THE LFED AND LNED CONJECTURES FOR LAURENT POLYNOMIAL ALGEBRAS

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Abstract. Let $R$ be an integral domain of characteristic zero, $x = (x_1, x_2, \ldots, x_n)$ $n$ commutative free variables, and $A_n := R[x^{-1}, x]$, i.e., the Laurent polynomial algebra in $x$ over $R$. In this paper we first classify all locally finite or locally nilpotent $R$-derivations and $R$-$E$-derivations of $A_n$, where by an $R$-$E$-derivation of $A_n$ we mean an $R$-linear map of the form $\text{Id}_{A_n} - \phi$ for some $R$-algebra endomorphism $\phi$ of $A_n$. In particular, we show that $A_n$ has no nonzero locally nilpotent $R$-derivations or $R$-$E$-derivations. Consequently, the LNED conjecture proposed in [Z4] follows. We then show some cases of the LFED conjecture proposed in [Z4] for $A_n$. In particular, we show that both the LFED and LNED conjectures hold for the Laurent polynomial algebras in one or two commutative free variables over a field of characteristic zero.

1. Introduction

Let $R$ be a unital commutative ring and $A$ an $R$-algebra. We denote by $1_A$ or simply 1 the identity element of $A$, if $A$ is unital, and $I_A$ or simply I the identity map of $A$, if $A$ is clear in the context.

An $R$-linear endomorphism $\eta$ of $A$ is said to be locally nilpotent (LN) if for each $a \in A$ there exists $m \geq 1$ such that $\eta^m(a) = 0$, and locally finite (LF) if for each $a \in A$ the $R$-submodule spanned by $\eta^i(a)$ ($i \geq 0$) over $R$ is finitely generated. For each $R$-linear endomorphism $\eta$ of $A$ we denote by $\text{Im} \ \eta$ the image of $\eta$, i.e., $\text{Im} \ \eta := \eta(A)$, and $\text{Ker} \ \eta$ the kernel of $\eta$.

By an $R$-derivation $D$ of $A$ we mean an $R$-linear map $D : A \rightarrow A$ that satisfies $D(ab) = D(a)b + aD(b)$ for all $a, b \in A$. By an $R$-$E$-derivation $\delta$ of $A$ we mean an $R$-linear map $\delta : A \rightarrow A$ such that for...
all $a, b \in A$ the following equation holds:

\[(1.1) \quad \delta(ab) = \delta(a)b + a\delta(b) - \delta(a)\delta(b).\]

It is easy to verify that $\delta$ is an $R$-$\mathcal{E}$-derivation of $A$, if and only if $\delta = I - \phi$ for some $R$-algebra endomorphism $\phi$ of $A$. Therefore an $R$-$\mathcal{E}$-derivation is a special so-called $(s_1, s_2)$-derivation introduced by N. Jacobson [J] and also a special semi-derivation introduced by J. Bergen in [B]. $R$-$\mathcal{E}$-derivations have also been studied by many others under some different names such as $f$-derivations in [E1, E2] and $\phi$-derivations in [BFE, BV], etc.

Now let us recall the following notion of associative algebras introduced in [Z2, Z3]. Since all algebras in this paper are commutative, here we recall only the case for commutative algebras.

**Definition 1.1.** Let $A$ be a commutative $R$-algebra. An $R$-subspace $V$ of $A$ is said to be a Mathieu subspace (MS) of $A$ if for all $a, b \in A$ with $a^m \in V$ for all $m \geq 1$, we have $a^m b \in V$ for all $m \gg 0$.

Note that a MS is also called a Mathieu-Zhao space in the literature (e.g., see [DEZ, EN, EH], etc.), as first suggested by A. van den Essen [E3].

The introduction of this new notion is mainly motivated by the study in [M, Z1] of the well-known Jacobian conjecture (see [Ke, BCW, E2]). See also [DEZ]. But, a more interesting aspect of the notion is that it provides a natural generalization of the notion of ideals.

Next, we recall the cases of the so-called LFED and LNED conjectures proposed in [Z4] for commutative algebras. For the study of some other cases of these two conjectures, see [EWZ], [Z4], [Z7].

**Conjecture 1.2.** Let $K$ be a field of characteristic zero, $A$ a commutative $K$-algebra and $\delta$ a LF (locally finite) $K$-derivation or $K$-$\mathcal{E}$-derivation of $A$. Then the image $\text{Im} \delta$ of $\delta$ is a MS of $A$.

**Conjecture 1.3.** Let $K$ be a field of characteristic zero, $A$ a commutative $K$-algebra and $\delta$ a LN (locally nilpotent) $K$-derivation or $K$-$\mathcal{E}$-derivation of $A$. Then $\delta$ maps every ideal of $A$ to a MS of $A$.

Throughout the paper we refer Conjecture 1.2 as the (commutative) LFED conjecture, and Conjecture 1.3 the (commutative) LNED conjecture.

In this paper we study some cases of the two conjectures above for the Laurent polynomial algebra $\mathcal{A}_n := R[x^{-1}, x]$ in $n$ commutative free variables $x = (x_1, x_2, \ldots, x_n)$ over an integral domain $R$ of characteristic zero.
In section 2 we classify all LF or LN $R$-derivations of $A_n$ (see Propositions 2.1 and 2.3). In particular, we show that $A_n$ has no nonzero LN $R$-derivations or $R$-$\mathcal{E}$-derivations. Hence the LNED conjecture 1.3 holds (trivially) for $A_n$.

In section 3 we show first that the LFED conjecture 1.2 holds for all LF $R$-derivations of $A_n$ (See Theorem 3.1). We also show the LFED conjecture 1.2 with Condition (C1) on page 9 on $R$ for some LF $R$-$\mathcal{E}$-derivations of $A_n$ (See Lemmas 3.3, 3.5 and 3.7 and Proposition 3.12). In particular, we show that both the LFED and LNED conjectures hold for the Laurent polynomial algebras in one or two variables over a field of characteristic zero (see Theorem 3.14). For the other LF $R$-$\mathcal{E}$-derivations $\delta$ of $A_n$, we give an explicit description for $\text{Im} \, \delta$ (see Theorem 3.10) and reduce the LFED conjecture 1.2 for $\delta$ to a special case of a conjecture (see Conjecture 3.11) proposed in [Z4].

2. Classification of Locally Finite or Locally Nilpotent Derivations and $\mathcal{E}$-Derivations of $A_n$

Throughout this paper $R$ stands for an integral domain of characteristic zero and $x = (x_1, x_2, \ldots, x_n)$ $n$ commutative free variables. We denote by $R[x^{-1}, x]$ or $A_n$ the algebra of the Laurent polynomials in $x$ over $R$. The main purpose of this section is to classify all LF (locally finite) or LN (locally nilpotent) $R$-derivations and $R$-$\mathcal{E}$-derivations of $A_n$.

First, we fix the following notations that will be used in the rest of this paper. The notations introduced in the previous section will also be freely used.

