Asymptotic behavior of the smallest eigenvalue of matrices associated with completely even functions (mod $r$) *

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Abstract In this paper we present systematically analysis on the smallest eigenvalue of matrices associated with completely even functions (mod $r$). We obtain several theorems on the asymptotic behavior of the smallest eigenvalue of matrices associated with completely even functions (mod $r$). In particular, we get information on the asymptotic behavior of the smallest eigenvalue of the famous Smith matrices. Finally some examples are given to demonstrate the main results.

Keywords: Arithmetic progression; completely even function (mod $r$); tensor product; Dirichlet convolution; Dirichlet’s theorem; Mertens’ theorem; Cauchy’s interlacing inequalities.

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1. Introduction and statements of results

For any given arithmetical function $f$, we denote by $f(m, r)$ the function $f$ evaluated at the greatest common divisor $(m, r)$ of positive integers $m$ and $r$. Cohen [Co2] called the function $f(m, r)$ a completely even function (mod $r$). Let $1 \leq x_1 < \ldots < x_n < \ldots$ be a given arbitrary strictly increasing infinite sequence of positive integers. For any integer $n \geq 1$, let $S_n = \{x_1, \ldots, x_n\}$. Let $I$ be the function defined for any positive integer $m$ by $I(m) := m$. In 1876, Smith [S] published his famous theorem showing that the determinant of the $n \times n$ matrix $[I(x_i, x_j)]$ on $S_n = \{1, \ldots, n\}$ is the product $\prod_{k=1}^{n} \phi(k)$, where $\phi$ is Euler’s totient function. Smith also proved that if $S_n = \{1, \ldots, n\}$, then

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The nonsingularity of the matrix \( f \) is the Dirichlet convolution of \( f \) and \( \mu \). In 1972, Apostol [A2] extended Smith’s result. In 1986, McCarthy [Mc2] generalized Smith’s and Apostol’s results to the class of even functions of \( m \) (mod \( r \)), where \( m \) and \( r \) are positive integers. A complex-valued function \( \beta(m, r) \) is said to be an even function of \( m \) (mod \( r \)) if \( \beta(m, r) = \beta((m, r), r) \) for all values of \( m \) [Co1, Co2]. Clearly a completely even function (mod \( r \)) is an even function of \( m \) (mod \( r \)), but the converse is not true. In 1993, Bourque and Ligh [Bo-L1] extended the results of Smith, Apostol, and McCarthy. In 1999, Hong [Hon3] improved the lower bounds for the determinants of matrices considered by Bourque and Ligh ([Bo-L1]). In 2002, Hong [Hon4] generalized the results of Smith, Apostol, McCarthy and Bourque and Ligh to certain classes of arithmetical functions. Another kind of extension of Smith’s determinant were obtained by Codecá and Nair [CN] and Hilberdink [Hi].

Let \( \varepsilon \) be a real number. Wintner [W] proved in 1944 that \( \lim \sup_{n \to \infty} \Lambda_n(\varepsilon) < \infty \) if and only if \( \varepsilon > 1 \), where \( \Lambda_n(\varepsilon) \) denotes the largest eigenvalue of the matrix \( N_n \) defined as follows:

\[
\lambda_n(\varepsilon) := \left( \frac{(i,j)^{2\varepsilon}}{\varepsilon \cdot j^\varepsilon} \right)_{1 \leq i,j \leq n}.
\]

Let \( \lambda_n(\varepsilon) \) denote the smallest eigenvalue of the matrix \( N_n \). Lindqvist and Seip [LS] in 1998 use the work of [He-L-S] about Riesz bases to investigate the asymptotic behavior of \( \lambda_n(\varepsilon) \) and \( \Lambda_n(\varepsilon) \) as \( n \to \infty \). In particular, they got a sharp bound for \( \lambda_n(\varepsilon) \) and \( \Lambda_n(\varepsilon) \). In 2004, Hong and Loewy [Hon-Lo] made some progress in the study of asymptotic behavior of the eigenvalues of the \( n \times n \) matrix \( (\xi_\varepsilon(x_i, x_j)) \) on \( S_n \), where \( \xi_\varepsilon \) is defined for any positive integer \( m \) by \( \xi_\varepsilon(m) := m^\varepsilon \). It was proved in [Hon-Lo] that if \( 0 < \varepsilon \leq 1 \) and \( q \geq 1 \) is any fixed integer, then the \( q \)-th smallest eigenvalue of the \( n \times n \) matrix \( (\xi_\varepsilon(i, j)) \) defined on the set \( S_n = \{1, \ldots, n\} \) approaches zero as \( n \) tends to infinity. Recently, Hong and Lee [Hon-Le] studied the asymptotic behavior of the eigenvalues of the reciprocal power LCM matrices and made some progress while Hong [Hon11] got some results about asymptotic behavior of the largest eigenvalue of matrices associated with completely even functions (mod \( r \)). Notice also that Bhatia [Bh], Bhatia and Kosaki [Bh-K] and Hong [Hon12] considered infinite divisibility of matrices associated with multiplicative functions.

Given any set \( S \) of positive integers, we define the class \( \tilde{C}_S \) of arithmetical functions by

\[
\tilde{C}_S := \{ f : (f \ast \mu)(d') > 0 \text{ whenever } d'|x, \text{ for any } x \in S \}.
\]

For an arbitrary given strictly increasing infinite sequence \( \{x_i\}_{i=1}^{\infty} \) of positive integers, we define the class \( \tilde{C} \) of arithmetical functions by

\[
\tilde{C} := \{ f : (f \ast \mu)(d') > 0 \text{ whenever } d'|x, \text{ for any } x \in \{x_i\}_{i=1}^{\infty} \}.
\]

Let \( S_n = \{x_1, \ldots, x_n\} \) for any integer \( n \geq 1 \). Then it is clear that if \( f \in \tilde{C} \), then \( f \in \tilde{C}_{S_n} \).

In 1993, Bourque and Ligh ([Bo-L2]) showed that if \( f \in \tilde{C}_{S_n} \), then the matrix \( (f(x_i, x_j)) \) (abbreviated by \( (f(S_n)) \)) is positive definite. Hong ([Hon1]) improved Bourque and Ligh’s bounds for \( \det(f(S_n)) \) if \( f \in \tilde{C}_{S_n} \). In [Hon7] and [Hon9], Hong obtained several results on the nonsingularity of the matrix \( (f(S_n)) \). On the other hand, the \( n \times n \) matrix \( (f[x_i, x_j]) \)
integers containing at least one prime, and \( \mu \) is a completely multiplicative function such that \((f \ast \mu)(d')\) is a nonzero integer whenever \(d'|\lcm(S_n)\), then the matrix \((f(x_i, x_j))\) divides the matrix \((f[x_i, x_j])\) in the ring \(M_n(\mathbb{Z})\) of \(n \times n\) matrices over the integers. Note also that Hong [Hon5] showed that for any \(S_n\) of positive integers, \(\tilde{\mathcal{C}}\) and for any given strictly increasing infinite sequence \(\{x_i\}_{i=1}^\infty\) of positive integers we define the following natural class of arithmetical functions:

\[
\mathcal{C}_S := \{ f : (f \ast \mu)(d') \geq 0 \text{ whenever } d'|x, \text{ for any } x \in S \}.
\]

In the meantime, associated with an arbitrary given strictly increasing infinite sequence \(\{x_i\}_{i=1}^\infty\) of positive integers we define the following natural class of arithmetical functions:

\[
\mathcal{C} := \{ f : (f \ast \mu)(d') \geq 0 \text{ whenever } d'|x, \text{ for any } x \in \{x_i\}_{i=1}^\infty \}.
\]

Then it is easy to see that if \(f \in \mathcal{C}\), then \(f \in \mathcal{C}_S\). Clearly \(\tilde{\mathcal{C}}_S \subset \mathcal{C}_S\) for any given set \(S\) of positive integers, and \(\tilde{\mathcal{C}} \subset \mathcal{C}\) for any given strictly increasing infinite sequence \(\{x_i\}_{i=1}^\infty\) of positive integers. Obviously for any given set \(S\) of positive integers, \(\tilde{\mathcal{C}}_S\) and \(\mathcal{C}_S\) are closed under addition and with respect to Dirichlet convolution, and for any given strictly increasing infinite sequence \(\{x_i\}_{i=1}^\infty\) of positive integers, \(\tilde{\mathcal{C}}\) and \(\mathcal{C}\) are closed under addition and with respect to Dirichlet convolution. Note that \(\mu \notin \mathcal{C}_S\) for any given set \(S\) of positive integers containing at least one prime, and \(\mu \notin \mathcal{C}\) for any given strictly increasing infinite sequence \(\{x_i\}_{i=1}^\infty\) of positive integers containing at least one prime. However, we have the following result (Theorem 1.1 below).

Let \(c \geq 0\) be an integer. For any arithmetical function \(f\), define its \(c\)-th Dirichlet convolution, denoted by \(f^{(c)}\), inductively as follows:

\[
f^{(0)} := \delta \text{ and } f^{(c)} := f^{(c-1)} \ast f \text{ if } c \geq 1,
\]
where \( \delta \) is the function defined for any positive integer \( m \) by

\[
\delta(m) := \begin{cases} 
1, & \text{if } m = 1; \\
0, & \text{otherwise}.
\end{cases}
\]

Note that \( f \ast \delta = f \) for any arithmetical function \( f \) and

\[
f^{(c)} := \underbrace{f \ast \ldots \ast f}_{c \text{ times}}.
\]

For any integer \( c \geq 1 \), let

\[
\mathbb{Z}_{>0}^c := \{ (x_1, \ldots, x_c) : 0 < x_i \in \mathbb{Z}, \text{ for } i = 1, \ldots, c \}.
\]

**Theorem 1.1.** Let \( c \geq 1 \) and \( d \geq 0 \) be integers. If \( f_1, \ldots, f_c \) are distinct arithmetical functions and \((l_1, \ldots, l_c) \in \mathbb{Z}_{>0}^c \), then each of the following is true:

