THE $\bar{\partial}$ STEEPEST DESCENT METHOD FOR ORTHOGONAL POLYNOMIALS ON THE REAL LINE WITH VARYING WEIGHTS

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ABSTRACT. We obtain Plancherel-Rotach type asymptotics valid in all regions of the complex plane for orthogonal polynomials with varying weights of the form $e^{-NV(x)}$ on the real line, assuming that $V$ has only two Lipschitz continuous derivatives and that the corresponding equilibrium measure has typical support properties. As an application we extend the universality class for bulk and edge asymptotics of eigenvalue statistics in unitary invariant Hermitian random matrix theory. Our methodology involves developing a new technique of asymptotic analysis for matrix Riemann-Hilbert problems with nonanalytic jump matrices suitable for analyzing such problems even near transition points where the solution changes from oscillatory to exponential behavior.

1. Introduction

Let $V(x)$ be a real-valued function (an external field or potential) growing faster as $|x| \to \infty$ than $[\log(1 + x^2)]^{1+\epsilon}$ for some $\epsilon > 0$. In particular, this implies that all moments of the measure on $\mathbb{R}$ given by

$$d\nu_N(x) := e^{-NV(x)} \, dx$$

are finite. A measure of this form is said to be a varying weight due to the presence of the parameter $N$. This paper concerns the asymptotic behavior of polynomials orthogonal with respect to varying weights of the form (1). They are defined as follows. For $n = 0, 1, 2, \ldots$, let $p_n = p_n(z; N) = \kappa_{n,n} z^n + \cdots + \kappa_{n,0}, \kappa_{n,n} > 0$ be the (unique) polynomial of degree $n$ satisfying

$$\int_{\mathbb{R}} p_n(x) x^k d\nu_N(x) = 0, \quad \text{for } 0 \leq k < n,$$

(2)

and

$$\int_{\mathbb{R}} p_n(x)^2 d\nu_N(x) = 1.$$

(3)

The interest is in the behavior of the polynomials of degree $N$ and $N - 1$, where the integer $N$ is the same which appears in the measure of orthogonality, in the limit $N \to \infty$. We obtain a precise description of the polynomials $p_N(z; N)$ and $p_{N-1}(z; N)$ which is uniformly valid for all $z \in \mathbb{C}$, for all $N$ sufficiently large. This type of asymptotic description is often referred to as Plancherel-Rotach asymptotics, after the analysis of the Hermite polynomials in [21].

In the late 1990s new Riemann-Hilbert techniques originally developed for the asymptotic analysis of integrable nonlinear partial differential equations were applied to the asymptotic analysis of Riemann-Hilbert problems encoding systems of orthogonal polynomials with respect to varying weights on $\mathbb{R}$, first for the case of external fields of the form $V(x) = x^4 - \gamma x^2$ [2], and then for the case of general real analytic $V$ in [6, 7]. (See [3] for more information about Plancherel-Rotach type asymptotics for orthogonal polynomials prior to the use of Riemann-Hilbert techniques.) The Riemann-Hilbert method has been extended, and applied to various types of asymptotic questions in approximation theory (see, for example, [12, 15, 14, and 13]). With the exception of [12] and [15], all of these applications and extensions deal with weights that are real analytic.

The main purpose of this manuscript is to establish Plancherel-Rotach type asymptotics for orthogonal polynomials, when the external field $V$ possesses only two Lipschitz continuous derivatives, i.e. in the absence of analyticity. (The precise assumptions on the external field $V$ are most naturally described in terms of the equilibrium measure to be defined in subsection 1.3 below.) To obtain such a uniform asymptotic description
we present a new hybrid Riemann-Hilbert-∂ method of asymptotic analysis, that is a significant extension of the ∂-method introduced in [15] to analyze orthogonal polynomials on the unit circle. By contrast with that method, a fundamental new feature of orthogonal polynomials with varying weights on the real line is the presence of “transition points” (also known as endpoints of the equilibrium measure) in the neighborhood of which the asymptotic behavior exhibits a complicated transition from oscillatory to exponential behavior.

1.1. Application to random matrices. Among many applications of the asymptotic theory of orthogonal polynomials is the calculation of certain statistics of eigenvalues in random matrix theory. Unitary invariant ensembles of random matrices are described by probability measures of the form

\[ d\mathbb{P}_N(M) = \frac{1}{Z_N} e^{-N\text{Tr}(V(M))} dM, \]

defined on \( N \times N \) Hermitian matrices \( M \), where \( V(x) \) is an external field of the type described earlier. Here \( dM \) denotes Lebesgue measure on the algebraically independent entries:

\[ dM = \prod_{j=1}^{N} dM_{jj} \prod_{1 \leq j < k \leq N} d\text{Re}(M_{jk}) d\text{Im}(M_{jk}), \]

and \( Z_N \) is a normalization constant (partition function). One of the origins of the theory of random matrices in physics was the study of nuclear resonance levels in the 1950s. See [16] and the references contained therein for more information.

1.1.1. Connection with orthogonal polynomials. A remarkable connection to orthogonal polynomials was discovered in the late 1960s by Gaudin and Mehta [17]. The connection is the following formula for the density of the probability measure on eigenvalues induced by (4):

\[ \mathbb{P}_N(\lambda_1, \ldots, \lambda_N) = \frac{1}{N!} \det (K_N(\lambda_i, \lambda_j))_{1 \leq i,j \leq N}, \]

where the function \( K_N(x, y) \) is the so-called reproducing kernel of orthogonal polynomials:

\[ K_N(x, y) = e^{-N(V(x)+V(y))/2} \sum_{n=0}^{N-1} p_n(x)p_n(y), \]

the polynomials \( p_n(x) \) being defined in (2)–(3). It is a basic result of the theory that (6) indeed defines a probability measure on \( \mathbb{R}^N \).

From formula (6) one may effectively compute many statistical quantities involving the eigenvalues (see [16], and also [3]). Two examples are as follows:

- **Mean density of eigenvalues** \( \rho_1^{(N)}(\lambda) \) defined as

\[ \rho_1^{(N)}(\lambda) := \frac{d}{d\lambda} \mathbb{E}_N \left( \frac{1}{N} \# \{\text{eigenvalues } \lambda_j \text{ such that } \lambda_j < \lambda \} \right), \]

where \( \mathbb{E}_N(\cdot) \) denotes the expectation of \( \cdot \) with respect to the probability measure (4) or equivalently (6). This may be equivalently represented in terms of the orthogonal polynomials:

\[ (9) \quad \rho_1^{(N)}(\lambda) = \frac{1}{N} K_N(\lambda, \lambda). \]

- **Gap probabilities** \( F_{(a,b)} \) defined as

\[ (10) \quad F_{(a,b)} := \text{Prob} (\text{no eigenvalues in } (a, b)), \]

which may be equivalently represented in terms of a Fredholm determinant built out of the orthogonal polynomials:

\[ (11) \quad F_{(a,b)} = \det \left( 1 - K_N|_{L^2(a,b)} \right). \]

Here the integral operator \( K_N : L^2(a, b) \to L^2(a, b) \) possesses the integral kernel \( K_N(x, y) \):

\[ K_N h(x) = \int_a^b K_N(x, y)h(y) dy. \]
One important example of the gap probability described in [10] and [11] is the case that $b = \infty$, for then the gap probability coincides with the distribution function of the largest eigenvalue:

$$F(a, +\infty) = \text{Prob}(\lambda_{\text{max}} < a) = \det \left(1 - K_N|_{L^2(a, +\infty)}\right).$$

1.1.2. Asymptotic behavior as $N \to \infty$. A basic and important result concerning the $N \to \infty$ asymptotic behavior of random matrices is that the mean density of eigenvalues $\rho_N^{(N)}$ has a limit: for all $\lambda \in \mathbb{R}$,

$$\lim_{N \to \infty} \rho_N^{(N)}(\lambda) = \psi(\lambda).$$

Note: the Gaussian Unitary Ensemble (GUE) first studied by Wigner corresponds to $V(x) = x^2$, and in this case $\psi(\lambda) = \pi^{-1/2}e^{-\lambda^2}$, which is the famous Wigner semicircle law. It is well-known that the limit [14] exists for quite general $V(x)$. It is also known that if $V$ is real analytic, the convergence in [14] is uniform. For those nonanalytic $V$ for which existence of the limiting density $\psi(\lambda)$ can be established, the convergence implied by the statement [14] has only been proven in a weaker sense. One consequence of the present work is that the convergence in [14] is in fact uniform assuming only that the function $V$ possesses 2 Lipschitz continuous derivatives.

The function $\psi$ is also a well-known quantity in approximation theory, where it is referred to as the density of the equilibrium measure. The equilibrium measure is defined in subsection 1.3 (for the purposes of the current discussion one may take the parameter $c$ appearing in the definition of the equilibrium measure to be unity).

In many circumstances, the largest eigenvalue distribution has been shown to possess a limit as $N \to \infty$ known as the Tracy-Widom distribution, a distribution function expressible in closed form in terms of the Hastings-McLeod solution of the Painlevé II equation. The form of the asymptotic result is:

$$\lim_{N \to \infty} \text{Prob} \left( \lambda_{\text{max}} < \beta + (\lambda N)^{-2/3}s \right) = F_{\text{TW}}(s)$$

where the constant $\lambda$ depends on the external field $V$, $\beta = \sup(\text{supp}(\psi))$, and $F_{\text{TW}}(s)$ is the famous Tracy-Widom distribution.

Another fundamental object concerning the eigenvalues of random matrices is the limiting spacing distribution. Properly speaking, this is defined in terms of the spacing between ordered eigenvalues; however a “poor-man’s” version of this distribution is the following (easier to define) quantity:

$$Q(s) := \lim_{N \to \infty} \text{Prob} \left( \text{no eigenvalues in } (a, a + \frac{s}{N\rho_N^{(N)}(a)}) \right).$$

This limit is known to exist provided the external field is real analytic and provided that $a$ is such that $\psi(a) > 0$, and it turns out that the function $Q(s)$ which emerges in the limit is universal in that it does not depend on properties of the function $V$. Indeed, under the assumption that $V$ is real analytic, one has

$$Q(s) = \det \left(1 - S|_{L^2(0, s)}\right),$$

where $S$ is an integral operator on the interval $(0, s)$:

$$Sh(x) := \int_0^s \frac{\sin(\pi(x - y))}{\pi(x - y)}h(y) \, dy.$$

Via the connection to orthogonal polynomials explained earlier, the following asymptotic result concerning the reproducing kernel $K_N(x, y)$ built from the orthogonal polynomials implies [15]:

**Asymptotic Result 1.** There is a constant $\lambda$ so that for every $u, v \in \mathbb{R}$, we have

$$\lim_{N \to \infty} \frac{1}{(\lambda N)^{2/3}} K_N \left(\beta + \frac{u}{(\lambda N)^{2/3}}; \beta + \frac{v}{(\lambda N)^{2/3}}\right) = \frac{\Lambda(u)\Lambda'(v) - \Lambda(v)\Lambda'(u)}{u - v}.$$

Here $\Lambda$ denotes the unique solution to Airy’s equation $y'' = xy$ that is real, and that behaves as follows as $x \to +\infty$: $\Lambda'(x) \sim e^{-2x^{3/2}/(2\sqrt{\pi}x^{1/4})}$.

Similarly, the limit appearing in (16) is implied by the following result:
Asymptotic Result 2. For every $a$ with $\psi(a) > 0$, and every $u, v \in \mathbb{R}$, we have

\[
\lim_{N \to \infty} \frac{1}{N\psi(a)} K_N \left( a + \frac{u}{N\psi(a)}, a + \frac{v}{N\psi(a)} \right) = \frac{\sin(\pi(u - v))}{\pi(u - v)}.
\]

Asymptotic Result 1 was first established in the special case of the Gaussian Unitary Ensemble (i.e. $V(x) = x^2$) \cite{22}, using the classical Plancherel-Rotach asymptotics of Hermite polynomials \cite{21}. This was extended to quartic potentials of the form $V(x) = x^4 - \gamma x^2$ in \cite{2}, where furthermore Asymptotic Result 2 was also established. Because the polynomials associated with quartic $V$ are not known to possess elementary contour integral representations, the analysis of \cite{2} required a new method, namely the use of the Riemann-Hilbert formulation of orthogonal polynomials found in \cite{10}. Asymptotic Result 2 was established for general real analytic potentials $V$ in \cite{7}, and the asymptotic formulae for orthogonal polynomials given in \cite{7} were used to establish Asymptotic Result 1 in \cite{5}. The new strategy introduced in \cite{7} was a general method linking the equilibrium measure associated with $V$ to a so-called $g$-function enabling the use of the non-commutative steepest descent technique for Riemann-Hilbert problems originally invented by Deift and Zhou \cite{9} and extended in \cite{8}. Pastur and Shcherbina \cite{19} have also studied the problem of establishing Asymptotic Result 2 under the assumption that $V$ has three continuous derivatives.

As is clear from the above discussion, the historical trend is toward establishing Asymptotic Results 1 and 2 for more and more general external fields $V$. The program of universality in random matrix theory is concerned with determining the most general external fields $V$ under which such results hold true. In particular, it is of some interest to admit external fields that are not real analytic. As pointed out by Deift in \cite{4}, the steepest descent method that works so well for analytic $V$ cannot be easily applied to the nonanalytic case. The authors recently introduced a "$\bar{\partial}$ steepest descent method" applicable to some Riemann-Hilbert problems involving nonanalytic data, but as formulated in \cite{15} this method does not apply to the orthogonal polynomials described by conditions \cite{3} because the equilibrium measure is compactly supported and the endpoints of support obstruct the type of nonanalytic deformations involved in the method.

Among the applications of the results in this manuscript are proofs of Asymptotic Results 1 and 2 under weakened hypotheses on the external field $V$ (we require two Lipschitz continuous derivatives) via rigorous Plancherel-Rotach type asymptotics for orthogonal polynomials. Our method involves a hybrid "Riemann-Hilbert-$\bar{\partial}$ steepest descent method" generalizing the simpler method of \cite{15} to handle support endpoints. Our results hold under weaker conditions on $V$ than those under which Asymptotic Result 2 is considered in the paper \cite{19}, and to our knowledge we have the first proof that Asymptotic Result 1 holds in the absence of analyticity of $V$.

We remark that it is not necessary to first obtain asymptotics for the orthogonal polynomials themselves in order to deduce enough information about the reproducing kernels $K_N(x, y)$ to establish Asymptotic Results 1 and 2 for certain general external fields $V$. For example, a recently introduced method (based on a comparison principle for Christoffel functions) of Levin and Lubinsky has been quite successful in establishing Asymptotic Result 2 \cite{14} under extremely weak global conditions on the external field $V$ and stronger local conditions (but still far from analyticity) near the point $a$ of expansion in the spectrum. Also, Asymptotic Result 1 has been studied for real analytic $V$ without the use of orthogonal polynomials by Pastur and Shcherbina \cite{20}.

1.2. Essence of the $\bar{\partial}$ method. In the asymptotic analysis of Riemann-Hilbert problems there is an analog of contour deformations which plays a crucial role in identifying subsets of the plane which produce the dominant contribution to the Riemann-Hilbert problem’s solution. It is common to begin with a Riemann-Hilbert problem whose solution, $A$, is analytic off a given contour $\Sigma_A$, and to use explicit piecewise analytic quantities to define a new matrix $B$ solving a new equivalent Riemann-Hilbert problem in which the relevant contour $\Sigma_B$ is a deformation of the original contour $\Sigma_A$.

A fundamental obstacle to this procedure occurs when one requires the analytic extension from a given contour of a rapidly oscillating function whose phase possesses no analyticity properties. For the asymptotic analysis of orthogonal polynomials with varying weights on the real line, in which the external field $V$ possesses only finitely many derivatives, this is a central issue.
In this paper, the new approach which circumvents this problem is to depart from Riemann-Hilbert problems entirely, by introducing transformations that explicitly violate analyticity. Instead of Riemann-Hilbert problems, we then characterize our newly-defined matrix-valued function as the unique solution of a \( \overline{\partial} \) problem.

Given a smooth matrix-valued function \( \mathbf{W}(x, y) \) of compact support in \( \mathbb{R}^2 \), a \( \overline{\partial} \) problem is a first-order system of linear partial differential equations on \( \mathbb{R}^2 \) involving \( \mathbf{W}(x, y) \) as coefficients and the Cauchy-Riemann operator

\[
(21) \quad \overline{\partial} := \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)
\]

acting on the unknown. Here is a prototypical example.

**Problem 0 (Prototype).** Determine a \( 2 \times 2 \) matrix \( \mathbf{A}(x, y) \) for \( (x, y) \in \mathbb{R}^2 \) having the following properties:

- **Continuity.** \( \mathbf{A}(x, y) \) is a continuous function of \( x \) and \( y \) for \( x + iy \in \mathbb{C} \).
- **Deviation From Analyticity.** For \( x + iy \in \mathbb{C} \),

\[
(22) \quad \overline{\partial} \mathbf{A}(x, y) = \mathbf{A}(x, y) \mathbf{W}(x, y),
\]

(note that \( \mathbf{A}(x, y) \) is analytic off the support of \( \mathbf{W} \), because there one has \( \overline{\partial} \mathbf{A}(x, y) = 0 \).

- **Normalization.** The matrix \( \mathbf{A}(x, y) \) is normalized as follows:

\[
(23) \quad \lim_{x,y \to \infty} \mathbf{A}(x, y) = \mathbb{I}.
\]

Once one admits the possibility to use non-analytic extensions of functions originally defined on contours, one is faced with an overabundance of choices, and the issue becomes one of selecting, constructing, or otherwise establishing the existence of, an extension suitable for subsequent asymptotic analysis.

This idea actually yields an interesting approach to a classical result of asymptotic analysis. Given a real-valued function \( \theta : [-1, 1] \to \mathbb{R} \) satisfying \( \theta(0) = \theta'(0) = 0 \), \( \theta''(0) > 0 \), the problem is to provide a large \( n \) asymptotic description for the integral

\[
(24) \quad I(n) := \int_{-1}^{1} e^{i\theta(x)} \, dx.
\]

For convenience, let us assume that \( \theta''(x) \) is bounded and \( \theta''(x) \geq w > 0 \) for all \( x \in [-1, 1] \).

The usual approach to this problem (see, for example, [13]) involves many steps, including integration by parts, implicit variable changes, and Taylor expansions, the result of which is

\[
(25) \quad I(n) = \sqrt{\frac{2\pi}{n^{\theta''(0)}}} e^{i\pi/4} \left( 1 + \mathcal{O} \left( n^{-1/2} \right) \right).
\]

We may instead establish this in the following way, which elucidates certain aspects of the methods we use in the sequel. Let \( \Theta(x, y) \) represent an arbitrary extension of \( \theta(x) \), which satisfies \( \Theta(x, 0) = \theta(x) \). Then with the aid of Stokes’ theorem, we may write

\[
(26) \quad I(n) = -\int_{\Gamma} e^{i\Theta(x,y)} \, d(x + iy) + 2i \int_{A} \overline{\partial} e^{i\Theta(x,y)} \, dA,
\]

where \( \Gamma \) represents a contour in \( \mathbb{C} \) from 1 to \( -1 \) (different than the interval \([-1, 1]\)), and \( A \) represents the (oriented) area enclosed by the oriented contour formed by the union of \([-1, 1]\) with \( \Gamma \). See Figure 1.

Were the function \( \Theta(x, y) \) analytic, the double integral appearing on the right-hand side of (26) would not be present, and we could choose the contour \( \Gamma \) to be the contour of steepest descent. Although \( \Theta(x, y) \) cannot be analytic if only three derivatives of \( \theta(x) \) are assumed to exist, we nonetheless observe that the right-hand side of (26) is still independent of the choice of both the contour \( \Gamma \) and the particular extension \( \Theta(x, y) \). This begs the question: Can we pick the extension \( \Theta(x, y) \) and the contour \( \Gamma \) so that the right-hand side of (26) may be easily estimated? The answer is yes.

We take \( \Gamma \) to be the contour comprised of a vertical segment from \((1, 0)\) to \((1, 1)\), followed by the line segment connecting \((1, 1)\) to \((-1, -1)\), and ending with the vertical line segment from \((-1, -1)\) to \((-1, 0)\), and let \( A_+ \) and \( A_- \) denote the interior of the two triangles formed by this contour and the real interval \([-1, 1]\), \( A_+ \) in the first quadrant, and \( A_- \) in the third quadrant, exactly as illustrated in Figure 1. We
Figure 1. The contour $\Gamma$ and the oriented area $A = A_+ \cup A_-$ for the analysis of $I(n)$.

will explicitly construct an extension $\Theta(x, y)$ of $\theta(x)$ defined for $(x, y) \in \overline{A_+ \cup A_-}$ to satisfy the following conditions for some constants $K > 0$ and $k > 0$:

1. $\Theta(x, 0) = \theta(x)$, for $-1 \leq x \leq 1$.
2. $\Theta(x, x) = \frac{1}{2} \theta''(0)(x + ix)^2$, for $-1 \leq x \leq 1$.
3. $|\overline{\partial} \Theta(x, y)| \leq Ky^2$ for all $(x, y) \in A_+ \cup A_-$. 
4. $\text{Im}(\Theta(x, y)) \geq kxy$ for all $(x, y) \in A_+ \cup A_-$. 

