HARNAK INEQUALITIES AND HÖLDER ESTIMATES
FOR MASTER EQUATIONS

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Abstract. We show parabolic interior and boundary Harnack inequalities and local Hölder continuity for solutions to master equations of the form $(\partial_t + L)^s u = f$ in $\mathbb{R} \times \Omega$, where $L$ is a divergence form elliptic operator and $\Omega \subseteq \mathbb{R}^n$. To this end, we prove that fractional powers of parabolic operators $\partial_t + L$ can be characterized with a degenerate parabolic extension problem.

1. Introduction

Continuous time random walks are stochastic processes with discontinuous paths for which both the jumps and the time elapsed in between them are random. They are governed by generalized master equations that take the form

$$\int_0^{\infty} \int_{\mathbb{R}^n} (u(t-\tau, z) - u(t, x)) K(t, x, \tau, z) \, dz \, d\tau = f(t, x)$$

for $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, $n \geq 1$. Master equations are nonlocal equations both in space and time, taking into account the past (memory). These also appear in the phenomenon of osmosis, semipermeable membranes, diffusion models for biological invasions and the parabolic Signorini problem, see [2, 4, 7, 10, 11] and references therein. L. A. Caffarelli and L. Silvestre proved Hölder estimates for master equations in the whole space when the right hand side $f$ is bounded, see [4]. They assumed some structural conditions on the kernel $K$ that enforce regularity of $u$. On the other hand, the most basic master equation is given by the fractional powers of the heat operator $(\partial_t - \Delta)^s u = f$, $0 < s < 1$, and this case was analyzed in great detail in [11].

We study regularity estimates for master equations driven by fractional powers of parabolic operators of the form

$$H^s u(t, x) \equiv (\partial_t + L)^s u(t, x) = f(t, x) \quad 0 < s < 1$$

for $t \in \mathbb{R}$ and $x \in \Omega$, where $\Omega$ is a open subset of $\mathbb{R}^n$, $n \geq 1$, that may be unbounded, and $L$ is an elliptic operator subject to appropriate boundary conditions on $\partial \Omega$.

The precise definition of (1.1) is given in terms of the Fourier transform and the spectral resolution of $L$. In this paper we develop a semigroup approach that allows us to show that (1.1) is indeed a master equation. The definition of $(\partial_t + L)^s$ and nonlocal pointwise integro-differential formulas are given in Section 2.

In particular, the elliptic operators $L$ that we consider in (1.1) are the following.

(1) $L = -\text{div}(a(x) \nabla) + c(x)$ in a bounded domain $\Omega$ with homogeneous Dirichlet or Neumann (conormal) boundary condition. The matrix of coefficients $a(x)$ is assumed to be bounded, measurable, symmetric $a^{ij}(x) = a^{ji}(x)$ and uniformly elliptic:

$$0 < \Lambda^{-1} |\xi|^2 \leq a^{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2$$

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for every $\xi \neq 0$, for a.e. $x \in \Omega$, for some ellipticity constant $\Lambda > 0$. The potential function $c(x) \geq 0$ and $c(x) \in L^\infty(\Omega)$. If $c(x) = 0$ and $a(x) = I$, then we get $-\Delta_D$ and $-\Delta_N$, the Dirichlet and Neumann Laplacians, respectively.

(2) The harmonic oscillators $L = -\Delta + |x|^2$ and $L = -\Delta + |x|^2 - n \in \Omega = \mathbb{R}^n$.

(3) The Laguerre differential operator $L = \frac{1}{4}(-\Delta + |x|^2 + \sum_{i=1}^n \alpha_i \frac{x_i^2}{x_i^2} + \sum_{i=1}^n \frac{1}{x_i} (\alpha_i^2 \frac{x_i^2}{x_i^2} - 1/4))$, for $\alpha_i > -1$, in $\Omega = (0, \infty)^n$.

(4) The ultraspherical operator $L = -\frac{\partial^2}{\partial x^2} + \frac{\lambda(\lambda-1)}{\sin^2 x}$, for $\lambda > 0$, in $\Omega = (0, \pi)$.

(5) The Laplacian $-\Delta$ in $\Omega = \mathbb{R}^n$.

(6) The Bessel operator $L = -\frac{\partial^2}{\partial x^2} + \frac{\lambda(\lambda-1)}{x}$, for $\lambda > 0$, in $\Omega = (0, \infty)$.

In (2)–(6) the ellipticity constant is $\Lambda = 1$.

For master equations (1.1) with $L$ as in (1)–(6) above we prove parabolic interior and boundary Harnack inequalities, and local boundedness and parabolic Hölder regularity. For notation see Section 2.

**Theorem 1.1** (Parabolic interior Harnack inequality). Let $L$ be any of the elliptic operators in (1)–(6) and $0 < s < 1$. Let $B_{2r}$ be a ball of radius $2r$, $r > 0$, such that $B_{2r} \subset \subset \Omega$. There exists a constant $c > 0$ depending only on $n, s, \Lambda$ and $r$ such that if $u = u(t, x) \in \text{Dom}(H^s)$ is a solution to

\begin{align*}
H^s u &= 0 \quad \text{for } (t, x) \in R := (0, 1) \times B_{2r} \\
u &\geq 0 \quad \text{for } (t, x) \in (-\infty, 1) \times \Omega,
\end{align*}

then

$$\sup_{R^-} u \leq c \inf_{R^+} u$$

where $R^- := (1/4, 1/2) \times B_r$ and $R^+ := (3/4, 1) \times B_r$. Moreover, solutions $u \in \text{Dom}(H^s)$ to $H^s u = 0$ in $R$ are locally bounded and locally parabolically $\alpha$-Hölder continuous in $R$, for some exponent $0 < \alpha < 1$ depending on $n, \Lambda$ and $s$. More precisely, for any compact set $K \subset R$ there exists $C = C(c, K, R) > 0$ such that

$$\|u\|_{C^{\alpha/2, \alpha}(K)} \leq C\|u\|_{L^2(\mathbb{R} \times \Omega)}.$$

To present the parabolic boundary Harnack inequality, let $\Omega_0 \subset \Omega$ and $\bar{x} \in \partial \Omega_0$ such that $B_{2r}(\bar{x}) \subset \Omega$, for some $r > 0$ fixed. Suppose that, up to a translation and rotation, $B_{2r}(\bar{x}) \cap \partial \Omega_0$ can be represented as the graph of a Lipschitz function $g : \mathbb{R}^{n-1} \to \mathbb{R}$ in the $e_n = (0, \ldots, 0, 1)$-direction, such that $g$ has Lipschitz constant $M > 0$. Thus,

$$\Omega_0 \cap B_{2r}(\bar{x}) = \{(x', x_n) : x_n > g(x')\} \cap B_{2r}(\bar{x})$$

$$\partial \Omega_0 \cap B_{2r}(\bar{x}) = \{(x', x_n) : x_n = g(x')\} \cap B_{2r}(\bar{x}).$$

Fix a point $(t_0, x_0) \in (-2, 2) \times \Omega_0$ such that $t_0 > 1$.

**Theorem 1.2** (Parabolic boundary Harnack inequality). Let $L$ be any of the elliptic operators in (1)–(6) and $0 < s < 1$. Assume the geometric conditions on $\Omega_0$ and $\Omega$ described above. Then there exists a constant $C > 0$ depending on $n, \Lambda, r, M, s, t_0 - 1$ and $g$, such that if $u(t, x) \in \text{Dom}(H^s)$ is a solution to

\begin{align*}
H^s u &= 0 \quad \text{for } (t, x) \in (-2, 2) \times (\Omega_0 \cap B_{2r}(\bar{x})) \\
u &\geq 0 \quad \text{for } (t, x) \in (-\infty, 2) \times \Omega
\end{align*}

such that $u$ vanishes continuously on $(-2, 2) \times ((\Omega \setminus \Omega_0) \cap B_{2r}(\bar{x}))$ then

$$\sup_{(-1, 1) \times (\Omega \cap \partial B_r(\bar{x}))} u(t, x) \leq C u(t_0, x_0).$$
In addition, we also consider master equations of the form \( (1.1) \) where \( L \) is any of the following elliptic operators having gradient term.

(7) The Ornstein–Uhlenbeck operator \( L = -\Delta + 2x \cdot \nabla \) in \( \Omega = \mathbb{R}^n \) with the Gaussian measure.

(8) The Laguerre operators
\[
L = \sum_{i=1}^{n} \left(-x_i \frac{\partial^2}{\partial x_i^2} - (\alpha_i + 1) \frac{\partial}{\partial x_i} + \frac{x_i}{4} \right),
\]
\[
L = \frac{1}{4}(-\Delta + |x|^2 - \sum_{i=1}^{n} \frac{2\alpha_i + 1}{x_i} \frac{\partial}{\partial x_i}),
\]
\[
L = \sum_{i=1}^{n} \left(-x_i \frac{\partial^2}{\partial x_i^2} - \frac{\partial}{\partial x_i} + \frac{x_i}{4} + \frac{\alpha^2}{4x_i} \right),
\]
\[
L = \sum_{i=1}^{n} \left(-x_i \frac{\partial^2}{\partial x_i^2} - (\alpha_i + 1 - x_i) \frac{\partial}{\partial x_i} \right),
\]

for \( \alpha_i > -1 \) in \( \Omega = (0, \infty)^n \), with their corresponding Laguerre measures.

(9) The ultraspherical operator \( L = -\frac{d^2}{dx^2} - 2\lambda \cot x \frac{d}{dx} + \lambda^2 \), for \( \lambda > 0 \) in \( \Omega = (0, \pi) \) with the measure \( d\eta(x) = \sin^{2\lambda} x \, dx \).

(10) The Bessel operator \( L = -\frac{d^2}{dx^2} - 2\lambda \frac{d}{dx} \), for \( \lambda > 0 \) in \( \Omega = (0, \infty) \) with the measure \( d\eta(x) = x^{2\lambda} \, dx \).

We develop a \textit{transference method} for fractional powers of parabolic operators (see Section 2), that allows us to transfer the Harnack inequalities and Hölder estimates from Theorems 1.1 and 1.2 to master equations (1.1) involving the operators \( L \) in (7)–(10).

**Theorem 1.3.** Theorems 1.1 and 1.2 hold true for solutions \( u \) to \((\partial_t + L)^s u = f\), where \( L \) is any of the elliptic operators in (7)–(10).

The main tool to prove Theorems 1.1 and 1.2 is an extension problem characterization for the fractional operators \((\partial_t + L)^s\). Observe that the elliptic operators \( L \) in (1)–(10) have discrete and continuous spectrum in different Hilbert spaces. The extension problem we present not only works for these particular examples, but for any fractional operator of the form \((\partial_t + L)^s\), where \( L \) is a nonnegative linear operator in a Hilbert space \( L^2(\Omega) \) with some positive measure \( d\eta \). Then the definition of \((\partial_t + L)^s\) can be given in terms of the Fourier transform in the variable \( t \) and the spectral resolution of \( L \), see Section 2.

**Theorem 1.4 (Extension problem).** Let \( L \) be a normal nonnegative linear operator on \( L^2(\Omega) \) and \( H = \partial_t + L \). Let \( u \in \text{Dom}(H^s) \). For \( (t, x) \in \mathbb{R} \times \Omega \) and \( y > 0 \) we define
\[
U(t, x, y) = \frac{y^{2s}}{4\pi \Gamma(s)} \int_0^\infty e^{-y^2/(4t)} e^{-\tau H} u(t, x) \frac{d\tau}{\tau^{1+s}}
\]
\[
= \frac{1}{\Gamma(s)} \int_0^\infty e^{-r} e^{-\frac{2r}{y} H} u(t, x) \frac{dr}{r^{1-s}}
\]
\[
= \frac{1}{\Gamma(s)} \int_0^\infty e^{-y^2/(4r)} e^{-r H} (H^s u)(t, x) \frac{dr}{r^{1-s}}.
\]

Then \( U(\cdot, \cdot, y) \in \text{Dom}(H) \) for each \( y > 0 \), \( U \in C^\infty((0, \infty) ; L^2(\mathbb{R} \times \Omega)) \cap C([0, \infty) ; L^2(\mathbb{R} \times \Omega)) \) and \( U \in L^2((0, \infty) ; \text{Dom}(H), y^{1-2s} \, dy) \). Moreover, \( U \) is a solution to
\[
\left\{ \begin{array}{ll}
\langle H U, v \rangle = \left\langle \frac{1-2s}{y} \partial_y U + \partial_y U, v \right\rangle_{L^2(\mathbb{R} \times \Omega)} & \text{for } v \in \text{Dom}(H) \text{ and } y > 0 \\
\lim_{y \to 0^+} U(t, x, y) = u(t, x) & \text{in } L^2(\mathbb{R} \times \Omega)
\end{array} \right.
\]
such that
\[
\lim_{y \to \infty} \langle U, v \rangle_{L^2(\mathbb{R} \times \Omega)} = 0, \quad \text{for every } v \in L^2(\mathbb{R} \times \Omega)
\]
\[ \sup_{y > 0} |\langle y^{1-2s} \partial_y U, v \rangle_{L^2(\mathbb{R} \times \Omega)}| \leq C_s \|u\|_{H^s} \|v\|_{H^s}, \text{ for every } v \in \text{Dom}(H^s). \]

In addition, for every \( v \in \text{Dom}(H^s) \),
\[
- \frac{1}{2s} \lim_{y \to 0^+} \langle y^{1-2s} \partial_y U, v \rangle_{L^2(\mathbb{R} \times \Omega)} = \frac{|\Gamma(-s)|}{4\pi^s} \langle H^s u, v \rangle
= - \lim_{y \to 0^+} \left\langle \frac{U(\cdot, \cdot, y) - U(\cdot, \cdot, 0)}{y^{2s}}, v \right\rangle_{L^2(\mathbb{R} \times \Omega)}.
\]

In Section 2 we provide the precise definition of \((\partial_t + L)^s\) and show that if \( L \) is as in (1) then this is a nonlocal in space and time integro-differential operator. Section 3 contains the proof of the general parabolic extension problem (Theorem 1.4) and Section 4 explains how to apply it when \( L \) is an elliptic operator in divergence form. The proof of Theorems 1.1 and 1.2 are given in Section 5. Finally, the transference method and the proof of Theorem 1.3 are presented in Section 6.

