Calogero-Moser Models V: Supersymmetry and Quantum Lax Pair

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Abstract

It is shown that the Calogero-Moser models based on all root systems of the finite reflection groups (both the crystallographic and non-crystallographic cases) with the rational (with/without a harmonic confining potential), trigonometric and hyperbolic potentials can be simply supersymmetrised in terms of superpotentials. There is a universal formula for the supersymmetric ground state wavefunction. Since the bosonic part of each supersymmetric model is the usual quantum Calogero-Moser model, this gives a universal formula for its ground state wavefunction and energy, which is determined purely algebraically. Quantum Lax pair operators and conserved quantities for all the above Calogero-Moser models are established.

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1 Introduction

The supersymmetric generalisation of quantum Calogero-Moser models in terms of superpotentials is presented. It applies to all of the Calogero-Moser models based on the crystalllographic and non-crystalllographic root systems and with the degenerate potentials, i.e. the rational, hyperbolic and trigonometric potentials. The supersymmetric ground state is easy to obtain and has zero energy, and we are able to deduce a universal formula for the ground state wavefunction and ground state energy of the non-supersymmetric models. Our calculations involve the consideration of all the two-dimensional sub-root systems lying in the original one. Historically, the integrability of Calogero-Moser models [1, 2] was first discovered in the quantum mechanical models. As we will show in this paper, the quantum and classical integrability [3, 4, 5] are very closely related. In this paper, the generic case with the elliptic potentials will not be discussed. Supersymmetrisation and quantisation of Calogero-Moser models with elliptic potentials remains a great challenge.

For general background and the motivations for this paper, and for the physical applications of the Calogero-Moser models with various potentials to lower-dimensional physics, ranging from solid state to particle physics and supersymmetric gauge theories, we refer to our previous papers [3, 4] and references therein.

This paper is organised as follows. In section 2 we summarise the classical Calogero-Moser models to set the stage and introduce appropriate notation. One special property (2.31) of the Lax pair for the models with degenerate potentials is pointed out. This property will be essential in constructing the quantum Lax pair operators in section 4. In section 3 the supersymmetrisation of quantum Calogero-Moser models with degenerate potentials is presented, and we derive the formulas for the ground state wavefunction (3.51) and the ground state energy (3.42), (3.43) of the non-supersymmetric models. In section 4 we derive quantum Lax pair equations (4.22) and (4.23) for the non-supersymmetric models and deduce the quantum conserved quantities (4.24) and (4.25). Section 5 is for comments and discussion.

2 Calogero-Moser Models

In this section we briefly introduce the classical Calogero-Moser models along with appropriate notation and background for the main body of this paper. We consider only the
degenerate potentials, that is the rational (with/without harmonic force), hyperbolic and trigonometric potentials. In these cases the universal Lax pair operator \textit{without spectral parameter} is drastically simplified and some new features not shared by the most general Lax pair arise. These will become important for the quantum Lax pairs and conserved quantities to be discussed in section \textit{4}.

2.1 Model

A (generalised) Calogero-Moser model is a Hamiltonian system associated with a root system $\Delta$ of rank $r$, which is a set of vectors in $\mathbb{R}^r$ with its standard inner product, invariant under reflections in the hyperplane perpendicular to each vector in $\Delta$. In other words,

$$s_\alpha(\beta) \in \Delta, \quad \forall \alpha, \beta \in \Delta,$$

where

$$s_\alpha(\beta) = \beta - 2(\alpha \cdot \beta/|\alpha|^2)\alpha.$$  \hspace{1cm} (2.1)

Dual roots are defined by $\alpha^\vee = 2\alpha/|\alpha|^2$, in terms of which

$$s_\alpha(\beta) = \beta - (\alpha^\vee, \beta)\alpha.$$  \hspace{1cm} (2.2)

The set of reflections $\{s_\alpha, \alpha \in \Delta\}$ generates a group, known as a Coxeter group, or finite reflection group. The orbit of $\beta \in \Delta$ is the set of root vectors resulting from the action of the Coxeter group on it. The set of positive roots $\Delta_+$ may be defined in terms of a vector $U \in \mathbb{R}^r$, with $\alpha \cdot U \neq 0, \forall \alpha \in \Delta$, as those roots $\alpha \in \Delta$ such that $\alpha \cdot U > 0$. Given $\Delta_+$, there is a unique set of $r$ simple roots $\Pi = \{\alpha_j, j = 1, \ldots, r\}$ defined such that they span the root space and the coefficients $\{a_j\}$ in $\beta = \sum_{j=1}^r a_j \alpha_j$ for $\beta \in \Delta_+$ are all non-negative. The highest root $\alpha_h$, for which $\sum_{j=1}^r a_j$ is maximal, is then also determined uniquely. The subset of reflections $\{s_\alpha, \alpha \in \Pi\}$ in fact generates the Coxeter group. The products of $s_\alpha$, with $\alpha \in \Pi$, are subject solely to the relations

$$(s_\alpha s_\beta)^{m(\alpha, \beta)} = 1, \quad \alpha, \beta \in \Pi.$$  \hspace{1cm} (2.4)

The interpretation is that $s_\alpha s_\beta$ is a rotation in some plane by $2\pi/m(\alpha, \beta)$. The set of positive integers $m(\alpha, \beta)$ (with $m(\alpha, \alpha) = 1, \forall \alpha \in \Pi$) uniquely specify the Coxeter group.

The root systems for finite reflection groups may be divided into two types: crystallographic and non-crystallographic. Crystallographic root systems satisfy the additional
condition
\[ \alpha \cdot \beta \in \mathbb{Z}, \quad \forall \alpha, \beta \in \Delta. \quad (2.5) \]
which implies that the \( \mathbb{Z} \)-span of \( \Pi \) is a lattice in \( \mathbb{R}^r \) and contains all roots in \( \Delta \). These root systems are associated with simple Lie algebras: \( \{ A_r, r \geq 1 \} \), \( \{ B_r, r \geq 2 \} \), \( \{ C_r, r \geq 2 \} \), \( \{ D_r, r \geq 4 \} \), \( E_6 \), \( E_7 \), \( E_8 \), \( F_4 \) and \( G_2 \), and also \( \{ BC_r, r \geq 2 \} \) which combines the root systems \( B_r \) and \( C_r \). The Coxeter groups for these root systems are called Weyl groups. The remaining non-crystallographic root systems \( [3] \) are \( H_3 \), \( H_4 \), whose Coxeter groups are the symmetry groups of the icosahedron and four-dimensional 600-cell, respectively, and the dihedral group of order \( 2m \), \( \{ I_2(m), m \geq 4 \} \).

The dynamical variables of the Calogero-Moser model are the coordinates \( \{ q_j \} \) and their canonically conjugate momenta \( \{ p_j \} \), with the Poisson brackets
\[ \{ q_j, p_k \} = \delta_{jk}, \quad \{ q_j, q_k \} = \{ p_j, p_k \} = 0, \quad j, k = 1, \ldots, r. \quad (2.6) \]
These will be denoted by vectors in \( \mathbb{R}^r \)
\[ q = (q_1, \ldots, q_r), \quad p = (p_1, \ldots, p_r). \quad (2.7) \]
The Hamiltonian for the classical Calogero-Moser model is
\[ H_C = \frac{1}{2} p^2 + \frac{1}{2} \sum_{\rho \in \Delta} g_{|\rho|}^2 |\rho|^2 V(\rho \cdot q), \quad (2.8) \]
in which the real positive coupling constants \( g_{|\rho|} \) are defined on orbits of the corresponding Coxeter group, i.e. they are identical for roots in the same orbit. That is, for the simple Lie algebra cases \( g_{|\rho|} = g \) for all roots in simply-laced models and \( g_{|\rho|} = g_L \) for long roots and \( g_{|\rho|} = g_S \) for short roots in non-simply laced models. For the \( BC_r \) models there are three couplings, and in the \( I_2(m) \) models, there is one coupling if \( m \) is odd, and two if \( m \) is even (see section \( [3] \)). The \( H_3 \) and \( H_4 \) models have one coupling constant \( g_{|\rho|} = g \), since these root systems are simply-laced. (Exhibiting the factor \( |\rho|^2 \), rather than absorbing it into the coupling constant, is a convenience.) This then ensures that for any potential \( V \), the Hamiltonian is invariant under reflection of the phase space variables in the hyperplane perpendicular to any root
\[ q \rightarrow s_\alpha(q), \quad p \rightarrow s_\alpha(p), \quad \forall \alpha \in \Delta \quad (2.9) \]
with \( s_\alpha \) defined by \( (2.3) \).
The Lax pair operators that we will introduce will apply for the following degenerate potentials:

\[
V(\alpha \cdot q) = \begin{cases} 
1/(\alpha \cdot q)^2, \\
a^2/\sinh^2 a(\alpha \cdot q), \\
a^2/\sin^2 a(\alpha \cdot q), 
\end{cases}
\]

in which \(a\) is an arbitrary real positive constant, determining the period of the trigonometric potentials (and the imaginary period in the hyperbolic case, although this has less significance). They imply integrability for all of the Calogero-Moser models based on the crystallographic root systems. Those models based on the non-crystallographic root systems, the dihedral group \(I_2(m)\), \(H_3\), and \(H_4\), are integrable only for the rational potential. The rational potential models are also integrable if a confining harmonic potential

\[
\frac{1}{2} \omega^2 q^2, \quad \omega > 0
\]

is added to the Hamiltonian.

