Intersection theory on tropicalizations of toroidal embeddings

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Abstract

We show how to equip the cone complexes of toroidal embeddings with additional structure that allows to define a balancing condition for weighted subcomplexes. We then proceed to develop the foundations of an intersection theory on cone complexes including push-forwards, intersections with tropical divisors, and rational equivalence. These constructions are shown to have an algebraic interpretation: Ulirsch’s tropicalizations of subvarieties of toroidal embeddings carry natural multiplicities making them tropical cycles, and the induced tropicalization map for cycles respects push-forwards, intersections with boundary divisors, and rational equivalence. As an application, we prove a correspondence between the genus 0 tropical descendant Gromov–Witten invariants introduced by Markwig and Rau and the genus 0 logarithmic descendant Gromov–Witten invariants of toric varieties.

1. Introduction

The process of tropicalizing subvarieties of algebraic tori over a field with trivial valuation is at the heart of tropical geometry. Accordingly, a lot of work has been done to understand the structure of tropicalizations and the ways in which they reflect algebraic properties and constructions. The main idea of tropical geometry, namely, that of transforming algebraic geometry into discrete geometry, is based on the result that the tropicalization of a subvariety $Z \subseteq G_n^m$ is the support of a purely dim($Z$)-dimensional rational polyhedral fan in $\mathbb{R}^n$ [11]. It can be enriched with the structure of what is known as a tropical cycle by defining multiplicities on its maximal cones [56, 58]. The tropicalization tells us which torus orbits are met by the closure of $Z$ in a toric compactification of $G_n^m$ [59] and, in case these intersections are proper and the compactification is smooth, the intersection multiplicities [33]. This relation between intersection theory on toric varieties and tropical geometry has been extended by the development of tropical intersection theory on $\mathbb{R}^n$ [7] that incorporates the intersection rings of all normal toric compactifications of $G_n^m$ [22, 34, 53]. Being able to intersect tropically as well as algebraically, it has been studied in how far tropicalization respects intersections [49, 50] or other intersection theoretic constructions, as, for example, push-forwards [58] ([9, 49] in the case of non-trivial valuations).

When we want to apply tropical intersection theory as is to describe intersections on a non-toric variety $X$, we need to embed it into a toric variety. We then have to determine what it means for a cocycle on $X$ to be represented by tropical data on the tropicalization of the part $X_0$ of $X$, which is mapped to the big open torus. A widely accepted notion for this has been introduced by Katz [34], who applied it for $X$ equal to the moduli space of $n$-marked rational stable curves and $X_0$ the open subset of irreducible curves to relate algebraic and tropical $\psi$-classes. In [14], it has been used in the case where $X$ is the moduli space of relative stable maps to $\mathbb{P}^1$ and $X_0$ is the open subset corresponding to maps with irreducible domain to show that the tropical and algebraic genus 0 relative descendant Gromov–Witten invariants of $\mathbb{P}^1$ coincide. The common feature of these two cases is that there is a canonical choice for $X_0$ and the embedding into a toric variety is essentially determined by the embedding $X_0 \subseteq X$. 

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In fact, in both examples, \(X_0 \subseteq X\) is a toroidal embedding without self-intersections in the sense of [37], and the fan of the toric variety in which \(X\) embeds is equal to the cone complex \(\Sigma(X)\) associated to \((X, X_0)\), which is embedded using the technique of geometric tropicalization [30, 41]. Most toroidal embeddings cannot be embedded in toric varieties, but yet they are of increasing importance in tropical geometry appearing especially in the study of tropicalizations of moduli spaces [1, 13, 14, 51, 62].

In this paper, we study in how far the constructions and results known for tori and toric varieties mentioned in the first paragraph carry over to the more general setting of toroidal embeddings. Given a toroidal embedding \(X_0 \subseteq X\) without self-intersection, the open part \(X_0\) should replace the torus in the toric case. Similarly as in the toric situation, the first step of tropicalizing cycles in \(X_0\) is to define a set-theoretic tropicalization of subvarieties \(Z \subseteq X_0\), which should be a polyhedral subset of \(\Sigma(X)\). It has been suggested by Ulirsch [63] to do this by taking the image of the analytification \(X^\an \cap Z^\an\) under the natural tropicalization map \(\text{trop}_Z : X^\an \cap X_0^\an \to \Sigma(X)\). This map equals Thuillier’s retraction map [60] and can be seen as the natural extension of the ‘ord’-map \(X(k[t]) \cap X_0(k((t))) \to \Sigma(X)\), which already appeared in [37].

The next step is to assign weights to the maximal faces of the set-theoretic tropicalization. This can be done similarly as in the toric case. It follows from a result in [61] that the closure of a closed subvariety \(Z \subseteq X_0\) in any toroidal modification of \(X\) with sufficiently refined cone complex intersects all boundary strata properly. We will thus define the weights on the tropicalization by taking the intersection multiplicities of \(Z\) with the boundary in any such toroidal modification. We show that as in the toric case, this is well-defined up to refinement, yielding a cycle \(\text{Trop}_X(Z)\). Note, however, that one has to modify the arguments used in the toric case since one lacks the existence of Tevelev’s tropical compactifications [59].

A key role in the tropical intersection theory developed by Allermann and Rau [7] is played by the balancing condition, without which their tropical intersection products would not be well-defined. The balancing condition asks that around every codimension-1 cell of a tropical cycle, the weighted sum of the lattice normal vectors in the directions of the adjacent top-dimensional cells vanishes. This balancing condition does not make sense on an arbitrary cone complex \(\Sigma\). While the lattice normal vectors do still exist, we cannot take their weighted sum if they lie in different cones of \(\Sigma\). We are simply lacking the ambient space in which such a summation could take place. The remedy we propose here is to consider \textit{weakly embedded cone complexes}, that is, cone complexes together with a piecewise linear map into a vector space. The balancing condition can then be checked in the vector space. Cone complexes of toroidal embeddings always come with a natural weak embedding, the vector space coming from the invertible functions on the interior \(X_0\). We will show that for a closed subvariety \(Z \subseteq X_0\), the cycle \(\text{Trop}_X(Z)\) does indeed satisfy the balancing condition, that is, tropicalizations of subvarieties are tropical cycles.

Once one works with weakly embedded cone complexes and thus can make sense of balancing, some of the constructions known for embedded cone complexes carry over to the weakly embedded setup without significant changes. Among them are the push-forward of tropical cycles, and Allermann and Rau’s product of a tropical cycle with a piecewise linear function [7]. Piecewise linear functions on \(\Sigma(X)\) correspond to boundary divisors on \(X\), so, ideally, if \(D\) is a boundary divisor with associated piecewise linear function \(\psi\), and if \(Z \subseteq X_0\) is a subvariety, one has

\[
\text{Trop}_X(D \cdot Z) = \psi \cdot \text{Trop}_X(Z).
\]  

Unfortunately, this cannot work, as the cycle \(D \cdot Z\) does not meet the interior of \(X\) and we cannot tropicalize it in the way described above. We resolve this issue by working with a compactification of \(\Sigma(X)\), the \textit{extended cone complex} \(\Sigma(X)\). We will define a group of tropical cycles \(Z_\ast(\Sigma(X))\) by considering the natural stratification of \(\Sigma(X)\). We can then define
a tropicalization map $\Trop_X : Z_\ast(X) \to Z_\ast(\Sigma(X))$ and will show that equation (1.1) indeed holds. Note, however, that on the right-hand side we need to replace the Allermann–Rau product by a different construction (cf. Proposition 3.11). We also obtain a functoriality result for the tropicalization generalizing the Sturmfels–Tevelev multiplicity formula. We show in Theorem 4.10 that whenever $f : X \to Y$ is a toroidal morphism between two proper toroidal embeddings without self-intersection with induced morphism $\Trop(f) : \Sigma(X) \to \Sigma(Y)$, we have

$$\Trop(f) \ast \Trop_X(\alpha) = \Trop_Y(f \ast \alpha)$$

for all $\alpha \in Z_\ast(X)$.

