STOCHASTIC SOLUTION OF FRACTIONAL FOKKER-PLANCK EQUATIONS WITH SPACE-TIME-DEPENDENT COEFFICIENTS

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Abstract. This paper develops solutions of fractional Fokker-Planck equations describing sub-diffusion of probability densities of stochastic dynamical systems driven by non-Gaussian Lévy processes, with space-time-dependent drift, diffusion and jump coefficients, thus significantly extends Magdziarz and Zorawik’s result in [14]. Fractional Fokker-Planck equation describing subdiffusion is solved by our result in full generality from perspective of stochastic representation.

1. Introduction

Fractional Fokker-Planck equation has shown its applicati on in diverse scientific areas, including biology [9], physics [6], [28], finance [7]. For example, in physics, it has been broadly used to describe phenomena related to anomalous diffusion [27], [29]. Many different types of fractional Fokker-Planck equation have been solved in terms of probability density function (PDF) of corresponding process [20], [21], related researches are also growing rapidly in different branches, e.g., [15], [16], [17], [18], [22]. Recently, Magdziarz and Zorawik [14] provide solution to an extended type of Fractional Fokker-Planck equation.

To state the result, let’s introduce subordinator $T_\Psi(t)$ with Laplace transform

$$(1.1) \quad E(e^{-uT_\Psi(t)}) = e^{-t\Psi(u)},$$

where Laplace exponent

$$(1.2) \quad \Psi(u) = \int_0^\infty (1 - e^{-ux})\nu(dx),$$

$\nu$ is Lévy measure satisfying

$$(1.3) \quad \int_{\mathbb{R}-\{0\}} \min\{|x|^2, 1\}\nu(dx) < \infty$$

and $\nu((0, \infty)) = \infty$. Then, the first passage time process defined as

$$(1.4) \quad S_\Psi(t) = \inf\{\gamma > 0 : T_\Psi(\gamma) > t\}, t \geq 0,$$

is called the inverse subordinator.

Define the integro-differential operator $\Phi_t$ as

$$(1.5) \quad \Phi_t f(t) = \frac{\partial}{\partial t} \int_0^t M(t - y)f(y)dy,$$

where the function $f$ is smooth enough and kernel $M(t)$ is defined by its Laplace transform as

$$(1.6) \quad \tilde{M}(u) = \int_0^\infty e^{-ut}M(t)dt = \frac{1}{\Psi(u)}.$$

A stochastic process $X = \{X(t), t \geq 0\}$ is a Lévy process if (a) $X(0) = 0$, a.s., (b) $X$ has independent and stationary increments, (c) $X$ is stochastically continuous in time. $X^-(t)$ is used to denote left limit, $X^-(t) = \lim_{s \to t-} X(s)$.

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By Theorem 1.2.14 and Proposition 1.3.1 of [1], a Lévy process $X$ has characteristics $(b, A, \nu)$, that’s,

$$E(e^{iuX(t)}) = \exp \left[ t \left( ibu - \frac{1}{2} Au^2 + \int_{\mathbb{R} \setminus \{0\}} (e^{iuy} - 1 - iuy 1_B(y)) \nu(dy) \right) \right],$$  

where $\nu$ is Lévy measure. In the remaining part of this paper, for convenience, we will decompose a Lévy process $X(t)$ into 3 parts: drift, Brownian motion $B(t)$ and pure jump Lévy process $L(t)$.

Improving on the methods in the papers [5] [10] [12], Magdziarz and Zorawik [14] proved that the PDF of process $X(t) = Y^-(S_\Psi(t))$, where

$$dY(t) = F(Y^-(t), T_{\Psi}^-(t))dt + \sigma(Y^-(t), T_{\Psi}^-(t))dB(t) + E(T_{\Psi}^-(t))dL(t), t \geq 0,$$

$Y(0) = 0, T_{\Psi}(t) = 0,$

with $F(x, t), \sigma(x, t), E \in C^2(\mathbb{R}^2)$ satisfying Lipschitz condition, solves fractional Fokker-Planck equation

$$\frac{\partial q(x, t)}{\partial t} = \left[ -\frac{\partial}{\partial x} F(x, t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} \sigma^2(x, t) \right] \Phi_t q(x, t)$$

$$+ \int_{\mathbb{R} \setminus \{0\}} \left[ \Phi_t q(r, t) |_{r=x-E(t)y} - \Phi_t q(x, t) + E(t)y \frac{\partial}{\partial x} \Phi_t q(x, t) 1_B(y) \right] \nu(dy),$$

with $q(x, 0) = \delta(x)$, where $\delta(x)$ is an indicator function, $B = \{y, |y| < 1\}$.

This result extends the following celebrated fractional Fokker-Planck equation introduced by Metzler and Klafter [21] in 2000,

$$\frac{\partial q(x, t)}{\partial t} = _0D_t^{1-\alpha} \left[ -\frac{\partial}{\partial x} F(x, t) + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} \right] q(x, t),$$

with $\sigma > 0$ and $q(x, 0) = \delta(x)$, which describes anomalous diffusion in the presence of an space-dependent force $F(x)$. Note that the operator $_0D_t^{1-\alpha}$, $\alpha \in (0, 1)$ is fractional derivative of Riemann-Liouville type [25],

$$_0D_t^{1-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

for $f \in C^1([0, \infty))$. Magdziarz et al. [13] showed that the PDF of the subordinated process $X(t) = Y(S_\alpha(t))$ solves equation [1.10], where

$$dY(t) = F(Y(t))dt + \sigma dB(t), Y(0) = 0.$$

Here, let $D_\alpha$ be an $\alpha$-stable subordinator with Laplace transform $E[e^{-uD_\alpha(\gamma)}] = e^{-tu^\alpha}$, its inverse $S_\alpha(t)$ is defined as

$$S_\alpha(t) = \inf \{\gamma > 0 : D_\alpha(\gamma) > t\}.$$

Magdziarz and Zorawik’s result [14] also extends the following fractional Fokker-Planck equation introduced by Sokolov and Klafter [23] in 2009,

$$\frac{\partial q(x, t)}{\partial t} = \left[ -F(x) \frac{\partial}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} \right] _0D_t^{1-\alpha} q(x, t),$$

with $\sigma > 0$ and $q(x, 0) = \delta(x)$, where external force $F(t)$ is time-dependent. Its solution is obtained by PDF of subordinated process $X(t) = Y(S_\alpha(t))$, where

