Hierarchical EM algorithm for estimating the parameters of Mixture of Bivariate Generalized Exponential distributions

Arabin Kumar Dey
Debasis Kundu
Tumati Kiran Kumar

Abstract This paper provides a mixture modeling framework using the bivariate generalized exponential distribution. We study different properties of this mixture distribution. Hierarchical EM algorithm is developed for finding the estimates of the parameters. The algorithm takes very large sample size to work as it contains many stages of approximation. Numerical Results are provided for more illustration.

Keywords Joint probability density function; Bivariate Generalized Exponential distribution; Mixture distribution; Pseudo likelihood function; EM algorithm

A. K. Dey
Department of Mathematics
IIT Guwahati
Guwahati
Assam
Tel.: +91361-258-4620
E-mail: arabin@iitg.ac.in

D. Kundu
Department of Mathematics and Statistics,
IIT Kanpur,
Kanpur, India
E-mail: kundu@iitk.ac.in

Tumati Kiran Kumar
Amazon India,
Hyderabad,
E-mail: classykiran@gmail.com
1 Introduction

In this paper we study mixture of two bivariate generalized exponential distributions. We choose Marshall-Olkin type of bivariate generalized exponential distribution introduced by Gupta and Kundu [4] for this purpose. The distribution can be used to model a data set which is heterogeneous and non-negative in nature where some of components are equal. The main objective of this paper is to explore the issues related to estimation of the parameters for this bivariate mixture distribution through EM algorithm. We see the behavior of EM algorithm over different sample size and parameters. The calculation of the E and M step is little cumbersome. An estimation procedure through hierarchical EM algorithm helps us to provide a computationally efficient procedure to get the parameter values. The simulation study shows that the method works well mainly for large sample data. It fails to provide the proper estimate when sample size is not sufficiently large.

Mixture distribution plays an important role in modeling heterogeneous populations, see for example McLachlan and Peel [15]. A Mixture distribution can easily capture Multimodality. We can also bring the heavy tail behaviour by mixing two distributions. An extensive work has been done on a mixture of multivariate normal distributions, not much work has been done on a mixture of multivariate non-normal distributions. Recently mixture of bivariate Birnbaum Saunder distribution is introduced by Khosravi, Kundu and Jamalizadeh [8] to model the fatigue failure caused by cyclic loading. For some related work in this connection readers are referred to [17], [9]. Mixture of bivariate generalized exponential is not used so far to model mixture of bivariate life time data. It can be a good option to model such data sets.

The rest of the paper is organized as follows. In section 2, we provide the formulation of MBVGE distribution. Some important properties for MBVGE are stated in section 3. EM algorithm to compute the MLEs of the unknown parameters is provided in section 4. Discussion regarding Numerical Simulations and results are kept at Section 5. Finally we conclude the paper in section 6.

2 Formulation of MBVGE

The univariate Generalized Exponential (GE) distribution has the following cumulative density function (CDF) and probability density function (PDF) respectively for \( x > 0 \);

\[
F_{GE}(x; \alpha, \lambda) = (1 - e^{-\lambda x})^\alpha, \quad f_{GE}(x; \alpha, \lambda) = \alpha \lambda e^{-\lambda x} (1 - e^{-\lambda x})^{\alpha - 1}
\]

Here \( \alpha > 0 \) and \( \lambda > 0 \) are shape parameter and scale parameters. It is clear that for \( \alpha = 1 \), it coincides with the exponential distribution. From now on a GE distribution with the shape parameter \( \alpha \) and the scale parameter \( \lambda \) will be denoted by GE(\( \alpha \), \( \lambda \)). For brevity when \( \lambda = 1 \), we will denote it by GE(\( \alpha \)).
and for $\alpha = 1$, it will be denoted by $\text{Exp}(\lambda)$. From now on unless otherwise mentioned, it is assumed that $\alpha_1 > 0, \alpha_2 > 0, \alpha_3 > 0, \lambda > 0$.

Suppose $U_1 \sim \text{GE}(\alpha_1, \lambda), U_2 \sim \text{GE}(\alpha_2, \lambda)$ and $U_3 \sim \text{GE}(\alpha_3, \lambda)$ and they are mutually independent. Here ‘∼’ means follows or has the distribution. Now define $X_1 = \max\{U_1, U_3\}$ and $X_2 = \max\{U_2, U_3\}$. Then we say that the bivariate vector $(X_1, X_2)$ has a bivariate generalized exponential distribution with the shape parameters $\alpha_1, \alpha_2$ and $\alpha_3$ and the scale parameter $\lambda$. We will denote it by $\text{BVGE}(\alpha_1, \alpha_2, \alpha_3, \lambda)$. Now for the rest of the discussions for brevity, we assume that $\lambda = 1$, although the results are true for general $\lambda$ also. The $\text{BVGE}$ distribution with $\lambda = 1$ will be denoted by $\text{BVGE}(\alpha_1, \alpha_2, \alpha_3)$. Before providing the joint CDF or PDF, we first mention how it may occur in practice.

We know that if $U_{0,1}, U_{0,2}$ and $U_{0,3}$ are three independent random numbers, but follows $\text{GE}(\alpha_1, \lambda_1)$, $\text{GE}(\alpha_2, \lambda_1)$ and $\text{GE}(\alpha_3, \lambda_1)$ respectively, we can define $X_1 = \max\{U_{0,1}, U_{0,3}\}, X_2 = \max\{U_{0,2}, U_{0,3}\}$ which follows $\text{BVGE}(\alpha_1, \alpha_2, \alpha_3, \lambda_1)$.

