Derivation of the Maxwell-Schrödinger Equations: A note on the infrared sector of the radiation field

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Abstract

We slightly extend prior results about the derivation of the Maxwell-Schrödinger equations from the bosonic Pauli-Fierz Hamiltonian. More concretely, we show that the findings from [25] about the coherence of the quantized electromagnetic field also hold for soft photons with small energies. This is achieved with the help of an estimate from [3] which proves that the domain of the number of photon operator is invariant during the time evolution generated by the Pauli-Fierz Hamiltonian.

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I Introduction

In this short paper we derive the Maxwell-Schrödinger system of equations as an effective model describing a Bose-Einstein condensate of charged particles immersed in a coherent electromagnetic field. More precisely, we prove quantitatively that the Maxwell-Schrödinger system approximates well the many-body quantum evolution generated by the Pauli-Fierz Hamiltonian; provided that the total number of particles \( N \) is large, the particles are initially in a Bose-Einstein condensate and that the quantum nature of the field – quantified by the semiclassical parameter \( \hbar \) – is negligible. In particular, we focus on the combined regime \( N \sim \frac{1}{\hbar} \to +\infty \). Equivalently, the same effective dynamics approximates well \( N \to \infty \) bosons weakly interacting with a quantized electromagnetic field, see the discussion below.

This problem has already been studied by one of the authors, together with P. Pickl, in [25]. The main focus here is to build on the results and techniques introduced there, and to strengthen them by studying convergence for the photons’ reduced density matrix. In the previous work the quantum fluctuations around the coherent state of photons have been classified only by means of their energy. The extension to the reduced density matrix is physically relevant and mathematically nontrivial because the coherence of photons with small frequencies can not be shown by the energy of the electromagnetic field, due to its massless nature. We often refer to [25] throughout the paper, hopefully striking a good balance between being concise and being self-contained.

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I.1 The Maxwell-Schrödinger System of Equations

The Maxwell-Schrödinger system of equations describes the wave function \( \varphi \) of a quantum particle (with a nontrivial charge distribution \( \kappa \)) interacting with the classical electromagnetic field, described by the vector potential \( \mathbf{A} \) and the electric field \( \mathbf{E} = -\mathbf{\nabla}\kappa \). We choose the Coulomb gauge

\[
\nabla \cdot \mathbf{A} = 0 \tag{I.1}
\]

and this makes indeed \( \mathbf{A} \) and \( \mathbf{E} \) the only dynamical degrees of freedom of the field. Let us also preliminarily define the current

\[
\mathbf{j} = 2(\text{Im}(\bar{\varphi} \nabla \varphi) - \bar{\varphi}(\kappa \ast \mathbf{A}) \varphi) \tag{I.2}
\]

The Maxwell-Schrödinger system thus takes the form

\[
\begin{cases}
  i\partial_t \varphi = (-i\nabla - (\kappa \ast \mathbf{A}))^2 \varphi + \mathcal{V}[\varphi] \\
  \partial_t \mathbf{A} = -\mathbf{E} \\
  \partial_t \mathbf{E} = -\Delta \mathbf{A} - (1 - \nabla(\nabla \cdot (\Delta^{-1}))(\kappa \ast \mathbf{j})
\end{cases} \tag{I.3}
\]

where \( \mathcal{V}[\varphi] \) is an interaction term for the quantum particle. The choice of gauge, Coulomb’s, in this case, can be seen as a constraint, for it is preserved by the Maxwell-Schrödinger flow. A typical example for the particle interaction \( \mathcal{V}[\varphi] \) could be

\[
\mathcal{V}[\varphi] = (W + v \ast |\varphi|^2) \varphi , \tag{I.4}
\]

where \( W \) is an external potential and \( (v \ast |\varphi|^2) \varphi \) a nonlinear term, usually originating from a microscopic pair interaction. The Cauchy problem associated to (I.3) is obtained by fixing an initial datum \( (\varphi_0, \mathbf{A}_0, \mathbf{E}_0) \), subjected to the constraint \( \nabla \cdot \mathbf{A}_0 = 0 \). In order to do so, it is convenient to introduce the complex scalar fields \( (\alpha_0(\cdot, \lambda))_{\lambda=1,2} \) by defining

\[
\mathbf{A}_0(x) = \frac{1}{(2\pi)^{3/2}} \sum_{\lambda=1,2} \int d^3k \frac{1}{\sqrt{2|k|}} \epsilon_\lambda(k) \left( e^{ikx} \alpha_0(k, \lambda) + e^{-ikx} \overline{\alpha_0(k, \lambda)} \right) , \tag{I.5}
\]

\[
\mathbf{E}_0(x) = \frac{i}{(2\pi)^{3/2}} \sum_{\lambda=1,2} \int d^3k \sqrt{\frac{|k|}{2}} \epsilon_\lambda(k) \left( e^{ikx} \alpha_0(k, \lambda) - e^{-ikx} \overline{\alpha_0(k, \lambda)} \right) , \tag{I.6}
\]

where \( (\epsilon_\lambda(k))_{\lambda=1,2} \) are the polarization vectors satisfying

\[
\epsilon_\lambda(k) \cdot \epsilon_\mu(k) = \delta_{\mu \lambda} , \quad k \cdot \epsilon_\lambda(k) = 0 , \tag{I.7}
\]

that implement the Coulomb gauge. In fact, there is a unique such decomposition for any time, \( t \), i.e.,

\[
\begin{align*}
\mathbf{A}(x, t) &= \frac{1}{(2\pi)^{3/2}} \sum_{\lambda=1,2} \int d^3k \frac{1}{\sqrt{2|k|}} \epsilon_\lambda(k) \left( e^{ikx} \alpha(t, \lambda) + e^{-ikx} \overline{\alpha(t, \lambda)} \right) , \tag{I.8} \\
\mathbf{E}(x, t) &= \frac{i}{(2\pi)^{3/2}} \sum_{\lambda=1,2} \int d^3k \sqrt{\frac{|k|}{2}} \epsilon_\lambda(k) \left( e^{ikx} \alpha(t, \lambda) - e^{-ikx} \overline{\alpha(t, \lambda)} \right) . \tag{I.9}
\end{align*}
\]
that respects both the Coulomb gauge and $\dot{\mathbf{A}} = -\mathbf{E}$. This makes it possible to consider the equivalent system

$$
\begin{aligned}
\begin{cases}
  i\partial_t \varphi_t &= (-i\nabla - \kappa \ast \mathbf{A}(\cdot, t))^2 \varphi_t + \mathcal{V}[\varphi], \\
  i\partial_t \alpha_t(k, \lambda) &= |k|\alpha_t(k, \lambda) - \frac{4\pi}{|k|} F[\kappa](k) \epsilon_{\lambda}(k) \cdot F[j_t](k), \\
  \mathbf{A}(x, t) &= (2\pi)^{-3/2} \sum_{\lambda=1,2} \int d^3 k \frac{1}{\sqrt{2|k|}} \epsilon_{\lambda}(k) \left( e^{ikx} \alpha_t(k, \lambda) + e^{-ikx} \overline{\alpha_t(k, \lambda)} \right), \\
  j_t &= 2 \left( \text{Im}(\varphi_t^* \nabla \varphi_t) - |\varphi_t|^2 \kappa \ast \mathbf{A}(\cdot, t) \right)
\end{cases}
\end{aligned}
$$

