A Mathematica script for harmonic oscillator nuclear matrix elements arising in semileptonic electroweak interactions

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Semi-leptonic electroweak interactions in nuclei – such as $\beta$ decay, $\mu$ capture, charged- and neutral-current neutrino reactions, and electron scattering – are described by a set of multipole operators carrying definite parity and angular momentum, obtained by projection from the underlying nuclear charge and three-current operators. If these nuclear operators are approximated by their one-body forms and expanded in the nucleon velocity through order $|\vec{p}|/M$, where $\vec{p}$ and $M$ are the nucleon momentum and mass, a set of seven multipole operators is obtained. Nuclear structure calculations are often performed in a basis of Slater determinants formed from harmonic oscillator orbitals, a choice that allows translational invariance to be preserved. Harmonic-oscillator single-particle matrix elements of the multipole operators can be evaluated analytically and expressed in terms of finite polynomials in $q^2$, where $q$ is the magnitude of the three-momentum transfer. While results for such matrix elements are available in tabular form, with certain restriction on quantum numbers, the task of determining the analytic form of a response function can still be quite tedious, requiring the folding of the tabulated matrix elements with the nuclear density matrix, and subsequent algebra to evaluate products of operators. Here we provide a Mathematica script for generating these matrix elements, which will allow users to carry out all such calculations by symbolic manipulation. This will eliminate the errors that may accompany hand calculations and speed the calculation of electroweak nuclear cross sections and rates. We illustrate the use of the new script by calculating the cross sections for charged- and neutral-current neutrino scattering in $^{12}$C.

I. INTRODUCTION

A common task in nuclear physics is to connect an observable – a rate for $\beta$ decay, the diffraction pattern seen in inelastic electron scattering, etc. – back to the underlying nuclear structure physics. While this can always be done implicitly through numerical calculations, one of the attractive properties of the harmonic-oscillator shell model is that this connection can be made analytically, at least in the case of one-body observables [1]. That is, each observable can be expressed as a simple function involving finite polynomials in $q^2$, the square of the three-momentum transfer to the nucleus, with the polynomial coefficients depending on the one-body density matrix. This allows one to determine quickly how a given experimental result constrains the underlying nuclear physics (the one-body density matrix), and to identify the unconstrained degrees of freedom in the density matrix that will enter into predictions of other observables, and thus will have to be taken from models.

Specifically, electroweak processes such as $\beta$ decay, muon capture, charged- and neutral-current neutrino reactions, electron scattering, and photo-absorption and $\gamma$ decay are traditionally described as one-body processes, possibly with corrections added to account for exchange currents, final- or initial-state distortions of the lepton wave function, etc. That is, the charge and three-current nuclear operators are expressed as the sum of the currents of the constituent nucleons. As described below, the many-body matrix elements of such operators can then be expressed as sums over single-nucleon matrix elements. Some time ago harmonic oscillator matrix elements – that is, the coefficients of the polynomials mentioned above – were tabulated by Donnelly and Haxton [2], though with some restrictions to reduce the size of the compilation. The tables include single-nucleon states up to the closed shell at proton/neutron number 126, e.g., all oscillators shells for principal quantum number $N = 0$ to 5, as well as the $j = 13/2$ subshell for $N = 6$. ($N$ determines the harmonic oscillator energy, $(N + 3/2)\hbar\omega$.) The tables also restricted transitions to $|N_f - N_i| \leq 2$.

In today’s context, the main shortcoming of the tables is not these restrictions, but the need to transcribe the results when using them in calculations. This is tedious and provides opportunities for errors. Thus here we provide a Mathematica [3] script to generate equivalent results, without restrictions on the choice of quantum numbers. This script will allow users to efficiently manipulate results within Mathematica, so that cross sections, rates, and relationships between various observables can be obtained quickly and without error, in closed form.

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II. ONE-BODY OPERATORS AND THE DENSITY MATRICES

We consider electroweak nuclear multipole operators $\hat{O}_{J,T}$ that are one-body, e.g., have the first-quantized form $\sum_{i=1}^{A} O_{J,T}(i)$, and carry definite parity and angular momentum. This operators, which are formed from the single-nucleon currents and take into account the spatial extent of these currents, can then be evaluated in terms of the one-body density matrix,

$$
(J;T;\hat{O}_{J,T};J;T_i) = \sum_{|\alpha|,|\beta|} \psi_{JT}^{f,i}(|\alpha|,|\beta|)|\alpha| \langle \frac{1}{2};O_{J,T};\frac{1}{2}|\beta| \rangle.
$$

Here $(J;T)$, $(J_f;T_f)$ and $(J_i;T_i)$ are the angular momentum and isospin of the operator and of the final and initial nuclear states, respectively, $|\alpha| = \{n_{\alpha}(l_{\alpha}^0)j_{\alpha}\}$ represents the set of nonmagnetic single-nucleon spatial and spin quantum numbers, and $\langle \cdot | \cdot \rangle$ indicates a matrix element reduced in both angular momentum and isospin, i.e.,

$$
(J_f M_f; T_f M_f)|\tilde{O}_{J,T}|J_i M_i; T_i M_i) = (-1)^{J_f-M_f+T_f-M_f} \left( \begin{array}{ccc}
J_f & J_i & 0 \\
-M_f & M_i & 0
\end{array} \right) \left( \begin{array}{ccc}
T_f & T_i & 0 \\
-M_f & M_i & 0
\end{array} \right) (J_f;T_f;\hat{O}_{J,T};J_i;T_i).
$$

The sums extend over complete sets of quantum numbers $|\alpha|$ and $|\beta|$ – which would normally correspond to the single-nucleus basis used in constructing the Slater determinants for the initial and final nuclear states.

For a one-body operator $O$, Eq. (1) is exact and factors the operator physics – embodied in the single-particle matrix elements $\langle |\alpha|; \frac{1}{2};O_{J,T};\frac{1}{2}|\beta|; \frac{1}{2}\rangle$ – from the many-body nuclear physics, which is contained within the one-body density matrix

$$
\psi_{JT}^{f,i}(\alpha,\beta) = \frac{1}{|J||T|} \langle J_f;T_f;\hat{c}_{\alpha}^{\dagger} \otimes \hat{c}_{\beta}|J_i;T_i\rangle.
$$

Here the single-nucleon creation and annihilation operators carrying good angular momentum and isospin are defined by

$$
c_{\alpha}^{\dagger} = c_{|\alpha| m_{\alpha}; m_{t_{\alpha}}}^{\dagger},
\tilde{c}_{\alpha} = (-1)^{j_{\alpha} - m_{\alpha} + 1/2 - m_{t_{\alpha}}} c_{|\alpha| - m_{\alpha}; - m_{t_{\alpha}}},
$$

where $m_{\alpha}$ and $m_{t_{\alpha}}$ are the single-nucleon magnetic quantum numbers for angular momentum and isospin and $\alpha = \{|\alpha| m_{\alpha}; m_{t_{\alpha}}\}$.

In practice, of course, the density matrix of Eq. (3) is often not calculated from first principles: for the classical nuclear physics problem of bound nucleons interacting via a two-body (or two- plus three-body) potential, the Schrödinger-equation many-body problem may not be tractable. Instead, wave functions are taken from nuclear models, such as the shell model, where model assumptions restrict the single-particle states that can be occupied. An effective interaction, often determined empirically, is employed to account for the effects of the omitted degrees of freedom. This leads to a finite basis of Slater determinants for the many-body problem and thus a Hamiltonian problem that can be solved by direct diagonalization.

