Transversality of the logarithmic divergences in the Classical Finite Temperature $SU(N)$ Self-Energy

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We show that the logarithmic divergences that appear in the classical approximation of the finite temperature $SU(N)$ self-energy are transverse. We use the Ward identities in linear gauges and the fact that the superficial degree of divergence $d$ of a classical diagram only depends on the number of loops $\ell$ via $d = 2 - \ell$. We comment on the relevance of this result to the construction of a low-energy effective theory beyond hard thermal loops.

I. INTRODUCTION

As it is well known, low-momentum excitations in a high temperature plasma behave classically. Nevertheless, to study these excitations one cannot simply replace the quantum thermal field theory that describes them by a classical thermal field theory, because of the appearance of ultraviolet Rayleigh-Jeans type divergences. One should think of a classical thermal field theory as an effective theory at large scales (larger than the typical interparticle distance $\sim \hbar/T$). This involves the introduction of a cut-off $\Lambda$ in the momentum of the order of the temperature $\Lambda \sim T/\hbar$. The resulting cut-off dependences reflect directly the divergences of the classical theory and indicate its different high-momentum behavior with respect to the quantum theory. One might hope that for low-momentum correlation functions this different behavior does not play a role so that the physics involved at the cut-off scale $\sim T/\hbar$ is unimportant. However, this is not so as the high-momentum modes affect, through interactions, also the low-momentum sector of the theory in an essential way.

A general strategy to improve the classical theory is to include the dominant quantum contributions from the high-momentum (hard) modes. In this context it is important to understand the classical divergences, since they correspond to the dominant hard-mode contributions in the quantum theory. For instance, the linear divergences in classical non-abelian gauge theories have a one-to-one correspondence to the well-known hard thermal loops (HTLs). The fact that hard thermal loops have to be included in an effective theory for the soft (low-momentum) modes has been known since the work of Braaten and Pisarski. These hard thermal loops have the following remarkable properties, namely, they are gauge invariant, they satisfy abelian-like Ward identities and they allow a kinetic formulation in terms of a Vlasov equation.

Besides linear divergences, classical theories contain also logarithmic divergences starting at two-loop. (The explanation of how loops arise in a classical theory can be found in.) Higher-loop diagrams are superficially finite, although they may contain linear or logarithmic divergences in the form of one- or two-loop subdiagrams, respectively. Thus, we have two essential kinds of classical divergences, linear and logarithmic.

The relevance and the physical significance of the linear divergences (the HTLs) has been extensively studied, in particular, an effective action which incorporates them has been developed (see or for example).

A further step towards an effective theory beyond hard thermal loops would then be to include the logarithmic divergences (“log-divergences”) into the effective action. At this point some natural questions arise, namely: Do the log-divergences have the same or similar properties as the linear divergences, namely the hard thermal loops? What are the quantum contributions corresponding to the log divergences?

In any case, an extra term in the effective action containing the physics of the logarithmic divergences would enter the equations of motion of the fields as a current, which must be conserved for the effective theory to be consistent. In this letter we show that the logarithmically divergent part of the classical $SU(N)$ gauge self-energy is transverse, therefore providing current conservation at the level of two-point functions in the equations of motion. For that we will use the Ward identities for the self-energy at finite temperature and the results of.
where it was argued that the degree of divergence \( d \) of a classical diagram with \( \ell \) loops is \( d = 2 - \ell \) and also that there are no log divergences at one-loop.

II. TRANSVERSALITY OF LOGARITHMIC DIVERGENCES

Gauge invariance provides us with identities among different Green functions, the so-called Ward identities. In \( SU(N) \) gauge theories the Ward identity for the full retarded propagator \( D_{\mu\nu}^{\text{R,full}}(K) \) in the covariant gauge \( (\partial_\mu A^\mu = 0) \) is

\[
K^\mu K^\nu D_{\mu\nu}^{\text{R,full}} = -\alpha \quad (1)
\]
and in general, in linear gauges \( (F_\mu A^\mu = 0) \) it is given by

\[
F^\mu F^\nu D_{\mu\nu}^{\text{R,full}} = -\alpha, \quad (2)
\]
where \( \alpha \) is the gauge parameter in both cases. The identity is the same at \( T = 0 \) and \( T \neq 0 \) [4].

