GENERALIZED DEHN FUNCTIONS I

CHAD GROFT

Abstract. Let \( X \) be a finite CW complex or compact Lipschitz neighborhood retract, and let \( \tilde{X} \) be its universal cover; let \( M \) be a compact orientable manifold of dimension \( q \geq 2 \) and \( \partial M \neq \emptyset \). We establish the existence of an isoperimetric profile for functions \( f: M \to \tilde{X} \), in the metric and cellular senses, and show that they are equivalent up to scaling factors when \( X \) is a triangulated CLNR (for example a triangulated Riemannian manifold). We also show that two finite complexes \( X \) and \( Y \) have the same profiles up to scaling given the existence of a \( q \)-connected map between them.

Introduction

Let \( G \) be a group with finite presentation \( P = \langle X|R \rangle \), so that there is an epimorphism \( \pi: F(X) \to G \) whose kernel is the normal subgroup of \( F(X) \) generated by \( R \). If \( w \in F(X) \) is a word such that \( \pi(w) = e \), then \( w = (r_1 w_1)^{e_1} \ldots (r_N w_N)^{e_N} \) for some \( N \geq 0 \), \( r_i \in R \), \( w_i \in F(X) \), and \( e_i = \pm 1 \). The smallest possible \( N \) is called the filling volume for \( w \), or \( \text{FV}(w) \), and the Dehn function for the presentation \( P \) is \( \Phi_P: \mathbb{N} \to \mathbb{N} \),

\[
\Phi_P(n) = \sup \{ \text{FV}(w) : \pi(w) = e, \ |w| \leq n \},
\]

where \(|w|\) is the word length of \( w \) in the generators \( X \). \( \Phi_P \) is not exactly an invariant of \( G \); for example, the trivial group has presentations \( P = \langle | \rangle \) and \( Q = \langle x | x \rangle \), and \( \Phi_P \equiv 0 \) while \( \Phi_Q(n) = n \) for all \( n \). However, for any two finite presentations \( P \) and \( Q \) of the same group, we have

\[
\Phi_P(n) \leq A \cdot \Phi_Q(Bn) + Cn + D
\]

for some \( A, B, C, D \), and vice versa. This is an equivalence relation between functions, so—in the typical abuse of language—one often speaks of the Dehn function of \( G \). Dehn functions can vary widely, from linear (as with \( G \) trivial) to busy-beaver (as when \( G \) has word problem equivalent to the halting problem).

Burillo and Taback give a geometric interpretation to the Dehn function in [3]: Fix a compact Riemannian manifold \( M \). Let \( f: [0, \infty) \to [0, \infty) \) be the smallest function where, given a loop \( \gamma: S^1 \to \tilde{M} \) of length \( \ell \), there is a disk map \( \Gamma: D^2 \to \tilde{M} \) where \( \Gamma \upharpoonright S^1 = \gamma \) (we say that \( \Gamma \) fills \( \gamma \)) and where the area of \( \Gamma \) is at most \( f(\ell) \).

Then \( f \) is equivalent to the Dehn function of \( \pi_1(M) \). This tells us that part of the large-scale geometry of \( \tilde{M} \) is determined by \( \pi_1(M) \).

The argument is as follows: Triangulate \( M \), and extract a presentation \( \langle X|R \rangle \) for \( G = \pi_1(M) \) from the triangulation. An word \( w \in F(X) \) is represented by a loop \( \gamma_w \) in the 1-skeleton of \( M \), whose length is roughly a constant times \(|w|\); \( \gamma_w \) lifts to \( \tilde{M} \) iff it is nullhomotopic, which occurs iff \( \pi(w) = e \). Conversely, a loop \( \gamma \)
in $M$ may be deformed to the 1-skeleton, where it represents a word $w, \gamma \in F(X)$, and $\gamma$ lifts to $\tilde{M}$ iff $\pi(w, \gamma) = e$.

If a loop $\gamma$ is nullhomotopic, then $w, \gamma = (w_1^{\epsilon_1}) \cdot \ldots \cdot (w_N^{\epsilon_N})^{\epsilon_N}$, and there is a nullhomotopy of $\gamma$ which deforms $\gamma$ to the 1-skeleton of $\tilde{M}$, then traverses $N$ 2-cells of $\tilde{M}$ corresponding to the words $r_i^{\epsilon_i}$; the $\epsilon_i$ determine the direction in which the cell is traversed. This fact, combined with the finite area of the 2-cells, bounds $f$ in terms of $\Phi_G$. Conversely, if $\pi(w) = e$, then $\gamma_w$ is nullhomotopic, and the nullhomotopy may be deformed to a disk map $\Gamma: D^2 \to \tilde{M}^{(2)}$. Now $\Gamma$ represents an element $[\Gamma] \in \pi_2(M^{(2)}, M^{(1)})$, whose generators as a group correspond both to 2-cells in $M$ and to words $r_i^\epsilon$ where $r \in R$ and $v \in F(X)$. Since $\partial[\Gamma] = [\Gamma \cup S^1] = [\gamma_w] = w$, we can bound $FV(w)$, and therefore $\Phi_G$, in terms of $f$.

It should be noted that $M$ can be replaced by an arbitrary CW complex with finite 2-skeleton whose cells are given unit volume by convention.

Alonso, Wang, and Pride extend this idea to higher dimensions in \cite{1}. Say that a group $G$ is of type $F_q$ if there is a CW complex $X = K(G, 1)$ with finite $q$-skeleton. Thus $F_0$ is trivial, $F_1$ means “finitely generated”, $F_2$ means “finitely presented”, and so forth. For such an $X$, and for $q \geq 2$, there is a $q$-dimensional Dehn function which compares the volumes of cellular maps $f: S^{q-1} \to X$ to those of cellular extensions $g: D^q \to X$. (Here “volume” is interpreted as word length in, say, $\pi_q(X^{(q)}, X^{(q-1)})$; thus the case $q = 2$ gives us the classical Dehn function.) Alonso et. al. prove that this higher-dimensional Dehn function is independent of the specific $K(G, 1)$ chosen, up to equivalence.

In \cite{3} section 6.D] Gromov correctly asserts, but does not prove, the following: Let $M$ be a compact Riemannian manifold with $\pi_2(M) = \cdots = \pi_{q-1}(M) = 0$, so that $M$ is “approximately” a $K(G, 1)$. Let $f$ be the isoperimetric function for filling maps $S^{q-1} \to \tilde{M}$ with maps $D^q \to \tilde{M}$. Then $f$ is equivalent to the $q$-dimensional Dehn function on $\pi_1(M)$. Again we see an aspect of the large-scale geometry of $\tilde{M}$ which depends solely on the low-dimensional topology of $M$.

Gromov continues in \cite{3} remark 6.34\frac{1}{2}x(c)):

It is unclear how to approach the $q$-dimensional isoperimetric problem with $q \geq 3$ without the . . . condition $\pi_i = 0$ for $i \leq q - 1$ . . . . Also, one should be more specific here on the allowed topology of manifolds . . . involved in the definition. For example, one can stick to $(q - 1)$-spheres which are allowed to be filled in (spanned) by $q$-balls. Or, we can look at $(q - 1)$-tori filled by solid $q$-tori . . . .

In general, we could look at maps defined on any manifold $N^q$ with nonempty boundary; for convenience, we only consider $N$ connected and orientable, though a parallel theory for $N$ non-orientable doubtless exists. Specifically, given a map $f: \partial N \to \tilde{M}$, extend to a map $N \to \tilde{M}$ with as small a volume as possible, then bound this minimal volume in terms of the volume of $f$. Such functions are examined in \cite{2}, for example.

As a sort of limiting case, one can also consider $(q - 1)$-currents filled by $q$-currents; this version is seen in \cite{5} and \cite{8} in the case of highly connected spaces.

The question is in fact nontrivial if we allow $\tilde{M}$ not highly connected. This is most easily seen by considering $M = \mathbb{C}P^2$ under, say, the Fubini-Study metric. Topologically, $M$ is a CW complex consisting of a 4-cell attached to a 2-sphere by the Hopf map $\eta: S^3 \to S^2$; this map represents a generator of $\pi_3(S^2) = \mathbb{Z}$. If
\( f : S^3 \to \mathbb{C}P^2 \) is the composition of a representative map for \( n[\eta] \) with the inclusion \( S^2 \to \mathbb{C}P^2 \), then \( f \) has zero volume, but any filling map \( D^4 \to \mathbb{C}P^2 \) must have volume at least \( n \text{Vol}(\mathbb{C}P^2) \). Thus the isoperimetric function corresponding to maps \( S^3 \to \mathbb{C}P^2 \) and \( D^4 \to \mathbb{C}P^2 \) is infinite everywhere, whereas looking only at \( \pi_1(\mathbb{C}P^2) = e \), one would expect a finite-everywhere, even linear, function. If one replaces \( D^4 \) with \( \mathbb{C}P^2 \setminus B \) where \( B \) is a small open ball, one can fill \( f \) with a map of volume \( (n - 1) \text{Vol}(\mathbb{C}P^2) \), but one still obtains an infinite isoperimetric function. Contrast with the version involving chains; there are no nonzero cellular 3-chains, so the cellular isoperimetric function is zero everywhere. We therefore expect that a 3-dimensional current \( S \) can be filled with a 4-dimensional current \( T \) where \( M(T) \leq cM(S) + d \) with \( c, d \) constant; and indeed this occurs.