The joint cdf of $\text{BVGE}$ can be written as:

$$F_{X_1, X_2}(x_1, x_2) = F_{\text{GE}}(x_1; \alpha_1, \lambda_1)F_{\text{GE}}(x_2; \alpha_2, \lambda_1)F_{\text{GE}}(z; \alpha_3, \lambda_1)$$

if $x_1 < x_2$

$$= F_{\text{GE}}(x_1; \alpha_1 + \alpha_3, \lambda_1)F_{\text{GE}}(x_2; \alpha_2, \lambda_1)$$

if $x_1 > x_2$

$$= F_{\text{GE}}(x_1; \alpha_1)F_{\text{GE}}(x_2; \alpha_2 + \alpha_3, \lambda_1)$$

if $x_1 = x_2 = x$

Therefore the joint pdf of $(X_1, X_2)$ for $x_1 > 0$ and $x_2 > 0$, is:

$$f_\alpha(x_1, x_2) = f_{1\alpha}(x_1, x_2) \text{ if } 0 < x_1 < x_2 < \infty$$

$$= f_{2\alpha}(x_1, x_2) \text{ if } 0 < x_2 < x_1 < \infty$$

$$= f_{0\alpha}(x) \text{ if } 0 < x_1 = x_2 = x < \infty$$

where

$$f_{1\alpha}(x_1, x_2) = f_{\text{GE}}(x_1; \alpha_1 + \alpha_3, \lambda_1)f_{\text{GE}}(x_2; \alpha_2, \lambda_1)$$

$$= (\alpha_1 + \alpha_3)\alpha_2(1 - e^{-\lambda_1 x_1})^{\alpha_1 + \alpha_3 - 1}(1 - e^{-\lambda_1 x_2})^{\alpha_2 - 1}e^{-\lambda_1(x_1 + x_2)}$$

$$f_{2\alpha}(x_1, x_2) = f_{\text{GE}}(x_1; \alpha_1, \lambda_1)f_{\text{GE}}(x_2; \alpha_2 + \alpha_3, \lambda_1)$$

$$= (\alpha_1 + \alpha_3)\alpha_2(1 - e^{-\lambda_1 x_1})^{\alpha_1 + \alpha_3 - 1}(1 - e^{-\lambda_1 x_2})^{\alpha_2 - 1}e^{-\lambda_1(x_1 + x_2)}$$

$$f_{0\alpha}(x) = \frac{\alpha_3}{\alpha_1 + \alpha_2 + \alpha_3}f_{\text{GE}}(x; \alpha_1 + \alpha_2 + \alpha_3, \lambda_1)$$

Our aim is to study mixture of two bivariate generalized exponential distributions. Let $\text{BVGE}(\alpha_1, \alpha_2, \alpha_3, \lambda_1)$ and $\text{BVGE}(\beta_1, \beta_2, \beta_3, \lambda_2)$ be two independent bivariate generalized exponential distributions. We consider mixture of them with mixture proportion $p_0$ and $p_1$.

$$f(x_1, x_2) = p_0f_{\alpha}(x_1, x_2; \alpha_1, \alpha_2, \alpha_3, \lambda_1) + p_1f_{\beta}(x_1, x_2; \beta_1, \beta_2, \beta_3, \lambda_2)$$
Figure-1 shows surface and contour plots of probability density function for four different sets of parameters of MBVGE. They are as follows:

\[ \xi_1 : p = 0.6, \lambda_1 = 2, \lambda_2 = 1.5, \alpha_1 = 0.5, \alpha_2 = 0.4, \alpha_3 = 0.3, \beta_1 = 0.5, \beta_2 = 1.5, \beta_3 = 0.5 \]

\[ \xi_2 : p = 0.3, \lambda_1 = 1, \lambda_2 = 0.5, \alpha_1 = 1, \alpha_2 = 1.2, \alpha_3 = 1, \beta_1 = 1, \beta_2 = 1.4, \beta_3 = 2. \]

3 Properties

**Theorem 1** 1. Marginal distribution of MBVGE is mixture of univariate generalized exponential distribution.

**Theorem 2** Copula function of MBVGE can be written as mixture of two different copulas i.e.: \( C(u, v) = pC_1(u, v) + (1-p)C_2(u, v) \) where \( C_1(u, v) \) and \( C_2(u, v) \) are two different copula and can be provided by the following expressions:

\[
C_1(u, v) = \begin{cases} 
\frac{1}{\alpha_1 + \alpha_2} u^{\frac{1}{\alpha_1 + \alpha_2}} v^{\frac{1}{\alpha_2}} & \frac{1}{\alpha_1 + \alpha_2} u^{\frac{1}{\alpha_1 + \alpha_2}} \leq \frac{1}{\alpha_2} v^{\frac{1}{\alpha_2}} \\
\frac{1}{\alpha_2} u^{\frac{1}{\alpha_2}} v^{\frac{1}{\alpha_2}} & \frac{1}{\alpha_2} v^{\frac{1}{\alpha_2}} < \frac{1}{\alpha_2} v^{\frac{1}{\alpha_2}}
\end{cases}
\]
\[ C_2(u,v) = \begin{cases} \frac{\frac{\partial t}{\partial v} - \beta_2}{u^{\frac{1}{\alpha_1 + \alpha_2}} - \beta_2}, & u^{\frac{1}{\alpha_1 + \alpha_2}} < v^{\frac{1}{\alpha_2 + \beta_3}} \\ \frac{\frac{\partial t}{\partial v} - \beta_2}{u^{\frac{1}{\alpha_1 + \alpha_2}} > v^{\frac{1}{\alpha_2 + \beta_3}}}, & \end{cases} \]

