Error–Disturbance Relation in Stern–Gerlach Measurements

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Based on the recent development of universally valid reformulations of Heisenberg’s error–disturbance uncertainty relation, we study the error and disturbance of Stern–Gerlach measurements of a spin-1/2 particle. By rigorously solving Heisenberg equations of motion for the spin and orbital degrees of freedom passing through an inhomogeneous magnetic field and freely evolving to reach the screen, we determine the range of the possible values of the error and disturbance for arbitrary Stern–Gerlach apparatuses with the orbital degree prepared in an arbitrary Gaussian state. We compare it with the range for completely arbitrary apparatuses previously obtained by Branciard and one of the authors, to show that the range occupies a broad area tighter than the previously investigated range for the improperly directed projective measurements of neutron spin by Hasegawa and co-workers. We show the existence of orbital states in which the error is minimized by the screen at a finite distance from the magnet, in contrast to the standard far field description, and characterize those states by the position-momentum correlation and contractivity under free evolution.

I. INTRODUCTION

A fundamental feature of quantum measurement is nontrivial error–disturbance relations, first found in 1927 by Heisenberg\textsuperscript{1}, who, using the famous $\gamma$-ray microscope thought experiment, derived the relation

$$
\varepsilon(Q)\eta(P) \geq \frac{\hbar}{2}
$$

(1)

between the position measurement error, $\varepsilon(Q)$, and the momentum disturbance, $\eta(P)$, thereby caused. His formal derivation of this relation from the well-established relation

$$
\sigma(Q)\sigma(P) \geq \frac{\hbar}{2}
$$

(2)

for standard deviations $\sigma(Q)$ and $\sigma(P)$ due to Heisenberg\textsuperscript{1} for the minimum uncertainty wave packets and von Neumann\textsuperscript{6, p. 335} for arbitrary wave functions needs an additional assumption, a quantitative version of the repeatability hypothesis stating that in any measurement of observable $A$ with error $\varepsilon(A)$ the wave function collapses up to $\sigma(A) \leq \varepsilon(A)$\textsuperscript{2}. Although the repeatability hypothesis was supported at that time by his contemporaries, including Schrödinger\textsuperscript{4, p. 329}, Dirac\textsuperscript{5, p. 36}, and von Neumann\textsuperscript{6, p. 335}, this hypothesis has been completely abandoned in the modern quantum mechanics\textsuperscript{6}. Now, the state change caused by a measurement is generally described by a completely positive (CP) instrument, a family of CP maps summing to a trace-preserving CP map\textsuperscript{5}. In such a general description of quantum measurements, Heisenberg’s relation\textsuperscript{1} loses its universal validity, as revealed in the debate in the 1980s on the sensitivity limit for gravitational wave detection alleged due to Heisenberg’s error–disturbance relation\textsuperscript{1,2,14}. A universally valid error–disturbance relation for arbitrary pairs of observables was derived only recently by one of the authors\textsuperscript{15–17}, and has recently received considerable attention. The validity of this relation, as well as a stronger version of this relation\textsuperscript{18–21}, were experimentally tested with neutrons\textsuperscript{22–24} and with photons\textsuperscript{25–29}. Other approaches generalizing Heisenberg’s original relation\textsuperscript{1} can be found, for example, in\textsuperscript{30–32}, apart from the information theoretical approach\textsuperscript{33–34}.

Stern–Gerlach measurements are among the most important quantum measurements, and a number of theoretical analyses have been and are being published by many authors. In his famous textbook, Bohm\textsuperscript{35, p. 596} derived the wave function of a spin-1/2 particle that has passed through the Stern–Gerlach apparatus. In his argument, he assumed that the magnetic field points in the same direction everywhere and varies in strength linearly with the $z$-coordinate of the position as

$$
B = \begin{pmatrix}
0 \\
0 \\
B_0 + B_1 z
\end{pmatrix}
$$

(3)

However, as Bohm\textsuperscript{35, p. 594} pointed out, such a magnetic field does not satisfy Maxwell’s equations. Theoretical studies\textsuperscript{36,38} of Stern–Gerlach measurements with the magnetic field

$$
B = \begin{pmatrix}
-B_1 x \\
0 \\
B_0 + B_1 z
\end{pmatrix}
$$

(4)

satisfying Maxwell’s equations were performed only recently. According to these studies, if the magnetic field in the center of the beam is sufficiently strong, the precession of the spin component to be measured becomes small, and hence Bohm’s approximation\textsuperscript{3} holds.

Home et al.\textsuperscript{39} investigated the error of Stern–Gerlach measurements with respect to the distinguishing

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bility of apparatus states. As an indicator of the operational distinguishability of apparatus states, they used the error integral, which is equal to the probability of finding the particle in the spin-up state on the lower half of the screen. They analyzed the error integral in the case where the spin state of the particle just before the measurement is the eigenstate, $|\uparrow\rangle_z$, of $\sigma_z$ corresponding to the eigenvalue $+1$. Nevertheless, the trade-off between the error and disturbance in Stern–Gerlach measurements has not been studied in the literature, even though the subject would elucidate the fundamental limitations of measurements in quantum theory, as Heisenberg did with the $\gamma$-ray microscope thought experiment.

In this paper, we determine the range of the possible values of the error and disturbance for arbitrary Stern–Gerlach apparatuses, based on the general theory of the error and disturbance, which has recently been developed to establish universally valid reformulations of Heisenberg’s uncertainty relation. Throughout this paper, we consider an electrically neutral particle with spin-$1/2$. Following Bohm [33], we assume that the magnetic field of a Stern–Gerlach apparatus is represented by Eq. (3), which is assumed to be sufficiently strong. The particle is assumed to stay in the magnet from time 0 to time $\Delta t$. Only the one-dimensional orbital degree of freedom along the $z$-axis is considered. The kinetic energy is not neglected. The particle having passed through the magnetic field is assumed to evolve freely from time $\Delta t$ to $\Delta t + \tau$. The initial state of the spin of the particle is assumed to be arbitrary. The initial state of the orbital degree of freedom is such that mean values of the position and momentum are both 0. We study the error, $\varepsilon(\sigma_{z})$, in measuring $\sigma_{z}$ with a Stern–Gerlach apparatus and the disturbance, $\eta(\sigma_{z})$, caused thereby on $\sigma_{z}$ for the orbital degree of freedom to be prepared in a Gaussian pure state [40] in detail. We obtain the error–disturbance relation

$$\frac{\eta(\sigma_{z})^2 - 2}{2} \leq \exp \left[ -\text{erf}^{-1} \left( \frac{\varepsilon(\sigma_{z})^2 - 2}{2} \right) \right],$$

for Stern–Gerlach measurements, where erf$^{-1}$ represents the inverse of the error function $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} \exp(-s^2)ds$, to compare with Heisenberg’s error–disturbance uncertainty relation for spin measurements

$$\varepsilon(\sigma_{z})^2 \eta(\sigma_{z})^2 \geq 1,$$

which holds for measurements with statistically independent error and disturbance [14, 17], the error–disturbance relation

$$\left| \frac{\eta(\sigma_{z})^2 - 2}{2} \right| \leq 1 - \left( \frac{\varepsilon(\sigma_{z})^2 - 2}{2} \right)^2,$$

which holds for improperly directed projective measurements experimentally tested with neutron spin measurements conducted by Hasegawa and co-workers [22, 23], and the tight error–disturbance relation

$$\left| \frac{\eta(\sigma_{z})^2 - 2}{2} \right| \leq \sqrt{1 - \left( \frac{\varepsilon(\sigma_{z})^2 - 2}{2} \right)^2}$$

(8)

for the range of $(\varepsilon(\sigma_{z}), \eta(\sigma_{z}))$ values of arbitrary qubit measurements obtained by Branciard and one of the authors [18–20].

In section II the general theory of the error and disturbance is reviewed and Stern–Gerlach measurements are investigated in the Heisenberg picture in detail. In sections III and IV the error and disturbance of Stern–Gerlach measurements are derived. In section V the error–disturbance uncertainty relation for Stern–Gerlach measurements is derived. In section VI our research is compared with the previous research conducted by Home et al. [33]. Section VII presents the conclusion for this paper.

II. MEASURING PROCESS

A. Spin measurements

We consider measurements for a spin-1/2 particle, $\mathbf{S}$, and investigate the error–disturbance relation for the measurements of the $z$-component, $A = \sigma_{z}$, and the disturbance of the $x$-component, $B = \sigma_{x}$, of the spin. We suppose that the measurement starts at time $0$ and ends with the meter reading at time $t_0$. The measuring process, $\mathbf{M}$, determines the time evolution operator, $U$, of the composite system of system $\mathbf{S}$ prepared in arbitrary state $\rho$ plus probe $\mathbf{P}$ prepared in fixed vector state $|\xi\rangle$. In the Heisenberg picture we have the time evolution of the observables,

$$\sigma_z(0) = \sigma_z \otimes \mathbb{1}, \quad \sigma_z(t_0) = U^\dagger \sigma_z(0) U,$$

$$\sigma_x(0) = \sigma_x \otimes \mathbb{1}, \quad \sigma_x(t_0) = U^\dagger \sigma_x(0) U,$$

$$M(0) = \mathbb{1} \otimes M, \quad M(t_0) = U^\dagger M(0) U.$$  

The quantum root-mean-square (rms) error, $\varepsilon(\sigma_{z}) = \varepsilon(\sigma_{z}, \mathbf{M}, \rho)$, and quantum rms disturbance, $\eta(\sigma_{z}) = \varepsilon(\sigma_{x}, \mathbf{M}, \rho)$, are defined by

$$\varepsilon(\sigma_{z}) = \text{Tr}[((M(t_0) - \sigma_z(0))^2 \otimes |\xi\rangle \langle \xi|]^{1/2},$$

$$\eta(\sigma_{x}) = \text{Tr}[(\sigma_x(t_0) - \sigma_x(0))^2 \otimes |\xi\rangle \langle \xi]|^{1/2}.$$  

Since $\sigma_z^2 = \sigma_z = \mathbb{1}$ and $\sigma_x^2 = \sigma_x$, we obtain

$$\varepsilon(\sigma_{z})^2 = \sigma_{z}^2 = \mathbb{1}$$

and

$$\eta(\sigma_{z})^2 = \sigma_{z}^2 = \mathbb{1}.$$  

(12)

(13)

(14)
where
\[
D_{\sigma_x,\sigma_z} = \text{Tr}(\sqrt{\rho} \sigma_y \sqrt{\rho}), \tag{15}
\]
\[
\hat{e}(\sigma_z) = \sqrt{1 - \left(\frac{\hat{\eta}(\sigma_z)^2 - 2}{2}\right)^2}, \tag{16}
\]
\[
\hat{\eta}(\sigma_z) = \sqrt{1 - \left(\frac{\hat{\eta}(\sigma_z)^2 - 2}{2}\right)^2}, \tag{17}
\]
from the error–disturbance relation from Branciard \[18, 19\] and one of the authors \[20\]; see Eq. (A29) in the Appendix.

