Abstract. For \( k \geq 1 \), a \( k \)-colouring \( c \) of \( G \) is a mapping from \( V(G) \) to \( \{1, 2, \ldots, k\} \) such that \( c(u) \neq c(v) \) for any two adjacent vertices \( u \) and \( v \). The \( k \)-COLOURING problem is to decide if a graph \( G \) has a \( k \)-colouring. For a family of graphs \( \mathcal{H} \), a graph \( G \) is \( \mathcal{H} \)-free if \( G \) does not contain any graph from \( \mathcal{H} \) as an induced subgraph. Let \( C_s \) be the \( s \)-vertex cycle. In previous work (MFCS 2019) we examined the effect of bounding the diameter on the complexity of \( 3 \)-Colouring for \((C_3, \ldots, C_s)\)-free graphs and \( H \)-free graphs where \( H \) is some polyad. Here, we prove for certain small values of \( s \) that \( 3 \)-Colouring is polynomial-time solvable for \( C_s \)-free graphs of diameter 2 and \((C_4, C_s)\)-free graphs of diameter 2. In fact, our results hold for the more general problem \( \text{List } 3 \)-Colouring. We complement these results with some hardness result for diameter 4.

1 Introduction

Graph colouring is a well-studied topic in Computer Science due to its wide range of applications. A \( k \)-colouring of a graph \( G \) is a mapping \( c : V(G) \to \{1, \ldots, k\} \) that assigns each vertex \( u \) a colour \( c(u) \) in such a way that \( c(u) \neq c(v) \) for any two adjacent vertices \( u \) and \( v \) of \( G \). The aim is to find the smallest value of \( k \) (also called the chromatic number) such that \( G \) has a \( k \)-colouring. The corresponding decision problem is called COLOURING, or \( k \)-COLOURING if \( k \) is fixed, that is, not part of the input. As even \( 3 \)-COLOURING is \( \text{NP} \)-complete \cite{[17]} \(, k \)-COLOURING and COLOURING have been studied for many special graph classes, as surveyed in, for example, \cite{[1, 5, 10, 14, 16, 20, 24, 25, 28, 29]}. This holds in particular for hereditary classes of graphs, which are the classes of graphs closed under vertex deletion.

It is well known and not difficult to see that a class of graphs is hereditary if and only if it can be characterized by a unique set \( \mathcal{F}_G \) of minimal forbidden induced subgraphs. In particular, a graph \( G \) is \( H \)-free for some graph \( H \) if \( G \) does not contain \( H \) as an induced subgraph. The latter means that we cannot modify \( G \) into \( H \) by a sequence of vertex deletions. For a set of graphs \( \{H_1, \ldots, H_p\} \), a graph \( G \) is \((H_1, \ldots, H_p)\)-free if \( G \) is \( H_i \)-free for every \( i \in \{1, \ldots, p\} \).

We continue a long-term study on the complexity of 3-COLOURING for special graph classes. Let \( C_t \) and \( P_t \) be the cycle and path, respectively, on \( t \) vertices.

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The complexity of 3-COLOURING for $H$-free graphs has not yet been classified; in particular this is still open for $P_t$-free graphs for every $t \geq 8$, whereas the case $t = 7$ is polynomial [16]. For $t \geq 3$, let $C_{\geq t} = \{C_{t+1}, C_{t+2}, \ldots\}$. Note that for $t \geq 2$, the class of $P_t$-free graphs is a subclass of $C_{\geq t}$-free graphs. Recently, Gartland et al. [19] gave for every $t \geq 3$, a quasi-polynomial-time algorithm for 3-COLOURING on $C_{\geq t}$-free graphs. Rojas and Stein [26] proved in another recent paper that for every odd integer $t \geq 9$, 3-COLOURING is polynomial-time solvable for $(C_{\geq t-3}^{dd}, P_t)$-free graphs, where $C_{\geq t}^{dd}$ is the set of all odd cycles on less than $t$ vertices. This complements a result from [11], which implies that for every $t \geq 1$, 3-COLOURING, or more general List 3-COLOURING (defined later), is polynomial-time solvable for $(C_4, P_t)$-free graphs (see also [19]).

The graph classes in this paper are only partially characterized by forbidden induced subgraphs: we also restrict the diameter. The distance $\text{dist}(u, v)$ between two vertices $u$ and $v$ in a graph $G$ is the length (number of edges) of a shortest path between them. The diameter of a graph $G$ is the maximum distance over all pairs of vertices in $G$. Note that the $n$-vertex path $P_n$ has diameter $n - 1$, but by removing an internal vertex the diameter becomes infinite. Hence, for every integer $d \geq 2$, the class of graphs of diameter at most $d$ is not hereditary (whereas if $d = 1$ we obtain the class of complete graphs, which is hereditary).

For every $d \geq 3$, the 3-COLOURING problem for graphs of diameter at most $d$ is NP-complete, as shown by Mertzios and Spirakis [23] who gave a highly non-trivial NP-hardness construction for the case where $d = 3$. In fact they proved that 3-COLOURING is NP-complete even for $C_3$-free graphs of diameter 3 and radius 2. The complexity of 3-COLOURING for the class of all graphs of diameter 2 has been posed as an open problem in several papers [24][21][23][24].

On the positive side, Mertzios and Spirakis [23] gave a subexponential-time algorithm for 3-COLOURING on graphs of diameter 2. Moreover, as we discuss below, 3-COLOURING is polynomial-time solvable for several subclasses of diameter 2.

In order to explain this, we need some terminology.

A graph $G$ has an articulation neighbourhood if $G - (N(v) \cup \{v\})$ is disconnected for some $v \in V(G)$. The neighbourhoods $N(u)$ and $N(v)$ of two distinct (and non-adjacent) vertices $u$ and $v$ are nested if $N(u) \subseteq N(v)$. The graph $K_{1,r}$ denotes the $(r + 1)$-vertex star, that is, the graph with vertices $x, y_1, \ldots, y_r$ and edges $xy_i$ for $i = 1, \ldots, r$. The subdivision of an edge $uv$ in a graph removes $uv$ and replaces it with a new vertex $v$ and edges $uv, vw$. We let $K_{1,r}^\ell$ be the $\ell$-subdivided star, which is obtained from $K_{1,r}$ by subdividing one edge exactly $\ell$ times. A polyad is a tree where exactly one vertex has degree at least 3. The graph $S_{h,i,j}$, for $1 \leq h \leq i \leq j$, is the tree with one vertex $x$ of degree 3 and exactly three leaves, which are of distance $h$, $i$ and $j$ from $x$, respectively. Note that $S_{1,1,1} = K_{1,3}$. The diamond is obtained from the 4-vertex complete graph by deleting an edge.

The 3-COLOURING problem is polynomial-time solvable for:

- diamond-free graphs of diameter 2 with an articulation neighbourhood but without nested neighbourhoods [23];
- $(C_3, C_4)$-free graphs of diameter 2 [21];
• $K_{1,r}^2$-free graphs of diameter 2, for every $r \geq 1$ [21]; and
• $S_{1,2,2}$-free graphs of diameter 2 [21].

It follows from results in [8,13,18] that without the diameter-2 condition, 3-Colouring is NP-complete again in each of the above cases; in particular 3-Colouring is NP-complete for $C$-free graphs for any finite set $C$ of cycles.

Our Results

We aim to increase our understanding of the complexity of 3-Colouring for graphs of diameter 2. In [21] we mainly considered 3-Colouring for graphs of diameter 2 with some forbidden induced subdivided star. In this paper, we continue this study by focussing on 3-Colouring for $C_s$-free or $(C_s,C_t)$-free graphs of diameter 2 for small values of $s$ and $t$; in particular for the case where $s = 4$ (cf. the aforementioned polynomial-time result for $(C_4,P_t)$-free graphs). In fact we prove our results for a more general problem, namely List 3-Colouring, whose complexity for diameter 2 is also still open. A list assignment of a graph $G = (V,E)$ is a function $L$ that prescribes a list of admissible colours $L(u) \subseteq \{1,2,\ldots\}$ to each $u \in V$. A colouring $c$ respects $L$ if $c(u) \in L(u)$ for every $u \in V$. For an integer $k \geq 1$, if $L(u) \subseteq \{1,\ldots,k\}$ for each $u \in V$, then $L$ is a list $k$-assignment. The List $k$-Colouring problem is to decide if a graph $G$ with an list $k$-assignment $L$ has a colouring that respects $L$. If every list is $\{1,\ldots,k\}$, we obtain $k$-Colouring.

