First-order logic with self-reference

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Abstract

We consider an extension of first-order logic with a recursion operator that corresponds to allowing formulas to refer to themselves. We investigate the obtained language under two different systems of semantics, thereby obtaining two closely related but different logics. We provide a natural deduction system that is complete for validities for both of these logics, and we also investigate a range of related basic decision problems. For example, the validity problems of the two-variable fragments of the logics are shown coNexpTime-complete, which is in stark contrast with the high undecidability of two-variable logic extended with least fixed points. We also argue for the naturalness and benefits of the investigated approach to recursion and self-reference by, for example, relating the new logics to Lindström’s Second Theorem.

1 Introduction

This paper investigates an extension of first-order logic FO with an operator that allows formulas to refer to themselves. The idea is simple. We extend the syntax of FO by the following two rules:

1. If \( \varphi \) is a formula, then so is \( L\varphi \). Here \( L \) is a label symbol intuitively naming the formula \( \varphi \).

2. If \( L \) is a label symbol, then \( C_L \) is an atomic formula (a claim symbol) intuitively referring to the formula labelled by \( L \).

We interpret formulas via extending the standard game-theoretic semantics for FO by the rule that if an atom \( C_L \) is reached in the play of the semantic game, then the players jump back to the formula \( L\varphi \) and the game continues from there. Thereby the symbol \( L \) can indeed be seen as a naming or labelling operator that names \( \varphi \), while \( C_L \) is a claming operator claiming that \( \varphi \) holds. Other rules are precisely as in standard first-order logic, making our system a conservative extension of FO.

We give two alternative semantics to the obtained language, called bounded and unbounded semantics. In the unbounded semantics, the players continue until (if ever) an atomic first-order formula is reached, and the play is then won according to the same criteria as in FO. If the game play continues forever, neither player wins. The bounded semantics is similar, but there the players must commit to an integer
value giving the number of times formulas can be repeated, i.e., how many jumps from claim symbols \( C_L \) back to label symbols \( L \) are allowed. This forces all plays to be of finite duration. The winner is decided in the same way as in unbounded semantics and can occur only if an FO-atom is reached. If the players simply run out of time, then neither player wins the play.

Under the unbounded semantics, our logic is a fragment of the logic CL, or computation logic, introduced in [19] and discussed further in, e.g., [15], [21]. In addition to the looping operator studied in this paper, CL extends FO with the capacity to modify models by adding and deleting domain points as well as tuples of relations. Yet CL is a conservative extension of FO, giving the same interpretations to all first-order operators as FO via game-theoretic semantics. CL captures the class RE in the sense of descriptive complexity theory, that is, CL can define precisely the classes of finite models that are recursively enumerable [19]. In fact, more is true. We can associate Turing machines TM with formulas \( \varphi_{TM} \) so that

1. \( M \models \varphi_{TM} \) iff TM accepts the encoding of \( M \),
2. \( M \models \neg \varphi_{TM} \) iff TM rejects the encoding of \( M \),
3. \( \varphi_{TM} \) is indeterminate on \( M \) iff TM diverges on the encoding of \( M \).

Note here that the game-theoretic negation is strongly constructive: while \( M \models \varphi \) means that the proponent has a winning strategy in the game for \( M \) and \( \varphi \), negation is defined such that \( M \models \neg \varphi \) if the opponent has a winning strategy in the game. While FO is determined, CL has formulas that are not, so neither player has a winning strategy. This is necessary for capturing the full expressive power of Turing machines in the way CL does, creating an exact match also between indeterminacy of formulas and diverging computations. Verifiability of a formula in a model is of course matched with acceptance and falsifiability (in a model) with rejection, i.e., halting in a rejecting state. Thus there is a full symmetry between logic and computation, game-theoretic semantics being the key for achieving this. In this context, \( \neg \) is most naturally read to indicate falsifiability, although on first-order formulas, the involved mode of falsifiability collapses to classical negation. Note also that the self-reference mechanisms have immediate readings in natural language, so the framework produces formulas that have simple natural language counterparts. See [20] for further discussions on this.

We call the logics studied in the current paper SCL and BndSCL, for static computation logic and bounded static computation logic. The logic SCL follows the unbounded version of our semantics and is thus a fragment of CL also semantically, because CL is defined in [19] based on the unbounded semantics. The term static here refers to the fact that SCL and BndSCL do not modify the models under investigation, unlike CL. We note that quite naturally, we could alternatively refer to SCL and BndSCL, e.g., as non-well-founded FO under bounded and unbounded semantics.

The logics SCL and BndSCL also relate to several other formalisms studied in the literature. The looping mechanism is similar to that of the modal \( \mu \)-calculus [2], which becomes especially apparent when considering the game-theoretic approach to its semantics via parity games. The parity condition essentially allows for the \( \mu \)-calculus to be closed under classical negation. In contrast, the logics SCL and
BndSCL—being based on a strongly constructive game-theoretic negation—allow for indeterminate formulas but are based on simple reachability games that can be won only by ending up in an FO-atom—as in first-order logic. Nevertheless, fixed-point logics in general bear similarities to SCL and BndSCL.

Perhaps the best known fixed-point logic is LFP, or least fixed-point logic, see [22] for an early thorough approach to the formalism. In LFP, the use of negation is limited to guarantee monotonicity of the iterated operators. As negation can be used entirely freely in SCL and BndSCL, they are syntactically perhaps more closely related to partial fixed-point logic PFP than LFP. In PFP, a non-converging computation is interpreted as ⊥, thereby essentially forcing the involved procedure to converge (see, e.g., [3] for an introduction to PFP). In contrast, SCL and BndSCL allow for diverging formulas and the game-theoretic and coinductive approach to their semantics is not based on fixed points in any direct way. Much of the naturality of the setting stems from the strongly constructive negation, making verification of a negated formula in a model M equivalent to falsifying the formula in M. This feature is present even in CL, and also contrasts with intuitionistic logic, as double negation cancels in CL (and thus also in SCL and BndSCL). Having said all this, it nevertheless ought to be kept in mind that both SCL and BndSCL are conservative extensions of FO and thus negation indeed behaves entirely classically when restricting to FO-formulas.

The difference between BndSCL and SCL is that in BndSCL, the players must commit to a maximum number of times the self-referential formulas can be repeated when playing the semantic game. This idea, which is conceptually related to the difference between for-loops and while-loops, has been investigated in different forms in various different studies. The papers [5], [8], [10], [7] utilize bounded semantics in alternating-time temporal logic ATL and its variants. The results concern, e.g., identifying a hierarchy of fixed-parameter tractable variants of the extension ATL⁺ of ATL. The papers [6] and [9] develop deduction systems and tableaux for ATL under bounded semantics, and the articles [14], [15] device a bounded game-theoretic semantics for the modal µ-calculus and show it equivalent to the standard one. It is also shown that in the new setting, semantic games of the µ-calculus always end after a finite number of rounds, even in infinite models. The µ-calculus formula size games of [16] are based on this bounded semantics in an essential way. Also concerning bounded semantics, [15] studies the modal fragment of CL, called MCL, and observes that while it has PTIME-model checking and a nice bounded semantics with short game durations, it can nevertheless easily express PTIME-complete properties such as alternating reachability. Concerning yet further relevant works, we stress that there are numerous logics that relate to self-reference and recursion, too numerous to detail here. Relating to axiomatizations, the seminal work [24] on the µ-calculus should be mentioned. Concerning self-reference, [1] gives a general overview on the topic. It is also worth noting that various directions in infinitary logic, especially infinitely deep formulas, bear technical links to our work.

1.1 Contributions

One of the main aims of this paper is to provide a complete proof system for SCL and BndSCL. Interestingly, it turns out that both of these logics have the same
set of validities. Below we provide a natural deduction system that is complete for validities of the logics. We also show that if $\Sigma$ is a set of first-order formulas, then $\Sigma \models \varphi$ iff $\Sigma \vdash \varphi$ holds for both logics.

Furthermore, we investigate the expressive powers and computational properties of SCL and BndSCL and their fragments. We identify several interesting properties that are straightforward to express in SCL or BndSCL while not being expressible in FO. For example, it is easy to express in SCL that a linear order is well-founded, whence it is easy to define the intended model of arithmetic up to isomorphism in that logic. Concerning computational properties, perhaps most notably, we show that the two-variable fragments of SCL and BndSCL have coNexpTime-complete validity problems. This is in stark contrast with validity for two-variable logic with fixed points, which is highly undecidable, having been shown $\Pi^1_1$-hard in [11]. This nicely demonstrates the possibilities of using recursion in the way used in this article. In addition to positive results, we also show, for example, that the satisfiability problem of BndSCL is $\Sigma^0_2$-complete.

To better understand the features of SCL and BndSCL, we investigate their model theory. First, we establish that both of these logics have the countable downwards Löwenheim-Skolem property: if $\mathcal{A}$ is a model of $\varphi$, then $\mathcal{A}$ has a countable substructure which is also a model of $\varphi$. Secondly, we show that neither of these logics enjoys the Craig interpolation property. Finally, we investigate determinacy of sentences, i.e., the question whether a sentence $\varphi$ has the property that in every model, one of the players has a winning strategy in the semantic game for $\varphi$. Note that this is equivalent to asking whether $\varphi \lor \neg \varphi$ is valid. Interestingly, it turns out that $\varphi \lor \neg \varphi$ is valid precisely when $\varphi$ is equivalent to a first-order sentence. We also give an example demonstrating that the above correspondence fails if we restrict our attention to finite models.

We also investigate $\Pi^1_1$-relations as well as $\Sigma^0_{\omega+1}$-relations. A well-known theorem of Kleene states that over the standard structure of natural numbers $\mathbb{N}$, the class of inductive relations and $\Pi^1_1$-relations (or relations definable in universal second-order logic over $\mathbb{N}$) coincide [17]. In [13], an alternative characterisation of $\Pi^1_1$-relations was given in terms of programs in the programming language IND. Inspired by these characterisations, we give yet another characterisation of $\Pi^1_1$-relations by showing that they also coincide with the class of SCL-definable relations. We also study the class of BndSCL-definable relations and prove that they coincide with the class of $\Sigma^0_{\omega+1}$-relations. To the best of our knowledge, this is the first logical characterisation of $\Sigma^0_{\omega+1}$-relations. Furthermore, it sheds light on the expressive power of BndSCL.

As a final remark, we describe one of the most important results on SCL and BndSCL we have obtained. Firstly, these logics are way more expressive than FO. Secondly, they nevertheless have recursively enumerable sets of validities, and, as discussed above, the downward Löwenheim-Skolem property. This contrasts with Lindström’s Second Theorem, which states the the expressive power of an effectively regular logic with recursively enumerable validities and the downward Löwenheim-Skolem property should not exceed that of FO (see, e.g., [11]). The nice thing is that the only property that SCL and BndSCL lack in being effectively regular is closure under classical negation, and, they nevertheless are both closed under the
highly natural strong negation. Indeed, in the case of CL, the logic cannot be closed under classical negation, as CL captures RE. Furthermore, we stress once more that the game-theoretic negation is simply the plain classical negation when limiting to the first-order fragment.

2 Preliminaries

We denote the natural numbers by \(\mathbb{N}\), the integers by \(\mathbb{Z}\) and the positive integers by \(\mathbb{N}_+\). A linear order structure \((A, \prec^A)\) is a structure where \(\prec^A\) is a strict linear order over the domain set \(A\). A discrete order structure \((A, \prec^A)\) is a structure where \(\prec^A\) is a strict linear order over the domain \(A\) such that the following conditions hold.

1. The linear order has a minimum element \(0^A \in A\).
2. Each element \(a \in A\) has a unique successor element \(b \in A\) in the case \(a\) has a successor at all. That is, if there is some \(d \in A\) such that \(a \prec^A d\), then there exists an element \(b \in A\) such that \(a \prec^A b\) and for all \(c \in A \setminus \{a, b\}\), we have \(c \prec^A a\) or \(b \prec^A c\).

A finite sequence in \(V\) is a finite tuple \((v_1, \ldots, v_n)\) of elements \(v_i \in V\). The element \(v_n\) is the last element of the tuple. An \(\omega\)-sequence in \(V\) is an infinite tuple \((v_i)_{i \in \mathbb{N}_+} = (v_1, v_2, \ldots)\) of elements \(v_i \in V\). Here \(\omega\) denotes the first infinite ordinal. The element \(v_1\) is the first element of both \((v_1, \ldots, v_n)\) and \((v_1, v_2, \ldots)\). If \(p = (v_1, \ldots, v_n)\) and \(q = (u_1, \ldots, u_m)\) are finite sequences, their concatenation \((v_1, \ldots, v_n, u_1, \ldots, u_m)\) is denoted by \(p \cdot q\). A singleton sequence \((v)\) is identified with \(v\). We sometimes denote tuples with vector notation, e.g., \(\vec{v}\) denotes a tuple of elements \(v_i\).

In this paper, a directed graph \((V, E)\) is a structure where \(V\) is any (possibly infinite) set and \(E \subseteq V \times V\). Thus directed graphs are allowed to have reflexive loops, i.e., the set \(E\) may contain pairs \((v, v)\). A dead end in \((V, E)\) is an element \(v \in V\) such that there does not exist exists any \(u\) such that \((v, u) \in E\). A walk in a directed graph \((V, E)\) is either a finite or an \(\omega\)-sequence in \(V\) such that we have \((v_i, v_{i+1}) \in E\) for each pair \((v_i, v_{i+1}) \in E\) of subsequent elements in the sequence. We note that in the literature, walks are often defined as sequences of edges, but our definition is more convenient for this paper. A walk is a path if it does not repeat any element. For \(v \in V\), the set of finite (nonempty) walks with the first element \(v\) is denoted by \(V^*_{\text{walk}}(v)\).

A game arena is a tuple \((V_0, V_1, E)\) where \(V_0\) and \(V_1\) are any disjoint sets and \(E \subseteq V \times V\) for \(V := V_0 \cup V_1\). Intuitively, the arena is a platform for a two-player game where \(V_0\) is a set of positions for player 0 and \(V_1\) for player 1. In each position \(v \in V_0\) (respectively, \(v \in V_1\)), player 0 (respectively, player 1) chooses a node \(u\) such that \((v, u) \in E\) and the players then continue from the new position \(u\). We define here that a play on the arena is a maximal walk in \((V, E)\), where maximality means that the walk is either infinite (of length \(\omega\)) or finite with its

\[\text{We note that in this article, we prove this statement explicitly only for the variant of FO without constant and function symbols. This is to keep the work simple. However, it is trivial to extend our study to involve constants and function symbols.}\]
last element being a dead end. A **generalized winning condition** over the arena 
\((V_0, V_1, E) = (V, E)\) is a pair \((S_0, S_1)\) where \(S_0\) and \(S_1\) are sets of plays. The set 
\(S_0\) (respectively, \(S_1\)) lists the plays that player 0 (player 1) wins. Note that it is 
possible that neither of the players—or even both players—win a play.

A **game** is a triple that specifies a game arena, a **beginning position** (which 
is a node \(v \in V\)) and a generalized winning condition \((S_0, S_1)\) consisting of plays 
with the first position \(v\). In a game with first position \(v\), a **strategy** of player 0 
(respectively, player 1) is a function \(f : U \rightarrow V\), where \(U\) is the set of finite walks 
with the last element in \(V_0\) (respectively, in \(V_1\)) and the first element \(v\). A strategy \(f\) is **followed** in a play \(p\) if every prefix \(q\) of \(p\) with \(q \in \text{dom}(f)\) has the property that 
\(f(q \cdot f(q))\) is, likewise, a prefix of \(p\). We may also talk about following a strategy 
in a prefix of a play; the meaning of this is defined in the obvious way. A strategy 
of player 0 (respectively, player 1) is a **winning strategy** if every play where \(f\) is 
followed belongs to \(S_0\) (respectively, \(S_1\)). Letting \(S\) denote the set of all possible 
plays of a game, a strategy of player 0 (respectively, player 1) is a **non-losing strategy** 
if every play where \(f\) is followed belongs to \(S \setminus S_1\) (respectively, \(S \setminus S_0\)). 
A strategy \(f\) is **positional** if it depends only on the last position of its inputs, i.e., 
\(f(q \cdot v) = f(q' \cdot v)\) for all prefixes of plays \(q \cdot v\) and \(q' \cdot v\) in \(\text{dom}(f)\). Note that we 
can identify such a strategy \(f\) of player \(i \in \{0,1\}\) by the function 
\(g : V_i \rightarrow V\) such that \(g(v) = f(q \cdot v)\) for all \((q \cdot v) \in \text{dom}(f)\). A game is **determined** if precisely one 
player has a winning strategy in it. It is **positionally determined** if precisely one 
player has a positional winning strategy in it. Note that positional determinacy 
implies determinacy.

A **reachability game** for player \(i \in \{0,1\}\) is a game where the winning condition 
\(S_i\) of player \(i\) contains precisely the plays \((v_1, \ldots, v_n)\) where \(v_n\) is a dead end 
belonging to the opponent, i.e., \(v_n \in V_j\) for \(j \in \{0,1\} \setminus \{i\}\). The complement of 
\(S_i\) defines a **safety game** for the opponent \(j\), that is, if the set of plays \(S_i\) defines 
a reachability game for player \(i\), then the complement set of plays defines a safety 
game for the opponent of \(i\). The complement set \(S_j\) contains precisely those finite 
plays that end in a dead end \(v_n \in V_i\) for \(i\) and all infinite plays. Reachability games 
(and thus safety games) are sometimes represented by structures \((V, E, v_1, V_0, V_1)\) 
(or by close variants of this representation) in a natural way such that \(V = V_0 \cup V_1\), 
\(E \subseteq V \times V\) and \(v_1\) is the beginning position. In this representation, there is no 
need to encode all the plays leading to a win of player \(i\), as obviously only the final 
position of each finite play matters. The following result is well known and follows 
directly from, e.g., [12]. It states that in any reachability or safety game, precisely one of the 
players has a winning strategy, and that strategy can be assumed positional.

