Exact interior solutions for rigidly rotating stationary cylindrical fluids with azimuthal pressure

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Exact solutions to Einstein’s equations of general relativity representing interior spacetimes sourced by rigidly rotating stationary cylindrical fluids with azimuthally directed pressure are found. These solutions are bounded by a cylindrically symmetric hypersurface on which they are required to satisfy junction conditions. Elementary flatness and regularity conditions are imposed. The mathematical and physical properties of these solutions are analysed. A physical interpretation of their constant parameter is provided. This parameter distinguishes different spacetimes while determining their properties. This particular set of solutions is the second in a series involving three cases of complementary rigidly rotating stationary anisotropic fluids where each principal stress is, in turn, required to dominate. A first class of solutions exhibiting purely axial anisotropic pressure has already been presented in a previous article and is compared to the one disclosed here.

I. INTRODUCTION

Rotating cylindrically symmetric spacetimes have been extensively investigated for a number of different purposes \[1, 2\]. See also \[3\] for a recent review on cylindrical systems in general relativity (GR). Stationarity being an interesting simplifying assumption for starting the exploration of such a domain, it has been retained in 1932 by Lewis for the building of his solution \[4\]. The Lewis solution describes a vacuum spacetime gravitationally sourced by a cylinder of stationary matter rotating around its axis of symmetry. The constant parameters appearing in the metric functions can be real or complex. They are thus said to belong to the Weyl class or to the Lewis class, respectively. A vacuum spacetime gravitationally sourced by a cylinder of matter in translating motion along its symmetry axis is mathematically similar to a Lewis solution where the coordinates \(z\) and \(\phi\) have been exchanged. Their physical properties are however different \[5\].

In \[6\], nonvacuum stationary spacetimes sourced by a rigidly rotating cylindrical anisotropic fluid have been considered. This study has been extended to nonrigid rotation in \[7\], where the analysis of the rigid case has also been supplemented, while the gravitoelectromagnetic properties of its Weyl tensor have been extensively analysed.

In \[8\], the study of the anisotropic properties of the fluid source have been systematically undertaken by setting to vanish two among the three components of its principal stress. There, a double class of solutions to the Einstein field equations with purely axial stress has been exhibited and a number of its main properties have been analysed. In the present work, solutions of GR are displayed for the case of purely azimuthal stress. As the purely axial ones \[8\], these new solutions are exact and are thoroughly examined here where a number of their mathematical and physical features are studied.

To the author’s knowledge, this series of works constitute the first attempt to study exact anisotropic stationary rotating cylindrical spacetimes. Even though each equation of state proposed in turn is rather simple, as it is often the case when one starts the exploration of a new domain, they might however prove useful as an approximation to improve our cosmological and astrophysical understanding. In any case, these exact solutions to a new physical configuration should allow us to improve our knowledge of cylindrically symmetric fluids in GR.

The contents of the paper run as follows: the stress-energy tensor and the line element of interest are provided in section \[II\] In section \[III\] the field equations are displayed and their solutions are constructed. Then, the regularity and junction conditions are applied and yield the final form of the solutions. Some among their prominent mathematical and physical properties are analysed in section \[IV\] A discussion comparing the purely azimuthal pressure case studied here with the purely axial one analysed in \[8\] is displayed in section \[V\] The conclusions are disclosed in section \[VI\]

II. INTERIOR SPACETIME SOURCED BY THE ANISOTROPIC FLUID

The cylinder of fluid is stationary and rigidly rotating around its axis of symmetry. It is an anisotropic nondissipative fluid bounded by a cylindrical surface \(\Sigma\). Its principal stresses \(P_r, P_z\) and \(P_\phi\) satisfy the equation of state \(P_r = P_z = 0\), which allows one to write its stress-energy tensor – see (1) of \[2\] for its general expression – under the form

\[ T_{\alpha\beta} = \rho V_\alpha V_\beta + P_\phi K_\alpha K_\beta, \] (1)
with $\rho$, the energy density of the fluid, $V_\alpha$, its timelike 4-velocity, and $K_\alpha$, a spacelike 4-vector, verifying
\[ V^\alpha V_\alpha = -1, \quad K^\alpha K_\alpha = 1, \quad V^\alpha K_\alpha = 0. \tag{2} \]

With a spacelike Killing vector $\partial_z$ hypersurface orthogonal, such as to ease its subsequent matching to an exterior Lewis metric, the line element reads
\[ ds^2 = -f dt^2 + 2k dt d\phi + e^\mu (dr^2 + dz^2) + l d\phi^2, \tag{3} \]
with $f$, $k$, $\mu$, and $l$, real functions of the radial coordinate $r$ only, such as to allow for stationarity. Due to cylindrical symmetry, the coordinates are bound to conform to the following ranges:
\[ -\infty \leq t \leq +\infty, \quad 0 \leq r, \quad -\infty \leq z \leq +\infty, \quad 0 \leq \phi \leq 2\pi, \tag{4} \]
with the two limits of the coordinate $\phi$ topologically identified. These coordinates are denoted $x^0 = t$, $x^1 = r$, $x^2 = z$, and $x^3 = \phi$.

