Alternative variables for the dynamics of general relativity

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Abstract

A new form of the dynamical equations of vacuum general relativity is proposed. This form involves the canonical Hamiltonian structure but non canonical variables. The new field variables are the electric field $E^a_i$ and the magnetic field $B^a_i$, which emerge from the Ashtekar’s representation for canonical gravity. The constraint algebra is studied in terms of the new field variables and the counting of the degrees of freedom is done. The quantization is briefly outlined.

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I. INTRODUCTION

In 1986, the canonical approach to general relativity received new life by the introduction by Abhay Ashtekar of a new formulation [1]. (See also [2].) In this formulation one can use a (complex) $SO(3)$ spatial connection as coordinate for the gravitational phase space instead of the 3-metric introduced by Arnowitt, Deser and Misner (ADM) [3]. Ashtekar variables led to a considerable simplification of the constraints associated with the Hamiltonian formulation of Einstein’s theory. Indeed, Ashtekar’s constraints are polynomials in the canonical variables. Ashtekar’s canonical gravity allows some progress in the direction of a quantum theory of gravity.

On the other hand a common framework has emerged which extends the structure of Hamiltonian mechanics to infinite-dimensional systems. The Hamiltonian formulation is usually obtained from the Lagrangian formulation by means of the Legendre transformation, but in the case of fields this canonical procedure presents difficulties since not always the momentum densities are independent of the field variables, which is usually mended by the introduction of constraints. Nevertheless, it is possible to avoid these complications and give a Hamiltonian formulation for a given continuous system, without making reference to the Lagrangian formulation, if its evolution equations can be written in the form

$$\dot{\phi}_\alpha = D_{\alpha\beta} \frac{\delta H}{\delta \phi_\beta}, \quad (1)$$

where the field variables $\phi_\alpha \ (\alpha = 1, 2, ..., n)$ represent the state of the system, $H$ is a suitable functional of the $\phi_\alpha$, $\delta H/\delta \phi_\beta$ is the functional derivative of $H$ with respect to $\phi_\beta$, and the $D_{\alpha\beta}$ are, in general, differential operators of an arbitrary finite order with the coefficients depending on the variables $\phi_\alpha$ and their derivatives (which are also of a finite order). These operators must satisfy certain conditions that allow the definition of a Poisson bracket between functionals of the $\phi_\alpha$ (see, e.g., Refs. [4] and [5]). Here and henceforth a dot denotes differentiation with respect to the time and there is summation over repeated indices.

In the case of the source-free electromagnetic field, taking the components of the electric and the magnetic field as the field variables $\phi_\alpha$, the evolution equations, given by Maxwell’s equations, can be expressed in the form [11], without having to introduce the electromagnetic potentials and, therefore, without having to choose an specific gauge [4]–[5]. By contrast, in the standard Lagrangian formulation for the electromagnetic field, the field variables are
precisely the electromagnetic potentials. In Ref. [6] a Hamiltonian structure for the linearized Einstein vacuum field equations is found by using as Hamiltonian density an analog of the energy of the electromagnetic field. This Hamiltonian structure involves integral operators. Another Hamiltonian structure for this linearized theory is found in Ref. [7] by using another Hamiltonian.

In this paper we show that the evolution equations for the gravitational field, given by the Einstein vacuum field equations in an alternative representation derived from that of Ashtekar, can be expressed in a Hamiltonian form analogous to Eq. (1) with the canonical Hamiltonian structure and in terms of non canonical variables. This construction is not immediately obvious. In particular, the covariant derivative in the operators $D_{\alpha \beta}$, defined below, leads to some difficulties which will be addressed here. Furthermore, the gauge systems has not been systematically studied in terms of non canonical variables (see [8] and [9] for a review).

This paper is organized as follows. We start with a brief summary of the Hamiltonian formalism for gravity in the ADM variables. Then we analyze the change of variables leading to the Ashtekar formalism. In Sect. 4 the alternative form of the dynamical equations of vacuum general relativity is derived. A Poisson bracket is obtained and it is shown that it yields the expected relations between the Hamiltonian and any functional of the field. Then we review the Poisson algebra of the constraints. In Sect. 5 we sketch the quantization. We end the paper with the conclusions and a brief discussion of the prospects related to the alternative representation.

