Positive Consensus for Heterogeneous Multi-agent Systems *

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Abstract: In this paper, we consider a positive consensus problem for some high-order linear multi-agent systems. Compared with existing consensus results, the most distinct feature of the formulated problem is that the state variables of all heterogeneous agents are confined to the positive orthant. To solve this problem, we first present some auxiliary multi-agent systems as positive local reference generators and then incorporate the reference generator into some applicable decentralized tracking controller for each agent. The proposed two different kinds of distributed algorithms are proven to solve the positive consensus problem fulfilling certain prescribed pattern for this multi-agent system under switching topology. A simulation example is finally given to illustrate the effectiveness of our design.

Keywords: positive consensus, distributed control, multi-agent system, switching topology

1 Introduction

Over the past decades, there has been a tremendous expansion of the research on the multi-agent coordination problem due to its wide applications in sensor networks, robotics, and power systems. Particularly, the fundamental consensus problem has been intensively studied and generalized for various kinds of agent dynamics from integrators to general linear systems and also typical classes of nonlinear ones under different communication topologies, to name a few, [1–7].

In practical applications arising in the areas of chemical process industry, electronic circuit design, communication networks, and biology, we may face an important class of multi-agent systems composed of a group of positive subsystems, in which the state variables of these agents are confined to the positive orthant. Although we might view the whole positive multi-agent systems as a single but large-scale positive system and utilize existing results in [8–13], the coordination of positive multi-agent systems suffers from at least two extra difficulties compared with conventional designs for a single positive plant. First, the controller for each agent (as an individual input channel) is not allowed to use the full state of the whole multi-input multi-output systems and the information flow among the agents should be compatible with some prior (time-varying) structure. Second, we have to ensure the positivity of the state variables of each agent, which might have different dimensions and be affected by others. These two aspects together make both the coordination problem of multiple positive systems much more challenging. As a consequence, compared with the intensive research for conventional multi-agent systems, the distributed coordination results for general positive multi-agent systems are relatively few.

Meanwhile, integrators are typical positive systems. Thus, the classical integrator-type multi-agent systems are naturally interconnected positive multi-agent systems. Although the positive constraint is not required in designing rules for single-integrator multi-agent systems, the fundamental consensus dynamics are indeed positive. In [19], such a positive constraint was explicitly discussed in the consensus...
problem for some positive multi-agent systems. Necessary and sufficient conditions for the consensus of all agents were derived under the positive constraint. Note that the positive consensus problem was done in this work via solving a positive static output feedback stabilization problem for the whole multi-agent system. Further efforts along this technical line have been made in [20–23] using state feedback. Moreover, some authors considered dynamic rules to relax the conditions to ensure a positive consensus, e.g., [24–26]. However, all these positive consensus results are only derived for homogeneous multi-agent systems where all agents share the identical dynamics. Noticing the various kinds of conventional consensus results for different classes of multi-agent systems, it is natural for us to ask whether and how the positive consensus problem can be solved for heterogeneous multi-agent systems.

With this question, we will focus on a group of positive linear multi-agent systems where the agent dynamics are allowed to be different from each other in both system matrices and the dimension of state spaces. We aim at distributed rules for these high-order agents to reach an output consensus corresponding to some predefined pattern with their state variables being positive. Since we have to achieve the expected consensus goal and ensure the positive constraints for each high-order agent simultaneously, the considered positive consensus problem for these heterogeneous agents has some exclusive challenges in contrast to existing consensus results for conventional linear multi-agent systems or positive consensus for homogeneous multi-agent systems.

To overcome the difficulties brought by the positive constraint and heterogeneous agent dynamics, we are motivated by the existing designs in [27–30] and constructively present a two-step procedure. Depending upon different available information, both state feedback and output feedback distributed controllers are proposed for each agent to reach the expected patterned consensus even with switching topology.

Based on the aforementioned observations, the contributions of this paper can be summarized as follows:

- We formulate and solve the positive output consensus problem for a group of heterogeneous high-order multi-agent systems over switching topology. Compared with existing positive consensus results for homogeneous multi-agent systems, we remove the identical agent dynamics requirement and allow the agents to possess different positive dynamics.

- We extend existing positive consensus results to the case where the consensus trajectory can be of some prespecified dynamic pattern. In existing positive consensus results, the considered consensus trajectory is limited to some finite constant. Here we allow the output trajectory of each agent to reach a consensus on some admissible trajectory of another different positive linear system, which takes the conventional static positive consensus problem as special cases.