Some remarks are in order. For all of the root systems and for any choice of potential (2.10), the Calogero-Moser model has a hard repulsive potential \(\sim 1/(\alpha \cdot q)^2\) near the reflection hyperplane \(H_\alpha = \{q \in \mathbb{R}^r, \alpha \cdot q = 0\}\). The strength of the singularity is given by the coupling constant \(g_{[\alpha]}^2\) which is independent of the choice of the normalisation of the roots. This determines the form of the ground state wave function in the quantum version of the theory, as we will see in section 3. The repulsive potential is classically insurmountable. Thus the motion is always confined within one Weyl chamber. This feature allows us to constrain the configuration space to the principal Weyl chamber

\[
PW = \{q \in \mathbb{R}^r \mid \alpha \cdot q > 0, \quad \alpha \in \Pi\},
\]

without loss of generality. In the case of the trigonometric potential, the configuration space is further limited due to the periodicity of the potential to

\[
PW_T = \{q \in \mathbb{R}^r \mid \alpha \cdot q > 0, \quad \alpha \in \Pi, \quad \alpha_h \cdot q < \pi/a\},
\]

where \(\alpha_h\) is the highest root.

### 2.2 Lax Pair

Here we recapitulate the essence of the universal Lax pair operators for the Calogero-Moser models with degenerate potentials and without spectral parameter. For details and a full
exposition, see [4]. The Lax operators are

\[
L = p \cdot \hat{H} + X, \quad X = i \sum_{\rho \in \Delta_+} g_{|\rho|} (\rho \cdot \hat{H}) x(\rho \cdot q) \hat{s}_\rho, \tag{2.14}
\]

\[
\tilde{M} = i \sum_{\rho \in \Delta_+} g_{|\rho|} |\rho|^2 y(\rho \cdot q) \hat{s}_\rho, \tag{2.15}
\]
in which \(\{\hat{s}_\alpha, \alpha \in \Delta\}\) are the reflection operators of the root system. They act on a set of \(R^r\) vectors \(\mathcal{R} = \{\mu^{(k)} \in R^r, \ k = 1, \ldots, d\}\), permuting them under the action of the reflection group. The vectors in \(\mathcal{R}\) form a basis for the representation space \(V\) of dimension \(d\). The simplest and the most natural representation spaces of the Lax pair operators are provided by the set of all roots \(\Delta\) for the simply-laced root systems, and the set of short roots \(\Delta_S\) or the set of long roots \(\Delta_L\) for non-simply laced root systems. These give root type Lax pairs, [3]. Another class of simple representations are the so-called minimal type representations, for which \(\mathcal{R}\) consists of the weights belonging to a minimal representation, and which give minimal type Lax pairs [3].

The set of operators \(\{\hat{H}_j, j = 1, \ldots, r\}\) are defined as follows. If \(\hat{H}_j\) acts on a vector \(\mu^{(k)} \in \mathcal{R}\), the \(j\)-th component is returned:

\[
\hat{H}_j \mu^{(k)} = \mu_j^{(k)} \mu^{(k)}.
\]

These, along with the reflection operators, form the following operator algebra:

\[
[\hat{H}_j, \hat{H}_k] = 0, \tag{2.16}
\]

\[
[\hat{H}_j, \hat{s}_\alpha] = \alpha_j (\alpha^\vee \cdot \hat{H}) \hat{s}_\alpha, \tag{2.17}
\]

\[
\hat{s}_\alpha \hat{s}_\beta \hat{s}_\alpha = \hat{s}_{\alpha(\beta)}, \tag{2.18}
\]

\[
(\hat{s}_\alpha \hat{s}_\beta)^{m(\alpha, \beta)} = 1. \tag{2.19}
\]

The first relation (2.16) implies that the operators \(\{\hat{H}_j, j = 1, \ldots, r\}\) form an abelian subalgebra and relations (2.18) and (2.19) are just those for the finite reflection group associated with the root system \(\Delta\). The set of integers \(m(\alpha, \beta)\) are those appearing in the Coxeter relations (2.4) which characterise the reflection group.

The form of the function \(x\) depends on the chosen potential, and the function \(y\) and another function \(w\) to be used in section [3] are defined by

\[
y(u) \equiv \frac{d}{du} x(u), \quad \frac{dw(u)}{du}/w(u) \equiv x(u). \tag{2.20}
\]
They have definite parities:
\begin{align*}
x(-u) &= -x(u), \quad y(-u) = y(u), \quad w(-u) = -w(u), \quad (2.21)
\end{align*}
so that \( \mathcal{L} \) and \( \widetilde{M} \) are independent of the choice of positive roots \( \Delta_+ \). This also implies that the sums in (2.14), (2.13) may be extended to a sum over all roots if an additional factor of \( 1/2 \) is included in front of the sums since the summands are even under \( \rho \rightarrow -\rho \). The functions \( x \) and \( y \) are further related to each other, and to the potential function \( V \) occurring in the Hamiltonian via
\begin{align*}
V(u) = -y(u) = x^2(u) + a^2 \times \begin{cases} 0 & \text{rational} \\ -1 & \text{hyperbolic} \\ 1 & \text{trigonometric}. \end{cases} \quad (2.22)
\end{align*}
Note that these relations are only valid for the degenerate potentials (2.10) and in the Lax pair without spectral parameter. The following Table 1 gives these functions for each potential:

| Potential   | Function 1 | Function 2 | Function 3 |
|-------------|------------|------------|------------|
| rational    | \( u \)    | \( 1/u \)  | \(-1/u^2\) |
| hyperbolic  | \( \sinh au \) | \( a \coth au \) | \(-a^2/\sinh^2 au\) |
| trigonometric | \( \sin au \) | \( a \cot au \) | \(-a^2/\sin^2 au\) |

Table 1: Functions appearing in the Lax pair and superpotential.

The underlying idea of the Lax operator \( \mathcal{L} \), (2.14), is quite simple. As seen from (2.29), \( \mathcal{L} \) is a “square root” of the Hamiltonian. Thus one part of \( \mathcal{L} \) contains \( p \) which is not associated with roots and another part contains \( x(\rho \cdot q) \), a “square root” of the potential \( V(\rho \cdot q) \), which being associated with a root \( \rho \) is therefore accompanied by the reflection operator \( \hat{s}_\rho \).

It is straightforward to show that the Lax equation
\begin{align*}
\frac{d}{dt} L &= [L, \widetilde{M}], \quad (2.23)
\end{align*}
which divides into two parts as
\begin{align*}
\frac{d}{dt} X &= [p \cdot \hat{H}, \widetilde{M}], \quad (2.24)
\end{align*}
\begin{align*}
\frac{d}{dt} (p \cdot \hat{H}) &= [X, \widetilde{M}], \quad (2.25)
\end{align*}
is equivalent to the canonical equations of motion:
\begin{align*}
\dot{q}_j &= \frac{\partial \mathcal{H}_C}{\partial p_j} = p_j, \quad (2.26)
\end{align*}
\begin{align*}
\dot{p}_j &= -\frac{\partial \mathcal{H}_C}{\partial q_j} = -\frac{\partial}{\partial q_j} \left[ \sum_{\rho \in \Delta_+} \frac{1}{2} g_{\rho}^2 |\rho|^2 V(\rho \cdot q) \right]. \quad (2.27)
\end{align*}
For the details of the proof, see [4]. It is amusing to note that the Lax equation is rather symmetric in $X \leftrightarrow p \cdot \hat{H}$. In section 4 we will discuss the quantum version of these equations.

It is well-known that conserved quantities are given in terms of a representation $R$ of the operator $L$ as

$$\text{Tr}(L^n) \equiv \sum_{\mu \in R} (L^n)_{\mu\mu}, \quad n = 1, 2, \ldots,$$

(2.28)

in which $\mu$’s are the basis vectors of the representation $R$. In particular, the classical Hamiltonian (2.8) is given by

$$H_C = \frac{1}{2C_R} \text{Tr}(L^2) + \text{const},$$

(2.29)

where the constant $C_R$, which depends on the representation, is defined by

$$\text{Tr}(\hat{H}_j \hat{H}_k) \equiv \sum_{\mu \in R} (\hat{H}_j \hat{H}_k)_{\mu\mu} = \sum_{\mu \in R} \mu_j \mu_k = C_R \delta_{jk}. $$

(2.30)

Before closing this section let us remark on one special property of a representation matrix of the Lax operator $\tilde{M}$:

$$\sum_{\mu \in R} \tilde{M}_{\mu\nu} = \sum_{\nu \in R} \tilde{M}_{\mu\nu} = -i \mathcal{D} \equiv i \sum_{\rho \in \Delta^+} g|\rho|^2 y(\rho \cdot q) = -i \frac{1}{2} \sum_{\rho \in \Delta^+} g|\rho|^2 V(\rho \cdot q).$$

(2.31)

The quantity $\mathcal{D}$ is independent of the representation $R$. Thus we can define a new Lax operator $M$ by

$$M = \tilde{M} + i \mathcal{D} \times I, \quad I : \text{Identity operator},$$

(2.32)

which satisfies the relation

$$\sum_{\mu \in R} M_{\mu\nu} = \sum_{\nu \in R} M_{\mu\nu} = 0.$$  

(2.33)

The new Lax pair $L$ and $M$ gives the same classical equations of motion as above. The above property (2.33) has been known for the $A_r$ model Lax pair in the vector representation [8].