**Notations and Conventions:**

i) We denote by $R^*$ the set of units of $R$. For each $k \geq 1$ we let $\{e_i | 1 \leq i \leq k\}$ be the standard basis of $\mathbb{Z}^k$, $M_k(\mathbb{Z})$ the set of all $k \times k$ matrices with integer entries, and $I_k$ the $k \times k$ identity matrix.

ii) For each nonzero $f \in A_n$ we denote by $\deg f$ the (generalized) degree of $f(x)$, and $\text{ord} f$ the order of $f$, i.e., the minimum of all the degrees of the monomials in $f(x)$ with nonzero coefficients. Furthermore, we denote by $\text{Supp} f$ the support of $f$, i.e., the set of all $\alpha \in \mathbb{Z}^n$ such that the coefficient of $x^\alpha$ is not zero, and $\text{Poly} f$ the polytope of $f$, i.e., the convex subset of $\mathbb{R}^n$ spanned by $\text{Supp} f$.

iii) We denote by $\partial_i$ ($1 \leq i \leq n$) the $R$-derivation $\partial/\partial x_i$ of $A_n$. Then every $R$-derivation $D$ of $A_n$ can be written uniquely as
\[ \sum_{i=1}^{n} a_i(x) \text{ for some } a_i(x) \in A_n. \] For all \( k \geq 0 \), we let \( \mathcal{H}_k \) denote the \( R \)-subspace of \( A_n \) of homogeneous Laurent polynomials of the (generalized) degree \( k \).

We start with a classification of all LF or LN \( R \)-derivations of \( A_n \).

**Proposition 2.1.** Let \( D \) be an \( R \)-derivation of \( A_n \). Then

1) \( D \) is locally finite, if and only if \( D = \sum_{1 \leq i \leq n} q_i x_i \partial_i \) for some \( q_i \in R \).

2) \( D \) is locally nilpotent, (if and) only if \( D = 0 \), i.e., \( A \) has no nonzero locally nilpotent \( R \)-derivations.

**Proof:** 1) The \(( \Leftarrow \) part is obvious. To show the \(( \Rightarrow \) part we may obviously assume \( D \neq 0 \) and write \( D \) uniquely as \( D = \sum_{k=r}^{s} D_k \) with \( r \leq s \) and \( D_k \) \((r \leq k \leq s)\) an \( R \)-derivation of \( A_n \) such that \( D_k \mathcal{H}_i \subseteq \mathcal{H}_{i+k} \) for all \( i \in \mathbb{Z} \) and \( D_r, D_s \neq 0 \).

We first show \( r = s = 0 \), i.e., \( D = D_0 \). Below we give only a proof for \( s = 0 \) by using the degree of elements of \( A_n \). The proof for \( r = 0 \) is similar (by using the order of elements of \( A_n \), instead).

Assume \( s \neq 0 \). Then for all \( 1 \leq i \leq n \) and \( m \geq 1 \), \( D^m x_i \) and \( D^m x_i^{-1} \), if not zero, are homogeneous of the (generalized) degree \( ms + 1 \) and \( ms - 1 \), respectively. On the other hand, since \( D \) is LF, \( D^m x_i \) and \( D^m x_i^{-1} \) \((1 \leq i \leq n \text{ and } m \geq 1\) lie inside a finitely generated \( R \)-submodule of \( A_n \). Note that \( D^m x_i \) and \( D^m x_i^{-1} \) if not zero, are the leading terms of \( D^m x_i \) and \( D^m x_i^{-1} \), respectively. Therefore, there exists \( N \geq 1 \) such that \( D^m x_i = D^m x_i^{-1} = 0 \) for all \( 1 \leq i \leq n \) and \( m \geq N \).

Since \( D^x x_i = -x_i D x_i \) for each \( 1 \leq i \leq n \), we have that \( D x_i = 0 \), if and only if \( D x_i^{-1} = 0 \). Since \( D \neq 0 \), there exists \( 1 \leq i \leq n \) such that \( D x_i, D x_i^{-1} \neq 0 \). Let \( k \) (resp., \( \ell \)) be the least positive integer such that \( D^x x_i = 0 \) (resp., \( D^x x_i^{-1} = 0 \)).

If \( k + \ell \geq 3 \), then by the Leibniz rule we have

\[ 0 = D^{k+\ell-2} = D^{k+\ell-2}(x_i x_i^{-1}) = \binom{k+\ell-2}{k-1} (D^{k-1} x_i)(D^{\ell-1} x_i^{-1}). \]

Since \( A_n \) is an integral domain of characteristic zero, we have \( D^{k-1} x_i = 0 \) or \( D^{\ell-1} x_i^{-1} = 0 \). But this contradicts to either the choice of \( k \) or that of \( \ell \). Therefore \( k + \ell \leq 2 \), whence \( k = \ell = 1 \), i.e., \( D x_i = D x_i^{-1} = 0 \). Contradiction. Hence \( s = 0 \), as desired.

Now we may write \( D = D_0 = \sum_{i=1}^{n} a_i(x) \partial_i \) for some homogeneous \( a_i(x) \in A_n \) \((1 \leq i \leq n)\) of degree one. We need to show that for all \( 1 \leq i \leq n \) we have \( a_i(x) = q_i x_i \) for some \( q_i \in R \). But, up to the conjugation by a permutation (on \( x_i \)'s) automorphism of \( A_n \) it suffices to show only the case for \( a_1(x) \).
Write \( a_1(x) = b(x)\partial_1 + q_1 x_1 \partial_1 \) with \( q_1 \in R \) and \( b(x) \) homogeneous of degree one and independent on \( x_1 \). Then for all \( m \geq 1 \), it is easy to see that \( D^{m}x_1^{-1} \) as a Laurent polynomial in \( x_1 \) over \( R[x_i^{-1}, x_i \mid 2 \leq i \leq n] \) has the least degree (in \( x_1 \)) term \((-1)^{m}m!b^m(x)x_1^{-m-1}\). Since \( D \) is LF (again), we have \( b(x)^m x_1^{-m-1} = 0 \) for all \( m \gg 0 \), whence \( b(x) = 0 \) and statement 2) follows.

2) Let \( D \) be a LN (locally nilpotent) \( R \)-derivation of \( A_n \). Then \( D \) is LF and by statement 1) we have \( D = D_0 = \sum_{1 \leq i \leq n} q_i x_i \partial_i \) for some \( q_i \in R \). If \( D \neq 0 \), then \( q_j \neq 0 \) for some \( 1 \leq j \leq n \), and for all \( m \geq 1 \) we have \( D^{m}x_j^{-1} = (-1)^{m}q_j^{m}x_j^{-1} \neq 0 \). Contradiction. \( \square \)

**Remark 2.2.** By some similar arguments as that in the proof of Proposition 2.1 it is easy to see that Proposition 2.1 also holds for the Laurent polynomial algebra in noncommutative free variables over \( R \).

Next, we give a classification of all LF or LN \( R \)-\( \partial \)-derivations of \( A_n \). Let \( \phi \neq 0 \) be an \( R \)-algebra endomorphism of \( A_n \). Since \( \phi(1) \) is an idempotent of \( A_n \) and \( A_n \) is an integral domain, we have \( \phi(1) = 0 \) or 1. Since \( \phi(1) = 0 \) implies \( \phi = 0 \), we have \( \phi(1) = 1 \), i.e., \( \phi \) preserves the unity 1 of \( A_n \). Consequently, it also preserves the units of \( A_n \).

Since all units of \( A_n \) have the form \( qx^\alpha \) with \( q \in R^* \) and \( \alpha \in \mathbb{Z}^n \), we see that \( \phi(x_i) = q_ix_i^{\alpha_i} \) with \( q_i \in R^* \) and \( \alpha_i \in \mathbb{Z}^n \) for all \( 1 \leq i \leq n \). Set \( q := (q_1, q_2, \ldots, q_n) \) and for each \( 1 \leq i \leq n \) write \( \alpha_i = (\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in}) \) and form the \( n \times n \) matrix \( A := (\alpha_{ij}) \in M_n(\mathbb{Z}) \). For convenience, for the case \( \phi = 0 \) we simply let \( A = 0 \) and \( q_i = 0 \) \( (1 \leq i \leq n) \). For the rest of this paper we call the matrix \( A \) the **exponent matrix** of the \( R \)-algebra endomorphism of \( \phi \).

