Abstract

We propose discrete TBA equations for models with discrete spectrum. We illustrate our construction on the Calogero-Moser model and determine the discrete 2-body TBA function which yields the exact $N$-body Calogero-Moser thermodynamics. We apply this algorithm to the Lieb-Liniger model in a harmonic well, a model which is relevant for the microscopic description of harmonically trapped Bose-Einstein condensates in one dimension. We find that the discrete TBA reproduces correctly the $N$-body groundstate energy of the Lieb-Liniger model in a harmonic well at first order in perturbation theory, but corrections do appear at second order.
I. INTRODUCTION

It has been known for a long time that the spectrum of the Calogero model [1], defined here as particles on an infinite line interacting via $\alpha(\alpha-1)/(x_i-x_j)^2$ 2-body interactions, can be found [2] by the Bethe Ansatz (BA) which assumes periodic boundary conditions, and that its thermodynamics can be obtained, in the thermodynamic limit, by the Thermodynamic Bethe Ansatz (TBA) [3]. It is however not necessary to rely on the BA (or the TBA) to get the Calogero spectrum (or its thermodynamics). Indeed, the Calogero model is exactly solvable

- either by confining the particles in a harmonic well of frequency $\omega$: the Calogero-Moser model [4] with discretized harmonic well quantum numbers and energies, and Hamiltonian

$$H_N = -\frac{1}{2} \sum_{i=1}^{N} \frac{d^2}{dx_i^2} + \alpha(\alpha-1) \sum_{i<j} \frac{1}{(x_i-x_j)^2} + \frac{1}{2} \omega^2 x_i^2$$ (1)

- or by confining the particles in a periodic box of length $L$: the Calogero-Sutherland model [5] with discretized momenta and energies, and Hamiltonian

$$H_N = -\frac{1}{2} \sum_{i=1}^{N} \frac{d^2}{dx_i^2} + \alpha(\alpha-1) \left(\frac{\pi}{L}\right)^2 \sum_{i<j} \frac{1}{\sin^2 \left[\frac{\pi}{L} (x_i-x_j) \right]}$$ (2)

(the $1/\sin^2[\pi(x_i-x_j)/L]$ interactions are nothing but the periodic version of the infinite line interactions). It is not a surprise that Bethe ansatz equations yield the Calogero-Sutherland spectrum, since they also assume, as stressed above, periodic boundary conditions.

Both parameters $\omega$ and $L$ can be considered as long distance regulators, the thermodynamic limit, i.e. the infinite line limit, being obtained either by $\omega \to 0$ or $L \to \infty$, resulting in continuous momenta and energies.

The Calogero model describes particles with intermediate statistics, which is natural due to the topological (statistical) nature of the $1/(x_i-x_j)^2$ interaction in 1d. In the thermodynamic limit indeed [3], the Calogero thermodynamics realizes microscopically Haldane (Hilbert space counting) statistics [6]. Moreover, the Calogero model has been shown to be obtained as the vanishing magnetic field limit [8] of the lowest Landau level anyon model [9].
(LLL-anyon model) in the regime where the flux tubes carried by the anyons screen the flux of the external magnetic field (screening regime). Not surprisingly, the LLL-anyon model also realizes microscopically Haldane statistics, the Hilbert space counting argument being manifest here via a mean field argument (adding anyons screen the external magnetic field, and thus diminish the Landau degeneracy of the total -mean+external- magnetic field): thus a clear relation between Haldane [10] and anyon statistics [11].

Starting from the BA spectrum, and following Yang and Yang footsteps [3], one can compute à la TBA the thermodynamics of the Calogero model in the thermodynamic limit $L \to \infty$. The thermodynamical potential $\ln Z$ -where $Z = \sum_{N=0}^{\infty} z^N Z_N$ is the grand partition function- ends up to be those of a Fermi gas\footnote{There is an equivalent formulation in terms of a free Bose gas, namely}

$$\log Z = \frac{L}{2\pi} \int_{-\infty}^{\infty} dk \log [1 + z e^{-\beta \epsilon(k)}],$$

but with a 1-body energy $\epsilon(k)$ defined in terms of the free continuous 1-body quadratic spectrum $\epsilon_o(k) = k^2/2$ as

$$\beta \epsilon(k_1) = \beta \epsilon_o(k_1) - \frac{L}{2\pi} \int_{-\infty}^{\infty} dk_2 \Phi(k_1 - k_2) \log [1 + z e^{-\beta \epsilon(k_2)}]$$

In the Calogero case [12],

$$\Phi(k_1 - k_2) = \frac{2\pi}{L} (1 - \alpha) \delta(k_1 - k_2)$$

\footnote{There is an equivalent formulation in terms of a free Bose gas, namely}

$$\log Z = \frac{L}{2\pi} \int_{-\infty}^{\infty} dk \log \frac{1}{1 - ze^{-\beta \hat{\epsilon}(k)}},$$

but with a 1-body energy $\hat{\epsilon}(k)$ defined as

$$\beta \hat{\epsilon}(k_1) = \beta \epsilon_o(k_1) - \frac{L}{2\pi} \int_{-\infty}^{\infty} dk_2 \hat{\Phi}(k_1 - k_2) \log \frac{1}{1 - ze^{-\beta \hat{\epsilon}(k_2)}}$$

One has obviously

$$\hat{\Phi}(k_1 - k_2) - \phi(k_1 - k_2) = -\frac{2\pi}{L} \delta(k_1 - k_2)$$
is intimately related to the 2-body scattering angle, and encodes, if one thinks in terms of statistics, the statistical exclusion between two quantum states, here with the same momentum.

