Interacting spinor and scalar fields in Bianchi cosmology

Bijan Saha

Laboratory of Information Technologies
Joint Institute for Nuclear Research, Dubna
141980 Dubna, Moscow region, Russia

(Dated: October 30, 2018)

A self-consistent system of interacting spinor and scalar fields is considered within the scope of Bianchi type VI cosmological model filled with a perfect fluid. The contribution of the cosmological constant ($\Lambda$-term) is taken into account as well. Exact self-consistent solutions to the field equations are obtained for a special choice of spatial inhomogeneity and the interaction terms of the spinor and scalar fields. It has been found that some special choice of metric functions can give rise to a singularity-free solutions independent of the value and sign of the $\Lambda$ term. It is also shown that the introduction of a positive $\Lambda$, the most widespread kind of dark energy, leads to the rapid growth of the universe, while the negative one, corresponding to an additional gravitational energy gives rise to an oscillatory or non-periodic mode of expansion. The role of the spatial inhomogeneity in the evolution of the universe is clarified within the scope of the considered models.

PACS numbers: 04.20.Ha, 03.65.Pm, 98.80.Cq
Keywords: Spinor field, cosmological constant, Bianchi type-VI universe

I. INTRODUCTION

The role of the spinor field in the evolution of the Universe has been studied by a number of authors \cite{1, 2, 3, 4, 5, 6, 7}. A principal goal of those studies was to find out singularity-free solutions to the corresponding field equations. The gravitational field in these cases was given by an anisotropic Bianchi type-I (BI) cosmological model. It has been found that the introduction of the nonlinear spinor field in some special cases results in a rapid growth of the Universe. In light of that study many authors believe it is possible to consider the spinor field as one of the candidates to explain the late time accelerated mode of expansion \cite{8, 9, 10}.

Though as an anisotropic cosmological model cosmologists basically consider Bianchi type-I space-time, there are still a few other models that describe an anisotropic space-time and generate particular interest among physicists \cite{11, 12, 13, 14, 15, 16, 17, 18}. In Ref. \cite{12} methods of dynamical systems analysis were used to show that the presence of a magnetic field orthogonal to the two commuting Killing vector fields in any spatially homogeneous Bianchi type $VI_0$ vacuum solution to Einstein’s equation changes the evolution towards the singularity from collapse to bounce. The authors in Ref. \cite{13} studied the problem of isotropization of scalar field Bianchi models with an exponential potential(s). Other papers mentioned above are devoted to tilted perfect fluid solutions, chaotic singularities, and conditional symmetries.

*Electronic address: bijan@jinr.ru URL: http://www.jinr.ru/~bijan/
In a recent paper [19] we studied the self-consistent system of the nonlinear spinor field and an anisotropic inhomogeneous gravitational field in order to clarify the role of the spinor field nonlinearity and the space-time inhomogeneity in the formation of a singularity-free universe. As an anisotropic space-time we chose a Bianchi type-VI (BVI) model, since a suitable choice of its parameters yields a few other Bianchi models including BI and FRW universes. It can be noted that unlike the BI universe, the BVI space-time is inhomogeneous. Inclusion of inhomogeneity in the gravitational field significantly complicates the search for an exact solution to the system. The purpose of this paper is to study a self-consistent system of spinor, scalar and Bianchi type-VI gravitational fields in presence of a perfect fluid and a cosmological constant and study the role of corresponding material fields in the evolution of the universe.

II. BASIC EQUATIONS AND THEIR GENERAL SOLUTIONS

We shall investigate a self-consistent system of nonlinear spinor and Einstein gravitational fields. These two fields are to be determined by the following action:

$$\mathcal{S}(g; \psi, \bar{\psi}) = \int \mathcal{L} \sqrt{-g} d\Omega$$

with

$$\mathcal{L} = \mathcal{L}_g + \mathcal{L}_{sp} + \mathcal{L}_{sc} + \mathcal{L}_{int} + \mathcal{L}_{pf}.$$ 

The gravitational part of the Lagrangian (2.2), $\mathcal{L}_g$ is given by a Bianchi type-VI (BVI hereafter) space-time, whereas the terms $\mathcal{L}_{sp}$, $\mathcal{L}_{sc}$, and $\mathcal{L}_{int}$ describe the spinor and scalar field Lagrangian and an interaction between them, respectively. The term $\mathcal{L}_{pf}$ describes the Lagrangian density of the perfect fluid which minimally couples to the spinor and scalar fields through gravitational one.

A. Matter field Lagrangian

For a spinor field $\psi$, the symmetry between $\psi$ and $\bar{\psi}$ appears to demand that one should choose the symmetrized Lagrangian [20]. Keeping this in mind we choose the spinor field Lagrangian as

$$\mathcal{L}_{sp} = \frac{i}{2} \left[ \bar{\psi} \gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \gamma^\mu \psi \right] - M \bar{\psi} \psi.$$ 

Here $M$ is the spinor mass, $\nabla_\mu$ is the covariant derivatives acting on a spinor field as [21, 22]

$$\nabla_\mu \psi = \frac{\partial \psi}{\partial x^\mu} - \Gamma_\mu \psi, \quad \nabla_\mu \bar{\psi} = \frac{\partial \bar{\psi}}{\partial x^\mu} + \bar{\psi} \Gamma_\mu,$$

where $\Gamma_\mu$ are the Fock-Ivanenko spinor connection coefficients defined by

$$\Gamma_\mu = \frac{1}{4} \gamma^\alpha \left( \Gamma^\nu_{\mu \sigma} \gamma_\nu - \partial_\mu \gamma_\sigma \right).$$

The massless scalar field Lagrangian is chosen to be

$$\mathcal{L}_{sc} = \frac{1}{2} \varphi, \varphi^\alpha.$$
The interaction between the spinor and scalar fields is given by the Lagrangian \[ L_{\text{int}} = \frac{\lambda}{2} \Phi \alpha \Phi^\alpha F. \] (2.7)

Here \( \lambda \) is the self-coupling constant and \( F = F(I,J) \) is some arbitrary functions of invariants \( I = S^2 \) and \( J = P^2 \) generated from the real bilinear forms \( S = \bar{\psi} \psi \) and \( P = i \bar{\gamma}^\alpha \psi \) of the spinor field.

