ASYMPTOTIC CONES AND BOUNDARIES OF CAT(0) GROUPS
CURTIS KENT AND RUSSELL RICKS

Abstract. It is well known that the Tits boundary of a proper cocompact CAT(0) space embeds into every asymptotic cone of the space. We explore the relationships between the asymptotic cones of a CAT(0) space and its boundary under both the standard visual (i.e. cone) topology and the Tits metric. We show that the set of asymptotic cones of a proper cocompact CAT(0) space admits canonical connecting maps under which the direct limit is isometric to the Euclidean cone on the Tits boundary. The resulting projection from any asymptotic cone to the Tits boundary is determined by the visual topology; on the other hand, the visual topology can be recovered from the connecting maps between asymptotic cones. We also demonstrate how maps between asymptotic cones induce maps between Tits boundaries.

1. Introduction

Asymptotic cones and compactifications are two very different approaches to studying the coarse geometry of groups and spaces. In general, asymptotic cones of a metric space $X$ will not be locally compact and not preserve the local geometry of $X$ while compactifications often can be endowed with proper metrics compatible with the local geometry of $X$.

Intuitively, an asymptotic cone of a metric space $(X, \text{dist})$ is the limit of the metric spaces $(X, \text{dist}/d_n)$ where $\text{dist}/d_n$ is the metric on $X$ scaled by $1/d_n$. Asymptotic cones have the desirable property that quasi-isometric spaces have bi-Lipschitz asymptotic cones. However, an ultrafilter is required to guarantee the existence of the limit. As a consequence, they also have the drawback that a metric space can have uncountably many distinct asymptotic cones depending on the choice of ultrafilter and scaling sequence used in the construction.

There are many connections between the topological structure of the asymptotic cones of a finitely generated group and its combinatorial and algorithmic properties. For example, a group has polynomial growth if and only if it is virtually nilpotent, if and only if every asymptotic cone is locally compact [5, 7]. Also, a finitely generated group is finitely presented and has polynomial Dehn function if all of its asymptotic cones are simply connected [6].

The second approach to studying the coarse geometry of groups and spaces is via boundaries. The visual boundary is one natural compactification for proper CAT(0) and hyperbolic spaces. The visual boundary is the set of large-scale directions and is an isometry invariant of the space. In fact, for hyperbolic groups the visual boundary is a quasi-isometry invariant. However, Croke and Kleiner proved that a group can act geometrically on two CAT(0) spaces with non-homeomorphic visual boundaries [3]. Thus in the CAT(0) setting the visual boundary is not a quasi-isometry invariant.

The boundary of a CAT(0) space can also be endowed with a metric called the Tits metric which reflects the Euclidean structure of the CAT(0) space. With the Tits metric, the set of large-scale directions is no longer a compactification but in some sense better encodes the coarse Euclidean geometry of the space.

We will illustrate how the visual boundary, the Tits boundary, and the asymptotic cones of proper CAT(0) spaces relate. It is well known that the Tits boundary admits a canonical isometric embedding into every asymptotic cone. We will show that the set of asymptotic cones of a proper CAT(0) space determine the Tits boundary of the space.
**Theorem A** (Theorem 3.17). Let \( X \) be a proper cocompact CAT(0) space. The direct limit of asymptotic cones of \( X \) induced by the geodesic retraction on \( X \) is isometric to the Euclidean cone on the Tits boundary of \( X \). Moreover, the resulting projection maps onto the Euclidean cone are determined by the visual topology on the boundary.

Thus the set of asymptotic cones of \( X \) for a fixed ultrafilter determines the Tits boundary of \( X \). The connecting maps also give rise to an inverse limit, which is related to the countable ultraproduct of the Tits boundary.

**Theorem B** (Theorem 3.19). Let \( X \) be a proper cocompact CAT(0) space. The inverse limit of asymptotic cones of \( X \) induced by the geodesic retraction on \( X \) has an inverse limit metric and with this metric there exists a canonical isometric embedding of the Euclidean cone on the ultraproduct of the Tits boundary of \( X \) into the inverse limit.

We leave it as an open question whether or not this embedding is surjective.

In Section 4, we demonstrate how the asymptotic cones determine the visual topology on the boundary.

**Theorem C** (Theorem 4.2). The visual topology on the boundary of a proper cocompact CAT(0) space is determined by the geodesic retraction maps between asymptotic cones.

In Section 5, the direct limit is used to define continuous maps between Tits boundaries of quasi-isometric CAT(0) spaces, which when restricted to Morse geodesics gives a bijection. This gives an alternate proof that a CAT(0) group has a cut-point in some asymptotic cone if and only if it has cut-points in every asymptotic cone if and only if it has a periodic rank one element.

2. Visual boundary

A CAT(0) space is a uniquely geodesic metric space such that every geodesic triangle \( \Delta(x, y, z) \) is thinner than the corresponding comparison triangle \( \overline{\Delta(x, y, z)} \) in Euclidean \( \mathbb{R}^2 \). This generalizes the property of nonpositive curvature from Riemannian manifolds to the metric setting. We refer the reader to [2] for a more complete account.

The visual boundary of a complete CAT(0) space can be considered as either the set of equivalence classes of geodesic rays (equivalent if they are asymptotic) or the set of based geodesic rays. Here we will use the latter.

For a CAT(0) space \((X, \text{dist})\) and \(x, y \in X\), we will use \([x, y]\) to denote the unique unparameterized geodesic from \(x\) to \(y\), and \(B^\text{dist}_\epsilon(x)\) or \(B_\epsilon(x)\) to denote the open metric ball of radius \(\epsilon\) about \(x\).

**Definition 2.1** (Visual compactification and boundary). Let \( X \) be a CAT(0) metric space. For a fixed \( x_0 \in X \), let \( \partial X \) be the set of geodesic rays with basepoint \( x_0 \).

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For \( \alpha \in \partial X \) and \( \epsilon, R > 0 \), let

\[
\overline{U}(\alpha, R, \epsilon) = \{ \beta \in \partial X \mid \text{dist}(\alpha(R), \text{im}(\beta)) < \epsilon \} \cup \{ x \in X \mid [x_0, x] \cap B_\epsilon(\alpha(R)) \neq \emptyset \}.
\]

If \( X \) is a proper CAT(0) metric space (proper meaning closed balls are compact), then

\[
\{ \overline{U}(\alpha, R, \epsilon) \mid \alpha \in \partial X, \epsilon, R > 0 \} \cup \{ B_\epsilon(x) \mid x \in X, \epsilon > 0 \}
\]

is a basis for a compact topology on \( \overline{X} = X \cup \partial X \) which we will call the visual compactification of \( X \). Notice that \( \partial X \) is a closed subspace. The visual boundary of \( X \) is the set \( \partial X \) endowed with the subspace topology from \( X \) and will be denoted by \( \partial X \). The visual boundary has basis \( \{ U(\alpha, R, \epsilon) \mid \alpha \in \partial X, \epsilon, R > 0 \} \) where \( U(\alpha, R, \epsilon) = \overline{U}(\alpha, R, \epsilon) \cap \partial X \).
Definition 2.2 (Limit set). Let $X$ be a CAT(0) space and $A \subset X$. The limit set of $A$, denoted $\Lambda(A)$, is the closure of $A$ in the visual compactification intersected with the visual boundary, i.e. $\Lambda(A) = \text{cl}_X(A) \cap \partial_\infty X$ where $\text{cl}_X$ is the topological closure operator in $X$.

The visual boundary can be endowed with several natural metrics. Here we will use the following metric since it relates well to the metric on asymptotic cones. Fix $C > 0$. For two geodesics $\alpha, \beta : ([0, \infty), 0) \to (X, x_0)$ into a CAT(0) space $X$, let

$$\text{dist}_C(\alpha, \beta) = \frac{1}{\sup \{ t \mid \text{dist}(\alpha(t), \beta(t)) \leq C \}}.$$

Lemma 2.3. The function $\text{dist}_C$ is a metric on $\partial X$.

Proof. Notice that $\text{dist}_C(\alpha, \beta) \leq \frac{1}{n}$ implies that $\text{dist}(\alpha(t), \beta(t)) \leq \frac{C}{n}$ for all $t \leq n$ by the CAT(0) condition. Hence $\text{dist}_C(\alpha, \beta) = 0$ if and only if $\alpha = \beta$. Clearly $\text{dist}_C$ is reflexive and we are only left to show the triangle inequality holds for $\text{dist}_C$.

