UNIVERSALITY FOR OUTLIERS IN WEAKLY CONFINED
COULOMB-TYPE SYSTEMS

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Abstract. This work treats weakly confined particle systems in the plane, characterized by a large number of outliers away from a droplet where the bulk of the particles accumulate in the many-particle limit. We are interested in the asymptotic behaviour of outliers for two families of point processes: Coulomb gases confined by regular background measures at determinantal inverse temperature, and a class of random polynomials. We observe that the limiting outlier process only depends on the shape of the uncharged region containing them, and the global net excess charge. In particular, for a determinantal Coulomb gas confined by a sufficiently regular background measure, the outliers in a simply connected uncharged region converge to the corresponding Bergman point process. For a finitely connected uncharged region \( \Omega \), a family of limiting outlier processes arises, indexed by the (Pontryagin) dual of the fundamental group of \( \Omega \). Moreover, the outliers in different uncharged regions are asymptotically independent, even if the regions have common boundary points. The latter result is a manifestation of screening properties of the particle system.

1. Introduction and main results

1.1. Weakly confined Coulomb gases. Let \( \mu \) be a probability measure on the complex plane \( \mathbb{C} \). We are interested in a system of \( N \) particles of unit negative charge attracted to the (positive) charge distribution \( \kappa_N \mu \), where \( \kappa_N \) is such that \( N < \kappa_N \leq N + 1 \). More precisely, we consider the point process on \( \mathbb{C} \) induced by the Boltzmann-Gibbs measure

\[
d\mathbb{P}_N(z_1, \ldots, z_N) = \frac{1}{Z_N} e^{-2H_N(z)} dm_{\mathbb{C}^N}(z), \quad (z_1, \ldots, z_N) \in \mathbb{C}^N
\]

where the Hamiltonian \( H_N \) is given by

\[
H_N(z_1, \ldots, z_N) = -\sum_{i<j} \log |z_i - z_j| + \kappa_N \sum_j U_{\mu}(z_j), \quad (z_1, \ldots, z_N) \in \mathbb{C}^N,
\]

and where \( Z_N \) is a normalizing constant. Throughout, \( m_{\mathbb{C}^N} \) denotes the Lebesgue measure on \( \mathbb{C}^N \) and \( U_{\mu} \) is the logarithmic potential of the measure \( \mu \) defined as

\[
U_{\mu}(z) = \int \log |z - w| d\mu(w), \quad z \in \mathbb{C}.
\]

We denote by \( \Theta_N \) a random variable with law \( \mathbb{P}_N \). As the particles are interchangeable and almost surely distinct, we may abuse notation and think of the random vector as a point process \( \Theta_N = \{z_1, \ldots, z_N\} \) (see Appendix A).

The point process \( \Theta_N \) is sometimes called a weakly confined Coulomb gas or a jellium associated to \( \mu \). The idealized total background charge \( \kappa_N = N \) would correspond to global charge neutrality of the system, but we ask \( \kappa_N > N \) for the law \( \mathbb{P}_N \) to be well-defined without spatial truncation. This choice is discussed [19, Section 2.1] for \( \kappa_N = N + 1 \) and also in [8]. The terminology weakly confining is related to the growth of the potential at infinity \( U_{\mu}(z) = \log |z| + O(1) \) (cf. [12]).

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If the total background charge would instead be so large that \( \liminf_N N^{-1} \kappa_N > 1 \) and if \( \mu \) is regular enough with simply connected support, then the gas would be strongly confined to a compact set (the droplet), and with high probability all particles would be found within a distance \( N^{-\frac{1}{2}} \log N \) from that set, see [2].

For both the weakly and strongly confined models, as the number \( N \) of particles tends to infinity, the system arranges itself in equilibrium at the macroscopic level. This amounts to the convergence in law

\[
\frac{1}{N} \sum_{z \in \Theta_N} \delta_z \rightarrow \mu
\]

which reflects the fact that the particles are attracted by the charge distribution \( \mu \).

### 1.2. Bergman universality and independence

We are interested in the outlier point processes for the Coulomb gas \( \Theta_N \). Specifically, if \( \Omega \) is an uncharged region, i.e., a connected component of \( \mathbb{C} \setminus \text{supp}(\mu) \), we want to determine the asymptotic distribution of the point process \( \Theta_N \) restricted to \( \Omega \), namely

\[
\Phi_N = \Theta_N \cap \Omega, \quad N \geq 1.
\]

In the current setting of weak confinement, there tends to be a relatively large number of outliers, i.e., particles at a positive distance from \( \text{supp}(\mu) \). In the recent work [7], these are studied under the assumption that the measure \( \mu \) is radial. As an example, if \( \text{supp}(\mu) \) is the closed unit disk \( \overline{D} \) centered at the origin and \( \kappa_N = N + 1 \), then the outlier process converges to the unweighted Bergman process of the exterior disk \( D_{e} = \{ z \in \mathbb{C} : |z| > 1 \} \), i.e., the determinantal point process whose correlation kernel is the standard Bergman kernel of \( D_{e} \). If we take \( \kappa_N = N + \chi \) for \( \chi \in (0, 1] \), there is a family of Bergman point processes arising on \( D_{e} \) naturally indexed by a circle, see [11].

We will use methods from potential theory that require some regularities of \( U^\mu \). This is most sensitive near the boundary \( \partial \Omega \), but to avoid technicalities we require regularity also elsewhere. The main example we have in mind is the case where \( d\mu = \omega dm_D \) for some nice enough \( \omega \) and \( D \) but we allow for charges situated on curves as well. Specifically, we require that the support of \( \mu \) is a disjoint union, \( \text{supp}(\mu) = D \cup \Gamma \), where \( D \) is an
open set and where $\Gamma \cup \partial D$ is a finite disjoint union of closed analytic Jordan curves (see Figure 1). We use the standard notation $C^\infty$ and $C^\omega$ for the classes of infinitely differentiable and real-analytic functions, respectively, and denote the arc length measure on a disjoint union of curves $\gamma$ by $\sigma_\gamma$ and the restriction of the Lebesgue measure to $D$ by $m_D$.

**Definition 1.1** (Admissible probability measure). A probability measure $\mu$ is said to be *admissible* if its support equals $D \cup \Gamma$ for some $D$ and $\Gamma$ as above, and

$$d\mu = \omega \, dm_D + \rho \, d\sigma_{\partial D \cup \Gamma},$$

where $\omega \in C^\infty(D)$ and $\rho \in C^\omega(\partial D \cup \Gamma)$ have no common zeros on $\partial D$, $\rho$ has no zeros on $\Gamma$ and, in addition, $\omega$ is real-analytic in a neighbourhood of $\partial D$.

Notice that, since $\partial D \cup \Gamma$ is a finite union of closed curves in the plane, either $\text{supp}(\mu)$ or its complement is bounded. We begin with a simple version of our main result, where $\Omega$ is assumed to be a simply connected domain in the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

**Theorem 1.2** (Outliers in a simply connected region). Suppose $\mu$ is an admissible probability measure and let $\Omega$ be a connected component of $\hat{\mathbb{C}} \setminus \text{supp}(\mu)$. If $\Omega$ is simply connected and if $\kappa_N = N + 1$ we have the convergence

$$\Phi_N \longrightarrow \Pi_\Omega \text{ in law}$$

as $N \to \infty$, where $\Pi_\Omega$ is the determinantal point process in $\Omega$ with correlation kernel given by the Bergman kernel of $\Omega$.

The limiting process, called the *Bergman point process*, is described in Section 2.1. If we would like to consider the Coulomb gas process in a natural way on the sphere, then we have to make the choice $\kappa_N = N + 1$, since any other choice will add an extraneous charge at infinity.

**Remark 1.3** (Number of outliers). The expected number $E[\#(\Phi_N)]$ of points in $\Omega$ is, up to a logarithmic factor, of order $\sqrt{N}$ (a precise statement is proven in Section 6). This is comparable to the strongly confining case mentioned in Section 1.2. In the setting of weak confinement there are particles in $\Omega$ that stay at a fixed positive distance to the
boundary for arbitrarily large $N$ (white boxes in Figure 2), while for a strongly confined system there are no outliers in the limit.

Theorem 1.2 is a particular case of the next theorem. If $\Omega$ is multiply-connected, the outlier processes do not converge in general. Instead, there is a whole family of possible limiting processes. We identify these and characterize subsequential convergence.

**Theorem 1.4 (Outliers in general regions).** Suppose $\mu$ is an admissible probability measure and let $\Omega$ be a connected component of $\mathbb{C} \setminus \text{supp}(\mu)$. Let us fix $z_1, \ldots, z_l$ in the interior of each of the holes of $\Omega$, i.e., in the bounded connected components of $\mathbb{C} \setminus \overline{\Omega}$, and let $q_1, \ldots, q_l$ be the $\mu$-mass of each (closed) hole. If, along some subsequence,

\begin{equation}
(\epsilon^{2\pi i \kappa_N q_1}, \ldots, \epsilon^{2\pi i \kappa_N q_l}) \to (\epsilon^{2\pi i Q_1}, \ldots, \epsilon^{2\pi i Q_l})
\end{equation}

for some $Q = (Q_1, \ldots, Q_l) \in (\mathbb{R}/\mathbb{Z})^l$, then, along the same subsequence

$$\Phi_N \to \Pi_{\Omega, \mathbf{Q}}$$

in law,

where $\Pi_{\Omega, \mathbf{Q}}$ is the weighted Bergman point process on $\Omega$ associated to the weight

\begin{equation}
\omega_{\mathbf{Q}}(z) = \prod_{i=1}^l |z - z_i|^{-2Q_i}.
\end{equation}

Theorem 1.2 is a particular case of this result with $l = 1$ and $\kappa_N = N + 1$. Notice that by the uniqueness of the limit, the limiting point process does not depend on the particular choice of the points $(z_i)_{i=1}^l$ (for a direct argument, see Proposition 2.2 and Proposition 2.4). Notice also that if $\mathbf{Q} \neq \mathbf{Q}'$ as elements of the torus $(\mathbb{R}/\mathbb{Z})^l$, then the limiting processes are distinct (see Proposition 2.4).

Theorem 1.2 and Theorem 1.4 likely hold under less restrictive conditions on the measure $\mu$, in particular in the transversal direction to the boundary. In the radial case from [7], a similar result holds without any additional assumption on the density. For the case of outliers in an annulus, one can obtain the result by applying methods similar to [7, 11]. Finally, we would like to point out that Theorem 1.4 can be generalized to deal with line bundles over Riemann surfaces as in the framework described in [6]. In this geometric setting, the non-negativity condition of $\mu$ becomes a non-negativity condition on the curvature of the Hermitian line bundle and the limiting point processes obtained are similar to the ones described in Section 2.2.

When there are several uncharged components, especially if these have common boundary points, it is natural to ask if the limiting processes interact with each other. The following theorem answers this question.

**Theorem 1.5 (Independence between components).** Suppose $\mu$ is an admissible probability measure. Let $\Omega_1$ and $\Omega_2$ be any two connected components of $\mathbb{C} \setminus \text{supp}(\mu)$ and denote by $\Phi_{1,N}$ and $\Phi_{2,N}$ the associated outlier processes. Along a subsequence for which (1.2) holds for both components, with some $Q_1$ and $Q_2$, we have that

$$\Phi_{1,N} \Phi_{2,N} \to (\Pi_{\Omega_1, \mathbf{Q}_1}, \Pi_{\Omega_2, \mathbf{Q}_2})$$

in law,

where $\Pi_{\Omega_1, \mathbf{Q}_1}$ and $\Pi_{\Omega_2, \mathbf{Q}_2}$ are independent.

### 1.3. Outliers for random zeros.

The Bergman universality of the outliers processes goes beyond the weakly confined Coulomb gas model. In support of this claim, we will show that the zero set for one model of random polynomials exhibits the same feature. We focus on this particular model for simplicity but the result should hold in larger generality, for instance, for Weyl-type Gaussian polynomials defined with respect to smooth exponentially varying weights. In addition, the proof of Theorem 1.6 can be adapted to the setting of a finitely connected uncharged region. This is related to [20] which deals with zeros of random Laurent series with i.i.d. coefficients.
The model we consider belongs to a family of random polynomials introduced by Zeitouni and Zelditch in [28]. Denote by $\nu$ a compactly supported probability measure on $\mathbb{C}$, and fix an orthonormal basis $(Q_{k,N})_{k=0}^N$ for the space $\mathbb{C}_N[z]$ of polynomials of degree at most $N$, endowed with the inner product of $L^2(\mathbb{C},e^{-2N\nu}d\nu)$. The random polynomial of degree $N$ associated to $\nu$ is the linear combination of basis elements

$$P_N = \sum_{k=0}^N \xi_k Q_{k,N},$$

where $(\xi_k)_k$ is a sequence of i.i.d. standard complex Gaussians, $\xi_k \sim \mathcal{N}_\mathbb{C}(0,1)$. The law of $P_N$ does not depend on the specific orthonormal basis $(Q_{k,N})_{k=0}^N$ we choose. The zero set $\Psi_N$ of $P_N$ forms a point process in the plane, which shares many features with a weakly confined Coulomb gas whose background measure is $\nu$. This was noticed in [28] and is discussed more thoroughly in [7]. In particular, as $N$ tends to infinity, the points of $\Psi_N$ distribute according to the measure $\nu$ and there is a large number of outliers.

We consider the class of measures

$$d\nu = \rho d\sigma_T$$

where $\rho$ is a strictly positive and real-analytic density with respect to the arc length measure $\sigma_T$ on a simple closed analytic curve $\Gamma$. Let $\Omega_1$ and $\Omega_2$ denote the components of $\hat{\mathbb{C}} \setminus \Gamma$, and $\Xi_{j,N} = \Psi_N \cap \Omega_j$ the point process of outliers in $\Omega_j$.

**Theorem 1.6.** As $N \to \infty$, we have the convergence

$$(\Xi_{1,N}, \Xi_{2,N}) \longrightarrow (\Pi_{\Omega_1}, \Pi_{\Omega_2}) \quad \text{in law},$$

where the Bergman point processes $\Pi_{\Omega_1}$ and $\Pi_{\Omega_2}$ are independent.

We believe that Bergman universality for outliers should hold also for more general measures $\nu$ which are not supported on a curve. However, we will not pursue this here.

### 1.4. High level description of the approach.

Our approach relies on proving convergence of kernels: the correlation kernel of the determinantal point processes for the jellium, and the covariance kernel for the random polynomials. In each case, the locally uniform convergence towards the limiting Bergman or Szegő kernel implies the convergence of the point processes.

