Tensor product and irregularity for holonomic D-modules
Jean-Baptiste Teyssier

To cite this version:
Jean-Baptiste Teyssier. Tensor product and irregularity for holonomic D-modules. 2015. hal-01127705

HAL Id: hal-01127705
https://hal.archives-ouvertes.fr/hal-01127705
Preprint submitted on 7 Mar 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
TENSOR PRODUCT AND IRREGULARITY FOR
HOLONOMIC $\mathcal{D}$-MODULES

by

Jean-Baptiste Teyssier

Introduction

Let $X$ be a complex variety and let $D^b_{\text{hol}}(\mathcal{D}_X)$ be the derived category of complexes of $\mathcal{D}_X$-modules with bounded holonomic cohomology. It is known [Meb04, 6.2-4] that for a regular complex $^{(1)} \mathcal{M} \in D^b_{\text{hol}}(\mathcal{D}_X)$, the derived tensor product $\mathcal{M} \otimes^L_{\mathcal{O}_X} \mathcal{M}$ is regular. The goal of this note is to prove the following

**Theorem 1.** — Let $\mathcal{M} \in D^b_{\text{hol}}(\mathcal{D}_X)$ and suppose that $\mathcal{M} \otimes^L_{\mathcal{O}_X} \mathcal{M}$ is regular. Then $\mathcal{M}$ is regular.

The technique used in this text is similar to that used in [Tey14], and proceed by recursion on the dimension of $X$. The main tool is a sheaf-theoretic measure of irregularity [Meb90].

I thank the anonymous referee for a careful reading and constructive remarks on this manuscript. This work has been achieved with the support of Freie Universität/Hebrew University of Jerusalem joint post-doctoral program and ERC 226257 program.

0.1. For every morphism $f : Y \to X$ with $X$ and $Y$ complex varieties, we denote by $f^+ : D^b_{\text{hol}}(\mathcal{D}_X) \to D^b_{\text{hol}}(\mathcal{D}_Y)$ and $f_* : D^b_{\text{hol}}(\mathcal{D}_Y) \to D^b_{\text{hol}}(\mathcal{D}_X)$ the inverse image and direct image functors for $\mathcal{D}$-modules. We define $f^! := f^+[\dim Y - \dim X]$.

0.2. If $Z$ is a closed analytic subspace of $X$, we denote by $\text{Irr}^Z_X(\mathcal{M})$ the irregularity sheaf of $\mathcal{M}$ along $Z$ [Meb90].

---

1. that is, a complex with regular cohomology modules.
1. The proof

1.1. The 1-dimensional case. — We suppose that $X$ is a neighbourhood of the origin $0 \in \mathbb{C}$ and we prove the following

**Proposition 1.1.1.** — Let $\mathcal{M} \in D_{\text{hol}}^b(D_X)$ so that $H^k\mathcal{M}$ is a smooth connexion away from 0 for every $k \in \mathbb{N}$. If $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{M}$ is regular, then $\mathcal{M}$ is regular.

The complex

$$(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{M})(*0) \simeq \mathcal{M}(*0) \otimes_{\mathcal{O}_X} \mathcal{M}(*0)$$

is regular. Since we are in dimension one, the regularity of $H^k\mathcal{M}$ is equivalent to the regularity of $H^k\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{M}$. Thus, we can suppose that $\mathcal{M}$ is localized at 0. In particular, the $H^k\mathcal{M}$ are flat $\mathcal{O}_X$-modules, so the only possibly non zero terms in the Künneth spectral sequence

$$(1.1.2) \quad E_2^{pq} = \bigoplus_{i+j=q} \text{Tor}^p_{\mathcal{O}_X}(H^i \mathcal{M}, H^j \mathcal{M}) \Rightarrow H^{p+q}(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{M})$$

sit on the line $p = 0$. Hence, (1.1.2) degenerates at page 2 and induces a canonical identification

$$H^k(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{M}) \simeq \bigoplus_{i+j=k} (H^i \mathcal{M} \otimes_{\mathcal{O}_X} H^j \mathcal{M})$$

for every $k$. In particular, the module $H^i \mathcal{M} \otimes_{\mathcal{O}_X} H^j \mathcal{M}$ is regular for every $i$. Hence, one can suppose that $\mathcal{M}$ is a germ of meromorphic connexions at 0. By looking formally at 0, one can further suppose that $\mathcal{M}$ is a $C((x))$-differential module. In this case, the regularity of $\mathcal{M}$ is a direct consequence of the Levelt-Turrittin decomposition theorem [Sv00].