The contribution of the perfect fluid to the system is performed by means of its energy-momentum tensor, which acts as one of the sources of the corresponding gravitational field equations. So here we do not need to write the Lagrangian density \( L_{\text{pf}} \) explicitly. The reason for writing \( L_{\text{pf}} \) in Eqs. (2.1) and (2.2) is to underline that we are dealing with a self-consistent system. An interesting discussion on the action and Lagrangian for a perfect fluid can be found in Refs. [23, 24, 25].

### B. The gravitational field

The gravitational part of the Lagrangian in (2.2) has the form:

\[ L_{\text{grav}} = \frac{R}{2\kappa}, \] (2.8)

Here \( R \) is the scalar curvature and \( \kappa \) is Einstein’s gravitational constant. The gravitational field in our case is given by a BVI metric:

\[ ds^2 = dt^2 - a^2 e^{-2mz} dx^2 - b^2 e^{2nz} dy^2 - c^2 dz^2, \] (2.9)

with \( a, b, c \) being functions of time only. Here \( m, n \) are some arbitrary constants and the velocity of light is taken to be unity. It should be emphasized that the BVI metric models a universe that is anisotropic and inhomogeneous. A suitable choice of \( m, n \) as well as the metric functions \( a, b, c \) in the BVI metric given by (2.9) generates the following Bianchi-type universes: (i) for \( m = n \) the BVI metric transforms into a Bianchi-type V (BV) universe; (ii) for \( n = 0 \) the BVI metric transforms into a Bianchi-type III (BIII) universe; (iii) for \( m = n = 0 \) the BVI metric transforms into a Bianchi-type I (BI) universe and finally, (iv) for \( m = n = 0 \) and an equal scale factor in all three directions the BVI metric transforms into a FRW universe.

The metric (2.9) has the following nontrivial components of Riemann and Ricci tensors:

\[
\begin{align*}
R_{01}^{01} &= -\frac{a}{a}, & R_{02}^{02} &= -\frac{b}{b}, & R_{03}^{03} &= -\frac{c}{c}, \\
R_{12}^{12} &= -\frac{mn}{c^2} - \frac{a b}{a b}, & R_{13}^{13} &= \frac{m^2}{c^2} - \frac{c}{c a}, & R_{23}^{23} &= \frac{n^2}{c^2} - \frac{b}{b c}, \\
R_3^3 &= \left( \frac{a}{m} - \frac{n}{b} - \frac{(m-n) c}{c} \right), \\
R_0^0 &= - \left( \frac{\dot{a}}{a^2} + \frac{\dot{b}}{b^2} + \frac{\dot{c}}{c^2} \right), \\
R_1^1 &= - \left( \frac{\dot{a}}{a^2} + \frac{\dot{b}}{b^2} + \frac{\dot{c}}{c^2} \right), \\
R_2^2 &= - \left( \frac{\dot{b}}{b} + \frac{\dot{a}}{a b} + \frac{\dot{b}}{b c} - \frac{m^2 - mn}{c^2} \right), \\
R_3^3 &= - \left( \frac{\dot{c}}{c} + \frac{\dot{a}}{a c} + \frac{\dot{b} c}{b c} - \frac{m^2 + n^2}{c^2} \right).
\end{align*}
\]
We write the components of the Riemann and Ricci tensors as invariant characteristics of space-time which one needs to know in order to investigate the existence of a singularity (singular point) are composed of these tensors together with the metric tensor. Although in 4D Riemann space there are 14 independent invariants [5, 26], it is sufficient to study only three of them, namely the scalar curvature $I_1 = R$, $I_2 = R_{\mu\nu}R^{\mu\nu}$ and the Kretschmann scalar $I_3 = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ [27, 28]. From the Riemann and Ricci tensors written above one finds

$$I_1 = R = -\frac{2}{\tau} \left[ \dot{\tau} - \dot{a} \dot{b} c - \dot{a} \dot{b} \dot{c} - \frac{ab}{c} (m^2 - mn + n^2) \right],$$  \hspace{1cm} (2.10a)

$$I_2 = (R_{00})^2 + (R_{11})^2 + (R_{22})^2 + (R_{33})^2 + R_{03} R_{30},$$  \hspace{1cm} (2.10b)

$$I_3 = 4 [(R_{01})^2 + (R_{02})^2 + (R_{03})^2 + (R_{12})^2 + (R_{31})^2 + (R_{23})^2],$$  \hspace{1cm} (2.10c)

where we define

$$\tau = abc.$$  \hspace{1cm} (2.11)

From (2.10) it follows that $I_1 \propto 1/\tau$, $I_2 \propto 1/\tau^2$, and $I_3 \propto 1/\tau^2$. Note that the remaining 11 invariants are composed of two or more Ricci and/or Riemann tensors and hence are inversely proportional to $(\tau)^r$, where $r$ is the number of tensors in the corresponding invariant. Thus we see that at any space-time point where $\tau = 0$, the invariants $I_1, I_2, I_3$ become infinity; hence the space-time becomes singular at this point.

