RELATIVE FONTAINE-MESSING THEORY OVER POWER SERIES RINGS

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Abstract. Let $k$ be a perfect field of characteristic $p > 2$, $R := W(k)[[t_1, \ldots, t_d]]$ be the power series ring over the Witt vectors, and $X$ be a smooth proper scheme over $R$. The main goal of this article is to extend classical Fontaine-Messing theory ([FM87]) to the setting where the base ring is $R$. In particular, we obtain comparison theorems between torsion crystalline cohomology of $X/R$ and torsion étale cohomology in this setting.

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1. Introduction

Let $k$ be a perfect field of characteristic $p > 2$, $W(k)$ the Witt vectors over $k$, $R = W(k)[[t_1, \ldots, t_d]]$ the power series ring over $W(k)$ in $d$-variables, and $X$ a smooth proper scheme over $R$. The aim of this paper is to extend classical Fontaine-Messing theory as in [FM87] to the setting where the base is $R$.

1.1. Classical Fontaine-Messing theory. We begin by briefly recalling one of the main results of classical Fontaine-Messing theory ([FM87]). In order to simplify notation, we shall use $\varphi$ to denote various Frobenii (on $W_p k$-algebras) extending the arithmetic Frobenius on $W_p k$, i.e., $\varphi(x) = x^p \mod p$ for $x \in W(k)$. Let $K := W(k)[\frac{1}{p}]$, $\overline{K}$ be a fixed algebraic closure of $K$ and $G_K := \text{Gal}(\overline{K}/K)$. One of the key objects in classical Fontaine-Messing theory (and Fontaine-Laffaille theory in [FL82]) is a Fontaine-Laffaille module:

Definition 1.1.1. A Fontaine-Laffaille module over $W(k)$ is a triple $(M, F^i(M), \varphi_i)$ where

1. $M$ is a finite generated $W_n(k)$-module for some $n \geq 1$.
2. $F^i(M) \subset M$ is a $W_n(k)$-submodule satisfying:
   (a) $F^{i+1}(M) \subset F^i(M)$ for all $i$,
   (b) $F^0(M) = M$ and $F^{r+1}(M) = \{0\}$ for some $r$.
3. $\varphi_i : F^i(M) \to M$ is a $\varphi_{W(k)}$-semilinear map such that
   (a) $\varphi_i|_{F^{i+1}(M)} = p\varphi_{i+1}$,
   (b) $\sum_{i=0}^r \varphi_i(F^i(M)) = M$ as $W(k)$-modules.

We denote by MF$(W(k))$ the category whose objects are Fontaine-Laffaille modules over $W(k)$ and morphisms are $W(k)$-module morphisms compatible with $F^i$ and $\varphi_i$. Let MF$^{[0,r]}(W(k))$ denote the full subcategory consisting of Fontaine-Laffaille modules such that $F^{r+1}(M) = \{0\}$. These categories are known to satisfy many good properties including:

1. By [FL82, Proposition 1.8], MF$^{[0,r]}(W(k))$ is an abelian category.
2. A sequence $0 \to M \to M'' \to M' \to 0$ in MF$(W(k))$ is exact if and only if $0 \to F^i(M) \to F^i(M'') \to F^i(M') \to 0$ is exact for all $i$.

Let $A_{\text{cris}}$ denote the integral crystalline period ring over $W(k)$ equipped with the usual Frobenius $\varphi$ and filtration $F^i A_{\text{cris}} \subset A_{\text{cris}}$ (see §2.1 for details). For $i = 0, \ldots, p - 1$, we have $\varphi(F^i A_{\text{cris}}) \subset p^i A_{\text{cris}}$. In particular, since $A_{\text{cris}}$ is $\mathbb{Z}_p$-flat, this allows us to define $\varphi_i :
F^{i}A_{cris} \to A_{cris} by setting \( \varphi_{i}(x) = \frac{x^{p^{i}}}{p^{i}} \). Given an object \((M, F^{i}(M), \varphi_{i}) \in MF^{[0,1]}(W(k))\) with \( r \leq p - 1 \), let

\[
F^{r}(A_{cris} \otimes W(k) M) := \sum_{i=0}^{r} F^{i}A_{cris} \otimes W(k) F^{r-i}M
\]

and define \( \varphi_{r} : F^{r}(A_{cris} \otimes W(k) M) \to A_{cris} \otimes W(k) M \) via \( \varphi_{r} := \sum_{i=0}^{r} \varphi_{i} |_{F^{i}A_{cris}} \otimes \varphi_{r-i} |_{F^{r-i}M} \).

Now set

\[
T_{cris}(M) := (F^{r}(A_{cris} \otimes W(k) M))^{\varphi_{r}=1}(-r) = \{ x \in F^{r}(A_{cris} \otimes W(k) M) \mid \varphi_{r}(x) = x \}(-r).
\]

It can be shown that \( T_{cris} \) is independent of the choice of \( r \) and gives rise to a functor from \( p^{n} \)-torsion Fontaine-Laffaille modules to \( \mathbb{Z}/p^{n}\mathbb{Z}[G_{K}] \)-modules. The main result of Fontaine-Laffaille theory states that if \( r \leq p - 2 \), then \( T_{cris} \) is a fully faithful exact functor such that the essential image is stable under subquotients.

Given a smooth proper scheme \( X \) over \( W(k) \), let \( X_{n} := X \times_{\mathbb{Z}} \mathbb{Z}/p^{n}\mathbb{Z} \) and consider the crystalline cohomology group \( \mathcal{M}_{i} := \mathcal{H}^{i}_{cris}(X_{n}/W_{n}(k), \mathcal{O}_{X_{n}/W_{n}(k)}) \) with filtration \( F^{i} \mathcal{M}_{i} := \mathcal{H}^{i}_{cris}(X_{n}/W_{n}(k), J_{X_{n}/W_{n}(k)}^{[j]}), \) where \( J_{X_{n}/W_{n}(k)}^{[j]} \) denotes the usual the \( j \)-th divided powers ideal on the crystalline site. In [FMS7], Fontaine-Messing show that there exist \( \varphi_{W(k)} \)-semi-linear maps \( \varphi_{j} : \mathcal{H}^{i}_{cris}(X_{n}/W_{n}(k), J_{X_{n}/W_{n}(k)}^{[j]}) \to \mathcal{H}^{i}_{cris}(X_{n}/W_{n}(k), \mathcal{O}_{X_{n}/W_{n}(k)}) \) for \( i \leq p - 1 \). The main result of [FMS7] can now be summarized as follows.

**Theorem 1.1.1.** With notation as above,

1. For \( i \leq p - 1 \), \( \left( \mathcal{H}^{i}_{cris}(X_{n}/W_{n}(k), \mathcal{O}_{X_{n}/W_{n}(k)}), \{ \mathcal{H}^{i}_{cris}(X_{n}/W_{n}(k), J_{X_{n}/W_{n}(k)}^{[j]}), \varphi_{j} \right) \) is an object of \( MF^{[0,1]}(W(k)) \).

2. For \( i \leq p - 2 \), let \( M^{i} = (M^{i}, F^{i}(M^{i}), \varphi_{j}) \) denote the above object in \( MF^{[0,1]}(W(k)) \). Then there exists a natural \( G_{K} \)-isomorphism \( \iota : T_{cris}(M^{i}) \simeq \mathcal{H}^{i}_{cris}(X_{\mathbb{R}}, \mathbb{Z}/p^{n}\mathbb{Z}) \).

**Remark 1.1.2.** Although \( \mathcal{H}^{i-1}_{cris}(X/W_{n}(k), \mathcal{O}_{X/W_{n}(k)}) \) has the structure of a Fontaine-Laffaille module, its relation with \( \mathcal{H}^{i-1}_{cris}(X_{\mathbb{R}}, \mathbb{Z}/p^{n}\mathbb{Z}) \) is unknown.

1.2. **The main results.** Our main goal is to extend Theorem 1.1.1 to the setting where \( X \) is a smooth proper scheme over \( R \). We briefly explain our main results.

We define the Frobenius \( \varphi \) on \( R \) by \( \varphi(t_{i}) = (1 + t_{i})^{p} - 1 \). Let \( R_{n} = R/p^{n}R \) and \( \hat{\Omega}_{R} := \lim_{\to} \Omega_{(R/p^{n})/W(k)} \) be the module of \( p \)-adically continuous Kähler differentials (where \( \Omega_{(R/p^{n})/W(k)} \) is the module of Kähler differentials). Following [Fal89], we define a Fontaine-Laffaille module over \( R_{n} \):

**Definition 1.2.1.** A Fontaine-Laffaille module over \( R_{n} \) is a tuple \((M, F^{i}(M), \varphi_{i}, \nabla) \) where

1. \( M \) and \( F^{i}(M) \) are finite generated \( R_{n} \)-modules such that \( F^{i}(M) = M \) for all \( i \leq 0 \) and \( F^{i+1}(M) = 0 \) for some \( r \).

\(^{1}\)Note that we do not use the more standard choice \( \varphi(t_{i}) = t_{i}^{p} \) since our choice of \( \varphi \) ensures that certain embeddings of \( R \) into various period rings such as \( A_{cris}(R) \) are compatible with Frobenius. See §2.1 for details.
(2) for each $i$, $F^{i+1}(M)$ is a direct summand of $F^i(M)$ as $R$-modules, and $F^i(M) \simeq \bigoplus_j R/p^{a_j} R$ for some integers $a_j$'s.

(3) $\varphi_i : F^i(M) \otimes_{R,\varphi} R \to M$ is an $R$-linear map such that

(a) $\varphi_i|_{F^{i+1}(M) \otimes_{R,\varphi} R} = p\varphi_{i+1};$

(b) $\sum_{i=0}^r \varphi_i(F^i(M) \otimes_{R,\varphi} R) = M$ as $R$-modules;

(4) $\nabla : M \to M \otimes_R \hat{\Omega}_R$ is an integrable connection such that

(a) Griffiths-transversality holds: $\nabla(F^i(M)) \subset F^{i-1}(M) \otimes_R \hat{\Omega}_R$, where $\eta$ is a fixed geometric point. In this setting, we can construct the integral crystalline homology groups $H^i$ as in the classical setting (for this definition to make sense, we need to have a structure group $\mathrm{GL}(n, \mathbb{Z})$ over $\mathbb{Z}_p$).

(b) $\varphi_i$'s are parallel: $\nabla \circ \varphi_i = (\varphi_{i-1} \otimes_R d\varphi_1) \circ \nabla$ as a map from $F^i(M) \otimes_{R,\varphi} R$ to $M \otimes_R \hat{\Omega}_R$, where $d\varphi_1 := \frac{d\varphi}{p} : \hat{\Omega}_R \otimes_{R,\varphi} R \to \hat{\Omega}_R$.

Let $R'$ denote the union of normal $R$-sub-algebras $R'$ of a fixed separable closure of $R[\frac{1}{p}]$ such that $R'[\frac{1}{p}]$ is finite étale over $R[\frac{1}{p}]$. Set $G_R := \mathrm{Gal}(\overline{R}[\frac{1}{p}]/R[\frac{1}{p}]) = \pi^\mathrm{et}_1(\mathrm{Spec} R[\frac{1}{p}], \eta)$ where $\eta$ is a fixed geometric point. In this setting, we can construct the integral crystalline period ring $A_{\mathrm{cris}}(R)$ over $R$, which is equipped with a filtration $F_i A_{\mathrm{cris}}(R)$, maps $\varphi_i : F^i A_{\mathrm{cris}}(R) \to A_{\mathrm{cris}}(R)$, and a continuous $G_R$-action. Unlike the classical situation (where $R = W(k)$), there is no longer a canonical embedding of $R$ into $A_{\mathrm{cris}}(R)$. However, we can choose an explicit embedding $R \to A_{\mathrm{cris}}(R)$ which is compatible with Frobenius $\varphi$ (see §2.1).

Given a Fontaine-Laffaille module $(M, F^i(M), \varphi_i, \nabla)$ as above, we can define a $\mathbb{Z}_p[G_R]$-module

$$T_{\mathrm{cris}}(M) := (F^r(A_{\mathrm{cris}}(R) \otimes_R M))^{\varphi_r = 1} (-r)$$

as in the classical setting (for this definition to make sense, we need to have a structure theorem (Theorem 2.2.1) for Fontaine-Laffaille modules over $R_n$). We note that the connection $\nabla$ is essential to construct the $G_R$-action on $T_{\mathrm{cris}}(M)$. We refer the reader to §2.3 for details.

If $X$ is a smooth proper scheme over $R$ and $X_n := X \times_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$, then the torsion crystalline cohomology $H^i_{\mathrm{cris}}(X_n/R_n)$ comes equipped with a natural filtration given by the cohomology groups $H^i_{\mathrm{cris}}(X_n/R_n, \mathcal{J}^{[j]}_{X_n/R_n})$ and $\varphi_R$-semilinear maps $\varphi_j : H^i_{\mathrm{cris}}(X_n/R_n, \mathcal{J}^{[j]}_{X_n/R_n}) \to H^i_{\mathrm{cris}}(X_n/R_n)$. Furthermore, $H^i_{\mathrm{cris}}(X_n/R_n)$ comes equipped with a natural connection (Gauss-Manin connection) $\nabla : H^i_{\mathrm{cris}}(X_n/R_n) \to H^i_{\mathrm{cris}}(X_n/R_n) \otimes_R \hat{\Omega}_R$. Our main result is

**Theorem 1.2.1.** With notation as above,

1. For $i \leq p-2$, $M^i := \left( H^i_{\mathrm{cris}}(X_n/R_n), \{H^i_{\mathrm{cris}}(X_n/R_n, \mathcal{J}^{[j]}_{X_n/R_n})\}, \varphi_j, \nabla \right)$ is a Fontaine-Laffaille module over $R_n$.

2. For $i \leq p-2$, there exists a natural $G_R$-isomorphism $T_{\mathrm{cris}}(M^i) \simeq H^i_{\mathrm{ét}}(X_{\mathbb{Z}/p^n\mathbb{Z}})$.

1.3. **Historical remarks and our strategy.** Classical Fontaine-Messing theory has been generalized in various directions by several authors. In [Bre00], Breuil proves an
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of Theorem 1.1.1 in the setting of semi-stable schemes over $W(k)$. In [Car08], Caruso generalized Breuil’s result to the setting of possibly ramified base rings.

In [Fal89], Faltings generalized classical Fontaine-Messing theory to the relative setting. More precisely, Faltings considers a smooth scheme $Y$ over $W(k)$, and smooth proper schemes of relative dimension $g$ over $Y$. In this setting, Faltings develops a general theory of relative FL-modules on $X$ over $Y$, and allows more general coefficients (i.e. certain types of crystals $E$ on $X$ as coefficients besides $\mathcal{O}_{X/Y}$). In [Fal89] Thm. 6.2, Faltings establishes a Fontaine-Messing type isomorphism comparing the crystalline pushforward of $E$ to $Y$ with an etale analog, under the assumption that $i + g \leq p - 2$. If $f$ is projective, then this assumption can be weakened to $i \leq p - 2$ (see Remark after Thm. 6.2 loc. cit.). Moreover, Faltings proves analogous results in the semi-stable setting. The notion of a Fontaine-Laffaille module in Falting’s relative setting was revisited by D. Xu ([Xu19]), who obtained a topos theoretic framework for such objects. It is likely that an analogous framework also exists in our relative setting, where the base is a power series ring, but we do not pursue this here.

Remark 1.3.1. In Theorem 1.2.1 we obtain a period isomorphism without any restriction coming from the relative dimension i.e. we obtain an isomorphism for all $i \leq p - 2$ in the possibly non-projective but proper case (unlike in the relative setting of Faltings).

The main new idea for the proof of Theorem 1.2.1 is the observation that one can “descend” to a setting where the results classical Fontaine-Messing theory applicable in order to obtain the results in our relative setting. More precisely, let $R_g$ be the perfection of $\text{Frac}(R/pR)$ and let $R_g := W(k_g)$. One has an embedding $b_g : R \to R_g$ compatible with $\varphi$, and we set $X_g := X \times_R R_g$. On the other hand, consider the reduction modulo $t_i$ map $\bar{b} : R \to W(k)$ and let $X_s := X \times_R W(k)$. One of our key observations is that classical Fontaine-Messing theory applied to $X_g$ and $X_s$ implies the first part of Theorem 1.2.1 if $H^q_{\text{cris}}(X_{g,n}/W(k))$ and $H^q_{\text{cris}}(X_{s,n}/R_g)$ are both Fontaine-Laffaille modules with the same type, then $H^q_{\text{cris}}(X_n/R_n)$ is a Fontaine-Laffaille module over $R_n$. We use a similar strategy to establish an isomorphism between $T^q_{\text{cris}}(M^i)$ and $H^q_{\text{et}}(X_{\pi_{n,1}^i},\mathbb{Z}/p^n\mathbb{Z})$. Namely, we show that it suffices to establish a natural map $\iota : T^q_{\text{cris}}(M^i) \to H^q_{\text{et}}(X_{\pi_{n,1}^i},\mathbb{Z}/p^n\mathbb{Z})$ which is compatible with $G_R$-actions and such that the base change of $\iota$ to $R_g$ is the classical Fontaine-Messing comparison isomorphism $\iota_g : T^q_{\text{cris}}(M^i_g) \simeq H^q_{\text{et}}(X_{\pi_{n,1}^i},\mathbb{Z}/p^n\mathbb{Z})$.

To establish the comparison map $\iota$, we extend the method of Fontaine-Messing via syntomic cohomology to our base ring $R$. Classically, the key point is to show that $H^q_{\text{syn}}(X_{\mathcal{C}_{\mathcal{R}}}, S_{n,1}^{[i]}) \simeq H^q_{\text{et}}(X_{\mathcal{R}},\mathbb{Z}/p^n\mathbb{Z}(i))$, where $S_{n,1}^{[i]}$ are certain sheaves on the syntomic site defined via the divided power Frobenii. This is achieved by first constructing a morphism, and then a computation of $p$-adic vanishing cycles yields an isomorphism. Our relative setting, while philosophically similar, is complicated by the fact that the ring $R_n$ is not perfect. In particular, basic computations dealing with the relevant crystalline cohomology groups are difficult and, in particular, even the construction of $S_{n,1}^{[i]}$ in our setting becomes complicated. Our approach is to first descend to the intermediate ring
We deduce the desired results over an ideal, then there is an existence of \( \varphi \). In particular, we complete the proof that our objects are FL-modules by showing the scheme over Frac, which makes it difficult to connect these sheaves directly to the classically defined cycle' sheaves. In order to relate \( \acute{\text{e}} \text{tale} \) cohomology via syntomic cohomology, we consider the 'nearby setting. In particular, we consider analogous objects over the perfection \( R \) and then connect these sheaves directly to the classically defined cycle' sheaves. This will be in our forthcoming work.

We now briefly outline the content of the paper. In §2, we discuss (relative) period rings, Fontaine-Laffaille modules over \( R \), and the functor \( T_{\text{cris}} \) to Galois representations. In §2.1, we recall the definition and some basic results on the period rings \( A_{\text{cris}}(R) \) as well as \( \mathcal{O}A_{\text{cris}}(R) \). The latter ring comes equipped with the natural structure of a relative Fontaine-Laffaille module over \( R \), and allows one to give a natural construction of the functor \( T_{\text{cris}} \). In §2.2, following Faltings' approach in the relative case, we prove some basic structural results about our category of relative FL-modules over \( R \). One key result (Proposition 2.2.5) shows that \( M \) is an FL-module over \( R \) if and only if \( M \otimes_R W(k) \) and \( M \otimes_R R_g \) are FL-modules in the classical sense with the same type. In §2.3, we discuss the functor \( T_{\text{cris}} \) and explain how to pass to torsion galois representations.

In §3, we show that the data arising from crystalline cohomology of a smooth proper scheme over \( R \) gives rise to an FL-module. More precisely, we prove Theorem 1.2.1 (1) in §3 by assuming the existence of \( \varphi_i \) and \( \nabla \) satisfying the required properties.

In §4, we construct a morphism \( \iota : T_{\text{cris}}(M^i) \to H^i_{\text{et}}(X_{R[[A]}], \mathbb{Z}/p^n\mathbb{Z}) \) via syntomic cohomology following the strategy of Fontaine-Messing. Let \( \bar{X} := X \times_R \tilde{R} \) and consider the presheaves on \( (\bar{X}_n)^{\text{syn}} \)

\[ \mathcal{O}_n(U) = H^0_{\text{cris}}(U_n/R_n, \mathcal{O}_{\bar{X}_n/R_n}) \quad \text{and} \quad \mathcal{J}_n^{[j]}(U) := H^0_{\text{cris}}(U_n/R_n, \mathcal{J}_n^{[j]}_{\bar{X}_n/R_n}). \]

As in the classical setting, \( \mathcal{O}_n \) and \( \mathcal{J}_n^{[j]} \) are in fact sheaves and their (syntomic) cohomology computes \( H^i_{\text{cris}}(\bar{X}_n/R_n) \) and \( H^i_{\text{cris}}(\bar{X}_n/R_n, \mathcal{J}_n^{[j]}_{\bar{X}_n/R_n}) \) respectively. We note that, since \( \tilde{R} \) is not perfect, the computation of these cohomology groups is not as direct as in the classical setting. In particular, we consider analogous objects over the perfection \( \tilde{R} \) and then deduce the desired results over \( R_n \). These computations are carried out in §4.1 – 4.4. In particular, we complete the proof that our objects are FL-modules by showing the existence of \( \varphi_i \) and \( \nabla \) satisfying the required properties.

The cohomology groups \( H^i_{\text{cris}}(\bar{X}_n/R_n) \) have natural Gauss-Manin connection \( \nabla \) over \( R \). In order to relate \( \acute{\text{e}} \text{tale} \) cohomology via syntomic cohomology, we consider the 'nearby cycle' sheaves \( S_n^{[i]} := (\mathcal{J}_n^{[j]}_{\bar{X}_n/R_n})_{\varphi_i = 1, \nabla = 0} \). This is another point of departure from the classical case, which makes it difficult to connect these sheaves directly to the classically defined
sheaves $S_n^{[i]}$ when the base is $R_g$, where the latter sheaves are defined as $(J_n^{[i]})^\varphi_i = 1$. Once again, to relate these two sheaves, we pass to the intermediate ring $\tilde{R}$ and analogous sheaves defined in that setting; the latter sheaves can then be related directly to the those over $R$ and those over $R_g$. In §4.5, using the aforementioned ideas and constructions, we construct a natural morphism $T_{\text{cris}}(M^i) \to H^i_{\text{syn}}(\overline{X}_n, S_n^{[i]})$. In §4.6, following the idea of Fontaine-Messing, we construct a natural map $H^i_{\text{syn}}(\overline{X}_n, S_n^{[i]}) \to H^i_{\text{ét}}(X_{\overline{\mathbb{F}}_p^1}, \mathbb{Z}/p^n\mathbb{Z})$. This completes the construction of a required map $\iota$ and proves Theorem 1.2.1.

Finally, in §5 we discuss some applications of our relative Fontaine-Messing theory. A key benefit of the relative theory (in the current setting) is that it allows us to understand the full $G_R$-action on torsion étale cohomology. For example, in §5.1 we show that $H^i_{\text{ét}}(X_{\overline{\mathbb{F}}_p^1}, \mathbb{Q}_p)$ is crystalline for $i \leq p - 2$. In §5.2, we prove a type of local invariant cycle theorem: if $k = \tilde{k}$ and $X_k := X \times R k$, then $H^i_{\text{ét}}(X_k, \mathbb{Z}/p^n\mathbb{Z}) \simeq H^i_{\text{ét}}(X_{\overline{\mathbb{F}}_p^1}, \mathbb{Z}/p^n\mathbb{Z})^{G_R}$.

Finally, in §5.3, we consider the case when $K$ is finite totally ramified over $W(k)[\frac{1}{p}]$. For a proper smooth scheme $Y$ over $\mathcal{O}_K$, we prove that the length of crystalline cohomology and length of étale cohomology match when $Y$ admits a lift over $R$.

1.4. Notations. We reserve $\varphi$ to denote various Frobenius compatible with arithmetic Frobenius over $W(k)$. If necessary, we add a subscript to $\varphi$ to indicate the corresponding ring or module. For example, we sometimes write $\varphi_M : M \to M$ in order to avoid confusion. For any module $M$ and integer $n \geq 1$, we set $M_n := M \otimes \mathbb{Z}[1/n]$ and denote by $\hat{M}$ the $p$-adic completion of $M$. For any scheme $X$, denote $X_n := X \times_{\text{Spec} \mathbb{Z}} \text{Spec} \mathbb{Z}[1/n]$. Given a ring $A$ and $x \in A$, let $[x]$ denote its Teichmüller lift in $W(A)$. Let $\text{Cris}(X/S)$ be the (small) crystalline site of $X$ over $S$ and $\mathcal{O}_{X/S}$ denote the usual crystalline structure sheaf. We denote by $H^i_{\text{cris}}(X/S) := H^i_{\text{cris}}((X/S)_{\text{cris}}, \mathcal{O}_{X/S})$ the usual crystalline cohomology of the structure sheaf. Let $J_n^{[j]}$ denote the $j$-th PD ideal sheaf over $\text{Cris}(X/S)$. We reserve $\gamma_m(x)$ to denote the $m$-th divided power of $x$. Since most of our base schemes are affine schemes $\text{Spec}(A)$ with a $\mathbb{Z}_p$-algebra $A$, we sometimes use $A$ to denote $\text{Spec}(A)$ to simplify notation.

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2. Relative Fontaine-Laffaille modules and associated Galois representations

In this section, we extend the theory of Fontaine-Laffaille modules and their associated Galois representations to the base ring $R = W(k)[[t_1, \ldots, t_d]]$. The exposition follows
closely the relative case studied in [Fal89], but we also give several new proofs/results using a dévissage to the “generic fiber” technique.

2.1. Relative Period Rings. In this subsection, following [Bri08], we briefly recall the construction of crystalline period rings in our relative setting. We refer to loc. cit. for details. As in the introduction, let $k$ be a perfect field of characteristic $p > 2$, $R = W(k)[[t_1, \ldots, t_d]]$, and define the Frobenius endomorphism $\varphi : R \to R$ extending the natural Frobenius on $W(k)$ by setting $\varphi(t_i) = (1 + t_i)^p - 1$. We remark that this is not the usual lift obtained by setting $\varphi(t_i) = t_i^p$, but we will see later that our choice is convenient for constructing a Frobenius equivariant embedding of $R$ into the corresponding crystalline period ring. Denote by $\Omega_{R_n/W_n(k)}$ the module of Kähler differentials, and let $\hat{\Omega}_R := \varprojlim_n \Omega_{R_n/W_n(k)}$.

Let $R[\frac{1}{p}]_{\text{sep}}$ be a fixed separable closure of $R[\frac{1}{p}]$, and $\overline{R}$ denote the union of normal $R$-sub-algebras $R' \subset R[\frac{1}{p}]_{\text{sep}}$ such that $R'[\frac{1}{p}]$ is finite étale over $R[\frac{1}{p}]$. It follows that $\text{Spec}(\overline{R}[\frac{1}{p}])$ is a universal étale covering of $\text{Spec}(R[\frac{1}{p}])$, and $\overline{R}$ is the integral closure of $R$ in $\overline{R}[\frac{1}{p}]$. Let $G_R := \pi_1^\text{ét}(\text{Spec}(R[\frac{1}{p}]), \eta)$ where $\eta$ is a fixed geometric point. We may identify the latter with the group $\text{Gal}(\overline{R}/R)$ of $R$-linear automorphisms. Note that there is a natural embedding $O_{\overline{R}} \hookrightarrow \overline{R}$ compatible with the inclusion $W(k) \hookrightarrow R$.

Given a $p$-adically separated $\mathbb{Z}_p$-algebra $A$, set $A^p = \varprojlim_{x \to x^p} A$. By [BMS18] Lemma 3.2, the canonical map $\varprojlim_{x \to x^p} \hat{A} \to A^p$ is an isomorphism of multiplicative monoids. By functoriality of this construction, we obtain a natural embedding $O_{\overline{R}} \hookrightarrow \overline{R}$.

We have a natural surjective map $\theta : W(\overline{R}) \to \overline{R}$ which lifts the projection onto the first factor (and similarly for $O_{\overline{R}}$). For integers $n \geq 0$, we choose $p_n \in \overline{R}$ such that $p_0 = p$ and $p_{n+1}^p = p_n$. Let $\tilde{p} := (p_n)_{n \geq 0} \in \overline{R}$, and $[\tilde{p}] \in W(\overline{R})$ be its Teichmüller lift. Then $\ker(\theta)$ is generated by $\xi := [\tilde{p}] - p$. Similarly, choose $\tilde{1} + t_i = (z_{i,n})_{n \geq 0} \in \overline{R}$ where $z_{i,0} = 1 + t_i$ and $z_{i,n+1}^p = z_{i,n}$. Let $W(\overline{R})^{DP}$ (resp. $W(O_{\overline{R}}^{DP})$) be the divided power envelope of $W(\overline{R})$ (resp. $W(O_{\overline{R}})$) with respect to $\ker(\theta)$, and let $A_{\text{cris}}(R)$ (resp. $A_{\text{cris}}$) be its $p$-adic completion. We note that $\varphi$ extends naturally to $A_{\text{cris}}$ and $A_{\text{cris}}(R)$. Moreover, it is compatible with the natural embedding $A_{\text{cris}} \hookrightarrow A_{\text{cris}}(R)$.

Let $\lambda_1 : R \to W(\overline{R})$ be the embedding given by $t_i \mapsto [1 + t_i] - 1$. This embedding is $\varphi$-equivariant (recall $\varphi_R(t_i) = (1 + t_i)^p - 1$). In this way, we regard $A_{\text{cris}}(R)$ as an $R$-algebra compatible with the corresponding Frobenii.

Given $i \geq 0$, let $F^i A_{\text{cris}}(R) \subset A_{\text{cris}}(R)$ be the corresponding $i$-th divided power ideal ideal. By the description of $\ker(\theta)$ above, this is given by $p$-adic completion of the ideal generated by the divided powers $\gamma_j(\xi)$ for $j \geq i$. This gives a filtration on $A_{\text{cris}}(R)$. We note that by the proof of [Bri08], Proposition 6.1.4], $W(\overline{R})^{DP}$ is $p$-torsion free, and therefore $A_{\text{cris}}(R)$ is $p$-torsion free by [Sta18] Tag 0AGW. For each $i \leq p - 1$, since
\( \varphi(F^i \Lambda_{\text{cris}}(R)) \subset p^i \Lambda_{\text{cris}}(R) \), we can define \( \varphi_i := \varphi^{p^i} : F^i \Lambda_{\text{cris}}(R) \to \Lambda_{\text{cris}}(R) \). The constructions of \( F^i \) and \( \varphi_i \) on \( \Lambda_{\text{cris}}(R) \) are compatible with those on \( \Lambda_{\text{cris}} \).

