A REPETITION-FREE HYPERSEQUENT CALCULUS FOR FIRST-ORDER RATIONAL PAVELKA LOGIC

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Abstract. We present a hypersequent calculus $G^3L\forall$ for first-order infinite-valued Łukasiewicz logic and for an extension of it, first-order rational Pavelka logic; the calculus is intended for bottom-up proof search. In $G^3L\forall$, there are no structural rules, all the rules are invertible, and designations of multisets of formulas are not repeated in any premise of the rules. The calculus $G^3L\forall$ proves any sentence that is provable in at least one of the previously known hypersequent calculi for the given logics. We study proof-theoretic properties of $G^3L\forall$ and thereby provide foundations for proof search algorithms.

Keywords: many-valued logic, mathematical fuzzy logic, first-order infinite-valued Łukasiewicz logic, first-order rational Pavelka logic, proof theory, hypersequent calculus, proof search.

1. Introduction

First-order infinite-valued Łukasiewicz logic $L\forall$ and an extension of it by rational truth constants, first-order rational Pavelka logic $RPL\forall$, are among the fundamental fuzzy logics [1, 2, 3] and are considered in the given paper from the standpoint of proof search.

Hilbert-type calculi for the logics under consideration are widely used (see, e.g., [4, 2]), but such calculi are unfit for bottom-up proof search. For $L\forall$, we also known the hypersequent calculus $GL\forall$ [4, 5] with structural rules, which make it unsuitable for bottom-up proof search.

On the basis of the calculus $GL\forall$ from [4, 5] and tableau calculi from [6], in [7] we introduced hypersequent calculi $G^1L\forall$ and $G^2L\forall$ for the logic $RPL\forall$ and hence for $L\forall$. The calculi $G^1L\forall$ and $G^2L\forall$ do not have structural rules; the latter is a noncumulative variant of the former, which is cumulative, i.e., preserves the conclusion of each inference rule in its premises. Any $GL\forall$-provable sentence is provable in $G^1L\forall$; and any prenex $L\forall$-sentence is provable or unprovable in $GL\forall$, $G^1L\forall$, and $G^2L\forall$ simultaneously. Also in [7], a family of proof search algorithms is described; given a prenex $G^2L\forall$-provable sentence, such an algorithm constructs some proof for it in a tableau modification of the calculus $G^2L\forall$.

A defect of $G^2L\forall$ (which does not appear in proving prenex sentences) is that designations of multisets of formulas are repeated in each premise of two quantifier rules. The defect causes repeating some work during bottom-up proof search and prevented us from establishing desirable proof-theoretic properties for the calculus $G^2L\forall$, in particular, invertibility of one of its rules.

In the present paper, we introduce a noncumulative hypersequent calculus $G^3L\forall$ for the logic $RPL\forall$. There are no structural rules in the calculus; and designations
of multisets of formulas are not repeated in any premise of its rules. The last feature of the calculus allows us to call it and each of its rules repetition-free.

This paper is organized as follows. In the rest of this section, we define the syntax and semantics of the logics $L\forall$ and $RPL\forall$, as well as some notation. In Section 2 we formulate the calculus $G^3L\forall$ and prove its soundness. In Section 3 we establish the invertibility of all the rules of $G^3L\forall$ and show that any $G^3L\forall$-provable sentence is provable in $G^3L\forall$. In Section 4 we investigate transformations of $G^3L\forall$-proofs according to proof search tactics and thereby provide foundations for various proof search algorithms. In Section 5 we prove the mid-hypersequent theorem for $G^3L\forall$; show that any prenex $RPL\forall$-sentence is provable or unprovable in $G^3L\forall$, $G^2L\forall$, and $G^3L\forall$ simultaneously; and establish undecidability of $G^3L\forall$.

Let us describe the syntax and semantics of the logics under consideration. We fix an arbitrary signature, which may contain predicate and function symbols of any nonnegative arities.

Terms are defined in the standard manner. Atomic $L\forall$- and $RPL\forall$-formulas are predicate symbols with argument terms, as well as truth constants: in $L\forall$, the only truth constant $\bar{0}$; and in $RPL\forall$, truth constants $\bar{r}$ for all rational numbers $r \in [0, 1]$ (where $[0, 1]$ is an interval of real numbers). $L\forall$- and $RPL\forall$-formulas are built as usual from atomic $L\forall$- and $RPL\forall$-formulas, respectively, using the logical symbols: the binary connective $\rightarrow$ and the quantifiers $\forall, \exists$.

The notion of an interpretation $\langle D, \mu \rangle$ differs from the classical notion of the same name only in that the map $\mu$ takes each $n$-ary predicate symbol $P$ to a predicate $\mu(P) : D^n \rightarrow [0, 1]$. Given an interpretation $\langle D, \mu \rangle$, a valuation is a map of the set of all (individual) variables to the domain $D$ of the interpretation. For a valuation $\nu$, a variable $x$, and $d \in D$, by $\nu[x \mapsto d]$ we denote the valuation that may differ from $\nu$ only on $x$ and meets the condition $\nu(x) = d$.

The value $|t|_{M, \nu}$ of a term $t$ under an interpretation $M$ and a valuation $\nu$ is defined as usual. The truth value $|C|_{M, \nu}$ of an $RPL\forall$-formula $C$ under an interpretation $M = \langle D, \mu \rangle$ and a valuation $\nu$ is defined as follows:

1. $|\bar{r}|_{M, \nu} = r$;
2. $|P(t_1, \ldots, t_n)|_{M, \nu} = \mu(P)(|t_1|_{M, \nu}, \ldots, |t_n|_{M, \nu})$ for an $n$-ary predicate symbol $P$ and terms $t_1, \ldots, t_n$;
3. $|A \rightarrow B|_{M, \nu} = \min(1 - |A|_{M, \nu}, |B|_{M, \nu}, 1)$;
4. $\forall x |A|_{M, \nu} = \inf_{d \in D} |A|_{M, \nu[x \mapsto d]}$;
5. $\exists x |A|_{M, \nu} = \sup_{d \in D} |A|_{M, \nu[x \mapsto d]}$.

An $RPL\forall$-formula $C$ is called valid (also written $\models C$) if $|C|_{M, \nu} = 1$ for every interpretation $M$ and every valuation $\nu$.

Note that the logic $RPL\forall$ allows us to express partial truth of statements in the following way [1, Section 3.3]. Given a rational number $r \in [0, 1]$ and an $RPL\forall$-formula $A$, we have: (a) for a fixed interpretation $M$ and a fixed valuation $\nu$: $r \leq |A|_{M, \nu}$ iff $\bar{r} \rightarrow A\models 1$; (b) $r \leq |A|_{M, \nu}$ for every interpretation $M$ and every valuation $\nu$ iff $\bar{r} \models (\bar{r} \rightarrow A)$.

The result of substituting a term $t$ for all free occurrences of a variable $x$ in an $RPL\forall$-formula $A$ is denoted by $|A|_t$. By a proof in a calculus considered below, we understand a proof tree. The provability of an object $\alpha$ in a calculus $\mathcal{E}$ is denoted by $\vdash_{\mathcal{E}} \alpha$.

The calculi $GL\forall$, $G^1L\forall$, and $G^2L\forall$ are formulated in Sections 2.1, 2.2, and 3.1 of the paper [4].
2. The repetition-free calculus $G^3L\forall$ and its soundness

Basically, we obtain $G^3L\forall$ from the calculus $G^2L\forall$, defined in [7 Section 3.1], by replacing its rules $(\forall \Rightarrow)^2$ and $(\Rightarrow \exists)^2$ with repetition-free ones.

We will work with a fixed signature that includes a countable set of nullary function symbols called parameters.

Semipropositional variables defined in [7 Section 2.2] are now called semipositional variables of type 1 and are denoted by $p, p_0, p_1, \ldots$. In addition to them, we introduce a countable set of new words called semipositional variables of type 0 and denoted by $q, q_0, q_1, \ldots$.

The way of obtaining the new repetition-free rules and the role of semipositional variables used in them will be revealed in the proofs of Lemmas 1 and 2 below.

We define an hs-interpretation as an interpretation $\langle D, \mu \rangle$ in which the map $\mu$ additionally takes each semipositional variable of type 0 to a real number from $[0, +\infty)$ and each semipositional variable of type 1 to a real number from $(-\infty, 1]$.

Taking into account that by semipositional variables we now mean semipositional variables of both types, the following definitions and abbreviations given in [7 Section 2.2] preserve their forms: the definitions of an atom, a formula, a sequent, a hypersequent, a member of a sequent, an atomic sequent; the abbreviations $|\Gamma|_{M, \nu}$ and $\Gamma \Rightarrow \Delta|_{M, \nu}$ (for a finite multiset $\Gamma$ of formulas, a sequent $\Gamma \Rightarrow \Delta$, an hs-interpretation $M$, and a valuation $\nu$); the definitions of a true sequent (under an hs-interpretation and a valuation), a valid hypersequent (with the abbreviation $\models H$ for such a hypersequent $H$), sound and semantically invertible rules.

In the sequel, let the letters $A, B,$ and $C$ denote any RPL\forall-formulas, $F$ a formula, $\Gamma, \Delta, \Pi,$ and $\Sigma$ any finite multisets of formulas, $S$ a sequent, $G$ and $H$ any hypersequents, $t$ a closed term, $a$ a parameter; all these letters may have subscripts.

The inference rules of the calculus $G^3L\forall$ are:

\[
\begin{align*}
G \vdash |\Gamma, p \Rightarrow \Delta | B \Rightarrow p, A & \quad (\rightarrow \Rightarrow)^3, \\
G \vdash |\Gamma, A \Rightarrow B \Rightarrow \Delta & \quad (\Rightarrow \rightarrow)^3, \\
G \vdash |\Gamma, p \Rightarrow \Delta | \forall x A \Rightarrow p | [A]_x^\forall \Rightarrow p & \quad (\forall \Rightarrow)^3, \\
G \vdash |\Gamma \Rightarrow \forall x A \Rightarrow \Delta & \quad (\forall \Rightarrow)^3, \\
G \vdash |\Gamma \Rightarrow q, \Delta | [q \Rightarrow \exists x A]_x^\exists \Rightarrow p & \quad (\exists \Rightarrow)^3, \\
G \vdash |\Gamma \Rightarrow \exists x A, \Delta & \quad (\exists \Rightarrow)^3,
\end{align*}
\]

where $p$ (resp. $q$) does not occur in the conclusion of $(\rightarrow \Rightarrow)^3$ or $(\forall \Rightarrow)^3$ (resp. $(\exists \Rightarrow)^3$) and is called the proper semipositional variable of an application of the corresponding rule; $t$ is called the proper term of an application of $(\forall \Rightarrow)^3$ or $(\Rightarrow \exists)^3$; $a$ does not occur in the conclusion of $(\Rightarrow \forall)^3$ or $(\exists \Rightarrow)^3$ and is called the proper parameter of an application of the corresponding rule.