In this paper we establish technical results for isoperimetric profiles for maps \( M^q \to \tilde{X} \), where \( M \) is a fixed compact manifold with nonempty boundary of dimension \( q \geq 2 \), and \( \tilde{X} \) is a fixed compact Riemannian manifold.

In particular we show that, up to equivalence, this profile depends only on the “low-dimensional topology” of \( X \). Specifically, let \( Y \) be another compact Riemannian manifold, and let \( f : X \to Y \) be a continuous function where \( f_* : \pi_t(X) \to \pi_t(Y) \) is an isomorphism for \( 1 \leq t < q \). Then \( X \) and \( Y \) have the same profiles for any fixed \( M \). This idea is borrowed from \([1]\), in which such a function is constructed for \( X \) and \( Y \) both \( K(G,1) \)'s for a common group \( G \). Such spaces are homotopy equivalent, as is well known. This equivalence, along with a bound on the local volume of the homotopies, is how Alonso et al. establish the invariance of their Dehn functions. We extend their idea by noting that since only the \( q \)-skeleta of \( X \) and \( Y \) matter, the homotopy groups \( \pi_t \) for \( t \geq q \) can be changed essentially at will and must be irrelevant. In particular, this implies that these generalized Dehn functions are well-defined, up to equivalence, on groups \( G \) of type \( F_q \).

We also establish homological versions of these profiles, in which we replace maps from \( M \) with \( q \)-currents or \( q \)-chains, and prove similar theorems about them.

As when \( M = D^q \) and \( \tilde{X} \) is \((q-1)\)-connected, this is done by addressing a similar question where \( X \) is a CW complex, then connecting the two notions through a variant of the Deformation Theorem. There is one serious issue with doing so: given a function \( f : M \to X \), how does one determine the volume of \( f \)? To have a volume in terms of number of cells covered in some sense, we must at least have \( f[M] \subseteq X^{(q)} \) and \( f[\partial M] \subseteq X^{(q-1)} \), as in \([1]\) for the cases \( M = D^q \) and \( M = S^q \).

In the literature (for example \([2]\)), one sees the notion of an admissible map, i.e., a map \( f \) as above and for which \( f^{-1}[X \setminus X^{(q-1)}] \) is a disjoint union of open \( q \)-dimensional disks, each of which is mapped homeomorphically onto the interior of a \( q \)-cell of \( X \). The volume of \( f \), hereafter written \( \text{Vol} f \), is simply the number of disks. While many maps are admissible (certainly enough to define the relevant profiles), not all are. In particular, if a Lipschitz map \( f : M \to X \) has been deformed to a map \( f' : M \to X^{(q)} \), it is unlikely that \( f' \) will be admissible.

If \( f \) happens to be cellular for some triangulation of \( M \), then we may restrict to each \( q \)-cell \( \Delta \) of \( M \), take the word length in \( \pi_q(X^{(q)}, X^{(q-1)}) \) as the volume of \( f \mid \Delta \) (as is done in \([1]\)), and sum over all \( \Delta \). This is very useful for explicitly constructed maps, but again it is a bit much to ask from an arbitrary map.

Alternately, generalize the concept of “admissible” somewhat, and suppose that each component \( W \) of \( f^{-1}[X^{(q)} \setminus X^{(q-1)}] \), though not necessarily a disk, has smooth
boundary. Then, ignoring some boundary issues, \( f \upharpoonright W \) is a map from one orientable \( q \)-dimensional manifold with boundary to another (namely a \( q \)-cell of \( X \)). This map has a certain degree, taken as nonnegative; we can then sum this degree over all \( W \). A variant of this idea, which uses Čech cohomology, can be applied to any map \( f : M \to X^{(q)} \) with \( f[\partial M] \subseteq X^{(q-1)} \).

We spend a fair amount of effort showing that these definitions are equivalent for our purposes.

The paper is organized as follows. Section 1 defines the notions of volumes for maps and currents in our various spaces. Section 2 is concerned with insuring that the cellular definitions of volume for a map are equivalent up to homotopy, which allows us to formally define the filling volumes and isoperimetric profiles in section 3. We relate the metric and cellular profiles and notions of volume in section 4 if \( X \) is a compact Riemannian manifold with a triangulation, then the profiles for a covering space of \( X \) are the same up to equivalence.

Finally, we show in section 5 the cellular version of the result above: that if a map \( f : X \to Y \) exists which is a \( q \)-homotopy equivalence, then the \( q \)-dimensional profiles for \( X \) and \( Y \) are equivalent. In fact, the result is slightly stronger; the induced map \( f_* : \pi_q(X) \to \pi_q(Y) \) need not be onto. This may be seen as generalizing the result given in [1], where the \( q \)-dimensional profiles on a highly connected \( \tilde{X} \) are shown to depend essentially on \( \pi_1(X) \) alone.

If \( X \) is a compact space with \( \pi_1(X) = G \) and where \( \tilde{X} \) is \((q-1)\)-connected, one can apply these constructions and theorems to establish geometric group invariants for \( G \) which are apparently distinct from the \( q \)-dimensional Dehn function. In part II of the paper, we will see that the new profiles are equal for \( X \) a CW complex, and equal almost everywhere for \( X \) a manifold, provided \( q \geq 4 \). In the case \( q = 3 \) the profile obtained is dependent only on \( \partial M \). For \( q = 2 \) we have no positive results.

Except for some corrections and technical additions, this work has been published as a dissertation (see [7]). The author would like to thank Richard Schoen, under whose advice the dissertation was completed, and Alex Nabutovsky and Rina Rotman, for valuable criticism of the articles.

1. Definitions and technical lemmas

1.1. Conventions. For the cellular definitions and results, \( X \) will denote a connected CW complex. We assume in that each attaching map is cellular, or equivalently that there is a vertex in the range of each attaching map. For the metric definitions and results, \( X \) will denote a connected local Lipschitz neighborhood retract, or LLNR. (That is, \( X \) will be a subset of some \( \mathbb{R}^N \) with a neighborhood \( U \) and a locally Lipschitz retraction \( r : U \to X \). This is slightly more general than a manifold, and is the natural setting for most of our metric work.) In many cases \( X \) will be an LLNR with a Lipschitz triangulation, putting it in both categories.

\( q \) will always denote a positive natural number, and \( M \) will always be a compact orientable smooth manifold of dimension \( q \), possibly with boundary. If \( \partial M \) is nonempty, we will assume \( M \) is connected. In the case \( q = 1 \), \( \partial M \) will always be empty, so that \( M \) is a finite disjoint union of \( S^1 \)'s. We assume that \( M \) has a basepoint \( * \), and that \( \partial M \) contains \( * \) unless it is empty.

For convenience, we define \( \partial f := f \upharpoonright \partial M \) for any \( f : M \to X \).
1.2. **Volumes.** When $X$ is an LLNR, the volumes of currents in $X$ and Lipschitz maps to $X$ are well established. If $T$ is an integral $q$-current in $X$, we take its mass $M(T)$ as defined in [1] \S 4.1.7 as the relevant volume. If $f : M \to X$ is a Lipschitz map under some (any) metric on $M$, we define its volume as usual:

$$\text{Vol}_{\text{met}} f := \int_M \sqrt{\det(f^*g)}_{ij} \, dx^1 \wedge \cdots \wedge dx^q = \int_M |Jf(x)| \, dV_x$$

where $(x^1, \ldots, x^q)$ are local coordinates.

Now let $X$ be a CW complex. Here, all our volumes will be word lengths in various groups. Recall that, if $G$ is a group with generating set $S$, the length of $g \in G$ (which we denote $|g|$) is the least $n$ where $g = s_1^\epsilon \ldots s_n^\epsilon$ for some $s_i \in S$ and some $\epsilon_i = \pm 1$. By obvious convention, $|e| = 0$.

Let $C_q(X) = C_q^{CW}(X) = H_q(X^q, X^{(q-1)})$ be the free abelian group with basis $S$ in 1-1 correspondence with the $q$-cells of $X$. The map $\partial : C_q(X) \to C_{q-1}(X)$ is canonical. We define the volume of $c \in C_q(X)$ as the word length $|c|$ in the generators $S$.

As noted above, we only assign volumes to certain functions $f : M \to X$.

**Definition 1.** A continuous function $f : M \to X$ is **quasi-cellular** if it is a map $f : (M, \partial M, *) \to (X^0, X^{(q-1)}, X^{(q)})$. For such a function $f$, we define the **content** of $f$ as the set $\text{Cont}(f) := M \setminus f^{-1}[X^{(q-1)}]$.

If $f$ is quasi-cellular, so is $\partial f$. The basepoint condition is for convenience. The intuition behind $\text{Cont}(f)$ is that the part of $f$ mapping to the $(q-1)$-skeleton cannot contribute to the volume of the $q$-dimensional object $f$.

One might consider taking the volume of a quasi-cellular map $f$ to be the length of $f_*([M]) \in C_q(X)$, but this reduces to the volume of a chain, which we are trying to avoid. If it is necessary to cover the same cell in opposite orientations in order to fill a given map on $\partial M$, we want both of these to have positive volume, not cancel each other out.

