Abstract
In this chapter of the Oxford Handbook of Random Matrix Theory we introduce chiral Random Matrix Theories with the global symmetries of QCD. In the microscopic domain, these theories reproduce the mass and chemical potential dependence of QCD. The main focus of this chapter is on the spectral properties of the QCD Dirac operator and relations between chiral Random Matrix Theories and chiral Lagrangians. Both spectra of the anti-hermitian Dirac operator and spectra of the nonhermitian Dirac operator at nonzero chemical potential are discussed.


## Contents

1 **Introduction**
   - 1.1 Spontaneous Symmetry Breaking in RMT
   - 1.2 The QCD Partition Function

2 **QCD and Chiral Random Matrix Theory**
   - 2.1 Symmetries of QCD
   - 2.2 Spontaneous Symmetry Breaking
   - 2.3 Chiral Random Matrix Theory
   - 2.4 Chiral Lagrangian
   - 2.5 Generating Function for the Dirac Spectrum
   - 2.6 Chiral Random Matrix Theory and the Dirac Spectrum
   - 2.7 Integrability

3 **ChRMT at Nonzero Chemical Potential**
   - 3.1 Dirac Spectrum
   - 3.2 Low-Energy Limit of QCD and Phase Quenched QCD
   - 3.3 Chiral Lagrangian at Nonzero Chemical Potential
   - 3.4 Chiral Random Matrix Theories at $\mu \neq 0$
   - 3.5 Integrability of the Partition Function
   - 3.6 Spectral Density at $\mu \neq 0$ for QCD with Dynamical Quarks
   - 3.7 The Phase of the Feron Determinant
   - 3.8 QCD at Imaginary Chemical Potential

4 **Applications to Gauge Degrees of Freedom**

5 **Concluding Remarks**
## 1 Introduction

Applications of Random Matrix Theory (RMT) to the physics of strong interactions have a long history (see Chapter 2). RMT was introduced to nuclear physics by Wigner to describe the level spacing distribution of nuclei [Wig55]. This paper inspired a large body of early work on RMT which is summarized in the book by Porter [Por65]. An important conceptual discovery that emerged from this field is the large $N$ approximation. It first appeared in the work of Wigner, and became an integral part of QCD through the seminal work of ’t Hooft [tHo74], which showed that the limit of a large number of colors is dominated by planar diagrams. It was realized soon thereafter that combinatorial factors can be obtained from matrix integrals [Bre78]. This culminated in the random matrix formulation of quantum gravity in 2d, which is a sum over random surfaces that can be triangulated by planar diagrams. (see Chapter 30 and [DiF93] for a review).

The main focus of this chapter to apply RMT to spectra of the Dirac operator both at zero chemical potential, when the Dirac operator is Hermitian, and at nonzero chemical potential, when the Dirac operator is nonhermitian. Because the Euclidean Dirac operator can be interpreted as a Hamiltonian, this application is closer in spirit to the original ideas of Wigner than to the work of ’t Hooft. However, we have benefited greatly from the mathematical techniques that were developed for large $N$ QCD and 2d quantum gravity.

Applications of RMT to QCD have been reviewed extensively in the literature [Guh97b, Jan98b, Ver00, Ver05, Ake07]. These papers offer both additional details and different points of view. Phenomenological applications of RMT to QCD are not discussed in this chapter (see [Ver00, Ake07] for reviews).

In the first half of this chapter we introduce chiral random matrix theory and its applications to QCD at zero chemical potential. In the second half, chiral random matrix theories for QCD at nonzero chemical potential are discussed.

### 1.1 Spontaneous Symmetry Breaking in RMT

One of the essential features of Random Matrix Theory is spontaneous symmetry breaking. The real part of the resolvent

$$G(z) = \frac{1}{N} \left\langle \text{Tr} \frac{1}{z + D} \right\rangle,$$

(1)

where the average is over the probability distribution of the ensemble of $N \times N$ anti-Hermitian random matrices, $D$, can be expressed as the replica limit

$$\text{Re} \ G(z) = \lim_{n \to 0} \frac{1}{2nN} \frac{d}{dz} \log(Z_n(z)), \quad \text{with} \quad z \in \mathcal{R},$$

(2)
where generating function is defined by

$$Z_n(z) = \langle \det^n(D + z)\det^n(z - D) \rangle. \quad (3)$$

For $z = 0$, the generating function is invariant under $Gl(2n)$. This symmetry is broken spontaneously to $Gl(n) \times Gl(n)$ by a nonzero value of $\text{Re} \, G(z)$

$$\lim_{z \to 0} \lim_{N \to \infty} \text{Re} \, G(z) \neq 0. \quad (4)$$

The order of the limits is essential – the reverse order gives zero. This can be seen by expressing the resolvent in terms of eigenvalues of $D$.

When we integrate the resolvent over the contour $C$ in the complex plane that is the boundary of $[-\frac{\epsilon}{2} < \text{Re} \, z < \frac{\epsilon}{2}] \times [-\frac{1}{2}\Delta x < \text{Im} \, z < \frac{1}{2}\Delta x]$ we obtain

$$N \oint_C G(z)dz = 2\pi iN_{\Delta x} = 2\pi i\rho(0)\Delta x, \quad (5)$$

where $N_{\Delta x}$ is the number of eigenvalues enclosed by the contour and $\rho(0)$ is the average spectral density around zero. In the limit $\epsilon \to 0$ the l.h.s. is given by $2i\Delta x\text{Re} \, G(\frac{\epsilon}{2})$. The discontinuity of the resolvent and $\rho(0)$ are thus related by

$$\lim_{\epsilon \to 0} \text{Re} \, G(\frac{\epsilon}{2}) = \frac{\pi \rho(0)}{N}. \quad (6)$$

This formula is known as the Banks-Casher formula [Ban80]. It relates the order parameter for spontaneous symmetry breaking to the spectrum of the associated operator.

When the spectrum of $D$ is reflection symmetric, i.e. $D$ and $-D$ have the same spectrum, then $\det(z - D) = \det(D + z)$, and the generating function for the real part of the resolvent is given by

$$Z_n(z) = \langle \det^n(D + z) \rangle. \quad (7)$$

This is the case when the random matrix ensemble has an involutive symmetry, $ADA = -D$ with $A^2 = 1$, which is the case for the Dirac operator in QCD.

### 1.2 The QCD Partition Function

The Euclidean QCD partition function for $N_f$ quarks with mass $m_f$ is given by

$$Z_{\text{QCD}} = \prod_{f=1}^{N_f} \langle \det(D + m_f) \rangle. \quad (8)$$

The average is over gauge fields weighted by the Yang-Mills action. The gauge fields are elements of the Lie Algebra $SU(N_c)$ and can be in the fundamental
or adjoint representation. The theory that describes the strong interactions has $N_c = 3$ with gauge fields in the fundamental representation. The Dirac operator $D$ is a function of the “random” gauge fields.

Applications of RMT to QCD differ in several respects from other applications. First, the physical system itself is already a stochastic ensemble. Second, the QCD partition function is not a quenched average, but the fermion determinant describes the quark degrees of freedom. Third, the average is over Dirac operators with different rank. The reason is that, according to the Atiyah-Singer index theorem, the number of topological zero modes is equal to the topological charge of the gauge field configuration. Because Dirac operators with different number of topological zero modes turn out to have spectra with different statistical properties, we will treat each topological charge sector separately. Fourth, QCD is a quantum field theory that has to be regularized and renormalized. We notice that the low-lying Dirac spectrum is gauge invariant and renormalizable [Giu09].

2 QCD and Chiral Random Matrix Theory

In this section we construct a RMT with the global symmetries of QCD and determine the parameter range for which it is equivalent to the QCD.

2.1 Symmetries of QCD

To analyze the global symmetries of QCD we consider the Dirac operator for a finite chiral basis. Then it is given by a matrix with the block structure

$$D = \begin{pmatrix} 0 & iA \\ -iA^\dagger & 0 \end{pmatrix}, \quad (9)$$

where $A$ is a complex $N_+ \times N_-$ matrix. Therefore, all nonzero eigenvalues occur in pairs $\pm \lambda_k$. The number of zero eigenvalues is equal to $|N_+ - N_-|$ and is interpreted as the topological charge. Paired zeros may occur, but this set of measure zero and of no interest. The block structure of (9) is due to the axial $U(1)$ symmetry. The partition function also has a vector $U(1)$ symmetry related to the conservation of baryon charge. For $N_f > 1$ the Dirac operator is the direct sum of $N_f$ one-flavor Dirac operators so that QCD has the axial flavor symmetry $U_A(N_f)$ and the vector flavor symmetry $U_V(N_f)$.

For QCD with $N_c \geq 3$ and gauge fields in the fundamental representation there are no other global symmetries. Because $SU(2)$ is pseudoreal, for $N_c = 2$ the Dirac operator has an anti-unitary symmetry [Ver94a]

$$[UK, D] = 0 \quad (10)$$
with \( K \) the complex conjugation operator and \( U \) a fixed unitary matrix with \( U^2 = 1 \). Then it is always possible to find a basis for which \( A \) becomes real \[ \text{Dys62} \], and for \( m = 0 \), we have that \( \det D = \det^2 A \). For \( N_f \) massless flavors the quark determinant occurs in the partition function as \( \det^2 A \). This enlarges the flavor symmetry group to \( U(2N_f) \). The third case is when gauge fields are in the adjoint representation. Then the Dirac operator also has an anti-unitary symmetry but now \( U^2 = -1 \) \[ \text{Ver94a} \]. In this case it is possible to construct a basis for which the Dirac operator can be rearranged in self-dual quaternions \[ \text{Hal95b} \]. The flavor symmetry is also enlarged to \( U(2N_f) \).