An axiom of the calculus $G^3L\forall$ is an arbitrary hypersequent in which, for any hs-interpretation $M$ and any valuation $\nu$, there exists an atomic sequent that is true under $M$ and $\nu$. Note that axioms of $G^3L\forall$ can be recognized in much the same way as described in [7 Section 4.2].

A $G^3L\forall$-proof of (for) an RPL\forall-formula $A$ is a $G^3L\forall$-proof of the hypersequent $\Rightarrow A$. 

The following definitions and notation given at the end of [7, Section 2.2] carry over to the calculus $G^3L\forall$: the definitions of a backward application (or a counter-application) of a rule, a principal formula (sequent) occurrence, and an ancestor of a formula (sequent) occurrence; and the convention for designating a proof of a hypersequent over an occurrence of it in a proof tree.

Suppose $D$ is a $G^3L\forall$-proof, and $G$ is a hypersequent. To get a $G^3L\forall$-proof $D'$, in $D$, we rename all proper semipositional variables occurring in $G$ and all proper parameters occurring in $G$ to new distinct ones. Then by $D|G$ we denote the $G^3L\forall$-proof obtained from $D'$ by appending “|G” to each node hypersequent of $D'$. (For our use of such an abbreviation, it does not matter how we perform renaming above.)

**Lemma 1.** Each inference rules of the calculus $G^3L\forall$ is sound and semantically invertible.

**Proof.** From assertions (1)–(4) of Lemma 2 stated below, it follows that the rules $(\rightarrow\Rightarrow)^3$, $(\Rightarrow\rightarrow)^3$, $(\Rightarrow\forall)^3$, and $(\exists\Rightarrow)^3$ are sound and semantically invertible.

Any application of the rule $(\forall\Rightarrow)^3$ can be represented as two applications of the rules

$$\frac{G_0|\Gamma_0, \forall x A \Rightarrow \Delta_0 | \Gamma_0, [A]^x \Rightarrow \Delta_0}{G_0|\Gamma_0, \forall x A \Rightarrow \Delta_0} (\forall \Rightarrow)^0$$

and

$$\frac{G | \Gamma, p \Rightarrow \Delta | B \Rightarrow p}{G | \Gamma, B \Rightarrow \Delta} (\text{den}_1),$$

where $p$ does not occur in the conclusion of the last rule, as follows:

$$\frac{G | \Gamma, p \Rightarrow \Delta | \forall x A \Rightarrow p | [A]^x \Rightarrow p}{G | \Gamma, \forall x A \Rightarrow \Delta} (\forall \Rightarrow)^0.$$

By assertion (5) of Lemma 2, the rule $(\forall \Rightarrow)^0$ is sound; and it is semantically invertible, since its premise includes its conclusion. By assertion (5) of Lemma 2, the rule $(\text{den}_1)$ is sound and semantically invertible. So $(\forall \Rightarrow)^3$ is sound and semantically invertible.

Any application of the rule $(\Rightarrow\exists)^3$ can be represented as two applications of the rules

$$\frac{G_0|\Gamma_0 \Rightarrow \exists x A, \Delta_0 | \Gamma_0 \Rightarrow [A]^x \Rightarrow \Delta_0}{G_0|\Gamma_0 \Rightarrow \exists x A, \Delta_0} (\Rightarrow \exists)^0$$

and

$$\frac{G | \Gamma \Rightarrow q, \Delta | q \Rightarrow B}{G | \Gamma \Rightarrow B, \Delta} (\text{den}_0),$$

where $q$ does not occur in the conclusion of the last rule, thus:

$$\frac{G | \Gamma \Rightarrow q, \Delta | q \Rightarrow \exists x A | q \Rightarrow [A]^x \Rightarrow \Delta}{G | \Gamma \Rightarrow \exists x A, \Delta} (\Rightarrow \exists)^0.$$
For an hs-interpretation $M$, a semipositional variable $r$ of type 0 (resp. type 1), and a real number $r \in [0, +\infty)$ (resp. $r \in (-\infty, 1]$), by $M[r \mapsto r]$ we denote the hs-interpretation that interprets $r$ by $r$ and does not differ from $M$ in any other respect.

**Lemma 2.** Let $\Gamma$ and $\Delta$ be finite multisets of formulas; $A$ and $B$ be RPL-$\forall$-formulas; $y$ be a variable not occurring in $\Gamma$, $\Delta$, $A$, $B$; $M$ be an hs-interpretation with domain $D$; and $\nu$ be a valuation. Then:

1. $|\Gamma, A \rightarrow B \Rightarrow \Delta|_{M, \nu} \geq 0$ iff, for every $r \in (-\infty, 1]$, at least one of the inequalities $|\Gamma, p \Rightarrow \Delta|_{M[p \mapsto r], \nu} \geq 0$ or $|B \Rightarrow p|_{M[p \mapsto r], \nu} \geq 0$ holds;
2. $|\Gamma \Rightarrow \forall x A, \Delta|_{M, \nu} \geq 0$ iff $|\Gamma \Rightarrow |A|_{y}^{\nu}, \Delta|_{M, \nu[y \mapsto d]} \geq 0$ for every $d \in D$;
3. $|\Gamma \Rightarrow \exists x A, \Delta|_{M, \nu} \geq 0$ iff $|\Gamma \Rightarrow |A|_{y}^{\nu}, \Delta|_{M, \nu[y \mapsto d]} \geq 0$ for every $d \in D$;
4. $|\Gamma, \exists x A \Rightarrow \Delta|_{M, \nu} \geq 0$ iff $|\Gamma, [A]_{y}^{\nu} \Rightarrow \Delta|_{M, \nu[y \mapsto d]} \geq 0$ for some $d \in D$;
5. $|\Gamma, \forall x A \Rightarrow \Delta|_{M, \nu} \geq 0$ iff $|\Gamma \Rightarrow |A|_{y}^{\nu}, \Delta|_{M, \nu[y \mapsto d]} \geq 0$ for some $d \in D$;
6. $|\Gamma, B \Rightarrow \Delta|_{M, \nu} \geq 0$ iff, for every $r \in (-\infty, 1]$, at least one of the inequalities $|\Gamma, p \Rightarrow \Delta|_{M[p \mapsto r], \nu} \geq 0$ or $|B \Rightarrow p|_{M[p \mapsto r], \nu} \geq 0$ holds.

**Proof.** Assertions (1)–(6) stated above are proved similarly to assertions (1)–(6) in [2, Lemma 2].

Let us prove assertions (5) and (6). Denote $|\Gamma|_{M, \nu}$, $|\Delta|_{M, \nu}$, and $|B|_{M, \nu}$ by $\gamma$, $\delta$, and $b$, respectively; and notice that $0 \leq b \leq 1$.

Assertion (5) is equivalent to the following:

\[(5') \delta - \gamma + 1 < b \iff \left(5''\right) \delta - \gamma + 1 < r < b \text{ for some } r \leq 1.\]

It is clear that $(5'')$ implies $(5')$. If $(5')$ holds, then by the density of the set of all real numbers, both inequalities from $(5'')$ hold for some $r < b \leq 1$. Thus (5) holds.

Assertion (6) is equivalent to the following:

\[b < \gamma - \delta + 1 \iff b < r < \gamma - \delta + 1 \text{ for some } r \geq 0.\]

By the density of the set of all real numbers, the last equivalence holds and so does (6). \qed

**Theorem 1** (soundness of $G^{3}\mathcal{L}\forall$). If $\vdash_{G^{3}\mathcal{L}\forall} \mathcal{H}$, then $\models \mathcal{H}$.

**Proof.** All axioms of $G^{3}\mathcal{L}\forall$ are obviously valid, and all the inference rules of $G^{3}\mathcal{L}\forall$ are sound by Lemma [1].

Using the semantical invertibility of the propositional rules of $G^{3}\mathcal{L}\forall$ (see Lemma [1]), we can easily prove

**Proposition 1.** Let $\mathcal{H}$ be a quantifier-free hypersequent. If $\models \mathcal{H}$, then $\vdash_{G^{3}\mathcal{L}\forall} \mathcal{H}$.

### 3. Invertibility of the Rules of the Calculus $G^{3}\mathcal{L}\forall$ and its Relationship to the Calculus $G^{1}\mathcal{L}\forall$

Suppose $\mathcal{C}$ is a calculus. By $h(D)$ we denote the height of a (tree-like) $\mathcal{C}$-proof $D$. Let us recall some definitions (cf., e.g., [8, Section 3.4.1]).
A rule is called admissible for $\mathcal{C}$ if, for all applications $\mathcal{H}_1; \ldots; \mathcal{H}_k/\mathcal{H}$ of the rule and all $\mathcal{C}$-proofs $D_1$ of $\mathcal{H}_1, \ldots, D_k$ of $\mathcal{H}_k$, there exists a $\mathcal{C}$-proof $D$ of $\mathcal{H}$; the rule is called hp-admissible, or height-preserving admissible, for $\mathcal{C}$ if, in addition, the condition $h(D) \leq \max\{h(D_1), \ldots, h(D_k)\}$ holds. Everywhere in the sequel, the existence of such a proof $D$ means that it can be constructed if such proofs $D_1, \ldots, D_k$ are given.

A $k$-premise rule $\mathcal{R}$ is called invertible (resp. hp-invertible, or height-preserving invertible) in $\mathcal{C}$ if, for each $i = 1, \ldots, k$, the rule $\{\langle \mathcal{H}, \mathcal{H}_i \rangle \mid \langle \mathcal{H}_1, \ldots, \mathcal{H}_k, \mathcal{H} \rangle \in \mathcal{R}\}$ is admissible (resp. hp-admissible) for $\mathcal{C}$.

**Lemma 3.** The following rules are hp-admissible for the calculus $G^3LY$:

\[
\frac{G}{G} \vdash S \quad (\text{ew})^3, \quad \frac{G | \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{G | \Gamma_1 \Rightarrow \Delta_1 | \Gamma_2 \Rightarrow \Delta_2} \quad (\text{split})^3, \quad \frac{G | \Gamma \Rightarrow \Delta}{G | \Gamma, P \Rightarrow P, \Delta} \quad (\text{at} \Rightarrow \text{at})^3,
\]

where $P$ is an atom (i.e., an atomic RPL$\forall$-formula or a semipropositional variable).

