A QUESTION ABOUT BELYI’S THEOREM

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Abstract. I discuss a natural version of Belyi’s Theorem over \( F_q(T) \) and prove that the situation I describe is unique and rigid for \( q \geq 5 \) (in the sense described below).

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1. Preliminaries

Let \( F_q \) be a finite field of characteristic \( p > 0 \) with \( q \) elements and let \( A = F_q[T] \) be the polynomial ring in one variable \( T \) over \( F_q \). Let \( F = F_q(T) \) be its quotient field, fix an algebraic closure \( \overline{F} \) of \( F \) and the separable closure \( F_{\text{sep}} \) of \( F \) in \( \overline{F} \). Let \( F_\infty = F_q((\frac{1}{T})) \) be the completion of \( F \) at the valuation of \( F \) corresponding to \( \infty := 1/T \). Let \( C_\infty \) be the completion of an algebraic closure of \( F_\infty \). Let \( G = \text{GL}_2(A) \subset \text{GL}(F_\infty) \). Note \( G \) is a discrete subgroup of \( \text{GL}(F_\infty) \). Let \( \Gamma_1 = \text{SL}_2(A) \) and

\[
\Gamma_T = \{ g \in \text{GL}_2(A) : g \equiv 1 \mod (T) \}.
\]

Then one has inclusions of normal subgroups \( \Gamma_T \subset \Gamma_1 \subset G \). None of these groups are finitely generated and \( \Gamma_1 \) has uncountably many subgroups of finite index (see [11, Theorem 12, page 124]) and the set of congruence subgroups is countable so most of the subgroups of finite index are not congruence subgroups.

Let \( \Gamma \subset \Gamma_T \) be a subgroup of finite index. The set

\[
q\ell(\Gamma) = \left\{ a \in A : \left( \begin{array}{cc} 1 & a \\ 0 & 1 \end{array} \right) \in \bigcap_{g \in G} g\Gamma g^{-1} \right\}
\]

will be called the quasi-level of \( \Gamma \). It is not difficult to show that the \( q\ell(\Gamma) \) is an \( F_q \)-subspace of \( A = F_q[T] \) (see [7, Lemma 3.1 and its proof]) of finite codimension ([7, Lemma 3.3(ii)]). The level of \( \Gamma \), denoted here by \( \ell(\Gamma) \), is the largest ideal \( \ell(\Gamma) \subseteq q\ell(\Gamma) \) of \( A \) contained in the quasi-level of \( \Gamma \). In general the inclusions \( \ell(\Gamma) \subseteq q\ell(\Gamma) \) may be strict and the level may

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be zero. It is a theorem that a subgroup $\Gamma$ of finite index of level $\ell(\Gamma) = I$ is a congruence subgroup if and only if $\Gamma$ contains the full congruence subgroup of $G$ of classical level $I$ (see (7)) and its references for basic facts about level and quasi-level. In particular the notion of level generalizes the classical notion of level of a congruence subgroup of finite index of $\text{SL}_2(\mathbb{Z})$.

The group $\text{GL}(F_\infty)$ operates on the classical rigid analytic (i.e. in the sense of Tate) Drinfeld upper half plane $\mathcal{H}_{\text{rig}} = \mathbb{P}^1(C_\infty) - \mathbb{P}^1(F_\infty)$ (by Mobius transformations (see [2])) and $G$ operates as discrete group with finite stabilizers. By ([3, Page 50]) one sees that stabilizers of order prime to $p$ have eigenvalues in $\mathbb{F}_{q^2}$ and in particular the action of $\Gamma_T$ on $\mathcal{H}_{\text{rig}}$ is such that no point has stabilizers of $p'$-order (i.e. $\Gamma_T$ has no elliptic elements).

Consider the action of a subgroup $\Gamma \subseteq \Gamma_T$ of finite index on $\mathcal{H}_{\text{rig}}$ and consider the quotient $Y_T$ which is a rigid-analytic curve which may be compactified to obtain a rigid analytic space $X_\Gamma$ which is a projective (and hence also algebraizable curve over $\mathbb{C}_\infty$).

