CUTOFF STABILITY OF MULTIVARIATE GEOMETRIC BROWNIAN MOTION

GERARDO BARRERA, MICHAEL A. HÖGELE, AND JUAN CARLOS PARDO

Abstract. This article quantifies the asymptotic $\varepsilon$-mixing times, as $\varepsilon$ tends to 0, of a multivariate stable geometric Brownian motion with respect to the Wasserstein-Kantorovich-Rubinstein-2-distance. We study the cases of commutative drift and diffusion coefficient matrices.

1. Introduction

Geometric Brownian motion serves as an important class of model in mathematical finance (e.g. [26, 30, 33]), but it is also an important and well-studied mathematical object in its own right, see for instance [3, 13, 16, 18] and the references therein. A geometric Brownian motion in one dimension is the solution of a linear stochastic differential equation (SDE) in the Itô sense with multiplicative noise

$$dX_t(x) = aX_t(x)dt + bX_t(x)dW_t \quad \text{with} \quad X_0(x) = x,$$

where $a$ and $b$ are non-zero real numbers and $(W_t)_{t \geq 0}$ is a standard scalar Wiener process. SDE (1.1) can be solved explicitly and has the following shape

$$X_t(x) = \exp((a - (1/2)b^2)t + bW_t)x, \quad t \geq 0, \quad x \in \mathbb{R},$$

see for instance [25] Example 5.5 and [29] Example 5.1.1. If (1.1) is understood in the Stratonovich sense, where the correction term $\exp(-(1/2)b^2t)$ is contained in the stochastic (Stratonovich) integral, the solution has the shape $X_t(x) = \exp(at + bW_t)x, \quad t \geq 0, \quad x \in \mathbb{R}$. This simple exponential form may change drastically in higher dimensions if $a$ and $b$ are changed to deterministic square matrices $A$ and $B$, and it is only valid in special cases. In particular, a multivariate geometric Brownian motion in the Stratonovich sense with initial value $x \in \mathbb{R}^d, \quad d \in \mathbb{N}$, has the following shape

$$X_t(x) = \exp(Y_t)x \quad \text{with} \quad Y_t = At + BW_t, \quad t \geq 0,$$

only if and only if the deterministic matrices $A$ and $B$ commute.

By the Stratonovich change of variables formula, it is well-known that in case of commuting $A$ and $B$ the process $(X_t(x))_{t \geq 0}$ is the unique strong solution of the Stratonovich stochastic differential equation

$$dX_t(x) = AX_t(x)dt + BX_t(x) \circ dW_t, \quad t \geq 0, \quad X_0(x) = x.$$
Ultimately, this result goes back to the classical result that in a unital associative algebra of operators, the commutativity of two elements $U, V$ is necessary and sufficient for the following functional equality of the operator exponentials to be valid

$$\exp(U) \exp(V) = \exp(U + V),$$

while in general the Baker-Campbell-Hausdorff-Dynkin (BCHD) formula gives additional linear combinations of nested commutators of $U$ and $V$ in the exponent, see [17, Chapter 5]. In [20] the authors show a stochastic extension of this result, that is, a matrix exponential representation of type (1.2) for the unique strong solution of (1.3) for noncommuting matrices $A$ and $B$ formulated for equations in the Itô sense. See also [15, 24, 27, 34, 36]. For comparison, (1.3) in the Itô sense reads as follows

$$dX_t(x) = (A + (1/2)B^2)X_t(x)dt + BX_t(x)dW_t, \quad t \geq 0, \quad X_0(x) = x,$$

see [24] Chapter 4, p. 159. The shape of (1.2) changes considerably when $A$ and $B$ do not commute, that is, $[A, B] := AB - BA \neq 0$. In this case following [24] the exponent (without simplifications) is in general a nonlinear functional in $t$ and $W_t$:

$$X_t(x) = \exp(Y_t)x \quad \text{with} \quad t \geq 0,$$

where

$$Y_t = \left( A + \frac{1}{2}B^2 \right)t + BW_t + [B, A + \frac{1}{2}B^2] \left( \frac{1}{2}tW_t - \int_0^t W_sds \right) - \frac{1}{2}B^2t$$

$$+ [[A + \frac{1}{2}B^2, B], B] \left( \frac{1}{2} \int_0^t W_s^2ds - \frac{1}{2}W_t \int_0^t W_sds + \frac{1}{12}tW_t^2 \right)$$

$$+ [[A + \frac{1}{2}B^2, B], A + \frac{1}{2}B^2] \left( \int_0^t sW_sds - \frac{1}{2}t \int_0^t W_sds - \frac{1}{12}t^2W_t \right) + \cdots.$$

The missing remainder terms in the exponent $Y_t$ of $X_t(x)$ contains only terms that includes higher order nested commutators of $A$ and $B$, see Section 3.1 in [20].