**Proof.** 1. The rule (ew)$^3$ is obviously hp-admissible: if $D$ is a proof of a premise of the rule, then $D \mid S$ is a proof of its conclusion.

2. Let us establish the hp-admissibility of the rule (split)$^3$.

Suppose $S_0$ is a sequent occurrence in the root of a proof search tree $D_0$; then we say that an ancestor $S$ of the occurrence $S_0$ is augmentable unless $S$ is an ancestor of an occurrence $S'$ of a sequent $S'$ such that:

(i) $S'$ has the form (a) $B \Rightarrow p, A$, (b) $\forall x A \Rightarrow p$ or $[A]^x \Rightarrow p$, or (c) $q \Rightarrow \exists x A$ or $q \Rightarrow [A]^x$;

(ii) in $D_0$, $S'$ is a sequent occurrence in the premise of an application of the rule (a) $\Rightarrow \Rightarrow)^3$, (b) $\forall \Rightarrow)^3$, or (c) $\Rightarrow \exists)^3$, respectively; and

(iii) $S'$ is distinguished in the formulation of this rule.

Let $D_0$ be a proof for the premise $G | \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$ of the rule (split)$^3$. A tree $D$ is constructed from $D_0$ as follows: each occurrence $S$ of a sequent $S$ of the form $\Pi_1, \Pi_2 \Rightarrow \Sigma_1, \Sigma_2$ such that

(1) $S$ is an augmentable ancestor of the distinguished occurrence of the sequent $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$ in the root of $D_0$, and

(2) for each $i = 1, 2$ and each formula occurrence $F$ (as a sequent member) in $S$, if $F$ is contained in the distinguished occurrence $\Pi_i$ or $\Sigma_i$ in $S$, then $F$ is an ancestor of some formula occurrence contained in the distinguished occurrence $\Gamma_i$ or $\Delta_i$ in the root of $D_0$, is replaced by $\Pi_1 \Rightarrow \Sigma_1 | \Pi_2 \Rightarrow \Sigma_2$.

The rules of $G^3LY$ guarantee that, in a premise of a rule application, there is exactly one augmentable ancestor of the principalsequent occurrence. Therefore, when the tree $D$ is constructed, exactly one sequent occurrence in each node hypersequent of the proof $D_0$ is split into two sequents. Then it is easy to see that each application of a rule in $D_0$ is turned into an application of the same rule. Clearly, the hypersequent $G | \Gamma_1 \Rightarrow \Delta_1 | \Gamma_2 \Rightarrow \Delta_2$ is in the root of the tree $D$. Hence $D$ is a proof search tree for the conclusion of the rule (split)$^3$.

Let $L$ be a leaf of the tree $D_0$. Let $S$ be an occurrence of an atomic sequent $S$ in $L$ such that $S$ has the form $\Pi_1, \Pi_2 \Rightarrow \Sigma_1, \Sigma_2$, and $S$ and $S'$ meet conditions (1) and (2) above. Then the leaf of the tree $D$ obtained from $L$ contains the atomic sequents $\Pi_1 \Rightarrow \Sigma_1$ and $\Pi_2 \Rightarrow \Sigma_2$. So $D$ is a proof.

It remains to note that $h(D) = h(D_0)$. 
3. The rule $(\text{at}\rightarrow\text{at})^3$ is hp-admissible, since, given a proof $D$ for $G|\Gamma \Rightarrow \Delta$, we can construct a proof for $G|\Gamma, P \Rightarrow P, \Delta$ (with $P$ being an atom) in the following way. First, in $D$, rename all proper semipropositional variables and proper parameters of $D$ that occur in $P$ to new distinct ones. Next, in the resulting proof for $G|\Gamma \Rightarrow \Delta$, add the atom $P$ to the antecedent and succedent of each augmentable ancestor of the distinguished occurrence of the sequent $\Gamma \Rightarrow \Delta$ in the root. □

**Lemma 4.** All the inference rules of the calculus $G^3\forall$ are hp-invertible in it.

**Proof.** The rule $(\forall \Rightarrow)^3$ is hp-invertible, since we can obtain its premise from its conclusion using rules, which are hp-admissible (by Lemma 3):

\[
\frac{\mathcal{G}|\Gamma, \forall x A \Rightarrow \Delta}{\mathcal{G}|\Gamma, \forall x A, p \Rightarrow p, \Delta} \quad (\text{at}\Rightarrow\text{at})^3
\]

\[
\frac{\mathcal{G}|\Gamma, p \Rightarrow \Delta | \forall x A \Rightarrow p}{\mathcal{G}|\Gamma, p \Rightarrow \Delta | \forall x A \Rightarrow p} \quad (\text{split})^3
\]

\[
\frac{\mathcal{G}|\Gamma, p \Rightarrow \Delta | \forall x A \Rightarrow p | [A]^p_p \Rightarrow p}{\mathcal{G}|\Gamma, p \Rightarrow \Delta | \forall x A \Rightarrow p} \quad (\text{ew})^3.
\]

The hp-invertibility of the rule $(\Rightarrow \exists)^3$ is established very similarly.

The fact that all the inference rules of $G^3\forall$ are repetition-free allows us to demonstrate the hp-invertibility of the rules $(\Rightarrow \Rightarrow)^3$, $(\Rightarrow\rightarrow)^3$, $(\Rightarrow \forall)^3$, and $(\exists \Rightarrow)^3$ according to the classical scheme (see, e.g., [8, Proposition 3.5.4]). We give these demonstrations in full because later we will need to check that formal proofs constructed in them enjoy some properties.

**I.** Let us demonstrate that the rule $(\Rightarrow \Rightarrow)^3$ is hp-invertible. Toward this end, we show that, given a proof $D$ for a hypersequent of the form $\mathcal{G}|\Gamma, A \rightarrow B \Rightarrow \Delta$ and a semipropositional variable $p$ not occurring in the hypersequent, we can construct a proof $D'$ for $\mathcal{G}|\Gamma, p \Rightarrow \Delta | B \Rightarrow p, A$ with $h(D') \leq h(D)$. We proceed by induction on $h(D)$.

We can assume that $p$ does not occur in $D$ (otherwise replace all occurrences of $p$ in $D$ by a semipropositional variable of type 1 not occurring in $D$).

1. If $h(D) = 0$ (i.e., $D$ consists of a single axiom), then $\mathcal{G}$ is an axiom, hence so is $\mathcal{G}|\Gamma, p \Rightarrow \Delta | B \Rightarrow p, A$.

2. Let the root hypersequent $\mathcal{G}|\Gamma, A \rightarrow B \Rightarrow \Delta$ in $D$ be the conclusion of an application $R$ of a rule $\mathcal{R}$.

2.1. Suppose the principal formula occurrence in $R$ is the distinguished occurrence of $A \rightarrow B$. By $D_1$ denote the subtree of the root of $D$; $D_1$ is a proof for the premise of $R$. The premise has the form $\mathcal{G}|\Gamma, p_1 \Rightarrow \Delta | B \Rightarrow p_1, A$. Then replacing all occurrences of $p_1$ in $D_1$ by $p$ yields a proof $D'$ for $\mathcal{G}|\Gamma, p \Rightarrow \Delta | B \Rightarrow p, A$ with $h(D') < h(D)$.

2.2. Now suppose the principal formula occurrence in $R$ is not the distinguished occurrence of $A \rightarrow B$.

2.2.1. If $\mathcal{R}$ is a one-premise rule, the proof $D$ looks like this:

\[
\frac{D_1}{\mathcal{G}|\Gamma, A \rightarrow B \Rightarrow \Delta} \quad \mathcal{R}.
\]

By applying the induction hypothesis to the proof $D_1$, we construct a proof $D'_1$ for $\mathcal{G}_1|\Gamma_1, p \Rightarrow \Delta_1 | B \Rightarrow p, A$ with $h(D'_1) \leq h(D_1)$. By applying $\mathcal{R}$ to the root

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2 See the proofs of Lemma 7 and Theorem 3.
To establish the hp-invertibility of the rule \((\Rightarrow \forall)\), we show that, given a proof \(D\) for a hypersequent of the form \(\mathcal{G} \mid \Gamma \Rightarrow \forall x A, \Delta\) and a parameter \(a\) not in \(\Delta\), we construct a proof \(D_{a}^{\prime}\) for \(\mathcal{G} \mid \Gamma, \Delta_{a} \Rightarrow B \Rightarrow p, A\) with \(h(D_{a}^{\prime}) \leq h(D)\).

2.2.2. If \(R\) is a two-premise rule, i.e., the rule \((\Rightarrow \forall)\), then the proof \(D\) looks like this:

\[
\begin{array}{c}
D_{1} \\
\mathcal{G}_{1} \mid \Gamma_{1}, A \Rightarrow B, \Delta_{1} \\
D_{2} \\
\mathcal{G}_{2} \mid \Gamma_{2}, A \Rightarrow B, \Delta_{2} \\
\hline
\mathcal{G} \mid \Gamma, \Rightarrow B \Rightarrow \Delta \\
R.
\end{array}
\]

For each \(i = 1, 2\), by the induction hypothesis applied to the proof \(D_{i}\), we construct a proof \(D_{i}^{\prime}\) for \(\mathcal{G}_{i} \mid \Gamma_{i}, p \Rightarrow \Delta_{i} \Rightarrow B \Rightarrow p, A\) with \(h(D_{i}^{\prime}) \leq h(D_{i})\).

By applying \(R\) to the root hypersequents of the proofs \(D_{1}^{\prime}\) and \(D_{2}^{\prime}\), we get a proof \(D^{\prime}\) for \(\mathcal{G} \mid \Gamma, p \Rightarrow \Delta_{1} \Rightarrow B \Rightarrow p, A\) with \(h(D^{\prime}) \leq h(D)\).

II. In order to establish the hp-invertibility of the rule \((\Rightarrow \forall)\), we show that, given a proof \(D\) for a hypersequent of the form \(\mathcal{G} \mid \Gamma \Rightarrow A \Rightarrow B, \Delta\), we can construct a proof \(D^{\prime}\) for \(\mathcal{G} \mid \Gamma \Rightarrow \Delta\) and a proof \(D^{\prime\prime}\) for \(\mathcal{G} \mid \Gamma, A \Rightarrow B, \Delta\) such that \(h(D^{\prime}) \leq h(D)\) and \(h(D^{\prime\prime}) \leq h(D)\). We use induction on \(h(D)\).

