THE FOURIER TRANSFORM FOR TRIPLES OF QUADRATIC SPACES

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Abstract. Let $V_1, V_2, V_3$ be a triple of even dimensional vector spaces over a number field $F$ equipped with nondegenerate quadratic forms $Q_1, Q_2, Q_3$, respectively. Let $Y \subset \prod_{i=1}^{3} V_i$ be the closed subscheme consisting of $(v_1, v_2, v_3)$ such that $Q_1(v_1) = Q_2(v_2) = Q_3(v_3)$. One has a Poisson summation formula for this scheme under suitable assumptions on the functions involved, but the relevant Fourier transform was previously only defined as a correspondence. In the current paper we employ a novel global to local argument to prove that this Fourier transform is well-defined on the Schwartz space of $Y(A_F)$. To execute the global to local argument, we extend the Poisson summation formula to a broader class of test functions which necessitates the introduction of boundary terms. This is the first time a summation formula with boundary terms has been proven for a spherical variety that is not a Braverman-Kazhdan space.

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1. Introduction

Let $d_1, d_2, d_3$ be three positive even integers, let $V_i := \mathbb{Q}_{d_i}^d$, $V := \bigoplus_{i=1}^3 V_i$, and let $F$ be a number field. For each $i$, let $\mathcal{Q}_i$ be a nondegenerate quadratic form on $V_i(F)$. Let $Y \subset V$ be the subscheme whose points in an $F$-algebra $R$ are given by

$$Y(R) := \{(v_1, v_2, v_3) \in V(R) : \mathcal{Q}_1(v_1) = \mathcal{Q}_2(v_2) = \mathcal{Q}_3(v_3)\}.$$

In [GL19] the first author and Liu proved a Poisson summation formula for this space. The space $Y$ is a spherical variety, and hence the summation formula confirms a special case of conjectures of Braverman and Kazhdan, later investigated by L. Lafforgue and Ngô, and extended to spherical varieties by Sakellaridis [BK99, BK00, BK02, Laf14, Ngo20, Sak12]. It is the first summation formula for a spherical variety that is not a Braverman-Kazhdan space. Here a Braverman-Kazhdan space is the affine closure of $[P, P] \setminus G$ where $G$ is a reductive group and $P \leq G$ is a parabolic subgroup.

In this paper we prove that the Fourier transform on $Y$, originally defined as a correspondence, descends to an automorphism of the Schwartz space. Let us be more precise. Let $X^\circ := [P, P] \setminus \text{Sp}_6$ where $P \leq \text{Sp}_6$ is the Siegel parabolic and let $X = \overline{\text{Pl}}(X^\circ)$ be the corresponding Braverman-Kazhdan space (see (3.0.6)). As explained in §3, the Schwartz space $\mathcal{S}(X(\mathbb{A}_F))$ is defined and comes equipped with a Fourier transform $\mathcal{F}_X : \mathcal{S}(X(\mathbb{A}_F)) \rightarrow \mathcal{S}(X(\mathbb{A}_F))$. We define $\mathcal{S}(X(\mathbb{A}_F) \times V(\mathbb{A}_F))$ using the conventions in §2. For notational simplicity, we let

$$\mathcal{F}_X : \mathcal{S}(X(\mathbb{A}_F) \times V(\mathbb{A}_F)) \rightarrow \mathcal{S}(X(\mathbb{A}_F) \times V(\mathbb{A}_F))$$
be the automorphism given on pure tensors by $F_X(f_1 \otimes f_2) = F_X(f_1) \otimes f_2$. Let $Y^{\text{sm}} \subset Y$ be the smooth locus. In this paper we introduce the Schwartz space

$$S(Y(A_F)) := \text{Im}(I : S(X(A_F) \times V(A_F)) \to C^\infty(Y^{\text{sm}}(A_F)))$$

where $I$ is defined as in (1.1.3) below. The Poisson summation formula in [GL19] relies not on a Fourier transform from $S(Y(A_F))$ to itself, but a correspondence

$$S(X(A_F) \times V(A_F)) \xrightarrow{F_X} S(X(A_F) \times V(A_F))$$

$$S(Y(A_F)) \xrightarrow{\text{......}} S(Y(A_F)).$$

In the current paper we prove the following theorem:

**Theorem 1.1.** Assume $Y^{\text{sm}}(A_F)$ is nonempty. There is a unique $\mathbb{C}$-linear isomorphism $F_Y : S(Y(A_F)) \to S(Y(A_F))$ such that $I \circ F_X = F_Y \circ I$.

In other words, the dotted arrow in the diagram above can be replaced by $F_Y$ and the resulting diagram is commutative. Theorem 1.1 follows from Theorem 12.1 below. We prove in Proposition 11.2 below that $S(Y(F_v))$ is contained in $L^2(Y(F_v))$ (with respect to an appropriate measure) for all places $v$ of $F$. As an application of Theorem 1.1, in a follow-up paper [GHL21] with S. Leslie, we give an explicit formula for $F_Y$ and prove that it extends to a unitary operator in the nonarchimedean case. This will constitute a sound setup for harmonic analysis on $Y(F_v)$. We refer to [GK19, KM11] for analogous work when $Y$ is replaced by the zero locus of a single quadratic form. We would like to emphasize again that the setting of the current paper is a significant leap from the setting of these other papers because our space is not a Braverman-Kazhdan space.

We prove Theorem 1.1 via global-to-local argument. Though global-to-local arguments using summation formulae such as the trace formula are common in the literature, the authors do not know of another example where such a technique is used to define a unitary operator. The argument is contained in §12. To execute it, we prove a more flexible version of the Poisson summation formula of [GL19] that involves boundary terms. We also develop Fourier and harmonic analysis on $S(Y(A_F))$ to the point that we can take limits of functions. This work is of independent interest.

In remainder of the introduction, we state the Poisson summation formula we prove in this paper. Before stating it in full generality, we highlight a special case. Fix a nontrivial additive character $\psi : F \backslash A_F \to \mathbb{C}^\times$. For $1 \leq i \leq 3$, let

$$\rho_i := \rho_{i,\psi} : \text{SL}_2(A_F) \times S(V_i(A_F)) \to S(V_i(A_F))$$

be the Weil representation attached to $\psi$ and the $Q_i$. For a place $v$ of $F$, let

$$S_{iv} := \{ f \in S(V_i(F_v)) : \rho_i(g)f(0) = 0 \text{ for all } g \in \text{SL}_2(F_v) \},$$

$$S_{0v} := S_{1v} \otimes S_{2v} \otimes S_{3v}.$$
Thus restrictions of elements of $S_{0v}$ to $Y^{\text{sm}}(F_v)$ are elements of $S(Y(F_v))$ by Lemma 5.3. We check in Lemma 12.4 below that $S_{0v}$ is nonzero for finite $v \nmid 2$ (in fact, infinite-dimensional).

**Theorem 1.2.** Let $f \in S(Y(\mathbb{A}_F))$. Assume that there are finite places $v_1, v_2$ of $F$ such that $f = f_{v_1} f_{v_2} f_{v_1 v_2}$ where $f_{v_1}$ and $F_Y(f_{v_2})$ are restrictions of elements of $S_{0v_1}$ and $S_{0v_2}$, respectively. Then

$$\sum_{y \in Y(F)} f(y) = \sum_{y \in Y(F)} F_Y(f)(y).$$

This is similar to the main theorem of [GL19], but Theorem 1.2 has the additional benefit that the hypotheses and conclusion are given intrinsically in terms of the space $S(Y(\mathbb{A}_F))$ and not extrinsically in terms of the map $I : S(X(\mathbb{A}_F) \times V(\mathbb{A}_F)) \to S(Y(\mathbb{A}_F))$.

### 1.1. The boundary terms.

We now describe our main summation formula more precisely. Possibly confusing (but useful) notational conventions on Schwartz spaces are given in §2.

Let $G$ be the image of $\text{SL}_2^3$ under a natural embedding $\text{SL}_2^3 \to \text{Sp}_6$ (see (4.0.10)). The quasi-affine scheme $X^o = [P, P] \backslash \text{Sp}_6$ admits a natural right action

$$X^o \times G \to X^o.$$

Over a field of characteristic zero, there are five orbits in $X^o$ under the action of $G$. We fix representatives $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ as in §4 and let $G_\gamma$ be the stabilizer of $\gamma$ in $G$. The subscript $b$ stands for basepoint; $\gamma_b$ is a representative for the open orbit.

We have a Weil representation

$$\rho := \rho_1 \otimes \rho_2 \otimes \rho_3 : G(\mathbb{A}_F) \times S(V(\mathbb{A}_F)) \to S(V(\mathbb{A}_F)).$$

We will require the following assumption on $f = f_1 \otimes f_2 \in S(X(\mathbb{A}_F) \times V(\mathbb{A}_F))$: There are finite places $v_1, v_2$ of $F$ such that

$$f_1 = f_{v_1} f_{v_2} f_{v_1 v_2}$$

and

$$f_{v_1} \in C_c^{\infty}(X^o(F_{v_1})), \quad F_X(f_{v_2}) \in C_c^{\infty}(X^o(F_{v_2})), \quad \rho(g) f_2(v) = 0 \text{ for } v \nmid V^o(F), \text{ for all } g \in G(\mathbb{A}_F).$$

Here $V^o$ is the open subscheme of $V$ consisting of triples $(v_1, v_2, v_3)$ with each $v_i \neq 0$.

**Remark.** A similar condition on $f_2$ was assumed in [GL19]. We warn the reader that in loc. cit. the assertion that (5.0.7) implies (5.0.5) is false. Fortunately, this claim is never used in loc. cit.

Let $\Phi \in S(\mathbb{A}_F^2)$ and let $N_2 \leq \text{SL}_2$ be the unipotent radical of the Borel subgroup of upper triangular matrices. For $f = f_1 \otimes f_2 \in S(X(\mathbb{A}_F) \times V(\mathbb{A}_F))$ we define

$$I(f)(\xi) := \int_{G_{\gamma_0}(\mathbb{A}_F) \backslash G(\mathbb{A}_F)} f_1(\gamma_0 g) \rho(g) f_2(\xi) \, dg \quad \text{for } \xi \in Y^{\text{sm}}(\mathbb{A}_F),$$

and

$$I_0(f)(\xi) := \int_{G_{\gamma_0}(\mathbb{A}_F) \backslash G(\mathbb{A}_F)} f_1(\gamma_0 g) \rho(g) f_2(\xi) \, dg \quad \text{for } \xi \in \tilde{Y}_0(\mathbb{A}_F).$$
Here $\tilde{Y}_0$ (and $\tilde{Y}_i$ for $1 \leq i \leq 3$) is defined as in §4. For $\xi \in \tilde{Y}_i(\mathbb{A}_F)$ we set

$$I_i(f \otimes \Phi)(\xi, s) := \int_{G_N(\mathbb{A}_F) \backslash G(\mathbb{A}_F)} f_1(\gamma_i g) \left( \int_{N_2(\mathbb{A}_F) \backslash SL_2(\mathbb{A}_F)} \rho(\Delta_i(h)g) f_2(\xi) \Phi(x(0, 1)hp_i(g))|x|^{2s} |x| \, dh \right) dg$$

where $\Delta_i$ is defined as in (5.0.2) and $p_i$ is defined as in (5.0.3). In §4, certain quotient schemes $Y_i$ of $\tilde{Y}_i$ are also defined. Roughly, $Y_0(F)$ is a quotient of $\tilde{Y}_0(F) := \{(v_1, v_2, v_3) \in V^o(F) : Q_1(v_1) = Q_2(v_2) = Q_3(v_3) = 0\}$ by an action of $(F^\times)^2$ and $Y_i$ is the product of the zero locus of $Q_i$ in $V^o_i$ and the projective scheme cut out of $\mathbb{P}(V^o_{i-1} \times V^o_{i+1})$ by $Q_{i-1} - Q_{i+1}$, where the indices are understood “modulo 3” in the obvious sense. Here $\mathbb{P}(V^o_i \times V^o_{i+1})$ is the image of $V^o_i \times V^o_{i+1}$ in $\mathbb{P}(V_i \times V_{i+1})$.

The main summation formula proved in this paper is the following theorem:

**Theorem 1.3.** Assume that

$$(f = f_1 \otimes f_2, \Phi) \in \mathcal{S}(X(\mathbb{A}_F) \times V(\mathbb{A}_F)) \times \mathcal{S}(\mathbb{A}_F^2)$$

where $f$ satisfies (1.1.1) and (1.1.2), and $\hat{\Phi}(0) = 2\text{Vol}(F^\times \backslash \mathbb{A}_F^1)^{-1}$. One has

$$\sum_{\xi \in Y(F)} I(f)(\xi) + \sum_{\xi \in Y_0(F)} I_0(f)(\xi) + \text{Res}_{s=1} \sum_{i=1}^3 \sum_{\xi \in Y_i(F)} I_i(f \otimes \Phi)(\xi, s)$$

$$= \sum_{\xi \in Y(F)} I(\mathcal{F}_X(f))(\xi) + \sum_{\xi \in Y_0(F)} I_0(\mathcal{F}_X(f))(\xi) + \text{Res}_{s=1} \sum_{i=1}^3 \sum_{\xi \in Y_i(F)} I_i(\mathcal{F}_X(f) \otimes \Phi)(\xi, s).$$

Here $\hat{\Phi}$ denotes the Fourier transform of $\Phi$ normalized as in (6.0.3), and $(\mathbb{A}_F^\times)^1 < \mathbb{A}_F^\times$ is the subgroup of ideles of norm 1. When we speak of **boundary terms** in the paper, we mean the summands in Theorem 1.3 involving $I_0$ and $I_i$.

Of course it would be desirable to remove assumptions (1.1.1) and (1.1.2). It seems likely that (1.1.1) could be removed if one had a better understanding of the boundary terms in the summation formula for $X$ obtained in [GL21]. In particular, it would be desirable to obtain a suitable geometric interpretation of these terms. Perhaps (1.1.2) could be removed by applying an argument using Arthur truncation as in [Get22].

### 1.2. Outline

We now outline the sections of this paper. We state conventions for Schwartz spaces in §2. In §3 we recall and refine certain results from harmonic analysis on Braverman-Kazhdan spaces. In particular we prove a Plancherel formula for $\mathcal{S}(X(F_v))$ (see Proposition 3.9).

After this, we discuss the geometric preliminaries necessary for the study of $Y$ in §4. We turn in §5 to the definition of the local integrals necessary to prove our main summation formula, Theorem 1.3. We give a definition of the Schwartz space of $Y(F_v)$ in §5.1.
Theorem 1.3 is proved in §6. This summation formula is given in terms of infinite sums of Eulerian integrals, that is, integrals that factor along the places of $F$ (or residues of such expressions). The local integrals are computed in the unramified case in §8. The proof of Theorem 1.3 depends on bounds on local integrals that are deferred to §7 and §9. We discuss the $L^2$-theory in §11, and prove in particular that $S(Y(F_v)) < L^1(Y(F_v)) \cap L^2(Y(F_v))$ with respect to a natural measure. In §12 we construct the isomorphism $\mathcal{F}_Y$ as described above and prove Theorem 1.2. We have appended a list of symbols for the reader’s convenience.

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2. Schwartz spaces

This work involves several Schwartz spaces. Let $F$ be a local field. For a quasi-affine scheme $X$ of finite type over $F$, let $X^{\text{sm}} \subset X$ be the smooth locus. Any Schwartz space $\mathcal{S}(X(F))$ we discuss will be a space of functions on $X^{\text{sm}}(F)$. Functions in the Schwartz space need not be defined on all of $X(F)$ if $X$ is not smooth. We will not define Schwartz spaces of general quasi-affine schemes of finite type over $F$. In fact obtaining a good definition for general spherical varieties is an important open problem [Sak12]. In this subsection we explain the definition for smooth quasi-affine schemes and how to form Schwartz spaces of products $X(F) \times Y(F)$ given that the Schwartz spaces of each factor have been defined. We have modeled our approach on the treatment of the smooth case in [AG08].

Suppose $F$ is nonarchimedean. If $X$ is smooth, we set $\mathcal{S}(X(F)) := C^\infty_c(X(F))$. More generally, if we have already defined $\mathcal{S}(X(F))$ and $\mathcal{S}(Y(F))$, we set $\mathcal{S}(X(F) \times Y(F)) := \mathcal{S}(X(F)) \otimes \mathcal{S}(Y(F))$ (the algebraic tensor product).

Now assume that $F$ is archimedean. In the case $X$ is smooth, we define $\mathcal{S}(X(F)) = \mathcal{S}(\text{Res}_{F/\mathbb{R}}X(F))$ as in [ES18, Remark 3.2]. It is a Fréchet space. It is defined as a quotient of a nuclear space by a closed subspace, and hence is nuclear. In general, suppose that we have defined Schwartz spaces $\mathcal{S}(X(F))$ and $\mathcal{S}(Y(F))$ that are additionally Fréchet spaces. We then define $\mathcal{S}(X(F) \times Y(F)) := \hat{\mathcal{S}}(X(F)) \otimes \mathcal{S}(Y(F))$ where the hat denotes the (complete) projective topological tensor product. It is also a Fréchet space.

We warn the reader that in [ES18] there is a definition of a Schwartz space for any quasi-affine scheme of finite type over the real numbers. In the smooth case their definition...
coincides with ours. In the nonsmooth case it does not, essentially because functions in our Schwartz spaces need not extend to the singular set.

Finally we discuss the adelic setting. Let $X$ be a quasi-affine schemes of finite type over a number field $F$. Then for all finite sets $S$ of places of $F$, $X(\mathbb{A}_F^S)$ is defined as topological spaces \cite{Con12}. Assume $\mathcal{S}(X(F_v))$ is defined for all places $v$. If $S$ contains all infinite places, $\mathcal{S}(X(\mathbb{A}_F^S))$ will always be a restricted tensor product $\otimes_{v \notin S} \mathcal{S}(X(F_v))$ with respect to some basic functions $b_{X,v} \in \mathcal{S}(X(F_v))$ for almost all $v$; if $S$ is a set of infinite places of $F$, then $\mathcal{S}(X(\mathbb{A}_F^S)) := \hat{\otimes}_{v \in S} \mathcal{S}(X(F_v))$ is the (completed) projective topological tensor product. For general $S$, we put $\mathcal{S}(X(\mathbb{A}_F^S)) := \mathcal{S}(X(F_{\infty \setminus S})) \otimes \mathcal{S}(X(\mathbb{A}_F^{\infty \cup S}))$.

3. Braverman-Kazhdan spaces

In this section we study Schwartz spaces of certain Braverman-Kazhdan spaces. In particular we endow these Schwartz spaces with a Fréchet structure in the archimedean case. This refinement is necessary for continuity arguments.

Let $\text{Sp}_{2n}$ denote the symplectic group on a $2n$-dimensional vector space, and let $P \leq \text{Sp}_{2n}$, $M \leq P$ denote the usual Siegel parabolic and Levi subgroup. More specifically, for $\mathbb{Z}$-algebras $R$, set

\[
\text{Sp}_{2n}(R) := \left\{ g \in \text{GL}_{2n}(R) : g^t \left( I_n - I_n \right) g = \left( I_n - I_n \right) \right\},
\]

\[
M(R) := \left\{ \begin{pmatrix} A & \ast \\ -A^t & -1 \end{pmatrix} : A \in \text{GL}_n(R) \right\},
\]

\[
N(R) := \left\{ \begin{pmatrix} I_n & Z \\ 0 & I_n \end{pmatrix} : Z \in \text{gl}_n(R), Z^t = Z \right\},
\]

and $P = MN$. Apart from this section, we will only use the $n = 3$ case, but since it is no more difficult to treat the general case, we include it. We define a character

\[
\omega : M(R) \longrightarrow R^\times
\]

\[
\begin{pmatrix} A & \ast \\ -A^t & -1 \end{pmatrix} \longmapsto \det A.
\]

Let

\[
X^\circ := [P, P] \setminus \text{Sp}_{2n}.
\]

Let $\text{GSp}_{2n}$ denote the group of similitudes and let

\[
\nu : \text{GSp}_{2n} \longrightarrow \mathbb{G}_m
\]

denote the similitude norm. We note that there is a left action

\[
M^{ab}(R) \times \text{GSp}_{2n}(R) \times X^\circ(R) \longrightarrow X^\circ(R)
\]

\[
(m, g, x) \longmapsto m \left( I_n \nu(g) I_n \right) x g^{-1}.
\]

One has the Plücker embedding

\[
\text{Pl} : X^\circ \longrightarrow \wedge^n \mathbb{G}_a^{2n}
\]
given by taking the wedge product of the last \( n \) rows of a representative \( g \in \text{Sp}_{2n}(R) \) for \([P, P](R)g\). We denote by \( X \) the closure of \( \text{Pl}(X^\circ) \):

\[
X := \overline{\text{Pl}(X^\circ)}.
\]

It is an affine variety (in fact a spherical variety, for many more details see [Li18, §7.2]). As explained in loc. cit., \( X \) is the affine closure of \( X^\circ \).

3.1. The local Schwartz spaces. Let \( F \) be a local field. The Schwartz space \( S(X(F)) \) of \( X(F) \) was defined in [GHL21, §5], where it was denoted \( S(X_P(F)) \). For \( f \in C^\infty(X^\circ(F)) \) and \( g \in \text{Sp}_{2n}(F) \), and a quasi-character \( \chi : F^\times \to \mathbb{C} \), let

\[
f_{\chi_s}(g) := \int_{M_{ab}(F)} \delta_P(m)^{1/2} \chi_s(\omega(m)) f(m^{-1}g) \, dm
\]

be its Mellin-transform. Here \( \chi_s := \chi| \cdot |^s \) and \( \omega \) is defined as in (3.0.2). When this integral is well-defined, either because it converges absolutely or is meromorphically continued from a half-plane of absolute convergence, it is a section of

\[
I(\chi_s) := \text{Ind}_P^{\text{Sp}_{2n}}(\chi \circ \omega).
\]

Here the induction is normalized. We regard \( I(\chi_s) \) as a representation in the category of smooth representations (in other words we require sections to be smooth).

Let \( \psi : F \to \mathbb{C}^\times \) be a nontrivial additive character.

**Theorem 3.1.** [GL21, Theorem 4.4], [GHL21, §5.3] There is a linear automorphism

\[
\mathcal{F}_X := \mathcal{F}_{X,\psi} : S(X(F)) \longrightarrow S(X(F)).
\]

For \( f \in S(X(F)) \), the Fourier transform \( \mathcal{F}_X(f) \) is the unique function in \( S(X(F)) \) such that

\[
\mathcal{F}_X(f)_{\chi_s} = M_{w_0}^*(f_{\chi_s})
\]

for all (unitary) characters \( \chi \) and all \( s \in \mathbb{C} \) with \( \text{Re}(s) \geq 0 \).

Here

\[
M_{w_0}^* := M_{w_0,\psi}^* := \left( \gamma(s + \frac{1-n}{2}, \chi, \psi) \prod_{r=1}^{[n/2]} \gamma(2s - n + 2r, \chi^2, \psi) \right) M_{w_0} : I(\chi_s) \longrightarrow I(\chi_{-s})
\]

is the normalized intertwining operator of [GL21, (3.5)]. The Tate \( \gamma \)-factors depend on a choice of Haar measure on \( F \) which we always take to be the self-dual measure with respect to \( \psi \). In [GHL21, Corollary 6.10] one finds an explicit formula for \( \mathcal{F}_X \). We point out that \( M_{w_0} = \iota_{w_0} \circ R_{P|P^{\text{op}}} \) in the notation of loc. cit.
Let $F$ be an archimedean local field. For real numbers $A < B$, $p(x) \in \mathbb{C}[x]$ and meromorphic functions $f : \mathbb{C} \to \mathbb{C}$, let

$$V_{A,B} : = \{ s \in \mathbb{C} : A \leq \text{Re}(s) \leq B \},$$

(3.1.3) $$|f|_{A,B,p} : = \sup_{s \in V_{A,B}}|p(s)f(s)|.$$ To complete our discussion of $\mathcal{S}(X(F))$ we must endow it with a topology. Recall the $L$-factors $a_w(s, \eta)$ indexed by $w \in \{\Id, w_0\}$ from [GL21]. Consider the Lie algebra

$$\mathfrak{g} := \text{Lie}(M^{ab}(F) \times \text{Sp}_{2n}(F)),$$

viewed as a real Lie algebra. It acts on $C^\infty(X^\circ(F))$ via the differential of the action (3.0.4) and hence we obtain an action of $U(\mathfrak{g})$, the universal enveloping algebra of the complexification of $\mathfrak{g}$. Let $\hat{K}_{G_m}$ be a set of representatives for the (unitary) characters of $F^\times$ modulo equivalence, where $\chi$ is equivalent to $\chi'$ if and only if $\chi = |t|^t\chi'$ for some $t \in \mathbb{R}$. For all real numbers $A < B$, $w \in \{\Id, w_0\}$, $D \in U(\mathfrak{g})$, any polynomials $p_w \in \mathbb{C}[s]$ such that $p_w(s)a_w(s, \eta)$ has no poles for all $(s, \eta) \in V_{A,B} \times \hat{K}_{G_m}$ and compact subsets $\Omega \subset X^\circ(F)$, let

$$|f|_{A,B,w,p_w,\Omega,D} : = \sum_{\eta \in \hat{K}_{G_m}} \sup_{g \in \Omega} |M_w(D,f)\eta_w(g)|_{A,B,p_w}.$$ By definition of the Schwartz space [GHL21, §5] this is a seminorm on $\mathcal{S}(X(F))$ and the collection of these seminorms as $A, B, p_w, \Omega, D$ vary gives $\mathcal{S}(X(F))$ the structure of a locally convex space.