We prove quantitative abrupt convergence results between the law of the current state of the solution of (1.4) under certain commutative relations and its limiting distribution, such as a sort of abrupt convergence or cutoff phenomenon.

Recently, the concept of cutoff has been studied in dynamical systems in the life sciences [28] and machine learning [4]. Historically [1, 2, 11, 13, 22, 23], this dynamical feature emerged in the context of card-shuffling Markov chains, which implies a discrete setting, measured in terms of the total variation distance. However, in a continuous space and time setting, the total variation distance turns out to be cumbersome. Note that the total variation distance between two absolutely continuous distributions is proportional to the $L^1$-distance between their respective densities. In particular, the distance between two laws, one absolutely continuous (w.r.t. the Lebesgue measure) and the other one a Dirac measure is equal to 1. Consequently, a sequence of absolutely continuous laws never converges in total variation distance to a deterministic limit, which happens to occur in our setting. The previous defect of the total variation distance turns out to run deeper. In particular, Slutsky’s lemma is not valid for the total variation distance (see [6, Lemma 1.17]). In our continuous state space setting, this metric turns out to be topologically too fine in order to be meaningful, since it is discontinuous for discrete approximations of absolutely continuous laws, in particular, all those distances are maximally equal to 1. This defect is maybe best illustrated by the fact, that not even the DeMoivre-Laplace central limit theorem is valid in the total variation distance, however, it is obviously perfectly valid for other -topologically weaker- distances, such
as the Wasserstein-Kantorovich-Rubinstein or the Kolmogorov distance, see [19]. For a complete overview between different distances we refer to [31]. In the particular case of a deterministic limiting distribution, the Wasserstein-Kantorovich-Rubinstein-$p$-distance reduces to a simple $L^p$ distance, which in the case of $p = 2$, yields a quadratic form representation for (random) matrix exponentials to be studied.

The main result of this article, Theorem 2.3 establishes that for solutions of (1.3) the commutativity of $A$ and $B$ entails asymptotic $\varepsilon$-mixing times, as $\varepsilon \to 0$, in Wasserstein-Kantorovich-Rubinstein-2-distance of leading order $|\ln(\varepsilon)|$.

In what follows, we embed our findings in the context of the so-called cutoff phenomenon mentioned before. For each $\varepsilon > 0$, let $X^{x,\varepsilon}_t := (X^{x}_t(x_\varepsilon))_{t \geq 0}$ be a stochastic process (including the degenerate deterministic case) with values in the Polish space $E_\varepsilon$ and with initial position $x_\varepsilon \in E_\varepsilon$. Consider $\mathcal{M}_1(E_\varepsilon)$ the space of probability measures on $E_\varepsilon$ equipped with the distance $d_\varepsilon$. Assume that for each $\varepsilon > 0$ there exists $\mu^\varepsilon \in \mathcal{M}_1(E_\varepsilon)$ satisfying

$$
\lim_{t \to \infty} d_\varepsilon(\text{Law}(X^{x,\varepsilon}_t), \mu^\varepsilon) = 0.
$$

We say that a system $(X^{x,\varepsilon}, \mu^\varepsilon, d_\varepsilon)_{\varepsilon > 0}$ exhibits a profile cutoff phenomenon at a cutoff time $t_\varepsilon \to \infty$, as $\varepsilon \to 0$, and cutoff window $w_\varepsilon = o(t_\varepsilon)$, if the following limit exists

$$
\lim_{\varepsilon \to 0} d_\varepsilon(\text{Law}(X_0^{x,\varepsilon}(x_\varepsilon), \mu^\varepsilon) = \mathcal{P}(\rho) \quad \text{for all} \quad \rho \in \mathbb{R},
$$

and additionally $\mathcal{P}(\infty) = 0$ and $\mathcal{P}(-\infty) = D$, where

$$
D = \lim_{\varepsilon \to 0} \sup_{t \geq 0} \text{Diameter}(\mathcal{M}_1(E_\varepsilon), d_\varepsilon) \in (0, \infty].
$$