For this discussion, we focus on the jellium model and assume, for simplicity, that $Q = 0$. Then, the correlation kernel $K_N$ for the weakly confined jellium satisfies

$$\forall z \in \Omega, \quad K_N(z, z) \leq B_{\Omega}(z, z),$$

where $B_{\Omega}$ is the Bergman kernel of $\Omega$ (for general $Q$, see Lemma 2.5). To obtain a matching lower bound, we show that, for some orthonormal basis $(\psi_k)_{k \geq 1}$ of the Bergman space $A^2(\Omega)$ one can find, for every $k_0 > 0$, an orthonormal set $(P_{k,N})_k$ in $L^2(\mathbb{C}, e^{-2\kappa N\nu}dm_{\mathbb{C}})$ consisting of polynomials in $\mathbb{C}_{N-1}[z]$ such that, for any $k \leq k_0$,

$$|P_{k,N}|^2 e^{-2\kappa N\nu} \longrightarrow |\psi_k|^2$$

pointwise in $\Omega$. We stress that these polynomials are not the standard orthogonal polynomials (ordered by degree) but a family adapted to the basis $(\psi_k)_{k \geq 1}$.

The method is based on the approach developed in [14] (see also [16] for a less technical exposition), which loosely goes back to e.g. [6, 4] and even to [18]. It involves constructing polynomial-like functions with good orthogonality properties, and then performing a surgery using Hörmander’s $\bar{\partial}$-estimates to obtain true polynomials with small error. Due to the weak confinement, there are considerable simplifications in the most technical step of [14]. On the other hand, to allow us to consider bounded (cf. [15]) and multiply

\[\text{In particular, there is no need to construct the orthogonal foliation, because the behavior in the exterior $\Omega$ dominates in the weakly confined case, and there one may achieve exact orthogonality by construction.}\]
connected uncharged domains, we have to work with more flexible bases than the one given by standard orthogonal polynomials.

1.5. Further questions about edge scaling limits. The above results show universality for the outliers at any positive distance from the boundary, but do not reveal the microscopic structure near the edge of the support. Comparing with various situations where edge behavior of Coulomb gases are understood [3, 11, 14, 15, 26], one is led to think that these scaling limits should exist and that they ought to be universal also in the present case. If so, the scaling limits should be governed by the (heavy tailed) Bergman kernel for the measure $e^{-2w}dm$, where $w(z)$ is a potential for the blow-up $\nu$ of the measure $\mu$ at a boundary point, as found in [11] for the radial case. For admissible measures where the density $\rho$ vanishes, one would expect $\nu$ to be Lebesgue measure on a half-plane, and this scaling limit was identified already in [19, Section 2.2].

1.6. Notation. We write $\mathbb{T}, \mathbb{D}, \mathbb{R}$ and $\mathbb{C}$ for the unit circle, unit disk, real line and complex plane, respectively. The extended complex plane (or Riemann sphere) is denoted by $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. We recall the definition of the Wirtinger derivatives,

$$\partial = \partial_x = \frac{1}{2}(\partial_x - i\partial_y), \quad \bar{\partial} = \partial_x = \frac{1}{2}(\partial_x + i\partial_y).$$

The Cauchy-Riemann equations are encoded in the operator $\bar{\partial}$ so that a complex differentiable function $f$ on a complex domain is holomorphic if $\bar{\partial}f = 0$, and its complex derivative is given by $f'(z) = \bar{\partial}f(z)$.

We will make frequent use of standard notation to compare asymptotic quantities. Specifically $f = O(g)$ and $a \lesssim b$ mean that $f/g$ and $a/b$ are bounded under the limit procedure in question. The notation $f = o(g)$ means that $f/g \to 0$, while $a \asymp b$ means that $a \lesssim b$ and $b \lesssim a$ holds simultaneously. If, for two functions $f$ and $g$, the quotient $f/g$ is a non-zero constant, we write $f \asymp g$.

We denote the class of $x$ in $\mathbb{R}/\mathbb{Z}$ as

$$[x] = x \mod 1 \in \mathbb{R}/\mathbb{Z}.$$ 

If we have a sequence $x_N$ so that the classes converge $[x_N] \to [x]$ with $x \in [0, 1)$, then we choose to represent $[x_N]$ by the unique real number (also denoted $[x_N]$, and depending on $x$) with $x_N - [x_N] \in \mathbb{Z}$ and

$$[x_N] \in [x - \frac{1}{2}, x + \frac{1}{2}).$$

Thus, if $[x_N]$ is convergent as an element of the circle, there exists a sequence of integers $k_N$ such that $x_N = k_N + [x_N]$ and $[x_N] \to x \in [0, 1)$ as a real number. In other words, we cut open the circle $\mathbb{R}/\mathbb{Z}$ diametrically opposite to the limit point.

We will use the notation $L^2(\Omega, \rho)$ for the $L^2$ space with respect to the measure $\rho dm_\Omega$ where $m_\Omega$ is the restriction of the Lebesgue measures on $\mathbb{C}$ to $\Omega$.

Finally, the number of points in a finite set $E$ is denoted by $\#(E)$.

2. Background on Bergman spaces and potential theory

In this section we have collected preliminary material on Bergman spaces and potential theory. We begin in Section 2.1 by recalling the standard notion of Bergman spaces and Bergman kernels in the weighted setting. Then, in Section 2.2, we consider log-harmonic weights and define the notion of a Bergman point process that appear in our theorems. Section 2.3 is about an harmonic extension of the potential while Section 2.4 deals with finding an equivalent kernel.
2.1. Bergman spaces and kernels. Let $\mathcal{D} \subset \mathbb{C}$ be an open set and let $\rho$ be a continuous positive weight function on $\mathcal{D}$. The Bergman space $A^2_\rho(\mathcal{D})$, also denoted by $A^2(\mathcal{D}, \rho)$, is the collection of all holomorphic functions in $L^2(\mathcal{D}, \rho)$. It is well-known that $A^2_\rho(\mathcal{D})$ is a reproducing kernel Hilbert space, and we denote its reproducing kernel by $K_{\rho, \mathcal{D}}(z, w)$. More precisely, $K_{\rho, \mathcal{D}}$ is the kernel of the orthogonal projection of $L^2(\mathcal{D}, \rho)$ onto $A^2_\rho(\mathcal{D})$ and if $(\varphi_k)_{k \geq 1}$ is an orthonormal basis of $A^2_\rho(\mathcal{D})$ then

$$K_{\rho, \mathcal{D}}(z, w) = \sum_{k=1}^{\infty} \varphi_k(z)\varphi_k(w),$$

where the convergence in the series is locally uniform. We denote by $K_{\rho, \mathcal{D}}$ the weighted correlation kernel, which is given by

$$K_{\rho, \mathcal{D}}(z, w) = K_{\rho, \mathcal{D}}(z, w)\rho^2(z)\rho^2(w).$$

When the weight $\rho$ depends only on a parameter $N$, we will simply write $K_N$ and $K_N$ for the kernels, and $\| \cdot \|_N$ and $\langle \cdot, \cdot \rangle_N$ for the $L^2$-norm and inner product, respectively.

For weakly confining weights $e^{-2\kappa N U^\kappa}$, the Bergman space $A^2(\mathbb{C}, e^{-2\kappa N U^\kappa})$ coincides with the $N$-dimensional space $\mathbb{C}_{N-1}[z]$ of polynomials of degree at most $N - 1$ (see (3.1) below). If $\mathcal{D}$ is not the whole space, then there may of course be non-polynomial elements of $A^2(\mathcal{D}, e^{-2\kappa N U^\kappa})$, and we instead introduce the polynomial subspaces

$$A^2_N(\mathcal{D}, e^{-2\kappa N U^\kappa}) := A^2(\mathcal{D}, e^{-2\kappa N U^\kappa}) \cap \mathbb{C}_{N-1}[z].$$

2.2. Bergman point processes on an uncharged background. Let $\mathcal{D} \subset \mathbb{C}$ be an open set and now write $\rho = e^{-2V}$ for some continuous function $V$ on $\mathcal{D}$. We say that $\rho$ is log-harmonic if $V$ is harmonic, i.e., $\Delta V = 0$. By using Gauss’s law of electrostatics, $V$ can be thought of as a possible potential for an uncharged domain $\mathcal{D}$ which makes it plausible that the weights $e^{-2V}$ determine the outliers.

**Definition 2.1** (Bergman point process). The Bergman point process $\Pi_V$ associated to the harmonic function $V$ is the determinantal point process on $\mathcal{D}$ associated to the kernel $K_{e^{-2V}, \mathcal{D}}$ and to the Lebesgue measure restricted to $\mathcal{D}$.

For instance, if $\mathcal{D} = \Omega$ is an $l$-connected complex domain, we may take points $z_1, \ldots, z_l \in \mathbb{C} \setminus \Omega$, one in each hole of $\Omega$, and let $V(z) = \mathbf{Q} \cdot \log(\mathbf{z})$ for some $\mathbf{Q} \in \mathbb{R}^l$ and where we have defined

$$\log(\mathbf{z}) = \log_{z_1, \ldots, z_l}(\mathbf{z}) := \left( \log |z - z_1|, \ldots, \log |z - z_l| \right).$$

This gives us a weight of the form (1.3),

$$\omega_{\mathbf{Q}}(z) = e^{-2\mathbf{Q} \cdot \log(\mathbf{z})}.$$

One of the first questions that may arise by the Definition 2.1 is how much the law of $\Pi_V$ depends on $V$. The following proposition answers this question.

**Proposition 2.2** (Dependence on the potential). Let $\mathcal{D} \subset \mathbb{C}$ be an open set and let $V$ be an harmonic function on $\mathcal{D}$. Fix a zero-free holomorphic function $f$ on $\mathcal{D}$ and let $V_f = V - \log |f|$. Then,

$$\Pi_V = \Pi_{V_f}$$. 

In particular, $K_{e^{-2V}}(z, w) = K_{e^{-2V_f}}(z, w)$ for every $z \in \mathcal{D}$. Moreover, if $V$ and $\tilde{V}$ are two harmonic functions on $\mathcal{D}$ such that $\Pi_V = \Pi_{\tilde{V}}$ in law and if we suppose that $\Pi_V$ is not the empty point process, then

$$V = \tilde{V} - \log |f|$$

for some zero-free holomorphic function $f$ on $\mathcal{D}$. 
Proof. Let us first prove the first part. Suppose that $V$ is harmonic and $f$ is zero-free and holomorphic. By the definition of determinantal point processes, we need to show that $$\det(K_{e^{-2V}}(x_i, x_j)) = \det(K_{e^{-2V'}}(x_i, x_j))$$ for every $k \geq 1$ and any points $x_1, \ldots, x_k \in D$. But, since $e^{-2V} = |f|^2 e^{-2V}$, the map given by $$g \in L^2(D, e^{-2V}) \mapsto fg \in L^2(D, e^{-2V})$$ is a well-defined isometry that sends $A^2_{e^{-2V'}}(D)$ onto $A^2_{e^{-2V}}(D)$ so that the projections are conjugates of each other, which is written as $$K_{e^{-2V}}(z, w) = f(z)K_{e^{-2V'}}(z, w)f(w)^{-1}$$ for every $z, w \in D$. By the definition of $K$, this implies that $$K_{e^{-2V}}(z, w) = \frac{f(z)}{|f(z)|} K_{e^{-2V'}}(z, w) \frac{f(w)}{|f(w)|},$$ for every $z, w \in D$, which implies that the laws of $\Pi_V$ and $\Pi_{V'}$ are the same and that $K_{e^{-2V}}(z, z) = K_{e^{-2V'}}(z, z)$ for every $z \in D$.

For the second part, let $V$ and $\tilde{V}$ be two harmonic functions on $D$ such that $\Pi_V = \Pi_{\tilde{V}}$ in law. For simplification, we denote $K = K_{e^{-2V}}$ and $\tilde{K} = K_{e^{-2\tilde{V}}}$ which are holomorphic on the first coordinate and anti-holomorphic on the second coordinate. We also use the notation $K = K_{e^{-2V}}$ and $\tilde{K} = K_{e^{-2\tilde{V}}}$. The equality in law $\Pi_V = \Pi_{\tilde{V}}$ implies that $$\det(K_{\tilde{z}_i, z_j}^{1 \leq i, j \leq k}) = \det(\tilde{K}_{\tilde{z}_i, z_j}^{1 \leq i, j \leq k}).$$ for every $k \geq 1$ and $z_1, \ldots, z_k \in D$. In particular, we have that $K(z, z) = \tilde{K}(z, z)$ while $K(z, w)K(w, z) = \tilde{K}(z, z)\tilde{K}(w, w) - \tilde{K}(z, w)\tilde{K}(w, z)$ for every $z, w \in D$, which are the $k = 1$ and $k = 2$ cases. Since the kernels are Hermitian, these two cases imply that $|K(z, w)| = |\tilde{K}(z, w)|$ for every $z, w \in D$. Then, $$|K(z, w)|e^{-V(z)}e^{-V(w)} = |\tilde{K}(z, w)|e^{-\tilde{V}(z)}e^{-\tilde{V}(w)}$$ for every $z, w \in D$. Now, either $K$ and $\tilde{K}$ are zero everywhere or there is some $w_0 \in D$ such that the maps $z \mapsto K(z, w_0)$ and $z \mapsto \tilde{K}(z, w_0)$ are non-zero holomorphic functions whose zero sets coincide (and are discrete). The first case is discarded because $\Pi_V$ is not the empty process so that we can consider the meromorphic function $f$ defined by $$f(z) = e^{(V - \tilde{V})(w_0)}K(z, w_0)/\tilde{K}(z, w_0).$$ Then, (2.1) implies that $$|f(z)| = e^{(V - \tilde{V})(z)}$$ for every $z \in D$ such that $\tilde{K}(z, w_0) \neq 0$. Since $e^{V - \tilde{V}}$ is locally bounded, $f$ has only removable singularities so that it can be extended holomorphically to $D$ and, by continuity, $|f| = e^{V - \tilde{V}}$ on $D$. Taking logarithms completes the proof. \(\square\)

If we were to define Bergman point processes for more general weights $\omega = e^{-2V}$, Proposition 2.2 would still hold.