The identities can be written for the self energy \( \Pi_{\mu\nu} \) using the relation \( (D_{\text{full}})^{-1} = (D_0)^{-1} - \Pi \). Now, a difference arises for the two cases \( T = 0 \) and \( T \neq 0 \). This difference can be better seen in the covariant gauge. At zero temperature, \( \Pi_{\mu\nu} \) must be a linear combination of the two available tensors \( g_{\mu\nu} \) and \( K_\mu K_\nu \), which immediately leads to the result that the self-energy is transverse, namely \( K^\mu \Pi_{\mu\nu} = 0 \). However, at finite temperature the difference arises due to the presence of a heat bath with four-velocity \( U^\mu \). From (2) and the presence of \( U^\mu \) the Lorentz structure of \( D_{\mu\nu}^{\text{R,full}} \) allows different independent tensors combination of both \( K_\mu \) and \( U_\mu \), for instance \( g_{\mu\nu}, K_\mu K_\nu, U_\mu U_\nu \) and \( K_\mu U_\nu + U_\mu K_\nu \). More convenient are the dimensionless tensors \( A_{\mu\nu}, B_{\mu\nu}, C_{\mu\nu} \) and \( D_{\mu\nu} \) detailed in [10–14]. Here we will use the fact that both \( A_{\mu\nu} \) and \( B_{\mu\nu} \) are transverse to the four-momentum \( K^\mu \), i.e. \( K^\mu A_{\mu\nu} = 0 \) and \( K^\mu B_{\mu\nu} = 0 \), whereas \( C_{\mu\nu} \) and \( D_{\mu\nu} \) are not. Moreover, \( A_{\mu\nu} \) and \( B_{\mu\nu} \) are respectively transverse and longitudinal to the spatial momentum \( k \). With these tensors the self-energy can be written as

\[
\Pi_{\mu\nu} = \Pi_T A_{\mu\nu} + \Pi_L B_{\mu\nu} + \Pi_C C_{\mu\nu} + \Pi_D D_{\mu\nu}, \quad (3)
\]
where \( \Pi_T \) and \( \Pi_L \) denote the four-momentum transverse components (transverse and longitudinal to the spatial momentum respectively) and \( \Pi_C \) and \( \Pi_D \) are the non-transverse components. This decomposition of the self-energy in the above basis of tensors is not only valid for the covariant gauge, but also for the temporal and Coulomb gauges and, in general, for linear gauges that do not break rotational invariance [4].

From the Ward identity (1) and the decomposition (3) above one can derive a relation between the different transverse and non-transverse components of the self-energy [4]:

\[
[\Pi_C(K)]^2 = [K^2 + \Pi_L(K)]\Pi_D(K). \quad (4)
\]

We note that this result does not imply that the self-energy is transverse. Indeed, it is well known that already at one-loop the self-energy is not transverse [5]. However, remarkably, the hard thermal loop part of the self-energy is, i.e. \( K^\mu \Pi_{\mu\nu}^{\text{HTL}} = 0 \). This is due to the fact that HTLs satisfy abelian-like ward identities [1].

The Ward identity (1) will be a starting point in our discussion on the transversality of the divergent parts of the classical self-energy.

A. Linear divergences

Let us now consider the classical approximation of \( SU(N) \) gauge theory, which is obtained by taking the \( \hbar \to 0 \) limit of the quantum theory. The classical theory is expected to be a good approximation at low energies because the classical and low-energy limit of the Bose-Einstein distribution function \( n(\omega_k) \) yield the same result:

\[
n(\omega_k) = \frac{1}{e^{\hbar \omega_k/T} - 1} \quad \rightarrow \quad n_c(\omega_k) \equiv \frac{T}{\omega_k \hbar}, \quad (5)
\]

where \( \omega_k = |k| \) is the frequency at wavenumber \( k \). As mentioned in the introduction, the classical theory as an effective theory requires the introduction of a cut-off \( \Lambda \) which appears in the calculation of diagrams, reflecting directly the divergences of the theory.

The linearly divergent terms in the classical theory correspond to the HTLs in the quantum theory [4]. This is so because the HTLs are proportional to the \( \omega_k^2 \) where \( \omega_k \sim gT h^{-1/2} \) is the plasmon frequency. Thus, the HTLs behave as \( 1/\hbar \), which become the linear divergences in the classical theory as we take \( \hbar \to 0 \).