With the remarks and notations above it is easy to see that \( \phi(x^\beta) = q^\beta x^A \beta \) for all \( \beta \in \mathbb{Z}^n \). In particular, both \( \phi \) and \( I - \phi \) maps each monomial to a scalar multiple of a monomial. Therefore, \( I - \phi \) is LF, if and only if \( \phi \) is LF, if and only if for each \( \alpha \in \mathbb{Z} \) the set \( \{ \phi^m(x^\alpha) \mid m \geq 0 \} \) is finite.

**Proposition 2.3.** With the notations above, for all \( R \)-endomorphism \( \phi \) of \( A_n \) the following statements hold:

1) \( I - \phi \) is locally finite if and only if there exist \( 0 \leq k \leq n \), an invertible and finite order \( B \in M_{n-k}(\mathbb{Z}) \), and an invertible \( S \in M_n(\mathbb{Z}) \) such that

\[
A = S \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} S^{-1}.
\]

(2.1)

2) \( I - \phi \) is locally nilpotent (if and) only if \( \phi = I \), i.e., \( A_n \) has no nonzero locally nilpotent \( R \)-\( \partial \)-derivations.
Proof: 1) The \((\Leftarrow)\) part can be checked easily. To show the \((\Rightarrow)\) part, we assume \(A \neq 0\), for the case \(A = 0\) is trivial, and set for all \(m \geq 1\)
\[
\tilde{A}_m := I_{n \times n} + A + A^2 + \cdots + A^{m-1}.
\]
Then it is easy to see inductively that for all \(\alpha \in \mathbb{Z}^n\) and \(m \geq 1\) we have
\[
\phi^m(x^\alpha) = q^{\tilde{A}_m \alpha} x^{A^{m} \alpha}.
\]
Since \(I - \phi\) is LF, \(\phi\) is also LF. Then for each \(1 \leq i \leq n\) the set \(\{\phi^m(x_i) \mid m \geq 1\}\) is finite, whence by Eq. (2.3) there exists \(1 \leq k_i < \ell_i\) such that
\[
A^{k_i} e_i = A^{\ell_i} e_i,
\]
where \(e = (e_1, e_2, \ldots, e_n)\) is the standard basis of \(\mathbb{Z}^n\).

Multiplying a power of \(A\) to Eq. (2.4) for each \(1 \leq i \leq n\) we may assume \(k_i = k_j\) for all \(1 \leq i, j \leq n\). We denote this integer by \(k\) and set \(\ell = k + \prod_{i=1}^{n} (\ell_i - k)\). Then it is easy to see that \(A^{k_i} e_i = A^{\ell_i} e_i\) for all \(1 \leq i \leq n\), whence \(A^k = A^\ell\).

Let \(H\) be the subgroup of \(\mathbb{Z}^n\) formed by \(\alpha \in \mathbb{Z}^n\) such that \(A \alpha = 0\). If \(H = 0\), then \(A\) is of rank \(n\), and from the fact \(A^k = A^\ell\) with \(1 \leq k < \ell\), we see that \(A^{k-\ell} = I_n\). Hence the statement holds in this case. So we assume \(H \neq 0\).

By a well-known fact (e.g., see [11, Theorem 1.6]) on subgroups of free abelian groups there exists a basis \((\beta_1, \beta_2, \ldots, \beta_n)\) of \(\mathbb{Z}^n\) such that \(H\) is the (free) subgroup generated by \((d_1 \beta_1, d_2 \beta_2, \ldots, d_k \beta_k)\) for some \(1 \leq k < n\) and positive integers \(d_i\) \((1 \leq i \leq k)\). Since \(A(d_i \beta_i) = 0\) implies \(A \beta_i = 0\) for each \(1 \leq i \leq k\), we hence have \(d_i = 1\) for all \(1 \leq i \leq k\), and also \(k \leq n - 1\), for \(A \neq 0\). Therefore there exist \(k \times (n-k)\) matrix \(C\), and \(B \in M_{n-k}(\mathbb{Z})\) of rank \(n-k\), and an invertible \(S \in M_n(\mathbb{Z})\) such that
\[
A = S \begin{pmatrix} 0 & C \\ 0 & B \end{pmatrix} S^{-1}.
\]

Set \(D := \begin{pmatrix} 0 & C \\ 0 & B \end{pmatrix}\). Since \(D^m = \begin{pmatrix} 0 & CB^{m-1} \\ 0 & B^m \end{pmatrix}\) for all \(m \geq 1\), by the fact \(A^k = A^\ell\) proved above we have \(B^k = B^\ell\), which implies \(B^{k-\ell} = I_{n-k}\). Therefore \(B\) is invertible and of finite order. So we need only to show \(C = 0\).

Assume otherwise, i.e., \(C \neq 0\). Denote by \(\psi\) the \(R\)-algebra automorphism of \(A_n\) that maps \(x^\alpha\) to \(x^{S \alpha}\) for all \(\alpha \in \mathbb{Z}^n\). Replacing \(\phi\) by \(\psi^{-1} \phi \psi\) we may assume \(A = D\). Since \(\phi\) is LF, so is \(\phi^m\) for all \(m \geq 1\).
Replacing $\phi$ by $\phi^{k-\ell}$ we may assume $B = I_{n-k}$, i.e., $A = \begin{pmatrix} 0 & C' \\ 0 & I_{n-k} \end{pmatrix}$, where $C' = CB^{j-i-1} \neq 0$.

Now let $y = (x_1, x_2, \ldots, x_k)$, $z = (x_{k+1}, x_{k+1}, \ldots, x_n)$, and $\beta \in \mathbb{Z}^{n-k}$ such that $C\beta \neq 0$. Then for all $m \geq 1$, $\phi^m(z^\beta)$ up to an invertible multiplicative scalar is equal to $y^{mC'\beta}z^\beta$. Since $mC'\beta$ ($m \geq 1$) are all distinct, $\mathrm{Im} \, \phi$ contains infinitely distinct monomials, which contradicts to the assumption that $\phi$ is LF. Therefore $C' = 0$, and hence $C = 0$, as desired.

2) Assume that $I - \phi$ is LN. Then $I - \phi$ is also LF, and statement 1) holds for $I - \phi$. In particular, there are exist some $1 \leq k < \ell$ such that $A^k = A^\ell$.

We first assume $A = A^2$. Then for each $\alpha \in \mathbb{Z}^n$ there exists $N \geq 1$ such that $0 = (I - \phi)^N x^\alpha = x^\alpha + c x^\alpha$ for some $c \in R$, whence $A\alpha = \alpha$. Therefore $A = I_n$. Consequently, for all $\alpha \in \mathbb{Z}^n$ and $m \geq 1$, we have $\phi(x^\alpha) = q^a x^\alpha$ and $(I - \phi)^m x^\alpha = (1 - q^a)^m x^\alpha$. Since $I - \phi$ is LN, we have $q^a = 1$ for all $\alpha \in \mathbb{Z}$, whence $q_i = 1$ for all $1 \leq i \leq n$ and $\phi = 1$, as desired.