Note that if one denotes $y(k) = 1 + z e^{-\beta \epsilon(k)}$, which can be regarded, in view of (3), as the grand partition function at momentum $k$, then (4) can be rewritten as

$$y(k) - z e^{-\beta \epsilon_o(k)} y(k)^{1-\alpha} = 1$$

a particular case of Ramanujan equations [13].

As already said, equations of the type (3,6) were first obtained directly by i) considering the exact $N$-body Calogero spectrum in a harmonic well [6], or by considering the exact $N$-body LLL-anyon spectrum in a harmonic well [9], ii) and then taking the thermodynamic limit.

Now we might ask the following question: are the TBA equations (3,4) specific to the thermodynamic limit with continuous momenta and continuous dressed energies, or can they also describe the thermodynamic of the Calogero model in a harmonic well or in a periodic box with discretized energies? In other words, can we find a discretized version of the function $\Phi(k_1 - k_2)$ in (5) such that the harmonic well or periodic box Calogero thermodynamics narrow down to a set of defining equations analogous to (3,4)?

We will show that the thermodynamic of the Calogero model in a harmonic well can indeed be rewritten “à la TBA” in terms of a discretized function $\Phi$ which will encode the statistical Calogero exclusion between different discrete harmonic energy levels and which will, as it should, reproduce, in the thermodynamic limit $\omega \to 0$, $\Phi(k_1 - k_2)$ in (3). Not surprisingly, the same conclusion will be reached for the LLL-anyon model in a harmonic well, whose thermodynamics will obey the same TBA equations as the Calogero-Moser thermodynamics.

We will also argue that the same logic applies in the Calogero-Sutherland case, provided that a global shift of the bare quantum numbers is made in order to maintain a symmetric repartition of the dressed quantum numbers around zero.
Finally, we will look at possible applications of discrete TBA thermodynamics beyond the Calogero-Moser and harmonic LLL-anyon cases, by considering the Lieb-Liniger model in a harmonic well \[14\], \[15\]. This model is interesting because of its relevance to the description of one dimensional trapped Bose condensates \[13\]. We will show that the \(N\)-body groundstate energy is correctly reproduced at first order in perturbation theory by the discrete TBA equations, but corrections do appear at second order.

II. THE CALOGERO CASE

For a system with a discrete 1-body harmonic spectrum, the TBA equations (3,4) should rewrite quite generally as

\[
\log Z = \sum_{n=0}^{\infty} \log[1 + ze^{-\beta\epsilon(n)}],
\]

where the 1-body dressed energy \(\epsilon(n)\) should now be defined in terms of the 1-body 1d harmonic spectrum (bare spectrum) \(\epsilon_o(n) = (n+1/2)\omega, \; 0 \leq n\) as

\[
\beta\epsilon(n_1) = \beta\epsilon_o(n_1) - \sum_{n_2=0}^{\infty} \Phi_{n_1,n_2} \log[1 + ze^{-\beta\epsilon(n_2)}]
\]

(7,8) are just the discretized versions of (3,4). In (8), \(\Phi_{n_1,n_2}\) has to be understood as acting on the free harmonic spectrum, i.e as acting on the power series in \(z\) obtained from (8) by expanding \(y(n) = 1 + ze^{-\beta\epsilon(n)}\) as

\[
y(n) = \sum_{N=0}^{\infty} y_N(n) z^N
\]

The lowest order terms of (8) are

\[
y_0(n_1) = 1,
\]

\[
y_1(n_1) = e^{-\beta\epsilon_o(n_1)},
\]

\[
y_2(n_1) = \sum_{n_2=0}^{\infty} e^{-\beta\epsilon_o(n_1)} \Phi_{n_1,n_2} e^{-\beta\epsilon_o(n_2)},
\]

\[
y_3(n_1) = \sum_{n_2,n_3=0}^{\infty} e^{-\beta\epsilon_o(n_1)} \left( \Phi_{n_1,n_2} e^{-\beta\epsilon_o(n_2)} \Phi_{n_2,n_3} + \frac{1}{2} \Phi_{n_1,n_2} e^{-\beta\epsilon_o(n_2)} \Phi_{n_1,n_3} \right) e^{-\beta\epsilon_o(n_3)}
\]

\[\quad - \frac{1}{2} \sum_{n_2=0}^{\infty} e^{-\beta\epsilon_o(n_1)} \Phi_{n_1,n_2} e^{-2\beta\epsilon_o(n_2)},\] (10)
where 0 ≤ n_1, n_2, · · · and the summation should be taken for all possible independant integers n_2, n_3, · · ·.