Remark 2.3 (Flat line bundles). The notion of a log-harmonic measure makes sense on any open Riemann surface $M$ since the change of coordinates preserves the log-harmonicity. In this way, Definition 2.1 can be used to define an analogous family of Bergman point processes on $M$ and Proposition 2.2 also holds in this case. Moreover, the freedom described in Proposition 2.2 is closely related to the classification of flat Hermitian line bundles on $M$. This is not a coincidence since, by noticing that every line bundle is trivial on the open Riemann surface $M$, we may rephrase our setting by considering Hermitian line bundles on $M$ similarly to [6] and the log-harmonic measures would essentially play the role of flat Hermitian metrics. In particular, the set of equivalence classes obtained by making two log-harmonic measures $\mu$ and $\tilde{\mu}$ equivalent if and only if
\[ d\mu = |f|^2d\tilde{\mu} \] for some zero-free holomorphic function \( f \) on \( M \) is in one-to-one correspondence with the set of morphisms from the fundamental group \( \pi_1(M) \) to the unit circle \( U(1) \) (the Pontryagin dual of \( \pi_1(M) \)). This is stated for a particular case in Proposition 2.4.

Let us go back to the case of an \( l \)-connected complex domain \( \Omega \). Suppose, for simplicity, that the holes of \( \Omega \) have non-empty interior. Then, the logarithmic conjugation theorem of [5, p. 248] and the previous proposition tells us that, for every harmonic function \( V \) there exists a unique \( Q \in [0, 1)^l \) such that

\[ \Pi_V = \Pi_Q^{\log}. \]

Moreover, for general \( Q = (Q_1, \ldots, Q_l) \in \mathbb{R}^l \), the Bergman process \( \Pi_Q^{\log(z)} \) only depends on \( (e^{2\pi i Q_1}, \ldots, e^{2\pi i Q_l}) \) or, equivalently, on \( ([Q_1], \ldots, [Q_l]) \in (\mathbb{R} \setminus \mathbb{Z})^l \). For this reason, there is no harm in thinking \( Q \) as an element of \( (\mathbb{R} \setminus \mathbb{Z})^l \) and denote the Bergman process \( \Pi_Q^{\log(z)} \) by \( \Pi_Q \). Furthermore, if \( B_{0,Q}(z,w) = \mathcal{K}_{Q,\Omega}(z,w) \), we notice that \( B_{0,Q}(z,z) \) also depends only on \( ([Q_1], \ldots, [Q_l]) \) although, strictly speaking, the function of two variables \( B_{0,Q} \) depends on \( Q \in \mathbb{R}^l \).

Let \( C_\Omega \) denote the space of point configurations on \( \Omega \) (see Section A) and let \( \mathcal{P}(C_\Omega) \) be the set of probability measures on \( C_\Omega \) endowed with the weak topology.

**Proposition 2.4 (Continuous injective parametrization).** The function defined by \( (z, Q) \in \Omega \times (\mathbb{R} / \mathbb{Z})^l \mapsto B_{0,Q}(z,z) \) is continuous. Moreover, the map from \( (\mathbb{R} / \mathbb{Z})^l \) to \( \mathcal{P}(C_\Omega) \) that associates \( Q \in (\mathbb{R} / \mathbb{Z})^l \) to the law of \( \Pi_Q \) is injective and continuous.

**Proof.** The injectivity is a consequence of Proposition 2.2 together with the fact that \( \prod_i |z - z_i|^{Q_i} \) is not the modulus of a holomorphic function if \( Q_i \in (0, 1) \) for some \( i \in \{1, \ldots, l\} \). The continuity claims are a consequence of [21, Theorem 5.1]. \( \square \)

Finally, we state a lemma that gives us the first glance of why Bergman processes associated to log-harmonic measures should appear in the study of the process of outliers. We use the notations of Section 1. Take an \( l \)-connected component \( \Omega \) of \( \mathbb{C} \setminus \text{supp} (\mu) \), where \( \mu \) is admissible, and points \( z_1, \ldots, z_l \in \mathbb{C} \setminus \Omega \) in each of the holes of \( \Omega \). Denote the mass of the \( i \)-th hole of \( \Omega \) by \( q_i \).

**Lemma 2.5 (Monotonicity).** Denote \( \omega_N = e^{-2\kappa_N U^\mu} \) and \( \mathcal{K}_N = \mathcal{K}_{\omega_N, C} \). Then,

\[ \mathcal{K}_N(z,z) \leq B_{1,\kappa_N q}(z,z) \]

for every \( N \geq 1 \) and \( z \in \Omega \).

**Proof.** Let us begin by noticing that \( B_{1,\kappa_N q}(z,z) = \mathcal{K}_{\omega_N|_{\Omega}, \Omega}(z,z) \). This is a consequence of Lemma 2.8 together with Proposition 2.2. By denoting by \( \|f\|_N \) the norm of \( f \) in \( L^2(\mathbb{C}, \omega_N) \) and by \( \|f\|_{N,\Omega} \) the norm of \( f|_{\Omega} \) in \( L^2(\Omega, \omega_N|_{\Omega}) \), we can write the extremal representation of the kernels

\[ \mathcal{K}_N(z,z) = \sup_{f \in A_{\omega_N}^2} \frac{|f(z)|^2 \omega_N(z)}{\|f\|_N^2} \quad \text{and} \quad \mathcal{K}_{\omega_N|_{\Omega}, \Omega}(z,z) = \sup_{f \in A^2(\Omega, \omega_N|_{\Omega})} \frac{|f(z)|^2 \omega_N(z)}{\|f\|_{N,\Omega}^2} \]

Then, since \( \|f\|_N \geq \|f\|_{N,\Omega} \) and \( A_{\omega_N}^2 \subset A^2(\Omega, \omega_N|_{\Omega}) \), we find the sought inequality. \( \square \)

### 2.3. Harmonic extension of potentials

Assume that \( f \) is a holomorphic function on a domain \( \Omega \), and that for some \( \varepsilon > 0 \) and \( z_0 \in \partial \Omega \), \( I := \partial \Omega \cap \mathbb{D}(z_0, \varepsilon) \) is a real-analytic arc. By Carathéodory’s theorem, \( f \) extends continuously to \( I \). Suppose further that \( f \) maps \( \Omega \cap \mathbb{D}(z_0, \varepsilon) \) univalently onto a domain in unit disk, such that \( f(I) \) is a sub-arc of the circle. Then \( f \) extends to a univalent function of a neighbourhood of \( I \) (see, for instance, [22, Theorem 6.2]). In fact, similar extension properties hold for any holomorphic or harmonic function which is defined on one side of an analytic curve with real-analytic extension to the boundary. We will routinely apply such extensions, and will usually not differentiate between the original function and the extended one.
The only function we will deal with for which the extension property is not entirely obvious is the potential \( U^{µ} \).

**Proposition 2.6 (Harmonic extension).** Under the assumptions of Theorem 1.4, the harmonic function \( U^{µ}\vert_{Ω} \) extends harmonically past \( ∂Ω \) to a function \( U = U^{µ} \) on an open set \( Ω \), containing \( Π \).

**Proof.** First of all, notice that the statement is local. Indeed, suppose we show that for every \( z \in ∂Ω \) there exists a small open disk \( D \) centered at \( z \) and a harmonic function on \( D \) that coincides with \( U^{µ} \) on \( Ω \cap D \). By choosing small enough disks, we would have that if two of those disks \( D_{1} \) and \( D_{2} \) intersect then \( D_{1} \cap D_{2} \cap Ω \neq ∅ \). The extension is then obtained by using unique harmonic continuation to glue together the local extensions.

Now fix \( z \in ∂Ω \) and choose a parametrization of \( ∂Ω \) near \( z \), i.e., a holomorphic function \( ϕ : (-1,1) \times (-1,1) \to \mathbb{C} \) such that \( ϕ_{(-1,1) \times \{0\}} \) is a usual parametrization of \( ∂Ω \) around \( z \) with \( ϕ(0) = z \) and \( ϕ'(0) \neq 0 \). By taking a restriction of \( ϕ \) if necessary and reparametrizing, we have that \( ϕ((-1,1) \times (-1,0)) \subset \mathbb{C} \backslash Π \) and \( ϕ((-1,1) \times (0,1)) \subset Ω \). Let us use the notation \( A = ϕ((-1,1) \times (-1,0)) \) and write

\[
U^{µ} = U^{µ}\vert_{A} + U^{µ}\vert_{A^{c}}.
\]

Notice that \( U^{µ}\vert_{A^{c}} \) is already harmonic on \( ϕ((-1,1) \times (-1,1)) \) so that we only need to extend harmonically \( U^{µ}\vert_{A} \) to \( A \).

Now, the new setting is the following. We define the square \( S = (-1,1) \times (-1,1) \) and \( µ = µ^{Φ}(µ\vert_{A}) \) which is a measure on \( (\text{the redefined}) \ A = (-1,1) \times (-1,0] \). We have a function \( U : S \to \mathbb{R} \) such that \( ΔU = 2πµ \) and we wish to find a harmonic extension of \( U\vert_{(-1,1) \times (0,1)} \) to the open square \( S \). The first fact to notice is that being able to extend \( U \) is independent of \( U \) as soon as \( ΔU = 2πµ \). Indeed, if \( V \) were another function such that \( ΔV = 2πµ \) and if \( U \) were a harmonic extension of \( U\vert_{(-1,1) \times (0,1)} \), the function \( U - U + V \) would be a harmonic extension of \( V\vert_{(-1,1) \times (0,1)} \). So, we have the freedom to choose \( U \) and we will use the nice choice

\[
U(z) = U^{µ}(z) = \int \log |z - w|dµ(w).
\]

Recall that \( dµ = ωdm_{S} + ρ dm_{(-1,1)} \) for some functions \( ω : (-1,1) \times (-1,0) \to \mathbb{R} \) and \( ρ : (-1,1) \to \mathbb{R} \) which are real-analytic. Denote \( ω_{y}(x) = ω(x, y) \) and, for any \( y \in (-1,0) \), define the measure \( dµ_{y} = ω_{y} dm_{(-1,1) \times \{y\}} \). Denote by \( η \) the measure on \( (-1,1) \) such that \( dη = ρ dm_{(-1,1)} \). Then, since \( µ = \int_{-1}^{0} µ_{y}dy + η \), we have

\[
U(z) = \int_{-1}^{0} \left( \int \log |z - w|dµ_{s}(w) \right) ds + \int \log |z - w|dη(w).
\]

We assume, without loss of generality, that \( ω(x, y) = \sum_{m,n \geq 0} a_{m,n}x^{m}y^{n} \) for every \((x, y) \in (-1,1) \times (-1,0) \) and we extend it to \( ω : S \times (-1,1) \to \mathbb{C} \) by assuming, without loss of generality, that \( ω(z, y) = \sum_{m,n \geq 0} a_{m,n}(z - iy)^{m}y^{n} \) is convergent. Moreover, we can assume that the series defining \( ω \) converges in \( (2S) \times (-1,1) \) and that \( ρ \) can be extended to a holomorphic function on \( 2S \). Fix \( y \in (-1,0) \) and take a path \( γ(y) \) that connects \((-1, y) \) and \((1, y) \) avoiding \( S \). Define \( U \) by

\[
U(z) = \int_{-1}^{0} \left( \Re \int_{γ(y)} \log(z - w)ω_{y}(w)dw \right) dy + \Re \int_{γ(0)} \log(z - w)ρ(w)dw,
\]

where any branch of the logarithm was chosen. Since \( ω_{y} \) and \( ρ \) are holomorphic on \( S \), \( U(z) = U(z) \) for \( z \in S \setminus A \) and, since \( U \) is harmonic on \( S \), the proof is complete. \( \square \)
Proposition 2.7 (Expansion). Let $z_0 \in \partial \Omega$ and denote by $n_0$ the outward unit normal to $\Omega$ at $z_0$. Under the assumptions of Theorem 1.4, there exists non-negative constants $c_1 = c_1(z_0)$ and $c_2 = c_2(z_0)$, at least one of them strictly positive, such that
\[
(U^t - U^{t^0})(z_0 + tn_0) = c_0 t + c_1 t^2 + O(t^3), \quad 0 \leq t \leq t_0.
\]

Proof. Notice that the statement is local so that we may assume that everything happens in $(-1, 1) \times (-1, 1)$. We use the notation of the proof of Proposition 2.6. Let $I_y$ be the segment path from $(1, y)$ to $(-1, y)$ and define the concatenation $\Gamma(y) = \gamma(y) * I_y$. Then,
\[
\frac{\partial}{\partial v} (U(z) - U(z)) = - \int_{-1}^0 \left( \text{Re} \int_{\Gamma(v)} \log(z - w)\omega_y(w)dw \right) dy - \text{Re} \int_{\Gamma(0)} \log(z - w)\rho(w)dw,
\]
where the real part does not depend on the branch of the logarithm we have chosen, although the integrand is no longer well-defined as a holomorphic function. If $v = -i$ denotes the downward direction then, for every $z \in (-1, 1) \times (-1, 0),$
\[
\frac{\partial}{\partial v} (U(z) - U(z)) = - \int_{-1}^0 \left( \text{Re} \int_{\Gamma(v)} \frac{-i\omega_y(w)}{z - w}dw \right) dy - \text{Re} \int_{\Gamma(0)} \frac{-i\rho(w)}{z - w}dw
\]
\[
= 2\pi \int_{\text{Im}(z)}^0 \text{Re}(\omega_y(z))dy + 2\pi \text{Re}(\rho(z)).
\]
Furthermore, the second derivative is
\[
\frac{\partial^2}{\partial v^2} (U(z) - U(z)) = 2\pi \left( \text{Re}(\omega_{\text{Im}(z)}(z)) + \int_{\text{Im}(z)}^0 \text{Re} \left( -i \frac{d}{dz} \omega_y(z) \right) dy + \text{Re} \left( -i \frac{d}{dz} \rho(z) \right) \right).
\]
Taking $x \in (-1, 1)$ and letting $z$ go to $x$, we have that
\[
\frac{\partial}{\partial v} (U(x) - U(x)) = 2\pi \rho(x) \quad \text{and} \quad \frac{\partial^2}{\partial v^2} (U(x) - U(x)) = 2\pi \omega(x, 0),
\]
where the derivatives are taken using $x + tv$ with $t > 0$. This completes the proof. \hfill $\square$

2.4. A transformation of the kernel. We recall the following well-known result about harmonic conjugates in a multiply connected domain. We let $\Omega'$, $(\gamma_j)_j$ and $q = (q_j)_j$ be as in Section 2.3.

Lemma 2.8. There exists an holomorphic function $h$ on $\Omega'$ such that
\[
U^t(z) - q \cdot \log(z) = \text{Re} h(z),
\]
so that we have
\[
e^{-\kappa_N U^t(z)} = e^{-\kappa_N q \cdot \log(z)} \left| e^{-\kappa_N h(z)} \prod_{i=1}^l (z - z_i)^{-(\kappa_N q_i - \langle \kappa_N q_i, \rangle)} \right|.
\]

This result is a direct consequence of the so-called logarithmic conjugation theorem in [5, p. 248].

