Abstract

We have developed a general geometric treatment of the GCE valid for any stationary axisymmetric metric. The method is based on the remark that the world lines of objects rotating along spacially circular trajectories are in any case, for those kind of metrics, helices drawn on the flat bidimensional surface of a cylinder. Applying the obtained formulas to the special cases of the Kerr and weak field metric for a spinning body, known results for time delays and synchrony defects are recovered.

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1 Introduction

The gravitomagnetic clock effect (GCE) consists in the loss of synchrony of identical clocks carried around a massive spinning body, in opposite directions. This effect is a consequence of general relativity and its presence has been foreseen in connection with the so called gravitomagnetism, i.e. that part of the gravitational field which, in weak field approximation behaves as the magnetic part of the electromagnetic interaction [1]. The GCE has been considered as an interesting an promising means to test the general relativistic influence of the angular momentum of a mass on the structure of space time nearby, and in particular on the pace of clocks orbiting around the body [2] [3].

Actually the GCE is strictly akin to the Sagnac effect, which is a special relativistic effect induced by pure rotations, first considered as a purely classical effect by G. Sagnac [4], further on recognized in its real nature and studied by several author (see for instance [5] [6]).

A relevant aspect both of GCE and of Sagnac effect is their genuine geometrical character.

The present paper moves from the remark that the world line of an object steadily moving along a spacially circular trajectory around a symmetry axis of the gravitational field is a helix, to derive a general method for describing the GCE from different viewpoints in general terms, without being a priori limited to the Kerr metric or to its weak field limit as in [1], [7] and [8]. Finally the general method, when applied to the mentioned special cases, will reproduce the known results, but will allow also for an easier recognition of the situations
more viable for experimentally detecting time delays and synchrony defects.

2 General geometric features of the gravitomagnetic clock effect

As already said in the introduction the space time of a massive body steadily rotating about an axis at rest with respect to distant galaxies has a simple and interesting property: the world line of any object orbiting the central mass at a constant distance from the axis is a helix.

In other words, for any such object a cylinder (bidimensional surface) exists on which the world line is drawn; this surface is flat and when opened in a plane the world line becomes a simple straight line. This is true irrespective of the global curvature of space time along the fourdimensional trajectory.

Viewing things this way allows one to catch at a glance both the Sagnac and the GCE. Consider the $1 + 1$ cylinder opened in a plane, shown on figure 1.

Oblique lines are the developed helices; on the horizontal axis angular coordinates are reported, the vertical one shows a variable proportional to the inertial time of an observer at rest with the symmetry axis; points at $\phi = \pi$ are identified with points at $\phi = -\pi$ and the same $t$. Objects steadily rotating with different speeds are represented by different slope lines; it is evident that such lines cannot cross each other on the vertical of the origin (vertical dotted line on O) unless they are symmetric with respect to it, i. e. unless the
Figure 1: The figure shows the development on a plane of the world tube on which the world lines of steadily rotating objects are drawn. The helices are now straight lines; points at $\pi$ are identified with points at $-\pi$. Event $C$ corresponds with the conjunction (after a whole revolution) of two oppositely rotating objects; $A$ and $B$ are the respective completions of a revolution as seen by a "fixed star" observer; $D$ and $E$ are the equivalent events seen by a rotating observer.
speeds are the same in magnitude but oppositely directed. Similarly it is evident that also the intervals between two successive crosspoints (two conjunctions) are different if measured along the two lines, unless the slopes are symmetric: the proper times between conjunctions are different.

These simple facts are indeed the graphic explanation of the Sagnac effect: a rotating observer is in his turn represented by a helix in space time (straight line on fig. 1), whose interceptions with the world lines of the test bodies determine proper intervals and proper time differences typical of the effect. The situation is not different when a gravitational field is present, provided it possesses an axial symmetry and no angular momentum as it is the case for a Schwarzschild metric. Simply the gravitational field will affect the conversion between the proper time of the rotating probes and the coordinate time, but all the typical Sagnac phase effects will be there.

When an angular momentum must be accounted for, the scheme remains in principle the same, but a careful discussion of the viewpoint of different observers needs be made.

2.1 Graphic representation of the non zero angular momentum case

When the gravity source possesses an angular momentum two objects rotating in opposite directions with the same coordinate speed have again a conjunction at the same coordinate time and the same proper time. However two freely orbiting objects on the same circular
Figure 2: This scheme is the equivalent of fig. 1, but for the fact that now an angular momentum is present and consequently the $t$ and $\phi$ axes are no more perpendicular.

