Truncated total variation image denoising model based on fractional B-spline wavelet

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Abstract. In order to suppress the noise and keep the clear structure, edges and details of the image under noisy images, a truncated total variation (truncated TV) image denoising model based on the fractional wavelet domain is proposed. The fractional B-spline wavelet can well describe the features of fractional singularity such as fine textures in the image. The truncated TV model is based on the TV model, which solves the shortcomings of the TV model's large penalty amplitude and reduced image contrast. The model first performs the fractional B-spline wavelet transform on the image, applies truncated TV to the fractional wavelet domain to process the image, and then performs the inverse transform of the fractional B-spline wavelet transform to obtain the denoised image. The experimental results of image noise reduction show that compared with some existing image noise reduction methods, the proposed noise reduction model can effectively suppress noise and maintain the structure and details of the image. The application of the model to other fields in the future remains to be studied.

1. Introduction

The purpose of image noise reduction is to restore the original real image based on the existing degraded noisy image estimation, that is, to effectively maintain the important structural information in the image while removing noise. Image noise reduction methods are roughly divided into two categories: spatial domain methods and transform domain methods. The spatial domain noise reduction methods directly process the gray values of image pixels. The methods include L0 smoothing filtering methods [1], weighted least squares (WLS) [2], total variation (TV) method [3] and other filtering algorithms. In addition, the above-mentioned algorithms have also produced many variants, such as the truncated TV method [4]; the transform domain noise reduction method is the transformation coefficients of the image are processed in the transform domain, which is based on a certain sparseness of the image after a certain transformation. Commonly used are Fourier transform, discrete cosine transform, wavelet transform and multi-scale geometric analysis methods, etc., among which the wavelet threshold shrinkage method is a widely used method.

The TV method tends to produce gradients and reduce the quality of image restoration. Since the gradient is often a high-frequency component in the wavelet domain, the wavelet shrinkage method can suppress the gradient caused by the TV model to a certain extent [5]. Therefore, the combination of wavelet and TV model is an effective image denoising method [6]. In 2000, Unser proposed the fractional B-spline function [7], which extended the original integer order B-spline to the fractional order, and constructed a fractional B-spline wavelet. Fractional B-spline wavelets have good properties...
such as attenuation, two-scale equations and fractional approximation. The truncated TV model proposed in 2017 is based on the TV model. Aiming at the shortcomings of the TV model of large punishment and reduced image contrast, an improved model is proposed [4]. Based on the characteristics of the fractional B-spline wavelet transform and the truncated TV model, this paper combines the fractional B-spline wavelet transform with the truncated TV model, and proposes a truncated total variation image noise reduction model based on the fractional B-spline wavelet (referred to as Fractional B-spline-truncated TV model). Experiments apply the proposed method and several other image denoising methods to images containing noise. The results of comparison experiments show that the proposed method can remove noise more effectively and restore the real image more effectively.

2. Mathematical theory

2.1. Fractional B-spline wavelet
By analogy with the classical B-splines, we define the fractional causal B-splines by taking the \((\alpha + 1)\)th fractional difference of the one-sided power function [7]:

\[
\beta_{\alpha}(x) = \frac{1}{\Gamma(\alpha + 1)} \Delta_{\alpha}^{+ \alpha} \epsilon_{\alpha} = \frac{1}{\Gamma(\alpha + 1)} \sum_{k} (-1)^{\frac{\alpha+1}{k}} (x-k)_{\alpha},
\]

(1)

Where \(\Gamma(\alpha + 1) = \int_0^\infty x^{\alpha-1} e^{-x} dx\), \((x-k)_{\alpha} = (\max(x-k, 0))^\alpha\), \(\alpha = \frac{\Gamma(\alpha + 1)}{\Gamma(k+1)\Gamma(\alpha-k+1)}\).

Fractional B-spline function has attenuation, Riesz basis, approximation, and satisfies two-scale equations. Therefore, the fractional B-spline function cluster can form a multi-resolution analysis, so that the corresponding wavelet basis function can be constructed and the image can be fractional B-spline wavelet transform.

