A Tutorial to Sparse Code Multiple Access

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Abstract—Sparse Code Multiple Access (SCMA) is an enabling code-domain non-orthogonal multiple access (NOMA) scheme for massive connectivity and ultra low-latency in future machine-type communication networks. As an evolved variant of code division multiple access (CDMA), multiple users in SCMA are separated by assigning distinctive codebooks which display certain sparsity. At an SCMA receiver, efficient multiuser detection is carried out by employing the message passing algorithm (MPA) which exploits the sparsity of codebooks to achieve error rate performance approaching to that of the maximum likelihood receiver. Despite numerous research efforts on SCMA in recent years, a comprehensive and in-depth tutorial to SCMA is missing, to the best of our knowledge. To fill this gap and to stimulate more forthcoming research, we introduce the principles of SCMA encoding, codebook design, and MPA based decoding in a self-contained manner for layman researchers and engineers.

Index Terms—Factor Graphs, Message Passing Algorithm, Sparse Code Multiple Access, Non-Orthogonal Multiple Access, Codebook Design.

I. INTRODUCTION

The next-generation cellular systems are expected to support a wide range of vertical industries such as e-health, smart homes/cities, connected autonomous vehicles, and factories-of-future [1]. While this leads to ubiquitous digital data services anywhere and anytime, the explosive growth of communication devices and applications brings an increasingly congested spectrum which is one of the major challenges in the design of 5G communication networks and beyond.

Multiple access, as one of the core techniques of wireless communication, aims to enable multiple users to access the finite resources simultaneously in an effective manner. Legacy multiuser communication systems mostly use orthogonal multiple access (OMA) schemes in which multiple users are orthogonal to each other with respect to certain type of resources. In the past cellular networks, there have been time division multiple access (TDMA), frequency division multiple access (FDMA), code division multiple access (CDMA), and orthogonal frequency-division multiple access (OFDMA). As an example, in TDMA, each user is given a distinct time slot and no two users simultaneously share the same time slot. In every OMA scheme, the number of users being simultaneously served is upper bounded by that of the orthogonal resources.

In recent years, non-orthogonal multiple access (NOMA) has attracted tremendous research attention with a major aim of providing massive connectivity, lower communication latency, and higher spectral efficiency compared to OMA. In a NOMA system, two or more users are superimposed over an identical physical resource (e.g., power, frequency, time, or code) to provide overloading factor larger than one [2].

In 2013, sparse code multiple access (SCMA) was proposed by H. Nikopour and H. Baligh as a disruptive code-domain NOMA scheme [3]. SCMA is a generalized multiple access scheme building upon low-density signature CDMA (LDS-CDMA) [4], where the latter is a special case of CDMA in which several bits of each user are mapped to a symbol followed by sparse sequence spreading. By contrast, QAM mapping and spreading are merged together in SCMA and therefore several incoming bits (of certain user) can be directly mapped to a sparse complex vector (codeword) which is drawn from its associated sparse codebook.

A. Related Works

In general, there are two major research lines associated to SCMA: codebook (CB) design and multiuser detection (MUD). The goal of CB design is to design and optimize CBs with respect to certain channels (e.g., downlink vs uplink, Gaussian vs Rayleigh) in order to improve a number of system performance measures such as bit/packet error rate, spectral efficiency, and peak-to-average power ratio (PAPR).

After the first CB design work [5] in 2014, numerous research attempts have been made. In [6], new CBs are designed from M-order pulse-amplitude modulation (PAM) to maximize the sum rate of SCMA. In [7], the authors proposed a multi-dimensional CB based on constellation rotation and interleaving to maximize the minimum Euclidean distance between the codewords. In [8], novel CBs for both Rayleigh and Gaussian channels based on Star-QAM constellations have been developed with the aim of minimizing the error probability. Inspired by recent major advances in machine learning, new CBs have been found in [9] using deep-learning-based method by considering minimum bit error rate (BER) as the optimization criteria. In [10], golden angle modulation (GAM) constellations have been used for CBs which exhibit excellent error rate performances in both uplink and downlink Rayleigh fading channels. Recently, a low complexity construction algorithm for near-optimal CBs has been developed in [11].

Thanks to the sparsity of the codebooks, the multiuser signals can be efficiently detected and recovered with the aid of message passing algorithm (MPA) [4] whose error rate performance approaches to that of maximum a posteriori (MAP) detector. In MPA, the belief messages are passed along the
In this paper, the function \( f(x) \) is the of binary numbers and complex numbers, respectively. Also, \( f(x) \) expressed in log-domain is introduced by simplifying the exponent and multiplication operations to maximize and add operations, respectively. In [14], a partial marginalization based MPA is proposed for fixed low-complexity SCMA detection. [15] proposed a belief threshold-based MPA in which the belief messages updates for every user are checked in every iteration. This reduces the total number of computational operations when the number of iterations increases. In [16], a lookup table method is designed to simplify the function node update which ensures the stable convergence of MPA. In [17], a hybrid belief propagation and expectation propagation receiver was proposed for a downlink MIMO-SCMA system. In [18], a threshold-based edge selection and Gaussian approximation (ESGA) was proposed in which partial Gaussian approximation was applied dynamically for significant complexity reduction at the receiver.

### II. System Model

#### A. SCMA Encoding and Codebook Design

Consider an uplink synchronous SCMA system in which \( J \) users transmit data to the basestation (BTS) using \( K \) resource elements (e.g., time- or frequency-slots). In SCMA, the data/input bits of user is mapped to a complex codeword using the SCMA encoder. For instance, user \( j \) wants to transmit \( b_j \) bits. The encoder will map these \( b_j \) bits to a codeword \( m_j \) selected from a pre-defined codebook \( \text{CB}_j \) as shown below

\[
m_j = \text{CB}_j(b_j).
\]

where \( m_j \in \mathbb{C}^K, \text{CB}_j \) denotes the set of codewords of \( j \)th user.

An SCMA encoder has \( J \) layers and there is a specific codebook (CB) dedicated to each user. Assuming one layer per user and from here on ‘user’ and ‘layer’ are used interchangeably. The performance gain of SCMA over other NOMA schemes is strongly dependent on well designed sparse codebooks. The codebook of each user has its own sparsity pattern and can be written as a matrix of size \( K \times M \), where \( M \) denotes the number of codewords (i.e., columns of a CB matrix) allotted to a user. In a CB, each column vector (i.e., codeword) is sparse consisting of \( d_u \) non-zero elements at certain fixed resources elements (REs) pertinent to a specific user.

Albeit numerous SCMA CBs have been proposed in the literature, the optimal CB design remains an open challenge. The current designs are mostly sub-optimal which are based on a multi-stage approach. For the \( j \)th user, multi-dimensional codebooks can be expressed as

\[
\text{CB}_j = \mathbf{V}_j \Delta_j \mathbf{A}'_{MC}, \quad j = 1, 2, \ldots, J.
\]

where \( \mathbf{V}_j \in \mathbb{B}^{K \times d_u} \) denotes the binary mapping matrix, \( \mathbf{A}'_{MC} \) denotes the multi-dimensional mother constellation and \( \Delta_j \) refers to the constellation operator for the \( j \)th user, respectively. The mapping matrix is selected in such a way that each user has active transmissions over a few fixed REs only. The allotment of REs among users can be represented by a signature matrix \( \mathbf{F} = [\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_J] \), where \( \mathbf{f}_j = \text{diag}(\mathbf{V}_j \mathbf{V}_j^H) \). \( \text{F}_{jk} = 1 \) denotes that user \( j \) has active transmission over the \( k \)th RE. For a \((4,6)\) SCMA block having \( K = 4 \) REs and \( J = 6 \) users, one example of the signature matrix of size \( 4 \times 6 \) is given as

\[
\mathbf{F}_{4 \times 6} = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1
\end{bmatrix}.
\]
The binary mapping matrices corresponding to the six users are given below:

\[
V_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad V_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}
\]

\[
V_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad V_5 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad V_6 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}
\]

\[V_1\] indicates that user 1’s data are transmitted over the first, and the second REs. Similarly, \(V_2\) indicates that user 2’s data are transmitted over the first and the third REs, and so on. Once the mapping matrices are decided, one also requires mother constellation \(A'_{MC}\) and layer-specific operations, \(\Delta_j\).