**Lemma 3.2.** The space $\mathcal{S}(X(F))$ is a Fréchet space.

**Proof.** We first observe that we can replace the family of seminorms with a countable subfamily inducing the same topology. More specifically we can choose a countable basis of $U(\mathfrak{g})$, and restrict the $(A, B)$ to lie in the set $\{(-N, N) : N \in \mathbb{Z}_{>0}\}$. Since the poles of $a_w(s, \eta)$ can only occur at points in $\frac{1}{2}\mathbb{Z}$ (see [GL21, (3.4)]) we can similarly restrict our attention to a countable family of $p_w$. Finally we can restrict attention to a countable family of $\Omega$ by simply choosing a countable family of compact subsets of $X^\circ(F)$ whose union is $X^\circ(F)$.

The countable family of seminorms described above is separating by Mellin inversion (see [GL21, Lemma 4.3]). It follows that $\mathcal{S}(X(F))$ is Hausdorff and metrizable. It is also clear that it is complete. \hfill \square

Recall that we have a left action (3.0.4) of $M^{ab} \times \text{GSp}_{2n}$ on $X^\circ$. This yields an action on functions: for a function $f$ on $X^\circ(F)$ and $(m, g, x) \in M^{ab}(F) \times \text{GSp}_{2n}(F) \times X^\circ(F)$

$$L(m)R(g)f(x) : = f \left( m^{-1} \left( I_n \nu(g)^{-1} I_n \right) xg \right).$$

Using the formula for $\mathcal{F}_X$ from [GHL21, Corollary 6.10] one deduces the following lemma:
Lemma 3.3. If \((m, g) \in M^{ab}(F) \times \text{GSp}_{2n}(F)\) and \(f \in \mathcal{S}(X(F))\), then \(L(m)R(g)f \in \mathcal{S}(X(F))\). Moreover,

\[ \mathcal{F}_X(L(m)R(g)f) = |\nu(g)|^{n(n+1)/2} \delta_P^{-1}(m) L(m^{-1}) R(\nu(g)^{-1}g) \mathcal{F}_X(f). \]

\( \square \)

One checks the following lemma:

Lemma 3.4. The action of \(M^{ab}(F) \times \text{GSp}_{2n}(F)\) on \(\mathcal{S}(X(F))\) is smooth. When \(F\) is archimedean, it is continuous with respect to the Fréchet topology on \(\mathcal{S}(X(F))\).

We observe that the inclusions (see §3.3)

\[ C^\infty_c(X^\circ(F)) \hookrightarrow \mathcal{S}(X^\circ(F)) \hookrightarrow \mathcal{S}(X(F)) \]

are continuous in the archimedean case, where we give \(C^\infty_c(X^\circ(F))\) the usual topology for compactly supported smooth functions on a real manifold and \(\mathcal{S}(X^\circ(F))\) the topology explained in §2.

It is useful to explicitly state and prove a refinement of [GL21, Lemmas 5.1 and 5.7]. The group \(\text{Sp}_{2n}\) acts on \(\wedge^n \mathbb{G}_a^{2n}\) via its action on \(\mathbb{G}_a^{2n}\). For \(F\) archimedean, choose a \(K\)-invariant bilinear form \((\cdot, \cdot)\) on \(\wedge^n F^{2n}\) and set \(|x| := (x, x)^{[F: \mathbb{R}]/2}\). For \(F\) nonarchimedean, the standard basis of \(F^{2n}\) induces a canonical basis on \(\wedge^n F^{2n}\) (given by wedge products of the standard basis of \(F^{2n}\) in increasing order). Define the norm \(|x|\) on \(\wedge^n F^{2n}\) to be the maximum norm with respect to the naturally induced basis. This norm is invariant under \(\text{Sp}_{2n}(\mathcal{O})\) where \(\mathcal{O}\) is the ring of integers of \(F\). In any case, we set

\[ (3.0.6) \quad |g| := |\text{Pl}(g)| \]

where \(\text{Pl} : X^\circ \to \wedge^n \mathbb{G}_a^{2n}\) is the Plücker embedding.

Lemma 3.5. Let \(0 \leq \beta < \frac{1}{2}\). If \(F\) is nonarchimedean, then any \(f \in \mathcal{S}(X(F))\) satisfies

\[ |f(g)| \ll_{f, \beta} |g|^{-\frac{n+1}{2} + \beta}. \]

Moreover \(f(g) = 0\) for \(|g|\) sufficiently large in a sense depending on \(f\). If \(F\) is archimedean, then for each \(N \in \mathbb{Z}_{\geq 0}\) and \(D \in U(\mathfrak{g})\) there is a continuous seminorm \(\nu_{D,N,\beta}\) on \(\mathcal{S}(X(F))\) such that for \(f \in \mathcal{S}(X(F))\) one has

\[ |D.f(g)| \leq \nu_{D,N,\beta}(f) |g|^{-N \frac{n+1}{2} + \beta}. \]

Proof. Assume first that \(F\) is nonarchimedean. When \(\beta = 0\) the lemma is just [GL21, Lemma 5.1] and by inspecting the proof one obtains the refined estimate stated in the current lemma.

For the archimedean assertion, using Mellin inversion [GL21, Lemma 4.3], we write

\[ (\omega \overline{\omega}(m))^{-N} D.f(mk) = \delta_P^{1/2}(m) \sum_{\eta \in K_{\mathfrak{o} m}} \int_{\mathbb{R}^{n+\sigma}} (D.f)_{\eta, \mathfrak{g}^{1+2N[F: \mathbb{R}]^{-1}}} (k) \eta_{\mathfrak{g}}(\omega(m)) \frac{ds}{4\pi [F: \mathbb{R}] i}. \]
for $\sigma$ sufficiently large. The factor $a_{t}(s, \chi)$ is holomorphic in the half plane $\text{Re}(s) > -\frac{1}{2}$ and hence so is $a_t(s + \frac{2N}{F^{\infty}}, \chi)$. Thus using the fact that the seminorms (3.1.4) are finite, we can shift the contour to $\sigma = -\beta$ in (3.1.7) to see that it is bounded by

$$
\delta_{p}^{1/2}(m) \sum_{h \in \mathbb{R}} \int_{d\mathbb{R} - \beta} (D.f)_{\eta_{s+2N[F^{\infty}]-1}}(k) \eta_{h}(\omega(m)) \frac{ds}{4\pi[F^{\infty}]} \leq \delta_{p}^{1/2}(m) |\omega(m)|^{-\beta} 2(|f|_{A,B,Id,1,K,D} + |f|_{A,B,Id,s^{2},K,D})
$$

where $A := -\beta - \varepsilon + \frac{2N}{F^{\infty}}$ and $B := -\beta + \varepsilon + \frac{2N}{F^{\infty}}$ for some $\varepsilon < \frac{1}{2} - \beta$. Since $|mk| = |\omega(m)|^{-1}$ and $\delta_{p}(m) = |m|^{-(n+1)}$, we deduce the lemma in the archimedean case.

To prove Proposition 3.7 below, we require a more precise version of [GL21, Lemma 3.3]. Assume for the moment that $F$ is archimedean and let

$$(3.1.8) \quad \mu(z) := \frac{z}{(z^{\alpha})^{1/2}}$$

where in the denominator we mean the positive square root. Then any character of $F^{\times}$ can be written uniquely in the form $\chi = |t|^{\cdot} |t^{\alpha} \mu|$ where $t \in \mathbb{R}$, $\alpha \in \{0, 1\}$ when $F$ is real, and $\alpha \in \mathbb{Z}$ when $F$ is complex.

**Lemma 3.6.** Assume $F$ is archimedean. Let $A < B$ be real numbers, and for $w \in \{\text{Id}, w_0\}$ let $p_{w}, p'_{w} \in \mathbb{C}[x]$ be polynomials such that $p_{w}(s)a_{w}(s, \mu^{\alpha})$ and $p'_{w}(s)a_{w}(-s, \mu^{\alpha})$ are holomorphic and nonvanishing for all $\alpha$ as above and all $s \in V_{A,B}$. Then the quotients

$$
\frac{p'_{Id}(s)a_{Id}(-s, \mu^{\alpha})}{p_{w_{0}}(s)a_{w_{0}}(s, \mu^{\alpha})}, \quad \frac{p_{w_{0}}(s)a_{w_{0}}(s, \mu^{\alpha})}{p'_{Id}(s)a_{Id}(-s, \mu^{\alpha})}, \quad \frac{p_{Id}(s)a_{Id}(s, \mu^{\alpha})}{p'_{w_{0}}(s)a_{w_{0}}(-s, \mu^{\alpha})}, \quad \frac{p'_{w_{0}}(s)a_{w_{0}}(-s, \mu^{\alpha})}{p_{Id}(s)a_{Id}(s, \mu^{\alpha})}
$$

are bounded in $V_{A,B}$ by polynomials in $s$. \hfill \Box

The key difference between this lemma and [GL21, Lemma 3.3] is the uniformity in $\alpha$ of the bound. However, the proof of [GL21, Lemma 3.3] actually yields the stronger assertion of Lemma 3.6.

Let $K \leq \text{Sp}_{2n}(F)$ be a maximal compact subgroup and let

$$S(X(F), K) < S(X(F))$$

be the space of $K$-finite vectors. It is dense [War72, §4.4.3.1]. We prove the following lemma for use in the proof of Theorem 3.12:

**Proposition 3.7.** If $F$ is archimedean then the Fourier transform

$$\mathcal{F}_{X} : S(X(F)) \rightarrow S(X(F))$$

is continuous.
Proof. Let \( w \in \{ \text{Id}, w_0 \} \). Assume \( D = D_1 \otimes D_2 \) where \( D_1 \in U(\text{Lie}(M^{ab}(F))) \) and \( D_2 \in U(\text{sp}_{2n}(F)) \). By [GL21, Lemma 5.9] and Theorem 3.1,

\[
|M_w(D \mathcal{F}_X(f))_{\mu_2^*}(k)|_{A,B,p_w} = |M_w \mathcal{F}_X(D_2 f)_{\mu_2^*}(g)|_{A,B,p_{w_0}} = |M_w M_w^*(D_2 f^\wedge_{\mu_2^*})(g)|_{A,B,p_{2w}}.
\]

Here \( p_{\alpha,w}(s) \) is a polynomial function of \( s \) and \( \alpha \) divisible by \( p_w(s) \) that depends on \( D_1 \). Using the argument of [GL21, Lemma 3.4], but with Lemma 3.6 replacing [GL21, Lemma 3.3], this is bounded by a constant depending on \( A, B \) times

\[
\max(1, |\alpha|)^N |M_{w'}(D_2 f^\wedge_{\mu_2^*})(g)|_{A,B,p_{w'}}
\]

for some \( N \) and an appropriate \( p'_{w'} \) independent of \( \alpha \), where

\[
w' = \begin{cases} w_0, & \text{if } w = \text{Id}, \\
\text{Id}, & \text{if } w = w_0. \end{cases}
\]

This in turn is dominated by

\[
|M_{w'}(D' D_2 f^\wedge_{\mu_2^*})(g)|_{A,B,p_{w'}}
\]

for an appropriate differential operator \( D' \) (see [GL21, Lemma 5.9]). The lemma follows. \( \square \)

Let

\[(3.1.9) \quad m(x) := \begin{pmatrix} x^{-1} & I_{n-1} \\ I_{n-1} & x \end{pmatrix}.\]

Assume now that \( F \) is nonarchimedean. By the Iwasawa decomposition, a \( \mathbb{C} \)-vector space basis for \( C_c^\infty(x^\circ(F))^b_{2n}(\mathcal{O}) \) is given by the functions

\[(3.1.10) \quad \mathbb{1}_k := \mathbb{1}_{[P,P](F)m(w^k)}b_{2n}(\mathcal{O}) \]

for \( k \in \mathbb{Z} \). The space \( S(x^\circ(F))^b_{2n}(\mathcal{O}) \) contains \( C_c^\infty(x^\circ(F))^b_{2n}(\mathcal{O}) \) but it is larger. It contains, for example, the basic function

\[(3.1.11) \quad b_X := \sum_{(j_1,\ldots,j_{[n/2]},k) \in \mathbb{Z}^{[n/2]+1}} q^{2j_1+4j_2+\cdots+2[n/2]j_{[n/2]}+k+2j_1+\cdots+2j_{[n/2]}} \mathbb{1}_k \cdot \]

One has \( \mathcal{F}_X(b_X) = b_X \) [GL21, Lemma 5.4] provided that \( \psi \) is unramified and \( F \) is absolutely unramified.

It will be convenient to isolate another family of functions in this space. For \( c \in \mathbb{Z} \), let

\[
\mathbb{1}_c := \sum_{\alpha \geq c} \mathbb{1}_\alpha.
\]

Lemma 3.8. One has \( \mathbb{1}_c \in S(x^\circ(F))^b_{2n}(\mathcal{O}). \)

Proof. One has \( L(m(w^c)) \mathbb{1}_c = \mathbb{1}_{c}, \) so by Lemma 3.3 it suffices to show \( \mathbb{1}_0 \in S(x^\circ(F))^b_{2n}(\mathcal{O}). \) Since

\[
\left( \prod_{j=1}^{[n/2]} (1 - q^{2j}L(m(w^2))) \right) b_X = \mathbb{1}_0,
\]

...
we can apply Lemma 3.3 again to deduce the result. □

The usual Schwartz space of $F$ is dense in $L^2(F, dx)$ and the Fourier transform extends to an isometry of $L^2(F, dx)$. We now prove analogues of these statements in the current setting. We choose a positive right $Sp_{2n}(F)$-invariant Radon measure on $X^o(F)$ (it is unique up to scaling). Since $X^o(F) \subset X(F)$ is open and dense we extend by zero to obtain a measure on $X(F)$ and we can speak of $L^2(X(F))$.

**Proposition 3.9.** One has $\mathcal{S}(X(F)) < L^2(X(F))$. The Fourier transform $\mathcal{F}_X$ extends to yield an isometry of $L^2(X^o(F))$. For $f, f_1, f_2 \in L^2(X(F))$ one has

\begin{equation}
\mathcal{F}_X(f) = \mathcal{F}_X(L(m(-1)^{n+1})f),
\end{equation}

\begin{equation}
\int_{X(F)} \overline{f_1(x)} \mathcal{F}_X(f_2)(x) dx = \int_{X(F)} \overline{\mathcal{F}_X(f_1)(x)}(L(m(-1)^{n+1})f_2)(x) dx.
\end{equation}  

Before giving the proof we recall two lemmas. The first is an identity that was stated with a typo in [Ike92, (1.2.3)]:

**Lemma 3.10.** Assume that $\chi : F^\times \rightarrow \mathbb{C}^\times$ is a character and that $n = 1$. The operator

$$M^*_{w_0} \circ M^*_{w_0} : I(\chi_s) \rightarrow I(\chi_s)$$

is the identity. □

This is well-known and may be obtained via a standard argument.

**Lemma 3.11.** For any $n$ and any character $\chi : F^\times \rightarrow \mathbb{C}^\times$, the operator

$$M^*_{w_0} \circ M^*_{w_0} : I(\chi_s) \rightarrow I(\chi_s)$$

is multiplication by $\chi(-1)^{n+1}$.

**Proof.** This was stated incorrectly in [Ike92, Lemma 1.1]. The source of the error is the typo in [Ike92, (1.2.3)]. Upon correcting the typo using Lemma 3.10, the same argument proves the current lemma. □

**Remark.** The typos in [Ike92] corrected in Lemma 3.10 and 3.11 do not affect the main results of [Ike92] or their proofs. Moreover they do not affect [GL21], which makes use of results in [Ike92], except for the statement of [GL21, Lemma 4.6]. The correct statement is in Proposition 3.9 above.

**Proof of Proposition 3.9.** The inclusion $\mathcal{S}(X(F)) < L^2(X(F))$ is an easy consequence of the Iwasawa decomposition and Lemma 3.5.

For $f \in \mathcal{S}(X(F))$, assertion (3.1.12) is a consequence of Theorem 3.1 and Lemma 3.11. Taking the complex conjugate of the identity of Theorem 3.1, for $\sigma \geq 0$ one has

$$\overline{\mathcal{F}_X(f)}_{\chi_{\sigma+it}} = \overline{\mathcal{F}_X(f)}_{\chi_{-\sigma-it}} = M^*_{w_0}(f, \chi_{-\sigma-it})$$
\[
\gamma(-\sigma + it - \frac{n-1}{2}, \chi, \psi) \prod_{r=1}^{\lfloor n/2 \rfloor} \gamma(2(-\sigma + it) - n + 2r, \chi^2, \psi) M_{w_0}^*(\mathcal{F}_{X,\sigma + it}) \\
= \gamma(-\sigma - it - \frac{n-1}{2}, \overline{\chi}, \overline{\psi}) \prod_{r=1}^{\lfloor n/2 \rfloor} \gamma(2(-\sigma - it) - n + 2r, \overline{\chi^2}, \overline{\psi}) M_{w_0}^*(\mathcal{F}_{X,\sigma - it}) \\
= \chi(-1) M_{w_0}^*(\mathcal{F}_{X,\sigma - it}).
\]

Thus by Theorem 3.1, we deduce assertion (3.1.13).

In [BK02] Braverman and Kazhdan defined an isometry
\[
\mathcal{F}_{P,P^\text{op}} : L^2([P, P]\backslash G(F)) \rightarrow L^2([P^\text{op}, P^\text{op}]\backslash G(F)).
\]
One has \( \mathcal{F}_X = \iota_{w_0} \circ \mathcal{F}_{P,P^\text{op}} \) by [GHL21, (5.19)], where \( \iota_{w_0} \) is the isometry
\[
\iota_{w_0} : L^2([P^\text{op}, P^\text{op}]\backslash G(F)) \rightarrow L^2(X(F)) \\
f \mapsto (x \mapsto f(w_0^{-1} x)).
\]
It follows that \( \mathcal{F}_X \) is an isometry. The Plancherel formula (3.1.14) follows from (3.1.12), the unitarity of \( \mathcal{F}_X \), and a standard argument using a polarization identity. \( \square \)

### 3.2. The summation formula

We now revert to the global setting. Let \( F \) be a number field. Recall that \( \mathcal{S}(X(\mathbb{A}_F)) \) is defined in §2 as the restricted tensor product of \( \mathcal{S}(X(F_v)) \) with respect to the basic functions \( b_{X,v} \). Let \( \psi : F\backslash \mathbb{A}_F \rightarrow \mathbb{C}^\times \) be a nontrivial character. Since \( \mathcal{F}_{X,\psi_v}(b_{X,v}) = b_{X,v} \) if \( \psi_v \) is unramified and \( F_v \) is nonarchimedean and absolutely unramified, we have a global Fourier transform
\[
\mathcal{F}_X := \mathcal{F}_{X,\psi} := \bigotimes_v \mathcal{F}_{X,\psi_v} : \mathcal{S}(X(\mathbb{A}_F)) \rightarrow \mathcal{S}(X(\mathbb{A}_F)).
\]
Recall that \( \mathcal{S}(X^\circ(F_v)) < \mathcal{S}(X(F_v)) \) for all places \( v \) by Proposition 3.13.

**Theorem 3.12.** Assume that for some finite places \( v_1, v_2 \) (not necessarily distinct) one has \( f = f_{v_1} f_{v_2} \) and \( \mathcal{F}_X(f) = \mathcal{F}_X(f_{v_2} \mathcal{F}_X(f_{v_2})) \) with \( f_{v_1} \in C_c^\infty(X^\circ(F_{v_1})) \) and \( \mathcal{F}_X(f_{v_2}) \in C_c^\infty(X^\circ(F_{v_2})) \). Then
\[
\sum_{\gamma \in X(F)} f(\gamma) = \sum_{\gamma \in X(F)} \mathcal{F}_X(f)(\gamma).
\]

**Proof.** We may assume \( f = f_{\infty} f^\infty \) with \( f_{\infty} \in \mathcal{S}(X(F_{\infty})) \) and \( f^\infty \in \mathcal{S}(X(\mathbb{A}_F^\infty)) \). Let \( K_{\infty} < \text{Sp}_{2n}(F_{\infty}) \) be a maximal compact subgroup and let \( \mathcal{S}(X(F_{\infty}), K_{\infty}) \) be the space of \( K_{\infty} \)-finite functions. Assume first that \( f_{\infty} \in \mathcal{S}(X(F_{\infty}), K_{\infty}) \). Then the stated identity follows from [GL21, Theorem 1.1] and [GL19, Theorem 10.1].

We now argue by continuity to deduce the identity in general. Using the estimates in Lemma 3.5 and the convergence argument in [GL21, Lemma 6.4], it suffices to show the existence of a sequence \( f_{n,\infty} \in \mathcal{S}(X(F_{\infty}), K_{\infty}) \) such that \( f_{n,\infty} \rightarrow f_{\infty} \) and \( \mathcal{F}_X(f_{n,\infty}) \rightarrow \mathcal{F}_X(f_{\infty}) \) in the topology on \( \mathcal{S}(X(F_{\infty})) \). But this directly follows from the fact that \( \mathcal{S}(X(F_{\infty}), K_{\infty}) \) is dense in \( \mathcal{S}(X(F_{\infty})) \) and \( \mathcal{F}_X \) is continuous by Proposition 3.7. \( \square \)
We remark that Theorem 3.12 was already proved in [BK02], but with a different definition of the Schwartz space. At least at the nonarchimedean places, the two definitions should yield the same space of functions. At the archimedean places this is less clear. In any case, it is easier to just prove the theorem directly than to rigorously check the compatibility of the two definitions.

3.3. Containment of Schwartz spaces. In this subsection we prove the following proposition:

**Proposition 3.13.** One has $S(X^\circ(F)) < S(X(F))$. In the archimedean case the inclusion is continuous.

In the nonarchimedean case this is [GL21, Proposition 4.7].

**Remark.** In the archimedean case the weaker statement that $K_\infty$-finite functions in $C_\infty^c(X^\circ(F))$ are contained in $S(X(F))$ was asserted in [GL21, Proposition 4.7]. This is true, but the proof is incomplete. It relies on [GL21, Lemma A.2], which is false. Happily, this does not affect the rest of the results in [GL21] because the false assertion in [GL21, Lemma A.2] is not used elsewhere in the paper.

Let $T \leq \text{Sp}_{2n}$ be the maximal torus of diagonal matrices, let $W_{\text{Sp}_{2n}}$ be the Weyl group of $T$ in $\text{Sp}_{2n}$ and let $W_M$ denote the Weyl group of $T$ in $M$. Denote by $B \leq P$ the unique Borel subgroup consisting of matrices whose upper left $n \times n$ block is upper triangular. We let $\Phi_{\text{Sp}_{2n}}$ be the set of roots of $T$ in $\text{Sp}_{2n}$ and we define positive roots using $B$. We let $W_n$ be the complete set of representatives for $W_{\text{Sp}_{2n}}/W_M$ obtained by choosing the unique element of minimal length in each coset as follows (see [Ike92, §1] and [GPSR87, Part A, Lemma 5.1]). For each subset $\iota = \{i_1, i_2, \ldots, i_k\}$ of $\{1, 2, \ldots, n\}$ let

$$J := \{j_1, j_2, \ldots, j_{n-k}\} = \{1, 2, \ldots, n\} - \iota,$$

where $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_{n-k}$. Define an element $w_{\iota}$ of $W_{\text{Sp}_{2n}}$ by

$$
\begin{align*}
t_1 &\mapsto t_{j_1}, \\
\cdots & \\
t_n &\mapsto t_{j_n}, \\
t_{n-k+1} &\mapsto t_{i_k}^{-1}, \\
\cdots & \\
t_{n-k} &\mapsto t_{i_1}^{-1}, \\
t_n &\mapsto t_{i_1}^{-1},
\end{align*}
$$

where

$$
\begin{pmatrix}
t_1 \\
\cdots \\
t_n \\
t_{i_k}^{-1} \\
\cdots \\
t_{i_1}^{-1}
\end{pmatrix} \in T(F).
$$

In particular $w_0 := w_{\{1, \ldots, n\}}$ is the long Weyl element. For each $0 \leq k \leq n-1$ let $\iota_k = \{k+1, \ldots, n\}$ and let $t_n = \emptyset$. Moreover, set

$$
w_k := \begin{pmatrix}
I_k & 0 & 0 & 0 \\
0 & 0 & 0 & -\beta_{n-k} \\
0 & 0 & I_k & 0 \\
0 & \beta_{n-k} & 0 & 0
\end{pmatrix}.$$
where \( \beta_{n-k} \in \text{GL}_{n-k}(\mathbb{Z}) \) is the antidiagonal matrix. This is a representative for \( w_k \). One checks that \( \{w_k : 0 \leq k \leq n\} \) is a complete minimal set of representatives for \( W_M \backslash W_{\text{Sp}_{2n}} / W_M \). We leave the proof of the following lemma to the reader:

**Lemma 3.14.** Let \( C = Pw_0 Pw_0^{-1} \subset \text{Sp}_{2n} \), where \( C \) and \( \text{Sp}_{2n} \) are both regarded as schemes over an arbitrary field \( F \). One has

\[
\bigcup_{k=0}^{n} Cw_k = \text{Sp}_{2n}.
\]

\( \square \)

**Proof of Proposition 3.13.** We can and do assume \( F \) is archimedean. If \( f \in \mathcal{S}(X^\circ(F)) \), it is easy to see that the integral defining \( f_\chi \) is absolutely convergent for all \( \chi : F^\times \to \mathbb{C}^\times \) and \( s \in \mathbb{C} \). Thus \( f_\chi \) is a good section by [Ike92, Lemma 1.3]. We have to verify that for all real numbers \( A < B \), \( w \in \{\text{Id}, w_0\} \), \( D \in U(\mathfrak{g}) \), any polynomials \( p_w \in \mathbb{C}[s] \) such that

\[
(3.3.1) \quad p_w(s)a_\nu(s, \eta) \text{ has no poles for all } (s, \eta) \in V_{A,B} \times \hat{K}_{G_m}
\]

and compact subsets \( \Omega \subset X^\circ(F) \), one has \( |f|_{A,B,w,p_w,\Omega,D} < \infty \) and \( |f|_{A,B,w,p_w,\Omega,D} \) is continuous with respect to the topology on \( \mathcal{S}(X^\circ(F)) \). Since \( U(\mathfrak{g}) \) acts continuously on \( \mathcal{S}(X^\circ(F)) \), it suffices to verify this for \( D = \text{Id} \).