The following example of this is important to this note: for $\Gamma = \Gamma_T$, one has an isomorphism of rigid analytic spaces $\mathcal{H}_{\text{rig}}/\Gamma_T \simeq \mathbb{P}^1 - B$ where $B \subset \mathbb{P}^1$ is a closed subset consisting of $|B| = q+1 \geq 3$ points, and in this case moreover one has an isomorphism $X_{\Gamma_T} \simeq \mathbb{P}^1$ of (classical) rigid analytic spaces and the points $B$ in particular consists of the cusps of $\Gamma_T$ which are also defined over a finite separable extension of $F$ (for more on this see 2.2(14)).

If $X/C_\infty$ is a projective rigid analytic space, I say that $X$ is weakly-modular if $X$ is isomorphic as a rigid analytic space to $X_T$ for some subgroup $\Gamma$ of finite index in $\Gamma_T$. I say that $X$ is modular if $X$ is weakly modular and $\ell(\Gamma) \neq 0$. Note that in this case $0 \subsetneq \ell(\Gamma) \subsetneq A$ as $\Gamma \subset \Gamma_T$ (see 2.2(3)) for more on this hypothesis).

If $X$ is weakly modular and hyperbolic (i.e genus of $X$ is at least two) then $X$ is a Mumford curve over $\mathbb{C}_\infty$ and so admits a Schottky uniformization. In particular its (topological) fundamental group is a free group on $g$ generators where $g$ is the genus of $X$ (and $g \geq 2$ by hyperbolicity). Modularity condition is invariant under geometric automorphisms of $\Gamma_1$ (see 2.2(15) and 2.2(16)). It is possible to formulate a condition stronger than modularity and which I call classical modularity (see 2.2(20)).

2. The main question and several remarks

Notations and conventions of Section 1 remain in force.

**Question 2.1.** Suppose $X$ is geometrically connected, smooth, projective, hyperbolic curve defined over finite separable extensions of $E/F$ for which there exists a place $v$ of $E$ lying over $\infty$ of $F$ such that $X_v := X \times_F E_v$ is Mumford curve. Can one characterize $(X, v)$ such that $X_v$ is modular?

First of all there exist such curves (see 2.2(1)) so the class of such curves is non-empty. Any such curve is not isotrivial (see 2.2(11)). Moreover such an $X$ always admits a morphism $X \to \mathbb{P}^1$ which is defined over $E$ and unramified outside $B$ (see 2.2(18)). Modularity of $X_v$ is a rigid condition (see Theorem 3.1 (given below) and 2.2(3)). In fact the situation is completely rigid for $q \geq 5$ (see Theorem 3.1). For connection with classical Belyï's Theorem see [1] and 2.2(7) and 2.2(17). There is also a characteristic $p > 0$ version of Belyï's Theorem due to S. S. Abhyankar (1957) which predates Belyï's paper (see 2.2(8)).
Remark 2.2. I expect that there should exist a natural, affirmative answer to the above question (simple enough to be scribbled in the margin of your copy of Belyi’s classic paper\(^1\) ([1])). Following remarks explain the necessity of various hypothesis and conditions in the definitions made above.

1. If \( \Gamma \) is a congruence subgroup of level \( I \) then by ([2]) \( X_\Gamma \) is, hyperbolic except for a finite number of ideals of \( A \), and more importantly defined over a finite separable extension of \( F \). So the set of such curves is certainly non-empty.

2. As was also pointed out to me by Akio Tamagawa, \( \textbf{I do not address the question of whether or not } X_\Gamma \textbf{ is always defined over a finite separable extension of } F \) and in fact \( \textbf{I do not know how to prove this at the moment; but I do expect that } X_\Gamma \textbf{ is defined over a finite extension of } F \textbf{ whenever } \ell(\Gamma) \neq 0 \). In conversations Tamagawa sketched a very interesting method to prove this but this remains to achieved at the moment.

