A stringy perspective on the quantum integrable model/gauge correspondence

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Abstract

We present a string theory realization for the correspondence between quantum integrable models and supersymmetric gauge theories. The quantization results from summing the effects of fundamental strings winding around a compact direction. We discuss the examples of the $xxz$ gauge/Bethe correspondence and five-dimensional $\Omega$-deformed $\mathrm{SYM}$ on $M \times S^1$. 
1 Introduction

The deep relationship between supersymmetric gauge theories and integrable models has proven valuable for both disciplines since the seminal work of Donagi and Witten [1]. In the recent years this type of analysis relating (deformed) gauge theories to integrable models in various dimensions has found some remarkable realizations such as the AGT correspondence [2] or the BPS/Bethe correspondence [3, 4]. In the former, the partition function of $\mathcal{N} = 2$ SYM on $S^4$ is related to the conformal blocks of Liouville field theory; in the latter the ground states of a two-dimensional $\mathcal{N} = (2, 2)$ system with twisted masses are identified with the Bethe states of an integrable spin chain. Both examples have a natural string/M–theory realization that can be used to understand the underlying structure implying the correspondence of the supersymmetric and integrable models [5–12].

From the integrable model point of view, most systems admit a natural extension corresponding to passing from classical to quantum symmetry. The archetypal example is the xxx spin chain with $sl_2$ symmetry and its xxz generalization with $U_q(sl_2)$ symmetry\(^1\). It is natural to wonder if such a quantization on the integrable side has a natural counterpart on the supersymmetric side. The answer seems to be affirmative as shown by some examples in the recent literature [4, 15–18]. We will show that all these examples follow a common pattern. The original gauge theory on a manifold $M$ is deformed by adding a set of states that can be seen as the full Kaluza–Klein tower coming from the reduction of a higher-dimensional system on $M \times S^1$ (or, more generally, an $S^1$–fibration over $M$). The quantum parameter of the corresponding integrable model is proportional to $q \propto e^R$, where $R$ is the radius of the $S^1$.

The aim of this note is to introduce a string theoretical framework in which such theories can be realized in terms of D–branes in non-trivial spacetimes with fluxes including, in some frames, a constant $B$–field. Such a field is typically an equivalent description of non-commutative geometries, which are related to the notion of quantum geometry associated to quantum groups. We will also lift these brane configurations to M–theory, where the extra Kaluza–Klein modes are interpreted as a gas of M2–branes.

2 Kaluza–Klein modes, B–field and twisted masses

The gauge theory that we intend to realize is the lift to $\mathbb{R}^2 \times S^1$ of a two-dimensional gauge theory on $\mathbb{R}^2$ in which all the Kaluza–Klein excitations are kept explicitly. Concretely, if $R$ is the radius of the $S^1$, the two-dimensional theory includes an additional tower of states with mass

$$E_w = \frac{|w|}{R}, \quad w \in \mathbb{Z}.$$  \hspace{1cm} (2.1)

Even though there is a natural interpretation of these states in terms of winding modes of the string theory around the T–dual circle, the most naive realization of the gauge theory in terms of a DBI action fails to take these states into account as it is limited to the massless degrees of freedom in the theory.

\(^1\)This system has also appeared in connection to supersymmetry in [13, 14].
Let us consider more closely the three-dimensional picture. The Kaluza–Klein spectrum is reproduced if we break the invariance under translations on the $S^1$ by adding a corresponding twisted mass $\tilde{m} = 1/R$. The simplest way of writing such a theory [7] is to consider the fibration $S^1 \rightarrow \mathbb{R}^2 \times T^2 \rightarrow \mathbb{R}^2 \times S^1$ defined by the monodromy

$$x^2 \simeq x^2 + 2\pi R, \quad \left\{ \begin{array}{l} x^8 \simeq x^8 + 2\pi R_8, \\ x^2 \simeq x^2 + 2\pi R_8 \tilde{m}, \end{array} \right. \quad (2.2)$$

where $x^8$ parametrizes the fiber of radius $R_8$ and $x^2$ parametrizes the $S^1$ in the base.

