THERMOMAGNETIC SHOCK WAVES IN THE VORTEX STATE OF TYPE-II SUPERCONDUCTORS

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Abstract

The nonlinear dynamics of thermal and electromagnetic perturbations in the vortex state of type II superconductors is analyzed with account of dissipation and dispersion effects. A theoretical analysis shows that nonlinear thermal and electromagnetic dissipative waves (structures) may be formed under certain conditions on the sample surface. The structures possess a finite-amplitude and propagate at a constant velocity. The appearance of these structures is qualitatively described and the wave propagation velocity is estimated. The possibility of experimental observation of thermomagnetic shock waves is briefly discussed.

Introduction

Recent years a greater number of comparatively examples of spontaneous origin of spatial and temporary dissipative waves (structures) in unordered systems - self-organization processes - become well known. By the terminology of Prigozhin [1] dissipative structures are organized condition in the space and time, which can move over to the thermodynamic balance condition of only by jumping (as a result of kinetic phase transition). If the deflection of a nonlinear system from the balance exceeds a critical value, these conditions can become unstable. In this case the system moves over to the new mode and becomes a dissipative structure, which appears and exists due to the dissipative processes. There are different types of dissipative structures [2, 3].

At present, superconducting systems with high critical field strengths and current densities are widely implemented in various advanced technologies. However, successful operation of the superconducting materials is only possible provided that special measures are taken to prevent a system from the thermal or magnetic breakage of superconductivity and the transition to a normal state. For this reason, one of the main problems in the investigation of properties of superconductors is predicting of the superconducting state breakage caused by dissipative and nonlinear effects related to viscous motions of the magnetic flux. This explains the considerable interest in the study of dissipative and nonlinear effects in superconductors that has arisen in recent years.

The existence of essential nonlinear and dissipative effects, connected with Joule heating at the moving of magnetic flow, creates different regimes of automodel processes, describing the evolution of thermal and electromagnetic perturbations in the vortex (resistive) state of type-II superconductor. One of measured regimes is the propagation of
thermomagnetic waves (normal region - the region heated with the temperature, which is higher than the critical temperature $T_c$) [3]. In early investigations [3, 4] it was shown that the nonlinear stage of evolution of thermal and electromagnetic perturbations in superconductors is determined by the stationary profile in the form of a running wave.

In the present work the nonlinear dynamics of thermal and electromagnetic perturbations evolution in the vortex state of type II superconductors is investigated with the account of dissipation and dispersion effects. It was found the existence of nonlinear thermomagnetic waves, describing the final stage of evolution of thermal and electromagnetic perturbations in the vortex state of superconductors. An estimate for the width and the velocity of a propagating wave is obtained.

§1. Running wave in the vortex state of superconductors

The evolution of the thermal ($T$) and electromagnetic ($\vec{E}, \vec{H}$) perturbations in the vortex state of superconductors is described by the nonlinear equation of the thermal conductivity [4]

$$\nu \frac{dT}{dt} = \nabla [\kappa \nabla T] + \vec{j} \vec{E},$$ (1)

by the Maxwell equations

$$\text{rot} \vec{E} = -\frac{1}{c} \frac{d\vec{H}}{dt},$$ (2)

$$\text{rot} \vec{H} = \frac{4\pi}{c} \vec{j},$$ (3)

and by the equation of the vortex state

$$\vec{j} = \vec{j}_c(T, \vec{H}) + \vec{j}_r(\vec{E}),$$ (4)

here $\nu = \nu(T)$ and $\kappa = \kappa(T)$ are the heat capacity and the thermal conductivity respectively; $\vec{j}_c = \vec{j}_c(T, \vec{H})$ is the critical current density and $\vec{j}_r = \vec{j}_r(\vec{E})$ is the resistive current density.

The above system is essentially nonlinear because the right-hand part of Eq.(1) contains a term describing the Joule heating in the region of the resistive phase. Such a set (1)-(4) of nonlinear parabolic differential equations in partial derivatives has no exact analytical solution.

Let us consider a planar semi-infinite sample ($x > 0$) placed in external magnetic field $\vec{H} = (0, 0, H_e)$ growing at a constant rate $\frac{d\vec{H}}{dt} = \text{const}$. According to the Maxwell equation (2), there is a vortex electric field $\vec{E} = (0, E_e, 0)$ in the sample, directed parallel to the current density $\vec{j}$: $\vec{E} \parallel \vec{j}$; here $H_e$ is the amplitude of the external magnetic field and $E_e$ is the amplitude of the external electric field.

In this section we will discuss the question on possibility of existence of automodel solution to the initial system of equations (1) - (4) in private derivatives with independent variables $x$ and $t$ in the form of running impulse, describing the nonlinear evolution of thermal and electromagnetic perturbations in superconductors.
The system of equations, describing the dynamics of evolution of thermomagnetic perturbations in the vortex state of superconductors is essentially nonlinear and it is impossible to integrate them in this form. As well known that, it is difficult to solve the problem with initial and boundary conditions for nonlinear differential equations in private derivatives by common methods, so that we can not use the superposition principle. Therefore it is reasonable to search other ways of the solution. There is such method, which allows the transformation of resemblance, due to this the system of equations with private derivatives can be reduced to the system of common differential equations. Recent years more attention is given to investigation of such automodel solution [5]. The obtained by the method differential equations with common derivatives can be integrated in the close form or reduced to investigation of one equation, which can be solved either analytically or numerically.

In many problems of mathematical physics the automodel solution of running wave type is of great interest [5]. For important class of stationary running waves the distribution of characteristics of waves is still unchangeable on time. In accordance with this determination the solution in the form of running wave can be presented in the form

$$u = u(x \pm vt),$$

where $u$ - the characteristics of the event under consideration, $x$ - coordinate, $t$ - time, $v$ - the velocity of the running wave. The solution (5) presents a wave, running either in negative or positive direction of $x$ with constant velocity $v$.