1.2.1. **Alonso-Wang-Pride volume.** In the case $M = D^q$ or $M = S^q$ (which by the collapsing map $(D^q, S^{q-1}, *) \to (S^q, *, *)$ may be seen as a special case of the first), a definition already exists, introduced by Alonso, Wang, and Pride in [1]. They define the groups

$$K_t(X, v) = \pi_t(X^{(t)}, X^{(t-1)}, v), \quad t \geq 2,$$

$$K_1(X) = K_1(X, v) = \pi_1(X^{(1)}/X^{(0)}, *), \quad t = 1,$$

where $v$ is a 0-cell of $X$. For $t \geq 3$, $K_t(X, v)$ is a free $\pi_1(X^{(t-1)}, v)$-module, so it is a free abelian group with basis

$$S = \{ \gamma \ast \phi_\sigma : \gamma \in \pi_1(X^{(t-1)}, v), \sigma \text{ a } t\text{-cell} \}$$

where $\phi_\sigma$ is the homotopy class of the standard map $i : D^t \to X^{(i)}$ combined with some fixed path from a 0-cell on $\partial \sigma$ to $v$. The group $K_t(X, v)$ is independent of $v$ up to isomorphisms which preserve the set of generators and therefore the word length.

$K_2(X, v)$ is not generally abelian, but it is a free crossed $(\pi_1(X^{(1)}, v), \partial)$-module as in [14], and there is a set of group generators

$$S = \{ \gamma \ast \phi_\sigma : \gamma \in \pi_1(X^{(1)}, v), \sigma \text{ a } 2\text{-cell} \}$$
defined as above. \(X^{(1)}\) is path-connected, and associated to each path \(\gamma: v \rightarrow w\) is an isomorphism \(p_\gamma: K_2(X, v) \rightarrow K_2(X, w)\) which is a bijection between the sets of group generators. This is enough to make the definition of volume independent of \(v\), as noted in Lemma 1 of [1].

\(K_1(X)\) is a free group with generators corresponding to the 1-cells of \(X\).

**Definition 2.** If \(f: D^q \rightarrow X\) or \(f: S^q \rightarrow X\) is quasi-cellular, we define the AWP-volume of \(f\) as \(\text{Vol}_{\text{AWP}} f := ||[f]||\), where \([f]\) is the equivalence class of \(f\) in \(K_q(X, f(*))\). If \(M = S^1\), then take \(\text{Vol}_{\text{AWP}} f := ||[\pi \circ f]||\) where \(\pi\) is the natural projection \((X^{(1)}, X^{(0)}) \rightarrow (X^{(1)}/X^{(0)}, *)\).

1.2.2. *Triangulated volume.* Return to general \(M\), and let \(\tau\) be a \(C^1\) triangulation of \(M\) in the sense of [11]. Assume \(* \in \tau^{(0)}\).

**Definition 3.** A continuous map \(f: M \rightarrow X\) is \(\tau\)-cellular if it is a map \(f: (M, \tau^{(q-1)}, *) \rightarrow (X^{(q)}, X^{(q-1)}, X^{(0)})\).

A \(\tau\)-cellular map is automatically quasi-cellular.

If we restrict a \(\tau\)-cellular map \(f\) to a single \(q\)-simplex of \(\tau\), we obtain a quasi-cellular map \(D^q \rightarrow X\), which has an AWP-volume. By summing over the top-dimensional simplices of \(\tau\), we obtain a reasonable definition of volume for \(f\).

**Definition 4.** For \(\tau\) a \(C^1\) triangulation of \(M\) and \(f: M \rightarrow X\) a \(\tau\)-cellular map, the \(\tau\)-volume of \(f\) is

\[
\text{Vol}_\tau f := \sum_\Delta \text{Vol}_{\text{AWP}} (f | \Delta),
\]

where \(\Delta\) ranges over the top-dimensional simplices of \(\tau\).

**Definition 5.** For \(f: M \rightarrow X\) quasi-cellular, the triangulated volume of \(f\) is

\[
\text{Vol}_{\text{tr}} f := \inf_\tau \text{Vol}_\tau f,
\]

where \(\tau\) ranges over those triangulations such that \(f\) is \(\tau\)-cellular.

As we will see, we will often construct \(f\) cell by cell on a fixed triangulation. For such \(f\), this notion of volume is convenient. However, \(\text{Vol}_{\text{tr}} f\) is finite iff there is a triangulation \(\tau\) where \(\text{Cont}(f) \cap \tau^{(q-1)} = \emptyset\), which fails for a number of maps. Infinity is a poor reflection of \(\text{Vol}_{\text{met}} f\) in such cases.

1.2.3. *Cohomology volume.* Finally, we turn to (co)homology. Recall that our objection to using \(||f_*(\langle M \rangle)||\) as the volume of \(f\) was that different regions of \(M\) might overlap the same point with opposite orientations. We can minimize this issue by considering each component \(W\) of \(\text{Cont}(f)\) separately. Assuming that \(H_q(W, \partial W) \cong \mathbb{Z}\) for each \(W\), we can consider the degree of

\[
f_*: H_q(W, \partial W) \rightarrow H_q(X^{(q)}, X^{(q)} \setminus \sigma(W))\]

where \(\sigma(W)\) is the unique \(q\)-cell of \(X\) which meets \(f[W]\). The volume could then be a sum of degrees over components of \(\text{Cont}(f)\).

This scheme has two potential problems. One is that there may be cancellation within a single component \(W\). If \(W\) and \(f \upharpoonright W\) are sufficiently nice, we can deal with this, as we will see in the proof of lemma 2. A more fundamental problem is that we generally do not know that \(H_q(W, \partial W)\) is infinite cyclic. To fix this.

\(^1\)This should really be called "\(\tau\)-quasi-cellular", but that would be ugly.
problem, we replace homology by Čech cohomology, for which [11] Ch. XI, §6] is the standard reference.

If \( \sigma \) is a q-cell of \( X \), then \( \tilde{H}^q(X^{(\sigma)}, X^{(\sigma)} \setminus \sigma^\circ) \cong \mathbb{Z} \). If \( W \subseteq M^\circ \) is open and \( \{ W_\alpha : \alpha \in I \} \) is an indexed collection of its components, then each \( \tilde{H}^q(M, M \setminus W_\alpha) \) is infinite cyclic and the diagram \( \tilde{H}^q(M, M \setminus W_\alpha) \to \tilde{H}^q(M, M \setminus W) \), where \( \alpha \) ranges over \( I \), is a coproduct diagram. It follows that \( \tilde{H}^q(M, M \setminus W) \) is a free abelian group with basis \( S \cong I \); the coordinate corresponding to \( \alpha \) can be seen as the number of times that \( W_\alpha \) covers \( \sigma \).

**Definition 6.** For a quasi-cellular map \( f : M \to X \), and for each q-cell \( \sigma \) in \( X \), there is a map

\[
f^*_\sigma : \mathbb{Z} \cong \tilde{H}^q(X^{(\sigma)}, X^{(\sigma)} \setminus \sigma^\circ) \to \tilde{H}^q(M, M \setminus f^{-1}[\sigma^\circ]) \cong \bigoplus_{\alpha \in I_\sigma} \mathbb{Z},
\]

where \( I_\sigma \) indexes the components of \( f^{-1}[\sigma^\circ] \). The **cohomology volume** of \( f \) is

\[
\text{Vol}_{\text{coh}} f := \sum_{\sigma} \| f^*_\sigma([X^{(\sigma)}]) \|,
\]

where \( \sigma \) ranges over the q-cells of \( X \) and \([X^{(\sigma)}]\) generates \( \tilde{H}^q(X^{(\sigma)}, X^{(\sigma)} \setminus \sigma^\circ) \).

Note that \( \text{Vol}_{\text{coh}} f \) is always finite, since \( f[M] \) is compact and therefore meets the interior of only finitely many q-cells (see [10] theorem A.1). Also note that, if \( f \) is admissible, then \( \text{Vol} f = \text{Vol}_{\text{coh}} f \).

### 2. Homotopy “invariance” of volume

Let \( X \) be a CW complex for this section. It is annoying to have two or three notions of volume for a single quasi-cellular map \( f : M \to X \). Of course one cannot say in general that \( \text{Vol}_{\text{AWP}} f = \text{Vol}_{\text{tr}} f = \text{Vol}_{\text{coh}} f \). However, we have a substitute.

**Definition 7.** Let \( f, f' : M \to X \) be quasi-cellular. A homotopy \( H : f \simeq f' \) is **nice** if it fixes \( \partial M \) and \( \ast \) and if \( H[M \times [0, 1]] \subseteq X^{(q)} \). \( f \) and \( f' \) are called **nicely homotopic** if such an \( H \) exists.

Note that for \( f, f' : M \to X \) where \( M = D^q \) or \( M = S^q \), if \( f \) and \( f' \) are nicely homotopic, then \( \text{Vol}_{\text{AWP}} f = \text{Vol}_{\text{AWP}} f' \).

This section is devoted to these two lemmas:

**Lemma 1.** Let \( M = D^q \) or \( M = S^q \), and let \( f : M \to X \) be quasi-cellular. Then \( \text{Vol}_{\text{AWP}} f \leq \text{Vol}_{\text{coh}} f \), and \( \text{Vol}_{\text{coh}} g \leq \text{Vol}_{\text{AWP}} f \) for some \( g \) which is nicely homotopic to \( f \) and admissible.

