Properties of the Discrete Pulse Transform for Multi-Dimensional Arrays

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1 Introduction

This report presents properties of the Discrete Pulse Transform on multi-dimensional arrays introduced earlier in [1]. The main result given here in Lemma 2.1 is also formulated in [4, Lemma 21]. However, the proof, being too technical, was omitted there and hence it appears in full in this publication.

2 The Lemma

The lemma which follows deals with two technical aspects of the Discrete Pulse Transform of a function $f \in \mathcal{A}(\mathbb{Z}^d)$ (where $\mathcal{A}(\mathbb{Z}^d)$ denotes a vector lattice). The first is that the Discrete Pulse representation of a function $f$, given by

$$ f = \sum_{n=1}^{N} D_n(f), $$

can be written as the sum of individual pulses of each resolution layer $D_n(f)$. The second result in the lemma below indicates a form of linearity for the nonlinear LULU operators.

Lemma 2.1

Let $f \in \mathcal{A}(\mathbb{Z}^d)$, supp($f$) < $\infty$, be such that $f$ does not have local minimum sets or local maximum sets of size smaller than $n$, for some $n \in \mathbb{N}$. Then we have the following two results.

a) \hspace{1cm} (id - P_n)f = \sum_{i=1}^{\gamma^- (n)} \phi_{ni} + \sum_{j=1}^{\gamma^+ (n)} \varphi_{nj},

where $V_{ni} = \text{supp}(\phi_{ni}), i = 1, 2, ..., \gamma^-(n)$, are local minimum sets of $f$ of size $n$, $W_{nj} = \text{supp}(\varphi_{nj}), j = 1, 2, ..., \gamma^+(n)$, are local maximum sets of $f$ of size $n$, $\phi_{ni}$ and $\varphi_{nj}$ are negative and positive discrete pulses respectively, and we also have that

- $V_{ni} \cap V_{nj} = \emptyset$ and $\text{adj}(V_{ni}) \cap V_{nj} = \emptyset$, $i, j = 1, ..., \gamma^-(n), i \neq j$, (2)
- $W_{ni} \cap W_{nj} = \emptyset$ and $\text{adj}(W_{ni}) \cap W_{nj} = \emptyset$, $i, j = 1, ..., \gamma^+(n), i \neq j$, (3)
- $V_{ni} \cap W_{nj} = \emptyset$ $i = 1, ..., \gamma^-(n)$, $j = 1, ..., \gamma^+(n)$. (4)

b) For every fully trend preserving operator $A$

$$ U_n(id - AU_n) = U_n - AU_n, $$
$$ L_n(id - AL_n) = L_n - AL_n. $$
Proof.

a) Let \( V_{n1}, V_{n2}, ..., V_{n\gamma^-(n)} \) be all local minimum sets of size \( n \) of the function \( f \). Since \( f \) does not have local minimum sets of size smaller than \( n \), then \( f \) is a constant on each of these sets, by [4, Theorem 14]. Hence, the sets are disjoint, that is \( V_{ni} \cap V_{nj} = \emptyset, \ i \neq j \). Moreover, we also have

\[
\text{adj}(V_{ni}) \cap V_{nj} = \emptyset, \ i, j = 1, ..., \gamma^-(n).
\] (5)

Indeed, let \( x \in \text{adj}(V_{ni}) \cap V_{nj} \). Then there exists \( y \in V_{ni} \) such that \((x, y) \in r\). Hence \( y \in V_{ni} \cap \text{adj}(V_{nj}) \). From the local minimality of the sets \( V_{ni} \) and \( V_{nj} \) we obtain respectively \( f(y) < f(x) \) and \( f(x) < f(y) \), which is clearly a contradiction. For every \( i = 1, ..., \gamma^-(n) \) denote by \( y_{ni} \) the point in \( \text{adj}(V_{ni}) \) such that

\[
f(y_{ni}) = \min_{y \in \text{adj}(V_{ni})} f(y).
\] (6)

Then we have

\[
U_n f(x) = \begin{cases} 
 f(y_{ni}) & \text{if } x \in V_{ni}, \ i = 1, ..., \gamma^-(n) \\
 f(x) & \text{otherwise (by [4, Theorem 9])}
\end{cases}
\]

Therefore

\[
(id - U_n) f = \sum_{i=1}^{\gamma^-(n)} \phi_{ni}
\] (7)

where \( \phi_{ni} \) is a discrete pulse with support \( V_{ni} \) and negative value (down pulse).

Let \( W_{n1}, W_{n2}, ..., W_{n\gamma^+(n)} \) be all local maximum sets of size \( n \) of the function \( U_n f \). By [4, Theorem 12(b)] every local maximum set of \( U_n f \) contains a local maximum set of \( f \). Since \( f \) does not have local maximum sets of size smaller than \( n \), this means that the sets \( W_{nj}, \ j = 1, ..., \gamma^+(n) \), are all local maximum sets of \( f \) and \( f \) is constant on each of them. Similarly to the local minimum sets of \( f \) considered above we have \( W_{ni} \cap W_{nj} = \emptyset, \ i \neq j \), and \( \text{adj}(W_{ni}) \cap W_{nj} = \emptyset, \ i, j = 1, ..., \gamma^+(n) \). Moreover, since \( U_n(f) \) is constant on any of the sets \( V_{ni} \cup \{y_{ni}\}, \ i = 1, ..., \gamma^-(n) \), see [4, Theorem 14], we also have

\[
(V_{ni} \cup \{y_{ni}\}) \cap W_{nj} = \emptyset, \ i = 1, ..., \gamma^-(n), \ j = 1, ..., \gamma^+(n),
\] (8)

which implies [4].

