INTEGRAL GROUP RING OF THE MATHIEU SIMPLE GROUP $M_{12}$

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Abstract. We consider the Zassenhaus conjecture for the normalized unit group of the integral group ring of the Mathieu sporadic group $M_{12}$. As a consequence, we confirm for this group the Kimmerle's conjecture on prime graphs.

1. Introduction, conjectures and main results

Let $V(ZG)$ be the normalized unit group of the integral group ring $ZG$ of a finite group $G$. A long-standing conjecture of H. Zassenhaus (ZC) says that every torsion unit $u \in V(ZG)$ is conjugate within the rational group algebra $QG$ to an element in $G$.

For finite simple groups the main tool for the investigation of the Zassenhaus conjecture is the Luthar-Passi method, introduced in [17] to solve it for $A_5$. Later M. Hertweck in [14] extended the Luthar-Passi method and applied it for the investigation of the Zassenhaus conjecture for $PSL(2, p^n)$. The Luthar-Passi method proved to be useful for groups containing non-trivial normal subgroups as well. For some recent results we refer to [5, 7, 12, 14, 13, 15]. Also, some related properties and some weakened variations of the Zassenhaus conjecture can be found in [1, 18] and [3, 16].

First of all, we need to introduce some notation. By $\#(G)$ we denote the set of all primes dividing the order of $G$. The Gruenberg-Kegel graph (or the prime graph) of $G$ is the graph $\pi(G)$ with vertices labeled by the primes in $\#(G)$ and with an edge from $p$ to $q$ if there is an element of order $pq$ in the group $G$. In [16] W. Kimmerle proposed the following weakened variation of the Zassenhaus conjecture:

\[(KC) \quad \text{If } G \text{ is a finite group then } \pi(G) = \pi(V(ZG)).\]

In particular, in the same paper W. Kimmerle verified that (KC) holds for finite Frobenius and solvable groups. Note that with respect to the so-called $p$-version of the Zassenhaus conjecture the investigation of Frobenius groups was completed by M. Hertweck and the first author in [4]. In [6, 7, 8] (KC) was confirmed for sporadic simple groups $M_{11}$, $M_{23}$ and some Janko simple groups.

Here we continue these investigations for the Mathieu simple group $M_{12}$. Although using the Luthar-Passi method we cannot prove the rational conjugacy for torsion units of $V(ZM_{12})$, our main result gives a lot of information on partial
of these units. In particular, we confirm the Kimmerle’s conjecture for this group.

Let \( G = M_{12} \). It is well known (see [10, 11]) that \( |G| = 2^6 \cdot 3^3 \cdot 5 \cdot 11 \) and \( \exp(G) = 2^3 \cdot 3 \cdot 5 \cdot 11 \). Let

\[ C = \{ C_1, C_2a, C_2b, C_3a, C_3b, C_4a, C_4b, C_5a, C_5b, C_6a, C_6b, C_8a, C_8b, C_{10a}, C_{11a}, C_{11b} \} \]

be the collection of all conjugacy classes of \( M_{12} \), where the first index denotes the order of the elements of this conjugacy class and \( C_1 = \{1\} \). Suppose \( u = \sum \alpha_g g \in V(\mathbb{Z}G) \) has finite order \( k \). Denote by \( \nu_{nt} = \nu_{nt}(u) = \varepsilon_{C_{nt}}(u) = \sum_{g \in C_{nt}} \alpha_g \) the partial augmentation of \( u \) with respect to \( C_{nt} \). From S.D. Berman’s Theorem [2] one knows that \( \nu_1 = \alpha_1 = 0 \) and

\[ \sum_{C_{nt} \in C} \nu_{nt} = 1. \]

Hence, for any character \( \chi \) of \( G \), we get that \( \chi(u) = \sum \nu_{nt} \chi(h_{nt}) \), where \( h_{nt} \) is a representative of the conjugacy class \( C_{nt} \).

Our main result is the following

**Theorem 1.** Let \( G \) denote the Mathieu simple group \( M_{12} \). Let \( u \) be a torsion unit of \( V(\mathbb{Z}G) \) of order \( |u| \). The following properties hold:

(i) If \( |u| \notin \{12, 20, 24, 40\} \), then \( |u| \) coincides with the order of some element \( g \in G \).

