On VOAs Associated to Jordan Algebras of Type $C$

Hongbo Zhao

1 Introduction and Main Results

In is well known that for any $\mathbb{Z}_{\geq 0}$-graded vertex operator algebra (VOA) such that $V_0 = \mathbb{C}1, V_1 = \{0\}$, $V_2$ has a commutative (but not necessarily associative) algebra structure, with the operation given by $a \circ b = a(1)b$. This algebra $V_2$ is called the Griess algebra of $V$.

The case when the Griess algebra $V_2$ is isomorphic to a finite dimensional simple Jordan algebra is of particular interest to us. In [Lam96] and [Lam99], Lam constructed VOAs whose Griess algebras are simple Jordan algebras of Hermitian type. In [AM09], Ashihara and Miyamoto constructed a family of VOAs $V_{J,r}$ parametrized by $r \in \mathbb{C}$, whose Griess algebras are isomorphic to type $B$ simple Jordan algebras $J$.

It is well known in the theory of VOA that we have many ‘universal constructions’. The most important examples include the level $k$ universal VOA $V^k(g)$ associated to a finite dimensional simple Lie algebra $g$, and the universal Virasoro VOA $M(k,0)$ with central charge equals $k$. It is also important to study the irreducibility of these VOAs, and the cases when $V^k(g)$ and $M(k,0)$ are reducible are of particular interest, which is sometimes called ‘degenerate’.

Let $V^k(g)$ and $L(k,0)$ be the VOA simple quotients of $V^k(g)$ and $M(k,0)$ respectively, then in the ‘degenerate’ cases, these simple VOAs have many interesting properties. The irreducibility of the universal VOAs $V^k(g)$ and $M(k,0)$, together with the constructions and properties of the simple VOAs $V^k(g)$ and $L(c,0)$, has already been extensively studied in literatures for decades.

In our previous paper [Zha16b], we made a further study of the VOA $V_{J,r}$ constructed by Ashihara and Miyamoto. We showed that the VOAs $V_{J,r}$ are the ‘universal VOAs’ satisfying $V_0 = \mathbb{C}1, V_1 = \{0\}$, $V_2 \simeq J$, which resembles the VOAs $V^k(g)$ and $M(c,0)$ mentioned above. We also explicitly constructed the simple quotients $\tilde{V}_{J,r}$ for $r \in \mathbb{Z}_{\neq 0}$, using dual-pair type constructions. We also reprove that $V_{J,r}$ is simple (or equivalently, $\tilde{V}_{J,r} = \tilde{V}_{J,r}$) if and only if $r \notin \mathbb{Z}$ using a different method. It follows that the VOA constructed by Lam for a type $B$ simple Jordan algebra, is actually isomorphic to $V_{J,1}$.

The main result of this paper, is to extend the construction of Ashihara and Miyamoto for the VOA $V_{J,r}$, to an arbitrary Hermitian type Jordan algebra $J$. We study the simplicities of these VOAs $V_{J,r}$, and prove similar irreducibility...
results. We construct the corresponding simple quotients $\bar{V}_{J,r}$ explicitly using dual-pair type constructions, which are parallel to the results in \cite{Zhao16a}. We also show that any simple VOA constructed by Lam in \cite{Lam99}, is isomorphic to one of the simple quotients $\bar{V}_{J,1}$.

In particular, we have

**Theorem 1.** Let $J = J(W)$ be the type C Jordan algebra associated to a symplectic space $W$, $\dim(W) \geq 4$, then

1. For arbitrary complex number $r$, there is a VOA $V = V_{J,r}$ such that $V_0 = \mathbb{C}1$, $V_1 = \{0\}$. For the Greiss algebra $V_2$ we have

   $$V_2 \cong J,$$

   and the central charge of $V_{J,r}$ equals $-\dim(W)r$.

2. $V = V_{J,r}$ is generated by $V_2$.

3. $V_{J,r}$ is simple if and only if $r /\in \mathbb{Z}$.

The content of this article is divided into four parts. In Section 2 we briefly review some basic facts about finite dimensional simple Jordan algebras. In Section 3 we review the VOA $V_{J,r}$, where $J$ is a Type B Jordan Algebra, and our exposition is a little bit different from the one given in \cite{AM09}. It will be seen that neither the central extension, or normal ordering is needed in the construction. Similar to the approach given in Section 3, we construct the VOA $V_{J,r}$, where $J$ is a Type C Jordan Algebra in Section 4, and we prove (1) and (2) in Theorem 1, which is similar to the main results in \cite{AM09} and \cite{NS10}. Finally in Section 5, we prove the simplicity result for the VOA $V_{J,r}$, where $J$ is a Type C Jordan Algebra.

## 2 Finite Dimensional Simple Jordan Algebras and Jordan Frames

In this section, we recall some basic facts about finite dimensional simple Jordan algebras.

Recall that a Jordan algebra $J$ is a vector space, together with a bilinear map $\circ : J \times J \to J$ called Jordan product, satisfying

$$x \circ y = y \circ x,$$

$$(x \circ y) \circ (x \circ x) = x \circ (y \circ (x \circ x)),$$

for all $x, y \in J$. It follows from the definition that a Jordan algebra is commutative, but it is not necessarily associative.

For any associative algebra $A$, it has a structure of Jordan algebra, with the Jordan product given by:

$$x \circ y = \frac{1}{2}(xy + yx), \text{ for all } x, y \in A.$$
We call \( \{ J, \circ \} \) a special Jordan algebra if it is isomorphic to a Jordan subalgebra of \( \{ A, \circ \} \), for some associative algebra \( A \). Otherwise we say the Jordan algebra \( \{ J, \circ \} \) is exceptional.

It is well known that the finite dimensional simple Jordan algebras over an algebraically closed field of characteristic zero is classified by A. Albert in \([Alb47]\). When the ground field is \( \mathbb{C} \), there are five types of simple Jordan algebras, called type \( A, B, C, D \) and \( E \) Jordan algebras respectively. For details about the general theory of Jordan algebras, the readers can consult \([McC06], [FK94]\).

For our purpose, we only describe type \( A, B \) and \( C \) Jordan algebras here.

We realize type \( A \) and type \( B \) Jordan algebras using tensors. Let \( (h, (\cdot, \cdot)) \) be a \( d \)-dimensional vector space with a non-degenerate bilinear form \((\cdot, \cdot)\). Then \( h \otimes h \) has an associative algebra structure:

\[
(a \otimes b)(u \otimes v) = (b, u)a \otimes v,
\]

which induces a Jordan algebra structure on \( h \otimes h \):

\[
x \circ y = \frac{1}{2} (xy + yx), \text{ for all } x, y \in h \otimes h.
\]  

We call \( h \otimes h \) the type \( A \) Jordan algebra associated to \( h \), denoted by \( J_A(h) \), and we set

\[
\tilde{L}_{a,b} := a \otimes b \in h \otimes h.
\]

We further assume that the bilinear form \((\cdot, \cdot)\) is symmetric, and let \( J_B(h) \) be the Jordan subalgebra of \( J_A(h) \), which consists of symmetric square tensors:

\[
J_B(h) := S^2(h) = \text{span}\{L_{a,b} | a, b \in h\}, \quad L_{a,b} := a \otimes b + b \otimes a.
\]

We call \( J_B(h) \) the type \( B \) Jordan algebra associated to \( h \). It is easy to check that these definitions coincide with the definitions using matrices.

We realize type \( C \) Jordan algebras in a similar way. Let \( W \) be a \( 2d \)-dimensional space with a non-degenerated bilinear form \( \langle \cdot, \cdot \rangle \), then we have the corresponding type \( A \) Jordan algebra \( J_A(W) \). We further assume that \( \langle \cdot, \cdot \rangle \) is skew-symmetric, and we consider the Jordan subalgebra \( J_C(W) \) of \( J_A(W) \), which consists of anti-symmetric tensors:

\[
J_C(W) := \wedge^2(W) = \text{span}\{L_{a,b} | a, b \in W\}, \quad L_{a,b} := a \otimes b - b \otimes a.
\]

We call \( J_C(W) \) the type \( C \) Jordan algebra associated to \( W \). It is also easy to check that this definition is the same as the definition of type \( C \) Jordan algebras using skew-symmetric matrices.