3. By an important result (see [7, Theorem 6.8]) the set of subgroups \( \Gamma \subseteq \Gamma_1 \) of finite index and non-zero level (i.e \( \ell(\Gamma) \neq 0 \)) is countable. This is the reason why I define modularity to include a restriction on the level. My definition ensures \textit{rigidity}. Note that for SL\(_2\)(\( \mathbb{Z} \)) the level of any finite index subgroup is always non-zero and coincides with its quasi-level.

4. In particular the subgroups \( \Gamma \) with \( \ell(\Gamma) = (0) \subset A \) are uncountable,

5. while non-congruence subgroups of non-zero level are also countable (this only uses the fact that \( A \) is a PID, but see Theorem 3.1 and its proof).

6. In my emails to Deligne I had coined the term \textit{ac cuspidal subgroup} for \( \Gamma \) such that \( \ell(\Gamma) = (0) \). As the next remark notes classically there are no acuspidal subgroups. Acuspidal subgroups remain quite mysterious (to me).

7. Since every subgroup of finite index of SL\(_2\)(\( \mathbb{Z} \)) has non-zero level (see [7] and its references), one has a reformulation of classical Belyi’s Theorem as follows: every smooth, proper curve over finite extension of \( \mathbb{Q} \) is modular and conversely every compact, connected (uniformizable) complex analytic space of dimension one equipped with a morphism to \( \mathbb{P}^1 \) which is etale outside \( \mathcal{B} \) is modular.

8. There is a characteristic \( p > 0 \) analog of Belyi’s Theorem which was in fact discovered by S. S. Abhyankar in his 1957 paper on fundamental groups and has been rediscovered again and again by many people including myself (sometime in 1991–1992). \textit{Abhyankar’s Theorem} says: every smooth, proj. curve \( X/k \) over a field of characteristic \( p > 0 \) admits a morphism \( X \to \mathbb{P}^1 \) which is unramified outside one point. Abhyankar pointed out to me that he had discovered this version in his paper during one of our meetings at the Tata Institute.

9. Related to the notions quasi-level and level of a subgroup are the notions of \textit{cuspidal amplitude} and \textit{quasi-amplitude} (see [8]). Cuspidal amplitude is an ideal, quasi-amplitude is a subgroup of \( A \) but not an \( \mathbb{F}_q \)-subspace in general. The intersection of all quasi-amplitudes is the quasi-level and intersection of cusp-amplitudes is the level. So my hypothesis \( \ell(\Gamma) \neq 0 \) implies that all the cuspidal amplitudes of \( \Gamma \) are nonzero and that there are only finitely many ideals in the set of cuspidal amplitudes (see [8, Remark 2.4(ii)]). For an acuspidal subgroup the intersection of cuspidal amplitudes is zero. So in this case cusps have no common ideal of parabolic stabilizers.

10. The assumption that \( X \) be a Mumford curve at \( v \) is necessary as \( X_\Gamma \) is a Mumford curve.

\( ^1 \)my apologies to Pierre de Fermat and G. V. Belyi.
The assumption that $X$ is a Mumford curve at least at one place of $E$ implies that $X$ is not isotrivial.

Note that if $X_\Gamma$ is hyperbolic for some $\Gamma \subseteq \Gamma_T$ then $\Gamma$ has a free quotient on $g$ generators given by $\Gamma/N$ where $N$ is the normal subgroup generated by the stabilizers of the cusps.

I use $\Gamma_T$ in this note because $\Gamma_T$ has no elliptic elements as opposed to $\Gamma_1$ which has elliptic elements. This follows from ([3]) as pointed out earlier.

That $X_{\Gamma_T} \simeq \mathbb{P}^1$ as rigid analytic spaces follows from the genus calculation in ([4, Theorem 8.1]) for $\Gamma_T$.