2. DEFINITION AND INTEGRO-DIFFERENTIAL FORMULA

In this section we present the precise definition of \( H^s u(t, x) = (\partial_t + L)^s u(t, x) \) and show that in general this is a master operator.

Let \( L \) be a nonnegative normal linear operator on a Hilbert space \( L^2(\Omega) \) with some positive measure \( d\eta \). For concreteness and simplicity of the presentation, we will assume that \( L \) has discrete spectrum and \( d\eta \) is the Lebesgue measure. We can always obtain the general result by using the Spectral Theorem, the Fourier transform, the Hankel transform, the corresponding orthogonal expansions with respect to \( d\eta \), etc.

Therefore, suppose that \( L \) has a countable sequence of eigenvalues and eigenfunctions \( (\lambda_k, \phi_k)_{k \geq 0} \) such that \( 0 \leq \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty \) and so that \( \{\phi_k\}_{k \geq 0} \) forms an orthonormal basis of \( L^2(\Omega) \). In the case in which \( \lambda_0 = 0 \) (for instance, for the Neumann Laplacian) we assume that all the functions involved have zero integral mean over \( \Omega \).

With this, any function \( u(t, x) \in L^2(\mathbb{R} \times \Omega) \) can be written as
\[
u(t, x) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} \sum_{k=0}^{\infty} \tilde{u}_k(\rho) \phi_k(x) e^{i\rho \rho} d\rho,
\]
where
\[
u_k(t) = \int_{\Omega} u(t, x) \phi_k(x) dx
\]
and \( \tilde{u}_k(\rho) \) is the Fourier transform of \( u_k(t) \) with respect to the variable \( t \in \mathbb{R} \):
\[
\tilde{u}_k(\rho) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} u_k(t) e^{-i\rho t} dt.
\]

The domain of the operator \( H^s \equiv (\partial_t + L)^s , \, 0 \leq s \leq 1 \), is defined as
\[
\text{Dom}(H^s) = \left\{ u \in L^2(\mathbb{R} \times \Omega) : \|u\|^2_{H^s} := \int_{\mathbb{R}} \sum_{k=0}^{\infty} |i\rho + \lambda_k|^s |\tilde{u}_k(\rho)|^2 d\rho < \infty \right\}.
\]

This is a complex Hilbert space with norm \( ||\cdot||_{H^s} \), whose dual is denoted by \( \text{Dom}(H^s)^* \).

Moreover, \( \text{Dom}(H^s) \subset \text{Dom}(H^t) \) whenever \( 0 \leq s \leq t \leq 1 \). For \( u \in \text{Dom}(H^s) \) we define \( H^s u \in \text{Dom}(H^t)^* \) as acting on any \( v \in \text{Dom}(H^s) \) by
\[
\langle H^s u, v \rangle = \int_{\mathbb{R}} \sum_{k=0}^{\infty} (i\rho + \lambda_k)^s \tilde{u}_k(\rho) \tilde{v}_k(\rho) d\rho.
\]
where $\overline{u_k}(\rho)$ denotes the complex conjugate of $\hat{v}_k(\rho)$. We have
\[
\|u\|_{H^s}^2 = \langle H^{s/2}u, H^{s/2}u \rangle \quad \text{for any } 0 \leq s \leq 1.
\]
Notice that we need to appropriately decide which $s$-power of the complex number $(i\rho + \lambda_k)$ we are taking. We are able to clarify this by developing a semigroup technique, in which the Gamma function plays a crucial role. The method permits us to show that (2.1) is indeed a master equation, or nonlocal in space and time integro-differential operator, in divergence form. Observe as well that $\text{Dom}(H^s)$ encodes the boundary condition on $L$.

As the family of eigenfunctions $\{\phi_k\}_{k \geq 0}$ is an orthonormal basis of $L^2(\Omega)$, we can write the semigroup $\{e^{-\tau L}\}_{\tau \geq 0}$ generated by $L$ as
\[
\langle e^{-\tau L} \varphi, \psi \rangle_{L^2(\Omega)} = \sum_{k=0}^{\infty} e^{-\tau \lambda_k} \varphi_k \psi_k = \int_{\Omega} \int_{\Omega} W_\tau(x,z) \varphi(z) \psi(x) \, dz \, dx
\]
for any $\varphi, \psi \in L^2(\Omega)$, where $\varphi_k = \int_{\Omega} \varphi x_k \, dx$ and $\psi_k = \int_{\Omega} \psi x_k \, dx$. As it happens for all our examples of elliptic operators (1)–(10), we will always assume that the heat kernel for $L$ is symmetric and nonnegative:
\[
W_\tau(x,z) = W_\tau(z,x) \geq 0.
\]
Since $\partial_t$ and $L$ commute, we define, for any $u \in L^2(\mathbb{R} \times \Omega)$,
\[
e^{-\tau H} u(t,x) = e^{-\tau L}(e^{-\tau \partial_t} u)(t,x) = e^{-\tau L}(u(t - \tau, \cdot))(x)
\]
in the sense that, for any $v \in L^2(\mathbb{R} \times \Omega)$,
\[
\langle e^{-\tau H} u, v \rangle_{L^2(\mathbb{R} \times \Omega)} = \int_{\mathbb{R}} \sum_{k=0}^{\infty} e^{-\tau (i\rho + \lambda_k)} \overline{u_k(\rho)} \overline{v_k(\rho)} \, d\rho
\]
\[
= \int_{\mathbb{R}} \sum_{k=0}^{\infty} e^{-\tau \lambda_k} u_k(t - \tau) v_k(t) \, dt
\]
\[
= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\Omega} W_\tau(x,z) u(t - \tau, z) v(t, x) \, dz \, dx \, dt.
\]

**Lemma 2.1.** Let $0 < s < 1$. If $u \in \text{Dom}(H^s)$ then
\[
H^s u = \frac{1}{\Gamma(-s)} \int_{0}^{\infty} \left( e^{-\tau H} u - u \right) \frac{d\tau}{\tau^{1+s}}
\]
in the sense that, for any $v \in \text{Dom}(H^s)$,
\[
\langle H^s u, v \rangle = \frac{1}{\Gamma(-s)} \int_{0}^{\infty} \left( \langle e^{-\tau H} u, v \rangle_{L^2(\mathbb{R} \times \Omega)} - \langle u, v \rangle_{L^2(\mathbb{R} \times \Omega)} \right) \frac{d\tau}{\tau^{1+s}}.
\]

**Proof.** Let $u, v \in \text{Dom}(H^s)$. We will use the following numerical formula with the Gamma function that comes from performing the analytic continuation to $\text{Re}(z) > 0$ of the function that maps $t \in [0,\infty)$ to $t^s$, see [3][11].
\[
(i\rho + \lambda_k)^s = \frac{1}{\Gamma(-s)} \int_{0}^{\infty} \left( e^{-\tau (i\rho + \lambda_k)} - 1 \right) \frac{d\tau}{\tau^{1+s}}, \quad \rho \in \mathbb{R}.
\]
The integral above is absolutely convergent. Then, in (2.1) we have
\[
\langle H^s u, v \rangle = \int_{\mathbb{R}} \sum_{k=0}^{\infty} \left[ \frac{1}{\Gamma(-s)} \int_{0}^{\infty} \left( e^{-\tau (i\rho + \lambda_k)} - 1 \right) \frac{d\tau}{\tau^{1+s}} \right] \overline{u_k(\rho)} \overline{v_k(\rho)} \, d\rho.
\]
On one hand,
\[
\int_0^{1/|\rho+\lambda_k|} |e^{-\tau(\rho+\lambda_k)} - 1| \frac{d\tau}{\tau^{1+s}} \leq C|\rho+\lambda_k| \int_0^{1/|\rho+\lambda_k|} \tau^{-s} d\tau = C|\rho+\lambda_k|^s.
\]

On the other hand,
\[
\int_{1/|\rho+\lambda_k|}^{\infty} |e^{-\tau(\rho+\lambda_k)} - 1| \frac{d\tau}{\tau^{1+s}} \leq C \int_{1/|\rho+\lambda_k|}^{\infty} \tau^{-1-s} d\tau = C|\rho+\lambda_k|^s.
\]

Since \(u, v \in \text{Dom}(H^s)\), Fubini’s Theorem and (2.2) allow us to get the conclusion. \(\square\)

In the case when \(L\) is a divergence form elliptic operator as in (1), we can use the heat kernel to prove that \((\partial_t + L)^s\) is a master operator in divergence form.

**Theorem 2.2** (Master equation). Let \(L\) be as in (1) and \(0 < s < 1\). If \(u, v \in \text{Dom}(H^s)\) then
\[
\langle H^s u, v \rangle = \int_0^{\infty} \int_R \int_\Omega K_s(\tau, x, z)(u(t-\tau, x) - u(t-\tau, z))(v(t, x) - v(t, z)) \, dz \, dx \, dt \, d\tau
\]

\[
+ \int_0^{\infty} \left[ \int_R \int_\Omega \left(1 - e^{-\tau L} 1(x) \right) \frac{u(t, x) v(t, x)}{\left|\Gamma(-s)\right|^{1+s}} \, dx \, dt - \int_R \int_\Omega e^{-\tau L} 1(x) \frac{(u(t-\tau, x) - u(t, x)) v(t, x)}{\left|\Gamma(-s)\right|^{1+s}} \, dx \, dt \right] \, d\tau,
\]

where
\[
K_s(\tau, x, z) = \frac{W_\tau(x, z)}{2\left|\Gamma(-s)\right|^{1+s}}
\]

and
\[
e^{-\tau L} 1(x) = \int_\Omega W_\tau(x, z) \, dz.
\]

**Remark 2.3.** There are cases in which \(e^{-\tau L} 1(x) \equiv 1\). This occurs, for example, when \(L\) is the Neumann Laplacian \(-\Delta_N\), a divergence form elliptic operator on the whole space \(\Omega = \mathbb{R}^n\), or the Laplacian \(-\Delta\) on \(\mathbb{R}^n\). Under this condition on the heat kernel, if \(u\) and \(v\) are smooth functions with compact support then in Theorem 2.2 we get
\[
\langle H^s u, v \rangle = \int_0^{\infty} \int_R \int_\Omega K_s(\tau, x, z)(u(t-\tau, x) - u(t-\tau, z))(v(t, x) - v(t, z)) \, dz \, dx \, dt \, d\tau
\]

\[- \int_0^{\infty} \int_R \int_\Omega (D_{\text{left}})^s u(t, x) v(t, x) \, dx \, dt \, d\tau.
\]

The second integral term above is equal to
\[
- \int_R \int_\Omega (D_{\text{left}})^s u(t, x) v(t, x) \, dx \, dt
\]

where \((D_{\text{left}})^s\) denotes the fractional power of the derivative from the left, which coincides with the Marchaud fractional derivative, acting on the variable \(t \in \mathbb{R}\), see [3].

**Remark 2.4.** By using the Gaussian heat kernel estimates of \(W_\tau(x, z)\) (see for example [5, 6]) one can prove that the kernel \(K_s(\tau, x, z)\) in Theorem 2.2 satisfies the size estimates of the master equations considered in [4, 11].