In the following, several of our arguments will involve dévissage to the perfect case via a base change map. We fix some notations and recall some results to facilitate this dévissage. Let \( R_{(p)} \) denote the localization of \( R \) at \( (p) \) and denote by \( R_g \) the \( p \)-adic completion of \( \lim_{\varphi} R_{(p)} \). Let

\[
\begin{align*}
b_g : R &\to R_g \\
\end{align*}
\]

denote the resulting morphism and note that by construction it is compatible with Frobenius. By [Bri08] Lemma 7.1.8, the Frobenius endomorphism \( \varphi : R \to R \) is flat, and thus \( b_g : R \to R_g \) is flat. Let \( k_g := \lim_{\varphi} \Frac(R/pR) \). Then, by the universal property of \( p \)-adic Witt vectors, we have a Frobenius equivariant isomorphism \( R_g \cong W(k_g) \).

We now fix an embedding \( \overline{b}_g : \overline{R} \to \overline{R}_g \). Then \( \overline{b}_g \) induces an embedding \( \overline{R} \hookrightarrow \overline{R}_g \) and hence an embedding \( W(\overline{R}) \to W(\overline{R}_g) \). Since \( \ker(\theta) = (\xi) \) and \( W(\overline{R})^{\text{DP}} \) is \( p \)-torsion free (again by the proof of [Bri08] Proposition 6.1.4)), the induced map \( \Lambda_{\text{cris}}(R) \to \Lambda_{\text{cris}}(R_g) \) is injective and compatible with all the aforementioned structures.

Consider now the \( R \)-linear extension \( \theta_R : R \otimes_{W(k)} W(\overline{R}) \to \hat{R} \) of \( \theta \), and let \( \Omega \Lambda_{\text{cris}}(R) \) be the \( p \)-adic completion of the divided power envelope of \( R \otimes_{W(k)} W(\overline{R}) \) with respect to \( \ker(\theta_R) \). The ring \( \Omega \Lambda_{\text{cris}}(R) \) is equipped with a natural \( G_R \)-action, and the Frobenius \( \varphi_R \otimes \varphi_W \) extends to \( \Omega \Lambda_{\text{cris}}(R) \). We equip \( \Omega \Lambda_{\text{cris}}(R) \) with a filtration by setting \( F^i(\Omega \Lambda_{\text{cris}}(R)) \subset \Omega \Lambda_{\text{cris}}(R) \) for \( i \geq 1 \) to be the \( p \)-adic completion of the ideal generated by elements of the form \( \gamma_{j_1}(a_1) \cdots \gamma_{j_i}(a_i) \) with \( a_1, \ldots, a_i \in \ker(\theta_R) \) and \( j_1 + \cdots + j_i \geq i \). Note that \( \Omega \Lambda_{\text{cris}}(R) \) and \( \Lambda_{\text{cris}}(R) \) have natural \( G_R \)-actions induced by functoriality. We have a natural embedding \( \Lambda_{\text{cris}}(R) \hookrightarrow \Omega \Lambda_{\text{cris}}(R) \) compatible with the \( G_R \)-action, Frobenius, and filtration. Furthermore, we have a natural \( \Lambda_{\text{cris}}(R) \)-linear connection \( \nabla : \Omega \Lambda_{\text{cris}}(R) \to \Omega \Lambda_{\text{cris}}(R) \otimes_R \hat{\Omega}_R \).

Note that \( \lambda_1 \) induces an \( R \)-algebra structure on \( \Omega \Lambda_{\text{cris}}(R) \). On the other hand, setting \( \lambda_2 : R \to \Omega \Lambda_{\text{cris}}(R) \) be the map given by \( \lambda_2(r) = r \otimes 1 \) also gives an \( R \)-algebra structure. Note that this is different from the morphism induced by \( \lambda_1 \) above. In the following, we will make use of both morphisms (i.e. both \( R \)-algebra structures) in the study of Galois actions.

We end this section by recalling two structural results due to Brinon on the rings \( \Lambda_{\text{cris}}(R) \) and \( \Omega \Lambda_{\text{cris}}(R) \). Denote by \( \Lambda_{\text{cris}}(R)(X_1, \ldots, X_d) \) the \( p \)-adic completion of the divided power polynomial ring with variables \( X_i \) having coefficients in \( \Lambda_{\text{cris}}(R) \). Consider the \( \Lambda_{\text{cris}}(R) \)-linear map \( f : \Lambda_{\text{cris}}(R)(X_1, \ldots, X_d) \to \Omega \Lambda_{\text{cris}}(R) \) given by \( X_i \mapsto 1 + t_i \otimes 1 - 1 \otimes [1 + t_i] \). We record the following lemmas for future use:

**Lemma 2.1.1.** ([Bri08] Corollary 6.1.3]) For each integer \( n \geq 1 \), we have an isomorphism

\[
W_n(\overline{R})[\delta_0, \delta_1, \ldots]/(p\delta_0 - \xi^p, p\delta_{m+1} - \delta_m^p)_{m \geq 0} \iso A_{\text{cris}}(R)/p^n A_{\text{cris}}(R)
\]

where \( \delta_m \) maps to \( \gamma^{m+1}(\xi) \) (here, \( \gamma : x \mapsto \frac{x^p}{p} \)).
We note that the isomorphism of the lemma is \( W_n(\overline{R}) \)-linear. In particular, we may define a \( G_R \)-action, filtration and divided power Frobenii \( \varphi_i \) on the left hand side by transporting the corresponding structures on the right hand side via the isomorphism from the lemma.

**Lemma 2.1.2.** ([Brö08 Proposition 6.1.8]) The map \( f : A_{\text{cris}}(R) \langle X_1, \ldots, X_d \rangle \rightarrow \mathcal{O}A_{\text{cris}}(R) \) defined above is an \( A_{\text{cris}}(R) \)-linear isomorphism. In particular, it is an \( R \)-algebra morphism where the algebra structure on the right hand side is defined via \( \lambda_1 \).

**Remark 2.1.3.** Once again we may define a filtration, \( G_R \)-action, and \( \varphi_i \) on the left hand side by pulling back the corresponding structures on the right hand side. We can give an explicit description as follows:

1. Given \( g \in G_R \), we have \( g(X_i) = X_i - g([1 + t_i]) + [1 + t_i] \).
2. The filtration on the left hand side is given by
   \[
   F^r A_{\text{cris}}(R) \langle X_1, \ldots, X_d \rangle = \bigoplus_{\sum_{ij} \geq r} F^{i_0} A_{\text{cris}, n}(A)\gamma_{i_1}(X_1) \cdots \gamma_{i_d}(X_d).
   \]
3. \( \varphi(X_i) = (X_i + [1 + t_i])^p - [(1 + t_i)^p] \).
4. Let \( \pi : \mathcal{O}A_{\text{cris}}(R) \rightarrow A_{\text{cris}}(R) \) denote the composition of \( f^{-1} \) and \( q : A_{\text{cris}}(R) \langle X_1, \ldots, X_d \rangle \rightarrow A_{\text{cris}}(R) \) given by \( X_i \mapsto 0 \). We note that this is an \( R \)-linear map with either \( R \)-algebra structure (the one induced by \( \lambda_1 \) or \( \lambda_2 \)) on the \( \mathcal{O}A_{\text{cris}}(R) \).
5. Under the isomorphism \( f \), \( \nabla(\frac{\partial}{\partial t_i}) (X_j) \) is 1 if \( i = j \) and 0 if \( i \neq j \).

Finally, for integers \( n \geq 0 \), we fix \( \epsilon_n \in \overline{R} \) such that \( \epsilon_0 = 1 \), \( \epsilon_{n+1}^p = \epsilon_n \), \( \epsilon_1 \neq 1 \) and set \( \bar{\epsilon} := (\epsilon_n)_{n \geq 0} \in \overline{R}^\mathbb{N} \). A choice of \( \mathbb{Z}_p \)-basis \( x \) of \( \mathbb{Z}_p(1) \) gives a \( G_R \)-equivariant map \( \beta : \mathbb{Z}_p(1) \rightarrow F^1 A_{\text{cris}}(R) \) defined by setting \( \beta(x) := \log([\bar{\epsilon}]) \).

### 2.2. Relative Fontaine-Laffaille Modules.

Following [Fal89], we define the categories of Fontaine-Laffaille modules over \( R \) and study their basic properties. The arguments below are similar to those of [Fal89], with some modifications for our specific setting.

Denote by \( \text{MF}_{\text{big}}(R) \) the category whose objects consist of a \( p \)-power torsion \( R \)-module \( M \), a sequence of \( p \)-power torsion \( R \)-modules \( F^i(M) \), and sequences of \( R \)-linear maps \( \iota_i : F^i(M) \rightarrow F^{i-1}(M), \pi_i : F^i(M) \rightarrow M, \) and \( \varphi_i : F^i(M) \otimes_{R, \varphi} R \rightarrow M \) satisfying the following conditions:

1. The composition \( F^i(M) \xrightarrow{\iota_i} F^{i-1}(M) \xrightarrow{\pi_{i-1}} M \) is the map \( \pi_i : F^i(M) \rightarrow M \);
2. The map \( \pi_i : F^i(M) \rightarrow M \) is an isomorphism for \( i < 0 \);
3. The composition of \( \varphi_{i-1} \) with \( \iota_i : F^i(M) \rightarrow F^{i-1}(M) \) is \( p\varphi_i \).

Morphisms between such objects are compatible collections of \( R \)-linear maps between \( M \)'s and \( F^i(M) \)'s. Note that \( \text{MF}_{\text{big}}(R) \) is an abelian category.

**Remark 2.2.1.** An \( R \)-linear map \( \varphi_i : F^i(M) \otimes_{R, \varphi} R \rightarrow M \) is equivalent to a \( \varphi_R \)-semilinear map \( \varphi_i : F^i(M) \rightarrow M \). In the following, we will use these two notions of \( \varphi_i \) interchangeably.
Remark 2.2.2. Let \( \hat{M} \) be the colimit of the following diagram:

\[
\cdots \xrightarrow{i_{i+2}} F^{i+1}(M) \xleftarrow{XP} F^i(M) \xrightarrow{i_{i+1}} F^{i}(M) \xleftarrow{XP} F^{i-1}(M) \xrightarrow{i_{i}} \cdots
\]

Then condition (3) above is equivalent to requiring that \( \varphi_i \)'s induce an \( R \)-linear map

\[
\varphi_M : \hat{M} \otimes_{R,\varphi} R \to M.
\]

Let \( MF(R) \) be the full subcategory of \( MF_{big}(R) \) whose objects consist of tuples \((M, F^i(M), \varphi_i)\) such that \( M \) and \( F^i(M) \) are finitely generated \( R \)-modules, \( F^i(M) = \{0\} \) for \( i \gg 0 \), and \( \varphi_i \)'s induce an isomorphism \( \varphi_M : \hat{M} \otimes_{R,\varphi} R \simeq M \). We further denote by \( MF^{[a,b]}(R) \) the full subcategory of \( MF(R) \) consisting of objects such that \( F^{b+1}(M) = \{0\} \) and \( F^a(M) = M \). We refer to objects in \( MF(R) \) as Fontaine-Laffaille modules over \( R \) without connection \( \nabla \). We often drop “without connection \( \nabla \)” or “over \( R \)” if no confusion arises.

Let \( MF_{big}(R) \) denote the full sub-category of \( MF_{big}(R) \) consisting of objects such that \( F^{b+1}(M) = \{0\} \) and \( F^a(M) = M \). For any such object \( M \), \( \hat{M} \) can be defined explicitly by the right exact sequence

\[
\bigoplus_{i=a+1}^b F^i(M) \xrightarrow{\theta_M} \bigoplus_{i=a}^b F^i(M) \longrightarrow M \longrightarrow 0,
\]

where \( \theta_M((x_{a+1}, \ldots, x_b)) = (\ell_{a+1}(x_{a+1}), -px_{a+1} + \ell_{a+1}(x_{a+2}), \ldots, -px_{b-1} + \ell_b(x_b), -px_b) \). It follows that the functor \( M \mapsto \hat{M} \) is a right exact functor. Moreover, if \( A \) is a \( W(k) \)-algebra equipped with a Frobenius endomorphism compatible with that on \( W(k) \), then \( M \otimes_{W(k)} A \) carries a \( W(k) \)-algebra morphism compatible with Frobenius, then \( A \otimes_{W(k)} M \simeq A \otimes_{W(k)} \hat{M} \) (with \( F^i(A \otimes_{W(k)} M) := A \otimes_{W(k)} F^i(M) \)). Hence if \( M \) is an object in \( MF(R) \), so is \( M/p^iM \), and we have \( M_0 := M \otimes_R W(k) \in MF(W(k)) \) and \( M_g := M \otimes_R R_g \in MF(R_g) \).

We have the following structure result for Fontaine-Laffaille modules over \( R \). As in the classical case, one consequence of this result is that \( MF(R) \) is an abelian category.

**Theorem 2.2.1.** (cf. [Fal89, Theorem 2.1]) Let \((M, F^i(M), \varphi_i) \in MF(R)\). Then

1. All maps \( F^{i+1}(M) \to F^i(M) \to M \) are injections onto direct summands as \( R \)-modules. In particular, we can consider \( F^i(M) \)'s as submodules of \( M \) giving a finite filtration.
2. \( F^i(M) \) and \( F^i(M)/F^{i+1}(M) \) are isomorphic to a direct sum of \( R \)-modules of the form \( R/p^jR \).
3. If \( M, N \in MF(R) \), any map \( f : M \to N \) in \( MF(R) \) is strict with respect to the filtration. Namely, \( f(F^i(M)) = f(M) \cap F^i(N) \).

**Proof.** The proof of [Fal89, Theorem 2.1] applies almost verbatim for our base \( R \). For the convenience of the readers, we sketch the proof below.

If \( d = 0 \), then \( R = W(k) \). In that case, (2) and (3) are results from the classical Fontaine-Laffaille theory ([FLS82, Proposition 1.8]). For (1), first note that the length of \( \hat{M} \) is greater than or equal to the length of \( M \) for any \( M \in MF_{big}(R) \), with equality if and only if all maps \( F^{i+1}(M) \to F^i(M) \) are injective. This proves the first part of (1). The assertion about
direct summands in (1) follows by applying this fact to \( M/p^i M \)'s (which are objects in \( MF(R) \)).

Let \( m \subset R \) denote the maximal ideal. For the general case when \( d \geq 1 \), we may assume that all assertions hold up to \( m \)-torsion by the case \( d = 0 \). Assume first that \( p M = 0 \). For any finite \( R/pR \)-module \( L \), denote by \( \text{Fitt}_i(L) \) the \( i \)-th Fitting ideal of \( L \) over \( R/pR \). Under our assumptions, there exist integers \( a \leq b \) such that \( \widetilde{M} \) is the cokernel of the map

\[
\bigoplus_{a+1 \leq i \leq b} F^i(M) \xrightarrow{\theta_i} \bigoplus_{a \leq i \leq b} F^i(M).
\]

Let \( r_i \) denote the rank of \( F^i(M) \) at the generic point of \( R/pR \). Then \( r_i \) is the smallest subscript \( s \) for which \( \text{Fitt}_s(F^i(M)) \neq \{0\} \). If we let \( r = r_a \) be the rank of \( M \) and \( \widetilde{M} \) at the generic point, then

\[
\prod_{a \leq i \leq b} \text{Fitt}_{r_i}(F^i(M)) \supset \text{Fitt}_{r_a}(\widetilde{M}) \cdot \prod_{a+1 \leq i \leq b} \text{Fitt}_{r_i}(F^i(M)).
\]

On the other hand, since \( \varphi_M \) is an isomorphism, we have

\[
\text{Fitt}_{r_a}(F^a(M)) = \text{Fitt}_r(M) = \varphi_*(\text{Fitt}_r(\widetilde{M})) = \text{the ideal generated by } \varphi(\text{Fitt}_r(\widetilde{M})).
\]

Since gr \( m \) \( R \) has no zero divisors, this implies that \( \text{Fitt}_r(M) = R/pR \), and therefore \( M \) and \( \widetilde{M} \) are free over \( R/pR \). Since \( \widetilde{M} \) is the direct sum of the cokernels of the maps \( F^{i+1}(M) \to F^i(M) \), these cokernels are free over \( R/pR \), and the images are direct summands. Furthermore, since these maps are injective up to \( m \)-torsion, we deduce (1) and (2) in this case \( p M = 0 \) by decreasing induction on \( i \).

For the general case, we induct on the smallest integer \( e \) such that \( p^e \) annihilates all modules involved. Let \( M \in MF(R) \). Since \( \widetilde{M} \) defines a right exact functor of \( M \), the assertions hold for \( M/pM \). Thus, each map \( F^i(M)/pF^i(M) \to F^{i-1}(M)/pF^{i-1}(M) \) is injective. If we denote by \( N \subset M \) the subobject \( pM \), then the map \( \widetilde{N} \to \widetilde{M} \) is injective, identifying \( \widetilde{N} \) with \( p\widetilde{M} \). Since \( \varphi : R \to R \) is flat, \( \varphi_N \) is an isomorphism and \( N \in MF(R) \). By the induction hypothesis, (1) and (2) hold for \( N \). So \( F^{i-1}(N) \cong F^i(N) \oplus F^{i-1}(N)/F^i(N) \) and both \( F^{i-1}(N) \) and \( F^i(N) \) are direct sums of the form \( R/p^a R \). That is, \( F^{i-1}(N) \cong \bigoplus_{i=1}^n R/p^{a_i} R \) for some integers \( a_i \geq 1 \) and \( F^i(N) \cong \bigoplus_{i=1}^m R/p^{a_i} R \) for \( m \leq n \). Set \( Q = \bigoplus_{i=1}^n R/p^{a_i+1} R \cdot e_i \) and \( L = \bigoplus_{i=1}^m R/p^{a_i+1} R \cdot e_i \) with \( e_i \) being basis elements. We have an isomorphism \( h' : F^{i-1}(N) \xrightarrow{\cong} pQ \) which induces an isomorphism of \( F^i(N) \cong pL \).

Since \( F^i(M)/pF^i(M) \to F^{i-1}(M)/pF^{i-1}(M) \) is an injection of free \( R/pR \)-modules, we can lift \( h' \) to an \( R \)-module morphism \( h : F^{i-1}(M) \to Q \) such that \( h(F^i(M)) \subset L \). For the same reason, \( h'^{-1} : pQ \to F^{i-1}(N) \) lifts to an \( R \)-module morphism \( f : Q \to F^{i-1}(M) \) such that \( f(L) \subset F^i(M) \). The composite \( h \circ f : Q \to Q \) is a lift of \( h' \circ h'^{-1} = \text{Id} : pQ \to pQ \), so \( (h \circ f)(e_i) = e_i + x_i \) for some \( p \)-torsion elements \( x_i \in Q \). Since \( Q \cong \bigoplus_{i=1}^n R/p^{a_i+1} R \cdot e_i \), we deduce that \( h \circ f \) is an isomorphism. This implies that \( Q \) injects onto a direct summand of \( F^{i-1}(M) \), which we identify with \( Q \). Similarly, \( L \) is a direct summand of \( F^i(M) \). Since \( pF^{i-1}(M) = F^{i-1}(N) = pQ \), we have \( F^{i-1}(M)/pF^{i-1}(M) \cong Q/pQ \oplus F^{i-1}(M)/Q \),
and similarly $F^i(M)/pF^i(M) \cong L/pL \oplus F^i(M)/L$. Note that $F^i(M)/L \subset F^{i-1}(M)/Q \subset F^{i-1}(M)/pF^{i-1}(M)$ are all direct summands of free $R/pR$-modules. Therefore, $F^i(M) \cong L \oplus F^i(M)/L$ is a direct summand of $F^{i-1}(M) \cong Q \oplus F^{i-1}(M)/Q$. We conclude that $F^{i-1}(M) \cong F^i(M) \oplus F^{i-1}(M)/F^i(M)$, and both $F^i(M)$ and $F^{i-1}(M)/F^i(M)$ are direct sums of the form $R/p^nR$.

Finally, we prove assertion (3). For any $R$-module $Q$, we denote the base change $Q \otimes_{R, \varphi} R$ by $Q_R$. We first show that $L := N/f(M)$ is an object in MF($R$). Since the functor $M \mapsto \tilde{M}$ is right exact, $\tilde{M} \rightarrow \tilde{N} \rightarrow \tilde{L} \rightarrow 0$ is exact. Since $\tilde{M}_R \rightarrow M$ and $\tilde{N}_R \rightarrow N$ are isomorphisms, $\varphi_L : \tilde{L}_R \rightarrow L$ is an isomorphism. Hence $L = N/f(M)$ is an object in MF($R$). In particular, $F^i(L) = F^i(N)/f(F^i(M))$ is a submodule of $L = N/f(M)$. This forces $f(M) \cap F^i(N) = f(F^i(M))$.

Remark 2.2.3. The above theorem shows that any $M \in$ MF($R$) admits an adapted basis: There exist elements $e_i \in M$, positive integers $a_i$, and decreasing positive integers $m_i \geq m_{i+1} \geq \cdots$ so that if $F^i(M) \neq 0$, then

$$F^i(M) = \bigoplus_{j=1}^{m_i} R/p^{a_i}R \cdot e_j.$$  

Corollary 2.2.4. MF($R$) is an abelian subcategory of MF$_{\text{big}}$($R$).

In the following, we give another criterion for an object of MF$_{\text{big}}$($R$) to be in MF($R$). We say a $W(k)$-module $M$ and an $R_g$-module $M'$ have the same type if $M' \cong M \otimes_{W(k)} R_g$ as $R_g$-modules. Note that the natural projection map $b : R \rightarrow W(k)$ given by $t_i \mapsto 0$ is compatible with Frobenius.

Proposition 2.2.5. Suppose $M \in$ MF$_{\text{big}}$($R$) such that each $F^i(M)$ is finite as an $R$-module. Then $M$ is in MF($R$) if and only if

1. Both $M_0 := M \otimes_{R,b} W(k)$ and $M_g := M \otimes_R R_g$ are objects in MF($W(k)$) and MF($R_g$) respectively;
2. $M_0$ and $M_g$ have the same type.

Proof. If $M$ is an object in MF($R$) then we have $M_0 \in$ MF($W(k)$), $M_g \in$ MF($R_g$), and by Theorem 2.2.1 (2), $M_0$ and $M_g$ have the same type. For the converse statement, we first reduce to the case where $pM = \{0\}$. Consider the exact sequence

$$0 \rightarrow pM \rightarrow M \rightarrow M/pM \rightarrow 0.$$  

Since $M/pM \otimes_R W(k) \cong M_0/pM_0$ and $M/pM \otimes_R R_g \cong M_g/pM_g$, $M/pM$ satisfies the two conditions required in the proposition. Suppose $M/pM$ is in MF($R$). Then all $F^i(M/pM)$ are finite $R/pR$-free, so the sequence

$$0 \rightarrow pF^i(M) \otimes_R W(k) \rightarrow F^i(M_0) \rightarrow F^i(M/pM) \otimes_R W(k) \rightarrow 0$$  

is exact. Since MF($W(k)$) is an abelian category, $pM \otimes_R W(k)$ is an object in MF($W(k)$). Similarly, $pM \otimes_R R_g$ lies in MF($R_g$). Using the above exact sequence for $i \ll 0$, we deduce that $pM \otimes_R W(k)$ and $pM \otimes_R R_g$ have the same type. By induction on the length of
$M \otimes_R W(k)$ as a $W(k)$-module, we may assume $pM$ is an object in $\text{MF}(R)$. For any $R$-module $N$, denote $N \otimes_{R,\varphi} R$ by $N_R$. By the commutative diagram

\[
\begin{array}{cccccc}
\widetilde{pM}_R & \longrightarrow & \widetilde{M}_R & \longrightarrow & \widetilde{M}/pM_R & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & pM & \longrightarrow & M & \longrightarrow & pM/M \longrightarrow 0
\end{array}
\]

we conclude that $\varphi_M : \tilde{M}_R \rightarrow M$ is an isomorphism and $M \in \text{MF}(R)$.

Now it suffices to prove the proposition when $M$ is killed by $p$. Since $M_0 \otimes k g \simeq M_g$, $M$ is finite free over $R/pR$ by Nakayama’s Lemma. Pick an adapted basis $\bar{e}_1, \ldots, \bar{e}_n$ of $M_0$ so that $\bar{e}_1, \ldots, \bar{e}_{m_i}$ (with $m_i \leq n$) is a basis of $F^i(M_0)$. Let $e_1, \ldots, e_{m_i} \in F^i(M)$ be a choice of lifts of $\bar{e}_1, \ldots, \bar{e}_{m_i}$ and $e_{m_i+1}, \ldots, e_n \in M$ be a choice of lifts of $\bar{e}_{m_i+1}, \ldots, \bar{e}_n$. For the structure map $\iota_i : F^i(M) \rightarrow M$, we deduce from Nakayama’s Lemma that $e_1, \ldots, e_{m_i}$ generate $F^i(M)$ and $\iota_i(e_1), \ldots, \iota_i(e_{m_i}), e_{m_i+1}, \ldots, e_n$ form a basis of $M$. This forces $\iota_i$ to be injective and $e_1, \ldots, e_{m_i}$ to be linearly independent. Thus, $F^i(M)$ is a finite free $R/pR$-module and $F^i(M)$ is a direct summand of $M$ via $\iota_i$. Moreover, $\tilde{M}$ is a finite free $R/pR$-module with the same rank as that of $M$. Hence, $\varphi_M : \tilde{M}_R \rightarrow M$ is an isomorphism since $\varphi_M \otimes_R W(k)$ is an isomorphism.

\[
\text{Remark 2.2.6.} \text{ If } M \in \text{MF}(R), \text{ then } M \in \text{MF}^{[a,b]}(R) \text{ if and only if } M_0 \in \text{MF}^{[a,b]}(W(k)) \text{ if and only if } M_{\varphi} \in \text{MF}^{[a,b]}(R_{\varphi}).
\]

Finally, we define the category $\text{MF}_\nabla(R)$ (resp. $\text{MF}_\nabla, \text{big}(R)$). An object of $\text{MF}_\nabla(R)$ (resp. $\text{MF}_\nabla, \text{big}(R)$) is a tuple $(M, F^i(M), \varphi_i)$ in $\text{MF}(R)$ (resp. $\text{MF}_{\text{big}}(R)$) equipped with an integrable connection $\nabla : M \rightarrow M \otimes_R \hat{\Omega}_R$ such that

- Griffiths-transversality holds: $\nabla(F^i(M)) \subset F^{i-1}(M) \otimes_R \hat{\Omega}_R$,
- the maps $\varphi_i : F^i(M) \otimes_{R,\varphi} R \rightarrow M$ are parallel: $\nabla \circ \varphi = (\varphi_{i-1} \otimes_R d\varphi_1) \circ \nabla$ as a map from $F^i(M) \otimes_{R,\varphi} R$ to $M \otimes_R \hat{\Omega}_R$, where $d\varphi_1 := \frac{d\varphi}{p} : \hat{\Omega}_R \otimes_{R,\varphi} R \rightarrow \hat{\Omega}_R$.

Morphisms in $\text{MF}_\nabla(R)$ and $\text{MF}_\nabla, \text{big}(R)$ are $R$-linear maps compatible with all structures. Denote by $\text{MF}_\nabla^{[0,r]}(R)$ (resp. $\text{MF}_\nabla, \text{big}^{[0,r]}(R)$) the full subcategory of $\text{MF}_\nabla(R)$ (resp. $\text{MF}_\nabla, \text{big}(R)$) consisting of objects with $F^0(M) = M$ and $F^{r+1}(M) = 0$. In most cases in this paper, $r \leq p - 2$.

\[
\text{Definition 2.2.7.} \text{ A Fontaine-Laffaille module over } R \text{ is an object } (M, F^i(M), \varphi_i, \nabla) \in \text{MF}_\nabla(R).
\]

\[
\text{Remark 2.2.8.} \text{ This definition is slightly different from Definition 2.2.1 where we use a weaker statement } \sum_i \varphi_i(F^i(M)) = M \text{ instead of the stronger one } \varphi_M : M \otimes_{R,\varphi} R \simeq M. \text{ But the weaker statement still implies that } M_0 \text{ and } M_{\varphi} \text{ are objects in } \text{MF}(W(k)) \text{ and } \text{MF}(R_{\varphi}) \text{ respectively. It follows from Proposition 2.2.3 that the two definitions are equivalent.}
\]
2.3. Torsion Representations associated to Fontaine-Laffaille Modules. We now study Galois representations associated to Fontaine-Laffaille modules $M \in \text{MF}^{[0,r]}(R)$ with $r \leq p-2$. Recall that we defined embeddings $\lambda_1 : R \to \text{A}_{\text{cris}}(R)$ and $\lambda_2 : R \to \text{O}_{\text{cris}}(R)$ in §2.1. Let $\overline{M}_R := \text{O}_{\text{cris}}(R) \otimes_{\lambda_2,R} M$ and $F^\bullet \overline{M}_R := \bigoplus_{i=0}^r F^i \text{O}_{\text{cris}}(R) \otimes_{\lambda_2,R} F^{r-i} M \subset \overline{M}_R$.

**Remark 2.3.1.**  
(1) Note that for any $M \in \text{MF}^{[0,r]}(R)$, we have $\text{Tor}_1^\bullet(F^i \text{A}_{\text{cris}}(R), F^j M) = 0$ for each $i$ and $j$ since $\text{A}_{\text{cris}}(R)$ is $p$-torsion free and $F^j M$ is a direct sum of modules of the form $R/p^j R$.

(2) It follows that $\bigoplus_{i=0}^r F^i \text{O}_{\text{cris}}(R) \otimes_{\lambda_2,R} F^{r-i} M \to \overline{M}_R$ is injective.

We see that the map

$$\varphi_r := \sum_i \varphi_i|_{F^i \text{O}_{\text{cris}}(R) \otimes_{\lambda_2,R} \varphi_{r-i}|_{F^{r-i} M} : F^r \overline{M}_R \to \overline{M}_R$$

is well-defined. $\overline{M}_R$ has a natural $G_R$-action defined by the diagonal action where $G_R$ acts on $\text{O}_{\text{cris}}(R)$ as before and trivially on $M$. We also have an induced connection defined by $\nabla(a \otimes m) = \nabla \text{O}_{\text{cris}}(R)(a) \otimes m + a \otimes \nabla_M(m)$. Let

$$T_{\text{cris}}(M) := (F^r \overline{M}_R)^{\varphi_r = 1, \nabla = 0}(-r) = \{x \in F^r \overline{M}_R \mid \varphi_r(x) = x, \nabla(x) = 0\}(-r).$$

Note that $T_{\text{cris}}(M)$ is a $\mathbb{Z}_p[G_R]$-module.