(i). Let \( \{x_i\}_{i=1}^{\infty} \) be any given strictly increasing infinite sequence of positive integers. If \( f_1, \ldots, f_c \in \mathcal{C}_{S_n} \) (resp. \( f_1, \ldots, f_c \in \mathcal{C} \)) and \( l_1 + \ldots + l_c > d \), then we have \( f_1^{(l_1)} \ast \ldots \ast f_c^{(l_c)} \ast \mu^{(d)} \in \mathcal{C}_{S_n} \) (resp. \( f_1^{(l_1)} \ast \ldots \ast f_c^{(l_c)} \ast \mu^{(d)} \in \mathcal{C} \));

(ii). For any prime \( p \), we have

\[
(f_1^{(l_1)} \ast \ldots \ast f_c^{(l_c)} \ast \mu^{(d)})(p) = \sum_{i=1}^{c} l_i f_i(p) f_i(1)^{l_i-1} \prod_{j=1 \atop j \neq i}^{c} f_j(1)^{l_j} - d \prod_{i=1}^{c} f_i(1)^{l_i}.
\]

Furthermore, if \( f_1, \ldots, f_c \) are multiplicative, then we have

\[
(f_1^{(l_1)} \ast \ldots \ast f_c^{(l_c)} \ast \mu^{(d)})(p) = \begin{cases} 
\sum_{i=1}^{c} l_i f_i(p) - d & \text{if } f_i(1) = 1 \text{ for all } 1 \leq i \leq c, \\
0 & \text{if } f_i(1) = 0 \text{ for some } 1 \leq i \leq c.
\end{cases}
\]

We remark that if the condition \( l_1 + \ldots + l_c > d \) is suppressed, then Theorem 1.1 (i) fails to be true. For example, let \( c = l_1 = d = 1 \). Take \( f_1 = \phi \). Then \( \phi \in \mathcal{C}_{S} \) for any given set \( S \) of positive integers and \( \phi \in \mathcal{C} \) for any given strictly increasing infinite sequence \( \{x_i\}_{i=1}^{\infty} \) of positive integers. But we have \( f_1 \ast \mu = \phi \ast \mu \notin \mathcal{C}_{S} \) for any given set \( S \) of positive integers containing at least one even number and \( f_1 \ast \mu \notin \mathcal{C} \) for any given strictly increasing infinite sequence \( \{x_i\}_{i=1}^{\infty} \) of positive integers containing at least one even number because \((\phi \ast \mu^{(2)})(2) = -1\).

Using Theorem 1.1 as well as [Theorem 1, Hon1] and by a continuity argument, we can prove the following result.

**Theorem 1.2.** Let \( c \geq 1 \) and \( d \geq 0 \) be integers and \( S_n = \{x_1, \ldots, x_n\} \) be a set of \( n \) distinct positive integers. If \( f_1, \ldots, f_c \in \mathcal{C}_{S_n} \) are distinct and \((l_1, \ldots, l_c) \in \mathbb{Z}_{>0}^c \) satisfies \( l_1 + \ldots + l_c > d \), then each of the following is true:
on the set of positive integers and let $c, d \geq 1$ be integers and $c \geq 1$ and $d \geq 0$ be integers. Let $(l_1, ..., l_c) \in \mathbb{Z}_{\geq 0}$ satisfy $l_1 + … + l_c > d$ and $f_1, ..., f_c \in \mathcal{C}$ be distinct. In the present paper, we investigate the asymptotic behavior of the $q$-th smallest eigenvalue of the matrix $((f_1^{l_1}) \ast ... \ast f_c^{l_c}) \ast \mu_d(S_n))$. Let $\lambda_n^{(q)}(l_1, ..., l_c, d) \leq \lambda_n^{(q)}(l_1, ..., l_c, d)$ be the eigenvalues of the matrix $((f_1^{l_1}) \ast ... \ast f_c^{l_c}) \ast \mu_d(x_i, x_j)$ defined on the set $S_n = \{x_1, ..., x_n\}$. By Theorem 1.2 (ii) we have

\[ \lambda_n^{(q)}(l_1, ..., l_c, d) \geq 0. \]

But by Cauchy’s interlacing inequalities (see [Hor-J1] and a new proof of it, see [Hw]) we have

\[ \lambda_{n+1}^{(q)}(l_1, ..., l_c, d) \leq \lambda_n^{(q)}(l_1, ..., l_c, d). \]

Thus the sequence $\{\lambda_n^{(q)}(l_1, ..., l_c, d)\}_{n=q}^{\infty}$ is a non-increasing infinite sequence of nonnegative real numbers and so it is convergent. Namely, we have

**Theorem 1.3.** Let $\{x_i\}_{i=1}^{\infty}$ be an arbitrary given strictly increasing infinite sequence of positive integers. Let $c \geq 1$ and $d \geq 0$ be integers and $q \geq 1$ be an arbitrary integer. Let $f_1, ..., f_c \in \mathcal{C}$ be distinct and $(l_1, ..., l_c) \in \mathbb{Z}_{\geq 0}$ satisfy $l_1 + … + l_c > d$. Let $\lambda_n^{(1)}(l_1, ..., l_c, d) \leq \lambda_n^{(0)}(l_1, ..., l_c, d)$ be the eigenvalues of the $n \times n$ matrix $((f_1^{l_1}) \ast ... \ast f_c^{l_c}) \ast \mu_d((x_i, x_j))$ defined on the set $S_n = \{x_1, ..., x_n\}$. Then the sequence $\{\lambda_n^{(q)}(l_1, ..., l_c, d)\}_{n=q}^{\infty}$ converges and

\[ \lim_{n \to \infty} \lambda_n^{(q)}(l_1, ..., l_c, d) \geq 0. \]

Let $\{y_i\}_{i=1}^{\infty}$ be a strictly increasing infinite sequence of positive integers. We say that $f$ is increasing on the sequence $\{y_i\}_{i=1}^{\infty}$ if $f(y_i) \leq f(y_j)$ whenever $1 \leq i < j$. For an arbitrary strictly increasing infinite sequence $\{x_i\}_{i=1}^{\infty}$ of positive integers satisfying that $(x_i, x_j) = x$ for any $i \neq j$, where $x \geq 1$ is an integer, we have the following result.

**Theorem 1.4.** Let $x$ be a positive integer and $\{x_i\}_{i=1}^{\infty}$ be a strictly increasing infinite sequence of positive integers satisfying that for every $i \neq j$, $(x_i, x_j) = x$. Assume that $f \in \mathcal{C}$ and is increasing on the sequence $\{x_i\}_{i=1}^{\infty}$. Let $\lambda_n^{(1)}$ be the smallest eigenvalue of the $n \times n$ matrix $(f(x_i, x_j))$ defined on the set $\{x_1, ..., x_n\}$. Then each of the following holds:

(i) If $f(x) = 0$, or $f (x) > 0$ and $f(x_1) = f(x_2)$, then $\lambda_n^{(1)} = f(x_1) - f(x)$;
(ii). If \( f(x) > 0 \) and \( f(x_1) < f(x_2) \), then
\[
f(x_1) - f(x) < \lambda_n^{(1)} < f(x_1) - f(x) + \frac{f(x)}{1 + \sum_{i=2}^{n} f(x_i) - f(x_1)};
\]

(iii). If \( f(x_1) = 0 \) then \( \lambda_n^{(1)} = 0 \) for all \( n \geq 1 \). If \( f(x_1) > 0 \) and \( \sum_{i=1}^{\infty} \frac{1}{f(x_i)} = \infty \), then we have \( \lim_{n \to \infty} \lambda_n^{(1)} = f(x_1) - f(x) \).

From Theorem 1.4 we can deduce the following result.

**Theorem 1.5.** Let \( x \) be a positive integer. Let \( \{x_i\}_{i=1}^{\infty} \) be a strictly increasing infinite sequence of positive integers satisfying the following conditions:

(i). For every \( i \neq j \), \( (x_i, x_j) = x \);

(ii). \( \sum_{i=1}^{\infty} \frac{1}{x_i} = \infty \).

Let \( \lambda_n^{(1)} \) be the smallest eigenvalue of the \( n \times n \) matrix \((f(x_i, x_j))\) defined on the set \( S_n = \{x_1, \ldots, x_n\} \). If \( f \in C \) and is increasing on the sequence \( \{x_i\}_{i=1}^{\infty} \) and \( f(x_i) \leq Cx_i \) for all \( i \geq 1 \), where \( C > 0 \) is a constant, then we have \( \lim_{n \to \infty} \lambda_n^{(1)} = f(x_1) - f(x) \).

Let \( b \geq 1 \) be an integer. By the well-known Dirichlet’s theorem (see, for example, [A1] or [I-R]) there are infinitely many primes in the arithmetic progression \( \{1+bi\}_{i=0}^{\infty} \). In the following let
\[
p_1(b) < \ldots < p_n(b) < \ldots
\]

(1-1)
denote the primes in this arithmetic progression. Consequently, for the arithmetic progression case, we have the following result.

**Theorem 1.6.** Let \( a, b, c, q \geq 1 \) and \( d, e \geq 0 \) be any given integers. Let \( x_i = a + b(e+i-1) \) for \( i \geq 1 \). Let \( (l_1, \ldots, l_c) \in \mathbb{Z}_{\geq 0}^c \) satisfy \( l_1 + \ldots + l_c > d \). Let \( f_1, \ldots, f_c \in C \) be distinct, multiplicative and increasing on the sequence \( \{p_i(b)\}_{i=1}^{\infty} \), where \( p_i(b) \) (\( i \geq 1 \)) is defined by \( (1-1) \). Let \( \lambda_n^{(1)}(l_1, \ldots, l_c, d) \leq \ldots \leq \lambda_n^{(n)}(l_1, \ldots, l_c, d) \) be the eigenvalues of the \( n \times n \) matrix \( \left((f_1^{(l_1)} \ast \ldots \ast f_c^{(l_c)} \ast (a+bi, a+bj))\right) \) defined on the set \( \{a+be, a+b(e+1), \ldots, a+b(e+n-1)\} \).

(i). If \( (f_1^{(l_1)} \ast \ldots \ast f_c^{(l_c)} \ast (a+bi)) \) is defined on the set \( \{a+be, a+b(e+1), \ldots, a+b(e+n-1)\} \).

(ii). \( (f_1^{(l_1)} \ast \ldots \ast f_c^{(l_c)} \ast (a+bi)) \) is defined on the set \( \{a+be, a+b(e+1), \ldots, a+b(e+n-1)\} \).

(i). If \( (f_1^{(l_1)} \ast \ldots \ast f_c^{(l_c)} \ast (a+bi)) \) is defined on the set \( \{a+be, a+b(e+1), \ldots, a+b(e+n-1)\} \).