QCD in three Euclidean space-time dimensions does not have an involutive symmetry. In that case the flavor symmetry is \( U(N_f) \) for gauge fields in the fundamental representation. For two colors in the fundamental representation and for any number of colors \( \geq 2 \) in the adjoint representation the symmetry group is enlarged to \( O(2N_f) \) \[ \text{Mag99a} \] and \( Sp(2N_f) \) \[ \text{Mag99b} \], respectively.

## 2.2 Spontaneous Symmetry Breaking

The features that mostly determine the physics of QCD at low energy are spontaneous symmetry breaking and confinement. Because of confinement quarks and gluons do not appear in the physical spectrum so that QCD at low energy is a theory of the weakly interacting Goldstone modes associated with the spontaneous breaking of chiral symmetry.

According to the Vafa-Witten theorem \[ \text{Vaf83} \], global vector symmetries of vector-like gauge theories cannot be broken spontaneously. The order parameter for the breaking of the axial symmetry is the chiral condensate

\[
\Sigma \equiv |\langle \bar{\psi}^a \psi^a \rangle| = \lim_{m_a \to 0} \lim_{V \to \infty} \frac{1}{V} \left\langle \text{Tr} \frac{1}{D + m_a} \right\rangle = \frac{\pi \rho(0)}{V},
\]

with \( \rho(0)/V \) the spectral density of the Dirac operator per unit volume of space time. Because of the Banks-Casher relation, the absolute value of the chiral condensate is flavor independent, but its sign is determined by the sign of \( m_a \).

In Table 1 we give the symmetry breaking patterns \[ \text{Pes80, Vys85} \] for the theories mentioned above. We also give the breaking pattern for QCD in three dimensions, but refer to the literature for additional discussions \[ \text{Ver94b, Mag99a, Mag99b, Sza00, Dun02} \].

## 2.3 Chiral Random Matrix Theory

Since the global symmetries of the Dirac operator are a direct consequence of its block structure and the reality properties of its matrix elements, it should be clear how to construct a RMT with the same global symmetries: just replace the nonzero matrix elements by an ensemble of random numbers. Such chiral
Random Matrix Theory (chRMT) is defined by \[ \text{Shu92, Ver94a} \]

\[ Z_{\text{chRMT}}^{\beta,\nu} = \int DA \prod_{k} \det \begin{pmatrix} m_k & A \\ A^\dagger & m_k \end{pmatrix} P(A), \]  
where $A$ is an $N_+ \times N_-$ matrix, and the integration is over the real (and imaginary) parts of $A$. The reality classes are denoted by the Dyson index $\beta$ which is equal to the number of degrees of freedom per matrix element and is the same as in the corresponding QCD like theory (see Table I). The properties of the chRMT partition function do not depend on the details of the probability distribution. This is known as universality \[ \text{Bre95, Jac96a, Ake96, Guh97a} \] and justifies to simply average over a Gaussian distribution

\[ P(A) = c e^{-N\Sigma^2 \text{Tr}(A^\dagger A)} \]  
with $N = N_+ + N_-$. \[ (12) \]

Both the Vafa-Witten theorem \[ \text{Val83} \] and the Banks-Casher formula \[ (6) \] apply to chRMT. The global symmetry breaking pattern and the Goldstone manifold are therefore the same as in QCD. In chRMT, $N$ is interpreted as the volume of space-time. This corresponds to units where $N/V = 1$ so that $\Sigma$ can be written as $\Sigma/V$ and Eq. \[ (13) \] becomes dimensionally correct. Notice that the matrix elements of the Dirac operator and its eigenvalues have the dimension of mass. The normalization of \[ (13) \] is such that $\Sigma$ can be interpreted as the chiral condensate that satisfies the Banks-Casher relation.

### 2.4 Chiral Lagrangian

The low-energy limit of QCD is given by a chiral Lagrangian and necessarily has the name transformation properties as the QCD partition function. The Lorentz invariant chiral Lagrangian to $O(M)$ and $O(p^2)$ is given by

\[ L = \frac{1}{4} F^2 \text{Tr} \partial_\mu U \partial_\mu U^\dagger - \frac{1}{2} \Sigma \text{Tr} [MU^\dagger + M^\dagger U], \]  
where $U \in G/H$ with $G$ the global symmetry group that is spontaneously broken to $H$ (see Table I) and $F$ is the pion decay constant. In the domain

\[ M \ll \frac{\pi^2 F^2}{\Sigma L^2} \ll \frac{\pi^2 F^4}{\Sigma} \]  
the kinetic term factorizes from the chiral Lagrangian \[ \text{Gas87} \]. In this domain, known as the microscopic domain \[ \text{Ver94a} \], the mass dependence of the QCD partition function in the sector of topological charge $\nu$ is given by

\[ Z^{\nu}(M) = \int_{U \in G/H} dU \det U e^{-\frac{1}{2} \Sigma \text{Tr} [MU^\dagger + M^\dagger U]}. \]  
\[ (16) \]
Figure 1: Domains for the mass dependence of the QCD partition function.

The first inequality in (15) can be rewritten as \( 1/M\pi \gg L \), i.e. the pion Compton wavelength is much larger than the size of the box. The domain where QCD is described by the chiral Lagrangian (14) including the kinetic term, i.e. where \( M\pi \sim p \ll \Lambda_{\text{QCD}} \), will be called the chiral domain. The zero momentum partition function (16) and the chiral Lagrangian (14) are the first term of the \( \epsilon \) and \( p \) expansion [Gas87], respectively. Therefore these domains are also known as the \( \epsilon \) domain or \( p \) domain corresponding to a counting scheme where \( M \sim 1/V \) and \( M \sim 1/\sqrt{V} \), in this order.

ChRMT can be reformulated identically in terms of a nonlinear \( \sigma \)-model even for finite size matrices [Shu92, Jac96a] (see also chapter 7). In this formulation the random matrix partition function is given by an integral over two types of modes, would be Goldstone modes with mass \( \sim \sqrt{Nm} \) and massive modes with mass \( \sim \sqrt{N} \) (the number of integration variables does not depend

| Theory               | \( \beta \) | Symmetry G                  | Broken to H          | chRMT |
|----------------------|-------------|-----------------------------|----------------------|-------|
| Fundamental         | 2           | \( U(N_f) \times U(N_f) \)  | \( U(N_f) \)         | chGUE |
| \( N_c \geq 3, d = 4\) |             |                             |                      |       |
| Fundamental         | 1           | \( U(2N_f) \)               | \( Sp(2N_f) \)       | chGOE |
| \( N_c = 2, d = 4 \) |             |                             |                      |       |
| Adjoint             | 4           | \( U(2N_f) \)               | \( O(2N_f) \)        | chGSE |
| \( N_c \geq 2, d = 4\) |             |                             |                      |       |
| Fundamental         | 2           | \( U(N_f) \) \times U(N_f) \ /2 | \( U(N_f/2) \times U(N_f/2) \) | GUE   |
| \( N_c \geq 3, d = 3\) |             |                             |                      |       |
| Fundamental         | 1           | \( Sp(2N_f) \)              | \( Sp(N_f) \times Sp(N_f) \) | GOE   |
| \( N_c = 2, d = 3 \) |             |                             |                      |       |
| Adjoint             | 4           | \( O(2N_f) \)               | \( O(N_f) \times O(N_f) \) | GSE   |
| \( N_c \geq 2, d = 3\) |             |                             |                      |       |

Table 1: Classification of QCD like theories in three and four dimensions. The Dyson index, \( \beta \), is the number of degrees of freedom per matrix element (\( \beta = 1 \) and \( \beta = 4 \) are interchanged for staggered fermions).
on $N$). To leading order in $1/N$ and $m$ the Goldstone mode part factorizes from the partition function. Since the pattern of chiral symmetry breaking for QCD and chRMT is the same, their mass dependence is the same in the microscopic domain \[15\]. The correction terms are of $O(Nm^2)$, so that chRMT results are universal for $m \ll 1/\sqrt{N}$. With the identification of $N$ as $V$, this corresponds to the $\epsilon$ domain.

In lattice QCD, and QCD in general, the topological charge does not affect the block structure of the Dirac matrix but leads to nontrivial correlations between the matrix elements. Then it is natural to work at fixed $\theta$-angle (in fact at $\theta = 0$) with partition function given by $Z(m, \theta)$. However, properties of Dirac eigenvalues depend on the topological charge, and in comparing lattice QCD and chRMT, Dirac spectra are sorted accordingly. In chRMT, topology is included by means of the block structure of the Dirac matrix right from the start, and it is natural to work with fixed topological charge. The two partitions functions are related by

$$Z(m, \theta) = \sum_{\nu=-\infty}^{\infty} e^{i\nu\theta} Z^\nu(m). \quad (17)$$

Since the spectrum of the Dirac operator depends on the topological charge it makes sense to introduce the topological domain \[16\], as the domain where the average properties of the eigenvalues are sensitive to the topological charge. This domain is expected to coincide with the microscopic domain.