We stress that this is a universal property shared by all of the Lax matrices without spectral parameter for the degenerate potentials in any representation. The reason is that the $\mu\nu$ matrix element of the $\tilde{M}$ operator (2.15) reads

$$\tilde{M}_{\mu\nu} = \frac{i}{2} \sum_{\rho \in \Delta^+} g|\rho|^2 y(\rho \cdot q) (\hat{s}_\rho)_{\mu\nu},$$

(2.34)

in which

$$(\hat{s}_\rho)_{\mu\nu} = \delta_{\mu,\nu}(\rho) = \delta_{\nu,\rho}(\mu).$$

(2.35)
Since \( s_\rho(\nu) \) (\( s_\rho(\mu) \)) is always contained in the basis of the representation precisely once,

\[
\sum_{\mu \in \mathcal{R}} (\hat{s}_\rho)_{\mu\nu} = \sum_{\nu \in \mathcal{R}} (\hat{s}_\rho)_{\mu\nu} = 1
\] (2.36)

and so (2.31) is obtained. This also means that for \( \mu \neq \nu \), \( \tilde{M}_{\mu\nu} \) is either 0 or it consists of a single term (two terms in the BC\(_r\) case). For suppose two positive roots \( \rho \) and \( \sigma \) connect \( \mu \) and \( \nu \), then we obtain

\[
\mu = s_\rho(\nu) = \nu - (\rho^\vee, \nu)\rho
\]
\[
= s_\sigma(\nu) = \nu - (\sigma^\vee, \nu)\sigma.
\]

This would imply \( \rho \propto \sigma \) which then means \( \rho = \sigma \) (\( \rho = 2\sigma \) or \( \sigma = 2\rho \) in the BC\(_r\) case) since both are positive roots. The diagonal element \( \tilde{M}_{\mu\mu} \) contains contributions from all of the roots which are orthogonal to \( \mu \).

### 2.2.1 Rational potential with harmonic force

Here we give the Lax pair for the rational potential model with harmonic force. The Hamiltonian is

\[
\mathcal{H}_{C,\omega} = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 + \frac{1}{2} \sum_{\rho \in \Delta_+} g_{\rho}^2 \frac{|\rho|^2}{(\rho \cdot q)^2}.
\] (2.37)

The canonical equations of motion are equivalent to the following Lax equations for \( L^\pm \):

\[
\dot{L}^\pm = [L^\pm, \tilde{M}] \pm i\omega L^\pm,
\] (2.38)

in which (see section 4 of [4]) \( \tilde{M} \) is the same as before (2.13), and \( L^\pm \) and \( Q \) are defined by

\[
L^\pm = L \pm i\omega Q, \quad Q = q \cdot \hat{H},
\] (2.39)

with \( L, \hat{H} \) as earlier. If we define hermitian operators \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) by

\[
\mathcal{L}_1 = L^+ L^- \quad \text{and} \quad \mathcal{L}_2 = L^- L^+,
\] (2.40)

they satisfy Lax-like equations

\[
\dot{\mathcal{L}}_k = [\mathcal{L}_k, \tilde{M}], \quad k = 1, 2.
\] (2.41)

From these we can construct conserved quantities

\[
\text{Tr}(\mathcal{L}_n^1) = \text{Tr}(\mathcal{L}_2^n), \quad n = 1, 2, \ldots,
\] (2.42)
as before. It is elementary to check that the first conserved quantities give the Hamiltonian
\[ (2.37) \]
\[ \text{Tr}(L_1) = \text{Tr}(L_2) \propto \mathcal{H}_{C \omega} + \text{const.} \]
As in the other cases, the operator \( \tilde{M} \) can be replaced by \( M, \) \( (2.32) \), without changing the classical equations of motion. This then completes the presentation of the Lax pairs for all of the classical Calogero-Moser models with non-elliptic potentials.

3 Supersymmetrisation

3.1 Superpotential and Hamiltonian

In this section we show that all the Calogero-Moser models with degenerate potentials summarised in the previous section can be simply supersymmetrised in terms of superpotentials. The result is a quantum system with bosonic and fermionic variables. There are some pioneering works on supersymmetric Calogero-Moser models with degenerate potentials, mainly those based on \( A_r \) and other classical root systems \( [7, 8, 9] \). We shall not consider here the classical supersymmetric Calogero-Moser models, which have dynamical variables taking values in a Grassmann algebra, although these are interesting too.

The bosonic variables have, as before, \( 2r \) degrees of freedom:
\[ q = (q_1, \ldots, q_r), \quad p = (p_1, \ldots, p_r), \]  
with the canonical commutation relations
\[ [q_j, p_k] = i \delta_{jk}, \quad [q_j, q_k] = [p_j, p_k] = 0, \quad j, k = 1, \ldots, r. \]

The corresponding \( N = 2 \) supersymmetric quantum mechanical system requires additionally as many fermionic degrees of freedom:
\[ \psi = (\psi_1, \ldots, \psi_r), \quad \psi^* = (\psi_1^*, \ldots, \psi_r^*), \]
with the canonical anti-commutation relations
\[ \psi_j^* \psi_k + \psi_k \psi_j^* = \delta_{jk}, \quad \psi_j^* \psi_k + \psi_k \psi_j = \psi_j^* \psi_k^* + \psi_k^* \psi_j^* = 0, \quad j, k = 1, \ldots, r. \]

The bosonic and fermionic variables commute with each other:
\[ [q_j, \psi_k] = [q_j, \psi_k^*] = [p_j, \psi_k] = [p_j, \psi_k^*] = 0, \quad j, k = 1, \ldots, r. \]
We realize these variables as operators in the standard way, as acting on wavefunctions which lie in the tensor product of the Hilbert space of functions of $q$ and a $2^r$-dimensional fermionic Fock space. The momentum operator $p_j$ acts as

$$p_j = -i \frac{\partial}{\partial q_j}, \quad j = 1, \ldots, r.$$  

The fermionic variables $\psi$ and $\psi^*$ are respectively annihilation and creation operators, which are hermitian conjugates of each other. The bosonic variables $q$ will be restricted by the potential in the same way as in the classical models to lie in the regions $(2.12)$ or $(2.13)$.

The dynamics of a supersymmetric quantum mechanical system is determined by a superpotential $W(q) = W(q_1, \ldots, q_r) \in \mathbb{R}$. The two supercharges $Q$ and $Q^*$ are defined by

$$Q = \sum_{j=1}^{r} \psi_j^* \left( p_j + i \frac{\partial W}{\partial q_j} \right), \quad Q^* = \sum_{j=1}^{r} \psi_j \left( p_j - i \frac{\partial W}{\partial q_j} \right),$$  

(3.6)

and the supersymmetric Hamiltonian is given by

$$H_{SUSY} = \frac{1}{2} (QQ^* + Q^*Q),$$  

(3.7)

which is obviously positive semi-definite. They satisfy

$$Q^2 = Q^{*2} = 0, \quad [H_{SUSY}, Q] = [H_{SUSY}, Q^*] = 0.$$  

(3.8)

In terms of the superpotential, $H_{SUSY}$ reads

$$H_{SUSY} = \frac{1}{2} \sum_{j=1}^{r} \left( p_j^2 + \left( \frac{\partial W}{\partial q_j} \right)^2 \right) - \frac{1}{2} \sum_{j,k=1}^{r} [\psi_j^*, \psi_k] \frac{\partial^2 W}{\partial q_j \partial q_k},$$  

(3.9)

$$= H_B + H_F,$$  

(3.10)

in which the bosonic and fermionic parts are

$$H_B = \frac{1}{2} \sum_{j=1}^{r} \left( p_j^2 + \left( \frac{\partial W}{\partial q_j} \right)^2 \right) + \frac{1}{2} \sum_{j=1}^{r} \frac{\partial^2 W}{\partial(q_j)^2},$$  

(3.11)

$$H_F = - \sum_{j,k=1}^{r} \psi_j^* \psi_k \frac{\partial^2 W}{\partial q_j \partial q_k},$$  

(3.12)

Note the ordering of the fermionic variables in $H_F$, which is responsible for the last term in $H_B$. The Calogero-Moser dynamics is specified by the following choice of the superpotential

$$W(q) = \sum_{\rho \in \Delta_+} g_{|\rho|} \ln |w(\rho \cdot q)| + \left( -\frac{\omega}{2} q^2 \right), \quad g_{|\rho|} > 0, \quad \omega > 0,$$  

(3.13)
in which the function \( w \) is defined by (2.20) (see also Table 1), and the last term corresponds to the harmonic confining potential in the rational potential model, if present. It should be remarked that this superpotential is \textit{Coxeter-invariant}:

\[
\hat{s}_\rho W = W, \quad \forall \rho \in \Delta,
\]

in which the new reflection operator \( \hat{s}_\rho \) acts on a function \( f \) of \( q \) as follows

\[
(\hat{s}_\rho f)(q) = f(s_\rho(q)).
\]

We show that the bosonic part, \( \mathcal{H}_B \) of \( \mathcal{H}_{SUSY} \), (3.11), can be written as follows:

\[
\mathcal{H}_B = \mathcal{H}_C + \mathcal{H}_{qc} - E_0.
\]