The following two theorems summarize our main results.

**Theorem 1.** For $s \in \{5,6\}$, List 3-Colouring is polynomial-time solvable for $C_s$-free graphs of diameter 2.

**Theorem 2.** For $t \in \{3,5,6,7,8,9\}$, List 3-Colouring is polynomial-time solvable for $(C_4,C_t)$-free graphs of diameter 2.

The case $t = 3$ in Theorem 2 directly follows from the Hoffman-Singleton Theorem [12], which states that there are only four $(C_3,C_4)$-free graphs of diameter 2. The cases $t \in \{5,6\}$ immediately follows from Theorem 1. Hence, apart from proving Theorem 1 we only need to prove Theorem 2 for $t \in \{7,8,9\}$.

We prove Theorem 1 and the case $t = 7$ of Theorem 2 in Section 3 As we explain in the same section, all these results follow from the same technique, which is based on a number of (known) propagation rules. We first colour a small number of vertices and then start to apply the propagation rules exhaustively. This will reduce the sizes of the lists of the vertices. The novelty of our approach is the following: we can prove that the diameter-2 property ensures such a widespread reduction that each precolouring changes our instance into an instance of 2-List Colouring: the polynomial-solvable variant of List Colouring where each list has size at most 2 [7] (see also Section 2).

We proves the cases $t = 8$ and $t = 9$ of Theorem 2 in Section 4 using a refinement of the technique from Section 3. We explain this refinement in detail.
at the start of Section [4]. In short, in our branching, we exploit information from earlier obtained no-answers to reduced instances of our original instance \((G,L)\).

We complement Theorems [1] and [2] by the following result for diameter 4, whose proof can be found in Section [5].

**Theorem 3.** For every even integer \( t \geq 6 \), 3-Colouring is NP-complete on the class of \((C_4,C_6,\ldots,C_t)\)-free graphs of diameter 4.

Results of Damerell [6] imply that 3-Colouring is polynomial-time solvable for \((C_3,C_4,C_5,C_6)\)-free graphs of diameter 3 and for \((C_3,\ldots,C_8)\)-free graphs of diameter 4 [21]. We were not able to reduce the diameter in Theorem 3 from 4 to 3; see Section [6] for a further discussion, including other open problems.

\section{Preliminaries}

In this section we give some more terminology and notation. We also recall some useful result from the literature.

Let \( G = (V,E) \) be a graph. A vertex \( u \in V \) is **dominating** if \( u \) is adjacent to every other vertex of \( G \). For \( S \subseteq V \), the graph \( G[S] = (S,\{uv \mid u,v \in S \text{ and } uv \in E\}) \) denotes the subgraph of \( G \) induced by \( S \). The **neighbourhood** of a vertex \( u \in V \) is the set \( N(u) = \{ v \mid uv \in E \} \) and the **degree** of \( u \) is the size of \( N(u) \). For a set \( U \subseteq V \), we write \( N(U) = \bigcup_{u \in U} N(u) \setminus U \).

A **clique** is a set of pairwise adjacent vertices, and an **independent set** is a set of pairwise non-adjacent vertices. A graph is **complete** if its vertex set is a clique. We denote the complete graph on \( r \) vertices by \( K_r \). Recall that the diamond is the graph obtained from the \( K_4 \) after removing an edge. The **bull** is the graph obtained from a triangle on vertices \( x, y, z \) after adding two new vertices \( u \) and \( v \) and edges \( xu \) and \( yv \).

Let \( G \) be a graph with a list assignment \( L \). If \( |L(u)| \leq \ell \) for each \( u \in V \), then \( L \) is a **\( \ell \)-list assignment**. A list \( k \)-assignment is a \( k \)-list assignment, but the reverse is not necessarily true. The **\( \ell \)-List Colouring** problem is to decide if a graph \( G \) with an \( \ell \)-list assignment \( L \) has a colouring that respects \( L \). We use a known general strategy for obtaining a polynomial-time algorithm for List 3-Colouring on some class of graphs. That is, we will reduce the input to a polynomial number of instances of 2-List Colouring and use a well-known result due to Edwards.

**Theorem 4 ([7]).** The 2-List Colouring problem is linear-time solvable.

We also need an observation.

**Lemma 1.** Let \( G \) be a non-bipartite graph of diameter 2. Then \( G \) contains a \( C_3 \) or induced \( C_5 \).

**Proof.** As \( G \) is non-bipartite, \( G \) has an odd cycle. Let \( C \) be an odd cycle in \( G \) of minimum length. Then \( C \) is induced; otherwise we would find a shorter odd
cycle. For contradiction, suppose that $C$ has length at least 7. Consider two vertices $u$ and $v$ at distance 3 in $C$. Then $C$ contains a 4-vertex path $uxyv$ for some $x, y \in V(C)$. As $C$ is induced, $u$ and $v$ are non-adjacent. Hence, there exists a vertex $w$ not on $C$ that is adjacent to $u$ and $v$ (as $G$ has diameter 2). Then the subgraph of $G$ induced by $\{u, v, w, x, y\}$ contains a $C_3$ or an induced $C_5$, contradicting the minimality of $C$.  

\[\square\]

3 The Propagation Algorithm and Three Results

We present our initial propagation algorithm, which is based on a number of (well-known) propagation rules; we illustrate Rules 4 and 5 in Figures 1 and 2, respectively.

**Rule 1. (no empty lists)** If $L(u) = \emptyset$ for some $u \in V$, then return no.

**Rule 2. (not only lists of size 2)** If $|L(u)| \leq 2$ for every $u \in V$, then apply Theorem 4.

**Rule 3. (single colour propagation)** If $u$ and $v$ are adjacent, $|L(u)| = 1$, and $L(u) \subseteq L(v)$, then set $L(v) := L(v) \setminus L(u)$.

**Rule 4. (diamond colour propagation)** If $u$ and $v$ are adjacent and share two common non-adjacent neighbours $x$ and $y$ with $|L(x)| = |L(y)| = 2$ and $L(x) \neq L(y)$, then set $L(x) := L(x) \cap L(y)$ and $L(y) := L(x) \cap L(y)$ (so $L(x)$ and $L(y)$ get size 1).

**Rule 5. (bull colour propagation)** If $u$ and $v$ are the two degree-1 vertices of an induced bull $B$ of $G$ and $L(u) = L(v) = \{i\}$ for some $i \in \{1, 2, 3\}$ and moreover $L(w) \neq \{i\}$ for the degree-2 vertex $w$ of $B$, then set $L(w) := L(w) \cap \{i\}$.

![Fig. 1. Left: A diamond graph before applying Rule 4. Right: After applying Rule 4.](image-url)

We say that a propagation rule is safe if the new instance is a yes-instance of List 3-COLOURING if and only if the original instance is so. We make the following observation, which is straightforward (see also [15]).
Lemma 2. Each of the Rules 1–5 is safe and can be applied in polynomial time.

Consider again an instance \((G, L)\). Let \(N_0\) be a subset of \(V(G)\) that has size at most some constant. Assume that \(G[N_0]\) has a colouring \(c\) that respects the restriction of \(L\) to \(N_0\). We say that \(c\) is an \(L\)-promising \(N_0\)-precolouring of \(G\).