This means that the time scale $t_\varepsilon$ is a temporal threshold in the sense that surfing ahead wave-front yields small values while lagging behind the threshold sees maximal values. In the classical example of Aldous and Diaconis [11, 2] of card shuffling, $\varepsilon = 1/n$ where $n$ represents the size of the deck of cards, $E_\varepsilon = S_n$ the space of card permutations (shufflings), $d_\varepsilon$ is the usual non-normalized total variation distance and the diameter $D$ equals 1.

We stress that such a profile cutoff phenomenon can occur even for systems without an intrinsic parameter $\varepsilon$, that is, $x_\varepsilon = x$, $X^{x,\varepsilon} = X^x$, $\mu^\varepsilon = \mu$ and $E_\varepsilon = E$ except in the renormalized distance $d_\varepsilon$. In the case of $d_\varepsilon = d/\varepsilon$ (diameter $D = \infty$) with a fixed distance $d$ on $\mathcal{M}_1(E)$ the parameter $\varepsilon$ plays the role of an external parameter which quantifies the abrupt convergence of the non-normalized distance in the following sense: for small $\varepsilon > 0$

$$
d(X^x_{t_\varepsilon + p \cdot w_\varepsilon}, \mu) \overset{\varepsilon \to 0}{\approx} \varepsilon \cdot \mathcal{P}(\rho) \approx \begin{cases} 
\varepsilon \cdot 0 = 0, & \text{as } \rho \to \infty, \\
\varepsilon \cdot D = \infty, & \text{as } \rho \to -\infty.
\end{cases}
$$

We denote this special type of profile cutoff phenomenon for non-parametrized systems as profile cutoff stability, since it gives a precise description of the asymptotics.

In the case that $(d_\varepsilon(\text{Law}(X_0^{x,\varepsilon}(x_\varepsilon), \mu^\varepsilon))_{\varepsilon > 0}$ has more than one accumulation point, which is generically the case (see [3, Theorem 3.2]), the natural generalization of the concept of profile cutoff phenomenon is the notion of the so-called window cutoff phenomenon, that is,

$$
\lim_{\rho \to \infty} \liminf_{\varepsilon \to 0} d_\varepsilon(\text{Law}(X_0^{x,\varepsilon}(x_\varepsilon), \mu^\varepsilon) = D,
$$

and

$$
\lim_{\rho \to \infty} \limsup_{\varepsilon \to 0} d_\varepsilon(\text{Law}(X_0^{x,\varepsilon}(x_\varepsilon), \mu^\varepsilon) = 0.
$$
In other words, the time scale $t_\varepsilon$ still splits large values from small values in the sense that a growing backward deviation measured in $w_\varepsilon$-units from $t_\varepsilon$ yields a maximal distance while a growing forward deviation measured in $w_\varepsilon$-units from $t_\varepsilon$ gives small values. In the analogous setting of profile cutoff stability (1.5) the notion of window cutoff stability reads as follows: there exist functions $\hat{\Phi}, \tilde{\Phi} : \mathbb{R} \to [0, \infty)$ such that for small $\varepsilon > 0$

$$d(X_t^{x} + \rho w_\varepsilon, \mu) \begin{cases} \leq \varepsilon \cdot \tilde{\Phi}(\rho) \to 0, & \text{as } \rho \to \infty, \\ \geq \varepsilon \cdot \hat{\Phi}(\rho) \to \infty, & \text{as } \rho \to -\infty. \end{cases}$$

We refer to the introductions [5, 8, 9, 10] for further details.

**Notation:** A matrix $U \in \mathbb{R}^{d \times d}$ is called Hurwitz stable ($U < 0$, for short), if its spectrum $\text{spec}(U) \subset \mathbb{C}_-$ for the open left complex half plane $\mathbb{C}_-$. For $U \in \mathbb{R}^{d \times d}$ let $U^*$ be the adjoint matrix of $U$ with respect to the standard Euclidean inner product. We define the Lie bracket or commutator by $[U, V] := UV - VU$ for $U, V \in \mathbb{R}^{d \times d}$.

