Mutually avoiding paths in random media and largest eigenvalues of random matrices

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Recently, it was shown that the probability distribution function (PDF) of the free energy of a single continuum directed polymer (DP) in a random potential, equivalently of the height of a growing interface described by the Kardar-Parisi-Zhang (KPZ) equation, converges at large scale to the Tracy-Widom distribution. The latter describes the fluctuations of the largest eigenvalue of a random matrix, drawn from the Gaussian Unitary Ensemble (GUE), and the result holds for a DP with fixed endpoints, i.e. for the KPZ equation with droplet initial conditions. A more general conjecture can be put forward, relating the free energies of \( N > 1 \) non-crossing continuum DP in a random potential, to the \( N \)-th largest eigenvalues of the GUE. Here, using replica methods, we provide an important test of this conjecture by calculating exactly the right tails of both PDF’s and showing that they coincide for arbitrary \( N \).

**Introduction.** — Remarkable connections have emerged in the last decade between random matrix theory, growth models, and glassy systems. The celebrated Kardar-Parisi-Zhang (KPZ) equation [1] provides the simplest description for the growth of an interface in presence of noise. This equation sits at the center of a wide universality class [2], encompassing several models and physical systems, such as the polynuclear growth model (PNG) [3], the asymmetric exclusion processes (ASEP) [4–6] and Burgers turbulence [7].

Additionally, the height \( b(x,t) \) of the KPZ interface in \( d \) dimensions can be exactly mapped into (minus) the free energy of a directed polymer (DP) of length \( t \) in a quenched random potential in \( 1 + d \) dimension [8, 9]. The DP is one of the most straightforward realization of a glass, with applications including domain walls in magnets [10], vortex lines in superconductors [11], localization paths in Anderson insulators [12] and even to problems in biophysics [13] and economics [14].

The link between KPZ and DP has been particularly fruitful in \( d = 1 \), where a hidden integrable structure comes to light. In this case, several exact solutions, first for zero temperature [15], and later for finite temperature discrete [16–18] and continuum DP models [19, 20], unveiled an astounding connection: the probability distribution of the (scaled) KPZ height field \( h(x,t) \) coincides with the (scaled) distribution of the largest eigenvalue of a random matrix drawn from the famous Gaussian ensembles; this is the so-called Tracy-Widom (TW) distribution [25], recurring in broad variety of contexts [26]. In particular, if we define as \( \hat{Z}_1(t) \), the partition function in the continuum of a DP, to the one of the GUE spectrum around the edge.

In the present work, we consider an ensemble of \( N \) mutually-avoiding polymers, i.e. several directed paths constrained not to intersect one another, and competing to optimize their total energy in the same random media. We extend the study of the single polymer partition function \( \hat{Z}_1 \), to the one of \( N \) non-crossing paths \( \hat{Z}_N \). We build on a general method which we recently developed to treat any number \( N \) of DPs, but until now, only applied in the specific case \( N = 2 \) to analyze the non-crossing probability [29, 31].

Here we put forward the conjecture that the \( N \)-path free energy takes the form at large time [32]

\[
\ln \hat{Z}_N(t) \approx -Nt/12 + t^{1/3}\hat{\gamma}(N)
\]

where the random variable \( \hat{\gamma}(N) \) coincides in law with partial sum of the \( N \) largest eigenvalues \( \hat{\gamma}_1, \ldots, \hat{\gamma}_N \) of a
The validity of this conjecture for the continuum, finite temperature model, is suggested by an argument of universality [29, 30, 33], together with exact results on discrete DP models at zero temperature, specifically the last passage percolation model [34, 35] and the semi-discrete directed polymer [14, 26, 38].

Obviously showing the equality of the probability distribution functions (PDF) $P_N^{DP}(\zeta)$ and $P_N^{GUE}(\gamma)$ is a major challenge. Here we will provide a first test, by showing that their leading (stretched exponential order) tail approximant functions are identical. More precisely we will show that at large arguments

$$P_N^{DP}(\zeta) = \rho_N^{DP}(\zeta)(1 + O(e^{-a_N\zeta^{3/2}})) \quad (3)$$

$$P_N^{GUE}(\gamma) = \rho_N^{GUE}(\gamma)(1 + O(e^{-a_N\gamma^{3/2}})) \quad (4)$$

with $a_N,a'_N > 0$ and exactly the same function $\rho_N^{DP}(\gamma) = \rho_N^{GUE}(\gamma) = O(e^{-4a_N^{3/2}})$. Here and below $O(e^{-a_N\gamma^{3/2}})$ means at leading exponential accuracy. Note that the function $\rho_N^{GUE}(\gamma)$ is non-trivial, hence the coincidence is a strong hint for the conjecture to hold. For instance in the simpler case of $N = 1$, where the conjecture is known to hold, one has $\rho_N^{GUE}(\gamma) = \Lambda'(\gamma)^2 - \gamma \Lambda(\gamma)^2$. Likewise, we will provide (more complicated) formula for $N > 1$.

Note that non-intersecting Brownian motions (sometimes dubbed “watermelon configurations”) have already been put in relation with Airy processes and Tracy-Widom distributions [39]. These studies hold however in a very different context, in particular in the absence of any quenched disorder.

