Compatible Complex Structures on Twistor Spaces

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Let \((M,g)\) be a Riemannian 4-manifold. The twistor space \(Z \to M\) is a \(\mathbb{CP}^1\)-bundle whose total space \(Z\) admits a natural metric \(\tilde{g}\). The aim of this article is to study properties of complex structures on \((Z,\tilde{g})\) which are compatible with the \(\mathbb{CP}^1\)-fibration and the metric \(\tilde{g}\). The results obtained enable us to translate some metric properties on \(M\) in terms of complex properties on its twistor space \(Z\).

Introduction

Let \((M,g)\) be an oriented 4-dimensional Riemannian manifold (not necessarily compact). Due to the Hodge-star operator \(\star\), we have a decomposition of the bivector bundle \(\bigwedge^2 TM = \bigwedge^+ \oplus \bigwedge^-\). Here \(\bigwedge^\pm\) is the eigen-subbundle for the eigenvalue \(\pm 1\) of \(\star\). The metric \(g\) on \(M\) induces a metric, denoted by \(<,>\), on the bundle \(\bigwedge^2 TM\). Let \(\pi: Z = S(\bigwedge^+) \to M\) be the sphere bundle; the fiber over a point \(m \in M\) parameterizes the complex structures on the tangent space \(T_m M\) compatible with the orientation and the metric \(g\). It is the twistor space of the manifold \((M,g)\). Since the structural group of \(Z\) is \(SO(3) \subset \text{Aut}(\mathbb{CP}^1)\), we can thus put the complex structure of \(\mathbb{CP}^1\) on each fiber. On the other hand, the Levi-Civita connection on \((M,g)\) induces a splitting of the tangent bundle \(TZ\) into the direct sum of the horizontal and vertical distributions: \(TZ = H \oplus V\). Therefore, the twistor space \(Z\) admits a natural metric \(\tilde{g}\) defined by its restrictions to \(H\) and \(V\): we endow \(V\) with the Fubini-Study metric and \(H \simeq \pi^* TM\) with the pullback of the metric \(g\).

In this article we study some aspects of almost complex structures on \((Z,\tilde{g})\) which are Hermitian and extend the complex structure of the fibers. These structures will be called compatible almost complex structures on \((Z,\tilde{g})\). In particular, the integrability of two such structures means that the metric \(\tilde{g}\) is bihermitian \([\text{Pon97}], [\text{AGG99}]\).

To each morphism respecting the twistor fibration

\[
\begin{array}{ccc}
Z & \xrightarrow{f} & Z \\
\downarrow \pi & & \downarrow \pi \\
M & & \\
\end{array}
\]

we associate a compatible almost complex structure \(\tilde{J}_f\) on \((Z,\tilde{g})\) in the following way. Let \(z \in Z\) with \(\pi(z) = m \in M\), and write \(T_z Z = H_z \oplus V_z\). Here, \(V_z\) is the tangent space to the fiber \(\pi^{-1}(m) \simeq \mathbb{CP}^1\) and is therefore equipped with a complex structure. On the other hand, we endow \(H_z \simeq T_m M\) with the complex structure associated to the point \(f(z)\). Conversely, any compatible almost complex structure
\[ \mathbb{J} \text{ on } (Z, \tilde{g}) \text{ defines a unique morphism } f : Z \rightarrow Z \text{ respecting the fibration such that } J_f = \mathbb{J}. \]

The almost complex structure \( J_{Id} \) associated to the identity is the canonical twistor almost complex structure [AHS78]. If \( \sigma \) is the morphism of \( Z \) whose restriction to each fiber of \( \pi \) is the antipodal map of \( \mathbb{S}^2 \), we denote by \( J_{\sigma} \) the almost complex structure associated to \( \sigma \). Now, an almost complex manifold \((M, g, J_M)\) such that \( J_M \) is compatible with the orientation and the metric \( g \) defines a tautological section of \( Z \rightarrow M \). This section can be taken as the infinity section and we can therefore consider the constant morphism \( f = \infty \). The associated almost complex structure will be denoted by \( J_{\infty} \). Let \( \lambda \in \mathbb{C}^* \) and consider the morphism \( f = \lambda Id \) acting as \( \lambda Id \) in each fiber minus infinity (i.e. \( \mathbb{C}P^1 - \{ \infty \} \simeq \mathbb{C} \)) and preserving infinity. We denote by \( J_{\lambda Id} \) the corresponding almost complex structure on \( Z \).

The integrability of the structures \( J_{Id}, J_{\sigma}, J_{\infty}, J_{\lambda Id} \) are related to the curvature of the metric \( g \) on \( M \). Let \( R : \wedge^2 TM \rightarrow \wedge^2 TM \) be the curvature operator. The decomposition \( \wedge^2 TM = \wedge^+ \oplus \wedge^- \) allows us to write \( R \) in block matrix form as follows
\[
R = \begin{pmatrix}
A & tB \\
B & C
\end{pmatrix},
\]
where \( A = W^+ + \frac{1}{12} Id, C = W^- + \frac{1}{12} Id \), \( W^+ \) (resp. \( W^- \)) is the selfdual (resp. anti-selfdual) Weyl tensor, \( s \) is the scalar curvature and \( B \) the trace-free Ricci curvature [Bes87].

The main result of this article is the following:

**Theorem 1.** Let \((M, g)\) be an oriented Riemannian 4-manifold.

A) The complex structure \( \mathbb{J}_{\sigma} \) is never integrable.

B) The complex structure \( \mathbb{J}_{Id} \) is integrable if, and only if, \( g \) anti-selfdual (i.e. \( A \) is a homothety) [AHS78].

C) Let \( J_M \) be an almost complex structure on \( M \) compatible with the metric \( g \) and the orientation. The complex structure \( \mathbb{J}_{\infty} \) is integrable if, and only if:
   i) \( J_M \) is integrable;
   ii) the kernel of \( A \) contains the plane \( J_M^+ \subset \wedge^+ \) orthogonal to the line generated by \( J_M \).

D) Let \((M, g, J_M)\) be a Kählerian manifold. If \( \lambda \notin \{0, 1\} \), the complex structure \( \mathbb{J}_{\lambda Id} \) is integrable if, and only if, \((M, g, J_M)\) is scalar-flat Kähler (i.e. \( A = 0 \)).

E) Let \((M, g)\) be an anti-selfdual Riemannian manifold. Its scalar curvature is zero if, and only if, any \( m \in M \) has an open neighborhood \( U \) such that, over \( U \), \((Z, \tilde{g})\) admits a compatible complex structure different from \( \mathbb{J}_{Id} \).

The conditions i) & ii) of part C in the previous theorem are satisfied as soon as \((M, g, J_M)\) is Kähler. We show in section C that this Kählerian property is equivalent to the integrability of \( \mathbb{J}_{\infty} \) in the compact case. For a scalar-flat Kähler manifold \((M, g, J_M)\), the complex structures \( \mathbb{J}_{Id} \) [Gau81], \( \mathbb{J}_{\infty} \) and \( \mathbb{J}_{\lambda Id} \) are integrable and compatible with the metric \( \tilde{g} \) on \( Z \). This gives us a huge family of real 6-dimensional manifolds admitting a bihermitian metric.

Recall that the Penrose correspondence gives a dictionary between holomorphic properties of the twistor space \( Z \) and properties of the Riemannian manifold \((M, g)\). The above result can be viewed as a new paragraph of that dictionary. In particular, we deduce from it some new characterizations of Kählerian metrics, anti-selfdual scalar-flat metrics and scalar-flat Kähler metrics, in terms of twistor spaces.
The proof of Theorem 1 is split into five theorems, Theorem A,..., E, the proof of each being given in the corresponding labelled section.

In section D we explain how Theorem 1 can be used in order to build a 1-dimensional family of non conformal anti-selfdual metrics on a scalar-flat Kähler manifold, or a 1-dimensional family of biholomorphisms on its corresponding complex twistor space \((Z, J_M)\).

In section F we study more precisely the set of all compatible complex structures on the twistor space of a locally conformally Kähler manifold. Whereas on section G we will study the case of bielliptic surfaces.

We conclude the paper by giving a generalisation of this theorem to quaternionic Kähler manifolds of dimension \(4n\) for \(n > 1\).

**Notation**

We will use Einstein summation convention over repeated indices. The fiber of \(\pi : Z \rightarrow M\) over \(m \in M\) will be freely identified with \(\mathbb{S}^2\), \(\mathbb{CP}^1\) or \(SO(4)/U(2)\), the set of all complex structure on \(T_m M\). The bundle of bivectors \(\wedge^2 TM\) will be identified with the bundle of skew-symmetric endomorphisms of \(TM\), or to the bundle of 2-forms.

Let \((\theta_1, \theta_2, \theta_3, \theta_4)\) be an oriented \(g\)-orthonormal frame defined over an open set \(U\) of \((M, g)\). Define three linear operators \(I, J, K \in \text{End}(TM)\), over \(U\), by their matrix in the basis \((\theta_1, \ldots, \theta_4)\):

\[
I = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.
\]

Then, \((I, J, K)\) gives an oriented orthonormal basis over \(U\) of \(\wedge^+\) and therefore defines a trivialization of the twistor space \(\pi : Z \rightarrow M\) over \(U\):

\[
\pi^{-1}(U) \simeq U \times SO(4)/U(2).
\]

Let \((\theta_1^*, \ldots, \theta_4^*)\) be the local coframe dual to \((\theta_1, \ldots, \theta_4)\). Locally, the covariant derivative \(\nabla\) (on \(M\)) defined by the Levi-Civita connection of the metric \(g\) writes \(\nabla \theta_j = \Gamma^i_{jk} \theta_j^* \otimes \theta_k\). The \(\Gamma^i_{jk}\) are the Christoffel symbols of the connection \(\nabla\); they satisfy \(\Gamma^i_{jk} = -\Gamma^j_{ik}\).

