Ergotropy from indefinite causal orders

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We characterize the impact that the application of two consecutive quantum channels or their quantum superposition (thus, without a definite causal order) has on ergotropy, i.e. the maximum work that can be extracted from a system through a cyclic unitary transformation. First of all, we show that commutative channels always lead to a non-negative gain; non-commutative channels, on the other hand, can entail both an increase and a decrease in ergotropy. We then perform a thorough analysis for qubit channels and provide general conditions for achieving a positive gain on the incoherent part of ergotropy. Finally, we extend our results to $d$-dimensional quantum systems undergoing a pair of completely depolarizing channels.

I. INTRODUCTION

The nature of causality and space-time structure remains one of the key puzzles in physics and philosophy. Although in quantum mechanics a fixed causal structure is implied, its fundamental nature is in question, for example, because of difficulties in explaining non-local correlations [1, 2]. Recently, it has been questioned whether the laws of quantum mechanics, in particular the superposition principle, can be applied to causal relations [3–5]. This can lead to situations when the causal order of events is not defined. A simple scheme to achieve a superposition of causal orders is provided by the so-called quantum switch model (QSM), which comprises a qubit used to control the order of application of quantum operations [6]. The QSM was used to demonstrate experimentally the advantages for communication [7–9] and computing [10] resulting from indefinite causal ordering.

Very recent work has addressed the implications that superposition of causal orders might have for the performance of cooling cycles [11] and work-extraction games [12]. This is potentially very interesting as, in thermodynamics, causality is intimately connected with the second law, which constraints the work that could be extracted from a given medium. In particular, when dealing with work extraction and heat dissipation resulting from quantum processes, attention should be paid to the role played by quantum coherence present or created in the state of the work medium [13, 14], which appears to be key.

In this paper, we make use of the concept of ergotropy, i.e. the maximum work that can be extracted from the state of a system through a cyclic unitary transformation, to explore the interplay between quantum coherence and causal order in a QSM. Our choice of figure of merit is due to recently unveiled links between ergotropy and quantum coherence [15–18], which we combine here with the use of superposed causal ordering of channels. Our goal is to show that the latter embodies a resource for ergotropic games. When considering a QSM that realizes a superposition of causal ordering of two commutative channels, we find an enhancement of the maximum extractable work compared with the discarded quantum switch, corresponding to a causally separable occurrence of the channels. Interestingly, the amount of quantum coherence initially present in the system plays a crucial role in this protocol: for certain channels and system states, the QSM allows to enhance the extractable work only by consuming coherence. Our findings pave the way to the assessment of finite-time (dis)charging protocols of quantum batteries [19], where quantum operations able to seed quantum coherences in the state of a multi-particle medium appear to be advantageous, and which might benefit of the explicit use of indefinitely ordered processes.

The remainder of this paper is organized as follows. In Sec. II, we introduce the notion of ergotropy and discuss how it is changed for causally non-separable occurrences of channels in a QSM. Sec. III is devoted to the consideration of general commutative channels and conditions for a non-zero gain in ergotropy from their causally non-separable occurrence. In Sec. IV, we consider a two-level quantum system sent through two commutative channels and study the amount of extractable work resulting from a separable and non-separable causal order of channels, respectively. In Sec. V, we particularize our findings and address the gain in ergotropy for a causally non-separable occurrence of two identical depolarizing channels. In Sec. VI, we consider the example of a two-level quantum system sent through the commutative amplitude damping and phase flip channel and study the enhancement of ergotropic work, assessing the role that coherence plays. Such results are then compared to those corresponding to the case of non-commutative channels. Finally, in Sec. VII, we draw the conclusions and comment on potential further applications of our framework.

II. EXTRACTABLE WORK AND CAUSAL ORDERING

We consider a quantum system $S$ in the initial state $\rho_S$ and whose dynamics in a fixed time interval $t \in [0, \tau]$ is governed by the time-dependent Hamiltonian $\hat{H}_S(t)$ such that $\hat{H}_S(0) = \hat{H}_S(\tau) \equiv \hat{H}_S$. The maximum extractable work from $S$ via such
The cyclical process is in general given by the ergotropy [20]
\[
W(\hat{\rho}_S) = \max_{U \in \mathcal{U}_c} \text{Tr}\left[\hat{H}_S(\hat{\rho}_S - \hat{U}\hat{\rho}_S\hat{U}^\dagger)\right],
\]
where \(\mathcal{U}_c\) denotes the set of unitary transformations generated by the time-dependent Hamiltonian under consideration. Such quantity can be meaningfully split into two separate contributions [16]
\[
W(\hat{\rho}_S) = W_i(\hat{\rho}_S) + W_c(\hat{\rho}_S),
\]
where \(W_i(\hat{\rho}_S)\) denotes the incoherent contribution to the ergotropy, i.e. the maximum amount of work that can be extracted from \(\hat{\rho}_S\) without altering its quantum coherence, while \(W_c(\hat{\rho}_S)\) characterizes the coherent counterpart.