The possibility of reducing of the system of differential equations in private derivatives to the system of common differential equations in automodel derivatives greatly simplifies the problem in the mathematical point of view and in many cases allows to find exact analytical solution. Let us consider the problem about in what conditions the system of initial differential equations in private derivatives with independent variables $x$ and $t$ can be reduced to the system of differential equations in common derivatives, depending on only one variable $\xi(x,t)$. The solution of the system of equations (1)-(4) may be presented as a function of new automodel variable $\xi(x,t)$:

$$T = T[\xi(x,t)],$$

$$E = E[\xi(x,t)],$$

$$j = j[\xi(x,t)].$$

Substituting (6) into the system (1)-(4) gives, as a result of simple differentiation, the following system

$$\frac{d\xi}{dt} \left[ \nu \frac{dT}{d\xi} \right] = \kappa \left\{ \frac{d^2\xi}{dx^2} \frac{dT}{d\xi} + \left( \frac{d\xi}{dx} \right)^2 \frac{d^2T}{d\xi^2} \right\} + [j_c(T) + j_r(E)]E, \quad (7)$$

$$\frac{d^2\xi}{dx^2} \frac{dE}{d\xi} + \left( \frac{d\xi}{dx} \right)^2 \frac{d^2E}{d\xi^2} = \frac{4\pi}{c^2} \left[ \frac{dj_c}{dT} \frac{dT}{d\xi} + \frac{dj_r}{dE} \frac{dE}{d\xi} \right] \frac{d\xi}{dt}. \quad (8)$$
In order the system (7),(8) was only function from $\xi$ at the substation (6) it is required carrying out the following conditions:

\[
\frac{d\xi}{dt} = A(\xi), \quad (9)
\]

\[
\left(\frac{d\xi}{dx}\right)^2 = B(\xi) \frac{d\xi}{dt} = G(\xi), \quad (10)
\]

\[
\frac{d^2\xi}{dx^2} = C(\xi) \frac{d\xi}{dt}, \quad (11)
\]

where $A, C, G$ are functions to be determined below. Solving first two system of equations (9)-(11) we have a relation

\[
G(\xi) \frac{dA}{d\xi} = A(\xi) \frac{dG}{d\xi}. \quad (12)
\]

Hence just follows relationship between $G$ and $A$

\[
G(\xi) = \frac{1}{u} A(\xi), \quad (13)
\]

here $u$ is a free constant of integrating of the equation (12). From (9) and (10) follows that $\xi(x, t)$ must satisfy single-line equation in private derivation

\[
\frac{d\xi}{dt} = u \frac{d\xi}{dx}, \quad (14)
\]

the only solution of which is the function

\[
\xi(x, t) = F(x - ut). \quad (15)
\]

Using (15) we can immediately obtain

\[
G(\xi) = 1, \quad A(\xi) = -Fu, \quad C(\xi) = 0. \quad (16)
\]

It is possible to ensure $F = 1$ by the transformation of coordinates and times. Thereby, we find final automodel substitution

\[
\xi = x - ut, \quad (17)
\]

corresponding to solution of a running type wave [5].

\section*{§2. Differential equations for nonlinear $E$ and $H$ waves.}

For the automodeling solution of the form (17), describing a running wave moving at a constant velocity $v$ along the $x$ axis, the system of equations (1)-(4) takes the following form

\[
-\nu(T)v \frac{dT}{d\xi} = \frac{d}{d\xi} \left[ \kappa(T) \frac{dT}{d\xi} \right] + \left[j_c(T) + j_r(E)\right]E, \quad (18)
\]
$$\frac{d^2 E}{d\xi^2} = -\frac{4\pi v}{c^2} \frac{dj}{d\xi},$$  
(19)

$$\frac{dE}{d\xi} = \frac{v}{c} \frac{dH}{d\xi}.$$  
(20)

The thermal and electrodynamics boundary conditions for equations (18)-(20) are as follows:

$$T(\xi \to +\infty) = T_0, \quad \frac{dT}{d\xi}(\xi \to -\infty) = 0,$$

(21)

$$E(\xi \to +\infty) = 0, \quad E(\xi \to -\infty) = E_c,$$

here $T_0$ is the temperature of the cooling medium.

In this section in view of considerable analytical difficulties involved in solving the exact problem, we will restrict our consideration to the so-called Bean-London critical state model [6] by assuming that $j = j_c(T, H_e) = j_c(T)$ and

$$j_c(T) = j_0 - a(T - T_0),$$  
(22)

where $j_0$ is the equilibrium current density, $a$ is the thermal heat softening coefficient of the magnetic flux pinning force [7].

The characteristic field dependence of $j_c(E)$ in the region of sufficiently strong electric field ($E \geq E_f$) can be approximated by a piece-wise linear function $j_c \approx \sigma_f E$, where

$$\sigma_f = \frac{\eta c^2}{H \Phi_0} \approx \sigma_n \frac{H_{c2}}{H}$$

is the effective conductivity in the flux-flow regime; $\eta$ is the viscous coefficient, $\Phi_0 = \frac{\pi h c}{2 e}$ is the magnetic flux quantum, $\sigma_n$ is the conductivity in the normal state; $H_{c2}$ is the upper critical magnetic field, $E_f$ is the boundary of the linear area in the voltage-current characteristics of the sample [8]. In the region of small field strengths ($E \leq E_f$), we assume a relationship $j_r(E) \approx j_1 \ln \frac{E}{E_0}$ to be valid with $j_1$ being a characteristic local current density scatter (related to a pinning force inhomogeneity [9]) on the order of $j_1 \approx 0,01 j_c$; $E_0$=const. This relationship between $j$ and $E$ is due to a thermoactivated flux-creep. The thermoactivated flux-creep is a principal mechanism of the energy dissipation during the magnetic flux penetrates deeply into the sample.

