Universality in all-$\alpha'$ order corrections to BPS/non-BPS brane world volume theories

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Abstract

Knowledge of all-$\alpha'$ higher derivative corrections to leading order BPS and non-BPS brane actions would serve in future endeavor of determining the complete form of the non-abelian BPS and tachyonic effective actions. In this paper, we note that there is a universality in the all-$\alpha'$ order corrections to BPS and non-BPS branes. We compute amplitudes between one Ramond-Ramond $C$-field vertex operator and several SYM gauge/scalar vertex operators. Specifically, we evaluate in closed form string correlators of two-point amplitudes $A^{C\phi}$, $A^{CA}$, a three-point amplitude $A^{C\phi\phi}$, and a four-point amplitude $A^{C\phi\phi\phi}$. We carry out pole and contact term analysis. In particular we reproduce some of the contact terms and the infinite massless poles of $A^{C\phi\phi\phi}$ by SYM vertices obtained through the universality.
1 Introduction

D-branes are sources for closed string RR fields in string theory [1]. Many results on their properties have been obtained [2]. Without the higher derivative corrections in string theory, the action takes the Born-Infeld [3, 4] and Wess-Zumino (WZ) forms [5]. A recent comprehensive discussion on the higher derivative corrections to BPS and non-BPS effective actions can be found, e.g., in works of [6] and [7]. An attempt at finding universal higher derivative corrections was made [8], and higher derivative corrections to WZ couplings for non-BPS branes were obtained in [9] and [10]. Roles played by D-branes were reviewed, e.g., in [11] and [12].

There has been recent progress in string amplitude computation (see [13] for an early development) and its matching with the corresponding low energy DBI computation [6], where in the amplitude of one RR closed string vertex and three open string gauge field vertex operators has been computed in detail. A related analysis for higher derivative corrections of two scalar and two tachyon vertex operators for a non-BPS case to all orders of $\alpha'$ was carried out in [7]. Subsequently, the amplitude of one RR closed string vertex and two open string gauge fields and one open string scalar field vertex operator was computed in [8]. There, we found all higher derivative corrections to the two-gauge field and two-scalar field couplings in a closed form by explicit S-matrix computation. The all-$\alpha'$ order vertices of SYM were determined by analyzing the poles and contact terms.

One of the motivations behind our recent and present works is to better understand the full closed form of the non-abelian DBI action. The closed form of non-abelian DBI action has been evasive despite the continued efforts so far (see for instance [14, 15]). With an ultimate goal of determining the closed form of the non-abelian action, we continue collecting ”data”; we analyze another set of amplitudes involving one RR $C$-field vertex and a few scalar field vertex operators in closed form; after warming up with $<V_C V_\phi>$ and $<V_C V_\phi V_\phi>$ (we also compute $<V_C V_A>$), we analyze the case of $<V_C V_\phi V_\phi V_\phi>$. Although $<V_C V_\phi V_\phi V_\phi>$ was considered in [16], we have decided to revisit this amplitude for several reasons. Firstly, the final form of our amplitude computation is different from that of [16]. Several subtleties are involved in the computation; we will note them based on the recent work [6]. We present all details necessary to derive our results below. In addition, we carry out all infinite poles and some of the contact term analysis, and determine the corresponding SYM vertices. This
part of the analysis was not carried out in \[16\]. Our computation is based on Wick-like contractions given in \[17, 6\]. Secondly, the amplitude is relatively simple yet expected to serve as another test ground for universality, a property that we expect to be a useful asset in future understanding of the closed form of DBI action. The results of \[6, 7, 8\] suggest a method for finding all-\(\alpha'\) order corrections to BPS (and non-BPS) DBI actions. The method is based on a universality in the pattern of all-\(\alpha'\) order extension of the vertices in the BPS (and non-BPS) brane actions.\footnote{One may wonder if it is possible to find higher derivative corrections of BPS branes by applying T-duality to previously known results. T-duality in open string loop computations is subtle and is not as effective as in tree amplitudes \[21\] (in particular, see footnote 4 therein). Although we are considering tree-level diagrams, the amplitudes share certain attributes of open string loop diagrams due to the presence of a closed string. Instead of relying on T-duality, we note that there is a persisting pattern in the higher order derivative corrections and that one may use the pattern as a prescription for determining the forms of the higher order corrections. As we will discuss, the prescription works not only for BPS cases but also for non-BPS cases. Recall that the forms of vertex operators for tachyons are quite different from gauge/scalar field vertex operators. We will comment on T-duality related issues in the conclusion.}

Utilizing the pattern, we determine\footnote{See \[6\] for an earlier attempt.} closed-form higher derivative corrections to four-scalar interaction vertex, and confirm them with an explicit Feynman diagram computation.

The third reason for considering \(\langle V_C V_\phi V_\phi V_\phi \rangle\) is its relevance for Myers’ effect. We analyzed \(\langle V_C V_A V_\phi \rangle\) in \[8\], and new WZ couplings were obtained. The result has been applied in \[18\] to understand the M5 brane \(N^3\) entropy scaling. In particular, (31) of \[8\] was discussed in sec 4.2 of \[18\]. Similarly, some of the new WZ couplings that we find below are expected to play a role in the higher \(\alpha'\) contributions to the entropy of various brane configurations. In fact we have noticed that Myers’ terms can be realized in a peculiar manner from the coupling between open and closed strings. They come from the closed string coupling to a lower dimensional branes. The lower dimensional branes can be viewed as soliton solutions of the branes that one started with. We also have obtained solutions that represent dissolution of lower dimensional brane inside of higher dimensional brane which is a manifestation of Myers’ terms \[19\]. For \(D(-1)/D3\) system, it could be higher \(\alpha'\)-order dielectric effect that is responsible for the \(N^2\) behavior \[18\] and \[20\].

The rest of the paper is organized as follows. In section 2, we start by analyzing two-point amplitudes \(\langle V_C V_\phi \rangle\) and \(\langle V_C V_A \rangle\). Then we compute a three-point amplitude between one RR and two scalar fields \(\langle V_C V_\phi V_\phi \rangle\), and work out with contact terms. We determine the corresponding field theory vertices that reproduce all infinite contact terms.\footnote{The corresponding analysis for one RR, one gauge field and one scalar field was done in the appendix of \[8\].} In section 3, which contains the main result of this work, we carry out the analogous analysis for \(\langle V_C V_\phi V_\phi V_\phi \rangle\). First, we compute the string amplitude in closed form and analyze the
infinite poles. Then we review the infinite extension results of [7, 6, 8] in which various amplitudes of two tachyons and two scalar fields, four gauge fields, two gauge fields and two scalar fields were analyzed. We notice a persisting pattern in the form of higher derivative corrections in those results. Taking the pattern as a prescription we determine, with correct coefficients, the all-order higher derivative corrections to the SYM four-scalar couplings. We further comment on T-duality related aspects in the conclusion after summarizing the results of the present work.

2 Analysis of $<V_C V_{\phi}>, <V_C V_A>$ and $<V_C V_{\phi} V_{\phi}>$

We start by considering relatively simple amplitudes $<V_C V_{\phi}>, <V_C V_A>$, and $<V_C V_{\phi} V_{\phi}>$ as a warm-up, and give more details for $<V_C V_{\phi} V_{\phi}>$. After computing the string amplitudes, we determine through inspection the SYM vertices that reproduce the momentum expansion of the string amplitudes. There are patterns in the field theory vertices that reproduce the leading order poles and contact terms that allow one to find the all-order extensions of these vertices. The patterns were present in the previous works [6], [7], [8] and [10] and they persist in the present work. We will see the first example of the pattern in $<V_C V_{\phi} V_{\phi}>$ and more examples in section 3.

2.1 $<V_C V_{\phi}>$

Here we consider BPS amplitudes in flat space and set all background fields to zero. The (0)- and (-1)- picture vertex operators for the scalar fields, gauge fields and the RR $C$-field vertex operator in (-1)-picture are given by

\[ V_{\phi}^{(0)}(x) = \xi_i (\partial x^i(x) + \alpha' \kappa \cdot \psi \psi^i(x)) e^{\alpha' \kappa \cdot X(x)}, \]
\[ V_{\phi}^{(-1)}(y) = \xi_j \psi^j(y) e^{-\phi(y)} e^{\alpha' \kappa \cdot X(y)}, \]
\[ V_A^{(-1)}(x_1) = \xi_a \psi^a(x_1) e^{-\phi(x_1)} e^{\alpha' \kappa \cdot X(x_1)} \]
\[ V_C^{(-\frac{1}{2}, -\frac{1}{2})}(z, \bar{z}) = (P_- \mathcal{H}_{(n)} M_p)^{\alpha \beta} e^{-\phi(z)/2} S_{\alpha}(z) e^{i\frac{\alpha' \kappa \cdot X(z)}{2}} e^{-\phi(\bar{z})/2} S_{\beta}(\bar{z}) e^{i\frac{\alpha' \kappa \cdot X(\bar{z})}{2}}, \]

where $(k, q, p)$ are the momenta of the scalar field, gauge field and $C$-field. They should satisfy the on-shell condition $k^2 = q^2 = p^2 = 0$. Our notation is such that the spinorial indices should be raised by the charge conjugation matrix, $C^{\alpha \beta}$

\[ (P_- \mathcal{H}_{(n)})^{\alpha \beta} = C^{\alpha \beta} (P_- \mathcal{H}_{(n)})_{\delta \beta} \]

\footnote{We keep $\alpha'$ explicitly in this work. We can set $\alpha' = 2$ on the string theory side to simplify the computations. Some of our conventions were summarized, e.g., in Appendix A of [3].}
In particular, the traces are defined as the following
\[ \text{Tr} \left( P_- \mathbb{H} (n) M_p \gamma^k \right) \equiv (P_- \mathbb{H} (n) M_p)^{\alpha \beta} (\gamma^k C^{-1})_{\alpha \beta} \]
\[ \text{Tr} \left( P_- \mathbb{H} (n) M_p \Gamma_{jai} \right) \equiv (P_- \mathbb{H} (n) M_p)^{\alpha \beta} (\Gamma^{jai} C^{-1})_{\alpha \beta} \]  
(3)

where \( P_- \) is a projection operator and its definition is \( P_- = \frac{1}{2} (1 - \gamma^{11}) \), and the definition of RR field strength is
\[ \mathbb{H} (n) = \frac{a_n}{n!} H_{\mu_1 \ldots \mu_n} \gamma^{\mu_1} \ldots \gamma^{\mu_n} , \]
with \( n = 2, 4 \) for type IIA and \( n = 1, 3, 5 \) for type IIB. \( a_n = i \) for IIA and \( a_n = 1 \) for IIB theory. To work with standard holomorphic world sheet correlators, we embed the usual doubling trick. To see doubling trick and more details on correlation functions between spin operators and some currents and/or fermion fields we refer to Appendix A of [7]. One may use holomorphic correlators for the world-sheet fields \( X^\mu, \psi^\mu, \phi \)
\[ \langle X^\mu (z) X^\nu (w) \rangle = -\eta^{\mu \nu} \log (z - w) , \]
\[ \langle \psi^\mu (z) \psi^\nu (w) \rangle = -\eta^{\mu \nu} (z - w)^{-1} , \]
\[ \langle \phi (z) \phi (w) \rangle = -\log (z - w) . \]  
(4)

Note that in this paper we do not fix the over all signs of the equations.

