HARMONIC CURRENTS DIRECTED BY FOLIATIONS BY RIEMANN SURFACES

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ABSTRACT. We study local positive $dd^c$-closed currents directed by a foliation by Riemann surfaces near a hyperbolic singularity which have no mass on the separatrices. A theorem of Nguyễn says that the Lelong number of such a current at the singular point vanishes. We prove that this property is sharp: one cannot have any better mass estimate for this current near the singularity.

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1. INTRODUCTION

In theory of foliations by Riemann surfaces, directed positive $dd^c$-closed currents play a central role like invariant measures for a dynamic system, see e.g. [2, 4, 5, 6, 9, 11]. A fundamental problem is to understand such currents near the singularities of the foliations.

Let $\mathcal{F}$ be a foliation by Riemann surfaces near $0 \in \mathbb{C}^n$ such that $0$ is an isolated hyperbolic singularity. Let $T$ be a positive $dd^c$-closed current of bi-dimension $(1, 1)$ directed by $\mathcal{F}$. We assume that this current has no mass on the separatrices of the foliation at $0$. In [7, 9], Nguyễn proves that the Lelong number of $T$ at $0$ vanishes. Equivalently, the mass of $T$ in the polydisc $\delta \mathbb{D}^n := \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^2 : |z_i| < \delta\}$ satisfies

$$\|T\|_{\delta \mathbb{D}^n} = o(\delta^2) \quad \text{as} \quad \delta \to 0.$$ 

In this paper, we show that Nguyễn’s result is sharp: one cannot have a better estimate.

For simplicity, we consider the case of complex dimension $n = 2$. It is not difficult to extend the result to the higher dimension case. Here is our main theorem.

**Theorem 1.1.** Let $\mathcal{F}$ be a foliation by Riemann surfaces in a neighborhood of $0 \in \mathbb{C}^2$. Assume that $0$ is a hyperbolic singularity. Let $\varepsilon : [0, 1] \to \mathbb{R}^+$ be a continuous function such that $\varepsilon(0) = 0$. Then there exists a positive $dd^c$-closed $(1, 1)$-current $T$ in a neighborhood of $0 \in \mathbb{C}^2$ directed by $\mathcal{F}$ having no mass on the separatrices at $0$ such that

$$\|T\|_{\delta \mathbb{D}^2} \geq \varepsilon(\delta)\delta^2.$$ 

In particular, we do not have the general estimate

$$\|T\|_{\delta \mathbb{D}^2} \lesssim |\log \delta|^{-\alpha}\delta^2$$

for some $\alpha > 0$ in the local setting. This estimate is crucial in the study of global dynamics of foliations via Poincaré metric and random walk on leaves, see [8] for details.

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Let $F$ be a holomorphic vector field on a neighborhood of $0 \in \mathbb{C}^2$ which defines the foliation $\mathcal{F}$. Recall that $0$ is a hyperbolic singularity of $\mathcal{F}$ if we can choose $F$ so that

$$F = \eta z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} + \text{higher order terms}$$

with $\eta = a + ib$ and $b \neq 0$. By Poincaré-Dulac theorem, see e.g. [11], holomorphic vector fields are linearizable near hyperbolic singularities. We can change the local coordinate system $(z_1, z_2)$ so that

$$F = \eta z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}.$$ 

From now on, we use these new coordinates. Then the leaves of $\mathcal{F}$ can be described explicitly, see Section 2 below. The two axes $\{z_1 = 0\}$ and $\{z_2 = 0\}$ are called the separatrices of $\mathcal{F}$ at $0$.

We will construct an explicit positive $dd^c$-closed current $T$ having full mass on a leaf $L$ of $\mathcal{F}$. It is given by a positive harmonic function on $L$.

In Section 2, we will recall the standard parametrization of the leaves of $\mathcal{F}$ together with some properties that we need in this paper. We will also give the construction of the current $T$. In Section 3, we will show that $T$ satisfies our main theorem.

Throughout this paper, the symbols $\lesssim$ and $\gtrsim$ stand for inequalities up to a multiplicative constant. We write $\simeq$ if both hold. We denote by $\mathbb{H}, \mathbb{D}, \mathbb{D}, \delta \mathbb{D}$ the upper half-plane in $\mathbb{C}$, the unit disk, the disk of centre 0 and radius $\delta$, and the bidisc $\delta \mathbb{D} \times \delta \mathbb{D}$ respectively.