(ii) If \( |u| = 2 \), then the tuple of the partial augmentations of \( u \) belongs to the set

\[ \{ (\nu_{2a}, \nu_{2b}, \nu_{3a}, \nu_{3b}, \nu_{4a}, \nu_{4b}, \nu_{5a}, \nu_{5b}, \nu_{6a}, \nu_{6b}, \nu_{8a}, \nu_{8b}, \nu_{10a}, \nu_{11a}, \nu_{11b}) \in \mathbb{Z}^{14} | (\nu_{2a}, \nu_{2b}) \in \{ (0, 1), (2, -1), (1, 0), (3, -2), (-1, 2) \}, \nu_{kx} = 0, kx \notin \{2a, 2b\} \} \].

(iii) If \( |u| = 3 \), then the tuple of the partial augmentations of \( u \) belongs to the set

\[ \{ (\nu_{3a}, \nu_{3b}, \nu_{4a}, \nu_{4b}, \nu_{5a}, \nu_{5b}, \nu_{6a}, \nu_{6b}, \nu_{8a}, \nu_{8b}, \nu_{10a}, \nu_{11a}, \nu_{11b}) \in \mathbb{Z}^{14} | (\nu_{3a}, \nu_{3b}) \in \{ (0, 1), (2, -1), (1, 0), (3, -2), (-1, 2) \}, \nu_{kx} = 0, kx \notin \{3a, 3b\} \} \].

(iv) If \( |u| = 5 \), then \( u \) is rationally conjugate to some \( g \in G \);

(v) If \( |u| = 10 \), then the tuple of the partial augmentations of \( u \) belongs to the set

\[ \{ (\nu_{2a}, \nu_{2b}, \nu_{3a}, \nu_{3b}, \nu_{4a}, \nu_{4b}, \nu_{5a}, \nu_{5b}, \nu_{6a}, \nu_{6b}, \nu_{8a}, \nu_{8b}, \nu_{10a}, \nu_{11a}, \nu_{11b}) \in \mathbb{Z}^{14} | (\nu_{2a}, \nu_{2b}, \nu_{10a}) \in \{ (0, 0, 1), (1, 1, -1) \}, \nu_{kx} = 0, kx \notin \{2a, 2b, 10a\} \} \].

(vi) If \( |u| = 11 \), the tuple of the partial augmentations of \( u \) belongs to the set

\[ \{ (\nu_{11a}, \nu_{11b}) \in \mathbb{Z}^{14} | (\nu_{11a}, \nu_{11b}) \in \{ (0, 1), (2, -1), (1, 0), (-1, 2) \}, \nu_{kx} = 0, kx \notin \{11a, 11b\} \} \].

As an immediate consequence of part (i) of the Theorem we obtain

**Corollary 1.** If \( G = M_{12} \) then \( \pi(G) = \pi(V(\mathbb{Z}G)) \).
2. Preliminaries

The following result is a reformulation of the Zassenhaus conjecture in terms of partial augmentations of torsion units.

**Proposition 1.** (see [17] and Theorem 2.5 in [19]) Let \( u \in V(\mathbb{Z}G) \) be of order \( k \). Then \( u \) is conjugate in \( \mathbb{Q}G \) to an element \( g \in G \) if and only if for each \( d \) dividing \( k \) there is precisely one conjugacy class \( C \) with partial augmentation \( \varepsilon_C(u^d) \neq 0 \).

The next two results now yield that several partial augmentations are zero.

**Proposition 2.** (see [17] and Theorem 2.7 in [19]) Let \( u \) be a torsion unit of \( V(\mathbb{Z}G) \). Let \( C \) be a conjugacy class of \( G \). If \( a \in C \) and \( p \) is a prime dividing the order of \( a \) but not the order of \( u \) then \( \varepsilon_C(u) = 0 \).

**Proposition 3.** (see [12], Proposition 3.1; [14], Proposition 2.2) Let \( G \) be a finite group and let \( u \) be a torsion unit in \( V(\mathbb{Z}G) \). If \( x \) is an element of \( G \) whose \( p \)-part, for some prime \( p \), has order strictly greater than the order of the \( p \)-part of \( u \), then \( \varepsilon_x(u) = 0 \).

Another important restriction on partial augmentations is given by the next result, explained in details in [17] and [3][14].