Using such an extension and properties (C1) and (C2), the representation (26) may be rewritten as

\[
I(n) - e^{i\pi/4} \int_{-\sqrt{2}}^{\sqrt{2}} e^{-n\theta''(0)s^2/2} ds =
\]

\[
i \int_{0}^{-1} e^{in\Theta(-1, y)} dy + i \int_{1}^{0} e^{in\Theta(1, y)} dy
\]

\[
- 2n \int_{A_+} e^{in\Theta(x, y)} \overline{\Theta}(x, y) dx dy + 2n \int_{A_-} e^{in\Theta(x, y)} \overline{\Theta}(x, y) dx dy.
\]

The four integrals on the right-hand side may be estimated directly with the help of properties (C3) and (C4):

\[
| i \int_{0}^{-1} e^{in\Theta(-1, y)} dy | \leq \int_{0}^{1} e^{-n\text{Im}(\Theta(-1, -s))} ds \leq \int_{0}^{1} e^{-kn s} ds \leq \int_{0}^{+\infty} e^{-kn s} ds = \frac{1}{kn}.
\]

\[
| i \int_{1}^{0} e^{in\Theta(1, y)} dy | \leq \int_{0}^{1} e^{-n\text{Im}(\Theta(1, s))} ds \leq \int_{0}^{1} e^{-kn s} ds \leq \int_{0}^{+\infty} e^{-kn s} ds = \frac{1}{kn}.
\]
\[ \left| 2n \int_{A_\pm} e^{i\phi(x,y)} \overline{\varTheta(x,y)} \, dx \, dy \right| \leq 2n \int_{A_\pm} e^{-n \text{Im}(\phi(x,y))} \left| \overline{\varTheta(x,y)} \right| \, dx \, dy \]

\[ = 2K_n \int_{A_\pm} e^{-knxy^2} \, dx \, dy \]

\[ = 2K_n \int_{A_\pm} e^{-knxy^2} \, dx \, dy \leq 2K_n \int_0^\infty r \, dr \int_0^{\pi/4} d\theta \, e^{-knr^2 \cos(\theta) \sin(\theta) r^2 \sin^2(\theta)} \]

\[ = \frac{2K}{k^2} \int_0^\infty e^{-u^2} \, du \int_0^{\pi/4} d\theta \frac{d\theta}{\cos^2(\theta)}. \]

Therefore all terms on the right-hand side of (27) are \( O(n^{-1}) \) as \( n \to \infty \). Now, since

\[ e^{i\pi/4} \int_{-\sqrt{2}}^{\sqrt{2}} e^{-n \theta''(0)s^2/2} \, ds = \sqrt{\frac{2\pi}{n \theta''(0)}} e^{\pi i/4} + \text{exponentially small terms as } n \to \infty, \]

we have established (25) if we can find an extension \( \varTheta(x,y) \) of \( \theta(x) \) satisfying conditions (C1)-(C4).

The extension \( \varTheta(x,y) \) may be defined as follows. First let \( B(t) \) represent a “cut-off” or “bump” function which is infinitely differentiable and satisfies \( B(t) \equiv 0 \) for \( t < 0 \), and \( B(t) \equiv 1 \) for \( t > 1 \). More precisely, we assume that \( B : \mathbb{R} \to [0,1] \) is of class \( C^\infty(\mathbb{R}) \) and satisfies \( B(x) \equiv 0 \) for \( x \leq 0 \) and \( B(x) \equiv 1 \) for \( x \geq 1 \). An example of such a function is

\[ B(t) := \begin{cases} 
0, & x \leq 0 \\
\frac{1}{2} \tanh \left( \frac{t}{1-t^2} \right) + \frac{1}{2}, & 0 < x < 1 \\
1, & x \geq 1,
\end{cases} \]

but our analysis will never require the detail of this formula. Next define

\[ \varTheta_0(x,y) := \theta(x) + iy\theta'(x) + \frac{1}{2} (iy)^2 \theta''(x), \]

\[ \varTheta_0^\text{hol}(x,y) := \frac{1}{2} \theta''(0)(x + iy)^2. \]

Our extension \( \varTheta(x,y) \) is then defined via

\[ \varTheta(x,y) := \left[ 1 - B \left( \frac{y}{x} \right) \right] \varTheta_0(x,y) + B \left( \frac{y}{x} \right) \varTheta_0^\text{hol}(x,y). \]

Note that the function \( \varTheta_0(x,y) \) is an extension of \( \theta(x) \) that satisfies

\[ \overline{\varTheta_0(x,y)} = \frac{1}{4} (iy)^2 \theta''(x). \]

The function \( \varTheta_0(x,y) \) is a rectilinear version of the type of extension discussed in [15]. It does not match the desired quadratic on the diagonal part of the contour \( \Gamma \); we use the function \( B \) to smoothly deform this extension to the quadratic \( \varTheta_0^\text{hol}(x,y) \). Straightforward calculations show that \( \varTheta(x,y) \) defined in (34) satisfies the four conditions (C1)-(C4) described above. Indeed, \( \varTheta(x,0) = \varTheta_0(x,0) = \theta(x) \), so condition (C1) holds, and \( \varTheta(x,x) = \varTheta_0^\text{hol}(x,x) = \frac{1}{2} \theta''(0)(x + ix)^2 \), so condition (C2) holds. To confirm condition (C3), note first that \( \overline{\varTheta_0^\text{hol}(x,y)} \equiv 0 \), so using (35) we have

\[ \overline{\varTheta(x,y)} = - \left[ 1 - B \left( \frac{y}{x} \right) \right] \frac{1}{4} \theta''(x)y^2 + \frac{1}{2} B' \left( \frac{y}{x} \right) \left[ \frac{y}{x^2} - i \frac{1}{x} \right] \left[ \varTheta_0(x,y) - \varTheta_0^\text{hol}(x,y) \right], \]

and then by Taylor expansion

\[ \varTheta_0(x,y) - \varTheta_0^\text{hol}(x,y) = \frac{1}{3} \theta'''(\xi_1)x^3 + \frac{i}{2} \theta'''(\xi_2)x^2y - \frac{1}{2} \theta'''(\xi_3)xy^2, \]
for some numbers $\xi_1$, $\xi_2$, and $\xi_3$ in $[-1, 1]$, so since $\theta''$ is uniformly bounded and $0 \leq y/x \leq 1$ for $(x, y) \in A_+ \cup A_-$, we have
\begin{equation}
\Theta_0(x, y) - \Theta_0^{\text{hol}}(x, y) = \mathcal{O}(x^3),
\end{equation}
and so
\begin{equation}
\mathcal{J}\Theta(x, y) = \mathcal{O}(y^2) + B' \left( \frac{y}{x} \right) \mathcal{O}(x^2).
\end{equation}
Finally, since $B'(t) = \mathcal{O}(t^2)$ holds for all $t \in \mathbb{R}$, we have confirmed condition (C3). To check condition (C4), we calculate directly
\begin{equation}
\text{Im}(\Theta(x, y)) = \left[ 1 - B \left( \frac{y}{x} \right) \right] \text{Im}(\Theta_0(x, y)) + B \left( \frac{y}{x} \right) \text{Im}(\Theta_0^{\text{hol}}(x, y))
\end{equation}
\begin{equation}
= \left[ 1 - B \left( \frac{y}{x} \right) \right] y\theta'(x) + B \left( \frac{y}{x} \right) \theta''(0)xy
\end{equation}
\begin{equation}
= \left[ 1 - B \left( \frac{y}{x} \right) \right] xy\theta''(\xi) + B \left( \frac{y}{x} \right) \theta''(0)xy,
\end{equation}
for some number $\xi \in [-1, 1]$. Then since by assumption $\theta''(x) \geq w > 0$ holds for $x \in [-1, 1]$ and since $B : \mathbb{R} \to [0, 1]$, we have
\begin{equation}
\text{Im}(\Theta(x, y)) \geq \left[ 1 - B \left( \frac{y}{x} \right) \right] wxy + B \left( \frac{y}{x} \right) wxy = wxy,
\end{equation}
so condition (C4) is verified as well.

Now we will be starting with a $2 \times 2$ matrix $B$, which is the solution of a Riemann-Hilbert problem in which the jump matrix contains entries of the form $e^{in\theta(x)}$, with $\theta$ real, and possessing only 2 Lipschitz continuous derivatives. We will define an extension of $\theta$ in exactly the spirit of the above calculations, and use it to define a new matrix-valued function $D$, that is no longer analytic. The matrix-valued function $D$ will be characterized by a hybrid Riemann-Hilbert-$\mathcal{J}$ problem. The main point is this: our extension of $\theta$ will be chosen so that this hybrid Riemann-Hilbert-$\mathcal{J}$ problem succumbs to a large-$n$ asymptotic analysis.

1.3. The equilibrium measure and associated quantities. The so-called equilibrium measure associated with the function $V(x)$ and the ratio $c := N/n$ is well-known to be a key ingredient in large-degree asymptotics of the polynomial $p_n(z)$ of degree $n$ in the orthonormal system associated with the measure $\nu_N(x)$ defined in terms of $N$ and $V$ by
\begin{equation}
\frac{d\nu_N}{d\nu}(x) = \frac{p_N(x)}{\|p_N\|_2^2}.
\end{equation}
Here $c > 0$ is held fixed as $n$ (and hence also $N$) tends to infinity. Generally, given a real-valued field $V(x)$ defined for $x \in \mathbb{R}$ and a parameter $c > 0$, we may consider the following associated weighted energy of a positive charge (Borel measure) $\mu$ on the real line $\mathbb{R} \subset \mathbb{C}$:
\begin{equation}
E[\mu] := \int_{\text{supp}(\mu)} \int_{\text{supp}(\mu)} \log \left( \frac{1}{|x-y|} \right) d\mu(x) d\mu(y) + c \int_{\text{supp}(\mu)} V(x) d\mu(x).
\end{equation}
The equilibrium measure $\mu_*$ is defined to be the unique positive measure $\mu_*$ minimizing $E[\mu]$ subject to the constraint
\begin{equation}
\int_{\text{supp}(\mu)} d\mu(x) = 1.
\end{equation}
The equilibrium measure is equivalently characterized by the corresponding Euler-Lagrange variational conditions. There is a real constant $\ell$ (the Lagrange multiplier originating from the constraint (43)) such that
\begin{equation}
\frac{\delta E}{\delta \mu}_{\mu=\mu_*}(x) := 2 \int_{\text{supp}(\mu_*)} \log \left( \frac{1}{|x-y|} \right) d\mu_*(y) + cV(x) \equiv -\ell, \quad x \in \text{supp}(\mu_*),
\end{equation}
and
\begin{equation}
\frac{\delta E}{\delta \mu}_{\mu=\mu_*}(x) := 2 \int_{\text{supp}(\mu_*)} \log \left( \frac{1}{|x-y|} \right) d\mu_*(y) + cV(x) \geq -\ell, \quad x \notin \text{supp}(\mu_*).
\end{equation}
1.4. Assumptions on external field \( V \). We now impose several conditions on the external field, some of which are best described in terms of the equilibrium measure and its complex valued “log-transform” \( g(z) \) defined below in (46).

**Condition 0** (Smoothness of \( V \)). The external field \( V \) possesses two Lipschitz continuous derivatives.

A consequence of this is that the equilibrium measure is absolutely continuous with respect to Lebesgue measure, with continuous density \( \psi(x) \).

**Condition 1** (Support properties of \( \mu_* \)). We suppose that the external field \( V \) is such that the equilibrium measure is supported on a finite union of intervals, \( \bigcup_{j=1}^{G+1}[\alpha_j, \beta_j] \), with \( \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \beta_{G+1} \).

By convention for future convenience, we set \( \beta_0 := -\infty \) and \( \alpha_{G+2} := +\infty \).

To describe further conditions on \( V \) imposed via its equilibrium measure \( \mu_* \), we will require an auxiliary function \( g(z) \) analytic for \( z \in \mathbb{C} \setminus (-\infty, \beta_{G+1}) \) defined in terms of \( \mu_* \) by

\[
(46) \quad g(z) := \int \log(z-s) d\mu_*(s) = \int_{\alpha_1}^{\beta_{G+1}} \log(z-s) \psi(s) ds,
\]

where \( \psi(x) \) is the Radon-Nikodym derivative of \( \mu_* \), that is, \( d\mu_*(x) = \psi(x) dx \). Here we are choosing the branch cut of the integrand so that for each \( s \in \mathbb{R} \), \( \log(z-s) \) is an analytic function of \( z \) for \( z \in \mathbb{C} \setminus (-\infty, s] \) that is real-valued for \( z > s \), which ensures the claimed analyticity properties of \( g(z) \). In terms of \( g(z) \) the variational condition (44) becomes

\[
(47) \quad cV(x) - (g_+(x) + g_-(x)) = -\ell, \quad \alpha_j < x < \beta_j, \quad j = 1, \ldots, G+1,
\]

where \( g_+(x) \) and \( g_-(x) \) denote the boundary values taken by \( g(z) \) as \( z \to x \) with \( z \in \mathbb{C}_+ \) and \( z \in \mathbb{C}_- \) respectively. Also, Condition 1 and the reality of the equilibrium measure together imply that there are real constants \( \Omega_0, \ldots, \Omega_G \) such that

\[
(48) \quad g_+(x) - g_-(x) = \begin{cases} i\Omega_0, & x < \alpha_1, \\ i\Omega_j, & \beta_j < x < \alpha_{j+1}, \\ 0, & x > \beta_{G+1}, \end{cases} \quad j = 1, \ldots, G,
\]

and from the normalization (43) it follows further that \( \Omega_0 = 2\pi \). Furthermore, since \( \mu_* \) is a positive measure,

\[
(49) \quad \theta(x) := -i(g_+(x) - g_-(x))
\]

is real and nonincreasing for all \( x \in \mathbb{R} \), so in particular \( 2\pi = \Omega_0 > \Omega_1 > \cdots > \Omega_G > 0 \). Assuming that differentiation commutes with taking boundary values (this may be easily justified later) (47) and (48) imply that

\[
(50) \quad g'_+(x) + g'_-(x) = cV'(x), \quad \alpha_j < x < \beta_j, \quad j = 1, \ldots, G+1,
\]

\[
\quad g'_+(x) - g'_-(x) = 0, \quad x \in \mathbb{R} \setminus \text{supp}(\psi).
\]

In particular, \( g'(z) \) is an analytic function for \( z \in \mathbb{C} \setminus \text{supp}(\psi) \).

Next, for \( x \in \mathbb{R} \), define the real-valued function

\[
(51) \quad \phi(x) := cV(x) + \ell - g_+(x) - g_-(x).
\]

According to (44) and (45), we have \( \phi(x) \equiv 0 \) for \( x \in \text{supp}(\psi) \) and \( \phi(x) \geq 0 \) for \( x \in \mathbb{R} \setminus \text{supp}(\psi) \).

Finally, for \( j = 1, \ldots, G+1 \) we define functions \( h_{\alpha_j} : (\beta_{j-1}, \beta_j) \to \mathbb{R} \) and \( h_{\beta_j} : (\alpha_j, \alpha_{j+1}) \to \mathbb{R} \) by the formulae

\[
(52) \quad h_{\alpha_j}(x) := \begin{cases} \frac{-\phi(x)}{\sqrt{\alpha_j - x}}, & \beta_{j-1} < x < \alpha_j, \\ \frac{\theta(\alpha_j) - \theta(x)}{\sqrt{x - \alpha_j}}, & \alpha_j < x < \beta_j, \end{cases}
\]
and

\[
\begin{align*}
\Phi(x) &= \begin{cases} 
\theta(x) - \theta(\beta_j), & \alpha_j < x < \beta_j, \\
\phi(x), & \beta_j < x < \alpha_{j+1}.
\end{cases}
\end{align*}
\]

Under the assumption of Condition 0, the definition \((52)\) extends by continuity to \(x = \alpha_j\) and the definition \((53)\) extends by continuity to \(x = \beta_j\); moreover, these functions will all have one Lipschitz continuous derivative. Moreover, if \(x\) is bounded away from the support interval endpoints, \(h_{\alpha_j}(x)\) and \(h_{\alpha_j}(x)\) will have a second derivative that is also Lipschitz. This is shown in the Appendix in the case of \(G > 0\) but the same reasoning also works for \(G = 0\). Note that the nonnegativity of the equilibrium measure implies that \(h_{\alpha_j}(x) \geq 0\) and \(h_{\beta_j}(x) \geq 0\) for \(\alpha_j < x < \beta_j\), and the variational inequality \((45)\) implies that \(h_{\alpha_j}(x) \leq 0\) for \(\beta_j-1 < x < \alpha_j\) and that \(h_{\beta_j}(x) \leq 0\) for \(\beta_j < x < \alpha_{j+1}\).

Now we may state the rest of the conditions that we impose on the external field \(V\).

**Condition 2** (Strict inequalities and behavior at endpoints). For \(j = 1, \ldots, G+1\), we suppose that \(\psi(x) > 0\) for \(\alpha_j < x < \beta_j\) and that the functions \(h_{\alpha_j} : (\beta_j-1, \beta_j) \rightarrow \mathbb{R}\) and \(h_{\beta_j} : (\alpha_j, \alpha_{j+1}) \rightarrow \mathbb{R}\) defined by \((52)\) and \((53)\) satisfy the strict inequalities

\[
\begin{align*}
(54) \quad & h_{\alpha_j}(x) < 0 \text{ for } \beta_{j-1} < x < \alpha_j, \quad h_{\alpha_j}(x) > 0 \text{ for } \alpha_j < x < \beta_j, \quad h'_{\alpha_j}(\alpha_j) > 0, \\
(55) \quad & h_{\beta_j}(x) > 0 \text{ for } \alpha_j < x < \beta_j, \quad h_{\beta_j}(x) < 0 \text{ for } \beta_j < x < \alpha_{j+1}, \quad h'_{\beta_j}(\beta_j) < 0.
\end{align*}
\]

**Condition 3** (Single interval of support w.l.o.g.). We assume that \(G = 0\).

The analysis for the case of \(G > 0\) (i.e. more than one interval comprising the support of \(\mu_\ast\)) may be deduced in a straightforward manner from the case of \(G = 0\) (i.e. one interval comprising the support of \(\mu_\ast\)). So, in the course of our presentation of the details of the asymptotic analysis of the orthogonal polynomials, we will assume, without loss of generality, that \(G = 0\) and hence the equilibrium measure is supported on the single interval \([\alpha, \beta] = [\alpha_1, \beta_1]\).

1.5. **Statement of results.** Because of the complex-conjugation symmetry \(p_\ast(z^\ast) = p_\ast(z)\ast\), we only need to present asymptotic formulae for the orthogonal polynomials in the upper half-plane. While our methods yield asymptotic formulae valid throughout the whole complex plane, we will restrict our attention to the regions \(\Omega_+\) and \(\mathbb{C}_+ \cap S_j\) as indicated in Figure 2. We focus on these regions for simplicity and also because these are most important for applications to random matrix theory.

![Figure 2](image-url)

**Figure 2.** The regions \(\Omega_\pm\) of the complex plane surround the interval \((\alpha, \beta)\). The square regions \(S_\alpha\) centered at \(\alpha\) and \(S_\beta\) centered at \(\beta\) are shown with dashed boundaries. These squares have sides of length \(2\delta\) for some \(\delta > 0\), and the regions \(\Omega_\pm\) each have vertical thickness \(\delta\).

**Theorem 1.** Suppose the external field \(V\) satisfies Condition [4] through Condition [3] as described earlier. Let \(n, N \rightarrow \infty\) so that \(N/n \rightarrow c\) with \(0 < c < \infty\). Then, with \((\alpha, \beta)\) representing the support of the equilibrium measure \(\mu_\ast\), the following asymptotic descriptions are valid.
1. The asymptotic behavior of the leading coefficients \( \kappa_{n-1,n-1} \) and \( \kappa_{n,n} \) is given by (207).

2. For \( z \) within \( \Omega_+ \) but outside the squares \( S_\alpha \) and \( S_\beta \), the orthogonal polynomials \( p_n(z) \) and \( p_{n-1}(z) \) possess the asymptotic description (212) and (213) respectively (see also (215)–(216) and (221)–(224)).

3. For \( z \) within \( S_\beta \cap \mathbb{C}_+ \), the orthogonal polynomials \( p_n(z) \) and \( p_{n-1}(z) \) possess the asymptotic descriptions (225)–(226) respectively (see also (233)–(234), (237), (238), and (241)–(242)).

4. Asymptotic formulae for the derivatives of the orthogonal polynomials may be obtained by differentiating the leading-order asymptotics for the polynomials themselves, as described in Section 8.

These formulae describe the orthogonal polynomials in terms of the first column of the matrix \( A(z) \) given in (59), and the error terms therein are expressed in terms of the quantity \( \Delta_n \) defined by (202). To mediate between the orthogonal polynomials contained in the first column of \( A(z) \) and the orthonormal polynomials \( p_{n-1}(z) \) and \( p_n(z) \), one must normalize by the leading coefficients \( \kappa_{n-1,n-1} \) and \( \kappa_{n,n} \) respectively, whose asymptotic behavior for large \( n \) is given by (207). As mentioned at the end of the previous subsection, the assumption that the support is a single interval is for convenience of presentation only. Theorem 1 may be easily extended to more general settings. As an example, it straightforward to carry out all the details if one assumes only Conditions 1–2 of the previous subsection. The following Theorems, describing the application of our results to random matrix theory, emphasize this point.

**Theorem 2.** Suppose that the external field \( V \) satisfies Conditions 2 of subsection 1.4. Then Asymptotic Result 3 from Section 1.1 holds true.