## 2. Construction of harmonic current and some estimates

We will use the notation as in [3, Sections 4 and 5]. For simplicity, we also assume that $b > 0$ (if $b < 0$ we use the change $(z_1, z_2) \mapsto (z_2, z_1)$ and $F \mapsto \eta^{-1} F$ in order to reduce to the case $b > 0$). Define the annulus $A$ by

$$A := \{ \alpha \in \mathbb{C} : e^{-2nb} < |\alpha| \leq 1 \}$$

and the sector $S$ in the upper half-plane $\mathbb{H}$ by

$$S := \{ \zeta = u + iv : v > 0, bu + av > 0 \}.$$ 

For $\alpha \in \mathbb{C}^*$, consider the Riemann surface $L_\alpha$ immersed in $\mathbb{C}^2$ defined by

$$z_1 = \alpha e^{i\eta(\zeta + \log|\alpha|/b)} \quad \text{and} \quad z_2 = e^{i(\zeta + \log|\alpha|/b)} \quad \text{with} \quad \zeta = u + iv.$$ 

The map $\zeta \mapsto (z_1, z_2)$ is injective because $\eta \notin \mathbb{R}$. It is easy to check that $L_\alpha$ is tangent to the vector field $F$. Hence $L_\alpha$ is a parametrization of a leaf of $\mathcal{F}$ in $\mathbb{C}^2$.

For $\alpha_1, \alpha_2 \in A$, one can check that $L_{\alpha_1}$ and $L_{\alpha_2}$ are disjoint if $\alpha_1 \neq \alpha_2$. The union of all $L_\alpha, \alpha \in A$, is equal to $(\mathbb{C}^*)^2$. The intersection $L_\alpha := L_\alpha \cap \mathbb{D}^2$ is given by the same equations as $L_\alpha$ but with $\zeta \in S$. Since $L_\alpha$ is a connected submanifold of $(\mathbb{D}^*)^2$, it is a leaf of $\mathcal{F} \cap \mathbb{D}^2$.

Fix an $\alpha \in A$, and denote by $\pi : S \to L_\alpha$ the above parametrization of $L_\alpha$. From now on, we take $\alpha = 1$ for simplicity. In this case, we have

$$\pi(\zeta) := (e^{i\eta \zeta}, e^{i\zeta}) \quad \text{with} \quad \zeta = u + iv \in S.$$
Consider the biholomorphic map $\Phi : \mathbb{S} \to \mathbb{H}$ defined by

$$\Phi(\zeta) := \zeta^\gamma \quad \text{with} \quad \gamma > 1 \quad \text{and} \quad \tan \frac{\pi}{\gamma} = \frac{b}{a}.$$  

We use the coordinate $Z = U + iV := \Phi(\zeta)$ on $\mathbb{H}$.

We can replace the function $\varepsilon$ in the main theorem by a suitable larger function which is smooth on $(0, 1]$, strictly increasing and concave. Fix a large constant $A > 0$ and define the function $\tilde{H}(x)$ on $\mathbb{R}$ by

$$\tilde{H}(\pm t^\gamma) := \gamma^{-1}Ae^{-t^\gamma}(e^{-t}) \quad \text{for} \quad t \geq 0.$$  

We can easily check that

$$\int_{x \leq -t^\gamma} \tilde{H}(x)(-x)^{-1+1/\gamma}dx = \int_{t \geq t^\gamma} \tilde{H}(x)x^{-1+1/\gamma}dx = Ae(e^{-t}).$$  

Then we extend $\tilde{H}$ to a positive harmonic function, still denoted by $\tilde{H}$, on the upper half-plane $\mathbb{H}$ by using Poisson kernel, i.e.

$$\tilde{H}(U + iV) := \frac{1}{\pi} \int_\mathbb{R} \tilde{H}(x)\frac{V}{V^2 + (U - x)^2}dx \quad \text{for} \quad U + iV \in \mathbb{H}.$$  

Since $\tilde{H}(x) = \tilde{H}(-x)$, the function $\tilde{H}$ is symmetric: $\tilde{H}(U + iV) = \tilde{H}(-U + iV)$ for $U + iV \in \mathbb{H}$. Define the function $H := \tilde{H} \circ \Phi$ on $\mathbb{S}$.