**Theorem 2.1.** Reachability and safety games are positionally determined (even on 
infinite arenas).

Leaving games behind for now, we denote models by \(\mathfrak{A}, \mathfrak{B}\), and so on. The 
domain of a model is denoted by the corresponding Roman capital letter, so for 
example \(A\) denotes the domain of \(\mathfrak{A}\). An **assignment** for a model \(\mathfrak{A}\) is a function 
\(s : V \rightarrow A\) where \(V\) is some (often finite) set of variable symbols. Note that also \(\emptyset\) 
is an assignment (for empty \(V\)). An assignment mapping into a set \(A\) is called an 
\(A\)-assignment. An assignment that is otherwise as \(s\) but sends \(x\) to \(a\) is denoted by
A model $\mathfrak{A}$ (respectively, and assignment $s$) is $\varphi$-suitable if the vocabulary of $\mathfrak{A}$ contains the vocabulary of $\varphi$ (respectively, the domain of $s$ contains the free variables of $\varphi$). When the specification of $\varphi$ is sufficiently clear for and from the context under investigation, we may simply call $\mathfrak{A}$ and $s$ suitable.

In this article, the language of first-order logic FO includes equality and $\bot$ as primitives and contains the Boolean operators $\neg$, $\land$, $\lor$ and the quantifiers $\exists$ and $\forall$. We may use $\top$, $\rightarrow$ and $\leftrightarrow$ as abbreviations in the usual way. We limit to purely relational vocabularies for the sake of simplicity and brevity.\footnote{Indeed, this limitation could easily be lifted.}

Atomic formulas belonging to FO are called FO-atoms, or first-order atoms. This is to distinguish them from claim symbols, to be formally introduced later on. Universal second-order logic $\forall$SO is the fragment of second-order logic with formulas of the form $\forall X_1 \ldots \forall X_n \psi$ where $X_1, \ldots, X_n$ are second-order relation variables and $\psi$ is a formula of FO.

**Definition 2.2.** Let $\varphi$ be a formula of FO, $\mathfrak{A}$ a suitable model and $r$ a suitable assignment. We define the evaluation game $G(\mathfrak{A}, r, \varphi)$ as follows. The game has two players, Abelard and Eloise. The positions of the game are tuples $(\psi, s, \#)$, where $s$ is a $\psi$-suitable $\mathfrak{A}$-assignment and $\# \in \{+, -\}$. The game begins from the initial position $(\varphi, r, +)$ and it is then played according to the following rules.

- In a position $(\alpha, s, +)$, where $\alpha$ is an FO-atom, the play of the game ends and Eloise wins if $\mathfrak{A}, s \models \alpha$. Otherwise Abelard wins.
- In a position $(\alpha, s, -)$, where $\alpha$ is an FO-atom, the play of the game ends and Abelard wins if $\mathfrak{A}, s \models \alpha$. Otherwise Eloise wins.
- In a position $(\neg \psi, s, +)$, the game continues from the position $(\psi, s, -)$. Symmetrically, in a position $(\neg \psi, s, -)$, the game continues from the position $(\psi, s, +)$.
- In a position $(\psi \land \theta, s, +)$, Abelard chooses whether the game continues from the position $(\psi, s, +)$ or $(\theta, s, +)$.
- In a position $(\psi \land \theta, s, -)$, Eloise chooses whether the game continues from the position $(\psi, s, -)$ or $(\theta, s, -)$.
- In a position $(\psi \lor \theta, s, +)$, Eloise chooses whether the game continues from the position $(\psi, s, +)$ or $(\theta, s, +)$.
- In a position $(\psi \lor \theta, s, -)$, Abelard chooses whether the game continues from the position $(\psi, s, -)$ or $(\theta, s, -)$.
- In a position $(\forall x \psi, s, +)$, Abelard chooses some element $a \in A$ and the game continues from the position $(\psi, s[a/x], +)$.
- In a position $(\forall x \psi, s, -)$, Eloise chooses some element $a \in A$ and the game continues from the position $(\psi, s[a/x], -)$.
- In a position $(\exists x \psi, s, +)$, Eloise chooses some element $a \in A$ and the game continues from the position $(\psi, s[a/x], +)$.
• In a position \((∃xψ, s, -)\), Abelard chooses some element \(a ∈ A\) and the game continues from the position \((ψ, s[a/x], -)\).

If \(r\) is the empty assignment \(∅\) (and hence \(ϕ\) is a sentence), we may write \(G(\mathcal{A}, ϕ)\) instead of \(G(\mathcal{A}, ∅, ϕ)\).

**Definition 2.3.** Let \(ϕ\) be an FO-formula, \(\mathcal{A}\) a suitable model and \(s\) a suitable assignment. We define that \(ϕ\) is **true** (or **verifiable**) in \(\mathcal{A}\) under \(s\), denoted by \(\mathcal{A}, s |═ |ϕ\), iff Eloise has a winning strategy in the game \(G(\mathcal{A}, s, ϕ)\). If \(ϕ\) is a sentence, we may write \(\mathcal{A} |═ |ϕ\) if \(\mathcal{A}, ∅ |═ |ϕ\), where \(∅\) is the empty assignment. We then simply say that \(ϕ\) is true (or verifiable) in \(\mathcal{A}\).

The above specifies the standard **game-theoretic semantics** for FO. It is well known and easy to see that the above definition via evaluation games agrees with the standard Tarski semantics for FO, i.e., \(ϕ\) is true in \(\mathcal{A}\) under \(s\) according to the game-theoretic semantics iff the same holds in the sense of Tarski semantics.

We then extend the syntax of FO. Define the set \(LBS := \{L_i | i ∈ N\}\) of **label symbols**, and based on this, define the set \(RFS := \{C_{L_i} | L_i ∈ LBS\}\) of **reference symbols**, also called **claim symbols**. The syntax of the logics SCL and BndSCL is obtained by extending the formula construction rules of FO by the following rules.

• Each claim symbol \(C_{L} ∈ RFS\) is an atomic formula.

• If \(ϕ\) is a formula and \(L ∈ LBS\), then \(Lϕ\) is a formula.

Reference symbols can also be called **non-FO-atoms** or **looping atoms**, while the remaining atomic formulas are **FO-atoms**. Now, consider a formula \(ϕ\) and an occurrence \(C_{L}\) of a reference symbol in \(ϕ\). The **reference formula** of \(C_{L}\), denoted \(Rf(C_{L})\), is the subformula occurrence \(Lψ\) of \(ϕ\) such that there is a directed path from \(Lψ\) to \(C_{L}\) in the syntax tree of \(ϕ\), and \(L\) does not occur strictly between \(Lψ\) and \(C_{L}\) on that path. Note that the reference formula of \(C_{L}\) is unique if it exists at all. When we talk about reference formulas, we mean reference formula occurrences. A claim symbol occurrence \(C_{L}\) is in the **strict scope** of a label symbol occurrence \(L\) when \(C_{L}\) can be reached in the syntax tree from the node with \(L\) via a directed path that does not contain further occurrences of \(L\). For example, in \(LLC_{L}\), the atom \(C_{L}\) is in the strict scope of the second but not the first occurrence of \(L\). However, the atom is in the scope of both occurrences. A looping atom occurrence \(C_{L}\) is **free** in a formula if it is not in the scope (equivalently, strict scope) of any occurrence of \(L\). A label symbol occurrence \(L\) is **dummy** if there are no corresponding looping atoms \(C_{L}\) in the strict scope of \(L\). A formula \(ϕ\) is **regular** if the following conditions hold.

1. No label symbol \(L\) occurs more than once in it.

2. If an atom \(C_{L}\) occurs free in the formula, then the corresponding label symbol \(L\) does not occur anywhere in the formula.

The set of **subformulas** of a formula \(ϕ\) is denoted by \(Sf(ϕ)\). As usual in game-theoretic semantics, subformulas mean subformula occurrences, so for example \(P(x) ∨ P(x)\) has three subformulas, the disjunction itself and the left and the
right occurrences of $P(x)$. The set $\text{SF}_L(\varphi)$ of \textit{L-subformulas} (or \textit{label subformulas}) of $\varphi$ is the set of subformulas of type $L' \psi$ (where $L'$ is any label symbol) in $\varphi$. This includes $\varphi$ itself if $\varphi$ is of type $L' \varphi'$. We say that $\varphi$ is in \textbf{weak negation normal form} if the only negated subformulas of $\varphi$ are atomic formulas. We say that $\varphi$ is in \textbf{strong negation normal form} if the only negated subformulas of $\varphi$ are atomic FO-formulas.

The set of free variables of a formula $\varphi$ of SCL of BndSCL is defined inductively in the same way as for FO-formulas, with the following two additional rules.

- The set of free variables of a claim symbol $C$ is $\emptyset$.
- The free variables of $L\psi$ is the the same as that of $\psi$.

Analogously to the case for FO, a model $\mathfrak{A}$ and assignment $s$ are called suitable for $\varphi$ if $s$ interprets the free variables of $\varphi$ in $\mathfrak{A}$ while $\mathfrak{A}$ interprets the relation symbols in $\varphi$.

**Definition 2.4.** The semantics of SCL is given via a game that extends the game for FO by the following rules.

- In a position $(C_L, s, \#)$, where $\# \in \{-, +\}$, the game continues from the position $(L\psi, s, \#)$ where $L\psi$ is the reference formula of $C_L$. In the case there exists no such reference formula, the play of the game ends and neither of the players win the play.
- In a position $(L\psi, s, \#)$, where $\# \in \{+, -\}$, the game simply continues from the position $(\psi, s, \#)$.

The game for $\mathfrak{A}$, $\varphi$ and $s$ is denoted by $G_\infty(\mathfrak{A}, s, \varphi)$. If $s$ is the empty assignment $\emptyset$ (and hence $\varphi$ is a sentence), we may write $G_\infty(\mathfrak{A}, \varphi)$ instead of $G_\infty(\mathfrak{A}, \emptyset, \varphi)$.

Note that infinite plays are won by neither player. Winning occurs only if an FO-atom is reached, exactly as in first-order logic. The game $G_\infty(\mathfrak{A}, s, \varphi)$ is clearly a reachability game for Eloise, and thereby, by Theorem 2.1, Eloise has a positional winning strategy if and only if she has a general one. We note that this holds despite the fact that the underlying models are not required to be finite. Furthermore, the same claims hold for Abelard as well.

The semantics of SCL is formally defined as follows.

**Definition 2.5.** Let $\varphi$ be a formula of SCL, $\mathfrak{A}$ a suitable model and $s$ a suitable assignment. We define that $\varphi$ is \textbf{true} (or \textbf{verifiable}) in $\mathfrak{A}$ under $s$, denoted $\mathfrak{A}, s \models \varphi$, iff Eloise has a winning strategy in the game $G_\infty(\mathfrak{A}, s, \varphi)$. If $\varphi$ is a sentence, we may write $\mathfrak{A} \models \varphi$ if $\mathfrak{A}, \emptyset \models \varphi$. We then say that $\varphi$ is true (or verifiable) in $\mathfrak{A}$.

We also define two notions of equivalence for SCL

**Definition 2.6.** Let $\varphi$ and $\psi$ be a formulas of SCL. The formulas are \textbf{weakly equivalent} if the equivalence

$$\mathfrak{A}, s \models \varphi \iff \mathfrak{A}, s \models \psi$$

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holds for all $\mathfrak{A}$ and $s$ that are suitable with respect to both $\varphi$ and $\psi$. The formulas $\varphi$ and $\psi$ are strongly equivalent if they are weakly equivalent and the equivalence

$$\text{Abelard has a winning strategy in } G_\infty(\mathfrak{A}, s, \varphi)$$

$$\iff$$

$$\text{Abelard has a winning strategy in } G_\infty(\mathfrak{A}, s, \psi)$$

holds for all $\mathfrak{A}$ and $s$ that are suitable with respect to $\varphi$ and $\psi$.

It is easy to see that an alternative way to formulate strong equivalence of $\varphi$ and $\psi$ is to require that for any suitable $\mathfrak{A}$ and $s$, precisely one of the following three conditions hold.

1. Eloise has a winning strategy in both games $G_\infty(\mathfrak{A}, s, \varphi)$ and $G_\infty(\mathfrak{A}, s, \psi)$.
2. Abelard has a winning strategy in both $G_\infty(\mathfrak{A}, s, \varphi)$ and $G_\infty(\mathfrak{A}, s, \psi)$.
3. Neither of the players has a winning strategy in $G_\infty(\mathfrak{A}, s, \varphi)$, and the same holds also for the game $G_\infty(\mathfrak{A}, s, \psi)$.

To define the semantics of BndSCL, we next define two related games, the second one extending the first one. The intuitive idea is simply that in the beginning of each play, the players commit to some maximum duration of the play.

**Definition 2.7.** Let $\varphi$ be a formula of BndSCL, and suppose $\mathfrak{A}$ is a suitable model and $r$ a suitable assignment. Let $n \in \mathbb{N}$. We define the $n$-bounded evaluation game $G_n(\mathfrak{A}, r, \varphi)$ as follows. The game has two players, Abelard and Eloise. The positions of the game are tuples $(\psi, s, \#, m)$ where $s$ is a $\psi$-suitable $\mathfrak{A}$-assignment, $\# \in \{+, -\}$ and $m \in \mathbb{N}$ is a clock value (or iteration index) which will, informally speaking, tell how many times the game play can still jump from a looping atom to a label symbol. The game begins from the initial position $(\varphi, r, +, n)$. The game is then played according to rules that contain—in addition to rules which are analogous to the rules used in the evaluation game for FO—the following rules.

- In a position $(C_L, s, \#, n)$ we have two principal cases. If $n > 0$, then the game continues from the position $(\text{Rf}(C_L), s, \#, n - 1)$. If $n = 0$, then the play of the game ends and neither player wins the play.
- In a position $(L\psi, s, \#, n)$, the game simply moves to the position $(\psi, s, \#, n)$.

We then extend this game and thereby define the bounded evaluation game $G_\omega(\mathfrak{A}, r, \varphi)$ as follows. The game starts by Abelard picking a natural number $n' \in \mathbb{N}$. After this, Eloise picks a natural number $n \geq n'$. Then the game $G_n(\mathfrak{A}, r, \varphi)$ is played. Eloise wins $G_\omega(\mathfrak{A}, r, \varphi)$ if she wins the game $G_n(\mathfrak{A}, r, \varphi)$, and similarly, Abelard wins if he wins $G_n(\mathfrak{A}, r, \varphi)$.

If $r$ is the empty assignment $\emptyset$ (and hence $\varphi$ is a sentence), we may write $G_n(\mathfrak{A}, \varphi)$ and $G_\omega(\mathfrak{A}, \varphi)$ instead of $G_n(\mathfrak{A}, \emptyset, \varphi)$ and $G_\omega(\mathfrak{A}, \emptyset, \varphi)$, respectively.
We note that, concerning the results below, it would make no difference if Eloise instead of Abelard picked her natural number first, and Abelard would then pick some greater number. Or the players could pick their numbers independently, and then the larger one of these would be chosen. However, we shall formally follow the convention that Abelard picks first.

Both $G_n(\mathfrak{A}, r, \varphi)$ and $G_\omega(\mathfrak{A}, r, \varphi)$ are reachability games for Eloise, and thus, by Theorem 2.1, Eloise has a positional winning strategy if and only if she has a general one. The same claims hold for Abelard as well. The semantics of BndSCL is defined as follows.

Definition 2.8. We define that $\varphi$ is true (or boundedly verifiable) in $\mathfrak{A}$ under $r$, denoted $\mathfrak{A}, r \models_\omega \varphi$, iff Eloise has a winning strategy in the game $G_\omega(\mathfrak{A}, r, \varphi)$.

Definition 2.9. The notions of weak equivalence and strong equivalence are defined for BndSCL precisely as for SCL in Definition 2.6, but with respect to the bounded evaluation game this time.

Somewhat informally, when it is clear from the context that we are considering BndSCL, we may use the turnstile $\models$ instead of $\models_\omega$. Furthermore, when $\varphi$ is a sentence, we may drop $r$ and simply write $\mathfrak{A} \models \varphi$.

Let $L\psi$ be a subformula of $\varphi$. Now, consider renaming this $L$ and the corresponding atoms $C_L$ in $L\psi$ in the strict scope of this particular occurrence of $L$. Suppose the occurrence of $L$ and the corresponding atoms in its strict scope end up being renamed as $L'$ and $C_{L'}$. This renaming is safe if $L\psi$ does not contain free occurrences of $C_{L'}$, and (2) if $\psi$ already contains symbols $L'$, then none of the occurrences $C_L$ to be renamed are in the scope of such occurrences of $L'$. A regularisation or a formula $\varphi$ is a strongly equivalent (with respect to both SCL and BndSCL) formula obtained from $\varphi$ by safe renamings. This leaves free occurrences of label symbols as they are.

We already saw that when considering winning strategies for Eloise in semantic games, it does not matter whether we limit attention to positional or general ones, because having a general winning strategy implies having a positional one (and of course vice versa). The following theorem generalizes this observation.

Proposition 2.10. Let $\mathcal{G}$ denote any of the semantic games $G_\infty(\mathfrak{A}, r, \varphi)$, $G_\omega(\mathfrak{A}, r, \varphi)$, $G_n(\mathfrak{A}, r, \varphi)$. Then the following conditions hold.

1. Eloise (respectively, Abelard) has a positional winning strategy in $\mathcal{G}$ iff (s)he has a general one.

2. Eloise (respectively, Abelard) has a positional non-losing strategy in $\mathcal{G}$ iff (s)he has a general one.

Proof. $\mathcal{G}$ is clearly a reachability game for both players, and non-losing in $\mathcal{G}$ is equivalent to winning the corresponding safety game. Thus the claims of the proposition follow immediately from Theorem 2.1.

Due to this proposition, below we will almost exclusively consider positional strategies only, conceived as functions from game positions to related choices in the game.
When comparing two logics \( L_1 \) and \( L_2 \), we write \( L_1 \leq L_2 \) if for every formula \( \varphi_1 \) of \( L_1 \), there exists a formula \( \varphi_2 \) of \( L_2 \) such that for each suitable model \( \mathfrak{A} \) and assignment \( s \), the formula \( \varphi_1 \) is true in \( \mathfrak{A} \) under \( s \) iff \( \varphi_2 \) is true in \( \mathfrak{A} \) under \( s \). The strict ordering \( < \) is defined from \( \leq \) in the natural way.

