A proposal to classify the radian as a base unit in the SI

Peter J. Mohr and William D. Phillips
National Institute of Standards and Technology, Gaithersburg, MD 20899, USA

We propose that the SI be modified so that the radian is a base unit. Some of the details and consequences of such a modification are examined.

I. INTRODUCTION

1. Background

When the International System of Units (SI) was established by Resolution 12 of the General Conference on Weights and Measures (CGPM) in 1960, units were classified into three classes, base units, supplementary units, and derived units. The category of supplementary units consisted of the radian for plane angle and the steradian for solid angle.

In 1995, Resolution 8 of the 20th CGPM stated the decision “to interpret the supplementary units in the SI, namely the radian and the steradian, as dimensionless derived units, the names and symbols of which may, but need not, be used in expressions for other SI derived units, as convenient.”

2. Current status of plane angle and solid angle

As a result of the reclassification of the radian and steradian as dimensionless derived units, they are listed in the SI Brochure [1] in Table 3, “Coherent derived units in the SI with special names and symbols,” as the unit for plane angle expressed as m/m in terms of SI base units and as the unit for solid angle expressed as m²/m² in terms of SI base units, respectively. In a footnote to Table 3, it is stated that: “The radian and steradian are special names for the number one that may be used to convey information about the quantity concerned. In practice the symbols rad and sr are used where appropriate, but the symbol for the derived unit one is generally omitted in specifying the values of dimensionless quantities.”

In the above quotation, the italics are added for emphasis; we view the italicized statement as nonsense. Furthermore, this practice has led to errors in published results for physical quantities involving angles and frequencies.

3. Proposal

It is proposed here that angles be considered to have dimension (the dimension of angle) and that therefore the radian no longer be considered a dimensionless derived unit and instead be reclassified as a base unit. It should be considered the coherent SI unit for plane angle and for phase, sometimes called phase angle [2]. This has implications for the units of frequency, as explained below.

The steradian is then a derived unit with the unit rad².

II. SUPPLEMENTAL MATERIAL

1. Plane angle and phase as physical quantities

Plane angles and phase angles have properties similar to other measurable physical quantities. In particular, the value of a plane angle or phase angle θ can be written in the SI as

\[ \theta = \{\theta\} [\theta] \]

where \{θ\} is the numerical value of the angle in the unit radian and [θ] is the unit rad. For a plane angle, the numerical value of the angle between two intersecting straight lines is the ratio of the length of the arc s between the lines of a circle centered at the vertex of the angle to the radius of the circle \{θ\} = s/r and the unit is [θ] = rad. In the current SI, the unit [θ] may be either omitted or replaced by the number 1.

It is sometimes said that angles are dimensionless quantities, because (as suggested in the SI Brochure) the ratio s/r is a length divided a length and is therefore just a number. In fact, the number in curly brackets for any physical quantity is just a number, but that does not make the physical quantity itself dimensionless.

For example, consider a two-meter long table. The numerical value of the length of the table in meters is the ratio of the length of the table to the length of a meter stick. This does not mean that the length of the table is a dimensionless quantity; it has the dimension of length with the unit of meter. Similarly, this does not mean that angle is a dimensionless quantity; it has the dimension of angle with the unit of radian.

It is an essential tenet of this note that the radian should not be considered a dimensionless unit in the SI; it should have dimension angle in the SI; it is not derived from other base units, and therefore should be a base unit itself, rather than a derived unit as in the present SI.
2. Relations to non-SI units for angles

As with any physical quantity, angles may be expressed in units other than coherent SI units. However, units other than the radian cannot also be coherent SI units if the radian is. Some other units for angles are degrees, gradians, turns, cycles, and revolutions. Angles range anywhere from zero to a complete revolution; angles greater than π rad are sometimes called reflex angles. A complete revolution is an angle of 2π rad.

