A number-phase Wigner function

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Abstract

One of the most prominent quasiprobability functions in quantum mechanics is the Wigner function that gives the right marginal probability functions if integrated over position or momentum. Here we depart from the definition of the position-momentum Wigner function to, in analogy, construct a number-phase Wigner function that, if summed over photon numbers gives the correct phase distribution and integrated over phase gives the right photon distribution.

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Over the years, quantum phase space distributions have not only been useful tools that allow to transcribe operator equations into c-number partial differential equations, but have proved to have many other uses, for instance in the reconstruction of the quantum state of a cavity field [1] or the vibrational motion of ions [2]. Among the most important ones, one can mention the Wigner distribution function [3], the Q function [4] and the Glauber-Sudarshan P-function [5], which belong to a more general one-parameter family of quantum phase space distributions.

Because they have an importance on their own, they have been extensively studied, among others by Cahill and Glauber [6] and Agarwal and Wolf [7] and more recently by more authors in a theoretical [8] and experimental manner [2].

Here we would like to show that it can be constructed a Wigner distribution function for phase and number in an analogous way in which is defined for the position-momentum variables. We should point out that phase-number Wigner functions have been introduced by Vaccaro and Pegg [9] using the Pegg-Barnett [10] phase formalism; Luks and Perinova [11] in an enlarged Hilbert space and Vaccaro [12] by using the properties of the position-momentum Wigner function and applying them to the phase-number Wigner function [9].

It should be mentioned that, although in the latest the marginal probabilities for phase and number are correct, figures in phase space are not completely understood (for instance, a coherent state with phase $\phi = 0$ contains also a small "wave" at $\phi = \pi$ [12]).

We start by writing the Wigner function for position and momentum in the form

$$W(\alpha) = \int C(\beta) \exp(\alpha \beta^* - \alpha^* \beta) d^2 \beta,$$  \(1\)

where the characteristic function, $C(\beta)$ is defined as

$$C(\beta) = Tr[\hat{D}(\beta)\hat{\rho}],$$  \(2\)

with $\hat{\rho}$ the system’s density matrix, $\hat{D}(\beta) = \exp(\beta \hat{a}^\dagger - \beta^* \hat{a})$ the Glauber’s displacement operator, and $\hat{a}$ and $\hat{a}^\dagger$ the annihilation and creation operators of the quantised field, respectively. One can further write it in terms of the position and momentum operators such that the characteristic function times the Kernel reads
\[
\hat{C}(\beta_x, \beta_p) = e^{-i \sqrt{2}(\beta_x \hat{p} - \beta_p \hat{x})} e^{\alpha \beta^\ast - \alpha^\ast \beta},
\]
where we have set the frequency of the quantised electromagnetic field and \( \hbar \) equal to one.

In analogy to equation (3) we can define the function
\[
\hat{\tilde{C}}_{\bar{n} - \Phi}(k, \theta) = \frac{1}{2} Tr \left[ \left( \hat{D}_{\bar{n} - \Phi}(k, \theta) e^{-i(k \Phi - n \theta)} + c.c. \right) \hat{\rho} \right],
\]
where [13]
\[
\hat{D}_{\bar{n} - \Phi}(k, \theta) = e^{i \theta k} e^{-i \theta \hat{n}} (\hat{V}^\dagger)^k,
\]
with \( \hat{V}^\dagger = \sum_{k=0}^\infty |k + 1\rangle \langle k| \) the Susskind-Glogower operator [14]. Because there is not a well defined phase operator, one can not use an expression of the form \( \exp [i(k \Phi - \phi \hat{n})] \), and we use instead a "factorized" form in Eq. (5). Note that in order to produce a real Wigner function we added the complex conjugate in (4) (because \( n \) can not be a negative integer).

Eq. (1) does not have this problem because the integrations over \( \beta_x \) and \( \beta_p \) are from \(-\infty\) to \( \infty \). By writing the density matrix in the number state basis,
\[
\hat{\rho} = \sum_{m=0}^\infty \sum_{l=0}^\infty Q_{m,l} |m\rangle \langle l|,
\]
we obtain
\[
\hat{\tilde{C}}_{\bar{n} - \Phi}(k, \theta) = \frac{e^{i \theta k}}{2} \sum_{m=0}^\infty \sum_{l=0}^\infty Q_{m,l} Tr [(\hat{V}^\dagger)^k e^{-i \theta \hat{n}} |m\rangle \langle l|] e^{-i(k \Phi - n \theta)} + c.c.. \]

