INDECOMPOSABLE EXPPLICIT ABELIAN GROUP

SAHARON SHELAH

Dedicated to Laszlo Fuchs for his ninetieth birthday

Abstract. For every $\lambda$ we give an explicit construction of an Abelian group with no non-trivial automorphisms. In particular the group absolutely has no non-trivial automorphisms, hence is absolutely indecomposable. Earlier we knew a stronger existence theorem but only up to a quite large cardinal which was a necessary restriction. In another direction the construction does not use the axiom of choice.

Date: August 8, 2013.

2010 Mathematics Subject Classification. Primary: 03C60, 20K30; Secondary: 20K20, 13C99, 03E55, 03E75, 03C75.

Key words and phrases. Abelian groups, rigid, endo-rigid, indecomposable, construction, set theory, non-structure theory.

The author would like to thank the Israel Science Foundation for partial support of this research (Grant No. 1053/11). The author thanks Alice Leonhardt for the beautiful typing. Publication 1023.
§ 0(A). Background.

Laszlo Fuchs in [Fuc73], continuing work of Corner [Cor69], proved that there are torsion free indecomposable, moreover, endo-rigid Abelian groups $G$ of quite large cardinality (up to the first inaccessible) and ask if it can be done in all. Endo-rigid means that every endomorphism $f$ of $G$ is $x \mapsto \lambda$ or is $x \mapsto ax$ for some $a \in \mathbb{Z}$ (and is onto if $a \in \{1, -1\}$); indecomposable means that $G = G_1 \oplus G_2 \Rightarrow G_1 = 0 \lor G_2 = 0$. The indecomposability was the original question, but endo-rigid is much stronger.

It is very fitting that this work is dedicated to Laszlo: he has been the father of modern Abelian group theory; his book [Fuc73] makes me in 1973, start to work on Abelian groups, (in [Sh:44]); this work was motivated by thinking of a paper suitable to be contributed to a volume in his honour; and last but not least the problem on the existence of indecomposable and endo rigid Abelian groups was the first I had started to work on reading his book. Meanwhile Fuchs [Fuc74] has succeeded to prove existence of indecomposable Abelian groups up to the first measurable cardinal.

The question was solved by the author ([Sh:44]); and see on the subject Trlifaj-Göbel [GT12], but the proof was less explicit: it used stationary subsets of regular uncountable cardinals. We may wonder: is this non-effectiveness necessary? How can we phrase this as an explicit problem? Moreover, we call a family $G$ of Abelian groups endo-rigid when if $G_1, G_2 \in G$ and $h \in \text{Hom}(G_1, G_2)$ then $G_1 = G_2$ and $h$ is a multiplication by an integer. In fact the proof in [Fuc73] is by building by induction on $\lambda$ such family $G_\lambda$ of $2^\lambda$ Abelian groups each of cardinality $\lambda$.

We may look at model theory essentially replacing “isomorphic” by “almost isomorphic”, that is isomorphisms by potential isomorphisms, i.e. isomorphism in some forcing extension (= generic extension). In [Sh:12] we have suggested to reconsider a major theme in model theory, that of counting the number of isomorphism types. Recall that $M, N$ are almost-isomorphic if $M, N$ have (the same vocabulary and) the same $\mathbb{L}_{\infty, \aleph_0}$-theory, equivalently are isomorphic in some forcing extension. For a theory $T$ let $\dot{I}_{\infty, \aleph_0}(\lambda, T)$ be the number of models of $T$ of cardinality $\lambda$ up to almost isomorphism, i.e. $|\{M/\equiv_{\mathbb{L}_{\infty, \aleph_0}}: M \text{ a model of } T \text{ of cardinality } \lambda\}|$. This behaves nicely ([Sh:12]): if $T$ has cardinality $\leq \lambda$, is first order or is just $\subseteq \mathbb{L}_{\lambda^+, \aleph_0}$ of cardinality $\leq \lambda$ then $\dot{I}_{\infty, \aleph_0}(\lambda, T) \leq \theta < \mu \Rightarrow \dot{I}_{\infty, \aleph_0}(\mu, T) \leq \dot{I}_{\infty, \aleph_0}(\theta, T)$, (recently on $\dot{I}_{\infty, \aleph_0}(\cdot, T)$ for $\aleph_1$-stable $T$, see a work of Laskowski-Shelah [LwSh:1010]). In [Sh:12] we also define “$M$ is ai-rigid, i.e. $a \neq b \in M \Rightarrow (M, a) \not\equiv_{\mathbb{L}_{\infty, \aleph_0}} (N, a)$” and have downward LST theorem for it. Generally on almost isomorphism and $\mathbb{L}_{\infty, \aleph_0}$ see Barwise [Bar75]. Later Nadel [Nad94] ask more specifically about the number of torsion free Abelian groups up to being almost isomorphic. He suggested further to consider homomorphisms, in particular for Abelian groups; that is, maybe we cannot find absolutely-rigid Abelian groups of arbitrarily large cardinal. In fact Nadel approach was to look at old constructions, he pointed out that the original constructions of Fuchs in [Fuc73] were absolute and the ones in [Fuc74] [Sh:44] were not. Fuchs one used infinite products (which are explicit but not absolute) and [Sh:44] use stationary sets.