**Theorem 3**  
Tail Index of the Copula can be provided by the following relation:

\[ \lambda_L = P[Y < F_Y^{-1}(t)|X < F_X^{-1}(t)] = \lim_{t \to 0} \frac{C(t,t)}{t} = 0 \]

\[ \lambda_U = P[Y > F_Y^{-1}(t)|X > F_X^{-1}(t)] \]

\[ = 2 - \lim_{t \to 1} \frac{1 - C(t,t)}{1 - t} \]

\[ = 2 - p \frac{\alpha_2}{\alpha_1 + \alpha_2 + \alpha_3} - (1 - p) \frac{\beta_2}{\beta_1 + \beta_2 + \beta_3} \]

**Theorem 4**  
Hazard function for the distribution can be obtained from the relation:

\[ h(t_1, t_2) = \frac{f(t_1, t_2)}{S(t_1, t_2)} = \begin{cases} \frac{pf_{1,0}(x_1, x_2) + (1-p)f_{1,1}(x_1, x_2)}{pf_{1,0}(x_1, x_2) + (1-p)f_{1,1}(x_1, x_2)} & \text{if } x_1 < x_2 \\
\frac{pf_{2,0}(x_1, x_2) + (1-p)f_{2,1}(x_1, x_2)}{pf_{2,0}(x_1, x_2) + (1-p)f_{2,1}(x_1, x_2)} & \text{if } x_1 > x_2 \\
\frac{pf_{0,0}(x_1, x_2) + (1-p)f_{0,1}(x_1, x_2)}{pf_{0,0}(x_1, x_2) + (1-p)f_{0,1}(x_1, x_2)} & \text{if } x_1 = x_2 \end{cases} \]

Other definition of Hazard function for the distribution can be obtained from the relation:

\[ h(t_1, t_2) = \left[ -\frac{\partial}{\partial t_1} \ln S(t_1, t_2), -\frac{\partial}{\partial t_2} \ln S(t_1, t_2) \right] \]

**Theorem 5**

\[ P(X_1 \leq x_1|X_2 = x_2) = \begin{cases} \frac{pA_1 + (1-p)B_1}{pA_1 + (1-p)B_1} & x_1 < x_2 \\
\frac{pA_1 + (1-p)B_1}{pA_1 + (1-p)B_1} & x_1 > x_2 \\
\frac{pA_1 + (1-p)B_1}{pA_1 + (1-p)B_1} & x_1 = x_2 \end{cases} \]

where \( A_1 = (1 - e^{-\lambda x_1})^{\alpha_1 + \alpha_2}(1 - e^{-\lambda x_2})^{\alpha_2 + 1 - \beta_2}, \) \( B_1 = (1 - e^{-\lambda x_1})^{\beta_1 + \beta_2}(1 - e^{-\lambda x_2})^{\beta_2 - 1}, \) \( C_1 = (1 - e^{-\lambda x_1})^{\alpha_1}(\alpha_2 + \alpha_3)(1 - e^{-\lambda x_2})^{\alpha_2 + \alpha_3 - 1}, \) \( D_1 = (1 - e^{-\lambda x_1})^{\beta_1}(\beta_2 + \beta_3)(1 - e^{-\lambda x_2})^{\beta_2 + \beta_3 - 1}, \) \( E_1 = (\alpha_1 + \alpha_2 + \alpha_3)(1 - e^{-\lambda x_2})^{\alpha_1 + \alpha_2 + \alpha_3 - 1}, \) \( F_1 = (\beta_1 + \beta_2 + \beta_3)(1 - e^{-\lambda x_2})^{\beta_1 + \beta_2 + \beta_3 - 1}, \) \( A = (\alpha_2 + \alpha_3)(1 - e^{-\lambda x_2})^{\alpha_2 + \alpha_3 - 1}, \) \( B = (\beta_2 + \beta_3)(1 - e^{-\lambda x_2})^{\beta_2 + \beta_3 - 1}, \)
In stage -1, we introduce
Here we use multistage EM algorithm to construct the final pseudo-likelihood.

**4 Implementation of EM algorithm**

Here we use multistage EM algorithm to construct the final pseudo-likelihood. In stage -1, we introduce

\[
\tau = p^2 \frac{(\alpha_1 + \alpha_2)}{(\alpha_1 + \alpha_2 + \alpha_3)} + (1 - p)^2 \frac{\beta_1 + \beta_2}{(\beta_1 + \beta_2 + \beta_3)}
\]

\[
+ 2p(1 - p) \frac{\beta_2(\alpha_2 + \alpha_3)}{(2\beta_1 + \beta_2 + 2\beta_3)(\alpha_2 + \alpha_3) + \alpha_2(\beta_2 + \beta_3)}
\]

\[
+ 2p(1 - p) \frac{\alpha_2(\beta_2 + \beta_3)}{(2\alpha_1 + \alpha_2 + 2\alpha_3)(\beta_2 + \beta_3) + \alpha_2(\alpha_2 + \alpha_3)}
\]

\[
+ 2p(1 - p) \frac{\beta_1(\alpha_1 + \alpha_3)}{2(\beta_1 + \beta_2 + \beta_3)\alpha_1 + (2\beta_2 + \beta_1 + \beta_3)\alpha_3}
\]

\[
+ 2p(1 - p) \frac{\alpha_1(\beta_1 + \beta_3)}{2(\alpha_1 + \alpha_2 + \alpha_3)\beta_1 + (2\alpha_2 + \alpha_1 + \alpha_3)\beta_3 - 1}
\]