Let \(z \simeq (m, Q) \in \pi^{-1}(U)\) be a point of \(Z\) and write the tangent space as the direct sum of the horizontal and vertical tangent spaces: \(T_z Z = V_z \oplus H_z\). Denote by \(\hat{\theta} \in H_z \simeq T_m M\) the horizontal lift of \(\theta \in T_m M\). We then have [BN98]:

\[
\begin{cases}
V_z = \left\{ X \frac{\partial}{\partial Q} \mid X \in \text{End}(T_m M), \; ^tX = -X \text{ et } QX = -QX \right\} \\
H_z = \text{Vect} \left( \hat{\theta}_1(z), \ldots, \hat{\theta}_4(z) \right)
\end{cases}
\]

with

\[
\begin{align*}
\hat{\theta}_i(z) &= \theta_i(m) - [\Gamma^i_{jk}(m), Q] \frac{\partial}{\partial Q} \\
[\Gamma^i_{jk}(m), Q] \frac{\partial}{\partial Q} &= \left( \Gamma^i_{jk}(m) Q - Q \Gamma^j_{ik}(m) \right) \frac{\partial}{\partial Q} \in V_z.
\end{align*}
\]

**Remark:** The complex structure of rational curves on the fiber \(\pi^{-1}(m) \simeq \mathbb{S}^2\) at a point \(z = (m, Q)\) is given by the application [BN98]:

\[
V_z \simeq T_Q \mathbb{S}^2 \quad \rightarrow \quad V_z \simeq T_Q \mathbb{S}^2 \\
X \frac{\partial}{\partial Q} \quad \mapsto \quad QX \frac{\partial}{\partial Q}.
\]

For all \(A \in \text{so}(4) = \{ A \in \text{End}(TM) \mid ^tA = -A \}\) we can define the vertical vector field \(\hat{A} = [A, Q] \frac{\partial}{\partial Q}\). These vector fields will be called **basic**.
A) General results

In this section \((M,g)\) will be an oriented Riemannian 4-manifold. Results – and proofs – given here in dimension 4, can be easily adapted to quaternionic Kähler \(4n\)-manifolds and will be used in the last section of the paper.

To study the integrability of the almost complex structure \(J_f\) we need to compute the Nijenhuis tensor \(N\) of \(J_f\) [\(\text{NN57}\)]:

\[
N(X,Y) = [J_fX, J_fY] - J_f[J_fX,Y] - J_f[X,J_fY] - [X,Y] \quad \forall (X,Y) \in T_z \mathbb{Z}.
\]

The first necessary condition for the integrability of \(J_f\) appears in the next proposition.

**Proposition 1.** For any morphism \(f\) we have:

i) \(N(X,Y) = 0\) for all \(X,Y \in V_z\);

ii) let \(X, \theta \in V_z \times H_z\), then

- the vertical component of \(N(X,\theta)\) is zero
- the horizontal component of \(N(X,\theta)\) is zero if and only if the restriction of \(f\) to each fiber is holomorphic.

As \(\sigma\) is an anti-holomorphic involution on fibers we easily get:

**Theorem A.** The almost complex structure \(J_\sigma\) is never integrable.

**Proof of Proposition 1.** For any morphism \(f\), each fiber of \(\pi: \mathbb{Z} \longrightarrow M\) has the structure of \(\mathbb{CP}^1\). It follows immediately from [\(\text{NN57}\)] that \(N(X,Y) = 0\) for all \(X,Y \in V_z\).

Let \(\hat{X}\) be a basic vertical vector field and \(\pi^{-1}(m)\) be a fixed fiber. The restriction to that fiber of the application \(f\) is:

\[
f|_{\pi^{-1}(m)} : \mathbb{S}^2 \simeq \pi^{-1}(m) \quad \longrightarrow \quad \mathbb{S}^2 \simeq \pi^{-1}(m) \quad Q \quad \longrightarrow \quad f(Q)
\]

Observe that \([\hat{X},\hat{\theta}_i]\) is vertical when \(\hat{X}\) is. Since the action of the complex structure \(J_f\) on the fiber is equal to the rational curve structure, it does not depend on the fiber. We then have: \([J_f\hat{X},\hat{\theta}_i] = [Q\hat{X},\hat{\theta}_i] = Q[\hat{X},\hat{\theta}_i] = J_f[\hat{X},\hat{\theta}_i]\). This implies that, for \(i \in \{1, \ldots, 4\}\):

\[
N(\hat{X},\hat{\theta}_i) = [Q\hat{X}, f(Q)\hat{\theta}_i] - Q[Q\hat{X},\hat{\theta}_i] + J_f[\hat{X}, f(Q)\hat{\theta}_i] - [\hat{X}, \hat{\theta}_i]
\]

\[
= \left( (Q\hat{X}).f(Q) - f(Q)(\hat{X}.f(Q)) \right) \hat{\theta}_i = \left( df(Q) \hat{X} - f(Q)dfQf(\hat{X}) \right) \hat{\theta}_i
\]

where \(dfQf\) is the differential of \(f\) at \(Q \in \mathbb{S}^2\). The horizontal component of \(N(X,\theta)\) vanishes for all \((X,\theta) \in V_z \times H_z\) if and only if the restrictions of \(f\) to the fibers are holomorphic .

In the trivialization of \(\mathbb{Z} \longrightarrow M\) over an open set \(\mathcal{U}\), the morphism \(f\) can be written:

\[
f|_{\pi^{-1}(\mathcal{U})} : \mathcal{U} \times \mathbb{S}^2 \quad \longrightarrow \quad \mathcal{U} \times \mathbb{S}^2 \quad (x, Q) \quad \longrightarrow \quad (x, f(x, Q)).
\]
In order to simplify the notation we set $P = f(x, Q)$ and $[P^l_i]$ denotes the matrix, in the basis $(\theta_1, \ldots, \theta_4)$, of the operator $P$ viewed as an endomorphism of $TM$.

**Proposition 2.** Let $f$ be any morphism and $(m, Q) \in \mathbb{Z}$. Then, for all $i, j \in \{1, \ldots, 4\}$ one has:

i) the horizontal component of $N(\hat{\theta}_i, \hat{\theta}_j)$ can be written as $E(\theta_i, \theta_j) + F_{ij}$

ii) the vertical component of $N(\hat{\theta}_i, \hat{\theta}_j)$ can be written as $G(\theta_i, \theta_j) \frac{\partial}{\partial Q}$.

**Proof.** To finish the proof of the proposition we need the following lemma.

**Lemma 1.** The Lie bracket of $\hat{\theta}_i$ with $\hat{\theta}_j$ satisfies:

$$[\hat{\theta}_i, \hat{\theta}_j] = [\theta_i, \theta_j] - [R^k_{ij}, Q] \frac{\partial}{\partial Q}$$

**Proof of Lemma 1.** From $\hat{\theta}_i = \theta_i - [\Gamma^m_i, Q] \frac{\partial}{\partial Q}$ we can deduce that:

$$[\hat{\theta}_i, \hat{\theta}_j] = [\theta_i - [\Gamma^m_i, Q] \frac{\partial}{\partial Q}, \theta_j - [\Gamma^m_j, Q] \frac{\partial}{\partial Q}]$$

$$= [\theta_i, \theta_j] - [\theta_i, [\Gamma^m_j, Q] \frac{\partial}{\partial Q}] - [\theta_j, [\Gamma^m_i, Q] \frac{\partial}{\partial Q}] + [\theta_i, \theta_j] [\Gamma^m_i, Q] \frac{\partial}{\partial Q} - [\Gamma^m_i, \Gamma^m_j, Q] \frac{\partial}{\partial Q}$$

$$= (\Gamma^m_{ij} - \Gamma^m_{ji}) \theta_m - [\theta_i, \theta_j] [\Gamma^m_i, Q] \frac{\partial}{\partial Q} - (\Gamma^m_{ij} - \Gamma^m_{ji}) \theta_m - [\Gamma^m_i, \Gamma^m_j, Q] \frac{\partial}{\partial Q}$$

$$= [\Gamma^m_{ij} - \Gamma^m_{ji}] \theta_m - [R^k_{ij}, Q] \frac{\partial}{\partial Q}$$

$$= [\hat{\theta}_i, \hat{\theta}_j] - [R^k_{ij}, Q] \frac{\partial}{\partial Q} \square$$

We can now complete the proof of Proposition 1. The Nijenhuis tensor is given by

$$N(\hat{\theta}_i, \hat{\theta}_j) = [\mathbb{J}_f \hat{\theta}_i, \mathbb{J}_f \hat{\theta}_j] - \mathbb{J}_f \left( [\mathbb{J}_f \hat{\theta}_i, \hat{\theta}_j] + [\hat{\theta}_i, \mathbb{J}_f \hat{\theta}_j] \right) - [\hat{\theta}_i, \hat{\theta}_j],$$

where:

$$[\mathbb{J}_f \hat{\theta}_i, \mathbb{J}_f \hat{\theta}_j] = [P^l_i \hat{\theta}_i, P^l_j \hat{\theta}_j]$$

$$= P^l_i \hat{\theta}_i (P^l_j \hat{\theta}_j) \hat{\theta}_l - P^l_j \hat{\theta}_j (P^l_i \hat{\theta}_i) \hat{\theta}_l + P^l_i P^l_j \hat{\theta}_l \hat{\theta}_l$$

$$[\mathbb{J}_f \hat{\theta}_i, \hat{\theta}_j] + [\hat{\theta}_i, \mathbb{J}_f \hat{\theta}_j] = [P^l_i \hat{\theta}_i, \hat{\theta}_j] + [\hat{\theta}_i, P^l_j \hat{\theta}_j]$$

$$= -\hat{\theta}_j (P^l_i \hat{\theta}_i) \hat{\theta}_l + P^l_i [\hat{\theta}_i, \hat{\theta}_j] + \hat{\theta}_i (P^l_j \hat{\theta}_j) \hat{\theta}_r + P^l_j [\hat{\theta}_i, \hat{\theta}_r].$$
By Lemma 1 the horizontal component of the Nijenhuis tensor is:
\[
\mathcal{H}N(\hat{\theta}_i, \hat{\theta}_j) = P\hat{\theta}_i(P^*_{j}) \hat{\theta}_i - P\hat{\theta}_j(P^*_{i}) \hat{\theta}_i + P^*_{i}P^*_{j}[\hat{\theta}_i, \hat{\theta}_j] - P\left(-\hat{\theta}_j(P^*_{i}) \hat{\theta}_i + P^*_{i}[\hat{\theta}_i, \hat{\theta}_j] + \hat{\theta}_i(P^*_{j}) \hat{\theta}_j + P^*_{j}[\hat{\theta}_j, \hat{\theta}_i]\right).
\]

