ABSTRACT
This paper presents a method to detect and recognize symmetries in Boolean functions. The idea is to use information theoretic measures of Boolean functions to detect sub-space of possible symmetric variables. Coupled with the new techniques of efficient estimations of information measures on Binary Decision Diagrams (BDDs) we obtain promised results in symmetries detection for large-scale functions.

1. INTRODUCTION
Determining symmetries among groups of variables is important in logic synthesis \cite{1}, design verification and testing \cite{2}, as well as in problems of technology mapping, such as Boolean matching \cite{3, 4}. The effectiveness of the matching procedure can be increased if the groups of symmetric variables are known. The symmetry properties are used in different areas of logic design, namely, in decomposition and minimization \cite{5, 6}.

There are several techniques to recognize symmetries based on the following principles:

Manipulation of tabular data. The well-known algorithms explore the properties of symmetries to manipulate truth tables. For example, in \cite{7}, an efficient method to detect different types of symmetries based on numerical methods has been proposed.

Transformation into spectral domain. This principle exploits the properties of spectra to determine the symmetries in variables for a given function. The results on detecting symmetries in Hadamard, Haar and other transform bases have been reported \cite{8}.

Formal representation of symmetric functions (DTs and DDs, Reed-Muller expressions). In recent years, DDs have been used as an efficient data structure in circuit synthesis, and symmetry detection has become feasible for large-scale functions \cite{9, 10}.

In our approach, we consider information theoretic measures at the first phase of forming a sub-space of possible symmetric variables \cite{11}. At the second phase, we apply the exact (naive) technique to recognize symmetries in the obtained sub-space. DDs are used to calculate efficiently the spectrum of information measures \cite{12}.

2. THE TYPES OF SYMMETRY
There are four cofactors, namely, $f_{x_{i}x_{j}}$, $f_{x_{i}\overline{x}_{j}}$, $f_{\overline{x}_{i}x_{j}}$ and $f_{\overline{x}_{i}\overline{x}_{j}}$, for any pair of variables $\{x_{i}, x_{j}\}$ of a Boolean function $f = f(x_{1}, \ldots, x_{i}, \ldots, x_{j}, \ldots, x_{n})$ \cite{13}.

Definition 1 A function $f$ has non-equivalence symmetry (NE) in variables $\{x_{i}, x_{j}\}$, if it remains invariant when $x_{i}$ and $x_{j}$ (or $\overline{x}_{i}$ and $\overline{x}_{j}$) are interchanged: $f_{x_{i}x_{j}} = f_{\overline{x}_{i}\overline{x}_{j}}$.

Definition 2 A function $f$ has equivalence symmetry (E) in variables $\{x_{i}, x_{j}\}$, if it remains invariant when $x_{i}$ and $\overline{x}_{j}$ (or $\overline{x}_{i}$ and $x_{j}$) are interchanged: $f_{x_{i}x_{j}} = f_{\overline{x}_{i}\overline{x}_{j}}$.

Definition 3 A function $f$ has multiform symmetry (M) in variables $\{x_{i}, x_{j}\}$, if it is simultaneously NE- and E-symmetric in $\{x_{i}, x_{j}\}$.

Example 1 The following functions are symmetric:
(i) $f = x_{2} \oplus x_{3} \oplus x_{2}x_{3} \oplus x_{1}x_{2}x_{3}$ is NE-symmetric in $\{x_{2}, x_{3}\}$, and $f = x_{1}x_{3} \lor x_{1}x_{2}$ is NE-symmetric in $\{x_{2}, x_{3}\}$; (ii) $f = \overline{x}_{1} \oplus x_{2} \oplus \overline{x}_{1}x_{2}x_{3}$ is E-symmetric in $\{\overline{x}_{1}, x_{2}\}$; (iii) $f = x_{1}x_{2} \lor \overline{x}_{1}x_{2}$ is M-symmetric in $\{x_{1}, x_{2}\}$.

A function $f$ is partially symmetric with respect to subset $X_{I}$ of variables, $X_{I} \subseteq X$, if any permutation of variables in $X_{I}$ leaves $f$ unchanged. A function $f$ is totally symmetric if every pair of variables in the function is either NE- or E-symmetric.