Unitary rotations may be applied to a mother constellation to increase power variation among different users in order to reinforce the “near-far effect” for suppression/mitigation of mutliruser interference as well as to enhance the constellation shaping gain. Once the mother constellation is designed, layer-specific operations are applied to generate multiple CBs for different users. These operations may include phase rotation, complex conjugate, layer power offset, and dimensional permutation.

We provide an example of SCMA codebook design for \(J = 6, K = 4\) (therefore with overloading factor \(J/K = 1.5\)) and the number of users superimposed over each RE is \(d_f = 3\). Steps 1-3 involve designing of mother codebook and Step 4 discusses user-specific operations.

- **Example 1:**
  - **Step 1:** Initially a complex vector \(A_1\) is designed.

  \(A_1 = \{Y_m(1+i)\mid Y_m = 2m-1-M, \ m = 1, \cdots, M\}.\)

  For \(M = 4\), \(A_1 = \{-3(1+i), -1(1+i), 1(1+i), 3(1+i)\}\).

  - **Step 2:** For \(d_v\) non-zero elements in a codeword, \(A_{d_v} = U(d_v)A_1\), where \(U(d_v) = \text{diag}(1e^{i\theta d_v})\) is a phase rotation matrix, \(I\) is a \(d_v\)-dimensional vector of all ones and \(\theta_d = \frac{(d_v - 1)\pi}{d_c}\). Such an \(d_v\)-dimensional mother constellation is given below:

\[
A_{MC} = [A_1, \cdots, A_{d_v}]^T = \begin{bmatrix}
\lambda_{11} & \lambda_{12} & \cdots & \lambda_{1M} \\
\lambda_{21} & \lambda_{22} & \cdots & \lambda_{2M} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{d_v,1} & \lambda_{d_v,2} & \cdots & \lambda_{d_v,M}
\end{bmatrix}
\]

- **Step 3: Interleaving:** The elements of even dimensions of mother constellation \(A_{MC}\) are reordered, as interleaved codewords performs good in fading channel and PAPR of mother constellation also reduces with interleaving. For \(d_v = 2, M = 4\), elements of second row \(A_2 = \{\lambda_{21}, \lambda_{22}, \lambda_{23}, \lambda_{24}\}\) can be re-ordered as \(A'_2 = \{\lambda_{22}, \lambda_{24}, \lambda_{21}, \lambda_{23}\}\). The final mother constellation is \(A'_{MC} = [A_1, A'_2, A_3]^T\).

**Step 4: Generating codebook for different users.**

Once the mother constellation is ready, the SCMA codebook is generated for each user. The phase rotation angle \(\omega_u\) can be given as

\[
\omega_u = \exp \left(\frac{i\pi n}{d_v d_f}\right), \quad \text{for } u = 0, 1, \cdots, d_f - 1.
\]

These \(\omega_u\) can be assigned to the non-zero positions of the factor graph matrix by Latin order. For \(d_v = 2, M = 4, d_f = 3\) and \(J = 6\) users, we have \(\omega_0 = 1, \omega_1 = \exp \left(\frac{i\pi}{3}\right), \omega_2 = \exp \left(\frac{2i\pi}{3}\right)\).

The signature matrix with phase rotation angles is given by

\[
\mathcal{F} = \begin{bmatrix}
\omega_0 & \omega_1 & \omega_2 & 0 & 0 & 0 \\
\omega_1 & 0 & 0 & \omega_2 & \omega_0 & 0 \\
0 & \omega_2 & 0 & \omega_0 & 0 & \omega_1 \\
0 & 0 & \omega_0 & 0 & \omega_1 & \omega_2
\end{bmatrix}.
\]

The operator for the \(j\)th user is \(\Delta_j = \text{diag}(w_j)\), where \(w_j\) is the \(j\)th column of \(\mathcal{F}\) with only non-zero values.

With mother constellation \(A'_{MC}\), constellation operator \(\Delta_j\) and binary mapping matrix \(V_j\) for the \(j\)th user, codebook for the \(j\)th user can be generated using [2].

**B. System Model**

Consider a symbol-synchronous uplink SCMA system where \(J\) users communicate over \(K\) resource elements (REs). Let \(m_j\) be the transmitted codeword of the \(j\)th user, where \(m_j \in \mathbb{C}^{K \times 1}\) has cardinality \(M = 2^b\) with \(b\) denoting the number of bits per codeword. Let us consider a \((4,6)\) SCMA system with \(M = 4\), i.e., each codebook has 4 codewords that are mapped to two binary bits.

For an uplink SCMA system, let \(h_j = [h_{1j}, \cdots, h_{Kj}]^T\) be the effective channel fading coefficient for the \(j\)th user, where \(h_{kj}\) denotes the channel fading coefficient at the \(k\)th RE for the \(j\)th user. Let \(m_j = [C_{1,j}, \cdots, C_{K,j}]^T\) be the transmitted codeword of the \(j\)th user, where \(C_{kj}\) be the codeword element transmitted by the \(j\)th user on the \(k\)th RE. The received signal at the Base station is

\[
y = \sum_{j=1}^J \text{diag}(h_j)m_j + \mathbf{n}.
\]

where \(\mathbf{n} \in \mathbb{C}^{K \times 1}\) is the noise vector, each element of which can be modeled as complex Gaussian distribution \(\mathcal{CN}(0, \sigma^2)\). Due to the sparse nature of SCMA codebooks, non-zero values from \(d_f\) out of \(J\) number of users overlap over each RE and also each user data is transmitted on \(d_v < K\) REs. Let \(\xi_k\) be the set of users transmitting data over the \(k\)th RE and \(\zeta_j\) be the set of REs on which the \(j\)th user has active transmission, respectively. By definition, the numbers of elements in \(\xi_k\) and \(\zeta_j\) are \(d_f\) and \(d_v\), respectively. Thus, the received signal at the \(k\)th RE is

\[
y_k = \sum_{j \in \xi_k} h_{kj} C_{k,j}(m_j) + n_k, \quad \text{for } k = 1, 2, \cdots, K.
\]
— Example 2: Let us consider (4,6) SCMA block encoding as shown in Fig. 1 which corresponds to the signature matrix $F_{4 \times 6}$ (overloading factor $\lambda = J/K = 1.5$) as shown in [3]. In $F_{4 \times 6}$, each row corresponds a RE and each column corresponds to a user, respectively. Note that the number of users superimposing over one RE is 3, i.e., $d_f = 3$ and the number of non-zero values in a column is 2, i.e., $d_c = 2$. The received signal at the $k$th RE is

$$y_k = h_{k,1}C_{k,1}(m_1) + h_{k,2}C_{k,2}(m_2) + h_{k,3}C_{k,3}(m_3) + n_k,$$

for $k = 1, 2, \cdots, K$. (6)

where $m_1, m_2, m_3$ are the codewords of users which belong to set $\xi_k$. From $F_{4 \times 6}$, the set of users which interfere over the first RE is $\xi_1 = \{1, 2, 3\}$ and the set of REs at which the first user has active transmission is $\zeta_1 = \{1, 2\}$. Similarly, $\xi_2 = \{1, 4, 5\}$ implies that the first, the fourth, and the fifth users superimpose at the second RE and $m_1, m_2, m_3$ are the codewords sent by users of set $\xi_2$.

