Analytic and geometric aspects of Laplace operator on Riemannian manifold

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Abstract
In the past decade there has been a flurry of work at intersection of spectral theory and Riemannian geometry. In this paper we present some of recent results on abstract spectral theory depending on Laplace-Beltrami operator on compact Riemannian manifold. Also, we will emphasize the interplay between spectrum of operator and geometry of manifolds by discussing two main problems (direct and inverse problems) with an eye towards recent developments.

Keywords
Spectrum, eigenvalue, Laplacian, spectral geometry, isospectral manifolds.

AMS Subject Classification
47A10, 58C40, 53C20, 58J50, 58J53.

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1. Introduction

Let $M$ be a compact, connected Riemannian manifold. Let $\varphi \in L^2(M)$ space of all square integrable real value on $M$. We define Laplace-Beltrami operator $\Delta \varphi = -\text{divgrad}\varphi$ where div is divergence , grad is the gradient or simply we write $\Delta \varphi = -\nabla(\nabla \varphi)$ which is differential unbounded self-adjoint operator.

The inner product is defined by $\langle \varphi, \psi \rangle = \int_M \varphi \psi dV$, where $V$ is volume form of $M$. In local coordinates $\{x_i\}$, the Laplace-Beltrami is defined by

$$\Delta_g f = \frac{1}{\sqrt{g}} \sum_{jk} \frac{\partial}{\partial x_j} \left( g^{jk} \sqrt{g} \frac{\partial}{\partial x_k} f \right)$$

where $g = |g_{jk}|$, $g^{jk} = (g_{jk})^{-1}$, $f$ is smooth function on $M$. We will discern $\Delta$ with metric when Laplace operator is associated by metric, we write $\Delta_g$. Particulary in Euclidean case the form is written as $\Delta f = -\sum_j \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j} f$, $f$ is a smooth function on $R^n$.

Suppose that $M$ is compact Riemannian manifold, we will deal with a class of eigen value problems as follows

Closed problem $\Delta \varphi = \lambda \varphi$ in $M$ $\partial M = \phi$ 
Dirchlet problem $\Delta \varphi = \lambda \varphi$ in $M$ $\varphi_{|\partial M} = \phi$ 
Neumann problem $\Delta \varphi = \lambda \varphi$ in $M$ $\frac{\partial \varphi}{\partial N_{|\partial M}} = \phi$

Where $N$ is outward oriented unit vector field normal to boundary. The discrete set of all eigenvalue $\lambda_j$ with multiplicity $m_j$; $j = 1, 2, 3, \ldots$ is spectrum of $\Delta_g$ and its denoted by $spec(M)$ or $spec(\Delta_g)$.

$spec(M) = \{\lambda_j(M)\}$ such that $\Delta_g(\varphi_j) = \lambda_j \varphi_j$, $\varphi_j$ is called eigen function.

The relationship between geometric structure of manifolds and spectrum of differential operators created a new concept which is spectral geometry. In the case of Laplace -Beltrami operator on closed Riemannian manifold this field sets two questions.

(1) Direct problem
(2) Inverse problem
Direct problem discusses how spectrum can be determined from Riemannian manifold from this point on many inequalities have been established like Cheeger and Cheng inequality see [2].

Inverse problem seeks to identify features of geometry from information about Laplace’s spectrum, some results are appeared in inverse problem when Milnor [13] gave answer of the question that Kac posted see [10], the analogy of this question is ” Is the spectrum of associated on smooth function Laplacian determine the shape of manifold? “In general, Sunada rise to give examples which clarifies iso-Spectral manifolds for which it is used here. When there is need for a separate term, implies ”manifold without boundary,” which is the sense in Definition 2.1. The simple example is canonical metric when $H^1 M = \{ f \in L^2(M) \mid |f| \in L^2(M) \}$ for $f, g \in H^1(M)$, $\langle f, g \rangle = \int_M f g dV + \int_M \langle df, dg \rangle dV$ where $dV = \sqrt{g} dx^1 dx^2 \ldots dx^n$ is canonical measure of $M(g)$, $dx^1 dx^2 \ldots dx^n$ is standard Lebesgue measure of $\mathbb{R}^n$. $H^1_0(M) = \{ f \in H^1(M) \mid \forall f_0 \in C^\infty_0(M) \| f_0 - f \|_1 \rightarrow 0$ as $n \rightarrow \infty \}$ i.e. $H^1_0(M)$ is closure of $C^\infty$ in $H^1(M)$.