To take the example we will use in this paper, consider the case of a 0hω shell-model calculation for $^{12}$C. That is, the included space consists of all Slater determinants where the 1s shell is fully occupied and eight valence nucleons are distributed in all allowed ways among the twelve single-particle orbitals within the 1p shell. Initial and final nuclear states determined from such a calculation would both carry positive parity. Under this assumption the density matrix for a transition between two such states simplifies,

$$
\psi_{JT}^{f,i}(\alpha,\beta) \rightarrow \{\psi_{JT}^{f,i}(1p_{1/2},1p_{1/2}), \psi_{JT}^{f,i}(1p_{3/2},1p_{1/2}), \psi_{JT}^{f,i}(1p_{3/2},1p_{3/2}), \psi_{JT}^{f,i}(1s_{1/2},1s_{1/2})\};
\psi_{JT}^{f,i}(1p_{3/2},1p_{1/2}) \equiv \psi_{JT}^{f,i}(1p_{3/2},1p_{1/2}) \pm \psi_{JT}^{f,i}(1p_{1/2},1p_{3/2}); \psi_{JT}^{f,i}(1s_{1/2},1s_{1/2}) = 2\delta_{fi}\delta_{Ji} \delta_{T0}\delta_{T0}
$$

where $\psi_{JT}^{f,i}(1p_{3/2},1p_{1/2})$ are defined in anticipation of the time-reversal properties of the operators we will discuss below. Note that the $1s_{1/2}$ shell only contributes to $J=0$ and $T=0$ elastic transitions.

So far the single-particle basis has not been specified: in many cases it can remain unspecified through the calculation of the one-body density matrix. This would be the case in shell-model calculations where matrix elements of the
effective interaction are treated as parameters, determined by fitting energy levels, e.g., as Cohen and Kurath [4] did in the $1p$ shell. Such a shell model calculation would yield specific numerical values for the density matrix elements of Eq. (1). However, when Eq. (1) is invoked to evaluate matrix elements, generally a single-particle basis must be specified.

The harmonic oscillator basis is an attractive choice for this purpose because the single-particle matrix elements for electroweak interactions can be evaluated analytically and have a simple form involving polynomials in the magnitude of the three-momentum transfer to the nucleus. This is very useful for determining the functional form of electroweak response functions. Even if the density matrix is unknown, response functions can be expressed in a form where the density matrix elements are parameters. This allows one to impose experimental constraints on model calculations – to quickly determine what degrees of freedom in the density matrix are already constrained by experiment, so that these can be eliminated, reducing the model dependence in predicting other process. In an example we will discuss later, charged- and neutral-current neutrino reactions in $^{12}$C involving the triad of $J^T = 1^+, M_T = (−1, 0, 1)$ excited states, the density matrix elements are sharply constrained by results from electron scattering (and gamma decay), $\beta$ decay, and $\mu$ capture [5]. In some truncated model spaces there may be a sufficient number of experimental constraints to determine all of the contributing density matrix elements: this would then free one from dealing with nuclear models.

III. THE BASIC NUCLEAR OPERATORS AND THEIR HARMONIC OSCILLATOR MATRIX ELEMENTS

The semileptonic weak nuclear operators $\hat{O}_{J,M,T,M_T}$ of interest in this paper are one-body and can be expressed as a product of space-spin and isospin operators. In first quantization

$$\sum_{i=1}^{A} O^T_{J,i} I_{TM_T}(i),$$

(6)

where the space-spin operator $O^T_J$ includes an isospin label because it contains couplings, like magnetic moments, that have different strengths depending on the isospin, e.g., the isoscalar magnetic moments is not equal to the isovector magnetic moment. Semileptonic nuclear cross sections and rates can be expressed in terms of nuclear matrix elements reduced in angular momentum

$$\langle J_f || \hat{O}_{J,T,M_T} || J_i \rangle \rightarrow \langle J_f; T_f, M_{T_f}; || \hat{O}_{J,T,M_T} || J_i; T_i, M_{T_i} \rangle =

(-1)^{T_f-M_{T_f}} \left( T_f \begin{array}{ccc}
T & T & T \\
-M_{T_f} & M_{T_i} & M_{T_i}
\end{array} \right) \sum_{|\alpha|,|\beta|} \psi^{j,\alpha}_{j,j}(|\alpha|,|\beta|) \langle |\alpha||O^T_J|||\beta| \langle \frac{1}{2} ||I_T|| \frac{1}{2},

(7)

where the arrow in the first line indicates we are ignoring small Coulomb and other charge-independence-breaking terms, so that nuclear states can be labeled as eigenstates of good isospin (a common assumption in nuclear structure calculations). The second line follows from Eq. (1).

The one-body space-spin operators $O^T_J$ governing semileptonic electroweak interactions are derived in standard references [1] by expanding the vector and axial-vector charge and current operators through order $|p|/M$. They can be expressed in terms of the seven single-particle operators

$$M_j^{Mj}(qx)$$

$$\Delta_j^{Mj}(qx) \equiv M_j^{Mj}(qx) - \frac{1}{q} \vec{\nabla}$$

$$\Delta'_{j}^{Mj}(qx) \equiv -i \frac{1}{q} \vec{\nabla} \times M_j^{Mj}(qx) \cdot \frac{1}{q} \vec{\nabla} = [J]^{-1} \left( -J^{1/2} M_j^{Mj} (qx) + (J + 1)^{1/2} M_{j+1}^{Mj} (qx) \right) \cdot \frac{1}{q} \vec{\nabla}$$

$$\Sigma_j^{Mj}(qx) \equiv M_j^{Mj}(qx) \cdot \vec{\sigma}$$

$$\Sigma'_{j}^{Mj}(qx) \equiv -i \frac{1}{q} \vec{\nabla} \times M_j^{Mj}(qx) \cdot \vec{\sigma} = [J]^{-1} \left( -J^{1/2} M_{j+1}^{Mj} (qx) + (J + 1)^{1/2} M_{j-1}^{Mj} (qx) \right) \cdot \vec{\sigma}$$

$$\Sigma''_{j}^{Mj}(qx) \equiv \left[ \frac{1}{q} \nabla M_j^{Mj}(qx) \right] \cdot \vec{\sigma} = [J]^{-1} \left( (J + 1)^{1/2} M_{j+1}^{Mj} (qx) + J^{1/2} M_{j-1}^{Mj} (qx) \right) \cdot \vec{\sigma}$$

$$\Omega_j^{Mj}(qx) \equiv M_j^{Mj}(qx) \cdot \vec{\sigma} \cdot \frac{1}{q} \vec{\nabla},$$

(8)
where \( q \) is the magnitude of the three-momentum transferred to the nucleus and \([J] = \sqrt{2J + 1}\). Because of operator time-reversal properties it is helpful to replace \( \Omega \) by a new operator

\[
\Omega^{M_j}(q\mathbf{x}) \equiv \Omega^{M_j}(q\mathbf{x}) + \frac{1}{2}\Sigma^M_{j}(q\mathbf{x}).
\]

(9)

The multipole operators in Eqs. (8) are constructed from spherical Bessel functions, spherical harmonics, and vector spherical harmonics [6]

\[
M^M_j(q\mathbf{x}) \equiv j_J(q\mathbf{x})Y^M_j(\Omega_x)
\]

\[
M^M_{j\ell}(q\mathbf{x}) \equiv j_{\ell}(q\mathbf{x})Y^M_{j\ell l}(\Omega_x)
\]

(10)

where

\[
Y^M_{j\ell l}(\Omega_x) = [Y_L \otimes \mathbf{e}_1]_{JM_j} = \sum_{m\lambda}(Lm1\lambda|(L1)JM_j)r^{-\lambda}_{LM}(\Omega_x) e_{1\lambda}.
\]

(11)

The spherical unit vectors \( \mathbf{e}_{1\lambda} \) are defined by \( \mathbf{e}_{1\pm 1} = \mp(\mathbf{e}_x \pm i\mathbf{e}_y)/\sqrt{2} \) and \( \mathbf{e}_{10} = \mathbf{e}_z \).