III. DECODING

The process of extracting data from the noise corrupted received signal is called decoding and there are two types of decoding, namely hard and soft decoding. In some communication systems, conventional hard decoding is used, where each decision is taken at the receiver based on a certain threshold value. For instance, a simple hard decoding receiver decides that if the received value is greater than the threshold value, it will be decoded as 1, and 0 otherwise. However, it is not verified how close or far is the received value to the threshold before making the decision. An alternative approach is to obtain the estimated sequence as well as its reliability level where the latter indicates the ‘confidence’ we have in that estimated sequence. Such a decoding approach is called soft decoding [19]. In this way, more information can be extracted from the received data and estimation/detection can be carried out in an iterative and improved manner. We now present the following example to illustrate soft decoding.

— Example 3: (3,1) Repetition code:

Let $x$ be the bit to be transmitted. It is passed through the encoder which results in a codeword $c = [c_1, c_2, c_3] = [x, x, x]$ since it is a repetition code. After BPSK modulation, each “0” bit is converted to “1” and “1” to “−1”, respectively. These modulated symbols propagate through a noisy channel, and the received signal can be expressed as $y = [y_1, y_2, y_3]$ as shown in Fig. 2. Next we discuss both hard decoding and soft decoding to recover the transmitted bit and understand the advantages of the latter.

1) Hard Decoding

Once $y$ is received at the receiver, every $y_i \in y$ goes through a threshold stage as shown in Fig. 3. If $y_i > 0$, $d_i = 0$ otherwise $d_i = 1$, where $d_i \in d$. Let us assume that the received vector is $y = [0.02, -2, -0.4]$. Using 0 as the hard threshold, $y_1 = 0.02$ is decided as $d_1 = 0$ even though it is quite close to the threshold. Obviously, the confidence in this decision is very low. For $y_2 = -2$, based on the threshold $d_2 = 1$ is decided and since it is far from the threshold, the confidence in this decision is large. Similarly, for $y_3 = -0.4$, $d_3 = 1$ is decided. It is noted that the confidence level for $y_3$ is not as large as that for $y_2$, albeit larger than that for $y_1$. Thus, even though the bit sequence $d$ is estimated, still there is a lack of confidence that our decisions are correct.

Once the estimated bit sequence $\hat{d}$ is determined, the Hamming distance for all possibilities for the codeword $d$ and $c$ is calculated. Since it’s repetition code, $c = [x, x, x]$ can be either [000] or [111]. Then, $c$ with lower Hamming distance is chosen corresponding to every $d$. As we see in Table 1, first four values of 1st column are closer to $c = [000]$ and last four values are closer to $c = [111]$. Next, since its a repetition code, $\hat{x}$ is decided from $c$. (Fig. 1: An illustration of 4 x 6 SCMA Encoding.)
With the repetition code, the same information is transmitted multiple times. However, that information is not being efficiently utilized in hard decoding for the correct estimation.

2) **Soft Decoding**

For binary variables, the output vector is formed by three log-likelihood ratios (LLRs) \([L_1, L_2, L_3]\) which are all real values indicating the ‘confidence or reliability level’ that \(x_i\) bit is 0. The reliability of each received value is computed in order to improve the detection performance for the transmit data. Using Bayes’ rule,

\[
P(c_1 = 0 | y_1) = \frac{f(y_1 | c_1 = 0) P(c_1 = 0)}{f(y_1)}, \quad (7)
\]

\[
P(c_1 = 1 | y_1) = \frac{f(y_1 | c_1 = 1) P(c_1 = 1)}{f(y_1)}. \quad (8)
\]

Here, \(P(c_1 = 0 | y_1)\) indicates the probability that \(c_1 = 0\) given \(y_1\) and \(P(c_1 = 1 | y_1)\) indicates the probability that \(c_1 = 1\) given \(y_1\), respectively. \(P(c_1 = 0)\) and \(P(c_1 = 1)\) indicates the prior probabilities of the transmit bit, \(f(y_1 | c_1 = 0)\) indicates the conditional distribution of \(y_1\) given \(c_1 = 0\), \(f(y_1 | c_1 = 1)\) indicates the conditional distribution of \(y_1\) given \(c_1 = 1\) and \(f(y_1)\) indicates the probability distribution function of \(y_1\), respectively. By assuming BPSK modulation with equiprobable transmission, we have \(P(c_1 = 0) = P(c_1 = 1) = 1/2\). Using (7),

\[
P(c_1 = 0 | y_1) = \frac{f(y_1 | c_1 = 0)}{f(y_1 | c_1 = 1)}. \quad (9)
\]

For \(c_1 = 0, y_1 = 1 + \mathcal{N}(0, \sigma^2)\) and for \(c_1 = 1, y_1 = -1 + \mathcal{N}(0, \sigma^2)\). Thus, (9) can be calculated as

\[
P(c_1 = 0 | y_1) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{(y_1 - 1)^2}{2\sigma^2} \right) \quad \text{and} \quad P(c_1 = 1 | y_1) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{(y_1 + 1)^2}{2\sigma^2} \right) \quad \text{for } 0 \leq y_1 \leq 2 \sigma.
\]

The above calculation gives the likelihood ratio (or reliability) computed based on \(y_1\) alone. By recalling the principle of repeated coding, every bit is repeated three times, meaning that the reliability of the transmit bit can be improved based on the three received values \(y_1, y_2, y_3\). Therefore, the LLR of the transmit bit can be expressed as

\[
L_1 = \log \frac{P(c_1 = 0 | y_1, y_2, y_3)}{P(c_1 = 1 | y_1, y_2, y_3)} \quad (11)
\]

Again using Bayes’ rule,

\[
P(c_1 = 0 | y_1, y_2, y_3) = \frac{f(y_1, y_2, y_3 | c_1 = 0) P(c_1 = 0)}{f(y_1, y_2, y_3)} \quad (12)
\]

\[
P(c_1 = 1 | y_1, y_2, y_3) = \frac{f(y_1, y_2, y_3 | c_1 = 1) P(c_1 = 1)}{f(y_1, y_2, y_3)} \quad (13)
\]
Therefore, (11) can be simplified as

\[
L_1 = \log \left( \frac{f(y_1, y_2, y_3 | c_1 = 0)}{f(y_1, y_2, y_3 | c_1 = 1)} \right) = \log \left( \frac{\exp \left( -\frac{(y_1-1)^2}{2\sigma^2} \right)}{\exp \left( -\frac{(y_1+1)^2}{2\sigma^2} \right)} \cdot \frac{\exp \left( -\frac{(y_2-1)^2}{2\sigma^2} \right)}{\exp \left( -\frac{(y_2+1)^2}{2\sigma^2} \right)} \cdot \frac{\exp \left( -\frac{(y_3-1)^2}{2\sigma^2} \right)}{\exp \left( -\frac{(y_3+1)^2}{2\sigma^2} \right)} \right)
\]

\[
= \frac{y_1 + y_2 + y_3}{2\sigma^2}.
\]

(14)

Similar calculations can be done for \(L_2\) and \(L_3\). Here, \(\frac{1}{2\sigma^2}\) is a constant and can be omitted for simplicity. So, if \(y_1 + y_2 + y_3 > 0\), then \(\hat{c} = [000]\) is decided otherwise \(\hat{c} = [111]\). With soft decoding, information from all the received values is efficiently utilized. This way transmitted bits are estimated along with a confidence level.