Let us consider
\[ A^{C \phi} \sim \int dx_1 dz d\bar{z} \langle V_\phi (-1) (x_1) V_{RR} (-\frac{1}{2}, -\frac{1}{2}) (z, \bar{z}) \rangle , \]
(5)
where the open string vertex operators ( for this amplitude just \( x_1 \) ) should be inserted at the boundary of disk world sheet and the closed string vertex operator such that \( z = x + iy, \bar{z} = x - iy \) must be inserted inside of disk. After performing Wick contractions, one finds
\[ A^{C \phi} = -\left( \frac{\pi \mu_p}{4} \right) \xi_{1i} \text{Tr} \left( P_- \mathbb{H} (n) M_p \gamma^i \right) \]  
(6)

The amplitude has been normalized by multiplying \( \frac{\sqrt{2\pi}}{4} \mu_p \). The normalization is chosen to match with the field theory computation. There will be similar rescalings for other amplitudes below. Above we have used the Jacobian \( j = |x_{14} x_{45} x_{51}|, x_4 \equiv z, x_5 \equiv \bar{z} \). In the field theory, this amplitude is reproduced by the following coupling,
\[ S^{(1)} = \lambda \mu_p \int d^{p+1} \sigma \frac{1}{(p + 1)!} (-\varepsilon^\nu)^{a_0 \ldots a_p} \text{Tr} \left( \Phi^i \right) \partial_i C_{a_0 \ldots a_p}^{(p+1)} (\sigma) . \]  
(7)

The scalar field in the coupling above comes from the Taylor expansion (see section 5 of [7]). Note also that \( \mu_p \) is the RR charge of branes and \( (-\varepsilon^\nu)^{a_0 \ldots a_p} \) is the volume form parallel to the world volume of the brane. Eq.(7) can be rewritten as
\[ S^{(1)} = \lambda \mu_p \int d^{p+1} \sigma \frac{1}{(p + 1)!} (-\varepsilon^\nu)^{a_0 \ldots a_p} \text{Tr} \left( \Phi^i \right) H_{ia_0 \ldots a_p}^{(p+2)} (\sigma) . \]  
(8)
One can easily check that \( \mathcal{A} \) precisely reproduces the string theory amplitude of \( \mathcal{A} \). Our second example of a two-point amplitude is \( \langle V_C V_A \rangle \),

\[
\mathcal{A}^{CA} \sim \int dxdzd\bar{z} \langle V_A^{(-1)}(x) V_{RR}^{(-1/2, -1/2)}(z, \bar{z}) \rangle,
\]

The final form of the amplitude is given by

\[
\mathcal{A}^{CA} \sim 2^{-1/2} \xi_1 a \text{Tr} \left( P_- H(n) M_p \gamma^a \frac{\pi \mu_2^{1/2}}{4} \right)
\]

where the amplitude has been normalized with \( \frac{\pi \mu_2^{1/2}}{4} \). The result can be reproduced by field theory using a coupling between a RR \( (p-1) \)-form and one gauge field; it takes the form of a Wess-Zumino coupling,

\[
S^{(2)} = (2\pi \alpha') \mu_p \int d^{p+1} \sigma C^{(p-1)} \wedge F.
\]

This WZ action can be rewritten as

\[
S^{(2)} = i(2\pi \alpha') \mu_p \int d^{p+1} \sigma \frac{1}{(p)!} (e^v)^{a_0 \ldots a_p} H^{(p)} H_{a_0 \ldots a_{p-1}} \xi_{a_p}.
\]

Extracting the trace in (10) and considering the coupling (11) in field theory exactly reproduce the amplitude of \( \langle V_C V_A \rangle \) in (10).

### 2.2 The three-point function \( \langle V_C V.phi V.phi \rangle \)

The scattering amplitude between one RR and two scalar fields \( \langle V_C V.phi V.phi \rangle \) is given by

\[
\mathcal{A}^{C\phi \phi} \sim \int dx_1 dx_2 dx_4 dx_5 \langle V_\phi^{(-1)}(x_1) V_\phi^{(0)}(x_2) V_{RR}^{(-1/2, -1/2)}(z, \bar{z}) \rangle,
\]

After some algebra, one gets

\[
\mathcal{A}^{C\phi \phi} \sim \int dx_1 dx_2 dx_4 dx_5 (P_- H(n) M_p)^{\alpha \beta} \xi_{i i} \xi_{j i} x_4^{1/4}(x_{14} x_{15})^{-1/2} \times (I_1 + I_2)
\]

where \( I_1 \) and \( I_2 \) are given by

\[
I_1 = \langle :e^{i k_1.x(x_1)} :\partial X^j(x_2) e^{i k_2.x(x_2)} :e^{i \frac{2}{p}.X(x_4)} :e^{i \frac{2}{p}.D.X(x_5)} :, \rangle \times \langle S_\alpha(x_4) :S_\beta(x_5) :\psi^i(x_1) :> \]

\[
I_2 = \langle :e^{i k_1.x(x_1)} :e^{i k_2.x(x_2)} :e^{i \frac{2}{p}.X(x_4)} :e^{i \frac{2}{p}.D.X(x_5)} :, \rangle \times \langle S_\alpha(x_4) :S_\beta(x_5) :\psi^i(x_1) :\alpha'^i k_2a \psi^a \psi^j(x_2) :> \rangle.
\]

\[5\] The three-point amplitude \( \langle V_C V_A V_A \rangle \) and its infinite contact terms can be found in [22] in the current context.
Using the basic OPEs \[17, 9, 10\], the fermionic correlators are given by
\[
I_1^i \equiv <: S_\alpha(x_4) : S_\beta(x_5) : \psi^i(x_1) : = 2^{-1/2} x_{45}^{-3/4} (x_{14} x_{15})^{-1/2} (\gamma^i C^{-1})_{\alpha\beta}. \tag{15}\n\]
\[
<: S_\alpha(x_4) : S_\beta(x_5) : \psi^a \psi^j(x_1) : = -\frac{1}{2} x_{45}^{-1/4} x_{14}^{-1} x_{15}^{-1} (\Gamma^{ai} C^{-1})_{\alpha\beta}. \tag{16}\n\]
Generalizing this, the correlation function of two spin operators, one fermion field and one current was obtained in \[6\] according to which,
\[
I_2^{jai} \equiv <: S_\alpha(x_4) : S_\beta(x_5) : \psi^i(x_1) : \psi^a \psi^j(x_2) : > = \left\{ (\Gamma^{jai} C^{-1})_{\alpha\beta} + \frac{\alpha' \Re[x_{14} x_{25}]}{x_{12} x_{45}} (\eta^{ij} (\gamma^a C^{-1})_{\alpha\beta}) \right\} \times 2^{-3/2} x_{45}^{-1/4} (x_{14} x_{15})^{-1/2}. \tag{17}\n\]
Substituting the spin correlators above into the amplitude and working out the X correlators, one finds
\[
\mathcal{A}^{C菲} \sim \int dx_1 dx_2 dx_4 dx_5 (P_\mathcal{H}(n) M_p)^{\alpha\beta} I_1^{\xi_1 \xi_2} x_{45}^{-1/4} (x_{14} x_{15})^{-1/2} \times \left( I_1^i (a_1^j) + i \alpha' k_2 \Gamma^{jai} \right), \tag{18}\n\]
where
\[
I = |x_{12}|^{\alpha \xi_1, k_2} |x_{14} x_{15}|^{\alpha \xi_2, k_2} |x_{24} x_{25}|^{\alpha \xi_2, k_2} |x_{45}|^{\alpha \xi_2, k_2} - p_d^D p, \quad a_1^j = i p_j x_{54} x_{24} x_{25}. \tag{19}\n\]
Let us gauge-fix \(SL(2, R)\) invariance as follows:
\[
(x_1, x_2, x_4, x_5) = (x, -x, i, -i) \tag{20}\n\]
with which the jacobian takes \(J = -2i (1 + x^2)\). The amplitude now takes
\[
\mathcal{A}^{C菲} = \int_{-\infty}^{\infty} dx (x^2 + 1)^{2t-1} x^{-2t} (2 \xi_1 \xi_2) 2^{-1/2} (x_{14} x_{15})^{-1/2} \times \left[ -p_\mathcal{H}(n) M_p \gamma^i \right] \tag{21}\n\]
where \(t = \frac{\alpha'}{2} (k_1 + k_2)^2\). Note that the term \(\alpha' \Re[x_{14} x_{25}]\) does not contribute to the amplitude. This is because the integrand is odd and the integration is over the whole worldsheet. Carrying out the integration and using momentum conservation, the final result of the amplitude is shown to be
\[
\mathcal{A}^{C菲} = (2 \xi_1 \xi_2) 2^{-1/2} \pi^{1/2} \frac{\Gamma(-t + \frac{1}{2})}{\Gamma(-t + 1)} \times \left[ -p_\mathcal{H}(n) M_p \gamma^i \right] \tag{22}\n\]
Figure 1: The Feynman diagram corresponding to the amplitude of (21).

Let us consider the low energy expansion in which the Mandelstam variable \( t \) is sent to \( t \rightarrow 0 \). The limit is equivalent to taking \( \alpha' \rightarrow 0 \) on the string amplitudes. It turns out that the amplitude is non zero only for \( n = p + 2 \). It is obvious from the gamma functions that the desired amplitude has an infinite number of contact terms. The expansion of the \( \Gamma \)-function factors can be written as

\[
\sqrt{\pi} \frac{\Gamma(-t + \frac{1}{2})}{\Gamma(-t + 1)} = \pi \sum_{n=-1}^{\infty} c_n (t)^{n+1}
\]  

(22)

The first few coefficients, \( c_n \), are

\[
c_{-1} = 1, \quad c_0 = 2\ln(2), \quad c_1 = \frac{\pi^2}{6} + 2\ln(2)^2
\]

(23)

Let us find the SYM vertices that reproduce this string result. First we start with a Chern-Simons action. The minimal form of the vertex includes a bulk RR \((p+1)\)-form potential and the two world volume scalars,

\[
S^{(3)} = \frac{i \lambda^2 \mu_p}{2(p+1)!} \int d^{p+1}\sigma (\varepsilon^v)^{a_0 \ldots a_p} \text{Tr} (\Phi^i \Phi^j) \partial_i \partial_j C^{(p+1)}_{a_0 \ldots a_p} (\sigma)
\]

The scalars have come from the Taylor expansion. One can rewrite the coupling above as

\[
S^{(3)} = \frac{i \lambda^2 \mu_p}{2(p+1)!} \int d^{p+1}\sigma (\varepsilon^v)^{a_0 \ldots a_p} \text{Tr} (\Phi^i \Phi^j) \partial_i H_{a_0 \ldots a_p}^{(p+2)} (\sigma)
\]

(24)

Normalizing the amplitude (21) by \( \frac{\pi \mu_p^{21/2}}{4} \), it is possible to produce all the contact terms for the first term of the amplitude (21) by a higher derivative extension of the coupling.

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6 Given a string amplitude that involves scalar vertices, there are three ways to construct the corresponding field theory vertices. The first is through Wess-Zumino type interactions, and was proposed in the Myers' paper [4]. The second is to examine pull-back procedure. The third - which we call "Taylor expansion" - was mentioned for example in section 5 of [7]. The terms of the third type take the form of Taylor expansion but they do not arise from Wess-Zumino terms.
above,
\[ S^{(3)} = \frac{\lambda^2 \mu_p}{2(p + 1)!} \int d^{p+1} \sigma (\varepsilon^a)_{a_0, a_p} p^i H_{i a_0, a_p}^{(p + 2)} (\sigma) \]
\[ \times \sum_{n = -1}^{\infty} c_n (\alpha')^n \mathrm{Tr} (\partial_i \partial_{a_1} \Phi \partial^{a_1} \Phi^{a_2} \Phi^{a_p}) \]
\[ H^{(p + 2)} = dC^{(p + 1)}. \]
Now we are going to re-derive all infinite contact terms of the second term of (21). In order to do so, the following interaction vertex must be taken as well,
\[ S^{(4)} = i \frac{\lambda^2 \mu_p}{2(p)!} \int d^{p+1} \sigma (\varepsilon^a)_{a_0, a_p} \mathrm{Tr} (D_{a_0} \Phi^i D_{a_1} \Phi^j) C_{i j a_2, a_p}^{(p + 1)} (\sigma) \] (26)

The scalars come from pull-back. By considering the antisymmetric property of \((\varepsilon^a)_{a_0, a_p}\),
the higher derivative extension of (26) can be re-expressed as
\[ S^{(4)} = i \frac{\lambda^2 \mu_p}{2(p)!} \int d^{p+1} \sigma (\varepsilon^a)_{a_0, a_p} \sum_{n = -1}^{\infty} c_n (\alpha')^n \mathrm{Tr} (\partial_i \partial_{a_1} \Phi \partial^{a_1} \Phi^{a_2} \Phi^{a_p}) H_{i j a_2, a_p}^{(p + 2)} (\sigma) \] (26)

Since only the couplings between one RR and two scalar fields are relevant, one can replace the covariant derivatives on the scalar fields by their partial derivatives.