**Proposition 4.** (see [17][14]) Let either \( p = 0 \) or \( p \) a prime divisor of \( |G| \). Suppose that \( u \in V(\mathbb{Z}G) \) has finite order \( k \) and assume \( k \) and \( p \) are coprime in case \( p \neq 0 \). If \( z \) is a primitive \( k \)-th root of unity and \( \chi \) is either a classical character or a \( p \)-Brauer character of \( G \), then for every integer \( t \) the number

\[
\mu_t(u, \chi, p) = \frac{1}{t} \sum_{d | k} Tr_{\mathbb{Q}(z^d)/\mathbb{Q}}(\chi(u^d)z^{-dt})
\]

is a non-negative integer.

Note that if \( p = 0 \), we will use the notation \( \mu_t(u, \chi, * ) \) for \( \mu_t(u, \chi, 0) \).

Finally, we shall use the well-known bound for orders of torsion units.

**Proposition 5.** (see [9]) The order of a torsion element \( u \in V(\mathbb{Z}G) \) is a divisor of the exponent of \( G \).

3. Proof of the Theorem

Throughout this section we denote \( M_{12} \) by \( G \). The character table of \( G \), as well as the \( p \)-Brauer character tables, which will be denoted by \( \mathcal{B}E\mathcal{S}(p) \) where \( p \in \{2, 3, 5, 11\} \), can be found using the computational algebra system GAP [10]. For the characters and conjugacy classes we will use throughout the paper the same notation, indexation inclusive, as used in GAP.

Since the group \( G \) possesses elements of orders 2, 3, 4, 5, 6, 8, 10 and 11, first of all we investigate units of some of these orders (except the units of orders 4, 6 and 8). After this, by Proposition 5 the order of each torsion unit divides the exponent of \( G \), so it remains to consider units of orders 12, 15, 20, 22, 33 and 55. We prove that no units of all these orders, except for 12 and 20, do appear in \( V(\mathbb{Z}G) \).

Now we consider each case separately.

- Let \( u \) be an involution. By [11] and Proposition 2 we have that \( \nu_{2a} + \nu_{2b} = 1 \). Applying Proposition 4 to the character \( \chi_2 \), we get the following system

\[
\begin{align*}
\mu_0(u, \chi_2, *) &= \frac{1}{2}(-t_1 + 11) \geq 0; \\
\mu_0(u, \chi_2, 3) &= \frac{1}{2}(-2t_2 + 10) \geq 0;
\end{align*}
\]
where $t_1 = \nu_{2a} - 3\nu_{2b}$ and $t_2 = \nu_{2a} - \nu_{2b}$. Obviously, $t_1 \in \{2s + 1 \mid -6 \leq s \leq 5\}$ and $-5 \leq t_2 \leq 5$. From these restrictions and the requirement that all $\mu_i(u, \chi_j, p)$ must be non-negative integers we obtain six pairs $(\nu_{2a}, \nu_{2b})$ listed in part (ii) of Theorem 1. Note that checking conditions of Proposition 4 for all other combinations of $\chi_j$ and $p \in \{2, 3, 5, 11\}$, we will not get further restrictions on partial augmentations.

- Let $u$ be a unit of order 3. By (1) and Proposition 2 we get $\nu_{3a} + \nu_{3b} = 1$. By (2) we obtain only three solutions $(\nu_{3a}, \nu_{3b})$ listed in part (iii) of Theorem 1.

- Let $u$ be a unit of order 5. Using Propositions 2 and 3 we obtain that all partial augmentations except one are zero. Thus by Proposition 2 the proof of part (iv) of Theorem 1 is done.

Now we need to consider the six cases defined by part (ii) of Theorem 1.

Case 1. $\chi(u^5) = \chi(2a)$. Applying Proposition 4 we get the system of inequalities

\[
\begin{align*}
\mu_0(u, \chi_2, \ast) &= \frac{1}{10}(-4t_1 + 14) \geq 0; \\
\mu_0(u, \chi_4, \ast) &= \frac{1}{10}(4t_2 + 24) \geq 0; \\
\mu_0(u, \chi_7, \ast) &= \frac{1}{10}(4t_3 + 56) \geq 0;
\end{align*}
\]

where $t_1 = 2\nu_{3a} - \nu_{3b}$. Obviously, $t_1 \in \{-4, -1, 2, 5, 8\}$ and $t_2 = \nu_{3a} + \nu_{3b} = 1$ and the condition that $\mu_i(u, \chi_j, p)$ are non-negative integers, we obtain the five pairs $(\nu_{3a}, \nu_{3b})$ listed in part (iii) of Theorem 1.