**Remark 2.3.2.** The Tate twist $(-r)$ above assures that $T_{\text{cris}}$ independent of the choice of $r \leq p-2$ (see Remark 2.3.6).

We now give an alternate construction of $T_{\text{cris}}(M)$, which will turn out to be more convenient for comparison under base change via the morphism $b$. Let $\overline{M} := \text{A}_{\text{cris}}(R) \otimes_{\lambda_1,R} M$ and $F^\bullet \overline{M} := \bigoplus_{i=0}^r F^i \text{A}_{\text{cris}}(R) \otimes_{\lambda_1,R} F^{r-i} M \subset \overline{M}$. Define $\varphi_r : F^r \overline{M} \to \overline{M}$ similarly as above. The natural projection $\pi : \text{O}_{\text{cris}}(R) \to \text{A}_{\text{cris}}(R)$ given by $X_i \mapsto 0$ (see Remark 2.1.3) induces a map $\pi_M : \overline{M}_R \to \overline{M}$. It is easy to check that $\pi_M$ is compatible with $\varphi_r$ and maps $F^r \overline{M}_R$ to $F^r \overline{M}$.

**Lemma 2.3.3.** The morphism $\pi_M$ induces an isomorphism $T_{\text{cris}}(M) \simeq (F^r \overline{M})^{\varphi_r = 1}$ of $\mathbb{Z}_p$-modules.

**Proof.** Consider $\overline{M}_R^{\nabla = 0} := \{x \in \overline{M}_R \mid \nabla(x) = 0\}$, which is an $\text{A}_{\text{cris}}(R)$-module. Define $F^r \overline{M}_R^{\nabla = 0} := F^r \overline{M}_R \cap \overline{M}_R^{\nabla = 0}$. Note that $\varphi_r$ maps $F^r \overline{M}_R^{\nabla = 0}$ to $\overline{M}_R^{\nabla = 0}$ since $\varphi_r$'s are parallel. It suffices to prove $\overline{M}_R^{\nabla = 0} \subset \overline{M}_R$ is a section of $\pi_M$ so that $\pi_M : \overline{M}_R^{\nabla = 0} \to \overline{M}$ is an $\text{A}_{\text{cris}}(R)$-linear isomorphism which induces an isomorphism on $F^r$ and compatible with $\varphi_r$. To show this, for any $x \in M \subset \overline{M}_R$, set

$$(3) \hat{x} := \sum_I \gamma_I(-X) \nabla(\hat{\theta})^I(x).$$

Here $I = (i_1, \ldots, i_d)$ with $i_k \geq 0$ is a multi-index, and $\gamma_I(-X) = (-1)^{i_1 + \cdots + i_d} X_1^{i_1} \cdots X_d^{i_d}$ with $X_i := \overline{b_i}$ as in Lemma 2.1.2 and $\nabla(\hat{\theta})^I = \overline{\Theta}_1^{i_1} \cdots \Theta_d^{i_d}$ with $\overline{\Theta}_i = \nabla(\frac{\hat{b}_i}{\epsilon i}) : M \to M$ (note
that $\Theta_i$ and $\Theta_j$ commute as $\nabla$ is integrable). Using $\Theta_j(\gamma_l(-X)) = -\gamma_l(-X)$ where $I' = (i_1, \ldots, i_j - 1, \ldots, i_d)$, we see $\hat{x} \in \overline{M}_{\overline{M}}^{\nabla = 0}$. Let $M^\nabla := \{ \hat{x} \mid x \in M \} \subset \overline{M}_{\overline{M}}^{\nabla = 0}$. Clearly $\pi_M(M^\nabla) = M \subset \overline{M}$. It is easy to check that for any $a \in R$ and $x \in M \subset \overline{M}_R$, we have $\hat{a}\hat{x} = \pi(a)\hat{x} \in \overline{M}_R$ where $\pi(a) = \lambda_1(a) \in A_{\text{cris}}(R)$. Therefore, $M^\nabla$ is an $R$-module via $\lambda_1 : R \to A_{\text{cris}}(R)$, and $\pi_M : M^\nabla \to M$ is an isomorphism of $R$-modules. Consequently the composite of the maps $A_{\text{cris}}(R) \otimes_{\lambda_1, R} M^\nabla \to \overline{M}_{R}^{\nabla = 0} \subset \overline{M}_R$ is an isomorphism of $A_{\text{cris}}(R)$-modules. In particular, $A_{\text{cris}}(R) \otimes_{\lambda_1} M^\nabla \to \overline{M}_R^{\nabla = 0}$ is injective.

We check that $A_{\text{cris}}(R) \otimes_{\lambda_1} M^\nabla = \overline{M}_R^{\nabla = 0}$ as follows. Write $M \simeq \bigoplus_i R/p^{n_i}R \cdot e_i$ with $e_i \in M$. We then have $\overline{M}_R \simeq \bigoplus_i \mathcal{O}A_{\text{cris}, n_i}(R) \cdot \hat{e}_i \simeq \mathcal{O}A_{\text{cris}}(R) \otimes_{\lambda_1, R} M^\nabla$. Taking $\nabla = 0$ on both sides gives $\overline{M}_R^{\nabla = 0} = A_{\text{cris}}(R) \otimes_{\lambda_1} M^\nabla$, which shows that $\overline{M}_R^{\nabla = 0}$ is an $A_{\text{cris}}(R)$-linear section of $\pi_M$.

Since $\pi_M(F^i\overline{M}_R) \subset F^i\overline{M}$, we have $\pi_M(F^i\overline{M}_R^{\nabla = 0}) \subset F^i\overline{M}$. To prove that $\pi_M(F^i\overline{M}_R^{\nabla = 0}) = F^i\overline{M}$, it suffices to check if $x \in F^i\overline{M}$ then $\hat{x} \in F^i\overline{M}_R$. This easily follows from Griffiths transversality and $\gamma_l(-X) \in F^i\mathcal{O}A_{\text{cris}}(R)$. Finally, $\pi_M : M^\nabla \to \overline{M}$ is compatible with $\varphi_i$ because $\pi_M : \overline{M}_R \to \overline{M}$ is. This completes the proof. □

The previous lemma allows us to give an alternate description of $T_{\text{cris}}(M)$ as follows. First, we define a $G_R$-action on $\overline{M} = A_{\text{cris}}(R) \otimes_R M$ via the isomorphism $\pi_M : \overline{M}_R^{\nabla = 0} \to \overline{M}$. This $G_R$-action on $\overline{M}$ can be described explicitly as follows. For any $g \in G_R$, let $\beta_i(g) := g([1 + t_i]) - [1 + t_i] \in F^1A_{\text{cris}}(R)$ and set

$$g(a \otimes x) = \sum_i g(a) \gamma_l(\beta(g)) \otimes \nabla(\hat{e})^i(x), \forall a \in A_{\text{cris}}(R), \forall x \in M.$$  

Here $\gamma_l(\beta(g)) := \prod_i \beta_i(g)^{x_i}$. Since $g(X_i) = -\beta_i(g) + X_i$, we see from formula (3) that $\pi_M(g(\hat{x})) = \sum_i \gamma_l(\beta(g)) \nabla(\hat{e})^i(x)$ for any $x \in M$ as required.

Recall from (2.3.3) that $z_{i,n}$ is a fixed $p^n$-th root of $1 + t_i$, and is used to define $\hat{1 + t_i}$. Let $R_x = \bigcup_{n \geq 1} R[z_{i,n}, \ldots, z_{d,n}]$, which is a subextension in $\overline{R}$, and $G_x := \text{Gal}(\overline{R}_x[\frac{1}{p[1/R_x[\frac{1}{p}]]}] \subset G_R$. By the proof of Lemma (2.3.3) the $G_R$-action defined on $\overline{M} = A_{\text{cris}}(R) \otimes_R M$ via $\overline{M}_R^{\nabla = 0}$ satisfies the following properties:

- The equation (4) gives a well-defined $A_{\text{cris}}(R)$-semilinear $G_R$-action, that is, for $m \in A_{\text{cris}}(R) \otimes_R M$ and $a \in A_{\text{cris}}(R)$ we have $g(am) = g(a)g(m)$. Moreover, $M \subset (A_{\text{cris}}(R) \otimes_R M)^{G_x}$ as $\beta_i(g) = 0$ for $g \in G_x$;
- The $G_R$-action preserves filtration, i.e., $g(F^i(A_{\text{cris}}(R) \otimes_R M)) \subset F^i(A_{\text{cris}}(R) \otimes_R M)$, and commutes with $\varphi_i$.

Combining everything gives the following alternate description of $T_{\text{cris}}(M)$:

$$T_{\text{cris}}(M) = (F^r(A_{\text{cris}}(R) \otimes_R M))^{\varphi_r=1}(-r) = \{ x \in F^r(A_{\text{cris}}(R) \otimes_R M) \mid \varphi_r(x) = x \}(-r).$$

An object $M \in \text{MF}(R)$ and a $\mathbb{Z}_p$-module $L$ are said to have the same type if $M \cong L \otimes_{\mathbb{Z}_p} R$ as $R$-modules.
Theorem 2.3.1. Let $M \in MF_{\mathcal{V}}^{[0,r]}(R)$. Then $T_{\text{cris}}(M)$ is a finite $\mathbb{Z}_p$-module of the same type as $M$, and the map $\varphi_r - 1 : F^r(A_{\text{cris}}(R) \otimes R M) \to A_{\text{cris}}(R) \otimes_R M$ is surjective. Furthermore, the functor $T_{\text{cris}}(\cdot)$ is covariant. Let $M_1 := M/pM$ with the induced structure as a Fontaine-Laffaille module. Consider the following diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & T_{\text{cris}}(pM)(r) & \longrightarrow & T_{\text{cris}}(M)(r) & \longrightarrow & T_{\text{cris}}(M_1)(r) & \longrightarrow & 0 \\
& & ↓ & & ↓ & & ↓ & & ↓ \\
0 & \longrightarrow & F^r(A_{\text{cris}}(R) \otimes_R pM) & \longrightarrow & F^r(A_{\text{cris}}(R) \otimes_R M) & \longrightarrow & F^r(A_{\text{cris}}(R) \otimes_R M_1) & \longrightarrow & 0 \\
& & \phi_r-1 & & \phi_r-1 & & \phi_r-1 & & \phi_r-1 \\
0 & \longrightarrow & A_{\text{cris}}(R) \otimes_R pM & \longrightarrow & A_{\text{cris}}(R) \otimes_R M & \longrightarrow & A_{\text{cris}}(R) \otimes_R M_1 & \longrightarrow & 0 \\
\end{array}
\]

By Theorem 2.2.1 and Remark 2.2.3 the middle and bottom rows are exact. By devisage, in order to prove that $\varphi_r - 1$ is surjective, it suffices to consider the case $pM = 0$. Furthermore, assuming $\varphi_r - 1$ is surjective, we can deduce by the snake lemma that the top row is exact. Therefore, to show that $T_{\text{cris}}(M)$ is a finite $\mathbb{Z}_p$-module of the same type as $M$, it also suffices to consider the case $pM = 0$ by Theorem 2.2.1.

Suppose now that $pM = 0$. By Lemma 2.1.1

\[A_{\text{cris}}(R)/pA_{\text{cris}}(R) \simeq \overline{R}/p\overline{R} \left[\bar{\delta}_0, \bar{\delta}_1, \ldots]/(\bar{\delta}_0^p, \bar{\delta}_1^p, \ldots),\right]
\]

where $\bar{\delta}_i := \delta_i \mod p$. We then have $(F^r(M \otimes_R A_{\text{cris}}(R)))^{\varphi_r=1} = (F^r(M \otimes_R \overline{R}/p\overline{R}))^{\varphi_r=1}$.

We equip $\overline{R}/p\overline{R}$ with an $R/pR$-module structure via

\[R/pR \to R_\infty/pR_\infty \stackrel{\varphi^{-1}}{\longrightarrow} R_\infty/pR_\infty \to \overline{R}/p\overline{R},\]

and a filtration by setting $F^i(\overline{R}/p\overline{R}) := p^i\overline{R}/p\overline{R}$ (recall $p_1$ is chosen so that $p_1^p = p$). Define $\varphi_i$ on $F^i(\overline{R}/p\overline{R})$ by $\varphi_i(x) = (-1)^iy^p$ where $x = p^iy$. This gives $\overline{R}/p\overline{R}$ a structure of an object in MF_{big}(R). It is easy to check that $\theta \circ \varphi^{-1} : \overline{R}/p\overline{R} \to \overline{R}/p\overline{R}$ induces an isomorphism which is compatible with all structures. Hence, denoting $R_1 = R/pR$ and $\overline{R}_1 = \overline{R}/p\overline{R}$, it suffices to show that $\dim_{R_1}(F^r(M \otimes_R \overline{R}_1))^{\varphi_r=1} = \text{rank}_{R_1}M$ and that $\varphi_r - 1 : F^r(M \otimes_R \overline{R}_1) \to M \otimes_R \overline{R}_1$ is surjective.

Fix an adapted $R_1$-basis $e_1, \ldots, e_m$ of $M$ and integers $n_1, \ldots, n_m$ such that $n_i \leq r$ and $\{p_1^{n_i}e_i, \ i = 1, \ldots, m\}$ form a basis for $F^r(M \otimes_R \overline{R}_1)$. We have $\varphi_r(p_1^{n_1}e_1, \ldots, p_1^{n_m}e_m) = (e_1, \ldots, e_m)A$ for some $A \in \text{GL}_m(R_1)$ (note that $\varphi_r(p_1^i) = (-1)^i$). In particular, $x = \sum_i x_ip_1^{n_i}e_i \in F^r(M \otimes_R \overline{R}_1)^{\varphi_r=1}$ if and only if $X = (x_1, \ldots, x_m)^T$ satisfies the equation $AX^p = AX$, where $X^p := (x_1^p, \ldots, x_m^p)^T$ and $A := \langle p_1^{n_1}, \ldots, p_1^{n_m} \rangle$ denotes the diagonal matrix with $p_1^{n_1}, \ldots, p_1^{n_m}$ on the main diagonal.

Choose a lift of $A$ to an element in $\text{GL}_m(R)$, which we also denote by $A$ abusing the notation. Consider the equation $AX^p = AX$ defined over $R[p_1]$. We claim that the set of
solutions for $AX^p = \Lambda X$ corresponds bijectively to that of $AX^p \equiv \Lambda X \mod p$. Indeed, if $X_0$ is a solution of $AX^p \equiv \Lambda X \mod p$, then we can find $Y$ such that $X_0 + p^2 Y$ satisfies the equation $AX^p \equiv \Lambda X \mod p^2$ (since $n_i \leq p - 2$). We can construct inductively a compatible system of solutions to $AX^p \equiv \Lambda X \mod p^2$ and therefore obtain a solution $X$ of $AX^p = \Lambda X$. Furthermore, we also deduce that such a lift $X$ of $X_0$ is unique by considering the $p$-adic valuation.

Now let $B = R[p^i][x_1, \ldots, x_m]$/(AX$^p - \Lambda X$). The Jacobian of $AX^p - \Lambda A$ is $pA(x_1^{p-1}, \ldots, x_m^{p-1}) - \Lambda$, which can be written as $-\Lambda(I_n - p\Lambda^{-1}A')$. As $p\Lambda^{-1}A'$ has entries in $p^2 R[p^i][x_1, \ldots, x_m]$, $I_n - p\Lambda^{-1}A'$ is an invertible matrix in $\text{GL}_n(R[p^i][x_1, \ldots, x_m])$. It follows that the determinant of the Jacobian is a unit in $B$. Hence, $B$ is étale over $R[p^i]$ and the solutions $X$ for $AX^p = \Lambda X$ lie inside $\overline{R}$. Since the set of solutions of $AX^p = \Lambda X$ has size $p^m$, we have $\dim_{R_p}(F^r(M \otimes_R \overline{R}))^{\varphi_r = 1} = \text{rank}_{R_p} M$.

To show that $\varphi_r - 1$ is surjective, we apply a similar argument as above to the equation $AX^p = \Lambda X + Y$ with $Y$ having fixed entries in $\overline{R}$. Choose a finite $R$-algebra $S$ containing elements in $Y$ such that $S[\frac{1}{p}]$ is étale. Then $S[p^i][\frac{1}{p}]$ is surjective and the Jacobian of the sequence

\[0 \to M_1 \to M_2 \to M_3 \to 0 \text{ in } \text{MF}^{[0,r]}(R),\]

is exact for each $i$ and $j$. Hence, by Theorem 2.2.1 and Remark 2.2.3, the sequence

\[0 \to F^i A_{\text{crys}}(R) \otimes_R M_1 \to F^i A_{\text{crys}}(R) \otimes_R M_2 \to F^i A_{\text{crys}}(R) \otimes_R M_3 \to 0\]

is exact. Since $\varphi_r - 1$ is surjective, this implies that

\[0 \to T_{\text{crys}}(M_1) \to T_{\text{crys}}(M_2) \to T_{\text{crys}}(M_3) \to 0.\]

is exact.\hfill \square

Remark 2.3.4. One can also show that $T_{\text{crys}}(\cdot)$ is fully faithful and stable under subquotients.

Let $M \in \text{MF}^{[0,r]}(R)$ and recall that $M_g := M \otimes_{R,R_g} R_g$ is an object in $\text{MF}^{[0,r]}(R_g)$. We have a natural $G_{R_g}$-equivariant morphism $T_{\text{crys}}(M) \to T_{\text{crys}}(M_g)$.

Corollary 2.3.5. The natural map $T_{\text{crys}}(M) \to T_{\text{crys}}(M_g)$ is an isomorphism of $G_{R_g}$-representations.

Proof. By Theorem 2.3.1, both $T_{\text{crys}}(M)$ and $T_{\text{crys}}(M_g)$ have the same type as $M$. Therefore, it suffices to show that the natural map $T_{\text{crys}}(M) \to T_{\text{crys}}(M_g)$ is injective. Since $T_{\text{crys}}(\cdot)$ is exact, we may assume without loss of generality that $pM = 0$. In this case, as in the proof of Theorem 2.3.1, $T_{\text{crys}}(M)(r)$ can be identified with the set $\{X \in R^n | AX^p = \Lambda X\}$. Similarly, $T_{\text{crys}}(M_g)(r)$ can be identified with the set $\{X \in R^n_g | AX^p = \Lambda X\}$. Since the map $\overline{g} : R \to \overline{R_g}$ is injective, we conclude that $T_{\text{crys}}(M) \to T_{\text{crys}}(M_g)$ is injective.\hfill \square
Remark 2.3.6. The construction $T_{\text{cris}}(M)$ does not depend on the choice of $r$. That is, if $r \leq s \leq p - 2$, we have

$$(F^r(A_{\text{cris}}(R) \otimes_R M))^{\varphi_r = 1}(-r) \ cong (F^s(A_{\text{cris}}(R) \otimes_R M))^{\varphi_s = 1}(-s).$$

To see this, note that $\beta(x)^{s-r} \cdot (F^r(A_{\text{cris}}(R) \otimes_R M))^{\varphi_r = 1} \ cong (F^s(A_{\text{cris}}(R) \otimes_R M))^{\varphi_s = 1}$. Since both sides have the same type as $M$ by Theorem 2.3.1, $\beta(x)^{s-r} \cdot (F^r(A_{\text{cris}}(R) \otimes_R M))^{\varphi_r = 1} = (F^s(A_{\text{cris}}(R) \otimes_R M))^{\varphi_s = 1}$ as $\mathbb{Z}_p$-modules. As $\mathbb{Z}_p[G_R]$-modules, we have $\beta(x)^{s-r} \cdot \mathbb{Z}_p \ cong \mathbb{Z}_p(s - r)$.

3. Fontaine-Laffaille data and torsion crystalline cohomology

Let $X$ be a smooth proper scheme over $R = W(k)[[t_1, \ldots, t_d]]$, and for a fixed integer $n \geq 1$, let $X_n$ denote its base change over $R_n = R/p^nR$. The ideal $(p) \subset R$ has a canonical divided power structure, and we consider $R_n$ with the corresponding quotient divided power structure. Below, we let $M^i := H^i_{\text{cris}}(X_n/R_n, \mathcal{O}_{X_n/R_n})$ and $F^j(M^i) := H^i_{\text{cris}}(X_n/R_n, \omega_{X_n/R_n}^{[j]})$ denote the corresponding crystalline cohomology groups. One of our main results is to show that $(M^i, F^j(M^i))$ for $i \leq p - 2$ gives rise to an object of $\text{MF}^{[0,p-2]}(R)$. More precisely, we have the following theorem.

Theorem 3.0.1. Let $X$ be a smooth proper scheme over $R$.

1. For $i \leq p - 1$, there exist $\varphi_R$-semilinear maps $\varphi_j : F^j(M^i) \rightarrow M^i$ and a connection $\nabla : M^i \rightarrow M^i \otimes_R \hat{\Omega}_R$ such that the data $(M^i, F^j(M^i), \varphi_j, \nabla)$ is an object of $\text{MF}^{0,i}_R(R)$.

2. Let $k \subset k'$ be a field extension where $k'$ is perfect, $R' := W(k')$ equipped with the Witt vector Frobenius, and suppose we have a morphism $R \rightarrow R'$ of divided power rings compatible with Frobenius. Let $X' := X \times_R R'$. Then the natural morphism (induced by functoriality)

$$H^i_{\text{cris}}(X_n/R_n, \mathcal{O}_{X_n/R_n}) \rightarrow H^i_{\text{cris}}(X'_n/R'_n, \mathcal{O}_{X'_n/R'_n})$$

is compatible with filtration and $\varphi_j$.

3. For $i \leq p - 2$, the data $(M^i, F^j(M^i), \varphi_j)$ is an object of $\text{MF}^{[0,i]}(R)$. Hence, $(M^i, F^j(M^i), \varphi_j, \nabla)$ is an object of $\text{MF}^{[0,i]}_R(R)$.

We shall prove Theorem 3.0.1 (1) (2) in section 3.4. In this section, we prove part (3) of the theorem assuming parts (1) and (2). We first study the case $d = 1$ in the following subsection, and then consider the general case via inductive argument.

3.1. The case $d = 1$. In this subsection, we study the special case $R = W(k)[[t_1]]$. We begin with some preliminary remarks. Let $\tilde{b} : R \rightarrow W(k)$ denote the natural quotient morphism and $X_{s} := X \times_{R_1} W(k)$. By the base change theorem for crystalline cohomology (cf. Theorem 3.0.1 and Example 3.0.2), $\tilde{b}$ induces a natural injective map:
Proof of Theorem 3.0.1 (3) when $d = 5$.

Remark

Theorem A.0.1 implies that the induced map

$$\bar{\alpha} : F^j(M^i)/t_1 F^j(M^i) \hookrightarrow F^j(M^i) = H^i_{\text{cris}}(X_{s,n}/W_n(k), J^j_{X_{s,n}/W_n(k)}).$$

Proposition 3.1.1. With notations as above, $\bar{\alpha}$ is an isomorphism for all $0 \leq i \leq p - 2$.

Before proving the proposition, we explain how to deduce Theorem 3.0.1 (3) assuming the proposition. Let $b_g : R \to R_g$ be as before, and let $X_g := X \times_R R_g$. Since $b_g$ is flat, Theorem [A.0.1] implies that the induced map

$$\alpha_g : F^j(M^i) \otimes_{R_n} R_{g,n} \to F^j(M^i) := H^i_{\text{cris}}(X_{g,n}/R_{g,n}, J^j_{X_{g,n}/R_{g,n}})$$

is an isomorphism.

Remark 3.1.2. The maps $\alpha_g$ and $\bar{\alpha}$ are compatible with $\varphi_j$ by Theorem 3.0.1 (2).

Proof of Theorem 3.0.1 (3) when $d = 1$. By Proposition 2.2.5 and Remark 2.2.6, it suffices to prove that $M^i \otimes_R R_g$ and $M^i \otimes_R W(k)$ with the induced $\varphi_j$ and filtrations are objects in MF($R_g$) and MF($W(k)$) respectively, and that they have the same type. Since $\alpha_g$ is an isomorphism, it follows from Theorem 3.0.1 (1), (2) that $M^i \otimes_R R_g \cong H^i_{\text{cris}}(X_{g,n}/R_{g,n})$ as objects of MF($R_g$). On the other hand, the latter object is in MF($R_g$) by classical Fontaine-Messing theory (over $R_g$). Similarly, by Proposition 3.1.1, $M^i \otimes_R W(k)$ is in MF($W(k)$). On the other hand, $M^i \otimes_R W(k)$ (resp. $M^i \otimes_R R_g$) has the same type with $H^i_{\text{ét}}(X_{s,R}^\flat, \mathbb{Z}/p^n\mathbb{Z})$ (resp. $H^i_{\text{ét}}(X_{g,R_g}^\flat, \mathbb{Z}/p^n\mathbb{Z})$) again by the classical Fontaine-Messing theory. Since $H^i_{\text{ét}}(X_{s,R}^\flat, \mathbb{Z}/p^n\mathbb{Z}) \cong H^i_{\text{ét}}(X_{g,R_g}^\flat, \mathbb{Z}/p^n\mathbb{Z})$ as $\mathbb{Z}_p$-modules by smooth and proper base change theorems, we conclude that $M^i \otimes_R R_g$ and $M^i \otimes_R W(k)$ have the same type.

To prove Proposition 3.1.1 we need the following lemma. For a ring $A$ and a finite length $A$-module $N$, we denote by $\ell_A(N)$ the $A$-length of $N$.

Lemma 3.1.3. For any finite $R_n$-module $M$, we have

$$\ell_{W(k)}(M \otimes_{R_n} W(k)) \geq \ell_{W(k_g)}(M \otimes_{R_n} W(k_g)).$$

Proof. We shall proceed via induction on the minimal integer $h$ such that $p^h M[t_1] = 0$. If $h = 0$, then $M \otimes_R W(k_g) = 0$ and the desired inequality is trivial. If $h = 1$, then $M[t_1] \cong (M/pM)[\frac{1}{t_1}]$. Since $M/pM$ is a finite $R_1 = k[[t_1]]$-module, we have $M/pM \cong N_{\text{tor}} \oplus N_{\text{free}}$ where $N_{\text{tor}}$ is killed by a power of $t_1$ and $N_{\text{free}}$ is a free $k[[t_1]]$-module. Thus,

$$\ell_{W(k)}(M \otimes_R W(k)) \geq \ell_{W(k)}((M/pM) \otimes_R W(k)) \geq \ell_{W(k)}(N_{\text{free}} \otimes_R W(k)) = \ell_{W(k_g)}(M \otimes_R W(k_g)).$$

Here the last equality follows from $M \otimes_R W(k_g) \cong (M/pM) \otimes_R W(k_g) \cong N_{\text{free}} \otimes_R W(k_g)$.

For $h \geq 2$, consider the map $f : M \to (M/pM)[\frac{1}{t_1}]$ and the associated exact sequence

$$0 \to \ker(f) \to M \to f(M) \to 0.$$ 

Note that $\ker(f)[\frac{1}{t_1}]$ is killed by $p^{h-1}$, and $f(M)$ has no $t_1$-torsion. It follows that the following sequence

$$0 \to \ker(f) \otimes_R W(k) \to M \otimes_R W(k) \to f(M) \otimes_R W(k) \to 0.$$
is exact. By the inductive hypothesis, we have 
\[ \ell_{W(k)}(M \otimes_R W(k)) = \ell_{W(k)}(\ker(f) \otimes_R W(k)) + \ell_{W(k)}(f(M) \otimes_R W(k)) \]
\[ \geq \ell_{W(k)}(\ker(f) \otimes_R W(k_g)) + \ell_{W(k)}(f(M) \otimes_R W(k_g)) \]
\[ = \ell_{W(k)}(M \otimes_R W(k_g)). \]

Proof of Proposition 3.1.1. By crystalline base change theorem (cf. Example 3.1.2), \( \alpha \) is an isomorphism if and only if \( F^j(M^{i+1}) \) is \( \ell^1 \)-torsion-free. We first treat the case \( j = 0 \). Since \( \alpha \) is injective, Lemma 3.1.3 implies that 
\[ \ell_{W(k)}(M^j_g) \geq \ell_{W(k)}(M^j/\ell_1 M^j) \geq \ell_{W(k)}(M^j_g). \]

On the other hand, by Theorem 3.1.1 for \( X \) base changed to \( W(k) \) and \( R_g \), we have for \( 0 \leq i \leq p - 2 \) that 
\[ \ell_{W(k)}(M^i_j) = \ell_{Z_p}(H^i_{\text{ét}}(X_{s,R_w}, \mathbb{Z}/p^n \mathbb{Z})) = \ell_{Z_p}(H^i_{\text{ét}}(X_{g,R_w}^{\text{an}}[1/p], \mathbb{Z}/p^n \mathbb{Z})) = \ell_{W(k)}(M^i_g). \]

This forces \( \ell_{W(k)}(M^j_i) = \ell_{W(k)}(M^j/\ell_1 M^j) \) and therefore \( \alpha \) is bijective.

Now consider the case \( j > 0 \). We claim that \( F^j(M^{i+1}) \) is \( \ell^1 \)-torsion-free for \( 0 \leq i \leq p - 2 \). Suppose that \( F^j(M^{i+1}) \) has a non-trivial \( \ell_1 \)-torsion. In that case, since \( M^{i+1} \) is \( \ell_1 \)-torsion-free, the map \( F^j(M^{i+1}) \to M^{i+1} \) is not injective. Then the natural morphism \( F^j(M^{i+1})/\ell_1 F^j(M^{i+1}) \to M^{i+1}/\ell_1 M^{i+1} \) is also not injective, since \( M^{i+1} \) is \( \ell_1 \)-torsion-free and \( F^j(M^{i+1}) \) is \( \ell_1 \)-adically separated. Consider the following commutative diagram:

\[ \begin{array}{ccc}
F^j(M^{i+1})/\ell_1 F^j(M^{i+1}) \ar[r] & F^j(M^{i+1}) \\
M^{i+1}/\ell_1 M^{i+1} \ar[r] & M^{i+1} \\
\end{array} \]

Since \( (M^l, F^l(M^l), \varphi_j) \), for \( 0 \leq l \leq p - 1 \), is a Fontaine-Laffaille module over \( W(k) \) by [FMS7] Cor. 2.7, the right column is injective. But this contradicts that the left column is not injective, which proves the claim.