Consider the following $(1, 1)$-current on $\mathbb{D}^2$,

$$T := \pi_*(H[\mathbb{S}]).$$  

In the rest of this paper, we will show that $T$ is a current that satisfies Theorem 1.1.

As in [3], to simplify the computation, we will introduce some new variables. Firstly, let $\zeta^* := u^* + i = -a/b + i$ be the intersection point of $\{v = 1\}$ with the line $\{bu + av = 0\}$. Define

$$\rho := |\zeta^*|^\gamma = -\Phi(\zeta^*).$$  

We consider $\zeta = u + iv$ in the half-line $\{v = s\} \cap \mathbb{S}$ and the new variable

$$r := u/s - u^*.$$  

Note that $rs$ is comparable with the distance of $\zeta = u + iv$ to the edge $\{bu + av = 0\}$ of $\mathbb{S}$ and we have $bu + av = brs$ and $du = sdr$ on the half-line $\{v = s\} \cap \mathbb{S}$. Define also the variables

$$Z^* := s^{-\gamma}Z, \quad U^* := s^{-\gamma}U, \quad V^* := s^{-\gamma}V$$  

for $U + iV = \Phi(\zeta) \in \mathbb{H}$. Notice that $\zeta = s(\zeta^* + r)$ and hence $Z^* = U^* + iV^* = \Phi(\zeta^* + r)$. Finally, we write

$$x^* := s^{-\gamma}x \quad \text{for} \quad x \in \mathbb{R}.$$  

We need the following three lemmas for the new variables. See [3, Section 5] for the proofs.

**Lemma 2.1.** When $r \to 0$, we have $U^* = -\rho + O(r)$ and $V^* = \beta r + O(r^2)$ for some constant $\beta > 0$. Moreover, given a constant $N > 0$, we have for $0 \leq r \leq N$,

$$\text{dist}(x^*, U^* + iV^*)^2 \geq c_N \left[r^2 + \text{dist}(x^*, -\rho)^2\right],$$

where dist denotes the standard distance and $c_N > 0$ is a constant independent of $x'$.  

Lemma 2.2. When $r \to \infty$, we have $U' = r^{\gamma} + O(r^{\gamma-1})$ and $V' = \gamma r^{\gamma-1} + O(r^{\gamma-2})$.

Lemma 2.3. There is a constant $c > 0$ such that
\[
\int_0^\infty \frac{V'}{V^2 + (U' - x')^2} \, dr \leq c|x'|^{-1+1/\gamma} \quad \text{for } |x'| \geq 2\rho.
\]

We also need the following estimate.

Lemma 2.4. There is a constant $c > 0$ such that
\[
\int_{1/b}^\infty \frac{V'}{V^2 + (U' - x')^2} \, dr \geq cx'^{-1+1/\gamma} \quad \text{for } x' \geq 1.
\]

Proof. The lemma is clear when $x'$ is bounded by a constant. So it is enough to consider $x' \geq (1/b)^\gamma$. In this case, we have $x'^{1/\gamma} \geq 1/b$ and the considered integral is larger than the integral for $r$ between $x'^{1/\gamma}$ and $x'^{1/\gamma} + 1$. For those $r$, by using Lemma 2.2, we have $V' \simeq x'^{1-1/\gamma}$ and $r^\gamma = x' + O(x'^{1-1/\gamma})$ by mean value theorem. Hence $|U' - x'| \lesssim V'$. Therefore, we have
\[
\int_{1/b}^\infty \frac{V'}{V^2 + (U' - x')^2} \, dr \geq \int_{x'^{1/\gamma}}^{x'^{1/\gamma}+1} \frac{V'}{V^2} \, dr \simeq \int_{x'^{1/\gamma}}^{x'^{1/\gamma}+1} \frac{1}{x'^{1-1/\gamma}} \, dr = x'^{-1+1/\gamma}.
\]
The proof of the lemma is finished. \hfill \Box

3. Proof of the main theorem

Now we complete the proof of Theorem 1.1. Firstly, we prove that $T$ is a well-defined positive $dd^c$-closed $(1,1)$-current on $\mathbb{D}^2$.