### 2.5 Generating Function for the Dirac Spectrum

The generating function for the Dirac spectrum is also given by \[17\] with the determinant of the physical quarks contained in the average. Therefore, $z$ plays the role of a quark mass and the theory will have Goldstone bosons with squared mass $2z\Sigma/F^2$. Therefore, for physical quark masses, i.e. quark masses that remain fixed in the thermodynamic limit, we can choose \[18\] \[19\]

$$z \ll \frac{F^2}{\Sigma L^2} \equiv E_{\text{Th}}. \quad (18)$$

In this domain the $z$-dependence of the generating function at fixed topological charge $\nu$ is given by the zero momentum partition function \[16\] with quark masses equal to $z$. The energy scale in \[18\] is known as the Thouless energy. A similar conclusion was reached in \[20\]. The volume dependence of the Thouless energy has been confirmed by lattice simulations \[21\].

The number of eigenvalues that is described by chRMT scales as $E_{\text{Th}}/\Delta \lambda = F^2 L^2/\pi$. Since $F \sim \sqrt{N_c}$, this number increases linearly with $N_c$ \[22\].

The determinants containing $z$ have to be quenched which can be done by the replica trick (see chapter 8) or the supersymmetric method (see chapter
7). A supersymmetric version of the chiral Lagrangian is known and the zero momentum integral has been evaluated analytically [Osb98a, Dam98b].

2.6 Chiral Random Matrix Theory and the Dirac Spectrum

In an influential paper that motivated the introduction of chRMT, Leutwyler and Smilga [Leu92] proposed to expand the QCD partition function at fixed topology in powers of $m$ and equate the coefficients to the expansion of the same ratio for the low energy limit of QCD, i.e., for the zero momentum partition function. The sum rules are saturated by eigenvalues in the microscopic domain which appear in the combination $\lambda_k V$.

With eigenvalues that scale as $1/V$, the microscopic scaling limit of the spectral density can be defined as [Shu92, Ver94a]

$$\rho_s(z) = \lim_{V \to \infty} \frac{1}{V} \rho \left( \frac{z}{V} \right)$$

(19)

with the microscopic scaling variable defined by $z = \lambda V \Sigma$. For $z \ll \sqrt{V} \Lambda^2$ the microscopic spectral density is given by chRMT. For all three values of the Dyson index, it can be expressed in terms of the Bessel kernel [For93]

$$K_a(x, y) = \frac{x J_{a+1}(x) J_a(y) - y J_{a}(x) J_{a+1}(y)}{x^2 - y^2}.$$  

(20)

The microscopic spectral density for $\beta = 2$ is given by [Ver93]

$$\rho^{\beta=2,a}_s(x) = \lim_{y \to x} K_a(x, y) = \frac{1}{2} x (J_a^2(x) - J_{a+1}(x) J_{a-1}(x)),$$  

(21)

with $a = N_f + |\nu|$. The microscopic spectral density for $\beta = 1$ and $\beta = 4$ can be obtained by rewriting the partition function in terms of skew-orthogonal polynomials. For $\beta = 1$ we find [Ver93, For98, Ake08b]

$$\rho^{\beta=1,a}_s(z) = \frac{1}{4} J_a(z) + \frac{z}{4} \int_0^\infty dw w^a \left( \frac{1}{|z-w|} \left( \frac{1}{w} \frac{d}{dw} - \frac{1}{z} \frac{d}{dz} \right) (zw)^{-a+3/2} K_{a-2}(w, z) \right)$$

$$= \rho^{\beta=2,a}_s(z) + \frac{1}{2} J_a(|z|) \left( 1 - \int_0^{|y|} dt J_a(t) \right)$$

(22)

with $a = 2N_f + |\nu|$, and for $\beta = 4$ the result is [Nag95, Ake08b]

$$\rho^{\beta=4,a}_s(z) = 2z^2 \int_0^1 dw w^2 \int_0^1 dv (1-v^2)^{-1/2} K_a(2uz, 2uvz)$$

$$= \rho^{\beta=2,2a}_s(2z) - \frac{1}{2} J_{2a}(2z) \int_0^{2|z|} dt J_{2a}(t).$$

(23)

where $a = N_f + 2|\nu|$. Similar relations between the microscopic spectral density and the kernel for $\beta = 2$ exist for an arbitrary invariant probability potential.
Figure 2: The valence quark mass dependence of the chiral condensate. Lattice data obtained by the Columbia group [Cha95] are compared to analytical chRMT result given in Eq. (25).

[Sen98, Kle00], and can be exploited to show universality for \( \beta = 1 \) and \( \beta = 4 \) from the universality of the microscopic \( \beta = 2 \) kernel [Ake96] (see Chapter 6).

It is of interest to study the critical exponent of the spectral density at a critical deformation of the Dirac operator for which \( \rho(0) = 0 \) which is a different universality class. It is unlikely that this critical exponent is equal to the mean field value of 1/3 [Jac95]. Other critical exponents can be obtained by fine tuning the probability distribution [Ake97, Ake02a, Jan02].

There are a large number of lattice results for the microscopic Dirac spectrum. The microscopic spectral density (21) was first observed for lattice QCD Dirac spectra through the mass dependence of the resolvent [Ver95] defined as (earlier direct comparisons for the microscopic spectral density were obtained for gauge field configurations given by a liquid of instantons and anti-instantons [Ver94c])

\[
\Sigma(m_v) = \left( \sum_k \frac{1}{\lambda_k + m_v} \right).
\]  

The microscopic limit of \( \Sigma(m_v) \), obtained by replacing the sum by an integral over the microscopic spectral density, is given by a simple expression in terms of Bessel functions [Ver95]

\[
\frac{\Sigma(m_v)}{\Sigma} = \hat{m}_v \left[ I_{N_f+|\nu|}(\hat{m}_v)K_{N_f+|\nu|}(\hat{m}_v) + I_{N_f+|\nu|+1}(\hat{m}_v)K_{N_f+|\nu|+1}(\hat{m}_v) \right].
\]
In Fig. 2 we compare this result to the chiral condensate (24) obtained from lattice simulations [Cha94, Cha95] for two flavors and various values of the coupling constant. The average is over the Yang-Mills action and the fermion determinant with the masses in the fermion determinant kept fixed. Agreement is found with the quenched result because the physical quark masses are much larger than the microscopic scale. There is no dependence on the topological charge because the staggered lattice fermions are not close enough to the continuum limit (This point was investigated in more detail in [Dam99c, Far99]. More recently it was shown [Won04, Fol04, Fol05] that dependence on topology as predicted by chRMT is reproduced by staggered fermions if we are sufficiently close to the continuum limit). Similar lattice results have been obtained for $\beta = 1$ and $\beta = 4$ [Dam99b] together with the corresponding analytical results.

The first direct observation of the microscopic spectral density in QCD was made for staggered fermions with two colors in the fundamental representation [Ber97a] for which the Dyson index is $\beta = 4$ (see right panel of Fig. 3). Results for QCD with $N_c = 3$ were obtained in [Dam98a, Goc98] (Fig. 3, middle). The left panel of Fig. 3 is for staggered fermions in the adjoint representation [Edw99].

The distribution of the smallest Dirac eigenvalue is given by $P^{\beta\nu}_{\min}(s) = -E'(s)$ with $E(s)$ defined as the probability that there are no eigenvalues in the interval $[0, s)$. In Table 2 we summarize analytical results for the quenched case [Ede88, For93, Wil97, Dam00]. The result for $\nu = 0$ is particularly simple. Expressions for the $k$'th smallest eigenvalue at arbitrary quark mass, topological charge and Dyson index are known as well [Wil97, Nis98, Dam00]. A useful measure to compare lattice QCD results with chRMT predictions is the ratio of low lying eigenvalues [Giu03, Pod09]. Agreement with the topology and mass dependence has become an important tool in lattice QCD to test the lattice implementation of chiral symmetry and topology.
The joint probability distribution of chRMT only depends on \( N_f \) and \( \nu \) through the combination \( 2N_f + \beta\nu \). This property, known as flavor-topology duality \([Ver97]\), has been observed in lattice QCD Dirac spectra \([Fuk07]\).

Contrary to correlations of low-lying eigenvalues, bulk spectral correlations can be investigated by spectral averaging, and for large lattice volumes, the Dirac spectrum of a single gauge field configuration is sufficient to obtain statistically significant correlators. Excellent agreement with the Wigner-Dyson ensembles was obtained \([Hal95a]\) without sign of a Thouless energy. It turns out \([Guh98]\) that the Thouless energy scale is due to ensemble averaging. The conclusion is that there is no spectral ergodicity beyond the Thouless energy.

\[
e^{\beta\xi^2/8} P_{\min}(\zeta) = \begin{cases} \frac{1}{4}(2 + \zeta)e^{-\zeta/2} & \nu = 0 \\ \zeta I_3(\zeta) & \nu = 1 \\ \zeta^{(1-\nu)/2}P[(i-j)I_{i+j+3}(\zeta)] & \text{general } \nu \\ \frac{1}{2} I_2(\zeta) & \nu \text{ odd for } \beta = 1 \\ \zeta^{(\beta-\nu)/2} \det I_{i+j+2}(\zeta) & \beta = 2 \\ \zeta^{4\nu-3}(1 + \sum_j a_j (|\nu|)\zeta^j) & \beta = 4 \end{cases}
\]

Table 2: Results for the distribution of the smallest Dirac eigenvalue for the quenched case. In the last row \( a_j \) is an expression in terms of a sum over partitions of \( j \). For explicit expressions we refer to \([Ber98a]\).