The double integration over the whole phase space in (1) becomes here a sum and a single integration
\[
W(n, \phi) = \frac{1}{(2\pi)^2} \sum_{k=-n}^\infty \int_0^{2\pi} \hat{\tilde{C}}_{\bar{n} - \Phi}(k, \theta) d\theta.
\]
Inserting equation (7) into (8) we obtain
\[
W(n, \phi) = \frac{1}{4\pi} \sum_{k=-n}^\infty (Q_{n,n+k} e^{-ik\phi} + Q_{n+k,n} e^{ik\phi}).
\]
It is easy to show that integrating (9) over the phase \( \phi \)
\[
\int_0^{2\pi} W(n, \phi) d\phi = Q_{n,n} = P(n), \]
gives the photon distribution. And adding (9) over \( n \)
\[
\sum_{n=0}^{\infty} W(n, \phi) = \frac{1}{4\pi} \sum_{n=0}^{\infty} \sum_{k=-n}^{\infty} (Q_{n,n+k} e^{-ik\phi} + Q_{n+k,n} e^{ik\phi}).
\] (11)
that may be rewritten as (by doing \( m = k + n \))
\[
\sum_{n=0}^{\infty} W(n, \phi) = \frac{1}{2\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} Q_{n,m} e^{-i(m-n)\phi}.
\] (12)
produces the correct phase distribution. It is worth to note that for a number state \( |M\rangle \)
equation (9) reduces to \( W(n, \phi) = \delta_{nM}/2\pi \), i.e. it is different from zero only for \( n = M \) as it should be expected.

A. Coherent state

The phase-number Wigner function for a coherent state
\[
|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}} |m\rangle
\] (13)
is given by
\[
W(n, \phi) = \frac{e^{-|\alpha|^2}}{2\pi \sqrt{n!}} \sum_{k=0}^{\infty} \frac{\alpha^k \cos[(n-k)\phi]}{\sqrt{k!}}.
\] (14)
In Fig. (1) it is plotted the phase-number Wigner function for an amplitude \( \alpha = 4 \) and \( \phi = 0.5 \). Besides being always positive, it may be noted a smooth behavior.

B. Phase-number Wigner function for a Schrödinger cat

A Schrödinger cat, or a superposition of two coherent states may be given by the state
\[
|\psi_\alpha\rangle = \frac{1}{N_\alpha} (|\alpha\rangle + |-\alpha\rangle),
\] (15)
with \( N_\alpha \) is the normalization constant. By writing \( |\psi_\alpha\rangle \) in a number states basis, one finds the phase-number Wigner function as
\[
W(n, \phi) = \frac{e^{-|\alpha|^2}}{2\pi \sqrt{n!} N_\alpha} \sum_{k=0}^{\infty} \frac{\alpha^k (1 + (-1)^k \cos[(n-k)\phi]}{\sqrt{k!}}
\] (16)
In Fig. (2) Eq. (15) is plotted. The oscillatory behavior in the photon number and the negativity , both proper of a Schrödinger cat may be clearly observed. Also the two peaks of the coherent states.
C. A special superposition of number states

Let us consider the state

$$|\phi_M\rangle = \frac{1}{\sqrt{M+1}} \sum_{m=0}^{M} e^{im\phi_0} |m\rangle.$$  \hspace{0.5cm} (17)

This state tends to have a completely well defined phase as $M$ tends to infinity. For this state the phase-number Wigner function reads

$$W(n, \phi) = \frac{1}{2(M+1)\pi} \sum_{k=0}^{M} \cos[(n-k)(\phi - \phi_0)],$$  \hspace{0.5cm} (18)

that may be put in the form [15]

$$W(n, \phi) = \frac{1}{2(M+1)\pi} \cos\left[\left(\frac{M}{2} - n\right)(\phi - \phi_0)\right] \sin\left[\frac{M+1}{2}(\phi - \phi_0)\right] \csc\left(\frac{\phi - \phi_0}{2}\right).$$  \hspace{0.5cm} (19)

In Fig. (3) it is plotted (19) for $M = 20$ and $\phi_0 = 0.7$. It shows a well defined phase. It is also seen that as $\phi$ approaches the value $\phi_0$ the maximum value for the phase-number Wigner function for the state (17) is obtained. From (19) it may be shown that this value is $1/2\pi$. By adding over $n$ equation (19) the phase distribution is obtained

$$P(\phi) = \frac{1}{2(M+1)\pi} \sin^2\left[\frac{M+1}{2}(\phi - \phi_0)\right] \csc^2\left(\frac{\phi - \phi_0}{2}\right)$$  \hspace{0.5cm} (20)

that corresponds to the phase distribution for the state (17).

In conclusion we have constructed a phase-number Wigner function that gives the correct marginal probabilities if added over the photon number or integrated over phase. It was constructed as an analogy to the definition of the position-momentum Wigner function given in (1).

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FIG. 1: Phase-number Wigner distribution function for the state (13). The amplitude of the coherent state is $|\alpha| = 4$ and the phase is 0.5

FIG. 2: Phase-number Wigner distribution function for the Schrödinger cat state (15). $\alpha = 4$.

FIG. 3: Phase-number Wigner function for the quasi-phase state (17) for $M = 20$ and $\phi_0 = 0.7$. 

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Fig. 1/Moya-Cessa

$W(n, \phi)$
Fig. 2/Moya-Cessa

$W(n, \phi)$
Fig. 3/Moya-Cessa

$W(n, \phi)$