Let us now consider two completely positive and trace-preserving maps (which we refer to as channels) \(A[\hat{\rho}_S]\) and \(B[\hat{\rho}_S]\), acting on the state of the system in some – not necessarily causally definite – order. Further, let \(\{A_i\}\) and \(\{B_i\}\) denote the set of Kraus operators of the respective channels, i.e.
\[
\Gamma[\hat{\rho}_S] = \sum_i \hat{\Gamma}_i \hat{\rho}_S \hat{\Gamma}_i^\dagger, \quad (\Gamma \equiv A, B).
\]
If channels \(A\) and \(B\) occur in a causally separable order, the resulting map can be represented as the convex combination
\[
M_i[\hat{\rho}_S] = \lambda (A \circ B)[\hat{\rho}_S] + (1 - \lambda) (B \circ A)[\hat{\rho}_S], \quad \lambda \in [0, 1],
\]
where \(\Gamma_i \equiv \Gamma_{i,i}\) stands for the composition of channels \(\Gamma_{i,j}\). Hence, the maximum work that can be extracted from a system undergoing a causally separable combination of channels Eq. (4) is readily given by
\[
W^{\text{sep}}(\hat{\rho}_S) = \max_i W(M_i[\hat{\rho}_S]).
\]
The main goal of this work is to investigate whether causally non-separable combinations of \(A[\hat{\rho}_S]\) and \(B[\hat{\rho}_S]\) can lead to an enhancement of the ergotropy with respect to \(W^{\text{sep}}\). A simple paradigmatic setup realizing an indefinite causal order is the so-called QSM, which consists of a two-level system \(Q\) controlling the order of occurrence of quantum operations on system \(S\). Specifically, the system will undergo an evolution conditional to the state of the control qubit, of the general form
\[
M_Q[\hat{\rho}_S \otimes \hat{\rho}_Q] = \sum_{ij} \hat{K}_{ij}(\hat{\rho}_S \otimes \hat{\rho}_Q) \hat{K}_{ij}^\dagger,
\]
where the QSM Kraus operators are given by \(\hat{K}_{ij} = \hat{A}_i \hat{B}_j \otimes |0\rangle \langle 0| + \hat{B}_j \hat{A}_i \otimes |1\rangle \langle 1|\) and \(\hat{\rho}_Q\) is the state of the control qubit [cf. Fig. 1]. In this particular means that when the control qubit state is in \(|0\rangle\langle 0|\), then the composite channel \((A \circ B)[\hat{\rho}_S]\) is applied to \(\hat{\rho}_S\); analogously \((B \circ A)[\hat{\rho}_S]\) will be implemented to \(\hat{\rho}_S\) if the control qubit is in \(|1\rangle \langle 1|\). On the other hand, any superposition in the control qubit state inevitably results in a combination of quantum channels, not of the form of Eq. (4). By then performing a measurement on the control qubit and communicating the result, one can prepare – with probability \(p_a = \text{Tr}(\hat{\pi}_a M_Q[\hat{\rho}_S \otimes \hat{\rho}_Q])\) – the state of the system
\[
\hat{\rho}_{S|a} = \frac{1}{p_a} \text{Tr}_Q(\hat{\pi}_a M_Q[\hat{\rho}_S \otimes \hat{\rho}_Q]),
\]
where \(\hat{\pi}_a\) is an orthogonal projection operator in the Hilbert space of the control qubit. This naturally brings us towards the scheme put forward in Ref. [21] and extended in [22], which quantifies and characterizes the gain of ergotropy associated with the acquisition of information on \(S\). Such gain originates from the quantity dubbed *daemonic ergotropy* in light of the role played by the control qubit, which is akin to a Maxwell demon, and reads
\[
W^D(\hat{\rho}_S) = \sum_a p_a W(\hat{\rho}_{S|a}).
\]
Eq. (8) can also be expanded in terms of incoherent and coherent contributions as \(W^D = W^D_i + W^D_c\), where \(W^D_a = \sum_{\alpha} p_{a\alpha} W_{a\alpha}(\rho_a)\) with \(a = c, i\), respectively. Such splitting helps us seek the conditions under which \(W^D > W^{\text{sep}}\), which would signal theergotropic advantage provided by the indefinite causal order.
III. GENERAL COMMUTATIVE CHANNELS