Suppose that $\text{dist}_C(\alpha, \beta) = \frac{1}{n}$ and $\text{dist}_C(\beta, \gamma) = \frac{1}{m}$ for some $m, n \in \mathbb{R}$. For all $t \leq \min\{m, n\}$, we have $\text{dist}(\alpha(t), \gamma(t)) \leq C t\left(\frac{1}{m} + \frac{1}{n}\right)$ by the triangle inequality for dist. Hence $\text{dist}(\alpha(\frac{m}{m+n}), \gamma(\frac{m}{m+n})) \leq C$ which implies that

$$\text{dist}_C(\alpha, \gamma) \leq \frac{m+n}{mn} = \frac{1}{m} + \frac{1}{n} = \text{dist}_C(\alpha, \beta) + \text{dist}_C(\beta, \gamma). \quad \Box$$

As with $\text{dist}$ on $X$, when the metric is understood we will simply write $B_\epsilon(\alpha)$ for $B_\epsilon^{\text{dist}_C}(\alpha)$. Notice that $B_\epsilon^{\text{dist}_C}(\alpha) \subset U(\alpha, R, \epsilon)$ and $U(\alpha, 2/\epsilon, C) \subset B_\epsilon^{\text{dist}_C}(\alpha)$ which proves the following observation.

Lemma 2.4. The metric $\text{dist}_C$ induces the visual topology on $\partial X$.

Definition 2.5 (Tits/Angle boundaries). For points $x, y, z$ in a CAT(0) space $X$, we will let $\angle_x(y, z)$ denote the comparison angle at $x$ between $y$ and $z$. If $p_y : [0, a] \to X$ and $p_z : [0, b] \to X$ are the unique geodesics in $X$ from $x$ to $y$ and from $x$ to $z$ respectively, then the angle between $y$ and $z$ at $x$ is $\angle_x(y, z) = \lim_{t \to 0} \angle_x(p_y(t), p_z(t))$.

For $\alpha, \beta \in \partial X$, let

$$\angle(\alpha, \beta) = \lim_{t \to \infty} \angle_{x_0}(\alpha(t), \beta(t)) = \sup \{ \angle_{x_0}(\alpha(t), \beta(t')) \mid t, t' > 0 \}$$

and

$$\angle_{x_0}(\alpha, \beta) = \lim_{t \to 0} \angle_{x_0}(\alpha(t), \beta(t)) = \inf \{ \angle_{x_0}(\alpha(t), \beta(t')) \mid t, t' > 0 \}.$$

It is an exercise to show that $\angle(\cdot, \cdot)$ defines a locally geodesic metric on $\partial X$, which is called the angle metric and we will denote $\partial X$ with this metric by $\partial_\angle X$.

The path metric induced by $\angle(\cdot, \cdot)$ is called the Tits metric and is denoted by $\text{dist}_T(\cdot, \cdot)$. The Tits boundary of $X$ is $\partial X$ with this metric and will be denoted by $\partial_\infty X$. Note that $\text{dist}_T$ is an extended metric in the sense that it maps into $[0, \infty]$. The Tits distance between any two points in distinct path components of the angle boundary is infinity. We refer the interested reader to [2 Chapter I. 1, II.9] for complete details.

Proofs of the following standard lemmas can be found in [2 Proposition II.9.8] and [2 Proposition II.9.9].

Lemma 2.6. Let $X$ be a CAT(0) space. For $\alpha, \beta \in \partial X$,

$$2\sin \left( \frac{\angle(\alpha, \beta)}{2} \right) = \lim_{t \to \infty} \frac{\text{dist}(\alpha(t), \beta(t))}{t}.$$
Lemma 2.7 (Flat Sector Theorem). Let $X$ be a CAT(0) space. If $\alpha, \beta \in \partial X$ such that $\angle(\alpha, \beta) < \pi$ and $\angle(\alpha, \beta) = \angle_{\infty}(\alpha, \beta)$, then the convex hull of the geodesic rays $\alpha$ and $\beta$ is isometric to a sector in the Euclidean plane bounded by two rays which meet at an angle $\angle(\alpha, \beta)$.

3. Direct limits and inverse limits of asymptotic cones

Ultrafilters and Asymptotic cones.

Definition 3.1. A (non-principal) ultrafilter $\omega$ on a set $S$ is a finitely additive probability measure on the power set of $S$ with values in $\{0, 1\}$ such that $\omega(A) = 0$ for all finite subsets $A \subset S$. We will say that $A \subset S$ is $\omega$-large if $\omega(A) = 1$. A property $P$ holds $\omega$-almost surely if it holds on an $\omega$-large subset of $S$.

For $\{a_s \mid s \in S\} \subset X$, a subset of a topological space $X$ indexed by $S$, we will say that the ultralimit of $a_s$ is $x$, written $\lim^\omega a_s = x$, if for every open neighborhood $U$ of $x$ the set $\{s \mid a_s \in U\}$ is $\omega$-large. It is an exercise to show that every $S$-indexed subset of a compact space has a unique ultralimit and that ultralimits satisfy the standard properties of limits. If $\{a_s\} \subset X$ is $S$-indexed but has no ultralimit in $X$, we will say $a_s$ is $\omega$-divergent.

Definition 3.2 (Ultraproducts). Let $(X_n, \text{dist}_n)$ be a sequence of metric spaces, $\omega$ an ultrafilter on $\mathbb{N}$, and $e = (e_n) \in \prod_{n=1}^\infty X_n$. The ultraproduct of $(X_n, \text{dist}_n)$ is

$$\prod^\omega X_n = \{(x_n) \in \prod X_n \mid \text{for each } (x_n), \text{dist}_n(x_n, e_n) \text{ is uniformly bounded}\}/\sim$$

where $(x_n) \sim (y_n)$ if $\lim^\omega \text{dist}_n(x_n, y_n) = 0$. The ultraproduct has metric

$$\text{dist}((x_n), (y_n)) = \lim^\omega \text{dist}_n(x_n, y_n).$$

In general, the ultraproduct depends on both $e$ and $\omega$ and the sequence $e$ will be called the observation sequence for the ultraproduct.

We will use the simplified notation $\partial^\omega X = \prod^\omega \partial X$ and $\partial^\omega_T X = \prod^\omega \partial_T X$. Since $\partial^\omega X$ is bounded, $\partial^\omega_X$ is independent of the chosen basepoint but in the case of $\partial^\omega_T X$, there is an implied but unspecified choice of basepoint.

Let $\omega$ be an ultrafilter on $\mathbb{N}$ and $d = (d_n)$, an $\omega$-divergent sequence of positive real numbers (called a scaling sequence). An asymptotic cone of $X$ is $\prod^\omega_T (X, \text{dist}/d_n)$ and will be denoted by $\text{Con}^\omega(X, e, d)$.

Definition 3.3 (Euclidean cones). If $X$ is a metric space, let $\text{Cone}(X) = (\mathbb{R}^+ \times X)/\sim$ where $(0, x) \sim (0, x')$ for all $x, x' \in X$. When convenient, we will denote the equivalence class of $(t, x)$ in $\text{Cone}(X)$ by $tx$.

We can endow $\text{Cone}(X)$ with a metric by

$$\text{dist}^2(tx, t'x') = t^2 + (t')^2 - 2tt' \cos\left(\max\{\pi, \text{dist}(x, x')\}\right).$$

Write $(\text{Cone}(X))^\omega$ for $\prod^\omega \text{Cone}(X)$ with $e = (0x)$.

Proposition 3.4. Let $X$ be a complete CAT(0) space. The identity map from $\partial^\omega X$ to $\partial^\omega_T X$ induces an isometry from $\text{Cone}(\partial^\omega X)$ to $\text{Cone}(\partial^\omega_T X)$. The natural map from $\text{Cone}(\partial^\omega_T X)$ to $(\text{Cone}(\partial^\omega X))^\omega$ is an isometry. As well, $\partial^\omega_T X$ is homeomorphic to a path component of $\partial^\omega_X$.

Hence we will often identify $\text{Cone}(\partial^\omega X)$ with $\text{Cone}(\partial^\omega_T X)$.

Proof. The angle boundary, $\partial^\omega X$, is a CAT(1) space, see [2, Theorem II.9.13]. Hence the identity map from $\partial^\omega X$ to $\partial^\omega_T X$ is an isometry when restricted to an open ball of radius $\pi$. It is then immediate that $\text{Cone}(\partial^\omega X)$ is canonically isometric to $\text{Cone}(\partial^\omega_T X)$.
Let $F : \text{Cone}(\partial^\omega X) \to (\text{Cone}(\partial X))^\omega$ by $F(t, (\alpha_n)) = ((t, \alpha_n))$. It is an exercise to verify that $F$ is an isometry.