For the general case, let $r \geq 1$ be the order of the matrix $B$ in Eq. (2.1). Then by Eq. (2.1) we have $A^r = (A^\ell)^2$, and by Eq. (2.3) $\phi^r(x^\alpha) = q^{A\alpha} x^{A^\ell \alpha}$ for all $\alpha \in \mathbb{Z}^n$. Since $(I - \phi^r) = (\sum_{i=1}^{r-1} \phi^i)(I - \phi)$, $(I - \phi^r)$ is also LN. Applying the case $A = A^2$ shown above to the $R$-algebra endomorphism $\phi^r$ we get $\phi^r = 1$. Then by [Z4, Corollary 6.5] we have $\phi = I$

A more straightforward way to show the last step above is as follows.

For each $\alpha \in \mathbb{Z}$, we have $(I - \phi^r) x^\alpha = 0 = (I - \phi)^N x^\alpha$ for some $N \geq 1$. On the other hand, both the polynomials $(1 - t^r)$ and $(1 - t)^N$ are monic and have the “greatest common divisor” $1 - t$ in $\mathbb{C}[t]$. By the Euclidean algorithm it is easy to verify that there exist $u(t), v(t) \in \mathbb{Z}[t] \subseteq R[t]$ such that $(1 - t^r)u(t) + (1 - t)^N v(t) = 1 - t$. Substituting $\phi$ for $t$ in the equation above and then applying it to $x^\alpha$ we see that $(I - \phi)(x^\alpha) = 0$ for each $\alpha \in \mathbb{Z}$. Hence $\phi = I$, as desired. $\square$

3. The LFED and LNED Conjectures for $A_n$

In this section we consider the LFED conjecture [1,2] and the LNED conjecture [1,3] for the Laurent polynomial algebras over an integral domain of characteristic zero.

Throughout this section $R$ denotes an integral domain $R$ of characteristic zero, $x = (x_1, x_2, \ldots, x_n)$ commutative free variables and
\( \mathcal{A}_n = R[x^{-1}, x] \). All other notations and conventions introduced in the previous two sections will also be in force in this section.

First, we have the following:

**Theorem 3.1.** 1) The LNED conjecture \( L.3 \) holds for \( \mathcal{A}_n \).

2) The LFED conjecture \( L.2 \) holds for all LF \( R \)-derivations of \( \mathcal{A}_n \).

**Proof:** 1) follows trivially from Propositions 2.1 and 2.3, by which \( \mathcal{A}_n \) has no nonzero LN \( R \)-derivations or LN \( R \)-\( \mathcal{E} \)-derivations. 2) follows directly from Propositions 2.1 and the lemma below. \( \square \)

**Lemma 3.2.** Let \( D = \sum_{1 \leq i \leq n} a_i x_i \partial_i \) for some \( a_i \in R \). Then \( \text{Im } D := D(\mathcal{A}_n) \) is a MS of \( \mathcal{A}_n \).

The lemma for polynomial algebras \( K[x] \) over a field \( K \) of characteristic zero has been proved in Lemma 3.4 in \( [EWZ] \). It is easy to see that the proof there with some slight modifications also works for this more general case. So we skip the proof here.

Next, we consider the \( \mathcal{E} \)-derivation case of the LFED conjecture \( L.2 \) for \( \mathcal{A}_n \). We fix a LF \( R \)-algebra endomorphism \( \phi \) of \( \mathcal{A}_n \). Then by Proposition 2.3 and up to a conjugation of \( \phi \) we may assume that \( \phi \) maps \( x^\alpha (\alpha \in \mathbb{Z}^n) \) to \( q^\alpha x^{A\alpha} \), where \( q = (q_1, q_2, \ldots, q_n) \) with \( q_i \in R^* \) \((1 \leq i \leq n)\) and \( A \in M_n(\mathbb{Z}) \) is the exponent matrix of \( \phi \) given by

\[
A = \begin{pmatrix}
0 & 0 \\
0 & B
\end{pmatrix}
\]  

(3.1)

for some invertible \( B \in M_{n-k}(\mathbb{Z}) \) of finite order.

We start with the following simple case.

**Lemma 3.3.** Assume \( A = 0 \). Then \( \text{Im } (I - \phi) = \text{Ker } \phi \), which is the maximal ideal of \( \mathcal{A}_n \) generated by \( (x_i - q_i) \) \((1 \leq i \leq n)\).

**Proof:** Note that in this case \( \phi(x^\alpha) = q^\alpha \) for all \( \alpha \in \mathbb{Z}^n \). In particular, \( \phi^2 = \phi \). Then it is easy to check directly (or by the more general \( [Z4, \text{Proposition 5.2}] \)) that the lemma indeed holds. \( \square \)

Next we drive the following reduction.

**Lemma 3.4.** To show whether or not \( \text{Im } (I - \phi) \) is a MS, we may assume that the exponent matrix \( A \) of \( \phi \) is invertible and of finite order.

**Proof:** Let \( A \) be given as in Eq. (3.1). If \( k = 0 \), then there is nothing to show. If \( k = n \), i.e., \( A = 0 \), then by Lemma 3.3 \( \text{Im } (I - \phi) \) is an ideal of \( \mathcal{A} \), hence a MS of \( \mathcal{A} \). So we assume \( k \neq 1, n \).

Let \( J \) be the ideal of \( \mathcal{A}_n \) generated by \( x_i - q_i \) \((1 \leq i \leq k)\). Then \( \phi(J) = 0 \) and \( (I - \phi)(J) = J \), whence \( J \subseteq \text{Im } (I - \phi) \).
Let \( \bar{A}_n = A_n/J \) and \( \bar{\phi} \) be the \( R \)-algebra endomorphism of \( \bar{A}_n \) induced by \( \phi \). Then it can be readily verified that \( \operatorname{Im} (I - \phi)/J = \operatorname{Im} (I_{\bar{A}_n} - \bar{\phi}) \). Then by [Z3, Proposition 2.7] we have that \( \operatorname{Im} (I_{\bar{A}_n} - \bar{\phi}) \) is a MS of \( A_n \), if and only if \( \operatorname{Im} (I_{\bar{A}_n} - \bar{\phi}) \) is a MS of \( \bar{A}_n \). Note that \( \bar{A} \) as an \( R \)-algebra is isomorphic to the Laurent polynomial algebra in \( x_j \) \( (k + 1 \leq j \leq n) \) and the exponent matrix of \( \bar{\phi} \) is \( B \in M_{n-k}(\mathbb{Z}) \), which is invertible and of finite order. Hence the lemma follows. \( \square \)

From now on by Lemma 3.4 above we may and will assume that the exponent matrix \( A \) of \( \phi \) is invertible and of finite order. We denote by \( r \) the order of \( A \) and assume further
\[ \text{(C1)} \quad \mathbb{Q}(q) \subseteq R, \]
where \( \mathbb{Q}(q) \) is the field of fractions of \( \mathbb{Z}[q] \subseteq R \).

We also need to fix the following notations:
\begin{align*}
\text{(3.2)} & \quad \bar{A}_n := I_n + A + \cdots + A^{m-1} \quad \text{(for all } m \geq 1) \quad \text{and} \quad \bar{A} := I_n + A + \cdots + A^{r-1} \quad \text{for all } r \geq 1 \quad \text{and all } m \geq 1 \quad \text{in} \quad R. \\
\text{(3.3)} & \quad \bar{A}_m := I_n + A + \cdots + A^m \quad \text{(for all } m \geq 1) \\
\text{(3.4)} & \quad W := \{ \alpha \in \mathbb{Z}^n \mid q^{\bar{A} \alpha} = 1 \}.
\end{align*}

**Lemma 3.5.** If the abelian subgroup \( W \) of \( \mathbb{Z}^n \) has rank \( n \) (e.g., when \( q^i = 1 \) for all \( 0 \leq i < n \) are all roots of unity). Then \( \operatorname{Im} (I - \phi) \) is a MS of \( A_n \).