**III. CONSTRUCTION OF THE NEW SOLUTION**

In the case of rigid rotation, a frame corotating with the fluid can be chosen \[6–8\]. Thus, the 4-velocity of the fluid can be written as
\[ V^\alpha = v \delta^\alpha_0, \tag{5} \]
with $v$ a function of $r$ only. Therefore, the timelike condition for $V^\alpha$ displayed in (2) reads
\[ f v^2 = 1. \tag{6} \]

The spacelike 4-vector $K^\alpha$ satisfying conditions (2) can be written as
\[ K^\alpha = -\frac{k v}{D} \delta^\alpha_0 - \frac{f v}{D} \delta^\alpha_3, \tag{7} \]
where the function $D(r)$ is defined as
\[ D^2 = fl + k^2. \tag{8} \]

A. Field equations

Using (5)–(8) into (1), the components of the stress-energy tensor corresponding to the five nonvanishing components of the Einstein tensor are obtained, and the following five field equations for the spacetime inside $\Sigma$ can be written as
\[ G_{00} = \frac{e^{-\mu}}{2} \left[ -f \mu'' - 2f \frac{D''}{D} + f'' - f' \frac{D'}{D} + \frac{3f(f' l' + k'^2)}{2D^2} \right] = \kappa \rho f, \tag{9} \]
\[ G_{03} = \frac{e^{-\mu}}{2} \left[ k \mu'' + 2k \frac{D''}{D} - k'' + k' \frac{D'}{D} - \frac{3k(f' l' + k'^2)}{2D^2} \right] = -\kappa \rho k, \tag{10} \]
\[ G_{11} = \frac{\mu' D'}{2D} + \frac{f' l' + k'^2}{4D^2} = 0, \tag{11} \]
\[ G_{22} = \frac{D''}{2D} - \frac{\mu' D'}{2D} - \frac{f' l' + k'^2}{4D^2} = 0, \tag{12} \]
\[ G_{33} = \frac{e^{-\mu}}{2} \left[ l \mu'' + 2l \frac{D''}{D} - l'' + l' \frac{D'}{D} - \frac{3l(f' l' + k'^2)}{2D^2} \right] = \frac{\kappa}{f} (\rho k^2 + P_\phi D^2), \tag{13} \]
where the primes stand for differentiation with respect to $r$. 
B. Conservation of the stress-energy tensor – Bianchi identities

The equation for the conservation of the stress-energy tensor is analogous to the Bianchi identity:

\[ T^\beta_{\alpha;\beta} = 0. \]  \hfill (14)

From (1), one obtains

\[ T^{\alpha \beta} = \rho V^{\alpha}V^{\beta} + P_\phi K^{\alpha}K^{\beta}, \]  \hfill (15)

where \( V^{\alpha} \) is given by (6), and the spacelike vector \( K^{\alpha} \) is given by (7), which are inserted into (15). Using (3) and (6), the Bianchi identity (14) reduces to

\[ T^\beta_{1;\beta} = \frac{1}{2} (\rho + P_\phi) \frac{f'}{f} - P_\phi \frac{D'}{D} = 0. \]  \hfill (16)

With \( h(r) \) defined as \( h(r) \equiv P_\phi(r)/\rho(r) \), it can be written as

\[ \frac{1}{2} (1 + h) \frac{f'}{f} - h \frac{D'}{D} = 0. \]  \hfill (17)

C. Junction conditions

The interior spacetime being stationary, the vacuum outside the cylinder bounded by the hypersurface \( \Sigma \) has to be also stationary. Therefore, since the metric functions describing the interior spacetime are real, the Weyl class of the Lewis solution is chosen to represent the vacuum exterior. Darmois’ junction conditions [9] are thus applied to both inside and outside metrics on the boundary.

The resulting conditions have already been displayed in [6–8] for metric (3). The main one, which will be applied here, implies that the radial component of the pressure should vanish on \( \Sigma \). This is indeed the case here, since it is imposed on the whole interior spacetime by the equation of state of the fluid implying \( P_r = 0 \) everywhere.

D. Solutions to the field equations

Adding both field equations (11) and (12) gives

\[ D'' = 0, \]  \hfill (18)

which can be integrated as

\[ D = c_1 r + c_2 \]  \hfill (19)

where \( c_1 \) and \( c_2 \) are integration constants.