The fact that HTLs are transverse indicates that the linearly divergent terms should also be so. This can be checked by making use of (1). Consider the case \( K^2 = K_\mu K^\mu \neq 0 \) and let us start at one-loop. Since \( \Pi_C = 0 \) at tree level, it begins at order \( O(g^2) \) (one-loop). Now, from (1) we notice that \( \Pi_D \) should start at \( O(g^4) \) (two-loops). The two-loop contribution \( \Pi_D^{(2)} \) (the super-script denotes loop order) is superficially log divergent, containing at most a linear subdvergence. Hence by (1) we see that the one-loop contribution \( \Pi_C \) cannot have a linear divergence. Thus, at one-loop both \( \Pi_C \) and \( \Pi_D \) vanish, and therefore,

\[
k^\mu \Pi_{\mu\nu}^{\text{HTL, lin}} = 0, \quad (6)
\]
as we expected from our considerations of hard thermal loops above. In fact, this is another way of showing that HTLs are transverse.
B. Logarithmic divergences

At one-loop there are no logarithmic divergences \( \Pi^1 \), so we consider the case of two-loop, i.e. \( O(g^4) \). We first split the two-loop self-energy component \( \Pi_D^{[2]} \) in a logarithmically divergent part, a part that may contain a linear subdivergence and a finite part:

\[
\Pi_D^{[2]} = \Pi_D^{[2], \text{log}} + \Pi_D^{[2], \text{sublin}} + \Pi_D^{[2], \text{fin}} .
\]

(7)

We insert this expression into the Ward identity (3). The terms in \( \Pi_C \) at the right-hand side that match with the \( O(g^4) \) at the left-hand side are those corresponding to one-loop, which does not have logarithmic divergences, and therefore

\[
\Pi_D^{[2], \text{log}} = 0 .
\]

(8)

We saw already that \( \Pi_C \) does not contain a linear divergence, thus also

\[
\Pi_D^{[2], \text{sublin}} = 0 .
\]

(9)

Next, we consider \( \Pi_C \). Analogously to (7), we split it in terms of the different types of divergences

\[
\Pi_C^{[2]} = \Pi_C^{[2], \text{log}} + \Pi_C^{[2], \text{sublin}} + \Pi_C^{[2], \text{fin}} .
\]

(10)

We use the Ward identity (3) at \( O(g^8) \), which we may write as

\[
\left( \Pi_C^{[2], \text{log}} + \Pi_C^{[2], \text{sublin}} + \Pi_C^{[2], \text{fin}} \right)^2 + 2\Pi_C^{[1], \text{fin}}\Pi_C^{[3]} = K^2\Pi_C^{[4]} + \Pi_L^{[1]}\Pi_D^{[3]} + \Pi_L^{[2]}\Pi_D^{[2]} .
\]

(11)

We now focus on the terms that could lead to an logarithmic divergence. We keep from (11) terms proportional to \( (\log \Lambda)^2 \). This results in

\[
\left( \Pi_C^{[2], \text{log}} \right)^2 + 2\Pi_C^{[1], \text{fin}}\Pi_C^{[3]} = K^2\Pi_C^{[4]} + \Pi_L^{[1]}\Pi_D^{[3]} .
\]

(12)

As a consequence of (8), \( \Pi_C^{[2]} \) does not contain a logarithmic divergence, therefore the last term on the r.h.s. of (11) does not contribute to (12). Let us consider the products of one- and three-loop contributions. Since at one-loop there are no log-divergences, the three-loop diagrams must contain a double log-divergence for this products to contribute. Now, schematically, the expression for a three-loop diagram is

\[
\Pi^{[3]}(P) = g^6T^3 \int \frac{d^3K}{(2\pi)^3} \frac{d^3K'}{(2\pi)^3} \frac{d^3K''}{(2\pi)^3} f^{[3]}(K, K', K'', P) ,
\]

(13)

where \( K, K' \) and \( K'' \) are the internal momenta, \( P \) is the external momentum, \( g \) the coupling constant and \( T \) the temperature. The result after the integration over any two arbitrary internal momenta can be regarded as either an expression for two disjunct one-loop diagrams or a two-loop diagram, with external lines depending on the other momenta. Consider for example the three-loop diagram in figure 1.

![FIG. 1. A three-loop diagram (a) with the two-loop subdiagram (b)](image)

In the case that the integration over two arbitrary internal momenta (\( K' \) and \( K'' \) in Fig. 1) corresponds to a two-loop diagram (as in Fig. 1b), it can at most give a single logarithmic divergence, which we denote as \( \log \Lambda \). When it does, the integration over the remaining momentum (\( K \) in Fig. 1) cannot give an extra \( \log \Lambda \), since the superficial degree of divergence of the total diagram is \( d = 2 - \ell = -1 \). In the case that the integration over \( K' \) and \( K'' \) does not give a log-divergence, the integration over \( K \) could still lead to one \( \log \Lambda \). Hence, a three-loop diagram can at most give a single log-divergence. Therefore, the product of one- and three-loop diagrams cannot contribute to (12).