1. If \(h(D) = 0\), then \(\mathcal{G}\) is an axiom, and so are \(\mathcal{G} \mid \Gamma \Rightarrow \Delta\) and \(\mathcal{G} \mid \Gamma, A \Rightarrow B, \Delta\).

2. Let the root hypersequent \(\mathcal{G} \mid \Gamma \Rightarrow A \Rightarrow B, \Delta\) in \(D\) be the conclusion of an application \(R\) of a rule \(R\).

2.1. If the principal formula occurrence in \(R\) is the distinguished occurrence of \(A \Rightarrow B\), then the subtrees of the root of \(D\) are the desired proofs.

2.2. Suppose the principal formula occurrence in \(R\) is not the distinguished occurrence of \(A \Rightarrow B\).

2.2.1. In the case the rule \(R\) is one-premise, the proof \(D\) looks like this:

\[
\begin{array}{c}
D_{1} \\
\mathcal{G}_{1} \mid \Gamma_{1} \Rightarrow A \Rightarrow B, \Delta_{1} \\
\hline
\mathcal{G} \mid \Gamma \Rightarrow A \Rightarrow B, \Delta \\
R.
\end{array}
\]

Using the induction hypothesis, from the proof \(D_{1}\), we construct a proof \(D_{1}^{\prime}\) for \(\mathcal{G}_{1} \mid \Gamma_{1} \Rightarrow \Delta_{1}\) and a proof \(D_{1}^{\prime\prime}\) for \(\mathcal{G}_{1} \mid \Gamma_{1}, A \Rightarrow B, \Delta_{1}\) such that \(h(D_{1}^{\prime}) \leq h(D_{1})\) and \(h(D_{1}^{\prime\prime}) \leq h(D_{1})\).

Applying \(R\) to the root hypersequent of the proof \(D_{1}^{\prime}\) gives a proof \(D^{\prime}\) for \(\mathcal{G} \mid \Gamma \Rightarrow \Delta\) with \(h(D^{\prime}) \leq h(D)\), and applying \(R\) to the root hypersequent of the proof \(D_{1}^{\prime\prime}\) gives a proof \(D^{\prime\prime}\) for \(\mathcal{G} \mid \Gamma, A \Rightarrow B, \Delta\) with \(h(D^{\prime\prime}) \leq h(D)\).

2.2.2. In the case the rule \(R\) is two-premise, the proof \(D\) looks like this:

\[
\begin{array}{c}
D_{1} \\
\mathcal{G}_{1} \mid \Gamma_{1} \Rightarrow A \Rightarrow B, \Delta_{1} \\
D_{2} \\
\mathcal{G}_{2} \mid \Gamma_{2} \Rightarrow A \Rightarrow B, \Delta_{2} \\
\hline
\mathcal{G} \mid \Gamma \Rightarrow A \Rightarrow B, \Delta \\
R.
\end{array}
\]

For each \(i = 1, 2\), by the induction hypothesis applied to the proof \(D_{i}\), we construct a proof \(D_{i}^{\prime}\) for \(\mathcal{G}_{i} \mid \Gamma_{i} \Rightarrow \Delta_{i}\) and a proof \(D_{i}^{\prime\prime}\) for \(\mathcal{G}_{i} \mid \Gamma_{i}, A \Rightarrow B, \Delta_{i}\) such that \(h(D_{i}^{\prime}) \leq h(D_{i})\) and \(h(D_{i}^{\prime\prime}) \leq h(D_{i})\).

Next, by applying \(R\) to the root hypersequents of the proofs \(D_{1}^{\prime}\) and \(D_{2}^{\prime}\), we obtain a proof \(D^{\prime}\) for \(\mathcal{G} \mid \Gamma \Rightarrow \Delta\) with \(h(D^{\prime}) \leq h(D)\).

Finally, applying \(R\) to the root hypersequents of the proofs \(D_{1}^{\prime\prime}\) and \(D_{2}^{\prime\prime}\) yields a proof \(D^{\prime\prime}\) for \(\mathcal{G} \mid \Gamma, A \Rightarrow B, \Delta\) with \(h(D^{\prime\prime}) \leq h(D)\).

III. To establish the hp-invertibility of the rule \((\Rightarrow \forall)\), we show that, given a proof \(D\) for a hypersequent of the form \(\mathcal{G} \mid \Gamma \Rightarrow \forall x A, \Delta\) and a parameter \(a\) not in \(\Delta\), we construct a proof \(D_{a}^{\prime}\) for \(\mathcal{G} \mid \Gamma, \Delta_{a} \Rightarrow B \Rightarrow p, A\) with \(h(D_{a}^{\prime}) \leq h(D)\).
occurrence in the hypersequent, we can construct a proof $D'$ for $G | \Gamma \Rightarrow [A]_a^\pi, \Delta$ with $h(D') \leq h(D)$. This is done by induction on $h(D)$.

We can assume that $a$ does not occur in $D$ (otherwise replace all occurrences of $a$ in $D$ by a parameter not occurring in $D$).

1. If $h(D) = 0$, then $G$ is an axiom and so is $G | \Gamma \Rightarrow [A]_a^\pi, \Delta$.

2. Let the root hypersequent $G | \Gamma \Rightarrow \forall x A, \Delta$ in $D$ be the conclusion of an application $R$ of a rule $\mathcal{R}$.

2.1. Suppose the principal formula occurrence in $R$ is the distinguished occurrence of $\forall x A$. By $D_1$ denote the subtree of the root of $D$; $D_1$ is a proof for the premise of $R$. The premise has the form $G | \Gamma \Rightarrow [A]_a^\pi, \Delta$. By replacing all occurrences of $a_1$ in $D_1$ by $a$, we get a proof $D'$ for $G | \Gamma \Rightarrow [A]_a^\pi, \Delta$ with $h(D') < h(D)$.

2.2. Next, suppose the principal formula occurrence in $R$ is not the distinguished occurrence of $\forall x A$.

2.2.1. If $\mathcal{R}$ is one-premise, the proof $D$ looks like this:

\[
\frac{D_1}{G | \Gamma \Rightarrow \forall x A, \Delta ^1 \mathcal{R}}
\]

Using the induction hypothesis, we transform the proof $D_1$ into a proof $D'_1$ for $G_1 | \Gamma _1 \Rightarrow [A]_a^\pi, \Delta _1$ such that $h(D'_1) \leq h(D_1)$. By applying $\mathcal{R}$ to the root hypersequent of the proof $D'_1$, we have a proof $D'$ for $G | \Gamma \Rightarrow [A]_a^\pi, \Delta$ with $h(D') \leq h(D)$.

2.2.2. If $\mathcal{R}$ is two-premise, the proof $D$ looks like this:

\[
\frac{D_1 \quad D_2}{G | \Gamma \Rightarrow \forall x A, \Delta ^2 \mathcal{R}}
\]

For each $i = 1, 2$, by the induction hypothesis, we transform the proof $D_i$ into a proof $D'_i$ for $G_i | \Gamma _i \Rightarrow [A]_a^\pi, \Delta _i$ such that $h(D'_i) \leq h(D_i)$.

By applying $\mathcal{R}$ to the root hypersequents of the proofs $D'_1$ and $D'_2$, we obtain a proof $D'$ for $G | \Gamma \Rightarrow [A]_a^\pi, \Delta$ with $h(D') \leq h(D)$.

IV. The hp-invertibility of the rule $(\exists \Rightarrow)^3$ is established very similarly to the hp-invertibility of the rule $(\Rightarrow \forall)^3$, see item III. □

**Remark 1.** We know the following about whether the inference rules of the calculus $G^2 \mathcal{L}_\forall$ are invertible in it. The rules $(\forall \Rightarrow)^2$ and $(\Rightarrow \exists)^2$ are hp-invertible because, for each of them, its premise includes its conclusion. Using arguments like those given in the proof of Lemma 3, we can establish the hp-invertibility of the rules $(\Rightarrow \Rightarrow)^2$, $(\Rightarrow \vee)^2$, and $(\exists \Rightarrow)^2$. However, we do not know whether the rule $(\Rightarrow \Rightarrow)^2$ is invertible.

**Lemma 5.** The following rule is hp-admissible for the calculus $G^3 \mathcal{L}_\forall$:

\[
\frac{G | S | S}{G | S} \text{ (ec)}^3.
\]

**Proof.** We show that a proof $D$ for a hypersequent of the form $G | S | S$ can be transformed into a proof $\tilde{D}$ for $G | S$ with $h(\tilde{D}) \leq h(D)$. We proceed by induction on $h(D)$.

1. If $h(D) = 0$, then the hypersequents $G | S | S$ and $G | S$ are axioms.
2. Let the root hypersequent \( \mathcal{G} \mid S \mid S \) in \( D \) be the conclusion of an application \( R \) of a rule \( \mathcal{R} \).

2.1. If the principal sequent occurrence in \( R \) is not one of the two occurrences of \( S \) distinguished in \( \mathcal{G} \mid S \mid S \), then we apply the induction hypothesis to the proof for each premise of \( R \) and next use \( \mathcal{R} \) to obtain the desired proof for \( \mathcal{G} \mid S \).

2.2. Otherwise, we are to treat each inference rule of \( G^3 \mathcal{L} \forall \) as \( \mathcal{R} \). However, all these cases are similar to one another. So we treat only the case where \( \mathcal{R} \) is \( (\forall \Rightarrow)^3 \). Then the proof \( D \) has the form:

\[
D_1
\]

\[
\begin{array}{c}
\mathcal{G} \mid \Gamma, p \Rightarrow \Delta \mid \forall x A \Rightarrow p \mid [A]^{r}_f \Rightarrow p \mid \Gamma, \forall x A \Rightarrow \Delta \\
\mathcal{G} \mid \Gamma, \forall x A \Rightarrow \Delta \mid \Gamma, \forall x A \Rightarrow \Delta
\end{array}
\]

\( (\forall \Rightarrow)^3 \).

Since the rule \( (\forall \Rightarrow)^3 \) is hp-invertible (see Lemma 4), given the proof \( D_1 \), we can find a proof \( D'_1 \) for

\[
\mathcal{G} \mid \Gamma, p \Rightarrow \Delta \mid \forall x A \Rightarrow p \mid [A]^{r}_f \Rightarrow p \mid \Gamma, p_1 \Rightarrow \Delta \mid \forall x A \Rightarrow p_1 \mid [A]^{r}_f \Rightarrow p_1,
\]

where \( p_1 \) does not occur in the root hypersequent of \( D_1 \) and \( h(D'_1) \leq h(D_1) \).