Example 2 The following functions are symmetric:
(i) $f = \overline{x}_{1} \oplus x_{2} \oplus \overline{x}_{1}x_{2}x_{3}$ is partially symmetric, i.e. $E\{x_{1}, x_{2}\}$; (ii) $f = x_{1}x_{2} \lor \overline{x}_{1}x_{2}$ and $f = x_{1}x_{2} \oplus x_{1}x_{3} \oplus x_{2}x_{3}$ are totally symmetric.

We summarize these types of symmetry in Table 1.
2.1. Information theory notations

In order to quantify the content of information for a finite field of events \( A = \{a_1, a_2, \cdots, a_n\} \) with probabilities distribution \( \{p(a_i)\}, i = 1, 2, \cdots, n \), Shannon introduced the concept of entropy \( [10] \): \( H(A) = -\sum_{i=1}^{n} p(a_i) \cdot \log p(a_i) \), where \( \log \) denotes the base 2 logarithm. For two finite fields of events \( A \) and \( B \) with probability distribution \( \{p(a_i)\}, i = 1, 2, \cdots, n \), and \( \{p(b_j)\}, j = 1, 2, \cdots, m \), probability of the joint occurrence of \( a_i \) and \( b_j \) is joint probability \( p(a_i, b_j) \), and there is conditional probability, \( p(a_i|b_j) = p(a_i, b_j)/p(b_j) \). The conditional entropy of \( A \) given \( B \) is defined by \( H(A|B) = -\sum_{i=1}^{n} \sum_{j=1}^{m} p(a_i, b_j) \cdot \log p(a_i|b_j) \).

**Example 3** Let us calculate the entropy of a Boolean function \( f \) given by its truth column vector \([1100000111000010]\): \( H(f) = -6/16 \log_2 6/16 - 10/16 \log_2 10/16 = 0.95 \) bit/pattern. The conditional entropy with respect to variable \( x_1 \) is \( H(f|x_1) = -5/16 \log_2 5/16 - \log_2 8/16 - 3/16 \log_2 3/16 - \log_2 8/16 = 0.95 \) bit/pattern.

### 3. INFORMATION THEORETIC MEASURES IN SYMMETRY DETECTION

In this section, we focus on detecting different types of symmetries \((NE^-, E^-, M^-)\) and total symmetries) by information theoretic measures via design of decision diagram.

#### 3.1. Detection of Non-equivalent Symmetry

**Lemma 1** A Boolean function \( f \) is \( NE^-\)-symmetric in \( \{x_i, x_j\} \), if \( f_{x_i} = f_{\overline{x}_i} \) and \( f_{x_j} = f_{\overline{x}_j} \).

**Theorem 2** If a Boolean function \( f \) is \( NE^-\)-symmetric in \( \{x_i, x_j\} \), then \( H(f|x_i) = H(f|x_j) \) and \( H(f_{x_i}) = H(f_{x_j}) \).

**Theorem 3** The condition \( H(f|x_i) = H(f|x_j) \) from the Theorem 2 is necessary but not sufficient to detect \( NE^-\)-symmetry.

#### 3.2. Detection of Equivalent Symmetry

It is easy to show that a Boolean function \( f \) is \( E^-\)-symmetric in \( \{x_i, x_j\} \), if \( f_{x_i} = f_{x_j} \) and \( f_{x_j} = f_{\overline{x}_j} \). \( E^-\)-symmetry condition \( f_{x_i, x_j} = f_{x_i, \overline{x}_j} \) implies \( f_{x_i} = f_{\overline{x}_i} \) and \( f_{x_j} = f_{\overline{x}_j} \). Nodes with \( E^-\)-symmetric variables are placed together through design of DDs (Figure 1).

**Property 1** If a Boolean function \( f \) is \( E^-\)-symmetric in \( \{x_i, x_j\} \), then \( H(f|x_i) = H(f|x_j) \) and \( H(f_{x_i}) = H(f_{x_j}) \).