C. Field equations

Let us now write the field equations corresponding to the action (2.1).

Variation of Eq. (2.1) with respect to the spinor field $\psi(\bar{\psi})$ gives the following spinor field equations:

$$i \gamma^\mu \nabla_\mu \psi - M \psi + D \psi + G i \gamma^5 \psi = 0,$$  \hspace{1cm} (2.12a)

$$i \nabla_\mu \bar{\psi} \gamma^\mu - \bar{D} \bar{\psi} - G i \bar{\psi} \gamma^5 = 0,$$  \hspace{1cm} (2.12b)

where we use the notation

$$D = \lambda S \phi, \alpha \phi, \alpha \frac{\partial F}{\partial I} = \frac{\lambda}{2} \phi, \alpha \phi, \alpha \frac{\partial F}{\partial S}, \quad G = \lambda P \phi, \alpha \phi, \alpha \frac{\partial F}{\partial J} = \frac{\lambda}{2} \phi, \alpha \phi, \alpha \frac{\partial F}{\partial P}.$$  

Since the nonlinearity in the foregoing equations is generated by the interacting scalar field, Eqs. (2.12) can be viewed as spinor field equations with induced nonlinearity.

Variation of Eq. (2.1) with respect to the scalar field yields the following scalar field equation:

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} \left( \sqrt{-g} g^{\nu\mu} (1 + \lambda F) \phi, \mu \right) = 0.$$  \hspace{1cm} (2.13)

Finally, varying Eq. (2.1) with respect to metric tensor $g_{\mu\nu}$ one finds the Einstein’s field equations. On account of the $\Lambda$ term they have the form

$$R^\nu_\mu - \frac{1}{2} \delta^\nu_\mu R = \kappa T^\nu_\mu + \delta^\nu_\mu \Lambda.$$  \hspace{1cm} (2.14)
where $R^\mu_\nu$ is the Ricci tensor, $R$ is the Ricci scalar, and $T^\mu_\nu$ is the energy-momentum tensor of the matter fields. In our case, where space-time is given by a BVI metric (2.9), the equations for the metric functions $a, b, c$ read

\[
\begin{align*}
\ddot{b} + \frac{\dot{c}}{c} + \frac{\dot{b}}{b} - \frac{n^2}{c^2} &= \kappa T_{1}^1 + \Lambda, \\
\ddot{c} + \frac{\dot{a}}{a} + \frac{\dot{c}}{c} &= \kappa T_{2}^2 + \Lambda, \\
\ddot{a} + \frac{\dot{b}}{b} + \frac{\dot{ab}}{a} &= \kappa T_{3}^3 + \Lambda, \\
\dot{a} - \frac{\dot{b}}{b} - \frac{\dot{a}b}{a} &= \kappa T_{0}^0 + \Lambda, \\
\frac{m}{a} - \frac{n}{b} - \frac{mn}{c} &= \kappa T_{3}^0.
\end{align*}
\] (2.15)

Here over dots denote differentiation with respect to time ($t$). The energy-momentum tensor of the material field $T^\nu_\mu$ is given by

\[
T^\nu_\mu = T^\nu_\mu^{(\text{sp})} + T^\nu_\mu^{(\text{sc})} + T^\nu_\mu^{(\text{int})} + T^\nu_\mu^{(\text{pf})}.
\] (2.16)

Here $T^\nu_\mu^{(\text{sp})}$ is the energy momentum tensor of the spinor field defined by

\[
T^\rho_\mu^{(\text{sp})} = i\frac{g^{\rho\nu}}{4}\left(\bar{\psi}\gamma_\mu \nabla_\nu \psi + \bar{\psi}\gamma_\nu \nabla_\mu \psi - \nabla_\mu \bar{\psi}\gamma_\nu \psi - \nabla_\nu \bar{\psi}\gamma_\mu \psi\right) - \delta^{(\mu}_{\rho}\mathcal{L}_{\text{sp}}.
\] (2.17)

The term $\mathcal{L}_{\text{sp}}$ in view of Eq. (2.12) takes the form

\[
\mathcal{L}_{\text{sp}} = -(\mathcal{D}S + \mathcal{D}P).
\] (2.18)

The energy momentum tensor of the scalar field is given by

\[
T^\nu_\mu^{(\text{sc})} = \varphi_\mu \varphi^\nu - \delta^\nu_{\mu}\mathcal{L}_{\text{sc}}.
\] (2.19)

For the interaction field we find

\[
T^\nu_\mu^{(\text{int})} = \lambda F \varphi_\mu \varphi^\nu - \delta^\nu_{\mu}\mathcal{L}_{\text{int}}.
\] (2.20)

$T^\nu_\mu^{(\text{pf})}$ is the energy momentum tensor of a perfect fluid. For a Universe filled with a perfect fluid, in a comoving system of reference such that $u^\mu = (1, 0, 0, 0)$ we have

\[
T^\nu_\mu^{(\text{pf})} = (p + \epsilon)u_\mu u^\nu - \delta^\nu_{\mu} p = (\epsilon, -p, -p, -p).
\] (2.21)

The energy $\epsilon$ and the pressure $p$ of the perfect fluid obey the following equation of state:

\[
p = \zeta \epsilon,
\] (2.22)

where $\zeta$ is a constant and lies in the interval $\zeta \in [0, 1]$. Depending on its numerical value, $\zeta$ describes the following types of Universes [29]: (i) $\zeta = 0$ (dust Universe); (ii) $\zeta = 1/3$ (radiation Universe); (iii) $\zeta \in (1/3, 1)$ (hard Universes) and (iv) $\zeta = 1$ (Zel’dovich Universe or stiff matter).
Here we note once again that the perfect fluid is minimally coupled to the system. Being one of its sources, the perfect fluid leaves its trace on the gravitational field which, in turn, influences the behavior of the spinor and scalar fields.