**Proof.** The calculations to deduce (20) from the asymptotic results concerning the orthogonal polynomials are by now standard, and we refer the reader to [16] for the details. (See also [6].) \( \square \)

**Theorem 3.** Suppose that the external field \( V \) satisfies Conditions 0–2 of subsection 1.4. Then Asymptotic Result 7 from Section 1.1 holds true.

**Proof.** We again refer the reader to [16] for the details of this calculation. (See also [5].) \( \square \)

1.6. **Notation.** We will indicate complex conjugation with an asterisk: \( z^* \). All matrices are written boldface with the notable exception of the identity matrix \( I \) and the Pauli matrices:

\[
\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

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2. **Orthogonal Polynomials, Riemann-Hilbert Problems, and Equilibrium Measures**

2.1. **Characterization of orthogonal polynomials via a Riemann-Hilbert problem.** Let \( N > 0 \) be a parameter, and let \( V(x) \) be a real-valued function satisfying merely the conditions set down in the beginning of Section 2.

The following Riemann-Hilbert problem [11] is known to characterize the polynomials \( \{p_n\}_{n=0}^\infty \) orthonormal with respect to the measure \( \nu_N \) given in (1), and defined by the conditions (2)–(3).

**Riemann-Hilbert Problem 1.** Find a \( 2 \times 2 \) matrix \( A(z) = A(z; n, N) \) with the properties:

- **Analyticity.** \( A(z) \) is analytic for \( z \in \mathbb{C} \setminus \mathbb{R} \), and takes continuous boundary values \( A_+(x), A_-(x) \) as \( z \) tends to \( x \) with \( x \in \mathbb{R} \) and \( z \in \mathbb{C}_+, \ z \in \mathbb{C}_- \).

- **Jump Condition.** The boundary values are connected by the relation

\[
A_+(x) = A_-(x) \begin{pmatrix} 1 & e^{-NV(x)} \\ 0 & 1 \end{pmatrix}.
\]
Normalization. The matrix $A(z)$ is normalized at $z = \infty$ as follows:

\begin{equation}
\lim_{z \to \infty} A(z) \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} = I.
\end{equation}

It was discovered in [10] that Riemann-Hilbert Problem 1 characterizes polynomials orthogonal with respect to $\nu_N$. The connection between these orthogonal polynomials and the solution of Riemann-Hilbert Problem 1 is the following:

\begin{equation}
A(z) = \begin{pmatrix} \frac{1}{\kappa_{n,n}} p_n(z) & \frac{1}{2\pi i \kappa_{n,n}} \int_{\mathbb{R}} p_n(s) e^{-NV(s)} ds \\ -2\pi i \kappa_{n-1,n-1} p_{n-1}(z) & -\kappa_{n-1,n-1} \int_{\mathbb{R}} p_{n-1}(s) e^{-NV(s)} ds \end{pmatrix}.
\end{equation}

Note that (59) implies in particular that

\begin{equation}
\kappa^2_{n-1,n-1} = -\frac{1}{2\pi i} \lim_{z \to \infty} z^{-(n-1)} A_{21}(z) \quad \text{and} \quad \kappa^2_{n,n} = -\frac{1}{2\pi i} \lim_{z \to \infty} z^{-(n+1)} A_{12}(z)^{-1}.
\end{equation}

These relationships provide a useful avenue for asymptotic analysis of the orthogonal polynomials in the limit $n \to \infty$; it is sufficient to carry out a rigorous asymptotic analysis of Riemann-Hilbert Problem 1.

2.2. Use of the equilibrium measure. Given the function $g(z)$ defined by (46), we introduce an explicit change of dependent variable into Riemann-Hilbert Problem 1. Set

\begin{equation}
B(z) := e^{-n\ell \sigma_3/2} A(z) e^{-ng(z)\sigma_3} e^{n\ell \sigma_3/2}.
\end{equation}

It follows from the properties of $A(z)$ set out in Riemann-Hilbert Problem 1 that the matrix $B(z)$ is analytic for $z \in \mathbb{C} \setminus \mathbb{R}$ and satisfies the normalization condition

\begin{equation}
\lim_{z \to \infty} B(z) = I.
\end{equation}

The boundary values $B_+(x)$ and $B_-(x)$, taken on the real axis as $z \to x$ from the upper and lower half-planes respectively, are continuous functions of $x \in \mathbb{R}$ related by the following jump condition:

\begin{equation}
B_+(x) = B_-(x) \begin{pmatrix} e^{-n\ell(x)-g_-(x)} & e^{n\ell(x)-\ell(x)} \\ 0 & e^{n\ell(x)-g_-(x)} \end{pmatrix},
\end{equation}

where $\theta(x)$ is defined by (49) and $\phi(x)$ is defined by (51). From (48) and (49), we see that for $x < \alpha$ and $x > \beta$ (recall that without loss of generality we are assuming that $\text{supp}(\psi) = [\alpha, \beta]$) this jump condition can be equivalently written in the form

\begin{equation}
B_+(x) = B_-(x) \begin{pmatrix} e^{-n\phi(x)} & 0 \\ 0 & e^{n\phi(x)} \end{pmatrix}, \quad x < \alpha \text{ or } x > \beta.
\end{equation}

Similarly, from (44), we see that for $\alpha < x < \beta$ the jump condition takes the form

\begin{equation}
B_+(x) = B_-(x) \begin{pmatrix} e^{-\ell(x)} & 0 \\ 0 & e^{\ell(x)} \end{pmatrix}, \quad \alpha < x < \beta.
\end{equation}

3. Extensions of $\theta(x)$ and $\phi(x)$

In this section we will define extensions from certain intervals of the real axis of the functions $\theta(x)$ and $\phi(x)$. We shall assume throughout the conditions on the external field $V$ described in the Introduction.
3.1. Existence of extensions with required properties.

**Lemma 1** (Extension of \(\theta(x)\)). Suppose that \(c > 0\) is held fixed as \(n \to \infty\) so that the real-valued function \(\theta(x)\) is independent of \(n\). There exists a function \(\Theta(x, y)\) that satisfies the following:

**Property 1:** Domain, smoothness, and boundary behavior. The function \(\Theta(x, y)\) is defined for \(\alpha < x < \beta\) and \(|y| < \delta\) for some \(\delta > 0\). In its domain of definition, \(\Theta(x, y)\) and the partial derivatives \(\Theta_x(x, y)\) and \(\Theta_y(x, y)\) are all continuous and uniformly bounded. Moreover, if \(x + iy\) is bounded away from both \(\alpha\) and \(\beta\), the second partial derivatives \(\Theta_{xx}(x, y)\), \(\Theta_{xy}(x, y)\), and \(\Theta_{yy}(x, y)\) are also continuous and bounded. The function \(\Theta(x, y)\) is an extension of the real-valued function \(\theta(x)\) in the sense that

\[
\Theta(x, 0) \equiv \theta(x), \quad \alpha < x < \beta.
\]

**Property 2:** Behavior near the real axis. There exist finite constants \(K > 0\) and \(k > 0\), such that the following three estimates hold true:

\[
|\Theta(x, y)| \leq K|y||x + iy - \alpha|^{1/2}|x + iy - \beta|^{1/2}, \quad \alpha < x < \beta, \quad |y| < \delta.
\]

\[
\text{Im}(\Theta(x, y)) \leq -ky^{3/2}, \quad \alpha < x < \beta, \quad 0 \leq y < \delta.
\]

\[
\text{Im}(\Theta(x, y)) \geq k|y|^{3/2}, \quad \alpha < x < \beta, \quad -\delta < y \leq 0.
\]

**Property 3:** Behavior near \(\alpha\) and \(\beta\). The function

\[
G_\alpha(x, y) := \frac{2\pi - \Theta(x, y)}{(x + iy - \alpha)^{3/2}}
\]

extends continuously to \(x + iy = \alpha\) with the limiting value \(G_\alpha(\alpha, 0) = \Theta'_\alpha(0) > 0\). Moreover,

\[
G_\alpha(x, y) = \Theta'_\alpha(0) + O(|x + iy - \alpha|), \quad \text{as } x + iy \to \alpha,
\]

and

\[
G_\alpha(x, \pm(x - \alpha)) \equiv \Theta'_\alpha(\alpha), \quad \alpha \leq x \leq \alpha + \delta.
\]

Similarly, the function

\[
G_\beta(x, y) := \frac{\Theta(x, y)}{((\beta - (x + iy))^{3/2}}
\]

extends continuously to \(x + iy = \beta\), and the limiting value \(G_\beta(\beta, 0) = -\Theta'_\beta(0) > 0\). Moreover,

\[
G_\beta(x, y) = -\Theta'_\beta(0) + O(|x + iy - \beta|), \quad \text{as } x + iy \to \beta,
\]

and

\[
G_\beta(x, \pm(\beta - x)) \equiv -\Theta'_\beta(\beta), \quad \beta - \delta \leq x \leq \beta.
\]

**Lemma 2** (Extension of \(\phi(x)\)). Suppose that \(c > 0\) is held fixed as \(n \to \infty\) so that the real-valued function \(\phi(x)\) is independent of \(n\). Then there exists a function \(\Phi(x, y)\) that satisfies the following:

**Property 1:** Domain, smoothness, and boundary behavior. The function \(\Phi(x, y)\) is defined in two rectangles: \(R_\alpha\) given by \(\alpha - 2\delta < x < \alpha\) with \(0 \leq y < \delta\) and \(R_\beta\) given by \(\beta < x < \beta + 2\delta\) with \(\delta < y \leq 0\) for some \(\delta > 0\). In its domain of definition, \(\Phi(x, y)\) and the partial derivatives \(\Phi_x(x, y)\) and \(\Phi_y(x, y)\) are all continuous and uniformly bounded. Moreover, if \(x + iy\) is bounded away from both \(\alpha\) and \(\beta\), the second partial derivatives \(\Phi_{xx}(x, y)\), \(\Phi_{xy}(x, y)\), and \(\Phi_{yy}(x, y)\) are also continuous and bounded. The function \(\Phi(x, y)\) is an extension of the real-valued function \(\phi(x)\) in the sense that

\[
\Phi(x, 0) \equiv \phi(x), \quad \beta < x < \beta + 2\delta \quad \text{and} \quad \alpha - 2\delta < x < \alpha.
\]

**Property 2:** Behavior near the real axis. There exist finite constants \(K > 0\) and \(k > 0\), such that the following estimates hold true:

\[
|\Phi(x, y)| \leq K|y||x + iy - \alpha|^{1/2}|x + iy - \beta|^{1/2}, \quad (x, y) \in R_\alpha \cup R_\beta,
\]

\[
\text{Re}(\Phi(x, y)) \geq k|x + iy - \alpha|^{3/2} \quad \text{for } (x, y) \in R_\alpha \quad \text{and} \quad \text{Re}(\Phi(x, y)) \geq k|x + iy - \beta|^{3/2} \quad \text{for } (x, y) \in R_\beta.
\]
3.2.1. \( x, \alpha, \Theta \)

Definition of the extension \( \Theta(x, y) \)

This follows immediately.)

**Property 3: Behavior near \( \alpha \) and \( \beta \).** The function

\[
H_{\alpha}(x, y) := \frac{\Phi(x, y)}{(\alpha - (x + iy))^{3/2}}
\]

extends continuously to \( x + iy = \alpha \) with the limiting value \( H_{\alpha}(\alpha, 0) = h'_{\alpha}(\alpha) > 0 \). Moreover,

\[
H_{\alpha}(x, y) = h'_{\alpha}(\alpha) + O(|x + iy - \alpha|), \quad \text{as } x + iy \to \alpha.
\]

and

\[
H_{\alpha}(x, \pm(\alpha - x)) \equiv h'_{\alpha}(\alpha), \quad \alpha - \delta \leq x \leq \alpha.
\]

Similarly, the function

\[
H_{\beta}(x, y) := \frac{\Phi(x, y)}{(x + iy - \beta)^{3/2}}
\]

extends continuously to \( x + iy = \beta \), and the limiting value satisfies \( H_{\beta}(\beta, 0) = -h'_{\beta}(\beta) > 0 \). Moreover,

\[
H_{\beta}(x, y) = -h'_{\beta}(\beta) + O(|x + iy - \beta|^{k/2}), \quad \text{as } x + iy \to \beta,
\]

and

\[
H_{\beta}(x, \pm(\beta - x)) \equiv -h'_{\beta}(\beta), \quad \beta \leq x \leq \beta + \delta.
\]

3.2. **Proofs of Lemmas [1] and [2]** In this subsection we construct suitable extensions \( \Theta(x, y) \) and \( \Phi(x, y) \) by further developing a strategy used in [15]. We will use the following notation generalizing the “bump” function \( B(\cdot) \) introduced in Section [1] for an interval \([a, b]\),

\[
B_{[a,b]}(x) := B \left( \frac{x-a}{b-a} \right).
\]

This \( C^{(\infty)}(\mathbb{R}) \) function maps \( \mathbb{R} \) onto the interval \([0, 1]\) with \( B_{[a,b]}(x) \equiv 0 \) for \( x \leq a \) and \( B_{[a,b]}(x) \equiv 1 \) for \( x \geq b \).

3.2.1. **Proof of Lemma [7]: extension of \( \Theta(x) \).**

Definition of the extension \( \Theta(x, y) \). First, we define functions \( \Theta_{\alpha,0}(x, y) \) and \( \Theta_{\beta,0}(x, y) \) by

\[
\Theta_{\alpha,0}(x, y) := 2\pi - ((x+iy) - \alpha)^{1/2}[h_{\alpha}(x) + i(h_{\alpha}(x + y) - h_{\alpha}(x))], \quad x < \beta \text{ and } x + y < \beta,
\]

and

\[
\Theta_{\beta,0}(x, y) := (\beta - (x+iy))^{1/2}[h_{\beta}(x) + i(h_{\beta}(x + y) - h_{\beta}(x))], \quad x > \alpha \text{ and } x + y > \alpha.
\]

Here, the function \( h_{\alpha}(x) = h_{\alpha}(x) \) is defined by \(|52|\) for \(-\infty < x < \beta\) and the function \( h_{\beta}(x) = h_{\beta}(x) \) is defined by \(|53|\) for \( \alpha < x < +\infty \).

\(<\text{Remark:}\) The extensions of the functions \( h_{\alpha} \) and \( h_{\beta} \) within the square brackets in \(|87|\) and \(|88|\) respectively are Cartesian versions of the polar-coordinate extensions discussed in [15], further generalized with the use of difference quotients in place of derivatives. >

Next, we define analytic approximations of \( \Theta_{\alpha,0}(x, y) \) and \( \Theta_{\beta,0}(x, y) \) valid for \( x + iy \approx \alpha \) and \( x + iy \approx \beta \) respectively:

\[
\Theta_{\alpha,0}^{\text{hol}}(x, y) := 2\pi - h'_{\alpha}(\alpha)((x+iy) - \alpha)^{3/2}, \quad x > \alpha \text{ or } y \neq 0,
\]

and

\[
\Theta_{\beta,0}^{\text{hol}}(x, y) := -h'_{\beta}(\beta)(\beta - (x+iy))^{3/2}, \quad x < \beta \text{ or } y \neq 0.
\]

In precisely the spirit of the simple example described in Section [1] we may combine the two types of extensions with the help of an appropriate angular bump function:

\[
\Theta(x, y) := B \left( \frac{y}{x - \alpha} \right) \Theta_{\alpha,0}^{\text{hol}}(x, y) + \left[ 1 - B \left( \frac{y}{x - \alpha} \right) \right] \Theta_{\alpha,0}(x, y),
\]
and

$$
\Theta_\beta(x, y) := B\left(\frac{y}{x - \beta}\right) \Theta^\text{hol}_{\beta,0}(x, y) + \left[1 - B\left(\frac{y}{x - \beta}\right)\right] \Theta_{\beta,0}(x, y).
$$

For short we will occasionally write

$$
B_{\text{ang}, \alpha} := B\left(\frac{y}{x - \alpha}\right) \quad \text{and} \quad B_{\text{ang}, \beta} := B\left(\frac{y}{x - \beta}\right).
$$

Finally, letting

$$
a := \alpha + \frac{1}{3}(\beta - \alpha) \quad \text{and} \quad b := \beta - \frac{1}{3}(\beta - \alpha)
$$

so that $[a, b] \subset (\alpha, \beta)$, we may smoothly glue these two extensions together through the vertical strip $a < x < b$ in the $(x, y)$-plane:

$$
\Theta(x, y) := B_{[a, b]}(x) \Theta_\beta(x, y) + \left[1 - B_{[a, b]}(x)\right] \Theta_\alpha(x, y).
$$

This will be our extension of the function $\theta(x)$ from the interior $(\alpha, \beta)$ of the support interval. Taking into account the supports of $B_{[a, b]}(x)$ and $1 - B_{[a, b]}(x)$ and comparing with the regions of definition of $\Theta_{\alpha,0}(x, y)$, $\Theta_{\beta,0}(x, y)$, $\Theta^\text{hol}_{\alpha,0}(x, y)$, and $\Theta^\text{hol}_{\beta,0}(x, y)$, we see that whenever $\delta < (\beta - \alpha)/3$, $\Theta(x, y)$ is well-defined on the rectangle $R$ given by the inequalities $\alpha < x < \beta$ and $|y| < \delta$.

**Remark:** It turns out that if we replace both angular bump functions $B_{\text{ang}, \alpha}$ and $B_{\text{ang}, \beta}$ by the constant function $B \equiv 1$, then the extension obtained only involves the functions $\Theta_{\alpha,0}(x, y)$ and $\Theta_{\beta,0}(x, y)$ glued together through the vertical strip $a < x < b$, and this simpler function satisfies all of the desired properties with the exception of (75) and (72) from Property 3. The purpose of the angular bump functions is to smoothly deform the simpler extension into one that satisfies these additional conditions (without ruining any of the other conditions, of course). ▷

**Confirmation of Property 1.** To confirm Property 1, we note that under the assumptions in force, both functions $h_\alpha$ and $h_\beta$ have one Lipschitz continuous derivative throughout their respective domains of definition, which implies that $\partial_x \Theta_{\alpha,0}(x, y)$, $\partial_y \Theta_{\alpha,0}(x, y)$, $\partial_x \Theta_{\beta,0}(x, y)$, and $\partial_y \Theta_{\beta,0}(x, y)$ are all continuous and uniformly bounded throughout the rectangle $R$ of definition of $\Theta(x, y)$. Since $\Theta^\text{hol}_{\alpha,0}(x, y)$ and $\Theta^\text{hol}_{\beta,0}(x, y)$ are analytic functions for $(x, y) \in R$, their first partial derivatives are certainly continuous. Then, since $\Theta(x, y)$ is constructed from these more elementary functions with the help of $C^{(\infty)}$ bump functions, it is clear that $\Theta(x, y)$, $\Theta_x(x, y)$, and $\Theta_y(x, y)$ are all continuous and uniformly bounded throughout $R$. As $h_\alpha(x)$ and $h_\beta(x)$ have a second Lipschitz derivative for $x$ bounded away from $\alpha$ and $\beta$, similar arguments show that $\Theta_{xx}(x, y)$, $\Theta_{xy}(x, y)$, and $\Theta_{yy}(x, y)$ are continuous and bounded for $x + iy$ bounded away from $\alpha$ and $\beta$. Furthermore, for $\alpha < x < \beta$,

$$
\Theta(x, 0) = B_{[a, b]}(x) \Theta_\beta(x, 0) + \left[1 - B_{[a, b]}(x)\right] \Theta_\alpha(x, 0)
$$

$$
= B_{[a, b]}(x) \Theta_\beta(x, 0) + \left[1 - B_{[a, b]}(x)\right] \Theta_{\beta,0}(x, 0)
$$

$$
= B_{[a, b]}(x) \theta(x) + \left[1 - B_{[a, b]}(x)\right] \theta(x)
$$

$$
= \theta(x),
$$

so $\Theta(x, y)$ is indeed an extension of $\theta(x)$ from the interval $(\alpha, \beta)$ to the rectangle $R$.

