Stochastic Partial Differential Equations
Driven by Fractional Lévy Noises

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Abstract In this paper, we investigate stochastic partial differential equations driven by multi-parameter anisotropic fractional Lévy noises, including the stochastic Poisson equation, the linear heat equation, and the quasi-linear heat equation. Well-posedness of these equations under the fractional noises will be addressed. The multi-parameter anisotropic fractional Lévy noise is defined as the formal derivative of the anisotropic fractional Lévy random field. In doing so, there are two folds involved. First, we consider the anisotropic fractional Lévy random field as the generalized functional of the path of the pure jump Lévy process. Second, we build the Skorohod integration with respect to the multi-parameter anisotropic fractional Lévy noise by white noise approach.

Keywords Stochastic partial differential equation; White noise analysis; Pure-jump Lévy process; Generalized Lévy random field; Anisotropic fractional Lévy random field; Anisotropic fractional Lévy noise

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1 Introduction

The study on fractional processes started from the fractional Brownian motion introduced by Kolmogrov [8] and popularized by Mandelbrot and Van Ness [13]. The self-similarity and long-range dependence properties make the fractional Brownian motion suitable to model driving noises in different applications such as hydrology. However, to capture the large jumps and to model the higher variability phenomena, it is natural to consider more general long-range dependent processes. Replacing a Brownian motion with a Lévy process to define a fractional process becomes more and more popular (see, e.g., [4] [5] [6] [7] [10] [14]). Particularly, the fractional Lévy process is

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long-range dependent and its one-dimensional distribution is infinitely divisible. **Furthermore,** in order to use the fractional Lévy processes to model the higher variability phenomena, it is **imperative** to investigate the stochastic calculus based on fractional Lévy processes and study the related stochastic differential equation. Along the line, the works presented in [9] and [14] are concerned with the stochastic integral for deterministic integrands with respect to fractional Lévy processes; In [1], the authors investigate the Skorohod integral for fractional Lévy process whose underlying Lévy process has finite moment of any order; In [11], the authors study stochastic (ordinary) differential equations driven by fractional Lévy noises.

Lokka and Proske [12] developed the white noise calculus for pure jump Lévy process by viewing it as an element in the Poisson space. According to [12], on the Poisson space every square integrable functional of the path of a pure jump Lévy process has a chaos expansion in terms of the Charlier polynomials with respect to Poisson random measure. Thus, a so-called S-transformation can be used to characterize a generalized stochastic distribution. Moreover, by the definition of the multi-parameter fractional Lévy random field, we can consider it as a square integrable functional of the path of a pure jump Lévy process. Therefore, the multi-parameter fractional Lévy random field has a chaos expansion, which implies that we can use the white noise calculus of the pure jump Lévy process given by [12] to handle its stochastic integration. **Well-posedness of these equations will be addressed.**

Motivated by the white noise analysis for pure jump Lévy process given by Lokka and Proske [12], we define the stochastic integration with respect to the anisotropic fractional Lévy random fields. Furthermore, based on the integration, we investigate several kinds of stochastic partial differential equations driven by anisotropic fractional Lévy noises including Poisson equation, linear heat equation and quasi-linear heat equation.

This paper is organized as follows: In Section 2, we recall the basic results about the white noise analysis of the square integrable pure jump Lévy process given by A. Lokka and F. N. Proske [12]; Based on S-transformation, the multi-parameter fractional Lévy noises are introduced in Section 3 as the formal derivative of anisotropic fractional Lévy random fields, and the Skorohod integral with respect to multi-parameter fractional Lévy noise is built. After the preparation, we investigate stochastic Poisson equation driven by $d$-parameter fractional Lévy noises in Section 4; In Section 5, we investigate stochastic linear heat equation driven by anisotropic fractional Lévy noises. In Section 6, under Lipschitz and linear conditions we obtain a unique solution for stochastic quasi-linear heat equation driven by anisotropic fractional Lévy noises.

2 White noise calculus for pure jump Lévy process

**For convenience to readers and citation,** in this section, we recall some basic results of white noise analysis of the pure jump Lévy process on the Poisson
space given by Lokka and Proske [12].

First, we recall the construction of the Poisson space which has the similar properties as the classical Schwartz space. Let $\xi_n$ denote the $n$'th Hermite function, the set of Hermite functions $\{\xi_n\}_{n \in \mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R})$. Denote by $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of rapidly decreasing $C^\infty$-functions on $\mathbb{R}^d$ and by $\mathcal{S}'(\mathbb{R}^d)$ the space of tempered distributions. The nuclear topology on $\mathcal{S}'(\mathbb{R}^d)$ is induced by the pre-Hilbertian norms

$$
\|\phi\|_p := \sum_{\alpha=(\alpha_1,\ldots,\alpha_d) \in \mathbb{N}^d} (1+\alpha)^{2p}\|\xi_\alpha\|_{L^2(\mathbb{R}^d)}^2,\ p \in \mathbb{N},
$$

where $(1+\alpha)^{2p} = \prod_{i=1}^{d}(1+\alpha_i)^{2p}, \xi_\alpha(x_1,\ldots,x_d) = \prod_{i=1}^{d}\xi_{\alpha_i}(x_i), \mathbb{N} = \mathbb{N}\setminus\{0\}$. Let $U = \mathbb{R}^d \times \mathbb{R}_0$, where $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$, define

$$
\mathcal{S}(U) := \{\phi \in \mathcal{S}(\mathbb{R}^{d+1}) : \phi(x_1,\ldots,x_d,0) = 0\}.
$$

$\mathcal{S}(U)$ is a closed subspace of $\mathcal{S}(\mathbb{R}^{d+1})$, thus it is a countably Hilbertian nuclear algebra endowed with the topology induced by the norms $\|\cdot\|_p$, and its dual $\mathcal{S}'(U) \supset \mathcal{S}'(\mathbb{R}^{d+1})$. For $\Phi \in \mathcal{S}(U)$, $\Phi \in \mathcal{S}'(U)$, the action of $\Phi$ on $\phi$ is given by $\langle \Phi, \phi \rangle = \int_U \Phi(x)\phi(x)d\lambda^{(d+1)}(x)$, $\lambda^d$ is the Lebesgue measure on $\mathbb{R}^d$. Assume that $\nu$ is the Lévy measure on $\mathbb{R}_0$ satisfying

$$
\int_{\mathbb{R}_0} |x|^2 d\nu(x) < \infty. \tag{2.1}
$$

Denote $\pi$ the measure on $U$ given by $\pi = \lambda^d \times \nu$. By Lemma 2.1 of [12], there exists an element denoted by $1 \otimes \dot{\nu}$ in $\mathcal{S}'(U)$ such that

$$
\langle 1 \otimes \dot{\nu}, \phi \rangle = \int_U \phi(x)\pi(dx), \phi \in \mathcal{S}(U). \tag{2.2}
$$

In a generalized sense, $1 \otimes \dot{\nu}$ is the Randon-Nikodým derivative of $\pi$ with respect to the Lebesgue measure.