In the algebro-geometric intersection theory, as worked out in [20], it is crucial to consider cycles modulo rational equivalence. If one wants to understand the intersection theory of toroidal varieties tropically, it is therefore very useful to be able to tropicalize cycle classes, rather than just cycles. To this end, we introduce a notion of rational equivalence for cycles on a weakly embedded extended cone complex $\Sigma$, giving rise to a Chow group $A_\ast(\Sigma)$. We show that it has the following properties.

(a) There exists a degree morphism $\deg : A_0(\Sigma) \to \mathbb{Z}$ (cf. Definition 3.21).

(b) If $f : \Sigma \to \Delta$ is a morphism of weakly embedded extended cone complexes, then the push-forward of cycles respects rational equivalence, that is, it induces a morphism $f_* : A_\ast(\Sigma) \to A_\ast(\Delta)$ (cf. Proposition 3.20).

(c) If $\psi$ is a piecewise linear function on $\Sigma$ that is locally a pullback of an integer linear function via the weak embedding, then the Allermann–Rau intersection pairing with $\psi$ respects rational equivalence. In fact, modulo rational equivalence it coincides with the intersection pairing appearing in equation (1.1) (cf. Proposition 3.22).

We can then show that for every proper toroidal embedding without self-intersection $X$, tropicalization induces a morphism

$$\Trop_X : A_\ast(X) \to A_\ast(\Sigma(X)),$$

which respects degrees and commutes with push-forwards and intersections with certain boundary divisors. We also show that $\Trop_X$ is an isomorphism if $X$ is a smooth projective toric variety or a suitable compactification of the complement of an hyperplane arrangement (cf. Corollary 4.31).

In the last section of the paper, we will apply our results to two toroidal moduli spaces. The first one is the space $M_{0,n} \subseteq \overline{M}_{0,n}$ of $n$-marked rational stable curves. The modular description of $\overline{M}_{0,n}$ endows it with $n$-divisor classes $\psi_i \in \text{Pic}(\overline{M}_{0,n})$. We will show that the tropicalization of these $\psi$-classes in our sense recovers the definition given in [44] (which was based on Mikhalkin’s definition [48]). Invoking the machinery that we develop in Sections 3 and 4 gives a very conceptual proof of the fact that top-dimensional intersections of the $\psi_i$ have the same degrees as top-dimensional intersections of the tropical $\psi$-classes.

Afterward, we will turn to the moduli spaces of genus 0 logarithmic stable maps with specified torically transverse contact order with the boundary. The weakly embedded cone complexes associated to these spaces are the tropical moduli spaces of tropical stable maps introduced in [24], subdivided suitably [51]. Then, again, our machinery for toroidal embeddings yields an equality of intersection numbers, more precisely of logarithmic and tropical genus 0 ancestor descendant Gromov–Witten invariants of toric varieties. Here, ‘ancestor’ means that we use the $\psi$-classes pulled back from $\overline{M}_{0,n}$ via the forgetful morphism. We note that this statement has been proven independently by Mandel and Ruddat [43] with completely different methods. They also show that the ancestor descendants coincide with the usual descendants, so that our result, in fact, yields an equality of logarithmic and tropical descendant Gromov–Witten invariants.
We conclude this introduction by discussing how our intersection-theoretic constructions relate to the previously existing tropical intersection theory. However, before doing so, we want to stress that the main focus of this paper is not merely to generalize tropical intersection theory in one way or another, but to develop tropical intersection theory hand in hand with a tropicalization procedure for toroidal embeddings: every tropical construction in Section 3 is related to its algebraic counterpart in Section 4 by the tropicalization map.

To begin with, there is Allermann and Rau’s intersection theory for tropical cycles in $\mathbb{R}^n$. As already mentioned above, some of their constructions, like their intersection pairing with piecewise integral linear functions and push-forwards carry over to our setting directly, and while recalling these construction in some detail, we will sometimes quote their results and leave minor adjustments of proofs to the reader. The two key distinctions between our considerations and theirs is that Allermann and Rau do not work with extended cone complexes, that is, do not consider strata at infinity, but therefor allow polyhedral complexes as domains of their tropical cycles instead of mere fan cycles as we do. They do define a notion of rational equivalence, yet it is fundamentally different from ours. In fact, in [6], it is shown that their notion of rational equivalence is trivial when restricted to fan cycles. To obtain our notion of rational equivalence, one really needs to include strata at infinity into the theory. This feature is present in the work of Shaw [54] and Meyer [45]. Shaw mainly considers locally matroidal tropical manifolds, which differs from our setup in that we do not consider local charts, but instead a global projection of the complex. But even if the cycle groups she considers differ from ours, our definition of rational equivalence is analogous to hers. The work of Meyer treats Kajiwara’s and Payne’s tropical toric varieties. These are the prototypical examples of weakly embedded extended cone complexes. Meyer defines intersection pairings of Cartier divisors and cycles on these tropical toric varieties. These are the prototypical examples of weakly embedded extended cone complexes. Meyer defines intersection pairings of Cartier divisors and cycles on these tropical toric varieties in a way mixing the definitions of our products ‘$\cdot$’ and ‘$\cup$’. He also gives an operational definition of rational equivalence that is designed to respect push-forwards. Although not obvious, using the results of this paper, one can show that Meyer’s definition of rational equivalence coincides with ours when restricting it to fan cycles.

2. Cone complexes and toroidal embeddings

The purpose of this section is to briefly recall the definitions of cone complexes and extended cone complexes. We refer to [37, Section II.1; 1, Section 2] for further details. We will also introduce weakly embedded cone complexes, the main objects of study in the next section, and show how to obtain them from toroidal embeddings.