**Confirmation of Property 2.** To confirm Property 2, first note that from (95) we have

$$
\overline{\partial} \Theta = \overline{\partial} B_{[a, b]} \cdot (\Theta_\beta - \Theta_\alpha) + B_{[a, b]} \overline{\partial} \Theta_\beta + \left[1 - B_{[a, b]}\right] \overline{\partial} \Theta_\alpha.
$$

Now, since $\Theta_\alpha(x, y)$ and $\Theta_\beta(x, y)$ are both extensions from $(\alpha, \beta)$ of the same function $\theta(x)$, and since they are both uniformly Lipschitz for $x + iy$ bounded away from $\alpha$ and $\beta$ (this is where $\overline{\partial} B_{[a, b]}$ is nonzero), the first term on the right-hand side is supported in $a < x < b$ and is $O(|y|)$. Therefore we certainly have

$$
\left|\overline{\partial} B_{[a, b]} \cdot (\Theta_\beta - \Theta_\alpha)\right| \leq K|y||x + iy - \alpha|^{1/2}|x + iy - \beta|^{1/2}
$$

15
for some constant $K > 0$. It therefore remains to estimate $\overline{\partial}\Theta_\alpha$ for $x + iy$ bounded away from $\beta$ and $\overline{\partial}\Theta_\beta$ for $x + iy$ bounded away from $\alpha$. Since $\overline{\partial}\Theta^{\text{hol}}_{\alpha,0}(x, y) \equiv 0$ and $\overline{\partial}\Theta^{\text{hol}}_{\beta,0}(x, y) \equiv 0$, we see that

$$
\overline{\partial}\Theta_\alpha = \overline{\partial} B_{\text{ang}, \alpha} \cdot (\Theta^{\text{hol}}_{\alpha,0} - \Theta_\alpha) + [1 - B_{\text{ang}, \alpha}] \overline{\partial}\Theta_{\alpha,0}
$$

(109)

and since the inequality $|x - \beta| > |y|$ holds wherever the derivative of the bump function in this formula is nonzero,

$$
\overline{\partial} B_{\alpha, \beta} \cdot (\Theta^{\text{hol}}_{\beta,0} - \Theta_\beta) = \int_{\beta}^{x} \overline{\partial} B_{\text{ang}, \beta} \cdot (\Theta^{\text{hol}}_{\beta,0} - \Theta_\beta) \, ds,
$$

we have

$$
\Theta^{\text{hol}}_{\beta,0}(x, y) = \Theta_\beta(x, y) = -[(\beta - (x + iy))^{1/2}] \left[ (1 - i) \int_{\beta}^{x} [\overline{\partial} B_{\text{ang}, \beta} \cdot (\Theta^{\text{hol}}_{\beta,0} - \Theta_\beta)] \, ds \right],
$$

so, since $h_\beta(x)$ has one Lipschitz continuous derivative, there are constants $K_1 > 0$ and $K_2 > 0$ such that

$$
|\Theta^{\text{hol}}_{\beta,0}(x, y) - \Theta_{\alpha, \beta}(x, y)| \leq |x - \beta + iy|^{1/2} \left[ K_1 (x - \beta)^2 + K_2 (x - \beta + y)^2 \right]
$$

(104)

$$
= |x - \beta + iy|^{1/2} \left[ 2\overline{K}(x - \beta)^2 + 2K_2 y(x - \beta) + K_2 y^2 \right]
$$

$$
\leq |x - \beta + iy|^{1/2} \left[ 2\overline{K}(x - \beta)^2 + 2K_2 y|x - \beta| + K_2 y^2 \right],
$$

where $\overline{K} := (K_1 + K_2)/2$. Using again the inequality $|x - \beta| > |y|$ (since we are going to multiply by $\overline{\partial} B_{\text{ang}, \beta}$), we therefore have

$$
|\Theta^{\text{hol}}_{\beta,0}(x, y) - \Theta_{\beta,0}(x, y)| \leq |x - \beta + iy|^{1/2} \left[ 2\overline{K}(x - \beta)^2 + 3K_2 y|x - \beta| \right].
$$

Since we have both $|B'(t)| \leq C$ and $|B'(t)| \leq C|t|$ for some constant $C > 0$, it follows from (101) and (105) that

$$
\overline{\partial} B_{\text{ang}, \beta} \cdot (\Theta^{\text{hol}}_{\beta,0} - \Theta_{\beta,0}) = O \left( |x + iy - \beta|^{-1/2} y|x - \beta| \right)
$$

(106)

$$
= O \left( |y||x + iy - \beta|^{1/2} \right),
$$

where we have used $|x - \beta| \leq |x + iy - \beta|$ in the last step. Furthermore,

$$
\overline{\partial}\Theta_{\beta,0}(x, y) = (\beta - (x + iy))^{1/2} \overline{\partial} [h_\beta(x) + \overline{\partial} (h_\beta(x + y) - h_\beta(x))] = \frac{1}{2}(\beta - (x + iy))^{1/2} \left[ h_\beta'(x + y) - h_\beta'(x) \right],
$$

so since $h_\beta'(x)$ is uniformly Lipschitz near $\beta$ and $1 - B_{\text{ang}, \beta}$ is bounded, we also have

$$
[1 - B_{\text{ang}, \beta}] \overline{\partial}\Theta_{\beta,0} = O \left( |y||x + iy - \beta|^{1/2} \right).
$$

Therefore, for $x + iy$ bounded away from $\alpha$ we have

$$
|\overline{\partial}\Theta_{\beta}(x, y)| \leq K|y||x + iy - \alpha|^{1/2}|x + iy - \beta|^{1/2}
$$

(109)

for some constant $K > 0$. In a completely analogous fashion we see that for $x + iy$ bounded away from $\beta$ we have

$$
|\overline{\partial}\Theta_{\alpha}(x, y)| \leq K|y||x + iy - \alpha|^{1/2}|x + iy - \beta|^{1/2}.
$$

(110)
Combining these results with (108) we complete the proof that
\[ |\mathfrak{F}(x,y)| \leq K|y||x+iy-\alpha|^{1/2}|x+iy-\beta|^{1/2}. \]

Now consider \( \text{Im}(\Theta(x,y)) \). Since all of the bump functions \( B_{\alpha,b}, B_{\text{ang},\alpha}, \) and \( B_{\text{ang},\beta} \) are real-valued, it will suffice to analyze \( \text{Im}(\Theta_{\alpha,0}(x,y)) \) and \( \text{Im}(\Theta_{\alpha,0}(x,y)) \) for \( x+iy \in R \) bounded away from \( \beta \) and to analyze \( \text{Im}(\Theta_{\beta,0}(x,y)) \) and \( \text{Im}(\Theta_{\beta,0}(x,y)) \) for \( x+iy \in R \) bounded away from \( \alpha \). Writing \( \beta-(x+iy) = |\beta-(x+iy)|e^{i\phi} \) with \( |\phi| < \pi/2 \), we have the exact formulae

\[
\begin{align*}
(112) \quad \text{Im}(\Theta_{\beta,0}(x,y)) &= |\beta-(x+iy)|^{1/2} y \\
&\quad \cdot \left[ h_{\beta}(x+y) - h_{\beta}(x) \frac{h_{\beta}(x+y) - h_{\beta}(x)}{y} \cos \left( \frac{\phi}{2} \right) - \frac{1}{2} \frac{h_{\beta}(x)}{\beta-x} \left( 1 - \tan^{2} \left( \frac{\phi}{2} \right) \right) \cos \left( \frac{\phi}{2} \right) \right],
\end{align*}
\]

and

\[
\begin{align*}
(113) \quad \text{Im}(\Theta_{\beta,0}(x,y)) &= |\beta-(x+iy)|^{1/2} y \\
&\quad \cdot \left[ \frac{1}{2} h_{\beta}(\beta) \left( 4 \cos \left( \frac{\phi}{2} \right) - \sec \left( \frac{\phi}{2} \right) \right) \right].
\end{align*}
\]

Since Condition 2 requires that \( h_{\beta}'(\beta) < 0 \), the condition that \( |\phi| < \pi/2 \) immediately implies that

\[
(114) \quad \frac{1}{2} h_{\beta}(\beta) \left( 4 \cos \left( \frac{\phi}{2} \right) - \sec \left( \frac{\phi}{2} \right) \right) < \frac{\sqrt{2}}{2} h_{\beta}(\beta) < 0.
\]

To analyze the terms in the square brackets in (112) requires a little more work. Suppose first that \( |\beta-(x+iy)| < \epsilon_{1} \). Then, as \( \epsilon_{1} \to 0 \), we have both

\[
(115) \quad \frac{h_{\beta}(x+y) - h_{\beta}(x)}{y} \to h_{\beta}'(\beta) < 0 \quad \text{and} \quad -\frac{1}{2} \frac{h_{\beta}(x)}{\beta-x} \to \frac{1}{2} h_{\beta}(\beta) < 0,
\]

where the inequalities follow from Condition 2. So \( \epsilon_{1} \) may be taken to be small enough that \( |\beta-(x+iy)| < \epsilon_{1} \) implies both

\[
(116) \quad \frac{h_{\beta}(x+y) - h_{\beta}(x)}{y} < \frac{1}{2} h_{\beta}'(\beta) < 0 \quad \text{and} \quad -\frac{1}{2} \frac{h_{\beta}(x)}{\beta-x} < \frac{1}{4} h_{\beta}'(\beta) < 0.
\]

Therefore, since \( |\phi| < \pi/2 \) implies both

\[
(117) \quad \frac{\sqrt{2}}{2} < \cos \left( \frac{\phi}{2} \right) \quad \text{and} \quad 0 < 1 - \tan^{2} \left( \frac{\phi}{2} \right),
\]

we see that \( |\beta-(x+iy)| < \epsilon_{1} \) implies

\[
(118) \quad \frac{h_{\beta}(x+y) - h_{\beta}(x)}{y} \cos \left( \frac{\phi}{2} \right) - \frac{1}{2} \frac{h_{\beta}(x)}{\beta-x} \left( 1 - \tan^{2} \left( \frac{\phi}{2} \right) \right) \cos \left( \frac{\phi}{2} \right) < \frac{\sqrt{2}}{4} h_{\beta}'(\beta) < 0.
\]

On the other hand, if we suppose that \( |\beta-(x+iy)| > \epsilon_{1}/2 \), but that \( x > a \) and \( |y| < \epsilon_{2} \) for some \( \epsilon_{2} > 0 \), then as \( \epsilon_{2} \to 0 \) we have both

\[
(119) \quad \phi \to 0 \quad \text{and} \quad \frac{h_{\beta}(x+y) - h_{\beta}(x)}{y} \to h_{\beta}'(x).
\]

Now, from (53), for \( \alpha < a < x < \beta \) we have

\[
(120) \quad h_{\beta}'(x) - \frac{1}{2} \frac{h_{\beta}(x)}{\beta-x} = \frac{\theta'(x)}{\sqrt{\beta-x}} = -\frac{2\pi \psi(x)}{\sqrt{\beta-x}} < -k_{1}
\]

with some constant \( k_{1} > 0 \) as a consequence of Condition 2 and the square-root vanishing of \( \psi(x) \) as \( x \uparrow \beta \). Therefore, by choosing \( \epsilon_{2} \) sufficiently small we will have

\[
(121) \quad \frac{h_{\beta}(x+y) - h_{\beta}(x)}{y} \cos \left( \frac{\phi}{2} \right) - \frac{1}{2} \frac{h_{\beta}(x)}{\beta-x} \left( 1 - \tan^{2} \left( \frac{\phi}{2} \right) \right) \cos \left( \frac{\phi}{2} \right) < -\frac{1}{2} k_{1}.
\]
as long as \(|\beta - (x + iy)| > \epsilon_1/2, x > a, \) and \(|y| < \epsilon_2. To combine these estimates, note that if \(\delta > 0\) is sufficiently small, the part of the rectangle \(R\) given by the inequalities \(\alpha < a < x < \beta\) and \(|y| < \delta\) consists of points \((x, y)\) for which either \(|\beta - (x + iy)| < \epsilon_1\) or \(|y| < \epsilon_2, so (118) and (121) may be combined to give

\[
\frac{h_{\beta}(x + y) - h_{\beta}(x)}{y} \cos \left(\frac{\phi}{2}\right) - \frac{1}{2} \cdot \frac{h_{\beta}(x) - 1 - \tan^2 \left(\frac{\phi}{2}\right)}{\beta - x} \cos \left(\frac{\phi}{2}\right) < -k_2 < 0
\]

for \((x, y) \in R, when

\[
k_2 := \min \left\{ \frac{\sqrt{2}}{4} |h_{\beta}'(\beta)|, \frac{1}{2} k_1 \right\}.
\]

Finally, we may combine (114) with (122) to find that for \((x, y) \in R\) with \(x > a,\)

\[
\text{Im}(\Theta_{\beta,0}(x, y)) \leq -k_3 y|x + iy - \beta|^{1/2}|x + iy - \alpha|^{1/2}
\]

if \(y \geq 0\) and

\[
\text{Im}(\Theta_{\beta,0}(x, y)) \geq -k_3 y|x + iy - \beta|^{1/2}|x + iy - \alpha|^{1/2}
\]

if \(y \leq 0,\)

where

\[
k_3 := \min \left\{ k_2, \frac{\sqrt{2}}{2} |h_{\beta}'(\beta)| \right\}
\]

Completely analogous arguments show that for \((x, y) \in R\) with \(x < b,\)

\[
\text{Im}(\Theta_{\alpha,0}(x, y)) \leq -k_4 y|x + iy - \beta|^{1/2}|x + iy - \alpha|^{1/2}
\]

if \(y \geq 0\) and

\[
\text{Im}(\Theta_{\alpha,0}(x, y)) \geq -k_4 y|x + iy - \beta|^{1/2}|x + iy - \alpha|^{1/2}
\]

if \(y \leq 0,\)

for some constant \(k_4 > 0.\) Letting \(k = \min\{k_3, k_4\} > 0\) and using the fact that \(\Theta(x, y)\) is a convex combination of \(\text{Im}(\Theta_{\alpha,0}(x, y)), \text{Im}(\Theta_{\alpha,0}^b(x, y)), \text{Im}(\Theta_{\beta,0}(x, y)), \) and \(\text{Im}(\Theta_{\beta,0}^b(x, y))\) through the various bump functions involved in the definition, it follows that

\[
\text{Im}(\Theta(x, y)) \leq -k y|x + iy - \beta|^{1/2}|x + iy - \alpha|^{1/2}, \quad y \geq 0
\]

\[
\text{Im}(\Theta(x, y)) \leq -k y|x + iy - \beta|^{1/2}|x + iy - \alpha|^{1/2}, \quad y \leq 0
\]

holds for \((x, y) \in R\) if the thickness parameter \(\delta\) of the rectangle \(R\) is sufficiently small. Now, if \(x \leq (\alpha + \beta)/2\) then \(|x + iy - \beta|\) is bounded away from zero while \(|x + iy - \alpha| \geq |y|, \) and if \(x \geq (\alpha + \beta)/2\) then \(|x + iy - \alpha|\) is bounded away from zero while \(|x + iy - \beta| \geq |y|). Combining these observations with (129) yields (68) and (69).

Confirmation of Property 3. To confirm that \(\Theta(x, y)\) satisfies Property 3, note that

\[
G_{\beta,0}(x, y) := \frac{\Theta_{\beta,0}(x, y)}{(\beta - (x + iy))^{3/2}} = \frac{h_{\beta}(x) + i(h_{\beta}(x + y) - h_{\beta}(x))}{(\beta - (x + iy))^{3/2}}
\]

is continuous near \((x, y) = (\beta, 0)\) and satisfies

\[
G_{\beta,0}(x, y) = -h_{\beta}'(\beta) + \mathcal{O}(\frac{\beta - (x + iy)}{\beta - (x + iy)}))
\]

because \(h_{\beta}'(x)\) is Lipschitz continuous. Similarly,

\[
G_{\beta,0}^b(x, y) := \frac{\Theta_{\beta,0}^b(x, y)}{(\beta - (x + iy))^{3/2}} \equiv -h_{\beta}'(\beta) > 0
\]
so since near \( x + iy = \beta \) (that is, for \( x > b \)) \( G_\beta(x,y) \) is a convex combination of \( G_{\beta,0}(x,y) \) and \( G_{\beta,0}^{\text{hol}}(x,y) \), the requirement (74) on \( G_\beta(x,y) \) given in Property 3 is met. And since for \( |y| \geq \beta - x \) we have \( G_\beta(x,y) = G_{\beta,0}^{\text{hol}}(x,y) \), we also confirm the requirement (75). Similar calculations show that \( G_\alpha(x,y) \) satisfies the requirements (71) and (72).

3.2.2. Proof of Lemma 2: extension of \( \phi(x) \). The construction of a suitable extension of \( \phi(x) \) follows the same general procedure as the construction above of \( \Theta(x,y) \). We give all details of the construction, after which it is straightforward to follow the reasoning given in the proof of Lemma 1 to establish that Properties 1, 2, and 3 are satisfied.

We first define

\[
\Phi_{\alpha,0}(x,y) := -(\alpha - (x + iy))^{1/2} [h_\alpha(x) + i (h_\alpha(x) - h_\alpha(x))], \quad x < \beta \text{ and } x + y < \beta,
\]

and

\[
\Phi_{\beta,0}(x,y) := -((x + iy) - \beta)^{1/2} [h_\beta(x) + i (h_\beta(x) - h_\beta(x))], \quad x > \alpha \text{ and } x + y > \alpha.
\]

The analytic approximations of these functions valid for \( x + iy \approx \alpha \) and \( x + iy \approx \beta \) respectively are

\[
\Phi_{\alpha,0}^{\text{hol}}(x,y) := h_\alpha'(\alpha)(\alpha - (x + iy))^{3/2}, \quad x < \alpha \text{ or } y \neq 0,
\]

and

\[
\Phi_{\beta,0}^{\text{hol}}(x,y) := -h_\beta'(\beta)((x + iy) - \beta)^{3/2}, \quad x > \beta \text{ or } y \neq 0.
\]

Since the rectangles \( R_\alpha \) and \( R_\beta \) are disjoint, there is no need to merge functions defined near \( x + iy = \alpha \) with functions defined near \( x + iy = \beta \), so we may simply define

\[
\Phi(x,y) := \begin{cases} 
B \left( \left| \frac{y}{\alpha - x} \right| \right) \Phi_{\alpha,0}^{\text{hol}}(x,y) + \left[ 1 - B \left( \left| \frac{y}{\alpha - x} \right| \right) \right] \Phi_{\alpha,0}(x,y), & (x,y) \in R_\alpha \\
B \left( \left| \frac{y}{x - \beta} \right| \right) \Phi_{\beta,0}^{\text{hol}}(x,y) + \left[ 1 - B \left( \left| \frac{y}{x - \beta} \right| \right) \right] \Phi_{\beta,0}(x,y), & (x,y) \in R_\beta.
\end{cases}
\]

4. An equivalent Riemann-Hilbert-\( \overline{\Phi} \) Problem.

The jump condition satisfied by the boundary values taken by \( B(z) \) on \((\alpha, \beta)\) can be written in the factored form:

\[
B_+(x) = B_-(x) \left( \begin{array}{cc} 1 & 0 \\ e^{in\theta(x)} & 1 \end{array} \right) \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ e^{-in\theta(x)} & 1 \end{array} \right).
\]

Consider the contour \( \Sigma \) illustrated in Figure 3.

\[\text{Figure 3. The oriented contour } \Sigma \text{ consists of the real intervals } (-\infty, \alpha - 2\delta), (\alpha, \beta), (\beta + 2\delta, +\infty), \text{ along with the indicated contour segments } \Sigma_\alpha \text{ connecting } \alpha - 2\delta \text{ to } \alpha, \Sigma_+ \text{ and } \Sigma_- \text{ connecting } \alpha \text{ to } \beta, \text{ and } \Sigma_\beta \text{ connecting } \beta \text{ to } \beta + 2\delta. \text{ All contour segments are oriented left-to-right, and all nonhorizontal segments have slopes } \pm 1.\]

Let \( \Theta(x,y) \) be any extension of \( \theta(x) \) having all three properties described in Lemma 1, and let \( \Phi(x,y) \) be any extension of \( \phi(x) \) having all three properties described in Lemma 2. We define a matrix \( D(x,y) \) for
\[ z = x + iy \in \mathbb{C} \setminus \Sigma \text{ relative to the domains } \Omega_+, \Omega_-, \Omega_\alpha, \text{ and } \Omega_\beta \text{ shown as shaded regions in Figure 3 as follows. Set} \]
\[
D(x, y) := B(x + iy) \begin{pmatrix} 1 & 0 \\ -e^{-in\Theta(x, y)} & 1 \end{pmatrix}, \quad x + iy \in \Omega_+,
\]
\[
D(x, y) := B(x + iy) \begin{pmatrix} 1 & 0 \\ e^{in\Theta(x, y)} & 1 \end{pmatrix}, \quad x + iy \in \Omega_-,
\]
\[
D(x, y) := B(x + iy) \begin{pmatrix} 1 & -e^{-n\Phi(x, y)} \\ 0 & 1 \end{pmatrix}, \quad x + iy \in \Omega_\alpha,
\]
\[
D(x, y) := B(x + iy) \begin{pmatrix} 1 & e^{-n\Phi(x, y)} \\ 0 & 1 \end{pmatrix}, \quad x + iy \in \Omega_\beta,
\]
and for all remaining \( z \in \mathbb{C} \setminus \Sigma \), we set \( D(x, y) := B(x + iy). \)

Because it is explicitly related to \( B(x + iy) \) and hence to \( A(x + iy) \), the matrix \( D(x, y) \) will solve Riemann-Hilbert-\( \partial \) problem to be defined below. Define the jump matrix \( V_D(z) \) for \( z \in \Sigma \) as follows:
\[
V_D(z) := \begin{pmatrix} 1 & e^{-n\Phi(x)} \\ 0 & 1 \end{pmatrix}, \quad z = x < \alpha - 2\delta \text{ and } z = x > \beta + 2\delta,
\]
\[
V_D(z) := \begin{pmatrix} 1 & e^{-n\Phi(x, y)} \\ 0 & 1 \end{pmatrix}, \quad z = x + iy \in \Sigma_\alpha \cup \Sigma_\beta,
\]
\[
V_D(z) := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad z = x \in (\alpha, \beta),
\]
\[
V_D(z) := \begin{pmatrix} 0 & 1 \\ e^{-n\Phi(x, y)} & 0 \end{pmatrix}, \quad z = x + iy \in \Sigma_+,
\]
\[
V_D(z) := \begin{pmatrix} 1 & 0 \\ e^{in\Theta(x, y)} & 1 \end{pmatrix}, \quad z = x + iy \in \Sigma_-.
\]

Also, define the auxiliary matrix \( W_0(x, y) \) as follows.
\[
W_0(x, y) := \begin{pmatrix} 0 & 0 \\ ine^{-in\Theta(x, y)}\partial\Theta(x, y) & 0 \end{pmatrix}, \quad x + iy \in \Omega_+,
\]
\[
W_0(x, y) := \begin{pmatrix} 0 & 0 \\ ine^{in\Theta(x, y)}\partial\Theta(x, y) & 0 \end{pmatrix}, \quad x + iy \in \Omega_-,
\]
\[
W_0(x, y) := \begin{pmatrix} 0 & ne^{-n\Phi(x, y)}\partial\Phi(x, y) \\ 0 & 0 \end{pmatrix}, \quad x + iy \in \Omega_\alpha,
\]
\[
W_0(x, y) := \begin{pmatrix} 0 & 0 \\ 0 & ne^{-n\Phi(x, y)}\partial\Phi(x, y) \end{pmatrix}, \quad x + iy \in \Omega_\beta.
\]

For all remaining \((x, y) \in \mathbb{R}^2\), we set \( W_0(x, y) := 0 \). Note that \( W_0(x, y) \) so-defined is compactly supported. From the properties of the matrix \( B(z) \) inherited via the substitution \([61]\) from properties of the matrix \( A(z) \) contained in the statement of Riemann-Hilbert Problem \([\text{I}]\) it follows that \( D(x, y) \) solves the following hybrid Riemann-Hilbert-\( \partial \) problem:

**Riemann-Hilbert-\( \partial \) Problem 1.** Find a \( 2 \times 2 \) matrix \( D(x, y) \) with the properties:

**Continuity.** \( D(x, y) \) is a continuous function of \( x \) and \( y \) for \( x + iy \in \mathbb{C} \setminus \Sigma \) taking continuous boundary values \( D_+(x, y) \) (respectively \( D_-(x, y) \)) on \( \Sigma \) from the left (respectively right).