Denote $L^2(U,\pi)$ by the space of all square integrable functions on $U$ with respect to $\pi$, let $\langle \cdot, \cdot \rangle_\pi$ be the inner product on $L^2(U,\pi)$ and $|\cdot|_\pi$ the corresponding norms on this space.

Define $\mathcal{N}_\pi := \{\phi \in \mathcal{S}(U) : |\phi|_\pi = 0\}$, then $\mathcal{N}_\pi$ is a closed ideal of $\mathcal{S}(U)$. Let $\mathcal{F}(U)$ be the space $\mathcal{F}(U) := \mathcal{S}(U)/\mathcal{N}_\pi$ endowed with the topology induced by the system of norms $\|\phi\|_{p,\pi} := \inf_{\psi \in \mathcal{N}_\pi} \|\phi + \psi\|_p$, then $\mathcal{F}(U)$ is a nuclear algebra. Let $\mathcal{F}'(U)$ be the dual of $\mathcal{F}(U)$, and for $p \in \mathbb{N}$, let $\mathcal{F}_p(\mathbb{R})$ denote the completion of $\mathcal{F}(U)$ with respect to the norm $\|\cdot\|_{p,\pi}$, $\mathcal{F}'_p(U)$ denote the dual of $\mathcal{F}_p(U)$. $\mathcal{F}(U)$ is the projective limit of $\{\mathcal{F}_p(U), p > 0\}$, and $\mathcal{F}'(U)$ is the inductive limit of $\{\mathcal{F}'_p(U), p > 0\}$. $\mathcal{F}(U)$ has similar nice properties as the classical Schwartz space, so Lokka and Proske [12] introduced it as the probability space to construct the white noise analysis for
pure jump Lévy process.

**Theorem 2.1** (Lokka and Proske [12])  
(1) There exists a probability measure $\mu_\pi$ on $\mathcal{F}(U)$ such that

$$
\int_{\mathcal{F}(U)} e^{i\langle \omega, \phi \rangle} d\mu_\pi(\omega) = \exp\{ \int_U (e^{i\phi(x)} - 1) d\pi(x) \}, \forall \phi \in \mathcal{F}(U). \tag{2.3}
$$

(2) Moreover, there exists a $p_0 \in \mathbb{N}$ such that $1 \otimes \hat{\nu} \in \mathcal{F}_{p_0}(U)$, and a natural number $q_0 > p_0$ such that the imbedding operator $\mathcal{F}_{q_0}(U) \hookrightarrow \mathcal{F}_{p_0}(U)$ is Hilbert-Schmidt and $\mu_\pi(\mathcal{F}_{-q_0}(U)) = 1$.

From now on, for all $q_0, p_0$ are described in the Theorem 2.1. Set $\Omega = \mathcal{F}(U)$ and $P = \mu_\pi$ given by Theorem 2.1, Lokka and Proske [12] give the infinite dimensional calculus for pure jump measure on $(\Omega, P)$, and all of our following discussion is based on this probability space.

Let $C_n(\cdot)$ be the Charlier polynomials given by [12], especially for $n = 1, C_1(\omega) = \omega - 1 \otimes \hat{\nu}, \omega \in \Omega$.

**Lemma 2.2** (Lokka and Proske [12]) For all $m, n \in \mathbb{N}, \varphi^{(n)} \in \mathcal{F}(U)^{\otimes n}, \psi^{(m)} \in \mathcal{F}(U)^{\otimes m}$, $(\otimes$ denotes the symmetrized tensor product), the following orthogonality relation holds,

$$
\int_{\mathcal{F}(U)} \langle C_n(\omega), \varphi^{(n)} \rangle \langle C_m(\omega), \psi^{(m)} \rangle d\mu_\pi(\omega) = \begin{cases} 
0, & n \neq m \\
n!(\varphi^{(n)}, \psi^{(n)})_\pi, & n = m.
\end{cases}
$$

Since $\mathcal{F}(U)$ is dense in $L^2(U)$, for $f \in L^2(U)$, there exists a sequence of functions $f_n \in \mathcal{F}(U)$ such that $f_n \rightarrow f$ in $L^2(U, \pi)$ as $n \rightarrow \infty$. Define $\langle C_1(\omega), f \rangle$ by $\lim_{n \rightarrow \infty} \langle C_1(\omega), f_n \rangle$ (limit in $L^2(\mu_\pi)$), the definition is independent of the choice of approximating sequence, and by Lemma 2.2 the following isometry holds

$$
\int_{\mathcal{F}(U)} \langle C_1(\omega), f \rangle^2 d\mu_\pi(\omega) = \int_{\mathcal{F}(U)} (\omega - 1 \otimes \hat{\nu}, f)^2 d\mu_\pi(\omega) = |f|^2_\pi. \tag{2.4}
$$

For any Borel sets $\Lambda_1 \subset \mathbb{R}^d$ and $\Lambda_2 \subset \mathbb{R}_0$ such that the 0 is not in the closure of $\Lambda_2$, define the random measure

$$
N(\Lambda_1, \Lambda_2) := \langle \omega, 1_{\Lambda_1 \times \Lambda_2} \rangle, \tilde{N}(\Lambda_1, \Lambda_2) := \langle \omega - 1 \otimes \hat{\nu}, 1_{\Lambda_1 \times \Lambda_2} \rangle.
$$

From the characterization function of $\mu_\pi$, it is easy to deduce that $N$ is a Poisson random measure, and $\tilde{N}$ is the corresponding compensated measure. The compensator of $N(\Lambda_1, \Lambda_2)$ is given by $\langle 1 \otimes \hat{\nu}, 1_{\Lambda_1 \times \Lambda_2} \rangle$ which is equal to $\pi(\Lambda_1 \times \Lambda_2)$. Moreover,

$$
\int_U \phi(s, x) \tilde{N}(ds, dx) = \langle \omega - 1 \otimes \hat{\nu}, \phi \rangle, \phi \in L^2(U, \pi). \tag{2.5}
$$