(ii). If \( (f_1^{(l_1)} \ast \ldots \ast f_c^{(l_c)} \ast (a+bi)) \) is defined on the set \( \{a+be, a+b(e+1), \ldots, a+b(e+n-1)\} \).

(iii). If \( \lambda_n^{(1)}(l_1, \ldots, l_c, d) \leq \lambda_n^{(n)}(l_1, \ldots, l_c, d) \), then for any large enough \( n \) we have \( \lambda_n^{(q)}(l_1, \ldots, l_c, d) = 0 \);

(iii). In particular, if for each \( 1 \leq j \leq c \), there is a positive constant \( C_j \) such that \( f_j(p_i(b)) \leq C_j p_i(b) \) for all \( i \geq 1 \), then we have \( \lim_{n \to \infty} \lambda_n^{(q)}(l_1, \ldots, l_c, d) = 0 \).

Furthermore, applying again Cauchy’s interlacing inequalities, it follows from Theorems 1.3 and 1.6 that the following result holds.

**Theorem 1.7.** Let \( a, b, c, q \geq 1 \) and \( d, e \geq 0 \) be any given integers. Let \( \{x_i\}_{i=1}^{\infty} \) be any given strictly increasing infinite sequence of positive integers which contains the arithmetic
progression \{a + bi\}_{i=1}^{\infty} as its subsequence. Let \(l_1, ..., l_c \in \mathbb{Z}_{>0}\) satisfy \(l_1 + ... + l_c > d\). Let \(f_1, ..., f_c \in \mathcal{C}\) be distinct, multiplicative and increasing on the sequence \(\{p_i(b)\}_{i=1}^{\infty}\), where \(p_i(b)\) \((i \geq 1)\) is defined by (1-1). Let \(\lambda_n^{(1)}(l_1, ..., l_c, d) \leq ... \leq \lambda_n^{(m)}(l_1, ..., l_c, d)\) be the eigenvalues of the \(n \times n\) matrix \(((f_1^{(l_1)} \ast ... \ast f_c^{(l_c)} \ast \mu^{(d)})(x_i, x_j))\) defined on the set \(S_n = \{x_1, ..., x_n\}\).

(i). If \((f_1^{(l_1)} \ast ... \ast f_c^{(l_c)} \ast \mu^{(d)})(p_i(b)) = 0\) for some \(i \geq 1\), then for any large enough \(n\) we have \(\lambda_n^{(q)}(l_1, ..., l_c, d) = 0\);

(ii). If \((f_1^{(l_1)} \ast ... \ast f_c^{(l_c)} \ast \mu^{(d)})(p_i(b)) \neq 0\) for all \(i \geq 1\) and \(\sum_{i=1}^{\infty} \frac{1}{f_1(p_i(b))^{*} + ... + f_c(p_i(b))} = \infty\), then we have \(\lim_{n \to \infty} \lambda_n^{(q)}(l_1, ..., l_c, d) = 0\);

(iii). In particular, if for each \(1 \leq j \leq c\), there is a positive constant \(C_j\) such that \(f_j(p_i(b)) \leq C_j p_i(b)\) for all \(i \geq 1\), then we have \(\lim_{n \to \infty} \lambda_n^{(q)}(l_1, ..., l_c, d) = 0\).

As a special case we have the following theorem.

**Theorem 1.8.** Let \(a, b, c, q \geq 1\) and \(d, e \geq 0\) be any given integers such that \(c \geq d\). Let \(\{x_i\}_{i=1}^{\infty}\) be any given strictly increasing infinite sequence of positive integers which contains the arithmetic progression \(\{a + bi\}_{i=1}^{\infty}\) as its subsequence. Let \(\lambda_n^{(1)}(c, d) \leq ... \leq \lambda_n^{(m)}(c, d)\) be the eigenvalues of the \(n \times n\) matrix \(((f^{(c)} \ast \mu^{(d)})(x_i, x_j))\) defined on the set \(S_n = \{x_1, ..., x_n\}\). Let \(f \in \mathcal{C}\) be multiplicative and increasing on the sequence \(\{p_i(b)\}_{i=1}^{\infty}\), where \(p_i(b)\) \((i \geq 1)\) is defined by (1-1).

(i). If \((f^{(c)} \ast \mu^{(d)})(p_i(b)) = 0\) for some \(i \geq 1\), then for any large enough \(n\) we have \(\lambda_n^{(q)}(c, d) = 0\);

(ii). If \((f^{(c)} \ast \mu^{(d)})(p_i(b)) \neq 0\) for all \(i \geq 1\) and \(\sum_{i=1}^{\infty} \frac{1}{f^{(c)}(p_i(b))^{*} + f^{(d)}(p_i(b))} = \infty\), then for any given integer \(q \geq 1\), we have \(\lim_{n \to \infty} \lambda_n^{(q)}(c, d) = 0\);

(iii). In particular, if \(f(p_i(b)) \leq C p_i(b)\) for all \(i \geq 1\), where \(C > 0\) is a constant, then for any given integer \(q \geq 1\), we have \(\lim_{n \to \infty} \lambda_n^{(q)}(c, d) = 0\).

**Corollary 1.9.** Let \(\lambda_n^{(1)} \leq ... \leq \lambda_n^{(m)}\) be the eigenvalues of the \(n \times n\) matrix \((f(i, j))\) defined on the set \(S_n = \{1, ..., n\}\). If \(f\) is an increasing multiplicative function satisfying \((f \ast \mu)(y) \geq 0\) and \(f(y) \leq C y\) for all positive integers \(y\), where \(C > 0\) is a constant, then for any given integer \(q \geq 1\), we have \(\lim_{n \to \infty} \lambda_n^{(q)}(c, d) = 0\).

This paper is organized as follows. The details of the proofs of Theorems 1.1-1.2 and 1.4-1.6 will be given in Section 2. In Section 3 we give some examples to illustrate our results. The final section is devoted to some open questions.

Throughout this paper, we let \(E_n\) denote the \(n \times n\) matrix with all entries equal to 1. For the basic facts about arithmetical functions, the readers are referred to [A1], [N] or [Mc1]. For a comprehensive review of papers related to the matrices associated with arithmetical functions not presented here, we refer to [Hon-Le] and [Hon-Lo] as well as the papers listed there.

2. The proofs of Theorems 1.1-1.2 and 1.4-1.6
First we prove Theorem 1.1.

Proof of Theorem 1.1. Clearly to prove Theorem 1.1 it suffices to prove that for any prime \( p \) and for any integer \( l \geq 1 \) and any (not necessarily distinct) arithmetical functions \( g_1, \ldots, g_l \), we have

\[
(g_1 * \ldots * g_l * \mu^{(d)})(p) = \sum_{i=1}^{l} g_1(1) \ldots g_{i-1}(1) g_i(p) g_{i+1}(1) \ldots g_l(1) - d g_1(1) \ldots g_l(1),
\]  

(2-1)

and if \( g_1, \ldots, g_l \in \mathcal{C}_S \) (resp. \( g_1, \ldots, g_l \in \mathcal{C} \)) and \( l > d \), then we have

\[
g_1 * \ldots * g_l * \mu^{(d)} \in \mathcal{C}_S \) (resp. \( g_1 * \ldots * g_l * \mu^{(d)} \in \mathcal{C} \)).
\]

(2-2)

Furthermore, if \( g_1, \ldots, g_l \) are multiplicative, then we have

\[
(g_1 * \ldots * g_l * \mu^{(d)})(p) = \begin{cases} 
\sum_{i=1}^{l} g_i(p) - d & \text{if } g_i(1) = 1 \text{ for all } 1 \leq i \leq l; \\
0 & \text{if } g_i(1) = 0 \text{ for some } 1 \leq i \leq l.
\end{cases}
\]

(2-3)

By the definition of Dirichlet convolution we have

\[
\begin{align*}
(g_1 * \ldots * g_l * \mu^{(d)})(p) &= \sum_{r_1 \ldots r_l = p} g_1(r_1) \ldots g_l(r_l) \mu(\bar{r}_1) \ldots \mu(\bar{r}_d) \\
&= \sum_{l=1}^{d} g_1(1) \ldots g_{i-1}(1) g_i(p) g_{i+1}(1) \ldots g_l(1) \mu(1)^d + d g_1(1) \ldots g_l(1) \mu(p) \mu(1)^{d-1} \\
&= \sum_{l=1}^{d} g_1(1) \ldots g_{i-1}(1) g_i(p) g_{i+1}(1) \ldots g_l(1) - d g_1(1) \ldots g_l(1).
\end{align*}
\]

So (2-1) is proved. Further, if \( f \) is multiplicative, then we have \( f(1)^2 = f(1) \). So we have \( f(1) = 1 \), or 0. Thus (2-3) follows immediately.

Now consider (2-2). Since the proof for the case \( g_1, \ldots, g_l \in \mathcal{C} \) is completely similar to that of the case \( g_1, \ldots, g_l \in \mathcal{C}_S \), we only need to show (2-2) for the case \( g_1, \ldots, g_l \in \mathcal{C}_S \).

In the following let \( g_1, \ldots, g_l \in \mathcal{C}_S \) and \( l > d \). Now for any \( x \in S \) and any \( r \mid x \), since \( l \geq d + 1 \), we have

\[
((g_1 * \ldots * g_l * \mu^{(d)})) \mu)(r) = (g_1 * \ldots * g_l * \mu^{(d+1)})(r) = ((g_1 * \mu) * \ldots *(g_{d+1} * \mu) * g_{d+2} \ldots * g_l)(r) = \sum_{r_1 \ldots r_l = x} (g_1 * \mu)(r_1) \ldots (g_{d+1} * \mu)(r_{d+1}) g_{d+2}(r_{d+2}) \ldots g_l(r_l)
\]

(2-4)

For \( 1 \leq i \leq d + 1 \), since \( g_i \in \mathcal{C}_S \) and \( r_i \mid x \), we have

\[
(g_i * \mu)(r_i) \geq 0.
\]

(2-5)
On the other hand, for \( d + 2 \leq j \leq l \), \( g_j \in \mathcal{C}_{S_n} \) together with \( r_j | x \) implies that
\[
g_j(r_j) = \sum_{d' | r_j} (g_j * \mu)(d') \geq 0. \tag{2 - 6}
\]

From (2-4)-(2-6) we then deduce that
\[
((g_1 * ... * g_l * \mu(d)) * \mu)(r) \geq 0.
\]

Thus (2-2) holds. This completes the proof of Theorem 1.1. \( \square \)

To prove Theorem 1.2 we need a result from [Hon1].