**Lemma 2.** Let \( f : M \to X \) be quasi-cellular. Then \( f \) is nicely homotopic to \( g \) where \( \text{Vol}_{\text{tr}} g = \text{Vol}_{\text{tr}} f \) for some \( \tau \) where, for each q-cell \( \Delta \), \( g[\Delta] \) meets the interior of at most one cell \( \sigma \); if there is such a \( \sigma \), then \( g \) is smooth on \( g^{-1}[\sigma^\circ] \) and covers each point of \( \sigma^\circ \) at most \( |\text{deg}(g \setminus \Delta)| \) times. Also \( f \) is nicely homotopic to \( h \) where \( \text{Vol}_{\text{coh}} h \leq \text{Vol}_{\text{tr}} f \) and \( h \) is admissible.

Note that any quasi-cellular map \( f : M \to X \) actually maps into a finite subcomplex \( Y \) of \( X^{(q)} \). If each q-cell \( \sigma \) of \( Y \) has center point \( p_\sigma \), then, by a uniform continuity argument, at most finitely many of the components of \( \text{Cont}(f) \) meet the set \( f^{-1} \{ p_\sigma : \sigma \text{ a q-cell} \} \). For every other component \( U \), there is a homotopy rel
$\partial U$ from $f \upharpoonright U$ to a map $g_U : U \to X^{(q-1)}$. Attach these homotopies to the identity homotopy on the rest of $M$ to obtain $H : f \simeq g$. This $H$ gives us the following:

**Lemma 3.** Any quasi-cellular $f : M \to X$ is nicely homotopic to $g : M \to X$ where $\text{Vol}_{\text{coh}} g = \text{Vol}_{\text{coh}} f$, $\text{Vol}_{\text{AWP}} g = \text{Vol}_{\text{AWP}} f$ when this expression makes sense, $\text{Vol}_{\text{coh}} g \leq \text{Vol}_{\text{AWP}} f$, and $\text{Cont}(g)$ has finitely many components.

The statement for $\text{Vol}_{\text{AWP}}$ is clear. The $\text{Vol}_{\text{coh}}$ statement follows from the fact that $\text{Vol}_{\tau} g = \text{Vol}_{\tau} f$ for any $\tau$ where $\text{Vol}_{\tau} f$ is finite, which in turn follows from the $\text{Vol}_{\text{AWP}}$ statement applied to each $q$-cell of $\tau$. The $\text{Vol}_{\text{coh}}$ equivalence follows from the commutative diagram

$$
\begin{array}{ccc}
\hat{H}^q(X, X \setminus e^0) & \xrightarrow{g^*} & \hat{H}^q(M, M \setminus g^{-1}[e^0]) \\
q^* & = & q^*
\end{array}
$$

where $e$ is any $q$-cell of $X$ and $\iota$ is the canonical inclusion (note that $q^*$ is a length-preserving embedding).

In fact, we can make lemma 3 stronger.

**Lemma 4.** In lemma 3 above, such a $g$ exists where the boundary of $\text{Cont}(g)$ is a smooth submanifold of $M$ and where, for each component $W$ of $\text{Cont}(g)$, the restriction $g \upharpoonright \overline{W}$ factors as $i \circ g_W$, where $i : D^n \to X$ canonically maps $D^n$ onto $\sigma(W)$.

**Proof.** Assume WLOG that $\text{Cont}(f)$ has finitely many components. Consider each $q$-cell as a unit disk with the standard differential structure and geometry. By a mollification, $f$ is homotopic rel $f^{-1}[X^{(q-1)}]$ to a quasi-cellular map $g_1$ which is smooth on $\text{Cont}(g_1) = \text{Cont}(f)$. Clearly $\text{Vol}_{\text{coh}} g_1 = \text{Vol}_{\text{coh}} f$, etc. For $x \in X^{(q)}$, let $\rho(x) = \text{dist}(x, X^{(q-1)}) \in [0, 1]$. Then there exist arbitrarily small $\epsilon > 0$ which are regular values of $\rho \circ g_1$.

For a given $\epsilon$, compose $g_1$ with a homotopy of each $\sigma$ which deformation retracts an $\epsilon$-neighborhood of $\partial \sigma$ onto $\partial \sigma$ and scales the rest of $\sigma$ uniformly, resulting in a map $g_2 : M \to X$. Then $\text{Cont}(g_2) \subseteq \text{Cont}(g_1)$. If $\epsilon$ is chosen as above, then $\partial \text{Cont}(g_2) = (\rho \circ g_1)^{-1}[\epsilon]$ is a smooth submanifold of $M$. For any component $W$ of $\text{Cont}(g_2)$, the restriction $g_2 \upharpoonright \overline{W}$ factors through the map $i : D^n \to X$ as in the statement because $g_1 \upharpoonright \overline{W}$ maps to the slightly smaller closed ball of radius $1 - \epsilon$.

We do not yet know that $\text{Vol}_{\text{coh}} g_2 \leq \text{Vol}_{\text{coh}} g_1$, because we do not know that each component $V$ of $\text{Cont}(g_1)$ contains at most one component of $\text{Cont}(g_2)$. We may achieve this by further homotopy. First, by a uniform continuity argument, at most finitely many components of $V \cap \text{Cont}(g_2)$ cover the center point of $\sigma(V)$; on the rest, we homotope $g_2$ to a map $g_3$ into $\partial \sigma(V)$ as before. Enumerate the components of $W = V \cap \text{Cont}(g_3)$ as $W_1, \ldots, W_n$, and suppose $n > 1$. We may homotope $g_3$ into a new map $h$ for which $V \cap \text{Cont}(h)$ has $n - 1$ components, iterate this construction to get at most one component, and repeat this iteration for each component $V$ to get a map $g$. That $\text{Vol}_{\text{coh}} g = \text{Vol}_{\text{coh}} g_1$ may be seen by a commutative diagram similar to that in lemma 3.

To construct $h$, let $T = (V \setminus W)^\circ$. There is some component of $T$ which shares boundary points with both $W_1$ and $W_j$ for some $j > 1$; otherwise the union of $W_1$ and the adjacent components of $T$ is a nontrivial component of $V$, but $V$ is by assumption connected. Note that $\partial W_1 \cap \partial W_j = \emptyset$. Let $\gamma : [0, 1] \to V$ be a path transverse to $\partial W_1 \cap \partial T$ at 0 and transverse to $\partial W_j \cap \partial T$ at 1, and where
Let $W_f$ be the remaining components. This proves the case. By lemma 4, we may assume that $\text{Cont}(f)$ has finitely many components, which lowers $\text{Vol}_{\text{coh}} f$ and keeps $\text{Vol}_{\text{AWP}} f$ the same. After finitely many such reductions, we have a map which represents a reduced word in the generators of $K_1(X)$, and it is clear that $\text{Vol}_{\text{AWP}} f = \text{Vol}_{\text{coh}} f$ is the number of remaining components. This proves the case.

Next, take $q \geq 3$. First adjust $f$ as in lemma 4. Also assume $\pi_1(X) = 0$; if this is not the case, lift $f$ to a map $\tilde{f} : M \to \tilde{X}$, which has the same volumes. Let $W = \text{Cont}(f)$, and let $W_1, \ldots, W_k$ be the components of $W$. Consider the

The third line is by the direct sum decomposition of $\check{H}^q(M, M \setminus W)$ by the components of $W$. The fourth follows by excision and a homotopy equivalence. The last is easily seen through the universal coefficient theorem for cohomology, which says in this case that the natural transformation $\check{H}^q \cong H^q \to \text{Hom}(H_q(\gamma), \mathbb{Z})$ is an isomorphism.

**Proof of lemma 4.** The second part of the lemma is easier. The homotopy class of $f$ is an element of $\pi_q(X^{(q)}, X^{(q-1)}, v)$ for some $v \in X^{(0)}$ (or is equivalent to an element of $\pi_1(X^{(1)}, X^{(0)}, *)$). This group has a standard generating set and a standard composition law; express $[f]$ in as few generators as possible, and let $f'$ be the map so obtained. From inspection one sees that $\text{Vol}_{\text{coh}} f' = \text{Vol}_{\text{AWP}} f' = \text{Vol}_{\text{AWP}} f$.

There is a homotopy $H : f' \simeq f$ where $H \upharpoonright S^{q-1} \times [0, 1]$ maps into $X^{(q-1)}$. Now let $g : D^q \to X$ where $g(r, \theta) = f'(2r, \theta)$ for $r \leq 1/2$ and $g(r, \theta) = H(\theta, 2r - 1)$ for $r \geq 1/2$. $f'$ and $g$ have the same volumes, and $f$ is nicely homotopic to $g$.

