Reducts of the Generalized Random Bipartite Graph

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Abstract Let \( \Gamma \) be the generalized random bipartite graph that has two sides \( R_l \) and \( R_r \) with edges for every pair of vertices between \( R_l \) and \( R_r \) but no edges within each side, where all the edges are randomly colored by three colors \( P_1, P_2, P_3 \). In this paper, we investigate the reducts of \( \Gamma \) that preserve \( R_l \) and \( R_r \), and classify the closed permutation subgroups in \( \text{Sym}(R_l) \times \text{Sym}(R_r) \) containing the group \( \text{Aut}(\Gamma) \). Our results rely on a combinatorial theorem of Nešetřil-Rödl and the strong finite submodel property of the random bipartite graph.

1 Introduction

As in [6], a reduct of a structure \( \Gamma \) is a structure with the same underlying set as \( \Gamma \) in some relational language, each of whose relation is \( \emptyset \)-definable in the original structure. If \( \Gamma \) is \( \omega \)-categorical, then a reduct of \( \Gamma \) corresponds to a closed permutation subgroup in \( \text{Sym}(\Gamma) \) (the full symmetric group on the underlying set of \( \Gamma \) that contains \( \text{Aut}(\Gamma) \) (the automorphism group of \( \Gamma \)). Two interdefinable reducts are considered to be equivalent. That is, two reducts of a structure \( \Gamma \) are equivalent if they have the same \( \emptyset \)-definable sets, or, equivalently, they have the same automorphism groups. There is a one-to-one correspondence between equivalence classes of reducts \( N \) and closed subgroups of \( \text{Sym}(\Gamma) \) containing \( \text{Aut}(\Gamma) \) via \( N \mapsto \text{Aut}(N) \) (see [6]).

There are currently a few \( \omega \)-categorical structures whose reducts have been explicitly classified. In 1977, Higman classified the reducts of the structure \( (\mathbb{Q}, <) \) (see Higman [3]). In 2008, Markus Junker and Martin Ziegler classified the reducts of expansions of \( (\mathbb{Q}, <) \) by constants and unary predicates (see M. Junker [4]). Simon Thomas showed that there are finitely many reducts of the random graph (Thomas [5]) in 1991, and of the random hypergraphs ([6]) in 1996. In 1995 James Bennett proved similar results for the random...
tournament, and for the random $k$-edge coloring graphs (Bennett [1]). In 2011, I investigated the reducts of the random bipartite graph that preserve sides. Equivalently, we analyze the closed subgroups of $\text{Sym}(R_l) \times \text{Sym}(R_r)$ containing $\text{Aut}(\Gamma)$.

In this paper, we consider the generalized random bipartite graphs, i.e. complete random bipartite graphs with $k$ colors $P_1, P_2, P_3$ on $R_l \times R_r$ such that $P_1 \cup P_2 \cup P_3 = R_l \times R_r$ and $P_i \cap P_j = \emptyset$ if $i \neq j$. The appropriate language $L_3$ for such structures can be taken to have two unary relations, $R_l$ and $R_r$, and 3 binary relations $P_1, P_2, P_3$. For convenience we consider a graph $\Gamma = (V, R_l, R_r, P_1, P_2, P_3)$, where $R_l, R_r \subseteq V$ and $P_1, P_2, P_3 \subseteq R_l \times R_r$. Then $\Gamma$ is a bipartite graph having 3 cross-types if it satisfies the following set $B_3$ of axioms:

1. $\exists x R_l(x)$
2. $\exists x R_r(x)$
3. $\forall x \forall y (P_i(x, y) \rightarrow (R_l(x) \land R_r(y))), i = 1, \ldots, 3$
4. $\forall x \forall y (P_i(x, y) \rightarrow \neg P_j(x, y)), i \neq j$
5. $\forall x \forall y (R_l(x) \land R_r(y) \rightarrow (P_1(x, y) \lor P_2(x, y) \lor P_3(x, y)))$
6. $\forall x ((R_l(x) \lor R_r(x)) \land \neg (R_l(x) \land R_r(x)))$.

**Definition 1.1.** A countable bipartite graph $\Gamma$ having 3 cross-types $\Gamma$ is random if it satisfies the extension properties $\Theta_n$ for all $n \in \mathbb{N}$:

(\Theta_n): For any finite pairwise disjoint $X_{i1}, X_{i2}, X_{i3} \subseteq R_l$ and finite pairwise disjoint $X_{r1}, X_{r2}, X_{r3} \subseteq R_r$, each of size at most $n$,

(a) there exists a vertex $v \in R_l$ such that $P_1(v, x)$ for every $x \in X_{ri}, i = 1, \ldots, 3$.
(b) there exists a vertex $w \in R_r$ such that $P_i(x, w)$ for every $x \in X_{ri}, i = 1, \ldots, 3$.

The $\Theta_n$’s are first-order sentences, and the axioms in Definition 1.1 together with the $\{\Theta_n\}_{n \in \mathbb{N}}$ form a complete and $\omega$-categorical theory. It can be shown that a 3-colored random bipartite graph exist. It is countable and unique up to isomorphism. It is also easy to show that the 3-colored random bipartite graph is homogeneous by a back-and-forth argument. In the rest of paper, the we use $\Gamma$ to denote the 3-colored random bipartite graph, unless otherwise mentioned.

Notice that with three cross-types, the definition of switch is more complicated because the permutation group $S_3$ is not commutative. From now on, we let $\text{Sym}_{\{l, r\}}(\Gamma)$ denote $\text{Sym}(R_l) \times \text{Sym}(R_r)$.

**Definition 1.2.** Given $\sigma \in S_3$ and a vertex $v \in R_l$, a switch on $v$ according to $\sigma$ is a permutation $\pi \in \text{Sym}_{\{l, r\}}(\Gamma)$ such that for any $(a, b) \in R_l \times R_r$ and for $i = 1, 2, 3$,

- if $v = a$, then $P_i(a, b) \rightarrow P_{\sigma(i)}(\pi(a), \pi(b))$;
- otherwise, $P_i(a, b) \rightarrow P_i(\pi(a), \pi(b))$.

Similarly we define a switch w.r.t. $v \in R_r$.

**Definition 1.3.** Given $\sigma \in S_3$ and $A \subseteq \Gamma$, a switch on $A$ according to $\sigma$ is a permutation $\pi \in \text{Sym}_{\{l, r\}}(\Gamma)$ such that for any $(a, b) \in R_l \times R_r$ and for $i = 1, 2$ or $3$,

- if $(a, b)$ has exactly one entry from $A$, then $P_i(a, b) \rightarrow P_{\sigma(i)}(\pi(a), \pi(b))$;
- if $(a, b)$ has both entries in $A$, then $P_i(a, b) \rightarrow P_{\sigma^2(i)}(\pi(a), \pi(b))$.
Definition 1.4. If \( X \subseteq \{l, r\} \) and \( H \leq S_3 \), then \( S^H_X(\Gamma) \) is the closed subgroup of \( \text{Sym}_{\{l,r\}}(\Gamma) \) generated as a topological group by \( \text{Aut}(\Gamma) \) together with all \( \pi \in \text{Sym}_{\{l,r\}}(\Gamma) \) such that there exists a vertex \( v \in R_i \) for \( i \in X \) and \( \sigma \in H \) such that \( \pi \) is a switch w.r.t. \( v \) according to \( \sigma \).

Thus the candidates for the reducts are \( S^H_{\{l\}}(\Gamma), S^H_{\{r\}}(\Gamma) \) and \( S^H_{\{l,r\}}(\Gamma) \), where \( H \) is one of the subgroups of the permutation group \( S_3 \):
\[
\{1\}, \{1, (12)\}, \{1, (13)\}, \{1, (23)\}, \{(1), (123), (132)\} \text{ and } S_3.
\]

We begin the analysis of reducts of \( \Gamma \), the random bipartite graph with three cross-types, by indicating which reducts are essentially new; these will be the ones we call irreducible.

Definition 1.5. Let \( G \) be a closed subgroup of \( \Gamma \) in \( \text{Sym}(\Gamma) \).

We say \( G \) is reducible if for some \( k \in \{1, 2, 3\} \), \( G \) contains every map \( g \in \text{Sym}(\Gamma) \) which preserves \( P_k \). This means that \( G \) is blind to the distinction between the other two cross-types. If \( G \) is not reducible, then we say \( G \) is irreducible.

So if \( G \) is reducible, then \( G \) can be viewed as a reduct of the bipartite graph with two edges, as already classified in the previous chapter.

Here is the main result of this paper:

Theorem 1.6. If \( G \) is an irreducible closed subgroup such that \( \text{Aut}(\Gamma) \leq G \leq \text{Sym}_{\{l,r\}}(\Gamma) \), then \( G = (S^H_{\{l\}}(\Gamma), S^H_{\{r\}}(\Gamma)) \) where \( H_1, H_2 \leq S_3 \). If \( G \subset \text{Sym}_{\{l,r\}}(\Gamma) \), then \( H_1 = H_2 \) unless one of the two groups is trivial.

Here is how the rest of the paper is organized. In section 2, we show that the 3-colored random bipartite graph has the strong finite submodel property; in section 3, we study the relations preserved by the groups \( S_X(\Gamma) \), where \( X \subseteq \{l, r\} \). In section 3, and in section 4, we discuss a technique term \((m \times n)\)-analysis and prove its existence for the random bipartite graph. These prepare us to give an explicit classification of the closed subgroups of \( S_{\{l,r\}}(\Gamma) \) containing \( \text{Aut}(\Gamma)^* \) in the rest of the paper. In section 5, we prove the first part of Theorem 2.1, which says that the closed subgroups of \( S_{\{l,r\}}(\Gamma) \) containing \( \text{Aut}(\Gamma)^* \) are \( \text{Aut}(\Gamma)^*, S_{\{l\}}(\Gamma), \text{ and } S_{\{r\}}(\Gamma), \text{ and } S_{\{l,r\}}(\Gamma) \). Then in section 6 we show there is no other proper closed subgroup between \( S_{\{l,r\}}(\Gamma) \) and \( \text{Sym}_{\{l,r\}}(\Gamma) \), which completes the proof of Theorem 2.1.

