Moving Stationary State of Exciton-Phonon Condensate in Cu$_2$O

D. Roubtsov and Y. Lépine

Groupe de Recherche en Physique et Technologie des Couches Minces,
Departement de Physique, Université de Montréal,
C.P. 6128, succ. Centre-ville, Montreal, PQ, H3C 3J7, Canada
e-mail: roubtsod@phycn.umontreal.ca

Abstract
We explore a simple theoretical model to describe the properties of Bose condensed para-excitons in Cu$_2$O. Taking into account the exciton-phonon interaction and introducing a coherent phonon part of the moving condensate, we derive the dynamic equations for the exciton-phonon condensate. Within the Bose approximation for excitons, we discuss the conditions for the moving inhomogeneous condensate to appear in the crystal. We calculate the condensate wave function and energy and a collective excitation spectrum in the semiclassical approximation. The stability conditions of the moving condensate are analyzed by use of Landau arguments, and two critical velocities appear in the theory. Finally, we apply our model to describe the recently observed interference between two coherent exciton-phonon packets in Cu$_2$O.

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1 Introduction
Excitons in semiconductor crystals [1] and nanostructures [2] are a very interesting and challenging object to search for the process of Bose Einstein condensation (BEC). Nowadays there is a lot of experimental evidence that the optically inactive para-excitons in Cu$_2$O can form a highly correlated state, or the excitonic Bose Einstein condensate [1],[3],[4]. A moving condensate of para-excitons in a 3D Cu$_2$O crystal turns out to be spatially inhomogeneous in the direction of motion, and the registered velocities of coherent exciton packets turn out to be always less, but approximately equal to the longitudinal sound speed of the crystal [5].

Analyzing recent experimental [3],[5],[6] and theoretical [7]-[10] studies of BEC of excitons in Cu$_2$O, we can conclude that there are essentially two different stages of this
process. The first stage is the kinetic one, with the characteristic time scale of $10 \sim 20$ ns. At this stage, a condensate of long-living para-excitons begins to be formed from a quasi-equilibrium degenerate state of excitons ($\mu \neq 0$, $T_{\text{eff}} > T_{\text{latt}}$) when the concentration and the effective temperature of excitons in a cloud meet the conditions of Bose-Einstein Condensation [1]. Note that we do not discuss here the behavior of ortho-excitons (with the lifetime $\tau_{\text{ortho}} \approx 30$ ns) and their influence on the para-exciton condensation process. For more details about the ortho-excitons in Cu$_2$O, ortho-para-exciton conversion, etc. see [3],[4],[11].

The most intriguing feature of the kinetic stage is that formation of the para-exciton condensate and the process of momentum transfer to the para-exciton cloud are happening simultaneously. It seems that nonequilibrium acoustic phonons (appearing at the final stage of exciton cloud cooling) play the key role in the process of momentum transfer. Indeed, the theoretical results obtained in the framework of the “phonon wind” model [8],[12] and the experimental observations [3],[4],[5] are the strong arguments in favor of this idea. To the authors’ knowledge, there are no realistic theoretical models of the kinetic stage of para-exciton condensate formation where quantum degeneracy of the initial exciton state and possible coherence of nonequilibrium phonons pushing the excitons would be taken into account. Indeed, the condensate formation and many other processes involving it are essentially nonlinear ones. Therefore, the condensate, or, better, the macroscopically occupied mode, can be different from $n(k = 0) \gg 1$, and the language of the states in $k$-space and their occupation numbers $n(k)$ may be not relevant to the problem, see [13].

In this study, we will not explore the stage of condensate formation. Instead, we investigate the second, quasi-equilibrium stage, in which the condensate has already been formed and it moves through a crystal with some constant velocity and characteristic shape of the density profile. In theory, the time scale of this “transport” stage, $\Delta t_{\text{tr}}$, could be determined by the para-exciton lifetime ($\tau_{\text{para}} \approx 13$ $\mu$s [1]). In practice, it is determined by the characteristic size $\ell$ of a high-quality single crystal available for experiments:

$$\Delta t_{\text{tr}} \approx \ell / c_l \approx 0.5 - 1.5$ \mu$s \ll \tau_{\text{para}},$$

where $c_l$ is the longitudinal sound velocity.

We assume that at the “transport” stage, the temperature of the moving packet (condensed + noncondensed particles) is equal to the lattice temperature,

$$T_{\text{eff}} = T_{\text{latt}} < T_c.$$ 

Then we can consider first the simplest case of $T = 0$ and disregard the influence of all sorts of nonequilibrium phonons (which appear at the stages of exciton formation, thermalization [3]) on the formed moving condensate.

Any theory of the exciton BEC in Cu$_2$O has to point out some physical mechanism(s) by means of which the key experimental facts can be explained. (For example, the condensate moves without friction within a narrow interval of velocities localized near $c_l$, and the shape of the stable macroscopic wave function of excitons resembles soliton profiles [1].) Here we explore a simple model of exciton-phonon condensate. In this case, the general structure of the Hamiltonian of the moving exciton packet and the lattice phonons is
Here $\hat{\psi}$ is the Bose-field operator describing the excitons, $\hat{u}$ is the field operator of lattice displacements, $\hat{\pi}$ is the momentum density operator canonically conjugate to $\hat{u}$, $v$ is the exciton packet velocity and, finally, $P$ is the momentum operator. Note that the Hamiltonian (1) is written in the reference frame moving with the exciton packet, i.e. $x \to x - vt$ and $v = \text{const}$ is the packet velocity.

2 3D Model of Moving Exciton-Phonon Condensate

To derive the equations of motion of the field operators (and generalize these equations to the case of $T \neq 0$), it is more convenient, however, to start from the Lagrangian. In the proposed model, the Lagrangian density has the form

$$\mathcal{L} = \frac{i\hbar}{2} (\hat{\psi}^\dagger \partial_t \hat{\psi} - \partial_t \hat{\psi}^\dagger \hat{\psi}) + \frac{m}{2} (\partial_x \hat{\psi}^\dagger \partial_x \hat{\psi} - \hat{\psi}^\dagger \partial_x \hat{\psi} - \hat{\psi} \partial_x \hat{\psi}) -$$

$$- E_g \hat{\psi}^\dagger \hat{\psi} - \frac{\hbar^2}{2m} \nabla \hat{\psi}^\dagger \nabla \hat{\psi} - \frac{\nu_0}{2} (\hat{\psi}^\dagger (x, t))^2 (\psi(x, t))^2 - \frac{\nu_1}{3} (\hat{\psi}^\dagger (x, t))^3 (\psi(x, t))^3 +$$

$$+ \frac{\rho}{2} (\partial_t \hat{u})^2 - \frac{\rho \rho_1}{2} \partial_x \hat{u}, \partial_y \hat{u}, \partial_z \hat{u} - \rho v \partial_t \hat{u} \partial_x \hat{u} + \frac{\rho v^2}{2} (\partial_x \hat{u})^2 -$$

$$- \sigma_0 \hat{\psi}^\dagger (x, t) \hat{\psi} (x, t) \nabla \hat{u} (x, t),$$

(2)

where $m$ is the exciton “bare” mass ($m = m_e + m_h \simeq 3m_e$ for 1s excitons in Cu$_2$O). $\nu_0$ is the exciton-exciton interaction constant ($\nu_0 > 0$ that corresponds to the repulsive interaction between para-excitons in Cu$_2$O [14]), $\rho$ is the crystal density, $\sigma_0$ is the exciton-longitudinal phonon coupling constant, and $\mathbf{v} = (v, 0, 0)$. The energy of a free exciton is $E_g + \hbar^2 \mathbf{k}^2 / 2m$. Although the validity of the condition [15] $n\tilde{a}_B^3 \ll 1$ ($\tilde{a}_B$ is the exciton Bohr radius) makes it possible to disregard all the multiple-particle interactions with more than two participating particles in $\hat{H}_{ex}$, we leave the hard-core repulsion term originated from the three-particle interaction in $\mathcal{L}$, i.e. $\nu_1 \neq 0$ and $0 < \nu_1 \ll \nu_0$. For simplicity’s sake, we take all the interaction terms in $\mathcal{L}$ in the local form and disregard the interaction between the excitons and transverse phonons. The packet velocity $v \equiv v_1$ is one of the parameters of the theory, and we will not take into account the excitonic normal component and velocity, ($T = 0$).

