INJECTIVITY, GLOBAL AND ALMOST GLOBAL STABILITY OF HURWITZ VECTOR FIELDS.

ÁLVARO CASTAÑEDA AND VÍCTOR GUÍNEZ

Abstract. We give, in dimension three, a family of vector fields that are examples to Weak Markus–Yamabe Conjecture and simultaneously counterexamples to Markus–Yamabe Conjecture. Furthermore, we construct a family of almost Hurwitz vector fields such that the origin is almost globally stable using the novel concept of Ranter’s density functions.

1. Introduction

One of the central problems on dynamical systems is to determine conditions under which certain points or sets are attractors for some dynamics, that is, the orbits of the points converge in the future to them. In the case of continuous-time, that is, flows associated to vector fields, an infinitesimal condition that ensures that a singular point is a local attractor is the negativity of the real part of eigenvalues at that point.

Motivated for this simple observation, in [12], L. Markus and H. Yamabe establish their well known global stability conjecture.

Markus–Yamabe Conjecture (MYC). Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a $C^1$–vector field with $F(0) = 0$. If for any $x \in \mathbb{R}^n$ all the eigenvalues of the Jacobian of $F$ at $x$ have negative real part, then the origin is a global attractor of the system $\dot{x} = F(x)$.

Recall that vector fields that satisfies the hypothesis of MYC are called Hurwitz vector fields.

It is known that the MYC is true when $n \leq 2$ and false when $n \geq 3$ (see [9]). In dimension two the proofs, both the polynomial case (G. Meisters and C. Olech in [14]) as the $C^1$–case (R. Feßler in [9], A.A. Glutsyuk in [10] and C. Gutiérrez in [11]) are based on a remarkable result of C. Olech ([16]) where the author showed that MYC (in dimension two) is equivalent to

Weak Markus–Yamabe Conjecture (WMYC). Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a $C^1$–Hurwitz. Then $F$ is injective.

Then WMYC is true when $n = 2$ and to the best of our knowledge is still open for $n \geq 3$.

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We give a family of polynomial Hurwitz vector fields for which the WMYC is true and proves that some vector fields belonging to this family are counterexamples to MYC, emphasizing that the result of Olech is no longer valid in dimension three. This family is inspired on the results of L.A. Campbell about the Jacobian Conjecture on $\mathbb{R}^n$ (see [2]).

On the other hand, a new tool (density functions) introduced by A. Rantzer in [18], gives sufficient conditions to obtain the almost global stability of an equilibrium point for a $C^1$ vector field in $\mathbb{R}^n$ (all trajectories, except for a zero Lebesgue measure set of initial states, converge to the equilibrium point). In the context of satisfying certain conditions except for a zero Lebesgue measure set, recently, in [17] the authors have studied, in dimension two, almost Hurwitz vector fields, i.e. vector fields that satisfy the Hurwitz condition outside of a zero measure set. The main result in [17] is: almost Hurwitz vector fields with the origin a hyperbolic singular point are all topologically equivalent to the radial vector field $\dot{x} = -x$.

Both concepts are relate, in dimension three, by P. Monzón and R. Potrie which give the first example of an almost Hurwitz vector field $X$ where the origin is almost globally stable of the differential system generated for $X$. In this article, inspired by this example, we construct a family of almost Hurwitz vector fields such that the origin is almost globally stable using the novel concept of density function introduced by A. Rantzer.

The paper is organized as follows. In section 2 we give examples to WMYC and counterexamples to MYC. In section 3 we construct a family of almost Hurwitz vector fields in dimension three that are perturbations of a harmonic oscillator with the origin an almost globally stable singular point. We pointed out that the example 3.3 of [15] belongs this family.