The GUE ensemble. — To fix the notation we take the GUE specified by the measure $\propto \exp(-\text{Tr} H^2)(dH)$, where $H$ is a complex $N \times N$ hermitian matrix. For large $N$, the support of the spectrum concentrates in $(-\sqrt{2N}, \sqrt{2N})$. Nevertheless, there is a finite probability for the eigenvalues $\lambda_1 > \ldots > \lambda_N$ to fall outside this interval. In particular, introducing the rescaled eigenvalues $\hat{\gamma}_i = (\lambda_i - \sqrt{2N})\sqrt{2N}^{-1/6}$, the mean spacing for the variables close to $\hat{\gamma}_i$ becomes of order unity. In the limit $N \to \infty$, this results in a well-defined determinantal point process, characterized by the correlation functions [40, 41]

$$r_N(x_1, \ldots, x_N) = \text{det}[K_{\lambda}(x_i,x_j)]_{i,j=1}^N \quad (5)$$

for the density probability that there is a scaled eigenvalue in each interval $[x_i,x_i+dx_i]$, $i = 1, \ldots, N$. Here, the Airy Kernel has been introduced as

$$K_{\lambda}(x,y) = \int_0^\infty dw \, \text{Ai}(x+w) \, \text{Ai}(y+w) \quad (6)$$

From this expression, a simple application of the inclusion-exclusion principle permits expressing the joint probability distribution for the $N$-largest (rescaled) eigenvalues [22] (non-vanishing and normalized to unity in the domain $\gamma_1 > \ldots > \gamma_N$)

$$p_N(\gamma_N) = \frac{1}{N!} \prod_{i=1}^N \int_{-\infty}^\infty d\gamma_i \, d\gamma_{N}\cdots d\gamma_1 \, \det[\Lambda(\gamma+v_j) \Lambda(\gamma+v_k)]_{j,k=1}^N \quad (7)$$

where the bold symbol $\gamma_N$ stands for $\gamma_1, \ldots, \gamma_N$ (and similarly for $\mathbf{x}_p$). In the particular case $N = 1$, this expression can be recast as the derivative of a Fredholm determinant: $p_1(\gamma) \equiv f_2(\gamma)$ is the GUE Tracy-Widom function. Setting $f_2(\gamma) = \gamma F_2(\gamma)/\gamma_3$, the cumulative distribution function $F_2(\gamma)$ is expressed as

$$F_2(\gamma) = \text{Det}(1 - \Pi, K_{\lambda}, \Pi_\gamma) \quad (8)$$

with $\Pi_\gamma$ the projector onto $[\gamma_1, +\infty)$.

Sums of largest eigenvalues. — We now introduce the partial sum of the $N$-largest eigenvalues, defined as

$$\hat{\gamma}^N = \hat{\gamma}_1 + \ldots + \hat{\gamma}_N \quad (9)$$

and in the following we will omit the superscript $N$ when not explicitly necessary. The probability distribution $P_N^{GUE}(\gamma)$ for this quantity can be inferred from Eq. [7]

$$P_N^{GUE}(\gamma) = \frac{1}{N!} \int d\gamma_1 \cdots d\gamma_N \delta(\gamma - \sum_{k=1}^N \gamma_k) p_N(\gamma_N) \quad (10)$$

It is useful to introduce the double-sided Laplace Transform (LT) of $P_N^{GUE}(\gamma)$ as

$$\tilde{P}_N^{GUE}(u) := \text{Det}(u^\gamma) = \int_{-\infty}^{\infty} d\gamma \, P_N^{GUE}(\gamma) e^{u\gamma} \quad (11)$$

We are interested in the right-tail $\gamma > 1$, which governs the integral when $u$ is large. Because of the behavior of the tail $K_{\lambda}(\hat{\gamma}_i, \hat{\gamma}_j) \sim e^{-\frac{1}{2}(\hat{\gamma}_i^{3/2} + \hat{\gamma}_j^{3/2})}$, this regime is dominated by the configuration minimizing the sum $\sum_i \hat{\gamma}_i$ at fixed $\gamma = \sum_i \hat{\gamma}_i$: this suggests that large values of the sum $\gamma$ require all the $N$ largest eigenvalues to be of the same order of magnitude, i.e. $\gamma_k \approx \gamma/N$. Then, in order to estimate the tail $\rho_N^{GUE}(\gamma)$ defined by [3] of the distribution of the sum in [10], we can limit the expansion in [7] to the first term $p_N(\gamma_N) \approx r_N(\gamma_N)$. Rearranging the determinant in $r_N(\gamma_N)$, we obtain

$$\rho_N^{GUE}(u) = \frac{1}{N!} \prod_{i=1}^N \int_{-\infty}^\infty dv_i \, \det[\int_{-\infty}^\infty d\gamma e^{u\gamma} \text{Ai}(\gamma+v_j) \text{Ai}(\gamma+v_k)]_{j,k=1}^N \quad (12)$$

which, after some simple manipulations [42] leads to our main result

$$\rho_N^{GUE}(u) = \frac{e^{\frac{N+2}{2}u}}{\pi^{N/2}N!} \prod_{i=1}^N \int_{v_i > 0} e^{-2v_i \gamma} \det[\frac{e^{-\frac{(v_j-v_k)^2}{2\gamma}}}{\sqrt{\gamma}}]_{j,k=1}^N \quad (13)$$
which generalizes the remarkably simple $N = 1$ result for the LT of the tail of the Tracy-Widom distribution

$$\hat{\rho}^\text{GUE}_{N=1}(u) = \frac{\sin^2}{2\pi^{1/2}u^2}$$

For $N > 1$ it takes the general form

$$\hat{\rho}^\text{GUE}_N(u) = \frac{e^{N\beta^2} G(N + 1)}{2N^2(2\pi)^N\sum^N_{j=1}} Q_N \left( \frac{1}{u^3} \right)$$

where $Q_N(z) = 1$ and $Q_N(z)$ admits a series expansion around $z = 0$, since at large large $u$, this last determinant can be computed explicitly. The Laplace inversion of gives the general form of the tail function $\hat{\rho}^\text{GUE}_N(\gamma)$ where the leading behavior at large $\gamma$ is apparent (with $R_N(\gamma) = 1$)