To prove the first statement, first take $q = 1$, so that $f : (S^1, *) \to (X^{(1)}, X^{(0)})$. By lemma 3 we may assume that $\text{Cont}(f)$ has finitely many components. By further homotopy we can eliminate any component which does not traverse an edge; this does not change $\text{Vol}_{\text{AWP}} f$ or $\text{Vol}_{\text{coh}} f$. If any two adjacent components traverse the same edge in opposite directions, then yet further homotopy can eliminate these components, which lowers $\text{Vol}_{\text{coh}} f$ and keeps $\text{Vol}_{\text{AWP}} f$ the same. After finitely many such reductions, we have a map which represents a reduced word in the generators of $K_1(X)$, and it is clear that $\text{Vol}_{\text{AWP}} f = \text{Vol}_{\text{coh}} f$ is the number of remaining components. This proves the case.
where $D$ component of $\text{Cont}(f)$ of $\text{Cont}(\sigma)$ is the same as $f$. Homotopy, we can assume that every component $\gamma$ smooth closed curve. $W$ the boundary of any given $S^q-1$ is connected, and the pairs $(D^q, S^{q-1})$ and $(X(q), X(q-1))$ are both $(q-1)$-connected, so by the Hurewicz isomorphism theorem (see [13 §7.5]) the horizontal arrows are isomorphisms. The isomorphism $H_q(D^q, D^q \setminus W) \cong \mathbb{Z}^k$ may be seen as the product of the $H_q(D^q, D^q \setminus W)$ or as a direct sum diagram from the $H_q(W, \partial W)$. The arrow from $H_q(D^q, S^{q-1})$ to $H_q(D^q, D^q \setminus W)$ is the identity on every factor of $\mathbb{Z}^k$. Thus

$$\text{Vol}_{\text{AWP}} f = |f_\ast[D^q]|$$

$$= \left| \sum_\sigma (f_\ast : H_q(D^q, S^{q-1}) \rightarrow H_q(X(q), X(q) \setminus \sigma)) [D^q] \right|$$

$$\leq \sum_\sigma \left| (f_\ast : H_q(D^q, S^{q-1}) \rightarrow H_q(X(q), X(q) \setminus \sigma)) [D^q] \right|$$

$$= \sum_j \left| f_\ast : H_q(W, \partial W) \rightarrow H_q(X(q), X(q) \setminus (\sigma(W))) \right|$$

$$= \sum_j \left| \text{deg}(f : W, \partial W) \rightarrow (X(q), X(q) \setminus (\sigma(W))) \right|$$

$$= \text{Vol}_{\text{coh}} f.$$

Finally, let $q = 2$. First adjust $f$ as in lemma [4]. Next we show that, up to nice homotopy, we can assume that every component $W$ of $\text{Cont}(f)$ is a disk. Initially, the boundary of any given $W$ has finitely many components, each of which is a smooth closed curve. $W$ is connected and bounded, so there must be a single component $\gamma \subseteq \partial W$ which surrounds $W$ and all the other components $\gamma_i$ of $\partial W$, and no $\gamma_i$ lies inside any other $\gamma_j$.

Proceed as in the proof of lemma [4] except “in reverse”: connect each $\gamma_i$ to $\gamma$ by some path $\delta_i$ in $W$, then homotope $f$ to a new function $f'$ which is mostly the same as $f$, but maps into $X(q-1)$ in a narrow strip around each $\delta_i$. Then $\text{Cont}(f') \subseteq \text{Cont}(f)$ and each component of $\text{Cont}(f)$ contains a unique component of $\text{Cont}(f')$, which by earlier reasoning implies $\text{Vol}_{\text{coh}} f' = \text{Vol}_{\text{coh}} f$. Moreover, each component of $\text{Cont}(f')$ has connected boundary, hence is a disk.

Now choose a point $x \in \partial D^2$ and draw $k-1$ loops starting at $x$, separating $D^2$ into $k$ regions, each containing a single component of $\text{Cont}(f')$. Each of these regions is topologically a disk, from which it becomes clear that

$$[f'] = \gamma_1 * (n_1 \phi_{\sigma(1)}) + \cdots + \gamma_k * (n_k \phi_{\sigma(k)})$$

where $\gamma_i$ is the composition of $f'$ with a path from $x$ to a point on the $i$th disk, $\sigma(i)$ is the cell to which this disk maps, and $n_i$ is the degree of the map between
disks. From there we see that
\[ \text{Vol}_{\text{coh}} f' = \sum_{i=1}^{k} |n_i| \geq ||f'|| = \text{Vol}_{\text{AWP}} f' = \text{Vol}_{\text{AWP}} f. \]

Proof of lemma\[2\] Again the second statement is easier. If \( \text{Vol}_{\text{tr}} f < \infty \), choose \( \tau \) where \( \text{Vol}_{\text{tr}} f = \text{Vol}_{\text{tr}} g \). Apply lemma[1] on each \( q \)-cell of \( \tau \) to obtain \( h \), nicely homotopic to \( f \), where \( \text{Vol}_{\text{coh}} h \leq \text{Vol}_{\text{tr}} f = \text{Vol}_{\text{tr}} f \). If \( \text{Vol}_{\text{tr}} f = \infty \), let \( h = f \).

For the first statement, first adjust \( f \) as in lemma[3]. Then
\[ \text{Vol}_{\text{coh}} f = \sum_{W} |\deg f_W| \]
where \( W \) ranges over the components of \( \text{Cont}(f) \) and \( f_W : (\overline{W} \setminus \partial W) \to (D^q, S^{q-1}) \) as in the conclusion of lemma[3]. Let \( \tau_W \) triangulate \( W \) for each \( W \), and let \( \tau \) extend the \( \tau_W \) to triangulate \( M \).

First we note that each \( f_W \) is homotopic relative to \( \partial W \) to a map \( g_W \) which maps all of \( M \), except for the interior of some \( q \)-cell \( \Delta \) of \( \tau_W \), to \( S^{q-1} \). First consider the special case where \( \tau_W \) consists of 2 \( q \)-cells \( \Delta_1 \) and \( \Delta_2 \) which intersect along a \((q-1)\)-face \( F \). It is not hard to see that \( 1_W : W \to W \) is homotopic (relative to \( \partial W \)) to a map which maps \( \Delta_2 \) onto \( W \) and \( \Delta_1 \) onto \( \partial \Delta_1 \setminus F \). Composing this homotopy with \( f_W \) gives \( g_W \).

For the general case, let \( G \) be the undirected graph whose vertices are the \( q \)-cells of \( \tau_W \) and where \( \{ \Delta, \Delta' \} \) is an edge iff \( \Delta \cap \Delta' \) is an \((q-1)\)-cell. Since \( W \) is a connected manifold, \( G \) is a connected graph. Let \( T \) be a spanning tree for \( G \). Proceed as follows: Let \( \Delta_1 \) be a leaf of \( T \), and let \( \Delta_2 \) be the unique vertex where \( \{ \Delta_1, \Delta_2 \} \) is an edge of \( T \). Apply the above homotopy to \( \Delta_1 \cup \Delta_2 \), then remove \( \Delta_1 \) and its edge from \( T \). Repeat until \( T \) has a single vertex. We have built a homotopy from \( f_W \) to some \( g_W \), and the key \( q \)-cell \( \Delta \) is precisely the vertex remaining in \( T \). By further homotopy, we can make \( g_W \) smooth on \( g^{-1}(D^q) \) and cover each point of \( (D^q)^c \) minimally.

We now build the homotopy from \( f \) to \( g \). On each \( W \), compose the homotopy from \( f_W \) to \( g_W \) with the characteristic map of the disk into \( X \) to obtain a homotopy from \( f \restriction W \) to \( g \restriction W \). Outside of \( \text{Cont}(f) \), take the constant homotopy. Then
\[ \text{Vol}_{\text{tr}} g \leq \text{Vol}_{\text{tr}} f = \sum_{W} |\deg f_W| = \text{Vol}_{\text{coh}} f \]
and the rest of the claims hold as well.

\[ \square \]

3. Filling volumes and profiles

Recall that \( \mathbf{I}(X) \) is the space of integral \( t \)-dimensional currents in \( X \), while \( C^{0,1}(M, X) \) is the space of Lipschitz maps from \( M \) to \( X \).

Definition 8. Let \( X \) be a Riemannian manifold. If \( S \in \mathbf{I}_{q-1}(X) \), then the current filling volume of \( S \) is
\[ \text{FV}_{\text{cur}}(S) := \inf \{ M(T) : T \in \mathbf{I}_q(X), \partial T = S \} \].
If \( f \in C^{0,1}(\partial M, X) \), then the Lipschitz filling volume of \( f \) through \( M \) is
\[ \text{FV}_{\text{met}}^{X,M}(f) := \inf \{ \text{Vol}_{\text{met}}(h) : h \in C^{0,1}(M, X), \partial h = f \} \].
Now let \( X \) be a CW complex. If \( c \) is a \((q - 1)\)-chain in \( X \), then the chain filling volume of \( c \) is

\[
FV_{\text{ch}}(c) := \inf \{ \|b\| : b \in C_q(X), \partial b = c \}.
\]

If \( f : \partial M \to X \) is quasi-cellular, then the cellular filling volume of \( f \) through \( M \) is

\[
FV^{X,M}_{\text{cell}}(f) := \inf \{ \text{Vol}_{\text{coh}}(g) : g : M \to X \text{ quasi-cellular}, \partial g = f \}
\]

\[
= \inf \{ \text{Vol}_{\text{tr}}(g) : g : M \to X \text{ quasi-cellular}, \partial g = f \}
\]

\[
= \inf \{ \text{Vol}(g) : g : M \to X \text{ admissible}, \partial g = f \};
\]

in the case \( M = D^q \), also

\[
= \inf \{ \text{Vol}_{AWP}(g) : g : D^q \to X \text{ quasi-cellular}, \partial g = f \}.
\]

The various definitions for \( FV_{\text{cell}} \) are equivalent by an application of lemmas 1 and 2. Note that \( FV^{X,D^q}_{\text{cell}} \) is the filling volume defined on [1, p. 87].