**Lemma 2.1.** ([Theorem 1, Hon1]) Let \( S_n = \{x_1, ..., x_n\} \) be a set of \( n \) distinct positive integers. If \( g \in \tilde{\mathcal{C}}_{S_n} \), then we have
\[
\det(g(x_i, x_j)) \geq \prod_{k=1}^{n} \sum_{d' | x_k} (g * \mu)(d') \geq \prod_{k=1}^{n} f(x_k).
\]

We can now prove Theorem 1.2.

**Proof of Theorem 1.2.** By Theorem 1.1 (i), to show Theorem 1.2 we only need to show that if \( f \in \mathcal{C}_{S_n} \), then each of the following is true:
\[
(i'). \quad \prod_{k=1}^{n} \sum_{d' | x_k} (f * \mu)(d') \leq \det(f(x_i, x_j)) \leq \prod_{k=1}^{n} f(x_k);
\]
\[
(ii'). \quad \text{The } n \times n \text{ matrix } (f(x_i, x_j)) \text{ is positive semi-definite.}
\]

First we show the inequality on the left-hand side of (i'). Let \( f \in \mathcal{C}_{S_n} \). Pick \( \epsilon > 0 \) and \( \bar{f} \in \mathcal{C}_{S_n} \). Then it is easy to see that \( f + \epsilon \bar{f} \in \tilde{\mathcal{C}}_{S_n} \). For an arithmetical function \( g \) and \( 1 \leq k \leq n \), let
\[
\alpha_g(x_k) := \sum_{d' | x_k} (g * \mu)(d') \geq \sum_{d' | x_k} \sum_{x_t, x_t < x_k} (g * \mu)(d') \geq \sum_{d' | x_k} \sum_{x_t, x_t < x_k} (g * \mu)(d')
\]

By Lemma 2.1 we have
\[
\det((f + \epsilon \bar{f})(x_i, x_j)) \geq \prod_{k=1}^{n} \alpha_g(x_k). \tag{2 - 7}
\]

Note that both sides of (2-7) are polynomials in \( \epsilon \). Moreover, the constant coefficients of the left and right hand sides are, respectively, \( \det(f(x_i, x_j)) \) and \( \prod_{k=1}^{n} \alpha_f(x_k) \). Since (2-7) holds for any \( \epsilon > 0 \), letting \( \epsilon \to 0 \) the left-hand side of (i') is proved.

For any \( 1 \leq l \leq n \), since \( f \in \mathcal{C}_{S_n} \), then the inequality on the left-hand side of (i') implies that the determinant of any principal submatrix of \( (f(x_i, x_j)) \) is nonnegative. This concludes part (ii'). From (ii') the inequality on the right-hand side of (i') follows.
immediately. Hence the proof of Theorem 1.2 is complete. \qed

The following result is known.

**Lemma 2.2.** Let $n \geq 1$ be an integer and let $a_1, \ldots, a_n \in R$, where $R$ is an arbitrary commutative ring. Then we have

$$\det(E_n + \text{diag}(a_1 - 1, \ldots, a_n - 1)) = \prod_{i=1}^{n}(a_i - 1) + \sum_{1 \leq i_1 < \ldots < i_{n-1} \leq n} \prod_{j=1}^{n-1}(a_{i_j} - 1).$$

In order to show Theorem 1.4 we need also the following lemma.

**Lemma 2.3.** Let $\{r_i\}_{i=1}^{\infty}$ be an increasing infinite sequence of real numbers satisfying $r_1 \geq 1$ and let $\lambda_n^{(1)}$ be the smallest eigenvalue of the $n \times n$ matrix $E_n + \text{diag}(r_1 - 1, \ldots, r_n - 1)$. Then each of the following holds:

(i). If $r_1 = r_2$, then $\lambda_n^{(1)} = r_1 - 1$;

(ii). If $r_1 < r_2$, then

$$r_1 - 1 < \lambda_n^{(1)} < r_1 - 1 + \frac{1}{1 + \sum_{i=2}^{n} \frac{1}{r_i - r_1}};$$

(iii). If $\sum_{i=1}^{\infty} \frac{1}{r_i} = \infty$, then $\lim_{n \to \infty} \lambda_n^{(1)} = r_1 - 1$.

**Proof.** Clearly part (iii) follows immediately from parts (i) and (ii). In what follows we show parts (i) and (ii).

Write

$$F_n := E_n + \text{diag}(r_1 - 1, \ldots, r_n - 1).$$

Note that $F_n$ is positive semi-definite. Consider its characteristic polynomial $\det(\lambda I_n - F_n)$. By Lemma 2.2 we have

$$(-1)^n \det(\lambda I_n - F_n) = \det(E_n + \text{diag}(r_1 - 1, \ldots, r_n - 1))$$

$$= \prod_{i=1}^{n}(r_i - \lambda - 1) + \sum_{1 \leq i_1 < \ldots < i_{n-1} \leq n} \prod_{j=1}^{n-1}(r_{i_j} - \lambda - 1).$$

(2-8)

We then deduce that if $\lambda < r_1 - 1$, then

$$(-1)^n \det(\lambda I_n - F_n) > 0$$

and thus

$$\det(\lambda I_n - F_n) \neq 0.$$ 

So we have $\lambda_n^{(1)} \geq r_1 - 1$.

If $r_1 = r_2$, then by (2-8) we have

$$(\lambda - r_1 + 1)\det(\lambda I_n - F_n).$$
It follows that $\lambda_n^{(1)} = r_1 - 1$ and this concludes part (i).

Now let $r_2 > r_1$. From (2-8) we deduce

$$(-1)^n \det((r_1 - 1)I_n - F_n) > 0.$$ 

This implies that $\lambda_n^{(1)} > r_1 - 1$. On the other hand, we have

$$F_n = (r_1 - 1)I_n + E_n + \text{diag}(0, r_2 - r_1, ..., r_n - r_1).$$

Let $\tilde{\lambda}_n^{(1)}$ be the smallest eigenvalue of the $n \times n$ matrix $E_n + \text{diag}(0, r_2 - r_1, ..., r_n - r_1)$. Then we have

$$\lambda_n^{(1)} = r_1 - 1 + \tilde{\lambda}_n^{(1)}.$$

(2-9)

Since $r_1 < r_2$, the proofs of Lemma 2.2 and Corollary 2.3 of [Hon-Lo] yield

$$\tilde{\lambda}_n^{(1)} < \frac{1}{1 + \sum_{i=2}^{n} \frac{1}{r_i - r_1}}.$$  

(2-10)

So the right-hand side of the inequalities in part (ii) follows immediately from (2-9) and (2-10). The proof of Lemma 2.3 is complete. □

We are now ready to prove Theorem 1.4.

**Proof of Theorem 1.4.** By $f \in C$ we have

$$f(x) = \sum_{d | x} (f \ast \mu)(d) \geq 0.$$ 

If $f(x) = 0$, then $(f(x_i, x_j)) = \text{diag}(f(x_1), ..., f(x_n))$. Since $f$ is increasing on the sequence $\{x_i\}_{i=1}^{\infty}$, we have $f(x_i) \geq f(x_1)$ for $1 \leq i \leq n$. Thus $\lambda_n^{(1)} = f(x_1)$. So Theorem 1.4 (i) is true in this case. Now let $f(x) \neq 0$, so $f(x) > 0$. Obviously we have

$$\frac{1}{f(x)}(f(x_i, x_j)) = E_n + \text{diag}\left(\frac{f(x_1)}{f(x)} - 1, ..., \frac{f(x_n)}{f(x)} - 1\right).$$

(2-11)

For $1 \leq i \leq n$, let $r_i = \frac{f(x_i)}{f(x)}$. Since $f$ is increasing on the sequence $\{x_i\}_{i=1}^{\infty}$, $\{r_i\}_{i=1}^{\infty}$ is an increasing infinite sequence of real numbers. Since $x | x_1$ and $f \in C$, we have

$$f(x_1) - f(x) = \sum_{d | x_1, d \neq 1} (f \ast \mu)(d) \geq 0.$$ 

So $f(x_1) \geq f(x)$, namely, $r_1 \geq 1$. Let $\bar{\lambda}_n^{(1)}$ be the smallest eigenvalue of the $n \times n$ matrix $\frac{1}{f(x)}(f(x_i, x_j))$ defined on the set $S_n = \{x_1, ..., x_n\}$.

Suppose first $f(x_1) = f(x_2)$. Then $r_1 = r_2$. Thus by (2-11) and Lemma 2.3 (i) we have

$$\bar{\lambda}_n^{(1)} = \frac{f(x_1)}{f(x)} - 1.$$
Theorem 1.4 (i) in this case then follows immediately from the fact that

\[ \lambda_n^{(1)} = f(x) \cdot \bar{\lambda}_n^{(1)}. \]  

(2 - 12)

This completes the proof of Theorem 1.4 (i).

Let now \( f(x_1) < f(x_2) \), i.e. \( r_1 < r_2 \). By (2-11) and Lemma 2.3 (ii) we have

\[ \frac{f(x_1)}{f(x)} - 1 < \bar{\lambda}_n^{(1)} < \frac{f(x_1)}{f(x)} - 1 + \frac{1}{1 + \sum_{i=2}^{n} f(x_i)} \]

So, by (2-12) part (ii) of Theorem 1.4 follows.