We start by reducing to an estimate involving a single \( \eta \). Let \( D_1 \) (and \( \overline{D}_1 \) when \( F \) is complex) be the generators of \( U(\text{Lie}(M^\text{ab}(F)) \) given in [GL21, (4.2) and (4.3)], respectively. Every element of \( \hat{K}_{G_m} \) is in the equivalence class of \( \mu^\alpha \) for \( \alpha = 0, 1 \) when \( F \) is real and \( \alpha \in \mathbb{Z} \) when \( F \) is complex. Here \( \mu \) is defined as in (3.1.8). If \( F \) is complex, \( A,B,p_w \)

\[
|M_w(D_1^N\overline{D}_1^Nf)_{\mu_\alpha^g}(g)|_{A,B,p_w}
\]

by [GL21, Lemma 5.9]. This provides us with an estimate on \( M_w(f)_{\mu_\alpha^g}(g) \) as a function of \( \alpha \). Using this estimate, we see that to prove \( |f|_{A,B,w,p_w,\Omega,1} \) is finite for all \( f \in \mathcal{S}(X^\circ(F)) \), it suffices to prove that for each \( p_w \) satisfying (3.3.1) there is a continuous seminorm \( \nu \) on \( \mathcal{S}(X^\circ(F)) \) such that

\[
\sup_{g \in \Omega} |M_w f_{\mu_\alpha^g}(g)|_{A,B,p_w} \leq \nu(f)
\]

for all \( f \in \mathcal{S}(X^\circ(F)) \) and \( \alpha \). Here and below the seminorm \( \nu \) is allowed to depend on \( A,B,w,p_w,\Omega \). In fact, it is enough to show that there is a continuous seminorm \( \nu \) on \( \mathcal{S}(X^\circ(F)) \) that

\[
(3.3.2) \quad |M_w f_{\mu_\alpha^g}(w_0)|_{A,B,p_w} \leq \nu(f).
\]
Indeed, let $\Omega' \subset \text{Sp}_{2n}(F)$ be a compact set whose projection to $X^\circ(F)$ is $\Omega$. Then assuming we have a seminorm $\nu$ as just described we have

$$\sup_{g \in \Omega} |M_{w_0} f_{\mu_2'}(g)|_{A,B,p_w} = \sup_{g \in \omega_0^{-1} \Omega} |M_w(R(g)f)_{\mu_2'}(w_0)|_{A,B,p_w} \leq \sup_{g \in \omega_0^{-1} \Omega} \nu(R(g)f)$$

and $\sup_{g \in \omega_0^{-1} \Omega} \nu(R(g)f)$ is another continuous seminorm.

Since (3.3.2) is clear for $w = \text{Id}$, we are left with the $w = w_0$ case. We will roughly follow the strategy of [PSR87, Lemma 4.1], but we cannot immediately reduce to functions having support in the big cell. Let $C$ be the image of $Pw_0Pw_0^{-1}$ in $X^\circ$. Choose a tempered partition of unity subordinate to the cover of $X^\circ(F)$ given by Lemma 3.14, that is, choose tempered functions $t_k$ supported in $C(F)w_k$ such that $\sum_{k=0}^n t_k = 1$ and $t_k f \in \mathcal{S}(C(F)w_k)$ for all $f \in \mathcal{S}(X^\circ(F))$ [ES18, Proposition 3.14]. Then

$$|M_{w_0} f_{\mu_2'}(w_0)|_{A,B,p_w} \leq \sum_{k=0}^n |M_{w_0}(t_k f)_{\mu_2'}(w_0)|_{A,B,p_w}.$$

Hence we can and do assume that $f$ is supported in $Cw_k$ for some fixed $0 \leq k \leq n$. Now

$$M_{w_0} f_{\mu_2'}(w_0) = \int_{N(F)} \int_{M_{1b}(F)} \delta_P(m)^{1/2} \mu_2^\alpha(\omega(m)) f(m^{-1}w_0^{-1}w_0) dmdn$$

$$= \int_{\text{Sym}^2(F^n)} \int_{(\text{SL}_n\setminus \text{GL}_n)(F)} \mu_2^\alpha(\det A) f((AZ A)^{-1}w_k) dAdZ.$$

Here we have taken a change of variables $(A, Z) \mapsto (A^t, -\beta_n Z \beta_n)$. The notation is a reminder that the image of an element of $\text{Sp}_{2n}(F)$ in $X^\circ(F)$ depends only on the bottom $n$ rows of the matrix.

We have an isomorphism of $F$-schemes

$$\iota : \text{GL}_n \times \text{Sym}^2(G_{\alpha}^n) \rightarrow Cw_k$$

$$(A, Z) \mapsto (A^t A)^{-1} w_0 \left( I_n - \beta_n Z \beta_n \right) w_0^{-1} w_k = (AZ A^t)^{-1} w_k.$$

Write $Z = \left( \begin{array}{cc} u & x \\ x & y \end{array} \right)$ where $(u, x, y) \in \text{Sym}^2(F^k) \times M_{k \times (n-k)}(F) \times \text{Sym}^2(F^{n-k})$, and let

$$A' = A'(x, y) := \left( \begin{array}{cc} I_k x & \beta_n \times \\ 0 & y & \beta_n^{-1} \end{array} \right).$$

Then if $A'$ is invertible, we have

$$\left( \begin{array}{c} * \\ A \end{array} \right) = \left( \begin{array}{cc} A^{-1} & * \\ A^t & * \end{array} \right) \left( \begin{array}{cc} u & 0 \\ x & \beta_n \end{array} \right) w_k$$

$$= \left( \left( AA' \right)^{-1} A \right) \left( \begin{array}{cc} u - xy^{-1}x^t & 0 \\ x & \beta_n^{-1} \end{array} \right) \left( \begin{array}{cc} I_k & 0 \\ \beta_n & 0 \end{array} \right) w_k.$$

Thus

$$M_{w_0} f_{\mu_2'}(w_0)$$

$$= \int \mu_2^\alpha(\det A) f \left( \left( \left( AA' \right)^{-1} A \right) \left( \begin{array}{cc} u - xy^{-1}x^t & 0 \\ x & \beta_n^{-1} \end{array} \right) \left( \begin{array}{cc} I_k & 0 \\ \beta_n & 0 \end{array} \right) w_k \right) dAdudxdy$$
\[
\int \mu^\alpha_{s+(n+1)/2}(\det A)f \left( \left( A^{-t} \right) \left( \begin{matrix} u & x & k \\ x & y & 0 \\ 0 & 0 & I_{n-k} \end{matrix} \right) w \right) \mu^{-\alpha}_{s-(n+1)/2}((-1)^{n-k} \det \beta_{n-k}y)dAdudxdy
\]
\[
\int \mu^\alpha_{s+(n+1)/2}(\det A)f \left( \left( A^{-t} \right) \left( \begin{matrix} u & x & k \\ x & y & 0 \\ 0 & 0 & I_{n-k} \end{matrix} \right) w \right) \mu^{-\alpha}_{s-(n+1)/2}((-1)^{n-k} \det \beta_{n-k}y)dAdudxdy,
\]
where the integrals are over \((\text{SL}_n/\text{GL}_n)(F) \times \text{Sym}^2(F^k) \times M_{k \times (n-k)}(F) \times \text{Sym}^2(F^{n-k})\). Here we have used that \(d(y^{-1}) = |\det y|^{-n+k-1}dy\). Now consider the differential operator
\[
\partial := \det(\partial_{ij})
\]
where \((\partial_{ij})\) is the unique symmetric \((n-k) \times (n-k)\) matrix of partial differential operators satisfying
\[
\partial_{ij} = \begin{cases} \frac{\partial f}{\partial z_i}, & \text{if } i = j; \\ \frac{1}{2} \frac{\partial f}{\partial z_i}, & \text{if } i > j. \end{cases}
\]
When \(F\) is complex, we view these as holomorphic differential operators. Then for \(y \in \text{Sym}^2(F^{n-k})\) we have
\[
\partial(\det y)^s = \prod_{i=0}^{n-k-1} \left( s + \frac{i}{2} \right) (\det y)^{s-1}
\]
(see, e.g. [CSS13, Theorem 2.2]).

Applying integration by parts \(m\) times we have
\[
p_{m,\alpha}(s) \int \mu^\alpha_{s-(n+1)/2}(\det y)\mu^\alpha_{s+(n+1)/2}(\det A)f \left( \left( A^{-t} \right) \left( \begin{matrix} u & x & k \\ x & y & 0 \\ 0 & 0 & I_{n-k} \end{matrix} \right) w \right) dAdudxdy
\]
\[
= \int \mu^\alpha_{s-(n+1)/2+m}(\det y)\mu^\alpha_{s+(n+1)/2}(\det A)(\partial^m(\partial))m f \left( \left( A^{-t} \right) \left( \begin{matrix} u & x & k \\ x & y & 0 \\ 0 & 0 & I_{n-k} \end{matrix} \right) w \right) dAdudxdy
\]
where \(p_{m,\alpha}(s) \in \mathbb{C}[s]\) has zeros only in \(\frac{1}{2}\mathbb{Z}\). Here by convention \(\overline{\partial}\) is the identity operator when \(F\) is real, and we are letting \(\partial\) and \(\overline{\partial}\) act on \(f\) viewed as a function of \(y \in \text{Sym}^{n-k}(F)\). We observe that the bottom integral converges absolutely for \(\text{Re}(s) + m > \frac{n+1}{2}\), and thus provides us with a holomorphic continuation of \(p_{m,\alpha}(s)M_{w_0}f_{\mu^\alpha_2}(w_0)\) to this range. Moreover, if \(A + m > \frac{n+1}{2}\), then for all \(p \in \mathbb{C}[s]\) one has
\[
|M_{w_0}f_{\mu^\alpha_2}(w_0)|_{A,B,ppm,\alpha} \leq \nu(f)
\]
for some continuous seminorm \(\nu\) on \(\mathcal{S}(X^\circ(F))\) depending on \(p, m, A, B\).

Assume henceforth that \(A + m > \frac{n+1}{2}\). Since the zeros of \(p_{m,\alpha}\) are located in \(\frac{1}{2}\mathbb{Z}\), by slightly decreasing \(A\) and increasing \(B\) if necessary, we are free to assume that no zeros of \(p_{m,\alpha}\) are on the lines \(\text{Re}(s) = A\) or \(\text{Re}(s) = B\) for all \(\alpha\). Let
\[
\Omega := \{ s \in V_{A,B} : |\text{Im}(s)| < 1, A < \text{Re}(s) < B \}.
\]
Again using the fact that the zeros of \(p_{m,\alpha}\) are located in \(\frac{1}{2}\mathbb{Z}\), we have
\[
\max_{s \in V_{A,B} - \Omega} \frac{1}{p_{m,\alpha}(s)} \ll m
\]
where the implied constant is independent of $\alpha$. Assume now that $p_{w_0}$ satisfies (3.3.1). Since $M_{w_0} f_{\mu_2}$ is a good section [Ike92, Lemmas 1.2 and 1.3], the maximum modulus principal implies
\[
\sup_{s \in V_{A,B}} |p_{w_0}(s) M_{w_0} f_{\mu_2}(w_0)| \leq \sup_{s \in V_{A,B} - \Omega} |p_{w_0}(s) M_{w_0} f_{\mu_2}(w_0)| \\
\leq |M_{w_0} f_{\mu_2}(w_0)|_{A,B, p_{w_0} p_m, \alpha} \max_{s \in V_{A,B} - \Omega} \frac{1}{p_m(s)} \\
\leq \nu(f)
\]
for some continuous seminorm $\nu$ on $\mathcal{S}(X^o(F))$ depending on $p_{w_0}, m, A, B$. Here in the last inequality have used (3.3.4) and (3.3.5). This implies (3.3.2). \hfill $\square$

4. Groups and Orbits

For this section, $F$ is a field of characteristic zero. For $1 \leq i \leq 3$, let $V_i = \mathbb{G}_a^{d_i}$ where $d_i$ is even and let $\mathcal{Q}_i$ be a nondegenerate quadratic form on $V_i(F)$. Let $\mathcal{Q} := \mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3$. We put
\[
V_i^o = V_i - \{0\} \text{ and } V^o = V_1^o \times V_2^o \times V_3^o
\]
and we let $V' \subset V$ be the open subscheme consisting of $(v_1, v_2, v_3)$ such that no two $v_i$ are zero. For an $F$-algebra $R$, recall that
\[
Y(R) = \{(y_1, y_2, y_3) \in V(R) : \mathcal{Q}_1(y_1) = \mathcal{Q}_2(y_2) = \mathcal{Q}_3(y_3)\}.
\]
We observe that $Y^{sm} = Y \cap V'$. We let $Y^{ani} \subset Y$ be the open complement of the vanishing locus of $\mathcal{Q}_i$ (it is independent of $i$).

We let $GO_{\mathcal{Q}_i}$ be the similitude group of $(V_i, \mathcal{Q}_i)$ and let $\nu : GO_{\mathcal{Q}_i} \to \mathbb{G}_m$ be the similitude norm. We then set
\[
H(R) := \{(h_1, h_2, h_3) \in GO_{\mathcal{Q}_1}(R) \times GO_{\mathcal{Q}_2}(R) \times GO_{\mathcal{Q}_3}(R) : \nu(h_1) = \nu(h_2) = \nu(h_3)\},
\]
and define
\[
\lambda : H(R) \longrightarrow R^x
\]
\[
(h_1, h_2, h_3) \longmapsto \nu(h_1).
\]
Let
\[
\tilde{Y}_0(R) := \{(y_1, y_2, y_3) \in V^o(R) : \mathcal{Q}_1(y_1) = \mathcal{Q}_2(y_2) = \mathcal{Q}_3(y_3) = 0\}
\]
and let $Y_0$ be the (quasi-affine) quotient of $\tilde{Y}_0$ by $\mathbb{G}_m \times \mathbb{G}_m$, acting via the restriction of the action
\[
\mathbb{G}_m(R) \times \mathbb{G}_m(R) \times V(R) \longrightarrow V(R)
\]
\[
(a_1, a_2, (v_1, v_2, v_3)) \longmapsto (a_1 v_1, a_2 v_2, (a_1 a_2)^{-1} v_3).
\]
This quotient can be constructed by taking the affine closure \( \tilde{Y}_0 \) of \( \tilde{Y}_0 \) in \( V \) and viewing \( Y_0 \) as an open subscheme of the GIT quotient of \( \tilde{Y}_0 \) by \( \mathbb{G}_m \times \mathbb{G}_m \). We observe that \( Y_0 \) is a geometric quotient of \( \tilde{Y}_0 \).

For \( 1 \leq i \leq 3 \), we define the scheme

\[
(4.0.7) \quad \tilde{Y}_i(R) := \{ (y_1, y_2, y_3) \in V^\circ(R) : Q_{i-1}(y_{i-1}) = Q_{i+1}(y_{i+1}) \text{ and } Q_i(y_i) = 0 \}.
\]

Here the indices are taken modulo 3 in the obvious sense. Let \( Y_1 \) be the quotient of \( \tilde{Y}_1 \) by \( \mathbb{G}_m \) acting via the restriction of the action

\[
(4.0.8) \quad \mathbb{G}_m(R) \times V(R) \to V(R)
\]

\[
(a, (v_1, v_2, v_3)) \mapsto (v_1, av_2, av_3).
\]

This is nothing but the product over \( F \) of the quasi-affine scheme cut out by \( Q_1 \) in \( V_1^\circ \) and the quasi-projective scheme cut out of \( \mathbb{P}(V_2^\circ \times V_3^\circ) \) by \( Q_2 = Q_3 \). The schemes \( Y_2 \) and \( Y_3 \) are defined similarly. Thus

\[
(4.0.9) \quad Y_0 := \tilde{Y}_0/\mathbb{G}_m^2 \quad \text{and} \quad Y_i := \tilde{Y}_i/\mathbb{G}_m
\]

where the quotients are defined as above. Using Hilbert’s theorem 90, we deduce the following lemma:

**Lemma 4.1.** The maps \( \tilde{Y}_0(F)/(F^\times)^2 \to Y_0(F) \) and \( \tilde{Y}_i(F)/F^\times \to Y_i(F) \) are bijective. \( \square \)

We often identify \( \text{SL}_2^3(R) \) with the subgroup \( G(R) \leq \text{Sp}_6(R) \) defined as follows:

\[
(4.0.10) \quad G(R) = \left\{ \begin{pmatrix} a_1 & a_2 & b_1 & b_2 & b_3 \\
1 & c_1 & a_3 & d_1 & d_2 & d_3 \\
& 1 & c_2 & c_3 & d_2 & d_3 \end{pmatrix} \in \text{GL}_6(R) : a_i d_i - b_i c_i = 1 \text{ for } 1 \leq i \leq 3 \right\}.
\]

We give a set of representatives for

\[ X^\circ(F)/G(F) \]

and the corresponding stabilizers. Let

\[
(4.0.11) \quad \gamma_b := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \gamma_i := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^{i-1} \quad \text{for } 1 \leq i \leq 3.
\]

All four matrices are in \( \text{Sp}_6(\mathbb{Z}) \). By [GL19, Lemmas 2.1 and 2.2], the matrices \( \gamma_i \) together with the identity matrix, denoted by \( \gamma_0 = \text{Id} \), form a minimal set of representatives of \( X^\circ(F)/G(F) \) (strictly speaking, we have chosen different representatives for the \( \gamma_i \) orbits than in [GL21], but this does not affect the validity of [GL19, Lemmas 2.1 and 2.2]). For \( \gamma \in X^\circ(F) \), let \( G_\gamma \leq G \) be the stabilizer of \( \gamma \) under the right action.
Lemma 4.2. [GL19, Lemma 2.3] One has

\[ G_{\gamma_b}(R) := \{((1_{\frac{t_1}{1}}), (1_{\frac{t_2}{1}}), (1_{\frac{t_3}{1}})) : t_1, t_2, t_3 \in R, t_1 + t_2 + t_3 = 0\}, \]

\[ G_{Id}(R) := \{((b_{-1}^{-1} t_1, b_{-1}^{-1} t_2, b_{-1}^{-1} t_3)) : t_1, t_2, t_3 \in R, b_1, b_2, b_3 \in R^x, b_1 b_2 b_3 = 1\}, \]

\[ G_{\gamma_1}(R) := \{(1_t, I_2, (1_{\frac{1}{-1}})) : t \in R, g \in SL_2(R)\}, \]

\[ G_{\gamma_2}(R) := \{((1_{\frac{1}{-1}}) g (1_{\frac{1}{-1}}), (1_{\frac{1}{-1}})) : t \in R, g \in SL_2(R)\}, \]

\[ G_{\gamma_3}(R) := \{((1_{\frac{1}{-1}}) g (1_{\frac{1}{-1}}), (1_{\frac{1}{1}})) : t \in R, g \in SL_2(R)\}. \]

\[ \square \]

5. Local functions

In this section, we define the local integrals required to state our summation formula and prove some of their basic properties. Let \( F \) be a local field of characteristic zero. We use the conventions on Schwartz spaces explained in §2. For each of the 5 orbits of \( G(F) \) in \( X^0(F) \) given in Lemma 4.2, we will define a family of integrals.

For \( f = f_1 \otimes f_2 \in S(X(F)) \otimes S(V(F)) \), let

\[ I(f) (y) = \int_{G_{\gamma_b}(F) \setminus G(F)} f_1(\gamma_b g) \rho(g) f_2(y) dg, \quad y \in Y^{sm}(F) \]

(5.0.1)

\[ I_0(f) (y) = \int_{N_3(F) \setminus G(F)} f_1(g) \rho(g) f_2(y) dg, \quad y \in \tilde{Y}_0(F). \]

Here the stabilizers \( G_{\gamma} \) are computed in Lemma 4.2. These are integrals attached to the \( G(F) \)-orbit of \( \gamma_b \) and \( \gamma_0 = \text{Id} \), respectively.

Let

\[ \Delta_i : SL_2 \longrightarrow G \]

be defined by

\[ \Delta_i(h) := \begin{cases} (I_2, h, (1_{\frac{1}{-1}}) h (1_{\frac{1}{-1}})) & \text{for } i = 1, \\
((1_{\frac{1}{-1}}) h (1_{\frac{1}{-1}}), I_2, h) & \text{for } i = 2, \\
(h, (1_{\frac{1}{-1}}) h (1_{\frac{1}{-1}}), I_2) & \text{for } i = 3. \end{cases} \]

(5.0.2)

Moreover let

\[ p_i : G(R) \longrightarrow SL_2(R) \]

\[ (g_1, g_2, g_3) \mapsto g_{i+1} \]

(5.0.3)

where the indices are taken modulo 3 in the obvious sense.
We need one more piece of data to define the integrals attached to the other orbits. Let \( \Phi \in \mathcal{S}(F^2) \). For \( y \in \tilde{Y}_i(F) \), \( 1 \leq i \leq 3 \) and \( s \in \mathbb{C} \) with \( \text{Re}(s) > 0 \), we define
\[
(5.0.4) \quad I_i(f \otimes \Phi)(v, s)
= \int_{G_{\gamma_i}(F) \backslash G(F)} f_1(\gamma_i g) \int_{N_2(F) \backslash \text{SL}_2(F)} \int_{F^\times} \rho(\Delta_i(h)g) f_2(v) \Phi(x(0,1)hp_i(g)) |x|^{2s} dx \times dh dg.
\]

In \( \S 8 \) we will compute these integrals in the unramified setting. We prove that these integrals are absolutely convergent and bound them in the nonarchimedean case in \( \S 7 \) and in the archimedean case in \( \S 9 \). We observe that in the nonarchimedean case \( I(f) \), \( I_0(f) \), and \( I(f \otimes \Phi) \) are defined for all \( f \in \mathcal{S}(X(F) \times V(F)) \) by bilinearity, and in the archimedean case the same is true using the estimates in \( \S 9 \).

5.1. The Schwartz space of \( Y \). In Propositions 7.1 and 9.3, we will show that \( I(f) \) in fact extends to a smooth function on \( Y^{\text{sm}}(F) \). With this in mind, we define
\[
(5.1.1) \quad \mathcal{S}(Y(F)) := \text{Im}(I : \mathcal{S}(X(F) \times V(F)) \to C^\infty(Y^{\text{sm}}(F))).
\]

This is the Schwartz space of \( Y(F) \). We observe that [GL19, Lemma 4.3] implies in particular that the natural action of \( H(F) \) on \( C^\infty(Y^{\text{sm}}(F)) \) preserves \( \mathcal{S}(Y(F)) \).

**Lemma 5.1.** Let \( F \) be an archimedean local field. The kernel of the map
\[
I : \mathcal{S}(X(F) \times V(F)) \to C^\infty(Y^{\text{sm}}(F))
\]

is closed.