In other words one should look at the lift of the three-dimensional theory to $\mathbb{R}^2 \times T^2$, where $T^2$ is the torus with parameter $\tau = R_8/(1 + i)$.

We have three equivalent descriptions of the same physical system:

1. a two-dimensional gauge theory on $\mathbb{R}^2$ with an additional tower of modes of mass $E_w = |w|/R$
2. a three-dimensional gauge theory on $\mathbb{R}^2 \times S^1$ with twisted mass $\tilde{m} = 1/R$ for the translations on $S^1$
3. a four-dimensional gauge theory on $\mathbb{R}^2 \times T^2$ on a torus with parameter $\tau = R_8/R (1 + i)$, where $R_8$ is an auxiliary parameter that is consistently sent to zero in the other pictures.

Our strategy is as follows. We first realize the theory in the third picture in terms of the effective action for a D$^3$–brane and then we work our way up to the other pictures using T–duality. The result will be an explicit form for the three-dimensional theory and a winding string interpretation for the extra modes of the two-dimensional theory.

Let us start with picture 3. In order to realize the $\mathbb{R}^2 \times T^2$ theory as a D–brane effective action, we must start with a bulk of the type $\mathbb{R}^2 \times T^2$, where the torus has the parameter $\tau = R_8/R (1 + i)$. Consider flat coordinates $x^\mu, \mu = 0, \ldots, 9$ with the identifications

$$x^2 \simeq x^2 + 2\pi R, \quad \left\{ \begin{array}{l} x^8 \simeq x^8 + 2\pi R_8, \\ x^2 \simeq x^2 + 2\pi R_8 \tilde{m}. \end{array} \right. \quad (2.3)$$

It is convenient to introduce a new $(2\pi R)$–periodic coordinate $y$,

$$y = x^2 - x^8, \quad (2.4)$$

such that the bulk metric takes the form of a fibration of $x^8$ over $\mathbb{R}^8 \times S^1$:

$$ds^2 = dx_{01345679}^2 + \frac{1}{2} dy^2 + 2 \left( dx^8 + \frac{1}{2} dy \right)^2. \quad (2.5)$$

The four-dimensional gauge theory 3 is realized as the DBI action for a D$^3$–brane extended in $(x^0, x^1, y, x^8)$ and its gauge coupling is $g^2 = \pi e^{\Phi_0}$, where $\Phi_0$ is the constant value of the bulk dilaton. The advantage of this coordinate system is that the periodicities of $x^8$ and $y$ are independent and we can decouple the (unphysical) modes around
As $x^8$ by T–dualizing and taking the $R_8 \to 0$ limit.

After T–duality in $x^8$ we move to picture (2) where the connection of the $S^1$–fibration has turned into a B–field. The bulk fields are given by

$$ds^2 = dx_{01345679}^2 + \frac{1}{2} dy^2 + \frac{1}{2} (d\tilde{x}^8)^2,$$

$$B = \frac{1}{2} dy \wedge d\tilde{x}^8,$$

$$e^{-\Phi} = \sqrt{2} e^{-\Phi_0}.$$  

A few remarks are in order.

1. The metric still has the form $\mathbb{R}^8 \times T^2$, but now the torus is rectangular, with independent identifications $y \simeq y + 2\pi R$ and $\tilde{x}^8 \simeq \tilde{x}^8 + 2\pi \alpha'/R_8$;

2. a constant $B$–field has appeared. Its field strength vanishes and in this it differs from the field that appears in the (related) fluxtrap description of the $\Omega$–deformation. The fundamental difference is that in the case at hand we introduce twisted masses to mod out a $U(1)$ symmetry that acts freely on the geometry;

3. no supersymmetries are broken by the identifications and T–duality, once more because we are considering a freely-acting orbifold.