3 Universality and analysis of \(< V_C V_\phi V_\phi V_\phi >\)

In this section, we compute \(< V_C V_\phi V_\phi V_\phi >\) and analyze its infinite poles and some of its contact terms. The corresponding low energy field theory vertices are determined by universality, and are subsequently shown to match the field theory poles and contact terms with those of the string amplitude. Because of the similarities between the present amplitudes and the amplitudes that were considered in the previous works, the present analysis shares some parts of the computations with the earlier works. However, most of final results concerning contact interactions cannot be derived, for example, by applying T-duality to the previous results as we will discuss in the main body.

More specifically, the amplitude of \(< V_C V_A V_A V_A >\) was analyzed in [3], and one might wonder whether the amplitude of \(< V_C V_\phi V_\phi V_\phi >\) could be derived by applying T-duality. After computing \(< V_C V_\phi V_\phi V_\phi >\), we will take up this issue. For example, we will see that the amplitude of \(< V_C V_A V_A V_A >\) has some extra terms that are absent in \(< V_C V_A V_A V_A >\).
As a comment we were not able to reproduce all contact terms of the four-point amplitude $< V_C V_\phi V_\phi V_\phi >$ with usual pull-back. This is a hint that pull-back must be modified \[24, 25\].

Although we could find the field theory vertices that reproduce the entire pole terms and some of the contact terms of the string result for our four-point function $< V_C V_\phi V_\phi V_\phi >$, more work is required to find the vertices that would reproduce all contact terms.

### 3.1 Computation of $< V_C V_\phi V_\phi V_\phi >$

Let us turn to our main result, the analysis of the amplitude of one closed string RR field and three scalar fields,

$$\mathcal{A}^{C_{\phi\phi}} \sim \int dx_1 dx_2 dx_3 dz d\bar{z} \langle V_\phi^{(-1)}(x_1)V_\phi^{(0)}(x_2)V_\phi^{(0)}(x_3)V^{(-\frac{3}{2})}_{RR}(z, \bar{z}) \rangle, \quad (27)$$

To obtain a precise result we must apply correct Wick and modified Wick-like contraction (see \[6, 7\]). With various Wick contractions, this amplitude reduces to

$$\mathcal{A}^{C_{\phi\phi}} \sim \int dx_1 dx_2 dx_3 dz d\bar{z} (P_H \mathcal{M}_p)_{\alpha\beta} \xi_1 \xi_2 \xi_3 x_{45}^{-1/4} (x_{14} x_{15})^{-1/2} \times (I_1 + I_2 + I_3 + I_4) \text{Tr} (\lambda_1 \lambda_2 \lambda_3), \quad (28)$$

for a particular ordering that we call 123 ordering. The explicit expressions for $I$’s are

- \[I_1 = \langle e^{\alpha' i k_1 X(x_1)} \partial X^j (x_2) e^{\alpha' i k_2 X(x_2)} \partial X^k (x_3) e^{\alpha' i k_3 X(x_3)} : e^{\frac{1}{2} i p.D.X(x_4)} e^{\frac{1}{2} i p.D.X(x_5)} : \rangle \times \langle S_\alpha(x_4) : S_\beta(x_5) : \psi^j(x_1) : \rangle,\]

- \[I_2 = \langle e^{\alpha' i k_1 X(x_1)} : e^{\alpha' i k_2 X(x_2)} \partial X^k (x_3) e^{\alpha' i k_3 X(x_3)} : e^{\frac{1}{2} i p.D.X(x_4)} e^{\frac{1}{2} i p.D.X(x_5)} : \rangle \times \langle S_\alpha(x_4) : S_\beta(x_5) : \psi^j(x_1) : \rangle,\]

- \[I_3 = \langle e^{\alpha' i k_1 X(x_1)} \partial X^j (x_2) e^{\alpha' i k_2 X(x_2)} e^{\alpha' i k_3 X(x_3)} : e^{\frac{1}{2} i p.D.X(x_4)} e^{\frac{1}{2} i p.D.X(x_5)} : \rangle \times \langle S_\alpha(x_4) : S_\beta(x_5) : \psi^j(x_1) : \rangle,\]

- \[I_4 = \langle e^{\alpha' i k_1 X(x_1)} : e^{\alpha' i k_2 X(x_2)} e^{\alpha' i k_3 X(x_3)} : e^{\frac{1}{2} i p.D.X(x_4)} e^{\frac{1}{2} i p.D.X(x_5)} : \rangle \times \langle S_\alpha(x_4) : S_\beta(x_5) : \psi^j(x_1) : \rangle,\]

Using the results of \[10, 7, 8\], one can show that

\[I_5^{ji} = \langle S_\alpha(x_4) : S_\beta(x_5) : \psi^j(x_1) : \psi^{ij}(x_2) : \rangle = \left\{ (\Gamma^{jai} C^{-1})_{\alpha\beta} + \alpha' \text{Re} \left[ \frac{\eta^{ij} (-\gamma^a C^{-1})_{\alpha\beta}}{x_{12} x_{45}} \right] \right\} \times 2^{-3/2} x_{45}^{-1/4} (x_{14} x_{15})^{-1/2}. \quad (30)\]

\[8\] It might be a hint that pull-back may need to be modified. We will discuss this issue in the conclusion section.
The correct result for the correlation function between two spin operators, two currents and one worldsheet fermion was obtained in [3],

\[
I_6^{kbjai} = \langle : S_\alpha(x_4) : S_\beta(x_5) :: \psi^i(x_1) : \psi^{a_i}(x_2) : \psi^{b_i}(x_3) > \\
= \left\{ (\Gamma^k b j a C^{-1})_{\alpha \beta} + \alpha r_1 \frac{Re[x_{24}x_{25}]}{x_{12}x_{45}} + \alpha r_2 \frac{Re[x_{24}x_{25}]}{x_{13}x_{45}} + \alpha r_3 \frac{Re[x_{24}x_{35}]}{x_{23}x_{45}} + \alpha^2 r_4 \times \left( \frac{Re[x_{24}x_{35}]}{x_{23}x_{45}} \right)^2 + \alpha^2 r_5 \left( \frac{Re[x_{24}x_{25}]}{x_{12}x_{45}} \times \frac{Re[x_{24}x_{35}]}{x_{23}x_{45}} \right) + \alpha^2 r_6 \left( \frac{Re[x_{14}x_{25}]}{x_{13}x_{45}} \right) \right\} 2^{-5/2} x_{45}^3 (x_{24}x_{25}x_{34}x_{35})^{-1}(x_{14}x_{15})^{-1/2},
\]

(31)

where

\[
\begin{align*}
  r_1 &= \left( \eta^{ij} (\Gamma^{kba} C^{-1})_{\alpha \beta} \right), \\
  r_2 &= \left( \eta^{jk} (\Gamma^{bja} C^{-1})_{\alpha \beta} \right), \\
  r_3 &= \left( \eta^{ab} (\Gamma^{kji} C^{-1})_{\alpha \beta} + \eta^{jk} (\Gamma^{bai} C^{-1})_{\alpha \beta} \right), \\
  r_4 &= \left( (-\eta^{ab} \eta^{jk}) (\gamma^i C^{-1})_{\alpha \beta} \right), \\
  r_5 &= \left( (-\eta^{ji} \eta^{ab}) (\gamma^k C^{-1})_{\alpha \beta} \right), \\
  r_6 &= \left( \eta^{ik} \eta^{ab} (\gamma^i C^{-1})_{\alpha \beta} \right).
\end{align*}
\]

(32)

Inserting the spin correlators above in the amplitude and performing contractions over X, one finds:

\[
A^{C\phi\phi} \sim \int dx_1 dx_2 dx_3 dx_4 dx_5 (P^- \mathcal{H} (n) M_p)_{\alpha \beta} I_{\xi_1 \xi_2 \xi_3} \xi_4 x_{45}^{-1/4} (x_{14}x_{15})^{-1/2} \\
\times \left\{ I_7 (-\eta^{jk} x_{23} + a_1^1 a_2^k) + a_2^k a_3^i + a_4^i a_4^k - \alpha^2 k_{2a} k_{3b} I_6^{kbjai} \right\} \mathrm{Tr} (\lambda_1 \lambda_2 \lambda_3),
\]

(33)

where \( I_6^{kbjai} \) is given in [31], and

\[
\begin{align*}
  I &= |x_{12}|^{\alpha_2 k_1, k_2} |x_{13}|^{\alpha_2 k_1, k_3} |x_{14}x_{15}|^{\alpha_2 k_2, k_3} |x_{23}|^{\alpha_2 k_2, k_1} |x_{24}x_{25}|^{\alpha_2 k_2, p} |x_{34}x_{35}|^{\alpha_2 k_2, p} |x_{45}|^{\alpha_2 p, D, p}, \\
  a_1^i &= ip^j x_{54}, \\
  a_2^k &= ip^k x_{54}, \\
  a_3^i &= \alpha^i k_{2a} I_5^{a i}, \\
  a_4^k &= \alpha^i k_{3b} 2^{-3/2} x_{45}^{1/4} (x_{34}x_{35})^{-1}(x_{14}x_{15})^{-1/2} \\
  &\times \left\{ (\Gamma^{kba} C^{-1})_{\alpha \beta} + \alpha^i \frac{Re[x_{14} x_{35}]}{x_{13} x_{45}} \left( \eta^{ik} (\gamma^b C^{-1})_{\alpha \beta} \right) \right\}, \\
  I_7 &= \langle : S_\alpha(x_4) : S_\beta(x_5) : \psi^i(x_1) : >= 2^{-1/2} x_{45}^{-3/4} (x_{14}x_{15})^{-1/2} (\gamma^i C^{-1})_{\alpha \beta}.
\end{align*}
\]

(34)
Using the integral presented in, e.g., [26] and [7], one can rewrite the amplitude (33) as

\[ A_{C\phi\phi} = A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8 + A_9 + A_{10} \]  \hspace{1cm} (35)

with

\[ A_1 \sim -2^{-1/2} \xi_{1i}\xi_{2j}\xi_{3k} \left[ k_{3b} k_{2a} \text{Tr} \left( P_- H^{(n)} M_p \Gamma^{kbi} \right) - k_{3b} p^i \text{Tr} \left( P_- H^{(n)} M_p \Gamma^{kbi} \right) \right. \]

\[ - k_{2a} p^b \text{Tr} \left( P_- H^{(n)} M_p \Gamma^{jai} \right) + p^j p^b \text{Tr} \left( P_- H^{(n)} M_p \gamma^i \right) \right] L_1, \]

\[ A_2 \sim 2^{-1/2} \left\{ 2 \xi_{1i} \xi_{2j} k_{2a} k_{3b} \xi_{3k} \text{Tr} \left( P_- H^{(n)} M_p \Gamma^{kba} \right) \right\} \]

\[ A_3 \sim 2^{-1/2} \left\{ \xi_{1i} \xi_{2j} \xi_{3k} \text{Tr} \left( P_- H^{(n)} M_p \Gamma^{jia} \right) \right\} L_2, \]

\[ A_4 \sim -2^{-1/2} \left\{ 2 \xi_{3j} \xi_{1i} k_{2a} k_{3b} \xi_{2k} \text{Tr} \left( P_- H^{(n)} M_p \Gamma^{bja} \right) \right\} L_4, \]

\[ A_5 \sim 2^{-1/2} \left\{ 2 \xi_{3j} k_{2a} k_{3b} \xi_{1i} \text{Tr} \left( P_- H^{(n)} M_p \Gamma^{bai} \right) \right\} L_5, \]

\[ A_6 \sim 2^{1/2} L_{33} \left\{ - k_{2a} p^k \xi_{1i} \xi_{2j} \xi_{3k} \text{Tr} \left( P_- H^{(n)} M_p \gamma^a \right) \right\}, \]

\[ A_7 \sim 2^{1/2} L_3 \left\{ k_{3b} p^i \xi_{1i} \xi_{2j} \xi_{3k} \text{Tr} \left( P_- H^{(n)} M_p \gamma^b \right) \right\}, \]