2. Examples to WMYC and Counterexamples to MYC.

As we pointed out in [3, Proposition 2.3], any vector field in dimension three of the form $X = \lambda I + (P, Q, 0)$ where $\lambda \neq 0$,

$$P(x, y, z) = -b(z) f(a(z) x + b(z) y) + c(z) \quad \text{and}$$

$$Q(x, y, z) = a(z) f(a(z) x + b(z) y) + d(z)$$

with $a, b, c, d \in \mathbb{R}[z]$ and $f \in \mathbb{R}[z][t]$ is injective and thus satisfies the conclusion of WMYC. In fact, if we consider $\gamma = \frac{1}{X}$, we can explicitly find its inverse $X^{-1} = \gamma I + (R, S, 0)(x, y, z)$, where

$$R = b(\gamma z) f\left(\gamma\{a(\gamma z) x + b(\gamma z) y - a(\gamma z)c(\gamma z) - b(\gamma z)d(\gamma z)\}\right) - c(\gamma z)$$

$$S = -a(\gamma z) f\left(\gamma\{a(\gamma z) x + b(\gamma z) y - a(\gamma z)c(\gamma z) - b(\gamma z)d(\gamma z)\}\right) - d(\gamma z)$$

Remark 2.1. (i) If in (1) $\lambda < 0$ and the degree of $f$ is 1, then the vector field $X$ is an example to WMYC and MYC. (see [3] Theorem 2.5)

(ii) If we consider $\lambda = 1$, then $X = I + (P, Q, 0)$ is an example to

Jacobian Conjecture on $\mathbb{R}^n$. Every polynomial map $X : \mathbb{R}^n \to \mathbb{R}^n$ such that $\det JX = 1$ is a biyective map with a polynomial inverse.

Notice that the nonlinear part of the vector field $X$, i.e., $(P, Q, 0)$ as in [11] has nilpotent Jacobian matrix with rows and linearly dependent over $\mathbb{R}$. With respect
to find explicit inverse to vector field of the form $X = I + H$ such that the nonlinear part has nilpotent Jacobian matrix with rows and linearly independent over $\mathbb{R}$, we have a first result of L.A. Campbell in [2], the author inspired by the Jacobian Conjecture on $\mathbb{R}^n$ gives the inverse of the map with unipotent Jacobian matrix (all eigenvalues equal to 1)

$$I + H = (x + H_1, y + H_2, z + H_3)$$

where

$$H_1(x, y, z) = \phi(y - x^2),$$
$$H_2(x, y, z) = z + 2x\phi(y - x^2),$$
$$H_3(x, y, z) = -(\phi(y - x^2))^2,$$

where $\phi \in \mathbb{R}[t]$ with $t = y - x^2$. The inverse of this map is

$$I + N = (x + N_1, y + N_2, z + N_3)$$

where

$$N_1(x, y, z) = -\phi(y - x^2 - z),$$
$$N_2(x, y, z) = -z - 2x\phi(y - x^2 - z) + \phi(y - x^2 - z)^2,$$
$$N_3(x, y, z) = (\phi(y - x^2 - z))^2.$$

We shall extend this procedure to obtain the inverse of the map $\lambda I + M$ where $\lambda \in \mathbb{R} - \{0\}$ and the components $M_i, i = 1, 2, 3$ of $M$ are

$$M_1(x, y, z) = \psi(y - ax - bx^2),$$
$$M_2(x, y, z) = z + (a + 2bx)\psi(y - ax - bx^2),$$
$$M_3(x, y, z) = -b\psi(y - ax - bx^2)^2.$$

where $\psi \in \mathbb{R}[t]$ with $t = y - ax - bx^2$ and $a, b \neq 0$. Thus, by considering

$$u = \lambda x + \psi(y - ax - bx^2)$$
$$v = \lambda y + z + (a + 2bx)\psi(y - ax - bx^2)$$
$$w = \lambda z - b\psi(y - ax - bx^2)^2$$

is easy obtain that

$$\gamma v - \gamma au - \gamma^2 bu^2 - \gamma^2 w = y - ax - bx^2$$

where $\gamma = \frac{1}{\lambda}$. Now, put $\Phi = \psi(y - ax - bx^2) = \psi(-\gamma au + \gamma v - \gamma^2 bu^2 - \gamma^2 w)$. Thus $\lambda I + M = (\lambda x + \Phi, \lambda y + z + (a + 2bx)\Phi, \lambda z - b\Phi)$ and we can obtain the inverse of this map and which is denoted by

$$(\gamma I + P) = (\gamma x + P_1, \gamma y + P_2, \gamma z + P_3)$$

where
Proof. Under the linear change of coordinates \((u,v,w) = (u,v,w)_{\gamma y - \gamma ax - \gamma^2 bx^2 - \gamma^2 z)\)  