Recall that the ultraproduct $\partial^\omega X$ depends on a fixed observation sequence $e = (e_n)$ and is the set of sequences of geodesic rays $(\alpha_n)$ such that $\text{dist}_T(\alpha_n, e_n)$ is uniformly bounded $\omega$-almost surely. Thus for $(\alpha_n), (\beta_n) \in \partial^\omega X$, $\text{dist}(\alpha_n, \beta_n) < M \omega$ almost surely for some $M$ and the $\omega$-limit of the geodesics from $\alpha_n$ to $\beta_n$ gives a path from $(\alpha_n)$ to $(\beta_n)$ in $\partial^\omega X$. Therefore the identity map from $\partial^\omega X$ to $\partial^\omega X$, takes $\partial^\omega X$ into a path component of $\partial^\omega X$. Suppose that $(\gamma_n)$ is any element of $\partial^\omega X$ contained in the same path component as $e$. Since $\partial^\omega X$ is locally geodesic, there exists a rectifiable path from $(\gamma_n)$ to $e$. Thus $\text{dist}(\gamma_n, e)$ is uniformly bounded $\omega$-almost surely by the length of this rectifiable path, which implies that $(\gamma_n) \in \partial^\omega X$. Therefore $\partial^\omega X$ is a path component of $\partial^\omega X$.

**Corollary 3.5.** For a complete $\text{CAT}(0)$ space, $X$, the following are equivalent.

(i) The natural embedding of $\text{Cone}(\partial^\omega X)$ into $\text{Cone}(\partial^\omega X)$ is surjective.

(ii) The diameter of $\partial_T X$ is bounded.

**Definition 3.6.** Fix $x_0 \in X$. For a fixed scaling sequence $d = (d_n)$ and a non-principal ultrafilter $\omega$, we will define $\Psi_d^\omega : \text{Cone}(\partial^\omega X) \to \text{Con}^\omega(X, (x_0), d)$ by

$$\Psi_d^\omega(t, (\alpha_n)) = (\alpha_n(td_n)).$$

Similarly we define $\Psi_d : \text{Cone}(\partial_{\omega} X) \to \text{Con}^\omega(X, (x_0), d)$ by

$$\Psi_d(t, (\alpha_n)) = (\alpha(td_n)).$$

It is an exercise to see that for fixed $(\alpha_n) \in \partial_{\omega} X$ and $\alpha \in \partial_{\omega} X$ that $\Psi_d^\omega(t, (\alpha_n))$ and $\Psi_d(t, \alpha)$ are geodesics rays in $\text{Con}^\omega(X, (x_0), d)$ based at $(x_0)$.

Thus we have the following induced maps of boundaries:

$$\Psi_d^\omega : \partial^\omega_{\omega} X \to \partial_{\omega} \left(\text{Con}^\omega(X, (x_0), d)\right)$$

by

$$\Psi_d^\omega(\alpha_n)(t) = \Psi_d^\omega(t, (\alpha_n))$$

and $\Psi_d : \partial_T X \to \partial_T \left(\text{Con}^\omega(X, (x_0), d)\right)$ by

$$\Psi_d(\alpha)(t) = \Psi_d(t, \alpha).$$

Notice that for a constant sequence of geodesics, we have $\Psi_d^\omega(t, (\alpha)) = \Psi_d(t, \alpha)$. Thus when convenient, we will identify $\text{Cone}(\partial_{\omega} X)$ with its canonical diagonal embedding in $\text{Cone}(\partial^\omega X)$ and consider $\Psi_d$ as a restriction of $\Psi_d^\omega$.

**Proposition 3.7.** The maps $\Psi_d^\omega$ and $\Psi_d^\omega$ are non-expansive. If $X$ is proper and cocompact, then $\Psi_d^\omega$ and $\Psi_d$ are surjective.

**Proof.** By construction of the Euclidean cone, the natural map $\psi : \text{Cone}(\partial X) \to X$ given by $\psi(t, \alpha) = \alpha(t)$ is 1-Lipschitz. It follows that $\Psi_d^\omega$ is 1-Lipschitz.

Let $(\alpha_n), (\beta_n) \in \partial^\omega_{\omega} X$. For any $s > 0$,

$$\text{dist}(((\alpha_n), (\beta_n)) = \lim_{n} \omega_{\omega}(\alpha_n, \beta_n) = \lim_{n} \omega_{\omega} \lim_{t \to \infty} \text{dist}_{x_0}(\alpha_n(t), \beta_n(t))$$

$$\geq \lim_{n} \omega_{\omega} \text{dist}_{x_0}(\alpha_n(sd_n), \beta_n(sd_n))$$

$$= \text{dist}_{x_0}(\Psi_d(\alpha_n)(s), \Psi_d(\beta_n)(s)).$$

Since this holds for any $s > 0$,

$$\text{dist}(((\alpha_n), (\beta_n)) \geq \lim_{s \to \infty} \text{dist}_{x_0}(\Psi_d(\alpha_n)(s), \Psi_d(\beta_n)(s)) = \text{dist}(\Psi_d(\alpha_n), \Psi_d(\beta_n)).$$
If \( X \) is proper and admits a cocompact action, by [4] there is some constant \( K \) (depending only on \( X \) and the action) such that for every \( (x_n) \in \text{Con}^\omega(X, (x_0), d) \) there exists a sequence of geodesic rays \( \alpha_n \) based at \( x_0 \) such that \( \alpha_n(0) = x_0 \) and \( \text{dist}(\alpha_n(t_n d_n), x_n) \leq K \) where \( t_n \) converges \( \omega \)-almost surely to \( t = \text{dist}(x_0, (x_n)) \). Thus \( \Psi^\omega_d(t, (\alpha_n)) = (x_n) \), i.e. \( \Psi^\omega_d \) is surjective.

Showing that \( \Psi^\omega_d \) is surjective requires a bit more work. Fix \( \alpha \) a geodesic in \( \text{Con}^\omega(X, (x_0), d) \) based at \( (x_0) \). Since \( \Psi^\omega_d \) is surjective, there exist \( (\alpha_n) \in \partial^\omega_X \) such that \( \Psi^\omega_d(k, (\alpha_n)) = \tilde{\alpha}(k) \). For \( k \in \mathbb{N} \), fix a representative \( (x_n^k) \) of \( \tilde{\alpha}(k) \).

Let

\[
A_i = \left\{ n \mid \text{dist}(\alpha_i^k(jd_n), x_n^k) < \frac{d_n}{i} \text{ for all } j \leq i \right\}.
\]

Notice that \( A_i \) is \( \omega \)-large for each \( i \) and forms a nested sequence. Let \( D_n = \{ i \leq n \mid n \in A_i \} \) and \( m_n = \max D_n \) if \( D_n \neq \emptyset \) and 1 otherwise. Then \( m_n \) diverges \( \omega \)-almost surely and

\[
\text{dist}(\alpha_n^{m_n}(m_n d_n), x_n^k) < \frac{d_n}{m_n}.
\]

Since \( m_n \) diverges \( \omega \)-almost surely, this together with the CAT(0) condition shows that for \( \omega \)-almost all \( n \)

\[
\text{dist}(\alpha_n^{m_n}(kd_n), x_n^k) < \frac{d_n}{l}.
\]

for any fixed \( k, l \in \mathbb{N} \).

Thus \( \Psi^\omega_d(\alpha_n^{m_n}) = \tilde{\alpha} \) and \( \Psi^\omega_d \) is surjective.

\[\square\]

**Lemma 3.8.** For fixed \( \alpha, \beta \in \partial X \),

\[
\angle(\Psi_d(\alpha), \Psi_d(\beta)) = \angle(\alpha, \beta) = \angle_{(x_0)}(\Psi_d(\alpha), \Psi_d(\beta)).
\]

In particular if \( \angle(\alpha, \beta) < \pi \), then \( \Psi_d(\alpha) \) and \( \Psi_d(\beta) \) bound a Euclidean sector in \( \text{Con}^\omega(X, (x_0), d) \) for every scaling sequence \( d \) and every ultrafilter \( \omega \).

**Proof.** By the Flat Sector Theorem, the second conclusion of the lemma will follow from the first.

\[
\angle(\Psi_d(\alpha), \Psi_d(\beta)) = \lim_{t \to \infty} \angle_{(x_0)}(\Psi_d(t, \alpha), \Psi_d(t, \beta))
\]

\[
= \lim_{t \to \infty} \lim_{n} \angle_{x_0}(\alpha(t d_n), \beta(t d_n))
\]

\[
= \lim_{t \to \infty} \angle(\alpha, \beta) = \angle(\alpha, \beta)
\]

\[
\angle_{(x_0)}(\Psi_d(\alpha), \Psi_d(\beta)) = \lim_{t \to 0} \angle_{(x_0)}(\Psi_d(t, \alpha), \Psi_d(t, \beta))
\]

\[
= \lim_{t \to 0} \lim_{n} \angle_{x_0}(\alpha(t d_n), \beta(t d_n))
\]

\[
= \lim_{t \to 0} \angle(\alpha, \beta) = \angle(\alpha, \beta)
\]

\[\square\]

**Proposition 3.9.** The map \( \Psi_d \) is an isometric embedding. Thus \( \Psi^\omega_d \) restricted to the diagonal is an isometric embedding.