Similarly,

**Theorem 6** Expression for Kendal’s tau can be obtained using its copula form as

\[
\tau = 6p \frac{(\alpha_1 + \alpha_2)}{2(\alpha_1 + \alpha_2 + \alpha_3) + \alpha_3} + 6(1 - p) \frac{(\beta_1 + \beta_2)}{2(\beta_1 + \beta_2 + \beta_3) + \beta_3} - 3.
\]

\[
\begin{aligned}
\text{Theorem 6} & \quad \text{Expression for Kendal’s tau can be obtained using its copula form as} \\
& \quad \tau = p^2 \frac{(\alpha_1 + \alpha_2)}{(\alpha_1 + \alpha_2 + \alpha_3)} + (1 - p)^2 \frac{\beta_1 + \beta_2}{(\beta_1 + \beta_2 + \beta_3)} \\
& \quad + 2p(1 - p) \frac{\beta_2(\alpha_2 + \alpha_3)}{(2\beta_1 + \beta_2 + 2\beta_3)(\alpha_2 + \alpha_3) + \alpha_2(\beta_2 + \beta_3)} \\
& \quad + 2p(1 - p) \frac{\alpha_2(\beta_2 + \beta_3)}{(2\alpha_1 + \alpha_2 + 2\alpha_3)(\beta_2 + \beta_3) + \alpha_2(\alpha_2 + \alpha_3)} \\
& \quad + 2p(1 - p) \frac{\beta_1(\alpha_1 + \alpha_3)}{2(\beta_1 + \beta_2 + \beta_3)\alpha_1 + (2\beta_2 + \beta_1 + \beta_3)\alpha_3} \\
& \quad + 2p(1 - p) \frac{\alpha_1(\beta_1 + \beta_3)}{2(\alpha_1 + \alpha_2 + \alpha_3)\beta_1 + (2\alpha_2 + \alpha_1 + \alpha_3)\beta_3 - 1}
\end{aligned}
\]

**4 Implementation of EM algorithm**

Here we use multistage EM algorithm to construct the final pseudo-likelihood. In stage -1, we introduce

\[
Z_i = \begin{cases} 
1 & \text{if } (x_{i1}, x_{i2}) \sim f_\alpha(x_{i1}, x_{i2}) \\
0 & \text{if } (x_{i1}, x_{i2}) \sim f_\beta(x_{i1}, x_{i2})
\end{cases}
\]

Depending on observations lying on $I_0, I_1$ and $I_2$, we can defne three parts of posterior distribution of $Z_i$, as $p_{0,0i}$, $p_{0,1i}$ and $p_{0,2i}$ respectively.

Therefore,

\[
p_{0,0i} = P(Z_i = 1|(X_1, X_2) \in I_0) = \frac{p_{0,0}(x_1, x_2)}{p_{0,0}(x_1, x_2) + p_{1,0}(x_1, x_2)} \quad \text{for } i = 1, \ldots, n_0
\]

\[
p_{0,1i} = P(Z_i = 1|(X_1, X_2) \in I_1) = \frac{p_{0,1}(x_1, x_2)}{p_{0,1}(x_1, x_2) + p_{1,1}(x_1, x_2)} \quad \text{for } i = 1, \ldots, n_1
\]

\[
p_{0,2i} = P(Z_i = 1|(X_1, X_2) \in I_2) = \frac{p_{0,2}(x_1, x_2)}{p_{0,2}(x_1, x_2) + p_{1,2}(x_1, x_2)} \quad \text{for } i = 1, \ldots, n_2
\]
We also take \( p_{1,0i} = (1 - p_{0,0i}), p_{1,1i} = (1 - p_{0,1i}) \) and \( p_{1,2i} = (1 - p_{0,2i}) \).

In second stage we take the missing information as the maximum between the the pair of observations corresponding to \((X_1, X_2)\). Therefore we introduce \((\Delta_{0,0}, \Delta_{0,1})\) if we assume \((X_1, X_2) \sim f_\alpha(\cdot)\) and \((\Delta_{1,0}, \Delta_{1,1})\) for \((X_1, X_2) \sim f_\alpha(\cdot)\) as described in [4] i.e. \(\Delta_{1,0} = 1\) or \(3\) if \(U_{0,1} > U_{0,3}\) or \(U_{0,1} < U_{0,3}\) and \(\Delta_{0,1} = 2\) or \(3\) if \(U_{0,2} > U_{0,3}\) or \(U_{0,2} < U_{0,3}\). If \(\gamma_1 = (\alpha_1, \alpha_2, \alpha_3, \lambda_1)\), fractional mass \((u_{0,1}(\gamma), u_{0,2}(\gamma))\) [We denote simply as \(u_{0,1}, u_{0,2}\)] assign to ‘pseudo observation’ \((x_1, x_2)\) is the conditional probability that the random vector \((\Delta_{0,1}, \Delta_{0,2})\) takes the values \((1, 2)\) or \((3, 2)\) respectively given that \(X_1 < X_2\).

Similarly, if \((x_1, x_2) \in I_2\), we form the pseudo observations by introducing fractional mass \(u_{0,1}\) and \(u_{0,2}\) which is the conditional distribution that the random vector \((\Delta_{0,0}, \Delta_{0,1})\) takes the values \((1, 2)\) and \((1, 3)\) respectively given that \(X_1 > X_2\).

We can show \(u_{0,1} = \frac{\alpha_1}{\alpha_1 + \alpha_3}\) and \(u_{0,2} = \frac{\alpha_3}{\alpha_1 + \alpha_3}\) whereas \(w_{0,1} = \frac{\alpha_2}{\alpha_2 + \alpha_3}\) and \(w_{0,2} = \frac{\alpha_2}{\alpha_2 + \alpha_3}\).