$$dY(t) = F(T_\alpha(t))dt + \sigma dB(t), Y(0) = 0.$$
One of the main results of this paper is that the PDF of \( X(t) = Y^-(S_\Psi(t)) \), where
\[
dY(t) = F(Y^-(t), T_{\Psi}^{-}(t))dt + \sigma(Y^-(t), T_{\Psi}^{-}(t))dB(t) + h(Y^-(t), T_{\Psi}^{-}(t))dL(t), \quad t \geq 0, \\
Y(0) = 0, T_{\Psi}(t) = 0,
\]
solves the following fractional Fokker-Planck equation, which involves fractional Laplacian operator,
\[
\frac{\partial q(x, t)}{\partial t} = \left[ -\frac{\partial}{\partial x} F(x, t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} \sigma^2(x, t) - (-\Delta)^{\frac{\alpha}{2}} (\text{sgn}(h(x,t))|h(x,t)|^\alpha) \right] \Phi_t q(x, t),
\]
with \( q(x, 0) = \delta(x) \), note that \( \alpha \in (0, 2) \) and \( L(t) \) is the \( \alpha \)-stable Lévy process.

Furthermore, in section 3, we extend Magdziarz and Zorawik’s result \([14] \) by solving
\[
\frac{\partial}{\partial t} w(x, t) = \left[ -\frac{\partial}{\partial x} (F(x, t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} \sigma^2(x, t) + T^{l+}) \right] \Phi_t w(x, t),
\]
with \( q(x, 0) = \delta(x) \), where
\[
T^{l+} f(x, t) = \int_{R-\{0\}} \sum_{k=1}^{\infty} \frac{(-r)^k}{k!} \frac{\partial^k}{\partial y^k} (h(x, t)^k f(x, t)) \mathbb{1}_B(r, t) r \frac{\partial}{\partial y} (h(x, t) f(x, t)) \nu(dr),
\]
for any \( f(x, t) \in C_0^\infty(R^2) \).

Note that coefficient \( E(t) \) of pure jump Lévy process in \([1.19] \) is time-dependent, while coefficient \( h(x, t) \) of pure jump Lévy process in \([1.18] \) is space-time-dependent. Thus, \([1.19] \) is a special case of \([1.18] \) when \( h(x, t) \) only depends on time \( t \). For more details on this, see remark \([5.9] \).

In the remaining of this paper, necessary concepts will be given in the Preliminaries section; in the Main Results section, we will solve 3 different fractional Fokker-Planck equations involving operators of \( \alpha \)-stable, symmetric and general Lévy processes, respectively, one by one.

2. Preliminaries

Let \( X = \{ X(t), t \geq 0 \} \) be a Lévy process with characteristics \( (b, A, \nu) \), by Theorem 6.7.4 of \([1] \), it has infinitesimal generator
\[
Af(x) = b \frac{\partial}{\partial x} f(x) + \frac{1}{2} \frac{\partial^2}{\partial x^2} f(x) + \int_{R-\{0\}} \left[ f(x + y) - f(x) - y \frac{\partial}{\partial x} f(x) \mathbb{1}_B(y) \right] \nu(dy),
\]
for each \( f \in C_0^\infty(R), \ x \in R \).

The following Lévy processes and their generators will be used in this paper later.

An \( \alpha \)-stable Lévy process \( X(t) \) has characteristics \( (0, 0, \nu) \) and Lévy symbol \( \eta(u) = -|u|^{\alpha}, \ \alpha \in (0, 2) \), see example 3.3.8 of \([1] \), and infinitesimal generator
\[
Af(x) = \int_{R-\{0\}} [f(x + y) - f(x)] \nu(dy) = -(-\Delta)^{\alpha/2} f(x),
\]
where \( \nu(dy) = \frac{C_{\alpha} dy}{|y|^{\alpha+1}}, \ C_{\alpha} = \frac{\pi^{\frac{\alpha}{2}}}{\Gamma(1 - \frac{\alpha}{2})} \).

A symmetric Lévy process \( X(t) \) has characteristics \( (0, 0, \nu) \) where \( \nu \) is symmetric Lévy measure, that’s, \( \nu(B) = \nu(-B) \), for each \( B \subset R \), and infinitesimal generator
\[
Af(x) = \int_{R-\{0\}} [f(x + y) - f(x)] \nu(dy).
\]

A general Lévy process \( X(t) \) with characteristics \( (0, 0, \nu) \) has infinitesimal generator
\[
Af(x) = \int_{R-\{0\}} \left[ f(x + y) - f(x) - y \frac{\partial}{\partial x} f(x) \mathbb{1}_B(y) \right] \nu(dy),
\]
Theorem 3.2. Suppose that the standard Brownian motion $F$ where the function $\alpha$ analyze the cases when the jump process is $3.4$ for each $t$ and $\alpha$. Let $G(w) = \nu((w, \infty))$, define the following operator $L^* \nu$ and $L^* \nu$ before analyzing general Lévy process, since the results for the cases of $3.5$ for each $x$. In fact, such $X(t)$ is a pure jump Lévy process since drift and Brownian motion parts are gone as $b = A = 0$.

Let $G(w) = \nu((w, \infty))$, define the following operator $14$

$$\Theta_w g(w) = \int_0^w G(w - z)g(z)dz,$$

with Laplace transform of its kernel

$$\tilde{G}(u) = \frac{\Psi(u)}{u}.$$  

3. Main Results

In this section, we will solve fractional Fokker-Planck equations with infinitesimal generator of Lévy processes with space-time-dependent coefficients for drift, diffusion and jump parts. We analyze the cases when the jump process is $\alpha$-stable and symmetric Lévy process, respectively, before analyzing general Lévy process, since the results for the cases of $\alpha$-stable and symmetric Lévy process are more explicit.