**Proof.** If \( \angle(\alpha, \beta) < \pi \), the geodesics \( \Psi_d(\alpha) \) and \( \Psi_d(\beta) \) bound a Euclidean sector by Lemma 3.8 which implies that the metric on the Euclidean cone and the metric in the asymptotic cone agree. If \( \angle(\alpha, \beta) = \pi \) then \( \text{dist}((s, \alpha), (t, \beta)) = s + t = \text{dist}(\Psi_d(s, \alpha), \Psi_d(t, \beta)) \). Thus \( \Psi_d \) is an isometric embedding.
Definition 3.10. The ultrafilter \( \omega \) induces a total order on the set of scaling sequences by 
\( (d'_n) = d' \preceq d = (d_n) \) if \( d'_n \leq d_n \) \( \omega \)-almost surely.

Lemma 3.11. Every countable set of scaling sequences is bounded.

Proof. Let \( \{d' = (d'_n)\} \) be a countable set of scaling sequences. Let \( \overline{d} = (\overline{d}_n) \) where \( \overline{d}_n = \max\{d'_n \mid i \leq n\} \). It is immediate that \( d' \preceq \overline{d} \) for all \( i \).

It takes more care to find a lower bound. For each \( i \in \mathbb{N} \), let 
\[ A_i = \{n \in \mathbb{N} \mid d'_n \geq i \text{ for all } j \leq i\}. \]

Since each scaling sequence is \( \omega \)-divergent, \( A_i \) is the finite intersection of finitely many \( \omega \)-large set and, hence, \( A_i \) is \( \omega \)-large. The sets \( A_i \) form a nested sequence, i.e. \( A_1 \supset A_2 \supset \cdots \), that we will use to defined \( d \).

Let \( d_n = 1 \) for \( n \in \mathbb{N} \setminus A_1 \), an \( \omega \)-small set. For \( n \in A_1 \setminus A_{i-1} \), let \( d_n = \min\{d'_n \mid j \leq i\} \) and let \( \overline{d} = (d'_n) \). Thus \( d_n \leq d'_n \) on \( A_i \), which implies that \( d \preceq d' \) for each \( i \). The sequence \( d \) is \( \omega \)-divergent since \( d_n \geq i \) for all \( n \in A_i \).

Lemma 3.12. For every countable subset \( C \subset \partial^ω X \), there exists a scaling sequence \( d = (d_n) \) such that for all \( (d'_n) = d \preceq d \), \( \Psi^ω_d \) restricted to \( \text{Cone}(C) \) is an isometric embedding. In particular; there exists a scaling sequence \( d \) such that if \( (\alpha_n), (\beta_n) \in C \), then \( \angle (\overline{\Psi}_d^ω(\alpha_n), \overline{\Psi}_d^ω(\beta_n)) = \angle_{(x_0)}(\overline{\Psi}_d^ω(\alpha_n), \overline{\Psi}_d^ω(\beta_n)) \) for all \( d \preceq d \).

Notice that this implies that if \( \lim_n \omega \angle(\alpha_n, \beta_n) < \pi \), then \( \overline{\Psi}_d^ω(\alpha_n) \) and \( \overline{\Psi}_d^ω(\beta_n) \) bound a Euclidean sector for all \( d \preceq d \).

Proof. Let \( C \) be a countable subset of \( \partial^ω X \) and fix \( (\alpha_n), (\beta_n) \in C \). Then there exits \( d_n \) such that \( \angle_{x_0}(\alpha_n(\sqrt{d_n}), \beta_n(\sqrt{d_n})) \geq \angle(\alpha_n, \beta_n) - 1/n \). Thus for any \( (d_n) = d \preceq d \) and \( t > 0 \), we have 
\[ \angle_{x_0}(\alpha_n(t\sqrt{d_n}), \beta_n(t\sqrt{d_n})) \geq \angle_{x_0}(\alpha_n(\sqrt{d_n}), \beta_n(\sqrt{d_n})) = \angle(\alpha_n, \beta_n) - 1/n \omega \]-almost surely. Then

\[
\angle(\overline{\Psi}_d^ω(\alpha_n), \overline{\Psi}_d^ω(\beta_n)) = \lim_{t \to \infty} \angle_{(x_0)}(\Psi^ω_d(t, (\alpha_n)), \Psi^ω_d(t, (\beta_n))) \\
= \lim_{t \to \infty} \lim_{n \to \infty} \omega \angle_{x_0}(\alpha_n(t\sqrt{d_n}), \beta_n(t\sqrt{d_n})) \\
\leq \lim_{t \to \infty} \lim_{n \to \infty} \omega \angle(\alpha_n, \beta_n) = \lim_{n \to \infty} \omega \angle(\alpha_n, \beta_n)
\]

\[
\angle_{(x_0)}(\overline{\Psi}_d^ω(\alpha_n), \overline{\Psi}_d^ω(\beta_n)) = \lim_{t \to 0} \angle_{(x_0)}(\Psi^ω_d(t, (\alpha_n)), \Psi^ω_d(t, (\beta_n))) \\
= \lim_{t \to 0} \lim_{n \to \infty} \omega \angle_{x_0}(\alpha_n(t\sqrt{d_n}), \beta_n(t\sqrt{d_n})) \\
\geq \lim_{t \to 0} \lim_{n \to \infty} \omega \angle(\alpha_n, \beta_n) - 1/n = \lim_{n \to \infty} \omega \angle(\alpha_n, \beta_n)
\]

Thus \( \angle(\overline{\Psi}_d^ω(\alpha_n), \overline{\Psi}_d^ω(\beta_n)) = \angle_{(x_0)}(\overline{\Psi}_d^ω(\alpha_n), \overline{\Psi}_d^ω(\beta_n)) \).

Case 1: \( \lim_n \omega \angle(\alpha_n, \beta_n) < \pi \). Then the geodesics \( \overline{\Psi}_d^ω(\alpha_n), \overline{\Psi}_d^ω(\beta_n) \) bound a Euclidean sector which implies that 
\( \text{dist}(\Psi^ω_d(t, (\alpha_n)), \Psi^ω_d(s, (\beta_n)) = \text{dist}(\angle(\alpha_n, \beta_n)) \).

Case 2: \( \lim_n \omega \angle(\alpha_n, \beta_n) = \pi \). As before, we have 
\( \text{dist}(\angle(\alpha_n, \beta_n)) \).

Thus for every pair of elements of \( C \) there exists a scaling sequence satisfying the conclusion of the lemma. Since the set of pairs from \( C \) is also countable, Lemma 3.11 implies that there exists a sequence \( d \) which will satisfy the conclusion of the lemma. □
**Definition 3.13.** Define $\Xi : X \times \mathbb{R}^+ \to X$ by

$$\Xi(x,t) = \begin{cases} x & \text{for } t \geq \operatorname{dist}(x_0,x) \\ y \in [x_0,x] & \text{for } t \leq d(x_0,x) \end{cases}.$$  

Notice that $\Xi$ is the canonical geodesic retraction of $X$ to $x_0$.

Fix scaling sequences $(d'_n) = d' \leq d = (d_n)$. Then $\Xi$ defines a map $\Theta_{d'}^d : \operatorname{Con}^\omega(X,(x_0),d) \to \operatorname{Con}^\omega(X,(x_0),d')$ by $\Theta_{d'}^d((x_n)) = (\Xi(x_n,\frac{d_n}{\omega} \operatorname{dist}(x_n,x_0))).$

**Lemma 3.14.** Let $X$ be a CAT(0) metric space. Then $\Theta_{d'}^d$ is a well-defined 1-Lipschitz map which preserves distance to the observation point of the asymptotic cone.

If, in addition, $X$ is proper and cocompact; then $\Theta_{d'}^d$ is surjective and $\Theta_{d'}^d \circ \Psi_{d'}^d = \Psi_d^d$ for all $(d'_n) = d' \leq d = (d_n)$.