**Definition 2.2.** $S^m = \{ x \in \mathbb{R}^{m+1} \mid \| x \| = r \}$ $m-$ sphere is smooth manifold. $T^m = S^1 \times \ldots \times S^1$ $m-$ dim tours (closed surface defined as product of $m$ circles).

**Figure (1)**

**Definition 2.3.** Tangent vector is the derivation on $C^\infty(M)$ $X : C^\infty(M) \rightarrow R$ such satisfies Leibnitz rule $X(fg) = X(f)g(x) + f(x)X(g)$ for $f, g \in C^\infty(M), x \in M$ the set of all derivations is $n$- dimensional tangent denoted by $T_x(M)$ the disjoint union of tangent spaces is tangent bundle $TM = \bigcup_{x \in M} T_x M$.

**Definition 2.4.** Riemannian metric is a smooth map $g : T_x(M) \times T_y(M) \rightarrow R$ which associates each point $x \in M$ by scalar product $g(x)(\cdot, \cdot)$. Riemannian manifold is smooth and is denoted by $(M,g)$.

**Example 2.5.** Example of Riemannian Metric Space

The simple example is canonical metric when $M = R^n$ for $x \in R^n, X, Y \in T_x(R^n)$ $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$ $< X, Y > = \sum_{i=1}^n x_i y_i$

**Definition 2.6.** Vector field of manifold is a smooth map $M \rightarrow TM$ (i.e. $x \rightarrow T_x(M)$).

Definitions of some spaces that we will use later $C^\infty_0(M) = \{ f : M \rightarrow R \mid f$ smooth $\}$ $C^0(M) = \{ f \in C^\infty(M) \mid \text{supp}(f)$ is compact $\}$

A space of all square integrable real valued functions on $M$ is $L^2(M)$ such that inner product on $L^2(M)$ is defined as $< f_1, f_2 > = \int_M f_1(x) f_2(x) dV$ for $f_1, f_2 \in L^2(M)$

**Definition 2.7.** Laplace-Beltrami $\Delta_g$ is unbounded, self-adjoint operator on $C^\infty(M)$ takes the formula $\Delta_g f = -\text{div} \text{grad} f$ where $\text{div}$ is divergence of vector field $V$ of $M$ given by $\text{div} V = \frac{1}{\sqrt{g}} \sum_{i=1}^n \frac{\partial}{\partial x^i} (\sqrt{g} V^i)$, grad is the gradient such that $\text{grad} f = \sum_i g^{ij} \frac{\partial f}{\partial x^j}$ where $V = \sum_i V^i \frac{\partial}{\partial x^i}$ $g = \det g_{ij}$

Laplace Beltrami is denoted by $\Delta_g$ with the form $\Delta_g f = -\text{div} \text{grad} f = \frac{1}{\sqrt{g}} \sum_{i,j=1}^n g^{ij} \frac{\partial}{\partial x^i} (\sqrt{g} g^{ij} \frac{\partial}{\partial x^j} f), f : M \rightarrow R, g^{ij} = (g_{ij})^{-1}$.

**Definition 2.8.** The discrete set of eigenvalues $(\lambda_j)$ which satisfies the equation $\Delta_g(\phi_j) = \lambda_j(\phi_j)$ where $\phi_j \in C^\infty(M), j = 1, 2, \ldots$ is called spectrum of $\Delta_g$ and denoted by $\text{spec}(M) = \{ \lambda_j(M) \}, \phi_j$ is called eigen function.

We need some definitions of sobolev spaces $H^1 M = \{ f \in L^2(M) \mid |f| \in L^2(M) \}$ for $f, g \in H^1(M)$ $\langle f, g \rangle = \int_M f g dV + \int_M \langle df, dg \rangle dV$ where $dV = \sqrt{g} dx^1 dx^2 \ldots dx^n$ is canonical measure of $(M,g)$, $dx^1 dx^2 \ldots dx^n$ is standard Lebesgue measure of $\mathbb{R}^n$. $H^1_0(M) = \{ f \in H^1(M) \mid \forall f_0 \in C^\infty_0(M) \| f_0 - f \|_1 \rightarrow 0$ as $n \rightarrow \infty \}$ i.e. $H^1_0(M)$ is closure of $C^\infty$ in $H^1(M)$.