(1.10)

with initial datum $(\varphi_0, \alpha_0(\cdot, 1), \alpha_0(\cdot, 2))$. As it will be clarified shortly, the latter appear naturally as the effective counterparts of the microscopic dynamical variables. Note that the energy functional of the Maxwell-Schrödinger system is given by

$$
\mathcal{E}_M[\varphi, \alpha] := \|(-i\nabla - (\kappa \ast \mathbf{A})) \varphi\|^2 + \frac{1}{2} \langle \varphi, (v \ast |\varphi|^2) \varphi \rangle + \sum_{\lambda=1,2} \int d^3 k |k| |\alpha(k, \lambda)|^2
$$

(1.11)

with $\mathbf{A}$ being defined in analogy to (1.5). Global well-posedness for the Maxwell-Schrödinger system with $\mathcal{V}[\varphi] = v \ast |\varphi|^2 \varphi$, $\kappa(x) = e \delta(x)$ (where $e$ is the electric charge of the Schrödinger particle), and $v(x) = \frac{e^2}{|x|^2}$ has been proven in [4, 29]. We will also consider only the case $\mathcal{V}[\varphi] = v \ast |\varphi|^2 \varphi$, but we may require the charge distribution $\kappa$ to be extended, in order to well-define the microscopic system, as discussed below. Typical examples of charge distributions that we will consider are of the form

$$
\kappa(x) = \frac{e}{\sigma^3 (2\pi)^{3/2}} e^{-\frac{x^2}{2\sigma^2}},
$$

(1.12)

representing a charged particle with total charge $e \in \mathbb{R}$, distributed in a Gaussian fashion ("smoothed" spherical distribution of "diameter" $\sigma$); or

$$
\kappa(x) = e \frac{\mathcal{F}[1_{|\cdot| \leq \Lambda}](x)}{(2\pi)^{3/2}},
$$

(1.13)

where $\mathcal{F}[\cdot]$ stands for the Fourier transform, representing a sharp cutoff in momentum space, with total charge $e \in \mathbb{R}$. Let us remark that globally neutral particles can be considered, as long as they have a nontrivial charge distribution: for example,

$$
\kappa(x) = \begin{cases}
\frac{e}{(2\pi)^{3/2}} & \text{if } x = (x_1, x_2, x_3), \ x_1 \geq 0 \\
-\frac{e}{(2\pi)^{3/2}} & \text{if } x = (x_1, x_2, x_3), \ x_1 < 0
\end{cases},
$$

(1.14)

yields null total charge but nontrivial dipole, quadrupole, etc. interactions with the electromagnetic field.

If the charge distribution is not concentrated in a single point, and the potential $v$ represents an electrostatic mean-field self-interaction, then the form of the latter changes as well: a physically sensible choice would be $v = \kappa \ast \frac{1}{|\cdot|^2} \ast \kappa$. We will allow some liberty in the choices of $\kappa$ and $v$; the specific requirements on the two will be made precise in Assumption 1.1 below. Concerning global well-posedness, let us remark that compared to the literature [4, 29] our choices for $\kappa$ and $v$ will be, at most, “better” (i.e., more regular) and therefore do not affect the proof in any way since both $\kappa$ and $v$ act by convolution in the equation.

**Proposition I.1** (4). *The Maxwell-Schrödinger system in Coulomb’s gauge (1.3) (with variables $\varphi, \mathbf{A}, \mathbf{E}$) is globally well-posed in $H^1 \times H^1 \times L^2$. More precisely,*
1. (Regular solutions – [29]) For every
\[(\varphi_0, A_0, E_0) \in H^2 \times H^2 \times H^1,\]
there exists a unique global solution
\[(\varphi, A) \in (C^0(\mathbb{R}, H^2) \cap C^1(\mathbb{R}, L^2)) \times (C^0(\mathbb{R}, H^2) \cap C^1(\mathbb{R}, H^1) \cap C^2(\mathbb{R}, L^2))\]
of the Cauchy problem associated to (1.3).

2. (Rough solutions) For every
\[(\varphi_0, A_0, E_0) \in H^1 \times H^1 \times L^2,\]
there exists a unique global solution
\[(\varphi, A) \in C^0(\mathbb{R}, H^1) \times (C^0(\mathbb{R}, H^1) \cap C^1(\mathbb{R}, L^2))\]
of the Cauchy problem associated to (1.3), being the unique strong limit of a sequence of regular solutions in (1), whose initial data approximate the rough initial datum in \(H^1 \times H^1 \times L^2\).

3. (Continuous dependence on initial data) The solutions \((\varphi, A)\) in (2) depend continuously on the initial datum \((\varphi_0, A_0, E_0) \in H^1 \times H^1 \times L^2\).

For \(m \in \mathbb{R}\), let \(h_m\) denote the weighed \(L^2(\mathbb{R}^3) \otimes \mathbb{C}^2\)-space with norm
\[\|\alpha\|_{h_m} = \left(\sum_{\lambda=1,2} \int d^3k (1 + |k|^2)^m |\alpha(k,\lambda)|^2\right)^{1/2}.\] (1.15)

Throughout this work we will rely on the following statement which results almost immediately from Proposition 1.1 (see Appendix A).

Corollary I.2. Let \(| \cdot |^{-1/2} F[k] \in L^2(\mathbb{R}^3, \mathbb{C})\). For every initial datum \((\varphi_0, \alpha_0) \in H^2(\mathbb{R}^3, \mathbb{C}) \times (h_{3/2} \cap h_{-3/2})\) the Maxwell-Schrödinger system (1.11) has a unique global solution in \(H^2(\mathbb{R}^3, \mathbb{C}) \times h_{3/2}\).