Fix $Q$ and denote by $P_0$ the almost complex structure on $TM$, over $\mathcal{U}$, defined by $P_0(m) = f(m, Q)$. Then:
\[
\mathcal{H}N(\hat{\theta}_i, \hat{\theta}_j) = P_0\hat{\theta}_i(P^*_{j}) \hat{\theta}_i - P_0\hat{\theta}_j(P^*_{i}) \hat{\theta}_i + P^*_{i}P^*_{j}[\hat{\theta}_i, \hat{\theta}_j] - P_0\left(-\hat{\theta}_j(P^*_{i}) \hat{\theta}_i + P^*_{i}[\hat{\theta}_i, \hat{\theta}_j] + \hat{\theta}_i(P^*_{j}) \hat{\theta}_j + P^*_{j}[\hat{\theta}_j, \hat{\theta}_i]\right).
\]

The vertical component of the Nijenhuis tensor is:
\[
\mathcal{V}N(\hat{\theta}_i, \hat{\theta}_j) = \left(R_{i,j}, Q - d^i d^j R_{i,j}, Q - Q\left(-d^i d^j R_{i,j}, Q - P^*_{i}d^j R_{i,j}, Q\right)\right) \frac{\partial}{\partial Q}
\]
\[
= G(\theta_i, \theta_j) \frac{\partial}{\partial Q}.
\]

In order to prove Theorem 1 we need to study the tensor $G$ and we set:
\[
\begin{align*}
G_1(\theta_i, \theta_j, P) &= \theta_i \wedge \theta_j - P\theta_i \wedge P\theta_j \\
G_2(\theta_i, \theta_j, P) &= P\theta_i \wedge \theta_j + \theta_i \wedge P\theta_j.
\end{align*}
\]

An easy computation gives the following lemma.

**Lemma 2.** Let $(\theta_1, \ldots, \theta_4)$ be an oriented orthonormal frame over an open set $\mathcal{U}$ and $(I, J, K)$ be the associated basis of $\wedge^+$. Then we have:
\[
\begin{align*}
I &= G_1(\theta_1, \theta_2, J) = G_1(\theta_1, \theta_2, K) \\
J &= G_1(\theta_1, \theta_3, I) = G_1(\theta_1, \theta_3, K) \\
K &= G_1(\theta_1, \theta_4, I) = G_1(\theta_1, \theta_4, J) \\
0 &= G_1(\theta_1, \theta_2, I) = G_1(\theta_1, \theta_3, J) = G_1(\theta_1, \theta_4, K) \\
G_1(\theta_1, \theta_2, aI + bJ + cK) &= (1 - a^2)I - abJ - acK \\
G_2(\theta_i, \theta_j, P) &= PG_1(\theta_i, \theta_j, P).
\end{align*}
\]

**B) The case where $f$ is the identity.**

In this section we give a proof of (the well known) part B of Theorem 1:

**Theorem B [AHS78].** *The complex structure $\mathcal{J}_{1d}$ is integrable if and only if $A$ is a homothety.*

The fact that $A$ is a homothety is equivalent to saying that the selfdual Weyl tensor $W^+$ is zero. In that case the metric is said to be anti-selfdual.

**Proof.** In the local trivialization $\pi^{-1}(\mathcal{U}) \simeq \mathcal{U} \times \mathbb{C}P^1$ of the previous section the morphism $f = Id$ when restricted to fibers is a holomorphic map, which only depends on the second variable. By Proposition 1 we know that it is sufficient to study $N(\hat{\theta}_i, \hat{\theta}_j)$. We have:
\[
F_{ij} = -Q_{i}^*[\Gamma^r_{ij}, Q] \frac{\partial}{\partial Q_{j}} \hat{\theta}_i + Q_{j}^*[\Gamma^r_{ij}, Q] \frac{\partial}{\partial Q_{i}} \hat{\theta}_i - Q(\Gamma_{ij}, Q) \hat{\theta}_i - Q(\Gamma_{ij}, Q) \hat{\theta}_i.
\]
Using $[\Gamma^i_{ij}, Q] = [\nabla_{\theta_i}Q, Q] = \nabla_{\theta_i}Q$ one gets:

\[
d\pi(F_{ij}) = - (\nabla_{\theta_i}Q)\theta_j + (\nabla_{\theta_j}Q)\theta_i - Q\left((\nabla_{\theta_i}Q)\theta_i - (\nabla_{\theta_j}Q)\theta_j\right)
\]

\[
= -\nabla_{\theta_i} Q \theta_j + Q\nabla_{\theta_j} \theta_i + \nabla_{\theta_i} Q \theta_j - Q\nabla_{\theta_j} \theta_i
\]

\[
- Q\nabla_{\theta_i} \theta_i + \nabla_{\theta_j} \theta_i + Q\nabla_{\theta_i} \theta_j + \nabla_{\theta_j} \theta_j
\]

\[
= -E(\theta_i, \theta_j).
\]

The horizontal component of $N(\hat{\theta}_i, \hat{\theta}_j)$ is then zero. The vertical component is:

\[
G(\theta_i, \theta_j) = \left[R\left(\theta_i \wedge \theta_j - Q\theta_i \wedge Q\theta_j\right) + QR\left(\theta_i \wedge Q\theta_j + Q\theta_i \wedge \theta_j\right), Q\right].
\]

But $Q$ preserves the orientation, hence:

\[
\begin{cases}
\theta_i \wedge \theta_j - Q\theta_i \wedge Q\theta_j \in \wedge^+ TmM \\
\theta_i \wedge Q\theta_j + Q\theta_i \wedge \theta_j \in \wedge^+ TmM.
\end{cases}
\]

Recall that the matrix of the curvature operator $R$ has the following splitting:

\[
R = \begin{pmatrix}
A & T \\
B & C
\end{pmatrix}
\]

Since the elements of $\wedge^+$ of $\wedge^-$ commute $[\text{AHSS78}]$, the component $A$ in the matrix $R$ is the only one which matters in the computation of $G(\theta_i, \theta_j)$. By Lemma 2, one has the equality:

\[
(\theta_i \wedge \theta_j - Q\theta_i \wedge Q\theta_j) + Q(\theta_i \wedge Q\theta_j + Q\theta_i \wedge \theta_j) = 0, \quad \forall i, j \in TmM.
\]

Therefore, if the matrix $A$ is a homothety the Nijenhuis tensor of $J_{str}$ is zero.

Conversely, assume that $J_{str}$ is integrable. We have noticed that the orthonormal frame $(\theta_1, \ldots, \theta_4)$ over $U$ defines an oriented orthonormal basis $(I, J, K)$ of $\wedge^+$ over $U$. Since $G(\theta_i, \theta_j) = 0$ for all $i, j \in \{1, \ldots, 4\}$, Lemma 2 implies:

- at the point $(m, I)$, $G(\theta_1, \theta_3) = [A(J) + IA(K), I] = 0$
- at the point $(m, J)$, $G(\theta_1, \theta_2) = [A(I) + JA(-K), J] = 0$
- at the point $(m, K)$, $G(\theta_1, \theta_2) = [A(I) + KA(J), K] = 0$.

Since $(I, J, K)$ is an oriented orthonormal basis, it follows from $IJ = -JI = K$ that relations of the following type hold:

\[
[A(J), I] = 2 < A(J), K > J - 2 < A(J), J > K.
\]

From the previous system we then deduce the following one:

\[
\begin{align*}
< A(J), J > &= - < IA(K), J > &= < A(K), K > \\
< A(J), K > &= - < IA(K), K > &= - < A(K), J > \\
< A(I), I > &= - < JA(-K), I > &= < A(K), K > \\
< A(I), K > &= - < JA(-K), K > &= - < A(K), I > \\
< A(I), I > &= - < KA(J), I > &= < A(J), J > \\
< A(I), J > &= - < KA(J), J > &= - < A(J), I >
\end{align*}
\]

But the matrix $A$ in the basis $(I, J, K)$ is symmetric, thus $A$ is a homothety. \hfill \square

C) The case when $f$ is constant

Integrability theorem

In this section we give a proof of part C of Theorem 1.

**Theorem C.** Let $(M, g, J_M)$ be an almost complex manifold such that $J_M$ is compatible with the orientation and the metric. The complex structure $J_\infty$ is integrable if and only if:
i) $J_M$ is integrable;

ii) the kernel of $A$ contains the subspace $J_M^+ \subset \Lambda^+$ orthogonal to the line generated by $J_M$ (i.e. $J_M^+ \subset \ker(A)$).

Notice that the integrability condition is not conformal on $g$. Moreover, when $J_\infty$ is integrable, it gives to the twistor projection $\pi: (Z, J_\infty) \to (M, J_M)$ the structure of a holomorphic $\mathbb{C}P^1$-bundle.

For a complex manifold $(M, g, J_M)$ we have a decomposition $\mathbb{C} \otimes TM = T^{1,0} \oplus T^{0,1}$ into $\pm i$ eigenspaces of $J_M$. We then obtain:

$$\left\{ \begin{array}{c}
\mathbb{C} \otimes \Lambda^+ = \mathbb{C}J_M \oplus \mathbb{C}(\Lambda^{2,0} \oplus \Lambda^{2,0}) \\
\mathbb{C} \otimes \Lambda^- = \{ \psi \in \Lambda^{1,1} \mid <\psi, J_M> = 0 \}
\end{array} \right. \text{ where } \left\{ \begin{array}{c}
\Lambda^{2,0} = T^{1,0} \wedge T^{1,0} \\
\Lambda^{1,1} = T^{1,0} \wedge T^{0,1}
\end{array} \right.$$

Condition ii) says that $(\Lambda^{2,0} \oplus \Lambda^{2,0}) \subset \ker(A)$. For a Kählerian manifold the curvature $R$ may be viewed as a symmetric endomorphism of $\Lambda^{1,1}$, so in some orthonormal basis compatible with these decompositions we have $A = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and

$$W^+ = \begin{bmatrix} \frac{a}{2} & 0 & 0 \\ 0 & -\frac{a}{2} & 0 \\ 0 & 0 & -\frac{a}{2} \end{bmatrix}.$$ We then have the following result:

**Proposition 3.** For any Kählerian manifold $(M, g, J_M)$ the complex structure $J_\infty$ on $(Z, \hat{g})$ is integrable. Furthermore, if $(M, g, J_M)$ is Kähler and the scalar curvature of $g$ is never zero, then $J_\infty$ and $\mathbb{J}_{-\infty}$ (the compatible complex structure on $(Z, \hat{g})$ associated to $-J_M$) are the only compatible complex structures on $(Z, \hat{g})$.