Proposition 3.1. $T$ is a positive $(1,1)$-current of finite mass in $\mathbb{D}^2$ supported by $\overline{T}_1 = L_1 \cup \{z_1 z_2 = 0\}$.

Proof. The positivity of $T$ is clear. In order to show that $T$ is a current, it is enough to check for every smooth $(1,1)$-form $\phi$ with compact support in $\mathbb{D}^2$ that $\int_S H(\zeta)\pi^*(\phi)$ is meaningful. For this purpose, we only need to show that
\[
\int_S H(\zeta)\pi^*(dd^c\|z\|^2) < +\infty.
\]
Indeed, this inequality also shows that $T$ has finite mass. It is clear that $T$ has full mass on $L_1$ and its support is $\overline{T}_1$.

By a direct computation, using (2.1), we have
\[
\pi^*(idz_1 \wedge d\overline{z}_1) = (a^2 + b^2)e^{-2(bu+av)} id\zeta \wedge d\overline{\zeta}
\]
and
\[
\pi^*(idz_2 \wedge d\overline{z}_2) = e^{-2v} id\zeta \wedge d\overline{\zeta}.
\]
Recall that $dd^c = \frac{i}{\pi} \partial \overline{\partial}$. So we get
\[
\pi^*(dd^c\|z\|^2) = \frac{1}{\pi} ((a^2 + b^2)e^{-2(bu+av)} + e^{-2v}) id\zeta \wedge d\overline{\zeta}.
\]
Note that on the half sector $S_1 := \{bu + av \geq v\} \cap S$, we have $e^{-2(bu+av)} \leq e^{-2v}$. Moreover, the equation $bu + av = v$ is equivalent to $br = 1$ in $S$, which means that the intersection
point of \( \{bu + av = s\} \) with \( \{v = s\} \) corresponds to \( r = 1/b \) for any \( s \). So using that \( id\zeta \wedge d\zeta = 2du \wedge dv \) and the variables \( s, r \) introduced in Section 2, we get
\[
\int_{S_1} H(\zeta)\pi^*(dd^c ||z||^2) \lesssim \int_{S_1} H(\zeta)e^{-2\nu}id\zeta \wedge d\zeta = 2 \int_0^\infty e^{-2s}\left( \int_{1/b}^\infty s\tilde{H}(U + iV)dr \right)ds.
\]
For the integral inside the parentheses, using Poisson formula and the variables \( U', V', x' \) defined in Section 2, we have
\[
\int_{1/b}^\infty s\tilde{H}(U + iV)dr = \int_{1/b}^\infty \frac{1}{\pi} \int_{\mathbb{R}} s\tilde{H}(x)\frac{V}{V^2 + (U - x)^2}dxdr = \frac{1}{\pi} \int_{\mathbb{R}} \tilde{H}(x)s^{-\gamma}(\int_{1/b}^\infty \frac{V'}{V'^2 + (U' - x')^2}dr)dx.
\]
For \( |x'| \geq 2\rho \), by Lemma 2.3, we obtain
\[
\int_{1/b}^\infty \frac{V'}{V'^2 + (U' - x')^2}dr \lesssim |x'|^{-1+1/\gamma}.
\]
For \( |x'| < 2\rho \), we show that a similar estimate holds. For this purpose, we only need to consider \( r \) large enough. By Lemma 2.2 we have \( |U' - x'| \geq U' \). Thus,
\[
\int_{1/b}^\infty \frac{V'}{V'^2 + (U' - x')^2}dr \lesssim \int_{1/b}^\infty \frac{V'}{V'^2 + U'^2}dr \lesssim \int_{1/b}^\infty \frac{1}{r^{\gamma + 1}}dr \lesssim |x'|^{-1+1/\gamma}
\]
because \( |x'|^{-1+1/\gamma} \) is bounded from below by \( (2\rho)^{-1+1/\gamma} \). Therefore,
\[
\int_{1/b}^\infty s\tilde{H}(U + iV)dr \lesssim \int_{\mathbb{R}} \tilde{H}(x)s^{-\gamma}|x'|^{-1+1/\gamma}dx = \int_{\mathbb{R}} \tilde{H}(x)|x|^{-1+1/\gamma}dx.
\]
By (2.2), the last integral is finite.