The equations of motion can be easily derived by the standard variational method from the following condition:

$$\delta S = \delta \int dt \, dx \, \mathcal{L} (\hat{\psi}^\dagger (x, t), \hat{\psi}(x, t), \hat{u}(x, t)) = 0.$$

Indeed, after transforming the Bose-fields $\hat{\psi}^\dagger$ and $\hat{\psi}$ by

$$\hat{\psi}(x, t) \to \exp(-iE_g t/\hbar) \exp(imvx/\hbar) \hat{\psi}(x, t),$$

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we can write these equations as follows:

\[(i\hbar \partial_t + mv^2/2)\hat{\psi}(x, t) = 0\]

\[= \left( -\frac{\hbar^2}{2m} \Delta + \nu_0 \hat{\psi}^\dagger \hat{\psi}(x, t) + \nu_1 \hat{\psi}^\dagger \hat{\psi}^2(x, t) \right) \hat{\psi}(x, t) + \sigma_0 \nabla \hat{u}(x, t) \hat{\psi}(x, t), \quad (3)\]

\[(\partial_t^2 - c_l^2 \Delta - 2v_0 \partial_x + v_2 \partial_x^2)\hat{u}(x, t) = \rho^{-1} \sigma_0 \nabla (\hat{\psi}^\dagger \hat{\psi}(x, t)). \quad (4)\]

We assume that the condensate of excitons exists. This means that the following representation of the exciton Bose-field holds: \(\hat{\psi} = \psi_0 + \delta \hat{\psi}\). Here \(\psi_0 \neq 0\) is the classical part of the field operator \(\hat{\psi}\) or, in other words, the condensate wave function, and \(\delta \hat{\psi}\) is the fluctuational part of \(\hat{\psi}\), which describes noncondensed particles.

One of the important objects in the theory of BEC is the correlation functions of Bose-fields. The standard way to calculate them in this model (the excitonic function \(\langle \psi(x, t) \psi^\dagger(x', t') \rangle\), for example,) can be based on the effective action or the effective Hamiltonian approaches [16]. Indeed, one can, first, integrate over the phonon variables \(u_i\) get the expression for \(S_{\text{eff}}(\psi, \psi^\dagger)\) and, second, use \(S_{\text{eff}}\) (or \(\hat{H}_{\text{eff}}\)) to derive the equations of motion for \(\psi_0\), \(\delta \hat{\psi}\), correlation functions, etc..

In this work we do not follow that way; instead, we treat excitons and phonons equally [1],[7]. This means that the displacement field \(\hat{u}\) can have a nontrivial classical part too, i.e. \(\hat{u} = u_0 + \delta \hat{u}\) and \(u_0 \neq 0\), and the actual moving condensate can be an exciton-phonon one, i.e. \(\psi_0(x, t) \cdot u_0(x, t)\). Then the equation of motion for the classical parts of the fields \(\hat{\psi}\) and \(\hat{u}\) can be derived by use of the variational method from \(\mathcal{L} = \mathcal{L}(\psi, \psi^\dagger, u)\), where all the fields can be considered as the classical ones. Eventually we have

\[ (i\hbar \partial_t + mv^2/2)\psi_0(x, t) = 0 \]

\[= \left( -\frac{\hbar^2}{2m} \Delta + \nu_0 |\psi_0|^2(x, t) + \nu_1 |\psi_0|^4(x, t) \right)\psi_0(x, t) + \sigma_0 \nabla u_0(x, t) \psi_0(x, t), \quad (5)\]

\[ (\partial_t^2 - c_l^2 \Delta - 2v_0 \partial_x + v_2 \partial_x^2)u_0(x, t) = \rho^{-1} \sigma_0 \nabla (|\psi_0|^2(x, t)). \quad (6)\]

Notice that deriving these equations we disregarded the interaction between the classical (condensate) and the fluctuational (noncondensate) parts of the fields. That is certainly a good approximation for \(T = 0\) and \(T \ll T_c\) cases [13].

In this article a steady-state of the condensate is the object of the main interest. In the co-moving frame of reference, the condensate steady-state is just the stationary solution of Eqs. (3), (4) and it can be taken in the form

\[\psi_0(x, t) = \exp(-i\mu t) \exp(i\varphi)\phi_0(x), \quad u_0(x, t) = q_0(x),\]

where \(\phi_0\) and \(q_0\) are the real number functions, and \(\varphi = \text{const}\) is the (macroscopic) phase of the condensate wave function. (This phase can be taken zeroth if only a single condensate is the subject of interest.) Then, the following equations have to be solved:

\[\hat{\mu} \phi_0(x) = \left( -\frac{\hbar^2}{2m} \Delta + \nu_0 \phi_0^2(x) + \nu_1 \phi_0^4(x) \right) \phi_0(x) + \sigma_0 \nabla q_0(x) \phi_0(x), \quad (7)\]
The first (isotropic) term in (11) causes the renormalization of the exciton-exciton interaction (8). Substituting \( \nabla \sigma_0 \phi^2_o(x) \) can be represented as follows:

\[
\nabla \sigma_o(x) = \left( -\frac{\lambda^2}{3} - \frac{2}{3}\frac{\sigma_0}{\rho c^2_l} \phi^2_o(x) + \frac{\lambda^3}{4\pi} (1 - \lambda^{-2}) \int \mathcal{F}(x - x') \frac{\sigma_0}{\rho c^2_l} \phi^2_o(x') \, dx', \right)
\]

where \( \lambda^2 = c_t^2/(c_t^2 - v^2) \), and \( \mathcal{F} \) can be expressed in terms of the Green function of Eq. (8). Substituting \( \nabla \sigma_0 \) in Eq. (7), we rewrite the latter in the following form:

\[
\tilde{\mu} \phi_o(x) = \left( -\frac{\hbar^2}{2m} \Delta + \nu_0 \phi^2_o(x) + \int U_{\text{eff}}(x - x') \phi^2_o(x') \, dx' + \nu_1 \phi^4_o(x) \right) \phi_o(x),
\]

where the effective exciton-exciton interaction \( U_{\text{eff}} \) is induced by the lattice (\( q_o \neq 0 \)). It can be represented as follows:

\[
U_{\text{eff}}(x) = \left( -\frac{\lambda^2}{3} - \frac{2}{3}\frac{\sigma_0}{\rho c^2_l} \delta(x) + \frac{\sigma_0^2}{4\pi \rho (c_t^2 - v^2)} \frac{v^2}{c_t(c_t^2 - v^2)^{1/2}} \left( \frac{3\lambda^2 x^2}{(\lambda^2 x^2 + y^2 + z^2)^{5/2}} - \frac{1}{(\lambda^2 x^2 + y^2 + z^2)^{3/2}} \right). \right)
\]

The first (isotropic) term in (11) causes the renormalization of the exciton-exciton interaction constant \( \nu_0 > 0 \):

\[
\nu_0 \rightarrow \nu_{\text{eff}} = \nu(v; c_t, \sigma_0). \tag{12}
\]

Note that \( \nu_{\text{eff}} \) can be positive or negative depending on the value of \( v \). The second term in the effective potential (11) is strongly anisotropic. Moreover, on the cylinder \( y^2 + z^2 = \varepsilon^2 \), its value is negative in the vicinity of \( x = 0 \) and positive at the large scales of \( x \) (\( \|x\| \)).