One may replace $\Gamma_T$ by other degree one prime ideals of $A$ and the genus of the quotient is unchanged. The group of automorphisms of $A$ given by $T \mapsto T + \alpha$ ($\alpha \in \mathbb{F}_q$) acts transitively on the set of degree one prime ideals so the description given above are isomorphic if we replace $\Gamma_T$ by its image under this group of automorphisms.

Moreover (here one uses that $A = \mathbb{F}_q[T]$) there are no non-congruence subgroups $\Gamma$ with $l(\Gamma) = (f(T))$ where $f(T)$ has degree $\leq 1$. Hence every geometric group automorphism (i.e. an automorphism of $\Gamma_1$ which is contained in the group generated by inner automorphisms and automorphisms induced from ring automorphisms of $A$) of $\Gamma_1$ maps $\Gamma_T$ to a congruence subgroup whose level has degree equal to degree of $l(\Gamma_T) = (T)$. So the description given above is really independent of the choice of $\Gamma_T$ and invariant under geometric automorphisms of $\Gamma_1$.

Proofs of Belyi’s Theorem show that it is more natural to work with a subgroup $\Gamma \subset \text{SL}_2(\mathbb{Z})$ of finite index which has no elliptic elements whose action on the classical upper half-plane provides a quotient isomorphic to $\mathbb{P}^1 - B$ where $B$ consists exactly of three cusps of $\Gamma$. So the situation described above bears a close parallel to Belyi’s Theorem hence the title.

Also note that given a curve $X$ over a finite separable extension $E/F$ as in Question 1.1, then there always exists a morphism to $X \to \mathbb{P}^1$ over a finite separable extension of $E$ which is unramified outside $B$. This is proved as follows: Choose some morphism to $\mathbb{P}^1$ defined over $E$ (this exists by Noether normalization). By the usual argument of Belyi’s proof one can enlarge the branch locus so that all the branch points of this morphism contained in $\mathbb{P}^1$ are defined over this field extension. Now apply an automorphism of $\mathbb{P}^1$ so that the place $v$ at which $X$ is a Mumford curve is mapped to $\infty$. The remaining points are collapsed to $0 \in \mathbb{P}^1 - \infty = \mathbb{A}^1 \simeq \mathbb{G}_a$ by quotienting by the additive subgroup generated by the remaining points. This gives a morphism $X \to \mathbb{P}^1$ which is unramified outside $0, \infty$. After passing to a finite separable extension apply an automorphism of $\mathbb{P}^1$ which maps $0, \infty$ to two points of $B$. This gives a morphism $X \to \mathbb{P}^1$ over the given base field which is unramified outside $B$. So every curve $X$ over a finite separable extension of $F$ and which is a Mumford curve at some valuation of the base field always admits a morphism $X \to \mathbb{P}^1$ (after passage to some finite separable extension) which is unramified outside $B$.

Let me note (as was pointed out to me by Berkovich) that $\mathcal{S}_r \to \mathcal{Y}_r$ is not a topological or analytic covering (even if one works with Berkovich spaces) because of the parabolic elements of $\Gamma$ which have order $p$. As pointed out to me by Mochizuki, this is not tempered covering either because of presence of $p$-torsion in $\Gamma$ (note that quotients of $\mathcal{S}_r$ by free groups are tempered coverings). In particular Mochizuki pointed out that the tempered fundamental group does not track such quotients.
So this raises the question which fundamental group tracks such discrete coverings of \( \mathfrak{f}^{\text{rig}} \)? Presumably one should could simply take étale coverings \( \mathfrak{f}^{\text{rig}} \to \mathfrak{f}^{\text{rig}} \) and their quotients by discrete groups and this should be “fundamental group” of the sort which one can use in this context.