**Lemma 5.** If \( f, g : \partial M \to X \) are quasi-cellular and nicely homotopic, then

\[
FV^{X,M}_{\text{cell}}(f) = FV^{X,M}_{\text{cell}}(g).
\]

**Proof.** Let \( F : M \to X \) be quasi-cellular with \( \partial F = f \), and let \( H : f \simeq g \) be a nice homotopy. There is a homeomorphism

\[
\phi : M \cong M' = \left[ M \cup (\partial M \times [0,1]) \right] / \{ x \sim (x,0) : x \in \partial M \}
\]

because \( \partial M \) has a collar neighborhood; \( F \) and \( H \) together form a continuous map \( F' \) on \( M' \), and \( G = F' \circ \phi \) is a quasi-cellular map filling \( g \) with \( \text{Vol}_{\text{coh}} G = \text{Vol}_{\text{coh}} F \). This proves \( FV^{X,M}_{\text{cell}}(f) \geq FV^{X,M}_{\text{cell}}(g) \); the reverse is similar. \( \square \)

**Definition 9.** Let \( X \) be a Riemannian manifold. The current profile of \( X \) in dimension \( q \) is the function \( \Phi^{X,q}_{\text{cur}} : [0, \infty) \to [0, \infty] \) where

\[
\Phi^{X,q}_{\text{cur}}(v) := \sup \{ FV_{\text{cur}}(\partial T) : T \in I_q(X), M(\partial T) \leq v \}.
\]

The metric profile of \( X \) for \( M \) is the function \( \Phi^{X,M}_{\text{met}} : [0, \infty) \to [0, \infty] \) where

\[
\Phi^{X,M}_{\text{met}}(v) := \sup \{ FV^{X,M}_{\text{met}}(\partial h) : h \in C^{0,1}(M, X), \text{Vol}_{\text{met}}(\partial h) \leq v \}.
\]

Now let \( X \) be a CW complex. The chain profile of \( X \) in dimension \( q \) is the function \( \Phi^{X,q}_{\text{ch}} : \mathbb{N} \to \mathbb{N} \cup \{ \infty \} \) where

\[
\Phi^{X,q}_{\text{ch}}(n) := \sup \{ FV_{\text{ch}}(\partial b) : b \in C_q(X), \|\partial b\| \leq n \}.
\]

The cellular profile of \( X \) for \( M \) is the function \( \Phi^{X,M}_{\text{cell}} : \mathbb{N} \to \mathbb{N} \cup \{ \infty \} \) where

\[
\Phi^{X,M}_{\text{cell}}(n) := \sup \{ FV^{X,M}_{\text{cell}}(\partial f) : f : M \to X \text{ quasi-cellular}, \text{Vol}_{\text{coh}} \partial f \leq n \}
\]

\[
= \sup \{ FV^{X,M}_{\text{cell}}(\partial f) : f : M \to X \text{ quasi-cellular}, \text{Vol}_{\text{tr}} \partial f \leq n \}
\]

\[
= \sup \{ FV^{X,M}_{\text{cell}}(\partial f) : f : M \to X \text{ quasi-cellular}, \text{Vol} \partial f \leq n \};
\]

in the case \( \partial M = S^{q-1} \), also

\[
= \sup \{ FV^{X,M}_{\text{cell}}(\partial f) : f : M \to X \text{ quasi-cellular}, \text{Vol}_{\text{AWP}} \partial f \leq n \}.
\]
The definitions for $\Phi_{X,M}^{X,M}$ are equivalent by an application of lemmas 1, 2, and 5.

Note that $\Phi_{X,D}^{X,D}$ is the $q$-dimensional Dehn function defined on $[1, p. 90]$.

These functions are the generalized Dehn functions we wish to study. Our positive results will state that two such functions are equal, or equal almost everywhere, or equivalent as follows.

**Definition 10.** Let $f, g: [0, \infty) \to [0, \infty]$ be weakly increasing. $f$ is quasi-bounded by $g$, written $f \lesssim g$, if there exist $A, B > 0$ and $C, D \geq 0$ where $f(x) \leq A \cdot g(Bx) + Cx + D$ for all $x \geq 0$.

We say $f$ and $g$ are quasi-equivalent, written $f \approx g$, if $f \lesssim g \lesssim f$.

A routine check shows that $\lesssim$ is a preorder, making $\approx$ an equivalence relation.

We extend these concepts to functions $f$ on $\mathbb{N}$ by extending $f$ to $[0, \infty)$ so that $f(x) = f(\lfloor x \rfloor)$. Ideas similar to $\lesssim$ and $\approx$ appear throughout the literature.

4. **Metric and cellular profiles**

For this section, let $\pi: X \to Y$ be a covering map, where $Y$ is a compact Lipschitz neighborhood retract (CLNR; say a Riemannian manifold) with a Lipschitz triangulation $\tau$. Both the metric structure and the triangulation may be lifted through $\pi$ to $X$, so that all of the profiles in section 3 are defined for $X$. As we might expect, and will prove in this section, the profiles corresponding to a manifold $M$ are equivalent, as are the homological profiles in each dimension $q$. Briefly, this occurs because the two notions of volume for a quasi-cellular map are roughly equivalent, and any Lipschitz map $f$ may be deformed into a quasi-cellular map without changing $\Vol_{\text{met}} f$ by more than a constant factor.

**Theorem 1.** $\Phi_{X,q}^{X,q} \approx \Phi_{X,q}^{X,q}$ for any $q \geq 2$ and $\Phi_{X,M}^{X,M} \approx \Phi_{X,M}^{X,M}$ for any $M$.

Thus there is a “metric independence” result:

**Corollary 1.** If $Y$ is a closed Riemannian manifold (with boundary), then the profiles $\Phi_{X,M}^{X,M}$ and $\Phi_{X,q}^{X,q}$ are independent of the metric on $Y$, up to quasi-equivalence.

We spend the rest of the section proving theorem 1. First note that any triangular $q$-chain in $X$ is also a $q$-current, and there are constants $0 < C \leq D$ (the minimum and maximum volume of a $q$-simplex, respectively) where

$$C \Vol_{\text{ch}} T \leq M(T) \leq D \Vol_{\text{ch}} T$$

for all triangular $q$-chains $T$. A similar statement for quasi-cellular maps from $M$ is harder to state and prove.

**Lemma 6.** For every quasi-cellular map $f: M \to X$, $C \Vol_{\text{coh}} f \leq \Vol_{\text{met}} f$ if $f$ is Lipschitz. Moreover, there is a Lipschitz map $g \simeq f$ rel $\partial M$ so that $\Vol_{\text{coh}} g = \Vol_{\text{coh}} f$ and $\Vol_{\text{met}} g \leq D \Vol_{\text{coh}} g$.

**Proof.** First assume for simplicity that $f$ is $C^1$ on $\text{Cont}(f)$. We start with the area formula

$$\Vol_{\text{met}} f = \sum_{\sigma} \int_{\sigma} N(f; y) \, dh(y).$$

If $M$ is a measurable subset of $\mathbb{R}^q$, this follows from case (2) of [10 theorem 3.2.3]; for general $M$, use a partition-of-unity argument. Now almost everywhere in any
Integrate over the $q$-cell $\sigma$ and sum over all cells $X$ is a $C^1$ function $g$ with the same Lipschitz constant as $f$ so that $g = f$ except on a set $S$ where $mS$ is arbitrarily small (assume a background metric). If we only modify $f$ at points $x$ where $\text{dist}(f(x), X^{(q-1)}) > \epsilon > 0$, and make $mS$ small enough, we can guarantee that $\text{Vol}_{coh} f$ does not change, and obtain a lower bound $\text{Vol}_{met} f \geq C(1 - \epsilon)^q \text{Vol}_{coh} f - \|f\|_{0,1}^q mS$. Now take $\epsilon, mS \to 0$.

For the second part, choose $g \simeq f$ and a suitable triangulation of $M$ as in lemma\textsuperscript{[2]} We need only check the last condition. Restrict $g$ to a single simplex $\Delta$. If $g|\Delta| \subseteq X^{(q-1)}$, then $\text{Vol}_{met}(g \restriction \Delta) = 0$. Otherwise $g|\Delta|$ meets the interior of at most one cell $\sigma$, and we have

$$\text{Vol}_{met}(g \restriction \Delta) = \int_{\sigma^*} N(g, y) \, dh(y) = |g^*([\Delta])| \text{Vol} \sigma \leq D|g^*([\Delta])|.$$  

Sum over the simplices $\Delta$ to obtain $\text{Vol}_{met} g \leq D \text{Vol}_{coh} g$. \square

Next we must show that any map can be made quasi-cellular without changing its volume by more than a constant factor. In other words, we need the Deformation Theorem, except with functions in place of currents.

**Lemma 7.** There exists a constant $C$ where, for all $f \in C^{0,1}(M, X)$, there exists a Lipschitz homotopy $H : f \simeq f'$ where $f'[M] \subseteq X^{(q)}$ and where

$$\text{Vol}_{met} f' \leq C \text{Vol}_{met} f,$$

$$\text{Vol}_{met} \partial f' \leq C \text{Vol}_{met} \partial f,$$

and $\text{Vol}_{met} H \leq C \text{Vol}_{met} f$.

Moreover, whenever $f(x) \in X^{(q)}$, we have $H(t, x) = f(x)$, and in particular $f'(x) = f(x)$.