For “endo-rigid” the answer is that we cannot construct when some specific mild large cardinal exists by Eklof-Shelah [EkSh378], see Eklof-Mekler [EM02].
INDECOMPOSABLE EXPLICIT ABELIAN GROUP

Ch.IV,§3,pg.487], i.e. \( \lambda = \kappa(\omega) \) the first \( \omega \)-Erdős cardinal. If \( \lambda \geq \kappa(\omega) \) then for every sequence \( (\langle G_\alpha, a_\alpha \rangle : \alpha < \lambda) \) for some \( \alpha < \beta < \lambda \), in some \( V^P \), \( G_\alpha \) is embeddable into \( G_\beta \): moreover if \( x_\gamma \in G_\gamma \) for \( \gamma < \lambda \) then for some \( \alpha < \beta < \lambda \), in some \( V^P \) there is an embedding of \( G_\alpha \) into \( G_\beta \) mapping \( x_\alpha \) to \( x_\beta \), (so \( (\forall \alpha)(G_\alpha = G) \) is allowed). This explains why [Fuc74] gets only indecomposable (not endo-rigid).

It was claimed there ([EkSh:678], [EM02]) that for every \( \lambda \) there are absolutely indecomposable Abelian groups, but the proof was withdrawn.

A problem left open was solved by Göbel-Shelah [GbSh:880]: if \( \lambda < \kappa(\omega) \) then there are absolutely endo-rigid Abelian groups and using it Fuchs-Göbel [?] does much more. That is, for smaller \( \lambda \), there is a family of \( 2^\lambda \) Abelian groups of cardinality \( \lambda \) which is absolutely endo-rigid. It is explicitly pointed out there that this gives absolutely an indecomposable Abelian group in any such cardinal.

All this still left open the question about the existence of indecomposable ones; we have made several wrong tries.

Another interpretation of “more explicit construction” is “provable without the axiom of choice”. We may also ask for more: no epimorphism (for monomorphisms we cannot). Also there are many works on such problems on \( R \)-modules and we may wonder on the situation for \( R \)-modules.

§ 0(B). The Results.

Our main result is that there is an explicit construction of Abelian groups of any cardinality \( \lambda \) which are absolutely indecomposable, moreover, absolutely has no non-trivial epimorphism. Also the axiom of choice is not needed and we get \( 2^\lambda \) many, pairwise absolutely non-isomorphic. We deal with modules but only as long as it does not complicate the proof.

However, note that (by absoluteness)

\((*)\) assume \( V \) is a model of \( ZF \) only. If in \( L \), \( G \) is absolutely endo-rigid (or indecomposable) \textbf{then} this holds also in \( V \).

So by [GbSh:880] we can deduce the existence of an endo-rigid \( G \) of cardinality \( \lambda \) when \( \lambda \) is not too large. Similarly here.

**Remark 0.1.** Clearly we can use only finitely many primes, and weaken the demand on being primes and in \( R \), but we delay this. It seems that we may look for \( R \)-modules with distinguished finitely many (or just four) submodules as in [CM90], [GbSh:880] and [FG08] and characterization of the ring of onto endomorphisms and consider non-well orderable rings, but again this is delayed.

§ 0(C). Preliminaries.

**Notation 0.2.** 1) \( R \) denotes a ring with unit, i.e. \( 1_R \).
2) Let \( \tau_R \) be the vocabulary of any left \( R \)-module.
3) \( R^+ = R \setminus \{0_R\} \).
4) Let \( Q = Q_R \) be the field of quotients of \( R \) when \( R \) is commutative torsion free.
5) For \( M \) and \( R \)-module let \( \text{inv}(R) = \{a \in R : a \text{ is invertible}\} \) and \( \text{epi}(M) = \{a \in R : aM = M\} \).
Definition 0.3. 1) We say $R$ is torsion free when $a \cdot b = 0 = a = 0 \lor b = 0$.
2) We say an $R$-module is torsion free when $M \models "ax = 0"$ implies $a = 0_R \lor x = 0_M$.

Definition 0.4. 1) For $M$ an $R$-module we say $X \subseteq M$ is pure when $ax \in X$ and $a \neq 0_R$ implies $x \in X$.
1A) Similarly for a torsion free $R$-module, $R$ a torsion free ring.
2) For a torsion free Abelian group $G$ and $A \subseteq G$ let $PC(A)$ be the minimal pure subgroup of $G$ which includes $A$.
3) For a formula $\varphi(x)$ in the vocabulary $\tau$ and $\tau$-model $M$, let $\varphi(M) = \{ a \in M : M \models "\varphi[a]" \}$.
4) For an $R$-module $M$ and $X \subseteq M$ the set affinely generated by $A$ in $M$ is $\{ \sum_{\ell \leq n} a_\ell z_\ell : a_\ell \in R, z_\ell \in X$ for $\ell \leq n$ and $\sum_{\ell \leq n} a_\ell = 1_R \}$.
5) We say $\bar{x} = \langle x_s : s \in I \rangle$ is a basis of the $R$-module $M$ when:
   - $x_s \in M \setminus \{0_M\}$ and $\bar{x}$ is with no repetitions
   - if $\langle s_\ell : \ell \leq n \rangle$ and $\ell \leq n \Rightarrow a_\ell \in R^+$ then $\sum_{\ell \leq n} a_\ell s_\ell \neq 0$
   - $M$ is the pure closure of $\sum_{s \in I} Rx_s$.