Replacing all occurrences of \( p_1 \) in \( D'_1 \) by \( p \) yields a proof \( D''_1 \) for

\[
\mathcal{G} \mid \Gamma, p \Rightarrow \Delta \mid \forall x A \Rightarrow p \mid [A]^{r}_f \Rightarrow p \mid \Gamma, p \Rightarrow \Delta \mid \forall x A \Rightarrow p \mid [A]^{r}_f \Rightarrow p;
\]

whence using the induction hypothesis three times, we get a proof \( \tilde{D} \) for

\[
\mathcal{G} \mid \Gamma, p \Rightarrow \Delta \mid \forall x A \Rightarrow p \mid [A]^{r}_f \Rightarrow p
\]

such that \( h(\tilde{D}_1) \leq h(D''_1) \leq h(D_1) \).

Finally, by applying \( (\forall \Rightarrow)^3 \) to the root hypersequent of the proof \( \tilde{D}_1 \), we obtain the desired proof \( \hat{D} \) for \( \mathcal{G} \mid \Gamma, \forall x A \Rightarrow \Delta \) with \( h(\hat{D}) \leq h(D) \).

**Lemma 6.** Each inference rule of the calculus \( G^3 \mathcal{L} \forall \) is admissible for the calculus \( G^3 \mathcal{L} \forall \).

**Proof.** An application of the rule \( (\Rightarrow \Rightarrow)^1 \), \( (\Rightarrow \Rightarrow)^1 \), \( (\Rightarrow \forall)^1 \), or \( (\exists \Rightarrow)^1 \) of \( G^3 \mathcal{L} \forall \) can be represented as an application of the corresponding rule of \( G^3 \mathcal{L} \forall \) followed by an application of the rule \( (ec)^3 \). E.g., an application of \( (\Rightarrow \Rightarrow)^1 \) is represented thus:

\[
\frac{\mathcal{G} \mid \Gamma \Rightarrow A \Rightarrow B, \Delta \mid \Gamma \Rightarrow \Delta; \quad \mathcal{G} \mid \Gamma \Rightarrow A \Rightarrow B, \Delta \mid \Gamma, A \Rightarrow B, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow A \Rightarrow B, \Delta \mid \Gamma \Rightarrow A \Rightarrow B, \Delta} \tag{\( \Rightarrow \Rightarrow)^3 \}
\]

By Lemma 5 the rule \( (ec)^3 \) is admissible for \( G^3 \mathcal{L} \forall \). So these four rules of \( G^3 \mathcal{L} \forall \) are admissible for \( G^3 \mathcal{L} \forall \).

The rule \( (\forall \Rightarrow)^1 \) is admissible for \( G^3 \mathcal{L} \forall \), since an application of it can be represented as several applications of rules, which are admissible for \( G^3 \mathcal{L} \forall \) (by Lemmas 3 and 5), as follows:

\[
\begin{array}{c}
\mathcal{G} \mid \Gamma, \forall x A \Rightarrow \Delta \mid \Gamma, [A]^{r}_f \Rightarrow \Delta \\
\mathcal{G} \mid \Gamma, \forall x A \Rightarrow \Delta \mid \Gamma, p \Rightarrow p, [A]^{r}_f \Rightarrow p \Rightarrow p, \Delta (at \Rightarrow at)^3 \\
\mathcal{G} \mid \Gamma, \forall x A \Rightarrow \Delta \mid \Gamma, p \Rightarrow \Delta \mid [A]^{r}_f \Rightarrow p \Rightarrow p, \Delta \Rightarrow \Delta (split)^3 \\
\mathcal{G} \mid \Gamma, \forall x A \Rightarrow \Delta \mid \Gamma, p \Rightarrow \Delta \mid \forall x A \Rightarrow p \mid [A]^{r}_f \Rightarrow p \Rightarrow p \Rightarrow p (ew)^3 \\
\mathcal{G} \mid \Gamma, \forall x A \Rightarrow \Delta \mid \Gamma, \forall x A \Rightarrow \Delta \Rightarrow \Delta (ec)^3,
\end{array}
\]
where \( p \) does not occur in the top hypersequent.

The rule \((\Rightarrow \exists)\) is treated similarly to \((\forall \Rightarrow)\).

\[ \square \]

**Theorem 2.** Suppose \( \mathcal{H} \) is a hypersequent of the calculus \( G^1 \mathcal{L} \forall \). If \( \vdash_{G^1 \mathcal{L} \forall} \mathcal{H} \), then \( \vdash_{G^3 \mathcal{L} \forall} \mathcal{H} \).

**Proof.** All axioms of \( G^1 \mathcal{L} \forall \) are axioms of \( G^3 \mathcal{L} \forall \), and all the inference rules of \( G^1 \mathcal{L} \forall \) are admissible for \( G^3 \mathcal{L} \forall \) by Lemma \[6\]. \[ \square \]

4. **Transforming \( G^3 \mathcal{L} \forall \)-proofs according to tactics**

As in \[7\], to organize bottom-up \( G^3 \mathcal{L} \forall \)-proof search, we can use an auxiliary algorithm \( t \), called a (proof search) tactic, that takes a proof search tree \( D \) as input and returns either

(a) the message \( t(D) \) indicating that no leaf hypersequent of \( D \) contains any logical symbol, or

(b) a non-atomic \( RPL \forall \)-formula occurrence \( t(D) \) (as a sequent member) in a leaf hypersequent of \( D \).

By a result of a backward rule application to a proof search tree \( D \) according to a tactic \( t \), we mean \( D \) if \( t(D) \) is not a formula occurrence; otherwise, a proof search tree obtained from \( D \) by a backward application of a (uniquely determined) rule of \( G^3 \mathcal{L} \forall \) to the occurrence \( t(D) \). We say that a proof search tree (in particular, a proof) \( D \) for \( \mathcal{H} \) can be constructed according to a tactic \( t \) if \( D \) can be obtained from \( \mathcal{H} \) by a finite number of backward rule applications according to \( t \).

For a tactic \( t \) and a hypersequent \( \mathcal{H} \), let \( D^t_H \) be a tree obtained from \( \mathcal{H} \) by an infinite number of backward applications according to \( t \). Call a tactic \( t \) fair if, for each hypersequent \( \mathcal{H} \), each branch \( B \) of the tree \( D^t_H \), and each non-atomic \( RPL \forall \)-formula occurrence \( F \) (as a sequent member) on \( B \), there is a backward application to some ancestor of \( F \) on \( B \).

Now we state a theorem that allows us to justify the use of any fair tactic for bottom-up proof search.

**Theorem 3.** Suppose \( \mathcal{G} \) is a \( G^3 \mathcal{L} \forall \)-provable hypersequent, and \( t \) is a fair tactic. Then some \( G^3 \mathcal{L} \forall \)-proof of \( \mathcal{G} \) can be constructed according to \( t \).

Before proving this theorem, we establish the following lemma, which helps us to make one step in transforming a \( G^3 \mathcal{L} \forall \)-proof according to a tactic.

**Lemma 7.** Suppose \( D \) is a \( G^3 \mathcal{L} \forall \)-proof for \( \mathcal{H} \), and \( F \) is a non-atomic \( RPL \forall \)-formula occurrence (as a sequent member) in \( \mathcal{H} \). Then a \( G^3 \mathcal{L} \forall \)-proof \( \hat{D} \) of the form

\[
\begin{array}{c}
\hat{D}_1 \\
\hat{H}_1 \\
\mathcal{H}
\end{array} \quad \text{or} \quad
\begin{array}{c}
\hat{D}_1 \\
\hat{H}_1 \\
\hat{D}_2 \\
\hat{H}_2 \\
\mathcal{H}
\end{array}
\]

can be constructed such that:

(1) \( F \) is the principal formula occurrence in the lowest backward application in \( \hat{D} \), and \( h(\hat{D}_1) \leq h(D) \) for each \( i \);

(2) if \( F \) is the principal formula occurrence in the lowest backward application in \( D \), then \( \hat{D} \) is the same as \( D \);
if \( h(D) > 0 \) and the principal formula occurrence \( F_0 \) in the lowest backward application in \( D \) differs from \( F \), then, for each \( i \), the ancestor of \( F_0 \) in \( \hat{H}_i \) is the principal formula occurrence in the lowest backward application in \( \hat{D}_i \).

Proof. If \( F \) is the principal formula occurrence in the lowest backward application in \( D \), then we immediately take \( D \) as \( \hat{D} \), and assertions (1) of the lemma clearly hold.

Suppose \( F \) is not the principal formula occurrence in the lowest backward application in \( D \). Then assertion (2) of the lemma is trivially true. Let \( R \) be the only inference rule that can be applied backward to the occurrence \( F \) in \( H \).

Using the construction in the proof of the hp-invertibility of \( R \) (see Lemma 4), from the proof \( D \) for \( H \), we construct proofs \( \hat{D}_i \) (for \( i = 1 \) or \( i = 1, 2 \)) for all the premises of a backward application of \( R \) to the occurrence \( F \) in \( H \), and we have \( h(\hat{D}_i) \leq h(D) \).

Now, by applying \( R \) to the root hypersequents of the proofs \( \hat{D}_i \), we obtain a proof \( \hat{D} \) of \( H \) for which assertion (1) of the lemma holds.

After examining the construction in the proof of the hp-invertibility of \( R \) (see Lemma 4), we are sure that \( \hat{D} \) satisfies assertion (3) of the lemma being proved.

Proof of Theorem 3. Fix a \( G^3L \forall \)-proof \( D_0 \) for \( G \) and transform it according to \( t \) in stages. The result of each stage will be some \( G^3L \forall \)-proof \( D \) for \( G \) consisting of

(a) a proof search tree \( D^1 \) that has the common root with \( D \) and is constructed according to \( t \), and which is called the transformed part of \( D \), as well as

(b) a finite number of proof trees whose roots are leaves of \( D^1 \), and each of which is called a nontransformed part of \( D \).

Define the transformed part of the initial proof \( D_0 \) to be its root, and the only nontransformed part of it to be \( D_0 \) itself.

We use induction on the maximal height \( H(D) \) of the nontransformed parts of the current proof \( D \) being transformed.

1. If \( H(D) = 0 \), then \( D \) is the required proof.

2. Suppose \( H(D) > 0 \) and \( D^1 \) is the transformed part of \( D \).

2.1. To obtain a proof \( \hat{D} \) (with its transformed part \( \hat{D}^1 \)) as a result of the stage, we carry out some finite number \( N \) of backward applications to the transformed part of the current proof (which is \( D \) initially) according to the fair tactic \( t \). We choose such a number \( N \) so that, for each branch \( B \) of \( \hat{D}^1 \) and each non-atomic RPL\( \forall \)-formula occurrence \( F \) (as a sequent member) in the node of \( \hat{D}^1 \) that was a leaf of \( D^1 \) and is on \( B \) now, there is a backward application to some ancestor of \( F \) on \( B \).