The above argument can be repeated for the four-loop contribution to the self-energy. In this case there are four internal momenta. The result after integration over any three given internal momenta can be regarded as the expression for a disjunct two and one-loop diagram or three disjunct one-loop diagrams. Therefore it can at most give a single logarithmic divergence, and since a four-loop contribution to the self-energy is finite, the remaining integration cannot give an extra log-divergence and as in the case above, \( \Pi^{[4]} \) cannot contribute to (12). Thus, we find from (12) that

\[
\Pi_C^{[2], \text{log}} = 0 .
\]

(14)

Note that we cannot say that the two-loop contribution to \( \Pi_C \) containing a linear divergence from one-loop subdiagrams equals zero, as we could for \( \Pi_D \).

Since both \( \Pi_C^{[2]} \) and \( \Pi_D^{[2]} \) vanish, then we conclude that the logarithmically divergent part of the two-loop classical self-energy is transverse

\[
K^\mu\Pi_{\mu\nu}^{[2], \text{log}} = 0 .
\]

(15)

Our argument does not hold for three- or higher-loop log-divergences. However, since those diagrams are superficially finite, the divergences can only come through two-loop subdiagrams and in general, we do not expect...
the whole contribution to the self-energy to be transverse. The important point is that, as we mentioned in the introduction, the essential divergences that appear in the classical theory are linear (at one-loop) and logarithmic (at two-loop) and they are both transverse.

III. CONCLUSIONS AND OUTLOOK

Here we showed that the logarithmically divergent parts of the classical finite temperature $SU(N)$ self-energy are transverse. This also holds for the hard thermal loops, which correspond to classical linear divergences. Therefore we see that all divergences appearing in the classical self-energy are transverse. We would like to comment on the importance of this result regarding the construction of a low-energy effective theory beyond the HTL approximation.

Consider for instance the effective action which results from integrating out the hard modes with momenta $P > \mu$, with $\mu$ an intermediate scale such that $\omega_{pl} < \mu < T/\hbar$. In a $h$ or high $T$ expansion the effective action for the soft modes would schematically be written as

$$\Gamma_{\text{eff}} = g^2 T \left( \frac{T}{\hbar} - c_1 \mu \right) \Gamma_{\text{HTL}} + (g^2 T)^2 \log \left( \frac{c_2 T}{\hbar \mu} \right) \Gamma_{\text{log}} + S_{\text{cl}} + O \left( g^2 \hbar, \frac{\omega_{pl}}{\mu}, \frac{h\mu}{T} \right),$$

where $c_1$ and $c_2$ are constants that depend on the regularization scheme. The first term in the expansion, which corresponds to the HTLs, is proportional to $\hbar^{-1}$, being thus linearly divergent in the classical limit $\hbar \to 0$. The second term is proportional to $\log (T/\hbar)$ and so corresponds to the logarithmic divergences in the classical theory. The third term is the classical action and the other terms in the expansion are unimportant contributions in either a high $T$ or classical regime. Thus we see that in a $h$ or high-temperature expansion the next-to-leading order terms are given by the classical log-divergences.

A consistent scheme to include hard-mode contributions beyond HTLs in the classical theory seems to require the inclusion of terms that diverge in the classical limit. An effective action of the form (16) gives rise to currents in the equations of motion for the classical $SU(N)$ gauge field

$$\delta_{\text{cl}} S_{\text{cl}} = j_{\text{HTL}} + j_{\text{log}},$$

where $j_{\text{HTL}} \sim \delta_{\text{cl}} \Gamma_{\text{HTL}}$ and $j_{\text{log}}^{\mu} \sim \delta_{\text{cl}} \Gamma_{\text{log}}^{\mu}$. The current $j_{\text{HTL}}^{\mu}$ generated by the HTLs is conserved. For consistency, it is necessary that the current $j_{\text{log}}^{\mu}$ generated by the log-divergences is also conserved. Our result (15) shows that the logarithmic divergent part of the self-energy is transverse, which implies that $j_{\text{log}}^{\mu}$ is indeed conserved. We stress that this is a special property at finite temperature that should not generally be expected, and in fact, this result encourages the study of the classical logarithmic divergences towards the construction of a feasible low-energy effective theory beyond hard thermal loops.

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