I.2 The Microscopic Model: Pauli-Fierz Hamiltonian

The microscopic model corresponding to the Maxwell-Schrödinger system with mean-field self-interaction \(v \ast |\varphi|^2 \varphi\) consists of many identical nonrelativistic particles – obeying Bose-Einstein condensation – interacting among themselves by means of a weak pair potential and with a quantized electromagnetic field in Coulomb’s gauge. Contrarily to the “classical” case, the microscopic model is known to be well-defined only for extended charges. Let us start by defining a Hilbert space \(\mathcal{H}_h^{(N)}\) depending on two parameters \(N \in \mathbb{N}, h \in \mathbb{R}^+\) as follows:
\[\mathcal{H}_h^{(N)} := L^2(\mathbb{R}^{3N}) \otimes \Gamma_h(L^2(\mathbb{R}^3) \otimes \mathbb{C}^2),\] (1.16)
where \(L^2(\mathbb{R}^{3N})\) is the natural Hilbert space of \(N\) identical bosons (the subscript \(s\) indicates symmetry under the interchange of variables) and \(\Gamma_h\) is the second quantization functor associating to any (pre-)Hilbert space \(h = L^2(\mathbb{R}^3) \otimes \mathbb{C}^2\) the corresponding Fock representation of the Canonical Commutation Relations \([a_h(f), a_h^*(g)] = \hbar(f, g)_h\), with \(\hbar\) a semiclassical parameter measuring the degree of noncommutativity of the quantum field. The Fock representation is the natural one to describe noninteracting or regularized quantum field theories,
the latter being the case here with \( \mathcal{H} := L^2(\mathbb{R}^3) \otimes \mathbb{C}^2 \). With this interpretation, the limits \( N \to \infty \) and \( \hbar \to 0 \) describe respectively the regimes in which the bosons are many and the quantum effects of the field are negligible. The time evolution is dictated by the Schrödinger equation
\[
i \partial_t \Psi_{N, \hbar}(t) = H_{N, \hbar} \Psi_{N, \hbar}(t) \, ,
\]
where the Hamiltonian \( H_{N, \hbar} \), called Pauli-Fierz Hamiltonian, is given by
\[
H_{N, \hbar} = \sum_{j=1}^{N} \left( -i \nabla_j - \mu_{N, \hbar} \mathbf{\hat{A}}_\kappa(x_j) \right)^2 + g_N \sum_{1 \leq j < k \leq N} v(x_j - x_k) + \frac{1}{\hbar} H_f \, ,
\]
where \( \mu_{N, \hbar} \) describes the coupling strength between the particles and the field, \( g_N \) the coupling strength between the particles, \( \kappa \) and \( v \) are the charge distribution and the pair potential introduced previously,
\[
H_f = \sum_{\lambda=1,2} \int d^3k |k| a^\dagger_{\kappa}(k, \lambda) a_{\kappa}(k, \lambda)
\]
is the field’s kinetic energy, with \( a^\dagger_{\kappa}(k, \lambda) \) the polarized creation and annihilation operators satisfying the CCR
\[
[a_{\kappa}(k, \lambda), a^\dagger_{\kappa}(k', \lambda')] = \hbar \delta_{\lambda \lambda'} \delta(k - k') \, , \quad [a_{\kappa}(k, \lambda), a_{\kappa}(k', \lambda')] = [a^\dagger_{\kappa}(k, \lambda), a^\dagger_{\kappa}(k', \lambda')] = 0 \, ,
\]
and
\[
\mathbf{\hat{A}}_\kappa(x) = \sum_{\lambda=1,2} \int d^3k \frac{\mathbf{F}[\kappa](k)}{\sqrt{2|k|}} \epsilon_\lambda(k) \left( e^{ikx} a_{\kappa}(k, \lambda) + e^{-ikx} a^\dagger_{\kappa}(k, \lambda) \right)
\]
the smeared quantized electromagnetic vector potential in Coulomb’s gauge. Let us remark accordingly. A possible choice is given by \( \mu_{N, \hbar} = g_N = 0 \), but physically different choice is given by \( \mu_{N, \hbar} = 1, g_N = \frac{1}{N} \). In this regime the electromagnetic field becomes classical inverse proportionally to the increasing number of bosons. At the same time the coupling between the particles and field is of order one, while the coupling between pairs of particles becomes weak (of order \( \frac{1}{N^2} \)). A mathematically equivalent but physically different choice is given by \( N \to \infty, \hbar = 1, \mu_{N, \hbar} = \frac{1}{N}, g_N = \frac{1}{N} \). Here, the physical interpretation is of many bosons that interact weakly both with the quantized electromagnetic field (coupling of order \( \frac{1}{\sqrt{N}} \)) and among themselves (pair coupling of order \( \frac{1}{N} \)). Our result reads as follows:

\(1\)To simplify the notation we assume \( \mathcal{F}[\kappa](k) \in \mathbb{R} \) for all \( k \in \mathbb{R}^3 \). Theorem 11.2 equally applies if \( \mathcal{F}[\kappa] \) is complex valued. In this case, \( \mathbf{\hat{A}}_\kappa(x) = \sum_{\lambda=1,2} \int d^3k \frac{\mathbf{F}[\kappa](k)}{\sqrt{2\hbar k}} \epsilon_\lambda(k) \left( \mathcal{F}[\kappa](k)e^{ikx} a_{\kappa}(k, \lambda) + \mathcal{F}[\kappa](k)e^{-ikx} a^\dagger_{\kappa}(k, \lambda) \right) \).

I.3 Scaling regime

Our aim is to prove that the Maxwell-Schrödinger system emerges in some limit \( N \to \infty \) and/or \( \hbar \to 0 \), as an effective model of the microscopic Pauli-Fierz dynamics. This is true only if we couple the parameters \( N, \hbar \) suitably, and choose the coupling constants \( \mu_{N, \hbar}, g_N \) accordingly. A possible choice is given by \( N \to \infty, \hbar = \frac{1}{N}, \mu_{N, \hbar} = 1, g_N = \frac{1}{N} \). In this regime the electromagnetic field becomes classical inverse proportionally to the increasing number of bosons. At the same time the coupling between the particles and field is of order one, while the coupling between pairs of particles becomes weak (of order \( \frac{1}{N^2} \)). A mathematically equivalent but physically different choice is given by \( N \to \infty, \hbar = 1, \mu_{N, \hbar} = \frac{1}{N}, g_N = \frac{1}{N} \). Here, the physical interpretation is of many bosons that interact weakly both with the quantized electromagnetic field (coupling of order \( \frac{1}{\sqrt{N}} \)) and among themselves (pair coupling of order \( \frac{1}{N} \)). Our result reads as follows:

Provided that we choose an initial microscopic state that is “close enough” to a non-interacting state representing a complete condensate and a coherent field of minimal uncertainty, then at any time \( t \geq 0 \) the evolution keeps the state “close” (in the same sense as above)
to an analogous configuration in which the one-particle wave function and the argument of the coherent field have been evolved by the coupled Maxwell-Schrödinger equations.

The described scaling regime has been considered in earlier works for the Nelson model with ultraviolet cutoff [11, 10, 11, 24], the renormalized Nelson model [2] and the Fröhlich model [22]. In [24] the Nelson model with ultraviolet cutoff has been studied in a limit of many weakly interacting fermions. The classical behavior of quantum fields has also been proven in different scaling regimes [3, 5, 6, 7, 8, 9, 12, 13, 14, 16, 17, 19, 21, 26, 28, 34]. We also would like to mention [32] which derives the Maxwell-Schrödinger equations in a nonrigorous manner by neglecting certain terms in the Pauli-Fierz Hamiltonian.

II Main Result

From now on, we will keep $N$ as the single parameter and choose $\hbar = 1$, $\mu_{N, h} = \frac{1}{\sqrt{N}}$, $g_N = \frac{1}{N}$.

We will use the notations $\mathcal{H}^{(N)} = \mathcal{H}^{(N)}_{\mathbf{h}}$, $\mathcal{F}_p = \Gamma_1(L^2(\mathbb{R}^3) \otimes \mathbb{C}^2)$ with vacuum $\Omega$, $H_N = H_{N, 1}$ and $\Psi_N = \Psi_{N, 1}$. Concerning the interaction potential and charge distribution we will make the assumptions.