Excluding variables $T(\xi)$ and $H(\xi)$ from Eqs. (18) and (20), and taking into account the boundary conditions (21), we obtain an equation describing the electric field $E(\xi)$ distribution ($E$-wave):

$$\frac{d^2 E}{d\xi^2} + \left[4\pi v \frac{dj_r}{dE} \frac{dE}{d\xi} + \frac{4\pi v^2}{c^2} a \frac{N(T) - N(T_0)}{\kappa(T)} \right] - \frac{aE^2}{2\kappa(T)} = 0,$$

(23)

where the dependency $T = T \left( E, \frac{dE}{d\xi} \right)$ is defined by expression (18), (19) and have the form

$$T = T \left( E, \frac{dE}{d\xi} \right) = T_0 + \frac{1}{a} \left[ j_0 + j_r(E) + \frac{4\pi v}{c^2} \frac{dE}{d\xi} \right].$$  
(24)
Here \( N(T) = \int_0^T \nu(T) dT \). The velocity of \( E \)-wave is determined by the Eq. (23) with account of the boundary conditions

\[
v_E^2 = \frac{c^2}{8\pi} N \left[ T_0 + \frac{1}{a} \left[ j_e(T) + j_r(E) \right] \right] - N(T_0).
\]

Using Eqs. (20) and (23) one can easily find the expression for distribution of the magnetic field \( H \) in the case of \( H \)-wave

\[
\frac{d^2H}{d\xi^2} + \frac{4\pi v}{c^2} \left[ \frac{dj_r}{dE} \bigg|_{E=vH/c} + \frac{c^2}{a} \frac{N(T) - N(T_0)}{\kappa(T)} \right] - \frac{av}{2c \kappa(T)} H^2 = 0.
\]

The velocity \( v_H \) of \( H \)-wave is connected with its amplitude \( H_e \) by the following expression

\[
N(T) - N(T_0) = \frac{H_e^2}{8\pi}.
\]

The expression (27) shows the adiabatic nature of the \( H \)-wave propagation: the magnetic energy, accumulated during decay of the field, is spent for heating the superconductor in the region close to the wave front. Therefore, depending on the character of the external electrodynamics conditions on the surface of the sample there may exist two type (\( E \) and \( H \)) of nonlinear stationary waves in the superconductor.

§3. Thermomagnetic shock waves

The first integrals of equations (18)-(20) have the form

\[
-\nu(T - T_0) = \kappa \frac{dT}{d\xi} - \frac{c^2}{8\pi} E^2,
\]

\[
J = -\frac{c^2}{4\pi v} \frac{dE}{d\xi},
\]

\[
E = \frac{v}{c} H,
\]

(an integrating constants in Eqs. (28)-(30) are equal to zero on the strength of the boundary conditions at \( \xi = -\infty \)). Using integrals (28), (29) and excluding variable \( T(\xi) \) we can get the following differential equation for the distribution of \( E(\xi) \):

\[
\frac{d^2E}{dz^2} + \beta(1 + \tau) \frac{dE}{dz} + \beta^2 \tau E - \frac{E^2}{2E_e} = 0.
\]

Here the following dimensionless parameters are introduced: \( z = \frac{\xi}{L}, \quad \beta = \frac{vt_c}{L}, \quad L = \frac{cH_e}{4\pi j_0} \) is the depth of the magnetic field penetration into the superconductor, \( \tau = \frac{D_t}{D_m} \) is
the parameter describing the ratio of the thermal $D_t = \frac{\kappa}{\nu}$ to the magnetic $D_m = \frac{c^2}{4\pi\sigma_d}$ diffusion coefficient, $t_\kappa = \frac{\nu L^2}{\kappa}$ is the time of thermal diffusion, and $E_\kappa = \frac{\kappa}{aL^2}$ is a constant parameter.

One can see that the mathematical model described by Eq. (31) can be considered as a decreasing nonlinear oscillator at the presence of a friction force

$$F_t = -\beta(1 + \tau) \frac{dE}{dz},$$

(32)

(where $z$ is the analogue of time, $E$ is the coordinate of a "material point") with a potential $U(E)$

$$U(E) = -\frac{E^3}{6E_\kappa} + \beta^2 \frac{E^2}{2}.$$  

(33)

The analysis of the phase plane of Eq. (31) shows that there are two equilibrium points in it: $E_0$ is a stable node and $E_1$ is a saddle node. These two equilibrium points $(E_0, E_1)$ are separated in the phase plane by the separatrix $AB$ (Fig.1). At $z \to +\infty$ the material point is in the point $E_1$, and at $z \to -\infty$ it moves to the point $E_0$. The transition from one equilibrium state to another one occurs only monotonically. This solution by joining these two equilibrium points may be presented in the form of the shock-wave-type with amplitude $E_e$ running at a constant velocity $v_E$ (Fig.2).

Let us consider the solution of the Eq (31), corresponding to the limiting case $\tau \ll 1$ (hard superconductors). It follows from the $\tau \ll 1$ condition that the thermomagnetic perturbation develops nearly adiabatically [8]. Therefore in this approximation the thermal conductivity is negligible and corresponding terms, containing dissipative effects can be neglected in Eqs. (28)-(30). Then for the linear region of the current-voltage characteristic the solution to Eq. (31) can be presented in the form

$$E(z) = \frac{E_1}{2} \left[ 1 - \tanh \frac{\beta \tau}{2(1 + \tau)} (z - z_0) \right],$$

(34)

Here $z_0$ is a constant. The condition $\tau \ll 1$ means that the magnetic flux diffusion $D_t$ is considerable faster than that of the thermal diffusion $D_m$. Then we can assume that the spatial scale of the variation of the magnetic field penetration $L_E$ is essentially larger than the corresponding thermal scale $L_T$, therefore the spatial derivatives $\frac{d^n E}{dz^n}$ contain the small parameter $\frac{L_T}{L_E} \ll 1$. It is easy to check fairness of this approach by differentiating Eq. (31):

$$\frac{d^2 E}{dz^2} \left( \beta \frac{dE}{dz} \right)^{-1} \tau \ll 1.$$  

(35)

Maximum inaccuracy of given approximations is of the order of $\frac{\tau}{(1 + \tau)^2}$. For example, for $\tau = 1$ it is 25 %, and in the limiting case $\tau \to 0$ or $\tau \to \infty$ is very small.

The expression (34) describes the structure of the thermomagnetic shock wave, penetrating into the superconductor (see, Fig.2). Using the boundary conditions for $E$-wave
one can easily find the wave velocity

\[ v_E = \frac{L}{t_\kappa} \left[ \frac{E_0}{2\tau E_\kappa} \right]^{1/2}, \]  

with amplitude \( E_e \) and with front width

\[ \delta z = 16 \left( 1 + \tau \right)^{1/2} \left[ \frac{E_\kappa E_e}{E_e} \right]^{1/2}. \]  

(37)

Numerical estimations give the value \( v_E = 10^1 \div 10^3 \cdot \frac{\text{sm}}{\text{sec}} \) and \( \delta z = 10^{-1} \div 10^{-2} \) for \( \tau = 1 \).