If the TBA cluster coefficients obtained from expanding log \( Z = \sum b_n z^n \)

\[
\begin{align*}
  b_1 &= \sum_{n=0}^{\infty} y_1(n), \\
  b_2 &= \sum_{n=0}^{\infty} \left[ y_2(n) - \frac{1}{2} y_1^2(n) \right], \\
  b_3 &= \sum_{n=0}^{\infty} \left[ y_3(n) - y_1(n) y_2(n) + \frac{1}{3} y_1^3(n) \right],
\end{align*}
\]

(11)

have to match against the Calogero-Moser cluster coefficients

\[
b_1 = e^{-\frac{2\omega}{1-e^{-\beta \omega}}}, \quad b_2 = e^{\frac{\beta \omega}{2(1-e^{-2\beta \omega})} - \frac{1}{2} \frac{1}{1-e^{-2\beta \omega}}}, \quad \ldots,
\]

\( \Phi_{n_1,n_2} \) should be defined as

\[
\Phi_{n_1,n_2} = P_{n_1,n_2}(\alpha) - P_{n_1,n_2}(\alpha = 1)
\]

(12)

where \( P_{n_1,n_2}(\alpha) \) projects the two independant quantum numbers 0 ≤ n_1, n_2 on dressed quantum numbers which, not surprisingly, obey exclusion statistics. More precisely, evaluating in (7,8,10) expressions of the type

\[
\sum_{n_2=0}^{\infty} P_{n_1,n_2}(\alpha) e^{-\beta \epsilon(n_2)},
\]

(13)

\( P_{n_1,n_2}(\alpha) \) amounts, \( n_1 \) being given, to the shift

\[
n_2 \to n_1 + n_2 + \alpha, \quad n_2 ≥ 0
\]

(14)

and the summation over \( n_2 \) is replaced by the summation over \( n_2 ≥ 0 \). In other words, in terms of the independant quantum numbers 0 ≤ n_1, n_2, denoting \( n_1 = n_1, n_2 = n_1 + \tilde{n}_2 \), \( P_{n_1,n_2}(\alpha) \) means \( n_1 \to n_1 \), and \( n_2 \to n_2 + \alpha \), where 0 ≤ \( n_1 \leq n_2 \) are now bosonic quantum numbers. Therefore \( P_{n_1,n_2}(0) \) projects 0 ≤ n_1, n_2 onto bosonic quantum numbers 0 ≤ \( n_1 \leq n_2 \), whereas \( P_{n_1,n_2}(1) \) projects 0 ≤ n_1, n_2 onto fermionic quantum numbers. Note that in (12) subtracting \( P_{n_1,n_2}(\alpha = 1) \) is simply a matter of convention, i.e. as stressed above, a fermionic thermodynamical potential \( \Phi \) with a spectrum which has to coincide with the bare spectrum when \( \alpha = 1 \) -the Bose convention would yield \( \tilde{\Phi}_{n_1,n_2} = P_{n_1,n_2}(\alpha) - P_{n_1,n_2}(\alpha = 0) \).
More generally notice that (12) allows to rewrite (8) as

\[ y(n_1) - z e^{-\beta \epsilon(o)(n_1)} \prod_{n_2=0}^{\infty} y(n_1 + \tilde{n}_2 + \alpha) = 1 \]  

which can be viewed as the discretized version of (3).

Going one step further one gets

\[ \prod_{\tilde{n}_2=0}^{\infty} y(n_1 + \tilde{n}_2) = \prod_{\tilde{n}_2=0}^{\infty} y(n_1 + \tilde{n}_2 + 1) + z e^{-\beta \epsilon(o)(n_1)} \prod_{\tilde{n}_2=0}^{\infty} y(n_1 + \tilde{n}_2 + \alpha) \]  

which in turn, taken at \( n_1 = 0 \), rewrites as

\[ Z = z e^{-\beta \epsilon(o)(0)} \prod_{\tilde{n}_2=0}^{\infty} y(\tilde{n}_2 + \alpha) + \prod_{\tilde{n}_2=0}^{\infty} y(\tilde{n}_2 + 1) \]  

Equation (17) can be interpreted by saying that either the first particle is in the ground-state at energy \( \frac{1}{2} \omega \) and then the next particle is in the energy level at least higher than \( (\frac{1}{2} + \alpha) \omega \), or the groundstate is vacant and the first particle is in the energy level at least higher than \( (\frac{1}{2} + 1) \omega \). One verifies recursively that \( Z \) in (17) is identical to the grand partition of the Calogero-Moser model with \( N \)-body spectrum

\[ E_N = \omega \sum_{i=1}^{N} \left( n_i + \alpha(i - 1) \right) + \frac{1}{2} \]  

where \( 0 \leq n_1 \leq n_2 \leq \ldots \). In terms of the bare independent quantum numbers \( 0 \leq n_1, n_2, \ldots \), one has \( n_i \rightarrow n_i + \alpha(i - 1) = n_i-1 + \tilde{n}_i + \alpha(i - 1) \) with \( \tilde{n}_i \geq 0 \). This is indeed a BA “like” spectrum, i.e. in terms of the dressed quantum numbers \( n'_i = n_i + \alpha(i - 1) \), one has \( n'_i = n_i + \alpha \sum_{j \neq i} \theta(n'_i - n'_j) \). In the 2-body case, it indeed amounts to \( n_1 \rightarrow n_1, n_2 \rightarrow n_1 + \tilde{n}_2 + \alpha \), i.e. to the the action of the projector \( P_{n_1,n_2}(\alpha) \) on the independent quantum numbers \( 0 \leq n_1, n_2 \).

