Reconfiguring colourings of graphs with bounded maximum average degree

Carl Feghali
Department of Informatics,
University of Bergen,
Bergen, Norway
carl.feghali@uib.no

Abstract

The reconfiguration graph $R_k(G)$ of the $k$-colourings of a graph $G$ has as vertex set the set of all possible $k$-colourings of $G$ and two colourings are adjacent if they differ in exactly one vertex of $G$. Let $d, k \geq 1$ such that $k \geq d + 1$. We prove that for every $\epsilon > 0$ and every graph $G$ with $n$ vertices and maximum average degree $d - \epsilon$, $R_k(G)$ has diameter $O(n(\log n)^d)$. 
Let $k$ be a positive integer. A $k$-colouring of a graph $G$ is a function $f : V(G) \rightarrow \{1, \ldots, k\}$ such that $f(u) \neq f(v)$ whenever $(u,v) \in E(G)$. The reconfiguration graph $R_k(G)$ of the $k$-colourings of a graph $G$ has as vertex set the set of all possible $k$-colourings of $G$ and two colourings are adjacent if they differ on the colour of exactly one vertex of $G$.

Given an integer $d$, a graph $G$ is $d$-degenerate if every subgraph of $G$ contains a vertex of degree at most $d$. Expressed differently, $G$ is $d$-degenerate if there exists an ordering $v_1, \ldots, v_n$ of the vertices in $G$, called an $s$-degenerate ordering, such that each $v_i$ has at most $d$ neighbours $v_j$ with $j < i$. The maximum average degree of a graph $G$ is defined as 

$$\max \left\{ \frac{2|E(H)|}{|V(H)|} : H \subseteq G \right\}.$$ 

In particular, if $G$ has maximum average degree less than some positive integer $d$, then $G$ is $(d-1)$-degenerate.

Consider the following conjecture of Cereceda [3].

**Conjecture 1.** For every integers $k$ and $\ell$, $\ell \geq k + 2$, and every $k$-degenerate graph $G$ on $n$ vertices, $R_\ell(G)$ has diameter $O(n^2)$.

The conjecture appears difficult to prove or disprove, with the case $k = 1$ only being known despite some efforts; for a recent exposition on the conjecture and the results around it see [4, 1]. The most important breakthrough is Theorem 1 in [1] due to Bousquet and Heinrich, which addresses a number of cases for Conjecture 1, generalising several existing results. For instance, it is shown in [1] that there exists a constant $c > 0$ independent of $k$ such that $R_\ell(G)$ has diameter $(cn)^{k+1}$ for every $\ell \geq k + 2$.

The purpose of this note is to prove the following theorem.

**Theorem 1.** Let $d, k \geq 1$ such that $k \geq d + 1$. For every $\epsilon > 0$ and every graph $G$ with $n$ vertices and maximum average degree $d - \epsilon$, $R_k(G)$ has diameter $O(n(\log n)^d)$.

Theorem 1 is a generalisation of [2, Theorem 2]. In particular, it has the following immediate consequences. By Euler’s formula, planar graphs, triangle-free planar graphs and planar graphs of girth 5 have maximum average degrees strictly less than, respectively, 6, 4 and $7/2$. Hence Theorem 1 affirms (and is stronger than) Conjecture 1 for planar graphs of girth 5 but is one colour short of confirming the conjecture for planar graphs and triangle-free planar graphs. It nevertheless generalises some best known existing results. More precisely, our theorem subsumes both [2, Corollary 5] and [1, Theorem 1] restricted to planar graphs, as well as [2, Corollary 7] and [6, Corollary 1].

Past results addressing the conjecture for planar graphs can be found in [2, 4, 6, 1]. Our method of proof can be seen as a combination of the ones found in [1, 3].

2
Let us prove the theorem. First, some definitions and lemmas.

**Definition 1.** Given a graph $G$, a colouring $\alpha$ of $G$ and a subgraph $H$ of $G$, let $\alpha^H$ denote the restriction of $\alpha$ to $H$.