II. ADM FORMALISM

Spacetime can be considered as a 4-manifold $M$, arising as a result of the time evolution of a three-dimensional space-like hypersurface $\Sigma$. The manifold $M$ is assumed to be orientable, and have the global topology $\Sigma \times \mathbb{R}$. $\mathbb{R}$ is the real line. We assume that $\Sigma$ is compact without boundary. The dynamical variables are the Riemannian 3-metric tensor field $q_{ab}$, and the tensor density field of the conjugate momenta $p^{ab}$, which are linearly related to the extrinsic curvature tensor $K_{ab}$ of the hypersurface,

$$p^{ab} = -q^{1/2}(K^{ab} - q^{ab}K),$$

\( (2) \)
where $q^{ab}$ is the inverse matrix to $q_{ab}$, $K = q^{ab}K_{ab}$, $q = \det(q_{ab})$, and the latin indices $a, b, \ldots$ label spatial coordinates and run over the values 1, 2, 3. These indices are raised and lowered by means of $q_{ab}$. (See e.g. Ref. [10] for a nice treatment of this formulation.)

The dynamic equations are generated by the Hamiltonian

$$H = \int \left( N\mathcal{H} + N^b\mathcal{H}_b \right) d^3x,$$

which is a linear combination of the (scalar and vectorial) constraints

$$\mathcal{H} = q^{1/2} \left( -3R + q^{-1}p^{ab}p_{ab} - \frac{1}{2}q^{-1}p^2 \right),$$

$$\mathcal{H}^a = -2q^{1/2}D_b \left( q^{-1/2}p^{ab} \right),$$

and by the canonical Poisson bracket

$$\{q_{ab}(x), p^{cd}(y)\} = \delta^c_{(a} \delta^d_{b)} \delta^3(x - y),$$

so that

$$\dot{q}_{ab} = \{q_{ab}, H\}, \quad \dot{p}^{ab} = \{p^{ab}, H\}.$$

In Eqs. (4) and (5) $p = p^{a}_{a} = p^{ab}q_{ab}$, and $D_a$ is the torsion-free covariant derivative compatible with $q_{ab}$, with Riemann curvature tensor $2D_{[a}D_{b]}v_c \equiv R_{abc}{}^{d}v_d$, where $v_a$ is an arbitrary covector on $\Sigma$. $3R$ is the Ricci scalar of this curvature. The scalar $N$ is known as the lapse and $N^a$ is a vector on $\Sigma$ and is usually referred to as the shift vector; they should be viewed as Lagrange multipliers.

Explicitly the dynamical equations (7) are given by

$$\dot{q}_{ab} = \frac{\delta H}{\delta p^{ab}} = 2q^{-1/2}N \left(p_{ab} - \frac{1}{2}q_{ab}p\right) + 2D_{(a}N_{b)},$$

and

$$\dot{p}^{ab} = -\frac{\delta H}{\delta q_{ab}} = -Nq^{1/2} \left( 3R^{ab} - \frac{1}{2}Rq^{ab} \right) + q^{1/2}D_c \left( q^{-1/2}N^c p^{ab} \right) - 2Nq^{-1/2} \left( p^{ac}p_{c}{}^b - \frac{1}{2}p_{ab} \right) - 2p^{c(a}D_cN^{b)} + \frac{1}{2}Nq^{-1/2}q^{ab} \left( p_{cd}p^{cd} - \frac{1}{2}p^2 \right) + q^{1/2}(D^aD^bN - q^{ab}D^cD_cN),$$

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where boundary terms have been ignored. Equations (4), (5), (8) and (9) are equivalent to the vacuum Einstein equation, \( R_{\alpha\beta} = 0 \) (\( \alpha, \beta = 0, 1, 2, 3 \), here). One can explicitly reconstruct the four-dimensional space-time geometry in arbitrary coordinates \( X^\alpha \). For a more thorough treatment of this Hamiltonian formalism, see Ref. [11].

III. ASHTEKAR FORMALISM

The original literature on Ashtekar’s variables uses the language of \( SU(2) \) spinors. We have preferred to avoid this language, and use \( SO(3) \)-valued variables. The translation from \( SO(3) \)-valued variables to \( SU(2) \) spinors is illustrated clearly in Ref. [12]. Moreover, in this section our convention is closer to that of Ref. [13].

Instead of the metric tensor \( q_{ab} \) we introduce the triad \( e^i_a \), such that the spatial metric is given by

\[
q_{ab} = e^i_a e^j_b \delta_{ij} = e^i_a e_{bi} \quad (10)
\]

Latin indices \( i, j, ... = 1, 2, 3 \), from the middle of the alphabet, are \( SO(3) \) indices. They are raised and lowered with the Kronecker delta \( \delta_{ij} \). The inverse matrices to the triad are denoted by \( e^i_a \), hence, \( e^i_a e^j_b = \delta^i_a \), and \( e^i_a e^a_j = \delta^i_j \). Since \( q^{cb} q_{ba} = q^{cb} e^j_b e^i_a = \delta^c_a \), the inverse matrix can be obtained by raising the index with the help of \( q^{cb} \), \( e^c_i = e^c_i = q^{cb} e^i_b \). The position of the internal index \( i \) is irrelevant. It is also not difficult to verify that

\[
q^{ab} = e^a_i e^b_i, \quad q = \det(e^a_i e_{bi}) = (\det(e^a_i))^2 \equiv e^2. \quad (11)
\]