### 2.7 Integrability

For \( \beta = 2 \) the partition function of invariant random matrix theories can be interpreted as a partition function of noninteracting fermions which is an integrable system. This is the reason that the low energy QCD partition function, the unitary matrix integral \((10)\), obeys a large number of remarkable relations. The \( N_f \) flavor partition function in the sector of topological charge \( \nu \) can be written as \([Bro81a, Bro81b, Guh96, Jac96b]\)

\[
Z_{N_f}^\nu(m_1, \ldots, m_{N_f}) = \frac{\det[x_k^{k-1} I_{\nu}^{(k-1)}(x_k)]_{k,l=1,\ldots,N_f}}{\Delta(x_k^2)}, \quad x_k = m_k V \Sigma. \tag{26}
\]

In the limit \( x_k \to x \) this partition function reduces to a Hankel determinant

\[
Z_{N_f}^\nu = \det[(x\partial_x)^k I_{\nu}(x)]_{k,l=0,\ldots,N_f-1}. \tag{27}
\]

Applying the Sylvester identity \([For02]\) relating the determinant of a matrix to co-factors gives the Toda lattice equation \([Kan02, Spl03a]\)

\[
(x\partial_x)^2 \log Z_{N_f}^\nu(x) = 2N_f x^2 \frac{Z_{N_f}^{\nu+1}(x) Z_{N_f}^{\nu-1}(x)}{Z_{N_f}^\nu(x)^2}. \tag{28}
\]

After taking the replica limit of this recursion relation we arrive at the following compact expression for the resolvent \([Spl03a]\)

\[
x\partial_x x G(x) = \lim_{N_f \to 0} \frac{1}{N_f} \partial_x \log Z_{N_f}^\nu(x) = 2x^2 Z_{1}^{\nu}(x) Z_{-1}^{\nu}(x). \tag{29}
\]
This factorized form is a general property of the spectral density and correlation functions of RMT’s with $\beta = 2$. Application of the replica limit to a discrete recursion relation does not require analyticity in the replica variable and this way problems with the replica limit can be circumvented [Ver85, Kan02] (see Chapter 8).

As a consequence of integrability relations, the zero momentum partition function satisfies Virasoro constraints. They provide efficient way to determine the coefficients of the small mass expansion [Dam99a, Dal01].

In addition to the relations discussed above we would like to mention the following relations: i) The Toda lattice equation can be formulated for finite size random matrices by exploiting the properties of orthogonal polynomials [Ake04]. ii) In the microscopic domain the partition functions in 3 and 4 dimensions are related by [Ake99, Ake00b, And04]

$$Z_{\text{QCD}}^{2N_f}(\{x_k\}) = Z_{N_f}^{\nu=-1/2}(\{x_k\}) Z_{N_f}^{\nu=1/2}(\{x_k\}), \quad (30)$$

iii) $k$-point spectral correlation functions can be expressed into partition functions with $\beta k$ additional flavors [Ake98]. iv) The correlation functions of invariant RMTs can be expressed in terms of the two-point kernel. This leads to consistency relations between various partition functions [Ake98]. v) Interpreting the quark mass as an additional eigenvalue leads to relations between correlators of massive and massless partition functions [Ake00a].

### 3 ChRMT at Nonzero Chemical Potential

An important application of chRMT is to QCD at nonzero chemical potential $\mu$. In that case the Dirac operator is given by

$$D(\mu) = D(\mu = 0) + \mu \gamma_0. \quad (31)$$

Since $D(\mu = 0)$ is anithermitian, $D(\mu \neq 0)$ has no hermiticity properties, and its eigenvalues are scattered in the complex plane. Because the determinant of the Dirac operator is complex, the QCD partition function at $\mu \neq 0$ is the average of a complex weight, and unless the chemical potential is small, it cannot be simulated by Monte-Carlo methods. For that reason chRMT has been particular helpful to answer questions that could not be addressed otherwise. In particular, the following issues have been clarified: i) The nature of the quenched approximation [Ste96b]. ii) The relation between the chiral condensate and the spectrum of the Dirac operator for QCD with dynamical quarks [Osb05]. iii) The expectation value of the phase of the fermion determinant [Sp06]. iv) The low-energy limit of phase-quenched QCD and the spectrum of the Dirac operator [Tou00]. v) The geometry of the support of the Dirac spectrum [Tou00, Osb08b].
The QCD partition function at nonzero chemical potential is given by

\[ Z_{\text{QCD}} = \left\langle \prod_{k=1}^{N_f} \det(D + m_k + \mu_k \gamma_0) \right\rangle. \] (32)

where the average is over the Yang-Mills action. The chemical potential for different flavors is generally different. Two important special cases are: \( \mu_k = \mu \) when \( \mu \) is the baryon chemical potential, and the case for an even number of flavors \( N_f = 2n \) with \( \mu_k = \mu \) for \( k = 1, \ldots, n \) and \( \mu_k = -\mu \) for \( k = n+1, \ldots, 2n \). In the second case the partition function is positive definite because

\[ \det(D + m - \mu \gamma_0) = \det(D^\dagger + m + \mu \gamma_0) = \det^*(D + m + \mu \gamma_0). \] (33)

For \( n = 1 \), \( \mu \) can be interpreted as an isospin chemical potential [Son00]. Since the determinant appears together with its complex conjugate, this partition function is also known as the phase quenched two-flavor partition function.

### 3.1 Dirac Spectrum

A particular useful tool for studying the spectrum of a nonhermitian operator is the resolvent [1]. The spectral density is given by

\[ \rho(z, z^*) = \left\langle \sum_k \delta^2(z - \lambda_k) \right\rangle = \frac{1}{\pi} \frac{d}{dz^*} G(z). \] (34)

and can be interpreted as the two-dimensional electric field at \( z \) of charges located at \( \lambda_k \). When the eigenvalues are on the imaginary axis this picture illustrates that \( G(z) \) has a discontinuity when \( z \) crosses the imaginary axis. When eigenvalues are not constrained by Hermiticity, because of level repulsion, they will scatter into the complex plane. Using the electrostatic analogy, the resolvent will be continuous. If \( z \) is outside the spectrum, \( G(z) \) is analytic in \( z \).

The resolvent cannot be expressed in terms of Eq. (7) but rather as

\[ G(z) = \lim_{n \to 0} \frac{1}{n} \frac{d}{dz} Z_n(z, z^*), \] (35)

where \( Z_n(z, z^*) \) is the phase quenched partition function [Ste96b]

\[ Z_n(z, z^*) = \langle \det^n(D + z)\det^n(D^\dagger + z^*) \rangle. \] (36)

Lattice QCD Dirac spectra were first calculated in [Bar86]. As remarkable features we note that the spectrum is approximately homogeneous, and that it has a sharp edge which both are explained by chRMT.
3.2 Low-Energy Limit of QCD and Phase Quenched QCD

According to the definition of the grand canonical partition function, the free energy at low temperature does not depend on the chemical potential until it is equal to the lightest physical excitation (per unit charge) with charge conjugate to $\mu$. For QCD this implies that the chiral condensate at zero temperature does not depend on $\mu$ until $\mu = m_N/3$ (with $m_N$ the nucleon mass). The Dirac spectrum, is $\mu$ dependent, though, which seems to violate the Banks-Casher relation [Ban80]. This problem is known as the 'Silver Blaze Problem' [Coh03].

At nonzero isospin chemical, $\mu_I$, the critical chemical potential is equal to $\mu_I = m_\pi/2$. Beyond this point, pions will Bose condense. For light quarks, this phase transition can be studied by chiral perturbation theory, and for quark masses in the microscopic domain it is described by chRMT. At nonzero $\mu_I$, the 'Silver Blaze Problem' is that at zero temperature the chiral condensate remains constant until $\mu_I = m_\pi/2$, while the spectral density depends on $\mu_I$. The solution is easy: according to the electrostatic analogy, the 'electric field', i.e. the chiral condensate, is constant outside a homogeneously charged strip. This implies that the width of the strip is determined by the relation $\mu_I = m_\pi/2$ [Gib86, Tou00]. Indeed, in terms of eigenvalues, the critical point is when the quark mass hits the boundary of the spectrum.

3.3 Chiral Lagrangian at Nonzero Chemical Potential

Chiral symmetry remains broken at small nonzero chemical potential. Therefore, also in this case, the low-energy limit of QCD is given by a theory of weakly interacting Goldstone bosons. As is the case at zero chemical potential, the $U_L(N_f) \times U_R(N_f)$ invariance of the partition function is broken spontaneously to $U_V(N_f)$. The invariance properties of QCD should also hold for the Lagrangian that describes the low-energy limit of QCD. In particular, because the chemical potential is an external vector potential, it only enters in the combination of the covariant derivative [Kog99]

$$\nabla_\nu U = \partial_\nu U - [B_\nu, U], \quad \text{with} \quad B_\nu = \text{diag}(\{\mu_k\})\delta_\nu,0. \quad (37)$$

Together with the mass term, the $O(p^2)$ chiral Lagrangian is thus given by

$$\mathcal{L} = \frac{F^2}{4} \nabla_\nu U \nabla_\nu U^\dagger - \frac{1}{2} \Sigma \text{Tr}(MU + MU^\dagger). \quad (38)$$

Eq. (38) shows that the chiral Lagrangian is determined by two constants. Since the Dirac spectrum at $\mu \neq 0$ is also determined by two constants, the eigenvalue density and the width of the spectrum, we can extract the low energy constants from the geometry of Dirac spectrum.