The proof will show that the result is locally true. In other terms, for a Kählerian manifold whose scalar curvature is non zero there are, even locally, only two compatible complex structures on its twistor space.

**Proof.** The first part being a consequence of Theorem C, we only need to prove the second part of the proposition. Let $J_f$ be a compatible complex structure on $(Z, \hat{g})$ and assume that the scalar curvature of $(M, g, J_M)$ is never zero. One can build an orthonormal basis $(I, J, K)$ of $\Lambda^+$ over an open set $U$ as follows. Setting $I = J_M$, pick any unitary vector $J$ orthonormal to $I$ and define $K = IJ$. For any $m \in U$, there exists $(a, b, c) \in S^2$ such that $f(m, J) = aI + bJ + cK$. But, as $(M, g, J_M)$ is Kähler, in this basis we have $A = \begin{bmatrix} \frac{a}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Let $\theta_1$ be a unitary vector field defined over $U$; set $\theta_2 = I\theta_1$. As $J_f$ is integrable, $G(\theta_1, \theta_2)$ is identically zero on $U$. In particular, at the point $(m, J)$ we obtain:

$$G(\theta_1, \theta_2) = 0 \Rightarrow [A \left( (1 - a^2)I - abJ - acK \right) + JA(cI - bK), J] = 0 \Rightarrow \left[ (1 - a^2)\frac{a}{2}I, J \right] = (1 - a^2)\frac{a^2}{2}K.$$ Therefore $a = \pm 1$, that is $f(m, J) = \pm I$ for all $J$ orthonormal to $I$. Since $f$ must be holomorphic in the fibers we get that $f$ is constant, equal to $I$ or $-I$. □

**Proof of Theorem C.** By Proposition 1, it is sufficient to check that $N(\hat{\theta}_i, \hat{\theta}_j) = 0$. As $f$ is constant on fibers we always have $F_{ij} = 0$. Therefore: $\mathbb{J}_\infty$ integrable
\[ \iff E(\theta_1, \theta_2) = G(\theta_1, \theta_2) = 0 \iff \{ J_M \text{ integrable and } G(\theta_i, \theta_j) = 0 \}. \]

For all \( \theta_i, \theta_j \in T M \) we have \( \theta_i \wedge \theta_j - J_M \theta_i \wedge J_M \theta_j \in J_M^\perp \). Consequently, if \( J_M^\perp \subset \ker(A) \) we obtain \( G(\theta_i, \theta_j) = 0 \) for all \( \theta_i, \theta_j \in TM \).

Conversely, suppose that \( J_M^\perp \) is integrable. Set \( J_0 = J_M \). Locally over an open set \( U \) one can complete \( \{ J_0 \} \) to get an oriented orthonormal basis \( (I_0, J_0, K_0) \) of \( \bigwedge^+ \). Let \( \theta_1 \) be a unitary vector field defined over \( U \); set \( \theta_2 = I_0 \theta_1 \). If \( G = 0 \), then, for all \( m \in U \) and \( Q \in \pi^{-1}(m) \), Lemma 2 implies that at the point \((m, Q)\):

\[ G(\theta_1, \theta_2) = [A(I_0) + QA(-K_0), Q] = 0. \]

In particular, for \( Q = A(K_0) \), we have \([A(I_0), A(K_0)] = 0\) and it follows that \( A(K_0) = cA(I_0)\) for some constant \( c \). The former equation yields:

\[ \forall Q \in \pi^{-1}(m), \quad 0 = [A(I_0) + QA(-K_0), Q] = (Id - cQ)[A(I_0), Q] \implies A(I_0) = 0. \]

Therefore \( J_0^\perp = \text{Vect}(I_0, K_0) \subset \ker A \).

Recall that we have a characterization of an integrable almost complex structure \( J_M \) on \( M \) in terms of the twistor space and one of the Kählerian complex structures.

**Proposition (see, for example, [Sal85, BDB88])**. Let \( J_M \) be a Hermitian almost complex structure on \((M, g)\). Then:

- \( J_M \) is integrable if and only if the associated section of the twistor space, \( s : (M, J_M) \to (Z, J_1^M) \), is almost holomorphic, that is: the differential \( ds \) satisfies \( ds \circ J_M = J_1^M \circ ds \);

- \( J_M \) is Kählerian if and only if \( s \) is an horizontal section, that is to say: the tangent space of the submanifold \( s(M) \subset Z \) is included in the horizontal distribution.

It is well known that the existence of a Kählerian metric on a compact complex surface \((M, J_M)\) is equivalent for the first Betti number \( b_1 \) to be even [Miy74, Siu83, Lam99]. Theorem C gives a new characterization of compact Kählerian manifolds in terms of compatible complex structures on the associated twistor spaces.

**Proposition 4.** A compact almost Hermitian 4-dimensional manifold \((M, g, J_M)\) is Kählerian if and only if \( J_\infty \) is integrable.

In section E we will deduce from that proposition a characterisation of compact scalar-flat Kähler manifolds in terms of compatible complex structures on \((Z, \tilde{g})\) (cf. Proposition 8).

**Proof.** Let \( \theta \) be the Lee form of \((M, g, J_M)\) defined by \( dJ_M = -2\theta \wedge J_M \), where \( J \in \bigwedge^+ \) is viewed as a 2-form. Denote by \( \kappa \) the conformal scalar curvature, which is related to the scalar curvature \( s \) by \( \kappa = s + 6(\delta \theta - |\theta|^2) \). The condition \( J_M^\perp \subset \ker A \) is equivalent to the following: the selfdual Weyl tensor \( W^+ \) is degenerate (meaning that, in every point, two of the eigenvalues coindice) and the scalar curvature of \((M, g)\) is equal to the conformal scalar curvature \[AG\]. This is also equivalent to \( \kappa = |\theta|^2 \). Integrating this expression over \( M \) gives \( \theta = 0 \) by the Brochner-Grenth theorem. But \((M, g, J_M)\) is Kähler if and only if \( \theta \) vanishes identically.

**Corollary 1.** Assume that a compact 4-dimensional manifold \((M, g)\) admits two almost complex structures \( J_1 \neq \pm J_2 \) compatible with the metric and the orientation.
Then the associated compatible almost complex structures $\mathbb{J}_\infty$, $\mathbb{J}_\infty$ on $(Z, \tilde{g})$ are integrable if and only if $\{J_1, J_2\}$ spans a hyperkähler structure on $(M, g)$.

**Proof.** By Proposition 4, $\mathbb{J}_\infty$ and $\mathbb{J}_\infty$ are integrable if and only if $J_1$ and $J_2$ are Kähler. As $J_1 \neq \pm J_2$, then $J_1$ is different from $\pm J_2$ everywhere. The holonomy of $g$ reduces to $U(2)$ by $J_1$ and further to $SU(2)$ by $J_2$. This says that $g$ is hyperkähler.

---

**Study of the manifold $(Z, \mathbb{J}_\infty)$**

Any scalar-flat Kähler manifolds $(M, g, J_M)$ is automatically anti-selfdual [Gau81]. For such a manifold we can put two natural complex structures on its twistor space: $\mathbb{J}_{I\!d}$ and $\mathbb{J}_\infty$. The next proposition shows that these complex structures are never deformation of each other.

**Proposition 5.** If $(M, g, J_M)$ is a scalar-flat Kähler manifold, the complex structure $\mathbb{J}_\infty$ on $Z$ is never a deformation of the complex structure $\mathbb{J}_{I\!d}$.

**Proof.** It is sufficient to show that $(Z, \mathbb{J}_{I\!d})$ and $(Z, \mathbb{J}_\infty)$ do not have the same Chern classes. Let $h$ be the generator of the second cohomology group $H^2(\mathbb{C}P^1, Z) \simeq \mathbb{Z}$. By Leray-Hirsch theorem’s [BTS82] the cohomology ring of $Z$ is a $H^*(M, \mathbb{R})$-module generated by $h$ with relation $4h^2 = 3\tau + 2\chi$, where $\tau$ and $\chi$ are the signature and the Euler characteristic of $M$. Denote by $c_1(J_M)$ the first Chern class of the manifold $(M, J_M)$. Under this notation we have:

\[
\begin{align*}
  c(\mathbb{J}_{I\!d}) &= 1 + 4h + 3\tau + 3\chi + 2h\chi \quad \text{[Hit81]} \\
  c(\mathbb{J}_\infty) &= (1 + 2h)(1 + c_1(J_M) + \chi) \\
  &= 1 + 2h + c_1(J_M) + 2h c_1(J_M) + \chi + 2h\chi.
\end{align*}
\]

If the complex structures were deformations of each other, they would have the same Chern numbers: $c_1(\mathbb{J}_{I\!d})^3 = 16(3\tau + 2\chi)h = c(\mathbb{J}_\infty)^3 = 8(3\tau + 2\chi)h$. This forces $3\tau + 2\chi = 0$. Let $\mu_g$ be the volume form on $M$ associated to the metric $g$; by the Gauss-Bonnet formula [AW43], [Hir66]:

\[
3\tau + 2\chi = \frac{1}{4\pi^2} \int_M 2\|W^+\| + \frac{1}{24} s^2 - 2\|B\|^2 \mu_g = -\frac{1}{2\pi^2} \int_M \|B\|^2 \mu_g.
\]

Thus, $3\tau + 2\chi = 0$ implies $B = 0$. As the scalar curvature of $(M, g)$ is supposed to be zero, the manifold $(M, g, J_M)$ would be Ricci-flat, hence $c_1(J_M) = 0$. Therefore the first Chern classes of $(Z, \mathbb{J}_{I\!d})$ and of $(Z, \mathbb{J}_\infty)$ are different and these two manifolds are never deformations of each other.

