Quaternion offset linear canonical transform in one-dimensional setting

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Abstract
In this paper, we introduce quaternion offset linear canonical transform of integrable and square integrable functions. Moreover, we show that the proposed transform satisfies all the respective properties like inversion formula, linearity, Moyal’s formula, product theorem and the convolution theorem.

Keywords Offset linear canonical transform · Quaternion offset linear canonical transform · Moyal’s formulla · Convolution

Mathematics Subject Classification Primary 42C40 · Secondary 42C15 · 47G10 · 42A38 · 42B10

1 Introduction
The classical Integral transform has been generalized to the six-parameter \((A, B, C, D, p, q)\) transform called the offset linear canonical transform (OLCT). For a matrix parameter \(\Lambda = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}\), the OLCT of any signal \(f\) is defined as

\[
\mathcal{O}_\Lambda[f](w) = \int f(t) \mathcal{K}_\Lambda(t, w) dt,
\]

where \(\mathcal{K}_\Lambda(t, w)\) denotes the kernel of the OLCT and is given by

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\[ K_A(t, w) = \frac{1}{\sqrt{2\pi B}} \exp \left\{ \frac{i}{2B} \left( A t^2 - 2t(w - p) - 2w(Dp - Bq) + D(w^2 + p^2) \right) \right\} \]

with \( AD - BC = 1 \).

It is here worth to mention that if \( B = 0 \) then the OLCT defined by (1.1) is simply a time scaled version off multiplied by a linear chirp. Hence, without loss of generality, in this paper we assume \( B \neq 0 \).

Looking at the core of OLCT, one can derive it as a time-shifted and frequency-modulated child of the parent linear canonical transform (LCT) [1, 2]. On the application side OLCT is similar to LCT but due to its two extra parameters \( p \) and \( q \), it is more general and flexible than parental LCT. Hence it has gained more popularity in optics, signal and image processing. For more details, we refer to [35, 27].

In the prospect of signal processing, one can consider that any signal processing tool converts the time-domain signals into frequency-domain. Further in signal processing, convolution of two functions [79] is a most useful tool in constructing a filter for denoising the given noisy signals (see [10]).

In past decades, hypercomplex algebra has become a leading area of research with its applications in color image processing, image filtering, watermarking, edge detection and pattern recognition (see [11–18]). The Cayley-Dickson algebra of order four is labeled as quaternions which has wide applications in optical and signal processing. The extension of Fourier transform in quaternion algebra is known as quaternion Fourier transform (QFT) [19] which is said to be the substitute of the commonly used two-dimensional Complex Fourier Transform (CFT). The QFT has wide range of applications see [20, 21].

The quaternionic offset linear canonical transform (QOLCT) can be defined as a generalization of the quaternionic linear canonical transform (QLCT) and has been studied in [22]. Here the authors derive the relationship between the QOLCT and the quaternion Fourier transform (QFT). Moreover, they proved the Plancherel formula, and some properties related to the QOLCT. For more details we refer to [23–36].

But to the best of our knowledge, theory about one-dimensional quaternion OLCT (1D-QOLCT) is still in its infancy. Therefore it is worthwhile to study the theory of 1D-QOLCT which can be productive for signal processing theory and applications. In this paper, our main objectives are to introduce the novel integral transform called the one-dimensional quaternion offset linear canonical transform (1D-QOLCT) and study its properties, such as inversion formula, linearity, Moyal’s formula, convolution theorem and product theorem 1D-QOLCT.

This paper is organized as follows: In Sect. 2, we summarize the general definitions and basic properties of quaternions. In Sect. 3, we introduce 1D-QOLCT and obtain various properties linearity, Moyal’s formula, convolution and product theorem of the proposed transform.
2 Preliminaries

2.1 Quaternions

Let $\mathbb{R}$ and $\mathbb{C}$ be the usual set of real numbers and set of complex numbers, respectively. The division ring of quaternions in the honor of Hamilton, is denoted by $\mathbb{H}$ and is defined as

$$\mathbb{H} = \{ h_0 + e_1 h_1 + e_2 h_2 + e_3 h_3 : h_0, h_1, h_2, h_3 \in \mathbb{R} \} = \{ z_1 + e_2 z_2 : z_1, z_2 \in \mathbb{C} \} (\text{Cayley Dickson form})$$

where $e_1, e_2, e_3$ satisfy Hamilton’s multiplication rule

$$e_1 e_2 = -e_2 e_1 = e_3, \quad e_2 e_3 = -e_3 e_2 = e_1, \quad e_3 e_1 = -e_1 e_3 = e_2$$