**Proof.** The maps $\Theta_{d'}^d$ are well-defined 1-Lipschitz maps, since $\Xi|_{X \times \{t\}}$ is 1-Lipschitz every $t$. By construction, $\Theta_{d'}^d((x_n))$ has a representative $(x'_n)$ where $\operatorname{dist}(x'_n,x_0) = \frac{d'_n}{\omega} \operatorname{dist}(x_n,x_0)$. Thus

$$\operatorname{dist}(\Theta_{d'}^d((x_n)),(x_0)) = \lim_{d'_n} \frac{\operatorname{dist}(x'_n,x_0)}{d'_n} = \lim_{d'_n} \frac{\operatorname{dist}(x_n,x_0)}{d_n} = \operatorname{dist}((x_n),(x_0)).$$

Notice that $\Theta_{d'}^d \circ \Psi_{d'}^d(t,(\alpha_n)) = \Theta_{d'}^d((\alpha_n(td_n))) = (\alpha_n(td'_n)) = \Psi_d^d(t,(\alpha_n))$. If $X$ is also proper and cocompact, then $\Psi_{d'}^d$ is surjective, which implies that $\Theta_{d'}^d$ must also be surjective. \hfill $\Box$

Thus $\Theta_{d'}^d$, while not an isometry, does send geodesics in $\operatorname{Con}^\omega(X,(x_0),d)$ based at $(x_0)$ to geodesics in $\operatorname{Con}^\omega(X,(x_0),d')$ based at $(x_0)$. When convenient, we can consider the function $\Theta_{d'}^d$ induces on the boundaries of $\operatorname{Con}^\omega(X,(x_0),d)$ and $\operatorname{Con}^\omega(X,(x_0),d')$.

**Corollary 3.15.** Let $X$ be a proper cocompact CAT(0) space. Then $\overline{\Theta}_{d'}^d : \partial_\omega \operatorname{Con}^\omega(X,(x_0),d) \to \partial_\omega \operatorname{Con}^\omega(X,(x_0),d')$ by $\overline{\Theta}_{d'}^d \circ \overline{\Psi}_{d'}^d(\alpha_n) = \overline{\Psi}_{d'}^d(\alpha_n)$ is a well-defined surjective non-expansive map for all $d' \leq d$.

**Proof.** Let $(d'_n) = d' \leq d = (d_n)$. If $\overline{\Psi}_{d'}^d(\alpha_n) = \overline{\Psi}_{d}^d(\beta_n)$, then $(\alpha_n(td_n)) = (\beta_n(td'_n))$ for all $t$ which implies that $(\alpha_n(td'_n)) = (\beta_n(td'_n))$ for all $t$. Hence, $\overline{\Psi}_{d'}^d(\alpha_n) = \overline{\Psi}_{d'}^d(\beta_n)$. Therefore $\overline{\Theta}_{d'}^d$ is well-defined.

Then

$$\operatorname{dist}(\overline{\Theta}_{d'}^d \circ \overline{\Psi}_{d'}^d(\alpha_n),\overline{\Theta}_{d'}^d \circ \overline{\Psi}_{d'}^d(\beta_n)) = \operatorname{dist}(\overline{\Psi}_{d'}^d(\alpha_n),\overline{\Psi}_{d'}^d(\beta_n)) = \lim_{t \to \infty} \angle(x_0)(\overline{\Psi}_{d'}^d(t,(\alpha_n)),\overline{\Psi}_{d'}^d(t,(\beta_n))) = \lim_{t \to \infty} \angle(x_0)(\Theta_{d'}^d \circ \overline{\Psi}_{d'}^d(t,(\alpha_n)),\Theta_{d'}^d \circ \overline{\Psi}_{d'}^d(t,(\beta_n))) \leq \lim_{t \to \infty} \angle(x_0)(\overline{\Psi}_{d}^d(t,(\alpha_n)),\overline{\Psi}_{d}^d(t,(\beta_n))) = \operatorname{dist}(\overline{\Psi}_{d}^d(\alpha_n),\overline{\Psi}_{d}^d(\beta_n))$$

implies that $\overline{\Theta}_{d'}^d$ is a non-expansive map. Since $\overline{\Psi}_{d'}^d$ is surjective, so is $\overline{\Theta}_{d'}^d$. \hfill $\Box$

Thus $\left(\operatorname{Con}^\omega(X,(x_0),d),\Theta_{d'}^d\right)$ is a directed system and can now consider the direct limit $\lim\left(\operatorname{Con}^\omega(X,(x_0),d),\Theta_{d'}^d\right)$ and the inverse limit $\lim\left(\operatorname{Con}^\omega(X,(x_0),d),\Theta_{d'}^d\right)$. Before computing the direct limit, we require the following lemma.
Lemma 3.16. Let $X$ be a proper CAT(0) space. For every $(\alpha_n) \in \partial^\omega_X$, there exits a scaling sequence $d$ such that $\Psi^\omega_d(\alpha_n) = \Psi^\omega_d(\alpha)$ for all $\bar{d} \leq d$ where $\alpha = \lim^\omega \alpha_n$ (the limit is taken in $\partial^\infty X$). Thus $\Psi^\omega_d(t, (\alpha_n)) = \Psi^\omega_d(t, (\alpha))$ for all $t$ and all $d' \leq d$.

Proof. Let $\alpha = \lim^\omega \alpha_n$ where the limit is taken in the compact space $\partial^\infty X$. Let $d^\infty_n = \max\{t \mid \text{dist}(\alpha(t), \alpha_n(t)) \leq C, \text{for some fixed } C > 0\}$. Then $d^\infty_n$ converges to $\alpha$, $\omega$-almost surely, the sequence $d_n$ diverges $\omega$-almost surely. Let $(d_n^\infty) = d' \leq d = (d_n)$. Then

$$\dist(\Psi^\omega_d(\alpha_n)(t), \Psi^\omega_d(\alpha)(t)) = \dist((\alpha_n(td_n^\infty), (\alpha(td_n^\infty)))$$

$$= \lim^\omega \text{dist}(\alpha_n(td_n^\infty), (\alpha(td_n^\infty)))$$

$$\leq \lim^\omega \text{dist}(\alpha_n(d_n^\infty), (\alpha(d_n^\infty)))$$

$$\leq \lim^\omega C/d_n^\infty = 0.$$

Theorem 3.17. Let $X$ be a proper cocompact CAT(0) space. Then $\lim (\text{Con}^\omega(X, (x_0), d), \Theta^d_d)$ is isometric to $\text{Cone}(\partial^\infty_T X)$. Moreover, for each scaling sequence $d$ the projection map

$$\Theta_d : \text{Con}^\omega(X, (x_0), d) \rightarrow \text{Cone}(\partial^\infty_T X)$$

induced by $\Theta^d_d$ is determined by the equation $\Theta_d \circ \Psi^\omega_d(t, (\alpha_n)) = (t, \lim^\omega \alpha_n)$.

Proof. Lemma 3.14 together with Lemma 3.9 implies that $\Psi_d$ induces an isometric embedding into $\lim (\text{Con}^\omega(X, (x_0), d), \Theta^d_d)$ and Lemma 3.16 implies that this embedding is surjective.

The final statement of the theorem follows from Lemma 3.16.

Remark 3.18. We can now identify $\text{Cone}(\partial^\infty_T X)$ with $\lim (\text{Con}^\omega(X, (x_0), d), \Theta^d_d)$.

Theorem 3.19. Let $X$ be a proper cocompact CAT(0) space. Then $\lim (\text{Con}^\omega(X, (x_0), d), \Theta^d_d)$ is a metric space with the metric induced by the metric on $\text{Con}^\omega(X, (x_0), d)$ and $\text{Cone}(\partial^\infty_T X)$ isometrically embeds into $\lim (\text{Con}^\omega(X, (x_0), d), \Theta^d_d)$.

Proof. Let $x^d = (x_n^d), y^d = (y_n^d) \in \text{Con}^\omega(X, (x_0), d)$ such that $\Theta^d_d(x^d) = x^d$ and $\Theta^d_d(y^d) = y^d$ for all $d' \leq d$. The function assigning to each scaling sequence $d$ the real number $\text{dist}(x^d, y^d)$ is an increasing function of $d$ that is bounded by the constant $\text{dist}(x^d, (x_0)) + \text{dist}(y^d, (x_0))$, which is independent of $d$ by Lemma 3.14. Thus $\lim d \rightarrow \infty \text{dist}(x^d, y^d)$ exists and we can define a metric on $\lim (\text{Con}^\omega(X, (x_0), d), \Theta^d_d)$ by $\rho((x^d), (y^d)) = \lim d \rightarrow \infty \text{dist}(x^d, y^d)$. It is trivial to verify that $\rho$ does define a metric. Lemma 3.14 implies that the functions $\Psi^\omega_d$ induce a well-defined map into the inverse limit that is an isometric embedding by Lemma 3.12.