Having introduced the notation and the key figure of merit for our investigation, we assume the control qubit to be in the initial state $|\phi, \alpha\rangle = \sqrt{\frac{1}{2}} |0\rangle + e^{i\alpha\sqrt{1-\phi^2}} |1\rangle$, with $\phi \in [0, 1]$ and $\alpha \in [0, 2\pi]$. Upon introducing the projector $\hat{\pi}_{\phi,\alpha} = |\phi, \alpha\rangle \langle \phi, \alpha |$, the action of the Eq. (6) gives

$$M_Q[\hat{\rho}_S \otimes \hat{\pi}_{\phi,\alpha}] = \phi [(A \circ B)[\hat{\rho}_S] \otimes |0\rangle \langle 0|] + (1 - \phi)(B \circ A)[\hat{\rho}_S] \otimes |1\rangle \langle 1|) + e^{i\alpha\sqrt{1-\phi^2}} (\chi[\hat{\rho}_S] \otimes |1\rangle \langle 0|) + h.c.,$$

where $h.c.$ stands for the hermitian conjugate and where $\chi[\hat{\rho}_S] = \sum_{ij} \hat{A}_j \hat{B}_j \hat{\rho}_S \hat{B}_j^\dagger \hat{A}_j^\dagger$ is the so-called cross-map, which emerges from the quantum coherence in the state $|\phi, \alpha\rangle$ of the control qubit. For commutative channels, i.e. such that $A \circ B = B \circ A$, this expression significantly simplifies to

$$M_Q[\hat{\rho}_S \otimes \hat{\pi}_{\phi,\alpha}] = (A \circ B)[\hat{\rho}_S] \otimes (|\phi\rangle \langle 0| + (1 - \phi)|1\rangle \langle 1|) + e^{i\alpha\sqrt{1-\phi^2}} \chi[\hat{\rho}_S] \otimes |1\rangle \langle 0| + h.c.,$$

thus leading to the following reduced state of the system

$$\hat{\rho}'_S = \text{Tr}_Q(M_Q[\hat{\rho}_S \otimes \hat{\pi}_{\phi,\alpha}]) = (A \circ B)[\hat{\rho}_S].$$

Such state corresponds to the causally separable application of the channels $A[\hat{\rho}_S]$ and $B[\hat{\rho}_S]$, thus in accordance with Eq. (4).

On the other hand, one can now perform a control qubit state in a different basis, e.g. the one spanned by

$$|\phi', \alpha'\rangle = \sqrt{\frac{1}{2}} |0\rangle + e^{i\alpha'} \sqrt{1 - \phi^2} |1\rangle, \quad |\phi', \alpha'\rangle = -e^{-i\alpha'} \sqrt{1 - \phi^2} |0\rangle + e^{i\alpha'} |1\rangle,$$

with $\phi' \in [0, 1]$ and $\alpha' \in [0, 2\pi]$. According to Eq. (7), this leads to the un-normalized conditional state

$$\hat{\rho}'_{S[\phi', \alpha]} = [\phi \phi' + (1 - \phi)(1 - \phi')] (A \circ B)[\hat{\rho}_S] + e^{i(\alpha + \alpha')} \sqrt{\phi \phi' (1 - \phi)(1 - \phi')} \chi[\hat{\rho}_S] + h.c.,$$

$$\hat{\rho}'_{S[\phi', \alpha', \pm]} = [\phi (1 - \phi') + (1 - \phi) \phi'] (A \circ B)[\hat{\rho}_S] - e^{i(\alpha - \alpha')} \phi \phi' (1 - \phi)(1 - \phi') \chi[\hat{\rho}_S] + h.c.$$

By discarding the control qubit, we extract the work $W^{\text{sep}}(\hat{\rho}_S)$, which is the ergotropy of $(A \circ B)[\hat{\rho}_S]$. Therefore, the projective measurement invoked here results in a gain $\delta W = W^D - W^{\text{sep}}$ that is non-negative for any control qubit state $|\phi, \alpha\rangle$ and any projective measurement of the control qubit [21]. Notice that the cross-map $\chi[\hat{\rho}_S]$, which highlights the contribution of the quantum coherence in the state of the control qubit to the conditional states (13), can be decomposed in the following form,

$$\chi[\hat{\rho}_S] = \pm (A \circ B)[\hat{\rho}_S] + \chi^\text{ec}[\hat{\rho}_S],$$

where

$$\chi^\text{ec}[\hat{\rho}_S] = \sum_{ij} \hat{A}_j \hat{B}_j \hat{\rho}_S \hat{B}_j^\dagger \hat{A}_j^\dagger_2, \quad \chi^\text{ec}[\hat{\rho}_S] = \sum_{ij} \hat{A}_j \hat{B}_j \hat{\rho}_S \hat{B}_j^\dagger \hat{A}_j^\dagger_2.$$  

and $[\ldots]_z$ refers to a commutator or an anticommutator, respectively. This means that if the Kraus operators belonging to different channels all commute or all anti-commute, i.e., $\hat{A}_j \hat{B}_j = \pm \hat{B}_j \hat{A}_j$, then $\chi^\text{ec}[\hat{\rho}_S] = 0$, and the output of the cross-map $\chi[\hat{\rho}_S]$ becomes proportional to the separable one $(A \circ B)[\hat{\rho}_S]$. In other words, a measurement performed on the control qubit prepares the same state $(A \circ B)[\hat{\rho}_S]$ as if the control qubit was discarded. This allows us to draw our first important conclusion, namely that the daemonic ergotropy for channels $A$ and $B$ with mutually commutative or anti-commutative Kraus operators, is equal to the ergotropy for causally separable channels [cf. Eq. (5)], i.e.

$$W^D_{\text{comm}}(\hat{\rho}_S) = W((A \circ B)[\hat{\rho}_S]) \sum_a \rho_a = W^{\text{sep}}(\hat{\rho}_S).$$

Stated otherwise, no gain $\delta W$ in ergotropy can be achieved in the case of commutative or anti-commutative channels. Hence, the observation of an extra gain in the case of commutative channels witnesses the presence of quantum signatures in a thermodynamic system. While this might remind of similar considerations that could be made in regard to contextuality-related issues [cf. Ref. [23]], any potential link with our observation above should be carefully scrutinized.