A. MAP Detection

The objective of an optimal detector is to minimize the probability of error \((P(e))\) for the transmitted bit sequence \(x\), i.e., to minimize the mismatch between transmitted bits \((x)\) and estimated bits \((\hat{x})\).

\[
\min P(e) = \min P(x \neq \hat{x}).
\]

(15)

Given all the possible observations \(y\) and the channel coefficient matrix \(H\), we have

\[
P(e) = \int P(e|y)f(y)dy,
\]

(16)

where \(P(e|y)\) indicates the probability of error given received vector \(y\) and \(f(y)\) indicates the probability density function (pdf) of \(y\), respectively. It is noted from (16) that \(P(e)\) is proportional to \(P(e|y)\). Since \(x\) is the desired signal, we have

\[
P(e|y) = 1 - P(x|y),
\]

(17)

where \(P(x|y)\) denotes the probability of \(x\) given the vector \(y\). Thus, to minimize the probability of error, \(P(x|y)\) needs to be maximized and this detection scheme is termed as maximum a-posteriori (MAP) detection. Using Bayes’ rule,

\[
P(x|y) = \frac{f(y|x)P(x)}{f(y)},
\]

(18)

where \(f(y|x)\) indicates the conditional pdf of \(y\) given \(x\) and \(f(y)\) indicates the pdf of \(y\), respectively. Since \(f(y)\) is constant for all the values that \(x\) can take and assuming the prior probabilities of \(P(x)\) are identical for different \(x\), so

\[
P(x|y) \propto f(y|x). \]

(19)

Thus, the \(x\) which maximizes \(f(y|x)\) is chosen to minimize the probability of error and this detection scheme is called the maximum likelihood (ML) detection. In short, the MAP detection is equivalent to the ML detection when prior probabilities take the identical value.

MAP Detection in SCMA:

For an SCMA decoder, given \(y\) and the channel matrix \(H\) at the receiver, the detected multiuser codeword \(\hat{X} = [\hat{m}_1, \hat{m}_2, \ldots, \hat{m}_J]\) is given as

\[
\hat{X} = \arg \max_{m_j \in A_j, j} p(X|y),
\]

(20)

where \(m_j\) represents the codeword transmitted by the \(j\)th user and \(A_j\) represents the set of codewords allotted to the \(j\)th user, i.e., the \(j\)th user’s codebook. The transmitted codeword of each user can be estimated by maximizing its a-posteriori probability mass function (pmf). This can be calculated by taking the marginal of the joint a-posteriori pmf defined in (20). So, the objective is to detect the codeword transmitted by each user one by one.

\[\circ\text{ Marginalization:}\]

It is known that that if \(v\) is a function of a number of variables, then \(v\) with respect to a specific variable can be obtained by carrying out marginalization with respect to that variable.

\[\text{Example 4: Let } v(\alpha_1, \alpha_2, \ldots, \alpha_j, \ldots, \alpha_J) \text{ be a function (also called a global function) of } J \text{ number of variables, then the marginalized function with respect to } \alpha_j \text{ variable is given as}
\]

\[
v(\alpha_j) = \sum_{\alpha_1} \cdots \sum_{\alpha_{j-1}} \sum_{\alpha_{j+1}} \cdots \sum_{\alpha_J} v(\alpha_1, \ldots, \alpha_J),
\]

(21)

i.e., the summation with respect to all variables except for \(\alpha_j\). It can be written in a more compact form as

\[
v(\alpha_j) = \sum_{\sim \alpha_j} v(\alpha_1, \alpha_2, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_J).
\]

(22)

where \(\sim \alpha_j\) denotes that the summation is performed with respect to all the variables of the global function except for \(\nu_j\).

Similar to (22), the marginalization with respect to \(m_j\) in (20) leads us to symbol by symbol detection for the \(j\)th user which can be given as

\[
\hat{m}_j = \arg \max_{m_j \in A_j} \sum_{\sim \hat{m}_j} p(X|y).
\]

(23)

Again using Bayes’ rule, we have

\[
\hat{m}_j = \arg \max_{m_j \in A_j} \sum_{\sim \hat{m}_j} f(y|X)P(X).
\]

(24)

For SCMA system, assume that the noise components corresponding to the \(K \) REs are identically distributed and independent of each other. As a result, we have

\[
f(y|X) = \prod_{k=1}^{K} f(y_k|X),
\]

(25)

where \(k = 1, \ldots, K\) denotes the REs and \(y_k\) is the received signal at the \(k\)th RE. Consequently, (24) becomes

\[
\hat{m}_j = \arg \max_{m_j \in A_j} \left( \sum_{\sim \hat{m}_j} \left( P(X) \prod_{k=1}^{K} f(y_k|X) \right) \right)
\]

for \(j = 1, \ldots, J\).  

(26)
Solving the marginal product of functions (MPF) problem in (26) with brute force will lead to prohibitively high complexity and hence may be infeasible when number of users is large. Thanks to the sparsity of SCMA codebooks, we will show in the sequel that low-complexity message passing algorithm (MPA) can be leveraged with the aid of factor graph to solve the MPF problem in (26) with near-optimal performance.

B. Factor Graphs

A graph is composed of vertices (or nodes) and edges. Two nodes are connected with an edge when there is some relationship between them. Different types of graphs are used to model problems in areas such as computer science, biology, physics, etc. A widely known graph for modelling communication and signal processing problems is called a bipartite graph. In this graph, total nodes can be divided in two sets and no two nodes within a set are connected to each other. Fig. 4. shows a bipartite graph having two set of nodes, set A and set B. It is noted that no two nodes of a set are connected.

![Bipartite Graph](image)

A factor graph is an undirected bipartite graph in which one set of nodes is called variable nodes (VNs) and the other called function nodes (FNs). An edge is connected between a variable node and a function node if that particular variable is an argument of that function. Factor graph shows how a global function can be represented in terms of simpler local functions (denoted by FNs) and can also help in computing marginal distribution with respect to single variable using sum-product algorithm (SPA). Next, some examples are illustrated on how a global function can be factorized and how factorization helps in reducing the computation complexity.

- **Example 5:** Let \( \gamma \) be a global function of three variables \((a, b, c)\) such that \( \gamma = a^2 + ab + a c \). Let \( a = 1, b = 3, c = 4 \), then it can be seen from Fig. 5(a) that function \( \gamma \) can be computed in 11 steps (including inserting values into the variables). By rearranging the global function to \( \gamma = a(a + b + c) \), \( \gamma \) can be computed in 7 steps (Fig. 5(b)). Thus, by simplifying a function, its computation complexity can be reduced.

- **Example 6:** Joint distribution of two variables \((a, b)\) can also be simplified in conditional and prior probability distribution functions.

\[
\begin{align*}
  f(a, b) &= f_1(a|b)f_2(b) \\
           &= f_1'(a)f_2(b) .
\end{align*}
\]

(27)

Factor graphs are also widely used to model a number of communication problems. Fig. 6 shows the factor graph of a communication channel. Let \( \mathbf{x} = [x_1, x_2, \ldots, x_J] \) be the transmit vector and \( \mathbf{y} = [y_1, y_2, \ldots, y_K] \) be the received vector, respectively. Assuming noise components are independent of each other, then

\[
  f(\mathbf{y}|\mathbf{x}) = \prod_{k=1}^{K} f(y_k|x_k) .
\]

(28)

It is noted that each received symbol \( y_k \) depends on certain elements of \( \mathbf{x} \). Let say \( y_k \) depends on \( x_1, x_2, x_j \) elements, then there will be edge connecting FN corresponding to \( f(y_k|x_1, x_2, x_j) \) with VNs \( x_1, x_2, x_j \). In Fig. 6 every VN denotes the data from one transmit point and every FN \( k \) denotes the likelihood function \( f(y_k|\mathbf{x}) \).
C. Message Passing Algorithm

Message passing algorithm (MPA) is an algorithm to conduct inference from graphical models by passing belief messages between the nodes. The following example may be helpful in understanding how inference can be obtained from a graph [19].