The coordinate \( r \) can then be rescaled from a factor \( c_1 \), which gives

\[ D = r + c_2. \]  \hfill (20)

Since, up to now, only five independent differential equations have been made available for six unknowns, i.e., the four functions \( f, k, e^{\phi}, \) and \( l \), the energy density \( \rho \), and the pressure defined either by \( P_\phi \) or by \( h \) – this last option being retained here – the set of equations needs to be closed by an additional gauge choice. The choice retained here, as in [8], is

\[ \frac{f'}{f} = \frac{2h'}{1 - h}, \]  \hfill (21)

which can be integrated as

\[ f = \frac{c f}{(1 - h)^2}. \]  \hfill (22)
where \( c_f \) is an integration constant. Inserting (21) into the Bianchi identity (17), one obtains

\[
\frac{1 + h}{h(1 - h)} h' = \frac{D'}{D},
\]

(23)

which can be written as

\[
h' \frac{1}{h} + \frac{2h'}{1 - h} = \frac{D'}{D},
\]

(24)

and then, integrated as

\[
D = \frac{h}{c_5(1 - h)^2},
\]

(25)

c_5 \text{ being another integration constant.}

Now, (20) inserted into (25) yields

\[
\frac{h}{(1 - h)^2} = c_5 r + c_2 c_5.
\]

(26)

Thus, the coordinate \( r \) can, once again, be rescaled, now from a factor \( c_5 \), and then (26) becomes

\[
\frac{h}{(1 - h)^2} = r + c_2 c_5.
\]

(27)

Now, (9) combined with (10) gives

\[
(kf' - fk')' = 0,
\]

(28)

which can be integrated as [6]

\[
kf' - fk' = 2cD,
\]

(29)

where \( 2c \) is an integration constant and where the factor 2 is chosen for further convenience. Considered as a first-order ordinary differential equation for \( k(r) \), (29) possesses as a general solution

\[
k = f \left( c_4 - 2c \int_0^r \frac{D(v)}{f(v)^2}dv \right) - \frac{c_4}{f_0},
\]

(30)

where \( c_4 \) is a new integration constant and \( f_0 \) is the value of the function \( f \) on the axis where \( r = 0 \). With expressions (22) for \( f \) and (24) for \( D \) inserted, (30) becomes

\[
k = \frac{c_f}{(1 - h)^2} \left[ c_4 - \frac{2c}{c_f} \int_0^h h(v)(1 - h(v))^2dv \right] - \frac{c_4}{f_0}.
\]

(31)

Now, differentiating (27) with respect to \( r \) yields

\[
h' = \frac{(1 - h)^3}{1 + h},
\]

(32)

which is used to make a change of integration variable in (31) such as to obtain

\[
k = \frac{c_f}{(1 - h)^2} \left[ c_4 - \frac{2c}{c_f} \int_{h_0}^h \frac{(1 + v)}{1 - v}dv \right] - \frac{c_4}{f_0},
\]

(33)

where \( h_0 \) is the value of \( h \) on the axis of symmetry. This equation can be integrated as

\[
k = \frac{c_f}{(1 - h)^2} \left\{ c_4 - \frac{2c}{c_f} \left[ \frac{h_0^2 - h_0^2}{2} + 2(h_0 - h) + 2 \ln \left( \frac{1 - h_0}{1 - h} \right) \right] \right\} - \frac{c_4}{f_0}.
\]

(34)
The metric function \( l \) thus follows from (8) as
\[
l = \frac{h^2}{c_f(1 - h)^2} - \frac{(1 - h)^2}{c_f} \left\{ \frac{c_f}{(1 - h)^2} \left[ \frac{c_f}{2} \left( 1 + h^2 \right) + 2(h - h_0) + 2 \ln \left( \frac{1 - h_0}{1 - h} \right) \right] - \frac{c_f}{f_0} \right\}^2. \tag{35}
\]