By \((1), (2), (3), \) and \((4)\), it is easy to check the following explicit formulas about Jordan products in type \( A, B \) and \( C \) Jordan algebras, which are useful in later discussions:

\[
L_{a,b} \circ L_{u,v} = \frac{1}{2} (b, u)L_{a,v} + \frac{1}{2} (b, v)L_{a,u} + \frac{1}{2} (a, u)L_{b,v} + \frac{1}{2} (a, v)L_{b,u}.
\]
for all $L_{a,b}, L_{u,v} \in \mathcal{J}_B(\mathfrak{h})$;

\[ L_{a,b} \circ L_{u,v} = \frac{1}{2}(b,u)L_{a,v} - \frac{1}{2}(b,v)L_{a,u} - \frac{1}{2}(a,u)L_{b,v} + \frac{1}{2}(a,v)L_{b,u}, \quad (5) \]

for all $L_{a,b}, L_{u,v} \in \mathcal{J}_C(W)$;

\[ \tilde{L}_{a,b} \circ \tilde{L}_{u,v} = \frac{1}{2}(b,u)\tilde{L}_{a,v} + \frac{1}{2}(a,v)\tilde{L}_{b,u}, \]

for all $\tilde{L}_{a,b}, \tilde{L}_{u,v} \in \mathcal{J}_A(\mathfrak{h})$.

We write down the Jordan frame of type $B$ and type $C$ Jordan algebras for later use. Recall that the Jordan frame of a simple Jordan algebra [FK94] is a set of idempotents $u_1, \ldots, u_n$ such that each $u_i$ is not a sum of two non-zero mutually orthogonal idempotents, and

\[ \sum u_i = e, \quad u_i \circ u_j = \delta_{i,j} u_i \]

for all $i, j$, where $e$ is the identity element in the Jordan algebra. For type $B$ and type $C$ Jordan algebras it is easy to find out the corresponding Jordan frames. Let $\{e_1, \ldots, e_d\}$ be an orthonormal basis of $\mathfrak{h}$, and $\{\psi_1, \ldots, \psi_n, \psi_1^*, \ldots, \psi_n^*\}$ be a symplectic basis of $W$ such that

\[ \langle \psi_i^*, \psi_j \rangle = \delta_{i,j}, \quad \langle \psi_i^*, \psi_j^* \rangle = \langle \psi_i, \psi_j \rangle = 0 \]

for all $1 \leq i, j \leq n$. Then we check that the Jordan frame of $\mathcal{J}_B(\mathfrak{h})$ can be given by

\[ \{\frac{1}{2}L_{e_i, e_i} | i = 1, \ldots, d\}, \]

and for the Jordan algebra $\mathcal{J}_C(W)$:

\[ \{L_{\psi_i, \psi_i^*} | i = 1, \ldots, n\}. \]

It will be seen later that the elements in the Jordan frame for each case corresponds to a set of mutually orthogonal Virasoro elements in the Greiss algebras, which will play a role in analyzing the corresponding VOA structures.

3 Construction of the VOA $V_{\mathcal{J}, r}$, Where $\mathcal{J}$ is a Type $B$ Jordan Algebra

In this section we review the VOA $V_{\mathcal{J}, r}$ constructed by Ashihara and Miyamoto, Where $\mathcal{J}$ is a Type $B$ Jordan Algebra. We explain the corresponding Lie algebra $\mathcal{L}$ as a central extension of another familiar infinite dimensional Lie algebra. This approach is slightly different from the original one given in [AM09], but they are essentially the same. Our approach is very similar to the one given by Kac and Radul in [KR96], and it is easy to see the analogy between $V_{\mathcal{J}, r}$ and the universal VOA $M_c$ given in [KR96]. The construction described in this section
will also inspire construction for type $C$ Jordan algebra, which will be given in Section 3.

We fix the notation that $\mathfrak{h}$ denotes a $d$-dimensional vector space with a non-degenerate symmetric bilinear form $(\cdot, \cdot)$, $\{e_1, \ldots, e_d\}$ be an orthonormal basis of $\mathfrak{h}$, and $J$ be the type $B$ Jordan algebra $J_B(\mathfrak{h})$. We recall the following infinite dimensional Lie algebra $\hat{\mathfrak{h}}$ associated to $\mathfrak{h}$:

$$\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c.$$ 

The Lie bracket over $\hat{\mathfrak{h}}$ is given by:

$$[a(m), b(n)] = m(a, b)\delta_{m+n, 0}c, \quad [x, c] = 0, \text{ for all } x \in \hat{\mathfrak{h}}.$$ 

It is well known that $\hat{\mathfrak{h}}_{-} \overset{def}{=} \mathfrak{h} \otimes \mathbb{C}t^{-1}[t^{-1}]$ is a commutative Lie subalgebra of $\hat{\mathfrak{h}}$. The Fock space $S(\hat{\mathfrak{h}}_{-}) \simeq U(\hat{\mathfrak{h}}_{-}) \cdot 1$ is a left $U(\hat{\mathfrak{h}})$-module, and $S(\hat{\mathfrak{h}}_{-})$ has a vertex operator algebra structure [FLM88]. We denote this VOA by $\mathcal{H}(\mathfrak{h})$.

Given a vector space $W$ with an antisymmetric bilinear form $\langle \cdot, \cdot \rangle$, the symmetric square $S^2(W)$ is a Lie algebra, and for $ab, uv \in S^2(W)$, the Lie bracket is given by

$$[ab, uv] = \langle b, u \rangle av + \langle a, u \rangle bv + \langle b, v \rangle ua + \langle a, v \rangle ub.$$ (6)

In particular, if $W$ is a finite dimensional symplectic space, the symmetric square $S^2(W)$ is isomorphic to the finite dimensional symplectic Lie algebra $\mathfrak{sp}(W)$. The corresponding Lie bracket is the same as (6), and this construction is essentially related to the Oscillator representations.

We apply this to the case

$$\tilde{W}_\infty = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}].$$

It is obvious that there is an anti-symmetric bilinear form $\langle \cdot, \cdot \rangle'$ on the space of Laurent polynomials $\mathbb{C}[t, t^{-1}]$

$$\langle t^m, t^n \rangle' = m\delta_{m+n, 0},$$

therefore $\tilde{W}_\infty$ also has an anti-symmetric bilinear form $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle \otimes \langle \cdot, \cdot \rangle'$

$$\langle a(m), b(n) \rangle = m\delta_{m+n, 0},$$

which is the same as the value of $[a(m), b(n)]$ by taking $c = 1$. Hence $S^2(\tilde{W}_\infty)$ is also a Lie algebra with the bracket given by (6). We note that $\langle \cdot, \cdot \rangle$ restricted to the following subspace

$$W_\infty \overset{def}{=} \mathfrak{h} \otimes \mathbb{C}[t] \bigoplus \mathfrak{h} \otimes \mathbb{C}t^{-1}[t^{-1}]$$
is non-degenerate, and the Lie subalgebra \( S^2(W_\infty) \) is actually isomorphic to \( \mathfrak{sp}_\infty \), the Lie algebra of infinite symplectic matrices with finitely non zero entries [Kac94]. This has been discussed in [Zha16b], and it plays a key role in proving the simplicity of \( V_{J,r} \).

By [8], we write down the explicit commutation relation for all \( a(m)b(n), u(p)v(q) \in S^2(W_\infty) \), \( m, n \in \mathbb{Z} \), which is useful for computations:

\[
[a(m)b(n), u(p)v(q)] = n\delta_{a+p,0}(b, u)a(m)v(q) + m\delta_{m+p,0}(a, u)b(n)v(q) + n\delta_{a+q,0}(b, v)u(p)a(m) + m\delta_{m+q,0}(a, v)u(p)b(n).
\] 

(7)

Let \( \mathcal{L}(\mathfrak{h}) \) denote the following direct sum of Lie algebras

\[ \mathcal{L}(\mathfrak{h}) \overset{\text{def.}}{=} S^2(\hat{\mathfrak{w}}_\infty) \bigoplus \mathbb{C}K, \]

where \( K \) is the central element in \( \mathcal{L}(\mathfrak{h}) \). Define

\[ \mathfrak{B}_+ \overset{\text{def.}}{=} \text{span}\{L_{a,b}(m, n) | n \geq 0 \text{ or } m \geq 0\}, \]

\[ \mathcal{L}_- \overset{\text{def.}}{=} \text{span}\{L_{a,b}(m, n) | m, n < 0\}, \]

\[ \mathcal{L}_+ \overset{\text{def.}}{=} \mathfrak{B}_+ \bigoplus \mathbb{C}K. \]

Then we have a decomposition of \( \mathcal{L}(\mathfrak{h}) \):

\[ \mathcal{L}(\mathfrak{h}) = \mathcal{L}_- \bigoplus \mathcal{L}_+ = \mathcal{L}_- \bigoplus \mathfrak{B}_+ \bigoplus \mathbb{C}K. \]

As mentioned in [Zha16b], \( \mathcal{L}_+ \) is certain ‘parabolic subalgebra’ of \( \mathcal{L}(\mathfrak{h}) \) [KR93].