It has been argued that at sufficient large chemical potential QCD will be in a color-flavor locked phase with spontaneously broken color-flavor symmetry.
In this phase the chiral condensate vanishes, and next order terms in the chiral expansion, which are quadratic in the quark mass, have to be taken into account. Universal results are obtained by scaling the Dirac eigenvalues with $\sqrt{V}$ and Leutwyler-Smilga sum rules have been derived both for QCD with three colors [Yan09] and QCD with two colors [Kan09].

### 3.4 Chiral Random Matrix Theories at $\mu \neq 0$

In a suitably normalized chiral basis the Dirac operator at nonzero chemical potential has the block structure

$$D(\mu) = \begin{pmatrix} 0 & id + \mu \\ id\dagger + \mu & 0 \end{pmatrix}. \tag{39}$$

A chiral random model at nonzero chemical potential is obtained [Ste96b] by replacing the matrix elements of $d$ and $d\dagger$ by an ensemble of random numbers exactly as in section 2.3. Also in this case, because of expected universality [Ake02b], it is justified to simplify the model by choosing a Gaussian distribution.

As is the case for $\mu = 0$, we can distinguish three different nonhermitian chRMTs [Hal97], with complex matrix elements ($\beta = 2$), with real matrix elements ($\beta = 1$), and with self-dual quaternion matrix elements ($\beta = 4$). They apply to the same cases as discussed in table 1. The full classification of nonhermitian ensembles is based on the Cartan classification of symmetric spaces [Zir96, Ber01, Mag07].

The random matrix model (39) is not unique. Adding $\mu$ in a different way results in the same chiral Lagrangian as long as the invariance properties of the matrix model remain the same. One drawback of the model (39) is that the overall unitary invariance has been lost so that methods that rely on the joint probability distribution of eigenvalues cannot be used. A model that does have a representation in terms of eigenvalues is defined by [Osb04]

$$D = \begin{pmatrix} 0 & id + \mu C \dagger \\ id\dagger + \mu C & 0 \end{pmatrix}, \tag{40}$$

where $C$ and $d$ are complex random matrices with the same distribution.

A rerun of the arguments of [Gas87, Dam06] in the microscopic domain

$$m^2V \ll 1 \quad \text{and} \quad \mu^4V \ll 1, \tag{41}$$

shows that the partition function corresponding to the chiral Lagrangian (38) factorizes into a zero momentum part and a nonzero momentum part. The chemical potential and mass dependence reside in the zero momentum part.

In the microscopic domain, invariance properties of QCD at $\mu \neq 0$ that rely on global symmetries can also hold for chRMT at $\mu \neq 0$ and give the same...
invariant terms in the zero momentum sector. Therefore, in this domain, the QCD partition function is given by chRMT at $\mu \neq 0$. Mean field studies only involve the zero momentum part of the chiral Lagrangian. Therefore, mean field results [Kog00] can also be derived from chRMT.

Applying the above arguments to the generating function for the Dirac spectrum we obtain the zero momentum partition function

$$Z^\nu_{n}(z,z^*;\mu) = c \int_{U \in U(2n)} dU \det U e^{-\frac{4\mu^2 F^2}{\Sigma} \text{Tr}[U,B][U^\dagger,B] + \frac{1}{2} \Sigma \text{Tr}(MU+MU^\dagger)}$$

with $B = \Sigma_3$ and $M = \begin{pmatrix} z & 0 \\ 0 & z^* \end{pmatrix}$. (42)

At the mean field level the resolvent is independent of the replica index and the partition function can be analyzed for $n = 1$. For the resolvent we find,

$$G(z) = \Sigma \quad \text{and} \quad G(z) = \frac{\Sigma^2(z+z^*)}{4\mu^2 F^2} \quad (43)$$

for $\text{Re} \, z > 2 \mu^2 F^2/\Sigma$ and $\text{Re} \, z < 2 \mu^2 F^2/\Sigma$, respectively. The eigenvalues are therefore distributed homogeneously inside a strip with width $4F^2\mu^2/\Sigma$. In agreement with lattice simulations [Bar86], the eigenvalue density has a sharp edge whereas the resolvent is continuous at this point. If $z$ and $z^*$ are interpreted as quark masses, the squared mass of the corresponding Goldstone bosons is equal to $m^2_G = (z + z^*)\Sigma/\Sigma$. The condition $\text{Re} \, z < 2 \mu^2 F^2/\Sigma$ can then be written as $\mu = m_G/2$, in agreement with physical considerations.

### 3.5 Integrability of the Partition Function

Remarkably, as was the case $\mu = 0$, the zero momentum partition function (42) can be rewritten in terms of a Hankel like determinant [Spl03a, Spl03b]

$$Z^\nu_{n}(z,z^*;\mu) = D_n(zz^*)^{n(1-n)} \det[(z\partial_z)^{k}(z^*\partial_{z^*})^l Z^\nu_{k+l}(z,z^*,\mu)]_{k,l=0,1,...,n-1}. \quad (44)$$

This form responsible for the integrable structure of the partition function (42). Most notably, it satisfies the Toda lattice equation

$$z\partial_z z^* \partial_{z^*} \log Z^\nu_{n}(z,z^*,\mu) = \frac{\pi n}{2}(zz^*)^2 \frac{Z_{n+1}^\nu(z,z^*,\mu)Z_{n-1}^\nu(z,z^*,\mu)}{[Z_n^\nu(z,z^*,\mu)]^2}. \quad (45)$$

which is obtained by applying the Sylvester identity to the determinant in (44). This equation can be extended to imaginary chemical potential and the two-point correlation function [Spl03b, Dam05, Dam06]. For imaginary chemical potential there is no transition to a Bose condensed state and the $n$-dependent part of the free energy vanishes after differentiation with respect to the masses. The nontrivial result in the free energy is $O(n^2)$ which gives the two-point correlation function.
The replica limit of the Toda lattice equation results in the spectral density
\[ \rho(z, z^*, \mu) = \lim_{n \to 0} \frac{1}{n \pi} \frac{d}{dz} \frac{d}{dz^*} \log Z_n^\nu(z, z^*, \mu) = \frac{1}{2} z z^* Z_n^\nu(z, z^*, \mu) Z_{n-1}^{\nu}(z, z^*, \mu), \]  
which was derived in \[\text{[Spl03b]}\]. The fermionic partition function can be obtained by an explicit evaluation of the integral over \( U(2) \). The result is given by
\[ Z_1^\nu(z, z^*, \mu) = \frac{1}{\pi} e^{2VF_2 \mu^2} \int_0^1 d\lambda \lambda e^{-2VF_2 \mu^2 \lambda^2} I_\nu(\lambda z \Sigma) I_\nu(\lambda z^* \Sigma). \]  
(47)

The evaluation of the bosonic partition is more complicated. The inverse complex conjugated determinants can only be represented as a Gaussian integral after combining them into a Hermitian matrix. In order to obtain a convergent integral, the Hermitian matrix has to be regularized by a mass \( \sim \epsilon \). This procedure is known as Hermitization \[\text{[Jan97, Fei97]}\]. It turns out that the partition function is logarithmically divergent in \( \epsilon \). This divergence is due to a single eigenvalue close to \( z \) and is present even if \( z \) is outside the support of the Dirac spectrum \[\text{[Spl03b, Spl08]}\]. Because of the Vandermonde determinant, the probability of finding two eigenvalues close to \( z \) does not converge. The partition function for one pair of conjugate quarks can also be written as an integral over Goldstone bosons. Instead of an integral over \( U(2) \), using an extension of the Ingham-Siegel integral \[\text{[Fyo01]}\], we obtain an integral over the noncompact manifold of positive definite Hermitian matrices \( Q \)
\[ Z_{n-1}^\nu(z, z^*, \mu) = \lim_{\epsilon \to 0} C_\epsilon \int \frac{dQ}{\det^2 Q} \theta(Q) e^{\text{Tr} \left[ \frac{V}{2} \Sigma^T (Q-IQ^{-1}) - \frac{V}{4} F_2^2 \mu^2 \lambda^2 \right]} I_\nu(\lambda z \Sigma) I_\nu(\lambda z^* \Sigma), \]  
with
\[ I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \zeta = \begin{pmatrix} \epsilon & z \\ z^* & \epsilon \end{pmatrix}. \]  
(48)

The integral over \( Q \) can be performed analytically resulting in
\[ Z_{n-1}^\nu(z, z^*, \mu) = \frac{C_{n-1} e^{-V \mu^2 F_2^2}}{4 \mu^2 F_2 V} e^{\frac{V \Sigma^2 (x^2 + y^2)}{4 \mu^2 F_2^2}} K_\nu \left( \frac{V \Sigma^2 (x^2 + y^2)}{4 \mu^2 F_2^2} \right). \]  
(49)
In Fig. 4 we compare the expression for the spectral density to quenched lattice simulations [Wet04]. The lattice data in the left panel of Fig. 4 are in the strong-nonhermiticity domain where \( \text{Re}(z)\Sigma/2\mu^2F^2 < 1 \) and \( \mu^2F^2V \gg 1 \). In this domain the analytical result can be simplified to [Spl03b, Ver05]

\[
\rho_{N_f=0,\nu}(z, z^*, u) = \frac{2}{\pi} u^2 z z^* K_\nu(u z^* z) I_\nu(u z^* z), \tag{50}
\]

with \( u \) defined as

\[
u = \frac{\Sigma^2V}{4\mu^2F^2}. \tag{51}
\]

The topological index of the staggered lattice Dirac operator at relatively strong coupling in the left panel of Fig. 4 is zero. Lattice results for nonzero topological charge have been obtained using the Bloch-Wettig overlap Dirac operator [Blo06] and are compared to the analytical expression (46) in the right panel of Fig. 4.