From (2.5) we find the following spinor connections for the metric (2.9)
\[ \Gamma_0 = 0, \quad \Gamma_1 = \frac{1}{2} \tilde{\gamma}^1 [\tilde{\alpha} \tilde{\gamma}^0 - m \frac{a}{c} \tilde{\gamma}^3] e^{-mc}, \quad \Gamma_2 = \frac{1}{2} \tilde{\gamma}^2 [\tilde{b} \tilde{\gamma}^0 + n \frac{b}{c} \tilde{\gamma}^3] e^{nz}, \quad \Gamma_3 = \frac{1}{2} \tilde{c} \tilde{\gamma}^3 \tilde{\gamma}^0. \]

It is easy to show that
\[ \gamma^\mu \Gamma^\mu = -\frac{1}{2} \frac{\tau}{\tau} \tilde{\gamma}^0 + \frac{m-n}{2c} \tilde{\gamma}^3. \]

The Dirac matrices \( \gamma^\mu(x) \) of the curved space-time are connected with those of Minkowski space-time as follows:
\[ \gamma^0 = \bar{\gamma}^0, \quad \gamma^1 = \bar{\gamma}^1 e^{mc/a}, \quad \gamma^2 = \bar{\gamma}^2 e^{nz}, \quad \gamma^3 = \bar{\gamma}^3 / c, \]
with
\[ \bar{\gamma}^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \bar{\gamma}^1 = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \bar{\gamma}^5 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \]
where \( \sigma^i \) are the Pauli matrices:
\[ \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

Let us consider the spinors and the scalars to be functions of \( t \) only, i.e.,
\[ \psi = \psi(t), \quad \phi = \phi(t) \quad (2.23) \]
Under this assumption for the scalar field from (2.13) we find
\[ \phi = C_{sc} \int \frac{dt}{\tau (1 + \lambda \mathcal{F})}, \quad C_{sc} = \text{const.} \quad (2.24) \]
For the spinor field from (2.12) we obtain
\[ \tilde{\gamma}^0 \left( \dot{\psi} + \frac{t}{2\tau} \psi \right) - \left( \frac{m-n}{2c} \right) \tilde{\gamma}^3 \psi + i \Phi \psi + \mathcal{G} \tilde{\gamma}^5 \psi = 0, \quad (2.25a) \]
\[ \left( \dot{\psi} + \frac{t}{2\tau} \psi \right) \tilde{\gamma}^0 - \left( \frac{m-n}{2c} \right) \tilde{\gamma}^3 - i \Phi \tilde{\psi} - \mathcal{G} \tilde{\gamma}^5 = 0. \quad (2.25b) \]
Here we define \( \Phi = M - \mathcal{G} \). Let us introduce a new function
\[ u_j(t) = \sqrt{\tau} \psi_j(t). \]
Then for the components of the nonlinear spinor field from (2.25), one obtains
\[ \dot{u}_1 + i \Phi u_1 - \left[ \frac{m-n}{2c} + \mathcal{G} \right] u_3 = 0, \quad (2.26a) \]
\[ \dot{u}_2 + i \Phi u_2 + \left[ \frac{m-n}{2c} - \mathcal{G} \right] u_4 = 0, \quad (2.26b) \]
\[ \dot{u}_3 - i \Phi u_3 - \left[ \frac{m-n}{2c} - \mathcal{G} \right] u_1 = 0, \quad (2.26c) \]
\[ \dot{u}_4 - i \Phi u_4 + \left[ \frac{m-n}{2c} + \mathcal{G} \right] u_2 = 0. \quad (2.26d) \]
Using the spinor field equations (2.12) and (2.25), it can be shown that the bilinear spinor forms

\begin{align*}
S &= \bar{\psi}\psi = \bar{v}v, \\
P &= i\bar{\psi}\gamma^5\psi = i\bar{v})^5v, \\
A^0 &= \bar{\psi}\gamma^5\gamma^0\psi = \bar{v}\gamma^5\gamma^0v, \\
A^3 &= \bar{\psi}\gamma^5\gamma^3\psi = \bar{v}\gamma^5\gamma^3v, \\
V^0 &= \bar{\psi}\gamma^0\psi = \bar{v}\gamma^0v, \\
V^3 &= \bar{\psi}\gamma^3\psi = \bar{v}\gamma^3v, \\
Q^{30} &= i\bar{\psi}\gamma^3\gamma^{30} = i\bar{v}\gamma^3\gamma^{30}v, \\
Q^{21} &= \bar{\psi}\gamma^2\gamma^{30} = \bar{v}\gamma^2\gamma^{30}v,
\end{align*}

obey the following system of equations:

\begin{align*}
\dot{S} - 2\mathcal{G}A^0 &= 0, \\
\dot{P} - 2\Phi A^0 &= 0, \\
\dot{A}^0 - \frac{m-n}{c}A^3 + 2\Phi P + 2\mathcal{G}S &= 0, \\
\dot{A}^3 - \frac{m-n}{c}A^0 &= 0, \\
\dot{V}^0 - \frac{m-n}{c}V^3 &= 0, \\
\dot{V}^3 - \frac{m-n}{c}V^0 + 2\Phi Q^{30} - 2\mathcal{G}Q^{21} &= 0, \\
\dot{Q}^{30} - 2\Phi V^3 &= 0, \\
\dot{Q}^{21} + 2\mathcal{G}V^3 &= 0,
\end{align*}

where we use the notation \( F_0 = \tau F \). Combining these equations and taking the first integral one gets

\begin{align*}
(S_0)^2 + (P_0)^2 + (A^0)^2 - (A^3)^2 &= C_1 = \text{const.}, \\
(V^3)^2 + (Q^{30})^2 + (Q^{21})^2 - (V^0)^2 &= C_2 = \text{const.}
\end{align*}