\(P_1(x,y,z) = -\gamma \psi(\gamma y - \gamma ax - \gamma^2 bx^2 - \gamma^2 z)\)  

\(P_2(x,y,z) = -\gamma^2 z - \gamma(a + 2\gamma bx)\psi(\gamma y - \gamma ax - \gamma^2 bx^2 - \gamma^2 z)\)  

\(P_3(x,y,z) = \gamma b(\psi(\gamma y - \gamma ax - \gamma^2 bx^2 - \gamma^2 z))^2\)  

(5)

Notice that the map \(X = \gamma I + P\) has the following properties:  

(P1) The Jacobian matrix \(JP\) is nilpotent and their rows are linearly independent over \(R\).  

(P2) \(P(x,y,z)\) has the form \((u(x,y,z),v(x,y,z),h(u(x,y,z),v(x,y,z)))\).  

(P3) \(\text{deg}_x(uA) = \text{deg}_x(vB)\) for all \(k \geq 1\) where \(\psi(t) = A_1 t + \cdots A_k t^k\) and \(t = y - x^2 - z\), with \(A = v_t u_z - u_t v_z\) and \(B = v_y u_z - u_y v_z\).  

(P4) Under the linear change of coordinates \((u,v,w) = (x,y - z,z)\) the vector field \(X\) is transformed into \(\hat{X} = I + \hat{P}\) where \(P\) has the property \((P2)\) however \(\text{deg}_x(uA) \neq \text{deg}_x(vB)\).  

Thus this map \(X = \lambda I + M\) with \(M\) as in \([4]\), tailored for our purposes, gives us a large family of examples to WMYC on \(R^3\).

Theorem 2.2. The vector field \(X\) with \(\gamma < 0\) satisfies the conclusion of WMYC.

Remark 2.3. We can also formulate a result about the Jacobian Conjecture on \(R^n\): the polynomial map \(I + M\) where \(\psi(t)\) is a polynomial of degree \(k \geq 1\) is injective and has explicit polynomial inverse \(I + P\).

On the other hand, we know that by \([4]\) Corollary 4.1 any vector field  

\[ Y = \lambda I + (u(x,y,z),v(x,y,z),h(u(x,y,z),v(x,y,z))) \]

where the rows of \(JH\) are linearly independent over \(R\) under the condition \(\text{deg}_x(uA) \neq \text{deg}_x(vB)\), modulo a linear change of coordinates, has the form  

\[ Y(x,y,z) = \lambda(x,y,z) + (0,v_1 z,0) + \]

\[ g(t)(1,-(b_1 + 2v_1 \alpha x),\alpha g(t)) \]

with \(t = y + b_1 x + v_1 \alpha x^2\) and \(v_1 \alpha \neq 0\), and \(g \in R[t]\) with \(g(0) = 0\) and \(\text{deg}_t g(t) \geq 1\). In \([3]\) Theorem 3.5 we have proved that any vector field of the form \((6)\) with \(g(t) = A_1 t + A_2 \frac{t^2}{2}\) satisfying the hypothesis of MYC has orbits that escape to infinity. As we have emphasized, the vector field \(X = \gamma I + P\) with \(P\) as in \([5]\) not satisfied the condition over the degrees (see \((P3)\)). However we can see that this vector field has orbits that escape to infinity and thus is a vector field that is an example to WMYC and counterexample to MYC, emphasizing that in dimension three both conjectures are not equivalent as if it is in dimension two.