In the case where
\[
\langle \sigma_z \rangle = 0, \tag{18}
\]
relation (A32) is reduced to the tight relation
\[
(\hat{e}(\sigma_z)^2 - 2)^2 + (\hat{\eta}(\sigma_z)^2 - 2)^2 \leq 4 \tag{19}
\]
as shown in FIG. 1; see Eq. (A39) in the Appendix.

![FIG. 1. \(e(\sigma_z)^2 - \eta(\sigma_z)^2\) plot of tight error–disturbance relation for spin measurements in the state satisfying Eq. (A39).](image)

**B. Stern–Gerlach measurements**

Let us consider a measurement of the spin component of an electrically neutral spin-1/2 particle with a Stern–Gerlach apparatus. A particle moving along the y-axis passes through an inhomogeneous magnetic field, and then the orbit is deflected depending on the spin component of the particle along the direction of the magnetic field. This situation is illustrated in FIG. 2. To analyze this measurement, we make the following assumptions.

(i) The magnetic field points everywhere in the z-axis.

(ii) The strength of the magnetic field increases proportional to the z-coordinate,
\[
B_z = B_0 + B_1 z, \tag{20}
\]

![FIG. 2. Illustration of the experimental setup for a Stern–Gerlach measurement. The relations between the length and time interval are \(L_2 = v_y \Delta t\) and \(L_3 = v_y \tau\).](image)

To describe the measuring process, \(M\), of a Stern–Gerlach measurement, measured system \(S\) is taken as the spin degree of freedom described by the two-dimensional state space, \(H_s\), with the Pauli operators, \(\sigma_x, \sigma_y, \sigma_z\), describing the \(x, y, z\)-components of the spin, respectively, of the spin 1/2 particle. Probe system \(P\) is taken as the orbital degree of freedom in the \(z\)-direction described by the Hilbert space, \(K\), of wave functions with position \(Z\) and momentum \(P\) satisfying the canonical commutation relation
\[
[Z, P] = i\hbar. \tag{21}
\]
The particle enters the magnetic field at time 0, emerges out of the magnetic field at time \(\Delta t\), and freely evolves until time \(\Delta t + \tau\) at which the particle reaches the screen and the observer can measure the meter observable, \(M\), that assigns +1 or −1 depending on the particle \(z\)-coordinate, \(Z\), as \(M = f(Z)\), with function \(f\) such that
\[
f(z) = \begin{cases} 
-1 & \text{if } z \geq 0, \\
+1 & \text{otherwise.} 
\end{cases} \tag{22}
\]
Thus, the measuring process starts at time 0, when system S is in any input state ρ and probe P is prepared in state |ψ⟩, and ends up at time $t_0 = \Delta t + \tau$. The time evolution operator, $U = U(\Delta t + \tau)$, of the composite system $S + P$ during the measurement is determined by the time-dependent Hamiltonian, $H$, of the particle given by

$$H = \left\{ \begin{array}{ll}
\mu \sigma_z \otimes (B_0 + B_1 Z) + \frac{1}{2m} \mathbb{I} \otimes P^2 & (0 \leq t \leq \Delta t), \\
\frac{1}{2m} \mathbb{I} \otimes P^2 & (\Delta t \leq t \leq \Delta t + \tau),
\end{array} \right. $$

(23)

where $\mu$ denotes the magnetic moment of the particle and $m$ denotes the mass of the particle. By solving the Schrödinger equation, we obtain the time evolution operator, $U(t)$, of $S + P$ for $0 \leq t \leq \Delta t + \tau$ by

$$U(t) = \exp \left\{ \frac{t}{i\hbar} \left[ \mu \sigma_z \otimes (B_0 + B_1 Z) + \frac{1}{2m} \mathbb{I} \otimes P^2 \right] \right\} \exp \left\{ \frac{t - \Delta t}{2ihm} \mathbb{I} \otimes P^2 \right\} \times \exp \left\{ \frac{\Delta t}{i\hbar} \left[ \mu \sigma_z \otimes (B_0 + B_1 Z) + \frac{1}{2m} \mathbb{I} \otimes P^2 \right] \right\} \left( \Delta t \leq t \leq \Delta t + \tau \right).$$

(24)

To describe the time evolution of composite system $S + P$ in the Heisenberg picture, we introduce Heisenberg operators for $0 \leq t \leq \Delta t + \tau$ as

$$Z(0) = \mathbb{I} \otimes Z, \quad Z(t) = U(t)^\dagger Z(0) U(t),$$

$$P(0) = \mathbb{I} \otimes P, \quad P(t) = U(t)^\dagger P(0) U(t),$$

$$\sigma_j(0) = \sigma_j \otimes \mathbb{I}, \quad \sigma_j(t) = U(t)^\dagger \sigma_j(0) U(t),$$

$$H(0) = H, \quad H(t) = U(t)^\dagger P(0) U(t),$$

where $j = x, y, z$. We also use the matrix representations of Pauli operators as

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. $$

(25)

To consider the time evolution of these observables from time $t = \Delta t$ to time $\Delta t + \tau$, suppose $\Delta t \leq t \leq \Delta t + \tau$. By the Heisenberg equation of motion, position operator $Z(t)$ satisfies

$$\frac{d}{dt} Z(t) = \frac{1}{i\hbar} [Z(t), \frac{1}{2m} P(t)^2] = \frac{1}{m} P(t).$$

(26)

Thus, we have

$$Z(t) = Z(\Delta t) + \frac{1}{m} \int_{\Delta t}^t P(t') dt'.$$

(27)

In contrast, $P(t)$ does not change since $[P(t), H(t)] = 0$. Consequently, we have

$$Z(t) = Z(\Delta t) + \frac{t - \Delta t}{m} P(\Delta t),$$

$$P(t) = P(\Delta t).$$

(28)

(29)

Since $\sigma_x(t)$ and $\sigma_y(t)$ commute with $H(t)$, we have

$$\sigma_x(t) = \sigma_x(\Delta t), \quad \sigma_y(t) = \sigma_y(\Delta t).$$

(30)

To describe the observables at time $t = \Delta t$ in terms of the observables at time $t = 0$, suppose $0 \leq t \leq \Delta t$. With the Heisenberg equations of motion, we obtain

$$\frac{d}{dt} Z(t) = \frac{1}{i\hbar} [Z(t), H(t)] = \frac{1}{m} P(t),$$

and

$$Z(\Delta t) = Z(0) + \frac{1}{m} \int_0^{\Delta t} P(t) dt.$$

(31)

(32)

On the other hand, we have

$$\frac{d}{dt} P(t) = \frac{1}{i\hbar} [P(t), H(t)] = -\mu B_1 \sigma_z(t).$$

(33)

Now, $\sigma_z(t)$ commutes with Hamiltonian $H(t)$. Hence, we have

$$\sigma_z(t) = \sigma_z(0).$$

(34)

Consequently, we have

$$P(t) = P(0) - \mu B_1 t \sigma_z(0),$$

$$Z(t) = Z(0) + \frac{t}{m} P(0) - \mu B_1 t^2 \sigma_z(0).$$

(35)

(36)

Therefore, we have

$$Z(\Delta t + \tau) = Z(0) + \frac{\Delta t + \tau}{m} P(0) - \mu B_1 \frac{\Delta t + \tau}{2m} \sigma_z(0),$$

$$P(\Delta t + \tau) = P(0) - \mu B_1 \Delta t \sigma_z(0),$$

$$\sigma_z(\Delta t + \tau) = \sigma_z(0).$$

(37)

(38)

(39)

Next, we calculate the $x$- and $y$-components of the spin of the particle at time $t = \Delta t + \tau$. Since Hamiltonian $H(t)$ from time $t = \Delta t$ to time $\Delta t + \tau$ commutes with $\sigma_x(t)$ and $\sigma_y(t)$, we have

$$\sigma_x(t) = \sigma_x(\Delta t),$$

$$\sigma_y(t) = \sigma_y(\Delta t).$$

(40)

(41)

If $\Delta t \leq t \leq \Delta t + \tau$, and it suffices to calculate $\sigma_x(\Delta t)$ and $\sigma_y(\Delta t)$. Suppose $0 \leq t \leq \Delta t$. By the Heisenberg equations of motion, we have

$$\frac{d}{dt} \sigma_x(t) = \frac{1}{i\hbar} [\sigma_x(t), H(t)]$$

$$= \frac{1}{i\hbar} \left[ \sigma_x(t), \frac{P(t)^2}{2m} + \mu (B_0 + B_1 Z(t)) \sigma_z(t) \right]$$

$$= \frac{\mu}{i\hbar} (B_0 + B_1 Z(t)) (-2i\sigma_y(t))$$

$$= -2\frac{\mu}{\hbar} (B_0 + B_1 Z(t)) \sigma_y(t).$$

(42)
Similarly, we have
\[
\frac{d}{dt} \sigma_y(t) = \frac{1}{i\hbar} \left[ \sigma_y(t), H(t) \right]
= \frac{1}{i\hbar} \left[ \sigma_y(t), \frac{P(t)^2}{2m} + \mu (B_0 + B_1 Z(t)) \sigma_z(t) \right]
= \frac{\mu}{i\hbar} (B_0 + B_1 Z(t)) (2i\sigma_x(t))
= 2\mu \frac{t}{\hbar} (B_0 + B_1 Z(t)) \sigma_x(t).
\]
(43)

Now, let us introduce \( \sigma_+ \) and \( \sigma_- \) by
\[
\sigma_+(t) = \frac{1}{\sqrt{2}} (\sigma_x(t) + i\sigma_y(t)),
\]
(44)
\[
\sigma_-(t) = \frac{1}{\sqrt{2}} (\sigma_x(t) - i\sigma_y(t)).
\]
(45)