Now we consider a map from \( \mathcal{L}(\mathfrak{h}) \) to \( U(\mathfrak{h})/(e-1) \). Let \( (e-1) \) be the two-sided ideal of \( U(\mathfrak{h}) \) generated by \( e-1 \). We consider the following map

\[ \iota: a(m)b(n) \mapsto \frac{1}{2}(a(m)b(n) + b(n)a(m)), S^2(\hat{\mathfrak{w}}_\infty) \to U(\mathfrak{h}). \]

It is checked that

\[ c\alpha([x, y]) = [\iota(x), \iota(y)] \]

for all \( x, y \in S^2(\hat{\mathfrak{w}}_\infty) \). In particular \( \iota \) is a Lie algebra embedding if we take \( c = 1 \), and \( \iota \) becomes a map from \( S^2(\hat{\mathfrak{w}}_\infty) \) to \( U(\mathfrak{h})/(e-1) \). We remark that the \( \frac{1}{2} \) on the left hand side guarantees that \( \iota \) is indeed a Lie algebra embedding.

Hence the Lie algebra \( S^2(\hat{\mathfrak{w}}_\infty) \) also acts on the Fock space \( S(\mathfrak{h}_-) \) by

\[ (a(m)b(n)) \cdot v \overset{\text{def.}}{=} \frac{1}{2}(a(m)b(n) + b(n)a(m))v \]

for any \( a(m)b(n) \in S^2(\hat{\mathfrak{w}}_\infty) \) and \( v \in S(\mathfrak{h}_-) \). We also note that \( \mathcal{L}_+ \cap \mathfrak{sp}_\infty \) is also a parabolic subalgebra of \( \mathfrak{sp}_\infty \), and \( \mathbb{C}1 \) is a one-dimensional \( \mathcal{L}_+ \cap \mathfrak{sp}_\infty \)-module. More explicitly we have

\[ (a(m)b(n)) \cdot 1 = \frac{1}{2}|m|\delta_{m+n,0}(a, b)1 \]

(8)
for all \(a(m)b(n) \in \mathcal{L}_+ \cap sp_\infty\).

Set
\[
L_{a,b}(m,n) \overset{\text{def.}}{=} \frac{1}{2}a(m)b(n) - \frac{1}{4}|m|\delta_{m+n,0}(a,b)K \in \mathcal{L}(\mathfrak{h}).
\]

We note that \(L_{a,b}(m,n)\) is well defined and
\[
L_{a,b}(m,n) = L_{b,a}(n,m), \quad \mathcal{L}(\mathfrak{h}) = \text{span}\{L_{a,b}(m,n), K \mid a,b \in \mathfrak{h}, m,n \in \mathbb{Z}\}.
\]

We also note that if \(K\) acts as the identity, then the ‘vacuum vector’ in \(S(\hat{\mathfrak{h}}_-)\), then \(C_1\) is a one dimensional \(\mathcal{L}_+\)-module such that
\[
L_{a,b}(m,n) \cdot 1 = 0 \text{ for all } m \geq 0 \text{ or } n \geq 0. \quad (9)
\]

The definition of \(L_{a,b}(m,n)\) and the Lie algebra \(\mathcal{L} = \mathcal{L}(\mathfrak{h})\) are actually the same as the ones given in [Zha16b] and [AM09], although they looks different.

The key point is that when restricts to \(\mathcal{L}_+\), we can define a new one dimensional \(\mathcal{L}_+\)-module related to (9) such that the central element \(K\) acts as \(r\) where \(r\) is arbitrary complex number. Therefore we are able to define a \(\mathcal{L}(\mathfrak{h})\)-module \(M_r\) which is induced from this one dimensional \(\mathcal{L}_+\)-module. We describe it explicitly as follows. Let \(Cv_r\) be the one dimensional \(\mathcal{L}_+\)-module spanned by the element \(v_r\) such that
\[
xv_r = 0, \quad \text{for all } x \in \mathfrak{B}_+, \quad Kv_r = rv_r.
\]

It is easy to check that this indeed gives an \(\mathcal{L}_+\)-module. Equivalently, \(Cv_r\) is also a one dimensional \(sp_\infty \cap \mathcal{L}_+\)-module such that
\[
(a(m)b(n)) \cdot v_r \overset{\text{def.}}{=} \frac{r}{2}|m|\delta_{m+n,0}(a,b)v_r
\]
for all \(a(m)b(n) \in sp_\infty \cap \mathcal{L}_+\), which can be compared with [13]. Then we have an \(\mathcal{L}(\mathfrak{h})\)-module \(M_r\) which is induced from this one dimensional \(\mathcal{L}_+\)-module:
\[
M_r \overset{\text{def.}}{=} U(\mathcal{L}(\mathfrak{h})) \otimes_{U(\mathcal{L}_+)} Cv_r \cong U(\mathcal{L}_-)1
= \text{span}\{L_{a_1,b_1}(-m_1,-n_1) \cdots L_{a_k,b_k}(-m_k,-n_k) \cdot v_r \mid m_i, n_i \in \mathbb{Z}_{\geq 1}, a_i, b_i \in \mathfrak{h}\}. \quad (10)
\]

We also note that \(M_r\) can be viewed as an induced \(sp_\infty\)-module
\[
M_r = U(sp_\infty) \otimes_{U(sp_\infty) \cap \mathcal{L}_+} Cv_r,
\]
and this will be used when discussing the simplicity of the VOA.

For \(a,b \in \mathfrak{h}\), define the operators \(L_{a,b}(l)\) and the fields \(L_{a,b}(z)\)
\[
L_{a,b}(l) \overset{\text{def.}}{=} \sum_{k \in \mathbb{Z}} L_{a,b}(-k+l-1,k), \quad L_{a,b}(z) \overset{\text{def.}}{=} \sum_{l \in \mathbb{Z}} L_{a,b}(l)z^{-l-1}. \quad (11)
\]
It is proved in [AM09] that these fields are mutually local. Therefore by reconstruction theorem [Kac98], these mutually local fields generate a vertex algebra:

\[ V_{\mathcal{J}, r} \overset{\text{def.}}{=} \text{span}\{L_{a_1, b_1}(m_1) \cdots L_{a_k, b_k}(m_k) \cdot v_r | m_i \in \mathbb{Z}, a_i, b_i \in \mathfrak{h}\}, \]

and \( v_r \) being the ‘vacuum’. The vertex algebra \( V_{\mathcal{J}, r} \) is actually a VOA. The Virasoro element \( \omega \) is given by:

\[ \omega = \sum_k L_{e_k, e_k}(-1, -1) \cdot v_r, \]

and \( L(0) = \omega(1) \) gives a gradation on \( V_{\mathcal{J}, r} \)

\[ V_{\mathcal{J}, r} = \bigoplus_{k \geq 0} (V_{\mathcal{J}, r})_k. \]

We also check that

\[ (V_{\mathcal{J}, r})_0 = C1, \quad (V_{\mathcal{J}, r})_1 = \{0\}, \]

and the Griess algebra \((V_{\mathcal{J}, r})_2\) is isomorphic to \( \mathcal{J} \):

\[ L_{a, b}(-1, -1) \cdot 1 \mapsto L_{a, b} \overset{\text{def.}}{=} a \otimes b + b \otimes a. \]

It was further shown in [NS10] that \( V_{\mathcal{J}, r} = M_r \), but we use \( V_{\mathcal{J}, r} \) instead of \( M_r \) to emphasize the corresponding VOA structure.

We give a set of mutually orthogonal Virasoro elements which corresponds to the Jordan frame. We note that the Griess algebra \((V_{\mathcal{J}, r})_2\) is isomorphic to \( \mathcal{J} \):

\[ \mathcal{J}(\mathfrak{h}) \overset{\text{def.}}{=} \{L_{e_i, e_i}, L_{e_i, e_j} | 1 \leq i, j \leq n, i \neq j\}. \]

It is easy to check that the following subset

\[ F(\mathfrak{h}) \overset{\text{def.}}{=} \{L_{e_i, e_i} | 1 \leq i \leq d\} \]

of \( \mathcal{J}(\mathfrak{h}) \) consists of mutually orthogonal Virasoro elements by a direct computation. Each element in \( F(\mathfrak{h}) \) is a Virasoro element of central charge 1, and the half of these elements are bijective to the the elements in the Jordan frame through the isomorphism between \( \mathcal{J}(W) \) and the Greiss algebra \((V_{\mathcal{J}, r})_2\):

\[ L_{e_i, e_i} \mapsto \frac{1}{2} L_{e_i, e_i}, \quad \mathcal{J}(\mathfrak{h}) \rightarrow (V_{\mathcal{J}, r})_2. \]

We remark that the normalized factor \( \frac{1}{2} \) is used to guarantee that this is an algebra homomorphism.