The remainder of this paper is organized as follows: We first introduce some preliminaries on our notations and positive systems in Section 2. Then we present the formulated positive consensus problem in Section 3. The main results are given in Section 4. We also provide a simulation example to illustrate the effectiveness of our algorithms in Section 5 along with conclusions in Section 6.

2 Preliminary

In this section, we introduce some preliminaries on our notations and positive systems.

Let \( \mathbb{R}^n \) be the \( n \)-dimensional real space. Denote by \( \mathbb{R}^{n \times m} \) the set of all \( n \times m \) matrices with all entries in \( \mathbb{R} \). Let \( I_n \) be the \( n \)-dimensional identity matrix. \( \mathbf{1} \) (or \( \mathbf{0} \)) denotes an all-one (or all-zero) matrix or
vector with proper dimensions. \( \text{col}(a_1, \ldots, a_n) = [a_1^T, \ldots, a_n^T]^T \) for column vectors \( a_i \) (\( i = 1, \ldots, n \)). Let diag\((b_1, \ldots, b_n)\) represent an \( n \times n \) diagonal matrix with diagonal elements \( b_1, \ldots, b_n \). For matrices \( B_1, \ldots, B_n \), we use \( \text{blkdiag}(B_1, \ldots, B_n) \) to represent the block diagonal matrix with diagonal blocks \( B_1, \ldots, B_n \). For a vector \( x \) (or matrix \( A \)), \( ||x|| \) (or \( ||A|| \)) denotes its Euclidean (or spectral) norm.

Let \( \mathbb{R}^n_+ \) be the nonnegative orthant. Denote the set of all \( m \times n \) matrices with each entry in \( \mathbb{R}_+ \) by \( \mathbb{R}^{m \times n}_+ \). We say such matrices are nonnegative and adopt the notation \( A > 0 \) (\( \geq 0 \)). If, in addition, \( A \) has at least one positive entry, we say \( A \) is positive (\( A > 0 \)). If all the entries are positive, we say \( A \) is strictly positive (\( A \gg 0 \)). \( A < 0 \) (or \( \leq 0 \)) if \( -A > 0 \) (\( \geq 0 \)). The nonnegativity, positivity, and strict positivity of vectors can be defined likewise. For a square matrix \( A \in \mathbb{R}^{n \times n} \), it is said to be Hurwitz if all its eigenvalues have negative real parts and be Metzler if the off-diagonal entries are nonnegative.

Consider the following (time-varying) linear system:

\[
y = C(t)x, \quad \dot{x} = A(t)x + B(t)u
\]

(1)

where \( x \) is the \( n \)-dimensional state vector, \( u \) is the \( p \)-dimensional control input, and \( y \) is an \( l \)-dimensional output vector. Here, \( A(t), B(t), C(t) \) are system matrices with compatible dimensions. We say this system is (internally) positive if for any nonnegative initial condition \( x(0) \geq 0 \) and \( u(t) \geq 0 \), it holds that \( x(t) \geq 0 \) and \( y(t) \geq 0 \) for \( t \geq 0 \). It is said to be a Metzler system if \( A(t) \) is Metzler and \( B(t), C(t) \geq 0 \) for almost all \( t \geq t_0 \). When (1) is time-invariant, we say it is positively stabilizable if there exists a gain matrix \( K \in \mathbb{R}^{n \times n} \) such that \( A + BK \) is Metzler and Hurwitz. It is positively detectable if \( (A^T, C^T) \) is positively stabilizable.

Here is a lemma modified from [9].

**Lemma 1** If (1) is a Metzler system then it is positive. Conversely, if (1) is positive and \( A(t), B(t), C(t) \) are continuous, then (1) is a Metzler system.

### 3 Problem Formulation

In this paper, we consider a multi-agent system consisting of \( N \) positive dynamic agents of the following form:

\[
y_i = C_i x_i, \quad \dot{x}_i = A_i x_i + B_i u_i, \quad i = 1, \ldots, N
\]

(2)

where \( x_i \in \mathbb{R}^{n_i}, u_i \in \mathbb{R}^{m_i}, \) and \( y_i \in \mathbb{R}^l \) are state, input, and output of agent \( i \). Moreover, we assume the triple \( (C_i, A_i, B_i) \) is both positively stabilizable and positively detectable for each \( i = 1, \ldots, N \).