2.1. Cone complexes

An (integral) cone is a pair $(\sigma, M)$ consisting of a topological space $\sigma$ and a lattice $M$ of real-valued continuous functions on $\sigma$ such that the product map $\sigma \to \text{Hom}(M, \mathbb{R})$ is a homeomorphism onto a strongly convex rational polyhedral cone. By abuse of notation, we usually just write $\sigma$ for the cone $(\sigma, M)$ and refer to $M$ as $M^\sigma$. Furthermore, we will write $N^\sigma = \text{Hom}(M, \mathbb{Z})$ so that $\sigma$ is identified with a rational cone in $N^\sigma_\mathbb{R} = N^\sigma \otimes_\mathbb{Z} \mathbb{R}$. We will also use the notation $M^\sigma_\mathbb{R}$ for the semigroup of non-negative functions in $M$, and $N^\sigma_\mathbb{R}$ for $\sigma \cap N^\sigma$. Giving an integral cone is equivalent to specifying a lattice $N$ and a full-dimensional strongly convex rational polyhedral cone in $N_\mathbb{R}$. A subspace $\tau \subseteq \sigma$ is called a subcone of $\sigma$ if the restrictions of the functions in $M^\sigma$ to $\tau$ make it a cone. Identifying $\sigma$ with its image in $N^\sigma_\mathbb{R}$, this is the case if and only if $\tau$ itself is a rational polyhedral cone in $N^\sigma_\mathbb{R}$. A morphism between two integral cones $\sigma$ and $\tau$ is a continuous map $f: \sigma \to \tau$ such that $m \circ f \in M^\tau$ for all $m \in M^\sigma$. The product $\sigma \times \tau$ of two cones $\sigma$ and $\tau$ in the category of cones is given by the topological space $\sigma \times \tau$, together with the functions

$$m + m': \sigma \times \tau \to \mathbb{R}, \quad (x, y) \mapsto m(x) + m'(y).$$
for $m \in M^{\sigma}$ and $m' \in M^\tau$. The notation already suggests that we can identify $M^{\sigma} \oplus M^\tau$ with $M^{\sigma \times \tau}$ via the isomorphism $(m, m') \mapsto m + m'$.

An (integral) cone complex $(\Sigma, |\Sigma|)$ consists of a connected topological space $|\Sigma|$ and a finite set $\Sigma$ of integral cones that are closed subspaces of $|\Sigma|$, such that their union is $|\Sigma|$, every face of a cone in $\Sigma$ is in $\Sigma$ again, and the intersection of two cones in $\Sigma$ is a union of common faces. A subset $\sigma \subseteq |\Sigma|$ is in $\Sigma$ again, and the intersection of two cones in $\Sigma$ is a union of common faces. Whenever $\sigma \subseteq |\Sigma|$ is called a subcone of $\Sigma$ if there exists some $\delta \in \Sigma$ such that $\tau \subseteq \delta$ is a subcone of $\sigma$. A morphism $f : \Sigma \rightarrow \Delta$ of cone complexes is a continuous map $|\Sigma| \rightarrow |\Delta|$ such that for every $\sigma \in \Sigma$, there exists $\delta \in \Delta$ such that $f(\sigma) \subseteq \delta$, and the restriction $f|_{\sigma} : \sigma \rightarrow \delta$ is a morphism of cones.

A subdivision of a cone complex $\Sigma$ is a cone complex $\Sigma'$ such that $|\Sigma'|$ is a subspace of $|\Sigma|$ and every cone of $\Sigma'$ is a subcone of $\Sigma$. The subdivision is called proper if $|\Sigma'| = |\Sigma|$.

The reason why the tropical intersection theory developed in [7] fails to generalize to cone complexes is that when gluing two cones $\sigma$ and $\sigma'$ along a common face, there is no natural lattice containing both $N^{\sigma}$ and $N^{\sigma'}$, hence making it impossible to speak about balancing. To remedy this, we make the following definition that interpolates between fans and cone complexes:

**Definition 2.1.** A weakly embedded cone complex is a cone complex $\Sigma$, together with a lattice $N^\Sigma$, and a continuous map $\varphi_\Sigma : |\Sigma| \rightarrow N^\Sigma_{R}$ that is integral linear on every cone of $\Sigma$.

We recover the notion of fans by imposing the additional requirements that $\varphi_\Sigma$ is injective, and the lattices spanned by $\varphi_\Sigma(N^\Sigma_{\varphi_\Sigma(\sigma)})$ for $\sigma \in \Sigma$ are saturated in $N^\Sigma$. In this case, we also call $\Sigma$ an embedded cone complex. On the other hand, the notion of cone complexes is recovered by setting $N^\Sigma = 0$.

Given a weakly embedded cone complex $\Sigma$, we write $M^\Sigma = \text{Hom}(N^\Sigma, \mathbb{Z})$ for the dual of $N^\Sigma$, and for $\sigma \in \Sigma$, we write $N^\Sigma_{\sigma}$ for what is usually denoted by $N^\Sigma_{\varphi_\Sigma(\sigma)}$ in toric geometry; that is, $N^\Sigma_{\sigma} = N^\Sigma \cap \text{Span} \varphi_\Sigma(\sigma)$. Furthermore, we will frequently abuse notation and write $\varphi_\Sigma$ for the induced morphisms $N^{\sigma} \rightarrow N^\Sigma$ and $N^\Sigma_{R} \rightarrow N^\Sigma_{F}$.

A morphism between two weakly embedded cone complexes $\Sigma$ and $\Delta$ is composed of a morphism $\Sigma \rightarrow \Delta$ of cone complexes and a morphism $N^\Sigma \rightarrow N^\Delta$ of lattices forming a commutative square with the weak embeddings.

Let $\Sigma$ be a cone complex, and let $\tau \in \Sigma$. For every $\sigma \in \Sigma$ containing $\tau$, let $\sigma / \tau$ be the image of $\sigma$ in $N^\Sigma_{F} / N^\Sigma_{\tau}$. Whenever $\sigma$ and $\sigma'$ are two cones containing $\tau$ such that $\sigma'$ is a face of $\sigma$, the cone $\sigma' / \tau$ is a naturally identified with a face of $\sigma / \tau$. Gluing the cones $\sigma / \tau$ for $\sigma \subseteq \sigma' \subseteq \Sigma$ along these face maps produces a new cone complex, the star $S_{\Sigma}(\tau)$ (or just $S(\tau)$ if $\Sigma$ is clear) of $\Sigma$ at $\tau$. If $\Sigma$ comes with a weak embedding, then the star is naturally a weakly embedded cone complex again. Namely, for every cone $\sigma$ of $\Sigma$ containing $\tau$, there is an induced integral linear map $\sigma / \tau \rightarrow (N^\Sigma / N^\Sigma_{\sigma})_{R} =: N^\Sigma_{\sigma \tau}$, and these maps glue to give a continuous map $\varphi_{S(\tau)} : |S(\tau)| \rightarrow N^\Sigma_{\sigma \tau}$.

The product $\Sigma \times \Delta$ of two cone complexes $\Sigma$ and $\Delta$ in the category of cone complexes is the cone complex with underlying space $|\Sigma| \times |\Delta|$ and cones $\sigma \times \delta$ for $\sigma \in \Sigma$ and $\delta \in \Delta$. If both $\Sigma$ and $\Delta$ are weakly embedded, then their product in the category of weakly embedded cone complexes is the product of the cone complexes together with the weak embedding $\varphi_{\Sigma \times \Delta} = \varphi_{\Sigma} \times \varphi_{\Delta}$.