**Definition 2.** Let $G$ be a graph, and let $k$ be a nonnegative integer. A subset $S \subseteq V(G)$ is a $k$-independent set of $G$ if $S$ is an independent set of $G$ and every vertex of $S$ has degree at most $k$ in $G$.

**Definition 3.** Given positive integers $s$ and $t$, $G$ is said to have degree depth $(s, t)$ if there exists a partition $\{V_1, \ldots, V_t\}$ of $V(G)$, called an $s$-degree partition, such that $V_1$ is an $s$-independent set of $G$ and, for $i \in \{2, \ldots, t\}$, $V_i$ is an $s$-independent set of $G - \bigcup_{j=1}^{i-1} V_j$.

Let $G$ be a graph of degree depth $(s, t)$ and with $s$-degree partition $\{V_1, \ldots, V_t\}$.

**Definition 4.** An ordering $v_1, \ldots, v_n$ of $V(G)$ is said to be embedded in $\{V_1, \ldots, V_t\}$ if, for every pair $(v_j, v_i) \in V(G) \times V(G)$ such that $v_i \in V_p$ and $v_j \in V_q$, $j < i$ implies $p \leq q$.

Notice that such an ordering is an $s$-degenerate ordering of $G$.

Let $H$ be a subgraph of $G$ such that $V(H) \subseteq \bigcup_{j=1}^{h} V_j$ for some index $h \in \{1, \ldots, t\}$. In the next definition, we shall slightly abuse Definition 3.

**Definition 5.** $H$ is said to have degree depth $(s', t)$ if, for each index $j \in \{1, \ldots, h\}$ and each $v \in V(H) \cap V_j$, vertex $v$ has at most $s'$ neighbours in $\bigcup_{i=j+1}^{t} V_i$.

Informally speaking, the degree depth of $H$ is $(s', t)$ if each vertex of $H$ has at most $s'$ neighbours in $G$ that occur in an independent partite set later in the ordering.

**Lemma 1.** Let $s, t \geq 0$, and let $G$ be a graph of degree depth $(s, t)$. Any $(s + 2)$-colouring of $G$ can be recoloured to some $(s + 1)$-colouring of $G$ by $O(s^2 t^{s+1})$ recolourings per vertex.

**Proof.** Let $\{V_1, \ldots, V_t\}$ be an $s$-degree partition of $V(G)$, and let $v_1, \ldots, v_n$ be an ordering of $V(G)$ that is embedded in $\{V_1, \ldots, V_t\}$.

Let $\alpha$ be an $(s + 2)$-colouring of $G$, and let $h \in \{1, \ldots, t\}$ be the smallest index such that $V_h$ contains a vertex with colour $s + 2$ under $\alpha$. Let $W$ denote the subset of vertices of $V_h$ with colour $s + 2$. For each colour $a \in \{1, \ldots, s + 1\}$, define $W_a$ to be the subset of $W$ whose vertices have no neighbour earlier in the ordering with colour $a$. More formally,

$$W_a = \{v_i \in W : \alpha(v_j) \neq a \text{ for all neighbours } v_j \text{ of } v_i \text{ with } j < i\},$$

and notice that

$$W = \bigcup_{i=1}^{s+1} W_i.$$
since each $v_i \in V(G)$ has at most $s$ neighbours $v_j$ with $j < i$ and there are $s + 2$ colours.

Define $U = \bigcup_{i=1}^{h-1} V_i$.

**Claim 1.** For each $a \in \{1, \ldots, s + 1\}$, there is a sequence of recolourings such that
- each vertex of $U \cup W_a$ is recoloured $O((2t)^s)$ times,
- no vertex of $V(G) \setminus (U \cup W_a)$ is recoloured,
- at the end of the sequence, no vertex of $U \cup W_a$ has colour $s + 2$.