Trajectories and opposite directions do so in general at different angular coordinate speeds, have conjunctions at different proper times and cross the line of sight of a distant inertial observer at different coordinate times. The image given in fig. 1 may be retained but for the fact that now the axes of $t$'s and $\phi$'s are no longer orthogonal: the surface on which the world lines of circularly rotating objects are drawn, are still flat and the plane representation of the situation is shown on fig. 2.

Everything may be described by the use of simple methods of bidimensional Minkowskian geometry. The reference frame is the one drawn in the figure. The world line of a rotating
object may still be written as
\[ t = \frac{\phi}{\omega} \]  
(1)

provided it passes through the origin event. We assume that when \( \omega > 0 \) the object is corotating with the source of the gravitational field; the reverse when \( \omega < 0 \). Considering the rotation symmetry, we must complement (1) with the condition that, when the running event reaches the borders of fig. 1 or fig. 2 the representative angle bounces back \((\omega > 0)\) or ahead \((\omega < 0)\) by a \(2\pi\) term. In other words the world line becomes
\[ t = \frac{\phi \pm 2\pi}{\omega} \]  
(2)
where the + sign corresponds to corotation and the − one to counter-rotation. After one more turn an additional \(2\pi\) is introduced, and so on.

Equipped with these simple definitions and rules we immediately see that the coordinate time for a complete revolution (distant observer viewpoint) which is of course \( T = \frac{2\pi}{|\omega|} \) corresponds to the proper time interval of the revolving object
\[ \tau = \frac{1}{c} \sqrt{g_{tt}T^2 + 2g_{t\phi}2\pi T + g_{\phi\phi}4\pi^2} \]  
(3)
\[ = \frac{2\pi}{c|\omega|} \sqrt{g_{tt} + 2g_{t\phi}\omega + g_{\phi\phi}\omega^2} \]

In flat space time this would be \( \tau = T\sqrt{1 - \beta^2} \); in the general case the presence of a term containing \( g_{t\phi} \) accounts for the non orthogonality of the reference axes (polar coordinates are understood).

It is useful to work out the position of the first intersection event (conjunction) of two objects endowed with different angular velocities \( \omega_1 \) and \( \omega_2 \); let us assume by default that
ω_1 > 0 and ω_1 > ω_2. If it is also ω_2 > 0 the first conjunction is found when the equation

\[ \frac{\phi + 2\pi}{\omega_1} = \frac{\phi}{\omega_2} \]

is satisfied, i.e. when

\[
\begin{align*}
\phi &= \phi_a = 2\pi \frac{\omega_2}{\omega_1 - \omega_2} \\
t &= t_a = \frac{2\pi}{\omega_1 - \omega_2}
\end{align*}
\]

The proper times of the two objects at the conjunction are

\[ \tau_{1,2} = \frac{2\pi}{c (\omega_1 - \omega_2)} \sqrt{g_{tt} + 2g_{t\phi}\omega_{1,2} + g_{\phi\phi}\omega_{1,2}^2} \]

corresponding to a synchrony defect

\[ \delta \tau_{12a} = (\tau_1 - \tau_2)_a \]

\[ = \frac{2\pi}{c (\omega_1 - \omega_2)} \left( \sqrt{g_{tt} + 2g_{t\phi}\omega_1 + g_{\phi\phi}\omega_1^2} - \sqrt{g_{tt} + 2g_{t\phi}\omega_2 + g_{\phi\phi}\omega_2^2} \right) \quad (4) \]

If it is ω_2 < 0 and −ω_2 > ω_1 the values are

\[
\begin{align*}
\phi_b &= 2\pi \frac{\omega_1}{\omega_1 - \omega_2} \\
t_b &= t_a \\
\delta \tau_{12b} &= \delta \tau_{12a}
\end{align*}
\]

Finally, when ω_2 < 0 and it is −ω_2 ≤ ω_1

\[
\begin{align*}
\phi_c &= 2\pi \frac{\omega_1 + \omega_2}{\omega_1 - \omega_2} = \phi_a + \phi_b \\
t_c &= \frac{4\pi}{\omega_1 - \omega_2} = 2t_a \\
\delta \tau_{12c} &= 2\delta \tau_{12a}
\end{align*}
\]

\[ \text{7} \]
A relevant situation is that of circular geodetic motion at a constant coordinate radius \( r \).

