A new approach to modified-gravity models

Sayan K. Chakrabarti∗
Theory Division, Saha Institute of Nuclear Physics, 1/AF Bidhannagar, Kolkata 700064, India

Emmanuel N. Saridakis†
Department of Physics, University of Athens, GR-15771 Athens, Greece

Anjan A. Sen‡
Center For Theoretical Physics, Jamia Millia Islamia, New Delhi 110025, India

We investigate $f(R)$-gravity models performing the ADM-slicing of standard General Relativity. We extract the static, spherically-symmetric vacuum solutions in the general case, which correspond to either Schwarzschild de-Sitter or Schwarzschild anti-de-Sitter ones. Additionally, we study the cosmological evolution of a homogeneous and isotropic universe, which is governed by an algebraic and not a differential equation. We show that the universe admits solutions corresponding to acceleration at late cosmological epochs, without the need of fine-tuning the model-parameters or the initial conditions.

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I. INTRODUCTION

Recently, there has been a tremendous thrust in research activities in cosmology, due to the wealth of data from various experiments that are already available and more that are anticipated in the near future. Amongst others, the discovery of the late time acceleration of the universe has been particularly fascinating for cosmologists, particle physicists and string theorists alike. In particular, the significant improvement of the data-statistics (from 50 to more than 300 Supernova(SnIa) [1] has made the aforementioned result indisputable. Furthermore, complementary probes like Baryon Acoustic Oscillation [2] and Cosmic Microwave Background Radiation measurements, [3] offer additional evidences.

Although the simplest candidate of the acceleration mechanism is the cosmological constant, one can construct various “field” models in order to incorporate this mysterious “dark energy” In particular, one can use a canonical scalar field (quintessence) [4], a scalar field with non-standard kinetic term (k-essence) [5] or with a negative sign of the kinetic term (phantom) [6], the combination of quintessence and phantom in a unified model named quintom [7], scalar fields non-minimally coupled to gravity [8] or simple barotropic fluids with specific pressure-form such as Chaplygin gas [9] (for nice reviews on dark energy models, see [10]).

Instead of using field dark energy constructions, an alternative approach is to modify gravity itself. The Dvali-Gabadadze-Poratti (DGP) [11], the Cardassian [12] and the Shtanov-Sahni [13] models follow this direction, lying in particular in the higher-dimensional sub-class. However, instead of using extra dimensions, one can insert higher-order curvature invariants in the usual Einstein-Hilbert action, with the simple consideration the so-called $f(R)$-gravity models [14], that is the addition of Ricci-scalar functions. Such models wish to alleviate the non-renormalizability of gravity, and acquire theoretical justification from low-energy string theory. Alternatively, since higher-order terms can be related to non-minimally coupled scalar degrees of freedom, these models are equivalent to scalar-tensor constructions. Concerning their cosmological implications, they can describe both inflation as well as the late-time acceleration. However, most of $f(R)$-gravity models do not manage to pass the observational and theoretical tests (solar system, neutron stars and binary pulsar constraints), giving also rise to an unusual matter dominated epoch and leading to significant fine-tunings [15]. Moreover, even if improved versions (like the “Chameleon mechanism” [16]) manage to satisfy the above constraints, one can still face problems in the strong gravity regime, due to the curvature singularity that is inevitable at the nonlinear level [17].

Although higher time-derivatives can be beneficial in making gravity renormalizable, they also lead to ghosts. However, the recently developed Horava gravity wishes to act as a power-counting renormalizable, Ultra-Violet (UV) complete theory of gravity [18], without possessing the full diffeomorphism invariance of General Relativity but only a subset that is manifest in the Arnowitt, Deser and Misner (ADM) slicing. There has been a large amount of effort in examining and extending its properties, as well as exploring its cosmological implications [19].

Motivated by these, in the present work we are interested in investigating $f(R)$ gravity models performing the ADM slicing of standard General Relativity, that is its (3+1)-decomposition based on the Hamiltonian formulation [20]. In particular, we wish to extract the
static, spherically-symmetric vacuum solution for general \(f(R)\)-models under ADM decomposition, and study the homogeneous and isotropic cosmological solutions which present late-time acceleration. The paper is organized as follows: In section II we present a brief introduction on ADM (3+1)-decomposition and we write the gravitational action for \(f(R)\)-gravity models. In section III we derive the static, spherically-symmetric vacuum solutions for general \(f(R)\) models under ADM slicing. In section IV we apply this approach to cosmological frameworks, investigating universe evolutions that experience late-time acceleration. Finally, section V is devoted to the summary of the obtained results.