3.1. Fractional Fokker-Planck equation with $\alpha$-stable Lévy generator.

Definition 3.1. Let $L(t)$ be $\alpha$-stable Lévy process satisfying the following SDE

$$dL(t) = \frac{\partial}{\partial x} \nu(x, t)dx + \int_{\mathbb{R}} \left[ e^{iux} - 1 - iux \mathbb{1}_B(x) \right] \nu(dx),$$

where Lévy symbol $\eta(u) = \int_{\mathbb{R}} \left[ e^{iux} - 1 - iux \mathbb{1}_B(x) \right] \nu(dx)$, with

$$\nu(dx) = C_\alpha \frac{dx}{|x|^{1+\alpha}}, \quad B = \{x, |x| < 1\}.$$

Theorem 3.2. Suppose that the standard Brownian motion $B(t)$, the $\alpha$-stable Lévy process $L(t)$ as defined in definition $\alpha$ and subordinator $T_\Psi(t)$ are independent, where $T_\Psi(t)$ has Laplace exponent $\Psi(u)$ and its inverse $S_\Psi(t)$. Let $Y(t)$ be the solution of the stochastic equation

$$dY(t) = F(Y^-(t), T_\Psi(t))dt + \sigma(Y^-(t), T_\Psi(t))dB(t) + h(Y^-(t), T_\Psi(t))dL(t), t \geq 0,$$

$Y(0) = 0$, $T_\Psi(0) = 0$

where the function $F(x, t)$, $\sigma(x, t)$, $h(x, t) \in C^2(\mathbb{R}^2)$ satisfy the Lipschitz condition. Assume that the PDF of the process $(Y(t), T_\Psi(t))$, $p_t(y, z)$ exists. Furthermore, we assume that

$$\frac{\partial}{\partial t} p_t(y, z), \frac{\partial}{\partial y} p_t(y, z), \frac{\partial^2}{\partial y^2} p_t(y, z)$$

exist,

$$\int_0^t \int_0^{t_0} \left| \frac{\partial}{\partial t} p_s(x, t) \right| dsdt < \infty$$

for each $t_0 > 0$,

$$\int_{x_1}^{x_2} \int_0^\infty \left| \frac{\partial}{\partial x} p_s(x, t) \right| dsdx < \infty, \quad \int_{x_1}^{x_2} \int_0^\infty \left| \frac{\partial^2}{\partial x^2} p_s(x, t) \right| dsdx < \infty,$$

for each $x_1, x_2 \in \mathbb{R}$ and

$$\int_0^\infty \int_{\mathbb{R}^{+}} \left| p_s(x + y, t) \frac{sgn(h(x + y, t))|h(x + y, t)|^\alpha}{|y|^{1+\alpha}} - p_s(x, t) \frac{sgn(h(x, t))|h(x, t)|^\alpha}{|y|^{1+\alpha}} \right| C_\alpha dyds < \infty.$$
Proof. This proof uses methods in [5] and [14] with crucial changes.

Equation (3.2) can be represented as the following stochastic equations
\[ dY(t) = F(Y^-(t), Z^-(t))dt + \sigma(Y^-(t), Z^-(t))dB(t) \]
(3.6)
\[ + h(Y^-(t), Z^-(t)) \int_B x \tilde{N}(dt, dx) + h(Y^-(t), Z^-(t)) \int_{B^c} x N(dt, dx) \]
\[ dZ(t) = dT_\Psi(t). \]

By Theorem 6.7.4 of [1], the infinitesimal generator of the process \((Y(t), Z(t))\) that operates on functions \(f \in C^2_0(\mathbb{R}^2)\) is
\[
\Gamma f(y, z) = F(y, z) \frac{\partial}{\partial y} f(y, z) + \frac{1}{2} \sigma^2(y, z) \frac{\partial^2}{\partial y^2} f(y, z) \]
(3.7)
\[ + \int_{\mathbb{R} - \{0\}} \left[ f(y + xh(y, z), z) - f(y, z) - \mathbb{1}_B(x) xh(y, z) \frac{\partial}{\partial y} f(y, z) \right] \nu(dx) \]
\[ + \int_0^\infty [f(y, z + u) - f(y, z)] \mu(du), \]
where \(\mu\) is the Lévy measure of \(T_\Psi(t)\).

Decompose \(\Gamma = A + T_\alpha\), where
\[
Af(y, z) = F(y, z) \frac{\partial}{\partial y} f(y, z) + \frac{1}{2} \sigma^2(y, z) \frac{\partial^2}{\partial y^2} f(y, z), \]
(3.8)
\[ + \int_0^\infty [f(y, z + u) - f(y, z)] \mu(du) \]
\[ T_\alpha f(y, z) = \int_{\mathbb{R} - \{0\}} \left[ f(y + xh(y, z), z) - f(y, z) - \mathbb{1}_B(x) xh(y, z) \frac{\partial}{\partial y} f(y, z) \right] \nu(dx). \]

By setting \(E(t) = 0\) in (2.17) of [14],
\[
A^+ p_t(y, z) = -\frac{\partial}{\partial y} (F(y, z) p_t(y, z)) + \frac{\partial^2}{\partial y^2} \left( \frac{1}{2} \sigma^2(y, z) p_t(y, z) \right) - \frac{\partial}{\partial z} \Theta z p_t(y, z), \]
(3.9)
where \(A^+\) is the Hermitian adjoint of \(A\).

Next, \(T_\alpha^+\) is derived below
\[
T_\alpha f(y, z) = \int_{\mathbb{R} - \{0\}} \left[ f(y + xh(y, z), z) - f(y, z) \right] C_\alpha \frac{dx}{|x|^{1+\alpha}} \]
(3.10)
\[ = \text{sgn}(h(y, z)) |h(y, z)|^\alpha \int_{\mathbb{R} - \{0\}} \left[ f(y + x, z) - f(y, z) \right] C_\alpha \frac{dx}{|x|^{1+\alpha}} \]
\[ = \text{sgn}(h(y, z)) |h(y, z)|^\alpha \left[ -(-\Delta)^{\alpha/2} f(y, z) \right]. \]

Note that the second equation is the result of change of variable.

Since the infinitesimal generator of \(\alpha\)-stable Lévy process is self-adjoint,
\[
\int_{\mathbb{R}^2} f(y, z) T_\alpha^+ p_t(y, z) dydz = \int_{\mathbb{R}^2} p_t(y, z) T_\alpha f(y, z) dydz \]
(3.11)
\[ \Rightarrow \int_{\mathbb{R}^2} p_t(y, z) \text{sgn}(h(y, z)) |h(y, z)|^\alpha [ -(-\Delta)^{\alpha/2} f(y, z) ] dydz \]
\[ \Rightarrow \int_{\mathbb{R}^2} f(y, z) [ -(-\Delta)^{\alpha/2} (p_t(y, z) \text{sgn}(h(y, z)) |h(y, z)|^\alpha) ] dydz. \]
Then, we get
\begin{align}
H \text{ of } & \int_{0}^{t} \frac{1}{2} \sigma^2(y, z) p_t(y, z) dt = \Gamma + p_t(y, z) \int_{0}^{t} \frac{1}{2} \sigma^2(y, z) p_t(y, z) dt
\end{align}