We introduce the transformed correlation kernel
\[
\hat{K}_N(z, w) = e^{-i\kappa_N \text{Im}(h(z))}K_N(z, w)e^{i\kappa_N \text{Im}(h(w))}, \quad (z, w) \in \Omega'.
\]
In the appendix we recall some basic properties of determinantal processes, including the fact (A.1) that if $c : \Omega \to \mathbb{R}$ is a continuous function, then the kernels $K(z, w)$ and
\[
\hat{K}_N(z, w) := e^{ic(z)}K(z, w)e^{-ic(w)},
\]
determine the same determinantal point process. Hence, the restrictions of the kernels $\hat{K}_N$ and $K_N$ to $\Omega \times \Omega$ induce the same process.
We obtain a holomorphic kernel after division by \((\omega[\kappa_N q](z)\omega[\kappa_N q](w))^{1/2}\), which is locally uniformly bounded in \(N\). So, we put
\[
\mathcal{K}_N(z, w) = \hat{K}_N(z, w)\omega[\kappa_N q](z)\omega[\kappa_N q](w).
\]
This allows us to use a normal family argument to deduce precompactness of the family of transformed kernels whenever \((|\kappa_N q|)_{N \in \mathbb{N}}\) converges.

3. Convergence of weighted polynomial bases

3.1. Construction of basis functions. Fix an admissible measure \(\mu\) and let \(\Omega\) be an \(l\)-connected uncharged domain let \(q_1, \ldots, q_l\) be the \(\mu\)-mass of each hole in \(\Omega\) and let \(z_1, \ldots, z_l\) be fixed points in each of the holes. We recall from Section 1.6 that the number \([\kappa_N q_j] \in \mathbb{R}/\mathbb{Z}\) is represented by the unique real number \((\text{also denoted }[\kappa_N q_j])\) such that \([\kappa_N q_j] - [\kappa_N q_j] \in \mathbb{Z}\) and \([\kappa_N q_j] \in [q - \frac{1}{2}, q + \frac{1}{2})\), where \(q\) is a representative of the limit. To simplify matters, we abuse notation and write that \((|\kappa_N q|)_{N \in \mathbb{N}}\) converges locally uniformly towards \(Q \in [0, 1]^l\), which may only be true along a subsequence \(L\).

The point process \(\Psi_N\) is the determinantal process associated to the correlation kernel \(\kappa_N\) described in Section 2.1. The outlier process is given by the restricted kernel \(\hat{K}_N(z, w)\).

To prove the convergence of the point process \(\Psi_N\) towards the Bergman point process \(B_{\Omega, Q}\), due to Proposition A.3, it is sufficient to show that the kernels \(\hat{K}_N\) converge uniformly on any compact set of \(\Omega \times \Omega\) towards \(B_{\Omega, Q}\). This is equivalent to asking that the modified holomorphic kernels \(\mathcal{K}_N\) defined in (2.2) converge locally uniformly towards the holomorphic Bergman kernel \(B_{\Omega, Q}\). The next normal families lemma takes advantage of the holomorphicity of \(\mathcal{K}_N\) to obtain precompactness. Once that is established, we only need to show that there is only one possible limit point.

**Lemma 3.1** (Normal families). The sequence \((\mathcal{K}_N)_{N \in \mathbb{N}}\) of modified holomorphic kernels is precompact for the compact-open topology.

**Proof.** The kernels in the sequence \((\mathcal{K}_N)_{N \in \mathbb{N}}\) are analytic in \(z, \bar{w}\). By Montel’s theorem for holomorphic functions in several variables, it is enough to show that \((\mathcal{K}_N)_{N \in \mathbb{N}}\) is uniformly bounded on any compact subset of \(\Omega \times \Omega\). By the reproducing kernel property and the Cauchy-Schwarz inequality, for any \((z, w) \in \Omega \times \Omega\),
\[
|\mathcal{K}_N(z, w)|^2 \leq \mathcal{K}_N(z, z)\mathcal{K}_N(w, w).
\]
So, it is sufficient to prove that the kernels are uniformly bounded on the diagonal.

Combining Lemma 2.5 and Proposition 2.4 gives, on any compact set \(K\)
\[
\mathcal{K}_N(z, z) \leq B_{\Omega, [\kappa_N q]}(z, z) \leq \sup_{(z, Q) \in (\mathbb{R}/\mathbb{Z})^l \times K} B_{\Omega, Q}(z, z) < \infty.
\]

From here on, all we have to do is to show that if \([\kappa_N q]\) converges to \(Q\), then we have convergence of holomorphic kernels along the diagonals
\[
\forall z \in \Omega, \lim_{N \to \infty} \mathcal{K}_N(z, z) = B_{\Omega, Q}(z, z).
\]
Indeed, if \(\mathcal{K}\) is the limit of \(\mathcal{K}_N:\)
\[
\mathcal{K}(z, w) = \lim_{N \to \infty} \mathcal{K}_N(z, w).
\]
then \(\mathcal{K}\) coincides with the kernel \(B_{\Omega, Q}\) on the diagonal of \(\Omega \times \Omega\), and the next lemma shows that this is enough to complete the proof.

**Lemma 3.2** (Lemma 2.5.1 from [17], p.30). Let \(L(z, w)\) be analytic in \(z\) and anti-analytic in \(w\) (i.e. analytic in \(\bar{w}\)) for \((z, w) \in \Omega \times \Omega\). If \(L(z, z) = 0\ \forall z \in \Omega\), then \(L(z, w) = 0\ \forall z, w \in \Omega\).
The rest of the proof is devoted to the proof of the pointwise convergence of the diagonal restrictions of the holomorphic kernels \( K_N \) towards \( B_{\Omega, Q} \). In the next lemma, we construct a family of orthonormal bases in which we will express the weighted Bergman kernels \( B_{\Omega, [k,N,Q]} \).

**Proposition 3.3.** There exists an orthonormal basis \( (\psi_k)_k \) of \( A^2(\Omega, e^{-2Q \cdot \log(z)}) \), such that moreover \( \psi_k \in A^2(\Omega, e^{-2Q_N \cdot \log(z)}) \). If \( (\psi_k,N)_k \) is the orthonormal set obtained by the Gram-Schmidt procedure in \( A^2(\Omega, e^{-2Q_N \cdot \log(z)}) \), we have

(i) For any \( z \in \Omega \), \( \psi_{k,N}(z) \xrightarrow{N \to \infty} \psi_k(z) \);
(ii) For any fixed \( k_0 \in \mathbb{N} \), there exists an open set \( \Omega' \) containing \( \overline{\Omega} \) such that all the functions \( \psi_{k,N} \) and \( \psi_k \) for \( k \leq k_0 \) extend holomorphically to \( \Omega' \).
(iii) The functions \( \psi_{k,N} \) are locally uniformly bounded in \( N \).

**Remark 3.4.** In the above construction, we only claim to obtain an orthonormal set \( (\psi_{k,N})_k \subset A^2(\Omega, e^{-2Q_N \cdot \log(z)}) \), as opposed to a full orthonormal basis. Indeed, the space \( A^2(\Omega, e^{-2Q_N \cdot \log(z)}) \) can be larger in general. However, this is all we need for our proof.

We start a lemma which ensures that there exists an orthonormal basis \( (\phi_k)_k \in \mathbb{N} \) of \( A^2(\Omega) \) consisting of certain rational functions. This is essentially an \( L^2 \)-version of the Mergelyan theorem in a smooth setting, of which related versions have appeared previously (see e.g. [10] and the references therein).

**Lemma 3.5.** Let \( \Omega \) be an \( l \)-connected planar domain with analytic boundary and let \( z_1, \ldots, z_l \) be fixed points in each hole of \( \Omega \). Then rational functions with poles at \( z_1, \ldots, z_l \) form a dense subspace of \( A^2(\Omega, e^{-2Q \cdot \log(z)}) \).

**Proof of Lemma 3.5.** Suppose that \( \Omega \) is bounded. The other case can be treated similarly\(^2\). We denote by \( T_1, \ldots, T_l \) the holes of \( \Omega \), i.e., the connected components of \( \mathbb{C} \setminus \Omega \). We use the notation \( \Omega_j = \mathbb{C} \setminus T_j \) for \( j \in \{1, \ldots, l\} \) and define also \( \Omega_{l+1} = \Omega \cup T_1 \cup \cdots \cup T_l \).

Denote by \( f \) an arbitrary element of \( A^2(\Omega, e^{-2Q \cdot \log(z)}) \). By Cauchy’s integral formula, \( f \) can be decomposed as

\[
  f(z) = \sum_{j=1}^{l+1} \frac{1}{2\pi i} \int_{C_j} \frac{f(w)}{w-z} \, dw = \sum_{j=1}^{l+1} g_j(z).
\]

where for \( 1 \leq j \leq l+1 \), the \( C_j \) are small inward perturbations towards \( \Omega \) of the connected components of \( \partial \Omega \), taken with the standard orientation. By deforming the contours \( C_j \), the functions \( g_j \) are seen to be holomorphic on \( \Omega_j \).

We will show that \( g_j \) can be approximated by certain rational functions. We treat the cases \( 1 \leq j \leq l \) and \( j = l+1 \) separately.

**Case 1.** We fix an arbitrary index \( j \) with \( 1 \leq j \leq l \). Since for all \( k \neq j \), the function \( g_k \) is locally bounded in a neighbourhood of \( \partial \Omega_j \), it follows that \( g_j = f - \sum_{k \neq j} g_k \) lies in \( A^2(\Omega_j, |z-z_j|^{-2\alpha}) \) where \( \alpha \) is some arbitrary fixed strictly positive number. Indeed, near \( \partial \Omega_j \), we use the uniform boundedness of \( g_k \) for \( k \neq j \) along with the fact that the \( f \) lies in the space \( A^2(\Omega, e^{-2Q \cdot \log(z)}) \), and near infinity we use the facts that \( |g_j| \) decays like \( 1/|z| \) and that \( \alpha \) is strictly positive.

There exists a conformal mapping \( \phi_j : \Omega_j \to \mathbb{D} \) which extends holomorphically past the boundary, where we choose the normalization \( \phi_j(\infty) = 0 \). Below we will prove that \( g_j \) can be arbitrarily well approximated in \( A^2(\Omega_j, |z-z_j|^{-2\alpha}) \) by functions of the form \( \phi_j^k \).

\(^2\)Specifically, by taking \( z_0 \in \mathbb{C} \setminus \Omega \), considering the domain \( \{1/(z-z_0) : z \in \Omega \} \), and modifying the weight to account for the possible pole at infinity. This induces a singularity of the \( \phi_j^k \) at an interior point, but this does not affect the argument.
for appropriate powers $k \in \mathbb{Z}$. But then by Runge’s theorem, since $\phi_j$ is holomorphic in a neighbourhood of $\Omega_j$, the function $\phi_j$ can be approximated in the uniform norm on a neighbourhood of $\overline{\Omega}_j$ by rational functions with poles at $z_j$. Finally, since $\Omega$ is bounded, the same statement holds in the norm in the space $A^2(\Omega, e^{-2Q\log(z)})$.

It remains to show that $\{\phi^k \cdot \phi_j : k \geq k_0\}$ spans the space $A^2(\Omega_j, |z - z_j|^{-2\alpha})$. But this follows from the classical fact that polynomials are dense in $A^2(\mathbb{D}, \omega)$ for weights which are smooth and positive in a neighbourhood of $\partial \mathbb{D}$, which is equivalent to the classical statement that polynomials are dense in $A^2(\mathbb{D})$ (see e.g. [13]).

Case 2. This case is similar, but we approximate $g$ by polynomials. \hfill \Box

Below, we need means to control basis elements of $A^2(\Omega, \rho)$ at infinity, in the case when $\Omega$ is unbounded and $\rho$ is a positive weight with $\rho(z) \approx |z|^\alpha$ as $z \to \infty$, for some $\alpha \in \mathbb{R}$. We use the following intuitive known lemma.

**Lemma 3.6.** Let $\mathbb{D}$ be the open unit disk and let $g : \mathbb{D} \setminus \{0\} \to \mathbb{C}$ be holomorphic. If $\int_\mathbb{D} |g(z)|^2 dm(z) < \infty$ then $g$ admits a holomorphic extension to $\mathbb{D}$.

**Proof.** It can be obtained by using a Laurent series expansion. We may also conclude it by subharmonicity properties. \hfill \Box

In particular, for any $\gamma \in \mathbb{R}$, there exists $\varepsilon > 0$ such that if $\int_\mathbb{D} |z|^\gamma |g(z)|^2 dm(z) < \infty$ then $|z|^\gamma |g(z)|^2 \leq C/|z|^{2-\varepsilon}$ for some constant $C > 0$ and near $z = 0$. Indeed, we may write $\gamma = 2k + \xi$ with $\xi \in (-2, 0]$ and $k \in \mathbb{Z}$ so that $\int_\mathbb{D} |z|^k g(z)|^2 \leq \int_\mathbb{D} |z|^\gamma |g(z)|^2 < \infty$. Lemma 3.6 implies that $|z^k g(z)|^2$ is bounded around zero which tells us that

$$|z|^\gamma |g(z)|^2 = |z^\xi z^k g(z)|^2 \leq C|z|^\xi = \frac{C}{|z|^{2-\varepsilon}}$$

for some constants $C, \varepsilon > 0$ and near $z = 0$.

From the previous argument, for a positive weight $\rho$ satisfying $\rho(z) \approx |z|^\alpha$ as $z \to \infty$, there exists $\varepsilon = \varepsilon(\alpha) > 0$ such that if $f \in A^2(\Omega, \rho)$ then

$$(3.1) \quad |f(z)|^2 \rho(z) \lesssim |z|^{-2-\varepsilon}.$$  

Indeed, for $g(z) = f(1/z)$ we would have $\int |g(z)|^2 |z|^{-\alpha-4} dm(z) < \infty$ which would imply that $|g(z)|^2 |z|^{-\alpha-4} \leq C/|z|^{2+\varepsilon}$. By going back to $f$, this is precisely (3.1).