Conversely, for non-commutative channels, there will be a $\lambda_{\text{max}}$ that gives the maximum ergotropy $W^{\text{sep}}$ [cf. Eq. (5)]. In that case, the sign of $\delta W$ depends also on the control state of the control qubit $|\phi, \alpha\rangle$. In particular, for $\phi = \lambda_{\text{max}}$ and regardless of the choice of $\alpha$, we obtain $\rho'_S|\phi = \lambda_{\text{max}}\rangle$ [cf. Eq. (11)]. Therefore, also in this case $\delta W \geq 0$ and thus we find a daemonic gain.

IV. ERGOTROPY OF A QUBIT

Let us now apply the above formalism to a paradigmatic QSM. In order to fix the ideas and provide as clear-cut results as possible, we start by considering a two-level quantum system $S$, with Hamiltonian $H_S = \text{Diag}[\epsilon_1, \epsilon_2]$, initially in the generic state

$$\hat{\rho}_S = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}, \quad (\rho_{11} + \rho_{22} = 1).$$

For simplicity, we assume that the energy eigenvalues $\epsilon_k$ are sorted in increasing order as $\epsilon_1 < \epsilon_2$. Without loss of generality, we can re-scale such energies so that $\epsilon_2 = 1$ and $\epsilon_1 = 0$. Following Ref. [16], the incoherent and coherent contributions to the ergotropy can be expressed as

$$W_i(\hat{\rho}_S) = \max(0, \delta \rho),$$

$$W_c(\hat{\rho}_S) = \frac{1}{2} \left( \eta - \sqrt{\eta^2 - 4|\rho_{12}|^2} \right),$$

where $\eta = \sqrt{2P(\hat{\rho}_S) - 1}$ is a function of the purity $P(\hat{\rho}_S) = \text{Tr} (\hat{\rho}_S^2)$ of the state of the system and $\delta \rho = \rho_{22} - \rho_{11}$ denotes the population imbalance of the state under scrutiny.

Let us first focus on the general conditions under which an incoherent daemonic gain is achieved. When the state of the
control qubit is discarded, the causal order of the channels is separable and leads to the following incoherent extractable work

\[ W_{i}^{\text{sep}}(\hat{\rho}_{S}) = \max\{0, \delta \rho'\}, \quad (19) \]

where \( \delta \rho' = \rho_{S,22}^' - \rho_{S,11}^' \) is the population imbalance of state \( \hat{\rho}_{S}^' \) in Eq. (11).

On the other hand, if one performs a projective measurement of the control qubit over the basis Eq. (12) the system is transformed into one of the conditional states in Eq. (13), which allows to extract the amount of incoherent work

\[ W_{i}^{D}(\hat{\rho}_{S}) = \max\{0, [\phi \phi' + (1 - \phi)(1 - \phi')]\delta \rho' + 2\sqrt{\phi \phi' (1 - \phi)(1 - \phi')} \cos(\alpha + \alpha')\xi\} \]
\[ + \max\{0, [\phi(1 - \phi') + (1 - \phi')\phi']\delta \rho' - 2\sqrt{\phi \phi' (1 - \phi)(1 - \phi')} \cos(\alpha - \alpha')\xi\}, \quad (20) \]

where \( \xi = \chi_{22} - \chi_{11} \) with \( \chi_{ij} \equiv \chi[\hat{\rho}_{S}]_{ij} \) one of the entries of the matrix representing \( \chi[\hat{\rho}_{S}] \). We are interested in the case when the incoherent daemonic gain is maximal, and this can be achieved by taking the control-qubit state \([1/2,0]\) and performing projective measurements upon it over the optimal basis \(\{1/2,0\}, \{1/2,\pi\}\) = \(\{|+,|-\}\). In this case, from Eq. (13) with \( \phi = \phi' = 1/2 \) and \( \alpha = \alpha' = 0 \), we obtain

\[ \hat{\rho}_{S}^' = \frac{1}{2}(A \circ B)[\hat{\rho}_{S}] + \frac{1}{4}(\chi[\hat{\rho}_{S}] + \chi^*[\hat{\rho}_{S}]) \quad (21) \]

and the corresponding incoherent extractable work

\[ W_{i}^{D}(\hat{\rho}_{S}) = \frac{1}{2} \left( \max\{0, \delta \rho' + \xi\} + \max\{0, \delta \rho' - \xi\} \right). \quad (22) \]

A positive daemonic gain is therefore achieved if and only if \( |\xi| > |\delta \rho'| \). This result implies that, in order to be thermodynamically advantageous, the action of the channels should increase the bias between the populations of the energy eigenstates of the Hamiltonian of the system, in line with an intuitive expectation based on the physical interpretation of the incoherent work. In what follows, we will keep considering the control-qubit state \([1/2,0]\) and projective measurements over the optimal basis \(\{|+,|-\}\).