Let \( \bar{V}_{\mathcal{J}, 1} \) be the VOA constructed in [Lam96] which is obtained by taking the \(-1\) fixed point of \( S(\mathfrak{h}) \), where \(-1\) denotes the action induced by multiplying \(-1\) on \( \mathfrak{h} \). In [Zha10b] we showed that \( \bar{V}_{\mathcal{J}, 1} \) is the simple quotient of \( V_{\mathcal{J}, 1} \). We
note that the properties of \( V_{\mathcal{J},r} \) are quite similar to \( \tilde{V}_{\mathcal{J},1} \), therefore we can study \( V_{\mathcal{J},r} \) by comparing it with \( \tilde{V}_{\mathcal{J},1} \). For example, for \( a(-1)b \in \hat{V}_{\mathcal{J},1} \subseteq S(\mathfrak{h}_-) \), the corresponding field is given by

\[
Y(a(-1)b, z) = \sum_{l \in \mathbb{Z}} (a(-1)b)(l)z^{-l-1} = \sum_{k, l \in \mathbb{Z}} :a(-k + l - 1)b(k) : z^{-l-1}.
\]

If we replace \( a(-1)b \) with \( 2L_{a,b} \) and \( :a(m)b(n) : \) with \( 2L_{a,b}(m, n) \) in the above, it is exactly the same as (11). We can also get some other identities starting from the identities in \( V_{\mathcal{J},1} \) (and therefore in \( S(\mathfrak{h}_-) \)). The difference appears when the central element \( c \) arises in the computation. For example, for \( a(-1)b, u(-1)v \in \hat{V}_{\mathcal{J},1} \), we have

\[
(a(-1)b, u(-1)v) = (a(-1)b)(1)u(-1)v = (a, u)(b, v) + (a, v)(b, u),
\]

while for \( 2L_{a,b}, 2L_{u,v} \in V_{\mathcal{J},r} \), we have

\[
(2L_{a,b}, 2L_{u,v}) = (2L_{a,b})(1)2L_{u,v} = r(a, u)(b, v) + r(a, v)(b, u) = r(a(-1)b, u(-1)v).
\]

In [Zha16a] we computed the correlation functions of generating fields in \( V_{\mathcal{J},r} \). If we view the correlation function as a function of \( r \), the result is a polynomial function of \( r \).

4 Construction of the VOA \( V_{\mathcal{J},r} \), Where \( \mathcal{J} \) is a Type C Jordan Algebra

Let \( r \) be arbitrary complex number. In this section, we construct the VOA \( V_{\mathcal{J},r} \) satisfying the property that \( (V_{\mathcal{J},r})_0 = C_1 \), \( (V_{\mathcal{J},r})_1 = \{0\} \), and \( (V_{\mathcal{J},r})_2 \) is isomorphic to the type \( C \) Jordan algebra \( \mathcal{J}_C(W) \).

Recall that in Section 3, we realize the type \( B \) simple Jordan algebra \( \mathcal{J}_B(\mathfrak{h}) \) using the symmetric square space \( S^2(\mathfrak{h}) \), and the type \( C \) simple Jordan algebra \( \mathcal{J}_C(W) \) is realized in a similar way using the symmetric wedge \( \wedge^2(W) \). The analogy between these two types of Jordan algebras indicates the construction of \( V_{\mathcal{J},r} \) for \( \mathcal{J} \simeq \mathcal{J}_C(W) \), which is almost parallel to Section 3.

In this section, we fix the notation that \( W \) denotes a \( 2n \)-dimensional symplectic space with the symplectic form \( \langle \cdot, \cdot \rangle \), and \( \mathcal{J} \simeq \mathcal{J}_C(W) \). We cite the following formulas for a given VOA \( V \) without proof:

\[
[x(m), y(n)] = \sum_{j \geq 0} \binom{m}{j} (x(j)y)(m + n - j),
\]

\[
(x(n)y)(l) = \sum_{i \geq 0} \binom{n}{i} (-1)^i (x(-i + n)y(i + l) - (-1)^n y(-i + n + l)x(i)),
\]

for all \( x, y \in V \) and \( m, n, l \in \mathbb{Z} \).
The details can be found, for example, in [Kac98]. These formulas will be frequently used in this section. Here we note that the combinatorial coefficients means

\[
\binom{m}{i} \overset{\text{def.}}{=} \frac{m \times \cdots (m-i+1)}{i \times \cdots 1}
\]

for all \(m \in \mathbb{Z}, i \in \mathbb{Z}_{\geq 1}\), and

\[
\binom{m}{0} = 1
\]

for all \(m \in \mathbb{Z}\) by convention. An elementary calculation shows that the recursion relation

\[
\binom{m}{i} = \binom{m-1}{i-1} + \binom{m-1}{i}
\]

holds for all \(m \in \mathbb{Z}, i \in \mathbb{Z}_{\geq 1}\).

Given a vector space \(V\) with a symmetric bilinear form \((\cdot, \cdot)\), the space \(\wedge^2(V)\) is a Lie algebra such that for \(ab, uv \in \wedge^2(W)\), the Lie bracket is given by

\[
[a, b] = (b, u)av - (a, u)bv + (b, v)ua - (a, v)ub.
\]

In particular when \(V\) is finite dimensional and \((\cdot, \cdot)\) is non-degenerate, \(\wedge^2(V)\) is isomorphic to the finite dimensional simple Lie algebra \(\mathfrak{so}(V)\). It is well known that this is a special case of the spinor construction of irreducible \(\mathfrak{so}(V)\)-modules.

We apply this to \(\widetilde{V}_\infty = W \otimes \mathbb{C}[t, t^{-1}]\).

Recall the anti-symmetric bilinear form \((\cdot, \cdot)'\) over the space of Laurent polynomials \(\mathbb{C}[t, t^{-1}]\) given in Section 3. It is obvious that \(\widetilde{V}_\infty\) has a symmetric bilinear form \((\cdot, \cdot) = (\cdot, \cdot) \otimes (\cdot, \cdot)';\)

\[
(a(m), b(n)) = m\langle a, b \rangle \delta_{m+n,0},
\]

hence \(\wedge^2(\widetilde{V}_\infty)\) is a Lie algebra with the bracket given by (16). The following explicit formula for the commutation relation, which can be easily derived from (16), is useful for later computations:

\[
[a(m)b(n), u(p)v(q)]
\]

\[
= n\delta_{m+p,0}\langle b, u \rangle a(m)v(q) - m\delta_{m+p,0}\langle a, u \rangle b(n)v(q)
\]

\[
- n\delta_{m+q,0}\langle b, v \rangle u(p)a(m) + m\delta_{m+q,0}\langle a, v \rangle u(p)b(n),
\]

for all \(a(m)b(n) \in \wedge^2(\widetilde{V}_\infty), m, n \in \mathbb{Z}\).

We also note that the symmetric bilinear form \((\cdot, \cdot)\) restricted to the subspace

\[
V_\infty \overset{\text{def.}}{=} W \otimes \mathbb{C}[t] \bigoplus W \otimes \mathbb{C}[t^{-1}]
\]

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is non-degenerate, and the Lie subalgebra \( \bigwedge^2 (V_{\infty}) \) is actually isomorphic to \( \mathfrak{so}_{\infty} \), the Lie algebra of infinite orthogonal matrices [Kac94].

We also recall some basic facts about an infinite dimensional Lie superalgebra associated to \( W \), and the corresponding SVOA called the symplectic Fermion SVOA. Note that there is a Lie superalgebra \( \hat{W} \) associated to \( W \):
\[
\hat{W} = \hat{V}_{\infty} \oplus Cc.
\]
The even part \( \hat{W}_0 \) and the odd part \( \hat{W}_1 \) are
\[
\hat{W}_0 = Cc, \quad \hat{W}_1 = \hat{V}_{\infty},
\]
and the Lie superbracket is given by
\[
[a(m), b(n)] = (a(m), b(n))c = m(a, b) \delta_{m+n,0} c, \quad [x, c] = 0, \quad \text{for all } x \in \hat{W},
\]
where \( a(m) \overset{\text{def.}}{=} a \otimes t^m, a \in W \). It follows that
\[
\hat{W}_- \overset{\text{def.}}{=} W \otimes \mathbb{C} t^{-1} [t^{-1}]
\]
is a super-commutative Lie sub-superalgebra of \( \hat{W} \). The enveloping algebra \( U(\hat{W}) \) is essentially certain kind of infinite dimensional Clifford algebra, and the Fock space \( \bigwedge (\hat{W}_-) \cdot 1 \cong \bigwedge (\hat{W}_-) \) is a left \( U(\hat{W}) \)-module. The space \( \bigwedge (\hat{W}_-) \) has a SVOA structure, and the field associated to \( x = x(-1) \cdot 1 \) is:
\[
Y(x, z) = \sum_k x(k) z^{-k-1}, \quad x \in \hat{W}.
\]
This is called the ‘symplectic Fermion’ SVOA in literatures [Abe07], denoted by \( \mathcal{A}(W) \).