- Let $u$ be a unit of order 5. Using Propositions 2 and 3 we obtain that partial augmentations except one are zero. Thus by Proposition 2 the proof of part (iv) of Theorem 1 is done.

Case 2. $\chi(u^5) = \chi(2b)$. Again by using Proposition 4 we obtain

\[
\begin{align*}
\mu_0(u, \chi_2, \ast) &= \frac{1}{10}(-4t_1 + 18) \geq 0; \\
\mu_0(u, \chi_4, \ast) &= \frac{1}{10}(4t_2 + 20) \geq 0; \\
\mu_0(u, \chi_7, \ast) &= \frac{1}{10}(4t_3 + 56) \geq 0;
\end{align*}
\]

where $t_1, t_2$ and $t_3$ are defined as in the previous case. From this, it follows that $t_1 \in \{-3, 2\}$, $t_2 \in \{-5, 0, 5\}$ and $t_3 \in \{-14, -9, -4, 1, 6, 11\}$. Only three 4-tuples $(-1, 1, 0, 1), (0, 1, 0, 0)$ and $(1, 1, 0, -1)$ may satisfy these restrictions and the condition for $\mu_i(u, \chi_j, p)$ to be non-negative integers. After considering additional inequalities

\[
\begin{align*}
\mu_1(u, \chi_4, \ast) &= \frac{1}{10}(4\nu_{2a} + \nu_{5a} - \nu_{10a} + 15) \geq 0; \\
\mu_0(u, \chi_4, 3) &= \frac{1}{10}(12\nu_{2a} - 4\nu_{2b} - 8\nu_{10a} + 14) \geq 0,
\end{align*}
\]

we can eliminate two more solutions, and so there remains only $(1, 1, 0, -1)$. 

Case 3. \( \chi(u^5) = -2\chi(2a) + 3\chi(2b) \). As above, by Proposition \([4]\) we obtain that

\[
\begin{align*}
\mu_1(u, \chi_2, *) &= \frac{1}{10}(-t_1 - 1) \geq 0; & \mu_5(u, \chi_2, *) &= \frac{1}{10}(t_1 + 4) \geq 0; \\
\mu_0(u, \chi_4, *) &= \frac{1}{10}(4t_2 + 12) \geq 0; & \mu_2(u, \chi_4, *) &= \frac{1}{10}(-t_2 + 7) \geq 0; \\
\mu_0(u, \chi_7, *) &= \frac{1}{10}(4t_3 + 56) \geq 0; & \mu_5(u, \chi_7, *) &= \frac{1}{10}(-4t_3 + 44) \geq 0,
\end{align*}
\]

where \( t_1, t_2 \) and \( t_3 \) are defined as in the previous case. This yields \( t_1 = -1 \), \( t_2 \in \{-3, 7\} \) and \( t_3 \in \{-14, -9, -4, 1, 6, 11\} \), and there are no solutions satisfying these restrictions and the condition for \( \mu_i(u, \chi_j, p) \) to be non-negative integers.

Case 4. \( \chi(u^5) = 2\chi(2a) - \chi(2b) \). Again, for the same \( t_1, t_2 \) and \( t_3 \) we have

\[
\begin{align*}
\mu_0(u, \chi_2, *) &= \frac{1}{10}(-4t_1 + 10) \geq 0; & \mu_2(u, \chi_2, *) &= \frac{1}{10}(t_1 + 5) \geq 0; \\
\mu_0(u, \chi_4, *) &= \frac{1}{10}(4t_2 + 28) \geq 0; & \mu_5(u, \chi_4, *) &= \frac{1}{10}(-4t_2 + 12) \geq 0; \\
\mu_0(u, \chi_7, *) &= \frac{1}{10}(4t_3 + 56) \geq 0; & \mu_5(u, \chi_7, *) &= \frac{1}{10}(-4t_3 + 44) \geq 0.
\end{align*}
\]

We have that \( t_1 = -5, t_2 \in \{-7, -2, -3\} \) and \( t_3 \in \{-14, -9, -4, 1, 6, 11\} \), and, by the same arguments as in the previous case, we have no solutions.