**Jump Conditions.** The boundary values are connected by the relation
\[
D_+(x, y) = D_-(x, y)V_D(x + iy), \quad x + iy \in \Sigma.
\]
Deviation From Analyticity. For \(x + iy \in \mathbb{C}\),
\[
\tilde{\mathcal{D}}(x, y) = \mathcal{D}(x, y)W_0(x, y).
\]
(Note that in particular \(\tilde{\mathcal{D}}(x, y) = 0\) for \(x + iy \notin \Omega_+ \cup \Omega_\alpha \cup \Omega_\beta\).)

Normalization. The matrix \(\mathcal{D}(x, y)\) is normalized as follows:
\[
\lim_{x, y \to \infty} \mathcal{D}(x, y) = I.
\]

5. Construction of a Global Approximation to \(\mathcal{D}(x, y)\)

In this section we will build a global approximation to \(\mathcal{D}(x, y)\) by considering a Riemann-Hilbert problem obtained from Riemann-Hilbert-\(\overline{\partial}\) problem 1 by ignoring the \(\overline{\partial}\) component of the problem:

Riemann-Hilbert Problem 2. Find a \(2 \times 2\) matrix \(\tilde{\mathcal{D}}(z)\) with the properties:

Analyticity. \(\tilde{\mathcal{D}}(z)\) is an analytic function for \(z \in \mathbb{C} \setminus \Sigma\) taking continuous boundary values \(\tilde{\mathcal{D}}_+(z)\) (respectively \(\tilde{\mathcal{D}}_-(z)\)) on \(\Sigma\) from the left (respectively right).

Jump Conditions. The boundary values are connected by the relation
\[
\tilde{\mathcal{D}}_+(z) = \tilde{\mathcal{D}}_-(z)V_D(x, y), \quad z = x + iy \in \Sigma.
\]

Normalization. The matrix \(\tilde{\mathcal{D}}(z)\) is normalized as follows:
\[
\lim_{z \to \infty} \tilde{\mathcal{D}}(z) = I.
\]

Riemann-Hilbert Problem 2 has been obtained from Riemann-Hilbert-\(\overline{\partial}\) Problem 1 in an ad-hoc fashion, and even though \(\mathcal{D}(x, y)\) clearly exists, it is not immediately clear that a solution \(\tilde{\mathcal{D}}(z)\) to Riemann-Hilbert Problem 2 exists. Theorem 4 below asserts that a unique solution exists, and describes important asymptotic properties of \(\tilde{\mathcal{D}}(z)\). The Theorem describes the asymptotic behavior of the solution in three different regions of the complex plane: two square domains \(S_\alpha\) and \(S_\beta\) of side-length \(2\delta\), with \(S_\alpha\) centered at \(\alpha\) and \(S_\beta\) centered at \(\beta\), and one exterior domain, \(\mathbb{C} \setminus (S_\alpha \cup S_\beta)\). We further subdivide each square into four regions according to the contour \(\Sigma\) as indicated in Figure 4.

![Figure 4](https://via.placeholder.com/150)

**Figure 4.** The squares \(S_\alpha\) (left) and \(S_\beta\) (right), each subdivided into four regions as indicated.

Note that according to Property 3 in Lemma 1 and Lemma 2, the restriction of the jump matrix \(V_D(z)\) to \(\Sigma \cap S_\alpha\) and \(\Sigma \cap S_\beta\) is piecewise analytic. Indeed, if we define
\[
u_\alpha = u_\alpha(z) := [h'_\alpha(\alpha)]^{2/3}(\alpha - z), \quad u_\beta = u_\beta(z) := [-h'_\beta(\beta)]^{2/3}(z - \beta),
\]
with the positive two-thirds power meant in each case, then we have
\[
V_D(z) = \begin{pmatrix} 1 & e^{-nu_\alpha^{2/3}} \\ 0 & 1 \end{pmatrix}, \quad z \in \Sigma_\alpha \cap S_\alpha,
\]
\begin{align}
V_D(z) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad z \in (\alpha, \beta) \cap S_\alpha, \\
V_D(z) &= \begin{pmatrix} 1 & 0 \\ e^{nu_\beta/2} & 1 \end{pmatrix}, \quad z \in \Sigma_+ \cap S_\alpha, \\
V_D(z) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad z \in (\alpha, \beta) \cap S_\beta, \\
V_D(z) &= \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \quad z \in (\alpha, \beta) \cap S_\beta.
\end{align}

For all \( z \in \Sigma \) outside of the squares \( S_\alpha \) and \( S_\beta \), with the notable exception of the interval \( (\alpha + \delta, \beta - \delta) \) where \( V_D(z) \) is a constant matrix, the jump matrix \( V_D(z) \) decays exponentially to the identity matrix as \( n \to \infty \), as a consequence of both the variational inequality \( \phi(x) > 0 \) for \( x < \alpha - 2\delta \) and \( x > \beta + 2\delta \) and also the inequalities on \( \text{Im}(\Theta(x,y)) \) in Property 2 of Lemma \( \text{[4]} \) and the inequality on \( \text{Re}(\Phi(x,y)) \) in Property 2 of Lemma \( \text{[5]} \).

In a way that is by now quite standard (see \( \text{[6]} \) \( \text{[7]} \)), these facts suggest an explicit model for \( \tilde{D}(z) \) that we will call \( \tilde{D}(z) \) and that we will now define. Let \( \gamma(z) \) be the function analytic in \( \mathbb{C} \setminus [\alpha, \beta] \) determined by the conditions
\begin{equation}
\gamma(z)^4 = \frac{z - \beta}{z - \alpha}, \quad \text{and} \quad \lim_{z \to \infty} \gamma(z) = 1,
\end{equation}
and let \( U \) denote the unitary eigenvector matrix for \( V_D(z) \) on \( (\alpha, \beta) \):
\begin{equation}
U := \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\pi/4} & e^{i\pi/4} \\ e^{-i\pi/4} & e^{-i\pi/4} \end{pmatrix}.
\end{equation}

Then, we set
\begin{align}
\tilde{D}(z) &= U \gamma(z)^{\sigma_3} U^\dagger = \begin{pmatrix} \frac{1}{2} (\gamma(z) + \gamma(z)^{-1}) & \frac{1}{2i} (\gamma(z) - \gamma(z)^{-1}) \\ -\frac{1}{2i} (\gamma(z) - \gamma(z)^{-1}) & \frac{1}{2} (\gamma(z) + \gamma(z)^{-1}) \end{pmatrix}, \quad z \in \mathbb{C} \setminus (S_\alpha \cup S_\beta), \\
\tilde{D}(z) &= -\sqrt{2\pi} U \left( \frac{3n}{4} \right)^{\sigma_3/6} \gamma(z)^{\sigma_3 u_\alpha(z)} \sigma_1/4 \sigma_2 M(u_\alpha(z)) \sigma_3 e^{nu_\alpha(z)^{3/2}\sigma_3/2}, \quad z \in S_\alpha, \\
\tilde{D}(z) &= \sqrt{2\pi} U \left( \frac{4}{3n} \right)^{\sigma_3/6} \gamma(z)^{\sigma_3 u_\beta(z)} \gamma(z)^{-\sigma_3/4} M(u_\beta(z)) e^{nu_\beta(z)^{3/2}\sigma_3/2}, \quad z \in S_\beta,
\end{align}
where \( M(u) \) is defined as follows with \( \xi := \left( \frac{3n}{4} \right)^{2/3} u.
\begin{align}
M(u) &= \begin{pmatrix} e^{-3\pi i/4} A_1' (\xi) & e^{11\pi i/12} A_1' (\xi e^{-2\pi i/3}) \\ e^{-\pi i/4} A_1 (\xi) & e^{11\pi i/12} A_1 (\xi e^{-2\pi i/3}) \end{pmatrix}, \quad -\pi < \arg(u) < \frac{3\pi}{4}, \\
M(u) &= \begin{pmatrix} e^{-5\pi i/12} A_1' (\xi e^{2\pi i/3}) & e^{11\pi i/12} A_1' (\xi e^{-2\pi i/3}) \\ e^{7\pi i/12} A_1 (\xi e^{2\pi i/3}) & e^{11\pi i/12} A_1 (\xi e^{-2\pi i/3}) \end{pmatrix}, \quad \frac{3\pi}{4} < \arg(u) < \pi, \\
M(u) &= \begin{pmatrix} e^{11\pi i/12} A_1' (\xi e^{-2\pi i/3}) & e^{5\pi i/12} A_1' (\xi e^{2\pi i/3}) \\ e^{\pi i/12} A_1 (\xi e^{-2\pi i/3}) & e^{5\pi i/12} A_1 (\xi e^{2\pi i/3}) \end{pmatrix}, \quad -\pi < \arg(u) < -\frac{3\pi}{4}, \\
M(u) &= \begin{pmatrix} e^{-3\pi i/4} A_1' (\xi) & e^{7\pi i/12} A_1' (\xi e^{2\pi i/3}) \\ e^{-\pi i/4} A_1 (\xi) & e^{7\pi i/12} A_1 (\xi e^{2\pi i/3}) \end{pmatrix}, \quad -\frac{3\pi}{4} < \arg(u) < -\frac{\pi}{4}.
\end{align}
Here $\text{Ai}(\xi)$ denotes the Airy function, the unique solution of $y'' - \xi y = 0$ with the asymptotic behavior
\begin{equation}
\text{Ai}(\xi) = \frac{e^{-2\xi^{3/2}/3}}{2^{1/4}\sqrt{\pi}} (1 + \mathcal{O}(|\xi|^{-3/2})) \quad \text{and} \quad \text{Ai}'(\xi) = -\frac{\xi^{1/4}e^{-2\xi^{3/2}/3}}{2\sqrt{\pi}} (1 + \mathcal{O}(|\xi|^{-3/2}))
\end{equation}
as $\xi \to \infty$ with $\arg(\xi) \in (-\pi, \pi)$.

\textbf{Remark:} For those readers familiar with the notation of the paper [7], we make the following clarification. The matrix $M(u)$ defined here may be expressed in the form
\begin{equation}
M(u) = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} 0 & -i \\ 1 & 0 \end{pmatrix} \mathcal{P} \left( \left( \frac{i}{2} \right)^{2/3} u \right) e^{-nu^{3/2}\sigma_3/2}.
\end{equation}
where the local parametrix $\mathcal{P}(\xi)$ is as defined in [7], equations (1.36)–(1.40).

The point of introducing the matrix $\hat{D}(z)$ is the following.

\textbf{Theorem 4.} Assume the conditions on the external field $V$ stated in the Introduction. Let $n, N \to \infty$ so that $N/n \to c$ with $0 < c < \infty$. Then for $n$ sufficiently large, there is a unique solution $\hat{D}(z)$ to Riemann-Hilbert Problem 3 which possesses the following global asymptotic description:
\begin{equation}
\hat{D}(z) = \left( I + \mathcal{O} \left( \frac{1}{n\sqrt{1 + |z|^2}} \right) \right) D(z),
\end{equation}
uniformly with respect to $z \in \mathbb{C}$ as $n \to \infty$.

\textbf{Proof.} Let $F(z) := \hat{D}(z)D(z)^{-1}$. It is easy to see from the properties of $\hat{D}(z)$ required by the conditions of Riemann-Hilbert Problem 3 and the explicit formulae given for the matrix $D(z)$ in various parts of the complex plane that $F(z)$ is a matrix that is required to have the following properties. Firstly, $F(z)$ is analytic at least for $z \in \mathbb{C} \setminus \Sigma^F$, where $\Sigma^F$ is the union of $\Sigma$ and the boundaries of the square regions $S_\alpha$ and $S_\beta$, and $F(z)$ takes continuous boundary values on $\Sigma^F$. Secondly, the boundary values satisfy $F_+(z) = F_-(z)V_F(z)$ for some jump matrix function $V_F(z)$ defined on $\Sigma^F$ that is explicitly calculable in terms of $V_D(z)$ and the boundary values taken on $\Sigma^F$ by $D(z)$. Thirdly, $F(z)$ must tend to the identity matrix as $z \to \infty$. In other words, these three facts show that $F(z)$ satisfies its own Riemann-Hilbert problem.

The Riemann-Hilbert problem satisfied by $F(z)$ is of a particularly convenient type: it is a “small-norm” problem in the sense that the jump matrix $V_F(z)$ is a small perturbation of the identity matrix in a suitable space of matrix-valued functions on the contour $\Sigma^F$. In fact, it is easy to check by direct calculation that $V_F(z) \equiv I$ for $\Sigma^F \cap (S_\alpha \cup S_\beta)$. This is a direct consequence of Property 3 in Lemma 1 and Lemma 2 characterizing respectively the extensions $\Theta(x, y)$ and $\Phi(x, y)$ on these portions of the contour $\Sigma$, and of the identity
\begin{equation}
\text{Ai}(\xi) + e^{-2\pi i/3}\text{Ai}(\xi e^{-2\pi i/3}) + e^{2\pi i/3}\text{Ai}(\xi e^{2\pi i/3}) \equiv 0.
\end{equation}
An even easier calculation shows that $V_F(z) \equiv I$ for $\alpha + \delta < z < \beta - \delta$. With the use of the asymptotic formulae (173), one sees that on the boundaries of the two squares $S_\alpha$ and $S_\beta$, $V_F(z) - I$ is uniformly $\mathcal{O}(n^{-1})$, and on all remaining parts of $\Sigma^F$ one finds (in part by the estimates (68)–(69) on $\text{Im}(\Theta(x, y))$ in Property 2 of Lemma 1 and the estimate (78) on $\text{Re}(\Phi(x, y))$ in Property 2 of Lemma 2) that $V_F(z) - I$ is uniformly exponentially small as $n \to \infty$ and also decays rapidly as $z \to \infty$.

Since for our purposes we need to control the size of $F(z) - I$ right up to the contour $\Sigma^F$, we need to formulate the Riemann-Hilbert problem for $F(z)$ in an appropriate space in which the boundary values of $F(z)$ are Hölder continuous with some exponent $\alpha \in (0, 1]$. To do this we need to observe that as a consequence of our assumptions on the external field $V$ and the corresponding smoothness of $\Theta(x, y)$ and $\Phi(x, y)$ described in Property 1 of Lemma 1 and Lemma 2 and also as a consequence of the piecewise analyticity of the comparison matrix $D(z)$, the jump matrix $V_F(z)$ is sufficiently smooth on a sufficiently (piecewise) smooth contour that the Hölder version of the small-norm theory applies. The result is that as $n \to \infty$, $F(z)$ exists uniquely in the space of matrices with Hölder-continuous boundary values, and also $F(z) - I = \mathcal{O}(n^{-1})$ holds uniformly throughout the complex $z$-plane.

A detailed account of the existence theory for small-norm Riemann-Hilbert problems is discussed, for example, in [6]. Specific information relevant to the application of small-norm theory in Hölder spaces can be found in Appendix A of [11].
An important property of \( \hat{D}(z) \) is that for all \( z \) where it is defined, \( \det(\hat{D}(z)) \equiv 1 \). Therefore, \( \hat{D}(z) \) and its inverse \( \hat{D}(z)^{-1} \) have comparable bounds in any matrix norm. From (166) one may then see that \( \hat{D}(z) \) and its inverse are bounded as \( n \to \infty \) uniformly for \( z \in \mathbb{C} \setminus (S_\alpha \cup S_\beta) \). On the other hand, from (168) and (167) together with the definition (169)–(172) of \( M(u) \) one sees that \( \hat{D}(z) = \mathcal{O}(n^{1/6}) \) and \( \hat{D}(z)^{-1} = \mathcal{O}(n^{1/6}) \) hold for \( z \in S_\alpha \cup S_\beta \), although for our purposes a more useful estimate coming from the same formulae is that

\[
\hat{D}(z) = \mathcal{O}(|z - \alpha|^{-1/4}|z - \beta|^{-1/4}) \quad \text{and} \quad \hat{D}(z)^{-1} = \mathcal{O}(|z - \alpha|^{-1/4}|z - \beta|^{-1/4})
\]

holds uniformly for \( z \) in bounded sets (the constants implicit in the order relations are independent of both \( n \) and \( z \)).

The construction of \( \hat{D}(z) \) is one part of the argument where the details are somewhat different for \( G > 0 \) than for \( G = 0 \). To handle the case with more than one interval of support one must replace the definition of \( \hat{D}(z) \) for \( z \in \mathbb{C} \setminus (S_\alpha \cup S_\beta) \) with a matrix constructed from Riemann theta functions for hyperelliptic curves of nonzero genus modeled by two copies of the complex \( z \)-plane identified along cuts made on the real axis in the support intervals of the equilibrium measure \( \mu_+ \). Full details may be found, for example, in [7]. The key property of uniform boundedness of \( \hat{D}(z) \) away from the endpoints of the support intervals remains valid in this more general case.

6. A \( \bar{\partial} \) Problem and Existence Theorem

Having constructed \( \hat{D}(x + iy) \), we now define \( E(x, y) \) via

\[
E(x, y) := \hat{D}(x, y)\hat{D}(x + iy)^{-1}.
\]

It is immediately clear that \( E(x, y) \) is continuous in \( \mathbb{C} \). It is straightforward to compute the \( \bar{\partial} \) derivative of \( E(x, y) \), and we learn that \( E(x, y) \) solves \( \bar{\partial} \) Problem 1 below. We define a “dressed” version of the matrix \( W_0(x, y) \) as follows:

\[
W(x, y) := \hat{D}(x + iy)W_0(x, y)\hat{D}(x + iy)^{-1}, \quad (x, y) \in \mathbb{R}^2.
\]

\( \bar{\partial} \) Problem 1. Find a \( 2 \times 2 \) matrix \( E(x, y) \) with the properties:

- Continuity. \( E(x, y) \) is a continuous function of \( x \) and \( y \) for \( (x, y) \in \mathbb{R}^2 \).
- Deviation From Analyticity. For \( (x, y) \in \mathbb{R}^2 \),

\[
\bar{\partial}E(x, y) = E(x, y)W(x, y).
\]

(Note that in particular \( \bar{\partial}E(x, y) = 0 \) for \( (x, y) \notin \Omega_+ \cup \Omega_- \cup \Omega_\alpha \cup \Omega_\beta \).)

- Normalization. The matrix \( E(x, y) \) is normalized as follows:

\[
\lim_{x,y \to \infty} E(x, y) = I.
\]

In view of the normalization condition (181) and the fact that \( W(x, y) \equiv 0 \) outside some compact set, we may invert the \( \bar{\partial} \) operator in (180) with the help of the Cauchy kernel:

\[
E(x, y) = I + K E(x, y),
\]

where

\[
K E(x, y) := -\frac{1}{\pi} \int_{\mathbb{R}^2} \left( ((u + iv) - (x + iy))^{-1} E(u, v)W(u, v) du dv \right.
\]

It is a basic fact that if \( E(x, y) \) satisfies the integral equation (182) then \( E(x, y) \) also solves \( \bar{\partial} \) Problem 1 in fact the integral equation (182) is equivalent to \( \bar{\partial} \) Problem 1.

In this section we will show that the integral operator \( K \), when considered in the space \( L^\infty(\mathbb{R}^2) \), has norm bounded by \( C n^{-1/3} \log(n) \) for some \( C > 0 \). This implies that the integral equation (182) may be solved by Neumann series.