Remark 3.1.4. The previous proof uses that \( (M^p_{p-1}, F^j(M^p_{p-1}), \varphi_j) \) is a Fontaine-Laffaille module. It is not known whether \( T_{\text{cris}}(M^p_{p-1}) \) is isomorphic to \( H^p_{\text{ét}}(X_{s,R}, \mathbb{Z}/p^n \mathbb{Z}) \), which we do not need here.

Remark 3.1.5. The proof of Proposition 3.1.1 does not use the \( \varphi_j \)-structure on \( F^j(M^i) \). In particular, the proof does not use the \( \varphi \)-structure on \( R \). This is important for the proof of Proposition 3.2.1 which proceeds via induction on \( d \) and base changing along maps \( R \to W(k_g)[[s_1, \ldots, s_m]] \) with \( m \leq d - 2 \) to reduce to the case \( d = 1 \). In the constructions below, those base change maps may not be compatible with Frobenius. But we can still do the induction since Proposition 3.1.1 does not use the \( \varphi \) structure on \( R \).
3.2. The general case. Now we consider the general case \( d \geq 1 \). Consider the base change map \( R \to R_d := R/t_d R = W(k)[t_1, \ldots , t_{d-1}] \), and let \( X_d = X \times_R R_d \). For each \( j \geq 0 \), we have a long exact sequence

\[
\cdots \to H^i_{\text{cris}}(X_n/R_n, J^j_{X_n/R_n}) \longrightarrow H^i_{\text{cris}}(X_{d,n}/R_{d,n}, J^j_{X_{d,n}/R_{d,n}}) \longrightarrow H^{i+1}_{\text{cris}}(X_n/R_n, J^j_{X_n/R_n}) \longrightarrow \cdots
\]

Let \( F^j(M^i) \) and \( F^j(M^i_d) \) denote \( H^i_{\text{cris}}(X_n/R_n, J^j_{X_n/R_n}) \) and \( H^i_{\text{cris}}(X_{d,n}/R_{d,n}, J^j_{X_{d,n}/R_{d,n}}) \) respectively. The following analog of Proposition 3.1.1 is the main result of this section.

**Proposition 3.2.1.** For \( 0 \leq i \leq p-2 \), \( F^j(M^i)/t_d F^j(M^i) \cong F^j(M^i_d) \); or equivalently, \( F^j(M^{i+1}) \) is \( t_d \)-torsion-free.

Given the previous proposition, the proof of Theorem 3.0.1 (3) (for any \( d \geq 1 \)) proceeds exactly as in the case \( d = 1 \), which will not be repeated here. To simplify notation, we only treat the case \( j = 0 \) of Proposition 3.2.1 since the proof for \( j > 0 \) is exactly the same. Note that if Proposition 3.2.1 holds for \( t_d \), then the analogous statement with \( t_d \) replaced by any of the \( t_i \)'s will also hold. We shall proceed via induction on \( d \), with the base case \( d = 1 \) proved in the previous subsection. Assume \( d \geq 2 \). We begin with some preliminaries.

The long exact sequence (7) for \( j = 0 \) gives an injective map \( \bar{\alpha} : M^i/t_d M^i \hookrightarrow M^i_d \), and we have

\[
\text{coker}(\bar{\alpha}) \cong N := \{ x \in M^{i+1} \mid t_d x = 0 \}.
\]

Note that \( N \) is an \( R_d \)-module. Let \( U \) be the \( p \)-adic completion of \( R_d[1/t_d] \). The following lemma will be a key input in the proof of the previous proposition.

**Lemma 3.2.2.** With notations as above, \( U \otimes_{R_d} N = 0 \).

Before proving the lemma, we explain how to prove the proposition assuming the lemma. Let \( N' := \{ x \in M^{i+1} \mid t_d^m x = 0 \text{ for some } m \} \). Then \( N \subset N' \subset M^{i+1} \). Furthermore, the map \( N'/t_d N' \to M^{i+1}/t_d M^{i+1} \subset M^{i+1}_d \) is injective.

**Proof of Proposition 3.2.1.** We give a proof of the proposition for the case \( j = 0 \). The argument for the case \( j > 0 \) is the same. We need to show that \( N = 0 \). Suppose \( N \neq 0 \). Since \( N \) is \( p \)-power torsion and \( U \otimes_{R_d} N = 0 \), we have \( t_1^s N = 0 \) for some integer \( s \geq 1 \). On the other hand, since \( N \subset N' \), we have \( N'/N \neq 0 \) and thus \( N'/t_d N' \neq 0 \) by Nakayama’s lemma. For any \( x \in N' \), define \( o(x) = \min \{ m \mid t_d^m x = 0 \} \). Let \( x_0 \in N' \setminus t_d N' \) such that \( o(x_0) = \min \{ o(x) \mid x \in N' \setminus t_d N' \} \), and let \( m_0 = o(x_0) \). Then \( t_d^{m_0-1} x_0 \) is a non-zero element in \( N \). Since \( t_1^s t_d^{m_0-1} x_0 = 0 \), we have \( t_1^s x_0 \in N' \). By the minimality of \( o(x_0) \), we have \( t_1^s x_0 \in t_d N' \). Then the image of \( x_0 \) in \( N'/t_d N' \subset M^{i+1}_d \) is a non-zero element killed by \( t_1^s \). This contradicts the inductive hypothesis that \( M^{i+1}_d = H^{i+1}_{\text{cris}}(X_{d,n}/R_{d,n}, \mathcal{O}_{X_{d,n}/R_{d,n}}) \) is \( t_1 \)-torsion-free. \( \square \)

To prove the lemma, we will apply base change maps \( b : R \to A \) to the exact sequence (7) for various rings \( A \) as follows. Abusing the notation, we will denote \( (X \times_R A)/A \) by

Lemma 3.2.3. \( \alpha \)

Following lemma in the proof of Lemma 3.2.2.

\[ \text{Lemma 3.2.3.} \]

(1) \( \hat{U}_q \cong O_q[[s_1, \ldots, s_m]] \) where \( O_q \) is a Cohen ring whose maximal ideal is \( (p) \) and \( m \leq d - 2 \). There exists a Frobenius endomorphism \( \varphi \) on \( O_q \) lifting the \( p \)-power Frobenius on \( O_q/(p) \).

(2) Let \( k_q := \lim_{\varphi} O_q/(p) \). Then there exists a unique map \( f : O_q \to W(k_q) \) which lifts the map \( O_q \to k_q \) and is compatible with (given choices of) Frobenius. Furthermore, \( f \) is faithfully flat.

Proof. Note that \( \hat{U}_q \) is a complete regular local ring such that \( (p) \) is a prime ideal. Hence, by the Cohen structure theorem, we have \( \hat{U}_q \cong O_q[[s_1, \ldots, s_m]] \) for some Cohen ring \( O_q \).

Since the Krull dimension of \( \hat{U}_q \) is less than \( d - 1 \), we have \( m \leq d - 2 \). Moreover, \( R/pR \) has a \( p \)-basis given by \( \{t_1, \ldots, t_d\} \), which is also a \( p \)-basis of \( \hat{U}_q \). In particular, \( O_q \) has a finite \( p \)-basis, and thus there exists a lift of Frobenius endomorphism \( \varphi : O_q \to O_q \). This proves (1).

For (2), since \( O_q \) is \( p \)-torsion free, we obtain a unique lift \( f : O_q \to W(k_q) \) commuting with given choices of Frobenius by Cartier’s Dieudonné-Dwork lemma. Since \( W(k_q) \) is \( p \)-torsion-free and \( f \) is a local homomorphism, \( f \) is faithfully flat.

Proof of Lemma 3.2.2. Consider the base change of (7) along the morphism \( b : R \to U[t_d] \) given by sending \( t_i \) to \( t_i \) for each \( i \). Since the map \( b \) is flat, we obtain

\[ \alpha_b = U[t_d] \otimes_{b, R} \alpha : H^i_{\text{cris}}(X/U_n[t_d])/t_d \to H^i_{\text{cris}}(X/U_n). \]

It suffices to prove that \( \alpha_b \) is an isomorphism. To see this, note that the base change map is functorial and apply it to the cartesian square:

\[ \begin{array}{ccc} X_U & \to & X_{U[t_d]} \\ \downarrow & & \downarrow \\ X_{R_d} & \to & X \end{array} \]

For each maximal ideal \( q \) of \( U \), we consider the natural flat base change \( U[t_d] \to \hat{U}_q[t_d] \) applied to the exact sequence (7) to obtain \( \alpha_{b,q} = \hat{U}_q[t_d] \otimes_{U[t_d]} \alpha_b \). We need to show that \( \alpha_{b,q} \) is an isomorphism. Consider further a base change map \( \hat{U}_q \to V_q := W(k_q)[[s_1, \ldots, s_m]] \) obtained from \( f : O_q \to W(k_q) \) as in Lemma 3.2.3 applied to (7). Since \( \hat{U}_q \to V_q \) is faithfully flat, \( \alpha_{b,q} \) is an isomorphism if and only if

\[ V_q[t_d] \otimes_{U_q[t_d]} \alpha_{b,q} : H^i_{\text{cris}}(X/V_q[t_d])/t_d \to H^i_{\text{cris}}(X/V_q). \]

is an isomorphism. By the inductive hypothesis, the above map is indeed an isomorphism, which proves Lemma 3.2.2. \( \square \)
4. Construction of Relative Fontaine-Laffaille data and the comparison map

As before, let $\overline{X} := X \times_R \overline{R}$. The aim of this section is to construct a morphism $T_{\text{cris}}(H_{\text{cris}}(X_n/R_n)) \to H^i_{\text{ét}}(X_{\overline{R}[\frac{1}{p}]}, \mathbb{Z}/p^n\mathbb{Z})$ which is compatible with $G_R$-actions. Along the way, we also complete the proof of Theorem 3.0.1 (1) (2); we show in particular the existence of a Gauss-Manin connection $\nabla$ and $\varphi_i$ satisfying the conditions for a relative Fontaine-Laffaille module. Following Fontaine-Messing, we use syntomic cohomology $H^i_{\text{syn}}(\overline{X})$ to relate $A_{\text{cris}}(R) \otimes_R H^i_{\text{cris}}(X_n/R_n)$ and $H^i_{\text{ét}}(X_{\overline{R}[\frac{1}{p}]}, \mathbb{Z}/p^n\mathbb{Z})$. Unfortunately, the computation of syntomic(-crystalline) cohomology in the relative setting is complicated by the fact that our base ring $R$ is not perfect (as opposed to the classical case over $W(k)$). We circumvent this issue by systematically making base change from $R$ to an intermediate ring $\tilde{R}$ (to be defined below) and descending back. To save space, we sometimes abuse notation and denote by $A$ the affine scheme $\text{Spec}(A)$ if no confusion arises. For example, we sometimes denote $H^i_{\text{cris}}(\text{Spec}(A)/\text{Spec}(R_n))$ by $H^i_{\text{cris}}(A/R_n)$.

4.1. The ring $\tilde{R}_n$. Recall that $R_\infty := \bigcup_{k \geq 0} R[(1+t_1)^{(k)}, \ldots, (1+t_d)^{(k)}]$ where $(1+t_i)^{(k)}$ is a fixed choice of $p^k$-th root of $(1+t_i)$ and $(1+t_i)^{(k+1)} = (1+t_i)^{(k)}$. Its $p$-adic completion $\tilde{R}_\infty$ is a subring inside $\overline{R}$, and $\tilde{R}_\infty \subset \overline{R}^p$. We embed $R$ into $W(\tilde{R}_\infty)$ via $t_i \mapsto [1+t_i]_1 - 1$; this is compatible with the earlier choice of embedding $\lambda_1 : R \hookrightarrow W(\overline{R})$. Recall that for any $p$-adically complete and separated ring $A$, there is a functorial morphism $\theta : W(A^p) \to A$ as defined in [BMS18, Section 3.1]. In particular, we have a commutative diagram:

$$
\begin{array}{ccc}
R & \longrightarrow & W(\tilde{R}^p_\infty) \\
\downarrow \text{id} & & \downarrow \theta \\
R & \longrightarrow & \tilde{R}_\infty \\
\downarrow \text{id} & & \downarrow \theta \\
\overline{R} & \longrightarrow & \overline{R}
\end{array}
$$

To see that the middle column is an isomorphism, first note that the inclusion $R \subset \tilde{R}_\infty$ induces an isomorphism $\tilde{R}_1 \simeq \tilde{R}^p_\infty$ where $\tilde{R}_1$ denotes the perfection of $R_1$. Since $\tilde{R}_\infty/(p) \simeq \tilde{R}_1$, the middle column mod $p$ is an isomorphism, and hence the middle column is an isomorphism. We set $\tilde{R} := W(\tilde{R}_1) \simeq W(\tilde{R}^p_\infty)$.

Lemma 4.1.1. With notations as above, the inclusion $R \hookrightarrow \tilde{R}$ is faithfully flat.

Proof. We first show $\tilde{R}$ is flat over $R$. Note that $\tilde{R}/(p) \simeq \tilde{R}_1$ is flat over $R_1$. By Lemma 15.27.5 [Sta18, Tag 06LD], it suffices to show that $\text{Tor}^R_1(\tilde{R}, R/pR) = 0$, which is clear since $\tilde{R} = W(\tilde{R}_1)$ has no $p$-torsion. Moreover, for the maximal ideal $m$ of $R$, we have $m\tilde{R} \neq \tilde{R}$. Thus, $\tilde{R}$ is faithfully flat over $R$. \hfill $\square$

Given a ring $A$ of char $p$, we mean by its perfection the ring $\tilde{A} := \lim_{\varphi:A \to A} \varphi$. This is a perfect ring which is universal for maps into perfect rings.
Remark 4.1.2. From the commutative diagram above, we obtain embeddings $\hat{R} \hookrightarrow \bar{R}$ and $\hat{R} \hookrightarrow W(\bar{R})$.

4.2. Comparison of crystalline and Syntomic cohomology. In this subsection, we interpret $H^i_{\text{cris}}(X_n/R_n)$ and $H^i_{\text{cris}}(\overline{X}_n/R_n)$ with their filtrations as certain syntomic cohomology groups with filtrations. Given a scheme $Y$ over $S$, we denote by $\text{CRIS}(Y/S)_{\text{SYN}}$ (resp. $\text{SYN}(Y)$) the big crystalline-syntomic site (resp. big syntomic site). Let $(Y/S)_{\text{CRIS-SYN}}$ and $(Y)_{\text{SYN}}$ denote the corresponding topoi. Abusing notation, we will denote by $O_{Y/S}$ (resp. $J_{Y/S}$) the corresponding structure sheaf (resp. divided power ideals) in the crystalline-syntomic site. We refer to Appendix B for a recollection of the definitions and some general facts on the syntomic and crystalline-syntomic sites and topoi.

We consider the sites $\text{CRIS}(Y_n/S_n)_{\text{SYN}}$ in the following two settings:

1. We set $Y = X$ and $S_n = \text{Spec}(R_n)$,
2. We set $Y = \overline{X}$. In this case, the base $S_n$ is either $\text{Spec}(W_n(k))$, $\text{Spec}(\hat{R}_n)$, or $\text{Spec}(R_n)$.

There is a natural morphism of topoi $u_{Y_n/S_n} : (Y_n/S_n)_{\text{CRIS-SYN}} \to (Y_n/S_n)_{\text{SYN}}$; we set $u := u_{Y_n/S_n}$ to simplify the notation. We begin by showing that the higher direct images under $u$ of the sheaves $J_{Y_n/S_n}^{[r]}$ vanish in any of the settings stated above. In the case $Y = X$, $S = \text{Spec}(W(k))$ and $r = 0$, the result is proved in [FM87, Proposition II.1.3]. The proof below follows closely that of [Bau92, Proposition 1.17], which generalizes the result of Fontaine-Messing to coefficients in quasi-coherent crystals. We note that, while the sheaves $J_{Y_n/S_n}^{[r]}$ are not crystals, the vanishing of higher direct images still holds in this case (with essentially the same proof).

Theorem 4.2.1. With notation as above, $\mathbb{R}^i u_*(J_{Y_n/S_n}^{[r]}) = 0$ for all $i > 0$ and $r \geq 0$.

Proof. It suffices to show that the sheaf associated to the pre-sheaf

$$U \to F(U) := H^i((Y_n/S_n)_{\text{CRIS-SYN}}/\hat{U}, J_{Y_n/S_n}^{[r]}), \ i > 0$$

vanishes in the syntomic topology on $Y_n$ for all $i > 0$. Here $\hat{U}$ is the pull back of the sheaf represented by $U$. On the other hand, $F(U)$ can be identified with $H^i_{\text{cris}}(U/S_n, J_{U/S_n}^{[r]})$ (see Remark B.0.3 and Corollary B.0.13). Since this is a local statement, we may assume $U = \text{Spec}(A)$ where $\text{Spec}(A)$ is flat over $Y_n$. As in the proof of [Bau92, Proposition 1.17], it suffices to show there exists a family of faithfully flat syntomic morphisms $\{V_i \to U\}_{i \in I}$ such that $\lim_{i \to I} F(V_i) = 0$. We will show below that the same construction as in loc. cit. also works in our case. Namely, let $A_0 = A$ and $A_{i+1} = A_1((T_a)_{a \in A_1})/(T_a^p - a)_{a \in A_1})$. Each $A_i$ is a faithfully flat syntomic cover of $A$, and we are reduced to showing that $\lim_{i \to I} F(A_i) = F(A) = 0$, where $A := \lim_{i \to I} A_i$. The Frobenius on $A/pA$ is surjective by construction. Now the theorem follows from the explicit calculation of $H^i_{\text{cris}}(A/S_n, J_{A/S_n}^{[r]})$ below in Proposition 4.2.1 and 4.2.4. □
To finish the proof of Theorem 4.2.1 it suffices to show \(H^i_{\text{cris}}(A/S_n, j^{[r]}_{A/S_n}) = 0\) for all \(i > 0\) and \(S_n\)-algebras \(A\) such that the Frobenius on \(A/pA\) is surjective. We first prove this vanishing in the case when \(S_n = \text{Spec}(W_n(k))\) or \(\text{Spec}(\tilde{R}_n)\). In this setting, the base ring is perfect and the construction of a final object in the corresponding crystalline site is simple. In the case of \(S_n = \text{Spec}(W_n(k))\), this is due to Tamme (cf. [Bau92, Proposition 1.16]). On the other hand, the case of \(S_n = \text{Spec}(\tilde{R}_n)\) is slightly more complicated, and is studied separately in Proposition 4.2.4.

**Proposition 4.2.1.** Let \(S_n = \text{Spec}(W_n(k))\) or \(\text{Spec}(\tilde{R}_n)\). Let \(\text{Spec}(A)\) be an \(S_n\)-affine scheme such that the Frobenius \(\varphi\) on \(A/pA = A_1\) is surjective. Then \(H^i_{\text{cris}}(A/S_n, j^{[r]}_{A/S_n}) = 0\) for all \(i > 0\).

**Proof.** Abusing notation, we also denote by \(S_n\) the underlying ring of the affine scheme \(S_n\). To compute the relevant crystalline cohomology groups, we will construct a final object for \(\text{CRIS}(A/S_n)\). More precisely, we shall only work with the subcategory of affine objects, which is sufficient for computing the relevant crystalline cohomology. Let \(B\) be an \(A\)-algebra and \(C\) be an \(S_n\)-linear PD-thickening of \(B\). We first construct a morphism \(\theta_{C,S_n}: W_n(A_1) \to C\) of \(S_n\)-algebras as follows. Given \(x = (x_0, x_1, \ldots, x_{n-1}) \in W_n(A_1)\) with \(x_i = (x_i^{(m)})_{m \geq 0} \in A_1\) and \(x_i^{(m)} \in A_1\), let \(y_i \in C\) be a lift of \(x_i^{(n)} \in A_1\). Set
\[
\theta_{C,n}(x) := \sum_{i=0}^{n-1} p^i y_i^{p^{n-i}}.
\]
Since \(C\) is a PD-thickening of \(B\), it is standard to check that \(\theta_{C,n}\) does not depend on the choice of the lift \(y_i\) of \(x_i^{(n)}\). We claim that \(\theta_{C,n}\) is a ring homomorphism. Consider the diagram
\[
\begin{array}{ccc}
W_n(A_1) & \xrightarrow{\alpha} & W_n(A_1) \\
\downarrow & & \downarrow \\
W_n(A_1) & \leftrightarrow & W_n(C) \xrightarrow{\beta} C
\end{array}
\]
where \(\beta\) is defined by the ghost component of \(W_n(C)\) and \(\alpha\) by sending \(x\) to \((x_0^{(n)}, \ldots, x_{n-1}) \in W_n(A_1)\). It is clear that \(\alpha\) is a ring homomorphism. \(\theta_{C,n}\) is the “composite” of the above ring homomorphisms, and thus a ring homomorphism.

From the construction, we obtain the following commutative diagram:
\[
\begin{array}{ccc}
W_n(A_1) & \xrightarrow{\theta_{C,n}} & C \\
\downarrow^{\theta_{A,n}} & & \downarrow^q \\
A & \xrightarrow{f} & B
\end{array}
\]
We claim that \(\theta_{C,n}\) is the unique map making the above diagram commute. More precisely, if there exists another ring homomorphism \(g: W_n(A_1) \to C\) so that \(q \circ g = f \circ \theta_{A,n}\) then \(\theta_{C,n} = g\). To prove this, note that \(x = \sum p^i [\varphi^{-1}(x_i)]\) for any \(x = (x_0, \ldots, x_{n-1}) \in W_n(A_1)\). So it suffices to show \(g([x_0]) = \theta_{C,n}([x_0]) = y_0^n\) where \(y_0 \in C\) is a lift of \(f(\varphi^{-1}(x_0))\) mod \(p \in B_1\). Let \(z = \varphi^{-n}(x_0)\). Then \(g([z])\) is a lift of \(f(\theta_{A,n}([z])) \mod p \in B_1\). Since
$f(\theta_{A,n}(\lfloor z \rfloor)) = f(x_0^{(n)}) \mod p$, both $g(\lfloor z \rfloor)$ and $y_0$ are lifts of $f(x_0^{(n)}) \mod p$. Hence we have $g(\lfloor x_0 \rfloor) = (g(\lfloor z \rfloor))_p^n = \theta_{C,n}(\lfloor x_0 \rfloor)$.

Next, we claim that the above diagram is compatible with $S_n$-algebra structures. Since $A_1$ is an $S_1$-algebra, we have a natural map $S_1^i \rightarrow A_1^n$. The map $\theta_0 : S_1^i \rightarrow S_1$ given by $(x^{(n)})_{n \geq 0} \mapsto x^{(0)}$ is an isomorphism since $S_1$ is perfect. We have $S_n = W_n(S_1)$, so the morphism $\theta_0$ gives $W_n(A_1^i)$ an $S_n$-algebra structure. To see that $\theta_{C,n}$ is an $S_n$-algebra, let $x = (x_0, \ldots, x_{n-1}) \in S_n = W_n(S_1)$ with $x_i \in S_1$. Then $x = \sum_{i=0}^{n-1} p^i[\varphi^{-i}(x_i)]$. To show $\theta_{C,n}(x) = x$, it thus suffices to consider the case when $x = [x_0]$, which is clear from the construction.

We now show $\theta_{A,n} : W_n(A_1^i) \rightarrow A$ is surjective by induction on $n$. Since $A_1^b$ is perfect, we have $W_n(A_1^i)/p^{n-1}W_n(A_1^i) \cong W_{n-1}(A_1^i)$. Moreover, from the construction, $\theta_{A,n}$ mod $p^{n-1} = \theta_{A,n-1}$. Note that $\theta_{A,n}$ when $n = 1$ is surjective since $\varphi : A_1 \rightarrow A_1$ is surjective. Suppose $\theta_{A,n-1}$ is surjective. Then for any $x \in A$, there exists $y \in W_n(A_1^i)$ such that $\theta_{A,n}(y) = x + p^{n-1}z$ for some $z \in A$. Pick $w \in W_n(A_1^i)$ so that $\theta_{A,n}(w) = z \mod p$. Then $\theta_{A,n}(y - p^{n-1}w) = x$, and therefore $\theta_{A,n}$ is surjective.

Finally, let $W_n^{PD}(A_1^i)$ be the divided power envelope of $W_n(A_1^i)$ with respect to the ideal $\ker \theta_{A,n}$. From the above discussion and the universal property of divided power envelopes, we see that $W_n^{PD}(A_1^i)$ is the desired final object. Thus,

$$H^r_{\text{cris}}(A/S_n, J_{A/S_n}^{[r]}) = H^r_{\text{zar}}(\text{Spec}(W_n^{PD}(A_1^i)), J_{W_n^{PD}(A_1^i)}^{[r]})$$

where $J_{W_n^{PD}(A_1^i)}^{[r]}$ denotes the $r$-th PD ideal of $W_n^{PD}(A_1^i)$. In particular, $H^0_{\text{cris}}(A/S_n, J_{A/S_n}^{[r]}) = 0$ for $i > 0$.

Remark 4.2.2. From the above proof, we also obtain

$$H^0_{\text{cris}}(A/S_n, J_{A/S_n}^{[r]}) \cong J_{W_n^{PD}(A_1^i)}^{[r]}.$$  

Remark 4.2.3. The proof above uses the assumption that $S_n = \text{Spec}(R_n)$ is perfect to define a natural $S_n$-algebra structure on $W_n^{PD}(A_1^i)$ and to show that $\theta_{C,n}$ is an $S_n$-algebra map.

We now prove an analog of Proposition 4.2.1 in the setting where $S_n = \text{Spec}(R_n)$. We begin by constructing a final object in the full subcategory of affine objects of $\text{CRIS}(A/R_n)$. Let $W_n(A_1^i)_R := R \otimes_{W(k)} W_n(A_1^i)$. Suppose $(\text{Spec}(B), \text{Spec}(C))$ is an object in $\text{CRIS}(A/R_n)$. We can consider $(\text{Spec}(B), \text{Spec}(C))$ as an object in $\text{CRIS}(A/W_n(k))$. Then by the proof of Proposition 4.2.1, we have a unique map $\theta_{C,n} : W_n(A_1^i)_R \rightarrow C$ such that diagram (5) commutes, and both $\theta_{A,n}$ and $\theta_{C,n}$ are compatible with $W_n(k)$-algebra structures. Thus, $\theta_{A,n}$ and $\theta_{C,n}$ naturally extend to maps $\theta_{A,n} : W_n(A_1^i)_R \rightarrow A$ and $\theta_{C,n} : W_n(A_1^i)_R \rightarrow C$. The analog of diagram (5) obtained from replacing $W_n(A_1^i)$ by $W_n(A_1^i)_R$ still commutes, and all such morphisms are compatible with $R_n$-algebra structures. It remains to show $\theta_{C,n}$ is unique in the sense that if there exists another $R_n$-algebra
morphism \( g : W_n(A_1^p) \to C \) satisfying \( g \circ g = f \circ \theta_{A,n} \) then \( g = \theta_{C,n} \). If we restrict the diagram to \( W_n(A_1^p) \), then we get back diagram \((8)\). Thus, \( g \) restricted to \( W_n(A_1^p) \) is \( \theta_{C,n} \) by the uniqueness. Since \( g \) is \( R \)-linear, we have \( g = \theta_{C,n} \). Let \( \mathcal{O}_{\text{cris},n}(A) \) be the divided power envelope with respect to the ideal \( \ker(\theta_{A,n}) \) in \( W_n(A_1^p)_R \). Then \((\text{Spec} A, \text{Spec}(\mathcal{O}_{\text{cris},n}(A))) \) is the desired final object, and we conclude the following:

**Proposition 4.2.4.** Suppose \( A \) is an \( R_n \)-algebra such that Frobenius \( \varphi \) on \( A/pA = A_1 \) is surjective. Then for each \( i > 0 \) and \( r \geq 0 \), we have \( H^0_{\text{cris}}(A/R_n, J^{[r]}_{A/R_n}) = 0 \). Furthermore,

\[
H^0_{\text{cris}}(A/R_n, J^{[r]}_{A/R_n}) \cong F^r \mathcal{O}_{\text{cris},n}(A)
\]

where \( F^r \mathcal{O}_{\text{cris},n}(A) \) denotes the \( r \)-th PD-ideal of \( \mathcal{O}_{\text{cris},n}(A) \).

**Corollary 4.2.5.**

1. For \( r \geq 0 \), we have canonical isomorphisms

\[
H^0_{\text{cris}}(\mathcal{R}_n/R_n, J^{[r]}_{\mathcal{R}_n/R_n}) \cong F^r \mathcal{O}_{\text{cris}}(R)/(p^n)
\]

where \( F^r \mathcal{O}_{\text{cris}}(R) \) denotes the \( r \)-th PD-ideal of \( \mathcal{O}_{\text{cris}}(R) \).

2. For all \( i > 0 \) and \( r \geq 0 \), we have \( H^i_{\text{cris}}(\mathcal{R}_n/R_n, J^{[r]}_{\mathcal{R}_n/R_n}) = 0 \).

**Remark 4.2.6.** Note that in the situations of Proposition 4.2.1 and 4.2.4, \( W_n^{\text{PD}}(A_1^p) \) and \( \mathcal{O}_{\text{cris},n}(A) \) have natural Frobenius endomorphisms induced by the natural Witt-vector Frobenius and \( \varphi \).

Consider an \( \tilde{R}_n \)-algebra \( A \) such that \( \varphi \) on \( A_1 \) is surjective. We make some observations on the structure of \( \mathcal{O}_{\text{cris},n}(A) \) which will be useful later. We first relate it to the ring \( A_{\text{cris},n}(A) := W^{\text{PD}}_n(A_1^p) \) and prove an analog of Lemma 2.1.2. For each \( i = 1, \ldots, d \), recall that we chose \( 1+t_i \in \tilde{R}_1 \cong \tilde{R}_1^p \) satisfying \( 1+t_i^{(0)} = 1+t_i \). Denote also by \( \mathcal{R}_n \) the corresponding image under the induced map \( \tilde{R}_1^p \to A_1^p \). Let \( A_{\text{cris},n}(A)\langle X_1, \ldots, X_d \rangle \) be the divided power polynomial ring in the variables \( X_i \) with coefficients in \( A_{\text{cris},n}(A) \), and let \( \theta_{A,n} : A_{\text{cris},n}(A)\langle X_1, \ldots, X_d \rangle \to A \) be the map extending that on \( A_{\text{cris},n}(A) \) given by \( X_i \to 0 \). Consider the \( A_{\text{cris},n}(A) \)-linear map

\[
f : A_{\text{cris},n}(A)\langle X_1, \ldots, X_d \rangle \to \mathcal{O}_{\text{cris},n}(A)
\]

given by \( X_i \to 1 + t_i \otimes 1 - 1 \otimes [1+t_i] \). Note that \( f \) is compatible with \( \theta_{A,n} \).