**Proof of Proposition 3.3.** Let $Q_N$ be the sequence of elements of $(\mathbb{R}/\mathbb{Z})^l$ which converge towards $Q \in [0, 1)^l$. We first notice that for $N$ large enough

$$A^2(\Omega, e^{-2Q\log(z)}) \subset A^2(\Omega, e^{-2Q_N\log(z)}).$$

Indeed, if $\Omega$ is bounded, this is simply the fact that $\log(z)$ is bounded on $\Omega$. If instead $\Omega$ contains a neighbourhood of infinity, (3.1) implies that for any $f \in A^2(\Omega, e^{-2Q\log(z)})$,

$$|f(z)|^2 e^{-2Q\log(z)} \leq C|z|^{-2-\varepsilon},$$

for a fixed $\varepsilon$, so that since $\|Q - Q_N\|_1 < \varepsilon$ for $N$ large enough, $f$ belongs to the space $A^2(\Omega, e^{-2Q_N\log(z)})$ as well.

Denote by $(\psi_k)_{k \in \mathbb{N}}$ an orthonormal basis of $A^2(\Omega, e^{-2Q\log(z)})$ consisting of rational functions with poles at the $z_j$, the existence of which is guaranteed by Lemma 3.5. We define $(\psi_{k,N})_{k \in \mathbb{N}}$ as the orthonormal set obtained from the Gram-Schmidt process applied to $(\psi_k)_{k \in \mathbb{N}}$ in the topology of $L^2(\Omega, e^{-2Q_N\log(z)})$. All these functions are defined on $\mathbb{C} \setminus \{z_1, \ldots, z_l\}$.

It only remains to establish the convergence

$$\psi_{k,N} \to \psi_k$$

as $N \to \infty$ and the locally uniform boundedness in $N$ of $\psi_{k,N}$. But this would be immediate if we can show that the Gram matrix $G_N = (\langle \psi_k, \psi_m \rangle_{L^2(\Omega, \omega_{Q_N})})_{1 \leq j, k \leq k_0}$
satisfies \( G_N = I + o(1) \) as \( N \to \infty \), using that the functions \( \psi_k \) are locally uniformly bounded for the last point. But the statement about the Gram matrices, in turn, follows from the bound (3.1), the convergence \( Q_N \to Q \) and the dominated convergence theorem. The proof is complete. \( \square \)

3.2. Quasipolynomials and \( \bar{\partial} \)-surgery. Let \( \psi_{k,N} \) be a family such as the one in Proposition 3.3 for \( Q_N = [\kappa_N q] \). We use it to construct an orthonormal basis \( (P_{k,N})_{0 \leq k \leq N-1} \) of the polynomial Bergman space \( A^2_N(\Omega,e^{-2\kappa_N U^\mu}) \) which satisfy, for any fixed \( k \),

\[
\forall z \in \Omega, \quad \lim_{N \to \infty} |P_{k,N}(z)| e^{-\kappa_N U^\mu(z)} = |\psi_k(z)| e^{-Q \cdot \log(z)}.
\]

Denote by \( \chi \) a smooth cut-off function which vanishes in a neighbourhood of \( \mathbb{C} \setminus \bar{\Omega} \), and is identically one in a neighbourhood of \( \Omega \). See Figure 3.

**Definition 3.7** (Approximately orthogonal quasipolynomials). For any \( N,k \in \mathbb{N} \) we define the function \( F_{k,N} : \mathbb{C} \to \mathbb{C} \) by

\[
F_{k,N}(z) = \chi(z)\psi_{k,N}(z)e^{\kappa_N b(z)} \prod_{i=1}^{l}(z - z_i)^{\kappa_N q_i - [\kappa_N q_i]}.
\]

Notice that \( |F_{k,N}(z)| = |\psi_{k,N}(z)| e^{\kappa_N U^\mu(z)} e^{-Q \cdot \log(z)} \) for \( z \in \Omega \). The following lemma shows that the \( F_{k,N} \) are approximately orthogonal in the space \( L^2(\mathbb{C},e^{-2\kappa_N U^\mu}) \) and behave like elements of \( \mathbb{C}_{N-1}[z] \) at infinity. In fact, we will soon show that \( F_{k,N} \) lies very close to a polynomial.

**Lemma 3.8.** For any \( N,k \in \mathbb{N} \), the function \( F_{k,N} \) belongs to \( L^2(\mathbb{C},e^{-2\kappa_N U^\mu}) \). Moreover, for any fixed \( k,m \in \mathbb{N} \),

\[
\langle F_{k,N}, F_{m,N} \rangle_N = \int_{\mathbb{C}} F_{k,N}(z) \overline{F_{m,N}(z)} e^{-2\kappa_N U^\mu(z)} dm(z) \xrightarrow{N \to \infty} \delta_{k,m}.
\]

**Proof.** By the definition of \( \psi_{k,N} \) we find that

\[
\int_{\Omega} F_{k,N}(z) \overline{F_{m,N}(z)} e^{-2\kappa_N U^\mu(z)} dm_C(z) = \delta_{km}.
\]

On the set \( \Omega' \setminus \bar{\Omega} \), Proposition 2.6 implies that \( U^\mu - U^\mu > 0 \). In view of Proposition 3.3 part (iii), the functions \( \psi_{k,N} \) are locally uniformly bounded in \( N \), so the same holds for
the functions $|F_{k,N}|^2 e^{-2\kappa NU}$. We may hence apply the dominated convergence theorem to obtain

$$\int_{\Omega^c \setminus \Omega} |F_{k,N}(z)|^2 e^{-2\kappa NU}(z) \, dm(z)$$

$$= \int_{\Omega^c \setminus \Omega} \chi(z) \psi_{k,N}(z)^2 e^{-2\kappa N(U(z)-U_\Omega(z))} e^{-2Q\log(z)} \to 0$$

as $N$ tends to infinity. Hence, we can conclude that

$$\langle F_{k,N}, F_{m,N} \rangle_N \xrightarrow{N \to \infty} \delta_{km}.$$ 

This completes the proof. \hfill \Box

Turning to the $\bar{\partial}$-surgery, we have the following,

**Proposition 3.9** ($\bar{\partial}$-surgery and Hörmander estimates). For any fixed $k \in \mathbb{N}$, there exists a sequence of smooth functions $(v_{k,N})_{N \in \mathbb{N}}$ defined on $\mathbb{C}$ with $\|v_{k,N}\|_N^2 \to 0$ as $N \to \infty$, such that $Q_{k,N} := F_{k,N} - v_{k,N}$ is a polynomial of degree at most $N - 1$ and we have the pointwise convergence

$$Q_{k,N}(z)e^{-\kappa_N b(z)} \prod_{i=1}^l (z - z_i)^{-(\kappa_N q_i - [\kappa_N q_i])} \xrightarrow{N \to \infty} \psi_k(z)$$

for $z \in \Omega$.

This claim is essential in our proof. It follows from Hörmander’s classical $L^2$-estimate for the $\bar{\partial}$ operator, with some minor adjustments which we proceed to outline. The first adjustment (this version is taken from [4]) gives control of the growth of the solution. Recall that a $C^{1,1}$ function is a derivable function whose derivative is locally Lipschitz.

**Lemma 3.10** (Hörmander estimate). Let $\phi$ be a $C^{1,1}$ subharmonic function on $\mathbb{C}$ such that $\phi(z) \geq (1 + c) \log |z|^2$ for some constant $c > 0$ and for $|z|$ large enough. Let $K \subset \mathbb{C}$ be a compact set where $\phi$ is strictly subharmonic. For any $f \in L^\infty(\mathbb{C})$ supported on $K$, there exists a solution $u : \mathbb{C} \to \mathbb{C}$ to the equation $\bar{\partial}u = f$ that satisfies

$$\int_{\mathbb{C}} |u|^2 e^{-\phi} \, dm \leq 2 \int_K |f|^2 e^{-\phi} \frac{\Delta \phi}{\bar{\partial} \phi} \, dm.$$ 

**Proof.** This follows directly from Hörmander’s classical result for a $C^{1,1}$ function $\phi$ that satisfies $\Delta \phi > 0$ on $\mathbb{C}$ (which gives the the conclusion of Lemma 3.10 without the factor 2 on the right-hand side). Indeed, the function $\phi : \mathbb{C} \to \mathbb{R}$ defined by

$$\phi_\varepsilon(z) = (1 - \varepsilon)\phi(z) + \varepsilon \log(1 + |z|^2).$$

meets the conditions of Hörmander’s theorem. Then, for any $\varepsilon > 0$ there exists a solution $u = u_\varepsilon$ to the equation $\bar{\partial}u = f$ that satisfies

$$\int |u|^2 e^{-\phi_\varepsilon} \, dm \leq \int |f|^2 e^{-\phi} \frac{\Delta \phi}{\bar{\partial} \phi} e^{\varepsilon(\phi - \log(1 + |z|^2))} \, dm \leq \frac{3}{2} \int_K |f|^2 e^{-\phi} \frac{\Delta \phi}{\bar{\partial} \phi} \, dm.$$ 

To complete the proof, it is enough to show that for small $\varepsilon > 0$, we have

$$\int |u|^2 e^{-\phi} \, dm \leq \frac{4}{3} \int |u|^2 e^{-\phi_\varepsilon} \, dm.$$ 

For this, we can use that, for $|z|$ large enough,

$$\phi(z) = \phi_\varepsilon(z) + \varepsilon(\phi(z) - \log(1 + |z|^2))$$

$$\geq \phi_\varepsilon(z) + \varepsilon(\phi(z) - (1 + \frac{1}{2}) \log |z|^2)$$

$$\geq \phi_\varepsilon(z) + \varepsilon \frac{1}{2} \log |z|^2.$$
We may thus split this integral according to $C = \mathbb{D}(0, R) \cup (\mathbb{C} \setminus \mathbb{D}(0, R))$, and use $\phi_\varepsilon = \phi + o(1)$ on the bounded set and $\phi \geq \phi_\varepsilon + O(1)$ on the unbounded set. This completes the proof.

We cannot apply Lemma 3.10 as stated with the potential $\phi = 2 \kappa_N U_\mu$ directly, so we introduce a new potential $V$ which satisfies the required regularity assumptions, whose existence and properties are given in the following Lemma.

**Lemma 3.11.** There exists a $C^2$ subharmonic function $V : \mathbb{C} \to \mathbb{R}$ such that

- (i) the inequality $V(z) \leq U_\mu(z)$ holds for $z \in \mathbb{C}$,
- (ii) for $z \in \Omega$ we have equality $V(z) = U_\mu(z)$,
- (iii) for $z \in \Omega'$ it holds that $U_\mu(z) \leq V(z)$, and
- (iv) there exists $\varepsilon_1, \varepsilon_2 > 0$ such that $\Delta V(z) \geq \delta > 0$ whenever $\varepsilon_1 \leq d(z, \Omega) \leq \varepsilon_2$.

**Proof.** For simplicity of notation, we will prove the case where $\Omega$ is unbounded. Denote $D = \mathbb{C} \setminus \Omega$ and recall that the domain of $\mathcal{U}_\mu$ is $\Omega' \supset \Omega$. We know that $\mathcal{U}_\mu - U_\mu = 0$ on $\Omega$. Notice also that there exists a compact set $E$ such that $\mathbb{C} \setminus \Omega' \subset E \subset D$ and $\nabla(\mathcal{U}_\mu - U_\mu(z)) \neq 0$ for every $z \in E$. Indeed, by Proposition 2.7, if the (outer) normal derivative of $\mathcal{U}_\mu - U_\mu$ at a boundary point is not positive, then $\mathcal{U}_\mu - U_\mu$ is strictly convex in the normal direction to that point. Either way, the derivative of $\mathcal{U}_\mu - U_\mu$ is non-zero near the boundary and, moreover, the function $\mathcal{U}_\mu - U_\mu$ is increasing in the normal direction to the boundary. Hence, the implicit function theorem tells us that we may choose $E$ such that, for some $t_0 > 0$ small enough, the level sets

$$\Gamma_t = \{ z \in D \setminus E : U_\mu(z) - U_\mu(z) = t \}, \quad 0 \leq t \leq t_0$$

is formed by $t$ closed curves, with $\Gamma_0 = \partial\Omega$ and smoothly varying with $t$. Denote by $D_t$ the open subset of $D$ enclosed by $\Gamma_t$, and for $t \in (0, t_0]$, define $V_t : \mathbb{C} \to \mathbb{R}$ by

$$V_t(z) = \begin{cases} 
\mathcal{U}_\mu(z) & \text{if } z \notin D_t \\
U_\mu(z) - t & \text{if } z \in D_t
\end{cases}.$$ 

Notice that $V_t$ is smooth on $\mathbb{C} \setminus \Gamma_t$. In addition, we have

$$\Delta V_t = 1_{D_t, \mu} + f_t d\sigma_{\Gamma_t},$$

where $f_t$ is the jump of the normal derivative across $\Gamma_t$, i.e.

$$f_t = \partial_{n^+} U_\mu + \partial_{n^-} U_\mu,$$

where $n^+$ and $n^-$ are the unit normals pointing towards $\Gamma_t$ from inside and outside $D_t$, respectively. By construction, the function $V_t$ satisfies the inequalities $V_t \leq U_\mu$ everywhere and $V_t \geq U_\mu$ on $\Omega'$. Moreover, for $t > 0$ the densities $f_t$ vary smoothly, and since $\Gamma_t$ are a disjoint foliation of $D \setminus D_0$, there exists one smooth positive function $f(z)$ defined on $D \setminus E$ so that $f_t = f|_{\Gamma_t}$.