**Assumption II.1.** The (repulsive) interaction potential $v$ is a positive, real, and even function satisfying

$$\|v\|_{L^2 + L^\infty(\mathbb{R}^3)} = \inf_{v=v_1+v_2} \{\|v_1\|_{L^2(\mathbb{R}^3)} + \|v_2\|_{L^\infty(\mathbb{R}^3)}\} < +\infty.$$  (II.1)

The charge distribution $\kappa$ with Fourier transform $\mathcal{F}[\kappa]$ satisfies

$$(| \cdot |^{-1} + | \cdot |^{1/2}) \mathcal{F}[\kappa] \in L^2(\mathbb{R}^3).$$  (II.2)

In order to state our result we define for $\Psi_N \in \mathcal{H}^{(N)}$ the one-particle reduced density matrix of the charged particles $\gamma_{\Psi_N} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ by

$$\gamma_{\Psi_N}^{(1, 0)} := \text{Tr}_{2, \ldots, N} \text{Tr}_{\mathcal{F}_p} |\Psi_N\rangle \langle \Psi_N|,$$  (II.3)

where $\text{Tr}_{2, \ldots, N}$ denotes the partial trace over the coordinates $x_2, \ldots, x_N$ and $\text{Tr}_{\mathcal{F}_p}$ is the trace over Fock space. In addition, we introduce the number of photon operator

$$\hat{N} = \sum_{\lambda=1, 2} \int d^3 k \ a^\dagger(k, \lambda) a(k, \lambda)$$  (II.4)

and the unitary Weyl operator

$$W(f) = \exp \left( \sum_{\lambda=1, 2} \int d^3 k \ f(k, \lambda) a^\dagger(k, \lambda) - \overline{f(k, \lambda)} a(k, \lambda) \right) \text{ with } f \in \mathfrak{h}. \quad \text{ (II.5)}$$

Our result is the following.

**Theorem II.2.** Let $v$ and $\kappa$ satisfy Assumption II.1 $(\varphi_0, \alpha_0) \in H^2(\mathbb{R}^3, \mathbb{C}) \times (\mathfrak{h}^{1/2} \cap \mathfrak{h}^{-1/2})$ with $\|\varphi_0\|_{L^2(\mathbb{R}^3)} = 1$ and $\Psi_{N, 0} \in D(H_N) \cap D(\mathcal{N}^{1/2})$ such that $\|\Psi_{N, 0}\|_{\mathcal{H}^{(N)}} = 1$. Define

$$a_N := \text{Tr}_{L^2(\mathbb{R}^3)} \gamma_{\Psi_{N, 0}}^{(1, 0)} - |\varphi_0\rangle \langle \varphi_0|$$  (II.6)

$$b_N := N^{-1} (W^{-1}(\sqrt{N} \alpha_0) \Psi_{N, 0} \Psi_{N, 0} W^{-1}(\sqrt{N} \alpha_0))_{\mathcal{H}^{(N)}}$$  (II.7)

$$c_N := \|N^{-1} H_N - \mathcal{E}_M [\varphi_0, \alpha_0]\|_{\mathcal{H}^{(N)}}^2.$$  (II.8)


Let \((\varphi_t, \alpha_t)\) and \(\Psi_{N,t}\) be the unique solutions of \((\text{II}.10)\) and \((\text{II}.11)\), respectively. Then, there exists a monotone increasing function \(C(s)\) of the norms \(\|\varphi_s\|_{H^2(\mathbb{R}^3)}\), \(\|\cdot|1/2\alpha_s\|_b\), \(\|v\|_{L^2+L^\infty(\mathbb{R}^3)}\) and \(\|(|\cdot|^{-1/2} + |\cdot|^{-1}) \mathcal{F}[\varphi]\|_{L^2(\mathbb{R}^3)}\) such that

\[
\text{Tr}_{L^2(\mathbb{R}^3)}|\gamma_{N,t}^{(1,0)} - |\varphi_t\rangle\langle \varphi_t|| \leq \sqrt{a_N + b_N + c_N + N^{-1}} e^{|t|/C(s)},
\]

\[(\text{II}.9)\]

\[
N^{-1}\langle W^{-1}(\sqrt{N} \alpha_t)\Psi_{N,t}, NW^{-1}(\sqrt{N} \alpha_t)\Psi_{N,t}\rangle_{\mathcal{H}(N)} \leq (a_N + b_N + c_N + N^{-1}) e^{|t|/ds C(s)},
\]

\[(\text{II}.10)\]

for any \(t \geq 0\). In particular, for \(\Psi_{N,0} = \varphi_0^N \otimes W(\sqrt{N} \alpha_0)\Omega\) one obtains

\[
\text{Tr}_{L^2(\mathbb{R}^3)}|\gamma_{N,t}^{(1,0)} - |\varphi_t\rangle\langle \varphi_t|| \leq N^{-1/2} C(0) e^{|t|/ds C(s)},
\]

\[(\text{II}.11)\]

\[
N^{-1}\langle W^{-1}(\sqrt{N} \alpha_t)\Psi_{N,t}, NW^{-1}(\sqrt{N} \alpha_t)\Psi_{N,t}\rangle_{\mathcal{H}(N)} \leq N^{-1} C(0) e^{|t|/ds C(s)},
\]

\[(\text{II}.12)\]

**Remark II.3.** Let \(\gamma_{N,t}^{(0,1)}\) be the one-particle reduced density matrix of the photons with integral kernel

\[
\gamma_{N,t}^{(0,1)}(k, \lambda; k', \lambda') = N^{-1}\langle \Psi_{N,t}, a^*(k', \lambda') a(k, \lambda) \Psi_{N,t}\rangle_{\mathcal{H}(N)}.
\]

\[(\text{II}.13)\]

By similar means as in \([25]\) Lemma 5.3 one obtains

\[
\text{Tr}_b|\gamma_{N,t}^{(0,1)} - |\alpha_t\rangle\langle \alpha_t|| \leq \max_{j=1,2} \left(a_N + b_N + c_N + N^{-1}\right)^{3/2} \left(1 + \|\alpha_t\|_b\right) e^{|t|/ds C(s)}
\]

\[(\text{II}.14)\]

from \((\text{II}.10)\) and

\[
\text{Tr}_0|\gamma_{N,t}^{(0,1)} - |\alpha_t\rangle\langle \alpha_t|| \leq N^{-1/2} \left(1 + \|\alpha_t\|_b\right) C(0) e^{|t|/ds C(s)}
\]

\[(\text{II}.15)\]

for initial product states \(\Psi_{N,0} = \varphi_0^N \otimes W(\sqrt{N} \alpha_0)\Omega\) from \((\text{II}.12)\).