2.2. We carry out each backward application to a formula occurrence \( F \) (chosen by \( t \)) in a leaf of the transformed part \( D^1 \) of the current proof \( D \) for \( G \) as follows. Let \( D^n \) be the nontransformed part of \( D \) whose root is this leaf, and \( \hat{H} \) be the root hypersequent of \( D^n \). By Lemma 7, given the proof \( D^n \) and the occurrence \( F \) in \( \hat{H} \), we construct a proof \( \hat{D}^n \) of the form

\[
\begin{array}{c}
\hat{D}^n \\
\hat{H}_1 \\
\hat{H}_2
\end{array}
\]

or

\[
\begin{array}{c}
\hat{D}^n \\
\hat{H}_1 \\
\hat{H}_2
\end{array}
\]

\[
\begin{array}{c}
\hat{D}^n \\
\hat{H}_1 \\
\hat{H}_2
\end{array}
\]

3 Roughly speaking, the lowest backward application in \( D \) goes one level up in \( \hat{D} \).
Properties P1–P4 can be verified in a straightforward way.

(4) if $F$ is the principal formula occurrence in the lowest backward application in $D^n$, then $\hat{D}_i^n$ is the same as $D^n$, and hence $h(\hat{D}_i^n) < h(D^n)$ for each $i$;

(5) if $h(D^n) > 0$ and the principal formula occurrence $F_0$ in the lowest backward application in $D^n$ differs from $F$, then, for each $i$, the ancestor of $F_0$ in $H_i$ is the principal formula occurrence in the lowest backward application in $\hat{D}_i^n$.

Next, we replace the subtree $D^n$ in $D$ by $\tilde{D}_i^n$. Finally, the lowest backward application in $D^n$ is included in the transformed part of the resulting proof for $G$.

Thus $H(\tilde{D}) < H(D)$. By the induction hypothesis applied to $\tilde{D}$, we construct a proof of $G$ according to $t$.

5. The mid-hypersequent theorem for $G^3\mathbf{Ly}$ and its consequences

We say that a $G^3\mathbf{Ly}$-proof is a mid-hypersequent proof if in it all applications of propositional rules are above all applications of quantifier rules.

To transform some $G^3\mathbf{Ly}$-proofs into mid-hypersequent ones, we will use the following properties (P1–P4), which express permutability of adjacent rule applications. In each of these properties, the resulting proof is displayed after the initial one. From now on, if a formula (or sequent) occurrence in the conclusion of a rule application is in boldface, then the occurrence is the principal one in the application.

Properties P1–P4 can be verified in a straightforward way.

P1. Let $R_1$ and $R_2$ be any one-premise inference rules of $G^3\mathbf{Ly}$, except the case where $R_1 \in \{(\Rightarrow \forall)^3, (\exists \Rightarrow)^3\}$ and $R_2 \in \{(\Rightarrow \exists)^3, (\forall \Rightarrow)^3\}$.

If $R_1$ is $(\Rightarrow \exists)^3$, $(\forall \Rightarrow)^3$, or $(\exists \Rightarrow)^3$, $R_2$ is $(\Rightarrow \forall)^3$ or $(\Rightarrow \exists)^3$, and the above case is excluded, then we can perform the following transformation:

\[
\begin{align*}
&\quad D \\
\frac{G | \Gamma,F_1 \Rightarrow F_2,\Delta | H_1 | H_2 | R_2}{G | \Gamma, A_1 \Rightarrow A_2, \Delta | R_1} & \frac{G | \Gamma,F_1 \Rightarrow F_2,\Delta | H_1 | H_2 | R_1}{G | \Gamma, A_1 \Rightarrow A_2, \Delta | R_2}
\end{align*}
\]

For a hypersequent that is at the bottom of an appropriate initial proof and has the form

$G | \Gamma, A_1, A_2 \Rightarrow \Delta$, $G | \Gamma \Rightarrow A_1, A_2, \Delta$, or $G | \Gamma, A_2 \Rightarrow A_1, \Delta$,

we can carry out a transformation similar to that just given.

E.g., if $R_1$ is $(\Rightarrow \exists)^3$ and $R_2$ is $(\Rightarrow \forall)^3$, then the initial and resulting proofs look like:
A. S. GERASIMOV

\[
\begin{array}{c}
D \\
D \\
D \\
D \\
D \end{array}
\]

\[
\begin{align*}
\sequent{G | \Gamma, p \Rightarrow [C^e_{\alpha}]_1, \Delta | B \Rightarrow p, A} & \quad (\Rightarrow \forall)^3 \\
\sequent{G | \Gamma, p \Rightarrow \forall x C, \Delta | B \Rightarrow p, A} & \quad (\Rightarrow \forall)^3 \\
\sequent{G | \Gamma, A \Rightarrow B \Rightarrow \forall x C, \Delta} & \quad (\Rightarrow \forall)^3 \\
\sequent{G | \Gamma, A \Rightarrow B \Rightarrow \forall x C, \Delta} & \quad (\Rightarrow \forall)^3
\end{align*}
\]

**P2.** Let rules \(\mathcal{R}_1\) and \(\mathcal{R}_2\) be as in the first paragraph of P1. Then we can perform this transformation:

\[
\begin{align*}
\sequent{G | \mathcal{H}_1 | \mathcal{H}_2 | \mathcal{R}_2} & \quad \sequent{G | \mathcal{H}_1 | \mathcal{H}_2 | \mathcal{R}_1} \\
\sequent{G | \mathcal{H}_1 | \mathcal{H}_2 | \mathcal{R}_1} & \quad \sequent{G | \mathcal{H}_1 | \mathcal{H}_2 | \mathcal{R}_2}
\end{align*}
\]

E.g., if \(\mathcal{R}_1\) is \((\Rightarrow \forall)^3\) and \(\mathcal{R}_2\) is \((\Rightarrow \forall)^3\), then the initial and resulting proofs have the forms:

\[
\begin{align*}
\sequent{G | \Gamma_1, p \Rightarrow \Delta_1 | B \Rightarrow p, A | \Gamma_2 \Rightarrow [C^e_{\alpha}]_1, \Delta_2} & \quad (\Rightarrow \forall)^3 \\
\sequent{G | \Gamma_1, p \Rightarrow \Delta_1 | B \Rightarrow p, A | \Gamma_2 \Rightarrow \forall x C, \Delta_2} & \quad (\Rightarrow \forall)^3 \\
\sequent{G | \Gamma_1, A \Rightarrow B \Rightarrow \Delta_1 | \Gamma_2 \Rightarrow \forall x C, \Delta_2} & \quad (\Rightarrow \forall)^3 \\
\sequent{G | \Gamma_1, A \Rightarrow B \Rightarrow \Delta_1 | \Gamma_2 \Rightarrow \forall x C, \Delta_2} & \quad (\Rightarrow \forall)^3
\end{align*}
\]

**P3.** Let \(\mathcal{R}\) be any one-premise inference rule of \(G^3 \forall\).

If \(\mathcal{R}\) is \((\Rightarrow \forall)^3\), \((\forall \Rightarrow)^3\), or \((\exists \Rightarrow)^3\), then under the conditions stated below, we can carry out the following transformation:

\[
\begin{align*}
\sequent{D_1} & \quad \sequent{D_2} \\
\sequent{G | \Gamma, F_1 \Rightarrow \Delta | \mathcal{H}_1} & \quad \sequent{G | \Gamma, F_2, A \Rightarrow B, \Delta | \mathcal{H}_2} \quad \sequent{G | \Gamma, C \Rightarrow \Delta} \quad \sequent{G | \Gamma, C \Rightarrow B, \Delta} \quad \sequent{G | \Gamma, C \Rightarrow A \Rightarrow B, \Delta}
\end{align*}
\]

\[
\begin{align*}
\sequent{D_1} & \quad \sequent{D_2} \\
\sequent{G | \Gamma, F_2 \Rightarrow \Delta | \mathcal{H}_2} & \quad \sequent{G | \Gamma, F_2, A \Rightarrow B, \Delta | \mathcal{H}_2} \quad \sequent{G | \Gamma, F_2 \Rightarrow A \Rightarrow B, \Delta} \quad \sequent{G | \Gamma, C \Rightarrow A \Rightarrow B, \Delta}
\end{align*}
\]

If \(\mathcal{R}\) is \((\Rightarrow \forall)^3\) or \((\Rightarrow \exists)^3\), then under the conditions stated below, we can perform a similar transformation with a bottom hypersequent of the form \(\sequent{G | \Gamma \Rightarrow C, A \Rightarrow B, \Delta}\).

For both the transformations, two conditions must hold. First, if \(\mathcal{R}\) is \((\forall \Rightarrow)^3\) or \((\Rightarrow \exists)^3\), then the proper terms of the three displayed applications of \(\mathcal{R}\) are the same. Second, we construct the proof \(D'_1\) thus:

(a) Suppose \(\mathcal{R}\) is \((\exists \Rightarrow)^3\) or \((\Rightarrow \forall)^3\), \(a_1\) and \(a_2\) are the proper parameters of the two applications of \(\mathcal{R}\) displayed in the initial proof on the left and right, respectively; then: \(D'_1 = D_1\) if \(a_1 = a_2\); otherwise, we obtain the proof \(D'_1\) (for the root hypersequent of \(D_1\)) from \(D_1\) by replacing all occurrences of \(a_2\) with a parameter...
not occurring in \(D_1\), and next, we get the required proof \(D'_1\) from \(\tilde{D}_1\) by replacing all occurrences of \(a_1\) with \(a_2\).

(b) If \(\mathcal{R}\) is \((\rightarrow)\), \((\forall \Rightarrow)\), or \((\Rightarrow \exists)\), then we obtain \(D'_1\) from \(D_1\) as in (a), but instead of parameters, we use semipropositional variables of the type corresponding to the rule \(\mathcal{R}\).