**Logical consequence** is defined in the usual way for both BndSCL and SCL. That is, for a set of formulas \( \Sigma \), we write \( \Sigma \models \varphi \) iff for any suitable \( \mathfrak{A} \) (i.e., any \( \mathfrak{A} \) interpreting the vocabulary in \( \Sigma \) and \( \varphi \)) and any suitable \( s \) (i.e., an assignment interpreting the free variables in \( \Sigma \) and \( \varphi \)), it holds that if \( \mathfrak{A}, s \models \psi \) for all \( \psi \in \Sigma \), then \( \mathfrak{A}, s \models \varphi \). A formula \( \varphi \) is said to be **valid** if \( \emptyset \models \varphi \). We usually drop \( \emptyset \) and write \( \models \varphi \). Concerning deduction systems, we also define the notation \( \Sigma \vdash \psi \) in the usual way. That is, given a deduction system \( S \), we write \( \Sigma \vdash \varphi \) to mean that \( \varphi \) can be inferred in \( S \) from the premises taken from \( \Sigma \). We write \( \vdash \varphi \) to denote that \( \emptyset \vdash \varphi \). The deduction system \( S \) is **complete for validities** if the implication

\[ \models \varphi \Rightarrow \vdash \varphi \]

holds for all formulas of the logic investigated. The system \( S \) is **strongly complete** if

\[ \Sigma \models \varphi \Rightarrow \Sigma \vdash \varphi \]

holds for all formulas \( \varphi \) and any formula set \( \Sigma \) of the logic studied. The system \( S \) is **complete for first-order premise sets** if the above implication holds in restriction to those cases where \( \Sigma \) must be a set of first-order formulas. Note that \( \varphi \) does not need to be first-order. The system \( S \) is **sound** if

\[ \Sigma \vdash \varphi \Rightarrow \Sigma \models \varphi \]

holds for all formulas \( \varphi \) and formula sets \( \Sigma \) of the logic investigated.

### 3 Properties of BndSCL

In this section we prove some basic properties related to BndSCL. Now, the **diameter** of a directed graph \( G = (V, E) \) is

\[ \text{diam}(G) = \sup \{d(v, u) \mid v, u \in V\}, \]

if the supremum exists, and otherwise it is \( \infty \). Here \( d(u, v) \) denotes the directed distance between \( u \) and \( v \) which is formally defined as follows. We let \( d(w, w') = 0 \), and for all \( w' \in V \setminus \{w\} \), we define \( d(w, w') \) to be the smallest number \( k + 1 \in \mathbb{N}_+ \) such that we have \( d(w, w'') = k \) for some \( w'' \in V \) such that \( Eu''w' \). If no such number \( k + 1 \) exists, then \( d(w, w') = \infty \). In this paper, a cycle in a directed graph is a finite sequence \( (u_1, \ldots, u_k) \) of nodes such that the following conditions hold.

1. \( k > 1 \)
2. \( Eu_iu_{i+1} \) for all \( i < k \)
3. \( u_1 = u_k \)
4. \( u_i \neq u_j \) for all distinct indices \( i, j \in \{1, \ldots, k - 1\} \)

The following proposition lists examples of properties definable in BndSCL which are not expressible in FO. Showing that these properties are not definable in FO is an easy exercise in using Ehrenreucht-Fraïssé games.
Proposition 3.1. The following classes are definable in BndSCL.

1. The class of directed graphs that contain a cycle.

2. The class of directed graphs with a finite diameter.

Proof. To prove the first claim, consider the sentence
\[ \varphi_{\text{cycle}} := \exists x \exists y (x = y \land L(Eyx \lor \exists z(Eyz \land \exists y(y = z \land C_L))) \]
where the purpose of the innermost quantifier \( \exists y \) is to redefine \( y \) to point at the same element as \( z \). Intuitively, this enables us to “move \( y \)” along the nodes of the candidate directed cycle. Now, it is easy to check that Eloise has a winning strategy in \( G_\omega(G,\varphi_{\text{cycle}}) \) iff \( G \) contains a cycle.

To prove the second claim, consider the following sentence.
\[ \varphi_{\text{diam}} := \forall x \forall y (x = y \lor L(Exy \lor \exists z(Exz \land \exists x(x = z \land C_L))) \].
At first sight it might seem that \( \varphi_{\text{diam}} \) expresses that there is a directed path between any two vertices in the underlying graph and possibly no finite limit on the lengths of such paths, but since the players are required to declare initial clock values in the beginning of the semantic game for BndSCL, the formula is actually saying that there is some \( n \in \mathbb{N} \) so that between any two vertices, there exists a path of length at most \( n \). In other words, it expresses the fact that the diameter of the underlying graph is finite.

We remark that since BndSCL can expresses the fact that the diameter of the underlying graph is finite, it follows easily that BndSCL does not have compactness theorem; otherwise one could use a standard compactness argument to construct a directed graph that is not connected (and hence does not have a finite diameter) but is nevertheless a model of \( \varphi_{\text{diam}} \).

The following lemmas are straightforward.

Lemma 3.2. Let \( \varphi \) be a formula of BndSCL, and let \( n_1, n_2 \in \mathbb{N} \) such that \( n_1 < n_2 \). If Eloise has a winning strategy in \( G_{n_1}(\mathcal{A},s,\varphi) \), then she has one also in \( G_{n_2}(\mathcal{A},s,\varphi) \).

Proof. It is easy to see that the strategy for \( G_{n_1} \) can be simulated in \( G_{n_2} \).

Lemma 3.3. Let \( \varphi \) be a sentence of BndSCL. Now Eloise has a winning strategy in the bounded evaluation game \( G_\omega(\mathcal{A},s,\varphi) \) iff there exists \( n \in \mathbb{N} \) such that Eloise has a winning strategy in the \( n \)-bounded evaluation game \( G_n(\mathcal{A},s,\varphi) \).

Proof. Immediate by the definitions and Lemma 3.2.

By the above Lemma, we have \( \mathcal{A},s \models_\omega \varphi \) iff Eloise has a winning strategy in \( G_n(\mathcal{A},s,\varphi) \) for some \( n \). Therefore, if we are only interested in whether or not a formula of BndSCL holds in a model \( \mathcal{A} \) under some assignment \( s \), we can consider evaluation games where only Eloise declares the initial clock value \( n \) and then the game \( G_n(\mathcal{A},s,\varphi) \) is played. We can ignore the rule that also Abelard declares a clock value. It is then easy to see that any two formulas are weakly equivalent with respect to the standard semantics of BndSCL iff they are weakly equivalent with respect to the new semantics. However, it is straightforward to show that the same does not hold for strong equivalence. The following gives a concrete example of how the alternative semantics affects the semantic games.
Example 3.4. Consider the sentence
\[ \varphi := \neg \exists x (P x \land \exists y (Rxy \land (Qy \lor \exists y (y = x \land C L)))], \]
and let \( A \) be any suitable model with the property that there exists an \( R \)-path \( \pi \) between the element belonging to \( P^A \) and an element belonging to \( Q^A \), and the length (number of edges) of the shortest path from an element in \( P^A \) to an element in \( Q^A \) is greater than one. Now, in the game \( G_\omega (A, \varphi) \), Abelard has the following winning strategy: choose \( n' \) so that it is at least, say, the length of \( \pi \), and then play along \( \pi \) the evaluation game \( G_n (A, \varphi) \) where \( n \geq n' \) is the number selected by Eloise. However, if Eloise is the only player who has influence on the initial clock value, then she has a trivial strategy which guarantees that she does not lose the game: set the clock value to zero. This shows that while Abelard has a winning strategy in \( G_\omega (A, \varphi) \), Eloise can easily prevent Abelard from winning in the alternative game. Nevertheless, even in the alternative game, Eloise surely has no winning strategy, as having one would imply she has one also in the standard game for BndSCL.

Finally, note that our standard semantics for BndSCL has the property that \( A, s |\models \neg \varphi \) iff the (opponent player) Abelard has a winning strategy in \( G_\omega (A, \varphi) \). This is a desirable property that gives an interpretation to negation, and this property does not hold as such in the alternative semantics where Abelard does not declare clock values.

3.1 Approximants of formulas

A standard technique in the study of logics with fixed points (such as the modal \( \mu \)-calculus) is the use of approximants, which evaluate the fixed point only up to some fixed bound. Even though BndSCL is not really based on fixed points in any direct way, a natural notion of an approximant can be defined also for BndSCL. However, this notion differs from the corresponding notion for fixed point logics in a number of crucial ways. For example, we will have to adjust the approximants to take care of the number of negations encountered in related plays of the semantic game. These issues will become transparent below when we give the related formal definitions. Intuitively, the \( n \)th approximant for a formula of BndSCL will describe an evaluation game where the initial clock value is set to \( n \). Before giving the definition of approximants, we provide the following auxiliary definition.

Definition 3.5. Let \( \varphi \in \text{BndSCL} \). If \( \varphi \) is not a regular formula, we let \( \varphi' \) denote a regularisation of it. To keep the current definition deterministic, we suppose \( \varphi' \) is obtained from \( \varphi \) in some systematic way. If \( \varphi \) is already regular then \( \varphi' = \varphi \). Now, the \( n \)th unfolding (or \( n \)-unfolding) of \( \varphi \), denoted by \( \Psi^n \varphi \), is defined inductively as follows.

1. The zeroeth unfolding \( \Psi^0 \varphi \) of \( \varphi \) is defined to be the formula \( \varphi' \).

2. The \( (k + 1) \)st unfolding \( \Psi^{k+1} \varphi \) is the formula obtained from the \( k \)th unfolding \( \Psi^k \varphi \) by replacing every looping atom \( C L \) in \( \Psi^k \varphi \) by the corresponding reference formula \( Rf(C_L) \) in \( \Psi^k \varphi \) (if the reference formula exists, i.e., if \( C_L \) is not free).

We now define the notion of an approximant.
Lemma 3.9. Let $\varphi$ be a formula of BndSCL. We define the $n$th approximant (or $n$-approximant) $\Phi^n_\varphi$ of $\varphi$ to be the FO-formula obtained from the $n$th unfolding $\Psi^n_\varphi$ by removing all the label symbols and replacing each occurrence of each looping atom by

1. $\bot$ if the occurrence of the atom is positive in $\Psi^n_\varphi$,
2. $\top$ if the occurrence is negative in $\Psi^n_\varphi$.

Example 3.7. Consider the sentence $LC_L \in \text{BndSCL}$. Every approximant of this sentence is just $\bot$.

Example 3.8. Consider the sentence $\varphi_{\text{cycle}} := \exists x \exists y (x = y \land L(Eyx \lor \exists z(Eyz \land \exists y(y = z \land C_L)))$ of BndSCL from Proposition 3.1 that defines the class of directed graphs that contain a cycle. Its first two approximants are

$$\Phi^0 = \exists x \exists y (x = y \land (Eyx \lor \exists z(Eyz \land \exists y(y = z \land \bot)))$$

and

$$\Phi^1 = \exists x \exists y (x = y \land (Eyx \lor \exists z(Eyz \land \exists y(y = z \land \chi))))$$

where $\chi = Eyx \lor \exists z(Eyz \land \exists y(y = z \land \bot))$.

The zeroth approximant expresses that there exists a reflexive loop (i.e., a cycle of length one), while the first approximant asserts that there exists a cycle of length at most two. In general, the $n$th approximant expresses that there exists a cycle of length at most $n + 1$.

The following lemma gives a natural characterization of approximants.

Lemma 3.9. Let $\varphi \in \text{BndSCL}$ be a formula. Now Eloise has a winning strategy in the $n$-bounded game $G_n(\mathfrak{A}, s, \varphi)$ iff she has a winning strategy in $G(\mathfrak{A}, s, \Phi^n_\varphi)$.

Proof. First notice that by the construction of $\Phi^n_\varphi$, the game trees of the games $G_n(\mathfrak{A}, s, \varphi)$ and $G(\mathfrak{A}, s, \Phi^n_\varphi)$ are essentially identical all the way from the root to the level with positions of type $(C_L, r, +, 0)$ and $(C_L, r, -, 0)$ in $G_n(\mathfrak{A}, s, \varphi)$. The position in $G(\mathfrak{A}, s, \Phi^n_\varphi)$ that corresponds to the position $(C_L, r, +, 0)$ of $G_n(\mathfrak{A}, s, \varphi)$ is $(\bot, r, +)$, and similarly, the position corresponding to $(C_L, r, -, 0)$ is $(\top, r, -)$. Eloise would not win any play entering any of such positions. Therefore no winning strategy of Eloise will lead to a play that enters such positions.

Now, if Eloise has a winning strategy in one of the games $G_n(\mathfrak{A}, s, \varphi)$ and $G(\mathfrak{A}, s, \Phi^n_\varphi)$, she can simulate that strategy in the other game. Each play where Eloise follows a winning strategy will end with her winning in some position involving some FO-atom $\delta$. The simulated play in the other game will then, likewise, end with her win in a position with $\delta$. Notice that when simulating a winning strategy for $G(\mathfrak{A}, s, \Phi^n_\varphi)$ in order to win $G_n(\mathfrak{A}, s, \varphi)$, every non-winning position $(C_L, r, +, 0)$ (respectively, $(C_L, r, -, 0)$) will indeed be avoided because Eloise will not enter the corresponding position $(\bot, r, +)$ (resp., $(\top, r, -)$) in $G(\mathfrak{A}, s, \Phi^n_\varphi)$ because such a position would be losing for her. A corresponding principle holds in the direction where Eloise simulates a strategy from $G_n(\mathfrak{A}, s, \varphi)$ to win a game on the approximant. □
Lemma 3.10. If $A \models \Phi^n_\varphi$, then $A \models \Phi^{n'}_\varphi$ for all $n' \geq n$.

Proof. Combine Lemma 3.9 with Lemma 3.2.

3.2 Applications of approximants

We will now show that BndSCL translates into $\mathcal{L}_{\omega_1\omega}$, i.e., the extension of FO that allows for countably infinite conjunctions and disjunctions.

Theorem 3.11. Let $\varphi$ be a formula of BndSCL. Then we have

$A, s \models \omega \varphi$ if and only if $A, s \models \bigvee_{n \in \mathbb{N}} \Phi^n_\varphi$.

Proof. By Lemma 3.3, we have $A, s \models \omega \varphi$ if and only if there exists some $n \in \mathbb{N}$ such that Eloise has a winning strategy in $G_n(A, s, \varphi)$. By Lemma 3.9, the latter condition is equivalent to there existing some $n$ such that $A \models \Phi^n_\varphi$.

We can use theorem 3.11 to prove undefinability results for BndSCL, and as an example of this, we next prove that graph connectivity is not definable in BndSCL.

Proposition 3.12. The class of (finite and infinitary) connected graphs is not definable in BndSCL.

Proof. Let $G_1 = (\mathbb{Z}, \{(n, n + 1) \mid n \in \mathbb{Z}\} \cup \{(n + 1, n) \mid n \in \mathbb{Z}\})$, and let $G_2$ be the disjoint union of two copies of $G_1$. Suppose that there exists a sentence $\varphi \in \text{BndSCL}$ which is true in a graph if and only if the graph is connected. Then $G_1 \models \omega \varphi$ and hence $G_1 \models \Phi^n_\varphi$ for some $n \in \mathbb{N}$. Now, it is straightforward to show that $G_1$ and $G_2$ are elementarily equivalent (for example using Ehrenfeucht-Fraïssé games). Thus we have $G_2 \models \Phi^n_\varphi$, which implies that $G_2 \models \omega \varphi$ by Theorem 3.11. This is a contradiction.

We then relate validities of BndSCL to approximants.

Theorem 3.13. Let $\varphi \in \text{BndSCL}$ be a formula. Now $\varphi$ is valid if and only if $\Phi^n_\varphi$ is valid for some $n \in \mathbb{N}$.

Proof. The direction from right to left follows directly from Theorem 3.11. Suppose then that none of the sentences $\Phi^n_\varphi$ is valid. We claim that then

$\Sigma := \{-\Phi^n_\varphi \mid n \in \mathbb{N}\}$

is satisfiable. By compactness, it suffices to show that all finite subsets of $\Sigma$ are satisfiable. But this is clear, since each sentence $-\Phi^n_\varphi$ is now satisfiable (due to no $\Phi^n_\varphi$ being valid) and we have $-\Phi^n_\varphi \models -\Phi^{n'}_\varphi$ for all $n' < n$ by Lemma 3.10. Since $\Sigma$ is satisfiable, we have $A, s \models \Sigma$ for some $A$ and $s$. Thus Eloise cannot have a winning strategy in $G_n(A, s, \varphi)$ for any $n \in \mathbb{N}$ by Lemma 3.9. Therefore $\varphi$ is not valid.

Since the set of valid FO-sentences is recursively enumerable, the following corollary is immediate.

Corollary 3.14. The set of valid formulas of BndSCL is recursively enumerable.
Since BndSCL is not closed under contradictory negation, the above corollary leaves open the complexity of its satisfiability problem. Using approximants, we can determine the exact complexity of the satisfiability problem of BndSCL. Before that, we give some auxiliary definitions.

The **grid** is the structure $\mathbf{G} := (\mathbb{N} \times \mathbb{N}, H, V)$ where the binary relations

$$H := \{ ((i,j), (i + 1, j)) \mid i, j \in \mathbb{N} \}$$

and

$$V := \{ ((i,j), (i,j + 1)) \mid i, j \in \mathbb{N} \}$$

are suggestively called the **horizontal** and **vertical successor relation**. By $G^*$ we denote the expansion $(\mathbb{N} \times \mathbb{N}, H, V, H^*, V^*)$ of $\mathbf{G}^*$ where $H^*$ and $V^*$ are the transitive closures of $H$ and $V$.

Now consider a deterministic Turing machine $M$ with one-way infinite tape and one read-write head. Denote the set of states of $M$ by $Q$. Let $S_t$ and $S_i$ be, respectively, the sets of tape and input symbols of $M$, and denote $S := S_t \cup S_i$. Let $s \in S^*$ be an input to $M$. Let $\tau$ denote the vocabulary that contains the relation symbols of $G^*$ and additionally a unary relation symbol $P_u$ for each $u \in S \cup Q$. The **computation table** $\Sigma_{M,s}$ of $M$ with the input $s$ is the expansion of $\mathbf{G}^*$ defined as follows.

1. Consider the point $(i,j)$ of $\Sigma_{M,s}$. For each predicate $P_u \in S$, we define $P_u$ to be true at the point $(i,j)$ of $\Sigma_{M,s}$ iff at time $j$, the tape cell $i$ contains the symbol $u$. Thus, intuitively, the tape of $M$ at time $j$ is encoded by the $j$th row of the structure $\Sigma_{M,s}$. Note that the input $s$ is of course encoded to the beginning cells of row 0.

2. We define $P_u \in Q$ to be true at the point $(i,j)$ of $\Sigma_{M,s}$ iff at time $j$, the read-write head of $M$ is at the cell $i$ and the current state of $M$ is $u$.