Case 5. \( \chi(u^5) = 3\chi(2a) - 2\chi(2b) \). Again, we obtain that

\[
\begin{align*}
\mu_0(u, \chi_2, *) &= \frac{1}{10}(-4t_1 + 6) \geq 0; & \mu_2(u, \chi_2, *) &= \frac{1}{10}(t_1 + 1) \geq 0; \\
\mu_1(u, \chi_4, *) &= \frac{1}{10}(t_2 + 3) \geq 0; & \mu_5(u, \chi_4, *) &= \frac{1}{10}(-4t_2 + 8) \geq 0; \\
\mu_0(u, \chi_7, *) &= \frac{1}{10}(4t_3 + 56) \geq 0; & \mu_5(u, \chi_7, *) &= \frac{1}{10}(-4t_3 + 44) \geq 0,
\end{align*}
\]

so \( t_1 = -1, t_2 = -3 \) and \( t_3 \in \{-14, -9, -4, 1, 6, 11\} \), which yields no solutions by the same arguments.

Case 6. \( \chi(u^5) = -\chi(2a) + 2\chi(2b) \). Similarly, we get

\[
\begin{align*}
\mu_1(u, \chi_2, *) &= \frac{1}{10}(-t_1 + 3) \geq 0; & \mu_5(u, \chi_2, *) &= \frac{1}{10}(4t_1 + 8) \geq 0; \\
\mu_0(u, \chi_4, *) &= \frac{1}{10}(4t_2 + 16) \geq 0; & \mu_5(u, \chi_4, *) &= \frac{1}{10}(-4t_2 + 24) \geq 0; \\
\mu_0(u, \chi_7, *) &= \frac{1}{10}(4t_3 + 56) \geq 0; & \mu_5(u, \chi_7, *) &= \frac{1}{10}(-4t_3 + 44) \geq 0,
\end{align*}
\]

so \( t_1 = 3, t_2 \in \{-4, 1, 6\} \) and \( t_3 \in \{-14, -9, -4, 1, 6, 11\} \). As before, in this case too we have no solutions. Thus, part (v) of the Theorem is proved.

- Let \( u \) be a unit of order 11. By (1) and Proposition (2) we get \( \nu_{11a} + \nu_{11b} = 1 \). Applying Proposition (4) to characters \( \chi_4 \) we obtain

\[
\begin{align*}
\mu_1(u, \chi_4, *) &= \frac{1}{11}(6\nu_{11a} - 5\nu_{11b} + 16) \geq 0; \\
\mu_2(u, \chi_4, *) &= \frac{1}{11}(-5\nu_{11a} + 6\nu_{11b} + 16) \geq 0; \\
\mu_1(u, \chi_4, 3) &= \frac{1}{11}(7\nu_{11a} - 4\nu_{11b} + 15) \geq 0; \\
\mu_2(u, \chi_4, 3) &= \frac{1}{11}(-4\nu_{11a} + 7\nu_{11b} + 15) \geq 0,
\end{align*}
\]

which admits only the four integer solutions listed in part (vi) of the Theorem, such that \( \mu_1(u, \chi_4, *) \), \( \mu_2(u, \chi_4, *) \) and \( \mu_2(u, \chi_4, 3) \) are non-negative integers.

- Let \( u \) be a unit of order 15. By (1) and Proposition (2) we have that

\[
\nu_{3a} + \nu_{3b} + \nu_{5a} = 1.
\]
Since $|u^5| = 3$, for any character $\chi$ of $G$ we need to consider five cases, defined by part (iii) of the Theorem. Put

\[ (\alpha, \beta, \gamma, \delta, \kappa) = \begin{cases} 
(45, 45, 19, 16, 8), & \text{if } \chi(u^5) = \chi(3a); \\
(51, 42, 13, 22, 11), & \text{if } \chi(u^5) = \chi(3b); \\
(39, 48, 25, 10, 5), & \text{if } \chi(u^5) = 2\chi(3a) - \chi(3b); \\
(33, 51, 31, 4, 2), & \text{if } \chi(u^5) = 3\chi(3a) - 2\chi(3b); \\
(57, 39, 7, 28, 14), & \text{if } \chi(u^5) = -\chi(3a) + 2\chi(3b). 
\end{cases} \]