Using superbosonization [Hac95, Bun07, Bas07], the fermionic and bosonic partition function can be combined into a supersymmetric partition function [Bas07] which can be used to derive the low-energy limit of the generating function of the QCD Dirac spectrum at \( \mu \neq 0 \).

### 3.6 Spectral Density at \( \mu \neq 0 \) for QCD with Dynamical Quarks

Although the spectral density of the Dirac operator for QCD with dynamical quarks at \( \mu \neq 0 \) was first derived using complex orthogonal polynomials [Osb04], a simpler expression is obtained from the Toda lattice equation [Ake04],

\[
\rho_{N_f}^{\nu}(z, z^*, \mu) \sim z z^* \prod_{f=1}^{N_f} (m_f^2 - z^2) \frac{Z^{n=1}(z, z^*, \{m_f\}, \mu) Z^{N_f}(z, z^*, \{m_f\}, \mu)}{Z^{N_f}(\{m_f\})}. \tag{52}
\]

In Fig. 5 we show a 3d plot of its real part. In addition to a flat region there is a strongly oscillating region with oscillations with an amplitude that increase exponentially with the volume and period that goes like \( \sim 1/V \). This region is absent in the quenched or phase-quenched case and is responsible for the discontinuity in the chiral condensate. As we can see from Fig. 5, we can distinguish three phases in the Dirac spectrum. A phase with no eigenvalues, a phase with a constant eigenvalue density, and a phase with a strongly oscillating eigenvalue density. These phases can be obtained [Osb08b] by means of a mean field study of a partition function with masses \( m, z \) and \( z^* \), similar to the analysis of 3 flavor QCD at nonzero chemical potentials [Kog01]. The phase diagram is shown in the right panel of in Fig. 5.

As we argued before, in the microscopic domain, the chiral condensate does not depend on the chemical potential. Both the flat region and oscillating...
Figure 5: Left: The real part of the spectral density of the QCD Dirac operator for one-flavor QCD at $\mu \neq 0$. For better illustration the z-axis has been clipped. Right: The phase diagram of the Dirac spectrum

region give rise to a $\mu$ dependent contribution to the chiral condensate, but the $\mu$-dependence cancels in their sum [Osb05]. This solves the “Silver Blaze Problem” [Coh03] and can be explained [Osb08a] in terms of orthogonality relations of the complex orthogonal polynomials.

3.7 The Phase of the Fermion Determinant

ChRMT can be used to study the complex phase of the fermion determinant. The average phase factor may be calculated with respect to the quenched, the phase quenched, or the two-flavor partition function. The phase quenched average is given by

$$\langle e^{2i\theta} \rangle_{1+1^*} \equiv \frac{\det(D + m + \mu \gamma_0)}{\det(D^\dagger + m + \mu \gamma_0)}_{1+1^*} = \frac{Z_{1+1}}{Z_{1+1^*}}.$$  \hspace{1cm} (53)

Since $Z_{1+1}$ does not depend on $\mu$, the latter ratio follows immediately from the expression from $Z_{1+1^*}$ given in [42].

The quenched average phase factor can be re-written in terms of a determinant of complex orthogonal polynomials. In the microscopic domain, the result is the sum of a polynomial in $\mu^2$ and a part with an essential singularity at $\mu = 0$. Therefore it cannot be obtained from analytical continuation from an imaginary chemical potential which is polynomial in $\mu^2$ [Dam05, Spl06, Blo08].

3.8 QCD at Imaginary Chemical Potential

Random matrix models at imaginary chemical potential are obtained by replacing $\mu \rightarrow i\mu$ in the Dirac operator. Then the Dirac operator becomes anti-hermitian with all eigenvalues on the imaginary axis. There are two ways of introducing an imaginary chemical potential, either as a multiple of the identity
or as a multiple of a complex random matrix ensemble. Both models have the same symmetry properties and lead to the same universal partition function in the microscopic domain. Spectral correlation functions can be obtained by means of the Toda lattice equation [Dam06], or in the second model, by means of the method of bi-orthogonal polynomials [Ake08a].

Parametric correlations of Dirac spectra in the microscopic domain depend on two low-energy constants, $F$ and $\Sigma$, which can be extracted from correlations of lattice QCD Dirac spectra [Dam06, Deg07, Ake08a].

4 Applications to Gauge Degrees of Freedom

The Eguchi-Kawai model [Egu82] is the lattice Yang-Mills partition function with all links in the same direction identified. In the large $N_c$ limit this model is an integral over $U(N_c)$-matrices. Although the original hope, that Wilson loops of Yang-Mills theory are given by this reduced theory is incorrect, the model continues to attract a considerable amount of attention.

For $d = 4$ the Eguchi-Kawai model cannot be solved analytically, but for $d = 2$ it is known as the Brezin-Gross-Witten model [Bre80, Gro80]

$$Z = \int_{U \in U(N_c)} dU e^{\frac{1}{g^2} \text{Tr}(U + U^\dagger)}.$$ (54)

and is identical to the zero momentum partition function (16) (see Chapter 17 for a discussion of such group integrals). In the large $N_c$ limit this model undergoes a third order phase transition at $g^2 N_c = 2$.

In the large $N_c$ limit eigenvalues of Wilson loops can be analyzed by means of RMT methods. It was shown in [Dur80] that Wilson loops in two dimensions undergo a phase transition for $N_c \rightarrow \infty$ at a critical value of the length of the loop. In one phase, the eigenvalues of the Wilson loop are distributed homogeneously over the unit circle, whereas in the other phase, they are localized at zero. This phase transition has been observed in lattice QCD simulations [Nar06] and has been analyzed in terms of shock solutions of the Burgers equation [Bla08, Bla09, Neu08]. It can be studied by analyzing the eigenvalue distribution of products of unitary matrices [Gud03, Loh08].

5 Concluding Remarks

Random Matrix Theory has changed our perspective of the QCD Dirac spectrum. Before the advent of chiral Random Matrix Theory, the discrete structure of the Dirac spectrum was viewed as random noise that will go away in the continuum limit. Now we know that Dirac eigenvalues show intricate correlations that are determined by chiral random matrix theory with one or two low-energy
constants as parameters. This implies that we can extract the chiral condensate
and the pion decay constant from the distribution of individual eigenvalues.

Chiral Random Matrix Theory primarily applies to the Dirac spectrum,
and therefore we have to distinguish QCD at zero chemical potential, when the
Dirac operator is anti-Hermitian, and QCD at nonzero chemical potential with
a nonhermitian Dirac operator. In the Hermitian case the statistical proper-
ties of the low-lying Dirac eigenvalues are completely determined by the chiral
condensate. In the nonhermitian case they are determined by two parameters,
the chiral condensate, and the pion decay constant. Since the nonhermitian
Dirac spectrum has the geometry of a strip, the two low-energy constants are
determined by the eigenvalue density and the width of the spectrum.

For imaginary chemical potential, the Dirac operator is Hermitian. Al-
though spectral correlations are completely determined by the chiral conden-
sate, this is not the case for parametric correlations, which are correlations of
eigenvalues for two different values of the chemical potential. They require both
the chiral condensate and the pion decay constant as input parameters.

Chiral Random Matrix Theory applies to the low-lying Dirac spectrum.
The scale is set by the momentum dependent terms in the chiral Lagrangian.
Physically, it is the scale of the quark mass for which the Compton wave length
of the Goldstone bosons is much larger the size of the box. The scale of the
chemical potential should also be well below the inverse box size.

There are two mechanisms to explain confinement in QCD: by condensation
of monopoles, or by the disorder of gauge fields. The success of Random Matrix
Theory points to the second mechanism. It our hope that the work discussed
in this chapter will contribute to the solution of this problem.

ACKNOWLEDGMENTS: This work was supported by U.S. DOE Grant No.
DE-FG-88ER40388. Poul Damgaard and Kim Splittorff are thanked for a criti-
cal reading of the manuscript.

References

[Ake96] G. Akemann, P. Damgaard, U. Magnea and S. Nishigaki, Nucl. Phys.
B 487, 721 (1997) [arXiv:hep-th/9609174].

[Ake97] G. Akemann, P. Damgaard, U. Magnea and S. M. Nishigaki, Nucl.
Phys. B 519, 682 (1998) [arXiv:hep-th/9712006].

[Ake98] G. Akemann and P. Damgaard, Phys. Lett. B 432, 390 (1998)
[arXiv:hep-th/9802174].

[Ake99] G. Akemann and P. Damgaard, Nucl. Phys. B 576, 597 (2000)
[arXiv:hep-th/9910190].
[Ake00a] G. Akemann and E. Kanzieper, Phys. Rev. Lett. **85**, 1174 (2000) [arXiv:hep-th/0001188].

[Ake00b] G. Akemann, D. Dalmazi, P. Damgaard and J. Verbaarschot, Nucl. Phys. B **601**, 77 (2001) [arXiv:hep-th/0011072].

[Ake02a] G. Akemann and G. Vernizzi, Nucl. Phys. B **631**, 471 (2002) [arXiv:hep-th/0201165].

[Ake02b] G. Akemann, Phys. Lett. B **547**, 100 (2002) [arXiv:hep-th/0206086].

[Ake04] G. Akemann, J. C. Osborn, K. Splittorff and J. J. M. Verbaarschot, Nucl. Phys. B **712**, 287 (2005) [arXiv:hep-th/0411030].