**2.2. Extended cone complexes**

In the image of an integral cone $\sigma$ under its canonical embedding into $\text{Hom}(M^{\sigma}, \mathbb{R})$ are exactly those morphisms $M^{\sigma} \rightarrow \mathbb{R}$ that are non-negative on $M^{\sigma}_{\geq 0}$. Therefore, $\sigma$ is canonically identified with the set $\text{Hom}(M^{\sigma}_{\geq 0}, \mathbb{R}_{\geq 0})$ of morphisms of monoids. This identification motivates the definition of the extended cone $\sigma = \text{Hom}(M^{\sigma}_{\geq 0}, \mathbb{R}_{>0})$ of $\sigma$, where $\mathbb{R}_{>0} = \mathbb{R}_{\geq 0} \cup \{\infty\}$, and the
topology on \( \overline{\sigma} \) is that of pointwise convergence. The cone \( \sigma \) is an open dense subset of \( \overline{\sigma} \), and \( \overline{\sigma} \) is compact by Tychonoff’s theorem. If \( v \in \overline{\sigma} \) is an element of this compactification, the sets \( \{ m \in M_\tau^\sigma \mid \langle m, v \rangle \in \mathbb{R}_{\geq 0} \} \) and \( \{ m \in M_\tau^\sigma \mid \langle m, v \rangle = 0 \} \) generate two faces of \( \sigma^\tau = \mathbb{R}_{\geq 0} M_\tau^\sigma \subseteq M_\tau^\sigma \), and these are dual to two comparable faces of \( \sigma \). In this way, we obtain a stratification of \( \overline{\sigma} \), the stratum corresponding to a pair \( \tau \geq \tau' \) of faces of \( \sigma \) being

\[
F_\sigma^\tau(\tau, \tau') = \left\{ v \in \overline{\sigma} \mid \langle m, v \rangle \in \mathbb{R}_{\geq 0} \Leftrightarrow m \in (\tau')^\perp \cap M_\tau^\sigma, \langle m, v \rangle = 0 \Leftrightarrow m \in \tau^\perp \cap M_\tau^\sigma \right\}.
\]

Denote by \( F_\sigma(\tau, \tau') \) the subset of \( \overline{\sigma} \) obtained by relaxing the second condition in the definition of \( F_\sigma^\tau(\tau, \tau') \) and allowing the vanishing locus of \( v \) to be possibly larger than \( \tau^\perp \cap M_\tau^\sigma \). Since \( (\tau')^\perp \cap M_\tau^\sigma \) is canonically identified with the dual lattice of \( N_\tau^\sigma/N_\tau^\sigma \), we can identify \( F_\sigma(\tau, \tau') \) with the image \( \tau/\tau' \) of \( \tau \) in \( (N_\tau^\sigma/N_\tau^\sigma)_\mathbb{R} \). This gives \( F_\sigma(\tau, \tau') \) the structure of an integral cone.

Every morphism \( f : \tau \to \sigma \) of cones induces a morphism \( \overline{f} : \tau \to \overline{\sigma} \) of extended cones, and if \( f \) identifies \( \tau \) with a face of \( \sigma \), then this extended morphism maps \( \tau \) homeomorphically onto \( \bigcup_{\tau \subseteq \tau'} F_\sigma(\tau, \tau') = F(\tau, 0) \). Every subset of \( \overline{\sigma} \) occurring like this is called an extended face of \( \overline{\sigma} \).

Given a cone complex \( \Sigma \), we can glue the extensions of its cones along their extended faces according to the inclusion relation on \( \Sigma \) and obtain a compactification \( \overline{\Sigma} \) of \( \Sigma \), which is called the extended cone complex associated to \( \Sigma \). For a cone \( \tau \in \Sigma \), the cones \( \sigma/\tau = F_\sigma(\sigma, \tau) \), where \( \sigma \in \Sigma \) with \( \tau \subseteq \sigma \), are glued in \( \overline{\Sigma} \) exactly as in the construction of \( S(\tau) \), and therefore there is an identification of \( S(\tau) \) with a locally closed subset of \( \overline{\Sigma} \), which extends naturally to an identification of the extended cone complex \( \overline{S}(\tau) \) with a closed subset of \( \overline{\Sigma} \). With these identifications, we see that \( \overline{\Sigma} \) is stratified by the stars of \( \Sigma \) at its various cones.

For every morphism \( f : \Sigma \to \Delta \) between two cone complexes, there is an induced map \( \overline{f} : \overline{\Sigma} \to \overline{\Delta} \), and we call any map arising that way a \textit{morphism of extended cone complexes}. If both \( \Sigma \) and \( \Delta \) are weakly embedded, we require \( f \) to respect these embeddings. Whenever \( \sigma \in \Sigma \), and \( \delta \in \Delta \) is the minimal cone of \( \Delta \) containing \( f(\sigma) \), there is an induced morphism \( S_f(\sigma) : \overline{S}_\Sigma(\sigma) \to \overline{S}_\Delta(\delta) \) of cone complexes. It is easily checked that this describes \( \overline{f} \) on the stratum \( \overline{S}_\Sigma(\sigma) \), that is, that the diagram

\[
\begin{array}{ccc}
S_\Sigma(\sigma) & \longrightarrow & \overline{S}_\Sigma(\sigma) \\
S_f(\sigma) \downarrow & & \downarrow S_f(\sigma) \\
S_\Delta(\delta) & \longrightarrow & \overline{S}_\Delta(\delta)
\end{array}
\]

is commutative. If \( \Sigma \) and \( \Delta \) are weakly embedded, and \( f \) respects the weak embeddings, then so does \( S_f(\sigma) \).

In general, we define a \textit{morphism of extended cone complexes} between \( \overline{\Sigma} \) and \( \overline{\Delta} \) to be a map \( \overline{\Sigma} \to \overline{\Delta} \), which factors through a finiteness-preserving morphism to \( \overline{S}_\Delta(\delta) \) for some \( \delta \in \Delta \). If \( \Sigma \) and \( \Delta \) are weakly embedded, we additionally require the finiteness-preserving morphism to respect the weak embeddings.

2.3. \textit{Toroidal embeddings}

Before we start with the development of an intersection theory on weakly embedded cone complexes, we want to point out how to obtain them from toroidal embeddings as this will be our primary source of motivation and intuition. A \textit{toroidal embedding} is pair \((X_0, X)\) consisting of a normal variety \( X \) and a dense open subset \( X_0 \subseteq X \) such that the open immersion \( X_0 \to X \) formally locally looks like the inclusion of an algebraic torus \( T \) into a \( T \)-toric variety. More precisely, this means that for every closed point \( x \in X \), there exists an affine toric variety \( Z \), a closed point \( z \in Z \), and an isomorphism \( \hat{O}_{X, x} \cong \hat{O}_{Z, z} \) over the ground field \( k \) that identifies the ideal of \( X \setminus X_0 \) with that of \( Z \setminus Z_0 \), where \( Z_0 \) denotes the open orbit of \( Z \). A toroidal
embedding is called strict, if all components of $X \setminus X_0$ are normal. In this paper, we will only consider strict toroidal embeddings and therefore omit the ‘strict’.

