Collective Field Theory of the Fractional Quantum Hall Edge State and the Calogero-Sutherland Model

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Using hydrodynamic collective field theory approach we show that one-particle density matrix of the \( \nu = 1/m \) fractional quantum Hall edge state interpolates between chiral Luttinger liquid behavior \( \langle \psi^\dagger(r)\psi(0) \rangle \sim r^{-m} \) and Calogero-Sutherland model behavior \( \langle \psi^\dagger(r)\psi(0) \rangle \sim r^{-(m+1)/2} \) as the droplet width is varied continuously. Low-energy excitations are described by \( c = 1 \) conformal field theory of a compact boson of radius \( \sqrt{m} \). The result suggests complementary relation between the two-dimensional quantum Hall droplet and the one-dimensional Calogero-Sutherland model.

Recently there has been renewed interest to the Calogero-Sutherland (CS) model of one-dimensional (1-d) fermions with long-range interactions \( \mathcal{H} \) as an exactly soluble Luttinger liquid. The model may be viewed as an ideal gas of anyons as the interaction is purely statistical \( \mathcal{H} \). In addition dynamical correlation functions of the model have been calculated \( \mathcal{H} \). The CS model also exhibits many similarities with the fractional quantum Hall effect (FQHE) such as the fractional statistics properties of the quasi-holes \( \mathcal{H} \), hierarchical extensions \( \mathcal{H} \) and \( W_\infty \) symmetry \( \mathcal{H} \). As such one naturally inquire whether there exists any deep relation between the FQHE and the CS model.

The FQHE is two-dimensional while the CS model is a 1-d system. Nevertheless it is possible to show that the FQHE restricted to the lowest Landau level (LLL) is directly related to the CS model. For a high external magnetic field \( B \) the FQHE system undergoes a ‘dimensional reduction’ so that the LLL wave function may be interpreted as wave function of a 1-d system \( \mathcal{H} \). Noting that the commutation relations of the guiding center coordinates \( [X,Y] = i/B \) 1-d interpretation of the LLL wave functions is obtained in the basis \( |x\rangle \) of the LLL: \( X|x\rangle = x|x\rangle \), \( Y|x\rangle = p_x/B|x\rangle \). In this representation the X coordinate is diagonalized while the Y coordinate is interpreted as the conjugate momentum to \( X \). In addition operators acting on the LLL can be interpreted as the operators of the 1-d fermion system. Therefore the FQHE droplet may be viewed as occupied phase space of the 1-d fermions.

Using the 1-d representation of the LLL wave functions, the Laughlin state on a cylinder of circumference \( L \) is expressed as \( \mathcal{H} \)

\[
\langle s_1 \cdots s_N | \Psi \rangle = e^{-\pi \sqrt{m} \sum_i \left( \frac{\pi \sqrt{m}}{L} \right)^2 \prod_{i<j} \left( e^{i 2 \pi s_i / L} - e^{i 2 \pi s_j / L} \right)^m. \tag{0.1}
\]

Width of the FQH droplet is \( \delta y = 2 \pi \hbar N m / B L = 2 m \hbar k_F / B \) where \( k_F = \pi N / L \). In the large \( B \) limit, the droplet becomes narrow \( (\delta y \to 0) \) and the Laughlin state coincides exactly with the ground state of the Sutherland model. (The total momentum is shifted to zero in the definition of the above Laughlin state.)

Long distance behavior of the fermion one-particle density matrix for the Laughlin state Eq. \( \mathcal{H} \) is controlled by the boundary excitations of the FQHE since there are no other gapless excitations in the bulk. On general ground Wen \( \mathcal{H} \) obtained the edge state correlation function: \( \langle \psi^\dagger(r)\psi(0) \rangle \sim r^{-m} \) where the filling factor is \( \nu = 1/m \).