### 2 Strong Finite Submodel Property

In this section, we use the notion of the Strong Finite Submodel Property (SFSP) initially introduced by Thomas in [6], and we prove that the random 3-colored bipartite graph has the SFSP. This property provides a powerful tool when it comes to the proof in the later sessions.

Definition 2.1 ([6]). A countable infinite structure \( \mathcal{M} \) has the **Strong Finite Submodel Property (SFSP)** if \( \mathcal{M} = \bigcup_{i \in \mathbb{N}} \mathcal{M}_i \) is a union of an increasing chain of substructures \( \mathcal{M}_i \) such that

1. \( |\mathcal{M}_i| = i \) for each \( i \in \mathbb{N} \); and
(2) for any sentence $\varphi$ with $\mathcal{M} \models \varphi$, there exists $N \in \mathbb{N}$ such that $\mathcal{M}_i \models \varphi$ for all $i \geq N$.

Here we choose a specific chain of bipartite graphs $\Gamma_i$ such that $\Gamma = \bigcup \Gamma_i$ where $|\Gamma_i| = i$, $\Gamma_i \subset \Gamma_{i+1}$ for $i \in \mathbb{N}$, and

- if $i$ is even, then $|\Gamma_i \cap R| = |\Gamma_i \cap R_r|$;
- otherwise, $|\Gamma_i \cap R| = |\Gamma_i \cap R_r| + 1$.

Thus for any sentence $\varphi$ true in $\Gamma$, there is an $j_\varphi$ such that $i > j_\varphi$ implies $\varphi$ is true in $\Gamma_i$.

**Theorem 2.2.** The countable random 3-colored bipartite graph $\Gamma$ has the SFSP.

Theorem 2.2 is a consequence of the Borel–Cantelli Lemma, as follows below:

**Definition 2.3** ([6]). If $\{A_n\}_{n \in \mathbb{N}}$ is a sequence of events in a probability space, then $\bigcap_{n \in \mathbb{N}} \bigcup_{n \leq k \in \mathbb{N}} A_k$ is the event that consists of realization of infinitely many of $A_n$, denoted by $\lim \inf A_n$.

**Lemma 2.4** (Borel–Cantelli, Billingsley [2]). Let $\{A_n\}_{n \in \mathbb{N}}$ be a sequence of events in a probability space. If $\sum_{n=0}^{\infty} P(A_n) < \infty$, then $P(\lim \inf A_n) = 0$.

**Proof of Theorem 2.2** Since the extension properties $\Theta_n$’s axiomatize the random 3-colored bipartite graph $\Gamma$ and $\Theta_i$ implies $\Theta_{i-1}$ for all $i \in \mathbb{N}$, for every sentence $\varphi$ true in $\Gamma$, there exists some $k \in \mathbb{N}$ such that $\Theta_k$ holds if and only if $\varphi$ holds. Let $\Omega$ be the probability space of all countable bipartite graphs $(S, R_l, R_r, P_1, P_2, P_3)$, where $|R_l| = |R_r| = \omega$ and every cross-edge $E \in R_l \times R_r$ has the cross-type $P_1, P_2$ or $P_3$ on it independently with probability $\frac{1}{3}$. For each $n \in \mathbb{N}$ with $n \geq k$, let $A_n$ be the event that a 3-colored bipartite subgraph $S_n \in [S]^n$ does not satisfy the extension property $\Theta_k$. We consider two cases: $n$ is even ($n=2m$), and $n$ is odd ($n=2m+1$). Then by simple computation,

$$P(A_{2m}) \leq 2 \cdot \left( \frac{m}{k} \right) \left( \frac{m-k}{k} \right) \left( \frac{m-2k}{k} \right) \left( 1 - \left( \frac{1}{3} \right)^{3k} \right)^{m-3k},$$

and

$$P(A_{2m+1}) \leq 2 \cdot \left( \frac{m+1}{k} \right) \left( \frac{m+1-k}{k} \right) \left( \frac{m+1-2k}{k} \right) \left( 1 - \left( \frac{1}{3} \right)^{3k} \right)^{m-3k}.$$

Notice that $\sum_{n=0}^{\infty} P(A_n) = \sum_{m=0}^{\infty} P(A_{2m}) + \sum_{m=0}^{\infty} P(A_{2m+1})$, we have

$$\sum_{n=0}^{\infty} P(A_n) \leq 4 \cdot \sum_{m=0}^{\infty} \left( \frac{m+1}{k} \right) \left( \frac{m+1-k}{k} \right) \left( \frac{m+1-2k}{k} \right) \left( 1 - \left( \frac{1}{3} \right)^{3k} \right)^{m-3k}$$

where $\binom{n}{k}$ is the number of combinations of $n$ objects taken $k$ at a time. Let

$$C_m = \left( \frac{m+1}{k} \right) \left( \frac{m+1-k}{k} \right) \left( \frac{m+1-2k}{k} \right) \left( 1 - \left( \frac{1}{3} \right)^{3k} \right)^{m-3k}.$$ \hspace{1cm} (1)

Then $\lim_{m \to +\infty} C_{m+1}/C_m = (1 - \frac{1}{3})^3 < 1$. By the ratio test for infinite series, we have $\sum_{m=0}^{\infty} C(m)$ converges, and so does $\sum_{n=0}^{\infty} P(A_n)$. Thus by Lemma 2.4, $P(\lim \inf A_n) = 0$. So there exists a 3-colored bipartite graph $S \in \Omega$ and an integer $N$ such that for all $n \geq N$, $S_n \in [S]^n$ satisfies the extension property $\Theta_k$, and so $\varphi$. Notice that the choice of $S$ ensures that $S$ is countable and satisfies all the axioms for the random bipartite graph. Hence $S$ is isomorphic to $\Gamma$. Then $\Gamma$ has the SFSP, which completes the proof of Theorem 2.2. \hfill $\Box$
Similarly, we can show as in Theorem 2.2 that

**Proposition 2.5.** The countable random \( k \)-colored bipartite graph \( \Gamma \) has the strong finite submodel property (SFSP).

For the remainder of this Chapter we will restrict our attention to the case \( k = 3 \). We expect that our results to generalize to arbitrary \( k \), but we have not organized the details for the more general results at this stage.

### 3 Candidates for Irreducible Closed Groups

In this section, we will discuss the candidates for irreducible closed groups.

Motivated by the colorings defined in Bennett’s thesis [1], we may define a class of edge colorings \( \chi : [A]^2 \rightarrow \{l, r, P_1, P_2, P_3\} \) for a bipartite graph \( A \subseteq \Gamma \) as follows, where \( \{a, b\} \in [A]^2 \):

- if \( \{a, b\} \in [R_l]^2 \), then \( \chi(a, b) = l \);
- if \( \{a, b\} \in [R_r]^2 \), then \( \chi(a, b) = r \);
- if \( \{a, b\} \in R_l \times R_r \) and \( P_i(a, b) \) for some \( i = 1, 2, 3 \), then \( \chi(a, b) = P_i \).

**Definition 3.1.** Let \( A_1 \) be a bipartite graph with the edge coloring \( \chi_{1} \), and \( A_2 \) be a bipartite graph with the edge coloring \( \chi_{2} \) where \( \chi \)'s are defined as above. If \( |A_1| = |A_2| \) and \( A_1 \cap R_l = A_2 \cap R_l \), then the edge coloring \( \chi_{2} \) is a permutation of the edge coloring \( \chi_{1} \) if there is some vertex bijection \( \varphi : A_1 \rightarrow A_2 \) preserving \( R_l, R_r \) and some permutation \( \sigma \in S_3 \) such that for every cross-edge \( (a, b) \in (A_1 \times A_1) \cap (R_l \times R_r) \), \( \chi_{1}(a, b) = \sigma(\chi_{2}(\varphi(a), \varphi(b))) \). That is, \( P_i(a, b) \) implies \( P_{\sigma(i)}(\varphi(a), \varphi(b)) \) for \( i = 1, 2, 3 \).

**Definition 3.2.** Let \( A \) be a bipartite graph, and \( \chi_{1}, \chi_{2} \) be edge colorings on \( [A]^2 \) as above. Then the edge coloring \( \chi_{2} \) is homogeneous w.r.t. the coloring \( \chi_{1} \) if for any \( (a, b), (a', b') \in (A \times A) \cap (R_l \times R_r) \), \( \chi_{2}(a, b) = \chi_{2}(a', b') \) implies \( \chi_{1}(a, b) = \chi_{1}(a', b') \).

**Claim 3.1.** If \( A \) is a bipartite graph, \( \chi_{1} \) and \( \chi_{2} \) are edge colorings on \( [A]^2 \) defined as above. If \( \chi_{2} \) is homogenous w.r.t. \( \chi_{1} \) but is not a permutation of \( \chi_{1} \), then there must be two distinct colors \( P_i \) and \( P_j \) and some color \( P_k \) \( (i, j, k \in \{1, 2, 3\}) \) such that for any \( (x, y) \subseteq (A \cap R_l) \times (A \cap R_r) \), \( \chi_{2}(x, y) = P_i \) or \( \chi_{2}(x, y) = P_j \) implies \( \chi_{1}(x, y) = P_k \).