Every toroidal embedding $X$ has a canonical stratification. If $E_1, \ldots, E_n$ are the components of $X \setminus X_0$, then the strata are given by the connected components of the sets

$$\bigcap_{i \in I} E_i \setminus \bigcup_{j \notin I} E_j,$$

where $I$ is a subset of $\{1, \ldots, n\}$.

The combinatorial open subset $X(Y)$ of $X$ associated to a stratum $Y$ is the union of all strata of $X$ containing $Y$ in their closure. This defines an open subset of $X$, as it is constructible and closed under generalizations. Furthermore, the stratum $Y$ defines the following lattices, semigroups, and cones:

$$M^Y = \{\text{Cartier divisors on } X(Y) \text{ supported on } X(Y) \setminus X_0\},$$

$$N^Y = \text{Hom}(M^Y, \mathbb{Z}),$$

$$M^Y_+ = \{\text{Effective Cartier divisors in } M^Y\},$$

$$N^Y_\mathbb{R} \supseteq \sigma^Y = (\mathbb{R}_{\geq 0} M^Y_+)^\vee.$$

If $X$ is unclear, we write $M^Y(X)$, $N^Y(X)$, and $\sigma^Y_X$. Whenever a stratum $Y$ is contained in the closure of a stratum $Y''$, the morphism $N^{Y''} \to N^{Y'}$ induced by the restriction of divisors maps injectively onto a saturated sublattice of $N^{Y'}$ and identifies $\sigma^{Y''}$ with a face of $\sigma^{Y'}$. Gluing along these identifications produces the cone complex $\Sigma(X)$. We refer to [37, Section II.1] for details.

In accordance with the toric case, we will write $O(\sigma)$ for the stratum of $X$ corresponding to a cone $\sigma \in \Sigma(X)$. Its closure will be denoted by $V(\sigma)$, and we will abbreviate $X(O(\sigma))$ to $X(\sigma)$.

Define $M^X = \Gamma(X_0, O_X^*)/k^*$, and let $N^X = \text{Hom}(M^X, \mathbb{Z})$ be its dual. For every stratum $Y$ of $X$, we have a morphism

$$M^X \to M^Y, \ f \mapsto \text{div}(f)|_{X(Y)} ,$$

which induces an integral linear map $\sigma^Y \to N^X_\mathbb{R}$. Obviously, these maps glue to give a continuous function $\varphi_X : |\Sigma(X)| \to N^X_\mathbb{R}$ that is integral linear on the cones of $\Sigma(X)$. In other words, we obtain a weakly embedded cone complex naturally associated to $X$, which we again denote by $\Sigma(X)$.

**Example 2.2.** (a) Let $\Sigma$ be a fan in $N_\mathbb{R}$ for some lattice $N$, and let $X$ be the associated normal toric variety. Let $M = \text{Hom}(N, \mathbb{Z})$ be the dual of $N$ and $T = \text{Spec} k[M]$ the associated algebraic torus. By definition, $T \subseteq X$ is a toroidal embedding. The components of the boundary $X \setminus T$ are the $T$-invariant divisors $D_\rho$ corresponding to the rays $\rho \in \Sigma(1)$, and the strata of $X$ are the $T$-orbits $O(\sigma)$ corresponding to the cones $\sigma \in \Sigma$. The combinatorial open subsets of $X$ are precisely its $T$-invariant affine opens. For every $\tau \in \Sigma$, the isomorphism

$$M/(M \cap \tau^+) \to M^{O(\tau)}, \ [m] \mapsto \text{div}(\chi^m),$$

where $\chi^m$ denotes the character associated to $m$, induces identifications of $N^{O(\tau)}$ with $N_\tau$ and $\sigma^{O(\tau)}$ with $\tau$. After identifying $M$ and $M^X$ via the isomorphism

$$M \to M^X = \Gamma(T, O_X^*)/k^*, \ m \mapsto \chi^m ,$$

we see that the image of $\sigma^{O(\tau)}$ in $N_\mathbb{R}$ under $\varphi_X$ is precisely $\tau$. We conclude that $\Sigma(X)$ is an embedded cone complex, naturally isomorphic to $\Sigma$.

(b) For a non-toric example, consider $X = \mathbb{P}^2$ with open part $X_0 = X \setminus H_1 \cup H_2$, where $H_i = \{x_i = 0\}$. Since $X$ is smooth, $\Sigma(X)$ is naturally identified with the orthant ($\mathbb{R}_{\geq 0}^2$), whose
restricted to the combinatorial open subset

It is well known that the character lattice of the open torus embedded in the toric variety $Y$ can be naturally identified with $M \cap \tau^\perp$. With this identification, the upper right morphism

\[ \chi \mapsto \chi^m, \quad m \mapsto \text{div}(\chi^m), \]

where $\chi^m$ is the character associated to $m$, induces a commutative diagram

\[
\begin{array}{ccc}
M & \cong & M \cap \tau^\perp \\
\downarrow & & \downarrow \\
M^{Y'}(X) & \cong & M^{Y'}(X) \cap (\sigma_Y)^\perp \rightarrow M^{Y'}(\overline{Y}).
\end{array}
\]

It follows directly from the definitions that whenever $Y$ is a stratum of a toroidal embedding $X$, the embedding $Y \subseteq \overline{Y}$ is toroidal again. In the following lemma, we compare the weakly embedded cone complex of $\overline{Y}$ with the star of $\Sigma(X)$ at $\sigma_Y$.

**Lemma 2.3.** Let $X_0 \subseteq X$ be a toroidal embedding, and let $Y$ and $Y'$ be two strata of $X$ such that $Y' \subseteq \overline{Y}$. Then the sublattice $M^{Y'}(X) \cap (\sigma_Y)^\perp$ consists precisely of the divisors in $M^{Y'}(X)$ whose support does not contain $Y$. The natural restriction map

\[ M^{Y'}(X) \cap (\sigma_Y)^\perp \rightarrow M^{Y'}(\overline{Y}) \]

is an isomorphism identifying $M^{Y'}(X) \cap (\sigma_Y)^\perp$ with $M^{Y'}(\overline{Y})$. Therefore, it induces an isomorphism $\sigma^{Y'}_Y \cong \sigma^{Y'}_X$ of cones. These isomorphisms glue and give an identification of $\Sigma(\overline{Y})$ with $\Sigma(X)(\sigma^X_Y)$. Furthermore, there is a natural morphism $N^{\Sigma(\overline{Y})} \rightarrow N^{\Sigma(X)}(\sigma^X_Y)$ respecting the weak embeddings.