Finally we show part (iii). If \( f(x_1) = 0 \), then we have \( f(x) = 0 \) because \( f(x_1) \geq f(x) \geq 0 \). Then by part (i) we have \( \lambda_n^{(1)} = 0 \) for \( n \geq 1 \). Thus Theorem 1.4 (iii) holds in this case. If \( f(x_1) > 0 \) and \( \sum_{i=1}^{\infty} \frac{1}{f(x_i)} = \infty \), then part (iii) in this case follows immediately from parts (i) and (ii). So part (iii) of Theorem 1.4 is proved. \( \square \)

Proof of Theorem 1.5. If \( f(x_1) = 0 \), then by Theorem 1.4 (iii) we have \( \lambda_n^{(1)} = 0 \) for all \( n \geq 1 \). So Theorem 1.5 is true in this case. Now let \( f(x_1) > 0 \). Then \( f(x_i) \geq f(x_1) > 0 \) for all \( i \geq 1 \) because \( f \) is increasing on the sequence \( \{x_i\}_{i=1}^{\infty} \). Since \( f(x_i) \leq Cx_i \) for all \( i \geq 1 \), we have \( 0 < f(x_i) \leq Cx_i \) and so \( \frac{1}{f(x_i)} \geq \frac{1}{Cx_i} \) for all \( i \geq 1 \). But by condition (ii), \( \sum_{i=1}^{\infty} \frac{1}{x_i} = \infty \). Thus we have \( \sum_{i=1}^{\infty} \frac{1}{f(x_i)} = \infty \). The result in this case then follows immediately from Theorem 1.4 (iii). \( \square \)

Definition. ([Hon-Lo]) Let \( e \) and \( r \) be positive integers. Let \( X = \{x_1, ..., x_e\} \) and \( Y = \{y_1, ..., y_r\} \) be two sets of distinct positive integers. Then we define the tensor product (set) of \( X \) and \( Y \), denoted by \( X \circ Y \), by

\[ X \circ Y := \{x_1y_1, ..., x_1y_r, x_2y_1, ..., x_2y_r, ..., x_ey_1, ..., x_ey_r\}. \]

Lemma 2.4. Let \( f \) be a multiplicative function. Let \( e \) and \( r \) be positive integers. Let \( X = \{x_1, ..., x_e\} \) be a set of \( e \) distinct positive integers such that for any \( 1 \leq i \neq j \leq e \), \( (x_i, x_j) = 1 \). Let \( Y = \{y_1, ..., y_r\} \) be a set of \( r \) distinct positive integers such that for any \( 1 \leq i \neq j \leq r \), \( (y_i, y_j) = 1 \). Assume that for all \( 1 \leq i \leq e, 1 \leq j \leq r, \ (x_i, y_j) = 1 \). Then the following equality holds:

\[ (f(X \circ Y)) = (f(X)) \otimes (f(Y)). \]

Proof. Since \( f \) is multiplicative, we have \( f(1) = 0 \) or \( f(1) = 1 \). If \( f(1) = 0 \), then \( f(z) = 0 \) for every integer \( z \geq 1 \) because \( f \) is multiplicative. Hence we have \( (f(X \circ Y)) = (f(X)) \otimes (f(Y)) = O_{er} \), the \( er \times er \) zero matrix. So the result holds in this case. Assume now that \( f(1) = 1 \). Then we have

\[ (f(X)) = \begin{pmatrix} f(x_1) & 1 & \ldots & 1 \\ 1 & f(x_2) & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & f(x_e) \end{pmatrix} \]

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and
\[ (f(Y)) = \begin{pmatrix} f(y_1) & 1 & \ldots & 1 \\ 1 & f(y_2) & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & f(y_r) \end{pmatrix}. \]

Since \( f \) is multiplicative, we deduce that
\[ f(x_1y_1, x_2y_2) = \begin{cases} f(x_{i_1})f(y_{j_1}) & \text{if } i_1 = i_2 \text{ and } j_1 = j_2, \\ f(y_{j_1}) & \text{if } i_1 \neq i_2 \text{ and } j_1 = j_2, \\ f(x_{i_1}) & \text{if } i_1 = i_2 \text{ and } j_1 \neq j_2, \\ 1 & \text{if } i_1 \neq i_2 \text{ and } j_1 \neq j_2. \end{cases} \]

Thus letting \( Y_f = (f(Y)) \) gives
\[ (f(X \odot Y)) = \begin{pmatrix} f(x_1)Y_f & Y_f & \ldots & Y_f \\ Y_f & f(x_2)Y_f & \ldots & Y_f \\ \vdots & \vdots & \ddots & \vdots \\ Y_f & Y_f & \ldots & f(x_e)Y_f \end{pmatrix} = \begin{pmatrix} f(x_1) & 1 & \ldots & 1 \\ 1 & f(x_2) & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & f(x_e) \end{pmatrix} \otimes Y_f = (f(X)) \otimes (f(Y)) \]
as required. \( \square \)

**Remark.** If \( f \) is not multiplicative, then Lemma 2.2 may fail to be true. For instance, let \( X = \{1, 2\} \) and \( Y = \{3, 5\} \). Then \( X \odot Y = \{3, 5, 6, 10\} \). Let \( f \) be the arithmetical function defined by \( f(l) = l \) for \( l \neq 10 \) and \( f(10) = 9 \). Then \( f \) is not multiplicative since \( f(10) \neq f(2)f(5) \). On the other hand, we have \((f(X)) \otimes (f(Y)))_{44} = 10 \) and \((f(X \odot Y)))_{44} = 9 \). This implies that \((f(X \odot Y)) \neq (f(X)) \otimes (f(Y)) \).

We are now in a position to prove Theorem 1.6.

**Proof of Theorem 1.6.** First we prove part (i). For convenience we let \( h := f_1^{(l_1)} \ast \ldots \ast f_c^{(l_c)} \ast \mu^{(d)} \) and \( h(p_i(b)) = 0 \) for some \( i \geq 1 \). Then \( (p_i(b), a+be) = 1 \) or \( (p_i(b), a+b(e+1)) = 1 \). Otherwise we have \( p_i(b) \mid (a+be) \) and \( p_i(b) \mid (a+b(e+1)) \). It implies \( p_i(b) \mid b \) and so \( p_i(b) \leq b \). This is absurd since \( p_i(b) \geq 1+b \). We may let \( (p_i(b), a+b(e+j)) = 1 \), where \( j = 0, \text{ or } 1 \). For any integer \( m \geq q \), let \( \nu_m^{(l)}(l_1, \ldots, l_c, d) \leq \ldots \leq \nu_m^{(m)}(l_1, \ldots, l_c, d) \) be the eigenvalues of the \( m \times m \) matrix \( (h(V_m)) \) defined on the set

\[ V_m := \{(a+b(e+j))p_i(b), (a+b(e+j))p_i(b)p_{i+w+1}(b), \ldots, (a+b(e+j))p_i(b)p_{i+w+m-1}(b)\}, \]

where \( w \geq 0 \) and \( a+b(e+j) < p_{i+w+1}(b) < \ldots < p_{i+w+m-1}(b) \). Clearly \( h \) is multiplicative since \( f_1, \ldots, f_c \) and \( \mu \) are multiplicative. For each \( 1 \leq l \leq m-1 \), since \( p_i(b), a+b(e+j) \)
and \( p_{i+w+l}(b) \) are mutually coprime, and note also that \( h(p_i(b)) = 0 \), we have \( h((a + b(e + j))p_i(b)p_{i+w+l}(b)) = h(a + b(e + j))h(p_i(b))h(p_{i+w+l}(b)) = 0 \). Thus we have \( h(V) = O_{m \times m} \), the \( m \times m \) zero matrix. So we have \( \nu_m^{(q)}(l_1, \ldots, l_c, d) = 0 \) for all \( 1 \leq i \leq m \). But by Cauchy’s interlacing inequalities we have for any large enough \( n \),

\[
\lambda_n^{(q)}(l_1, \ldots, l_c, d) \leq \nu_m^{(q)}(l_1, \ldots, l_c, d).
\]

On the other hand, Theorem 1.2 (ii) gives \( \lambda_n^{(q)}(l_1, \ldots, l_c, d) \geq 0 \). So we have \( \lambda_n^{(q)}(l_1, \ldots, l_c, d) = 0 \). This completes the proof of part (i) of Theorem 1.6.

From now on we assume that \( h(p_i(b)) \neq 0 \) for all \( i \geq 1 \). Next we prove Theorem 1.6 (ii) for the case \( l_1 = c = 1 \) and \( d = 0 \). Then we have \( h = f_1 \).

Let \( \{1 + bt_i\}_{i=0}^\infty \) be the sequence consisting of all those elements in the sequence \( \{1 + bi\}_{i=0}^\infty \) which are coprime to \( a + be \). So \( 1 + bt_i, a + be \) = 1 for all \( i \geq 0 \). Then this is an infinite sequence because it contains the set of all primes strictly greater than \( a + be \) in \( \{1 + bi\}_{i=1}^\infty \), which is infinite by Dirichlet’s theorem. For the arithmetic progression \( \{a + bi\}_{i=e}^\infty \), consider its subsequence \( \{a + b(e + (a + be)t_i)\}_{i=0}^\infty = \{(a + be)(1 + bt_i)\}_{i=0}^\infty \).

For any integer \( m \geq 1 \), let \( \gamma^{(1)}_m \leq \ldots \leq \gamma^{(m)}_m \) be the eigenvalues of the \( m \times m \) matrix \( (f_1(W_m)) \) defined on the set \( W_m := \{a + be, (a + be)(1 + bt_1), \ldots, (a + be)(1 + bt_{m-1})\} \) and let \( \tilde{\gamma}^{(1)}_m \leq \ldots \leq \tilde{\gamma}^{(m)}_m \) be the eigenvalues of the \( m \times m \) matrix \( (f_1(\tilde{W}_m)) \) defined on the set \( \tilde{W}_m := \{1, 1 + bt_1, \ldots, 1 + bt_{m-1}\} \).

Since \( f_1 \) is multiplicative and \( (a + be, 1 + bt_i) = 1 \), we have \( (f_1(W_m)) = f_1(a + be)(f_1(\tilde{W}_m)) \). So we have \( \gamma^{(i)}_m = f_1(a + be)\tilde{\gamma}^{(i)}_m \) for \( 1 \leq i \leq m \). In particular,

\[
\gamma^{(q)}_m = f_1(a + be)\tilde{\gamma}^{(q)}_m. \tag{2-13}
\]

Now let \( m_n \) be the largest integer \( l \) such that

\[
t_{l-1} \leq \left\lfloor \frac{n}{a + be} \right\rfloor,
\]

where \( \lfloor x \rfloor \) denotes the largest integer \( \leq x \). Clearly \( m_n \to \infty \) as \( n \to \infty \). Choose \( n \) so that \( m_n \geq q \).