We aim at effective controllers for the agents such that this multi-agent system can asymptotically reach a positive output consensus specified by the following pattern:

\[
y_0 = C_0 x_0, \quad \dot{x}_0 = A_0 x_0
\]

(3)

with internal state \( x_0 \in \mathbb{R}^{n_0} \) and output \( y_0 \in \mathbb{R}^l \). In other words, the output trajectory of each agent is expected to reach some solution to this differential equation simultaneously. Since the agents are positive, we assume this consensus pattern is also positive with nontrivial modes.

**Assumption 1** The matrix \( A_0 \) is Metzler with no eigenvalues having negative real parts.

We are interested in distributed designs for this problem and assume the agents can share their own information with others. For this purpose, we utilize an undirected graph \( G = \{N, E, A\} \) to represent
the allowed information flow among them with node set \( N = \{1, \ldots, N\} \), edge set \( E \subset N \times N \), and the symmetric adjacency matrix \( A = [a_{ij}]_{N \times N} \). When agents \( i \) and \( j \) can exchange information, there is an edge between them in this graph \( G \) and \( a_{ij} = a_{ji} = 1 \). Otherwise, \( a_{ij} = 0 \). Node \( i \)'s neighbor set is defined as \( N_i = \{ j \mid (j, i) \in E \} \). We denote \( N^0_i = N_i \cup \{ i \} \). Moreover, we consider the case when the communication topology may be time-varying. To describe the communication constraint precisely, we denote all possible communication graphs among these agents by \( \{G_1, \ldots, G_p\} \) with \( P = \{1, 2, \ldots, p\} \). Consider a strictly increasing sequence of positive constants \( \{t_i\} \) with \( t_0 = 0 \) and \( \lim_{i \to \infty} t_i = \infty \). We suppose \( t_{i+1} - t_i \geq \tau > 0 \) for any \( i = 0, 1, \ldots \) as that in [35]. This sequence divides \([0, \infty)\) into some contiguous time intervals \([t_i, t_{i+1})\). Define a switching signal \( \sigma(t) : [0, \infty) \to P \). It is time-dependent and piecewise constant. During each \([t_i, t_{i+1})\), all the agents can share their information according to graph \( G_{\sigma(t_i)} \).

The considered positive consensus problem in this paper is then formulated as follows.

**Problem 1** Given a multi-agent system [2], graph \( G_p \), and the consensus pattern [3], find distributed controllers of the following form:

\[
    u_i = f_i(t, x_j, \eta_j), \quad \dot{\eta}_i = g_i(t, x_j, \eta_j), \quad j \in N^0_i(t)
\]

with proper smooth functions \( f_i, g_i \) and a compensator \( \eta_i \in \mathbb{R}^{n_0} \) such that, for any initial point \( x_i(0) \geq 0 \), the closed-loop system [2] and [4] satisfies

1) The trajectory of \( x_i(t) \) is always nonnegative, i.e., \( x_i(t) \geq 0 \) for any \( t \geq 0 \).

2) The outputs of these agents reach a patterned consensus specified by [3] in the sense that there exists a constant \( x_{i0} \in \mathbb{R}^{n_0} \) such that, \( e_i(t) \triangleq x_i(t) - y_i(t) \) converges to 0 as \( t \) tends to \( \infty \) with \( y_i(t) = C_0 x_i(t) \) and \( x_i(0) \) the corresponding trajectory of [3] starting from \( x_i(0) = x_{i0} \).

The formulated problem has been extensively discussed in literature without the first requirement. In this case, this problem recovers the well-studied output consensus or synchronization problem [3][28][29][32][34]. Since we have to ensure a positive state constraint simultaneously, the problem is inherently nonlinear. In fact, only a few important attempts have been made in literature for a special case when \( A_0 = 0 \) (i.e., the consensus pattern is a constant point) and the agents are with identical dynamics under the name of positive consensus [19][23][20]. Compared with these results, this formulation takes positive state constraints, general consensus pattern, and heterogeneous agent dynamics together, which will make the problem much more challenging.

Before the main results, we make the two extra assumptions to ensure the solvability of our problem as follows.