II. ADM (3+1)-DECOMPOSITION AND \(f(R)\) GRAVITY MODELS

We start by considering the ADM decomposition of the four dimensional metric [28]:

\[
ds^2 = -N^2dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt).
\]

(1)

Here \(N\) and \(N^i\) are the lapse function and shift vectors respectively, while \(g_{ij}\) is the induced metric on the 3-dimensional space-like hypersurface for a fixed time. Indices of all projected tensors can be lowered (raised) by \(g_{ij}\) (\(g^{ij}\)).

For the space-like hypersurface with fixed time, extrinsic curvature is defined as

\[
K_{ij} = \frac{1}{2N}(g_{ij} - \nabla_i N_j - \nabla_j N_i),
\]

(2)

where \(\text{dot}\) represents derivative with respect to time and \(i, j = 1, 2, 3\). Under such decomposition,

\[
R^{(4)} = R^{(3)} + K_{ij}K^{ij} - K^2,
\]

(3)

where \(R^{(4)}\) and \(R^{(3)}\) are the four and three-dimensional Ricci scalars respectively. Hence, the Einstein-Hilbert action can now be written as

\[
S = \frac{1}{16\pi G} \int dt d^3x \sqrt{g^3} (R^{(3)} + K_{ij}K^{ij} - K^2) + S_m,
\]

(4)

where \(g^3\) is the determinant of the three-dimensional space-like hypersurface, and \(S_m\) accounts for the matter content of the universe. Generalizing the above action, one can write the action for an \(f(R)\)-gravity model as

\[
S = \frac{1}{16\pi G} \int dt d^3x \sqrt{g^3} f(R^{(3)} + K_{ij}K^{ij} - K^2) + S_m.
\]

(5)

The action is in general function of \(N\), \(N^i\) and \(g_{ij}\) and hence varying the action with respect to these quantities leads to the equations of motion. In our subsequent calculations we assume \(f(R^{(4)}) = R^{(4)} + F(R^{(3)})\), thus the above action becomes,

\[
S = \frac{1}{16\pi G} \int dt d^3x \sqrt{g^3} [R^{(3)} + K_{ij}K^{ij} - K^2] + F(R^{(3)} + K_{ij}K^{ij} - K^2) + S_m.
\]

(6)

III. STATIC SPHERICALLY SYMMETRIC SOLUTIONS

We are looking for the static, spherically-symmetric vacuum solutions of the aforementioned general \(f(R)\) gravity models, under ADM decomposition. In this case the metric (1) writes:

\[
ds^2 = -N^2(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2d\Omega^2,
\]

(7)

that is \(K_{ij} = 0\) and \(R^{(3)} = -\frac{2}{r^2} [f'(r) + f(r) - 1]\). Setting \(f(r) - 1 = X(r)\), we obtain \(f'(r) + f - 1 = rX'(r) + X(r)\). Thus, defining \(\mathcal{H}(r)\) as

\[
R^{(3)} = \frac{2}{r^2} [X(r)r]' \equiv \mathcal{H}(r),
\]

(8)

the action (6) after angular integration reads

\[
S = \frac{1}{16\pi G} \int dt d^3x \frac{N(r)r^2}{\sqrt{f(r)}} [\mathcal{H}(r) + F(\mathcal{H}(r))].
\]

(9)

Varying (9) with respect to \(N(r)\) and setting \(\delta S/\delta N(r) = 0\) we obtain

\[
\mathcal{H}(r) + F(\mathcal{H}(r)) = 0.
\]

(10)

This is an algebraic equation for \(\mathcal{H}(r)\) depending on the functional form of \(F\). One can always solve this equation to get

\[
\mathcal{H}(r) = \text{constant},
\]

(11)

which depends upon the functional form of \(F(\mathcal{H}(r))\). Denoting the above constant by the parameter \(\beta\), we can obtain

\[
\mathcal{H}(r) \equiv - \frac{2}{r^2} [X(r)r]' = \beta.
\]

\[
\Rightarrow X(r) = - \frac{\beta r^2}{6} + \frac{A}{r},
\]

(12)

where \(A\) is the integration constant, set from now on to \(-2M\) with \(M\) a new constant. Thus,

\[
f(r) = 1 - \frac{2M}{r} - \frac{\beta r^2}{6}.
\]

(13)

Variation of (9) with respect to \(f(r)\) leads to

\[
\frac{d}{dr} \left( \frac{\partial L}{\partial f'} \right) - \frac{\partial L}{\partial f} = 0,
\]