By (2.2) and (2.3), we have

$$
\int_{\mathcal{F}(U)} e^{i\langle \omega - 1 \otimes \hat{\nu}, \phi \rangle} d\mu_\pi(\omega) = \exp\{ \int_U (e^{i\phi(x)} - 1 - i\phi(x)) d\pi(x) \}, \forall \phi \in \mathcal{F}(U). \tag{2.6}
$$
Denote $\mathcal{B}(\mathbb{R}^d)$ the Borel $\sigma$-algebra on $\mathbb{R}^d$, for $S \in \mathcal{B}(\mathbb{R}^d)$, define $X(S)$ by
\[
X(S)(\omega) = \langle C_1(\omega), \phi_S \rangle = \langle \omega - 1 \otimes \nu, \phi_S \rangle,
\]
where $\phi_S(x_1, \ldots, x_d, x_{d+1}) = 1_S(x_1, \ldots, x_d) \times x_{d+1}$. By (2.6), we have
\[
\int_{\mathcal{F}^1(\mathbb{U})} e^{iX(\omega)} d\mu_x(\omega) = \exp\left\{ \int_{\mathbb{U}} (e^{i1_S(x_1, \ldots, x_d \times x_{d+1}) - 1 - i1_S(x_1, \ldots, x_d) \times x_{d+1}) d\pi(x) \right\}
\]
\[
= \exp\left\{ \int_{S \times \mathbb{R}_0} (e^{ix_{d+1}} - 1 - ix_{d+1}) d\nu(x_{d+1}) d\lambda^d(x_1, \ldots, x_d) \right\}
\]
\[
= \exp\{ Leb(S) \int_{\mathbb{R}_0} (e^{iy} - 1 - iy) d\nu(y) \},
\]
where $Leb(S)$ is the Lebesgue measure of $S$, then \{ $X(S), S \in \mathcal{B}(\mathbb{R}^d)$ \} is a real-valued random measure on $\mathbb{R}^d$. For $f \in L^2(\mathbb{R}^d)$, define
\[
\hat{X}(f)(\omega) = \langle C_1(\omega), \hat{f} \rangle = \langle \omega - 1 \otimes \nu, \hat{f} \rangle,
\]
where $\hat{f}(x_1, \ldots, x_d, x_{d+1}) = f(x_1, \ldots, x_d) \times x_{d+1}$. By (2.6), we obtain
\[
\int_{\mathcal{F}^1(\mathbb{U})} e^{i\hat{X}(f)(\omega)} d\mu_x(\omega) = \exp\left\{ \int_{\mathbb{U}} (e^{i\hat{f}(x)} - 1 - i\hat{f}(x)) d\pi(x) \right\}.
\]
Hence, we can write formally
\[
\hat{X}(f) = \int_{\mathbb{R}^d} f(x) dX(x), f \in L^2(\mathbb{R}^d).
\]
Moreover, by (2.4),
\[
E_{\mu} (\hat{X}(f))^2 = \|f\|^2_2 \int_{\mathbb{R}_0} |x|^2 d\nu(x).
\]
Now we recall the space of the stochastic distribution functions defined by [12]. Define the space $\mathcal{P}(\mathcal{F}(\mathbb{U})) = \{ f : \mathcal{F}(\mathbb{U}) \to \mathbb{C}, f(\omega) = \sum_{n=0}^{N} \langle \omega^{\otimes n}, \phi^{(n)} \rangle, \omega \in \mathcal{F}(\mathbb{U}), \phi^{(n)} \in \mathcal{F}(\mathbb{U})^{\otimes n}, N \in \mathbb{N}, \}$ is called a continuous polynomial function if $f \in \mathcal{P}(\mathcal{F}(\mathbb{U}))$ and it admit a unique representation of the form
\[
\hat{f}(\omega) = \sum_{n=0}^{\infty} (C_n(\omega), f_n), f_n \in \mathcal{F}(\mathbb{U})^{\otimes n}.
\]
For any number $p \geq q_0$, define the Hilbert space $(\mathcal{H})$ as the completion of $\mathcal{P}(\mathcal{F}(\mathbb{U}))$ with respect to the norm
\[
\|f\|^2_{p,1} = \sum_{n=0}^{\infty} (n!)^2 \|f_n\|^2_{p,\pi}.
The corresponding inner product is

\[(f, g)_{p,1} = \sum_{n=0}^{\infty} (n!)^2 ((f_n, g_n))_{p,\pi},\]

where \((f, g)_{p,\pi}\) denote the inner product on \(\tilde{\mathcal{S}}_p(U)^\otimes n\). Obviously, \((\mathcal{S})^1_{p+1} \subset (\mathcal{S})^1_p\), [12] define \((\mathcal{S})^1\) as the projective limit of \{\((\mathcal{S})^1_p, p \geq q_0\}\}, and \((\mathcal{S})^1\) is a nuclear Fréchet space which can be densely imbedding in \(L^2(\mu_\pi)\). Denote \((\mathcal{S})^{-1}_{-p}\) as the dual of \((\mathcal{S})^1_p\), \((\mathcal{S})^{-1}\) as the inductive limit of \{\((\mathcal{S})^{-1}_{-p}, p \geq q_0\}\} which is equal to the dual of \((\mathcal{S})^1\). \(F \in (\mathcal{S})^{-1}\) if and only if \(F\) admit an expansion

\[F(\omega) = \sum_{n=0}^{\infty} (C_n(\omega), F_n), F_n \in \tilde{\mathcal{S}}'(U)^\otimes n,\]

and there exists a \(p \geq q_0\) such that

\[\|F\|_{-p,-1}^2 = \sum_{n=0}^{\infty} \|F_n\|_{-p,\pi}^2 < \infty.\]

For \(F \in (\mathcal{S})^{-1}, f \in (\mathcal{S})^1,\)

\[\langle\langle F, f \rangle\rangle = \sum_{n=0}^{\infty} n! (F_n, f_n)_{\pi},\]

\(\langle\langle \cdot, \cdot \rangle\rangle\) is an extension of the inner product on \(L^2(\mu_\pi)\). \((\mathcal{S})^1\) is called space of stochastic test functions, \((\mathcal{S})^{-1}\) is called space of stochastic distribution functions, they are pairs of dual spaces and \((\mathcal{S})^{-1} \subset L^2(\mu_\pi) \subset (\mathcal{S})^1\).

Next we recall the S-transform given by [12] which can transform stochastic distribution functions to deterministic functionals. Let

\[\tilde{e}(\phi, \omega) := \exp(\langle \omega, \ln(1 + \phi) \rangle - \langle 1 \otimes \dot{\nu}, \phi \rangle),\]

it is analytic as a function of \(\phi \in \tilde{\mathcal{S}}_{q_0}\) satisfying \(\phi(x) > -1\) for all \(x \in U\). Moreover, it has the following chaos expansion,

\[\tilde{e}(\phi, \omega) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle C_n(\omega), \phi^{\otimes n} \rangle.\]

Denote \(U_p := \{\phi \in \tilde{\mathcal{S}}(U) : \|\phi\|_{p,\pi} < 1\}\), by (2.13), [12] proved that \(\tilde{e}(\phi, \omega) \in (\mathcal{S})^1_{-1}\) if and only if \(\phi \in U_p\).