$$\hat{\rho}^\text{GUE}_N(\gamma) = \frac{e^{N\beta^2} G(N + 1)}{2N^2(2\pi)^N\sum^N_{j=1}} R_N(\gamma)$$

where $G(x)$ is the Barnes function. The function $R_N(\gamma)$ can be obtained from subdominant orders in a saddle point expansion and has the form of a double series in $1/\gamma$ and $1/\gamma^{3/2}$. In Fig. 2 we compare these predictions with the empirical distribution for $N = 2$. Note that the exact form of $\hat{\rho}^\text{GUE}_N(\gamma)$ is a major improvement compared to the naive approximation for the tail obtained by setting $R_N(\gamma) = 1$ in [16]. From considerations of Airy function asymptotics it is easy to see that the corrections in [3] to $\hat{\rho}^\text{GUE}(\gamma)$ itself, calculated as above, are indeed subdominant by $O(e^{-a_N\gamma^3/2})$ with $a_N = \frac{2}{3} N^{-3/2}$.

Mutually avoiding directed polymers. — We introduce the partition function of a directed polymer with fixed endpoints $x, y$

$$\hat{Z}_N(x; y|t) = \int_{x(0)=x}^{x(t)=y} Dx e^{-t \int_0^t dx \left[ (\frac{\partial}{\partial x})^2 - \sqrt{2}\delta(\tau(x), \tau) \right]}$$

in a given realization of a random potential with white-noise correlations $\delta_x(x, t) \eta(x, t') = \delta(x-x') \delta(t-t')$. In the following, to simplify the notation we rescale time and space and set $\tilde{c} = 1$. Considering $N$ polymers starting respectively at $x = x_1, \ldots, x_N$ and arriving at $y = y_1, \ldots, y_N$ the partition function constrained to non-intersecting paths can be expressed, using [14], as a single determinant

$$\hat{Z}_N^N(x; y|t) = \det[\hat{Z}_N(x_i; y_j|t)]_{i,j=1}^N$$

This expression involves arbitrary space dependence; in order to simplify it, we consider therefore the limit where all the initial/final points coincide: $x_i = y_i = \epsilon u_i$. In the limit $\epsilon \to 0$, $\hat{Z}_N^N(x; y|t) \approx \frac{N(N+1)}{\epsilon^2} \prod_{i<j}(u_i - u_j)^2 \hat{Z}_N(t)$, where

$$\hat{Z}_N(t) = \det\left[ \hat{\rho}^{-1}_x \delta_y^{-1} \hat{Z}_N(x; y|t)_{|x=y=0} \right]_{i,j=1}^N$$

This random variable will be our quantity of interest. Its integer moments can be treated in the framework of the nested Bethe ansatz (NBA) [29] and of Macdonald processes [27]. As shown in [29], both methods lead to an expansion in terms of a sum over eigenstates of the (integrable) quantum Hamiltonian associated to the attractive $\delta$-Bose gas, i.e., the Lieb-Liniger model. In particular, using a residue expansion of the contour-integral formula of [16], one obtains a series over integer partitions:

$$\hat{Z}_N(t)^m = \sum_{n_1 \geq \ldots \geq n_N} \frac{n!}{n_1! (2\pi)^{n_s} \prod_{i<j}(n_1 - n_2)^2} \phi(k, m) \mathcal{B}_N, m[k, m]$$

where $(m_1, \ldots, m_N)_n$ indicates sum over all integers $m_j \geq 1$ whose sum equals $\sum_{i=1}^n m_j = n = nN$ and the energy of the string configuration has the form $E[k, m] = \sum_{j=1}^N m_j k_j^2 + \frac{1}{12}(m_j - m_j)^2$. Eq. (20) can be interpreted as an expansion over Lieb-Liniger eigenstates composed by $n_s$ strings of sizes $m_1, \ldots, m_N$. Then, the factor $\phi(k, m)$ can be obtained from the normalization of the string eigenstates and has the form [19] [27]

$$\phi(k, m) = \prod_{1 \leq i < j \leq n_s} \frac{(k_i - k_j)^2 + (m_i - m_j)^2/4}{(k_i - k_j)^2 + (m_i + m_j)^2/4}$$

The factor $\mathcal{B}_N, m[k, m]$ encodes the non-crossing constraint and contains all the dependence on $N$ and $m$. It is expressed by introducing $(\mu_{jk} = \mu_j - \mu_k)$

$$\mathcal{B}_N, m[\mu] = \frac{1}{N!} m! \sigma_{\mu} \left[ \prod_{1 \leq i < j \leq n_s} (\mu_{jk} - \mu_{jk}) h(\mu_{jk}) \right] \prod_{1 \leq j < k \leq n_s} f(\mu_{jk})$$

where the functions $h(u) = u(1 + u) + f(u) = u/(1 + u) + f(\lambda)$ and $\sigma_{\lambda} W(\lambda) = \sum_{\lambda} W(\lambda)/\lambda$ is the symmetrization of $W(\lambda)$ over the variables $\lambda$. Then, to obtain $\mathcal{B}_N, m[k, m]$,
one needs to specialize the $n$ variables $\mu = \{\mu_1, \ldots, \mu_n\}$ with $\{\tilde{k}_1, \tilde{k}_2 + i, \ldots, k_1 + i(m_1 - 1), \tilde{k}_2, k_2 + i, \ldots\}$ and $\tilde{k}_j = k_j - (m_j - 1)/2$.