Here \( \mathcal{H}_C \) is the classical Hamiltonian (2.8), interpreted as a quantum operator, and \( \mathcal{H}_{qc} \) is the \textquotedblleft quantum correction\textquotedblright{} term derived from the last term of (3.11):

\[
\frac{1}{2} \sum_{j=1}^{r} \frac{\partial^2 W}{\partial (q_j)^2} = \frac{1}{2} \sum_{\rho \in \Delta_+} g_{|\rho|} |\rho|^2 y(\rho \cdot q) + \left( -\frac{r \omega}{2} \right)
\]

\[
= \frac{1}{2} \sum_{\rho \in \Delta_+} g_{|\rho|} |\rho|^2 V(\rho \cdot q) + \left( -\frac{r \omega}{2} \right),
\]

\[
\mathcal{H}_{qc} = -\frac{1}{2} \sum_{\rho \in \Delta_+} g_{|\rho|} |\rho|^2 V(\rho \cdot q) = -\mathcal{D}.
\]

In deriving (3.18) from (3.17), the relation (2.22) is used, and recall \( \mathcal{D} \) is defined by (2.31). The constant \( r \omega / 2 \) becomes a part of the constant \( E_0 \), which is the \textit{ground state energy} of the bosonic Hamiltonian \( \mathcal{H}_C + \mathcal{H}_{qc} \), since we shall see that both \( \mathcal{H}_{SUSY} \) and \( \mathcal{H}_F \) annihilate the ground state, so \( \mathcal{H}_B \) must do so too.

In order to show (3.16) we need to evaluate \( \sum_j (\partial W/\partial q_j)^2 \). Firstly we have

\[
\frac{\partial W}{\partial q_j} = \sum_{\rho \in \Delta_+} g_{|\rho|} \frac{w'(\rho \cdot q)}{w(\rho \cdot q)} \rho_j + (-\omega q_j) = \sum_{\rho \in \Delta_+} g_{|\rho|} x(\rho \cdot q) \rho_j + (-\omega q_j)
\]

so

\[
\sum_j \left( \frac{\partial W}{\partial q_j} \right)^2 = \sum_{\rho \in \Delta_+} g_{|\rho|}^2 |\rho|^2 x(\rho \cdot q)^2 + (\omega^2 q^2)
\]

\[
+ \sum_{\rho,\sigma \in \Delta_+ \rho \neq \sigma} g_{|\rho|} g_{|\sigma|} (\rho \cdot \sigma) x(\rho \cdot q) x(\sigma \cdot q) + (-2 \omega \sum_{\rho \in \Delta_+} g_{|\rho|} x(\rho \cdot q) \rho \cdot q).
\]

The first line of (3.20) gives the potential terms of the classical Hamiltonian (2.8) up to a constant. Secondly we show that the terms in the second line of (3.20) sum up to a constant.
depending on the root system and the choice of potential. The terms proportional to $\omega$ exist only for the rational potential $x(u) = 1/u$ and thus they give rise to a constant solely determined by the root system:

$$- 2\omega \sum_{\rho \in \Delta^+} g_{[\rho]} x(\rho \cdot q)(\rho \cdot q) = - 2\omega \sum_{\rho \in \Delta^+} g_{[\rho]} \cdot 1 = - \omega \times \left\{ \begin{array}{ll} gN & \text{simply-laced}, \\ gS N_S + gL N_L & \text{non-simply laced}, \end{array} \right.$$  

(3.21)

in which $N$ is the total number of roots in $\Delta$ and $N_S$ ($N_L$) is the number of short (long) roots in $\Delta$.

The main task is to evaluate

$$\sum_{\rho \neq \sigma \in \Delta^+, R_\psi = s_\rho s_\sigma} g_{[\rho]} g_{[\sigma]} (\rho \cdot \sigma) x(\rho \cdot q) x(\sigma \cdot q),$$  

(3.22)

where each distinct pair of roots gives two equal contributions. This we do by decomposing it into two-dimensional planes specified by two roots $\rho$ and $\sigma$:

$$f_\Theta(q) = \sum_{\rho \neq \sigma \in \Delta^+, R_\psi = s_\rho s_\sigma} g_{[\rho]} g_{[\sigma]} (\rho \cdot \sigma) x(\rho \cdot q) x(\sigma \cdot q).$$  

(3.23)

This quantity must be evaluated for a fixed sense of rotation $R_\psi = s_\rho s_\sigma$ and all roots appearing in it are in the two-dimensional sub-root system $\Theta = \{ \kappa, \kappa \in (\Delta \cap \text{span}(\rho, \sigma)) \}$ with positive roots $\Theta_+ \equiv \Theta \cap \Delta_+$. There is a reverse rotation $R_{-\psi} = s_\sigma s_\rho$ in the same plane which gives the same contribution as $f_\Theta(q)$ as it is obtained by $\rho \leftrightarrow \sigma$. The roots belonging to each two-dimensional plane constitute the positive roots of a two-dimensional sub-root system of the original set of roots $\Delta$. The only possible two-dimensional root systems are $A_1 \times A_1$, $A_2$, $B_2$, $G_2$, and $I_2(m)$. Table 2 shows the two-dimensional sub-root systems appearing in the root systems of finite reflection groups. The $A_1 \times A_1$ root system has been omitted since its corresponding quantity (3.23) is always zero. It should be stressed that the quantities $f_\Theta$ are determined by the two-dimensional sub-root systems only and not by where they are embedded in the entire root system.

Let us evaluate them in order. We first consider the $A_2$ sub-root system with $\alpha$ and $\beta$ simple roots, which are of the same length. Thus the coupling dependence factorises and we obtain

$$2f_{A_2}(q)/(g^2|\rho|^2) = x((\alpha + \beta) \cdot q) x(\alpha \cdot q) + x(\beta \cdot q) x((\alpha + \beta) \cdot q)$$

$$- x(\alpha \cdot q) x(\beta \cdot q).$$  

(3.24)
For the rational potential we have, immediately

\[
2f_{A_2}(q) = a^2 g^2 |\rho|^2 \begin{cases} 
0 & \text{rational} \\
1 & \text{hyperbolic} \\
-1 & \text{trigonometric}
\end{cases} = a^2 g^2 \sum_{\rho \neq \sigma \in A_2^+} (\rho \cdot \sigma) \begin{cases} 
0 & \text{rational} \\
1 & \text{hyperbolic} \\
-1 & \text{trigonometric}
\end{cases}
\]  

(3.25)

It is also elementary to evaluate (3.24) for the hyperbolic and trigonometric potentials by using the addition theorems for cot and coth functions. Combining the results:

\[
2f_{A_2}(q) = a^2 g^2 |\rho|^2 \begin{cases} 
0 & \text{rational} \\
1 & \text{hyperbolic} \\
-1 & \text{trigonometric}
\end{cases} = a^2 g^2 \sum_{\rho \neq \sigma \in A_2^+} (\rho \cdot \sigma) \begin{cases} 
0 & \text{rational} \\
1 & \text{hyperbolic} \\
-1 & \text{trigonometric}
\end{cases}
\]  

(3.26)

in which \( a \) is the parameter in the potential (2.10). The sums for the \( B_2 \) and \( G_2 \) sub-root systems may be written in terms of the short and long simple roots, \( \alpha \) and \( \beta \), respectively:

\[
2f_{B_2}(q)/(g_{SL}|\rho_L|^2) = - x(\alpha \cdot q) x(\beta \cdot q) + x((\alpha + \beta) \cdot q) x((2\alpha + \beta) \cdot q)
\]

\[
+ x(\alpha \cdot q) x((2\alpha + \beta) \cdot q) + x((\alpha + \beta) \cdot q) x(\beta \cdot q),
\]  

(3.27)

\[
2f_{G_2}(q)/(g_{SL}|\rho_L|^2) = - x(\alpha \cdot q) x(\beta \cdot q) + x(\alpha \cdot q) x((3\alpha + \beta) \cdot q)
\]

\[
+ x((2\alpha + \beta) \cdot q) x((3\alpha + \beta) \cdot q) + x((2\alpha + \beta) \cdot q) x((3\alpha + 2\beta) \cdot q)
\]

\[
+ x((\alpha + \beta) \cdot q) x((3\alpha + 2\beta) \cdot q) + x((\alpha + \beta) \cdot q) x(\beta \cdot q).
\]  

(3.28)

The \( G_2 \) root system consists of six long roots and six short roots, and the sets of long and short roots have the same structure as the \( A_2 \) roots, scaled and rotated by \( \pi/6 \). The contributions from the long (short) roots only are accounted for by \( f_{A_2} \). The above \( f_{G_2} \) denotes the contribution from the cross terms between the long and short roots.

| Root System | Sub-root Systems |
|-------------|------------------|
| \( A_r, r \geq 1 \) | \( A_2 \) |
| \( B_r, r \geq 2 \) | \( A_2, B_2 \) |
| \( C_r, r \geq 2 \) | \( A_2, B_2 \) |
| \( D_r, r > 3 \) | \( A_2 \) |
| \( BC_r, r \geq 2 \) | \( A_2, B_2 \) |
| \( E_6, E_7, E_8 \) | \( A_2 \) |
| \( F_4 \) | \( A_2, B_2 \) |
| \( G_2 \) | \( A_2, G_2 \) |
| \( I_2(m) \) | \( I_2(k) \)† |
| \( H_3 \) | \( A_2, I_2(5) \) |
| \( H_4 \) | \( A_2, I_2(5) \) |