[Ake07] G. Akemann, Int. J. Mod. Phys. A **22**, 1077 (2007) [arXiv:hep-th/0701175].

[Ake08a] G. Akemann and P. Damgaard, JHEP **0803**, 073 (2008) [arXiv:0803.1171 [hep-th]].

[Ake08b] G. Akemann and P. Vivo, J. Stat. Mech. P09002 (2008).

[And04] T. Andersson, P. Damgaard and K. Splittorff, Nucl. Phys. B **707**, 509 (2005) [arXiv:hep-th/0410163].

[Ban80] T. Banks and A. Casher, Nucl. Phys. B **169**, 103 (1980).

[Bar86] I. Barbour, N. Behilil, E. Dagotto, F. Karsch, A. Moreo, M. Stone and H. Wyld, Nucl. Phys. B **275**, 296 (1986).

[Bas07] F. Basile and G. Akemann, JHEP **0712**, 043 (2007) [arXiv:0710.0376 [hep-th]].

[Ber97a] M. Berbenni-Bitsch, S. Meyer, A. Schafer, J. Verbaarschot and T. Wettig, Phys. Rev. Lett. **80**, 1146 (1998) [arXiv:hep-lat/9704018].

[Ber98a] M. Berbenni-Bitsch, S. Meyer and T. Wettig, Phys. Rev. D **58**, 071502 (1998).

[Ber98b] M. Berbenni-Bitsch et al. Phys. Lett. B **438**, 14 (1998) [arXiv:hep-ph/9804439].

[Ber99] M. Berbenni-Bitsch, M. Gockeler, H. Hehl, S. Meyer, P. E. L. Rakow, A. Schafer and T. Wettig, Phys. Lett. B **466**, 293 (1999) [arXiv:hep-lat/9907014].

[Ber01] D. Bernard and A. LeClair, [arXiv:cone-mat/0110649].
[Bla08] J. Blaizot and M. Nowak, Phys. Rev. Lett. 101, 102001 (2008) [arXiv:0801.1859 [hep-th]].

[Bla09] J. Blaizot and M. Nowak, arXiv:0902.2223 [hep-th].

[Blo06] J. Bloch and T. Wettig, Phys. Rev. Lett. 97, 012003 (2006) [arXiv:hep-lat/0604020].

[Blo08] J. Bloch and T. Wettig, JHEP 0903, 100 (2009) [arXiv:0812.0324 [hep-lat]].

[Bre78] E. Brezin, C. Itzykson, G. Parisi and J. Zuber, Commun. Math. Phys. 59, 35 (1978).

[Bre80] E. Brezin and D. Gross, Phys. Lett. B 97, 120 (1980).

[Bre95] E. Brezin, S. Hikami and A. Zee, Nucl. Phys. B 464, 411 (1996) [arXiv:cond-mat/9511104].

[Bro81a] R. Brower and M. Nauenberg, Nucl. Phys. B 180, 221 (1981).

[Bro81b] R. Brower, P. Rossi and C. I. Tan, Nucl. Phys. B 190, 699 (1981).

[Bun07] J. Bunder, K. Efetov, V. Kravtsov, O. Yevtushenko and M. Zirnbauer, J. Stat. Phys. 129, 809 (2007) [arXiv:0707.2932v1 [cond-mat.mes-hal]].

[Cha94] S. Chandrasekharan, Nucl. Phys. Proc. Suppl. 42, 475 (1995) [arXiv:hep-lat/9412070].

[Cha95] S. Chandrasekharan and N. H. Christ, Nucl. Phys. Proc. Suppl. 47, 527 (1996) [arXiv:hep-lat/9509095].

[Coh03] T. Cohen, Phys. Rev. Lett. 91, 222001 (2003) [arXiv:hep-ph/0307089].

[Dal01] D. Dalmazi and J. Verbaarschot, Phys. Rev. D 64, 054002 (2001) [arXiv:hep-th/0101035].

[Dam98a] P. Damgaard, U. Heller and A. Krasnitz, Phys. Lett. B 445, 366 (1999) [arXiv:hep-lat/9810060].

[Dam98b] P. Damgaard, J. Osborn, D. Toublan and J. Verbaarschot, Nucl. Phys. B 547, 305 (1999) [arXiv:hep-th/9811212].

[Dam99a] P. Damgaard and K. Splittorff, Nucl. Phys. B 572, 478 (2000) [arXiv:hep-th/9912146].
[Dam99b] P. H. Damgaard, R. G. Edwards, U. M. Heller and R. Narayanan, Phys. Rev. D 61, 094503 (2000) [arXiv:hep-lat/9907016].

[Dam99c] P. H. Damgaard, U. M. Heller, R. Nielsen and K. Rummukainen, Phys. Rev. D 61, 014501 (1999) [arXiv:hep-lat/9907019].

[Dam00] P. Damgaard and S. Nishigaki, Phys. Rev. D 63, 045012 (2001) [arXiv:hep-th/0006111].

[Dam05] P. Damgaard, U. Heller, K. Splittorff and B. Svetitsky, Phys. Rev. D 72, 091501 (2005) [arXiv:hep-lat/0508029].

[Dam06] P. Damgaard, U. Heller, K. Splittorff, B. Svetitsky and D. Toublan, Phys. Rev. D 73, 105016 (2006) [arXiv:hep-th/0604054].

[Deg07] T. DeGrand and S. Schaefer, Phys. Rev. D 76, 094509 (2007) [arXiv:0708.1731 [hep-lat]].

[DiF93] P. Di Francesco, P. Ginsparg and J. Zinn-Justin, Phys. Rep. 254, 1 (1995) [arXiv:hep-th/9306153].

[Dun02] G. Dunne and S. Nishigaki, Nucl. Phys. B 654 (2003) 445 [arXiv:hep-ph/0210219].

[Dur80] B. Durhuus and P. Olesen, Nucl. Phys. B 184, 406 (1981).

[Dys62] F. Dyson, J.Math. Phys. 3, 1199 (1962).

[Ede88] A. Edelman, SIAM J. Matrix. Anal.Appl. 9, 543 (1988).

[Edw99] R. Edwards, U. Heller and R. Narayanan, Phys. Rev. D 60, 034502 (1999) [arXiv:hep-lat/9901015].

[Egu82] T. Eguchi and H. Kawai, Phys. Rev. Lett. 48, 1063 (1982).

[Far99] F. Farchioni, I. Hip and C. Lang, Phys. Lett. B 471, 58 (1999) [arXiv:hep-lat/9909132].

[Fei97] J. Feinberg and A. Zee, Nucl. Phys. B 504, 579 (1997) [cond-mat/9703087].

[Fol04] E. Follana, A. Hart and C. Davies [HPQCD Collaboration and UKQCD Collaboration], Phys. Rev. Lett. 93, 241601 (2004) [arXiv:hep-lat/0406010].

[Fol05] E. Follana, A. Hart, C. Davies and Q. Mason [HPQCD Collaboration and UKQCD Collaboration], Phys. Rev. D 72, 054501 (2005) [arXiv:hep-lat/0507011].
[For02] P. Forrester and N. Witte, Comm. P. Appl. Math. 55, 679 (2002).

[For93] P. Forrester, Nucl. Phys. B402, 709 (1993).

[For98] P.J. Forrester, T. Nagao and G. Honner, Nucl. Phys. B533, 601 (1999).

[Fod09] Z. Fodor, K. Holland, J. Kuti, D. Nogradi and C. Schroeder, [arXiv:0907.4562 [hep-lat]].

[Fuk07] H. Fukaya et al. Phys. Rev. Lett. 98, 172001 (2007) [arXiv:hep-lat/0702003].

[Fyo01] Y. Fyodorov, Nucl. Phys. B 621, 643 (2002) [arXiv:math-ph/0106006].

[Gas87] J. Gasser and H. Leutwyler, Phys. Lett. 188B, 477 (1987).

[Gib86] P. Gibbs, Glasgow Preprint 86-0389 (1986).

[Giu03] L. Giusti, M. Luscher, P. Weisz and H. Wittig, JHEP 0311, 023 (2003) [arXiv:hep-lat/0309189].

[Giu09] L. Giusti and M. Luscher, JHEP 0903, 013 (2009) [arXiv:0812.3638 [hep-lat]].

[Goc98] M. Gockeler, H. Hehl, P. Rakow, A. Schafer and T. Wettig, Phys. Rev. D 59, 094503 (1999) [arXiv:hep-lat/9811018].

[Gro80] D. Gross and E. Witten, Phys. Rev. D 21, 446 (1980).

[Gud03] E. Gudowska-Nowak, R. Janik, J. Jurkiewicz and M. A. Nowak, Nucl. Phys. B 670, 479 (2003) [arXiv:math-ph/0304032].

[Guh96] T. Guhr and T. Wettig, J. Math. Phys. 37, 6395 (1996) [arXiv:hep-th/9605110].

[Guh97a] T. Guhr and T. Wettig, Nucl. Phys. B 506, 589 (1997) [arXiv:hep-th/9704055].

[Guh97b] T. Guhr, A. Muller-Groeling and H. Weidenmuller, Phys. Rept. 299, 189 (1998) [arXiv:cond-mat/9707301].

[Guh98] T. Guhr, J. Ma, S. Meyer and T. Wilke, Phys. Rev. D 59, 054501 (1999) [arXiv:hep-lat/9806003].