Now, using (8) into (29) one obtains, after some calculations,
\[
f'' + k'^2 \frac{2D^2}{fD} = f'D' \frac{f'}{fD} - f'^2 \frac{2c^2}{f^2}, \tag{36}
\]
which, inserted into (11), gives
\[
\frac{1 + h}{2h} \frac{f'}{f} \mu' + \frac{1}{2h} f'' + \frac{2c^2}{f^2} = 0. \tag{37}
\]
where (17) is inserted to yield
\[
\frac{1 + h}{2h} \frac{f'}{f} \mu' + \frac{1}{2h} f'' + \frac{2c^2}{f^2} = 0. \tag{38}
\]
Then, (21) and (22) inserted into (38) give
\[
\frac{(1 + h)h'}{h(1 - h)} \mu' + \frac{2h'^2}{h(1 - h)^2} + \frac{2c^2}{c_f^2} (1 - h)^4 = 0. \tag{39}
\]
Now, using (32) into the last term of (39) in such a way as to homogenise the equation with respect to first derivatives, yields
\[
\frac{(1 + h)h'}{h(1 - h)} \mu' + \frac{2h'^2}{h(1 - h)^2} + \frac{2c^2}{c_f^2} (1 + h)^2 h'^2 = 0, \tag{40}
\]
that can be written as
\[
\mu' = -\frac{h'}{1 - h} - \frac{h'}{1 + h} + \frac{2c^2}{c_f^2} \left[ hh' + 2h' - \frac{2h'}{1 - h} \right], \tag{41}
\]
which can be integrated as
\[
\mu = \ln \frac{(1 - h)}{1 + h} + \frac{4c^2}{c_f^2} \ln(1 - h) + \frac{c^2}{c_f^2} (h^2 + 4h) + \ln c_\mu, \tag{42}
\]
where \( c_\mu \) is another integration constant, and that yields
\[
e^\mu = c_\mu \frac{(1 - h)^{1 + \frac{4c^2}{c_f^2}}}{(1 + h)} \exp \left[ \frac{c^2}{c_f^2} h(4 + h) \right]. \tag{43}
\]

E. Axisymmetry, regularity and elementary flatness conditions

To represent an axisymmetric spacetime the solution thus obtained from the integration of the field equations must verify \( l = 0 \), where \( \frac{h}{r} \) denotes the value of the function taken on the rotation axis, i.e., for \( r = 0 \) \[2\ 10\]. With (35) inserted into this condition, together with \( f_0 = c_f/(1 - h_0)^2 \), this equation yields
\[
l = 0 = \frac{h_0^2}{c_f(1 - h_0)^2} + \frac{c_4(1 - h_0)^4}{c_f^2} = 0. \tag{44}
\]
With \( f > 0 \), owing to the signature of the metric, (22) imposes \( c_f > 0 \), which gives, by inspection of (44),

\[
h_0 = c_4 = 0. \tag{45}
\]

Remark that \( h_0 = 0 \) implies that the azimuthal pressure vanishes on the axis. This was not the case for a purely axial pressure [8], but here the situation is different. Indeed, while, in the axial case, the pressure \( P_z \) can always be uniquely defined on the axis where it is not obligatorily vanishing, a non zero azimuthal pressure is undefined there. For, as the coordinate \( \phi \) can take any value from 0 to \( 2\pi \), the angular direction of the principal stress component \( P_\phi \) can be any on the axis. This drawback disappears if \( P_\phi \) is set to vanish there, as it is imposed here by (45).

The regularity condition, which ensures Lorentzian geometry, i.e., ‘elementary flatness’, in the vicinity of the axis reads [2]

\[
t_\alpha t^\alpha = 1, \tag{46}
\]

which, in the case of metric \( \Theta \), can be written as [6]

\[
e^{-\mu l'^2} = 1. \tag{47}
\]

Now, inserting (35), (43) and (45) into (47), one obtains, with the help of (32),

\[
c_f c_\mu = 1. \tag{48}
\]

Then, evaluating both expressions for \( D \), (20) and (25), on the axis, yields

\[
c_2 = 0. \tag{49}
\]

To avoid singularities on the axis, spacetime flatness is demanded in its vicinity which imposes the following constraints on the metric functions [6–8]

Implementing the first one as \( f_0 \equiv 1 \) yields, with (22) and the constraint on \( h_0 \) in (45) inserted,

\[
c_f = 1. \tag{50}
\]

The condition on the second metric function, which reads \( k_0 \equiv 0 \), is automatically fulfilled from (45) inserted into (34) and does not imply any new constraint on the integration constants. Then, inserting (43) and (45) into the flatness condition for the third metric function on the axis, i.e., \( e^\mu_0 \equiv 1 \), one obtains

\[
c_\mu = 1. \tag{51}
\]

Thus (48) is satisfied.

As regards the vanishing of the function \( l \), it has already been discussed above and has yielded (45).

Finally, the regularity condition and the flatness of the spacetime in the vicinity of the axis impose \( l' = 0 \) [6]. Differentiating (35) with respect to \( r \) and inserting there (45) and (50), it is easy to see that this demand is satisfied without the need for any additional constraints on the integration constants of the solution.