The possibility of the existence of specific and to some extent unexpected solutions of Eq. (10) follows from the fact that the effective two-particle interaction between excitons can be attractive at small distances between the particles and repulsive at large distances. For example, the wave function \( \phi_o(x) \) may become strongly localized in 3D space because of this attraction.

If \( v > c_t \), the formulas for \( \nabla \sigma_o(x) \) and \( U_{\text{eff}}(x) \) are very similar to those obtained in the case of \( v < c_t \). For instance, the solution of Eq. (8) can be written as follows:

\[
\nabla \sigma_o(x) = \left( \frac{\tilde{\lambda}^2}{3} - \frac{2}{3}\frac{\sigma_0}{\rho c^2_l} \phi^2_o(x) + \frac{\tilde{\lambda}}{2\pi} (1 + \tilde{\lambda}^2) \int \mathcal{F}'(x - x') \frac{\sigma_0}{\rho c^2_l} \phi^2_o(x') \, dx'. \right)
\]

(Here \( \tilde{\lambda}^2 = c_t^2/(v^2 - c_t^2) \)). Again, the renormalization (12) takes place and a nonisotropic part of \( U_{\text{eff}}'(x) \) appears in Eq. (11). However, the effective two-particle interaction between excitons remains repulsive, i.e. \( \nu_0 \delta(x) + U_{\text{eff}}'(x) > 0 \).
3 Effective 1D Model for the Condensate Wave Function

Solving Eqs. (7),(8) in 3D space seems to be a difficult problem (see Eqs. (10),(11)). However, these equations can be essentially simplified if we assume that the condensate is inhomogeneous along the $x$-axis only, that is $\phi_o(x) = \phi_o(x)$ and $q_o(x) = (q_o(x), 0, 0)$. Such an effective reduction of dimensionality transforms the difficult integro-differential equation (10) into a rather simple differential one, and obtained in this way the effective 1D model for the condensate wave function $\phi_o$ conserves all the important properties of the “parent” 3D model. Indeed, if $v < c_l$, the following equations stand for the condensate:

$$\tilde{\mu}\phi_o(x) = (-\hbar^2/2m)\partial_x^2 + \nu_{\text{eff}}\phi_o^2(x) + \nu_1\phi_o^4(x)\phi_o(x),$$

$$\partial_x q_o(x) = \text{const} - (\sigma_0/\rho(c_l^2 - v^2))\phi_o^2(x),$$

where $\nu_{\text{eff}} = \nu_0 - \sigma_0^2/\rho(c_l^2 - v^2)$. If $v > c_l$, Eq. (14) describes the excitonic part of the condensate, but with the enhanced effective repulsion $\nu'_{\text{eff}} = \nu_0 + \sigma_0^2/\rho(v^2 - c_l^2)$.

The effective two-particle interaction constant $\nu_{\text{eff}}$ is negative if the velocity of the condensate lies inside the interval $v_o < v < c_l$, where

$$v_o = \sqrt{c_l^2 - (\sigma_0^2/\rho\nu_0)}$$

(16)
can be called the first ‘critical’ velocity in the model. (Note that outside this interval $\nu_{\text{eff}} > 0$.)

The estimate value of the threshold velocity $v_o$ can be obtained from the following formula:

$$v_o \simeq c_l\sqrt{1 - (C_c - C_v)^2/(8\pi \text{Ry}^* \text{Ry} \gamma^3)},$$

where $C_c - C_v$ is the relative volume deformation potential of a semiconductor, $(\sigma_0 \simeq C_c - C_v)$, $\text{Ry}^*$ and $\text{Ry}$ are the exciton and atom Rydberg energies, $\gamma = a_B/a_l$ and $a_l$ is the lattice constant. (The repulsive exciton-exciton interaction is taken in the form $\nu_0 \simeq 4\pi\hbar^2 a_{\text{ex}}/m$.) In the case of Cu$_2$O oxide, we have $v_o \simeq (0.5 \sim 0.7)c_l$.

In this study we will consider the case of $v_o < v < c_l$ in detail. If the sign of the effective two-particle interaction can vary, an extra (repulsive) term should be included into the Hamiltonian of a many-particle system to insure the finiteness of a steady-state wave function or the absence of wave function collapse in dynamic processes. In our case, it is the term $\nu_1(\phi_i)^3\phi_i^3$, and $\nu_1 > 0$ is supposed to be the smallest energy parameter in the theory. In the framework of the considered 1D model, however, a finite steady-state solution of Eqs. (14),(15) can be obtained without accounting for the “hard core” repulsion term $\nu_1\phi_i^4$. Indeed, we can write out the corresponding solution as follows:

$$\phi_o(x) = \Phi_o \cosh^{-1}(\beta\Phi_o x), \quad \partial_x q_o(x) = -\nu_0/(\rho(c_l^2 - v^2))\Phi_o^2 \cosh^{-2}(\beta\Phi_o x),$$

$$\tilde{\mu} = \nu_0\Phi_o^2/2 - (\sigma_0^2/\rho(c_l^2 - v^2))\Phi_o^2/2 < 0,$$

$$\beta = \beta(v) = \sqrt{mv_o v^2 - \nu_0^2}/\hbar^2 c_l^2 - v^2.$$
The amplitudes of the exciton and phonon parts of the condensate, $L_0 = (\beta(v)\Phi_o)^{-1}$, and the value of the effective chemical potential $\bar{\mu}$ depend on the normalization of the exciton wave function $\phi_o(x)$. We normalize it in 3D space assuming that the characteristic width of the packet in the $(y, z)$ plane is finite and the cross-section area of the packet can be made equal to the cross-section area $S$ of a laser beam. Then we can write this condition as follows:

$$\int |\psi_0|^2(x, t) \, dx = S \int \phi_o^2(x) \, dx = N_o,$$

where $N_o$ is the number of condensed excitons. Immediately, we get the following results:

$$\Phi_o = \frac{N_o}{2S} \beta(v), \quad L_0 = \left(\frac{N_o}{2S} \beta^2(v)\right)^{-1},$$

$$\bar{\mu} = -\frac{\nu_o v^2 - \nu_o^2}{2} \left(\frac{N_o}{2S} \beta(v)\right)^2 = -\frac{\hbar^2}{2m} L_0^{-2}.$$

The important “nonlinear” property of the obtained solution (17) is the dependence of the amplitudes of exciton and phonon parts of the condensate on the velocity $v$ and the number of excitons in the condensate, $N_o$. For example,

$$\Phi_o^2 = \Phi_o^2(N_o, v) \sim N_o^2 \left(\frac{v^2 - \nu_o^2}{\epsilon^2 - v^2}\right)$$

stands for the exciton amplitude. Notice that the characteristic width of the condensate and its velocity are not independent parameters, see (21). For estimates, the formula for $L_0$ can be rewritten as follows:

$$L_0^{-1} \simeq 2\tilde{a}_B^{-1}(na_B^3)^{1/2} \left(\frac{v^2 - \nu_o^2}{\epsilon^2 - v^2}\right)^{1/2},$$

where $n$ is the average density of excitons in the soliton state. Although $\tilde{a}_B^{-1}$ in this formula is multiplied by a small factor, this factor is not small enough to show quantitative agreement with the experimentally observed value of $L_0$, $2L_{\text{exp}} \simeq 10^{-1} \sim 10^{-2} \text{cm}$. It seems to be reasonable that a theory with nonzero temperature ($T > |\bar{\mu}|$) will provide a more realistic value of the effective size of the packet.