The standard way to extract the PDF of the random variable $\tilde{c}$ in Eq. (14) from the knowledge of the moments in Eq. (20) is to introduce a generating function by

$$g_N(s) = \sum_{m=0}^{\infty} \frac{(-1)^m e^{-s \lambda_m}}{m!} \mathcal{Z}_N(t)^m = \exp(-e^{-\lambda s + t/\lambda} \tilde{c})$$

(23)

where $x$ is related to $s$ by $xe^{-\frac{Nt}{4}} = e^{-\lambda s}$ and we introduce the rescaled time $\lambda = (Nt/4)^{1/3}$. Eq. (23) has two advantages: i) it lifts the constraint over the sum of $m_i$ in Eq. (20); ii) in the limit $t \to \infty$, $g_N(s) \to \text{Prob}(\tilde{c} < (N/4)^{1/3} s)$, i.e. the cumulative distribution function (CDF) of the random variable $\tilde{c}$. Unfortunately, even without the constraint, it is difficult to perform the sum (23) exactly for $N > 1$: indeed, already obtaining a closed expression for $B_{N,m}^{\mu}(\mu)$ is a non-trivial task, which, apart from $B_1, m, [\mu] = 1$, has been overcome only for $N = 2$ [31], where however the sum in Eq. (20) remains an open challenge.

Fortunately however, we can still deal with (23) by replacing each moment with its asymptotics at large time. Although this does not give the exact large time behavior of $g_N(s)$, it is sufficient, as discussed below, to obtain the exact tail behavior of the PDF of $\tilde{c}$. Indeed, such properties already appear in the studies of the case $N = 1$.

At large times $t$ and for fixed $N, m$, the sum in (20) is dominated by the configurations $\mathbf{m}$ with smallest energy $E[\mathbf{k}, \mathbf{m}]$. In general, the energy $E[\mathbf{k}, \mathbf{m}]$ will be minimized by the configurations with the largest possible $m_i$. For a single polymer $N = 1$, this simply translates into $n_s = 1$ and $m_1 = n$. However, for $N > 1$, this configuration gives a vanishing contribution: a general property of $B_{N,m}^{\mu}(\mu, \mathbf{m})$ is that it vanishes on any configuration with at least one $m_j > m$ [31]. This condition has the simple physical interpretation: a bound state (i.e. a string) cannot be formed joining particles which have been constrained to avoid each other. Surprisingly, this property is sufficient to completely determine the value of $B_{N,m}^{\mu}(\mu, \mathbf{m})$ on the lowest energy configuration with non-vanishing contribution, which is the one consisting of a set $N$ $m$-strings, i.e. $n_s = N$ and every $m_j = m$ [43]. Combining Eq. (21) with Eq. (22) on this configuration, we have (omitting now the trivial dependence on $m_j = m$, and noting $k_{ij} = k_i - k_j$)

$$\Phi[k]B_{N,m}^{\mu}[k] = \frac{m^{LN}}{(mN)!} \prod_{1 < i < j < N} (-i)_{k_{ij}} m(i k_{ij})_m$$

(24)

where $(x)_m$ indicates the Pochhammer symbol. Inserting in the formula for the $m$-th moment (20) and keeping only the configuration $m_j = m$, $j = 1, \ldots, n_s$ with $n_s = N$, one finds $\mathcal{Z}_{N,m}^{(0)}(t)$, defined as the leading contribution at large $t$ and fixed $N, m$, to $\mathcal{Z}_N(t)^m$ in Eq. (20) (see [43]).

We now calculate $g_N^{(0)}(s) = \sum_{m=0}^{\infty} (-1)^m x^m \mathcal{Z}_{N,m}^{(0)}(t)$. In order to deal with the summation over $m$ we follow two steps: i) we use the Airy trick [19, 20] to get rid of the factor $m^3$ in the exponent:

$$\int_{-\infty}^{\infty} dy \Phi(y) e^{iyw} = e^{w^3/3};$$

(25)

ii) we rewrite the sum over $m$ using the Mellin-Barnes representation

$$\sum_{m \geq 1} (-1)^m f(m) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dz \frac{f(z)}{z}$$

(26)

where $c \in [0, 1]$ has to be chosen such that the function $f(z)$ does not have singularities for $\Re(z) > c$. After some manipulations (see [33]), one arrives at

$$g_N^{(0)}(s) \equiv 1 - \frac{1}{N!} \int_{-\infty}^{\infty} \frac{dk_i}{2\pi} \int_{0}^{\infty} dy \Phi(y) e^{\sum_{i} k_i^2 + s}$$

\begin{align*}
&\int_{-\infty}^{e^{-1+i\infty}} dz e^{N \sum_{j,k} \text{det} \left(\left[\frac{1}{2z+i k_j k_k}ight]_{j,k=1}^{N}\right)}.
\end{align*}

(27)

We observe how a nice determinantal structure emerges at this level, reminiscent of the $N \times N$ determinant appearing in Eq. (12). To compare further, we obtain the PDF by differentiating with respect to $s$ and we take again the Laplace transform

$$\rho_N^{DP}(u) = \exp(u \tilde{c}) = \int_{-\infty}^{\infty} ds \partial_s g_N^{(0)}(s) e^{(\frac{u}{N})^{1/3}\lambda_m}$$

(28)

The integral over $s$ in Eq. (28) can now be computed by a simple variation of Eq. (25):

$$\int_{-\infty}^{\infty} ds \partial_s g_N^{(0)}(s) e^{\frac{1}{N}\lambda_m} = -\tilde{u} e^{-\tilde{u} \mathcal{S} + e^{\frac{1}{N}}}$$

(29)

where in order to simplify the notation we set $\tilde{u} \equiv (N/4)^{1/3} u$. When inserting this equality back in Eq. (28), the integral over $y$ can be easily performed as $\epsilon > 0$ and leads to a simple pole in at $z = \tilde{u} \sqrt{N}$. This allows us to perform the integral over $z$, by closing the contour in the positive $\Re(z)$ half-plane and arrive at

$$\rho_N^{DP}(u) = e^{\frac{1}{N} \sum_{i=1}^{N} k_{ij}} \prod_{i=1}^{N} e^{\frac{1}{2\tilde{u} +ik_{ij}}} \text{det} \left[\frac{1}{2\tilde{u} + ik_{jk}}\right]_{j,k=1}^{N}$$