In [8], other requirements have been imposed on the derivatives of the metric functions in the vicinity of the symmetry axis. Those are suitable for the case of fluids with purely axial pressure, since the pressure does not vanish on the axis. In the case of purely azimuthal pressure, a kind of singularity appears at the limit at the axis where the derivatives with respect to \( r \) are not obligatorily well defined. It is indeed easy to verify that \( f' = 0 \) is not satisfied by the solution displayed in Sec. III F. However, as advocated in [12], when one adapts the coordinate \( \phi \) to the Killing vector generating the axial isometry of some spacetime, which is done here, the metric is apparently singular at \( r = 0 \), but this is just a coordinate singularity.

F. The energy density

After a rescaling of the coordinate \( r \) from a factor \( c_5 \) allowed by (26) to give (27), the intermediate function \( D \) represented by (25) becomes

\[
D = \frac{h}{(1 - h)^2}. \tag{52}
\]
and the equation of state function $h$ follows as a double solution of the second degree in $h$ equation (26) where (49) has been inserted, which reads

$$h = 1 + \frac{1}{2r} + \epsilon \sqrt{\frac{1}{r} + \frac{1}{4r^2}},$$

(53)

where $\epsilon = \pm 1$. Notice that, since this expression is obtained after dividing by $r$, it is only valid for $r \neq 0$.

Now, to derive an expression for the density $\rho$, we insert the metric functions, the intermediate function $D$, and their derivatives into (9) written, with the help of (11) and (18), as

$$- \mu'' + \frac{f''}{f} - \frac{f''D'}{fD} - 3\frac{\mu'D'}{D} = 2\kappa \epsilon \rho.$$  

(54)

Equation (41) with (50), together with (32), yields

$$\mu'' = 4(1 - h)^4 \left[ \frac{2}{(1 + h)^3} - \frac{c^2}{2}(1 - 3h) \right],$$

(55)

while (21), (22), (32) and (50) give,

$$\frac{f''}{f} = -2 \frac{(1 - h)^5}{(1 + h)^3},$$

(56)

and (32) inserted into (23) yields

$$\frac{D'}{D} = \frac{(1 - h)^2}{h},$$

(57)

all of which are carried over in (54). Then, after some straightforward calculations, one obtains

$$[3 + 2c^2h(1 + h)] \frac{h''}{1 - h} + \left[ 1 + 8h + 4c^2h(2 + 4h + h^2) \right] \frac{h'^2}{h(1 - h)^2} = 2\kappa \epsilon \rho.$$  

(58)

It is easy to verify that the same calculation done with (13) instead of (9) gives indeed the same result.

Then, from (32), one can compute the second derivative of $h$ with respect to $r$, that reads

$$h'' = -2 \frac{(1 - h)^5(2 + h)}{(1 + h)^3},$$

(59)

which, inserted, together with (32) and (53), into (58), yields

$$\rho = \frac{2}{\kappa} \frac{(1 - h)^3 - 4c^2}{(1 + h)^3} \left[ 2c^2 + \frac{(1 - h)}{h(1 + h)^3} \right] \exp \left[ -c^2h(4 + h) \right].$$

(60)

G. **Behaviour of the function $h(r)$**

Differentiating (53) with respect to the coordinate $r$, one obtains, for $r \neq 0$,

$$h' = -\frac{1}{2r^2} \left[ 1 + \epsilon \frac{1 + 2r}{\sqrt{1 + 4r^2}} \right].$$

(61)

It is obvious that this first derivative of $h(r)$ never vanishes, hence it keeps once for all the same sign as the one measured at any location in spacetime. However, its precise behaviour depends on the sign of $\epsilon$ and this dependence is examined below.

1. **Case (i)**

If $\epsilon$ in (53) and (61) is assumed to be positive, $h'(r)$ should be everywhere negative. Thus $h$ should be a monotonically decreasing function of $r$ from the vicinity of $r = 0$, where $h = 0$, to $r = r_\Sigma$. This would imply that $h$ should be negative, which is inconsistent with (53) that imposes $h > 0$ for $\epsilon > 0$. This case is therefore ruled out.
2. Case (ii)

If $\epsilon$ in (53) and (61) is assumed to be negative, the behaviour of $h(r)$ depends on the sign of $\sqrt{1 + 4r} - (1 + 2r)$. Since $r > 0$, this expression is negative, and therefore, from (61) $h'$ is positive and from (53) $h$ is positive. The solutions displayed in this article are thus only valid for a positive pressure and the ratio $h$ of $P_\phi$ over the energy density $\rho$ is monotonically increasing from the axis to the boundary $\Sigma$.