**Definition 2.9.** Let $M$ be a smooth Manifold and $\Gamma(TM) = \text{space of all vector fields of } M$. The connection $\nabla$ on
M is bilinear map $\Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(TM)$ defined by $(X, Y) \rightarrow \nabla_X Y$, $\nabla$ is said to be Levi-Civita if

(1) For all smooth vector fields $X, Y$ of $M$ the next relation hold $\nabla_X Y - \nabla_Y X = [X, Y]$

(2) $\nabla$ is compatible with metric $g$, i.e for all smooth vector fields $X, Y, Z$.

(3) $Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$

Where $[X, Y]$ is Lie bracket given by $[X, Y](f) = X(Y(f)) - Y(X(f))$, for all $f \in C^\infty(M)$. We can regrade each smooth vector field $X$ on manifold $M$ as differential operator on $C^\infty(M)$.

**Definition 2.10. Curvature on Riemannian manifold**

Let $\nabla$ be Levi-Civita connection associated to the metric $g$, let $X, Y$ be smooth vector fields on $M$. The curvature $R(X, Y)$ is a map from the set of all vector fields on $M$ into itself, so for $U$ vector field.

$R(X, Y)U = \nabla_X \nabla_Y U - \nabla_Y \nabla_X U - \nabla_{[X, Y]} U$

For four vector fields we have the formula $R(X, Y, Z, W) = g(R(X, Y), Z, W)$. 

In this paper we will use Ricci curvature by the equality

$\text{Ricci}_i(V, V) = \sum_{j=1}^{n} R(V, e_j, V, e_i)$ where $V \in T_x(M), x \in M, (e_i)_{i=1, 2, \ldots, n}$, orthogonal basis of vector space $T_x(M)$, also we refer to scalar curvature $R(x)$ of $M$ by $R(x) = \sum_{i=1}^{n} \text{Ricci}_i(e_i, e_i) \in R$.

**3. Standard Result About Spectrum**

Let $M$ be a compact Riemannian manifold with boundary $\partial M$ (possibly empty) suppose one of mentioned eigenvalue problems, then

(1) Spectrum consists of infinite sequence of eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \ldots$

Where $0$ is not an eigenvalue of Dirichlet problem. Each eigenvalue has finite multiplicity and eigenvectors corresponding to distinct eigenvalues are $L^2$ orthogonal.

(2) Each eigenvalue is $C^\infty$ smooth analytic.

**4. Properties and Estimates**

Given a compact Riemannian Manifold $(M, g)$ we can find the spectrum $\{\lambda_k(M)\}_{k \geq 0}$ of $M$, this question comes under Direct problem.

In fact, we can discern that the explicit computation of spectrum is not easy task, there are few examples where the spectrum of manifold is known, like (sphere, flat tori, balls), for this reason will describe some of estimates of spectrum:

The aim is to find $a_k$ and $b_k$ depending on geometrical invariants $a_k \leq \lambda_k \leq b_k$ we will focus on the boundary of $\lambda_1$ for that we will introduce Min-Max principle

$R(f) = \frac{\int_M \Delta f \, dv}{\int_M f^2 \, dv} = \frac{\int_M \Delta f \, dv}{\int_M f^2 \, dv}$ is called Rayleigh form on $(M, g)$

If $f$ is eigenfunction associated to eigenvalue then it takes the formula

$R(f) = \frac{\int_{\Omega} \Delta f \, dv}{\int_{\Omega} f^2 \, dv} = \lambda$

$f \in H^1(M)$ in case of Neumann problem, $f \in H^1_0(M)$ in case of Dirichlet problem.

**Definition 4.1. minimax principle**

For each spectral problem and for $k \geq 1$ we have

$\lambda_k(M) = \min_{E \subset H^1_0(M)} \max \{ R(f) \}$ for Dirichlet problem

and $\lambda_k(M) = \min_{E \subset H^1(M)} \max \{ R(f) \}$ for Neumann problem.

For ease, we will serve the explanation of min max principle for another operator

For example, let take matrix of dim $2$, $A = \begin{pmatrix} 4 & -2 \\ -2 & 7 \end{pmatrix}$

both operators $\triangle$ and $A$ are self-adjoint.

The eigenvalues are $\lambda_1 = 3, \lambda_2 = 8$ with corresponding normalized eigenvector $f_1 = \frac{1}{\sqrt{3}}(2, 1), f_2 = \frac{1}{\sqrt{2}}(-1, 2)$

$R(f) = \frac{\langle Af, f \rangle}{\langle f, f \rangle}$, the Rayleigh quotient formula is constant in any subspace $E$ of dimension 1 and 2.