Further we have

\[
L_n U_n f(x) = \begin{cases} 
 U_n f(z_{nj}) & \text{if } x \in W_{nj}, \ j = 1, ..., \gamma^+(n) \\
 U_n f(x) & \text{otherwise}
\end{cases}
\]

where \( z_{nj} \in \text{adj}(W_{nj}), \ j = 1, ..., \gamma^+(n) \), are such that \( U_n f(z_{nj}) = \max_{z \in \text{adj}(W_{nj})} U_n f(z) \). Hence

\[
(id - L_n) U_n f = \sum_{j=1}^{\gamma^+(n)} \varphi_{nj}
\] (9)
where \( \varphi_{nj} \) is a discrete pulse with support \( W_{nj} \) and positive value (up pulse). Thus we have shown that

\[
(id - P_n)f = (id - U_n)f + (id - L_n)U_nf = \sum_{i=1}^{\gamma^-(n)} \phi_{ni} + \sum_{j=1}^{\gamma^+(n)} \varphi_{nj}.
\]

b) Let the function \( f \in A(\mathbb{Z}^d) \) be such that it does not have any local minimum or local maximum sets of size less than \( n \). Denote \( g = (id - AU_n)(f) \). We have

\[
g = (id - AU_n)(f) = (id - U_n)(f) + ((id - A)U_n)(f).
\]

As in a) we have that (7) holds, that is we have

\[
(id - U_n)(f) = \sum_{i=1}^{\gamma^-(n)} \phi_{ni},
\]

where the sets \( V_{ni} = \text{supp}(\phi_{ni}), i = 1, ..., \gamma^-(n) \), are all the local minimum sets of \( f \) of size \( n \) and satisfy (2). Therefore

\[
g = \sum_{i=1}^{\gamma^-(n)} \phi_{ni} + ((id - A)U_n)(f).
\]

Furthermore,

\[
U_n(f)(x) = \begin{cases} 
  f(x) & \text{if } x \in \mathbb{Z}^d \setminus \bigcup_{i=1}^{\gamma^-(n)} V_{ni} \\
  v_i & \text{if } x \in V_{ni} \cup \{y_{ni}\}, i = 1, ..., \gamma^-(n),
\end{cases}
\]

where \( v_i = f(y_{ni}) = \min_{y \in \text{adj}(V_{ni})} f(y) \). Using that \( A \) is fully trend preserving, for every \( i = 1, ..., \gamma^-(n) \) there exists \( w_i \) such that \( ((id - A)U_n)(f)(x) = w_i, x \in V_{ni} \cup \{y_{ni}\} \). Moreover, using that every adjacent point has a neighbor in \( V_{ni} \) we have that \( \min_{y \in \text{adj}(V_{ni})} ((id - A)U_n)(f)(y) = w_i \). Considering that the value of the pulse \( \phi_{ni} \) is negative, we obtain through the representation (12) that \( V_{ni}, i = 1, ..., \gamma^-(n), \) are local minimum sets of \( g \).

Next we show that \( g \) does not have any other local minimum sets of size \( n \) or less. Indeed, assume that \( V_0 \) is a local minimum set of \( g \) such that \( \text{card}(V_0) \leq n \). Since \( V_0 \cup \text{adj}(V_0) \subset \mathbb{Z}^d \setminus \bigcup_{i=1}^{\gamma^-(n)} V_{ni} \) it follows from (12) that \( V_0 \) is a local minimum set of \( ((id - A)U_n)(f) \). Then using that \( (id - A) \) is neighbor trend preserving and using [4, Theorem 17] we obtain that there exists a local minimum set \( W_0 \) of \( U_n(f) \) such that \( W_0 \subseteq V_0 \). Then applying again [4, Theorem 17] or [4, Theorem 12] we obtain that there exists a local minimum set \( \tilde{W}_0 \) of \( f \) such that \( \tilde{W}_0 \subseteq W_0 \subseteq V_0 \). This inclusion implies that \( \text{card}(\tilde{W}_0) \leq n \). Given that \( f \) does not have local minimum sets of size
less than \( n \) we have \( \text{card}(\tilde{W}_0) = n \), that is \( \tilde{W}_0 \) is one of the sets \( V_{ni} \) - a contradiction. Therefore, \( V_{ni}, i = 1, ..., \gamma^{-}(n) \), are all the local minimum sets of \( g \) of size \( n \) or less. Then using again (7) we have

\[
(id - U_n)(g) = \sum_{i=1}^{\gamma^{-}(n)} \phi_{ni}
\]

(13)

Using (11) and (13) we obtain

\[
(id - U_n)(g) = (id - U_n)(f)
\]

Therefore

\[
(U_n(id - AU_n))(f) = U_n(g) = g - (id - U_n)(f)
\]

\[
= (id - AU_n)(f) - (id - U_n)(f)
\]

\[
= (U_n - AU_n)(f).
\]

This proves the first identity. The second one is proved in a similar manner. □

References

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