We now construct the function $V$ by integration

$$V(z) = \int_0^{t_0} V_t(z) \varphi(t) dt,$$

where $\varphi$ is a smooth positive function with $\varphi(0) = 0$ and such that $\int \varphi(t) dt = 1$. This construction mollifies the measures $\Delta V_t$, so that $\Delta V$ is $C^\infty$-smooth. It follows that $V$ is a $C^\infty$-smooth function which satisfies (i)-(iii). In view of Proposition 2.7 the functions $f_t$ are uniformly bounded from below by a positive constant in regions of the form $\{ z \in \mathbb{C} : \varepsilon_1 \leq d(z, \Omega) \leq \varepsilon_2 \}$ for $0 < \varepsilon_1 < \varepsilon_2$ small enough. It follows that $\Delta V$ meets the desired positivity condition (iv). \qed

We are now ready for the proof of our proposition on $\partial$-surgery.
Proof of Proposition 3.9. By Lemma 3.11, there exists a $C^2$ function $V : \mathbb{C} \to \mathbb{R}$ and a constant $\delta > 0$ such that

$$V \geq U^\mu$$

and such that $U^\mu \geq V$ everywhere and $V|_{\Omega} = U^\mu|_{\Omega}$. By Lemma 3.10 applied with $\phi = 2\kappa_N V$ and

$$\partial F_{k,N} = (\partial \phi) \psi_{k,N} e^{\kappa_N h} \prod_{i=1}^l (z - z_i)^{\kappa_N q_i - \lfloor \kappa_N q_i \rfloor},$$

there exists a function $v_{k,N}$ such that $\partial v_{k,N} = \partial F_{k,N}$ and

$$\int_C |v_{k,N}|^2 e^{-2\kappa_N U^\mu} \, dm \leq \int_C |\partial F_{k,N}|^2 e^{-2\kappa_N V} \, dm$$

(3.3)

$$\leq \int_{\text{supp} \nabla \chi} \frac{1}{2\kappa_N \delta} |\partial \phi| |\psi_{k,N}|^2 e^{-2\kappa_N (V - U^\mu)} e^{B \log dm}.$$  

The important thing to notice is that $\partial \chi$ (which is essentially the gradient $\nabla \chi$) is supported where the weight $e^{-2\kappa_N (V - U^\mu)}$ is exponentially small. Since the $\psi_{k,N}$ are bounded independently of $N$, the right-hand side of (3.3) tends to zero as $N$ goes to infinity, and in view of the elementary inequality

$$\int_C |v_{k,N}|^2 e^{-2\kappa_N U^\mu} \, dm \leq \int_C |v_{k,N}|^2 e^{-2\kappa_N V} \, dm$$

we have obtained that $\|v_{k,N}\|_N^2 \to 0$.

In particular, $\int_C |v_{k,N}|^2 e^{-2\kappa_N U^\mu} \, dm_C < \infty$ so that, if $Q_{k,N} := F_{k,N} - v_{k,N}$, we also have $\int_C |Q_{k,N}|^2 e^{-2\kappa_N U^\mu} \, dm_C < \infty$. The bound (3.1) implies that there exists a constant $C > 0$ such that

$$|Q_{k,N}(z)| \leq C |z|^{\kappa_N} \leq C |z|^{\kappa_N + \varepsilon/2} \leq C |z|^{N - \varepsilon/2}$$

for $|z|$ large enough. Since $\partial Q_{k,N} = \partial F_{k,N} - \partial v_{k,N} = 0$, we get that $Q_{k,N}$ is entire and, since $|Q_{k,N}|$ grows at most like $|z|^{\kappa_N - \varepsilon/2}$ (in fact like $|z|^{-1}$), Liouville’s theorem implies that $Q_{k,N}$ is a polynomial of degree less or equal than $N - 1$.

The holomorphicity of $v_{k,N}|_{\Omega}$ implies that $|v_{k,N}|^2 e^{-2\kappa_N U^\mu}$ is subharmonic on $\Omega$. Then, for any fixed $z \in \Omega$ and any $\varepsilon > 0$ such that $D(z, \varepsilon) \subset \Omega$, we may write

$$\int_{D(z, \varepsilon)} |v_{k,N}(z)|^2 e^{-2\kappa_N U^\mu(z)} \, dm_C(z) \leq \frac{1}{\pi \varepsilon^2} \int_{D(z, \varepsilon)} |v_{k,N}(z)|^2 e^{-2\kappa_N U^\mu(z)} \, dm_C(z)$$

$$\leq \frac{1}{\pi \varepsilon^2} \|v_{k,N}\|_N^2 \to 0.$$  

This implies that

$$|Q_{k,N}(z) e^{-\kappa_N h(z)} \prod_{i=1}^l (z - z_i)^{-(\kappa_N q_i - \lfloor \kappa_N q_i \rfloor)} - \psi_k(z)| = |v_{k,N}(z)|^2 e^{-2\kappa_N U^\mu(z)} \to 0$$

as $N \to \infty$, which is what we wanted.  \hfill \Box

Recall that, by (3.1) and Liouville’s theorem, $A^2(\mathbb{C}, e^{-2\kappa_N U^\mu}) = \mathbb{C}_{N-1}[z]$ as sets.

Lemma 3.12 (The adapted orthonormal basis). For any fixed $k_0 \in \mathbb{N}$, there exists an orthonormal family $(P_{k,N})_{k \leq k_0}$ in $A^2(\mathbb{C}, e^{-2\kappa_N U^\mu})$, such that for any $k \leq k_0$, we have

$$P_{k,N}(z) e^{-\kappa_N h(z)} \prod_{i=1}^l (z - z_i)^{-(\kappa_N q_i - \lfloor \kappa_N q_i \rfloor)} \to \psi_k(z).$$
for every $z \in \Omega$. In particular,
\[ |P_{k,N}(z)|^2 e^{-2\kappa_N U''(z)} \xrightarrow{N \to \infty} |\psi_k(z)|^2 e^{-2Q \log(z)}. \]

**Proof.** Recall the holomorphic functions $F_{k,N}$ and the $\bar{\partial}$-corrections $v_{k,N}$ from Proposition 3.9. Since
\[
\langle F_{k,N}, F_{m,N} \rangle_N \xrightarrow{N \to \infty} \delta_{km} \quad \text{and} \quad \|v_{k,N}\|_N \xrightarrow{n \to \infty} 0,
\]
we have
\[
\langle Q_{k,N}, Q_{m,N} \rangle_N \xrightarrow{N \to \infty} \delta_{km},
\]
where $Q_{k,N}$ are the polynomials $Q_{k,N} = F_{k,N} - v_{k,N}$ from Proposition 3.9. In other words, for any fixed positive integer $k_0$, the Gram matrix $(\langle Q_{k,N}, Q_{m,N} \rangle_N)_{1 \leq k, m \leq k_0}$ of the family $(Q_{k,N})_{k \leq k_0}$ converges towards the identity matrix, hence for $N$ large enough, it is invertible and the family $(Q_{k,N})_{1 \leq k \leq k_0}$ is linearly independent. Moreover, if $(P_{k,N})_{k \leq k_0}$ is the Gram-Schmidt orthonormalization of $(Q_{k,N})_{k \leq k_0}$, the fact that the Gram matrix of $(Q_{k,N})_{k \leq k_0}$ converges let us conclude the proof. This can be seen, for instance, by induction or by using an explicit formula such as the following. Fix $k \geq k_0$ and let $D_k^{(N)}$ be the determinant of the Gram matrix of $(Q_{k,N})_{k \leq k}$ and let $D_k^{(N)}$ be the determinant of the Gram matrix of $(Q_{k,N})_{k \leq k-1}$. Then,
\[
P_{k,N}(z) = \frac{1}{\sqrt{D_k^{(N)} D_{k-1}^{(N)}}} \begin{vmatrix} \langle Q_{1,N}, Q_{1,N} \rangle_N & \langle Q_{2,N}, Q_{1,N} \rangle_N & \cdots & \langle Q_{k,N}, Q_{1,N} \rangle_N \\ \langle Q_{1,N}, Q_{2,N} \rangle_N & \langle Q_{2,N}, Q_{2,N} \rangle_N & \cdots & \langle Q_{k,N}, Q_{2,N} \rangle_N \\ \vdots & \vdots & \ddots & \vdots \\ \langle Q_{1,N}, Q_{k-1,N} \rangle_N & \langle Q_{2,N}, Q_{k-1,N} \rangle_N & \cdots & \langle Q_{k,N}, Q_{k-1,N} \rangle_N \\ Q_{1,N}(z) & Q_{2,N}(z) & \cdots & Q_{k,N}(z) \end{vmatrix}.
\]
By multiplying by $e^{-\kappa_N h_N(z)} \prod_{i=1}^{N} (z - z_i)^{-\kappa_N [Q_i]}$, using the linearity of the determinant in the last row and using the convergence of the inner products, Proposition 3.9 allows us to conclude the proof. □

### 4. PROOFS OF THE MAIN RESULTS

#### 4.1. Proof of Theorem 1.4
We finish the proof of the theorem by completing the family $(P_{k,N})_{k \leq k_0}$ to an orthonormal basis $(P_{k,N})_{k \leq N-1}$ of $A^2(\mathbb{C}, e^{-2\kappa_N U''})$. We obtain
\[
\sum_{k=0}^{k_0} |P_{k,N}(z)|^2 e^{-2\kappa_N U''(z)} \leq \sum_{k=0}^{N-1} |P_{k,N}(z)|^2 e^{-2\kappa_N U''(z)} = K_N(z, z)
\]
and, since
\[
\sum_{k=0}^{k_0} |\psi_k(z)|^2 e^{-2Q \log(z)} = \lim_{N \to \infty} \sum_{k=1}^{k_0} |P_{k,N}(z)|^2 e^{-2\kappa_N U''(z)},
\]
we obtain for any $z \in \Omega$
\[
\sum_{k=1}^{k_0} |\psi_k(z)|^2 e^{-2Q \log(z)} \leq \liminf_{N \to \infty} K_N(z, z).
\]
By taking $k_0 \to \infty$, we find that
\[
\mathcal{B}_{\Omega, Q}(z, z) = \sum_{k=1}^{\infty} |\psi_k(z)|^2 e^{-2Q \log(z)} \leq \liminf_{N \to \infty} K_N(z, z)
\]
and in view of the convergence $[\kappa_N q] \to Q$ along the subsequence $\mathcal{L}$, Lemma 2.5 and Proposition 2.4 together imply that $\limsup_{N \in \mathcal{L}, N \to \infty} K_N(z, z)$, which completes the proof.
4.2. Proof of Theorem 1.2. If $\Omega$ is a simply connected domain, then this theorem is an immediate application of Theorem 1.4. If $\Omega$ is the unbounded component of $\mathbb{C} \setminus \text{supp}(\mu)$ and is simply connected in $\hat{\mathbb{C}}$ (for instance, the complement of a single Jordan domain), then we apply Theorem 1.4, and we remark that the $\mu$-mass of the hole is $q_1 = 1$, hence $(N + 1)q_1$ it is an integer and the sequence $[N + 1]$ is identically equal to zero.

4.3. Proof of Theorem 1.5. The proof of the independence between components is general and elementary. Recall from the proof of Theorem 1.4 that, in view of (1.2), the kernels associated to $\Phi_{1,N}$ and $\Phi_{2,N}$ (called here $K_{1,N}$ and $K_{2,N}$) converge uniformly on compact subsets of $\Omega_1 \times \Omega_1$ and $\Omega_2 \times \Omega_2$ towards two limiting Bergman kernels $B_1$ and $B_2$. If we show that the kernel of the determinantal point process

$$\Phi_{N,\Omega_1 \cup \Omega_2} = \Theta_N \cap (\Omega_1 \cup \Omega_2),$$

converges uniformly on compact subsets of $\Omega_1 \cup \Omega_2$ towards the kernel

$$B_{1,2}(z, w) = \begin{cases} B_1(z, w) & \text{if } (z, w) \in \Omega_1 \times \Omega_1 \\ B_2(z, w) & \text{if } (z, w) \in \Omega_2 \times \Omega_2 \\ 0 & \text{if } (z, w) \in (\Omega_1 \times \Omega_2) \cup (\Omega_2 \times \Omega_1), \end{cases}$$

then the asymptotic independence of the point processes follows (see Proposition A.3) since $B_{1,2}$ corresponds to the kernel of the union of independent determinantal point processes associated to $B_1$ and $B_2$ as shown in Proposition A.4 of the Appendix.

Here, as in section 2.1, the kernel $\mathcal{K}_N$ associated to $\Phi_{N,\Omega_1 \cup \Omega_2}$ is holomorphic in the first coordinate and anti-holomorphic in in the second coordinate. Thanks to the reproducing kernel property and Cauchy-Schwarz inequality, we obtain that

$$\forall z \in \Omega_1, \forall w \in \Omega_2, \quad |\mathcal{K}_N(z, w)|^2 \leq \mathcal{K}_N(z, z) \mathcal{K}_N(z, w).$$

This allows us to apply Montel’s Theorem, and to show that any subsequence of the sequence $K_N$ has at least one further subsequence converging uniformly on compact subsets of $(\Omega_1 \cup \Omega_2) \times (\Omega_1 \cup \Omega_2)$. On $\Omega_1 \times \Omega_1$ and $\Omega_2 \times \Omega_2$, we have already established locally convergence of the kernels. The proof is complete if we can show that for any $z \in \Omega_1$, we have

$$\int_{\mathbb{C} \setminus \Omega_1} |\mathcal{K}_N(z, w)|^2 dm(w) \xrightarrow{N \to \infty} 0.$$ 

To see why this convergence holds, let $S$ be a compact subset of $\Omega_1$. Then

$$\mathcal{K}_N(z, z) = \int_S |\mathcal{K}_N(z, w)|^2 dm(w) \geq \int_S |\mathcal{K}_N(z, z)|^2 dm(w) + \int_{\Omega_1} |\mathcal{K}_N(z, w)|^2 dm(w).$$

Since we know that $\mathcal{K}_N$ converges uniformly towards $B_1$ on the compact $S$, we get

$$B_1(z, w) \geq \int_S |B_1(z, w)|^2 dm(w) + \limsup_{N \to \infty} \int_{\mathbb{C} \setminus \Omega_1} |\mathcal{K}_N(z, w)|^2 dm(w).$$

Now, let $S$ grow to fill up the set $\Omega_1$, and notice that by the monotone convergence theorem we get

$$B_1(z, z) \geq \int_{\Omega_1} |B_1(z, w)|^2 dm(w) + \limsup_{N \to \infty} \int_{\mathbb{C} \setminus \Omega_1} |\mathcal{K}_N(z, w)|^2 dm(w),$$

which, since $B_1$ is a reproducing kernel on $\Omega_1$ yields

$$B_1(z, z) = \int_{\Omega_1} |B_1(z, w)|^2 dm(w),$$

which completes the proof.
5. Outliers for random polynomials

5.1. A word on Hardy spaces. Denote by $\Omega$ a plane region bounded by finitely many disjoint analytic Jordan curves, and let $\eta$ be a measure on $\partial\Omega$ with a continuous and strictly positive density with respect to the arc length measure. We define the Hardy space $H^2(\Omega, \eta)$ as the completion of the set of holomorphic functions on $\Omega$ that can be extended holomorphically to a neighbourhood of $\overline{\Omega}$ with the norm

$$ \|f\|_{H^2(\Omega, \eta)}^2 := \int_{\partial\Omega} |f|^2\,d\eta. $$

Hardy spaces are usually defined by the finiteness of certain $L^2$-maximal functions, but since we only consider regular domains, the above simple definition is enough. The Hardy space $H^2(\Omega, \eta)$ is a reproducing kernel Hilbert space, with a Hermitian holomorphic reproducing kernel $S_{\Omega, \eta}(z, w)$ defined on $\Omega \times \Omega$, known as the Szegő kernel of $(\Omega, \eta)$. If the measure is simply arc length measure $\sigma_{\partial\Omega}$, we write $H^2(\Omega)$ and $S_{\Omega}$ for the space and the kernel, respectively. We will denote the $N+1$-dimensional polynomial Hardy space $\mathbb{C}_N[z] \subset H^2(\Omega, \eta)$ by $H^2_N(\Omega, \eta)$.