4. Visual boundary

We have seen that $\Psi^\omega_d : \partial^\omega_T X \rightarrow \text{Con}^\omega(X, (x_0), d)$ gives a parametrization of $\text{Con}^\omega(X, (x_0), d)$ which converges to $\text{Cone}(\partial^\infty_T X)$ as we allow $d$ to decrease. Thus $\{\text{Con}^\omega(X, (x_0), d), \Theta^d_d\}_{d, d'}$ completely determines the Tits boundary $\partial^\infty_T(X)$. We now wish to understand the visual boundary $\partial^\infty X$ in terms of $\{\text{Con}^\omega(X, (x_0), d), \Theta^d_d\}_{d, d'}$.

Lemma 4.1. Let $X$ be a proper CAT(0) space and $(\alpha_n)$ a sequence in $\partial X$. If $\Psi^\omega_d(t_0, (\alpha_n)) = \Psi_d(t_0, \alpha_0)$ for some scaling sequence $d$, some $t_0 > 0$ and some $\alpha_0 \in \partial X$, then $\lim^\omega \alpha_n = \alpha_0$. 


Proof. Suppose that $\Psi^\omega_d(t, (\alpha_n)) = \Psi_d(t, \alpha_0)$ for some $t > 0$, then $\text{dist}(\alpha_n(t_0d_n), \alpha_0(t_0d_n)) = o(d_n)$. Fix $C, \epsilon > 0$. Let $A$ be the $\omega$-large set such that $\text{dist}(\alpha_n(t_0d_n), \alpha_0(t_0d_n)) = C\epsilon t_0d_n$ for all $n \in A$. Then the CAT(0) inequality implies that $\text{dist}(\alpha_n(\frac{1}{\epsilon}), \alpha_0(\frac{1}{\epsilon})) \leq C$, and therefore $\text{dist}(\alpha_n, \alpha_0) \leq \epsilon$, for all $n \in A \cap B$ where $B$ is the $\omega$-large set such that $d_n \geq \frac{1}{\epsilon t_0}$ for all $n \in B$. Thus $\lim^\omega \alpha_n = \alpha_0$.

Theorem 4.2. Let $X$ be a proper CAT(0) space. For a sequence $(\alpha_n)$ in $\partial X$, $\alpha_n$ converges to $\alpha_0$ in the visual boundary of $X$ if and only if for every bijection $\sigma : \mathbb{N} \rightarrow \mathbb{N}$, there exists a scaling sequence $d$ such that $\Psi^\omega_d(t, (\alpha_{\sigma(n)})) = \Psi_d(t, \alpha_0)$ for all $t$.

Notice that $\Psi^\omega_d(t, (\alpha_{\sigma(n)})) = \Psi_d(t, \alpha_0)$ for all $t$ is equivalent to $\Theta_d \circ \Psi^\omega_d(t, (\alpha_{\sigma(n)})) = (t, \alpha_0)$ for all $t$.

Proof. If $\alpha_n$ converges to $\alpha_0$ in the visual boundary, then $\lim^\omega \alpha_{\sigma(n)} = \alpha_0$ for all bijections $\sigma$ and the forward implication then follows for Lemma 3.16. Thus we need only show that if for every bijection $\sigma : \mathbb{N} \rightarrow \mathbb{N}$, there exists a scaling sequence $d$ such that $\Psi^\omega_d(t, (\alpha_{\sigma(n)})) = \Psi_d(t, \alpha_0)$ for all $t$, then $\alpha_n$ converges to $\alpha_0$ in the visual boundary.

Suppose that there existed a subsequence $n_i$ such that $\alpha_{n_i}$ converges to $\beta$. We may assume that $B = \{n_i\}$ has infinite complement in $\mathbb{N}$. Let $A$ be an $\omega$-large subset of $\mathbb{N}$ with infinite complement. Let $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ be a bijection sending $B$ to $A$.

Then $\lim^\omega \alpha_{\sigma(n)} = \beta$. By hypothesis, there also exists a scaling sequence $d$ such that $\Psi^\omega_d(t, (\alpha_{\sigma(n)})) = \Psi_d(t, \alpha_0)$ for all $t$ which by Lemma 4.1 implies that $\lim^\omega \alpha_{\sigma(n)} = \alpha_0$. Thus $\alpha_0 = \beta$.

It is then an exercise to use the compactness of $\partial\infty X$ to show that $\alpha_n$ converges to $\alpha_0$, if every convergent subsequence of $\alpha_n$ converges to $\alpha_0$. □

5. Quasi-isometries

A map $f : X \rightarrow Y$ of metric spaces is a $(\lambda, C)$-quasi-isometric embedding if

$$\frac{1}{\lambda} \text{dist}(x, y) - C \leq \text{dist}(f(x), f(y)) \leq \lambda \text{dist}(x, y) + C$$

for every $x, y \in X$

and is $C$-quasi-surjective if $Y \subset N_C(\text{im}(f))$, i.e., the open $C$-neighborhood of $\text{im}(f)$ is all of $Y$. A $(\lambda, C)$-quasi-isometry is a $C$-quasi-surjective $(\lambda, C)$-quasi-isometric embedding. A $C$-quasi-inverse of a map $f : X \rightarrow Y$ is a map $g : Y \rightarrow X$ such that $f \circ g$ is $C$-close to the identity on $Y$ and $g \circ f$ is $C$-close to the identity on $X$. It is a standard exercise to show that every quasi-isometry $f : X \rightarrow Y$ admits a quasi-inverse $g : Y \rightarrow X$, which is itself a quasi-isometry. A quasi-geodesic ray in $Y$ is a quasi-isometric embedding of $\mathbb{R}^+ \rightarrow Y$.

Definition 5.1. Let $\omega$ be a nonprincipal ultrafilter on $\mathbb{N}$ and $d = (d_n)$ be an $\omega$-divergent sequence of positive real numbers. Suppose that $f : X \rightarrow Y$ is a $(\lambda, C)$-quasi-isometric embedding. Then $f$ naturally induces a $\lambda$-Lipschitz map $f^\omega : \text{Con}^{\omega}(X, (x_n), d) \rightarrow \text{Con}^{\omega}(Y, (f(x_n)), d)$, defined by $f^\omega((x_n)) = (f(x_n))$. If $f$ is a quasi-isometry, $f^\omega$ is bi-Lipschitz.

On the other hand, a quasi-isometric embedding (or even a quasi-isometry) $f : X \rightarrow Y$ of proper CAT(0) spaces does not induce a canonical map from $\partial X$ to $\partial Y$. The reason is that if $\alpha : \mathbb{R}^+ \rightarrow X$ is a geodesic ray, then $f \circ \alpha$ is a quasi-geodesic ray, which in general will not have a unique limit point in $\partial\infty Y$ (see Proposition 5.3). However, since $Y$ is proper, the pointwise limit $f^d(\alpha) = \lim f(\alpha(0), f(\alpha(d)))$ is a geodesic ray in $Y$. This defines a map $f^d : \partial X \rightarrow \partial Y$.

In general, $f^d$ will depend on both $\omega$ and the scaling sequence $d$.

Remark 5.2. The functions $\Theta_d, \Theta^d, \Psi_d, \Psi^\omega_d$ are defined for all CAT(0) spaces and have domains and ranges which depend on the chosen CAT(0) space. In most cases we will allow the chosen CAT(0) space to change without changing our notation for the functions $\Theta_d, \Theta^d, \Psi_d, \Psi^\omega_d$. When
Proposition 5.6. Let $f : X \to Y$ be a quasi-isometric embedding of proper CAT(0) spaces, $\omega$ a nonprincipal ultrafilter on $\mathbb{N}$, and $d = (d_n)$ an $\omega$-divergent sequence of positive real numbers. For every $\alpha \in \partial X$ there exists a scaling sequence $\tilde{d} = (\tilde{d}_n)$ such that for all $d' \preceq \tilde{d}$,
\[
\Theta_{d'}^d \circ f^\omega \circ \Psi_d(1, \alpha) = \Psi_{d'}(1, f^d(\alpha)).
\]
In particular,
\[
\Theta_d \circ f^\omega \circ \Psi_d(1, \alpha) = (1, f^d(\alpha)).
\]

Proof. Let $f : X \to Y$ be a quasi-isometric embedding of proper CAT(0) spaces, $\omega$ a nonprincipal ultrafilter on $\mathbb{N}$, and $d = (d_n)$ an $\omega$-divergent sequence of positive real numbers. Fix $\alpha \in \partial X$ and let $\gamma_n$ be the geodesic from $f \circ \alpha(0)$ to $f \circ \alpha(d_n)$. Let $\tilde{d}_n = \max\{t \mid \text{dist}(f^d(\alpha)(t^2), \gamma_n(t^2)) \leq 1\}$. Notice that $\tilde{d}_n$ is $\omega$-divergent since $\gamma_n$ converges to $f^d(\alpha)$ by definition.