**Proof.** Adapted from [5], starting on page 223, which is itself adapted from the classic proofs in [6] and [12]. WLOG assume the simplices of $X$ are standard. Suppose $f[M] \subseteq X^{(k)}$ with $k > q$. It suffices to show that we can deform $f$ to $f'$ where $f'[M] \subseteq X^{(k-1)}$, the estimates above hold for some constant $C$, and those points already lying in $X^{(k-1)}$ are unmoved; from there we simply iterate the deformation to $X^{(k-2)}$, etc., until our map lies in $X^{(q)}$. Further, we can perform this deformation separately on the interior of each $k$-simplex $\Delta$ and glue the results together by local finiteness of $(X, \tau)$.

Thus, fix $\Delta$ and let $S = f^{-1}[\Delta^0]$. Let $u_0$ be the barycenter of $\Delta$, and let $r = \text{dist}(u_0, \partial \Delta)$. For every $u \in B := B(u_0, r/4)$, let $\pi_u : \Delta \setminus \{u\} \to \Delta \setminus B(u, r/2)$
be the identity outside the ball $B(u, r/2)$ and radial projection inside. The function $f_u = \pi_u \circ (f \mid S)$ is defined for almost all $u$, and
\[
\text{Vol}_\text{met} f_u = \int_{S \setminus f^{-1}[B(u, r/2)]} f_u(x) \, dx + \int_{f^{-1}[B(u, r/2)]} f_u(x) \, dx \\
\leq \text{Vol}_\text{met}(f \mid S) + \int_{f^{-1}[B(u, r/2)]} \frac{Jf(x)(r/2)^q}{|f(x) - u|^q} \, dx.
\]
If we integrate over all $u \in B$, we have
\[
\int_B (\text{Vol}_\text{met} f_u) \, du \\
\leq \text{Vol}(B) \text{Vol}_\text{met}(f \mid S) + \int_B \int_{f^{-1}[B(u, r/2)]} \frac{Jf(x)(r/2)^q}{|f(x) - u|^q} \, dx \, du \\
\leq c_1 \text{Vol}_\text{met}(f \mid S) + \left( \int_{f^{-1}[B(u_0, 3r/4)]} \frac{Jf(x)}{|f(x) - u|^q} \right) \left( \int_{B(f(x), r/2)} |u - f(x)|^{-q} \chi_B(u) \, du \right) \\
\leq c_1 \text{Vol}_\text{met}(f \mid S) + \left( \frac{r}{2} \right)^q \left( \int_{S} Jf(x) \, dx \right) \left( \int_{B(0, r/2)} |v|^{-q} \, dv \right)
\]
since $\int |v|^{-q} \, dv$ is finite over balls in $\mathbb{R}^k, k > q$. (In this case $c$ is dependent only on $k$ and $q$.) It follows that, if we define for $v > 0$
\[
A_v := \{ u \in B : \text{Vol}_\text{met} f_u > v \text{Vol}_\text{met}(f \mid S) \},
\]
then $mA_v \leq c/v$. We can perform similar estimates for $\partial f$ and for the radial homotopies $h_u$; if
\[
B_v := \{ u \in B : \text{Vol}_\text{met}(\partial f)_u > v \text{Vol}_\text{met}(\partial f \mid (S \cap \partial M)) \}
\]
and
\[
C_v := \{ u \in B : \text{Vol}_\text{met} h_u > v \text{Vol}_\text{met}(f \mid S) \},
\]
then $mB_v \leq c'/v$ and $mC_v \leq c''/v$ for some $c', c''$ depending on $k$ and $q$.

Thus, if we fix $v > (c + c' + c'')/(\text{Vol}(B))$ (once again $v = v(k, q)$), then there is some $u \in B$ where $\text{Vol}_\text{met} f_u \leq v \text{Vol}_\text{met}(f \mid S)$, $\text{Vol}_\text{met}(\partial f)_u \leq v \text{Vol}_\text{met}(\partial f \mid S)$, and $\text{Vol}_\text{met} h_u \leq v \text{Vol}_\text{met}(f \mid S)$. Since $\text{dist}(x_0, f_u[S]) \geq r/4$, we may further deform $f_u$ to $f'_u : S \to \partial \Delta$ by a radial homotopy, and all the volumes are appropriately bounded. 

**Lemma 8.** There is a constant $C$ where, for $f \in C^{0,1}(M, X)$, there is a Lipschitz homotopy $H : f \simeq f'$ where $f'$ is quasi-cellular and
\[
\text{Vol}_\text{met} f' \leq C(\text{Vol}_\text{met} f + \text{Vol}_\text{met} \partial f),
\]

\[
\text{Vol}_\text{met} \partial f' \leq C \text{Vol}_\text{met} \partial f,
\]

and $\text{Vol}_\text{met} H \leq C \text{Vol}_\text{met} f$.

**Proof.** First produce a homotopy $H' : f \simeq f''$ as in lemma 7, then produce a homotopy $J : \partial f'' \to j$, also as in lemma 7. We obtain $f'$ by attaching $J$ to $f''$ by a collar neighborhood, which yields the estimate on $\text{Vol}_\text{met} f'$. $\partial f' = j$ gives the
estimate on $V_{\text{met}} f'$, and $f'' \simeq f'$ by a homotopy which lives entirely inside $f'[M]$ and therefore has zero volume; so $V_{\text{met}} H = V_{\text{met}} H'$ gives the last estimate. \hfill \square

We can probably strengthen lemma $8$ to remove the $V_{\text{met}} \partial f$ in the first estimate, a la $12$.

**Lemma 9.** There exist constants $0 < C \leq D$ where, for all $f : \partial M \to X^{(q-1)}$ quasi-cellular and Lipschitz,

$$C FV_{\text{cell}}^{X,M}(f) \leq FV_{\text{met}}^{X,M}(f) \leq DFV_{\text{cell}}^{X,M}(f).$$

**Proof.** For the first inequality, let $h \in C^{0,1}(M, X)$ where $\partial h = f$. By lemma $7$ there is a map $h' \in C^{0,1}(M, X^{(q)})$ where $\partial h' = f$ and $V_{\text{met}} h' \leq C_1 V_{\text{met}} h$. By lemma $6$ there is a constant $c$ where $c V_{\text{coh}} h' \leq V_{\text{met}} h'$. Thus let $C = c/C_1$. The second inequality follows directly from lemma $8$. \hfill \square

The corresponding results for currents carry over directly. We state them without proof.

**Lemma 10.** There is a constant $C$ where, for all $T \in I_q(X)$, there exist currents $P, S \in I_q(X)$ and $R \in I_{q+1}(X)$ where $T - P = \partial R + S$, supp $P \subseteq X^{(q)}$, and

$$M(P) \leq CM(T), \quad M(\partial P) \leq CM(\partial T),$$

$$M(R) \leq CM(T), \quad M(S) \leq CM(\partial T).$$

If supp $\partial T \subseteq X^{(q)}$, then $\partial S = 0$ (i.e., $\partial P = \partial T$).

**Lemma 11.** As above, but the conclusion includes supp $\partial P \subseteq X^{(q-1)}$ and the first volume estimate becomes

$$M(P) \leq C(M(T) + M(\partial T)).$$

**Lemma 12.** There exist constants $0 < C \leq D$ where, for any $q$-cycle $T$ of $X$,

$$C FV_{\text{ch}}(T) \leq FV_{\text{cell}}(T) \leq DFV_{\text{ch}}(T).$$

**Proof of theorem $1$** We prove the second statement. Let $f : \partial M \to X$ be quasi-cellular; we may perturb it slightly to make it Lipschitz without changing $V_{\text{coh}} f$. Then by lemmas $6$ and $9$

$$FV_{\text{cell}}^{X,M}(f) \leq c FV_{\text{met}}^{X,M}(f) \leq c \Phi_{\text{met}}^{X,M}(V_{\text{met}} f) \leq c \Phi_{\text{met}}^{X,M}(D V_{\text{coh}} f).$$

It follows that for all $n$,

$$\Phi_{\text{cell}}^{X,M}(n) \leq c \Phi_{\text{met}}^{X,M}(D n).$$

Now let $f \in C^{0,1}(\partial M, X)$. By lemma $7$ there is a Lipschitz homotopy $H : f \simeq f'$ where $f'$ is quasi-cellular, $V_{\text{met}} f' \leq C V_{\text{met}} f$, and $V_{\text{met}} H \leq C V_{\text{met}} f$. We can attach $H$ to any filling function for $f'$ to get a filling function for $f$, so that

$$FV_{\text{met}}^{X,M}(f) \leq FV_{\text{met}}^{X,M}(f') + C V_{\text{met}} f \leq DFV_{\text{cell}}^{X,M}(f') + C V_{\text{met}} f$$

$$\leq D \Phi_{\text{cell}}^{X,M}(V_{\text{coh}} f') + C V_{\text{met}} f \leq D \Phi_{\text{cell}}^{X,M}(c V_{\text{met}} f') + C V_{\text{met}} f$$

$$\leq D \Phi_{\text{cell}}^{X,M}(c C V_{\text{met}} f) + C V_{\text{met}} f$$

and for any $x \geq 0$,

$$\Phi_{\text{met}}^{X,M}(x) \leq D \Phi_{\text{cell}}^{X,M}(c C x) + C x.$$
5. Profiles and $q$-homotopy equivalence

In the usual proofs that the Dehn function $\delta^k$ of a group $G$ is well-defined up to equivalence, one starts with two $K(G,1)$s $X$ and $Y$ with finite $(k+1)$-skeleta. Without loss of generality, $Y$ is a subcomplex of $X$, in fact a deformation retraction. By choosing the homotopy $(\text{rel } X)$ between $1_X$ and $r: X \to Y$ to be cellular, one obtains a way to deform maps $M \to \tilde{X}$ into maps $M \to \tilde{Y}$ while changing their volume by at most a constant factor. This allows one to show that $\delta^k_X \approx \delta^k_Y$.