Let us introduce the momenta \( p^a_i \) conjugate to the triad. They satisfy the equations

\[
\{e^a_i(x), p^b_j(y)\} = \delta^b_j \delta^a_i \delta^3(x - y), \quad (12)
\]

and can be easily related to the momenta \( p^{ab} \) by means of

\[
p^a_i = 2p^{ab} e_{bi}. \quad (13)
\]

It now turns out that part of the Poisson brackets for the ADM variables has been modified:

\[
\{q_{ab}(x), p^{cd}(y)\} = \delta^c_a \delta^d_b \delta^3(x - y), \quad \{q_{ab}(x), q_{cd}(y)\} = 0, \quad (14)
\]

while

\[
\{p^{ab}(x), p^{cd}(y)\} = \frac{1}{4} (q^{ac} j^{bd} + q^{ad} j^{bc} + q^{bc} j^{ad} + q^{bd} j^{ac}) \delta^3(x - y), \quad (15)
\]

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where
\[ J^{ab} = \frac{1}{4}(e^{ai}p^b_i - e^{bi}p^a_i) = J^{[ab]}, \] (16)

To preserve the correspondence between Poisson structures, one has to impose the three constraints \( J^{ab} = 0 \), which also ensures the conservation of the number of degrees of freedom (a symmetric tensor \( q_{ab} \) is defined by six numbers at each point, while the triad matrix \( e_a^i \) contains nine independent components). These additional constraints generate \( SO(3) \) rotations (which leave \( q_{ab} \) invariant) and can be represented equivalently in the form (see e.g. Ref. [13])
\[ \mathcal{J}^i = \epsilon^{ijk}p^a_j e_{ab} = 0, \] (17)
where \( \epsilon^{ijk} \) is the totally antisymmetric Levi-Civita symbol (\( \epsilon^{123} = 1 \)).

Thus, the constraint \( \mathcal{J}^i \) implements the condition that \( p_{ab} \), considered now as a derived quantity, is symmetric
\[ p^{ab} = \frac{1}{4}(p^a_i e^b_i + p^b_i e^a_i). \] (18)

In terms of \((e^a_i, p^a_i)\), the Hamiltonian becomes
\[ H = \int \left( N\mathcal{H} + N^a\mathcal{H}_a + N_i\mathcal{J}^i \right) d^3x, \] (19)
where \( \mathcal{H}, \mathcal{H}_a \) are given by (11), (15), with \( q_{ab} \) and \( p^{ab} \) considered here as derived quantities, and we have annexed the additional constraint with the Lagrange multiplier \( N_i \).

Clearly, the choice of \((e^a_i, p^a_i)\) as the canonical variables is not unique. In view of the transition to the Ashtekar variables that we make below, it is more convenient to use the variables \((E^a_i, K_a^i)\) defined by
\[ E^a_i \equiv ee^a_i, \quad K_a^i \equiv K_{ab}e^b_i + J_{ab}e^b_i, \] (20)
where \( K_{ab} = K_{(ab)} \) is the extrinsic curvature, and \( J_{ab} \) is given by (16). Then
\[ \{E^a_i(x), K_b^j(y)\} = \frac{1}{2} \delta^a_b \delta^i_j \delta^3(x - y), \] (21)
\[ \{E^a_i(x), E^b_j(y)\} = 0, \quad \{K_a^i(x), K_b^j(y)\} = 0. \] (22)

In Ref. [1], Ashtekar proposed a transformation that allowed one to represent the density of the gravitational Hamiltonian as a polynomial in canonical variables. The transformation is canonical, up to a surface term. Since we are considering a closed \( \Sigma \) without boundary, this term vanishes.
Ashtekar also introduced a complex parametrization in which the new variables are represented as

\[ A^i_a = \frac{1}{2} \epsilon^{ijk} e^b_k D_a e^j_b + i K^i_a. \]  

(23)

In this parametrization, we have

\[ \{ E^a_i(x), A^j_b(y) \} = i \delta^a_b \delta^j_i \delta^3(x - y), \]  

(24)

\[ \{ E^a_i(x), E^b_j(y) \} = 0, \quad \{ A^i_a(x), A^j_b(y) \} = 0. \]  

(25)

For any two functionals in phase space \( F(E, A), G(E, A) \), the Poisson bracket is thus given by

\[ \{ F, G \} \equiv i \int \left( \frac{\delta F}{\delta E^a_i} \frac{\delta G}{\delta A^a_i} - \frac{\delta F}{\delta A^a_i} \frac{\delta G}{\delta E^a_i} \right) d^3x. \]  