**Proof:** Let \( H = \mathbb{Z}^n/W \). Then \( H \) is a finite abelian group. Let \( d = |H| \). Then \( d\alpha \in W \) for all \( \alpha \in \mathbb{Z}^n \). Let \( s = rd \). Then \( A^s = I_n \) and \( \bar{A}_s = d\bar{A} \). By Eqs. (2.3) and (3.2)-(3.4) we have for all \( \alpha \in \mathbb{Z}^n \)
\[ \phi^s(x^\alpha) = q^{\bar{A} \alpha \cdot x^\alpha} = q^{d\bar{A} \alpha \cdot x^\alpha}. \]
Therefore \( \phi^s = I \), and by [Z4, Corollary 5.5] the lemma follows. \( \square \)

Before we show the next lemma, let us recall the following remarkable Duistermaat-van der Kallen Theorem [DK].

**Theorem 3.6.** Let \( K \) be a field of characteristic zero, \( z = (z_1, \ldots, z_n) \) commutative free variables and \( M \) the \( K \)-subspace of \( K[z^{-1}, z] \) of the Laurent polynomials with no constant term. Then for each nonzero \( f \in K[z^{-1}, z] \) such that \( f^m \in M \) for all \( m \geq 1 \) we have \( 0 \not\in \text{Poly } f \). Consequently, \( M \) is a MS of \( K[z^{-1}, z] \).

**Lemma 3.7.** Assume \( W = 0 \) (e.g., when \( q^\alpha \neq 1 \) for all \( 0 \neq \alpha \in \mathbb{Z}^n \)). Then

\[ \text{without the condition (C1) the LFED conjecture 1.2 may not be true for } I - \phi, \]
e.g., see [Z4, Example 2.8].
1) \( \text{Im}(I - \phi) \) is the \( R \)-subspace spanned by all Laurent polynomial without constant terms.

2) \( \text{Im}(I - \phi) \) is a MS of \( A_n \).

**Proof:** 1) For all \( \alpha \in \mathbb{Z}^n \), by Eq. (2.3) we have \( \phi^r(x^\alpha) = q^{\hat{\alpha}} x^\alpha \), whence \((I - \phi^r)(x^\alpha) = (1 - q^{\hat{\alpha}}) x^\alpha \). Since \( W = 0 \), \( q^{\hat{\alpha}} \neq 1 \) for all \( 0 \neq \alpha \in \mathbb{Z}^n \). Then by Condition (C1), \( x^\alpha \in \text{Im}(I - \phi^r) \) for all \( 0 \neq \alpha \in \mathbb{Z}^n \).

Furthermore, since \((I - \phi^r) = (I - \hat{\phi}) \sum_{i=1}^{r-1} \phi^i \), we have \( \text{Im}(I - \phi^r) \subseteq \text{Im}(I - \hat{\phi}) \). Hence \( x^\alpha \in \text{Im}(I - \hat{\phi}) \) for all \( 0 \neq \alpha \in \mathbb{Z}^n \). Since \( \text{Im}(I - \phi) \) is a homogeneous \( R \)-subspace of \( A_n \) and \( 1 \notin \text{Im}(I - \hat{\phi}) \), the statement follows.

2) Let \( K_R \) be the field of fractions of \( R \) and \( f \in A_n \) such that \( f^m \in \text{Im}(I - \phi) \) for all \( m \geq 1 \). Viewing \( f \) as an element of \( K_R[x^{-1}, x] \) and by Theorem 3.6 we get \( 0 \notin \text{Poly} f \). Hence for all \( g \in A_n \) we have \( 0 \notin \text{Poly}(f^m g) \) when \( m \gg 0 \). Therefore \( \text{Im}(I - \phi) \) is a MS of \( A_n \). \( \Box \)

From now on we assume \( W \neq 0 \). By a well-known fact (e.g., see [H, Theorem 1.6]) on subgroups of free abelian groups there exists a basis \((\alpha_1, \alpha_2, \ldots, \alpha_n)\) of \( \mathbb{Z}^n \) such that \( W \) is freely spanned by \((d_1\alpha_1, d_2\alpha_2, \ldots, d_k\alpha_n)\) for some \( 1 \leq k \leq n \) and positive integers \( d_i \) with \( 1 \leq d_2 \leq d_3 \cdots \leq d_k \).

By conjugating an automorphism of \( A_n \) to \( \phi \) we may further assume that the basis \((\alpha_1, \alpha_2, \ldots, \alpha_n)\) is the standard basis \((e_1, e_2, \ldots, e_n)\) of \( \mathbb{Z}^n \). Furthermore, we also fix the following notations for the rest of this section:

i) Denote by \( E_1 \) (resp., \( E_2 \)) the subgroup of \( \mathbb{Z}^n \) generated by \( e_i \) with \( 1 \leq i \leq k \) (resp., \( k + 1 \leq i \leq n \)).

ii) Set \( \ell = n - k; y = (y_1, y_2, \ldots, y_k) \) with \( y_i = x_i \) (\( 1 \leq i \leq k \)); and \( z = (z_1, z_2, \ldots, z_\ell) \) with \( z_j = x_{k+j} \) (\( 1 \leq j \leq \ell \)).

iii) Set \( \mathcal{B}_1 := R[y_1, y_2, \ldots, y_k] \) and \( \mathcal{B}_2 := R[z_1, z_2, \ldots, z_\ell] \).

iv) Set \( \phi_1 := \phi|_{\mathcal{B}_1} \) and \( \phi_2 := \phi|_{\mathcal{B}_2} \).

**Lemma 3.8.** With the notations fixed above the following statements hold:

1) for all \( \alpha \in \mathbb{Z}^n \), \( q^{\hat{\alpha}} \) is a root of unity, if and only if \( \alpha \in E_1 \).

2) there exist \( A_1 \in M_k(\mathbb{Z}) \), \( A_2 \in M_{n-k}(\mathbb{Z}) \) and a \( k \times (n-k) \) matrix \( B \) with integer entries such that

\[
A_1' = I_k, \quad A_2' = I_{n-k}, \quad A = \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix}
\]

3) \( \phi(\mathcal{B}_1) \subseteq \mathcal{B}_1 \), and \( \mathcal{B}_1 \cap \text{Im}(I - \phi) = \text{Im}(I_{\mathcal{B}_1} - \phi_1) \)
Proof: 1) ($\Rightarrow$) Since $\alpha \in E_1$, $d_k\alpha \in W$. Then $1 = q^{\tilde{A}(d_k\alpha)} = q^{d_k\tilde{A}\alpha} = (q^{\tilde{A}\alpha})^{d_k}$, whence $q^{\tilde{A}\alpha}$ is a root of unity.

To show the ($\Leftarrow$) part, since $\mathbb{Z}^n = E_1 \oplus E_2$, it suffices to show that for any $0 \neq \beta \in E_2$, $q^{\tilde{A}\beta}$ is not a root of unity.