Returning to the laboratory reference frame, we can write the condensate wave function in the form:

$$\psi_0(x, t) \cdot u_o(x, t) \delta_{ij} = \exp \left(-i \left(E_g + \frac{mv^2}{2} - |\bar{\mu}|\right) t\right) \exp(imvx) \times$$

$$\times \Phi_o \cosh^{-1}(L_0^{-1}(x - vt)) \cdot \left(Q_o - Q_o \tanh(L_0^{-1}(x - vt))\right),$$

where we count the exciton energy from the bottom of the crystal valence band; $2Q_o(N_o, v)$ is the amplitude of the phonon part of condensate and $Q_o \propto \Phi_o$. To calculate the energy of the moving condensate within the Lagrangian approach, (see Eq. (23), we
have to integrate the zeroth component of the energy-momentum tensor $T^0_0$ over the spatial coordinates,

$$T^0_0(x, t) = E_g \psi^\dagger \psi + \frac{\hbar^2}{2m} \nabla \psi^\dagger \nabla \psi + (\nu_0/2)(\psi^\dagger)^2 \psi^2 + \frac{\rho v^2}{2} (\partial_x u_0)^2 + \frac{\rho v^2}{2} \partial_j u_k \partial_j u_k + \sigma_0 \psi^\dagger \psi \nabla u.$$

We do not take into account the small correction to this energy due to the quantum depletion of the condensate as well as the term $(\nu_1/3) \phi_0^3$ in $T^0_0$. Then the result reads:

$$E_o = \int d^3 r \, T^0_0 = E_{ex} + E_{int} + E_{ph} =$$

$$= N_o \left( E_g + \frac{mv^2}{2} \right) - N_o \left( |\tilde{\mu}| + (\nu_0/3) \Phi_0^2 \right) + \frac{c_i^2 + v^2}{3(c_i^2 - v^2)} N_o \frac{\sigma_0^2}{\rho(c_i^2 - v^2)} \Phi_0^2.$$  

The value $|\tilde{\mu}|$ is a rather small parameter,

$$|\tilde{\mu}| \simeq 6 \text{Ry}^*(na_{ex})^3 (v^2 - v_0^2)/(c_i^2 - v^2),$$

and the energy of the phonon part of the condensate is estimated as $E_{ph} \leq (5 \sim 6) N_o |\tilde{\mu}|$. However, the back surface of the crystal will experience some pressure when the condensate reaches this surface and the excitons are destroyed near it. The estimation of the maximum value of the pressure in a pulse is as follows:

$$p_m \simeq 10 \sigma_0 \Phi_0^2 \simeq 10^{-2} \sim 10^{-3} \text{J/cm}^{-3}$$

and the main contribution to this value comes from the phonon part. Indeed, one can see that the exciton-phonon condensate carries a nonzeroth momentum $P_{ox} = P_{ex,x} + P_{ph,x}$:

$$P_{ox} = \int d^3 x (h/2i) (\phi_0^* \partial_x \phi_0(x, t) - \partial_x \phi_0^* \phi_0(x, t) - \rho \partial_x u_0(x, t) \partial_x u_0(x, t) - \rho \partial_x u_0(x, t) \partial_x u_0(x, t) =$$

$$= \int d^3 x m v \Phi_0^2(x) + \rho v \left( \frac{\sigma_0}{(c_i^2 - v^2) \rho} \Phi_0^2(x) \right)^2.$$

### 4 Low-Lying Excitations of Exciton-Phonon Condensate

Although the condensate wave function $\phi_o(x) \cdot q_o(x)$ was obtained in the framework of the effective 1D model, (but normalized in 3D space), we will use this solution as a classical part in the 3D field operator decomposition:

$$\hat{\psi}(x, t) = \exp(-i\mu t)(\phi_o(x) + \delta \hat{\psi}(x, t)),$$  

$$\hat{u}_j(x, t) = q_o(x) \delta_{ij} + \delta \hat{u}_j(x, t),$$

where $\mu = \tilde{\mu} - mv^2/2$. Substituting the field operators of the form (24), (25) into the Lagrangian density (2), we can write the later in the following form:

$$\mathcal{L} = \mathcal{L}_o(e^{-i\mu t}\phi_o(x), q_o(x)\delta_{ij} + \delta \hat{u}_j(x, t)) + \mathcal{L}_2(\delta \hat{\psi}^\dagger(x, t), \delta \hat{\psi}(x, t), \delta \hat{u}(x, t)) + ...,$$  

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where $L_o$ stands for the classical part of $L$, and $L_2$ is the bilinear form in the $\delta$-operators. As the classical parts of the field operators satisfy the equality $\delta S_o[\psi_0^*, \psi_0, u_0] = 0$, the linear form in the “$\delta$-operators” vanishes in (23).

In the simplest (Bogoliubov) approximation [13], [24], $L \approx L_o + L_2$ and, hence, the bilinear form $L_2$ defines the equations of motion for the fluctuating parts of the field operators. (To derive them we use the variational method: $\delta S_2 = \delta \int L_2(\delta \psi^\dagger, \delta \psi, \delta u) \, dx \, dt = 0$.) As a result, these equations are linear and can be written as follows:

$$i\hbar \partial_t \delta \hat{\psi}(x, t) = \left(-\frac{\hbar^2}{2m} \Delta + |\vec{\mu}| + \left\{ 2\nu_0 - \frac{\sigma_0^2}{\rho(c_l^2 - v^2)} \right\} \phi_o^2(x) + 3\nu_1 \phi_o^4(x) \right) \delta \hat{\psi}(x, t) +$$

$$+ (\nu_0 \phi_o^2(x) + 2\nu_1 \phi_o^4(x)) \delta \hat{\psi}^\dagger(x, t) + \sigma_0 \phi_o(x) \nabla \delta \hat{u}(x, t),$$

$$\left(\partial_t^2 - c_l^2 \Delta - 2\nu \partial_x \partial_x + v^2 \partial_x^2 \right) \delta \hat{u}(x, t) = \rho^{-1} \sigma_0 \nabla \left( \phi_o(x) \left( \delta \hat{\psi}(x, t) + \delta \hat{\psi}^\dagger(x, t) \right) \right)$$

(27)

The same approximation can be performed within the Hamiltonian approach. Indeed, decomposition of the field operators near their nontrivial classical parts leads to the decomposition of the Hamiltonian (I) itself, and – as it was done with the Lagrangian – only the classical part of $\hat{H}$, $H_o$, and the bilinear form in the fluctuating fields, $\hat{H}_2$, are left for examination:

$$\hat{H} \approx H_o(\psi_0^*, \psi_0, \pi_0, u_0) + H_2(\delta \hat{\psi}^\dagger, \delta \hat{\psi}, \delta \hat{\pi}, \delta \hat{u}).$$

(29)

In this approximation, the Hamiltonian (23) can be diagonalized and rewritten in the form:

$$\hat{H} = H_o(e^{-i\mu t} \phi_o(x), q_o(x)) + \delta E_o + \sum_s \hbar \omega_s \hat{a}_s^\dagger \hat{a}_s.$$

(30)

Here, $\delta E_o$ is the quantum correction to the energy of the condensate and the index $s$ labels the elementary excitations of the system. The operators $\hat{a}_s^\dagger$, $\hat{a}_s$ are the Bose ones, and they can be represented by linear combinations of the exciton and displacement field operators:

$$\hat{a}_s = \int d\mathbf{x} \left( U_s(x) \delta \hat{\psi}(x) + V_s^*(x) \delta \hat{\psi}^\dagger(x) + X_{s,j}(x) \delta \hat{u}_j(x) + Y_{s,j}(x) \delta \hat{\pi}_j(x) \right)$$

(31)

$$\hat{a}_s^\dagger = \int d\mathbf{x} \left( U_s^*(x) \delta \hat{\psi}^\dagger(x) + V_s(x) \delta \hat{\psi}(x) + X_{s,j}^*(x) \delta \hat{u}_j(x) + Y_{s,j}^*(x) \delta \hat{\pi}_j(x) \right)$$

(32)

Note that by analogy with the exciton-polariton modes in semiconductors [21] the excitations of the condensate (23) can be considered as a mixture of the exciton- and phonon-type modes, but in this model the phonons come from fluctuations of the $u_0(x, t)$-part of the condensate.