(30)

We now check that this expression is equivalent to Eq. (14). Indeed, expanding the determinant in a sum over the permutation group $\mathcal{S}_N$ of $N$ elements through the Leibniz formula and introducing auxiliary variables $v_1, \ldots, v_N$, we have

$$\text{det} \left[\frac{1}{2\tilde{u} + ik_{jk}}\right] = \sum_{\nu \in \mathcal{S}_N} (-1)^{\nu_p} \prod_{j=1}^{N} e^{-2\tilde{u} v_j + i(k_{ij} - k_{pj}) v_j}$$

(31)
where $\sigma_P$ is the signature of $P$. We can now easily perform the gaussian integrals over the $k_1, \ldots, k_N$ variables and, relabeling $P \rightarrow P^{-1}$ in the sum, one obtains exactly the expansion of the determinant in Eq. (32) (see (33) for more details), i.e. the two Laplace transforms coincide

$$\rho^{(0)}_{\text{GUE}}(s) = \rho^{(0)}_{\text{DP}}(s).$$

Via a Laplace inversion, this shows our main statement, below Eq. (2), namely that the two PDF exactly coincide in the tails, i.e.

$$\rho^{(0)}_{\text{GUE}}(\gamma) = \rho^{(0)}_{\text{DP}}(\gamma).$$

Note that we have assumed that the restriction to the $N$-string states, gives the exact tail of the PDF of $\zeta$ at large time, in other words that

$$\lim_{t \rightarrow +\infty} g^{(0)}_{N\pm}(s)|_{s=\pm(4/N)^{1/3}} \approx 1 - \int_{\zeta}^{+\infty} d\zeta' \rho^{(0)}_{\text{GUE}}(\zeta'),$$

and that the neglected terms give a contribution subdominant by $O(e^{-a_N\zeta^{1/3}})$ as in (4). This however can be justified by examining the contributions of the remaining states, which necessarily contain a larger number of strings. As in the case of $N = 1$, these lead to a larger number of Airy functions, hence to subdominant asymptotics.

**Conclusion.** — We analyzed a general correspondence between random variables arising in very different contexts of statistical mechanics: on the one hand, the sum of the $N$ largest eigenvalues in the GUE and on the other, the free energy of $N$ non-crossing directed polymers in a $d = 1 + 1$ random media. We provided a striking indication that these two quantities have the same distributions for any $N$, by comparing the tails of their PDF’s at large positive values. Indeed, the perfect agreement found between the Laplace transforms associated to the leading stretched exponential decays implies the non-trivial matching of an infinite series of coefficients. This naturally extends the well-known $N = 1$ case, where the single-polymer free energy, in turn the KPZ height, is the signature of $\rho^{(0)}_{\text{GUE}}(\zeta)$ as in (4). This however can be justified by examining the contributions of the remaining states, which necessarily contain a larger number of strings. As in the case of $N = 1$, these lead to a larger number of Airy functions, hence to subdominant asymptotics.

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Supplementary Material for EPAPS
Mutually avoiding paths in random media and largest eigenvalues of random matrices

Here we give additional details about the calculations presented in the letter.

Tail of the partial sums of GUE edge eigenvalues

Let us start from the definition in Eq. (11), where \( \rho_N(\gamma) \) is obtained by (9), replacing \( P_N^{GUE}(\gamma_N) \rightarrow r_N(\gamma_N) \).

\[
N! \tilde{\rho}_N^{GUE}(u) = \prod_i \int_{-\infty}^{\infty} d\gamma_i e^{\gamma_i u} \det[K_{\lambda}(\gamma_i, \gamma_j)]_{i,j=1}^N = \sum_{P \in \mathcal{S}_N} (-1)^P \int_{\gamma_i \in \mathbb{R}} \prod_j e^{\gamma_j u} \text{Ai}(\gamma_j + v_j) \text{Ai}(\gamma_j + v_j) = \sum_{P \in \mathcal{S}_N} (-1)^P \int_{\gamma_i \in \mathbb{R}} \prod_j e^{\gamma_j u} \text{Ai}(\gamma_j + v_j) \text{Ai}(\gamma_j + v_j) = \int_{\gamma_i \in \mathbb{R}} \prod_j e^{\gamma_j u} \text{Ai}(\gamma_j + v_j) \text{Ai}(\gamma_j + v_j) \]

which is Eq. (12) in the main text. Using the identity

\[
\int_{-\infty}^{\infty} dq \text{Ai}(q^2 + v + v') e^{i q (v-v')} = 2^{2/3} \pi \text{Ai}(2^{1/3} v) \text{Ai}(2^{1/3} v')
\]

the integral over \( \gamma \) can be performed thanks to the integral formula in Eq. (25). Finally, the integral over \( k \) reduces to a gaussian integral and leads to

\[
N! \tilde{\rho}_N^{GUE}(u) = \frac{e^{-u^3/3} u^{-3/2}}{\pi^{N/2} N!} \prod_{i=1}^N \int_{v_i > 0} e^{-2 v_i} \det \left[ e^{-(v_i - v_j)^2/2} \right]_{i,j=1}^N \]

after rescaling \( v_j \rightarrow 2 v_i / u \), which is Eq. (13) in the main text.