\[ A_8 \sim 2^{1/2} L_6 \left\{ \xi_{2j} \text{Tr} \left( P_- H^{(n)} M_p \gamma^j \right) \left( u t \xi_{1i} \xi_{3k} \right) \right\}, \]

\[ A_9 \sim 2^{1/2} L_6 \left\{ \xi_{3k} \text{Tr} \left( P_- H^{(n)} M_p \gamma^k \right) \left( u s \xi_{1i} \xi_{2j} \right) \right\}, \]

\[ A_{10} \sim 2^{1/2} L_6 \left\{ \xi_{1i} \text{Tr} \left( P_- H^{(n)} M_p \gamma^i \right) \left( t s \xi_{3j} \xi_{2k} \right) \right\} \]  \hspace{1cm} (36)

where

\[ L_1 = (2)^{-2(t+s+u+1)} \frac{\Gamma(-u + \frac{1}{2}) \Gamma(-s + \frac{1}{2}) \Gamma(-t + \frac{1}{2}) \Gamma(-t - s - u + 1)}{\Gamma(-u - t + 1) \Gamma(-t - s + 1) \Gamma(-s - u + 1)}, \]

\[ L_2 = (2)^{-2(t+s+u)} \frac{\Gamma(-u + 1) \Gamma(-s + 1) \Gamma(-t) \Gamma(-t - s - u + 1)}{\Gamma(-u - t + 1) \Gamma(-t - s + 1) \Gamma(-s - u + 1)}, \]

\[ L_{22} = (2)^{-2(t+s+u)} \frac{\Gamma(-u + 1) \Gamma(-s + 1) \Gamma(-t + 1) \Gamma(-t - s - u + 1)}{\Gamma(-u - t + 1) \Gamma(-t - s + 1) \Gamma(-s - u + 1)}, \]

\[ L_{33} = (2)^{-2(t+s+u)} \frac{\Gamma(-u + 1) \Gamma(-s + 1) \Gamma(-t) \Gamma(-t - s - u + \frac{1}{2})}{\Gamma(-u - t + 1) \Gamma(-t - s + 1) \Gamma(-s - u + 1)}, \]

\[ L_3 = (2)^{-2(t+s+u)} \frac{\Gamma(-u + 1) \Gamma(-s) \Gamma(-t + 1) \Gamma(-t - s - u + \frac{1}{2})}{\Gamma(-u - t + 1) \Gamma(-t - s + 1) \Gamma(-s - u + 1)}, \]

\[ L_4 = (2)^{-2(t+s+u)} \frac{\Gamma(-u + 1) \Gamma(-s) \Gamma(-t + 1) \Gamma(-t - s - u + \frac{1}{2})}{\Gamma(-u - t + 1) \Gamma(-t - s + 1) \Gamma(-s - u + 1)}, \]
where
\[ s = -\frac{\alpha'}{2} (k_1 + k_3)^2, \quad t = -\frac{\alpha'}{2} (k_1 + k_2)^2, \quad u = -\frac{\alpha'}{2} (k_2 + k_3)^2 \]

\[ \mathcal{A}^{C\phi\phi} \] can be further simplified to
\[ \mathcal{A}^{C\phi\phi} = \mathcal{A}_1' + \mathcal{A}_2' + \mathcal{A}_3', \]  
where
\[ \mathcal{A}_1' \sim 2^{-1/2} 2\xi_{1i} \xi_{2j} \xi_{3k} (t + s + u) L_1 \left[ k_{3b} k_{2a} \text{Tr} \left( \mathcal{H} (n) M_p \Gamma^{kbjai} \right) - k_{3b} p^j \text{Tr} \left( \mathcal{H} (n) M_p \Gamma^{kbj} \right) \right] - k_{2a} p^b \text{Tr} \left( \mathcal{H} (n) M_p \Gamma^{jai} \right) + p^b p^k \text{Tr} \left( \mathcal{H} (n) M_p \Gamma^j \right) \]
\[ \mathcal{A}_2' \sim 2^{-1/2} L_2' \left[ 2 u s \xi_{1i} \xi_{2j} k_{2a} k_{3b} \xi_{3k} \text{Tr} \left( \mathcal{H} (n) M_p \Gamma^{kba} \right) - u s t \xi_{1i} \xi_{2j} \xi_{3k} \text{Tr} \left( \mathcal{H} (n) M_p \Gamma^{kji} \right) - 2 u t s \xi_{1i} k_{2a} k_{3b} \xi_{2j} \text{Tr} \left( \mathcal{H} (n) M_p \Gamma^{bij} \right) + 2 u s t \xi_{1i} k_{2a} k_{3b} \xi_{3k} \text{Tr} \left( \mathcal{H} (n) M_p \Gamma^{bai} \right) - 2 u s t \xi_{1i} k_{2a} p^k \xi_{2j} \xi_{3k} \text{Tr} \left( \mathcal{H} (n) M_p \Gamma^a \right) + 2 u t s k_{2a} p^j \xi_{1i} \xi_{2j} \xi_{3k} \text{Tr} \left( \mathcal{H} (n) M_p \Gamma^b \right) \right] \]
\[ \mathcal{A}_3' \sim 2^{-1/2} L_1' \left[ \xi_{1i} \text{Tr} \left( \mathcal{H} (n) M_p \Gamma^j \right) (t s \xi_{3k} \xi_{2j}) + \left[ 1 \leftrightarrow 2 \right] + \left[ 1 \leftrightarrow 3 \right] \right]. \]

The functions \( L_1', L_2' \) are given as follows:
\[ L_1' = (2)^{-2(t+s+u)} \pi \frac{\Gamma(-u) \Gamma(-t + \frac{1}{2}) \Gamma(-s + \frac{1}{2}) \Gamma(-t - s - u)}{\Gamma(-u - t + 1) \Gamma(-t - s + 1) \Gamma(-s - u + 1)}, \]
\[ L_2' = (2)^{-2(t+s+u)} \pi \frac{\Gamma(-u) \Gamma(-s) \Gamma(-t) \Gamma(-t - s - u + \frac{1}{2})}{\Gamma(-u - t + 1) \Gamma(-t - s + 1) \Gamma(-s - u + 1)} \]

For the amplitude of two fermions and three massless scalar vertex operators, there is no correlation between \( < \partial X^j (x_2) e^{2ik \cdot X(x)} > \) and \( < \partial X^k (x_3) e^{2ik \cdot X(x)} > \); one can easily see that the terms \( \mathcal{A}_6, \mathcal{A}_7, \) and all terms of \( \mathcal{A}_1 \) except its first term identically vanish. Since \( \mathcal{H} (n), M_p \) and \( \Gamma^{kji} \) are totally antisymmetric combinations of the Gamma matrices, it follows that the amplitude is nonzero only for \( n = p + 4, \ n = p + 2, \) and \( p = n \). From the poles of the gamma functions, one can easily see that the scattering amplitude has infinite massless and massive poles. In order to compare this with the field theory, which has massless fields, one must expand the amplitude so that the massless poles of the field theory remain while
all massive poles rearrange themselves in the form of contact interactions. The low energy expansion is carried out by sending all Mandelstam variables to zero.

As stated in the introduction, the above results cannot be entirely derived by applying T-duality to the previous result that was obtained for $< V_C V_A V_A >$ in [6]. All the terms that contain the transverse components of the momentum $p^i$ such as $p^i p^k$, the second and third terms in $A'_1$, and also the last two terms in $A'_2$ in (40), are not present in the corresponding result of [6]. We will ponder this issue in the conclusion.

Let us momentum-expand the string amplitude above in the $\alpha' \to 0$ limit,

$$s \to 0, \quad t \to 0, \quad u \to 0. \quad (42)$$

The Mandelstam variables satisfy

$$s + t + u = -p_a p^a. \quad (43)$$

Note that the combination $stL'_1$ has appeared in eq. (40) and that $L'_1$ is symmetric under the interchange of $(u, t, s)$. For the $st L'_1$ term, the proper expansion is

$$stL'_1 = -\pi^{5/2} \left( \sum_{n=0}^{\infty} c_n (s + t + u)^n \sum_{n,m=0}^{\infty} c_{n,m} [s^n t^m + s^m t^n] \frac{1}{(t + s + u)} + \sum_{p,n,m=0}^{\infty} f_{p,n,m} (s + t + u)^p [(s + t)^n (st)^m] \right), \quad (44)$$

When considering $suL'_1$, the proper expansion for $suL'_1$ is such that $t \leftrightarrow u$ in the expansion above. Similarly when considering $tuL'_1$, the proper expansion for $tuL'_1$ is such that $s \leftrightarrow u$ in (44). The expansion for $L'_2$ can be similarly summarized:

$$suL'_2 = -\pi^{3/2} \sum_{n=-1}^{\infty} b_n \left( \frac{1}{u} (u + s)^{n+1} \right) + \sum_{p,n,m=0}^{\infty} e_{p,n,m} u^p (s u)^n (s + u)^m$$

$$stL'_2 = -\pi^{3/2} \sum_{n=-1}^{\infty} b_n \left( \frac{1}{t} (t + s)^{n+1} \right) + \sum_{p,n,m=0}^{\infty} e_{p,n,m} t^p (s t)^n (s + t)^m$$

$$tuL'_2 = -\pi^{3/2} \sum_{n=-1}^{\infty} b_n \left( \frac{1}{s} (u + t)^{n+1} \right) + \sum_{p,n,m=0}^{\infty} e_{p,n,m} s^p (s u)^n (s + u)^m \quad (45)$$

where some of the coefficients $b_n$, $e_{p,n,m}$, $c_n$, $c_{n,m}$ and $f_{p,n,m}$ are

$$b_{-1} = 1, \quad b_0 = 0, \quad b_1 = \frac{1}{6} \pi^2, \quad b_2 = 2 \zeta(3), \quad c_0 = 0, \quad c_1 = -\frac{\pi^2}{6}. \quad (46)$$

$^9$ The constraint (43) implies that $p_a p^a \to 0$. Also note that it is known that for amplitudes including tachyon and RR, the expansion makes sense only for a constant value of $p^a p_a \to \frac{1}{4}$. [7].
\[ e_{2,0,0} = e_{0,1,0} = 2\zeta(3), e_{1,0,0} = \frac{1}{6}\pi^2, e_{1,0,2} = \frac{19}{60}\pi^4, e_{0,0,2} = 6\zeta(3), \]
\[ c_2 = -2\zeta(3), c_{1,1} = \frac{\pi^2}{6}, c_{0,0} = \frac{1}{2}, c_{3,1} = c_{1,3} = \frac{2}{15}\pi^4, c_{2,2} = \frac{1}{5}\pi^4, \]
\[ c_{1,0} = c_{0,1} = 0, c_{3,0} = c_{0,3} = 0, c_{2,0} = c_{0,2} = \frac{\pi^2}{6}, c_{1,2} = c_{2,1} = -4\zeta(3), c_{4,0} = c_{0,4} = \frac{1}{15}\pi^4, \]

The coefficients \( b_n \) appeared in the momentum expansion of the S-matrix element of one RR, three gauge field vertex operators [6]. The function of \( L' \) has infinite massless scalar poles in the \((t+s+u)\)-channel (in contrast with the amplitude of \( <V_C V_A V_A> \), which has infinite gauge but not scalar poles) and \( L_2 \) has infinite massless gauge poles in \( t,-s \)-and \( u \)-channels.

### 3.2 Universality in all-\( \alpha' \) order higher derivative corrections of non-BPS and BPS branes

Below we determine all higher derivative corrections of non-BPS and BPS branes by utilizing the pattern that appeared in previous works that we now review.