Since the $\alpha$-operators (see [8]) evolve in time as simply as

$$\hat{\alpha}_s(t) = e^{-i\omega_s t} \hat{\alpha}_s \quad \hat{\alpha}_s^\dagger(t) = e^{i\omega_s t} \hat{\alpha}_s^\dagger,$$

these operators (and the frequencies $\{\omega_s\}$) are the eigenvectors (and, correspondingly, the eigenvalues) of the equations of motion (27), (28) obtained within the Lagrangian method.
Then, the time dependent "\(\delta\)-operators" in (27), (28) can be represented by the following linear combinations of the \(\alpha\)-operators:

\[
\delta \hat{\psi}(\mathbf{x}, t) = \sum_s u_s(\mathbf{x}) \hat{\alpha}_s e^{-i\omega_s t} + v_s^*(\mathbf{x}) \hat{\alpha}_s^\dagger e^{i\omega_s t},
\]

(33)

\[
\delta \hat{\psi}^\dagger(\mathbf{x}, t) = \sum_s u_s^*(\mathbf{x}) \hat{\alpha}_s^\dagger e^{i\omega_s t} + v_s(\mathbf{x}) \hat{\alpha}_s e^{-i\omega_s t},
\]

(34)

\[
\delta \hat{u}_j(\mathbf{x}, t) = \sum_s C_{s,j}(\mathbf{x}) \hat{\alpha}_s e^{-i\omega_s t} + C_{s,j}^*(\mathbf{x}) \hat{\alpha}_s^\dagger e^{i\omega_s t}.
\]

(35)

Substituting this ansatz (which is a generalization of the u-v Bogoliubov transformation) into Eqs. (27), (28), we obtain the following coupled eigenvalue equations [9]:

\[
(\hat{\hat{L}}(\Delta) - \hbar \omega_s) u_s(\mathbf{x}) + \left(\nu_0 \phi_o^2(x) + 2\nu_1 \phi_0^4(x)\right)v_s(\mathbf{x}) + \sigma_0 \phi_o(x) \nabla C_s(x) = 0
\]

(36)

\[
\left(\nu_0 \phi_o^2(x) + 2\nu_1 \phi_0^4(x)\right)u_s(x) + (\hat{\hat{L}}(\Delta) + \hbar \omega_s) v_s(x) + \sigma_0 \phi_o(x) \nabla C_s(x) = 0
\]

(37)

\[-\rho^{-1} \sigma_0 \nabla \left(\phi_o(x) u_s(x)\right) - \rho^{-1} \sigma_0 \nabla \left(\phi_o(x) v_s(x)\right) + \left[(-i\omega_s - \nu \partial_x)^2 - c_1^2 \Delta\right] C_s(x) = 0,
\]

(38)

where \(\hat{\hat{L}}(\Delta) = (-\hbar^2/2m) \Delta + |\mu| + |2\nu_0 - \sigma_0^2/\rho(c_1^2 - \nu^2)| \phi_o^2(x) + 3\nu_1 \phi_0^4(x).

To simplify investigation of the characteristic properties of the different possible solutions of Eqs. (36)-(38), we subdivide the excitations (33)-(35) into two major parts, the inside-excitations and the outside-ones. The inside-excitations are localized merely inside the packet area, i.e. \(|\mathbf{x}| < L_0\) and \(\phi_o^2(x) \approx \text{const}\), whereas the outside-excitations propagate merely in the outside area, i.e. \(|\mathbf{x}| > (1 \sim 2)L_0\) and \(\phi_o^2(x) \approx \phi_o^2 \exp(-2|x|/L_0) \rightarrow 0\).

### 4.1 Outside-Excitations

For the outside collective excitations, the asymptotics of the low-lying energy spectrum can be found easily. Indeed, if we assume that \(\phi_o^2(x) \approx 0\) in the outside packet area, the equations (27) and (28) begin to be uncoupled. Then, Eq. (27) describes the excitonic branch of the outside-excitations with the following dispersion law in the co-moving frame:

\[
\hbar \omega_{\text{ex}}(\mathbf{k}) \approx |\mu| + (\hbar^2/2m)k^2, \quad u_k(\mathbf{x}) \sim e^{i\mathbf{kx}}, \quad v_k(\mathbf{x}) \approx 0,
\]

(39)

and Eq. (28) describes the phonon branch and yields the spectrum \(\omega_{\text{ph}}(\mathbf{k}) = c_l|\mathbf{k}|\) in the laboratory frame of reference. Then, the exciton field operator, which describes the exciton condensate with the one outside excitation, has the following form:

\[
\psi(\mathbf{x}, t) \sim \exp(-i(E_g + mv^2/2 - |\mu|)t) \exp(i\mathbf{mvx}) \phi_o(x - vt) + \exp(-i(E_g + mv^2/2 - |\mu|)t) \exp(i\mathbf{mvx}) \left\{\exp(-i(|\mu| + \hbar k^2/2m + k_x v)t) \exp(ikx) u_k\right\}
\]

It is easy to see that such a collective excitation, \(\omega_{\text{ex}} = |\mu| + \hbar k^2/2m + k_x v\), can be interpreted as an exciton with the energy \(E_g + \hbar^2 k^2/2m\), where \(\hbar k_j = \hbar k_j + mv\delta_{ij}\). Then
we can compare the condensate energy \( E_o(N_o) \) and the energy of the condensate with one outside excitation,

\[
E_o(N_o - 1) + E_{exc}(k') + E_{ph}(k') \approx E_o(N_o) - \partial N o E_o(N_o) + (E_g + \hbar^2 k'^2/2m) + \hbar c |k'| \approx \frac{1}{2} \left( \frac{E_g + \hbar^2 k'^2}{2m} + \hbar c |k'| \right) > E_o(N_o) \text{ in } k \to 0 \ (k' \neq 0) \text{ limit.}
\]

Note that the energy (and the momentum) of the phonon part of the condensate changes after exciton emission. We assume that the transformation \( N_o \to N_o - 1 \) (or emission of an outside exciton) corresponds to the situation when the outside exciton and the outside phonon(s) appear together, and the phonon is emitted with the energy compensating the change of \(-\delta E_{ph}\) in the phonon part of the condensate.

However, the condensate collective excitations are uncoupled only in the \( k \to 0 \) limit, i.e. \( \lambda = 2\pi/k \gg L_0 \). It follows from the structure of Eqs. (36)-(38) that the coupling between the outside phonon and exciton branches is originated from the condensate “surface” area, i.e. from the scale \( L_0 < |x| < 3L_0 \). Indeed, \( \phi_o(x) \) and \( \phi'_o(x) \sim \pm \phi_o(x)L_0^{-1} \) cannot be put equal zero in this area, and the “particle”- and the “hole”-type components of the exciton operators, namely \( u_s \sim e^{ikx} \) and \( v_s^* \sim e^{-ikx} \), should be both different from zero and spatially modulated in the surface area, at least for the excitations with \( \lambda < 2L_0 \). We left this question for future investigations.
4.2 Inside-Excitations