[Hac95] G. Hackenbroich and H. Weidenmueller, Phys. Rev. Lett. 74, 4118 (1995).
[Hal95a] A. Halasz and J. Verbaarschot, Phys. Rev. Lett. 74, 3920 (1995) [arXiv:hep-lat/9501025].

[Hal95b] A. Halasz and J. Verbaarschot, Phys. Rev. D 52, 2563 (1995) [arXiv:hep-th/9502096].

[Hal97] M. Halasz, J. Osborn and J. Verbaarschot, Phys. Rev. D 56, 7059 (1997) [arXiv:hep-lat/9704007].

[Jac95] A. Jackson and J. Verbaarschot, Phys. Rev. D 53, 7223 (1996) [arXiv:hep-ph/9509324].

[Jac96a] A. Jackson, M. Sener and J. Verbaarschot, Nucl. Phys. B 479, 707 (1996) [arXiv:hep-ph/9602225].

[Jac96b] A. Jackson, M. Sener and J. Verbaarschot, Phys. Lett. B 387, 355 (1996) [arXiv:hep-th/9605183].

[Jan97] R. Janik, M. Nowak, G. Papp and I. Zahed, Acta Phys. Polon. B 28, 2949 (1997) [arXiv:hep-th/9710103].

[Jan98a] R. Janik, M. Nowak, G. Papp and I. Zahed, Phys. Rev. Lett. 81, 264 (1998) [arXiv:hep-ph/9803289].

[Jan98b] R. Janik, M. Nowak, G. Papp and I. Zahed, Acta Phys. Polon. B 29, 3957 (1998) [arXiv:hep-ph/9812376].

[Jan02] R. Janik, Nucl. Phys. B 635, 492 (2002) [arXiv:hep-th/0201167].

[Kan09] T. Kanazawa, T. Wettig and N. Yamamoto, JHEP 0908, 003 (2009) [arXiv:0906.3579 [hep-ph]].

[Kle00] B. Klein and J. Verbaarschot, Nucl. Phys. B 588, 483 (2000) [arXiv:hep-th/0004119].

[Kog99] J. Kogut, M. Stephanov and D. Toublan, Phys. Lett. B 464, 183 (1999) [arXiv:hep-ph/9906346].

[Kog00] J. Kogut, M. Stephanov, D. Toublan, J. Verbaarschot and A. Zhitnitsky, Nucl. Phys. B 582, 477 (2000) [arXiv:hep-ph/0001171].

[Kog01] J. Kogut and D. Toublan, Phys. Rev. D 64, 034007 (2001) [arXiv:hep-ph/0103271].
[Leh09] C. Lehner, M. Ohtani, J. Verbaarschot and T. Wettig, arXiv:0902.2640 [hep-th].

[Leu92] H. Leutwyler and A. Smilga, Phys. Rev. D 46, 5607 (1992).

[Loh08] R. Lohmayer, H. Neuberger and T. Wettig, JHEP 0811, 053 (2008) arXiv:0810.1058 [hep-th].

[Mag99a] U. Magnea, Phys. Rev. D 61, 056005 (2000) arXiv:hep-th/9907096.

[Mag99b] U. Magnea, Phys. Rev. D 62, 016005 (2000) arXiv:hep-th/9912207.

[Mag07] U. Magnea, J. Phys. A: Math. Theor. 41, 045203 (2008) arXiv:0707.0418 [math-ph].

[Nag95] T. Nagao and P. Forrester, Nucl. Phys. B 435, 401 (1995).

[Nar04] R. Narayanan and H. Neuberger, Nucl. Phys. B 696, 107 (2004) arXiv:hep-lat/0405025.

[Nar06] R. Narayanan and H. Neuberger, Phys. Lett. B 646, 202 (2007) arXiv:hep-lat/0612006.

[Neu08] H. Neuberger, Phys. Lett. B 670 (2008) 235 arXiv:0809.1238 [hep-th].

[Nis98] S. Nishigaki, P. Damgaard and T. Wettig, Phys. Rev. D 58, 087704 (1998) arXiv:hep-th/9803007.

[Osb98a] J.C. Osborn, D. Toublan and J. J. M. Verbaarschot, Nucl. Phys. B 540, 317 (1999) arXiv:hep-th/9806110.

[Osb98b] J. Osborn and J. Verbaarschot, Phys. Rev. Lett. 81, 268 (1998) arXiv:hep-ph/9807490.

[Osb04] J. Osborn, Phys. Rev. Lett. 93, 222001 (2004) arXiv:hep-th/0403131.

[Osb05] J. Osborn, K. Splittorff and J. Verbaarschot, Phys. Rev. Lett. 94, 202001 (2005) arXiv:hep-th/0501210.

[Osb08a] J. Osborn, K. Splittorff and J. Verbaarschot, Phys. Rev. D 78, 065029 (2008) arXiv:0805.1303 [hep-th].

[Osb08b] J. Osborn, K. Splittorff and J. Verbaarschot, Phys. Rev. D 78, 105006 (2008) arXiv:0807.4584 [hep-lat].

[Pes80] M. Peskin, Nucl. Phys. B175, 197 (1980).
[Por65] C. Porter, *Statistical Theory of Spectra: Fluctuations*, Academic Press, New York, 1965.

[Sen98] M. Sener and J. Verbaarschot, Phys. Rev. Lett. **81**, 248 (1998) [arXiv:hep-th/9801042].

[Shu92] E. Shuryak and J. Verbaarschot, Nucl. Phys. A **560**, 306 (1993) [arXiv:hep-th/9212088].

[Son00] D. Son and M. Stephanov, Phys. Rev. Lett. **86**, 592 (2001) [arXiv:hep-ph/0005225].

[Spl03a] K. Splittorff and J. Verbaarschot, Phys. Rev. Lett. **90**, 041601 (2003) [arXiv:cond-mat/0209594].

[Spl03b] K. Splittorff and J. Verbaarschot, Nucl. Phys. B **683**, 467 (2004) [arXiv:hep-th/0310271].

[Spl06] K. Splittorff and J. Verbaarschot, Phys. Rev. Lett. **98**, 031601 (2007) [arXiv:hep-lat/0609076].

[Spl08] K. Splittorff, J. Verbaarschot and M. Zirnbauer, Nucl. Phys. B **803**, 381 (2008) [arXiv:0802.2660 [hep-th]].

[Ste96b] M. Stephanov, Phys. Rev. Lett. **76**, 4472 (1996) [arXiv:hep-lat/9604003].

[Sza00] R. Szabo, Nucl. Phys. B **598**, 309 (2001) [arXiv:hep-th/0009237].

[tHo74] G. ’t Hooft, Nucl. Phys. B **75**, 461 (1974).

[Tou00] D. Toublan and J. Verbaarschot, Nucl. Phys. B **603**, 343 (2001) [arXiv:hep-th/0012144].

[Vaf83] C. Vafa and E. Witten, Nucl. Phys. B **234**, 173 (1984).

[Ver85] J. Verbaarschot and M. Zirnbauer, J. Phys. A **18**, 1093 (1985).

[Ver93] J. Verbaarschot and I. Zahed, Phys. Rev. Lett. **70**, 3852 (1993) [arXiv:hep-th/9303012].

[Ver94a] J. Verbaarschot, Phys. Rev. Lett. **72**, 2531 (1994) [arXiv:hep-th/9401059].

[Ver94b] J. Verbaarschot and I. Zahed, Phys. Rev. Lett. **73**, 2288 (1994) [arXiv:hep-th/9405005].

[Ver94c] J. Verbaarschot, Nucl. Phys. B **427**, 534 (1994) [arXiv:hep-lat/9402006].
| Reference | Author(s) | Journal/Conference | Year | Volume | Pages | URL |
|-----------|-----------|--------------------|------|--------|-------|-----|
| [Ver95]   | J. Verbaarschot | Phys. Lett. B | 1996 | 368    | 137   | [arXiv:hep-ph/9509369](http://arxiv.org/abs/hep-ph/9509369) |
| [Ver97]   | J. Verbaarschot | [arXiv:hep-th/9710114](http://arxiv.org/abs/hep-th/9710114) |
| [Ver00]   | J. Verbaarschot and T. Wettig | Ann. Rev. Nucl. Part. Sci. | 2000 | 50    | 343   | [arXiv:hep-ph/0003017](http://arxiv.org/abs/hep-ph/0003017) |
| [Ver05]   | J. Verbaarschot | [arXiv:hep-th/0502029](http://arxiv.org/abs/hep-th/0502029) |
| [Vys85]   | M. Vysotskii, Y. Kogan and M. Shifman | Sov. J. Nucl. Phys. | 1985 | 42    | 318   |
| [Wet04]   | T. Wettig | Private communication | 2004 |       |       |
| [Wig55]   | E. Wigner | Ann. Math. | 1955 | 62    | 548   |
| [Wil97]   | T. Wilke, T. Guhr and T. Wettig | Phys. Rev. D | 1998 | 57    | 6486  | [arXiv:hep-th/9711057](http://arxiv.org/abs/hep-th/9711057) |
| [Won04]   | K. Y. Wong and R. M. Woloshyn | Phys. Rev. D | 2005 | 71    | 094508 | [arXiv:hep-lat/0412001](http://arxiv.org/abs/hep-lat/0412001) |
| [Yam09]   | N. Yamamoto and T. Kanazawa | Phys. Rev. Lett. | 2009 | 103   | 032001 | [arXiv:0902.4533](http://arxiv.org/abs/0902.4533) [hep-ph] |
| [Zir96]   | M. Zirnbauer | J. Math. Phys. | 1996 | 37    | 4986  |