Now let us solve the spinor field equations (2.26). From the first and the third equations of the system (2.26) one finds

\[ \dot{u}_{13} = (\mathcal{G} - Q)u_{13}^2 - 2i\Phi u_{13} + (\mathcal{G} + Q), \]

where, we denote \( u_{13} = u/\sqrt{3} \) and \( Q = [m-n]/2c \). Equation (2.29) is of the Riccati type [30] with variable coefficients. Transformation \[ 31 \]

leads from the general Riccati equation (2.29) to a second order linear one, namely,

\[ (\mathcal{G} - Q)\psi_{13} + [2i\Phi(\mathcal{G} - Q) - \mathcal{G} + \Phi]\psi_{13} + (\mathcal{G} - Q)^2(\mathcal{G} + Q)\psi_{13} = 0. \]

Sometimes it is easier to solve a linear second order differential equation than a first order nonlinear equation. Here we give a general solution to (2.29). For this purpose we rewrite (2.29) in the form

\( \dot{w}_{13} = (\mathcal{G} - Q)w_{13}^2 e^{-2i\int\Phi(t)dt} + (\mathcal{G} + Q)e^{2i\int\Phi(t)dt}, \)

where we set \( u_{13} = w_{13}\exp[-2i\int\Phi(t)dt] \). This is an inhomogeneous nonlinear differential equation for \( w_{13} \). Then we have the homogeneous part of (2.32), i.e.,

\[ \dot{w}_{13} = (\mathcal{G} - Q)w_{13}^2 \exp\left(-2i\int\Phi(t)dt\right) \]
reads
\[ w_{13} = - \left[ \int (\mathcal{G} - Q) \exp \left( -2i \int \Phi(t) dt \right) dt + C \right]^{-1}, \quad (2.34) \]

where \( C \) is an arbitrary constant. Then the general solution to the inhomogeneous Eqn. (2.32) can be presented as
\[ w_{13} = - \left[ \int (\mathcal{G} - Q) \exp \left( -2i \int \Phi(t) dt \right) dt + C(t) \right]^{-1}, \quad (2.35) \]

with the time dependent parameter \( C(t) \) to be determined from
\[ \dot{C} = \left[ \int (\mathcal{G} - Q) \exp \left( -2i \int \Phi(t) dt \right) dt + C(t) \right]^{2} (\mathcal{G} + Q)e^{2i \int \Phi(t) dt}. \quad (2.36) \]

Thus given a concrete nonlinear term in the Lagrangian and the solutions of the Einstein equations, one finds the relation between \( u_1 \) and \( u_3 \) (\( u_2 \) and \( u_4 \) as well), hence the components of the spinor field.

Now we study the Einstein equations (2.15). In doing so, we write the components of the energy-momentum tensor, which in our case read
\[ T^0_0 = MS + \frac{1}{2} \dot{\phi}^2 (1 + \lambda F) + \epsilon, \quad (2.37a) \]
\[ T^1_1 = T^2_2 = T^3_3 = \mathcal{G}S + \mathcal{G}P - \frac{1}{2} \dot{\phi}^2 (1 + \lambda F) - p. \quad (2.37b) \]

Let us demand the energy-momentum tensor to be conserved, i.e.,
\[ T^\nu_{\nu, \mu} = T^\mu_\nu, \Gamma^\nu_\beta \mu + \Gamma^\beta_\nu \mu T^\mu_\beta - \Gamma^\beta_\nu \mu T^\mu_\beta = 0. \quad (2.38) \]

Taking into account that \( T^\nu_\mu \) is a function of \( t \) only and \( T^1_1 = T^2_2 = T^3_3 \), from (2.38) we find
\[ \frac{\dot{T}^0_0}{\tau} + \frac{\dot{T}^1_1}{\tau} \left( T^0_0 - T^1_1 \right) = 0. \quad (2.39) \]

In view of the scalar field equation and \( \Phi \dot{\mathcal{G}} - \mathcal{G} \dot{P} = 0 \) which follows from (2.27) the Eq. (2.39) yields
\[ \dot{\epsilon} = - \frac{\dot{\tau}}{\tau} \left( \epsilon + p \right). \quad (2.40) \]

Further using the equation of state (EOS) (2.22) for \( \epsilon \) and \( p \) one finds
\[ \epsilon = \frac{\epsilon_0}{\tau^{1 + \xi}}, \quad p = \frac{\xi \epsilon_0}{\tau^{1 + \xi}}. \quad (2.41) \]

Let us return to Eqs. (2.15). In view of (2.37), from (2.15e) one obtains the following relation between the metric functions \( a, b, c \):
\[ \left( \frac{a}{c} \right)^m = N \left( \frac{b}{c} \right)^n, \quad N = \text{const.} \quad (2.42) \]
Subtracting (2.15a) from (2.15b) we find

\[ \frac{d}{dt} \left[ \tau \frac{d}{dt} \{ \ln \left( \frac{a}{b} \right) \} \right] = \frac{m^2 - n^2}{c^2} \tau. \]  

(2.43)

Analogously, subtraction of (2.15a) from (2.15c) and (2.15b) from (2.15c) gives

\[ \frac{d}{dt} \left[ \tau \frac{d}{dt} \{ \ln \left( \frac{a}{c} \right) \} \right] = -\frac{mn + n^2}{c^2} \tau, \]  

(2.44)

and

\[ \frac{d}{dt} \left[ \tau \frac{d}{dt} \{ \ln \left( \frac{b}{c} \right) \} \right] = -\frac{mn + m^2}{c^2} \tau, \]  

(2.45)

respectively. It can be shown that, in view of (2.27) and (2.42), the Eqs. (2.43), (2.44), and (2.45) are interchangeable.