and

$$e_1^2 = e_2^2 = e_3^2 = -1$$

Every member of $\mathbb{H}$ is known as quaternion. In quaternion algebra addition, multiplication, conjugate and absolute value of quaternions are defined by

$$(a_1 + e_2 a_2) + (b_1 + e_2 b_2) = (a_1 + b_1) + e_2 (a_2 + b_2),$$
$$(a_1 + e_2 a_2)(b_1 + e_2 b_2) = (a_1 b_1 - \bar{a}_2 b_2) + e_2 (a_2 b_1 + \bar{a}_1 b_2),$$

$$(a_1 + e_2 a_2)^c = \overline{a_1} - e_2 \overline{a}_2,$$

$$|a_1 + e_2 a_2| = \sqrt{|a_1|^2 + |a_2|^2},$$

here $\overline{a}_k$ is the complex conjugate of $a_k$ and $|a_k|$ is the modulus of the complex number $a_k, k = 1, 2$. For all $a = a_1 + e_2 a_2, \quad b = b_1 + e_2 b_2 \in \mathbb{H}$, the following properties of conjugate and modulus and multiplicative inverse are well known.

$$(a^c)^c = a, \quad (a + b)^c = a^c + b^c, \quad (ab)^c = b^c a^c,$$

$$|a|^2 = a a^c = |a_1|^2 + |a_2|^2, \quad |ab| = |a||b|,$$

$$a^{-1} = \frac{\overline{a}}{|a|^2}.$$

We denote $L^p(\mathbb{R}, \mathbb{H})$, the Banach space of all quaternion-valued functions $f$ satisfying

$$\|f\|_p = \left( \int |f_1(t)|^p + |f_2(t)|^p dt \right)^{1/p} < \infty, \quad p = 1, 2.$$

And on $L^2(\mathbb{R}, \mathbb{H})$ the inner product $\langle f, g \rangle = \int f(t)[g(t)]^c dt$, where integral of a quaternion valued function is defined by $\int(f_1 + e_2 f_2)(t) dt = \int f_1(x) dt + e_2 \int f_2(x) dt$, whenever the integral exists.
3 Quaternion one-dimensional offset linear canonical transform

In this section we will introduce the definition of quaternion one-dimensional offset linear canonical transform (1D-QOLCT) by using \[37\,39\]. Prior to that we note \(e_1, e_2\) and \(e_3\) (or equivalently \(i, j, k\)) denote the three imaginary units in the quaternion algebra.

**Definition 3.1** The 1D-QOLCT of any signal \(f \in L^1(\mathbb{R}, \mathbb{H})\) with respect a matrix parameter \(\Lambda = (A, B, C, D, p, q)\) is defined by

\[
O_{\Lambda}^{e_2}[f(t)](w) = \int f(t)K_{\Lambda}^{e_2}(t, w)dt \tag{3.1}
\]

where

\[
K_{\Lambda}^{e_2}(t, w) = \frac{1}{\sqrt{2\pi Be_2}}\exp\left\{ \frac{e_2}{2B} \left( 4t^2 - 2t(w - p) - 2w(Dp - Bq) + D(w^2 + p^2) \right) \right\}
\]

With \(AD - BC = 1\). Now we can find that if \(f(t)\) is real-valued signal in (3.1), then we can interchange the kernel in Definition 3.1.

By appropriately choosing parameters in \(\Lambda = (A, B, C, D, p, q)\) the 1D-QOLCT (3.1) gives birth to the following existing time-frequency transforms:

- For \(\Lambda = (0, 1, -1, 0, 0, 0)\), the 1D-QOLCT (3.1) boils down to the quaternion one-dimensional Fourier Transform [37].
- For \(\Lambda = (A, B, C, D, 0, 0)\) the 1D-QOLCT (3.1) reduces to the Quaternion one-dimensional Linear Canonical Transform [39].
- For \(\Lambda = (\cos \theta, \sin \theta, -\sin \theta, \cos \theta, 0, 0)\) the 1D-QOLCT (3.1) reduces to the Quaternion one-dimensional fractional Fourier Transform [38].