Thus, the nonlinear stage of evolution of thermal and electromagnetic perturbations in the vortex state of superconductors is determined by the stationary profile in the form of running wave. It is seen that, the transition from superconducting state to normal can occur by the expanding of stationary thermomagnetic wave, which is caused by the balance between nonlinear and dissipative effects, caused by Joule heating at the viscous motion of Abrikosov vortices inside superconductor.

§ 4. Nonlinear waves in the flux-creep regime of hard superconductors

As mentioned in Sec.2 in the viscous flux-flow regime the electric conductivity \( \sigma_f \) is independent of the electric field \( E \), the profile of thermomagnetic waves is independent of the background electric field in the sample. In the region of small electric field strengths the current-voltage characteristic of the sample has a nonlinear part and it is determined by the following logarithmic dependence

\[ j_r(E) \approx j_1 \ln \frac{E}{E_0}. \]

The current-voltage characteristics of hard superconductor is described by the last equation over a wide interval of temperatures \( T \) and magnetic fields \( H \). This factor may considerably affect the character of nonlinear thermomagnetic wave propagation in the sample. In this section we consider a problem of thermomagnetic wave profile, with account the nonlinear current-voltage curve of superconductor. Taking into account the nonlinear dependence \( j_r(E) \), we can represented Eq. (24) in the following form

\[ T = T_0 + \frac{1}{a} \left[ j_0 + j_1 \ln \frac{E}{E_0} + \frac{4\pi v c}{c^2} \right] \frac{dE}{dz}. \]

Substituting the last expression into Eq. (18), and taking into account the boundary conditions (21), we obtain an equation describing the distribution of \( E \) - wave:

\[ \frac{d^2 E}{dz^2} + \beta \left[ \frac{1}{\sigma_d(E)E'} \right] \frac{dE}{dz} + \beta^2 \tau \left[ \frac{j_0}{\sigma_d(E)} + \frac{j_1}{\sigma_d(E)} \ln \frac{E}{E_0} \right] = \frac{E^2}{2 E_\kappa}. \]  

(38)

Here \( \sigma_d = \sigma_d(E) \) is the differential conductivity in the flux-creep regime [8]. The corresponding equation of state is obtained using the relationship [10]

\[ \Omega E^2 = X(E) = 1 + \frac{j_1}{j_0} \ln \frac{E}{E_0}, \]  

(39)
where $\Omega = 2j_0\sigma_{d}^{-1}(E_0)E_\kappa \beta^2 \tau$ is a constant. As seen from the plots of $X(E)$ versus $E$ (Fig.3), there exists a single intersection point of the curves $y = \Omega E^2$ and $y = 1 + \frac{j_1}{j_0} \ln \frac{E}{E_0}$, which corresponds to a single stable equilibrium state. The stability of this state is determined by the sign of the derivative $\frac{d^2 E}{dz^2}$ in the vicinity of the equilibrium point. The wave velocity $v_E$ can be determined from Eq. (39) using for the boundary conditions and have the form of

$$v_E = \frac{L}{t_\kappa} E_e \left[ 2\sigma_d(E) \left( 1 + \frac{j_1}{j_0} \ln \frac{E_e}{E_0} \right) \right]^{-1/2}.$$ (40)

Now we can use Eq. (30) and readily derive an equation describing the field distribution for a nonlinear $H$-wave. The wave velocity $v_H$ of the $H$-wave is given by the formula

$$v_H = \frac{cE}{H_e} \exp \left[ \frac{\sigma_d(E)}{2\tau j_1} E_\kappa \left( \frac{LH_e}{ct_\kappa} \right)^2 - \frac{j_0}{j_1} \right].$$ (41)

It is seen that the wave velocity increases exponentially with the amplitude $H_e$. For a sufficiently small amplitude $H_e < H_a$, where

$$H_a = \frac{c t_\kappa}{L} \left[ \frac{2\tau j_0 E_\kappa}{\sigma_d(E)} \right]^{1/2},$$ (42)

the wave velocity is negligible small, which corresponds to the case of the thermoactivated flux creep. For $H_e = H_a$ the wave propagates at a finite constant velocity $v_H$. The expression

$$E = E_0 \exp \left[ \frac{\sigma_d(E)}{2\tau j_1} E_\kappa \left( \frac{LH_e}{ct_\kappa} \right)^2 - \frac{j_0}{j_1} \right].$$ (43)

represents the intensity of the spontaneous electrical field caused by the thermally activated magnetic flux-creep inside the region warmed up during wave propagation. In this case the maximum heating of the superconductor in the region immediately close to the wave front is described by the relationship

$$\frac{T - T_0}{T_0} = \Theta = \frac{H^2_e}{8\pi \nu T_0}.$$ 

Let us now estimate the maximum heating $\Theta$ from the following typical values of the physical parameters of the sample: $H_e = 10^3 \cdot Gs$, $\nu = 10^5 \cdot erg/(cm^3 \cdot K)$, $T_0 = 4.5 \cdot K$, we have $\Theta \approx 0.2 \sim 0.5$. At this value of the heating the wave velocity is of the order of $v_H \approx 10 \sim 100 \cdot sm/sek$. Finally it should be note that taking into account the nonlinear dependence of the current density $j$ on the electric field strength $E$ should not qualitatively change the main results, since the character of the equilibrium state on the phase plane remains essentially the same.