Note that (12) implies that the \( N \)-body partition function \( Z_N \) obtained from (7) as

\[ Z_N = \prod_{n=0}^{\infty} y_{N_n}(n), \quad 0 \leq N_n, \quad \sum_n N_n = N \]  

has, using (10) to all orders, the simple factorized form

\[ Z_N = \sum_{n_1, \ldots, n_N=0}^{\infty} P_{n_1,n_2}(\alpha)P_{n_2,n_3}(\alpha) \cdots P_{n_{N-1},n_N}(\alpha) e^{-\beta(\epsilon(o)(n_1) + \epsilon(o)(n_2) + \ldots + \epsilon(o)(n_N))} \]  

Note that (12) implies that the \( N \)-body partition function \( Z_N \) obtained from (7) as

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has, using (10) to all orders, the simple factorized form

\[ Z_N = \sum_{n_1, \ldots, n_N=0}^{\infty} P_{n_1,n_2}(\alpha)P_{n_2,n_3}(\alpha) \cdots P_{n_{N-1},n_N}(\alpha) e^{-\beta(\epsilon(o)(n_1) + \epsilon(o)(n_2) + \ldots + \epsilon(o)(n_N))} \]
In particular in the 2-body case

\[ Z_2 - Z_2|_{Fermi} = \sum_{n_1,n_2=0}^{\infty} \left( P_{n_1,n_2}(\alpha) - P_{n_1,n_2}(\alpha = 1) \right) e^{-\beta(e_0(n_1) + e_0(n_2))} \]  

(21)

and thus in the thermodynamic limit \(\omega \to 0\)

\[ Z_2 - Z_2|_{Fermi} = \left( \frac{L^2}{2\sqrt{\pi}} \right)^2 \int_{-\infty}^{\infty} dk_1 dk_2 \Phi(k_1 - k_2) e^{-\beta(\frac{k_1^2}{4} + \frac{k_2^2}{4})} \]  

(26)

where \(\Phi\) is given in (5).

As far as the LLL-anyon model in a harmonic well [9] is concerned, one finds that (7,8,12) are unchanged, to the exception of the 1-body energy which now reads \(e_0(n) = \omega_t n + \omega_c\), \(\omega_c = eB/2\), \(\omega_t = \sqrt{\omega_c^2 + \omega^2}\), and the statistical anyonic parameter has to be understood as being \(-\alpha\), i.e. the screening regime where the flux \(\phi = -\alpha \phi_o\) (\(\phi_o\) is the quantum of flux) carried by each anyon is antiparallel to the external magnetic field.

One can easily convince oneself that in the thermodynamic limit \(\omega \to 0\), both the Calogero and LLL-anyon TBA thermodynamics narrow down to

\[ \frac{1}{2\sinh \frac{\beta \omega}{2}} \sum_{\tilde{n}_2 = 2l \geq 0} \left( e^{-\beta\omega(\tilde{n}_2 + \frac{1}{2} + \alpha)} - e^{-\beta\omega(\tilde{n}_2 + \frac{1}{2} + 1)} \right) \]  

(22)

and (26) as

\[ Z_2 - Z_2|_{Fermi} = \left( \frac{L}{2\sqrt{\pi}} \right)^2 \int_{-\infty}^{\infty} dK e^{-\beta \frac{K^2}{4}} \int_{-\infty}^{\infty} dk \Phi(2k) e^{-\beta k^2} \]  

(23)

Since, in the thermodynamic limit \([17]\) for the \(N\)-th cluster coefficient, \(\frac{1}{\beta \omega} \to \frac{L}{\sqrt{2\pi} \beta \sqrt{N}}\), one infers that in the 2-body case

\[ \frac{1}{2\sinh \frac{\beta \omega}{2}} \to \frac{L}{2\pi} \int_{-\infty}^{\infty} dK e^{-\beta \frac{K^2}{4}} \]  

(24)

and therefore one should have

\[ \sum_{\tilde{n}_2 = 2l \geq 0} \left( e^{-\beta\omega(\tilde{n}_2 + \frac{1}{2} + \alpha)} - e^{-\beta\omega(\tilde{n}_2 + \frac{1}{2} + 1)} \right) \to \frac{L}{2\pi} \int_{-\infty}^{\infty} dk \Phi(2k) e^{-\beta k^2} \]  

(25)

a result that can be trivially checked by direct computation.
\[
\log Z = \int_0^\infty d\epsilon_o \rho_o(\epsilon_o) \ln(1 + ze^{-\beta \epsilon(\epsilon_o)})
\]

where the dressed energy \( \epsilon(\epsilon_o) \) is implicitly defined à la TBA in terms of the bare energy \( \epsilon_o \) as

\[
\beta \epsilon = \beta \epsilon_o - \int_0^\infty d\epsilon' \Phi(\epsilon, \epsilon') \ln(1 + ze^{-\beta \epsilon'})
\]

and

\[
\Phi(\epsilon, \epsilon') = (1 - \alpha) \delta(\epsilon - \epsilon')
\]

Here, \( \rho_o(\epsilon_o) \) is the 1-body density of states of the bare spectrum of the system considered, i.e. in the Calogero case the 1d free density of states \( \rho_o(\epsilon_o) = L/\sqrt{2\epsilon_o} \), and, in the LLL-anyon case, the 2d LLL density of states, \( \rho_o(\epsilon_o) = (BV/\phi_o)\delta(\epsilon_o - \omega_c) \), where \( V \) is now the infinite surface of the 2d plane.