H. Final form of the solution

Implementing the rescalings and constraints described above leads to the final form of the solution, which can be summarized as

$$f = \frac{1}{(1 - h)^2},$$  \hspace{1cm} (62)

$$e^\mu = \frac{(1 - h)^{1 + 4c^2}}{(1 + h)} \exp \left[c^2 h(4 + h)\right],$$  \hspace{1cm} (63)

$$k = \frac{c}{(1 - h)^2} \left[h^2 + 4h + 4 \ln(1 - h)\right],$$  \hspace{1cm} (64)

$$l = \frac{1}{(1 - h)^2} \left[h^2 - c^2 \left[h^2 + 4h + 4 \ln(1 - h)\right] - 4 c^2 \ln(1 - h)\right],$$  \hspace{1cm} (65)

$$D = \frac{h}{(1 - h)^2} = r,$$  \hspace{1cm} (66)

$$\rho = \frac{2}{\kappa(1 - h)^3 - 4c^2} \left[2c^2 + \frac{(1 - h)}{h(1 + h)^3} \exp \left[-c^2 h(4 + h)\right]\right],$$  \hspace{1cm} (67)

$$h = 1 + \frac{1}{2r} - \sqrt{\frac{1}{r} + \frac{1}{4r^2}},$$  \hspace{1cm} (68)

Recall that, since (68) is obtained after dividing by $r$, this expression is only valid for $r \neq 0$.

IV. PHYSICAL PROPERTIES OF THE SOLUTION

A. Hydrodynamical scalars, vectors, and tensors

The timelike 4-vector $V_\alpha$ can be invariantly decomposed into three independent parts $\mathbb{I}$ $\mathbb{R}$: the acceleration vector,

$$\dot{V}_\alpha = V_{\alpha;\beta} V^\beta,$$  \hspace{1cm} (69)

the twist or rotation tensor,

$$\omega_{\alpha\beta} = V_{[\alpha;\beta]} + \dot{V}_{[\alpha} V_{\beta]},$$  \hspace{1cm} (70)

and the shear tensor,

$$\sigma_{\alpha\beta} = V_{(\alpha;\beta)} + \dot{V}_{(\alpha} V_{\beta]}.$$  \hspace{1cm} (71)
Since a detailed account of their properties has already been displayed in [7, 8], only the main expressions needed for further analyses will be given here. With the timelike 4-vector given by (5), the nonzero component of the acceleration vector is

\[ \dot{V}_1 = \frac{1}{2} \frac{f'}{f}, \]  

(72)

which becomes, with (24) and (52) inserted,

\[ \dot{V}_1 = \frac{(1 - h)^2}{1 + h}. \]  

(73)

The modulus of the acceleration vector follows as

\[ \dot{V}^\alpha \dot{V}_\alpha = \frac{1}{4} \frac{f'^2}{f^2} e^{-\mu}, \]  

(74)

which becomes, with (24) and (63) inserted,

\[ \dot{V}^\alpha \dot{V}_\alpha = \frac{(1 - h)^3 - 4 h^3}{1 + h} \exp \left( -c^2 h (4 + h) \right). \]  

(75)

The rotation scalar defined as

\[ \omega^2 = \frac{1}{2} \omega^{\alpha\beta} \omega_{\alpha\beta}, \]  

(76)

takes the form [8]

\[ \omega^2 = \frac{c^2}{f^2 e^\mu}, \]  

(77)

which becomes, after inserting (62) and (63),

\[ \omega^2 = c^2 (1 - h)^3 - 4 c^2 (1 + h) \exp \left( -c^2 h (4 + h) \right). \]  

(78)

As already remarked in [8, 7], rigid rotation implies a vanishing shear.

B. Sign constraints and signature of the metric

A sign constraint emerges from \( \ln(1 - h) \) appearing in \( k \) and \( l \) as expressed by (64) and (65), respectively. It implies \( h < 1 \).

Since \( h \) is positive, this compels the amplitude of the pressure to be everywhere smaller than that of the density, in units where the velocity of light in vacuum is set to \( c = 1 \) and where the gravitational constant is reduced to \( \kappa \).

Moreover, the weak energy condition \( \rho > 0 \) implies

\[ (1 - h)^3 - 4 c^2 \left[ 2 c^2 + \frac{(1 - h)}{h(1 + h)^3} \right] > 0. \]  

(80)

Since the constraint from the logarithm function applies, which gives \( 0 < h < 1 \), it is easy to verify that the inequality (80) is always verified.

Now, to reproduce the proper signature pertaining to the metric, i.e., \((- + + +)\), every function as occurring in (3) must be positive definite. The functions \( f \) and \( e^\mu \) as given by (62) and (63), respectively, are positive by construction, as in the case of purely axial pressure [8].