§5. The structure of waves

The nature of the structure of the nonlinear $E$ (or $H$) - wave can be determined by investigating the asymptotical behaviour at $z \to \pm \infty$ of solutions for $E(z)$ in the
close vicinity of the equilibrium points \(E = E_e\). Linearizing the equation (38) near the equilibrium point (weak-nonlinear wave) it can be represented in the following form

\[
\frac{d^2 E}{dz^2} + \beta(1 + \tau) \frac{dE}{dz} + \left[1 - \frac{\sigma_d(E_e)E_e}{2j_0(T_0, E_e)}\right] \frac{(E - E_e)}{2E_\kappa} = 0. \tag{44}
\]

The analysis of the phase plane \((E, \frac{dE}{dz})\) of Eq.(44) shows that there is a single equilibrium point, which is a saddle under the condition

\[
j_0(T_0, E_e) > \frac{1}{2}\sigma_d(E_e)E_e.
\]

The spatial scale of the variation of the solutions of Eq.(44), obviously, determines the width of the wave front \(\Delta z_0\):

\[
\delta_{z_0} = \left[1 - \frac{\sigma_d(E_e)E_e}{2j_0(T_0, E_e)}\right]^{-1} [(1 + \tau)\beta]^{-1}. \tag{45}
\]

An asymptotical solution of the Eq. (44) can be found in the field of \(z \to +\infty (E \to 0)\) using the voltage-current characteristics

\[
j \to E^{1/n}; \quad (n >> 1), \tag{46}
\]

in the flux-creep regime \(E << E_0\) of the following form

\[
E(z) = E_1 \exp(-n\beta z). \tag{47}
\]

The total width of the wave front is determined obviously, by the value \(\Delta z_0\) since in the region of \(z > 0\) (on the strength of the condition \(n >> 1\)) the amplitude of the wave sharply tends to zero (on the scale \(\Delta z^* \approx \frac{1}{n\beta} << \Delta z_0\)). It is seen that the width of the \(E\)-wave front is determined mainly by the largest of the parameter heat capacity \(\nu\) and thermal conductivity \(\kappa\) coefficients \((\beta \sim \frac{\kappa}{\nu})\).

§6. Waves with the finite amplitude

It is noticeable that the evolution of perturbations of temperature \(T(x, t)\) and fields \(E(x, t)\) and \(H(x, t)\) essentially is determined by the equation of resistive state (4) and physical parameters of the sample. It should be noted that inclusion of the temperature dependences of the parameters \(\kappa\) and \(\nu\) substantially complicates analytical calculations of the wave propagation dynamics that is described by the system of Eqs. (28)- (30). In most cases, the changes in the local values of these parameters in the sample can be considered negligible comparatively to the characteristic scale of temperature variations. Hence, we can take these parameters to be constant. Indeed, the investigation revealed that the thermal conductivity almost does not affect the character of the stationary wave propagation. This stems from the fact that the thermal flux \(\kappa \frac{dT}{dz}\) vanishes at stationary
points of the system at \( z \to \pm \infty \). However, the temperature dependence of the heat capacity should be taken into account. Such a dependence is represented as \( \nu \approx \nu_0 \left[ \frac{T}{T_0} \right]^3 \) over a wide range of temperatures \([8]\). Then Eq. (28) can be written in the following form

\[-\nu_0 v \left[ \frac{T - T_0}{4T_0} \right]^4 = \kappa \frac{dT}{dz} - \frac{c^2}{8\pi v} E^2.\]

By eliminating variable \( T(z) \) from the last relationship and employing the boundary conditions (21), we obtain

\[
\frac{d^2 E}{dz^2} - 2\pi \nu T_0 v^2 E \kappa^2 \left[ \left( 1 + \frac{\sigma f E}{a T_0} + \frac{c^2}{4\pi a v T_0} \frac{dE}{dz} \right)^4 - 1 \right] + \beta \frac{dE}{dz} = \frac{E^2}{2E_\kappa}. \tag{48}\]

According to the qualitative theory \([11]\), the equilibrium states are found from the condition

\[2\pi \nu_0 T_0 v^2 E \kappa^2 \left[ \left( 1 + \frac{\sigma f E}{a T_0} \right)^4 - 1 \right] = E^2. \tag{49}\]

An evident property of system (49) is the absence of closed curves that are fully composed of the phase trajectories in the phase plane \( (E, \frac{dE}{dz} = P) \). The proof of this statement can be based on the Bendixson criterion \([10]\). The number of stationary points (one or three) and their type are determined by the parameter

\[W = 2\pi \nu_0 T_0 v^2 E \kappa^2. \tag{50}\]

The three equilibrium points \( E = 0, E = E_1 \) and \( E = E_2 \) correspond to the condition \( W < W_k = \frac{1}{2} \) (Fig.4). There is only one singular point \( E_0 = 0 \) at \( W > W_k \). The parabola and the quadratic curve in Eq. (49) are tangent at \( W = W_k \); i.e., this condition corresponds to the coincidence \( E_1 = E_2 = E^* = \frac{6 a T_0}{\nu_0 a^2 c^2} \). The direct solution of Eq. (49) yields the following waves:

1) \( E_{1,2} = E^*[1 + 2, 2(W_k - W)^{1/2}] \), at \( \left( \frac{W_k - W}{W_k} \right) << 1 \);

2) \( E_1 = 8\pi \frac{\sigma f \nu_0 v^2}{a c^2} \), \( E_2 = (2\pi)^{1/2} \frac{\sigma f \nu_0 c^2}{a^2 v} \gg E_1 \), at \( W_k >> W \). \tag{51}\]

Analysis of the phase plane \((E, P)\) shows that the points \( E_0 = 0 \) and \( E = E_2 \) are stable nodes and that \( E = E_1 \) is a saddle. In addition to the separatrix \( E_1 E_0 \), the set (49) has the separatrix \( E_1 E_2 \) connecting the points \( E_1 \) and \( E_2 E_1 \). This means that two types of waves with amplitudes \( \Delta E = E_1 \) and \( \Delta E = E_2 - E_1 \) can exist in the superconductor (Fig.5). Evidently, wave I has an amplitude of the order \( E_k \) at \( W \to W_k \); its velocity
is determined by equality (49) at $E = E_1$. Equation (49) has two stationary points at $W << W_k$: $E_0 = 0$ is a stable node and $E_1 = 2\beta^2 \tau E_\kappa$ is a saddle. The separatrix that connects these two equilibrium states corresponds to a "difference"-type solution with amplitude $E_e$, which is related to the wave velocity $v_E$ and the wave front width $\Delta z$ by the equations (36) and (37), respectively. Wave II has a small amplitude at $W \rightarrow W_k$

$$\frac{\Delta E}{E_k} = 4,4(W_k - W)^{1/2} << 1,$$

and its velocity is inversely proportional to the amplitude at $W << W_k$. Such an exotic dependence of $v_E$ on $\Delta E = E_e$ most likely means that the waves of this type are unstable. Finally it is noticeable that observation of the second-type waves becomes possible in finite-sized samples with asymmetric boundary conditions. The above investigations prove the possibility of applying the results obtained to high-temperature superconductors cooled to liquid-nitrogen temperatures ($T = 77$ K), providing that the values of the physical parameters of the sample are known.