(26)

Changing the variables in the Hamiltonian leads to the expression

\[ H = i \int \left( -\frac{i}{2} \mathcal{N} S + \frac{1}{2} N^a \mathcal{V}_a + N^i \mathcal{G}_i \right) d^3x, \]  

(27)

where

\[ \mathcal{G}_i(A, E) \equiv \mathcal{D}_a E^a_i \equiv i \epsilon^{abc} J_{ab} c^i = 0, \]  

(28)

\[ \mathcal{V}_a(A, E) \equiv E^b_i F^i_{ab} = 0, \]  

(29)

\[ S(A, E) \equiv E^a_i E^b_j F_{abk} \epsilon^{ijk} = 0. \]  

(30)

are the (Gauss, vectorial and scalar) constraints, \( \mathcal{N} = e^{-1} N \) and \( \epsilon^{abc} \) is the totally antisymmetric Levi-Civita symbol \( (\epsilon^{123} = 1) \).

The covariant derivative \( \mathcal{D}_a \) is defined by

\[ \mathcal{D}_a v_i \equiv \partial_a v_i + \frac{1}{2} \epsilon_{ijk} A_a^j v^k. \]  

(31)

The curvature of the connection \( A^i_a \) can be found from

\[ 2 \mathcal{D}_a \mathcal{D}_b v_i = \frac{1}{2} \epsilon_{ijk} F^j_{ab} v^k, \]  

(32)

hence

\[ F^i_{ab} = \partial_a A^i_b - \partial_b A^i_a + \frac{1}{2} \epsilon_{ijk} A_a^j A_b^k. \]  

(33)
The evolution equations for the canonical variables are obtained taking the Poisson bracket of the variables with the Hamiltonian (27), and, neglecting boundary terms, they are given by

\[
\dot{A}_a^i(x) = \{A_a^i, H\} = -iN\epsilon_{ijk}E_b^jF_{ab}^k + \frac{1}{2}N^bF_{ba}^i, \tag{34}
\]

\[
\dot{E}_a^i(x) = \{E_a^i, H\} = iD_b(N\epsilon_{ijk}E^{[a]j}E_b^k) + D_b(N^bE^{[a]}_i). \tag{35}
\]

A simplification is evident in the equations of motion.

IV. THE ALTERNATIVE HAMILTONIAN FORMULATION

In this section, we shall review the alternative Hamiltonian formulation for general relativity that emerges from Ashtekar’s canonical gravity and from the Hamiltonian formulation outlined in the Introduction which is wider than the one derived from the Lagrangian formulation. This Hamiltonian formulation is based in the fact that the time evolution of the field variables \(\phi_\alpha\) can be written in the form \(\dot{}\). Clearly, for a candidate Hamiltonian operator \(D_{\alpha\beta}\) [c.f. Eq. (1)], the correct expression for the corresponding Poisson bracket has the form

\[
\{F, G\} = \int \frac{\delta F}{\delta \phi_\alpha} D_{\alpha\beta} \frac{\delta G}{\delta \phi_\beta} d^3x, \tag{36}
\]

where \(F\) and \(G\) are functionals. Of course, the Hamiltonian operator \(D_{\alpha\beta}\) must satisfy certain further restrictions in order for (36) to be a true Poisson bracket. The condition that \(D_{\alpha\beta} = -D_{\beta\alpha}^\dagger\), where \(D_{\alpha\beta}^\dagger\) is the adjoint of \(D_{\alpha\beta}\) and the bar denotes complex conjugation, is equivalent to the antisymmetry of the Poisson bracket (i.e., \(\{F, G\} = -\{G, F\}\)). The other condition on the Poisson bracket is the Jacobi identity; when the \(D_{\alpha\beta}\) are constants, this condition is automatically satisfied, but in other cases one has to verify that this identity is satisfied [4].

The Poisson bracket (36), while formally correct, fails to incorporate boundary effects, and needs to be slightly modified when discussing solutions over bounded domains (see e.g. Refs. [14] and [15] for this point). However, we have assumed that \(\Sigma\) is compact without boundary here.
Using the fact that
\[ \phi_\alpha(y, t) = \int \delta_\alpha^\beta \delta^3(\mathbf{x} - \mathbf{y}) \phi_\beta(\mathbf{x}, t) d^3x \] (37)
it follows that
\[ \frac{\delta \phi_\alpha(y, t)}{\delta \phi_\beta(\mathbf{x}, t)} = \delta_\alpha^\beta \delta^3(\mathbf{x} - \mathbf{y}), \] (38)

therefore, from Eq. (36), one gets
\[ \{\phi_\alpha(\mathbf{x}, t), \phi_\beta(\mathbf{y}, t)\} = D_\alpha^\beta \delta^3(\mathbf{x} - \mathbf{y}). \] (39)