**Proof of Theorem 2.2.** For \(u, v \in \text{Dom}(H^s)\) we have, by Lemma 2.1, up to the multiplicative constant \(1/\Gamma(-s)\),
\[
\langle H^s u, v \rangle = \int_0^{\infty} \left( (e^{-\tau L} u(\cdot - \tau, \cdot), v(\cdot, \cdot))_{L^2(\mathbb{R} \times \Omega)} - \langle u, v \rangle_{L^2(\mathbb{R} \times \Omega)} \right) \frac{d\tau}{\tau^{1+s}}
\]
By adding (2.4) and (2.5), we get that, up to the multiplicative constant 1
above are also equal to

\[ x \]

By exchanging the roles of \( x \) and \( z \) and using that \( W_\tau(z,x) = W_\tau(x,z) \), the integrals
above are also equal to

\[
\begin{align*}
- \int_\mathbb{R} \int_\Omega W_\tau(x,z)(u(t-\tau,z) - u(t-\tau,x))v(t,x) \, dx \, dz \, dt \\
+ \int_\mathbb{R} \int_\Omega (e^{-\tau^L_1(x)}u(t-\tau,x) - u(t,x))v(t,x) \, dx \, dt.
\end{align*}
\]

By adding (2.4) and (2.5), we get that, up to the multiplicative constant \( 1/|\Gamma(-s)| \),

\[
2\langle H^*u, v \rangle = \int_0^\infty \left[ \int_\mathbb{R} \int_\Omega W_\tau(x,z)(u(t-\tau,x) - u(t-\tau,z))(v(t,x) - v(t,z)) \, dz \, dx \, dt \\
+ 2 \int_\mathbb{R} \int_\Omega (u(t,x) - e^{-\tau^L_1(x)}u(t-\tau,x))v(t,x) \, dx \, dt \right] \frac{d\tau}{\tau^{1+s}}.
\]

Observe that

\[
\begin{align*}
\left| \int_0^\infty \int_\Omega W_\tau(x,z) \int_\mathbb{R} (u(t-\tau,x) - u(t-\tau,z))(v(t,x) - v(t,z)) \, dt \, dz \, dx \, \frac{d\tau}{\tau^{1+s}} \right| \\
= \left| \int_0^\infty \int_\Omega \int_\mathbb{R} e^{i\tau\rho}(\hat{u}(\rho,x) - \hat{u}(\rho,z))(\hat{v}(\rho,x) - \hat{v}(\rho,z)) \, d\rho \, dz \, dx \, \frac{d\tau}{\tau^{1+s}} \right| \\
\leq \int_\mathbb{R} \int_\Omega \left| \hat{u}(\rho,x) - \hat{u}(\rho,z) \right| \left| \hat{v}(\rho,x) - \hat{v}(\rho,z) \right| \left[ \int_0^\infty W_\tau(x,z) \frac{d\tau}{\tau^{1+s}} \right] \, dz \, d\rho < \infty
\end{align*}
\]

because, obviously, \( u, v \in L^2(\mathbb{R}; \text{Dom}(L^s)) \) (see [5] for the description of \( \text{Dom}(L^s) \)). Therefore, we can write \( \langle H^*u, v \rangle \) as the sum of

\[
\frac{1}{2|\Gamma(-s)|} \int_0^\infty \int_\mathbb{R} \int_\Omega W_\tau(x,z)(u(t-\tau,x) - u(t-\tau,z))(v(t,x) - v(t,z)) \, dz \, dx \, dt \, \frac{d\tau}{\tau^{1+s}}
\]

and

\[
\frac{1}{|\Gamma(-s)|} \int_0^\infty \int_\mathbb{R} \int_\Omega (u(t,x) - e^{-\tau^L_1(x)}u(t-\tau,x))v(t,x) \, dx \, dt \, \frac{d\tau}{\tau^{1+s}}.
\]

The conclusion readily follows from here. \( \square \)

3. Proof of Theorem 1.4

We begin with an important preliminary result.
Lemma 3.1. Let $0 < s < 1$. Denote by $K_\nu(z)$ the modified Bessel function of the second kind and order $\nu$. For $y > 0$ and $\lambda \in \mathbb{C}$ with $\text{Re}(\lambda) > 0$ we define

\[
I_s(y, \lambda) = \frac{2^{1-s}}{\Gamma(s)} y^\lambda \sqrt{\lambda}^s K_s(y\sqrt{\lambda})
\]

\[
= \frac{1}{\Gamma(s)} \int_0^\infty e^{-t} t^{-s/2} e^{-t\lambda} dt
\]

\[
= \frac{y^{2s}}{4^s \Gamma(s)} \int_0^\infty e^{-y^2/(4r)} e^{-r\lambda} dr
\]

\[
= \frac{1}{\Gamma(s)} \int_0^\infty e^{-y^2/(4r)} e^{-\tau y\lambda} \frac{dr}{\tau^{1-s}}.
\]

The integrals are absolutely convergent. Fix any $s$ and $\lambda$ as above. Then

1. $I_s(y, \lambda)$ is a smooth function of $y \in (0, \infty)$.
2. For each $y > 0$, $I_s(y, \lambda)$ satisfies the equation

\[
\lambda u - \frac{1 - 2s}{y} \partial_y u - \partial_{yy} u = 0.
\]

3. $\lim_{y \to 0^+} I_s(y, \lambda) = 1$.

4. \[-y^{1-2s} \partial_y I_s(y, \lambda) = \frac{\Gamma(1-s)}{4^{s-1/2} \Gamma(s)} \lambda^s I_{1-s}(y, \lambda).\]

5. The following estimates hold:

   5.a $|I_s(y, \lambda)| \leq 1$.

   5.b There is a constant $C_s > 0$ such that

   \[
   |I_s(y, \lambda)| \leq C_s |y\lambda|^{1/2} e^{-\cos(\arg(\lambda)/2)|y\lambda|^{1/2}} \text{ as } y \to \infty.
   \]

   5.c There is a constant $C_s > 0$ such that

   \[
   |\lambda I_s(y, \lambda)| + |\frac{1}{y} \partial_y I_s(y, \lambda)| + |\partial_{yy} I_s(y, \lambda)| \leq C_s \frac{|\lambda|^s}{y^{2-2s}} \text{ for every } y > 0.
   \]

6. The function $I_s(\lambda, y)$ is the unique $C^\infty$ solution to such that

\[
\lim_{y \to 0} I_s(y, \lambda) = 1, \quad \lim_{y \to \infty} I_s(y, \lambda) = 0, \quad \text{and} \quad y^{1-2s} \partial_y I_s(y, \lambda) \in L^\infty_y([0, \infty)).
\]

Proof. It is well known that for $\nu$ arbitrary (see [9, eq. (5.10.25)])

\[
K_\nu(z) = \frac{1}{2} \left( \frac{z}{2} \right)^\nu \int_0^\infty e^{-t} e^{-z^2/4t} t^{\nu-1} dt \text{ for } |\arg z| < \frac{\pi}{4}.
\]

As $K_\nu = K_{-\nu}$ we get the second identity in (3.1). The third one follows from the change of variables $r = y^2/(4t)$. The last one for $\lambda > 0$ is obtained from the third one via the change of variables $\tau = y^2/(4r\lambda)$, and the general case of $\text{Re}(\lambda) > 0$ follows from the case of $\lambda > 0$ by analytic continuation.

Now (1) is easy to check by differentiating under the integral sign. Indeed, since

\[
|\partial_y(y^{2s} e^{-y^2/(4\tau)})| = \left| 2sy^{2s-1} - \frac{y^{2s+1}}{2\tau} \right| e^{-y^2/(4\tau)} \leq Cy^{2s-1} e^{-y^2/(4\tau)},
\]

we get

\[
\partial_y I_s(y, \lambda) = \int_0^\infty \partial_y \left( \frac{y^{2s}}{4^s \Gamma(s)} e^{-y^2/(4r)} \right) e^{-r\lambda} \frac{dr}{r^{1+s}}.
\]

Similarly for higher order derivatives. For (2) we can use integration by parts to get

\[
\lambda I_s(y, \lambda) = -\frac{y^{2s}}{4^s \Gamma(s)} \int_0^\infty e^{-y^2/(4r)} \partial_r e^{-r\lambda} \frac{dr}{r^{1+s}}
\]
implies (5.1). We immediately obtain (4). Observe that (5.b) and with a nonnegative, symmetric heat kernel, as in Section 2. Recall that all the functions involved have zero spatial mean. The general case follows by using the Spectral Theorem or the spectral resolution of the corresponding operator (like the Fourier transform or the Hankel transform). Details are left to the interested reader.

Proof of Theorem 1.4. Let us denote \( U(y) = U(\cdot, \cdot, y) \), for \( y > 0 \), where \( U \) is given by (1.2). Since

\[
\frac{y^{2s}}{4^s \Gamma(s)} \int_0^\infty \frac{e^{-y^2/(4r)}}{r^{1+s}} \frac{dr}{r^{1+s}} = 1
\]

we find that, for any \( v = v(t, x) \in L^2(\mathbb{R} \times \Omega) \),

\[
\|U(y) v\|_{L^2(\mathbb{R} \times \Omega)} \leq \frac{y^{2s}}{4^s \Gamma(s)} \int_0^\infty e^{-y^2/(4r)} \|e^{-\tau H} u\|_{L^2(\mathbb{R} \times \Omega)} \|v\|_{L^2(\mathbb{R} \times \Omega)} \frac{dr}{r^{1+s}}
\]

\[
\leq \|u\|_{L^2(\mathbb{R} \times \Omega)} \|v\|_{L^2(\mathbb{R} \times \Omega)}
\]

so that

\[
(3.5) \quad (U(y), v)_{L^2(\mathbb{R} \times \Omega)} = \frac{y^{2s}}{4^s \Gamma(s)} \int_0^\infty e^{-y^2/(4r)} \langle e^{-\tau H} u, v \rangle_{L^2(\mathbb{R} \times \Omega)} \frac{dr}{r^{1+s}} < \infty.
\]
In particular, for each $y > 0$, $U(y) \in L^2(\mathbb{R} \times \Omega)$, with

$$||U(y)||_{L^2(\mathbb{R} \times \Omega)} \leq ||u||_{L^2(\mathbb{R} \times \Omega)}.$$ 

In addition, by using (2.2) and (3.1) from Lemma 3.1,

$$\langle U(y), v \rangle_{L^2(\mathbb{R} \times \Omega)} = \int_\mathbb{R} \sum_{k=0}^{\infty} \hat{u}_k(\rho) \overline{\hat{v}_k(\rho)} I_s(y, i\rho + \lambda_k) \, d\rho$$

and

$$U(y) = \frac{1}{(2\pi)^{1/2}} \int_\mathbb{R} \sum_{k=0}^{\infty} \hat{u}_k(\rho) I_s(y, i\rho + \lambda_k) e^{i\rho t} \, d\rho.$$ 

Next, by using Lemma 5.1 parts (5.a) and (5.c),

$$\int_\mathbb{R} \sum_{k=0}^{\infty} |i\rho + \lambda_k||\hat{u}_k(\rho)|^2 I_s(y, i\rho + \lambda_k) \, d\rho \leq \frac{C_s}{y^{2s}} \int_\mathbb{R} \sum_{k=0}^{\infty} |i\rho + \lambda_k|^s|\hat{u}_k(\rho)|^2 \, d\rho < \infty,$$

we get that $U(y) \in \text{Dom}(H)$ for each $y > 0$. Then, for any $v \in \text{Dom}(H)$, (see (2.1))

$$\langle HU(y), v \rangle = \int_\mathbb{R} \sum_{k=0}^{\infty} \hat{u}_k(\rho) \overline{v_k(\rho)} (i\rho + \lambda_k) I_s(y, i\rho + \lambda_k) \, d\rho.$$ 

Let us check that $U \in C^\infty((0, \infty); L^2(\mathbb{R} \times \Omega))$ and that, for any $k \geq 1$,

$$\partial_y \langle U(y), v \rangle_{L^2(\mathbb{R} \times \Omega)} = \langle \partial_y U(y), v \rangle_{L^2(\mathbb{R} \times \Omega)}.$$ 

Indeed, first notice that

$$(e^{-\tau H} u, v)_{L^2(\mathbb{R} \times \Omega)} \leq e^{-\tau \lambda_i} ||u||_{L^2(\mathbb{R} \times \Omega)} ||v||_{L^2(\mathbb{R} \times \Omega)}$$

where $i = 0$ if $\lambda_0 \neq 0$ and $i = 1$ if $\lambda_0 = 0$. Here we have used that

$$||e^{-\tau H} u||^2_{L^2(\mathbb{R} \times \Omega)} = \sum_{k=i}^{\infty} e^{-2\tau \lambda_k} \int_\mathbb{R} |u_k(t-\tau)|^2 \, dt \leq e^{-2\tau \lambda_i} ||u||^2_{L^2(\mathbb{R} \times \Omega)}.$$ 

By using (3.3),

$$\int_0^\infty \left| \partial_y \left( \frac{y^{2s}}{4\Gamma(s)} e^{-y^2/(4\tau)} \right) (e^{-\tau H} u, v)_{L^2(\mathbb{R} \times \Omega)} \right| \frac{d\tau}{\tau^{1+s}} \leq C_s y^{2s-1} ||u||_{L^2(\mathbb{R} \times \Omega)} ||v||_{L^2(\mathbb{R} \times \Omega)} \int_0^\infty e^{-\tau \lambda_i} e^{-y^2/(4\tau)} \frac{d\tau}{\tau^{1+s}}$$

so we can differentiate under the integral sign in (3.5). Similarly it can be done for higher order derivatives and we get $U(y) \in C^\infty((0, \infty); L^2(\mathbb{R} \times \Omega))$.