From Eqs. (42) and (43), we have
\[
\frac{d}{dt} \sigma_\pm(t) = \pm \frac{2\mu i}{\hbar} \left[ B_0 + B_1 (U^\dagger(t)Z(0)U(t)) \right] \sigma_\pm(t).
\]
(46)

Let
\[
\gamma_\pm(t) = U(t)\sigma_\pm(t) = \exp \left[ \frac{H(0)}{i\hbar} t \right] \sigma_\pm(t).
\]
(47)

The left-hand side (LHS) and right-hand side (RHS) of Eq. (46) satisfy
\[
\text{LHS} = \frac{d}{dt} U(-t)\gamma_\pm(t)
= -\frac{H(0)}{i\hbar} U(-t)\gamma_\pm(t) + U(-t)\frac{d}{dt} \gamma_\pm(t),
\]
(48)

\[
\text{RHS} = \pm \frac{2\mu i}{\hbar} \left[ U^\dagger(t) (B_0 + B_1 Z(0)) U(t) \right] \gamma_\pm(t)
= \pm \frac{2\mu i}{\hbar} U(-t) (B_0 + B_1 Z(0)) \gamma_\pm(t).
\]
(49)

Hence, we have
\[
\frac{d}{dt} \gamma_\pm(t) = \left[ \frac{H(0)}{i\hbar} \pm \frac{2\mu i}{\hbar} (B_0 + B_1 Z(0)) \right] \gamma_\pm(t).
\]
(50)

The solution of the above differential equation is given by
\[
\gamma_\pm(t) = \exp \left\{ \frac{it}{\hbar} \left[ -H(0) \pm 2\mu (B_0 + B_1 Z(0)) \right] \right\} \gamma_\pm(0).
\]
(51)

Since \( \gamma_\pm(0) = \sigma_\pm(0) \), we have
\[
\sigma_\pm(t) = \exp \left\{ \frac{it}{\hbar} H(0) \right\}
\times \exp \left\{ \frac{it}{\hbar} \left[ -H(0) \pm 2\mu (B_0 + B_1 Z(0)) \right] \right\} \sigma_\pm(0).
\]
(52)

Using the Baker-Campbell-Hausdorff formula [41], we have
\[
\exp A \exp B = \exp \left[ (A + B) + \frac{1}{2} [A, B] \right.
+ \frac{1}{12} \left[ [A, B], B \right] - \left[ [A, B], A \right] + \cdots \right].
\]
(53)

Hence, for
\[
A = \frac{it}{\hbar} H(0),
\]
(54)
\[
B = \frac{it}{\hbar} \left[ -H(0) \pm 2\mu (B_0 + B_1 Z(0)) \right],
\]
(55)

we have
\[
[A, B] = \left[ \frac{it}{\hbar} H(0), \frac{it}{\hbar} \left( -H(0) \pm 2\mu (B_0 + B_1 Z(0)) \right) \right]
= -\left\{ \frac{t^2}{\hbar^2} \left[ \frac{1}{2m} P(0)^2, \pm 2\mu (B_0 + B_1 Z(0)) \right] \right\}
= \pm \frac{2i\mu B_1 t^2}{m\hbar} P(0),
\]
(56)

\[
[[A, B], A] = \left[ \pm \frac{2i\mu B_1 t^2}{m\hbar} P(0), \frac{it}{\hbar} H(0) \right]
= \pm \frac{2i\mu B_1 t^3}{m\hbar^2} \left[ P(0), \mu (B_0 + B_1 Z(0)) \sigma_z(0) \right]
= \pm \frac{2i\mu^2 B_1^3 t^3}{m\hbar^3} \sigma_z(0),
\]
(57)

\[
[[A, B], B] = \left[ \mp \frac{2i\mu B_1 t^2}{m\hbar} P(0), \frac{it}{\hbar} \left( -H(0) \pm 2\mu (B_0 + B_1 Z(0)) \right) \right]
= \pm \frac{2i\mu^2 B_1^3 t^3}{m\hbar^3} \sigma_z(0)
= \pm \frac{2i\mu^2 B_1^3 t^3}{m\hbar^3} \left( 2 \mp \sigma_z(0) \right).
\]
(58)

The commutators of the higher orders “…” in Eq. (53) are 0 since the third commutators, \([ [A, B], A ] \) and \([ [A, B], B ] \), commute with \( A \) and \( B \), respectively.

Let
\[
R(t) = \frac{\mu^2 B_1^3 t^3}{3m\hbar},
\]
(59)
\[
S(t) = \frac{2\mu t}{\hbar} \left[ B_0 + B_1 \left( Z + \frac{t}{2m} P \right) \right].
\]
(60)

We have
\[
\sigma_\pm(t) = \exp i \left\{ [R(t) \pm S(t)] \mathbb{I} \mp R(t) \sigma_z(0) \right\} \sigma_\pm(0).
\]
(61)

Since
\[
\sigma_+(0) = \frac{1}{\sqrt{2}} (\sigma_z(0) + i\sigma_y(0)) = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix},
\]
(62)
\[
\sigma_-(0) = \frac{1}{\sqrt{2}} (\sigma_z(0) - i\sigma_y(0)) = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix},
\]
(63)
we have
\[
\sigma_+(t) = \begin{pmatrix} \exp iS(t) & 0 \\ 0 & \exp i(S(t) + 2R(t)) \end{pmatrix} \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix} = \exp iS(t)\sigma_+(0),
\]
\[
\sigma_-(t) = \begin{pmatrix} \exp i(-S(t) + 2R(t)) & 0 \\ 0 & \exp -iS(t) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2} \end{pmatrix} = \exp -iS(t)\sigma_-(0).
\]
Therefore, \( \sigma_x(t) \) and \( \sigma_y(t) \) from time \( t = 0 \) to time \( t = \Delta t \) are
\[
\sigma_x(t) = \frac{1}{\sqrt{2}}(\sigma_+(t) + \sigma_-(t)) = \begin{pmatrix} 0 & \exp iS(t) \\ \exp -iS(t) & 0 \end{pmatrix},
\]
\[
\sigma_y(t) = -\frac{i}{\sqrt{2}}(\sigma_+(t) - \sigma_-(t)) = \begin{pmatrix} 0 & -i\exp iS(t) \\ i\exp -iS(t) & 0 \end{pmatrix}.
\]

III. ERROR

Let us consider the quantum rms error of a Stern–Gerlach measurement, \( M \), of the z-component, \( \sigma_z(0) \), of the spin at time 0 using the meter observable, \( M(\Delta t + \tau) = f(Z(\Delta t + \tau)) \), introduced in Section II. The noise operator, \( N \), of this measurement is given by
\[
N = M(\Delta t + \tau) - \sigma_z(0).
\]
Initial state \( \rho \) of spin \( S \) is supposed to be an arbitrary state with the matrix
\[
\rho = \sum_{i=1}^{n} p_i \left( \begin{array}{cc} c_i^+ c_i^+ & c_i^+ c_i^- \\ c_i^- c_i^+ & c_i^- c_i^- \end{array} \right),
\]
where \( \{p_i\}_{i=1}^{n} \) is a sequence of positive numbers such that \( \sum_{i=1}^{n} p_i = 1 \) and \( c_i^\pm \) are complex numbers where \( |c_i^+|^2 + |c_i^-|^2 = 1 \) for all \( i \), so that the initial state of the composite system \( S + P \) is given by \( \rho \otimes |\xi\rangle \langle \xi| \), where \( |\xi\rangle \) is a fixed but arbitrary wave function describing the initial state of the orbital degree of freedom, \( P \). Then, the error, namely, the quantum rms error, of this measurement of \( \sigma_z \) is given by
\[
\varepsilon(\sigma_z) = \sqrt{\langle N^2 \rangle_{\rho \otimes |\xi\rangle \langle \xi|}},
\]
where we abbreviate \( \text{Tr}[A\rho] \) as \( \langle A \rangle_{\rho} \) for observable \( A \) and density operator \( \rho \).

Let
\[
U_i = \exp \left[ \frac{t}{2i\hbar} P^2 \right],
\]
\[
g_0 = \frac{\mu B_1 \Delta t}{m} \left( \tau + \frac{\Delta t}{2} \right).
\]
From Eq. [77], we have
\[
Z(\Delta t + \tau) = 1 \otimes (U_{\Delta t + \tau} U_{\Delta t + \tau} - g_0 \sigma_z \otimes 1) = U_{\Delta t + \tau} \left( Z - g_0 0 \right) U_{\Delta t + \tau}.
\]
Thus, we have
\[
N = M(\Delta t + \tau) - \sigma_z(0) = f(Z(\Delta t + \tau)) - \sigma_z(0)
\]
\[
= U_{\Delta t + \tau} \left( f(Z - g_0) - 1 \right) f(Z + g_0) + 1 U_{\Delta t + \tau}
\]
\[
= 2U_{\Delta t + \tau} \left( -\chi_+(Z - g_0) \right) \sigma_z \left( Z + g_0 \right) U_{\Delta t + \tau},
\]
where
\[
\chi_+(z) = \begin{cases} 1 & \text{if } z \geq 0, \\ 0 & \text{otherwise}, \end{cases}
\]
\[
\chi_-(z) = 1 - \chi_+(z),
\]
\[
f(z) = 1 - 2\chi_+(z).
\]
It follows that
\[
N^2 = M(\Delta t + \tau) - \sigma_z(0)
\]
\[
= 4U_{\Delta t + \tau} \left( \chi_+(Z - g_0) 0 \right) \chi_- (Z + g_0) U_{\Delta t + \tau}.
\]
Therefore, we have
\[
\varepsilon(\sigma_z)^2 = \langle N^2 \rangle_{\rho \otimes |\xi\rangle \langle \xi|}
\]
\[
= \sum_{i=1}^{n} p_i \left( \left( c_i^+ c_i^- \right) \otimes \chi_+(Z - g_0) \right) \left( c_i^- c_i^+ \right) \otimes \chi_-(Z + g_0) \right)
\]
\[
= 4 \sum_{i=1}^{n} p_i \left| c_i^+ \right|^2 \left| \chi_+(Z - g_0) U_{\Delta t + \tau} |\xi\rangle \right|^2 + \left| c_i^- \right|^2 \left| \chi_-(Z + g_0) U_{\Delta t + \tau} |\xi\rangle \right|^2.
\]
Consequently, we have
\[
\varepsilon(\sigma_z)^2 = 4 \sum_{i=1}^{n} p_i \left( \left| c_i^+ \right|^2 \int_{g_0}^{\infty} |U_{\Delta t + \tau} |\xi\rangle |^2 dz + \left| c_i^- \right|^2 \int_{-\infty}^{-g_0} |U_{\Delta t + \tau} |\xi\rangle |^2 dz \right).
\]
IV. DISTURBANCE

