Slow modes in passive advection

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Abstract

The anomalous scaling in the Kraichnan model of advection of the passive scalar by a random velocity field with non-smooth spatial behavior is traced down to the presence of slow resonance-type collective modes of the stochastic evolution of fluid trajectories. We show that the slow modes are organized into infinite multiplets of descendants of the primary conserved modes. Their presence is linked to the non-deterministic behavior of the Lagrangian trajectories at high Reynolds numbers caused by the sensitive dependence on initial conditions within the viscous range where the velocity fields are more regular. Revisiting the Kraichnan model with smooth velocities we describe the explicit solution for the stationary state of the scalar. The properties of the probability distribution function of the smeared scalar in this state are related to a quantum mechanical problem involving the Calogero-Sutherland Hamiltonian with a potential.

1 Introduction

One of the basic open problems in fully developed hydrodynamical turbulence is the understanding of the origin of observed violations of the Kolmogorov scaling. The violations indicate presence of strong short-distance intermittency in the turbulent cascade, i.e. of frequent occurrence of large fluctuations on short distances. Recently some progress has been achieved in the understanding of the analogous problem for the passive advection of a scalar quantity by a random velocity field. The scalar is known to exhibit strong short-distance intermittency even if such is absent in the velocity field. In
the simplest model of the passive scalar, due to Kraichnan [2], one assumes a Gaussian
distribution of time-decorrelated and spatially non-smooth velocities. The anomalous
scaling of the scalar in this model was related in references [3][4][5] to zero modes of
differential operators describing the stochastic evolution of the flow. In the present
paper we elaborate on this idea showing that the short-distance intermittency of the
scalar is due to the presence of slow collective modes in the otherwise super-diffusive
evolution of the (quasi)-Lagrangian trajectories of fluid particles. We show that in the
Kraichnan model the slow modes, reminiscent of resonances in multi-body problems,
are organized into infinite multiplets of descendants with the zero modes playing the
role of primary objects. This structure might indicate the presence of hidden infinite
symmetries in the Kraichnan problem.

The other important feature of the Lagrangian flow in non-smooth velocities is its
intrinsically probabilistic character: the Lagrangian trajectories of the fluid particles
behave randomly even in a fixed velocity field. This phenomenon appears to be closely
related to the presence of the slow modes in the stochastic flow of fluid particles. In
more realistic velocity fields which are regularized on the viscous scale the effective
stochasticity of the fluid trajectories is due to their sensitive dependence on initial
conditions on scales shorter than the viscous one. We expect both phenomena: the
presence of resonant slow modes in the Lagrangian flow and the non-deterministic
character of the fluid trajectories, to be present in more realistic high Reynolds number
velocity ensembles and to be responsible for their intermittency.

The version of the Kraichnan model with smooth velocity fields, relevant for the
description of the distances smaller than the viscous scale, has been intensively studied,
see [3] and [7] to [13]. We return to this case developing further the tools of harmonic
analysis used first for this model in [3] and [8]. These tools allow a fast calculation of
the Lyapunov exponents for the flow of fluid particles found first in [1] and [14]. We
also explicitly construct the stationary state of the scalar relating its functional Fourier
transform to a certain Schrödinger operator on the symmetric space $SL(d)/SO(d)$
where $d$ is the space dimension. In particular, we compute the exponential decay
rate of the probability distribution functions $p(\theta)$ of smeared scalar values obtaining in
three dimensions (and above) a result different from that of [9][10][13]. The discrepancy
is traced to the contribution of correlations of different (pairs of) fluid trajectories
disregarded in [9][10][13]. For the exponential decay rate of the Fourier transform of
$p(\theta)$ our results reproduce fully the calculations of [9][14] and confirm their semiclassical
interpretation proposed in [13].

The paper is organized as follows. In Sect. 2 we present the Kraichnan model and
obtain its solution employing a path integral formalism. Sect. 3 recalls briefly the anal-
ysis of [3] and [13] establishing anomalous scaling of the scalar by perturbative analysis
of the scaling zero modes of operators governing the flow. The physical interpretation
of the zero modes as scaling structures conserved in mean is the subject of Sect. 4. Sect.
5 discusses the collective slow modes of the random flow of fluid particles. The analytic
origin of the slow modes is unraveled in more technical Sect. 6. Sect. 7 describes the
intricacies of the probabilistic description of fluid trajectories. Finally, Sects. 8 and 9
study in detail the case of Kraichnan model with smooth velocity field elaborating on the earlier results of [3] and of [7] to [13]. Appendix A explicitly analyzes the slow modes in the relative motion of two fluid particles in non-smooth velocity field. Appendix B contains some more details on the smooth velocity case related to the results of [9] and [12].

2 Kraichnan model of passive scalar

Let us consider an advection of a scalar quantity $T(t, x)$ (the temperature) in $d$ space dimensions. The time evolution of $T$ is governed by the linear equation

$$\frac{\partial}{\partial t} T + v \cdot \nabla T - \kappa \Delta T = f \quad (2.1)$$

where $v(t, x)$ is the incompressible ($\nabla \cdot v = 0$) velocity field of the advecting fluid, $\kappa$ is the diffusion constant and $f(t, x)$ is a given source term. Denote by $R(t, x; t_0, x_0)$ the solution of the homogeneous equation

$$(\partial_t + v \cdot \nabla - \kappa \Delta) R(t, t_0) = 0 \quad (2.2)$$

with the initial condition

$$R(t_0, x; t_0, x_0) = \delta(x - x_0). \quad (2.3)$$

We shall call $R(t, x; t_0, x_0)$ the evolution kernel and the corresponding operator $R(t, t_0)$ the evolution operator. The solution of Eq. (2.1) has the form

$$T(t, x) = \int R(t, x; t_0, y) T(t_0, y) dy + \int_{t_0}^{t} \int R(t, x; s, y) f(s, y) ds dy \quad (2.4)$$

with $T(t_0)$ being the initial configuration of $T$ at time $t_0$.

There exists a functional integral formula for the evolution kernel which, for sufficiently regular $v$, may be easily given a rigorous sense as an integral with respect to the Wiener measure with density:

$$R(t, x; t_0, x_0) = \int_{x(t_0) = x_0}^{x(t) = x} e^{-\frac{1}{4\kappa} \int_{t_0}^{t} ds [x(s) - v(s, x(s))]^2} D x \quad (2.5)$$

where $x \equiv \frac{dx}{dt}$. It will be useful to rewrite the above functional integral as a phase space one:

$$R(t, x; t_0, x_0) = \int_{x(t_0) = x_0}^{x(t) = x} e^{-\int_{t_0}^{t} ds [\kappa p(s)^2 + i p(s) \cdot (x(s) - v(s, x(s)))]} D x D p \quad (2.6)$$

with the Gaussian integral over the unconstrained paths $s \mapsto p(s)$ reproducing the previous integral. From the functional integral representations it is clear that when $\kappa \to 0$ then

$$R(t, x; t_0, x_0) \to \delta(x - x(t; t_0, x_0)) \quad (2.7)$$
where \( x(t; t_0, x_0) \) is the Lagrangian trajectory of the fluid particle satisfying the equations

\[
\dot{x} = v(t, x), \quad x(t_0) = x_0.
\]  

(2.8)

Indeed, we may set \( \kappa = 0 \) in the phase space integral \((2.6)\) and the \( p \)-integral gives then a delta function(al) concentrated on the Lagrangian trajectory. For small positive \( \kappa \), on the other hand, \( R(t, x; t_0, x_0) \) is the probability distribution function (p.d.f.) of the endpoint of a small Brownian motion around the Lagrangian trajectory. Such a Brownian motion \( x_\beta(t; t_0, x_0) \) with a drift is a solution of the stochastic ordinary differential equation (ODE)

\[
dx = v(t, x) \, dt + \kappa \, d\beta, \quad x(t_0) = x_0
\]

(2.9)

with \( \beta(t) \) denoting the Brownian motion without drift. Thus

\[
R(t, x; t_0, x_0) = E(\delta(x - x_\beta(t; t_0, x_0)))
\]

(2.10)

where \( E(\cdot) \) denotes the expectation with respect to the Wiener measure of \( \beta \). Eq. \((2.10)\) is another form of Eq. \((2.5)\).

We shall be interested in the situation when both velocities \( v \) and source \( f \) in Eq. \((2.1)\) are random so that Eq. \((2.1)\) is a stochastic PDE. In order to solve such a stochastic equation, we should define the evolution kernel \( R(t, x; t_0, x_0) \) as a random, \( v \)-dependent process or, in plain English, be able to compute various expectation values of \( R \)'s like

\[
\mathcal{P}_n(t, x; t_0, x_0) \equiv \langle \prod_{i=0}^n R(t_i, x_i; t_{0,i}, x_{0,i}) \rangle = E(\prod_{i=0}^n \delta(x_i - x_{\beta_i}(t_i; t_{0,i}, x_{0,i})))
\]

(2.11)

where \( t \equiv (t_1, \ldots, t_n), \ x \equiv (x_1, \ldots, x_n) \) and similarly for \( t_0, \ x_0 \) and where \( E(\cdot) \) denotes the expectation w.r.t. \( \beta_i \)'s and \( v \). It is clear from the second expression for \( \mathcal{P}_n(t, x; t_0, x_0) \) that it gives the joint p.d.f. of the ends \( x_{\beta_i}(t_i; t_{0,i}, x_{0,i}) \) of \( n \) Brownian motions (independent for given \( v \)) around the Lagrangian trajectories starting at times \( t_{0,i} \) from points \( x_{0,i} \). The \( \kappa \to 0 \) limit of \( \mathcal{P}_n(t, x; t_0, x_0) \) (if exists) should simply give the joint p.d.f. of the endpoints \( x(t_i; t_{0,i}, x_{0,i}) \) of \( n \) Lagrangian trajectories \(^1\).

Let us assume that the velocity is a Gaussian stationary field with mean zero and covariance

\[
\langle v^\alpha(t_1, x_1) v^\beta(t_2, x_2) \rangle = D^\alpha\beta(t_{12}, x_{12})
\]

(2.12)

where \( t_{12} \equiv t_1 - t_2, \ x_{12} \equiv x_1 - x_2 \) and \( \partial_\alpha D^\alpha\beta(t, x) = 0 \) to assure the incompressibility. Employing the phase space path integral representation \((2.3)\) and performing the Gaussian functional integration over \( v \), we obtain \(^2\)

\[
\mathcal{P}_n(t, x; t_0, x_0) = \left\langle \int e^{-\sum_i \int_{x_i(t_0) = x_{0,i}}^{x_i(t_1) = x_i} ds_i [\kappa p(s_i)^2 + i p(s_i) \cdot (x_i(s_i) - v(s_i, x_i(s_i)))]} \right\rangle Dx Dp
\]

\(^1\)K.G. thanks Ya. Sinai for attracting his attention to the statistics of Lagrangian trajectories

\(^2\)similar expressions appeared in \([14],[15]\)
stationary processes. Imposing also the zero initial condition for $T$ in passing from the path integral to the expression for compare Eq. (2.6). Note that, due to incompressibility, there is no ordering ambiguity obtains using Eq. (2.4) the following expression for the correlators of i.e. that follows from the semigroup law for the heat kernels.

Equation (2.1) in the time decorrelated case. Of course, the composition property (2.15) may take expressions (2.18) as defining the evolution operators for the stochastic PDE

If, following Kraichnan [2], we assume that $v(t,x)$ is also decorrelated in time, i.e. that

$$D^{\alpha \beta}(t,x) = \delta(t) \, D^{\alpha \beta}(x),$$

then formula (2.13) for $P_n(t,x; t_0, x_0)$ may be further simplified. Let as set all $t_i$ equal to $t$ and all $t_{0i}$ equal to $t_0$. We shall denote the corresponding $P_n$ by $P_n(t,x; t_0, x_0)$ (the general case can be reconstructed from the special one for $v$’s delta-correlated in time). The fundamental property of the p.d.f.’s $P_n(t,x; t_0, x_0)$ for the time-decorrelated velocities (not necessarily Gaussian) is the composition property

$$\int P_n(t,x; s,y) P_n(s,y; t_0, x_0) \, dy = P_n(t,x; t_0, x_0).$$

From expression (2.13) we obtain, assuming relation (2.14),

$$P_n(t,x; t_0, x_0) = \int e^{-\frac{1}{2} \sum_{i,j} \int_{t_{0j}}^{t_j} ds_i \int_{t_{0j}}^{t_j} ds_j \, D^{\alpha \beta}(s_{i-} - s_{j-}, x_i(s_i) - x_j(s_j)) \, p^\alpha_i(s_i) p^\beta_j(s_j)} \, Dx \, Dp. \tag{2.16}$$

It is easy to see that the right hand side is a phase space path integral expression for the heat kernel (dynamical Green function) of the 2nd order (positive, elliptic) differential operator

$$\mathcal{M}_n = -\frac{1}{2} \sum_{i,j=1}^{n} \, D^{\alpha \beta}(x_{ij}) \, \partial_{x_i}^\alpha \partial_{x_j}^\beta - \kappa \sum_{i=1}^{n} \Delta_{x_i},$$

i.e. that

$$P_n(t,x; t_0, x_0) = e^{-(t-t_0)\mathcal{M}_n(x, x_0)}, \tag{2.18}$$

compare Eq. (2.6). Note that, due to incompressibility, there is no ordering ambiguity in passing from the path integral to the expression for $\mathcal{M}_n$. Rigorously minded person may take expressions (2.13) as defining the evolution operators for the stochastic PDE equation (2.1) in the time decorrelated case. Of course, the composition property (2.15) follows from the semigroup law for the heat kernels.

Let us now go back to the passive scalar. Assume that both $v$ and $f$ are independent stationary processes. Imposing also the zero initial condition for $T$ at $t_0 = -\infty$, we obtain using Eq. (2.4) the following expression for the correlators of $T$:

$$\langle \prod_{i=0}^{n} T(t_i, x_i) \rangle = \prod_{i=1}^{n} \int_{-\infty}^{t_i} ds_i \int dy_i \, P_n(t,x; s,y) \langle \prod_{i=1}^{n} f(s_i, y_i) \rangle. \tag{2.19}$$
It should be clear that if a stationary state of the scalar is generated for large time and independent of the initial conditions (say, decaying at infinity) then its correlation functions should be given by the above equation. Hence the importance of understanding the behavior of the p.d.f.’s $P_n(t, x; t_0, x_0)$.

Assume now that the source $f$ (independent of $v$) is a Gaussian process with mean zero and covariance

$$
\langle f(t_1, x_1) f(t_2, x_2) \rangle = \delta(t_{12}) C(x_{12})
$$

(2.20)

where $C$ is a positive definite test function. In this case and for the Gaussian, time decorrelated velocities, Eqs. (2.19) simplify permitting an inductive calculation of the stationary correlation functions of the scalar. Let us see how this works for equal time correlators. We may consider only the even-point functions of $T$,

$$
T(x) \equiv \langle T(t, x_1) \cdots T(t, x_2n) \rangle
$$

and for the 2-point function we obtain

$$
\mathcal{F}_2(x_{12}) = \int_{-\infty}^{t} ds \int d\mathbf{y} \, e^{-(t-s)/M_2(x, \mathbf{y})} C(y_{12}) = \int d\mathbf{y} \, M_2^{-1}(x, \mathbf{y}) C(y_{12})
$$

(2.21)

and for the 4-point function

$$
\mathcal{F}_4(x) = \sum_{1 \leq i<j \leq 4} \int_{-\infty}^{t} ds \int_{-\infty}^{s} ds' \int d\mathbf{y} \int d\mathbf{z} \, e^{-(t-s)/M_4(x, \mathbf{y})} e^{-(s-s')/M_2(y_{1,\cdots,i}, z_{i,j})} C(z_{12})
$$

$$
\mathcal{F}_4(x) = \sum_{1 \leq i<j \leq 4} \mathcal{M}_4^{-1}(x, \mathbf{y}) \mathcal{F}_2(y_{1,\cdots,i}, y_{4}) C(y_{ij}) d\mathbf{y}.
$$

(2.22)

Similar arguments for the $2n$-point function give

$$
\mathcal{F}_{2n}(x) = \sum_{1 \leq i<j \leq 2n} \mathcal{M}_{2n}^{-1}(x, \mathbf{y}) \mathcal{F}_{2n-2}(y_{1,\cdots,n}, y_{2n}) C(y_{ij}) d\mathbf{y}.
$$

(2.23)

The above equations permit an inductive calculation of the stationary equal time correlation functions of $T$ with the use of the (static) Green functions $\mathcal{M}_n^{-1}(x, \mathbf{y})$ of operators $\mathcal{M}_n$.