On the other hand the fermion correlation for the CS model is given by \( \mathcal{H} \) \( \langle \psi^\dagger(r)\psi(0) \rangle \sim r^{-(m+1)/2} \). Since the Laughlin state coincides with the ground state of the CS model in the narrow channel limit the fermion correlation for edge state of the FQHE system must interpolate its behavior from \( r^{-m} \) to \( r^{-(m+1)/2} \) when the width of the FQHE droplet is adiabatically narrowed by changing either the strength of the magnetic field or the electron number density.

In this letter we show explicitly that the correlation exponent interpolates as anticipated above. Our starting point is the hydrodynamic description of the FQH droplet dynamics using collective field theory. It is a straightforward generalization of the integer QHE case \( \mathcal{H} \). It has been known that the \( \nu = 1/m \) FQHE may be interpreted as arising due to condensation of bosonized fermion \( \psi \) once appropriate flux of Chern-Simons gauge field is attached to it \( \mathcal{H} \). The Lagrangian reads
\[ \mathcal{L} = \bar{\psi} \Pi_0 \psi - \frac{1}{2m_0} \bar{\psi} (\Pi^x + i \Pi^y) (\Pi^x - i \Pi^y) \psi + \mathcal{L}_{CS} \]  

(0.2)

where \( m_0 \) is the fermion mass, \( \Pi_0 = i \partial_0 - A_0 - a_0, \Pi_i = -i \partial_i - A_i - a_i, i = 1, 2 \) and \( \mathcal{L}_{CS} = -(1/4\pi m) \epsilon^{\mu\nu\rho\sigma} a_\mu \partial_\nu a_\rho \). External magnetic field and mechanical potential are described by \( A \) \((-B = \nabla \times A < 0)\) and \( A_0(x,y) \) respectively. In a strong magnetic field the low-energy excitations are restricted to the LLL. Projection to the LLL is made by taking \( m_0 \to 0 \) and yields [16]

\[ \Pi_x - i \Pi_y \psi = 0. \]  

(0.3)

Next we change the bosonized fermion fields \( \psi, \bar{\psi} \) to hydrodynamic fields \( \rho, \theta \) through \( \psi = \sqrt{\rho} e^{i\theta} \). The phase field \( \theta \) consists of a regular part and a singular part: \( \theta = \theta_{\text{reg}} + \theta_{\text{sing}} \). The singular part originates from vortex configurations:

\[ \partial^\mu \theta_{\text{sing}} = \rho^\mu, \quad \nabla \times \mathbf{v} = 2\pi \rho_V, \quad \nabla \cdot \mathbf{v} = 0 \]  

(0.4)

where \( \rho_V \) denotes density of the vortices. Acting \( (\partial_x + i \partial_y) \) on the LLL condition Eq. (0.3) yields two constraint equations in the gauge \( \nabla \cdot \mathbf{A} = 0 \)

\[ \nabla \times \mathbf{v} + \nabla \times \mathbf{A} + \nabla \cdot \mathbf{a} = 0, \]  

(0.5)

\[ \nabla^2 \theta_{\text{reg}} + \nabla \cdot \mathbf{a} = 0. \]  

(0.6)

Imposing Eq. (0.3) the Lagrangian is written as

\[ L = -\int d^2x \left[ \rho(\dot{\theta} + A_0) + \frac{1}{4\pi m} \epsilon^{ij} a_i a_j \right] \]  

(0.7)

where we have integrated over \( a_0 \) and obtained the constraint \( \nabla \times \mathbf{a} = 2\pi \rho m \). Eq. (0.4) implies that the regular part of the phase in Eq. (0.7) cancels the last term. Therefore the reduced Lagrangian of the FQH droplet is

\[ \mathcal{L} = -(v_0 + A_0)\rho \]  

(0.8)

supplemented by the subsidiary condition

\[ B + \frac{1}{2} \nabla^2 \ln \rho - 2\pi \rho_V - 2\pi \rho m = 0. \]  

(0.9)

To investigate low-energy, gapless excitations of the FQH droplet we now quantize the hydrodynamic collective field theory derived above. Ground state of the droplet is determined by the external potential \( A_0 \). Following [14] we choose the potential as \( A_0(x,y) = B^2 y^2/2 + v(x) - \mu \). In the large magnetic field limit Eq. (0.9) is solved approximately by

\[ \rho(x,y,t) = \frac{B}{2\pi m} \theta(y_+(x,t) - y) \theta(y - y_-(x,t)). \]  

(0.10)

Boundary shape of the static ground state droplet depends on the 1-d potential \( v(x) \). The dynamical degrees of freedom are the droplet boundaries \( y_\pm(x,t) \) fluctuating near the ground state configuration.