In the simplest case of the canonical variables \( \phi_\alpha = (q^i, p_i) \), like the ADM and Ashtekar variables, the operator \( D \equiv (D_\alpha^\beta) \) is the antisymmetric matrix
\[ D = \begin{pmatrix} 0 & \delta_i^j \\ -\delta^i_j & 0 \end{pmatrix}. \] (40)

Of course, the ADM and Ashtekar variables have two indices and the fundamental canonical Poisson brackets are given by (6) and (24).

Coming back, now, to the case of general relativity, we wish to use \( E^a_i \) and the “magnetic field” \( B^a_i \) as new variables, rather than \( E^a_i \) and \( A^a_i \). First, we define
\[ B^a_i \equiv \epsilon^{abc} F_{bci} = \epsilon^{abc} \left( 2 \partial_b A_{ci} + \frac{1}{2} \epsilon^{ijk} A_{bj} A_{ck} \right), \] (41)

then one finds that
\[ \dot{B}^a_i = \{B^a_i, H\} = 2 \epsilon^{abc} D_b \dot{A}_{ci}. \] (42)

From Eq. (41) it should be stated that given \( A_{ai} \), one can calculate \( B^a_i \), but, can this relation be inverted? The answer is no in general.

Let us consider the possibility of describing the configuration space of the system using \( B^a_i \) rather than \( A_{ai} \), which will be necessary in order to write the connection from the covariant derivative in terms of \( B^a_i \) and their partial derivatives.

For the non-Abelian theory is generically possible to view the Bianchi identity
\[ D_a B^{ai} = \partial_a B^{ai} + \frac{1}{2} \epsilon^{ijk} A_{aj} B^{ai}_k = 0, \] (43)
as a linear relation between \( B^{ai} \) and \( A_{ai} \) [which is compatible with (41)] to be solved for \( A_{ai} \), thus, \( B^{ai} \) can be used as a variable.

Now, it is possible that two or more gauge inequivalent non-Abelian potentials \( A_{ai} \) generate the same field \( B^{ai} \), which is known as the Wu-Yang ambiguity [16]. But there exist
some examples in the context of $SU(2)$ gauge theories \[17\], in which the correspondence between $A_{ai}$ and $B_{ai}$ modulo gauge is made, but some conditions on $B_{ai}$ are necessary (see also \[16\]). (One condition is that the $3 \times 3$ matrix $B_{ai}$ satisfies $\det B_{ai} \neq 0$.)

Therefore, in what follows we will suppose that it is possible to write $A = A(B)$, (we can consider the conditions on $B_{ai}$ given in \[17\] for $SO(3)$ theories, for instance).

On the other hand, we can express the evolution equations in terms of the variables $E_{ai}$ and $B_{ai}$ only. Equation (35) remains unchanged and Eq. (34) can be rewritten as

$$\dot{A}_{ai}(E, B) = -\frac{i}{2} N \epsilon_{ijk} E_{bj} \epsilon_{abc} B_{ck} + \frac{1}{4} N^b \epsilon_{bac} B^c_i,$$

(44)

where we have used the fact that $2F_{ab}^i = \epsilon_{abc} B_{ci}$ [cf. Eq. (42)]. Thus, we have that [cf. Eq. (42)]

$$2 \epsilon_{abc} D_b \dot{A}_{ci} = -2i \epsilon_{ijk} D_b (N E^{|a| j} B^{b|k}) - D_b (N^{|a} B^{|b} i).$$

(45)

Therefore, we have an alternative set of equations of evolution for the gravitational field equivalent to Eqs. (35) and (34), given by

$$\dot{E}^a_i = i D_b (N \epsilon_{ijk} E^{|a| j} B^{b|k}) + D_b (N^{|b} E^{|a} i),$$

(46)

$$\dot{B}^a_i = -2i D_b (N \epsilon_{ijk} E^{|a| j} B^{b|k}) - D_b (N^{|a} B^{|b} i).$$

(47)

These equations are more symmetric than Eqs. (35) and (34), and in some sense they are analogous to the Maxwell equations. Therefore, in terms of the variables $E^a_i$ and $B^a_i$, which are not canonical, the equations of evolution for vacuum general relativity take an interesting form. However, this is not sufficient. What is needed is a Hamiltonian structure that defines a Poisson bracket and a Hamiltonian which involves the constraints and generates the evolution equations (46) and (47). We also need that the constraint algebra of the constraints closes.