(14)

where

\[
L = \frac{N(r)r^2}{\sqrt{f}} \left[ \mathcal{H}(f, f', r) + F(\mathcal{H}(f, f', r)) \right].
\]

(15)

A straightforward calculation gives

\[
\frac{\partial L}{\partial f} = - \frac{1}{2} \frac{N(r)r^2}{f^{3/2}} \left[ \mathcal{H} + F(\mathcal{H}) \right]
\]

\[
+ \frac{N(r)r^2}{\sqrt{f}} \left( \frac{\partial \mathcal{H}}{\partial R^{(3)}} + \frac{\partial F(\mathcal{H})}{\partial R^{(3)}} \right) \frac{\partial R^{(3)}}{\partial f},
\]

(16)
which under \( \text{(10)} \) leads to
\[
\frac{\partial L}{\partial f'} = \frac{2N(r)}{\sqrt{f}} \left[ 1 + \frac{\partial F(H)}{\partial R^{(3)}} \right].
\] (17)

Similarly, we acquire
\[
\frac{\partial L}{\partial f'} = \frac{N(r)r^2}{\sqrt{f}} \left[ \frac{\partial H}{\partial R^{(3)}} + \frac{\partial F(H)}{\partial R^{(3)}} \right] \frac{\partial R^{(3)}}{\partial f'}
\]
\[
= -\frac{2N(r)r}{\sqrt{f}} \left[ 1 + \frac{\partial F(H)}{\partial R^{(3)}} \right].
\] (18)

In conclusion, inserting \( \text{(10)}, \text{(18)} \) into \( \text{(14)} \), and using that \( 1 + \frac{\partial F(H)}{\partial R^{(3)}} \) is constant (arising from the solution for \( f(r) \) in \( R^{(3)} \)) we finally obtain
\[
\frac{d}{dr} \left( \frac{N(r)}{\sqrt{f}} \right) = 0 \Rightarrow N^2(r) = f(r).
\] (19)

This result, together with equation \( \text{(13)} \), shows that the static, spherically-symmetric vacuum solution for a general \( f(R) \) model under ADM decomposition is either a Schwarzschild de-Sitter or Schwarzschild anti-de-Sitter one, depending upon the choice of \( \beta \).

**IV. HOMOGENEOUS AND ISOTROPIC COSMOLOGICAL EVOLUTION**

Here we assume that the background is homogeneous and isotropic and the spatial 3-hypersurface is flat \( (k = 0) \):
\[
ds^2 = -N^2(t)dt^2 + a^2(t)(dx^2 + dy^2 + dz^2),
\] (20)
where \( a(t) \) is the scale factor and \( N(t) \) is the lapse function. In order to present an example of explicit cosmological solutions, we consider the form for \( f(R) \) proposed by Starobinsky in \( \text{(21)} \):
\[f(R) = \lambda R_0 \left[ \left( 1 + \left( \frac{R}{R_0} \right)^2 \right)^{-n} - 1 \right], \] (21)

where \( \lambda, R_0 \) and \( n \) are the model parameters, and from now on, \( R \) denotes the four dimensional Ricci scalar. The advantage of choice \( \text{(21)} \) is that it accepts a theoretical justification, but one could also use a different ansatz at will. Using this choice for \( f(R) \), and under the metric \( \text{(20)} \), the action \( \text{(6)} \) becomes
\[
S = \frac{1}{16\pi G} \int dt d^3x N \sqrt{g^{(3)}} \left\{ \lambda R_0 \left[ \left( 1 + \frac{36H^4}{N^4R_0^2} \right)^{-n} - 1 \right] \right\} \left\{ \frac{6H^2}{N^2} \right\} + S_m,
\] (22)

where \( H \equiv \dot{\delta} \) is the Hubble parameter. Finally, as usual, the energy density and pressure for the matter field are respectively defined as
\[
\rho_m = \frac{1}{\sqrt{g^{(3)}}} \frac{\delta S_m}{\delta N}, \]
\[
g_{ij}p_m = -\frac{2}{N} \sqrt{g^{(3)}} \frac{\delta S_m}{\delta g^{ij}}.
\] (23)

Variation of action \( \text{(22)} \) with respect to \( N \), and the subsequent fixing \( N = 1 \) (as it is usual in cosmological applications of the “foliation preserving” framework), leads to the “effective” Friedmann equation
\[
H^2 + \frac{\lambda R_0}{6} \left[ \left( 1 + \frac{36H^4}{R_0^2} \right)^{-n} - 1 \right] + \frac{24\lambda nH^4}{R_0} \left[ \left( 1 + \frac{36H^4}{R_0^2} \right)^{-n+1} \right] = \frac{8\pi G}{3} \rho_m.
\] (25)