Observe that, by the first equation in (3.1),

$$\int_0^\infty y^{1-2s} ||U||^2_{H^1} \, dy = \int_0^\infty y^{1-2s} \int_\mathbb{R} \sum_{k=0}^{\infty} |i\rho + \lambda_k||\hat{u}_k(\rho)|^2 I_s(y, i\rho + \lambda_k) \, d\rho \, dy$$

$$= \int_\mathbb{R} \sum_{k=0}^{\infty} |i\rho + \lambda_k||\hat{u}_k(\rho)|^2 \int_0^\infty y^{1-2s} |I_s(y, i\rho + \lambda_k)|^2 \, dy \, d\rho$$

$$\leq C_s \int_\mathbb{R} \sum_{k=0}^{\infty} |i\rho + \lambda_k|^{1+4s} ||\hat{u}_k(\rho)||^2 \int_0^\infty y |K_s(y\sqrt{i\rho + \lambda_k})|^2 \, dy \, d\rho.$$ 

To estimate the integral in $dy$, let $r = y\sqrt{i\rho + \lambda_k}$ and $\theta = \arg(\sqrt{i\rho + \lambda_k})$, hence

$$\int_0^\infty y |K_s(y\sqrt{i\rho + \lambda_k})|^2 \, dy = |i\rho + \lambda_k|^{-1} \int_0^\infty r |K_s(re^{i\theta})|^2 \, dr \leq C_s |i\rho + \lambda_k|^{-1},$$
In the last inequality we used the fact that
\( K_s(z) \sim C_s z^{-s} \) as \( z \to 0 \), and \( K_s(z) \sim z^{-1/2} e^{-z} \) as \( z \to \infty \), see [9]. Then,
\[
\int_0^\infty y^{1-2s} \| U \|^2 \, dy \leq C_s \int_\mathbb{R} \sum_{k=0}^\infty |i\rho + \lambda_k|^s |\hat{u}_k(\rho)|^2 \, d\rho = C_s \| u \|_{H^s}^2 < \infty
\]
so \( U \in L^2((0, \infty) ; \text{Dom}(H), y^{1-2s} \, dy) \).

For \( v \in \text{Dom}(H) \), by Lemma 3.1, we have that
\[
\langle HU(y), v \rangle = \int_\mathbb{R} \sum_{k=0}^\infty \hat{u}_k(\rho) \hat{v}_k(\rho) (i\rho + \lambda_k) I_s(y, i\rho + \lambda_k) \, d\rho
\]
\[
= \int_\mathbb{R} \sum_{k=0}^\infty \hat{u}_k(\rho) \hat{v}_k(\rho) \left( \frac{1-2s}{y} \partial_y + \partial_{yy} \right) I_s(y, i\rho + \lambda_k) \, d\rho
\]
\[
= \left( \left( \frac{1-2s}{y} \partial_y + \partial_{yy} \right) U(y), v \right)_{L^2(\mathbb{R} \times \Omega)}.
\]

By Lemma 3.1 and Dominated Convergence Theorem,
\[
\lim_{y \to 0^+} \langle U(y), v \rangle_{L^2(\mathbb{R} \times \Omega)} = \int_\mathbb{R} \sum_{k=0}^\infty \hat{u}_k(\rho) \hat{v}_k(\rho) \, d\rho = \langle u, v \rangle_{L^2(\mathbb{R} \times \Omega)}
\]
and
(3.8)
\[
\langle -y^{1-2s} \partial_y U(y), v \rangle_{L^2(\mathbb{R} \times \Omega)} = \frac{\Gamma(1-s)}{4^{s-1/2} \Gamma(s)} \int_\mathbb{R} \sum_{k=0}^\infty (i\rho + \lambda_k)^s \hat{u}_k(\rho) \hat{v}_k(\rho) I_{1-s}(y, i\rho + \lambda_k) \, d\rho
\]
\[
\to -\frac{\Gamma(1-s)}{4^{s-1/2} \Gamma(s)} \langle H^s u, v \rangle, \quad \text{as } y \to 0^+.
\]

Now, for every \( v \in \text{Dom}(H^s) \), since \( I_s(0, i\rho + \lambda_k) = 1 \),
\[
\frac{1}{y^{2s}} \langle U(y) - U(0), v \rangle_{L^2(\mathbb{R} \times \Omega)} = \int_\mathbb{R} \sum_{k=0}^\infty \hat{u}_k(\rho) \hat{v}_k(\rho) \frac{I_s(y, i\rho + \lambda_k) - 1}{y^{2s}} \, d\rho.
\]

From the third equation in (3.1), (3.4) and (2.3), we get
\[
\frac{I_s(y, i\rho + \lambda_k) - 1}{y^{2s}} = \frac{1}{4^s \Gamma(s)} e^{-y^2/4(t)} \left( e^{-\tau(i\rho + \lambda_k)} - 1 \right) \frac{d\tau}{\tau^{1+s}}
\]
\[
\to \frac{\Gamma(-s)}{4^s \Gamma(s)} (i\rho + \lambda_k)^s, \quad \text{as } y \to 0^+.
\]

Moreover, by applying Lemma 3.1(4) and (5.a),
\[
\frac{|I_s(y, i\rho + \lambda_k) - 1|}{y^{2s}} \leq \frac{1}{y^{2s}} \int_0^y |\partial_r I_s(r, i\rho + \lambda_k)| \, dr
\]
\[
\leq C_s \frac{y^{2s}}{2^s} |\partial_r I_s(r, i\rho + \lambda_k)| \, dr = C_s |i\rho + \lambda_k|^s.
\]

Thus, as \( u, v \in \text{Dom}(H^s) \), by Dominated Convergence Theorem,
\[
\lim_{y \to 0^+} \frac{1}{y^{2s}} \langle U(y) - U(0), v \rangle_{L^2(\mathbb{R} \times \Omega)} = \frac{\Gamma(-s)}{4^s \Gamma(s)} \int_\mathbb{R} \sum_{k=0}^\infty (i\rho + \lambda_k)^s \hat{u}_k(\rho) \hat{v}_k(\rho) \, d\rho
\]
\[
= \frac{\Gamma(-s)}{4^s \Gamma(s)} \langle H^s u, v \rangle.
\]
For any $v \in L^2(\mathbb{R} \times \Omega)$, by (3.6) and Lemma 3.1 we have
\begin{equation}
|\langle U(y), v \rangle_{L^2(\mathbb{R} \times \Omega)}| \leq \|u\|_{L^2(\mathbb{R} \times \Omega)} \|v\|_{L^2(\mathbb{R} \times \Omega)} \frac{y^{2s}}{4s \Gamma(s)} \int_0^\infty e^{-\tau \lambda_i} e^{-\frac{\tau}{\tau + s}} \, d\tau,
\end{equation}
where $i = 0$ if $\lambda_0 \neq 0$ and $i = 1$ if $\lambda_0 = 0$. Since $I_s(y, \lambda_i) \to 0$ as $y \to \infty$, we get that $U$ weakly vanishes as $y \to \infty$.

If $v \in \text{Dom}(H^s)$ then we see from Lemma 3.1(5) and (3.8) that
\[ |\langle y^{1-2s} \partial_y U, v \rangle_{L^2(\mathbb{R} \times \Omega)}| \leq C_s \|u\|_{H^s} \|v\|_{H^s}, \quad \text{for all } y \geq 0. \]

\[ \square \]

4. Extension problem for parabolic operators in divergence form

In this section we specialize the extension characterization for $(\partial_t + L)^s$ in Theorem 1.4 to the case when $L$ is a divergence form elliptic operator.

Let $\Omega \subset \mathbb{R}^n$ be a (possibly unbounded) domain and
\[ Lu = -\text{div}(a(x)\nabla u) + c(x)u \quad \text{in } \Omega, \]
where $a(x) = (a^{ij}(x))$ is a bounded, measurable, symmetric matrix defined in $\Omega$, satisfying the uniform ellipticity condition, that is, for some $\Lambda > 0$
\[ \Lambda^{-1} |\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda |\xi|^2 \]
for a.e. $x \in \Omega$, for all $\xi = (\xi_i)_{i=1}^n \in \mathbb{R}^n$, and $c(x) \in L^\infty(\Omega)$. Let $f \in L^2(\Omega)$. For $u \in L^2(\Omega)$, $Lu = f$ in $\Omega$ in the weak sense means that $\nabla u \in L^2(\Omega)$, $c^{1/2}u \in L^2(\Omega)$ and
\[ \int_{\Omega} a(x)\nabla u \nabla v \, dx + \int_{\Omega} c(x)uv \, dx = \int_{\Omega} f v \, dx, \]
for every $v \in C_0^\infty(\Omega)$. We also assume appropriate boundary conditions on $\partial \Omega$ so that $L$ has a countable family of nonnegative eigenvalues and eigenfunctions $(\lambda_k, \phi_k)_{k=0}^\infty$ such that the set $\{\phi_k\}_{k=0}^\infty$ forms an orthonormal basis for $L^2(\Omega)$. As before, if the first eigenvalue $\lambda_0 = 0$ then we assume that all the functions involved have zero spatial mean. In particular,
\[ L\phi_k = \lambda_k \phi_k \quad \text{for all } k \geq 0 \text{ in the weak sense.} \]

Therefore, if we define
\[ H^1_0(\Omega) = \text{Dom}(L) = \left\{ u \in L^2(\Omega) : \sum_{k=0}^\infty \lambda_k |u_k|^2 < \infty \right\}, \]
where $u_k = \int_{\Omega} u \phi_k \, dx$, then, for any $u, v \in H^1_0(\Omega)$,
\[ \int_{\Omega} a(x)\nabla u \nabla v \, dx + \int_{\Omega} c(x)uv \, dx = \sum_{k=0}^\infty \lambda_k u_k v_k. \]

The operators listed in (1)–(4) in the Introduction satisfy the conditions above.

Now, the extension equation takes the form
\[ \partial_t U = y^{-(1-2s)} \text{div}_{x,y}(y^{1-2s}B(x)\nabla_{x,y} U) - c(x)U, \]
where
\[ B(x) = \begin{bmatrix} a(x) & 0 \\ 0 & 1 \end{bmatrix} \]
is also uniformly elliptic. Let us denote $D = \{(x, y) : x \in \Omega, \; y > 0\} \subset \mathbb{R}^{n+1}$. The weight $\omega(x, y) = |y|^{1-2s}$ belongs to the Muckenhoupt class $A_2(\mathbb{R}^{n+1})$. Define $H^1_{L, y}(D)$ as the set of functions $w = w(x, y) \in L^2(D, y^{1-2s} \, dx \, dy)$ such that

$$[w]^2_{H^1_{L, y}(D)} := \int_{\Omega} \int_{0}^{\infty} y^{1-2s} \left( a(x) \nabla w \nabla + c(x) w^2 \right) dx \, dy + \int_{\Omega} \int_{0}^{\infty} y^{1-2s} |\partial_y w|^2 dx \, dy$$

$$= \int_{0}^{\infty} y^{1-2s} \sum_{k=0}^{\infty} \lambda_k |w_k(y)|^2 dy + \int_{\Omega} \int_{0}^{\infty} y^{1-2s} |\partial_y w|^2 dx \, dy < \infty,$$

where $w_k(y) = \int_{\Omega} w(x, y) \phi_k(x) \, dx$, under the norm

$$\|w\|^2_{H^1_{L, y}(D)} = \|w\|^2_{L^2(D, y^{1-2s} \, dx \, dy)} + [w]^2_{H^1_{L, y}(D)}.$$

**Theorem 4.1.** Consider the extension problem in Theorem 1.4 with $L$ is as above. Then $U$, defined in (1.2), belongs to $L^2(\mathbb{R}; H^1_{L, y}(D)) \cap C^\infty((0, \infty); L^2(\mathbb{R} \times \Omega)) \cap C([0, \infty); L^2(\mathbb{R} \times \Omega))$ and for any fixed $y > 0$ and $v \in C_c^\infty(\mathbb{R} \times \Omega)$,

$$\langle HU, v \rangle = \int_{\mathbb{R}} \int_{\Omega} \left( \frac{1-2s}{y} \partial_y + \partial_y \right) U v \, dx \, dt = y^{2s-1} \int_{\Omega} \int_{\Omega} \partial_y (y^{1-2s} \partial_y U)v \, dt \, dx.$$}

In particular, $U$ is a weak solution to the parabolic extension problem

$$\partial_t U = y^{-(1-2s)} \text{div}_{x,y}(y^{1-2s} B(x) \nabla_{x,y} U) - c(x) U.$$}

*In other words, for any $V(t, x, y) \in C_c^\infty(\mathbb{R} \times \Omega \times [0, \infty))$,

$$\int_{\mathbb{R}} \int_{\Omega} U \partial_t V \, dx \, dt = \int_{\mathbb{R}} \int_{\Omega} \left( a(x) \nabla_x U \nabla_x V + c(x) UV \right) dx \, dt$$

$$- \int_{\mathbb{R}} \int_{\Omega} \left( \frac{1-2s}{y} \partial_y + \partial_y \right) UV \, dx \, dt.$$}