2.2. Truncated Total Variation (Truncated TV)
Truncated TV is represented as [4]

\[
U(x) = \int (T(u) + T(u)) dx dy
\]

(2)

\[
T(u) = \begin{cases} \|u\|_p, & \|u\| < \varepsilon \\ \varepsilon, & \text{else} \end{cases}
\]

(3)

Where \((u, u)\) is the gradient of \(u\). Truncated TV only penalizes gradients whose amplitudes are smaller than the threshold \(\varepsilon\), while for those whose magnitudes are greater than \(\varepsilon\), the proposed penalty doesn’t penalize them. The truncated TV is non-convex and difficult to optimize due to its “truncated” shape. But equation (3) can be re-represented using \(l_1\) regularization. Taking \(\varepsilon\) as the parameter, the \(T(x)\) defined in equation (3) is equivalent to

\[
\phi(u, l) = \min\{\varepsilon \|l\|_1 + \|u-l\|_1\}
\]

(4)

Where \(\|l\|_p\) is zero power operator, i.e. \(\|l\|_p = 1\) if \(l \neq 0\), else \(\|l\|_p = 0\).

3. Truncated total variation image denoising model based on fractional B-spline wavelet
Combining the fractional B-spline wavelet transform and truncated TV, the noise reduction model in this paper is shown in equation (5):

\[
u = \arg\min_{u, l_1, l_2} \|W_{\text{raw}} f - W_{\text{raw}} u\|_2 + \alpha \|\phi(W_{\text{raw}} u, l_1)\|_1 + \alpha \|\phi(W_{\text{raw}} u, l_1)\|_1
\]

(5)

The first term on the right side of the above formula is the data fidelity term, and the second term is the fractional B-spline-truncated TV term. Where \(f\) is the noise-containing image to be processed, \(u\) is the image after noise reduction, \(W_{\text{raw}}\) is the fractional B-spline wavelet transform, and \(\alpha\) is the parameter.
of the truncated TV regular term defined in the wavelet domain, \( \| \phi(W_{u_{1}, l_{1}}) + \phi(W_{u_{2}, l_{2}}) \|_{h} \) defined as:

\[
\left\| \phi(W_{u_{1}, l_{1}}) + \phi(W_{u_{2}, l_{2}}) \right\|_{h} = \sum_{i,j} \left( \min \left| \phi(w_{i,j}) + |W_{u_{1}, l_{1}}| + \min \left| \phi(w_{i,j}) + |W_{u_{2}, l_{2}}| \right| \right) \]

(6)

Where \( \phi(W_{u_{1}, l_{1}}) = \min \left| \phi(w_{i,j}) + |W_{u_{1}, l_{1}}| \right| \) and \( \phi(W_{u_{2}, l_{2}}) = \min \left| \phi(w_{i,j}) + |W_{u_{2}, l_{2}}| \right| \).

Directly minimizing functional equation (5) is difficult because it involves \( l_{1} \) and \( l_{2} \) penalty terms. We adopt an alternating optimization strategy with split Bregman framework [8]. We introduce two dual variables \( b_{1} \) and \( b_{2} \), corresponding to \( W_{u_{1}, l_{1}} \) and \( W_{u_{2}, l_{2}} \) respectively, and re-express the objective function as follows:

\[
\min \left\{ \sum_{i,j} \|W_{f_{i,j}} - W_{u_{1}, l_{1}}\|^2 + \alpha \sum_{i,j} \|b_{1,i,j}\|^2 + \|b_{2,i,j}\|^2 \right\} + \frac{\lambda}{2} \sum_{i,j} \left( b_{1,i,j} - W_{u_{1}, l_{1}} + b_{2,i,j} - W_{u_{2}, l_{2}} \right)^2
\]

(7)

where \( b_{1} = W_{u_{1}, l_{1}} - l_{1}; b_{2} = W_{u_{2}, l_{2}} - l_{2} \).