Some relations to the non-SI units are

\[180^\circ = \pi \text{ rad}\] \hspace{1cm} (2)
\[100 \text{ grad} = \frac{\pi}{2} \text{ rad}\] \hspace{1cm} (3)
\[1 \text{ turn} = 2\pi \text{ rad}\] \hspace{1cm} (4)
\[1 \text{ cycle} = 2\pi \text{ rad}\] \hspace{1cm} (5)
\[1 \text{ revolution} = 2\pi \text{ rad}\] \hspace{1cm} (6)

3. Periodic phenomena

A physical phenomenon that repeats in time over a regular time interval is periodic or cyclic. One repetition of the phenomenon that happens in a period is a cycle. The rate of repetitions is a physical quantity called the frequency. The numerical value of the frequency will depend on the units in which it is expressed.

We recommend that the SI states that the value of any frequency of a periodic phenomenon include the complete unit such as rad/sec or Hz, but never just s^{-1}. An example is a turning bicycle wheel. If the wheel rotates by 2 radians in one second, its rotational frequency in SI units is 2 rad/s. Another example is electromagnetic radiation, where the electric field vector undergoes a periodic repetition of one million cycles per second. This frequency is

\[\nu = 1 \text{ MHz},\] \hspace{1cm} (7)

where MHz is one million cycles per second. Since the electric field vector may be described as

\[E \sin \omega t,\] \hspace{1cm} (8)

where the time dependence of the vector repeats when \(\omega t\) increases by 2π. (Here we take \(\omega t\) to represent the numerical value of the product.) If \(t\) is expressed in seconds, then \(\omega\) must be expressed in the coherent unit rad/s:

\[\omega = 2\pi \times 10^6 \text{ rad/s}.\] \hspace{1cm} (9)

4. Difference with the current SI

There are two principal differences between the formulation above and the current SI. One consequence of writing plane angles and phase angles as quantities with explicit units is that an ambiguity present in the current SI is prevented. In particular, in the current SI, since the SI unit radian is considered a dimensionless unit, it may be omitted. The same is true for the non-SI unit "cycle," which is omitted when the currently permitted replacement Hz \(\rightarrow \text{s}^{-1}\) is made. As a result of these prescriptions, the units on both sides of Eq. \(\ref{cycle}\) may be omitted leading to an inconsistent result.

The other principal difference that results from regarding angles as quantities with units that may not be dropped is that Hz, i.e., cycles/second, explicitly includes the non-SI unit cycle and is therefore not a coherent SI unit. Instead, radians/second is the coherent SI unit for frequency. As a consequence, Hz should be considered a unit “permitted for use with the SI.”

5. Mathematics vs physics

In texts and monographs on mathematics, particularly calculus and complex variable theory, there is little or no discussion of units. Lengths and angles are simply numbers with no units and with no mention of meters or radians. As a result, mathematical functions such as trigonometric functions, the exponential function, spherical Bessel functions, spherical harmonics, or any number of other mathematical functions are defined as functions of a dimensionless real or complex number.

On the other hand, in physics, it is necessary to include units in order to make contact with measurements. The values of physical quantities are given by a number times a unit. Thus when, for example, the sine function is used in physics, the argument often is written as a quantity whose units are understood to be radians, as in Eq. \(\ref{cycle}\). However, since the sine function can be defined by its Taylor series, and the terms must be homogeneous in their units, the argument is necessarily a number with no unit.

This leads to a problem with the use of the radian as a unit in the argument of functions that are defined as having dimensionless arguments. An unambiguous resolution of this conflict would be to write the argument of mathematical functions as the numerical value of the angle. That is, to write the sine function in Eq. \(\ref{cycle}\) as \(\sin \{\omega t\}\), where \(\omega t = \{\omega t\}\) rad.

However, in scientific publications, functions are written with arguments without the curly brackets, and it would be too disruptive to the common practice to insist that curly brackets should be present. So a compromise would be to recommend that when angles are present in mathematical functions and expressed in units of radians, the curly brackets be omitted. By the same token, it should also be recognized that when quantities that are the arguments of functions such as the sine function are used in equations outside of those functions, the radian unit must be explicitly restored.
6. Illustrative example

A simple example that illustrates the points made above is the physics of a harmonic oscillator. The equation for a harmonic oscillator is

\[ m \frac{d^2}{dt^2} x(t) + kx(t) = 0, \]  