Then we deduce that \( \int_{S_1} H(\zeta)\pi^*(dd^c ||z||^2) < \infty \). We can repeat the argument above for the other half sector \( S_2 := \{bu + av \leq v\} \cap S \) and get \( \int_{S_2} H(\zeta)\pi^*(dd^c ||z||^2) < \infty \), which finishes the proof of this proposition. Note that using the symmetric property of \( \tilde{H} \), we can also see that the integral on \( S_1 \) is similar to the one on \( S_2 \) and can be treated in the same way.

**Proposition 3.2.** The current \( T \) is \( dd^c \)-closed.

**Proof.** Let \( Q_s \) be the parallelogram in \( S \) limited by \( bu + av = s \) and \( v = s \). We have
\[
T = \lim_{s \to \infty} T_s \quad \text{with} \quad T_s := \pi_*(H[Q_s]).
\]

For a smooth function \( \phi \) with compact support in \( \mathbb{D}^2 \), we need to show that \( \langle T_s, dd^c \phi \rangle \to 0 \) as \( s \) tends to infinity.

Since \( \phi \) is compactly supported in \( \mathbb{D}^2 \), there exists a positive constant \( \lambda \) such that the support of \( \pi^*(\phi) \) is contained in the sector \( S^* := \{v > \lambda, bu + av > \lambda\} \) which is contained in \( S \). Define \( Q^* := Q_s \cap S^* \).

By Stokes formula, we have
\[
\langle T_s, dd^c \phi \rangle = \langle H d[Q^*_s], \pi^*(d^c \phi) \rangle + \langle dH \wedge [Q^*_s], \pi^*(d^c \phi) \rangle.
\]
We show that the first term in (3.2) tends 0. Observe that \( d[Q^*_s] = [E_s] + [E'_s] \), where \( E_s \subset \{v = s\} \) is the horizontal edge and \( E'_s \subset \{bu + av = s\} \) is the vertical edge of \( Q^*_s \).
inside $S^*$ with suitable orientations. We will only show that $\langle H[E_s], \pi^*(d^c\phi) \rangle$ tends to 0. A similar property for $\{E^*_s\}$ can be obtained in the same way.

Note that $d^c\phi$ is a combination with bounded coefficients of $dz_i$ and $dz_\zeta$. We also have

$$\pi^*(dz_1) = i\rho e^{\rho \zeta} d\zeta, \quad \pi^*(dz_\zeta) = -i\overline{\eta} e^{-i\overline{\eta} \zeta} d\zeta$$

and

$$\pi^*(dz_2) = i e^{i\zeta} d\zeta, \quad \pi^*(d\overline{\zeta}) = -i e^{-i\zeta} d\zeta.$$  

Observe that $|e^{i\rho \zeta}| = |e^{-i\rho \overline{\eta} \zeta}| = e^{-(bu+av)}$ and $|e^{i\zeta}| = |e^{-i\zeta}| = e^{-v}.

Since we are working with $\zeta = u + iv \in E_s$, we have that $e^{-v} \leq e^{-(bu+av)}$ and $d\zeta = d\overline{\zeta} = du$. So $\pi^*(d^c\phi)$ is equal to $e^{-(bu+av)} du$ times a bounded function. We only have to check that

$$\int_{E_s} H(\zeta) e^{-(bu+av)} du \to 0.$$  

Observe that $E_s = \{ \lambda < bu + av \leq s, v = s \}$. Moreover, we have $bu + av = bs\rho$ on the half-line $\{ v = s \} \cap S$. Hence the considered integral is equal to

$$\int_{\lambda < bu+av \leq s} H(\zeta) e^{-(bu+av)} du = \int_{\lambda/(bs)}^{1/b} \tilde{H}(U + iV)e^{-bs\rho} s dr$$

$$= \frac{1}{\pi} \int_{\mathbb{R}} \tilde{H}(x) \left( \int_{\lambda/(bs)}^{1/b} s e^{-bs\rho} \frac{V}{V^2 + (U - x)^2} dr \right) dx$$

$$= \frac{1}{\pi} \int_{\mathbb{R}} \tilde{H}(x) \left( \int_{\lambda/(bs)}^{1/b} s^{1-\gamma} e^{-bs\rho} \frac{V'}{V'^2 + (U' - x')^2} dr \right) dx$$

$$\lesssim \int_{\mathbb{R}} \tilde{H}(x) s^{1-\gamma} \left( \int_{\lambda/(bs)}^{1/b} (bs\rho)^{1-1/\gamma} \frac{V'}{V'^2 + (U' - x')^2} dr \right) dx.$$  