For $d' = (d'_n) \preceq (\tilde{d}_n)$, we have
\[
\Theta_{d'}^d \circ f^\omega \circ \Psi_d(1, \alpha) = \Theta_{d'}^d(f \circ \alpha(d_n)) = \Theta_{d'}^d(\gamma_n(d_n)) = (\gamma_n(d'_n))(f^d(\alpha)(d'_n)) = \Psi_{d'}(1, f^d(\alpha)).
\]

Example 5.7. Let $G$ be the group introduced by Croke and Kleiner in [3] that acts properly cocompactly on CAT(0) spaces $X$ and $Y$, which have non-homeomorphic visual boundaries. Then their exists $G$-equivariant quasi-isometries $f : X \to Y$ and $g : Y \to X$ such that both $f \circ g$ and $g \circ f$ are uniformly close to the identity map. Since $G$ has rank one, the set of Morse geodesics of both $\partial_\infty X$ and $\partial_\infty Y$ are dense.
If for some $d$, both $f^d$ and $g^d$ induced continuous maps on the visual boundary, then Lemma 5.13 would imply that $f^d \circ g^d$ is the identity on a dense set of $\partial_\infty Y$ and hence the identity function on all of $\partial_\infty Y$. Similarly, $g^d \circ f^d$ would be the identity function on $\partial_\infty X$. Thus $f^d$ would give a homeomorphism from $\partial_\infty X$ to $\partial_\infty Y$, a contradiction. Thus, for every $d$, at least one of $f^d : \partial_\infty X \to \partial_\infty Y$ and $g^d : \partial_\infty Y \to \partial_\infty X$ is not continuous.

Alternatively, Ruane and Bowers give an example of two actions of a group $G$ on the product of a tree with $\mathbb{R}$ such that the $G$-equivariant induced quasi-isometry does not induce a continuous map of the boundary $[1]$.

In [8], it is shown that the limit set of a geodesic under a quasi-isometry can be any connected, compact subset of Euclidean space. One might hope that for the sufficiently small scaling sequences the maps $f^d$ would tend to choose a favorite limit point, i.e. $f^d$ stabilize for a fixed $\alpha$ and sufficiently small scaling sequence $d$. However the following proposition illustrates that this is not the case.

**Proposition 5.8.** Let $f : X \to Y$ be a quasi-isometric embedding of proper CAT(0) spaces and fix $\alpha \in \partial X$. Then $\Lambda(\text{im}(f \circ \alpha)) = \{ f^d(\alpha) \mid d \text{ is a scaling sequence}\}$. For every $\gamma \in \Lambda(\text{im}(f \circ \alpha))$ and scaling sequence $d$, there exist scaling sequences $d', \tilde{d}$ such that $d' \preceq d \preceq \tilde{d}$ and $f^d(\alpha) = \gamma = f^{d'}(\alpha)$.

**Proof.** Let $f : X \to Y$ be a quasi-isometric embedding of CAT(0) spaces and fix a scaling sequence $d = (d_n)$. Let $\gamma \in \Lambda(\text{im}(f \circ \alpha))$, i.e. $\gamma$ is in the limit set of $f \circ \alpha$. Then there exist $\tilde{d}_n$ such that $f \circ \alpha(\tilde{d}_n)$ converges (not just $\omega$-almost surely) to $\gamma$ in $\overline{\Psi}$ which implies that $\text{lim}_\omega [f \circ \alpha(0), f \circ \alpha(\tilde{d}_n)] = \gamma$.

Let $d''_n = \max\{ \tilde{d}_i \mid d_i \leq \tilde{d}_n \}$ and $\tilde{d}_n = \min\{ d_i \mid d_i \geq d_n \}$. Then for $d' = (d''_n)$ and $\tilde{d} = (\tilde{d}_n)$, we have $d' \preceq d \preceq \tilde{d}$ and it is elementary to check that $f^{d'}(\alpha) = \gamma = f^{\tilde{d}}(\alpha)$. □

**Theorem 5.9.** Let $X$ be a proper cocompact CAT(0) space such that $\partial_T X$ is compact. Then $\partial_T (X)$ is homeomorphic to $\partial_\infty X$ and if there exists a quasi-isometry $f : X \to Y$, then $f^d$ is a bi-Lipschitz equivalence from $\partial_T X$ to $\partial_T Y$.

**Proof.** There is always a continuous bijection from $\partial_T X$ to $\partial_\infty X$. If $\partial_T X$ is compact, then this is a homeomorphism.

Suppose there exists a quasi-isometry $f : X \to Y$. Fix a scaling sequence $d = (d_n)$. If $\partial_T X$ is compact, then $\partial_T^r X$ is isometric to $\partial_T X$ and $\text{Con}(\partial_T^r X)$ is locally compact. Thus both $\text{Con}^\omega(X, (x_0), d)$ and $\text{Con}^\omega(Y, (f(x_0)), d)$ are proper metric spaces, since $\Psi_d^\omega$ is a 1-Lipschitz surjection. The isometry $\Psi_d(1, \cdot)$ embeds $\partial_T Y$ as a closed subset of a closed ball of the proper metric space, $\text{Con}^\omega(Y, (f(x_0)), d)$. Thus $\partial_T Y$ is also compact and $\partial_T^\omega Y = \partial_T Y$.

Thus $\Psi_d$ is a surjection when viewed as a map from $\text{Con}(\partial_T X)$ to $\text{Con}^\omega(X, (x_0), d)$ or as a map from $\text{Con}(\partial_T Y)$ to $\text{Con}^\omega(Y, (f(x_0)), d)$. By Lemma 5.8, $\Theta_d$ is an isometry when restricted to the image of $\Psi_d$. Thus $f^d = \Theta_d \circ f^\omega \circ \Psi_d|_{\{1\} \times \partial_T X}$ is bi-Lipschitz.

The *Hausdorff distance* between two subsets $A, B$ of a metric space $X$ is

$$\text{dist}_H(A, B) = \max\left\{ \sup_{a \in A} \inf_{b \in B} \text{dist}(a, b), \sup_{b \in B} \inf_{a \in A} \text{dist}(a, b) \right\}.$$  

**Lemma 5.10.** Let $\alpha$ and $\beta$ be quasi-geodesic rays in an arbitrary metric space. If $\text{im}(\alpha) \subset \mathcal{N}_M(\text{im}(\beta))$ for some $M$, then the Hausdorff distance $\text{dist}_H(\text{im}(\alpha), \text{im}(\beta))$ between $\text{im}(\alpha)$ and $\text{im}(\beta)$ is finite.

**Proof.** Let $X$ be a metric space, and let $\alpha$ and $\beta$ be $(L', C')$-quasi-geodesic and $(L, C)$-quasi-geodesic rays, respectively, in $X$ such that the image $\text{im}(\alpha) \subset \mathcal{N}_M(\text{im}(\beta))$ for some $M$. 

We need to show there exists \( M' \) such that \( \text{im}(\beta) \subset \mathcal{N}_{M'}(\text{im}(\alpha)) \). Let \( t_0 = 0 \) and, for \( n \in \mathbb{N} \), fix \( t_n \) such that \( \text{dist}(\beta(t_n), \alpha(n)) \leq M \). Notice that \( t_n \) is unbounded which implies that the union of the closed intervals between \( t_n \) and \( t_{n+1} \) is \( \mathbb{R}^\geq \). By the triangle inequality, \( \text{dist}(\beta(t_n), \beta(t_{n+1})) \leq 2M + (L' + C') \) which implies that \( |t_n - t_{n+1}| \leq L(2M + L' + C' + C) \).

Thus for every \( t \) in the interval between \( t_n \) and \( t_{n+1} \), we have \( \text{dist}(\beta(t), \beta(t_n)) \leq L^2(2M + L' + C' + C) + C + M \).

Therefore \( \text{im}(\beta) \subset \mathcal{N}_{L^2(2M+L'+C'+C)+C+M}(\text{im}(\alpha)) \). This proves the lemma. \( \square \)

**Definition 5.11** (Morse). A (quasi-)geodesic \( \gamma \) is called a Morse (quasi-)geodesic, if for every \( L \geq 1 \) and \( C \geq 0 \) there exists an \( M = M(L, C) \) such that every \( (L, C) \)-quasi-geodesic with endpoints on \( \gamma \) remains \( M \)-close to \( \gamma \).

**Lemma 5.12.** Let \( f : X \rightarrow Y \) be a \((L, C)\)-quasi-isometry of proper CAT(0) spaces. If \( \alpha \) is a Morse ray in \( \partial X \), then \( f^d(\alpha) \) is a Morse ray in \( \partial Y \) and there exists an \( M \) such that \( \text{dist}_H(f \circ \alpha, f^d(\alpha)) < M \).