**Remark II.4.** In \([25]\) Theorem \(\text{II}2\) was proven for the charge distribution \((\text{II}.13)\) and with the number operator \(N\) in \((\text{II}.7)\), \((\text{II}.10)\) and \((\text{II}.12)\) being replaced by the field energy \(H_f\). Because of Markov’s inequality,

\[
\sum_{\lambda=1,2} \int_{|k| \geq I} d^3k a^*(k, \lambda) a(k, \lambda) \leq I^{-1} H_f,
\]

one can use the field energy to conclude that the quantum fluctuations around the coherent state are subleading for all photons with \(|k| \geq I\). For sufficiently small \(a_N\), \(b_N\) and \(c_N\) one can choose \(I \sim N^{-a}\) with \(a < 1\). However, this choice does not provide information about the coherence of soft photons with frequencies below this threshold.

### III Proof of the result

The rest of the article outlines the proof of Theorem \((\text{II}.2)\). We will proceed as follows:

1. We define a functional \(\beta[\Psi_N, \varphi, \alpha]\) which measures if the charges of the many-body state \(\Psi_N\) form a Bose-Einstein condensate with condensate wave function \(\varphi\) and if the photons are in a coherent state with mean photon number \(N \|\alpha\|_b^2\).
2. Next, we show that the domain of $\beta$ contains the solutions $(\varphi_t, \alpha_t)$ of (I.10) and $\Psi_{N,t}$ of (I.17) from Theorem (II.2) for all $t \geq 0$.

3. Afterwards, we compute the change of $\beta[\Psi_{N,t}, \varphi_t, \alpha_t]$ in time.

4. Finally, we control the growth of $\beta[\Psi_{N,t}, \varphi_t, \alpha_t]$ with the help of Grönwall’s inequality. This concludes the proof.

In doing so, we will rely on the findings from [25] and rather explain how the original proof of [23] has to be adapted. Most of the modifications are necessary to show the invariance of the domain in Step 2 and to compute the time derivative of $\beta$ in Step 3.

### III.1 Definition of the functional

We define a functional which consists of three parts.

**Definition III.1.** For $\varphi \in L^2(\mathbb{R}^3)$ we define $p_1^\varphi : L^2(\mathbb{R}^{3N}) \to L^2(\mathbb{R}^{3N})$ by

$$p_1^\varphi f(x_1, \ldots, x_N) := \varphi(x_1) \int d^3x_1 \varphi(x_1) f(x_1, \ldots, x_N)$$

and $q_1^\varphi := 1_{L^2(\mathbb{R}^{3N})} - p_1^\varphi$. Now, let $\Psi_N \in \mathcal{D}(H_N) \cap \mathcal{D}(N^{1/2})$, $\varphi \in H^1(\mathbb{R}^3)$, $\alpha \in \mathfrak{h}_\mathbb{R}$. Then,

$$\beta^a(\Psi_N, \varphi) := \langle \Psi_N, q_1^\varphi \otimes 1_{F_p} \Psi_N \rangle_{H^{(N)}},$$

$$\beta^b(\Psi_N, \alpha) := \sum_{\lambda=1,2} \int d^3k \left( \frac{a(k, \lambda)}{\sqrt{N}} - \alpha(k, \lambda) \right) \Psi_N, \left( \frac{a(k, \lambda)}{\sqrt{N}} - \alpha(k, \lambda) \right) \Psi_N \rangle_{H^{(N)}},$$

$$\beta^c(\Psi_N, \varphi, \alpha) := \langle \left( H_N - \mathcal{E}_M[\varphi, \alpha] \right) \Psi_N, \left( H_N - \mathcal{E}_M[\varphi, \alpha] \right) \Psi_N \rangle_{H^{(N)}}$$

and the functional $\beta : (\mathcal{D}(H_N) \cap \mathcal{D}(N^{1/2})) \times H^1(\mathbb{R}^3) \times \mathfrak{h}_{\mathbb{R}} \to \mathbb{R}_0^+$ is defined as $\beta := \beta^a + \beta^b + \beta^c$.

The functional $\beta^a$ measures if the charges of the many-body state are in a Bose-Einstein condensate (we refer to [20] [30] for a comprehensive introduction). Its relation to the trace norm distance of the one-particle reduced density matrix is given by (see, e.g. [24] Lemma 5.3)

$$\beta^a(\Psi_N, \varphi) \leq \text{Tr}_{L^2(\mathbb{R}^3)}|\gamma^{(1,0)}_{\Psi_N,0} - |\varphi||^2 \leq \sqrt{8} \beta^a(\Psi_N, \varphi).$$

The functional $\beta^b$ quantifies the fluctuations of $\Psi_N$ around the coherent state $W(\sqrt{N}\alpha)\Omega$. Using property (II.20) of the Weyl operators it can be written as

$$\beta^b(\Psi_N, \alpha) = N^{-1}\langle W^{-1}(\sqrt{N}\alpha)\Psi_N, N^{-1}W^{-1}(\sqrt{N}\alpha)\Psi_N \rangle_{H^{(N)}},$$

showing that it is the same quantity as usually considered in the coherent state approach [31]. While $\beta^a$ and $\beta^b$ measure the deviation of $\Psi_N$ from the product state $\varphi^{\otimes N} \otimes W(\sqrt{N}\alpha)\Omega$ the functional $\beta^c$ is introduced for technical reasons. It quantifies the fluctuations of the many-body energy per particle around the energy of the Maxwell-Schrödinger system.

In the original proof of [25] the functional $\beta$ was considered with $\beta^b(\Psi_N, \alpha)$ being replaced by

$$\tilde{\beta}^b(\Psi_N, \alpha) := \sum_{\lambda=1,2} \int d^3k |k| \left( \frac{a(k, \lambda)}{\sqrt{N}} - \alpha(k, \lambda) \right) \Psi_N, \left( \frac{a(k, \lambda)}{\sqrt{N}} - \alpha(k, \lambda) \right) \Psi_N \rangle_{H^{(N)}}$$

$$= N^{-1}\langle W^{-1}(\sqrt{N}\alpha)\Psi_N, H_N W^{-1}(\sqrt{N}\alpha)\Psi_N \rangle_{H^{(N)}}.$$
This definition has the advantage that it can be defined for many-body states \( \Psi_N \) in the domain \( \mathcal{D}(H_N) = (L^2(\mathbb{R}^3, \mathbb{C}) \otimes \mathcal{F}) \cap \mathcal{D}(H_f) \) which is invariant under the time evolution \( e^{-iH_N t} \). The additional difficulties with respect to \[20\] actually originate from the fact that \( \mathcal{D}(H_f) \) is not contained in the domain of the Pauli-Fierz Hamiltonian. On the contrary \( \mathcal{D}(H_N) \) does not allow to investigate the coherence of photons with small frequencies because the factor \(|k|\) in the integral on the right hand side of \((III.3)\) suppresses contributions from photons with small energies.