**Definition 2.3** [12] Let \(F \in (\mathcal{S})^{-1}_{-p}, \xi \in U_p\), the S-transform of \(F\) is defined by

\[S(F)(\xi) := \langle\langle F, \tilde{e}(\xi, \omega) \rangle\rangle.\]

For example, if \(F = \sum_{n=0}^{\infty} (C_n(\omega), F_n) \in (\mathcal{S})^{-1}_{-p}, \xi \in U_p\), then \(S(F)(\xi) = \sum_{n=0}^{\infty} \langle C_n(\omega), F_n, \xi^{\otimes n} \rangle_{\pi}.\)
Denote $\mathcal{U} = Hol(0)$ the algebra of germs of functions that are holomorphic in a neighborhood of 0. The S-transform is isomorphic between $(\mathcal{S})^{-1}$ and $\mathcal{U}$.

**Theorem 2.4** ([12]) If $F \in (\mathcal{S})^{-1}$, then $S(F) \in \mathcal{U}$. Conversely, if $G \in \mathcal{U}$, there is a uniquely defined distribution $F \in (\mathcal{S})^{-1}$ such that $G = S(F)$ on some neighborhood of 0 in $(\mathcal{S})^{-1}$.

Since $f, g \in \mathcal{U}$, then $fg \in \mathcal{U}$, then by Theorem 2.4, the following definition of Wick product is well-defined.

**Definition 2.5** ([12]) Let $F, G \in (\mathcal{S})^{-1}$, define the Wick product $F \diamond G$ of $F$ and $G$ by

$$F \diamond G = S^{-1}(S(F)S(G)).$$

The Wick exponential of $F \in (\mathcal{S})^{-1}$ denoted by $\exp^\diamond(F)$ is defined by

$$\exp^\diamond(F) := \sum_{n=0}^{\infty} \frac{1}{n!} F^{\diamond n}, \quad (2.14)$$

whenever $\sum_{n=0}^{\infty} \frac{1}{n!} F^{\diamond n} \in (\mathcal{S})^{-1}$. In this case,

$$S(\exp^\diamond X)(\eta) = \exp[(SX)(\eta)]. \quad (2.15)$$

Last recall the Skorohod integral of pure jump processes given by [12]. Let $F : \mathbb{U} \to (\mathcal{S})^{-1}$ be the random fields with chaos expansion

$$F(x) = \sum_{n=0}^{\infty} \langle C_n(\omega), F_n(\cdot, x) \rangle,$$

where $F_n(\cdot, x) \in \mathcal{S}'(\mathbb{U})^{\otimes n}$ and $\|F(x)\|_{-p, -1} < \infty$, for some $p > 0$. Let $\mathbb{L}$ denote the set of all $F : \mathbb{U} \to (\mathcal{S})^{-1}$ such that $\widehat{F}_n \in \mathcal{S}'(\mathbb{U})^{\otimes (n+1)}(\mathcal{F})$ is the symmetrization of $F_n$ and $\sum_{n=0}^{\infty} |\widehat{F}_n|^2_{p, \pi} < \infty$ for some $p > 0$.

**Definition 2.6** ([12]) (Skorohod integral) For $F \in \mathbb{L}$, define the Skorohod integral $\delta(F)$ by

$$\delta(F) := \sum_{n=0}^{\infty} \langle C_{n+1}(\omega), \widehat{F}_n \rangle.$$

From the assumption on $\mathbb{L}$, we see that $\delta(F) \in (\mathcal{S})^{-1}$. For the predictable integrands, the Skorohod integral coincides with the usual Itô-type integral with respect to the compensated Poisson random measure.

**Proposition 2.7** ([12]) If $F \in \mathbb{L}$, then $\delta(F) \in (\mathcal{S})^{-1}$ for some $p > 0$ and

$$S\delta(F)(\xi) = \int_{\mathbb{U}} SF(x)(\xi)\xi(x)\pi(dx), \xi \in U_p. \quad (2.16)$$

3 Anisotropic fractional Lévy noises
In this section, we define the anisotropic fractional Lévy random field which can be considered as a generalized functional of the path of the pure jump Lévy random field and according to the result of section 2 we give its S-transformation. Moreover, based on the S-transformation of the anisotropic fractional Lévy random field, we define its formal derivative as \( d \)-parameter fractional Lévy noise.

Let \( \beta = (\beta_1, \ldots, \beta_d), 0 < \beta_k < \frac{1}{2}, k = 1, 2, \ldots, d, f \in \mathcal{S}(\mathbb{R}^d), \Gamma(\beta) = \prod_{k=1}^d \Gamma(\beta_k), x^\beta = \prod_{k=1}^d x_k^{\beta_k}, x = (x_1, \ldots, x_d), \) the multi-variate fractional integral operator of Liouville-type is defined by Samko et al.[16]:

\[
I_{\alpha_{1}}^{\beta_{1}} \cdots I_{\alpha_{d}}^{\beta_{d}} f(x) := \frac{1}{\Gamma(\beta)} \int_{\mathbb{R}_+^d} \frac{f(x - y) dy}{y^{1-\beta}}, \quad (3.1)
\]

\[
I_{\alpha_{1}}^{\beta_{1}} \cdots I_{\alpha_{d}}^{\beta_{d}} f(x) := \frac{1}{\Gamma(\beta)} \int_{\mathbb{R}_+^d} \frac{f(x + y) dy}{y^{1-\beta}}. \quad (3.2)
\]

**Theorem 3.1** (Samko et al.[16]) The operator \( I_{\alpha_{1}}^{\beta_{1}} \cdots I_{\alpha_{d}}^{\beta_{d}} \) is bounded from \( L^\mathfrak{p}(\mathbb{R}^d) \) to \( L^\mathfrak{q}(\mathbb{R}^d) \) with \( \mathfrak{p} = (p_1, \ldots, p_d), \mathfrak{q} = (q_1, \ldots, q_d) \) if and only if

\[
1 < p_k < \frac{1}{\beta_k}, q_k = \frac{p_k}{1 - \beta_k p_k}, k = 1, 2, \ldots, d,
\]

where \( L^\mathfrak{p} \) is the Banach space of functions with mixed norm

\[
\|f\|_{\mathfrak{p}} = \left\{ \int_{\mathbb{R}} \cdots \left\{ \int_{\mathbb{R}} |f(s_1, \ldots, s_d)|^{p_1 ds_1} \right\}^{\frac{p_2}{d}} \cdots \right\}^{\frac{p_d}{d-1}} ds_d \frac{1}{\mathfrak{d}_d} < \infty. \quad (3.3)
\]

Especially, for \( p_1 = \ldots = p_d = p, \) \( L^p(\mathbb{R}^d) \) is equal to \( L^p(\mathbb{R}^d). \)

Take \( q_1 = \ldots = q_d = 2, p_k = \frac{1}{2+\beta_k}, k = 1, 2, \ldots, d, \) we deduce that for \( \beta = (\beta_1, \ldots, \beta_d), 0 < \beta_k < \frac{1}{2}, k = 1, \ldots, d, \) the operator \( I_{\alpha_{1}}^{\beta_{1}} \cdots I_{\alpha_{d}}^{\beta_{d}} : \mathcal{S}(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \) is continuous. Hence, by (2.8), we can define the generalized anisotropic fractional Lévy random field as follows:

**Theorem 3.2** For \( \beta = (\beta_1, \ldots, \beta_d), 0 < \beta_k < \frac{1}{2}, k = 1, \ldots, d, \)

\[
\hat{X}^\beta(f) := \hat{X}(I_{\alpha_{1}}^{\beta_{1}} \cdots I_{\alpha_{d}}^{\beta_{d}} f), \quad f \in \mathcal{S}(\mathbb{R}^d) \quad (3.4)
\]

is a tempered real-valued generalized random field. We refer it as the generalized anisotropic fractional Lévy random field.