In contrast to the incoherent contribution, the coherent counterpart of the ergotropy in Eq. (18) has a more sophisticated behavior and crucially depends on the action of the channels \(A\) and \(B\) on the coherence of the system’s state. Hence, we leave its analysis for the following sections, where the specific realizations of the channels \(A\) and \(B\) are considered.

V. IDENTICAL CHANNELS: COMPLETELY DEPOLARIZING CHANNEL

While the considerations above refer to generic channels, it is informative to address explicitly exemplary cases. First we consider a simple case of \(A\) and \(B\) being not only commutative but identical channels, i.e., \(A[\hat{\rho}_{S}] = B[\hat{\rho}_{S}]\). Interestingly, even this scenario can lead to a thermodynamically non-trivial result in terms of daemonic ergotropy gain. In light of the formalism put forward in Sec. III, in fact, whenever the quantum channel \(A[\hat{\rho}_{S}]\) has non-commutative Kraus operators, a causally non-separable application of two identical copies of it leads to a finite daemonic gain according to Eq. (8).

In order to better illustrate this point, let us consider the explicit example of fully depolarizing channels. Since the output of the latter is a maximally mixed state, i.e.

\[ D[\hat{\rho}_{S}] = \frac{1}{4} \sum_{i=1}^{4} \hat{U}_{i} \hat{\rho}_{S} \hat{U}_{i}^\dagger = \frac{1}{2}. \quad (23) \]

with \(\hat{U}_{i}\) being a set of orthogonal unitary operators, no ergotropic work can be extracted from a system sent through such a channel. This further implies that, seen as a communication channel, it cannot be used for effective information transmission. However, if the occurrence of each channel’s copy is controlled by the control qubit, then some communication and thermodynamic tasks such as classical information communication [24] and cooling cycles [11] can be effectively performed. In the QSM, we have \(A = B = D\) and, moreover, \(A \circ B[\hat{\rho}_{S}] = D[\hat{\rho}_{S}]\). Straightforward calculations lead to the following expression for the the cross-map Eq. (14)

\[ \chi[\hat{\rho}_{S}] = \frac{\hat{\rho}_{S}}{4}, \quad (24) \]

and to

\[ \hat{\rho}_{S}^' = \frac{1}{4} \pm \frac{\hat{\rho}_{S}}{8}, \quad (25) \]

for the un-normalized conditional states after the measurement of the control qubit onto the basis \(\{|+,|-\}\). Since the output state in Eq. (23) is maximally mixed, no ergotropic work can be extracted from it. This means that, for a causally separable occurrence of the copies of \(D[\hat{\rho}_{S}]\), the corresponding ergotropy in Eq. (5) is zero, \(W_{i}^{\text{sep}}(\hat{\rho}_{S}) = 0\), and the gain in ergotropy is simply the daemonic ergotropy, \(\delta W = W^{D}\). In particular, the incoherent contribution Eq. (22) to the daemonic ergotropy coming from the quantum switch is non-zero and equal to

\[ W_{i}^{D}(\hat{\rho}_{S}) = \frac{|\delta \rho|}{8}, \quad (26) \]

i.e., proportional to the population imbalance \(\delta \rho\) of the system’s state \(\hat{\rho}_{S}\). On similar footing, the coherent contribution
to the daemonic ergotropy is also finite and reads

$$W_c^D(\hat{\rho}_S) = \frac{1}{8} \sqrt{\langle \delta \rho \rangle^2 + 4|\rho_{12}|^2}.$$  (27)

We furthermore see that this term is always non-negative and, as expected, non-zero whenever $\rho_{12} \neq 0$. Combining these results, we find that the total daemonic gain for the state sent through two copies of $D[\hat{\rho}_S]$ controlled by the control qubit $Q$ is given by

$$W^D(\hat{\rho}_S) = \frac{1}{8} \sqrt{\langle \delta \rho \rangle^2 + 4|\rho_{12}|^2}.$$  (28)

This physically means that a causally non-separable occurrence of two identical completely depolarizing channels allows for an effective work extraction if the state of the system has a non-zero coherence or if it has a population bias, i.e., $\delta \rho \neq 0$. Notice that for the maximal coherence of $\hat{\rho}_S$, i.e., $\rho_{12} = \sqrt{\rho_{11}\rho_{22}}$, we have one obtains the values

$$W_c^D(\hat{\rho}_S) = \frac{1}{8} \left(1 - |\delta \rho|\right) = \frac{1}{4} \min \{\rho_{11}, \rho_{22}\},$$
$$W^D(\hat{\rho}_S) = \frac{1}{8},$$

so that the total gain in ergotropy is the same for any pure state of the system.