Exactly in the similar line we can define \((\Delta_{1,0}, \Delta_{1,1})\) for second type of bivariate generalized exponential distribution and we denote four conditional probabilities as \((u_{1,1}, u_{1,2})\) and \((w_{1,1}, w_{1,2})\) where \(u_{1,1} = \frac{\alpha_1}{\alpha_1 + \alpha_3}, u_{1,2} = \frac{\alpha_2}{\alpha_2 + \alpha_3}, w_{1,1} = \frac{\alpha_3}{\alpha_2 + \alpha_3}\) and \(w_{1,2} = \frac{\beta_3}{\beta_2 + \beta_3}\).

Therefore first step log-likelihood can be written as

\[
\mathcal{L} = \log \prod_{i=1}^{n} [p f_\alpha(x_{1i}, x_{2i})]^{z_i} [(1 - p) f_\beta(x_{1i}, x_{2i})]^{1-z_i}
\]

In the second step we use complete information in \(\log(f_\alpha(x_{1i}, x_{2i}))\) and \(\log(f_\beta(x_{1i}, x_{2i}))\) by introducing \((\Delta_{0,0}, \Delta_{0,1})\) and \((\Delta_{1,0}, \Delta_{1,1})\) respectively. In the calculation of pseudo-likelihood we only need to take care of the proper usage of posterior of \(z_i\) given the data \((x_{1i}, x_{2i})\). Formulation of the E-step and M-step is shown in the subsequent subsections.

4.1 Formulation of E-step

The form of pseudo-likelihood can be written as follows:

\[ l_{\text{pseudo}}(\alpha_1, \alpha_2, \alpha_3, \lambda_1, \beta_1, \beta_2, \beta_3, \lambda_2) \]

= contribution from 1st part of pseudo likelihood

+ contribution from 2nd part of pseudo likelihood
First part of pseudo likelihood

\[
\begin{align*}
&= \left[ \sum_{i=1}^{n_0} p_{0,0i} + \sum_{i=1}^{n_1} p_{0,1i} + \sum_{i=1}^{n_2} p_{0,2i} \right] \ln p + \left[ \sum_{i=1}^{n_0} p_{1,0i} + \sum_{i=1}^{n_1} p_{1,1i} + \sum_{i=1}^{n_2} p_{1,2i} \right] \ln(1 - p) \\
&+ \sum_{i \in I_0} p_{0,0i} \ln(\alpha_3) + \sum_{i \in I_0} p_{0,0i} \ln \lambda_1 + (\alpha_1 + \alpha_2 + \alpha_3 - 1) \sum_{i \in I_0} p_{0,0i} \ln(1 - e^{-\lambda_1 y_i}) \\
&- \lambda_1 \sum_{i \in I_0} p_{0,0ix_{1i}} + u_{0,1} \left[ \sum_{i \in I_1} p_{0,1i} \ln \alpha_1 + 2 \sum_{i \in I_1} p_{0,1i} \ln \lambda_1 - \lambda_1 \sum_{i \in I_1} p_{0,1ix_{1i}} \\
&+ (\alpha_1 + \alpha_3 - 1) \sum_{i \in I_2} p_{0,1i} \ln(1 - e^{-\lambda_1 x_{1i}}) \right] \\
&+ \left( \alpha_2 + \alpha_3 - 1 \right) \sum_{i \in I_2} p_{0,2i} \ln(1 - e^{-\lambda_2 x_{2i}}) + u_{0,1} \left[ \sum_{i \in I_2} p_{0,2i} \ln \alpha_2 + 2 \sum_{i \in I_2} p_{0,2i} \ln \lambda_1 - \lambda_1 \sum_{i \in I_2} p_{0,2ix_{2i}} \\
&+ (\alpha_2 + \alpha_3 - 1) \sum_{i \in I_2} p_{0,2i} \ln(1 - e^{-\lambda_2 x_{2i}}) \right] \\
&= \left[ \sum_{i=1}^{n_0} p_{0,0i} + \sum_{i=1}^{n_1} p_{0,1i} + \sum_{i=1}^{n_2} p_{0,2i} \right] \ln p + \left[ \sum_{i=1}^{n_0} p_{1,0i} + \sum_{i=1}^{n_1} p_{1,1i} + \sum_{i=1}^{n_2} p_{1,2i} \right] \ln(1 - p) \\
&+ \sum_{i \in I_0} p_{0,0i} \ln(\alpha_3) + \sum_{i \in I_0} p_{0,0i} \ln \lambda_1 + (\alpha_1 + \alpha_2 + \alpha_3 - 1) \sum_{i \in I_0} p_{0,0i} \ln(1 - e^{-\lambda_1 y_i}) \\
&- \lambda_1 \sum_{i \in I_0} p_{0,0ix_{1i}} + u_{0,1} \left[ \sum_{i \in I_1} p_{0,1i} \ln \alpha_1 + 2 \sum_{i \in I_1} p_{0,1i} \ln \lambda_1 - \lambda_1 \sum_{i \in I_1} p_{0,1ix_{1i}} \\
&+ (\alpha_1 + \alpha_3 - 1) \sum_{i \in I_2} p_{0,1i} \ln(1 - e^{-\lambda_1 x_{1i}}) \right] \\
&+ \left( \alpha_2 + \alpha_3 - 1 \right) \sum_{i \in I_2} p_{0,2i} \ln(1 - e^{-\lambda_2 x_{2i}}) + u_{0,1} \left[ \sum_{i \in I_2} p_{0,2i} \ln \alpha_2 + 2 \sum_{i \in I_2} p_{0,2i} \ln \lambda_1 - \lambda_1 \sum_{i \in I_2} p_{0,2ix_{2i}} \\
&+ (\alpha_2 + \alpha_3 - 1) \sum_{i \in I_2} p_{0,2i} \ln(1 - e^{-\lambda_2 x_{2i}}) \right] \\
&- \lambda_1 \sum_{i \in I_0} p_{0,0ix_{1i}} + (\alpha_1 + \alpha_2 + \alpha_3 - 1) \sum_{i \in I_0} p_{0,0ix_{1i}} \ln(1 - e^{-\lambda_1 x_{1i}})
\end{align*}
\]
Second part of pseudo likelihood
\[ + \sum_{i \in I_0} p_{1,0i} \ln(\beta_3) + \sum_{i \in I_0} p_{1,0i} \ln \lambda_2 + (\beta_1 + \beta_2 + \beta_3 - 1) \sum_{i \in I_0} p_{1,0i} \ln(1 - e^{-\lambda_2 w_i}) \]
\[ - \lambda_2 \sum_{i \in I_0} p_{1,0i} x_{1i} + u_{1,1i} \sum_{i \in I_1} p_{1,1i} \ln \beta_1 + 2 \sum_{i \in I_1} p_{1,1i} \ln \lambda_2 - \lambda_2 \sum_{i \in I_1} p_{1,1i} x_{1i} \]
\[ + (\beta_1 + \beta_3 - 1) \sum_{i \in I_1} p_{1,1i} \ln(1 - e^{-\lambda_2 x_{1i}}) \]
\[ + u_{1,2i} \sum_{i \in I_1} p_{1,1i} \ln \beta_3 + 2 \sum_{i \in I_1} p_{1,1i} \ln \lambda_2 - \lambda_2 \sum_{i \in I_1} p_{1,1i} x_{1i} \]
\[ + (\beta_1 + \beta_3 - 1) \sum_{i \in I_1} p_{1,1i} \ln(1 - e^{-\lambda_2 x_{1i}}) + \sum_{i \in I_1} p_{1,1i} \ln \beta_2 \]
\[ - \lambda_2 \sum_{i \in I_1} p_{1,1i} x_{2i} + (\beta_2 - 1) \sum_{i \in I_1} p_{1,1i} \ln(1 - e^{-\lambda_2 x_{2i}}) + u_{1,1i} \sum_{i \in I_2} p_{1,2i} \ln \beta_2 \]
\[ + 2 \sum_{i \in I_2} p_{1,2i} \ln \lambda_2 - \lambda_2 \sum_{i \in I_2} p_{1,2i} x_{2i} + (\beta_2 + \beta_3 - 1) \sum_{i \in I_2} p_{1,2i} \ln(1 - e^{-\lambda_2 x_{2i}}) \]
\[ + u_{1,2i} \sum_{i \in I_2} p_{1,2i} \ln \beta_3 + 2 \sum_{i \in I_2} p_{1,2i} \ln \lambda_2 - \lambda_2 \sum_{i \in I_2} p_{1,2i} x_{2i} \]
\[ + (\beta_2 + \beta_3 - 1) \sum_{i \in I_2} p_{1,2i} \ln(1 - e^{-\lambda_2 x_{2i}}) + \sum_{i \in I_2} p_{1,2i} \ln \beta_1 \]
\[ - \lambda_2 \sum_{i \in I_2} p_{1,2i} x_{1i} + (\beta_1 - 1) \sum_{i \in I_2} p_{1,2i} \ln(1 - e^{-\lambda_2 x_{1i}}) \]