### III.2 Invariance of the domain

Throughout the rest of the article \((\varphi_t, \alpha_t)\) and \(\Psi_{N,t}\) denote the solutions of \((II.10)\) and \((II.17)\) from Theorem \((II.2)\). In this section we show that \((\Psi_{N,t}, \varphi_t, \alpha_t) \in \mathcal{D}(H_N) \cap \mathcal{D}(\mathcal{N}^{1/2}) \times H^2(\mathbb{R}^3 \times \mathbb{R}^+) \) for all \( t \geq 0 \). The condition on the Maxwell-Schrödinger solutions is satisfied because of Corollary \((II.2)\). While \( \mathcal{D}(H_N) \) is invariant under the evolution of the Pauli-Fierz Hamiltonian, due to Stone’s theorem, the invariance of \( \mathcal{D}(\mathcal{N}^{1/2}) \) is less clear because the photon number is not conserved during the time evolution. The next statement, however, displays that the number of photons can be controlled by the energy of the system\(^2\).

**Lemma III.2.** Let \( \Psi_{N,0} \in \mathcal{D}(H_N^{1/2}) \cap \mathcal{D}(\mathcal{N}^{1/2}) \). Then, there exists a constant \((\text{depending on } N \text{ and the choice of } \kappa)\) such that

\[
\left\| \mathcal{N}^{1/2} e^{-iH_N t} \Psi_{N,0} \right\| \leq \left\| \mathcal{N}^{1/2} \Psi_{N,0} \right\| + C t^{1/2} \left\| (H_N + C)^{1/2} \Psi_{N,0} \right\| \quad \text{for all } t \geq 0. \tag{III.6}
\]

This implies \( e^{-iH_N t} \mathcal{D}(H_N^{1/2}) \cap \mathcal{D}(\mathcal{N}^{1/2}) = \mathcal{D}(H_N^{1/2}) \cap \mathcal{D}(\mathcal{N}^{1/2}) \).

**Proof of Lemma \((III.2)\)** In the following, we use the notation

\[
a(g) = \sum_{\lambda=1,2} \int d^3k \overline{g(k, \lambda)} a(k, \lambda), \quad a^*(g) = \sum_{\lambda=1,2} \int d^3k g(k, \lambda) a^*(k, \lambda) \tag{III.7}
\]

and

\[
G_x(k, \lambda) = \mathcal{F}[\kappa](k) \frac{1}{\sqrt{2|k|}} \epsilon_{\lambda}(k) e^{-i k x}. \tag{III.8}
\]

The vector potential can then be written as \( \hat{A}_x(x) = a(G_x) + a^*(G_x) \). Recall the standard estimates for the annihilation and creation operators

\[
\|a(g)\Psi\| \leq \|g\|_b \left\| \mathcal{N}^{1/2} \Psi \right\| ,
\]

\[
\|a^*(g)\Psi\| \leq \|g\|_b \left\| (\mathcal{N} + 1)^{1/2} \Psi \right\| ,
\]

\[
\|a(g)\Psi\| \leq \left\| | \cdot |^{-1/2} \right\|_b \left\| H_f^{1/2} \Psi \right\| ,
\]

\[
\|a^*(g)\Psi\| \leq \left\| (1 + | \cdot |^{-1/2}) \right\|_b \left\| (H_f + 1)^{1/2} \Psi \right\| . \tag{III.9}
\]

\(^2\)Inequality \((III.6)\) was originally proven by Fumio Hiroshima and appeared in a slightly different form (for the second instead of the first moment of the number operator) in \[3\] Proposition 3.11. We would like to thank Fumio Hiroshima for sharing his notes with us. The proof is presented again for the convenience of the reader.
Let $\Psi_{N,0} \in \mathcal{D}(H_N^{1/2}) \cap \mathcal{D}(N^{1/2})$, $\Psi_{N,t} = e^{-iH_N t}\Psi_{N,0}$, $\delta \geq 0$ and consider the bounded operator $N_\delta = N e^{-\delta N}$. Using
\[
[N_\delta, H_N] = 2 \sum_{j=1}^{N} [N_\delta, N^{-1/2} \hat{A}_\kappa(x_j)] \cdot i \nabla_j + N^{-1} \sum_{j=1}^{N} [N_\delta, \hat{A}_\kappa(x_j)] \hat{A}_\kappa(x_j)
+ N^{-1} \sum_{j=1}^{N} \hat{A}_\kappa(x_j) [N_\delta, \hat{A}_\kappa(x_j)]
\] (III.10)
and the Cauchy-Schwarz inequality we estimate
\[
\frac{d}{dt} \left\| N_\delta^{1/2} \Psi_{N,t} \right\|^2 = i \langle N_\delta \Psi_{N,t}, [H_N, N_\delta] \Psi_{N,t} \rangle
\leq CN^{-1/2} \sum_{j=1}^{N} \left\| [N_\delta, \hat{A}_\kappa(x_j)] \Psi_{N,t} \right\| \left( \left\| i \nabla_j \Psi_{N,t} \right\| + N^{-1/2} \left\| \hat{A}_\kappa(x_j) \Psi_{N,t} \right\| \right).
\] (III.11)
By means of the canonical commutation relations and the shifting property of the number operator we get
\[
[N_\delta, \hat{A}_\kappa(x)] = \left[ N \left( 1 - e^{-\delta} \right) - e^{-\delta} \right] e^{-\delta N} a(G_x) + \left[ N \left( e^{-\delta} - 1 \right) + 1 \right] e^{-\delta(N-1)} a^*(G_x).
\] (III.12)
Using $e^{-\delta N} \leq 1, N(1 - e^{-\delta})e^{-\delta N} \leq 1, N(e^{-\delta} - 1)e^{-\delta(N-1)}|_{N \geq 1} \leq 3$ and (III.9) we obtain
\[
\left\| [N_\delta, \hat{A}_\kappa(x)] \Psi \right\| \leq C \left\| \left( | \cdot |^{-1/2} + | \cdot |^{-1} \right) \mathcal{F}[\kappa] \right\|_{L^2(\mathbb{R}^3)} \left\| (H_f + 1)^{1/2} \Psi \right\|.
\] (III.13)
Hence,
\[
\frac{d}{dt} \left\| N_\delta^{1/2} \Psi_{N,t} \right\|^2 \leq C \left\| \left( | \cdot |^{-1/2} + | \cdot |^{-1} \right) \mathcal{F}[\kappa] \right\|^2_{L^2(\mathbb{R}^3)} \left\| (H_f + 1)^{1/2} \Psi_{N,t} \right\|^2
+ \langle \Psi_{N,t}, \sum_{j=1}^{N} - \Delta_j \Psi_{N,t} \rangle
\leq C \left( 1 + \left\| \left( | \cdot |^{-1/2} + | \cdot |^{-1} \right) \mathcal{F}[\kappa] \right\|^2_{L^2(\mathbb{R}^3)} \right) \left\| (H_N^{(0)} + 1)^{1/2} \Psi_{N,t} \right\|^2
\] (III.14)
with $H_N^{(0)} = - \sum_{j=1}^{N} \Delta_j + H_f$. Note that there exists a constant $C(N, \kappa)$ dependent on the number of particles and the choice of $\kappa$ such that $\left\| (H_N^{(0)} + 1)^{1/2} \Psi \right\| \leq C(N, \kappa) \left\| (H_N + C)^{1/2} \Psi \right\|$ holds for all $\Psi \in \mathcal{D}(H_N^{1/2})$. This fact follows from $\mathcal{D}(H_N^{(0)}) = \mathcal{D}(H_N) = (H^2(\mathbb{R}^3, \mathcal{C}) \otimes \mathcal{F}_p) \cap \mathcal{D}(H_f)$ and the closed graph theorem [18] Theorem 1.3 and Corollary 1.4. In that regard note that $(H_N^{(0)})^{1/2} ((H_N + C)^{1/2} + i)^{-1}$ is a closed operator. We consequently obtain
\[
\left\| N_\delta^{1/2} \Psi_{N,t} \right\|^2 \leq \left\| N_\delta^{1/2} \Psi_{N,0} \right\|^2 + C(N, \kappa) \int_{0}^{t} ds \left\| (H_N + C)^{1/2} \Psi_{N,s} \right\|.
\] (III.15)
By the spectral theorem and monotone convergence
\[
\lim_{\delta \to 0} \left\| N_\delta \Psi \right\|^2 = \lim_{\delta \to 0} \int_{0}^{\infty} \lambda^2 e^{-2\lambda t} \langle \Psi, dE(\lambda) \Psi \rangle = \int_{0}^{\infty} \lambda^2 \langle \Psi, dE(\lambda) \Psi \rangle = \left\| \mathcal{N} \Psi \right\|^2.
\] (III.16)
Together with Stone’s theorem this shows the claim. \qed
Lemma III.3. Let \((\varphi_t, \alpha_t)\) and \(\Psi_{N,t}\) be the solutions of (I.10) and (I.17) from Theorem (II.2) and \(\beta^b\) be defined as in Definition III.7. Then