**Remark 1** Property \( \overline{1} \) is necessary, but not sufficient to detect \( E^-\)-symmetry.
Example 5 Consider a Boolean function \( f \) of three variables given by its truth column vector \([11100011]\). The information measures and cofactors for this function are shown below.

| \( x \)  | \( H(f_x) \) | \( H(f_{\overline{x}}) \) | \( f \) | \( f_{\overline{x}} \) |
|-------|-------------|-----------------|-----|----------------|
| \( x_1 \) | 0.81        | 0.81            | \( x_1 \) | [1110] [0011] |
| \( x_2 \) | 1           | 0.81            | \( x_2 \) | [1100] [1011] |
| \( x_3 \) | 0.81        | 1               | \( x_3 \) | [1101] [0011] |

Following information theoretic measures, we expect \( E \)-symmetry in \( \{x_1, \overline{x}_2\} \) and in \( \{x_2, \overline{x}_3\} \). Really, the function \( f \) is \( E \)-symmetric in \( \{x_1, \overline{x}_2\} \), because \( f_{\overline{x}_1} = f_{x_2} \) and \( f_{x_1} = f_{\overline{x}_2} \). But it is not \( E \)-symmetric in \( \{x_2, \overline{x}_3\} \) because \( f_{\overline{x}_2} \neq f_{x_3} \). The function can be represented by the following AND/OR expression: \( f = \overline{x}_1 \cdot x_2 \lor x_1 \cdot x_2 \lor \overline{x}_1 \cdot x_2 \land \overline{x}_3 \).

### 3.3. Detection of Multiform and Totally Symmetries

**Property 2** If a Boolean function \( f \) is \( M \)-symmetric in \( \{x_i, x_j\} \), then \( H(f|x_i) = H(f|x_j) \) and \( H(f|x_i) = H(f|x_j) = H(f_{\overline{x}_i}) \).

A part of the DT (nodes primitive) to be constructed for a function \( f \) that is \( M \)-symmetric in variables \( x_i, x_j \) is given in Figure 1.

Example 6 (Continuation of Example 5) The function \( f \) is \( M \)-symmetric in \( \{x_1, x_2\} \), because \( H(f_{\overline{x}_1}) = H(f_{x_2}) = H(f_{\overline{x}_1}) \).

**Property 3** If a Boolean function \( f \) is totally symmetric, then \( H(f_{x_1}) = H(f_{x_2}) = \ldots = H(f_{x_n}) = H(f_{\overline{x}_n}) \).

Example 7 Consider a Boolean function \( f \) of three variables given by its truth column vector \([00010111]\). The entropy measures and cofactors for the function are presented in the tables below.

| \( x \)  | \( H(f_x) \) | \( H(f_{\overline{x}}) \) | \( f \) | \( f_{\overline{x}} \) |
|-------|-------------|-----------------|-----|----------------|
| \( x_1 \) | 0.81        | 0.81            | \( x_1 \) | [0001] [0111] |
| \( x_2 \) | 0.81        | 0.81            | \( x_2 \) | [0001] [0111] |
| \( x_3 \) | 0.81        | 0.81            | \( x_3 \) | [0001] [0111] |

The information theoretic measures are equal, so the function is totally symmetric: \( f = \overline{x}_1 \cdot x_2 \cdot x_3 \lor x_1 \cdot \overline{x}_2 \cdot x_3 \lor x_1 \cdot x_2 \cdot \overline{x}_3 \lor x_1 \cdot x_2 \cdot x_3 \).

### 3.4. BDD Based Technique for Calculation of Information Measures

Our technique for exact computation of information measures for Boolean functions represented in the form of BDDs exploits the following. The conditional entropy \( H(f|x) \) of the function \( f \) with respect to the variable \( x \) can be simplified using the theorem below:

\[
H(f|x) = H(f_{x=0}) \cdot p(x=0) + H(f_{x=1}) \cdot p(x=1)
\]

It means that for calculation of conditional entropy we need to compute the entropy of each sub-function. In this case probability must be assigned to every node in BDD in order to distribute the desired output probability to the root.

Example 8 The conditional entropy of the function \( f = \overline{x}_3 \cdot x_2 \lor x_1 \) be: \( H(f) = 0.96 \) bit. The conditional entropy of the function \( f \) given \( x_2 \) be: \( H(f|x_2) = 0.5 \) bit and \( H(f|x_2=0) + 0.41 \cdot 0.5 = 0.91 \) bit (Figure 2). The same manipulation yields: \( H(f|x_1) = 0.41 \) bit and \( H(f|x_3) = 0.91 \) bit. The conditional entropy of the function \( f \) given a set of variables \( \{x_1, x_2\} \) be: \( H(f|x_1x_2) = 0.25 \) bit.