**Proof:** The proof is the same to that of Theorem 3.3 of [10], we omit here.

In fact, by (2.8), for \( f \in \mathcal{S}(\mathbb{R}^d), \) \( \hat{X}^\beta(f) \) can be represented as

\[
\hat{X}^\beta(f) = \langle C_1, K^\beta f \rangle,
\]

where

\[
(K^\beta f)(x_1, \ldots, x_d, x_{d+1}) = I_{\alpha_{1}}^{\beta_{1}} \cdots I_{\alpha_{d}}^{\beta_{d}} f(x_1, \ldots, x_d) \times x_{d+1}.
\]
Moreover, by (2.9), we get
\[
\int_{\mathcal{U}} e^{iX''(f)(\omega)} d\mu_\pi(\omega) = \exp\left\{ \int_U (e^{i(K^{\beta}f)(x)} - 1 - i(K^{\beta}f)(x)) d\pi(x) \right\}. \tag{3.5}
\]
Since \( I_{-\ldots-1}^{\beta} \in L^2(\mathbb{R}^d), \bar{t} \in \mathbb{R}^d_+ \), by (2.8) we can define the anisotropic fractional Lévy random field as follows:

**Definition 3.3** The anisotropic fractional Lévy random field is defined by

\[
X^{\bar{t}}_t := \dot{X}(I_{-\ldots-1}^{\beta}[0,\bar{t}]), \bar{t} = (t_1, \ldots, t_d), t_i \geq 0, i = 1, 2, \ldots, d \tag{3.6}
\]

(3.6) can be represented as

\[
X^{\bar{t}}_t = \int_{\mathbb{R}^d} I_{-\ldots-1}^{\beta}[0,\bar{t}](s) dX(s) = \int_{-\infty}^{t_1} \ldots \int_{-\infty}^{t_d} \prod_{k=1}^{d} (t_k - s_k)^{\beta_k} - (-s_k)^{\beta_k} dX(s). \tag{3.7}
\]

From (3.7) we see that the fractional integral parameters along different time axis are different, thus the fractional Lévy random field \( \{X^{\bar{t}}_t, \bar{t} \in \mathbb{R}^d_+\} \) is anisotropic.

Since \( I_{-\ldots-1}^{\beta} \in L^2(\mathbb{R}^d), \bar{t} \in \mathbb{R}^d_+ \), \( X^{\bar{t}}_t \) has the following representation

\[
X^{\bar{t}}_t = \delta (K^{\beta}1_{[0,\bar{t}]}). \tag{3.8}
\]

Thus, by (2.16), we get the \( S \)-transform of anisotropic fractional Lévy random field

\[
SX^{\bar{t}}_t(\eta) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} y \eta(s, y) I_{-\ldots-1}^{\beta}[0,\bar{t}](s, y) \nu(dy) d\bar{s}, \eta \in U_p, p > p_0. \tag{3.9}
\]

On the other hand, by the following fractional integral by parts formula of operator \( I_{+\ldots+}^{\beta} \):

\[
\int_{\mathbb{R}^d} f(\bar{s}) I_{+\ldots+}^{\beta} g(\bar{s}) d\bar{s} = \int_{\mathbb{R}^d} g(\bar{s}) I_{-\ldots-}^{\beta} f(\bar{s}) d\bar{s}, f, g \in \mathcal{S} (\mathbb{R}) \tag{3.10}
\]

which can be extended to \( f \in L^p(\mathbb{R}), g \in L^r(\mathbb{R}) \) with \( p_i > 1, r_i > 1 \) and \( \frac{1}{p_i} + \frac{1}{r_i} = 1 + \beta_i, i = 1, \ldots, d \), (3.9) can be written as

\[
SX^{\bar{t}}_t(\eta) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 1_{[0,\bar{t}]}(s) y I_{+\ldots+}^{\beta} \eta(\cdot, y)(s) \nu(dy) d\bar{s} \tag{3.11}
\]

\[
= \int_0^{t_1} \ldots \int_0^{t_d} \left\{ \int_{\mathbb{R}^d} y I_{+\ldots+}^{\beta} \eta(\cdot, y)(\bar{s}) \nu(dy) \right\} d\bar{s}.
\]

Hence,

\[
\frac{\partial^d}{\partial t_1 \ldots \partial t_d} SX^{\bar{t}}_t(\eta) = \int_{\mathbb{R}^d} y I_{+\ldots+}^{\beta} \eta(\cdot, y)(\bar{t}) \nu(dy), \eta \in U_p, p > p_0. \tag{3.12}
\]
We denote \( \dot{X}_t^\beta \) the fractional Lévy noise in the following sense:

\[
SX_t^\beta(\eta) = \frac{\partial^d}{\partial t_1 \ldots \partial t_d} SX_t^\beta(\eta) = \int_{\mathbb{R}_0^d} y t^{\beta - 1} \eta(s, y) (\mathcal{I}) \nu(dy), \eta \in U_p, p > p_0, \bar{t} \in \mathbb{R}_+^d.
\]

(3.13)

Next we prove that \( \dot{X}_t^\beta \) is a generalized stochastic distribution function and it has a chaos representation:

**Theorem 3.4** \( \dot{X}_t^\beta \in (\mathcal{S})^{-1}_p \) for all \( p > \max\{1, p_0\} \) and

\[
\dot{X}_t^\beta = \langle C_1, \lambda_{\mathcal{T}} \rangle,
\]

(3.14)

where

\[
\lambda_{\mathcal{T}}(\eta, y) = \frac{y(t - u)^{\beta - 1}}{\Gamma(\beta)} = \frac{y \prod_{k=1}^d (t_k - u_k)^{\beta_k - 1}}{\Gamma(\beta)}
\]

**Proof:** We first show that \( \langle C_1, \lambda_{\mathcal{T}} \rangle \in (\mathcal{S})^{-1}_p \) for all \( p > \max\{1, p_0\} \). By the estimate

\[
\int_{\mathbb{R}} (t - u)^{\beta - 1} \xi_n(u)du \leq Cn^{\frac{\beta}{2} - \frac{3}{4}}.
\]