In the case of the Calogero model in a periodic box -the Calogero-Sutherland model-, one can still propose (7) and (8), but now the 1-body dressed energy \( \epsilon(n) \) should be defined in terms of the 1-body spectrum in a 1d periodic box of length \( L \), \( \epsilon_o(n) = k^2/2 \), with discretized momentum \( k = 2\pi n/L \) and \( n \) positive, null or negative integer.

However, and contrary to the harmonic case, the very fact that the bare quantum numbers in a periodic box can be of both signs lead to some adjustments. If one looks at the \( N \)-body Calogero-Sutherland spectrum\(^3\)

\[
E_N = \frac{1}{2} \left( \frac{2\pi}{L} \right)^2 \sum_{i=1}^N \left( n_i + \alpha(i-1) - \frac{\alpha(N-1)}{2} \right)^2
\]

where \( n_1 \leq n_2 \leq \ldots \), one finds that in terms of the bare quantum numbers \( n_i \rightarrow n_i + \alpha(i-1) - \alpha(N-1)/2 = n_{i-1} + \tilde{n}_i + \alpha(i-1) - \alpha(N-1)/2 \) with \( \tilde{n}_i \geq 0 \). This is quite similar to the Calogero-moser spectrum, up to a global shift, \( n_i \rightarrow n_i - \alpha(N-1)/2 \), a \( N \)-dependant periodic boundary condition adjustment insuring that the dressed spectrum

\(^3\)with a BA spectrum \( n_i' = n_i + \frac{\alpha}{2} \sum_{j \neq i} sign(n_i' - n_j') \)
remain symmetric around 0 in order to minimize the $N$-body energy. In the 2-body case
$n_1 \to n_1 - \alpha/2, n_2 \to n_1 + \tilde{n}_2 + \alpha/2$, it amounts to the the action of the Calogero-Moser
projector $P_{n_1,n_2}(\alpha)$ as given in (12) on the a priori two independent quantum numbers $n_1, n_2$, again up to the 2-body shift $n_{1,2} \to n_{1,2} - \alpha/2$. This being considered, altogether with the fact
that the Calogero-Moser and Calogero-Sutherland models originate from the same model, up to a long distance regularisation, it is natural to take for both models the same TBA function (12) to obtain, in view of (20), the correct Calogero-Sutherland $N$-body partition function, but in addition the shift $n_i \to n_i - \alpha(N - 1)/2$ has to be made a posteriori.

At this point, one can remark that in all cases studied so far, the Calogero-Moser model, as well as the Calogero-Sutherland model up to periodic boundary conditions adjustments, and their thermodynamic limit, the Calogero model, the TBA functions $\Phi_{n_1,n_2}$ and $\Phi(k_1 - k_2)$ are intimately related to the relative 2-boson density of states for the problem at hand. Indeed, in a harmonic well, the spectrum for a relative particle with bosonic statistics and interacting with a Calogero potential at the origin is

$$\epsilon = \omega(n + \frac{1}{2} + \alpha) \quad 0 \leq n \quad (31)$$

with $n$ even, i.e. with symmetric eigenstates under $x \to -x$.

Let us first consider, in the thermodynamic limit, the Calogero model: when $\omega \to 0$, the relative 2-boson density of states reads

$$\rho_\alpha(\epsilon) - \rho_{\alpha=1}(\epsilon) = \frac{1 - \alpha}{2} \delta(\epsilon) \quad (32)$$

It rewrites in terms of the relative momentum $k$ such that $\epsilon = k^2$

$$\rho_\alpha(k) - \rho_{\alpha=1}(k) = \frac{1 - \alpha}{2} \delta(k) \quad (33)$$

Now one has to map the relative 2-body momentum $k$ on the “momentum” $k_2 - k_1$ the function $\Phi(k_2 - k_1)$ is concerned with. Since $k_2 - k_1 = 2k$, one gets for the density of states in terms of $k_2 - k_1$

$$\rho_\alpha\left(\frac{k_2 - k_1}{2}\right) - \rho_{\alpha=1}\left(\frac{k_2 - k_1}{2}\right) = (1 - \alpha)\delta(k_2 - k_1) \quad (34)$$
i.e. precisely (34) up to a factor $2\pi/L$.

When $\omega$ is kept finite, the same logic applies: the relative spectrum (31) yields

$$n \rightarrow n + \alpha$$

(35)

One has yet to map the relative 2-body bosonic even quantum number $n$ on the “quantum number” $n_2 - n_1$ that the function $\Phi_{n_1,n_2}$ is concerned with. For a given 2-body energy, i.e. for $\mathbf{n}_1 + \mathbf{n}_2$ given, one has $n_2 - n_1 = n$ -then the center of mass quantum number is $2n_1$, or $n_2 - n_1 = n + 1$ -then the center of mass quantum number is $2n_1 + 1$, depending if $n_2 - n_1$ is even or odd. One finds that $n \rightarrow n + \alpha$ rewrites, in terms of the independent $n_1, n_2$ as $n_1 \rightarrow n_1, n_2 \rightarrow n_1 + \tilde{n}_2 + \alpha$, where now $\tilde{n}_2 = n$ or $\tilde{n}_2 = n + 1$, i.e. any positive integer. Then (35) is indeed identical to the action of $\Phi_{n_1,n_2}$ in (12).