Example 7: Fig. 7(a) shows a group of students standing in a line. The objective is to count total number of students and pass this information to each student. The constraint is that one student can communicate with maximum two neighboring students (front and back) at a time. To solve this problem, the approach proposed by MPA is, that when one student receives a count from one side, he/she adds 1 to it (to indicate the student’s presence) and pass it to the other side. The algorithm starts from each end of the line where the counter starts from 1 and increases by 1 as the message passes through each student. Messages pass in both directions of the line. If a student receives a count of \( n \) from one side and \( m \) from another side, then the total students present are \( n + m + 1 \) (as shown in Fig. 7(b)). In this way, every student obtains the information of total number of students present in the line. Such type of problems can also be modelled with the help of a graph, where each node represents a student and every connecting edge represents a communication link between them.

D. Message Passing Between Nodes

In this subsection, we introduce how messages are passed between function nodes (FNs) and variable nodes (VNs) of a factor graph using Sum-product algorithm (SPA) (a version of Message passing algorithm). In SCMA systems, each VN denotes one data layer and each FN denotes the likelihood function at the resource element (RE). Therefore, the total number of VNs is equal to the total number of layers/users and the total FNs equals the total REs present.

Suppose the transmitted bits are \( \mathbf{c} = [c_1, c_2, \cdots, c_N] \) and received bits are \( \mathbf{y} = [y_1, y_2, \cdots, y_N] \), then the aim is to compute the \textit{a posteriori} probability (APP) of bit \( c_i \), i.e., \( P(c_i = 0|\mathbf{y}) \). Using Bayes’ rule, the APP ratio with regard to \( c_i \) can be converted into likelihood ratio as follows:

\[
P(c_i = 0|\mathbf{y}) \frac{P(c_i = 1|\mathbf{y})}{f(\mathbf{y}|c_i = 0)} = f(\mathbf{y}|c_i = 1).
\]

Taking natural logarithm, we obtain the Log-likelihood ratio (LLR) of \( c_i \) below

\[
\text{LLR}(c_i) = \log \left( \frac{f(\mathbf{y}|c_i = 0)}{f(\mathbf{y}|c_i = 1)} \right).
\]

If \( \text{LLR}(c_i) < 0 \), then \( c_i = 1 \) is decoded otherwise.

Message passing in a factor graph using SPA is an iterative process if the factor graph has cycles (closed loops) present in it. In every iteration, there are two steps. In Step 1, a belief message is passed from a variable node (VN) to a function node (FN) and in Step 2, the message is passed from an FN to a VN, respectively. These two steps are discussed in detail as follows:

- Step 1: Suppose there is a VN \( j_1 \) which has connections with 3 FNs with indices \( k_1, k_2, k_3 \) as shown in Fig. 8. To pass a message from VN \( j_1 \) to FN \( k_2 \), firstly VN \( j_1 \) multiplies all the messages received from its neighboring nodes except FN \( k_2 \) (i.e., \( k_1 \) and \( k_3 \)) and then transfer the output to FN \( k_2 \).

\[
n_{j_1 \rightarrow k_2} = n_{k_1 \rightarrow j_1} \ n_{k_3 \rightarrow j_1}.
\]

Here, \( n_{j_1 \rightarrow k_2} \) indicates the belief message from VN \( j_1 \) to FN \( k_2 \). The outgoing message from a VN is in the form of either \( P(c_i = 0|\mathbf{y}) \) or APP ratio or likelihood ratio.

- Step 2: In this step, a belief message is passed from an FN to a VN. Consider a FN \( k_1 \) which has three neighboring VNs \( (j_1, j_2, j_3) \) as shown in Fig. 9. To send a belief message from FN \( k_1 \) to VN \( j_2 \), FN \( k_1 \) first collects all the messages from its neighboring nodes except for VN \( j_2 \). These received messages are multiplied with the local function \( (f_{k_1}(j_1, j_2, j_3)) \) associated with FN \( k_1 \) and then the resulting function is marginalized with respect to VN \( j_2 \). After marginalization, the resulting message to be sent to VN \( j_2 \) can be expressed as

\[
n_{k_1 \rightarrow j_2} = \sum_{j_3} (f_{k_1}(j_1, j_2, j_3) \ n_{j_1 \rightarrow k_1} \ n_{j_3 \rightarrow k_1}).
\]

Here, \( f_{k_1}(j_1, j_2, j_3) \) indicates the local function of FN \( k_1 \) and message \( n_{k_1 \rightarrow j_2} \) indicates \( P(\text{function } k_1 \text{ is satisfied } | \text{ messages received at FN } k_1) \), respectively. Similarly, if a belief message needs to be passed from FN \( k_1 \) to VN \( j_3 \), then the belief message from the VNs \( j_1 \) and \( j_2 \) (all VNs except the one to which message needs to be passed) is considered as extrinsic information.

In this way, belief messages are passed from every connected pair of FN and VN in both directions (FN \( \leftrightarrow \) VN). Assuming that the messages passed through the factor graph are statistically independent, exact APPs can be computed. In the case of cyclic factor graph (i.e., factor graph with cycles in it), independent assumption is true for initial iterations only. However in practice, simulations have shown that MPA provides very effective results for computing APPs or LLRs even in cyclic factor graphs when the size of each cycle is large enough [20].

E. Sum-Product Algorithm

In the previous sub-section, we discussed how messages are passed from FN to VN and vice-versa. In this sub-section, we will explain with the help of an example how messages are passed across a factor graph and the resultant inference obtained. Consider a factor graph having \( K \) function nodes (FNs) and \( J \) variable nodes (VNs). The sum-product algorithm (SPA) starts by passing messages from leaf nodes. Let say a node has a degree (number of edges connected to it) \( d \), then it will remain idle until messages have arrived on \( d - 1 \) edges.
can be carried out as follows: let symbols that can be sent from VN $V_N$ through to the FN $F_N$, where $\psi_j$ and $\xi_k$ be the set of nodes directly connected to $V_N j$ and $F_N k$, respectively. Let $A_j$ be the set of information symbols that can be sent from $V_N j$ and $a_j \in A_j$. Next, SPA can be carried out as follows:

1) Passing message from $F_N k$ to $V_N j$:

$$n_{k \rightarrow j}(a_j) = \sum_{r} \psi(R) \prod_{r \in R \setminus \{j\}} n_{r \rightarrow k}(a_r)$$

for $a_j \in A_j$, (29)

where $\psi(R)$ is the likelihood function associated with FN $k$ and $r \in R \setminus \{j\}$ denotes all the VNs of $R$ except $V_N j$, respectively. Also, $n_{r \rightarrow k}(a_r)$ denotes the message from the VN $r$ to the FN $k$ corresponding to symbol $a_r$.

2) Passing message from $V_N j$ to $F_N k$:

$$n_{j \rightarrow k}(a_j) = \prod_{d \in \zeta_j \setminus \{k\}} n_{d \rightarrow j}(a_j) \quad \text{for} \quad a_j \in A_j. \quad (30)$$

where $d \in \zeta_j \setminus \{k\}$ denotes the FNs in $\zeta_j$ except the $k$th FN. The message is normalized to ensure that the sum of all the probabilities is equal to 1.

$$n_{j \rightarrow k}(a_j) = \frac{\prod_{d \in \zeta_j \setminus \{k\}} n_{d \rightarrow j}(a_j)}{\sum_{a_j} \prod_{d \in \zeta_j \setminus \{k\}} n_{d \rightarrow j}(a_j)} \quad \text{for} \quad a_j \in A_j. \quad (31)$$

The above algorithm makes use of the sum and product operations and hence it is called sum-product algorithm. The detailed explanation of (29) and (30) is given in [21]. It is noted that (30) is simpler than (29) because there is no local function associated with a VN. Next, we discuss how messages are passed over a factor graph using SPA and the resultant marginal function obtained with respect to each variable.