*from which it follows that

$$\int_{0}^{\infty} \int_{\Omega} y^{1-2s} U \partial_t V \, dx \, dt \, dy = \int_{0}^{\infty} \int_{\Omega} y^{1-2s} \left( B(x) \nabla_{x,y} U \nabla_{x,y} V + c(x) UV \right) \, dx \, dt \, dy$$

$$- \frac{\Gamma(1-s)}{4s-1/2\Gamma(s)} \langle H^s u, V(t, x, 0) \rangle.$$}

**Proof.** Let us first check that $U(t, x, y) \in L^2(\mathbb{R}; H^1_{L, y}(D))$. We found in (3.5) that

$$\|U(y)\|_{L^2(\mathbb{R} \times \Omega)} \leq \|u\|_{L^2((0, \infty))} I_s(y, \lambda_i)$$

where $i = 0$ if $\lambda_0 \neq 0$ and $i = 1$ if $\lambda_0 = 0$. Then, from (3.1),

$$\int_{0}^{\infty} y^{1-2s} \|U(y)\|^2_{L^2(\mathbb{R} \times \Omega)} \, dy \leq C_s \|u\|^2_{L^2((0, \infty))} \int_{0}^{\infty} y^{1-2s} (y \sqrt{\lambda_i})^{2s} K_s^2(y \sqrt{\lambda_i}) \, dy$$

$$= C_s \|u\|^2_{L^2((0, \infty))} \lambda_i^{s-1} \int_{0}^{\infty} r K_s^2(r) \, dr < \infty.$$}

In the last inequality we used (3.7). We are left to show that

$$\int_{\mathbb{R}} \int_{0}^{\infty} y^{1-2s} \sum_{k=0}^{\infty} \lambda_k |U_k(t, y)|^2 \, dy \, dt + \int_{\mathbb{R}} \int_{0}^{\infty} \int_{\Omega} y^{1-2s} |\partial_y U(t, x, y)|^2 \, dx \, dy \, dt < \infty,$$

where, for any $k \geq i$,

$$U_k(t, y) = \langle U(t, \cdot, y), \phi_k(\cdot) \rangle_{L^2(\Omega)}$$

$$= \frac{y^{2s}}{4^s \Gamma(s)} \int_{0}^{\infty} e^{-y^2/(4\tau)} \left( e^{-\tau L} u(t - \tau, \cdot), \phi_k(\cdot) \right)_{L^2(\Omega)} \frac{d\tau}{\tau^{1+s}}.$$
Therefore, as done in (4.2),

\[ \int_{\mathbb{R}} |U_k(t, y)|^2 \, dt \leq \|u_k\|_{L^2(\mathbb{R})}^2 |I_s(y, \lambda_k)|^2. \]

Next, observe that

\[ U \] \[ V \]

\[ \int_{\mathbb{R}} y^{1-2s} \sum_{k=0}^{\infty} \lambda_k |U_k(t, y)|^2 \, dy \, dt \leq \sum_{k=0}^{\infty} \lambda_k \|u_k\|_{L^2(\mathbb{R})}^2 \int_{\mathbb{R}} y^{1-2s} (y/\lambda_k)^{2s} K_s(y/\lambda_k) \, dy \]

\[ \leq C_s \sum_{k=0}^{\infty} \lambda_k^s \|u_k\|_{L^2(\mathbb{R})}^2 < \infty. \]

Next, observe that

\[ \partial_y U(t, x, y) = C_s y^{2s-1} \sum_{k=0}^{\infty} \int_{\mathbb{R}} \tilde{u}_k(\rho)(i\rho + \lambda_k)^s I_{1-s}(y, i\rho + \lambda_k)e^{i\rho t} \, d\rho \phi_k(x) \]

and then

\[ \|\partial_y U\|_{L^2(\mathbb{R} \times \Omega)}^2 = C_s y^{2s} \sum_{k=0}^{\infty} \int_{\mathbb{R}} |\tilde{u}_k(\rho)|^2 |i\rho + \lambda_k|^{1+s} |K_{1-s}(y, i\rho + \lambda_k)|^2 \, d\rho. \]

Hence,

\[ \int_{\mathbb{R}} y^{1-2s} \|\partial_y U\|_{L^2(\mathbb{R} \times \Omega)}^2 \, dy \]

\[ = C_s \sum_{k=0}^{\infty} \int_{\mathbb{R}} |\tilde{u}_k(\rho)|^2 |i\rho + \lambda_k|^{1+s} \int_{0}^{\infty} y|K_{1-s}(y, i\rho + \lambda_k)|^2 \, dy \, d\rho. \]

To estimate the integral in \(dy\), we write \( r = y|\sqrt{i\rho + \lambda_k}| \) and \( \theta = \arg(\sqrt{i\rho + \lambda_k}) \) to get

\[ \int_{0}^{\infty} y|K_{1-s}(y, i\rho + \lambda_k)|^2 \, dy = \frac{1}{|i\rho + \lambda_k|} \int_{0}^{\infty} r|K_{1-s}(r e^{i\theta})|^2 \, dr \leq \frac{C_s}{|i\rho + \lambda_k|}, \]

because of (3.7). Whence,

\[ \int_{\mathbb{R}} y^{1-2s} \|\partial_y U\|_{L^2(\mathbb{R} \times \Omega)}^2 \, dy \leq C_s \sum_{k=0}^{\infty} \int_{\mathbb{R}} |\tilde{u}_k(\rho)|^2 |i\rho + \lambda_k|^s \, d\rho < \infty. \]

Thus \( U(t, x, y) \in L^2(\mathbb{R}; H^1_{1,y}(D)) \), as desired.

Let \( V \in C_{c}^{\infty}(\mathbb{R} \times \Omega \times [0, \infty)) \). The action of \( \partial_t U \) on \( V \) is given by

\[ \partial_t U(V) = - \int_{\mathbb{R}} U \partial_t V \, dt \]

for a.e. \((x, y) \in \Omega \times [0, \infty)\). For a fixed \( y \), we already know that

\[ \langle HU, V \rangle = \int_{\Omega} \int \left( \frac{1-2s}{y} \partial_y y + \partial_{yy} \right) U V \, dt \, dx = y^{2s-1} \int_{\mathbb{R}} \int \partial_y (y^{1-2s} \partial_y U) V \, dt \, dx. \]

But now,

\[ \langle HU, V \rangle = - \int_{\mathbb{R}} \sum_{k=0}^{\infty} \tilde{u}_k(\rho) I_s(y, i\rho + \lambda_k) i\rho \overline{V_k(\rho, y)} \, d\rho \]

\[ + \int_{\mathbb{R}} \sum_{k=0}^{\infty} \lambda_k \tilde{u}_k(\rho) I_s(y, i\rho + \lambda_k) V_k(\rho, y) \, d\rho \]
To conclude, (4.1) follows.

Let us multiply (4.1) by $y^{1-2s}$ and integrate in $dy$ to obtain

$$\int_0^\infty \int_\Omega y^{1-2s} \partial_t U(V) \, dx \, dt \, dy = \int_0^\infty \int_\Omega y^{1-2s} (a(x) \nabla_x U \nabla_x V + c(x) UV) \, dx \, dt \, dy$$

Thus, (4.1) follows.

Let us multiply (4.1) by $y^{1-2s}$ and integrate in $dy$ to obtain

$$\int_0^\infty \int_\Omega y^{1-2s} \partial_t U(V) \, dx \, dt \, dy = \int_0^\infty \int_\Omega y^{1-2s} (a(x) \nabla_x U \nabla_x V + c(x) UV) \, dx \, dt \, dy$$

Let $0 < a < b < \infty$. Since $U \in C^\infty((0, \infty); L^2(\mathbb{R} \times \Omega))$ we can apply Fubini’s Theorem and integration by parts to get

$$\int_a^b \int_\Omega y^{1-2s} \left( \frac{1-2s}{y} \partial_y + \partial_{yy} \right) U V \, dx \, dt \, dy$$

By letting $a \to 0$ and $b \to \infty$, we have

$$\int_0^\infty \int_\Omega y^{1-2s} \left( \frac{1-2s}{y} \partial_y + \partial_{yy} \right) U V \, dx \, dt \, dy$$

To conclude,

$$\lim_{y \to 0} \int_\Omega y^{1-2s} \partial_y UV \, dx \, dt = \lim_{y \to 0} \int_\Omega y^{1-2s} \partial_y U(V(t, x, y) - V(t, x, 0)) \, dx \, dt$$

$$+ \lim_{y \to 0} \int_\Omega y^{1-2s} \partial_y UV(t, x, 0) \, dx \, dt$$

$$= 0 - \frac{\Gamma(1-s)}{4^{s-1/2} \Gamma(s)} \langle H^s u, V(\cdot, \cdot, 0) \rangle,$$

where for the last identity we have used (4.1), the fact that $V \in C^\infty_c(\mathbb{R} \times \Omega \times [0, \infty))$ and Dominated Convergence Theorem. Indeed,

$$\left| \int_\Omega \int y^{1-2s} \partial_y U(V(t, x, y) - V(t, x, 0)) \, dx \, dt \right|^2 \leq C_s \|u\|_{H^s}^2 \|V(\cdot, \cdot, y) - V(\cdot, \cdot, 0)\|_{H^s}^2$$

$$\leq C_{s,A} \|u\|_{H^s}^2 \|V(\cdot, \cdot, y) - V(\cdot, \cdot, 0)\|_{H^s}^2$$

$$\leq C_{s,A} \|u\|_{H^s}^2 \left\{ \|V(\cdot, \cdot, y) - V(\cdot, \cdot, 0)\|_{L^2(\mathbb{R} \times \Omega)}^2 + \int_\Omega \int |\partial_t (V(t, x, y) - V(t, x, 0))|^2 \, dx \, dt \right\}$$

$$+ \int_\Omega |\nabla_x (V(t, x, y) - V(t, x, 0))|^2 \, dx \, dt + \int_\Omega |c(x)||V(t, x, y) - V(t, x, 0)|^2 \, dx \, dt \right\}$$

$$\to 0 \quad \text{as} \quad y \to 0.$$
Lemma 4.2 (Reflection extension). Let $L$ and $U$ be as in Theorem 4.1. Let $\Omega_0 \subset \Omega$ be a bounded domain and $(T_0, T_1) \subset \mathbb{R}$. Suppose that
\[
\lim_{y \to 0^+} \langle y^{1-2s} \partial_y U, V \rangle_{L^2(\mathbb{R} \times \Omega)} = 0
\]
for all $V \in C^\infty_0((T_0, T_1) \times \Omega_0 \times ]0, \infty[)$. Fix $Y_0 > 0$. Then, the even extension $\tilde{U}$ of $U$ in the variable $y$, defined by
\[
\tilde{U}(t, x, y) = \begin{cases} U(t, x, y) & \text{for } 0 \leq y < Y_0 \\ U(t, x, -y) & \text{for } -Y_0 < y < 0 \end{cases}
\]
is a weak solution to the degenerate parabolic equation
\[
\partial_t \tilde{U} = |y|^{-(1-2s)} \text{div}_{x,y}(|y|^{1-2s}B(x)\nabla_{x,y} \tilde{U}) - c(x)\tilde{U}
\]
in $(T_0, T_1) \times \Omega_0 \times (-Y_0, Y_0)$.

Proof. Let $V \in C^\infty_c((T_1, T_2) \times \Omega_0 \times (-Y_0, Y_0))$. We shall prove that
\[
\int_{T_0}^{T_1} \int_{-Y_0}^{Y_0} \int_{\Omega_0} |y|^{1-2s} \tilde{U} \partial_t V \, dx \, dy \, dt = \int_{T_0}^{T_1} \int_{-Y_0}^{Y_0} \int_{\Omega_0} |y|^{1-2s} (B(x)\nabla_{x,y} \tilde{U} \nabla_{x,y} V + c(x)\tilde{U} V) \, dx \, dy \, dt.
\]
Let $\delta > 0$. From (4.11), for any $y > 0$,
\[
\int_{\mathbb{R}} \int_{\Omega} U \partial_t V \, dx \, dt = \int_{\mathbb{R}} \int_{\Omega} (a(x)\nabla_x U \nabla_x V + c(x)UV) \, dx \, dt - \int_{\mathbb{R}} \int_{\Omega} |y|^{2s-1} \partial_y (|y|^{1-2s} \partial_y U) \, dx \, dt.
\]
By multiplying this equation by $|y|^{1-2s}$, integrating in $y \in (\delta, Y_0)$, and using integration by parts we get
\[
\int_{T_0}^{T_1} \int_{\Omega} |y|^{1-2s} \tilde{U} \partial_t V \, dx \, dy \, dt = \int_{T_0}^{T_1} \int_{\delta}^{Y_0} \int_{\Omega_0} |y|^{1-2s} (B(x)\nabla_{x,y} \tilde{U} \nabla_{x,y} V + c(x)\tilde{U} V) \, dx \, dy \, dt + \int_{T_0}^{T_1} \int_{\Omega_0} \delta^{1-2s} \partial_y U(t, x, \delta) V(t, x, \delta) \, dx \, dt.
\]
From here we readily get
\[
\int_{T_0}^{T_1} \int_{\delta < |y| < Y_0} \int_{\Omega_0} |y|^{1-2s} \tilde{U} \partial_t V \, dx \, dy \, dt = \int_{T_0}^{T_1} \int_{\delta < |y| < Y_0} \int_{\Omega_0} |y|^{1-2s} (B(x)\nabla_{x,y} \tilde{U} \nabla_{x,y} V + c(x)\tilde{U} V) \, dx \, dy \, dt + \int_{T_0}^{T_1} \int_{\Omega_0} \delta^{1-2s} \partial_y U(t, x, y)|_{y=\delta} V(t, x, -\delta) \, dx \, dt + \int_{T_0}^{T_1} \int_{\Omega_0} \delta^{1-2s} \partial_y U(t, x, \delta) V(t, x, \delta) \, dx \, dt.
\]
The conclusion follows by taking $\delta \to 0$ in this last identity. \qed
Remark 4.3. If the differential operator \( L \) has continuous spectrum then all the previous results are still valid.