A **generalized ordered grid** is a structure $\mathfrak{T} := (A \times B, H, V, H^*, V^*)$ such that the following conditions hold.

1. The sets $A$ and $B$ are domains of two discrete order structures $(A, <^A)$ and $(B, <^B)$. Both $(A, <^A)$ and $(B, <^B)$ have a minimum element. The relation $H \subseteq (A \times B) \times (A \times B)$ is the **horizontal successor relation**

   $$H := \{ ((a,b), (a',b)) \mid b \in B \text{ and } a' \text{ is the } <^A \text{-successor of } a \}.$$  

   Similarly, $V$ is the **vertical successor relation**

   $$V := \{ ((a,b), (a,b')) \mid a \in A \text{ and } b' \text{ is the } <^B \text{-successor of } b \}.$$  

2. $H^*$ is the relation

   $$\{ ((a,b), (a',b)) \mid b \in B \text{ and } a <^A a' \}$$

and $V^*$ the relation

$$\{ ((a,b), (a,b')) \mid a \in A \text{ and } b <^B b' \}.$$  

Our next aim is to define the notion of a **generalized computation table**. These are structures that resemble computation tables but are built on generalised ordered grids. To define the notion, we let $M$ denote, as above, a deterministic Turing machine with a one-way infinite tape and one read-write head. The set of
states of $M$ is denoted by $Q$. We let $S_t$ and $S_i$ be the sets of tape and input symbols of $M$, and we define $S := S_t \cup S_i$. We let $\tau$ be the vocabulary containing the relation symbols of generalized ordered grids and additionally a unary relation symbol $P_u$ for each $u \in S \cup Q$.

Now, consider an infinite discrete order structure $(A, <^A)$ with a minimum element $0^A$. Suppose $A' \subseteq A$ is a prefix set for $(A, <^A)$. Then a function $s : A' \to S_i$ is a \textbf{generalized input} for $M$ and $(A, <^A)$, the intuition being that $s$ labels some prefix of $(A, <^A)$ with the input symbols of $M$. Let $(B, <^B)$ be an infinite discrete order structure with a minimum element $0^B$. Now consider the structure $\Sigma_{M, s}$ defined as follows.

1. $\Sigma_{M, s}$ expands the generalized grid $\Sigma = (A \times B, H, V, H^*, V^*)$ to the vocabulary $\tau$.

2. For each $b \in B$, there exists precisely one cell $(a, b) \in A \times B$ such that for some $u \in Q$, the predicate $P_u$ is satisfied at $(a, b)$. Furthermore, if some $P_u$ is satisfied at the cell $(a, b)$, then no other $P_{u'}$ with $u' \in Q$ is satisfied at that cell. Intuitively, all this simply means that the following two conditions hold.
   - At the computation stage indexed by (the row) $b \in B$, the read-write head is located in the cell $a \in A$.
   - At that computation stage, the machine $M$ is in state $u \in Q$.

3. The first cells of the row $0^B$ are indexed according to the generalized input $s$ in the natural way. In other words, the (possibly infinitary) input to $M$ is as given by the generalized input $s$.

4. At each point $(a, b)$ of the structure excluding row zero, the truth of the predicates $P_u$ with $u \in S$ are determined by neighbouring cells in the previous row, i.e., the cells $(a', b'), (a, b')$, $(a'', b')$ such that $(a, b')V(a, b)$ and $(a', b')H(a, b')$ and $(a, b')H(a'', b')$. The truth of the predicates is determined in the natural way according to the computation of $M$. Note that these predicates can change their truth value only if the predicates indicating the position of the read-write head is in the vicinity. The predicates corresponding to the read-write head are similarly locally related to the computation of the machine and relate to the other symbols in the correct way as allowed by $M$. The read-write head begins from the first cell in the beginning of computation.

Such a structure is a \textbf{generalized computation table} for $M$ and $s$. A generalized computation table for $M$ is a structure that is a generalized computation table for $M$ and some generalized input $s$.

\textbf{Lemma 3.15.} Let $M$ be a deterministic one-way Turing machine with one read-write head. The class of generalized computation tables for $M$ is definable by an FO-sentence.

We are now ready to prove the following.

\textbf{Theorem 3.16.} The satisfiability problem for BndSCL is $\Sigma_2^0$-complete.
Proof. We prove the upper bound first. By Lemma 3.11, a given sentence \( \varphi \) of BndSCL is satisfiable iff there exists some \( n \) such that the approximant \( \Phi^n \varphi \) is satisfiable. Therefore, and since the satisfiability problem for FO is in \( \Pi^0_1 \), satisfiability for BndSCL can be defined in \( \Sigma^0_2 \).

Next we shift our attention to the lower bound. Given a Turing machine \( M \), we use \( L(M) \) to denote the set of strings accepted by \( M \). The following problem is known to be \( \Sigma^0_2 \)-hard [18]: given a Turing machine \( M \), determine whether \( L(M) \) is finite. To prove the \( \Sigma^0_2 \)-hardness of the satisfiability problem of BndSCL, we give a recursive mapping \( M \mapsto \varphi_M \) such that the BndSCL-sentence \( \varphi_M \) is satisfiable iff \( L(M) \) is finite.

Fix a Turing machine \( M \). Without loss of generality, we may assume that the vocabulary of \( M \) is \{0, 1\}. To simplify the construction of \( \varphi_M \), instead of \( M \) we consider a Turing machine \( M^* \) which, when given \( 1^n \) as an input a string, does the following.

- \( M^* \) begins the process of enumerating the set of binary strings of length at least \( n \).
- During the enumeration process, \( M^* \) halts if it encounters a string accepted by \( M \).

Clearly

\[ L(M) \cap \{0, 1\}^{\geq n} = \varnothing \iff M^* \text{ does not halt on the input } 1^n. \]

Thus

\[ L(M) \text{ is finite } \iff \text{ for some } n \geq 0, M^* \text{ does not halt on the input } 1^n. \]

The sentence \( \varphi_M \) will express the right hand side of this equivalence.

The sentence \( \varphi_M \) will be true in precisely those models that are isomorphic to some generalized computation table \( \Sigma_{M^*, 1^n} \) that encodes a non-halting computation with the input being a finite string on bits 1. Now, by Lemma 3.15, there exists an FO-formula \( \psi_{M^*} \) that defines the class of generalized computation tables for \( M^* \). The formula \( \varphi_M \) will be a conjunction

\[ \psi_{M^*} \land \psi' \land \chi \]

such that the following conditions hold.

1. \( \psi' \) is an FO-sentence making sure that the computation of \( M^* \) does not halt.

2. \( \chi \) is a sentence of BndSCL asserting that there exists some \( n \in \mathbb{N} \) such that the input to the computation is \( 1^n \).

While writing \( \psi' \) in FO is straightforward, the attempt to define \( \chi \) in FO (rather than BndSCL) runs into the challenge of specifying that only finite input strings \( 1^n \) (of arbitrary lengths) are allowed. Thereby we resort to the expressive capacities of BndSCL for defining \( \chi \). Now, to define \( \chi \). Recall that \( P_0 \) and \( P_1 \) are unary predicates that encode the bits 0 and 1, respectively. We first give the following auxiliary formulas.

- **column_zero(x) := \( \neg \exists y H^* y x \)**
\[ \text{row}_0(x) := \neg \exists y \forall^* yx \]

\[ \chi_0 := \neg \exists x (\text{row}_0(x) \land P_0(x)) \]

We then define the formula \( \chi \) as follows.

\[ \chi := \chi_0 \land \forall x \left( \neg \text{row}_0(x) \lor \neg P_1(x) \lor L(\text{column}_0(x) \lor \exists y (Hyx \land \exists x (x = y \land C_L))) \right). \]

It is easy to show that \( \varphi_M \) is as required. \( \square \)

4 Results on SCL

In this section we investigate SCL. We first list some interesting properties that are definable in SCL.

**Proposition 4.1.** The following classes of models are definable in SCL.

1. The class of connected graphs (i.e., graphs such that for all vertices \( u \) and \( v \neq u \), there is a directed path from \( u \) to \( v \)). We do not limit attention to finite graphs here.

2. The class of well-founded linear order structures.

**Proof.** To establish the first claim, we notice that clearly the following sentence of SCL is true in a graph \((V, E)\) iff the graph is connected:

\[ \forall x \forall y (x = y \lor L(Exy \lor \exists z (Exz \land \exists x (x = z \land C_L))) \). \]

To establish the second claim of the theorem, we claim that the sentence

\[ \forall x \forall y (\neg y < x \lor \forall x (\neg x = y \land C_L)) \]

of SCL is true in a linear order structure \((A, <)\) iff the linear order is well-founded. Indeed, it is easy to see that if \((A, <)\) is well-founded, then the game can always be won by Eloise. On the other hand, if \((A, <)\) is not well-founded, then Abelard can force the play of the game to last for infinitely many rounds. \( \square \)

4.1 The validity problem of SCL

Our next aim is to prove that the set of validities of SCL is recursively enumerable. As in the case of BndSCL, our arguments make extensive use of approximants introduced for BndSCL (as opposed to SCL). As the following example demonstrates, we will have to use these approximants in a slightly more sophisticated way.

**Example 4.2.** Consider the sentence

\[ \varphi = \forall x \forall y (\neg y < x \lor \forall x (\neg x = y \land C_L)) \]

from Proposition 4.1. The proposition shows that this sentence is true in a linear order \((A, <)\) according to SCL iff the linear order is well-founded. Now, the sentence \( \neg \Phi^n_\varphi \), where \( \Phi^n_\varphi \) is the \( n \)-approximant of \( \varphi \), is true on a linear order structure \((A, <)\) iff < has a decreasing sequence of at least \( n + 1 \) nodes. Hence \((\mathbb{N}, <)\) is a model for the set \( \{ \varphi \} \cup \{ \neg \Phi^n_\varphi \mid n \in \mathbb{N} \} \) according to SCL.
The above example demonstrates that there are sentences $\varphi$ of the logic SCL and models $\mathfrak{A}$ for which Eloise has a winning strategy in the game $G_\infty(\mathfrak{A}, \varphi)$, but not in any of the games $G_n(\mathfrak{A}, \varphi)$. Nevertheless, we can still establish the following compactness-like property for SCL.

**Lemma 4.3.** Let $\varphi$ be a formula of SCL. Suppose that for every $n \in \mathbb{N}$, there exists a model $\mathfrak{A}_n$ and an assignment $s_n$ so that Eloise does not have a winning strategy in the game $G_n(\mathfrak{A}_n, s_n, \varphi)$. Then there exists a model $\mathfrak{A}$ and an assignment $s$ so that Eloise does not have a winning strategy in the game $G_\infty(\mathfrak{A}, s, \varphi)$.

**Proof.** We may assume that the domains of the assignments contain precisely the set of variables occurring in $\varphi$, free and bound. Thus, we can clearly even assume that the assignments have the domain $\emptyset$ by extending the underlying vocabulary by finitely many constant symbols. We note that we strictly speaking consider purely relational vocabularies, so these constants ultimately need to be encoded by relation symbols. We shall below further extend our vocabulary also by function symbols, and these also need to be encoded by relation symbols. All these encodings, however, are straightforward, so we can make the related assumptions without further discussion. Now, as $s = \emptyset$, the formula $\varphi$ is a sentence. We may also assume that $\varphi$ makes no use of $\lor$ or $\forall$, as these can be defined via using the operators $\neg$, $\land$, $\exists$. Finally, we may assume, without loss of generality, that $\varphi$ does not have free looping atoms, as such an atom can be replaced by $LC_L$.

Now, suppose indeed that for every $n \in \mathbb{N}$, there exists a model $\mathfrak{A}_n$ so that Eloise does not have a winning strategy in the game $G_\infty(\mathfrak{A}_n, s_n, \varphi)$. Based on this, we will construct a countable first-order theory $\Sigma$ such that from any model of $\Sigma$, we can read off a model $\mathfrak{A}$ and a non-losing strategy $\sigma$ for Abelard in $G_\infty(\mathfrak{A}, \varphi)$. This implies, in particular, that Eloise does not have a winning strategy in the game $G_\infty(\mathfrak{A}, \varphi)$. To show that the theory $\Sigma$ is satisfiable, we will use compactness of FO.

We will start defining $\Sigma$ by first specifying the underlying vocabulary $\tau$ which will be finite but extend the vocabulary of $\varphi$. So, we first put into $\tau$ every relation symbol occurring in $\varphi$. We assume that $\{x_1, \ldots, x_k\}$ is the set of all variables that occur in the sentence $\varphi$. We then define the set

$$S := \{N(\psi, \#) \mid \psi \in Sf(\varphi), \# \in \{+, -\}\} \cup \{D, W, <\},$$

of relation symbols, where each of the symbols $N(\psi, \#)$ is a $(k+1)$-ary relation symbol, $<$ is a binary relation symbol and $W$ and $D$ are unary relation symbols. We add the symbols in $S$ into $\tau$. Note that the subscript $(\psi, \#)$ of $N(\psi, \#)$ encodes partial information on positions of evaluation games for $\varphi$, namely, $(\psi, \#)$ lists the current subformula occurrence $\psi$ being played and $\# \in \{+, -\}$ encodes whether Eloise or Abelard is currently the verifier in the game. Each predicate $N(\psi, \#)$ has arity $k + 1$, and the first $k$ elements of a tuple of $N(\psi, \#)$ encode the current assignment in the evaluation game while the $(k+1)$st element will encode (together with $<$) a measure of how many times positions with claim atoms $CL$ have been visited. The symbol $D$ will encode the domain of the models $\mathfrak{A}_n$ and $W$ the (disjoint) domain of $\sigma$. The details will be formalized below. To conclude the definition of $\tau$, we will add the constant symbol $d$ and the unary function symbol $f$ to $\tau$.

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3Syntactically identical subformulas that are in different parts of $\varphi$ can be distinguished by some convention whose details are not of importance here.
We then proceed with the definition of the theory Σ. First we need a sentence saying that \( < \) is a discrete linear order over the set encoded by the unary predicate \( W \), and that \( d \) is the minimum element of this ordering. We will also need the sentence
\[
\forall x (\exists y (x < y) \rightarrow (x < f(x) \land \forall y (x < y \rightarrow (f(x) < y \lor y = f(x))))),
\]
which expresses that \( f \) maps each element to its immediate successor with respect to \(<\), if such an element exists. Furthermore, we need a sentence ensuring that the intersection of \( D \) and \( W \) is empty.

We then write axioms for the relations \( N(\psi,\#) \). The intuition is that tuples in the interpretation of \( N(\psi,\#) \) encode non-losing positions for Abelard in the evaluation game, i.e., positions that are not winning for Eloise. Thus the axioms of \( \Sigma \) will encode natural safety restrictions on the evaluation game.

First, for every FO-atom \( \alpha := \alpha(x_{i_1}, \ldots, x_{i_j}) \) with variables in \( \{x_1, \ldots, x_k\} \), where \( (x_{i_1}, \ldots, x_{i_j}) \) lists the variables of \( \alpha \), we add the sentences
\[
\forall x_1 \ldots x_k \forall c (N(\alpha,+)(x_1, \ldots, x_k, c) \rightarrow \neg \alpha(x_{i_1}, \ldots, x_{i_j}))
\]
and
\[
\forall x_1 \ldots x_k \forall c (N(\alpha,-)(x_1, \ldots, x_k, c) \rightarrow \alpha(x_{i_1}, \ldots, x_{i_j}))
\]
to the theory \( \Sigma \). Such sentences guarantee that \( N(\alpha,+ \land \chi) \) and \( N(\alpha,- \land \chi) \) will not contain positions involving atoms such that Abelard directly loses the evaluation game in those positions.

For every subformula \( \psi \land \chi \), we add to \( \Sigma \) the formulas
\[
\forall x_1 \ldots x_k \forall c (N(\psi \land \chi,+)(x_1, \ldots, x_k, c) \rightarrow (N(\psi,+)(x_1, \ldots, x_k, c) \lor N(\chi,+)(x_1, \ldots, x_k, c)))
\]
and
\[
\forall x_1 \ldots x_k \forall c (N(\psi \land \chi,-)(x_1, \ldots, x_k, c) \rightarrow (N(\psi,-)(x_1, \ldots, x_k, c) \land N(\chi,-)(x_1, \ldots, x_k, c))).
\]
For every subformula \( \exists x_i \psi \), we add the following formulas
\[
\forall x_1 \ldots x_k \forall c (N(\exists x_i \psi,+)(x_1, \ldots, x_k, c) \rightarrow \forall x_i (D(x_i) \rightarrow N(\psi,+)(x_1, \ldots, x_k, c)))
\]
and
\[
\forall x_1 \ldots x_k \forall c (N(\exists x_i \psi,-)(x_1, \ldots, x_k, c) \rightarrow \exists x_i (D(x_i) \land N(\psi,-)(x_1, \ldots, x_k, c)))
\]
to the theory \( \Sigma \). To cover negation, we add to \( \Sigma \) the formulas
\[
\forall x_1 \ldots x_k \forall c (N(\neg \psi,+)(x_1, \ldots, x_k, c) \rightarrow (N(\neg \psi,-)(x_1, \ldots, x_k, c)))
\]
and
\[
\forall x_1 \ldots x_k \forall c (N(\neg \psi,-)(x_1, \ldots, x_k, c) \rightarrow (N(\neg \psi,+)(x_1, \ldots, x_k, c))).
\]

We will also need two axioms for claim symbols \( C_L \). For every subformula \( C_L \) with the reference formula \( L\psi \), we add to \( \Sigma \), for every \( \# \in \{+, -, \} \), the formula
\[
\forall x_1 \ldots x_k \forall c ((N(C_L,\#)(x_1, \ldots, x_k, c) \land \exists y (c < y)) \rightarrow \exists z (W(z) \land f(c) = z \land N(L\psi,\#)(x_1, \ldots, x_k, z))).
\]
Similarly, for every subformula \( L\psi \) and \( \# \in \{+, -, \} \), we add to \( \Sigma \) the formula
\[
\forall x_1 \ldots x_k \forall c (N(L\psi,\#)(x_1, \ldots, x_k, c) \rightarrow N(\psi,\#)(x_1, \ldots, x_k, c)).
\]
We will also need the following formula which encodes the initial position of the game.

$$\exists x_1 \ldots x_k N_{(\psi, +)}(x_1, \ldots, x_k, d).$$

Now, if $\Phi$ denotes the (finite) set of axioms that we have listed above, then as our theory $\Sigma$ we choose the following set

$$\Phi \cup \{\theta_n^d \mid n \in \mathbb{N}\}$$

where each $\theta_n^d$ expresses that $W$ has at least $n$ elements $u$ such that $d < u$. Therefore, in any model of $\Sigma$, the sequence

$$d, f(d), f(f(d)), \ldots$$

is an infinite ascending sequence of elements with respect to $\prec$.