---

When $(M, g, J_M)$ is a complex spin manifold, Hitchin has shown that there exists a holomorphic line bundle $L \rightarrow M$ such that $L \otimes L = K_M$ is the canonical line bundle [Hit74]. Then, the twistor space $Z$ can be identified, in a $C^\infty$-way, to the projectivization bundle $\mathbb{P}(L \oplus L^*)$ [Sak83]. By this construction we see that the manifold $Z \simeq \mathbb{P}(L \oplus L^*)$ admits a natural complex structure denoted by $\mathbb{J}$. When $(M, g, J_M)$ is not spin, but only complex, the bundle $L \oplus L^*$ exists only locally. Nevertheless, the projectivization $\mathbb{P}(L \oplus L^*)$ still exists globally, due to the fact that the transition functions on $L \oplus L^*$ are well defined holomorphic maps up to sign. In general $\mathbb{J}$ is not a compatible complex structure on $(Z, \tilde{g})$.

Now, if $(M, g, J_M)$ satisfies the conditions of Theorem C, we can put another complex structure on its twistor space, namely $\mathbb{J}_\infty$. The question is then to determine the relationship between the manifolds $(Z, I)$ and $(Z, \mathbb{J}_\infty)$. In that direction we have the following result.
Proposition 6. Let \((M, g, J_M)\) be a manifold satisfying conditions of Theorem C (i.e. \(I_\infty\) integrable). The complex structures \(I\) and \(I_\infty\) on \(Z\) are deformations of each other: there exists on \(Z\) a path of integrable complex structures \(I_t, t \in [0, 1],\) connecting \(I\) to \(I_\infty.\)

By combining this result and [Ts087, Theorem 4.1] we obtain another proof of Proposition 5.

Proof. In an appropriate local trivialization of the bundle \(Z \rightarrow M,\) the almost complex structure \(I\) on \(U \times S^2\) can be identified with the product structure \(J_M \times J_{CP^2}.\) Let \((\theta_1, \theta_2, \theta_3, \theta_4)\) be an oriented orthonormal frame defined over \(U\) providing this trivialization. Set \(\theta_{i,t} = \theta_i - t[\theta_i, Q]\frac{\partial}{\partial Q}\) for \(t \in [0, 1].\) The subspace \(H_i = \text{Vect}(\dot{\theta}_{1,t}, \ldots, \dot{\theta}_{4,t})\) is in direct sum with the vertical distribution \(V_z\) and can be glued into a global distribution over \(Z.\) Define the almost complex structure \(J_i\) on \(\pi^{-1}U\) as follows: endow \(V_z\) with the complex structure of the fibers (complex projective lines) and pull back on \(H_i \simeq T_m M\) the complex structure \(J_M.\) Then, \(J_i\) is a path of almost complex structures from \(I\) to \(I_\infty.\) The integrability of \(J_i\) is shown in the same way as that of \(I_\infty.\) \(\square\)

D) The case where \(f\) is a homothety

Integrability theorem

In this section we prove part D of Theorem 1.

Theorem D. Let \((M, g, J_M)\) be a Kählerian manifold. For all complex \(\lambda \notin \{0, 1\}\) the almost complex structure \(J_{\lambda M}\) is integrable if and only if \((M, g, J_M)\) is scalar-flat Kähler (i.e. \(A = 0\)).

The condition \(A = 0\) is equivalent to saying that the metric \(g\) is Hermitian scalar-flat and anti-selfdual. These metrics are called optimal by LeBrun because they are absolute minimizers of the functional \(K(g) = \int_M |R|^2 \text{dvol} [\text{LeB}].\) Let \((M, g, J_M)\) be a compact scalar-flat Kähler manifold and \(c_1(M)\) be the real first Chern class of \((M, J_M).\) Two possibilities may occur [La82]. Either \(c_1(M) = 0\) and \((M, g, J_M)\) is then finitely covered by a hyperkähler surface, i.e. a flat torus or a \(K3\)-surface with Ricci-flat Kähler metric [Boy86], [Pon91]. Or \(c_1(M) < 0,\) in which case \((M, g)\) is a ruled surface [KL97], i.e. \((M, g)\) is obtained by blowing up \(m\) points on a \(CP^2\)-bundle over a Riemann surface of genus \(\gamma.\) The condition \(c_1(M) < 0\) gives a lower bound on the number of points \(m\) to be blown up: namely \(m \geq 9\) when \(\gamma = 0,\) \(m \geq 1\) when \(\gamma = 1\) and there is no restriction for \(\gamma > 1.\) Conversely we have:

Theorem [KL97]. A ruled surface \(M\) has a blow-up \(\tilde{M}\) which is a scalar-flat Kähler surface. Moreover, any further blow up of \(\tilde{M}\) admits a scalar-flat Kähler metric.

For simply connected manifold we have the following result:

Theorem [KL97, LeB]. Let \(M\) be a smooth compact simply connected 4-manifold. If \(M\) admits a scalar-flat Kähler structure, then \(M\) is diffeomorphic to a \(K3\)-surface or to the connected sum \(CP^2 \# kCP^2\) for some \(k \geq 10.\) Conversely, if \(M\) is a \(K3\)-surface or is diffeomorphic to \(CP^2 \# kCP^2\) for some \(k \geq 14,\) then it admits a scalar-flat Kähler metric.
**Proof of Theorem D.** By Propositions 1 & 2, if \( A = 0 \) it is enough to show that \( E(\theta_i, \theta_j) + F_{ij} = 0 \) to get the integrability of \( J_{\lambda Id} \). Let \( z \in \pi^{-1}(m) \) be a point of \( Z \) over \( m \in M \). Let \( \theta_1, \theta_2 \) be two unitary vector fields, defined on an open set \( U \) of \( M \), such that \( \nabla_m |_{m} \theta_1 = \nabla_m |_{m} \theta_2 = 0 \) and \( \theta_2 \in (\theta_1, J_M \theta_1)^{\perp} \). As \( J_M \) is parallel, the vector fields \( \theta_1 = J_M \theta_1 \) and \( \theta_2 = J_M \theta_2 \) complete the family \((\theta_1, \theta_2)\) to give an orthonormal basis such that \( \nabla_m |_{m} \theta_i = 0 \) for all \( i \in \{1, \ldots, 4\} \). Hence \( F_{ij} |_{m} = 0 \), since \( \Gamma_{ij}^k (m) = 0 \) for all \( i,j,k \in \{1, \ldots, 4\} \). Moreover, this frame gives an oriented orthonormal basis \((I, J, K)\) of \( \Lambda^+ \), and therefore a local trivialization of \( Z \) over the open set \( U \), where \( \infty \) coincides with \( J \). It follows that the restriction of \( f \) to the fibers does not depend on the second variable:

\[
\pi^{-1}(U) \cong U \times \mathbb{S}^2 \quad \xrightarrow{f} \quad \pi^{-1}(U) \cong U \times \mathbb{S}^2 \quad (x, Q) \quad \xrightarrow{\pi} \quad (x, f(Q))
\]

Thus \( E(\theta_i, \theta_j) |_{m} = 0 \) and \( J_{\lambda Id} \) is integrable.

Conversely, assume that \( J_{\lambda Id} \) is integrable. From Proposition 3 one deduces that the scalar curvature must be zero, hence \( A = 0 \).

**Study of the manifold \((Z, J_{\lambda Id})\)**

We know that the almost complex structure \( J_{\lambda Id} \) on \( Z \) is integrable if and only if the metric \( g \) is anti-selfdual. In that case the twistor space \( Z \) is a complex 3-manifold. The fibers of the projection \( \pi : Z \rightarrow M \) are rational curves with normal bundles isomorphic to \( \mathcal{O}(1) \oplus \mathcal{O}(1) \). On each fiber the antipodal map \( \sigma \) is an antiholomorphic free involution. Observe that this construction only depends on the conformal class of the metric \( g \). The converse holds: an arbitrary conformal anti-selfdual 4-manifold can be constructed from a complex 3-manifold \( Z \) as soon as \( Z \) admits a free antiholomorphic involution \( \sigma \) and a foliation by \( \sigma \)-invariant rational curves, each of which having \( \mathcal{O}(1) \oplus \mathcal{O}(1) \) as normal bundle [Pen76, AHS78]. This is the Penrose correspondence.

Thus if \( (M, g, J_M) \) is a scalar-flat Kähler surface we have that, for \( \lambda \in \mathbb{C} - \{0, 1\} \), the complex 3-manifold \((Z, J_{\lambda Id})\) is the twistor space of \((M, g_{\lambda})\) for some anti-selfdual metric \( g_{\lambda} \).

At least two cases may occur. Firstly: all the \((Z, J_{\lambda Id})\) are biholomorphic to \( J_{Id} \), thus there exists a 1-dimensional family of biholomorphism of \((Z, J_{Id})\). We will see in section G that this is the case for any bi-elliptic surface (quotient of a flat torus). Secondly: none of the \((Z, J_{\lambda Id})\) is biholomorphic, thus we have a 1-dimensional family of non conformal anti-selfdual metrics on \( M \). For example, if one blows-up at least 14 points in \( \mathbb{CP}^2 \), equipped with the Fubini-Study metric \( g \), one gets \((\mathbb{CP}^2, k\mathbb{CP}^2, J_M)\) for some \( k \geq 14 \). This manifold admits a scalar-flat Kähler metric [LeB] but doesn’t have any non trivial conformal application, thus its twistor doesn’t have any biholomorphism. Therefore the complex structure \((Z, J_{\lambda Id})\) defines a 1-dimensional family of anti-selfdual metrics:

**Proposition 7.** There exist a 1-dimensional family of non conformal scalar-flat Kähler metrics on \((\mathbb{CP}^2, k\mathbb{CP}^2, J_M)\) for every \( k \geq 14 \).

**Proof.** There exists a 1-dimensional family of non conformal anti-selfdual metrics
\(g_\lambda\) on \((\mathbb{C}P^2, J_M)\). But \(g_\lambda\) is Hermitian, thus in the conformal class of \(g_\lambda\) there exists a scalar-flat Kähler metric \([\text{Boy88}]\).