\[
\beta^b (\Psi_{N,t}, \alpha_t) - \beta^b (\Psi_{N,0}, \alpha_0) = 2\text{Re} \int_0^t ds \sum_{\lambda=1,2} \int d^3k \left[ \langle \Psi_{N,s}, i \frac{4\pi^3}{|k|} \mathcal{F}[\mathcal{G}_s]N(t) \rangle \right. \\
+ 2\langle \Psi_{N,s}, i \frac{\mathcal{F}[\mathcal{G}_s]N(t)}{2|k|} \rangle \mathcal{G}_s \left. \mathcal{G}_s \right] e^{ikx}(i \nabla + N^{-1/2} \hat{A}_s(x_1)) \left( \frac{a(k, \lambda)}{\sqrt{N}} - \alpha_s(k, \lambda) \right) \Psi_{N,s} \right] . \tag{III.17}
\]

Proof of Lemma III.3. Let \(\beta^b(t) = N^{-1} \langle W^{-1} (\sqrt{N} \alpha_t) \Psi_{N,t}, N \delta W^{-1} (\sqrt{N} \alpha_t) \Psi_{N,t} \rangle \) with \(N \delta\) being defined as in the proof of Lemma II.2. Using that \(W^{-1} (\sqrt{N} \alpha_t)\) is strongly differentiable in \(t\) from \(D N^{1/2}\) to \(H(N)\) with (see [15, Lemma 3.1])

\[
\frac{d}{dt} W^{-1} (\sqrt{N} \alpha_t) = \left( N^{1/2} \left( a(\alpha_t) - a^* (\alpha_t) - Ni \text{Im} \langle \alpha_t, \hat{\alpha}_t \rangle \right) \right) W^{-1} (\sqrt{N} \alpha_t) \tag{III.19}
\]

and

\[
W^{-1} (f) a(k, \lambda) W (f) = a(k, \lambda) + f(k, \lambda) \tag{III.20}
\]

we obtain that the time derivative of the fluctuation vector \(\xi_{N,t} = W^{-1} (\sqrt{N} \alpha_t) \Psi_{N,t}\) is given by \(\frac{d}{dt} \xi_{N,t} = -i \mathcal{G}(t) \xi_{N,t}\) with

\[
\mathcal{G}(t) = N^{-1} \sum_{1 \leq j < k \leq N} v(x_j - x_k) + H_f + N \| | \cdot | \|^2 \| + N \text{Im} \langle \alpha_t, \hat{\alpha}_t \rangle \| + N^{1/2} \left( a(|\alpha| \cdot i \hat{\alpha}_t) + a^* (|\alpha| \cdot \hat{\alpha}_t) \right) + N \sum_{j=1}^N \left( x_j - N^{-1/2} \hat{A}_s(x_j) - A_s(x_j, t) \right)^2 . \tag{III.21}
\]

In analogy to (III.12) and the subsequent discussion one shows

\[
\lim_{\delta \to 0} \langle \xi_{N,t}, [a(f), \mathcal{N}_\delta] - a(f) \rangle \xi_{N,t} \rangle = 0 \tag{III.22}
\]

and

\[
\lim_{\delta \to 0} \left\| \left[ \hat{A}_s(x_j), \mathcal{N}_\delta \right] - a \left( G_{x_j} - a^* \left( G_{x_j} \right) \right) \right\| \xi_{N,t} \| = 0 \tag{III.23}
\]

for \(f \in \mathfrak{h}\) and \(\xi_{N,t} \in D(H_N) \cap D(N^{1/2})\). Together with

\[
\frac{d}{dt} \beta^b(t) = i N^{-1} \langle \xi_{N,t}, [\mathcal{G}(t), \mathcal{N}_\delta] \xi_{N,t} \rangle \tag{III.24}
\]
this leads to
\[
\lim_{\delta \to 0} \frac{d}{dt} \beta^\delta(t) = 2N^{-3/2} \text{Im} \left( \sum_{j=1}^{N} (a(G_x) - a^*(G_x)) \left( -i \nabla_j - N^{-1/2} \hat{A}_\kappa(x_j) - A_\kappa(x_j, t) \right) \xi_{N,t} \right) \\
+ iN^{-1/2} \left( \text{Im} \left( a(\cdot | \alpha_t - i \hat{\alpha}_t) - a^*(\cdot | \alpha_t - i \hat{\alpha}_t) \right) \xi_{N,t} \right).
\]

(III.25)
The claim then follows by Duhamel’s formula, monotone convergence and straightforward manipulations using (III.20), $[\nabla_j, \hat{A}_\kappa(x_j)] = 0$ and $[\nabla_j, G_{x_j}(k, \lambda)] = 0$.

From [25], Section 6.2 and Section 6.4 and Duhamel’s formula we immediately obtain the following.

**Lemma III.4.** Let $(\phi_t, \alpha_t)$ and $\Psi_{N,t}$ be the solutions of (II.10) and (II.17) from Theorem (II.2) and $\beta^\alpha$, $\beta^\beta$ be defined as in Definition (III.7). Then,
\[
\beta^\alpha (\Psi_{N,t}, \phi_t) - \beta^\alpha (\Psi_{N,0}, \phi_0) = -2 \int_0^t ds \left[ 2 \text{Re} \left( \langle \Psi_{N,s}, p_1^\phi \left( N^{-1/2} \hat{A}_\kappa(x_1) - A_\kappa(x_1, s) \right) \cdot \nabla_1 q_1^\phi \Psi_{N,s} \rangle 
+ \text{Im} \left( \langle \Psi_{N,s}, p_1^\phi \left( N^{-1} \hat{A}_\kappa^2(x_1) - A_\kappa^2(x_1, s) \right) q_1^\phi \Psi_{N,s} \rangle 
+ \text{Im} \left( \langle \Psi_{N,s}, p_1^\phi \left( (N-1)N^{-1} v(x_1 - x_2) - (v \ast |\phi_t|^2)(x_1) \right) q_1^\phi \Psi_{N,s} \rangle \right). \right.
\]

(III.26)
Moreover,
\[
\beta^\beta (\Psi_{N,t}, \phi_t, \alpha_t) = \beta^\beta (\Psi_{N,0}, \phi_0, \alpha_0)
\]
(III.27)
due to energy conservation.