By Cauchy’s interlacing inequalities

\[
\lambda_n^{(q)}(1, 0) \leq \gamma^{(q)}_m, \tag{2-14}
\]

and by (2-13) and (2-14),

\[
\lambda_n^{(q)}(1, 0) \leq f_1(a + be)\tilde{\gamma}^{(q)}_{m_n}. \tag{2-15}
\]
We claim that \( \lim_{m \to \infty} \tilde{z}_{m}^{(q)} = 0 \). Then we have \( \lim_{n \to \infty} \lambda_{n}^{(q)}(1,0) = 0 \) as desired. It remains to prove the assertion which will be done in the following.

Let \( p_1 < p_2 < \ldots \) denote the primes in the sequence \( \{1 + bt_i\}_{i=0}^{\infty} \). Then \( \{p_i(b)\}_{i=s}^{\infty} \subset \{p_i\}_{i=1}^{\infty} \), where \( p_{s-1}(b) \leq a + be < p_s(b) \), \( s \geq 1 \) is an integer and \( p_0(b) := 1 \). Let

\[
Q := \{p_i(b)\}_{i=1}^{\infty} \setminus \{p_i\}_{i=1}^{\infty}.
\]

Then \( Q \) is a finite set. Since \( f_1(p_i(b)) \neq 0 \) for all \( i \geq 1 \), we have \( \sum_{p \in Q} \frac{1}{f_1(p)} < \infty \). So by the assumption \( \sum_{i=1}^{\infty} \frac{1}{f_1(p_i)} = \infty \) we have

\[
\sum_{i=1}^{\infty} \frac{1}{f_1(p_i)} = \infty. \tag{2 - 16}
\]

For \( i \geq 1 \), let \( \pi_i = p_{q-1+i} \). Then \( p_{q-1} < \pi_1 < \ldots \). Since \( q \) is a fixed number, it follows from (2-16) that

\[
\sum_{i=1}^{\infty} \frac{1}{f_1(\pi_i)} = \infty. \tag{2 - 17}
\]

Now let \( r \geq 2 \) be an arbitrary integer and let

\[
P_q := \{1, p_1, \ldots, p_{q-1}\}, \quad T_r := \{1, \pi_1, \ldots, \pi_{r-1}\}.
\]

It is clear that the matrices \( (f_1(P_q)) \) and \( (f_1(T_r)) \) are positive semi-definite. Consider the tensor product set \( P_q \odot T_r \). Note that the entries in the set \( P_q \odot T_r \) are not arranged in increasing order, but the eigenvalues of the corresponding matrix do not depend on rearranging those entries. Since \( f_1 \) is multiplicative, by Lemma 2.4 we have

\[
(f_1(P_q \odot T_r)) = (f_1(P_q)) \otimes (f_1(T_r)).
\]

Let \( \delta_q^{(1)} \leq \ldots \leq \delta_q^{(q)} \) and \( \tilde{\lambda}_r^{(1)} \leq \ldots \leq \tilde{\lambda}_r^{(r)} \) be the eigenvalues of the matrix \( (f_1(P_q)) \) defined on the set \( P_q \) and the matrix \( (f_1(T_r)) \) defined on the set \( T_r \) respectively. Then it is known (see [Hor-J2]) that the eigenvalues of the tensor product matrix \( (f_1(P_q)) \otimes (f_1(T_r)) \) are given by the set

\[
\left\{ \delta_q^{(i)} \cdot \tilde{\lambda}_r^{(j)} \right\}_{1 \leq i \leq q, 1 \leq j \leq r}.
\]

Notice that

\[
\delta_q^{(1)} \cdot \tilde{\lambda}_r^{(1)} \leq \ldots \leq \delta_q^{(q)} \cdot \tilde{\lambda}_r^{(1)} \tag{2 - 18}
\]

Clearly the sequence \( \{1+bt_i\}_{i=0}^{\infty} \) is closed under the usual multiplication. So the tensor product set \( P_q \odot T_r \subset \{1+bt_i\}_{i=0}^{\infty} \). For any integer \( r \geq 2 \), define an integer \( m_r \) by

\[
m_r := \frac{p_{q-1} \cdot \pi_{r-1} - 1}{b} + 1.
\]

Then \( P_q \odot T_r \subset \{1+bt_i\}_{i=0}^{m_r-1} \). Thus the matrix \( (f_1(P_q \odot T_r)) \) defined on \( P_q \odot T_r \) is a principal submatrix of the \( m_r \times m_r \) matrix \( (f_1(1+bt_i, 1+bt_j)) \) defined on the set
\{1, 1 + bt_1, \ldots, 1 + bt_{m_r - 1}\}. Let \(\bar{\lambda}_{qr}^{(1)} \leq \ldots \leq \bar{\lambda}_{qr}^{(qr)}\) be the eigenvalues of \((f_1(P_q \otimes T_r))\). Then by Cauchy’s interlacing inequalities we have

\[
\bar{\gamma}_{m_r}^{(q)} \leq \bar{\lambda}_{qr}^{(q)} .
\]  
(2 - 19)

But by (2-18)

\[
\bar{\lambda}_{qr}^{(q)} \leq \delta_q^{(q)} \cdot \bar{\lambda}_r^{(1)} .
\]  
(2 - 20)

So it follows from (2-19) and (2-20) that

\[
\bar{\gamma}_{m_r}^{(q)} \leq \delta_q^{(q)} \cdot \bar{\lambda}_r^{(1)} .
\]  
(2 - 21)

On the other hand, in Theorem 1.4, if we choose \(x = x_1 = 1\) and \(x_i = \pi_i - 1\) for \(i \geq 2\), then by (2-17) the conditions of Theorem 1.4 are satisfied. It then follows immediately from Theorem 1.4 that

\[
\lim_{r \to \infty} \bar{\lambda}_r^{(1)} = 0 .
\]  
(2 - 22)

But by Theorem 1.3 we have that the subsequence \(\{\bar{\gamma}_{m_r}^{(q)}\}_{r=1}^{\infty}\) of the sequence \(\{\bar{\gamma}_{m}^{(q)}\}_{m=1}^{\infty}\) converges and

\[
\lim_{r \to \infty} \bar{\gamma}_{m_r}^{(q)} \geq 0.
\]  
(2 - 23)

Hence by (2-21)-(2-23), \(\lim_{r \to \infty} \bar{\gamma}_{m_r}^{(q)} = 0\). Finally, again by Theorem 1.2, the desired result \(\lim_{m \to \infty} \bar{\gamma}_{m}^{(q)} = 0\) follows immediately. The claim is proved and this completes the proof of part (ii) of Theorem 1.6 for the case \(l_1 = c = 1\) and \(d = 0\).

Now consider part (ii) for the general case. In the case \(l_1 = c = 1\) and \(d = 0\), we replace \(f_1\) by \(h = f_1^{(l_1)} \ast \ldots \ast f_c^{(l_c)} \ast \mu^{(d)}\). Since \(f_1, \ldots, f_c \in \mathcal{C}\) and \(l_1 + \ldots + l_c > d\), by Theorem 1.1 (i) we have \(h \in \mathcal{C}\). Thus \(h(p_i(b)) \geq 0\) for all \(i \geq 1\). So by the assumption \(h(p_i(b)) \neq 0\) for all \(i \geq 1\) we have that \(h(p_i(b)) > 0\) for all \(i \geq 1\). Note that \(h\) is multiplicative. On the other hand, \(h\) is increasing on the sequence \(\{p_i(b)\}_{i=1}^{\infty}\) because of the formula in Theorem 1.1 (ii). It remains to prove that

\[
\sum_{i=1}^{\infty} \frac{1}{h(p_i(b))} = \infty .
\]  
(2 - 24)

But Theorem 1.1 (ii) tells us

\[
h(p_i(b)) = \sum_{j=1}^{c} l_j f_j(p_i(b)) - d < \sum_{j=1}^{c} l_j f_j(p_i(b)) ,
\]

So we have

\[
\sum_{i=1}^{\infty} \frac{1}{h(p_i(b))} \geq \sum_{i=1}^{\infty} \frac{1}{\sum_{j=1}^{c} l_j f_j(p_i(b))} \geq \frac{1}{l} \sum_{i=1}^{\infty} \frac{1}{\sum_{j=1}^{c} f_j(p_i(b))} ,
\]  
(2 - 25)

where \(l = \max_{1 \leq j \leq c} l_j\). Hence (2-24) follows immediately from (2-25) and the condition of Theorem 1.6 (ii). So Theorem 1.6 (ii) for the general case follows from Theorem 1.6 (ii) for the case \(l_1 = c = 1\) and \(d = 0\). The proof of Theorem 1.6 (ii) is complete.
Example 3.2. Let \( f = \xi_\varepsilon \), where \( \xi_\varepsilon \) is defined as in the introduction and \( \varepsilon \) is a real number. Then \( \xi_\varepsilon \) is increasing on any strictly increasing infinite sequence, and completely multiplicative if \( \varepsilon \geq 0 \). Let \( J_\varepsilon := \xi_\varepsilon \ast \mu \). Then \( J_\varepsilon(1) = 1 \) and for any integer \( m > 1 \),

\[
J_\varepsilon(m) = m^\varepsilon \prod_{p|m} \left( 1 - \frac{1}{p^\varepsilon} \right) \geq 0
\]

if \( \varepsilon \geq 0 \). Thus \( \xi_\varepsilon \in \mathcal{C}_S \) for any set \( S \) of positive integers and so \( \xi_\varepsilon \in \mathcal{C} \) for any given strictly increasing infinite sequence \( \{x_i\}_{i=1}^\infty \) of positive integers if \( \varepsilon \geq 0 \). For integers \( c > d \geq 0 \), let \( \lambda_n^{(1)}(c, d) \leq \ldots \leq \lambda_n^{(m)}(c, d) \) be the eigenvalues of the \( n \times n \) matrix \( ((\xi_\varepsilon^{(c)} \ast \mu^{(d)})(x_i, x_j)) \) defined on the set \( S_n = \{x_1, \ldots, x_n\} \).