The single-particle basis \( |\alpha\rangle = |n\ell_\alpha(l_{\alpha}/2)j_{\alpha}m_\alpha\rangle \) of Eq. (1) has the general coordinate-space form

\[
R_{n\ell_\alpha l_{\alpha}j_{\alpha}}(x)|Y_n(\Omega_x) \otimes \xi_{j\ell/2}j_{j\ell}m_\alpha\rangle
\]

(12)

where \( \xi_{j\ell/2} \) is the Pauli spinor. The angular and spin portions of the single-particle operator matrix elements can then be performed, leaving only matrix elements between unspecified radial wave functions,

\[
\langle n'(l'_{\frac{1}{2}})j'||M_J(q\mathbf{x})|n(l_{\frac{1}{2}})j\rangle = \frac{1}{\sqrt{4\pi}}(-)^{l+1/2} |l'\rangle [l' \rangle [j'] \rangle [J] \left\{ \begin{array}{ccc}
  l' & J & 1/2 \\
  l & 0 & 0 \\
\end{array} \right\} \langle n'l'j' | j_J(\rho) | nlj \rangle
\]

\[
\langle n'(l'_{\frac{1}{2}})j'||M_{j\ell}(q\mathbf{x}) \cdot \vec{\sigma}|n(l_{\frac{1}{2}})j\rangle = \frac{1}{\sqrt{4\pi}}(-)^{l+1/2} |l'\rangle [l' \rangle [j] \rangle [L] \left\{ \begin{array}{ccc}
  l' & J & 1/2 \\
  l & 0 & 0 \\
\end{array} \right\} \langle n'l'j' | j_L(\rho) | nlj \rangle
\]

\[
\times \left\{ -(l + 1)^{1/2} |l + 1\rangle \left\{ \begin{array}{ccc}
  L & 1 & j \\
  l' & 0 & 0 \\
\end{array} \right\} \langle n'l'j' | j_L(\rho) \left( \frac{d}{d\rho} - \frac{l}{\rho} \right) | nlj \rangle
\]

\[
+ l^{1/2} |l - 1\rangle \left\{ \begin{array}{ccc}
  L & 1 & j \\
  l' & 0 & 0 \\
\end{array} \right\} \langle n'l'j' | j_L(\rho) \left( \frac{d}{d\rho} + \frac{l + 1}{\rho} \right) | nlj \rangle
\}
\]

\[
\langle n'(l'_{\frac{1}{2}})j'||M_J(q\mathbf{x})\vec{\sigma} \cdot \vec{\nabla}|n(l_{\frac{1}{2}})j\rangle = \frac{1}{\sqrt{4\pi}}(-)^{l+1/2} |l'\rangle [l' \rangle [j'] \rangle [J] \left\{ \begin{array}{ccc}
  l' & J & 1/2 \\
  l & 0 & 0 \\
\end{array} \right\} \langle n'l'j' | j_J(\rho) \left( \frac{d}{d\rho} - \frac{l}{\rho} \right) | nlj \rangle
\]

\[
\times \left\{ -\delta_{j,l+1/2} \langle n'l'j' | j_J(\rho) \left( \frac{d}{d\rho} + \frac{l + 1}{\rho} \right) | nlj \rangle + \delta_{j,l-1/2} \langle n'l'j' | j_J(\rho) \left( \frac{d}{d\rho} + \frac{l + 1}{\rho} \right) | nlj \rangle \right\}.
\]

(13)

Here the dimensionless coordinate \( \rho \equiv qx \), while the radial matrix elements are defined by

\[
\langle n'l'j'|\theta(\rho)|nlj\rangle \equiv \int x^2dxR^*_{n'l'j'}(x)\theta(\rho)R_{nlj}(x)
\]

(14)

for

\[
\theta(\rho) = \begin{cases}
  j_J(\rho) \\
  j_J(\rho) \left( \frac{d}{d\rho} - \frac{l}{\rho} \right) \\
  j_J(\rho) \left( \frac{d}{d\rho} + \frac{l + 1}{\rho} \right)
\end{cases}
\]

(15)

To complete the evaluation a specific choice for the single-particle radial wave functions must be made. The harmonic oscillator is an attractive choice, convenient both for nuclear structure structure reasons (for example, in
certain model spaces spurious center-of-mass degrees of freedom can be removed exactly) and because the radial matrix elements of Eqs. (13) can be evaluated analytically. The properly normalized radial wave functions are

$$R_{nlj}(x) \equiv R_{nl}(x) = \frac{2 e^{x}}{\beta^{3} (n+l+\frac{1}{2})} \frac{d^{n-l-1/2}}{dx^{n-l-1}} \left[ x^{n+l-1} e^{-x} \right],$$

(16)

where $z \equiv (x/b)^{2}$ and $b$ is the oscillator size parameter. We employ a notation where the nodal quantum number $n = (N - l)/2 + 1$, where the principal quantum number $N = 0,1,2,...$ Thus $n = 1,2,3...$ A given harmonic oscillator state can be labeled by $n, l, j$ or, equivalently, $N, j$: as $N$ determines the parity of the shell, knowledge of $N$ and $j$ uniquely determines $l$. Our Mathematica script uses $N, j$ to identify single-nucleon states.

The radial integrals appearing in Eqs. (13) can be evaluated analytically for harmonic oscillator states,

$$\langle n' l' j'(\rho) | n l \rangle = \frac{2^{L} e^{-y}}{(2L + 1)!!} y^{l'/2} e^{-y} \sqrt{(n-1)! (n'-1)!} \Gamma(n+l+\frac{1}{2}) \Gamma(n+l+\frac{1}{2}) \sum_{k=0}^{n-1} \sum_{k'=0}^{n'-1} \frac{(-1)^{k+k'}}{k! k'!} \frac{\Gamma\left(\frac{1}{2} (l' + L + 2k + 2k' + 3)\right)}{\Gamma\left[ l + k + \frac{3}{2} \right] \Gamma\left[ k + \frac{3}{2} \right]} \left[ \frac{1}{2} (l + l' + L + 2k + 2k' + 2) \right]$$

$$\times \frac{\Gamma\left(\frac{1}{2} (L + l' - 2k - 2k' - 1)\right)}{\Gamma\left[ l + k + \frac{3}{2} \right] \Gamma\left[ k + \frac{3}{2} \right]} \left[ \frac{1}{2} (L + l' - 2k - 2k' + 1)\right]$$

$$\times \left[ 2F_{1} \left( \frac{1}{2} (L + l' - 2k - 2k' - 1); L + \frac{3}{2}; y \right) \right]$$

$$\times \left[ 2F_{1} \left( \frac{1}{2} (L + l' - 2k - 2k' + 1); L + \frac{3}{2}; y \right) \right]$$

$$\times \frac{1}{2F_{1} \left( \frac{1}{2} (L + l' - 2k - 2k' + 1); L + \frac{3}{2}; y \right)}$$

$$\times \frac{1}{2F_{1} \left( \frac{1}{2} (L + l' - 2k - 2k' + 1); L + \frac{3}{2}; y \right)}$$

where $y = (qb/2)^{2}$. Here $2F_{1}$ is the confluent hypergeometric function

$$\alpha \left( \beta; y \right) = 1 + \frac{\alpha}{\beta} + \frac{\alpha (\alpha + 1) y^{2}}{\beta (\beta + 1) 2!} + ...$$

(20)

In the present application $\alpha$ is a nonpositive integer, so that this series is a polynomial of order $-\alpha$ in $y$.

Thus the single-nucleon harmonic-oscillator matrix elements of the seven basic operators are completely determined in terms of simple functions. One finds

$$\langle n' (l' \frac{1}{2}) j' | T_{j} (q \xi) | n (l \frac{1}{2}) j \rangle = \frac{1}{\sqrt{4\pi}} y^{(j-K)/2} e^{-y} p(y),$$

(21)

where $K = 2$ for the normal parity operators $M, \Delta', \Sigma'$; $K' = 1$ for the abnormal parity operators $\Delta, \Sigma, \Sigma$, $\Omega$ (or $\Omega'$). The tabulations of Ref. [2] provide the polynomials $p(y)$ for such matrix elements, with the restrictions described previously. The Mathematica script accompanying this paper generates the full expression in Eq. (21), without restrictions on the quantum numbers and in a form that can be easily manipulated within Mathematica to generate analytical expressions for form factors, rates, etc.

An important property related to the time-reversal properties of these operators is

$$\langle n' (l' \frac{1}{2}) j' | T_{j} (q \xi) | n (l \frac{1}{2}) j \rangle = (-1)^{l} \langle n (l \frac{1}{2}) j | T_{j} (q \xi) | n' (l' \frac{1}{2}) j' \rangle$$

(22)

where $\lambda = j' - j$ for the operators $M, \Delta, \Sigma$, and $\Sigma'$, and $\lambda = j' + j$ for the operators $\Delta', \Sigma$, and $\Omega'$. The operator $\Omega'$ of Eq. (9) was defined because $\Omega$ lacks such a simple transformation property. This “turn around” property is important because it leads to a dependence of observables of density matrix element combinations such as $\psi_{j,T}^{\pm} (1p_{3/2}, 1p_{1/2})$. 