To simplify the calculation of inside-excitation spectrum (see Eqs. (36)-(38)) we will use the semiclassical approximation \[20\]. In this approximation, the excitations can be labeled by the wave vector \(k\) in the co-moving frame, and the following representation holds:

\[
\psi_s(x) = u_k(x)e^{i\varphi_k(x)}, \quad v_s(x) = v_k(x)e^{i\varphi_k(x)}, \quad C_{s,j}(x) = C_{k,j}(x)e^{i\varphi_k(x)},
\]

where the phase \(\varphi_k(x) \approx \varphi_0 + kx\), and \(u_k(x), v_k(x), C_{j,k}(x)\) are assumed to be smooth functions of \(x\) in the inside condensate area. Notice that the \(k\)- and \(x\)-representations are mixed here. This means that the operator nature of the fluctuating fields is factually dismissed within the semiclassical approximation. However, the orthogonality relations between \(u_k\) and \(v_k\), and, hence, between \(u_k\) and \(v_k\) come from the Bose commutation relations between the operators \(\alpha_s\) and \(\alpha_s^\dagger\) \[19\], \[20\]. For example, Eq. (33) is modified as follows:

\[
\delta \psi(\mathbf{x},t) \simeq \int \frac{d\mathbf{k}}{(2\pi)^3} u_k(\mathbf{x})e^{i\varphi_k(\mathbf{x})}e^{-i\omega_k(\mathbf{x})t} + v_k(\mathbf{x})e^{-i\varphi_k(\mathbf{x})}e^{i\omega_k(\mathbf{x})t},
\]

and the inside-excitation part of the elementary excitation term in (30),

\[
\sum_s \cdots \approx \sum_{s,\text{out}} + \sum_{s,\text{surf}} + \sum_{s,\text{in}} \cdots,
\]

can be written in the form

\[
\sum_{s,\text{in}} \hbar \omega_s \hat{\alpha}_s^\dagger \hat{\alpha}_s \simeq \int \frac{d\mathbf{k} \, d\mathbf{x}}{(2\pi)^3} \hbar \omega_k(\mathbf{x}) n_k(\mathbf{x}).
\]

Note that the semiclassical energy \(\hbar \omega_k(x)\) of the inside-excitation mode is supposed to be a smooth function of \(x\), i.e. at least as smooth as \(\phi_0^2(x) \approx \text{const}\) in the “inside” approximation.

Although the low-lying excitations cannot be properly described within the semiclassical approximation, we apply it here to calculate the low energy asymptotics of the spectrum. In fact, all the important properties of these excitations can be understood within this approach.

Substituting (33) into Eqs. (36)-(38), we transform these differential equations into the algebraic ones (\(L(-\mathbf{k}^2) = \Delta(-\mathbf{k}^2)\)):

\[
(L(-\mathbf{k}^2) - \hbar \omega_k) u_k(x) + \left(\nu_0 \phi_0^2(x) + 2\nu_1 \phi_0^4(x)\right) v_k(x) + \sigma_0 \phi_0(x)i\mathbf{k}C_k(x) = 0, \quad (43)
\]

\[
\left(\nu_0 \phi_0^2(x) + 2\nu_1 \phi_0^4(x)\right) u_k(x) + (L(-\mathbf{k}^2) + \hbar \omega_k) v_k(x) + \sigma_0 \phi_0(x)i\mathbf{k}C_k(x) = 0 \quad (44)
\]

\[
\rho^{-1} \sigma_0 \phi_0(x)i\mathbf{k} u_k(x) + \rho^{-1} \sigma_0 \phi_0(x)i\mathbf{k} v_k(x) + \left[(\omega_k + vk_x)^2 - \mathbf{c}^2 k^2\right] C_{k,j}(x) = 0. \quad (45)
\]

After some straightforward algebra, we can write out the equation for the exciton-phonon excitation spectrum:

\[
\left((\omega_k + vk_x)^2 - \mathbf{c}^2 k^2\right) \times
\]

\[
\left[(\hbar \omega_k)^2 - (L(-\mathbf{k}) - (\nu_0 \phi_0^2(x) + 2\nu_1 \phi_0^4(x))) (L(-\mathbf{k}) + (\nu_0 \phi_0^2(x) + 2\nu_1 \phi_0^4(x)))\right] =
\]
\[
\begin{align*}
&= \left( L(-k) - (\nu_0 \phi_0^2(x) + 2\nu_1 \phi_0^4(x)) \right) \frac{2\sigma_0^2}{\rho c_i^2} \phi_0^2(x) (c_i^2 k^2). \\
\end{align*}
\]

Note that within the semiclassical description of the inside-excitations, the low energy limit means \( k \to k_0 \) where \( k_0 \) is the momentum cut-off,

\[
(h^2/2m)k_0^2 \approx |\tilde{\mu}| = (h^2/2m)L_0^{-2}.
\]

The inequality \( k > k_0 \approx L_0^{-1} \) ensures the function \( \hbar \omega_k(x) \) in (46) to be real and positive. Indeed, only the excitations with the wave lengths \( \lambda < (2 \sim 3)L_0 \) can be considered as the inside ones. The presence of the “hard core” terms, \( \text{const} \nu_1 \phi_0^4(x) \), leads to a slight renormalization of the value of the momentum cut-off. However, this renormalization does not factually change the characteristic properties of the possible solutions of Eq. (46). We will mark the “hard core” terms by \( \epsilon_+ > 0 \). For example, in the low energy limit,

\[
k \approx k_0 + \delta k, \quad \delta k \to 0,
\]

the exciton part of the l.h.s. of Eq. (46) – i.e. the formula inside the square brackets – can be reduced to the form:

\[
(h \omega_k)^2 - \left( \frac{\hbar^2(k^2 - k_0^2)}{2m} + F(x) + \epsilon_+ \right) \left( \frac{\hbar^2(k_0^2 - k^2)}{2m} + F(x) + 2\nu_0 \phi_0^2(x) + \epsilon_+ \right),
\]

where \( F(x) = (\sigma_0^2/\rho c_i^2 - \nu_0) (\Phi_o^2 - \phi_0^2(x)) \), and the following two estimates hold: \( F(x) \approx 2|\tilde{\mu}|/(2x/L_0)^2 \) at \( x \sim 0 \) and \( F(x) \approx 2|\tilde{\mu}| \) at \( x > \pm 2L_0 \).

There are two different types of the inside-excitations, the longitudinal excitations and the transverse ones. The later have the wave vectors \( k \) perpendicular to the \( x(v) \)-direction. Although Eq. (46) can be solved exactly for the transverse excitation spectrum [22], taking into account the coupling term changes the values of excitation energies slightly, and the excitations can be approximately considered as of the pure exciton or phonon types. Then we have the acoustic phonon dispersion low for the phonon branch and the following spectrum for the exciton branch:

\[
(h \omega_{ex,k_\perp})^2 \approx \left( \frac{\hbar^2}{2m}(k^2 - k_0^2) + \epsilon_+ \right) \left( \frac{\hbar^2}{2m}(k^2_0 - k_0^2) + 2\nu \phi_0^2(x) + \epsilon_+ \right) \approx \frac{\hbar^2}{2m}(k^2 - k_0^2) 2\nu \phi_0^2(x) + 2\nu \phi_0^2(x) \epsilon_+ < \left( |\tilde{\mu}| + (h^2/2m)k_0^2 \right)^2.
\]

The smooth function \( \omega_{k_\perp}(x) > 0 \) has a gap when \( k \to k_0 \), but unlike the case of the outside excitations, the gap value is determined by the “hard core” repulsion term and is much less than \( |\tilde{\mu}| \). Furthermore, if we let (formally) the \( x \) coordinate in (18) change in the area of \( |x| > L_0 \), the dispersion low \( h \omega_{ex,k_\perp}(x) \) reproduces the outside excitation asymptotics, \( |\tilde{\mu}| + (h^2/2m)k_0^2 \). However, inside the condensate, we obtain a strong deviation of the collective excitation spectrum from the simple excitonic one.