Definition 0.5. 1) We say a group or any structure $M$ is absolutely rigid in $\psi(x)$ or in $\psi(M)$ when: $\psi(x) \in \text{L}_{\infty, \aleph_0}$ and in every forcing extension $V^\mathbb{P}$ of $V$, every automorphism of $M$ is trivial on $\psi(M)$ which means it is $x \mapsto cx$ for $x \in \psi(M)$ where $c \in \mathbb{Q}_R$ and so necessarily $c\psi(M) = \psi(M)$.
1A) We add “strictly” when above the $c \in R$ is invertible (in $R$).
2) If above $\psi(x) = (x = x)$ so $\psi(M) = M$ then we may omit the “in $\psi(M)$”.
3) For an $R$-module $M$ we say $M$ is semi-rigid in $X \subseteq M$ when for every automorphism $f$ of $M$ we have $x \in X \Rightarrow f(x) \in PC_M(\{x\})$. We may write $\bar{x} = \langle x_\alpha : \alpha < \alpha_* \rangle$ instead $\{x_\alpha : \alpha < \alpha_* \}$. We define “absolutely semi-rigid in $X$” similarly.
§ 1. Constructing absolutely rigid Abelian groups

Below we can choose \( R = \mathbb{Z} \) and the assumptions on \( R \) we use are chosen just such that the proof is not more complicated.

**Main Claim 1.1.** \( M \) is absolutely semi-rigid in \( \bar{x} \) when:

\((*)\) (a) \( R \) is a commutative torsion free ring (so with unit \( 1_R \)), so \( a, b \in R^+ \Rightarrow a \cdot b \in R^+ \)

(b)\( \alpha \) \( \psi_1(x), \psi_2(x) \in \mathbb{L}_{\infty, \aleph_0}(\tau_R) \)

(c)\( M \) is an \( R \)-module, torsion free

(d)\( M \) is a submodule of \( M \) with universe \( \psi_1(M) \) and \( M_1 \cap M_2 = \{0\} \)

(e)\( \bar{x} = (x_\alpha : \alpha < \lambda) \) is a basis of \( M_1 \) so with no repetitions

(f) \( \bar{g} = (g_\alpha : \alpha < \lambda, i < \lambda) \) is a basis of \( M_2 \)

\( \gamma \) \( \bar{x} \) is a basis of \( M_1 \) so with no repetitions

\( \gamma \) \( \bar{g} = (g_\alpha : \alpha < \lambda) \) where \( g_\alpha : \lambda \to \lambda \)

\( \beta \) \( \alpha_0 < \ldots < \alpha_n < \lambda \) and \( k \leq n \) then \( \text{for some } \varepsilon < \lambda, \varepsilon \in \text{Rang}(g_{\alpha_k}) \) but \( \ell \leq n \land \ell \neq k \Rightarrow \varepsilon \notin \text{Rang}(g_{\alpha_k}) \)

\( \gamma \) \( \text{Rang}(g_{\alpha}) \) is unbounded in \( \lambda \)

\( \delta \) \( \text{if } \varepsilon < \lambda \text{ then } \varphi_2(M) = \text{PC}_M(\{x_\alpha + y_{\alpha,i} : \alpha < \lambda, i < \lambda \text{ are such that } g_\alpha(i) \geq \varepsilon \}) \).

**Proof.** So assume that \( P \) is a forcing notion and in \( V^P \) we have an automorphism \( f \) of \( G \) such that \( f|\psi_1(M) \) is not the identity. Now \( \{x_\alpha : \alpha < \lambda\} \) is a basis of \( \psi_1(M) \), i.e. of \( M_1 \) and \( M \) is torsion free so also \( \psi_1(M) \) is torsion free, hence

\( \exists_1 f \{x_\alpha : \alpha < \lambda\} \) is not the identity.