The strategy to prove this is quite straightforward: because the singularity of the Cauchy kernel is integrable in \( \mathbb{R}^2 \), the basic estimate is:

\[
\|K E\|_{L^\infty(\mathbb{R}^2)} \leq \|H\|_{L^\infty(\mathbb{R}^2)} \sup_{(x, y) \in \mathbb{R}^2} \left[ \frac{1}{\pi} \int_{\mathbb{R}^2} \|W(u, v)\|_{L^\infty(\mathbb{R}^2)} du dv \right].
\]
where \( \|W(u, v)\| \) is a pointwise matrix norm, i.e. a norm of the matrix \( W \) evaluated at \((u, v)\). Since \( W \) is uniformly bounded and has compact support, this immediately implies that \( K \) is a bounded operator on \( L^\infty(\mathbb{R}^2) \). The goal is then to prove that the integral appearing on the right hand side of (184) is small, and for this the \((u, v)\) dependence of \( \|W(u, v)\| \) will be essential.

**Theorem 5.** There is a unique solution \( E \) to Problem 1 which possesses the following uniform asymptotic description, valid for all \((x, y) \in \mathbb{R}^2\):

\[
E(x, y) = I + O \left( \frac{\log(n)}{n^{1/4} \sqrt{1 + x^2 + y^2}} \right),
\]

where the constant implicit in the order notation is independent of \( n \) and \( z \) and depends only on the external field \( V \) and the constant \( c \).

**Proof.** We begin by describing the asymptotic behavior of \( W(x, y) \). According to (179), \( W(x, y) \) is obtained from \( W_0(x, y) \) by conjugation, so we start by making the following two observations about \( W_0(x, y) \). Firstly, since \( ||0|| \) together with (177) implies that

\[
(185)
I = I + \int_{B} \int \frac{\|W_0(u, v)\|}{|u + iv - (x + iy)|} du dv \leq Kn \int_{B} \int \frac{|v| e^{-kn|v|^{3/2}}}{|u + iv - (x + iy)|} du dv
\]

Secondly, as a consequence of the definition of \( W_0(x, y) \) given in (148)–(151) and Property 2 in Lemma 1 describing \( \Theta(x, y) \), \( \text{Im}(\Theta(x, y)) \), \( \text{Re}(\Phi(x, y)) \) for \((x, y)\) in the support of \( W_0 \), we may assume that

\[
(187) \quad \|W_0(u, v)\| \leq Kn e^{-kn|v|^{3/2}} |v|^2, \quad w = u + iv \in B,
\]

for some constants \( K > 0 \) and \( k > 0 \). Using the definition (179) of \( W(x, y) \) in terms of \( W_0(x, y) \), Theorem 4 together with (177) implies that

\[
(188) \quad \|W(u, v)\| \leq Kn e^{-kn|v|^{3/2}}, \quad w = u + iv \in B.
\]

Of course \( \|W(u, v)\| \equiv 0 \) for \( u + iv \notin B \). Therefore, we have

\[
(189) \quad I(x, y, v) := \int_{\alpha-2\delta}^{\beta+2\delta} \frac{du}{\sqrt{(u-x)^2 + (v-y)^2}} \leq C \log \left( 1 + \frac{1}{|v-y|} \right), \quad (x, y) \in \mathbb{R}^2, \quad v \in \mathbb{R}.
\]

Indeed, since \(|u + iv - (x + iy)| \geq |v-y| \), on the one hand we have

\[
(191) \quad I(x, y, v) \leq \int_{\alpha-2\delta}^{\beta+2\delta} \frac{du}{|v-y|} = \frac{\beta - \alpha + 4\delta}{|v-y|} = \frac{C_1}{|v-y|}.
\]

We will use this estimate when \(|v-y| \) is large. On the other hand, for \(|v-y| \) small we have the following. Firstly, since \(|u + iv - (x + iy)| \geq |u-x| \),

\[
(192) \quad x \leq \alpha - 2\delta - 1 \text{ or } x \geq \beta + 2\delta + 1 \quad \Rightarrow \quad I(x, y, v) \leq \int_{\alpha-2\delta}^{\beta+2\delta} \frac{du}{|u-x|} \leq \int_{\alpha-2\delta}^{\beta+2\delta} du = \beta - \alpha + 4\delta.
\]

Secondly, we have may evaluate \( I(x, y, v) \) explicitly as

\[
(193) \quad I(x, y, v) = \text{arcsinh} \left( \frac{\beta + 2\delta - x}{|v-y|} \right) - \text{arcsinh} \left( \frac{\alpha - 2\delta - x}{|v-y|} \right),
\]

from which it follows that

\[
(194) \quad \alpha - 2\delta - 1 \leq x \leq \alpha - 2\delta < \beta + 2\delta \quad \Rightarrow \quad I(x, y, v) \leq \text{arcsinh} \left( \frac{\beta + 2\delta - x}{|v-y|} \right) \leq \text{arcsinh} \left( \frac{\beta - \alpha + 4\delta + 1}{|v-y|} \right),
\]
\[ \alpha - 2\delta \leq x \leq \beta + 2\delta \implies I(x, y, v) = \arcsinh \left( \frac{\beta + 2\delta - x}{|v - y|} \right) + \arcsinh \left( \frac{x - \alpha + 2\delta}{|v - y|} \right) \leq 2 \arcsinh \left( \frac{\beta - \alpha + 4\delta}{|v - y|} \right), \]

and
\[ \alpha - 2\delta < \beta + 2\delta \leq x \leq \beta + 2\delta + 1 \implies I(x, y, v) \leq \arcsinh \left( \frac{x - \alpha + 2\delta}{|v - y|} \right) \leq \arcsinh \left( \frac{\beta - \alpha + 2\delta + 1}{|v - y|} \right). \]

The estimates (192) and (194)–(196) may be combined for \( |v - y| \) sufficiently small to give
\[ I(x, y, v) \leq C_2 \log \left( \frac{1}{|v - y|} \right). \]

This estimate is useful for \( |v - y| \) small. Taking (197) with (191) yields (190).

Using (190) in (189) and extending the integration from \( |v| \leq \delta \) to \( v \in \mathbb{R} \) yields
\[ \int_{\mathbb{R}^2} \frac{\|W(u, v)\|}{|(u + iv) - (x + iy)|} \, du \, dv \leq Kn \int_{-\infty}^{\infty} |v|e^{-kn|v|^{3/2}} \log \left( 1 + \frac{1}{|v - y|} \right) \, dv \]

for some modified positive constant \( K > 0 \) independent of \( (x, y) \in \mathbb{R}^2 \) and \( n \). Now rescaling the integration variable by \( v = n^{-2/3}s \) gives
\[ n \int_{-\infty}^{\infty} |v|e^{-kn|v|^{3/2}} \log \left( 1 + \frac{1}{|v - y|} \right) \, dv = n^{-1/3} \int_{-\infty}^{\infty} |s|e^{-k|s|^{3/2}} \log \left( 1 + \frac{n^{2/3}}{|s - n^{2/3}y|} \right) \, ds \]
\[ \leq n^{-1/3} \int_{-\infty}^{\infty} |s|e^{-k|s|^{3/2}} \log \left( n^{2/3} + \frac{n^{2/3}}{|s - n^{2/3}y|} \right) \, ds \]
\[ = \frac{2}{3} n^{-1/3} \log(n) \int_{-\infty}^{\infty} |s|e^{-k|s|^{3/2}} \, ds + n^{-1/3} \int_{-\infty}^{\infty} |s|e^{-k|s|^{3/2}} \log \left( 1 + \frac{1}{|s - n^{2/3}y|} \right) \, ds. \]

The first integral is clearly finite and independent of \( n \), and since by Cauchy-Schwarz,
\[ \int_{-\infty}^{\infty} |s|e^{-k|s|^{3/2}} \log \left( 1 + \frac{1}{|s - n^{2/3}y|} \right) \, ds \leq \left[ \int_{-\infty}^{\infty} s^2e^{-2k|s|^{3/2}} \, ds \right]^{1/2} \left[ \int_{-\infty}^{\infty} \log \left( 1 + \frac{1}{|s|} \right)^2 \, ds \right]^{1/2} \]

the second integral is bounded by a finite quantity independent of \( n \). This proves that
\[ \sup_{(x, y) \in \mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\|W(u, v)\|}{|(u + iv) - (x + iy)|} \, du \, dv \leq Cn^{-1/3} \log(n) \]

holds for some constant \( C > 0 \) independent of \( n \), for \( n \) sufficiently large.

The Neumann series for the integral equation (182) corresponding to \( \mathcal{P} \) Problem 1 therefore converges in \( L^\infty(\mathbb{R}^2) \) for sufficiently large \( n \), and the estimate (185) follows immediately. \( \square \)

7. LARGE-\( n \) ASYMPTOTICS FOR \( \mathcal{A}(z) \) AND THE ORTHOGONAL POLYNOMIALS

We will restrict our attention to \( \Omega_+ \) in “the bulk” (i.e. away from the endpoints \( \alpha \) and \( \beta \)) and also to the upper half-plane in the vicinity of the endpoint \( \beta \). Considerations for \( z \) near \( \alpha \) are nearly identical, and we will omit them for the sake of brevity. Since the orthogonal polynomials being considered have real coefficients, the asymptotic behavior for \( z \) in the lower half-plane, \( \mathbb{C}_- \), may always be obtained by complex conjugation. In this section and the next section we will use the notation
\[ \Delta_n := n^{-1/3} \log(n). \]
7.1. **Asymptotics of the leading coefficients.** The leading coefficients $\kappa_{n-1,n}$ and $\kappa_{n,n}$ are obtained from $A(z)$ using (60). In a neighborhood of $z = \infty$, the matrix $A(z)$ is related to $E(x,y)$ via

\[
A(z) = e^{n\ell \sigma_3/2}E(x,y)\tilde{D}(z)e^{ng(z)\sigma_3}e^{-n\ell \sigma_3/2}.
\]

Using (166), (175), and (185), we get

\[
A_{12}(z) = \frac{1}{2i}e^{n(\ell-g(z))} \left( \gamma(z) - \gamma(z)^{-1} + O\left( \frac{\Delta_n}{z} \right) \right)
\]

\[
A_{21}(z) = -\frac{1}{2i}e^{n(g(z)-\ell)} \left( \gamma(z) - \gamma(z)^{-1} + O\left( \frac{\Delta_n}{z} \right) \right),
\]

where the error terms are valid for $z$ near $z = \infty$. Now, from (16) we have

\[
g(z) = \log(z) + O(z^{-1}), \quad z \to \infty,
\]

and from (164) and the condition that $\gamma(z) \to 1$ as $z \to \infty$ we easily obtain

\[
\gamma(z) = 1 - \frac{1}{4z}(\beta - \alpha) + O(z^{-2}), \quad z \to \infty.
\]

Therefore, using (61), (139), and (178). According to (59), the first column of $A$ is related to the Riemann-Hilbert Problem [4] is related to $E(x,y)$ via

\[
A(z) = e^{n\ell \sigma_3/2}E(x,y)\tilde{D}(z) \begin{pmatrix} 1 & 0 \\ e^{-in\Theta(x,y)} & 1 \end{pmatrix} e^{ng(z)\sigma_3}e^{-n\ell \sigma_3/2}, \quad z = x + iy \in \Omega_+,
\]

as follows from (61), (139), and (178). According to (59), the first column of $A(z)$ contains the orthogonal polynomials:

\[
A_{11}(z) = \frac{1}{\kappa_{n,n}}p_n(z), \quad \text{and} \quad A_{21}(z) = -\frac{2\pi i}{\kappa_{n-1,n-1}}p_{n-1}(z).
\]

Via (208) these may be expressed as follows:

\[
A_{11}(z) = \left[ E_{11}(x,y) \left( \tilde{D}_{11}(z) + e^{-in\Theta(x,y)}\tilde{D}_{12}(z) \right) + E_{12}(x,y) \left( \tilde{D}_{21}(z) + e^{-in\Theta(x,y)}\tilde{D}_{22}(z) \right) \right] e^{ng(z)},
\]

\[
A_{21}(z) = \left[ E_{21}(x,y) \left( \tilde{D}_{11}(z) + e^{-in\Theta(x,y)}\tilde{D}_{12}(z) \right) + E_{22}(x,y) \left( \tilde{D}_{21}(z) + e^{-in\Theta(x,y)}\tilde{D}_{22}(z) \right) \right] e^{n(g(z)-\ell)}.
\]

Using the asymptotic estimate (185) of $E(x,y) - \mathbb{1}$, the relation (175) between $\tilde{D}(z)$ and $\hat{D}(z)$, and the explicit formula (166) for $\hat{D}(z)$ valid for $z \in \Omega_+$, straightforward manipulations yield

\[
A_{11}(z) = e^{n(g(z)-i\Theta(x,y)/2)}a(z) \left[ \cos \left( \frac{1}{2} \left( n\Theta(x,y) - \varphi(z) \right) \right) (1 + O(\Delta_n)) 
+ \sin \left( \frac{1}{2} \left( n\Theta(x,y) + \varphi(z) \right) \right) O(\Delta_n) \right]
\]

\[
A_{21}(z) = -ie^{n(g(z)-\ell-i\Theta(x,y)/2)}a(z) \left[ \sin \left( \frac{1}{2} \left( n\Theta(x,y) + \varphi(z) \right) \right) (1 + O(\Delta_n)) 
+ \cos \left( \frac{1}{2} \left( n\Theta(x,y) - \varphi(z) \right) \right) O(\Delta_n) \right],
\]
where
\[(214)\quad a(z) := \frac{\sqrt{\beta - \alpha}}{(z - \alpha)^{1/4}(\beta - z)^{1/4}} \quad \text{and} \quad \varphi(z) := \arcsin \left( \frac{2z - (\alpha + \beta)}{\beta - \alpha} \right)\]
are both functions analytic for \(z \in \mathbb{C} \setminus (\mathbb{R} \setminus [\alpha, \beta])\). One apparent difficulty with these asymptotic formulae is that they involve an extension \(\Theta(x, y)\) of \(\theta(x)\) that is completely arbitrary except that it must satisfy Properties 1–3 of Lemma 1. (While our proof of Lemma 1 was by construction, there was no assertion of uniqueness, and indeed there are many extensions \(\Theta(x, y)\) having the required properties.) On the other hand, it is also easy to see that the differences between various extensions \(\Theta(x, y)\) may be absorbed into the error terms. For example, if we fix \(x \in (\alpha, \beta)\) and fix \(y > 0\) sufficiently small so as to be in the region \(\Omega_+\), then by Property 2 of Lemma 1 we have \(\Im(\Theta(x, y)) \leq -ky\) for some \(k > 0\) (here we are using the fact that \(x\) is bounded away from \(\alpha\) and \(\beta\), so \(212\) and \(213\) become
\[(215)\quad A_{11}(z) = \frac{1}{2} e^{n g(z)} a(z) e^{-i\varphi(z)/2} (1 + O(\Delta_n)), \quad y \gg n^{-1},\]
and
\[(216)\quad A_{21}(z) = -\frac{1}{2} i e^{n (g(z) - \ell)} a(z) e^{-i\varphi(z)/2} (1 + O(\Delta_n)), \quad y \gg n^{-1}.\]
On the other hand, if we suppose that \(x \in (\alpha, \beta)\) is fixed and \(y = O(n^{-1})\), then using Property 1 of Lemma 1 and the Mean Value Theorem we have
\[(217)\quad \Theta(x, y) = \theta(x) + \Theta_y(x, \xi_1)y, \quad 0 \leq \xi_1 \leq y\]
and using \(21\) to eliminate \(\Theta_y\) in terms of \(\Theta_x\) and \(\overline{\Theta}\), this becomes
\[(218)\quad \Theta(x, y) = \theta(x) + i \Theta(x, \xi_1)y - 2\overline{\Theta}(x, \xi_1)y, \quad 0 \leq \xi_1 \leq y.\]
Since \(x\) is bounded away from \(\alpha\) and \(\beta\), Property 1 of Lemma 1 guarantees further that \(\Theta_{xy}\) is continuous, so another application of the Mean Value Theorem gives
\[(219)\quad \Theta_x(x, \xi)y = \Theta_x(x, 0)y + \Theta_{xy}(x, \xi_2)y\xi_1 = \theta'(x)y + \Theta_{xy}(x, \xi_2)y\xi_1.\]
Finally, using Property 2 of Lemma 1 to control \(\overline{\Theta}\) and using the assumption that \(y = O(n^{-1})\) we find
\[(220)\quad \Theta(x, y) = \theta(x) + i \theta'(x)y + O(n^{-2}).\]
It follows that \(212\) and \(213\) become
\[(221)\quad A_{11}(z) = e^{n(g(z) - i\theta(x)/2 + \theta'(x)y/2)} a(x) \left[ \cos \left( \frac{1}{2} (n\theta(x) + i n\theta'(x)y - \varphi(x)) \right) + O(\Delta_n) \right]\]
and
\[(222)\quad A_{21}(z) = -ie^{n(g(z) - \ell - i\theta(x)/2 + \theta'(x)y/2)} a(x) \left[ \sin \left( \frac{1}{2} (n\theta(x) + i n\theta'(x)y + \varphi(x)) \right) + O(\Delta_n) \right].\]
In particular, for \(y = 0+\) we can write these in the form
\[(223)\quad A_{11}(x) = e^{n(cV(x) + \ell - \phi(x))/2} a(x) \cos \left( \frac{1}{2} (n\theta(x) - \varphi(x)) \right) + O \left( \Delta_n e^{n(cV(x) + \ell - \phi(x))/2} \right)\]
and
\[(224)\quad A_{21}(x) = -ie^{n(cV(x) - \ell - \phi(x))/2} a(x) \sin \left( \frac{1}{2} (n\theta(x) - \varphi(x)) \right) + O \left( \Delta_n e^{n(cV(x) - \ell - \phi(x))/2} \right).\]
7.3. Asymptotics of the orthogonal polynomials at the edge. Next, suppose that \( z \in S^I_\beta \) (see Figure 4). Then similar calculations in which the formulae (167) and (170) are used to find \( \hat{D}(z) \) yield

\[
A_{11}(z) = \sqrt{\pi} e^{\eta g(z)} [F^1_n(z) (1 + \mathcal{O}(\Delta_n)) + F^2_n(z) \mathcal{O}(\Delta_n)],
\]
\[
A_{21}(z) = \sqrt{\pi} e^{\eta g(z)-\ell} [F^2_n(z) (1 + \mathcal{O}(\Delta_n)) + F^1_n(z) \mathcal{O}(\Delta_n)],
\]
where

\[
F^1_n(z) := n^{1/6} w(z) \left[ e^{-i\pi/3} e^{nu_\beta(z)3/2} Ai' \left( \left( \frac{3n}{4} \right)^{2/3} u_\beta(z) e^{2\pi i/3} \right) + e^{i\pi/3} e^{-nu_\beta(z)3/2} e^{-i\Theta(x,y)} Ai' \left( \left( \frac{3n}{4} \right)^{2/3} u_\beta(z) e^{-2\pi i/3} \right) \right]
\]
\[
F^2_n(z) := n^{1/6} w(z) \left[ e^{-5i\pi/6} e^{nu_\beta(z)3/2} Ai' \left( \left( \frac{3n}{4} \right)^{2/3} u_\beta(z) e^{2\pi i/3} \right) + e^{i\pi/6} e^{-nu_\beta(z)3/2} e^{-i\Theta(x,y)} Ai' \left( \left( \frac{3n}{4} \right)^{2/3} u_\beta(z) e^{-2\pi i/3} \right) \right],
\]

and

\[
w(z) := \left( \frac{3}{4} \right)^{1/6} [-h_\beta'(\beta)]^{1/6} (z - \alpha)^{1/4}.
\]

Now it will also be useful to have the asymptotic behavior of \( A_{11}(z) \) and \( A_{21}(z) \) for \( z \in \mathbb{S}^I_\beta \cap \mathbb{C}_+ \) (see Figure 4), and for this purpose we note that for such \( z \) we have \( B(z) = D(z) \), so in place of (208) we have instead

\[
A(z) = e^{\eta \sigma_3/2} E(x,y) \hat{D}(z) e^{\eta g(z)\sigma_3} e^{-\eta \sigma_3/2}, \quad z = x + iy \in \mathbb{C} \setminus (\Omega_+ \cup \Omega_- \cup \Omega_\alpha \cup \Omega_\beta).
\]

Therefore, for such \( z \):

\[
A_{11}(z) = \left[ E_{11}(x,y) \hat{D}_{11}(z) + E_{12}(x,y) \hat{D}_{21}(z) \right] e^{n g(z)},
\]
\[
A_{21}(z) = \left[ E_{21}(x,y) \hat{D}_{11}(z) + E_{22}(x,y) \hat{D}_{21}(z) \right] e^{n (g(z)-\ell)}.
\]

Supposing that \( z \in S^I_\beta \cap \mathbb{C}_+ \), we may now proceed by using (167) and (169) to find \( \hat{D}(z) \), with the result that

\[
A_{11}(z) = \sqrt{\pi} e^{\eta g(z)} e^{nu_\beta(z)3/2} \left[ G^1_n(z) (1 + \mathcal{O}(\Delta_n)) + G^2_n(z) \mathcal{O}(\Delta_n) \right],
\]
\[
A_{21}(z) = \sqrt{\pi} e^{n (g(z)-\ell)} e^{nu_\beta(z)3/2} \left[ G^2_n(z) (1 + \mathcal{O}(\Delta_n)) + G^1_n(z) \mathcal{O}(\Delta_n) \right],
\]

where

\[
G^1_n(z) := n^{1/6} w(z) Ai' \left( \left( \frac{3n}{4} \right)^{2/3} u_\beta(z) \right) - n^{-1/6} w(z)^{-1} Ai' \left( \left( \frac{3n}{4} \right)^{2/3} u_\beta(z) \right),
\]
\[
G^2_n(z) := -i n^{1/6} w(z) Ai' \left( \left( \frac{3n}{4} \right)^{2/3} u_\beta(z) \right) - i n^{-1/6} w(z)^{-1} Ai' \left( \left( \frac{3n}{4} \right)^{2/3} u_\beta(z) \right).
\]