**Assumption 2** Each graph \( G_p \) is connected.

**Assumption 3** For each \( i = 1, 2, \ldots, N \), there exist constant matrices \( X_i \in \mathbb{R}^{n_i \times n_0} \) and \( U_i \in \mathbb{R}^{n_i \times 1} \) such that

\[
    A_i X_i + B_i U_i - X_i A_0 = 0
    \]

\[
    C_i X_i - C_0 = 0
\]

Assumption [2] is made to ensure the connectivity of the communication graphs. Under this assumption, the Laplacian \( L_p \) of graph \( G_p \) is positive semidefinite with a simple zero eigenvalue. Assumption [3] is known as the solvability of the regulator equations for plant [2] with an exosystem [3] in the terminology of output regulation [35]. Similar conditions have been widely used in the multi-agent literature, e.g., [27][33][36].
4 Main Result

This section is devoted to the design of effective distributed controllers for each agent to reach the expected positive consensus specified by (3).

4.1 Two-step design scheme

To hurdle the corresponding difficulties, we present a two-step design to solve our problem as follows: We will first develop an auxiliary multi-agent system as local reference generators for each agent to satisfy the expected consensus pattern. Then the final distributed controller to solve the formulated positive consensus problem is given by incorporating the reference generator into some decentralized tracking controller for each agent (2).

For the first step, we consider the following auxiliary multi-agent system with the identical system matrices as the expected pattern dynamics (3):

\[ \dot{\tilde{w}}_i = A_0 w_i + I_n v_i \]  

where \( w_i \in \mathbb{R}^n \) is the virtual state and \( v_i \) the virtual input. Consider the typical distributed controller for agent (2) as 

\[ v_i = \mu \sum_{j=1}^{N} a_{ij}(t)(w_j - w_i). \]

The resultant reference generator is thus given as

\[ w_{av}(t) = \frac{\sum_{i=1}^{N} w_i(t)}{N}, \quad \tilde{w}_i(t) = w_i(t) - w_{av}(t), \quad \text{and} \quad \tilde{w} = \text{col}(\tilde{w}_1, \ldots, \tilde{w}_N). \]

It can be verified that

\[ \dot{\tilde{w}}_{av} = A_0 w_{av} \]  

with an initial point \( w_{av}(0) = \frac{\sum_{i=1}^{N} w_i(0)}{N} \). In other words, the trajectory \( w_{av}(t) \) is an admissible solution to the expected pattern dynamics (3).

Since there are a finite number of graphs fulfilling Assumption 2 \( \lambda_{\min} \{ \lambda_2(L_p) \} \) is well-defined and strictly greater than 0. Here is a key lemma on the performance of virtual positive multi-agent system (7).

**Lemma 2** Suppose Assumptions 1, 2 holds. Let \( \mu \geq \frac{\|A_0\|}{\lambda} + 1 \). Then, along the trajectory of system (7), it holds that \( w_i(t) \geq 0 \) and \( \|\tilde{w}_i(t)\| \leq \|\tilde{w}(0)\|e^{-\lambda t} \) for any initial condition \( w_i(0) \geq 0 \).

**Proof.** To prove this theorem, we put the full multi-agent system (6) into a compact form:

\[ \dot{\tilde{w}} = (I_N \otimes A_0 - \mu L_{\sigma(t)} \otimes I_n_0)w \]

where \( w = \text{col}(w_1, \ldots, w_N) \). Since \( A_0 \) and \(-L_p\) are Metzler for any \( p \in \mathcal{P} \), one can conclude that \((I_N \otimes A_0 - \mu L_p \otimes I_n_0)\) is also Metzler for any positive constant \( \mu \). Then, we can conclude that the positivity of \( w(t) \) for \( t \geq 0 \) by Lemma 1 from any initial point \( w_i(0) \geq 0 \).

Next, we prove the positive consensus of \( w_i(t) \). Recalling equation (5), it is sufficient for us to prove the convergence of \( \tilde{w}_i(t) \) towards \( 0 \) when \( t \to \infty \). Or equivalently, we will prove the exponential stability of the following error system at the origin:

\[ \dot{\tilde{w}} = (I_N \otimes A_0 - \mu L_{\sigma(t)} \otimes I_n_0)\tilde{w} \]  