**Proof** By the definition of homogeneous and permutation colorings, if \( \chi \) is not a permutation then it must collapse two colors. \( \square \)

**Definition 3.3.** Let \( G \) be an irreducible closed subgroup of \( Sym(\Gamma) \). The pair \( (R, \alpha) \), where \( R \) is a finite subset of \( \Gamma \) and \( \alpha \in \Gamma \backslash R \), is sufficiently complex w.r.t. \( G \) if the following hold:

1. For any \( g \in G \) and \( c \in \{P_1, P_2, P_3\} \), there is a cross-edge \( \langle a, b \rangle \in (g(R) \cap R_l) \times (g(R) \cap R_r) \) such that \( c(a, b) \). (\( R \) witnesses all the cross-types.)
2. If there is some \( g \in G \) such that \( g \upharpoonright R \cup \{\alpha\} \) is a switch w.r.t. \( \alpha \in R_l \) permuting \( P_i \) according to \( \sigma \), then some (hence all) switches \( f \) w.r.t. singletons in \( R_l \) according to \( \sigma \) are also in \( G \). (\( R \cup \{\alpha\} \) witnesses which switches w.r.t. \( R_l \) are not in \( G \).)
Theorem 3.5

We prove that there exist $\phi$ and a vertex coloring $(3)$. There exists some $\alpha \in R_l$ such that $(R, \alpha)$ satisfies Property (2), and there is some $\beta \in R_r$ such that $(R, \beta)$ satisfies Property (3).

Claim 3.2. If $R$ is sufficiently complex and $S \supseteq R$, then $S$ is sufficiently complex.

Proof

First, for any $g \in G$, and any $c \in \{P_i\}$ for $i = 1, 2, 3$, since $R$ is sufficiently complex, there always exists one cross-edge $(a, b) \in g(R) \subseteq g(S)$ such that $c(a, b)$. So $S$ has the property (1). Second, suppose $\alpha \in \Gamma \setminus R$ and $(R, \alpha)$ is sufficiently complex. If there exists some $g \in G$ such that $g \mid S \cup \{\alpha\}$ is a switch w.r.t. $\alpha \in R_l$ according to $\sigma$, then $g \mid R \cup \{\alpha\}$ is a switch w.r.t. $\alpha$ according to $\sigma$. Since $(R, \alpha)$ is sufficiently complex, some (hence all) switches $f$ w.r.t. singletons according to $\sigma$ are in $G$, showing that condition (2) holds. Similarly we can prove $(S, \alpha)$ has the property (3).

Theorem 3.5. If $G$ is an irreducible closed subgroup of $Sym(\Gamma)$ containing $Aut(\Gamma)$, then there is a pair $(R, \alpha)$ which is sufficiently complex w.r.t. $G$.

Proof We prove that there exist $R_0$ with the property (1), $(R_1, \alpha)$ where $\alpha \in R_l$ with the property (2), and $(R_2, \alpha)$ where $\alpha \in R_r$ with the property (3).

Property (1). Suppose there is no such $R_0$ in $\Gamma$: i.e. for any $R \subseteq \Gamma$, there exists some $g \in G$ such that the cross-edges of $g(R)$ have fewer than three cross-types. Then if $\Gamma = \bigcup \Gamma_i$ is our nice enumeration as in the strong finite submodel property, there exists a sequence $\{f_i\}_{i \in \mathbb{N}} \subseteq G$ such that $f_i(\Gamma_i)$ has fewer than three cross-types. Since there are only three cross-types, but infinitely many $\{f_i\}$, then there exists some $c \in \{P_i\}$ and $\{f_{i_1}\} \subseteq \{f_i\}$ such that $f_{i_1}(\Gamma_{i_1})$ has no cross-type $c$. Hence for every finite $B \subseteq \Gamma$, there is some $g \in G$ such that $g(B)$ has no cross-edge with cross-type $c$ on it. For each $i \in \mathbb{N}$, we define an edge coloring $\chi : [\Gamma_i]^2 \rightarrow \{l, r, P_1, P_2, P_3\}$ for every $\{a, b\} \in [\Gamma_i]^2$ by

- if $\{a, b\} \subseteq R_l$, then $\chi(a, b) = l$;
- if $\{a, b\} \subseteq R_r$, then $\chi(a, b) = r$;
- if $(a, b) \in R_l \times R_r$ and $P_1(a, b)$, then $\chi(a, b) = P_1$;
- if $(a, b) \in R_l \times R_r$ and $P_2(a, b)$, then $\chi(a, b) = P_2$;
- if $(a, b) \in R_l \times R_r$ and $P_3(a, b)$, then $\chi(a, b) = P_3$.

and a vertex coloring $\varphi : \Gamma_i \rightarrow \{L, R\}$ for every $a \in \Gamma_i$ by

- if $a \in R_l$, then $\varphi(a) = L$;
- if $a \in R_l$, then $\varphi(a) = R$.

Note: This “sufficiently complex” concept is different from that in Chapter 3.
Let \((\Gamma, \chi, \varphi)\) be the \(\alpha\)-pattern \(P\). By the Nešetřil-Rödl Theorem, there is some bipartite graph \(B_i \subset \Gamma\) such that for every partition \(F\) on \(B_i\), there is \(\Gamma'_i \subset B_i\) such that

1. \(\Gamma'_i\) has the \(\alpha\)-pattern \(P\) (hence \(\Gamma'_i \cong \Gamma_i\));
2. \(\Gamma'_i\) is \(F\)-homogeneous.

Now we choose \(N \in \mathbb{N}\) such that when \(j \geq N\), \(\Gamma_j\) has all colors. Now let \(g_i \in G\) be such that \(g_i(B_i)\) has no cross-edge with cross-type \(F\) and let \(F = \chi \circ g_i\). Then \(F = \chi \circ g_i : |B_i|^2 \rightarrow \{l, r, P_1, P_2, P_3\} \setminus \{c\}\). Since \(g(\Gamma'_i)\) has no cross-edge with cross-type \(c\), the coloring \(\chi\) is not a permutation of \(\chi \circ g_i\): so by Claim 3.1 there must be distinct cross-types \(P_m\) and \(P_n\) and some cross-type \(P_l\) such that for every \((a, b) \in (R_i \times R_r) \cap \Gamma'_i\), \(\chi(x, y) = P_m\) or \(\chi(x, y) = P_n\) implies \(\chi \circ g_i(x, y) = P_l\), i.e. \(P_m(x, y)\) or \(P_n(x, y)\) implies \(P_l(g_i(x), g_i(y))\). WLOG, let \(\Gamma_i\) replace \(\Gamma'_i\).

Let \(X, Y \subset \Gamma\) be finite bipartite subgraphs with \(|X \cap R_i| = |Y \cap R_i| (i = l, r)\) and \(f : X \rightarrow Y\) such that \(f\) preserves \(P_k, R_l\) and \(R_r\) where \(k \neq m, n, n\). For \(X\), there are some \(N_x \in \mathbb{N}\) such that \(\Gamma(N_x) \supset X\). By a similar argument as above, there is some \(g_X \in G\) such that for every \((a_x, b_x) \in X \cap (R_l \times R_r)\) with \(P_m(a_x, b_x) \lor P_n(a_x, b_x)\), we have \(P_l(g_X(a_x), g_X(b_x))\). Similarly, for \(Y\), there is some \(N_y \in \mathbb{N}\) such that \(\Gamma(N_y) \supset Y\). By a similar argument as above, there is some \(g_Y \in G\) such that for every \((a_y, b_y) \in Y \cap (R_l \times R_r)\) with \(P_m(a_y, b_y) \lor P_n(a_y, b_y)\), \(P_l(g_Y(a_y), g_Y(b_y))\). Thus there is an isomorphism \(\sigma : g_X(X) \rightarrow g_Y(Y)\). Hence \(g_Y \circ f = \sigma \circ g_X\), hence \(f = g_Y^{-1} \circ \sigma \circ g_X\) and then \(f \in G | X\). Since \(X\) and \(Y\) are arbitrary finite bipartite subgraphs of \(\Gamma\) and \(G\) is closed, so for any \(f \in Sym(\Gamma)\) preserving \(P_k\) for some \(k \in \{1, 2, 3\}, f \in G\). By Definition 1.5, \(G\) is reducible, a contradiction with our assumption.

**Property (2):** Suppose there is no such \((R_l, \alpha)\) in \(\Gamma\), i.e. for any finite bipartite \(R \subset \Gamma\) and any \(\alpha \in (\Gamma \setminus R) \cap R_l\), there exist some \(\sigma \in S_3\) and \(g \in G\) such that \(g | R \cup \{\alpha\}\) is a switch w.r.t. \(\alpha\) according to \(\sigma\), but there is no \(f \in G\) such that \(f\) is a switch w.r.t. \(\alpha\) according to \(\sigma\). Let \(\Gamma = \Gamma \cup R_l\), as in the SFSP, then there exists a sequence \(\{f_i, \sigma_i\} \subset G\) such that \(f_i \mid (\Gamma \cup \{\alpha\})\) is a switch w.r.t. a single vertex \(\alpha\) of \(R_l\) according to \(\sigma_i\). Since \(S_3\) is finite but \(\{f_i\}\) is infinite, we have \(\{f_i\} \subseteq \{f_i\}\) and some \(\sigma\) such that \(f_i \mid (\Gamma \cup \{\alpha\})\) is a switch w.r.t. a single vertex \(\alpha\) of \(R_l\) according to \(\sigma\). Since \(G\) is closed, there exists a switch w.r.t. \(\alpha\) according to \(\sigma\) as in \(G\). But this contradicts the assumption.

**Property (3):** Similar to the previous proof of **Property (2).**

Now we choose \(R = R_0 \cup R_1 \cup R_2\). By Claim 3.2, the set \(R\) is sufficiently complex. 

**Lemma 3.6.** Suppose \(H_1, H_2 \leq S_3\). If \(G = \langle S_{(l)}^{H_1}, S_{(r)}^{H_2} \rangle\) is an irreducible closed subgroup in \(Sym(\Gamma)\), then for any \(f \in H_1\) and any \(g \in H_2\) it is the case that \(f \circ g = g \circ f\).