- Example 8: Consider a global function $\phi(j_1,j_2,j_3,j_4)$ and the set of values $j_i$ can take is $A_{j_i}$ and $a_{j_i} \in A_{j_i}$. The global function can be written as a product of local functions (each having argument as subsets of $\{j_1,j_2,j_3,j_4\}$) as

$$\phi(j_1,j_2,j_3,j_4) = k_1(j_1)k_2(j_2,j_3)k_3(j_1,j_3,j_4). \quad (32)$$

There is one marginal function associated with each variable, i.e. $\phi_i(a_{j_i})$ for $i = 1, 2, 3, 4$. The factor graph corresponding to (32) is shown in Fig. 10. It has 4 VNs corresponding to 4 variables ($j_1,j_2,j_3,j_4$) and 3 FNs corresponding to 3 local functions ($k_1,k_2,k_3$), respectively. An edge will be connected between a VN and a FN if that variable is an argument of that local function. For example, for $k_1(j_1)$, an edge will be present between FN $k_1$ and VN $j_1$. The passing of messages using SPA in the factor graph of Fig. 10 is explained in the following steps:
Fig. 10: Factor graph for the product $k_1(j_1)k_2(j_2)k_3(j_3,j_4)$.

- Step 1: SPA starts from the leaf nodes, i.e. $k_1, j_2$ and $j_4$.

$$n_{k_1\rightarrow j_1}(a_{j_1}) = k_1(a_{j_1}) \quad \forall \ a_{j_1} \in A_{j_1},$$
$$n_{j_2\rightarrow k_2}(a_{j_2}) = 1 \quad \forall \ a_{j_2} \in A_{j_2},$$
$$n_{j_4\rightarrow k_3}(a_{j_4}) = 1 \quad \forall \ a_{j_4} \in A_{j_4}.$$

The initial message from a VN to FN can be 1 as default.

- Step 2: VN $j_1$ is connected to two FNs $k_1$ and $k_3$, i.e., VN $j_1$ has degree 2. Since VN $j_1$ has received message from FN $k_1$ in Step 1 (i.e. it has received message from $d-1$ edges), it can now pass message to FN $k_3$.

$$n_{j_1\rightarrow k_3}(a_{j_1}) = n_{k_1\rightarrow j_1}(a_{j_1}) \quad \forall \ a_{j_1} \in A_{j_1},$$
$$n_{k_2\rightarrow j_3}(a_{j_3}) = \sum_{a_{j_2} \in A_{j_2}} (k_2(a_{j_2}, a_{j_3})),$$
$$n_{j_2\rightarrow k_2}(a_{j_3}) \quad \forall \ a_{j_3} \in A_{j_3}.$$

To pass message from FN $k_2$ to VN $j_3$, the message from VN $j_2$ to FN $k_2$ is multiplied with the local function of FN $k_2$ and then summation is performed with respect to all variables of FN $k_2$ except VN $j_3$. This results in an outgoing message in terms of variable $j_3$ only. When the belief message is passed from FN $k_2$ to VN $j_3$, the message from VN $j_2$ to FN $k_2$ is called extrinsic information.

- Step 3: Once FN $k_3$ has received messages from VNs $j_1$ and $j_4$ (all neighbors except VN $j_3$), it will then generate and pass a message to VN $j_3$. Here, $n_{j_3\rightarrow k_3}(a_{j_3})$ and $n_{j_4\rightarrow k_3}(a_{j_4})$ are considered as extrinsic information.

$$n_{k_3\rightarrow j_3}(a_{j_3}) = \sum_{a_{j_1} \in A_{j_1}} \sum_{a_{j_4} \in A_{j_4}} \left( k_3(a_{j_1}, a_{j_3}, a_{j_4}) \times n_{j_1\rightarrow k_3}(a_{j_1}) n_{j_4\rightarrow k_3}(a_{j_4}) \right) \quad \forall \ a_{j_3} \in A_{j_3},$$
$$n_{j_3\rightarrow k_3}(a_{j_3}) = n_{k_2\rightarrow j_3}(a_{j_3}) \quad \forall \ a_{j_3} \in A_{j_3}.$$  

- Step 4:

$$n_{k_3\rightarrow j_1}(a_{j_1}) = \sum_{a_{j_3} \in A_{j_3}} \sum_{a_{j_4} \in A_{j_4}} \left( k_3(a_{j_1}, a_{j_3}, a_{j_4}) \times n_{j_3\rightarrow k_3}(a_{j_3}) n_{j_4\rightarrow k_3}(a_{j_4}) \right) \quad \forall \ a_{j_1} \in A_{j_1},$$
$$n_{k_3\rightarrow j_4}(a_{j_4}) = \sum_{a_{j_3} \in A_{j_3}} \sum_{a_{j_4} \in A_{j_4}} \left( k_3(a_{j_1}, a_{j_3}, a_{j_4}) \times n_{j_3\rightarrow k_3}(a_{j_3}) n_{j_4\rightarrow k_3}(a_{j_4}) \right) \quad \forall \ a_{j_4} \in A_{j_4},$$
$$n_{j_3\rightarrow k_2}(a_{j_3}) = n_{k_3\rightarrow j_3}(a_{j_3}) \quad \forall \ a_{j_3} \in A_{j_3}.$$

- Step 5:

$$n_{k_2\rightarrow j_2}(a_{j_2}) = \sum_{a_{j_3} \in A_{j_3}} k_2(a_{j_2}, a_{j_3}) n_{j_3\rightarrow k_2}(a_{j_3}) \quad \forall \ a_{j_2} \in A_{j_2},$$
$$n_{j_1\rightarrow k_1}(a_{j_1}) = n_{k_3\rightarrow j_1}(a_{j_1}) \quad \forall \ a_{j_1} \in A_{j_1}.$$

The algorithm continues until messages are passed through all the edges in both directions. The order of messages generated when SPA is applied over the factor graph of Fig. 10 is shown in Fig. 11.

**Termination**: The marginal function of a VN can be computed by multiplying all the received messages at the respective VN. For instance, the neighbouring nodes of VN $j_1$ are FNs $k_1$ and $k_3$, respectively. Therefore, the product of messages received from $k_1$ and $k_3$ will be the resultant message at the VN $j_1$. Similarly, the marginal function can be generated for other VNs.

$$\phi_1(a_{j_1}) = n_{k_3\rightarrow j_1}(a_{j_1}) n_{k_1\rightarrow j_1}(a_{j_1}) \quad \forall \ a_{j_1} \in A_{j_1},$$
$$\phi_2(a_{j_2}) = n_{k_2\rightarrow j_2}(a_{j_2}) \quad \forall \ a_{j_2} \in A_{j_2},$$
$$\phi_3(a_{j_3}) = n_{k_2\rightarrow j_3}(a_{j_3}) n_{k_3\rightarrow j_3}(a_{j_3}) \quad \forall \ a_{j_3} \in A_{j_3},$$
$$\phi_4(a_{j_4}) = n_{k_3\rightarrow j_4}(a_{j_4}) \quad \forall \ a_{j_4} \in A_{j_4}.$$