### 4. ALGORITHM AND EXPERIMENTAL RESULTS

We propose an algorithm to detect symmetries of Boolean functions using BDDs, called \textit{InfoRECSym – DD} (\textit{I}nf \textit{f} \textit{o} \textit{R} \textit{E} \textit{C} \textit{S} \textit{ym} \textit{—} \textit{D} \textit{D}) (Information \textit{RE} \textit{C} \textit{O} \textit{g} \textit{n} \textit{i} \textit{r} \textit{n} \textit{i} \textit{e} of \textit{S} \textit{ym} \textit{m} \textit{e} \textit{t} \textit{i} \textit{r} \textit{i} \textit{i} \textit{e} \textit{r} \textit{i} \textit{y} \textit{s}). A sketch of the algorithm is given in Figure 3.

The original program presented in [8] was modified to detect possible symmetries groups using binary decision diagrams representation. Table 3 contains a fragment of our results comparing to the strategy published by Tsai et al. [11]. We use the notation \((S,N)\), where \( S \) means the size of a symmetric group (the number of symmetric variables), and \( N \) means the number of symmetric groups.

### 5. CONCLUDING REMARKS

This paper addresses a method for detecting and recognizing different types of symmetries (totally, partially \( NE, E, M \)) in Boolean functions. The method is based on a variety of information measures computed on decision diagrams.
Table 2: Results of \textit{InfoRecSym − DD} in symmetry detection

| I/O nr. (S, N) | Time (S, N) | Time |
|---------------|------------|------|
| cm82          | 5/1 (3,1) (2,2) 0.044 | (3,1) (2,2) 0.00 |
| 511m          | 5/3 (2,2) 0.113 | (4,1) 0.113 |
| z4ml          | 7/1 (3,1) (2,2) 0.113 | (3,1) (2,2) 0.00 |
| 4x            | 10/66 (4,1) (3,1) (2,1) 0.275 | (8,1) (4,1) 0.13 |
| 13/x          | 13/13 (10,1) (2,1) 0.113 | (10,1) (2,1) 1.05 |
| apex6         | 13/41 (10,1) (2,1) 0.691 | (10,1) (2,1) 0.48 |
| des           | 18/120 (4,1) (2,7) 3.602 | (13,1) (3,1) (17,1) 11.16 |
| apex7         | 19/8 (8,1) (3,1) (2,2) 4.917 | (8,1) (3,1) (2,2) 0.02 |

\textbf{Input}: Decision Diagram of a function \(f\)

\textbf{Output}: The pairs of symmetric variables \(\{x_i, x_j\}\) with detected types of symmetries

\textit{InfoRECSym − DD}(f)

\begin{verbatim}
for(All pairs \(\{x_i, x_j\}\))
    if \((H(f|x_i) = H(f|x_j))\)
        \begin{align*}
        & if (H(f|\overline{x_i}) = H(f|\overline{x_j})) \\
        & \quad if (f|\overline{x_i} = f|\overline{x_j} \text{ and } f_{x_i} = f_{x_j}) \\
        & \quad \text{then } f \text{ is } NE\text{-symmetric in } \{x_i, x_j\} \\
        & if (H(f|\overline{x_i}) = H(f|\overline{x_j})) \\
        & \quad if (f_{x_i} = f_{x_j} \text{ and } f_{\overline{x_i}} = f_{\overline{x_j}}) \\
        & \quad \text{then } f \text{ is } E\text{-symmetric in } \{x_i, \overline{x_j}\} \\
        & if (f \text{ is } NE\text{-}, E\text{-symmetric simultaneously}) \\
        & \quad \text{then } f \text{ is } M\text{-symmetric in } \{x_i, \overline{x_j}\} \\
        \end{align*}
    if (All pairs are either NE\text{-} or E\text{-}symmetric)
    then \(f\) is totally symmetric function.
\end{verbatim}

Figure 3: Sketch of the algorithm \textit{InfoRECSym − DD}

6. REFERENCES

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