**Proof** Suppose not, then there exist \(f \in H_1, g \in H_2\) such that \(f \circ g \neq g \circ f\). Then for \(\gamma = g^{-1} \circ f^{-1} \circ g \circ f\), there must be some \(c \in \{1, 2, 3\}\) such that \(\gamma(c) \neq c\). Choose \(x \in R_l, y \in R_r\) such that \(P\)(\(x, y\)). We construct an \(h \in G = \langle S_{(l)}^{H_1}, S_{(r)}^{H_2} \rangle\) which will be a composition \(h = g_4 \circ g_3 \circ g_2 \circ g_1\) of four switches \(g_1, g_3 \in S_{(l)}^{H_1}\), \(g_2, g_4 \in S_{(r)}^{H_2}\) on single vertices. First let \(g_1\)
be a switch w.r.t. $x$ according to $f$, then let $g_2$ be a switch w.r.t. $g_1(y)$ according to $g$. Then let $g_3$ be a switch w.r.t. $g_2g_1(x)$ according to $f^{-1}$ and finally let $g_4$ be a switch w.r.t. $g_3g_2g_1(y)$ according to $g^{-1}$. Then for every $(a, b) \in R_t \times R_t$, if $(a, b) \neq (x, y)$, then $P_c(a, b) \implies P_c(h(a), h(b))$ but $P_c(x, y) \implies -P_c(h(x), h(y))$. Hence for any finite bipartite $A \subset \Gamma$, we can construct a $g_A \in G$ such that $g_A(A)$ has one fewer edge with cross-type $c$. By repeating this process, we can find some $\overline{\gamma} \in G$ such that $\overline{\gamma}A$ has no edge with cross-type $c$. Then $\Gamma$ cannot contain a sufficiently complex set. By Theorem 3.5, $G$ is reducible, a contradiction. This completes the proof of Lemma 3.6.

In particular we have:

\[
\langle S_{(t)}^{(12)}, S_{(r)}^{(123)} \rangle \text{ is reducible since } (12)(123) = (23) \neq (13) = (123)(12).
\]

\[
\langle S_{(t)}^{(13)}, S_{(r)}^{(123)} \rangle \text{ is reducible since } (13)(123) = (12) \neq (23) = (123)(13).
\]

\[
\langle S_{(t)}^{(23)}, S_{(r)}^{(123)} \rangle \text{ is reducible since } 23(123) = (13) \neq (12) = (123)(23).
\]

\[
\langle S_{(t)}^{(12)}, S_{(r)}^{(13)} \rangle \text{ is reducible since } (12)(13) = (132) \neq (123) = (13)(12).
\]

\[
\langle S_{(t)}^{(12)}, S_{(r)}^{(23)} \rangle \text{ is reducible since } (12)(23) = (232) \neq (123) = (23)(12).
\]

\[
\langle S_{(t)}^{(13)}, S_{(r)}^{(23)} \rangle \text{ is reducible since } (13)(23) = (123) \neq (132) = (12)(13).
\]

**Lemma 3.7.** The group $S_{(t,r)}^{S_A} = \langle S_{(t)}^{S_A}, S_{(r)}^{S_A} \rangle$ is the full symmetric group $\text{Sym}_{(t,r)}(\Gamma)$.

**Proof** It is enough to show that for any $(n \times m)$-bipartite $A \subset \Gamma$ where $n, m \in \mathbb{N}$, there exists some $g_A \in S_{(t,r)}^{S_A}$ such that $g_A(A)$ has only a single cross-type $P_1$. Then for any two $(n \times m)$-bipartite graphs $B, C$, we can find an automorphism $\sigma$ of $\Gamma$ sending $g_B(B)$ to $g_C(C)$, since each of these two subgraphs has only $P_1$ as cross-type. Then the map $f = g_C^{-1} \circ \sigma \circ g_B$ takes $B$ to $C$, and $f \in S_{(t,r)}^{S_A}$. Then $S_{(t,r)}^{S_A} = \text{Sym}_{(t,r)}(\Gamma)$.

WLOG, suppose $A$ has three cross-types: $P_1, P_2$ and $P_3$. Let $f, g \in S_A$, $f = (123)$ and $g = (12)$. Then $f \circ g \neq g \circ f$ and let $\gamma = g^{-1} \circ f^{-1} \circ g \circ f (= (123))$. Using the similar argument as in the proof of Lemma 3.6, for every finite bipartite subgraph $A \subset \Gamma$ we can construct some $g \in S_{(t,r)}^{S_A}$ such that $g(A)$ has no cross-edge with cross-type $P_2$. Similarly, using the similar argument as in the proof of Lemma 3.6, we can construct some $f \in S_{(t,r)}^{S_A}$ such that $f(g(A))$ has no cross-edge with cross-type $P_3$. That is, for every finite bipartite subgraph $A \subset \Gamma$, there exists $f \circ g \in S_{(t,r)}^{S_A}$ such that $h(A)$ has only a single cross-type $P_1$. This completes the proof of this Lemma.
Note that when $\sigma \in S_3$ is nontrivial, $S_{\{l\}}^{<\sigma>} = \langle Aut(\Gamma), h \rangle$ where $h$ is a switch w.r.t. some subset of $R_l$ according to $\sigma$.

**Lemma 3.8.** If $H_1 \neq H_2$ are non-trivial subgroups of $S_3$, then $\langle S_{\{l\}}^{H_1}, S_{\{l\}}^{H_2} \rangle = S_{\{l\}}^{S_3}$.

**Proof.** Let $H_1 = <\sigma_1>$ and $H_2 = <\sigma_2>$ for $\sigma_1, \sigma_2 \in S_3$, $\sigma_1 \neq \sigma_2$ and $\sigma_1, \sigma_2 \neq (1)$. There exist $f_1 \in S_{\{l\}}^{H_1}$ which is a switch w.r.t. some vertex in $R_l$ according to $\sigma_1$ such that $S_{\{l\}}^{H_1} = \langle Aut(\Gamma), f_1 \rangle$, and $f_2 \in S_{\{l\}}^{H_2}$ which is a switch w.r.t. some vertex in $R_l$ according to $\sigma_2$ such that $S_{\{l\}}^{H_2} = \langle Aut(\Gamma), f_2 \rangle$. Note that every two distinct proper subgroups of $S_3$ generate the whole $S_3$. Then every switch w.r.t. a single vertex in $R_l$ according to $\sigma \in S_3$ is generated by the elements in $Aut(\Gamma)$ together with two additional elements $f_1$ and $f_2$. Thus $\langle S_{\{l\}}^{H_1}, S_{\{l\}}^{H_2} \rangle = \langle Aut(\Gamma), f_1, f_2 \rangle$ is the closed group generated by $Aut(\Gamma)$ and all the switches w.r.t. single vertex in $R_l$ according to some $\sigma \in S_3$, which is $S_{\{l\}}^{S_3}$ by definition. □

**Lemma 3.9.** Let $S = S_{\{l\}}^{<(12)>}(\Gamma) \cup S_{\{l\}}^{<(13)>}(\Gamma) \cup S_{\{l\}}^{<(23)>}(\Gamma) \cup S_{\{l\}}^{<(123)>}(\Gamma)$, and $\sigma \in S_{\{l\}}^{S_3}(\Gamma) \setminus S$, then $\langle Aut(\Gamma), \sigma \rangle = S_{\{l\}}^{S_3}(\Gamma)$.

**Proof.** Recall that $S_{\{l\}}^{H}(\Gamma)$ for $H \leq S_3$ is generated by compositions of switches on singletons in $R_l$ together with automorphisms. We will show that any element which is in $S_{\{l\}}^{S_3}(\Gamma) \setminus S$ can be modified to produce elements in two distinct subgroups $S_{\{l\}}^{H_1}(\Gamma)$ and $S_{\{l\}}^{H_2}(\Gamma)$ where $H_1 \neq H_2$. Hence we can get all of $S_{\{l\}}^{S_3}(\Gamma)$ as closure. Let $\sigma \in S_{\{l\}}^{S_3}(\Gamma)$ but not in any of $S_{\{l\}}^{H}$ for every $H < S_3$. By considering the action of $\sigma$ on each edge, we can find sets $\{A_1, A_2, A_3, A_4, A_5\}$ where $A_i \subseteq R_l$ for $1 \leq i \leq 5$ and $A_j \cap A_k = \emptyset$ for $j \neq k$ ($A_i$ could be empty for $1 \leq i \leq 5$, but there are at least two distinct $j, k \in \{1, 2, 3, 4, 5\}$ such that $A_j$ and $A_k$ are nonempty) such that $\sigma$ sends the cross-types on the cross-edges with exactly one endpoint in $A_1$ to the cross-types dictated by (12), the cross-types on the cross-edges with exactly one endpoint in $A_2$ change the cross-types according to (13), the cross-types on the cross-edges with exactly one endpoint in $A_3$ change the cross-types according to (23), the cross-types on the cross-edges with exactly one endpoint in $A_4$ change the cross-types according to (123) and the cross-types on the cross-edges with exactly one endpoint in $A_5$ change the cross-types according to (132).

**Case 1** Assume $0 < |A_i| < \omega$ for $i = 1, \ldots, 5$.

Step 1: There exists some $f_1 \in Aut(\Gamma)$ such that $f_1(\sigma(A_i)) = A_i$. Let $h_1 = \sigma \circ f_1 \circ \sigma$, then $h_1$ is a switch w.r.t. $A_4$ according to (132) and w.r.t. $A_5$ according to (123).

Step 2: Similarly, there exists some $f_2 \in Aut(\Gamma)$ such that $f_2(\sigma(A_i)) = A_i$ for $i = 1, \ldots, 5$, and let $h_2 = h_1 \circ f_2$, then $h_2$ is a switch w.r.t. $\sigma(A_4)$ by (132) and w.r.t. $\sigma(A_5)$ by (123). Let $h_3 = h_2 \circ \sigma$, then $h_3$ is a switch w.r.t. $A_1$ according to (12), w.r.t. $A_2$ according to (13) and w.r.t. $A_3$ according to (23).