**Proof.** Let $D_1, \ldots, D_s$ be the boundary divisors containing $Y'$, and assume that they are labeled such that $D_1, \ldots, D_r$ are the ones containing $Y$. Let $u_i$ be the primitive generator of the ray $\sigma^D_i$. For a Cartier divisor $D = \sum_i a_i D_i \in M^{Y'}(X)$, we have $\langle D, u_i \rangle = a_i$ by [37, p. 63]. It follows that $M^{Y'}(X) \cap (\sigma_Y)^\perp$ contains exactly those divisors of $M^{Y'}(X)$ with $a_1 = \cdots = a_r = 0$, which are precisely those whose support does not contain $Y$. These divisors can be restricted to the combinatorial open subset $\overline{Y}(Y')$ of $\overline{Y}$, yielding divisors in $M^{Y'}(\overline{Y})$. To show that the restriction map is an isomorphism, we may reduce to the toric case by choosing a local toric model at a closed point of $Y$ and using [37, II, §1, Lemma 1]. So, assume that $X = U_\sigma$ is an affine toric variety defined by a cone $\sigma \subseteq N_\mathbb{R}$ in a lattice $N$, the stratum $Y' = O(\sigma)$ is the closed orbit, and $Y = O(\tau)$ for a face $\tau \prec \sigma$. Further reducing to the case in which $X$ has no torus factors, we may assume that $\sigma$ is full-dimensional. Let $M$ be the dual of $N$. The isomorphism

\[ M \rightarrow M^{Y'}(X), \quad m \mapsto \text{div}(\chi^m), \]

where $\chi^m$ is the character associated to $m$, induces a commutative diagram

\[
\begin{array}{ccc}
M & \cong & M \cap \tau^\perp \\
\downarrow & & \downarrow \\
M^{Y'}(X) & \cong & M^{Y'}(X) \cap (\sigma_Y)^\perp \rightarrow M^{Y'}(\overline{Y}).
\end{array}
\]
in the diagram sends a character in $M \cap \tau^\perp$ to its associated principal divisor. Thus, it is an isomorphism, which implies that the restriction map $M_+^Y(X) \cap (\sigma_+^Y)^\perp \to M_+^Y(Y)$ is an isomorphism as well. Both $M_+^Y(X) \cap (\sigma_0^X)^\perp$ and $M_+^Y(Y)$ correspond to $M \cap \tau^\perp \cap \sigma_1^Y$; hence, they get identified by the restriction map. Dualization induces an isomorphism of the dual cone $\sigma^\perp_0$ of $M_+^Y(Y)$ and the dual cone $\sigma^\perp_1/\sigma_1^Y$ of $M_+^Y(X) \cap (\sigma_0^X)^\perp$. If $Y''$ is a third stratum of $X$ such that $Y'' \subseteq Y'$, then the diagram

$$
\begin{array}{c}
M_+^Y(X) \cap (\sigma_0^X)^\perp \\
\downarrow \\
M_+^Y(X) \cap (\sigma_0^Y)^\perp
\end{array}
\begin{array}{c}
M_+^Y(Y) \\
\downarrow \\
M_+^Y(Y')
\end{array}
$$

is commutative because all maps involved are restrictions to open or closed subschemes. It follows that the isomorphisms of cones glue to an isomorphism $\Sigma(Y) \cong S_{\Sigma(X)}(\sigma_0^X)$. To see that this isomorphism respects the weak embeddings, note that $M_{S_{\Sigma(X)}(\sigma_0^X)}^Y$ is equal to the sublattice $(N^X_0)^\perp$ of $N^X$ by definition. It consists exactly of those rational functions that are invertible on $X(Y)$. Hence, they can be restricted to $Y'$, giving a morphism

$$
M_{S_{\Sigma(X)}(\sigma_0^X)}^Y \to M_+^Y.
$$

For any rational function $f$ on $X$ that is invertible on $Y$, we have $\text{div}(f|_\tau) = \text{div}(f)|_\tau$. It follows directly from this equality that the identification $\Sigma(Y) \to S_{\Sigma(X)}(\sigma_0^X)$ is, in fact, a morphism of weakly embedded cone complexes.

A toroidal morphism between two toroidal embeddings $X$ and $Y$ is a dominant morphism $X \to Y$ of varieties that can be described by toric morphisms in local toric models (see [4] for details). A toroidal morphism $f : X \to Y$ induces a finiteness-preserving morphism $\text{Trop}(f) : \Sigma(X) \to \Sigma(Y)$ of extended cone complexes. The restrictions of $\text{Trop}(f)$ to the cones of $\Sigma(X)$ are dual to pulling back Cartier divisors. From this, we easily see that $\text{Trop}(f)$ can be considered as a morphism of weakly embedded extended cone complexes by adding to it the data of the linear map $N^X \to N^Y$ dual to the pullback $\Gamma(Y_0, O_0^Y) \to \Gamma(X_0, O_0^X)$. We call a morphism $f : X \to Y$ subtoroidal if it factors as $X \xrightarrow{f'} V(\sigma) \xrightarrow{i} Y$, where $f'$ is toroidal, and $\sigma \in \Sigma(Y)$. By Lemma 2.3, the closed immersion $i$ induces a morphism $\text{Trop}(i) : \Sigma(V(\sigma)) \to \Sigma(Y)$ of weakly embedded extended cone complexes, namely, the composite of the canonical morphism $\Sigma(V(\sigma)) \to S_{\Sigma(V(\sigma))}(\sigma)$ and the inclusion of $S_{\Sigma(V(\sigma))}(\sigma)$ in $\Sigma(Y)$. Thus, we can define $\text{Trop}(f) : \Sigma(X) \to \Sigma(Y)$ as the composite $\text{Trop}(i) \circ \text{Trop}(f')$.

A special class of toroidal morphisms is given by toroidal modifications. They are the analogs of the toric morphisms resulting from refinements of fans in toric geometry. For every subdivision $\Sigma'$ of the cone complex $\Sigma$ of a toroidal embedding $X$, there is a unique toroidal modification $X \times_{\Sigma} \Sigma' \to X$ whose tropicalization is the subdivision $\Sigma' \to \Sigma$ [37]. This modification maps $(X \times_{\Sigma} \Sigma')_0$ isomorphically onto $X_0$. Modifications are compatible with toroidal modifications in the sense that if $f : X \to Y$ is toroidal, and $\Sigma'$ and $\Delta'$ are subdivisions of $\Sigma(X)$ and $\Sigma(Y)$, respectively, such that $\text{Trop}(f)$ induces a morphisms $\Sigma' \to \Delta'$, then $f$ lifts to a toroidal morphism $f' : X \times_{\Sigma(X)} \Sigma' \to Y \times_{\Sigma(Y)} \Delta$ [4, Lemma 1.11].

Remark 2.4. The notation for toroidal modifications is due to Kazuya Kato [32] and is not as abusive as it may seem, see [23, Proposition 9.6.14; 3, Corollary 4.4.3].

3. Intersection theory on weakly embedded cone complexes

In what follows we will develop the foundations of a tropical intersection theory on weakly embedded cone complexes. Our constructions are motivated by the relation of algebraic and
tropical intersection theory as well as by the well-known constructions for the embedded case studied in [7]. In fact, those of our constructions that work primarily in the finite part of the cone complex are natural generalizations of the corresponding constructions for embedded complexes. Intersection theoretical constructions involving boundary components at infinity have been studied in the setup of tropical manifolds [47, 54] and for Kajiwara’s and Payne’s tropical toric varieties [45]. The latter is closer to our setup, yet definitions and proofs vary significantly from ours.