For this purpose, we introduce two matrices \( M_1 \in \mathbb{R}^{N \times 1} \) and \( M_2 \in \mathbb{R}^{N \times (N-1)} \). Here, \( M_1 = \frac{1}{\sqrt{N}} \) and \( M_2 \) be a matrix such that \( M_1^T M_2 = 0 \), \( M_2^T M_2 = I_{N-1} \), and \( M_1 M_1^T + M_2 M_2^T = I_N \). Let \( \tilde{w}_1 = (M_1^T \otimes I_n_0)\tilde{w} \).
and \( \bar{w}_2 = (M_2^T \otimes I_{n_0}) \bar{w} \). It can be verified that \( \dot{\bar{w}}_1 = (M_1^T \otimes A_0) \bar{w} = 0 \) and \( \bar{w}_1(t) \equiv 0 \) by the definition of \( w_{ai}(t) \) and the property \( 1^T L_p = 0 \) under Assumption 2. This further implies that

\[
\dot{\bar{w}}_2 = [I_{N-1} \otimes A_0 - \mu (M_1^T L_{\sigma(t)} M_2) \otimes I_{n_0}] \bar{w}_2
\]

It can be verified that the matrix \( M_1^T L_{\sigma(t)} M_2 \) is positive definite with only real eigenvalues \( 0 < \lambda_2(L_{\sigma(t)}) \leq \cdots \leq \lambda_N(L_{\sigma(t)}) \) under Assumption 2.

Let us choose a candidate of common Lyapunov function \( V_{\bar{w}} = \frac{1}{2} \bar{w}^T \bar{w} \) for the switched positive linear system (3). Then we check the evolution of \( V_{\bar{w}} \) over each time interval. It can easily be seen that \( V_{\bar{w}} = \frac{1}{2} \bar{w}_1^T \bar{w}_1 + \frac{1}{2} \bar{w}_2^T \bar{w}_2 = \frac{1}{2} \bar{w}_1^T \bar{w}_1 + \frac{1}{2} \bar{w}_2^T \bar{w}_2 \) along the trajectory of the error system (3). We assume \( \sigma(t) = p \) during \([t_k, t_{k+1})\). Since \( M_1^T L_p M_2 \) is positive definite, there must be a unitary matrix \( U_p \in \mathbb{R}^{(N-1) \times (N-1)} \) such that \( U_p^T [M_1^T L_{\sigma(t)} M_2] U_p = D_p \) where \( D_p \) is a diagonal matrix with the eigenvalues \( \lambda_2(L_p), \ldots, \lambda_N(L_p) \) on the diagonal. Let \( \bar{w}_2 = (U_p^T \otimes I_{n_0}) \bar{w}_2 \). During this time interval, we have

\[
\dot{\bar{w}}_2 = (I_{N-1} \otimes A_0 - \mu D_p \otimes I_{n_0}) \bar{w}_2
\]

Note that \( I_{N-1} \otimes A_0 - \mu D_p \otimes I_{n_0} \) is a block diagonal matrix with diagonal blocks of the form \( A_0 - \mu \lambda_i(L_p) I_{n_0} \) for \( i = 2, \ldots, N \). We use the fact that \( \bar{w}_1 \bar{w}_2 = \bar{w}_1^T \bar{w}_2 \) and set \( \mu \) as in the theorem condition. The time derivative of \( V_{\bar{w}} \) along the trajectory of (3) satisfies

\[
\dot{V}_{\bar{w}} = \bar{w}_1 (I_{N-1} \otimes A_0 - \mu D_p \otimes I_{n_0}) \bar{w}_2 \\
\leq (||A_0|| - \mu \lambda_2(L_p)) ||\bar{w}_2||^2 \\
\leq -2 \lambda_2 V_{\bar{w}}
\]

It is verified that this inequality holds for any \( p \in P \) and thus holds over \([0, \infty)\). According to Theorem 3.1 in [13], \( \bar{w}_1(t) \) exponentially converges to zero as \( t \to \infty \). Moreover, solving this inequality gives \( V_{\bar{w}}(t) \leq V_{\bar{w}}(0) e^{-2\lambda_2 t} \). It follows then

\[
||\bar{w}_1(t)|| \leq ||\bar{w}(t)|| = \sqrt{2 V_{\bar{w}}(t)} \leq ||\bar{w}(0)|| e^{-\lambda_2 t}
\]

The proof is thus completed.

According to this lemma, the virtual multi-agent system (3) will reach a positive consensus specified by pattern (3) if letting \( \mu \) be large enough. With this auxiliary agent as a local reference generator for agent (2), we will embed the auxiliary agent into some reference tracking controller for agent (2) to solve the formulated positive patterned consensus problem in the next subsection.