(3.15)

from section 4 of [3], where \( C \) is a certain constant independent of \( t \),

\[
\|\langle C_1, \lambda_{\mathcal{T}} \rangle\|_{-1, -p}^2 = \frac{\int_{\mathbb{R}} |y|^2 d\nu(y)}{\Gamma(\beta)} \sum_{\alpha = (\alpha_1, \ldots, \alpha_d) \in N_0^d} (\alpha + 1)^{-2p} \langle \delta^{\beta - 1}, \xi_\alpha \rangle_{L^2(\mathbb{R}^d)}^2
\]

\[
= A \sum_{\alpha = (\alpha_1, \ldots, \alpha_d) \in N_0^d} \prod_{k=1}^d (\alpha_k + 1)^{-2p} \langle \delta^{\beta - 1}, \xi_\alpha \rangle_{L^2(\mathbb{R}^d)}^2
\]

\[
\leq AC \sum_{\alpha = (\alpha_1, \ldots, \alpha_d) \in N_0^d} \prod_{k=1}^d (\alpha_k + 1)^{-2p + \frac{3}{4} - \beta_k}
\]

\[
= AC \prod_{k=1}^d \sum_{\alpha_k = 1}^\infty (\alpha_k + 1)^{-2p + \frac{3}{4} - \beta_k}
\]

\[
< +\infty, \text{for } p > \max\{1, p_0\},
\]

(3.16)

where \( A = \frac{\int_{\mathbb{R}} |y|^2 d\nu(y)}{\Gamma(\beta)} \) is a positive constant. Thus \( \langle C_1, \lambda_{\mathcal{T}} \rangle \in (\mathcal{S})^{-1}_p \) for all \( p > \max\{1, p_0\} \). Next, we prove (3.14) holds. In fact,

\[
(\mathcal{T} \dot{f} - \ldots \dot{\delta}_t)(\bar{s}) = \frac{1}{\Gamma(\beta)} \int_{\mathbb{R}_+^d} \frac{\delta_t(\bar{s} + \bar{u})d\bar{u}}{\bar{u}^{1-\beta}} = \frac{(\bar{t} - \bar{s})^{\beta - 1}}{\Gamma(\beta)}. \]

(3.17)
Taking S-transform of \( \langle C_1, \lambda T \rangle \),

\[
S\langle C_1, \lambda T \rangle (\eta) = \int_{\mathbb{R}^d} \int_{\mathbb{R}_0^+} \frac{y(\bar{t} - \bar{s})^{\beta - 1}}{\Gamma(\beta)} \eta(\bar{s}, y) \nu(dy) d\bar{s}
\]

\[
= \int_{\mathbb{R}^d} \int_{\mathbb{R}_0^+} y I_{\gamma} \delta_t(\bar{s}) \eta(\bar{s}, y) \nu(dy) d\bar{s}
\]

\[
= \int_{\mathbb{R}^d} \int_{\mathbb{R}_0^+} y \delta_t(\bar{s}) I_{\gamma} \eta(\cdot, y)(\bar{s}) \nu(dy) d\bar{s}
\]

\[
= \int_{\mathbb{R}_0^+} y I_{\gamma} \delta_t \eta(\cdot, y)(\bar{s}) \nu(dy) d\bar{s}, \quad \eta \in U_p, p > p_0.
\]

Hence, by (3.13), (3.14) holds when take S-transfrom by two sides. \( \square \)

Now we define the Skorohod integral for \((\mathcal{S})^{-1}\)-valued processes with respect to \(X^\beta\). First, we define \((\mathcal{S})^{-1}\)-valued integrals as follows:

**Definition 3.5** Suppose \( F : \mathbb{R}^d_+ \rightarrow (\mathcal{S})^{-1} \) is a given function such that \( \langle \langle F(x), f \rangle \rangle \in L^1(\mathbb{R}^d_+, dx) \) for all \( f \in (\mathcal{S}) \), then \( \int_{\mathbb{R}^d_+} F(x) dx \) is defined to be the unique element of \((\mathcal{S})^{-1}\) such that

\[
\langle \langle \int_{\mathbb{R}^d_+} F(x) dx, f \rangle \rangle = \int_{\mathbb{R}^d_+} \langle \langle F(x), f \rangle \rangle dx.
\]

(3.18)

**Definition 3.6** Suppose that \( F : \mathbb{R}^d_+ \rightarrow (\mathcal{S})^{-1} \) such that \( F(\bar{s}) \diamond \dot{X}^{\beta}_s \) is \( d\bar{s} \)-integrable in \((\mathcal{S})^{-1}\). Then we define the Skorohod integral of \( F \) with respect to \( X^\beta \) by

\[
\delta^{\beta}(F) := \int_{\mathbb{R}^d_+} F(\bar{s}) \delta \dot{X}^{\beta}_s := \int_{\mathbb{R}^d_+} F(\bar{s}) \diamond \dot{X}^{\beta}_s d\bar{s}.
\]

(3.19)

In particular, if \( A \subset \mathbb{R}^d_+ \) is a Borel set, then

\[
\int_A F(\bar{s}) \delta \dot{X}^{\beta}_s := \int_{\mathbb{R}_0^+} 1_A(\bar{s}) F(\bar{s}) \diamond \dot{X}^{\beta}_s d\bar{s}.
\]

(3.20)

By definition 3.6, we get

**Proposition 3.7** Let \( F : \mathbb{R}^d_+ \rightarrow (\mathcal{S})^{-1} \) be Skorohod integrable with respect to \( X^\beta \), \( Y \in (\mathcal{S})^{-1} \), then

\[
Y \circ \delta^{\beta}(F) = \delta^{\beta}(Y \circ F),
\]

the equation holds whenever one side exists.

4 The stochastic Poisson equation driven by d-parameter fractional Lévy noise

In this section, we investigate the stochastic Poisson equation driven by d-parameter fractional Lévy noise:

\[
\begin{cases}
\Delta U(x) = -\dot{X}^{\beta}_x, & x \in D, \\
U(x) = 0, & x \in \partial D.
\end{cases}
\]

(4.1)
where \( \Delta = \sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2} \) is the Laplace operator in \( \mathbb{R}^d \), \( D \subset \mathbb{R}_+^d \) is a given domain with regular boundary and \( \tilde{X}^\beta_t \) is the d-parameter fractional Lévy noise.