Theorem 2.4. Consider the vector field \(X = \gamma I + P\) with \(\gamma < 0\) and where \(\psi(t) = A_1 t + A_2 \frac{t^2}{2}\) with \(A_2 \neq 0\). Then \(X\) has orbits that escape to infinity.

Proof. Under the linear change of coordinates  

\((u,v,w) = (\gamma x - \psi(t),t,-\gamma^3(z + bx^2))\)  

where \(t = \gamma y - \gamma ax - \gamma^2 bx^2 - \gamma^2 z\), the vector field \(X\) is written as
$X(x, y, z) = \gamma(x, y, z) + (-\psi'(y)(\gamma y + z), z, -b\gamma^2 x^2)$.  

To find orbits of $X$ that escape to infinity, we first make the coordinate change  
\[
(u, v, w) = \frac{1}{z}(x, y, 1).
\]

If $Y$ is the vector field $X$ in the new coordinates, then $Z = wY$ is defined by  
\[
Z(u, v, w) = (b\gamma^2 u^3 - (A_1 w + A_2 v)(\gamma v + 1), +b\gamma^2 u^2 v + w, \gamma w(-w + b\gamma^2 u^2)).
\]

For $w \neq 0$, the vector fields $Y$ and $Z$ have the same orbits. Moreover, for $w > 0$ (resp. $w < 0$), the orbits of $Y$ and $Z$ have the same (resp. inverse) orientation.

Now, we apply the blow-up  
\[
(s, q, p) = \left(u, \frac{v}{u^3}, \frac{w}{u^5}\right).
\]

If $Y_1$ is the vector field $Y$ in the new coordinates, then  
\[
Y_1(s, q, p) = A(s, q, p)(s, -3q, -5p) + (0, p + b\gamma^2 q, \gamma p(b\gamma - ps^3))
\]
with $A(s, q, p) = b\gamma^2 - (A_1 ps^2 + A_2 q)(\gamma qs^3 + 1)$.

We must calculate the singularities of $Z_1$ over the plane $s = 0$. Is easy to see that the singularities are  
\[
(0, 0, 0), \quad \left(0, \frac{3b\gamma^2}{3A_2}, 0\right), \quad \text{and} \quad \left(0, \frac{4b\gamma^2}{5A_2}, -\frac{8b^2\gamma^4}{25A_2}\right).
\]

The Jacobian matrix of $Z_1$ at $(0, \frac{3b\gamma^2}{3A_2}, 0, 0)$ has eigenvalues  
\[
\mu_1 = \frac{4b\gamma^2}{5}, \quad \mu_2 = 2b\gamma^2.
\]

Therefore, if $b > 0$ (resp. $b < 0$), this singularity is a repeller (resp. attractor) of the vector field $Z_1$. Given an initial condition $(s(0), q(0), p(0))$ sufficiently close to the singularity, with $s(0)p(0) > 0$ (resp. $s(0)p(0) < 0$) for $b > 0$ (resp. $b < 0$), we obtain an orbit of the original vector field $X$ that escapes positively to infinity. \hfill \Box

On the other hand, with respect to an one of the open problem on MYC, in \cite{1} the authors found a Hurwitz analytic vector field (injective) in dimension $n \geq 4$ with a non-constant periodic solution. In dimension $n = 3$ there no exists counterexamples to MYC with a non-constant periodic solution, all counterexamples to this conjecture are polynomial vector fields that have an orbit that scape to infinity. In fact, in \cite{13}, G. Meisters says that such a non-constant periodic orbit is surely impossible for polynomial Hurwitz systems in dimension $n \geq 3$. (This remains to be proved.) A tool for answer this problem would be studied of dynamics of the vector fields of the form

\begin{align*}
\dot{x} &= -y + \varepsilon P(x, y, z) \\
\dot{y} &= x + \varepsilon Q(x, y, z) \\
\dot{z} &= \varepsilon R(x, y, z)
\end{align*}  

where $P, Q, R$ are polynomials of degree $k \geq 1$ and $\varepsilon \neq 0$. However, we can see that this kind of vector fields have different behaviours and not necessarily we can obtain a polynomial Hurwitz vector field in dimension three with a non-constant
periodic orbit. In fact, there exist examples where the origin is global attractor for a Hurwitz polynomial vector field of this form.