E.g., if \(\mathcal{R}\) is \((\forall \Rightarrow)\), then the initial and resulting proofs look like:

\[
\begin{array}{c}
\frac{G|\Gamma, \forall x C \Rightarrow \Delta}{G|\Gamma, \forall x C \Rightarrow \Delta} \quad \frac{G|\Gamma, \forall x C \Rightarrow A \Rightarrow B, \Delta}{G|\Gamma, \forall x C \Rightarrow \Delta} \quad \frac{G|\Gamma, \forall x C \Rightarrow \Delta}{G|\Gamma, \forall x C \Rightarrow A \Rightarrow B, \Delta}
\end{array}
\]

\[
\begin{array}{c}
\frac{G|\Gamma, p_2 \Rightarrow \Delta}{G|\Gamma, p_2 \Rightarrow \Delta} \quad \frac{G|\Gamma, p_2 \Rightarrow B, \Delta}{G|\Gamma, p_2 \Rightarrow \Delta} \quad \frac{G|\Gamma, \forall x C \Rightarrow p_2 | C| \Rightarrow p_2}{G|\Gamma, \forall x C \Rightarrow \Delta} \quad \frac{G|\Gamma, \forall x C \Rightarrow \Delta}{G|\Gamma, \forall x C \Rightarrow A \Rightarrow B, \Delta}
\end{array}
\]

\[
\begin{array}{c}
D_1' \\
D_2
\end{array}
\]

P4. Let \(\mathcal{R}\) be any one-premise inference rule of \(G^3L\forall\). Then we can carry out the transformation:

\[
\begin{array}{c}
\frac{G|\Gamma_1, p_1 \Rightarrow \Delta}{G|\Gamma_1, p_1 \Rightarrow \Delta} \quad \frac{G|\Gamma_1, p_2 \Rightarrow B, \Delta}{G|\Gamma_1, p_2 \Rightarrow \Delta} \quad \frac{G|\Gamma_1, \forall x C \Rightarrow p_2 | C| \Rightarrow p_2}{G|\Gamma_1, \forall x C \Rightarrow \Delta}
\end{array}
\]

Here all the principal formula occurrences in the three displayed applications of \(\mathcal{R}\) represent the same formula; in the case where \(\mathcal{R}\) is \((\forall \Rightarrow)\) or \((\Rightarrow \exists)\), the additional condition is the same as in P3; and the proof \(D'_1\) is constructed from \(D_1\) as in P3.

E.g., if \(\mathcal{R}\) is \((\Rightarrow \forall)\), then the initial and resulting proofs have the forms:

\[
\begin{array}{c}
\frac{G|\Gamma_1 \Rightarrow \Delta_1 | \Gamma_2 \Rightarrow [C|_{a_2} \Rightarrow \Delta_2}{G|\Gamma_1 \Rightarrow \Delta_1 | \Gamma_2 \Rightarrow \forall x C, \Delta_2} \quad \frac{G|\Gamma_1, A \Rightarrow B, \Delta_1 | \Gamma_2 \Rightarrow [C|_{a_2} \Rightarrow \Delta_2}{G|\Gamma_1, A \Rightarrow B, \Delta_1 | \Gamma_2 \Rightarrow \forall x C, \Delta_2}
\end{array}
\]

\[
\begin{array}{c}
\frac{G|\Gamma_1 \Rightarrow A \Rightarrow B, \Delta_1 | \Gamma_2 \Rightarrow \forall x C, \Delta_2}{G|\Gamma_1 \Rightarrow A \Rightarrow B, \Delta_1 | \Gamma_2 \Rightarrow \forall x C, \Delta_2}
\end{array}
\]

\[
\begin{array}{c}
D_1' \\
D_2
\end{array}
\]

\[
\begin{array}{c}
\frac{G|\Gamma_1 \Rightarrow \Delta_1 | \Gamma_2 \Rightarrow [C|_{a_2} \Rightarrow \Delta_2}{G|\Gamma_1 \Rightarrow \Delta_1 | \Gamma_2 \Rightarrow [C|_{a_2} \Rightarrow \Delta_2} \quad \frac{G|\Gamma_1, A \Rightarrow B, \Delta_1 | \Gamma_2 \Rightarrow \forall x C, \Delta_2}{G|\Gamma_1, A \Rightarrow B, \Delta_1 | \Gamma_2 \Rightarrow \forall x C, \Delta_2}
\end{array}
\]

\[
\begin{array}{c}
\frac{G|\Gamma_1 \Rightarrow A \Rightarrow B, \Delta_1 | \Gamma_2 \Rightarrow \forall x C, \Delta_2}{G|\Gamma_1 \Rightarrow A \Rightarrow B, \Delta_1 | \Gamma_2 \Rightarrow \forall x C, \Delta_2}
\end{array}
\]

Theorem 4 (the mid-hypersequent theorem for \(G^3L\forall\)). Let \(\mathcal{H}\) be a hypersequent in which each member of each sequent is a prenex \(RPL\forall\)-formula or a semipropositional variable. Then any \(G^3L\forall\)-proof \(D\) for \(\mathcal{H}\) can be transformed into a mid-hypersequent \(G^3L\forall\)-proof \(\tilde{D}\) for \(\mathcal{H}\); moreover, \(Q(\tilde{D}) \leq Q(D)\), where \(Q(D)\) is the number of quantifier rule applications in a \(G^3L\forall\)-proof \(D\).
Proof. For a $G^3Lv$-proof $D$ and a propositional rule application $R$ in $D$, let $O(R)$ be the number of quantifier rule applications above $R$, and $O(D)$ be the sum of $O(R)$ over all propositional rule applications $R$ in $D$.

We proceed by induction on $O(D)$, where $D$ is a given proof for $H$.

1. If $O(D) = 0$, then $D$ is the desired proof.

2. Otherwise, choose an application $R_0$ of a (propositional) rule $R_0$ in $D$ such that $O(R_0) > 0$ and no application $R'$ with $O(R') > 0$ is above $R_0$.

2.1. Suppose $R_0$ is $(\to\to)^3$. By $R_1$ denote the (quantifier) rule application that stands immediately above the application $R_0$. We permute $R_0$ and $R_1$ using transformation P1 or P2, and next, by the induction hypothesis, we obtain the desired proof.

2.2. Now suppose $R_0$ is $(\to\to)^3$, and the proof for the conclusion $H_0$ of the application $R_0$ looks like:

$$\frac{D_1 \quad D_2}{\frac{H_1 \quad H_2}{H_0}} R_0.$$ 

Then the lowest application in $D_1$ or $D_2$, say for definiteness the lowest application $R_2$ in $D_2$, is an application of a quantifier rule $R$.

By the induction hypothesis, we can transform $D_1$ into a mid-hypersequent proof $D_1'$ for $H_1$ such that $Q(D_1') \leq Q(D_1)$. In the proof $D$ (for $H$), we replace the subtree $D_1$ by $D_1'$, thus obtaining a proof $D'$ for $H$.

Let the principal formula occurrence $F_2$ in $R_2$ (which is a formula occurrence in $H_2$) be an ancestor of an occurrence $F_0$ in $H_0$. The formulas $A$ and $B$ in $H_2$ that originate from the principal occurrence of $A \to B$ in $R_0$ are quantifier-free. Therefore the occurrence $F_0$ has an ancestor $F_i$ in $H_1$, and all $F_i$ $(i = 0, 1, 2)$ represent the same formula.

Using the construction in the proof of the hp-invertibility of the rule $R$ (see Lemma [4]), from the proof $D_1'$ for $H_1$, we construct a proof $D''_1$ for the premise of an application $R_1$ of $R$ with $H_1$ as the conclusion and $F_1$ as the principal formula occurrence. Here if $R$ is $(\forall \to)^3$ or $(\to \exists)^3$, then the proper term of the application $R_1$ (of $R$) is taken to be the proper term of the application $R_2$ (of $R$). Let $D''_1$ be the proof (for $H_1$) obtained from the proof $D_1'$ by the application $R_1$.

Given the mid-hypersequent proof $D_1'$, it is not hard to see that $D''_1$ is also a mid-hypersequent proof (i.e., $O(D''_1) = 0$) and $Q(D''_1) \leq Q(D_1')$. Then obviously, $O(D''_1) = 0$ and $Q(D''_1) \leq Q(D_1') + 1$.

Next, in the proof $D'$ (for $H$), we replace the subtree $D_1'$ by $D''_1$ and get a proof $D''$ for $H$. Using transformation P3 or P4, in $D''$ we permute the application $R_0$ (of the two-premise rule $R_0$) and the applications $R_1$ and $R_2$ (of the quantifier rule $R$), which stand immediately above $R_0$; and we have a proof $D'''$ for $H$ as a result.

From $O(D''') = 0$, $Q(D''') \leq Q(D_1') + 1 \leq Q(D_1) + 1$, and the forms of transformations P3 and P4, it follows that $O(D''') \leq O(D)$ and $Q(D''') \leq Q(D)$. Then by the induction hypothesis, we can construct the desired proof from $D'''$.

Remark 2. In contrast to Theorem [5] above, Theorems 10 and 18 in [7] (i.e., the mid-hypersequent theorems for $G^3Lv$ and $G^4Lv$) require an initial hypersequent to be of the form $\to A$, where $A$ is a prenex RPLv-formula.

Theorem 5. Suppose $A$ is a prenex RPLv-formula. Then the following are equivalent: (1) $\vdash_{G^3Lv} A$, (2) $\vdash_{G^4Lv} A$, (3) $\vdash_{G^3Lv} A$. 

\[ \Rightarrow q_1 \mid q_1 \Rightarrow q_1 \mid q_3 \Rightarrow \exists x \forall y \exists z B(x, y, z) \mid q_4 \Rightarrow q_4 \]
\[ | q_4 \Rightarrow q_4 \mid q_4 \Rightarrow \exists z B(t_3, a_2, z) \mid q_5 \Rightarrow B(t_3, a_2, t_3) \]
\[ \Rightarrow q_1 \mid q_1 \Rightarrow q_4 \mid q_3 \Rightarrow \exists x \forall y \exists z B(x, y, z) \mid q_4 \Rightarrow q_4 \mid q_3 \Rightarrow \exists z B(t_3, a_2, z) \]
\[ | q_1 \Rightarrow q_1 \mid q_4 \Rightarrow \exists z B(t_1, a_1, z) \mid q_2 \Rightarrow B(t_1, a_1, t_2) \]
\[ \Rightarrow q_1 \mid q_1 \Rightarrow q_3 \mid q_4 \Rightarrow \exists x \forall y \exists z B(x, y, z) \mid q_4 \Rightarrow q_4 \mid q_3 \Rightarrow \exists z B(t_3, a_2, z) \]
\[ | q_1 \Rightarrow q_2 \mid q_4 \Rightarrow \exists z B(t_1, a_1, z) \mid q_2 \Rightarrow B(t_1, a_1, t_2) \]
\[ \Rightarrow q_1 \mid q_1 \Rightarrow q_1 \mid q_3 \Rightarrow \exists x \forall y \exists z B(x, y, z) \mid q_4 \Rightarrow q_4 \mid q_3 \Rightarrow \exists z B(t_3, a_2, z) \]
\[ | q_1 \Rightarrow q_2 \mid q_2 \Rightarrow \exists z B(t_1, a_1, z) \mid q_2 \Rightarrow B(t_1, a_1, t_2) \]
\[ \Rightarrow q_1 \mid q_1 \Rightarrow q_1 \mid q_3 \Rightarrow \exists x \forall y \exists z B(x, y, z) \mid q_4 \Rightarrow q_4 \mid q_3 \Rightarrow \exists z B(t_3, a_2, z) \]
\[ | q_1 \Rightarrow q_1 \mid q_4 \Rightarrow \exists z B(t_1, a_1, z) \mid q_2 \Rightarrow B(t_1, a_1, t_2) \]
\[ \Rightarrow q_1 \mid q_1 \Rightarrow q_1 \mid q_3 \Rightarrow \exists x \forall y \exists z B(x, y, z) \mid q_4 \Rightarrow q_4 \mid q_3 \Rightarrow \exists z B(t_3, a_2, z) \]
\[ | q_1 \Rightarrow q_2 \mid q_2 \Rightarrow \exists z B(t_1, a_1, z) \mid q_2 \Rightarrow B(t_1, a_1, t_2) \]