From (64), one can see that, once (79) is fulfilled, \( k \) is positive provided

\[ p(h) \equiv c \left[ h^2 + 4 h + 4 \ln(1 - h) \right] > 0. \]  

(81)
Table 1. Sign constraints

| $h$   | 0       | $h > 0$ | $h = 0$ |
|-------|---------|---------|---------|
| $c$   | $c \to \infty$ | $c > 0$ | $c \to 0$ |

$c^2 < \left(\frac{h^2}{h^2 + 4h + 4 \ln(1 - h)}\right)^2$ 

| $h = 1$ |
|---------|

**TABLE I:** Constraints on the parameters of the solutions issued from sign requirements.

As regards $l$ given by (65), an analogous constraint implies

$$q(h) \equiv h^2 - c^2 \left[ h^2 + 4h + 4 \ln(1 - h) \right]^2 > 0. \quad (82)$$

With the requirement $0 < h < 1$, the constraint $\text{(S1)}$ imposes $c > 0$.

Finally, the inequality $\text{(S2)}$ imposes another constraint that reads

$$c^2 < \left(\frac{h^2}{h^2 + 4h + 4 \ln(1 - h)}\right)^2. \quad (83)$$

When approaching the axis of symmetry, i.e., $h \to 0$, the right hand side of $\text{(S2)} \to \infty$. Hence the amplitude of $c$ is unbounded there. However, when approaching the upper limit $h \to 1$, $c^2 \to 0$ and the solution is ruled out. This implies that the upper limit $h = 1$ lies definitely off the domain of definition of the solution. Anyhow, in many configurations, the value of $h$ is bounded by that reached at the boundary $\Sigma$ before approaching the limit $h = 1$.

These constraints are summarized in table I.

C. Singularities

The solutions displayed here exhibit two possible singular behaviours.

One might occur for $h = +1$ where the metric functions $f$, $k$, and $l$ diverge, while $e^\mu$ vanishes. The radial coordinate $r$, given by (66), diverges also for this value of $h$. However, since $r$ is bounded by the radius of the cylinder $\Sigma$, this confirms that $h = +1$, should it determine or not a genuine singularity, is never reached inside the domain of application of the spacetime.

Another singularity might occur for $h = r = 0$, i.e., on the axis. Here, the metric functions $k$ and $l$ vanish and the density diverges. However, as it has been stressed in [12] and discussed in section III E, the axis of symmetry should be considered as a mere coordinate singularity due to the behaviour of the coordinate $\phi$.

Moreover, it has already been noticed that both potential singularities occur for values of $h$ which delimit the interval of definition studied in section IV B. They can therefore be excluded from the domain of definition of the solution, as it has been proposed in this section, and thus the solution becomes singularity-free. Hence, a spacetime with $P_\phi = 0$ everywhere cannot be considered as belonging to this class of solutions which therefore does not possess any dust limit, as do not the solutions applying to fluids with purely axial pressure [8].

D. Interpretation of the parameters

The mathematical and physical properties of the new solution displayed here through (62)–(68) depend only on one parameter, the integration constant $c$.

Owing to (78) where $h$ is set to vanish, this parameter represents the amplitude of the rotation scalar on the axis. This interpretation is akin to that for fluids with an axially directed pressure displayed in [8]. We recall that since this parameter $c$ is the same as the one appearing in the Lewis metric for the exterior spacetime [13], its absolute value can be interpreted, in the vacuum framework also, as the amplitude of the rotation scalar of the interior gravitational source considered on the axis of symmetry. Moreover, it has been shown, in section IV B, how its amplitude is bounded by an expression implying $h$.

Considering (69) and (61), it has been stressed in section III E that an azimuthally directed pressure is obligatorily positive. Now, a positive pressure is the case most generally encountered in the physics of standard fluids which might therefore provide an interesting domain of application for such solutions.
V. COMPARISON WITH THE PURELY AXIAL PRESSURE CASE

Solutions describing spacetimes sourced by a stationary rigidly rotating cylindrical fluid with purely axial pressure have been displayed in [8]. The solutions presented here, corresponding to an analogous fluid, but with an azimuthally directed principal stress, are now compared to that previously studied there.

As regards the field equations, only the right hand sides made out off the stress-energy tensor components are different. However, the three first ones, corresponding to \(T_{00}, T_{03}, \) and \(T_{11}\) are strictly equal. This does not prevent their outcome, the metric functions, to be however different.

Indeed, the analogous gauge choice made for both cases, i.e.,

\[
f = \frac{(1 - h_0)^2}{(1 - h)^2},
\]

ends up with very different expressions for the other metric functions, see (62) – (65) in above Sec. compared to (68) – (71) in [8]. Moreover, as \(P_0\) is bond here to vanish on the axis, which is not the case for \(P_z\), even (64) ends up as two different functions, (62) here and (68) in [8]. All the other metric functions are more or less drastically different in each case.