(20) It is possible, and even tempting, to consider the following stronger modularity condition: I say that \( X_T \) is classically modular if \( \Gamma \subseteq \Gamma_T \) is a subgroup of finite index such that \( q^t(\Gamma) = t(\Gamma) \) and \( t(\Gamma) \neq 0 \). This implies the intersection of all cuspidal amplitudes and quasi-amplitudes are equal to the level. If \( \Gamma \subseteq \Gamma_T \) is a congruence subgroup then \( \Gamma \) is classically modular (more generally any congruence subgroup of \( \Gamma_1 \) is classically modular) and any \( \Gamma \subset \text{SL}_2(\mathbb{Z}) \) certainly is classically modular. At any rate classically modular \( \Gamma \subseteq \Gamma_T \) are also countable. But it seems worth keeping the hypothesis as minimal as possible. So I chose to work with modularity (as opposed to classical modularity).

(21) It is tempting to speculate that every hyperbolic, smooth, compact, uniformizable, strictly analytic space (in the sense of Berkovich) of dimension one over \( \mathbb{C}_\infty \) is in fact weakly modular. But at this point this seems too wild to be true.

(22) As was pointed out to me by Mochizuki, there is also an analog of Belyi’s Theorem over finite fields (see [10, 12]): which asserts that \( X \) is defined over a finite field if and only if there is a morphism \( X \to \mathbb{P}^1 \) which is tamely ramified outside \( 0, 1, \infty \). Note that if \( q = 2 \) then the set \( |\mathcal{B}| = 3 \), but the morphism \( X \to \mathbb{P}^1 \) is wildly ramified over points of \( \mathcal{B} \). In another context, in my construction of the Drinfeld analog of category of Thakur’s function field multi-zeta values, I have observed that \( q = 2 \) presents presently a distinctly puzzling behavior and this also surfaces in the present context: for \( q = 2, |\mathcal{B}| = 3 \) means that \( \mathcal{B} \) can be mapped bijectively into any three-point set in \( \mathbb{P}^1 \).

3. Uniformizational rigidity of Drinfeldian domains

Before proceeding further let me introduce some additional terminology. Let \( B/\mathbb{F}_q \) be an \( \mathbb{F}_q \) algebra. I say that \( B \) is a Drinfeldian domain if \( B = H^0(C - \{x\}, \mathcal{O}_C) \) for some geometrically connected, smooth, proper curve \( C \) over \( \mathbb{F}_q \) and where \( x \in C \) is a closed point. Morphisms of Drinfeldian domains \( f : B \to B' \) is a morphism of smooth, proper curves \( f : C \to C' \) with \( f(x) = x' \). I say that a Drinfeldian domain \( B \) is uniformizationally rigid if \( \Gamma = \text{GL}_2(\mathbb{Z}) \) has only countably many subgroups of finite index and non-zero level (note that \( \text{SL}_2(\mathbb{Z}) \) is uniformizationally rigid). By [2] any Drinfeldian domain gives rise to many “modularity scenarios” such as the one sketched above (arising from arbitrary Drinfeldian domain \( B \)). However my next result (see Theorem 3.1), which strengthens the rigidity theorem of ([7, Theorem 6.8]), shows that for \( q \geq 5 \) there is, up to isomorphism, one and only one uniformizationally rigid Drinfeldian domain:

**Theorem 3.1.** If \( q \geq 5 \) then any uniformizationally rigid Drinfeldian domain \( B \) is isomorphic to \( A = \mathbb{F}_q[T] \).

**Proof.** Let \( B \) arise from the datum \((C, x)\) as above. By ([7, Theorem 6.8]) \( B \) is uniformizationally rigid if and only if the class group of \( B \) is trivial. So it suffices to prove that if \( B \) is a Drinfeldian domain with trivial class group then \( B \simeq \mathbb{F}_q[T] \). Let \( K \) be the quotient field of \( B \). Let \( \mathfrak{p}_x \) be the unique valuation of \( K \) such that \( B \) is ring of \( S = \{\mathfrak{p}_x\}\)-integers of \( K \). Let \( Cl(K) \), \( Cl(B) \) denote class groups of \( K \) and \( B \) respectively (recall that class group of \( K \)
is by definition equal to $\text{Pic}^0(C)(\mathbb{F}_q)$. Then by ([5, 9]) one has an exact sequence