**Lemma 4.2.7.** The map \( f \) as above is an isomorphism.

**Proof.** Consider the map \( \bar{g} : R_1 = k[[t_1, \ldots, t_d]] \to A_1^p \) given by \( t_i \mapsto 1 + t_i - 1 \). This indeed gives a ring homomorphism since for any \( \sum_I a_I t^I \in R_1 \) with \( a_I \in k \) and \( I = (i_1, \ldots, i_d) \) a multi-index, we have

\[
\bar{g}(\sum_I a_I t^I) = \sum_I a_I \prod_{j=1}^d (1 + t_j - 1)^{i_j} \in A_1^p
\]
with \((\tilde{g}(\sum_{i} a_{i}t_{i}^{l}))^{(0)} = \sum_{i} a_{i}t_{i}^{l} \in A_1\). By Cartier’s Dieudonné-Dwork lemma, there exists a unique lifting \(g : R \rightarrow W(A_1)\) of \(\tilde{g}\) compatible with Frobenius. Note that \(\varphi\) on \(W(A_1)\) is an isomorphism since \(A_1\) is perfect. Since \(g\) is compatible with Frobenius, it follows that \(g\) is given by \(t_i \mapsto [1 + t_i] - 1\). Denote by \(g_n : R \rightarrow W_n(A_1)\) the induced map mod \(p^n\).

Now, let \(h : R \rightarrow \mathcal{O}_{\text{cris}, n}(A) \langle X_1, \ldots, X_d \rangle\) be the \(W(k)\)-linear map given by \(t_i \mapsto X_i + [1 + t_i] - 1\). This gives a ring homomorphism since \(g_n\) as above is a ring homomorphism and \(X_i^{l} = 0\) for \(|l| > 0\). Denote also by \(h : R \otimes_{W(k)} W_n(A_1) \rightarrow \mathcal{O}_{\text{cris}, n}(A) \langle X_1, \ldots, X_d \rangle\) the \(W_n(A_1)\)-linear extension. Since \(h\) commutes with \(\theta_{A,n}\), it extends uniquely to a map \(h : \mathcal{O}_{\text{cris}, n}(A) \rightarrow \mathcal{O}_{\text{cris}, n}(A) \langle X_1, \ldots, X_d \rangle\). Note that \(h \circ f\) is the identity on \(\mathcal{O}_{\text{cris}, n}(A) \langle X_1, \ldots, X_d \rangle\) since it maps \(X_i\) to \(X_i\). Similarly, \(f \circ h\) is the identity on \(\mathcal{O}_{A_n}(A)\) since it maps \(t_i\) to \(t_i\). □

**Remark 4.2.8.** Suppose \(A\) and \(B\) are \(\mathcal{R}_n\)-algebras such that \(\varphi\) on \(A_1\) and \(B_1\) are surjective. Let \(g \in G_R\), and suppose we have a \(g\)-semi-linear morphism \(A \rightarrow B\) of \(\mathcal{R}_n\)-algebras. By functoriality, it induces a \(W_n(k)\)-algebra map \(\mathcal{O}_{\text{cris}, n}(A) \rightarrow \mathcal{O}_{\text{cris}, n}(B)\) and an \(R_n\)-algebra map \(\mathcal{O}_{\text{cris}, n}(A) \rightarrow \mathcal{O}_{\text{cris}, n}(B)\), both of which are compatible with Frobenius. Under the isomorphisms \(\mathcal{O}_{\text{cris}, n}(A) \cong \mathcal{O}_{\text{cris}, n}(A)[X_1, \ldots, X_d]\) and \(\mathcal{O}_{\text{cris}, n}(B) \cong \mathcal{O}_{\text{cris}, n}(B)[X_1, \ldots, X_d]\) of the previous lemma, the induced map \(\mathcal{O}_{\text{cris}, n}(A) \rightarrow \mathcal{O}_{\text{cris}, n}(B)\) sends \(X_i\) to \(g(X_i) = X_i + [1 + t_i] - g([1 + t_i])\), and is therefore compatible with the \(\mathcal{O}_{A_n}(A)\)-linear (resp. \(\mathcal{O}_{A_n}(B)\)-linear) connection given by \(\nabla(X_i) = dt_i\).

**4.3. Fontaine-Laffaille data associated to \((\mathcal{O}_{\text{cris}, n}(A), F^r \mathcal{O}_{\text{cris}, n}(A))\).** Let \(f_1, \ldots, f_c\) be elements in \(\mathcal{R}[x_1, \ldots, x_m]\) whose images in \(\mathcal{R}_n[x_1, \ldots, x_m]\) (which, by abusing notation, we also denote by \((f_1, \ldots, f_c)\)) generate a Koszul-regular ideal, and suppose that \(\mathcal{R}_n[x_1, \ldots, x_m]/(f_1, \ldots, f_c)\) is flat over \(\mathcal{R}_n\). Let \(A = \mathcal{R}_n[\frac{x_1}{p}, \ldots, \frac{x_m}{p}] / (f_1, \ldots, f_c)\) and note that Frobenius on \(A_1\) is surjective. In this section, we show that \((\mathcal{O}_{\text{cris}, n}(A), F^r \mathcal{O}_{\text{cris}, n}(A))\) can be equipped with the structure of a relative Fontaine-Laffaille module functorially in \(A\). In particular, we construct divided power Frobenii \(\varphi_c\) and a connection \(\nabla\). We begin by making some standard observations on the filtration \(F^r \mathcal{O}_{\text{cris}, n}(A)\).

**Lemma 4.3.1.** With notations as above,

(1) For any \(r \geq 0\) and \(1 \leq m < n\), we have an exact sequence

\[
0 \longrightarrow J^{[r]}_{W_n^p(A_1^r)} \xrightarrow{p^m} J^{[r]}_{W_n^p(A_1^r)} \mod p^m J^{[r]}_{W_n^p(A_1^r)} \longrightarrow 0.
\]

(2) We have an analogous exact sequence in the setting where \(\mathcal{R}\) is replaced by \(\hat{\mathcal{R}}\), and with \(A\) constructed in an analogous manner.

**Proof.** Consider \(C = \mathcal{R}[\frac{x_1}{p}, \ldots, \frac{x_m}{p}]\) with the ideal \(I = (f_1, \ldots, f_c)\). By [BMS18 Lemma 3.10], the \(p\)-adic completion of \(C\) is a perfectoid, and the kernel of \(\theta_{C,n} : W_n(C_1) \rightarrow C_n\) is generated by \(\xi\). Let \(\theta'\) be the composite \(\theta' : W_n(C_1^0) \rightarrow C_n \rightarrow A\), which is surjective.
Hence, we obtain the commutative diagram:

\[
\begin{array}{ccc}
W_n(C_1^p) & \xrightarrow{\theta_{C,n}} & C_n \\
\downarrow{\cong} & & \downarrow{\cong} \\
W_n(A_1^p) & \xrightarrow{\theta_{A,n}} & A
\end{array}
\]

In particular, \( \ker(\theta_{A,n}) = \tilde{I} W_n(A_1^p) \). The natural map from the PD-envelope \( D_{W_n(C_1^p)}(\tilde{I}) \) of \( \tilde{I} \) in \( W_n(C_1^p) \) (over the PD-ring \( (\mathbb{Z}/(p^n), (p)) \)) to \( W_n^{\text{PD}}(A_1^p) \) is an isomorphism of PD-algebras by [BO78, Remark 3.20 (7)]. Thus, \( J_{W_n^{\text{PD}}(A_1^p)}^r \) for any \( r \geq 0 \) is \( \mathbb{Z}/p^n\mathbb{Z} \)-flat by Lemma 4.3.2 below, and statement (1) follows.
For (2), the above arguments go through after following modifications. Let $C = \widehat{R}[x_1^{\frac{1}{n}}, \ldots, x_m^{\frac{1}{n}}]$ with the ideal $I = (f_1, \ldots, f_c)$. Note in this case that $\theta_{C,n} : W_n(C_1^0) \to C_n$ is an isomorphism since $C_1$ is perfect. Let $\tilde{I} = (\tilde{f}_1, \ldots, \tilde{f}_c)W_n(C_1^0)$ and $P = (f_1^0, \ldots, f_c^0)C_1^0$. It can be checked easily that there is an isomorphism between $A_1^0$ and the $P$-completion of $C_1^0$ compatible with the natural map $C_1^0 \to A_1^0$, and that this induces an isomorphism between $W_n(A_1^0)$ and the $\tilde{I}$-completion of $W_n(C_1^0)$. Hence, $\ker(\theta_{A,n}) = \tilde{I}W_n(A_1^0)$, and (2) follows similarly as above. \hfill \square

**Lemma 4.3.2.** Let $B$ be a flat $\mathbb{Z}/p^n\mathbb{Z}$-algebra, $I \subset B$ a finitely generated Koszul-regular ideal, and suppose that $B/I$ is flat over $\mathbb{Z}/p^n\mathbb{Z}$. Then the PD-envelope $D_B(I)$ and its $r$-th divided power ideal $J^{[r]}$ for any $r \geq 1$ are $\mathbb{Z}/p^n\mathbb{Z}$-flat.

**Proof.** By [Sta18: Tag 068Q], there exists a faithfully flat smooth morphism $\text{Spec}(B') \to \text{Spec}(B)$ such that $IB'$ is generated by a regular sequence. Since $D_B(I) \otimes_B B' \cong D_{B'}(IB')$ by [BO78: Proposition 3.21], we can reduce to the case where $I \subset B$ is generated by a regular sequence $(f_1, \ldots, f_m)$.

Write $\text{Spec}(B) = \varprojlim \text{Spec}(B_\alpha)$ as an inverse limit where $B_\alpha$’s are finitely generated flat $\mathbb{Z}/p^n\mathbb{Z}$-algebras, and similarly write $\text{Spec}(B/I) = \varprojlim \text{Spec}(B_\alpha/I_\alpha)$ as a limit of finitely generated flat $\mathbb{Z}/p^n\mathbb{Z}$-algebras where $I_\alpha$ denotes the pull-back of $I$ to $B_\alpha$. By [Gro67: 19.8.2], the morphism $\text{Spec}(B/I) \hookrightarrow \text{Spec}(B)$ is a regular immersion if and only if $\text{Spec}(B_\alpha/I_\alpha) \hookrightarrow \text{Spec}(B_\alpha)$ is a regular immersion for sufficiently large $\alpha$. Since divided power envelopes are compatible with direct limits, we may assume that $B$ is noetherian.

Consider the natural cartesian diagram

$$
\begin{array}{ccc}
\text{Spec}(B/I) & \longrightarrow & \text{Spec}(B) \\
\downarrow & & \downarrow \theta \\
\text{Spec}(\mathbb{Z}/p^n\mathbb{Z}) & \longrightarrow & \mathbb{A}^m_{\mathbb{Z}/p^n\mathbb{Z}}
\end{array}
$$

where the bottom horizontal map is given by $T_i \mapsto 0$ and right vertical by $T_i \mapsto f_i$. Since the fiber over the origin is flat and the top horizontal map is a regular immersion, by [Gro67: 19.2.4], we deduce that for every point $x \in \text{Spec}(B/I)$, the map $\theta$ is flat in a neighborhood of $x$. Thus, there is an open neighborhood $U \subset \text{Spec}(B)$ of $\text{Spec}(B/I)$ such that the resulting diagram

$$
\begin{array}{ccc}
\text{Spec}(B/I) & \longrightarrow & U \\
\downarrow & & \downarrow \\
\text{Spec}(\mathbb{Z}/p^n\mathbb{Z}) & \longrightarrow & \mathbb{A}^m_{\mathbb{Z}/p^n\mathbb{Z}}
\end{array}
$$

is cartesian and the right vertical map is flat. We have isomorphisms of PD-algebras

$$D_B(I) \cong D_U(I) \cong D_{\mathbb{A}^m_{\mathbb{Z}/p^n\mathbb{Z}}((T_1, \ldots, T_d))} \otimes_{\mathbb{A}^m_{\mathbb{Z}/p^n\mathbb{Z}}} U.$$
Since $D_{A_{Z/p^n Z}}((T_1, \ldots , T_d))$ and its $r$-th divided power ideals are flat over $\mathbb{Z}/p^n\mathbb{Z}$, the lemma holds.

\begin{corollary}
For any $r \geq 0$ and $1 \leq m < n$, we have an exact sequence
\[ 0 \longrightarrow F^r \mathcal{O} A_{\text{cris}, n-m}(A) \xrightarrow{xp^m} F^r \mathcal{O} A_{\text{cris}, n}(A) \mod p^m \longrightarrow F^r \mathcal{O} A_{\text{cris}, m}(A) \longrightarrow 0. \]
\end{corollary}

\begin{proof}
The isomorphism $f$ of Lemma 4.2.7 is compatible with filtrations given by the divided power ideals, since $f$ is compatible with $\theta_{A,n}$. In particular, we have
\[ F^r \mathcal{O} A_{\text{cris}, n}(A) = \bigoplus_{\sum_i j_i = r} F^{i_0} A_{\text{cris}, n}(A) \gamma_{i_1}(X_1) \cdots \gamma_{i_d}(X_d) + F^{r+1} \mathcal{O} A_{\text{cris}, n}(A). \]
Here $\gamma_i$’s denote the corresponding divided power operations. So the assertion follows from Lemma 4.3.1.
\end{proof}

We now explain the construction of $\varphi_r$ and connection on $\mathcal{O} A_{\text{cris}, n}(A)$. Note that both $A_{\text{cris}, n}(A)$ and $\mathcal{O} A_{\text{cris}, n}(A)$ have natural Frobenius induced by those on $W_n(A^n_1)$ and $R_n$. We have $\varphi(J^{[r]}_{W_n^p(A^n_1)}) \subset p^w W_n^p(A^n_1)$ for any $0 \leq r \leq p - 1$. Define $\varphi_r : F^r A_{\text{cris}, n}(A) \to A_{\text{cris}, n}(A)$ for $0 \leq r \leq p - 1$ as follows. First, given $x \in F^r A_{\text{cris}, n}(A)$, choose a lift $\hat{x} \in F^r A_{\text{cris}, n+r}(A)$. We have $\varphi(\hat{x}) = p^r \hat{y}$ for some $\hat{y} \in A_{\text{cris}, n+r}(A)$. Define $\varphi_r(x) := \hat{y} \mod p^r \in A_{\text{cris}, n}(A)$. It follows from Lemma 4.3.1 that $\varphi_r$ is well-defined. Similarly, we define $\varphi_r : F^r \mathcal{O} A_{\text{cris}, n}(A) \to \mathcal{O} A_{\text{cris}, n}(A)$ using Corollary 4.3.3.

Consider the natural connection
\[ \nabla : A_{\text{cris}, n}(A)\langle X_1, \ldots , X_d \rangle \to A_{\text{cris}, n}(A)\langle X_1, \ldots , X_d \rangle \otimes_R \hat{\Omega}_R \]
given by setting $\nabla(A_{\text{cris}, n}(A)) = 0$ and $\nabla(X_i) = dt_i$. By Lemma 4.2.7, this defines a natural connection $\nabla : \mathcal{O} A_{\text{cris}, n}(A) \to \mathcal{O} A_{\text{cris}, n}(A) \otimes_R \hat{\Omega}_R$. The following corollary is immediate.

\begin{corollary}
Let $0 \leq r \leq p - 1$.
\begin{enumerate}
\item The morphism $\varphi_r : F^r \mathcal{O} A_{\text{cris}, n}(A) \to \mathcal{O} A_{\text{cris}, n}(A)$ is $\varphi$-semilinear, and for $r \leq p-2$,
\[ \varphi_r|_{F^{r+1} \mathcal{O} A_{\text{cris}, n}(A)} = p \varphi_{r+1}. \]
We have a similar statement for $A_{\text{cris}, n}(A)$.
\item The connection $\nabla$ on $\mathcal{O} A_{\text{cris}, n}(A)$ is a flat connection such that:
\begin{enumerate}
\item $\nabla(F^r \mathcal{O} A_{\text{cris}, n}(A)) \subset F^{r-1} \mathcal{O} A_{\text{cris}, n}(A) \otimes_R \hat{\Omega}_R$.
\item $\nabla \circ \varphi_r = (\varphi_{r-1} \otimes d\varphi_1) \circ \nabla$ (when $r = 0$, $\varphi_{-1}$ is understood to be $p \varphi_0$).
\end{enumerate}
\end{enumerate}
The above structures $(\mathcal{O} A_{\text{cris}, n}(A), F^r \mathcal{O} A_{\text{cris}, n}(A), \varphi_r, \nabla)$ are functorial in $A$.
\end{corollary}

\begin{remark}
Let $(f_1, \ldots , f_e) \subset \overline{R}_n[x_1, \ldots , x_m]$ be a Koszul-regular sequence such that $\overline{R}_n[x_1, \ldots , x_m]/(f_1, \ldots , f_e)$ is flat over $\overline{R}_n$. Then we may choose lifts $f_i$ in $\overline{R}[x_1, \ldots , x_m]$, and Lemma 4.3.1 and previous corollary hold for algebras $A$ as above. This observation is useful below as we work with syntomic sites modulo $p^n$.
\end{remark}
4. The divided power sheaves $\mathcal{J}^{[r]}_{Y_n/S_n}$ on the syntomic site. In this section, we make some observations on the divided power sheaves $\mathcal{J}^{[r]}_{Y_n/S_n}$ on the (small) syntomic site. Recall that $X$ is a smooth proper scheme over $R$. We will work with $Y/S$ where $Y = \mathcal{X}$ and $S$ is either $\text{Spec}(R)$, $\text{Spec}(\hat{R})$ or $\text{Spec}(W(k))$, or where $Y = X$ and $S = \text{Spec}(R)$.

Given integers $n > 0$ and $r \geq 0$, consider the following presheaf on the small syntomic site $(Y_n)_{\text{syn}}$:

$$\mathcal{J}^{[r]}_{Y_n/S_n} : U \mapsto H^0_{\text{cris}}(U/S_n, j^{[r]}_{U_n/S_n}), \forall U \in (Y_n)_{\text{syn}}.$$  

**Lemma 4.4.1.** With notations as above,

1. $\mathcal{J}^{[r]}_{Y_n/S_n}$ is a sheaf on $(Y_n)_{\text{syn}}$.
2. We have isomorphisms

$$H^i_{\text{cris}}(Y_n/S_n, j^{[r]}_{Y_n/S_n}) \cong H^i_{\text{syn-cr}}(Y_n/S_n, j^{[r]}_{Y_n/S_n}) \cong H^i_{\text{syn}}(Y_n, \mathcal{J}^{[r]}_{Y_n/S_n}).$$

**Proof.** The second part is a consequence of the first part, Corollary B.0.9, and Theorem 4.2.1. The first part follows from the fact that $\mathcal{J}^{[r]}_{Y_n/S_n}$ is the push-forward of the analogous divided power sheaf on the corresponding syntomic-crystalline site.

We now explain how to define $\varphi_r$ (for $0 \leq r \leq p-1$) and $\nabla$ on $\mathcal{J}^{[r]}_{Y_n/S_n}$ when $S = \text{Spec}(R)$. We first consider the case when $Y = \mathcal{X}$. Let $C_{Y_n} \subset (Y_n)_{\text{syn}}$ denote the full subcategory of affine schemes $\text{Spec}(A)$ over $Y_n$ such that $A \cong \mathcal{R}_n[x_1, \ldots, x_m]/I$ where $I$ is a Koszul-regular ideal. Since $\mathcal{X}$ is syntomic (in fact smooth) over $\mathcal{R}_n$, in order to define a morphism of sheaves on $(Y_n)_{\text{syn}}$, it suffices to construct such a morphism when restricted to this subcategory.

Consider $\text{Spec}(A_{(0)}) \in C_{Y_n}$ where $A_{(0)} \cong \mathcal{R}_n[x_1, \ldots, x_m]/(f_1, \ldots, f_c)$ such that $(f_1, \ldots, f_c)$ is Koszul-regular, and let $A_{(i)} := \mathcal{R}_n[x_1^{1/p^i}, \ldots, x_m^{1/p^i}]/(f_1, \ldots, f_c)$. Let $A = \varinjlim_i A_{(i)} \cong \mathcal{R}_n[x_1^{1/p^i}, \ldots, x_m^{1/p^i}]/(f_1, \ldots, f_c)$. By Lemma 4.4.1 we have an exact sequence:

$$0 \rightarrow \mathcal{J}^{[r]}_{Y_n/S_n}(A_{(0)}) \rightarrow \mathcal{J}^{[r]}_{Y_n/S_n}(A) \rightarrow \mathcal{J}^{[r]}_{Y_n/S_n}(A \otimes A_{(0)} A).$$

Note that $\mathcal{J}^{[r]}_{Y_n/S_n}(A) = \varinjlim_i \mathcal{J}^{[r]}_{Y_n/S_n}(A_{(i)}) = H^0_{\text{cris}}(A/S_n, j^{[r]}_{A/S_n})$. By Corollary 4.3.4 $\varphi_r$ and $\nabla$ are well-defined on both $\mathcal{J}^{[r]}_{Y_n/S_n}(A)$ and $\mathcal{J}^{[r]}_{Y_n/S_n}(A \otimes A_{(0)} A)$, since $A \otimes A_{(0)} A \cong \mathcal{R}_n[x_1^{1/p^i}, \ldots, x_m^{1/p^i}, y_1^{1/p^i}, \ldots, y_m^{1/p^i}]/(f_1(x), \ldots, f_c(x), f_1(y), \ldots, f_c(y))$. By functoriality, $\varphi_r$ and $\nabla$ are compatible with the maps $\mathcal{J}^{[r]}_{Y_n/S_n}(A) \rightarrow \mathcal{J}^{[r]}_{Y_n/S_n}(A \otimes A_{(0)} A)$. Thus, we have induced Frobenius $\varphi_r : \mathcal{J}^{[r]}_{Y_n/S_n}(A_{(0)}) \rightarrow \mathcal{O}_{Y_n/S_n}(A_{(0)})$ and connection $\nabla : \mathcal{O}_{Y_n/S_n} \rightarrow \mathcal{O}_{Y_n/S_n} \otimes_R \Omega_R$ compatible with the exact sequence above.

We claim that this construction is functorial for maps $h : A_{(0)} \rightarrow B_{(0)}$ in $C_{Y_n}$, and so does not depend on the presentation $A_{(0)} = \mathcal{R}_n[x_1, \ldots, x_m]/(f_1, \ldots, f_c)$. First note that
Thus, \( \hat{\varphi} \) lifts \( \hat{h} \) to a map \( \hat{A} : \mathcal{R}_n[y_1, \ldots, y_m]/(g_1, \ldots, g_s, z_1 - \hat{h}(x_1), \ldots, z_m - \hat{h}(x_m)) \). Via a diagram chase, we deduce that using the presentation \( \mathcal{R}_n[y_1, \ldots, y_m]/(g_1, \ldots, f_c, x_{m+1} - a) \) for \( A(0) \) gives the same \( \varphi_r \) and \( \nabla \). Now, if \( B(0) \cong \mathcal{R}_n[y_1, \ldots, y_m] \), then there exists a map \( \hat{h} : \mathcal{R}_n[x_1, \ldots, x_m] \to \mathcal{R}_n[y_1, \ldots, y_m] \) which lifts \( h \). Furthermore, we may use the presentation \( B(0) \cong \mathcal{R}_n[y_1, \ldots, y_m, x_1, \ldots, x_m]/(g_1, \ldots, g_s, z_1 - \hat{h}(x_1), \ldots, z_m - \hat{h}(x_m)) \). Then \( \hat{h} \) extends to a morphism

\[
\hat{h} : A \to B := \mathcal{R}_n[y_1^{\frac{1}{r}}, \ldots, y_m^{\frac{1}{r}}, z_1^{\frac{1}{r}}, \ldots, z_m^{\frac{1}{r}}]/(g_1, \ldots, g_s, z_1 - \hat{h}(x_1), \ldots, z_m - \hat{h}(x_m)).
\]

This \( \hat{h} \) induces a map \( \mathcal{J}^{[r]}_{Y_n/S_n}(A) \to \mathcal{J}^{[r]}_{Y_n/S_n}(B) \) which is compatible with the map \( \mathcal{J}^{[r]}_{Y_n/S_n}(A(0)) \to \mathcal{J}^{[r]}_{Y_n/S_n}(B(0)) \) induced by \( h \), and a diagram chase gives the desired functoriality. This completes the construction of \( \varphi_r \) and \( \nabla \) when \( Y = \overline{X} \), and by construction they satisfy the properties analogous to those stated in Corollary 4.3.1. We also remark that these constructions are natural in \( Y_n \).

When \( Y = X \), we construct Frobenius and connection on \( \mathcal{J}^{[r]}_{Y_n/S_n} \) in an analogous manner, starting with \( A(0) = \mathcal{R}_n[x_1, \ldots, x_m]/(f_1, \ldots, f_c) \) (where \( f_i \)'s form a regular sequence), and setting \( A = \mathcal{R}_n[x_1^{\frac{1}{r}}, \ldots, x_m^{\frac{1}{r}}]/(f_1, \ldots, f_c) \). Note that \( \mathcal{R} \) is a direct limit of syntomic covers of \( R \), and therefore \( A \) is a direct limit of syntomic covers of \( A(0) \).

The following corollary summarizes the above discussion.

**Corollary 4.4.2.** Suppose \( S_n = \text{Spec}(R_n) \), and \( Y_n = \overline{X}_n \) or \( X_n \). Let \( 0 \leq r \leq p - 1 \) and \( 0 \leq i \leq p - 1 \).

1. There are natural (in \( Y_n \)) morphisms \( \varphi_r : \mathcal{J}^{[r]}_{Y_n/S_n} \to \mathcal{O}_{Y_n/S_n} \) in \( (Y_n)_{\text{syn}} \) such that \( \varphi_r \) is \( \varphi \)-semilinear, and for \( r \leq p - 2 \), \( \varphi_{r+1} |_{\mathcal{J}^{[r+1]}_{Y_n/S_n}} = p \varphi_{r+1} \).

2. There is a natural flat connection \( \nabla : \mathcal{O}_{Y_n/S_n} \to \mathcal{O}_{Y_n/S_n} \otimes_R \hat{\Omega}_R \), such that:
   
   - (a) \( \nabla(\mathcal{J}^{[r]}_{Y_n/S_n}) \subseteq \mathcal{J}^{[r-1]}_{Y_n/S_n} \otimes_R \hat{\Omega}_R \).
   - (b) \( \nabla \circ \varphi_r = (d \varphi_1 \otimes \varphi_{r-1}) \circ \nabla \) (when \( r = 0 \), \( \varphi_{-1} \) is understood to be \( p \varphi_0 \)).

3. There are natural morphisms \( \varphi_r : \text{H}^i_{\text{cris}}(Y_n/S_n, \mathcal{J}^{[r]}_{Y_n/S_n}) \to \text{H}^i_{\text{cris}}(Y_n/S_n, \mathcal{O}_{Y_n/S_n}) \otimes_R \hat{\Omega}_R \), satisfying the analogous properties as in (1) and (2), and thus

\[
(\text{H}^i_{\text{cris}}(Y_n/S_n, \mathcal{O}_{Y_n/S_n}), \text{H}^i_{\text{cris}}(Y_n/S_n, \mathcal{J}^{[r]}_{Y_n/S_n}), \varphi_r, \nabla)
\]

is an object of \( \text{MF}_{\nabla, \text{big}}(R) \). If \( Y_n = X_n \), each \( \text{H}^i_{\text{cris}}(Y_n/S_n, \mathcal{J}^{[r]}_{Y_n/S_n}) \) is finite as an \( R_n \)-module.
Proof. It remains to prove the last assertion. By passing to cohomology and using Lemma 4.4.1, we see that the resulting data \((H^i_{\text{cris}}(Y_n/S_n, \mathcal{O}_{Y_n/S_n}), H^i_{\text{cris}}(Y_n/S_n, \mathcal{J}^{[r]}_{Y_n/S_n}, \varphi_r, \nabla))\) is an object of \(\text{MF}_{\text{\textbullet}, \text{big}}(R)\). When \(Y_n = X_n\), each \(H^i_{\text{cris}}(Y_n/S_n, \mathcal{J}^{[r]}_{Y_n/S_n})\) is a finite \(R_n\)-module by [Fal99, Theorem 14] since \(X\) is proper and smooth over \(R\). □

Remark 4.4.3. When \(Y = \overline{X}\) and \(S\) is either \(\text{Spec}(\tilde{R})\) or \(\text{Spec}(W(k))\), we construct \(\varphi_r\) for \(0 \leq r \leq p - 1\) on \(\mathcal{J}^{[r]}_{Y_n/S_n}\) in an analogous manner. As before, these construction are functorial in \(Y_n\).

Remark 4.4.4. For \(X_n/R_n\), one can also prove the existence of \(\varphi_r\) and \(\nabla\) as in the previous corollary via de Rham cohomology. In particular, one can identify crystalline cohomology with de Rham cohomology, and directly construct \(\varphi_r\). In this case, \(\nabla\) is the usual Gauss-Manin connection. One can show that these two constructions coincide.

Completion of proof of Theorem 3.0.1. The first part of the theorem is contained in the previous corollary. It remains to prove the second part. Suppose we have a cartesian diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
\text{Spec}(R') & \longrightarrow & \text{Spec}(R)
\end{array}
\]

as in Theorem 3.0.1 (2). First note that it suffices to show that the following induced diagram of sheaves on the syntomic site commutes:

\[
\begin{array}{ccc}
\mathcal{J}^{[r]}_{X_n/R_n} & \xrightarrow{f_*} & f_* \mathcal{J}^{[r]}_{X'_n/R'_n} \\
\downarrow \varphi_r & & \downarrow f_*(\varphi_r) \\
\mathcal{O}_{X_n/R_n} & \xrightarrow{f_*} & f_* \mathcal{O}_{X'_n/R'_n}
\end{array}
\]

Assuming that the previous diagram commutes, the functoriality of the comparison isomorphisms between syntomic and crystalline cohomology (see Remark 3.0.4) allows us to conclude the desired result. We can check this locally on objects of the category \(\mathcal{C}_{X_n} \subset (X_n)_{\text{syn}}\). Let \(A_0 \in \mathcal{C}_{X_n}\) be as above, and similarly for \(A\). Then \(A'_0 := A_0 \otimes_{R_n} R'_n \in \mathcal{C}_{X'_n}\) and we have \(A' = A \otimes_{R_n} R'_n\). By construction of \(\varphi_r\), it suffices to show that the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{J}^{[r]}_{X_n/R_n}(A) & \xrightarrow{\varphi_r} & \mathcal{J}^{[r]}_{X'_n/R'_n}(A') \\
\downarrow & & \downarrow \\
\mathcal{O}_{X_n/R_n}(A) & \longrightarrow & \mathcal{O}_{X'_n/R'_n}(A')
\end{array}
\]

Moreover, by definition of \(\varphi_r\), it suffices to show that the corresponding diagram with \(\varphi_r\) replaced by \(\varphi\) commutes. We may further reduce to the setting where frobenius on \(A/p\) is...
surjective. More precisely, we can consider $A$ as in the paragraph before Corollary 4.4.2. But then compatibility of Frobenius follows from the fact that $R \rightarrow R'$ is compatible with Frobenius and Witt vector functoriality.