Consider, for example, \( L = -\Delta \) in \( \Omega = \mathbb{R}^n \). We can use Fourier transform \( \mathcal{F} \) in the variables \( t \) and \( x \) to define the operator \((\partial_t + L)^s\) as

\[
\langle (\partial_t - \Delta)^s u, v \rangle_{L^2(\mathbb{R}^{n+1})} = \int_{\mathbb{R}} \int_{\mathbb{R}^n} (i\rho + |\xi|^2)^s \mathcal{F}u(\rho, \xi)\mathcal{F}v(\rho, \xi) \, d\xi \, d\rho.
\]

The analogous to the expression

\[
u^s = \sum_{k=0}^{\infty} u_k(t)\phi_k(x)
\]

in this case is just

\[
u^s = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{u}(t, \xi) e^{i\xi x} \, d\xi
\]

where the Fourier transform is taken in the variable \( x \) by leaving \( t \) fixed. The eigenvalues and eigenfunctions \((\lambda_k, \phi_k)_{k=0}^{\infty}\) are replaced by \( (|\xi|^2, e^{ix\xi})_{\xi \in \mathbb{R}^n} \).

Consider another one, the Bessel operator \( L = -\frac{d^2}{dx^2} + \frac{\lambda(\lambda-1)}{x^2} \), for \( \lambda > 0 \), in \( \Omega = (0, \infty) \). In this case we can use Hankel transform in \( x \) and Fourier transform in \( t \). Let \( \phi_y(x) = (yx)^{\nu/2}J_{\nu-1/2}(yx) \), \( x, y > 0 \), where \( J_\nu \) denotes the Bessel function of the first kind with order \( \nu \). Then \( L\phi_y(x) = y^2 \phi_y(x) \) and the eigenvalues and eigenfunctions \((\lambda_k, \phi_k)_{k=0}^{\infty}\) are replaced by \((y^2, \phi_y(x))_{y>0}\). The Hankel transform in the variable \( x \) is defined as

\[
\mathcal{H}u(t, y) = \int_0^{\infty} u(t, x)\phi_y(x) \, dx
\]

and, since \( \mathcal{H}^{-1} = \mathcal{H} \), we can write

\[
u^s = \int_0^{\infty} \mathcal{H}u(t, y)\phi_y(x) \, dy.
\]

With this, we can let

\[
\langle (\partial_t + L)^s u, v \rangle = \int_0^{\infty} \int_0^{\infty} (i\rho + y^2)^s \mathcal{H}u(\rho, y)\mathcal{H}v(\rho, y) \, dy \, d\rho.
\]

Similarly, Lemma 4.2 holds in all these cases.

5. Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1 Consider the extension \( U \) of \( u \) given by Theorems 1.4 and 4.1. If \( u \geq 0 \) in \((0, 1) \times \mathbb{R}^n\) then, since the heat kernel for \( L \) is nonnegative, the first formula in

\[
(\mathcal{L}u)^s = \sup_{R^-} u(t, x, 0) \leq \sup_{R^+} U(t, x, y) \leq C_H \inf_{R^+} U(t, x, y) \leq C_H \inf_{R^+} U(t, x, 0) = C_H \inf_{R^+} U(t, x).
\]

Now we prove the local boundedness and Hölder estimates on \( u \). By using the results in \( \mathbb{S} \) we get that \( \tilde{U} \) is locally bounded and locally parabolically Hölder continuous of order \( 0 < \alpha < 1 \) in \( R \). Let \( K \) be a compact subset of \( R \). We have

\[
\|\tilde{U}\|_{L^\infty(K \times (-1,1))} \leq C \|\tilde{U}\|_{L^2(R \times (-2,2))} = 2C \|U\|_{L^2(R \times (0,2))}.
\]
Since \( \|U\|_{L^2(R \times (0,2))} \leq C\|u\|_{L^2(R \times \Omega)} \), we obtain
\[
\|u\|_{L^\infty(K)} \leq \|\tilde{U}\|_{L^\infty(K \times (-1,1))} \leq C\|u\|_{L^2(R \times \Omega)}.
\]
Next, from the local H"older continuity of \( \tilde{U} \),
\[
[u]_{C^{\alpha/2,\alpha}(1)} = [\tilde{U}]_{C^{\alpha/2,\alpha}(K \cap y=0)} \leq C\|\tilde{U}\|_{L^\infty(K \times (-1,1))} \leq C\|u\|_{L^2(R \times \Omega)}.
\]

\[\square\]

**Remark 5.1.** If in Theorem 1.1 we substitute \( B_{2r} \) by an open set and \( B_r \) by a compact set contained in the open set, the result remains valid and the constant \( c \) also depends on both sets.

**Proof of Theorem 1.1** For simplicity, and without loss of generality, we will assume that \( \tilde{x} = 0 \). Let \( \tilde{U} \) be the reflection in \( y \) of the extension \( U \) of \( u \). By Lemma 1.2, \( \tilde{U} \) is a nonnegative weak solution to
\[
(t, x, y) \in (-2, 2) \times (B_{2r}(0) \cap \Omega_0) \times (-2r, 2r)
\]
that vanishes continuously in \( (t, x, y) \in (-2, 2) \times ((\Omega \setminus \Omega_0) \cap B_{2r}(0)) \times \{0\} \).

As a first step we flatten the boundary of \( \Omega_0 \) inside \( B_{2r}(0) \). We use a bi-Lipschitz transformation \( \Psi \) such that \( \Psi(0) = 0 \) and \( \Psi(\Omega_0 \cap B_{2r}(0)) = \Omega_1 \), where \( \Omega_1 \) is a new domain with flat boundary at \( x_n = 0 \), which can be extended as constant in \( t \) and \( y \). Without loss of generality we can assume that the flat part of \( B_{2r}(0) \cap \mathbb{R}^n_+ \) is the flat part of the new domain \( \Omega_1 \). Then the transformed function \( \tilde{U}_1 := \tilde{U} \circ \Psi^{-1} \) satisfies the same type of degenerate parabolic equation with bounded measurable coefficients in the domain \((-2, 2) \times (\mathbb{R}^n_+ \cap B_{2r}(0)) \times (-2r, 2r)\) and vanishes continuously on \((-2, 2) \times ((\mathbb{R}^n \setminus \mathbb{R}^n_+) \cap B_{2r}(0)) \times \{0\}\).

As a second step, we define a transformation which maps \( \mathbb{R}^{n+1} \setminus \{x_n \leq 0, y = 0\} \) into \( \mathbb{R}^{n+1} \cap \{x_n > 0\} \) and is extended to be constant in \( t \). This construction is standard, see [11]. After this transformation is performed, we obtain a function \( \tilde{U}_2 \) that solves again a degenerate parabolic equation with bounded measurable coefficients in the domain \((-2, 2) \times (\mathbb{R}^n_+ \cap B_{2r}(0)) \times (-2r, 2r)\) and vanishes continuously for \((t, x, y) \in (-2, 2) \times \{(x', 0, y) : (x')^2 + y^2 < (2r)^2\}\).

Now we can apply the boundary Harnack inequality of Ishige [8] to \( \tilde{U}_2 \) to get
\[
\sup_{(-1,1) \times (\Omega \cap B_r(0))} u(t, x) = \sup_{(-1,1) \times (\mathbb{R}^n_+ \cap B_r(0))} \tilde{U}_2(t, x, 0) \leq C\tilde{U}_2(t_0, \tilde{x}_0, 0) = u(t_0, x_0),
\]
where \( \tilde{x}_0 \) is the point obtained from \( x_0 \) via the two transformations. \[\square\]

**Remark 5.2.** If in Theorem 1.2 we substitute \( B_{2r}(\tilde{x}) \) by an open set and \( B_r(\tilde{x}) \) by another open subset of the first one, the result remains still valid and the constant \( C \) also depends on both open sets.

## 6. Transference Method and proof of Theorem 1.3

In this section we assume that
\[
Lu = -\text{div}(a(x)\nabla u) + c(x)u \quad \text{in} \quad \Omega
\]
is an operator as in Section 4.

### 6.1. Change of variables
Let \( \Omega \subset \mathbb{R}^n \) be a domain and \( h : \Omega \to \tilde{\Omega} \) be a smooth change of variables from \( x \in \Omega \) into \( \tilde{x} = h(x) \in \tilde{\Omega} \), that is, \( h \) is one-to-one, onto and differentiable with inverse \( h^{-1} : \tilde{\Omega} \to \Omega \) differentiable as well. We denote by \( J_h(x) = |\det \nabla h(x)| \), for \( x \in \Omega \), and \( J_{h^{-1}}(\tilde{x}) = |\det \nabla h^{-1}(\tilde{x})| \), for \( \tilde{x} \in \tilde{\Omega} \). Let us define the change of variables application
\[
W : L^2(\tilde{\Omega}, J_{h^{-1}}d\tilde{x}) \to L^2(\Omega, dx)
\]
as

\[ W(\tilde{f})(x) = \tilde{f}(h(x)) \quad \text{for } x \in \Omega. \]

Then \( W \) is one-to-one, onto and, for any \( f \in L^2(\Omega, dx) \),

\[ W^{-1}(f)(\tilde{x}) = f(h^{-1}(\tilde{x})), \quad \tilde{x} \in \tilde{\Omega}. \]

It is readily seen that

\[ \|W \tilde{f}\|_{L^2(\Omega, dx)} = \|\tilde{f}\|_{L^2(\tilde{\Omega}, J_{h^{-1}}d\tilde{x})}. \]

Let \( \{\phi_k\}_{k=0}^{\infty} \) be the orthonormal basis of \( L^2(\Omega, dx) \) consisting of eigenfunctions of \( L \).

We claim that \( \{\tilde{\phi}_k := W^{-1}\phi_k\}_{k=0}^{\infty} \) is an orthonormal basis of \( L^2(\tilde{\Omega}, J_{h^{-1}}d\tilde{x}) \). Indeed, by changing variables,

\[ \int_{\tilde{\Omega}} \tilde{\phi}_k(\tilde{x})\phi_k(x)J_{h^{-1}}(\tilde{x}) \, d\tilde{x} = \int_{\Omega} \phi_k(x)\phi_k(x) \, dx = \delta_{kk}. \]

Also, if \( \tilde{f} \in L^2(\tilde{\Omega}, J_{h^{-1}}d\tilde{x}) \) is orthogonal to each \( \tilde{\phi}_k \) then

\[ 0 = \int_{\tilde{\Omega}} \tilde{f}(\tilde{x})\phi_k(x)J_{h^{-1}}(\tilde{x}) \, d\tilde{x} = \int_{\Omega} W(\tilde{f})(x)\phi_k(x) \, dx \]

for all \( k \geq 0 \), which gives \( \tilde{f} = 0 \), and the orthonormal set \( \{\tilde{\phi}_k\}_{k=0}^{\infty} \) is complete in \( L^2(\tilde{\Omega}, J_{h^{-1}}d\tilde{x}) \).