IV. RELATIONSHIP WITH SEMILEPTONIC PROCESSES IN NUCLEI

To illustrate the use of the Mathematica script, we will consider various charged-current and neutral-current neutrino reactions from the $J^\pi T = 0^+ 0$ ground state of $^{12}\text{C}$:

\[
^{12}\text{C}(g.s.)(\nu_e, e^-)\ 12\text{N}(g.s.)
\]
\[
^{12}\text{C}(g.s.)(\nu, \nu)\ 12\text{C}(15.11 \text{ MeV})
\]
\[
^{12}\text{C}(g.s.)(\bar{\nu}, \bar{\nu})\ 12\text{C}(15.11 \text{ MeV})
\]
\[
^{12}\text{C}(g.s.)(\nu, \nu)\ 12\text{C}(12.7 \text{ MeV})
\]
\[
^{12}\text{C}(g.s.)(\bar{\nu}, \bar{\nu})\ 12\text{C}(12.7 \text{ MeV})
\]

The first four transitions are isovector, populating the triad of $J^\pi T = 1^+ 1$ states formed by the ground states of $^{12}\text{N}$ ($M_T = 1$) and $^{12}\text{B}$ ($M_T = -1$) and the 15.11 MeV ($M_T = 0$) isovector analog state in $^{12}\text{C}$. The last two are isoscalar, populating the $J^\pi T = 1^+ 0$ 12.7 MeV state in $^{12}\text{C}$. The four-momentum transfer for these reactions is defined by

\[
q^\mu = P_i^\mu - P_f^\mu = k^\mu - k^\mu = (\omega, \hat{q})
\]

where $P_i^\mu$, $P_f^\mu$, $k^\mu$, and $k^\mu$ are the four-momenta of the initial nucleus, final nucleus, incident neutrino, and scattered lepton (electron/positron or neutrino/antineutrino). The charged-current cross section in the extreme relativistic limit can be written [1]

\[
\left( \frac{d\sigma}{d\Omega} \right)_{\nu_e} = \frac{2}{\pi} G_F^2 \cos^2 \theta_e F(Z, \epsilon) \frac{2}{2J_i + 1} \cos^2 \frac{\theta}{2} \left\{ \sum_{J_f = 0}^{\infty} |\langle J_f | \hat{\mathcal{M}}_J + \hat{\mathcal{L}}_J | J_i \rangle|^2 \right. \\
+ \left. \left[ -\frac{q^2}{2q^2} + \tan^2 \frac{\theta}{2} \right] \sum_{J_f = 1}^{\infty} \sum_{J_m = J_f}^{\infty} \sum_{J_n = -J_m}^{J_m} \left| \langle J_f | \hat{T}^{\text{mag}}_{J;J_m,J_n} | J_i \rangle \right|^2 \right\}^{1/2} \sum_{J_f = 1}^{\infty} \left| \langle J_f | \hat{T}^{\text{mag}}_{J;J_m,J_n} | J_i \rangle \langle J_f | \hat{T}^{\text{el}}_{J;J_m,J_n} | J_i \rangle^* \right| \}
\]

where $G_F \cos \theta_e$ is the weak coupling constant, $\theta$ the angle between the incident neutrino and outgoing electron, $q^2 = \omega^2 - q^2$, and $\epsilon$ is the energy of the outgoing electron/positron. The function $F(Z, \epsilon)$ corrects the electron phase space for the effects of Coulomb distortion in the field of the daughter nucleus of charge $Z$. This expression assumes $\epsilon \gg m_\nu$ and neglects nuclear recoil.

The operators $\hat{\mathcal{M}}_J$, $\hat{\mathcal{L}}_J$, $\hat{T}^{\text{el}}_J$, and $\hat{T}^{\text{mag}}_J$ are the familiar charge, longitudinal, transverse electric, and transverse magnetic projections of the weak charge-changing hadronic current,

\[
\hat{J}_\mu^\pm = \hat{J}_\mu^1 \pm \hat{J}_\mu^2
\]

where $\hat{J}_\mu^i$, $i = 1, 2, 3$ are the three components of an isovector. The weak current is made up of vector and axial-vector components

\[
\hat{J}_\mu^i = \hat{J}_\mu^i + \hat{J}_\mu^5.
\]

Multipole projections of this current are taken to exploit the angular momentum and parity quantum labels that are normally carried by the initial and final nuclear states. The multipole operators are defined by

\[
\hat{\mathcal{M}}_{J;M;T} = \int dx \ M_J^M (q \cdot x) (T;M;T) = \hat{\mathcal{M}}_{J;M;T} + \hat{\mathcal{M}}^5_{J;M;T}
\]
\[
\hat{L}_{J;M;T} = \frac{i}{q} \int dx \ \nabla \cdot M_J^M (q \cdot x) \cdot \hat{J}^\mu (x;M;T) = \hat{L}_{J;M;T} + \hat{L}^5_{J;M;T}
\]
\[
\hat{T}_{J;M;T} = \frac{1}{q} \int dx \ \nabla \times M_J^M (q \cdot x) \cdot \hat{J}^\mu (x;M;T) = \hat{T}_{J;M;T} + \hat{T}^5_{J;M;T}
\]
\[
\hat{T}^{\text{mag}}_{J;M;T} = \int dx \ M_J^M (q \cdot x) \cdot \hat{J}^\mu (x;M;T) = \hat{T}^{\text{mag}}_{J;M;T} + \hat{T}^{\text{mag}5}_{J;M;T}
\]

where we have separated the vector and axial-vector (with superscript ‘5’) contributions in Eqs. (27). The operators $\hat{\mathcal{M}}_J$, $\hat{L}_J$, $\hat{T}_J$, and $\hat{T}^{\text{mag}}_J$ are normal parity operators, that is, $\Delta \pi = (-1)^J$. The operators $\hat{\mathcal{M}}_J^5$, $\hat{L}_J^5$, $\hat{T}_J^5$, and $\hat{T}^{\text{mag}5}_J$ are abnormal parity, $\Delta \pi = (-1)^{J+1}$. Also note that, for a conserved vector current, $\hat{L}_J$ can be eliminated as

\[
\hat{L}_{J;M;T} = \frac{\omega}{q} \hat{\mathcal{M}}_{J;M;T}.
\]
Although contributions from two-body (exchange) currents can be included, more commonly the nuclear charges and currents are modeled as the sum of the one-body contributions from the constituent protons and neutrons. For a single free nucleon one has the following general forms for the matrix elements of the vector and axial-vector isovector and currents are modeled as the sum of the one-body contributions from the constituent protons and neutrons. For a momentum dependence of $F_\tau$ assuming pion-pole dominance and the Goldberger-Treiman relationship [1]. The isospin raising/lowering operator

$$M_{J\to J^\pm}(qx) = F_{1\pm}^\uparrow(q^2)M_{J^\pm}(qx)\tau_\pm$$

$$T_{J\to J^\pm}^{el}(qx) = \frac{q}{M_N}(F_{1\pm}^\uparrow(q^2)A_{J^\pm}(qx) + \frac{1}{2}\mu^{(1)}(q^2)\Sigma_{J^\pm}(qx))\tau_\pm$$

$$T_{J\to J^\pm}^{ax}(qx) = -\frac{iq}{M_N}(F_{1\pm}^\uparrow(q^2)A_{J^\pm}(qx) - \frac{1}{2}\mu^{(1)}(q^2)\Sigma_{J^\pm}(qx))\tau_\pm$$

$$M_{J\to J^\pm}(qx) = \frac{iq}{M_N}(F_{A\pm}^\uparrow(q^2)\Sigma_{J^\pm}(qx) + \frac{1}{2}\omega F_{P\pm}^\uparrow(q^2)\Sigma_{J^\pm}(qx))\tau_\pm$$

$$T_{J\to J^\pm}^{el5}(qx) = i(F_{A\pm}^\uparrow(q^2) - \frac{q^2}{2M_N}F_{P\pm}^\uparrow(q^2))\Sigma_{J^\pm}(qx)\tau_\pm$$

$$T_{J\to J^\pm}^{ax5}(qx) = F_{A\pm}^\uparrow(q^2)\Sigma_{J^\pm}(qx)\tau_\pm$$

$$T_{J\to J^\pm}^{el0}(qx) = F_{A\pm}^\uparrow(q^2)\Sigma_{J^\pm}(qx)\tau_\pm$$

where we have omitted second-class scalar (which violates conservation of the vector current) and axial-tensor couplings. [Our notation in this paper follows Bjorken and Drell [2], including gamma matrices, metric, and current definitions, with the exception of the definition of the form factors, which follows that of Ref. [1].]