Our first example of the pattern is the case of two tachyons and two scalar fields on the world volume of \( N \) non-BPS D-branes [7]. Given the leading order vertices as follows

\[
2T_p(\pi\alpha')^3\text{STr} \left( m^2T^2(D_a\phi^iD^a\phi_i) + D^aTD_aTD_a\phi^iD^a\phi_i - 2D_a\phi^iD\phi_iD^\mu TD^\alpha T \right)
\]

the all-order vertices turned out to be

\[
\mathcal{L} = -2T_p(\pi\alpha')(\alpha')^{2+n+m} \sum_{n,m=0}^{\infty} (\mathcal{L}_1^{nm} + \mathcal{L}_2^{nm} + \mathcal{L}_3^{nm} + \mathcal{L}_4^{nm})
\]

where

\[
\mathcal{L}_1^{nm} = m^2\text{Tr} \left( a_{n,m}[D_{nm}(T^2D_a\phi^iD^a\phi_i) + D_{nm}(D_a\phi^iD^a\phi_iT^2)] \\
+ b_{n,m}[D'_{nm}(TD_a\phi^iTD^a\phi_i) + D'_{nm}(D_a\phi^iTD^a\phi_iT)] + h.c. \right)
\]

\[
\mathcal{L}_2^{nm} = \text{Tr} \left( a_{n,m}[D_{nm}(D^aTD_aTD_a\phi^iD^a\phi_i) + D_{nm}(D_a\phi^iD^a\phi_iD^\alpha TTD_aT)] \\
+ b_{n,m}[D'_{nm}(D^aTD_a\phi^iD^a\phi_i) + D'_{nm}(D_a\phi^iD_aT^a\phi_iD^\alpha T)] + h.c. \right)
\]

\[
\mathcal{L}_3^{nm} = -\text{Tr} \left( a_{n,m}[D_{nm}(D^\beta TD_aTD^\mu\phi^iD\beta\phi_i) + D_{nm}(D^\mu\phi^iD\beta\phi_iD^\beta TD_aT)] \\
+ b_{n,m}[D'_{nm}(D^\beta TD^\mu\phi^iD\beta\phi_i) + D'_{nm}(D^\mu\phi^iD\beta TD_a\phi_iD^\beta T)] + h.c. \right)
\]

\[
\mathcal{L}_4^{nm} = -\text{Tr} \left( a_{n,m}[D_{nm}(D^\beta TD^\mu TTD_a\phi^iD\mu\phi_i) + D_{nm}(D^\beta \phi^iD^\mu TTD_a\phi_iD^\beta T)] \\
+ b_{n,m}[D'_{nm}(D^\beta TD\phi^iD^\mu TD\phi_i) + D'_{nm}(D\phi^iD^\mu TD^\beta\phi_iD^\beta T)] + h.c. \right)
\]
where
\[
\mathcal{D}_{nm}(EFGH) \equiv D_{b_1} \cdots D_{b_m} D_{a_1} \cdots D_{a_n} EFD_{a_1} \cdots D_{a_n} G D_{b_1} \cdots D_{b_m} H,
\]
\[
\mathcal{D}_{nm}'(EFGH) \equiv D_{b_1} \cdots D_{b_m} D_{a_1} \cdots D_{a_n} E D_{a_1} \cdots D_{a_n} F G D_{b_1} \cdots D_{b_m} H.
\] (49)

The crucial step seems to extract the symmetric trace in terms of the ordinary trace and apply the higher derivative corrections \(\mathcal{D}_{nm}, \mathcal{D}_{nm}'\) on it.

The second example is the amplitude of two tachyons and two gauge fields on the worldvolume of non-BPS branes \([10]\). One just applies \(\mathcal{D}_{nm}, \mathcal{D}_{nm}'\) on the couplings between two tachyons and two gauge fields.\(^{10}\) Now let us turn to BPS systems.

The third example is four-gauge field couplings
\[
- T_p (2\pi \alpha')^4 S \text{Tr} \left( -\frac{1}{8} F_{bd} F_{df} F_{fh} F_{hb} + \frac{1}{32} (F_{ab} F_{ba})^2 \right). \tag{50}
\]
The closed form of the higher derivative corrections of four-gauge fields to all orders of \(\alpha'\) (which must be added to DBI) was shown \([6]\) to be
\[
(2\pi \alpha')^4 \frac{1}{8 \pi^2} T_p (\alpha')^{n+m} \sum_{m,n=0}^{\infty} (\mathcal{L}_{nm}^5 + \mathcal{L}_{nm}^6 + \mathcal{L}_{nm}^7), \tag{51}
\]
with
\[
\mathcal{L}_{5}^{nm} = -\text{Tr} \left( a_{n,m} \mathcal{D}_{nm} F_{bd} F_{df} F_{fh} F_{hb} + b_{n,m} \mathcal{D}_{nm}' F_{bd} F_{df} F_{fh} F_{hb} + h.c. \right),
\]
\[
\mathcal{L}_{6}^{nm} = -\text{Tr} \left( a_{n,m} \mathcal{D}_{nm} F_{bd} F_{df} F_{fh} F_{hh} + b_{n,m} \mathcal{D}_{nm}' F_{bd} F_{df} F_{fh} F_{hh} + h.c. \right),
\]
\[
\mathcal{L}_{7}^{nm} = \frac{1}{2} \text{Tr} \left( a_{n,m} \mathcal{D}_{nm} F_{ab} F_{cd} F_{cd} + b_{n,m} \mathcal{D}_{nm}' F_{ab} F_{cd} F_{cd} + h.c. \right),
\]
where the higher derivative operators \(D_{nm}\) and \(D_{nm}'\) are defined in (49). These couplings are exact up to total derivative terms and these corrections have been checked by explicit computations of the amplitude of one RR and three gauge fields \([6]\).

The fourth example \([7]\) is two scalar and two gauge field couplings,
\[
- \frac{T_p (2\pi \alpha')^4}{2} S \text{Tr} \left( D_a \phi^i D^b \phi_l F_{ab} F_{bc} - \frac{1}{4} (D_a \phi^i D^a \phi_i F_{bc} F_{bc}) \right). \tag{52}
\]
After implementing the crucial step mentioned above, the closed form of the higher derivative corrections of two scalars and two gauge fields (which must be added to DBI)\(^{8}\) turned out to be
\[
(2\pi \alpha')^4 \frac{1}{2 \pi^2} T_p (\alpha')^{n+m} \sum_{m,n=0}^{\infty} (\mathcal{L}_{8}^{nm} + \mathcal{L}_{9}^{nm} + \mathcal{L}_{10}^{nm}), \tag{53}
\]
\(^{10}\)The prescription works for brane anti-brane systems as well. The only subtlety in this case is that after applying \(\mathcal{D}_{nm}, \mathcal{D}_{nm}'\), all \(b_{n,m}\) must be rendered to \(-b_{n,m}\) \([27]\).
\[ L_{8}^{nm} = - \text{Tr} \left( a_{n,m} D_{nm} [D_{a} \phi^{i} D^{b} \phi_{i} F^{ac} F_{bc} + b_{n,m} D'_{nm} [D_{a} \phi^{i} F^{ac} D^{b} \phi_{i} F_{bc} + h.c.] \right), \]
\[ L_{9}^{nm} = - \text{Tr} \left( a_{n,m} D_{nm} [D_{a} \phi^{i} D^{b} \phi_{i} F_{bc} F^{ac} + b_{n,m} D'_{nm} [D_{a} \phi^{i} F_{bc} D^{b} \phi_{i} F^{ac} + h.c.] \right), \]
\[ L_{10}^{nm} = \frac{1}{2} \text{Tr} \left( a_{n,m} D_{nm} [D_{a} \phi^{i} D^{a} \phi_{i} F^{bc} F_{bc} + b_{n,m} D'_{nm} [D_{a} \phi^{i} F_{bc} D^{a} \phi_{i} F^{bc} + h.c.] \right), \]

As usual, the above couplings are valid up to total derivative terms and terms such as \( \partial_{a} \partial^{a} F F D \phi D \phi \) that vanish on-shell.

The examples above suggest that there exists a regularity in the higher derivative expansions. One can formulate a prescription based on this regularity. In order to find all infinite higher derivative corrections we must find the S-matrix element of desired amplitudes which are either non-BPS or BPS amplitudes. The next step is using the relation between Mandelstam variables. In other words we must rewrite the amplitudes such that all poles can be seen in a clear way. The third step is finding leading couplings from tachyonic DBI or DBI action. The last step is to express the symmetric trace in terms of the ordinary trace and apply the higher derivative corrections \( D_{nm}, D'_{nm} \) (as appeared in (49)) on it.

Let us apply the prescription to the current case, the higher derivatives vertices of four-scalar fields. The first simple massless scalar pole is reproduced by the non-abelian kinetic terms of the scalar field [28] [6],

\[ - T_{p} (2 \pi \alpha')^{4} \text{St} \left( \frac{1}{4} D_{a} \phi^{i} D_{b} \phi_{i} D^{b} \phi^{j} D^{a} \phi_{j} + \frac{1}{8} (D_{a} \phi^{i} D^{a} \phi_{i})^{2} \right) \]  \hspace{1cm} (54)

Applying our prescription, one can easily determine their higher derivative forms by noting universality property that was present in the previous works as follows

\[ (2 \pi \alpha')^{4} \frac{1}{4 \pi^{2}} T_{p} (\alpha')^{n+m} \sum_{m,n=0}^{\infty} \left( L_{11}^{nm} + L_{12}^{nm} + L_{13}^{nm} \right) \]  \hspace{1cm} (55)

\[ L_{11}^{nm} = - \text{Tr} \left( a_{n,m} D_{nm} [D_{a} \phi^{i} D_{b} \phi_{i} D^{b} \phi^{j} D^{a} \phi_{j} + b_{n,m} D'_{nm} [D_{a} \phi^{i} D^{b} \phi_{i} D_{b} \phi_{j} D^{a} \phi_{j} + h.c.] \right) \]
\[ L_{12}^{nm} = - \text{Tr} \left( a_{n,m} D_{nm} [D_{a} \phi^{i} D_{b} \phi_{i} D^{a} \phi^{j} D^{b} \phi_{j} + b_{n,m} D'_{nm} [D_{b} \phi^{i} D^{b} \phi_{i} D_{a} \phi_{j} D^{a} \phi_{j} + h.c.] \right) \]
\[ L_{13}^{nm} = \text{Tr} \left( a_{n,m} D_{nm} [D_{a} \phi^{i} D^{a} \phi_{i} D_{b} \phi^{j} D^{b} \phi_{j} + b_{n,m} D'_{nm} [D_{a} \phi^{i} D_{b} \phi^{j} D^{a} \phi_{i} D^{b} \phi_{j} + h.c.] \right) \]  \hspace{1cm} (56)

By comparing (56) and (51), we see that the corresponding equations in (56) and (51) have different numerical coefficients. We will comment on this in the conclusion. We now turn to verification of (56).
3.3 Pole analyses

We analyze the infinite massless poles of the string amplitude (39) for the relevant cases (i.e., \( n = p + 2 \) (infinite massless scalar poles) and \( n = p \) case (infinite gauge field poles)), and show that the vertices together with appropriately chosen WZ vertices reproduce them.

3.3.1 Massless scalar poles for \( n = p + 2 \) case

Working out the trace and using special expansions in the previous section, the massless scalar poles of the string amplitude take

\[
16\pi^3\mu_p \epsilon_{a_0\ldots a_p} H^{(p+2)} a_{a_0\ldots a_p} \sum_{n,m=0}^\infty c_{n,m} \left( u s \xi_3 \xi_1 \xi_2 [s^m u^n + s^n u^m] + u t \xi_2 \xi_1 [t^m u^n + t^n u^m] + t s \xi_3 \xi_2 [s^m t^n + s^n t^m] \right)
\]

where we have normalized the amplitude by \((2^{1/2} \pi^{1/2} \mu_p)\).

\[
\phi_1 \quad \phi_2 \quad \phi \quad C_{p+1} \quad \phi_3
\]

Figure 2: The Feynman diagram corresponding to the amplitudes (57).