Step 3: There exists $f_3 \in Aut(\Gamma)$ such that $f_3(A_4) \cap A_1 \neq \emptyset$ but $f_3(A_4) \cap A_i = \emptyset$ for $i = 2, 3$, and $f_3(A_5) \cap A_2 \neq \emptyset$ but $f_3(A_5) \cap A_i = \emptyset$
for $i = 1, 3$ and $f_3(A_5) \cap f_3(A_4) = \emptyset$. There exists $f_4 \in Aut(\Gamma)$ such that $f_4(h_3 \circ f_3(A_4)) = A_1$ for $j = 4, 5$ and $f_4 \circ h_3(\{A\}) \cap A_5 = \emptyset$, $f_4 \circ h_3(\{A\}) \cap A_4 = \emptyset$. Let $h_4 = h_1 \circ f_4 \circ h_3$. By the definition of $h_3, f_4, h_1$ and the fact that $(132)(12) = (23)$ and $(123)(13) = (23)$. Now $h_4$ is a switch w.r.t. $A_1 \setminus f_3(A_4) \subset A_1$ according to (12), w.r.t. $A_2 \setminus f_3(A_5) \subset A_2$ according to (13), and w.r.t. $f_3(A_4) \setminus A_1$ according to (132), w.r.t. $f_3(A_5) \setminus A_2$ according to (123), and w.r.t. $A_3 \cup (A_1 \cap f_3(A_4)) \cup (A_2 \cap f_3(A_5)) \supset A$ according to (23).

Case 1: Let $f$ according to (12), w.r.t. $(A)$ and $f$ that $f \in Aut(\Gamma)$, we have $f(\{A\}) = (23)$. Now follow Step 1 – 3, we can get a switch w.r.t. $A_1$ according to (12), w.r.t. $A_2$ according to (13) and w.r.t. $A_3$ according to (23).

Step 5: Since $|A_i| < \omega$ for $i = 1, 2$, we can follow steps 1 – 4 finitely many times to obtain a sequence of $A_1, A_2, A_3 \subset \Gamma$, ending with $A_1^N = \emptyset$ and $A_2^N = \emptyset$. Then we now have a switch $g$ w.r.t. $A_3^N$ according to (23).

Similarly, we produce a switch $h$ w.r.t. some subset of $R_i$ according to (13). Then $\langle Aut(\Gamma), \sigma \rangle = \langle Aut(\Gamma), g, h, \sigma \rangle = \langle S_{<23}^{<13}(\{I\}), S_{<23}^{<13}(\{I\}) \rangle$. By Lemma 3.8 $\langle Aut(\Gamma), g, h, \sigma \rangle = S_{<23}^{<13}(\{I\})$, and this completes the proof of Case 1.

Case 2: If all $A_k$’s are finite but some $A_k$ is empty, then we follow the proof above, only with fewer steps.

Case 3: If there exists some $k$ with $|A_k| = \omega$, then since $A_k$ is the “limit” of its finite approximations $A_k'$, we can deal with its finite subsets $A_k'$ with the method in Case 1, then take the limit (which must lie in the closure).

This completes the proof of Lemma 3.9.

**Theorem 3.10.** Let $G$ be an irreducible closed subgroup of $S_{<3}^{<13}(\Gamma)$ containing $Aut(\Gamma)$. Then there exists $H \leq S_3$ such that $G = S_H^{\Gamma}(\Gamma)$.

**Proof** Let $G$ be an irreducible closed proper subgroup of $S_{<3}^{<13}(\Gamma)$ containing $Aut(\Gamma)$. If there is some nontrivial $H \leq S_3$ such that $G \leq S_H^{\Gamma}(\Gamma)$, then if $G$ is not $Aut(\Gamma)$, we have $G = S_H^{\Gamma}(\Gamma)$. Suppose not, i.e. $G \subset S_H^{\Gamma}(\Gamma)$. Then every $g \in G \setminus Aut(\Gamma)$ is a composition of switches w.r.t. permutations in $H$. If $|H| = 2$, then $g$ is also a switch w.r.t. some subset $A \subseteq R_1$ according to some $\sigma \in H$. Then by Lemma ?? $G \geq \langle Aut(\Gamma), g \rangle = S_H^{\Gamma}(\Gamma)$, contradicting the assumption that $G \subset S_H^{\Gamma}(\Gamma)$. If $|H| = 3$, we can use an argument similar to that in Lemma 3.9 to get some $g' \in \langle Aut(\Gamma), g \rangle$ such that $g' \in G \setminus Aut(\Gamma)$ is a switch w.r.t. some subset $A \subseteq R_1$ according to some $\sigma \in H$. Then by Lemma ?? $G \geq \langle Aut(\Gamma), g' \rangle = S_H^{\Gamma}(\Gamma)$, contradicting with the assumption that $G \subset S_H^{\Gamma}(\Gamma)$.

If such $H$ does not exist, then there exists some element $f \in G$ but $f \notin S_K^{<13}(\Gamma)$ for any proper $K \leq S_3$. Then by Lemma 3.9, there exist at least two nontrivial $H_1, H_2 \leq S_3$ with $H_1 \neq H_2$ such that $f \in \langle S_K^{<13}(\Gamma), S_{\{I\}}^{<13}(\Gamma) \rangle$, thus $G \geq S_{\{I\}}^{S_3}(\Gamma)$, a contradiction.
4 Switch Groups as Irreducible Closed Subgroups

Now we show that the various switch groups are exactly the irreducible closed subgroups of $\text{Sym}(\Gamma)$ where $\Gamma$ is the random bipartite graph having three cross-types.

**Lemma 4.1.** Let $G$ be an irreducible closed subgroup of $\text{Sym}_{\{l,r\}}(\Gamma)$ containing $\text{Aut}(\Gamma)$, and suppose there exist a finite bipartite subgraph $R \subset \Gamma$ and $\alpha \in \Gamma \setminus R$ such that

1. $(R, \alpha)$ is a sufficiently complex subgraph with respect to $G$,
2. for any $\pi \in G$ such that $\pi \upharpoonright R$ is an isomorphism, $\pi \upharpoonright R \cup \{\alpha\}$ is a switch w.r.t. $\alpha$ according to some $\sigma \in S_3$ ($\sigma$ could be the identity).

Then $G = \langle S_{\{l\}}^H(\Gamma), S_{\{r\}}^H(\Gamma) \rangle$ for some $H_1, H_2 \leq S_3$. If $G \subset \text{Sym}_{\{l,r\}}(\Gamma)$, then $H_1 = H_2$ unless one of the two groups is trivial.

**Proof** Since $G \cap S_{\{l\}}^{S_3}(\Gamma)$ is an irreducible closed subgroup of $S_{\{l\}}^{S_3}(\Gamma)$, by Theorem 3.10 there exists $H_1 \leq S_3$ such that $S_{\{l\}}^{H_1}(\Gamma) = G \cap S_{\{l\}}^{S_3}(\Gamma)$. Similarly, there exists $H_2 \leq S_3$ such that $S_{\{r\}}^{H_1}(\Gamma) = G \cap S_{\{r\}}^{S_3}(\Gamma)$. Now let $\pi \in G$ be given. Choose $N$ large enough such that for our fixed enumeration $\Gamma = \cup \Gamma_i$, if $i \geq N$ then

1. $\Gamma_i$ is sufficiently complex,
2. for any $x, y, z \in \Gamma_i$ with $x$ on the same side as $\alpha$ and $x \neq y, z$, there exists subgraphs $R_y$ and $R_z$ of $\Gamma_i$ with $y \in R_y, z \in R_z, x \notin R_y \cup R_z, (R_y, x) \cong (R, \alpha) \cong (R_z, x)$, and with cross-edges between $x$ and $R_y \cap R_z$ of all three cross-types.

We can find an $N$ such that (1) holds by Theorem 3.5 and SFSP. By the extension property of $\Gamma$, (2) holds in $\Gamma_i$, and hence by the Strong Finite Submodel Property, (2) holds for $\Gamma_i$ for all large $i$.

Now look at $\Gamma_N$, and define the coloring $\chi$ on $[\Gamma_N]^{\leq 2}$ by

1. for every $x \in \Gamma_N$,
   - if $x \in R_l$, then $\chi(x) = L$;
   - if $x \in R_r$, then $\chi(x) = R$.
2. for every $\{a, b\} \in [\Gamma_N]^2$,
   - if $\{a, b\} \in [R_l]^2$, then $\chi(a, b) = l$;
   - if $\{a, b\} \in [R_r]^2$, then $\chi(a, b) = r$;
   - if $\{a, b\} \in R_l \times R_r$ and $P_3(a, b)$, then $\chi(a, b) = P_3$;
   - if $\{a, b\} \in R_l \times R_r$ and $P_2(a, b)$, then $\chi(a, b) = P_2$;
   - if $\{a, b\} \in R_l \times R_r$ and $P_3(a, b)$, then $\chi(a, b) = P_3$.

Let $(\Gamma_N, \chi)$ be the $\alpha$-pattern $P$. By the Nešetřil-Rödl theorem, there exists a $\alpha$-Pattern $Q$, with the underlying set $X$, such that for any partition $F : [X]^2 \to \{P_1, P_2, P_3, l, r\}$, there exists $\Gamma_N \subseteq X$ such that

- $(\Gamma_N, \chi)$ has the $\alpha$-pattern $P$ (hence $\Gamma_N \cong \Gamma_N$);
- $(\Gamma_N, \chi)$ is $F$-homogeneous.

We define $F = \chi \circ \pi$ on $[\Gamma]^{\leq 2}$ for every $\{a, b\} \in [\Gamma]^{\leq 2}$ by

- if $\{a, b\} \in [R_l]^2$, then $F(a, b) = l$;
- if $\{a, b\} \in [R_r]^2$, then $F(a, b) = r$;
Such that any vertex $b$ cross-edges between $g_i$.Lemma
Now we show that there are no other irreducible closed subgroups.