Solving Eq. (0.10) for \( \theta_{\text{sing}} \) the first term of Eq. (0.8) reads

\[ L_0 = -\int d^2x \, v_0 \rho = 2\pi \int \rho(x) G(x - x') \dot{\rho}_V(x') d^2x d^2x' \]  

(0.11)

where \( G \) is the Green’s function satisfying \( \nabla^2 G = 0, \nabla \times \nabla G(x - x') = \delta^2(x - x') \). Using Eq. (0.10) \( L_0 \) becomes

\[ L_0 = \frac{B^2}{4\pi m} \int d^2x \int d^2x' \, \theta(x - x') \times [y_+(x) y_+(x') - y_-(x) y_-(x')]. \]  

(0.12)

From \( L_0 \) we obtain commutation relations for \( \rho(x), \rho_V(x) \) and, in turn, \( y_\pm(x,t) \)

\[ [y_\pm(x), y_\pm(x')]_{\text{ET}} = \mp i \frac{2\pi m}{B^2} \delta'(x - x'). \]  

(0.13)
Physically \( y_\pm(x, t) \) generate the left and the right moving chiral edge state excitations respectively. The second term of Eq. (0.8) is the Hamiltonian

\[
H = \int \rho A_0 = \frac{B}{2\pi m} \int dx \left[ \frac{B^2}{6} \left( y_+^2(x) - y_-^2(x) \right) + (v(x) - \mu)(y_+(x) - y_-(x)) \right].
\] (0.14)

For small amplitude the Hamiltonian measures elastic curvature energy of \( y_\pm(x, t) \).

From the above collective field theory we now construct operators for low-energy quasi-particle excitations along the boundaries of the FHQ droplet. This is most conveniently described in terms of the left- and right-moving chiral boson fields \( \varphi_\pm \)

\[
y_\pm(x) = \frac{\sqrt{m} \partial \varphi_\pm(x)}{B}. \] (0.15)

The propagator of \( \varphi_\pm \) is obtained from the Eq. (0.12)

\[
\langle \varphi_\pm(x) \varphi_\pm(x') \rangle = -\ln(x - x'). \] (0.16)

Eq. (0.13) gives the commutation relations for \( \varphi \)

\[
[\varphi_\pm(x), \varphi_\pm(x')] = \pm i\pi \text{sign}(x - x'). \] (0.17)

The charge density at the upper and lower chiral boundaries are measured by \( (B/2\pi m)y_\pm(x) \). Thus an operator \( V(x) \) carrying the upper and lower chiral charge \( Q_\pm \) must satisfy

\[
[y_\pm(x), V(x')] = \frac{2\pi m}{B} Q_\pm \delta(x - x')V(x'). \] (0.18)

Statistics of \( V(x) \) is defined by the exchange phase \( \theta \)

\[
V(x)V(x') = \exp(i\theta)V(x')V(x). \] (0.19)

If \( V(x) \) is fermionic, \( \theta \) is quantized as \( (2l + 1)\pi \) for integer-valued \( l \). Let us consider operators of the form

\[
V_{p,q}(x) = : \exp(-ip\sqrt{m}(\varphi_+(x) - \varphi_-(x))/2) - iq(\varphi_+(x) + \varphi_-(x))/\sqrt{m} :. \] (0.20)