If $\Sigma$ is satisfiable in a model $\mathfrak{A}^+$, then, in the submodel of $\mathfrak{A}^+$ induced by the set $D^{\mathfrak{A}^+}$, Abelard can survive the semantic game for $\varphi$ without losing. This submodel (or its reduct to the vocabulary of $\varphi$) is then the model $\mathfrak{A}$ required by the current theorem.

To show that $\Sigma$ is indeed satisfiable, it suffices, by compactness of FO, to show that each finite subset of $\Sigma$ is satisfiable. To this end, it clearly suffices to show that for every $n \in \mathbb{N}$, the set

$$\Sigma_n := \Phi \cup \{\theta_n^m \mid 1 \leq m \leq n\}$$

is satisfiable. Now, we have assumed that there exists a model $\mathfrak{A}_n$ for each $n \in \mathbb{N}$ such that Eloise does not have a winning strategy in the game $G_n(\mathfrak{A}_n, \varphi)$. Thus, by Theorem 2.1 Abelard has a positional non-losing strategy in $G_n(\mathfrak{A}_n, \varphi)$. We create a model of $\Sigma_n$ as follows.

1. We take a copy of $\mathfrak{A}_n$ and interpret $D$ to correspond to the domain of $\mathfrak{A}_n$.

2. We take a disjoint (from $D$) set of size at least $n + 1$ and interpret $W$ to correspond to that set. We also interpret the symbols $\prec$, $d$, $f$ over the set $W$ so that $d$ becomes the minimum element of the order $\prec$ and $f$ runs step by step from $d$ upwards along $\prec$.

3. The predicates $N_{(\psi, \#)}$ are interpreted in the natural way according to the non-losing strategy of Abelard over $\mathfrak{A}_n$. Note that the $(k + 1)$st elements of the tuples of $N_{(\psi, \#)}$ run along the order $\prec$.

Thereby we create a model for $\Sigma_n$ for each $n \in \mathbb{N}$, as required.

By the above lemma, if a formula of SCL is valid, then already one of the approximants $\Phi_n^\varphi$ is. It is easy to see that also the converse holds.

**Lemma 4.4.** Let $\varphi$ be a formula of SCL, and let $\mathfrak{A}$ be a suitable model and $s$ a suitable assignment. If Eloise has a winning strategy in the game $G_n(\mathfrak{A}, s, \varphi)$, then she has a winning strategy in the game $G_\infty(\mathfrak{A}, s, \varphi)$.

**Proof.** As the argument for Lemma 3.2 the proof is based on simulating strategies. Indeed, in $G_\infty(\mathfrak{A}, s, \varphi)$, by using a non-positional strategy, Eloise can keep track of the number of times the players have visited a position where the formula is a looping atom. In particular, she can thereby pretend that she is playing the game $G_n(\mathfrak{A}, s, \varphi)$. Thus, by Lemma 2.10 she also has a positional winning strategy in the game $G_\infty(\mathfrak{A}, s, \varphi)$. \qed
Theorem 4.5. Let $\varphi$ be a formula of SCL. Now $\varphi$ is valid if and only if for some $n \in \mathbb{N}$, the approximant $\Phi^*_n \varphi$ is valid.

Proof. The claim follows directly from Lemmas 3.9, 4.4 and 4.3. Note that formally 3.9 refers to BndSCL but is in fact independent of the the difference between SCL and BndSCL. $\square$

Corollary 4.6. The set of valid sentences of SCL is recursively enumerable.

Remark 4.7. Perhaps surprisingly, Theorems 3.13 and 4.5 imply that the set of valid sentences of BndSCL and SCL coincide, i.e., $\varphi$ is valid with respect to bounded semantics iff it is valid with respect to unbounded semantics. This will be strengthened to concern first-order premise sets below when we discuss completeness.

4.2 Translating SCL into $\forall\mathcal{SO}$

We now show how SCL can be translated into universal second-order logic $\forall\mathcal{SO}$. After that we will use this translation to determine the complexity of the validity problems of the two-variable fragments of BndSCL and SCL.

By SCL$^k$ and FO$^k$ we mean, respectively, the $k$-variable fragments of SCL and FO. We start with the following result, which is also of independent interest.

Theorem 4.8. Let $\varphi(x) \in$ SCL$^k$ be a formula, and let $\tau$ denote the vocabulary of $\varphi(x)$. Then there exists a vocabulary $\tau' \supseteq \tau$ and a formula $\Psi(x)$ of FO$^k$ with the following properties.

1. If $A$ is a $\tau$-model such that Eloise does not have a winning strategy in the game $G_\infty(A, s, \varphi)$, then $A$ can be expanded to a $\tau'$-model $A'$ so that $A', s \models \Psi(x)$.

2. If $A'$ is a $\tau'$-model so that $A, s \models \Psi(x)$, then Eloise does not have a winning strategy in the game $G_\infty(A \upharpoonright \tau, s, \varphi)$.

Furthermore, $\Psi(x)$ can be computed from $\varphi(x)$ in polynomial time.

Proof. Suppose that $\{x_1, \ldots, x_k\}$ contains the set of variables occurring in $\varphi(x)$, including the bound variables. The vocabulary $\tau'$ will contain, in addition to the relation symbols in $\varphi$, the relation symbols

$$\{N(\psi, \#) \mid \psi \in \text{Sf}(\varphi), \# \in \{+, -\}\}$$

where each symbol $N(\psi, \#)$ is $k$-ary. The proof is similar to the proof of Lemma 4.3: the relation symbols will be used to encode positions that are safe for Abelard, by which we mean that Eloise cannot force a win from such positions.

The formula $\Psi(x)$ will be a conjunction of axioms for the relations $N(\psi, \#)$ which intuitively speaking encode “safety conditions” for Abelard in a natural way. For instance, for every subformula $\exists x_1 \psi$ of $\varphi(x)$, we use the formulas

$$\forall x_1 \ldots x_k (N(\exists x_1 \psi, +)(x_1, \ldots, x_k) \rightarrow \forall x_1 N(\psi, +)(x_1, \ldots, x_k))$$

and

$$\forall x_1 \ldots x_k (N(\exists x_1 \psi, -)(x_1, \ldots, x_k) \rightarrow \exists x_1 N(\psi, -)(x_1, \ldots, x_k)).$$

As an another example, for every subformula $C_L$ with reference formula $\text{Ref}(C_L)$, we use for both $\# \in \{+, -\}$ the formula

$$\forall x_1 \ldots x_k N(C_L, \#)(x_1, \ldots, x_k) \rightarrow N(\psi, \#)(x_1, \ldots, x_k).$$
It is clear that such axioms can be written in FO for every pair \((\psi, \#)\).

Theorem 4.8 immediately yields a polynomial time algorithm for translating formulas of SCL\(^k\) to formulas of \(\forall\text{SO}^k\), i.e., to formulas \(\forall X_1 \ldots \forall X_n \psi\) where the first-order part \(\psi\) is a formula of FO\(^k\).

**Corollary 4.9.** Every formula of \(\varphi(\overline{x}) \in \text{SCL}^k\) can be translated in polynomial time to an equivalent formula \(\Psi(\overline{x}) \in \forall\text{SO}^k\).

**Proof.** Given a formula \(\varphi(\overline{x}) \in \text{SCL}^k\), compute the formula \(\Psi(\overline{x}) \in \text{FO}^k\) given by Theorem 4.8. This can clearly be computed in polynomial time from \(\varphi(\overline{x})\). If \(\sigma\) denotes the set of relation symbols that occur in \(\Psi(\overline{x})\) but not in \(\varphi(\overline{x})\), then \(\varphi(\overline{x})\) is equivalent to the second-order formula

\[
\neg (\exists R)_{R \in \sigma} \Psi(\overline{x})
\]

and thereby to the formula

\[
(\forall R)_{R \in \sigma} \neg \Psi(\overline{x})
\]

of \(\forall\text{SO}^k\).

Unsurprisingly, the containment implied by Corollary 4.9 turns out to be strict.

**Theorem 4.10.** SCL < \(\forall\text{SO}\) in relation to expressive power.

**Proof.** To establish that \(\forall\text{SO} \not\leq \text{SCL}\), we show that the class of finite structures over the empty vocabulary is not definable in SCL. Aiming for a contradiction, suppose that the property of finiteness is definable. Let \(\varphi \in \text{SCL}\) be a sentence defining this class. Let \(k\) be the number of variables occurring in \(\varphi\). Define \(\mathfrak{A}\) to be the model over the empty vocabulary and with the domain \(\{0, \ldots, k\}\) with precisely \(k+1\) elements. Thus \(\mathfrak{A} \models \varphi\) and hence Eloise has a winning strategy \(\sigma\) in the game \(G_\infty(\mathfrak{A}, \varphi)\). Now, let \(\mathfrak{B} = \mathbb{N}\). To derive a contradiction, one can show that \(\sigma\) can be used to design a winning strategy \(\sigma'\) for Eloise in the game \(G_\infty(\mathfrak{B}, \varphi)\). The idea is to simulate the “identity information” allowed by \(\sigma\) in the game played with \(\sigma'\).

We next formulate this more formally.

Let \(X \subseteq \{x_1, \ldots, x_k\}\), and let \(S_1\) and \(S_2\) be nonempty sets. Consider two assignments \(s_1 : X \rightarrow S_1\) and \(s_2 : X \rightarrow S_2\). We say that the assignments are similar if for all \(x, y \in \{x_1, \ldots, x_k\}\), we have \(s_1(x) = s_1(y)\) iff \(s_2(x) = s_2(y)\). We say that two positions in two semantic games are assignment-similar if the assignment functions in the positions are similar. Now, it is easy to see that we can define \(\sigma'\) based on \(\sigma\) such that every play according to \(\sigma'\) corresponds to a play according to \(\sigma\) such that the two plays simultaneously realize assignment similar positions in every round. Therefore \(\sigma'\) is a winning strategy in \(G_\infty(\mathfrak{B}, \varphi)\).

We can use Corollary 4.9 to determine the exact complexity of the validity problem of SCL\(^2\).

**Theorem 4.11.** The validity problem of SCL\(^2\) is coNExpTime-complete.

**Proof.** The lower bound follows directly from the corresponding lower bound for two-variable logic FO\(^2\). For the upper bound, we first note that the translation given in the proof of Theorem 4.10 is polynomial and preserves the number of variables being used. Hence sentences of SCL\(^2\) are translated efficiently into equivalent
sentences of ∀SO². Now, obviously a sentence \( \psi \) of ∀SO is valid if and only if the first-order part \( \chi \) of \( \psi \) is valid. Thus the complexity of the validity problem of ∀SO² is the same as FO², namely coNExpTime-complete. Hence also the validity problem of SCL² is in coNExpTime.

Since we have already observed that BndSCL and SCL have the same set of valid sentences, we have the following further corollary, where BndSCL² denotes the two-variable fragment of BndSCL.

**Corollary 4.12.** The validity problem of BndSCL² is coNExpTime-complete.

We note that, by Corollary 4.9, the fact that the validity problem of SCL is recursively enumerable follows immediately from the validity problem of ∀SO being, likewise, recursively enumerable. However, the value of having proved the validity problem of SCL recursively enumerable via Theorem 4.5 lies in the fact that the theorem relates SCL-validities to structurally similar validities of first-order logic (namely, approximants of the SCL-formulas). We have already used this fact to deduce that the set of valid sentences of BndSCL and SCL coincide. This fact will be crucial also in the next section, where we design an axiomatization which is complete for BndSCL as well as SCL.

## 5 A complete axiomatization

In this section we develop a proof system for BndSCL and SCL that is complete for validities and of course sound. Recalling that both of these logics have the same validities, the same system works for both of them. However, formally speaking, we will consider the case of SCL first.

In fact, more than just completeness for validities will be achieved. We shall show that our deduction system is actually complete for first-order premise sets for both SCL and BndSCL, thereby also establishing that the two logics coincide in their logical consequence relations in restriction to FO premise sets.

### 5.1 A system of natural deduction

The lack of classical negation must be carefully taken into account when designing a proof system for SCL and BndSCL. For example, the law of excluded middle fails to be valid in these logics, as demonstrated already by simple sentences such as \( LC_L \lor \neg LC_L \) and even \( CL \lor \neg CL \). To define a suitable deduction system, we first fix some rules. Let \( \land \)Intro denote the rule which we may conveniently denote also by \( \land, \psi \mapsto \varphi \land \psi \). Let \( \land \)Elim1 and \( \land \)Elim2 be the rules \( \varphi \land \psi \mapsto \varphi \) and \( \varphi \land \psi \mapsto \psi \), respectively, and let \( \lor \)Intro1 and \( \lor \)Intro2 denote \( \varphi \mapsto \varphi \lor \psi \) and \( \varphi \lor \psi \mapsto \varphi \lor \psi \). Let \( \lor \)Elim be the rule

\[
\frac{\varphi \lor \psi}{\chi \lor \psi \lor \varphi} \quad (\lor \text{Elim})
\]

Let \( \bot \)Intro be the rule \( \varphi \land \neg \varphi \mapsto \bot \) and \( \bot \)Elim the rule \( \varphi \mapsto \varphi[\psi_1/\bot, \ldots, \psi_k/\bot] \) which reads as follows. We begin with a formula \( \varphi \) in strong negation norm (recall
the definition from the preliminaries). We first identify a finite number $k$ of occurrences of $\bot$ that are not in the scope of any negations, and then we simultaneously replace them, respectively, by arbitrarily chosen formulas $\psi_1, \ldots, \psi_k$. Here the star above $\mapsto$ reminds the reader that the rule has the side condition that the replaced occurrences of $\bot$ must not be in the scope of any negations and that $\varphi$ should be in strong negation normal form. Now, the deduction system $S$ we shall use is defined as follows.

1. We include $\land$Intro, $\land$Elim1, $\land$Elim2, $\lor$Intro1, $\lor$Intro2, $\lor$Elim, $\bot$Elim and $\bot$Intro in $S$.

2. We include enough further rules in $S$ so that together with the above rules, our system becomes strongly complete for the language of FO as specified in this paper and remains sound for SCL and BndSCL. There are many ways of doing this, and this is straightforward to do also concerning soundness, as where necessary, we can add rules with side conditions that limit their use to first-order formulas.

3. We include the six recursion operator rules and six duality rules specified below.

The recursion operator rules are the following, with explanations on notation and side conditions given after the list.

\[
\begin{align*}
\varphi[L\psi] & \quad \varphi[L\psi] \\
\varphi[L\psi/C_L] & \quad \varphi[L\psi/\psi/C_L] \\
\varphi[L\psi] & \quad \varphi[L\psi] \\
\varphi[L\psi/\neg C_L] & \quad \varphi[L\psi] \\
\varphi[L\psi] & \quad \varphi[L\psi/C_L] \\
\varphi[L\psi/\psi/C_L] & \quad \varphi[L\psi/C_L]
\end{align*}
\]

Firstly, both of the rules $LSubst1$ and $LSubst2$ must satisfy the side-condition that the formula on top of the horizontal line is regular. We shall see that the rule $LC_L$Rename will in fact enable modifying formulas so that they indeed become regular.

When using the rule $LSubst1$, we begin with a formula $\varphi[L\psi]$ that has a subformula occurrence $L\psi$. We transform that occurrence $L\psi$ to $L\psi/L\psi/C_L$, that is, to a formula obtained from $L\psi$ by replacing some occurrences of $C_L$ by $L\psi$ itself. Any subset of the occurrences of $C_L$ in $L\psi$ can be replaced; the curly brackets indicate that indeed any set of atoms $C_L$ in $L\psi$ can be chosen to be replaced. As an example of using the rule, if $L\psi = L(P(x) \land C_L)$, then $L\psi/L\psi/C_L$ can be the formula $L(P(x) \land L(P(x) \land C_L))$.

Notice that the rule allows us to modify a subformula occurrence $L\psi$ of $\varphi[L\psi]$, so we have a deep inference rule, as its use is not limited to the modification of the main operator of $\varphi[L\psi]$. Furthermore, $LSubst1$ is bidirectional, which is indicated by the upright double arrow and means that we can use the rule in the standard way as well as in the reverse direction. More rigorously, when using a
bidirectional rule, we can (1) use the rule in the standard downward fashion, and (2) if we can syntactically produce a formula $\varphi_{\text{bottom}}$ from $\varphi_{\text{top}}$ in the standard way, then we are also allowed to produce $\varphi_{\text{top}}$ from $\varphi_{\text{bottom}}$.

The rule $L\text{Subst}2$ replaces an occurrence of $L\psi$ in $\varphi[L\psi]$ by a formula $L\psi\{\psi/C_L\}$ obtained from $L\psi$ by replacing some looping atoms $C_L$ by $\psi$. Again we are free to choose any subset of the atoms $C_L$ in the occurrence $L\psi$ to be replaced. The rule $L\text{Dual-Intro}$ modifies $L\psi$ so that some literals $\neg C_L$ in the strict scope or $L$ are replaced by $\neg C_{L'}$ and a new corresponding label symbol $L'$ is introduced. We require that $L'$ is a fresh symbol not occurring anywhere in $\varphi[L\psi]$. Once again any subset of the set of literals $\neg C_L$ in the strict scope of the $L$ in the occurrence $L\psi$ can be replaced.

The rule $L\text{Dummy-Intro-Elim}$ allows us to introduce (and eliminate in the upward direction) a dummy label symbol $L$, i.e., a symbol with no looping atoms $C_L$ in its strict scope. The rule $L\text{C}_L\text{Rename}$ replaces the occurrence $L\psi$ by the formula $L'\psi\{\{C_{L'/C_L}\}\}$ where we have renamed $L$ to $L'$ and replaced all atoms $C_L$ in the strict scope of $L$ by $C_{L'}$. The double brackets indicate that indeed all occurrences of $C_L$ in the strict scope or the particular instance of $L$ must be renamed. The symbol $L'$ can be any label symbol, as long as (1) the formula $L\psi$ does not have any occurrences of $C_{L'}$ that are free in $\psi$ and (2) if $\psi$ already contains symbols $L'$, none of the occurrences of $C_L$ (that are to be replaced) are in the scope of such occurrences of $L'$. So the renaming procedure is safe. Note that while we defined $L\text{C}_L\text{Rename}$ as bidirectional, this is redundant, as the top-to-bottom direction already covers what can be achieved by the reverse direction. Finally, $L\text{C}_L\text{Free-Elim}$ (which is not a bidirectional rule) allows us to replace a free looping atom $C_L$ with any formula. A free $C_L$ is an atom not in the scope of any occurrence of $L$.