Since \( \Gamma^+ = A^+ + T^+ \) and \( \frac{\partial}{\partial t} p_t(y, z) = \Gamma^+ p_t(y, z) \), we have
\begin{align}
\frac{\partial}{\partial t} p_t(y, z) = & \Gamma^+ p_t(y, z) = A^+ p_t(y, z) + T^+ p_t(y, z) \\
\end{align}

Next, we establish the relationship between \( q(x, t) \) and \( p_t(y, z) \), the probability density functions of \( X(t) \) and \( Y(t), Z(t) \), respectively. Let \( w \) denote a random path of stochastic process, for each fixed interval \( I \), define indicator function
\begin{align}
\mathbb{1}_I(x) &= \begin{cases} 
1 & \text{if } x \in I, \\
0 & \text{otherwise.}
\end{cases}
\end{align}

and the auxiliary function
\begin{align}
H_t(s, w, u) &= \begin{cases} 
\mathbb{1}_I(Y^-(s, w)) & \text{if } Z^-(s, w) \leq t \leq Z^-(s, w) + u, \\
0 & \text{otherwise.}
\end{cases}
\end{align}

Then, we get
\begin{align}
\int_{t} q(x, t) dx = E[\mathbb{1}_I(X(t, w))].
\end{align}

Let \( \Delta Z(t, w) = Z(t, w) - Z^-(t, w) \) and \( s = S_{\Phi}(t, w) = \inf\{ \tau > 0 : T_{\Phi}(\tau) > t \} \), then
\begin{align}
H_t(S_{\Phi}(t, w), \Delta Z(t, w)) = \mathbb{1}_I(X(t, w)) \quad \text{and}
\end{align}

\begin{align}
H_t(S_{\Phi}(t, w), \Delta Z(t, w)) &= \begin{cases} 
\mathbb{1}_I(Y^-(S_{\Phi}(t, w), w)) & \text{if } Z^-(S_{\Phi}(t, w), w) \leq t \leq Z(S_{\Phi}(t, w), w), \\
0 & \text{otherwise.}
\end{cases}
\end{align}

Since \( T_{\Phi}(t, w) \) is a subordinator, \( T_{\Phi}(S_{\Phi}(t, w), w) \leq t \leq T_{\Phi}(S_{\Phi}(t, w), w) \) is always true, thus
\begin{align}
H_t(S_{\Phi}(t, w), \Delta Z(t, w)) &= \mathbb{1}_I(Y^-(S_{\Phi}(t, w), w)).
\end{align}

To avoid \( Z(t, w) \) being a compound Poisson process, we set \( \nu([0, \infty) = \infty \), see Remark 27.3 and 27.4 of [26], thus, jumping times of \( Z(t, w) \) are dense in \([0, \infty) \) almost surely, see Theorem 21.3 of [26]. Then, we can derive that
\begin{align}
H_t(s, w, \Delta Z(t, w)) = 0, \text{ if } s \neq S_{\Phi}(t, w)
\end{align}

Since if \( s < S_{\Phi}(t, w) \), as \( Z(t) = T_{\Phi}(t) \) is a subordinator, \( Z(s) < Z^-(S_{\Phi}(t, w)) \leq t \). Similarly, if \( s > S_{\Phi}(t, w) \), then \( Z(s) > Z(S_{\Phi}(t, w)) \geq t \). In both cases, \( H_t(s, w, \Delta Z(t, w)) = 0 \).

Hence,
\begin{align}
\mathbb{1}_I(X(t, w)) = \sum_{s > 0} H_t(s, w, \Delta Z(t, w)).
\end{align}
By Compensation Formula in Ch. XII, Proposition (1.10) of [24],

\[
E \left[ \sum_{s>0} H_t(s, w, \Delta Z(s, w)) \right] = E \left[ \int_0^\infty \int_0^\infty H_t(s, w, u) \nu(du)ds \right] \\
= E \left[ \int_0^\infty \int_0^\infty 1_I(Y(s, w)) 1_{[Z(s-w), Z(s-w)+u]}(t) \nu(du)ds \right] \\
= E \left[ \int_0^\infty 1_I(Y(s, w)) \int_0^\infty 1_{[t-z, \infty]}(u) 1_{[0, t]}(z) \nu(du)ds \right] \\
(3.20) \\
= E \left[ \int_0^\infty 1_I(Y(s, w)) 1_{[0, t]}(Z(s, w)) \nu(t - Z(s, w), \infty)ds \right] \\
= E \left[ \int_0^\infty 1_I(Y(s, w)) 1_{[0, t]}(Z(s, w)) G(t - Z(s, w)ds \right] \\
= \int_0^\infty \int_0^t G(t - z) p_s(y, z) dz dy \\
= \int_0^\infty \Theta_t p_s(y, z) dz dy.
\]

By (3.16), (3.19) and (3.20), we have

\[
\int_I q(x, t) dx = \int_I \int_0^\infty \Theta_t p_s(y, t) dz dy.
\]

By the arbitrariness of interval \( I \),

\[
q(x, t) = \int_0^\infty \Theta_t p_s(y, t) dz.
\]

Next we claim that

\[
\frac{\partial}{\partial t} \Theta_t p_s(x, t) = \Theta_t \frac{\partial}{\partial t} p_s(x, t).
\]

To see this, let \( p_s(x, u) \) and \( G(u) \) be the Laplace transform \( (t \to u) \) of \( p_s(x, t) \) and \( g(t) \), respectively. Then Laplace transform of \( \frac{\partial}{\partial t} \Theta_t p_s(x, t) \) is given as

\[
\mathcal{L} \left[ \frac{\partial}{\partial t} \Theta_t p_s(x, t) \right] = u \mathcal{L} [\Theta_t p_s(x, t)] - \Theta_0 p_s(x, 0) \\
= u \mathcal{L} \left[ \int_0^t G(t - z) p_s(x, z) dz \right] \\
= u G(u) P_s(x, u).
\]

On the other hand, since \( T_\Phi(0) = 0 \text{ a.s.} \), Laplace transform of \( \Theta_t \frac{\partial}{\partial t} p_s(x, t) \) is as below,

\[
\mathcal{L} \left[ \Theta_t \frac{\partial}{\partial t} p_s(x, t) \right] = \mathcal{L} \left[ \int_0^t G(t - z) \frac{\partial}{\partial t} p_s(x, z) dz \right] \\
= G(u) \mathcal{L} \left[ \frac{\partial}{\partial t} p_s(x, t) \right] \\
= G(u) \{ u P_s(x, u) - p_s(x, 0) \} \\
= u G(u) P_s(x, u).
\]