We now study the asymptotic behavior at large \( u \). Since the determinant is symmetric and vanishes whenever \( v_i = v_j \), we have at the leading order in \( 1/u \)

\[
\det \left[ e^{-(v_i - v_j)^2/2} \right]_{i,j=1}^N \approx \frac{1}{G(N+1)} \prod_{i < j} 2 (v_i - v_j)^2 / u^3 \quad .
\]

In this expression, the prefactor can be fixed by setting \( v_i = i \) and computing explicitly the left-hand side

\[
e^{-\frac{2}{u^2}} \sum_i v_i^2 \det \left[ e^{\frac{2 x_i}{u^2}} \right]_{i,j=1}^N = e^{-\frac{2}{u^2}} \sum_i (i^2 + 1) \approx 2^{N(N-1)/2} u^{-3N(N-1)/2} \prod_{i < j} (j-i) \quad .
\]

Then, by comparing with the right-hand side and using that \( \prod_{i \leq j} (j-i) = G(N+1) \), we arrive at Eq. (S4). Note that (S4) is a particular case of the more general identity for any function \( f(x,y) \), differentiable near zero

\[
\det (f(\epsilon x_i, \epsilon x_j))_{i,j=1}^N = e^{N(N-1)} \prod_{i \leq j} \frac{(x_i - x_j)^2}{G(N+1)^2} \det \left[ \partial_x^{j-i} \partial_y^{j-i} f(x,y) \right]_{i,j=1}^N \Big|_{\epsilon = 0} \quad .
\]

valid to leading order in small \( \epsilon \). Indeed, to that order, it is equivalent to insert \( f(x,y) = e^{2xy/u^3} \) whose determinant of derivatives is simply \( G(N+1) \). This equation has been used in the text to arrive at (19).

The integral in (13) can be computed again using the Selberg integral. In particular we have

\[
\prod_{i=1}^n \int_0^{\infty} dz_i z_i^{\alpha-1} e^{-k z_i} \prod_{1 \leq i < j \leq n} |z_i - z_j|^{2\gamma} = \prod_{j=0}^{n-1} \frac{\Gamma(\alpha + j \gamma) \Gamma(1 + (j+1) \gamma)}{k^{\alpha + (n-1) \gamma} \Gamma(1 + \gamma)}
\]

which for \( \gamma = \alpha = 1 \) and \( k = 2 \), simply reduces to \( G(N+1) G(N+2) 2^{-N^2} \). When inserted in (13), it leads to Eq. (15) in the main text.
Leading contribution to the \(m\)-th moment: term with the smallest number of strings, \(n_s = N\).

Let us find the leading non-vanishing contribution \(B_{N,m}[k,m]\). The first non-vanishing contribution comes from the configuration composed by \(N\) \(m\)-strings, i.e. \(n_s = N\) and \(m_j = m\). We come now to the problem of evaluating \(B_{N,m}(\mu)\) for this particular configuration. From its definition, one can show that it is a symmetric polynomial in the variables \(k_j\), hence of degree \(N(N-1)m\). It is strongly constrained by the fact that \(B_{N,m}(\mu) = 0\) whenever \(\mu\) contains a sequence \(k,k+i,..k+im\) for some \(k\). This implies that \(B_{N,m}(\mu) = 0\) when \(k_i = k_i + ip\) with \(p = 1,..m\). This implies that

\[
B_{N,m}(k) = \frac{m!}{(mn)!} \prod_{1 \leq i < j \leq N} \prod_{p=1}^{m} ((k_i - k_j)^2 + p^2) \tag{S8}
\]

where the normalization is fixed by the limit where all \(k_j\) are large, in which case the evaluation of (22) is a simple exercise in combinatorics. The energy and normalization factor for this configuration have the form

\[
E(k,m) = m \sum_{j=1}^{N} k_j^2 + \frac{1}{12} (m - m^3) \quad , \quad \Phi(k,m) = \prod_{1 \leq i < j \leq N} \frac{(k_i - k_j)^2}{(k_j - k_j)^2 + m^2}. \tag{S9}
\]

Multiplying the second equation with (S8) one finds equation (24) in the text with \((x)_{m} = x(x+1)\cdots(x+m-1) = \Gamma(x+m)/\Gamma(x)\). Inserting in (20) keeping only the configuration \(m_j = m\), \(j = 1,\ldots,n_s\) with \(n_s = N\) one finds

\[
Z_{N,m}^{(0)}(t) = Z_{N}(t)^m_{|m_s=m} = \frac{m! e^{-\frac{2}{\sqrt{\pi}}(m-m^3)}}{N!m^N(2\pi)^N} \prod_{j=1}^{N} \int_{-\infty}^{+\infty} dk_j e^{-mk^2_j} \prod_{1 \leq i < j \leq N} \prod_{p=0}^{m-1} \left[ (k_i - k_j)^2 + p^2 \right]. \tag{S10}
\]

Note that this is the exact contribution to the \(m\)-th moment for all \(t\) of the \(N\) \(m\)-string state, which is also the state with the lowest number of strings \(n_s = N\).

What is the significance of the contribution of this state? before discussing that point, let us recall the analysis for \(N = 1\). Then (S10) reduces to

\[
Z_{1,m}^{(0)}(t) = \frac{m! e^{-\frac{2}{\sqrt{\pi}}(m-m^3)}}{m(2\pi)^2} \int_{-\infty}^{+\infty} dk e^{-mk^2} = \frac{m! e^{-\frac{2}{\sqrt{\pi}}(m-m^3)}}{m^{3/2} \sqrt{4\pi t}} \tag{S11}
\]

which is the well known single string contribution \(n_s = 1\) associated to the droplet initial condition. Note that it is valid for all times \(t\). It contains information about the right tail of the PDF, \(P(H,t)\), of the KPZ height, where we denote \(H = h(0,t) + \frac{t}{12}\). More precisely it contains two types of information (i) the right tail of the TW distribution for typical fluctuations, i.e. large values of \(H\) within the regime \(H \sim t^{1/3}\), (ii) the form of \(P(H,t)\) in the large deviation regime, i.e. for atypically large fluctuations \(H \sim t\). In the large deviation regime, from (S11) one finds (see the Supp Mat in Ref. [49]) that at large \(t\)

\[
\ln P(H,t) \simeq -t^{4/3} \frac{4}{3} z^{3/2} - \ln t - \chi(z) + o(1) \quad , \quad z = H/t \quad \text{fixed}. \tag{S12}
\]

where \(\chi(z)\) encodes the first subleading correction [19]

\[
\chi(z) = \ln(4\pi) + \frac{1}{2} \ln z - \ln \left( \Gamma(2\sqrt{z}) \right) \tag{S13}
\]