If \( u \in \text{Dom}(L) \) and we define \( \tilde{u} = W^{-1}u = u \circ h^{-1} \) then we can write \( u = W\tilde{u} = \tilde{u} \circ h \) and the change rule gives

\[ u_{x_i}(x) = \sum_{k=1}^{n} \tilde{u}_{\tilde{x}_k}(h(x))(\nabla h(x))_{ki} \]

where \((\nabla h(x))_{ki} = \left( \frac{\partial h_k(x)}{\partial x_i} \right)_{ki}\) denotes the \( ki \)-th entry of the matrix \( \nabla h(x) \). From the definition of the action of \( L \) on \( u \) we have, for any \( v \in \text{Dom}(L) \),

\[ \langle Lu, v \rangle = \int_{\Omega} \left( \sum_{i,j=1}^{n} \frac{a^{ij}(x)u_{x_i}(x)v_{x_j}(x) + c(x)u(x)v(x)}{dx} \right) \, dx \]

\[ = \int_{\Omega} \left[ \sum_{k,k'=1}^{n} \left( \sum_{i,j=1}^{n} a^{ij}(x)(\nabla h(x))_{ki}(\nabla h(x))_{kj} \right) \tilde{u}_{\tilde{x}_k}(h(x))\tilde{v}_{\tilde{x}_j}(h(x)) + c(x)u(x)v(x) \right] \, dx \]

\[ = \int_{\tilde{\Omega}} (\tilde{a}(\tilde{x})\nabla \tilde{u} \nabla \tilde{v} + \tilde{c}(\tilde{x})\tilde{u}\tilde{v})J_{h^{-1}}(\tilde{x}) \, d\tilde{x} \]

where

\[ \tilde{a}^{kl}(\tilde{x}) = \sum_{i,j=1}^{n} a^{ij}(h^{-1}(\tilde{x}))((\nabla h(h^{-1}(\tilde{x})))_{ki}(\nabla h(h^{-1}(\tilde{x})))_{kj} \]

and

\[ \tilde{c}(\tilde{x}) = c(h^{-1}(\tilde{x})). \]

With this identity we define a new operator \( \tilde{L} \) in the following way. Let \( \tilde{u}, \tilde{v} \in L^2(\tilde{\Omega}, J_{h^{-1}}d\tilde{x}) \) such that \( u = W\tilde{u} \) and \( v = W\tilde{v} \) belong to \( \text{Dom}(L) \). We define

\[ \langle \tilde{L}\tilde{u}, \tilde{v} \rangle := \langle Lu, v \rangle. \]

With this, \( (\lambda_k, \tilde{\phi}_k)_{k=0}^{\infty} \) are the eigenvalues and eigenfunctions of \( \tilde{L} \), where \( \lambda_k \) are the eigenvalues of \( L \). Moreover,

\[ \text{Dom}(\tilde{L}) = \left\{ \tilde{u} \in L^2(\tilde{\Omega}, J_{h^{-1}}d\tilde{x}) : \sum_{k=0}^{\infty} \lambda_k \tilde{a}^2_k < \infty \right\}, \]
Therefore, for any $v$ define the multiplication operator for $\tilde{\mathcal{L}}$ eigenfunctions of $\mathcal{L}$. With this, then we can formally write

$$\langle \tilde{L}u, \tilde{v} \rangle = \langle L(W\tilde{u}), (W\tilde{v}) \rangle = \langle W^{-1}LW\tilde{u}, \tilde{v} \rangle,$$

or

$$\tilde{L} = W^{-1} \circ L \circ W.$$

6.2. Multiplication operator. Let $M = M(x) \in C^\infty(\Omega)$ be a positive function. We define the multiplication operator

$$U : L^2(\Omega, M(x)^2 dx) \to L^2(\Omega, dx)$$

as

$$U(\tilde{u})(x) = M(x)\tilde{u}(x),$$

for $\tilde{u} \in L^2(\Omega, M(x)^2 dx)$. If $\{\phi_k\}_{k=0}^\infty$ is the orthonormal basis of $L^2(\Omega, dx)$ consisting of eigenfunctions of $L$ then $\{\tilde{\phi}_k = U^{-1}\phi_k\}_{k=0}^\infty$ is an orthonormal basis of $L^2(\Omega, M(x)^2 dx)$. Now given $u \in \text{Dom}(L)$ we define $\tilde{u}(x) = U^{-1}u(x) = M(x)^{-1}u(x)$, so that

$$u_{x_i}(x) = M(x)\tilde{u}_{x_i}(x) + M_{x_i}(x)\tilde{u}(x).$$

Therefore, for any $v \in \text{Dom}(L)$,

$$\langle Lu, v \rangle = \int_{\Omega} \left( a^{ij}(x)u_{x_i}v_{x_j} + c(x)uv \right) dx$$

$$= \int_{\Omega} \left[ a^{ij}(x) \left( \tilde{u}_{x_i} + \frac{M_{x_i}(x)}{M(x)}\tilde{u} \right) \left( \tilde{v}_{x_j} + \frac{M_{x_j}(x)}{M(x)}\tilde{v} \right) + c(x)\tilde{u}\tilde{v} \right] M(x)^2 dx.$$

This allows us to define the operator $\tilde{L}$ in the following way. For $\tilde{u}, \tilde{v} \in L^2(\Omega, M(x)^2 dx)$ such that $u = U(\tilde{u}) = M \cdot \tilde{u}$ and $v = U(\tilde{v}) = M \cdot \tilde{v}$ are in $\text{Dom}(L)$, we define

$$\langle \tilde{L}\tilde{u}, \tilde{v} \rangle := \langle Lu, v \rangle.$$

With this, $\{\lambda_k, \tilde{\phi}_k\}_{k=0}^\infty$ are the eigenvalues and eigenfunctions of $\tilde{L}$, where $\lambda_k$ are the eigenvalues of $L$. Whence,

$$\text{Dom}(\tilde{L}) = \left\{ \tilde{u} \in L^2(\Omega, M(x)^2 dx) : \sum_{k=0}^\infty \lambda_k\tilde{u}_k^2 < \infty \right\},$$

where $\tilde{u}_k = \int_{\Omega} \tilde{u}\tilde{\phi}_k M(x)^2 dx = \int_{\Omega} u\phi_k dx = u_k$. Observe that

$$\int_{\Omega} U(\tilde{u})(x)v(x) dx = \int_{\Omega} \tilde{u}(x)U^{-1}v(x)M(x)^2 dx.$$

Then we can formally write

$$\langle \tilde{L}\tilde{u}, \tilde{v} \rangle = \langle L(U\tilde{u}), (U\tilde{v}) \rangle = \langle U^{-1}LU\tilde{u}, \tilde{v} \rangle,$$

or

$$\tilde{L} = U^{-1} \circ L \circ U.$$
6.3. Composition of multiplication and change of variables. We consider the following composition of the multiplication operator $U$ with the change of variables operator $W$:

$$U \circ W : L^2(\tilde{\Omega}, M(h^{-1}(\tilde{x}))^2 J_{h^{-1}} d\tilde{x}) \to L^2(\Omega, dx).$$

Notice that if $\tilde{f} \in L^2(\tilde{\Omega}, M(h^{-1}(\tilde{x}))^2 J_{h^{-1}} d\tilde{x})$ then

$$\int_{\tilde{\Omega}} |(U \circ W)\tilde{f}|(x)|^2 dx = \int_{\tilde{\Omega}} |\tilde{f}(\tilde{x})|^2 M(h^{-1}(\tilde{x}))^2 J_{h^{-1}}(\tilde{x}) d\tilde{x}.$$  

By using a similar technique as we used in cases of $W$ and $U$ separately, we can define a new operator $\tilde{L}$ in the following way. For $\tilde{u}, \tilde{v} \in L^2(\tilde{\Omega}, M(h^{-1}(\tilde{x}))^2 J_{h^{-1}} d\tilde{x})$ such that $u := (U \circ W)\tilde{u}$ and $v := (U \circ W)\tilde{v}$ are in $\text{Dom}(L)$ we let

$$\langle \tilde{L}\tilde{u}, \tilde{v} \rangle = \langle Lu, v \rangle.$$

By proceeding as in the previous cases we can formally write

$$\tilde{L} = (U \circ W)^{-1} \circ L \circ (U \circ W).$$

6.4. Transference method from $(\partial_t + L)^{\alpha}$ to $(\partial_t + \tilde{L})^{\alpha}$. Now we consider the parabolic operators $H = \partial_t + L$ and $\tilde{H} = \partial_t + \tilde{L}$, where $L$ and $\tilde{L}$ are as above. If $\tilde{u} = \tilde{u}(t, \tilde{x})$ is a function of $t \in \mathbb{R}$ and $\tilde{x} \in \tilde{\Omega}$ then the composition operator will act on $\tilde{u}$ by leaving the variable $t$ fixed:

$$(U \circ W)\tilde{u}(t, x) = M(x)\tilde{u}(t, h(x)), \text{ for } x \in \Omega,$$

so that

$$U \circ W : L^2(\mathbb{R}, dt; L^2(\tilde{\Omega}, M(h^{-1}(\tilde{x}))^2 J_{h^{-1}} d\tilde{x})) \to L^2(\mathbb{R}, dt; L^2(\Omega, dx)) = L^2(\mathbb{R} \times \Omega).$$

Recall that

$$\text{Dom}(H) = \left\{ u \in L^2(\mathbb{R} \times \Omega) : \int_{\mathbb{R}} \sum_{k=0}^{\infty} |(i\rho + \lambda_k)||\hat{u}_k(\rho)|^2 d\rho < \infty \right\}$$

and that, for $u \in \text{Dom}(H)$ any $v \in C_c^\infty(\mathbb{R} \times \Omega)$,

$$\langle Hu, v \rangle_{L^2(\mathbb{R} \times \Omega)} = \int_{\mathbb{R}} \int_{\Omega} \left( -u \partial_t v + \sum_{i,j=1}^{n} a^{ij}(x)u_{x_i}(t, x)v_{x_j}(t, x) + c(x)u(t, x)v(t, x) \right) dx dt.$$

Now, for $\tilde{u} \in L^2(\mathbb{R}, dt; L^2(\tilde{\Omega}, M(h^{-1}(\tilde{x}))^2 J_{h^{-1}} d\tilde{x}))$ such that $u := (U \circ W)\tilde{u} \in \text{Dom}(H)$, and $v := (U \circ W)\tilde{v}$, we define the parabolic operator

$$\langle \tilde{H}\tilde{u}, \tilde{v} \rangle := \langle Hu, v \rangle.$$

As a matter of fact, we can write,

$$\langle Hu, v \rangle_{L^2(\mathbb{R} \times \Omega)} = \int_{\mathbb{R}} \int_{\Omega} \left[ -M(x)\tilde{u}(t, h(x))M(x)\tilde{v}_t(t, h(x)) + \sum_{i,j=1}^{n} a^{ij}(x)M_{x_i}(x)\tilde{u}(t, h(x)) + \sum_{k=1}^{n} M(x)\tilde{u}_{x_k}(t, h(x))(\nabla h(x))_{k_i} \right] \times \left( M_{x_j}(x)\tilde{v}(t, h(x)) + \sum_{\ell=1}^{n} M(x)\tilde{v}_{x_\ell}(t, h(x))(\nabla h(x))_{\ell_j} \right) + c(x)M(x)\tilde{u}(t, h(x))M(x)\tilde{v}(t, h(x)) \right] dx dt$$

$$= \int_{\mathbb{R}} \int_{\Omega} \left[ -\tilde{u}\tilde{v}_t \right] dx dt.$$
Whence, we can formally write

\[ e^{i\lambda} \sum_{i,j=1}^{n} a^{ij}(h^{-1}(\tilde{x})) \left( \frac{M_{ij}(h^{-1}(\tilde{x}))}{M(h^{-1}(\tilde{x}))} \tilde{u} + \sum_{k=1}^{n} \tilde{u}_{\tilde{x}k}(\nabla h(h^{-1}(\tilde{x})))_{ki} \right) \]

\times \left( \frac{M_{ij}(h^{-1}(\tilde{x}))}{M(h^{-1}(\tilde{x}))} \tilde{v}(t, \tilde{x}) + \sum_{l=1}^{n} \tilde{v}_{\tilde{x}l}(\nabla h(h^{-1}(\tilde{x})))_{lj} \right) \]

\[ + c(h^{-1}(\tilde{x})) \tilde{u} \tilde{v} \]

By using a similar argument as before we can formally write

\[ \tilde{H} = (U \circ W)^{-1} \circ H \circ (U \circ W). \]

Next, for \( u \in \text{Dom}(H) \) set \( u_k(t) = \int_{\Omega} u \phi_k \, dx \), and write

\[ u(t, x) = \sum_{k=0}^{\infty} u_k(t) \phi_k(x). \]

We know from the previous discussion that \( (\lambda_k, \tilde{\phi}_k)_{k=0}^{\infty} \) is the family of eigenvalues and eigenfunctions of \( \tilde{L} \), where

\[ \tilde{\phi}_k(\tilde{x}) = \frac{1}{M(h^{-1}(\tilde{x}))} \phi_k(h^{-1}(\tilde{x})) \quad \text{for} \ \tilde{x} \in \tilde{\Omega}. \]

So if \( u(t, x) \in L^2(\mathbb{R}, dt; L^2(\tilde{\Omega}, M(h^{-1}(\tilde{x}))^2 J_{h^{-1}} d\tilde{x})) \), then

\[ \tilde{u}(t, \tilde{x}) = \sum_{k=0}^{\infty} \tilde{u}_k(t) \frac{1}{M(h^{-1}(\tilde{x}))} \phi_k(h^{-1}(\tilde{x})). \]