In our algorithms we first determine a set \(N_0\) of constant size and consider every \(L\)-promising \(N_0\)-precolouring of \(G\). That is, we modify \(L\) into a list assignment \(L_c\) with \(L_c(u) = \{c(u)\}\) (where \(c(u) \in L(u)\)) for every \(u \in N_0\) and \(L_c(u) = L(u)\) for every \(u \in V(G) \setminus N_0\). We then apply Rules 1–5 on \((G, L_c)\) exhaustively, that is, until none of the rules can be applied anymore. This is the propagation algorithm and we say that it did a full c-propagation. The propagation algorithm may output yes or no (when applying Rules 1 or 2); else it will output unknown.

If the algorithm returns yes, then \((G, L)\) is a yes-instance of list 3-Colouring by Lemma 2. If it returns no, then \((G, L)\) has no \(L\)-respecting colouring coinciding with \(c\) on \(N_0\), again by Lemma 2. If the algorithm returns unknown, then \((G, L)\) may still have an \(L\)-respecting colouring that coincides with \(c\) on \(N_0\). In that case the propagation algorithm did not apply Rule 1 or 2. Hence, it modified \(L_c\) into a list assignment \(L'_c\) of \(G\) such that \(L'_c(u) \neq \emptyset\) for every \(u \in V(G)\) and at least one vertex \(v\) of \(G\) still has a list \(L'_c(v)\) of size 3, that is, \(L'_c(v) = \{1, 2, 3\}\). We say that \(L'_c\) (if it exists) is the c-propagated list assignment of \(G\).

After performing a full c-propagation for every \(L\)-promising \(N_0\)-precolouring \(c\) of \(G\) we say that we performed a full \(N_0\)-propagation. We say that \((G, L)\) is \(N_0\)-terminal if after the full \(N_0\)-propagation one of the following cases hold:

1. for some \(L\)-promising \(N_0\)-precolouring, the propagation algorithm returned yes;
2. for every \(L\)-promising \(N_0\)-precolouring, the propagation algorithm returned no.

Note that if \((G, L)\) is \(N_0\)-terminal for some set \(N_0\), then we have solved List 3-Colouring on instance \((G, L)\). The next lemma formalizes our approach.

Lemma 3. Let \((G, L)\) be an instance of list 3-Colouring. Let \(N_0\) be a subset of \(V(G)\) of constant size. Performing a full \(N_0\)-propagation takes polynomial time. Moreover, if \((G, L)\) is \(N_0\)-terminal, then we have solved List 3-Colouring on instance \((G, L)\).

Proof. The first part of the lemma follows from the facts that (i) each application of each rule is safe and takes polynomial time by Lemma 2 (ii) if a rule does
not return a yes-answer or no-answer, then it reduces the list size of at least one vertex and the latter can happen at most $3|V|$ times; and (iii) the number of $L$-promising $N_0$-precolourings of $G$ is at most $3^{|N_0|}$, which is a constant as $N_0$ has constant size. The second part of the lemma follows from the definition of a full $N_0$-propagation and Lemma 2.

\[ \square \]

We now prove our first three results on List 3-Colouring for diameter-2 graphs. The first result generalizes a corresponding result for 3-Colouring in [21].

**Theorem 5.** List 3-Colouring can be solved in polynomial time for $C_5$-free graphs of diameter at most 2.

**Proof.** Let $G = (V,E)$ be a $C_5$-free graph of diameter 2 with a list 3-assignment $L$. We first check in polynomial time if $G$ is bipartite. Suppose that we find that $G$ is bipartite, say with partition classes $A$ and $B$. As $G$ has diameter 2, we find that $G$ must be complete bipartite. This implies that either $A$ or $B$ must be monochromatic. For each $i \in \bigcap_{u \in A} L(u)$ (which might be empty) we set $L(u) = \{i\}$ for every $u \in A$ and $L(v) := L(v) \setminus \{i\}$ for every $i \in B$ and apply Theorem 4. If we do not find a colouring respecting $L$, then we reverse the role of $A$ and $B$ and perform the same step.

Now suppose that we find that $G$ is not bipartite. If $G$ contains a $K_4$, then $G$ is not 3-colourable, and hence, $(G,L)$ is a no-instance of List 3-Colouring. We can check this in $O(|V|^4)$ time. From now on we assume that $G$ is $K_4$-free and non-bipartite. The latter implies that $G$ must have a triangle or an induced $C_5$, due to Lemma 1. As $G$ is $C_5$-free, it follows that $G$ has at least one triangle.

**Fig. 3.** Left: Examining the situation in the proof of Theorem 5 where a vertex $u \in N_2$ does not belong to $T$; we show that $y_1, y_2, y_3$ and $u$ either form a $K_4$ or we would find an induced $C_5$ (both of these cases are not possible). Right: A situation where $u \in T$.

Let $C$ be a triangle in $G$. We write $N_0 = V(C) = \{x_1, x_2, x_3\}$, $N_1 = N(V(C))$ and $N_2 = V(G) \setminus (N_0 \cup N_1)$. As $N_0$ has size 3, we can apply a full $N_0$-propagation in polynomial time by Lemma 3. By the same lemma we are done if we can prove that $(G,L)$ is $N_0$-terminal. We prove this claim below after first showing a structural result.
As $G$ has diameter 2, for every $i \in \{1, 2, 3\}$, it holds that every vertex in $N_2$ has a neighbour in $N_1$ that is adjacent to $x_i$. Now let $T$ consist of all vertices of $N_2$ that have a neighbour in $N_1$ that is adjacent to exactly two vertices of $N_0$.

**Claim 1.** $N_2 = T$.

We prove Claim 1 as follows. Let $u \in N_2$. For contradiction, assume $u \notin T$. If $u$ has a neighbour $y \in N_1$ adjacent to every $x_i$, then $G$ contains a $K_4$, a contradiction. Hence, as $u \notin T$, we find that $u$ must have three distinct neighbours $y_1, y_2, y_3$, such that for $i \in \{1, 2, 3\}$, it holds that $N(y_i) \cap N_0 = \{x_i\}$. If $\{y_1, y_2, y_3\}$ is a clique, then $G$ has a $K_4$ on vertices $u, y_1, y_2, y_3$, a contradiction. Hence, we may assume without loss of generality that $y_1$ and $y_2$ are non-adjacent. However, then $\{u, y_1, x_1, x_2, y_2\}$ induces a $C_5$ in $G$, another contradiction. See also Figure 3. We conclude that $T = N_2$. This proves Claim 1.

Now, for contradiction, assume that $(G, L)$ is not $N_0$-terminal. Then there must exist an $L$-promising $N_0$-precolouring $c$ for which we obtain the $c$-propagated list assignment $L'$. By definition of $L'$, we find that $G$ contains a vertex $u$ with $L'_c(u) = \{1, 2, 3\}$. Then $u \notin N_0$, as every $v \in N_0$ has $L'_c(v) = \{c(v)\}$. Moreover, $u \notin N_1$, as vertices in $N_1$ have a list of size at most 2 after applying Rule 3. Hence, we find that $u \in N_2$. As $N_2 = T$ by Claim 1, we find that $u \in T$. From the definition of $T$ it follows that $u$ has a neighbour $v \in N_1$ with two neighbours in $N_0$. By Rule 3 we find that $|L'_c(v)| = 1$. By the same rule, this implies that $|L'_c(u)| \leq 2$, a contradiction. We conclude that $(G, L)$ is $N_0$-terminal. \(\square\)

![Fig. 4. The situation in the proof of Theorem 6](image)

**Theorem 6.** List 3-Colouring can be solved in polynomial time for $C_6$-free graphs of diameter at most 2.

**Proof.** Let $G = (V, E)$ be a $C_6$-free graph of diameter 2 with a list 3-assignment $L$. If $G$ is $C_5$-free, then we apply Theorem 5. If $G$ contains a $K_4$, then $G$ is not
3-colourable and hence, \((G, L)\) is a no-instance of List 3-Colouring. We check these properties in polynomial time. So, from now on, we assume that \(G\) is a \(K_4\)-free graph that contains an induced 5-vertex cycle \(C\), say with vertex set \(N_0 = \{x_1, \ldots, x_5\}\) in this order. Let \(N_1\) be the set of vertices that do not belong to \(C\) but that are adjacent to at least one vertex of \(C\). Let \(N_2 = V \setminus (N_0 \cup N_1)\) be the set of remaining vertices.