**Proof.** It is immediate that \( f \circ \alpha \) is Morse. Hence for every \( n \), \( [f \circ \alpha(0), f \circ \alpha(d_n)] \subset \mathcal{N}_{M'}(f \circ \alpha) \) for some \( M' \) depending only on \( L, C \), which implies that \( \text{im}(f^d(\alpha)) \subset \mathcal{N}_{M+1}(\text{im}(f \circ \alpha)) \). By Lemma 5.10 \( f \circ \alpha \) and \( f^d(\alpha) \) are finite Hausdorff distance apart. Since any quasi-geodesic ray that is finite Hausdorff distance from a Morse quasi-geodesic ray is also Morse, \( f^d(\alpha) \) is a Morse geodesic ray at \( f(x_0) \). \( \square \)

**Lemma 5.13.** Let \( f : X \rightarrow Y \) be a \((L, C)\)-quasi-isometry of proper CAT(0) metric spaces with \( C \)-quasi-inverse \( g : Y \rightarrow X \). If \( \alpha \) is a Morse ray in \( \partial X \), then \( f^d(\alpha) = f^d(\alpha) = g^d \circ f^d(\alpha) = \alpha \) for all scaling sequences \( d, d' \).

**Proof.** Since \( \text{dist}_H(f \circ \alpha, f^d(\alpha)), \text{dist}_H(f \circ \alpha, f^d(\alpha)) \) are both finite then \( f^d(\alpha) \) and \( f^d(\alpha) \) are asymptotic geodesics based at \( f(x_0) \). Hence \( f^d(\alpha) = f^d(\alpha) \).

Since \( g \) is a quasi-isometry and \( f^d(\alpha) \) is a Morse geodesic, \( \text{dist}_H(g \circ f^d(\alpha), g^d \circ f^d(\alpha)) \) and \( \text{dist}_H(g \circ f(\alpha), g \circ f(\alpha)) \) are both finite. Since \( g \) is a quasi-inverse of \( f \), \( \text{dist}_H(\alpha, g \circ f(\alpha)) \) is also finite. Thus \( g^d \circ f^d(\alpha) \) and \( \alpha \) are asymptotic, and therefore equal. \( \square \)

Although Morse rays are well-behaved under quasi-isometries, other rays are generally not.

**Example 5.14.** Consider the family of maps \( f : X \rightarrow X \) on \( X = \mathbb{R}^2 \) given in polar coordinates by \( f(r, \theta) = (r, \theta + h(r)) \), where \( h \) is differentiable on \( r > 0 \). (The case \( h(r) = \log(r) \) is the classic “logarithmic spiral” map.) A short calculation reveals the Jacobian of \( f \) at \((x, y) = (r \cos \theta, r \sin \theta)\) is, up to multiplying on the left and right by orthogonal matrices,

\[
\begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ r h'(r) & 1 \end{pmatrix}
\]

for \((x, y) \neq (0, 0)\). Thus we see that \( f \) is Lipschitz if and only if \( r h'(r) \) is bounded. If so, \( f^{-1} \) is also clearly well-defined and Lipschitz, so \( f \) is bi-Lipschitz and therefore a quasi-isometry.

Again, the case \( h(r) = \log(\log(r)) \) also has the property that \( f^d \) can map any point in \( \partial X \) to any other point of \( \partial X \) by adjusting \( d \).

The case \( h(r) = \log(\log(r)) \) also has the property that \( f^d \) can map any point in \( \partial X \) to any other by choosing the appropriate \( d \). However, in this case, for every \( d \) the induced bi-Lipschitz map \( f^w : \text{Con}^w(X, (x_0), d) \rightarrow \text{Con}^w(X, (f(x_0)), d) \) is an isometry, since \( r h'(r) \rightarrow 0 \) as \( r \rightarrow \infty \).

More examples can be constructed from the above by using the following general lemma.

**Lemma 5.15.** Let \( Y \) be a geodesic metric space and \( r : Y \rightarrow X \) be a locally isometric covering map. If \( \tilde{f} : Y \rightarrow Y \) is a continuous lift of an \( L \)-Lipschitz map \( f : X \rightarrow X \), then \( \tilde{f} \) is \( L \)-Lipschitz.
**Proposition 5.18.** Let \( f : X \to Y \) be a proper CAT(0) space. Then the following are equivalent.

(i) The Tits distance from \( \alpha \) to \( \beta \) is finite.

(ii) For every (or some) \( d \) and any \( i > 0 \), \( \Psi_d(t, \alpha), \Psi_d(t, \beta) \) are contained in the same component of \( \text{Con}^\omega(X, (x_0), d) \setminus \{(x_0)\} \).

(iii) For every (or some) \( d \) and any \( i > 0 \), \( \Psi_d(t, \alpha), \Psi_d(t, \beta) \) are contained in the same component of \( \text{Con}^\omega(X, (x_0), d) \setminus \{(x_0)\} \).

**Proof.** The implication (i) \(\implies\) (ii) follows immediately from Lemma 3.8 and (ii) \(\implies\) (iii) is trivial. Thus we need only show (iii) \(\implies\) (i).

Let \( \gamma : [0, 1] \to \text{Con}^\omega(X, (x_0), d) \setminus \{(x_0)\} \) be a path from \( \Psi_d(1, \alpha) \) to \( \Psi_d(1, \beta) \). Since \( \text{dist}(\text{im}(\gamma'), (x_0)) > 0 \), we can use the geodesic retraction of \( \text{Con}^\omega(X, (x_0), d) \setminus \{(x_0)\} \) to find a path \( \gamma : [0, 1] \to \text{Con}^\omega(X, (x_0), d) \setminus \{(x_0)\} \) from \( \Psi_d(t_0, \alpha) \) to \( \Psi_d(t_0, \beta) \) such that \( \text{dist}(\gamma(s), (x_0)) = t_0 \) for all \( s \).

Then we can fix \( k \) sufficiently large such that \( \text{dist}(\gamma\left(\frac{1}{k}\right), \gamma\left(\frac{i}{k}\right)) < t_0 \) for all \( 0 \leq i \leq k - 1 \). Fix \( \alpha_i \) a geodesic from \( x_0 \) to \( x_i \) such that \( (x_i) = \gamma\left(\frac{i}{k}\right) \). Let \( \alpha_i = \lim_{n \to \infty} \alpha_i^n \). Let \( \alpha = \alpha_0 \) and \( \beta = \alpha_k \). Fix \( l \in \mathbb{N} \). Then there exists an \( \omega \)-large set, \( A \), such that \( \text{dist}(\alpha_i(l), \alpha_i^\omega(l)) < 1 \) and \( \text{dist}(\alpha_i^n(t_0d_n), \alpha_i^{n+1}(t_0d_n)) \leq \frac{1}{2}t_0d_n \) for all \( n \in A \) and for \( 0 \leq i \leq k \).

Thus for any \( n \in A \) such that \( t_0d_n > l \), we have

\[
\frac{\frac{1}{2}t_0d_n + 2}{t_0d_n} \geq \frac{\text{dist}(\alpha_i(l), \alpha_i^\omega(l))}{l}
\]
by the CAT(0) inequality. Thus \( \angle(\alpha_i, \alpha_{i+1}) < \pi \) and \( \text{dist}_T(\alpha, \beta) < k\pi \).

Corollary 5.19. Let \( X \) be a proper CAT(0). Then \( \text{Con}^\omega(X, (x_0), d) \) has a cut-point if and only if the Tits diameter of \( X \) is infinite.

Since, under a cocompact action, having infinite Tits diameter is equivalent to having a periodic Morse geodesic, we have the following.

Corollary 5.20. Let \( X \) be a proper cocompact CAT(0) space. If \( \text{Con}^\omega(X, (x_0), d) \) has a cut-point, then all asymptotic cones of \( X \) have cut-points and \( X \) contains a periodic Morse geodesic.

6. Questions

Theorem 3.19 shows that the limit of asymptotic cones contains a canonical copy of \( \text{Cone}(\partial_\omega X) = \left(\text{Cone}(\partial_T X)\right)^\omega \) for proper cocompact CAT(0) spaces. However, it is not clear even in simple cases, for example where \( X \) is a tree, if this embedding is surjective.

Question 6.1. Is the inverse limit \( \lim \left(\text{Con}^\omega(X, (x_0), d), \Theta^d\right) \) isometric to \( \text{Cone}(\partial_\omega^\omega X) \)?

Example 5.7 shows that in general \( f^d \) cannot be a homeomorphism. However, it would be interesting to consider if \( f^d \) is a homotopy equivalence or if \( f^d \) preserves homotopy groups.

Question 6.2. Let \( f : X \to Y \) be a quasi-isometry of proper cocompact CAT(0) spaces. Does the continuous map \( f^d : \partial_T X \to \partial_T Y \) induce an injective homomorphism of homotopy groups?

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