Notice that

\[
\int_0^{t_0} G(u) du = \int_0^{t_0} \int_{(u, \infty)} \nu(dw) du = \int_{(0, \infty)} \min(w, t_0) \nu(dw) = K < \infty,
\]
by assumption (3.3) and (3.23), we derive that

\[
\int_0^t \int_0^\infty \left| \frac{\partial}{\partial t} \Theta p_s(x,t) \right| dsdt = \int_0^t \int_0^\infty \left| \Theta_t \frac{\partial}{\partial t} p_s(x,t) \right| dsdt \\
= \int_0^t \int_0^\infty \left| \int_0^t G(t-u) \frac{\partial}{\partial z} p_s(x,z) \right| du dsdt \\
\leq \int_0^t \int_0^\infty \left| \int_u^t G(t-u) \frac{\partial}{\partial z} p_s(x,z) \right| dudsdt \\
= \int_0^t \int_0^\infty \frac{\partial}{\partial z} p_s(x,z) \int_u^t G(t-u) dudsdt \\
\leq K \int_0^t \int_0^\infty \frac{\partial}{\partial z} p_s(x,z) \| dsdu < \infty.
\]

(3.27)

Thus, we can put differentiation on both side of (3.22) and move it inside the integral on the righthand side as below,

\[
\frac{\partial}{\partial t} q(x,t) = \int_0^\infty \frac{\partial}{\partial t} \Theta p_s(x,t) ds.
\]

(3.28)

Next, we claim that

\[
\int_0^\infty p_s(x,t) ds = \Phi_t q(x,t).
\]

(3.29)

By Fubini theorem,

\[
q(x,t) = \int_0^\infty \Theta p_s(x,t) ds = \int_0^t \int_0^\infty G(t-z)p_s(x,z)dzds \\
= \int_0^t G(t-z) \int_0^\infty p_s(x,z)dz = \Theta_t \int_0^\infty p_s(x,t) ds,
\]

(3.30)

thus,

\[
\int_0^\infty p_s(x,t) ds = \Theta_t^{-1} q(x,t).
\]

(3.31)

To prove \( \Theta_t^{-1} = \Phi_t \), let \( \tilde{M}(u), \tilde{G}(u) \) and \( \tilde{Q}(x,u) \) be the Laplace transform of \( M(t) \), \( G(t) \) and \( q(x,t) \), respectively. Since, by \( \text{[1.2]} \)

\[
\int_0^\infty e^{-ut} G(t) dt = \int_0^\infty \int_(t,\infty) e^{-ut} \nu(ds) dt = \int_0^\infty \frac{1-e^{as}}{u} \nu(ds) = \frac{\Psi(u)}{u},
\]

and

\[
\mathcal{L}[q(x,z)] = \mathcal{L}[G(t)] \mathcal{L}[\Theta_t^{-1} q(x,t)],
\]

(3.33)

we have

\[
\mathcal{L}[\Theta_t^{-1} q(x,t)] = \mathcal{L}[q(x,z)] = \frac{\tilde{Q}(x,u)}{\tilde{G}(u)} = \frac{u}{\tilde{Q}(x,u)} \Psi(u).
\]

(3.34)
Also
\[
\mathcal{L} [\Phi_t q(x, t)] = \mathcal{L} \left[ \frac{d}{dt} \int_0^t M(t-y) q(x, y) dy \right]
\]
(3.35)
\[
= u \mathcal{L} \left[ \int_0^t M(t-y) q(x, y) dy \right] - 0
\]
\[
= u \tilde{M}(u) \tilde{Q}(x, u) = u \frac{\tilde{Q}(x, u)}{\Psi(u)}.
\]
this shows \( \Phi_t q(x, t) = \Theta_t^{-1} q(x, t) \), hence \( \int_0^\infty p_s(x, t) ds = \Phi_t q(x, t) \).

Since \( \lim_{s \to \infty} p_s(x, t) = 0 \) and \( p_0(x, t) = \mathbb{1}_{(0,0)}(x, t) \), by (3.13), (3.28) and (3.29),
\[
\frac{\partial}{\partial t} q(x, t) = \int_0^\infty \left[ \frac{\partial^2}{\partial x^2} \left( \frac{1}{2} \sigma^2(x, t)p_s(x, t) \right) - \frac{\partial}{\partial x} (F(x, t)p_s(x, t)) \right. \\
- (-\Delta)^{\alpha/2} (p_s(x, t)(\text{sgn}(h(x, t))|h(x, t)|^\alpha)) - \left. \frac{\partial}{\partial s} p_s(x, t) \right] ds
\]
(3.36)
\[
= \int_0^\infty \left[ \frac{\partial^2}{\partial x^2} \left( \frac{1}{2} \sigma^2(x, t)p_s(x, t) \right) - \frac{\partial}{\partial x} (F(x, t)p_s(x, t)) \right] ds \\
+ \int_{\mathbb{R} - \{0\}} \left[ \int_0^\infty p_s(x+y, t) \frac{\text{sgn}(h(x+y, t))|h(x+y, t)|^\alpha}{|y|^{1+\alpha}} \right. \\
- p_s(x, t) \frac{\text{sgn}(h(x, t))|h(x, t)|^\alpha}{|y|^{1+\alpha}} \left. \right] C_\alpha dy ds \\
= \frac{\partial^2}{\partial x^2} \left( \frac{1}{2} \sigma^2(x, t) \int_0^\infty p_s(x, t) ds \right) - \frac{\partial}{\partial x} (F(x, t) \int_0^\infty p_s(x, t) ds) \\
- (-\Delta)^{\alpha/2} \left( (\text{sgn}(h(x, t))|h(x, t)|^\alpha) \int_0^\infty p_s(x, t) ds \right) \\
- \frac{\partial^2}{\partial x^2} (\frac{1}{2} \sigma^2(x, t) \Phi_t q(x, t)) - \frac{\partial}{\partial x} (F(x, t) \Phi_t q(x, t)) \\
- (-\Delta)^{\alpha/2} (\text{sgn}(h(x, t))|h(x, t)|^\alpha) \Phi_t q(x, t)
\]
In summary,
\[
\frac{\partial}{\partial t} q(x, t) = \left[ \frac{1}{2} \frac{\partial^2}{\partial x^2} \sigma^2(x, t) - \frac{\partial}{\partial x} F(x, t) - (-\Delta)^{\alpha/2} (\text{sgn}(h(x, t))|h(x, t)|^\alpha) \right] \Phi_t q(x, t)
\]
(3.37)
\[
\frac{\partial}{\partial t} q(x, t) = \left[ \frac{1}{2} \frac{\partial^2}{\partial x^2} \sigma^2(x, t) - \frac{\partial}{\partial x} F(x, t) - (-\Delta)^{\alpha/2} (\text{sgn}(h(x, t))|h(x, t)|^\alpha) \right] \Phi_t q(x, t)
\]
\[
\square
\]
3.2. Fractional Fokker-Planck equation with symmetric Lévy generator. As is known, \( \alpha \)-stable Lévy process is a special case of symmetric Lévy process, this subsection solves fractional Fokker-Planck equation associated with symmetric Lévy process with space-time-dependent coefficient, which extends result of previous subsection.