Next, the crucial point

\[ \exists_2 \text{ for every } \alpha < \lambda, f(x_\alpha) \in \text{PC}(\{x_\alpha\}). \]

Why? \( f \) is an automorphism of \( M \) hence \( M \models \psi_1[f(x_\alpha)] \land f(x_\alpha) \neq 0 \). Recalling that \( \bar{x} \) is a basis of \( M_1 \) we can find \( a_0 < \ldots < a_n < \lambda \) and \( a_0, \ldots, a_n, b \in R^+ \) such that

\[ (*)_{2.1} \text{ bf}(x_\alpha) = a_0 x_{\alpha_0} + \ldots + a_n x_{\alpha_n}. \]

If \( n = 0 \land a_0 = \alpha \) we are done so assume that this fails hence for some \( k \leq n \) we have \( \alpha_k \neq \alpha \). There is \( \varepsilon \) such that

\[ (*)_{2.2} \varepsilon < \lambda \text{ and } \varepsilon \in \text{Rang}(g_{\alpha_k}) \text{ but } \varepsilon \notin \text{Rang}(g_{\alpha_k}) \text{ and } \ell \leq n \land \ell \neq k \Rightarrow \varepsilon \notin \text{Rang}(g_{\alpha_k}). \]

Why? If \( \alpha \in \{\alpha_\ell : \ell \leq n\} \) we apply clause \((*)_{2.1}(\beta)\) to \( \langle \alpha_\ell : \ell \leq n \rangle \) and \( k \) to find \( \varepsilon \). If \( \alpha \notin \{\alpha_\ell : \ell \leq n\} \) let \( \alpha_{n+1} = \alpha \) and apply clause \((*)_{2.2}(\beta)\) to \( \langle \alpha_\ell : \ell \leq n + 1 \rangle \) and \( k \) to find \( \varepsilon \).

For \( \zeta < \lambda \) and \( z \in M_1 \) let \( A^M_{\zeta,z} := \{t \in M_2 : z + t \in N_\zeta\} \).

\footnote{We do not ask, e.g. \( g_{\alpha_\ell}(\varepsilon) > g_{\alpha_\ell}(\varepsilon) \) for \( \ell \leq n, \ell \neq k! \)}
\((*)_{2.3} \) \(A^M_{\zeta,z} \) is definable in \(M\) from \(z\)

hence

\((*)_{2.4} \) \(f\) maps \(A^M_{\zeta,x_\alpha}\) onto \(A^M_{\zeta,f(x_\alpha)}\) for \(\zeta < \lambda, \alpha < \lambda\).

Next

\((*)_{2.5} \) \(A^M_{\zeta,x_\alpha} = A^M_{\zeta+1,x_\alpha} \).

[Why? Recall \(\varphi_\zeta(M) = N_\zeta\) is the sub-module of \(M\) which is \(\text{PC}(\{x_\beta + y_\beta,i : \beta, i < \lambda, g_\alpha(i) \geq \zeta\})\) hence is a sub-module of \(M_1 + M_2\). For every \(\zeta < \lambda\) and \(t \in M_2\) we have \(t \in A^M_{\zeta,x_\alpha} \Leftrightarrow x_\alpha + t \in N_\zeta \Rightarrow M \models \varphi_\zeta(x_\alpha + t) \Leftrightarrow x_\alpha + t \in \text{PC}(\{x_\beta + y_\beta,i : \beta < \lambda, i < \lambda\})\).

As \(\langle x_\zeta, \gamma < \lambda \rangle \varphi_\gamma(y_\gamma,i : \beta, i < \lambda\) is independent in \(M\), necessarily also \(\langle x_\zeta + y_\zeta,i : \beta, i < \lambda\) is independent in \(M\) so for \(t \in M_2\) we get \(t \in A^M_{\zeta,x_\alpha} \Leftrightarrow x_\alpha + t \in \text{PC}(\{x_\alpha + y_\alpha,i : g_\alpha(i) \geq \zeta\})\). But for \(\zeta \in \{\varepsilon, \varepsilon + 1\}\) in the right side we get the same condition (as \(\varepsilon \notin \text{Rang}(g_\alpha)\)). So the left sides are equivalent too, i.e. \(t \in A^M_{\zeta,x_\alpha} \Rightarrow t \in A^M_{\zeta+1,x_\alpha}\) as promised.]

\((*)_{2.6} \) (a) if \(c \in R^+\) and \(x \in M_1\) and \(A^M_{\varepsilon,x} = A^M_{\varepsilon+1,x}\) then \(A^M_{\varepsilon,x} = A^M_{\varepsilon+1,x}\) e.g., for \((c,x) = (b,x_\alpha)\)

(b) \(A^M_{\varepsilon,bx_\alpha} = A^M_{\varepsilon+1,bx_\alpha}\).

[Why? Clause (a) holds by the proof of \((*)_{2.5}\) replacing \(x_\alpha\) by \(bx_\alpha\) except in “\(\text{PC}(\{x_\alpha + y_\alpha,i : g_\alpha(i) \geq \zeta\})\)” recalling \(M\) is torsion free. Clause (b) then follows.]

\((*)_{2.7} \) \(A^M_{\varepsilon,x} \neq A^M_{\varepsilon+1,x}\) when \(x = \sum_{\ell \leq n} a_\ell x_{a_\ell}\).