In the case of the longitudinal excitations, the mode interaction is non-negligible in the low energy limit. For instance, the “exciton” root of Eq. (46) with the nonzer
exists if \( k_x > 2.5 L_0^{-1} \). Therefore there are no distinct exciton and phonon modes, and the cases \( k_x > 0 \) and \( k_x < 0 \) are different because of different position of the “bare” phonon root on the energy axis. Then, on the energy axis \( \hbar \omega \), the modified phonon spectrum is located higher than the phonon frequencies and the modified exciton spectrum is located lower than \( \hbar \omega_{ex, k_x}^{(o)} \). Yet, like the case of transverse excitations, the same inequality and the same (formal) asymptotics are valid for the lower branch of the spectrum:

\[
0 < \omega_{k_x} < |\mu| + (\hbar^2/2m)k_x^2.
\]

The approximate formulas for the longitudinal spectrum are too cumbersome to be presented here. However, the phonon component of the excitonic longitudinal excitation, \( C_{k,1}(x) \neq 0 \), can be found approximately from Eq. (13) by use of the Bogoliubov form for the wave functions \( u_k^2(x) \) and \( v_k^2(x) \):

\[
\begin{align*}
u_k^2(x) &\approx \left( \frac{1}{V_{eff}} \right) \frac{L(-k^2) + \hbar \omega_k}{2 \hbar \omega_k}, \\
u_k^2(x) &\approx \left( \frac{1}{V_{eff}} \right) \frac{L(-k^2) - \hbar \omega_k}{2 \hbar \omega_k},
\end{align*}
\]

where \( L(-k^2) \approx \hbar^2(k^2 - k_0^2)/2m + \nu \delta\phi^2(x) + \epsilon_+ \). The effective condensate volume \( V_{eff} \approx 4SL_0 \) is used to normalize the \( u \)- and \( v \)-wave functions of the inside excitations, \( \int dx (|u_k|^2 - |v_k|^2) = 1 \). Subsequently, we get

\[
C_{k,1}(x) \approx \rho^{-1} \sigma_0 \delta\phi(x) i k_x (u_k(x) + v_k(x)).
\]

One can use the zeroth approximation, Eq. (18), for \( \omega_{k_x} \) in (19),(20). Then we can roughly estimate the maximum value of \( |C_k(x)|^2 \) in the low energy limit \( (k \to 2.5L_0^{-1}, \omega \ll c|k_x|) \):

\[
|C_k|^2 \simeq \left( |\mu|/\rho(c_1^2 - v^2) V_{eff} \right) L_0^2 \ll L_0^2.
\]

To investigate the stability of the moving condensate in relation to the creation of inside excitations, we can calculate the energy of the condensate with the one inside excitation described by the following set: \( k, \omega_k, u_k \) and \( v_k \), and \( C_k \). Although such an excitation was defined in the co-moving frame, calculations should be done in the laboratory frame. Returning to the lab frame, we represent the exciton and phonon field functions as follows:

\[
\phi_o(x - vt, t) \rightarrow \phi_o(x - vt, t) + \exp(-i(E_g + mv^2/2 - |\mu|)t) \exp(imvx) \delta\Psi(x, t),
\]

\[
\delta\Psi(x, t) = u_k(x - vt)e^{ikx}e^{-i(\omega_k + k_x)vt} + v_k(x - vt)e^{-ikx}e^{i(\omega_k + k_x)vt},
\]

\[
u_o(x - vt) \rightarrow u_o(x - vt) + C_k(x - vt) \exp(i kx) \exp(-i(\omega_k + k_x)vt) + c.c..\]

As the inside excitation is considered as a fluctuation, the number of particles in the condensate and its energy can be estimated as \( N_o = \int \! dx \delta\psi^\dagger \delta\psi \) and \( E_o(N_o) - \partial \partial N_o \int \! dx \delta\psi^\dagger \delta\psi \), respectively.

The zeroth component of the energy-momentum tensor can be represented in the form

\[
T^0_0 = T^0_0(\phi_o, u_o) + T^0_0(\delta\Psi, \delta\Psi, \delta u | \phi_o, u_o),
\]

14
where the first part gives the condensate energy $E_o$ and the second part will give the
energy of the inside excitation, $E_{in}$. After substitution of (52),(53) into $E_{in} = \int dx T_0^{(2)}$, it can be rewritten as follows:

$$E_{in} = \int dx (E_g + mv^2/2 - |\vec{\mu}|) \delta \bar{\psi}^\dagger \delta \bar{\psi} + \int dx \hbar(\omega_k + k_x v)(|u_k|^2 - |v_k|^2) + 2 \rho|C_k|^2(\omega_k + v k_x)^2.$$  (54)

The first term appearing in (54), $\delta \bar{\psi}^\dagger \delta \bar{\psi} \rightarrow |u_k|^2 + |v_k|^2$, vanishes in the following formula for the total energy:

$$E_o + E_{in} \approx E_o(N_o) + (2|\vec{\mu}| + \nu_0 \Phi_o^2) \int dx \delta \bar{\psi}^\dagger \delta \bar{\psi} + \int dx \hbar(\omega_k + k_x v)(|u_k|^2 - |v_k|^2) -$$

$$- \frac{c_l^2 + v^2}{c_l^2 + v^2} \frac{\sigma_o^2}{\rho(c_l^2 - v^2)} \Phi_o^2 \int dx \delta \bar{\psi}^\dagger \delta \bar{\psi} + \int dx 2 \rho|C_k|^2(\omega_k + v k_x)^2.$$  (55)

Here the last two terms compensate each other approximately, and all the interesting
effects come form the exciton part of (55). Although $\omega_k + k_x v$ can be negative, its negative
value can be compensated by the term $2|\vec{\mu}| + \nu_0 \Phi_o^2$ if the velocity of the condensate is close
to $c_l$ or the exciton concentration is high enough. Therefore, there is a second critical
velocity $v_c$ in the theory. If the velocity of the condensate is less then the velocity $v_c$ and
$v_o < v_c < c_l$, the condensate is unstable in relation to creation of the inside excitations,
i.e. $E_o + E_{in}(k_x, v) < E_o$ in the lab frame. To estimate the value of $v_c$, we solve the
following equation:

$$(3 \sim 5)\hbar v L_0^{-1}(v) \approx \frac{\sigma_o^2}{\rho(c_l^2 - v^2)} \Phi_o^2(v),$$

which can be reduced to $(3 \sim 5)\hbar v \approx \frac{\sigma_o^2}{\rho(c_l^2 - v^2)} \frac{N_o}{2S}$. For example, for the packets
with the exciton concentration of $n \approx 10^{14} \sim 10^{15}$ cm$^{-3}$, the estimate is as follows:

$$\frac{c_l - v_c}{c_l} \approx \frac{\sigma_o^2}{\rho c_l^2} \frac{N_o}{2S} \frac{0.1}{\hbar c_l} \approx 0.1 \sim 0.3.$$  (56)

### 5 Interference between Two Moving Packets

There are at least two interesting problems that can be examined in the framework of the
proposed model (Eqs. (54)). The first problem is the investigation of the condensate
steady-state and its stability, calculation of the low-lying excitation spectrum, etc. (Some
part of this program was presented in the previous sections.) The second one is the
investigation of interference between two moving packets [23]. In this case the problem is
essentially nonstationary. Indeed, the amplitude and the shape of the resultant moving
packet are expected to change in time [24]. These effects, however, can be considered
theoretically by a numerical solving of Eqs. (54) with the proper initial conditions.
(We will disregard the influence of noncondensed particles, the condensate depletion and
nonequilibrium phonons on the dynamic processes being considered.) For example, if two
“input” packets have the same velocity ($v_1 = v_2 = v$) and shape, we can write the initial
conditions in the form

$$\psi_0(x, t = 0) \cdot u_0(x, t = 0) = \phi_o(x) \cdot q_o(x) + e^{i\delta \tau} \phi_o(x + v \tau) \cdot q_o(x + v \tau),$$  (57)
where $\delta \varphi = \text{const}$ is the macroscopical phase difference between the two “input” condensates, and $\tau = \text{const}$ is the time delay between them.