**F. SCMA Decoding**

In this section, with the help of the signature matrix of SCMA systems, we design the factor graph and use SPA to detect the symbols transmitted by each user. In a factor graph, each VN denotes an SCMA user and each FN denotes a resource element. In the signature matrix $\mathbf{F}_{4 \times 6}$ given in (3), there are $d_v$ ones in each column and $d_f$ number of ones in each row, respectively. In $\mathbf{F}_{4 \times 6}$, every column corresponds to a user and every row corresponds to a RE, respectively. The first column of $\mathbf{F}_{4 \times 6}$ indicates first user and it has non-zero values at first and second row. This means that data of first user is transmitted on the first and second RE, so there is an edge between VN $j_1$ and FN $k_1$ and, VN $j_1$ and FN $k_2$, respectively. The second row of $\mathbf{F}_{4 \times 6}$ indicates second RE and it has non-zero values at first, fourth and fifth position. This means that data of first, fourth and fifth user overlaps on second RE and thus there will be edges connecting the FN $k_2$ with VN $j_1$, with VN $j_4$ and with VN $j_5$, respectively as shown in Fig. 12.
The algorithm is discussed in a detailed manner as follows:

1) SPA for SCMA Decoding: The objective is to detect the symbols transmitted by each user, i.e., compute \( \mathbf{y} \) using SPA and this can be carried out in the following steps:

1) Initialization.
2) Passing of messages between FNs and VNs.
3) Termination and selection of codewords.

The algorithm is discussed in a detailed manner as follows:

**Step 1: Initialization**

Assuming received vector \( \mathbf{y} \) and channel coefficient matrix \( \mathbf{H} \) are known at receiver, firstly the likelihood ratio is computed at each FN. Let us assume a FN \( l \) where data of three users corresponding to set \( \xi_l = \{ v_1, v_2, v_3 \} \) superimpose. Therefore, the likelihood function at FN \( l \) becomes \( f(y_l|m_1, m_2, m_3, N_0) \), where, \( m_1, m_2, m_3 \) are the codewords transmitted by users belonging to set \( \xi_l \), respectively. Assume the set of codewords allotted to user \( v \) is denoted as \( \mathcal{A}_v \). The likelihood function of FN \( l \) is given as

\[
\begin{align*}
    & f(y_l|m_1, m_2, m_3, N_0) = \exp \left( -\frac{1}{N_0} ||y_l - (h_{11} C_{l,1}(m_1) + h_{12} C_{l,2}(m_2) + h_{13} C_{l,3}(m_3)) ||^2 \right) \\
    & \text{for } m_1 \in \mathcal{A}_1, m_2 \in \mathcal{A}_2, m_3 \in \mathcal{A}_3. \tag{33}
\end{align*}
\]

Here, \( C_{l,v}(m_v) \) denotes the codeword element transmitted by the \( v \)th user when sending \( m_v \) codeword on the \( l \)th RE. In total, \( KM^3 \) values are stored for function \( f(y_l|m_1, m_2, m_3, N_0) \). For an uncoded SCMA system, let us assume equal prior probability for each codeword, i.e., \( P(m_1) = P(m_2) = P(m_3) = \frac{1}{M} \). Therefore, the initial message passed from VN \( v_1, v_2, v_3 \) to the \( l \)th FN is

\[
    n_{v_1 \rightarrow l}^{\text{init}}(m_1) = n_{v_2 \rightarrow l}^{\text{init}}(m_2) = n_{v_3 \rightarrow l}^{\text{init}}(m_3) = \frac{1}{M}. \tag{34}
\]

**Step 2: Passing of messages between FNs and VNs.**

(i) From Function Node to Variable Node:

Let us assume \( \xi_l = \{ v_1, v_2, v_3 \} \), where \( v_1, v_2, v_3 \) denotes the three users connected to RE \( l \), respectively. To pass the message from the FN to one user, information received on the FN from the other two users may be regarded as extrinsic information.

Message passed from FN \( l \) to VN \( v_1 \) is given as

\[
    n_{l \rightarrow v_1} (m_1) = \sum_{m_2 \in \mathcal{A}_2} \sum_{m_3 \in \mathcal{A}_3} \left( f(y_l|m_1, m_2, m_3, N_0) \right) n_{v_2 \rightarrow l}(m_2) n_{v_3 \rightarrow l}(m_3) \text{ for } m_1 \in \mathcal{A}_1. \tag{35}
\]

In equation (35), messages from the two VNs \( n_{v_2 \rightarrow l}(m_2) \) and \( n_{v_3 \rightarrow l}(m_3) \) are multiplied with the local likelihood function of \( l \)th FN and then marginalized with respect to \( v_1 \). Similarly, message passed from FN \( l \) to VNs \( v_2 \) and \( v_3 \), respectively are
Message from FN $l$ to VN $v_2$ is shown graphically in Fig. [13] The message transmitted from FN $l$ to VN $v$ indicates the guess of what signal is received at FN $l$ for all possible values of VN $v$.

(ii) From Variable Node to Function Node:
Let us assume $\zeta_v = \{l_1, l_2\}$, where $l_1$ and $l_2$ are the FNs connected to VN $v$.

![Fig. 14: Message Passing from VN to FN.](image)

Message sent from VN $v$ to FN $l_1$ and $l_2$, respectively are

$$n_{v\to l_1}(m_v) = \text{normalize}(p_a(m_v) \ n_{l_1\to v}(m_v))$$

for $m_v \in \Lambda_v$.  

(38)

$$n_{v\to l_2}(m_v) = \text{normalize}(p_a(m_v) \ n_{l_2\to v}(m_v))$$

for $m_v \in \Lambda_v$.  

(39)

where $p_a$ denotes the prior probability of user $v$ and $n_{v\to l_2}$ indicates the updates VN $v$ received from the other FNs connected to $v$. Here, normalization is necessary in order to ensure that each belief falls in the range $[0,1]$. The normalization is conducted similar to (31) and so, (38) can also be written as

$$n_{v\to l_1}(m_v) = \frac{p_a(m_v) n_{l_1\to v}(m_v)}{\sum_{m_v} n_{l_1\to v}(m_v)}$$

for $m_v \in \Lambda_v$.  

(40)

The message from the VN $v$ to FN $l_2$ is shown graphically in Fig. [14]. Since factor graph has cycles in it, Step 2 is repeated for each iteration. This continues until no considerable change is observed in the belief computed at each VN. The number of iterations for which step 2 is repeated is denoted as $N_{iter}$.

• Step 3: Termination and selection of codewords.
After repeating Step 2 for $N_{iter}$ number of iterations, the final belief is computed at each VN which is the product of the prior probability and messages from the neighboring FNs of each VN.

$$I_v(m_v) = p_a(m_v) n_{l_1\to v}(m_v) n_{l_2\to v}(m_v)$$

for $m_v \in \Lambda_v$.  