§7. Stability of nonlinear shock waves

In Section 3, it was demonstrated that, depending on the surface conditions, stationary nonlinear thermomagnetic shock waves of two types, $E$ and $H$, may exist in the superconductor sample. In this Section the stability of nonlinear shock waves with respect to small thermal and electromagnetic perturbations in a hard and composite superconductors is studied. It is shown that spatially bounded solutions may correspond only to the perturbations decaying with time, which implies stability of the nonlinear thermomagnetic wave.

In order to study the stability of a nonlinear wave with respect to small perturbations, it is convenient to write a solution to Eqs. (1)-(4) in the following form:

$$T(z, t) = T(z) + \delta T(z, t) \exp \left[ \frac{\lambda t}{t_\kappa} \right],$$

$$E(z, t) = E(z) + \delta E(z, t) \exp \left[ \frac{\lambda t}{t_\kappa} \right].$$

(53)

Here $T(z)$ and $E(z)$ are the stationary solutions and $\delta T, \delta E$ are small perturbations; $\lambda$ is a parameter to be determined. From solution (53), one can see that the characteristic time of thermal and electromagnetic small perturbations $t_j$ is of the order of $t_\kappa/\lambda$. Substituting (53) into Eqs. (1)-(4) and linearizing for small perturbations $\left(\delta T(T(x), \delta E(x) << 1\right)$ we obtain a differential equations for the distributions $\delta T$ and $\delta E$ in the following form

$$\nu \frac{\lambda}{t_\kappa} \delta T = \frac{\kappa}{L^2} \frac{d^2 \delta T}{dz^2} + [j(z) + \sigma_f E(z)] \delta E - aE(z) \delta T,$$

(54)

$$\frac{1}{L^2} \frac{d^2 \delta E}{dz^2} = \frac{4\pi \lambda}{c^2 t_\kappa} [\sigma_f \delta E - a \delta T].$$

(55)
Eliminating the variable $\delta T$ by using the relationship (55) and substituting into Eq.(54), we obtain a fourth-order differential equation with variable coefficients for the distribution of small electromagnetic perturbation $\delta E$:

$$\frac{d^4 \delta E}{dz^4} + \beta \frac{d^3 \delta E}{dz^3} - \left[ \lambda (1 + \tau) + \frac{E(z)}{E_\kappa} \right] \frac{d^2 \delta E}{dz^2} - \lambda \tau \beta \frac{d \delta E}{dz} + \lambda [\lambda \tau - B(z)] \delta E = 0,$$

(56)

Here $B(z) = \frac{j(z)}{\sigma f E_\kappa}$ characterizes the spontaneous heating of superconductor caused by a small perturbations of $\delta T$ and $\delta E$. The condition of existence of a non-trivial solutions of Eq. (56) combined with boundary conditions (21) allows to define the spectrum of eigenvalues of $\lambda$ and the stability of the thermomagnetic wave in the sample, accordingly. From the expression (56) just follows that if $\text{Re} \lambda \leq 0$, then spatially bounded solutions of $\delta T$ and $\delta E$ at infinity corresponds only a damping perturbations on the time. This problem is complicated, and its analytical solution cannot be found in a closed form. Let us consider the problem of the thermomagnetic wave stability in the limiting case of the hard ($\tau << 1$) and the composite ($\tau >> 1$) superconductors, which is presented a greatest practical interest.

§7a. Hard superconductors ($\tau << 1$)

As well known (see, e.g.,[8]), the magnetic flux variations in a hard superconductors occur at a much greater rate as compared to those of the heat transfer, so that $\tau << 1$. The adiabatic character of the perturbations development leads to the predominance of magnetic flux diffusion over heat diffusion in the sample, $D_t << D_m$. In this case, as seen from solution (53), the characteristic times of temperature $t_\kappa$ and electromagnetic field $t_m$ perturbations have to satisfy the inequalities $t_j << t_\kappa$ ($\lambda >> 1$) and $t_j >> t_m$ ($\lambda \tau << 1$). Assuming that $\tau << 1$ in the limit of $\lambda >> \frac{\nu_2}{\kappa}$, which corresponds to a "fast" instability [4], we obtain the equation for determining the eigenvalues of $\lambda$

$$\frac{d^2 \delta E}{dz^2} + \beta \tau \frac{d \delta E}{dz} + [B(z) - \lambda \tau] \delta E = 0,$$

(57)

In hard superconductors with $\tau << 1$, the instability threshold depends on the thermal boundary conditions at the surface of the sample only slightly. Therefore, the electrodynamic boundary conditions at the boundaries of the current-carrying layer ($z = +\infty, z = -\infty$) can be neglected and one can keep only the thermal boundary conditions to Eq. (57). Using the substitution of variables of the type

$$j(z) = -\frac{c^2}{4\pi v} \frac{dE}{dz} = -\frac{c^2}{4\pi v} \frac{\beta^2 \tau E_\kappa}{\cosh^2 [\beta \tau (z - z_0)]},$$

(58)

Eq. (57) can be rewritten in the following form [12]

$$\frac{d^2 \epsilon}{dz^2} + \left[ \frac{2}{\cosh^2} - \Lambda \right] \epsilon = 0.$$

(59)
Here
\[ \Psi(z) = \delta E \exp \left[ -\frac{\beta \tau}{2} (z - z_0) \right], \]
\[ y = \frac{\beta \tau}{2} (z - z_0), \quad \Lambda = \lambda \tau. \]

The Eq. (59) with the help substitution
\[ \psi(x) = (1 - x^2)^{\frac{\epsilon}{2}} \Phi(x); \quad 1 - x = 2s, \]
\[ x = \theta y, \quad 1 - x^2 = \cosh^2 y, \]
\[ \epsilon = \sqrt{\Lambda}, \quad P = 1, \]
can be reduced to a standard hypergeometric equation [13]
\[ s(1 - s) \frac{d^2 \Phi}{ds^2} - [\epsilon + 1 - 2s(\epsilon + 1)] \frac{d\Phi}{ds} - [\epsilon(\epsilon - 1) - p(p - 1)] \Phi = 0. \quad (60) \]