3.1. Minkowski weights, tropical cycles, and tropical divisors

For the definitions of Minkowski weights and tropical cycles, we need the notion of lattice normal vectors. Let $\tau$ be a codimension 1 face of a cone $\sigma$ of a weakly embedded cone complex $\Sigma$. We denote by $u_{\sigma/\tau}$ the image under the morphism $N^\sigma/N^\tau \to N^\Sigma/N^{\Sigma_{\tau}}$ induced by the weak embedding of the generator of $N^\sigma/N^\tau$ that is contained in the image of $\sigma \cap N^\sigma$ under the projection $N^\sigma \to N^\sigma/N^\tau$, and call it the lattice normal vector of $\sigma$ relative to $\tau$. Note that lattice normal vectors may be equal to 0.

**Definition 3.1.** Let $\Sigma$ be a weakly embedded cone complex. A $k$-dimensional Minkowski weight on $\Sigma$ is a map $c: \Sigma_{(k)} \to \mathbb{Z}$ from the $k$-dimensional cones of $\Sigma$ to the integers such that it satisfies the balancing condition around every $(k-1)$-dimensional cone $\tau \in \Sigma$: if $\sigma_1, \ldots, \sigma_n$ are the $k$-dimensional cones containing $\tau$, then

$$\sum_{i=1}^n c(\sigma_i)u_{\sigma_i/\tau} = 0 \quad \text{in } N^\Sigma/N^{\Sigma_{\tau}}.$$ 

The $k$-dimensional Minkowski weights naturally form an abelian group, which we denote by $M_k(\Sigma)$.

We define the group of tropical $k$-cycles on $\Sigma$ by $Z_k(\Sigma) = \lim M_k(\Sigma')$, where $\Sigma'$ runs over all proper subdivisions of $\Sigma$. If $c$ is a $k$-dimensional Minkowski weight on a proper subdivision of $\Sigma$, we denote by $[c]$ its image in $Z_k(\Sigma)$. The group of tropical $k$-cycles on the extended complex $\Sigma$ is defined by $Z_k(\Sigma) = \bigoplus_{\sigma \in \Sigma} Z_k(S(\sigma))$. We will write $M_k(\Sigma) = \bigoplus_k M_k(\Sigma)$ for the graded group of Minkowski weights, and similarly $Z_k(\Sigma)$ and $Z_k(\Sigma')$ for the graded groups of tropical cycles on $\Sigma$ and $\Sigma'$, respectively.

The support $|A|$ of a cycle $A = [c] \in Z_k(\Sigma)$ is the union of all $k$-dimensional cones on which $c$ has non-zero weight. This is easily seen to be independent of the choice of $c$.

**Remark 3.2.** In the definition of cycles, we implicitly used that a proper subdivision $\Sigma'$ of a weakly embedded cone complex $\Sigma$ induces a morphism $M_k(\Sigma) \to M_k(\Sigma')$. The definition of this morphism is clear, yet a small argument is needed to show that balancing is preserved. But this can be seen similarly as in the embedded case [7, Lemma 2.11].

**Remark 3.3.** Minkowski weights are meant as the analogs of cocycles on toroidal embeddings, whereas tropical cycles are the analogs of cycles. To see the analogy, consider the toric case. There, the cohomology group of a normal toric variety $X$ corresponding to a fan $\Sigma$ is canonically isomorphic to the group of Minkowski weights on $\Sigma$ [22]. On the other hand, for a subvariety $Z$ of $X$, this isomorphism cannot be used to assign a Minkowski weight to the cycle $[Z]$ unless $X$ is smooth. In general, we first have to take a proper transform of $Z$ to a smooth toric modification of $X$, and therefore only obtain a Minkowski weight up to refinements of fans, that is, a tropical cycle. That this tropical cycle is well-defined has been shown, for example, in [33, 58].
Example 3.4. Consider the weakly embedded cone complex from Example 2.2b) and let us denote it by $\Sigma$. Let $\sigma$ be its maximal cone, and $\rho_1 = \mathbb{R}_{\geq 0} e_1$ and $\rho_2 = \mathbb{R}_{\geq 0} e_2$ its two rays. The two lattice normal vectors $u_{\rho_1/\rho_1}$ and $u_{\rho_2/\rho_2}$ are equal to 0. Therefore, the balancing condition for 2-dimensional Minkowski weights is trivial and we have $M_2(\Sigma) = \mathbb{Z}$. In dimension 1, we have $u_{\rho_1/\rho_1} = 1$ and $u_{\rho_2/\rho_1} = -1$. Hence, every balanced 1-dimensional weight must have equal weights on the two rays, which implies $M_1(\Sigma) = \mathbb{Z}$. In dimension 0, there is, of course, no balancing to check and we have $M_0(\Sigma) = \mathbb{Z}$ as well.

Cocycles on a toroidal embedding can be pulled back to closures of strata in its boundary. This works for Minkowski weights as well.

**Proposition 3.5 (Pullbacks of Minkowski weights).** Let $\Sigma$ be a weakly embedded cone complex, let $\gamma$ be a cone of $\Sigma$, let $k \geq \dim \gamma$, and let $c \in M_k(\Sigma)$ be a Minkowski weight. Then the induced weight on $S(\Sigma)$ that assigns $c(\sigma)$ to the $(k - \dim \gamma)$-dimensional cone $\sigma/\gamma \in S\Sigma(\gamma)$ is a Minkowski weight again. We denote it by $i^*c$, where $i: S\Sigma(\gamma) \to \Sigma$ is the inclusion map. If $k < \dim \gamma$, we set $i^*c = 0$.

**Proof.** We have to check the balancing condition. Whenever we are given an inclusion $\tau \leq \sigma$ of two cones of $\Sigma$ that contain $\gamma$, we obtain an inclusion $\tau/\gamma \leq \sigma/\gamma$ of cones in $S\Sigma(\gamma)$ of the same codimension. By construction, the restriction of the weak embedding $\varphi_{S\Sigma(\gamma)}$ to $\tau/\gamma$ is equal to the map $\tau/\gamma \to (N^\Sigma/N^\Sigma_\tau)_R = N^S(\tau/\gamma)$ induced by $\varphi_{\Sigma}$. It follows that there is a natural isomorphism $N^\Sigma/N^\Sigma_\tau \cong N^S(\tau)/N^S(\tau/\gamma)$. Since $N^\sigma/\gamma = N^\sigma/N^\gamma$ by construction, we also have a canonical isomorphism $N^\sigma/N^\tau \cong N^{\sigma/\gamma}/N^{\tau/\gamma}$. These isomorphisms fit into a commutative diagram

$$
\begin{array}{ccc}
N^\sigma/N & \longrightarrow & N^\Sigma/N^\Sigma_\tau \\
\downarrow \cong & & \downarrow \cong \\
N^\sigma/\gamma/N^{\tau/\gamma} & \longrightarrow & N^S(\gamma)/N^S(\tau/\gamma).
\end{array}
$$

If $\tau$ has codimension 1 in $\sigma$, we see that the image of the lattice normal vector $u_{\sigma/\tau}$ in $N^S(\gamma)/N^S(\tau/\gamma)$ is the lattice normal vector $u_{(\sigma/\gamma)/(\tau/\gamma)}$. It follows immediately from these considerations that the induced weight $i^*c$ on $S(\gamma)$ satisfies the balancing condition. \[\square\]