**Lemma 1.1** (The asymptotics of Hurwitz stable matrix exponentials). For $Q \in \mathbb{R}^{d \times d}$ with $Q < 0$ we have the following. For any $y \in \mathbb{R}^d$, $y \neq 0$, there exist $q := q(y) > 0$, $\ell := \ell(y)$, $m := m(y) \in \{1, \ldots, d\}$, $\theta_1 := \theta_1(y), \ldots, \theta_m := \theta_m(y) \in \mathbb{R}$ and linearly independent vectors $v_1 := v_1(y), \ldots, v_m := v_m(y) \in \mathbb{C}^d$ such that

$$\lim_{t \to \infty} \left| \frac{e^{\ell t} \exp(tQ)}{t^{\ell-1}} y - \sum_{k=1}^{m} e^{it\theta_k} v_k \right| = 0,$$

and there are constants $K_0 := K_0(y) > 0$ and $K_1 := K_1(y) > 0$ such that

$$K_0 \leq \lim_{t \to \infty} \left| \sum_{k=1}^{m} e^{it\theta_k} v_k \right| \leq \lim_{t \to -\infty} \left| \sum_{k=1}^{m} e^{it\theta_k} v_k \right| \leq K_1,$$

where $i$ denotes the imaginary unit.

The preceding lemma is the main tool of this article. The lemma is established as Lemma B.1 in [3], p. 1195-1196, and proved there.

The manuscript is organized as follows. The main result, Theorem 2.3, and its proof are given in Section 2. In addition, the stronger property of profile cutoff stability is established in Corollary 2.7 and asymptotics for the so-called mixing times are given in Corollary 2.7.

## 2. CUTOFF CONVERGENCE FOR COMMUTING MATRICES $A$ AND $B$

We start with the hypotheses on the coefficient matrices $A$ and $B$.

**Hypothesis 2.1** (Normality of $B$). The matrix $B$ is normal, that is, $[B, B^*] = O$.

**Hypothesis 2.2** (Commutativity of $A$ and $B$). The matrices $A$ and $B$ commute, that is, $[A, B] = O$.

We note that by the Fuglede-Putnam-Rosenblum Theorem, [32, 12.16 Theorem], Hypothesis 2.1 and Hypothesis 2.2 imply $[A, B^*] = O$.

**Theorem 2.3** (Window cutoff stability for commuting coefficients). Let Hypothesis 2.1 and Hypothesis 2.2 be satisfied and assume that

$$Q = A + (B + B^*)^2/4 < 0.$$
Then for any $x \in \mathbb{R}^d$, $x \neq 0$, there are constants $q := q_x > 0$ and $\ell := \ell_x \in \mathbb{N}$ such that for
\begin{equation}
(2.1) \quad t_\varepsilon := \frac{|\ln(\varepsilon)|}{q} + \frac{(\ell - 1) \ln(|\ln(\varepsilon)|)}{q}, \quad \varepsilon > 0,
\end{equation}
and any $w > 0$ it follows the following window cutoff convergence
\begin{align*}
\lim_{\rho \to \infty} \lim_{\varepsilon \to 0} \frac{\mathbb{E}[|X_{t_\varepsilon + \rho w}(x)|^2]}{\varepsilon^2} = 0 \quad \text{and} \quad \lim_{\rho \to \infty} \lim_{\varepsilon \to 0} \frac{\mathbb{E}[|X_{t_\varepsilon + \rho w}(x)|^2]}{\varepsilon^2} = \infty.
\end{align*}

**Remark 2.4.** This result is the finite dimensional analogue of Theorem 5.1 for constant noise intensity in [17].

**Proof.** Hypothesis 2.2 implies by [25], Section 3.4 (iii) or [20], Theorem 1, the representation $X_t(x) = \exp(tA + W_t B)x$. Since $[A, B] = O$, it follows that
\[ [tA, W_t B] = tW_t[A, B] = O. \]

Hence the BCHD formula, yields $X_t(x) = \exp(W_t B) \exp(tA)x$ such that
\[ \mathbb{E}[|X_t(x)|^2] = x^* \exp(tA^*) \mathbb{E}[\exp(W_t B^*) \exp(W_t B)] \exp(tA)x. \]