### 4.2 Solvability of positive consensus

We first choose a gain matrix \( K_{1i} \in \mathbb{R}^{1 \times n_i} \) such that \( A_i + B_i K_{1i} \) is Metzler and Hurwitz. Such matrix \( K_{1i} \) indeed exists for agent (2) according to Lemma 6 in [10] and can be practically determined by some linear programming as shown in the references therein. Let \( K_{2i} = U_i - K_{1i} X_i \) under Assumption 3.

The full information controller \( u_i = K_{1i} x_i + K_{2i} x_0 \) is able to drive the output of agent \( i \) to track the reference \( y_0(t) = C_0 x_0(t) \) by Theorem 1.7 in [35].

We combine the preceding reference generator (7) with this full information regulator together and present the final controller for multi-agent system (2) as follows:

\[
\begin{align*}
    u_i &= K_{1i} x_i + K_{2i} w_i \\
    \dot{w}_i &= A_0 w_i + \mu \sum_{j=1}^{N} a_{ij}(t)(w_j - w_i), \quad i \in \mathcal{N}
\end{align*}
\]
with $\mu$ specified as above. It can be verified that the controller is indeed of the form (11) and thus is distributed.

We summarize the main result of this paper as follows.

**Theorem 1** Suppose Assumptions 1–3 hold. The formulated positive consensus problem with pattern (3) for multi-agent system (2) can be solved by distributed controllers of the form (10).

**Proof.** Under the theorem conditions, the composite system of (2) and (10) can be written out as

$$
\begin{align*}
\dot{x}_i &= (A_i + B_iK_{1i})x_i + B_iK_{2i}w_i \\
\dot{w}_i &= A_0w_i + \mu \sum_{j=1}^{N} a_{ij}(t)(w_j - w_i), \quad i \in N \\
y_i &= C_ix_i
\end{align*}
$$

Viewing the $K_{2i}w_i$ as the control input of the $x_i$-subsystem, it is apparently positive. Since $K_{i2} = U_i - K_{1i}X_i \geq 0$ and $w_i \geq 0$ by Lemma 2, the control input for $x_i$-subsystem is nonnegative. Thus for any $x_i(0) \geq 0$, it must hold that $x_i(t) \geq 0$ for any $t \geq 0$. Then, we are left to show that there exists some $x_{00} \geq 0$ such that $e_i(t)$ converges towards $0$ as $t$ tends to $\infty$.

By our two-step design procedure and Lemma 2, we directly set $x_{00} = w_{00}(0)$. Then, $e_i(t) = C_i\dot{x}_i(t) - C_0w_{00}(t)$. Performing the coordinate transformation $\tilde{x}_i = x_i - X_iw_{00}$, we have

$$
\begin{align*}
\dot{\tilde{x}}_i &= (A_i + B_iK_{1i})\tilde{x}_i + B_iK_{2i}\tilde{w}_i \\
\dot{\tilde{w}} &= (I_N \otimes A_0 - \mu L_{\sigma(t)} \otimes I_{\alpha_0})\tilde{w} \\
e_i &= C_i\tilde{x}_i
\end{align*}
$$

(11)
The rest proof is a direct application of the input-to-state stability of the $\tilde{x}_i$-subsystem. Here, we like to provide some elementary self-contained proofs as follows.

Since $A_i + B_iK_{1i}$ is Hurwitz by the choice of $K_{1i}$, there must be a unique positive definite matrix $P_i \in \mathbb{R}^{n_i \times n_i}$ satisfying the following Lyapunov equation:

$$(A_i + B_iK_{1i})^TP_i + P_i(A_i + B_iK_{1i}) = -2I_{n_i}$$

Let us choose $V_i = \tilde{x}_i^TP_i\tilde{x}_i + \iota_iV_\omega$ with a constant $\iota_i > 0$ to be determined later. It is positive definite with respect to $\tilde{x}_i$ and $\tilde{w}$. Recalling the proofs in Lemma 2 we take its derivative along the trajectory of (11) and obtain

$$
\begin{align*}
\dot{V}_i &= 2\tilde{x}_i^TP_i[(A_i + B_iK_{1i})\tilde{x}_i + B_iK_{2i}\tilde{w}_i] + \iota_iV_\omega \\
&\leq -2||\tilde{x}_i||^2 + 2||P_iB_iK_{2i}||||\tilde{x}_i||||\tilde{w}_i|| - 2\iota_i\Delta V_\omega
\end{align*}
$$