Proof

Since $\Gamma_N$, $\chi$ is $F$-homogeneous, $\chi$ is homogeneous w.r.t. $\chi \circ \pi$. Since $\Gamma_N$ is sufficiently complex and $\Gamma_N \cong \Gamma$, $\Gamma_N$ witnesses all the cross-types. Hence $\chi \circ \pi$ does not lose any cross-types. By Claim 3.1, we have $\chi \circ \pi$ is a permutation of $\chi$. So $\pi$ is a switch w.r.t. all the vertices in $R_i$ according to some $\sigma \in S_3$. Since $\pi \in G \cap S_{\Lambda_i}^{3}(\Gamma)$, such $\sigma \in H_1$. Hence $\pi \in S_{\Lambda_i}^{3}(\Gamma) \mid \Gamma_N$.

Since $\Gamma_N$ is sufficiently complex, it witnesses which switches are in $G$, so there must be some $\varphi \in G \cap S_{\Lambda_i}^{3}(\Gamma)$ such that $\varphi^{-1} \circ \pi$ is an isomorphism on $\Gamma_N$. Let $\pi_1 = \varphi^{-1} \circ \pi$; we show that $\pi_1$ is a switch map.

Write $\Gamma_{N+1} = \Gamma_N \cup \{ x \}$. WLOG, we suppose $x \in R_i$. If $\pi_1 \mid \Gamma_{N+1}$ is an isomorphism, then it is trivially a switch. If not, then let $y, z \in \Gamma_N$ be arbitrary. By the choice of $N$ there exists subgraphs $y \in R_y \subseteq \Gamma_N$ and $z \in R_z \subseteq \Gamma_N$ such that there are all cross-types between $\{ x \}$ and $R_y \cap R_z$. $\pi_1 \mid \Gamma_N$ is an isomorphism, and $R_y, R_z, (R_y, x), (R_z, x)$ are sufficiently complex with $R_y \cong R$ and $R_z \cong R$. Since $\pi_1 \mid R_y$ is an isomorphism, $\pi_1 \mid R_y \cup \{ x \}$ is a switch w.r.t. $x$ according to some $\sigma_1 \in H_1$. Similarly, $\pi_1 \mid R_z \cup \{ x \}$ is a switch w.r.t. $x$ according to some $\sigma_2 \in H_1$. But $\sigma_1 = \sigma_2$ since $\pi_1 \mid R_y \cup \{ x \}$ has to agree with $\pi_1 \mid R_z \cup \{ x \}$ on $R_y \cap R_z$. Since $y, z$ are arbitrary, $\pi_1 \mid \Gamma_{N+1}$ is a switch w.r.t. $x$ according to some $\sigma' \in S_3$. $(R_y, x)$ witnesses the fact that this switch is in $G$, so there exists some $\varphi_1 \in G$ such that $\varphi_1$ is a switch w.r.t. $x$ according to $\sigma'$. Since $\varphi_1 \in G \cap S_{\Lambda_i}^{3}(\Gamma)$, such $\sigma' \in H_1$. Hence $\varphi_1^{-1} \circ \pi_1$ is an isomorphism on $\Gamma_{N+1}$. Let $\pi_2 = \varphi_1^{-1} \circ \pi_1$.

Write $\Gamma_{N+2} = \Gamma_{N+1} \cup \{ x' \}$ for $x' \in R_i$. Similarly, we get $\pi_2 \mid \Gamma_{N+2}$ is a switch w.r.t. $x'$. So there exists some $\varphi_2 \in G$ and $\varphi_2 \mid \Gamma_{N+2}$ is a switch w.r.t. $x'$ and $\varphi_2 \circ \pi_2$ is an isomorphism on $\Gamma_{N+2}$. By induction, $\pi \mid \Gamma_{N+k}$ is a composition of switches on vertices in $\Gamma_{N_k} \setminus \Gamma_N$. Since $\langle S_{\Lambda_i}^{3}(\Gamma), S_{\Lambda_i}^{2}(\Gamma) \rangle$ is closed, $\pi$ is a switch. We have shown that it is a composition of switches on the vertices in $R_i$ and switches on the vertices in $R_r$. Since the choice of $\pi$ is arbitrary, $G \subseteq \langle S_{\Lambda_i}^{3}(\Gamma), S_{\Lambda_i}^{2}(\Gamma) \rangle$. But $S_{\Lambda_i}^{3}(\Gamma), S_{\Lambda_i}^{2}(\Gamma) \leq G$, so we have $G = \langle S_{\Lambda_i}^{3}(\Gamma), S_{\Lambda_i}^{2}(\Gamma) \rangle$. If $G \subset Sym_{(i,r)}(\Gamma)$, then by Lemma 3.8, $H_1 = H_2$ unless one of them is trivial.

5 Switches Are the Only Nontrivial Closed Groups

Now we show that there are no other irreducible closed subgroups.

Lemma 5.1. Let $\Gamma = \sqcup \Gamma_i$ as in SFSP, and let $G$ be an irreducible closed subgroup of $Sym_{(i,r)}(\Gamma)$ containing $Aut(\Gamma)$. There is an integer $N$ such that $i \geq N$ implies

$(\star)$ If $\alpha \in \Gamma_i$ and $g \in G$ is an isomorphism on $\Gamma \setminus \{ \alpha \}$, then there are cross-edges between $g(\Gamma \setminus \{ \alpha \})$ and $g(\alpha)$ with all cross-types.

Proof Let $R$ be sufficiently complex. By the extension properties $\Theta_n$ for $n \in \mathbb{N}$, the following sentence is true in $\Gamma$:

$(\star)$ For any $\alpha \in \Gamma$ and any bipartite subgraph $B \subset \Gamma$ with $|B| = |R|$ and any vertex $b \in B$ on the same side as $\alpha$, there is some embedding $\varphi : B \rightarrow \Gamma$ such that $\varphi(b) = \alpha$. 

By SFSP, there exists \( N \in \mathbb{N} \) such that when \( i \geq N \), \((\ast)\) is true for \( \Gamma_i \).

Suppose for any \( M \in \mathbb{N} \), there exist \( i \geq M \) and some \( g \in G \) which is an isomorphism on \( \Gamma_i \setminus \{\alpha\} \), but for some \( c \in \{P_1, P_2, P_3\} \), there is no cross-edge in \( g(\Gamma_i) \) with endpoint \( g(\alpha) \) having the cross-type \( c \). For any subgraph \( B \subset \Gamma \) with \( |B| = |R| \) and any \( b \) in the same side as \( \alpha \), there is an isomorphism \( \varphi \in \text{Aut}(\Gamma) \) with \( \varphi(B) \leq \Gamma_i \) such that \( \varphi(b) = \alpha \). If \( f = g \circ \varphi \in G \), then \( f(B) \) has no cross-edge with cross-type \( c \) and endpoint \( f(b) \). By a composition of such maps, one for each vertex of \( B \) in the same side as \( \alpha \), we have a \( f^* \) such that \( f^*(B) \) has no cross-type \( c \). Since \( R \) is a special case of \( B \), we can find a \( f^*_d \in G \) such that \( f^*_d(R) \) has no cross-edge with cross-type \( c \). But \( R \) is sufficiently complex, and witnesses the fact that \( G \) is irreducible, and we have reached a contradiction.

\( \square \)

**Lemma 5.2.** Let \( \Gamma = \cup \Gamma_i \) given by SFSP, and let \( G \) be an irreducible closed group of \( \text{Sym}_{l,r}(\Gamma) \) containing \( \text{Aut}(\Gamma) \). Then there is an integer \( N \) such that for any \( i \geq N \):

1. There is no \( g \in G \) which is an isomorphism on \( \Gamma_i \) except for its effect on one cross-edge.
2. For every \( \alpha \in \Gamma \), there is no \( g \in G \) which is an isomorphism on \( \Gamma_i \setminus \{\alpha\} \) and for which there exists a switch \( f \) w.r.t. \( g(\alpha) \) such that \( f \circ g \) is an isomorphism on \( \Gamma_i \) except for exactly one cross-edge.

**Proof** Let \((C, \gamma)\) where \( C \subset \Gamma \) and \( \gamma \in \Gamma \setminus C \) be sufficiently complex. Since \( \Gamma \) has the extension property, the following is true in \( \Gamma \):

(a) for any \(|C|\)-graph \( C' \) with three cross-types, and any cross-edge \((a, b)\) of \( \Gamma \), and any cross-edge \((a', b') \in C'\) with the same cross-type as \((a, b)\), there is an embedding \( \varphi : C' \rightarrow \Gamma \) such that \( \varphi(a', b') = (a, b) \).

(b) For any two vertices \( \alpha, \beta \) on the same side of \( \Gamma \) as \( \gamma \), there is a finite subgraph \( A \subset \Gamma \) such that \( \alpha, \beta \notin A \) and \( A \cong C \) can be extended to \( A \cup \{\alpha\} \cong C \cup \{\gamma\} \).

By SFSP, there exists some \( N \in \mathbb{N} \) such that \( i \geq N \) implies that \( \Gamma_i \) has the properties \((a)\) and \((b)\). We show that the same \( N \) will satisfy Lemma 5.2.

Since \((2)\) implies \((1)\) when a switch is trivial, we suppose \((2)\) does not hold, i.e. for any \( N \in \mathbb{N} \) there are \( i \geq N, g \in G \), and \( f \) such that \( g \) is an isomorphism on \( \Gamma_i \setminus \{\alpha\} \), \( f \) is a switch w.r.t. \( g(\alpha) \) and \( f \circ g \) is an isomorphism except on one cross-edge of \( \Gamma_i \). One endpoint must be \( \alpha \), let the other endpoint be \( \beta \). By \((b)\) above, there exists some \( A \cong C \) and \( \alpha, \beta \notin A \) such that \( A \cup \{\alpha\} \cong C \cup \{\gamma\} \).