Taking into account that \( \tau = abc \), from (2.42) we can write \( a \) and \( b \) in terms of \( c \), such that

\[ a = \left[ N \tau^n c^{m-2n} \right]^{1/(m+n)}, \]  

(2.46)

and

\[ b = \left[ \tau^m c^{n-2m} / N \right]^{1/(m+n)}. \]  

(2.47)

Thus we find \( a \) and \( b \) in terms of \( c \) and \( \tau \). In doing so we employed only four out of five Einstein equations, leaving (2.15d) unused. Addition of (2.15a), (2.15b), (2.15c), and (2.15d), multiplied by 3 gives

\[ \frac{\ddot{\tau}}{\tau} = 2 \frac{m^2 - mn + n^2}{c^2} + \frac{3\kappa}{2} [T^0_0 + T^1_1] + 3\Lambda, \]  

(2.48)

which in view of (2.37) takes the form

\[ \frac{\ddot{\tau}}{\tau} = 2 \frac{m^2 - mn + n^2}{c^2} + \frac{3\kappa}{2} [MS + Ds + Dp + (1 - \zeta) \epsilon_0 / \tau^{1+\zeta}] + 3\Lambda. \]  

(2.49)

As one sees, we only have one equation with two unknowns. In order to resolve this problem, we have to assume \( c \) as a function of \( \tau \) (or vice versa). Given a concrete form of the spinor field nonlinearity one finds the solution of (2.49). This is exactly what we do in the next section.

### III. ANALYSIS OF THE RESULT

In the preceding section we derived equations for the spinor, scalar and gravitational fields and their general solutions. Comparing the equation with those in a BI universe (see e.g., Ref. [5, 6]) we conclude that introduction of inhomogeneity in gravitational (through \( m \) and \( n \)) imposes additional restriction on the metric functions. In fact, Eq. (2.15e) which connects the metric functions \( a, b, c \) among themselves, does not figure in the BI universe. In the foregoing sections we obtained the solutions to the field equations in terms of \( \tau \), whereas, the equations for \( \tau \) contains the function \( c \) explicitly, i.e., we have just one equation for two unknowns \( \tau \) and \( c \). In order to resolve this we have to impose some additional condition relating \( \tau \) and \( c \). Though this assumption imposes some restrictions on the metric functions, though leaving the space-time anisotropic.
Let us consider the case when \( F = F(I) = F(S) \) only. In this case from (2.27) we obtain

\[
S = C_0 / \tau,
\]

with \( C_0 \) being some arbitrary integration constant. The components of the spinor field can be obtained from (2.34) setting \( \mathcal{G} = 0 \) and \( Q = (m - n)/2c \). As far as \( F = F(J) \) is concerned, the volume scale \( \tau \) can be obtained from the corresponding equations setting spinor mass \( M = 0 \), while the for the components of the spinor field we need to set \( Q = (m - n)/2c \) and \( \Phi = 0 \) in (2.34). In this case from (2.27) one finds \( P = D_0 / \tau \), with \( D_0 \) being some arbitrary integration constant. Beside this, we assume \( c = c(\tau) \). The Eq. (2.49) then can be written as

\[
\dot{\tau} = \mathcal{F}(\tau, q),
\]

where we define

\[
\mathcal{F}(\tau, q) = 2(m^2 - mn + n^2) \frac{\tau}{c^2} + \frac{3\kappa}{2} [M + \mathcal{D} + \frac{1 - \zeta}{\tau^\zeta}] + 3\Lambda \tau.
\]

Here \( q \) is a set of problem parameters, namely, \( q = \{m, n, M, \lambda, \eta, \zeta, \Lambda\} \). Equation (3.2) admits the following first integral,

\[
\dot{\tau} = \sqrt{2[E - U(\tau, q)]},
\]

with the potential

\[
U(\tau, q) = - \{4(m^2 - mn + n^2) \int \frac{\tau d\tau}{c^2} + 3\kappa[M\tau + \int \mathcal{D}d\tau + \tau^{1 - \zeta}] + 3\Lambda \tau^2 \}. \tag{3.5}
\]

From a mechanical point of view, Eq. (3.2) can be interpreted as an equation of motion of a single particle with unit mass under the force \( \mathcal{F}(\tau, q) \). In (3.4) \( E \) is the integration constant which can be treated as an energy level, and \( \mathcal{U} (\tau, q) \) is the potential of the force \( \mathcal{F}(\tau_1, \tau) \). We solve Eq. (3.2) numerically using Runge-Kutta method. The initial value of \( \tau \) is taken to be a reasonably small one, while the corresponding first derivative \( \dot{\tau} \) is evaluated from (3.4) for a given \( E \).

In what follows we solve Eq. (3.2) for (i) \( c = \tau \) and (ii) \( c = \sqrt{\tau} \).

A. Case with \( c = \tau \)

Let us consider this case in details for different types of spinor analyze field nonlinearity.