By (2) we obtain the system of inequalities

\[ \mu_0(u, \chi_6, *) = \frac{1}{15}(24\nu_{3b} + \alpha) \geq 0; \quad \mu_5(u, \chi_6, *) = \frac{1}{15}(-12\nu_{3b} + \beta) \geq 0. \]

It follows that the integral solution is $\nu_{3b} \in \{-1, 4\}$ if $(\alpha, \beta) = (39, 48)$, and

\[ \nu_{3b} = \begin{cases} 
0, & \text{if } (\alpha, \beta) \in \{(45, 45), (57, 39)\}; \\
1, & \text{if } (\alpha, \beta) = (51, 42); \\
3, & \text{if } (\alpha, \beta) = (33, 51). 
\end{cases} \]

Again by (2) we obtain the system of inequalities

\[ \mu_0(u, \chi_2, *) = \frac{1}{15}(16\nu_{3a} - 8\nu_{3b} + 8\nu_{5a} + \gamma) \geq 0; \quad \mu_0(u, \chi_4, *) = \frac{1}{15}(-16\nu_{3a} + 8\nu_{3b} + 8\nu_{5a} + \delta) \geq 0; \quad \mu_1(u, \chi_2, *) = \frac{1}{15}(2\nu_{3a} - \nu_{3b} + \nu_{5a} + \kappa) \geq 0, \]

which has no integral solutions such that $(\alpha, \beta, \gamma, \delta, \kappa)$ from (3), $\nu_{3b}$ from (4) and $\mu_i(u, \chi_2, *)$ are non-negative integers.

• Let $u$ be a unit of order 22. By (1) and Proposition 2 we have

\[ \nu_{2a} + \nu_{2b} + \nu_{11a} + \nu_{11b} = 1. \]

Since $u^{11}$ has order 2 and $u^2$ has order 11, by (ii) and (v) of the Theorem we have six and four different partial augmentations for these orders, respectively. Thus we need to consider altogether $6 \cdot 4 = 24$ cases. For any character $\chi$ of $G$ put

\[ \mathfrak{R}_\chi = \{ \chi(11a), \chi(11b), 2\chi(11a) - \chi(11b), -\chi(11a) + 2\chi(11b) \}; \]

\[ (\alpha, \beta) = \begin{cases} 
(10, 12), & \text{if } \chi(u^2) \in \mathfrak{R}_\chi, \chi(u^{11}) = \chi(2a); \\
(14, 8), & \text{if } \chi(u^2) \notin \mathfrak{R}_\chi, \chi(u^{11}) = \chi(2b); \\
(22, 0), & \text{if } \chi(u^2) \in \mathfrak{R}_\chi, \chi(u^{11}) = -2\chi(2a) + 3\chi(2b); \\
(6, 16), & \text{if } \chi(u^2) \notin \mathfrak{R}_\chi, \chi(u^{11}) = 2\chi(2a) - \chi(2b); \\
(2, 20), & \text{if } \chi(u^2) \in \mathfrak{R}_\chi, \chi(u^{11}) = 3\chi(2a) - 2\chi(2b); \\
(18, 4), & \text{if } \chi(u^2) \notin \mathfrak{R}_\chi, \chi(u^{11}) = -2\chi(2a) + 2\chi(2b). 
\end{cases} \]

By (2) we obtain that

\[ \mu_0(u, \chi_2, *) = \frac{1}{15}(-10 \cdot (\nu_{2a} - 3\nu_{2b}) + \alpha) \geq 0; \quad \mu_{11}(u, \chi_2, *) = \frac{1}{15}(10 \cdot (\nu_{2a} - 3\nu_{2b}) + \beta) \geq 0. \]

It is easy to check, that if $(\alpha, \beta) \notin \{(10, 12), (22, 0)\}$, then this system of inequalities has no integral solutions such that $\mu_i(u, \chi_2, *)$ are non-negative integers.

If $(\alpha, \beta) = (10, 12)$, then from the last system of inequalities we obtain that $\nu_{2a} = 3\nu_{2b} + 1$, and put $(\gamma, \delta) = (50, 60)$. If $(\alpha, \beta) = (2, 20)$, then from the last system of inequalities we get $\nu_{2a} = 3\nu_{2b} - 2$, and put $(\gamma, \delta) = (26, 84).$
Again, using \(^2\) in both cases of values of \((\alpha, \beta)\) we have
\[
\begin{align*}
\mu_0(u, \chi_8, \ast) &= \frac{1}{12}(50 \nu_2a - 70 \nu_2b + \gamma) \\
\mu_11(u, \chi_8, \ast) &= \frac{1}{12}(50 \nu_2a - 70 \nu_2b + \delta)
\end{align*}
\]
which has no integer solution such that \((\gamma, \delta) \in \{(50, 60), (26, 84)\}\) and \(\mu_1(u, \chi_8, \ast)\) are non-negative integers.