**Figure 1.** The G^3Lv-proof search tree $\bar{D}_3$

\[ \Rightarrow \exists x \forall y \exists z B(x, y, z) \mid \Rightarrow \exists z B(t_3, a_2, z) \mid \Rightarrow B(t_3, a_2, t_3) \mid \Rightarrow B(t_3, a_2, t_4) \]
\[ | \Rightarrow \exists z B(t_1, a_1, z) \mid \Rightarrow B(t_1, a_1, t_2) \]
\[ \Rightarrow \exists x \forall y \exists z B(x, y, z) \mid \Rightarrow \exists z B(t_3, a_2, z) \mid \Rightarrow B(t_3, a_2, t_4) \]
\[ | \Rightarrow \exists z B(t_1, a_1, z) \mid \Rightarrow B(t_1, a_1, t_2) \]

**Figure 2.** The G^2Lv-proof search tree $\bar{D}_2$

**Proof.** (1) and (2) are equivalent by Theorem 15 in [7]. (3) follows from (1) by Theorem 2. We will show that (3) implies (2).

In view of Theorem 4, it is enough to transform any mid-hypersequent G^3Lv-proof $D_3$ for $A$ into some G^2Lv-proof for $A$. Let $\bar{D}_3$ be the G^3Lv-proof search tree for $A$ consisting of all quantifier rule applications in $D_3$; $\mathcal{H}_3$ be the only top hypersequent in $D_3$; and $\mathcal{H}_3$ be the hypersequent that is obtained from $\mathcal{H}_3$ by removing all sequents containing quantifiers.

To avoid cumbersome notation, first we will perform the transformation in the case when $A$ has the form $\exists x \forall y \exists z B(x, y, z)$ (where $B(x, y, z)$ is a quantifier-free RPLV-formula, and $x, y, z$ are distinct variables), and $\bar{D}_3$ has the form given in Figure 1 then we will explain why a similar transformation can be carried out in the general case. The result of simultaneously replacing all occurrences of $x, y, z$ in $B(x, y, z)$ with terms $s_1, s_2, s_3$, respectively, is denoted by $B(s_1, s_2, s_3)$.

From $\bar{D}_3$ we can construct the G^2Lv-proof search tree $\bar{D}_2$ given in Figure 2 by starting with the hypersequent $\Rightarrow A$ and applying the rules $(\Rightarrow \exists)^2$ and $(\Rightarrow \forall)^2$ backward, according to how the rules $(\Rightarrow \exists)^3$ and $(\Rightarrow \forall)^3$ are applied backward in
Let $\mathcal{H}_2$ be the top hypersequent in $\bar{D}_2$; and $\bar{H}_2$ be the hypersequent consisting of all quantifier-free sequents of $\mathcal{H}_2$.

To complete our proof in the case being considered, it remains to show that $\vdash_{G^2L\forall} \bar{H}_2$. For this, it is sufficient to establish that $\vdash \bar{H}_3$ implies $\vdash \bar{H}_2$. Indeed, $\vdash_{G^3L\forall} \bar{H}_3$ and the soundness of $G^3L\forall$ (see Theorem 1) guarantee that $\vdash \bar{H}_3$. If we prove that the latter implies $\vdash \bar{H}_2$, then first we will obtain $\vdash_{G^2L\forall} \bar{H}_2$ by the completeness of $G^2L\forall$ for quantifier-free hypersequents (see Proposition 14 in [7]), and next we will get $\vdash_{G^2L\forall} \bar{H}_2$ because a rule similar to the rule $(\text{ew})^3$ in Lemma 3 is admissible for $G^2L\forall$.

The hypersequent $\bar{H}_2$ has the form:

$$\Rightarrow B(t_3, a_2, t_5) | \Rightarrow B(t_3, a_2, t_4) | \Rightarrow B(t_1, a_1, t_2);$$

and the hypersequent $\bar{H}_3$ has the form:

$$\Rightarrow q_1 | q_1 \Rightarrow q_3 | q_3 \Rightarrow q_4 | q_4 \Rightarrow q_5 | q_5 \Rightarrow B(t_3, a_2, t_5)
\quad | q_4 \Rightarrow B(t_3, a_2, t_4) | q_1 \Rightarrow q_2 | q_2 \Rightarrow B(t_1, a_1, t_2).$$

For a hypersequent $\mathcal{H}$, we write $\not\models \mathcal{H}$ to denote that $\mathcal{H}$ is not valid.

The condition $\not\models \bar{H}_2$ is equivalent to the existence of an interpretation $M_2$ and a valuation $\nu_2$ such that these three inequalities hold:

$$1 > |B(t_3, a_2, t_5)|_{M_2, \nu_2}, \quad 1 > |B(t_3, a_2, t_4)|_{M_2, \nu_2}, \quad 1 > |B(t_1, a_1, t_2)|_{M_2, \nu_2}.$$

The condition $\not\models \bar{H}_3$ is satisfied iff there exist an interpretation $M_3$ and a valuation $\nu_3$ for which all these inequalities hold:

$$1 > |q_1|_{M_3, \nu_3} > |q_3|_{M_3, \nu_3} > |q_4|_{M_3, \nu_3} > |q_5|_{M_3, \nu_3} > |B(t_3, a_2, t_5)|_{M_3, \nu_3},
\quad |q_4|_{M_3, \nu_3} > |B(t_3, a_2, t_4)|_{M_3, \nu_3},
\quad |q_1|_{M_3, \nu_3} > |q_2|_{M_3, \nu_3} > |B(t_1, a_1, t_2)|_{M_3, \nu_3}.$$

Clearly, $\not\models \bar{H}_2$ implies $\not\models \bar{H}_3$, as required in the given case.

In the general case, it is obvious that from $\bar{D}_3$ we can similarly construct a $G^2L\forall$-proof search tree $\bar{D}_2$. Then the assertion $\not\models \bar{H}_2$ implies $\not\models \bar{H}_3$ follows from the next observation, which is easily justified using induction on the height of the tree $\bar{D}_3$.

We can represent the hypersequent $\bar{H}_3$ as a directed acyclic graph by associating, to each sequent member in $\bar{H}_3$, a unique vertex and, to each sequent of the form $F_1 \Rightarrow F_2$, an edge from $F_1$ to $F_2$. In this graph, there is exactly one source, and all vertices corresponding to RPL\forall-formulas are sinks. The condition $\not\models \bar{H}_3$ is equivalent to the existence of an interpretation $M_3$ and a valuation $\nu_3$ such that, for each edge $F_1 \Rightarrow F_2$, the inequality $|F_1|_{M_3, \nu_3} > |F_2|_{M_3, \nu_3}$ holds and so does the inequality $1 > |F|_{M_3, \nu_3}$ for the source $F$ of the graph.

**Theorem 6.** Let $\mathcal{G}$ be a signature such that the validity problem for existential sentences of classical logic over $\mathcal{G}$ is undecidable. Then the $G^3L\forall$-provability problem for existential $L\forall$-sentences over $\mathcal{G}$ is undecidable.

**Proof.** By Theorem 21 in [7], the corresponding problem for $G^2L\forall$ is undecidable; and the result follows by Theorem 5. □
6. Conclusion

For the logics $L^\forall$ and $RPL^\forall$, we presented the hypersequent calculus $G_3^{\forall L}$, whose rules are repetition-free and hp-invertible.

Theorem 2 established above and Theorem 4 and Proposition 11 both given in [7] ensure that any $GL^\forall$, $G_1^{\forall L}$-, or $G_2^{\forall L}$-provable hypersequent is provable in $G_3^{\forall L}$. By Theorem 5 in the present paper, any prenex $RPL^\forall$-formula is provable or unprovable in $G_1^{\forall L}$, $G_2^{\forall L}$, and $G_3^{\forall L}$ simultaneously. From Theorem 5 stated above and Theorem 17 given in [7], it follows that any prenex $L^\forall$-formula is $GL^\forall$-provable iff it is $G_3^{\forall L}$-provable.

In essentially the same manner as in [7, Section 4], we can formulate a free-variable tableau modification $T_3^{\forall L}$ of the calculus $G_3^{\forall L}$ and describe a family of $T_3^{\forall L}$-proof search algorithms parameterized by a fair tactic. Then Theorem 3 (on constructing $G_3^{\forall L}$-proofs according to fair tactics) will allow us to establish that any algorithm of the family constructs some $T_3^{\forall L}$-proof for any $G_3^{\forall L}$-provable sentence (and so for any $GL^\forall$-provable sentence).

Among problems for further research are the following.

1. Find out whether every $L^\forall$-sentence (resp. $RPL^\forall$-sentence) provable in $G_3^{\forall L}$ is provable in $GL^\forall$ (resp. in $G_2^{\forall L}$).

2. Investigate how complexity of formal proofs varies in passages from one of the calculi mentioned to another.

3. Describe a nontrivial class $\mathcal{C}$ of hypersequent calculi in syntactic terms, with every calculus of $\mathcal{C}$ having the proof-theoretic properties established for $G_3^{\forall L}$. Cf., e.g., [7], which gives sufficient conditions for several properties of some sequent calculi, in particular, for invertibility of inference rules.

4. Develop a method for obtaining sound calculi of the class $\mathcal{C}$, for first-order many-valued logics meeting some semantic conditions. Cf. [10], which solves a somewhat similar problem for a certain class of propositional many-valued logics.

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