If we assume that \( \zeta := (3n/4)^{2/3} u_\beta(z) \) is bounded, then Property 3 of Lemma I guarantees that \( e^{-i\Theta(x,y)} = e^{2\zeta^2/3} (1 + O(n^{-2/3})) \), and so with the use of (170) we see that (225) and (226) agree, respectively, with (233) and (234) up to error terms; we therefore have

\[
A_{11}(z) = e^{\eta g(z)} \left[ \sqrt{\pi} w(\beta) n^{1/6} e^{2\zeta^2/3} Ai(\zeta) + \mathcal{O} \left( n^{1/6} \Delta_n \right) \right]
\]
and

\begin{equation}
A_{21}(z) = e^{n(g(z)-\ell)} \left[ -i\sqrt{\pi}w(\beta)n^{1/6}e^{2cV^{3/2}/3}A_i(\zeta) + O\left(n^{1/6} \Delta_n\right) \right]
\end{equation}

for \( \zeta \) bounded with \( 0 \leq \arg(\zeta) \leq \pi \), where

\begin{equation}
z = \beta + (\lambda n)^{-2/3} \zeta, \quad \lambda := \frac{3}{4} \left[ -i\ell']_{\beta}(\beta) \right]^{-1}.
\end{equation}

Moreover, we may observe that for \( z \) near \( \beta \), and \( z \in \mathbb{C}_+ \), the following local expansion holds true:

\begin{equation}
g(z) + \frac{2}{3n} \zeta^{3/2} = g(z) + \frac{1}{2} u_\beta(z)^{3/2} = \frac{eV(\beta) + \ell}{2} + \frac{cV'(\beta)}{2} (z - \beta) + O((z - \beta)^2).
\end{equation}

(This is proved in the Appendix under the conditions on the external field \( V \) in force in this paper.) Therefore, (237) and (238) may also be written as

\begin{equation}
A_{11} \left( \beta + (\lambda n)^{-2/3} \zeta \right) = n^{1/6} e^{n(cV(\beta) + \ell)/2} e^{n^{1/3} cV'(\beta) \lambda^{-2/3} \zeta/2} \sqrt{\pi} w(\beta) A_i(\zeta) + O \left( n^{1/6} \Delta_n e^{n(cV(\beta) + \ell)/2} e^{n^{1/3} cV'(\beta) \lambda^{-2/3} \zeta/2} \right)
\end{equation}

and

\begin{equation}
A_{21} \left( \beta + (\lambda n)^{-2/3} \zeta \right) = -i n^{1/6} e^{n(cV(\beta) - \ell)/2} e^{n^{1/3} cV'(\beta) \lambda^{-2/3} \zeta/2} \sqrt{\pi} w(\beta) A_i(\zeta) + O \left( n^{1/6} \Delta_n e^{n(cV(\beta) - \ell)/2} e^{n^{1/3} cV'(\beta) \lambda^{-2/3} \zeta/2} \right).
\end{equation}

Here we used the fact that the error term in (240) is \( O(n^{-4/3}) \) for \( |\zeta| \) bounded, so the dominant terms in the errors come from (237) and (238).

8. Asymptotics for the Derivative and Applications to Random Matrix Theory

We have established large-\( n \) asymptotics uniform with respect to \( z \) for the matrix \( A(z) \), for all \( z \in \mathbb{C} \). In particular, \( A_{11}(z) \) and \( A_{21}(z) \) possess asymptotic descriptions in a neighborhood of the interval \([\alpha, \beta]\). In this section we derive asymptotic descriptions for the derivatives \( A'_{11}(z) \) and \( A'_{21}(z) \), both in “the bulk”, i.e. for \( x \in (\alpha, \beta) \) as well as near the endpoints \( \alpha \) and \( \beta \). (In fact we will only consider the endpoint \( \beta \).

Our aim is to obtain derivative asymptotics in order to establish bulk and edge universality for unitarily invariant matrix models with external fields that possess only two Lipschitz continuous derivatives, as described in the Introduction. It is by now well-known (see, for example, [16]) that if one obtains an asymptotic description of the orthogonal polynomials and their derivatives of the form which we obtain here, then the corresponding asymptotic formulae for the reproducing kernels follows and exhibits universality (independence of details of the external field \( V \)), and so we will omit these details.

8.1. Analysis of derivatives in the bulk. Let \( x \in (\alpha, \beta) \) be bounded away from the endpoints as \( n \to \infty \). Since \( A_{11}(z) \) and \( A_{21}(z) \) are polynomials and hence entire functions, we may express their derivatives at \( x \) by Cauchy’s integral formula:

\begin{equation}
A'_{j1}(x) = \frac{1}{2\pi i} \oint \frac{A_{j1}(s) \, ds}{(s - x)^2}, \quad j = 1, 2.
\end{equation}

Moreover, since (see 209) \( A_{11}(z) \) and \( iA_{21}(z) \) have real coefficients, we may use complex-conjugation symmetry and the reality of \( x \) to write

\begin{equation}
A'_{11}(x) = \frac{1}{\pi} \text{Im} \left( \int_{\Gamma} \frac{A_{11}(s) \, ds}{(s - x)^2} \right) \quad \text{and} \quad A'_{21}(x) = \frac{1}{\pi i} \text{Re} \left( \int_{\Gamma} \frac{A_{21}(s) \, ds}{(s - x)^2} \right),
\end{equation}

where \( \Gamma \) is any path of integration that begins on the real axis to the right of \( x \) and terminates on the real axis to the left of \( x \), and that avoids the singularity at \( x \) by passing through the upper half-plane. For our
calculations, we will take the path \( \Gamma \) to be a semicircle of radius \( n^{-1} \) centered at \( s = x \): \( s = x + n^{-1}e^{i\omega} \) for \( 0 < \omega < \pi \). Thus we have

\begin{align}
A'_{11}(x) &= \frac{n}{\pi} \Im \left( \int_0^\pi A_{11}(x + n^{-1}e^{i\omega})i e^{-i\omega} \, d\omega \right), \\
A'_{21}(x) &= \frac{n}{\pi i} \Re \left( \int_0^\pi A_{21}(x + n^{-1}e^{i\omega})i e^{-i\omega} \, d\omega \right).
\end{align}

We will now substitute from (212) and (213), but first we write them in a more suitable form. Since \( a(z) \) is bounded for \( z \in \Gamma \), and since \( g(x + iy) \), \( \Theta(x, y) \), and \( \varphi(x + iy) \) are all differentiable, Taylor expansion about \( x = 0 \) shows that for \( z \in \Gamma \) we have both

\begin{align}
A_{11}(z) &= M_1(x, y) + \mathcal{O} \left( \Delta_n e^{ng_+(x)} \right) \quad \text{and} \quad e^{n\ell}A_{21}(z) = M_2(x, y) + \mathcal{O} \left( \Delta_n e^{ng_+(x)} \right),
\end{align}

where \( g_+(x) \) is the boundary value taken by \( g(z) \) as \( z \to x \) from \( \mathbb{C}_+ \), and where \( (z = x + iy) \)

\begin{align}
M_1(x, y) &= e^{a(g(z) - i\theta(x, y)/2)} a(z) \cos \left( \frac{1}{2}(n\Theta(x, y) - \varphi(z)) \right)
\end{align}

and

\begin{align}
M_2(x, y) &= -ie^{a(g(z) - i\theta(x, y)/2)} a(z) \sin \left( \frac{1}{2}(n\Theta(x, y) + \varphi(z)) \right).
\end{align}

One important observation is that \( M_1(x, y) \) and \( iM_2(x, y) \) have real boundary values taken on the real axis from the upper half-plane. Indeed, the analytic functions \( a(z) \) and \( \varphi(z) \) are real for real \( z \), Property 1 of Lemma 1 implies \( \Theta(x, 0) = \theta(x) \in \mathbb{R} \), and furthermore by (49).

\begin{align}
2g_+(x) - i\Theta(x, 0) = 2g_+(x) - i\theta(x) = g_+(x) + g_-(x) \in \mathbb{R}, \quad x \in (\alpha, \beta).
\end{align}

Using (247) in (245) and (246) gives

\begin{align}
A'_{11}(x) &= \frac{n}{\pi} \Im \left( \int_0^\pi M_1(x + n^{-1} \cos(\omega), n^{-1} \sin(\omega))i e^{-i\omega} \, d\omega \right) + \mathcal{O} \left( \frac{n}{\Delta_n} e^{ng_+(x)} \right), \\
e^{n\ell}A'_{21}(x) &= \frac{n}{\pi i} \Re \left( \int_0^\pi M_2(x + n^{-1} \cos(\omega), n^{-1} \sin(\omega))i e^{-i\omega} \, d\omega \right) + \mathcal{O} \left( \frac{n}{\Delta_n} e^{ng_+(x)} \right).
\end{align}

We begin our analysis by integrating by parts: since for \( M = M_1 \) or \( M = M_2 \),

\begin{align}
\int_0^\pi M(x + n^{-1} \cos(\omega), n^{-1} \sin(\omega))i e^{-i\omega} \, d\omega = -\int_0^\pi M(x + n^{-1} \cos(\omega), n^{-1} \sin(\omega)) \frac{d}{d\omega} (e^{-i\omega}) \, d\omega \\
= M(x - n^{-1}, 0) + M(x + n^{-1}, 0) \\
+ \frac{n}{2} \int_0^\pi M_y(x + n^{-1} \cos(\omega), n^{-1} \sin(\omega)) \cos(\omega) e^{-i\omega} \, d\omega \\
- \frac{1}{n} \int_0^\pi M_x(x + n^{-1} \cos(\omega), n^{-1} \sin(\omega)) \sin(\omega) e^{-i\omega} \, d\omega,
\end{align}

we see that upon taking the imaginary part (for \( A'_{11}(x) \)) or real part (for \( A'_{21}(x) \)), the boundary terms vanish:

\begin{align}
\frac{n}{\pi} \Im \left( \int_0^\pi M_1(x + n^{-1} \cos(\omega), n^{-1} \sin(\omega))i e^{-i\omega} \, d\omega \right) \\
= \Im \left( \int_0^\pi M_{1y}(x + n^{-1} \cos(\omega), n^{-1} \sin(\omega)) \cos(\omega) e^{-i\omega} \, d\omega \right) \\
- \Im \left( \int_0^\pi M_{1x}(x + n^{-1} \cos(\omega), n^{-1} \sin(\omega)) \sin(\omega) e^{-i\omega} \, d\omega \right),
\end{align}

\( \text{31} \).
and

\[ \text{(255)} \quad n \text{Re} \left( \int_{0}^{\pi} M_2(x + n^{-1} \cos(\omega), n^{-1} \sin(\omega)) i e^{-i \omega} d\omega \right) \]

\[ = \text{Re} \left( \int_{0}^{\pi} M_{2y}(x + n^{-1} \cos(\omega), n^{-1} \sin(\omega)) \cos(\omega) e^{-i \omega} d\omega \right) \]

\[ - \text{Re} \left( \int_{0}^{\pi} M_{2x}(x + n^{-1} \cos(\omega), n^{-1} \sin(\omega)) \sin(\omega) e^{-i \omega} d\omega \right). \]

Moreover, using (21) to eliminate \( M_{jy} \) in favor of \( M_{jx} \) and \( \overline{\partial}M_j \) for \( j = 1, 2 \), these become

\[ \text{(256)} \quad n \text{Im} \left( \int_{0}^{\pi} M_1(x + n^{-1} \cos(\omega), n^{-1} \sin(\omega)) i e^{-i \omega} d\omega \right) \]

\[ = \text{Re} \left( \int_{0}^{\pi} M_{1x}(x + n^{-1} \cos(\omega), n^{-1} \sin(\omega)) d\omega \right) \]

\[ - 2 \text{Re} \left( \int_{0}^{\pi} \overline{\partial}M_1(x + n^{-1} \cos(\omega), n^{-1} \sin(\omega)) \cos(\omega) e^{-i \omega} d\omega \right), \]

and

\[ \text{(257)} \quad n \text{Re} \left( \int_{0}^{\pi} M_2(x + n^{-1} \cos(\omega), n^{-1} \sin(\omega)) i e^{-i \omega} d\omega \right) \]

\[ = - \text{Im} \left( \int_{0}^{\pi} M_{2x}(x + n^{-1} \cos(\omega), n^{-1} \sin(\omega)) d\omega \right) \]

\[ + 2 \text{Im} \left( \int_{0}^{\pi} \overline{\partial}M_2(x + n^{-1} \cos(\omega), n^{-1} \sin(\omega)) \cos(\omega) e^{-i \omega} d\omega \right). \]

By splitting cosines and sines into exponentials, we may write \( M_1(x, y) \) and \( M_2(x, y) \) in the form

\[ M_1(x, y) := u^-(z)e^{ng(z)} + u^+(z)e^{ng(z) - i\Theta(x, y)} \]

and

\[ M_2(x, y) := u^-(z)e^{ng(z) - i\Theta(x, y)} - u^+(z)e^{ng(z)}, \]

where

\[ u^\pm(z) := \frac{1}{2}a(z)e^{\pm i\varphi(z)/2}. \]

One reason for writing \( M_1(x, y) \) and \( M_2(x, y) \) in this way is to explicitly display their dependence on \( n \); indeed \( u^\pm(z) \), \( g(z) \), and \( \Theta(x, y) \) are independent of \( n \). Using the fact that \( u^\pm(z) \) and \( g(z) \) are all analytic functions of \( z = x + iy \) on the contour \( \Gamma \), we have

\[ \overline{\partial}M_1(x, y) = -iu^+(z)e^{ng(z) - i\Theta(x, y)}\overline{\Theta}(x, y) \]

and

\[ \overline{\partial}M_2(x, y) = -iu^-(z)e^{ng(z) - i\Theta(x, y)}\overline{\Theta}(x, y). \]

Now, \( u^\pm(z) \) are bounded functions, and according to Property 2 of Lemma 1 \( \overline{\partial}\Theta(x, y) = O(n^{-1}) \) for \( x + iy \in \Gamma \). Since also \( g(z) \) and \( \Theta(x, y) \) are differentiable and (by Property 1 of Lemma 1) \( \Theta(x, 0) = \theta(x) \in \mathbb{R} \), we finally learn that

\[ \overline{\partial}M_1(x, y) = O \left( e^{ng^+(x)} \right) \]

and

\[ \overline{\partial}M_2(x, y) = O \left( e^{ng^+(x)} \right), \quad x + iy \in \Gamma. \]

Therefore, (251) and (252) may be written in the form

\[ A_{11}^t(x) = \frac{1}{\pi} \text{Re} \left( \int_{0}^{\pi} M_{1x}(x + n^{-1} \cos(\omega), n^{-1} \sin(\omega)) d\omega \right) + O \left( e^{ng^+(x)} \right) \]

and

\[ e^{nt}A_{21}(x) = \frac{i}{\pi} \text{Im} \left( \int_{0}^{\pi} M_{2x}(x + n^{-1} \cos(\omega), n^{-1} \sin(\omega)) d\omega \right) + O \left( e^{ng^+(x)} \right). \]
Now, each term in $M_1(x, y)$ and $M_2(x, y)$ is of the form $u^\pm(z)e^{n(g(z)-i\sigma \Theta(x, y))}$ where either $\sigma = 0$ or $\sigma = 1$, and

$$
\frac{\partial}{\partial x} \left[ u^\pm(z)e^{n(g(z)-i\sigma \Theta(x, y))} \right] = \left[ u^\pm(z) + nu^\pm(z) \left(g'(z) - i\sigma \Theta'(z, y)\right)\right] e^{n(g(z)-i\sigma \Theta(x, y))},
$$

so evaluating for $z = x + n^{-1}e^{i\omega}$ we may expand the result for large $n$. Using Property 1 of Lemma 1 to assert second-order differentiability of $\Theta(x, y)$ in the bulk and Property 2 of the same Lemma to guarantee that $\partial \Theta(x, 0) = 0$, we finally arrive at

$$
\frac{\partial}{\partial x} \left[ u^\pm(z)e^{n(g(z)-i\sigma \Theta(x, y))} \right] \bigg|_{z=x+n^{-1}e^{i\omega}} = nu^\pm(x) \left(g'(x) - i\sigma \theta'(x)\right) e^{n(g(x)-i\sigma \theta(x))} e^{n\pi\omega} + O \left(e^{n\pi\omega}\right),
$$

where $q := g'(x) - i\sigma \theta'(x)$ is independent of $\omega$. Since for any complex number $q$

$$
\int_0^\pi e^{n\pi\omega} d\omega = \pi,
$$

we obtain from (263) and (264) that

$$
A'_{11}(x) = \text{Re} \left( nu^- (x) g'_+(x) e^{ng_+(x)} + nu^+ (x) \left(g'_+(x) - i\theta'(x)\right) e^{ng_+(x) - i\theta(x)}\right) + O \left(e^{ng_+(x)}\right)
$$

(268)

$$
= \text{Re} \left( \frac{d}{dx} M_1(x, 0+)\right) + O \left(e^{ng_+(x)}\right)
$$

$$
= \frac{d}{dx} \left[ e^{n(cV(x)+\tilde{\ell} - \phi(x)/2)} a(x) \cos \left(\frac{1}{2} (n\theta(x) - \varphi(x))\right)\right] + O \left(e^{n(cV(x)+\tilde{\ell} - \phi(x)/2)}\right)
$$

and

$$
A'_{21}(x) = i\text{Im} \left( nu^- (x) \left(g'_+ - i\theta'(x)\right) e^{ng_+(x) - i\theta(x)} - nu^+ (x) g'_+ e^{ng_+(x)}\right) + O \left(e^{ng_+(x) - \tilde{\ell}}\right)
$$

(269)

$$
= i\text{Im} \left( e^{-n\tilde{\ell}} \frac{d}{dx} M_2(x, 0+)\right) + O \left(e^{ng_+(x) - \tilde{\ell}}\right)
$$

$$
= \frac{d}{dx} \left[ -i e^{n(cV(x)-\tilde{\ell} - \phi(x)/2)} a(x) \sin \left(\frac{1}{2} (n\theta(x) + \varphi(x))\right)\right] + O \left(e^{n(cV(x)-\tilde{\ell} - \phi(x)/2)}\right).
$$

Comparing these results with (223) and (224) shows that the asymptotic formulae for the derivatives of the orthogonal polynomials on the real axis in the bulk may be obtained from the corresponding asymptotic formulae for the polynomials themselves by differentiating the leading terms.

### 8.2. Analysis of derivatives at the edge.

We will now apply similar considerations to the asymptotic formulae (237)–(238), to obtain asymptotics for derivatives of the orthogonal polynomials that are valid for $z$ in a vicinity of the endpoints $\beta$ and $\alpha$. We will present the details for the endpoint $z = \beta$, as the argument for the behavior near $z = \alpha$ is entirely similar. More precisely, our aim is to establish asymptotic formulae for the quantities

$$
\frac{d}{d\zeta} A_{11} \left(\beta + (\lambda n)^{-2/3} \zeta\right) \quad \text{and} \quad \frac{d}{d\zeta} A_{21} \left(\beta + (\lambda n)^{-2/3} \zeta\right), \quad \lambda := \frac{3}{4} \left[-h'_0(\beta)\right]^{-1},
$$

which are what one needs to establish universality of the distribution of the largest eigenvalue in Hermitian random matrix theory.

Let

$$
\tau_1(\zeta) := e^{-n(cV(\beta)+\tilde{\ell})/2} e^{-n^{1/3} cV'(\beta)\lambda^{-2/3} \zeta/2} A_{11} \left(\beta + (\lambda n)^{-2/3} \zeta\right),
$$

(271)

$$
\tau_2(\zeta) := e^{-n(cV(\beta)-\tilde{\ell})/2} e^{-n^{1/3} cV'(\beta)\lambda^{-2/3} \zeta/2} A_{21} \left(\beta + (\lambda n)^{-2/3} \zeta\right).
$$

(272)

According to (241) and (242), these may be expressed as

$$
\tau_1(\zeta) = n^{1/6} \sqrt{\pi} u(\beta) A_1(\zeta) + O \left(n^{1/6} \Delta_n\right)
$$

(273)
These may also be written as edge may be obtained by differentiating the leading terms of the corresponding formulae for the polynomials which, upon comparing with (241) and (242), show that asymptotic formulae for derivatives valid at the point (281)

and

(274) \[ \tau_2(\zeta) = -in^{1/6}\sqrt{\pi}w(\beta)Ai(\zeta) + O\left(n^{1/6}\Delta_n\right) \]

for \(|\zeta|\) bounded.