Remark 5.1. Assume that $\partial\Omega$ is a single closed analytic curve. Let $s = s_\rho$ be a holomorphic function on $\Omega$ whose real part equals $\frac{1}{2} \log \rho$ on the boundary $\partial\Omega$. Such a function is often referred to as a Szegő function of $\rho$. The function $s$ allows us to extend $\rho$ to a log-harmonic function on $\Omega$ by putting $\rho(z) = |e^{2s(z)}| = e^{2 \Re \, s(z)}$. This implies that multiplication operator $f \mapsto fe^{-s}$ induces an isometric isomorphism between $H^2(\Omega)$ and $H^2(\Omega, \rho \, d\sigma)$. In particular the weighted Szegő kernels satisfy the relation:

$$ \forall z, w \in \Omega, \quad S_{\Omega}(z, w) = S_{\Omega, \rho}(z, w)e^{s(z) + \overline{s(w)}}. $$

5.2. Proof of Theorem 1.6. We begin by recalling some preliminaries.

Premilinaries. Without loss of generality we may consider the case when $\Omega$ is bounded. The unbounded case then follows by applying an appropriate inversion (see, for instance, [7, Theorem 1.14]).

The Szegő kernel $S_{\Omega}$ is the covariance kernel for the Gaussian analytic function

$$ P(z) = \sum_{k=0}^{\infty} \xi_k \psi_k(z), \quad \xi_k \text{ i.i.d., } \xi_k \sim \mathcal{N}_{\mathbb{C}}(0, 1), $$

where $(\psi_k)_{k \geq 0}$ is any orthonormal basis of $H^2(\Omega)$. By Remark 5.1, it follows that the zero set of $P$ has the same distribution as the zero set of $P_\rho$, obtained similarly as a random linear combination of an orthonormal basis for the weighted Hardy space $H^2(\Omega, \rho \, d\sigma)$. To prove Theorem 1.6 it would be enough to show that the covariance kernel for the random polynomial (1.4), converges locally uniformly towards the Szegő kernel $S_{\Omega, \rho}$ (see, for instance, [24, Section 2]). Strictly speaking, this is not the case for our model, but it can be arranged by multiplying the sum (1.4) by zero-free holomorphic functions. To be specific, we introduce a holomorphic function $h$ on $\Omega$ whose real part equals $U'$ on $\partial\Omega$, and consider the random analytic function

$$ f_N(z) = \left( \sum_{k=0}^{N} \xi_k Q_{k, N}(z) \right) e^{-N h(z)}, \quad \xi_k \text{ i.i.d., } \xi_k \sim \mathcal{N}_{\mathbb{C}}(0, 1). $$

Notice that $f_N$ is a random linear combination of an orthonormal system in the Hardy space $H^2(\Omega, \rho \, d\sigma)$. Denote the covariance kernel of $f_N$ by $K_N(z, w)$, which is given by

$$ K_N(z, w) = \sum_{k=0}^{N} Q_{k, N}(z) \overline{Q_{k, N}(w)} e^{-N (h(z) + \overline{h(w)})}. $$

3Embedded into the set of holomorphic functions on $\Omega$ by using, for instance, Cauchy’s integral formula.
Structure of the proof. From here, the structure of the proof is very similar to the one of Theorem 1.4. We start by showing an inequality relating the covariance kernels and the Szegő kernel on the diagonal, similar to Lemma 2.5. From this key inequality, we can mimic Lemmas 3.1 and 3.2 to reduce the local uniform convergence to the pointwise convergence of the kernels on the diagonal of $\Omega \times \Omega$. To obtain the pointwise convergence of the kernels, we construct, for fixed $k_0 > 0$, an orthonormal set $(R_{k,N})_{k \leq k_0}$ of $H_N(\Omega, e^{-2NU}\,d\nu)$ such that, for any $k \leq k_0$ and any fixed $z \in \Omega$,
\[
|\psi_k(z)|^8 e^{-2NU\nu(z)} \xrightarrow{N \to \infty} |\psi_k(z)|^8.
\]
Remembering that we can express the covariance kernels by using any orthonormal basis of $H_N(\Omega, e^{-2NU\rho}\,d\sigma)$, we obtain a lower bound which matches the already known upper bound.

Monotonic bound and normal families. We observe that
\[
K_N(z, z) = \sup_{p \in H^2_{\nu}(\Omega, e^{-2NU\nu}\,d\nu)} \frac{|p(z)|^2 e^{-2NU\nu(z)}}{\|p\|^2_N} \leq \sup_{p \in H^2(\Omega, e^{-2NU\nu}\,d\nu)} \frac{|p(z)|^2 e^{-2NU\nu(z)}}{\|p\|^2_N},
\]
where, in this section, $\|\cdot\|_N$ refers to the norm in $L^2(\partial \Omega, e^{-2NU\nu}\,d\nu)$. On the other hand, we may notice that, since $p \in H^2(\Omega)$ if and only if $pe^{-2NU\nu} \in H^2(\Omega, e^{-2NU\nu})$,
\[
S_{\Omega, \nu}(z, z) = \sup_{f \in H^2(\Omega, \nu)} \frac{|f(z)|^2}{\|f\|^2_{H^2(\Omega, \nu)}} = \sup_{f \in H^2(\Omega, \nu)} \frac{|f(z)e^{-2NU\nu}(z)|^2}{\|f\|^2_{H^2(\Omega, \nu)}}
\]
which implies that, for every $z \in \Omega$,
\[
\forall N > 0, \forall z \in \Omega, \quad K_N(z, z) \leq S_{\Omega, \nu}(z, z).
\]
From this equation, we can mimic Lemmas 3.1 and 3.2 to reduce the local uniform convergence to pointwise convergence on the diagonal of $\Omega \times \Omega$.

Construction of the good orthonormal basis. Let $(\psi_k)_k$ be an orthonormal basis of the space $H^2(\Omega, \nu)$, consisting of functions holomorphic on a slightly larger set $\Omega'$, containing $\Omega$. For instance, one may take $\psi_k = \sqrt{\phi^k} e^{-\phi}$, where $\phi$ is a fixed conformal map of $\Omega$ onto the unit disk $\mathbb{D}$, which can be extended past the boundary of $\Omega$. We define the function $F_{k,N}$ on $\mathbb{C}$ by
\[
F_{k,N} = \chi \psi_k e^{2NU\nu}
\]
where $\chi$ denotes a smooth non-negative cut-off function which is identically one in a neighbourhood of $\overline{\Omega}$ and vanishes identically on a neighbourhood of $\mathbb{C} \setminus \Omega'$. By construction, $F_{k,N}$ is a holomorphic function on a neighbourhood of $\overline{\Omega}$ and $(F_{k,N})_k$ is an orthonormal family in the weighted Hardy space $H^2(\Omega, e^{-2NU\nu}\,d\nu)$. Indeed, this follows from the identity
\[
F_{k,N} \overline{F_j,N} e^{-2NU\nu} \,d\nu = \psi_k \overline{\psi_j} \,d\nu,
\]
which holds since $\chi$ is identically one on $\Gamma$ and since, by construction,
\[
e^{-2NU\nu} = |e^{-2NU\nu}|^2.
\]
Now, choose a $C^2$ subharmonic function $V : \mathbb{C} \to \mathbb{R}$, harmonic on a neighbourhood of $\overline{\Omega}$, that coincides with $U\nu$ on $\overline{\Omega}$, satisfies $\lim \sup_{z \to \infty} \{V(z) - \log |z|\} < \infty$ and such that there is a $\delta > 0$ with
\[
\Delta V(z) \geq \delta \text{ and } V(z) \geq \Re h(z) \text{ whenever } \nabla \chi(z) \neq 0.
\]
This function can be constructed by the proof of Lemma 3.11 (taking \( \varphi \) from (3.2) to be zero on a neighbourhood of 0). By Lemma 3.10 applied with \( \varphi = 2NV \) and
\[
\overline{\partial}F_{k,N} = (\overline{\partial} \chi) \psi k e^{N b},
\]
there exists a function \( v_{k,N} \) such that \( \overline{\partial}v_{k,N} = \overline{\partial}F_{k,N} \) and
\[
\int_C |v_{k,N}|^2 e^{-2NV} dm \leq \int_C \frac{|\overline{\partial}F_{k,N}|^2}{2N\Delta V} e^{-2NV} dm
\]
\[
\leq \int_{\text{supp} \nabla \chi} |\overline{\partial} \chi|^2 |\psi_k|^2 e^{-2(N(V - Re b))} dm \xrightarrow{N \to \infty} 0.
\]
In particular, due to the harmonicity of \( V \) and the holomorphicity of \( v_{k,N} \) on a neighbourhood of \( \Omega \) we have also that \( v_{k,N} e^{-NV} \xrightarrow{N \to \infty} 0 \) uniformly on \( \Omega \). Then,
\[
R_{k,N} = F_{k,N} - v_{k,N} \text{ is holomorphic on } \mathbb{C} \text{ and } \int_C |R_{k,N}|^2 e^{-2NV} dm < \infty
\]
which implies that \( R_{k,N} \) is a polynomial of degree less or equal than \( N - 2 \). Moreover,
\[
R_{k,N} e^{-N b} - \psi_k \xrightarrow{N \to \infty} 0
\]
uniformly on \( \Omega \). In particular, we have that
\[
\int_{\mathbb{C}} R_{k,N} R_{m,N} e^{-2NU^* \nu} d\nu \xrightarrow{N \to \infty} \int_{\mathbb{C}} \psi_k \overline{\psi}_m d\nu = \delta_{k,m}.
\]
Fix \( k_0 > 0 \) and let \( (P_{k,N})_{k=0}^{k_0} \) be the Gram-Schmidt orthonormalization of \( (R_{k,N})_{k=0}^{k_0} \). This can be done since, by (5.2), \( (P_{k,N})_{k=0}^{k_0} \) is a linear independent family for \( N \) is large enough. By extending \( (P_{k,N})_{k=0}^{k_0} \) to an orthonormal basis of \( H_N^2(\Omega, e^{-2NU^* \nu} d\nu) \) we see that
\[
\sum_{k=0}^{k_0} |P_{k,N}(z)|^2 e^{-2NU^* \nu(z)} \leq \mathcal{K}_N(z, z).
\]
By (5.1) and (5.2), \( P_{k,N} e^{-N b}(z) \to \psi_k(z) \) as \( N \to \infty \) for every \( z \in \Omega \) so that
\[
\sum_{k=0}^{k_0} |\psi_k(z)|^2 \leq \liminf_{N \to \infty} \mathcal{K}_N(z, z)
\]
for every \( z \in \Omega \) and, by letting \( k_0 \to \infty \), we complete the proof.

**Independence.** The proof of the independence is exactly the same as the proof of Theorem 1.5, because the Szegő kernels also have the reproducing property. We omit the details.

6. **The number of outliers**

In this section, we assume that \( \mu \) is supported on a bounded simply connected domain \( D \) with analytic boundary. Assume that \( \mu \) has real-analytic density \( \omega \) with respect to \( m \) restricted to \( D \), which extends past \( \partial D \) to a positive real-analytic density on a larger set \( \tilde{D} \). We denote by \( \tilde{\mu} \) the measure \( \tilde{\omega} dm_{\tilde{D}} \).

Recall the notation \( a \lesssim b \) from Section 1.6.

**Proposition 6.1.** Under the above assumptions, the expected number
\[
\eta_N(\Omega) = \mathbb{E} \left[ \#(\Phi_N) \right]
\]
of outliers in the domain \( \Omega \) satisfies, as \( N \to \infty \),
\[
\sqrt{N} \lesssim \eta_N(\Omega) \lesssim \sqrt{N} \log N.
\]
As mentioned above, the expected number of particles in $\Omega$ is close to the corresponding number for the strongly confined case. The difference is that in the present model, there are outliers at a positive distance as well.

Our argument below relies on a Lemma which we have not managed to locate in the literature but which may be derived by well-known ideas contained in [1], [4] and [6]. We also provide an independent proof sketch.

6.1. A strongly confining potential. We will use this extended measure to construct a strongly confining potential $Q$, such that $\mu$ is the weighted equilibrium measure for $Q$. To achieve this, we let $h$ be the harmonic function on $D$ which equals $U^\mu - U^\tilde{\mu}$ on the boundary. Then, similarly as in the proof of Proposition 2.6, one finds that $h$ extends harmonically past $\partial D$ to a larger set. By possibly shrinking it, we may assume that this set is $\tilde{D}$. Repeating the argument in the proof of Proposition 2.7 we also find that for $z \in \partial D$ and $t > 0$, 

$$(U^\tilde{\mu} + h - U^\mu)(z + tn_z) = \pi \omega(z)t^2 + O(t^3),$$

where $n_z$ is the outward pointing unit normal to $\partial D$ at $z$, and the implicit constant is uniform for $z \in \partial D$. We let $D_0$ denote the component of the sub-level set

$$\{z \in \mathbb{C} : (U^\tilde{\mu} + h - U^\mu)(z) \leq \delta_0\},$$

containing $D$, where $\delta_0$ is small enough to ensure that $D_0 \subset \tilde{D}$ and that for all $z \in D_0$

$$(Q - U^\mu)(z) \geq c_0 d(z, D)^2$$

for some strictly positive constant $c_0$. We define a potential $Q$ by putting $Q = U^\tilde{\mu} + h$ on $D_0$, and outside $D_0$ we just require that $Q$ is $C^2$-smooth and that for some $c_1 > 0$ we have

$$Q(z) - U^\mu(z) \geq c_1 \log(1 + |z|^2)$$

for $z \in \mathbb{C} \setminus D_0$. This is possible to achieve e.g. by extending $Q$ to equal $U^\mu + \delta_0$ outside $D_0$ and smoothing.