The self-similarity in law $W_t \overset{d}{=} \sqrt{t}W_1$, $W_1 \overset{d}{=} \mathcal{N}(0, 1)$ standard normal, the diagonalization of the symmetric matrix $(B + B^*)^2$ and Hypothesis 2.1 imply
\[ \mathbb{E} [\exp(W_t B^*) \exp(W_t B)] = \mathbb{E} [\exp(\sqrt{t}W_1(B^* + B))] = \exp \left( t \frac{(B^* + B)^2}{2} \right). \]

Note that $[B, B^*] = [A, B] = [A, B^*] = O$ and thus
\[ [A, (B + B^*)^2] = (B + B^*)[A, (B + B^*)] + [A, (B + B^*)](B + B^*) = O \]
and also $[A^*, (B + B^*)^2] = O$. Hence, we obtain
\begin{equation}
(2.2) \quad \exp(tA^*) \mathbb{E}[\exp(W_t B^*) \exp(W_t B)] \exp(tA) = \exp(tQ^*) \exp(tQ).
\end{equation}

Consequently, $\mathbb{E}[|X_t(x)|^2] = |\exp(tQ)x|^2$. Since $Q < 0$, Lemma 1.1 yields for $\exp(tQ)x$ the $x$-dependent parameters $q > 0$, $\ell, m \in \{1, \ldots, d\}$, $\theta_1, \ldots, \theta_m \in \mathbb{R}$ and vectors $v_1, \ldots, v_m \in \mathbb{C}^d$ such that
\begin{equation}
(2.3) \quad 0 < K_0 \leq \lim_{t \to \infty} \left| \sum_{j=1}^m e^{i\theta_j v_j} \right| \leq \lim_{t \to \infty} \left| \sum_{j=1}^m e^{i\theta_j v_j} \right| \leq K_1 < \infty.
\end{equation}

Hence for $t_\varepsilon$ given in (2.1) and for fixed $\rho \in \mathbb{R}$, $s_\varepsilon := t_\varepsilon + \rho w$, we have by a straightforward calculation that
\[ \lim_{\varepsilon \to 0} \frac{e^{-qs_\varepsilon s_\varepsilon^{\ell-1}}}{\varepsilon} = \frac{e^{-q\rho w}}{q^{\ell-1}}. \]
Consequently, the right-hand side of (2.3) and an application of Lemma 1.1 to \( \exp(s Q) \) imply
\[
\lim_{\rho \to \infty} \lim_{\varepsilon \to 0} \frac{\mathbb{E}[|X_t + \rho w(x)|^2]}{\varepsilon^2} = \lim_{\rho \to \infty} \lim_{\varepsilon \to 0} \left( \frac{e^{-qs_x s^{-1}} e^{qs_x}}{s^{-1}} \right)^2 \exp(s Q) x \right) \leq \lim_{\rho \to \infty} \lim_{\varepsilon \to 0} \left( \frac{e^{-q \rho w}}{q^{-1}} \right)^2 \lim_{\varepsilon \to 0} \sum_{j=1}^m \left| e^{i \theta_j t s_j} \right|^2 
\]
\[
= \lim_{\rho \to \infty} \left( \frac{e^{-q \rho w}}{q^{-1}} \right)^2 \lim_{\varepsilon \to 0} \sum_{j=1}^m \left| e^{i \theta_j t s_j} \right|^2 = \lim_{\rho \to \infty} K^2 \left( \frac{e^{-q \rho w}}{q^{-1}} \right)^2 = 0.
\]
The proof of the lower bound of (2.4) follows analogously changing \( \lim \) by \( \lim \) and using the left inequality of (2.3). This finishes the proof. \( \square \)

**Corollary 2.5** (Profile cutoff stability).
Assume the hypotheses and notation of Theorem 2.3. Additionally, assume that \( A \) is diagonalizable. Then for any \( x \in \mathbb{R}^d, x \neq 0 \), there are constants \( q := q_x > 0 \) and \( v := v_x \in \mathbb{R}^d, v \neq 0 \), such that for any \( w > 0 \) and \( \rho \in \mathbb{R} \) it follows that
\[
\lim_{\varepsilon \to 0} \mathbb{E} |X_t(x + \rho w(x))|^2 = e^{-q \rho w} |v|.
\]