Let us consider the quantum rms disturbance, $\eta(\sigma_x)$, for the $x$-component of the spin in Stern–Gerlach measurements. The disturbance operator, $\sigma_x$, is given by

$$D = \sigma_x(\Delta t + \tau) - \sigma_x(0).$$

From Eqs. (40) and (66),

$$D = \left( \begin{array}{cc} 0 & \exp i S(\Delta t) - 1 \\ \exp -i S(\Delta t) - 1 & 0 \end{array} \right).$$

Consequently, we have

$$D^2 = \mathbb{1} \otimes (2 - 2 \cos S(\Delta t)).$$

Thus, we have

$$\eta(\sigma_x)^2 = 2 - 2 \left\langle \cos \left\{ \frac{2\mu\Delta t}{\hbar} \left[ B_0 + B_1 \left( Z + \frac{\Delta t}{2m} P \right) \right] \right\} \right\rangle_\xi.$$

V. ERROR AND DISTURBANCE FOR GAUSSIAN STATES

Let us consider the error and disturbance in Stern–Gerlach measurements under the condition that the orbital state of the particle is in the family $\mathcal{G}$ of Gaussian states given by

$$\mathcal{G} = \left\{ \xi \in L^2(\mathbb{R}) \mid \xi(\lambda(z) = A \exp[-\lambda z^2], \int_{-\infty}^{\infty} |\xi(\lambda)|^2 dz = 1, \lambda \in \mathbb{C}, \text{Re}(\lambda) > 0 \right\}.$$ (85)

This family of states consists of all Gaussian pure states [40], whose mean values of the position and momentum are both 0. For simplicity, it is assumed that the spin state of the particle is in the eigenstate of the spin component, $\sigma_y$. It is easy to minimize the error of the measurement with respect to the mean values of the position and momentum. In particular, $\mathcal{G}$ is the family of optimal states for the measurement among the Gaussian pure states if the spin state of the particle is the eigenstate of $\sigma_y$. We remark that the equality in the Schrödinger inequality (see Eq. (B1) in Appendix B1) holds for any state $\xi$ in $\mathcal{G}$, i.e.,

$$\langle Z^2 \xi | P^2 \xi \rangle - \frac{1}{4} \langle Z, P \rangle^2 \xi = \frac{\hbar^2}{4}.$$

Here, we use the abbreviation $\langle A \rangle_\xi = \langle \xi | A | \xi \rangle$. The converse also holds, that is, any state $\xi$ satisfying $\langle P \rangle_\xi = \langle Z \rangle_\xi = 0$ and Eq. (86) belongs to $\mathcal{G}$.

Let us consider the range of the error and disturbance of Stern–Gerlach measurements. Let

$$V(\psi, t) = \left\langle \left( \frac{Z + t P}{m} \right)^2 \right\rangle_{\psi}.$$ (87)

for any orbital state $\psi$. For disturbance $\eta(\sigma_x)$, from Eq. (84),

$$\eta(\sigma_x)^2 = \frac{1}{\hbar} \int_{-\infty}^{\infty} \exp \left( -2 \frac{z^2}{V(\xi, \Delta t/2)} \right) \cos \left( \frac{2\mu\Delta t}{\hbar} \left( B_0 + B_1 z \right) \right) dz.$$ (85)

From the above formula, the disturbance is determined by $V(\xi, \Delta t/2)$ and the parameters of the magnet if the orbital state is in $\mathcal{G}$. Now, for a fixed constant, $v$, let us find the error for state $\xi_\lambda$ in $\mathcal{G}$ and time interval $\Delta t$ satisfying $V(\xi_\lambda, \Delta t/2) = v$. In the following, we fix the time interval, $\Delta t$.

From Eq. (80), we have

$$\varepsilon(\sigma_z)^2 = 4 \int_{0}^{\infty} |U_{\lambda t + \tau} \xi_\lambda(z)|^2 dz = 4 \int_{0}^{\infty} \exp(-w^2) dw.$$ (90)

Here, we use the relation $|c_+| = |c_-|$, which is obtained from the assumption that the mean value of the $z$-component of the spin of the particle is 0. Equation (80) shows that the error is minimized by maximizing the lower limit of the integration $g_0/\sqrt{2V(\xi_\lambda, \Delta t + \tau)}$. First, we fix state $\xi_\lambda$ and focus on time interval $\tau$. Setting $W_{\xi_\lambda}(\tau) = g_0/\sqrt{2V(\xi_\lambda, \Delta t + \tau)}$, it is easy to verify that if

$$m \langle \{Z, P \} \rangle_{\xi_\lambda} + \langle P^2 \rangle_{\xi_\lambda} \Delta t < 0$$

holds, then $W_{\xi_\lambda}(\tau)$ assumes maximum value

$$W_{\xi_\lambda}(\tau_0) = \frac{\sqrt{2V(\xi_\lambda, \Delta t/2)} \mu B_1 \Delta t}{\hbar}$$

at

$$\tau = \tau_0 = \frac{4m^2 \langle Z^2 \rangle_{\xi_\lambda} + 3m \langle \{Z, P \} \rangle_{\xi_\lambda} \Delta t + 2 \langle P^2 \rangle_{\xi_\lambda} \Delta t^2}{2 \left( m \langle \{Z, P \} \rangle_{\xi_\lambda} + \langle P^2 \rangle_{\xi_\lambda} \Delta t \right)}.$$ (91)
On the other hand, if condition \( [90] \) does not hold, the supremum of \( W_{\xi\lambda}(\tau) \) is given by
\[
\sup_{\tau \geq 0} W_{\xi\lambda}(\tau) = \lim_{\tau \to \infty} W_{\xi\lambda}(\tau) = \frac{\mu B_1 \Delta t}{\sqrt{2}} \langle P^2 \rangle_{\xi\lambda}^{-1/2}.
\]
(93)

Next, let us consider the maximization of \( W_{\xi\lambda}(\tau) \) with respect to state \( \xi\lambda \). For any pair of orbital states, \( \psi \) and \( \phi \), in \( \mathcal{G} \) satisfying \( V(\psi, \Delta t/2) = v \) and \( V(\phi, \Delta t/2) = w \), respectively, if \( \psi \) satisfies condition \( [90] \), then
\[
W_\psi(\tau_0) \geq \lim_{\tau \to \infty} W_\phi(\tau)
\]
(94)
holds. This inequality follows immediately from the fact that the Kennard inequality \( [2] \) holds for any two canonically conjugate observables that satisfy the canonical commutation relation. Therefore, we obtain the supremum of \( W_{\xi\lambda}(\tau) \) with respect to state \( \xi\lambda \) and time interval \( \tau \) as
\[
\sup_{\text{Re}(\lambda) > 0, \tau \geq 0} W_{\xi\lambda}(\tau) = \frac{\sqrt{2\mu B_1 \Delta t}}{\hbar}.
\]
(95)

Although the above argument is for finding the range of the error and disturbance that Stern–Gerlach measurements can achieve, it contains one more important assertion. That is, the calculation suggests that the error of Stern–Gerlach measurements is minimized by placing the screen at a finite distance from the magnet under the condition represented by \( [90] \), in contrast to the conventional assumption that the error is minimized by placing the screen at infinity. If a state in \( \mathcal{G} \) satisfies condition \( [90] \), then the correlation term \( [11] \),
\[
\left\{ \left\{ Z - \langle Z \rangle_{\xi\lambda}, P - \langle P \rangle_{\xi\lambda} \right\} \right\}_{\xi\lambda}
\]
, is negative, and this leads to a narrowing of the standard deviation of the position of the particle during the free evolution (see Appendix B.3). Such a class of states is introduced by Yuen \( [11] \) and they are known as contractive states.

Let us return to the problem of finding the range of the values of the error and disturbance that Stern–Gerlach measurements can assume. Now, setting \( W_0 = \sqrt{2\mu B_1 \Delta t} \), the disturbance and the infimum of the error under the condition that \( V(\lambda, \Delta t/2) = v \) for fixed \( \Delta t \) and \( v \) are
\[
\eta_x^2 = 2 - 2 \exp \left(-W_0^2\right) \cos \frac{2\mu \Delta t B_0}{\hbar},
\]
(96)
\[
\inf \varepsilon_x^2 = \frac{4}{\sqrt{\pi}} \int_{-W_0}^{\infty} \exp(-w^2) dw,
\]
(97)
respectively. By varying the parameter of the magnet, \( B_0 \), we obtain the range of the disturbance as
\[
2 - 2 \exp \left(-W_0^2\right) \leq \eta_x^2 \leq 2 + 2 \exp \left(-W_0^2\right).
\]
(98)

We obtain the range of the disturbance and the infimum of the error of Stern–Gerlach measurements for each constant, \( v \). By varying \( v \), we obtain the range of the error and disturbance as the inequalities
\[
\frac{\eta_x^2(\sigma_x) - 2}{2} \leq \exp \left\{ -\left[ \text{erf}^{-1} \left( \frac{\varepsilon_x^2(\sigma_x) - 2}{2} \right) \right]^2 \right\},
\]
(99)
\[
0 \leq \varepsilon_x^2(\sigma_x) \leq 2,
\]
(100)
where \( \text{erf}^{-1} \) represents the inverse of the error function \( \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} \exp(-s^2) ds \). The square of the error varies from 0 to 2 since \( W_0 \) is positive.