Definition 3.3. Let \( L(t) \) be a symmetric Lévy process satisfying the following SDE
\[
dL(t) = \int_B x \tilde{N}(dt, dx) + \int_{B^c} x \tilde{N}(dt, dx)
\]
(3.38)
where Lévy symbol $\eta(u) = \int_{\mathbb{R}} \{\cos(u, x) - 1\} \nu(dx)$ and $\nu$ is a symmetric Lévy measure.

**Theorem 3.4.** Suppose that the standard Brownian motion $B(t)$, the symmetric Lévy process $L(t)$ as defined in definition 3.3, and subordinator $T_\Psi(t)$ are independent, where $T_\Psi(t)$ has Laplace exponent $\Psi(u)$ and its inverse $S_\Psi(t)$. Let $Y(t)$ be the solution of the stochastic equation

\[
\begin{equation}
\frac{dY(t)}{dt} = F(Y(t), T_\Psi(t)) dt + \sigma(Y(t), T_\Psi(t)) dB(t) + h(Y(t), T_\Psi(t)) dL(t), \quad t \geq 0,
\end{equation}
\]

where the function $F(x, t)$, $\sigma(x, t)$, $h(x, t) \in C^2(\mathbb{R}^2)$ satisfy the Lipschitz condition. Assume that the PDF of the process $(Y(t), T_\Psi(t))$, $p(t, y, z)$ exists. Furthermore, we assume that $\frac{\partial}{\partial y} p_t(y, z)$, $\frac{\partial}{\partial y} p_t(y, z)$, $\frac{\partial^2}{\partial y^2} p_t(y, z)$ exist,

\[
\int_0^\infty \int_0^\infty \left| \frac{\partial}{\partial t} p_s(x, t) \right| dsdt < \infty
\]

for each $t_0 > 0$,

\[
\int_1^x \int_0^\infty \left| \frac{\partial}{\partial x} p_s(x, t) \right| dsdx < \infty, \quad \int_1^x \int_0^\infty \left| \frac{\partial^2}{\partial x^2} p_s(x, t) \right| dsdx < \infty,
\]

for each $x_1, x_2 \in \mathbb{R}$ and

\[
\int_0^\infty \int_1^x |p_0(x + r, t) - p_0(x, t)| \nu'(dr)ds < \infty.
\]

for each $x \in \mathbb{R}$, where $\nu'(B) = \nu(\{x; x h(y, z) \in B\})$.

Then, the PDF of the process $X(T) = Y^{-}(S_\Psi(t))$ is the weak solution of fractional Fokker-Planck equation

\[
\frac{\partial q(x, t)}{\partial t} = \left[ -\frac{\partial}{\partial x} F(x, t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} \sigma^2(x, t) + T^s \right] \Phi \nu q(x, t),
\]

with $q(x, 0) = \delta(x)$, where

\[
T^s f(x, t) = \int_{\mathbb{R}} f(x + r, t) - f(x, t) \nu'(dr),
\]

for any $f(x, t) \in C^2_0(\mathbb{R}^2)$.

**Proof.** We follow the same steps as in the proof of Theorem 5.2 and modify the part related to $T_\alpha$. Let $L(t)$ be a symmetric Lévy process as defined in definition 3.3, it has self-adjoint infinitesimal generator

\[
(T f)(y) = \int_{\mathbb{R}} [f(y + x) - f(y)] \nu(dx)
\]

for each $f \in C^2_0(\mathbb{R})$, and Lévy symbol

\[
\eta(u) = \int_{\mathbb{R}} [\cos(u, x) - 1] \nu(dx).
\]

Define $L_h(t) = L(t) h(Y(t), Z(t))$, since theorem 3.4 involves $h(Y(t), T_\Psi(t)) dL(t)$ instead of just $dL(t)$, by Proposition 11.10 of [20] and Corollary 3.4.11 of [1], $L_h(t)$ has Lévy symbol

\[
\eta_h(u) = \int_{\mathbb{R}} [\cos(u, x) - 1] \nu'(dx),
\]

also $\nu'$ is symmetric, hence $L_h$ has self-adjoint generator

\[
T^s f(y, z) = \int_{\mathbb{R}} [f(y + x, z) - f(y, z)] \nu'(dx).
\]
It follows that
\[
\int_{\mathbb{R}^2} p_t(y, z) T^s f(y, z) dy dz = \int_{\mathbb{R}^2} f(y, z) T^s p_t(y, z) dy dz
\]
(3.49)
\[
= \int_{\mathbb{R}^2} f(y, z) \int_{\mathbb{R} - \{0\}} [p_t(y + x, z) - p_t(y, z)] \nu'(dx) dy dz,
\]
so,
\[
T^s p_t(y, z) = \int_{\mathbb{R} - \{0\}} [p_t(y + x, z) - p_t(y, z)] \nu'(dx)
\]
(3.50)

Since \( \Gamma^+ = A^+ + T^{s+} \) and \( \frac{\partial}{\partial t} p_t(y, z) = \Gamma^+ p_t(y, z) \), we have
\[
\frac{\partial}{\partial t} p_t(y, z) = \Gamma^+ p_t(y, z)
\]
(3.51)
\[
= - \frac{\partial}{\partial y} (F(y, z) p_t(y, z)) + \frac{\partial^2}{\partial y^2} \left( \frac{1}{2} \sigma^2(x, t) p_t(y, z) \right) - \frac{\partial}{\partial z} \Theta z p_t(y, z)
\]
\[
+ \int_{\mathbb{R} - \{0\}} [p_t(y + x, z) - p_t(y, z)] \nu'(dx)
\]