**Proof.** Note that \( [A, BB^*] = B[A, B^*] + B^*[A, B] = O \) and \( [A, B^* B] = B^*[A, B] + B[A, B^*] = O \). Hence \( [A, (B + B^*)^2] = O \) and \( (B + B^*)^2 \) is diagonalizable. Since \( A \) is diagonalizable, there is a joint base of eigenvectors for \( A \) and \( (B + B^*)^2 \) in (2.2). For given \( Q \) and \( x \neq 0 \), Lemma 1.1 yields the existence of \( x \)-dependent parameters \( q > 0 \) and \( v \in \mathbb{R}^d, v \neq 0 \), such that \( |e^{\theta t} \exp(tQ)x - v| \) tends to 0 as \( t \to \infty \), which implies the desired result in (2.4). \( \square \)

**Remark 2.6.** We stress that the Dirac measure at zero, \( \delta_0 \), is invariant for the dynamics (1.3) and hence \( \mathbb{E}[|X_t(x)|^2] = \mathcal{W}_2^2(X_t(x), \delta_0) \), where \( \mathcal{W}_2 \) is the standard Wasserstein-Kantorovich-Rubinstein distance of order 2, see [37]. Moreover, the map
\[
t \to \mathcal{W}_2(X_t(x), \delta_0)
\]
is known to be non-increasing, see for instance Lemma B.3 (Monotonicity) in [12].

The following result connects the cutoff stability with the notion of mixing times with respect to the Wasserstein-Kantorovich-Rubinstein-2-distance and the respective cutoff phenomenon in the sense of Levin, Peres and Wilmer given in Chapter 18 of [23], see the definition by (18.3) and Lemma 18.1.

**Corollary 2.7** (Asymptotic \( \varepsilon \)-mixing time).
Assume the hypotheses and notation of Theorem 2.3. Given \( \delta > 0 \) we define the \( \delta \)-mixing time as follows.
\[
\tau^\varepsilon_\delta (\delta) := \inf \left\{ t \geq 0 : \frac{\mathbb{E} [ |X_t(x)|^2 ]}{\varepsilon^2} \leq \delta \right\}.
\]
Then for any \( M > \delta \) it follows that
\[
\lim_{\varepsilon \to 0} \frac{\tau^\varepsilon_\delta (\delta)}{\tau^\varepsilon_\delta (M - \delta)} = 1 \quad \text{and} \quad \lim_{\varepsilon \to 0} \frac{\tau^\varepsilon_\delta (\delta)}{t_\varepsilon} = 1.
\]
Proof. We start by noticing that Theorem 2.3 implies
\[
\lim_{\varepsilon \to 0} \frac{\mathbb{E}[|X_{c,t_\varepsilon}(x)|^2]}{\varepsilon^2} = \begin{cases} 
\infty & \text{if } c \in (0, 1), \\
0 & \text{if } c > 1.
\end{cases}
\] 
Let $\delta > 0$ be fixed and choose $c > 1$. Then (2.7) yields the existence of $\varepsilon_0 := \varepsilon_0(\delta, c) > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ it follows
\[
\frac{\mathbb{E}[|X_{c,t_\varepsilon}(x)|^2]}{\varepsilon^2} \leq \delta.
\] 
By (2.5) we infer $\tau^x_\varepsilon(\delta) \leq c \cdot t_\varepsilon$ for all $\varepsilon \in (0, \varepsilon_0)$. Conversely, for any $M > \delta$ and $1/c \in (0, 1)$ there exists $\varepsilon_1 := \varepsilon_1(M, \delta, c) > 0$ such that for all $\varepsilon \in (0, \varepsilon_1)$ it follows
\[
\frac{\mathbb{E}[|X_{(1/c),t_\varepsilon}(x)|^2]}{\varepsilon^2} > M - \delta
\] 
and by (2.5) we infer $(1/c) \cdot t_\varepsilon \leq \tau^x_\varepsilon(M - \delta)$ for all $\varepsilon \in (0, \varepsilon_1)$. Therefore, we have
\[
\limsup_{\varepsilon \to 0} \frac{\tau^x_\varepsilon(\delta)}{\tau^x_\varepsilon(M - \delta)} \leq c^2
\] 
for all $c > 0$. Sending $c \to 1$ we obtain the upper bounds in (2.6). The lower bounds follow similarly.

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