The results for the depolarizing channel (23) can be easily generalized to a $d$-dimensional system with Hamiltonian $H_S = \text{Diag}[\epsilon_1, \ldots, \epsilon_d]$, where the action is described by the map

$$D[\hat{\rho}_S] = \frac{1}{d^2} \sum_{i=1}^{d^2} U_i \hat{\rho}_S U_i^\dagger = \frac{1}{d}.\quad (30)$$

Clearly, if the channels occur in a causally separable order, we obtain $(A \circ B)[\hat{\rho}_S] = \frac{1}{d}$ and $W^\text{sep}(\hat{\rho}_S) = 0$. Conversely, for the pure state $|\phi, \alpha\rangle$ of the control qubit $Q$, we have that

$$\hat{\rho}_{S|\alpha}^\text{in} = \frac{1}{2d} \pm \hat{\rho}_S^\text{in}.$$

(31)

The incoherent contribution is

$$W_i^D(\hat{\rho}_S) = \frac{1}{2d^2} \sum_k \epsilon_k (\rho_{kk}^\text{in} - \rho_{kk}^\text{ex}),$$

(32)

where $\rho_{kk}^\text{in}$ ($\rho_{kk}^\text{ex}$) are the populations of $\rho$ in decreasing (increasing) order and the energies $\epsilon_k$ are understood to be arranged in increasing order. If there is coherence among the energy eigenstates, then the gain will be increased by the amount $W_c^D > 0$ given by

$$W_c^D(\hat{\rho}_S) = \frac{1}{2d^2} \sum_k \epsilon_k (r_k^\text{in} - r_k^\text{ex}),$$

(33)

where $r_k^\text{in}$ ($r_k^\text{ex}$) are the eigenvalues of $\rho$ in decreasing (increasing) order. For a pure state, this reduces to

$$W_c^D(\hat{\rho}_S) = \frac{\epsilon_d - \epsilon_1}{2d^2}.\quad (34)$$

VI. NON-IDENTICAL CHANNELS: AMPLITUDE DAMPING AND DEPHASING

We continue our analysis by generalizing the above scenario to non-identical quantum channels, either describing a generalized amplitude damping or a phase-flip. The former is described by the following Kraus operators,

$$\hat{A}_0 = \sqrt{p} \left(\begin{array}{cc} 1 & 0 \\ 0 & \sqrt{1-p} \end{array}\right), \quad \hat{A}_1 = \sqrt{p} \left(\begin{array}{cc} 0 & \sqrt{1-p} \\ 0 & 0 \end{array}\right),$$
$$\hat{A}_2 = \sqrt{1-p} \left(\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right), \quad \hat{A}_3 = \sqrt{1-p} \left(\begin{array}{cc} 0 & 0 \\ 0 & \sqrt{1-p} \end{array}\right).$$  (35)

while the latter by

$$\hat{B}_0 = \sqrt{q} \mathbf{1}, \quad \hat{B}_1 = \sqrt{1-q} \hat{a}_z,$$  (36)

with $\mathbf{1}$ being the identity matrix and $\hat{a}_k$ ($k = x, y, z$) the $k$-Pauli matrix. The parameters $\gamma, \gamma_q \in [0, 1]$ quantify the strength of the channel, while $p \in [0, 1]$ determines the chance that incoherent damping (described by $\hat{A}_{0,1}$) or pumping (resulting from the application of $\hat{A}_{2,3}$) occur. Importantly, channels $A$ and $B$ commute, a fact which, in light of Eq. (10), implies that an initial state $\hat{\rho}_S$ is mapped into

$$(A \circ B)[\hat{\rho}_S] = (1 - \gamma) \left(\begin{array}{cc} \rho_{11} & 0 \\ 0 & \rho_{22} \end{array}\right) + \gamma \left(\begin{array}{cc} 0 & 0 \\ 0 & 1 - p \end{array}\right)$$
$$- (1 - 2q) \sqrt{1 - \gamma} \left(\begin{array}{cc} 0 & \rho_{12} \\ \rho_{21} & 0 \end{array}\right).$$  (37)

For $\gamma = 1$, $q = 1/2$ or $\rho_{12} = 0$, the state in Eq. (37) has no quantum coherence. In turn, this means that the contribution to the separable ergotropy is fully incoherent $W^\text{sep} = W^\text{sep}_i$.

Taking a generic state of the control qubit and assuming arbitrary projective measurements, we achieve Eq. (13) with the following contribution from the quantum coherence in the state of the control qubit

$$\chi[\hat{\rho}_S] = (A \circ B)[\hat{\rho}_S] - \gamma(1 - q) \left(\begin{array}{cc} p(1 + \delta \rho) & 0 \\ 0 & 0 \end{array}\right),$$
$$- (1 - p)(1 - \delta \rho)$$.  (38)

which bears dependence on the population imbalance $\delta \rho$ of the initial state. Notice that if $\delta \rho = 1 - 2p$, we get the trivial case $\chi^\text{inc}[\hat{\rho}_S] = 0$ due to Eq. (15). This means that the populations and the eigenvalues of each conditional state will be shifted by the same amount, and the resulting gain $\delta W$ in ergotropy is zero.