To study this case let us start from a metric whose non-zero elements are \( g_{tt}, g_{t\phi}, g_{rr}, g_{r\theta}, g_{\theta\theta} \) and \( g_{\phi\phi} \); all of these elements depend on \( r \) and \( \theta \) only. Imposing the conditions \( r = \text{constant}, \theta = \text{constant} = \pi/2 \) with a symmetry such that all the metric elements are extremal for the chosen \( \theta \) value, the equations of geodesic motion lead to the expression

\[
g_{\phi\phi, r} \omega^2 + 2 g_{t\phi, r} \omega + g_{tt, r} = 0 \tag{7}
\]

Commas mean partial differentiation with respect to the variable after them.

Angular velocities of (spacely) circular geodesic motion are then

\[
\omega_{\pm} = -g_{t\phi, r} \pm \sqrt{g_{t\phi, r}^2 - g_{\phi\phi, r} g_{tt, r}/g_{\phi\phi, r}} \tag{8}
\]

This can be written

\[
\omega_{1,2} = \omega_0 \pm \omega_* \tag{9}
\]

where \( \omega_0 = -g_{t\phi, r}/g_{\phi\phi, r} \) and \( \omega_* = \sqrt{\omega_0^2 - g_{tt, r}/g_{\phi\phi, r}} \).

Using (9) and arranging summations so that \( \omega_1 > \omega_2 \), we can calculate the synchrony defect at conjunction for two freely counter-orbiting objects.

To end this section let us still consider the situation as viewed by an observer who rotates with an angular velocity \( \Omega \) of his own. In the proper time of this observer the revolution period of a prograde orbiting object is deduced from (4) with \( \omega_1 = \omega_0 + \omega_* \) and \( \omega_2 = \Omega \) (\( \omega_1 > \Omega > 0 \)), obtaining

\[
\tau_+ = \frac{2\pi}{c (\omega_1 - \Omega)} \sqrt{g_{tt} + 2 g_{t\phi} \Omega + g_{\phi\phi} \Omega^2}
\]
\[ \tau_+ = \frac{2\pi}{c(\omega_0 + \omega_\ast - \Omega)} \sqrt{g_{tt} + 2g_{t\phi}\Omega + g_{\phi\phi}\Omega^2} \]

The retrograde case (now \( \omega_1 = \Omega \) and \( \omega_2 = \omega_0 - \omega_\ast \)) corresponds to a revolution time

\[ \tau_- = \frac{2\pi}{c(\Omega - \omega_2)} \sqrt{g_{tt} + 2g_{t\phi}\Omega + g_{\phi\phi}\Omega^2} \]

Then the proper time difference between the conjunctions with the observer will be

\[ \delta\tau = \frac{2\pi}{c(\omega_0 + \omega_\ast - \Omega)} \sqrt{g_{tt} + 2g_{t\phi}\Omega + g_{\phi\phi}\Omega^2} \]

(10)

\[ = \frac{4\pi}{c\left(\omega_\ast^2 - (\Omega - \omega_0)^2\right)} \sqrt{g_{tt} + 2g_{t\phi}\Omega + g_{\phi\phi}\Omega^2} \]

The same quantity, expressed in terms of coordinate times, is

\[ \delta t = 2\pi \left(\frac{1}{\omega_0 + \omega_\ast - \Omega} - \frac{1}{\Omega + \omega_\ast - \omega_0}\right) = \frac{c\delta\tau}{\sqrt{g_{tt} + 2g_{t\phi}\Omega + g_{\phi\phi}\Omega^2}} \]

Finally it must be remarked that an inertial distant observer finds also a difference in revolution times between pairs of freely counter-rotating objects. One obtains

\[ \delta T = 2\pi \left(\frac{1}{\omega_0 + \omega_\ast} - \frac{1}{\omega_0 - \omega_\ast}\right) = 4\pi \frac{\omega_\ast}{\omega_\ast^2 - \omega_0^2} \]

(11)

An interesting category of observers are the so called locally non rotating observers (LNRO) or Bardeen observers. An LNRO is an observer who does not rotate with respect to matter radially falling towards the central mass; when the latter is spinning it drags, in a sense, the space time around it (though the image of a ”drag” is not really appropriate as pointed out in [3]), so that a locally ”non rotating” observer is actually seen as rotating from
another inertial far away observer (distant stars); its angular velocity is \( \Omega_{LNRO} = -g_{t\phi}/g_{\phi\phi} \) and its motion is in general non geodesic. In fig. 2 \( \Omega_{LNRO} \) is a measure of the slope of the cylinder.