We simplify notation by setting $\mathcal{J}^{[r]}_n := \mathcal{J}^{[r]}_{\mathbb{X}_n/W_n(k)}$, $\mathcal{J}^{[r]}_n := \mathcal{J}^{[r]}_{\mathbb{X}_n/R_n}$, and as before $\mathcal{J}^{[0]}_{Y_n/S_n} = \mathcal{O}_{Y_n/S_n}$. The natural inclusions $W(k) \subset R \subset \hat{R}$ and functoriality of crystalline cohomology induce the following morphisms on the syntomic site of $\mathbb{X}_n$:

$$\mathcal{J}^{[r]}_n \rightarrow \mathcal{J}^{[r]}_n \rightarrow \mathcal{J}^{[r]}_n.$$

**Proposition 4.4.5.** Let $0 \leq r \leq p - 2$.

1. $i$ is injective and compatible with $\varphi_r$. Furthermore, it induces an isomorphism $\mathcal{J}^{[r]}_n \cong (\mathcal{J}^{[r]}_n)^{\nabla = 0}$.

2. $j$ is surjective and compatible with $\varphi_r$. Moreover, $j \circ i$ is the identity map.

**Proof.** It suffices to prove the assertions at the level of sections on Spec$A$ where $A$ is assumed to be as in Proposition 4.2.1 and Proposition 4.2.4. Using the isomorphism $f$ in Lemma 4.2.7 to identify these local sections, $i$ and $j$ can be identified with the morphisms $A_{\text{cris},n}(A) \rightarrow A_{\text{cris},n}(A)\langle X_1, \ldots, X_d \rangle \rightarrow A_{\text{cris},n}(A)$ where the first map is the natural inclusion and the second map is induced by $X_i \mapsto 0$. Both maps are compatible with $\varphi_r$, and

$$(A_{\text{cris},n}(A)\langle X_1, \ldots, X_d \rangle)^{\nabla = 0} = A_{\text{cris},n}(A).$$

The following lemma will be used later to build the comparison map between crystalline and étale cohomology.

**Lemma 4.4.6.** For $0 \leq r \leq p - 2$, the map $\varphi_{r - 1} : \mathcal{J}_{n}^{[r]} \rightarrow \mathcal{O}_n$ is surjective.

**Proof.** By Lemma 4.3.1 we are reduced to showing the desired surjectivity for the case $n = 1$. Let $U = \text{Spec}(A)$ with $A$ as in Lemma 4.3.1. In this case, $\varphi_{r - 1} : \mathcal{J}_{1}^{[r]}(U) \rightarrow \mathcal{O}_1(U)$ can be identified with $\varphi_{r - 1} : J_{W_1^{PD}(A_1^p)}^{[r]} \rightarrow W_1^{PD}(A_1^p)$. It suffices to show for any $y \in W_1^{PD}(A_1^p)$, there exists a direct limit of syntomic $A$-algebras $B$ such that $\varphi$ on $B$ is surjective (we are assuming $n = 1$) and that there exists $x \in J_{W_1^{PD}(B_1^p)}^{[r]}$ satisfying $\varphi_r(x) - x = y$. Moreover, it suffices to find such an algebra $B$ and $x \in J_{W_1^{PD}(B_1^p)}^{[r]}$ so that $\varphi_r(x) - x \equiv y \mod J_{W_1^{PD}(B_1^p)}^{[r-1]}$ since $\varphi_r$ on $J_{W_1^{PD}(B_1^p)}^{[r-1]}$ is 0.

We are reduced to showing that given $y = (y^{(i)})_{i \geq 0} \in A_1^p \cong W_1(A_1^p)$, we can find $B$ and $x$ as above such that $\varphi_r(x) - x \equiv y \mod J_{W_1^{PD}(B_1^p)}^{[r-1]}$. Let $(q_i)_{i \geq 0} \in A_1^p$ be a non-zero element with $q_0 = -p$, and let $z = [(q_i)] + p \in W_1(A_1^p)$. Then $z' \in J_{W_1^{PD}(A_1^p)}^{[r]}$ and $\varphi_r(z') - 1 \in J_{W_1^{PD}(A_1^p)}^{[r-1]}$. Let $B^{(0)} = A[X]/(X^p - q_1 X - y^{(1)})$, which is syntomic over $A$,
and consider \( B = \lim_{\rightarrow} B^{(j)} \) where \( B^{(j+1)} := B^{(j)}[(T_b)_{b \in B^{(j)}}/((T^p_b - b)_{b \in B^{(j)}})] \) for \( j \geq 0 \). Let a polynomial \( w = (w^{(i)})_{i \geq 0} \in B^{(j)} \) such that \( w^{(1)} \in B \) is the image of \( X \) in \( B \), and let \( x = x^{(r)}w \in J_{W_1^p(\mathcal{B}^j)}^{[r]} \). Since \( w^{(1)} \) in \( B \) satisfies the equation \( X^p - q^n X - y^{(1)} = 0 \), we conclude that \( \varphi_r(x) - x \equiv y \mod J_{W_1^p(\mathcal{B}^j)}^{[r]} \).

The cohomology groups \( H^i_{\text{syn}}(\overline{X}_n, \mathcal{J}_n^{[r]}) \) have a natural \( G_R \)-action, and there are several equivalent ways to define this action. Given an element \( g \in G_R \), we have a natural induced morphism of the corresponding syntomic topoi, and moreover a natural morphism \( g^*(\mathcal{J}_n^{[r]}) \to \mathcal{J}_n^{[r]} \) or equivalently a morphism \( \mathcal{J}_n^{[r]} \to g_* \mathcal{J}_n^{[r]} \). The latter morphism is given by an application of crystalline functoriality. We recall an explicit construction of this action in order to see that this morphism is \( \varphi_r \)-compatible. Consider \( U_0 = \text{Spec}(A_0) \) with \( A_0 = \mathcal{O}_{\mathcal{R}_n}[x_1, \ldots, x_m]/I \), where \( I \) is a Koszul-regular ideal and \( A_0 \) is flat over \( \mathcal{O}_{\mathcal{R}_n} \). For each \( g \in G_R \), \( g(I) \subseteq \mathcal{O}_{\mathcal{R}_n}[x_1, \ldots, x_m] \) is a Koszul-regular ideal and \( g(A_0) := \mathcal{O}_{\mathcal{R}_n}[x_1, \ldots, x_m]/g(I) \) is \( \mathcal{O}_{\mathcal{R}_n} \)-flat. By functoriality of crystalline cohomology, the \( g \)-semi-linear map \( A_0 \to g(A_0) \) of \( \mathcal{O}_{\mathcal{R}_n} \)-algebras induces a morphism \( \mathcal{J}_n^{[r]}(U_0) \to \mathcal{J}_n^{[r]}(g(U_0)) \) of \( R_n \)-algebras, where \( g(U_0) := \text{Spec}(g(A_0)) \). We have a similar statement for \( U = \text{Spec}(A) \) with \( A = \mathcal{O}_{\mathcal{R}_n}[[x_1, \ldots, x_m]]/I \). By Remark 4.2.8, the induced map \( \mathcal{J}_n^{[r]}(U) = F^r \mathcal{O}A_{\text{cris}, n}(A) \to \mathcal{J}_n^{[r]}(g(U)) = F^r \mathcal{O}A_{\text{cris}, n}(g(A)) \) is compatible with \( \varphi_r \) and connection. By a similar argument as above using exact sequences of the form

\[
\mathcal{J}_n^{[r]}(U_0) \longrightarrow \mathcal{J}_n^{[r]}(U) \longrightarrow \mathcal{J}_n^{[r]}(U \times_{U_0} U) \, ,
\]

we conclude that the natural induced \( G_R \)-action on \( H^i_{\text{syn}}(\overline{X}_n, \mathcal{J}_n^{[r]}) \) is compatible with \( \varphi_r \) and connection.

Remark 4.4.7. The \( G_R \)-action can also be defined directly over the crystalline-syntomic site or the crystalline site via functoriality. These are all compatible with the action defined above by the comparison isomorphisms between these cohomology groups.

For \( 0 \leq r \leq p - 2 \), consider the following sheaf on \( (\overline{X}_n)_{\text{syn}} \):

\[
S_n^{[r]} := (\mathcal{J}_n^{[r]})_{\varphi_r = 1, \nabla = 0}.
\]

Since \( G_R \)-action on \( H^i_{\text{syn}}(\overline{X}_n, \mathcal{J}_n^{[r]}) \) is compatible with \( \varphi_r \) and connection, we have a natural induced \( G_R \)-action on \( H^i_{\text{syn}}(\overline{X}_n, S_n^{[r]}) \). By Proposition 4.4.5, we have \( S_n^{[r]} = (\mathcal{J}_n^{[r]})_{\varphi_r = 1} \), and by Lemma 4.4.6 we have an exact sequence over \( (\overline{X}_n)_{\text{syn}} \):

\[
0 \longrightarrow S_n^{[r]} \longrightarrow \mathcal{J}_n^{[r]} \bigg|\bigg|_{\varphi_r = 1} \longrightarrow \mathcal{O}_n \longrightarrow 0. \tag{9}
\]

We shall use this sequence in the following sections to construct the desired comparison map from crystalline to etale cohomology.
4.5. **Construction of the comparison map, Part 1.** We now construct a natural $G_R$-equivariant morphism for $i \leq r \leq p - 2$:

$$ t_R : T_{\text{cris}}(H^i_{\text{cris}}(X_n/R_n))(r) \to H^i_{\text{syn}}(X_n, S^{[r]}_n). $$

First, consider the natural $G_R$-equivariant ‘Kunneth’ morphisms:

$$ \alpha'_R : H^0_{\text{cris}}(\tilde{R}_n/R_n) \otimes_R H^i_{\text{cris}}(X_n/R_n) \to H^i_{\text{cris}}(\overline{X}_n/R_n) $$

and

$$ \beta'_R : \bigoplus_{j=0}^r \left( H^0_{\text{cris}}(\tilde{R}_n/R_n, J^{[j]}_{\tilde{R}_n/R_n}) \otimes_R H^i_{\text{cris}}(X_n/R_n, J^{[r-j]}_{X_n/R_n}) \right) \to H^i_{\text{cris}}(\overline{X}_n/R_n, J^{[r]}_{X_n/R_n}). $$

By Corollary 4.2.3 and Lemma 4.4.1 we obtain $G_R$-equivariant morphisms:

$$ \alpha_R : OA_{\text{cris}, n}(R) \otimes_R H^i_{\text{syn}}(X_n, \mathcal{O}_{X_n/R_n}) \to H^i_{\text{syn}}(\overline{X}_n, \mathcal{O}_{\overline{X}_n/R_n}), $$

$$ \beta_R : \bigoplus_{j=0}^r \left( \mathcal{F}^j OA_{\text{cris}, n}(R) \otimes_R H^i_{\text{syn}}(X_n, J^{[r-j]}_{\overline{X}_n/R_n}) \right) \to H^i_{\text{syn}}(\overline{X}_n, J^{[r]}_{\overline{X}_n/R_n}). $$

By construction, $\varphi_r$ and $\nabla$ on $\mathcal{J}^{[r]}_{Y_n/S_n}$ are functorial in $Y_n$, and so the morphisms $\alpha_R$ and $\beta_R$ are compatible with these structures. When $i \leq p - 2$, by Theorem 3.0.1, $M := (H^i_{\text{cris}}(X_n/R_n), H^i_{\text{cris}}(X_n/R_n, J^{[j]}_{X_n/R_n}), \varphi_j, \nabla)$ is an object in $\mathcal{M}_r^{[0,i]}(R)$, and thus admits an adapted basis by Remark 2.2.3. It follows that $\beta_R$ induces the following map (which we also denote by $\beta_R$):

$$ \beta_R : F'(OA_{\text{cris}, n}(R) \otimes_R H^i_{\text{cris}}(X_n/R_n)) \to H^i_{\text{syn}}(\overline{X}_n, J^{[r]}_{\overline{X}_n/R_n}). $$

A similar construction involving $\tilde{R}_n$, $\tilde{X} := \tilde{R} \times_R X$, and $\tilde{\mathcal{J}}^{[r]}_n$ gives natural maps

$$ \alpha_{\tilde{R}} : A_{\text{cris}, n}(R) \otimes_R H^i_{\text{cris}}(X_n/R_n) \simeq A_{\text{cris}, n}(R) \otimes_{\tilde{R}_n} H^i_{\text{cris}}(\tilde{X}_n/\tilde{R}_n) \to H^i_{\text{cris}}(\overline{X}_n/\tilde{R}_n) \simeq H^i_{\text{syn}}(\overline{X}_n, \tilde{\mathcal{O}}_n), $$

$$ \beta_{\tilde{R}} : F'(A_{\text{cris}, n}(R) \otimes_R H^i_{\text{cris}}(X_n/R_n)) \to H^i_{\text{cris}}(\overline{X}_n/\tilde{R}_n, J^{[r]}_{\overline{X}_n/\tilde{R}_n}) \simeq H^i_{\text{syn}}(\overline{X}_n, \tilde{\mathcal{J}}^{[r]}_n). $$

Here we use the isomorphism $H^i_{\text{cris}}(\tilde{X}_n/\tilde{R}_n) \simeq \tilde{R} \otimes_R H^i_{\text{cris}}(X_n/R_n)$, which follows from Theorem [A.0.1] and Lemma 4.1.1.

Consider the natural map from $T_{\text{cris}}(M)$ to the kernel of the map

$$ \varphi_r - 1 : H^i_{\text{syn}}(\overline{X}_n, \tilde{\mathcal{J}}^{[r]}_n)^{\nabla=0} \to H^i_{\text{syn}}(\overline{X}_n, \mathcal{O}_n)^{\nabla=0}. $$

We obtain the commutative diagram

$$ \begin{array}{c}
\begin{array}{ccc}
T_{\text{cris}}(M)(r) & \to & H^i_{\text{syn}}(\overline{X}_n, \tilde{\mathcal{J}}^{[r]}_n)^{\nabla=0} \\
\downarrow & & \downarrow \\
K & \to & H^i_{\text{syn}}(\overline{X}_n, \tilde{\mathcal{J}}^{[r]}_n)^{\nabla=0} \\
\downarrow & & \downarrow \\
& & H^i_{\text{syn}}(\overline{X}_n, \tilde{\mathcal{O}}_n) \\
\end{array}
\end{array} $$

where $K$ denotes the kernel of the bottom horizontal map $\varphi_r - 1$. 

Proposition 4.5.1. For $i \leq r \leq p - 2$, we have a natural isomorphism $H^i_{\text{syn}}(\overline{X}, S_n^{[r]}) \simeq K$ such that the induced morphism $\iota_R : T_{\text{cris}}(M)(r) \to H^i_{\text{syn}}(\overline{X}, S_n^{[r]})$ is compatible with $G_R$-actions.

Proof. The exact sequence (9) yields the long-exact sequence

$$\cdots \to H^i_{\text{syn}}(\overline{X}, S_n^{[r]}) \to H^i_{\text{syn}}(\overline{X}, \mathcal{J}_n^{[r]}) \xrightarrow{\varphi_r^{-1}} H^i_{\text{syn}}(\overline{X}, \mathcal{O}_n) \to H^{i+1}_{\text{syn}}(\overline{X}, S_n^{[r]}) \to \cdots.$$ 

It suffices to prove that $\varphi_r - 1$ is surjective for $0 \leq i \leq p - 3$ and $i \leq r \leq p - 2$. Consider the commutative diagram

$$F^*(A_{\text{cris}, n}(R) \otimes_{\mathbb{R}_n} H^i_{\text{cris}}(X_n/R_n)) \xrightarrow{\varphi_r^{-1}} A_{\text{cris}, n}(R) \otimes_{\mathbb{R}_n} H^i_{\text{cris}}(X_n/R_n)$$

By Theorem 2.3.1, $\varphi_r - 1$ in the top row is surjective. Since $\alpha_{\mathbb{R}}$ is an isomorphism by Lemma 4.5.2 below, we conclude that the bottom row is surjective.

Thus, we obtain a morphism $\iota_R : T_{\text{cris}}(M)(r) \to H^i_{\text{syn}}(\overline{X}, S_n^{[r]})$. It remains to check that $\iota_R$ is compatible with $G_R$-actions. Note that this is not obvious a priori because $\mathcal{J}_n^{[r]}$ does not have a natural $G_R$-action (although it has a natural $G_X$-action). By Proposition 4.4.5, we can identify $\mathcal{J}_n^{[r]} \simeq \mathcal{J}_n^{[r]}$ and get the following commutative diagram

$$T_{\text{cris}}(M)(r) \xrightarrow{\delta} H^i_{\text{syn}}(\overline{X}, \mathcal{F}_n^{[r]}) \xrightarrow{\varphi_r - 1} H^i_{\text{syn}}(\overline{X}, \mathcal{J}_n^{[r]})$$

In particular, the $G_R$-equivariant map $T_{\text{cris}}(M)(r) \to H^i_{\text{syn}}(\overline{X}, \mathcal{J}_n^{[r]})$ factors through $\iota_R$ with $H^i_{\text{syn}}(\overline{X}, S_n^{[r]}) \to H^i_{\text{syn}}(\overline{X}, \mathcal{J}_n^{[r]})$ being injective, and thus $\iota_R$ is $G_R$-equivariant.

Lemma 4.5.2. For $i \leq p - 3$, $\alpha_{\mathbb{R}}$ is an isomorphism.

Proof. The usual base change map in crystalline cohomology (see Appendix A) gives rise to a hypertor spectral sequence with

$$E_2^{m,q} = H^m(A_{\text{cris}, n}(R) \otimes_{\mathbb{R}_n} H^q_{\text{cris}}(\tilde{\mathcal{X}}_n/\tilde{R}_n, \mathcal{O}_{\tilde{X}_n/\tilde{R}_n}))$$

which converges to

$$H^m_{\text{cris}}(\overline{X}_n/\tilde{R}_n, \mathcal{O}_{\overline{X}_n/\tilde{R}_n}).$$

Note that the $E_2^{m,q}$ terms are concentrated in the second quadrant, and after reindexing cochain complexes to chain complexes, we have

$$\text{Tor}_m^{\tilde{R}_n}(A_{\text{cris}, n}(R), H^q_{\text{cris}}(\tilde{\mathcal{X}}_n/\tilde{R}_n, \mathcal{O}_{\tilde{X}_n/\tilde{R}_n})) = H^{-m}(A_{\text{cris}, n}(R) \otimes_{\mathbb{R}_n} H^q_{\text{cris}}(\tilde{\mathcal{X}}_n/\tilde{R}_n, \mathcal{O}_{\tilde{X}_n/\tilde{R}_n})).$$
For \( i \leq p - 2 \), \( H^i_{\text{cris}}(X_n/R_n, \mathcal{O}_{X_n/R_n}) \) is a Fontaine-Laffaille module, so by Theorem 2.2.1 it is a direct sum of \( R/p^m R \) for some integers \( m \)'s. By the flat base change \( R_n \to \hat{R}_n \), \( H^i_{\text{cris}}(\hat{X}_n/\hat{R}_n, \mathcal{O}_{\hat{X}_n/\hat{R}_n}) \) is a direct sum of \( \hat{R}/p^m \hat{R} \) for some integers \( m \)'s when \( q \leq p - 2 \). Since \( A_{\text{cris},n}(R) \) is \( \mathbb{Z}/p^n \mathbb{Z} \)-flat, we have \( E^m_q = 0 \) when \( q \leq p - 2 \) and \( m < 0 \). Hence, \( \alpha_{\hat{R}} \) is an isomorphism when \( i \leq p - 3 \).

Remark 4.5.3. We do not know whether \( A_{\text{cris}}(R) \) is flat over \( \hat{R} \) in general, in which case \( \alpha_{\hat{R}} \) would be an isomorphism for any \( i \).

4.6. Construction of the comparison map, Part 2. Following the approach of Fontaine-Messing, we explain how to obtain a natural \( G_R \)-equivariant morphism

\[
H^i_{\text{syn}}(\mathcal{X}_n, \mathcal{S}_n^{[r]}) \to H^i_{\text{ét}}(X_{\mathbb{F}[1/p]}/\mathbb{Z}/p^n \mathbb{Z}(r)).
\]

For any \( \mathbb{Z} \)-module \( A \), we denote by \( \hat{A} \) its \( p \)-adic completion. For \( \overline{X} = X \times_R \mathbb{F} \) as before, let \( \hat{X} \) be the corresponding \( p \)-adic formal scheme. We shall construct a diagram of topoi

\[
(\mathcal{X}_n)_{\text{syn}} \hookrightarrow \hat{X}_{\text{syn-ét}} \xrightarrow{i} \mathcal{X}_{\text{syn-ét}} \xleftarrow{j} (X_{\mathbb{F}[1/p]})_{\text{ét}}
\]

and a sheaf \( \mathcal{S}_n^{[r]} \) on \( \mathcal{X}_{\text{syn-ét}} \) whose restriction to \( \hat{X}_{\text{syn-ét}} \) is \( \iota_* (\mathcal{S}_n^{[r]}) \) and whose pull-back to \( (X_{\mathbb{F}[1/p]})_{\text{ét}} \) is \( \mathbb{Z}/p^n \mathbb{Z}(r) \).

This will allow us to construct the desired comparison map. The constructions below follow the approach of Fontaine-Messing very closely. The main technical modification is that instead of passing to the rigid analytic generic fiber in defining the site \( \hat{X}_{\text{syn-ét}} \), we work directly with restricted power series algebras.

We begin by defining the \textit{syntomic-étale} site of the \( p \)-adic formal scheme \( \hat{X} \). For any morphism of \( p \)-adic affine formal schemes \( g : \text{Spf} \mathcal{B} \to \text{Spf} \mathcal{A} \) over \( \hat{X} \), we say \( g \) is syntomic if for each integer \( m \geq 1 \), the associated morphism \( A_m \to B_m \) is syntomic. We say \( g \) has étale generic fiber if it is of finite presentation, and the associated ring map \( A[\frac{1}{p}] \to B[\frac{1}{p}] \) is flat and the universal finite differential module \( \Omega_{\mathcal{B}[\frac{1}{p}]/A[\frac{1}{p}]}^f \) is 0. Note that if \( g \) has étale generic fiber, then \( \mathcal{B} \) has a presentation \( \mathcal{B} = \mathcal{A} \langle X_1, \ldots, X_m \rangle / (P_1, \ldots, P_m) \) such that the Jacobian \( J = \det \left( \frac{\partial P_i}{\partial X_j} \right) \) is invertible in \( B[\frac{1}{p}] \). The objects of the (small) syntomic-étale site of \( \hat{X} \) are defined to be the morphisms of \( p \)-adic formal schemes \( \mathcal{Y} \to \hat{X} \) which are locally quasi-finite, syntomic, and have étale generic fiber. The topology is generated by surjective families of open immersions and finite surjective families of locally quasi-finite syntomic morphisms having étale generic fibers with source and target both affine schemes. From the immersion \( \mathcal{X}_n \to \hat{X} \), we have a natural morphism of topoi

\[
\iota : (\mathcal{X}_n)_{\text{syn}} \to \hat{X}_{\text{syn-ét}}
\]

where \( \iota_* \) is given by \( \iota_*(F)(Y) = F(Y_n) \).

\footnote{In the following, we work with the small sites.}
Lemma 4.6.1. For any \( n \geq 1 \), \( \iota_* \) is an exact functor.

Proof. By [Sta18 Tag 0570], it suffices to prove that if \( \mathcal{A} \) is a \( p \)-adically complete \( p \)-torsion free \( R \)-algebra and \( f : \mathcal{A}_n \to B \) is a syntomic quasi-finite morphism, then locally \( B \) can be lifted to a \( p \)-adically complete \( R \)-algebra \( B' \) and \( f \) can be lifted to \( \tilde{f} : \mathcal{A} \to B' \) such that \( \text{Spf} B' \to \text{Spf} \mathcal{A} \) is syntomic, quasi-finite with \( \acute{e}t \)al generic fiber. By [Sta18 Tag 0DWJ], locally \( B \) can be written as \( B = \mathcal{A}_n[X_1, \ldots, X_m]/(g_1, \ldots, g_m) \) such that \( \mathcal{A}_n \to B \) is a relative global complete intersection. Consider the syntomic map \( B \to B' := \mathcal{A}_n[Y_1, \ldots, Y_m]/(g_1(Y_{p^{n+1}}), \ldots, g_m(Y_{p^{n+1}})) \) given by \( X_i \mapsto Y_i^{p^{n+1}} \), and let \( B' = \mathcal{A}(Y_1, \ldots, Y_m)/(\tilde{g}_1(Y_{p^{n+1}}) + p^nY_1, \ldots, \tilde{g}_m(Y_{p^{n+1}}) + p^nY_m) \) where \( \mathcal{A}(Y_1, \ldots, Y_m) \) denotes the \( p \)-adic completion of \( \mathcal{A}[Y_1, \ldots, Y_m] \) and \( \tilde{g}_i \) is a lift of \( g_i \). Note that \( \mathcal{B}/p^n\mathcal{B} = B' \).

Letting \( \tilde{g}_i = \tilde{g}_i(Y_{p^{n+1}}) + p^nY_i \), we claim that \( (\tilde{g}_1, \ldots, \tilde{g}_m) \) is Koszul-regular in \( \mathcal{A}(Y_1, \ldots, Y_m) \). For the Koszul complex \( K_{\mathcal{A}}(\tilde{g}_1, \ldots, \tilde{g}_m) \), we have \( H_i(K_{\mathcal{A}}(\tilde{g}_1, \ldots, \tilde{g}_m))/p^nH_i(K_{\mathcal{A}}(\tilde{g}_1, \ldots, \tilde{g}_m)) = 0 \) for \( i > 0 \) since \( \mathcal{A}_n \to B' \) is syntomic and \( \mathcal{A}(Y_1, \ldots, Y_m) \) is \( p \)-torsion free. Since the terms in \( K_{\mathcal{A}}(\tilde{g}_1, \ldots, \tilde{g}_m) \) is \( p \)-adically complete, it follows that \( H_i(K_{\mathcal{A}}(\tilde{g}_1, \ldots, \tilde{g}_m)) = 0 \) for \( i > 0 \). Moreover, \( B' \) is \( p \)-torsion free by [Sta18 Tag 068M].

Hence, the map \( \tilde{f} : \mathcal{A} \to B' \) is quasi-finite, syntomic, and has \( \acute{e}t \)al generic fiber. \( \square \)

For \( \iota : (\mathcal{X}_n)_\text{syn} \to (\mathcal{X})_\text{syn-\acute{e}t} \), we write \( S_n^{[r]} \) for \( \iota_*S_n^{[r]} \) and \( (\mathcal{J}_n^{[r]})_{\varphi=1} \) for \( \iota_*((\mathcal{J}_n^{[r]})_{\varphi=1}) \) to simplify notations. By the previous lemma, we have

\[
H^i((\mathcal{X}_n)_\text{syn}, S_n^{[r]}) \cong H^i((\mathcal{X})_\text{syn-\acute{e}t}, S_n^{[r]}),
\]

and similarly for \( (\mathcal{J}_n^{[r]})_{\varphi=1} \).

We define the syntomic-\( \acute{e}t \)ale site of \( \mathcal{X} \) so that the objects are morphisms \( f : U \to \mathcal{X} \) which are syntomic, locally quasi-finite, and \( f \otimes_R \mathcal{O}_p[\frac{1}{p}] \) is \( \acute{e}t \)al. The topology is generated by surjective families of open immersions of finite surjective families of locally quasi-finite syntomic morphisms with source and target both affine. By definition, we have a natural morphism of topoi \( j : (X_{\mathcal{O}_p}[\frac{1}{p}])_\text{\acute{e}t} \to (\mathcal{X})_\text{syn-\acute{e}t} \). Similarly, we have a natural morphism of topoi \( i : (\mathcal{X}_{\text{syn-\acute{e}t}})_\text{synt-\acute{e}t} \to (\mathcal{X})_\text{synt-\acute{e}t} \) as follows. If Spec\( B \) is an affine object of the syntomic-\( \acute{e}t \)ale site of \( \mathcal{X} \), then \( i_*F(\text{Spec} B) := F(\text{Spf} B) \). We obtain a description of \( i_* \) via the following lemma.

Lemma 4.6.2. Let Spec\( \mathcal{A} \) be Zariski open in \( \mathcal{X} \), and \( \mathcal{B} \) a \( p \)-adically complete algebra equipped with a syntomic, quasi-finite morphism \( \tilde{f} : \mathcal{A} \to \mathcal{B} \) with \( \acute{e}t \)al generic fiber. Then there exists a syntomic, quasi-finite morphism \( \hat{f} : \mathcal{A} \to \mathcal{B} \) with \( \acute{e}t \)al generic fiber such that \( \hat{B} = \mathcal{B} \) and \( \hat{f} = \tilde{f} \).

Proof. Choose a presentation \( \mathcal{B} = \mathcal{A}(X_1, \ldots, X_m)/(P_1, \ldots, P_m) \) such that \( J = \det(\partial_{P_i}/\partial X_j) \in \mathcal{A}(X_1, \ldots, X_m) \) is invertible in \( \mathcal{B}[\frac{1}{p}] \). Then there exist \( a, a_1, \ldots, a_m \in \mathcal{A}(X_1, \ldots, X_m) \) such
that

\[ Ja = p^r + \sum_{i=1}^{m} a_i P_i \]

for some integer \( r \geq 0 \). For each \( i = 1, \ldots, m \), let \( Q_i \in A[X_1, \ldots, X_m] \) such that \( Q_i \equiv P_i \mod p^{2r+1} \) in \( \hat{A}\langle X_1, \ldots, X_m \rangle \). Let \( B = A[X_1, \ldots, X_m]/(Q_1, \ldots, Q_m) \). Note that by a similar argument as in the proof of Lemma \[ \ref{4.6.1} \] \((Q_1, \ldots, Q_m) \subset \hat{A}\langle X_1, \ldots, X_m \rangle \) is Koszul-regular and \( \hat{B} \) is \( p \)-torsion free. We claim that there exists a unique \( \hat{A} \)-algebra map \( h: B \to \hat{B} \) which is the identity map mod \( p^{3r+1} \).