We complete the square and have

$$
\dot{V}_i \leq -||\tilde{x}_i||^2 - 2(\iota_i\Delta - ||P_iB_iK_{2i}||^2)V_\omega
$$

Letting $\iota_i \geq \frac{2}{\Delta} \max\{||P_iB_iK_{2i}||^2, 1\}$ implies

$$
\dot{V}_i \leq -||\tilde{x}_i||^2 - ||\tilde{w}||^2
$$

Recalling Theorem 3.1 in [15], we conclude that $V_i(t)$ and $\tilde{x}_i(t)$ both exponentially converge to $0$ as $t \to \infty$. As a result, the tracking error $e_i(t) = C_i\tilde{x}_i(t)$ also converges to $0$ exponentially fast. This means
that the formulated positive consensus problem for the multi-agent system (2) is indeed solved by the distributed controller (10) and the consensus confirms the pattern (3). The proof is thus complete.

In many circumstances, the state $x_i$ may not be available for us. In this case, we present an output feedback extension for (10) as follows to solve our problem:

$$u_i = K_{1i} \xi_i + K_{2i} w_i$$

$$\dot{\xi}_i = (A_i - K_{3i} C_i) \xi_i + B_i w_i + K_{3i} y_i$$

$$\dot{w}_i = A_0 w_i + \mu \sum_{j=1}^{N} a_{ij}(t)(w_j - w_i), \quad i \in N$$

(12)

where $K_{3i}$ is a chosen gain matrix such that $(A_i - K_{3i} C_i)$ is Metzler and Hurwitz.

The effectiveness of this controller is given as follows.

**Theorem 2** Suppose Assumptions hold. The formulated positive consensus problem with pattern (3) for multi-agent system (2) can be solved by distributed output feedback controllers of the form (12) with $\xi_i(0) = 0$.

**Proof.** The convergence part is similar as that of Theorem 1 and we only have to show $x_i(t) \geq 0$ under the controller (12). Let $\bar{x}_i = x_i - \xi_i$ and put the full composite closed-loop systems as follows:

$$\dot{x}_i = (A_i + B_i K_{1i}) x_i - B_i K_{1i} \bar{x}_i + B_i K_{2i} w_i$$

$$\dot{\bar{x}}_i = (A_i - K_{3i} C_i) \bar{x}_i$$

$$\dot{w}_i = A_0 w_i + \mu \sum_{j=1}^{N} a_{ij}(t)(w_j - w_i), \quad i \in N$$

$$y_i = C_i x_i$$

Since $\bar{x}_i(0) = x_i(0) - \xi_i(0) = x_i(0) \geq 0$, we have $\bar{x}_i(t) \geq 0$ by Lemma 1 and the choice of $K_{3i}$. Then $-B_i K_{1i} \bar{x}_i(t) \geq 0$ from the nonpositivity of $K_{1i}$. Viewing $-B_i K_{1i} \bar{x}_i + B_i K_{2i} w_i$ as the input for the $x_i$-subsystem, we have $x_i(t) \geq 0$ for any $t \geq 0$ according to Lemma 1. The proof is thus complete.

**Remark 1** The two theorems confirm the solvability of the formulated positive consensus problem for multi-agent system (2) by distributed state/output feedback controllers. Compared with conventional consensus results in [1,3,5,29,32,33], we take both the positive constraint and general consensus pattern into consideration and can be taken as their extended counterpart for positive high-order multi-agent systems.

**Remark 2** Compared with existing positive consensus results derived for homogeneous multi-agent systems in [19–26], we have removed such limitations and present a two-step design for this problem. This enables us to handle heterogeneous multi-agent systems whose dynamics can be different from each other in both system matrices and orders over switching communication topologies. Moreover, the expected consensus trajectory of the multi-agent system is allowed to be of a more general prespecified pattern including nonnegative static consensus as special cases.
Figure 1: The communication graphs in our example.