A standard non-relativistic reduction of the matrix elements through order $1/M_N$, in which momenta are interpreted as gradients operating within the nucleus, leads to expressions for the multipole operators of Eqs. (27) in terms of the seven basic single-particle operators defined in Eqs. (8):

$$F_{P\pm}^\uparrow(q^2) \sim \frac{2M_N F_{P\pm}^\uparrow(0)}{m_n^2 - q^2},$$

assuming pion-pole dominance and the Goldberger-Treiman relationship [1]. The isospin raising/lowering operator $\tau_\pm$ is related to the spherical projections of isospin Pauli operator $\tau_{1m}$ by

$$I_{\tau=1}^{\pm} = \frac{\tau_1 \pm \tau_2}{2} \equiv \tau_\pm = \pm \frac{1}{\sqrt{2}}\tau_{1\pm}.$$ 

**Charged-current neutrino scattering off $^{12}$C**: Now we consider the example of charged-current neutrino scattering off the $J^\pi T = 0^+ 0$ $^{12}$C ground state, leading to the $1^+ 1$ grounds states of $^{12}$B and $^{12}$N. We treat these states as isospin eigenstates, ignoring isospin breaking due to electromagnetic or other charge-dependent interactions. Thus Eq. (24) simplifies because of the angular momentum and parity constraints imposed by the $0^+ 0 \to 1^+ 1$ transition and because the matrix elements can be reduced in isospin,
\[ \left( \frac{d\sigma}{d\Omega} \right)_{\text{elastic}} = \frac{2e^2}{3\pi} \frac{G_F^2}{q^2} \cos^2 \theta_e F(Z, \epsilon) \cos^2 \frac{\theta}{2} \left\{ \langle 1^+; 1^+; 1^+; 0^+; 0^+ \rangle \right\}^2 \]

\[ + \left[ -\frac{q^2}{2q^2} + \tan^2 \frac{\theta}{2} \right] \left\{ \langle 1^+; 1^+; 1^+; 0^+; 0^+ \rangle \right\}^2 \]

\[ = 2\tan \frac{\theta}{2} \left[ \frac{q^2}{q^2} + \tan^2 \frac{\theta}{2} \right] \left\{ \langle 1^+; 1^+; 1^+; 0^+; 0^+ \rangle \right\}^{1/2} \]

\[ \right\} \] \hspace{1cm} (33)

where a factor of 1/3 results from isospin reduction,

\[ \langle 1; 1 \pm 1 | \hat{O}_{1;1} | 0; 0 \rangle = \frac{1}{\sqrt{3}} \langle 1; 1^+ | \hat{O}_{1;1} | 0; 0 \rangle. \] \hspace{1cm} (34)

Eqs. (33) and (34) thus determine the cross section, once the doubly-reduced single-particle matrix elements are evaluated. Eqs. (1) and (32) yield

\[ \langle 1^+; 1^+; 1^+; 0^+; 0^+ \rangle = \sum_{|\alpha|,|\beta|} \psi_{j=1; T=1}^i (|\alpha|, |\beta|) \langle \frac{1}{2}; 0 | \hat{O}_{1;1} | |\alpha|, |\beta| \rangle \]

\[ \sum_{|\alpha|,|\beta|} \psi_{j=1; T=1}^i (|\alpha|, |\beta|) \langle |\alpha|, |\beta| | O_{j}^i | |\beta| \rangle = \sqrt{3} \sum_{|\alpha|,|\beta|} \psi_{j=1; T=1}^i (|\alpha|, |\beta|) \langle |\alpha|, |\beta| | O_{j}^i | |\beta| \rangle \] \hspace{1cm} (35)

where we have used \( \langle \frac{1}{2}; 0 | \hat{O}_{1;1} | |\beta| \rangle = \sqrt{6} \). The remaining spin-spatial matrix element can then be evaluated with the Mathematica program, by virtue of Eq. (30). As discussed early, we truncate the density matrix to the 1p shell and assume a harmonic oscillator basis, but otherwise keep the elements of the density matrix arbitrary: In the next section we will describe how the Mathematica code can be used to provide the needed matrix element, once this truncation has been made. The result is an analytic expression for the cross section that encompasses any shell-model calculation restricted in this way.

\[ \left( \frac{d\sigma}{d\Omega} \right)_{\text{elastic}} = \frac{2e^2}{3\pi} \frac{G_F^2}{q^2} \cos^2 \theta_e F(Z, \epsilon) \cos^2 \frac{\theta}{2} e^{-2y} \]

\[ \left\{ \left[ -\frac{q^2}{3} F_A^i \psi_{j=1; T=1}^i (|\alpha|, |\beta|) \psi_{31}^\pm \frac{\omega}{q} F_{P}^{(1)} \left( -\frac{\sqrt{2}}{3} (1 + 2y) \psi_{11} - \frac{4}{3} (1 - y) \psi_{33}^\pm \frac{2\sqrt{2}}{3} (1 - \frac{2}{3} y) \psi_{33} \right) \right]^2 + \right. \]

\[ \left. \left[ -\frac{q^2}{2q^2} + \tan^2 \frac{\theta}{2} \right] \left\{ \left[ -\frac{q^2}{3} F_A^i \psi_{j=1; T=1}^i (|\alpha|, |\beta|) \psi_{31}^\pm \frac{\omega}{q} F_{P}^{(1)} \left( -\frac{\sqrt{2}}{3} (1 + 2y) \psi_{11} - \frac{4}{3} (1 - y) \psi_{33}^\pm \frac{2\sqrt{2}}{3} (1 - \frac{2}{3} y) \psi_{33} \right) \right]^2 + \right. \]

\[ \left. \left. \frac{q^2}{M^2} \left[ F_1^{(1)} \left( -\frac{2}{3} (1 + 2y) \psi_{11} - \frac{4\sqrt{2}}{3} (1 - y) \psi_{33}^\pm \frac{2\sqrt{2}}{3} (1 - \frac{2}{3} y) \psi_{33} \right) \right]^2 \right\} \right. \]

\[ \left. \right\} \] \hspace{1cm} (36)

where it is understood that all single-nucleon couplings are described by form factors, e.g., \( F_A^{(1)} = F_A^{(1)} (q_\mu^2) \). In this expression we used the short-hand notation for the density matrix of \( \psi_{2j_\alpha, 2j_\beta} \equiv \psi_{j=1; T=1}^i (|\alpha|, |\beta|) \). That is, the final and initial nuclear state labels \( f, i \) are suppressed, as are the \( J, T \) of the transition, since these quantum numbers are uniquely determined for the transition \( (J = 1, T = 1) \).
Neutral current scattering to the $1^+ 1$ 15.11 MeV state of $^{12}C$: The hadronic neutral current is

$$\hat{J}_\mu^{NC} = \hat{J}_\mu^3 - 2 \sin^2 \theta_W J_\mu^{em}$$  \hspace{1cm} (37)$$

where $\hat{J}_\mu^3$ is third isospin component of the weak current of Eq. (25), $\hat{J}_\mu^{em}$ is the electromagnetic current, and $\theta_W$ the Weinberg angle. Because of the presence of $\hat{J}_\mu^3$, the neutral current includes an isoscalar contribution. Eqs. (25) and (37) allow one to compare the isospin dependence of charged and neutral-current isovector transitions,

$$I_{T=1}^\pm = \tau_\pm \leftrightarrow I_{T=1}^{NC} = \left[ 1 - 2 \sin^2 \theta_W \right] \frac{\tau_3}{2} \text{ vector current}$$

$$I_{T=1}^{NC} = \frac{\tau_3}{2} \text{ axial -- vector current}$$  \hspace{1cm} (38)$$