Note that the integer moments themselves a-priori allow to determine \(\chi(z)\) only for \(z = z_m = m^2/4\) for \(m \in \mathbb{N}^+\). However the resummation of these contributions into a generating function, i.e. \(g^{(0)}(s)\) as performed here in the text, allows to obtain \(\chi(z)\) for all \(z\) (see the derivation in [19]). In the limit \(z \to 0\) using that \(\Gamma(2\sqrt{z}) \simeq 1/(2\sqrt{z})\) this expression matches with the right tail of the TW distribution, i.e. \(P(H,t) \simeq \frac{1}{2\sqrt{\pi t}} f_2\left(\frac{H}{2\sqrt{t}}\right)\) with \(f_2(x) \simeq \frac{1}{2\sqrt{\pi}} e^{-\frac{4}{3} x^{3/2}}\) for \(x \to +\infty\).

Here, we extend this analysis to \(N > 1\). The first property of (S10) is that it gives the leading behavior of the \(m\)-th moment at fixed \(m, N\) and large \(t\), up to an exponential correction, for \(m \geq 2\), more precisely

\[
\bar{Z}_N(t)^{m} = Z_{N,m}^{(0)}(t)(1 + O(e^{-\frac{4}{3} m(m-1)t})) \tag{S14}
\]

where the subdominant terms come from lowest "excitations" with a larger number of strings, i.e. such that \(n_s = N + 1\) (with \(m_j = m\), \(j = 1,..N-1\), \(m_N = m - 1\), \(m_{N+1} = 1\)).
The dominant term in the limit of large $t$ in Eq. (S10) has the form
\[ Z_N(t)^m = \frac{(m-1)!N^d e^{-\frac{Nt}{2\pi}(m-3)}}{(\sqrt{2\pi})^N(2mt)^{N^2/2}}. \] (S15)

Indeed, since $k_j$ scales as $1/\sqrt{t}$, this term can be obtained by replacing in the product $\prod_{p=1}^{m-1} \frac{1}{(k_i - k_j)^2 + p^2}$. The remaining integral over the $k_j$ can then be performed using the Mehta integral
\[ \int_{-\infty}^{\infty} dp_j e^{-\frac{p^2}{2}} \prod_{1\leq j \leq N} (p_i - p_j)^2 = (2\pi)^{N/2} G(N+2). \]

From a saddle point argument similar to the one given in [49], one finds from the formula (S15) that the PDF, $P_N(H,t)$ of the variable $H = Nt/12 + \ln \hat{Z}_N(t)$, should take the form
\[ \ln P_N(H,t) \simeq -4 \int \frac{dz}{\sqrt{N}} e^{-\frac{3}{2} - a_N \ln t - \chi_N(z)} + o(1), \quad z = H/t \text{ fixed}. \] (S16)

in the large deviation regime. Indeed
\[ \bar{m}H = t \int dz e^{-\left(\frac{4}{\sqrt{N}} z^2 - m_2 z - m_3 \right) - a_N \ln t - \chi_N(z)} \simeq t^2 \sqrt{2\pi m N} e^{Nt/m - \chi_N(z)} \] (S17)

where the saddle point is located at $z_m = Nm^2/4$. Matching with the formula (S15) we find
\[ a_N = \frac{1 + N^2}{2} \] (S18)
\[ \chi_N(z) = -N^2 \ln \left(\frac{\sqrt{\frac{z}{N}}}{2} + \frac{N^2}{2} \right) + \frac{1}{2} \ln(4\sqrt{\frac{z}{N}} + 1) + \frac{N}{2} \ln(2\pi) - \ln(N+1) \] (S19)

which is valid a priori for $z = z_m$ for $m \in \mathbb{N}^+$. One could conjecture that it remains the correct analytic continuation for arbitrary $z > 0$. To confirm or infirm this conjecture one would need to analyze the formula (S21) for the generating function $g^0(s)$ in the regime where $s \sim t^{1/3}$, as was done for $N = 1$ in [49]. We leave this for future work.

In the limit $z \to 0$ one should be able to match with the tail of the typical values for $H \sim t^{1/3}$ which has been calculated in this paper. Such a check can be performed on formula (S15) directly. Assuming the simplest analytic continuation ($m-1)! \to \Gamma(m)$ in that formula one can check explicitly that the Laplace transform $\hat{\rho}_N(u)$ obtained in [15] when evaluated at the argument $u = m^{1/3}$ recovers exactly the small $m$ limit of (S15).