Figure 2: Performance of (10) for static consensus.
5 Simulation

In this section, we consider a heterogeneous four-agent system of the form (2) to illustrate the effectiveness of our preceding designs. Suppose the system matrices are as follows:

\[
A_i = \begin{bmatrix}
-2 & 1 & 1 \\
1 & -3 & 0 \\
1 & 1 & -1
\end{bmatrix}, \quad B_i = \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}, \quad C_i = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}^T, \quad i = 1, 2
\]

and

\[
A_i = \begin{bmatrix}
-2 & 1 \\
0 & 0 \\
0 & 0
\end{bmatrix}, \quad B_i = \begin{bmatrix}
0 \\
1 \\
1
\end{bmatrix}, \quad C_i = \begin{bmatrix}
0 \\
1 \\
1
\end{bmatrix}^T, \quad i = 3, 4
\]

It can be verified by Lemma 1 that these agents are indeed positive. Moreover, they are positively stabilizable and detectable according to Theorem 8 in [10]. Suppose the communication graph is alternatively switching between \(G_1\) and \(G_2\) given in Fig. 1 every 10 seconds. Assumption 2 is then fulfilled.

We first consider the static consensus case with

\[
A_0 = 0, \quad C_0 = 1
\]

In this case, these agents are expected to reach a consensus at some positive constants. We can numerically solve the regulator equations and obtain

\[
X_i = \begin{bmatrix}
1.0000 \\
0.3333 \\
1.6667
\end{bmatrix}, \quad U_i = 0.3333, \quad i = 1, 2
\]

\[
X_i = \begin{bmatrix}
1.0000 \\
2.0000
\end{bmatrix}, \quad U_i = 0, \quad i = 3, 4
\]

Hence Assumption 3 holds as well. We can resort to Theorem 1 to solve this problem by controller (10).

For simulations, we choose \(K_{11} = K_{12} = [0 \ 0 \ -1], K_{13} = K_{14} = [0 \ -1], \) and \(\mu = 3.\) With \(w_i(0) = i\) and all other initial conditions randomly generated between 0 and 10, we list the simulation results in Fig. 2. It can be found that all outputs of these positive agents reach a consensus on \(y_0 = 2.5\) while all components of \(x_1(t)\) and \(x_3(t)\) are nonnegative as we expected.

Next, we consider a nontrivial consensus pattern with

\[
A_0 = \begin{bmatrix}
0.01 & 0.01 \\
0 & 0
\end{bmatrix}, \quad C_0 = \begin{bmatrix}
1 \\
1
\end{bmatrix}^T
\]

(13)

In this case, we can numerically solve the regulator equations (5) and have

\[
X_i = \begin{bmatrix}
1.0000 & 1.0000 \\
0.3322 & 0.3322 \\
1.6778 & 1.6778
\end{bmatrix}, \quad U_i = \begin{bmatrix}
0.3439 \\
0.3623
\end{bmatrix}^T, \quad i = 1, 2
\]

\[
X_i = \begin{bmatrix}
0.4975 & 0.4975 \\
1.0000 & 1.0000
\end{bmatrix}, \quad U_i = \begin{bmatrix}
0.0100 \\
0.0100
\end{bmatrix}^T, \quad i = 3, 4
\]

Then Assumption 3 is confirmed. In this case, we resort to Theorem 2 and employ the output feedback controller (12) to achieve a positive consensus with pattern (13).
In the simulations, we use the same matrices $K_1$, $K_2$, and parameter $\mu$ as above. Choose $K_{31} = K_{32} = [0 0 1]^T$ and $K_{33} = K_{34} = [1 1]^T$. Let $w_i(0) = \text{col}(i - 0.5, i)$, $\xi_i(0) = 0$. The rest of initial conditions are randomly generated between 0 and 10. The simulation results are shown in Fig. 3. It can be found that all agents reach a positive output consensus even the expected consensus trajectory tends to be unbounded in this case. At the same time, the components of $x_1$ and $x_3$ are observed to stay in the positive orthant. These observations verify the effectiveness of the output feedback controller (10) to solve the positive consensus problem for heterogeneous multi-agent system (2).

6 Conclusion

We have formulated and solved the positive consensus problem for a group of heterogeneous positive multi-agent systems. To handle the positive constraint and heterogeneous agent dynamics, we have proposed a two-step design method. By constructively building a local reference generator for each agent, we have developed two different kinds of effective rules for these agents to reach a consensus having the expected dynamic pattern while their states fulfill the positive constraints even under switching communication topologies. In the future, we may consider the same problem but for uncertain nonlinear positive multi-agent systems with more general communication graphs.
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