As \(N_0\) has size 5, we can apply a full \(N_0\)-propagation in polynomial time by Lemma 3. By the same lemma we are done if we can prove that \((G, L)\) is \(N_0\)-terminal. We prove this claim below.

For contradiction, assume that \((G, L)\) is not \(N_0\)-terminal. Then there must exist an \(L\)-promising \(N_0\)-precolouring \(c\) for which we obtain the \(c\)-propagated list assignment \(L'\). By definition of \(L'\) we find that \(G\) contains a vertex \(v\) with \(L'_i(v) = \{1, 2, 3\}\). Then \(v \notin N_0\), as every \(u \in N_0\) has \(L'_i(u) = \{c(u)\}\). Moreover, \(v \notin N_1\), as vertices in \(N_1\) have a list of size at most 2 after applying Rule 3. Hence, we find that \(v \in N_2\).

We first note that some colour of \(\{1, 2, 3\}\) appears exactly once on \(N_0\), as \(|N_0| = 5\). Hence, we may assume without loss of generality that \(c(x_1) = 1\) and that \(c(x_i) \in \{2, 3\}\) for every \(i \in \{2, 3, 4, 5\}\).

As \(G\) has diameter 2, there exists a vertex \(y \in N_1\) that is adjacent to \(x_1\) and \(v\). As \(L'_i(y) = \{1, 2, 3\}\) and \(c(x_1) = 1\), we find that \(L'_i(y) = \{2, 3\}\). As \(c(x_i) \in \{2, 3\}\) for every \(i \in \{2, 3, 4, 5\}\), the latter means that \(y\) is not adjacent to any \(x_i\) with \(i \in \{2, 3, 4, 5\}\). Hence, as \(G\) has diameter 2, there exists a vertex \(z \in N_1\) with \(z \neq y\), such that \(z\) is adjacent to \(x_3\) and \(v\). We assume without loss of generality that \(c(x_3) = 3\) and thus \(c(x_2) = c(x_4) = 2\) and thus \(c(x_5) = 3\). As \(L'_i(v) = \{1, 2, 3\}\) and \(c(x_3) = 3\), we find that \(L'_i(z) = \{1, 2\}\). Hence, \(z\) is not adjacent to any vertex of \(\{x_1, x_2, x_4\}\). Now the set \(\{x_1, x_2, x_3, z, v, y\}\) forms a cycle on six vertices. As \(G\) is \(C_6\)-free, this cycle cannot be induced. Hence, the above implies that \(y\) and \(z\) must be adjacent; see also Figure 4.

As \(G\) has diameter 2, there exists a vertex \(w \in N_1\) that is adjacent to \(x_4\) and \(v\). As both \(y\) and \(z\) are not adjacent to \(x_4\), we find that \(w \notin \{y, z\}\). As \(L'_i(w) = \{1, 2, 3\}\) and \(c(x_4) = 2\), we find that \(L'_i(w) = \{1, 3\}\). As \(c(x_1) = 1\) and \(c(x_3) = c(x_5) = 3\), the latter implies that \(w\) is not adjacent to any vertex of \(\{x_1, x_3, x_5\}\). Consequently, \(w\) must be adjacent to \(y\), as otherwise the 6-vertex cycle with vertex set \(\{x_1, x_3, x_4, w, v, y\}\) would be induced, contradicting the \(C_6\)-freeness of \(G\). We refer again to Figure 4 for a display of the situation.

If \(w\) and \(z\) are adjacent, then \(\{v, w, y, z\}\) induces a \(K_4\), contradicting the \(K_4\)-freeness of \(G\). Hence, \(w\) and \(z\) are not adjacent. Then \(\{v, w, y, z\}\) induces a diamond, in which \(w\) and \(z\) are the two non-adjacent vertices. However, as \(L'_i(w) = \{1, 3\}\) and \(L'_i(z) = \{1, 2\}\), our algorithm would have applied Rule 4. This would have resulted in lists of \(w\) and \(z\) that are both equal to \(\{1, 3\} \cap \{1, 2\} = \{1\}\). Hence, we obtained a contradiction and conclude that \((G, L)\) is \(N_0\)-terminal.

Theorem 7 is proven in a similar way as Theorem 9.

**Theorem 7.** List 3-Colouring can be solved in polynomial time for \((C_4, C_7)\)-free graphs of diameter 2.
Proof. Let \( G = (V, E) \) be a \( C_4 \)-free graph of diameter 2 with a list 3-assignment \( L \). If \( G \) is \( C_5 \)-free, then we apply Theorem 9. Hence, we may assume that \( G \) contains an induced 5-vertex cycle \( C \), say with vertex set \( N_0 = \{x_1, \ldots, x_5\} \) in this order. As before, we let \( N_1 \) be the set of vertices that do not belong to \( C \) but that are adjacent to at least one vertex of \( C \). We also let \( N_2 = V \setminus (N_0 \cup N_1) \) denote the set of remaining vertices again.

As \( N_0 \) has size 5, we can apply a full \( N_0 \)-propagation in polynomial time by Lemma 3. By the same lemma we are done if we can prove that \( (G, L) \) is \( N_0 \)-terminal. We prove this claim in exactly the same way in which we proved a similar claim in the proof of Theorem 6 except for the following differences:

1. instead of using the 6-vertex set \( \{x_1, x_2, x_3, z, v, y\} \) we use the 7-vertex set \( \{x_1, x_5, x_4, x_3, z, v, y\} \) after observing that \( z \) cannot be adjacent to \( x_5 \) due to the \( C_4 \)-freeness of \( G \), and
2. instead of using the 6-vertex set \( \{x_1, x_5, x_4, w, v, y\} \) we use the 7-vertex set \( \{x_1, x_2, x_3, x_4, w, v, y\} \) after observing that \( w \) cannot be adjacent to \( x_2 \), again due to the \( C_4 \)-freeness of \( G \).

We refer again to Figure 4 for a display of the situation. \( \square \)

4 The Extended Propagation Algorithm and Two Results

For our next two results, we need a more sophisticated method. Let \((G, L)\) be an instance of List 3-Colouring. Let \( p \) be some positive constant. We consider each set \( N_0 \subseteq V(G) \) of size at most \( p \) and perform a full \( N_0 \)-propagation. Afterwards we say that we performed a full \( p \)-propagation. We say that \((G, L)\) is \( p \)-terminal if after the full \( p \)-propagation one of the following cases hold:

1. for some \( N_0 \subseteq V(G) \) with \( |N_0| \leq c \), there is an \( L \)-promising \( N_0 \)-precolouring \( c \) such that the propagation algorithm returns yes; or
2. for every set \( N_0 \subseteq V(G) \) with \( |N_0| \leq c \) and every \( L \)-promising \( N_0 \)-precolouring \( c \), the propagation algorithm returns no.

We can now prove the following lemma.

Lemma 4. Let \((G, L)\) be an instance of List 3-Colouring and \( p \geq 1 \) be some constant. Performing a full \( p \)-propagation takes polynomial time. Moreover, if \((G, L)\) is \( p \)-terminal, then we have solved List 3-Colouring on instance \((G, L)\).

Proof. For every set \( N_0 \subseteq V(G) \), a full \( N_0 \)-propagation takes polynomial time by Lemma 3. Then the first statement of the lemma follows from this observation and the fact that we need to perform \( O(n^p) \) full \( N_0 \)-propagations, which is a polynomial number, as \( p \) is a constant.