Now, the duality rules are the following.

\[
\begin{align*}
\frac{\varphi[-(\psi \land \chi)]}{\varphi[\neg(\psi \lor \neg \chi)]} & \quad \frac{\varphi[-(\psi \lor \chi)]}{\varphi[\neg(\psi \land \neg \chi)]} & \quad \frac{\varphi[-\psi]}{\varphi[\neg \psi]} \\
\frac{\varphi[-\forall x \psi]}{\varphi[\exists x \neg \psi]} & \quad \frac{\varphi[-\exists x \psi]}{\varphi[\forall x -\psi]} & \quad \frac{\varphi[-L\psi]}{\varphi[L\neg \psi\{\neg \{C_{L'/C_L}\}\}]} \\
\end{align*}
\]

In the last rule, an occurrence of $\neg L\psi$ is replaced by $L\neg \psi\{\neg \{C_{L'/C_L}\}\}$, where $\psi\{\neg C_{L'/C_L}\}$ is obtained from $\psi$ by replacing every $C_L$ (which is in the strict scope of the $L$ of our occurrence $L\psi$) by $\neg C_{L'}$.

It is straightforward to show soundness for $\mathcal{S}$, so we skip that proof here for the sake of brevity.

### 5.2 Completeness

Let $\varphi$ be a formula of $\mathcal{SCL}$. Recall that we say that $\varphi$ is in weak negation normal form if the only negated subformulas of $\varphi$ are atomic formulas. We say that $\varphi$ is in strong negation normal form if the only negated subformulas of $\varphi$ are atomic FO-formulas. For example $P(x) \land L\neg C_L$ is in weak but not strong negation normal form because $C_L$ is not an FO-atom.
Lemma 5.1. For any $\varphi \in \text{SCL}$, there exists a formula $\varphi^*$ in strong negation normal form such that $\varphi \vdash \varphi^*$ and $\varphi^* \vdash \varphi$.

Proof. Firstly, we note that eliminating a negation from a free but negated $C_L$ can be done by $C_L$Free-Elim and the duality rule for double negation. It is also easy to reverse this effect by $C_L$Free-Elim.

Now, to prove the claim of the lemma, we will also use the rules $L$Subst2 and $L$Dual-Intro together with the duality rules. It follows directly from the duality rules that for any formula $\beta$, there exists a formula $\beta^*$ in weak negation normal form such that $\beta \vdash \beta^*$ and $\beta^* \vdash \beta$. Thus we assume that $\varphi$ is a formula in weak negation normal form, and our first goal is to show how to modify $\varphi$ in a deduction so that we get rid the possible negations in front of looping atoms $C_L$ in $\varphi$, thereby ending up with a formula in strong negation normal form.

Now, suppose $L\psi$ is a subformula of $\varphi$ such that the following conditions hold.

1. $\neg C_L$ occurs in $L\psi$ at least once in the strict scope of the main operator $L$ of $L\psi$.

2. There are no label symbols in $\psi$ that are referred to by a negated looping atom, i.e., there is no subformula $L_0\alpha$ in $\psi$ such that $\alpha$ contains the literal $\neg C_{L_0}$ in the scope of the main operator $L_0$ of $L_0\alpha$.

Then we call $L\psi$ an innermost-level switching loop formula, or more shortly, an ILSL-formula. To eliminate all negated looping atoms from $\varphi$, we first use the rule $L$Dual-Intro to every ILSL-formula $L\psi$ of $\varphi$, thereby replacing $L\psi$ by

$$L' L\psi \{ \{ \neg C_{L'} \}/ \neg C_L \}$$

where $L'$ is a fresh label symbol. Intuitively, this just renames the literals $\neg C_L$ to $\neg C_{L'}$. Then, denoting $L' L\psi \{ \{ \neg C_{L'} \}/ \neg C_L \}$ by $L' L\psi'$, we use $L$Subst2 to transform $L' L\psi'$ to $L' L\psi' \{ L\psi'/ C_{L'} \}$ where we choose to replace every occurrence of $C_{L'}$ by $L\psi'$. As indeed every such occurrence becomes replaced, we denote $L' L\psi' \{ L\psi'/ C_{L'} \}$ by $L' L\psi' \{ L\psi'/C_{L'} \}$. Now, notice that each atom $C_{L'}$ occurs negated in $\psi'$, so we will therefore denote $L' L\psi' \{ L\psi'/ C_{L'} \}$ by $L' L\psi' \{ \neg L\psi'/ \neg C_{L'} \}$. Note then that due to the formulas $\neg L\psi'$ that replace the literals $\neg C_{L'}$, the formula $L' L\psi' \{ \neg L\psi'/ \neg C_{L'} \}$ is no longer in weak negation normal form. Thus the next step is to push negations to the atomic level using the duality rules, including elimination of double negations. Hence we obtain the formula $L' L\psi' \{ \{ L\psi'_d \}/ \neg C_{L'} \}$ where $L\psi'_d$ is the formula we ultimately obtain from $\neg L\psi'$ via the duality rules. Now, we claim $L\psi'_d$ contains no literals $\neg C_{L'}$ or $\neg C_L$. This is due to the following observations. Firstly, recall from above that each atom $C_{L'}$ occurs negated in $\psi'$ while none of the atoms $C_L$ does. Thereby, when we apply the duality rule (for label symbols) to $\neg L\psi'$, we obtain a formula $L' \neg \psi''$ where all the atoms $C_{L'}$ and $C_L$ occur in $\psi''$ with a single negation directly in front of them. Thus the transformation of $L' \neg \psi''$ to $L\psi'_d$ has the desired effect that $L\psi'_d$ contains no literals $\neg C_{L'}$ or $\neg C_L$.

This way we have eliminated negated literals $\neg C_L$ from the ILSL-formulas $L\psi$ of $\varphi$. The obtained formula $\chi$ may still contain further literals $\neg C_{L''}$, but we may simply repeat the procedure described above, starting from the ILSL-formulas of $\chi$. Altogether, the strategy is to repeat the procedure sufficiently many times until there no longer exist any ILSL-formulas. The process ultimately terminates since
after each repetition, the ILSL-formulas of the newly obtained formula $\chi'$ will be closer to the main connective (i.e., closer to the root of the syntax tree of $\chi'$) than after the previous repetition. Indeed, the number of repetitions needed is clearly bounded above by the maximum nesting depth of label symbols in the original formula $\varphi$. We let $\varphi_0^*$ denote the final formula obtained from the procedure. Now, $\varphi_0^*$ can still contain free negated literals $C_L$, but as discussed in the beginning of the current proof, these can be eliminated by $C_L$-Free-Elim and the duality rule for double negation. We let $\varphi^*$ denote the formula obtained from $\varphi_0^*$ after also the possible negations in front of free looping atoms have been eliminated.

We then discuss the converse deduction from $\varphi^*$ to $\varphi$. First, note that all the inferences used to obtain $\varphi_0^*$ from $\varphi$ used bidirectional rules only, so we can reverse the inferences and therefore $\varphi_0^* \vdash \varphi$. Thus it suffices to show that $\varphi^* \vdash \varphi_0^*$. Now, our inference $\varphi_0^* \vdash \varphi^*$ above simply removed the possible negations in front of free occurrences of looping atoms. As discussed in the beginning of our proof, this last step can be reversed simply by $C_L$-Free-Elim (replacing $C_L$ by $\neg C_L$).

Recall the definition of the $n$-approximants $\Phi^n_\varphi$ of a formula $\varphi$. We are now ready to prove the following lemma.

**Lemma 5.2.** Let $\varphi \in \text{SCL}$ be in strong negation normal form and let $n \in \mathbb{N}$. Then $\Phi^n_\varphi \vdash \varphi$.

**Proof.** Suppose $\varphi \in \text{SCL}$ is in strong negation normal form. In our argument below we can assume that $\varphi$ does not have free looping atoms or dummy labels, and furthermore, $\varphi$ is regular. This can be seen as follows. Suppose we have proved $\Phi^n_\varphi' \vdash \varphi^*$ where $\varphi^*$ is obtained from a regularisation $\varphi'$ of $\varphi$ by removing dummy labels and replacing free looping atoms by $\bot$. Firstly, we have $\Phi^n_\varphi' = \Phi^n_\varphi$ and thus $\Phi^n_\varphi \vdash \varphi^*$. Secondly, we have $\varphi^* \vdash \varphi'$ by $\bot$Elim and $L$Dummy-Intro-Elim. Finally, we have $\varphi' \vdash \varphi$ by $LC_L$-Rename. Thus we can indeed make the simplifying assumption that $\varphi$ is regular and has neither free looping atoms nor dummy labels.

Now, let us define a sequence $\varphi_0, \ldots, \varphi_m$ of formulas where $\varphi_0 = \varphi$ and $\varphi_i \vdash \varphi_{i+1}$ for each $i < m$. The idea is to replace looping atoms by corresponding reference formulas. To obtain $\varphi_{i+1}$ from $\varphi_i$, do the following. Suppose there are $\ell$ looping atom occurrences in $\varphi_i$. We enumerate these looping atoms, with the aim of replacing them in the order of enumeration with corresponding reference formulas, one by one\footnote{Strictly speaking, we enumerate the paths from the root of the syntax tree of $\varphi_i$ to the occurrences of looping atoms, because the atoms themselves may become renamed several times during the next $\ell$ steps of our procedure. When we talk about the $j$th atom in the enumeration, we mean the atom whose path from the root of the syntax tree of the current formula is the same as the path of the $j$th atom in $\varphi_i$.} More formally, we define a sequence of $\ell$ operations that produce formulas $\varphi_{i,0}, \ldots, \varphi_{i,\ell}$ such that $\varphi_{i,0} = \varphi_{i,0}$ and $\varphi_{i,\ell} = \varphi_{i+1}$ and we have $\varphi_{i,j} \vdash \varphi_{i,j+1}$ for each $j < \ell$. Each of the $\ell$ operations will replace a looping atom by a corresponding reference formula in a way to be specified as follows.

1. Suppose we have already obtained $\varphi_{i,j}$ (if $j = 0$, then $\varphi_{i,j} = \varphi_i$). First use $LC_L$-Rename sufficiently many times to obtain a regular variant $\varphi_{i,j}^*$ of $\varphi_{i,j}$.

2. Now, let $C_L$ denote the $j$th atom occurrence in our enumeration of the looping atoms, that is, we let $C_L$ denote the looping atom occurrence in $\varphi_{i,j}^*$ that
corresponds to the \( j \)th atom in the original enumeration. The set of looping atoms of \( \varphi_{i,j}^* \) can be different from that of \( \varphi_{i,j} \), but here we are indeed replacing the atom occurrences in \( \varphi_{i,j}^* \) that correspond to the original looping atom occurrences in \( \varphi_i \) (recall Footnote 4). Now, we use the rule \( L\text{Subst}1 \) in the top-to-bottom direction to replace the \( j \)th looping atom occurrence by the reference formula which that atom has in \( \varphi_{i,j}^* \). Note that regularity of \( \varphi_{i,j}^* \) is required to enable us to use \( L\text{Subst}1 \).

This way we obtain the formula \( \varphi_{i+1} \), and thus ultimately the formula \( \varphi_m \), essentially by replacing looping atoms by corresponding reference formulas.

We have now established that \( \varphi_0 \vdash \varphi_m \) by using \( L\text{Subst}1 \) and \( L\text{C}\text{Rename} \). Clearly \( L\text{C}\text{Rename} \) has the property that if we can deduce \( \beta \) from \( \alpha \) by using the rule, then we can also deduce \( \alpha \) from \( \beta \) via \( L\text{C}\text{Rename} \) (whence we could have defined the rule as bidirectional). Furthermore, \( L\text{Subst}1 \) is bidirectional by definition. Therefore we conclude that also \( \varphi_m \vdash \varphi_0 \), that is, \( \varphi_m \vdash \varphi \).

Now, intuitively \( \varphi_m \) was obtained from \( \varphi = \varphi_0 \) by repeated substitution of looping atoms by corresponding reference formulas. By making \( m \) large enough, we obtain a formula \( \Psi^* := \varphi_m \) that can also be obtained in an alternative way from the \( n \)-enfolding \( \Psi^n_\varphi \) of \( \varphi \) by the following two steps (note here that we do not claim that the second one of the steps can be reproduced by using our deduction rules):

1. We first rename label symbols and looping atoms of \( \Psi^n_\varphi \) in a suitable way, obtaining a formula \( \Psi^n_0^* \).

2. We then replace all the looping atom occurrences \( C_{L_1}, \ldots, C_{L_p} \) in \( \Psi^n_0^* \) by suitable formulas \( \chi_1, \ldots, \chi_p \), thereby ending up with \( \Psi^* \).

The informal key intuition is simply that we can view \( \Psi^* = \varphi_m \) as an extension of a renaming of the \( n \)-unfolding \( \Psi^n_\varphi \). We next aim to show that \( \Phi^n_\varphi \vdash \Psi^* \) where \( \Phi^n_\varphi \) is the \( n \)-approximant of \( \varphi \). This is done as follows.

Firstly, as \( \varphi \) is in strong negation normal form, so is the \( n \)-unfolding \( \Psi^n_\varphi \). Thus the \( n \)-approximant \( \Phi^n_\varphi \) is by definition obtained from \( \Psi^n_\varphi \) by replacing all the looping atoms by \( \bot \) and then deleting all label symbols \( L \). Now recall from above the formula \( \Psi^n_0^* \) obtained from \( \Psi^n_\varphi \) by renaming label symbols and looping atoms in \( \Psi^n_\varphi \). Beginning from the approximant \( \Phi^n_\varphi \), we can reintroduce the corresponding label symbols (but not looping atoms) by using the rule \( L\text{Dummy-Intro-Elim} \), thus obtaining a formula \( \Phi^* \) which is otherwise as \( \Psi^n_0^* \) but has atoms \( \bot \) in the place of the looping atoms \( C_{L_1}, \ldots, C_{L_p} \) of \( \Psi^n_0^* \). Then, recalling the formulas \( \chi_1, \ldots, \chi_p \), we can replace the atoms \( \bot \) in \( \Phi^* \) (corresponding to the atoms \( C_{L_1}, \ldots, C_{L_p} \) in \( \Psi^n_0^* \)) by the formulas \( \chi_1, \ldots, \chi_p \), thereby obtaining the formula \( \Psi^* \). This step can be done using the rule \( \bot\text{Elim} \). Thus we have \( \Phi^n_\varphi \vdash \Psi^* \).

Now, as \( \Phi^n_\varphi \vdash \Psi^* \), is suffices to show that \( \Psi^* \vdash \varphi \) to conclude our proof. But we have already essentially shown this. Indeed, we defined above that \( \Psi^* := \varphi_m \) for a suitably large \( m \). Furthermore, we explicitly proved above that \( \varphi_m \vdash \varphi \) for any \( m \). Thus we have \( \Psi^* \vdash \varphi \), as required. \( \square \)

**Theorem 5.3.** Let \( \varphi \) be a formula of \( \text{SCL} \) or \( \text{BndSCL} \). If \( \varphi \) is valid, then \( \vdash \varphi \).

**Proof.** Suppose that \( \varphi \) is a valid formula of \( \text{SCL} \). Let \( \varphi^* \) be the negation normal form variant of \( \varphi \) guaranteed to exist by Lemma 5.1. Since \( \varphi \) is valid, so is \( \varphi^* \). By
Remark 3.17 and Theorem 3.13 this implies that $\Phi^\varphi_n$ is valid for some $n \in \mathbb{N}$. Since our proof calculus is complete for first-order formulas, we have $\vdash \Phi^\varphi_n$. By Lemma 5.2 we thus have $\vdash \varphi^\ast$. Hence we have $\vdash \varphi$ by Lemma 5.1 concluding the case for SCL. The case for BndSCL now follows from Remark 3.17.

We the show that if $\varphi$ is a formula of SCL or BndSCL and $\Sigma$ is a set of FO-formulas, then we have $\Sigma \models \varphi$ iff $\Sigma \vdash \varphi$, i.e., we have completeness with respect to FO premise sets.

**Lemma 5.4.** Let $\varphi$ be a formula of BndSCL or SCL and let $\Sigma$ be a set of FO-formulas. Suppose that $\Sigma \models \varphi$. Then there exists a finite $\Sigma_0 \subseteq \Sigma$ so that $\Sigma_0 \models \varphi$.

**Proof.** We first consider the case where $\varphi$ is in BndSCL. Suppose that for every finite $\Sigma_0 \subseteq \Sigma$, there exists some $\mathfrak{A}$ and $s$ so that $\mathfrak{A}, s \models \Sigma_0$ but $\mathfrak{A}, s \not\models \varphi$. By Theorem 5.11 for all such $\mathfrak{A}$ and $s$, we have $\mathfrak{A} \models \neg \Phi^\varphi_n$ for every $n \in \mathbb{N}$. Thus, by compactness of FO, we can deduce that $\Sigma \cup \{\neg \Phi^\varphi_n \mid n \in \mathbb{N}\}$ is satisfiable. This implies, by Theorem 5.11 that $\Sigma \not\models \varphi$, contradicting the assumption that $\Sigma \models \varphi$.

We then consider the case where $\varphi$ is in SCL. Suppose that for every finite $\Sigma_0 \subseteq \Sigma$, there exist $\mathfrak{A}$ and $s$ so that $\mathfrak{A}, s \models \Sigma_0$ but $\mathfrak{A}, s \not\models \varphi$. Every such model has an expansion $\mathfrak{A}'$ to a larger vocabulary so that $\mathfrak{A}', s \models \Psi$ where $\Psi$ is the formula promised by Theorem 4.7.8. Thus, by compactness of FO, we see that $\Sigma \cup \{\Psi\}$ is satisfiable. But if some $\mathfrak{B}$ and $s$ satisfy this theory, then—due to the properties of the formula $\Psi$—Elise does not have a winning strategy in the game $G^\ast_\infty(\mathfrak{B}', s, \varphi)$ where $\mathfrak{B}'$ is the restriction of $\mathfrak{B}$ to the vocabulary of $\varphi$. Thus $\Sigma \not\models \varphi$. \hfill $\square$

**Lemma 5.5.** Let $\varphi$ be a formula of BndSCL or SCL. Let $\Sigma$ be a finite set of FO-formulas. Now $\Sigma \vdash \varphi$ if and only if $\vdash (\neg \land \Sigma) \lor \varphi$.