The claim implies the lemma: applying the sequence described in Claim 1 for each $a \in \{1, \ldots, s + 1\}$, we obtain a colouring in which colour $s + 2$ is not used in $\bigcup_{j=1}^{s} V_j$ by $O(s(2t)^s)$ recolourings. The smallest index $h'$ such that $V_{h'}$ contains a vertex with colour $s + 2$ has now increased; hence at most $t$ such repetitions are needed to obtain a colouring in which colour $s + 2$ is not used in $G$, so each vertex is recoloured $O(s2^st^{s+1})$ times and the lemma follows.

**Proof of Claim.** Let $G^* = G[U]$. Thus $G^*$ has degree depth $(s', t)$ for some $s' \in \{0, \ldots, s\}$. To prove the claim, we use induction on $s'$. The base case $s' = 0$ is trivial so we can assume that $s' = s > 0$ and that Claim 1 and hence the lemma holds for each subgraph $H$ of $G^*$ of degree depth $(s-1, t)$.

Let $u_1, \ldots, u_k$ be an ordering of the vertices of $U$ that is embedded in $\{V_1, \ldots, V_{h-1}\}$. Let us first try to recolour immediately, whenever possible, each vertex of $U$ to colour $s + 2$ starting with $u_1$ and moving forward towards $u_k$. Let $\gamma$ denote the resulting colouring, let $S = \{\gamma(v) = s + 2 : v \in V(G)\}$ and let $H = G[U \setminus S]$.

**Subclaim 1.** $H$ has degree depth $(s-1, t)$.

**Proof of Subclaim.** By our choice of $h$, each vertex $u \in U \cap V_p$ for some $p \in \{1, \ldots, h-1\}$ either satisfies $\gamma(u) = s + 2$ or has a neighbour $u' \in V_q$ for some $q \in \{p+1, \ldots, t\}$ such that $\gamma(u') = s + 2$. This implies the subclaim. □

By the subclaim and our induction hypothesis, we can recolour the restriction $\gamma^H$ of $\gamma$ to $H$ to some colouring $\zeta^H$ of $H$ in which colour $a$ is not used in $H$ by $O(2^{s-1}t^s)$ recolourings per vertex of $H$ such that no vertex of $V(G) \setminus V(H)$ is recoloured (this sequence of recolourings does not use colour $s + 2$ so we need not worry about adjacencies between $H$ and $S$).

Let $\zeta$ be the colouring of $G$ such that $\zeta(v) = \zeta^H(v)$ if $v \in V(H)$ and $\zeta(v) = \gamma(v)$ if $v \in V(G) \setminus V(H)$. From $\zeta$, we can now immediately recolour each vertex of $W_a$ to colour $a$. It remains to recolour each vertex of $U$ to a colour distinct from $s + 2$. To do so, we simply repeat the above steps with the roles of $a$ and $s + 2$ interchanged. This takes again $O(2^{s-1}t^s)$ recolourings per vertex of $H$. Hence each vertex of $H$ is recoloured in total $O(2^st^s)$ times. This proves the claim and hence completes the proof of the lemma. □
We can prove our final lemma, from which Theorem \ref{thm:main} follows easily.

Lemma 2. Let \( s, t \geq 0 \), and let \( G \) be a graph on \( n \) vertices with degree depth \( (s, t) \). Then \( R_{s+2}(G) \) has diameter \( O(ns(2t)^{s+1}) \).

Proof. The proof is completely standard. We proceed by induction on \( s \). The base case \( s = 0 \) is trivial, so we can assume that \( s > 0 \) and that the lemma holds for graphs with degree depth \( (s - 1, t) \). Let \( \alpha \) and \( \beta \) be \((s + 2)\)-coloured graphs of \( G \), and let \( \{V_1, \ldots, V_t\} \) be an \( s \)-degree partition of \( G \). It suffices to show that we can recolour \( \alpha \) to \( \beta \) by \( O(s(2t)^{s+1}) \) recolourings per vertex. By Lemma 1 we can recolour \( \alpha \) to some \((s + 1)\)-colouring \( \alpha_1 \) of \( G \) and \( \beta \) to some \((s + 1)\)-colouring \( \beta_1 \) of \( G \) by \( O(s2^{s+1}) \) recolourings per vertex.