**Proof.** For any \( N \geq 0 \), the Cauchy-Schwartz inequality implies that \( |I(f_1 \otimes f_2)|(y) \) is bounded by the product of the square-roots of the following two integrals:
\[
(5.1.2) \quad \int_{G_{\gamma_0}(F) \backslash G(F)} |f_1|^2(\gamma_0 g) \max(|\gamma_0 g|, 1)^{2N} dg,
\]
\[
(5.1.3) \quad \int_{G_{\gamma_0}(F) \backslash G(F)} \max(|\gamma_0 g|, 1)^{-2N} |\rho(g)f_2|^2(y) dg.
\]

Now \( G_{\gamma_0}(F) \backslash G(F) \) is dense in \( X^0(F) \) and hence the right \( \text{Sp}_0(F) \)-invariant positive Radon measure on \( X^0(F) \) agrees with \( dg \), at least after scaling by a positive real constant. We continue to denote this measure on \( X^0(F) \) by \( dg \). Thus (5.1.2) is equal to
\[
(5.1.4) \quad \int_{X^0(F)} |f_1|^2(g) \max(|g|, 1)^{2N} dg.
\]

Let \( \nu_{d_4,N+1,0} \) be defined as Lemma 3.5, where \( \text{Id} \) is the identity in \( U(g) \). Using the decomposition of the measure \( dg \) afforded by the Iwasawa decomposition and Lemma 3.5, we see that (5.1.4) is bounded by \( \|f_1\|^2_2 + c^2 \nu_{d_4,N+1,0}(f_1)^2 \), where \( c \) is a positive constant independent of \( f_1 \).
On the other hand, by the Iwasawa decomposition \((5.1.3)\) equals
\[
\int_{(F^\times)^3 \times F} \max(m(t, a), 1)^{-2N} \int_{K^3} |\rho(k)f_2|^2 \langle a^{-1}y \rangle dk \left( \prod_{i=1}^3 |a_i|^{2-d_i} \right) d^x adt.
\]
Here
\[
(5.1.5) \quad m(t, a) := \max(|ta_1a_2a_3|, |a_1|, |a_2|, |a_3|, |a_1^{-1}a_2a_3|, |a_2^{-1}a_1a_3|, |a_3^{-1}a_1a_2|)
\]
(see the proof of [GL19, Proposition 7.1] for more details). Taking \(N \) sufficiently large and applying Lemma 9.2 in the special case \(D = \text{Id}, r = e_i = 0\) we deduce that the linear form \(f \mapsto I(f)(y)\) is continuous for every \(y \in Y^{\text{sm}}(F)\). The kernel in the statement of the lemma is the intersection of the kernels of these continuous linear forms.  

We endow \(S(Y(F)) = S(X(F) \times V(F))/\ker I\) with the quotient topology (which is Fréchet).

The integrals \(I(f)\) depend on the choice of additive character \(\psi\) used to define the Weil representation \(\rho_\psi\). We write \(I_\psi(f)\) for \(I(f)\) defined using the Weil representation \(\rho_\psi\). Let \(\gamma(\mathcal{Q}, \psi) := \prod_{i=1}^3 \gamma(\mathcal{Q}_i, \psi)\) be the product of the Weil indices.

**Lemma 5.2.** Let \(c \in F^\times\) and \(\psi_c(x) := \psi(cx)\). Then
\[
I_{\psi_c}(f_1 \otimes f_2)(y) = \frac{\gamma(\mathcal{Q}_c, \psi_c)}{\gamma(\mathcal{Q}, \psi)} |c|^{-2 + \sum_{i=1}^3 d_i/2} I_\psi \left( L(m(c^{-1})) R(c^t I_3) f_1 \otimes f_2 \right) (cy).
\]
In particular, the Schwartz space \(S(Y(F))\) is independent of the choice of \(\psi\).

**Proof.** Let \(B_2 \leq \text{SL}_2\) be the Borel subgroup of upper triangular matrices and let
\[
w_0 = ((-1^1), (-1^1), (-1^1)) \in \text{SL}_2^3(F).
\]
Since \(N^3_2(F)w_0B_2^3(F)\) is dense in \(\text{SL}_2^3(F)\), we have
\[
I_{\psi_c}(f_1 \otimes f_2)(y) = \int_{F^3 \times F^3 \times (F^\times)^3} f_1(\gamma_b(1^1) w_0(1^1) (a_{-1})) \rho_{\psi_c}(1^1) w_0(1^1) (a_{-1})) f_2(y) dtdxd^xa.
\]
Observe that
\[
\rho_{\psi_c}(1^1) w_0(1^1) (a_{-1})) f_2(y)
\]
\[
= \psi(ct \mathcal{Q}(y)) \gamma(\mathcal{Q}, \psi_c) \int_{V(F)} \rho_{\psi_c}(1^1) (a_{-1}) f_2(u) \prod_{i=1}^3 \psi(cu^t I_3 y_i) du_i
\]
\[
= \psi(c^{-1} t \mathcal{Q}(cy)) \gamma(\mathcal{Q}, \psi_c) \int_{V(F)} \rho_\psi(a_{-1}) f_2(u) \prod_{i=1}^3 \psi(cx_i \mathcal{Q}_i(u_i)) \psi(u^t_i I_3 c y_i) du_i
\]
\[
= |c|^{\sum_{i=1}^3 d_i/2} \gamma(\mathcal{Q}, \psi_c) \gamma(\mathcal{Q}, \psi)^{-1} \rho_\psi(1^1 c^{-1} t) w_0(1^1 (a_{-1})) f_2(cy).
\]
Here \(J_i\) is the matrix of \(\mathcal{Q}_i\). The factor of \(|c|^{\sum_{i=1}^3 d_i/2}\) appears because we have to renormalize the self-dual Haar measures with respect to \(\psi_c\) so that they are self-dual with respect
to $\psi$. Taking a change of variables $t \mapsto ct$, $x_i \mapsto c^{-1}x_i$, we see that $I_{\psi\chi}(f_1 \otimes f_2)(y)$ is $|c|^{\frac{d_1+1}{2}}\gamma(\mathcal{Q}, \psi_c)\gamma(\mathcal{Q}, \psi)^{-1}$ times

$$|c|^{-2} \int_{F \times F^3 \times (F \times)^3} f_1(\gamma_b (1 c t) w_0 (1 c^{-1} x) (a^{-1} a)) \rho_\psi ((1 c -1) w_0 (1 x^{-1} x) (a^{-1} a)) f_2(cy) dtdxdx a$$

$$= |c|^{-2} \int_{F \times F^3 \times (F \times)^3} f_1(\gamma_b (1 c t) (1 x) w_0 (1 x^{-1} x) (a^{-1} a)) \rho_\psi ((1 c^{-1} x) w_0 (1 x^{-1} x) (a^{-1} a)) f_2(cy) dtdxdx a$$

$$= |c|^{-2} I_{\psi\chi}(L(m(c^{-1}))) R(\mathcal{F}_L) f_1 \otimes f_2(cy).$$

The fact that the Schwartz space is preserved now follows from Lemma 3.3 and [GL19, Lemma 4.3].

For $F$ archimedean or nonarchimedean, let

$$\mathcal{S} := \text{Im}(\mathcal{S}(V(F)) \longrightarrow C^\infty(Y^{\text{sm}}(F)))$$

where the implicit map is restriction of functions. We observe that $C_c^\infty(Y^{\text{sm}}(F)) < \mathcal{S}$. Moreover, we have the following result:

**Lemma 5.3.** One has

$$\mathcal{S} = \text{Im} \left( I : C_c^\infty(x_0 \text{SL}_2^3(F)) \otimes \mathcal{S}(V(F)) \longrightarrow C^\infty(Y^{\text{sm}}(F)) \right)$$

$$= \text{Im} \left( I : \mathcal{S}(x_0 \text{SL}_2^3(F) \times V(F)) \longrightarrow C^\infty(Y^{\text{sm}}(F)) \right),$$

where the tensor product is algebraic. In particular, $\mathcal{S} < \mathcal{S}(Y(F))$.

**Proof.** For $f \in \mathcal{S}(V(F))$, choose $\Phi_i \otimes f_i \in C_c^\infty(G(F)) \otimes \mathcal{S}(V(F))$ for $1 \leq i \leq n$ such that

$$\sum_{i=1}^n \int_{G(F)} \Phi_i(g) \rho(g) f_i dg = f.$$ 

In the nonarchimedean case it is clear that this is possible, and in the archimedean case it follows from a well-known theorem of Dixmier-Malliavin. Then for $y \in Y(F)$ one has

$$(5.1.6) \sum_{i=1}^n \int_{G_\gamma(F) \setminus G(F)} \int_{G_\gamma(F)} \Phi_i(ng) dn \rho(g) f_i(y) dg = f(y).$$

Let $\gamma_b G(F) \cong G_\gamma(F) \setminus G(F)$ be the orbit of $\gamma_b$ in $X(F)$; it is open and dense in $X^0(F)$. Since $C_c^\infty(X^0(F)) < \mathcal{S}(X(F))$ by Proposition 3.13, we have $C_c^\infty(\gamma_b G(F)) < \mathcal{S}(X(F))$ and hence

$$C_c^\infty(\gamma_b G(F)) \otimes \mathcal{S}(V(F)) < \mathcal{S}(X(F) \times V(F)).$$

We conclude that

$$\mathcal{S} \leq I (C_c^\infty(\gamma_b G(F)) \otimes \mathcal{S}(V(F))) \leq I (\mathcal{S}(\gamma_b G(F) \times V(F))).$$

Thus to complete the proof it suffices to observe that $I(\mathcal{S}(\gamma_b G(F) \times V(F))) \leq \mathcal{S}$. $\square$
We now revert to the adelic setting, bearing in mind the conventions on Schwartz spaces explained in §2. Let $F$ be a number field. The obvious global analogue of (5.0.1) yields a map

$$S(Y(\mathbb{A}_F)) := \text{Im}(I : S(X(\mathbb{A}_F) \times V(\mathbb{A}_F)) \to C^\infty(Y^{\text{sm}}(\mathbb{A}_F))).$$

To check that this is well-defined, one uses the computation of the **basic function for $Y_{F_v}$**

$$b_{Y,v} := I(b_{X,v} \otimes 1_{V(\mathcal{O}_v)})$$

in Proposition 8.1 below. We define

$$S(Y(F_\infty)) := \text{Im}(I : S(X(F_\infty) \times V(F_\infty)) \to C^\infty(Y^{\text{sm}}(F_\infty))).$$

The map $I$ has closed kernel by a trivial modification of the proof of Lemma 5.1 and we give $S(Y(F_\infty))$ the quotient topology. Hence $S(Y(F_\infty))$ is the projective topological tensor product $\hat{\otimes}_v S(Y(F_v))$. We then have

$$S(Y(\mathbb{A}_F)) = \hat{\otimes}_v S(Y(F_v)) \otimes \bigotimes_{v \neq \infty}' S(Y(F_v))$$

where the restricted tensor product is taken with respect to the $b_{Y,v}$.

### 6. The summation formula

Our goal in this section is to prove our main summation formula, Theorem 1.3, modulo some convergence statements that we prove later in the paper. We require the following assumptions on $f = f_1 \otimes f_2 \in S(X(\mathbb{A}_F) \times V(\mathbb{A}_F))$: There are finite places $v_1, v_2$ of $F$ (not necessarily distinct) such that

\begin{align*}
(6.0.1) & \quad f_1 = f_{v_1} f_{v_2} f^{v_1 v_2} \quad \text{and} \quad f_{v_1} \in C_c^\infty(X^\circ(F_{v_1})), \quad \mathcal{F}_X(f_{v_2}) \in C_c^\infty(X^\circ(F_{v_2})) \\
(6.0.2) & \quad \rho(g) f_2(v) = 0 \quad \text{for} \quad v \notin V^\circ(F), \quad \text{for all} \quad g \in G(\mathbb{A}_F).
\end{align*}

We will also require that $\Phi \in S(\mathbb{A}_F^\times)$ satisfies $\hat{\Phi}(0) = 2\text{Vol}(F^\times \setminus (\mathbb{A}_F^\times)^{-1})^{-1}$, where

\begin{align*}
(6.0.3) & \quad \hat{\Phi}(x,y) = \int_{\mathbb{A}_F^\times} \Phi(t_1, t_2) \psi(x t_1 + y t_2) dt_1 dt_2
\end{align*}

and $(\mathbb{A}_F^\times)^1 < \mathbb{A}_F^\times$ is the subgroup of ideles of norm 1.

We prove Theorem 1.3 in this section assuming the absolute convergence statements given in Propositions 10.2 and 10.5. We will indicate precisely when these propositions are used below. After this section, much of the remainder of the paper is devoted to proving these convergence statements.

Computing formally one has

\begin{align*}
\int_{G(F) \setminus G(\mathbb{A}_F)} \sum_{\gamma \in X(F)} f_1(\gamma g) \Theta f_2(g) dg \\
= \sum_{\gamma \in X^\circ(F) / G(F)} \int_{G_\gamma(F) \setminus G(\mathbb{A}_F)} f_1(\gamma g) \Theta f_2(g) dg
\end{align*}
\[
(6.0.4) \quad = \sum_{\gamma \in X^\circ(F)/G(F)} \int_{G_{\gamma}(A_F) \backslash G(A_F)} f_1(\gamma g) \int_{[G_{\gamma}]} \Theta f_2(g_1 g) dg_1 dg.
\]

The set \(X^\circ(F)/G(F)\) has 5 elements represented by \(\gamma_6, \gamma_i, 1 \leq i \leq 3\) and \(\gamma_0 = \text{Id}\) in the notation of (4.0.11). The stabilizers are given explicitly by Lemma 4.2, and we will use this lemma without further comment below.

We start with the \(\gamma_6\) contribution. It is computed as in the proof of [GL19, Theorem 5.3]:

\[
\int_{G_{\gamma_6}(A_F) \backslash G(A_F)} f_1(\gamma_6 g) \int_{[G_{\gamma_6}]} \Theta f_2(g_1 g) dg_1 dg = \sum_{\xi \in V(F)} I(f)(\xi).
\]

Strictly speaking, the proof of [GL19, Theorem 5.3] assumed \(f_1\) was finite under a maximal compact subgroup of \(\text{Sp}_6(F_\infty)\), but the same proof is valid given our work in §3.

We now turn to the \(\text{Id}\) term. Using the definition of the Weil representation, we have that this term is

\[
\int_{G_{\text{Id}}(A_F) \backslash G(A_F)} f_1(g) \int_{[G_m \times G_m]} \sum_{\xi \in V(F)} \phi_{\xi}(\xi_1 = \xi_2 = \xi_3 = 0) \rho \left( \begin{array}{ccc} a_1 & a_2 & (a_1 a_2)^{-1} \\ a_1 & a_2 & (a_1 a_2)^{-1} \end{array} \right) g \right) f_2(\xi) d\alpha^\times
\]

\[
\int_{G_{\text{Id}}(A_F) \backslash G(A_F)} f_1(g) \int_{\mathbb{A}_F^\times \times \mathbb{A}_F^\times} \sum_{\xi \in Y_0(F)/(F^\times)^2} \phi_{\xi}(\xi_1 = \xi_2 = \xi_3 = 0) \rho \left( \begin{array}{ccc} a_1 & a_2 & (a_1 a_2)^{-1} \\ a_1 & a_2 & (a_1 a_2)^{-1} \end{array} \right) g \right) f_2(\xi) d\alpha^\times
\]

Here \((F^\times)^2\) acts as in (4.0.6).

Thus using Lemma 4.1 we conclude that the above is equal to

\[
\int_{G_{\text{Id}}(A_F) \backslash G(A_F)} f_1(g) \int_{\mathbb{A}_F^\times \times \mathbb{A}_F^\times} \sum_{\xi \in Y_0(F)} \phi_{\xi}(\xi_1 = \xi_2 = \xi_3 = 0) \rho \left( \begin{array}{ccc} a_1 & a_2 & (a_1 a_2)^{-1} \\ a_1 & a_2 & (a_1 a_2)^{-1} \end{array} \right) g \right) f_2(\xi) d\alpha^\times
\]

\[
= \int_{N_2^2(A_F) \backslash G(A_F)} f_1(g) \sum_{\xi \in Y_0(F)} \rho(g) f_2(\xi) dg
\]

\[
= \sum_{\xi \in Y_0(F)} I_0(f)(\xi),
\]

where \(N_2 \leq \text{GL}_2\) is the unipotent radical of the standard Borel subgroup of upper triangular matrices. This formal computation is justified by Proposition 10.2.

We finally turn to the \(\gamma_i, 1 \leq i \leq 3\), terms. Let \(\Phi \in \mathcal{S}(A_F^\times)\) be a function satisfying \(\Phi(0) = 2\text{Vol}(F^\times/(A_F^\times)^1)^{-1}\). We prove in Proposition 10.5 below that the sum

\[
(6.0.5) \quad \sum_{\xi \in Y_1(F)} I_i(f \otimes \Phi)(\xi, s)
\]
converges absolutely and defines a holomorphic function of $s$ for $\text{Re}(s)$ sufficiently large. Moreover, we show that it admits a meromorphic continuation to the $s$ plane and its residue at $s = 1$ is

$$\int_{G_{\gamma}(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} f_1(\gamma i g) \int_{[G_{\gamma}]} \sum_{\xi \in V(F)} \rho(hg) f_2(\xi) dh dg.$$ 

Thus altogether we have shown that

$$\sum_{\gamma \in X(F)} f_1(\gamma g) \Theta f_2(g) dg = \sum_{\xi \in Y_0(F)} I(f) + \sum_{\xi \in Y_0(F)} I_0(f) + \text{Res}_{s=1} \sum_{i=1}^3 \sum_{\xi \in Y_i(F)} I_i(f \otimes \Phi)(\xi, s).$$

On the other hand by Theorem 3.12

$$\int_{G(F)\backslash G(\mathbb{A}_F)} \sum_{\gamma \in X(F)} f_1(\gamma g) \Theta f_2(g) dg = \int_{G(F)\backslash G(\mathbb{A}_F)} \sum_{\gamma \in X(F)} F_\chi(f_1)(\gamma g) \Theta f_2(g) dg.$$ 

Replacing $f_1$ by $F_\chi(f_1)$ in the argument above we see that this is

$$\sum_{\xi \in Y(F)} I(F_\chi(f))(\xi) + \sum_{\xi \in Y_0(F)} I_0(F_\chi(f))(\xi) + \text{Res}_{s=1} \sum_{i=1}^3 \sum_{\xi \in Y_i(F)} I_i(F_\chi(f) \otimes \Phi)(\xi, s).$$

Thus assuming the absolute convergence statements in Propositions 10.2 and 10.5 we have proved Theorem 1.3.

7. Bounds on integrals in the nonarchimedean case

Assume that $F$ is a characteristic zero nonarchimedean local field. For $v_i \in V_i(F) = F^{d_i}$, we let

$$(7.0.1) \quad \text{ord}(v_i) \quad (\text{resp. } |v_i|)$$

be the minimum of the orders (resp. maximum of the norms) of the entries of $v_i$ with respect to the standard basis on $V_i(F)$. Thus $|v_i| = q^{-\text{ord}(v_i)}$. We have natural induced bases on $V_i(F) \otimes V_j(F)$ and $V_1(F) \otimes V_2(F) \otimes V_3(F)$ and we define $|v_i \otimes v_j|$, etc., similarly.

Fix functions

$$(f = f_1 \otimes f_2, \Phi) \in \mathcal{S}(X(F) \times V(F)) \times \mathcal{S}(F^2).$$

We bound the integrals attached to these functions that appeared in the proof of Theorem 1.3. These bounds will be used to deduce the absolute convergence statements of Propositions 10.2 and 10.5 below. All implicit constants in this section are allowed to depend on $f \otimes \Phi$.

First we pause to justify an assertion made in §5.1:

**Proposition 7.1.** We have $\mathcal{S}(Y(F)) \subset C^\infty(Y^{sm}(F))$. 
Proof. Fix \( v = (v_1, v_2, v_3) \in Y^{sm}(F) \) and let \( v' \in Y^{sm}(F) \). By symmetry we can assume that \( |v_2||v_3| \neq 0 \). It suffices to show
\[
\int_{G_{\gamma_0}(F) \backslash G(F)} |f_1(\gamma_0 g)||\rho(g)f_2(v) - \rho(g)f_2(v')|dg
\]
is zero for \( |v - v'| \) sufficiently small. We can choose \( \kappa_v \in \mathbb{R}_{>0} \) such that if \( |v - v'| < \kappa_v \) then \( |v'_i| = |v_i| \) for \( 2 \leq i \leq 3 \), and \( |v_1| = |v'_1| \) if \( v_1 \neq 0 \). For the remainder of the proof we assume \( |v - v'| < \kappa_v \). By the Cauchy-Schwartz inequality and Lemma 3.5, the integral above is bounded by
\[
\|f_1\|_2 \left( \int_{G_{\gamma_0}(F) \backslash G(F)} 1_{\geq c}(\gamma_0 g)|\rho(g)f_2(v) - \rho(g)f_2(v')|^2dg \right)^{1/2}
\]
for some \( c \in \mathbb{Z} \).

By the Iwasawa decomposition, it suffices to show that for all \( c \in \mathbb{Z} \) the integral
\[
(7.0.2) \quad \int_{m(t,a) \leq q^{-c}} \left( \int_{K^3} |\rho(k)f_2(a^{-1}v) - \psi(-tQ(v) + tQ(v'))\rho(k)f_2(a^{-1}v')|^2dk \right) \frac{d^x adt}{\prod_{i=1}^3 |a_i|^{d_i-2}}
\]
is zero for \( |v - v'| \) sufficiently small, where \( m(t,a) \) is defined as in (5.1.5). The integral (7.0.2) is supported in the set of \( a_i \) such that \( |v_i|q^{-N} \leq |a_i| \leq q^{-c} \) for each \( i \) for some \( N \) depending on \( f_2 \). Since \( m(t,a) \leq q^{-c} \), we have additionally \( q^{-2N+c}|v_2||v_3| \leq q^c|a_2||a_3| \leq |a_1| \leq q^{-c} \). We have assumed that \( |v_2||v_3| \neq 0 \); hence the support of the integral, as a function of \( a \), lies in a compact subset of \((F^\times)^3\) independent of \( v' \) (since \( |v - v'| < \kappa_v \)). Thus the integral over \( t \) has support in a set that is independent of \( v' \). In particular, if \( |v - v'| \) is sufficiently small, then \( \psi(-tQ(v) + tQ(v')) = 1 \), so (7.0.2) becomes
\[
\int_{m(t,a) \leq q^{-c}} \left( \int_{K^3} |\rho(k)f_2(a^{-1}v) - \rho(k)f_2(a^{-1}v')|^2dk \right) \left( \prod_{i=1}^3 |a_i|^{2-d_i} \right) d^x adt.
\]
Since the vector space \( \langle \rho(k)f_2 \rangle_{k \in K} \) is finite dimensional, and the integral over \( a \) is supported in a compact set independent of \( v' \) (since \( |v - v'| < \kappa_v \)), for \( v' \) close enough to \( v \), the integral above vanishes.

Proposition 7.2. For \( v \in V^c(F) \), one has
\[
\int_{N_2^\circ(F) \backslash G(F)} |f_1(g)\rho(g)f_2(v)|dg \ll \prod_{i=1}^3 |v_i|^{-d_i/2}.
\]
The integral is supported in the set of \( v \) satisfying \( |v_1 \otimes v_2 \otimes v_3| \ll 1 \). The function \( I_0(f)(v) \) satisfies the same bounds on its magnitude and support.
Proof. We decompose the Haar measure using the Iwasawa decomposition to see that the integral in the proposition is equal to
\[
\int_{N_2(F)\backslash G(F)} |f_1(g)\rho(g)f_2(v)| \, dg
\]
\[
= \int_{(F^\times)^3 \times K^3} |f_1((a^{-1})^k)\rho((a^{-1})^k)f_2(v)| \left( \prod_{i=1}^{3} |a_i|^{2d_i} \, d^\times a_i \right) \, dk
\]
(7.0.3)
\[
= \int_{(F^\times)^3 \times K^3} |f_1((a^{-1})^k)\rho(k)f_2(a^{-1}v)| \left( \prod_{i=1}^{3} |a_i|^{2-d_i/2} \right) \, d^\times a_i \, dk.
\]
Now
\[
|((a^{-1})^k)| = |a_1a_2a_3|.
\]
By Lemma 3.5, (7.0.3) is bounded by a constant times
(7.0.4)
\[
\int_{|a_1a_2a_3| \ll 1} \tilde{f}_2(a^{-1}v) \left( \prod_{i=1}^{3} |a_i|^{-d_i/2} \right) \, d^\times a_i,
\]
where
(7.0.5)
\[
\tilde{f}_2(v) := \int_{K^3} |\rho(k)f_2(v)| \, dk.
\]
Since \(\tilde{f}_2\) is compactly supported, we have that \(|v_i| \ll_{f_2} |a_i|\) for \(1 \leq i \leq 3\). Therefore (7.0.4) is bounded by a constant times
\[
\prod_{i=1}^{3} \int_{|v_i| \ll |a_i|} |a_i|^{-d_i/2} \, d^\times a_i \ll \prod_{i=1}^{3} |v_i|^{-d_i/2}.
\]
Moreover, the support of (7.0.4) as a function of \(v\) satisfies \(|v_1||v_2||v_3| \ll |a_1a_2a_3| \ll 1\), as claimed.