To prove the claim, we construct a compatible system of \( \hat{A} \)-algebra maps \( h_n: B \to \hat{B}/p^n\hat{B} \) for each integer \( n \geq 3r + 1 \) inductively as follows. Let \( h_{3r+1} \) be given by \( X_i \mapsto X_i \). Suppose for \( n \geq 3r + 1 \), \( h_n \) is given by \( X_i \mapsto X_i + u_i \) where \( u_i \in p^{2r+1} \hat{B} \). Then \( P_i(X_1 + u_1, \ldots, X_m + u_m) \in p^n\hat{B} \) for each \( i \). By Taylor series expansion, we have

\[ P_i(X + u + \mu) = P_i(X + u) + \sum_{l=1}^{m} \frac{\partial P_i}{\partial X_l}(X + u) \cdot \mu_l + \sum_{|b|\geq 2} \frac{\partial^b P_i}{\partial X^b}(X + u) \cdot \frac{\mu^b}{b!} \]

where \( b = (b_1, \ldots, b_m) \in \mathbb{N}^m \) denotes a multi-index in the last term. From equation (11), there exist \( \mu_1, \ldots, \mu_m \in p^{n-r}\hat{B} \), uniquely determined mod \( p^{n-r} \hat{B} \), such that \( P_i(X + u) + \sum_{l=1}^{m} \frac{\partial P_i}{\partial X_l}(X + u) \cdot \mu_l \in p^{n+1}\hat{B} \), and thus \( P_i(X + u + \mu) \in p^{n+1}\hat{B} \) for each \( i \). This proves the claim. Similarly, there exists a unique \( \hat{A} \)-algebra map \( \hat{B} \to B \) which is the identity mod \( p^{3r+1} \), and therefore \( h: B \to \hat{B} \) is an isomorphism.

Since \( B \) is finitely presented over \( A \) and \( \hat{B} \) is \( p \)-torsion free, we have that \( B \) is \( p \)-torsion free. Let \( j \in B \) be the jacobian. Then for some \( g, b \in B \), we have \( jg = p^r + p^{r+1}b = p^r(1 + pb) \). If we replace \( B \) by \( B[\frac{1}{1+pb}] \), then the map \( f: A \to B \) has étale generic fiber. Furthermore, since \( A \to B \) is syntomic, \( A/pA \to B/pB \) is flat, and thus \( f: A \to B \) is flat.

It follows easily that \( f: A \to B \) satisfies the desired properties.

We may now define \( i^*F \) as the sheafification of the presheaf \( B \mapsto \varprojlim F(B') \), where \( \varprojlim F(B') \) is the filtered colimit of the commutative diagrams

\[
\begin{array}{ccc}
B & \longrightarrow & B' \\
\downarrow & & \downarrow \\
B & \longrightarrow & B
\end{array}
\]

such that \( B \) is a fixed choice of a ring satisfying the conditions in Lemma \[ \ref{4.6.2} \]. \( B \to B' \) is étale, and the map \( B' \to B \) induces an isomorphism \( \hat{B}' \cong \hat{B} \). We claim that \( \varprojlim F(B') \) does not depend on the choice of \( B \), and \( i^*F \) is well-defined. For this, suppose \( B \) and \( B' \) are \( A \)-algebras both satisfying the conditions in Lemma \[ \ref{4.6.2} \] and let \( B'' \subset B \) be the \( A \)-algebra given by the image of the induced map \( B \otimes_A B' \to B \). Then we claim that the maps \( B \to B'' \) and \( B' \to B'' \) are étale, and \( B'' \) also satisfies the conditions in
Lemma 4.6.2] First, note that for each $n \geq 1$, the induced map $B/p^nB \to B''/p^nB''$ is an isomorphism, and $B'' \cong B$. In particular, $A/pA \to B''/pB''$ is flat, and $B''$ is $p$-torsion free as $B'' \subset B$. Write $\overline{R} = \lim_{\eta \to R} R_{\eta}$ where the limit goes over finite normal $R$-algebras $R_{\eta} \subset \overline{R}$ such that $R_{\eta}[1/p]/R[1/p]$ is étale. Then there exists an index $\alpha$, finitely generated flat $R_{\alpha}$-algebra $A_{\alpha}$, and finitely generated $A_{\alpha}$-algebra $B_{\alpha}$ such that $A = \lim_{\eta \to \alpha} A_{\alpha} \otimes_{R_{\alpha}} R_{\eta}$ and $B'' \cong \lim_{\eta \to \alpha} B_{\alpha} \otimes_{R_{\alpha}} R_{\eta}$. For indices $\eta \geq \alpha$, denote $A_{\eta} = A_{\alpha} \otimes_{R_{\alpha}} R_{\eta}$ and $B_{\eta} = B_{\alpha} \otimes_{R_{\alpha}} R_{\eta}$.

Then there exists an index $\beta \geq \alpha$ such that $A_{\beta}/pA_{\beta} \to B_{\beta}/pB_{\beta}$ is flat. Furthermore, the natural map $\text{Tor}_1^A(B_{\eta}/pA_{\eta}) \otimes_{R_{\eta}} R_{\alpha} \to \text{Tor}_1^A(B_{\alpha}/pA_{\alpha})$ is surjective for any $\lambda \geq \eta \geq \beta$. Since $\text{Tor}_1^A(B_{\gamma}/pA_{\gamma}) = 0$, we have $\text{Tor}_1^A(B_{\gamma}/pA_{\gamma}) = 0$ for some $\gamma \geq \beta$. Then by the local criterion of flatness, $B_{\gamma}$ is flat over $A_{\gamma}$, and so $B''$ is flat over $A$. By a similar argument, we deduce that $B''$ is flat over both $B$ and $B'$. Since $(B \otimes_A B')[\frac{1}{p}]$ is étale over $B[\frac{1}{p}]$ and $(B \otimes_A B')[\frac{1}{p}] \to B''[\frac{1}{p}]$ is a closed immersion, $B''[\frac{1}{p}]$ is unramified over $B[\frac{1}{p}]$. Since $B/pB \cong B''/pB''$, $B''$ is unramified over $B$ and so it is étale over $B$. Similarly, $B''$ is étale over $B'$.

We denote $B^h := \lim_{\eta \to \beta} B_{\eta}$ where the colimit is taken over the diagrams as above. Note that $B^h$ is the henselization of $B$ with respect to $(p)$ as defined in [Ray70].

**Proposition 4.6.3.** The functor $G \mapsto (i^*G, j^*G, \alpha)$ is an equivalence of categories between the category of sheaves on $\overline{X}_{\text{syn-ét}}$ to the category of triples $(F, H, \alpha)$ where $F$ is a sheaf on $(\overline{X})_{\text{syn-ét}}$, $H$ a sheaf on $(\overline{X}_{\overline{R}[\frac{1}{p}]})_{\text{ét}}$, and $\alpha : F \to i^*j^*_sH$ a morphism.

Furthermore, $i_*^*$ is exact, and the functor $j_!$ from the category of sheaves on $(\overline{X}_{\overline{R}[\frac{1}{p}]})_{\text{ét}}$ to that on $\overline{X}_{\text{syn-ét}}$ given by $j_!(\alpha) := (0, j^*_sG, 0)$ is exact. For any sheaf $G$ on $\overline{X}_{\text{syn-ét}}$, the sequence

$$0 \to j_!j^*G \to G \to i_*i^*G \to 0$$

is exact.

**Proof.** Note that for any sheaf $F$ on $(\overline{X})_{\text{syn-ét}}$, we have $i^*i_*F \cong F$ and $j^*i_*F \cong 0$. And for any sheaf $H$ on $(\overline{X}_{\overline{R}[\frac{1}{p}]})_{\text{ét}}$, $j^*j_!H \cong H$.

For a sheaf $G$ on $\overline{X}_{\text{syn-ét}}$, consider the commutative diagram

$$\begin{array}{ccc}
G & \longrightarrow & i_*i^*G \\
\downarrow & & \downarrow \\
\overline{j}_*j^*G & \longrightarrow & i_*i^*\overline{j}_*j^*G
\end{array}$$

We claim that it is Cartesian. Consider the natural morphism $G \to i_*i^*G \times (i_*i^*j_*\overline{j}^*G) j_*\overline{j}^*G$. Since $i^*$ and $j^*$ are exact,

$$i^*(i_*i^*G \times (i_*i^*j_*\overline{j}^*G) j_*\overline{j}^*G) \cong i^*i_*i^*G \times (i_*i^*j_*\overline{j}^*G) i^*j_*\overline{j}^*G \cong i^*G \times i_*j_*\overline{j}^*G \cong i^*G,$$

$$j^*(i_*i^*G \times (i_*i^*j_*\overline{j}^*G) j_*\overline{j}^*G) \cong j^*i_*i^*G \times (j_*i_*j_*\overline{j}^*G) j_*\overline{j}^*G \cong 0 \times 0 \cong 0.$$
Thus, the morphism $G \rightarrow i_*i^*G \times_{(i_*i^*j_*j^*G)} j_*j^*G$ is an isomorphism. This proves that the diagram is Cartesian, which implies the first statement.

The exactness of $i_*$ follows from its definition. The remaining statements are direct consequences. \hfill \square

We now construct a sheaf $S_n^{[r]}$ on $\text{X}_{\text{syn-}\acute{e}t}$ such that $i^*S_n^{[r]} = S_n^{[r]}$ and $j^*S_n^{[r]} = \mathbb{Z}/p^n\mathbb{Z}(r)$. By Proposition 4.6.3, this is equivalent to defining a map $\alpha : S_n^{[r]} \to i^*j_*\mathbb{Z}/p^n\mathbb{Z}(r)$. For a map $\tilde{f} : \tilde{A} \to \mathcal{B}$ with $f : A \to B$ as in Lemma 4.6.2, we need to define a functorial map $S_n^{[r]}(\mathcal{B}) \to \mathbb{Z}/p^n\mathbb{Z}(r)(B^h[\frac{1}{p}])$. If we write $\overline{R} = \varprojlim_{\eta} R_\eta$ as above, then there exists an index $\lambda$, finitely generated smooth $\mathcal{R}_\lambda$-algebra $A_\lambda$, and finitely generated $A_\lambda$-algebra $B_\lambda$ with $f_\lambda : A_\lambda \to B_\lambda$ such that $A \cong \varprojlim_{\eta} A_\lambda \otimes_{R_\eta} R_\eta$ and $B \cong \varprojlim_{\eta} B_\lambda \otimes_{R_\eta} R_\eta$, and $f_\lambda$ is syntomic and quasi-finite having étale generic fiber inducing $f$. Moreover, by taking large enough $R_\lambda$, we can assume that $\lambda R_\lambda$ contains a primitive $p^n$-th root of 1.

Note that $S_n^{[r]}(\mathcal{B}) = S_n^{[r]}(\mathcal{B}/p^n) = \varprojlim_{\eta} S_n^{[r]}(B_\eta/p^n)$, and $\mathbb{Z}/p^n\mathbb{Z}(r)(B^h[\frac{1}{p}]) = \varprojlim_{\eta} \mathbb{Z}/p^n\mathbb{Z}(r)(B^h_\eta[\frac{1}{p}])$. Denote $A' = A_\lambda$ and $B' = B_\lambda$. Since $\pi_0(\text{Spec}(B')^{h[\frac{1}{p}]}) = \pi_0(\text{Spec}(\hat{B}'[\frac{1}{p}]))$ by [FR70, Cor. 4.4], it suffices to construct a functorial morphism $S_n^{[r]}(B'/p^n) \to \mathbb{Z}/p^n\mathbb{Z}(r)(\hat{B}'[\frac{1}{p}])$.

Since $\hat{B}'[\frac{1}{p}]$ is étale over $\tilde{A}'[\frac{1}{p}]$ and Noetherian, we can write $\hat{B}'[\frac{1}{p}] \cong \prod_{a=1}^l B'_a$ where each $B'_a$ is an integral domain which is regular and thus normal. Each $B'_a$ is an affinoid algebra over $\overline{R}[\frac{1}{p}]$, and we denote by $\hat{B}'_a$ the subring of $B'_a$ consisting of elements with spectral norm $\leq 1$. Note that $\hat{B}'_a$ is integrally closed in $B'_a$ and thus is a normal domain. We have a natural map $S_n^{[r]}(B'/p^n) \to \prod_{a=1}^l S_n^{[r]}(B'_a/p^n)$. On the other hand, $\mathbb{Z}/p^n\mathbb{Z}(r)(\hat{B}'[\frac{1}{p}]) \cong \prod_{a=1}^l (\mu_{p^n}(\overline{R}[\frac{1}{p}])^{\otimes r})_a$.

Let $E = \text{Frac}(A')$ equipped with the $p$-adic valuation, and let $C$ be the $p$-adic completion of the algebraic closure of $\hat{E}$. For each $a = 1, \ldots, l$, choose an embedding $\hat{B}'_a \to C$, which induces an embedding $\hat{B}'_a \to \mathcal{O}_C$. This induces a map $S_n^{[r]}(\hat{B}'_a/p^n) \to S_n^{[r]}(\mathcal{O}_C/p^n)$. Note that $C$ contains $\overline{R}[\frac{1}{p}]$. We claim that $S_n^{[r]}(\mathcal{O}_C/p^n) = \mu_{p^n}(\overline{R}[\frac{1}{p}])^{\otimes r}$. Indeed, we have $S_n^{[r]}(\mathcal{O}_C) = H^0_{\text{cris}}(\mathcal{O}_{C,n}/R_n, J_{\mathcal{O}_{C,n}/R_n}^{[r]}\varphi_r = 1, \nabla = 0).$ Since $\varphi : \mathcal{O}_C/p \to \mathcal{O}_C/p$ is surjective, we have by Proposition 4.2.3 and Lemma 4.2.7

$$S_n^{[r]}(\mathcal{O}_C) = H^0_{\text{cris}}(\mathcal{O}_{C,n}/R_n, J_{\mathcal{O}_{C,n}/R_n}^{[r]}\varphi_r = 1, \nabla = 0) = (F^r A_{\text{cris},n}(\mathcal{O}_C))^{\varphi_r = 1} = \mu_{p^n}(C)^{\otimes r}$$

where the last equality uses the well-known result in [Fon82]. We obtain the desired morphism $\alpha$ in this way.

**Proposition 4.6.4.** The natural map $H^*(\overline{X}_{\text{syn-}\acute{e}t}, S_n^{[r]}) \to H^*(\hat{X}_{\text{syn-}\acute{e}t}, S_n^{[r]})$ induced by $i$ is an isomorphism.
Proof. By Proposition 4.6.3 we need to show that $H^*(\overline{X}_{\text{syn-ét}}, j_! \mathbb{Z}/p^n \mathbb{Z}) = 0$. We have an exact sequence
\[ 0 \to j_!(\mathbb{Z}/p^n \mathbb{Z}) \to \mathbb{Z}/p^n \mathbb{Z} \to i^*(\mathbb{Z}/p^n \mathbb{Z}) \to 0. \]
Since $i^*$ is exact, it suffices to show that the natural map
\[ H^*(\overline{X}_{\text{syn-ét}}, \mathbb{Z}/p^n \mathbb{Z}) \to H^*(\widetilde{\overline{X}}_{\text{syn-ét}}, \mathbb{Z}/p^n \mathbb{Z}) \]
is an isomorphism.

By [Sta18, Tag 0DDU], the natural map
\[ H^*(\overline{X}_\text{ét}, \mathbb{Z}/p^n \mathbb{Z}) \to H^*(\overline{X}_{\text{syn-ét}}, \mathbb{Z}/p^n \mathbb{Z}) \]
induced by the Leray spectral sequence for $\overline{X}_{\text{syn-ét}} \to \overline{X}_\text{ét}$ is an isomorphism since the syntomic-étale topology on $\overline{X}_{\text{syn-ét}}$ is finer than the étale topology and coarser than the flat topology. Similarly, we have $H^*(\overline{X}_\text{ét}, \mathbb{Z}/p^n \mathbb{Z}) \cong H^*(\overline{X}_{\text{syn-ét}}, \mathbb{Z}/p^n \mathbb{Z})$. On the other hand, the natural map $H^*(\overline{X}_\text{ét}, \mathbb{Z}/p^n \mathbb{Z}) \to H^*(\widetilde{\overline{X}}_{\text{ét}}, \mathbb{Z}/p^n \mathbb{Z})$ is an isomorphism by the proper base change theorem.

By the previous proposition, we obtain a map
\[ H^*_\text{syn}(\overline{X}_n, S^{[r]}_n) \to H^*_\text{ét}(\overline{X}_{\overline{R}[\frac{1}{p}]}, \mathbb{Z}/p^n \mathbb{Z}(r)) \]
which is compatible with $G_R$-actions.

4.7. Proof of the main theorem. In summary, for $i \leq r \leq p - 2$, we have constructed a map
\[ \iota : T_{\text{cris}}(H^i_{\text{cris}}(X_n/R_n))(r) \to H^i_{\text{syn}}(\overline{X}_n, S^{[r]}_n) \to H^i_{\text{ét}}(\overline{X}_{\overline{R}[\frac{1}{p}]}, \mathbb{Z}/p^n \mathbb{Z}(r)) \]
which is compatible with $G_R$-actions. On the other hand, consider the morphism $b_g : R \to R_g$ (which factors through $\overline{R}$) together with a choice of an extension $\overline{b}_g : \overline{R} \to \overline{R}_g$. Setting $X_g := X \times_R R_g$ (viewed over $R_g$), we similarly obtain a $G_{R_g}$-equivariant map
\[ \iota_g : T_{\text{cris}}(H^i_{\text{cris}}(X_{g,n}/R_{g,n}))(r) \to H^i_{\text{syn}}(\overline{X}_{g,n}, S^{[r]}_{g,n}) \to H^i_{\text{ét}}(\overline{X}_{\overline{R}_g[\frac{1}{p}]}, \mathbb{Z}/p^n \mathbb{Z}(r)). \]
The map $\overline{R} \to \overline{R}_g$ induces a map of topoi $i : (\overline{X}_{g,n})_{\text{syn}} \to (\overline{X}_n)_{\text{syn}}$, and we have a natural map $\mathcal{J}_{\overline{X}_{g,n}/\overline{R}_n} \to i_* \mathcal{J}_{\overline{X}_{g,n}/\overline{R}_n}$. By a similar argument as in the proof of Theorem 3.0.1 (2), this map is compatible with $\varphi_r$. Therefore, we obtain a natural map $H^i_{\text{syn}}(\overline{X}_n, S^{[r]}_n) \to H^i_{\text{syn}}(\overline{X}_{g,n}, S^{[r]}_{g,n})$. The construction of the comparison map is functorial, and we therefore have the following commutative diagram:

\[
\begin{array}{ccc}
T_{\text{cris}}(H^i_{\text{cris}}(X_n/R_n))(r) & \xrightarrow{\iota} & H^i_{\text{ét}}(\overline{X}_{\overline{R}[\frac{1}{p}]}, \mathbb{Z}/p^n \mathbb{Z}(r)) \\
\downarrow & & \downarrow \\
T_{\text{cris}}(H^i_{\text{cris}}(X_{g,n}/R_{g,n}))(r) & \xrightarrow{\iota_g} & H^i_{\text{ét}}(\overline{X}_{\overline{R}_g[\frac{1}{p}]}, \mathbb{Z}/p^n \mathbb{Z}(r))
\end{array}
\]
Note that our construction for \( \iota_g \) is the same as that of [FM87], and therefore \( \iota_g \) is an isomorphism. By Theorem A.0.1 and Corollary 2.3.5, the left vertical map is an isomorphism. As the right vertical is also an isomorphism by the smooth and proper base change for étale cohomology, we conclude that \( \iota \) is an isomorphism. This completes the proof of the main theorem.

5. Applications

In this section, we study some applications of Fontaine-Messing theory over \( R \) analogous to those in the original setting of Fontaine-Messing (see [FM87 §3, 4]).

5.1. Crystalline property of \( \mathcal{H}^i_{\text{ét}}(X_{\mathcal{R}(p)}; \mathbb{Q}_p) \). As before, we consider a smooth proper scheme \( X \) over \( R \). Let \( \mathcal{H}^i_{\text{ét}}(X_{\mathcal{R}(p)}, \mathbb{Z}_p) \) := \( \lim_n \mathcal{H}^i_{\text{ét}}(X_{\mathcal{R}(p)}; \mathbb{Z}/p^n\mathbb{Z}) \), \( T^i := \mathcal{H}^i_{\text{ét}}(X_{\mathcal{R}(p)}; \mathbb{Z}/p) / \text{torsion} \) and \( \mathcal{H}^i_{\text{cris}}(X_{\mathcal{R}(p)}; \mathbb{Q}_p) := \mathcal{H}^i_{\text{ét}}(X_{\mathcal{R}(p)}; \mathbb{Q}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \). Let \( \mathcal{H}^i_{\text{cris}}(X/R) := \lim_n \mathcal{H}^i_{\text{cris}}(X_n/R_n) \) and \( \mathcal{M}^i := \mathcal{H}^i_{\text{cris}}(X/R) / \text{torsion} \). Recall that we say a finite continuous \( \mathbb{Q}_p \)-representation \( V \) of \( \mathcal{G}_R \) is crystalline if the natural injective map

\[
\alpha_{\text{cris}} : D^{\cris}(V) \otimes R[p] \mathcal{O}_{\text{cris}}(R) \to V \otimes_{\mathbb{Q}_p} \mathcal{O}_{\text{cris}}(R)
\]

is an isomorphism, where \( \mathcal{O}_{\text{cris}}(R) := \mathcal{O}_{A_{\text{cris}}(R)}[\frac{1}{p}, \frac{1}{p}] \) with \( \beta = \log([\ell]) \) as defined in Section 2.1, and \( D^{\cris}(V) := (V \otimes_{\mathcal{O}_{\text{cris}}(R)} \mathcal{O}_{\text{cris}}(R))^G_R \) (see [Br10] Section 8).

**Theorem 5.1.1.** For \( i \leq p - 2 \), the natural map

\[
\mathcal{H}^i_{\text{ét}}(X_{\mathcal{R}(p)}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_{\text{cris}}(R)[\frac{1}{p}] \to \mathcal{H}^i_{\text{cris}}(X/R) \otimes_R \mathcal{O}_{\text{cris}}(R)[\frac{1}{p}]
\]

is injective, and its cokernel is killed by \( \beta^i \). In particular, \( \mathcal{H}^i_{\text{ét}}(X_{\mathcal{R}(p)}, \mathbb{Q}_p) \) is a crystalline representation of \( \mathcal{G}_R \).

**Proof.** By our main theorem with \( r = i \) (Theorem 2.2.1 and Theorem 2.3.1), we have

\[
T^i(i) = T^{\text{cris}}(M^i)(i) := F^i(\mathcal{O}_{A^{\text{cris}}(R)} \otimes_R M^i)^{a=1, v=0} \subset \mathcal{O}_{A^{\text{cris}}(R)} \otimes_R M^i,
\]

and \( M^i \) is finite \( R \)-free with the same type as \( T^i(i) \). Choose an \( R \)-basis \( \{e_1, \ldots, e_m\} \) of \( M^i \) and a \( \mathbb{Z}_p \)-basis \( \{f_1, \ldots, f_m\} \) of \( T^i \). Let \( A = (a_{ij}) \) be the \( m \times m \) matrix with entries in \( \mathcal{O}_{A^{\text{cris}}(R)} \) such that \( (f_1, \ldots, f_m) = (e_1, \ldots, e_m)A \). We claim that there exists an \( m \times m \) matrix \( B = (b_{ij}) \) with entries in \( \mathcal{O}_{A^{\text{cris}}(R)}[\frac{1}{p}] \) such that \( AB = \beta^i I_m \).

To prove the claim, we first project \( \overline{M}_R := \mathcal{O}_{A^{\text{cris}}(R)} \otimes_{\lambda_2} M^i \) to \( \overline{M} := A^{\text{cris}}(R) \otimes_{\lambda_2} M \) via \( \pi_M \) as in Lemma 2.3.3. By the same argument as in the proof of Lemma 2.3.3, we obtain the following commutative diagram:

\[
\begin{array}{ccc}
T^i(i) & \longrightarrow & \overline{M}_R \\
\downarrow & & \downarrow \cong \\
\overline{M} & \longrightarrow & M
\end{array}
\]
Moreover, $\mathcal{M}_R = \mathcal{M}_R^{\nabla = 0} \otimes_{\Lambda_{\text{cris}}(R)} O_{\text{cris}}(R)$. So it suffices to prove the claim after replacing $e_j$’s and $f_j$’s by $\pi_M(e_j)$’s and $\pi_M(f_j)$’s respectively, and $A$ by $\pi(A)$ (the corresponding matrix with entries in $A_{\text{cris}}(R)$).

We now have

$$T^i(i) \cong F^i(A_{\text{cris}}(R) \otimes_R M^i)^{\varphi = 1} \subset A_{\text{cris}}(R) \otimes_R M^i$$

with $Z_p$-basis $\{f_j\}$ of $T^i$ and an $R$-basis $\{e_j\}$ of $M^i$ with $(f_1, \ldots, f_m) = (e_1, \ldots, e_m)A$. It suffices to show there exists an $m \times m$ matrix $B$ with entries in $A_{\text{cris}}(R)[\frac{1}{p}]$ such that $AB = \beta^i I_m$. By Remark 2.2.3 we may assume $\{e_1, \ldots, e_m\}$ is an adapted basis, so that $e_j \in F^{r_j}M^i \setminus F^{r_j+1}M^i$ for $r_j \leq r_{j+1} \leq i$. Then the structure $(M^i, F^j(M^i), \varphi_j)$ corresponds to an invertible matrix $D \in \text{GL}_m(R)$ via $(\varphi_{r_1}(e_1), \ldots, \varphi_{r_n}(e_m)) = (e_1, \ldots, e_m)D$. Since

$$(f_1, \ldots, f_m) = (e_1, \ldots, e_m)A \subset F^i(A_{\text{cris}}(R) \otimes R M)^{\varphi = 1},$$

we conclude that $D\varphi_{i-r}(A) = A$, where $\varphi_{i-r}(A)$ has $(k, l)$-entry $\varphi_{i-r}(a_{kl})$ with $a_{kl} \in F^{i-r_k}A_{\text{cris}}(R)$ being $(k, l)$-entry of $A$.

Consider $(N, F^jN, \varphi_j)$ defined as follows: $N$ is a finite free $R$-module with basis $\{e_1^\gamma, \ldots, e_m^\gamma\}$, and $F^jN$’s are direct summands of $N$ as $R$-modules such that $e_j^\gamma \in F^{i-r_j}N \setminus F^{i-r_j+1}N$. Define $\varphi_j$-structure via $(\varphi_{i-r_1}(e_1^\gamma), \ldots, \varphi_{i-r_m}(e_m^\gamma)) = (e_1^\gamma, \ldots, e_m^\gamma)(D^T)^{-1}$. Then $N/p^n$ is an object of $\text{MF}^{[0, \beta]}(R)$ for each $n$. In particular, $F^i(N \otimes_R A_{\text{cris}}(R))^{\varphi = 1}$ is finite $Z_p$-free with the same type as $N$ by the proof of Theorem 2.3.1 (note that the proof does not need the existence of $\nabla$). Similarly as above, we then have a matrix $Z = (z_{kl})_{m \times m}$ with $z_{kl} \in F^{r_k}A_{\text{cris}}(R)$ such that $(D^{-1})^T \varphi(Z) = Z$, where $(k, l)$-entry of $\varphi(Z)$ is $\varphi_{r_k}(z_{kl})$. Now consider $Z^TA = (y_{kl})_{m \times m}$. We have $y_{kl} \in F^A_{\text{cris}}(R)$, and

$$\varphi(Z^TA) = \varphi(Z^T) \varphi(A) = Z^T D N' \Lambda D^{-1} A = p^i Z^T A$$

where $\Lambda'$ and $\Lambda$ are diagonal matrices whose $(k, l)$-entries are $p^{r_k}$ and $p^{i-r_k}$ respectively. Since $\varphi(\beta) = p^\beta$ and $y_{kl} \in F^A_{\text{cris}}(R)$, we have $y_{kl} = \beta^w_{kl}$ for some $w_{kl} \in \mathbb{Q}_p$ by [Bri08 Corollary 6.2.19]. Let $W$ be the $m \times m$ matrix $(w_{kl})$ so that $Z^T A = \beta^W$.

For $M_g := M \otimes_{R, b_g} R_g$ and $N_g := N \otimes_{R, b_g} R_g$, $M_g/p^n$ and $N_g/p^n$ are objects in $\text{MF}^{[0, \alpha]}(R_g)$ for each $n \geq 1$. Moreover, $T_{\text{cris}}(M^i) \cong T_{\text{cris}}(M_g)$ as $G_{R_g}$-representations by Corollary 2.3.5. By classical Fontaine-Laffaille theory, we have $T^i(i) \otimes_{Z_p} B_{\text{cris}}(R_g) \cong M_g \otimes_{R_g} B_{\text{cris}}(R_g)$. In particular, $\det A \neq 0$. Similarly, $\det Z \neq 0$, and thus $\det W \neq 0$. Then we can let $B = W^{-1}Z^T$, which proves the first part of the statement.

Now the natural map

$$T^i(i) \otimes_{Z_p} O_{\text{cris}}(R) \to M^i \otimes_R O_{\text{cris}}(R)$$

is an isomorphism. Hence, for $V = H^i_{\text{et}}(X_{\mathbb{Q}_p}, \mathbb{Q}_p)$, we have $D_{\text{cris}}(V) = M^i[\frac{1}{p}]$ by [Bri08 Proposition 6.2.9], and $\alpha_{\text{cris}}$ is an isomorphism.

5.2. A local invariant cycle theorem. We assume $k = \bar{k}$ in this subsection. Let $X_k := X \times_R k$. By Theorem 1.2.1, $M := H^i_{\text{cris}}(X_n/R_n)$ is an object of $\text{MF}^{[0, \alpha]}(R)$ if $i \leq p - 2$. We write $\varphi = \varphi_0 : M \to M$. 