This is not a surprise, scattering 2-body phase shifts are known to be linked to the 2-body density of states via S-matrix arguments [18].

III. THE LIEB-LINIGER CASE

In the Lieb-Liniger model in the thermodynamic limit, the same conclusion happens to be true. The model, defined as

$$H_N = -\frac{1}{2} \sum_{i=1}^{N} \frac{d^2}{dx_i^2} + c \sum_{i<j} \delta(x_i - x_j)$$

(36)

is solvable by Bethe ansatz [14] and has a TBA thermodynamics [3] obtained from

$$\Phi(k_1 - k_2) = \frac{1}{L} \frac{2c}{(k_2 - k_1)^2 + c^2}$$

(37)

It interpolates between the Bose ($c = 0$) and Fermi ($c = \infty$) thermodynamics and describes particles with intermediate statistics [15]. For a relative particle with bosonic statistics interacting with a $\delta$ potential at the origin the density of states is

$$\rho_c(\epsilon) - \rho_{c=\infty}(\epsilon) = \frac{1}{4\pi \sqrt{\epsilon \epsilon + c^2}}$$

(38)

which in terms of $\epsilon = k^2$, $k > 0$ rewrites as
\[ \rho_c(k) - \rho_{c=\infty}(k) = \frac{1}{2\pi k^2 + c^2} \]  

(39)

Now, one has again to map \( k \) on the “momentum” \( k_2 - k_1 \) the function \( \Phi(k_1, k_2) \) is concerned with, i.e. \( k_2 - k_1 = 2k \), and since \( k_2 - k_1 \) can be either positive or negative, one gets for the density of states in terms of \( k_2 - k_1 \)

\[ \rho_c\left(\frac{k_2 - k_1}{2}\right) - \rho_{c=\infty}\left(\frac{k_2 - k_1}{2}\right) = \frac{1}{2\pi} \frac{2c}{(k_2 - k_1)^2 + c^2} \]

(40)

i.e. nothing but (37), again up to a factor \( 2\pi/L \).

If one follows the same line of reasoning which was operative in the Calogero-Moser case to obtain the discrete TBA function \( \Phi_{n_1, n_2} \) from the relative 2-body spectrum, one might try, for the Lieb-Liniger model in a harmonic well, discrete TBA thermodynamics equations with a TBA function \( \Phi_{n_1, n_2} \) deduced from the 2-body relative bosonic spectrum in a harmonic well. It rewrites as

\[ \epsilon = \omega \left( n + \frac{1}{2} + \frac{2}{\pi} \arctan \left( \frac{c}{2\sqrt{2}\omega} \Gamma\left(\frac{n+1}{2} + \frac{1}{4}\right) \right) \right) 0 \leq n \]

(41)

with \( n \) even, i.e. as \( \epsilon = \omega(n + \frac{1}{2} + f_c(n)) \) with

\[ 0 \leq f_c(n) = \frac{2}{\pi} \arctan \left( \frac{c}{2\sqrt{2}\omega} \Gamma\left(\frac{n+1}{2} + \frac{f_c(n)}{2}\right) \right) \leq 1 \]

(43)

and interpolates between the relative 2-body bosonic \( (c = 0, f_0(n) = 0, g_0(n) = 1) \) and fermionic \( (c = \infty, f_\infty(n) = 1, g_\infty(n) = 0) \) spectra.

Therefore let us try for the Lieb and Liniger in a harmonic well the discrete TBA function

\[ \Phi_{n_1, n_2} = P_{n_1, n_2}(c) - P_{n_1, n_2}(c = \infty) \]

(44)

should be defined in terms of \( P_{n_1, n_2}(c) \) such that, \( n_1 \) being left unchanged,

\[ 0 \leq g_c(n) = \frac{2}{\pi} \arctan \left( \frac{2\sqrt{2}\omega}{c} \Gamma\left(\frac{n+3}{2} - \frac{g_c(n)}{2}\right) \right) \leq 1 \]

(42)

\[ \text{Equivalently, starting from the Fermi spectrum by rewriting } f_c(n) = 1 - g_c(n) \]
\[ n_2 \to n_1 + \tilde{n}_2 + f_c(\tilde{n}_2) \quad (45) \]

if \( \tilde{n}_2 \geq 0 \) is even, and

\[ n_2 \to n_1 + \tilde{n}_2 + f_c(\tilde{n}_2 - 1) \quad (46) \]

if \( \tilde{n}_2 \) is odd. Note again that subtracting \( P_{n_1,n_2}(c = \infty) \) in (44) originates, as in the Calogero case, from the fermionic convention (obviously \( P_{n_1,n_2}(c = \infty) = P_{n_1,n_2}(\alpha = 1) \)).