1.2. Proof of theorem 1 in higher dimension. — We proceed by recursion on the dimension of $X$ and suppose that $\dim X > 1$. For every point $x \in X$ taken away from a discrete set of points $S \subset X$, one can find a smooth hypersurface $i : Z \to X$ passing through $x$ which is non characteristic for $\mathcal{M}$. Since regularity is preserved by inverse image, the complex

$$i^+ \mathcal{M} \otimes_{\mathcal{O}_X} i^+ \mathcal{M}$$

is regular. By recursion hypothesis, we deduce that $i^+ \mathcal{M}$ is regular. From [Tey14, 3.3.2], we obtain

$$\text{Irr}^*_S(\mathcal{M}) \simeq \text{Irr}^*_S(i^+ \mathcal{M}) = 0$$

Since regularity can be punctually tested [Meb04, 6.2-6], we deduce that $\mathcal{M}$ is regular away from $S$. In what follows, one can thus suppose that $X$ is a neighbourhood of the origin $0 \in \mathbb{C}$ and that $\mathcal{M}$ is regular away from 0.

Let us suppose that 0 is contained in an irreducible component of $\text{Supp} \mathcal{M}$ of dimension $> 1$. Let $Z$ be an hypersurface containing 0 and satisfying the conditions

1. $Z \cap \text{Supp} \mathcal{M}$ has codimension 1 in $\text{Supp} \mathcal{M}$.
2. The modules $H^k \mathcal{M}$ are smooth away from $Z$.

2. That is, $\text{Supp}(H^k \mathcal{M})$ is smooth away from $Z$ and the characteristic variety of $H^k \mathcal{M}$ away from $Z$ is the conormal bundle of $\text{Supp}(H^k \mathcal{M})$ in $X$. 

(3) \( \dim \text{Supp} \, R\Gamma_{[Z]} M < \dim \text{Supp} \, M \).

Such an hypersurface always exists by [Meb04, 6.1-4]. According to the fundamental criterion for regularity [Meb04, 4.3-17], the complex \( M(\ast Z) \) is regular. From the local cohomology triangle

\[
R\Gamma_{[Z]} M \longrightarrow M \longrightarrow M(\ast Z) \to^1 \]

we deduce that one is left to prove that \( R\Gamma_{[Z]} M \) is regular. There is a canonical isomorphism

\[
(\ast) \quad R\Gamma_{[Z]} M \otimes^L_{O_X} R\Gamma_{[Z]} M \cong R\Gamma_{[Z]} (M \otimes^L_{O_X} M) \]

Since \( R\Gamma_{[Z]} \) preserves regularity, the left hand side of (1.2.1) is regular. So one is left to prove theorem 1 for \( R\Gamma_{[Z]} M \), with \( \dim \text{Supp} \, R\Gamma_{[Z]} M < \dim \text{Supp} \, M \). By iterating this procedure if necessary, one can suppose that the components of \( \text{Supp} \, M \) containing 0 are curves. We note \( C := \text{Supp} \, M \). At the cost of restricting the situation to a small enough neighbourhood of 0, one can suppose that \( C \) is smooth away from 0. Let \( p : \tilde{C} \longrightarrow X \) be the composite of normalization map for \( C \) and the canonical inclusion \( C \hookrightarrow X \). By Kashiwara theorem [HTT00, 1.6.1], the canonical adjunction [Meb89, 7.1]

\[
(\ast \ast) \quad p_+p^! M \longrightarrow M
\]

is an isomorphism away from 0. So the cone of (1.2.2) is supported at 0. Hence, it is regular. One is then left to show that \( p_+p^! M \) is regular. Since regularity is preserved by proper direct image, we are left to prove that \( p^! M \) is regular. There is a canonical isomorphism

\[
(\ast \ast \ast) \quad p^! M \otimes^L_{O_X} p^! M \cong p^!(M \otimes^L_{O_X} M)
\]

So the left hand side of (1.2.3) is regular and one can apply 1.1.1, which concludes the proof of theorem 1.

References

[HTT00] R. Hotta, K. Takeuchi, and T. Tanisaki, \textit{\( \mathcal{D} \)-Modules, Perverse Sheaves, and Representation Theory}, vol. 236, Birkhäuser, 2000.

[Meb89] Z. Mebkhout, \textit{Le formalisme des six opérations de Grothendieck pour les \( \mathcal{D} \)-modules cohérents}, vol. 35, Hermann, 1989.

[Meb90] Z. Mebkhout, \textit{Le théorème de positivité de l’irrégularité pour les \( \mathcal{D}_X \)-modules}, The Grothendieck Festschrift III, vol. 88, Birkhäuser, 1990.

[Meb04] Z. Mebkhout, \textit{Le théorème de positivité, le théorème de comparaison et le théorème d’existence de Riemann}, Éléments de la théorie des systèmes différentiels géométriques, Cours du C.I.M.P.A., Séminaires et Congrès, vol. 8, SMF, 2004.

[Sv00] M.T Singer and M. van der Put, \textit{Galois Theory of Linear Differential Equations}, Grundlehren der mathematischen Wissenschaften, vol. 328, Springer, 2000.

[Tey14] J.-B. Teissier, \textit{Sur une caractérisation des \( \mathcal{D} \)-modules holonomes réguliers}, submitted (2014).