Now \( g \upharpoonright A \cup \{\alpha\} \) is a switch with the same permutation of cross-types as that of \( f^{-1} \). But \( A \cup \{\alpha\} \cong C \cup \{\gamma\} \) which witnesses which switches are in \( G \). So \( f^{-1} \in G \), hence \( f \in G \). Let \( h = f \circ g \). Then \( h \in G \), but this leads to a contradiction since for any \(|C|\)-graph \( C' \) having three cross-types, and any cross-edge \((a', \beta') \in C' \cap (R_i \times R_r) \) with \( P_i(\alpha, \beta) \) and \( P_i(\alpha', \beta') \) for some \( i \in \{1, 2, 3\} \), by \((a)\) there is an embedding \( \varphi : C' \rightarrow \Gamma \) such that \( \varphi(a', \beta') = (\alpha, \beta) \). Since \( \Gamma \) is homogeneous, there is some \( \Phi \in \text{Aut}(\Gamma) \) such that \( \Phi \upharpoonright C' = \varphi \). Now let \( h_1 = h \circ \Phi \). Then \( h_1 \in G \) and \( h_1 \upharpoonright C' \) is an isomorphism except on the cross-edge \((\alpha', \beta') \). Hence \( h(C') \) has one fewer cross-edge with cross-type \( P_i \) on it. If there are \( n \) many cross-edges with cross-type \( P_i \) in \( C' \), then by repeating this argument \( n \) times, we can
construct $h_n \in G$ such that $h_n(C')$ has no cross-edge with cross-type $P_i$. Since $C$ is a special case of $C'$, we have the same result for $C$. But since $C$ is sufficiently complex, it witnesses all the cross-types in $C$ and we have a contradiction, and $N$ is as desired. \hfill \Box

Given a bipartite graph $A \subseteq \Gamma$, we define a class of vertex colorings $\Phi : A \rightarrow \{L, \overline{P}, P_2, P_3\}$ for every $x \in A$ by
- if $x \in R_l$, then $\Phi(x) = L$;
- if $x \in R_r$, then $\Phi(x) = \overline{P}_i$ for some $i = 1, 2, 3$.

**Definition 5.3.** Let $A_1$ be a bipartite graph with vertex coloring $\Phi_1$, and let $A_2$ be a bipartite graph with vertex coloring $\Phi_2$, where $\Phi_1$ is defined as above and $\Phi_2 = \Phi \circ g$ for some $g \in Sym(A_1)$ preserving $R_l, R_r$. If $|A_1| = |A_2|$ and $|A_1 \cap R_l| = |A_2 \cap R_l|$, then the vertex coloring $\Phi_2$ is a permutation of the vertex coloring $\chi_1$ if there is some vertex bijection $\varphi : A_1 \rightarrow A_2$ preserving $R_l, R_r$ and some permutation $\sigma \in S_3$ such that for any vertex $x \in R_r$, $\Phi_1(x) = \sigma(\Phi_2(\varphi(x)))$.

**Definition 5.4.** Let $A$ be a bipartite graph, and $\Phi_1, \Phi_2$ be vertex colorings on $A$ defined in Definition 5.3. Then the vertex coloring $\Phi_2$ is homogeneous w.r.t. the coloring $\Phi_1$ if for any $x, x' \in A \cap R_r$, $\Phi_2(x) = \Phi_2(x') \Longrightarrow \Phi_1(x) = \Phi_1(x')$.

**Claim 5.1.** If $A$ is a bipartite graph, $\Phi_1, \Phi_2$ are the vertex colorings on $A$ defined as above, and $\Phi_2$ is homogenous w.r.t. $\chi_1$ but is not a permutation of $\Phi_1$, then there must be two distinct colors $\overline{P}_i$ and $\overline{P}_j$ and some color $\overline{P}_k$ ($i, j, k \in \{1, 2, 3\}$) such that for any $x \in A \cap R_r$, $\Phi_2(x) = \overline{P}_i$ or $\Phi_2(x) = \overline{P}_j$ implies $\Phi_1(x) = \overline{P}_k$.

**Proof** This follows immediately from the definitions of homogeneous and permutation colorings. \hfill \Box

Given a bipartite graph $A \subseteq \Gamma$, we define an edge coloring $\chi : A \rightarrow \{l, r, P_1, P_2, P_3\}$ for every $\{a, b\} \in A$ by
- if $\{a, b\} \subseteq R_l$, then $\chi(a, b) = l$;
- if $\{a, b\} \subseteq R_r$, then $\chi(a, b) = r$;
- if $a \in R_l, b \in R_r$ and $P_1(a, b)$, then $\chi(a, b) = P_1$;
- if $a \in R_l, b \in R_r$ and $P_2(a, b)$, then $\chi(a, b) = P_2$;
- if $a \in R_l, b \in R_r$ and $P_3(a, b)$, then $\chi(a, b) = P_3$.

**Lemma 5.5.** Let $\mathcal{A}$ be the class of bipartite subgraphs having at least one vertex in $R_i$ ($i = l, r$) in $\Gamma$ with edge coloring $\chi : |V|^2 \rightarrow \{l, r, P_1, P_2, P_3\}$ defined above, and with vertex coloring $\Phi : V \rightarrow \{L, \overline{P}, P_2, P_3\}$ defined above. For any finite $A_1 \in \mathcal{A}$, there is a finite $A_2 \in \mathcal{A}$ such that for any vertex coloring $\Psi : A_2 \rightarrow \{L, \overline{P}, P_2, P_3\}$, which is not a permutation of $\Phi$, then there exists $A'_1 \in \mathcal{A}$ with $\langle A_1, \Phi, \chi \rangle \cong \langle A'_1, \Phi, \chi \rangle$, $A'_1 \subseteq A_2$, and one of the following properties:

(a) There is a color $t \in \{\overline{P}_1, \overline{P}_2, P_3\}$ such that $\Psi(x) \neq t$ for every vertex $x \in A'_1 \subseteq A_2$.

(b) There is some vertex coloring $\Psi' : A'_1 \rightarrow \{L, \overline{P}, P_2, P_3\}$, which is a permutation of $\Psi$, and differs from $\Phi : A'_1 \rightarrow \{L, \overline{P}, P_2, P_3\}$ on exactly one vertex in $R_r$. 

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Proof

We choose $M \in \mathbb{N}$ such that for any $i \geq M$, and for any $v \in \Gamma_i$, there exists $B \subset \Gamma_i$ with $v \in B$ such that $B \cong A_1$. We can do this because of the extension property and SFSP. Now we choose $N \in \mathbb{N}$ such that $N \geq M$ and $\Gamma_N \supseteq A_1$. Since for any $v \in \Gamma_N$, there exists an isomorphic copy of $A_1$ containing $v$, we can extend the colorings $\Phi, \chi$ on $A_1$ to the colorings on the whole $\Gamma_N$. We call them $\Phi, \chi$ to simplify the notation. Now let $(\Gamma_N, \Phi)$ be our $P$-pattern. Then by the Neˇ setˇ ril-R¨ odl Theorem, there exists a finite $A_2 \in \mathfrak{A}$ such that for any partition function $F = \Psi : A_2 \rightarrow \{L, \overline{P}_1, \overline{P}_2, \overline{P}_3\}$, there is $\Gamma_N' \subset A_2$ satisfying

- $\Gamma_N'$ has the $\alpha$-pattern $P$ (hence $\Gamma_N' \cong \Gamma_N$);
- $(\Gamma_N', \Phi)$ is $\Psi$-homogeneous.

Since $\Gamma_N' \cong \Gamma_N$, we can find a graph $A_1' \subset \Gamma_N'$ such that $\langle A_1', \Phi, \chi \rangle \cong \langle A_1, \Phi, \chi \rangle$. Now we have $(\Gamma_N', \Phi)$ is $\Psi$-homogeneous, i.e. if $\Phi(a) = \Phi(b)$, then $\Psi(a) = \Psi(b)$ for $a, b \in \Gamma_N'$.

If $\Psi$ is not a permutation of $\Phi$ on $\Gamma_N'$, then by Claim 5.1 there must exist some color $t \in \{\overline{P}_1, \overline{P}_2, \overline{P}_3\}$ such that $\Psi(v) \neq t$ for every $v \in \Gamma_N'$. Since $A_1' \subset \Gamma_N'$, the same result holds for $A_1'$. Then Property (1) holds.

If $\Psi$ is a permutation of $\Phi$ on $\Gamma_N'$, then since $\Psi$ is not a permutation of $\Phi$ on $A_2 \supseteq \Gamma_N'$, there must exist some $\Gamma_k$ for $k \geq N$ such that $\Psi$ is a permutation of $\Phi$ on $\Gamma_k$, but $\Psi$ is not a permutation of $\Phi$ on $\Gamma_{k+1}$. Let $\Gamma_{k+1} = \Gamma_k \cup \{v\}$. Note that $v \in R_k$, since $\Psi(x) = \Phi(x)$ for every $x \in R_k$. There must exist a subgraph $B \subset \Gamma_k$ such that $v \in B$ and $f : B \cong A_1$ preserves the vertex coloring $\Phi$ and the edge coloring $\chi$. Hence we have $\Psi \upharpoonright (B \backslash \{v\})$ is a permutation of $\Phi \upharpoonright (B \backslash \{v\})$, but $\Psi \upharpoonright B$ is not a permutation of $\Phi \upharpoonright B$. Let $f_1$ be an isomorphism from $A_1$ to $A_1'$, and let $\Psi' = \Psi \circ f_1 \circ f_1^{-1}$. Then we have the Property (2). This completes the proof of Lemma 5.5.

Similarly, given a bipartite graph $A \subseteq \Gamma$, we define a class of vertex colorings $\overline{\Phi} : A \rightarrow \{R, \overline{P}_1, \overline{P}_2, \overline{P}_3\}$ by for every $x \in A$,

- if $x \in R_x$, then $\Phi(x) = R$;
- if $x \in R_i$, then $\Phi(x) = \overline{P}_i$ for some $i = 1, 2, 3$.