By using Lagrangian multipliers, equation (8) can be converted to an unconstraint problem:

\[
\min \left\{ \sum_{i,j} \|W_{f_{i,j}} - W_{u_{1}, l_{1}}\|^2 + \alpha \sum_{i,j} \|b_{1,i,j}\|^2 + \|b_{2,i,j}\|^2 \right\} + \frac{\lambda}{2} \sum_{i,j} \left( b_{1,i,j} - W_{u_{1}, l_{1}} + b_{2,i,j} - W_{u_{2}, l_{2}} \right)^2
\]

(9)

By introducing Bregman distance, equation (9) becomes:

\[
\min \left\{ \sum_{i,j} \|W_{f_{i,j}} - W_{u_{1}, l_{1}}\|^2 + \alpha \sum_{i,j} \|b_{1,i,j}\|^2 + \|b_{2,i,j}\|^2 \right\} + \frac{\lambda}{2} \sum_{i,j} \left( b_{1,i,j} - W_{u_{1}, l_{1}} + b_{2,i,j} - W_{u_{2}, l_{2}} \right)^2
\]

(10)

The above joint minimizing problem can be solved alternately by decoupling it into several subproblems.

4. Experimental results and analysis

4.1. Experimental data

Figure 1. Comparison of the desired effect of the image to be processed and the noise reduction achieved.

Use MATLAB R2017a software to realize the real image noise reduction experiment on the computer side (Intel(R)Core(TM)i3-5005U CPU @2.00GHz, 4.00GB). The CT reconstructed image of pigeons of real biological samples is used for image denoising experiment, as shown in figure 1, the left is the ideal denoised image, and the right is the noisy image. After comparing the images and partially zooming in, it can be seen that the image contains obvious noise, which makes the internal structure unclear, the detail texture is blurred, and there are obvious striped artifacts on the periphery.
4.2. Evaluation Index
Quantitative evaluation uses peak signal-to-noise ratio (PSNR) and mean square error (MSE) to evaluate the quality of denoised images. The evaluation index PSNR is usually used to evaluate the quality of an image after denoising compared with the original image. The higher the PSNR, the better the denoising effect. Define two values, one is the MSE, the other is the PSNR, the formula is as follows:

\[
MSE(x, y) = \frac{1}{M \times N} \sum_{i=1}^{M} \sum_{j=1}^{N} (x_{ij} - y_{ij})^2
\]

\[
PSNR(x, y) = 10 \log_{10} \left( \frac{\text{Peak}}{\text{MSE}} \right) = 20 \log_{10} \left( \frac{\text{Peak}}{\sqrt{\text{MSE}}} \right)
\]

Among them, \(x\) represents the reference image, \(y\) represents the noise reduction image, \(M \times N\) is the size of \(x\) and \(y\); \(x_{ij}\) and \(y_{ij}\) represent the pixel intensity of \(x\) and \(y\) in a certain pixel \((i, j)\), and \(\text{Peak}\) represents the maximum pixel intensity in the normalized image (Usually the gray level of the image, generally the value is 255). The general value range of PSNR: 20–40, the larger the value, the better the image quality.

4.3. Experimental results
Apply fractional B-spline-truncated TV model, existing TV and truncated TV method process the noisy image to obtain the denoised image.

![Figure 2. Comparison of noise reduction results.](image)

Intuitively from figure 2, when the image to be processed is degraded due to noise, several noise reduction methods can reduce the image noise. After TV and truncated TV noise reduction processing, the internal structure of the image is clearer, but the detailed texture will be lost. The method in this paper can have the advantages of truncated TV and fractional B-spline wavelet transform noise reduction. While effectively reducing noise, it keeps the image structure and detail texture, and the outer edges are more clear.

The quantitative results are shown in table 1. The PSNR and MSE of the noisy image to be processed are 22.3229 and 380.8818, respectively. Adjust all the parameters of several noise reduction algorithms to obtain high PSNR and low MSE. It can be clearly seen that the fractional B-spline-truncated TV model has higher PSNR and lower MSE, better noise reduction and better preservation of image structure.
Table 1. PSNR and MSE of several methods of noise reduction image.

| Method                              | PSNR  | MSE       |
|-------------------------------------|-------|-----------|
| TV                                  | 22.5424 | 362.1094 |
| Truncated TV                        | 22.5537 | 361.1696 |
| Fractional B-spline-truncated TV    | 26.3019 | 152.3685 |

5. Conclusion
In this article, we propose a truncated total variation noise reduction model based on the fractional wavelet domain to suppress noise and maintain the structural details and edges of the image. Through the noise reduction processing of noisy images, the experimental results show that compared with other methods, the model can effectively suppress noise, keep the edges and textures clear, the processed image effect is better, and can maintain a higher PSNR and lower MSE.

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