[Why? First, \(x \in M_1 = \psi_1(M)\); second, let \((i_\ell : \ell \leq n)\) be such that \(\ell \leq n \Rightarrow i_\ell \notin \lambda\) and \(g_\alpha(i_k) = \varepsilon\) and \(\ell \leq n, \ell \neq k \Rightarrow g_\alpha(i_k) > \varepsilon\); this is possible: for \(\ell = k\) by the choice of \(k\) and \(\varepsilon\), and for \(\ell \neq k\) because \(\text{Rang}(g_\beta)\) is an unbounded subset of \(\lambda\).

So

- \(x_{a_\ell} + y_{a_\ell,i_\ell} \notin N_\varepsilon\) for \(\ell \leq n\)
- \(x_{a_\ell} + y_{a_\ell,i_\ell} \notin N_{\varepsilon+1}\) for \(\ell \leq n, \ell \neq k\); recalling \(\varepsilon \notin \text{Rang}(\alpha_k)\)
- \(x_{a_k} + y_{a_k,i_k} \notin N_{\varepsilon+1}\) recalling \(g_\alpha(i_k) = \varepsilon\).

Hence as \(a_k \neq 0_R^\varepsilon\):

- \(\sum_{\ell \leq n} a_\ell x_{a_\ell} + \sum_{\ell \leq n} a_\ell y_{a_\ell,i_\ell} \notin N_{\varepsilon+1}\) \(\setminus N_\varepsilon\).

Hence recalling \(x = \sum_{\ell \leq n} a_\ell x_{a_\ell}\) we conclude

- \(\Sigma a_\ell y_{a_\ell,i_\ell} \in A^M_{\varepsilon,x} \setminus A^M_{\varepsilon+1,x}\).

So \((*)_{2.7}\) holds indeed.]

But recalling that \(bf(x_\alpha) = \sum_{\ell \leq n} a_\ell y_{a_\ell,i_\ell}\) so by \((*)_{2.7}\)

\((*)_{2.8} \) \(bf(x_\alpha) \in A^M_{\varepsilon,x} \setminus A^M_{\varepsilon+1,x}\)

and this contradicts \((*)_{2.6}(b)\), because \(f(bx_\alpha) = bfg(x_\alpha)\), so we are done proving \(\boxplus_2\) hence the main claim. \(\square\)
Claim 1.2. $M$ is absolutely semi-rigid in $\psi_1(M)$ when

(*) as in [1,1] but replacing clause (d)(\(\beta\)) by (d)(\(\beta\)''), replacing clause (f) by (f)' and adding clause (g) where:

(d) \((\beta)'\) \(\langle y_{\alpha,i} : \alpha < \lambda, i < \lambda \rangle y_*\) is a basis of \(M_2\)

(f)' if \(\varepsilon < \lambda\) then \(\varphi_\varepsilon(M) = \text{PC}_M(\{x_\alpha + y_\alpha, : \alpha < \lambda, i < \lambda\})\)

(g) \(\varphi_\lambda(x) \in \mathbb{L}_{\infty,\aleph_0}(\tau_R)\) and \(N_\lambda := \varphi_\lambda(M) = \text{PC}_M(\{x_\alpha + y^\ast : \alpha < \lambda\})\) so a pure sub-module.

Proof. Let \(\mathbb{P}\) and \(f\) be as in the proof of Claim [1,1]. By the proof of Claim [1,1] we have \(\alpha < \lambda \Rightarrow f(x_\alpha) \in \text{PC}_M(\{x_\alpha\})\).

Let \(a, b \in R^+\) be such that \(f(ax_0) = bx_0\), easily it suffices to prove:

\(\oplus\) if \(\alpha < \lambda\) then \(f(ax_\alpha) = bx_\alpha\).

Why \(\oplus\) holds? If not, there are \(a_\ast, b_1, b_2 \in R^+\) and \(\alpha_1 < \alpha_2\) such that \(f(a_\ast x_\alpha_1) = b_2 x_\alpha_1\) for \(\ell = 1, 2\) and \(\alpha_1 < \alpha_2, b_1 \neq b_2\). So \(f(a_\ast x_\alpha_1 - a_\ast x_\alpha_2) = b_1 x_\alpha_1 - b_2 x_\alpha_2\). Also we know \(x_\alpha_1 + y_\ast \in \varphi_\lambda(M)\) hence \(a_\ast x_\alpha_1 - a_\ast x_\alpha_2 \in \varphi_\lambda(M)\) hence \(a_\ast b_1 x_\alpha_1 - a_\ast b_2 x_\alpha_2 = f(a_\ast x_\alpha_1 - a_\ast x_\alpha_2) \in \varphi_\lambda(M)\). But \(x_\alpha_1 + y_\ast \in \varphi_\lambda(M)\) hence \(a_\ast b_1 x_\alpha_1 + a_\ast b_2 y_\ast \in \varphi_\lambda(M)\) for \(\ell = 1, 2\) so by subtracting \((a_\ast b_1 x_\alpha_1 - a_\ast b_2 x_\alpha_2) + (a_\ast b_1 - a_\ast b_2)y_\ast \in \varphi_\lambda(M)\).