**Proposition 2.5.** Let \( X(x, y, z) = (-y - \varepsilon x, x - \varepsilon y, \varepsilon(-z + x^2 + y^2)) \) a Hurwitz vector field. Then the origin is a global attractor for the system generated for \( X \).

**Proof.** The vector field \( X \) is linear quasi-homogeneous with weights \((\alpha_1, \alpha_2, \alpha_3) = (1, 1, 2)\). Since \( \alpha_i \alpha_j > 0 \) for all \( i, j = 1, 2, 3 \), and the origin is locally asymptotically stable. By [7, Proposition 2.2], the origin is global attractor for the differential system generated for \( X \).

\[ \square \]

As discussed in the following section, there exist other results about the vector field of the form (7) which lead us to recall the novel concepts of Almost Hurwitz vector field and Almost global stability. These results are far from an affirmative answer to the Meisters's problem in dimension three.

### 3. Almost Global Stability of an Almost Hurwitz Vector Fields.

Notice that the counterexamples in dimension three to MYC (see [3], [5]) have a open set of initial states of positive measure such that the trajectories not converge to the origin. Thus, the origin is not almost global attractor in the sense following.

**Definition 3.1.** Consider the differential equation

\[ \dot{x} = f(x) \]

where \( f : \mathbb{R}^n \to \mathbb{R}^n \) such that \( f \in C^1 \) and \( f(0) = 0 \). We say the origin is almost global attractor if all trajectories, except for a zero Lebesgue measure set of initial states, converge to the origin.

Thus we are led to posing the following.

**Question.** There exist Hurwitz vector fields such that the origin is almost global attractor ?

To a first approximation to an answer to this problem, we must recall the following concepts.

**Definition 3.2.** A vector field \( X : \mathbb{R}^n \to \mathbb{R}^n \) is called almost Hurwitz vector field if is Hurwitz except in a zero Lebesgue measure set.

Both concepts have been studied recently. Indeed, in [17], the authors have studied planar vector fields which are almost Hurwitz proving that if the vector field \( X \) is almost Hurwitz and if the origin is a hyperbolic singularity then \( X \) is topologically equivalent to the radial vector field. In [15], the authors give the first example of an almost Hurwitz vector field \( X \) in dimension three where the origin is almost global stable of the differential system generated for \( X \). Namely, the example is the following:

\[ X(x, y, z) = (y - 2xz^2, -x - 2yz^2, -z^3). \]

The main tool, in [15], used by the authors for show the almost global stability of the origin is the novel concept of *density functions*. 
**Theorem 3.5.** All trajectories converge to the origin, i.e., the origin is almost globally stable.

**Theorem 3.4.** Given the differential system

\[ \dot{x} = f(x) \]

where \( f \in C^1 \), \( f(0) = 0 \), suppose there exists a density function \( \rho : \mathbb{R}^n - \{0\} \rightarrow [0, +\infty) \) such that \( \rho(x) f(x) / \|x\| \) is integrable on \( \{x \in \mathbb{R}^n : \|x\| \geq 1\} \). Then, almost all trajectories converge to the origin, i.e., the origin is almost globally stable.

Now, we present our main result of this section.

**Theorem 3.5.** Consider the real numbers \( \alpha > 1 \), \( c < a < 0 \leq b \), and \( d \geq 3 \) odd such that \( \alpha > \frac{a + c}{2a} \) and \( 2d \leq 2\alpha + 1 \). Let the vector field

\[ X(x, y, z) = (y, -x, 0) + ((ax + by)R(z)^{d-1}, (-bx + cy)R(z)^{d-1}, -R(z)^d) \]

where \( R(z) \) is a function of the class \( C^1 \) such that \( R(0) = 0 \). If

(A1) \( 0 < R'(z) \leq \frac{a(1-2\alpha)c}{d} \) for all \( z \in \mathbb{R} \).