Since \(\tau_1(\zeta)\) and \(\tau_2(\zeta)\) are entire functions that are real for real \(\zeta\), just as in the analysis in the bulk we may express the derivatives \(\tau'_1(\zeta)\) and \(\tau'_2(\zeta)\) in terms of Cauchy’s formula as

(275) \[ \tau'_1(\zeta) = \frac{1}{\pi} \text{Im} \left( \int_{\Gamma} \frac{\tau(\xi) d\xi}{(\xi - \zeta)^2} \right) \]

and

(276) \[ \tau'_2(\zeta) = \frac{1}{\pi i} \text{Re} \left( \int_{\Gamma} \frac{\tau(\xi) d\xi}{(\xi - \zeta)^2} \right) \]

as long as \(\zeta \in \mathbb{R}\), where \(\Gamma\) is any path in the upper half-plane from the real axis to the right of \(\zeta\) to another point on the real axis to the left of \(\zeta\). But, since the dominant terms in (273) and (274) are entire functions of \(\zeta\), and since \(|\xi - \zeta|\) is bounded away from zero on the contour \(\Gamma\) of finite length, it follows from a residue calculation that

(277) \[ \tau'_1(\zeta) = \frac{d}{d\zeta} \left(n^{1/6}\sqrt{\pi}w(\beta)Ai(\zeta)\right) + O\left(n^{1/6}\Delta_n\right) \]

and

(278) \[ \tau'_2(\zeta) = \frac{d}{d\zeta} \left(-in^{1/6}\sqrt{\pi}w(\beta)Ai(\zeta)\right) + O\left(n^{1/6}\Delta_n\right). \]

Therefore,

(279) \[ \frac{d}{d\zeta} A_{11} \left(\beta + (\lambda\zeta)^{-2/3}\right) = \frac{1}{2} n^{1/2} e^{\lambda n^{1/3}} e^{n^{1/3}e^{\lambda n^{1/3}}(\beta + \lambda n^{1/3})} e^{n^{1/3}e^{\lambda n^{1/3}}(\beta)\lambda^{-2/3}\zeta/2} \sqrt{\pi}w(\beta)Ai(\zeta) \]

\[ + O\left(n^{1/2}\Delta_n e^{n^{1/3}e^{\lambda n^{1/3}}(\beta)\lambda^{-2/3}\zeta/2}\right) \]

(280) \[ \frac{d}{d\zeta} A_{21} \left(\beta + (\lambda\zeta)^{-2/3}\right) = -\frac{1}{2} n^{1/2} e^{\lambda n^{1/3}} e^{n^{1/3}e^{\lambda n^{1/3}}(\beta - \lambda n^{1/3})} e^{n^{1/3}e^{\lambda n^{1/3}}(\beta)\lambda^{-2/3}\zeta/2} \sqrt{\pi}w(\beta)Ai(\zeta) \]

\[ + O\left(n^{1/2}\Delta_n e^{n^{1/3}e^{\lambda n^{1/3}}(\beta)\lambda^{-2/3}\zeta/2}\right). \]

These may also be written as

(281) \[ \frac{d}{d\zeta} A_{11} \left(\beta + (\lambda\zeta)^{-2/3}\right) = \frac{d}{d\zeta} \left[n^{1/6} e^{n^{1/3}e^{\lambda n^{1/3}}(\beta)} e^{n^{1/3}e^{\lambda n^{1/3}}(\beta)\lambda^{-2/3}\zeta/2} \sqrt{\pi}w(\beta)Ai(\zeta)\right] \]

\[ + O\left(n^{1/2}\Delta_n e^{n^{1/3}e^{\lambda n^{1/3}}(\beta)\lambda^{-2/3}\zeta/2}\right) \]

(282) \[ \frac{d}{d\zeta} A_{21} \left(\beta + (\lambda\zeta)^{-2/3}\right) = \frac{d}{d\zeta} \left[-in^{1/6} e^{n^{1/3}e^{\lambda n^{1/3}}(\beta)} e^{n^{1/3}e^{\lambda n^{1/3}}(\beta)\lambda^{-2/3}\zeta/2} \sqrt{\pi}w(\beta)Ai(\zeta)\right] \]

\[ + O\left(n^{1/2}\Delta_n e^{n^{1/3}e^{\lambda n^{1/3}}(\beta)\lambda^{-2/3}\zeta/2}\right), \]

which, upon comparing with (241) and (242), show that asymptotic formulae for derivatives valid at the edge may be obtained by differentiating the leading terms of the corresponding formulae for the polynomials themselves.

Appendix: Convex External Fields

Everywhere in this Appendix we shall assume that (i) the external field grows sufficiently rapidly as \(|x| \to \infty\), and (ii) the external field is strictly convex, and possesses \(d\) continuous derivatives, with \(d \geq 2\).

In a separate discussion below, we will consider the specific situation that \(V\) possesses just two Lipschitz continuous derivatives. The main results are summarized in Lemma [3] at the end of this Appendix.

The assumed growth and strict convexity of \(V(x)\) and the positivity of \(c\) implies that \(\mu_\ast\) is compactly supported and absolutely continuous with respect to Lebesgue measure, with support consisting of a single
interval \([\alpha, \beta]\) for some real \(\alpha < \beta\). To obtain a formula for \(\mu_*\) in this case, we consider the auxiliary function \(g(z)\) defined in (40), analytic for \(z \in \mathbb{C} \setminus (-\infty, \beta]\) (in the present case the integral is taken over the single interval \([\alpha, \beta]\)). In terms of \(g(z)\) the variational condition (44) becomes (47) which we rewrite here:

\[
\psi(x) = \psi(x), \quad x \in (\alpha, \beta)
\]

(283)

where \(g_+(x)\) and \(g_-(x)\) denote the boundary values taken by \(g(z)\) as \(z \to x\) with \(z \in \mathbb{C}_+\) and \(z \in \mathbb{C}_-\) respectively.

Assuming that differentiation commutes with taking boundary values (47) and (48) imply that

\[
g_+(x) + g_-(x) = cV'(x), \quad x \in (\alpha, \beta)
\]

(284)

\[
g_+(x) - g_-(x) = 0, \quad x \in \mathbb{R} \setminus (\alpha, \beta).
\]

In particular, \(g'(z)\) is an analytic function for \(z \in \mathbb{C} \setminus [\alpha, \beta]\). To find \(g'(z)\) from these conditions, we introduce the function \(R(z)\) satisfying \(R(z)^2 = (z - \alpha)(z - \beta)\) such that \(R(z)\) is analytic for \(z \in \mathbb{C} \setminus [\alpha, \beta]\) and \(R(z) = z + \mathcal{O}(1)\) as \(z \to \infty\). Setting \(g'(z) = f(z)R(z)\) for some new unknown function \(f(z)\), we find that like \(g'(z)\) and \(R(z)\), \(f(z)\) is an analytic function of \(z\) for \(z \in \mathbb{C} \setminus [\alpha, \beta]\), and that its boundary values taken on \((\alpha, \beta)\) from the upper and lower half-planes satisfy the relation

\[
f_+(x) - f_-(x) = \frac{cV'(x)}{R'(x)}, \quad \alpha < x < \beta.
\]

(285)

Since \(g'(z) = 1/z + \mathcal{O}(1/z^2)\) as \(z \to \infty\), it follows that \(f(z) = 1/z^2 + \mathcal{O}(1/z^3)\) as \(z \to \infty\), and hence

\[
f(z) = \frac{c}{2\pi i} \int_{\alpha}^{\beta} \frac{V'(s) ds}{(s - z)R_+(s)}.
\]

(286)

Considering (286) for large \(z\), we see that

\[
c \int_{\alpha}^{\beta} \frac{V'(s) ds}{R_+(s)} = 0, \quad c \int_{\alpha}^{\beta} \frac{sV'(s) ds}{R_+(s)} = -2\pi i.
\]

(287)

These two equations determine the endpoints \(\alpha\) and \(\beta\). With \(\alpha\) and \(\beta\) chosen so that the equations (287) hold, we may obtain a formula, in terms of a Cauchy principal value integral, for the density \(\psi(x)\) of the equilibrium measure \(\mu_*\) valid in the support interval \(\alpha \leq x \leq \beta\):

\[
\psi(x) = \frac{cR_+(x)}{2\pi^2} \int_{\alpha}^{\beta} \frac{V'(s) ds}{(s - x)R_+(s)}.
\]

(288)

Defining a real-valued function \(h(x)\) for \(x \in \mathbb{R}\) by the formula

\[
h(x) := \frac{i}{\pi} \int_{\alpha}^{\beta} \frac{V'(s) - V'(x)}{s - x} \frac{ds}{R_+(s)},
\]

(289)

it is straightforward to verify that

\[
\psi(x) = \frac{c}{2\pi i} R_+(x) h(x), \quad \alpha < x < \beta,
\]

(290)

and that

\[
\phi(x) := cV(x) + \ell - g_+(x) - g_-(x) = \begin{cases} -c \int_{\alpha}^{x} R(s) h(s) ds, & x < \alpha \\ c \int_{x}^{\beta} R(s) h(s) ds, & x > \beta. \end{cases}
\]

(291)

From the assumption that \(V(x)\) is \(d\) times continuously differentiable, we see that \(h(x)\) is \(d - 2\) times continuously differentiable. Also, since \(i/R_+(x)\) is positive for \(\alpha < x < \beta\) and

\[
\frac{V'(s) - V'(z)}{s - z} = \int_{0}^{1} V''(ts + (1 - t)z) dt,
\]

(292)
the assumption of convexity of $V(x)$ implies that $h(x)$ is strictly positive for all $x \in \mathbb{R}$. Also, since $g_+(\beta) - g_-(\beta) = 0$ and since for $\alpha < x < \beta$ we have $g_+'(x) - g_-'(x) = -2\pi i \psi(x)$, we conclude that

$$\theta(x) := -i(g_+(x) - g_-(x)) = -ic \int_x^\beta R_+(s) h(s) \, ds, \quad \alpha < x < \beta.$$  

We may now use (291) to verify (from the positivity of $h(x)$ and the facts that $R(x) > 0$ for $x > \beta$ while $R(x) < 0$ for $x < \alpha$) the strict inequality $\phi(x) > 0$ for $x < \alpha$ and $x > \beta$, as required by Condition 2. Also, from (293) we see that $0 < \theta(x) < 2\pi$ and $\psi(x) > 0$ (i.e. $\theta'(x) < 0$) both hold strictly for $\alpha < x < \beta$ as required by Condition 2. Moreover, since $h(s)$ is $d - 2$ times continuously differentiable, $\theta(x)$ is $d - 1$ times continuously differentiable for $\alpha < x < \beta$.

One may prove (240) as follows. From the identity $g'(z) = R(z)f(z)$, we have that for $z$ near $\beta$,

$$f(z) = \frac{cV'(\beta)}{2R(z)} + \frac{c}{2\pi i} \int_\alpha^\beta \frac{V'(s) - V'(\beta)}{(s-\beta)R_+(s)} \, ds + \frac{cV''(\beta)}{2R(z)}(z-\beta) \int_\alpha^\beta \frac{V'(s) - V'(\beta)}{s-\beta} \, ds + \frac{cV''(\beta)}{2R(z)}(z-\beta)^2 \int_\alpha^\beta \frac{V'(s) - V'(\beta)}{(s-\beta)R_+(s)} \, ds + \frac{cV''(\beta)}{2R(z)}(z-\beta)^2 \int_\alpha^\beta \frac{V'(s) - V'(\beta)}{(s-\beta)R_+(s)} \, ds.$$  

Recalling that $2g(\beta) - cV(\beta) - \ell = 0$, we learn that

$$g(z) = \frac{cV(\beta)}{2} + \frac{\ell}{2} + \frac{cV''(\beta)}{2} (z-\beta) + \left(\frac{c}{2\pi i} \int_\alpha^\beta \frac{V'(s) - V'(\beta)}{(s-\beta)R_+(s)} \, ds\right) \int_\alpha^z \frac{V'(s) - V'(\beta)}{s-\beta} \, ds + \frac{cV''(\beta)}{4} (z-\beta)^2 \int_\alpha^z \frac{V'(s) - V'(\beta)}{(s-\beta)R_+(s)} \, ds.$$  

Now, recalling that

$$\frac{1}{2} u_\beta(z)^{3/2} = -\frac{h'_\beta(\beta)}{2} (z-\beta)^{3/2},$$

and

$$h_\beta(x) = \frac{\theta(x)}{\sqrt{\beta-x}} = -i \frac{g_+(x) - g_-(-x)}{\sqrt{\beta-x}}, \quad \alpha < x < \beta,$$

we may use (295) to obtain the value of $h'_\beta(\beta)$:

$$h'_\beta(\beta) = \frac{2c\sqrt{\beta} - \ell}{3\pi i} \int_\alpha^\beta \frac{V'(s) - V'(\beta)}{(s-\beta)R_+(s)} \, ds < 0,$$

and then the expansion (240) follows by adding (295) and (296).

**Regularity for $V''$ Lipschitz.** It is also straightforward to derive a formula for $g''(z)$. This is a useful exercise if one assumes only that $V''$ is Lipschitz continuous, which we do throughout this subsection.

By further differentiation, (17) and (48) imply that

$$g''_+(x) + g''_-(x) = cV''(x), \quad x \in (\alpha, \beta)$$

and

$$g''_+(x) - g''_-(x) = 0, \quad x \in \mathbb{R} \setminus (\alpha, \beta).$$

In particular, $g''(z)$ is an analytic function for $z \in \mathbb{C} \setminus [\alpha, \beta]$. To find $g''(z)$ from these conditions, we set $g''(z) = F(z)/R(z)$ for some new unknown function $F(z)$, we find that like $g''(z)$ and $R(z)$, $F(z)$ is an analytic function of $z$ for $z \in \mathbb{C} \setminus [\alpha, \beta]$, and that its boundary values taken on $(\alpha, \beta)$ from the upper and lower half-planes satisfy the relation

$$F_+(x) - F_-(x) = cV''(x)R_+(x), \quad \alpha < x < \beta.$$  

It follows that $F(z)$ is a function of the form

$$F(z) = \frac{c}{2\pi i} \int_\alpha^\beta \frac{V''(s)R_+(s)}{(s-z)} \, ds.$$
Since $\psi''(x) = -2\pi i (g''_+(x) - g''_-(x))$ for $\alpha < x < \beta$, we obtain

$$
\psi'(x) = -\frac{2\pi i}{R_+(x)} (F_+(x) + F_-(x))
$$

$$
(302)
\quad = -\frac{2c}{R_+(x)} \int_\alpha^\beta \frac{V''(s) - V''(x)}{s - x} R_+(s) \, ds + \frac{2\pi i c V''(x)}{R(x)} \left( x - \frac{\alpha + \beta}{2} \right),
$$

from which it follows that $\sqrt{(x - \alpha)(\beta - x)}\psi'(x)$ is bounded uniformly for $\alpha < x < \beta$. In other words,

$$
|\psi'(x)| \leq C \sqrt{(x - \alpha)(\beta - x)}, \quad \alpha < x < \beta.
$$

From the above considerations we have the following formula, which is valid for $x < \alpha$ and also for $x > \beta$:

$$
\phi''(x) = \frac{ic}{\pi R(x)} \int_\alpha^\beta \frac{V''(s) - V''(x)}{s - x} R_+(s) \, ds + \frac{cV''(x)}{R(x)} \left( x - \frac{\alpha + \beta}{2} \right),
$$

which in turn implies that $|x - \alpha|^{1/2}|x - \beta|^{1/2}\phi''(x)$ is bounded on any compact subset of $\mathbb{R}$. (Of course, this quantity diverges as $|x| \to \infty$, with $x \in \mathbb{R}$.)

On the other hand, one may also consider the quantity $\tilde{F}(z) := g(z)/R(z)$, for which the following identity can be shown to hold true:

$$
\tilde{F}_\pm(z) = \left( \int_\alpha^\beta \frac{ds}{R(s)(s - z)} \right) \pm \frac{c}{\pi i} \int_\alpha^\beta \frac{V(s) - V(z)}{s - z} \, ds + \frac{cV(z) + \ell}{2R_\pm(z)}, \quad z \in \mathbb{R}.
$$

One direct consequence of this last identity is that

$$
g_+(z) - g_-(z) = R_+(z) \left[ 2 \int_\infty^\alpha \frac{ds}{R(s)(s - z)} + \frac{c}{\pi i} \int_\alpha^\beta \frac{V(s) - V(z)}{s - z} \, ds \right], \quad z \in (\alpha, \beta],
$$

with both quantities appearing within the square brackets on the right hand side of (306) possessing at least one Lipschitz continuous derivative for all $z \in [\alpha + \epsilon, \beta]$ for any $\epsilon > 0$. Similarly, we have

$$
g_+(z) + g_-(z) - cV(z) - \ell = R(z) \left[ 2 \int_\infty^\alpha \frac{ds}{R(s)(s - z)} + \frac{c}{\pi i} \int_\alpha^\beta \frac{V(s) - V(z)}{s - z} \, ds \right], \quad z \in [\beta, \infty),
$$

and again the quantities appearing within the square brackets on the right hand side of (307) possess at least one Lipschitz continuous derivative for all $z \in [\beta, \infty)$. The behavior near $z = \alpha$ is slightly more subtle, but using the identity

$$
\int_\infty^\alpha \frac{ds}{R(s)(s - z)} = \frac{\pi i}{R(z)} + \int_\infty^\beta \frac{ds}{R(s)(s - z)},
$$

where $\hat{R}(z) = \text{sgn}(\text{Im}(z)) R(z)$ is the function which coincides with $R(z)$ in $\mathbb{C}_+$, and is analytic in $\mathbb{C} \setminus ((-\infty, \alpha] \cup [\beta, \infty))$. Indeed, using (308), the identity (306) becomes

$$
g_+(z) - g_-(z) = 2\pi i + R_+(z) \left[ 2 \int_\infty^\beta \frac{ds}{R(s)(s - z)} + \frac{c}{\pi i} \int_\alpha^\beta \frac{V(s) - V(z)}{s - z} \, ds \right], \quad z \in [\alpha, \beta],
$$

and once again the quantity within the square brackets on the right-hand side of (309) possesses at least one Lipschitz continuous derivative. Similarly, the identity (307) can be rewritten, in light of (308), as follows:

$$
g_+(z) + g_-(z) - cV(z) - \ell = R(z) \left[ 2 \int_\infty^\beta \frac{ds}{R(s)(s - z)} + \frac{c}{\pi i} \int_\alpha^\beta \frac{V(s) - V(z)}{s - z} \, ds \right], \quad z \in (-\infty, \alpha],
$$

the quantity within the square brackets again possessing one Lipschitz continuous derivative.

We summarize the results of this Appendix with the following Lemma.

**Lemma 3.** Suppose that the external field $V$ possesses two Lipschitz continuous derivatives, is strictly convex, and grows faster than $|\log(1 + x^2)|^{1+\epsilon}$ for some $\epsilon > 0$. Then the density $\psi(x)$ of the equilibrium measure $\mu_*$ is supported on a single interval, $[\alpha, \beta]$. On this interval, the function $\psi$ has the following properties.
The function $\psi$ may be expressed via (300) with $h$, defined in (289), being Lipschitz continuous on $[\alpha, \beta]$.

The function $\psi$ has one derivative, which satisfies the bound (303).

In vicinities of the endpoints $\beta$ and $\alpha$, the related function $\theta$ (recall $\theta'(x) = -2\pi\psi(x)$) satisfies (cf. (307) and (309))

$$
\theta(x) = 2\pi \int_\epsilon^\beta \psi(s) ds = \begin{cases}
-iR_+(x)\hat{h}_\beta(x), & x \in (\alpha + \epsilon, \beta) \\
2\pi + iR_+(x)\hat{h}_\alpha(x), & x \in (\alpha, \beta - \epsilon),
\end{cases}
$$

for some small $\epsilon > 0$, with $\hat{h}_\beta$ and $\hat{h}_\alpha$ being positive functions on $(\alpha, \beta)$. In addition, $\hat{h}_\beta$ possesses one Lipschitz continuous derivative on $[\alpha, \beta]$, and $\hat{h}_\alpha$ possesses one Lipschitz continuous derivative on $[\alpha, \beta]$.

On the complementary set $(-\infty, \alpha) \cup (\beta, \infty)$, the following properties hold true:

- The quantity $\phi(x) := cV(x) + \ell - g_+(x) - g_-(x)$ possess two derivatives. The first derivative $\phi'(x)$ may be obtained from (291), with $h$ being Lipschitz continuous on $(-\infty, \alpha) \cup [\beta, \infty)$.

- The second derivative $\phi''(x)$ satisfies the inequality

$$
|x - \alpha|^{1/2}|x - \beta|^{1/2}|\phi''(x)| \leq C
$$
on any compact subset of $\mathbb{R}$.

- In vicinities of the endpoints $\alpha$ and $\beta$, the function $\phi(x)$ satisfies (cf. (307) and (310))

$$
\phi(x) = \begin{cases}
-R(x)\hat{h}_\beta(x), & x \in [\beta, \infty) \\
R(x)\hat{h}_\alpha(x), & x \in (-\infty, \alpha],
\end{cases}
$$

with $\hat{h}_\beta$ and $\hat{h}_\alpha$ being extensions, to $[\beta, \infty)$ and $(-\infty, \alpha]$ respectively, of the functions of the same name, formerly defined on $(\alpha, \beta)$ and $[\alpha, \beta)$ respectively. These extensions possess one Lipschitz continuous derivative as well. The function $\hat{h}_\beta$ is strictly negative on all of $(\beta, \infty)$, and the function $\hat{h}_\alpha$ is strictly positive on all of $(-\infty, \alpha)$.

Note that the functions $h_\alpha(x)$ and $h_\beta(x)$ used in the main text are simply related to $\hat{h}_\alpha(x)$ and $\hat{h}_\beta(x)$ as follows:

$$
h_\alpha(x) = \sqrt{\beta - x} \hat{h}_\alpha(x) \quad \text{and} \quad h_\beta(x) = \sqrt{x - \alpha} \hat{h}_\beta(x).
$$

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