A. Hamiltonian and Hamiltonian structure

In order to express the alternative set of evolution equations in the Hamiltonian form \[11\], we introduce the Hamiltonian

$$H = i \int d^3 x \left( -\frac{i}{2} N S + \frac{1}{2} N^a \mathcal{V}_a + N^i \mathcal{G}_i \right)$$

(48)
where, now,

\[ V_a(E, B) \equiv \frac{1}{2} \epsilon_{abc} E^b_i B^c_i = 0, \]  
\[ S(E, B) \equiv \frac{1}{2} \epsilon_{abc} E^a_i E^b_i B^c_i \epsilon^{ijk} = 0, \]  
\[ G_i(E, B) \equiv D_a E^a_i, \]  

are the constraints. The Hamiltonian \[ (48) \] is the same of Ashtekar \[ cf. Eq. (27) \], but, now, in the Gauss constraint we consider \( A = A(B) \).

On the other hand, Eqs. \((46)\) and \((47)\) can be written in the Hamiltonian form

\[ \dot{E}^a_i = D^{ab}_{ij} \frac{\delta H}{\delta B^b_j}, \quad \dot{B}^a_i = -D^{ab}_{ij} \frac{\delta H}{\delta E^b_j} \]  

where

\[ D^{ab}_{ij} \equiv -2i\epsilon^{abc} D_c \delta_{ij} \equiv -2i\epsilon^{abc} \left( \partial_c \delta_{ij} + \frac{1}{2} \epsilon_{ikl} A^k_c \delta^l_j \right) \]  

(53)

and \( H \) is given by \((48)\).

In this case, the matrix differential operator \( D = (D_{\alpha\beta}) \) \[ cf. Eq. (1) \] can be seen in a schematic form (forgetting for a moment the internal indices \( i, j \)) as

\[ D = (D_{\alpha\beta}) = \begin{pmatrix} 0 & \epsilon^{abc} D_c \\ -\epsilon^{abc} D_c & 0 \end{pmatrix}, \]  

(54)

\( (\alpha, \beta = 1, 2, \ldots, 6 \) here).

Making use of the \( D^{ab}_{ij} \) given by Eq. \((53)\), a Poisson bracket between any pair of functionals of the field \( F(E, B) \) and \( G(E, B) \) can be defined as

\[ \{ F, G \}_n \equiv \int \left( \frac{\delta F}{\delta E^a_i} D^{ab}_{ij} \frac{\delta G}{\delta B^b_j} - \frac{\delta F}{\delta B^a_i} D^{ab}_{ij} \frac{\delta G}{\delta E^b_j} \right) d^3 x, \]  

where the subscript \( n \) \( (\)non canonical variables\( ) \) is introduced to distinguish it from the canonical Poisson bracket.

Integrating by parts one can see that the bracket \((55)\) is antisymmetric up to a surface term; since we are considering a closed \( \Sigma \) without boundary this term vanishes. Equivalently, the matrix differential operator \( D \) is skew-adjoint \[ i.e. D^\dagger = -D \].

In order to prove the Jacobi identity for this Poisson bracket we will use the methods of functional multi-vectors given in Ref. \[ \text{[4]} \]. In such case the Jacobi identity is equivalent to the condition that the functional tri-vector

\[ \Psi \equiv \frac{1}{2} \int [\theta \wedge \text{prv}_{D\theta}(D) \wedge \theta] dv \]  

(56)
vanishes. Here θ is some functional uni-vector and \( \text{pr}_{Dθ}(D) \) is the prolongation of the evolutionary vector field \( v_{Dθ} \) (with characteristic \( Dθ \)) acting on the skew-adjoint differential operator \( D \). For the computation of \( \text{pr}_{Dθ}(D) \) it is necessary to write \( D \) in terms of the field variables, \( E^a_i \) and \( B^a_i \), only, i.e. to write \( A^i_a \) in terms of \( B^a_i \). In any (possible) case, the relation \( A = A(B) \) does not involve differential operators, then \( \text{pr}_{Dθ}(D) \) turns out to be some uni-vector, \( ϑ \), that does not involve differential operators. Thus

\[
Ψ = \frac{1}{2} \int [θ ∧ ϑ ∧ θ]dv = 0
\]  

(by the antisymmetry of the wedge product). Hence, the operators \( D^{ab}_{ij} \) define a Hamiltonian structure, or, equivalently, a Poisson bracket.

Also one can see that the antisymmetry and the Jacobi identity of the Poisson bracket (55) follows from the fact that the Hamiltonian structure is the canonical one, i.e., from the canonical Poisson bracket (26), by using the fact of that

\[
\frac{δ}{δA^i_a} = 2ε^{abc}D_b \frac{δ}{δB^c_i},
\]

which follows from the chain rule, one can see that (integrating by parts)

\[
\{F, G\} = \{F, G\}_n
\]

[c.f. Eqs. (26) and (55)]. Therefore, the Hamiltonian structure is the canonical one, only the variables are new. Thus, in that follows, we will use the subscript \( n \) in order to point out that we are using non canonical variables.