Let us consider another meter, \( M' \):
\[
M' = -M.
\]
(101)

Then, the relation between errors \( \varepsilon(\sigma_z) \) and \( \varepsilon'(\sigma_z) \) determined by the two meters, \( M \) and \( M' \), respectively, is
\[
\varepsilon'(\sigma_z)^2 = 4 - \varepsilon(\sigma_z)^2.
\]
(102)

Therefore, the range of the error and disturbance of Stern–Gerlach measurements is
\[
\left| \frac{\eta_x^2(\sigma_x) - 2}{2} \right| \leq \exp \left\{ -\left[ \text{erf}^{-1} \left( \frac{\varepsilon_x^2(\sigma_x) - 2}{2} \right) \right]^2 \right\}
\]
(103)

The plot of this region is shown in FIG. 3. For comparison, the figure shows the plot of the boundary of the tight error–disturbance relation for general spin measurement found by Branciard \( [18] \) for pure states and subsequently generalized to mixed states by Ozawa \( [21] \).
\[
\left| \frac{\eta_x^2(\sigma_x) - 2}{2} \right| \leq \sqrt{1 - \left( \frac{\varepsilon_x^2(\sigma_x) - 2}{2} \right)^2}.
\]
(104)
This relation is known as the general error–disturbance relation between the error of measurements of an observable, $A$, by a meter observable, $M$, and thereby caused disturbance of an observable $B$ satisfying $A^2 = B^2 = I$ and $M^2 = I$. The initial state is chosen to be the same as in Stern–Gerlach measurements considered in this paper. From this plot, we conclude that the range of the error and disturbance for Stern–Gerlach measurements considered in this paper occupies most of the possible range of the error and disturbance for general spin measurements. Here, the range of the error and disturbance for Stern–Gerlach measurements is also compared with Heisenberg’s uncertainty relation $\frac{\varepsilon}{2} \geq 1$, (105) and the theoretical estimate of the experiment conducted by Hasegawa and co-workers [22] (black line),

$$\frac{\eta(\sigma_z)^2 - 2}{2} \leq 1 - \left(\frac{\varepsilon(\sigma_z)^2 - 2}{2}\right)^2.$$  (106)

The enlarged plot is shown in FIG. 4.

**VI. COMPARISON WITH "ASPECTS OF NONIDEAL SERN–GERLACH EXPERIMENT AND TESTABLE RAMIFICATIONS"**

Home et al. [39] discussed the same error of Stern–Gerlach measurements as our paper does for similar conditions. Therefore, their paper is among the papers preceding ours. We consider in what sense their paper is related to ours, and we compare its results with ours. They derived the wave function of a particle in the Stern–Gerlach apparatus under the following conditions.

(i) The magnetic field is oriented along the $z$-axis everywhere, and the gradient of the $z$-component of the magnetic field is non-zero only in the $z$-direction.

(ii) The initial orbital state is a Gaussian state whose mean values of the position and momentum, and the correlation term of the particle in the wave function are all zero.

(iii) Unlike Bohm’s discussion [35], the kinetic energy of the particle in the magnetic field is not neglected.

Based on their argument, they discussed the distinguishability of the value of the measured observable by observing the probe system directly in Stern–Gerlach measurements. To consider this problem, they introduced the two indices,

$$I := \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi_+^*(x, \tau) \psi_-(x, \tau) dx \right|, \quad (107)$$

$$E(t) := \int_{-\infty}^{0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\psi_+(x, t)|^2 dx dy dz, \quad (108)$$

where $\psi_\pm$ are the wave functions of the particle in the Schrödinger picture whose spin $z$-components are $\pm 1/2$, respectively. The origin of time is taken to be the moment when the particle enters the Stern–Gerlach magnet. $\tau$ is the time at which the particle emerges from the Stern–Gerlach magnet ($\tau$ corresponds to $\Delta t$ in our notation), and $t$ is any time after emerging from the Stern–Gerlach magnet ($\tau$ corresponds to $\Delta t + \tau$ in our notation). Namely, they adopted the inner product, $I$, of the two wave functions with different spin directions, and the probability, $E(t)$, of finding the particle with the spin $z$-components of $+1/2$ and $-1/2$ within the lower and upper half planes, respectively, at time $t$. They concluded that $I$ always vanishes whenever $E(t)$ vanishes, but that $E(t)$ does not necessarily vanish even when $I$ vanishes.

We discuss the relationship between their paper and ours. The relationship between the quantities $E(t)$ and $\varepsilon(\sigma_z)$ is

$$\varepsilon(\sigma_z)^2 = 4E(t). \quad (109)$$

However, the error we obtained has a rich theoretical background [21], and thus we can infer more consequences from our result about Stern–Gerlach measurements than from theirs. We compare their research with ours as follows.

(i) Their set up and approximation are the same as ours and they used the same Hamiltonian as in our research.

(ii) In both papers, the orbital state of the particle is assumed to be the pure state where the mean values of its position and momentum are zero. We assume that the correlation term of a Gaussian pure state is not necessarily zero, whereas they assumed that the orbital state is a Gaussian pure state with no correlation.

(iii) We evaluate the tradeoff between the error and disturbance, whereas they compared the error with...
VII. CONCLUSION

We studied the performance of Stern–Gerlach measurements from the viewpoint of the error and disturbance. Consequently, we obtained the formulas for the error and disturbance of Stern–Gerlach measurements under the condition that the magnetic field points in the same direction everywhere. From the process of deriving the error and disturbance, we know that the error of Stern–Gerlach measurements arises from the spread of the wave function, and that the disturbance of the measurement arises from the precession of the spin due to the magnetic field. Furthermore, we obtained the range of the error and disturbance of Stern–Gerlach measurements under the condition that the state of the probe system is a Gaussian pure state. This revealed that the lower bound of the range of the error and disturbance of Stern–Gerlach measurements is less than the theoretical estimation for neutron spin measurements by Erhart et al. [22], and that the error–disturbance relation for a Stern–Gerlach measurement can break the Heisenberg uncertainty relation. It is interesting to see that Stern–Gerlach measurements occupy a broad area in the general range with a characteristic shape. Furthermore, it is revealed that there is an initial probe system state in which the error is minimized after a finite time interval of free evolution, in contrast to the conventional assumption that the error is minimized after an infinite time. These new results may overthrow the established theory of Stern–Gerlach measurements. Therefore, the experiment should be verified experimentally.

We assumed that the orbital state of the particle is a Gaussian state when we consider the range of the error and disturbance tradeoff. However, the range of the error and disturbance in Stern–Gerlach measurements for general states is not revealed, and we should determine the relation. In addition, we should analyze more realistic models of Stern–Gerlach measurements; for example, a model using the magnetic field satisfying Maxwell’s equations or a model considering the decoherence of the particle during the measuring process.

VIII. ACKNOWLEDGEMENT

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Appendix A: Error and disturbance in quantum measurements

In this section, we review the general theory of the error and disturbance in quantum measurements referring to [17,21].

1. Classical root-mean-square error

Let us consider the classical case first. Recall the root-mean-square (rms) error introduced by Gauss [43]. Consider a measurement of value $x$ of quantity $X$ by actually observing value $y$ of a meter quantity, $Y$. Then, the error of this measurement is given by $y - x$. If these quantities obey a joint probability distribution, $\mu(x,y)$, then the rms error, $\varepsilon_G(\mu)$, is defined as

$$\varepsilon_G(\mu) = \left( \sum_{x,y} (y - x)^2 \mu(x,y) \right)^{1/2}. \quad (A1)$$

2. Quantum measuring processes

In this paper, we consider only finite-type quantum measurements, such that the measured system is described by a finite dimensional Hilbert space and that the apparatus has only a finite number of possible outcomes. We consider a quantum system, $S$, described by a finite dimensional Hilbert space, $\mathcal{H}$. We assume that every measuring apparatus for system $S$ has its own output variable, $x$. The statistical properties of the apparatus, $A(x)$, having output variable $x$ are determined by (i) probability distribution $Pr\{x = m||\rho\}$ of $x$ for input state $\rho$, and (ii) output state $\rho_{x=m}$, given the outcome, $x = m$.

A measuring process of an apparatus $A(x)$ measuring $S$ is specified by a quadruple, $M = (K, |\xi\rangle, U, M)$, consisting of a Hilbert space, $K$, describing probe system $P$, a state vector, $|\xi\rangle$, in $K$ describing the initial state of $P$, a unitary operator, $U$, on $\mathcal{H} \otimes K$ describing the time evolution of composite system $S + P$ during the measuring interaction, and an observable, $M$, called the meter observable, of $P$ describing the meter of the apparatus $S$.

3. Operational description

For any measuring process $M$, its instrument is defined as a completely positive map valued function, $\mathcal{I}$, given by

$$\mathcal{I}(m)\rho = \text{Tr}_K[(1 \otimes P^M(m))U(\rho \otimes |\xi\rangle \langle \xi|)U^\dagger]. \quad (A2)$$

for any state $\rho$ and real number $m$, where $\text{Tr}_K$ is the partial trace over $K$ and $P^M(m)$, the spectral projection of $M$ corresponding to real number $m$. The statistical
properties of $A(x)$ are determined by the instrument, $I$, of $M$ as
\[ \Pr\{x = m \| \rho\} = \text{Tr}[I(m)\rho], \tag{A3} \]
\[ \rho(x=m) = \frac{I(m)\rho}{\text{Tr}[I(m)\rho]}. \tag{A4} \]

4. Heisenberg picture

In measuring process $M$, suppose that the measuring interaction is turned on from time $t = 0$ to time $t = t_0$. To describe the time evolution of composite system $S + P$ in the Heisenberg picture, let $A(0) = A \otimes I$ and $M(t_0) = U^{\dagger}(I \otimes M)U$.

Then, $POVM$ II of $M$ is defined as
\[ \Pi(m) = \langle \xi | P^{M(t_0)}(m) | \xi \rangle. \tag{A5} \]

The $n$-th moment operator of $\Pi$ for $n = 1, \ldots, n$ is defined by
\[ \hat{P}^{(n)} = \langle \xi | M(t_0)^n | \xi \rangle. \tag{A6} \]

The dual non-selective operation, $T^*$, of $M$ is defined by
\[ T^*(A) = \langle \xi | A(t_0) | \xi \rangle \tag{A7} \]
for any observable $A$ of $S$.