Here, the even and odd integrals can be presented as
\[ \Phi_1 = F(\epsilon - 1, \epsilon + 2, \epsilon + 1, s), \quad (61) \]
\[ \Phi_2 = s^{-\epsilon} F(-1, 2, 1 - \epsilon, s), \quad (62) \]
where F is the hypergeometric function [13]. Returning to the variable y we have
\[ \Psi_1(y) = (\chi y)^{-\epsilon} F \left( \epsilon - 1, \epsilon + 2, \epsilon + 1, \frac{1 - \theta y}{2} \right) \quad (63) \]
\[ \Psi_2(y) = (\theta y - \epsilon) \exp(\epsilon y) F \left( -1, 2, 1 - \epsilon, \frac{1 - \theta y}{2} \right) \quad (64) \]

The function must be finite at the singular point s = 1. The \( \epsilon \) values for which \( \Phi_1 \) is finite evidently correspond to a discrete spectrum: \( \epsilon_i = 0, 1 \) or \( \lambda_i = -\tau^{-1}, 0 \). The function \( \Phi_2 \) is finite only for \( \epsilon = 0 \). Any solution in the form of a running wave is characterized by translation symmetry, which implies that a ”perturbed” stationary profile \( E(z) \) determined by Eq. (57) corresponds to the eigenvalue of the ground state \( \lambda_0 = 0 \).

Differentiating Eq. (31) with respect to \( z \), one may readily see that \( \frac{dE_0}{dz} \) is an eigenfunction corresponding to \( \lambda_0 = 0 \). Indeed, the perturbation \( \delta E = \frac{dE_0}{dz} \) essentially represents a small wave displacement. Thus, we may suggest that the function \( \frac{dE_0}{dz} \) exponentially tends to zero for \( z \to +\infty \) and the corresponding eigenvalue is zero. Therefore, the problem cannot possess positive eigenvalues and \( \text{Re} \lambda_i < 0 \). This result implies that the wave is stable with respect to relatively small thermal \( \delta T \) and electromagnetic \( \delta E \) fluctuations. The analysis of the second linearly-independent solution (64) leads to the same conclusion.

§7b. Composite superconductors (\( \tau >> 1 \))
In composite superconductors, an efficient value of effective electric conductivity $\sigma_f$ is much above than in hard superconductors. As a result any intensity of the thermal and the electromagnetic small perturbations is valid an inequality $\tau >> 1$. It may be assumed that induced the normal current $\sigma_f E$ caused by raising of temperature compensates a droping of the critical current $j_c(T)$ and obviously, impediments of an entry of a magnetic flow into sample. On the other hand in composites the thermomagnetic instability development is accompanied by "slow" perturbations with characteristic time of a growth $t_\kappa << t_j << t_m$ ($\lambda << 1$) and consequently, $t_m >> t_\kappa \lambda (\lambda \tau >> 1)$ [8]. In the approximation $\lambda << \frac{\nu v^2}{\kappa}$ from Eq. (56) we obtain the following expressions for determining the eigenvalues $\lambda$

$$\frac{d^2 \delta E}{dz^2} + \left[ \beta - \frac{E(z)}{\beta \tau E_\kappa} \right] \frac{d \delta E}{dz} - \left[ \lambda \left( 1 + \frac{E(z)}{\beta^2 \tau E_\kappa} \right) - \frac{1}{\tau} B(z) \right] \delta E = 0, \quad (65)$$

The last equation with the help substitution $$\Psi(z) = \delta E(z) ch(y - z_0),$$
can be reduced to a canonical form [12]

$$[H - \Lambda] \Psi = 0, \quad (66)$$

$$H = \frac{d^2}{dy^2} + U(y),$$

$$U(y) = -1 + \frac{2}{\alpha^2} + \Lambda_1 \alpha^2; \quad y = \frac{\beta}{\alpha^2} (z - z_0), \quad \Lambda_1 = \lambda \beta^{-2}. \quad (67)$$

The stationary wave is stable if the eigenvalues of the operator $H$ include the negative $\Lambda_1 < 0$. As a boundary conditions for Eq. (66) serves the condition that the solutions $\Psi(y)$ are bounded at infinity. The odd term in Eq. (67) describes the effect of the thermal mode on the dynamics of electromagnetic perturbations in the superconductor. Using the standard procedure [12] we can readily find the exact solution of Eq. (66) in the form

$$\Psi(y) = (1 - \alpha^2)^{p+\frac{1}{2}} \left( 1 + \alpha^2 \right)^{q+\frac{1}{2}} \frac{1}{2} F \left( \alpha, \beta, \gamma, \frac{1 - \alpha^2}{2} \right). \quad (68)$$

Here a constant parameters $\alpha$, $\beta$, $\gamma$ are determined by the following relationships

$$\alpha = p + q + 3,$$

$$\beta = p + q,$$

$$\gamma = 2p + 1.$$

and

$$p = \sqrt{1 + \Lambda_1}, \quad q = \sqrt{1 + 3\Lambda_1}.$$

It can be seen that the spectrum of eigenfunctions in (68) is continuous [12]. Analysis of the asymptotic behaviour of solution (68) at $y \to \pm \infty$ shows that $\delta E(y)$ is bounded.
only at \( \Lambda_1 < 0 \). Indeed, one of the partial solutions of Eq. (66), remaining bounded at \( y \to +\infty \), has the form

\[
\delta E(y) \approx \exp(\omega_p y) \times F(p + q + 2, p + q - 1, 2p + 1, \exp(-2y)); \quad 1 - 2p = \omega_p. \tag{69}
\]

It is obvious that at \( y \to +\infty \) (\( \exp(-2y) \to 0 \)) to the asymptotic (68) corresponds to exponentially decaying at positive \( \omega_p \) and growing wave as \( \exp(i\omega_p y) \) at negative \( \omega_p \). It may be assumed that the obtained solution exhibits regular asymptotic behaviour at \( y \to +\infty \) i.e., the function \( \delta E(y) \) is bounded only at those eigenvalues for which \( \Lambda_1 < 0 \).