Table 2: Two-dimensional sub-root systems. \( A_1 \times A_1 \) is not included. †: \( k \) divides \( m \).
Again it is not difficult to evaluate
\[
2f_{B_2}(q) = a^2 g_{SL} |\rho_L|^2 \times \begin{cases} 
0 & \text{rational} \\
2 & \text{hyperbolic} \\
-2 & \text{trigonometric}
\end{cases}
\]

\[= a^2 g_{SL} \sum_{\rho \neq \sigma \in B_2^+} (\rho \cdot \sigma) \times \begin{cases} 
0 & \text{rational} \\
1 & \text{hyperbolic} \\
-1 & \text{trigonometric}
\end{cases}
\]

(3.29)

and
\[
2f_{G_2}(q) = a^2 g_{SL} |\rho_L|^2 \times \begin{cases} 
0 & \text{rational} \\
4 & \text{hyperbolic} \\
-4 & \text{trigonometric}
\end{cases}
\]

\[= a^2 g_{SL} \sum_{\rho, \sigma \in G_2^{+\text{Long, Short}}} (\rho \cdot \sigma) \times \begin{cases} 
0 & \text{rational} \\
1 & \text{hyperbolic} \\
-1 & \text{trigonometric}
\end{cases}
\]

(3.30)

The corresponding sums for the dihedral root systems $I_2(m)$ (with rational potential) are different for odd $m$ (simply-laced) and even $m$ (non-simply laced):
\[
f_{I_2(m)}(q) = g^2 \sum_{j \neq k}^m (\rho_j \cdot q)(\rho_k \cdot q), \quad m : \text{odd}
\]

\[
= g^2 \sum_{j \neq k}^m (\rho_j \cdot q)(\rho_k \cdot q) + g_0^2 \sum_{j \neq k}^m (\rho_j \cdot q)(\rho_k \cdot q)
\]

\[+ 2g_e g_0 \sum_{j \neq k}^m (\rho_j \cdot q)(\rho_k \cdot q), \quad m : \text{even}
\]

(3.31)

in which $g_e$ and $g_0$ are the coupling constants for the even and odd roots. In all cases all the roots are chosen to have the same length $|\rho_j|^2 = 1$, and are parametrised as
\[
\rho_j = (\cos(j\pi/m), \sin(j\pi/m)), \quad j = 1, \ldots, 2m.
\]

(3.32)

It is elementary to show that the sums vanish. For example, for odd $m$, we have
\[
f_{I_2(m)}(q) = \frac{g^2}{|q|^2} \sum_{j \neq k}^m \frac{\cos(\frac{j-k}{m}\pi)}{\cos(t - \frac{j}{m}\pi) \cos(t - \frac{k}{m}\pi)}, \quad q = |q|(\cos t, \sin t),
\]

(3.33)

which is meromorphic and periodic in $t$, with period $\pi$ and it is exponentially decreasing at $t \to \pm i\infty$. It has possible simple poles at $t = j\pi/m + \pi/2$, $j = 1, \ldots, m$. However, its residue at $t = j\pi/m + \pi/2$ vanishes
\[
-\sum_{k=1}^m \frac{\cos(\frac{j-k}{m}\pi)}{\cos(\frac{\pi}{2} + \frac{j-k}{m}\pi)} = \sum_{k=1}^m \cot(\frac{j-k}{m}\pi) = 0.
\]
in which \( \sum' \) means that \( k = j \) term should be omitted. Thus we find \( f_{I_2(m)}(q) = 0 \) for odd \( m \). A similar calculation and result holds for even \( m \).

The ground state energy \( \mathcal{E}_0 \) in (3.16) depends on the root system \( \Delta \) and the choice of the potential \( V \). It has two terms

\[
\mathcal{E}_0 = \mathcal{E}_1 + \mathcal{E}_2. \quad (3.35)
\]

The former, \( \mathcal{E}_1 \), comes from the diagonal part coming from the difference of \( x^2 \) in (3.20) and \( V \) (see (2.22)) and the additional term in (3.17) and (3.21):

\[
\mathcal{E}_1 = \begin{cases} 
0 & \text{rational} \\
\omega \left( \frac{r}{2} + \sum_{\rho \in \Delta_+} g_\rho \right) & \text{rational with harmonic potential} \\
\frac{a^2}{2} \sum_{\rho \in \Delta_+} g_\rho^2 |\rho|^2 \times & \begin{cases} 
-1 & \text{hyperbolic} \\
1 & \text{trigonometric}. 
\end{cases}
\end{cases} \quad (3.36)
\]

The latter, \( \mathcal{E}_2 \), is the constant term coming from (3.22). From (3.26), (3.29) and (3.30) we obtain a universal formula

\[
\mathcal{E}_2 = \frac{a^2}{2} \sum_{\rho \neq \sigma \in \Delta_+} g_{|\rho|} g_{|\sigma|} (\rho \cdot \sigma) \times \begin{cases} 
0 & \text{rational with/without harmonic potential} \\
-1 & \text{hyperbolic} \\
1 & \text{trigonometric}. 
\end{cases} \quad (3.37)
\]

For actual evaluation of \( \mathcal{E}_2 \) we need to know how many two-dimensional root systems are contained in the root system \( \Delta \). The list is as follows:

\[
\begin{align*}
A_r & : \left( \frac{r+1}{3} \right) \times A_2, \\
B_r & : 4 \left( \frac{r}{3} \right) \times A_2^{long} + \left( \frac{r}{2} \right) \times B_2, \\
C_r & : 4 \left( \frac{r}{3} \right) \times A_2^{short} + \left( \frac{r}{2} \right) \times B_2, \\
D_r & : 4 \left( \frac{r}{3} \right) \times A_2, \\
E_6 & = 120 \times A_2, \\
E_7 & = 336 \times A_2, \\
E_8 & = 1120 \times A_2, \\
F_4 & : 16 \times A_2^{short} + 16 \times A_2^{long} + 18 \times B_2, \\
G_2 & : 1 \times G_2, \\
BC_r & : 4 \left( \frac{r}{3} \right) \times A_2 + \left( \frac{r}{2} \right) \times B_2^{short-medium} + \left( \frac{r}{2} \right) \times B_2^{medium-long}. 
\end{align*} \quad (3.38)
\]
The non-crystallographic root systems are not listed since the constant terms are zero in these cases. We list $E(\Delta, \text{trig.})/a^2$ for various root systems in Table 3.

| $\Delta$ | $E(\Delta, \text{trig.})/a^2$ |
|----------|-------------------------------|
| $A_r$    | $\frac{|\rho|^2 g^2}{2} \left( \frac{r + 1}{3} \right)$ |
| $B_r$    | $2|\rho_S|^2 g_L \left( \frac{r}{3} \right) g_L + \left( \frac{r}{2} \right) g_S$ |
| $C_r$    | $2|\rho_S|^2 g_S \left( \frac{r}{3} \right) g_S + \left( \frac{r}{2} \right) g_L$ |
| $D_r$    | $2|\rho|^2 \left( \frac{r}{3} \right) g^2$ |
| $E_6$    | $60|\rho|^2 g^2$ |
| $E_7$    | $118|\rho|^2 g^2$ |
| $E_8$    | $560|\rho|^2 g^2$ |
| $F_4$    | $4|\rho_S|^2 \left( 2g_S^2 + 4g_L^2 + 9g_S g_L \right)$ |
| $G_2$    | $|\rho_S|^2 \left( g_S^2 + 3g_L^2 + 12g_S g_L \right) / 2$ |
| $BC_r$   | $2|\rho_S|^2 g_M \left( \frac{r}{3} \right) g_M + \left( \frac{r}{2} \right) g_S + 2 \left( \frac{r}{2} \right) g_L$ |

Table 3: The part of the ground state energy $E_2/a^2$ for the trigonometric potential.

We arrive at the following explicit forms of the bosonic and fermionic Hamiltonians $H_B$ (3.10) and $H_F$ (3.12):

$$H_B = \frac{1}{2} p^2 + \frac{1}{2} \sum_{\rho \in \Delta_+} g_{|\rho|} (g_{|\rho|} - 1)|\rho|^2 V(\rho \cdot q) + \left( \frac{\omega^2}{2} q^2 \right) - E_0, \quad (3.40)$$

$$H_F = \sum_{\rho \in \Delta_+} g_{|\rho|} (\rho \cdot \psi^*)(\rho \cdot \psi) V(\rho \cdot q) + (\omega \psi^* \cdot \psi). \quad (3.41)$$

For the hyperbolic and trigonometric cases $E_0 = E_1 + E_2$ is expressed succinctly as:

$$E_0 = \frac{a^2}{2} \left( \sum_{\rho \in \Delta_+} g_{|\rho|} \rho \right)^2 \times \left\{\begin{array}{ll}
-1 & \text{hyperbolic} \\
1 & \text{trigonometric}.
\end{array}\right. \quad (3.42)$$

For the rational potential cases the ground state energy $E_0$ is

$$E_0 = \left\{\begin{array}{ll}
0 & \text{rational} \\
\omega \left( \frac{r}{2} + \sum_{\rho \in \Delta_+} g_{|\rho|} \right) & \text{rational with harmonic potential}.
\end{array}\right. \quad (3.43)$$

The bosonic Hamiltonian $H_B$ (3.40) has the same form as the classical Hamiltonian (2.8) with only one replacement

$$g_{|\rho|} \rightarrow g_{|\rho|} (g_{|\rho|} - 1), \quad (3.44)$$
which is essential for quantum integrability as we will see shortly. It should be remarked
that the mechanism which guarantees

\[ \mathcal{H}_C = \frac{1}{2} \sum_{j=1}^{r} \left( p_j^2 + \left( \frac{\partial W}{\partial q_j} \right)^2 \right) + \text{const.} \]  

is the same one that guarantees the consistency of the classical Lax equation (2.23) \[4\].