(i). By Theorem 1.4 (ii) we get: If \( \varepsilon > 0 \) and \( S_n \) satisfies that for every \( 1 \leq i \neq j \leq n \), \( (x_i, x_j) = x \), then we have

\[
x_1^\varepsilon - x_1^\varepsilon \leq \lambda_n^{(1)}(1, 0) < x_1^\varepsilon - x_1^\varepsilon + \frac{x_1^\varepsilon}{1 + \sum_{i=2}^{n} \frac{x_i^\varepsilon}{x_i^\varepsilon - x_1^\varepsilon}};
\]

(ii). ([Hon-Lo]) By Theorem 1.4 (iii) we get: For any given strictly increasing infinite sequence \( \{x_i\}_{i=1}^\infty \) consisting of all but finitely many primes, we have \( (x_i, x_j) = 1 \) for every \( i \neq j \), and by Mertens' theorem ([Me]) we have \( \sum_{i=1}^{\infty} \frac{1}{x_i^\varepsilon} = \infty \) if \( \varepsilon \leq 1 \). So if \( 0 \leq \varepsilon \leq 1 \), then we have \( \lim_{n\to\infty} \lambda_n^{(1)}(1, 0) = x_1^\varepsilon - 1 \).

(iii). By Theorem 1.8 we get: For any given strictly increasing infinite sequence \( \{x_i\}_{i=1}^\infty \) of positive integers which contains the arithmetic progression \( \{a + bi\}_{i=1}^\infty \) as its subsequence, where \( a, b \geq 1 \) and \( e \geq 0 \) are integers, if \( 0 \leq \varepsilon \leq 1 \), then for any given integer \( q \geq 1 \), we have \( \lim_{n\to\infty} \lambda_n^{(q)}(c, d) = 0 \).

Example 3.2. Let \( f = J_\varepsilon \), where \( \varepsilon \) is a real number and \( J_\varepsilon \) is defined in Example 3.1. Note that if \( \varepsilon \) is a positive integer, then \( J_\varepsilon \) becomes Jordan's totient function (see, for example, [A1], [Mc1] or [Mu]). Clearly \( J_\varepsilon \ast \mu \) is multiplicative and \( (J_\varepsilon \ast \mu)(1) = 1 \). It is easy to see that if \( p \) is an odd prime number and \( \varepsilon \geq \frac{\log 2}{\log 3} \), then \( (J_\varepsilon \ast \mu)(p) = p^\varepsilon - 2 \geq 0 \). For any prime \( p \) and integer \( l \geq 2 \), we have \( (J_\varepsilon \ast \mu)(p^l) = p^{(l-2)e}(p^\varepsilon - 1)^2 > 0 \). Thus \( J_\varepsilon \in \mathcal{C}_S \) for any set \( S \) of positive odd numbers and so \( J_\varepsilon \in \mathcal{C} \) for any given strictly increasing infinite sequence \( \{x_i\}_{i=1}^\infty \) of positive odd numbers if \( \varepsilon \geq \frac{\log 2}{\log 3} \). On the other hand, if \( \varepsilon \geq 0 \), then for any primes \( 3 \leq p_1 < p_2 \), we have \( J_\varepsilon(p_1) = p_1^\varepsilon - 1 \leq p_2^\varepsilon - 2 = J_\varepsilon(p_2) \) and for any integer \( m \geq 2 \), we have \( J_\varepsilon(m) \leq m^\varepsilon \). For integers \( c > d \geq 0 \), let \( \lambda_n^{(1)}(c, d) \leq \ldots \leq \lambda_n^{(m)}(c, d) \) be the eigenvalues of the \( n \times n \) matrix \( ((J_\varepsilon^{(c)} \ast \mu^{(d)})(x_i, x_j)) \) defined on the set \( S_n = \{x_1, \ldots, x_n\} \).
(i). By Theorem 1.4 (ii) we get: For any given strictly increasing infinite sequence 
\{x_i\}_{i=1}^{\infty} consisting of all but finitely many odd primes, if \(\frac{\log 2}{\log p_i} \leq \varepsilon < 1\), then we have
\[
x_1^\varepsilon - 2 < \lambda_n^{(1)}(1,0) < x_1^\varepsilon - 2 + \frac{1}{1 + \sum_{i=2}^{\infty} \frac{1}{x_i^\varepsilon - x_1^\varepsilon}}.
\]
Furthermore by Theorem 1.5, \(\lim_{n \to \infty} \lambda_n^{(1)}(1,0) = x_1^\varepsilon - 2\).

(ii). By Theorem 1.8 we get: For any given strictly increasing infinite sequence 
\{x_i\}_{i=1}^{\infty} of positive odd numbers which contains the arithmetic progression 
\{a + bi\}_{i=\varepsilon}^{\infty} as its subsequence, where \(a, b \geq 1\) and \(e \geq 0\) are integers, if \(\frac{\log 2}{\log 3} \leq \varepsilon < 1\), then for any given integer \(q \geq 1\), we have \(\lim_{n \to \infty} \lambda_n^{(q)}(c,d) = 0\).

**Example 3.3.** Let \(f = \sigma_\varepsilon := \xi_\varepsilon \ast \xi_0\), where \(\varepsilon\) is a real number. Then for any positive integer \(m\) we have

\[
\sigma_\varepsilon(m) = \sum_{d|m} d^\varepsilon.
\]

The function \(d(m) = \sigma_0(m)\) is the usual divisor function. The function \(\sigma(m) = \sigma_1(m)\) gives the sum of the divisors of \(m\). Clearly \(\sigma_\varepsilon\) is multiplicative. Since \(\sigma_\varepsilon \ast \mu = \xi_\varepsilon \ast \xi_0 \ast \mu = \xi_\varepsilon\), we have \((\sigma_\varepsilon \ast \mu)(m) = m^\varepsilon > 0\) for any integer \(m \geq 1\). So \(\sigma_\varepsilon \in C\) for any given strictly increasing infinite sequence \(\{x_i\}_{i=1}^{\infty}\) of positive integers. Obviously if \(\varepsilon \geq 0\) and \(p_1 < p_2\) are primes, then \(\sigma_\varepsilon(p_1) = 1 + p_1^\varepsilon \leq 1 + p_2^\varepsilon = \sigma_\varepsilon(p_2)\). For integers \(c > d \geq 0\), let \(\lambda_n^{(1)}(c,d) \leq \ldots \leq \lambda_n^{(n)}(c,d)\) be the eigenvalues of the \(n \times n\) matrix \(((\sigma_\varepsilon^{(c)} \ast \mu^{(d)})(x_i,x_j))\) defined on the set \(S_n = \{x_1,...,x_n\}\).

(i). By Theorems 1.4 (ii) we get: For any given strictly increasing infinite sequence 
\(\{x_i\}_{i=1}^{\infty}\) consisting of all the primes in \(\mathbb{Z}^+\) except finitely many of them, if \(\varepsilon > 0\), then we have
\[
x_1^\varepsilon < \lambda_n^{(1)}(1,0) < x_1^\varepsilon + \frac{1}{1 + \sum_{i=2}^{\infty} \frac{1}{x_i^\varepsilon - x_1^\varepsilon}}.
\]
Furthermore by Theorem 1.4 (iii), if \(0 \leq \varepsilon \leq 1\), then we have \(\lim_{n \to \infty} \lambda_n^{(1)}(1,0) = x_1^\varepsilon\).

(ii). By Theorem 1.8 we get: For any given strictly increasing infinite sequence \(\{x_i\}_{i=1}^{\infty}\) of positive integers which contains the arithmetic progression \(\{a+bi\}_{i=\varepsilon}^{\infty}\) as its subsequence, where \(a, b \geq 1\) and \(e \geq 0\) are integers, since
\[
\sum_{i=1}^{\infty} \frac{1}{\sigma_\varepsilon(p_i(b))} = \sum_{i=1}^{\infty} \frac{1}{p_i(b)^\varepsilon + 1} \geq \frac{1}{2} \sum_{i=1}^{\infty} \frac{1}{p_i(b)^\varepsilon}
\]
if \(\varepsilon \geq 0\), we deduce that if \(0 \leq \varepsilon \leq 1\)
\[
\sum_{i=1}^{\infty} \frac{1}{\sigma_\varepsilon(p_i(b))} = \infty.
\]
Then for any given integer \(q \geq 1\), if \(0 \leq \varepsilon \leq 1\), we have \(\lim_{n \to \infty} \lambda_n^{(q)}(c,d) = 0\).
Example 3.4. Let $f = \psi_\varepsilon$, where $\varepsilon$ is a real number and $\psi_\varepsilon$ is defined for any positive integer $m$ by

$$
\psi_\varepsilon(m) := \sum_{d|m} d^\varepsilon |\mu(m/d)|.
$$

The function $\psi_1$ is called Dedekind’s function (see, for instance, [Mc1]). Clearly $\psi_\varepsilon$ is multiplicative. Then for any positive integer $m$ we have

$$
\psi_\varepsilon(m) = m^\varepsilon \prod_{p|m} (1 + \frac{1}{p^\varepsilon}) = \frac{J_{2\varepsilon}(m)}{J_{\varepsilon}(m)}.
$$

Thus for any positive integer $l$ and any prime $p$, we have

$$
(\psi_\varepsilon * \mu)(p^l) = \begin{cases} p^\varepsilon, & \text{if } l = 1; \\ p^{l-2}\varepsilon(p^{2\varepsilon} - 1), & \text{if } l \geq 2. \end{cases}
$$

If $\varepsilon \geq 0$, then $\psi_\varepsilon \in \mathcal{C}$ for any given strictly increasing infinite sequences $\{x_i\}_{i=1}^\infty$ of positive integers. For integers $c > d \geq 0$, let $\lambda_n^{(1)}(c, d) \leq \ldots \leq \lambda_n^{(c)}(c, d)$ be the eigenvalues of the $n \times n$ matrix $((\psi_\varepsilon^{(c)} * \mu^{(d)})(x_i, x_j))$ defined on the set $S_n = \{x_1, ..., x_n\}$.

(i). By Theorem 1.4 (ii) we get: For any given strictly increasing infinite sequence $\{x_i\}_{i=1}^\infty$ consisting of all the primes in $\mathbb{Z}^+$ except finitely many of them, if $\varepsilon > 0$, then we have

$$
x_1^\varepsilon < \lambda_n^{(1)}(1, 0) < x_1^\varepsilon + \frac{1}{1 + \sum_{i=2}^n \frac{1}{x_i - x_{i-1}}}. $$

Furthermore by Theorem 1.4 (iii), if $0 \leq \varepsilon \leq 1$, then we have $\lim_{n \to \infty} \lambda_n^{(1)}(1, 0) = x_1^\varepsilon$.