3 Zero mode dominance

We shall be interested in the case where the spatial part $D^{\alpha\beta}(x)$ of the $v$-covariance has the form

$$
D^{\alpha\beta}(x) = D_0 \delta^{\alpha\beta} - d^{\alpha\beta}(x)
$$

(3.1)

where $d^{\alpha\beta}(x)$ scales with power $2 - \gamma$,

$$
d^{\alpha\beta}(x) \sim |x|^{2-\gamma},
$$

(3.2)
for small $|x|$. Here $0 \leq \gamma \leq 2$ is a fixed parameter. The tensorial form of $d^{\alpha \beta}(x)$ is fixed for small $|x|$ by the incompressibility condition $\partial_\alpha d^{\alpha \beta}(x) = 0$:

$$d^{\alpha \beta}(x) \equiv \frac{D}{d-1} \left( (d+1-\gamma) \delta^{\alpha \beta} |x|^{2-\gamma} - (2-\gamma) x^\alpha x^\beta |x|^{-\gamma} \right) \equiv d^{\alpha \beta}_{sc}(x) \quad (3.3)$$

where $D$ is a constant. For $0 < \gamma < 2$, one may take

$$\mathcal{D}^{\alpha \beta}(x) \sim \int \frac{e^{-ikx}}{(k^2 + m^2)^{(d+2-\gamma)/2}} (\delta^{\alpha \beta} - \frac{k^\alpha k^\beta}{k^2}) \, dk \quad (3.4)$$

where $m$ is an infrared regulator. Relations (3.2) and (3.3) hold then for $m|x| \ll 1$. When $m \to 0$, $d^{\alpha \beta}(x)$ tends to the scaling form $d^{\alpha \beta}_{sc}(x)$ but $\mathcal{D}_0$ diverges like $O(m^{\gamma-2})$.

The Gaussian distribution with covariance given by Eqs. (2.14) and (3.1) is relatively far from a realistic description of the statistics of turbulent flows. First, it excludes the velocity intermittency, i.e., more frequent occurrence of large deviations of velocity differences than in the normal distribution. Such occurrence characterizes short scales in the inertial interval of the turbulent cascade. Second, the time decorrelation is a brutal approximation since one observes scale-dependent time correlations in turbulent flows. The power-law growth of the velocity difference covariance

$$\langle (v^\alpha(t_1, x_1) - v^\alpha(t_1, x_2))(v^\beta(t_2, x_1) - v^\beta(t_2, x_2)) \rangle = 2 \delta(t_{12}) d^{\alpha \beta}(x_{12})$$

$$\sim \delta(t_{12}) |x_{12}|^{2-\gamma} \quad (3.5)$$

mimics, however, the expected behavior in the turbulent cascade (the Kolmogorov value of the scaling exponent corresponds to $\gamma = \frac{2}{3}$ since time appears to scale like length to power $\gamma$ in the model). The point is that even the velocity distributions far from realistic, as the one described above, induce strongly intermittent scalar distributions and the purpose of the Kraichnan model is to understand this phenomenon in the simplest context.

Operators $\mathcal{M}_n$ may be rewritten in the form

$$\mathcal{M}_n = \sum_{1 \leq i < j \leq n} d^{\alpha \beta}(x_{ij}) \partial_i \partial_j - \kappa \sum_{i=1}^n \Delta x_i - \frac{1}{2} \mathcal{D}_0 \left( \sum_{i=1}^n \partial x_i \right)^2 \quad (3.6)$$

where the last term drops out in the action on translationally invariant functions. We shall, somewhat pedantically, denote the operator $\mathcal{M}_n$ acting in the translational invariant sector by $\mathcal{M}_n$. We shall view $\mathcal{M}_n$ as an operator in the reduced space $L^2(\mathbb{R}^{d_n})$, with $d_n \equiv (n-1)d$. This is the space of functions of the difference variables $x_{in} \equiv x_i - x_n$, square-integrable with the measure $d'x \equiv dx_1 \cdots dx_{(n-1)n}$. The heat kernel of $\mathcal{M}_n$,

$$e^{-(t-t_0)\mathcal{M}_n}(x, x_0) = \int_{\mathbb{R}^d} \mathcal{P}_n(t, x + a; t_0, x_0) \, da \equiv \mathcal{P}_n(t, x; t_0, x_0), \quad (3.7)$$

with $a \equiv (a, \ldots, a)$, gives the joint p.d.f. of the differences $x_{in}$ of the Lagrangian trajectories starting at points $x_0$ (or, equivalently, the joint p.d.f. of the Lagrangian
trajectories in the quasi-Lagrangian picture [10]). It is translationally invariant separately in $\mathbf{x}$ and $\mathbf{x}_0$.

In the limit $m \to 0$ when $d^{\alpha\beta}(x)$ takes the scaling form [13] but $\mathcal{D}_0$ diverges, operator $M_n$, unlike $\mathcal{M}_n$, tends to the limit which for $\kappa = 0$ coincides with the scaling operator

$$
M_{n}^{sc} = \sum_{1 \leq i < j \leq n} d_{sc}^{\alpha\beta}(x_{ij}) \partial_{x_i^\alpha} \partial_{x_j^\beta}
$$

of scaling dimension $-\gamma$. $M_{n}^{sc}$ is a positive singular elliptic differential operator of the $2^{nd}$ order in $L^2(\mathbb{R}^{d_n})$. By a simple self-consistent analysis one may convince oneself that, at least for $\gamma$ close to 2, $e^{-tM_n}(x, y)$ and $M_n^{-1}(x, y)$ converge pointwise when $m \to 0$ and $\kappa \to 0$ to the heat kernel $e^{-tM_{n}^{sc}}(x, y)$ and the Green function $(M_{n}^{sc})^{-1}(x, y)$, respectively. The latter should satisfy bounds that may be inferred from a semi-classical analysis of the path integral expressions (2.16) with $\mathcal{D}^{\alpha\beta}$ replaced by $d_{sc}^{\alpha\beta}$ and $\kappa$ set to zero. In the limit $\gamma \to 2$, $d_{sc}^{\alpha\beta}(x)$ tends for non-zero $x$ to a constant times $\delta^{\alpha\beta}$ and $M_{n}^{sc}$ becomes proportional to the $d_n$-dimensional Laplacian. When $\gamma$ is close to 2, the heat kernel $e^{-tM_n^{sc}}(x, y)$ and the Green function $(M_{n}^{sc})^{-1}(x, y)$ differ little from the heat kernel and the Green function of the Laplacian. In particular, $e^{-tM_{n}^{sc}}(x, y)$ is finite everywhere and $(M_{n}^{sc})^{-1}(x, y)$ is infinite only when $x = y$. We expect this to hold for all $\gamma > 0$. When $2 - \gamma$ is small, the behaviors of $(M_{n}^{sc})^{-1}(x, y)$ around $x = y$ and at infinity differ from those of the Green function of the $d_n$-dimensional Laplacian by $O(2 - \gamma)$ modifications of the power laws. All that implies that the pointwise limits $m \to 0$ and $\kappa \to 0$ of the equal time correlators of $T$ given by Eqs. (2.23) exist, at least for $\gamma$ close to 2, and are given by the version of the same equations employing the scaling Green functions $(M_{n}^{sc})^{-1}(x, y)$ (with $dy$ replaced by $d'y$). From now on, we shall deal only with these limits and with the scaling operators $M_{n}^{sc}$ and shall drop the superscript ”sc”.

We are interested in the behavior of the equal time correlators of $T$, especially in their scaling properties, in the situation when the source acts only on large distances, i.e., in our Gaussian model, when the spatial part $\mathcal{C}$ of the covariance of $f$ is almost constant. We may study this regime by replacing $\mathcal{C}(x)$ by $\mathcal{C}_L(x) \equiv \mathcal{C}(x/L)$ and by examining the large $L$ behavior of the equal time correlators $\mathcal{F}_{2n} \equiv \mathcal{F}_{2n,L}$. The following result, which may be referred to as the zero mode dominance, has been described in [3] and [5]: at $\gamma$ sufficiently close to 2 and at $m, \kappa = 0$,

$$
\mathcal{F}_{2n,L}(\mathbf{x}) = A_{C} L^{\rho_{2n}} \mathcal{F}_{2n}^{0}(\mathbf{x}) + O(L^{-2 + O(2 - \gamma)}) + \ldots
$$

for $n > 1$. Above, $A_{C}$ is a non-universal amplitude (a constant depending on the shape of the source covariance $\mathcal{C}$) and $\rho_{2n} = \frac{2(n-1)}{d+2}(2 - \gamma) + O((2 - \gamma)^2)$ is a universal (i.e. $\mathcal{C}$-independent) anomalous exponent. $\mathcal{F}_{2n}^{0}$ is the scaling (translationally invariant) zero mode of $M_{2n}$,

$$
\mathcal{F}_{2n}^{0}(\lambda\mathbf{x}) = \lambda^{\gamma n - \rho_{2n}} \mathcal{F}_{2n}^{0}(\mathbf{x}), \quad M_{2n} \mathcal{F}_{2n}^{0} = 0.
$$

³to our knowledge, such bounds have not been obtained in the mathematical literature and constitute an open mathematical problem
\[ \mathcal{F}_{2n}^0(x) = S x_{12}^2 x_{34}^2 \cdots x_{2n-1,2n}^2 + \mathcal{O}(2 - \gamma) + [\ldots] \] (3.11)

where \( S \) is the symmetrization operator. The contribution to \( \mathcal{F}_{2n}^0 \) proportional to \( 2 - \gamma \) is also known up to \([\ldots]\) terms \([6]\). A similar analysis was performed in \([4]\) and \([17]\) for large space dimensions \( d \).

The main implication of the relation (3.9) is the anomalous scaling of the \( n > 1, \gamma \) close to 2 structure functions \( S_{2n,L}(x) \equiv \langle (T(t,x) - T(t,0))^2 \rangle \). At \( m, \kappa = 0 \) and for \(|x|/L \ll 1\),

\[ S_{2n,L}(x) \sim L^{2n} |x|^{\gamma_n - 2n}. \] (3.12)

The above behavior contradicts the simple dimensional prediction \( S_{2n}(x) \sim |x|^{\gamma_n} \) which holds only for the 2-point function.

Let us sketch the argument leading to the result (3.9), based on applying the Mellin transform to select the dominant contributions for large \( L \). It will be convenient to work with a version of operators \( M_n \) of scaling dimension zero

\[ N_n = R_n^{\gamma/2} M_n R_n^{\gamma/2} \] (3.13)

where \( R_n^2 \equiv \sum_{i<j} (x_i - x_j)^2 \). \( N_n \) is also a positive (unbounded) operator in \( L^2(\mathbb{R}^{d_n}) \). Since it commutes with the self-adjoint generator of dilations

\[ D_n = \frac{1}{i} (\sum_i x_i^2 \partial_{x_i} + \frac{d_n}{2}), \] (3.14)

it is partially diagonalized by the Mellin transform of the translationally invariant functions

\[ f(x) \rightarrow \hat{f}(\sigma, \tilde{x}) = \int_0^\infty \lambda^{-\sigma - 1} f(\lambda \tilde{x}) \, d\lambda. \] (3.15)

The map (3.15) is a unitary transformation, diagonalizing \( D_n \), between

\[ L^2(\mathbb{R}^{d_n}) \quad \text{and} \quad L^2(\{\text{Re} \sigma = -\frac{d_n}{2}\}) \otimes L^2(S^{d_n-1}) \]

where \( S^{d_n-1} \) is composed of points \( \tilde{x} = x/R_n \) in the space \( \mathbb{R}^{d_n} \) of difference variables. In the language of the Mellin transform, \( N_n \) becomes a family \( \hat{N}_n(\sigma) \) of operators in \( L^2(S^{d_n-1}) \). In particular,

\[ (N_n^{-1} f)(\sigma, \tilde{x}) = \int_{S^{d_n-1}} \hat{N}_n^{-1}(\sigma; \tilde{x}, \tilde{y}) \hat{f}(\sigma, \tilde{y}) \, d\tilde{y}. \] (3.16)

\(^4\text{technically, } N_n, \text{ as well as } M_n, \text{ may be defined as the Friedrichs extension of its restriction to smooth functions with compact support, vanishing around the diagonals } x_i = x_j\)
where the Mellin-transformed Green function $\hat{N}_n^{-1}(\sigma; \hat{x}, \hat{y})$ satisfies the hermiticity relation

$$\hat{N}_n^{-1}(\sigma; \hat{y}, \hat{x}) = \hat{N}_n^{-1}(-d_n - \sigma; \hat{x}, \hat{y}). \quad (3.17)$$

It is a meromorphic function of $\sigma$ with simple poles for generic $\gamma$. Around the poles

$$\hat{N}_n^{-1}(\sigma - \frac{\gamma}{2}; \hat{x}, \hat{y}) \simeq \frac{1}{\sigma - \sigma_i} f_i(\hat{x}) g_i(\hat{y}) \quad (3.18)$$

where $f_i$ are the scaling zero modes of $M_n$ of scaling dimension $\sigma_i$ and $g_i$ are similar modes with scaling dimensions $-d_n + \gamma - \sigma_i$, both in $L^2(S^{d_n-1})$. Although operator $M_n$ has continuous spectrum when considered as a positive operator in $L^2(\mathbb{R}^{d_n})$, it induces an operator $\hat{N}_n(\sigma - \frac{\gamma}{2})$ in $L^2(S^{d_n-1})$ with a discrete spectrum when acting on scaling functions with a scaling dimension $\sigma$. The scaling zero modes occur at discrete values $\sigma_i$ of $\sigma$ for which zero belongs to the spectrum.

It is easy to see from the inductive equations (2.23) that

$$\mathcal{F}_{2n,L}(x) = L^{n\gamma} \mathcal{F}_{2n,1}(x/L) \quad (3.19)$$

and that, with the use of the Mellin transform, these equations may be rewritten as

$$\mathcal{F}_{2n,L}(x) = L^{\gamma n} \int_{\text{Re} \sigma = -\frac{d_n}{2} + \frac{\gamma}{2}} \frac{d\sigma}{2\pi i} \left( R_n/L \right)^\sigma \hat{N}_n^{-1}(\sigma - \frac{\gamma}{2}; \hat{x}, \hat{y}) \cdot (\mathcal{F}_{2n-2,1} \otimes \mathcal{C})^\gamma(\sigma - \gamma, \hat{y}) \, d\hat{y}. \quad (3.20)$$

Shifting the integration contour to $\text{Re} \sigma = \gamma n + 2 - \mathcal{O}(2 - \gamma)$, we obtain

$$\mathcal{F}_{2n,L}(x) = -\sum_i L^{\gamma n - \sigma_i} R_n^{\sigma_i} \int_{\text{Res} = \sigma_i} \hat{N}_n^{-1}(\sigma - \frac{\gamma}{2}; \hat{x}, \hat{y}) \cdot (\mathcal{F}_{2n-2,1} \otimes \mathcal{C})^\gamma(\sigma - \gamma, \hat{y}) \, d\hat{y} + \mathcal{O}(L^{-2+\mathcal{O}(2 - \gamma)}) \quad (3.21)$$

where the sum runs over the poles $\sigma_i$ in the strip

$$-\frac{d_n}{2} + \frac{\gamma}{2} < \text{Re} \sigma_i < \gamma n + 2 - \mathcal{O}(2 - \gamma) \quad (3.22)$$

and the last term, suppressed for large $L$, comes from the shifted contour. There are two types of poles: those coming from $(\mathcal{F}_{2n-2,1} \otimes \mathcal{C})^\gamma$ and those in the Green function $\hat{N}_n^{-1}$. The first ones contribute either to $\ldots$ or to $\mathcal{O}(L^{-2+\mathcal{O}(2 - \gamma)})$ in Eq. (3.21) and are not interesting for us (at least for $\gamma$ close to 2). The second ones are related to the scaling zero modes of $M_n$, see Eq. (3.18). Only rotationally invariant (if $\mathcal{C}$ has the same property) zero modes symmetric under permutations of points and square-integrable on $S^{d_n-1}$ contribute to $\mathcal{F}_{2n,L}$. Such zero modes may be studied for $\gamma$ close to 2 by perturbative analysis of discrete-spectrum operators $M_n$ acting on scaling functions or, equivalently, of operators $\hat{N}_n(\sigma)$ acting in $L^2(S^{d_n-1})$. (Recall that for $\gamma = 2$, $M_n$ becomes the $d_n$-dimensional Laplacian). The result is that, for $\gamma$ close to 2, all but one zero modes in the strip (3.22) contribute $\ldots$ terms. $\mathcal{F}_{2n}^0$ is the exception and
it has scaling dimension \( \sigma_0 = \gamma n - \rho_{2n} \). We expect essentially the same picture with the zero mode domination of correlation functions to persist for all \( \gamma > 0 \). One of the open problems is whether there are other non-\([\ldots]\) zero modes entering the strip \((3.22)\) for smaller \( \gamma \) and whether, if they cross, they may produce pairs of zero modes with complex scaling dimensions. For \( \gamma = 0 \) the singularities in the inverse symbols of operators \( M_n \) become strong enough to induce continuous spectrum of operators \( \hat{N}_n(\sigma) \) and the picture of zero mode dominance has to be somewhat modified \([3][12]\).