**Proof.** Suppose $\Sigma \vdash \varphi$. Thus $\land \Sigma \vdash \varphi$ by $\land$Elim1 and $\land$Elim2. Therefore $\land \Sigma \vdash (\neg \land \Sigma) \lor \varphi$ by $\lor$Intro2. On the other hand, we have $\neg \land \Sigma \vdash (\neg \land \Sigma) \lor \varphi$ by $\lor$Intro1. As our system is complete for first-order logic, we have $\vdash (\neg \land \Sigma) \lor \land \Sigma$. Combining this with the above established facts that $\land \Sigma \vdash (\neg \land \Sigma) \lor \varphi$ and $\neg \land \Sigma \vdash (\neg \land \Sigma) \lor \varphi$, we conclude by $\lor$Elim that $\vdash (\neg \land \Sigma) \lor \varphi$.

Suppose $\vdash (\neg \land \Sigma) \lor \varphi$. We need to show that $\Sigma \vdash \varphi$. We have $\Sigma \vdash \land \Sigma$ by $\land$Intro. Thus $\Sigma \cup \{\neg \land \Sigma\} \vdash \varphi$ due to $\land$Intro and $\land$Elim. As also $\Sigma \cup \{\varphi\} \vdash \varphi$, we have $\Sigma \cup \{\neg \land \Sigma\} \lor \varphi \vdash \varphi$ by $\lor$Elim. Thus, as we have assumed that $\vdash (\neg \land \Sigma) \lor \varphi$, we have $\Sigma \vdash \varphi$. \hfill $\square$

**Theorem 5.6.** Let $\varphi$ be a formula of BndSCL or SCL. Let $\Sigma$ be a set of FO-formulas. Now $\Sigma \models \varphi$ iff $\Sigma \vdash \varphi$.

**Proof.** The right-to-left direction follows from soundness. For the other direction, suppose that $\Sigma \models \varphi$, where $\Sigma$ is a set of FO-sentences and $\varphi$ is a formula of either BndSCL or SCL. By Lemma 5.4 there exists a finite set $\Sigma_0 \subseteq \Sigma$ so that $\Sigma_0 \models \varphi$, i.e., $(\neg \land \Sigma_0) \lor \varphi$ is valid. By Theorem 5.3 we have that $\vdash (\neg \land \Sigma_0) \lor \varphi$. Using Lemma 5.5 we have $\Sigma_0 \vdash \varphi$, and hence $\Sigma \vdash \varphi$. \hfill $\square$

Now note that we can express in SCL that a linear order is well-founded, and this can clearly be used to define $(\mathbb{N}, +, \times, 0, 1)$ up to isomorphism with a single sentence of SCL. Therefore we cannot upgrade the above theorem so that $\Sigma$ is a
set of SCL-formulas, as the equivalence $\Sigma \models \varphi \iff \Sigma \vdash \varphi$ would imply that true arithmetical FO-sentences would form a recursively enumerable set.

A similar limitation holds for BndSCL. To see this, let $\varphi_<$ be a formula defining that $<$ is a strict discrete linear order with end points, and $S$ is the corresponding successor order. Now, consider the following formula (where we use “min” and “max” as constants that indicate the end points; it is clear that the constants can be eliminated in order to keep the vocabulary entirely relational):

$$\varphi_\leq \land (\text{min} = \text{max} \lor \exists x (S(\text{min}, x) \land L(x = \text{max} \lor \exists y (S(x, y) \land \exists x (x = y \land C_L))))).$$

The formula, let us denote it by $\psi$, essentially states that there is a finite path from min to max, which implies that the domain of the underlying model must be finite (without imposing any finite upper bound on its size). Now, let $R$ denote a fresh binary relation. It is easy to see that, for every $\varphi$ in the first-order language over the vocabulary $\{R\}$, we have that $\psi \models \varphi$ iff $\varphi$ is valid over the class of finite $\{R\}$-models. It follows quite directly from Trakhtenbrot’s theorem that validity over finite $\{R\}$-models is $\Pi^0_1$-complete, which implies that the consequences of $\psi$ form a set that is not recursively enumerable.

6 Some model theory of BndSCL and SCL

The purpose of this section is to present preliminary results on the model theory of BndSCL and SCL. Given that both of these logics are non-compact, it is unlikely that they have as rich model theory as, say, FO. However, both of these logics can be seen as fragments of infinitary logics, which in turn do admit nice model theories (even though they are also often non-compact). This gives us hope that one could also develop nice model theories for BndSCL and SCL.

6.1 Löwenheim-Skolem

We say that a logic $\mathcal{L}$ has countable downwards Löwenheim-Skolem property, if every sentence $\varphi$ of $\mathcal{L}$ has the following property: if $\mathfrak{A} \models \varphi$, then $\mathfrak{A}$ has a countable substructure $\mathfrak{B}$ which is also a model of $\varphi$. As advertised in the introduction, both BndSCL and SCL have the countable downwards Löwenheim-Skolem property.

We start by establishing this for BndSCL, for which it follows almost directly from the fact that FO has the countable downwards Löwenheim-Skolem property.

**Theorem 6.1.** Let $\varphi$ be a sentence of BndSCL and suppose that $\mathfrak{A} \models \varphi$. Then there exists a countable substructure $\mathfrak{B}$ of $\mathfrak{A}$ such that $\mathfrak{B} \models \varphi$.

**Proof.** Suppose that $\mathfrak{A} \models \varphi$. Theorem 3.11 implies that $\mathfrak{A} \models \bigvee_{n \in \mathbb{N}} \Phi^n_\varphi$. Hence $\mathfrak{A} \models \Phi^n_\varphi$, for some $n \in \mathbb{N}$. Since $\Phi^n_\varphi$ is a sentence of FO, we know that there exists a countable substructure $\mathfrak{B}$ of $\mathfrak{A}$ such that $\mathfrak{B} \models \Phi^n_\varphi$. Using theorem 3.11 again, we conclude that $\mathfrak{B} \models \varphi$. \hfill $\Box$

In the case of SCL it turns out that we can adapt the standard proof that FO has the countable downwards Löwenheim-Skolem property.

**Theorem 6.2.** Let $\varphi$ be a sentence of SCL and suppose that $\mathfrak{A} \models \varphi$. Then there exists a countable substructure $\mathfrak{B}$ of $\mathfrak{A}$ such that $\mathfrak{B} \models \varphi$. 33
Proof. To simplify notation, we may assume that \( \varphi \) has only quantifiers \( \exists \) by writing \( \neg \exists \neg \) instead of \( \forall \) in the usual way. Suppose that \( \varphi \) has a model \( \mathfrak{A} \) so that Eloise has a winning strategy \( \sigma \) in the game \( \mathcal{G}_\infty(\mathfrak{A}, \varphi) \). We may assume the strategy is positional by Lemma 2.1. We want to construct a countable model \( \mathfrak{B} \) so that Eloise has a winning strategy also in the game \( \mathcal{G}_\infty(\mathfrak{B}, \varphi) \).

Pick an arbitrary \( b \in \mathfrak{A} \). We define a sequence of sets \( (B_n)_{n \in \mathbb{N}} \) inductively as follows.

1. \( B_0 = \{ b \} \).
2. \( B_{n+1} = B_n \cup \{ d \mid \sigma((\exists x \psi, s, +)) = d, \text{ where } \text{ran}(s) \subseteq B_n \} \).

Let \( \mathfrak{B} \) be the substructure of \( \mathfrak{A} \) induced by the set \( \bigcup_{n \in \mathbb{N}} B_n \). \( \mathfrak{B} \) is clearly countable.

It is easy to see that \( \sigma \), or more precisely its restriction to the set of positions occurring in \( \mathcal{G}_\infty(\mathfrak{B}, \varphi) \), is also a winning strategy for Eloise also in \( \mathcal{G}_\infty(\mathfrak{B}, \varphi) \). Indeed, as long as Eloise follows it in \( \mathcal{G}_\infty(\mathfrak{B}, \varphi) \), which is possible by the definition of \( \mathfrak{B} \), Eloise will eventually reach—after a finite number of rounds—a winning position.

6.2 Craig interpolation property

A logic \( \mathcal{L} \) has the **Craig interpolation property**, if the following holds for every two sentences \( \varphi \) and \( \psi \) of \( \mathcal{L} \): if \( \varphi \models \psi \), then there exists a third sentence \( \theta \in \mathcal{L} \) called an **interpolant**, such that \( \varphi \models \theta \models \psi \) and \( \theta \) contains only those relation symbols that occur in both of the sentences \( \varphi \) and \( \psi \). We will next establish that neither BndSCL nor SCL has the Craig interpolation property. These results should be contrasted with the fact that several infinitary logics, such as \( \mathcal{L}_{\omega_1 \omega} \), do enjoy the Craig interpolation property.

We start by establishing that the class of finite structures of even size is not definable in neither BndSCL nor in SCL.

**Proposition 6.3.** For every sentence \( \varphi \) of either BndSCL or SCL there exists a finite structure \( \mathfrak{A} \) of even size and a finite structure \( \mathfrak{B} \) of odd size such that

\[
\mathfrak{A} \models \varphi \Rightarrow \mathfrak{B} \models \varphi.
\]

**Proof.** Since the expressive power of BndSCL and SCL coincides over finite models, it suffices to consider the case of BndSCL. Let \( \varphi \) be an arbitrary sentence of BndSCL. Suppose that \( \varphi \) contains \( k \) distinct variables. By Theorem 3.11 we know that \( \varphi \) is equivalent with the sentence

\[
\Phi := \bigvee_{n \in \mathbb{N}} \Phi^n_{\varphi}
\]

of \( \mathcal{L}_{\omega_1 \omega}^k \). Consider now the models \( \mathfrak{A} \) and \( \mathfrak{B} \), where both are models over the empty vocabulary with domains \( \{0, \ldots, 2k\} \) and \( \{0, \ldots, 2k + 1\} \) respectively. Now, it is easy to show using pebble games that these structures can not be distinguished via a sentence of \( \mathcal{L}_{\omega_1 \omega}^k \). In particular, if \( \mathfrak{A} \models \Phi \), then \( \mathfrak{B} \models \Phi \). This in turn entails that if \( \mathfrak{A} \models \varphi \), then \( \mathfrak{B} \models \varphi \).

We will next establish the failure results. Our counterexample is inspired by the standard counterexample which shows that over finite models FO does not enjoy
Craig interpolation property. It is not surprising that a similar example could be made to work also in our case, since both BndSCL and SCL can projectively define the class of finite structures (which FO can not do, since it has compactness).

Theorem 6.4. Neither BndSCL nor SCL has the Craig interpolation property.

Proof. Recall the sentence $\psi$ that we introduced in the end of Section 5.2:

$$\phi < \land (\min = \max \lor \exists x (S(\min, x) \land L(x = \max \lor \exists y (S(x, y) \land \exists x(x = y \lor C_L))))).$$

The main properties of this sentence were the following.

1. If $A \models \psi$, then $A$ is finite.

2. If $A$ is a finite structure over a vocabulary which is disjoint from that of $\psi$, then it has an extension $\tilde{A}$ such that $\tilde{A} \models \psi$.

Both of these properties hold regardless of whether we are using bounded or unbounded semantics.

Now consider the sentences

\[
\chi_1 := \theta_1 \land \forall x \exists y (x \neq y \land E_1(x, y) \land \forall z (E_1(x, z) \rightarrow (x = z \lor y = z)))
\]

and

\[
\chi_2 := \theta_2 \land \exists x (\forall y (x \neq y \rightarrow \neg E_2(x, y)) \\
\land \forall y (y \neq x \rightarrow \exists z (y \neq z \land E_1(y, z) \land \forall w (E_1(y, w) \rightarrow (y = w \lor z = w))))
\]

where $\theta_i$, for $i \in \{1, 2\}$, expresses that $E_i$ is an equivalence relation. Note that the common vocabulary of $\chi_1$ and $\chi_2$ is the empty vocabulary. It is easy to see that $\chi_1$ expresses that $E_1$ is an equivalence relation where each equivalence class contains precisely two elements, while $\chi_2$ is expressing that $E_2$ is an equivalence relation where there exists one equivalence class with one element while every other equivalence class has precisely two elements.

Clearly $\psi \land \chi_1 \models \neg \chi_2$, since $A \models \psi \land \chi_1$ entails that $|A|$ is finite and even, while $A \models \chi_2$ would entail that $|A|$ is either infinite or even. We now claim that there exists no interpolant between $\psi \land \chi_1$ and $\chi_2$ either in BndSCL or in SCL.

Aiming for a contradiction, suppose that $\theta$ is a sentence of either BndSCL or SCL over the empty vocabulary which is an interpolant between $\psi \land \chi_1$ and $\chi_2$. Let $A$ and $B$ be the structures promised by Proposition 6.3. $A$ clearly has an extension $\tilde{A}$ such that $\tilde{A} \models \psi \land \chi_1$. Since $\theta$ was interpolant, we have that $\tilde{A} \models \theta$, which implies that $A \models \theta$, since $\theta$ was a sentence over the empty vocabulary. Thus $B \models \theta$. Now, $B$ clearly has an extension $\tilde{B}$ for which $\tilde{B} \models \chi_2$, since $B$ had odd size. But now also $\tilde{B} \models \theta$, which is a contradiction, since $\theta \models \neg \chi_2$.

6.3 Sentences that are determined everywhere

We have seen several examples of sentences of BndSCL and SCL which can define properties of classes of models which are not definable by any sentence of FO. In each case one can make the observation that the relevant sentence has a model in which it is non-determined, i.e., neither player has a winning strategy. For instance, the SCL sentence which defined the class of well-founded linear orders is non-determined in any model which contains an infinite descending sequence.
This raises the following question: if a sentence of BndSCL or SCL defines a class of models which is not definable by any FO-sentence, must it be non-determined in some model? The answer turns out to be positive in both cases; if a sentence of either BndSCL or SCL is determined everywhere, then it is in fact (strongly) equivalent to one of its approximants.

**Theorem 6.5.** Suppose that \( \varphi \) is a sentence of either BndSCL or SCL, which is determined everywhere. Then \( \varphi \) is equivalent to a sentence of FO and more specifically it is equivalent with its \( n \)th approximant, for some \( n \).

**Proof.** We will first consider the case where \( \varphi \) is a sentence of BndSCL. Since \( \varphi \) is determined everywhere, the sentence \( \varphi \lor \neg \varphi \) is valid, which implies — together with Theorem 3.13 — that \( \Phi_{\varphi \lor \neg \varphi}^{n} \) is valid. Note that \( \Phi_{\varphi \lor \neg \varphi}^{n} \) is the same as \( \Phi_{\varphi}^{n} \lor \Phi_{\neg \varphi}^{n} \). Now, we claim that \( \Phi_{\varphi}^{n} \) is in fact equivalent with \( \varphi \). Recall that Theorem 3.11 implies that \( \Phi_{\psi}^{n} \models \psi \), for every sentence \( \psi \). Thus in particular \( \Phi_{\varphi}^{n} \models \varphi \). Concerning the other direction \( \varphi \models \Phi_{\varphi}^{n} \), we note that since \( \Phi_{\varphi}^{n} \lor \Phi_{\neg \varphi}^{n} \) is valid, \( \neg \Phi_{\varphi}^{n} \models \Phi_{\neg \varphi}^{n} \models \neg \varphi \). Hence \( \varphi \) is equivalent with \( \Phi_{\varphi}^{n} \).

The case where \( \varphi \) is a sentence of SCL can be handled analogously. The main differences are that instead of Theorem 3.13 we use Theorem 4.5, and the fact that \( \Phi_{\psi}^{n} \models \psi \) holds for every sentence \( \psi \) of SCL follows from lemmas 4.4 and 3.9 instead of Theorem 3.11.

We now make two remarks concerning the question of to what extent our result can be made effective. We start by determining the exact complexity of the problem of determining whether a given sentence of BndSCL or SCL is determined everywhere.

**Proposition 6.6.** The problem of determining whether a given sentence \( \varphi \) of either BndSCL or SCL is determined everywhere is \( \Sigma^{0}_{1} \)-complete.

**Proof.** Let \( L \in \{ \text{BndSCL, SCL} \} \). We have seen in the previous sections that the set of valid sentences of \( L \) is a recursively enumerable set. This fact already implies that the set of sentences of \( L \) which are determined everywhere is a recursively enumerable set; an effective procedure can simply go through the list of valid sentences of \( L \), and print the sentence \( \varphi \) whenever it encounters the sentence \( \varphi \lor \neg \varphi \).

For the lower bound we will reduce the validity problem of FO to the problem of determining whether a sentence of \( L \) is determined everywhere. Let \( \varphi \in \text{FO} \) be a sentence. We claim that \( \varphi \) is valid iff the sentence

\[
\psi_{\varphi} := \varphi \lor C_{L}
\]

is determined everywhere. First, if \( \varphi \) is valid, then \( \psi_{\varphi} \) is determined everywhere, because it is a valid sentence. Conversely, if \( \psi_{\varphi} \) is determined everywhere, then \( \varphi \) must be valid, since \( C_{L} \) is non-determined in every model.

An immediate corollary of the above result is that the problem of determining whether a given sentence of BndSCL or SCL is strongly equivalent to a sentence of FO is also \( \Sigma^{0}_{1} \)-complete.

**Corollary 6.7.** The problem of determining whether a given sentence \( \varphi \) of either BndSCL or SCL is strongly equivalent to a sentence of FO is \( \Sigma^{0}_{1} \)-complete.
Proof. Let $L \in \{\text{BndSCL}, \text{SCL}\}$. We have already established that a sentence of $L$ is strongly equivalent with a sentence of FO if and only if it is determined everywhere. Thus the claim follows from Proposition 6.6. 

To complement these results, we note that if $\varphi$ is determined everywhere, then we can effectively recover a sentence of FO which is equivalent with $\varphi$. This follows from the observation that if $\varphi$ is equivalent to some FO-sentence $\psi$, then for some $n \in \mathbb{N}$ we have that $(\varphi \leftrightarrow \Phi^n_\varphi)$ is a valid sentence of $L$, since $\varphi$ was determined everywhere. This allows us, together with the fact that the set of valid sentences of $L$ is recursively enumerable, recover $\Phi^n_\varphi$ effectively.