(10)

where \( m \) is the mass and \( x(t) \) is the coordinate of the body and \( k \) is the spring constant of the spring. The solution for the motion of the body is

\[ x(t) = x_0 \sin \left\{ \sqrt{\frac{k}{m}} t + \phi \right\}, \]  

(11)

where \( x_0 \) is the amplitude of the motion and \( \phi \) is a phase that depends on the boundary condition. The curly brackets emphasize that the argument of the sine function is the numerical value of the enclosed physical quantities. The frequency of the oscillations is

\[ \text{frequency} = \sqrt{\frac{k}{m}} = \left\{ \sqrt{\frac{k}{m}} \right\} \text{s}^{-1}. \]  

(12)

In the current SI, where both \( \text{rad/s} \) and Hz may be expressed as \( \text{s}^{-1} \), this could be taken to mean both

\[ \text{frequency} = \left\{ \sqrt{\frac{k}{m}} \right\} \text{Hz} \quad \text{(incorrect)} \]  

(13)

and

\[ \text{frequency} = \left\{ \sqrt{\frac{k}{m}} \right\} \text{rad s}^{-1} \quad \text{(correct)}. \]  

(14)

The choice of Eq. (14) as the correct expression of the frequency is evident from the rule (see above) that when the argument of a trigonometric function (or the imaginary argument of an exponential function) is used outside of such functions, the unit radian must be explicitly restored.

7. On the nature of units

In this brief note, we have proposed that the radian be viewed as a base unit in the SI. In this section, we clarify some issues relating to units.

Values of most physical quantities \( q \) can be written as

\[ q = \{q\}[q], \]  

(15)

where \( \{q\} \) is the numerical value of the quantity expressed in the unit \([q]\). As seen in Eq. (11), angles fall into this category. Another category of physical quantity, examples of which are the proton-electron mass ratio and the fine-structure constant, are indeed dimensionless and unitless. They are simply numbers. Angles do not fall into this category.

Concerning units for frequency, some of the ambiguity and confusion over frequency arises from the idea that frequency may be classified into types of frequency with different units being used for different types. The prime example is Hz vs rad/s which might be considered different types of frequency and so are expressed in different units. However, frequency is a general concept and classifying types of frequency leads to confusion. For frequency, there is one coherent unit, which is rad/s, and which applies to all types of frequency whether cycles of electromagnetic radiation or rotations of a wheel or heartbeats. In the latter case, the role of radians per second is easily seen by considering a Fourier decomposition of the pattern of the heartbeats and realizing that the fundamental frequency can be viewed as the rate of change of the phase of a sine function with the argument being the numerical value of the phase angle expressed in rad.

This is analogous to energy, for example. There are different types of energy, mechanical energy, energy of electromagnetic radiation, gravitational potential energy, but these are recognized to be various forms of energy, and so they all have the same unit, joule. Of course, this is based on physical principles and may be counterintuitive to people unfamiliar with these principles, but nevertheless is well understood to be true.

While in our proposal the radian becomes a base unit and angles have dimension, we recognize that historic and common practice often drop the unit radian where it would otherwise appear. With the exception of frequencies and only if no confusion can result, we would continue to allow the unit radian to be dropped.

In the new SI, expected to be adopted in 2018, all units are defined by giving an exact value to a fundamental physical constant, although not all the definitions strictly adhere to that designation. In the case of angle, one could specify the unit radian by fixing the numerical value of a right angle in radians by writing

\[ \theta_\perp = \frac{\pi}{2} \text{ rad}, \]  

(16)

where \( \theta_\perp \) a right angle.

This note focuses on angles, radians and related quantities, because the ambiguity in the SI associated with these concepts has often led to errors and confusion. Other similar quantities, such as counting units, may need to be addressed in a similar manner, but this is beyond the scope of this note.

Finally, we note that in the new SI, the distinction between base units and derived units may be unnecessary. The same may be true of the distinction between dimensions and units.
[1] BIPM. *Le Système international d’unités (SI)*. Bureau International des Poids et Mesures, Sèvres, France, 8th edition, 2006.

[2] Here the term phase refers to a physical phase as in quantum mechanics or electrodynamics. In mathematics, particularly complex variable theory, phase is regarded as a dimensionless number as is length.