One may also read the \( O(2 - \gamma) \) contribution to the anomalous exponent \( \rho_{2n} \) from the \( O((2 - \gamma) \ln L) \) term in the expansion of \( \mathcal{F}_{2n,L} \) into powers of \( 2 - \gamma \), similarly as in the \( \varepsilon \)-expansion for critical phenomena one obtains anomalous exponents from logarithmic divergences. In the latter case, the renormalization group which exponentiates the divergent logarithms provides an explanation why it is reasonable to extract information from badly divergent expansions. In our argument, the Mellin transform analysis played a similar role exponentiating the logarithms of \( L \). One may show \([18]\) that there is an (inverse) renormalization group picture of the advection problem hidden behind the above analogy. The renormalization group for the passive scalar eliminates inductively the long-distance modes, unlike in critical phenomena where it is based on subsequent elimination of the short-distance degrees of freedom.

4 Conserved scaling structures

In view of the domination of the equal time correlators of the scalar by the scaling zero modes of operators \( M_n \), it is important to understand the physical interpretation of such modes. It is, in fact, very simple:

**zero modes are scaling structures preserved in mean by the flow.**

Indeed. Recall that \( e^{-tM_n(x,x_0)} = P_n(t,x;0,x_0) \) and it describes the probability that the differences of \( n \) Lagrangian trajectories starting at time 0 from points \( x_0 \) are at time \( t \) equal to \( x_0 \). The mean value of a translationally invariant function \( f(x) \) of positions of \( n \) fluid particles at time \( t \) is then equal to

\[
\langle f \rangle_{t,x_0} \equiv \int_\mathbb{R}^d f(x) P_n(t,x;0,x_0) \, dx' = \int_\mathbb{R}^d f(x) e^{-tM_n(x,x_0)} \, dx'.
\]

(4.1)

Differentiating the right hand side w.r.t. \( t \), we obtain

\[
\int_\mathbb{R}^d f(x) M_n x \, dx' = \int_\mathbb{R}^d M_n f(x) e^{-tM_n(x,x_0)} \, dx
\]

(4.2)

where we have integrated by parts twice. If \( f \) is a zero mode of \( M_n \) then the right hand side vanishes and, consequently, the mean (4.1) is constant and

\[
\langle f \rangle_{t,x_0} = f(x_0).
\]

(4.3)

In fact, the story is a little bit more complicated. The zero modes with \( \text{Re} \sigma_i > -d_n + \gamma \) are true zero modes. However the ones with the real part of their dimension
≤ −d_n + γ are not. For them, M_n f is a contact term supported at the origin. Such contact terms may give non-zero contributions to the right hand side of Eq. (4.2) or to the boundary terms in the integration by parts, depending on the interpretation. The zero modes with the scaling dimensions belonging to the strip (3.22) are true zero modes and hence they describe scaling structures of the flow conserved in mean. As was mentioned before, the translationally invariant zero modes that are square-integrable on S^{d_n−1} come in pairs (f_i, g_i) corresponding to scaling dimensions σ_i and −d_n + γ − σ_i (we may assume that Reσ_i ≥ −d_n/2 + γ/2). For γ close to 2 there are no zero modes square integrable on S^{d_n−1} with dimensions in the strip −d_n + γ < Reσ < 0. We expect this to hold for any γ > 0. In that situation f_i are the true zero modes and they have non-negative real parts of the scaling dimension whereas g_i are the false zero modes with real parts of dimension ≤ −d_n + γ and with M_n g_i being contact terms. Our claim about the absence of zero modes in the strip −d_n + γ < Reσ < 0 may seem paradoxical if we recall that the multi-body structure of operators M_n assures that zero modes of M_{n−p} are also annihilated by M_n. Indeed, the (false) zero modes of M_{n−p} with Reσ ≤ −d_{n−p} + γ may lie in the strip −d_{n−p}/2 + γ < Reσ < 0. However, the resulting zero modes of M_n are not in L^2(S^{d_n−1}) and do not contribute to the poles of the Green function ˜N_n−1(˜x, ˜y) and hence to the right hand side of Eq. (3.21).

It should be stressed that the behavior (4.3) is atypical. For a general translationally invariant, scaling function with (say, positive) dimension σ and for any time τ > 0,

\[ \langle f \rangle_{t, x_0} = \left( \frac{1}{\tau} \right)^{\sigma} \int f(x) \ e^{-\tau M_n(x, (\frac{\tau}{\gamma})^{\frac{1}{2}} x_0)} \ d'x \]  

(4.4)

as it is easy to see with the use of the scaling property

\[ e^{-\lambda \gamma t M_n(\lambda x, \lambda x_0)} = \lambda^{-d_n} e^{-t M_n(x, x_0)}. \]  

(4.5)

It follows that, typically,

\[ \langle f \rangle_{t, x_0} \sim t^{\frac{\sigma}{\gamma}}. \]  

(4.6)

The behavior (4.6) characterizes a super-diffusion where the square distances between points grow faster than linearly in time. A slower behavior requires vanishing of \( f f(x) \ e^{-\tau M_n(x, 0)} \ d'x \). Note that the exponent of the growth diverges when γ → 0.

It is easy to understand the origin of the behavior (4.6). The stochastic process described by the probabilities P_n(t, x; 0, x_0) = e^{-t M_n(x, x_0)} may be viewed as a diffusion with the diffusion coefficient proportional to the power 2 − γ of the distance between the particles. When particles separate they diffuse faster and faster which results in the super-diffusive behavior with mean distance square growing proportionally to \( t^{2/\gamma} \). On the other hand, on small distances the diffusion is slow and particles which get close spend relatively long time together. Since \( \langle f \rangle_{0, x_0} = f(x_0) \), the time t after which \( \langle f \rangle_{t, x_0} \) reaches, say, twice its original value behaves like \( O(f(x_0)^{\frac{\sigma}{2}}) \), i.e. it goes slower

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K.G. thanks E. Balkovsky, G. Falkovich and V. Lebedev for a discussion of this point
to zero with the diminishing separation between the initial points than for the standard diffusion at $\gamma = 2$.

We have seen that the scaling zero modes of $M_n$ correspond to **conserved collective modes** of the super-diffusion with the transition probabilities $P_n(t; x; t_0, x_0)$. Existence of such conserved modes is nothing exceptional. They are present already in the standard diffusion. For example,

$$
\int \left[ x_{12}^2 - x_{13}^2 \right] e^{t \Delta(x, x_0)} \, dx = x_{0,12}^2 - x_{0,13}^2
$$

(4.7)

and is time independent, although $\int x_{ij}^2 e^{t \Delta(x, x_0)} \, dx$ behaves like $O(t)$. Another example is

$$
\int \left[ x_{12}^2 x_{34}^2 - \frac{d}{2(d+2)}(x_{12}^4 + x_{34}^4) \right] e^{t \Delta(x, x_0)} \, dx = x_{0,12}^2 x_{0,34}^2 - \frac{d}{2(d+2)}(x_{0,12}^4 + x_{0,34}^4).
$$

(4.8)

Under symmetrization, the first conserved mode vanishes whereas the second one gives the zero mode of $\Delta$ whose $(2 - \gamma)$-perturbation dominates the 4-point function of the scalar for $\gamma$ close to 2.

### 5 Some physics: short-distance behavior of fluid particles

The zero mode dominance of the structure functions of the scalar is due to the appearance of such modes in the asymptotics of the Green functions $M_n^{-1}(x, y)$. Indeed, with the use of the Mellin transform, one may write (in the reduced space):

$$
M_n^{-1}(x/L, y) = \int_{\Re \sigma = \frac{d_n}{2} + \frac{\gamma}{2}} \frac{d\sigma}{2\pi i} R_n(x/L)^\sigma \tilde{N}_n^{-1}(\sigma - \frac{\gamma}{2}, \hat{x}, \hat{y}) R_n(y)^{-d_n + \gamma - \sigma},
$$

(5.1)

compare to Eq. (3.20). Pushing the integration contour more and more to the right and using Eq. (3.18) to control the residues of the poles, we obtain for $y \neq 0$ the asymptotic large $L$ expansion:

$$
M_n^{-1}(x/L, y) = \sum_i L^{-\sigma_i} f_i(x) \overline{g_i(y)}
$$

(5.2)

with $\Re \sigma_i > -\frac{d_n}{2} + \frac{\gamma}{2}$ or, as we expect, with $\Re \sigma_i \geq 0$. Although it has a similar form to the eigen-function expansion of an operator with discrete spectrum, it has little to do with the spectral decomposition of $M_n^{-1}$. Since $M_n$ is a positive operator in $L^2(\mathbb{R}^{d_n})$ with continuous spectrum coinciding with the positive real line, the spectral decomposition of $M_n^{-1}$ is a continuous integral involving the generalized eigen-functions of $M_n$. The scaling zero modes $f_i$ or $g_i$ of scaling dimensions $\sigma_i$ and $-d_n + \gamma - \sigma_i$, respectively, are square-integrable on $S^{d_n-1}$ but are not generalized eigen-functions of $M_n$ (except for $\sigma_i = 0$). They are rather analogous to resonances in many-body problems with the plane of complex $\sigma$ replacing that of complex energies and $\sigma$ with real part equal to $-\frac{d_n}{2}$ corresponding to real energies.\footnote{This analogy is somewhat loose, since the poles in $\sigma$ live in the first sheet, probably only on the real axis.} Note that due to the hermiticity
and to the overall scaling of the Green function \( M_n^{-1}(x, y) \),
\[
M_n^{-1}(\lambda x, \lambda y) = \lambda^{\gamma-d_n} M_n^{-1}(x, y),
\] (5.3)
the expansion (5.2) may be also rewritten as
\[
M_n^{-1}(Lx, y) = \sum_i L^{-d_n+\gamma-\sigma_i} g_i(x) f_i(y)
\] (5.4)
so that the zero modes \( g_i \) with the scaling dimensions \(-d_n+\gamma-\sigma_i\) of real part less than \(-\frac{d_n+\gamma}{2}\) (or even \(-d_n+\gamma\)) dominate the large distance behavior of the Green function of \( M_n \). Expansion (5.4) may be also obtained directly from Eq. (5.1) by pushing the \( \sigma \)-integration contour to the left.

It is not difficult to see directly that the functions \( f_i \) appearing in expansion (5.2) have to describe scaling structures conserved by the flow. Indeed,
\[
\int M_n^{-1}(x/L, y) e^{-\tau M_n}(x, x_0) \, d'x = \sum_i L^{-\sigma_i} g_i(y) \int f_i(x) e^{-\tau M_n}(x, x_0) \, d'x
\] (5.5)
if we insert expansion (5.2) into the left hand side. But on the other hand,
\[
\partial_\tau \int M_n^{-1}(x/L, y) e^{-\tau M_n}(x, x_0) \, d'x = \partial_\tau L^{d_n-\gamma} \int M_n^{-1}(x, Ly) e^{-\tau M_n}(x, x_0) \, d'x
\]
\[
= -L^{d_n-\gamma} e^{-\tau M_n}(x_0, Ly)
\] (5.6)
where we have used the scaling property (5.3). The (reduced space) heat kernel on the right hand side decays in \( L \) faster than any power for \( y \neq 0 \). Comparing the latter expression to relation (5.5), we infer that \( \partial_\tau \int f_i(x) e^{-\tau M_n}(x, x_0) \, d'x \) has to vanish and hence \( f_i \), a function with scaling dimension \( \sigma_i \), is conserved in mean by the Lagrangian flow. This gives another proof of the statement (4.3).

Since the Green function is given by the time integral of the heat kernel,
\[
M_n^{-1}(x, y) = \int_0^\infty e^{-\tau M_n} \, dt,
\] (5.7)
one may also expect to see the zero modes in the asymptotic behavior of the probabilities \( P_n(t, x; 0, x_0) = e^{-t M_n}(x, x_0) \). Assume an asymptotic expansion
\[
P_n(t, x/L; 0, x_0) = \sum_j L^{-\rho_j} \phi_j(x) \, \psi_j(t, x_0),
\] (5.8)
with \( \text{Re} \rho_j \geq 0 \), describing asymptotics of the probabilities that the Lagrangian trajectories will come at time \( t \) close together. We could expect that functions \( \phi_j \) are again zero modes of \( M_n \). To verify whether this is the case, consider the integral
\[
\int P_n(t, x/L; 0, y) e^{-\tau M_n}(x, x_0) \, d'x = \sum_j L^{-\rho_j} \psi_j(t, y) \int \phi_j(x) e^{-\tau M_n}(x, x_0) \, d'x.
\] (5.9)
The scaling dimension of chain of equations \( \tau \) with the previous reasoning. Note that Eq. (5.12) implies that the functions \( \phi \) may appear, subsequent application of \( M \) may produce a homogeneous zero mode of \( M \) after a finite number of steps. Hence functions \( \phi \) must be organized into towers of descendants \( \phi_{i,p} \) based at zero modes \( f_i \equiv \phi_{i,0} \) of \( M \) and satisfying the chain of equations

\[
M_n \phi_{i,p} = \phi_{i,p-1}, \quad p = 1, \ldots .
\]

The scaling dimension of \( \phi_{i,p} \) is \( \sigma_i + \gamma p \). Since \( M_n^{p+1} \phi_{i,p} = 0 \), it follows that the \((p+1)^{th}\) time derivative of

\[
\int \phi_{i,p}(x) \ e^{-\tau M_n(x,x_0)} \ d'x
\]

vanishes so that the above integral is a polynomial in \( \tau \) of degree \( p \), in accordance with the previous reasoning. Note that Eq. (5.12) implies that the functions \( \psi_{i,p} \) corresponding to \( \phi_{i,p} \) satisfy

\[
\psi_{i,p} = -\partial_t \psi_{i,p-1} = M_n \psi_{i,p-1}.
\]
Summarizing: the asymptotics of probabilities of the Lagrangian trajectories to get close together is dominated by the towers of slow collective modes of the super-diffusion:

$$P_n(t, x/L; 0, x_0) = \sum_{p=0,1,...} L^{-\sigma_1 - \gamma p} \phi_{i,p}(x) \overline{\psi_{i,p}(t, x_0)}. \quad (5.16)$$

Since $P_n(t, x; 0, x_0) = P_n(t, x_0; 0, x)$, expansion (5.16) may be also rewritten as

$$P_n(t, x; 0, x_0/L) = \sum_{p=0,1,...} L^{-\sigma_1 - \gamma p} \overline{\psi_{i,p}(t, x)} \phi_{i,p}(x_0). \quad (5.17)$$

giving the asymptotics of the probabilities of the Lagrangian trajectories starting very close. The leading term on the right hand side is equal to $\overline{\psi_{0,0}(t, x)} = e^{-tM_n(x, 0)}$ and it corresponds to the constant zero mode $\phi_{0,0} = f_0 = 1$.