**Definition 3.2** (*Inversion*) The inverse of a 1D-QOLCT with parameter \(\Lambda = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \) is given by a 1D-QOLCT with parameter

\[
\Lambda^{-1} = \begin{bmatrix} D & B \\
-C & A \end{bmatrix} \begin{bmatrix} \quad p \\ \quad q \end{bmatrix}
\]

as

\[
f(t) = \{O_{\Lambda}^{e_2}\}^{-1}[O_{\Lambda}^{e_2}[f]](t) = \int O_{\Lambda}^{e_2}[f](w)K_{\Lambda^{-1}}^{e_2}(w, t)dw \tag{3.2}
\]

where \(K_{\Lambda^{-1}}^{e_2}(w, t) = K_{\Lambda}^{e_2}(t, w) = \overline{K}_{\Lambda}^{e_2}(t, w)\)

**Definition 3.3** Let \(f = f_1 + e_2f_2\) be a quaternion valued signal in \(L^1(\mathbb{R}, \mathbb{H})\), then the quaternion offset linear canonical transform is defined as
\[ O^h_\lambda[f(t)](w) = O^h_\lambda[f_1(t)](w) + e_2 O^h_\lambda[f_2(t)](w). \]  

(3.3)

By above definition, it is consistent with the offset linear canonical transform on \( L^1(\mathbb{R}, \mathbb{C}) \). Now it is clear from the definition of quaternion offset linear canonical transform and the properties of offset linear canonical transform on \( L^1(\mathbb{R}, \mathbb{H}) \), that \( O^h_\lambda(O^h_\lambda[f]) = O^h_{\lambda_1}[f] \) and \( \{O^h_\lambda[f]\}^{-1} = O^h_{\lambda^{-1}}[f] \) for every signal \( f \in L^1(\mathbb{R}, \mathbb{H}) \).

**Theorem 3.4** The quaternion offset linear canonical transform \( O^h_\lambda \) is \( \mathbb{H} \)-linear on \( L^1(\mathbb{R}, \mathbb{H}) \).

**Proof** Let us consider two quaternion signals \( f = f_1 + e_2 f_2 \) and \( g = g_1 + e_2 g_2 \) in \( L^1(\mathbb{R}, \mathbb{H}) \), now by the linearity of \( O^h_\lambda \) on \( L^1(\mathbb{R}, \mathbb{C}) \), we obtain

\[
O^h_\lambda[f + g] = O^h_\lambda[(f_1 + e_2 f_2) + (g_1 + e_2 g_2)] \\
= O^h_\lambda[(f_1 + g_1) + e_2 (f_2 + g_2)] \\
= O^h_\lambda[f_1] + O^h_\lambda[g_1] + e_2 (O^h_\lambda[f_2] + O^h_\lambda[g_2]) \\
= (O^h_\lambda[f_1] + e_2 O^h_\lambda[f_2]) + (O^h_\lambda[g_1] + e_2 O^h_\lambda[g_2]) \\
= O^h_\lambda[f] + O^h_\lambda[g].
\]

Now to prove \( \mathbb{H} \)-linearity, we let \( q = q_1 + e_2 q_2 \in \mathbb{H} \) and \( f = f_1 + e_2 f_2 \in L^1(\mathbb{R}, \mathbb{H}) \) be arbitrary, then we have

\[
O^h_\lambda[e_2 f] = O^h_\lambda[e_2(f_1 + e_2 f_2)] \\
= O^h_\lambda[e_2 f_1 - f_2] \\
= e_2 O^h_\lambda[f_1] - O^h_\lambda[f] \\
= e_2 (O^h_\lambda[f_1] + e_2 O^h_\lambda[f_2]) \\
= e_2 O^h_\lambda[f].
\]

Therefore,

\[
O^h_\lambda[q f] = O^h_\lambda[q_1 f] + O^h_\lambda[e_2 q_2 f] \\
= q_1 O^h_\lambda[f] + e_2 q_2 O^h_\lambda[f] \\
= (q_1 + e_2 q_2) O^h_\lambda[f] \\
= q O^h_\lambda[f].
\]

Which completes proof. \( \square \)

**Theorem 3.5** (Moyal’s formula) Let \( f, g \in L^1(\mathbb{R}, \mathbb{H}) \cap L^2(\mathbb{R}, \mathbb{H}) \) be two signals functions with \( O^h_\lambda[f] \in L^1(\mathbb{R}, \mathbb{H}), \langle f, g \rangle = \langle O^h_\lambda[f], O^h_\lambda[g] \rangle \).