According to the usual approach, one could also vary the action \( \text{(22)} \) with respect to the second variable \( g_{ij} \). This procedure, although straightforward, leads to a complicated result, which forbids its physical use. Alternatively, we prefer to use the matter conservation equation:
\[
\rho_m + 3H(\rho_m + p_m) = 0,
\] (26)
assuming for simplicity, and without loss of generality, the matter to be dust \( (p_m = 0) \), which is definitely justified for late-time cosmological behavior. In this case, \( \text{(26)} \) leads to the usual evolution \( \rho_m = \rho_m(0) \alpha^{-3} \), with \( \rho_m(0) \) the matter energy density at present, where the scale factor is fixed to 1. Therefore, inserting this formula into \( \text{(25)} \) we result to
where we have defined $\alpha \equiv \frac{n}{H_0}$ with $H_0$ the current $H$-value and $\Omega_{m0}$ the matter density-parameter at present. Thus, the parameter $\alpha$ accounts for the modification of gravity. Finally, note that imposing (27) at present, that is taking $\alpha = 1$ and $H = H_0$, allows for the elimination of the parameter $\lambda$ in favor of $\alpha$, $n$ and $\Omega_{m0}$, namely:

$$\lambda = \frac{\Omega_{m0} - 1}{\alpha \left[ 1 + \frac{36H_0^4}{R_0^2} \right]^{-n} - 1} + \frac{24\lambda n}{\alpha} \left( \frac{H}{H_0} \right)^4 \left[ 1 + \frac{36H_0^4}{R_0^2} \right]^{-(n+1)} = \Omega_{m0}a^{-3}. \quad (27)$$

Equation (27) is the modified Friedmann equation of the modified-gravity model at hand, and contains all the cosmological information of the system. It presents the significant advantage that it does not contain any higher-order time-derivatives that appear in the usual approach of $f(R)$-gravity models. Instead, and due to ADM decomposition, it is just an algebraic equation for the Hubble-parameter $H(a)$, although not simple form. Therefore, one can examine its solutions for various values of $n$, which in turn can be used to construct all the observable quantities like deceleration parameter, luminosity distance, angular diameter distance etc.

In particular, knowledge of $H(a)$ allows for a straightforward calculation of $dH(a)/da = H'(a)$, while for every quantity $Q$ we obtain $\dot{Q} = Q'(a)aH(a)$. Therefore, for the deceleration parameter $q \equiv -\ddot{a}/(aH(a))^2$ we acquire:

$$q(a) = -1 - \frac{a}{H(a)}H'(a). \quad (29)$$

As usual, $q < 0$ corresponds to $\ddot{a} > 0$ that is to an accelerating universe, while $q > 0$ corresponds to a decelerating one. Finally, $q < -1$ corresponds to $\dot{H} > 0$, which is the case of a super-accelerating universe $[22]$.

As it is usual for modified gravity models, the role of matter is crucial for the determination of the cosmological behavior. For instance, in the complete absence of matter, that is setting $\Omega_{m0}$ to zero, (27) implies immediately that $H$ is independent of $a$, that is $H'(a) = 0 = \dot{H}$ and $q = -1$ which is just what is expected in this case. In the following we focus on the realistic case where $\Omega_{m0} \approx 0.3$.

As can be seen from (27), even for $n = 1$ the equation for $H(a)$ is of high order, giving rise to many solution branches, the number of which increases fast with increasing $n$. Some of these solution-branches lead to imaginary $H(a)$ and thus are not physical. Additionally, one gets solution-branches that lead to divergencies in finite scale factors in the past, which must also be omitted. We are interested in those branches that have $H(a) > 0$ at all $a$ and $q(a) < 0$ at large $a$, that is corresponding to an expanding universe which accelerates at late cosmological epochs. One significant advantage of the aforementioned procedure is that since we do not solve any differential equation, we do not have to face the discussion of the initial-condition determination. We only have to fix the values of $H_0$, $\Omega_{m0}$ at present, and then the system is fully determined for particular values of $n$ and $\alpha$.

Let us first investigate the $n = 1$ solution subclass. In fig. 1 we depict the deceleration parameter $q(a)$ as it is given by (29) for the numerically obtained solution of the algebraic equation (27) for $n = 1$ with $\Omega_{m0} = 0.3$. The curves correspond to four values of the parameter $\alpha$.