The above results have an important physical significance for the dynamics of the scalar. Recall the expression (1.4) for the time-dependence of the mean value $\langle f \rangle_{t, x_0}$ of a function $f$ of scaling dimension $\sigma$. Inserting the relation (5.17) to the right hand side of Eq. (1.4) we obtain the asymptotic expansion of $\langle f \rangle_{t, x_0}$ for large $t$:

$$\langle f \rangle_{t, x_0} = \sum_{p=0,1,...} \left( \frac{t}{t_0} \right)^{\sigma-q} \phi_{i,p}(x_0) \int f(x) \overline{\psi_{i,p}(\tau, x)} d'x. \quad (5.18)$$

The leading term corresponds to the constant zero mode. This term dominates for large $t$ if $\int f(x) e^{-tM_n(x, 0)} d'x \neq 0$. However if $f = \phi_{l,q}$ then, as we have seen above, $\langle \phi_{l,q} \rangle_{t, x_0}$ is a polynomial of order $q$ in $t$. Consequently, $\int \phi_{l,q}(x) \overline{\psi_{i,p}(\tau, x)} d'x$ has to vanish unless $\frac{\sigma-q}{\gamma} + q - p$ is an integer between 0 and $q$. Hence $\langle f \rangle_{t, x_0}$ for $f$ equal to a slow mode $\phi_{l,q}$ with a positive scaling dimension is dominated by subleading terms on the right hand side of Eq. (5.18).

To see the lower order terms\footnote{K.G. thanks M. Vergassola for the discussion of this point} for a generic scaling function $f$ for which $\langle f \rangle_{t, x_0} \sim \frac{t^q}{t_{0}},$ it is enough to compare the mean values $\langle f \rangle_{t, x_0}$ for two different $x_0$. For example, subtracting $\langle f \rangle_{t, x_0}$ for two values of $x_{0,1}$ gets rid of the contribution of the constant mode. Denote by $\delta_{y_0,m, y_0',m}$ the operator which performs the subtraction on functions $h$ of $x_{0,m}$:

$$\delta_{y_0,m, y_0',m} h = h(y_0, m) - h(y_0', m). \quad (5.19)$$

Subtracting subsequently at two different values of $x_{0,m}$ for $m = 1, \ldots, n - 1$, and setting $\prod_{m=1}^{n-1} \delta_{y_0,m, y_0',m} \equiv \delta_{y,y'}$, we obtain

$$\delta_{y,y'} \langle f \rangle_{t, \cdot} = \sum_{p=0,1,...} \left( \frac{t}{t_0} \right)^{\sigma-q-p} \delta_{y,y'} \phi_{i,p} \int f(x) \overline{\psi_{i,p}(\tau, x)} d'x \quad (5.20)$$
where the primed sum omits the contributions of the slow modes which do not depend on all variables. Hence the non-constant slow modes dominate the relative motion of groups of Lagrangian trajectories starting from different initial configurations. In particular, the relative motion is slower than the super-diffusive spread of the trajectories. This supports the interpretation of the slow modes as resonance-type objects in the motion of Lagrangian trajectories. The slow modes depending on less variables correspond to resonances in fewer-particle channels which drop out under the subtractions.

6 Some mathematics: structure of the multi-body operators \( M_n \)

To understand analytically the origin of the asymptotic expansion (5.16), let us examine closer operators \( M_n \). We shall work in the reduced space \( \mathcal{H} \equiv L^2(\mathbb{R}^{d_n}) \). \( M_n \) is a positive, unbounded, self-adjoint operator in \( \mathcal{H} \). Let

\[
(U_s f)(x) = e^{s d_n/2} f(e^s x).
\]

Operators \( U_s \) form a unitary version of the 1-parameter group of dilations in \( \mathcal{H} \) with the self-adjoint operator \( D_n \) of Eq. (3.14) as its generator:

\[
U_s = e^{isD_n}.
\]

\( U_s \) preserve the domain of \( M_n \) and

\[
U_s M_n U_s^{-1} = e^{-\gamma s} M_n.
\]

Denote by \( X_n \) the natural logarithm of operator \( M_n \): \( X_n = \ln M_n \). \( X_n \) is an unbounded self-adjoint operator on \( \mathcal{H} \) with the domain invariant under \( U_s \) and the whole real line as the spectrum. The relation (6.3) is equivalent to

\[
U_s X_n U_s^{-1} = X - \gamma s,
\]

i.e. to a strong form of the canonical commutation relation

\[
[D_n, X_n] = i\gamma.
\]

Since under the Mellin transform (3.13) \( D_n \) becomes the multiplication operator by \( \frac{1}{\gamma}(\sigma + \frac{d_n}{2}) \), \( X_n \) must be unitarily equivalent to \( \gamma \partial_\sigma \) by virtue of the von Neumann Theorem on representations of the canonical commutation relations. More exactly, there exists a one-parameter family \( \hat{U}_n(\sigma) \), \( \Re \sigma = -\frac{d_n}{2} \), of unitary operators in \( L^2(S^{d_n-1}) \), unique modulo a right multiplication by a \( \sigma \)-independent unitary operator, such that

\[
(X_n f)(\sigma, \cdot) = \gamma \hat{U}_n(\sigma) \partial_\sigma \hat{U}_n^{-1}(\sigma) \hat{f}(\sigma, \cdot)
\]

for \( \Re \sigma = -\frac{d_n}{2} \). Equivalently,

\[
(M_n f)(\sigma, \cdot) = \hat{U}_n(\sigma) e^{\gamma \partial_\sigma} \hat{U}_n^{-1}(\sigma) \hat{f}(\sigma, \cdot)
\]
or, noting that the operator $\partial_\sigma$ corresponds to the multiplication by $-\ln R_n$ in the language of the original functions,

$$M_n = U_n R_n^{-\gamma} U_n^{-1}$$  \hspace{1cm} (6.8)

where

$$(U_n f)(\sigma, \cdot) = \hat{U}_n(\sigma) \hat{f}(\sigma, \cdot).$$  \hspace{1cm} (6.9)

This is the promised structural result about operators $M_n$. For the heat kernels, we obtain

$$e^{-tM_n} = U_n e^{-t R_n^{-\gamma} U_n^{-1}}.$$  \hspace{1cm} (6.10)

What is the relation of the expression (6.8) to the representation

$$M_n = R_n^{-\gamma/2} N_n R_n^{-\gamma/2}$$  \hspace{1cm} (6.11)

used before, with $N_n$ an operator commuting with $D_n$? The comparison of the two equations gives

$$N_n = R_n^{\gamma/2} U_n R_n^{-\gamma} U_n^{-1} R_n^{\gamma/2} \quad \text{or} \quad N_n^{-1} = R_n^{-\gamma/2} U_n R_n^{\gamma} U_n^{-1} R_n^{-\gamma/2}.$$  \hspace{1cm} (6.12)

Let us suppose that the family of operators $\hat{U}_n(\sigma)$ has a meromorphic continuation to the complex plane of $\sigma$ with no poles in the strip

$$-\frac{d_n}{2} \leq \Re \sigma \leq -\frac{d_n}{2} + \frac{\gamma}{2}$$  \hspace{1cm} (6.13)

(this will prove consistent with our zero mode analysis). Then $R_n^{-\gamma/2} U_n R_n^{\gamma/2}$ becomes under the Mellin transform the operator

$$e^{\frac{\gamma}{2} \partial_\sigma} \hat{U}_n(\sigma) e^{-\frac{\gamma}{2} \partial_\sigma} = \hat{U}_n(\sigma + \frac{\gamma}{2}).$$

The unitarity of $\hat{U}_n$ for $\Re \sigma = -\frac{d_n}{2}$ implies that

$$\hat{U}_n(\sigma) \hat{U}_n(-d_n - \overline{\sigma})^* = \hat{U}_n(-d_n - \overline{\sigma})^* \hat{U}_n(\sigma) = 1$$  \hspace{1cm} (6.14)

and that $\hat{U}_n^{-1}(\sigma)$ also possesses a meromorphic continuation. Operator $R_n^{\gamma/2} U_n^{-1} R_n^{-\gamma/2}$ becomes $\hat{U}_n^{-1}(\sigma - \frac{\gamma}{2})$ under the Mellin transform and Eq. (6.12) gives rise to the relation

$$\hat{N}_n^{-1}(\sigma - \frac{\gamma}{2}) = \hat{U}_n(\sigma) \hat{U}_n^{-1}(\sigma - \gamma).$$  \hspace{1cm} (6.15)

If $\hat{N}_n^{-1}(\sigma - \frac{\gamma}{2})$ has a pole, see Eq. (3.18), then the simplest possibility is that either $\hat{U}_n(\sigma_i - \gamma)$ is regular and then

$$\hat{U}_n(\sigma) = \frac{1}{\sigma - \sigma_i} |f_i \rangle \langle g_i| \hat{U}_n(\sigma_i - \gamma) + O(1)$$  \hspace{1cm} (6.16)
when \( \sigma \to \sigma_i \) or \( \hat{U}_n(\sigma_i) \) is regular and
\[
\hat{U}_n(\sigma_i) = \frac{1}{\sigma - \sigma_i} |f_i\rangle \langle g_i| \hat{U}_n(\sigma - \gamma) + \mathcal{O}(\sigma - \sigma_i) \tag{6.17}
\]
with \( \langle g_i| \hat{U}_n(\sigma - \gamma) = \mathcal{O}(\sigma - \sigma_i) \). In the last case, multiplying by \( \hat{U}_n(-d_n - \sigma_i)^* \) from the left and by \( \hat{U}_n(-d_n - \sigma + \gamma)^* = \hat{U}^{-1}(\sigma - \gamma) \) from the right hand side and taking adjoints, we obtain
\[
\hat{U}_n(-d_n - \sigma + \gamma) = \frac{1}{\sigma - \sigma_i} |g_i\rangle \langle f_i| \hat{U}_n(-d_n - \sigma_i) + \mathcal{O}(1), \tag{6.18}
\]
i.e. relation (6.16) for \( \sigma_i \) replaced by \(-d_n + \gamma - \sigma_i\) and corresponding to the twin zero mode \( g_i \) of \( f_i \). We expect the behavior (6.17) if \( f_i \) is less singular at the origin than \( g_i \) and the behavior (6.16) in the opposite case (in Appendix A, this is established for \( \hat{U}_2 \)). Rewrite Eq. (6.16) as
\[
\hat{U}_n(\sigma + \gamma p) = \hat{N}^{-1}_n(\sigma + \gamma(p - \frac{1}{2})) \hat{U}_n(\sigma + \gamma(p - 1)). \tag{6.19}
\]
Assume that \( \hat{N}^{-1}_n(\sigma_i + \gamma(p - \frac{1}{2})) \) is regular for \( p = 1, 2, \ldots \), i.e. that there are no zero modes of \( M_n \) (square-integrable on \( S^{d_n-1} \)) with scaling dimensions differing by multiplicity of \( \gamma \). This should hold for generic \( \gamma \). From Eq. (6.19) for \( p = 0 \) and from relation (6.16) we infer that
\[
\hat{U}_n(\sigma + \gamma) = \frac{1}{\sigma - \sigma_i} \hat{N}^{-1}_n(\sigma + \gamma) |f_i\rangle \langle g_i| \hat{U}_n(\sigma_i - \gamma) + \mathcal{O}(1)
\equiv \frac{1}{\sigma - \sigma_i} |\phi_{i,1}\rangle \langle g_i| \hat{U}_n(\sigma_i - \gamma) + \mathcal{O}(1) \tag{6.20}
\]
for \( \sigma \to \sigma_i \). By induction on \( p \), it follows then that
\[
\hat{U}_n(\sigma + \gamma p)) = \frac{1}{\sigma - \sigma_i} |\phi_{i,p}\rangle \langle g_i| \hat{U}_n(\sigma_i - \gamma) + \mathcal{O}(1) \tag{6.21}
\]
where
\[
\phi_{i,p} = \hat{N}^{-1}_n(\sigma_i + \gamma(p - \frac{1}{2})) \phi_{i,p-1} \tag{6.22}
\]
Note that Eqs. (6.22) may be rewritten as the chain of relations (5.13) for the scaling functions \( \phi_{i,p}(x) \equiv R^{\sigma_i+\gamma p} \phi_{i,p} (\hat{x}) \) where \( \phi_{i,0} \equiv f_i \) is the zero mode of \( M_n \) of scaling dimension \( \sigma_i \). By virtue of the assumption that there are no zero modes of \( M_n \) of scaling dimension \( \sigma_i + \gamma p \), the tower of descendants \( \phi_{i,p} \) is uniquely determined for each zero mode \( f_i \).

For \( f \) a test function vanishing near the origin, Eq. (5.13) may be rewritten as
\[
(e^{-tM_n} f)(\sigma, \hat{x}) = \int d\tilde{y} \hat{U}_n(\sigma; \hat{x}, \tilde{y}) \int_0^\infty d\lambda \lambda^{-\sigma-1} e^{-t\lambda} \frac{d\sigma'}{2\pi i} \lambda^\sigma' (\hat{U}_n^{-1} f)(\sigma', \tilde{y}). \tag{6.23}
\]

\[8\text{for non-generic } \gamma \text{ the situation may be slightly more complicated with mixing of different towers.}\]
After shifting the \( \sigma' \)-integration contour infinitesimally to the left we may perform the \( \lambda \)-integral. The inverse Mellin transform of the resulting expression gives

\[
(e^{-tM_n} f)(\hat{x}/L) = \frac{1}{\gamma} \int \frac{d\sigma}{2\pi i} L^{-\sigma} \int d\hat{y} \tilde{U}_n(\sigma; \hat{x}, \hat{y}) \int \frac{d\sigma'}{2\pi i} \frac{t^{\sigma'-\sigma}}{\gamma} \Gamma\left(\frac{\sigma-\sigma'}{\gamma}\right) (\tilde{U}_n^{-1} f)(\sigma', \hat{y}).
\] (6.24)

By moving the \( \sigma \)-integration contour further and further to the right\( ^9 \), we obtain from Eq. (6.24) the asymptotic expansion

\[
(e^{-tM_n} f)(\hat{x}/L) = \sum_{p=0,1,...} L^{-\sigma_i-\gamma p} \phi_{i,p}(\hat{x}) \bar{\psi}_{i,p} \] (6.25)

where the sum is over scaling dimensions of zero modes of \( M_n \) satisfying \( \text{Re} \sigma_i > -\frac{dn}{2} \) and where

\[
\bar{\psi}_{i,p} = -\frac{1}{\gamma} \int d\hat{y} \overline{(U_n(\sigma_i-\gamma) g_i)(\hat{y})} \cdot \int \frac{d\sigma'}{2\pi i} \frac{t^{\sigma'-\sigma}}{\gamma} \Gamma\left(\frac{\sigma_i-\sigma'}{\gamma} + p\right) (\tilde{U}_n^{-1} f)(\sigma', \hat{y}).
\] (6.26)

Eq. (6.25) is an integrated version of expansion (5.16) (at least for \( \gamma \) close to 2, there are no scaling zero modes square-integrable on \( S^{d_n-1} \) with \(-d_n < \text{Re} \sigma_i < 0 \) and, as mentioned before, we expect this to hold for all positive \( \gamma \)).

7 Lagrangian flow

There are some subtle points in the above discussion of the motion of fluid particles. When stating that, for \( m, \kappa = 0 \), \( P_n(t, x; 0, x_0) \) is the joint p.d.f. of the differences of the endpoints of \( n \) Lagrangian trajectories, we have silently assumed that such trajectories, or at least their differences, make sense as random processes defining the Lagrangian flow on the probability space of \( v \)'s. A straightforward consequence of such an assumption is the relation

\[
P_n(t, x; 0, x_0) \bigg|_{x_{0,k}=x_{0,k+1}=...=x_{0,n}} = P_k(t, x'; 0, x_0') \prod_{i=k}^n \delta(x_{in})
\] (7.1)

where \( x' = (x_1, \ldots, x_k) \) and similarly for \( x_0' \). Eq. (7.1) expresses the elementary property that the joint p.d.f. of coinciding random variables is concentrated on the diagonal. In particular, \( P_n(t, x; 0, 0) \) should be proportional to the \( d_n \)-dimensional delta-function. But the heat kernels \( e^{-tM_n}(x, x_0) \) do not have this property at least for \( \gamma \) close to 2 and, expectedly, for all \( \gamma > 0 \). Instead they are regular when \( x_0 \to 0 \). How exactly \( P_n(t, x; 0, x_0/L) \) fails to become the delta-function when \( L \) goes to infinity is described

\( ^9 \)Note that the poles of of the \( \Gamma \)-function do not contribute
by the asymptotics (5.17) dominated by the slow collective modes of the stochastic evolution of the Lagrangian trajectories. Hence, even if all joint p.d.f.’s $P_n$ of the differences of Lagrangian trajectories make sense as given by the $m, \kappa = 0$ heat kernels $e^{-tM_n(x,x_0)}$, the differences of Lagrangian trajectories do not exist as random processes for $\gamma > 0$. Note that for $\kappa$ positive we should not expect the behavior (7.1) since the Brownian motions starting from $x_{0,i}$ are different for different $i$’s even if they wiggle around the same Lagrangian trajectory. The system behaves as if the wigglings were present even for $\kappa = 0$ (see more on that below).