This argument works just as well for maps from any $M^q$ into $\tilde{X}$ and $Y$. Also, $X$ and $Y$ may be any two homotopy equivalent complexes with finite $q$-skeleta, not necessarily $K(G,1)$s; and we need not use the universal covers, only covers corresponding to the same subgroup of $\pi_1(X) \cong \pi_1(Y)$. Add to this that the cells in dimensions higher than $q$ are irrelevant, and we have the following.

**Theorem 2.** Let $q \geq 2$, let $X$ and $Y$ be connected CW complexes with $X(q)$ and $Y(q)$ finite, and suppose there is a continuous map $f: Y \to X$ where the induced map $f_*: \pi_t(Y,*) \to \pi_t(X,f(\ast))$ is an isomorphism for $1 \leq t < q$. Let $\tilde{Y}$, $\tilde{X}$ be covering spaces of $Y$ and $X$ corresponding to subgroups $G \subseteq \pi_1(Y,*)$ and $f_*[G] \subseteq \pi_1(X,f(*)$) respectively. Then $\Phi_{\text{cell}}^{\tilde{X},M} \cong \Phi_{\text{cell}}^{\tilde{Y},M}$ for all $q$-dimensional $M$, and $\Phi_{\text{ch}}^{\tilde{X},q} \cong \Phi_{\text{ch}}^{\tilde{Y},q}$.

The special case where $M = D^q$ (which makes the covering spaces irrelevant) and $\pi_t(X) = \pi_t(Y) = 0$ for $1 < t < q$ is an easy corollary to the major theorems in [1].

To see that isomorphic fundamental groups are not sufficient (as they are in [1]), consider for each $n \geq 1$ the spaces $X = S^{2n}$ (with one 0-cell and one $2n$-cell) and $Y = S^{2n} \sqcup_a D^{4n}$, where $a: S^{4n-1} \to S^{2n}$ represents an element $a \in \pi_{4n-1}(S^{2n})$ with infinite order. Then $\Phi_{\text{cell}}^{X,D^{4n}} \equiv 0$, since any map $D^{4n} \to S^{2n}$ has zero volume. By contrast, $\Phi_{\text{cell}}^{Y,D^{4n}} \equiv \infty$, as any map $S^{4n-1} \to Y(2n) = S^{2n}$ representing $ka$ has volume 0 and filling volume $|k|$.

**Proof of theorem 2.** We show that $\Phi_{\text{cell}}^{\tilde{X},M} \cong \Phi_{\text{cell}}^{\tilde{Y},M}$; the reasoning for $\Phi_{\text{ch}}$ is analogous. WLOG assume $f$ is cellular, and an inclusion (consider the mapping cylinder $M_f$). By adjoining $(q+1)$-cells to $X$, we may kill $\pi_q(X)$ without changing $\Phi_{\text{cell}}^{\tilde{X},M}$ or $\Phi_{\text{ch}}^{\tilde{X},q}$, so also assume $f$ is $q$-connected. In other words, we reduce to the case of a $q$-connected pair $(X,A)$ with $X(q)$ finite, $\pi: \tilde{X} \to X$ a covering space, and $\tilde{A} = \pi^{-1}[A]$ the corresponding covering space of $A$.

Let $j: (X(q),A) \to (X,A)$ be the inclusion map. $(X,A)$ is $q$-connected, so there is a (cellular) homotopy $h: j \simeq g$ to some cellular map $g: X(q) \to A$, and both $g$ and $h$ fix $A$. Let $K := \sup_{\sigma} \text{Vol}(h \upharpoonright \sigma)$, which is finite because only finitely many $\sigma$ exist. Assume WLOG that $g \upharpoonright \sigma$ is admissible for each $(q-1)$-cell $\sigma$ of $X(q) \times I$. Let $L := \sup_{\sigma} \text{Vol}(g \upharpoonright \sigma)$, which is also finite. $g$ lifts to a map $\tilde{g}: \tilde{X(q)} \to A$ which fixes $\tilde{A}$, and $h$ lifts to a homotopy $\tilde{h}: \tilde{X(q)} \times I \to \tilde{X}$ from the inclusion $\tilde{j} \to \tilde{g}$, also fixing $\tilde{A}$.

To establish $\Phi_{\text{cell}}^{\tilde{X},M} \cong \Phi_{\text{cell}}^{\tilde{A},M}$, fix $n$ and let $\omega = \partial \psi$ where $\psi: M \to \tilde{X}$ and $\omega$ are both admissible and $\text{Vol}_{\omega} \leq n$. Then $\omega' := \tilde{g} \circ \omega: \partial M \to \tilde{A}$ is admissible, $\omega' = \partial(\tilde{g} \circ \psi)$ in $\tilde{A}$, and $\text{Vol}_{\omega'} \leq In$. Similarly $\omega'' := h \circ (\omega \times 1_I): \partial M \times I \to X(q)$ is an admissible homotopy from $\omega$ to $\omega'$, and $\text{Vol}_{\omega''} \leq Kn$. Let $\phi': M \to \tilde{A}$ fill $\omega'$.
where $\text{Vol} \phi' = FV_{cell}^{\tilde{X},M}(\omega')$. By attaching $\phi'$ to $\omega''$, we obtain an admissible filling map for $\omega$, and we see that
\[
FV_{cell}^{\tilde{X},M}(\omega) \leq \text{Vol}(\phi') + \text{Vol}(\omega'') \leq FV_{cell}^{\tilde{X},M}(\omega') + Kn \leq \Phi_{cell}^{\tilde{X},M}(Ln) + Kn.
\]
We take the supremum over all such $\omega$ to see that
\[
\Phi_{cell}^{\tilde{X},M}(n) \leq \Phi_{cell}^{\tilde{X},M}(Ln) + Kn,
\]
so in particular $\Phi_{cell}^{\tilde{X},M} \leq \Phi_{cell}^{\tilde{X},M}$.

To establish $\Phi_{cell}^{\tilde{X},M} \leq FV_{cell}^{\tilde{X},M}$, fix $n$ and let $\omega = \partial \psi$ where $\psi : M \to \tilde{A}$ and $\omega$ are admissible and $\text{Vol} \omega \leq n$. Let $\phi : M \to \tilde{X}$ fill $\omega$ where $\text{Vol} \phi = FV_{cell}^{\tilde{X},M}(\omega)$. Then $\phi' = g \circ \phi : M \to \tilde{A}$ is admissible and $FV_{cell}^{\tilde{A},M}(\omega) \leq \text{Vol} \phi' \leq K \text{Vol} \phi \leq K \Phi_{cell}^{\tilde{X},M}(n)$. Taking the supremum over all such $\omega$, we see that
\[
\Phi_{cell}^{\tilde{X},M}(n) \leq K \Phi_{cell}^{\tilde{X},M}(n),
\]
and therefore $\Phi_{cell}^{\tilde{X},M} \leq \Phi_{cell}^{\tilde{X},M}$.

There is a metric version of theorem 2 which is an easy consequence of it and of theorem 1.

**Corollary 2.** Let $q \geq 2$, let $X$ and $Y$ be connected triangulable CLNRs (say connected compact Riemannian manifolds), and suppose there is a continuous map $f : Y \to X$ where $f_* : \pi_1(Y,*) \to \pi_1(X,f(*))$ is an isomorphism for $1 \leq t < q$. Let $\tilde{Y}$, $\tilde{X}$ be covering spaces of $Y$ and $X$ corresponding to subgroups $G \subseteq \pi_1(Y,*)$ and $f_*[G] \subseteq \pi_1(X,f(*))$ respectively. Then $\Phi_{cell}^{\tilde{X},q} \approx \Phi_{cell}^{\tilde{Y},q}$, and $\Phi_{\text{met}}^{\tilde{X},M} \approx \Phi_{\text{met}}^{\tilde{Y},M}$ for all $q$-dimensional $M$.

6. FURTHER QUESTIONS

Intuitively, the large-scale geometry of a universal covering space $\tilde{X}$ with $X$ compact should be captured by $\pi_1(X)$. Thus there should be a natural way of taming the infinities noted in section 3 such that the “tamed” versions of $\Phi_{cell}^{\tilde{X},M}$ etc. will depend only on $\pi_1(X)$ up to quasi-equivalence. We do not see any way to do so; on the other hand, we also do not know of any compact spaces $X$, $Y$ where $\pi_1(X) \cong \pi_1(Y)$, $\Phi_{cell}^{\tilde{X},M}(n) < \infty$ and $\Phi_{cell}^{\tilde{Y},M}(n) < \infty$ for all $n$, but $\Phi_{cell}^{\tilde{X},M} \neq \Phi_{cell}^{\tilde{Y},M}$.

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