Proposition 7.3. For \(r > 0\), as a function of \(v \in V^\circ(F)\), the integral
\[
\int_{(N_2(F)\times \Delta_1(SL_2(F))\backslash G(F)} |f_1(\gamma_1 g)|
\]
\[
\times \int_{N_2(F)\backslash SL_2(F)} \int_{F^\times} |\rho((I_2, h, \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix})g) f_2(v)\Phi(x(0, 1)hp_1(g))| |x|^{2r} \, d^\times x \, dh \, dg
\]
has support in \(|v_1| \ll 1\). It is bounded by a constant times
\[
C'^{r} \zeta(2r)\zeta(2r + d_2/2 - 1)|v_1|^{-d_1/2} |v_3|^{1-d_3/2} \max(|v_1||v_3|, |v_2|)^{1-d_2/2-2r}
\]
\[
\times (C + \max(0, \ord(v_1 \times v_3) - \ord(v_2)) + \zeta(2r + d_2/2 - 1)).
\]
for some constant \(C, C' > 0\). If \(r = \Re(s)\), the function \(I_1(f \otimes \Phi)(v, s)\) satisfies the same bounds on its magnitude and support.
Proof. The integral in the proposition is equal to
\[
\int_{N_2(F)\backslash SL_2(F)} \int_{N_2(F)\backslash SL_2(F) \times SL_2(F)} |f_1 (\gamma_1(g_1, g_2, I_2)) | \\
\times \int_{F^\times} |\rho ((g_1, h g_2, (1 -1) h (1 -1))) f_2(v) \Phi(x(0,1)h g_2)||x|^{2r}d^x dxg_1dg_2dh.
\]
We change variables $g_2 \mapsto h^{-1}g_2$ to see that this is
\[
\int_{N_2(F)\backslash SL_2(F)} \int_{N_2(F)\backslash SL_2(F) \times SL_2(F)} |f_1 (\gamma_1(g_1, h^{-1}g_2, I_2)) | \\
\times \int_{F^\times} |\rho ((g_1, g_2, (1 -1) h (1 -1))) f_2(v) \Phi(x(0,1)g_2)||x|^{2r}d^x dxg_1dg_2dh.
\]
Since $\Delta_1(SL_2(F))$ is in the stabilizer of $\gamma_1$, this is
\[
\int_{N_2(F)\backslash SL_2(F)} \int_{N_2(F)\backslash SL_2(F) \times SL_2(F)} |f_1 (\gamma_1(g_1, g_2; (1 -1) h (1 -1))) | \\
\times \int_{F^\times} |\rho ((g_1, g_2, (1 -1) h (1 -1))) f_2(v) \Phi(x(0,1)g_2)||x|^{2r}d^x dxg_1dg_2dh.
\]
Now decomposing the Haar measure using the Iwasawa decomposition we see that this is
\[
\int_{(F^\times)^3 \times F \times K^3 \times F^\times} |f_1 (\gamma_1 (\left( \begin{array}{cc} a_1^{-1} & a_1 \\ 1 & 1 \end{array} \right) k_1, \left( \begin{array}{cc} a_2^{-1} & a_2 \\ 1 & 1 \end{array} \right) k_2, \left( \begin{array}{cc} a_3^{-1} & a_3 \\ 1 & 1 \end{array} \right) k_3 ))| \\
\times |\rho (\left( \begin{array}{cc} a_1^{-1} & a_1 \\ 1 & 1 \end{array} \right) k_1, \left( \begin{array}{cc} a_2^{-1} & a_2 \\ 1 & 1 \end{array} \right) k_2, \left( \begin{array}{cc} a_3^{-1} & a_3 \\ 1 & 1 \end{array} \right) k_3 )) f_2(v) \Phi((0, xa_2)k_2)| \\
\times |x|^{2r}|a_1a_2a_3|^{2r}d^x xtdx \alpha_1 \alpha_2 \alpha_3 dk_1dk_2dk_3.
\]
Now
\[
|\gamma_1 (\left( \begin{array}{cc} a_1^{-1} & a_1 \\ 1 & 1 \end{array} \right) k_1, \left( \begin{array}{cc} a_2^{-1} & a_2 \\ 1 & 1 \end{array} \right) k_2, \left( \begin{array}{cc} a_3^{-1} & a_3 \\ 1 & 1 \end{array} \right) k_3 ))| \\
= \max(|a_1|, |a_1a_2a_3^{-1}|, |a_1a_2^{-1}a_3|, |a_1a_2a_3t|) \\
=: m'(t,a).
\]
Taking a change of variable $x \mapsto xa_2^{-1}$, by Lemma 3.5 the integral above is bounded by a constant times
\[
q^{2nr} \zeta(2r) \int_{(F^\times)^3 \times F \times m'(t,a) \ll 1} m'(t,a)^{-2r} \tilde{f}_2(a^{-1}v)dt|a_2|^{-2r} \prod_{i=1}^{3} |a_i|^{2-d_i/2}d^x a_i
\]
for some $n \in \mathbb{Z}$, where $\tilde{f}_2$ is defined as in (7.0.5). For some $N \in \mathbb{Z}_{\geq 0}$ sufficiently large, we can write the integral here as
\[
\sum_{k=-N}^{\infty} q^{2k} \int \tilde{f}_2(a^{-1}v)dt|a_2|^{-2r} \prod_{i=1}^{3} |a_i|^{2-d_i/2}d^x a_i
\]
where the integral is over \( t, a \) such that \( m'(t, a) = q^{-k} \). This is bounded by

\[
\sum_{k=-N}^{\infty} \int q^{k} \tilde{f}_2(a^{-1}v) |a_2|^{-2r} \prod_{i=1}^{3} |a_i|^{1-d_i/2} d^x a_i,
\]

where the integral is now over \( a \) such that \( m'(a) := \max(|a_1|, |a_1 a_2 a_3^{-1}|, |a_1 a_2^{-1} a_3|) \leq q^{-k} \).

Taking a change of variables \( a_1 \mapsto \varpi^k a_1 \), one arrives at

\[
(7.0.6) \quad \sum_{k=-N}^{\infty} \int q^{kd_1/2} \tilde{f}_2 \left( \frac{v_1}{\varpi^k a_1}, \frac{v_2}{a_2}, \frac{v_3}{a_3} \right) |a_2|^{-2r} \prod_{i=1}^{3} |a_i|^{1-d_i/2} d^x a_i
\]

where the integral is now over \( a_1, a_2, a_3 \) such that

\[
1 \geq \max(|a_1|, |a_1 a_2 a_3^{-1}|, |a_1 a_2^{-1} a_3|).
\]

The bound on the support as a function of \( v_1 \) is now obvious. We also deduce that if \( a \) is in the support of the integral for a given \( v \), then

\[
|v_3| \ll |a_3| \leq |a_1|^{-1} |a_2| \ll |v_1|^{-1} |a_2|.
\]

Thus for some \( C, C' > 0 \) depending on \( f_2 \), (7.0.6) is bounded by a constant times

\[
|v_1|^{-d_1/2} \int_{|v_1| \ll |a_3| \ll |v_1|^{-1} |a_2|, |v_2| \ll |a_2|} |a_3|^{1-d_3/2} |a_2|^{1-d_2/2-2r} d^x a_3 d^x a_2
\]

\[
\ll_{d_3} |v_1|^{-d_1/2} |v_3|^{1-d_3/2} \max(|v_1||v_3|, |v_2|) \left( C + \ord(v_1 \otimes v_3) - \ord(a_2) \right) |a_2|^{1-d_2/2-2r} d^x a_2
\]

\[
\leq |v_1|^{-d_1/2} |v_3|^{1-d_3/2} C^{1-d_2/2-2r} \max(|v_1||v_3|, |v_2|)^{1-d_2/2-2r}
\]

\[
\times \int_{1 \leq |a_2|} \left( C + \ord(v_1 \otimes v_3) - \min(\ord(v_1 \otimes v_3), \ord(v_2)) - \ord(a_2) \right) |a_2|^{1-d_2/2-2r} d^x a_2
\]

\[
\leq |v_1|^{-d_1/2} |v_3|^{1-d_3/2} C^{1-d_2/2-2r} \max(|v_1||v_3|, |v_2|)^{1-d_2/2-2r} \zeta(2r + d_2/2 - 1) \times (C + \max(0, \ord(v_1 \otimes v_3) - \ord(v_2)) + \zeta(2r + d_2/2 - 1)).
\]

8. The unramified calculation

For this section, \( F \) is a local field of residual characteristic \( p \) with ring of integers \( \mathcal{O} \) that is unramified over \( \mathbb{Q}_p \). We let \( \psi : F \to \mathbb{C}^\times \) be an unramified nontrivial character. Let

\[
(8.0.1) \quad \chi_{\mathcal{O}}(a_1, a_2, a_3) := \chi_{\mathcal{O}_1}(a_1) \chi_{\mathcal{O}_2}(a_2) \chi_{\mathcal{O}_3}(a_3)
\]

where \( \chi_{\mathcal{O}} \) is the (quadratic) character attached to \( \mathcal{O} \), as in [GL19, §3.1]. We assume that \( \chi_{\mathcal{O}} \) is unramified and \( 1_{V(\mathcal{O})} \) is fixed under \( \rho(K^3) \) where \( K = \text{SL}_2(\mathcal{O}) \). Recall \( 1_c \) defined in (3.1.10) and the basic function

\[
b_X := \sum_{j,k=0}^{\infty} q^{2j} 1_{k+2j}.
\]
In this section we give formulae for the unramified functions $I(b_X \otimes 1_{V(O)})(v), I_0(b_X \otimes 1_{V(O)})(v), I_i(b_X \otimes 1_{V(O)} \otimes 1_{\Omega^2})(v, s)$ for $1 \leq i \leq 3$.

8.1. The open orbit. For the reader’s convenience, we state the formula for

\[(8.1.1) \quad b_Y := I(b_X \otimes 1_{V(O)})\]

given by [GL19, Proposition 6.3]:

**Proposition 8.1.** For $v \in Y^{sm}(F)$, one has

\[b_Y(v) = \sum_{j=0}^{\infty} \int \mathbb{1}_O \left( \frac{Q(v)}{a_1a_2a_3a^4j} \right) \mathbb{1}_V(O) \left( \frac{v}{a(2)^j} \right) \chi_Q(a) \prod_{i=1}^{3} \left( \frac{|a_i|}{q^{2j}} \right)^{1-d_i/2} d^\times a\]

where the integral is over $a_1, a_2, a_3 \in \mathcal{O}$ satisfying

\[\max(|a_1^{-1}a_2a_3|, |a_2^{-1}a_1a_3|, |a_3^{-1}a_1a_2|) \leq 1.\]

\[\Box\]

8.2. The identity orbit. By a minor modification of the proof of Proposition 7.2 above, we obtain the following proposition:

**Proposition 8.2.** Suppose $v = (v_1, v_2, v_3) \in \tilde{Y}_0(F)$. Assume moreover that $|v_1| = |v_2| = 1$. One has

\[I_0(b_X \otimes 1_{V(O)})(v) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} q^{2j} \int \mathbb{1}_V(O)(a^{-1}v)\chi_Q(a) \prod_{i=1}^{3} |a_i|^{2-d_i/2} d^\times a_i\]

where the integral is over those $a_1^{-1}, a_2^{-1}, a_3 \in \mathcal{O}$ such that $|a_1a_2a_3| = q^{-k-2j}$. As a function of $v_3$, the integral is supported in $V_3(\mathcal{O})$. Let $\epsilon > 0$. For $v \in V^o(F)$ with $|v_1| = |v_2| = 1$, we have that

\[\int_{N^2_2(F) \setminus G(F)} |b_X (g) \rho (g) \mathbb{1}_V(O)(v)| dg \leq C|v_3|^{-d_3/2-\epsilon} \mathbb{1}_{V_5}(\mathcal{O})(v_3)\]

for some constant $C > 0$ depending on $\epsilon$, which equals 1 for $q$ sufficiently large.

\[\Box\]

8.3. The other orbits. The following assertions can be proved by an easy refinement of the argument proving Proposition 7.3:

**Proposition 8.3.** Suppose $v = (v_1, v_2, v_3) \in \tilde{Y}_1(F)$. For $\text{Re}(s) > 0$ one has

\[I_1(b_X \otimes 1_{V(O)} \otimes 1_{\Omega^2})(v, s) = \zeta(2s) \sum_{j=0}^{\infty} \int \mathbb{1}_O \left( \frac{Q_2(v_2)}{a_1a_2a_3} \right) \mathbb{1}_V(O) \left( \frac{v_1}{a_2^{2j}a_1}, \frac{v_2}{a_2}, \frac{v_3}{a_3} \right) \chi_Q(a) \prod_{i=1}^{3} |a_i|^{1-d_i/2} d^\times a_i\]

where the integral is over $a_1 \in F^\times \cap \mathcal{O}, a_2, a_3 \in F^\times$ such that $|a_1|^{-1} \geq \max (|a_2a_3^{-1}|, |a_2^{-1}a_3|)$.

\[\Box\]
Lemma 8.4. For \( v = (v_1, v_2, v_3) \in V^o(F) \) and \( r > 0 \), the integral
\[
\int_{(N_2(F) \times \Delta_1(SL_2(F))) \setminus G(F)} |b_X(\gamma_1 g)| \bullet \int_{N_2(F) \setminus SL_2(F)} |\rho((I_2, h, (1 - 1) h (1 - 1)) g) \mathbb{1}_{V(\mathcal{O})}(v) \mathbb{1}_{\mathcal{O}^2}((0, x) hp_1(g))| x |^{2r} d^x x dhdg
\]
vanishes unless \( v_1 \in V_1(\mathcal{O}) \). It is bounded by
\[
C \zeta(2r) \zeta(2r + d_2/2 - 1) |v_1|^{-d_1/2} |v_3|^{-d_3/2} \max (|v_1 \otimes v_3|, |v_2|)^{1-d_2/2-2r} \\
\times (\max(0, \text{ord}(v_1 \otimes v_3) - \text{ord}(v_2)) + \zeta(2r + d_2/2 - 1))
\]
for some constant \( C > 0 \) which equals 1 for \( q \) sufficiently large. Thus if \( \text{Re}(s) = r \), the function \( I_1(b_X \otimes \mathbb{1}_{V(\mathcal{O})} \otimes \mathbb{1}_{\mathcal{O}^2})(v, s) \) admits the same bounds on its magnitude and support. □

An expression for the integrals \( I_2 \) and \( I_3 \) and corresponding bounds and supports can be obtained by symmetry.

9. Bounds on integrals in the archimedean case

In this section \( F \) is an archimedean local field. We estimate the local integrals defined in §5. The bounds obtained in this section will be used to prove Propositions 10.2 and 10.5, the absolute convergence statements used in the proof of Theorem 1.3. As usual, the bounds in the archimedean case are slightly harder to prove than in the nonarchimedean case, but the basic outline of the proofs is the same.

We let
\[
|(a_1, \ldots, a_d)| := \max\{|a_j| : 1 \leq j \leq d_i\}
\]
for \((a_1, \ldots, a_d) \in V_i(F)\). We moreover fix
\[
(f = f_1 \otimes f_2, \Phi) \in \mathcal{S}(X(F) \times V(F)) \times \mathcal{S}(F^2)
\]

The following is a rephrasing of [GL19, Lemma 8.1]:

Lemma 9.1. Let \( A, B \in \mathbb{R}_{\geq 0}, C \in \mathbb{R}_{\geq 0} \) and let \( x \in F^\times \). Assume \( A > B \) and \( A \neq B + C \). One has
\[
\int_{F^\times} \max(|a^{-1}x|, 1)^{-A}|a|^{-B} \max(|a|, 1)^{-C} da^x \ll_{A,B,C} \max(|x|, 1)^{-\min(A-B,C)} |x|^{-B}.
\]

This will be used several times below.
9.1. **The open orbit.** Recall that $V'(F) \subset V(F)$ is the subset of vectors $(v_1, v_2, v_3)$ such that no two $v_i$ are zero.

**Lemma 9.2.** Given $r, e_i, N \in \mathbb{R}_{\geq 0}$, $D \in U(\text{Lie}(V(F)))$, let $M : (F^\times)^3 \times F \times K^3 \times V(F) \to \mathbb{R}$ be the function

\[
M(a, t, k, v) := \max(m(t, a), 1)^{-2N|t|^r}|D\rho(k)f_2|^2(a^{-1}v) \left(\prod_{i=1}^{3} |a_i|^{2-d_i-e_i}\right),
\]

where $m(t, a)$ is defined as (5.1.5). For $v \in V'(F)$, there is a compact neighborhood $U$ of $v$, and a continuous integrable function $M'$ on $(F^\times)^3 \times F \times K^3$ such that for all $v' \in U$

\[
M(a, t, k, v') \leq M'(a, t, k).
\]

Moreover, given $N_i \in \mathbb{Z}_{\geq 0}$ there exists a continuous seminorm $\nu'$ on $S(V(F))$ such that

\[
\int_{(F^\times)^3 \times F \times K^3} M(a, t, k, v)dkd^\times dt \leq \nu'(f_2)^2 \left\{ \prod_{i=1}^{3} \max(|v_i|, 1)^{-2N_i|v_i|^{1-d_i-e_i-r}}, \quad \text{if } v \in V^\circ(F), \right.
\]

\[
\left. \prod_{i \neq j} \max(|v_i|, 1)^{-2N_i|v_i|^{2-d_i-e_i-2r-d_j-e_j}}, \quad \text{if } v_j = 0, \right\}
\]

provided $N \geq 5 \max\{N_i, d_i + e_i + r\}$.

**Proof.** By the continuity of Weil representation and compactness of $K$, for any $C_1, C_2, C_3 \in \mathbb{Z}_{\geq 0}$, there exists a continuous seminorm $\nu_{D,C_1,C_2,C_3}$ on $S(V(F))$ such that for all $(k, v) \in K^3 \times V(F)$ we have

\[
|D\rho(k)f_2|(v) \leq \nu_{D,C_1,C_2,C_3}(f_2) \left(\prod_{i=1}^{3} \max(|v_i|, 1)^{-C_i}\right).
\]

Let $U$ be a compact neighborhood of $v$ such that for $v' \in U$, if $v_i \neq 0$ then $v'_i \neq 0$. Choose $v' \in U$ with minimum norm. Put

\[
M'(a, t, k) := \nu_{D,C_1,C_2,C_3}(f_2)^2 \max(m(t, a), 1)^{-2N|t|^r} \prod_{i=1}^{3} \max(|a_i^{-1}v'_i|, 1)^{-2C_i|a_i|^{2-d_i-e_i}}.
\]

Then $M(a, t, k, v) \leq M'(a, t, k)$ for all $v \in U$. Thus to prove the lemma it suffices to show that for all $v \in V'(F)$ one has

\[
\int_{(F^\times)^3 \times F} \max(m(t, a), 1)^{-2N|t|^r} \left(\prod_{i=1}^{3} \max(|a_i^{-1}v'_i|, 1)^{-2C_i|a_i|^{2-d_i-e_i}}\right) dt \leq \left\{ \prod_{i=1}^{3} \max(|v_i|, 1)^{-2N_i|v_i|^{1-d_i-e_i-r}}, \quad \text{if } v \in V^\circ(F); \right.
\]

\[
\left. \prod_{i \neq j} \max(|v_i|, 1)^{-2N_i|v_i|^{2-d_i-e_i-2r-d_j-e_j}}, \quad \text{if } v_j = 0, \right\}
\]
provided that \( N \geq 5 \max_i \{N_i, d_i + e_i + r\} \). We break the integral into \( m(t, a) \leq 1 \) and \( m(t, a) > 1 \). Suppose \( v \in V^\circ(F) \). In the range \( m(t, a) \leq 1 \), we have \(|t| \leq |a_1a_2a_3|^{-1} \) and \(|a_i| \leq 1 \) for all \( i \). Therefore the integral is bounded by

\[
\int_{|a| \leq 1} \prod_{i=1}^{3} \max(|a_i^{-1}v_i|, 1)^{-2C_i}|a_i|^{1-d_i-e_i-r}d^x a
\]

(9.1.4) \[
\leq \prod_{i=1}^{3} \int_{\mathbb{F}^x} \max(|a_i|, 1)^{-2N_i} \max(|a_i^{-1}v_i|, 1)^{-2C_i}|a_i|^{1-d_i-e_i-r}d^x a_i.
\]

In the range \( m(t, a) > 1 \), applying the inequality

(9.1.5) \[
m(t, a)^4 \geq \max(|ta_1a_2a_3|, 1) \max(|a_1|, 1) \max(|a_2|, 1) \max(|a_3|, 1),
\]

the contribution of this part of the integral is bounded by

\[
\int_{(\mathbb{F}^x)^3} \max(|ta_1a_2a_3|, 1)^{-N/2} |ta_1a_2a_3|^r \left( \prod_{i=1}^{3} \max(|a_i|, 1)^{-N/2} \max(|a_i^{-1}v_i|, 1)^{-2C_i}|a_i|^{2-d_i-e_i-r} \right) d^x a d^x t d^x d
\]

\[
\ll_N \prod_{i=1}^{3} \int_{\mathbb{F}^x} \max(|a_i|, 1)^{-N/2} \max(|a_i^{-1}v_i|, 1)^{-2C_i}|a_i|^{1-d_i-e_i-r}d^x a_i,
\]

provided \( N > 2r + 2 \). Since \( N \geq 4 \max_i N_i \) by assumption the integral is bounded by (9.1.4).

The assertion then follows from Lemma 9.1 by setting \( A = 2C_i, B = r + d_i + e_i - 1, C = 2N_i \) and choosing \( C_i \) so that \( A - B > C \) for each \( i \).

Now assume \( v \in V'(F) - V^\circ(F) \). By symmetry we may assume \( v_1 = 0 \). Let

\[
|a| := \max_{1 \leq i \leq 3} (|a_i|).
\]

If \( m(t, a) \leq 1 \) then \(|t| \leq |a_1a_2a_3|^{-1}, |a| \leq 1, |a_2a_3| \leq |a_1| \). Therefore the contribution of \(|m(t, a)| \leq 1 \) to the integral (9.1.3) is bounded by

(9.1.6) \[
\int_{|a| \leq 1, |a_2a_3| \leq |a_1|} |a_1|^{1-d_1-e_1-r} \prod_{i=2}^{3} \max(|a_i^{-1}v_i|, 1)^{-2C_i}|a_i|^{1-d_i-e_i-r}d^x a_1d^x a_2d^x a_3
\]

\[
\ll \prod_{i=2}^{3} \int_{\mathbb{F}^x} \max(|a_i|, 1)^{-2N_i} \max(|a_i^{-1}v_i|, 1)^{-2C_i}|a_i|^{2-d_i-e_i-2r-d_1-e_1}d^x a_i.
\]

For \( m(t, a) > 1 \) we have the inequality

\[
m(t, a)^5 \geq \max(|ta_1a_2a_3|, 1) \max(|a_1|, 1) \max(|a_2|, 1) \max(|a_3|, 1) \max(|a_1^{-1}a_2a_3|, 1).
\]

Thus the contribution of \( m(t, a) > 1 \) to (9.1.3) is bounded by a constant depending on \( N \) times

(9.1.7) \[
\int_{(\mathbb{F}^x)^3} \max(|a_1^{-1}a_2a_3|, 1)^{-2N/5} \left( \prod_{i=1}^{3} \max(|a_i|, 1)^{-2N/5} \max(|a_i^{-1}v_i|, 1)^{-2C_i}|a_i|^{1-d_i-e_i-r} \right) d^x a
\]
since $N > 5r + 5$. The contribution of $|a_2 a_3| \leq |a_1|$ to (9.1.7) is dominated by (9.1.6) since $N \geq 5 \max_i N_i$. In the range $|a_2 a_3| \geq |a_1|$, one has that
\[
\int_{|a_1| \leq |a_2 a_3|} \max(|a_1|, 1)^{-2N/5} |a_1|^{1-d_1-e_1-r+2N/5} d^x a_1 \ll_{N, e_1, r} \min(|a_2 a_3|, 1)^{1-d_1-e_1-r+2N/5}.
\]
Since $2N/5 \geq d_1 + e_1 + r$, we deduce that the integral (9.1.7) is also dominated by (9.1.6). The assertion now follows from Lemma 9.1 by setting $A = 2C_i$, $B = 2r + d_1 + e_1 + d_i + e_i - 2$, $C = 2N_i$ and choosing $C_i$ so that $A - B > C$ for $i = 2, 3$. \hfill \Box

**Proposition 9.3.** We have $S(Y(F)) < C^\infty(Y^{\text{sm}}(F))$. Moreover, for $f \in S(Y(F))$ and $D \in U(\text{Lie}(V(F)))$,
\[
|Df(v)| \left( \prod_{i=1}^{3} \max(|v_i|, 1)^{N_i} |v_i|^{(d_i + \deg D - 1)/2} \right)
\]
is bounded on $Y^{\text{sm}}(F)$ for all $N_i \in \mathbb{Z}$.

**Proof.** Let $v_0 \in Y^{\text{sm}}(F)$ and $D \in U(\text{Lie}(V(F)))$. Let $\Delta : F \to F^3$ be the diagonal embedding. Using the notation of Lemma 9.2 there is a neighborhood $U$ of $v_0$ such that the expression
\[
(9.1.8) \quad |f_1(\gamma_b(1\Delta(t)) (a^{-1}) a k) D\rho(1\Delta(t)) (a^{-1}) a k) f_2(v)|
\]
is dominated by a finite sum of functions of the form
\[
|f_1(\gamma_b(1\Delta(t)) (a^{-1}) a k)| \max(m(t, a), 1)^N M(a, t, k, v) \frac{1}{2}|a|^{-1}
\]
where $M(a, t, k, v)$ is defined using various parameters $f_2$, $e_i$, $r$ depending on $D$. We recall that
\[
m(t, a) = |\gamma_b(1\Delta(t)) (a^{-1}) a k|
\]
by (5.1.5). Thus applying the Cauchy-Schwartz inequality we have
\[
\int_{F \times F^3 \times K} |f_1(\gamma_b(1\Delta(t)) (a^{-1}) a k) D\rho(1\Delta(t)) (a^{-1}) a k) f_2(v)|
\]
\[
\leq \left( \int_{G_b(F) \setminus G(F)} |f_1(\gamma_b g)| \max(|\gamma_b g|, 1)^N \frac{1}{2} dg \right)^{1/2} \left( \int_{F \times F^3 \times K} M(a, t, k, v) \frac{1}{2} \right)^{1/2}.
\]
The left integral converges by the argument in the proof of Lemma 5.1, and the right converges by Lemma 9.2. To obtain the bound in the lemma one simply keeps track of which parameters $r$ and $e_i$ are required in the argument above in terms of $\deg D$.