The first massless poles are reproduced by the non-abelian kinetic terms of the scalar field (54). Now we want to show that those higher derivative corrections with the correct coefficient (55) reproduce all infinite scalar poles in the \((s + t + u)\)-channel of the string theory S-matrix. Below, we show that the rest of the poles are reproduced by

\[
\lambda \mu_p \int d^{p+1}\sigma \frac{1}{(p + 1)!} (\epsilon^v)_{a_0\ldots a_p} \text{Tr} (\phi^i) H^{(p+2)}_{i a_0\ldots a_p} (\sigma)
\]

Let us consider the field theory amplitude of one R-R field and three scalars for the \( p+2 = n \) case. In Feynman rules (we use the Feynman gauge), it is given by

\[
\mathcal{A} = V_i^i (C_{p+1}, \phi) C^{ij}_{\alpha\beta} (\phi) V_j^j (\phi, \phi_1, \phi_2, \phi_3),
\]

\[\text{Figure 2: The Feynman diagram corresponding to the amplitudes (57).}\]

\[\text{The case } n = p + 4 \text{ has only contact terms but no poles.}\]
where

\[ G^{ij}_{\alpha\beta}(\phi) = \frac{-i\delta_{\alpha\beta}\delta^{ij}}{T_p(2\pi\alpha')^2k^2} = \frac{-i\delta_{\alpha\beta}\delta^{ij}}{T_p(2\pi\alpha')^2(t + s + u)}, \]

\[ V^i_\alpha(C_{p+1}, \phi) = i(2\pi\alpha')(p+1)^{(\varepsilon^\nu)_{a_0...a_p}} H^{i(\rho+2)}_{a_0...a_p} \text{Tr} (\lambda_\alpha). \]  

(60)

where \( k \) is the off-shell momentum of the scalar field and \( k^2 \) has been replaced by \( (t + s + u) \) in the propagator in the first equation of (60). Note that \( \text{Tr} (\lambda_\alpha) \) is just nonzero for the abelian generator \( \lambda_\alpha \). Since the off-shell scalar field (i.e., the field \( \phi \) in (59)) is abelian, we must consider 12 (as opposed to the full 24) cyclic permutations of the vertices for the given \( \text{Tr} (\lambda_1\lambda_2\lambda_3) \) ordering. After including the permutations, one obtains the higher derivative vertex \( V^3_\beta(\phi, \phi_1, \phi_2, \phi_3) \) from the higher derivative couplings in (55) as follows:

\[ V^j_\beta(\phi, \phi_1, \phi_2, \phi_3) = \text{Tr} (\lambda_1\lambda_2\lambda_3\lambda_\beta) I_9 [V^j_1(\phi, \phi_1, \phi_2, \phi_3) + V^j_2(\phi, \phi_1, \phi_2, \phi_3) + V^j_3(\phi, \phi_1, \phi_2, \phi_3)] \]

where

\[ I_9 = \frac{1}{4\pi^2}(\alpha')^n+m(a_n,m + b_{n,m})(2\pi\alpha')^4T_p \]

(62)

and

\[ V^j_1(\phi, \phi_1, \phi_2, \phi_3) = \frac{\xi_1\xi_2\xi_3}{2} (k_3 \cdot k_1)^m(k_1 \cdot k_2)^n + (k_3 \cdot k)^n(k_1 \cdot k_3)^m + (k_1 \cdot k_2)^n(k_1 \cdot k_2)^m \]

\[ + (k \cdot k_3)^m(k \cdot k_2)^n + (k \cdot k_2)^m(k_2 \cdot k_1)^n + (k_3 \cdot k_1)^m(k_2 \cdot k_1)^n \]

\[ + (k_2 \cdot k)^m(k_3 \cdot k)^n + (k_3 \cdot k_1)^m(k_3 \cdot k)^n \],

(63)

\[ V^j_2(\phi, \phi_1, \phi_2, \phi_3) = \frac{\xi_1\xi_2\xi_3}{2} (k_2 \cdot k_1)^m(k_2 \cdot k_2)^n + (k_2 \cdot k_1)^m(k_1 \cdot k)^n + (k_2 \cdot k_3)^n(k_1 \cdot k)^n \]

\[ + (k_1 \cdot k_2)^n(k_3 \cdot k_2)^m + (k_3 \cdot k_2)^n(k_3 \cdot k)^n + (k \cdot k_1)^n(k_3 \cdot k)^n \]

\[ + (k_3 \cdot k)^m(k_1 \cdot k)^n + (k_2 \cdot k_1)^n(k_1 \cdot k)^n \]

\[ + (k_3 \cdot k)^m(k_1 \cdot k)^n \]  

\[ + (k_2 \cdot k_1)^n(k_1 \cdot k)^n \],

\[ + (k_3 \cdot k)^m(k_1 \cdot k)^n \]

(63)

\[ + (k_3 \cdot k)^m(k_1 \cdot k)^n \]

\[ + (k_3 \cdot k)^m(k_1 \cdot k)^n \]

\[ + (k_3 \cdot k)^m(k_1 \cdot k)^n \]

\[ + (k_3 \cdot k)^m(k_1 \cdot k)^n \]

The 12 possible cyclic permutations of the vertices are associated with different orderings of generators inside the trace:

\[ \text{Tr} (\lambda_1\lambda_2\lambda_3\lambda_\beta), \text{Tr} (\lambda_1\lambda_2\lambda_\beta\lambda_3)\text{Tr} (\lambda_1\lambda_\beta\lambda_2\lambda_3), \]

(61)

\[ \text{Tr} (\lambda_2\lambda_3\lambda_1\lambda_\beta)\text{Tr} (\lambda_2\lambda_3\lambda_\beta\lambda_1), \text{Tr} (\lambda_2\lambda_\beta\lambda_1\lambda_3) \]

\[ \text{Tr} (\lambda_3\lambda_\beta\lambda_1\lambda_2), \text{Tr} (\lambda_3\lambda_1\lambda_2\lambda_\beta)\text{Tr} (\lambda_3\lambda_1\lambda_\beta\lambda_2), \]

\[ \text{Tr} (\lambda_\beta\lambda_1\lambda_2\lambda_3)\text{Tr} (\lambda_\beta\lambda_3\lambda_1\lambda_2), \text{Tr} (\lambda_\beta\lambda_2\lambda_3\lambda_1) \]

for the given 123 ordering of the amplitude.
\[ V^j_{\phi}(\phi_1, \phi_2, \phi_3) = \frac{u s}{2} \xi_1^j \xi_2 \left( (k \cdot k_1)^m (k_3 \cdot k_1)^n + (k \cdot k_2)^m (k_2 \cdot k)^n + (k_1 \cdot k_3)^m (k \cdot k_1)^n \\
+ (k_1 \cdot k_3)^m (k_3 \cdot k_2)^n + (k_3 \cdot k_1)^m (k_3 \cdot k_2)^n + (k_1 \cdot k_2)^m (k_3 \cdot k_2)^n \\
+ (k_1 \cdot k)^m (k_2 \cdot k)^n + (k_2 \cdot k_3)^m (k_3 \cdot k)^n \right) \]

The coefficients \( a_{n,m} \) and \( b_{n,m} \) are identical to those that were computed in [6] for the case of four-gauge fields amplitude. We list some of them for convenience and self-containedness of the paper:

\[
\begin{align*}
a_{0,0} &= -\frac{\pi^2}{6}, \ b_{0,0} = -\frac{\pi^2}{12}, \ a_{1,0} = 2\zeta(3), \ a_{0,1} = 0, \ b_{0,1} = -\zeta(3), \ a_{1,1} = a_{0,2} = -7\pi^4/90, \\
a_{2,2} &= (83\pi^6 - 7560\zeta(3)^2)/945, \ b_{2,2} = -(23\pi^6 - 15120\zeta(3)^2)/1890, \ a_{1,3} = -62\pi^6/945, \\
a_{2,0} &= -4\pi^4/90, \ b_{1,1} = -\pi^4/180, \ b_{0,2} = -\pi^4/45, \ a_{0,4} = -31\pi^6/945, \ a_{4,0} = -16\pi^6/945, \\
a_{1,2} &= a_{2,1} = 8\zeta(5) + 4\pi^2\zeta(3)/3, \ a_{0,3} = 0, \ a_{3,0} = 8\zeta(5), \ b_{1,3} = -(12\pi^6 - 7560\zeta(3)^2)/1890, \\
a_{3,1} &= (-52\pi^6 - 7560\zeta(3)^2)/945, \ b_{0,3} = -4\zeta(5), \ b_{1,2} = -8\zeta(5) + 2\pi^2\zeta(3)/3, \\
b_{0,4} &= -16\pi^6/1890.
\end{align*}
\]

Substituting \( k_1 \cdot k = k_2 k_3 - (k^2)/2 \), \( k_3 \cdot k = k_2 k_1 - (k^2)/2 \) and \( k_2 \cdot k = k_1 k_3 - (k^2)/2 \), one finds the following massless scalar poles:

\[
32\pi \mu_p \frac{e^{a_0 \cdots a_p} H^{(p+2)}}{(p+1)! (s + t + u)} \Tr (\lambda_1 \lambda_2 \lambda_3) \sum_{n,m=0}^{\infty} (a_{n,m} + b_{n,m}) \left( u s \xi_{3i} \xi_{1j} \xi_{2k} [s^m u^n + s^n u^m] \\
+ u t \xi_{2i} \xi_{1j} \xi_{3k} [t^m u^n + t^n u^m] + t s \xi_{1i} \xi_{3j} \xi_{2k} [s^m t^n + s^n t^m] \right)
\]

Let us compare this with the massless scalar poles of the string theory amplitude [57]. We have chosen several values of \( n, m \). Note that for simplicity common factors of both string and field theory have been omitted. For \( n = m = 0 \), the amplitude (65) has the following coefficient

\[-2(a_{0,0} + b_{0,0}) = -2\left(-\frac{\pi^2}{6} + \frac{\pi^2}{12}\right) = \frac{\pi^2}{2}\]

The string amplitude has a corresponding term with a numerical factor of \((\pi^2 c_{0,0})\). It indeed matches with the field theory result. At the order of \( \alpha' \), the field theory result (65) has a term with the coefficient \((a_{0,1} + a_{0,1} + b_{1,0} + b_{0,1})\). This vanishes as the corresponding string theory term vanishes with the coefficient \( \pi^2 (c_{1,0} + c_{0,1}) \). At the order of \((\alpha')^2\), the amplitude

\footnote{Contact terms are produced as well when the terms \( k^2 \) in the vertex [65] get canceled against the \( k^2 \) in the denominator of the scalar field propagator. We will not consider them explicitly (for more details see section 7.2 of [7]).}
(65) has the following coefficient

\[-2(a_{0,2} + a_{2,0} + b_{0,2} + b_{2,0}) \left( u s \xi_3 \xi_1 \xi_2 [s^2 + u^2] + ut \xi_2 \xi_1 \xi_3 [t^2 + u^2] + ts \xi_{1i} \xi_3 \xi_2 [s^2 + t^2] \right) \]

\[-2(a_{1,1} + b_{1,1}) \left( u s \xi_3 \xi_1 \xi_2 [2su] + ut \xi_2 \xi_1 \xi_3 [2tu] + ts \xi_{1i} \xi_3 \xi_2 [2st] \right) \]

\[= \frac{\pi^4}{3} \left( u s \xi_3 \xi_1 \xi_2 [s^2 + u^2] + ut \xi_2 \xi_1 \xi_3 [t^2 + u^2] + ts \xi_{1i} \xi_3 \xi_2 [s^2 + t^2] \right) \]

\[+ \frac{\pi^4}{6} \left( u s \xi_3 \xi_1 \xi_2 [2su] + ut \xi_2 \xi_1 \xi_3 [2tu] + ts \xi_{1i} \xi_3 \xi_2 [2st] \right) \]

and the string result has

\[\pi^2 (c_2,0 + c_0,2) \left( u s \xi_3 \xi_1 \xi_2 [s^2 + u^2] + ut \xi_2 \xi_1 \xi_3 [t^2 + u^2] + ts \xi_{1i} \xi_3 \xi_2 [s^2 + t^2] \right) \]

\[+ \pi^2 c_{1,1} \left( u s \xi_3 \xi_1 \xi_2 [2su] + ut \xi_2 \xi_1 \xi_3 [2tu] + ts \xi_{1i} \xi_3 \xi_2 [2st] \right) \]

The latter becomes equal to the former upon using the coefficients in (66). At the order of \(\alpha'^2\), field theory amplitude has two different terms. The first one has the coefficient of \((a_{3,0} + a_{0,3} + b_{0,3} + b_{3,0})\) which is zero and the corresponding term on the string theory side has a \(\pi^2 (c_{0,3} + c_{3,0})\) coefficient which is again zero in accordance with field theory. The second term has the following coefficient

\[-2(a_{1,2} + a_{2,1} + b_{1,2} + b_{2,1}) = -8\pi^2 \zeta(3)\]

and it again matches with the corresponding coefficient in the string amplitude, \(\pi^2 (c_{2,1} + c_{1,2}) = -8\pi^2 \zeta(3)\).