(41)

The probability for each codeword is calculated at each VN and the codeword with highest probability becomes the estimated codeword for that user. This is one method of estimating the transmitted symbol for each user. Another method is to compute bit LLR from $I_v(m_v)$. Let there be $b$ bits per symbol, then the $b_{th}$ bit LLR for VN $v$ is

$$LLR^v_{b_{th}} = \log \frac{P(b_1 = 0)}{P(b_1 = 1)} = \log \frac{\sum_{m_v | b_1 = 0} I_v(m_v)}{\sum_{m_v | b_1 = 1} I_v(m_v)}.$$  

(42)

In case of $M = 4$, there are two bits per symbol.
Let $\Lambda_v = \{m^1_v, m^2_v, m^3_v, m^4_v\}$ denote the 4 codewords corresponding to 4 symbols $\{s^1, s^2, s^3, s^4\}$ that can be sent by user $v$. Using (41), final belief corresponding to each of the four codewords is calculated. Then, the LLR of the first bit is given as

$$LLR^v_{b_{th}} = \log \frac{P(b_1 = 0)}{P(b_1 = 1)} = \log \frac{I_v(m^1_v) + I_v(m^2_v)}{I_v(m^3_v) + I_v(m^4_v)}.$$  

(43)

It can be seen from Fig. [15] that first bit $b_1$ is 0 for symbols $s^1$ and $s^2$ and is 1 for symbols $s^3$ and $s^4$, respectively. As such, the ratio of the belief computed corresponding to codewords $m^3_v$ and $m^4_v$ and $m^1_v$ and $m^2_v$ gives the bit LLR of the first bit. Consequently if $P(b_1 = 1) > P(b_1 = 0)$ or if $LLR^v_{b_{th}} < 0$, then ‘1’ is decoded for 1st bit of the symbol otherwise 0. Similarly for second bit,
if \( LLR_{b_2}^v < 0 \), then ‘1’ is decoded for the second bit of the symbol.

\[
LLR_{b_2}^v = \log \left( \frac{I_v(m_1^1) + I_v(m_3^3)}{I_v(m_2^2) + I_v(m_4^4)} \right).
\]  \hspace{1cm} (44)

2) Max-Log-SPA for SCMA Decoding:

The SPA algorithm discussed above computes LLR in probability domain. Although, it is simpler than MAP algorithm, it still has complexity issues mainly because of the message passing from FN to VN \((O(N_{\text{iter}}KM^{2d}))\). It involves number of product and exponential operations rendering it computationally intensive. Also, exponential operations involve large dynamic range which puts a burden on storage. Thus, similar to the max-log-MAP algorithm \(^{22}\), max-log-SPA algorithm \(^{23}\) can be used to further simplify the MPA.

To this end, the SPA algorithm is moved into the logarithm domain and Jacobian logarithm can also be used to avoid exponential operations, which further simplifies the algorithm, i.e.,

\[
\log(\exp(a_1) + \cdots + \exp(a_j)) \approx \max(a_1, \ldots, a_j).
\]  \hspace{1cm} (45)

The max-log-SPA algorithm for (4,6) SCMA block is illustrated in the following steps:

- **Step 1: Initialization.**
  Given \( y \) and \( \mathbf{H} \), initial value of LLR is calculated.

\[
\log(f(y_l|m_1, m_2, m_3, N_0)) = \frac{-1}{N_0} ||y_l - (h_{l1}C_{l1}(m_1) + h_{l2}C_{l2}(m_2) + h_{l3}C_{l3}(m_3))||^2
\]

for \( m_1 \in \mathbb{A}_1, m_2 \in \mathbb{A}_2, m_3 \in \mathbb{A}_3. \)  \hspace{1cm} (46)

The initial message from VN \( v \) to FN \( l \) is

\[
\eta_{v \rightarrow l}^{\text{init}} = \log(\eta_{v \rightarrow l}^{\text{init}}) = \log\left(\frac{1}{M}\right).
\]  \hspace{1cm} (47)

- **Step 2: Passing of Messages between FNs and VNs.**
  - (i) From Function Node to Variable Node.
    Applying natural logarithm to \(^{35}\). Let \( a_i \) be
    \[
    a_i = \log(f(y_l|m_1, m_2, m_3, N_0)) - \log(I_v(m_1)) - \log(I_v(m_2)) - \log(I_v(m_3))
    = \log(f(y_l|m_1, m_2, m_3, N_0)) - \log(I_v(m_1)) - \log(I_v(m_2)) - \log(I_v(m_3))
    = \log(f(y_l|m_1, m_2, m_3, N_0)) - \log(I_v(m_1)) - \log(I_v(m_2)) - \log(I_v(m_3)).
    \]

Using \( a_i \) as given in \(^{45}\), now message passed from the \( l \)th FN to VN \( v_1 \) is given as:

\[
\eta_{l \rightarrow v_1}^{\text{init}}(m_1) = \max_{m_2, m_3} \left( \log(f(y_l|m_1, m_2, m_3, N_0)) + \eta_{v_2 \rightarrow l}(m_2) + \eta_{v_3 \rightarrow l}(m_3) \right)
\]  \hspace{1cm} (48)

Similarly, message from FN \( l \) to VN \( v_2 \) and \( v_3 \), respectively are

\[
\eta_{l \rightarrow v_2}^{\text{init}}(m_2) = \max_{m_1, m_3} \left( \log(f(y_l|m_1, m_2, m_3, N_0)) + \eta_{v_1 \rightarrow l}(m_1) + \eta_{v_3 \rightarrow l}(m_3) \right)
\]  \hspace{1cm} (49)

(ii) From Variable Node to Function Node.

Since each user is sending data on two REs, message passed from the \( r \)th VN to FNs \( l_1 \) and \( l_2 \) in logarithm domain, respectively are (applying log to \(^{38, 39}\))

\[
\eta_{v \rightarrow l_1}(m_v) = \log\left(\frac{1}{M}\right) + \eta_{l_2 \rightarrow v}(m_v)
\]  \hspace{1cm} (50)

Using \(^{42}\) and \(^{45}\), bit LLR of \( b_i \)th bit is given as

\[
LLR_{b_i}^v = \max_{m_v, b_i = 0} \left( \log(I_v(m_v)) - \max_{m_v, b_i = 1} \left( \log(I_v(m_v)) \right) \right).
\]  \hspace{1cm} (54)

The LLR of first bit of the symbol is

\[
LLR_{b_1}^v = \max_{m_1^1, m_2^2} \left( \log(I_v(m_1^1)) + \log(I_v(m_2^2)) \right)
\]  \hspace{1cm} (55)

If \( LLR_{b_1}^v < 0 \), then ‘1’ is decoded for the first bit of the symbol otherwise 0. Similarly, bit LLR for the second bit is given as

\[
LLR_{b_2}^v = \max_{m_1^1, m_2^2} \left( \log(I_v(m_1^1)) + \log(I_v(m_2^2)) \right)
\]  \hspace{1cm} (56)
Again if \( LLR_{b_2} \) is less than 0, ‘1’ is decoded for the second bit of the symbol otherwise zero.

We can see that in max-log-MPA, exponential operations are no longer needed and more than 90\% of multiplication operations are removed with respect to SPA. Although, it involves many addition operations but overall it has much lower complexity than SPA.

Fig. 16 shows average bit error rate (BER) obtained of SCMA for M=4 and M=16 in Rayleigh Channel using SPA and Max-Log-SPA algorithms. It can be observed that there is negligible difference in the performance of the two algorithms. In Fig. 16 we compare it with the BER performance of OMA scheme with diversity order 1 and 2 which has no multi-user interference. It can be noted that in SCMA data of six users is being transmitted using four subcarriers and suffers multi-user interference, still its performance is comparable with OMA scheme of diversity order 2. Also, it was found in simulations that the run time in Max-Log-SPA is 20\% and 30\% lesser than SPA for M=4 and M=16 respectively. Therefore, practically Max-Log-SPA is a preferable solution due to its low complexity and near-optimal performance.

### IV. Conclusions

In this paper, we have provided a systematic self-contained tutorial to SCMA which is a disruptive code-domain NOMA scheme for the enabling of massive connectivity. SCMA uses carefully designed sparse codebooks for significant constellation shaping gain and anti-interference capability as compared to previously proposed code-domain NOMA techniques. In comparison to the complex MAP decoding, iterative message passing algorithm gives rise to efficient MPA decoding with significantly reduced complexity. Thus, SCMA represents a promising solution for providing better quality of service to users (with overloading factor greater than 1), low latency and high spectral efficiency. It is anticipated that this paper will provide a quick and comprehensive understanding of SCMA to motivate more researchers towards this research topic.

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