To prove that $S(Y(F)) < C^\infty(Y^{\text{sm}}(F))$ we apply the Leibniz integral rule. To justify its application, we require a bound on $\int_{G_b(F) \setminus G(F)} |f_1(\gamma_b g) D\rho(g) f_2(v)| dg$ that is uniform in a small neighborhood of a given $v \in Y^{\text{sm}}(F)$. Choose a compact neighborhood $U$ of $v$ as in
the proof of Lemma 9.2. Then by Lemma 9.2 for \( v \in U \) one has \( M(a, t, k, v) \leq M'(a, t, k) \), defined as in (9.1.2). It suffices to show

\[
\int_{F \times (F \times)^3 \times K} |f_1(\gamma \left( 1, \Delta(\delta) \right) \left( a^{-1} \right) k) \max(m(t, a), 1)^N |M'(a, t, k)^{1/2}|a|^{-1}|a|^2 d^x adtk < \infty.
\]

This follows from (9.1.3) and the argument above.

\[ \square \]

Remark. By mimicking the proof above one can also bound \( Df(v) \) when \( v_i = 0 \) for some \( i \).

9.2. The identity orbit.

**Proposition 9.4.** Let \( v \in V^\circ(F) \). Given a positive integer \( N' \) and \( \epsilon > 0 \), there are continuous seminorms \( \nu \) on \( S(X(F)) \) and \( \nu' \) (depending on \( N', \epsilon \)) on \( S(V(F)) \) such that one has the bound

\[
\int_{N^2(F) \times G(F)} |f_1(g)\rho(g)f(v)|dg \leq \nu(f_1)\nu'(f_2) \max\{ |v_1| |v_2| |v_3|, 1\}^{-N'} \prod_{i=1}^3 |v_i|^{-d_{i}/2-\epsilon}.
\]

The function \( I_0(f)(v) \) admits the same bound.

**Proof.** By symmetry, we may assume \( d_1 \geq d_2 \geq d_3 \). Recall the seminorms \( \nu_{D,N,\beta} \) mentioned in Lemma 3.5. Arguing as in the proof of Proposition 7.2, we see that the integral in the proposition is bounded by \( \max(\nu_{4d,N,0}(f_1), \nu_{4d,0,0}(f_1)) \) times

\[
\int_{(F \times)^3} |a_1a_2a_3|^{-2} \max(|a_1a_2a_3|, 1)^{-N} \tilde{f}_2(a^{-1}v) \prod_{i=1}^3 |a_i|^{-d_{i}/2} d^x a_i,
\]

where \( \tilde{f}_2 \) is defined as in (7.0.5). By the continuity of Weil representation and compactness of \( K \), for any \( N_1, N_2, N_3 \in \mathbb{Z}_{\geq 0} \), there exists a continuous seminorm \( \nu' \) depending on \( N_1, N_2, N_3 \) such that the integral is bounded by \( \nu'(f_2) \) times

\[
\int_{(F \times)^3} \max(|a_1a_2a_3|, 1)^{-N} \prod_{i=1}^3 \max(|a_i^{-1}v_i|, 1)^{-N_i} |a_i|^{-d_{i}/2} da_i^x
\]

\[
= \int_{(F \times)^3} \max(|a_1|, 1)^{-N} \max(|a_1^{-1}(a_2a_3)v_1|, 1)^{-N_1} |a_1|^{-d_{1}/2} |a_2a_3|^{d_{1}/2} \prod_{i=2}^3 \max(|a_i^{-1}v_i|, 1)^{-N_i} |a_i|^{-d_{i}/2} da_i^x
\]

Here we have taken a change of variables \( a_1 \mapsto (a_2a_3)^{-1}a_1 \). For the remainder of the proof all implicit constants are allowed to depend on \( N_1, N_2, N_3, N \), and we assume

\[
(9.2.1) \quad N_i - d_{i}/2 > N_{i-1} - d_{i-1}/2
\]

for each \( i \) where \( N_0 = d_0 = 0 \).

Taking \( N > N_1 + d_{1}/2 \) and applying Lemma 9.1 with \( A = N_1, B = d_{1}/2 \) and \( C = N \) to the \( a_1 \) integral we see that the above is bounded by

\[
\int_{(F \times)^2} \max(|a_2a_3v_1|, 1)^{-N_1+d_{1}/2} |a_2a_3v_1|^{-d_{1}/2} \prod_{i=2}^3 \max(|a_i^{-1}v_i|, 1)^{-N_i} |a_i|^{(d_{1}-d_{i})/2} d^x a_i
\]
Here we have taken a change of variables $a_2 \mapsto a_3^{-1}a_2$. The integral

\begin{equation}
\int_{F_x} \max(|a_2^{-1}a_3v_2|, 1)^{-N_2} \max(|a_3^{-1}v_3|, 1)^{-N_3} |a_3|^{(d_2-d_3)/2} da_2 \times da_3^x.
\end{equation}

breaks into the sum of four integrals

\[
\int_{|a_3| \leq \min\left(\frac{|a_2|}{|v_2|}, |v_3|\right)} + \int_{\min\left(\frac{|a_2|}{|v_2|}, |v_3|\right) < |a_3| \leq \frac{|a_2|}{|v_2|}} + \int_{|a_3| < \max\left(\frac{|a_2|}{|v_2|}, |v_3|\right)} + \int_{\max\left(\frac{|a_2|}{|v_2|}, |v_3|\right) \leq |a_3|}
\]

and this is bounded by a constant (depending only on $N_2, N_3$) times

\[
|v_3|^{-N_3} \min\left(\frac{|a_2|}{|v_2|}, |v_3|\right)^{N_3+(d_2-d_3)/2} + G_{d_2,d_3}(a_2, v_2, v_3) + \left(\frac{|a_2|}{|v_2|}\right)^{N_2} \times \left(\left|v_3\right|^{-N_3} \left(\max\left(\frac{|a_2|}{|v_2|}, |v_3|\right)^{N_3-N_2+(d_2-d_3)/2} - \left(\frac{|a_2|}{|v_2|}\right)^{N_3-N_2+(d_2-d_3)/2}\right) + \max\left(\frac{|a_2|}{|v_2|}, |v_3|\right)^{-N_2+(d_2-d_3)/2}\right)
\]

where

\[
G_{d_2,d_3}(a_2, v_2, v_3) = \begin{cases} \left(\frac{|a_2|}{|v_2|}\right)^{(d_2-d_3)/2} - \min\left(\frac{|a_2|}{|v_2|}, |v_3|\right)^{(d_2-d_3)/2}, & \text{if } d_2 \neq d_3; \\
\log\left(\frac{|a_2|}{|v_2|}\right) - \log \min\left(\frac{|a_2|}{|v_2|}, |v_3|\right), & \text{if } d_2 = d_3.
\end{cases}
\]

Thus (9.2.3) is bounded by a constant times

\[
F(a_2, v_2, v_3) := \begin{cases} \left|v_3\right|^{(d_2-d_3)/2} \left(\frac{|a_2|}{|v_2||v_3|}\right)^{N_2}\left(\frac{|a_2|}{|v_2||v_3|}\right)^{(d_2-d_3)/2}, & \text{if } |a_2| < |v_2||v_3|; \\
\left|v_3\right|^{(d_2-d_3)/2} \left(\frac{|a_2|}{|v_2||v_3|}\right)^{(d_2-d_3)/2}, & \text{if } |a_2| \geq |v_2||v_3| \text{ and } d_2 \neq d_3 \\
1 + \log\left(\frac{|a_2|}{|v_2||v_3|}\right), & \text{if } |a_2| \geq |v_2||v_3| \text{ and } d_2 = d_3.
\end{cases}
\]

Thus the original integral (9.2.2) is bounded by a constant times

\[
|v_1|^{-d_1/2} \int_{F_x} \max(|a_2v_1|, 1)^{-N_1+d_1/2} |a_2|^{-d_2/2} F(a_2, v_2, v_3) da_2 \times
\]

\[
= \left(\prod_{i=1}^{3} |v_i|^{-d_i/2}\right) \int_{F_x} \max(|a_2|, 1)^{-N_1+d_1/2} |a_2|^{-d_2/2} \min(|a_2|, 1)^{N_2} F'(a_2) da_2^x,
\]

where $c = |v_1||v_2||v_3|$, and

\[
F'(a_2) = \begin{cases} \max(|a_2|, 1)^{(d_2-d_3)/2}, & \text{if } d_2 \neq d_3; \\
\log(\max(|a_2|, 1)) + 1, & \text{if } d_2 = d_3.
\end{cases}
\]

Here we have changed variables $a_2 \mapsto a_2(|v_2||v_3|)$. The assertion of the proposition now follows from taking a change of variable $a_2 \mapsto a_2^{-1}$ and applying Lemma 9.1 with $A = N_1 - d_1/2, B = \epsilon < 1/2, C = N_2 - d_2/2 - \epsilon$. \hfill \Box
9.3. The other orbits.

**Proposition 9.5.** Let \( r = \text{Re}(s) > 0 \) and \( N \in \mathbb{Z}_{>0} \), and assume \( N > \max(2r + d_2/2 - 2, d_3/2 - 2) \). For \( v \in V^*(F) \), there are continuous seminorms \( \nu \) on \( S(X(F)) \) and \( \nu' \) on \( S(V(F)) \), depending on \( N \), such that

\[
\int_{(N_2(F) \times \Delta_1(\text{SL}_2(F))) \setminus G(F)} |f_1(\gamma_1 g)| \\
\times \int_{N_2(F) \setminus \text{SL}_2(F)} \int_{\mathbb{F}_x} \rho((I_2, h, (1 - 1) h (1 - 1)) g) f_2(v) \Phi((0, x) hp_1(g)) |x|^{2r} d^x x dhdg
\leq \Psi(2r) \nu(f_1) \nu'(f_2) |v_1|^{N-d_1/2} \max(|v_2|, |v_3|)^{-2r-d_2/2-d_3/2+2}
\]

where \( \Psi : \mathbb{R}_{>0} \to \mathbb{R} \) is an analytic function. The function \( I_1(f \otimes \Phi)(v, s) \) admits the same bound.

**Proof.** By Lemma 3.5 one has \( |f_1(g)| \leq \nu_{t,0}(f_1)|g|^{-2-N} \) for any \( N \geq 0 \). Thus arguing as in Proposition 7.3, we see that the integral is bounded by

\[
(9.3.1)
\nu_{t,0}(f_1) \int_{(F \times)^3 \times F} m'(t, a)^{-2-N} \left( \int_{\mathbb{K}^3 \times F} |\rho(k_1, k_2, k_3) f_2(a^{-1} v) \Phi((0, x) k_2)| |x|^{2r} d^x x dk_1 dk_2 dk_3 \right)
\times dt |a_2|^{-2r} \prod_{i=1}^3 |a_i|^{2-d_i/2} d^x a_i,
\]

where \( m'(t, a) = \max(|a_1|, |a_1 a_2 a_3^{-1}|, |a_1 a_2^{-1} a_3|, |a_1 a_2 a_3 t|) \). For simplicity we assume \( f_2 = \otimes_{i=1}^3 f_{2i} \). The general case merely requires more annoying notation. Applying the Cauchy-Schwartz inequality on the second copy of \( K \), the inner integral is bounded by

\[
\left( \prod_{i=1,3} \int_K |\rho(k_i) f_{2i}(a_i^{-1} v_i)| dk_i \right) \left( \int_K |\rho(k_2) f_{22}(a_2^{-1} v_2)|^2 dk_2 \right)^{1/2}
\times \int_{\mathbb{F}_x} \left( \int_K |\Phi((0, x) k_2)|^2 dk_2 \right)^{1/2} |x|^{2r} d^x x.
\]

The last factor is \( \Psi(2r) \) for an appropriate analytic function \( \Psi : \mathbb{R}_{>0} \to \mathbb{R} \). By the continuity of the Weil representation and compactness of \( K \), for any \( N_1, N_2, N_3 \in \mathbb{Z}_{>0} \), there exists a continuous seminorm \( \nu'_{N_1, N_2, N_3} \) on \( S(V(F)) \) such that the integral (9.3.1) is bounded by \( \Psi(2r) \) times

\[
\nu'_{N_1, N_2, N_3}(f_2) \int_{(F \times)^3 \times F} m''(t, a)^{-N-2} |a_2|^{-2r} \prod_{i=1}^3 \max(|a_i^{-1} v_i|, 1)^{-N_i} |a_i|^{2-d_i/2} da^x dt.
\]

From now on all implicit constants are allowed to depend on \( N_1, N_2, N_3, N \). Let

\[
m''(t, a) := \max(1, |a_2 a_3^{-1}|, |a_2^{-1} a_3|, |a_2 a_3 t|) = \max(|a_2 a_3^{-1}|, |a_2^{-1} a_3|, |a_2 a_3 t|).
\]
We assume without loss of generality that $N_1 > N + d_1/2$. We then write the integral above as the product of

$$\int_{F^\times} |a_1|^{-N-d_1/2} \max(|a_1^{-1}v_1|, 1)^{-N_1} da_1^x \ll |v_1|^{-N-d_1/2}$$

and

$$\int_{(F^\times)^2 \times F} m''(t, a)^{-N-2r} |a_2|^{-2r} \prod_{i=2}^3 \max(|a_i^{-1}v_i|, 1)^{-N_i} |a_i|^{2-d_i/2} da_2^x da_3^x dt. \tag{9.3.2}$$

Now consider (9.3.2). We write it as the sum of

$$\int_{|a_2| \geq |a_3|} |a_2|^{-N-d_2/2-2r} \max(|a_2^{-1}v_2|, 1)^{-N_2}$$

$$\times \int_F \max(|a_3^{-1}|, |a_3t|)^{-N-2} \max(|a_3^{-1}v_3|, 1)^{-N_3} |a_3|^{2-d_3/2} da_3^x da_2^x dt \tag{9.3.3}$$

and

$$\int_{|a_3| > |a_2|} |a_3|^{-N-d_3/2} \max(|a_3^{-1}v_3|, 1)^{-N_3}$$

$$\times \int_F \max(|a_2^{-1}|, |a_2t|)^{-N-2} \max(|a_2^{-1}v_2|, 1)^{-N_2} |a_2|^{2-d_2/2-2r} da_2^x da_3^x dt \tag{9.3.4}$$

Executing the $t$ integral in (9.3.3), we see that it is bounded by a constant times

$$\int_{|a_2| \geq |a_3|} |a_2|^{-N-d_2/2-2r} \max(|a_2^{-1}v_2|, 1)^{-N_2} \max(|a_3^{-1}v_3|, 1)^{-N_3} |a_3|^{2+N-d_3/2} da_3^x da_2^x \tag{9.3.5}$$

$$= \int_{|a_2| \geq |a_3|} |a_2|^{2-d_2/2-d_3/2-2r} \max(|a_2^{-1}v_2|, 1)^{-N_2} \max(|a_3^{-1}a_2^{-1}v_3|, 1)^{-N_3} |a_3|^{2+N-d_3/2} da_3^x da_2^x,$$

where the latter equation is obtained by taking a change of variables $a_3 \mapsto a_2 a_3$. Similarly (9.3.4) is bounded by a constant times

$$\int_{|a_3| > 1} |a_3|^{-N-d_3/2} \max(|a_2^{-1}a_3^{-1}v_3|, 1)^{-N_3} \max(|a_2^{-1}v_2|, 1)^{-N_2} |a_2|^{2-d_2/2-d_s/2-2r} da_3^x da_2^x \tag{9.3.6}$$

Carrying out the integral over $a_3$ directly in (9.3.5) we see that it is bounded by a constant times

$$\int_{F^\times} |a_2|^{2-d_2/2-d_3/2-2r} \max(|a_2^{-1}v_2|, 1)^{-N_2} \max(|a_2^{-1}v_3|, 1)^{-N_3} da_3^x da_2^x.$$

provided $N - d_3/2 + 2 > 0$. This bound is also valid for (9.3.6) provided $N + d_3/2 > N_3$. The integral above is bounded by a constant times

$$\max(|v_2|, |v_3|)^{-2r-d_2/2-d_3/2+2}$$

provided $N_i > 2r + d_2/2 + d_3/2 - 2$ for $i = 2, 3$. \qed
10. Absolute convergence

In this section, we prove the absolute convergence statements that make the proof of the summation formula in §6 rigorous. Fix a number field $F$. For the remainder of the section, we fix

$$(f = f_1 \otimes f_2, \Phi) \in \mathcal{S}(X(\mathbb{A}_F) \times V(\mathbb{A}_F)) \times \mathcal{S}(\mathbb{A}_F^2).$$

All implicit constants are allowed to depend on $f \otimes \Phi$. For $y_i \in V_i^\circ(\mathbb{A}_F)$, we let

$$|y_i| := \prod_v |y_i|_v.$$

**Lemma 10.1.** Let $1/2 > \epsilon > 0$ and a finite set of places $S$ containing the infinite places be given. For $y \in V^\circ(\mathbb{A}_F)$ such that $|y_1|_v = |y_2|_v = 1$ for all $v \not\in S$, there exists a Schwartz function $\Psi \in \mathcal{S}((V_1 \otimes V_2 \otimes V_3)(\mathbb{A}_F))$ (depending on $S, \epsilon$) such that

$$(10.0.1) \quad \int_{(N_2)^3(\mathbb{A}_F) \setminus G(\mathbb{A}_F)} |f_1(g)\rho(g) f_2(y)| \, dg \leq \Psi(y_1 \otimes y_2 \otimes y_3) \prod_{i=1}^3 |y_i|^{-d_i/2-\epsilon}.$$ 

The function $I_0(f)$ satisfies the same bound.

**Proof.** This follows from the local bounds in Propositions 7.2, 8.2, and 9.4. \qed

Let $\mathbb{G}_m^2$ act on $V^\circ$ via the restriction of the action (4.0.6).

**Proposition 10.2.** The sum

$$\sum_{\xi \in V^\circ(F)/(F^\times)} \int_{N_2^3(\mathbb{A}_F) \setminus G(\mathbb{A}_F)} |f_1(g)\rho(g) f_2(\xi)| \, dg,$$

is finite.

**Proof.** Let

$$\{a_j \subset \mathcal{O} : 1 \leq j \leq k\}$$

be a set of representatives for the ideal classes of $\mathcal{O}$, the ring of integers of $F$. For every $\xi_i \in V_i(F)$, we can choose an $\alpha \in F^\times$ such that $\alpha \xi_i \in V_i(\mathcal{O})$ and the greatest common denominator $\gcd(\alpha \xi_i)$ of the coefficients of $\alpha \xi_i$ is $a_j$ for some $1 \leq j \leq k$. Using this observation, we see that the sum in the proposition is bounded by a constant times

$$\sum_{j_1,j_2=1}^k \sum_{\gcd(v_1) = a_{j_1}, \gcd(v_2) = a_{j_2}} \int_{N_2^3(\mathbb{A}_F) \setminus G(\mathbb{A}_F)} |f_1(g)\rho(g) f_2(\xi)| \, dg.$$ 

Here $(\mathcal{O}^\times)^2 < (F^\times)^2$ acts via the action (4.0.6). Thus it suffices to fix a pair of ideals $b_1$ and $b_2$ and prove convergence of the sum

$$\sum_{\xi \in V_1(\mathcal{O}) \times V_2(\mathcal{O}) \times V_3^\circ(F)/(\mathcal{O}^\times)^2} \int_{N_2^3(\mathbb{A}_F) \setminus G(\mathbb{A}_F)} |f_1(g)\rho(g) f_2(\xi)| \, dg.$$
Let $S$ be a finite set of places including the infinite places such that $b_i\hat{\mathcal{O}}^S = \hat{\mathcal{O}}^S$ for each $i$. Then by Lemma 10.1, there exists $\beta \in F^\times$ and $\Psi \in \mathcal{S}(\mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \mathcal{V}_3)(\mathbb{A}_F)$ such that the sum above is bounded by

$$\sum_{\xi \in V_1(O) \times V_2(O) \times \beta^{-1}(V_3^0(F) \cap V_3(O))} \Psi(\xi_1 \otimes \xi_2 \otimes \xi_3 \prod_{i=1}^3 |\xi_i|^{-d_i/2-\epsilon}) \leq \sum_{\xi \in S} \Psi(\xi).$$

Here we have used the fact that, by the product rule,

$$|\xi| \geq 1.$$  \hfill \Box

**Lemma 10.3.** Let constants $c > 1/2 > \epsilon > 0$ be given. For $1/2 + \epsilon < r < c$ and an integer $N > \max(d_1/2 + d_3/2 - 2, 2r + d_1/2 + d_2/2 - 2)$ there exists a Schwartz function $\Psi \in \mathcal{S}(\mathcal{V}_1(\mathbb{A}_F))$ (depending on $\epsilon, c$) such that

$$\int_{G_{\gamma_1}(\mathbb{A}_F) \backslash G(\mathbb{A}_F)} |f_1(\gamma g)| \int_{N_2(\mathbb{A}_F) \backslash \text{SL}_2(\mathbb{A}_F)} |\rho((I_2, h, (1 - 1) h (1 - 1)) g) f_2(y) \Phi((0, x) h p_1(g))| x^{2r} |d^\times x| dhdg \leq \Psi(y)|y_1|^{-N}|y_3|^{1-d_3/2-\epsilon} \prod_v \max(|y_2|_v, |y_3|_v)^{1-2r-d_2/2+\epsilon}.$$

The function $I_1(f \otimes \Phi)(y, s)$ defines a holomorphic function of $s$ in the strip $\frac{1}{2} + \epsilon < \Re(s) < c$ for each $y$ and admits the same bound with $r = \Re(s)$.

**Proof.** Let $S$ be a finite set of places including the infinite places such that $f_1^S = b_1^S$, $f_2^S = \mathbb{1}_{V_1(\mathcal{O}^S)}$ is fixed by $\rho(\text{SL}_2(\mathcal{O}_2^S))$ and $\Phi^S = 1(\mathbb{O}_2^S)$. Assume moreover that $\psi_v$ is unramified and $F_v$ is absolutely unramified for $v \notin S$. Using Lemma 8.4 and Propositions 7.3 and 9.5, for any given integers $N_1 > 0$, $N > \max(d_1/2 + d_3/2 - 2, 2r + d_2/2 + d_1/2 - 2)$, there exist $\Psi^\infty \in \mathcal{S}(\mathcal{V}_1(\mathbb{A}_F^\infty))$ and a positive constant $C$ depending on $\epsilon$ and $c$, such that the integral is bounded by a constant depending on $N_1, N, \epsilon, c$ times

$$\Psi^\infty(y_1) \prod_{v|\infty} \max(|y_1|_v, 1)^{-N_1}|y_1|_v^{-N} \prod_{v|\infty} \max(|y_2|_v, |y_3|_v)^{-2r-d_2/2-d_3/2+2} \times \prod_{v|\infty} \zeta_v(2r) \zeta_v(2r+d_2/2-1)|y_1|_v^{-d_1/2}|y_3|_v^{-d_3/2} \max(|y_1 \otimes y_3|_v, |y_2|_v)^{1-d_2/2-2r}$$

$$\times (C_v + \max(0, \text{ord}_v(y_1 \otimes y_3) - \text{ord}_v(y_2)) + \zeta_v(2r + d_2/2 - 1))$$

where $C_v \in \{C, 0\}$ and $C_v = 0$ for almost all $v$. Since $1/2 + \epsilon < r$, we have

$$C_v + \max(0, \text{ord}_v(y_1 \otimes y_3) - \text{ord}_v(y_2)) + \zeta_v(2r + d_2/2 - 1) \ll \min(|y_1 \otimes y_3|_v, |y_2|_v)^{-\epsilon} \zeta_v(2r + d_2/2 - 1)$$

$$= |y_1|_v^{-\epsilon} |y_3|_v^{-\epsilon} \max(|y_1 \otimes y_3|_v, |y_2|_v) \zeta_v(2r + d_2/2 - 1).$$
for all finite \( v \). Here the implied constant is equal to 1 for \( q_v \) sufficiently large in a sense independent of \( y \). Thus (10.0.3) is bounded by a constant depending on \( c \) and \( \epsilon \) times
\[
\Psi^\infty(y_1) \prod_{v|\infty} \max(|y_1|_v, 1)^{-N_1} |y_1|^{-N} \max(|y_2|_v, |y_3|_v)^{-2r-d_2-2d_3/2+2} \\
\times \prod_{v|\infty} |y_1|_v^{-d_1/2-\epsilon} |y_3|_v^{-d_3/2-\epsilon} \max(|y_1 \otimes y_2|_v, |y_3|_v)^{-d_4/2-2r+\epsilon}.
\]
(10.0.4)

For a finite place \( v \), if \( \Psi^\infty(y_1) \neq 0 \) then \( |y_1|_v \leq C'_v \) for some constant \( C'_v \geq 1 \), which is 1 for almost all \( v \), and hence
\[
\max(|y_1 \otimes y_3|_v, |y_2|_v) \geq C'^{-1}_v |y_1|_v \max(|y_3|_v, |y_2|_v).
\]

Thus (10.0.4) is bounded by a constant times
\[
\Psi^\infty(y_1) \prod_{v|\infty} \max(|y_1|_v, 1)^{-N_1} |y_1|^{-N} \max(|y_2|_v, |y_3|_v)^{-2r-d_2-2d_3/2+2} \\
\times \prod_{v|\infty} |y_1|_v^{-d_1/2-d_2/2-2r} |y_3|_v^{-d_3/2-\epsilon} \max(|y_2|_v, |y_3|_v)^{-d_4/2-2r+\epsilon}.
\]
(10.0.5)

The desired inequality follows from \( \max(|y_2|_v, |y_3|_v) \geq |y_3|_v \). \( \square \)

To study the sum of the boundary terms involving \( I_i \), we require the following:

**Lemma 10.4.** Let \( \Lambda \subset O^x \) be a finite index subgroup. There is a constant \( C_\Lambda \geq 1 \) depending on \( \Lambda \) such that for all \( \alpha \in F^x \), there exists \( u \in \Lambda \) such that
\[
C^{-1}_\Lambda(N\alpha)^{[F_p:Q]} < |u\alpha|_v < C_\Lambda(N\alpha)^{[F_p:Q]}
\]
for all \( v|\infty \).