The reduction in isospin between nuclear states (compare Eq. (34)) yields

$$\langle 1;10||\hat{O}_{1;10}||0;00 \rangle = \frac{1}{\sqrt{3}} \langle 1;1;\hat{O}_{1;1};0;0 \rangle,$$

while the single-particle matrix element for $\tau_3/2$ (compare Eq. (35)) yields

$$\langle 1^+;1;\hat{O}_{1;1}^{(1);}0^+;0 \rangle = \sum_{[\alpha],\beta} \psi_{1;1}^{f,i} ([\alpha], \beta) \langle [\alpha]; \frac{1}{2};\hat{O}_{1;1}^{(1);}||\beta]; \frac{1}{2} \rangle =$$

$$\frac{1}{\sqrt{2}} \sum_{[\alpha],\beta} \psi_{1;1}^{f,i} ([\alpha], \beta) \langle [\alpha] || O_{1;1}^{(1)} || \beta] \langle \frac{1}{2} || \tau] \langle \frac{1}{2} \rangle = \frac{\sqrt{6}}{2} \sum_{[\alpha],\beta} \psi_{1;1}^{f,i} ([\alpha], \beta) \langle [\alpha] || O_{1;1}^{(1)} || \beta] \langle \frac{1}{2} || \tau] \langle \frac{1}{2} \rangle.$$  \hspace{1cm} (40)$$

Thus there is an overall factor of $1/2$ in the cross section due to isospin, relative to charged-current result in $^{12}C$.

Finally, the cross section for neutral-current scattering of neutrinos off nuclei can be obtained from Eq. (24) by replacing $\epsilon$ with $\epsilon_\nu$, the energy of the scattered neutrino, and by removing $F(Z, \epsilon)$ and $\cos^2 \theta_c$. It follows

$$\left( \frac{d\sigma}{d\Omega} \right)^{^{12}C(\nu,\nu')^{^{12}C}(1^+)} = \frac{e^2}{4\pi^2} G_F^2 \cos^2 \frac{\theta}{2} e^{-2y} \left\{ \left[ \frac{q}{M} F_A^{(1)} \psi_{31} + \frac{q}{M} \left( \frac{F_A^{(1)} - q^2 F_P^{(1)}}{M} \right) \left( \frac{-\sqrt{2}}{3} (1 + 2y) \psi_{11} - \frac{4}{3} (1 - y) \psi_{31} + \frac{2\sqrt{2}}{3} (1 - \frac{2}{5} y) \psi_{33} \right) \right]^2 + \frac{q^2}{2q^2} \frac{\tan^2 \theta}{2} \left\{ \left[ \frac{F_A^{(1)}}{3} \left( \frac{2}{3} (1 - 2y) \psi_{11} - \frac{4\sqrt{2}}{3} (1 - \frac{y}{2}) \psi_{31} + \frac{2\sqrt{10}}{3} (1 - \frac{4}{5} y) \psi_{33} \right) \right]^2 + \frac{q^2}{M^2} \left( 1 - 2 \sin^2 \theta_W \right)^2 \times \left[ \left( \frac{2}{3} \psi_{11} + \frac{\sqrt{2}}{3} \psi_{31} + \frac{\sqrt{10}}{3} \psi_{33} \right) \right] + \frac{\mu^{(1)}}{2} \left( \frac{2}{3} (1 - 2y) \psi_{11} - \frac{4\sqrt{2}}{3} (1 - \frac{y}{2}) \psi_{31} + \frac{2\sqrt{10}}{3} (1 - \frac{4}{5} y) \psi_{33} \right) \right] \right\} \right\}$$

$$\pm 2 \tan \frac{\theta}{2} \left[ \left[ \frac{q}{q^2} + \frac{\tan^2 \theta}{2} \right]^{1/2} F_A^{(1)} \left( \frac{1 - 2 \sin^2 \theta_W}{2} \right) \left( \frac{2}{3} (1 - 2y) \psi_{11} - \frac{4\sqrt{2}}{3} (1 - \frac{y}{2}) \psi_{31} + \frac{2\sqrt{10}}{3} (1 - \frac{4}{5} y) \psi_{33} \right) \times \left[ \left( \frac{2}{3} \psi_{11} + \frac{\sqrt{2}}{3} \psi_{31} + \frac{\sqrt{10}}{3} \psi_{33} \right) \right] + \frac{\mu^{(1)}}{2} \left( \frac{2}{3} (1 - 2y) \psi_{11} - \frac{4\sqrt{2}}{3} (1 - \frac{y}{2}) \psi_{31} + \frac{2\sqrt{10}}{3} (1 - \frac{4}{5} y) \psi_{33} \right) \right] \right\} \right\} \right\} \right\} \right\}$$

(41)

where, of course, both the Coulomb correction and effects of the Cabibbo angle have been removed.

Neutral current scattering to the $1^+ 0$ 12.7 MeV state $^{12}C$: The transition to the $1^+ 0$ 12.7 MeV state of $^{12}C$ is generated by the isoscalar neutral current which, according to Eq. (37), involves

$$I_{T=0}^{NC} = -(2 \sin^2 \theta_W \frac{1}{2}) \text{ vector current}$$

$$I_{T=0}^{NC} = 0 \text{ axial -- vector current}$$  \hspace{1cm} (42)$$
That is, the isoscalar neutral current is purely vector which, for example in $^{12}$C, thus limiting the contributing multipoles to $T_{J=1}^{mag}$. The isospin factors are trivial

$$
\langle 1;00|\hat{O}_{1;00}|0;00 \rangle = \langle 1;0;0|\hat{O}_{1;0}|0;0 \rangle,
$$

while the single-particle matrix element for the unit isospin operator is

$$
\langle 1^+;0;0|\hat{O}_{1;0}^{(1);0^+};0 \rangle = \sum_{|\alpha|,|\beta|} \psi_{f,i}^{|\alpha|,|\beta|}(1)\langle |\alpha|,|\beta|;1;0;\frac{1}{2} |\hat{O}_{1;0}^{(1);|\beta|};\frac{1}{2} \rangle = \sum_{|\alpha|,|\beta|} \psi_{f,i}^{|\alpha|,|\beta|}(1)|\alpha|,|\beta|;1;0;\frac{1}{2} \langle |\alpha|,|\beta|;1;0;\frac{1}{2} |O_{1;0}^{(1)}||\beta\rangle
$$

It follows that the cross section is

$$
\left( \frac{d\sigma}{d\Omega} \right)_{^{12}C(y,\nu')^{12}C(1^+0)} = \frac{e^2 G_F^2}{\pi} \sin^4 \theta_W \cos^2 \frac{\theta}{2} a^{-2y} \left[ -\frac{q_\mu^2}{2q^2} + \tan^2 \frac{\theta}{2} \frac{q^2}{M^2} \right] F_1^{(0)} \left( \frac{2}{3} \psi_{11} + \frac{\sqrt{2}}{3} \psi_{31} + \frac{\sqrt{10}}{3} \psi_{33} \right) + \frac{\mu^{(0)}}{2} \left( \frac{2}{3} \psi_{11}(1-2y) - \frac{4}{3} \psi_{33}(1-\frac{4}{5}y) \right) \right]^2
$$

The isoscalar vector couplings are functions of $q_\mu^2$, with $F_1^{(0)}(0) = 1$ and $\mu^{(0)}(0) = 0.88$.

## V. REDUCED MATRIX ELEMENT FUNCTIONS IN MATHEMATICA

The Mathematica package *SevenOperators* has been developed and made available online \[8\] to replace and extend the tabulated results of Ref. \[2\]. The main motivation for this package is to make calculations of semileptonic electroweak cross sections and decay rates far less tedious: users can couple it to cross section and rate formulas, explore various truncations of the density matrix, determine density-matrix constraints imposed by known rates or cross sections, etc. Such closed-form expressions previously would have required tedious hand calculations. The *SevenOperators* package has been cross-checked against Ref. \[2\], including the production and posting of tables in the form of Ref. \[2\].