**Proof** For \( f, g \in L^1(\mathbb{R}, \mathbb{H}) \cap L^2(\mathbb{R}, \mathbb{H}) \) with \( O^h_\lambda[f] \in L^1(\mathbb{R}, \mathbb{H}), \)
\[
\langle f, g \rangle = \int f(t)[g(t)]^c dt
\]
\[
= \int \int \mathcal{O}_A^H[f](w)K_{\alpha}^{\epsilon_2}(w, t)dw[g(t)]^c dt \quad \text{(by (3.2))}
\]
\[
= \int \int \mathcal{O}_A^H[f](w)K_{\alpha}^{-\epsilon_2}(t, w)[g(t)]^c dw dt
\]
\[
= \int \mathcal{O}_A^H[f](w) \left\{ \int K_{\alpha}^{\epsilon_2}(t, w)[g(t)]^c dt \right\} dw
\]
\[
= \int \mathcal{O}_A^H[f](w) \left\{ \int g(t)K_{\alpha}^{\epsilon_2}(t, w)dt \right\}^c dw
\]
\[
= \int \mathcal{O}_A^H[f](w) \{ \mathcal{O}_A^H[g](w) \}^c dw
\]
\[
= \langle \mathcal{O}_A^H[f], \mathcal{O}_A^H[g] \rangle.
\]
Which completes proof. \(\Box\)

**Lemma 3.6** For \(f \in L^p(\mathbb{R}, \mathbb{C}), p = 1, 2\); we have \(\mathcal{O}_A^H[f](w) = \mathcal{O}_{A^{-1}}^H[f](w)\).

**Proof** It follows from definition 3.1 that

\[
\mathcal{O}_A^H[f](w) = \int f(t)K_{\alpha}^{\epsilon_2}(t, w)dt
\]
\[
= \int \overline{f(t)K_{\alpha}^{\epsilon_2}(t, w)} dt
\]
\[
= \int f(t)K_{\alpha}^{-\epsilon_2}(t, w)dt
\]
\[
= \int f(t)K_{\alpha}^{\epsilon_2}(w, t)dt
\]
\[
= \mathcal{O}_{A^{-1}}^H[f](w).
\]

Which completes the proof. \(\Box\)

**Remark 3.7** The Lemma 3.6 can also be written as \(\mathcal{O}_{A^{-1}}^H[f](w) = \mathcal{O}_A^H[f](w)\).

**Definition 3.8** For \(f \in L^2(\mathbb{R}, \mathbb{H})\) and \(g \in L^1(\mathbb{R}, \mathbb{H})\), define

\[
(f * g) = (f_1 * g_1 - \mathcal{O}_{A^{-2}}^H[f_2 * g_2]) + e_2(f_2 * g_1 + \mathcal{O}_{A^{-2}}^H[f_1 * g_1]),
\]
where \(*\) is the proposed definition of convolution.

**Lemma 3.9** Under the assumptions of Definition 3.8, we have

\[
\mathcal{O}_A^H[f * g(t)](u) = \mathcal{O}_A^H[f](u)\mathcal{O}_A^H[g](u) \exp \left\{ \frac{e_2}{2B} (2w(Dp - Bq) - Dw^2) \right\}
\]

\[\Box\] Springer
Theorem 3.10  (Convolution theorem) Let \( f, g \) be two given signal functions such that \( f \in L^2(\mathbb{R}, \mathbb{H}) \) and \( g \in L^1(\mathbb{R}, \mathbb{H}) \), then for all \( w \in \mathbb{R} \), we have

\[
\mathcal{O}_A^H[f * g](w) = \mathcal{O}_A^H[f](w)\mathcal{O}_A^H[g](w) \exp\left\{\frac{e_2}{2B} (2w(Dp - Bq) - Dw^2)\right\}
\]  

(3.6)

Proof  By applying Definition 3.8 and Lemma 3.9, we have

\[
\mathcal{O}_A^H[f * g](w) = \mathcal{O}_A^H[(f_1 * g_1 - \mathcal{O}_A^H[f_2 * g_2])](w) + e_2 \mathcal{O}_A^H[(f_2 * g_1 + \mathcal{O}_A^H[f_1 * g_1])](w)
\]

\[
= \mathcal{O}_A^H[f_1](w)\mathcal{O}_A^H[g_1](w) \exp\left\{\frac{e_2}{2B} (2w(Dp - Bq) - Dw^2)\right\}
\]

\[
- \mathcal{O}_A^H\mathcal{O}_A^H[f_2](w)\mathcal{O}_A^H[g_2](w) \exp\left\{\frac{e_2}{2B} (2w(Dp - Bq) - Dw^2)\right\}
\]