Now suppose that \((G, L)\) is \( p \)-terminal. First assume that for some \( N_0 \subseteq V(G) \) with \( |N_0| \leq c \), there exists an \( L \)-promising \( N_0 \)-precolouring \( c \), such that the propagation algorithm returns yes. Then \((G, L)\) is a yes-instance due to Lemma 2. Now assume that for every set \( N_0 \subseteq V(G) \) with \( |N_0| \leq c \) and every \( L \)-promising
Claim 1. For each induced vertices of \( V(C) \), the propagation algorithm returns no. Then \((G, L)\) is a no-instance. This follows from Lemma 2 combined with the observation that if \((G, L)\) was a yes-instance, the restriction of a colouring \( c \) that respects \( L \) to any set \( N_0 \) of size at most \( p \) would be an \( L \)-promising \( N_0 \)-precolouring of \( G \).

\[ \square \]

In our next two algorithms, we perform a full \( p \)-propagation for some appropriate constant \( p \). If we find that an instance \((G, L)\) is \( p \)-terminal, then we are done by Lemma 4. In the other case, we exploit the new information on the structure of \( G \) that we obtain from the fact that \((G, L)\) is not \( p \)-terminal.

**Theorem 8.** LIST 3-COLOURING can be solved in polynomial time for \((C_4, C_6)\)-free graphs of diameter 2.

**Proof.** Let \( G = (V, E) \) be a \((C_4, C_6)\)-free graph of diameter 2 with a list 3-assignment \( L \). If \( G \) is \( C_6 \)-free, then we apply Theorem 2. If \( G \) contains a \( K_4 \), then \( G \) is not 3-colourable and hence, \((G, L)\) is a no-instance of LIST 3-COLOURING.

We check these properties in polynomial time. So, from now on, we assume that \( G \) is a \( K_4 \)-free graph that contains at least one induced cycle on six vertices.

We set \( p = 6 \) and perform a full \( p \)-propagation. This takes polynomial time by Lemma 2. By the same lemma, we have solved LIST 3-COLOURING on \((G, L)\) if \((G, L)\) is \( p \)-terminal. Suppose we find that \((G, L)\) is not \( p \)-terminal.

We first prove the following claim.

**Claim 1.** For each induced 6-vertex cycle \( C \), the propagation algorithm returned no for every \( V(C) \)-promising colouring \( c \) that assigns the same colour \( i \) on two vertices of \( C \) that have a common neighbour on \( C \).

We prove Claim 1 as follows. Consider an induced 6-vertex cycle \( C \), say with vertex set \( N_0 = \{x_1, \ldots, x_6\} \) in this order. Let \( N_1 \) be the set of vertices that do not belong to \( C \) but that are adjacent to at least one vertex of \( C \). Let \( N_2 = V \setminus (N_0 \cup N_1) \) be the set of remaining vertices. For contradiction, let \( c \) be a \( V(C) \)-promising colouring that assigns two vertices of \( C \) with a common neighbour on \( C \) the same colour, say \( c(x_1) = 1 \) and \( c(x_3) = 1 \), such that a full \( c \)-propagation does not yield a no output. As \((G, L)\) is not \( p \)-terminal, this means that we obtained the \( c \)-propagated list assignment \( L'_c \). By definition of \( L'_c \) we find that \( G \) contains a vertex \( v \) with \( L'_c(v) = \{1, 2, 3\} \). Then \( v \notin N_0 \), as every \( u \in N_0 \) has \( L'_c(u) = \{c(u)\} \). Moreover, \( v \notin N_1 \), as vertices in \( N_1 \) have a list of size at most 2 after applying Rule 3. Hence, we find that \( v \in N_2 \).

As \( G \) has diameter 2, there exists a vertex \( y \in N_1 \) that is adjacent to both \( v \) and \( x_1 \). As \( c(x_1) = 1 \), we find that \( c(x_2) \in \{2, 3\} \) and \( c(x_6) \in \{2, 3\} \). As \( c(x_3) = 1 \), we find that \( c(x_4) \in \{2, 3\} \). Hence, \( y \) is not adjacent to any vertex of \( \{x_2, x_4, x_6\} \); otherwise \( y \) would have a list of size 1 due to Rule 3 and by the same rule, \( v \) would have a list of size 2. We note that \( y \) is not adjacent to \( x_3 \) or \( x_5 \) either, as otherwise \( \{x_1, x_2, x_3, y\} \) or \( \{x_1, x_6, x_5, y\} \) induces a \( C_4 \), contradicting the \( C_4 \)-freeness of \( G \).

As \( G \) has diameter 2 and \( yx_3 \notin E \), there exists a vertex \( y' \in N_1 \setminus \{y\} \) that is adjacent to both \( v \) and \( x_3 \). By the same arguments as above, \( y' \) is not adjacent to any vertex of \( \{x_1, x_2, x_4, x_6\} \). If \( y \) and \( y' \) are adjacent, then \( v \) would have list \( \{1\} \) due to Rule 5. Hence, \( y \) and \( y' \) are not adjacent. However, we now find
that \( \{x_1, y, v, y', x_3, x_4, x_5, x_6\} \) induces a \( C_8 \), contradicting the \( C_8 \)-freeness of \( G \); see also Figure 5. This proves Claim 1.

Fig. 5. The situation that is described in Claim 1 in the proof of Theorem 8: the set \( \{x_1, y, v, y', x_3, x_4, x_5, x_6\} \) induces a \( C_8 \), which is not possible.

Due to Claim 1, we know that if \( G \) has a colouring \( c \) respecting \( L \), then any such colouring \( c \) gives a different colour to every two non-adjacent vertices that are of distance 2 on some induced 6-vertex cycle. Hence, we can safely use the following new rule. To explain this, \( x_5 \) cannot get the same colour of both \( x_1 \) and \( x_3 \), which are both of distance 2 from \( x_5 \) on an induced \( C_6 \), thus \( x_5 \) must get the remaining colour, which is the colour of \( x_2 \). Moreover, an application of the new rule takes polynomial time. Note that we must also have that \( L(x_4) = L(x_1) \) and \( L(x_6) = L(x_3) \) but this will be irrelevant for our purposes.

**Rule-C6. (\( C_6 \) colour propagation)** Let \( C \) be an induced cycle on six vertices \( x_1, x_2, \ldots, x_6 \) in that order. If \( |L(x_1)| = |L(x_2)| = |L(x_3)| = 1 \), \( L(\{x_1, x_2, x_3\}) = \{1, 2, 3\} \) and \( L(x_2) \neq L(x_3) \), then set \( L(x_5) := L(x_2) \cap L(x_5) \) (so \( x_5 \) gets a list of size at most 1).

We can now do as follows. Consider an induced 6-vertex cycle \( C \) in \( G \), say on vertices \( x_1, \ldots, x_6 \) in that order. Then we may assume without loss of generality that if \( G \) has a colouring \( c \) that respects \( L \), then \( c(x_1) = 1, c(x_2) = 2, c(x_3) = 3, c(x_4) = 1, c(x_5) = 2 \) and \( c(x_6) = 3 \) (otherwise we can do some permutation of the colours). See also Figure 6.

We let again \( N_0 = \{x_1, \ldots, x_6\} \), \( N_1 \) be the set of vertices that do not belong to \( C \) but that are adjacent to at least one vertex of \( C \), and \( N_2 = V \setminus (N_0 \cup N_1) \) be the set of remaining vertices. We define a colouring \( c \) of \( G[N_0] \) by setting \( c(x_1) = 1, c(x_2) = 2, c(x_3) = 3, c(x_4) = 1, c(x_5) = 2 \) and \( c(x_6) = 3 \). We do a full \( c \)-propagation but now we also include the exhaustive use of Rule-C6. By combining Lemma 2 with the observation that Rule-C6 runs in polynomial time
and reduces the list size of at least one vertex, this takes polynomial time. By combining the same lemma with the fact that Rule-C6 is safe (due to Claim 1) and the above observation that every \( L \)-respecting colouring of \( G \) coincides with \( c \) on \( N_0 \) (subject to colour permutation), we are done if we can prove that the propagation algorithm either outputs yes or no.