A similar argument gives the following Lemma:

Lemma 5.6. Let $\mathfrak{A}$ be the class of bipartite subgraphs of $\Gamma$ having at least one vertex in $R_i$ ($i = l, r$) with edge coloring $\chi : [V]^2 \rightarrow \{l, r, P_1, P_2, P_3\}$ defined above and with vertex coloring $\overline{\Phi} : V \rightarrow \{R, \overline{P}_1, \overline{P}_2, \overline{P}_3\}$ defined above. For any finite $A_1 \in \mathfrak{A}$, there is a finite $A_2 \in \mathfrak{A}$ such that for any vertex coloring $\Psi : A_2 \rightarrow \{R, \overline{P}_1, \overline{P}_2, \overline{P}_3\}$, and which is not a permutation of $\overline{\Phi}$, there exists $A_1' \in \mathfrak{A}$ with $\langle A_1, \overline{\Phi}, \chi \rangle \cong \langle A_1', \overline{\Phi}, \chi \rangle$, $A_1' \subset A_2$, and one of the following properties:

- There is a color $t \in \{\overline{P}_1, \overline{P}_2, \overline{P}_3\}$ such that $\Psi(x) \neq t$ for every vertex $x \in A_1'$.
- There is some vertex coloring $\Psi' : A_1' \rightarrow \{R, \overline{P}_1, \overline{P}_2, \overline{P}_3\}$, which is a permutation of $\Psi$, and differs from $\overline{\Phi} : A_1' \rightarrow \{R, \overline{P}_1, \overline{P}_2, \overline{P}_3\}$ on exactly one vertex in $R_i$. 

Theorem 5.7. If $G$ is an irreducible closed subgroup such that $\text{Aut}(\Gamma) \leq G \leq \text{Sym}_{\{t, r\}}(\Gamma)$, then $G = (S^H_t(\Gamma), S^H_r(\Gamma))$ where $H_1, H_2 \leq S_3$. If $G \subset \text{Sym}_{\{t, r\}}(\Gamma)$, then $H_1 = H_2$ unless one of the two groups is trivial.

Proof We show that there is a sufficiently complex $(R, \alpha)$ with $R \subset \Gamma$ and $\alpha \in \Gamma \setminus R$, such that for any $g \in G$, if $g \mid R$ is an isomorphism, then $g \mid R \cup \{\alpha\}$ is a switch w.r.t. $\alpha$ according to some $\sigma \in S_3$. Then by Lemma 4.1 we are done.

Suppose that for any sufficiently complex $(R, w)$, there exists some $g \in G$ such that $g \mid R$ is an isomorphism and $g \mid R \cup \{w\}$ is not a switch w.r.t. $w$ according to any $\sigma \in S_3$. We eventually get a contradiction. WLOG, let $w \in R_t$. Choose $N$ such that $i \geq N$ implies $\Gamma_i$ is sufficiently complex and satisfies the conclusion of Lemma 5.1 and Lemma 5.2. Since $\Gamma_N$ is sufficiently complex, then there exists some $v \in \Gamma \setminus \Gamma_N$ such that $\Gamma_N, v)$ is sufficiently complex. Let $v \in R_t$, and let $R = \Gamma_N \cup \{v\}$. We define the coloring $\chi : (R \setminus \{v\})^2 \rightarrow \{t, r, P_1, P_2, P_3\}$ as above. The edges between $\{v\}$ and $R \setminus \{v\}$ also induce a vertex coloring $\Phi_v : R \setminus \{v\} \rightarrow \{L, \overline{P_1}, \overline{P_2}, \overline{P_3}\}$ given for every $a \in R \setminus \{v\}$ by

- if $a \in R_t$, then $\Phi_v(a) = L$;
- if $a \in R_r$ and $P_1(a, a)$, then $\Phi_v(a) = \overline{P_3}$;
- if $a \in R_r$ and $P_2(a, a)$, then $\Phi_v(a) = \overline{P_2}$;
- if $a \in R_r$ and $P_3(a, a)$, then $\Phi_v(a) = \overline{P_1}$.

That is, the color of the vertex $a$ is given by the cross-type of the cross-edge $(a, v)$.

By Lemma 5.5, given $(R \setminus \{v\}, \Phi_v, \chi)$, there are a subgraph $S \subset \Gamma$ with a vertex coloring $\Phi_\alpha : S \rightarrow \{L, \overline{P_1}, \overline{P_2}, \overline{P_3}\}$ and an edge coloring $\chi : [S]^2 \rightarrow \{t, r, P_1, P_2, P_3\}$ such that for any other vertex coloring $\Psi : S \rightarrow \{L, \overline{P_1}, \overline{P_2}, \overline{P_3}\}$, if $\Psi$ is not a permutation of $\Psi_\alpha$, there exists $R' \subset S$ such that $(R' \setminus \{v\}, \Phi_v, \chi) \cong (R', \Phi_\alpha, \chi)$, hence $R' \cong R \setminus \{v\}$, and one of the properties (a) and (b) in Lemma 5.5 holds. Since $\Gamma_N$ is random, there must exists a vertex $\alpha \in (\Gamma \setminus S) \cap R_t$ inducing the vertex coloring $\Phi_\alpha$.

Note that $(R', \Phi_\alpha, \chi) \cong (R \setminus \{v\}, \Phi_v, \chi)$, hence $(R \setminus \{v\}) \cup \{v\} \cong R' \cup \{\alpha\}$. Since $(R \setminus \{v\}, v)$ is sufficiently complex, so is $(R', \alpha)$. Then $S \cup \{\alpha\}$ is sufficiently complex since $S \supset R'$. By the assumption at the beginning of the proof, there exists some $g \in G$ which is an isomorphism on $S$, but $g \mid S \cup \{\alpha\}$ is not a switch w.r.t. $\alpha$ according to any $\sigma \in S_3$. Then the vertex $g(\alpha)$ induces a new coloring $\Phi_\alpha(g(\alpha))$ on $g(S)$, and $\Phi_{g(\alpha)}$ is not a permutation of $\Phi_\alpha$ since $g \mid S \cup \{\alpha\}$ is not a switch w.r.t. $\alpha$ according to any $\sigma \in S_3$. Since $g(S) \cong S$, $g(\alpha)$ also induces a new coloring $\Phi_\alpha(g(\alpha))$ on $S$. Let $\Psi = \Phi_{g(\alpha)}$. Then by Lemma 5.5 we can find $R' \subset S$ such that one of the following two possibilities holds:

(a) there is a color $t \in \{\overline{P_1}, \overline{P_2}, \overline{P_3}\}$ such that $\Phi_{g(\alpha)}(x) \neq t$ for any vertex $x \in R'$
(b) There is a coloring $R' \rightarrow \{L, \overline{P_1}, \overline{P_2}, \overline{P_3}\}$ which is a permutation of $\Phi_{g(\alpha)}$, but differs from $\Phi_\alpha$ on exactly one vertex in $R_r$.

Now $R \cong R' \cup \{\alpha\}$ since $(R', \Phi_\alpha) \cong (R \setminus \{v\}, \Phi_v)$. If (a) holds, then there is some $P_i, (i = 1, 2, 3)$ such that there is no edge between $g(\alpha)$ and $g(R')$ with the cross-type $P_i$, contrary to Lemma 5.1. If (b) holds, then $g \mid R' \cup \{\alpha\}$ differs
from a switch on exactly one cross-edge, contradicting Lemma 5.2. Similarly, if we assume \( w \in R_r \), we eventually get the contradiction by applying Lemma 5.6. This completes the proof of Theorem 5.7.

6 The Case When \( R_l \) and \( R_r \) Are Not Preserved

Next we do not assume that \( G \) preserves \( R_l \) and \( R_r \). Since an element of \( G \) either preserves \( R_l \) and \( R_r \) or switches \( R_l \) and \( R_r \), for every reduct \( S_X^H(\Gamma) \) where \( X \subseteq \{l, r\} \), and \( H \leq S_3 \) (which preserves \( R_l, R_r \)), we always have a corresponding reduct \( S_X^{H*}(\Gamma) \) which preserves the same relation as \( S_X^H(\Gamma) \) except it either preserves \( R_l \) and \( R_r \) or switches them.

Since the elements in \( G \) either preserve \( R_l \) and \( R_r \) or switch \( R_l \) and \( R_r \), we can introduce a relation of on the same side \( S(x, y) \) to replace \( R_l, R_r \) defined by \( S(x, y) \leftrightarrow (R_l(x) \land R_r(y)) \lor (R_r(x) \land R_l(y)) \), and a new binary relation \( P_i^* \) to replace \( P_i \) defined by \( P_i^*(x, y) \leftrightarrow P_i(x, y) \lor P_i(y, x) \) for \( i = 1, 2, 3 \). So if one forgets \( R_l, R_r \) but retains \( S(x, y) \) and replaces \( P_i \) by \( P_i^* \), then either the sides are preserved or switched. Since \( \Gamma \) satisfies the extension property \( \Theta_n \) for \( n \in \mathbb{N} \), we can construct \( \rho \in Sym(\Gamma) \) which exchanges the sets \( R_r \) and \( R_l \) and such that for every \( a \in R_l \) and every \( b \in R_r \), \( P_i(a, b) \rightarrow P_i(\rho(b), \rho(a)) \) where \( i = 1, 2, 3 \). Let \( S_X^{H*}(\Gamma) \) be \( \langle S_X^H(\Gamma), \rho \rangle \). Note that \( |S_X^{H*}(\Gamma)| = 2 \), and \( S_X^H(\Gamma) \leq S_X^{H*}(\Gamma) \).

**Theorem 6.1.** If \( G \) is an irreducible closed subgroup such that \( Aut(\Gamma) \leq G \leq Sym_{\{l, r\}}(\Gamma) \), then \( G = \langle S_X^{H_1}(\Gamma), S_X^{H_2}(\Gamma) \rangle \) or \( G = \langle S_X^{H_1}(\Gamma), S_X^{H*}(\Gamma) \rangle \) where \( H_1, H_2 \leq S_3 \). If \( G < Sym_{\{l, r\}}(\Gamma) \), then \( H_1 = H_2 \) unless one of the two groups is trivial.

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