The same mechanism plays an important role in the consistency of the quantum conserved
quantities, as we shall see in section \[4\].

### 3.2 Vacuum or ground state

Supersymmetric quantum mechanics provides the easiest way to construct the supersymmet-
ric vacuum, which also gives the ground state energy and eigenfunction of the pure bosonic
theory. The supersymmetric vacuum state \(|\text{vac}\rangle\) is annihilated by the supercharges

\[ \mathcal{Q}|\text{vac}\rangle = \mathcal{Q}^*|\text{vac}\rangle = 0, \]  

therefore it is an eigenstate of the supersymmetric Hamiltonian with zero energy

\[ \mathcal{H}_{SUSY}|\text{vac}\rangle = 0. \]  

In order to express \(|\text{vac}\rangle\) explicitly, let us introduce the state \(|0\rangle\) which is annihilated by all
of the fermionic annihilation operators:

\[ \psi_j|0\rangle = 0, \quad j = 1, \ldots, r. \]  

Let us suppose that

\[ |\text{vac}\rangle = \Phi_0(q)|0\rangle, \]  

in which \(\Phi_0(q)\) is yet to be determined. Then it satisfies \(\mathcal{Q}^*|\text{vac}\rangle = 0\) trivially. The other
condition \(\mathcal{Q}|\text{vac}\rangle = 0\) is fulfilled if \(\Phi_0\) satisfies

\[ \left( p_j + i \frac{\partial W}{\partial q_j} \right) \Phi_0 = 0, \quad j = 1, \ldots, r. \]  

A solution of (3.50) is given simply by

\[ \Phi_0(q) = e^{W(q)} = \prod_{\rho \in \Delta_+} |w(\rho \cdot q)|^{g_{\rho \ell}} e^{-\frac{1}{2}q^2}, \]  

where

\[ g_{\rho \ell} \]  

is the number of \(\rho\), the sum of which is equal to \(\ell\).
which is real and Coxeter invariant (3.14). The exponential factor $e^{-\frac{\omega^2}{2}q^2}$ exists only for the rational potential case with the harmonic confining force.

By substituting the above solution (3.51) into (3.47) and using the decomposition of the supersymmetric Hamiltonian (3.10) together with

$$H_F|vac\rangle = 0,$$

we obtain from $H_B|vac\rangle = 0$

$$\left(\frac{1}{2}p^2 + \frac{1}{2} \sum_{\rho \in \Delta_+} g|\rho| (g|\rho| - 1)|\rho|^2 V(\rho \cdot q) + \frac{\omega^2}{2}q^2\right) e^W = E_0 e^W. \tag{3.52}$$

In other words, the above solution (3.51) provides a ground state with energy $E_0$ of the pure bosonic model with Hamiltonian $H_C + H_{qc}$. It should be stressed that $E_0$ is determined purely algebraically using (3.35)–(3.39), without really applying the operator on the left hand side of (3.52) to the solution. In fact, one would need essentially the same calculation as above to show that $e^W$ is an eigenstate by direct application of the Hamiltonian operator.

Supersymmetry provides the simplest means to assert that it is the ground state. This type of ground state has been known for some time. It is derived by various methods, see for example [1, 2], and also by using supersymmetric quantum mechanics for the models based on classical root systems [7, 8]. Needless to say, our solution (3.51) provides a universal ground state solution for all the models considered in this paper.

The other states of the bosonic models can be obtained as eigenfunctions of a differential operator $\widetilde{H}_B$ obtained from $H_B$ by a similarity transformation:

$$\widetilde{H}_B = e^{-W} H_B e^W = e^{-W} \left(\frac{1}{2}p^2 + \frac{1}{2} \sum_{\rho \in \Delta_+} g|\rho| (g|\rho| - 1)|\rho|^2 V(\rho \cdot q) + \frac{\omega^2}{2}q^2 - E_0\right) e^W, \tag{3.53}$$

$$\widetilde{H}_B \phi_\lambda = \lambda \phi_\lambda \iff H_B \phi_\lambda e^W = \lambda \phi_\lambda e^W. \tag{3.54}$$

Obviously we have

$$\int_{PW\,(PW_T)} e^{2W(q)} dq = \begin{cases} \infty &: \text{rational and hyperbolic} \\
\text{finite} &: \text{trigonometric and rational with the harmonic potential,} \end{cases} \tag{3.55}$$

in which $PW$ and $PW_T$ denote that the integration is over the regions defined in (2.12) and (2.13). It should be remarked that the ‘ground state’ wavefunctions and ‘ground state’
energies in the non-normalisable cases \( i.e. \) the rational and hyperbolic potentials and, in particular, the negative ‘ground state’ energy of the latter) should not be taken at face value. In the rational (hyperbolic) case the wavefunction \( \Phi_0(q) = e^{W(q)} \) diverges polynomially (exponentially) for \( \alpha_h \cdot q \to +\infty \). A similar and better-known situation arises in the quantum mechanics of a free particle in one-dimension: \( H = p^2/2 \). It has an exponential ‘eigenstate’ with a negative energy:

\[
H\phi_0(q) = -\frac{k^2}{2}\phi_0(q), \quad \phi_0(q) = e^{kq}, \quad k \in \mathbb{R}.
\]

Naturally, most existing results in quantum Calogero-Moser models are for the models with normalisable states. There are also some results for the rational and hyperbolic models \([1, 11, 12]\). We will not discuss the eigenstates and spectra of (3.54) further. In the rest of this paper we will concentrate on the integrability structure of the quantum pure bosonic system (3.40).

4 Quantum Lax Pair Operators

In this section we present the formulation of the quantum Lax pair operators, which enables us to construct the quantum conserved quantities for the Calogero-Moser models based on all of the root systems (crystallographic and non-crystallographic) and for all of the degenerate potentials, as in the classical case given in section 2. We believe such a universal construction of the quantum Calogero-Moser Lax pair is new. We will write down the quantum equations of motion of the Calogero-Moser models in an equivalent matrix form, whose matrix elements are quantum operators. Surprisingly the difference between the classical and quantum Lax pair operators is very small, as we will see below.

The Lax pair operators for \textit{classical} Calogero-Moser models imply the quantum equations of motion, if they are interpreted as quantum operators. We start from the classical Hamiltonian

\[
H_C = \frac{1}{2}p^2 + \frac{1}{2} \sum_{\rho \in \Delta_+} g_{[\rho]}^2 |\rho|^2 V(\rho \cdot q), \quad (4.1)
\]

and consider its action as the quantum evolution operator. This is not at all strange, since the solvable Hamiltonians of the harmonic oscillator and the hydrogen atom are the same at the classical and quantum levels. The canonical equations of motion and quantum Heisenberg
The equations of motion are formally identical:

\[ \dot{q} = \{q, H_C\} = i[H_C, q] = p, \]

\[ \dot{p} = \{p, H_C\} = i[H_C, p] = -\frac{1}{2} \sum_{\rho \in \Delta_+} g^2_{|\rho|} |\rho|^2 V'(\rho \cdot q) \rho. \]

As shown in section 2, the classical equations of motion are equivalent to the Lax form:

\[ \frac{d}{dt} L = [L, \tilde{M}], \]

which is divided into two parts as

\[ \frac{d}{dt} X = [p \cdot \hat{H}, \tilde{M}], \]

\[ \frac{d}{dt} (p \cdot \hat{H}) = [X, \tilde{M}]. \]

The second equation corresponds to (4.3). Since only the \( q \) operators appear on the right hand side of (4.6), the quantum commutator \([X, \tilde{M}]\) is the same as the classical one, depending only on the matrix structure. We still have to consider the first equation which could be different from the classical one when \( p \) and \( q \) are non-commuting. For this purpose, let us evaluate \( dx(\rho \cdot q)/dt \) quantum mechanically:

\[ \frac{d}{dt} x(\rho \cdot q) = i [H_C, x(\rho \cdot q)] = i \left[ \frac{p^2}{2}, x(\rho \cdot q) \right] = \frac{1}{2} \left( (p \cdot \rho) y(\rho \cdot q) + y(\rho \cdot q) (p \cdot \rho) \right), \]

in which \( x' = y \) is used. The right hand side is Weyl (symmetrically) ordered.