**Theorem 4.1** The stochastic Poisson equation (4.1) has a unique continuous solution in \( (\mathcal{S})^{-1} \).

**Proof:** Based on the corresponding solution in the deterministic case (with \( \tilde{X}^\beta_t \) replaced by a bounded deterministic function), the solution of (4.1) will be

\[
U(x) = \int_D G(x, y) \tilde{X}^\beta_t dy,
\]

where \( G \) is the Dirichlet Laplacian. We first prove that \( U(x) \in (\mathcal{S})^{-1} \) for all \( p > \max\{1, p_0\} \). By (3.14), (4.2) can be written as

\[
U(x) = \int_D G(x, y) X^\beta_y dy = \int_D G(x, y) (C_1, \lambda_y) dy.
\]

Then by (3.16) and the fact that \( G(x, \cdot) \in L^1(\mathbb{R}^d) \), for \( p > \max\{1, p_0\} \),

\[
\|U(x)\|_{-1, -p} \leq \int_D \| (C_1, \lambda_y) \|_{-1, -p} \|G(x, y)\| dy < +\infty,
\]

that is \( U(x) = (C_1, \int_D G(x, y) \lambda_y dy) \in (\mathcal{S})^{-1} \) for all \( p > \max\{1, p_0\} \) and the same estimate gives that \( U(x) : \overline{D} \to (S)^{-1} \) is continuous. It is easy to show that

\[
\Delta U(x) = -\langle C_1, \lambda_x \rangle = -\tilde{X}^\beta_x, x \in D,
\]

Thus we finish the proof of the theorem. \( \square \)

### 5 The stochastic linear heat equation driven by d-parameter fractional Lévy noise

In this section, we consider the linear stochastic heat equation driven by d-parameter fractional Lévy noise:

\[
\begin{align*}
\frac{\partial}{\partial t}U(t, x) &= \frac{1}{2} \Delta U(t, x) + \tilde{X}^{\beta_0, \beta_1, \ldots, \beta_d}_{t, x}, x \in D, \\
U(0, x) &= 0, t > 0, x \in D, \\
U(t, x) &= 0, x \in \partial D.
\end{align*}
\]

where \( 0 < \beta_k < \frac{1}{2}, k = 0, 1, \ldots, d, \Delta = \sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2} \) is the Laplace operator in \( \mathbb{R}^d \), \( D \subset \mathbb{R}_+^d \) is a given domain with regular boundary and \( \tilde{X}^{\beta_0, \beta_1, \ldots, \beta_d}_{t, x} \) is the \( d+1 \)-parameter fractional Lévy noise.

Based on the corresponding solution in the deterministic case, we guess that

\[
U(t, x) = \int_0^t \int_D G_{t-s}(x, y) \tilde{X}^{\beta_0, \beta_1, \ldots, \beta_d}_{s, y} dy ds,
\]

(5.2)
where $G$ is the Green function of the heat operator. In fact, we can prove that $U$ is the unique strong solution.

**Theorem 5.1** The stochastic heat equation (5.1) has a unique strong solution in $U : [0, \infty) \times D \rightarrow (\mathcal{S})^{-1}$. The solution is

$$U(t, x) = \int_0^t \int_D G_{t-s}(x, y) \dot{X}^{\beta_0, \beta_1, \ldots, \beta_d}_{s, y} dy ds,$$

where $G$ is the Green function of the heat operator $\frac{\partial}{\partial t} - \frac{1}{2} \Box$, and (5.2) belongs to $C^1(\mathbb{R}) \cap C([0, \infty) \times D, (\mathcal{S})^{-1})$.

**Proof:** We first prove that $U(t, x) \in (\mathcal{S})^{-1-\rho}$ for all $p > \max\{1, p_0\}$.

$$U(t, x) = \int_0^t \int_D G_{t-s}(x, y) \dot{X}^{\beta_0, \beta_1, \ldots, \beta_d}_{s, y} dy ds = \int_0^t \int_D G_{t-s}(x, y) (C_1, \lambda_{s, y}) dy ds$$

Then by (3.16), for $p > \max\{1, p_0\}$,

$$\|U(t, x)\|_{-1, -p} \leq \int_0^t \int_D \|(C_1, \lambda_{s, y})\|_{-1, -p} G_{t-s}(x, y) dy ds < +\infty,$$

that is, $U(t, x) \in (\mathcal{S})^{-1-\rho}$ for all $p > \max\{1, p_0\}$, for all $t, x$ and

$$U(t, x) = (C_1, \int_0^t \int_D G_{t-s}(x, y) \lambda_{s, y} dy ds).$$

In fact, the estimate also shows that $U(t, x)$ is uniformly continuous function from $[0, T] \times \overline{D}$ into $(\mathcal{S})^{-1}$ for any $T < \infty$. Moreover, by the properties of the operator $G_{t-s}(x, y)$, we get from (5.2) that

$$\frac{\partial}{\partial t} U(t, x) - \frac{1}{2} \Box U(t, x) = \dot{X}^{\beta_0, \beta_1, \ldots, \beta_d}_{t, x} + \int_0^t \int_D \left( \frac{\partial}{\partial t} - \frac{1}{2} \Box \right) G_{t-s}(x, y) dy ds$$

So, $U(t, x)$ satisfies (5.1).

Moreover, we can prove that under some condition, the solution $U(t, x)$ of (5.1) is $L^2$-integrable.

**Theorem 5.2** If $2\beta_0 + \sum_{i=1}^d \beta_i + 1 > \frac{4}{\sigma}$, then $U(t, x) \in L^2(\Omega)$ for all $t \geq 0$, $x \in \overline{D}$.

**Proof:** From [2], we know that $G$ is smooth in $(0, \infty) \times D$ and that in $(0, \infty) \times D$,

$$|G_{u}(x, y)| \sim u^{-\frac{d}{2}} \exp\left(\frac{|x - y|^2}{\delta u}\right)$$

$$\left| \frac{\partial G_{u}(x, y)}{\partial y_i} \right| \sim u^{-\frac{d}{2}-1} |x_i - y_i| \exp\left(\frac{|x - y|^2}{\delta u}\right),$$

where $G$ is the Green function of the heat operator. In fact, we can prove that $U$ is the unique strong solution.
the notion $X \sim Y$ in $(0, \infty) \times D$ means that $\frac{1}{C} X \leq Y \leq CX$ for some positive constant $C < \infty$ depending only on $D$. By this result, we use the similar proof of Theorem 8.4.1 of [2] to verify the condition for $U(t, x) \in L^2$ for all $t \geq 0$, $x \in \overline{D}$.

$$
\mathbb{E}(U(t, x))^2 = \mathbb{E}\left(\int_0^t \int_D G_{t-s}(x, y)^2 dy ds\right)^2
$$

By this result, we have

$$
\mathbb{E}(U(t, x))^2 = \mathbb{E}\left(\int_0^t \int_D (I^{\overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_d} G_{t-s}(x, y))^2 dy ds\right)
$$

$$
\sim \int_0^t \int_0^t \int_0^t \int_0^t |(t - r)|^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{2(t-s)^2}} |(t - s)|^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{2(t-r)^2}} |r - s|^{2\overline{\beta}_0 - 1} \prod_{i=1}^d |y_i - z_i|^{2\overline{\beta}_i} dy_1 \ldots dy_d dz_1 \ldots dz_d ds dr
$$

$$
\sim \int_0^t \int_0^t \int_0^t \int_0^t |r|^{-\frac{d}{2}} s^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{2s^2}} e^{-\frac{|x-y|^2}{2r^2}} |r - s|^{2\overline{\beta}_0 - 1} \prod_{i=1}^d |y_i - z_i|^{2\overline{\beta}_i} dy_1 \ldots dy_d dz_1 \ldots dz_d ds dr
$$