After T–duality, the $D3$–brane turns into a $D2$–brane extended in $(x^0, x^1, y)$. The corresponding $DBI$ action in static gauge is similar to the one of the $\Omega$–deformation. The relevant part of the bosonic component reads

$$\mathcal{L} = \frac{1}{g_s^2} F^{\mu\nu} F_{\mu\nu} + \left( \partial_{\mu} \sigma_1 + \frac{1}{g} V^{\nu} F_{\nu\mu} \right)^2 + \ldots,$$  

where $V$ is the unit vector in the direction $y$,

$$V^\nu \partial_\nu = \partial_y,$$  

and $\sigma_1$ is the field that describes the motion of the brane in the direction $\tilde{x}^8$, which we are free to consider non-periodic after having taken the $R_8 \to 0$ limit (recall that $\tilde{x}^8$ has radius $\alpha'/R_8$). If the potential $A_\mu$ does not depend on $y$, the three-dimensional action is obtained by promoting the field $\sigma_1$ to a covariant derivative,

$$\sigma_1 \mapsto \sigma_1 + \frac{1}{g} A_y,$$  

as it was already discussed in [3]. What we find is another manifestation of the effect of a closed bulk $B$–field on the dynamics of a D–brane. In fact, this construction in which the brane is wrapped around a circle (the direction $y$) in the torus $(y, \tilde{x}^8)$ that supports the $B$–field is in some sense intermediate between the situation studied by Seiberg and Witten [19] in which the brane covers the full torus and the one studied by Douglas and Hull [20] where the brane is a point in the torus. In both cases the effective theories admit a natural interpretation in terms of non-commutative geometry.
Finally, we can move to picture (1) by T–dualizing in \( y \). The \( B \)-field vanishes, the dilaton takes the same value \( \Phi_0 \) as in frame (3) and the metric takes the form
\[
d^{s^2} = dx_0^2 + (dx^8 - dy)^2 + dy^2, \tag{2.10}
\]
where \( y \) is periodic with period \( 2\pi \alpha'/R \). The D2–brane turns into a D1–brane extended in \( (x^0, x^1) \). The original Kaluza–Klein modes of the gauge theory can now be understood as modes of the fundamental strings wound around \( y \). The energy of a string winding \( w \) times is then
\[
E_w = T\ell = \frac{1}{\alpha'} 2\pi \alpha' R w = \frac{w}{R}, \tag{2.11}
\]
in perfect agreement with Equation (2.1).

### 3 Applications

The construction that we have outlined so far can be combined with other deformations of gauge theories in various dimensions to obtain a string realization of examples of interest that have appeared in the literature in connection with quantum integrable models.

In [3] the authors study a \( U(N) / N = 2 \) two-dimensional gauge theory with twisted mass for the adjoint and a tower of Kaluza–Klein modes. They find that the supersymmetric ground states for the system are described by the Bethe Ansatz equations for an xxz spin chain with \( U_q(sl_2) \) symmetry with
\[
q = e^{-mR}, \tag{3.1}
\]
where \( m \) is the twisted mass of the adjoint field and \( R \) is the radius of the Kaluza–Klein reduction. The xxx theory (without Kaluza–Klein modes) was already realized in [7] in terms of D2–branes suspended between NS5–branes in the fluxtrap background. A similar construction can be repeated here. The equivalent of picture (3) is obtained by starting with a NS5–D4 system with NS5–branes extended in \( x^0, \ldots, x^3, x^8, x^9 \) and a D4 suspended between the NS5s and extended in \( x^0, x^1, x^6, x^8, x^9 \) with identifications
\[
x^9 \simeq x^9 + 2\pi R, \quad \left\{ \begin{array}{l}
x^8 \simeq x^8 + 2\pi R_8, \\
x^9 \simeq x^9 + 2\pi R_8, \\
\theta_2 \simeq \theta_2 + 2\pi R_8 m, \\
\theta_3 \simeq \theta_3 - 2\pi R_8 m, \end{array} \right. \tag{3.2}
\]
where \( \tan \theta_2 = x^3/x^2 \) and \( \tan \theta_3 = x^5/x^4 \). Following [7] we disentangle the periodicities by introducing the variables
\[
y = x^9 - x^8, \quad \phi_2 = \theta_2 - mx^8, \quad \phi_3 = \theta_3 + mx^8. \tag{3.3}
\]
After a T–duality in \( x^8 \) the D4 turns into a D3 and the effective description is a three-dimensional theory on \( \mathbb{R}^2 \times S^1 \) with real mass \( m \) for the adjoints and \( 1/R \) for the
translations in the $S^1$ (picture 2). A final T–duality in $y$ turns the brane configuration into an NS5–D2 system with extra modes corresponding to windings of the fundamental string in $\tilde{y}$ (picture 1).