### III.4 Controlling the growth of $\beta$ in time

In this section, we classify the growth of $\beta (\Psi_{N,t}, \phi_t, \alpha_t)$ in time. The first two inequalities of Theorem (II.2) then follow from (III.3), (III.4) and the statement below. By similar estimates as in [25], Chapter 7 one obtains (II.11) and (II.12).

**Lemma III.5.** Let $(\phi_t, \alpha_t)$ and $\Psi_{N,t}$ be the solutions of (II.10) and (II.17) from Theorem (II.2). Then, there exists a monotone increasing function $C(s)$ of the norms $\|\phi_s\|_{H^2(\mathbb{R}^3)}$, $\|\cdot|1/2\alpha_s\|_b$, $\|v\|_{L^2+L^\infty(\mathbb{R}^3)}$ and $\|(|\cdot|^{-1/2} + |\cdot|^{-1}) \mathcal{F}[\kappa]\|_{L^2(\mathbb{R}^3)}$ such that
\[
\beta (\Psi_{N,t}, \phi_t, \alpha_t) - \beta (\Psi_{N,0}, \phi_0, \alpha_0) \leq \int_0^t ds C(s) \left( \beta (\Psi_{N,s}, \phi_s, \alpha_s) + N^{-1} \right),
\]
(III.28)
\[
\beta (\Psi_{N,t}, \phi_t, \alpha_t) \leq \left( \beta (\Psi_{N,0}, \phi_0, \alpha_0) + N^{-1} \right) e^{\int_0^t ds C(s)}
\]
(III.29)
holds for any $t \geq 0$. 

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Sketch of the proof of Lemma III.5. Inequality (III.28) is proven analogously to [25, Lemma 6.10]. The similarity becomes obvious if one defines the auxiliary fields

\[
\hat{F}^+(x) := \frac{i}{\sqrt{2}} \sum_{\lambda=1,2} \int d^3k \, \epsilon_\lambda(k) e^{ikx} a(k, \lambda),
\]

\[
\hat{F}^-(x) := -\frac{i}{\sqrt{2}} \sum_{\lambda=1,2} \int d^3k \, \epsilon_\lambda(k) e^{-ikx} a^*(k, \lambda),
\]

\[
F^+(x, t) := \frac{i}{\sqrt{2}} \sum_{\lambda=1,2} \int d^3k \, \epsilon_\lambda(k) e^{ikx} \alpha_\lambda(k, \lambda),
\]

\[
F^-(x, t) := -\frac{i}{\sqrt{2}} \sum_{\lambda=1,2} \int d^3k \, \epsilon_\lambda(k) e^{-ikx} \alpha^*_\lambda(k, \lambda).
\]

By means of the cutoff function \( \eta \) with Fourier transform

\[
\mathcal{F}[\eta](k) = (2\pi)^{-3/2} |k|^{-1/2} \mathcal{F}[\epsilon](k)
\]

we can write the quantum and classical vector potentials as

\[
\hat{A}_\epsilon(x) = -i(\eta * \hat{F}^+)(x) + i(\eta * \hat{F}^-)(x),
\]

\[
A_\epsilon(x, t) = -i(\eta * F^+)(x, t) + i(\eta * F^-)(x, t).
\]

These relations are the analogue of [25, Lemma 6.1]. Thus if we replace \( \hat{E}_\epsilon^\sharp, E_\epsilon^\sharp \) in the original estimates of [25] by \( \hat{F}^\sharp, F^\sharp \) with \( \sharp \in \{-, +\} \) and use

\[
\int d^3y \left\| \left( N^{-1/2} \hat{F}^+(y) - F^+(y, t) \right) \Psi_{N, t} \right\|_{H^s(\mathbb{R}^3)}^2 = 4\pi^3 \beta'(t)
\]

we obtain (III.28) by similar means. This implies (III.29) because of Grönwall’s inequality.

\[\square\]

A Properties of (I.10)

Proof of Corollary I.2. For the initial data of the corollary we have that \((\varphi_0, A_0, E_0) \in H^2 \times H^2 \times H^1\). The existence of a unique global solution \((\varphi, \alpha)\) with \(\varphi_t \in H^2(\mathbb{R}^3)\) and \((| \cdot |^{1/2} + | \cdot |^{3/2}) \alpha \in \mathbb{B}\) then follows from Proposition I.1. In order to see that \(\alpha \in \mathbb{B}\) we bound the integral version of (I.10) by

\[
\|\alpha\|_\mathbb{B} \leq \|\alpha_0\|_\mathbb{B} + \int_0^t ds \left( \| | \cdot |\alpha_s \|_\mathbb{B} + C \left\| | \cdot |^{-1/2} \mathcal{F}[|\kappa| \epsilon \cdot \mathcal{F}[j_s]] \right\|_\mathbb{B} \right).
\]

(A.1)

Using Hölder’s inequality and Young’s inequality we get

\[
\left\| | \cdot |^{-1/2} \mathcal{F}[|\kappa| \epsilon \cdot \mathcal{F}[j_s]] \right\|_\mathbb{B} \leq C \left\| | \cdot |^{-1/2} \mathcal{F}[|\kappa|] \right\|_{L^2(\mathbb{R}^3)} \left( \|\varphi_s \nabla \varphi_s\|_{L^1(\mathbb{R}^3, C^3)} + \|\varphi_s^2 \kappa \ast A_s\|_{L^1(\mathbb{R}^3, C^3)} \right)
\]

\[
\leq C \left\| | \cdot |^{-1/2} \mathcal{F}[|\kappa|] \right\|_{L^2(\mathbb{R}^3)} \|\varphi_s\|_{H^1(\mathbb{R}^3)}^2 \left( 1 + \left\| (-\Delta)^{-1/4} \kappa \right\|_{L^2(\mathbb{R}^3)} \left\| (-\Delta)^{1/4} A_s\right\|_{L^2(\mathbb{R}^3)} \right)
\]

\[
\leq C \|\varphi_s\|_{H^1(\mathbb{R}^3)}^2 \left( \|A_s\|_{H^1(\mathbb{R}^3, C^3)} + 1 \right) \left( \left\| | \cdot |^{-1/2} \mathcal{F}[|\kappa|] \right\|_{L^2(\mathbb{R}^3)}^2 + 1 \right).
\]

(A.2)

With the help of Proposition I.1 we conclude that the right hand side of (A.1) is finite. This shows the claim.

\[\square\]
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