Similar as proof of Theorem 3.4, we have \( (3.28) \) and \( (3.29) \), plus \( \lim_{s \to \infty} p_s(x, t) = 0 \) and \( p_0(x, t) = \mathbb{1}_{(0,0)}(x,t) \),
\[
\frac{\partial}{\partial t} q(x, t) = \int_0^\infty \left[ - \frac{\partial}{\partial x} (F(x, t) p_s(x, t)) + \frac{\partial^2}{\partial x^2} \left( \frac{1}{2} \sigma^2(x, t) p_s(x, t) \right) - \frac{\partial}{\partial s} p_s(x, t) \right] ds
\]
\[
+ \int_0^\infty \int_{\mathbb{R} - \{0\}} [p_s(x + r, t) - p_s(x, t)] \nu'(dr) ds
\]
\[
= - \frac{\partial}{\partial x} \left[ F(x, t) \int_0^\infty p_s(x, t) ds \right] + \frac{\partial^2}{\partial x^2} \left[ \frac{1}{2} \sigma^2(x, t) \int_0^\infty p_s(x, t) ds \right]
\]
\[
+ \int_{\mathbb{R} - \{0\}} \left[ \int_0^\infty p_s(x + r, t) ds - \int_0^\infty p_s(x, t) ds \right] \nu'(dr)
\]
\[
= - \frac{\partial}{\partial x} \left[ F(x, t) \int_0^\infty p_s(x, t) ds \right] + \frac{\partial^2}{\partial x^2} \left[ \frac{1}{2} \sigma^2(x, t) \int_0^\infty p_s(x, t) ds \right] + T^s \int_0^\infty p_s(x, t) ds
\]
\[
= - \frac{\partial}{\partial x} \left[ F(x, t) \Phi_t w(x, t) \right] + \frac{\partial^2}{\partial x^2} \left[ \frac{1}{2} \sigma^2(x, t) \Phi_t q(x, t) \right] + T^s \Phi_t q(x, t)
\]
\[
= \left[ - \frac{\partial}{\partial x} F(x, t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} \sigma^2(x, t) + T^s \right] \Phi_t q(x, t)
\]
\]

\[\square\]

**Remark 3.5.** \( (1.17) \) is a special case of \( (3.43) \) when Lévy measure in Theorem 3.4 is defined as in Definition 3.7.

3.3. **Fractional Fokker-Planck equation with a general Lévy generator.** Now we solve fractional Fokker-Planck equation associated with a general Lévy process.

**Definition 3.6.** Let \( L(t) \) be a Lévy process satisfying the following SDE
\[
dL(t) = \int_B x \tilde{N}(dt, dx) + \int_{B^c} x \tilde{N}(dt, dx)
\]
(3.53)
with Lévy symbol $\eta(u) = \int_{\mathbb{R}} [e^{ixu} - 1 - iux \mathbb{1}_B(x)]\nu(dx)$, where $\nu$ is a Lévy measure.

**Theorem 3.7.** Suppose that the standard Brownian motion $B(t)$, the Lévy process $L(t)$ as defined in definition 3.6, and subordinator $T_{\Psi}(t)$ are independent, where $T_{\Psi}(t)$ has Laplace exponent $\Psi(u)$ and its inverse $S_{\Psi}(t)$. Let $Y(t)$ be the solution of the stochastic equation

$$
\begin{align*}
\frac{dY(t)}{dt} &= F(Y(t), T_{\Psi}(t))dt + \sigma(Y(t), T_{\Psi}(t))dB(t) + h(Y(t), T_{\Psi}(t))dL(t), \quad t \geq 0, \\
Y(0) &= 0, \quad T_{\Psi}(0) = 0
\end{align*}
$$

where the function $F(x, t), \sigma(x, t) \in C^2(\mathbb{R}^2)$, $h(x, t) \in C^\infty(\mathbb{R}^2)$ satisfy the Lipschitz condition. Assume that the PDF of the process $(Y(t), T_{\Psi}(t))$, $p_t(y, z)$ exists. Furthermore, we assume that $\frac{\partial}{\partial r}p_t(y, z), \frac{\partial^k}{\partial y^k}p_t(y, z), k \in \mathbb{N}^+$ exist,

$$
\int_0^{t_0} \int_0^\infty \left| \frac{\partial}{\partial t}p_s(x, t) \right| dsdt < \infty
$$

for each $t_0 > 0$,

$$
\int_0^{x_2} \int_0^\infty \left| \frac{\partial}{\partial x}p_s(x, t) \right| dsdx < \infty, \quad \int_0^{x_2} \int_0^\infty \left| \frac{\partial^2}{\partial x^2}p_s(x, t) \right| dsdx < \infty,
$$

for each $x_1, x_2 \in \mathbb{R}$ and

$$
\int_0^\infty \int_{\mathbb{R} - \{0\}} \sum_{k=1}^{\infty} \frac{(-r)^k}{k!} \frac{\partial^k}{\partial x^k}(p_s(x, t)h(x, t)^k) \nu(dr)ds < \infty.
$$

for each $x \in \mathbb{R}$.

Then, the PDF of the process $X(T) = Y^-(S_{\Psi}(t))$ is the weak solution of fractional Fokker-Planck equation

$$
\frac{\partial}{\partial t} q(x, t) = \left[ -\frac{\partial}{\partial x} F(x, t) + \frac{\partial^2}{\partial x^2} \frac{1}{2} \sigma^2(x, t) + T^{\downarrow+} \right] \Phi_t q(x, t),
$$

with $q(x, 0) = \delta(x)$, where

$$
T^{\downarrow+} f(x, t) = \int_{\mathbb{R} - \{0\}} \left[ \sum_{k=1}^{\infty} \frac{(-r)^k}{k!} \frac{\partial^k}{\partial x^k}(h(x, t)^k f(x, t) + \mathbb{1}_B(r, t)r \frac{\partial}{\partial x}(h(x, t)f(x, t)) \right] \nu(dr),
$$

for any $f(x, t) \in C^\infty_0(\mathbb{R}^2)$.

**Remark 3.8.** The operator $T^{\downarrow+}$ appears naturally in Sun and Duan [30]. Theorem 2-14 in [4] guarantees the existence of density $p_t(y, z)$ by requiring some regularity on coefficients as follows, $F(x, t), \sigma(x, t), h(x, t) \in C^3(\mathbb{R}^2)$ have bounded partial derivatives from order 0 to 3, $\sup_x |D^p_\nu(xh(x, t))| \in L^p(\mu, \nu)$ for all $p \geq 2$.

**Proof of Theorem 3.7.** We follow the same steps as in the proof of Theorem 3.2 and modify the part related to $T_{\Psi}$.