Assume otherwise, say, $1 = (q^{\tilde{A}\beta})^j = q^{\tilde{A}(j\beta)}$ for some $j \geq 1$. Then $j\beta \in W \subseteq E_1$. Hence $j\beta \in E_1 \cap E_2 = 0$ and $\beta = 0$. Contradiction.

2) Since $A^r = I_n$, we have $\tilde{A}A = A\tilde{A} = \tilde{A}$. For each $1 \leq i \leq k$, since $d_ie_i \in W$, we have

$$q^{\tilde{A}(d_ie_i)} = q^{\tilde{A}A(d_ie_i)} = q^{\tilde{A}(d_ie_i)} = 1.$$ 

Therefore $d_i(Ae_i) \in W$, whence $Ae_i \in E_1$, whence $AE_1 \subseteq E_1$. Consequently, $A$ has the form as the matrix in Eq. (3.5). The first two equations in Eq. (3.5) follow immediately from the last one and the fact $A^r = I_n$.

3) By statement 2) we immediately have $\phi(B_1) \subseteq B_1$ and $(I - \phi)(B_1) \subseteq B_1$. Hence $\text{Im}(I_{B_1} - \phi_1) \subseteq B_1 \cap \text{Im}(I - \phi)$. Conversely, let $b(y) \in B_1 \cap \text{Im}(I - \phi)$. Then there exists $f(x) \in A_n$ such that $(I - \phi)(f) = b(y)$.

Write $f(x) = a_0(y) + \sum_{\alpha \neq \beta \in E_2} a_{\alpha}(y)z^\beta$ with $a_{\alpha}(y) \in B_1$ ($\beta \in E_2$). Note that by statement 2) the projection of $A\beta$ on $E_2$ for each nonzero $\beta \in E_2$ is also nonzero, since $A_2$ in Eq. (3.3) is invertible. Hence $(I - \phi)$ preserves the $R$-subspace $\sum_{0 \neq \beta \in E_2} z^\beta B_1$, by which the term of $(I - \phi)(f)$ that lies in $B_1$ is equal to $(I - \phi)(a_0)$, whence $(I - \phi)(a_0) = b(y)$. Therefore $b(y) \in \text{Im}(I_{B_1} - \phi_1)$, and the statement follows. $\square$

**Lemma 3.9.** Set $s := rd_k$, $\tilde{q}_i := q^{\tilde{A}e_{i+k}}$ for all $1 \leq i \leq \ell = n - k$, and $\tilde{q} := (\tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_\ell)$. Then the following statements hold:

1. $\phi^s(z^\beta) = \tilde{q}^\beta z^\beta$ for all $\beta \in E_2$. In particular, $\phi^s(B_2) \subseteq B_2$.
2. For all $0 \neq \beta \in E_2$, $\tilde{q}^\beta$ is not a root of unity.
3. $\phi^s|_{B_1} = I_{B_1}$.

**Proof:** 1) Since $\rho \mid s$, we have $A^s = I_n$ and $\tilde{A}_s = d_k\tilde{A}$. Then by Eq. (2.3) we have for all $\alpha \in \mathbb{Z}^n$

$$\phi^s(x^\alpha) = q^{\tilde{A}_s x^\alpha} = q^{d_k\tilde{A}\alpha} x^\alpha,$$

from which by letting $\alpha = \beta \in E_2$ statement 1) follows.

2) Assume otherwise, then $q^{\tilde{A}\beta}$ is also a root of unity, since $\tilde{q}^\beta = q^{\tilde{A}_s\beta} = q^{d_k\tilde{A}\beta} = (q^{\tilde{A}\beta})^{d_k}$. By Lemma 3.8 1) we have $\beta \in E_1 \cap E_2$, whence $\beta = 0$. Contradiction.

3) By Lemma 3.8 2) we have $AE_1 \subseteq E_1$, whence $\tilde{A}_sE_1 \subseteq E_1$. Since $d_kE_1 \subseteq W$, by Eq. (3.6) the statement follows. $\square$
Now we give a complete description for $\text{Im} (I - \phi)$.

**Theorem 3.10.** $\text{Im} (I - \phi)$ is the $R$-subspace of $A_n$ consisting of all $f(x) \in A_n$ of the form

$$f(x) = a_0(y) + \sum_{0 \neq \beta \in E_2} a_{\beta}(y) z^{\beta}$$

with $a_{\beta}(y) \in B_1 \ (0 \neq \beta \in E_2)$ and $a_0(y) \in \text{Im} (I_{B_1} - \phi_1)$.

**Proof:** Let $s = rd_k$ and $q$ as in Lemma 3.9 by which we have $\phi^s |_{B_1} = I_{B_1}$. So we may view $\phi^s$ as a $B_1$-algebra endomorphism of $A_n = B_1[z^{-1}, z]$ such that $\phi^s(z^\beta) = q^\beta z^\beta$ for all $\beta \in E_2$. Since $q^\beta$ is not a root of unity for all $0 \neq \beta \in E_2$ and $Q[\tilde{q}] \subseteq Q[q] \subseteq R \subseteq B_1$, by Lemma 3.7 we have that $z^\beta B_1 \subseteq \text{Im} (I - \phi^s)$ for all $0 \neq \beta \in E_2$. Since $(1 - t) | (1 - t^k)$, we have $\text{Im} (I - \phi^s) \subseteq \text{Im} (I - \phi)$, whence $z^\beta B_1 \subseteq \text{Im} (I - \phi)$ for all $0 \neq \beta \in E_2$.

Now let $f(x) \in A_n$ and write $f(x) = a_0(y) + \sum_{0 \neq \beta \in E_2} a_{\beta}(y) z^{\beta}$ with $a_{\beta}(y) \in B_1 \ (\beta \in E_2)$. Then from the fact shown above we have that $f \in \text{Im} (I - \phi)$, if and only if $a_0(y) \in \text{Im} (I - \phi)$. Combining with Lemma 3.8 we further have that $f \in \text{Im} (I - \phi)$, iff $a_0(y) \in \text{Im} (I - \phi)(B_1) = \text{Im} (I_{B_1} - \phi_1)$. Hence the theorem follows. \hfill $\square$

One remark on Theorem above and the LFED conjecture [1.2] for $A_n$ is as follows.

Since $q_i \ (1 \leq i \leq k)$ are all roots of unity, by Lemma 3.5 we see that $\text{Im} (I_{B_1} - \phi_1)$ is a MS of $B_1$. Another way to see this fact is to apply Lemma 3.9 and [Z4] Corollary 5.5. Then by Proposition 2.3 and Theorem 3.10 we see that when $R$ is a field $K$ of characteristic zero, the LFED conjecture [1.2] for $A_n$ follows from the following conjecture, which is the case of [Z4] Conjecture 4.4 for commutative $K$-algebras.

**Conjecture 3.11.** Let $K$ be a field of characteristic zero and $z = (z_1, z_2, \ldots, z_n)$ $n$ commutative free variables. Let $A$ be a commutative $K$-algebra and $V$ a $K$-subspace of $A$. Set $\tilde{V}$ to be the $K$-subspace of the Laurent polynomial algebra $A[z^{-1}, z]$ consisting of the Laurent polynomials with the constant term in $V$. Then $\tilde{V}$ is a MS of $A[z^{-1}, z]$, if (and only if) $V$ is a MS of $A$.

Next, we use Theorem 3.10 to show two special cases of the LFED conjecture [1.2] for $A_n$.