To compare the jellium with a strongly confined Coulomb gas, we consider the Coulomb gas obtained by replacing $U^\mu$ by the potential $Q$ in (1.1). Notice that this new Coulomb gas has the same droplet $D = \text{supp}(\mu)$ as the weakly confined gas. Indeed, $U^\mu$ satisfies the Euler-Lagrange equations for the minimization of the weighted logarithmic energy with external field $Q$ among all probability measure on $\mathbb{C}$ (see [23, Chapter I, Theorem 1.3]). For every $N > 0$ we denote by $K_{N,Q} = K_{c_0^{-2}NQ,C}$, which is the kernel of the determinantal point process formed by this new Coulomb gas.

Objects pertaining to this new Coulomb gas with potential $Q$ will be denoted by subscripts $N$ and $Q$.

6.2. Two Lemmas on weighted Bergman kernels. Recall that $K_N = K_{c_0^{-2}\pi N,C}$. The kernels $K_N$ and $K_{N,Q}$ have the same bulk asymptotics. Specifically, we have the following.

Lemma 6.2. Denote by $D_N$ the set

$$D_N = \{z \in D : d(z, D^c) \geq N^{-\frac{1}{2}} \log N\}.$$

Then, for any $\kappa > 0$ there is a constant $C > 0$ such that for every $N > 0$ and $z \in D_N$, we have

$$|K_N(z, z) - K_{N,Q}(z, z)| \leq CN^{-\kappa}.$$ 

This may be obtained by repeating the argument in [1] for the potential $U^\mu$, which strictly speaking does not belong to the class of weights covered in that work. We will sketch an argument that shows directly that the kernels agree on $D_N$ up to small error terms.
Sketch of proof. Let $k_{N,w,Q}(z) = K_{N,Q}(z,w)$. Using the off-diagonal decay of the Bergman kernel one finds that for any $w \in D_N$ and any polynomial $p$ of degree at most $N - 1$, we have

$$
(6.3) \int_{\mathbb{C}\setminus B(w,\delta_N/2)} k_{N,w}(z) e^{-2\kappa N Q} dm(z) = O(N^{-\kappa} \|f\|_N),
$$

and using the fact that $U^* = Q$ on $\mathbb{D}(w, \delta_N/2)$, it is not difficult to show that

$$
\int_{\mathbb{D}(w,\delta_N/2)} k_{N,w}(z) e^{-2\kappa N U^*(z)} dm(z) = f(z) + O(N^{-\kappa} \|f\|_N)
$$
as $N \to \infty$. Using a local cut-off $\chi^\perp$ which vanishes in the disk $\mathbb{D}(w, \delta_N/2)$ and Hörmander’s $\bar{\partial}$-estimate with polynomial growth (Lemma 3.10), one may further show that $\chi^\perp k_{N,w,Q}$ may be approximated well by polynomials of degree at most $N - N\delta_N/2$. Combining this with the Bernstein-Walsh Lemma [4, Lemma 3.5], one finds that (6.3) holds also with $Q$ replaced by the equilibrium potential $U^*$, so that in fact we have the approximate reproducing property

$$
\int_{\mathbb{C}} k_{N,w}(z) f(z) e^{-2\kappa N U^*(z)} dm(z) = f(z) + O(N^{-\kappa} \|f\|_N)
$$
as $N \to \infty$. But by standard Hilbert space techniques, this implies that $k_{N,w,Q}$ lies very close to $k_{N,w}(z) := K_N(z,w)$.

We recall the following robust bound for the kernel $K_{N,Q}$.

**Lemma 6.3** (Proposition 3.6 in [4]). Under the above conditions, there exists a constant $C$ such that

$$
K_{N,Q}(z,z) \leq C Ne^{-2N(Q-U^*)}, \quad z \in \mathbb{C}
$$

for all $N \geq 1$.

### 6.3. Proof of Proposition 6.1.

We begin with the upper bound, which we will obtain by showing that there are at least $N - C_1 \log N \sqrt{N}$ points in the set $D_N$ and using the fact that there are $N$ points in total. For any point $z \in D_N$, the standard Bergman kernel bulk expansion of Lemma 6.2 applies, so on average the two processes has the same number of bulk points to leading order. But the strongly confined Coulomb gas has all but order $\sqrt{N} \log N$ points in $D_N$:

$$
\eta_{N,Q}(D_N) = \int_{D_N} K_{N,Q}(z,z) dm(z) \geq N - C_1 \sqrt{N} \log N,
$$

so we obtain

$$
\eta_N(D_N) = (1 + O(N^{-\kappa})) \eta_{N,Q}(D_N) \geq N - C_1 \sqrt{N} \log N,
$$

from which we deduce that $\eta_N(\Omega) \leq N - \eta_N(D_N) \lesssim \sqrt{N} \log N$, completing the proof of the upper bound.

For the lower bound, we use the inequality $U^* \leq Q$ together with the familiar monotonicity bound

$$
\sup \frac{|f(z)|^2}{\int |f|^2 e^{-2NQ_0} dm} \leq \sup \frac{|f(z)|^2}{\int |f|^2 e^{-2NQ_1} dm},
$$
valid for potentials $Q_0 \leq Q_1$ (the supremum being taken over all polynomials $f$) which yields the inequality for the kernels

$$
K_N(z,z) \leq e^{2N(Q-U^*)} K_{N,Q}(z,z), \quad z \in \mathbb{C}
$$

In particular, we may apply this bound inside $\text{supp}(\mu)$ where we find

$$
K_N(z,z) \leq K_{N,Q}(z,z), \quad z \in D.
But then the expected number of points in $D = \Omega^c$ satisfies
\[
\eta_N(D) = \int_D K_N(z, z) dm(z) \leq \int_D K_{N,Q}(z, z) dm(z) = N - \int \Omega K_{N,Q}(z, z) dm(z) \leq N - C_1 \sqrt{N}.
\]
This last claim follows from the bound $K_{N,Q}(z, z) \leq C N e^{-2N(Q-U^x)}$ of Lemma 6.3 together with the bounds (6.1) and (6.2), and an elementary integral estimate. But then the claim follows from the identity $\eta_N(\Omega) + \eta_N(D) = N$.

**Appendix A. Determinantal Point Processes**

In this section we recall some basic properties of determinantal point processes that are needed in this paper. More information may be found in [17] or [27].

Let $U$ be an open subset of $\mathbb{C}$ and denote by $\mathcal{C}_U$ the set of locally finite integer-valued positive measures $\mu$ on $U$. Any element $\mathcal{X}$ of $\mathcal{C}_U$ can be written as $\mathcal{X} = \sum_{\lambda \in \Lambda} \delta_{x_{\lambda}}$ for some locally finite family $(x_{\lambda})_{\lambda \in \Lambda}$ of points of $U$. This allows us to think of $\mathcal{X} \in \mathcal{C}_U$ as a multi-set and to use notations such as $\mathcal{X} = \{x : x \in \mathcal{X}\}$, $\sum_{x \in \mathcal{X}} f(x) = \int f d\mathcal{X}$ or $\#(A \cap \mathcal{X}) = \mathcal{X}(A)$ for simplicity and to help our intuition. Endow $\mathcal{C}_U$ with the smallest topology that makes the maps
\[
\mathcal{X} \in \mathcal{C}_U \mapsto \sum_{x \in \mathcal{X}} f(x)
\]
continuous, for every compactly supported continuous function $f : U \to \mathbb{R}$. By a point process on $U$ we mean a random element of $\mathcal{C}_U$, and by a determinantal point process we mean the kind of following point processes.

**Definition A.1** (Determinantal point process). Suppose that $\mu$ is an atomless locally finite positive measure on $U$ and let $K : U \times U \to \mathbb{C}$ be continuous and Hermitian, i.e., $K(z, w) = \bar{K}(w, z)$ for every $z, w \in U$. A point process $\mathcal{X}$ on $U$ is a determinantal point process associated to the kernel $K$ with respect to the measure $\mu$ if for every $K \geq 1$ and for every pairwise disjoint measurable sets $A_1, \ldots, A_k$,
\[
\mathbb{E}[\#(A_1 \cap \mathcal{X}) \ldots \#(A_k \cap \mathcal{X})] = \int_{A_1 \times \cdots \times A_k} \det (K(x_i, x_j)_{1 \leq i, j \leq k}) d\mu^{\otimes k}(x_1, \ldots, x_k).
\]

Here, the notation $\mu^{\otimes k}$ denotes the $k$-fold product measure of $\mu$ with itself.

There is an important freedom in the choice of $K$ and $\mu$ that we use throughout the article. Namely, if $f : U \to \mathbb{C} \setminus \{0\}$ is continuous and if we let
\[
(A.1) \quad \tilde{K}(z, w) = f(z)K(z, w)f(w) \quad \text{and} \quad \tilde{d}\mu = |f|^{-2} d\mu
\]
we may see, by using the definition, that a determinantal point process associated to $K$ with respect to $\mu$ is also associated to $\tilde{K}$ with respect to $\tilde{\mu}$.

A well-known example of a determinantal point process is the following.

**Proposition A.2** (Coulomb gases are determinantal at $\beta = 2$). Let $\mu$ be an atomless locally finite positive measure on $\mathbb{C}$ such that $\int |x|^{2(N-1)} d\mu(x) < \infty$. Then, the measure $\gamma$ on $\mathbb{C}^n$ defined by
\[
d\gamma(x_1, \ldots, x_N) = \prod_{i<j} |x_i - x_j|^2 d\mu^{\otimes N}(x_1, \ldots, x_N)
\]
is a finite measure. Moreover, if we denote $\mathbb{P}_N = \frac{n}{d\gamma}$, a random element $(X_1, \ldots, X_N)$ that follows the law $\mathbb{P}_N$ induces a determinantal point process $\{X_1, \ldots, X_N\}$ with respect to the reference measure $\mu$ and associated to the kernel $K$ defined as follows. Let $\mathbb{C}_{N-1}[z]$
be the set of polynomials of degree less or equal than $N - 1$ and let $\{p_1, \ldots, p_N\}$ be an orthonormal basis of $C_{N-1}[z] \subset L^2(\mathbb{C}, \mu)$. Then, $K$ is given by

$$K(z, w) = \sum_{k=1}^{\infty} p_k(z) \overline{p_k(w)}.$$

**Proof.** We first remark that the determinant $\det(K(x_j, x_k)_{1 \leq j, k \leq N})$ is proportional to $\prod_{i<j} |x_i - x_j|^2$ so the measure $\gamma$ is proportional to $\det(K(x_j, x_k)_{1 \leq j, k \leq N})d\mu^{\otimes N}$. Since $K$ is an orthogonal projection onto a space of dimension $N$ we have that

$$\int K(x, y)K(y, z) d\mu(y) = K(x, z)$$

and $\int K(x, x) d\mu(x) = N$. This implies (see Lemma 5.27 in [9] for details) that

$$\int \det(K(x_i, x_j)_{1 \leq i, j \leq k}) d\mu(x_k) = (N - k + 1) \det(K(x_i, x_j)_{1 \leq i, j \leq k-1}),$$

from which the theorem follows. 

Usually, we use Proposition A.2 with $d\mu = e^{-2V} dm_\mathbb{C}$ for some continuous function $V : \mathbb{C} \to \mathbb{R}$. In that case, by (A.1) we may choose $m = m_\mathbb{C}$ as the reference measure for the determinantal point process $\mathcal{X}$ and $K(z, w)e^{-V(z)}e^{-V(w)}$ as the kernel.

The next result can be found in [25, Proposition 3.10].

**Proposition A.3** (Uniform convergence of kernels). Let $U$ be an open set of $\mathbb{C}$ and, for every positive integer $N$, let $K_N : U \times U \to \mathbb{C}$ be a continuous Hermitian function. Fix an atomless locally finite positive measure $\mu$ on $\mathbb{C}$ and, for each $N$, suppose $\mathcal{X}_N$ is a determinantal point process associated to $K_N$ with respect to $\mu$. Then, if $K_N$ converges uniformly on compact sets to $K : U \times U \to \mathbb{C}$, there exists a determinantal point process $\mathcal{X}$ associated to $K$ with respect to $\mu$ such that

$$\mathcal{X}_N \overset{\text{law}}{\longrightarrow} \mathcal{X}.$$
On the other hand, if \( F_k(x_1,\ldots,x_k) = \det (K(x_i,x_j))_{i,j\in\{1,\ldots,k\}} \), we also have
\[
\int_{A_1\times\cdots\times A_k} F_k \, d\mu_k^\otimes k = \sum_{\sigma_1,\ldots,\sigma_k \in \{1,2\}} \int_{A_1^{(\sigma_1)}\times\cdots\times A_k^{(\sigma_k)}} F_k \, d(\mu_{\sigma_1} \otimes \cdots \otimes \mu_{\sigma_k}).
\]
Then, we only need to show that
\[
E[\#(A_1^{(\sigma_1)} \cap \mathcal{X}) \cdots \#(A_k^{(\sigma_k)} \cap \mathcal{X})] = \int_{A_1^{(\sigma_1)}\times\cdots\times A_k^{(\sigma_k)}} F_k \, d(\mu_{\sigma_1} \otimes \cdots \otimes \mu_{\sigma_k}).
\]
Since \( F_k \) is symmetric, we may suppose that there is \( m \in \{0,\ldots,k\} \) such that
\[
\sigma_1 = \cdots = \sigma_m = 1 \quad \text{and} \quad \sigma_{m+1} = \cdots = \sigma_k = 2.
\]
By the independence of \( \mathcal{X}_1 \) and \( \mathcal{X}_2 \), it holds that
\[
E[\#(A_1^{(\sigma_1)} \cap \mathcal{X}_1) \cdots \#(A_k^{(\sigma_k)} \cap \mathcal{X}_\sigma)] = E \left[ \#(A_1^{(1)} \cap \mathcal{X}_1) \cdots \#(A_m^{(1)} \cap \mathcal{X}_1) \right] E \left[ \#(A_{m+1}^{(2)} \cap \mathcal{X}_2) \cdots \#(A_k^{(2)} \cap \mathcal{X}_2) \right]
\]
\[
= \int_{A_1^{(1)}\times\cdots\times A_m^{(1)}} F_m \, d\mu_1 \otimes \cdots \otimes F_m \, d\mu_k \int_{A_{m+1}^{(2)}\times\cdots\times A_k^{(2)}} F_k \, d\mu_2 \otimes \cdots \otimes d\mu_k.
\]
Moreover, by the calculation of the determinant of a diagonal block matrix,
\[
F_m(x_1,\ldots,x_m) F_{k-m}(x_{m+1},\ldots,x_k) = F_k(x_1,\ldots,x_k)
\]
whenever \( x_1,\ldots,x_m \in U_1 \) and \( x_{m+1},\ldots,x_k \in U_2 \) which, by Fubini's theorem, allows us to conclude the proof. \( \square \)

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