Let $L = (L(t), t \geq 0)$ be a Lévy process as defined in definition 3.6, it has infinitesimal generator

$$
T^I f(y, z) = \int_{\mathbb{R} - \{0\}} \left[ f(y + xh(y, z), z) - f(y, z) + \mathbb{1}_B(x, z) xh(y, z) \frac{\partial}{\partial y} f(y, z) \right] \nu(dx)
$$

By equation (32) in [30],

$$
T^{\downarrow+} p_t(y, z) = \int_{\mathbb{R} - \{0\}} \left[ \sum_{k=1}^{\infty} \frac{(-x)^k}{k!} \frac{\partial^k}{\partial y^k}(h(y, z)^k p_t(y, z)) + \mathbb{1}_B(x, z) x \frac{\partial}{\partial y}(h(y, z)p_t(y, z)) \right] \nu(dx)
$$
Since \( \Gamma^+ = A^+ + T_t^+ \) and \( \frac{\partial}{\partial y} p_t(y, z) = \Gamma^+ p_t(y, z) \), we have
\[
\frac{\partial}{\partial t} p_t(y, z) = -\frac{\partial}{\partial y} (F(y, z)p_t(y, z)) + \frac{\partial^2}{\partial y^2} \left[ \frac{1}{2} \sigma^2(y, z)p_t(y, z) \right] - \frac{\partial}{\partial z} \Theta_p(y, z)
\]
\[
+ \int_{\mathbb{R} - \{0\}} \left[ \sum_{k=1}^{\infty} \frac{(-x)^k}{k!} \frac{\partial^k}{\partial y^k} (h(y, z)^k p_t(y, z)) + \mathbb{1}_B(x, z) \frac{\partial}{\partial y} (h(y, z)p_t(y, z)) \right] \nu(dx)
\]

Similar as proof of Theorem 3.5, we can derive \( 3.52 \) and \( 3.54 \), combined with \( \lim_{s \to \infty} p_s(x, t) = 0 \) and \( p_0(x, t) = 1_{\{0,0\}}(x, t) \),
\[
\frac{\partial}{\partial t} w(x, t) = \int_0^\infty \left[ \frac{\partial}{\partial x} \left( \Phi_t F(x, t) \right) p_s(x, t) \right] \nu(ds)
\]
\[
+ \sum_{k=1}^{\infty} \frac{(-x)^k}{k!} \frac{\partial^k}{\partial x^k} \left( h(x, t)^k \int_0^\infty p_s(x, t) ds \right) + \mathbb{1}_B(x, t) \frac{\partial}{\partial x} \left( h(x, t) \int_0^\infty p_s(x, t) ds \right) \nu(dr)
\]
\[
= \left[ -\frac{\partial}{\partial x} F(x, t) + \frac{\partial^2}{\partial x^2} \frac{1}{2} \sigma^2(x, t) \right] \Phi_t q(x, t)
\]
\[
+ \sum_{k=1}^{\infty} \frac{(-x)^k}{k!} \frac{\partial^k}{\partial x^k} \left( h(x, t)^k \Phi_t q(x, t) \right) + \mathbb{1}_B(x, t) r \frac{\partial}{\partial x} (h(x, t) \Phi_t q(x, t)) \nu(dr)
\]
\[
= \left[ -\frac{\partial}{\partial x} F(x, t) + \frac{\partial^2}{\partial x^2} \frac{1}{2} \sigma^2(x, t) + T^1 \right] \Phi_t w(x, t)
\]

\[\square\]

**Remark 3.9.** In Theorem 3.4, when the space-time-dependent coefficient \( h(x, t) \) of pure jump Lévy process only depends on time \( t \), say \( h(t) \), then
\[
T_t^+ \Phi_t q(x, t) = \int_{\mathbb{R} - \{0\}} \left[ \sum_{k=1}^{\infty} \frac{(-r)^k}{k!} \frac{\partial^k}{\partial x^k} (h(t)^k \Phi_t q(x, t)) + \mathbb{1}_B(x, t) r \frac{\partial}{\partial x} (h(t) \Phi_t q(x, t)) \right] \nu(dr)
\]
\[
= \int_{\mathbb{R} - \{0\}} \left[ \sum_{k=1}^{\infty} \frac{(-r h(t))^k}{k!} \frac{\partial^k}{\partial x^k} (\Phi_t q(x, t)) + \mathbb{1}_B(x, t) r h(t) \frac{\partial}{\partial x} \Phi_t q(x, t) \right] \nu(dr)
\]
\[
= \int_{\mathbb{R} - \{0\}} \left[ \Phi_t q(x - r h(t), t) - \Phi_t q(x, t) + \mathbb{1}_B(x, t) r h(t) \frac{\partial}{\partial x} \Phi_t q(x, t) \right] \nu(dr),
\]
thus, \( 3.38 \) becomes
\[
\frac{\partial}{\partial t} q(x, t) = \left[ -\frac{\partial}{\partial x} (F(x, t) + \frac{\partial^2}{\partial x^2} \frac{1}{2} \sigma^2(x, t)) \right] \Phi_t q(x, t),
\]
\[
+ \int_{\mathbb{R} - \{0\}} \left[ \Phi_t q(x - r h(t), t) - \Phi_t q(x, t) + \mathbb{1}_B(x, t) r h(t) \frac{\partial}{\partial x} \Phi_t q(x, t) \right] \nu(dr),
\]
which corresponds \( 1.9 \).
 Remark 3.10. The methods that Magdziarz and Zorawik used to calculate the adjoint of infinitesimal generator of the Lévy process when the coefficient of the Lévy noise depends only on time is substitution and integration by parts, (2.11) in [14]. Such a method does not work when coefficient of Lévy noise depends on both time and space; however, using the self-adjointness of the infinitesimal generator of symmetric Lévy process and the method in [30] for general Lévy process, we can figure out the adjoint operator for Lévy noise with space-time-dependent coefficients. Thus, Theorem 3.7 extends [14] and provides stochastic solution of fractional Fokker-Plank equation (3.58) describing subdiffusion in full generality.

Remark 3.11. Simulations of paths of stochastic processes play an important role in applications. Results in this paper provide a useful way for obtaining approximate solutions of fractional Fokker-Planck equations mentioned above. Using Monte Carlo methods based on realization of $X(t)$, our results can be used to approximate solutions of fractional Fokker-Planck equations (1.17), (3.43), and (3.58), see [8], [10], [11], [23]. Also our results can be used to obtain solution of equations with particle tracking methods, see [2], [4], [19].

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