The $\gamma = 0$ and $m, \kappa = 0$ case will be analyzed in Sects. 8 and 9 and was previously considered in [8] to [11]. The p.d.f. $P_2(t,x;0,x_0) = e^{-tM_2(x,x_0)}$ of the difference $x_{12}(t) = x(t)$ of two Lagrangian trajectories may be easily computed in this case and the result is the log-normal distribution [8] [10]

$$P_2(t,x;0,x_0) = \frac{r^{-d}}{\sqrt{4\pi D t}} e^{-\frac{1}{4D t} (\ln \frac{r}{x_0} - tD)^2} \tilde{k}_{D+1+i}(\tilde{x}, \tilde{x_0}) \quad (7.2)$$

where $r = |x|$, $\tilde{x} = x/r$ and similarly for $r_0$, $\tilde{x_0}$. $k_t(\tilde{x}, \tilde{x_0})$ denotes the heat kernel on the unit sphere in $d$ dimensions and it drops out in the rotationally invariant sector. The most important consequence of Eq. (7.2) is that $\frac{1}{t} \ln \frac{r}{r_0}$ is a Gaussian variable with covariance $2D/t$ tending to zero at large times and with mean $Dd$. The mean gives the Lyapunov exponent i.e. the rate of exponential growth in time of the distance $r$ between the Lagrangian trajectories. Note that

$$\int r^\sigma P_2(t,x;0,x_0) \, dx = r_0^\sigma e^{D\sigma(d+\sigma)t} \quad (7.3)$$

which should be contrasted with the super-diffusive behavior for $\gamma > 0$ described by Eq. (4.6). More generally,

$$\int f(x) P_2(t,x;0,x_0) \, dx = \int f(e^{u} r_0 \tilde{x}) \frac{1}{\sqrt{4\pi D t}} e^{-\frac{1}{4D t} (u-tDd)^2} \tilde{k}_{D+1+i}(\tilde{x}, \tilde{x_0}) \, du \, d\tilde{x}. \quad (7.4)$$

For any test function $f$ and for fixed $t$, the right hand side tends to $f(0)$ when $r_0 \to 0$, in accordance with the relation (7.3) and unlike for $\gamma > 0$. As it is easy to see from the above integral (or from Eq. (7.2)), the concentration of the p.d.f. $P_2(t,x;0,x_0)$ within $r < \eta$ is visible if $\eta \gg e^{Dd} r_0$. For the later use, note that for a small but non-zero $r_0$ and for a rotationally invariant test function $f$,

$$\int f(x) P_2(t,x;0,x_0) \, dx = \int f(e^{u} r_0) \frac{\sqrt{t}}{\sqrt{4\pi D t}} e^{-\frac{1}{4D t} (u-Dd)^2} \, du \underset{t \to \infty}{\to} 0. \quad (7.5)$$

The approach of [8] [11] was based on the observation that at $\gamma = 0$ the 2-point function of the velocity differences becomes

$$\langle (v^\alpha(t_1,x_1) - v^\alpha(t_1,x_2)) (v^\beta(t_2,x_1') - v^\beta(t_2,x_2')) \rangle = 2 \frac{D}{d-1} \delta(t_{12}) \cdot [(d + 1) \delta^\alpha\beta x_{12} \cdot x_{12}' - x_{12}^\alpha x_{12}'^\beta - x_{12}'^\alpha x_{12}^\beta]. \quad (7.6)$$
In particular, the first derivatives of $v$ have space-independent correlations. In other words, we may set

$$v(t, x_1) - v(t, x_2) = X(t) x_{12} \quad \text{or} \quad \partial_t v^\alpha(t, x) = X^{\alpha\gamma}(t)$$ (7.7)

where $X^{\alpha\beta}(t)$ is a Gaussian process with values in traceless matrices with mean zero and the 2-point function

$$\langle X^{\alpha\gamma}(t) X^{\delta\beta}(s) \rangle = 2^{\frac{D}{d-1}} \delta(t - s) [(d + 1) \delta^{\alpha\beta} \delta^{\gamma\delta} - \delta^{\alpha\gamma} \delta^{\beta\delta} - \delta^{\alpha\delta} \delta^{\beta\gamma}]$$ (7.8)

obtained by differentiating twice the right hand side of (7.6). It is easy to check directly that the above covariance is positive and that it is invariant under the adjoint action of $O(d)$, i.e. that $X$ and $kXk^{-1}$ have the same covariance for $k \in O(d)$. Eqs. (7.7) are equalities between Gaussian processes. Physically, they mean that for $m = 0$ and $\gamma = 0$ the velocity flow acts as a uniform, volume preserving strain and rotation, as far as the relative motions of fluid particles are concerned. The difference of two Lagrangian trajectories $x_{12}(t) \equiv x(t)$ should satisfy the linear (stochastic) ODE

$$dx = X(t) x dt, \quad x(0) = x_0$$ (7.9)

with a solution given formally by

$$x(t) = g_{t, t_0} x_0$$ (7.10)

where $g_{t, t_0}$ is the time-ordered exponential of an integral of independent matrices,

$$g_{t, t_0} = T \int_{t_0}^t X(s) ds,$$ (7.11)

of the type similar to the ones that appears in the theorems on products of independent equally distributed matrices [19] or in the one-dimensional Anderson localization [20]. The point is that $g_{t, t_0}$ may be defined as a random Markov process (a diffusion) with values in $SL(d)$. It has three basic properties:

1. $g_{t_2, t_1}$ and $g_{t_2 + \tau, t_1 + \tau}$ have the same distribution,
2. $g_{t_2, t_1} g_{t_1, t_0} = g_{t_2, t_0}$ a.e.,
3. $g_{t, t_0}$ is independent of $g_{t', t_0'}$ if $(t_0, t) \cap (t_0', t') = \emptyset$.

To define such a process, it is enough to give the (transition) probability distributions $p_{t-t_0}(g) dg$ of $g_{t, t_0}$ ($dg$ denotes the Haar measure on $SL(d)$) satisfying the composition law:

$$\int p_t(g) p_s(g^{-1} h) dg = p_{t+s}(h).$$ (7.12)

The $SO(d)$-invariance of the Lie-algebra-valued process $X$ imposes also the relation

$$p_t(k g k^{-1}) = p_t(g).$$ (7.13)

In Sect. 8 we identify $p_t$ with the heat kernel of a certain operator on $SL(d)$. 
The net outcome of that analysis is that for $\gamma = 0$, unlike for $\gamma > 0$, the differences of Lagrangian trajectories $x_{ij}(t) = g_{t,t_0}x_{0,ij}$ are well defined random variables. In particular, the knowledge of $p_t$ is all that is needed to compute the joint p.d.f.’s of $x_{ij}(t)$:

$$\int P_n(t, x; 0, x_0) f(x) \, d'x = \int_{SL(d)} p_t(g) f(gx_0) \, dg$$

(7.14)

for translationally invariant $f$. In fact, the above integrals uniquely determine $p_t$.

One of the consequences of the relation (7.14), closely related to the property (7.1), is that, for $\gamma = 0$, the stochastic evolution of the scalar $T$ defined by the $m, \kappa = 0$ flow preserves the Gibbs measure formally given as $e^{-\beta \int T^2 \mu}$ normalization. Indeed, the 2-point function of the scalar in this measure is

$$F_{Gibbs}^n(x_0) = (2^n)\frac{(2n)!}{2^n n! \beta^n} S \delta(x_{12}) \delta(x_{34}) \cdots \delta(x_{2n-1,2n})$$

(7.15)

(the odd functions vanish). But Eq. (7.14) implies the relation

$$\int P_{2n}(t, x; 0, x_0) F_{Gibbs}^n(x_0) \, d'x_0 = F_{Gibbs}^n(x)$$

(7.16)

i.e. the time invariance of the Gibbs measure correlations for $\gamma = 0$. This should be contrasted with the behavior for the $\gamma > 0$ case where the flux of the scalar energy towards high wavenumbers destroys the invariance of the Gibbs measure, see [21]. For $\gamma = 0$, the invariant Gibbs measure is nevertheless unstable under perturbations, as follows from relation (7.5). It has also little to do with the $\kappa \to 0$ limit of the stationary state of the scalar obtained in the presence of large scale forcing. The latter will be constructed in Sect. 9.

The mathematics of the difference between the $\gamma = 0$ and $\gamma > 0$ cases is simple. Eq. (2.8) requires that $v(t, x)$ be Lipschitz in $x$ for the uniqueness of solutions[10]. But the Gaussian $v$-measure with 2-point function (3.4) lives on $v$’s which are Hölder in $x$ with exponent $(2 - \gamma)/2$ (modulo logarithmic corrections) but not Lipschitz, except for $\gamma = 0$ where the velocity differences become smooth, as we have seen above. Hence, one should not expect uniqueness of Lagrangian trajectories even if the probabilistic description of them may be maintained but with violation of the property (7.1). Physically[11], the velocity covariance should be smoothed on the dissipative scale $\eta$ due to viscous effects so that it behaves as $\sim D \eta^{-\gamma} r^2$ for $r \ll \eta$, i.e. like the $\gamma = 0$ covariance with $D$ increased to $D \eta^{-\gamma}$. The Lagrangian trajectories diverge now exponentially in time as long as their distance is $\ll \eta$. Note, however, that for arbitrary small but fixed $r_0 = |x_0|$ one never sees concentration of the p.d.f. $P_2(t, x; 0, x_0)$ on scales smaller than $\eta$ if $\eta \leq e^{O(\tau \eta^{-\gamma})} r_0$, i.e. for $\eta$ sufficiently small. This explains in more physical terms

[10] recall the existence of two solutions with vanishing initial condition: $x(t) = (\pm t)^\gamma$ and $x = 0$, for the equation $x = x^{2\gamma}$.

[11] K.G. thanks G. Falkovich for a discussion of this point
why relation (7.1) fails when $\eta \to 0$ for $\gamma > 0$. The exponential divergence of trajectories closer than $\eta$ makes it impossible to maintain the concept of (differences of) individual trajectories in the inviscid limit $\eta \to 0$. Instead, we should talk about the $v$-dependent p.d.f. $P_n(t, x; 0, x_0 | v)$ whose average over the velocity ensemble reproduces $P_n(t, x; 0, x_0)$. It is worth noting that for positive diffusivity $\kappa$, when the deterministic equation (2.8) should be replaced by the stochastic ODE (2.9), although the problems with the non-uniqueness of the solutions persist, there exists a rigorous probabilistic treatment\footnote{we thank G. Eyink for pointing this out to us} allowing to define uniquely the transition probabilities $P_n(t, x; 0, x_0 | v)$ for Hölder continuous velocities [22]. Our analysis calls for an extension of such a treatment to the $\kappa = 0$ case.

8 Advection by smooth velocities and harmonic analysis

When $\gamma = 0$ and $m, \kappa = 0$, the Kraichnan model becomes exactly solvable as we will show now. That simplifications occur in this case has been noted before, see [3] and [7] to [13]. Our analysis is based on some observations by Shraiman and Siggia [3] [8]. As was noted in [3] and in [8], for $\gamma = 0$ the model has extra symmetries. The operators $M_n$ can be expressed in terms of the quadratic Casimir operators corresponding to an action of the groups $SL(d)$ and $SO(d)$ on the correlation functions. Let us explain what this means.

The group $SL(d)$ of real matrices of determinant 1 acts on functions $f$ on $\mathbb{R}^d$ on the left by $(L_g f)(x) = f(g^{-1}x)$. The infinitesimal form of this action is given by $\frac{d}{dt}|_{t=0} L e^{tA} f = A^{\alpha \beta} H_{\alpha \beta} f$ where $A$ is a traceless matrix (i.e. in the Lie algebra of $SL(d)$) and the generators $H_{\alpha \beta}$ are

$$H_{\alpha \beta} = -x^\alpha \partial_{x^\beta} + \frac{1}{d} \delta_{\alpha \beta} x^\gamma \partial_{x^\gamma} .$$

(8.1)

Similarly, on functions of $n$ $\mathbb{R}^d$ variables $(x_1, \ldots, x_n) = x$, we have the (diagonal) action

$$(L_g f)(x) = f(g^{-1}x)$$

(8.2)

with generators $H_{\alpha \beta} = \sum_i ( -x_i^\alpha \partial_{x_i^\beta} + \frac{1}{d} \delta_{\alpha \beta} x_i^\gamma \partial_{x_i^\gamma})$. The quadratic Casimir of $SL(d)$ is in terms of these generators

$$H^2 = \sum_{\alpha, \beta} H_{\alpha \beta} H_{\beta \alpha} .$$

(8.3)

The generators of the action of the $SO(d)$ subgroup are $J_{\alpha \beta} = H_{\alpha \beta} - H_{\beta \alpha}$ and the corresponding quadratic Casimir is

$$J^2 = -\frac{1}{2} \sum_{\alpha, \beta} J_{\alpha \beta}^2 .$$

(8.4)
The observation of Shraiman and Siggia was that when $\gamma, m = 0$ in the velocity covariance $d^{\alpha\beta}$ (3.3), the operator $M_n = \sum_{i<j} d^{\alpha\beta}(x_i - x_j)\partial_{x_i}^\alpha \partial_{x_j}^\beta$ becomes

$$M_n = \frac{D}{d-1}[(d + 1)J^2 - dH^2].$$

(8.5)

By definition of the action (8.2), the same formula holds also when we express $\sigma$ where we have performed the Gaussian integral over $t$ we obtain by integrating over $\sigma$ with $Re\sigma = -\frac{d}{2}$. It follows, in particular, that the spectrum $H^2$ acting in $L^2(\mathbb{R}^d)$ is $]-\infty, -\frac{(d-1)}{4}]$ and that of $M_2$ is $[\frac{D^2}{4}, \infty]$. Denoting $e^{-tJ^2} \equiv k_t$, we obtain

$$\int f(x) e^{-tM_2(x, x_0)} dx$$

$$= \int_{Re\sigma = -\frac{d}{2}} \frac{d\sigma}{2\pi i} \int_0^\infty dr \int d\hat{x} r^{-\sigma - 1} f(r\hat{x}) \tau_0^\sigma e^{tD\sigma(\sigma + d)} k_{D\sigma + d} \tau(i\hat{x}, \hat{x}_0)$$

$$= \int_0^\infty \frac{dr}{r} \int d\hat{x} f(r\hat{x}) e^{-\frac{1}{4\pi d} (\ln r - tDd)^2} k_{\frac{Dd}{2} + d} \tau(i\hat{x}, \hat{x}_0)$$

(8.7)

where we have performed the Gaussian integral over $\sigma$. The result (8.7) readily follows.