Let us represent the second linear-independent solution of Eq. (66) as:

\[
F(p + q + 2, p + q - 1, 2p + 1, s) = \frac{(2p + 1)(-2q)}{(p - q - 1)(p - q + 2)} \times \\
\times F(p + q + 2, p + q - 1, 2p + 1, 1 - s) + \\
\frac{(2p + 1)(2q)}{(p + q + 2)(p + q + 1)} (1 - s)^{-2q} F(p - q - 1, p - q + 2, -2q + 1, 1 - s). \tag{70}
\]

where \( \Gamma \) is the Euler’s function [13]. At \( y \to -\infty \) the following asymptotic representation is valid

\[
\delta E(y) \approx \exp(\omega_q y); \quad 1 - 2q = \omega_q. \tag{71}
\]

It can be seen that the space-limited thermomagnetic perturbations are damped both at \( \omega_p > 0 \) and at \( \omega_q > 0 \) i.e., \( \Lambda_1 < 0 \). Let us consider a some specific cases which follows from the properties of the asymptotic solutions of Eq. (66).

a) \( -\frac{1}{3} < \Lambda_1 < 0 \); then the parameters \( \omega_p > 0 \) and \( \omega_q > 0 \) can be chosen substantially. According to the (53) small perturbations damped exponentially with the passage of a time (see, [4]) (Fig.6).

b) \( -\frac{1}{3} > \Lambda_1 > 0 \); in this specific case the wave amplitude at the sufficiently large positive values \( y \) is damped exponentially and at the sufficiently large negative values of \( y \) it has the oscillating-damping profile with the wavelength \( L_q = \frac{\delta 2\pi}{Jm \cdot q} \) (Fig.7).

c) \( \Lambda_1 < -1 \); in this case to the solution corresponds oscillating-damping wave at \( y \to \pm \infty \) with the wavelength \( L_q = \frac{2\pi}{Jm \cdot q} \) and \( L_p = \frac{2\pi}{Jm \cdot p} \) (Fig.8).

Thus it is shown that only damped perturbations correspond to space-limited solutions, which means that the nonlinear wave is stable.

**Conclusion**

In conclusion note that the existence of essential nonlinear and dissipative effects, connected with Joule heating at the moving of magnetic flow, creates different regimes of automodel processes, describing the evolution of perturbations in the vortex state of type-II superconductor. One of measured regimes is the propagating of thermomagnetic waves (normal region - the region heated with the temperature, which is higher that the critical temperature \( T_c \)). The nonlinear stage of evolution of thermal and electromagnetic
perturbations in superconductors is determined by the stationary profile in the form of a running wave. It is seen that the transition from the superconducting state to the normal state can occur by expansion of a stationary thermomagnetic wave, which is caused by the balance between nonlinear effects, caused by Joule heating and dissipative effects. The results obtained make it possible to describe the final stage of evolution of the thermomagnetic instability in the vortex state of type-II superconductors. It is possible that two kinds of stationary thermomagnetic shock waves to exist, depending on the electrodynamic boundary conditions on the sample surface.

We should emphasis that this result was obtained for an arbitrary temperature dependence of the thermophysical parameters $\nu$ and $\kappa$ of superconducting material and for an arbitrary function $j(H)$. Moreover, since the system of equations (1)-(4) is invariant with respect to an arbitrary translation, the wave propagation condition can be found for an arbitrary critical current density dependence on $T$ and $H$.

Let us discuss the validity of the present approach. The applicability of the macroscopic description is ensured by the inequality $\Delta \xi >> d$, $d$ being the average distance between the Abrikosov vortices. The boundary conditions used are valid to describe the situation for a sample with a finite thickness $2L$ provided $\Delta \xi << L$. It is seen that the shock waves under consideration can be observed only for sufficiently thick samples.

The result obtained enables us to propose a possible version of thermomagnetic instability development in superconductors. The initial stage of flux jump is characterized by the exponential growth of both thermal and electromagnetic perturbations. The instability threshold as well as the instability increment are defined on the basis of the linear theory [14]. At a further stage the thermomagnetic shock 'H-wave' with the amplitude $H_e = H_j$ spreads inside the superconductor. Such a process results in a total penetration of the magnetic flux into the sample. The velocity $v_H$ to be found from equation (41), determines the time of the wave motion $t_w$ inside the sample of thickness $2L$

$$t_w \approx \frac{L}{v_H}$$

However, the normal transition occurring within the warmed region complicates the condition of shock-wave propagation. A more detailed description of a such two-phase configuration will be presented in a further report.

Finally, let us briefly discuss the possibility of experimental observation of thermomagnetic shock waves. The necessary condition to observe the phenomena predicted is to ensure sudden initial magnetic flux lattice motion within the superconductor. For example, a thermomagnetic 'E-wave' could be generated by initiating the magnetic flux motion in the presence of an external magnetic field varying with the rate $\dot{H}_e$. Then, a 'background' electric field of order $E_b \sim \frac{L}{c} \dot{H}_e$ arises inside the sample and directly determines the amplitude of an 'E-wave' shock wave. We would point out that thermomagnetic nonlinear waves may also be initiated by abrupt heat input leading to a sudden temperature increase $\delta T = T^* - T_0$ at the sample surface (accompanied by a simultaneous magnetic flux slip initiation inside the superconductor). The electrodynamic boundary conditions at the sample surface is determine the type (E or H - wave) of the propagating wave.

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FIGURE CAPTIONS

Fig.1. The phase plane of Eq. (31).
Fig.2. The structure of shock wave.
Fig.3. The phase plane of Eq. (39).
Fig.4. The equilibrium points of Eq. (49).
Fig.5. The phase plane of Eq. (49).
Fig.6. Asymptotical behaviour of waves at $-\frac{1}{3} < \Lambda_1 < 0$.
Fig.7. The wave amplitude at the sufficiently large positive values $y$ is damped exponentially and at the sufficiently large negative values of $y$ it has the oscillating-damping profile at $-\frac{1}{3} > \Lambda_1 > 0$.
Fig.8. Asymptotical behaviour of waves at $\Lambda_1 < -1$. 
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