$$0 \to D_1 \to \text{Cl}(K) \to \text{Cl}(B) \to D_2 \to 0$$

where the morphism in the middle maps a divisor of degree $\sum z n_z[z]$ (read modulo principal divisors) to the divisor $\sum_{z \neq x} n_z[z]$ (read modulo the image of principal divisors). The groups $D_1$ is the kernel of this homomorphism and hence $D_1$ is the subgroup of divisors of $K$ which are supported on $p_x$ and are of degree zero modulo its subgroup of principal divisors and by ([5, 9]), while $D_2$ is a certain cyclic subgroup. Firstly since there are no non-trivial divisors of degree zero supported on a single point $x$, it follows that this short exact sequence reduces to

$$0 \to \text{Cl}(K) \to \text{Cl}(B) \to D_2 \to 0.$$

As $\text{Cl}(B)$ has class number one it follows that $\text{Cl}(K) = 1$. As $q \geq 5$, by ([5, 6]) it follows that there is exactly one function field over $\mathbb{F}_q$ whose class number is one: namely $K = \mathbb{F}_q(T)$. Thus one deduces that $C \simeq \mathbb{P}^1$. From ([5, 9]) one deduces that $D_2 \neq 0$ if and only if $\deg(x) \neq 1$. Thus $\deg(x) = 1$ hence one has $(C, \{x\}) = (\mathbb{P}^1, \{x\})$ where $x$ is a closed point of degree one. Since $\deg(x) = 1$ there are $q + 1$ choices for $x$ and the corresponding domains $B$ are all isomorphic to $A = \mathbb{F}_q[T]$ and hence the result is established. □

Since the situation is unique and rigid for $q \geq 5$, it seems reasonable that curves in Question 2.1 should have a nice characterization.

**Remark 3.2.** If $q \leq 4$ there are exactly four Drinfeldian domains of class number one which are not isomorphic to $A = \mathbb{F}_q[T]$. For a complete list see ([7, Remark 6.1]). I thank Andreas Schweizer for this remark.

### 4. Acknowledgments

In late 2016 (or early 2017), I had an idea for a naive version of Belyi’s Theorem for $\mathbb{F}_q(T)$ while reading Shinichi Mochizuki’s paper on non-critical Belyi maps and I promptly wrote up my naive version and sent it off to a few people for comments. Thanks are due to Pierre Deligne for a stimulating correspondence. In particular, Deligne (and later Mochizuki as well), pointed out importance of rigidifying the data. In my emails to Deligne, I had explained briefly the solution to this issue which I found via ([7]), whose existence I had just become aware of at that time, and is presented here in my definition of modularity given above. Unfortunately soon after this Vladimir Berkovich in a gentle email pointed out a number of errors in my proof (which was based on his proof of the Mustafin conjecture) and I thank him for alerting me to a number of subtleties I had overlooked. I would also like to thank Brian Conrad for answering (insightfully) all my naive questions about Berkovich spaces with infinite patience.

After realizing my mistake, I had abandoned my quest to find an analog of Belyi’s theorem in this context and the details of this embarrassment promptly slipped out of my mind including the way to rigidify the data.

Since this quest had begun while reading Mochizuki’s paper on noncritical Belyi maps I felt that perhaps it would worth revisiting this issue upon my arrival in Kyoto for a sabbatical early this year (2018). During our conversations, Mochizuki also pointed out that data should be rigid and I realized that I had completely forgotten all the details about how to force rigidity. So I decided that I should write up the details and spent a few days trying to
retrace my approach and my observations as they may be useful to others who may want to take this up (*if at all*). In the course of this process I discovered Theorem 3.1.

It is a pleasure to thank Shinichi Mochizuki for his comments and answering my naive and even elementary questions about tempered fundamental groups, Belyi’s Theorem and other topics with great sagacity. Support and excellent hospitality from Research Institute for Mathematical Sciences, Kyoto University (RIMS) is gratefully acknowledged, as is my indebtedness to Shinichi Mochizuki for hosting my visit at RIMS.

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