Taking $f(x) = C_L(r) = C(r/L)$ with $C$ the rotationally invariant forcing covariance, we obtain by integrating over $t$ the expression for the 2-point function of $T$ at $\gamma = 0$:

$$\mathcal{F}_{2,L}(x) = (M^{-1}_2 C_L)(x) = \frac{1}{Dd} \left( \int_r^\infty C(\rho/L) \frac{d\rho}{\rho} + r^{-d} \int_0^r C(\rho/L) \rho^{d-1} d\rho \right).$$

(8.8)

Clearly $\mathcal{F}_{2,L}(x)$ is smooth for $x \neq 0$ and

$$\mathcal{F}_{2,L}(x) \cong -\frac{C(0)}{Dd} \ln (r/L) \quad \text{for small } r,$$

$$\mathcal{F}_{2,L}(x) \sim (r/L)^{-d} \quad \text{for large } r.$$ 

(8.9)

(8.10)

In order to solve Eqs. (2.22) for the higher-point functions of $T$ we need a representation for the Green function $M_n^{-1}$. This is obtained by relating $H^2$ and $J^2$ to the Casimirs $\mathcal{H}^2$ and $\mathcal{J}^2$ of the left action of $SL(d)$ and of $SO(d)$ on functions $F$ on $SL(d)$, given by $(\mathcal{L}_g F)(h) = F(g^{-1}h)$, or in the infinitesimal form by $\frac{d}{dt} |_{t=0} \mathcal{L}_{\epsilon^a} F = A^{\alpha\beta} \mathcal{H}_{\alpha\beta} F$. Note that $\mathcal{H}_{\alpha\beta}$ are skew-adjoint in the regular representation and that

$$(d + 1)\mathcal{J}^2 - d\mathcal{H}^2 = -\frac{d+2}{4} \sum_{\alpha\beta} (\mathcal{H}_{\alpha\beta} - \mathcal{H}_{\beta\alpha})^2 - \frac{d}{4} \sum_{\alpha\beta} (\mathcal{H}_{\alpha\beta} + \mathcal{H}_{\beta\alpha})^2$$

(8.11)
is a positive elliptic operator in $L^2(dg)$. In particular, it has the heat kernel

$$K_t(g, h) \equiv e^{-t\frac{D}{d-1}((d+1)\mathcal{J}^2-d\mathcal{H}^2)}(g, h)$$

(8.12)
satisfying $\int K_t(g, h) dh = 1$ and $K_t(g, h) = K_t(kg, kh) = K_t(gg', hg')$ for $k \in SO(d)$ and $g' \in SL(d)$. Assign to a translationally invariant function $f(x)$ and to $x$ a function $F_x(g) = f(gx)$ on $SL(d)$. The linear map $f \mapsto F$ intertwines the two actions of $SL(d)$:

$$(L_gf)(h) = (\mathcal{L}_g F_x)(h).$$

(8.13)

It follows that

$$\int e^{-tM_n(gx, y)} f(y) d'y = \int K_t(g, h) f(hx) dh.$$  

(8.14)

Comparing the above relation to Eq. (7.14) we conclude that

$$p_t(hg^{-1}) = K_t(g, h).$$

(8.15)

Clearly the basic properties (7.12) and (7.13) of the p.d.f. $p_t$ follow. In other words, we may identify $g_{t,t_0}$ as the diffusion process on group $SL(d)$ with the generator equal to $\frac{D}{d-1}((d+1)\mathcal{J}^2-d\mathcal{H}^2)$.

Integrating the relation (8.14) over $t$ we infer that

$$\int M_n^{-1}(gx, y) f(y) d'y = \int \mathcal{G}(g, h) f(hx) dh$$

where $\mathcal{G}$ is the integral kernel of $\left(\frac{D}{d-1}((d+1)\mathcal{J}^2-d\mathcal{H}^2)\right)^{-1}$. Applying iteratively identity (8.14) to Eq. (2.23) we end up with the expression

$$F_{2n}(x) = \sum_p F_{2n}(u_p)$$

(8.17)

where the sum runs through all ordered pairings $p = \{i_1, j_1\}, \ldots, \{i_n, j_n\}$ of $\{1 \ldots 2n\}$, $u_p = (x_{i_1j_1}, \ldots, x_{i_nj_n})$ and

$$F_{2n}(u) = \int \prod_{i=1}^n \mathcal{G}(g_{i-1}, g_i) C(g_iu_i) dg_i = \int \prod_{i=1}^n \tilde{\mathcal{G}}(g_{i-1}, g_i) C(g_iu_i) dg_i$$

(8.18)

where $g_0 = e$ and $\tilde{\mathcal{G}}(g, h) = \int_{SO(d)} \mathcal{G}(g, kh) dk$. The last equality follows by substituting $g_1 = k_1g'_1, g_2 = k_1k_2g'_2, g_n = k_1 \cdot k_n g'_n$, and using $\mathcal{G}(kg_1, kg_2) = \mathcal{G}(g_1, g_2)$ and $C(kx) = C(x)$.

The final reduction consists of identifying $\tilde{\mathcal{G}}$ with the Green function of the Laplace-Beltrami operator $\Delta$ on the homogeneous space $H_d \equiv SL(d)/SO(d)$. By definition, $\Delta$ coincides with the Casimir $\frac{d}{2} \mathcal{H}^2$ if we view functions on $H_d$ as functions on $SL(d)$ right-invariant under the action of $SO(d)$. Assign to a function $f$ on $H_d$ the function $g \mapsto \tilde{f}(g) = f(g^{-1})$. Clearly $\tilde{f}(kg) = \tilde{f}(g)$ and $(\mathcal{L}_g f)(h^{-1}) = \tilde{f}(hg) \equiv (\mathcal{R}_g \tilde{f})(h)$, i.e. the map $f \mapsto \tilde{f}$ intertwines the action of $SL(d)$ on the functions on $H_d$ with the right
regular action of $SL(d)$. Since the quadratic Casimirs of $SL(d)$ in the left-regular and in the right-regular representations coincide and $\mathcal{J}^2$ vanishes in the action on $\tilde{f}$, we infer that

$$-2d(\Delta f)(g^{-1}) = -d(\mathcal{H}^2 \tilde{f})(g) = ([((d + 1)\mathcal{J}^2 - d\mathcal{H}^2] \tilde{f})(g) \tag{8.19}$$

and that

$$\int G(g^{-1}, h) f(h) \, dh = \int G(g, h) \tilde{f}(h) \, dh = \int \tilde{G}(g, h) \tilde{f}(h) \, dh \tag{8.20}$$

where the function $G(g, h)$ on $H_d \times H_d$ represents the kernel of the operator $(-D'\Delta)^{-1}$ where $D' \equiv \frac{2D_d}{d-1}$. Thus $\tilde{G}(g, h) = G(g^{-1}, h^{-1})$ and Eq. (8.18) becomes

$$F_{2n}(u) = \int \prod_{i=1}^{n} G(g_{i-1}, g_i) C(g_{i}^{-1}u_i) \, d\nu(z) \tag{8.21}$$

Every matrix $g \in SL(d)$ can be uniquely represented as a product (the so called Iwasawa decomposition) $g = nak$ where $k \in SO(d)$, $n$ is upper triangular with 1 on the diagonal and $a$ is diagonal with positive entries. Thus one may parametrize the cosets $gSO(d)$ by $na$. For $d = 2$ we may write $a = \text{diag}(y_1^\frac{1}{2}, y_2^\frac{1}{2})$, $y > 0$, $n = (1 x 0 y_1^\frac{1}{2})$, $x \in \mathbb{R}$. The Haar measure $dg$ becomes $dg = y^{-2}dx dy dk$. The homogeneous space $H_2$ may be identified with the upper half-plane $H = \{z = x + iy \in \mathbb{C} | y > 0\}$. The action of $SL(2)$ on $H$ is given by the M"{o}bius transformations $(a b \begin{smallmatrix} c & d \end{smallmatrix}) z = \frac{az + b}{cz + d}$. Since $ki = i$, the identification maps the coset $gSO(2)$ to $gi = nau$. We shall denote $na \equiv g(z) = (y_1^\frac{1}{2} y_2^\frac{1}{2} x 0)$. The $SL(2)$-invariant measure on $H$ is $d\nu(z) = y^{-2}dx dy$ and the Laplace-Beltrami operator becomes

$$\Delta = y^2(\partial_y^2 + \partial_x^2). \tag{8.22}$$

The Green function $G$ is given by the explicit expression:

$$G(z, z') = \frac{1}{16D\pi} \ln \frac{(x - x')^2 + (y + y')^2}{(x - x')^2 + (y - y')^2}. \tag{8.23}$$

Eq. (8.21) may now be rewritten as

$$F_{2n}(u) = \int \prod_{i=1}^{n} G(z_{i-1}, z_i) C(g(z_i)^{-1}u_i) \, d\nu(z) \tag{8.24}$$

with $z_0 = i$. In Appendix B we study the integrals (8.24) in more detail. In particular we show that the leading singularities at coinciding points of the correlation functions of $T$ are given by a Gaussian expression, a sum of products of 2-point functions, confirming the analysis of [9][12].
For the dimension $d > 2$ one can proceed analogously. In the Iwasawa decomposition we parametrize $n$ by the off-diagonal entries, $x_\alpha, \alpha = 1, \ldots, \frac{d^2-d}{2}$, and write $a = e^{\phi}, \phi = \text{diag}(\phi_1, \ldots, \phi_d)$ with $\sum_i \phi_i = 0$. The Haar measure becomes in these variables
\[ dg = e^{\sum_{i<j}(\phi_j - \phi_i)} \prod_{i=1}^{d-1} d\phi_i \prod_\alpha dx_\alpha dk. \] (8.25)

$G$ is (proportional to) the Green function of the Laplace-Beltrami operator $\Delta$ on $SL(d)/SO(d)$. Explicitly, for $d = 3$ write $\phi = \frac{1}{2} \alpha \text{diag}(1, -1, 0) + \frac{1}{6} \beta \text{diag}(1, 1, -2)$. Then
\[ \Delta = e^{2\alpha^2} \partial_{x_1}^2 + e^{\alpha+\beta^2} \partial_{x_2}^2 + e^{\beta-\alpha}(\partial_{x_3} + x_1 \partial_{x_2})^2 + \partial_\alpha^2 + 3\partial_\beta^2. \] (8.26)
and $dg = e^{-\alpha-\beta} d\alpha d\beta dx_1 dx_2 dx_3 dk$. There does not seem to exist a very explicit expression for $G$ in $d > 2$. However, the singular behavior of $F_{2n}$ can be extracted again, see Appendix B.

Let us end this section by deriving the formula for the Lyapunov exponents of the Lagrangian trajectories, previously found in [11] by path-integral techniques. In [3] it was observed that $M_n$ may be also expressed using the quadratic Casimir of the action of $SL(n-1)$ with the generators
\[ G_{ij} = -x_{in}^\alpha \partial_{x_i^\alpha} + \frac{1}{n-1} \delta_{ij} x_{kn}^\alpha \partial_{x_k^\alpha} \] (8.27)
for $1 \leq i, j \leq n-1$. This action corresponds to the natural action of $SL(n-1)$ on the $i$-index of $x_{in} \equiv x_i - x_n$. Denoting by $G^2$ the quadratic Casimir $\sum_{i,j} G_{ij} G_{ji}$ and by $\Lambda$ the generator of dilations $x_i^\alpha \partial_{x_i^\alpha}$, one obtains [3]
\[ M_n = \frac{D}{d-1} \left[ (d+1)j^2 - dG^2 - \frac{d-n+1}{n-1} \Lambda (\Lambda + d_n) \right]. \] (8.28)

Let $\rho$ denote the volume spanned by vectors $x_{in}, i = 1, \ldots, n-1$, describing the time $t$ differences of the Lagrangian trajectories starting at time zero from points $x_0$:
\[ \rho = \sqrt{\det_{i,j} (x_{in} \cdot x_{jn})}. \] (8.29)

We would like to find the p.d.f. of $\rho$. Note that for a function $f(\rho)$,
\[ M_n f(\rho) = -\frac{(d-n+1)D}{(d-1)(n-1)} \Lambda (\Lambda + d_n) f(\rho) = -\frac{(n-1)(d-n+1)D}{d-1} \rho \partial_\rho(\rho \partial_\rho + d) f(\rho). \] (8.30)
This follows from Eq. (8.28) since $\rho$ is $SL(n-1)$- and $SO(d)$-invariant. Hence $M_n$ preserves the space of functions $f(\rho)$. Also
\[ \int |f(\rho)|^2 \prod_i dx_{in} = \text{const.} \int_0^\infty |f(\rho)|^2 \rho^{d-1} d\rho \] (8.31)
where \( \text{const.} = \int \delta(\rho-1) \prod_i dx_i \). Hence \( M_n \) in the action on \( f(\rho) \) is diagonalized by the Mellin transform \( f(\rho) \to \hat{f}(\sigma) = \int_0^\infty \rho^{-\sigma-1} f(\rho) \, d\rho \) (unitary for \( \Re \sigma = -\frac{d}{2} \)):

\[
(M_n f)(\sigma) = -\frac{(n-1)(d-n+1)}{d-1} \sigma (\sigma + d) \hat{f}(\sigma) \equiv -D(n) \sigma (\sigma + d) \hat{f}(\sigma). \tag{8.32}
\]

As in Eq. (8.7), we obtain

\[
\int f(\rho) P_n(t, x; 0, x_0) \, dx = \int_0^\infty \frac{d\rho}{\rho} \, f(\rho) \, \frac{1}{\sqrt{4\pi D(n) t}} \, e^{-\frac{2D(n)t}{t} \ln \rho - tD(n)d^2} \tag{8.33}
\]

Hence \( \frac{1}{t} \ln \rho \) is a Gaussian variable with covariance \( 2D(n)/t \) tending to zero at large times and with mean \( D(n)d \) which, by definition, is the sum of the \( (n - 1) \) largest Lyapunov exponents describing the effective separation of \( n \) Lagrangian trajectories. We infer that the \( n^{th} \) Lyapunov exponent is

\[
\lambda_n = (D(n+1) - D(n))d = \frac{1}{2}(d - 2n + 1)D'
\]

with \( d \) exponents equally spaced and symmetric with respect to the origin, confirming the result of [11].

9 Quadrature of the \( \gamma = 0 \) case

Let us explicitly construct the stationary state of the passive scalar advected by smooth Gaussian velocity with 2-point function (7.6). Relations (8.17) and (8.21) allow to write a compact expression for the generating function of the \( \gamma = 0 \) theory:

\[
\Phi(\chi) \equiv \langle e^{i \int \chi(x) T(x) \, dx} \rangle = \sum_{n=0}^\infty \frac{(-1)^n}{(2n)!} \sum_p \int F_{2n}(u_p) \prod i \chi(x_i) \, dx_i. \tag{9.1}
\]

Noting that all pairings give the same contribution to the \( x_i \) integral and that there are \( \frac{(2n)!}{2^n} \) of them we get

\[
\Phi(\chi) = \sum_{n=0}^\infty (-1)^n \int \prod_{i=1}^n G(g_{i-1}, g_i) V_\chi(g_i) \, dg_i \tag{9.2}
\]

where

\[
V_\chi(g) = \frac{1}{2} \int C(g^{-1}(x-y)) \chi(x) \chi(y) \, dx \, dy \tag{9.3}
\]

is a non-negative function on \( H_d \) bounded by \( V_\chi(e) = \frac{1}{2} C(0)(\int \chi)^2 \). Eq. (9.2) may be rewritten in the operator language as \( (D' \equiv \frac{2Dd}{d-1}) \)

\[
\Phi(\chi) = \sum_{n=0}^\infty (-1)^n \int \left[ \frac{1}{-D' \Delta} V_\chi \right]^n \delta(x) \tag{9.4}
\]
The sum on the right hand side involves the Neuman series for the operator \((-D'\Delta + V_\chi)^{-1}\), i.e. for the Laplacian on \(H_d\) perturbed by a potential. Resumming the series we obtain

\[
\Phi(\chi) = 1 - [(-D'\Delta + V_\chi)^{-1}V_\chi](e) \tag{9.5}
\]

which is an explicit expression for the characteristic functional of the stationary state of the \(\gamma = 0\) Kraichnan model.

Let us see that the right hand side of Eq. (9.5) makes sense. Using the Feynman-Kac formula expressing the perturbed heat kernel as an expectation \(E_g(\cdot)\) with respect to the Brownian motion on \(H_d\) with transition amplitudes \(e^{-D'\Delta}(g,h)\), starting at time zero at \(g\):

\[
e^{-t(-D'\Delta + V_\chi)}(g,h) = E_g\left(e^{-\int_0^t V_\chi(h(s)) ds} \delta_h(h(t))\right), \tag{9.6}
\]

we infer the bounds

\[
0 \leq e^{-t(-D'\Delta + V_\chi)}(g,h) \leq e^{tD'\Delta}(g,h), \tag{9.7}
\]

\[
0 \leq (-D'\Delta + V_\chi)^{-1}(g,h) \leq G(g,h). \tag{9.8}
\]

Since

\[
[(-D'\Delta + V_\chi)^{-1}V_\chi](e) = \int (-D'\Delta + V_\chi)^{-1}(e,h)V_\chi(h) \, dh, \tag{9.9}
\]

it follows that the latter integral is bounded by the smeared 2-point function

\[
\int G(e,h)V_\chi(h) \, dh = \frac{1}{2} \int \mathcal{F}_2(x_{12}) \chi(x_1) \chi(x_2) \, dx_1 \, dx_2 \tag{9.10}
\]

which is finite for test functions \(\chi\) e.g. from the Schwartz space \(\mathcal{S}(\mathbb{R}^d)\), see Eq. (8.8).