Let
\[
\mathcal{L}(W) \overset{\text{def.}}{=} \bigwedge^2 (\hat{V}_{\infty}) \bigoplus C K,
\]
where \( K \) is the central element:
\[
[x, K] = 0, \quad \text{for all } x \in \bigwedge^2 (\hat{V}_{\infty}).
\]
We note that there is a map
\[
\iota : a(m)b(n) \mapsto \frac{1}{2} (a(m)b(n) - b(n)a(m))
\]
from \( \bigwedge^2 (\hat{V}_{\infty}) \) to \( U(\hat{W}) \). It is easy to check that \( \iota \) satisfies
\[
\iota ([x, y]) = [\iota(x), \iota(y)]. \quad (18)
\]
In particular, if we view \( \iota \) as a map from \( \bigwedge^2 (\hat{V}_{\infty}) \) to \( U(\hat{W})/(c-1) \), then \( \iota \) is a Lie algebra embedding. Therefore \( \bigwedge^2 (\hat{V}_{\infty}) \) acts on the Fermionic Fock space \( \bigwedge (\hat{W}_-) \cdot 1 \).

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Set

\[ L_{a,b}(m,n) \overset{def.}{=} a(m)b(n) - \frac{1}{2} |m| \delta_{m+n,0} \langle a,b \rangle K \in \mathcal{L}(W). \]  

(19)

The element \( L_{a,b}(m,n) \) is also well defined and

\[ L_{a,b}(m,n) = -L_{b,a}(n,m), \quad \mathcal{L}(W) = \text{span}\{L_{a,b}(m,n), K \mid a,b \in W, m,n \in \mathbb{Z}\}. \]

Note that there is a sign change for \( L_{a,b}(m,n) \) when we exchange the pairs \((a,m), (b,n) \in W \times \mathbb{Z}\). Define

\[ \mathcal{B}_+ \overset{def.}{=} \text{span}\{L_{a,b}(m,n) \mid n \geq 0 \text{ or } m \geq 0\}, \]

\[ \mathcal{L}_- \overset{def.}{=} \text{span}\{L_{a,b}(m,n) \mid m,n < 0\}, \quad \mathcal{L}_+ \overset{def.}{=} \mathcal{B}_+ \bigoplus \mathbb{C}K. \]

Then we have a decomposition of \( \mathcal{L}(W) \):

\[ \mathcal{L}(W) = \mathcal{L}_- \bigoplus \mathcal{L}_+ = \mathcal{L}_- \bigoplus \mathcal{B}_+ \bigoplus \mathbb{C}K. \]

Similar to what we have done in the type B case, we can also construct a one dimensional \( \mathcal{L}_+ \)-module such that \( K \) acts as arbitrary complex number \( r \). Let \( C_v_r \) be the one dimensional \( \mathcal{L}_+ \)-module such that\n
\[ xv_r = 0, \quad \text{for all } x \in \mathcal{B}_+, Kv_r = rv_r. \]  

(20)

Then we construct \( M_r \) as an induced \( \mathcal{L}(W) \)-module:

\[ M_r \overset{def.}{=} U(\mathcal{L}(W)) \otimes_{U(\mathcal{L}_+)} C_v_r \cong U(\mathcal{L}_+)1 \]

\[ = \text{span}\{L_{a_1,b_1}(-m_1,-n_1) \cdots L_{a_k,b_k}(-m_k,-n_k) \cdot v_r \mid m_i,n_i \in \mathbb{Z}_{\geq 1}, a_i,b_i \in W\}. \]  

(21)

We note that \( C_v_r \) is also a one dimensional \( \mathfrak{so}_\infty \cap \mathcal{L}_+ \)-module. The action is given by

\[ (a(m)b(n)) \cdot v_r = \frac{r}{2} |m| \delta_{m+n,0} \langle a,b \rangle v_r \]  

(22)

for all \( a(m)b(n) \in \mathfrak{so}_\infty \cap \mathcal{L}_+ \) by (19), and \( M_r \) is also an induced \( \mathfrak{so}_\infty \)-module:

\[ M_r = U(\mathfrak{so}_\infty) \otimes_{U(\mathfrak{so}_\infty) \cap \mathcal{L}_+} C_v_r. \]

This will be used later in Section 6.

For \( a,b \in W \), define the operators \( L_{a,b}(l) \) and the fields \( L_{a,b}(z) \)

\[ L_{a,b}(l) \overset{def.}{=} \sum_{k \in \mathbb{Z}} L_{a,b}(-k+l-1,k), \quad L_{a,b}(z) \overset{def.}{=} \sum_{l \in \mathbb{Z}} L_{a,b}(l)z^{-l-1}. \]  

(23)

Then we have
Proposition 1. The formal power series $L_{a,b}(z)$, $a, b \in W$ are mutually local fields over $M_r$.

Proof of Proposition 1. The proof is essentially the same as the proof given in [AM09] for type $B$ case. First it easy to show that for a given $v \in M_r$, there exists an integer $N$ (depending on $a$, $b$ and $v$) such that

$$L_{a,b}(n)v = 0$$

for all $n \geq N$, hence $L_{a,b}(z)$ are all fields. We also check that for all $n \geq 0$,

$$L_{a,b}(n) \cdot 1 = 0.$$

Next we show the locality by comparing the fields $L_{a,b}(z)$ with the fields $Y(a(-1)b, z)$ in the symplectic Fermion SVOA $A(W)$. It is easy to check that for all $a, b \in W$ the $U(\hat{W})$-valued formal power series are mutually local:

$$(z - w)^2[a(z), b(w)] = 0.$$

By a variant of Dong’s lemma for Lie superalgebra valued formal power series [Kac98], for any $a, b, u, v \in W$, the $U(\hat{W})$-valued formal power series $Y(a(-1)b, z)$ and $Y(u(-1)v, z)$ are also mutually local:

$$(z - w)^4[Y(a(-1)b, z), Y(u(-1)v, z)] = 0.$$

By (18), (19), and (23) we have

$$c(z - w)^4\iota([L_{a,b}(z), L_{u,v}(w)]) = \iota(Y(a(-1)b, z)), \iota(Y(u(-1)v, w))] = 0.$$

as $U(\hat{W})$-valued formal power series. Because $\iota$ is an embedding and $cx = 0$ implies $x = 0$ in $U(\hat{W})$, therefore we have

$$(z - w)^4[L_{a,b}(z), L_{u,v}(w)] = 0,$$

and we conclude the proof of the locality of the fields. □

By Proposition 1 and the reconstruction theorem [Kac98], these mutually local fields generate a vertex algebra:

$$V_{\mathcal{J},r} \overset{\text{def}}{=} \text{span}\{L_{a_1,b_1}(m_1) \cdots L_{a_k,b_k}(m_k) \cdot 1 | m_i \in \mathbb{Z}, a_i, b_i \in \mathfrak{h}\}.$$

Now we check the remaining part of (1) in Theorem 1 about the VOA $V_{\mathcal{J},r}$. We note that to do the computations, we only need (17) and (19). For any $a, b, u, v \in W$, a direct computation shows that

$$(L_{a,b})(1)(L_{u,v}) = \langle b, u \rangle L_{a,v} - \langle b, v \rangle L_{a,u} - \langle a, u \rangle L_{b,v} + \langle a, v \rangle L_{b,u}.$$
Compared with (5) it is clear that

\[ L_{a,b} \mapsto L_{a,b}, \ (V_{\mathcal{J},r})_2 \mapsto J(W) \]

is an algebra isomorphism.