**Summation of the generating function**

We now perform the explicit derivation of Eq. (27) given in the text. We insert (S10) in
\[ g_N^0(s) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \bar{m}^{-m} Z_{N,m}(t) = \frac{N}{2\pi} \int_{-\infty}^{\infty} \frac{dz}{\sqrt{\pi z}} e^{\frac{z}{N}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} m! e^{\frac{Nz^2}{2}} i \bar{m}^{-m} \prod_{1 \leq i < j \leq N} \Gamma(m + ik_{ij}) \Gamma(\bar{m} - ik_{ij}) \] (S20)

where in the second line we used the Airy-trick introduced in Eq. (25) in the main text, with $w = m(Nt/4)^{1/3} = m\lambda$. We now use the Mellin-Barnes formula to perform the summation over $m$ in the large time limit as explained in Eq. (26). This amounts to replacing the integer variable $m$ with the integral over the complex $z$, i.e.
\[ g_N^0(s) = -\frac{N}{2\pi} \int_{-\infty}^{\infty} \frac{dz}{\sqrt{\pi z}} e^{\frac{z}{N}} \sum_{i=1}^{\infty} \frac{dz}{2\pi \sin(\pi z)} e^{\frac{z}{N} \sum_{i=1}^{\infty} \frac{1}{\lambda i} - \frac{1}{\bar{m} i} - \frac{1}{\bar{m} i}} \prod_{1 \leq i < j \leq N} \Gamma(z - ik_{ij}) \Gamma(z + ik_{ij}) \] (S21)

We now perform the changes of variables in the integrals $z \to z/\lambda$, $k_i \to k_i - k_i/\sqrt{N}(1/2\lambda)$ and then $y \to y + s + \sum_i k_i^2$ and take the large time limit: $\lambda \to \infty$:
\[ g_N^0(s) = -\frac{N^{1/2}}{2N!} \prod_{j=1}^{N} \int_{-\infty}^{\infty} \frac{dk_j}{2\pi} \int_{-\infty}^{\infty} \frac{dy}{2\pi} e^{\frac{y}{2} + \frac{1}{\lambda} \sum_{i=1}^{\infty} \frac{k_i^2}{\lambda i} + \frac{1}{\bar{m} i}} \prod_{1 \leq i < j \leq N} \left( \frac{2z^{1/2} + \sqrt{N}k_{ij}}{2\lambda} \right) \prod_{1 \leq i < j \leq N} \left( \frac{2z^{1/2} - \sqrt{N}k_{ij}}{2\lambda} \right) \] (S22)
Finally we use the following identity:

$$\prod_{1 \leq i < j \leq N} \frac{(k_i - k_j)^2N}{4z^2 + (k_i - k_j)^2N} = \det \left[ \frac{1}{i(k_i - k_j)\sqrt{N + 2z}} \right]_{i,j=1}^{N} (2z)^N$$

(S23)

and then we replace $z \to \sqrt{Nz}$ to obtain the Eq. (27) in the main text.

**Laplace transform of the DP free energy distribution**

We derived in the text the expression

$$g_N^{(0)}(s) \overset{t \to \infty}{=} 1 - \frac{1}{N!} \prod_{i=1}^{N} \int_{-\infty}^{+\infty} \frac{dk_i}{2\pi} \int_{0}^{\infty} dy (y + \sum_{i} k_i^2 + s) \int_{-\infty}^{+\infty} \frac{dz}{2\pi i z} e^{Nzy} \det \left[ \frac{1}{2z + ik_j} \right]_{j,k=1}^{N}. \quad (S24)$$

The Laplace transform defined in in (28), can then be performed using that:

$$\int_{\infty}^{-\infty} ds \Ai'(y + \sum_{i} k_i^2 + s)e^{(\frac{\pi}{2} + i)^{1/3}u s} = -\tilde{u} e^{-\tilde{u}(\Sigma, k_i^2 + y)} e^{\frac{N\tilde{u}^2}{2}}, \quad \tilde{u} \equiv (N/4)^{1/3}u. \quad (S25)$$

Then, substituting Eq. (S24) in Eq. (28), and using Eq. (S26) to perform the integral over $s$, we have, after integrating over $y$:

$$\rho_N^{DP}(u) = \tilde{u} e^{\frac{N\tilde{u}^2}{2}} N! \prod_{i=1}^{N} \int_{-\infty}^{+\infty} dk_i e^{-\tilde{u}k_i^2} \int_{-\infty}^{+\infty} \frac{dz}{2\pi i z} \frac{1}{\tilde{u} - \sqrt{Nz}} \det \left[ \frac{1}{2z + ik_j} \right]_{j,k=1}^{N} =$$

$$= e^{\frac{N\tilde{u}^2}{2}} \prod_{i=1}^{N} \int_{-\infty}^{+\infty} dk_i e^{-\tilde{u}k_i^2} \left( \sum_{P \in S_N} (-1)^P \prod_{j=1}^{N} \frac{1}{2\tilde{u}N^{-1/2} + ik_j} \right) =$$

$$= e^{\frac{N\tilde{u}^2}{2}} \prod_{i=1}^{N} \int_{-\infty}^{+\infty} dk_i e^{-\tilde{u}k_i^2} \left( \sum_{P \in S_N} (-1)^P \int_{0}^{\infty} \frac{dv_j e^{-2\tilde{u}N^{-1/2}v_j}}{2\pi} \int_{0}^{\infty} \frac{dv_j e^{-2\tilde{u}N^{-1/2}v_j + e^{-i(k_j - k_j)v_j}}}{2\pi} \right) =$$

$$= e^{\frac{N\tilde{u}^2}{2}} \prod_{i=1}^{N} \int_{v_i>0} dv_i e^{-v_iu^{3/2}} \det \left[ e^{-(v_j - v_i)^2} \right]_{j,k=1}^{N} \quad (S26)$$

where in the second line we integrated over $z$, by closing the contour at $\Re[z] > 0$ and in the fourth line we redefined the permutation as $P \to P^{-1}$. To obtain the last line we rescaled $v_j \to \sqrt{\tilde{u}} v_j$.

Now, upon the change of variable $v_i \to 2v_i/u^{3/2}$ one puts (S26) in exactly the same form as (13) in the text, hence showing that $\rho_N^{DP}(u) = \rho_N^{DP}(u)$. 