**Proof.** Let \( r_1, r_2 \) be the number of real and complex places of \( F \) respectively. Let
\[
\log_\infty : F^x_\infty \longrightarrow \mathbb{R}^{r_1+r_2} \\
t \longmapsto (\log |t|_v)_{v|\infty}
\]
be the usual logarithm map and let
\[
L := \{ x \in \mathbb{R}^{r_1+r_2} \mid \sum_{v|\infty} x_v = 0 \}
\]
be the usual trace zero hyperplane. Then \( \log_\infty(\alpha) - \log_\infty((N\alpha)^{[F_p:Q]}) \in L \). By Dirichlet’s unit theorem and the fact that \( \Lambda \subset O^x \) is of finite index, one can choose \( u \in \Lambda \) so that \( \log_\infty(u) + \log_\infty(\alpha) - \log_\infty((N\alpha)^{[F_p:Q]}) \) lies in a fundamental domain for the full rank lattice \( \log_\infty(\Lambda) \subset L \). It follows that there is a constant \( c_\Lambda \geq 0 \) such that for every \( \alpha \in F^x_\infty \), there exists \( u \in \Lambda \) such that
\[
-c_\Lambda < \log |u|_v + \log |\alpha|_v - \log((N\alpha)^{[F_p:Q]}) < c_\Lambda
\]
for all \( v|\infty \). Taking exponentials implies the statement of the lemma. \( \square \)
Proposition 10.5. If $f_2$ satisfies (6.0.2) and $r \gg 1$, the sum
\[
\sum_{\xi \in V_1 \times F(V_2 \times V_3)(F)} \int_{G_1(\mathbb{A}_F),G(\mathbb{A}_F)} |f_1(\gamma_1 g)|
\times \int_{N_2(\mathbb{A}_F) \times \mathbb{A}_F} \int_{\mathbb{A}_F^\times} |\rho((I_2, h, (1 -_1) h (1 -_1)) g) f_2(\xi) \Phi((0, x) h p_1(g))| |x|^2r d^x x dh dg
\]
is finite. Therefore,
\[
\sum_{\xi \in V_1 \times F} I_1(f \otimes \Phi)(\xi, s)
\]
defines a holomorphic function for $\text{Re}(s) \gg 1$. Moreover, it extends to a meromorphic function of $\mathbb{C}$, holomorphic except for possible simple poles at $s = 0$ and $s = 1$. One has
\[
\text{Res}_{s=1} \sum_{\xi \in V_1 \times F} I_1(f \otimes \Phi)(\xi, s)
\]
\[
= \frac{\text{Vol}(F^\times \times \mathbb{A}_F^\times)}{2} \int_{G_1(\mathbb{A}_F),G(\mathbb{A}_F)} f_1(\gamma_1 g) \int_{[G_1]} \sum_{\xi \in V_1 \times F} \rho(h g) f_2(\xi) dh dg.
\]
and
\[(10.0.6) \int_{G_1(\mathbb{A}_F),G(\mathbb{A}_F)} |f_1(\gamma_1 g)| \int_{[G_1]} \left| \sum_{\xi \in V_1 \times F} \rho(h g) f_2(\xi) dh \right| dg
\]
is absolutely convergent.

The corresponding assertions for the integrals $I_2(f \otimes \Phi)$ and $I_3(f \otimes \Phi)$ are valid by symmetry.

Proof. We use the notation introduced at the beginning of Proposition 10.2. Let $(V_2 \times V_3)^0$ be the complement of the origin in $V_2 \times V_3$. For each $\xi \in (V_2 \times V_3)^0(F)$ there exists an $\alpha \in F^\times$ such that $\alpha \xi \in (V_2 \times V_3)(\mathcal{O})$ and the greatest common divisor $\gcd(\alpha \xi)$ of the coefficients of $\alpha \xi$ is $a_j$ for some $1 \leq j \leq k$. Then the sum in the proposition is bounded by
\[
\sum_{j=1}^k \sum_{(\xi, \xi) \in V_1(\mathbb{A}_F) \times \mathbb{A}_F^\times \cap (V_2 \times V_3)(\mathcal{O})} \int_{G_1(\mathbb{A}_F),G(\mathbb{A}_F)} |f_1(\gamma_1 g)|
\times \int_{N_2(\mathbb{A}_F) \times \mathbb{A}_F^\times} \int_{\mathbb{A}_F^\times} |\rho((I_2, h, (1 -_1) h (1 -_1)) g) f_2(\xi, \xi) \Phi((0, x) h p_1(g))| |x|^2r d^x x dh dg.
\]
Here $\mathcal{O}^\times$ acts diagonally via scaling on $V_2(\mathcal{O}) \times V_3(\mathcal{O})$. It suffices to fix $a = a_j$ and prove the convergence of the corresponding summand. Choose $\epsilon$ sufficiently small. By Lemma 10.3 and (10.0.2), there exists $\Phi \in \mathcal{S}(V_1(\mathbb{A}_F))$ such that the sum is bounded by a constant times
\[
\sum_{(\xi, \xi) \in V_1(\mathbb{A}_F) \times \mathbb{A}_F^\times \cap (V_2 \times V_3)(\mathcal{O})} \Psi(\xi) |\xi|^{1-d_2/2-2r+\epsilon}.
\]
To show the sum is finite for \( r \gg 1 \), it suffices to show
\[
\sum_{\xi \in (V_2 \times V_3)^0(F) \cap (V_2 \times V_3)(\mathcal{O})/(\mathcal{O}^\times)} \prod_{v|\infty} |\xi|_v^{-r}
\]
is finite for \( r \gg 1 \). By Lemma 10.4, all but finitely many classes in \( \mathcal{O} \cap F^\times /\mathcal{O}^\times \) admit representatives \( \alpha \) such that \( |\alpha|_v \geq 1 \) for all \( v|\infty \), and thus the sum is bounded by a constant times
\[
\sum_{\xi \in (V_2 \times V_3)(\mathcal{O}) \cap (\mathcal{O}^\times)} \prod_{v|\infty} \text{max}(|\xi|_v, 1)^{-r}
\]
which is finite for \( r \gg 1 \). This completes the proof of the first claim of the proposition, and we deduce that \( \sum_{\xi \in Y_1(F)} I_1(f \otimes \Phi)(\xi, s) \) is holomorphic for \( \text{Re}(s) \gg 1 \).

To obtain the meromorphic continuation, we break down the integral \( \sum_{\xi \in Y_1(F)} I_1(f \otimes \Phi)(\xi, s) \) into two sums of the form
\[
\int_{G_{\gamma_1}(\mathbb{A}_F) \setminus G(\mathbb{A}_F)} f_1(\gamma_1) \int_{[\text{SL}_2]} \sum_{\xi \in V(F) \setminus \mathbb{Q}_1(\xi_1)=0} \rho((I_2, h, (1 -1) h (1 -1)) g) f_2(\xi)
\]
(10.0.7)
\[
\times \int \sum_{\delta \in B_2(F) \setminus \text{SL}_2(F)} \Phi(x(0, 1)\delta h p_1(g)) |x|^{2s} d^x x dhdg
\]
where the unspecified integral is over \( |x| \geq 1 \) or \( |x| \leq 1 \). The contribution of \( |x| \geq 1 \) converges for \( \text{Re}(s) \) large and hence converges for all \( s \). Using the Poisson summation formula on \( F^2 \), the contribution of \( |x| \leq 1 \) equals
\[
\int_{G_{\gamma_1}(\mathbb{A}_F) \setminus G(\mathbb{A}_F)} f_1(\gamma_1) \int_{[\text{SL}_2]} \sum_{\xi \in V(F) \setminus \mathbb{Q}_1(\xi_1)=0} \rho((I_2, h, (1 -1) h (1 -1)) g) f_2(\xi)
\]
\[
\times \int_{|x|\leq 1} \left( \sum_{\delta \in B_2(F) \setminus \text{SL}_2(F)} \widehat{\Phi}(x^{-1}(0, 1)\delta^{-1} h^{-1} p_1(g)^{-1}) |x|^{2s-2} d^x x + \widehat{\Phi}(0) |x|^{2s-2} - \Phi(0) |x|^{2s} d^x x \right) dhdg.
\]

An argument similar to the argument proving the holomorphy of the \( |x| > 1 \) contribution implies that the contribution of the sum over \( \delta \) defines an entire function of \( s \). For \( \text{Re}(s) \gg 1 \), the remaining contribution is
\[
\frac{\text{Vol}(F^\times \setminus (\mathbb{A}_F^\times)^1)}{2} \left( \frac{\widehat{\Phi}(0)}{s - 1} - \frac{\Phi(0)}{s} \right) \int_{G_{\gamma_1}(\mathbb{A}_F) \setminus G(\mathbb{A}_F)} f_1(\gamma_1) \int_{[G_{\gamma_1}]} \sum_{\xi \in V(F)} \rho(hg) f_2(\xi) dhdg.
\]
Assuming that (10.0.6) is convergent, this term admits a meromorphic continuation to the \( s \) plane, holomorphic except at \( s \in \{0, 1\} \) with poles and residues as specified. To obtain the convergence of (10.0.6) one begins with
\[
\int_{G_{\gamma_1}(\mathbb{A}_F) \setminus G(\mathbb{A}_F)} |f_1(\gamma_1) g| \int_{[G_{\gamma_1}]} \left| \sum_{\xi \in V(F)} \rho(hg) f_2(\xi) \right|
\]
\[ \times \int_{\delta \in B_2(F) \backslash \SL_2(F)} \Phi(x(0, 1)\delta h p_1(g))|x|^{2s} d^x \, dx \, dh \, dg \]

instead of (10.0.7), argues as before, and then observes that one obtains an equality between (10.0.6) times \( \frac{\text{Vol}(F \backslash \mathbb{A}_F^{(1)} \backslash \mathbb{A}_F^2)}{2} \left( \frac{\Phi(0)}{s-1} - \frac{\Phi(0)}{s} \right) \) and a sum that converges for \( \Re(s) \) large. The absolute convergence statement follows.

\[ \square \]

11. The \( L^2 \)-theory

We now discuss the \( L^2 \)-theory. Let \( F \) be a local field of characteristic zero. We assume throughout this section that \( Y^{\text{sm}}(F) \subset Y(F) \) is nonempty, and hence is dense in the Hausdorff topology [Pool17, Remark 3.5.76].

We first improve the bound in [GL19, Propositions 7.1 and 8.2]:

**Proposition 11.1.** Assume \( \frac{1}{2} > \beta \geq 0 \) and \( v \in V^\bullet(F) \). If \( F \) is nonarchimedean then

\[ (11.0.1) \int_{G_{\gamma b}(F) \backslash G(F)} |f_1(\gamma b)\rho(g)f_2(v)| \, dg \ll_{f_1,f_2} \prod_{i=1}^{3} |v_i|^{\beta/3-d_i/2+2/3}. \]

The integral as a function of \( v \) has support in \( \omega^{-N} V(O) \) for some \( N \in \mathbb{Z} \). If \( F \) is archimedean then, given \( N > 0 \), there is a continuous seminorm \( \nu_{\beta,N} \) on \( \mathcal{S}(X(F) \times V(F)) \) such that

\[ \int_{G_{\gamma b}(F) \backslash G(F)} |f_1(\gamma b)\rho(g)f_2(v)| \, dg \leq \nu_{\beta,N}(f_1 \otimes f_2) \prod_{i=1}^{3} \max(|v_i|, 1)^{-N} |v_i|^{\beta/3-d_i/2+2/3}. \]

The function \( I(f) \) satisfies the same bound and support constraint.

**Proof.** Assume for the moment that \( F \) is nonarchimedean. The bound on the support of the integral is part of [GL19, Proposition 7.1], so we only require the bound on the magnitude. By Lemma 3.5 and Iwasawa decomposition, the integral is bounded by a constant depending on \( \beta \) and \( f_1 \) times

\[ (11.0.2) \int_{m(t,a) \leq c} m(t,a)^{-2+\beta} \tilde{f}_2(a^{-1}v) \left( \prod_{i=1}^{3} |a_i|^{2-d_i/2} \right) d^x \, dt \, da, \]

for some constant \( c > 0 \), where \( \tilde{f}_2 \) and \( m(t,a) \) are defined as in (7.0.5) and (5.1.5) respectively. Observe that

\[ m(t,a) = |a_1 a_2 a_3| \max \left( |t|, |a_1|^{-2}, |a_2|^{-2}, |a_3|^{-2} \right). \]

Thus (11.0.2) is equal to

\[ (11.0.3) \int_{m(t,a) \leq c} \max \left( |t|, |a_1|^{-2}, |a_2|^{-2}, |a_3|^{-2} \right)^{\beta-2} \tilde{f}_2(a^{-1}v) \left( \prod_{i=1}^{3} |a_i|^{\beta-d_i/2} \right) d^x \, dt \, da. \]
Since  
\begin{equation}
\int_F \max(|t|, |a_1|^{-2}, |a_2|^{-2}, |a_3|^{-2})^{-2+\beta} dt \ll_\beta \min(|a_1|, |a_2|, |a_3|)^{2-2\beta} \leq |a_1 a_2 a_3|^{2/3-2\beta/3},
\end{equation}
the integral (11.0.3) is bounded by a constant times  
\[ \int_{(F^\times)^3} \tilde{f}_2(a^{-1}v) \left( \prod_{i=1}^3 |a_i|^{\beta/3-\delta_i/2+2/3} \right) d^\times a. \]
Taking a change of variables \( a_i \mapsto a_i \omega_{\text{ord}(v_i)} \), the above is bounded by a constant times  
\[ \prod_{i=1}^3 |v_i|^{\beta/3-\delta_i/2+2/3} \int_{q^{-N} \leq |a_i|} \left( \prod_{i=1}^3 |a_i|^{\beta/3-\delta_i/2+2/3} \right) d^\times a \]
for some \( N \) depending on \( f_2 \). The integral converges as \( d_i \geq 2 \) for all \( i \).

Now assume \( F \) is archimedean. By Lemma 3.5 and the argument above, there is a continuous seminorm \( \nu_{\beta,N'} \) on \( S(X(F)) \) such that the integral is bounded by \( \nu_{\beta,N'}(f_1) \) times  
\begin{equation}
\int_{(F^\times)^3 \times F} \max( |t|, |a_1|^{-2}, |a_2|^{-2}, |a_3|^{-2} )^{-2} \max( m(t,a), 1 )^{-N'} \tilde{f}_2(a^{-1}v) \left( \prod_{i=1}^3 |a_i|^{\beta/3-\delta_i/2} \right) dt d^\times a.
\end{equation}
Choose \( N' \) so that \( N' > 3N \), and observe  
\[ m(t,a)^3 \geq \max(|a_1|, 1) \max(|a_2|, 1) \max(|a_3|, 1). \]
Then for any \( M > 0 \), there is a continuous seminorm \( \nu'_M \) on \( S(V(F)) \) such that (11.0.5) is bounded by \( \nu'_M(f_2) \) times  
\[ \int_{(F^\times)^3 \times F} \max( |t|, |a_1|^{-2}, |a_2|^{-2}, |a_3|^{-2} )^{-2} \left( \prod_{i=1}^3 \max(|a_i|, 1)^{-M} \max(|a_i^{-1}v_i|, 1)^{-M} |a_i|^{\beta/3-\delta_i/2} \right) dt d^\times a. \]
The proposition then follows from (11.0.4) (which is still valid for \( F \) archimedean) and Lemma 9.1.

Let \( \Omega_V \) be a top degree form on \( V(F) \) such that \( |\Omega_V| \) is the Haar measure. We endow \( Y^{\text{sm}}(F) \) with the unique positive measure \( dy = |\Omega_V| \) such that \( d(Q_1 - Q_2) \land d(Q_2 - Q_3) \land \Omega_Y \) is \( \Omega_V \) on \( V(F) \). Since \( Y^{\text{sm}}(F) \subset Y(F) \) is dense, we can consider the \( L^p \) space  
\[ L^p(Y(F)) := L^p(Y(F), dy) := L^p(Y^{\text{sm}}(F), dy). \]

We observe that for \( r \in F^\times \), one has  
\[ r^{d_1 + d_2 + d_3} \Omega_V = \Omega_V(rv) \]
\[ = d(Q_1 - Q_2)(rv) \land d(Q_2 - Q_3)(rv) \land \Omega_Y(rv) \]
\[ = r^4 d(Q_1 - Q_2) \land d(Q_2 - Q_3) \land \Omega_Y(rv). \]
Thus

\[(11.0.6) \quad d(ry) = |r|^{d_1 + d_2 + d_3 - 4} dy.\]

**Proposition 11.2.** Let $0 < p \leq 2$. One has $\mathcal{S}(Y(F)) < L^p(Y(F))$ and the inclusion is continuous if $F$ is archimedean.

**Proof.** Let $f \in \mathcal{S}(Y(F))$. Since $\mathcal{S}(Y(F)) < C^\infty(\mathcal{T}_m^\infty(F))$ by Propositions 7.1 and 9.3, we have

\[
\int_{\mathcal{T}_m^\infty(F)} |f(y)|^p dy = \int_{\mathcal{T}_m^\infty(F)} |f(y)|^p dy.
\]

We will therefore bound the integral on the right.

Fix $\frac{1}{2} > \beta > 0$. Assume first that $F$ is nonarchimedean. By Proposition 11.1, for some $c \in \mathbb{Z}$ one has

\[
\int_{\mathcal{T}_m^\infty(F)} |f(y)|^p dy \ll_f \int_{\mathcal{T}_m^\infty(F) \cap \mathbb{C}_{V(C)}} \left( \prod_{i=1}^{3} |y_i|^{p(\beta/3-d_i)/2+2/3} \right) dy.
\]

Let $\alpha := p\beta + 4 - 4 \sum_{i=1}^{3} (1 - \frac{3}{2})d_i \geq 2\beta > 0$. Using the homogeneity property (11.0.6), the integral above is bounded by a constant depending on $c$ times

\[
\zeta(\alpha) \int_{\{|y| \leq 1\}} \left( \prod_{i=1}^{3} |y_i|^{\alpha/2+2} \right) dy.
\]

Here we could just write $|y| = 1$, but we have written $1 \leq |y| < 2$ so that we can use the same formula in both archimedean and nonarchimedean cases.

Now assume that $F$ is archimedean. Fix $N > \alpha$. Then by Proposition 11.1 there is a continuous seminorm $\nu_\beta$ on $\mathcal{S}(Y(F))$ such that

\[
\int_{\mathcal{T}_m^\infty(F)} |f(y)|^p dy \leq \nu_\beta(f)^p \sum_{j=1}^{\infty} \int_{\{|y| \leq 2^j\}} \left( \prod_{i=1}^{3} |y_i|^{\alpha/2+2} \right) dy
\]

\[
+ \nu_\beta(f)^p \sum_{j=0}^{\infty} \int_{\{|y| \leq 2^{j+1}\}} |y|^{-N} \left( \prod_{i=1}^{3} |y_i|^{\alpha/2+2} \right) dy.
\]

Using the homogeneity property (11.0.6) again, we see that this is

\[
\nu_\beta(f)^p \left( \sum_{j=1}^{\infty} 2^{-\alpha j} + \sum_{j=0}^{\infty} 2^{(\alpha-N)j} \right) \int_{\{|y| \leq 2\}} \left( \prod_{i=1}^{3} |y_i|^{\alpha/2+2} \right) dy.
\]

Thus for any $F$, we are reduced to showing that

\[
\int_{\{|y| \leq 2\}} \left( \prod_{i=1}^{3} |y_i|^{\alpha/2+2} \right) dy.
\]
is finite. By symmetry, it suffices to show that the integral

\[(11.0.7) \quad \int_{y \in Y^{an}(F): \max(|y_1|,|y_2|) \leq |y_3|, 1 \leq |y_3| < 2} \left( \prod_{i=1}^{3} |y_i|^{p(\beta/3-d_i/2+2/3)} \right) dy \]

is finite. After a change of variables, we can assume that \(Q_i(v_i) = v^t c_i v\) where

\[c_i = \left( \begin{array}{c} c_{i1} \\ \vdots \\ c_{id_i} \end{array} \right)\]

is diagonal. Write \(v_i = (v_{i1}, \ldots, v_{id_i})\). For any \(1 \leq j \leq d_1, 1 \leq k \leq d_2\), we have

\[\det \left( \frac{\partial_{y_{i1}}(Q_1-Q_2)}{\partial_{y_{i1}}(Q_2-Q_3)} \frac{\partial_{y_{i2}}(Q_1-Q_2)}{\partial_{y_{i2}}(Q_2-Q_3)} \right) = \det \left( \begin{array}{cc} 2c_{i1}v_{i1} & -2c_{i2}v_{i2} \\ 2c_{i3}v_{i3} & v_{i2} \end{array} \right) = \left| 4c_{i1}c_{i2} \right| |v_{i1}| |v_{i2}|.\]

Then for any \(j\) and \(k\) as above we have

\[(11.0.8) \quad \left( \prod_{i=1}^{3} |y_i|^{p(\beta/3-d_i/2+2/3)} \right) dy = \frac{\left( \prod_{i=1}^{3} |y_i|^{p(\beta/3-d_i/2+2/3)} \right) dy_1 dy_2 dy_3}{4c_{i1}c_{i2} |y_{i1}y_{i2}|} dy_{i1} dy_{i2}\]

outside a set of measure zero with respect to \(dy\). Here \(\frac{dy}{dy_{i1}} := dy_{i2} \ldots dy_{id_i}\), etc. and the values of \(|y_{1j}|\) and \(|y_{2k}|\) are given implicitly in terms of the other entries of \(y\). We can assume that \(j\) and \(k\) are chosen so that \(|y_{1j}| = |y_1|\) and \(|y_{2k}| = |y_2|\). Therefore, setting \(t_1 = |y_1|\) and \(t_2 = |y_2|\), we see that (11.0.7) is bounded by a constant times

\[\int_0^2 \int_0^2 \left( \int_{(x_1,x_2) \in F^{d_1-1} \times F^{d_2-1}, |x_1| \leq t_1, |x_2| \leq t_2} t_1^{p(\beta/3-d_1/2+2/3)-1} t_2^{p(\beta/3-d_2/2+2/3)-1} dx_1 dx_2 \right) dt_1 dt_2 < \infty.\]

\[\square\]

Let \(K \leq \text{Sp}_6(O)\) be a compact open subgroup and let \(L^2(X^o(F))^K\) denote the space of functions on \(X^o(F)\) that are right \(K\)-invariant and square-integrable.

**Lemma 11.3.** For \(f \in L^2(X^o(F))^K\) we have

\[|f(x)| \leq \frac{\|f\|}{|x|^2 \text{meas}(K)^{1/2}}\]

for any \(x \in X^o(F)\).

**Proof.** For \(f \in L^2(X^o(F))^K\) we have

\[\|f\|^2 = \sum_{\gamma \in X^o(F)/K} \frac{|f(\gamma)|^2 \text{meas}(K)}{\delta_p(\gamma)} = \sum_{\gamma \in X^o(F)/K} |f(\gamma)|^2 \text{meas}(K)|\gamma|^4.\]

\[\square\]

**Proposition 11.4.** For \(F\) nonarchimedean and \(f_2 \in S(V(F))\), the map \(I(\cdot \otimes f_2)\) extends to a continuous map

\[I(\cdot \otimes f_2) : L^2(X^o(F))^K \rightarrow L^2_{\text{loc}}(Y^{an}(F)).\]
Proof. Assume \( y \in Y^{\text{ani}}(F) \). By Lemma 11.3, for any \( f_1 \in S(X(F)) \) the integral \( I(f_1 \otimes f_2)(y) \) is bounded by \( \text{meas}(K)^{-1/2} \) times
\[
\|f_1\|_2 \int_{G_{\gamma b}(F) \backslash G(F)} |\gamma b g|^{-2} |\rho(g)f_2|(y)dg.
\]
The proposition thus follows from the argument proving Proposition 11.1 in the special case \( \beta = 0 \). \( \square \)

12. The Fourier transform

Let \( F \) be a number field.