The new variables satisfy the Poisson brackets relations

\[
\{E^a_i(x), E^b_j(y)\}_n = 0, \quad \{B^a_i(x), B^b_j(y)\}_n = 0
\]

and

\[
\{E^a_i(x), B^b_j(y)\}_n = -2iε^{abc}D_c δ_{ij} δ^3(x - y)
\]

\[
= -2iε^{abc} (\partial_c δ_{ij} + 1/2 δ_{ilj} A^l_c ) δ^3(x - y). \tag{61}
\]

The Poisson bracket (55) yields the expected relations between the Hamiltonian and any functional of the field. If \( F(E, B) \) is any functional of the field that does not depend explicitly on the time then Eqs. (55) and (52) give

\[
\{F, H\}_n = \int \left( \frac{δF}{δE^a_i} D^{ab}_{ij} \frac{δH}{δB^b_j} - \frac{δF}{δB^a_i} D^{ab}_{ij} \frac{δH}{δE^b_j} \right) d^3x
\]

\[
= \int \left( \frac{δF}{δE^a_i} \dot{E}^a_i + \frac{δF}{δB^a_i} \dot{B}^a_i \right) d^3x = \dot{F}, \tag{62}
\]

12
\[i.e., H \text{ generates time translations.}\]

**B. Constraint Poisson algebra**

The geometrical interpretation of the constraints is made easier by integrating them over the entire spatial slice as

\[C_{N^i} \equiv \int d^3x N^i \mathcal{G}_i(E, B),\]  

(63)

\[C'_{N^a} \equiv \frac{1}{2} \int d^3x N^a V_a(E, B),\]  

(64)

\[C_N \equiv -\frac{i}{2} \int d^3x N S(E, B).\]  

(65)

First we consider the Gauss constraint. By using Eq. (58) one has that

\[\{E^a_i, C_{N^i}\}_n = \frac{i}{2} \epsilon^{ijk} E^a_j N^k,\]  

\[\{B^a_i, C_{N^i}\}_n = \frac{i}{2} \epsilon^{ijk} B^a_j N^k,\]  

(66)

\[i.e., \text{Gauss constraint generates internal rotations on the variables.}\]

Next, let us consider the vector constraint (64). To obtain the generator of spatial diffeomorphisms, one has to add to Eq. (64) a multiple of Eq. (63). Given a shift vector \(N^a\), let us set

\[C_{N^a} \equiv \frac{1}{2} \int d^3x N^a V_a(E, B) - \frac{1}{2} \int d^3x N^b A_i^a D_a E^a_i.\]  

(67)

Then, by using the Poisson bracket (55) and Eq. (58), one can show that

\[\{E^a_i, C_{N^a}\}_n = -\frac{i}{2} \mathcal{L}_{N^a} E^a_i,\]  

\[\{B^a_i, C_{N^a}\}_n = -\frac{i}{2} \mathcal{L}_{N^a} B^a_i\]  

(68)

where \(\mathcal{L}_{N^a}\) is the Lie derivative, with respect to the vector field \(N^a\). Thus, \(C_{N^a}\) does indeed generate diffeomorphisms along the vector field \(N^a\).

Finally, let us consider the scalar constraint (65). In this case \(C_N\) has the following Poisson bracket relations

\[\{E^a_i, C_N\}_n = D_b (N \epsilon_{ijk} E^{|a|}[i] E^{b|k|}),\]  

\[\{B^a_i, C_N\}_n = -2 D_b (N \epsilon_{ijk} B^{|a|}[i] B^{b|k|})\]  

(69)

[\textit{cf. Eqs. (46) and (47)}].
We can now compute the Poisson bracket algebra. One readily obtains
\[\{C_{N^i}, C_{M^j}\}_n = -\frac{i}{2} C_{\epsilon^{ijk} N^j M^k}\]
\[\{C_{N^a}, C_{M^a}\}_n = \frac{i}{2} C_{\mathcal{L}_{N^a} M^a}\]
\[\{C_{N^a}, C_{N^i}\}_n = \frac{i}{2} C_{\mathcal{L}_{N^a} N^i}\]
\[\{C_{N^a}, C_N\}_n = \frac{i}{2} C_{\mathcal{L}_{N^a} N}\]
\[\{C_N, C_M\}_n = 0\]
\[\{C_N, C_{N^a}\}_n = -i C_K - \frac{i}{2} C_{K^a A^a_i},\]
(70)
where the vector $K$ is defined by $K^a = E^a_i E^b_i (N \partial_b M - M \partial_b N)$. Here we clearly see that the algebra closes. In the Dirac terminology, the algebra is first class.