(ii). By Theorem 1.8 we get: For any given strictly increasing infinite sequence $\{x_i\}_{i=1}^\infty$ of positive integers which contains the arithmetic progression $\{a+bi\}_{i=1}^\infty$ as its subsequence, where $a$, $b \geq 1$ and $e \geq 0$ are integers, in a same way as in Example 3.3, we can check that for $\varepsilon \geq 0$, $\psi_\varepsilon$ is increasing on the sequence $\{p_i(b)\}_{i=1}^\infty$ and if $0 \leq \varepsilon \leq 1$

$$
\sum_{i=1}^\infty \frac{1}{\psi_\varepsilon(p_i(b))} = \infty.
$$

Then for any given integer $q \geq 1$, if $0 \leq \varepsilon \leq 1$, we have $\lim_{n \to \infty} \lambda_n^{(q)}(c, d) = 0$.

Example 3.5. Let $f = \phi$, Euler’s totient function. Clearly $\phi$ and $\phi * \mu$ are multiplicative, and $\phi(1) = (\phi * \mu)(1) = 1$. For any prime $p$ we have $(\phi * \mu)(p) = \phi(p) = p - 1 = p - 2 \geq 0$, and for any integer $l \geq 2$ we have

$$
(\phi * \mu)(p^l) = \sum_{i=1}^l \phi(p^i)\mu(p^{l-i}) = \phi(p^l) - \phi(p^{l-1}) = p^{l-2}(p - 1)^2 > 0.
$$

Thus $\phi \in \mathcal{C}_S$ for any set $S$ of positive integers and so $\phi \in \mathcal{C}$ for any given strictly increasing infinite sequence $\{x_i\}_{i=1}^\infty$ of positive integers. Note that $\phi(p) = p - 1 \leq p$. So for any
primes \( p_1 < p_2, \phi(p_1) < \phi(p_2) \). For integers \( c > d \geq 0 \), let \( \lambda_n^{(1)}(c,d) \leq ... \leq \lambda_n^{(n)}(c,d) \) be the eigenvalues of the \( n \times n \) matrix \( ((\phi^{(c)}*\mu^{(d)})(x_i,x_j)) \) defined on the set \( S_n = \{x_1, ..., x_n\} \).

(i) By Theorem 1.4 (ii) we get: For any given strictly increasing infinite sequence \( \{x_i\}_{i=1}^{\infty} \) consisting of all the primes in \( \mathbb{Z}^+ \) except finitely many of them, we have

\[
x_1 - 2 < \lambda_n^{(1)}(1,0) < x_1 - 2 + \frac{1}{1 + \sum_{i=2}^{\infty} \frac{1}{x_i-x_1}}.
\]

Furthermore, by Theorem 1.5 we have \( \lim_{n \to \infty} \lambda_n^{(1)}(1,0) = x_1 - 2 \).

(ii) By Theorem 1.8 we get: For any given strictly increasing infinite sequence \( \{x_i\}_{i=1}^{\infty} \) of positive integers which contains the arithmetic progression \( \{a+bi\}_{i=\epsilon}^{\infty} \) as its subsequence, where \( a, b \geq 1 \) and \( e \geq 0 \) are integers, and for any given integer \( q \geq 1 \), we have \( \lim_{n \to \infty} \lambda_n^{(q)}(c,d) = 0 \).

**Example 3.6.** Let \( f_1 = \xi_\epsilon \) and \( f_2 = \phi \) be as in Examples 3.1 and 3.5 respectively. Clearly \( \xi_\epsilon \) and \( \phi \) are distinct and multiplicative. Note that \( \xi_\epsilon \) is increasing on any strictly increasing infinite sequence of positive integers if \( \epsilon \geq 0 \) and \( \phi \) is increasing on any subsequence of strictly increasing infinite sequence consisting of all the primes in \( \mathbb{Z}^+ \). By Examples 3.1 and 3.5 we know that \( \xi_\epsilon \in \mathcal{C} \) and \( \phi \in \mathcal{C} \) for any given strictly increasing infinite sequence \( \{x_i\}_{i=1}^{\infty} \) of positive integers if \( \epsilon \geq 0 \). For integers \( c_1 > 0, c_2 > 0 \) and \( d \geq 0 \), let \( \lambda_n^{(1)}(c_1,c_2,d) \leq ... \leq \lambda_n^{(n)}(c_1,c_2,d) \) be the eigenvalues of the \( n \times n \) matrix \( ((\xi_\epsilon^{(c_1)}*\phi^{(c_2)}*\mu^{(d)})(x_i,x_j)) \) defined on the set \( S_n = \{x_1, ..., x_n\} \). Since for any prime \( p \), we have \( \phi(p) \leq p \) and \( \xi_\epsilon(p) \leq p \) if \( \epsilon \leq 1 \), then by Theorem 1.7 we get: For any given strictly increasing infinite sequence \( \{x_i\}_{i=1}^{\infty} \) of positive integers which contains the arithmetic progression \( \{a+bi\}_{i=\epsilon}^{\infty} \) as its subsequence, where \( a, b \geq 1 \) and \( e \geq 0 \) are integers, if \( 0 \leq \epsilon \leq 1 \) and \( c_1 + c_2 > d \), then for any given integer \( q \geq 1 \), we have \( \lim_{n \to \infty} \lambda_n^{(q)}(c_1,c_2,d) = 0 \).

4. Open questions

Let \( \{x_i\}_{i=1}^{\infty} \) be an arbitrary strictly increasing infinite sequence of positive integers. For an integer \( n \geq 1 \), let \( S_n = \{x_1, ..., x_n\} \). Let \( c, q \geq 1 \) and \( d \geq 0 \) be given integers. Let \( \lambda_n^{(1)}(c,d) \leq ... \leq \lambda_n^{(n)}(c,d) \) be the eigenvalues of the \( n \times n \) matrix \( ((f^{(c)}*\mu^{(d)})(x_i,x_j)) \) defined on the set \( S_n \). It follows from Theorem 1.4 that if \( \{x_i\}_{i=1}^{\infty} \) is a strictly increasing infinite sequence of positive integers satisfying that for every \( i \neq j \), \( (x_i,x_j) = x_1 \) and \( f \in \mathcal{C} \) is increasing on the sequence \( \{x_i\}_{i=1}^{\infty} \) and \( \sum_{i=1}^{\infty} \frac{1}{f(x_i)} = \infty \), then \( \lim_{n \to \infty} \lambda_n^{(1)}(1,0) = 0 \). Then by Cauchy’s interlacing inequalities and Theorem 1.3 we know that for any given strictly increasing infinite sequence \( \{x_i\}_{i=1}^{\infty} \) of positive integers which contains a subsequence \( \{x_i'\}_{i=1}^{\infty} \) satisfying that for every \( i \neq j \), \( (x'_i,x'_j) = x'_1 \), if \( f \in \mathcal{C} \) (with respect to the whole sequence \( \{x_i\}_{i=1}^{\infty} \)) and \( f \) is increasing on the sequence \( \{x'_i\}_{i=1}^{\infty} \) and \( \sum_{i=1}^{\infty} \frac{1}{f(x'_i)} = \infty \), then \( \lim_{n \to \infty} \lambda_n^{(1)}(1,0) = 0 \) (Note that this holds when some \( f(x'_i) \) is 0). On the other hand, by Theorem 1.8 we know that for any given strictly increasing infinite sequence \( \{x_i\}_{i=1}^{\infty} \) of positive integers containing the arithmetic progression \( \{a+bi\}_{i=\epsilon}^{\infty} \) as its subsequence, if \( c > d \geq 0 \) and \( f \in \mathcal{C} \) is multiplicative and increasing on the sequence \( \{p_i(b)\}_{i=1}^{\infty} \) and
\[ \sum_{i=1}^{\infty} \frac{1}{f(p_i(b))} = \infty, \] where \( p_i(b) \) \((i \geq 1)\) is defined as in (1-1), then for any given integer \( q \geq 1 \), we have \( \lim_{n \to \infty} \lambda_n^{(q)}(c, d) = 0 \). First we would like to understand for what sequences \( \{x_i\}_{i=1}^{\infty} \), \( \lim_{n \to \infty} \lambda_n^{(1)}(c, d) = 0 \). Namely, we have the following question:

**Question 4.1.** Given any multiplicative function \( f \), and given nonnegative integers \( c \) and \( d \) such that \( c > d \), characterize all strictly increasing infinite sequences \( \{x_i\}_{i=1}^{\infty} \) of positive integers so that \( \lim_{n \to \infty} \lambda_n^{(1)}(c, d) = 0 \), where, as before, \( \lambda_n^{(1)}(c, d) \) is the smallest eigenvalue of the matrix \( ((f(c) \ast \mu(d))(x_i, x_j)) \) defined on the set \( S_n = \{x_1, ..., x_n\} \).

Consequently, we raise a further problem.

**Question 4.2.** The same as the previous question, with \( \lambda_n^{(1)}(c, d) \) is replaced by \( \lambda_n^{(q)}(c, d) \), where, as before, \( \lambda_n^{(q)}(c, d) \) is the \( q \)-th smallest eigenvalue of the matrix \( ((f(c) \ast \mu(d))(x_i, x_j)) \) defined on the set \( S_n = \{x_1, ..., x_n\} \).

In concluding this paper we propose the following question.

**Question 4.3.** Let \( c > d \geq 0 \) be given integers and \( \{x_i\}_{i=1}^{\infty} \) be an arbitrary strictly increasing infinite sequence of positive integers. Let \( \lambda_n^{(1)}(c, d) \) be the smallest eigenvalue of the \( n \times n \) matrix \( ((f(c) \ast \mu(d))(x_i, x_j)) \) defined on the set \( S_n = \{x_1, ..., x_n\} \). Assume that \( f \in \mathcal{C} \) is multiplicative. Are the following true:

(i). If \( f \) satisfies that \( f(x_i) \geq Cx_i^{c} \) for all \( i \geq 1 \), where \( \varepsilon > 1 \) and \( C > 0 \) are constants, do we have \( \lim_{n \to \infty} \lambda_n^{(1)}(c, d) > 0 \)?

(ii). If \( f \) satisfies that \( \sum_{i=1}^{\infty} \frac{1}{f(x_i)} < \infty \), do we have \( \lim_{n \to \infty} \lambda_n^{(1)}(c, d) > 0 \)?

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