**Proposition 3.12.** In terms of notation as in Lemma 3.8 assume $A_1 = \pm I_k$. Given $f \in A_n$ such that $f^m \in \text{Im} (I - \phi)$ for all $m \geq 1$ we have $0 \notin \text{Poly}_z f$, where $\text{Poly}_z f$ is the polytope of $f(x)(= f(y, z))$ as a
Laurent polynomial in \( z \) over \( \mathcal{B}_1 \). In particular, \( \text{Im} (I - \phi) \) is a MS of \( A_n \).

Proof: Note that the second statement follows from the first one, by a similar argument as in the proof of Lemma \ref{lem:2.7} 2).

To show the first statement, we assume otherwise and fix a nonzero \( f(x) \in A_n \) such that \( 0 \in \text{Poly}_z f(y, z) \) but \( f^m \in \text{Im} (I - \phi) \) for all \( m \geq 1 \). We drive a contradiction as follows.

First, for all \( a(y) = \sum_{\alpha \in E_1} c_\alpha y^\alpha \in \mathcal{B}_1 \) the following statements can be readily verified:

\( i \) if \( A_1 = I_k \), then \( a(y) \in \text{Im} (I_{\mathcal{B}_1} - \phi_1) \), if and only if \( c_\alpha = 0 \) for all \( \alpha \in E_1 \) such that \( q^\alpha = 1 \);

\( ii \) if \( A_1 = -I_k \), then \( \phi^2 = \phi \), and by \cite{Z4} Proposition 5.2] we have \( \text{Im} (I_{\mathcal{B}_1} - \phi_1) = \text{Ker} (I_{\mathcal{B}_1} + \phi_1) \). Then it is easy to see that \( a(y) \in \text{Im} (I_{\mathcal{B}_1} - \phi_1) \), if and only if \( c_{-\alpha} = -q^\alpha c_\alpha \) for all \( \alpha \in E_1 \).

Then by Theorem \ref{thm:3.10} and the properties \( i \) and \( ii \) above we have

\( iii \) if \( A_1 = \pm I \), then for a Laurent polynomial \( h(x) \in A_n \) we have that \( h^m \in \text{Im} (I - \phi) \) for all \( m \geq 1 \), if and only if \( q_i \) (1 \( i \leq n \)) and the coefficients of \( h(x) \) satisfy a certain sequence of polynomial equations over \( \mathbb{Z} \).

Let \( K \) be the field of fractions of the subring of \( R \) generated by \( q_i \) (1 \( i \leq n \)) and the coefficients of \( f(x) \). Then by Property \( iii \) above and a reduction similar as the one in Section 4 in \cite{FPYZ} we may assume that \( K \) is a subfield of the algebraic closure \( \overline{\mathbb{Q}} \) of \( \mathbb{Q} \). Therefore, for each prime \( p \geq 2 \) the \( p \)-valuation \( \nu_p \) of \( \mathbb{Q} \) can be extended to \( K \), which we will still denote by \( \nu_p \).

Applying Theorems \ref{thm:3.6} and \ref{thm:3.10} we may assume

\begin{align}
(3.8) \quad f(x) &= a_0(y) + \sum_{0 \neq \beta \in E_2} a_\beta(y) z^\beta, \\
(3.9) \quad a_0(y) &= \sum_{\alpha \in E_1} c_\alpha y^\alpha
\end{align}

such that \( a_0(y) \neq 0 \) and \( a_0(y) \in \text{Im} (I_{\mathcal{B}_1} - \phi_1) \).

Note that by Properties \( i \) and \( ii \) \( a_0(y) \) can not be a constant in \( K \), so we can fix a nonzero extremal point \( \lambda \in E_1 \) of \( \text{Poly}_y a_0(y) \). Let \( d \) be an even positive integer such that for all \( m \geq 1 \)

\begin{align}
(3.10) \quad q^{dm\lambda} &= 1, \\
(3.11) \quad c_{-\lambda}^{md} &\neq -q^{md} c_\lambda^{md}.
\end{align}
Note that such an integer $d$ does exist. This is because that, first, $q_i \ (1 \leq i \leq k)$ by Lemma 3.8) 1) are all roots of unity and, second, $c_\lambda \neq 0$ and, third, by Property ii) above $c_{-\lambda} = -q^\lambda c_\lambda$.

Therefore, by Properties i) and ii) above we have for all $m \geq 1$
\begin{equation}
(3.12)
\nonumber
a_0(y)^{md} \not\in \operatorname{Im}(I_{\mathbb{B}_1} - \phi_1).
\end{equation}

By Dirichlet’s prime number theorem there exist infinitely many $m \geq 1$ such that $p := md - 1$ is prime. Furthermore, it is well-known in Algebraic Number Theory (e.g., see [W, Theorem 4.1.7]) that for all but finitely such primes $p$, the values of $\nu_p$ at $q_i \ (1 \leq i \leq k)$ and the nonzero coefficients of $f(x)$ are equal to 0. For all primes $p$ with these properties we consider
\begin{equation}
(3.13)
\nonumber
f^p(x) \equiv a_0^p(y) + \sum_{\beta \in E_2} a_\beta^p(y) z^\beta \mod S_p[x^{-1}, x],
\end{equation}
where $S_p$ is the subring of $K$ formed by the elements $u \in K$ such that $\nu_p(u) \geq 1$.

Choosing $p$ large enough we assume $p\beta + \gamma \neq 0$ for all $0 \neq \beta, \gamma \in \operatorname{Poly}_z f(y, z)$. Then by Eq. (3.13) the constant term of $f^{p+1}$ (viewed as a Laurent polynomial in $B_1[z^{-1}, z]$) is equal to $a_0^{p+1}$ modular $S_p[y^{-1}, y]$.

By the choice of $p$ we have $p + 1 = md$. Since $f^{p+1}(x) = f^{md}(x) \in \operatorname{Im}(I_\phi)$, by Eq. (3.12) and Theorem 3.10 there exists a nonzero $b_{p+1}(x) \in S_p[x, x^{-1}]$ such that
\begin{equation}
(3.14)
\nonumber
a_0^{p+1}(x) + b_{p+1}(x) \in \operatorname{Im}(I_{\mathbb{B}_1} - \phi_1).
\end{equation}

Since $\lambda$ is an extremal point of $\operatorname{Poly}_y a_0(y)$, we have that $(p+1)\lambda$ is an extremal point of $\operatorname{Poly}_y a_0^{p+1}(y)$, and the coefficient of $x^{(p+1)\lambda}$ in $a_0^{p+1}(y)$ is equal to $c_\lambda^{p+1}$. Then by Eqs. (3.10), (3.11), (3.14), and Property i), ii) above we have $c_\lambda^{p+1} + u_{p+1} = 0$, where $u_{p+1}$ is the coefficient of $x^{(p+1)\lambda}$ in $b_{p+1}(x)$. Hence $0 \neq c_\lambda^{p+1} = -u_{p+1} \in S_p$, which implies $\nu_p(c_\lambda^{p+1}) \geq 1$. Therefore we also have $\nu_p(c_\lambda) \geq 1$. But this contradicts to the choice of $p$ with $\nu_p(c_\lambda) = 0$. \hfill \Box

It is worthy to point out explicitly the following two special cases of the proposition above.

**Corollary 3.13.** Assume $A = \pm I$, i.e., $\phi$ is the $R$-algebra automorphism of $A$ that maps $x_i$ to $q_i x_i$ for all $1 \leq i \leq n$, or $\phi$ maps $x_i$ to $q_i x_i^{-1}$ for all $1 \leq i \leq n$. Then $\operatorname{Im}(I_\phi)$ is a MS of $A$.