By the last two sentences \((a_\ast b_1 - a_\ast b_2)y_\ast \in \varphi_\lambda(M)\); but \(b_1 \neq b_2\) so \(a_\ast b_1 - a_\ast b_2 \neq 0\) hence \(y_\ast \in \varphi_\lambda(M)\) contradicting clause (g) of the assumption. \(\square\)

Conclusion 1.3. The model $M$ is an absolutely rigid when:

(*)(a) $R$ is a commutative torsion free ring with $1_R$

(b) $M$ is a torsion free $R$-module

(c) for some \(\bar{\psi} = (\psi_\ell : \ell < \iota_\ast)\) we have

(α) \(\psi_\ell \in L_{\infty,\aleph_0}(\tau_R)\)

(β) \(M_\ell = \psi(M)\) is a sub-module of \(M\)

(γ) \(M = \text{PC}_M(\bigcup M_\ell)\)

(δ) for every \(\ell\) there are \(j < \iota_\ast\) and \(\bar{\varphi}_\ell = \langle \varphi_\ell, \varepsilon : \varepsilon < \lambda \rangle\) such that \(M, \psi_\ell, \psi_j, \bar{\varphi}_\ell, \bar{\varphi}_\lambda\) satisfies the assumptions of [1,2] for \(M, \psi_1, \psi_2, \bar{\varphi}, \bar{\varphi}_\lambda\)

(ε) if \(c \in R^+\) is not invertible then for some \(\ell < \iota_\ast\) we have \(cM_\ell \neq M_\ell\)

(ζ) if \(0 \leq u \leq \iota_\ast\) then for some \(t_1 \in u_*\) and \(t_2 \in u \setminus u\) there are no automorphism \(\pi\) of \(\text{PC}_M(M_{t_1} \cup M_{t_2})\) and \(c_1 \neq c_2 \in \mathbb{Q}_R\) such that \(x \mapsto c_1x\) induce the automorphism \(\pi|_{M_\ell}\) of \(M_\ell\) for \(\ell = 1, 2\).

Proof. Let \(f\) be an automorphism of \(M\) in the universe \(\mathbb{V}_\mathbb{P}\) for some forcing notion \(\mathbb{P}\).

By [1,2] for every \(\ell < \iota_\ast\) for some \(c \in \mathbb{Q}_R\) we have \(c_* M_\ell = M\) and \(x \in M_\ell \Rightarrow f(x) = c_* x\). If \(\langle c_\ell : \ell < \iota_\ast\rangle\) is constant we are done by (*)\((c)\)(ε) so toward contradiction we assume it is not and so \(u := \langle \ell < c_* : c_\ell \neq c_0 \rangle\) is \(\neq \iota_\ast\) and \(\neq \emptyset\).

By (*)\((c)\)(ζ) we get a contradiction. \(\square\)

Theorem 1.4. Assume $R = \mathbb{Z}$ or just $R$ is a commutative torsion free ring with $1$ and has infinitely many primes. Then for every \(\lambda\) there is an absolutely rigid $R$-module of cardinality \(\lambda + |R|\).
Proof. Let \( p_\ell,q_\ell,n (\ell < \omega , n < \omega ) \) be pairwise distinct primes of \( R \).
For each \( \ell \) we let
\[
(*)_1 \quad \psi_\ell(x) = \bigwedge_n (p_\ell^n | x) , \text{ of course } p_\ell^n \text{ is the } n\text{-th power of } p_\ell \\
(*)_2 \text{ we define } \varphi_\ell,\varepsilon,k(x_0, x_1) \text{ by induction on }\varepsilon \text{ as follows} \\
(\text{a) if } \varepsilon = 0 : \varphi_\ell,\varepsilon,k(x_0, x_1) = (\psi_\ell,k(x_0) \land \psi_{\ell+1,k}(x_1) \land _m ((q_\ell,k)^m | (x_0 + x_1)) \\
(\text{b) if } \varepsilon > 0, \varphi_\ell,\varepsilon,k(x_0, x_1) = \bigwedge_\zeta < \varepsilon \varphi_\ell,\zeta,k(x_0, x_1) \land \bigwedge_\zeta <\varepsilon (\exists x_2)(\varphi_\ell,\varepsilon,k+1(x_1, x_2) \\
(*)_3 \text{ for } \varepsilon \leq \lambda \text{ let } \varphi_\ell,\varepsilon(x) = \bigvee_{b \in R} (\exists x_0, x_1) (bx = x_0 + x_1 \land \varphi_\ell,\varepsilon,0(x_0, x_1)).
\]

The rest should be clear. \\[\square\]

**Theorem 1.5.** (ZF)
1) For every cardinal \( \lambda \) there is such an absolutely rigid Abelian group of cardinality \( \lambda \).
2) Moreover, if \( A \subseteq \mathcal{P}(\lambda), \aleph_0 \leq |A| = |A|^{\omega} \), then there is such a group of power \( |A| \).
3) Similarly for \( R\)-modules when \( R \) is as in [1.4] and \( R \) is well orderable.