5. Measurement of observables

If observables $A(0)$ and $M(t_0)$ commute in the initial state, $\rho \otimes |\xi\rangle\langle\xi|$, that is,
\[ [P^{A(0)}(a), P^{M(t_0)}(m)](\rho \otimes |\xi\rangle\langle\xi|) = 0 \tag{A8} \]
for all $a, m \in \mathbb{R}$, then their joint probability distribution, $\mu(a, m)$, is defined as
\[ \mu(a, m) = \text{Tr}[P^{A(0)}(a)P^{M(t_0)}(m) \rho \otimes |\xi\rangle\langle\xi|)] \tag{A9} \]
and satisfies
\[ \text{Tr}[f(A(0), M(t_0))(\rho \otimes |\xi\rangle\langle\xi|)] = \sum_{a,m} f(a, m) \mu(a, m) \tag{A10} \]
for any polynomial $f(A(0), M(t_0))$ of $A(0)$ and $M(t_0)$.

We say that measuring process $M$ accurately measures observable $A$ in state $\rho$ if $A(0)$ and $M(t_0)$ are perfectly correlated in state $\rho \otimes |\xi\rangle\langle\xi|$, namely, one of the following two equivalent conditions holds.

(S) $A(0)$ and $M(t_0)$ commute in $\rho \otimes |\xi\rangle\langle\xi|$ and their joint probability distribution, $\mu$,
\[ \sum_{a, m : a = m} \mu(a, m) = 1. \tag{A11} \]

(W) For any $a, m \in \mathbb{R}$ with $a \neq m$,
\[ \text{Tr} \left[ \Pi(m)P^{A}(a) \rho \right] = 0. \tag{A12} \]

6. Quantum root-mean-square error

The noise operator $N(A, M)$ of the measuring process $M$ for measuring $A$ is defined as
\[ N(A, M) = M(t_0) - A(0). \tag{A13} \]

The quantum root-mean-square (q-rms) error, $\varepsilon(A, M, \rho)$, for measuring $A$ in $\rho$ by $M$ is defined as the rms of the noise operator, i.e.,
\[ \varepsilon(A, M, \rho) = \text{Tr}[N(A, M)^2(\rho \otimes |\xi\rangle\langle\xi|)]^{1/2}. \tag{A14} \]

One of the authors [21] showed that the q-rms error, $\varepsilon$, has the following properties.

(i) **Operational definability.** The q-rms error, $\varepsilon$, is definable by the POVM, $\Pi$, of measuring process $M$, observable $A$ to be measured, and initial state $\rho$ of the measured system $S$ as
\[ \varepsilon(A, M, \rho) = \text{Re} \text{Tr}[(A^2 - 2A\hat{\Pi}^{(1)} + \hat{\Pi}^{(2)})\rho]. \tag{A15} \]

(ii) **Correspondence principle.** In the case where $A(0)$ and $M(t_0)$ commute in $\rho \otimes |\xi\rangle\langle\xi|$, the relation
\[ \varepsilon(A, M, \rho) = \varepsilon_C(\mu) \tag{A16} \]
holds for the joint probability distribution $\mu$ of $A(0)$ and $M(t_0)$ in $\rho \otimes |\xi\rangle\langle\xi|$.

(iii) **Soundness.** If $M$ accurately measures $A$ in $\rho$, then $\varepsilon$ vanishes, i.e.,
\[ \varepsilon(A, M, \rho) = 0. \tag{A17} \]

(iv) **Completeness for dichotomic measurements.** In the case where $A^2 = \hat{\Pi}^{(2)} = I$, if $\varepsilon$ vanishes then $M$ accurately measures $A$ in $\rho$.

In this paper, we consider measurements of a spin component of a spin-1/2 particle using a dichotomic meter observable, $M$, i.e., $M^2 = I$. Thus, we consider only dichotomic measurements, for which the q-rms error satisfies properties (i)–(iv) above.

7. Disturbance of observables

For any observable $B$ of system $S$, denote by $\Pi_B$ the output POVM of $B$ by $M$ defined by
\[ \Pi_B(b) = T^*(E^B(b)) \tag{A18} \]
for every real number $b$. Then, we have
\[ \Pi_B(b) = \langle \xi | E^{B(t_0)}(b) | \xi \rangle \tag{A19} \]
for all $b$.

We say that measuring process $M$ does not disturb observable $B$ in state $\rho$ if $B(0)$ and $B(t_0)$ are perfectly correlated in state $\rho \otimes |\xi\rangle\langle\xi|$, namely, one of the following two equivalent conditions holds.
(i) $B(0)$ and $B(t_0)$ commute in $\rho \otimes |\xi\rangle \langle \xi|$ and their joint probability distribution, $\mu$, satisfies
\[
\sum_{b,b':k=b'} \mu(b,b') = 1. \quad (A20)
\]

(ii) For any $a, m \in \mathbb{R}$ with $a \neq m$,
\[
\text{Tr} \left[ \Pi_B(b') P_B^a(b) \rho \right] = 0. \quad (A21)
\]

8. Quantum root-mean-square disturbance

For any observable $B$ of system $S$, the disturbance operator, $D(B, M)$, for measuring process $M$ affecting observable $B$ is defined as the change of observable $B$ during the measurement,
\[
D(B, M) = B(t_0) - B(0). \quad (A22)
\]

Similar to the q-rms error, the q-rms disturbance $\eta(B, M)$ of $B$ in $\rho$ caused by $M$ is defined as the rms of the disturbance operator,
\[
\eta(B, M) = \text{Tr} \left[ D(B, M)^2 (\rho \otimes |\xi\rangle \langle \xi|) \right]^{1/2}. \quad (A23)
\]

The q-rms disturbance has properties analogous to the q-rms error.

(i) Operational definability. The q-rms disturbance, $\eta$, is definable by the non-selective operation, $T$, of measuring process $M$, observable $B$ to be disturbed, and initial state $\rho$ of the measured system $S$ as
\[
\eta(B, M, \rho) = \text{Re} \text{Tr} \left[ (B^2 - 2BT^*B + T^*[B^2]) \rho \right]. \quad (A24)
\]

(ii) Correspondence principle. In the case where $B(0)$ and $B(t_0)$ commute in $\rho \otimes |\xi\rangle \langle \xi|$, the relation
\[
\eta(B, M, \rho) = \varepsilon_B(\mu) \quad (A25)
\]
holds for joint probability distribution $\mu$ of $B(0)$ and $B(t_0)$ in $\rho \otimes |\xi\rangle \langle \xi|$.

(iii) Soundness. If $M$ does not disturb $B$ in $\rho$ then $\eta$ vanishes, i.e.,
\[
\eta(B, M, \rho) = 0. \quad (A26)
\]

(iv) Completeness for dichotomic observables. In the case where $B^2 = 1$, if $\varepsilon$ vanishes, then $M$ does not disturb $B$ in $\rho$.

9. Universally valid error–disturbance relations

In the following, we abbreviate $\varepsilon(A, M, \rho)$ as $\varepsilon(A)$ and $\eta(B, M, \rho)$ as $\eta(B)$, where no confusion may occur.

In 2003, one of the authors [15] derived the relation
\[
\varepsilon(A)\eta(B) + \varepsilon(A)\sigma(B) + \sigma(A)\eta(B) \geq \frac{|\langle \psi | [A, B] | \psi \rangle|^2}{2} \quad (A27)
\]
holding for any pair of observables $A, B$, state $|\psi\rangle$, and measuring process $M$. Later, Branciard [18] and one of the authors [20] obtained a stronger error–disturbance relation given by
\[
\varepsilon(A)^2 \sigma(B)^2 + \sigma(A)^2 \eta(B)^2 + 2\varepsilon(A)\eta(B)\sqrt{\sigma(A)^2 \sigma(B)^2 - D_{AB}^2} \geq D_{AB}^2, \quad (A28)
\]
where $D_{AB} = \frac{1}{2} \text{Tr}(\sqrt{\rho[B, A] \sqrt{\rho}})$. In the case where $A^2 = B^2 = 1$ and $M^2 = 1$, the above relation can be strengthened as
\[
\varepsilon(A)^2 + \hat{\eta}(B)^2 + 2\varepsilon(A)\hat{\eta}(B)\sqrt{1 - D_{AB}^2} \geq D_{AB}^2, \quad (A29)
\]
where
\[
\hat{\varepsilon}(A) = \sqrt{1 - \left(\frac{\varepsilon(A)^2 - 2}{2}\right)^2}, \quad (A30)
\]
\[
\hat{\eta}(B) = \sqrt{1 - \left(\frac{\eta(B)^2 - 2}{2}\right)^2}. \quad (A31)
\]

Since $\sigma_z^2 = 1$ and $M^2 = 1$, we obtain
\[
\varepsilon(\sigma_z)^2 + \hat{\eta}(\sigma_z)^2 + 2\varepsilon(\sigma_z)\hat{\eta}(\sigma_z)\sqrt{1 - D_{\sigma_z, \sigma_z}^2} \geq D_{\sigma_z, \sigma_z}^2, \quad (A32)
\]
where
\[
D_{\sigma_z, \sigma_z} = \text{Tr}(\sqrt{\rho\sigma_z \sqrt{\rho}}), \quad (A33)
\]
\[
\hat{\varepsilon}(\sigma_z) = \sqrt{1 - \left(\frac{\varepsilon(\sigma_z)^2 - 2}{2}\right)^2}, \quad (A34)
\]
\[
\hat{\eta}(\sigma_z) = \sqrt{1 - \left(\frac{\eta(\sigma_z)^2 - 2}{2}\right)^2}. \quad (A35)
\]

from the error–disturbance relation from Branciard [18] [19] and one of the authors [20]; see Eq. (A29) in the Appendix.

In the case where
\[
\langle \sigma_z \rangle_\rho = \langle \sigma_z \rangle_\rho = 0, \quad (A36)
\]
relation (A32) is reduced to the tight relation
\[
(\varepsilon(\sigma_z)^2 - 2)^2 + (\eta(\sigma_z)^2 - 2)^2 \leq 4 \quad (A37)
\]
as shown in FIG 1; see Eq. (A39) in the Appendix.

In the case where
\[
A = \sigma_z, \quad B = \sigma_x, \quad \langle \sigma_z(0) \rangle_\rho = \langle \sigma_x(0) \rangle_\rho = 0, \quad (A38)
\]
the above inequality (A29) is reduced to
\[
(\varepsilon(\sigma_z)^2 - 2)^2 + (\eta(\sigma_x)^2 - 2)^2 \leq 4. \quad (A39)
\]
Appendix B: Gaussian wave packets

In this appendix, we review relations between Gaussian states and inequalities. Let $Z$ and $P$ be the canonical position and momentum observables, respectively, of a one-dimensional quantum system. These observables satisfy the usual canonical commutation relation, $[Z,P] = i\hbar$. Here, we only consider a vector state denoted by $\psi$. However, some of the results in this appendix can easily be generalized to mixed states.