But

\[ \tilde{u}_k(t) = \int_{\tilde{\Omega}} \tilde{u}(t, \tilde{x}) \tilde{\phi}_k(\tilde{x}) M^2(h^{-1}(\tilde{x})) J_{h^{-1}} d\tilde{x} = \int_{\Omega} u(t, x) \phi_k(x) \, dx = u_k(t). \]

Hence,

\[ \langle \tilde{H} \tilde{u}, \tilde{v} \rangle = \langle Hu, v \rangle = \int_{\mathbb{R}} \int_{\Omega} (i\rho + \lambda_k) \tilde{u}(\rho) \tilde{\phi}_k(\rho) d\rho = \int_{\mathbb{R}} \sum_{k=0}^{\infty} (i\rho + \lambda_k) \tilde{u}_k(\rho) \tilde{\phi}_k(\rho) d\rho. \]

Therefore, for any \( 0 \leq s \leq 1 \),

\[ \langle \tilde{H}^s \tilde{u}, \tilde{v} \rangle = \int_{\mathbb{R}} \sum_{k=0}^{\infty} (i\rho + \lambda_k)^s \tilde{u}_k(\rho) \tilde{\phi}_k(\rho) d\rho = \langle H^s u, v \rangle. \]

Whence, we can formally write

\[ \tilde{H}^s = (U \circ W)^{-1} \circ H^s \circ (U \circ W). \]

**Theorem 6.1** (Transference method). *If Theorems 4.1 and 4.2 hold true for solutions \( u \in \text{Dom}(H^s) \) to \((\partial_t + L)^s u = 0\) then they also hold true for solutions \( \tilde{u} \in \text{Dom}(\tilde{H}^s) \) to \((\partial_t + \tilde{L})^s \tilde{u} = 0\).*

**Proof.** Let us first show how to transfer Theorem 4.1. Let \( \tilde{u} \in \text{Dom}(\tilde{H}^s) \) be a solution to

\[
\begin{cases}
\tilde{H}^s \tilde{u} = 0 & \text{in } (0, 1) \times \tilde{\Omega} \\
\tilde{u} \geq 0 & \text{in } (-\infty, 1) \times \tilde{\Omega},
\end{cases}
\]

for some open set \( \tilde{\Omega} \subset \tilde{\Omega} \). From the definition, \( \langle \tilde{H}^s \tilde{u}, \tilde{v} \rangle = \langle H^s u, v \rangle \), where \( u = (U \circ W) \tilde{u} \) and \( v = (U \circ W) \tilde{v} \). Then, by taking any \( v \in C^\infty_c((0, 1) \times O) \), where \( O = h^{-1}(\tilde{O}) \),
we can let \( \bar{v} = (U \circ W)^{-1}v \in C^\infty_c((0,1) \times \tilde{O}) \) and thus conclude that \( H^s u = 0 \) in \((0,1) \times h^{-1}(\tilde{O}) = (0,1) \times O \). Also \( u \geq 0 \) in \((-\infty,1) \times h^{-1}(\tilde{O}) = (-\infty,1) \times \Omega \). Let \( \tilde{J} \) be a compact subset of \( \tilde{O} \). Then \( h^{-1}(\tilde{J}) \) is a compact subset of \( O \) and, by Harnack inequality for \( H^s \), (see Remark 5.1),

\[
\sup_{(\frac{1}{4},\frac{1}{2})^{x \in h^{-1}(\tilde{J})}} u \leq C \inf_{(\frac{1}{4},\frac{1}{2})^{x \in h^{-1}(\tilde{J})}} u.
\]

Since \( M(x) \) is strictly positive, continuous and bounded in \( h^{-1}(\tilde{J}) \),

\[
\sup_{(\frac{1}{4},\frac{1}{2})^{x \in h^{-1}(\tilde{J})}} W\bar{u} \leq C' \inf_{(\frac{1}{4},\frac{1}{2})^{x \in h^{-1}(\tilde{J})}} W\bar{u}.
\]

The change of variable \( h \) is a smooth diffeomorphism, so that

\[
\sup_{(\frac{1}{4},\frac{1}{2})^{x \in \tilde{J}}} \bar{u} \leq C' \inf_{(\frac{1}{4},\frac{1}{2})^{x \in \tilde{J}}} \bar{u}.
\]

Thus Harnack inequality holds for \( H^s \). Let \( \tilde{K} \) be a compact subset of \((0,1) \times \tilde{O} \). Then \( K = h^{-1}(\tilde{K}) \) is a compact subset of \((0,1) \times O \) and \( u \) is parabolically Hölder continuous in \( K \) with

\[
\|u\|_{C^{\alpha/2}_{t,x}(K)} \leq C\|u\|_{L^2(R \times \Omega)} = C\|\bar{u}\|_{L^2(R,dt;L^2(\tilde{O},M(h^{-1}(\tilde{x}))^2J_{h^{-1}d\tilde{x}}))}.
\]

Notice that \( \bar{u}(t,\tilde{x}) = [(U \circ W)^{-1}u](t,\tilde{x}) = \frac{1}{M(h^{-1}(\tilde{x}))}u(t,h^{-1}(\tilde{x})) \), which, for any \((t_1,x_1) = (t_i,h^{-1}(\tilde{x}_i)) \in K, i = 1,2 \), gives

\[
|\bar{u}(t_1,\tilde{x}_1) - \bar{u}(t_2,\tilde{x}_2)| = \left| \frac{u(t_1,x_1)}{M(x_1)} - \frac{u(t_2,x_2)}{M(x_2)} \right| \leq C\|M^{-1}\|_{C^2_\alpha(K)}\|u\|_{C^{\alpha/2}_{t,x}(K)}d((t_1,x_1), (t_2,x_2))^\alpha
\]

\[
\leq C'\|\bar{u}\|_{L^2(R,dt;L^2(\tilde{O},M(h^{-1}(\tilde{x}))^2J_{h^{-1}d\tilde{x}}))}d((t_1,\tilde{x}_1), (t_2,\tilde{x}_2))^\alpha
\]

where \( d \) denotes the parabolic distance. In the last identity we used the fact that \( h^{-1} \) is a smooth diffeomorphism.

Let us next transfer the boundary Harnack inequality of Theorem [12]. Again, for simplicity and without loss of generality, we consider \( \tilde{x} = 0 \). Let \( \bar{u} \in \text{Dom}(\tilde{H}^s) \) be a solution to

\[
\begin{cases}
\tilde{H}^s \bar{u} = 0 & \text{in } (-2,2) \times (\tilde{O}_0 \cap \tilde{B}_{2r}(0)) \\
\bar{u} \geq 0 & \text{in } (-\infty,2) \times \tilde{O},
\end{cases}
\]

such that \( \bar{u} \) vanishes continuously on \((-2,2) \times (\tilde{O}_0 \cap \tilde{B}_{2r}(0)) \). Let \((t_0,\tilde{x}_0) \) be a fixed point in \((-2,2) \times \tilde{O}_0 \) such that \( t_0 > 1 \). Then \( \tilde{H}^s \bar{u} = 0 \) in \((-2,2) \times (\tilde{O}_0 \cap h^{-1}(\tilde{B}_{2r}(0))) \), where \( \tilde{O}_0 = h^{-1}(\tilde{O}_0) \), \( u \geq 0 \) in \((-\infty,2) \times \tilde{O} \) and, as \( h \) is a smooth diffeomorphism, we can also see that \( u = (U \circ W)\bar{u} \) vanishes continuously in \((-2,2) \times ((\tilde{O} \setminus \tilde{O}_0) \cap h^{-1}(\tilde{B}_{2r}(0))) \).

We assume, again for simplicity, that \( h(0) = 0 \) and let \( K = h^{-1}(\tilde{B}_r(0)) \). Then \( 0 \in K \) and \( K \) is compactly contained in \( h^{-1}(\tilde{B}_{2r}(0)) \). We know that (see Remark 5.2)

\[
\sup_{(-1,1) \times (\tilde{O}_0 \cap K)} u(t,x) \leq C u(t_0,x_0),
\]

for \( C > 0 \). Since \( M > 0 \) is bounded and continuous, and \( h \) is a smooth diffeomorphism,

\[
\sup_{(-1,1) \times (\tilde{O}_0 \cap \tilde{B}_{r}(0))} \bar{u}(t,\tilde{x}) \leq C' \bar{u}(t_0,\tilde{x}_0).
\]
Remark 6.2. As it was explained in Remark 4.3, one can check that if the differential operator $L$ has continuous spectrum, then all the previous transference results are still valid.

6.5. Proof of Theorem 1.3. With Theorem 6.1 at hand we prove Theorem 1.3. For details about transference in fractional elliptic PDEs see [12], and in Harmonic Analysis and Laguerre systems see [11].

6.5.1. Transference from (2) to (7). In this case, $H^s = (\partial_t - \Delta + |x|^2 - n)^s$ in $\mathbb{R} \times \Omega = \mathbb{R} \times \Omega$ with the Laguerre measure and with zero boundary condition at infinity whereas $H^s = (\partial_t - \Delta + 2 \kappa \cdot \nabla)^s$ in $\mathbb{R} \times \Omega = \mathbb{R} \times \mathbb{R}^n$ with Gaussian measure $\pi^{-n/4} e^{-|x|^2/2} dx$. For the transference we use $h(x) = x$ and $M(x) = \pi^{-n/4} e^{-|x|^2/2}$.

6.5.2. Transference from (3) to (8). In all these examples we have $\Omega = \Omega$. As it was explained in Remark 4.3, one can check that if the different measure $H^s = (\partial_t - \Delta + |x|)^s = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} + \frac{x_i}{4}$ with measure $x^{0 \cdot 1} \cdots x^{0 \cdot n} dx$, which is related to the Laguerre system $L^s_k$, we choose $h(x) = (x^0_1, x^0_2, \ldots, x^0_n)$ and $M(x) = 2^{n/2} x^{0 \cdot 1 + 1/2} \cdots x^{0 \cdot n + 1/2}$.

• For $H^s = (\partial_t + \sum_{i=1}^{n} (x_i - \kappa_i x_i^2) + (\alpha_i + 1) \frac{\partial}{\partial x_i} + \frac{x_i}{4})^s$ with measure $x^{\alpha \cdot 1} \cdots x^{\alpha \cdot n} dx$, which is related to the Laguerre system $L^s_k$, we choose $h(x) = (x^\alpha_1, x^\alpha_2, \ldots, x^\alpha_n)$ and $M(x) = 2^{(2 \alpha)/2} x^{\alpha \cdot 1 + 1/2} \cdots x^{\alpha \cdot n + 1/2}$.

• For $H^s = (\partial_t + \sum_{i=1}^{n} (\kappa_i x_i^2 - \frac{\partial}{\partial x_i} + \frac{x_i}{4} + \frac{\kappa_i^2}{4})^s$ with measure $x^{\alpha \cdot 1} \cdots x^{\alpha \cdot n} dx$, which is related to the Laguerre system $\psi^s_k$, we choose $h(x) = x \cdot \alpha$ and $M(x) = 2^{(2 \alpha)/2} x^{\alpha \cdot 1 + 1/2} \cdots x^{\alpha \cdot n + 1/2}$.

In the last case, we start with $H^s = (\partial_t - \frac{d^2}{dx^2} + \lambda(x_1 - sin x) e^{-|x|^2/2})^s$ in $\mathbb{R} \times \Omega = \mathbb{R} \times (0, \pi)$ with Lebesgue measure, and $H^s = (\partial_t - \frac{d^2}{dx^2} - 2 \alpha \cot \frac{x}{\alpha} + \lambda^2) e^{-|x|^2/2}$ in $\mathbb{R} \times \Omega = \mathbb{R} \times (0, \pi)$ with measure $\sin^2 \lambda x dx$. For the transference method we use $h(x) = x$ and $M(x) = \sin x$.

6.5.3. Transference from (4) to (9). In this case, $H^s = (\partial_t - \frac{d^2}{dx^2} + \frac{\lambda(x_1 - sin x)}{2})^s$ in $\mathbb{R} \times \Omega = \mathbb{R} \times (0, \pi)$ with Lebesgue measure, and $H^s = (\partial_t - \frac{d^2}{dx^2} - 2 \alpha \cot \frac{x}{\alpha} + \lambda^2) e^{-|x|^2/2}$ in $\mathbb{R} \times \Omega = \mathbb{R} \times (0, \pi)$ with measure $\sin^2 \lambda x dx$. For the transference method we use $h(x) = x$ and $M(x) = \sin^2 \lambda x$.

6.5.4. Transference from (6) to (10). In this case $\Omega = \tilde{\Omega} = (0, \infty)$, $H^s = (\partial_t - \frac{d^2}{dx^2} + \lambda^2 x) e^{-|x|^2/2}$ in $\mathbb{R} \times (0, \infty)$ with Lebesgue measure and $H^s = (\partial_t - \frac{d^2}{dx^2} - 2 \alpha \cot \frac{x}{\alpha} + \lambda^2)^s$ in $\mathbb{R} \times (0, \infty)$ with measure $x^2 \lambda dx$. For the transference method we use $h(x) = x$ and $M(x) = x^\lambda$.

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