Another situation that could be of importance for experimentation is the one of a rotating observer (angular speed \( \Omega \)) who sends with opposite but locally equal velocities (in the tangent space) two objects along his own path. If \( u_o, u_1 \) and \( u_2 \) are the fourvelocities of the observer and the two objects the condition for the equality of the velocities with respect to the observer is

\[
u_o \cdot u_1 = u_o \cdot u_2 \quad (12)
\]

Using (11) and the normalization condition for the fourvelocity of the observer (12) transforms into

\[
\frac{g_{tt} + 2g_{t\phi}\omega_2 + g_{\phi\phi}\omega_2^2}{g_{tt} + 2g_{t\phi}\omega_1 + g_{\phi\phi}\omega_1^2} = \left( \frac{g_{tt} + (g_{t\phi} + g_{\phi\phi}\Omega) \omega_2 + g_{t\phi}\Omega}{g_{tt} + (g_{t\phi} + g_{\phi\phi}\Omega) \omega_1 + g_{t\phi}\Omega} \right)^2
\]

Solving for \( \omega_2 \) one finds, besides the trivial solution \( \omega_2 = \omega_1 \), the relevant result

\[
\omega_2 = -\frac{g_{tt}\omega_1 - 2g_{tt}\Omega - g_{\phi\phi}\Omega^2\omega_1 - 2g_{t\phi}\Omega^2}{g_{tt} + 2g_{\phi\phi}\Omega\omega_1 - g_{\phi\phi}\Omega^2 + 2g_{t\phi}\omega_1} \quad (13)
\]

Combining (13) with (10) it is possible to find the time delay registered by the observer at the passing by him of the two apparently equal velocity objects. The result is

\[
\delta\tau = -\frac{4\pi}{c} \frac{g_{\phi\phi}\Omega + g_{t\phi}}{\sqrt{g_{\phi\phi}\Omega^2 + 2\Omega g_{t\phi} + g_{tt}}} \quad (14)
\]

As it can be seen \( \delta\tau \) does not depend on \( \omega_1 \) i.e. it is independent from the actual velocity of the objects with respect to the observer. In Minkowski space time (14) reproduces the formula of the Sagnac effect.
3 Special cases

The time delays and synchrony defects determined in the preceding section may be specialized to various different metric tensors. Two cases are particularly of interest either in principle or for practical reasons: the Kerr metric and the weak field approximation of the metric of a spinning object. The relevant metric elements \((\theta = \pi/2)\) are:

| Metric elements | Kerr | Weak field |
|-----------------|------|------------|
| \(g_{tt}\)      | \(c^2 \left(1 - 2\frac{G}{}\right)\) | \(c^2 \left(1 - 2\frac{G}{}\right)\) |
| \(g_{\phi\phi}\) | \(-2a^2\frac{GM}{} - r^2 - a^2 - r^2\) | |
| \(g_{t\phi}\)   | \(2a\frac{GM}{}\) | \(2a\frac{GM}{}\) |

Introducing these expressions into the formulas of the preceding section we obtain

\[
\omega_{0K} = \frac{aGMc}{a^2GM - c^2r^3}
\]

\[
\omega_{0wf} = -a\frac{G}{}c^3
\]

and

\[
\begin{aligned}
\omega_{sK} &= c\sqrt{\frac{GMc^2r^3}{a^2GM - c^2r^3}} \\
\omega_{swf} &= \sqrt{\frac{GM}{r^3}} \left(1 + \frac{a^2GM}{c^2r^3}\right)
\end{aligned}
\]

Combining these formulas we obtain

\[
\begin{aligned}
\omega_{1,2K} &= \frac{c}{a \pm c\sqrt{\frac{r^3}{GM}}} \\
\omega_{1,2wf} &= \sqrt{\frac{GM}{r^3}} \left(\pm 1 - \frac{a}{c} \sqrt{\frac{GM}{r^3}}\right)
\end{aligned}
\]