Now we give a set of mutually orthogonal Virasoro elements in \((V_{\mathcal{J},r})_2\), which corresponds to the elements in the Jordan frame of \(J(W)\) mentioned in Section 2. We note that the Greiss algebra \((V_{\mathcal{J},r})_2 \simeq \bigwedge^2(W)\) is \(n(2n-1)\) dimensional, which is spanned by

\[ J(W) \overset{def}{=} \{L_{\psi_i,\psi_i^*}, L_{\psi_i,\psi_j}, L_{\psi_i^*,\psi_j^*}, L_{\psi_j,\psi_k^*} \mid 1 \leq i, j, k \leq n, j \neq k\} \]

It is easy to check that the following subset

\[ F(W) \overset{def}{=} \{L_{\psi_i,\psi_i^*} \mid 1 \leq i \leq n\} \]

of \(J(W)\) consists of mutually orthogonal Virasoro elements by a direct computation. Each element in \(F(W)\) is a Virasoro element of central charge \(-2\), and the half of these elements are bijective to the elements in the Jordan frame through the isomorphism between \(J(W)\) and the Greiss algebra \(V_{\mathcal{J},r}\):

\[ \frac{1}{2}L_{\psi_i,\psi_i^*} \mapsto L_{\psi_i,\psi_i^*}, \ (V_{\mathcal{J},r})_2 \mapsto J_C(W) \]

Moreover, the sum of these elements

\[ \omega \overset{def}{=} \sum_i L_{\psi_i,\psi_i^*} \]

is also a Virasoro element, with central charge equals \(-2n\). The element \(\omega(1)\) gives the gradation of \(V_{\mathcal{J},r}\):

\[ V_{\mathcal{J},r} = \sum_{k \geq 0} (V_{\mathcal{J},r})_k, \]

where \(k\) is the eigenvalue of \(\omega(1)\). It follows easily from the construction of \(V_{\mathcal{J},r}\) that \((V_{\mathcal{J},r})_0 = \mathbb{C}1\) and \((V_{\mathcal{J},r})_1 = \{0\}\); hence the Greiss algebra \((V_{\mathcal{J},r})_2\) is indeed isomorphic to the type \(C\) Jordan algebra \(J(W)\), and we conclude the proof of (1) in Theorem 1.

Let \(\tilde{V}_{\mathcal{J},1}\) be the sub VOA of \(\bigwedge(W)\) by taking the \(-1\) fixpoint of \(\bigwedge(W)\), where \(-1\) denotes the action induced by multiplying \(-1\) on \(W\). It follows that \(V_{\mathcal{J},1}\) acts on \(\tilde{V}_{\mathcal{J},1}\) and \(\bigwedge(W)\) such that

\[ L_{a,b}(m,n) =: a(m)b(n) \circ, K = Id \]

for all \(a, b \in W, m, n \in \mathbb{Z}\), and \(\tilde{V}_{\mathcal{J},1}\) is essentially a quotient of \(\tilde{V}_{\mathcal{J},1}\). The VOA \(\tilde{V}_{\mathcal{J},1}\) is studied by Abe in [Abe07], and it will also plays a role when discussing the simplicity of \(V_{\mathcal{J},r}\).
We give some formulas to conclude this section, which are needed later. We note that for \( i \neq k \), \( L_{\psi_k, \psi_k^*} \) and \( L_{\psi_i, \psi_i^*} \) are two mutually orthogonal Virasoro elements in \( F(W) \), \( L_{\psi_i, \psi_i^*} \in J(W) \) is not an element in \( F(W) \). By (12) we have
\[
(x(0)y)(l) = [x(0), y(l)]
\]
for any \( x, y \) in a VOA \( V \) and \( l \in \mathbb{Z} \), hence it is easy to show that
\[
(L_{\psi_k, \psi_k^*}(-i - 1, -j)1)(l) = \frac{1}{i} [L_{\psi_k, \psi_k^*}(0), (L_{\psi_k, \psi_k^*}(-i, -j)1)(l)],
\]
and
\[
(L_{\psi_i, \psi_i^*}(-i, -j - 1)1)(l) = \frac{1}{j} [L_{\psi_i, \psi_i^*}(0), (L_{\psi_i, \psi_i^*}(-i, -j)1)(l)]
\]
for all \( i, j \geq 1 \). Apply this repeatedly together with (12), (13) and (23), we have the following identities:

**Lemma 1.** If \( L_{a,b} \in J(W) \) is not an element in \( F(W) \), then for all \( s \in \mathbb{Z} \) and \( i, j \geq 1 \) the following formula holds
\[
(L_{a,b}(-i, -j)1)(s + i + j - 1) = \sum_{k \in \mathbb{Z}} (-1)^{i+j} \binom{i + k - 1}{i - 1} \binom{j + s - k - 1}{j - 1} L_{a,b}(k, s - k).
\]

In particular when \( j = 1 \),
\[
(L_{a,b}(-i, -1)1)(s + i) = \sum_{k \in \mathbb{Z}} (-1)^{i-1} \binom{i + k - 1}{i - 1} L_{a,b}(k, s - k).
\]

We remark that (26) and (27) actually holds for any \( L_{a,b} \in J(W) \), but we do not need this fact later.

### 5 Properties of the VOA \( V_{\mathcal{J}, r} \), Where \( \mathcal{J} \) is a Type \( C \) Jordan Algebra

In this section, we study the property of the VOA \( V_{\mathcal{J}, r} \), where \( \mathcal{J} = \mathcal{J}_C(W) \). We will prove that \( V_{\mathcal{J}, r} \)-submodules of \( M_r \) are essentially the same as \( \mathcal{L}(W) \)-submodules of \( M_r \). This result is analogous to Proposition 3.4 in [NS10]. Therefore, the study of the \( V_{\mathcal{J}, r} \) can be reduced to the study of \( M_r = V_{\mathcal{J}, r} \) as an \( \mathcal{L}(W) \)-module. We will also show that (2) of Theorem 1 will be a corollary of Proposition 1 at the end of this section.

The proofs in this section are similar to the one in [NS10], and we just make some modifications. We first need the following lemma:

**Lemma 2.** For any \( m, n \in \mathbb{Z} \) and \( L_{a,b} \in J(W) \), but \( L_{a,b} \notin F(W) \), we have
\[
L_{a,b}(m, n)u \in \text{span}\{L_{a,b}(-p, -1)(l)u | a, b \in W; p \in \mathbb{Z}_{\geq 1}, l \in \mathbb{Z}\}.
\]
Proof of Lemma 2. Because $L_{a,b}(m,n)$ acts as zero on $M_r$ if $m = 0$ or $n = 0$, so we only need to consider the case $m, n \neq 0$. Because for a given element $u$, there exists an integer $t$ and a positive integer $N$ such that

$$L_{a,b}(j, s - j)u = 0$$

for all $s < t$ and $s > t + N$, therefore by \[27\]

$$(L_{a,b}(-i, -1)1)(s + i)u = \sum_{t \leq j \leq t + N} (-1)^{i-1} \binom{i + j - 1}{i - 1} L_{a,b}(j, s - j)u.$$ 

That is, $(L_{a,b}(-i, -1)1)(s + i)u$ is a linear combination of $L_{a,b}(j, s - j)u$.

Now we show that we can express $L_{a,b}(j, s - j)u$ in terms of $(L_{a,b}(-i, -1)1)(s + i)u, i = 1, \cdots N + 1$. It is enough to show that the $(N + 1) \times (N + 1)$ matrix $(a_{i,j}^{t,N})$, $1 \leq i, j \leq N + 1$ where

$$a_{i,j}^{t,N} = (-1)^{i-1} \binom{t + i + j - 2}{i - 1}$$

has non-zero determinant. This can be shown by elementary linear algebra as follows. Let $(a_{i,j}'), 1 \leq i, j \leq N + 1$ be the new matrix such that

$$a_{i,j}' = a_{i,j}^{t,N} - a_{i,j-1}^{t,N}, a_{i,1}' = a_{i,1}$$

for all $2 \leq j \leq N + 1$, that is, the $j$-th column of $(a_{i,j}')$ equals the $j$-th column of $(a_{i,j}^{t,N})$ subtracting the $j-1$-th column of $(a_{i,j}^{t,N})$, $2 \leq j \leq N + 1$, and the first column of $(a_{i,j}')$ equals the first column of $(a_{i,j}^{t,N})$. It follows from \[15\] that

$$a_{i,j}' = -a_{i-1,j}^{t,N}$$

for all $2 \leq i, j \leq N + 1$, and

$$a_{1,1}' = 1, a_{i,j}' = 0$$

for all $2 \leq j \leq N + 1$. By the property that a determinant of a given matrix is invariant under elementary operations, we see that

$$det((a_{i,j}^{t,N}'')) = det(a_{i,j}') = (-1)^N det(a_{i,j}^{t+1,N-1}).$$

It is also clear that when $N = 0$,

$$det((a_{i,j}^{t,0}')) = 1$$

for all $t \in \mathbb{Z}$, hence

$$det((a_{i,j}^{t,N}')) = (-1)^{\frac{N(N+1)}{2}} \neq 0,$$

and we conclude the proof. \[\square\]