Normal ordering is taken with respect to the chiral boson vacuum. It is easily seen that the chiral charge of this operator is \( Q_\pm = (p/2) \pm (q/m) \) and the statistical phase is \( \theta = 2\pi pq \). We have assumed that the elementary excitations are fermions of unit charge. Then \( p \) must be an integer and statistics \( 2\pi pq \) must be odd- or even-integer multiples of \( \pi \) for odd or even \( p \) respectively. Therefore we find that \( p, q \) satisfies the selection rule \( q = p/2 + \mathbb{Z} \). This agrees with the selection rule derived in [12]. Physical meaning of the operators \( V_{p,q} \) is as follows. Chiral charge quantum numbers \( Q_\pm \) imply that the operator \( V_{p,q}(x) \) adds \( p \) fermions and transfers \( q \) units of quasi-hole carrying a fractional charge \(-1/m\) from one boundary to the other. As such state created by this operator carries the momentum \( 2qk_F \). (If one quasi-hole is transferred \( q \) from one boundary to the other the momentum is changed by one \( m \)-th of that for one electron transfer. One electron transfer from one boundary to the other changes momentum \( 2mk_F \) since the maximum (minimum) momentum of the fractional quantum Hall state is \( \pm mk_F \).) We also obtain the scaling dimension

\[
h_{p,q} = \frac{p^2 m}{4} + \frac{q^2}{m} \] (0.21)

from the Hamiltonian Eq. (0.14) or, equivalently, from the operator product expansion

\[
\langle V(x)V(x') \rangle = (x - x')^{-2h_{p,q}}. \] (0.22)

The spectrum, the commutation relations and the Hamiltonian structure indicate that the hydrodynamic collective field theory is a \( c = 1 \) conformal field theory. The dynamical degrees of freedom is a compact boson \( \varphi = \varphi_+ + \varphi_- \).
\( \varphi_- \) of radius \( \sqrt{m} \). The spectrum also exhibits particle-hole duality interchanging \( \sqrt{m} \leftrightarrow 1/\sqrt{m} \) at least in the thermodynamic limit \([17]\).

We now investigate fermion correlations along the boundaries of the FQH droplet. The fermion correlation function (one-particle density matrix) is defined by

\[
\langle \psi^\dagger(x)\psi(x') \rangle = \sum_{\text{int}} \langle 0|\psi^\dagger(x)|\text{int}\rangle \langle \text{int}|\psi(x')|0 \rangle, \tag{0.23}
\]

where \(|\text{int}\rangle\) denotes intermediate states of charge \(-1\) and fermionic statistics. It is easy to see that such states are created by operators \([18]\) as

\[
|l\rangle = V_{-1,l+1/2}|0\rangle, \quad l \in \mathbb{Z}. \tag{0.24}
\]

This state carries momentum \((2l + 1)k_F\), statistics \(\theta = (2l + 1)\pi\) and scaling dimension

\[
h_{-1,l+1/2} = \frac{1}{4} \left( m + \frac{(2l + 1)^2}{m} \right). \tag{0.25}
\]

Since

\[
V_{-1,l+1/2} = : \exp[-\frac{i}{2}(\sqrt{m} + \frac{2l + 1}{\sqrt{m}})\varphi_+(x)] + \frac{i}{2}(\sqrt{m} - \frac{2l + 1}{\sqrt{m}})\varphi_-(x) : \tag{0.26}
\]

the state \(|l\rangle\) consists of a mixture of the two boundary states of different chiralities \(\varphi_+(x), \varphi_-(x)\).

We are interested in the behavior of the correlation function as the width of the FQH droplet is continuously changed. When the droplet width is wider than the magnetic length \(\sim 1/\sqrt{B}\), the two boundary states of opposite chiralities are decoupled each other. Therefore the intermediate states in \(0.23\) must be created only by one of the chiral bosons \(\varphi_+(x)\) or \(\varphi_-(x)\) at each edges. Unique choices of the appropriate operators in \(0.24\) are

\[
V_{-1,m/2} = : \exp[-i\sqrt{m}\varphi_+(x)] \tag{0.27}
\]

and

\[
V_{-1,-m/2} = : \exp[+i\sqrt{m}\varphi_-(x)]. \tag{0.28}
\]