In the figure (2), (3) the Rayleigh quotient has its minimum value when $f_1$ equals the value will be at $3$. we find $\max R(f)$ is 8 this exactly when $f = f_2$, $\lambda = 8$.
The map \( H \) is one of the important tools in studying the spectrum of Laplace operator on a compact Riemannian manifold. \( H \) is a heat kernel that helps to describe the spectrum of Laplace operator. One of the important tools in studying the spectrum is the spectral partition function.

**5. Application of heat kernel in Riemannian geometry**

**Definition 5.1. Spectral Partition Function**

Let \( (M, g) \) be a compact Riemannian manifold, consider we have unbounded operator (in our case) Laplace operator on \( C^\infty(M) \), let \( \lambda_k(M) \) be the spectrum of Laplace i.e. \( \text{spec}(\Delta_g) = \{\lambda_k\}_{|k|\geq 1} \).

One of the important tools in the study of spectrum of Laplace operator is the heat kernel. The map \( H(x,y,t) : M \times M \times \mathbb{R}_+ \to \mathbb{R} \) is heat kernel if:

1. \( H \) is \( C^3 \) in \( x,y,t \) variables, \( C^2 \) in \( y \), \( C^1 \) in \( t \).
2. \( \lim_{t \to 0^+} H(x,y,t) = \delta_y(y) \), \( \lim_{t \to +\infty} H(x,y,t)f(y)dy = f(x) \).
3. \( \frac{\partial H(x,y,t)}{\partial t} = \Delta_{\Delta_g} H(x,y,t) \) where \( f \) is compactly supported function.

Mathematical relationship between the kernel and spectrum of Laplace is regarded by \( \text{spec}(\Delta_g) = \sum_k e^{-\lambda_k t} \) as spectral invariant. The spectral partition function \( Z(t) = \sum_k e^{-\lambda_k t} \).

\[
\int_M H(x,y,t) dx = \int_M \sum_k e^{-\lambda_k t} \delta_y(y) \delta_x(x) dx = \int_M \sum_k e^{-\lambda_k t} e^2(x) dx = \sum_k e^{-\lambda_k t} \int_M e^2(x) dx = \sum_k e^{-\lambda_k t} \| e_k \|^2_2.
\]

**Definition 5.2. The Minakshisundaram-pleijel expansion**

Let \( (M, g) \) be closed Riemannian Manifold of dim \( n \), asymptotic expansion of heat trace is \( H(x,y,t) = (4\pi)^{\frac{n}{2}}(\alpha_1 t + \alpha_2 t^2 + \ldots) \). where \( \alpha_1 \) integral over \( M \) depend on curvature and covariant derivative, \( \alpha_2 \) is difficult to compute all formulas. \( \alpha_1 \) and \( \alpha_2 \).

\[
\alpha_0 = \text{vol}(M), \quad \alpha_1 = \frac{1}{4} \int_M S
\]

Where \( S \) is scalar curvature.

\[
\alpha_2 = \frac{1}{360} \int_M S^2 - 2(Ric)^2 - 10|K|^2 \quad \text{where Ric is Ricci curvature.}
\]

\( K \) is main curvature. By Minakshisundaram-pleijel expansion we can see that \( \text{dim, vol, scalar curvature is known by spectrum.} \)

**Note:** If \( t \) is two dim then by Gauss –Bonnet theorem we get Euler characteristic of \( M \) is also spectral invariant.

\( \alpha_0 = \text{vol}(M), \quad \alpha_1 = \frac{1}{2} \chi(M) \)

**Definition 5.3. Vardhan’s formula**

Vardhan’s formula is used to be another application of heat kernel in Riemannian geometry.

\[
\lim_{t \to 0^+} t \log E(x,y,t) = -\frac{d_M(x,y)^2}{4}
\]

where \( d_M(x,y) \) is Riemannian distance between \( x \) and \( y \). We can see from above examples that the spectrum invariant is defined in terms of Riemannian manifold \( (M, g) \).