**Proof.** Should be clear. \\[\square\]

So we can

**Theorem 1.6.** For every \( \lambda \), there are \( 2^\lambda \) absolutely rigid, pairwise absolutely non-isomorphic Abelian groups of cardinality \( \lambda \).

**Proof.** Obvious by the proof. \\[\square\]

Recall that we cannot exclude embeddings (= mono-morphisms). So we may wonder what about the epimorphisms?

**Claim 1.7.** If \( f \) is an endomorphism of \( M \) which maps \( \varphi_0(M) \) onto \( \varphi_0(M) \) then \( f |\varphi_0(M) \) is one-to-one provided that:

\[
(*) \begin{align*}
(a) & \text{ R is a commutative torsion free ring with } 1_R \\
(b) & \text{ M is a torsion free } R\text{-module} \\
(c) & \varphi_\varepsilon(x) \in \mathbb{L}_{\infty,\aleph_0}^\mathcal{P}(\tau_R) \text{, for } \varepsilon \leq \varepsilon_* \text{ where } ep \text{ stands for existential positive} \\
& \text{ (or just generated from the atomic formulas by } \exists \text{ and } \land ) \\
(d) & (\varphi_\varepsilon(M) : \varepsilon \leq \varepsilon_*) \text{ is } \subseteq\text{-decreasing continuous of sub-modules of } M \\
(e) & \varphi_{\varepsilon_*}(M) = \{0_M\} \\
(f) & \varphi_\varepsilon(M)/\varphi_{\varepsilon+1}(M) \text{ is torsion free of rank } 1, \text{ so isomorphic to some sub-} R\text{-module of } \mathbb{Q}_R \text{ (= the field of fractions of } R) \text{ considered as an } R\text{-module} \\
(g) & x_\varepsilon \in \varphi_\varepsilon(M) \setminus \varphi_{\varepsilon+1}(M) \text{ for } \varepsilon < \varepsilon_* \\
(h) & \varphi_0(M) = PC_M(\{x_\varepsilon : \varepsilon < \varepsilon_*\}).
\end{align*}
\]

**Remark 1.8.** E.g. for \( R = \mathbb{Z} \), we can use in (f) “finite rank torsion free”.

**Proof.** For \( \varepsilon < \varepsilon_* \) let \( y_\varepsilon := f(x_\varepsilon) \). As \( \varphi_\varepsilon(x) \in \mathbb{L}_{\infty,\aleph_0}(\tau_R) \) is existential positive, clearly \( f \) maps \( \varphi_\varepsilon(M) \) into \( \varphi_\varepsilon(M) \), hence \( y_\varepsilon \in \varphi_\varepsilon(M) \).

The main point is:
Why is \((\ast)\) sufficient?

If \(x \in \varphi_0(M) \setminus \{0_M\}\) then by clause (h) of the assumption for some \(b \in R^+, n\) and \(\varepsilon_0 < \ldots < \varepsilon_n < \varepsilon_\ast\) and \(a_\ell \in R^+\) for \(\ell \leq n\) we have \(M \models bx = \sum_{\ell \leq n} a_\ell x_{\varepsilon_\ell}\)

hence \(bf(x) = \sum_{\ell \leq n} a_\ell f(x_{\alpha_\ell}) = a_0 y_{\varepsilon_0} + \varphi_{\varepsilon_0+1}(M)\). As \(a_0 \in R^+\), by clause (g) clearly \(y_{\varepsilon_0} \in \varphi_{\varepsilon_0}(M_{\varepsilon_0}) \setminus \varphi_{\varepsilon_0+1}(M)\) and by clause (f) of the assumption, \(a_0 y_{\varepsilon_0} \not\in \varphi_{\varepsilon_0+1}(M)\) hence by the previous sentence \(bf(x) \neq 0_M\) hence \(f(x) \neq 0_M\). So we have proved that \(f\) maps any non-zero member of \(\varphi_0(M)\) into a non-zero member of \(\varphi_0(M)\), hence \(f|\varphi_0(M)\) is one-to-one as promised.

Why is \((\ast)\) true?

Toward contradiction, assume \(Y \neq \emptyset\) and let \(\zeta\) be the first member of \(Y\). As we are assuming “\(f\) maps \(\varphi_0(M)\) onto \(\varphi_0(M)\)” there is \(z \in \varphi_0(M)\) such that \(f(z) = x_\zeta\). As \(z \in \varphi_0(M)\) by clause (h) of the assumption of the claim we can find \(b \in R^+, n\) and \(\varepsilon_0 < \ldots < \varepsilon_n < \varepsilon_\ast\) and \(a_0, \ldots, a_n \in R^+\) such that \(M \models bz = \sum_{\ell \leq n} a_\ell x_{\varepsilon_\ell}\); now applying \(f\) we have \(M \models bx_\zeta = \sum_{\ell \leq n} a_\ell y_{\varepsilon_\ell}\).