The counting of the degrees of freedom is done as in the Ashtekar’s case. We have a 18-dimensional phase space. In that space we have seven constraints and we can fix seven gauge conditions. We are therefore left with a four-dimensional constraint-free space, which gives two degrees of freedom. Note that the Bianchi identity, $\mathcal{D}_a B^a_i = 0$, does not reduce the dimension of the phase space, since it is not a constraint.

V. SKETCHING QUANTIZATION

Now we sketch briefly the quantization by using the new Hamiltonian approach (we will not go into the details). In principle, it is entirely straightforward to quantize the theory. However, quantum gravity is still poorly understood, and we will be sketching a program that people hoped would lead to a theory of quantum gravity, but which has technical complications.

Quantization requires us to replace the variables ($E$ and $B$ in this case) by operators that act on the space of states of the theory. The wave-functionals that are annihilated by the constraints are the physical states of the theory. Notice that we do not yet have a Hilbert space. One needs to introduce an inner product on the space of physical states in order to compute expectation values and make physical predictions. We will ignore these questions for now and think of the states as arbitrary functionals $\psi[B]$.

We replace the classical variables $E^a_i$ and $B^a_i$ by the operators
\[\hat{E}^a_i(x)\psi[B] = 2i e^{abc} \hat{D}_b \frac{\delta}{\delta B^c_i (x)} \psi[B],\]
(71)
and

\[ \hat{B}^a_i(x) \psi[B] = B^a_i(x) \psi[B], \quad (72) \]

which have commutation relations

\[
\begin{align*}
[\hat{E}^a_i(x), \hat{E}^b_j(y)] &= 0, \\
[\hat{B}^a_i(x), \hat{B}^b_j(y)] &= 0, \\
[\hat{E}^a_i(x), \hat{B}^b_j(y)] &= 2i\epsilon^{abc}D_c\delta_{ij}\delta^3(x-y),
\end{align*}
\]

\[
(73, 74)
\]

analogous to the classical Poisson brackets relations (60) and (61).

With these operators at hand one can promote the constraints formally to operator equations if one picks a factor ordering. There are two factor orderings which have been explored: with the \(E'\)'s either to the right [19] or the left of the \(B'\)'s.

The problem, now, is to find the physical state space of functions \(\psi[B]\) that satisfy the constraints in quantum form.

For the factor ordering with \(E'\)'s to the left there exist the Chern-Simons state \(\psi_{CS}[A]\) (in Ashtekar representation), which satisfies

\[
\hat{G}_i\psi_{CS} = \hat{V}_a\psi_{CS} = \hat{S}\psi_{CS} = 0,
\]

\[
(75)
\]

whenever the cosmological constant \(\Lambda\) is nonzero. In this case the Gauss and vectorial constraints are unchanged, but the scalar constraint becomes [18]

\[
\hat{S}_\Lambda(E, B) = \frac{1}{2}\epsilon_{abc}\hat{E}^a_i\hat{E}^b_j\hat{E}^c_k\epsilon^{ijk} - \frac{\Lambda}{6}\epsilon_{abc}\hat{E}^a_i\hat{E}^b_j\hat{E}^c_k\epsilon^{ijk} = 0,
\]

\[
(76)
\]

which remains polynomial.

One expects that \(\psi_{CS}\) also satisfy the constraints in the new representation, however one must consider it as function of \(B\), since in its original form it is a function of \(A\). However, we end at this point and leave this problem for a possible next work.

VI. CONCLUSIONS AND PROSPECTS

We have shown that it is possible to write the dynamic equations of general relativity in terms of new variables, which are not canonical. We obtained a Poisson bracket (associated
with the canonical Hamiltonian structure) and it was shown that it yields the expected
relations between the Hamiltonian and any functional of the field. The constraint algebra
was studied in terms of the new field variables. The only disadvantage is that we cannot
write explicitly the connection $A$ in terms of $B$ in general and we are restricted to the cases
which it is possible (see e.g. Ref. [17]).

Usually the ADM formalism is considered as a metric representation and the Ashtekar
formalism as a connection representation; the formalism presented here can be considered
as a curvature representation to describe gravity. However, it is necessary to point out that
in this framework we do not have an action that leads to the new Hamiltonian formulation
of gravity.

We have replaced the new variables by operators, however we do not know at present
whether they could help in gaining information about quantum gravity. This is an interesting
problem for a future work.

Finally, we point out that a similar treatment to that followed here, is also applicable in
Yang-Mills theory [20].

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