(3.3)

We need to show that the expression in (3.3) tends to 0. For this purpose, we will split the integral into two parts corresponding to $|x'| \geq 2\rho$ and $|x'| < 2\rho$.

For $|x'| \geq 2\rho$, by Lemma 2.3 and using that $bs\rho > \lambda$ on $E_s$, we have

$$\int_{|x'| \geq 2\rho} \tilde{H}(x) s^{1-\gamma} \left( \int_{\lambda/(bs)}^{1/b} (bs\rho)^{1-1/\gamma} \frac{V'}{V'^2 + (U' - x')^2} dr \right) dx$$

$$\lesssim \int_{|x'| \geq 2\rho} \tilde{H}(x) s^{1-\gamma} |x'|^{-1+1/\gamma} dx = \int_{|x| \geq 2\rho s^{\gamma}} \tilde{H}(x) |x|^{-1+1/\gamma} dx.$$  

Since $\tilde{H}(x)|x|^{-1+1/\gamma}$ is integrable on $\mathbb{R}$ (see (2.2)) and $2\rho s^{\gamma} \to \infty$, the last integral tends to 0.

For $|x'| < 2\rho$, when $s \to \infty$, we have $\lambda/(bs) \to 0$. Using Lemma 2.1 it is enough to estimate

$$\int_{|x'| < 2\rho} \tilde{H}(x) s^{1-\gamma} \left( \int_{\lambda/(bs)}^{1/b} (bs\rho)^{-1/2} \frac{V}{r^2 + (\rho + x')^2} dr \right) dx$$

$$\lesssim \int_{|x'| < 2\rho} \tilde{H}(x) s^{1-\gamma} \left( \int_{\lambda/(bs)}^{1/b} (bs\rho)^{-1/2} dr \right) dx$$

$$\simeq \int_{|x| < 2\rho s^{\gamma}} \tilde{H}(x) s^{1-\gamma} dx = \int_{|x| < 2\rho s^{\gamma}} \tilde{H}(x) |x|^{-1+1/\gamma} \left( \frac{|x|}{s^{\gamma}} \right)^{-1/\gamma} dx.$$  

Note that the function inside the last integral converges pointwise to 0. Therefore, by Lebesgue dominated convergence theorem, the last integral goes to 0. Thus, the first term in (3.2) tends to 0.

Now we show that the second term in (3.2) tends to 0. Since \( d = \partial + \overline{\partial}, d^c = \frac{i}{2\pi}(\partial - \overline{\partial}) \) and \( H \) is harmonic, by Stokes formula, this term is equal to

\[
-\langle dH \wedge d^c[Q^*_s], \pi^*(\phi) \rangle = \frac{i}{2\pi} \langle \partial H \wedge d[Q^*_s], \pi^*(\phi) \rangle - \frac{i}{2\pi} \langle \overline{\partial} H \wedge d[Q^*_s], \pi^*(\phi) \rangle.
\]

Recall that \( d[Q^*_s] = [E_s] + [E_s'] \). We only consider the integral on the horizontal edge \( E_s \) for simplicity. Fix a point \( p \) on \( E_s \), and denote by \( d_{p,1} \) the distance from \( p \) to the line \( \{bu + av = 0\} \) and \( d_{p,2} \) the distance from \( p \) to the line \( \{v = 0\} \). It is not hard to see that \( d_{p,1} \simeq sr \) and \( d_{p,2} = s \). Denote by \( d_p := \min(d_{p,1}, d_{p,2}) \). Then \( H \) is harmonic on the open disc \( \mathbb{D}(p, d_p) \) of center \( p \) and radius \( d_p \), by Harnack’s inequality, for \( x \in \mathbb{D}(p, d_p) \) and \( d_x := \text{dist}(x, p) \) we have

\[
\frac{d_p - d_x}{d_p + d_x} H(p) \leq H(x) \leq \frac{d_p + d_x}{d_p - d_x} H(p).
\]