4.2 Formulation of M-step:

Now the ‘M’ step involves the maximization of the
\[ l_{\text{pseudo}}(\alpha_1, \alpha_2, \alpha_3, \lambda_1, \beta_1, \beta_2, \beta_3, \lambda_2, p) \]
with respect to all parameters. Taking derivative with respect to \(p\), yields,
\[ p = \frac{\sum_{i=1}^{n_0} p_{0,0i} + \sum_{i=1}^{n_1} p_{0,1i} + \sum_{i=1}^{n_2} p_{0,2i}}{n} \]

For fixed \(\lambda_1\) and \(\lambda_2\), the maximization of \(l_{\text{pseudo}}(\cdot)\) occurs at
\[ \hat{\alpha}_1(\lambda_1) = \frac{u_{0,1} \sum_{i \in I_1} p_{0,1i} + \sum_{i \in I_2} p_{0,2i}}{A} \]
\[ A = \sum_{i \in I_0} p_{0,0i} \ln(1-e^{-\lambda_1 x_{1i}}) + \sum_{i \in I_1} p_{0,1i} \ln(1-e^{-\lambda_1 x_{1i}}) + \sum_{i \in I_2} p_{0,2i} \ln(1-e^{-\lambda_1 x_{1i}}) \]

\[ \hat{\alpha}_2(\lambda_1) = -\frac{\sum_{i \in I_1} p_{0,1i} - u_{0,1} \sum_{i \in I_2} p_{0,2i}}{B} \]

\[ B = \sum_{i \in I_0} p_{0,0i} \ln(1-e^{-\lambda_1 x_{1i}}) + \sum_{i \in I_1} p_{0,1i} \ln(1-e^{-\lambda_1 x_{1i}}) + \sum_{i \in I_2} p_{0,2i} \ln(1-e^{-\lambda_1 x_{1i}}) \]

\[ \hat{\alpha}_3(\lambda_1) = \frac{-\sum_{i \in I_0} p_{0,0i} - u_{0,2} \sum_{i \in I_1} p_{0,1i} - u_{0,2} \sum_{i \in I_2} p_{0,2i}}{D} \]