\(\Phi\) defines a continuous positive-definite functional on \(\mathcal{S}(\mathbb{R}^d)\). The continuity of \(\Phi(\chi)\) w.r.t. \(\chi \in \mathcal{S}(\mathbb{R}^d)\) is easy: it follows by the Dominated Convergence Theorem from the Feynman-Kac representation of the perturbed Green function:

\[
(-D'\Delta + V_\chi)^{-1}(g,h) = \int_0^\infty dt \ E_g\left(e^{-\int_0^t V_\chi(h(s)) ds} \delta_h(h(t))\right). \tag{9.11}
\]

The positive definiteness:

\[
\sum_{r,s} \lambda_r \overline{\lambda_s} \Phi(\chi_r - \chi_s) \geq 0, \tag{9.12}
\]

is a little bit more complicated. Let us sketch its proof. Define first the positive definite characteristic functional

\[
\Phi_t(\chi) = \langle e^{i\int T(t,x) \chi(x) \, dx} \rangle \tag{9.13}
\]
of the time $t$ (quasi-Lagrangian) state of the scalar where $T(t,x) = \int_0^t f(s, g_{t,s}^{-1} x) \, ds$ is a functional of the forcing $f$ and of $g_{t,s}$. The above expression for $T(t,x)$ is obtained for the initial condition vanishing at $t_0 = 0$ in Eq. (2.4). The expectation in (9.13) is w.r.t. the Gaussian measure of the forcing and w.r.t. the measure of the diffusion process $g_{t,s}$. It is easy to see that $\int T(t,x) \chi(x) \, dx$ is square-integrable with respect to these measures. Performing the integration with respect to $f$, we obtain

$$\Phi_t(\chi) = \langle e^{-\int_0^t V_\chi(g_{t,s}) \, ds} \rangle. \tag{9.14}$$

The remaining expectation over $g_{t,s}$ is easy to calculate by expanding the exponential (the resulting series of expectations converges absolutely for finite $t$). The result is

$$\Phi_t(\chi) = 1 - \int_0^t [e^{-s(-D'\Delta + V_\chi)} V_\chi](e) \, ds. \tag{9.15}$$

Using the bound (9.7), it is easy to see that $\Phi_t(\chi)$ converge to $\Phi(\chi)$ when $t \to \infty$. Hence the positive definiteness of $\Phi$. Note that Eq. (9.15) may be rewritten by integration by parts and the Feynman-Kac formula (9.6) as

$$\Phi_t(\chi) = [e^{-t(-D'\Delta + V_\chi)} 1](e) = E_e \left( e^{-\int_0^t V_\chi(h(s)) \, ds} \right) \tag{9.16}$$

which follows also directly from Eq. (9.14) if we notice that the diffusion process $s \mapsto g_{t,t-s}$ on $SL(d)$ projects to the Brownian motion on $H_d$. The resulting alternative expressions for $\Phi$:

$$\Phi(\chi) = \lim_{t \to \infty} \int e^{-t(-D'\Delta + V_\chi)}(e, h) \, dh = E_e \left( e^{-\int_0^\infty V_\chi(h(s)) \, ds} \right). \tag{9.17}$$

relate $\Phi(\chi)$ to the long time behavior of the diffusion on the homogeneous space $H_d$ in the presence of a positive potential $V_\chi$ or to the low-energy properties of the Schrödinger operator $-D'\Delta + V_\chi$. They imply that $0 \leq \Phi(\chi) \leq 1$. Expressions (9.17) may also be obtained directly in the Martin-Siggia-Rose (MSR) [23] formal functional integral approach.

By Minlos Theorem, the normalized ($\Phi(0) = 1$), continuous, positive-definite functional $\Phi$ on $S(R^d)$ given by Eqs. (1.5) or (9.17) defines a unique probability measure $d\mu$ on $S'(R^d)$ s.t.

$$\Phi(\chi) = \int e^{i \int T(x) \chi(x) \, dx} \, d\mu(T). \tag{9.18}$$

d$\mu$ is the stationary state of the Kraichnan model for $\gamma = 0$ alluded to in Sect. 7. It is quite different from the Gibbs measure and quite non-Gaussian and is, indeed, supported by distributional configurations of the scalar since the correlation functions

$$F_{2n}(x) = \int T(x_1) \cdots T(x_{2n}) \, d\mu(T) \tag{9.19}$$

diverge logarithmically at coinciding points. The measure $d\mu$ contains all the joint p.d.f.’s of smeared scalar values $\int T(x) \chi(x) \, dx$. In particular, the function $p \mapsto \Phi(p\chi)$,
is the Fourier transform of the p.d.f. $p_\chi(\theta)$ of $\int T(x) \chi(x) dx$ whose behavior was studied in [13], see also [7][9][10].

$\Phi(p\chi)$ is a pointwise limit of the finite-time functions $\Phi_t(p\chi)$ which are entire in $p$. For $\Re p^2 \geq -b^2$, $b > 0$,

$$|\Phi_t(p\chi)| \leq \Phi_t(\pm ib\chi) = E_e \left( e^{b^2 \int_0^t V_{\chi}(h(s)) \, ds} \right) = \int e^{t(D' \Delta + b^2 V_\chi)}(e, h) \, dh = 1 + b^2 \int_0^t ds \int e^{s(D' \Delta + b^2 V_\chi)}(e, h) V_\chi(h) \, dh. \quad (9.20)$$

The Schrödinger operator $-D' \Delta - b^2 V_\chi$ with a negative potential may develop bound states. The right hand side of the inequality (9.20) grows with $t$ since the expression under the integrals is positive. If $e_b \equiv \inf\{\text{spec}(-D' \Delta - b^2 V_\chi)\} < 0$ then the growth is unbounded since $e^{s(D' \Delta + b^2 V_\chi)}(e, h) \sim e^{-es_b}$ for large $s$. On the other hand, for $e_b > 0$ the right hand side of (9.20) would be bounded uniformly in $t$ if $V_\chi$ were of compact support on $H_d$. $V_\chi$, however, does not have a compact support as a function on $H_d$ even if $C$ and $\chi$ do (if they do not vanish identically). It is, nevertheless, easy to see from the definition (9.3) that $V_\chi$ vanishes at infinity of $H_d$, i.e. that it gets arbitrarily small outside sufficiently big compact subsets of $H_d$. This is enough to assure a uniform bound for the right hand side of (9.20) as may be seen by the following argument which separates the behavior at infinity of $H_d$ from that in the interior. Write $V_\chi = V'_\chi + V''_\chi$ where $0 \leq V'_\chi \leq V_\chi$ and $V''_\chi$ has a compact support. By the Hölder inequality,

$$E_e \left( e^{b^2 \int_0^t V_{\chi}(h(s)) \, ds} \right) \leq E_e \left( e^{(1+\epsilon)b^2 \int_0^t V'_{\chi}(h(s)) \, ds} \right)^{\frac{1}{1+\epsilon}} E_e \left( e^{\frac{1+\epsilon}{\epsilon}b^2 \int_0^t V''_{\chi}(h(s)) \, ds} \right)^{\frac{\epsilon}{1+\epsilon}}. \quad (9.21)$$

If we choose $\epsilon$ small so that for $b' = (1 + \epsilon)^{\frac{1}{2}} b$ the relation $e_{b'} > 0$ still holds then the first expectation on the right hand side of inequality (9.21) is bounded uniformly in $t$ ($e_{b'}$ increases with decrease of $V'_\chi$). Choose the support of $V'_\chi$ so that $\frac{1+\epsilon}{\epsilon}b^2 V''_\chi \leq v_0 < \frac{D\epsilon^2}{4}$ where $v_0$ is a constant. Then

$$E_e \left( e^{\frac{1+\epsilon}{\epsilon}b^2 \int_0^t V''_{\chi}(h(s)) \, ds} \right) \leq 1 + \frac{1+\epsilon}{\epsilon} b^2 \int_0^t ds \int e^{-s(D' \Delta - v_0)}(e, h) V_\chi(h) \, dh$$

$$= 1 + \frac{1+\epsilon}{2\epsilon} b^2 \int_0^t ds \int e^{-s(M_2 - v_0)}(x_1 - y) C(y) \chi(x_1) \chi(x_2) \, dx_1 \, dx_2 \, dy \quad (9.22)$$

and the last expression is bounded uniformly in $t$ as may be easily seen from Eq. (8.7).

We infer that $\Phi(p\chi)$, as a limit of uniformly bounded analytic functions, is analytic in $p$ for $\Re p^2 > b^2$ but has a singularity at $p = \pm ib_0$ where $b_0$ is the positive number s.t. $e_{b_0} = 0$. The Cauchy bounds imply now that

$$\left| \frac{d^n}{dp^n} \Phi(p\chi) \right| \leq \frac{\text{const.}(\epsilon) n!}{1+|p|^n} \quad (9.23)$$

in any strip $|\text{Im} p| < b_0 - \epsilon$ for $\epsilon > 0$. Note also that $\lim_{|p| \to \infty} \Phi(p\chi) = 0$ by virtue of Eq. (9.17). Since

$$p_\chi(\theta) = \frac{1}{2\pi} \int e^{-ibp} \Phi(p\chi) \, dp, \quad (9.24)$$

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it is easy to show integrating by parts few times and moving the $p$-integration contour to $\text{Im} p = \pm (b_0 - \epsilon)$ that $p_\chi(\theta)$ is smooth except, possibly, at $\theta = 0$ and that

$$p_\chi(\theta) \leq \text{const.(\epsilon)} \ e^{-(b_0 - \epsilon)|\theta|}$$

(9.25)

for $|\theta| \geq O(1)$ and any positive $\epsilon$. Clearly, the same inequality fails for negative $\epsilon$ since it would imply analyticity of $\Phi(p\chi)$ at $p = \pm ib_0$. \textbf{In short}: the p.d.f. $p_\chi(\theta)$ of $\int T(x) \chi(x) \, dx$ has an exponential decay for large $|\theta|$ with the rate $b_0$ equal to the value of $b$ at which the ground state of $-D^2 \Delta - b^2 V_\chi$ crosses zero energy. Note that the rate $b_0$, as related to a bound state energy is not, in general, a semi-classical quantity.

For rotationally invariant $\chi$ which, for simplicity, we shall normalize so that $\int \chi = 1$, our operators on $H_d$ reduce to the ones on the double coset space $SO(d) \backslash SL(d) / SO(d)$. This space may be identified with the Cartan algebra of $L$ acting in $V$.

Note however that although $b_0$ were reinterpreted in [13] within the semiclassical approach. Our rigorous result about the decay rate $b_0$ of $p_\chi(\theta)$ disagrees for $d > 2$ with the result of [13] and with the instanton calculation of [13]. These papers obtain the value $b'_0 = d \sqrt{\frac{D}{2c(0)}}$ for the decay.
rate which is smaller than $b_0$ for $d > 2$. The point is that in [4] and [11] the function $V_\chi(g)$ of Eq. (9.23) was replaced by $V_x(g) = \text{const} \{ g^{-1} x \}$ with fixed $x \neq 0$. This simplifies the calculation of the expressions of Eq. (9.17) since only the distribution of $g_{t,s} x$ for one $x$ is needed. They become

$$\Phi'_x \equiv E_x \left( e^{-\int_0^\infty V_x(b(s)) \, ds} \right) = \lim_{t \to \infty} \int_{\mathbb{R}^d} e^{-t(M_2 + \text{const})} x, y \, dy$$  \hspace{1cm} (9.28)$$

and lead to the quantum mechanical problem analyzed in [4][10]. Upon the replacement of $V_x$ by $p^2 V_x$ one obtains a function $\Phi'_x(p)$ whose first singularity off the real axis is at $p = \pm ib$ with $b$ s.t. the ground state of $M_2 - \frac{1}{4} b^2 \mathcal{C}$ crosses zero energy. Since the spectrum of $M_2$ starts from $\frac{D d^2}{4}$, see the remark after Eq. (8.3), we indeed obtain, for $\mathcal{C} = \mathcal{C}_L$ and large $L$, the exponential decay rate $b_0'$ for the Fourier transform of $\Phi'_x(p)$.

The technical reason for the discrepancy with our exact calculation is that $V_x(g)$, unlike its smeared version $V_\chi(g)$, does not vanish at the infinity of $H_d$ and leads to a more singular behavior of the right hand side of Eq. (9.20). Another way to see it is that $\Phi'_x$ is given by a version of Eq. (9.1) with $\int \prod \chi(x_i) \, dx_i$ omitted and with $F_{2n}(u_p)$ replaced by the partition-independent contribution $\tilde{F}_{2n}(x, \ldots , x)$ corresponding the collinear configuration $u_p$ giving the most singular behavior when $u_p \to 0$ (see Appendix B). The smearing in Eq. (9.1) makes this behavior more regular. Our result persists, however, also if we replace $V_\chi(g)$ with $\tilde{V}_\psi(g) = \frac{1}{2} \int \mathcal{C}(g^{-1} x) \psi(x) \, dx$, if $\psi$ and $\mathcal{C}$ are non-negative function from $\mathcal{S}(\mathbb{R}^d)$, since $\tilde{V}_\psi(g)$ still vanishes at infinity. In particular, $\psi$ may vanish around the origin which shows that it is the smearing of collinearity, not the inclusion of coinciding points, which is responsible for the discrepancy between $b_0$ and $b_0'$. The lesson is that the correlation of (non-collinear pairs of) Lagrangian trajectories renders the smeared scalar less intermittent in more than two dimensions and should not be neglected.

It is easy to see that $\Phi(p\chi)$ decays exponentially for large real $p$. Denote by $\tau$ the first exit time of the Brownian motion on $H_d$ from a fixed neighborhood of $e$. The probability of a given value of $\tau$ is bounded by $e^{-\text{const}./\tau}$. Since $V = \int_0^\infty V_x(h(s)) \, ds \geq \text{const.} \tau$, the conditional expectation $E_x(e^{-p^2 V} \mid \tau)$ is bounded by $e^{-\text{const.} \cdot p^2 \tau}$. Hence the exponential decay of $E_x(e^{-p^2 V}) \leq \int_0^\infty e^{-\text{const.} \cdot (p^2 \tau + 1/\tau)} \, d\tau$. A more exact description of the decay follows from the path-integral integral representation of the expectation (9.17). The latter implies that the large $p$ behavior of $\Phi(p\chi)$ for real $p$, unlike the large $\theta$ behavior of $p\chi(\theta)$, is semi-classical:

$$\Phi(p\chi) \sim e^{-|p| \cdot S(g(\cdot))} \hspace{1cm} (9.29)$$

where $[0, \infty) \mapsto g(s)$ describes a trajectory (instanton) in $H_d$ minimizing the action

$$S(h(\cdot)) = \int_0^\infty \left( \frac{1}{2D} |h(s)|^2 + V_\chi(h(s)) \right) \, ds \hspace{1cm} (9.30)$$

for fixed initial value $h(0) = e$ (with $| \cdot |^2$ standing for the $SL(d)$-invariant metric on $H_d$). This is the same instanton as in the field theoretic MSR analysis of [13]. For

\footnote{we thank M. Chertkov for suggesting this interpretation}
rotationally invariant $\chi$, the problem reduces to the one on $SO(d) \backslash SL(d)/SO(d)$ with the action

$$S(\phi_i(\cdot)) = \int_0^\infty \left( \frac{1}{2D'} \sum \phi_i(s)^2 + V_\chi(\phi_i(s)) \right) ds$$  \hfill (9.31)

and with the initial value $\phi_i = 0$. In $d = 2$ the minimal value of $S$ is

$$\frac{1}{\sqrt{D'}} \int_0^\infty \sqrt{V_\chi(\phi)} \, d\phi > 0$$  \hfill (9.32)

where $\phi \equiv \phi_2 - \phi_1$. For $C(r)$ approximately constant up to $r \equiv L$, $V_\chi(\phi)$ is approximately constant up to $\frac{1}{d} \phi \equiv \ln L$ and then it decays to zero like $\sim e^{-\phi/2}$. Consequently, the exponential decay rate of $\Phi(p\chi)$ is approximately $\ln L \sqrt{\frac{C(0)}{2D}}$ for large $L$, in agreement with [9] and [13]. For $d > 2$ and large $L$ the minimum of $S$ is attained on the trajectory which in the region of constant potential goes in the direction $\sqrt{\frac{1}{n!}} (-1, \ldots, -1, d-1)$ and the value of the action is again $\approx \ln L \sqrt{\frac{C(0)}{2D}}$ up to lower order terms, as pointed out in [10] and [13]. The exponential decay of $\Phi(p\chi)$ implies that $p_\chi(\theta)$ is smooth also at zero.