In the simplest (quasi-1D) model, the following equations govern the dynamics of the two “input” packets:

$$(ih \partial_t + mv^2/2)\psi_0(x,t) = (-\frac{\hbar^2}{2m}\partial_x^2 + \nu_0|\psi_0|^2 + \nu_1|\psi_0|^4)\psi_0(x,t) + \sigma_0 \partial_x u_0(x,t) \psi_0(x,t)$$

$$(\partial_t^2 - (c_i^2 - v^2)\partial_x^2 - 2v\partial_t\partial_x)u_0(x,t) = \rho^{-1}\sigma_0 \partial_x |\psi_0|^2(x,t).$$

Then the initial conditions can be written in the explicit 1D form by using the exact solution of the model. Note that the amplitudes of the “input” condensates in the model have the same values.

Some predictions of the form of the expected solution can be made easily. Indeed, the shape and other characteristics of the steady-state solution will depend mainly on the value of the parameter $x_0/L_0$ where $x_0 = \nu \tau$ and $L_0$ is the characteristic condensate width. If $x_0/L_0 < 1 \sim 2$, the nonlinear interaction between the packets plays an important role in the process of total wave function formation. (Notice that the lower limit of $x_0$, $x_0^* = \tau^* v$, is defined by the time scale of formation of a condensate wave function.) It is reasonable to assume that in the limit of strong interaction between condensates, the $N_o + N_o$ excitons can form the single condensate wave function in the steady-state regime. Then this wave function can be written (in the laboratory frame) as follows:

$$\psi_0(x,t) \cdot u_0(x,t) \simeq \exp(-i(\tilde{\mu}(2N_o, \tilde{v}) + m\tilde{v}^2/2)t) \exp(im\tilde{v}x) \times \exp(i\tilde{\varphi})\phi_0(x - \tilde{v}t; 2N_o) \cdot q_o(x - \tilde{v}t; 2N_o).$$

As the dynamic equations conserve the energy,

$$E_{\text{in}}(N, v; x_0/L_0) = E_{\text{out}}(2N, \tilde{v}),$$

$\tilde{v}$ cannot be equal to $v$ in theory. Moreover, if the parameter $x_0/L_0$ is small enough, it can be only the approximate equality, $\tilde{v} \approx v$.

In the case of $x_0/L_0 \gg 1$, one can disregard the influence of the mutual nonlinear interaction on the dynamics of the packets. In this approximation, the packet moving in the crystal can be modeled by the following formula:

$$\psi_0(x,t) \cdot u_0(x,t) \simeq \exp(-i(\tilde{\mu}(N_o) - mv^2/2)t)\phi_0(x; N_o) \cdot q_o(x; N_o) + \exp(i\delta \varphi)\exp(-i(\tilde{\mu}(N_o) - mv^2/2)t)\phi_0(x + x_0; N_o) \cdot q_o(x + x_0; N_o).$$

An interesting and noninvestigated case in the interference problem is the condensate dynamics after posing nonsymmetric initial conditions. In fact, the amplitude and the velocity of the “input” packets can be different, for example, $N_2 > N_1$ and $v_2 > v_1$, $v_2 \parallel v_1$. We use here the experimental result that at $v > v_0$, the velocity of the condensate...
depends on the laser power or, equivalently, on the initial concentration of excitons, i.e. \( v = v(N) \). (Note that in theory the exciton amplitude \( \Phi_o \) is the function of \( N_o \) and \( v \).)

If the exciton concentration in the first packet, \( n_1 \), is close to the Bose condensation threshold value and the exciton concentration in the second packet, \( n_2 \), can be made \( \gg n_1 \), the velocity difference between condensates can reach \((0.2 \sim 0.3)c_1\). Then, in the reference frame moving with the first packet, the initial conditions can be taken as the following:

\[
\psi_0(x, t = 0) \cdot u_0(x, t = 0) = \phi_0(x; N_1) \cdot q_o(x; N_1) + \exp(i\delta \varphi) \exp(im \delta v x) \phi_0(x + x_0; N_2) \cdot q_o(x + x_0; N_2),
\]

(60)

where \( \delta v = v_2 - v_1, x_o = v_1 \tau \), and the second packet moves in this frame of reference. In the case of such the initial conditions, the regime of strong nonlinear interaction between the condensates is unavoidable. Following the logic of the soliton theory, we speculate that the steady-state solution may consist of two packets moving with different velocities and with different exciton concentrations. Roughly speaking, the “input” packets could exchange their places, i.e. the both packets survive after collision, and the first packet arrives at the “detecting” boundary of a crystal after the second one:

\[
\psi_0(x, t) \cdot u_0(x, t) \simeq \exp(-i\bar{\mu}(N_1) - mv_1^2/2)t) \phi_o(x; N_1) \cdot q_o(x; N_1) + \exp(-i(\bar{\mu}(N_2) - mv_1^2/2 + m\delta v^2/2)t) \exp(i\delta \bar{\varphi}) \exp(im \delta v x) \times \\
\times \phi_o(x + x_0 - \delta vt; N_2) \cdot q_o(x + x_0 - \delta vt; N_2).
\]

However, the hypothesis about the solitonic character of packet collisions in 3D needs both numerical and experimental evidence.

6 Conclusion

In this study, we considered a model within which the inhomogeneous excitonic condensate with a nonzero momentum can be investigated. The important physics we include in our model is the exciton-phonon interaction and the appearance of a coherent part of the crystal displacement field that makes the moving condensate of the exciton-phonon one. Then, the condensate wave function and its energy can be calculated exactly in the simplest quasi-1D model. We believe that the transport and other unusual properties of the coherent para-exciton packets in Cu_2O can be described in the framework of the proposed model properly generalized to meet more realistic conditions.

As the exciton-phonon interaction is very important in any processes involving excitons in lattices [23], we can speculate about the possibility of use of piezoelectrical transducers to pump acoustic waves into the system condensate + lattice. Moreover, the transducers could be used to register the phonon part of the coherent packet in experiments in which the condensate is formed by optically inactive excitons and phonons.

We showed that there are two critical velocities in the theory, \( v_0 \) and \( v_c \). The first one, \( v_0 \), comes from the renormalization of two particle exciton-exciton interaction due to phonons, and the inhomogeneous soliton state can be formed if \( v > v_0 \). The second
one, \( v_c \), comes from use of Landau arguments \[19\] for investigation of the dynamical stability / instability of the moving condensate. Within the semiclassical approximation for the condensate excitations, we found the condensate is unstable if \( v < v_c \). It is interesting to discuss the possibility of observation of such an instability when the condensate can be formed in the inhomogeneous state with \( v \neq 0 \), but with \( v_0 < v < v_c(n, v) \). Such a coherent packet has to disappear during its move through a single pure crystal used for experiments. As the shape of the moving packet depends on time, the form of the registered signal may depend on the crystal length changing from the solitonic to the standard diffusion density profile.

We did not concentrate on detailed investigation of excited states of the moving exciton-phonon condensate in this study. First, the possibility of their observation is an unclear question itself. Second, the stability conditions of the moving condensate – in relation to the creation of condensate excitations – can be derived from the low energy asymptotics of the excitation spectra at \( T \ll T_c \). However, the stability problem is not without difficulties \[22\],[25\]. One can easily imagine the situation when the condensate moves in a very high quality crystal, but with some impurity region carefully prepared in the middle of the sample. Then the impurities could bound the noncondensed excitons, which always accompany the condensate, and could mediate, for instance, the emission of the outside excitations. The last process may lead to depletion of the condensate and, perhaps, some other observable effects, such as damping, bound exciton PL, etc.. On the other hand, the inside excitations could manifest themselves at \( T \neq 0 \) by the effective enlargement of the packet length, \( L_0 \rightarrow L_{\text{eff}}, T \neq 0 \), or by interaction with external acoustic waves.

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