We also need the following lemma which can be derived from the previous two lemmas:
Lemma 3. For any \(a, b \in W, m, n \in \mathbb{Z}\) and \(u \in M_r\), we have
\[
\text{span}\{L_{a,b}(m,n)u| a, b \in W, m, n \in \mathbb{Z}_{\geq 1}\} 
\subseteq \text{span}\{L_{a_1,b_1}(l_1) \cdots L_{a_k,b_k}(l_k)u| a_i, b_i \in W, l_i \in \mathbb{Z}\}.
\]

Proof of Lemma 3. We first prove the case for \(L_{a,b} \in J(W)\) but \(L_{a,b} \notin F(W)\). It follows from (24) and (25) that for all \(m, n \geq 1\) and \(l \in \mathbb{Z}\) we have
\[
(L_{a,b}(-m, -n)1)(l) \in \text{span}\{L_{a_1,b_1}(l_1) \cdots L_{a_k,b_k}(l_k)u| a_i, b_i \in W, l_i \in \mathbb{Z}\} \quad (28)
\]
using induction on \(m\) and \(n\). Then by Lemma 2, it is clear that
\[
L_{a,b}(m,n)u \in \text{span}\{L_{a,b}(-p, -1)(l)u| a, b \in W, p \in \mathbb{Z}_{\geq 1}, l \in \mathbb{Z}\} 
\subseteq \text{span}\{L_{a_1,b_1}(l_1) \cdots L_{a_k,b_k}(l_k)u| a_i, b_i \in W, l_i \in \mathbb{Z}\}, \quad (29)
\]
for all \(m, n \in \mathbb{Z}\), so we finish the proof for this case.

Now we consider the case \(L_{a,b} \in F(W)\). It is enough to show that
\[
L_{\psi_1,\psi_1^*}(m,n)u \in \text{span}\{L_{a_1,b_1}(l_1) \cdots L_{a_k,b_k}(l_k)u| a_i, b_i \in W, l_i \in \mathbb{Z}\}.
\]
for all \(m, n \in \mathbb{Z}\). This follows easily from the fact that
\[
L_{\psi_1,\psi_1^*}(m,n) = [L_{\psi_1,\psi_1^*}(m,1), L_{\psi_2,\psi_1^*}(-1,n)]
\]
holds for all \(m, n \in \mathbb{Z}\), together with (29) shown above. Hence we conclude the proof of Lemma 3.

It can be seen from the proof that the condition \(\text{dim}(W) \geq 4\) is necessary, because when \(\text{dim}(W) = 2\), all elements in \(J(W)\) are Virasoro elements, hence we cannot find non Virasoro elements in \(J(W)\) to guarantee the first half in the proof of Lemma 3. Some discussion about the case \(\text{dim}(W) = 2\) are given, for example, in [Abe07].

The next proposition is an analogue of the Proposition 3.4 in [NS10], which will be used in next section.

Proposition 2. Suppose \(\text{dim}(W) \geq 4\), then any \(\mathcal{L}(W)\)-submodule of \(M_r\) is also a \(V_{\mathcal{L},r}\)-module. Conversely, any \(V_{\mathcal{L},r}\)-submodule of \(M_r\) is a \(\mathcal{L}(W)\)-module.

Proof of Proposition 1. First, we note that by (23), the ‘mode’ \(L_{a,b}(l)\) of the field \(Y(L_{a,b}, z)\) is an (infinite) sum of elements in \(\mathcal{L}(W)\). Suppose \(M\) is a \(\mathcal{L}(W)\)-submodule of \(M_r\), we have shown that for any \(l \in \mathbb{Z}\) and \(a, b \in W, L_{a,b}(l)M \subseteq M\) holds, because
\[
L_{a,b}(l)u = \sum_{|k| \leq N} L_{a,b}(-k + l - 1, k)u \subseteq M
\]
for some integer \(N\) depending on \(L_{a,b}(l)\) and \(u\). Therefore any \(\mathcal{L}(W)\)-submodule of \(M_r\) is also a \(V_{\mathcal{L},r}\)-module. The converse follows from Lemma 3.  
\[\square\]
Proof of (2) in Theorem 1: Using induction on \(k\), it follows easily from Lemma 3 that for any \(k \geq 1\),

\[
\text{span}\{L_{a_1,b_1}(-m_1,-n_1) \cdots L_{a_k,b_k}(-m_k,-n_k)| a_i, b_i \in W, m_i, n_i \in \mathbb{Z}_{\geq 1}\} 
\subseteq \text{span}\{L_{a_1,b_1}(l_1) \cdots L_{a_k,b_k}(l_k)| a_i, b_i \in W, l_i \in \mathbb{Z}\},
\]

which is exactly

\[M_r = V_{\mathcal{J},r}.\]

Therefore we conclude the proof of (2) in Theorem 1. \(\square\)

We remark that the analogue of Lemma 3 proved in this section can also be applied to simplify the proofs in [NS10] for the case \(\mathcal{J} = \mathcal{J}_B(\mathfrak{h})\).

6 Simplicity of the VOA \(V_{\mathcal{J},r}\), Where \(\mathcal{J}\) is a Type C Jordan Algebra

In this section we prove (3) of Theorem 1 about the simplicity of the VOA \(V_{\mathcal{J},r}\), where \(\mathcal{J} = \mathcal{J}(W)\). The method we use here is essentially the same as the approach given in [Zha16b].

Recall that the infinite dimensional Lie algebra \(\bigwedge^2(V_{\infty})\) (and therefore, \(\mathcal{L}(W)\)) contains the Lie subalgebra \(\bigwedge^1(V_{\infty})\), which is isomorphic to \(\mathfrak{so}_{\infty}\). We note that

\[\mathcal{I} \simeq \text{span}\{L_{a,b}(0,m)| a, b \in W, m \in \mathbb{Z}\}\]

acts as zero on \(M_r\), and

\[\mathcal{L}(W)/\mathcal{I} \simeq \mathfrak{so}_{\infty} \bigoplus \mathbb{C}c.\]

Hence, the study of \(M_r\) as an \(\mathcal{L}(W)\)-module is the same as the study of \(M_r\) as a \(\mathfrak{so}_{\infty}\)-module. Combine this with Proposition 1, we have

**Proposition 3.** \(V_{\mathcal{J},r}\) is a simple VOA if and only if \(M_r = V_{\mathcal{J},r}\) is simple as a \(\mathfrak{so}_{\infty}\)-module.

We first analyze the Lie algebra structure of \(\mathfrak{so}_{\infty}\). It is easy to choose countable number of vectors \(\{e_i|i \in \mathbb{Z}_{\geq 0}\}\) with \((e_i,e_j) = \delta_{i+j,0}\) such that they span \(V_{\infty}\), and for all \(i \neq 0\), there are some \(a, b \in W\) and \(j \in \mathbb{Z}_{\geq 1}\) such that

\[e_i = a(j), e_{-j} = b(-j), \langle a, b \rangle = 1.\]

Then we see that \(V_{\infty}\) has an increasing chain of subspaces \(V_N:\)

\[\{0\} = V_0 \subseteq V_1 \subseteq \cdots V_N \subseteq V_{\infty}, V_N = \text{span}\{e_i| i \leq N\}.\]

It is clear that the symmetric bilinear form restricted to \(V_N\) is non-degenerate, and \(\text{dim}(V_N) = 2N\). Because \(\bigwedge^2(V_N) \simeq \mathfrak{so}_{2N}\), there is an increasing chain of Lie algebras:

\[\mathfrak{so}_2 \subseteq \cdots \mathfrak{so}_{2N} \subseteq \mathfrak{so}_{\infty},\]
Set
\[ g^{(N)} \overset{df.}{=} \mathfrak{so}(2N) \simeq \text{span}\{e_k e_l | 1 \leq |k|, |l| \leq N, k \neq l\}, \]
where we write
\[ e_k e_l \overset{df.}{=} e_k \wedge e_l \in \mathfrak{so}_{2N} \]
for convenience. We note that
\[ g^{(N)} = g^{(N)}_+ \bigoplus h^{(N)} \bigoplus g^{(N)}_-, \]
where
\[ g^{(N)}_+ = \text{span}\{e_k e_l | k + l > 0\}, g^{(N)}_- = \text{span}\{e_k e_l | k + l < 0\}, \]
\[ h^{(N)} = \text{span}\{e_k e_l | k + l = 0\}. \]