As we observe, in all cases we do obtain acceleration at late times, with the transition to the accelerated phase realized at earlier times for larger $\alpha$-values, that is for more significant gravity-modification. It is interesting to notice that for large values of $\alpha$, the behavior of $q(a)$ can be non-monotonic (as can be seen in the $\alpha = 10$ curve), and going to even larger values ($\alpha = 20$) it brings deceleration at very late times. Since this scenario is not favored by observations we do not show it explicitly, but it is characteristic of the rich phenomenology and cosmological possibilities that our model presents. Finally, we mention that in all cases $q(a)$ is larger than $-1$, that is the imposed $f(R)$-ansatz does not seem to be able to lead to a super-accelerating universe.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig1.png}
\caption{The deceleration parameter $q(a)$ for an expanding universe for $n = 1$, with $\Omega_{m0} = 0.3$ and $H_0 = 1$, $\alpha_0 = 1$. The curves correspond to $\alpha = 0.1$ (black-solid), $\alpha = 1$ (red-dashed), $\alpha = 5$ (green-dotted) and $\alpha = 10$ (blue-dashed-dotted). The horizontal line marks the $q = 0$ bound.}
\end{figure}
In fig. 2 we depict the $q$-behavior for the $n = 2$ solution subclass. In this case, the curves for $\alpha = 0.1$ and $\alpha = 1$ have a small difference (not observed in the scale of the figure), and one needs to go to larger $\alpha$ in order to see a different cosmological evolution (which is achieved fast for $\alpha$ becoming larger that 1). As we see, we again obtain acceleration at late cosmological epochs. Note that for $\alpha = 5$, the transition to the acceleration phase is realized earlier than for $\alpha = 10$, as a result of the highly nonlinear behavior of equation (27).

In fig. 3 we depict the $q$-behavior for the $n = 3$ solution subclass. Similarly to the previous cases, we do obtain late-time acceleration, with $q(a)$ for large $\alpha$ being nonmonotonic. Finally, we mention that qualitatively similar results arise for larger values of $n$ too, but for simplicity we do not present them explicitly.

The aforementioned solutions correspond to $H(a) > 0$ at all $a$’s, that is to an expanding universe. However, for completeness we mention that the present model allows also for solutions that describe a contracting universe. Indeed, since (27) is a even equation for $H(a)$, we deduce that for every $H(a)$-solution, $-H(a)$ is also a solution. Therefore, while the investigated solutions of figures 1 to 3 possess $H(a) > 0$ at all scale factors, the same $q(a)$-behavior arise from the corresponding branches with $H(a) < 0$ at all $a$, that is a contracting universe.

V. CONCLUSIONS

In the present work we have studied $f(R)$-gravity models performing the ADM slicing of standard General Relativity, that is its (3+1)-decomposition based on the Hamiltonian formulation. This approach allows for an easier treatment of modified-gravity systems and for the extraction of their general theoretical and cosmological implications.

As a first application we derived the static, spherically-symmetric vacuum solutions for general $f(R)$-ansatze. As we saw, they correspond to either Schwarzschild de-Sitter or Schwarzschild anti-de-Sitter ones, depending upon the choice of a particular parameter $\beta$.

Concerning applications in a cosmological framework, we investigated the evolution of a homogeneous and isotropic flat universe. Imposing as a specific example a particular ansatz for $f(R)$, we showed that the Hubble parameter is given by an algebraic equation in terms of the scale factor, with a new parameter that determines the modification of gravity. This fact is an advantage since one does not need to discuss the initial condition, since all the information is included in the value of the matter density parameter at present. The system accepts many solution branches, the physical sub-class of which corresponds to either expanding or contracting universes. Furthermore, one can easily acquire solutions that correspond to acceleration at late cosmological epochs, in agreement with observations, and this is achieved without the need of fine tuning the model-parameters or the initial conditions. The model at hand presents rich cosmological behavior, and moreover one can calculate additional observables such as luminosity distance and angular diameter distance.

ADM decomposition, which is crucial in General Relativity and its possible quantum-gravitational extensions such as Hořava theory, proves very convenient in studying modified-gravity constructions in general.
One could apply this formalism in order to extract the black-hole properties, to study the gravitational wave spectrum, to examine the cosmological bounce, etc. Finally, one could use observational data in order to constraint the possible gravity modifications, both qualitatively and quantitatively. Such a research is left for a future investigation.

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