The other comparisons to all orders of \(\alpha'\) need not be done\(^\text{14}\). Therefore we could exactly reproduce the infinite massless scalar poles of the string theory amplitude of one RR and three scalar fields in the worldvolume of BPS branes. This confirms that we have obtained the higher derivative couplings of four scalars with the correct coefficients and they are exact up to terms that vanish on-shell.

### 3.3.2 Massless gauge field poles for \(p = n\) case

Working out the trace, it is possible to obtain all massless gauge poles in the string theory side as follows:

\[\mathcal{A}_2 = \pm 2^{-1/2} (2^{1/2} n^{1/2} \mu_p)^{32} 2p! L_2 \left( e^{a_0 \cdots a_{p-2} b a H_{ab}^{k(p)}} \left( 2 u s \xi_1 \xi_2 k_2 a k_{3b} \xi_{3k} + 2 u t \xi_3 \xi_1 k_2 a k_{3b} \xi_{2k} 

+ 2 s t \xi_3 \xi_2 k_{2a} k_{3b} \xi_{1k} \right) + p^k e^{a_0 \cdots a_{p-1}} H_{ab}^{k(p)} \right) \right) \]

\(^{14}\)The same checks were carried out for finding infinite massless poles of \(<V_C V_A V_A V_A>\) and \(<V_C V_A V_A>\), see [6] and [8].
where the amplitude is normalized by $2^{1/2} \pi^{1/2} \mu_p$. Substituting special expansion of \(15\) into the amplitude and keeping all the gauge field poles (but not the contact terms), one gets

\[
A_2 = \pm \frac{32}{2p!} \pi^2 \mu_p \left\{ \sum_{n=-1}^{\infty} \frac{1}{u} b_n (t + s)^{n+1} (2 \zeta_2 \zeta_3 k_{2a} k_{3b} k_{1k}) \epsilon^{a_0 \ldots a_{p-2} b a} H_{a_0 \ldots a_{p-2}}^{(p)}(u) + \sum_{n=-1}^{\infty} \frac{1}{t} b_n (u + s)^{n+1} \epsilon^{a_0 \ldots a_{p-2} b a} H_{a_0 \ldots a_{p-2}}^{(p)}(2 \zeta_1 \zeta_2 k_{2a} k_{3b} k_{1k}) + p^k \epsilon^{a_0 \ldots a_{p-1} a} H_{a_0 \ldots a_{p-1}}^{(p)}(u) \times (-2 k_{2a} \zeta_1 \zeta_2 k_{3k}) \right\} [2 \leftrightarrow 3] \right\} \text{Tr} (\lambda_1 \lambda_2 \lambda_3) (66)
\]

Let us examine the massless gauge poles. First, we will show that effective field theory reproduces the infinite massless gauge poles in the \(u\)-channel, i.e., the first term in \(66\). Then we reproduce the second and third terms in \(66\). Since the amplitudes in \(t\)- and \(s\)-channels are similar, we just reproduce all infinite massless gauge \(t\)-channel poles in detail. By interchanging the momentum and polarization of scalar fields \(2 \leftrightarrow 3\), one can find the other infinite massless gauge poles in the \(s\)-channel as well. The needed field theory vertex for the first term in \(66\) is

\[
S^{(5)} = i \lambda \mu_p \int \text{STr} \left( F P \left[ C^{(p-1)}(\sigma, \phi) \right] \right)
= i \lambda^2 \mu_p \int d^{p+1} \sigma \frac{1}{(p-1)!} (\varepsilon^v)^{a_0 \ldots a_p} \left[ \text{Tr} \left( F_{a_0 a_1} \partial_a \phi^k \right) C^{(p-1)}_{a_2 \ldots a_p}(\sigma) \right]. (67)
\]

Where the scalar field comes from pull-back (see section 5 of [7]). The partial derivatives on the scalar fields can be replaced by the covariant derivatives. The connection parts do not contribute because there is no external gauge field; therefore, the off-shell gauge field must come from the abelian field strength. With this vertex, the massless gauge poles in the \(u\)-channel are reproduced in the form

\[
\mathcal{A} = V_a^c(C_{p-1}, \phi_1, A) G_{\alpha \beta}^{ab}(A) V_{\beta}^b(A, \phi_2, \phi_3), (68)
\]

where the vertices and gauge field propagator are

\[
V_a^c(C_{p-1}, \phi_1, A) = \lambda^2 \mu_p \left( \varepsilon^v \right)^{a_0 \ldots a_{p-1} a} (H^{(p)})^k_{a_0 \ldots a_{p-2} k_{1k}} \text{Tr} (\lambda_1 \lambda_2 \lambda_3) \sum_{n=-1}^{\infty} b_n (a' k_{1k})^{n+1}, (69)
\]

with

\[
V_{\beta}^b(A, \phi_2, \phi_3) = iT_p (2 \pi a')^2 \xi_2 \xi_3 (k_2 - k_3)^b \text{Tr} (\lambda_2 \lambda_3 \lambda_\beta),
\]

\[
G_{\alpha \beta}^{ab}(A) = \frac{i \delta_{\alpha \beta} \delta^{ab}}{(2 \pi a')^2 T_p (k_2^2)},
\]
$k$ is the momentum of the abelian gauge field, $k^2 = (k_3 + k_2)^2 = -u$. The propagator is found from the kinetic term of the gauge field in the Born-Infeld action. The vertex of $V^b_\beta (A, \phi_2, \phi_3)$ has been obtained from the kinetic term of the scalar field $\frac{\lambda^2}{2} \text{Tr} (D_a \phi_1 D^a \phi^1)$. (A vertex similar to $V^b_\beta (A, \phi_2, \phi_3)$ was obtained in [7].) The simple massless poles of string amplitude indicate that the kinetic term of the scalar field has no higher derivative corrections; hence, the vertex $V^b_\beta (A, \phi_2, \phi_3)$ has no higher derivative correction either. The vertex $V^a_\alpha (C_{p-1}, \phi_1, A)$ must be derived from the higher derivative extensions of the WZ coupling (67) as follows

\[
S^{(5)} = i \lambda^2 \mu_p \int d^{p+1} \sigma \frac{1}{(p-1)!} \epsilon^{a_0 \cdots a_p} \times \sum_{n=-1}^{\infty} b_n (\alpha')^{n+1} \left[ \text{Tr} \left( \partial_{a_{m_0}} \cdots \partial_{a_{m_n}} F_{a_0 a_1} \partial^{a_{m_0}} \cdots \partial^{a_{m_n}} \partial_{a_2} \phi^k \right) C^{(p-1)}_{k_{a_3 \cdots a_p}} (\sigma) \right] \tag{70}
\]

Inserting this into the amplitude (68), one finds

\[
A = (2\pi \alpha')^2 \frac{\mu_p}{p! u} \epsilon^{a_0 \cdots a_{p-1} a} H_{k_{a_0 \cdots a_{p-2}} \xi} \text{Tr} \left( \lambda_1 \lambda_2 \lambda_3 \right) \sum_{n=-1}^{\infty} b_n \left( \frac{\alpha'}{2} \right)^{n+1} (s+t)^{n+1} \times \left( -2 (\xi_2 \cdot \xi_3) k \right)^{a_0 \cdots a_{p-1}} . \tag{71}
\]

These are precisely the $u$-channel massless poles of (66). Unlike the $p + 2 = n$ case in the previous section, here there are no residual contact terms.

\textbf{Figure 3} : The Feynman diagram corresponding to the massless gauge pole of the amplitude (66) in $u$-channel.

Having reproduced all infinite poles corresponding to the first term of (66), we turn to the rest of the terms (namely, we would like to reproduce the second and third terms of (66)). We quote those terms here:

\[
A = \pm \frac{32}{2p!} \pi^2 \mu_p \left\{ \sum_{n=-1}^{\infty} \frac{1}{n!} b_n (u + s)^{n+1} \left( \epsilon^{a_0 \cdots a_{p-2} a b} H_{a_0 \cdots a_{p-2} b}^k (2 \xi_1 \cdot \xi_2 k_2 a k_3 \xi_3 k) + p^k \epsilon^{a_0 \cdots a_{p-1} a} \right) \right\} \text{Tr} \left( \lambda_1 \lambda_2 \lambda_3 \right) \tag{72}
\]
Again it vanishes for the abelian group. We just kept the infinite massless poles in the t-channel and at the moment we disregard all contact terms. We now show that effective field theory will result in the infinite massless gauge poles in the t-channel. The corresponding effective field theory vertex is given by

\[
S^{(6)} = i\lambda \mu_p \int \overline{S} T \left( F_{[a} \partial_{b]} \partial_k C^{(p-1)}_{a_2...a_p}(\sigma) \right)
\]

Notice that the scalar field in (73) comes from Taylor expansion (for review, see section 5 of [7]). Now if we extract the field strength and take integration by parts we will have several terms such as the following:

\[
S^{(6)} = i\lambda \mu_p \int d^{p+1}\sigma \frac{1}{(p-1)!} (\varepsilon^v)^{a_0...a_p} \left( -A_{a_1} \partial_{a_0} \partial_k C^{(p-1)}_{a_2...a_p}(\sigma) - A_{a_1} \partial_k \partial_{a_0} C^{(p-1)}_{a_2...a_p}(\sigma) \right)
\]

Having taken into account the off-shell gauge field, writing the above coupling in momentum space and applying momentum conservation along the world volume of brane we can obtain the final form of the vertex of one off-shell gauge field, one RR $p-1$ form field and one external scalar field (which we labeled its polarization with $\xi_3$). The vertex $V^a_{\alpha}(C_{p-1}, \phi_3, A)$ should be obtained from the higher derivative extension of the WZ coupling (73) as

\[
S^{(6)} = i\lambda^2 \mu_p \int d^{p+1}\sigma \frac{1}{(p-1)!} (\varepsilon^v)^{a_0...a_p} \times \sum_{n=-1}^{\infty} b_n(\alpha')^{n+1} \left( \partial_{a_0} \cdots \partial_{a_n} F_{a_0 a_1} \partial^{a_0 a_1} \cdots \partial^{a_n} \phi^k \partial_k C^{(p-1)}_{a_2...a_p}(\sigma) \right)
\]

Therefore

\[
V^a_{\alpha}(C_{p-1}, \phi_3, A) = \frac{\lambda^2 \mu_p}{(p-1)!} (\varepsilon^v)^{a_0...a_{p-1}} \partial_{a_0} \cdots \partial_{a_{p-1}} F_{a_0 a_1} \partial^{a_0 a_1} \cdots \partial^{a_{p-1}} \phi^k \partial_k C^{(p-1)}_{a_0...a_{p-2}}(\sigma)
\]

Note that the above vertex is taken into account after applying higher derivative extensions in (74). The corresponding Feynman amplitude is now

\[
A = V^a_{\alpha}(C_{p-1}, \phi_3, A) G^{ab}_{\alpha \beta}(A) V^b_{\beta}(A, \phi_1, \phi_2),
\]

By interchanging $(2 \leftrightarrow 3)$ we find the other massless poles in the s-channel.
with

\[ V^b_\beta(A, \phi_1, \phi_2) = iT_p(2\pi \alpha')^2 \xi_1, \xi_2 (k_1 - k_2)^b \text{Tr} (\lambda_1 \lambda_2 \lambda_\beta), \]
\[ G^{ab}_{\alpha\beta}(A) = \frac{i\delta_{\alpha\beta} \delta^{ab}}{(2\pi \alpha')^2 T_p(k^2)}, \]

\( k^2 \) in the propagator is now, \( k^2 = (k_1 + k_2)^2 = -t \). Now applying momentum conservation we can reexpress \( V^b_\beta(A, \phi_1, \phi_2) \) as:

\[ V^b_\beta(A, \phi_1, \phi_2) = iT_p(2\pi \alpha')^2 \xi_1, \xi_2 (-2k_2 - k_3 - p)^b \text{Tr} (\lambda_1 \lambda_2 \lambda_\beta) \]

Replacing them in the amplitude (76), we find that the infinite massless gauge poles in the t-channel are reproduced as

\[ A = (2\pi \alpha')^2 \frac{\mu_p}{(p)! t} \epsilon^{a_0...a_{p-1}}b^k \xi_3 (\xi_2, \xi_1) \text{Tr} (\lambda_1 \lambda_2 \lambda_3) \sum_{n=-1}^{\infty} b_n \left( \frac{\alpha'}{2} \right)^{n+1} (s + u)^{n+1} \]
\[ \times \left( -2k_{2b} P^k H_{a_0...a_{p-1}}^{(p)} + 2k_{2b} k_{3a_{p-1}} H_{a_0...a_{p-2}}^{(p)} \right). \] (77)

These exactly are t-channel massless poles of the string theory amplitude (72). So we find that field theory computations are in exact agreement with the string amplitude at pole levels. Similar computations in s-channel also lead to agreement.