Theorem 12.1. Let \( v \) be a place of \( F \) and assume that \( Y^{\text{sm}}(F_v) \) is nonempty. There is a unique \( \mathbb{C} \)-linear isomorphism \( \mathcal{F}_Y : S(Y(F_v)) \to S(Y(F_v)) \) such that \( \mathcal{F}_Y \circ I = I \circ \mathcal{F}_X \). It is continuous if \( v \) is archimedean. In particular there is a commutative diagram

\[
\begin{array}{ccc}
S(X(F_v) \times V(F_v)) & \xrightarrow{\mathcal{F}_X} & S(X(F_v) \times V(F_v)) \\
\downarrow I & & \downarrow I \\
S(Y(F_v)) & \xrightarrow{\mathcal{F}_Y} & S(Y(F_v))
\end{array}
\]

We should pause to explain why this theorem is not obvious. For simplicity assume that \( v \) is a nonarchimedean place. Let
\[
C := S(X(F_v) \times V(F_v))_{\text{SL}_3(F_v)}
\]
denote the space of coinvariants. It is clear that the map \( I \) factors through \( C \) and yields a surjection \( C \to S(Y(F_v)) \). Since \( \mathcal{F}_X \) is equivariant under the action of \( \text{SL}_3(F_v) < \text{Sp}_6(F_v) \), it is clear that \( \mathcal{F}_X \) descends to define an automorphism of \( C \). However, it is not clear that the map \( C \to S(Y(F_v)) \) is injective. For instance, there are several orbits of \( \text{SL}_3(F_v) \) on \( X(F_v) \), but the map \( I \) depends only on the restriction of a function in \( S(X(F_v) \times V(F_v)) \) to one of these orbits.

Now that one has Theorem 12.1, many prior results can be stated more transparently. For example, by [GL19, Lemma 4.3] we have

Corollary 12.2. For any place \( v \) of \( F \), \( f \in S(Y(F_v)) \), and \( h \in H(F_v) \) one has that
\[
\mathcal{F}_Y(L(h)f) = |\lambda(h)|^{\sum_{i=1}^3 d_i/2-2}L \left( \frac{h}{\lambda(h)} \right) \mathcal{F}_Y(f).
\]

Unlike in other sections in this paper, we have not abbreviated \( F_v \) by \( F \). We really require both \( F \) and \( F_v \) in this section because we will use a global-to-local argument to prove Theorem 12.1. The global-to-local argument is fairly simple, and we invite the reader to skip to the proof of Theorem 12.1 to see the basic idea.
There is a somewhat hidden assumption on the base change $Y_{F_v}$ of $Y$ to $F_v$ in the statement of Theorem 12.1. Namely, we are assuming that $Y_{F_v}$ is the base change to $F_v$ of the scheme cut out of a triple of quadratic spaces over the number field $F$ by the simultaneous values of three quadratic forms. Since every characteristic zero local field is a localization of a number field, and every quadratic form over a local field is equivalent to the localization of a quadratic form over the corresponding number field this is no loss of generality.

We claim, moreover, that upon replacing $F$ by another number field if necessary, we can assume that $Y^{sm}(F) \neq \emptyset$. By a change of basis, we may assume $\mathcal{Q}_i$ is associated to the diagonal matrix $\text{diag}(c_{i1}, \ldots, c_{id_i})$. In the archimedean case, we can assume $c_{ij} \in \{\pm 1\}$. If $F_v = \mathbb{C}$, take $F = \mathbb{Q}(i)$ and if $F_v = \mathbb{R}$, take $F = \mathbb{Q}$. Now suppose $F_v$ is nonarchimedean. Let $p$ be the prime ideal corresponding to $v$. We may assume $c_{ij} \in \mathcal{O}_v$, the ring of integers of $F_v$, for all $i, j$. As $Y^{sm}(F_v)$ is nonempty, $Y^{sm}(F_v) \cap \mathcal{V}(\mathcal{O}_v)$ is nonempty. Note that $A := F^{sep} \cap \mathcal{O}_v$ is the henselization of an excellent discrete valuation ring $\mathcal{O}_p$ whose completion is $\mathcal{O}_v$ (see e.g., [Sta18, tag 07QS], [Liu02, Example 8.3.34]). Thus $A$ has the approximation property by [Art69, Theorem 1.10]. In particular, there exists $(y_1, y_2, y_3) \in Y^{sm}(F_v)$ such that each coordinate of $y_i$ is algebraic over $F$. Let $E$ be the field extension of $F$ adjoining the coordinates of the $y_i$. Then $E_w = F_v$ for some $w|v$ and $Y^{sm}(E) \neq \emptyset$. This justifies our claim.

Thus in proving Theorem 12.1, we can and do assume $Y^{sm}(F) \neq \emptyset$.

**Theorem 12.3.** If $Y^{sm}(F_v) \neq \emptyset$ for all $v$, then $Y^{sm}(F)$ is nonempty and has dense image in $Y^{sm}(F_v)$.

**Proof.** Since we have assumed $\dim V_i \geq 2$ and $\mathcal{Q}_i$ is nondegenerate for each $i$, this is a direct consequence of [CTS82, Corollaire in §4]. \qed

For a place $v$ of $F$, consider the linear map

$$T : \mathcal{S}(V_i(F_v)) \longrightarrow C^\infty(F_v)$$

$$f \longmapsto T(f)(\alpha) := \int_{V_i(F_v)} f(v)\psi(\alpha \mathcal{Q}_i(v))dv.$$ 

Let

$$(12.0.1) \quad \mathcal{S}_{i,v} = \{f \in \mathcal{S}(V_i(F_v)) \mid T(f) = 0 \text{ and } f(0) = 0\}.$$

Note that $(12.0.1)$ coincides with $(1.0.1)$ by the definition of the Weil representation. We set $\mathcal{S}_0 = \mathcal{S}_{1,v} \otimes \mathcal{S}_{2,v} \otimes \mathcal{S}_{3,v}$. Clearly, $\text{supp}(\rho(g)f) < V^c(F_v)$ for all $f \in \mathcal{S}_0$ and $g \in \text{SL}_3(F_v)$. The following lemma implies, in particular, $\mathcal{S}_0$ is nontrivial when $v$ is a finite place above an odd prime.

**Lemma 12.4.** If $v$ is a finite place lying above an odd prime then the kernel of $T$ is infinite dimensional.
Proof. By diagonalizing the quadratic form $Q_i$, we see it suffices to show the kernel of the linear map

$$T' : S(F_v) \to C^\infty(F_v)$$

$$f \mapsto T'(f)(\alpha) := \int_{F_v} f(x) \psi_v(\alpha x^2) dx$$

is infinite dimensional. Observe that if a nonzero $f$ lies in the kernel of $T'$, then so is the infinite dimensional vector space

$$\{x \mapsto f(x\varpi_v^n) : n \in \mathbb{Z}\}.$$ 

Therefore, it suffices to show $T'$ has nontrivial kernel. For $a \in O_v^\times$, let $U_a := a + \varpi_v O_v$. Since 2 does not divide the residual characteristic of $F_v$, the map $x \mapsto x^2$ induces a bijection $U_a \to U_a^2$. Therefore,

$$T'(1_{U_a})(\alpha) = \frac{dx(O_v)}{q_v} \psi_v(a^2 \alpha) 1_{\varpi_v^{N-1}O_v}(\alpha),$$

where $N$ is the smallest integer such that $\psi_v$ is trivial on $\varpi_v^N O_v$. In particular, $T'(1_{U_1}) = T'(1_{U_{-1}})$. Since $U_1$ and $U_{-1}$ are disjoint, the function $1_{U_1} - 1_{U_{-1}}$ is nonzero and lies in the kernel of $T'$.

\begin{lemma}
Let $v$ be a place where $Q_{iv}$ splits for all $1 \leq i \leq 3$. Suppose there exists $y_0 \in Y^{ani}(F_v)$ such that $I(\mathcal{F}_X(f_1) \otimes f_2)(y_0) = 0$ for all $f_1 \otimes f_2 \in C_c^\infty(\gamma_b G(F_v)) \otimes S_{0v}$. Then

$$I(\mathcal{F}_X(f_1) \otimes f_2) = 0$$

for any $f_1 \otimes f_2 \in C_c^\infty(\gamma_b G(F_v)) \otimes S_{0v}$.
\end{lemma}

Proof. Given $y \in Y^{ani}(F_v)$, choose $h \in H(F_v)$ such that $\lambda(h)h^{-1}y_0 = y$. Then for $f_1 \otimes f_2 \in C_c^\infty(\gamma_b G(F_v)) \otimes S_{0v}$,

$$I(\mathcal{F}_X(f_1) \otimes f_2)(y) = \frac{h}{\lambda(h)} I(\mathcal{F}_X(f_1) \otimes f_2)(y_0) = |\lambda(h)|^{-\frac{1}{2}} \sum_{i=1}^{d_v/2} I(\mathcal{F}(x))(y_0)$$

for some $f \in C_c^\infty(\gamma_b G(F_v)) \otimes S_{0v}$ by [GL19, Lemma 4.3]. Thus our hypothesis implies $I(\mathcal{F}_X(f_1) \otimes f_2)(y) = 0$. Since $I(\mathcal{F}_X(f_1) \otimes f_2)$ is continuous on $Y^{sm}(F_v)$ by Propositions 7.1 and 9.3, and $Y^{ani}(F_v) \subset Y^{sm}(F_v)$ is dense by [Poo17, Remark 3.5.76], we deduce the lemma.

\begin{lemma}
Let $v$ be a finite place where $Q_{iv}$ splits for $1 \leq i \leq 3$ and $S_{0v}$ is nontrivial. For a given $y \in Y^{ani}(F_v)$, there exists $f_1 \otimes f_2 \in C_c^\infty(\gamma_b G(F_v)) \otimes S_{0v}$ such that $I(\mathcal{F}_X(f_1) \otimes f_2)(y) \neq 0$.
\end{lemma}

Proof. Choose $f_1' \otimes f_2 \in S(X(F_v)) \otimes S_{0v}$ such that $I(f_1' \otimes f_2) \neq 0$. For example, we could take $f_1'$ to be the characteristic function of a sufficiently small neighborhood of $\gamma_b$ in $\gamma_b G(F_v)$. 

Choose a compact open subgroup $K \leq \text{Sp}_6(O)$ such that $f'_1$ is fixed by $K$. Finally, choose $f_{1n} \in C_c^\infty(\gamma_0 G(F_v))^K$ indexed by $n \in \mathbb{Z}_{>0}$ such that

$$\lim_{n \to \infty} f_{1n} = \mathcal{F}_X^{-1}(f'_1)$$

in $L^2(X^\circ(F))^K$. Then since $\mathcal{F}_X$ is an isometry of $L^2(X^\circ(F))^K$, we have $\mathcal{F}_X(f_{1n}) \to f'_1$ in $L^2(X^\circ(F))^K$. Since $I(\cdot \otimes f_2) : L^2(X^\circ(F))^K \to L^2_{\text{loc}}(Y^{\text{ani}}(F))$ is well-defined and continuous by Proposition 11.4, we deduce that

$$I(\mathcal{F}_X(f_{1n}) \otimes f_2) \to I(f'_1 \otimes f_2)$$

in $L^2_{\text{loc}}(Y^{\text{ani}}(F))$ and hence $I(\mathcal{F}_X(f_{1n}) \otimes f_2) \neq 0$ for $n$ large enough. The statement thus follows from Lemma 12.5.

**Proof of Theorem 12.1.** We first prove that if $I(f_v) = 0$ then $I(\mathcal{F}_X(f_v)) = 0$. Choose finite places $v_1$ and $2 \nmid v_2$ distinct from $v$ such that $\mathbb{Q}_{v_2}$ splits for $1 \leq i \leq 3$. Suppose that $f_{v_1} \in \mathcal{S}(X(F_{v_1}) \times V(F_{v_1}))$ is chosen so that $\mathcal{F}_X(f_{v_1}) \in C_c^\infty(\gamma_0 G(F_{v_1}) \times V(F_{v_1}))$ and $I(\mathcal{F}_X(f_{v_1})) \in C_c^\infty(Y^{\text{sm}}(F_{v_1}))$ and that $f_{v_2} \in C_c^\infty(\gamma_0 G(F_{v_2})) \otimes \mathcal{S}_{v_2}$. Moreover, choose $f^{v_1, v_2} \in \mathcal{S}(X(\mathbb{A}_F^{v_1, v_2}) \times V(\mathbb{A}_F^{v_1, v_2}))$. Then applying Theorem 1.3, we obtain

$$0 = \sum_{y \in Y^{\text{sm}}(F)} I(\mathcal{F}_X(f_v f_{v_1} f_{v_2} f^{v_1, v_2}))(y).$$

In particular, since $\mathcal{F}_X(f_{v_1}) \in C_c^\infty(\gamma_0 G(F_{v_1}) \times V(F_{v_1}))$ and $f_{v_2} \in C_c^\infty(\gamma_0 G(F_{v_2})) \otimes \mathcal{S}_{v_2}$, all of the boundary terms in the formula vanish.

We observe that $Y(F)$ is discrete in $Y(\mathbb{A}_F)$. Let $y_0 \in Y^{\text{ani}}(F)$. We claim that we can choose $f_{v_1} f_{v_2} f^{v_1, v_2}$ so that the right hand side is equal to $I(\mathcal{F}_X(f_{v_1} f_{v_2} f^{v_1, v_2}))(y_0)$ where $I(\mathcal{F}_X(f_{v_1} f_{v_2} f^{v_1, v_2}))(y_0) \neq 0$. Indeed, using the argument in the proof of Lemma 5.3, we can choose $f_{v_1}$ so that $I(\mathcal{F}_X(f_{v_1}))$ is any function in $C_c^\infty(Y^{\text{sm}}(F_{v_1}))$. Combining this with Lemma 5.3, the computation of the basic function in Proposition 8.1, and Lemma 12.6, we deduce the claim.

The claim implies that $I(\mathcal{F}_X(f_v))(y) = 0$ for all $y \in Y^{\text{ani}}(F)$. Since $Y^{\text{ani}}(F)$ is dense in $Y^{\text{sm}}(F_v)$ by [Poo17, Remark 3.5.76] and Theorem 12.3, we can use the continuity of $I(\mathcal{F}_X(f_v))$ (Propositions 7.1 and 9.3) to deduce that $I(\mathcal{F}_X(f_v)) = 0$.

We have shown that $\mathcal{F}_X(\ker I) \leq \ker I$. On the other hand $\mathcal{F}_X \circ \mathcal{F}_X = \text{Id}$ by Proposition 3.9, so $\ker I = \mathcal{F}_X \circ \mathcal{F}_X(\ker I) \leq \mathcal{F}_X(\ker I)$, hence $\mathcal{F}_X(\ker I) = \ker I$. This implies the theorem.

The Fourier transform $\mathcal{F}_{X, \psi}$ and $I := I_\psi$ depend on a choice of additive character $\psi$. The dependence of $I$ on $\psi$ is through its dependence on the Weil representation $\rho = \rho_\psi$. Thus $\mathcal{F}_Y$ also depends on $\psi$. We write $\mathcal{F}_{Y, \psi}$ when we need to indicate this dependence. Thus $\mathcal{F}_{Y, \psi}$
is determined by the relation

\begin{equation}
(12.0.2) \quad \mathcal{F}_{Y,\psi} \circ I_\psi = I_\psi \circ \mathcal{F}_{X,\psi}.
\end{equation}

**Corollary 12.7.** For \( f \in \mathcal{S}(Y(F_v)) \), we have

\begin{align*}
(12.0.3) \quad & \quad \mathcal{F}_\psi^2(f)(v) = f(v), \\
(12.0.4) \quad & \quad \overline{\mathcal{F}_{Y,\psi}(f)} = \mathcal{F}_{Y,\overline{\psi}}(f), \\
(12.0.5) \quad & \quad \mathcal{F}_{Y,\psi} = \mathcal{F}_{Y,\overline{\psi}}.
\end{align*}

**Proof.** The first equation \((12.0.3)\) is immediate from Proposition 3.9. As for \((12.0.4)\), by the explicit formula for \( \mathcal{F}_{X,\psi} \) given in \([\text{GHL21, Corollary 6.11}]\), for any \( f \in \mathcal{S}(X(F_v)) \) one has

\begin{equation}
(12.0.6) \quad \overline{\mathcal{F}_{X,\psi}(f_1)} = \mathcal{F}_{X,\overline{\psi}}(f_1).
\end{equation}

Moreover, we claim that for \( f_2 \in \mathcal{S}(V(F_v)) \) one has

\begin{equation}
(12.0.7) \quad \rho_\psi(g)f_2 = \rho_\overline{\psi}(g)f_2
\end{equation}

for all \( g \in \text{SL}_2^3(F_v) \). By the second corollary to \([\text{Wei64, Théorème 2}]\), the Weil index \( \gamma(\mathcal{Q}_i, \psi) \) satisfies the relation \( \gamma(\mathcal{Q}_i, \overline{\psi}) = \gamma(\mathcal{Q}_i, \psi) \). Using this fact, one checks \((12.0.7)\) by checking it on the same set of generators for \( \text{SL}_2^3(F_v) \) traditionally used to define the Weil representation (see \([\text{GL19, §3.1}]\), for example). Thus \((12.0.7)\) is valid. Hence for \( f_1 \otimes f_2 \in \mathcal{S}(X(F_v) \times V(F_v)) \) one has

\[ I_\psi(\mathcal{F}_{X,\psi}(f_1) \otimes f_2) = I_\overline{\psi}(\mathcal{F}_{X,\overline{\psi}}(f_1) \otimes f_2) = I_\overline{\psi}(\mathcal{F}_{X,\overline{\psi}}(f_1) \otimes f_2). \]

This implies \((12.0.4)\). The space \( \mathcal{S}(Y(F_v)) \) is independent of the character \( \psi \) by Lemma 5.2. Thus to show \((12.0.5)\), by \((12.0.3)\) it suffices to show \( \mathcal{F}_{Y,\psi} \circ \mathcal{F}_{Y,\overline{\psi}}(f) = f \) for functions \( f \) of the form \( I_\overline{\psi}(f_1 \otimes f_2) \). We compute

\begin{align*}
& \quad \mathcal{F}_{Y,\psi} \circ \mathcal{F}_{Y,\overline{\psi}}(I_\overline{\psi}(f_1 \otimes f_2)) \\
& = \mathcal{F}_{Y,\psi}(I_\overline{\psi}(\mathcal{F}_{X,\overline{\psi}}(f_1) \otimes f_2)) \\
& = \frac{\gamma(\mathcal{Q}, \overline{\psi})}{\gamma(\mathcal{Q}, \psi)} \mathcal{F}_{Y,\psi} \circ L(-1)I_\psi(L(m(-1)))R(-I_3I_3) \mathcal{F}_{X,\overline{\psi}}(f_1) \otimes f_2) \quad \text{(Lemma 5.2)} \\
& = \frac{\gamma(\mathcal{Q}, \overline{\psi})}{\gamma(\mathcal{Q}, \psi)} \mathcal{F}_{Y,\psi} \circ I_\psi(L(m(-1)))R(-I_3I_3) \mathcal{F}_{X,\overline{\psi}}(f_1) \otimes L(-1)f_2 \\
& = \frac{\gamma(\mathcal{Q}, \overline{\psi})}{\gamma(\mathcal{Q}, \psi)} I_\psi \circ \mathcal{F}_{X,\psi}(L(m(-1)))R(-I_3I_3) \mathcal{F}_{X,\overline{\psi}}(f_1) \otimes L(-1)f_2 \\
& = \frac{\gamma(\mathcal{Q}, \overline{\psi})}{\gamma(\mathcal{Q}, \psi)} I_\psi (R(I_3 - I_3) \mathcal{F}_{X,\psi}(L(m(-1))) \mathcal{F}_{X,\overline{\psi}}(f_1) \otimes L(-1)f_2) \quad \text{(Lemma 3.3)} \\
& = \frac{\gamma(\mathcal{Q}, \overline{\psi})}{\gamma(\mathcal{Q}, \psi)} I_\psi (R(I_3 - I_3) \mathcal{F}_{X,\overline{\psi}}(f_1) \otimes L(-1)f_2) \quad \text{(Proposition 3.9 and (12.0.6))}
\end{align*}
Here the last equality follows from the fact that $m(-1)[P, P](F) = (-I_6)[P, P](F)$. \hfill \Box

We now explain how to deduce Theorem 1.2 from Theorem 1.3 and Theorem 12.1.

**Proof of Theorem 1.2.** Given such $f$, we can choose $f_{v_i} \in C^\infty_c(\gamma_b G(F_{v_i}))$ for $i = 1, 2$ such that $I(f_{v_1} \otimes f_{v_2}) = f_{v_1}$ and $I(f_{v_2} \otimes f_{v_2}) = F_Y(f_{v_2})$ where $f_{2v_2}Y_{\text{sim}}(F_{v_2}) = F_Y(f_{v_2})$. Indeed, we can take $f_{v_i}$ to be a scalar multiple of the characteristic function of a sufficiently small neighborhood of $\gamma_b$ in $\gamma_b G(F_{v_i})$.

Moreover, choose $f'_{v_1v_2} \in S(X(A_{v_1}^{v_2}) \times V(A_{v_1}^{v_2}))$ such that $I(f'_{v_1v_2}) = f_{v_1}^{v_2}$. To deduce the theorem, we now apply Theorem 1.3 to $f' = (f_{v_1} \otimes f_{v_1})(F^{-1}_X(f_{v_2}) \otimes f_{v_2})f'_{v_1v_2}$. Assumption (6.0.1) is clearly valid, and (6.0.2) is valid by our hypotheses on $f_{v_1}$ and $F_Y(f_{v_2})$. By construction, the boundary terms vanish and the theorem is proved. \hfill \Box

**List of symbols**

- $b_X$: basic function on $X$ (3.1.11)
- $b_Y$: basic function on $Y$ (8.1.1)
- $f_{xs}$: local Mellin transform (3.1.1)
- $|f|_{A,B,p}$: seminorm (3.1.3)
- $|f|_{A,B,w,pw,\Omega,D}$: seminorm (3.1.4)
- $\mathcal{F}_X$: Fourier transform on $\mathcal{S}(X(F))$ (§3.1)
- $\mathcal{F}_Y$: Fourier transform on $\mathcal{S}(Y(F))$ (§12)
- $G$: $\text{SL}_2^3$ (4.0.10)
- $g$: Lie algebra of $M^{ab} \times \text{Sp}_{2n}$ (§3.1)
- $\gamma_i$: representatives of $X^\circ(F)/G(F)$ (4.0.11)
- $G_{\gamma_i}$: stabilizer of $\gamma_i$ in $G$ (§4)
- $|g|$: the norm of $g$ under the Plücker embedding (3.1.6)
- $H$: subgroup of a similitude group on $V$ (4.0.3)
- $I$: integral operator attached to the representative $\gamma_b$ (5.0.1)
- $I_0$: integral operator attached to the representative $\text{Id} = I_0$ (5.0.1)
- $I_i$: integral operator attached to the representative $\gamma_i$ (5.0.4)
- $L(m)R(g)$: action of $M^{ab}(F) \times \text{GSp}_6(F)$ on $\mathcal{S}(X(F))$ (3.1.5)
- $\lambda(h)$: similitude norm of $h$ (4.0.4)
- $m$: an isomorphism $M^{ab}(F) \to F^\times$ (3.1.9)
- $M$: Levi subgroup of $P$ (3.0.1)
- $N$: unipotent radical of $P$ (3.0.1)
\[ N_2 \] standard maximal unipotent subgroup in \( SL_2 \) §1.1
\[ \omega \] character of \( M \) (3.0.2)
\[ 1_k \] characteristic function of \( 1_{[P,P](F) \cap (x^k)Sp_n(O)} \) (3.1.10)
\[ P \] Siegel parabolic §3
\[ \text{Pl} \] Plücker embedding (3.0.5)
\[ (V, Q) \] \[ \prod_{i=1}^3 (V_i, Q_i) \] §1
\[ V^o \] \[ \prod_{i=1}^3 (V_i - \{0\}) \] (4.0.1)
\[ (V_i, Q_i) \] quadratic space of even dimension §1
\[ X \] affine closure of \( X^o \) (3.0.6)
\[ X^o \] Braverman-Kazhdan space (3.0.3)
\[ Y \] \[ \{(y_1, y_2, y_3) \in V : Q_1(y_1) = Q_2(y_2) = Q_3(y_3)\} \] (4.0.2)
\[ Y^{ani} \] anisotropic vectors in \( Y \) §4
\[ Y^{sm} \] smooth locus of \( Y \) §4
\[ Y_0 \] \[ \tilde{Y}_0 / \mathbb{G}_m^2 \] (4.0.9)
\[ Y_i \] \[ \tilde{Y}_i / \mathbb{G}_m \] (4.0.9)
\[ \tilde{Y}_0 \] vanishing locus of \( Q_1, Q_2, Q_3 \) in \( V^o \) (4.0.5)
\[ \tilde{Y}_i \] \[ \{(y_1, y_2, y_3) \in V^o : Q_i(y_{i-1}) = Q_{i+1}(y_{i+1}) \text{ and } Q_i(y_i) = 0\} \] (4.0.7)

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