\[ D = \sum_{i \in I_0} p_{0,0i} \ln(1-e^{-\lambda_1 x_{1i}}) + \sum_{i \in I_1} p_{0,1i} \ln(1-e^{-\lambda_1 x_{1i}}) + \sum_{i \in I_2} p_{0,2i} \ln(1-e^{-\lambda_1 x_{1i}}) \]

and \( \hat{\lambda}_1 \), which maximizes \( l_{pesudo}(\cdot) \) can be obtained as a solution of the following fixed point equation:

\[ g_1(\lambda_1) = \lambda_1 \]

where

\[ g_1(\lambda_1) = \frac{E_1}{F_1} \]

\[ E_1 = \sum_{i \in I_0} p_{0,0i} + 2u_{0,1} \sum_{i \in I_1} p_{0,1i} + 2u_{0,2} \sum_{i \in I_2} p_{0,2i} + 2u_{0,1} \sum_{i \in I_1} p_{0,1i} + 2u_{0,2} \sum_{i \in I_2} p_{0,2i} + 2u_{0,1} \sum_{i \in I_1} p_{0,1i} + 2u_{0,2} \sum_{i \in I_2} p_{0,2i} \]

\[ F_1 = \sum_{i \in I_0} p_{0,0i} x_{1i} - (\alpha_1 + \alpha_2 + \alpha_3 - 1) \sum_{i \in I_0} \frac{y_i p_{0,0i} e^{-\lambda_1 y_i}}{(1-e^{-\lambda_1 y_i})} \]

\[ + u_{0,1} \sum_{i \in I_1} p_{0,1i} x_{1i} - u_{0,1} (\alpha_1 + \alpha_3 - 1) \sum_{i \in I_1} p_{0,1i} \frac{e^{-\lambda_1 x_{1i}}}{(1-e^{-\lambda_1 x_{1i}})} \]

\[ + u_{0,2} \sum_{i \in I_1} p_{0,1i} x_{1i} - u_{0,2} (\alpha_1 + \alpha_3 - 1) \sum_{i \in I_2} p_{0,1i} \frac{x_{1i} e^{-\lambda_1 x_{1i}}}{(1-e^{-\lambda_1 x_{1i}})} \]

\[ + u_{0,1} \sum_{i \in I_2} p_{0,2i} x_{2i} - u_{0,1} (\alpha_2 + \alpha_3 - 1) \sum_{i \in I_2} p_{0,2i} \frac{x_{2i} e^{-\lambda_1 x_{2i}}}{(1-e^{-\lambda_1 x_{2i}})} \]

\[ + u_{0,2} \sum_{i \in I_2} p_{0,2i} x_{2i} - u_{0,2} (\alpha_2 + \alpha_3 - 1) \sum_{i \in I_2} p_{0,2i} \frac{x_{2i} e^{-\lambda_1 x_{2i}}}{(1-e^{-\lambda_1 x_{2i}})} \]

Similarly from the second part, we get
\[ \hat{\beta}_1(\lambda_2) = -\frac{u_{1,1} \sum_{i \in I_1} p_{1,1i} + \sum_{i \in I_2} p_{1,2i}}{G} \]

\[ G = \sum_{i \in I_0} p_{1,0i} \ln(1 - e^{-\lambda_2 x_{1i}}) + \sum_{i \in I_1} p_{1,1i} \ln(1 - e^{-\lambda_2 x_{1i}}) + \sum_{i \in I_2} p_{1,2i} \ln(1 - e^{-\lambda_2 x_{2i}}) \]

\[ \hat{\beta}_2(\lambda_2) = -\frac{\sum_{i \in I_0} p_{1,0i} - w_{1,1} \sum_{i \in I_2} p_{1,2i}}{H} \]

\[ H = \sum_{i \in I_0} p_{1,0i} \ln(1 - e^{-\lambda_2 x_{1i}}) + \sum_{i \in I_1} p_{1,1i} \ln(1 - e^{-\lambda_2 x_{1i}}) + \sum_{i \in I_2} p_{1,2i} \ln(1 - e^{-\lambda_2 x_{2i}}) \]

\[ \hat{\beta}_3(\lambda_2) = -\frac{\sum_{i \in I_0} p_{1,0i} - u_{1,2} \sum_{i \in I_1} p_{1,1i} - w_{1,2} \sum_{i \in I_2} p_{1,2i}}{K} \]

\[ K = \sum_{i \in I_0} p_{1,0i} \ln(1 - e^{-\lambda_2 x_{1i}}) + \sum_{i \in I_1} p_{1,1i} \ln(1 - e^{-\lambda_2 x_{1i}}) + \sum_{i \in I_2} p_{1,2i} \ln(1 - e^{-\lambda_2 x_{2i}}) \]

\[ \hat{\lambda}_2, \text{ which maximizes } l_{\text{pseudo}}(\cdot) \text{ can be obtained as a solution of the following fixed point equation:} \]

\[ g_1(\lambda_2) = \lambda_2 \]

\[ \text{where} \]

\[ g_1(\lambda_2) = \frac{K_1}{L_1} \]

\[ K_1 = \sum_{i \in I_0} p_{1,0i} + 2u_{1,1} \sum_{i \in I_1} p_{1,1i} + 2u_{1,2} \sum_{i \in I_2} p_{1,1i} + 2w_{1,1} \sum_{i \in I_1} p_{1,2i} + 2w_{1,2} \sum_{i \in I_2} p_{1,2i} \]