Next let us evaluate the matrix element

\[ [p \cdot \hat{H}, \tilde{M}]_{\mu\nu} = \left[ p \cdot \hat{H}, \frac{i}{2} \sum_{\rho \in \Delta_+} g_{|\rho|} |\rho|^2 y(\rho \cdot q) \hat{s}_\rho \right]_{\mu\nu}, \quad \mu, \nu \in \mathcal{R} \]

quantum mechanically. We find

\[ [p \cdot \hat{H}, \tilde{M}]_{\mu\nu} = \left( \frac{i}{2} \sum_{\rho \in \Delta_+} g_{|\rho|} |\rho|^2 (p \cdot \rho) (p \cdot \hat{H}) (\hat{s}_\rho)_{\mu\nu} \right) \]

\[ = \left( \frac{i}{2} \sum_{\rho \in \Delta_+} g_{|\rho|} |\rho|^2 \left( p \cdot s_\rho(\nu) y(\rho \cdot q) - y(\rho \cdot q) p \cdot \nu \right) (\hat{s}_\rho)_{\mu\nu} \right). \]
Note that
\[ s_\rho(\nu) = \nu - (\rho^\nu \cdot \nu)\rho, \quad p \cdot s_\rho(\nu) = p \cdot \nu - (\rho^\nu \cdot \nu)p \cdot \rho, \]
and it is easy to see that
\[
\begin{align*}
p \cdot \nu (i/2) \sum_{\rho \in \Delta_+} g_{|\rho|} |\rho|^2 y(\rho \cdot q) - (i/2) \sum_{\rho \in \Delta_+} g_{|\rho|} |\rho|^2 y(\rho \cdot q) p \cdot \nu &= (1/2) \sum_{\rho \in \Delta_+} g_{|\rho|} |\rho|^2 y'(\rho \cdot q) \rho \cdot \nu. \tag{4.9}
\end{align*}
\]
Thus we arrive at
\[
[p \cdot \hat{H}, \tilde{M}]_{\mu\nu} = (1/2) \sum_{\rho \in \Delta_+} g_{|\rho|} |\rho|^2 \left( -i(\rho^\nu \cdot \nu)(p \cdot \rho) y(\rho \cdot q) + y'(\rho \cdot q) \rho \cdot \nu \right) (\hat{s}_\rho)_{\mu\nu}. \tag{4.10}
\]
At this point we split \( p \cdot \rho = (1/2)p \cdot \rho + (1/2)p \cdot \rho \) and apply the second momentum operator to the function \( y \)
\[
(p \cdot \rho) y(\rho \cdot q) = -i|\rho|^2 y'(\rho \cdot q) + y(\rho \cdot q) (p \cdot \rho). \tag{4.11}
\]
Thus we arrive at
\[
[p \cdot \hat{H}, \tilde{M}]_{\mu\nu} = \frac{-i}{4} \sum_{\rho \in \Delta_+} g_{|\rho|} |\rho|^2 (\rho^\nu \cdot \nu) \left( (p \cdot \rho) y(\rho \cdot q) + y(\rho \cdot q)(p \cdot \rho) \right) + \sum_{\rho \in \Delta_+} g_{|\rho|} \left( -\rho^\nu \cdot \nu |\rho|^2/4 + \rho \cdot \nu/2 \right) y'(\rho \cdot q) (\hat{s}_\rho)_{\mu\nu}. \tag{4.12}
\]
The second line vanishes, since \( \rho^\nu = 2\rho/|\rho|^2 \). By using the formulas
\[
\rho^\nu \cdot s_\rho(\nu) = \rho^\nu \cdot (\nu - (\rho^\nu \cdot \nu)p) = -\rho^\nu \cdot \nu, \quad \rho^\nu \cdot \rho = 2,
\]
we obtain
\[
[p \cdot \hat{H}, \tilde{M}]_{\mu\nu} = \frac{i}{2} \sum_{\rho \in \Delta_+} g_{|\rho|} (\rho \cdot s_\rho(\nu)) \left( (p \cdot \rho) y(\rho \cdot q) + y(\rho \cdot q)(p \cdot \rho) \right) (\hat{s}_\rho)_{\mu\nu}
\]
\[
= \frac{i}{2} \sum_{\rho \in \Delta_+} g_{|\rho|} (\rho \cdot \hat{H}) \left( (p \cdot \rho) y(\rho \cdot q) + y(\rho \cdot q)(p \cdot \rho) \right) (\hat{s}_\rho)_{\mu\nu}
\]
\[
= \frac{i}{2} \sum_{\rho \in \Delta_+} g_{|\rho|} (\rho \cdot \hat{H}) \frac{d}{dt} x(\rho \cdot q) \hat{s}_\rho \right)_{\mu\nu}
\]
\[
= \frac{d}{dt} X_{\mu\nu} = i[H_C, X_{\mu\nu}]. \tag{4.13}
\]
Thus we have established that the first Lax equation has the same form at the quantum and classical levels:
\[
\frac{d}{dt} X = i[H_C, X] = [p \cdot \hat{H}, \tilde{M}]
\]
and that the quantum Lax equation as a whole has the same form as the classical one
\[
\frac{d}{dt} L = i[H_C, L] = [L, \tilde{M}]
\]
or to be more precise
\[
i[H_C, L_{\mu\nu}] = \sum_{\lambda \in \mathcal{R}} \left( L_{\mu\lambda} \tilde{M}_{\lambda\nu} - \tilde{M}_{\mu\lambda} L_{\lambda\nu} \right), \quad \mu, \nu \in \mathcal{R}.
\]
Similarly we obtain for the rational model with the harmonic force
\[
i[H_{C\omega}, L_{\mu\nu}^\pm] = \sum_{\lambda \in \mathcal{R}} \left( L_{\mu\lambda} \tilde{M}_{\lambda\nu}^\pm - \tilde{M}_{\mu\lambda} L_{\lambda\nu}^\pm \right) \pm i\omega L_{\mu\nu}^\pm, \quad \mu, \nu \in \mathcal{R}.
\]
From these it is straightforward to derive (recalling the definitions (2.40))
\[
\frac{d}{dt} L_n = i[H_C, L_n] = [L_n, \tilde{M}], \quad n = 1, 2, \ldots,
\]
\[
\frac{d}{dt} \mathcal{L}_k^n = i[H_{C\omega}, \mathcal{L}_k^n] = [\mathcal{L}_k^n, \tilde{M}], \quad k = 1, 2, \quad n = 1, 2, \ldots.
\]
However, the parallelism between the classical and quantum Lax equations ends here. These equations do not imply that Tr$L^n$ and Tr$\mathcal{L}_k^n$ are conserved. This is because the matrix elements of the quantum $L$ and $\tilde{M}$ operators do not commute and the cyclicity of the matrix trace is broken.

The remedy is simple. We adopt the bosonic Hamiltonian $H_B$ (3.40) obtained from the superpotential in the previous section and the Lax pair $L$ and $M$ instead of $L$ and $\tilde{M}$. The quantum Lax bracket
\[
[L, M] = [L, \tilde{M} + i\mathcal{D} \times I] = [L, \tilde{M}] + [p \cdot \hat{H}, i\mathcal{D} \times I]
\]
is different from the classical one, since the last term is no longer vanishing:
\[
[p \cdot \hat{H}, i\mathcal{D} \times I] = \frac{1}{2} \sum_{\rho \in \Delta_+} g_{\rho\rho} |\rho|^2 V'(\rho \cdot q) \rho \cdot \hat{H}.
\]
This term provides the necessary difference between the equations of motion of $\mathcal{H}_C$ and $\mathcal{H}_B$. Thus we have established for the Hamiltonian $\mathcal{H}_B$

$$
\frac{d}{dt}(L^n)_{\mu\nu} = i[\mathcal{H}_B, (L^n)_{\mu\nu}] = [L^n, M]_{\mu\nu}
$$

$$
= \sum_{\lambda \in \mathcal{R}} \left( (L^n)_{\mu\lambda} M_{\lambda\nu} - M_{\mu\lambda} (L^n)_{\lambda\nu} \right), \quad n = 1, \ldots, \quad (4.22)
$$

$$
\frac{d}{dt}(\mathcal{L}_k^n)_{\mu\nu} = i[\mathcal{H}_B, (\mathcal{L}_k^n)_{\mu\nu}] = [\mathcal{L}_k^n, M]_{\mu\nu}, \quad k = 1, 2, \quad n = 1, 2, \ldots, \quad (4.23)
$$

The above equations are operator equations, therefore they are valid for any Calogero-Moser models with any potentials and any representations of the Lax pairs.

We define quantum conserved quantities as the total sum (Ts) of all matrix elements of $L^n$ ($\mathcal{L}_k^n$, $k = 1, 2$):

$$
Q_n = \text{Ts}(L^n) \equiv \sum_{\mu, \nu \in \mathcal{R}} (L^n)_{\mu\nu}, \quad n = 1, \ldots, \quad (4.24)
$$

$$
Q_{\omega,k}^n = \text{Ts}(\mathcal{L}_k^n) \equiv \sum_{\mu, \nu \in \mathcal{R}} (\mathcal{L}_k^n)_{\mu\nu}, \quad k = 1, 2, \quad n = 1, \ldots. \quad (4.25)
$$

They are conserved thanks to the property of the $M$ operator (2.33):

$$
\sum_{\mu \in \mathcal{R}} M_{\mu\nu} = \sum_{\nu \in \mathcal{R}} M_{\mu\nu} = 0.
$$

Such quantum conserved quantities have been previously reported for some models based on $A_r$ root systems [8, 21]. It should be remarked that $\text{Ts}(\mathcal{L}_2^n)$ is no longer the same as $\text{Ts}(\mathcal{L}_1^n)$ due to quantum corrections. As we will show at the end this section, the quantum Hamiltonian $\text{Ts}(\mathcal{L}_2)$ differs from $\text{Ts}(\mathcal{L}_1)$ by a constant.