By the inequality (2.1) of [3],

$$
\int_\mathbb{R} \int_\mathbb{R} |f(x)||g(y)||x - y|^{2\beta - 1} \leq C \|f\| \|g\| \|x - y\|^{\frac{1-\beta}{1+\beta}}, 0 < \beta < \frac{1}{2},
$$

where $C$ is a positive constant, we have

$$
\prod_{i=1}^d \int_{\frac{1}{2}R}^{\frac{1}{2}R} \int_{\frac{1}{2}R}^{\frac{1}{2}R} e^{-\frac{|x_i - y_i|^2}{2s^2}} e^{-\frac{|x_i - z_i|^2}{2r^2}} (y_i - z_i)^{2\overline{\beta}_i - 1} dy_i dz_i
$$

$$
\leq \prod_{i=1}^d \left[ \int_{\frac{1}{2}R}^{\frac{1}{2}R} e^{-\frac{|x_i - y_i|^2}{2(\overline{\beta}_i + 1)s^2}} dy_i \right]^{\overline{\beta}_i + \frac{1}{2}} \left[ \int_{-\frac{1}{2}R}^{\frac{1}{2}R} e^{-\frac{|x_i - z_i|^2}{2(\overline{\beta}_i + 1)s^2}} dz_i \right]^{\overline{\beta}_i + \frac{1}{2}}
$$

$$
\sim (rs)^{\frac{1}{2}} \sum_{i=1}^d (\overline{\beta}_i + \frac{1}{2}),
$$

where $R$ is some constant such that $D \subset [-\frac{1}{2}R, \frac{1}{2}R]^d$. Substituting (5.5) into (5.3), we have

$$
\mathbb{E}(U(t, x))^2 \leq C \int_0^t \int_0^t (rs)^{\frac{1}{2}} \sum_{i=1}^d (\overline{\beta}_i + \frac{1}{2}) |r - s|^{2\overline{\beta}_0 - 1} < \infty
$$

if $2\overline{\beta}_0 + \sum_{i=1}^d \overline{\beta}_i + 1 > \frac{d}{2}$. Thus we complete the proof. \qed
6 The quasi-linear stochastic fractional heat equation driven by \(d\)-parameter fractional Lévy noise

In this section, we consider the following quasi-linear equation driven by \(d\)-parameter fractional Lévy noise:

\[
\begin{aligned}
\frac{\partial}{\partial t} U(t, x) &= \frac{1}{2} \Delta U(t, x) + f(U(t, x)) + \dot{X}^{\beta_0, \beta_1, \ldots, \beta_d}_{t, x}, \quad t > 0, x \in \mathbb{R}^d, \\
U(0, x) &= U_0(x), \quad x \in \mathbb{R}^d,
\end{aligned}
\]

(6.1)

where \(U_0(x)\) is a given bounded deterministic function on \(\mathbb{R}^d\), \(f: \mathbb{R} \rightarrow \mathbb{R}\) is a function satisfying

\[
|f(x) - f(y)| \leq L|x - y|, \forall x, y \in \mathbb{R},
\]

(6.2)

\[
|f(x)| \leq C(1 + |x|), \forall x \in \mathbb{R}.
\]

(6.3)

\(U(t, x)\) solves (6.1) if and only if it solves the following integral equation:

\[
U(t, x) = \int_{\mathbb{R}^d} U_0(y) G_t(x, y) dy + \int_0^t \int_{\mathbb{R}^d} f(U(s, y)) G_{t-s}(x, y) dy ds
\]

\[
+ \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x, y) dX^{\beta_0, \beta_1, \ldots, \beta_d}_{t, x}
\]

(6.4)

where

\[
G_{t-s}(x, y) = (t-s)^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{2(t-s)}}, \quad s < t, \ x, y \in \mathbb{R}^d,
\]

is the Green function for the heat operator \(\frac{\partial}{\partial t} - \frac{1}{2} \Delta\).

**Theorem 6.1** If

\[
\beta_i > \frac{1}{2} - \frac{1}{d}, \quad i = 1, \ldots, d,
\]

(6.5)

then there exist a unique solution \(U(t, x)\) for (6.1) such that \(U(t, x) \in L^2(\Omega)\) for all \(t \geq 0, \ x \in \mathcal{D}\).

**Proof:** Define

\[
V(t, x) = \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x, y) dX^{\beta_0, \beta_1, \ldots, \beta_d}_{t, x}
\]

Since (6.5) holds, by the similar arguments of Theorem 5.2, we can prove that \(V(t, x) \in L^2(\Omega)\) for all \(t \geq 0, \ x \in \mathcal{D}\), so \(V(t, x)\) exists as an ordinary random field. The existence of the solution now follows Picard iteration. Define

\[
U_0(t, x) = U_0(x)
\]

and iteratively

\[
U_{j+1}(t, x) = \int_{\mathbb{R}^d} U_0(y) G_t(x, y) dy + \int_0^t \int_{\mathbb{R}^d} f(U_j(s, y)) G_{t-s}(x, y) dy ds
\]

\[
+ \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x, y) dX^{\beta_0, \beta_1, \ldots, \beta_d}_{t, x}, \quad j = 0, 1, 2, \ldots
\]

(6.6)
Then by (6.3), $U_j(t, x) \in L^2(\mathbb{P})$ for all $j$. And by (6.2),
\[
\mathbb{E}[|U_{j+1}(t, x) - U_j(t, x)|^2] = \mathbb{E}\left[ \int_0^t \int_{\mathbb{R}^d} (f(U_j(s, y)) - f(U_{j-1}(s, y)))G_{t-s}(x, y)dyds \right]^2 \\
\leq L\mathbb{E}\left[ \int_0^t \int_{\mathbb{R}^d} |U_j(s, y) - U_{j-1}(s, y)|G_{t-s}(x, y)dyds \right]^2 \\
\leq L \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x, y)dyds \int_0^t \int_{\mathbb{R}^d} \mathbb{E}|U_j(s, y) - U_{j-1}(s, y)|^2G_{t-s}(x, y)dyds \\
\leq CT \int_0^t \sup_y \mathbb{E}|U_j(s, y) - U_{j-1}(s, y)|^2 ds \\
\leq CT \int_0^t \sup_y \mathbb{E}|U_1(s, y) - U_0(s, y)|^2 ds_{j-1} \ldots ds_1 ds \\
\leq A_T C_T^j \frac{T^j}{j!}
\]
for some constants $A_T, C_T$. It follows that the sequence $\{U_j(t, x)\}_{j=0}^\infty$ of random fields converges in $L^2(\mathbb{P})$ to a random field $U(t, x)$. Letting $j \to \infty$ in (6.6), we see that $U(t, x)$ is a solution of (6.1). The uniqueness follows by the Gronwall’s inequality.

\[\square\]

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