\section{10 Conclusions}

In this paper we have analyzed the stochastic dynamics of Lagrangian trajectories for Gaussian, time-decorrelated random velocity fields considered in the Kraichnan model of passive advection. We found that the dynamics is characterized by two related phenomena. First, the Lagrangian trajectories loose in the limit of high Reynolds numbers the deterministic sense for a fixed velocity realization due to their sensitive dependence on initial conditions. Second, their relative stochastic dynamics is dominated by slow resonance-type modes. The slow modes determine the average characteristics of the spread of Lagrangian trajectories responsible for the loss of their deterministic character. Both phenomena were essentially due to non-smoothness of the typical velocities signaled by fractional Hölder exponents in their spatial dependence. Since the turbulent velocities are non-smooth in the limit of high Reynolds numbers, we expect the two phenomena to persist for more realistic velocity ensembles and to continue to be responsible for the anomalous scaling. For the spatially smooth velocities, we calculated the Lyapunov exponents describing the sensitive dependence of the Lagrangian trajectories on initial conditions for distances smaller than the viscous scale. Using harmonic analysis on the symmetric spaces $SL(d)/SO(d)$ we also obtained in this case an explicit form of the characteristic functional of the stationary state of the passive scalar and exhibited an exponential decay of the p.d.f.’s of smeared values of the scalar relating the decay rate to the properties of the ground state of the Calogero-Sutherland Schrödinger operator with a potential.

\section{Appendix A}
We shall make explicit the structural result of Sect. 6 for the heat kernel $e^{-tM_2(x,x_0)}$. In the angular momentum $l=0,1,\ldots$ sector,

$$M_2 \equiv M_2(l) = -\frac{D}{r^{d-1}} \partial_r r^{d+1-\gamma} \partial_r + \frac{D(d+1-\gamma)}{d-1} l(d-2+l)r^{-\gamma} \quad (A.1)$$

which is a positive operator in $L^2(\mathbb{R}, du)$. The generalized eigen-function of $M_2(l)$ corresponding to eigenvalue $E \geq 0$ involves the Bessel function

$$\varphi_E(r) = r^{\frac{\gamma d}{2}} J_{\nu_l}^2 \left( \frac{2\sqrt{E/D}}{r} \right) \quad (A.2)$$

where

$$\nu_l = \frac{1}{4} \sqrt{(d-\gamma)^2 + 4\frac{d+1-\gamma}{d-1} l(d-2+l)} \quad (A.3)$$

The spectral decomposition of $M_2(l)$ has the form

$$M_2(l) = \int E |\varphi_E \rangle \langle \varphi_E| d\nu(E). \quad (A.4)$$

Since

$$(U_s \varphi_E)(r) \equiv e^{sd/2} \varphi_E(e^{s}r) = e^{\frac{2s}{r}} \varphi_{e^{s}E}(r), \quad (A.5)$$

we infer that

$$U_s M_2(l) U_s^{-1} = \int E |\varphi_E \rangle \langle \varphi_E| d\nu(e^{s}E). \quad (A.6)$$

Since, on the other hand, $U_s M_2 U_s^{-1} = e^{sM_2}$, see Eq. (6.3), it follows that

$$d\nu(e^{s}E) = e^{s}d\nu(E), \quad (A.7)$$

i.e. that $d\nu(E) = c dE$ for some positive constant $c$. Hence

$$c \int |\varphi_E \rangle \langle \varphi_E| dE = I \quad (A.8)$$

and for

$$\hat{f}(E) = \sqrt{c} \int_0^\infty \varphi_E(r) f(r) r^{d-1} dr, \quad (A.9)$$

we obtain

$$\int_0^\infty |\hat{f}(E)|^2 dE = \int_0^\infty |f(r)|^2 r^{d-1} dr. \quad \text{Substituting } E = e^{\gamma u}, \text{ we shall define} \quad (A.10)$$

$$\mathcal{V}_1 : L^2(\mathbb{R}, du) \rightarrow L^2(\mathbb{R}, du) \text{ is a unitary operator. Besides,}$$

$$\mathcal{V}_1 M_2(l)f(u) = e^{\gamma u} \mathcal{V}_1 f(u). \quad (A.11)$$
Let $V_2 : L^2([0, \infty[, r^{d-1}dr) \to L^2(\mathbb{R}, du)$ be another unitary operator defined by
\begin{equation}
(V_2 f)(u) = e^{-\frac{d}{2}u} f(e^{-u}).
\end{equation}

Note that
\begin{equation}
(V_i U_s f)(u) = (V_i f)(u - s), \quad i = 1, 2,
\end{equation}
so that $U_2 = V_1^{-1} V_2$ commutes with $U_s$. Besides,
\begin{equation}
M_2(l) = U_2 r^{-\gamma} U_2^{-1},
\end{equation}
as follows from Eq. (A.11). This is the relation (6.8) for $n = 2$. Since, explicitly,
\begin{equation}
(V_1 f)(u) = \sqrt{\gamma c} \int e^{\frac{2}{\gamma}(u-u')} J_{\nu l} \left( \frac{2}{\sqrt{\delta}} e^{\frac{2}{\gamma}(u-u')} \right) (V_2 f)(u') du',
\end{equation}
and the Mellin transform is the composition of $V_2$ and the Fourier transform, we obtain for $\Re \sigma = \frac{d}{2}$
\begin{align}
\hat{U}_2(\sigma)^{-1} &= \sqrt{\gamma c} \int e^{\left(\frac{d}{2} + \sigma\right)u} e^{\frac{2}{\gamma}u} J_{\nu l} \left( \frac{2}{\sqrt{\delta}} e^{\frac{2}{\gamma}u} \right) du \\
&\cdot \int x^{d+2\nu} J_{\nu l}(x) dx = \sqrt{\gamma c} D \left( \frac{d+2\nu}{2} \right) \frac{\Gamma\left(\frac{1}{2}(1 + \nu l + \frac{d+2\nu}{\gamma})\right)}{\Gamma\left(\frac{1}{2}(1 + \nu l + \frac{d+2\nu}{\gamma})\right)}.
\end{align}
The unitarity implies that $\gamma c D = 1$ so that, finally,
\begin{equation}
\hat{U}_2(\sigma) = \left(\frac{\gamma \sqrt{D}}{\gamma} \right)^{-\frac{d+2\nu}{2}} \frac{\Gamma\left(\frac{1}{2}(1 + \nu l - \frac{d+2\nu}{\gamma})\right)}{\Gamma\left(\frac{1}{2}(1 + \nu l + \frac{d+2\nu}{\gamma})\right)}.
\end{equation}
The right hand side has a meromorphic continuation to the complex plane of $\sigma$ with poles at
\begin{equation}
\sigma_{l,p} = -\frac{d-\gamma}{2} + \frac{\gamma}{2} \nu l + \gamma p
\end{equation}
for $p = 0, 1, \ldots$. Since the true (more regular at the origin) zero mode of $M_2(l)$ occurs at scaling dimension $\sigma_{l,0}$, this is exactly the analytic structure predicted for $\hat{U}_n(\sigma)$. The function $r^{\sigma_{l,p}}$ (multiplied by an angular term) represents a slow 2-point mode in the angular momentum $l$ sector.

Appendix B

Let us briefly consider the convergence properties of the integrals (8.24). Let $k_i \in SO(2)$ be rotation matrices s.t. $u_i = k_i(r_i, 0)$ where $r_i = |u_i|$. We have $g(z_i)^{-1} k_i =
\((k_i^{-1}g(z_i))^{-1} = k_i^r g(k_i^{-1}z_i)^{-1}\) for some \(k_i^r \in SO(2)\). Denoting \(C(k(r, 0)) = C(r)\), observing that \(|g(z_i)^{-1}(r_i, 0)| = r_i y_i^{-\frac{1}{2}}\) and using the \(SL(2)\) invariance of \(d\nu(z_i)\), we obtain

\[
F_{2n}(u) = \int G(i, z_1)G(\kappa_1 z_1, z_2) \cdots G(\kappa_{n-1} z_{n-1}, z_n) \prod_i C(r_i y_i^{-\frac{1}{2}}) d\nu(z_i) \quad (B.1)
\]

where \(\kappa_i = k_{i+1}^{-1} k_i\) is the rotation by the angle between \(u_{i+1}\) and \(u_i\). We shall study the behavior of \(F_{2n}\) as \(r_i\) tend to 0. The following is a useful relation:

\[
\int G(z, z')\, dx' = \frac{1}{4D} (y \theta(y' - y) + y' \theta(y - y')) \equiv G_0(y, y') \tag{B.2}
\]

For the 4-point function, noting that \(|\kappa_1 z_1| = \gamma_1 y_1\) where

\[
\gamma_1 = [(x_1 \sin \vartheta + \cos \vartheta)^2 + y_1^2 \sin^2 \vartheta]^{-1}, \tag{B.3}
\]

\(\vartheta\) being the angle between \(u_2\) and \(u_1\), we obtain

\[
F_4(u_1, u_2) = \int G(i, z_1) G_0(\gamma_1 y_1, y_2) C(r_1 y_1^{-\frac{1}{2}}) C(r_2 y_2^{-\frac{1}{2}}) y_1^{-2} \, dy_1 \, dy_2 \, d\nu(z_1) \tag{B.4}
\]

Consider first the case \(\vartheta = 0\) i.e. \(\gamma_1 = 1\). Then

\[
F_4(u_1, u_2) = \frac{1}{(4D)^2} \int (\theta(y_1 - 1) + y_1 \theta(1 - y_1))(y_1 \theta(y_2 - y_1) + y_2 \theta(y_1 - y_2)) \cdot C(r_1 y_1^{-\frac{1}{2}}) C(r_2 y_2^{-\frac{1}{2}}) y_1^{-2} \, dy_1 \, y_2^{-2} \, dy_2. \tag{B.5}
\]

Since \(C = C_L\) has rapid decay at infinity, the integrals are effectively cut to \(y_i > (r_i/L)^2\) and produce logarithms of \((r_i/L)\) as these ratios tend to zero. The most singular contribution is from \(y_1 \theta(1 - y_1) y_2 \theta(y_1 - y_2)\) term which yields \(4 \ln(r_1/L) \ln(r_2/L) - 2(\ln(r_1/L))^2\) if \(r_1 > r_2\) and \(2(\ln(r_2/L))^2\) if \(r_2 > r_1\). Thus

\[
F_{4,L}(u_1, u_2) + F_4(u_2, u_1) = \left(\frac{C(0)}{2D}\right)^2 \ln(r_1/L) \ln(r_2/L) + \text{less singular} \tag{B.6}
\]

for \(\vartheta = 0\).

For \(\vartheta \neq 0\), the \(y_2\) integral yields

\[
F_{4,L}(u_1, u_2) = \frac{1}{8\pi} \int \ln \left(\frac{x_1^2 + (y_1 + 1)^2}{x_1^2 + (y_1 - 1)^2}\right) C_L(r_1 y_1^{-\frac{1}{2}}) \left[\ln(\gamma_1) + \ln(y_1 r_2^{-2} L^2)\right] \cdot \vartheta(\gamma_1 y_1 - r_2^2/L^2) + B] \, y_1^{-2} \, dy_1 \, dx_1 \tag{B.7}
\]

where \(B\) is bounded. For the \(\ln(\gamma_1)\) term the only singularity is at \(y_1\) small and this term is bounded by \(\text{const.} |\ln(r_1/L)|\). The rest has same leading singularity as the \(\vartheta = 0\) calculation. Thus recognizing in \(B.6\) the 2-point singularities \(B.3\), we infer that the leading singularity of the 4-point function is Gaussian, the sum of products of 2-point functions.

The analysis of the general correlation is similar though tedious. When all the points are on the same line i.e. all the angles are zero, we can do all the \(x_i\) integrals
Let us here note only that for \( d \) near 2 the products of 2-point functions. Using the Mellin transform we may write
\[
\sum_{\pi} F_{2n}(u_{\pi}) = \prod_{i} \frac{1}{4\pi} \int_{0}^{1} C_{L}(r_{i}y_{i}^{-\frac{1}{2}}) y^{-1} dy + \ldots = \prod_{i} \frac{-C(0)}{2D} \ln(r_{i}/L) + \ldots \tag{B.8}
\]
where \( \ldots \) is less singular. Non-zero angles give again subleading contributions. The subsequent sum over unordered pairings gives the Gaussian expression for the leading short-distance singularity of \( F_{2n,L}(x) \) in terms of the singular contributions to the 2-point function, in agreement with the observations of [9] and [12].

Finally, for \( d > 2 \), to extract the leading singularity some bounds for \( G \) are needed. Let us here note only that for \( d = 3 \) if all the \( \kappa_{i} \) are identity, then the \( x \) integrals can again be done and the result is that \( G \) gets replaced by \( G_{0} \) where
\[
G_{0} = \text{const.} \left( -y_{1}^{2} \partial_{y_{1}}^{2} - 3y_{2}^{2} \partial_{y_{2}}^{2} \right)^{-1} \tag{B.9}
\]
on \( L^{2}((y_{1}y_{2})^{-2}dy_{1}dy_{2}) \) (we have put \( y_{1} = e^{\alpha}, y_{2} = e^{\beta} \) in (8.2)). The behavior of \( G_{0} \) near \( y_{i} = 0 \) is calculable and the leading singularity can again be shown to be given by products of 2-point functions. Using the Mellin transform we may write
\[
(G_{0}F)(y_{1}', y_{2}') = \text{const.} \int_{0}^{\infty} \frac{dt}{t} e^{-t} \int dy_{1}dy_{2} \left( \frac{y_{1}y_{2}}{y_{1}y_{2}} \right)^{\frac{1}{2}} e^{-\frac{t}{4t} \left( \log \frac{y_{1}}{y_{1}} \right)^{2} - \frac{1 - 2t}{12t} \left( \log \frac{y_{2}}{y_{2}} \right)^{2}} F(y_{1}, y_{2}) \tag{B.10}
\]
\( C(y_{2}^{-\frac{1}{2}}) \) in (B.4) is replaced by \( C(y_{1}^{-\frac{1}{2}}y_{2}^{-\frac{1}{2}}) \). Hence, let \( \rho = y_{1}'y_{2}' \) and let \( F(y_{1}, y_{2}) = f(\rho) \). Then \( (G_{0}F)(y_{1}', y_{2}') = \tilde{f}(\rho') \) with
\[
\tilde{f}(\rho') = \text{const.} \int_{0}^{\infty} \frac{dt}{t} e^{-t} \int du dv f(e^{-u-v} \rho) e^{u+3v} e^{-\frac{u}{t} - \frac{3v}{2t}} \tag{B.11}
\]
which after performing the \( u - v \) and the \( t \) integrals becomes \( \int g(\rho', \rho) f(\rho) d\rho \) with
\[
g(\rho', \rho) = \text{const.} \left[ \frac{1}{\rho} \theta(\rho' - \rho) + \frac{2\rho^{3}}{\rho^{4}} \theta(\rho - \rho') \right]. \tag{B.12}
\]
We may then write \( F_{2n} \) in the form
\[
F_{2n}(u) = \int g(1, \rho_{1}) \ldots g(\rho_{n-1}, \rho_{n}) \prod_{i=1}^{n} C(r_{i}\rho_{i}^{-1}) d\rho_{i} \tag{B.13}
\]
and the analysis of the singularities goes on as in the \( d = 2 \) case.

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