For contradiction, assume that the propagation algorithm returns unknown. Then we obtained the \( c \)-propagated list assignment \( L'_c \). By definition of \( L'_c \) we find that \( G \) contains a vertex \( v \) with \( L'_c(v) = \{1, 2, 3\} \). Then \( v \notin N_0 \), as every \( u \in N_0 \) has \( L'_c(u) = \{c(u)\} \). Moreover, \( v \notin N_1 \), as vertices in \( N_1 \) have lists of size at most 2 after applying Rule 3. Hence, we find that \( v \in N_2 \).

As \( G \) has diameter 2, there exists a vertex \( y \in N_1 \) that is adjacent to \( x_1 \) and \( v \). Hence, \( y \) is not adjacent to any vertex in \( \{x_2, x_3, x_5, x_6\} \); otherwise \( y \) would have a list of size 1 due to Rule 3 and by the same rule, \( v \) would have a list of size 2. As \( G \) has diameter 2 and \( yx_3 \notin E \), there exists a vertex \( y' \in N_1 \setminus \{y\} \) that is adjacent to \( x_3 \) and \( v \). By the same arguments as above, \( y' \) is not adjacent to any vertex in \( \{x_1, x_2, x_4, x_5\} \). If \( yy' \notin E \), then \( \{x_1, x_2, x_3, y', v, y\} \) induces a \( C_6 \). However, in that case we would have applied Rule-C6 and \( v \) would have had list \( \{2\} \). Hence, we find that \( y \) and \( y' \) are adjacent; see also Figure 6.

As \( G \) has diameter 2, \( yx_5 \notin E \) and \( y'x_5 \notin E \), there exists a vertex \( y'' \in N_1 \setminus \{y, y'\} \) that is adjacent to \( x_5 \) and \( v \). By using exactly the same arguments as above but now applied to \( y'' \) and to the pairs \( (y, y'') \) and \( (y', y'') \), respectively, we find that \( y'' \) is adjacent to both \( y \) and \( y' \). However, now the vertices \( v, y, y', y'' \) induce a \( K_4 \), contradicting the \( K_4 \)-freeness of \( G \) (see again Figure 6). We conclude that the propagation algorithm returned either yes or no.

\[ \square \]

**Theorem 9.** List 3-Colouring can be solved in polynomial time for \((C_4, C_5)\)-free graphs of diameter 2.
Proof. Let $G = (V, E)$ be a $(C_4, C_9)$-free graph of diameter 2 with a list 3-assignment $L$. If $G$ is $C_7$-free, then we apply Theorem 7. If $G$ contains a $K_4$, then $G$ is not 3-colourable and hence, $(G, L)$ is a no-instance of List 3-COLOURING. We check these properties in polynomial time. So, from now on, we assume that $G$ is a $K_4$-free graph that contains at least one induced cycle on seven vertices.

We set $p = 7$ and perform a full $p$-propagation. This takes polynomial time by Lemma 2. By the same lemma, we have solved List 3-COLOURING on $(G, L)$ if $(G, L)$ is $p$-terminal. Suppose we find that $(G, L)$ is not $p$-terminal.

We first prove the following claim.

**Claim 1.** For each induced $7$-vertex cycle $C$, the propagation algorithm returned no for every $L$-promising $V(C)$-colouring $c$ that assigns the same colour $i$ on two vertices of $C$ that have a common neighbour on $C$ and that gives every other vertex of $C$ a colour different from $i$.

We prove Claim 1 as follows. Consider an induced $7$-vertex cycle $C$, say with vertex set $N_0 = \{x_1, \ldots, x_7\}$ in this order. Let $N_1$ be the set of vertices that do not belong to $C$ but that are adjacent to at least one vertex of $C$. Let $N_2 = V \setminus (N_0 \cup N_1)$ be the set of remaining vertices. Let $c$ be an $L$-promising $V(C)$-colouring that assigns two vertices of $C$ with a common neighbour on $C$ the same colour, say $c(x_1) = 1$ and $c(x_3) = 1$, and moreover, that assigns every vertex $x_i$ with $i \in \{2, 4, 5, 6, 7\}$ colour $c(x_i) \neq 1$.

For contradiction, suppose that a full $c$-propagation does not yield a no output. As $(G, L)$ is not $p$-terminal, this means that we obtained the $c$-propagated list assignment $L'_c$. By definition of $L'_c$ we find that $G$ contains a vertex $v$ with $L'_c(v) = \{1, 2, 3\}$. Then $v \notin N_0$, as every $u \in N_0$ has $L'_c(u) = \{c(u)\}$. Moreover, $v \notin N_1$, as vertices in $N_1$ have a list of size at most 2 after applying Rule 3. Hence, we find that $v \in N_2$.

As $G$ has diameter 2, there exists a vertex $y \in N_1$ that is adjacent to both $v$ and $x_1$. Then $y$ is not adjacent to any $x_i$ with $i \in \{2, 4, 5, 6, 7\}$; in that case $y$ would have a list of size 1 (as each $x_i$ other than $x_1$ and $x_3$ is coloured 2 or 3) meaning that $L'_c(v)$ would have size at most 2. Hence, $y$ is not adjacent to $x_3$ either, as otherwise $\{y, x_1, x_2, x_3\}$ would induce a $C_4$. As $G$ has diameter 2, this means that there exists a vertex $y' \in N_1$ with $y' \neq y$ such that $y'$ is adjacent to both $v$ and $x_3$. By the same arguments we used for $y'$, we find that $x_3$ is the only neighbour of $y'$ on $C$.

If $yy'$ is an edge, then by Rule 5 $v$ would have had list $\{1\}$ instead of $\{1, 2, 3\}$. Hence, $y$ and $y'$ are not adjacent. However, now $\{y, y', x_3, x_4, x_5, x_6, x_7, x_1\}$ induces a $C_9$, a contradiction; see also Figure 7. This proves Claim 1.

Claim 1 tells us that if $G$ has a colouring $c$ respecting $L$, then $c$ only gives the same colour to two vertices $x$ and $x'$ that are of distance 2 on some induced 7-vertex cycle $C$ if there is a third vertex $x''$ that is of distance 2 from either $x$ or $x'$ on $C$ with $c(x'') = c(x') = c(x)$. Hence, we can safely use the following new rule, whose execution takes polynomial time (in this rule, $c(x_1) = c(x_6)$ is not possible: view $x_1$ as $x$ and $x_6$ as $x'$ and note that $x''$ can neither be $x_3$ or $x_4$).
We let again $N_0 = \{x_1, \ldots, x_7\}$, $N_1$ be the set of vertices that do not belong to $C$ but that are adjacent to at least one vertex of $C$, and $N_2 = V \setminus (N_0 \cup N_1)$ be the set of remaining vertices. We do a full $c$-propagation but now we also include the exhaustive use of Rule-C7. By combining Lemma 2 with the observation that Rule-C7 runs in polynomial time and reduces the list size of at least one vertex, this takes polynomial time. By combining the same lemma with the fact that Rule-C7 is safe (due to Claim 1) and the above observation that every $L$-respecting colouring of $G$ coincides with $c$ on $N_0$ (subject to colour permutation), we are done if we can prove that the propagation algorithm either outputs yes or no. We show that this is the case for each of the two possibilities (1) and (2) of $c$.

For contradiction, assume that the propagation algorithm returns unknown. Then we obtained the $c$-propagated list assignment $L'_c$. By definition of $L'_c$ we find that $G$ contains a vertex $v$ with $L'_c(v) = \{1, 2, 3\}$. Then $v \not\in N_0$, as every $u \in N_0$ has $L'_c(u) = \{c(u)\}$. Moreover, $v \not\in N_1$, as vertices in $N_1$ have a list of size at most 2 after applying Rule 3. Hence, we find that $v \in N_2$. We now need to distinguish between the two possibilities of $c$.