Next, we show the LFEN conjecture (and also the LNED conjecture) for the Laurent polynomial algebras over a field of characteristic zero in one or two commutative free variables.
Theorem 3.14. Let $K$ be a field of char zero and $A$ the Laurent polynomial algebra over $K$ in one or two commutative variables. Then both the LFED Conjecture 1.2 and LNED Conjecture 1.3 holds for $A$.

Proof: By Theorem 3.1 we only need to show the $K$-$\mathcal{E}$-derivation case for the LFED Conjecture 1.3. Furthermore, the one variable case follows from Corollary 3.13 above. So we may assume that $A = A_2$ with $R = K$, i.e., $A$ is the Laurent polynomial algebra over $K$ in two variables.

Let $\phi$ be a LF $K$-algebra endomorphism of $A_2$. We use the same notations as before, in particular, let $A$ be the exponent matrix of $\phi$. If the rank of $A$ is less than 2, then by Lemmas 3.3, 3.4 and Corollary 3.13 it is easy to see that $\text{Im} (I - \phi)$ also is a MS of $A_2$.

So we assume that $A$ is of rank 2. Then the case $k = 2$, i.e., $E_2 = \mathbb{Z}^2$, follows from Lemma 3.5; the case $k = 1$ follows from Proposition 3.12; and the case $k = 0$ follows from Lemma 3.7. $\square$

We end this section with the following corollary of the proof of Theorem 3.10, which can be proved by going through the proof of Theorem 3.10 with some slight modifications for the polynomial algebra $R[x]$ instead of $A_n = R[x^{-1}, x]$.

Corollary 3.15. Let $\phi$ be as in Theorem 3.10 and assume further that all entries of the exponent matrix $A$ of $\phi$ lie in $\mathbb{N}$. Then the following statements hold:

1) $\phi$ preserves the polynomial algebra $R[x]$ and its restriction $\psi := \phi |_{R[x]}$ is a locally finite $R$-derivation of $R[x]$.

2) $\text{Im} (I_{R[x]} - \psi)$ consists of all $f \in R[x]$ of the form

$$
(3.15) f(x) = a_0(y) + \sum_{0 \neq \beta \in E_2 \cap \mathbb{N}^n} a_\beta(y) z^\beta
$$

with $a_\beta(y) \in R[y]$ ($\beta \in E_2 \cap \mathbb{N}^n$) and $a_0(y) \in \text{Im} (I_{R[y]} - \phi R[y])$.

In particular, $\text{Im} (I_{R[x]} - \psi)$ is a MS of $R[x]$, i.e., the LFED conjecture 1.2 holds for the $R$-$\mathcal{E}$-derivation $\psi$ of $R[x]$.

References

[BCW] H. Bass, E. Connell and D. Wright, The Jacobian Conjecture, Reduction of Degree and Formal Expansion of the Inverse. Bull. Amer. Math. Soc. 7, (1982), 287–330.

[B] J. Bergen, Derivations in Prime Rings. Canad. Math. Bull. 26 (1983), 267-270.

[BFF] M. Brešar, A. Fošner and M. Fošner, A Kleinecke-Shirokov Type Condition with Jordan Automorphisms. Studia Math. 147 (2001), no. 3, 237-242.
[BV] M. Brešar and AR Villena, *The Noncommutative Singer-Wermer Conjecture and $\phi$-Derivations*. J. London Math. Soc. **66** (2002), 710-720.

[DEZ] H. Derksen, A. van den Essen and W. Zhao, *The Gaussian Moments Conjecture and the Jacobian Conjecture*. To appear in *Israel J. Math.*. See also arXiv:1506.05192 [math.AC].

[DK] J. J. Duistermaat and W. van der Kallen, *Constant Terms in Powers of a Laurent Polynomial*. Indag. Math. (N.S.) **9** (1998), no. 2, 221–231. [MR1691479].

[E1] A. van den Essen, *The Exponential Conjecture and the Nilpotency Subgroup of the Automorphism Group of a Polynomial Ring*. Prepublications. Univ. Autónoma de Barcelona, April 1998.

[E2] A. van den Essen, *Polynomial Automorphisms and the Jacobian Conjecture*. Prog. Math., Vol.190, Birkhäuser Verlag, Basel, 2000.

[E3] A. van den Essen, *Introduction to Mathieu Subspaces*. “International Short-School/Conference on Affine Algebraic Geometry and the Jacobian Conjecture” at Chern Institute of Mathematics, Nankai University, Tianjin, China. July 14-25, 2014.

[EH] A. van den Essen and L. C. van Hove, *Mathieu-Zhao Spaces*. To appear.

[EN] A. van den Essen and S. Nieman, *Mathieu-Zhao Spaces of Univariate Polynomial Rings with Non-zero Strong Radical*. J. Pure Appl. Algebra, **220** (2016), no. 9, 3300–3306.

[H] T. W. Hungerford, *Algebra*. Graduate Texts in Mathematics 73. Springer-Verlag, 1974.

[J] N. Jacobson, *Structure of Rings*. Amer. Math. Soc. Coll. Pub. **37**, Amer. Math. Soc. Providence R. I., 1956.

[EWZ] A. van den Essen, D. Wright and W. Zhao, *Images of Locally Finite Derivations of Polynomial Algebras in Two Variables*. J. Pure Appl. Algebra **215** (2011), no.9, 2130-2134. [MR2786603]. See also arXiv:1004.0521 [math.AC].

[FPYZ] J. P. Francoise, F. Pakovich, Y. Yomdin and W. Zhao, *Moment Vanishing Problem and Positivity: Some Examples*. Bull. Sci. Math., **135** (2011), no. 1, 10–32.

[Ke] O. H. Keller, *Ganze Gremona-Transformationen*. Monats. Math. Physik **47** (1939), no. 1, 299-306.

[M] O. Mathieu, *Some Conjectures about Invariant Theory and Their Applications*. Algèbre non commutative, groupes quantiques et invariants (Reims, 1995), 263–279, Sémin. Congr., 2, Soc. Math. France, Paris, 1997.

[W] E. Weiss, *Algebraic Number Theory*. McGrawHill Book Co., Inc., 1963. [MR0159805].

[Z1] W. Zhao, *Images of Commuting Differential Operators of Order One with Constant Leading Coefficients*. J. Alg. **324** (2010), no. 2, 231–247. [MR2651354]. See also arXiv:0902.0210 [math.CV].

[Z2] W. Zhao, *Generalizations of the Image Conjecture and the Mathieu Conjecture*. J. Pure Appl. Alg. **214** (2010), 1200–1216. See also arXiv:0902.0212 [math.CV].

[Z3] W. Zhao, *Mathieu Subspaces of Associative Algebras*. J. Alg. **350** (2012), no.2, 245-272. [MR2859886]. See also arXiv:1005.4260 [math.RA].

[Z4] W. Zhao, *Some Open Problems on Locally Finite or Locally Nilpotent Derivations and $E$-Derivations*. Preprint.
[Z5] W. Zhao, *Idempotents in Intersection of the Kernel and the Image of Locally Finite Derivations and E-derivations*. Preprint.

[Z6] W. Zhao, *The LFED and LNED Conjectures for Algebraic Algebras*. Preprint.

[Z7] W. Zhao, *Images of Ideals under Derivations and E-Derivations of Univariate Polynomial Algebras over a Field of Characteristic Zero*. Preprint.

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