1. Schrödinger inequality

For the variances of the position and momentum, the following inequality holds \[17\]:

$$\text{Var}_\psi(Z)\text{Var}_\psi(P) \geq \frac{\langle \{Z,P\}_\psi \rangle^2 + \hbar^2}{4}. \quad (B1)$$

Inequality \[B1\] is known as the Schrödinger inequality. The proof proceeds as follows. First, we consider the case $\langle Z \rangle_\psi = \langle P \rangle_\psi = 0$. Then, we have

$$\text{Im} \langle Z\psi, P\psi \rangle = \frac{1}{2i}\langle [Z,P]_\psi \rangle = \hbar/2 \quad (B2)$$

$$\text{Re} \langle Z\psi, P\psi \rangle = \frac{1}{2}\langle \{Z,P\}_\psi \rangle. \quad (B3)$$

Consequently, we have

$$\langle Z\psi, P\psi \rangle^2 = \frac{\langle \{Z,P\}_\psi \rangle^2 + \hbar^2}{4}. \quad (B4)$$

On the other hand, according to the Cauchy–Schwarz inequality,

$$\langle Z\psi, P\psi \rangle^2 \leq \langle Z^2 \rangle_\psi \langle P^2 \rangle_\psi = \text{Var}_\psi(Z)\text{Var}_\psi(P). \quad (B5)$$

Hence, the Schrödinger inequality \[B1\] holds if $\langle Z \rangle_\psi = \langle P \rangle_\psi = 0$ holds. We can obtain the proof for the general case by substituting $Z$ and $P$ into $Z - \langle Z \rangle_\psi$ and $P - \langle P \rangle_\psi$, respectively. This concludes the proof.

The equation in this inequality holds if and only if

$$\langle Z - \langle Z \rangle_\psi \rangle_\psi = c \langle P - \langle P \rangle_\psi \rangle_\psi \quad (B6)$$

for some complex number $c$. From the condition above, we obtain the differential equation for the wave function as

$$\frac{d}{dz} \psi(z) = -2k \left[ z - \left( \langle Z\psi \rangle + \frac{i}{2\hbar k} \langle P\psi \rangle \right) \right] \psi(z), \quad (B7)$$

where $k$ is a complex number. Therefore, we have

$$\psi(z) = A \exp \left( -k \left[ z - \left( \langle Z\psi \rangle + \frac{i}{2\hbar k} \langle P\psi \rangle \right) \right]^2 \right). \quad (B8)$$

where $A$ is a constant. Since the wave function should be normalizable, constant $k$ must satisfy $\text{Re} k > 0$.

2. Kennard inequality

The inequality, which is known as the Kennard inequality \[2\],

$$\text{Var}_\psi(Z)\text{Var}_\psi(P) \geq \frac{\hbar^2}{4}, \quad (B9)$$

can be derived from the Schrödinger inequality \[B1\]. The equality in Eq. \[B9\] holds if and only if $2\hbar k (Z - \langle Z \rangle_\psi) = (P - \langle P \rangle_\psi)$ for some positive real number $k$. A wave function $\psi$ satisfies the equality in the Kennard inequality \[B9\] if and only if $\psi$ has the form

$$\psi(z) = A \exp \left( -k \left[ z - \left( \langle Z\psi \rangle + \frac{i}{2\hbar k} \langle P\psi \rangle \right) \right]^2 \right) \quad (B10)$$

for some positive real number $k$. This wave function has the same form as that of Eq. \[B8\] except for the condition of the constant $k$. i.e., the constant $k$ in Eq. \[B10\] is a complex number with a positive real part whereas the constant $k$ in Eq. \[B10\] is a positive real number. The state in Eq. \[B10\] is known as the minimal uncertainty state.

3. Squeezed state

For any two complex numbers, $\mu$ and $\nu$, satisfying $|\mu|^2 - |\nu|^2 = 1$, squeezed operator $c_{\mu,\nu}$ is defined as

$$c_{\mu,\nu} := \mu a + \nu a^\dagger, \quad (B11)$$

where $a$ and $a^\dagger$ are the annihilation and creation operators, respectively.

$$a := \sqrt{\frac{m\omega}{2\hbar}} Z + i \sqrt{\frac{1}{2bm\omega}} P. \quad (B12)$$

Here, $m$ and $\omega$ are the mass and angular frequency of the corresponding harmonic oscillator, respectively. A coherent state \[48\] is defined as the eigenstate of the annihilation operator, $a$, in Eq. \[B12\]. A squeezed state \[49\] is defined as the eigenstate of squeezed operator $c_{\mu,\nu}$,

$$c_{\mu,\nu} \psi = \lambda \psi. \quad (B13)$$

By this definition, the wave function of every squeezed state satisfies the differential equation,

$$\left[ (\mu + \nu) \sqrt{\frac{m\omega}{2\hbar}} z + (\mu - \nu) \sqrt{\frac{\hbar}{2m\omega}} \frac{d}{dz} \right] \psi(z) = \lambda \psi(z). \quad (B14)$$

The solution of this differential equation is

$$\psi(z) := A \exp \left[ -\frac{m\omega \mu + \nu}{2\hbar \mu - \nu} \left( z - \sqrt{\frac{2\hbar}{m\omega \mu - \nu}} \right)^2 \right]. \quad (B15)$$
Hence, the equality in the Schrödinger inequality \[(B1)\] holds for squeezed states.

Next, let us consider the relation between these parameters and the mean values of the position and momentum. By comparing the two formulas, \[(B8)\] and \[(B15)\], we have

\[\langle Z \rangle_\psi + \frac{i}{\hbar} \frac{\mu - \nu}{\mu + \nu} \langle P \rangle_\psi = \sqrt{\frac{2\hbar}{m\omega}} \frac{\lambda}{\mu - \nu}. \quad (B16)\]

Taking the imaginary part, we have

\[\langle P \rangle_\psi = \sqrt{2\hbar m\omega} |\mu + \nu|^2 \text{Im} \left( \frac{\lambda}{\mu - \nu} \right), \quad (B17)\]

\[\langle Z \rangle_\psi = \sqrt{\frac{2\hbar}{m\omega}} \text{Re} \left( \frac{(\mu + \nu)(\mu^* - \nu^*)}{\mu - \nu} \lambda \right). \quad (B18)\]

Next, let us calculate the variances of the position and momentum and the correlation \(\langle \{Z, P\}_\psi \rangle\). Setting \(\tilde{z} = z - \langle Z \rangle_\psi\), we have

\[\text{Var}(Z) = |A|^2 \int_{-\infty}^{\infty} \exp \left( -\frac{\hbar}{m\omega} \frac{\mu - \nu}{\mu + \nu} (\tilde{z} + \langle Z \rangle_\psi)^2 \right) d\tilde{z}. \quad (B19)\]

To calculate the variance of the momentum, it is convenient to obtain the Fourier transform of the wave function, \(\tilde{\psi}(\tilde{z}) := \psi(\tilde{z} + \langle Z \rangle_\psi)\),

\[\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi \hbar}} \int_{-\infty}^{\infty} \tilde{\psi}(\tilde{z}) \exp(ip\tilde{z}/\hbar) d\tilde{z} = \hat{A} \exp \left[ -\frac{1}{\hbar m\omega} \frac{\mu - \nu}{\mu + \nu} (p - \langle P \rangle_\psi)^2 \right], \quad (B20)\]

where \(\hat{A}\) is the normalization constant. Consequently, we have

\[\text{Var}(P) = \langle (P - \langle P \rangle_\psi)^2 \rangle_\psi = |A|^2 \int_{-\infty}^{\infty} p^2 \exp \left[ -\frac{\hbar}{m\omega} \text{Re} \left( \frac{p - \langle P \rangle_\psi}{\mu + \nu} \right)^2 \right] dp = \frac{\hbar m\omega}{2} |\mu + \nu|^2. \quad (B21)\]

Finally, we calculate the correlation term,

\[\langle \{Z - \langle Z \rangle_\psi, P - \langle P \rangle_\psi\}_\psi \rangle = \langle \{Z - \langle Z \rangle_\psi, P\}_\psi \rangle = 2\text{Re} \langle \tilde{Z} \psi, P \psi \rangle = 2\text{Re} \left[ |A|^2 \hbar m\omega \right. \]

\[\times \int_{-\infty}^{\infty} \exp \left( -\frac{\hbar}{m\omega} \text{Re} \left( \frac{\mu + \nu}{\mu - \nu} \tilde{z}^2 \right) \right) d\tilde{z} \]

\[= 2\hbar \text{Im} (\mu^* \nu). \quad (B22)\]

The coherent state is defined as the eigenstate of the annihilation operator. Using the results of the calculation above, the corresponding wave function is

\[\psi(z) = A \exp \left[ -\frac{m\omega}{2\hbar} \left( z - \sqrt{\frac{2\hbar}{m\omega}} \lambda \right)^2 \right], \quad (B23)\]

where \(\lambda\) is the corresponding eigenvalue of the annihilation operator. Thus, every coherent state satisfies the equation in the Schrödinger inequality \[(B1)\] and the Ken-

nard inequality \[(B9)\].

Since \(\frac{\mu + \nu}{\mu - \nu}\) moves all over the right half-plane of the complex plane as \(\mu\) and \(\nu\) move all over the complex plane satisfying \(|\mu|^2 - |\nu|^2 = 1\), the union of all squeezed states and coherent states coincides with the states that satisfy the Schrödinger inequality \[(B1)\], namely, \(G\).