The gain $\delta W$ is maximized by the choice of $p = 1/2, \alpha = 0$, which maximizes the initial coherence in the state of the control qubit. The optimal projective measurement is the one performed over the basis $\{|+\rangle, |-\rangle\}$. Due to Eq. (38), the off-diagonal elements in the output state originate solely from the action of the channels $(A \circ B)[\hat{\rho}_S]$ themselves. Hence, only the conditional state

$$\hat{\rho}_{S|\alpha}^\text{in} = (A \circ B)[\hat{\rho}_S] - \frac{\gamma}{2} (1 - q) \left(\begin{array}{cc} p(1 + \delta \rho) & 0 \\ 0 & 0 \end{array}\right),$$
$$- (1 - p)(1 - \delta \rho)$$.  (39)
obtained by projecting upon $|+\rangle$ bears coherences, whereas
\[ \tilde{\rho}_{S|c} = \frac{Y}{2} (1 - q) \left( \begin{array}{cc} p(1 + \delta \rho) & 0 \\ (1 - p)(1 - \delta \rho) & 0 \end{array} \right) \]
(40)
is diagonal since the action of $(A \circ B)\{\tilde{\rho}_S\}$ is canceled out. In this way, solely (39) would contribute to the coherent part $W^D_1(\tilde{\rho}_S)$ of daemonic ergotropy. On the other hand, due to (37), for a maximal strength of the amplitude damping channel $A(\gamma = 1)$, a balanced phase flip channel $B(q = 1/2)$ or incoherent state $\tilde{\rho}_S$ of the system ($\rho_{12} = 0$), the corresponding daemonic ergotropy has a purely incoherent nature, and, thus, $W^D_1 = W^D_i$.

Focusing on the incoherent counterpart [Eq. (22)] of the ergotropy, we obtain
\[ \delta \rho' = \gamma (1 - 2p) + (1 - \gamma) \delta \rho, \]
(41)
\[ \zeta = \delta \rho' + \gamma (1 - q) \{ \delta \rho - (1 - 2p) \}. \]
(42)
This leads to the following condition that should be met in order to achieve incoherent gain
\[ |\delta \rho' + \gamma (1 - q) \{ \delta \rho - (1 - 2p) \}| > |\delta \rho'|, \]
(43)
which entails $\delta \rho \notin [-x_-, x_+]$, where we have introduced the population imbalances $x_\pm$, whose expressions against the values taken by $p$ are given in Table I. In particular, a more effective ergotropic work extraction can be performed for the initially passive states with $\delta \rho < -x_-$ being sent through the quantum switch. Notice that for $\gamma = 0$ ($q = 1$), channel $A$ ($B$) reduces to an identity map, whose Kraus operators commute with any other operator, so that $\chi^\text{nc}_c[\tilde{\rho}_S] = 0$ due to Eq. (15). Hence, coherence of the control qubit plays no role in this case, and no gain can be achieved.

We address the case of $q = 0$, which corresponds to the strongest possible action of the phase-flip channel. For $\gamma < 1$ and $q \neq 1/2$, when the strength of the amplitude damping channel $A$ is not maximal, and the phase flip channel $B$ is imbalanced, the quantum coherence in $\tilde{\rho}_S$ can contribute to $\delta W$. In contrast, the incoherent counterpart, the coherent contribution $\delta W_c = W^D_i - W^\text{sep}_c$ can be positive as well as negative. In particular, the gain in ergotropic work has an exclusively coherent origin if it is extracted from the states with the population imbalance lying in the interval $\delta \rho \in [-x_-, x_+]$. This means that, for $\rho_{12} = 0$, a non-zero gain $\delta W_c$ can be achieved only for the states with $\delta \rho \notin [-x_-, x_+]$.

In the presence of coherence in $\tilde{\rho}_S$ this condition reduces to $\delta \rho \neq 1 - 2p$, i.e., more work can be potentially extracted from the state $\tilde{\rho}_S$ sent through the quantum switch if its population imbalance differs from one between the incoherent damping and pumping of channel $A$. Otherwise, the amplitude damping channel does not change the relative population of the eigenstates in $\tilde{\rho}_S$, so that $\chi^\text{nc}_c[\tilde{\rho}_S] = 0$ due to Eq. (15), and no ergotropic gain $\delta W$ is acquired. For example, setting $p = 1/2$ makes the incoherent damping and pumping of channel $A$ equally important. In this case, quantum coherence in $\tilde{\rho}_S$ contributes poorly to the ergotropic gain $\delta W$, which, thus, is produced mainly by the incoherent counterpart $\delta W_i = W^D_i - W^\text{sep}_i$ and can be gained for the imbalanced states $\tilde{\rho}_S$, i.e., $\delta \rho \neq 0$.

