the magnitude of the sliding velocity, $\mu$ is the Coulomb friction coefficient, $n_d$ is the dimension of the configuration space, and $W_{fj} \in \mathbb{R}^{n_d \times D}$ is the friction contact wrench matrix that transforms the generalized velocity $v \in \mathbb{R}^{n_d}$ of the body along the tangential component of the friction cone. For a two-dimensional problem ($n_d = 3$, two translations and one rotation), at each contact point $j$, there are only two directions $\{t_j, -t_j\}$ of the friction component of the contact impulse, each perpendicular to the outward contact normal $c_{nj}$. The friction contact wrench matrix $W_{fj}$ is 

$$
W_{fj} = \begin{bmatrix}
t_j & -t_j & r_j \otimes t_j & -r_j \otimes t_j
\end{bmatrix},
$$

where $r_j$ is the vector from the body centroid to contact point $j$, and $\otimes$ is the two-dimensional equivalent of cross-product. In addition to these complementary constraints, we have equations of motion for the system:

$$
\bar{M} (\dot{v}^t - \dot{v}^{t-1}) - \bar{J}^T \dot{v}^t - \bar{W} n \bar{p}_n - \bar{W} f \dot{p}_f = \left( \bar{F}_b + \bar{F}_{ext} \right) \Delta t
$$

(4)

$$
\bar{J} \dot{v}^t = 0
$$

(5)

where superscript $t$ indicates value at time $t$. $\bar{p}_n = [p_{n1}, \ldots, p_{nk}]^T$ is the concatenated normal contact impulses for all the $K$ contact points in the assembly. Similarly $\bar{p}_f = [p_{f1}, \ldots, p_{fk}]^T$ is the concatenated friction impulses. $\bar{M}$ is the concatenated mass matrix, $\bar{W}_n = [W_{n1}, \ldots, W_{nk}]$ is the concatenated normal contact wrench matrix where $W_{nj} = [c_{nj}^T r_j \otimes c_{nj}]$, $\bar{W}_f = [W_{f1}, \ldots, W_{fk}]$ is the concatenated friction wrench matrix where $W_{fj}$ is the friction contact wrench matrix that transforms the generalized velocity at time $t$, $\Delta t$ is the time step of simulation, $\bar{F}_{ext}$ is the concatenated external generalized forces (input torques) for all the bodies in the system and $\bar{F}_b$ is the concatenated body forces, $\bar{J}$ is the concatenated constraint Jacobian matrix of the system. A gear rotating on its bearing is modelled as a gear hinged at its center and the corresponding constraint Jacobian matrix becomes,

$$
\bar{J} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},
$$

$\bar{J}$ is the concatenated constraint reaction impulses for all the parts. Thus the linear complementarity model [7] for multibody-multiple contact problems can be written as:

$$
0 \leq \begin{bmatrix}
\bar{A}_{nn} & \bar{A}_{nf} & \bar{A}_{ff} & \bar{E} & 0 \\
\bar{A}_{fn} & \bar{A}_{fn} & \bar{A}_{ff} & \bar{E} & 0 \\
\mu & -\bar{E}^T & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
\bar{p}_n^t \\
\bar{p}_f^t \\
\bar{b}_f^t \\
\bar{b}_n^t \\
\bar{\lambda}^t
\end{bmatrix} + \begin{bmatrix}
\dot{\bar{v}}_n^t \\
\dot{\bar{v}}_f^t \\
\dot{\bar{b}}_f^t \\
\dot{\bar{b}}_n^t \\
\bar{\lambda}^t
\end{bmatrix} = \begin{bmatrix}
\bar{p}_n^0 \\
\bar{p}_f^0 \\
\bar{b}_f^0 \\
\bar{b}_n^0 \\
\bar{\lambda}^0
\end{bmatrix} \perp \begin{bmatrix}
\bar{p}_n^t \\
\bar{p}_f^t \\
\bar{b}_f^t \\
\bar{b}_n^t \\
\bar{\lambda}^t
\end{bmatrix} \geq 0
$$

(6)

where $\bar{\lambda} = [\lambda_1, \ldots, \lambda_K]^T$ are concatenated $\lambda$s for all the $K$ contact points in the assembly, and $\bar{A}_{nn} = W_{n1}^T \bar{W} W_{n1}$, $\bar{A}_{nf} = W_{n1}^T \bar{W} W_{f1}$, $\bar{A}_{ff} = W_{f1}^T \bar{W} W_{f1}$, $\bar{A}_{fn} = W_{f1}^T \bar{W} W_{n1}$, $\bar{b}_f = \bar{W}_f^T \bar{W} \left( \begin{bmatrix} \bar{F}_b + \bar{F}_{ext} \end{bmatrix} \Delta t + \bar{M} \dot{v}_f^{t-1} \right)$, $\bar{b}_n = \bar{W}_n^T \bar{W} \left( \begin{bmatrix} \bar{F}_b + \bar{F}_{ext} \end{bmatrix} \Delta t + \bar{M} \dot{v}_n^{t-1} \right)$, $\bar{W} = \bar{M}^{-1} \left( I - J^T \bar{G}^{-1} J \bar{M}^{-1} \right)$, $\bar{G} = \bar{J} \bar{M}^{-1} J^T$, $\bar{\mu} = diag[\mu_1, \ldots, \mu_K]$, $\bar{E} = diag[e_1, \ldots, e_K]$, $\bar{v}_n = [v_{n1}, \ldots, v_{nk}]^T$, $\bar{p} = [p_{1}, \ldots, p_{K}]^T$, $\bar{\lambda} = [\tilde{\lambda}_1, \ldots, \tilde{\lambda}_K]^T$.