\[
+ e_2\left\{\mathcal{O}_A^H[f_2](w)\mathcal{O}_A^H[g_1](w) \exp\left\{\frac{e_2}{2B} (2w(Dp - Bq) - Dw^2)\right\}
\]

\[
+ \mathcal{O}_A^H\mathcal{O}_A^H[f_1](w)\mathcal{O}_A^H[g_1](w) \exp\left\{\frac{e_2}{2B} (2w(Dp - Bq) - Dw^2)\right\}\right\}
\]

\[
= \left\{\mathcal{O}_A^H[f_1]\mathcal{O}_A^H[g_1] - \mathcal{O}_A^H[f_2]\mathcal{O}_A^H[g_2]\right\}(w)
\]

\[
e_2\left\{\mathcal{O}_A^H[f_2]\mathcal{O}_A^H[g_1] - \mathcal{O}_A^H[f_1]\mathcal{O}_A^H[g_1]\right\}(w) \exp\left\{\frac{e_2}{2B} (2w(Dp - Bq) - Dw^2)\right\}
\]

Which completes the proof. \( \square \)

Definition 3.11  For \( f \in L^2(\mathbb{R}, \mathbb{H}) \) and \( g \in L^1(\mathbb{R}, \mathbb{H}) \), define

\[
(f \otimes g) = (f_2 \otimes g_2 + \mathcal{O}_A^H[f_1] \otimes [g_1]) + e_2(\mathcal{O}_A^H[f_1] \otimes [g_2] - f_2 \otimes g_1),
\]  

(3.7)

where \( \otimes \) is the proposed definition of convolution.

Lemma 3.12  Under the assumptions of definition 3.11, we have

\[
\left\{\mathcal{O}_A^H[f(t)] \otimes \mathcal{O}_A^H[g(t)](u) = \mathcal{O}_A^H[f(t)g(t) \exp\left\{\frac{e_2}{2B} (2w(Dp - Bq) - Dw^2)\right\}\right\}
\]

(3.8)

where \( \mathcal{O}_A^H[f] \otimes \mathcal{O}_A^H[g] = \mathcal{O}_A^H[fg] \)

Theorem 3.13  (Product theorem) Let \( f, g \) be two given signal functions such that \( f \in L^2(\mathbb{R}, \mathbb{H}) \) and \( g \in L^1(\mathbb{R}, \mathbb{H}) \), then for all \( w \in \mathbb{R} \), we have
\[ O_{\lambda}^{H}[g] = O_{\lambda}^{H}[f] \otimes O_{\lambda}^{H}[g]. \] (3.9)

**Proof** By Definition 3.11 and Lemma 3.12, we have

\[
O_{\lambda}^{H}[g] = O_{\lambda}^{H}[(f_1 - e_2f_2)(g_1 + e_2g_2)] \\
= O_{\lambda}^{H}[f_1g_1] + O_{\lambda}^{H}[f_2g_2] + e_2(O_{\lambda}^{H}[f_1g_2] - O_{\lambda}^{H}[f_2g_1]) \\
= O_{\lambda}^{H}[f_1] \otimes O_{\lambda}^{H}[g_1] + O_{\lambda}^{H}[f_2] \otimes O_{\lambda}^{H}[g_2] \\
+ e_2(O_{\lambda}^{H}[f_1] \otimes O_{\lambda}^{H}[g_2] - O_{\lambda}^{H}[f_2] \otimes O_{\lambda}^{H}[g_1]) \\
= O_{\lambda}^{H}[f_1] \otimes O_{\lambda}^{H}[g_1] + O_{\lambda}^{H}[f_2] \otimes O_{\lambda}^{H}[g_2] \\
+ e_2(O_{\lambda}^{H}[f_1] \otimes O_{\lambda}^{H}[g_2] - O_{\lambda}^{H}[f_2] \otimes O_{\lambda}^{H}[g_1]) \\
= O_{\lambda}^{H}[f] \otimes O_{\lambda}^{H}[g].
\]

Which completes the proof. \(\square\)

## 4 Conclusion

In this paper, we have proposed the definition of the novel integral transform known as the one-dimensional quaternion offset linear canonical transform (1D-QOLCT) which is embodiment of several well known signal processing tools. We then obtained Moyal’s formula, convolution theorem and product theorem for proposed transform. Our future work about convolution and correlation theorems for two-sided short-time offset linear canonical transform and uncertainty principles for short-time quaternion offset linear canonical transform is in progress.

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**Declarations**

**Conflict of interest** The authors have no conflict of interests.
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