Therefore for FQH droplet with widely separated boundaries the correlation function is given by

\[
\langle \psi^\dagger(r)\psi(0) \rangle \sim r^{-m} \cos(mk_Fr) + \cdots. \tag{0.29}
\]

This agrees with the edge state correlation function of the FQHE obtained previously by Wen \([1]\). As shown in the second paper in \([10]\) the 2-d fermion correlation function in the LLL and in the Landau gauge is expressed in terms of the 1-d fermion correlation as

\[
\langle \psi^\dagger_0(z, \bar{z})\psi_0(z', \bar{z}') \rangle \sim e^{-\frac{B}{2\hbar}[(y-i\hbar\partial_y/B)^2+(y'-i\hbar\partial_y/B)^2]} \times \langle \psi^\dagger(0)\psi(0) \rangle \tag{0.30}
\]

where \(\psi_0(z, \bar{z})\) is the LLL fermion operator. It is clear that the 2-d correlation in the long distance limit shows a power-law behavior only when \(y \) and \(y'\) lie on either one of the two boundaries \(y = y' = \pm mk_F/B\) of the FQH droplet. As one moves into the bulk from the boundaries, the 2-d correlation is exponentially damped.

Next consider the limit in which two edges of the FQH droplet are close each other of order of the magnetic length. We assume that the edge state fluctuations remain small so that topology change of the droplet does not take place. In this limit the two edge states with different chiralities interact strongly each other. Accordingly all the states with the correct quantum numbers \(|l\rangle\) can contribute to the intermediate states in Eq. \(0.24\). The long-distance correlation is given by the most relevant operator. Such an operator with the smallest scaling dimension is \(V_{-1,\pm 1/2}\). Since this operator has momentum \(k_F\) and scaling dimension \(h_{-1,\pm 1/2} = (m + 1/m)/4\), the fermion correlation function becomes

\[
\langle \psi^\dagger(r)\psi(0) \rangle \sim r^{-(m+1/m)/2} \cos(k_Fr). \tag{0.31}
\]
This is the Tomonaga-Luttinger liquid correlation behavior with $\cos(k_Fr)$ oscillation. The exponent is the same as the one found in [12] for the CS model. In deriving Eq. (0.31) we have assumed that there are no further mixing between the left- and the right-handed edge states by, for instance, impurities on the edges. Therefore we have shown that the fermion correlation function along the boundaries of the FQH droplet reproduces that of the chiral Luttinger liquid of the edge states and interpolates to the correlation function of the CS model as the droplet width is continuously narrowed.

Density correlation function is derived similarly. When the droplet width is wide the density correlations at either boundaries are

$$
\langle y_\pm(x)y_\pm'(x') \rangle = \frac{m}{B^2} \partial_x \partial_{x'} \langle \varphi_\pm(x)\varphi_\pm(x') \rangle = \frac{m}{B^2} \frac{1}{(x-x')^2}.
$$

(0.32)

Once the droplet width becomes narrow, the otherwise decoupled chiral edge excitations begin to mix each other. Since the state $\rho(x|0)$ has charge 0 and bosonic statistics, intermediate states in $\langle \rho(x)\rho(x') \rangle$ are created by operators $V_{0,l}$ for integer-valued $l$. This operator has $Q_\pm = \pm(l/m)$, $\theta = 0$, $h_{0,l} = l^2/m$ and momentum $2lk_F$. Therefore the density correlation function of the narrow FQH droplet is given by

$$
\langle \rho(r)\rho(0) \rangle \sim A_0 r^{-2} + \sum_{l=1}^{\infty} A_l r^{-2l^2/m} \cos(2lk_F r)
$$

(0.33)

and exhibits the expected $\cos(2lk_Fr)$ oscillations.

In this Letter, applying the hydrodynamic collective field theory approach to the FQH droplet dynamics, we have shown that the correlation functions of the FQHE edge state and the Calogero-Sutherland model are related each other. The 1-d representation of the Laughlin wave function Eq. (1.1) describes both the edge state correlation and detailed exposition of this Letter will be reported separately [19].

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