Introduce elements \( \epsilon_k \in (h^{(N)})^*, k = 1, \cdots, N \) such that:
\[ \epsilon_l (e^{-k} e_k) = -\delta_{k,l}. \]

The positive and negative roots with respect to the triangular decomposition are:
\[ \Phi^{(N)}_+ = \{+\epsilon_i + \epsilon_j | i < j\} \cup \{-\epsilon_i + \epsilon_j | i < j\}, \]
\[ \Phi^{(N)}_- = \{-\epsilon_i - \epsilon_j | i < j\} \cup \{+\epsilon_i - \epsilon_j | i < j\}. \]

The corresponding simple roots are:
\[ \Delta^{(N)} = \{\epsilon_1 + \epsilon_2\} \cup \{-\epsilon_i + \epsilon_{i+1} | 1 \leq i \leq N\}, \]
and the half sum of positive roots is:
\[ \rho^{(N)} = \frac{1}{2} \sum_{\alpha \in \Phi^{(N)}_+} \alpha = \sum_{1<i\leq N} (i-1)\epsilon_i. \]

Now we show that \( M_r \) is a generalized Verma module for the Lie algebra \( \mathfrak{so}_\infty \). Define:
\[ n^{(N)}_- \overset{df.}{=} \text{span}\{e_k e_l | k, l < 0\}, l^{(N)} \overset{df.}{=} \text{span}\{e_k e_l | k < 0, l > 0\}, \]
\[ u^{(N)} \overset{df.}{=} \text{span}\{e_k e_l | k, l > 0\}, p^{(N)} \overset{df.}{=} l^{(N)} \oplus u^{(N)}. \]
Then
\[ g^{(N)} = p^{(N)} \oplus n^{(N)}_- = l^{(N)} \oplus u^{(N)} \oplus n^{(N)}_- \]
We also define \( \Phi^{(N)}_I \) as follows:
\[ \Phi^{(N)}_I = \{-\epsilon_i + \epsilon_j | i < j\} \cup \{\epsilon_i - \epsilon_j | i < j\}. \]
then \(l^{(N)}\) is a direct sum of \(h^{(N)}\) and root spaces \((\mathfrak{g}^{(N)})_\alpha\), where \(\alpha \in \Phi^+_I\).

Define the 1-dimensional \(p^{(N)}\)-module of weight \(\lambda^{(N)} \in (h^{(N)})^*\) spanned by the element 1 such that:

\[
x \cdot 1 = 0, \quad h \cdot 1 = \lambda^{(N)}(h) \cdot 1 \quad \forall h \in h^{(N)}, \ x \in (\mathfrak{g}^{(N)})_\alpha, \alpha \in \Phi^+_I \cup \Phi^+_I.
\]

Then we obtain the following generalized Verma module \(M_I(\lambda^{(N)})\):

\[
M_I(\lambda^{(N)}) \overset{\text{def.}}{=} U(\mathfrak{g}^{(N)}) \otimes_{U(p^{(N)})} \mathbb{C} \cdot 1 \simeq U(n^{(N)}_-) \cdot 1.
\]

The module \(M_I(\lambda^{(N)})\) is a ‘generalized Verma module of scalar type’ (see for example [Hum08]), and by (22), it is easy to calculate \(\lambda^{(N)}\) explicitly:

\[
\lambda^{(N)} = -\frac{r}{2} \sum_{i=1}^{N} \epsilon_i.
\]

We note that have an embedding of \(\mathfrak{g}^{(N)}\)-modules

\[
M_I(\lambda^{(N)}) \hookrightarrow M_r,
\]

and we have an exhaustive filtration:

\[
\{0\} \subseteq M_I(\lambda^{(1)}) \subseteq \cdots \subseteq M_I(\lambda^{(N)}) \subseteq \cdots \subseteq M_r.
\]

Hence \(M_r = V_{\mathcal{J},r}\) is a ‘generalized Verma module of scalar type’ for \(\mathfrak{so}_\infty\).

We recall the following lemma on the simplicity of the scalar type generalized Verma module. This lemma is essentially due to Jantzen etc. [Hum08], and the reader can consult Theorem 9.12 (a):

**Lemma 4.** If \(\lambda^{(N)}\) is a weight for \(\mathfrak{g}^{(N)}\) and

\[
\langle \lambda^{(N)} + \rho^{(N)}, \beta^\vee \rangle \notin \mathbb{Z}_{>0}, \quad \forall \beta \in \Phi^+_I - \Phi^+_I,
\]

then \(M_I(\lambda^{(N)})\) is a simple \(\mathfrak{g}^{(N)}\)-module. Conversely, if \(M_I(\lambda^{(N)})\) is simple and \(\lambda^{(N)}\) is regular, that is

\[
\langle \lambda^{(N)} + \rho^{(N)}, \beta^\vee \rangle \neq 0
\]

for all \(\beta \in \Delta\), then (33) also holds.

It is easy to compute that

\[
\beta^\vee = e_{-k}e_k + e_{-l}e_l
\]

for \(\beta = \epsilon_k + \epsilon_l \in \Phi^+_I - \Phi^+_I = \{\epsilon_i + \epsilon_j | 1 \leq i < j \leq N\}\). Hence

\[
\langle \lambda^{(N)} + \rho^{(N)}, \beta^\vee \rangle = -r + k + l.
\]

It is obvious that if \(r \notin \mathbb{Z}\), then

\[
\langle \lambda^{(N)} + \rho^{(N)}, \beta^\vee \rangle \notin \mathbb{Z}_{>0}, \forall \beta \in \Phi^+_I - \Phi^+_I.
\]
By Lemma 2.4 $M_1(\lambda^{(N)})$ is simple as a $\mathfrak{g}^{(N)}$-module if $r \not\in \mathbb{Z}$.

We now conclude that $V_{\mathcal{J},r}$ is simple if $r \not\in \mathbb{Z}$ by contradiction. Suppose the contrary that $V_{\mathcal{J},r}$ is not simple for $r \not\in \mathbb{Z}$, then it has a proper $\mathfrak{so}_\infty$ submodule $M$. We deduce that $M \cap M_1(\lambda^{(N)})$ is a proper $\mathfrak{g}^{(N)}$-submodule of $M_1(\lambda^{(N)})$, for some $N$. This contradicts to the result that $M_1(\lambda^{(N)})$ is irreducible for all $N$ when $r \not\in \mathbb{Z}$. Hence we conclude the proof in one direction.

To prove the converse, we need to show that if $V_{\mathcal{J},r}$ is not simple, then $r \in \mathbb{Z}_{\not=0}$. This happens if for some $N$, $M_1(\lambda^{(N)})$ is an reducible $\mathfrak{g}^{(N)}$-module. A direct calculation shows that for the simple root $\beta = \epsilon_1 + \epsilon_2$ we have

$$\langle \lambda^{(N)} + \rho^{(N)}, \beta^\vee \rangle = -r + 1,$$

and

$$\langle \lambda^{(N)} + \rho^{(N)}, \beta'^\vee \rangle = 0$$

for other simple roots $\beta$. Therefore $\lambda^{(N)}$ is regular if and only if $r \neq 1$. It is also easy to show that for $r \in \mathbb{Z}_{\not=0,1}$, there exists an integer $N$ and $1 \leq k, l \leq N$ such that

$$\langle \lambda^{(N)} + \rho^{(N)}, \beta^\vee \rangle = -r + k + l \in \mathbb{Z}_{>0},$$

therefore by the second half of Lemma 4, $V_{\mathcal{J},r}$ is not simple when $r \in \mathbb{Z}_{\not=0,1}$.

It remains to show that $V_{\mathcal{J},1}$ is not simple. This follows from the fact that the VOA $\bar{V}_{\mathcal{J},1}$, is a quotient of $V_{\mathcal{J},1}$, and

$$V_{\mathcal{J},1} \neq \bar{V}_{\mathcal{J},1}$$

by comparing the dimensions of the corresponding graded vector spaces. Thus we conclude the proof of (3) in Theorem 1.

It is interesting to know the explicit construction of the simple quotients of $V_{\mathcal{J},r}$ for all $r \in \mathbb{Z}_{\not=0}$. This can be done using dual-pair type constructions similar to the one given in [Zha16b], and it will be given in a separate paper.

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School of Mathematics (Zhuhai), Sun Yat Sen University, Zhuhai, China.

*Email Address: hzhaoab@connect.ust.hk*