1. Spinor field with power law induced nonlinearity

Here we consider the case when the spinor field nonlinearity is given by a power law of \( S \). In doing so we set \( F = S^\eta \), with \( \eta \) being the power of nonlinearity. The right hand side of Eq. (3.2) in this case has the form

\[
\mathcal{F}(\tau, q) = 2 \frac{m^2 - mn + n^2}{\tau} + \frac{3\kappa}{2} \left[ MC_0 + \frac{\lambda C_s \eta \tau^{\eta - 1}}{(\tau^\eta + \lambda C_0^n)^2} + \frac{(1 - \zeta)\varepsilon_0}{\tau^\zeta} \right] + 3\Lambda \tau, \tag{3.6}
\]

with \( C_s = C_{se} C_0^n / 2 \). For the potential in this case we have

\[
\mathcal{U}(\tau, q) = - \{4(m^2 - mn + n^2) \ln \tau + 3\kappa[M C_0 \tau - \lambda C_s/(\tau^\eta + \lambda C_0^n) + \varepsilon_0 \tau^{1 - \zeta}] + 3\Lambda \tau^2 \}. \tag{3.7}
\]
We solve Eq. (3.2) with the right hand side given by (3.6). For simplicity further we set \( \kappa = 1, C_0 = 1, C_s = 1 \) and \( \epsilon_0 = 1 \). For numerical calculations we used the following values for problem parameters: \( m = 2, n = 1, M = 1, \lambda = 0.1, \zeta = 1/3, \eta = 3 \). Note that \( \eta = -3 \) gives almost the same result as in case of \( \eta = 3 \). For cosmological constants we used \( \Lambda = 0, +1, -1 \), respectively. The energy level \( E \) is taken to be zero \( (E = 0) \). It should be noted that in the present case the potential possesses an infinitely high barrier at \( \tau = 0 \), it means in the case at hand \( \tau \) is always positive, that is we have singularity-free solution. Note that for a given value of \( E \) the minimum value of \( \tau \) \( (\tau_{\text{min}}) \) should be greater or equal to the value of \( \tau \) at the point of intersection of \( E \) and \( \mathcal{U} \). Here we set \( E = 0 \), so \( \tau_{\text{min}} \geq \tau_{\text{int}} : \mathcal{U}(\tau_{\text{int}}) = 0 \). For simplicity here the initial value of \( \tau \) is taken to be unity. Moreover, in case of \( \Lambda < 0 \) which is responsible for additional gravitational energy the value of \( \tau \) is bound from above as well, i.e., in this case the value of \( \tau \) lies between the two points of intersection of \( E \) and \( \mathcal{U} \).

In Fig. 1 we plot the potential corresponding to (3.7). In Fig. 2 the evolution of \( \tau \) in case of a negative \( \Lambda \) is illustrated. As one sees, in this case the model allows an oscillatory mode of expansion. In Figs. 3 and 4 behavior of \( \tau \) for a nonnegative \( \Lambda \) is presented. As one sees, introduction of a positive \( \Lambda \), which is often used to model the dark energy, results in the rapid growth of the universe. It is evident from the Eqs. (3.2) and (3.6), the inhomogeneity plays crucial role at small \( \tau \). In case of expanding universe inhomogeneity becomes notable only at the early stage of evolution, while for a oscillatory mode it is notable at \( \tau = \tau_{\text{min}} \). Apparently from the expression (3.7) varying the value of inhomogeneity parameters \( m \) and \( n \) the height of the potential barrier in the vicinity of \( \tau = 0 \) can be manipulated, though it will be infinitely high at \( \tau = 0 \).

2. Spinor field with trigonometric induced nonlinearity

Here we consider the case when the spinor field nonlinearity is given by some trigonometric functions. Here we chose the following two cases with (i) \( F = \sin(S) \) and (i) \( F = \exp(S) \).
If $F = \sin(S)$, we have
\[
\mathcal{F}(\tau, q) = 2\frac{m^2 - mn + n^2}{\tau} + \frac{3}{2} \left[ M + \frac{\lambda \cos(1/\tau)}{\tau^2 (1 + \lambda \sin(1/\tau))^2} + \frac{1 - \zeta}{\tau^5} \right] + 3\Lambda \tau, \tag{3.8}
\]
and
\[
\mathcal{U}(\tau, q) = -\{4(m^2 - mn + n^2)\ln \tau + 3\kappa [M\tau - \lambda / 2(1 + \lambda \sin(1/\tau)) + \tau^{1-\xi}] + 3\Lambda \tau^2\}. \tag{3.9}
\]
For $F = \exp(S)$ one finds
\[
\mathcal{F}(\tau, q) = 2\frac{m^2 - mn + n^2}{\tau} + \frac{3}{2} \left[ M + \frac{\lambda \exp(1/\tau)}{\tau^2 (1 + \lambda \exp(1/\tau))^2} + \frac{1 - \zeta}{\tau^5} \right] + 3\Lambda \tau, \tag{3.10}
\]
and
\[
\mathcal{U}(\tau, q) = -\{4(m^2 - mn + n^2)\ln \tau + 3\kappa [M\tau - \lambda / (\lambda + \tau^{\eta}) + \tau^{1-\xi}] + 3\Lambda \tau^2\}. \tag{3.11}
\]
Both these cases were solved numerically. The overall behavior of the potential $\mathcal{U}$ and $\tau$ is almost the same as in case of a power law nonlinear term illustrated in the Figs. (1-4).

B. Case with $c = \sqrt{\tau}$

In this case we find the following picture of the force $\mathcal{F}(\tau, q)$ and potential $\mathcal{U}(\tau, q)$:
\[
\mathcal{F}(\tau, q) = 2(m^2 - mn + n^2) + \frac{3}{2} \left[ M + \frac{\lambda \eta \tau^{\eta-1}}{(\lambda + \tau^{\eta})^2} + \frac{1 - \zeta}{\tau^5} \right], \tag{3.12}
\]
and
\[
\mathcal{U}(\tau, q) = -\{4(m^2 - mn + n^2)\tau + 3[M\tau - \lambda / (\lambda + \tau^{\eta}) + \tau^{1-\xi}]\}. \tag{3.13}
\]
Unlike the previous case now $\tau$ can be trivial as well, thus giving rise to a spacetime singularity. Given the value of energy level $E$ in case of a negative $\Lambda$ the solutions may be both oscillatory and nonperiodic.