E) Metric properties on \(M\) in terms of compatible complex structures on \((Z, \tilde{g})\)

We can use the almost complex structures \(J_f\) to characterize some properties of the metric \(g\) on \(M\). Indeed, by (the well known) Theorem B we have that \(g\) is anti-selfdual if and only if \(J_{Id}\) is integrable. We showed that a compact almost Hermitian manifold \((M, g, J_M)\) is Kähler if and only if \(J_\infty\) is integrable; furthermore the integrability of \(J_{Id}\) and \(J_\infty\) is equivalent to \((M, g, J_M)\) scalar-flat Kähler (cf. Proposition 8).

When limiting to the case where \((M, g)\) is anti-selfdual, we can give a characterization of metrics which are scalar-flat in terms of compatible complex structures on \((Z, \tilde{g})\). According to the terminology of LeBrun this is a characterization of optimal metrics \([\text{LeB}]\).

**Theorem E.** Let \((M, g)\) be an anti-selfdual Riemannian manifold. The following are equivalent:

- the scalar curvature of \(g\) is flat;
- every \(m \in M\) has an open neighborhood \(U\) such that \(Z\) admits, over \(U\), an integrable compatible complex structure \(J_f\) for at least one (and then infinitely many) morphism(s) \(f \neq Id\).

In other words, if \((M, g)\) is an anti-selfdual metric with non zero scalar curvature then, even locally on \(Z\), the only integrable almost complex structure among the \(J_f\)'s is \(J_{Id}\). This result should be compared to the following result of Salamon:

**Proposition \([\text{Sal91}]\) (see also \([\text{Pon97}]\)).** A metric \(g\) on \(M\) is anti-selfdual if, and only if, locally around each point \(m \in M\) there exist infinitely many compatible complex structures on \((M, g)\).

In a similar direction, Pontecorvo gives a conformal characterization of scalar-flat Kähler manifolds among anti-selfdual Hermitian manifolds. Indeed, let \((M, g, J_M)\) be an anti-selfdual complex Hermitian manifold. The complex structure \(J_M\) on \(M\) defines a section \(s : Z \longrightarrow M\) \([\text{BDB88}]\), whose image will be noted \(\Sigma = s(M)\). Similarly, the hypersurface \(\Sigma = \sigma(\Sigma)\) of \(Z\) corresponds to the conjugate complex structure \(-J_M\). Let \(X\) be the divisor \(\Sigma + \Sigma\) in \(Z\) and consider the holomorphic line bundle \([X]\). Denote by \(K_Z\) be the canonical line bundle of \((Z, J_{Id})\).

**Proposition \([\text{Pon92a}]\).** Let \((M, g, J_M)\) be a Hermitian anti-selfdual manifold. The line bundles \([X]\) and \(-\frac{1}{2}K_Z\) are isomorphic if and only if \(g\) is conformal to a scalar-flat Kähler metric.

Notice that Theorem 1 and Proposition 3\&4 give a non conformal characterization of compact scalar-flat Kähler manifolds.

**Proposition 8.** Let \((M, g, J_M)\) be a compact almost Hermitian manifold. The following are equivalent:

- the metric \(g\) is scalar-flat Kähler;
- the compatible complex structures \(J_{Id}\) and \(J_\infty\) on \((Z, \tilde{g})\) are integrable;
the compatible complex structures $J_{\lambda \text{Id}}$ and $J_{\infty}$ on $(Z, \tilde{g})$ are integrable.

**Proof.** A Kählerian manifold $(M, g, J_M)$ is scalar-flat if and only if $g$ is anti-selfdual [Gau81]. Then, it follows from Proposition 3&4 and Theorem 1 that: \{ $J_{\infty}$ and $J_{\lambda \text{Id}}$ are integrable \} $\iff$ \{ $g$ is scalar-flat Kähler \} $\iff$ \{ $(M, g, J_M)$ is anti-selfdual Kähler \} $\iff$ \{ $J_{\infty}$ and $J_{\lambda \text{Id}}$ are integrable \}. $\square$

**Proof of Theorem E.** If $(M, g)$ is a scalar-flat anti-selfdual metric its twistor space is complex and $(M, g, J_M)$ admits, locally, at least one complex structure [Sal91]. Then Theorem C ensures that the locally defined almost complex structure $J_{\infty}$ on $Z$ is integrable. Actually, as soon as $(M, g, J_M)$ is scalar-flat there are, locally, infinitely many integrable complex structures $J_M$ on $M$, and so infinitely many integrable complex structures $J_{\infty}$ on $Z$.

Conversely, let $(M, g)$ be a manifold with an anti-selfdual metric $g$ having non zero scalar curvature. Let $f : Z \to Z$ be a morphism such that $J_f$ is integrable over an open set $U$. Let $(m, Q)$ be a point in $\pi^{-1}(U)$ and set $f(m, Q) = P$. According to our notation, if $\mathcal{U}$ is small enough we can build an orthonormal basis $(\theta_1, \ldots, \theta_4)$ of vector fields on $M$ such that $P = J = \theta_1 \wedge \theta_3 - \theta_2 \wedge \theta_4$. Then there exists $(a, b, c) \in S^2$ such that $Q = aI + bJ + cK$.

As $J_f$ is integrable, $G(\theta_1, \theta_2)$ vanishes everywhere. In particular, at the point $(m, Q)$ one obtains:

$$
G(\theta_1, \theta_2) = 0 = \frac{1}{16} [I - QK, Q] = \frac{2a}{16} \left( acI - c(1-b)J + (b(1-b) - a^2)K \right) \implies \begin{cases} \quad ac = 0 \\ \quad c(1-b) = 0 \\ b = a^2 + b^2 \end{cases}
$$

Therefore we have $Q = J = P$ for every $(m, Q) \in \pi^{-1}(U)$, that is to say $f = \text{Id}$. $\square$

**F) Compatible complex structure on locally conformally Kähler manifolds**

The aim of this section is to give a local description of the set $I$ of integrable compatible complex structures on the twistor space $(Z, \tilde{g})$ of a compact locally conformally Kähler (abbreviated in l.c.k.) manifold $(M, g, J_M)$. This condition is equivalent to $W^+$ being degenerate, which means that at each point of $M$ at least two eigenvalues of $W^+$ coincide.

We start by recalling the main results about the l.c.k. manifolds.

A result by Tricerri, generalizing the analogous result in the Kähler case, shows that it is enough to understand minimal complex surfaces.

**Proposition [Tri82].** A complex manifolds $(M, g, J_M)$ is l.c.k if and only if the blow-up of $M$ at a point is l.c.k.

When the first Betti number $b_1$ is even, a l.c.k. manifold is globally Kähler.

**Proposition [Vai80].** Every l.c.k. metric on a compact surface $(M, J_M)$ with even first Betti number is globally conformal Kähler.

When the first Betti number is odd and the Euler characteristic is zero, we have a classification due to Belgun, Gauduchon-Ornea, Tricerri, Vaisman.

**Proposition [Bel00].** The complete list of compact minimal l.c.k. surfaces with odd first Betti number and zero Euler characteristic is:
i) the properly elliptic surfaces (i.e. surfaces with \( \text{Kod}(M) = 1 \) and \( b_1 \) odd);
ii) the Kodaira surfaces (i.e. surfaces with \( \text{Kod}(M) = 0 \) and \( b_1 \) odd);
iii) the Hopf surfaces;
iv) the Inoue-Bombieri surfaces different from \( S_{n,u}^- \) with \( u \notin \mathbb{R} \).

When the first Betti number is odd and the Euler characteristic is non-zero, the only other possible case is that of surfaces of class VII with \( 0 < \chi = b_2 \) [BHPVdV04], for which there is (yet) no classification. (For some existence results see [FP05].)

Let \( J \) be a compatible almost complex structure on \((Z, \tilde{g})\). We say that \( J \) is semi-integrable if the vertical component of the Nijenhuis tensor is zero. Denote by \( I \) the set of semi-integrable\( I_1^2 \) \( \text{resp. } I_2^2 \) \( \text{integrable} \) compatible complex structures on \((Z, \tilde{g})\).

Propositions 1 and 2 give a necessary and sufficient condition for \( J \) to be semi-integrable, or integrable. The set \( I \) on an l.c.k. manifold \((M, g, J_M)\) depends on the spectrum of the operator \( A = W^+ + \frac{s}{12} \).

\[
A = W^+ + \frac{s}{12} \text{Id} = \begin{bmatrix}
\frac{\alpha}{12} & 0 & 0 \\
0 & \frac{\beta}{12} & 0 \\
0 & 0 & \frac{-\kappa}{12}
\end{bmatrix} + \begin{bmatrix}
\frac{x}{12} & 0 & 0 \\
0 & \frac{y}{12} & 0 \\
0 & 0 & \frac{s}{12}
\end{bmatrix} = \begin{bmatrix}
x & 0 & 0 \\
0 & y & 0 \\
0 & 0 & \frac{s}{12}
\end{bmatrix}.
\]

Moreover \( J_M \) is actually an eigenvector of \( W^+ \) for the simple eigenvalue \( \frac{s}{6} \).

**Theorem 2.** Let \((M, g, J_M)\) be a compact surface l.c.k., if we don’t have \( x = y = 0 \) we note \( \frac{s}{6} \in \mathbb{R} \). On an open set \( U \) of \( M \):

A) We have \( x = y = 0 \) if, and only if, on \( U \) one of the following equivalent conditions hold:

i) \((M, g, J_M)\) is scalar-flat Kähler.
ii) \( g \) anti-selfdual scalar-flat.
iii) The compatible complex structures \( J_{\text{Id}}, J_{\infty} \) and \( J_{\lambda \text{Id}} \) are integrable.
iv) The cardinal of \( I \) is infinity.

This is the case globally if, and only if, \((M, g, J_M)\) is a flat torus (or a quotient), a K3-surface with a Calabi-Yau metric (or a quotient), a \( \mathbb{C}P^1 \)-bundle over a Riemann surface \( \Sigma_\gamma \) of genus \( \gamma > 1 \) with the conformally flat Kähler metric which locally is a product of the \((+1)\)-curvature metric on \( \mathbb{C}P^1 \) and \((-1)\)-curvature metric on \( \Sigma_\gamma \) [Boy88], [Pon92b].