For negative results in inverse problem we will extend some of isospectral non-isometric Manifolds, first let’s define isometric and isospectral manifolds

**Definition 5.4. Two manifolds \( M, N \) are isometric if there is a diffeomorphism (diffeomorphism is a map between manifolds which is differentiable and has differentiable inverse) such that \( \text{Riemannian metric from } M \text{ pull back to metric on } N \).**

**Definition 5.5. Two closed Riemannian manifolds are said to be iso-spectral if the eigenvalues of their (Laplace-Beltrami operator) counted multiplicities coincide.**

**Note:** Isometric manifolds are Isospectral manifolds.

**6. Isospectral manifolds**

We will refer to outline of constructing isospectral manifolds

(*) Direct computation

(*) Representation theorem

(*) Riemannian submersion

**6.1 First Method : (Direct computation)**

There are a few examples of manifolds which we can compute the spectrum by direct computation one of them is rectangle with Dirichlet conditions and \([0,1] \times [0,1] \) is the domain in \( R^2 \), \( \sin(n_1 \pi x) \sin(n_2 \pi y) \) \( n_1, n_2 \in \mathbb{N}_+ \) form a Hilbert basis of \( \{ f \in L^2([0,1] \times [0,1]); f(1) = 0 \} \)

\( \text{spec}(M, g) = \{ \pi^2(n_1^2 + n_2^2) \} \) eigenvectors are \( \sin(n_1 \pi x) \sin(n_2 \pi y) \) \( n_1, n_2 \in \mathbb{N}_+ \) where \( M = [0,1] \times [0,1] \) is Riemannian manifold with canonical metric.

**Example 6.1. Milnor’s counter**

There are two lattices \( \Gamma, \Gamma' \) of \( R^{16} \) such that the associated tori \( T^{16}(\Gamma) \) and \( T^{16}(\Gamma') \) are isospectral, but not isometric.
The main idea is given by theorem 6.2. If \( G \) is a cocompact subgroup of \( T \), let in particular spec \( T \) be manifolds, are totally geodesic flat tori. Minimizing length and distance between points on manifold, the complement is horizontal space. Geodesics are curves which minimize length and distance between points on manifold, ker \( \Gamma \) bundle, we called ker \( \Gamma \) be Riemannian if \( \Gamma \) is a cocompact group acts on Riemannian manifold. This submersion is said to be representation equivalent then \( \rho \Gamma_1(f) = f \circ R_{\Gamma_1} \), where \( R_{\Gamma_1} \) right translation operator on \( \Gamma_1 \). We refer to \( \rho \Gamma_1(f) = f \circ R_{\Gamma_1} \), where \( R_{\Gamma_1} \) right translation operator on \( \Gamma_1 \) i.e. \( R_{\Gamma_1}(\Gamma x) = (\Gamma \alpha a) \) for all \( a \in \Gamma \) and \( f \in L^2(\Gamma_1 \mid G) \).

Theorem 6.2. If \( \Gamma_1 \Gamma_2 \) are representation equivalent subgroups of group \( G \), acts on \( G \) compact Riemannian Manifold then \( \text{spec}(\Gamma_1 \mid M) = \text{spec}(\Gamma_2 \mid M) \).

Note: We call the constructed isospectral manifolds by this method strongly isospectral manifold.

6.3 Third method : (Riemannian submersions method)

The recent development is Riemannian submersions method, the surjective differentiable map \( \pi : M \rightarrow N \) between differentiable manifolds is submersion, this submersion is said to be Riemannian if ker \( \pi^* \) is a submersion of \( T \) and \( f \) is a function on \( N \) in particular spec \( \pi^* \) = spec \( \pi^* \). The main idea is giving by theorem

Let \( T \) be a torus of dimension greater than one, suppose that \( M_1 \) and \( M_2 \) are compact Riemannian manifolds with induced Riemannian metrics, are totally geodesic flat tori. \( M_1 \) and \( M_2 \) are isospectral for every subtorus \( S \) and for codimension \( \leq 1 \) then \( M_1 \), \( M_2 \) are isospectral.

We will introduce example of isospectral manifolds. \( j : S^3 \times S^3 \rightarrow So(6), So(6) \) the space of skew-symmetric \( 6 \times 6 \) matrices.

\( (x, y) \in \mathbb{R}^3 \times \mathbb{R}^3, \ z \in \mathbb{R}^3 \) define \( j(z)(x, y) = (z \times x, z \times y) \) for \( f'((z)(x, y) = (z \times x, -z \times y) \)

Where \( u \times v \) denotes the vector cross product of \( u, v \in \mathbb{R}^3 \), so \( f \) is isospectral to \( f' \).

7. Conclusion

We covered most of analytic and geometric aspects of spectrum of Laplace on Riemannian Manifold by the solution of direct problems which is typified by Cheeger, Cheng, Szegö inequalities.

This paper mainly illustrates the inverse problems and the way to discover the geometry of Riemannian manifold from spectral data this study still a very active field of research till now.

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