Here as in previous case we set the following values for problem parameters: $m = 2$, $n = 1$, $M = 1$, $\lambda = 0.1$, $\xi = 1/3$, $\eta = 3$. For cosmological constants we used $\Lambda = 0, +1, -1$, respectively. For energy level $E$ we set $E = 0$ and $E = -1$. The evolution of $\tau$ in case of $\Lambda \geq 0$ is same as in previous case, while for $\Lambda < 0$ the evolution of $\tau$ is nonperiodic for $E \geq 0$ and oscillatory for $E < 0$. In Fig. 5 we plot the potential corresponding to (3.13). Contrary to the previous case it does not possess the infinitely high potential barrier at $\tau = 0$, which means in this case $\tau$ can be trivial as well. In Fig. 6 we illustrated the evolution of $\tau$ for a negative $\Lambda$ with two different values of $E$. Given the concrete value of $E$ the solution is either oscillatory or non-periodic. As far as inhomogeneity is concerned, in this case the part related to inhomogeneity can be added to the mass term and it plays the same role as the spinor mass in the evolution of the universe.

IV. CONCLUSION

A self-consistent system of interacting spinor and scalar fields within the framework of Bianchi type-VI (BVI) is studied in presence of a cosmological constant. Exact solutions of the spinor, scalar and gravitational field equations are obtained for some special choice of the spinor field nonlinearity. It is shown that introduction of a positive $\Lambda$ which is often used to model the dark energy results in a rapid growth of the universe, while a negative $\Lambda$ gives rise to an oscillatory or non-periodic mode of expansion. If the metric functions $a$ and $b$ are taken to be inverse to each other ($ab = 1$), we have a singularity free universe independent of the sign of the $\Lambda$ term.
[1] Yu.P. Rybakov, B. Saha, and G.N. Shikin, Commun. Theor. Phys. 3, 199 (1994).
[2] B. Saha and G.N. Shikin, J. Math. Phys. 38 (1997) 5305.
[3] B. Saha and G.N. Shikin, Gen. Relativ. Gravit. 29 (1997) 1099.
[4] Bijan Saha, Mod. Phys. Lett. A 16 (2001), 1287.
[5] Bijan Saha, Phys. Rev. D 64 (2001) 123501.
[6] Bijan Saha and T. Boyadjiev, Phys. Rev. D 69 (2004) 124010.
[7] C. Armendáriz-Picón and P.B. Greene, Gen. Relativ. Gravit. 35, 1637 (2003).
[8] M.O. Ribas, F.P. Devecchi, and G.M. Kremer, Phys. Rev. D 72 (2005) 123502.
[9] Bijan Saha, Gravitation & Cosmology 12 N. 2-3 (46-47), 215-218 (2006); [arXiv:gr-qc/0512050].
[10] Bijan Saha, Phys. Rev. D 74 124030 (2006).
[11] T. Singh and A.K. Agrawal, Int. J. Theor. Phys. 32, 1041 (1993).
[12] Marsha Weaver, Class. Quantum Grav. 17, 421 (2000).
[13] J. Ibáñez, R.J. van der Hoogen, and A.A. Coley, Phys. Rev. D 51, 928 (1995).
[14] J. Socorro and E.R. Medina, Phys. Rev. D 61, 087702 (2000).
[15] B.K. Berger, Class. Quantum Grav. 13, 1273 (1996).
[16] P.A. Apostolopoulos, e-print gr-qc/0310033.
[17] T. Christodoulakis, G. Kofinas, and G.O. Papadopoulos, Phys. Lett. B 514, 149 (2001).
[18] T. Christodoulakis and G.O. Papadopoulos, e-print gr-qc/0109058.
[19] Bijan Saha, Phys. Rev. D 69, 124006 (2004).
[20] T.W.B. Kibble, J. Math. Phys. 2, 212 (1961).
[21] V.A. Zhelnorovich, Spinor theory and its application in physics and mechanics (Nauka, Moscow, 1982).
[22] D. Brill and J. Wheeler, Rev. Mod. Phys. 29, 465 (1957).
[23] P.A.M. Dirac, General Theory of Relativity (John Wiley & Sons, Inc., Toronto, 1975).
[24] G.A. Milekhin, Izv. Akad. Nauk SSSR, Ser. Fiz. 26, 635 (1962).
[25] G.A. Milekhin, in Works of Intern. Conf. on Cosmic Rays I (USSR Academy of Science, Moscow, 1960), p. 223.
[26] N.V. Mitskevich, A.P. Efremov, and A.I. Nesterov, Dynamics of Fields in General Relativity (Energoatomizdat, Moscow, 1985).
[27] K.A. Bronnikov and G.N. Shikin, Gravitation & Cosmology 7, 231 (2001).
[28] S. Fay, Class. Quantum Grav. 17, 2663 (2000); e-print gr-qc/0309087.
[29] K.C. Jacobs, Astrophys. J. 153, 661 (1968).
[30] E. Kamke, Differentialgleichungen losungsmethoden und losungen (Akademische Verlagsgesellschaft, Leipzig, 1957).
[31] V.F. Zaitsev and A.D. Polyanin, A Handbook on Nonlinear Differential Equations (Nauka, Moscow, 1993).