Using that \( \lambda/(bs) < r \leq 1/2b \) on \( E_s \), we get that \( d_p \simeq sr \) on \( E_s \). Then by definition of derivative and taking \( x \to p \), we deduce that at the point \( p, \partial H \) is equal to \((sr)^{-1}Hd\zeta \times \) a bounded number and \( \overline{\partial} H \) is equal to \((sr)^{-1}Hd\zeta \times \) a bounded number. Moreover, for \( \zeta = u + iv \in E_s \), we have \( d\zeta = d\zeta = du \). Therefore, it is enough to show that

\[
\int_{E_s} H(\zeta)(sr)^{-1}du \to 0.
\]

Using the variables \( U', V' \) and \( x' \), we get

\[
\int_{E_s} H(\zeta)(sr)^{-1}du = \int_{\lambda/(bs)}^{1/b} \tilde{H}(U + iV)(sr)^{-1}sdv = \frac{1}{\pi} \int_{\mathbb{R}} \tilde{H}(x)s^{1-\gamma} \left( \int_{\lambda/(bs)}^{1/b} \frac{V'}{V' + (U' - x')^2}dv \right)dx.
\]

As for (3.3), we see that the last integral tends to 0 as \( s \) tends to infinity. This ends the proof of the proposition. \( \square \)

**Proposition 3.3.** For \( 0 < \delta < 1 \), we have \( \|T\|_{\delta\mathbb{D}^2} \geq \varepsilon(\delta)\delta^2 \).

**Proof.** Recall that \( \pi(\zeta) := (e^{i\eta\zeta}, e^{i\zeta}) \). So

\[
\pi^{-1}(\delta\mathbb{D}^2) = \{u + iv : bu + av > -\log \delta, v > -\log \delta\}.
\]

We define \( t := -\log \delta \). Hence

\[
\|T\|_{\delta\mathbb{D}^2} \simeq \int_{bu + av > t, v > t} H(\zeta)\pi^*(dd\zeta \parallel \zeta)^2).
\]

To prove the proposition, it suffices to bound the integral on \( S_1 \cap \{bu + av > t, v > t\} \) from below. Using (3.1) and the variables \( r \) and \( s \) introduced in Section\{2\}, the considered integral is equal to a constant times

\[
\int_{v > t, bu + av > v} H(\zeta)((a^2 + b^2)e^{-2(bu + av)} + e^{-2v})id\zeta \wedge d\zeta.
\]
The famous Siu's semicontinuity theorem [10] says that for every positive closed current $T$, we can show that $T$ has no mass on the separatrices at 0 because $L_1 \subset (\mathbb{D}^*)^2$. The proof of Theorem 1.1 is finished.

Remark 3.4. For every $\alpha \in \mathbb{A}$, we can construct a similar current $T_\alpha$ supported by $T_\alpha$. By taking an average of those currents, we can have a current $T$ satisfying Theorem 1.1 which is given by a smooth form on $\mathbb{D}^2$.  

Remark 3.5. Consider a general positive $dd^c$-closed current $T$ on $\mathbb{D}^2$ directed by $\mathcal{F}$. Assume that $T$ has no mass on the separatrices at 0. It is known that $T$ can be written as an average of currents $T_\alpha$ supported by $T_\alpha$ by mean of a positive measure on $\mathbb{A}$. As above, we can show that $T_\alpha$ is positive $dd^c$-closed for almost every $\alpha$.

Remark 3.6. The famous Siu's semicontinuity theorem [10] says that for every positive closed current $S$ on a complex manifold $X$, the upperlevel set $E_c := \{ \nu(S, \cdot) > c \}$ is an analytic subset of $X$ for every $c > 0$. It is well-known that this property is not true for positive $dd^c$-closed current. For the current $T$ we constructed, we have

$$\{ \nu(T, \cdot) > c \} = \{ \pi(z) : H(z) > c \},$$

which is an open set of $L_1$ and is Zariski dense in $\mathbb{D}^2$.

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