As regards the hydrodynamical tensors, vectors and scalars of these two fluids, their general expressions are different in each case, save the shear which always vanishes everywhere owing to rigid rotation. Moreover, the modulus of the acceleration vector vanishes on the axis in the axial pressure case, while it is unity in the azimuthal pressure one. Now, the amplitude of the rotation scalar on the axis is equal in both cases, axial and azimuthal, to the absolute value of the parameter \(c\). Its interpretation as a vorticity parameter of the vacuum Lewis-Weyl solution coming from the interior fluid source [13] is therefore confirmed.

The intervals of definition of \(h\) are significantly different. They cover part of the interval \(-1 < h < +1\), in the purely axial pressure case, see [8], but only part of \(0 < h < +1\) in the purely azimuthal one. Hence, a negative pressure is allowed in the axial case while only solutions with positive purely azimuthal pressure can occur here.

Three possible singular loci appear for axially directed pressure, \(h = -1, 0, +1\), while only two are present in the azimuthal case, \(h = 0, +1\). Their nature and relevance are however different.

Finally, it has been shown that none of these two sets of solutions possesses any dust limit, if one aims at singularity-free spacetimes.

VI. CONCLUSIONS

Following the investigations of the interior spacetimes sourced by stationary cylindrical anisotropic fluids initiated in [6, 8], the rigidly rotating fluid case with the particular equation of state \(P_r = P_z = 0\) has been examined here. A class of exact solutions to the field equations has thus been exhibited under the form of functions of \(h(r)\) for the metric and the density, with \(h\) defined as \(h(r) = P_0(r)/\rho(r)\), and an explicit expression for \(h\) as a function of \(r\) has been displayed. Of course, as usual, this solution is valid in a given system of coordinates which, however, has been chosen such as to allow a direct physical interpretation. In view of potential further uses for astrophysical purposes, the regularity conditions on the symmetry axis have been examined and discussed and the solution has been matched to the Weyl class of the Lewis vacuum solution on a cylindrical hypersurface \(\Sigma\) acting as a boundary for the fluid.

A number of physical and mathematical properties of these solutions have been analysed such as the hydrodynamical quantities obtained as functions of \(h(r)\). Two possible singularities have been identified. It has been shown that they can be dismissed from the definition intervals of the solution, yielding therefore singularity-free spacetimes.

For typographical purpose and to ease the comparison between this set of solutions and the one displayed in [8] for fluids with purely axial pressure, the expressions of interest have been presented here accordingly as functions of \(h\). If needed, it is however straightforward to obtain them as explicit functions of \(r\) through the use of (84).

The independent parameter \(c\) exhibited by this solution has been physically interpreted. It represents the amplitude of the vorticity of the fluid on the axis and corresponds to the parameter \(c\) of the exterior Lewis-Weyl vacuum which thus inherits a precise confirmation of its previous interpretation as a vorticity [13]. The pressure \(P_0\) which is here compelled to be positive can thus be considered as a genuine pressure occurring in a standard fluid. Moreover, its amplitude increases with respect to that of \(\rho\) from zero on the axis towards a maximum value reached on the boundary \(\Sigma\).

Therefore, this class of anisotropic fluid as a source for cylindrically symmetric spacetimes, which is the second simple case successfully integrated from the Einstein field equations in this regards, the first one having been displayed in [8], is another important step towards the understanding of anisotropy in GR from both a mathematical and a physical point of view.
The next step should be the study of the equation of state \( P_z = P_\phi = 0 \), i.e., the case of purely radial pressure, provided exact solutions exist in this case.

One must of course be aware that, at first sight, this endeavour could seem special from a physical point of view, especially since each class of solutions is displayed independently one of the others. Save perhaps in the case of purely axial pressure when some hints for direct physical applications have been given, but to which the following remark however applies, the rough equations of state considered in turn can be discussed. E.g., what about a fluid with purely azimuthal or axial pressure whose amplitude varies with the radial coordinate while the radial pressure remains null? The physical interest of the present approach is not to exhibit solutions to be considered at face value, but to provide a set of exact solutions from which anisotropy in cylindrical objects could be better understood. Of course, given the non-linearity of Einstein’s field equations, the use of solutions of this kind for the study of a generalised anisotropic fluid is not simple, but one can suspect that such exact solutions might be used as, e.g., starting points for numerical or perturbative approaches. In such designs, each principal stress might be no more required to be in turn the only non vanishing component, but merely to dominate the other two. Therefore the above remark formulated as a question should no more be considered as a drawback.

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