### 4. Contractive state

The contractive state is introduced by Yuen [11] as a squeezed state whose correlation term is negative. This state contracts during some period of time if it evolves freely. To see this, let us calculate the variance of the position in the Heisenberg picture. The position operator \(Z(t)\) at time \(t\) in the Heisenberg picture is

\[Z(t) = \exp \left[ -\frac{t}{2i\hbar} \frac{P(t)^2}{2} \right] Z(0) \exp \left[ -\frac{t}{2i\hbar} \frac{P(t)^2}{2} \right] = Z(0) + \frac{t}{m} P(0). \quad (B24)\]

Hence, we have

\[\text{Var}_\psi (Z(t)) = \left\langle \left( Z(0) + \frac{t}{m} P(0) - \langle Z(0) + \frac{t}{m} P(0) \rangle_\psi \right)^2 \right\rangle_\psi = \frac{t^2}{m^2} \text{Var}_\psi (P(0)) + \text{Var}_\psi (P(0)) \]

\[+ \frac{t}{m} \langle \{Z(0) - \langle Z(0) \rangle_\psi, P(0) - \langle P(0) \rangle_\psi \}_\psi \rangle. \quad (B25)\]

Therefore, if the state is a contractive state, the variance of the position contracts until the time

\[t = \frac{m}{2} \langle \{Z(0) - \langle Z(0) \rangle_\psi, P(0) - \langle P(0) \rangle_\psi \}_\psi \rangle. \quad (B26)\]

### 5. Summary

We have discussed the relationship between the inequalities and the subclasses of Gaussian states whose wave functions are of the form

\[\psi(z) = A \exp \left[ -k \left( z - \langle Z \rangle_\psi + \frac{i}{2\hbar k} \langle P \rangle_\psi \right)^2 \right] \quad (B27)\]
and obtained the relations shown in Table. FIG. 5 represents the inclusion relation between the subsets of the set of Gaussian wave packets.

| $k$ | Name of state          | Inequality whose equality holds          |
|-----|------------------------|-----------------------------------------|
| $\text{Re } k > 0$ | Squeezed         | Schrödinger                             |
| $\text{Re } k > 0$ and |                  |                                          |

FIG. 5. Inclusion relation of the subsets of wave functions. A wave function is in the yellow region if and only if the equality in the Kennard inequality holds. A wave function is in the yellow region if and only if the equality in the Kennard inequality holds.

### Appendix C: Time evolution of Gaussian wave packets

In this appendix, we discuss the time evolution of the probability density of a Gaussian wave packet during free evolution. The wave function under consideration is the Gaussian wave packet derived in the previous section,

$$
\psi(z) := A \exp \left[ -k z^2 \right],
$$

where $k$ is a complex number with a positive real part. For simplicity, we consider only the case in which the mean values of the position and momentum are zero. Applying the Fourier transform $\mathcal{F}$ successively, we obtain

$$
\mathcal{F}^{-1} \exp \left( \frac{t}{2i\hbar m^2} p^2 \right) \psi(z)
$$

$$
= \mathcal{F}^{-1} \exp \left( \frac{t}{2i\hbar m^2} p^2 \right) \mathcal{A} \exp \left( -\frac{p^2}{4\hbar k^2} \right)
$$

$$
= \mathcal{A} \sqrt{\frac{2\pi\hbar}{k^2}} \int_{-\infty}^{\infty} \exp \left[ \left( \frac{t}{2i\hbar m} - \frac{1}{k^2} \right) p^2 - ipz/\hbar \right] dp
$$

$$
= N \exp \left[ -\frac{z^2}{\left( k^2 - \frac{2\hbar m}{i} \right)} \right],
$$

where $N$ is the normalization constant. Thus, the probability density, $\Pr(z)$, at time $t$ has the form

$$
\Pr(z) = |N|^2 \exp \left( -rz^2 \right)
$$

for some positive real number $r$. That is, we have again obtained a Gaussian distribution. Since the variance of the Gaussian distribution is

$$
\langle (Z(t)^2) \psi \rangle = \left\langle \left( Z(0) + \frac{t}{m} P(0) \right)^2 \right\rangle,\quad (C4)
$$

we have

$$
\Pr(z) = |N|^2 \exp \left( -\frac{z^2}{2\langle (Z(0) + \frac{1}{m} P(0))^2 \rangle_{\psi}} \right).
$$

### Appendix D: Relationship between the Heisenberg picture and the Schrödinger picture

Let us consider the relationship between the Heisenberg picture and the Schrödinger picture. Consider the time evolution of quantum system $\mathbf{S}$ described by $\mathcal{H}$. Let $A$ be an observable of system $\mathbf{S}$ and state $\psi$. Denote by $E(A, \psi, t)$ the expectation value of the outcome of the measurement of observable $A$ at time $t$, provided that system $\mathbf{S}$ is in state $\psi$ at time 0. In the Schrödinger picture, state $\psi(t)$ evolves in time $t$ as a solution of the Schrödinger equation by the time evolution operator, $U(t)$, as $\psi(t) = U(t)\psi$ with the initial condition, $U(0) = 1$, so that $E(A, \psi, t) = \langle \psi(t), A\psi(t) \rangle$ holds. The unitary operator $U^{S}(t_2, t_1)$ describing the time evolution from time $t = t_1$ to $t = t_2$ ($t_1 \leq t_2$) in the Schrödinger picture is defined by

$$
U^{S}(t_2, t_1) = U(t_2)U^{\dagger}(t_1).
$$

Then, we have

$$
U^{S}(t_2, t_1)\psi(t_1) = \psi(t_2),
$$

$$
U^{S}(t_3, t_2)U^{S}(t_2, t_1) = U^{S}(t_3, t_1).
$$

In the Heisenberg picture, observable $A(t)$ evolves in time $t$ by the time evolution operator $U(t)$ as $A(t) =$
Let \( f(A, \psi, t) = \langle \psi, A(t)\psi \rangle \) holds. The unitary operator, \( U^H(t_2, t_1) \), describing the time evolution from time \( t = t_1 \) to \( t = t_2 \) \((t_1 \leq t_2)\) in the Heisenberg picture is defined by

\[
U^H(t_2, t_1) = U^\dagger(t_1)U(t_2).
\]

Then, we have

\[
U^H(t_2, t_1)^\dagger A(t_1)U^H(t_2, t_1) = A(t_2),
\]

\[
\alpha^H(t_3, t_2)\alpha^H(t_2, t_1) = \alpha^H(t_3, t_1),
\]

where

\[
\alpha^H(t_2, t_1)A = U^H(t_2, t_1)^\dagger AU^H(t_2, t_1).
\]

We have the following relations between the Schrödinger picture and the Heisenberg picture.

\[
U(t) = U^S(t, 0) = U^H(t, 0),
\]

\[
U^H(t_2, t_1) = U(t_1)^\dagger U^S(t_2, t_1) U(t_1).
\]

Let \( f(A_1, \ldots, A_n, t, s) \) be a function of observables \( A_1, \ldots, A_n \) and real numbers \( t, s \). If

\[
U^S(t_2, t_1) = f(A_1, \ldots, A_n, t_1, t_2),
\]

then

\[
U^H(t_2, t_1) = f(A_1(t_1), \ldots, A_n(t_1), t_1, t_2).
\]

**V. ON THE SUPREMUM OF THE FUNCTION \( W_\lambda(t) \)**

Let us consider the supremum of the function in section \( \nabla \)

\[
W_\lambda(\tau) = \alpha \left( \tau + \frac{\Delta t}{2} \right) \left[ a + b(\Delta t + \tau) + c(\Delta t + \tau)^2 \right]^{-1/2}.
\]

Here we put \( \alpha = \frac{\mu B_1 \Delta t}{\sqrt{2m}} \), \( a = \langle Z^2 \rangle \), \( b = \langle \{Z, P\}\rangle \), and \( c = \langle P^2 \rangle / m^2 \). The derivative of function \( W_\lambda(\tau) \) is

\[
\frac{d}{d\tau} W_\lambda(\tau) = \frac{\alpha}{4} \left[ a + b(\Delta t + \tau) + c(\Delta t + \tau)^2 \right]^{-3/2}
\]

\[
\times [2(b + c\Delta t)(\Delta t + \tau) + 4a + b\Delta t].
\]

Hence, \( W_\lambda(\tau) \) assumes the maximum value at \( \tau = \tau_0 = -\frac{4a + 3b\Delta t + 2c\Delta t^2}{2(b + c\Delta t)} \geq 0 \) if the following conditions hold.

(i) \( W'(0) > 0 \).
(ii) \( 2b + 2c\Delta t < 0 \).

Condition (i) holds automatically. In fact, (i) is equivalent to condition

\[
4a + 3b\Delta t + 2c\Delta t^2 \geq 0.
\]

Now let us consider function

\[
f(t) = 4a + 3bt + 2ct^2.
\]

This function assumes the minimum value at \( t = -\frac{3b}{4c} \),

\[
f(t) \geq f \left( -\frac{3b}{4c} \right) = \frac{32ac - 9b^2}{8c} = \frac{9}{8c} (4ac - b^2) - \frac{4ac}{8c} \geq \frac{9h^2}{8cm^2} - \frac{h^2}{8cm^2} = \frac{h^2}{cm^2} > 0.
\]

Therefore, condition (i) is satisfied automatically. Here, we use Schrödinger inequality \( \nabla \). Hence, if condition (ii) holds, function \( W_\lambda(\tau) \) assumes the maximum value at \( \tau = \tau_0 \geq 0 \). The maximum value of \( W_\lambda(\tau) \) for \( \tau \geq 0 \) is

\[
W_\lambda(\tau_0) = -\alpha \frac{4a + 2b\Delta t + c\Delta t^2}{2(b + c\Delta t)}
\]

\[
\times \left[ a + b(\Delta t + \tau_0) + c(\Delta t + \tau_0)^2 \right]^{-1/2} = \alpha \left( 4a + 2b\Delta t + c\Delta t^2 \right)^{1/2} (4ac - b^2)^{-1/2}
\]

\[
= \frac{2\alpha m}{h} \left[ a + b \frac{\Delta t}{2} + \frac{c}{2} \left( \frac{\Delta t}{2} \right)^2 \right]^{1/2}
\]

\[
= \frac{\sqrt{2} \mu B_1 \Delta t}{\sqrt{h}} \left[ \left( \frac{Z + \Delta t}{2} P \right) \right]^{1/2} \frac{1}{\xi_\lambda}.
\]

If condition (ii) does not hold, function \( W_\lambda(\tau) \) increases monotonically and we have

\[
\sup_{\tau \geq 0} W_\lambda(\tau) = \lim_{\tau \to \infty} W_\lambda(\tau) = \frac{\mu B_1 \Delta t}{\sqrt{2(P Z)\xi_\lambda}}.
\]
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