It is easy to check that the 2-body partition function is reproduced by the discrete TBA equations

\[ Z_2 - Z_2|_{\text{Fermi}} = \sum_{n_1=n_2=0}^{\infty} (P_{n_1,n_2}(c) - P_{n_1,n_2}(c = \infty)) e^{-\beta(\epsilon_0(n_1) + \epsilon_0(n_2))} \quad (47) \]

and thus, in the thermodynamic limit \( \omega \to 0 \),

\[ \sum_{\tilde{n}_2 = 2l \geq 0} \left( e^{-\beta(\tilde{n}_2 + \frac{1}{2} + f_c(\tilde{n}_2))} - (c = \infty) \right) \to \frac{L}{2\pi} \int_{-\infty}^{\infty} dk \Phi(2k) e^{-\beta k^2} \quad (48) \]

where the function \( \Phi \) is given in (37), a result that can be checked by direct computation, order by order in \( 1/c \). There are two independent dimensionless parameters, \( \beta \omega \) (thermodynamic limit) and \( \sqrt{\beta c} \) (“coupling constant”). Clearly, for a given coupling constant \( \sqrt{\beta c} \), looking at \( f_c(n) = 1 - g_c(n) \) in (43,42), one has to consider, in the thermodynamic limit \( \beta \omega \to 0 \), the spectrum close to the Fermi point \( (c = \infty) \),

\[ g_c(2l) = 4\sqrt{2} \sqrt{\beta \omega} \frac{\Gamma(l + \frac{3}{2})}{l!} + \ldots \quad (49) \]

from which (18) can be recovered, here at first order \( 1/(\sqrt{\beta c}) \).

Note also that in the thermodynamic limit for the relative spectrum, with \( (n + 1/2)\omega \to k^2 \), i.e. \( n\omega \) fixed, (16) becomes

\[ f_c(k) = \frac{2}{\pi} \arctan \frac{c}{2k} \quad (50) \]

which is indeed reminiscent of the Lieb and Liniger BA spectrum [14].

What about the \( N \)-body problem? A possible way to check the discrete TBA is to see if the perturbative TBA thermodynamics coincide with the exact (standard) Hamiltonian
perturbative thermodynamics \[19\], which can be computed with the Lieb and Liniger Hamiltonian from the Bose point \(c = 0\) (from the Fermi point \(c = \infty\) standard perturbation theory is meaningless). Perturbation theory yields

\[
\log Z = \log Z|_{Bose} + \sqrt{\beta c} \sum_{s,t=1}^{\infty} \frac{z^{s+t}}{4\sqrt{\pi}} \frac{\sqrt{\beta \omega}}{\sinh \frac{s \beta \omega}{2} \sinh \frac{t \beta \omega}{2} \sinh \frac{(s+t) \beta \omega}{2}} + \ldots
\]

Let us now consider the \(\sqrt{\beta c}\) expansion from the TBA point of view, again from the Bose point. One has to consider \(43\) at first order in \(\sqrt{\beta c}\)

\[
f_c(2l) = \frac{1}{\pi \sqrt{2} \sqrt{\beta \omega}} \frac{\Gamma(l + \frac{1}{2})}{l!} + \ldots
\]

and compute from the discrete TBA equations

\[
\log Z = \log Z|_{Bose} + \sum_{n_1=0}^{\infty} \frac{z \epsilon^{-\beta \omega(n_1 + \frac{1}{2})}}{1 - z \epsilon^{-\beta \omega(n_1 + \frac{1}{2})}} \sum_{n_2=0}^{\infty} \Phi_{n_1,n_2} \ln \frac{1}{1 - z \epsilon^{-\beta \omega(n_2 + \frac{1}{2})}} + \ldots
\]

where \(\Phi_{n_1,n_2} \ln \frac{1}{1 - z \epsilon^{-\beta \omega(n_2 + \frac{1}{2})}}\) means evaluating this expression at first order in \(\sqrt{\beta c}\) using \(52\). One finds

\[
\log Z = \log Z|_{Bose} + \beta c \sum_{s,t=1}^{\infty} \frac{z^{s+t}}{4\sqrt{2\pi} \sinh \frac{(s+t) \beta \omega}{2}} \left( \sqrt{\coth \frac{s \beta \omega}{2} - \coth \frac{t \beta \omega}{2} + \coth \frac{t \beta \omega}{2}} \right) + \ldots
\]

which coincides with \(51\) only in the limit \(\beta \omega \to \infty\), i.e. for a given \(\omega\), in the zero temperature limit\[5\], i.e the groundstate.

In fact, discrete TBA gives in the vanishing temperature limit direct information on the \(N\)-body groundstate energy : in the Calogero-Moser case, it is obtained, in the bosonic based formulation, by restricting the discrete TBA equations

\[
\log Z = \sum_{n_1=0}^{\infty} \log \frac{1}{1 - z \epsilon^{-\beta \omega(n_1)}},
\]

\[5\]The limit \(\omega \to \infty\) for a given temperature is not considered here. In this limit all particles are confined at \(x_i = 0\). But \(\delta\) interactions actually forbid this unless the effective coupling constant vanishes, which is precisely happening in the 2-body case \(43\). In other words the \(\omega \to \infty\) limit is the trivial bosonic limit.
\[
\beta \tilde{\epsilon}(n_1) = \beta \epsilon_\alpha(n_1) - \sum_{n_2=0}^{\infty} \Phi_{n_1,n_2} \log \frac{1}{1 - z e^{-\beta \tilde{\epsilon}(n_2)}}
\]

with \(\Phi_{n_1,n_2} = P_{n_1,n_2}(\alpha) - P_{n_1,n_2}(\alpha = 0)\) to the groundstate quantum numbers \(n_1 = 0\) and \(\tilde{n}_2 = 0\). One obtains

\[
E_{N}^{G.S.} = \omega \left( \frac{N}{2} + \frac{N(N-1)}{2} \alpha \right)
\]