In this section we give a basic description of *SevenOperators*, and provide useful technical notes. As an example of usage, we also show its application to the calculation of the $^{12}$C transitions, which were illustrated in the previous sections.

The package is available in two forms. The first is a notebook version (7operators.nb) that was developed on Mathematica 5.2. The second is a modular version (package.tar.gz) suitable for use with both the graphical and text-based versions of Mathematica. It consists in a .tar.gz archive containing a master file (7operators.m) and several input files, with self-explanatory names, each of them properly commented. The whole package is run by simply running the master file.

*SevenOperators* returns seven functions, which give the reduced matrix elements of the seven basic operators between single-particle states, in the form of Eq. (21). The functions have the variables $y, N', j', N, j, J$ as arguments, and their Mathematica syntax is given in Tables \[II\] and \[III\] for normal and abnormal parity operators respectively.

| Table I: The functions describing the reduced matrix elements of the normal parity operators, as implemented in *SevenOperators* Mathematica package \[8\]. For reference, next to each function we give the reduced matrix element evaluated in *Seven Operators* (see Eqs. \[8\]). Note that the labels $N$ and $j$ fully specify the standard space-spin nonmagnetic quantum numbers for a harmonic oscillator, $n(l 1/2)j$. | | |
|---|---|---|
| Mathematica function | Reduced matrix element |
| $M_J[y, \{N, j\}, \{N, j\}, J]$ | $(n' l' 1/2)j || M_J^{l' l} (qx) || n(l 1/2)j$ |
| $\Sigma y[y, \{N, j\}, \{N, j\}, J]$ | $(n' l' 1/2)j' || \Sigma y^{l' l} (qy) || n(l 1/2)j$ |
| $\Delta y[y, \{N, j\}, \{N, j\}, J]$ | $(n' l' 1/2)j' || \Delta y^{l' l} (qy) || n(l 1/2)j$ |
Notice that Mathematica has built in functions for the Wigner $3j$ and $6j$ symbols. For the practical reason of avoiding warning messages in the output, we explicitly imposed that a $3j$ symbol is put to zero when the conditions for its existence are not satisfied, while the built-in function is used otherwise. A similar procedure was used for $6j$ symbols.

There is no built-in function for the Wigner $9j$ symbols in Mathematica, therefore we had to build one, using the definition:

$$\left\{ \begin{array}{ccc} j_1 & j_2 & J_{12} \\ j_3 & j_4 & J_{34} \\ J_{13} & J_{24} & J \end{array} \right\} = \sum_g (-1)^{2g}(2g + 1) \left\{ \begin{array}{ccc} j_1 & j_2 & J_{12} \\ j_3 & j_4 & J_{34} \\ j_3 & j_4 & J \end{array} \right\} \left\{ \begin{array}{ccc} J_{34} & J & g \\ J_{24} & g & j_1 \\ j_2 & g & j_3 \end{array} \right\}.$$  \hspace{1cm} (46)

The code implements an explicit check of the conditions to have non-zero reduced matrix elements: (i) parity conservation, (ii) that the nodal numbers $n$ and $n'$ are positive: $n > 0$, $n' > 0$, and (iii) that $j, j', J$ satisfy the triangular inequality. Only if all three conditions are satisfied, the code proceeds with the evaluation of the function describing the reduced matrix element, otherwise the function is put to zero.

In Appendix A we include the text of a Mathematica notebook (also available online [8]) where SevenOperators used to calculate the first of the three examples we considered, $^{12}\text{C}(\nu_e, e^-)^{12}\text{N}(\text{g.s.})$, yielding the result shown in Eq. (46).

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[1] See J. D. Walecka, in *Muon Physics*, ed. by V. W. Hughes and C. S. Wu (Academic, New York, 1975), vol. 2, p. 113; T. DeForest, Jr., and J. D. Walecka, Advances in Phys. 15 (1966) 1; T. W. Donnelly and R. Peccei, Phys. Rep. 50 (1979) 1.
[2] T. W. Donnelly and W. C. Haxton, Atomic Data and Nuclear Data Tables 23 (1979) 103.
[3] Wolfram Research, Inc., Mathematica, version 6.0, Champaign, IL (2005); S. Wolfram, *The Mathematica Book*, 5th edition (Wolfram Media/Cambridge University Press) 2003. Also see http://www.wolfram.com/.
[4] S. Cohen and D. Kurath, Nucl. Phys. 73 (1965) 1.
[5] J. Dubach and W. C. Haxton, Phys. Rev. Lett. 41 (1978) 1453.
[6] A. R. Edmonds, *Angular Momentum in Quantum Mechanics*, third printing (Princeton University Press, Princeton, 1974).
[7] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, 1964).
[8] See http://www.int.washington.edu/users/lunardi/70.htm for the download of the SevenOperators package and for additional resources and references.

**Appendix A: Mathematica notebook calculation of $^{12}\text{C}(\nu_e, e^-)^{12}\text{N}(\text{g.s.})$**

This notebook illustrates the uploading of the package (specifically, uploading the master file of the modular version) and the use of the seven basic operator matrix elements to construct the nuclear multipole operators in Sec. IV. The application is the charge-current cross section $^{12}\text{C}(\nu_e, e^-)^{12}\text{N}(\text{g.s.})$, the first example discussed in this paper. The notebook describes points where a user will need to insert steps to account for the specific electroweak reaction, assumed one-body density matrix, isospin matrix elements, and allowed multipolarities. SevenOperators makes an otherwise tedious computation very immediate, eliminating much of the potential for algebraic errors. The final result in this notebook can be further manipulated to improve its form, but we have not done this here, in order to keep this appendix readable. Such final manipulations are also a matter of personal taste.
Example of use of SevenOperators: calculation of neutrino charged current cross section on $^{12}\text{C}$

- This notebook illustrates the use of the Mathematica package SevenOperators to calculate the neutrino charged current cross section for the reaction $^{12}\text{C}(\nu, e^-)^{12}\text{N}(\text{g.s.})$, as given in Eq. (33) of this paper. The final result is Eq. (36). Although we do a specific example, the discussion here points out those steps where a user must supply information to modify this script for other possible transitions and reactions.

First, we upload the package that computes the single-particle reduced space-spin matrix elements of the seven basic operators. The package must be placed in a directory searched by Mathematica, which one can determine with the command $\$Path$.

```
In[1]:= << 7o_master_file.m
```

- We input the onebody density matrix, which here is defined as the set of coefficients relating manybody matrix elements reduced in angular momentum and isospin to single nucleon matrix elements reduced in angular momentum and isospin. In this example calculation we truncate the density matrix expansion to the 1p shell, so that there are four independent density matrix elements. We denote these as $\psi_{1,1}$, $\psi_{1,2}$, $\psi_{2,2}$, and $\psi_{3,3}$, and we use below the relation $\psi_{3,1} = (\psi^+ + \psi^-)/2$ and $\psi_{1,3} = (\psi^+ - \psi^-)/2$, where $\psi^+$ and $\psi^-$ are the symmetric and antisymmetric combinations introduced in the paper because of operator time reversal properties. Because the transition being considered in this example picks out the $J$ and $T$ of the transition, it is sufficient to label the density matrix as $\psi_{2,2}$. Thus $\psi_{2,2}$ gives the density matrix element and the SevenOperators calling arguments. The $\psi_{2,2}$ enter symbolically, but could easily be replaced by numerical values.

```
In[2]:= $\psi = \{\{\psi_{1,1}, \{1, 1/2\}, \{1, 1/2\}\},
\{\psi^+ + \psi^\dagger\}/2, \{1, 1/2\}, \{1, 1/2\}\},
\{\psi^+ - \psi^\dagger\}/2, \{1, 1/2\}, \{1, 3/2\}\},
\{\psi_{2,2}, \{1, 1/2\}, \{1, 3/2\}\}\};
```
We have defined the density matrix in terms of doubly reduced
to doubly reduced matrix elements (which assumes good isospin labels),
but our charged current cross section formulae (either Eq. (33) or the general case Eq. (24)) involve
manybody matrix elements that are not reduced in isospin. Thus we can define a new density
matrix that relates singly reduced manybody matrix elements to doubly reduced manybody matrix
elements. The user in general will have to insert the proper Wigner Eckart 3 J symbol here,
as it depends on the isospins of the initial and final states and the isospin of the transition
(which could be isoscalar for electromagnetic or weak neutral current interactions). Our
example has \( T_f = M_{T_f} = 1 \), \( T = M_T = 1 \), and \( T_i = M_{T_i} = 0 \),
so the Wigner Eckart factor \((-1)^{T_f-M_{T_f}} \text{ThreeJSymbol} (T_f, M_{T_f}, T, M_T, T_i, M_{T_i}) = 1/\sqrt{3})
which we then fold into the density matrix.