Next we show that the quantum Hamiltonian $\mathcal{H}_B$ is obtained by taking the total sum of $L^2$ in a representation $\mathcal{R}$:

$$
\mathcal{H}_B \propto \text{Ts}(L^2).
$$

This is a necessary condition for the internal consistency of the quantum Lax pair operator formalism. We start from

$$
L^2 = (p \cdot \hat{H})^2 + (p \cdot \hat{H}X + X p \cdot \hat{H}) + X^2.
$$

For the diagonal operator $\hat{H}$, Ts and Tr are the same and the first term, $(p \cdot \hat{H})^2$, gives as in the classical theory $p^2 C_\mathcal{R}$. The next term reads

$$
p \cdot \hat{H}X + X p \cdot \hat{H}
$$

24
\[
\begin{align*}
&= p \cdot \hat{H} \left( i \sum_{\rho \in \Delta_+} g_{|\rho|} (\rho \cdot \hat{H}) x(\rho \cdot q) \hat{s}_{\rho} \right) + \left( i \sum_{\rho \in \Delta_+} g_{|\rho|} (\rho \cdot \hat{H}) x(\rho \cdot q) \hat{s}_{\rho} \right) p \cdot \hat{H} \\
&= \sum_{\rho \in \Delta_+} g_{|\rho|} y(\rho \cdot q)(\rho \cdot \hat{H})^2 \hat{s}_{\rho} \\
&\quad + i \sum_{\rho \in \Delta_+} g_{|\rho|} x(\rho \cdot q)(\rho \cdot \hat{H}) \left( (p \cdot \hat{H}) \hat{s}_{\rho} + \hat{s}_{\rho} (p \cdot \hat{H}) \right) .
\end{align*}
\] (4.26)

Here, the first term on the right hand side of (4.26) gives the same expression as \( \mathcal{H}_{qc} \), (3.18), as we have

\[
\sum_{\mu, \nu \in \mathcal{R}} \left( (\rho \cdot \hat{H})^2 \hat{s}_{\rho} \right)_{\mu\nu} = \sum_{\mu, \nu} (\rho \cdot \mu)^2 (\hat{s}_{\rho})_{\mu\nu} = \sum_{\mu} (\rho \cdot \mu)^2 = C_{\mathcal{R}}|\rho|^2 ,
\] (4.27)

in which the formula \( \sum_{\nu \in \mathcal{R}} (\hat{s}_{\rho})_{\mu\nu} = 1 \), (2.36), is used. The second sum in (4.26) vanishes, since we have

\[
\sum_{\mu, \nu} \left( \rho \cdot \hat{H} \left( p \cdot \hat{H} \hat{s}_{\rho} + \hat{s}_{\rho} p \cdot \hat{H} \right) \right)_{\mu\nu} \\
= \sum_{\mu, \nu} \rho \cdot s_{\rho} (\nu) \left( p \cdot s_{\rho} (\nu) + p \cdot \nu \right) (\hat{s}_{\rho})_{\mu\nu} \\
= - \sum_{\nu} (\rho \cdot \nu) p \cdot (2 \nu - (\rho^\nu \cdot \nu) \rho) = C_{\mathcal{R}}(-2 \rho \cdot p + \rho^\nu \cdot \rho \rho \cdot p) \\
= 0 ,
\] (4.28)

in which (2.36) is used again. Finally we show that \( T_s X^2 = \text{Tr} X^2 \), which is rather non-trivial since the off-diagonal terms \( (X^2)_{\mu\nu} \) are generally non-vanishing:

\[
(X^2)_{\mu\nu} = - \sum_{\rho, \sigma} g_{|\rho|} g_{|\sigma|} x(\rho \cdot q) x(\sigma \cdot q) \left( \rho \cdot \hat{H} \hat{s}_{\rho} \sigma \cdot \hat{H} \hat{s}_{\sigma} \right)_{\mu\nu} ,
\] (4.29)

in which

\[
\left( \rho \cdot \hat{H} \hat{s}_{\rho} \sigma \cdot \hat{H} \hat{s}_{\sigma} \right)_{\mu\nu} = \rho \cdot (s_{\sigma}(\nu)) \sigma \cdot \nu (\hat{s}_{\rho} \hat{s}_{\sigma})_{\mu\nu} .
\]

Since \( \sum_{\mu} (\hat{s}_{\rho} \hat{s}_{\sigma})_{\mu\nu} = 1 \) for \( \rho = \sigma \) and \( \rho \neq \sigma \), we obtain

\[
\sum_{\mu, \nu} \left( \rho \cdot \hat{H} \hat{s}_{\rho} \sigma \cdot \hat{H} \hat{s}_{\sigma} \right)_{\mu\nu} = - C_{\mathcal{R}} \rho \cdot \sigma ,
\] (4.30)

hence

\[
\sum_{\mu, \nu \in \mathcal{R}} (X^2)_{\mu\nu} = C_{\mathcal{R}} \sum_{\rho, \sigma} g_{|\rho|} g_{|\sigma|} (\rho \cdot \sigma) x(\rho \cdot q) x(\sigma \cdot q) ,
\] (4.31)

which is proportional to the \( (\partial W/\partial q)^2 \) term in (3.24). Thus we have established the announced result for the models without the harmonic potential:

\[
\frac{1}{2C_{\mathcal{R}}} T_s (L^2) = \mathcal{H}_B = \frac{1}{2} p^2 + \frac{1}{2} \sum_{\rho \in \Delta_+} g_{|\rho|} (|g_{|\rho|} - 1| |\rho|^2 V(\rho \cdot q) - \mathcal{E}_0 .
\] (4.32)
It should be emphasised that the mechanism which ensures the equality $T_s X^2 = \text{Tr} X^2$ is the same one which allows the introduction of supersymmetry in section 3.

Finally we establish that the Hamiltonian of the rational model with the harmonic potential is obtained in a similar way. To do this, we show that

$$H_B \propto \sum_{\mu, \nu \in \mathcal{R}} (L^+ L^-)_{\mu \nu} = \sum_{\mu, \nu \in \mathcal{R}} (L^- L^+)_{\mu \nu} + \text{const}, \quad (4.33)$$

in which

$$L^\pm = L \pm i\omega Q, \quad Q = q \cdot \hat{H}.$$  

We have

$$L^\pm L^\pm = (L \pm i\omega Q)(L \mp i\omega Q)$$

$$= L^2 + \omega^2 Q^2 \pm i\omega(QL - LQ), \quad (4.34)$$

and

$$QL - LQ = Q(p \cdot \hat{H} + X) - (p \cdot \hat{H} + X)Q$$

$$= \hat{H}_j \hat{H}_k \delta_{jk} + QX - XQ. \quad (4.35)$$

The second term is the same as in the classical theory

$$QX - XQ = i \sum_{\rho \in \Delta^+} g_{[\rho]} (\rho \cdot \hat{H})(\rho^\vee \cdot \hat{H}) \hat{s}_\rho. \quad (4.36)$$

Thus we arrive, using (4.34), at

$$\frac{1}{2C_\mathcal{R}} \sum_{\mu, \nu \in \mathcal{R}} (L^\pm L^\pm)_{\mu \nu} = \frac{1}{2} p^2 + \frac{1}{2} \sum_{\rho \in \Delta^+} g_{[\rho]} (g_{[\rho]} - 1) \frac{|\rho|^2}{(\rho \cdot q)^2} + \frac{\omega^2}{2} q^2 \mp \omega \left( r + \sum_{\rho \in \Delta^+} g_{[\rho]} \right), \quad (4.37)$$

which confirms (4.33). The constant term is the ground state energy (3.43).

5 Comments and Discussion

In this paper we have established in an elementary way how all of the Calogero-Moser models with degenerate potentials can be supersymmetrisated. As a by-product, universal formulas for the ground state energies and wavefunctions of the original (i.e. non-supersymmetric) quantum Calogero-Moser models are obtained. We have also given quantum Lax pair operators for these models and derived quantum conserved quantities. These results would constitute
a good starting point for the systematic study of quantum Calogero-Moser models, in particular, those based on the exceptional and non-crystallographic root systems. Besides the seminal work by Dunkl on the models based on the dihedral groups \([11]\), there are many works on the quantum \(G_2\) model, the exceptional Calogero-Moser model with the fewest (i.e. 2) degrees of freedom \([13]\) and some on the \(H_3\) and \(F_4\) models \([14]\). One advantage of our formulation is its universality and another is that it is independent of any specific choice of the realisation of the root systems.

Another merit of constructing quantum conserved quantities in terms of quantum Lax operators is that it becomes obvious that these conserved quantities are operators acting on functions in the configuration space, that is, either a Weyl chamber \([2.12]\) or a Weyl alcove \([2.13]\). The quantum theory we are discussing is the so-called first quantised theory. That is, the notion of identical particles and the associated statistics is non-existent. The symmetry properties of quantum solutions with respect to the action of the reflection operators will be discussed elsewhere.

For technical reasons, we have not developed the corresponding theory of commuting differential operators for these models \([15, 16, 17, 18]\). As is well-known, the theory of commuting differential operators would provide another method for constructing quantum conserved quantities. Analysis of the spectrum and eigenfunctions of Calogero-Moser models based on commuting differential operators \([14, 20, 21]\), shape-invariance \([18, 22, 23, 24]\) and quantum Lax pairs will be published elsewhere.

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