(A2) \( dR'(z) - \frac{2\alpha R(z)}{z} < a + c \) for all \( z \in \mathbb{R} - \{0\} \).

Then

(i) The vector field \( X \) is almost Hurwitz vector field (Hurwitz except in the plane \( z = 0 \)).

(ii) The origin is almost global stable for the differential system generated by \( X \). Furthermore, the density function associated to this vector field is

\[ \rho(x, y, z) = (x^2 + y^2 + z^2)^{-\alpha}. \]

**Proof:**

(i) We show that the vector field \( X \) is almost Hurwitz vector field by using the Routh–Hurwitz stability criterion. Notice that the Jacobian Matrix of \( X \) is

\[ JX(x, y, z) = \begin{pmatrix} aR(z)^{d-1} & 1 + bR(z)^{d-1} & * \\ -(1 + bR(z)^{d-1}) & cR(z)^{d-1} & * \\ 0 & 0 & -dR(z)^{d-1}R'(z) \end{pmatrix}, \]

the trace of \( JX \) is \( T = R(z)^{d-1}(a + c - dR'(z)) \), the determinant of \( JX \) is \( D = -dR(z)^{d-1}R'(z)(acR(z)^{2d-2} + (1 + bR(z)^{d-1})^2) \) and the sum of minors determinants is \( M = acR(z)^{2d-2} + (1 + bR(z)^{d-1})^2 - (a + c)dR(z)^{2d-2}R'(z) \). Thus is easy to see \( T \) and \( D \) are negative except on \( z = 0 \) where the trace and determinant are zero. The third condition of criterion requires the positivity of \( -T \cdot M + D \) which is equal to

\[ -(a + c)R(z)^{d-1}\{acR(z)^{2d-2} + (1 + bR(z)^{d-1})^2 - d(a + c)R(z)^{2d-2}R'(z) + (dR(z)^{d-1}R'(z))^2\}. \]
This condition is satisfied except on $z = 0$ where $-T \cdot M + D \equiv 0$. Therefore, we have proved the part (i).

(ii) We use for the proof of this item the Rantzer’s result (Theorem 3.4). Firstly, if we consider that the density function

$$\rho(x, y, z) = (x^2 + y^2 + z^2)^{-\alpha},$$

by the condition $2d \leq 2\alpha + 1$ we ensure the integrability of $\rho(x)\|X\|/\|x\|$ . Secondly, we will show that $\nabla \cdot (\rho X)(x)$ is positive almost everywhere in $\mathbb{R}^3$.

$$\nabla \cdot (\rho X)(x, y, z) = (x^2 + y^2 + z^2)^{-1-\alpha}R(z)^{-1+d}$$

$$\left\{ (a + c)(x^2 + y^2 + z^2) - 2(ax^2 + cy^2)\alpha + 2z\alpha R(z) - d(x^2 + y^2 + z^2)R'(z) \right\}$$

$$= (x^2 + y^2 + z^2)^{-1-\alpha}R(z)^{-1+d}$$

$$\left\{ x^2(a + c - dR'(z) - 2\alpha a) + y^2(a + c - dR'(z) - 2\alpha c) + 2\alpha z R(z) + z^2(a + c - dR'(z)) \right\}.$$ 

By the conditions (A1)-(A2) we have that $\nabla \cdot (\rho X)(x, y, z)$ is positive almost everywhere (except on $z = 0$).

Remark 3.6. (i) The vector field (9) satisfies this conditions (A1–A2) with $a, c = -2, b = 0, d = 3$ and $R(z) = z$. Notice that that the density function associated to the vector field (9) is $\rho(x, y, z) = (x^2 + y^2 + z^2)^{-4}$.

(ii) The origin is an isolated not asymptotically stable equilibrium point for this family of vector fields.

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E-mail address: castaneda@u.uchile.cl, victor.guinez@usach.cl