In the Lieb and Liniger case a similar approach gives

\[
E_{N}^{G.S.} = \omega \left( \frac{N}{2} + \frac{N(N-1)}{2} f_c(0) \right)
\]

a result which is consistent with the groundstate energy at first order in \(c\)

\[
E_{N}^{G.S.} = \omega \left( \frac{N}{2} + \frac{N(N-1)}{2} f_c^{(1)}(0)c + \ldots \right)
\]

where \(f_c^{(1)}(0) = 1/\sqrt{2\pi \omega}\) stands for the first order term in the expansion of \(f_c(0)\) in power of \(c\). However, second order standard perturbation theory gives corrections to the Calogero-Moser like energy \(\omega N(N-1)f_c(0)/2\). For example in the \(N = 3\) case

\[
E_{3}^{G.S.} = \omega \left( \frac{3}{2} + 3f_c^{(1)}(0)c + 3 \left( f_c^{(2)}(0) - \frac{1}{\pi \omega} \log \frac{4}{2 + \sqrt{3}} \right) c^2 + \ldots \right)
\]

where \(f_c^{(2)}(0) = -\frac{\log 2}{2\pi \omega}\) stands for the second order term in the expansion of \(f_c(0)\).

**IV. CONCLUSION**

We have shown how the Caloger-Moser thermodynamics can be rewritten in terms of discrete TBA equations. In the Calogero-Sutherland model, the same TBA equations were shown to be operative, up to a global shift of the bosonic quantum numbers. Since the Lieb-Liniger model shares common features with the Calogero model -BA solvability, TBA thermodynamics in the thermodynamic limit, intermediate statistics- it might also have, when considered in a harmonic well, a discrete TBA thermodynamics. We tried to illustrate this point of view by proposing discrete TBA equations for the harmonic Lieb and Liniger
model in analogy with the Calogero-Moser TBA thermodynamics. However the groundstate energy shows deviations from this TBA framework at second order in perturbation theory.

We leave to a further study to find analytical or numerical ways to improve and give a stronger basis to the discrete TBA thermodynamics for the Lieb-Liniger model, and in particular extract a useful information on the groundstate for a given density of particles. It would however certainly be interesting to understand more in detail the zero temperature limit of a system which is supposed to describe the physics of 1d Bose Einstein condensates in harmonic traps.

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REFERENCES

[1] F. Calogero, J. Math. Phys. 10, 2191 (1969); ibid. 12, 419 (1971).

[2] for a recent review on the subject and references see A. P. Polychronakos, in Les Houches Lecture Notes, Session LXIX, “Topological Aspects of Low-dimensional Systems” (1998) Ed. Springer; see also V. E. Korepin et al, “Quantum Inverse scattering method and correlation functions” (1993) Cambridge monographs on mathematical physics.

[3] C. N. Yang and C.P. Yang, J. Math. Phys. 10, 1115 (1969).

[4] J. Moser, Adv. Math. 16, 1 (1975).

[5] B. Sutherland, J. Math. Phys. 12, 246(1971); ibid. 251 (1971); Phys. Rev. A4, 2019 (1971); ibid. A5, 1372 (1972).

[6] S. B. Isakov, Int. J. Mod. Phys. A9, 2563 (1994).

[7] F. D. M. Haldane, Phys. Rev. Lett. 67, 937 (1991); M. C. Bergère, JMP 41, 7252 (2000).

[8] S. Ouvry, cond-mat/9907239, Phys. Lett. B (to be published)

[9] A. Dasnières de Veigy and S. Ouvry, Phys. Rev. Lett. 72, 600 (1994).

[10] Y.-S. Wu, Phys. Rev. Lett. 73, 922 (1994); S. B. Isakov, Mod. Phys. Lett. B8, 319 (1994).

[11] J. M. Leinaas, J. Myrheim, Nuovo Cimento B37, 1 (1977); F. Wilczek, Phys. Rev. Lett. 48, 1144 (1982); 49, 957 (1982).

[12] D. Bernard, in les Houches Lecture Notes, Session LXII (1994), Ed. North Holland; D. Bernard and Y.S. Wu, in Proc. 6th Nankai Workshop, eds. M.L. Ge and Y.S. Wu, World Scientific (1995).

[13] M. V. N. Murthy and R. Shankar, IMSc/99/01/01 report.

[14] E. H. Lieb and W. Liniger, Phys. Rev. 130, 1605 (1963).
[15] J. Myrheim, in Les Houches Lecture Notes, Session LXIX, “Topological Aspects of 
Low-dimensional Systems” (1998) Ed. Springer.

[16] M. Oshanii, Phys. Rev. Lett. 81, 938 (1998); D. S. Petrov et al, cond-mat 0006339; V. 
Dunjko et al, cond-mat/0103083.

[17] A. Dasnieres de Veigy and S. Ouvry, Phys. Rev. Lett. 75, 352 (1995); K. Olaussen, 
Trondheim preprint (1992)

[18] M. G. Krein, Matem. Sbornik, 33, 597 (1953); J. Friedel, Nuovo Cimento Suppl. 7, 287 
(1958); Philos. Mag. 43, 153 (1953).

[19] for the formalism used see A. Dasnieres de Veigy and S. Ouvry, Nucl. Phys. B 388 [FS], 
715 (1992).