\[ L_1 = \sum_{i \in I_0} p_{1,0i} x_{1i} - (\beta_1 + \beta_2 + \beta_3 - 1) \sum_{i \in I_0} \frac{y_{0} p_{1,0i} e^{-\lambda_2 y_{0i}}}{(1 - e^{-\lambda_2 y_{0i}})} \]

\[ + u_{1,1} \sum_{i \in I_1} p_{1,1i} x_{1i} - u_{1,1} (\beta_1 + \beta_3 - 1) \sum_{i \in I_1} p_{1,1i} x_{1i} \frac{e^{-\lambda_2 x_{1i}}}{(1 - e^{-\lambda_2 x_{1i}})} \]

\[ + u_{1,2} \sum_{i \in I_1} p_{1,1i} x_{1i} - u_{1,2} (\beta_1 + \beta_3 - 1) \sum_{i \in I_1} p_{1,1i} x_{1i} e^{-\lambda_2 x_{1i}} \]

\[ + \sum_{i \in I_1} p_{1,1i} x_{2i} - (\beta_2 - 1) \sum_{i \in I_1} p_{1,1i} x_{2i} e^{-\lambda_2 x_{2i}} \]

\[ + w_{1,1} \sum_{i \in I_2} p_{1,2i} x_{2i} - w_{1,1} (\beta_2 + \beta_3 - 1) \sum_{i \in I_2} p_{1,2i} x_{2i} e^{-\lambda_2 x_{2i}} \]

\[ - w_{1,2} (\beta_2 + \beta_3 - 1) \sum_{i \in I_2} p_{1,2i} x_{2i} e^{-\lambda_2 x_{2i}} \]

\[ + \sum_{i \in I_2} p_{1,2i} x_{1i} - (\beta_1 - 1) \sum_{i \in I_2} p_{1,2i} x_{1i} e^{-\lambda_2 x_{1i}} \]
5 Numerical Result

We use package R 3.2.3 to perform the estimation procedure. All the programs will be available on request to author. We have taken two different sets of parameters to conduct our simulation. These are $\alpha_1 = 1; \alpha_2 = 1.2; \alpha_3 = 1; \beta_1 = 1; \beta_2 = 1.4; \beta_3 = 2; \lambda_1 = 1; \lambda_2 = 0.5; p = 0.3. \alpha_1 = 0.5; \alpha_2 = 0.4; \alpha_3 = 0.3; \beta_1 = 0.5; \beta_2 = 1.5; \beta_3 = 0.5; \lambda_1 = 1; \lambda_2 = 0.5; p = 0.6$.

We take sample size as $n = 1000, 1500$. The procedure demands high sample size as it uses many stages of approximation. We start EM algorithm with random initial guesses at each iteration. Left side of the Table-1 shows the values of the parameters of parent distributions from which data is generated. We use stopping criteria as absolute value of likelihood changes with respect to previous likelihood at each iteration. The average estimates (AE), mean squared error (MSE) are reported based on 1000 replications. With a very small probability, algorithm is unable to find out the convergent point under this stopping criteria. As a remedy we stop the algorithm after 5000 iterations. This won’t affect the estimates and MSE much, because due to some reason, algorithm was unable to reach the convergence point. However it will roam around the actual values. Since major objective of EM is to extract some closer value of the original parameters, we observe in our simulation experiment that the goal will achieve without much affecting average estimates and mean square error. In practice we can use other optimization techniques taking the initial values of the parameters as the values that we have obtained using EM algorithm to get more perfect estimates.

6 Conclusion

In this paper we proposed hierarchical EM algorithm in mixture of two bivariate distributions. We formulate the mixtures of taking higher dimensional version of Generalized Exponential distribution proposed by Kundu and Gupta [3]. We observed that our algorithm is giving good results for large samples. Although MSE is on higher side for small sample. It can be a good guess for the choice of initial parameters in other optimization algorithm.

We can further extend this version in more generalized set-up or much larger class of distributions. The work is on progress.

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### Table 1

| Parameter Set | $n = 1500$ | $n = 1000$ | $n = 1500$ | $n = 1000$ |
|---------------|------------|------------|------------|------------|
| $\alpha_1$   | 1          | 0.5        | 1          | 0.5        |
| $\alpha_2$   | 1.2        | 0.4        | 1.2        | 0.4        |
| $\alpha_3$   | 1          | 0.3        | 1          | 0.3        |
| $\beta_1$    | 1          | 0.5        | 1          | 0.5        |
| $\beta_2$    | 1.4        | 1.5        | 1.4        | 1.5        |
| $\beta_3$    | 2          | 0.5        | 2          | 0.5        |
| $\lambda_1$  | 1          | 2          | 1          | 2          |
| $\lambda_2$  | 0.5        | 1.5        | 0.5        | 1.5        |
| $\rho$       | 0.3        | 0.6        | 0.3        | 0.6        |

| Average Estimates | 1.0902  | 1.0457  | 0.5039  | 0.5051  |
| Mean Square Error | 0.0119  | 0.0015  | 0.0017  | 0.00021 |
| Average Estimates | 1.2372  | 1.2556  | 0.4071  | 0.4076  |
| Mean Square Error | 0.0019  | 0.0015  | 0.0014  | 0.00168 |
| Average Estimates | 2.0006  | 1.3885  | 1.6028  | 1.4907  |
| Mean Square Error | 0.0006  | 0.0014  | 0.0048  | 0.00168 |
| Average Estimates | 1.0902  | 1.0046  | 0.50148 | 0.5047  |
| Mean Square Error | 0.00027 | 0.00046 | 0.00034 | 0.00034 |

Table 1 The Average Estimates (AE) and Mean Square Error (MSE)

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