In this problem, it is considered that gears are running in steady state, i.e., running at constant velocity and transmitting constant torque. This ensures static equilibrium condition at all contact configurations. Hence, the contact configurations are independent of each other as meshing progresses along the line of contact, and doing simultaneous static contact analysis at contact points is equivalent to continuous contact analysis. The steady state input-output torques are shown in Fig. 2. The $\bar{F}_{ext}$ in Eq. (4) for sun gear: $[0 \ 0 \ -T_1]^T$, for planet gears:
\[[0 \ 0 \ 0]^T \text{ and for ring gear: } [0 \ 0 \ -T_2]^T. \ T_1 \text{ and } T_2 \text{ are the respective torque values acting on sun and ring gear.}

After each time step $\Delta t$, we update the positions of gears, and use the LCP (Eq. (1) to Eq. (6)) to find the contact forces at the current time step. PATH solver [9] is used to solve the LCP.

### 2.3. Calculation of Wear

After obtaining contact forces, contact pressure values are computed using expression [10],

$$p_H = \frac{2F_{nt}}{\pi a_H}$$

where $p_H$ is the local contact pressure, $F_{nt}$ is the normal load transmitted at a particular point, $a_H$ is the semi-hertzian contact width. Archard’s wear theory [5] is applied to obtain the wear depth for each point on the gear tooth flank. Archard’s wear equations give relationship between contact pressure, sliding distance and wear.

$$\frac{h}{S} = k_w p_H$$

where $h$ is the wear depth, $S$ is the sliding distance the point is sliding against the interacting surface and $k_w$ is the dimensional wear constant.

### 3. Results and Discussion

The gear parameters considered for generating gear geometry are given in Table 1.

| Parameter                  | Sun  | Planet | Ring |
|----------------------------|------|--------|------|
| Profile                    | Involute | Involute | Involute |
| Type                       | Input gear | Idler gear | Output gear |
| Number of Teeth            | 24   | 36     | 96   |
| Module (mm)                | 2.5  | 2.5    | 2.5  |
| Pressure angle (degree)    | 20   | 20     | 20   |
| Pitch circle diameter (mm) | 60   | 90     | 240  |
| Tip circle diameter (mm)   | 65   | 95     | 235  |
| Base circle diameter (mm)  | 56.38| 84.57  | 225.53|
| Mass (mm$^2$)              | 2827.43 | 6361.725 | 45238.93 |
| Polar moment of inertia (mm$^4$) | 1.27e+6 | 6.44e+6 | 3.26e+8 |
| Face Width (mm)            | 15   | 15     | 15   |
| Modulus of Elasticity (N/mm$^2$) | 2.06e+5 | 2.06e+5 | 2.06e+5 |
| Poisson’s ratio            | 0.3  | 0.3    | 0.3  |
| Wear coefficient (mm$^2$N$^{-1}$) | 5e-10 | 5e-10 | 5e-10 |
| Torque (Nm)                | 3.75e+4 | -      | 1.5e+5 |
| Speed (rpm)                | 1500 | 1000   | 375  |
| Coulomb friction coefficient | 0.2 | 0.2    | 0.2  |

In linear complementarity model, multiple static contact analysis are performed for various contact points in the path of contact. Fig. 3a shows the contact force is shared equally between
Figure 3. Load on gear tooth per unit face width vs contact point radii, (a): Sun gear (b): Ring gear

Figure 4. Load ratio vs distance from pitch point, (a): Sun gear (b): Ring gear

Figure 5. Wear depth vs contact point radii, (a): Sun gear (b): Ring gear

gear teeth during double pair contact in sun gear. During single pair contact, the single gear tooth takes up all the contact load. A sudden drop in the contact load can be seen at pitch point which is because of the reversal of the friction force direction at pitch point. Fig. 3b shows a
single tooth takes up all the contact load during single pair contact in ring gear. The contact force is shared nearly equal between gear teeth during double pair contact. The small variations observed in this region is because of the changing contact force values at the meshing between sun and planet gears and also the reversal of friction force direction at the pitch point in case of sun gear as well as ring gear. Fig. 4 shows the load ratio versus distance from pitch point plots for sun and ring gear. The sliding velocity expressions are given as [5] [11]:

\[ v_s = (w_p + w_g) y \] \[ \text{[For meshing of external gears]} \] (9)

\[ v_s = (w_p - w_g) y \] \[ \text{[For meshing of external pinion and internal gear]} \] (10)

where \( v_s \) is sliding velocity, \( w_p \) and \( w_g \) are magnitudes of angular velocities of pinion and gear respectively and \( y \) is the distance along line of action between pitch point and arbitrary point chosen. From contact forces and sliding velocities, sliding wear is predicted using Archard's law (Eq. (8)) for different points on the gear tooth profile (Fig. 5). It can be observed that the overall wear depth for ring gear is smaller than that for the sun gear. This is because the sliding velocity between planet gears and ring gear is smaller than that between sun gear and planet gears. Both the wear plots show sudden change in wear due to sudden change in contact load near the region of transition from double pair contact to single pair contact. The wear at pitch point is zero due to pure rolling. The wear predicted goes on increasing as we go away from the pitch point due to increasing value of sliding velocity (Eq. (9) and Eq. (10)). On an average, the computation time required for the analysis of each contact configuration in planetary gear train is 0.2 seconds (with a system having configurations: 16GB RAM, Intel Core i5 processor and Windows 10 OS).

4. Conclusion

In present work, linear complementarity model is extended to a more complicated case of planetary gear train. The reaction forces at bearings are incorporated as bilateral constraints in LCP formulation. The model developed here is meant for rigid bodies. Future work will focus on the extension of linear complementarity model to gears as deformable bodies.

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