Using the package and the formulae in Eq. (30),
we calculate the space – spin matrix elements of the operators \( M_5, L_5, T_{el}, T_{\text{mag}} \). As the density matrix in this example connects the
manybody matrix elements to doubly reduced single nucleon matrix elements,
the user has to multiply the package results for the seven basic operators by the appropriate
single nucleon reduced isospin matrix element. Now for the charged current neutrino
reaction under consideration, the isospin operator is \( \tau_x = (\tau_1 + \tau_2)/2 = -\tau_{1,M_{T_f} = \pm 1}/\sqrt{2} \),
where \( \tau_{1,M_{T_f} = \pm 1} \) denotes the \( M_{T_f} = \pm 1 \) component of the isospin spherical tensor operator
(see the paper, if additional explanation is needed). We have already used the Wigner Eckart
theorem to remove magnetic isospin quantum numbers. As \(<1/2 \parallel \tau_1 \parallel 1/2 > = \sqrt{6})
we see that the single nucleon isospin factor is \(-\sqrt{3})
which we insert by hand below. Users will need to supply analogous factors, depending on the process
(charged current weak, neutral current weak, electromagnetic) being considered.

\[ Y_{1\text{e5}5, T_{\text{mag}} \text{J}} = \{(\{1, 1\} / \sqrt{3}, \{1, 2\}), \{1, 3\}\}, \]
\[ \{(\{2, 1\} / \sqrt{3}, \{2, 2\}), \{2, 3\}\}, \]
\[ \{(\{3, 1\} / \sqrt{3}, \{3, 2\}), \{3, 3\}\}, \]
\[ \{(\{4, 1\} / \sqrt{3}, \{4, 2\}), \{4, 3\}\} \];

\[ M_5[Y_{\text{~, } (Np_\text{-}, jp_\text{-}), (N_\text{-}, J_\text{-})}, J_\text{-}] := \]
\[ -\frac{\mathbf{q}}{\mathbf{M}} \left( P_{\text{f}}[1] \text{OmegaP} \left[ y, (Np, jp), (N, j), J \right] + \frac{1}{2} \text{w} P_{\text{f}}[1] \text{SigmaPP} \left[ y, (Np, jp), (N, j), J \right] \right) (-\sqrt{3}) \]

\[ L_5[Y_{\text{~, } (Np_\text{-}, jp_\text{-}), (N_\text{-}, J_\text{-})}, J_\text{-}] := \left( \frac{\mathbf{q}^2}{2 \mathbf{M}} \right) P_{\text{f}}[1] \text{SigmaPP} \left[ y, (Np, jp), (N, j), J \right] (-\sqrt{3}) \]
Now, we fold the density matrix with the corresponding doubly-reduced single nucleon matrix elements.

We calculate the three terms governing the nuclear response (see Eq. (33)), removing the common term $e^{-2y}$.

Now, one can compose the full expression of the cross section for $^{12}C(y, e^-)^{12}N(g.s.),$ reinserting $e^{-2y}$. 

The cross section is printed.

In[16]:= \text{xsection}[e, y]

Out[16]= \frac{1}{\pi} 2 e^{-y} c^2 \cos \left( \frac{\theta}{2} \right)^2 \cos \left( \theta_\perp \right)^2 \, G_f

\left( \left( \frac{1}{225 \, M \, \pi^2} \right) \left( 5 \, (-1 + 2 \, y) \, \psi_{[1,1]} + \sqrt{2} \left( \sqrt{5} \left( 5 - 4 \, y \right) \, \psi_{[3,3]} + 5 \, (-2 + y) \, \psi^- \right) \frac{\tan \left( \frac{\theta}{2} \right)}{2} \right) - \frac{q^2}{q^2} \tan \left( \frac{\theta}{2} \right) \right)

\left( 5 \, \psi_{[1,1]} \, ( (-1 + 2 \, y) \, \mu_{[1]} + 2 \, F_1[1] ) + \sqrt{2} \left( 5 \, \psi^- \, ( (-2 + y) \, \mu_{[1]} + F_1[1] ) + \sqrt{5} \, \psi_{[3,3]} \, ( (5 - 4 \, y) \, \mu_{[1]} + 5 \, F_1[1] ) \right) \right) \right)

\left( F_1[1] + 1 \, \frac{q^2}{2 \, q^2} \tan \left( \frac{\theta}{2} \right)^2 \right) \left( 5 \, \psi_{[1,1]} \, ( (-1 + 2 \, y) \, \mu_{[1]} + 2 \, F_1[1] ) + \sqrt{2} \left( 5 \, \psi^- \, ( (-2 + y) \, \mu_{[1]} + F_1[1] ) + \sqrt{5} \, \psi_{[3,3]} \, ( (5 - 4 \, y) \, \mu_{[1]} + 5 \, F_1[1] ) \right) \right)^2

\left( 5 \, \frac{\sqrt{2}}{\pi} \, (1 + 2 \, y) \, \omega \, \psi_{[1,1]} \, ( M \, F_1[1] - q^2 \, F_5[1] ) + 2 \, \sqrt{5} \, ( -5 + 2 \, y ) \, \omega \, \psi_{[3,3]} \, ( M \, F_1[1] - q^2 \, F_5[1] ) \right) -

5 \left( 3 \, q^2 \, \psi^- \, F_1[1] + 4 \, ( -1 + y ) \, \omega \, \psi^- \, ( M \, F_1[1] - q^2 \, F_5[1] ) \right)^2

- \left( 5 \, \psi_{[1,1]} \, ( (-1 + 2 \, y) \, \mu_{[1]} + 2 \, F_1[1] ) + \sqrt{2} \left( 5 \, \psi^- \, ( (-2 + y) \, \mu_{[1]} + F_1[1] ) + \sqrt{5} \, \psi_{[3,3]} \, ( (5 - 4 \, y) \, \mu_{[1]} + 5 \, F_1[1] ) \right) \right)

\left( F_1[1] + 1 \, \frac{q^2}{2 \, q^2} \tan \left( \frac{\theta}{2} \right)^2 \right) \left( 5 \, \psi_{[1,1]} \, ( (-1 + 2 \, y) \, \mu_{[1]} + 2 \, F_1[1] ) + \sqrt{2} \left( 5 \, \psi^- \, ( (-2 + y) \, \mu_{[1]} + F_1[1] ) + \sqrt{5} \, \psi_{[3,3]} \, ( (5 - 4 \, y) \, \mu_{[1]} + 5 \, F_1[1] ) \right) \right)^2

\left( 5 \, \frac{\sqrt{2}}{\pi} \, (1 + 2 \, y) \, \omega \, \psi_{[1,1]} \, ( M \, F_1[1] - q^2 \, F_5[1] ) + 2 \, \sqrt{5} \, ( -5 + 2 \, y ) \, \omega \, \psi_{[3,3]} \, ( M \, F_1[1] - q^2 \, F_5[1] ) \right) -

5 \left( 3 \, q^2 \, \psi^- \, F_1[1] + 4 \, ( -1 + y ) \, \omega \, \psi^- \, ( M \, F_1[1] - q^2 \, F_5[1] ) \right)^2

One can continue to manipulate this expression, according to one’s taste. It is straightforward to show this result is equivalent to Eq. (36).