The simple massless poles of string field show that the kinetic term of the scalar fields has no higher derivative corrections so the vertex \( V^b_\beta(A, \phi_1, \phi_2) \) has no higher derivative correction either.

### 3.4 Contact term analyses

Above we have successfully determined the field theory amplitudes that reproduce all of the poles of the string amplitude. We attempt to achieve the same for the contact terms below.

We have succeeded in the case of \( p + 4 = n \). However, as for the cases of \( n = p + 2, n = p \), we could not find the field theory vertices that reproduce the leading order contact terms nor the infinite extension. Perhaps this is a hint that the pull-back method may need modification.

#### 3.4.1 Contact terms for \( p + 4 = n \) case

The relevant part of the string amplitude can be rewritten as

\[ A_2 = \pi^2 \mu_p \xi_1, \xi_2, \xi_3 \text{Tr} (P_- H_{(n)} M_p \Gamma^{kji}) \]
\[ \times \left( \sum_{n=-1}^{\infty} b_n (u + s)^{n+1} - \sum_{p,n,m=0}^{\infty} e_{p,n,m} t^{p+1} (su)^n (s + u)^m \right). \] (78)
The contact terms of this amplitude can be reproduced by an infinite extension of a Wess-Zumino term. Let us reproduce all those contact terms using a vertex that contains a $(p+3)$-form Ramond-Ramond potential and three scalar fields. There are two relevant Wess-Zumino terms depending on whether the third scalar comes from Taylor expansion or pull-back. The first contribution is given by

$$S_7 = \frac{1}{2} \lambda^2 \mu_p \int d^{p+1}\sigma \frac{1}{(p+1)!} (\varepsilon^v)^{a_0\ldots a_p} \text{Tr} \left( [\Phi^i, \Phi^j] \Phi^l \right) \partial_l C_{ij[a_0\ldots a_p]}^{(p+3)}(\sigma).$$

The commutator comes from the exponential in Wess-Zumino action and the last scalar comes from Taylor expansion. The second Wess-Zumino term is given by

$$S_8 = \frac{1}{2} \lambda^2 \mu_p \int d^{p+1}\sigma \frac{1}{(p)!} (\varepsilon^v)^{a_0\ldots a_p} \text{Tr} \left( [\Phi^i, \Phi^j] \partial_{a_0} \Phi^l \right) C_{ij[a_1\ldots a_p]}^{(p+3)}(\sigma),$$

where the commutator comes from the exponential in Wess-Zumino action and the third scalar comes from pull-back. Applying integration by parts these two results can be combined to yield the final result,

$$S_9 = \frac{1}{3} \lambda^2 \mu_p \int d^{p+1}\sigma \frac{1}{(p+1)!} (\varepsilon^v)^{a_0\ldots a_p} \text{Tr} \left( \Phi^i \Phi^j \Phi^l \right) H_{ij[a_0\ldots a_p]}^{(p+4)}(\sigma),$$

where $S_9 = S_7 + S_8$ and $H^{(p+4)} = dC^{(p+3)}$. While the leading contact terms of the string amplitude $\langle VC V_3 V_3 V_3 \rangle$ are reproduced by (79), the rest of the contact terms require a higher derivative extension thereof. Evaluating the trace in (78), one can show that the first term in (78) can be reproduced by

$$\frac{\lambda^2 \mu_p}{3(p+1)!} \int d^{p+1}\sigma (\varepsilon^v)^{a_0\ldots a_p} \sum_{n=-1}^{\infty} b_n (\alpha')^{n+1} \text{Tr} \left( D^{a_0} \cdots D^{a_n} (\Phi^i \Phi^j) D_{a_0} \cdots D_{a_n} \Phi^k \right) H_{ij[a_0\ldots a_p]}^{(p+4)}.$$
terms in the second term of (78) can be reproduced by
\[-\frac{\lambda^2 \mu_p}{3(p+1)!} \int d^{p+1} \sigma (\varepsilon^v)^{a_0 \cdots a_p} \sum_{p,n,m=0}^{\infty} \delta_{p,n,m} \left( \alpha' \right)^{p+1}(\alpha')^{2n+m} H_{ijk \cdots a_p} \text{Tr} \left( (D^a D_a)^{p+1} D^{a_1} \cdots D^{a_m} \right) \times (D^{a_1} \cdots D^{a_n} \Phi^i D^{a_{n+1}} \cdots D^{a_{2n}} \Phi^j) D_{a_1} \cdots D_{a_{2n}} D_{a_1} \cdots D_{a_m} \Phi^k \right) \]

4 Conclusion

In this work, we have analyzed the amplitudes of \( <V_C V_A> \), \( <V_C V_\phi V_\phi> \), \( <V_C V_\phi V_\phi> \), \( <V_C V_\phi V_\phi> \), \( <V_C V_\phi V_\phi> \). We have found the field theory vertices that reproduce all infinite contact terms of two- and three-point amplitudes. We could produce all poles of the four-point function, but we could produce the contact terms only for \( p + 4 = n \) case. At the moment it is not entirely clear how to produce all infinite contact terms of \( <V_C V_\phi V_\phi> \) for \( p + 2 = n, p = n \) cases. Possibly the pull-back method may need modification.

We found universality in all order \( \alpha' \) higher derivative corrections of non-BPS and BPS branes and the universality played an important role in the determination of field theory vertices. Several remarks on T-duality are in order. T-duality can be straightforwardly employed to deduce a pure open string tree amplitude of scalar vertex operators from a tree amplitude of gauge field vertex operators. Once one considers an amplitude of a mixture of open and closed strings, direct computation is necessary because of the subtleties associated with T-duality. Two subtleties exist in the very construction of the RR C vertex operator in (1). First, the construction of the C vertex operator was such that one set of oscillators was used instead of two. The second issue - which was addressed in footnote 4 of [21] - is that the C vertex operator does not contain winding modes, and this must be related to the fact that we have pointed out above (42): the terms that contain \( p^i \) are absent in \( <V_C V_A V_A V_A> \).

We hope to be able to compute higher point amplitudes of various mixtures of open string and closed string states. Another more ambitious direction would be to make progress in the full form of the DBI action. We hope to report on these issues in the near future.

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\[ ^{16}\text{Perhaps this step may be some kind of analytic continuation.} \]
References

[1] J. Polchinski, “Dirichlet-Branes and Ramond-Ramond Charges,” Phys. Rev. Lett. 75, 4724 (1995) [arXiv:hep-th/9510017].

[2] J. Polchinski, “Lectures on D-branes,” [arXiv:hep-th/9611050].

[3] A. A. Tseytlin, “Born-Infeld action, supersymmetry and string theory,” [arXiv:hep-th/9908105]; “On non-abelian generalisation of the Born-Infeld action in string theory,” Nucl. Phys. B 501, 41 (1997) [arXix:hep-th/9701125].

[4] R. C. Myers, “Dielectric-branes,” JHEP 9912, 022 (1999) [arXiv:hep-th/9910053].

[5] M. Li, “Boundary states of D-branes and dy-strings,” Nucl. Phys. B 460, 351 (1996) [arXiv:hep-th/9510161]; M. R. Douglas, “Branes within branes,” [arXiv:hep-th/9512077].

[6] E. Hatefi, “On effective actions of BPS branes and their higher derivative corrections,” JHEP 1005, 080 (2010) [arXiv:1003.0314 [hep-th]].

[7] E. Hatefi, “On higher derivative corrections to Wess-Zumino and Tachyonic actions in type II super string theory,” to appear in PRD, [arXiv:1203.1329 [hep-th]].

[8] E. Hatefi and I. Y. Park, “More on closed string induced higher derivative interactions on D-branes,” Phys. Rev. D 85, 125039 (2012) [arXiv:1203.5553 [hep-th]].

[9] M. R. Garousi and E. Hatefi, “On Wess-Zumino terms of Brane-Antibrane systems,” Nucl. Phys. B 800, 502 (2008) [arXiv:0710.5875 [hep-th]].

[10] M. R. Garousi and E. Hatefi, “More on WZ action of non-BPS branes,” JHEP 0903 (2009) 008 [arXiv:0812.4216 [hep-th]].

[11] J. Polchinski, “String duality: A Colloquium,” Rev. Mod. Phys. 68, 1245 (1996) [hep-th/9607050].

[12] C. Vafa, “Lectures on strings and dualities,” [arXiv:hep-th/9702201].

[13] A. Hashimoto and I. R. Klebanov, “Scattering of strings from D-branes,” Nucl. Phys. Proc. Suppl. 55B, 118 (1997) [arXiv:hep-th/9611214].

[14] P. Koerber and A. Sevrin, “The NonAbelian D-brane effective action through order alpha-prime**4,” JHEP 0210, 046 (2002) [hep-th/0208044].
[15] A. Keurentjes, P. Koerber, S. Nevens, A. Sevrin and A. Wijns, “Towards an effective action for D-branes,” Fortsch. Phys. 53, 599 (2005) [hep-th/0412271].

[16] M. R. Garousi and R. C. Myers, “World volume potentials on D-branes,” JHEP 0011, 032 (2000) [hep-th/0010122].

[17] H. Liu and J. Michelson, “*t*-trek III: The search for Ramond-Ramond couplings,” Nucl. Phys. B 614, 330 (2001) [arXiv:hep-th/0107172].

[18] E. Hatefi, A. J. Nurmagambetov and I. Y. Park, “$N^3$ entropy of M5 branes from dielectric effect,” to appear in NPB, [arXiv:1204.2711 [hep-th]].

[19] E. Hatefi, A. J. Nurmagambetov and I. Y. Park, “Near-Extremal Black-Branes with $n^*3$ Entropy Growth,” [arXiv:1204.6303 [hep-th]].

[20] E. Hatefi, A. J. Nurmagambetov and I. Y. Park, work in progress.

[21] I. Y. Park, “One loop scattering on D-branes,” Eur. Phys. J. C 62, 783 (2009) [arXiv:0801.0218 [hep-th]].

[22] E. Hatefi, “Three Point Tree Level Amplitude in Superstring Theory,” Nucl. Phys. Proc. Suppl. 216, 234 (2011) [arXiv:1102.5042 [hep-th]].

[23] R. Medina, F. T. Brandt and F. R. Machado, “The open superstring 5-point amplitude revisited,” JHEP 0207, 071 (2002) [arXiv:hep-th/0208121].

[24] C. M. Hull, “Matrix theory, U duality and toroidal compactifications of M theory,” JHEP 9810, 011 (1998) [hep-th/9711179].

[25] H. Dorn, “NonAbelian gauge field dynamics on matrix D-branes,” Nucl. Phys. B 494, 105 (1997) [hep-th/9612120].

[26] A. Fotopoulos, “On (alpha')**2 corrections to the D-brane action for non-geodesic world-volume embeddings,” JHEP 0109, 005 (2001) [arXiv:hep-th/0104146].

[27] E. Hatefi, work in progress.

[28] M. R. Garousi and H. Golchin, “On higher derivative corrections of the tachyon action,” Nucl. Phys. B 800, 547 (2008) [arXiv:0801.3358 [hep-th]].