In Fig. 2 we represent the gain $\delta W$ with its incoherent and coherent contributions $\delta W_i = W^D_i - W^\text{sep}_i$ and $\delta W_c$. We note that the coherent gain can be larger or smaller than the coherent contribution for the separable causal order, and in the interval $\delta \rho \in [-x_-, x_+]$ the incoherent part is null, thus delivering a fully coherent gain. For $q \neq 0$ and $q \neq 1/2$ we have the same behavior until the parameters are such that $x_+ > 1$ (or $x_- > 1$), for which the gain goes to zero also at $\rho_{22} = 1$ (or $\rho_{11} = 1$) since both incoherent and coherent gains are zero.

On the other hand, by considering the bit flip channel with Kraus operators
\[ \hat{B}_0 = \sqrt{q} \hat{I}, \quad \hat{B}_1 = \sqrt{1 - q} \hat{\sigma}_x, \]
(44)
instead of the phase flip, we achieve two non-commutative channels. In this case, the gain $\delta W$ can be negative depending on the state $|\tilde{\phi}\rangle$. We focus on the case $\gamma = 1$ and $q = 0$ for giving an example of negative gain. We have
\[ (A \circ B)[\tilde{\rho}_S] = \text{Diag}[p, 1 - p], \]
\[ (B \circ A)[\tilde{\rho}_S] = 1 - (A \circ B)[\tilde{\rho}_S] \]
(45)
with $\text{Diag}[-]$ a diagonal matrix. Thus, for the separable case the ergotropy will be $W^\text{sep} = \max x_\pm \max \{0, (2\lambda_1 - 1)(1 - 2p)\}$ such that $\lambda_{\max} = 1$ if $p < 1/2$ or $\lambda_{\max} = 0$ if $p > 1/2$. Then

![TABLE I. The values of the population imbalance $x_\pm$, leading to the essentially coherent gain, in the ranges defined by the values taken by $p$.]

\begin{tabular}{|c|c|}
\hline
$p < 1/2$ & $p > 1/2$ \\
\hline
$x_+$ & $1 - 2p$ \\
$y(1 + q)(1 - 2p)$ & $2 - y(1 + q)$ \\
$x_-$ & $y(1 + q)(1 - 2p)$ \\
$2 - y(1 + q)$ & $|1 - 2p|$ \\
\hline
\end{tabular}

![FIG. 2. We plot the gain $\delta W$ and its coherent $\delta W_c$ and incoherent $\delta W_i$ contributions against the population imbalance $\delta \rho$. We have taken $\gamma = 1/2$, $p = 1/3$, $q = 0$ and $\rho_{12} = \sqrt{\rho_{11} \rho_{22}}$, i.e., the state $\tilde{\rho}_S$ has the maximal coherence and, hence, is pure. Notice that the gain is fully coherent in the interval $\delta \rho \in [-x_-, x_+]$.](image-url)
if $p > 1/2$ (or $p < 1/2$) it is enough to choose $\phi = 1$ (or $\phi = 0$) for acquiring a zero daemonic ergotropy, such that $\delta W = -W^{sep} < 0$.

VII. CONCLUSIONS AND OUTLOOK

We have investigated the daemonic ergotropy gain entailed by a causally non-separable application of quantum channels. In particular, we focused on the quantum switch (QS) model that realizes a quantum superposition of causal orders, and explored its advantages for assisted ergotropic work extraction in finite quantum systems. We showed that after sending a system through a QS that superimposes two commutative channels, and following the performance of a suitable projective measurement on the control qubit, one can achieve a non-zero gain in ergotropy compared to a causally separable occurrence of the same channels. Conversely, the gain can be negative (loss) if the quantum channels do not commute.

We have illustrated our findings within paradigmatic examples of channels controlled by the QS. Firstly, we have started by considering the simplest case of two identical completely depolarizing quantum channels, which by construction do not allow to extract ergotropic work from a system sent through them. In turn, a superposition of two such channels realized in the quantum switch model leads to a non-zero ergotropy. In this case, we showed that a causally non-separable application through a QSM lead to a finite gain in the daemonic ergotropy.

We have also extended our discussion by considering the case of non-identical quantum channels. In particular we considered the combination of an amplitude damping and a phase-flip channel within the QSM. Here we computed the daemonic ergotropy as well as their incoherent and coherent contributions. Our results have shown that the daemonic gain due to the incoherent contribution is always non-negative, whereas the coherent one can be positive as well as negative depending on the initial state of the system. Moreover, we have provided conditions concerning the initial system state $p_S$ in order to lead to a purely coherent gain in ergotropy: for such a scenario, the QSM outperforms leads to a gain with respect to the causally separable channel implementation. Recently, it was shown that the QSM model can be outperformed by more general models at enhancing classical and quantum channel capacity, albeit consuming the same resources as the switch [25, 26]. It will be interesting to explore the implications of such general models for the performance of work extraction schemes and put them in relations with the effects of quantum resources such as coherence.

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