Fig. 7. The situation that is described in Claim 1 in the proof of Theorem 9. The set $\{x_1, y, v, y', x_3, x_4, x_5, x_6, x_7\}$ induces a $C_9$, which is not possible.
Case 1. \( c(x_1) = 1, c(x_2) = 2, c(x_3) = 3, c(x_4) = 2, c(x_5) = 3, c(x_6) = 2 \) and \( c(x_7) = 3 \).

As \( G \) has diameter 2, there exists a vertex \( y \in N_1 \) that is adjacent to \( x_1 \) and \( v \). Hence, \( y \) is not adjacent to any vertex in \( \{x_2, \ldots, x_7\} \); otherwise \( y \) would have a list of size 1 due to Rule-3 and by the same rule, \( v \) would have a list of size 2. As \( G \) has diameter 2, there exists a vertex \( y' \in N_1 \) that is adjacent to \( x_4 \) and \( v \). By the same arguments as above, \( y' \) is not adjacent to any vertex of \( \{x_1, x_3, x_5, x_7\} \). The latter, together with the \( C_4 \)-freeness of \( G \), implies that \( y' \) is not adjacent to \( x_2 \) and \( x_6 \) either.

First suppose that \( yy' \in E \). Then \( \{x_1, x_7, x_6, x_5, x_4, y', y\} \) induces a \( C_7 \); see also Figure 8. As \( c(x_1) = 1, c(x_7) = 3, c(x_6) = 2 \) and \( c(x_5) = 3 \), we find that \( L_c(\{x_1, x_7, x_6\}) = \{1, 2, 3\} \) and \( L_c(x_5) = L_c(x_7) \). Then \( 1 \notin L_c(y') \), as otherwise the propagation algorithm would have applied Rule-C7. Moreover, \( 2 \notin L_c(y') \), as otherwise the propagation algorithm would have applied Rule-3. Hence, \( L_c(y') = \{3\} \). However, then \( |L_c(v)| \leq 2 \), again due to Rule-3, a contradiction.

Now suppose that \( yy' \notin E \). Then \( \{x_1, x_2, x_3, x_4, y', v\} \) induces a \( C_7 \). As \( c(x_1) = 1, c(x_2) = 2, c(x_3) = 3, c(x_4) = 2 \), we find that \( L_c(\{x_1, x_2, x_3\}) = \{1, 2, 3\} \) and \( L_c(x_4) = L_c(x_2) \). Then \( 1 \notin L_c(v) \) due to Rule-C7. This is a contradiction, as we assumed \( L_c(v) = \{1, 2, 3\} \). We conclude that the propagation algorithm returned either yes or no.

![Figure 8](image)

**Fig. 8.** The situation that is described in Case 1 in the proof of Theorem 9. If the edge \( yy' \) exists, then \( \{x_1, x_7, x_6, x_5, x_4, y', y\} \) induces a \( C_7 \) to which Rule-C7 should have been applied. Otherwise the vertices \( \{x_1, x_2, x_3, x_4, y', v\} \) induce such a \( C_7 \).

Case 2. \( c(x_1) = 1, c(x_2) = 2, c(x_3) = 3, c(x_4) = 1, c(x_5) = 3, c(x_6) = 2 \) and \( c(x_7) = 3 \).

As \( G \) has diameter 2, there is a vertex \( y \in N_1 \) adjacent to \( x_3 \) and \( v \). Hence, \( y \) is not adjacent to any vertex in \( \{x_1, x_2, x_4, x_6\} \); otherwise \( y \) would have a list of size 1 due to Rule-3 and by the same rule, \( v \) would have a list of size 2. As \( yx_4 \notin E \), we find that \( yx_5 \notin E \) either; otherwise \( \{y, x_3, x_4, x_5\} \) induces a \( C_4 \).
As $G$ has diameter 2, this means there is a vertex $y' \in N_1 \setminus \{y\}$ adjacent to $x_5$ and $v$. By the same arguments as above, $y'$ is not adjacent to any vertex of $\{x_1, x_2, x_4, x_6\}$. As $G$ is $C_4$-free, the latter implies that $y'x_3 \notin E$ and $y'x_7 \notin E$.

**Fig. 9.** The situation that is described in Case 2 in the proof of Theorem 9. The set $\{x_6, x_5, x_4, x_3, y, v, z\}$ induces a $C_7$ to which Rule-C7 should have been applied.

If $yy' \in E$, then $v$ would have a list of size at most 2 due to Rule 5. Hence, $yy' \notin E$. If $yx_7 \notin E$, this means that $\{x_1, x_3, x_4, x_5, x_7\}$ induces a $C_9$, which is not possible. Hence, $yx_7 \in E$.

To summarize, we found that $v$ has two distinct neighbours $y$ and $y'$, where $y$ has exactly two neighbours on $C$, namely $x_3$ and $x_7$, and $y'$ has exactly one neighbour on $C$, namely $x_5$. As $G$ has diameter 2, this means that there exists a vertex $z \in N_1$ with $z \notin \{y, y'\}$ that is adjacent to $x_6$ and $v$. Then $z$ is not adjacent to any vertex of $\{x_1, x_3, x_4, x_5, x_7\}$, as otherwise $z$ would have a list of size 1 due to Rule 3 and by the same rule, $v$ would have a list of size 2. If $zy \in E$, then $\{y, z, x_6, x_7\}$ induces a $C_4$, which is not possible. Hence, $zy \notin E$.

From the above, we find that $\{x_6, x_5, x_4, x_3, y, v, z\}$ induces a $C_7$; see also Figure 9. As $c(x_6) = 2$, $c(x_5) = 3$, $c(x_4) = 1$ and $c(x_3) = 3$, we find that $L_c(\{x_6, x_5, x_4\}) = \{1, 2, 3\}$ and $L_c(x_3) = L_c(x_5)$. Then $2 \notin L_c(v)$, due to Rule-C7. Hence, $|L_c(v)| \leq 2$, a contradiction. We conclude that the propagation algorithm returned either yes or no in Case 2 as well.

\[\Box\]

5 The Proof of Theorem 3

In this section we prove Theorem 3 which we restate below.

**Theorem 3 (restated).** For every even integer $t \geq 6$, 3-COLOURING is NP-complete on the class of $(C_4, C_6, \ldots, C_t)$-free graphs of diameter 4.

**Proof.** Note that the problem is readily seen to be in NP. To prove NP-hardness we modify the standard reduction for COLOURING from the NP-complete problem
Not-All-Equal 3-Satisfiability \cite{27}, where each variable appears in at most three clauses. So, given a CNF formula $\phi$, we first construct a graph $G$ as follows (see also Figure 10):

- add literal vertices $v_i$ and $v'_i$ for each variable $x_i$;
- add an edge between each $v_i$ and $v'_i$;
- add a vertex $z$ adjacent to every $v_i$ and every $v'_i$;
- for each clause $C_i$ add a triangle $T_i$ with clause vertices $c_{i1}, c_{i2}, c_{i3}$;
- fix an arbitrary order of the literals $x_{i1}, x_{i2}, x_{i3}$ of $C_i$ and for $j \in \{1, 2, 3\}$,
  add the edge $v_{ij}c_{ij}$ if $x_{ij}$ is positive and the edge $v'_{ij}c_{ij}$ if $x_{ij}$ is negative.

It is well known that $\phi$ has a truth assignment $\tau$ such that each clause contains at least one true literal and at least one false literal (call such a $\tau$ satisfying) if and only if $G$ has a 3-colouring. For completeness we give a proof below.

First suppose $\phi$ has a satisfying truth assignment. Colour vertex $z$ with colour 1, each true literal with colour 2 and each false literal with colour 3. Then, as each clause has a true literal and a false literal, each triangle $T_i$ has neighbours in two different colours. Hence, we can complete the 3-colouring.