Let \( v_1, \ldots, v_n \) be an ordering of \( V(G) \) that is embedded in \( \{V_1, \ldots, V_t\} \). Let us recolour \( \alpha_1 \) and \( \beta_1 \) to new colourings \( \alpha_2 \) and \( \beta_2 \) of \( G \) by trying to recolour, from \( \alpha_1 \) and \( \beta_1 \), immediately each vertex of \( G \) to colour \( s + 2 \) starting with \( v_1 \) and moving forward towards \( v_n \). Let \( S = \{ v \in V(G) : \alpha_2(v) = s + 2 (= \beta_2(v)) \} \). As argued in the proof of Lemma 1, the graph \( H = G - S \) has degree depth \((s - 1, t)\) so we can apply our induction hypothesis to recolour \( \alpha_2 \) to \( \beta_2 \) by \( O(s2t^s) \) recolourings per vertex (this sequence of recolourings does not use colour \( s + 2 \) so we need not worry about adjacencies between \( H \) and \( S \)).

Proof of Theorem \ref{thm:main}. Let \( H \) be any subgraph of \( G \), and let \( h = |V(H)| \). An independent set \( I \) of \( H \) is said to be special if \( I \) is a \((d - 1)\)-independent set of \( H \) such that \( |I| \geq \epsilon h/d^2 \). It was shown in \cite{5} that \( H \) contains a special independent set, but we include the short proof for completeness. Let \( S \) be the set of vertices of degree \( d - 1 \) or less in \( H \). The number of vertices of \( S \) is at least \( \epsilon h/d \) since otherwise

\[
\sum_{v \in H} \deg(v) \geq \sum_{v \in H-S} \deg(v) > d \left( h - \frac{\epsilon h}{d} \right) = (d - \epsilon)h,
\]

which contradicts the maximum average degree of \( G \). Let \( I \subseteq S \) be a maximal independent set of \( S \). Then every vertex of \( S - I \) has at least one neighbour in \( I \) and every vertex of \( I \) has at most \( d - 1 \) neighbours in \( S \). Therefore, \( |I| + (d - 1)|I| \geq |S| \geq \epsilon h/d \) and so \( I \) is a special independent set of \( H \), as required.

Therefore there exists a partition \( \{I_1, I_2, \ldots, I_\ell\} \) of \( V(G) \) such that \( I_1 \) is a special independent set of \( G \) and, for \( i \in \{2, \ldots, \ell\} \), \( I_i \) is a special independent set of \( G \setminus \left( \bigcup_{j=1}^{i-1} I_j \right) \). Hence \( G \) has degree depth \((d - 1, \ell)\). But \( \ell = f(n) \) satisfies the recurrence

\[
f(n) \leq f \left( n - \frac{\epsilon n}{d^2} \right) + 1,
\]
implying $\ell = O(\log n)$, by the master theorem. The theorem now follows by Lemma 1 with $t = \log n$ and $s = d - 1$. □

References

[1] N. Bousquet and M. Heinrich. A polynomial version of Cereceda’s conjecture. arXiv, 2019.

[2] N. Bousquet and G. Perarnau. Fast recoloring of sparse graphs. European Journal of Combinatorics, 52:1–11, 2016.

[3] L. Cereceda. Mixing graph colourings. PhD thesis, London School of Economics, 2007.

[4] E. Eiben and C. Feghali. Towards Cereceda’s conjecture for planar graphs. arXiv, 1810.00731, 2018.

[5] C. Feghali. Paths between colourings of sparse graphs. European Journal of Combinatorics, 75:169–171, 2019.

[6] C. Feghali. Reconfiguring 10-colourings of planar graphs. arXiv preprint arXiv:1902.02278, 2019.