Proposition 5.2.1. Assume that $k = \overline{k}$ and $i \leq p - 2$. Then

$$H^i_{\text{et}}(X_k, \mathbb{Z}/p^n\mathbb{Z}) \simeq H^i_{\text{cris}}(X_n/R_n)^{\varphi = 1} \simeq H^i_{\text{et}}(X_{\mathbb{Z}[\frac{1}{p}]}[\varphi], \mathbb{Z}/p^n\mathbb{Z})^{G_R}.$$

Proof. Let $X_0 := X \times_R W_n(k)$ and $\overline{M} := H^i_{\text{cris}}(X_0/W_n(k))$. By Proposition 3.2.1, the natural base change map induces a $\varphi$-compatible isomorphism $M \otimes_R W(k) \simeq \overline{M}$. Let $M^0 = R \otimes_{\mathbb{Z}_p} M^{\varphi = 1}$ and $\overline{M}^0 = W(k) \otimes_{\mathbb{Z}_p} \overline{M}^{\varphi = 1}$. Since $\varphi$ is $\nabla$-horizontal, we have $\nabla(M^{\varphi = 1}) = \{0\}$. Consider the natural map $\iota : M^0 \to M$. If we define $F^1M^0 = \{0\}$ and $\nabla$ on $M^0$ via $\nabla(M^{\varphi = 1}) = \{0\}$, then $M^0$ is an object in $\text{MF}_{\nabla, \text{big}}(R)$ and $\iota$ is a morphism in $\text{MF}_{\nabla, \text{big}}(R)$. In a similar manner, we define $\overline{M}^0$ and a morphism $\overline{i} : \overline{M}^0 \to \overline{M}$ in $\text{MF}_{\nabla, \text{big}}(W(k))$. We claim that $\overline{M}^0$ is a direct summand of $\overline{M}$ and hence $\overline{i}$ is an injection. To see this, note that by Fitting’s Lemma, we can write $\overline{M} \simeq \overline{M}^e \oplus \overline{M}^f$ where $\varphi$ acts on $\overline{M}^e := \bigcap_n \varphi^n(\overline{M})$ bijectively and $\overline{M}^{\varphi = 1} \subset \overline{M}^e$. By a standard Frobenius descent, we deduce that $\overline{M}^e \simeq W(k) \otimes_{\mathbb{Z}_p} \overline{M}^{\varphi = 1} = \overline{M}^0$.

Let $\mathfrak{m} := (t_1, \ldots, t_d) \subset R$ be the maximal ideal, and let $\pi : M \to \overline{M} = M/\mathfrak{m}M$ denote the natural projection. We claim that the induced map $\pi : M^{\varphi = 1} \to \overline{M}^{\varphi = 1}$ is an isomorphism; note this implies that $\overline{i} \simeq \iota \otimes_R W(k)$ and that $\iota$ is a morphism in $\text{MF}_{\nabla}(R)$. To prove the claim, first note that for $x, y \in M^{\varphi = 1}$, if $\pi(x) = \pi(y)$ then $x - y \in \mathfrak{m}M$ and $\varphi(x - y) = x - y$. So $x - y \in \varphi^j(\mathfrak{m})M$ for any $j > 0$, which implies $x - y = 0$. On the other hand, let $\bar{x} \in \overline{M}^{\varphi = 1}$ and choose $x \in M$ such that $\pi(x) = \bar{x}$. Consider $\bar{x} := x + \sum_{j=0}^{\infty} (\varphi^j(\varphi(x) - x)). \bar{x}$ diverges in $M$ since $\varphi(x) - x \in \mathfrak{m}M$, and $\pi(\bar{x}) = \bar{x}$ with $\varphi(\bar{x}) = \bar{x}$.

We now show $\ker(\iota) = 0$ as follows. Consider the image $\text{Im}(\iota)$ in the abelian category $\text{MF}_{\nabla}(R)$. Note that $\text{Tor}_1^R(W(k), N) = 0$ for any object $N$ in $\text{MF}_{\nabla}(R)$ since such an object is a direct sum $\oplus R/p^nR$. In particular, tensoring the exact sequence

$$0 \to \text{ker}(\iota) \to M^0 \to \text{Im}(\iota) \to 0$$

with $W(k)$ over $R$ gives the analogous exact sequence for $\overline{i}$. So $\text{Im}(\iota) \otimes_R W(k) \simeq \text{Im}(\overline{i})$, and $\ker(\iota) = 0$ by Nakayama’s lemma.

Thus, $M^0$ is a subobject of $M$. By Theorem 2.3.1, $T_{\text{cris}}(M^0)$ embeds into $T_{\text{cris}}(M)$. Since the map $\varphi$ on $M^{\varphi = 1}$ is trivial and $\nabla(M^{\varphi = 1}) = \{0\}$, we conclude that $T_{\text{cris}}(M^0)$ is a representation on which $G_R$ acts trivially, i.e., $M^{\varphi = 1} = T_{\text{cris}}(M^0) \hookrightarrow T_{\text{cris}}(M)^{G_K}$. Similarly, $T_{\text{cris}}(\overline{M}^0) \hookrightarrow T_{\text{cris}}(\overline{M})^{G_K}$.

On the other hand, since the classical Fontaine-Laffaille theory is stable under subobjects, there exists a subobject $N \subset \overline{M}$ such that $T_{\text{cris}}(N) \simeq T_{\text{cris}}(\overline{M})^{G_K}$. $T_{\text{cris}}(\overline{M})^{G_K}$ is the maximal sub-representation of $T_{\text{cris}}(\overline{M})$ on which $G_K$-acts trivially, and $\overline{M}^0$ is the maximal $W(k)$-submodule of $\overline{M}$ on which $\varphi$ acts bijectively. Thus, $N = \overline{M}^0$ and $T_{\text{cris}}(\overline{M})^{G_K} \simeq T_{\text{cris}}(\overline{M}^0) = \overline{M}^{\varphi = 1}$. 
From above observations, we obtain a commutative diagram:

\[
\begin{array}{ccc}
M^{\varphi=1} = T_{\text{cris}}(M^0) & \xrightarrow{\cong} & T_{\text{cris}}(M)^{G_R} \\
\downarrow \cong & & \downarrow \cong \\
\overline{M}^{\varphi=1} = T_{\text{cris}}(\overline{M}) & \rightarrow & T_{\text{cris}}(\overline{M})^{G_K}
\end{array}
\]

Thus, all maps in this diagram are isomorphisms. In particular,

\[
H^i_{\text{cris}}(X_0/W_n(k))^{\varphi=1} \simeq H^i_{\text{cris}}(X_n/R_n)^{\varphi=1} \simeq H^i_{\text{et}}(X_{R, \mathbb{F}_p}, \mathbb{Z}/p^n\mathbb{Z})^{G_R} \simeq H^i_{\text{et}}(X_{\mathbb{Q}_p}, \mathbb{Z}/p^n\mathbb{Z})^{G_K}.
\]

The result now follows from the well-known fact \(H^i_{\text{cris}}(X_0/W_n(k))^{\varphi=1} \simeq H^i_{\text{et}}(X_k, \mathbb{Z}/p^n\mathbb{Z})\) (see the proof of [Il79, Thm. 5.2]). \(\square\)

5.3. **Cohomological non-defectiveness.** Let \(k\) be a perfect field of characteristic \(p > 2\) as before. In this section, we consider a finite totally ramified extension \(K\) over \(W(k)[\frac{1}{p}]\) with ramification index \(e\). Let \(\mathcal{O}_K\) denote the ring of integers of \(K\). Let \(X\) be a proper smooth scheme over \(\mathcal{O}_K\), and \(X_k := X \times_{\mathcal{O}_K} k\). It is interesting to study the relationship between \(H^i_{\text{cris}}(X_k/W_n(k))\) and \(H^i_{\text{et}}(X_{\mathbb{Q}_p}, \mathbb{Z}/p^n\mathbb{Z})\). By [BM18] Thm. 1.1 (ii), we have

\[
(12) \quad \ell_{W(k)}(H^i_{\text{cris}}(X_k/W_n(k))) \geq \ell_{\mathbb{Q}_p}(H^i_{\text{et}}(X_{\mathbb{Q}_p}, \mathbb{Z}/p^n\mathbb{Z})).
\]

Moreover, it is known that the inequality above can be strict in general when \(ei \geq p - 1\) (see §2 loc. cit.). Motivated by the above inequality, we give the following definition.

**Definition 5.3.1.** If equality in (12) holds for \(X\) and \(i\), then we say \(X\) has cohomological non-defect in degree \(i\). Otherwise, we say \(X\) has cohomological defect in degree \(i\).

We say that \(X\) admits a model \(\mathcal{X}\) over \(R = W(k)[[t]]\) if there exists a smooth proper scheme \(\mathcal{X}\) over \(R\) and a map \(b: R \rightarrow \mathcal{O}_K\) such that \(X \simeq \mathcal{X} \times_{R,b} \mathcal{O}_K\).

**Proposition 5.3.2.** If \(X\) admits a model \(\mathcal{X}\) over \(R\), then \(X\) has cohomological non-defect in degree \(i\) for all \(i \leq p - 2\).

**Proof.** By our main theorem, \(H^i_{\text{cris}}(X_n/R_n)\) has the same type as \(H^i_{\text{et}}(X_{\mathbb{Q}_p}, \mathbb{Z}/p^n\mathbb{Z})\). By considering base change maps \(b: R \rightarrow \mathcal{O}_K\) and \(\overline{b}: R \rightarrow W(k)\) and from Proposition 3.1.1 we conclude that \(H^i_{\text{et}}(X_{\mathbb{Q}_p}, \mathbb{Z}/p^n\mathbb{Z})\) and \(H^i_{\text{cris}}(X_k/W_n(k))\) have the same type. In particular, they have the same length. \(\square\)

**Corollary 5.3.3.** If \(X\) has cohomological defect in degree \(i\) for \(i \leq p - 2\), then \(X\) does not admit a model over \(R\).

**Appendix A. Background on the Crystalline Site**

In this Appendix, we recall some basic facts about the crystalline site. The results are standard and recalled here only for the convenience of the reader.

We begin by recalling the definition of the big and small crystalline site. Let \((\Sigma, \mathcal{I}, \gamma)\) denote the data of a scheme \(\Sigma\) with a quasi-coherent divided power ideal \((\mathcal{I}, \gamma)\). Let \(X\) be
a $\Sigma$-scheme such that $p$ is locally nilpotent on $X$, and that the divided powers $\gamma$ extend to $X$. We shall moreover assume that the divided powers $\gamma$ extend to all $X$-schemes. The latter condition always holds when the ideal $I$ is principal, and this will be the case in all our applications.

**Definition A.0.1.** The big (resp. small) crystalline site CRIS$(X/\Sigma)$ (resp. cris$(X/\Sigma)$) is defined as follows (cf. [BBM82], 1.1.1):

1. The objects of CRIS$(X/\Sigma)$ are quadruples $(U, T, i, \delta)$ where $U$ is a $X$-scheme, $T$ is a $\Sigma$-scheme with $p$ locally nilpotent on $T$, $i: U \rightarrow T$ is a closed $\Sigma$-immersion, and $\delta$ is a divided power structure on the ideal of definition of $i$ compatible with $\gamma$. In particular, $i$ is a nil-immersion. Below, we shall denote such data simply by $(U, T)$. Morphisms are given by morphisms of the underlying schemes compatible with all of the above structures.
2. A morphism $(U', T') \rightarrow (U, T)$ is cartesian if the natural map $U' \rightarrow U \times_T T'$ is an isomorphism.
3. A morphism in CRIS$(X/\Sigma)$ is an open immersion if the morphism is cartesian and the induced map $T' \rightarrow T$ is an open immersion.
4. The topology on CRIS$(X/\Sigma)$ is generated by surjective families of open immersions.
5. The small site is defined to be the full subcategory of the big site where $U \rightarrow X$ is an open immersion, with the induced topology.

We now recall the standard base change theorem for crystalline cohomology. We shall denote by $(X/\Sigma)_{\text{cris}}$ the topos of abelian sheaves corresponding to the small crystalline site. We will work with the small site in this section, but the usual comparison theorem between the big and small sites allows one to conclude analogous results for the big site. Let $O_{X/\Sigma}$ denote the usual crystalline structure sheaf whose section over $(U, T)$ is given by $O_T(T)$. We have natural morphisms of topoi:

$$(X/\Sigma)_{\text{cris}} \xrightarrow{f_{X/\Sigma}} X_{\text{zar}} \xrightarrow{f} \Sigma_{\text{zar}}.$$  

Here $X_{\text{zar}}$ and $\Sigma_{\text{zar}}$ denote corresponding topoi of abelian sheaves on the (small) Zariski site. Let $f_X$ denote the composite. The morphism $f_{X/\Sigma}$ comes equipped with a natural splitting $i_{X/\Sigma} : X_{\text{zar}} \rightarrow (X/\Sigma)_{\text{cris}}$. By abuse of notation, let $O_X := i_{X/\Sigma, *} O_X$. We have a natural exact sequence of sheaves of $O_{X/\Sigma}$-modules:

$$0 \rightarrow J_{X/\Sigma} \rightarrow O_{X/\Sigma} \rightarrow O_X \rightarrow 0.$$  

We note that $J_{X/\Sigma}$ is a divided power ideal, and denote by $J_{X/\Sigma}^{(m)}$ the corresponding ideal of $m$-th divided powers.
Let \( u : (\Sigma', I', \gamma') \to (\Sigma, I, \gamma) \) be a PD-morphism, and set \( X' := X \times_{\Sigma} \Sigma' \). The cartesian diagram of schemes

\[
\begin{array}{ccc}
X' & \xrightarrow{u'} & X \\
\downarrow f' & & \downarrow f \\
\Sigma' & \xrightarrow{u} & \Sigma
\end{array}
\]

gives rise to a canonical base change morphism

\[ B_{X'/X} : \mathbb{L}u^* \mathbb{R}f_* E \to \mathbb{R}f'_* \mathbb{L}u'^*(E) \]

in \( \text{D}(\mathcal{O}_{\Sigma'}) \) for any \( E \in \text{D}(f^{-1}\mathcal{O}_\Sigma) \). Here \( \text{D}(\mathcal{O}_{\Sigma'}) \) and \( \text{D}(f^{-1}\mathcal{O}_\Sigma) \) denote the corresponding derived categories of sheaves in the Zariski topology.

By [Ber74, V Corollary 2.3.4, 2.3.7], there is a natural commutative diagram

\[ \begin{array}{ccc}
\mathbb{R}f_{X/\Sigma,*}(J_{X/\Sigma}^{[m]}) & \xrightarrow{\tau_{\leq m}\Omega_{X/\Sigma}^*} & \tau_{\leq m}\Omega_{X/\Sigma}^* \\
\downarrow & & \downarrow \\
\mathbb{R}u'_*(\mathbb{R}f_{X'/\Sigma',*}(J_{X'/\Sigma'}^{[m]})) & \xrightarrow{\tau_{\leq m}\Omega_{X'/\Sigma'}^*} & \tau_{\leq m}\Omega_{X'/\Sigma'}^*
\end{array} \]  

where the horizontal maps are isomorphisms if \( X/\Sigma \) is smooth. Applying the previous base change map to the left vertical map (and counit of adjunction \( \mathbb{L}u'^* \mathbb{R}u'_*(E) \to E \)) gives rise to a morphism

\[ B_{X'/X} : \mathbb{L}u^* \mathbb{R}f_* (\mathbb{R}f_{X/\Sigma,*}(J_{X/\Sigma}^{[m]})) \to \mathbb{R}f'_* (\mathbb{R}f_{X'/\Sigma',*}(J_{X'/\Sigma'}^{[m]})) \]

in \( \text{D}(\mathcal{O}_{\Sigma'}) \).

**Theorem A.0.1.** With notations as above, suppose that \( X \to \Sigma \) is a smooth morphism (quasi-separated and quasi-compact). Then the base change map \( B_{X'/X} \) is an isomorphism.

**Proof.** We can proceed exactly as in [BO78, Theorem 7.8]. Suppose first that \( X \) is a smooth affine scheme over \( \Sigma \). The horizontal maps in (13) are isomorphisms. Moreover, in this case, the base change map \( B_{X'/X} \) identifies with the usual base change map for coherent sheaves in the Zariski topology:

\[ u^* f_*(\sigma_{\geq m}\Omega_{X/\Sigma}^*) \to f'_*(\sigma_{\geq m}\Omega_{X'/\Sigma'}^*). \]

Here we used the compatibility of Kh"{a}ler differential with base change to identify \( u^* \Omega_{X/\Sigma}^* \) with \( \Omega_{X'/\Sigma'}^* \) (and similarly for its truncations). Moreover, since \( f \) is smooth, these complexes are flat and therefore derived pull-back maybe replaced by ordinary pull back. Similarly, since the morphisms are affine, we may replace derived push-forward by ordinary push-forward. On the other hand, the above base change map in the Zariski topology is known to be an isomorphism for affine morphisms. We may now proceed exactly as in the proof of [BO78, Theorem 7.8]. Briefly, we use a Cech cover and cohomological descent to reduce to the case of an affine morphism. \( \square \)
Example A.0.2. Let \( k \) be a perfect field of characteristic \( p \), \( R = W(k)[[t_1, \ldots, t_d]] \), and \( X \) be a proper smooth scheme over \( R \). Let \( R_n := R/p^nR \) and \( R'_n := W_n(k) \). Then we have a natural isomorphism:

\[
\mathbb{R}\Gamma((X_n/S_n)_{\text{cris}}, J_{X_n/S_n}^{[m]}) \otimes_{R_n} R'_n \to \mathbb{R}\Gamma((X'_n/S'_n)_{\text{cris}}, J_{X'_n/S'_n}^{[m]}),
\]

where \( S_n = \text{Spec}(R_n) \) and \( S'_n = \text{Spec}(R'_n) \).

We will also use a slightly more general form of the base change theorem, which we recall below for the convenience of the reader. Let \((\Sigma, J, \lambda)\) and \((\Sigma', J', \lambda')\) be divided power schemes with \( p \) nilpotent. We shall assume as before that the divided power structure extends to all schemes over \( \Sigma \) (resp. \( \Sigma' \)). Consider a commutative diagram:

\[
\begin{array}{ccc}
  X' & \longrightarrow & Y' \longrightarrow & \Sigma' \\
  \downarrow g' & & \downarrow g & \downarrow u \\
  X & \longrightarrow & Y & \longrightarrow & \Sigma.
\end{array}
\]

This gives rise to a commutative diagram of topoi:

\[
\begin{array}{ccc}
  (X'/\Sigma')_{\text{cris}} & \longrightarrow & (Y'/\Sigma')_{\text{cris}} \\
  \downarrow g'_{\text{cris}} & & \downarrow g_{\text{cris}} \\
  (X/\Sigma)_{\text{cris}} & \longrightarrow & (Y/\Sigma)_{\text{cris}}.
\end{array}
\]

Let \( E \) be a flat quasi-coherent crystal of \( \mathcal{O}_{X/\Sigma} \)-modules. Suppose \( f \) is a smooth morphism and the left square involving \( g \) and \( f \) is cartesian. Then by [Ber74, Theorem V 3.5.1], there is a natural base change isomorphism:

\[
\mathbb{L}g^*_{\text{cris}}(\mathbb{R}f_{\text{cris,}*}(E)) \to \mathbb{R}f'_{\text{cris,}*}(\mathbb{L}g'_{\text{cris}}(E))
\]

in the derived category of \( \mathcal{O}_{Y'/\Sigma'} \)-modules. We will apply this base change morphism to the following diagram for a proper smooth scheme \( X \) over \( R \):

\[
\begin{array}{ccc}
  \overline{X}_n & \longrightarrow & \text{Spec}(\overline{R}_n) \longrightarrow & \text{Spec}(\overline{R}_n) \\
  \downarrow g' & & \downarrow g & \downarrow u \\
  \tilde{X}_n & \longrightarrow & \text{Spec}(\tilde{R}_n) & \longrightarrow & \text{Spec}(\tilde{R}_n)
\end{array}
\]

Here \( \tilde{X}_n \) is a proper smooth \( \tilde{R}_n \)-scheme via \( f \) and \( \overline{X}_n = \tilde{X}_n \times_{\tilde{R}} \overline{R} \), and \( u \) is the identity map. We then have an isomorphism

\[
\mathbb{L}g^*_{\text{cris}}(\mathbb{R}f_{\text{cris,}*}(\mathcal{O}_{\overline{X}_n/\overline{R}_n})) \to \mathbb{R}f'_{\text{cris,}*}(\mathbb{L}g'_{\text{cris}}(\mathcal{O}_{\tilde{X}_n/\tilde{R}_n})).
\]

In particular, we have an isomorphism

\[
\mathbb{R}f_{\text{cris,}*}(\mathcal{O}_{\overline{X}_n/\overline{R}_n}) \otimes_{\overline{R}_n} \mathbb{R}g_{\text{cris,}*}(\mathcal{O}_{\overline{R}_n/\overline{R}_n}) \to \mathbb{R}f'_{\text{cris,}*}(\mathcal{O}_{\tilde{X}_n/\tilde{R}_n}).
\]
By Remark 4.2.2, we have $\mathbb{R}\Gamma((\tilde{R}_n/\tilde{R}_n)_{\text{cris}}, \mathcal{O}_{\tilde{X}_n/\tilde{R}_n}) = A_{\text{cris}, n}(R)$. Applying global section to the previous isomorphism gives an isomorphism

$$A_{\text{cris}, n}(R) \otimes_{\tilde{R}_n}^L \mathbb{R}\Gamma((\tilde{X}_n/\tilde{R}_n)_{\text{cris}}, \mathcal{O}_{\tilde{X}_n/\tilde{R}_n}) \simeq \mathbb{R}\Gamma((\tilde{X}_n/\tilde{R}_n)_{\text{cris}}, \mathcal{O}_{\tilde{X}_n/\tilde{R}_n}).$$

This gives rise to the following Tor-spectral sequence (see [Sta18, Tag 061Y], Example 15.60.4):

$$E_2^{m,q} := \text{Tor}_{\tilde{R}_n}^m (A_{\text{cris}, n}(R), H^q_{\text{cris}}(\tilde{X}_n/\tilde{R}_n)) \Rightarrow H^{m+q}_{\text{cris}}((\tilde{X}_n/\tilde{R}_n)_{\text{cris}}, \mathcal{O}_{\tilde{X}_n/\tilde{R}_n}).$$

**Appendix B. Background on the Syntomic and Crystalline-Syntomic sites**

In this appendix, we recall some basic facts about the crystalline-syntomic site, and its comparison with the syntomic site. The results presented here are standard and well-known to the experts. In particular, we do not claim any originality. We recall them for readers’ convenience due to the lack of an appropriate reference.

**Definition B.0.1.** Given a scheme $X$, let $\text{SYN}(X)$ (resp. $\text{syn}(X)$) denote the big (resp. small) syntomic site defined as follows (see [Bau92, Section 1]):

1. The objects of $\text{SYN}(X)$ are schemes over $X$, and the topology is generated by surjective families of open immersions and finite surjective families of syntomic morphisms with source and target both affine schemes.

2. $\text{syn}(X)$ is the full subcategory of $\text{SYN}(X)$ with syntomic structure map and induced topology.

We now recall the (big) crystalline-syntomic site. Let $(\Sigma, \mathcal{I}, \gamma)$ denote the data of a scheme $\Sigma$ with a quasi-coherent divided power ideal $(\mathcal{I}, \gamma)$. Let $X$ be a $\Sigma$-scheme such that $p$ is locally nilpotent on $X$, and the divided powers $\gamma$ extend to $X$. We shall moreover assume that the divided powers $\gamma$ extend to all $X$-schemes. The latter condition always holds when $\mathcal{I}$ is principal, which will be the case in all our applications.

**Definition B.0.2.** Let $\text{CRIS}(X/\Sigma)_{\text{SYN}}$ denote the (big) crystalline syntomic site defined as follows (see [Bau92, Definition 1.8]):

1. The underlying category is $\text{CRIS}(X/\Sigma)$.

2. A morphism $(U', T') \to (U, T)$ in $\text{CRIS}(X/\Sigma)$ is syntomic if it is cartesian and the map $T' \to T$ is syntomic.

3. The topology on $\text{CRIS}(X/\Sigma)_{\text{SYN}}$ is generated by surjective families of open immersions and finite surjective families of syntomic morphisms with base and target affine (i.e. objects $(U, T)$ with both $U$ and $T$ affine).

Let $X_{\text{SYN}}$ (resp. $X_{\text{syn}}$) denote the topos of sheaves associated to $\text{SYN}(X)$ (resp. $\text{syn}(X)$). Similarly, let $(X/\Sigma)_{\text{CRIS-SYN}}$ denote the topos associated to $\text{CRIS}(X/\Sigma)_{\text{SYN}}$.

**Remark B.0.3.** We have natural morphisms of topoi:
Remark B.0.4. The topoi $X_{\mathrm{SYN}}$ is functorial in $X$, and similarly $(X/\Sigma)_{\mathrm{CRIS-SYN}}$ is functorial in $X/\Sigma$. In particular, given a commutative diagram

$$
\begin{array}{ccc}
X' & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
\Sigma' & \rightarrow & \Sigma
\end{array}
$$

with $\Sigma$ and $\Sigma'$ as before (i.e., divided power schemes with $p$ nilpotent), we have morphisms of topoi $f_{\mathrm{cris,syn}} : (X'/\Sigma')_{\mathrm{CRIS-SYN}} \rightarrow (X/\Sigma)_{\mathrm{CRIS-SYN}}$ and $f_{\mathrm{syn}} : X'_{\Sigma'} \rightarrow X_{\Sigma}$. The construction of $f_{\mathrm{cris,syn}}$ is exactly analogous to the case of usual crystalline topoi. We also note that the resulting diagrams of topoi given by $u_{X/\Sigma}$ (resp. $\alpha$) commute.

Remark B.0.5. (1) The site $\mathrm{CRIS}(X/\Sigma)_{\mathrm{SYN}}$ is naturally a ringed site where the structure sheaf $\mathcal{O}_{S/\Sigma}$ is defined by

$$
\Gamma((U, T), \mathcal{O}_{X/\Sigma}) := \Gamma(T, \mathcal{O}_T).
$$

(2) Similarly, we define the divided powers ideal sheaves $J^{[k]}_{X/\Sigma}$.

We now recall a result on the higher direct images of $\alpha$ due essentially to [BBM82, Proposition 1.1.19]. Before stating the result, we make some preliminary remarks.

Remark B.0.6. (see [BBM82, 1.1.3]) Given a morphism of schemes $v : T' \rightarrow T$, there is a natural inverse image functor $v^{-1} : T_{\mathrm{syn}} \rightarrow T'_{\mathrm{syn}}$ (with a natural right adjoint denoted by $v_*$). Giving a sheaf on $\mathrm{CRIS}(X/\Sigma)_{\mathrm{SYN}}$ is equivalent to giving the the data of a sheaf $F_{(U, T)}$ on $T_{\mathrm{syn}}$, for each object $(U, T) \in \mathrm{CRIS}(X/\Sigma)_{\mathrm{SYN}}$ and for all morphisms $(u, v) : (U, T) \rightarrow (U', T')$, a morphism of sheaves $\rho_{(u, v)} : v^{-1}(F_{(U, T)}) \rightarrow F_{(U', T')}$ such that:

1. $\rho_{(Id, Id)} = Id$,
2. $\rho_{(u_*, v_*)} = \rho_{(u', v')}$, and
3. If $(u, v)$ is syntomic, then $\rho_{(u, v)}$ is an isomorphism.

Remark B.0.7. (see [BBM82, 1.1.18]) (1) Let $\alpha$ be as above. If $E$ is a sheaf on $\mathrm{CRIS}(X/\Sigma)$, then the pull back $\alpha^*(E)$ is by definition the sheaf associated to the presheaf $E$ on $\mathrm{CRIS}(X/\Sigma)_{\mathrm{SYN}}$.

(2) Suppose that $E$ is a sheaf of $\mathcal{O}_{X/\Sigma}$-modules on $\mathrm{CRIS}(X/\Sigma)$ such that for all $(U, T)$, $E_{(U, T)}$ is a quasi-coherent $\mathcal{O}_T$-module and for each syntomic morphism $(u, v) : (U', T') \rightarrow (U, T)$, the natural pull back map $v^*(E_{(U, T)}) \rightarrow E_{(U', T')}$ is an isomorphism of $\mathcal{O}_T$-modules.
Then by part (1) above and flat descent, the presheaf $E$ is a sheaf in the syntomic topology, and the presheaves underlying $\alpha^*(E)$ and $E$ are the same, i.e. $\alpha^*(E) = E$.

**Proposition B.0.8.** (see [BBM82 Proposition 1.1.19]) Let $E$ be an $\mathcal{O}_{X/\Sigma}$-module satisfying the conditions of Remark B.0.7 (2). Then for all $i > 0$, $\mathbb{R}^i\alpha_*(E) = 0$.

**Proof.** This is proved in *loc. cit.* with the fppf and etale topologies on CRIS$(X/\Sigma)$ instead of the syntomic topology. However, the same proof goes through verbatim in our setting. We briefly outline the steps:

1. As in *loc. cit.*, we are reduced to showing that the sheaf associated to the presheaf
   $$(U, T) \mapsto H^i((X/\Sigma)_{\text{CRIS-SYN}}/\tilde{T}, E)$$
   vanishes for $i > 0$. Here $\tilde{T}$ is the sheaf represented by $(U, T)$, and $(X/\Sigma)_{\text{CRIS-SYN}}/\tilde{T}$ is the corresponding topos of objects over $\tilde{T}$.
2. Since this is a local question, we may reduce to the case where $T$ is affine.
3. It suffices to show that the corresponding Cech cohomology groups vanish.
4. The Cech complex of cochains along a syntomic cover $(U, T) \to (U', T')$ can be computed by considering the analogous Cech complex of cochains for the syntomic cover $T' \to T$ with coefficients in $E_{(U,T)}$. Since syntomic morphisms are flat, the latter complex is exact by faithfully flat descent. □

**Corollary B.0.9.** With notations as above, $\mathbb{R}^i\alpha_*(J^k_{X/\Sigma}) = 0$ for all $i > 0$.

**Proof.** We must verify the hypotheses of Remark B.0.7 (2). For each $(U, T)$, $(J^k_{X/\Sigma})(U,T)$ is by definition the divided powers of the ideal defining the closed immersion $U \hookrightarrow T$. Let $(u, v) : (U', T') \to (U, T)$ be a syntomic morphism. Since both $u$ and $v$ are flat, the ideal of definition of $U'$ in $T'$ is the pull back $v^*((J_{X/\Sigma})(U,T))$, and $(J_{X/\Sigma})(U,T') = (J_{X/\Sigma})(U,T) \otimes_{\mathcal{O}_T} \mathcal{O}_{T'}$. Since divided power envelopes are compatible with flat base change by [BO78 Proposition 3.21], we have $v^*((J^k_{X/\Sigma})(U,T)) = (J^k_{X/\Sigma})(U',T')$. □

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