B) We have \( \frac{x}{y} = \infty \) if, and only if, on \( U \) one of the following equivalent conditions hold:

i) \((M, g, J_M)\) is Kähler with \( s \neq 0 \).
ii) \( I = I_{\frac{s}{6}} = \{ J_{-\infty}, J_{\infty} \} \).

This is the case globally on \( M \) if \((M, g, J_M)\) is Kähler-Einstein not Ricci-flat (that is a Fano manifolds or a manifold where the canonical line bundle is ample).

C) We have \( \frac{s}{6} \leq 1 \) if, and only if, on \( U \): \( I_{\frac{s}{6}} = \{ J_{\pm \cos \theta \text{Id}} \} \) where \( \cos \theta = \frac{x}{y} \).

D) We have \( \infty \neq \frac{s}{6} \geq 1 \) if, and only if, on \( U \): \( I_{\frac{s}{6}} = \{ J_{u_1 \text{Id}}, J_{u_2 \text{Id}} \} \) where \( u_1 = \frac{1 + \sin \theta}{\cos \theta}, \ u_2 = \frac{1 - \sin \theta}{\cos \theta} \) and \( \cos \theta = \left( \frac{s}{6} \right)^{-1} \).
Remark. We have $\frac{z}{y} = 1$ if, and only if, $(M, g, J_M)$ is anti-selfdual with $s \neq 0$. If it is the case globally then $(M, J_M)$ must be in class VII $[\text{Boy88}]$. We can find some global example of manifolds $(M, g, J_M)$ with arbitrary $\frac{z}{y}$ in $[\text{AM99}]$.

Proof of A. The multiplicity of the eigenvalue 0 of $A$ is equal to $3 \iff \kappa = s = 0 \iff (M, J_M, g)$ scalar-flat Kähler $\iff (M, J_M, g)$ anti-selfdual scalar-flat $[\text{Boy88}] \iff J_{Id}, J_\infty$ and $J_{Id}$ integrable by proposition 8. The equivalence with condition iv) will be a consequence of (the rest of the proof of) the theorem.

Proof of B. The multiplicity of the eigenvalue 0 of $A$ is equal to $2 \iff \kappa = s \neq 0 \iff (M, J_M, g)$ Kähler with $s \neq 0 \iff \mathcal{I} = \mathcal{I}_2 = \{J_\infty, J_\infty\}$ by Proposition 3.

Proof of C & D. In those cases the matrix of $A$ in a basis adapted to the decomposition $\mathbb{C} \otimes \Lambda^+ = \mathbb{C}J_M \oplus \Lambda_1^0 \oplus \Lambda_{0,1}$ is
\[
\begin{bmatrix}
  x & 0 & 0 \\
  0 & y & 0 \\
  0 & 0 & y
\end{bmatrix}
\] with $y \neq 0$. Let $f$ such that $J_f \in \mathcal{I}_2$, $(m, Q)$ be any point of $\mathcal{Z}$ and $(\theta_1, ..., \theta_4)$ be a local frame such that
\[
\begin{cases}
  J_M = \theta_1 \wedge \theta_2 + \theta_3 \wedge \theta_4 \\
  Q \in \text{Vect}(I, J)
\end{cases}
\] So there exist $(a, b, (\alpha, \beta, \gamma)) \in S^2$ such that $Q = aI + bJ$ and $P = f(Q) = \alpha I + \beta J + \gamma K$. In that case at the point $(m, Q)$ we have:
\[
G(\theta_1, \theta_2) = 0
\]
\[
\begin{aligned}
&\Rightarrow (a - \alpha)\gamma y a = 0 \\
&\Rightarrow \begin{cases}
  b(1 - a^2)x - b\beta y = a(a - \alpha)\beta y \\
  \gamma = 0 \\
  \beta bx = y(1 - a\alpha) \\
  a^2 + \beta^2 = 1
\end{cases}
\end{aligned}
\]
\[
\begin{aligned}
&\Rightarrow \begin{cases}
  \alpha = a \\
  \beta = \frac{a}{b} \\
  \beta^2 + \gamma^2 = b^2
\end{cases}
\end{aligned}
\]

The resolution of $G(\theta_1, \theta_3) = 0$ or $G(\theta_1, \theta_4) = 0$ gives the same system. Two cases can happen first $\sqrt{\frac{z}{y}} > 1$ then the second system doesn’t have any solution and the first one has two solutions. An easy computation enable us to verify that they correspond to $f_1 = u_1 I_d$ or $f_2 = u_2 I_d$.

On the other hand if $\sqrt{\frac{z}{y}} < 1$ then the second system gives two solutions which correspond to $f = e^{\pm \theta} I_d$, whereas the first system doesn’t have any solution:
\[
1 - a^2 = \beta^2 = \frac{y^2}{b^2}(1 - a\alpha)^2 > \frac{(1 - a\alpha)^2}{b^2}
\]
\[
\Rightarrow b^2 - b^2\alpha^2 > 1 + a^2\alpha^2 - 2a\alpha
\]
\[
\Rightarrow 0 > (\alpha - a)^2.
\]

When $\sqrt{\frac{z}{y}} = 1$ both systems give the same solutions.

G) Example

Let $T$ be a torus which is a quotient of $\mathbb{C}$ by the lattice $\mathbb{Z} \oplus i\mathbb{Z}$. Define $(M, g, I)$ to be the quotient of the complex flat torus $\mathbb{T}^2 = T \times T$ by the group $H = \mathbb{Z}/2\mathbb{Z}$ generated by an element $h$. If $(z_1, z_2) = (x_1 + ix_2, x_3 + ix_4)$ are the canonical coordinates on $\mathbb{C} \times \mathbb{C}$, we have:
\[
h(z_1, z_2) = \left( z_1 + \frac{1}{2}, -z_2 \right).
\]
The manifold \((M, g, I)\) is a bi-elliptic surface which is scalar-flat Kähler; denote by \(Z \rightarrow M\) its twistor space. In this section we will study in details this example, especially the integrability of \(J_f\). Thanks to Theorem 1, one knows that \(J_{Id}, J_\infty\) and \(J_{\lambda Id}\) are integrable.

Let \(\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}, \frac{\partial}{\partial x^4}\right)\) be the canonical basis of \(\mathbb{C}^2\) identified with \(\mathbb{R}^4\). This furnishes a basis of vector fields on \(\mathbb{T}^2\) and, consequently, a global trivialisation of its twistor space \(Z_0 \cong \mathbb{T}^2 \times \mathbb{S}^2\). Define another basis (on \(\mathbb{T}^2\)) by:

\[
\theta_1 + i\theta_2 = \frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2} \quad \text{and} \quad \theta_3 + i\theta_4 = e^{2i\pi x_1} \left(\frac{\partial}{\partial x^3} + i\frac{\partial}{\partial x^4}\right).
\]

Then, \((\theta_1, \theta_2, \theta_3, \theta_4)\) is a global basis on \(\mathbb{T}^2\) which goes down to a basis of \(M\). This defines a new trivialisation of \(Z_0\), denoted by \(\tilde{M} \times \mathbb{S}^2\). The manifold \(Z\) is the quotient of \(\tilde{M} \times \mathbb{S}^2\) by the group \(\tilde{H} \simeq \mathbb{Z}/2\mathbb{Z}\), generated by \(\tilde{h}\) acting as follows:

\[
\tilde{h}: \tilde{M} \times \mathbb{S}^2 \longrightarrow \tilde{M} \times \mathbb{S}^2 \quad \left(m, Q\right) \longmapsto (\tilde{h}(m), Q).
\]

Viewing \(\mathbb{S}^2\) as a subspace of \(\mathbb{R} \times \mathbb{C}\) with coordinates \((a, z)\), the identity map \(\Psi\) of \(Z_0\) has the following form in these trivialisations:

\[
\Psi: Z_0 \cong \mathbb{T}^2 \times \mathbb{S}^2 \longrightarrow Z_0 \cong \tilde{M} \times \mathbb{S}^2 \quad \xi \mapsto \left(m, (a, z)\right) \longmapsto (m, (a, e^{-2i\pi x_1} z)).
\]

The matrix, in both basis \(\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}, \frac{\partial}{\partial x^4}\right)\) and \((\theta_1, \theta_2, \theta_3, \theta_4)\), of the natural complex structure \(I\) on \(\mathbb{T}^2\) is equal to

\[
\begin{bmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{bmatrix}.
\]

According to our notation, this is the infinity section.

Endow \(Z_0\) with the complex structure of twistor space \(\mathbb{J}_{Id}\). As \((\mathbb{T}^2, I)\) is hyperkähler, the projection \(pr_2: Z_0 \cong \mathbb{T}^2 \times \mathbb{S}^2 \longrightarrow \mathbb{CP}^1\) is a holomorphic submersion [Boy88]. For \(n \in \mathbb{N}^*\) and \(\lambda \in \mathbb{C}^*\), consider the application \(f_n: \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1\) equal to \(\lambda z^n\). Then there exist two applications \(f_1, f_2\) depending only on \(|\lambda|\) such that:

\[
\begin{array}{ccc}
\mathbb{S}^2 & \xrightarrow{f_n} & \mathbb{S}^2 \\
(a, z) & \longmapsto & \left(f_1(a), \lambda f_2(a) z^n\right) \\
\mathbb{C} \cup \{\infty\} & \xrightarrow{f_n} & \mathbb{C} \cup \{\infty\} \\
U = \frac{z}{1 - a} & \longmapsto & \lambda U^n
\end{array}
\]

Introduce now the pull back \(Z_n = f_n^* Z_0\):

\[
\begin{array}{ccc}
\mathbb{CP}^1 & \xrightarrow{f_n} & \mathbb{CP}^1 \\
\mathbb{C} \cup \{\infty\} & \longmapsto & \mathbb{CP}^1 \\
\mathbb{Z}_{\lambda Id} & \longmapsto & \mathbb{Z}_{\lambda Id}
\end{array}
\]

Since the fibration \(Z_0 \longrightarrow \mathbb{CP}^1\) is topologically trivial, this is also the case for \(Z_n \longrightarrow \mathbb{CP}^1\). Therefore one can identify the manifold \(Z_n\) with \(\mathbb{T}^2 \times \mathbb{S}^2\) equipped
with a complex structure denoted by $J_n$. If one considers the morphism $\tilde{f}_n = \text{Id} \times f_n : T^2 \times S^2 \rightarrow T^2 \times S^2$, which respects the fibration, one has $J_n = \tilde{f}_n$.