Finally we will give an example which demonstrates that Theorem 6.5 fails if we restrict our attention to the class of finite models.

Example 6.8. Consider the following sentence

$$\varphi := \varphi_\prec \land \exists x \exists y (x = \text{min} \land y = \text{max} \land (S(x, y) \lor L \exists z \exists w (S(x, z) \land S(w, y) \land (S(z, w) \lor \exists x \exists y (x = z \land y = w \land C_L))))),$$

where $\varphi_\prec$ expresses that $\prec$ is a strict linear ordering of the domain, min and max are distinct elements that correspond to the smallest and the largest elements of $\prec$ and $S$ is the successor relation induced by $\prec$. It is easy to see that, regardless of whether we are using the bounded or unbounded game-theoretical semantics, $\varphi$ defines the class of linear orders of even size. It is well-known that this class is not FO-definable, and hence $\varphi$ is not equivalent to any sentence of FO.

Next we will show that $\varphi$ is determined everywhere. Suppose that $A$ is a suitable model for $\varphi$. If $A \not\models \varphi_\prec$, then Abelard clearly has a winning strategy. Suppose then that $A \models \varphi_\prec$, but $|A|$ is not an even number. To see that Abelard has a winning strategy also in this case, note that $\varphi$ describes a game where Eloise needs to move two pebbles along the successor relation induced by $\varphi_\prec$, the initial position of these pebbles being the smallest and the largest elements of $\varphi_\prec$. Now, if $|A|$ is not an even number, Eloise will eventually reach a position where there is only a single element between the two pebbles, which is a position that is outside her winning region, because she needs to maintain the condition that the first pebble is always placed on an element which is strictly smaller than the element to which the second pebble is placed. We note that if we are using bounded semantics, then Abelard additionally needs to make sure that the initial clock value for $L$ is large enough.

7 Definability over natural numbers

The purpose of this section is to characterise relations over natural numbers that are definable in SCL and in BndSCL over the standard structure $\mathbb{N}$ of natural numbers. We start by formally defining the classes $\Pi^1_1$ and $\Sigma^0_{\omega+1}$ starting with the former. A relation $X \subseteq \mathbb{N}^k$ is called $\Pi^1_1$ if there exists a formula

$$\forall X_1 \ldots \forall X_n \psi(x_1, \ldots, x_k)$$

of $\forall$SO such that for every $(m_1, \ldots, m_k) \in \mathbb{N}^k$ we have that $(m_1, \ldots, m_k) \in X$ if and only if $\mathbb{N} \models \psi(m_1, \ldots, m_k)$.

---

5Note that a sentence of $L$ might be weakly equivalent with a sentence of FO and yet be undetermined in some models. A concrete example of such a sentence is $\exists x \exists y R(x, y) \lor C_L$.
To define the class $\Sigma^0_{\omega+1}$, we start by fixing some (reasonable) Gödel numbering for the formulas of FO-arithmetic. What we mean by reasonable should become clear in our proofs. Now, consider the set

$$T := \{ \varphi \mid \varphi \in \text{FO is a sentence in prenex normal form and } N \models \varphi \}.$$ 

A relation $X \subseteq \mathbb{N}^k$ is called $\Sigma^0_{\omega+1}$, if there exists a $T$-computable relation $R$ such that

$$(x_1, \ldots, x_k) \in X \iff \exists y_1 \ldots \exists y_\ell R(x_1, \ldots, x_k, y_1, \ldots, y_\ell).$$

Here by $T$-computable, we mean that the relation can be computed by a Turing machine that has a distinct (oracle) tape where the characteristic function $\chi_T$ of $T$ is written down. In other words, the machine has access to a tape that contains the infinite sequence $\chi_T(0), \chi_T(1), \ldots$.

We note that an alternative — and perhaps a more standard — way of defining the class $\Sigma^0_{\omega+1}$ would be to use the set $\emptyset(\omega)$ instead of $T$ (for a formal definition of $\emptyset(\omega)$, see [23, p. 257]). Since the two sets are recursively isomorphic, meaning that there exists a computable bijection $f: \mathbb{N} \to \mathbb{N}$ so that $n \in T$ if and only if $f(n) \in \emptyset(\omega)$, the two definitions of $\Sigma^0_{\omega+1}$ coincide, see [23, p. 318]. However, in our case it is technically more convenient to work with the set $T$.

### 7.1 SCL-definable relations

By Theorem 4.10 we know that SCL is contained in $\forall SO$ over general structures, and hence every relation over $\mathbb{N}$ that is definable in SCL is also $\Pi^1_1$. To prove the converse direction, we will modify the proof of Kleene’s theorem as presented in the book [18].

**Lemma 7.1.** Every $\Pi^1_1$-relation over $\mathbb{N}$ is definable in SCL.

**Proof.** We start by observing that, over $\mathbb{N}$, it is routine to rewrite an arbitrary $\forall SO$ formula $\varphi(\overline{x})$ as a formula of the form

$$\forall f \exists y \varphi(\overline{x}, y),$$

where $f$ is an unary function and $\varphi(\overline{x}, y)$ is quantifier-free. Thus we need to show that for each such formula there exists — over $\mathbb{N}$ — an equivalent SCL formula. For simplicity, we will restrict our attention to the case where $\varphi$ contains a single free variable, i.e., we consider formulas where the quantifier-free part is of the form $\varphi(x, y)$.

The basic idea is now as follows. Given any function $f: \mathbb{N} \to \mathbb{N}$ and natural numbers $m, m' \in \mathbb{N}$, we can determine whether $\varphi(m, m')$ holds in $\mathbb{N}$ by considering $f \mid n$ — the restriction of $f$ to $\{0, \ldots, n-1\}$ — for some sufficiently large $n$. Thus the evaluation of $\forall f \exists y \varphi(x, y)$ can be formulated as the following game: Abelard picks natural numbers $f(0), f(1), f(2), \ldots$ until Eloise chooses to stop the game and evaluate the formula $\exists y \varphi(m, y)$ with the restriction of $f$ induced by the natural numbers that were chosen by Abelard.

Now we construct an SCL-formula $\theta(x)$ which essentially describes the above game. To do this, we will first need to fix some effective method of encoding tuples of natural numbers as a single natural number. A standard choice of encoding is

$$(n_1, \ldots, n_k) \mapsto \prod_{i=1}^{k} p_i^{n_i+1},$$

where $p_i$ is the $i$-th prime number. For example, $2, 3, 5, 7, 11, \ldots$ encodes the tuple $(0, 2, 0, 2, 0, 2)$ as $2^1 \cdot 3^1 \cdot 5^{-1} \cdot 7^1 \cdot 11^{-1} \cdots$. However, this encoding is not the only one that is effective. Alternative encodings include

$$n \mapsto 2^n,$$

which is also effective since it is a bijection from $\mathbb{N}$ to $2^{\mathbb{N}}$. This encoding is often referred to as the Cantor pairing function. However, for the purposes of this proof, any effective encoding is sufficient.
where $p_i$ denotes the $i$th prime number. Now consider the function $g : \mathbb{N}^2 \to \mathbb{N}$ defined by $(n_1, n_2) \mapsto n_1 \times p^{n_2+1}$, where $p$ is the smallest prime number which does not divide $n_1$ (in the case where $n_1 = 1$, we simply set $p$ to 2). Since $g$ is effectively computable, there exists a formula $\psi(x, y, z)$ of FO-arithmetic which defines it.

The formula $\theta(x)$ can now be defined as the formula
\[
\exists n \exists f [n = 1 \land f = 1 \land L(\exists y \varphi^*(n, f, x, y)) \\
\forall z \exists n' \exists f' (n' = n + 1 \land \psi(f, z, f') \land \exists n \exists f (n = n' \land f = f' \land C_L))]
\]
where $\varphi^*(n, f, x, y)$ is a formula which will be specified later, but, roughly speaking, it is false if the values chosen by Abelard are not sufficient to determine whether $\varphi(x, y)$ holds and true if $\varphi(x, y)$ is true when evaluated under the mapping determined by the values chosen by Abelard. Based on the above discussion, it should be clear that $\theta(x)$ is the desired formula.

To define the formula $\varphi^*(n, f, x, y)$ we proceed as follows. First, consider the mapping $h : \mathbb{N}^2 \to \mathbb{N}$ defined by
\[
\left( \prod_{i=1}^{k} p_{i+1}^{n_i}, j \right) \mapsto \begin{cases} 
    n_j - 1 & \text{if } 1 \leq j \leq k \text{ and } n_j \geq 1 \\
    0 & \text{otherwise}
\end{cases}
\]
Again, this mapping is clearly computable, and hence there exists a formula $\chi(x, y, z)$ of FO-arithmetic which defines it.

Consider now an arbitrary atomic formula $\alpha(x, y)$ of $\varphi(x, y)$. Our goal is to write, for every such formula $\alpha(x, y)$, a formula $\psi_\alpha(n, f, x, y)$ which essentially verifies that the values chosen by Abelard – which are encoded in the number $f$ – are indeed enough to determine whether $\alpha(x, y)$ holds. Having such formulas at hand, we will replace each such atomic formula $\alpha(x, y)$ of $\varphi(n, f, x, y)$ with the corresponding formula $\psi_\alpha(n, f, x, y)$. The resulting formula will then the desired formula $\varphi^*(n, f, x, y)$.

We start with a more concrete example. Consider an atomic formula $\alpha(x, y)$ that contains only the terms $x, y, f(x), f(f(x))$. Consider then the formula
\[
(x < n \to \exists (x, f(x, z) \land (z < n \to \exists w (\chi(f, z, w) \land \alpha(x, y)[z/f(x), w/f(f(x))])))
\]
The formula starts by verifying that $f$ encodes the value of $f(x)$, which it does as long as $x < n$, since then $f(x) + 1$ is the exponent of the $x$th prime number that divides $f$. Having verified this, the formula then stores $f(x)$ into the variable $z$. Next, the formula verifies that $f$ also encodes the value $f(f(x)) = f(z)$, and then stores $f(f(x))$ into the variable $w$. Finally, the formula verifies that $\alpha(x, y)[z/f(x), w/f(f(x))]$ holds. Clearly this formula could now be used as the formula $\psi_\alpha(n, f, x, y)$.

In general the atomic formula $\alpha(x, y)$ contains terms from the set
\[
\{x, f(x), \ldots, f^i(x), y, f(y), \ldots, f^s(y)\},
\]
which will make the resulting formulas $\psi_\alpha(n, f, x, y)$ more complicated. On the other hand, it is easy to see that the technique that was used in the above example generalizes also to handle this more general case.

The following is immediate.

**Theorem 7.2.** Over $\mathbb{N}$, SCL-definable relations and $\Pi^1_1$-relations coincide.
Proof. Combine Theorem 4.10 with Lemma 7.1.

7.2 BndSCL-definable relations

We will next establish that the class of BndSCL-definable relations and the \( \Sigma^0_{\omega+1} \)-relations coincide. We will start by establishing that every BndSCL-definable relation is \( \Sigma^0_{\omega+1} \).

Lemma 7.3. Every BndSCL-definable relation over \( \mathbb{N} \) is \( \Sigma^0_{\omega+1} \).

Proof. Suppose that \( X \subseteq \mathbb{N}^k \) is defined by the formula \( \varphi(\overline{x}) \in \text{BndSCL} \). By Theorem 4.11, we have that \( \overline{m} \in X \) iff for some \( n \in \mathbb{N} \) it is the case that \( \mathbb{N} \models \Phi^\varphi_n(\overline{m}) \). Let \( R(\overline{x}, y) \) be a relation which is satisfied by those pairs \( (\overline{m}, n) \in \mathbb{N}^{k+1} \) for which the \( n \)-th approximant of \( \varphi(\overline{x}) \) is satisfied by \( \overline{m} \) in \( \mathbb{N} \). Since \( \Phi^\varphi_n(\overline{x}) \) is, for every \( y \in \mathbb{N} \), a formula of FO-arithmetic which can be computed from \( \varphi(\overline{x}) \) when given \( y \), \( R(\overline{x}, y) \) is clearly \( T \)-computable. Hence \( \exists y R(\overline{x}, y) \) is a \( \Sigma^0_{\omega+1} \)-definition of \( X \). \( \square \)

To prove the converse direction, we will first show that in a certain technical sense the set \( T \) is itself BndSCL-definable.

Lemma 7.4. There exists a formula \( \theta(x) \in \text{BndSCL} \) so that for every sentence of FO-arithmetic in prenex normal form with quantifier-depth at most \( d \) we have that the following two conditions hold.

1. Eloise has a winning strategy in the game \( G_d(\mathbb{N}, r, \theta(x)) \) if and only if \( \mathbb{N} \models \varphi \).
2. Eloise has a winning strategy in the game \( G_d(\mathbb{N}, r, -\theta(x)) \) if and only if \( \mathbb{N} \not\models \varphi \).

Here \( r \) is an assignment for which \( r(x) = ^\gamma \varphi \).

Proof. Consider a sentence \( \varphi \) of FO-arithmetic which is in prenex normal form. There clearly exists a computable function which when given as input \( ^\gamma \varphi \) computes the length of the prefix of \( \varphi \). Furthermore, we can effectively determine from \( ^\gamma \varphi \) whether the \( i \)-th quantifier in the prefix of \( \varphi \) is universal. Let \( \psi_{\text{prefix}}(x, y) \) denote a formula defining the first function and let \( \psi_{\text{universal}}(x, i) \) denote a formula which is true iff the \( i \)-th quantifier in the sentence encoded by \( x \) is universal. Let \( \psi(x, y, z) \) denote a formula defining the function \( g : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) that we used in the proof of Lemma 7.1. Consider now the following formula \( \theta(x) \) of BndSCL

\[
\exists d \exists n \exists m [\psi_{\text{prefix}}(x, d) \land n = d \land m = 1 \land L((n = 0 \to \psi(x, m)) \land (n > 0 \to \\
(\psi_{\text{universal}}(x, d - (n - 1)) \to \exists n' \forall y \exists m'(n' = n - 1 \land \psi(m, y, m'))
\land \exists n \exists m(n = n' \land m = m' \land C_L)))] \land (\neg \psi_{\text{universal}}(x, d - (n - 1)) \to \exists n' \forall y \exists m'(n' = n - 1 \land \psi(m, y, m'))
\land \exists n \exists m(n = n' \land m = m' \land C_L))]},
\]

where \( \psi(x, m) \) is true iff the quantifier-free part of the sentence encoded by \( x \) is true under the assignment encoded by \( m \).

Now, roughly speaking, \( \theta(x) \) describes a game where Abelard and Eloise choose interpretations for variables that are being quantified in the formula encoded by \( x \). Eloise chooses values for the existentially quantified variables, while Abelard
chooses values for the universally quantified variables. After the players have chosen $d$ values, where $d$ is the length of the quantifier prefix of the input sentence, Eloise looses if the resulting assignment $m$ does not satisfy the quantifier-free part of the formula, and otherwise Abelard looses. (Note that there are no plays where neither Eloise nor Abelard wins.) It is straightforward to verify that $\theta(x)$ satisfies both conditions stated in the lemma.

Lemma 7.5. Every $\Sigma^0_{\omega+1}$-relation is BndSCL-definable.

Proof. Suppose that $X \subseteq \mathbb{N}^k$ is $\Sigma^0_{\omega+1}$. Thus there exists a $T$-computable relation $R(\overline{x}, y_1, \ldots, y_\ell)$ so that

$$\overline{n} \in X \iff \exists y_1 \ldots \exists y_\ell R(\overline{n}, y_1, \ldots, y_\ell).$$

Suppose that $M(\overline{x}, y_1, \ldots, y_\ell)$ is a Turing machine which computes $R$ when it has oracle access to the set $T$. Now (1) can be rewritten as

$$\overline{n} \in X \iff \exists y_1 \ldots \exists y_\ell \exists t \text{ “} M \text{ halts on input } (\overline{n}, y_1, \ldots, y_\ell) \text{ after } t \text{ steps”}$$

Observe that if $M$ halts after $t$ steps, then it could have only accessed the first $t$-bits on the oracle tape. This simple observation will play a crucial role in our proof.

A number $m$ is called $t$-good, if its prime factorization is of the form

$$\prod_{i=1}^{t} p_i^{e_i},$$

where $e_i \in \{1, 2\}$, for every $1 \leq i \leq t$. In other words, $m$ is $t$-good if it encodes a binary sequence of length $t$. Now the following relation is clearly computable:

“$m$ is $t$-good, for some $t \geq 1$, and $M$ halts after at most $t$-steps, if $(\overline{x}, y_1, \ldots, y_\ell)$ is on the input tape and the binary sequence encoded by $m$ is on the oracle tape.”

Let $\varphi_M(m, \overline{x}, y_1, \ldots, y_\ell)$ denote a formula of FO-arithmetic which defines this relation. Consider now the following formula of BndSCL

$$\exists y_1 \ldots \exists y_\ell \exists m[\varphi_M(m, \overline{x}, y_1, \ldots, y_\ell) \land \exists t(\psi_1(m, t) \land \forall i(1 \leq i \leq t \rightarrow (\neg \psi_2(m, i) \lor \theta(i)) \land (\psi_2(m, i) \lor \neg \theta(i))))]]$$

where $\theta$ is the formula given by Lemma 7.3, while the formulas $\psi_1$ and $\psi_2$ have the following meaning: $\psi_1(m, t)$ is true iff $m$ is $t$-good; and $\psi_2(m, i)$ is true iff the $i$th bit in the bit sequence encoded by $m$ is one. It is straightforward to verify that this formula defines the relation $X$.

The following is immediate.

Theorem 7.6. Over $\mathbb{N}$, BndSCL-definable relations and $\Sigma^0_{\omega+1}$-relations coincide.

Proof. Combine Lemma 7.5 with Lemma 7.3.

We conclude this section with the observation that it seems likely that one can generalize the proof of Corollary 7.6 to show that stronger variants of BndSCL are able to capture $\Sigma^0_\alpha$-relations for every computable ordinal $\alpha$. (We note that in the case $\alpha = \omega$ we have by definition that $\Sigma^0_\omega = \Delta^0_\omega$, i.e., $\Sigma^0_\omega$ is the class of arithmetical relations which is already captured by FO.) Here by stronger variants we mean variants where the initial value chosen by Eloise is not a natural number, but rather some computable ordinal. For instance, $\Sigma^0_{\omega+2}$-relations should be captured by the variant of BndSCL where the players can force the initial clock value to be $\omega$. 

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