In [21, 22] the authors evaluate the partition function for a $U(1)$ five-dimensional non-commutative theory on $\mathbb{R}^4 \times S^1$ in the $\Omega$–background and they find that it can be understood as a $q$–deformation of the standard $d = 4 \ N = 2$ theory with two $\epsilon$ parameters, where $q_i = \exp[\epsilon_i R]$. By now we can recognize this as the higher-dimensional analogue of the previous construction, i.e. as the effective theory in picture 2 of a D5–brane suspended between two NS5–branes. In picture 3 this corresponds to a D6–brane extended in $(x^0, \ldots, x^3, x^6, x^8, x^9)$ between two NS5–branes in a Melvin spacetime with identifications [8]

\begin{align}
\theta_1 &\approx \theta_1 + 2\pi R\epsilon_1, \\
\theta_2 &\approx \theta_2 + 2\pi R\epsilon_2, \\
\theta_3 &\approx \theta_3 + 2\pi R\epsilon_3,
\end{align}

where $\tan \theta_1 = x^1/x^0$ and in order to preserve supersymmetry, $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$. After introducing proper angular coordinates to disentangle the periodicities we can T–dualize in $x^8$ to go to picture 2, where the D5–brane describes the five-dimensional picture. A T–duality in the $S^1$–direction turns the system into picture 1 with a D4–brane suspended between NS5s with fundamental strings wrapping the dual circle. This latter system can be lifted directly to M–theory. If we consider the case in which $x^6$ is periodic, the D4–NS5 system lifts to an M5–brane extended on $\mathbb{R}_{\epsilon_1, \epsilon_2}$ and wrapped on a torus with a puncture. The fundamental strings are lifted to M2–branes wrapped on the torus and on an extra circle of radius proportional to $1/R$ as shown in Table 1.

An equivalent description for this gas of M2–branes can be obtained by using the $SL_2(\mathbb{Z}) \times SL_3(\mathbb{Z})$ symmetry of M–theory on $T^3$ generated by $\langle x^0, \tilde{x}^8, x^{10} \rangle$ that leaves the M5–brane invariant and turns the M2–branes wrapped on two cycles of the $T^3$ into Kaluza–Klein excitations [23]² along the third circle $\tilde{x}^8$.

In both descriptions we get the picture of the theory of an M5–brane embedded into a non-trivial spacetime, which cannot be described by a conventional six-dimensional gauge theory.

²The reduction of this gas of Kaluza–Klein modes leads to a D4/D0 system in type IIA. This realizes the theory as described in [24, 25].
In [15] the authors describe the correspondence between the partition function on $S^4 \times S^1$ and conformal blocks of quantum Liouville field theory with parameter $q = e^R$. This system can be seen as the picture 2 for a system analogous to the previous one, where the M5-brane is now wrapped on $S^4_{\epsilon_1,\epsilon_2} \times T^2$ and the Kaluza–Klein modes running on the $S^1$ correspond to a gas of M2-branes wrapping the torus and an external compact direction of radius proportional to $1/R$. A similar description is possible for the $S^5$ case: now we need to consider the fibration $S^1 \to S^5 \to \mathbb{P}^2$ and it is natural to conjecture that the system will be realized as an M5-brane wrapped on $\mathbb{P}^2_{\epsilon_1,\epsilon_2} \times T^2$ with a gas of M2-branes. An explicit realization of these systems is currently under investigation.

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