Case 1: \(\varepsilon_0 < \zeta\)

Note that \(bx_\zeta \in \varphi_\zeta(M) \subseteq \varphi_{\varepsilon_0+1}(M)\) and \(\ell > 0 \Rightarrow a_\ell y_{\varepsilon_\ell} \in \varphi_{\varepsilon_\ell}(M) \subseteq \varphi_{\varepsilon_0+1}(M)\) so as \(bx_\zeta = \sum_{\ell \leq n} a_\ell y_{\varepsilon_\ell}\) we get \(a_0 y_{\varepsilon_0} \in \varphi_{\varepsilon_0+1}(M)\) and as \(a_0 \in R^+\) by clause (f) we get \(y_{\varepsilon_0} \in \varphi_{\varepsilon_0+1}(M)\) hence contradiction to \(\varepsilon_0 < \zeta = \min(Y)\).

Case 2: \(\varepsilon_0 \geq \zeta\)

On the one hand as \(b \in R^+\) clearly \(bx_\zeta \not\in \varphi_{\zeta+1}(M)\). On the other hand \(\varepsilon_\ell > \zeta \Rightarrow a_\ell y_{\varepsilon_\ell} \in \varphi_{\varepsilon_\ell}(M) \subseteq \varphi_{\varepsilon_0+1}(M)\) and \(\varepsilon_\ell = \zeta \Rightarrow \ell = 0 \Rightarrow y_{\varepsilon_\ell} = y_\zeta \in \varphi_\zeta(M)\) hence \(\sum_{\ell \leq n} a_\ell y_{\varepsilon_\ell} \in \varphi_\zeta(M)\). Together we get a contradiction to \(M \models bx_\zeta = \sum_{\ell \leq n} a_\ell y_{\varepsilon_\ell}\). \(\square\)

**Conclusion 1.9.** Theorem 1.4, 1.5 we can strengthen “no automorphism” to “no endomorphism which is onto”.

**Proof.** Easy by 1.8 and the proof of 1.4, 1.5. \(\square\)
References

[Bar73] Jon Barwise, Back and forth through infinitary logic, Studies in Model Theory, Math. Assoc. Amer., Buffalo, N.Y., 1973, pp. 5–34, MAA Studies in Math., Vol. 8.
[Cor69] A. L. S. Corner, Endomorphism algebras of large modules with distinguished submodules, Journal of Algebra 11 (1969), 155–185.
[EM02] Paul C. Eklof and Alan Mekler, Almost free modules: Set theoretic methods, North-Holland Mathematical Library, vol. 65, North-Holland Publishing Co., Amsterdam, 2002, Revised Edition.
[FG08] Laszlo Fuchs and Rüdiger Göbel, Modules with absolute endomorphism rings, Israel J. Math. 167 (2008), 91–109.
[Fuc73] Laszlo Fuchs, Infinite Abelian Groups, vol. I, II, Academic Press, New York, 1970, 1973.
[Fuc73] ———, Infinite Abelian Groups, vol. I, II, Academic Press, New York, 1970, 1973.
[Fuc74] László Fuchs, Indecomposable abelian groups of measurable cardinalities, Symposia Mathematica, Vol. XIII (Convegno di Gruppi Abelianii, INDAM, Rome, 1972), Academic Press, London, 1974, pp. 233–244.
[GM90] Rüdiger Göbel and Warren May, Four submodules suffice for realizing algebras over commutative rings, Journal of Pure and Applied Algebra 65 (1990), 29–43.
[GT12] Rüdiger Göbel and Jan Trlifaj, Approximations and endomorphism algebras of modules, vol. i, ii, de Gruyter Expositions in Mathematics, Walter de Gruyter, 2012.
[Nad94] M.E. Nadel, Scott heights of abelian groups, Journal of Symbolic Logic 59 (1994), 1351–1359.
[Sh:11] Saharon Shelah, On the number of non-almost isomorphic models of $T$ in a power, Pacific Journal of Mathematics 36 (1971), 811–818.
[Sh:12] ———, The number of non-isomorphic models of an unstable first-order theory, Israel Journal of Mathematics 9 (1971), 473–487.
[Sh:44] ———, Infinite abelian groups, Whitehead problem and some constructions, Israel Journal of Mathematics 18 (1974), 243–256.
[EkSh:678] Paul C. Eklof and Saharon Shelah, Absolutely rigid systems and absolutely indecomposable groups, Abelian groups and modules (Dublin, 1998), Trends in Mathematics, Birkhäuser, Basel, 1999, math.LO/0010264, pp. 257–268.
[Sh:797] Saharon Shelah, Nice infinitary logics, Journal of the American Mathematical Society 25 (2012), 395–427, 1005.2806.
[Sh:F808] Absolutely indecomposable.
[GbSh:880] Ruediger Goebel and Saharon Shelah, Absolutely Indecomposable Modules, Proceedings of the American Mathematical Society 135 (2007), 1641–1649, 0711.3011.
[LwSh:1016] Michael C. Laskowski and Saharon Shelah, Borel completeness of some aleph-stable theories, Fundamenta Mathematicae.