We were wondering whether this complex structure goes down to $Z$, i.e.: does it commute with the action of the group $H$? We need to study $\Psi \circ \tilde{f}_n \circ \Psi^{-1}$:

\[
\begin{array}{cccc}
(T^2 \times S^2, J_n) & \xrightarrow{\tilde{f}_n} & (T^2 \times S^2, J_{Id}) \\
(m, (a, e^{2\pi i z})) & \Psi & (m, (f_1(a), \lambda f_2(a)(e^{2\pi i z})^n)) \\
\tilde{M} \times S^2, J_n) & \Psi \circ \tilde{f}_n \circ \Psi^{-1} & \tilde{M} \times S^2, J_{Id}) \\
(m, (a, z)) & \rightarrow & (m, (f_1(a), \lambda e^{2\pi i z} f_2(a)z^n))
\end{array}
\]

Thus, in the trivialisation of $Z_0 \cong \tilde{M} \times S^2$ associated to $(\theta_1, \theta_2, \theta_3, \theta_4)$, the complex structure $J_n$ is $\Psi \circ \tilde{f}_n \circ \Psi^{-1} = J_{\lambda e^{2\pi i z} x}$. It commutes with $H$ if and only if $n$ is odd. Moreover, for $n=1$, $\tilde{f}_1$ is a biholomorphism. We have proved the following:

**Proposition 9.** For all $\lambda \in \mathbb{C}^*$ the complex structures $J_{\lambda z}$ on $Z$ are biholomorphic. Furthermore, the compatible almost complex structures $J_{\lambda e^{2\pi i z} x}$ are integrable for odd $n$.

This proposition can be generalised to other bi-elliptic surfaces. A computation similar to the one made in Proposition 5 enables us to say that, for different integers $n$, these complex structures are not deformation of each other. This is consequence of the fact that they do not have the same Chern classes. Indeed, the first Chern class satisfies $c_1(J_{\lambda e^{2\pi i z} x}) = 2(n + 1)$. In [Del88], following an idea of LeBrun, we showed that for any hypercomplex manifold $M$ there exist infinitely many complex structures on its twistor space $Z \cong M \times S^2$ which are not deformation of each other. Recall that the only compact hypercomplex surfaces are the torus, the $K3$-surfaces and the quaternionic Hopf surfaces [Boy88]. The previous proposition can therefore be viewed as a generalisation of this result to bi-elliptic surfaces.

**H) Higher dimension**

The previous sections have focused on the 4-dimensional case. We now briefly give a generalization of Theorem 1 in higher dimension. Let $n > 1$ and $(M, g)$ be an oriented $4n$-dimensional Riemannian manifold, not necessarily compact. An almost hypercomplex structure on $(M, g)$ is a triple $(I, J, K)$ of almost complex structures compatible with the orientation and the metric, such that $IJ = -JI = K$. When $I, J, K$ are integrable one speaks about a hypercomplex structure. When they are Kähler one says that $(M, g)$ is hyperkähler.

An almost quaternionic structure $D$ on $(M, g)$ is a rank 3 subbundle $D \subset \text{End}(TM)$ which is locally spanned by an almost hypercomplex structure $H = (I, J, K)$; such a triple is called a local admissible basis. For $n > 1$, one says that $(M, g, D)$ is a quaternionic structure if there exists a torsion free connection $\nabla$ on $TM$ preserving $D$. If one can choose $\nabla$ to be the Levi-Civita connection, $(M, g, D)$ is called quaternionic Kähler. This is equivalent to saying that the holonomy group of $g$ is contained in $\text{Sp}(1)\text{Sp}(n)$ [Bes87].

A compatible almost complex structure on $(M, g, D)$ is a section $J_M$ of $D \rightarrow M$ such that $J_M^2 = -\text{Id}$. 
Let \((M, g, D)\) be a Riemannian almost quaternionic 4n-manifold. One can define a scalar product on \(D\) by saying that a local admissible basis of \(D\) is orthonormal. One can then define the twistor space \(Z \rightarrow M\), which is the unit sphere bundle of \(D\). This is a locally trivial bundle over \(M\) with fiber \(S^2\) and structure group \(SO(3)\). As in the introduction, one can define a natural metric \(\tilde{g}\) and look for the compatible almost complex structures on \((Z, \tilde{g})\) which are integrable. When \((M, g, D, J_M)\) is quaternionic Kähler with a compatible almost complex structure \(J_M\), its twistor space \((Z, \tilde{g})\) admits different compatible almost complex structures: \(J_\sigma, J_{Id}, J_\infty, J_{\lambda Id}\), defined as previously. The main result of this section is the following, where no hypothesis of compacity is made.

**Theorem 3.** Let \((M, g, D)\) be a quaternionic Kähler manifold.

A) The almost complex structure \(J_\sigma\) is never integrable.

B) The almost complex structure \(J_{Id}\) is always integrable [Sal82].

C) If \((M, g, D, J_M)\) is a compatible almost complex quaternionic Kähler manifold the almost complex structure \(J_\infty\) is integrable if, and only if:

i) \(J_M\) is integrable;

ii) \(g\) is scalar-flat.

D) If \((M, g, D, J_M)\) is a quaternionic Kähler manifold with a compatible Kählerian complex structure \(J_M\) then, for all \(\lambda \notin \{0, 1\}\), the complex structure \(J_{\lambda Id}\) is integrable if, and only if, \(g\) is scalar-flat.

E) Let \((M, g, D)\) be a quaternionic Kähler manifold. Then the scalar curvature is flat if, and only if, one (and then any) \(m \in M\) has an open neighborhood \(U\) such that \((Z, \tilde{g})\) admits over \(U\) an integrable compatible complex structure different from \(J_{Id}\).

Any quaternionic Kähler manifold which is scalar-flat is locally hyperkähler [Bes87]. Thus, part E of the previous theorem yields a characterization of locally hyperkähler manifolds among quaternionic Kähler’s in terms of twistor spaces.

It is possible to give a simpler version of that theorem in the compact case because of the following result.

**Proposition [Pon94].** In the compact case any compatible complex structure \(J_M\) on a quaternionic Kähler manifold \((M, g, D)\) is automatically scalar-flat Kähler.

In particular, in the compact case, Theorem 3 has the following corollary.

**Corollary 2.** Let \((M, g, D, J_M)\) be a compact quaternionic Kähler manifold with a compatible almost complex structure. Then \(J_M\) is integrable if, and only if, \(J_\infty\) is integrable. In this case \(J_{\lambda Id}\) is integrable for all \(\lambda \in \mathbb{C}^*\).

**Proof of Theorem 3.** Proposition 1 and Proposition 2 remain true in dimension \(4n\). Since \(\sigma\) is an antiholomorphic involution when restricted to the fibers, part A can be easily proved.

The proof of part B is the same as the one given in dimension 4. Notice first that \(d\sigma F_{ij} = -E(\theta_i, \theta_j)\) for all \((i, j) \in \{1, \ldots, 4n\}\). It remains to show that \(G(\theta_i, \theta_j) = 0\) for all \(i, j \in \{1, \ldots, 4n\}\). To get that result we use the following lemma.

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Let \( r(\cdot,\cdot) \) be the Ricci tensor. For all \((X,Y) \in TM\) one has:

\[
\begin{align*}
[R(X,Y),I] &= \gamma(X,Y)J - \beta(X,Y)K \\
[R(X,Y),J] &= -\gamma(X,Y)I + \alpha(X,Y)K \\
[R(X,Y),K] &= \beta(X,Y)I - \alpha(X,Y)J
\end{align*}
\]

with \( \alpha(X,Y) = \frac{2}{n+2}r(IX,X) \), \( \beta(X,Y) = \frac{2}{n+2}r(JX,X) \), \( \gamma(X,Y) = \frac{2}{n+2}r(KX,X) \).

Let \((m,I) \in Z\) and \((I,J,K)\) be a local admissible basis. Then Lemma 3 yields:

\[
G(\theta_i,\theta_j) = \frac{2s}{4(n+2)}\left(2g(K\theta_i,\theta_j) - 2g(K\theta_i,\theta_j)\right)J + \left(2g(J\theta_i,\theta_j) - 2g(J\theta_i,\theta_j)\right)K
\]

But any quaternionic Kähler manifold is Einstein \([\text{Ber}66]\), hence \( r = \frac{4}{s}g \), where \( s \) is the scalar curvature of \( g \). One then has, for all \((\theta_i,\theta_j)\):

\[
G(\theta_i,\theta_j) = \frac{2s}{4(n+2)}\left(2g(K\theta_i,\theta_j) - 2g(K\theta_i,\theta_j)\right)J + \left(2g(J\theta_i,\theta_j) - 2g(J\theta_i,\theta_j)\right)K
\]

Let \( f : Z \rightarrow Z \) be a morphism such that \( \theta_i \) is integrable over an open set \( \mathcal{U} \) of \( M \) in the basis \( \theta_i \). If \( \mathcal{U} \) is small enough there exists an orthonormal basis \( \theta_1, \ldots, \theta_{4n} \) and a local admissible basis \((I,J,K)\) such that \( P = J \). Write \( Q = aI + bJ + cK \) with \((a,b,c) \in S^2\).

As \( J \) is integrable we have \( G(\theta_1,\theta_2) = 0 \) everywhere. In particular at the point \((m,Q)\) :

\[
G(\theta_1,\theta_2) = 0
\]

Hence \( Q = J = P \) for any \((m,Q) \) is \( \pi^{-1}(\mathcal{U}) \), that is \( f = Id \).

The converse is the same as the one given in section E. Indeed, a quaternionic Kähler manifolds \((M,g,D)\) admits, locally, infinitely many compatible complex structures \( J_M \) (for example \([\text{AMP}98]\)). \(\square\)

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