Now suppose $G$ has a 3-colouring. Say $z$ is assigned colour 1. Then each literal vertex has either colour 2 or colour 3. Moreover, each $T_i$ must be adjacent to at least one literal vertex coloured 2 and to at least one literal vertex coloured 3. Hence, the truth assignment that sets literals whose vertices are coloured with colour 2 to be true and those coloured with colour 3 to be false is satisfying.

As every clause vertex is adjacent to a literal vertex and literal vertices are adjacent to $z$, every vertex has distance at most 2 from $z$. So $G$ has diameter 4.

We modify $G$ into a graph $G'$: for some $p \geq 0$, subdivide each edge $v_{ij}, c_{ij}$ and each edge $v'_{ij}, c_{ij}$ $p$ times and make each newly introduced vertex adjacent to $z$; see also Figure 10. Then $G'$ has a 3-colouring if and only if $G$ has a 3-colouring, as the new vertices will be alternatingly coloured by 2 and 3 if $z$ has colour 1. Moreover, $G'$ still has diameter 4, and it can be readily checked that every induced cycle...
of $G$ of length at most $p$ is either a $C_3$ (either a triangle $T_i$ or a triangle containing $z$) or a $C_5$ (which must contain $z$). As we can make $p$ arbitrarily large, the result follows. \hfill \QED

6 Conclusions

We proved that 3-COLOURABILITY is polynomial-time solvable for several subclasses of diameter 2 that are characterized by forbidding one or two small induced cycles. In order to do this we used a unified framework of propagation rules, which allowed us to exploit the diameter-2 property of the input graph. Our current techniques need to be extended to obtain further results (in particular, we cannot currently handle the increasing number of different 3-colourings of induced cycles of length larger than 9).

As open problems we pose: determine the complexity of 3-COLOURING and LIST 3-COLOURING for:

- graphs of diameter 2 (which we recall is a long-standing open problem)
- $C_t$-free graphs of diameter 2 for $s \in \{3, 4, 7, 8, \ldots \}$; and
- $(C_4, C_t)$-free graphs of diameter 2 for $t \geq 10$.

We also note that the complexity of $k$-COLOURING for $k \geq 4$ and COLOURING is still open for $C_3$-free graphs of diameter 2 (see also [21]).

Finally, we turn to the class of graphs of diameter 3. The construction of Mertzios and Spirakis [23] for proving that 3-COLOURING is NP-complete for $C_3$-free graphs of diameter 3 appears to contain not only induced subdivided stars of arbitrary diameter and with an arbitrary number of leaves but also induced cycles of arbitrarily length $s \geq 4$. Hence, we pose as open problems: determine the complexity of 3-COLOURING and LIST 3-COLOURING for $C_t$-free graphs of diameter 3 for $t \geq 4$ and $(C_4, C_t)$-free graphs of diameter 3 for $t \in \{3, 5, 6, \ldots \}$.

References

1. N. Alon. Restricted colorings of graphs. Surveys in combinatorics, London Mathematical Society Lecture Note Series, 187:1–33, 1993.
2. M. Bodirsky, J. Kára, and B. Martin. The complexity of surjective homomorphism problems - a survey. Discrete Applied Mathematics, 160(12):1680–1690, 2012.
3. F. Bonomo, M. Chudnovsky, P. Maceli, O. Schaudt, M. Stein, and M. Zhong. Three-coloring and list three-coloring of graphs without induced paths on seven vertices. Combinatorica, 38(4):779–801, 2018.
4. H. Broersma, F. V. Fomin, P. A. Golovach, and D. Paulusma. Three complexity results on coloring $P_k$-free graphs. European Journal of Combinatorics, 34(3):609–619, 2013.
5. M. Chudnovsky. Coloring graphs with forbidden induced subgraphs. Proc. ICM 2014, IV:291–302, 2014.
6. R. M. Damerell. On Moore graphs. Mathematical Proceedings of the Cambridge Philosophical Society, 74:227–236, 1973.
7. K. Edwards. The complexity of colouring problems on dense graphs. *Theoretical Computer Science*, 43:337–343, 1986.
8. T. Emden-Weinert, S. Hougardy, and B. Kreuter. Uniquely colourable graphs and the hardness of colouring graphs of large girth. *Combinatorics, Probability and Computing*, 7(04):375–386, 1998.
9. P. Gartland, D. Lokshavanov, M. Pilipczuk, M. Pilipczuk, and P. Rzążewski. Finding large induced sparse subgraphs in $C_{2t}$-free graphs in quasipolynomial time. *Proc. STOC 2021*, 330–341, 2021.
10. P. A. Golovach, M. Johnson, D. Paulusma, and J. Song. A survey on the computational complexity of colouring graphs with forbidden subgraphs. *Journal of Graph Theory*, 84(4):331–363, 2017.
11. P. A. Golovach, D. Paulusma, and J. Song. Coloring graphs without short cycles and long induced paths. *Discrete Applied Mathematics*, 167:107–120, 2014.
12. A. J. Hoffman and R. R. Singleton. On Moore graphs with diameter 2 and 3. *IBM Journal of Research and Development*, 5:497–504, 1960.
13. I. Holyer. The NP-completeness of edge-coloring. *SIAM Journal on Computing*, 10(4):718–720, 1981.
14. T. R. Jensen and B. Toft. *Graph coloring problems*. John Wiley & Sons, 1995.
15. T. Klímašová, J. Malík, T. Masářík, J. Novotná, D. Paulusma, and V. Slívová. Colouring $(P_5 + P_5)$-free graphs. *Algorithmica* 82:1833–1858, 2020.
16. J. Kratochvíl, Zs. Tuza, and M. Voigt. New trends in the theory of graph colorings: choosability and list coloring. *Proc. DIMATIA-DIMACS Conference*, 49:183–197, 1999.
17. L. Lovász. Coverings and coloring of hypergraphs. *Congressus Numerantium*, VIII:3–12, 1973.
18. V. V. Lozin and M. Kaminski. Coloring edges and vertices of graphs without short or long cycles. *Contributions to Discrete Mathematics*, 2(1), 2007.
19. V. V. Lozin and D. S. Malyshev. Vertex coloring of graphs with few obstructions. *Discrete Applied Mathematics*, 216:273–280, 2017.
20. D.S. Malyshev. The complexity of the vertex 3-Colorability problem for some hereditary classes defined by 5-vertex forbidden induced subgraphs. *Graphs and Combinatorics*, 33(4):1009–1022, 2017.
21. B. Martin, D. Paulusma, and S. Smith. Colouring $H$-free graphs of bounded diameter. *Proc. MFCS 2019, LIPIcs*, 138:1–14:14, 2019.
22. B. Martin, D. Paulusma, and S. Smith. Colouring graphs of bounded diameter in the absence of small cycles. *Proc. CIAC 2021, LNCS*, 12701, 367–380, 2021.
23. G. B. Mertzios and P. G. Spirakis. Algorithms and almost tight results for 3-colorability of small diameter graphs. *Algorithmica*, 74(1):385–414, 2016.
24. D. Paulusma. Open problems on graph coloring for special graph classes. *Proc. WG 2015, LNCS*, 9224:16–30, 2015.
25. B. Randerath and I. Schiermeyer. Vertex colouring and forbidden subgraphs – a survey. *Graphs and Combinatorics*, 20(1):1–40, 2004.
26. A. Rojas and M. Stein. 3-colouring $P_t$-free graphs without short odd cycles. *CoRR*, abs/2008.04845, 2020.
27. T. J. Schaefer. The complexity of satisfiability problems. *Proc. STOC 1978*, 216–226, 1978.
28. D. V. Sirotkin and D. S. Malyshev. On the complexity of the Vertex 3-Coloring problem for the hereditary graph classes with forbidden subgraphs of small size. *Journal of Applied and Industrial Mathematics*, 12:759–769, 2018.
29. Z. Tuza. Graph colorings with local constraints - a survey. *Discussiones Mathematicae Graph Theory*, 17(2):161–228, 1997.