- Let \(u\) be a unit of order 33. Obviously, \(\nu_{3a} + \nu_{3b} + \nu_{11a} + \nu_{11b} = 1\) by \(^1\) and Proposition\(^2\). Since \(u^{11}\) has order 3 and \(u^3\) has order 11, by (iii) and (v) of the Theorem we have five and four different partial augmentations, respectively. Thus we need to consider 20 cases, such that
\[
\begin{align*}
\chi(u^3) &\in \mathfrak{R}_\chi = \{-\chi(11a) + 2\chi(11b), \chi(11b), 2\chi(11a) - \chi(11b), \chi(11a)\}; \\
\chi(u^{11}) &\in \{-\chi(3a) + 2\chi(3b), \chi(3a), \chi(3b), 3\chi(3a) - 2\chi(3b), 2\chi(3a) - \chi(3b)\},
\end{align*}
\]
where \(\chi\) is a character of the group \(G\). Put
\[
(5) \quad (\alpha, \beta) = \begin{cases} (15,9), & \text{if } \chi(u^3) \in \mathfrak{R}_\chi, \chi(u^{11}) = \chi(3a); \\
(9,12), & \text{if } \chi(u^3) \in \mathfrak{R}_\chi, \chi(u^{11}) = \chi(3b); \\
(21,6), & \text{if } \chi(u^3) \in \mathfrak{R}_\chi, \chi(u^{11}) = 2\chi(3a) - \chi(3b); \\
(27,3), & \text{if } \chi(u^3) \in \mathfrak{R}_\chi, \chi(u^{11}) = 3\chi(3a) - 2\chi(3b); \\
(3,15), & \text{if } \chi(u^3) \in \mathfrak{R}_\chi, \chi(u^{11}) = -\chi(3a) + 2\chi(3b).
\end{cases}
\]
By \(^2\) we obtain the system of inequalities
\[
\begin{align*}
\mu_0(u, \chi_2, \ast) &= \frac{1}{33}(20 \cdot (2\nu_{3a} - \nu_{3b}) + \alpha) \geq 0; \\
\mu_{11}(u, \chi_2, \ast) &= \frac{1}{33}(10 \cdot (2\nu_{3a} - \nu_{3b}) + \beta) \geq 0.
\end{align*}
\]
It is easy to check that this system of inequalities has no integral solutions such that \((\alpha, \beta)\) from \(^3\) and \(\mu_i(u, \chi_2, \ast)\) are non-negative integers.

- Let \(u\) be a unit of order 55. By \(^1\) and Proposition\(^2\) we have that
\[
\nu_{5a} + \nu_{11a} + \nu_{11b} = 1.
\]
Since \(u^5\) has order 11 and by (v) of the Theorem we have four different partial augmentations, we need to consider the following four cases:
\[
\begin{align*}
\chi(u^5) &= \chi(11a); \\
\chi(u^5) &= \chi(11b); \\
\chi(u^5) &= -\chi(11a) + 2\chi(11b); \\
\chi(u^5) &= 2\chi(11a) - \chi(11b),
\end{align*}
\]
where \(\chi\) is a character of \(G\). Again by \(^2\) we get in all of these four cases
\[
\begin{align*}
\mu_0(u, \chi_2, \ast) &= \frac{1}{55}(40\nu_{5a} + 15) \geq 0; \\
\mu_1(u, \chi_2, \ast) &= \frac{1}{55}(\nu_{5a} + 10) \geq 0; \\
\mu_{11}(u, \chi_2, \ast) &= \frac{1}{55}(-10\nu_{5a} + 10) \geq 0.
\end{align*}
\]
Clearly, this system of inequalities has no integral solutions such that \(\mu_0(u, \chi_2, \ast)\), \(\mu_1(u, \chi_2, \ast)\) and \(\mu_{11}(u, \chi_2, \ast)\) are non-negative integers. The proof is done.

\[\square\]

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