It is clearly \(-\omega_2 > \omega_1\) consequently the synchrony defect between two counter-orbiting objects is obtained from (5) and (4). Let us directly calculate the result in weak field
approximation, keeping the first order (in $a$ and $GM/c^2$) terms only:

$$\delta \tau_{12} \simeq 6\pi \frac{GM}{c^2r} \frac{a}{c}$$  \hspace{1cm} (18)

From the view point of a distant inertial observer the difference in revolution times for the two counter-orbiting objects is obtained from (11) and (17):

$$\delta T = 4\pi \frac{a}{c}$$ \hspace{1cm} (19)

This exact result (in Kerr geometry) is remarkably independent both from the $r$ parameter of the orbit and from the gravity constant $G$. Actually in weak field approximation and for a spherical homogeneous mass it turns also to be independent from the very mass of the central object since then it is: $a = 2R^2\Omega_0/5c$ ($R$ is the radius of the body and $\Omega_0$ is its rotation speed).

From (3) we see that the readings of clocks attached to our two objects after what is seen as a complete revolution by the distant observer are

$$\tau_{1,2} = \frac{2\pi}{c\omega_{1,2}} \sqrt{g_{tt} + 2g_{t\phi}\omega_{1,2} + g_{\phi\phi}\omega_{1,2}^2}$$

The difference between these readings corresponds, in weak field approximation, to (19):

$$\tau_1 - \tau_2 \simeq 4\pi \frac{a}{c}.$$ It is also the value that would be found, with the same approximation, by a LNRO.

In WFA formula (14) becomes

$$\delta \tau \simeq 4\pi \frac{r^2\Omega - 2aG^M}{c^2} \sqrt{1 - 2G^M/c^2 - \frac{r^2\Omega^2}{c^2} + 4\Omega^4 a^2 G^M/c^2}$$

$$\simeq 4\pi \frac{r^2\Omega}{c^2} \left( 1 + \frac{G^M}{c^2r} + \frac{1}{2} \frac{r^2\Omega^2}{c^2} \right) - 8\pi \frac{a}{c} \frac{GM}{c^2r}$$  \hspace{1cm} (20)
There is a contribution to $\delta \tau$ depending on $a$ but not on $\Omega$ and reproducing the result obtained as first order relativistic correction to the Sagnac effect \cite{7}.

4 Conclusion

We have shown how a simple geometric vision of the world lines of steadily rotating objects in axisymmetric metrics endowed with angular momentum allows for a description and explanation of the GCE. The method evidences and recovers some interesting results that could lead to experimental verifications. Using satellites on circular trajectories one would find a synchrony defect between counter-orbiting identical clocks given (in WFA) by (18), which in the case of Earth is $\sim 10^{-16}$s. A much bigger effect is seen considering revolution times with respect to a fixed direction in space (with respect to fixed stars); in that case the two rotation directions correspond to differences in period length given exactly by (19) which for the Earth is $\sim 10^{-7}$s.

Another interesting possibility would be to work with a satellite (the observer) sending light signals in opposite directions along (non geodesic) closed paths; the relevant formula in this case would be \cite{20} with $\Omega$ given by (17), which produces

$$\delta \tau \simeq \pm \frac{4 \pi}{c^2} r^2 \sqrt{\frac{GM}{r^3}} - 12\pi \frac{aGM}{c^2r}$$

The upper (lower) sign corresponds to a prograde (retrograde) orbiting observer. The first term in (21) is the Sagnac effect, the second one is the correction induced by the angular momentum of the source of gravity. Again, in the case of the Earth, the correction is in the
order of $\sim 10^{-16}$s (a hundredth of a period for visible light).

Finally, in case of experiments on the surface of the Earth (non geodesic equatorial observer, equal speed objects/signals in opposite directions) the formula is again (20) with $\Omega$ coinciding with the angular speed of the Earth $\Omega_0$. Using the expression of an appropriate for this case, the formula reads

$$\delta \tau \simeq \frac{4\pi}{c^2} R^2 \Omega_0 \left( 1 + \frac{1}{5} \frac{G M}{c^2 R} + \frac{1}{2} \frac{R^2 \Omega_0^2}{c^2} \right)$$

The ”correction” originated by the angular momentum of the planet is still in the order of $\sim 10^{-16}$s.

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