**Definition 3.6.** Let $\Sigma$ be a weakly embedded cone complex. A Cartier divisor on $\Sigma$ is a continuous function $\psi: [\Sigma] \to \mathbb{R}$ that is integral linear on all cones of $\Sigma$. We denote the group of Cartier divisors by $\text{Div}(\Sigma)$. A Cartier divisor $\psi$ is said to be combinatorially principal (cp for short) if $\psi$ is induced by a function in $M^\Sigma$ on each cone. More precisely, $\psi$ is cp if and only if for every cone $\sigma \in \Sigma$, there exists $m \in M^\Sigma$ such that $\psi|\sigma = m \circ \varphi_{|\sigma}$. The subgroup of $\text{Div}(\Sigma)$ consisting of cp-divisors is denoted by $\text{CP}(\Sigma)$. Two divisors are called linearly equivalent if their difference is induced by a function in $M^\Sigma$, that is, equal to $m \circ \varphi_{|\sigma}$ for some $m \in M^\Sigma$ on all of $[\Sigma]$. We denote by $\text{ClCP}(\Sigma)$ the group of cp-divisors modulo linear equivalence.

**Remark 3.7.** Note that our usage of the term Cartier divisors is non-standard. Usually, piecewise integral linear functions on $\mathbb{R}^n$, or more generally embedded cone complexes, are called rational functions because they arise naturally as ‘quotients’, that is, differences, of tropical ‘polynomials’, that is, functions of the form $\mathbb{R}^n \to \mathbb{R}$, $x \mapsto \min\{\langle m, x \rangle \mid m \in \Delta\}$ for some finite set $\Delta \subseteq \mathbb{Z}^n$. As we lack the embedding, we chose a different analogy: Assuming that the cone complex $\Sigma$ comes from a toroidal embedding $X$, integer linear functions on a cone $\sigma$ correspond to divisors on $X(\sigma)$ supported on $X(\sigma) \setminus X_0$. Since the cones in $\Sigma$ and the
combinatorial open subsets of $X$ are glued accordingly, this induces a correspondence between $\text{Div}(\Sigma)$ and the group of Cartier divisors on $X$ supported away from $X_0$. Unlike in the toric case, a Cartier divisor on a combinatorial open subset $X(\sigma)$ supported away from $X_0$ does not need to be principal. In case it is, it is defined by a rational function in $\Gamma(X_0, \mathcal{O}_X^*)$. This means that the associated linear function on $\sigma$ is the pullback of a function in $M^\delta$, explaining the terminology combinatorially principal. Finally, linear equivalence is defined precisely in such a way that two divisors are linearly equivalent if and only if their associated divisors on $X$ are.

The functorial behavior of tropical Cartier divisors is as one would expect from algebraic geometry. They can be pulled back along finiteness-preserving morphisms of weakly embedded extended cone complexes, and along arbitrary morphisms if passing to linear equivalence. In the latter case, however, we need to restrict ourselves to cp-divisors.

**Construction 3.8 (Pullbacks of Cartier divisors).** Let $f: \Sigma \to \Sigma'$ be a morphism of weakly embedded extended cone complexes, and let $\psi$ be a divisor on $\Delta$. If $f$ is finiteness-preserving, then it follows directly from the definitions that $f^*\psi = \psi \circ f$ is a divisor on $\Sigma$. We call it the pullback of $\psi$ along $f$. Moreover, if $\psi$ is combinatorially principal, then so is $f^*\psi$. The map $f^*: \text{CP}(\Delta) \to \text{CP}(\Sigma)$ clearly is a morphism of abelian groups, and it preserves linear equivalence. Therefore, it induces a morphism $\text{ClCP}(\Delta) \to \text{ClCP}(\Sigma)$ which we again denote by $f^*$. If $f$ is arbitrary, then it factors uniquely as $\Sigma \xrightarrow{i} \Sigma' \xrightarrow{\delta} \Sigma$, where $\delta \in \Delta$, $\delta'$ is finiteness-preserving, and $i$ is the inclusion map. Hence, to define a pullback $f^*: \text{ClCP}(\Delta) \to \text{ClCP}(\Sigma)$, it suffices to construct pullbacks of divisor classes on $\Delta$ to $\Sigma'$. So, let $\psi$ be a cp-divisor on $\Delta$. Choose $\psi_\delta \in M^\delta$ such that $\psi|_\delta = \psi_\delta \circ \varphi_\delta^\Sigma|_\delta$, and denote $\psi' = \psi - \psi_\delta \circ \varphi_\delta^\Sigma$. Then $\psi'|_\delta = 0$ and thus for every $\sigma \in \Delta$ with $\delta \preceq \sigma$, there is an induced integral linear map $\overline{\psi}'_\sigma: \sigma/\delta \to \mathbb{R}$. These maps patch together to a cp-divisor $\overline{\psi}'$ on $\Sigma'(\delta)$. Clearly, $\overline{\psi}'$ depends on the choice of $\psi_\delta$, but different choices produce linearly equivalent divisors. Thus, we obtain a well-defined element $i^*\psi \in \text{ClCP}(\Sigma'(\delta))$. By construction, divisors linearly equivalent to 0 are in the kernel of $i^*$, so there is an induced integral linear map $\overline{\psi}'_\delta: \sigma/\delta \to \mathbb{R}$. Finally, we define the pullback along $f$ by $f^* := (f')^* \circ i^*$.

3.2. Push-forwards

Now we want to construct push-forwards of tropical cycles. To do so, we need to know that cone complexes allow sufficiently fine proper subdivisions with respect to given subcones.

**Lemma 3.9.** Let $\Sigma$ be a cone complex and $A$ a finite set of subcones of $\Sigma$. Then there exists a proper subdivision $\Sigma'$ of $\Sigma$ such that every cone in $A$ is a union of cones in $\Sigma'$.

**Proof.** If $\Sigma$ is a fan in $N_\mathbb{R}$ for some lattice $N$, it is well known how to obtain a suitable subdivision. Namely, we choose linear inequalities for every $\tau \in A$ and intersect the cones in $\Sigma$ with the half-spaces defined by these inequalities. In the general case, we may assume that $\Sigma$ is strictly simplicial by choosing a proper strictly simplicial subdivision $\Sigma'$ and intersecting all cones in $A$ with the cones in $\Sigma'$. Let $\varphi: |\Sigma| \to \mathbb{R}^{\Sigma(1)}$ be the weak embedding sending the primitive generator of a ray $\rho \in \Sigma(1)$ to the generator $e_\rho$ of $\mathbb{R}^{\Sigma(1)}$ corresponding to $\rho$. Let $\Delta$ be a complete fan in $\mathbb{R}^{\Sigma(1)}$ such that $\varphi(\sigma)$ is a union of cones of $\Delta$ for every $\sigma \in \Sigma \cup A$. We define $\Sigma'$ as the proper subdivision of $\Sigma$ consisting of the cones $\varphi^{-1}\delta \cap \sigma$ for $\delta \in \Delta$ and $\sigma \in \Sigma$ and claim that it satisfies the desired property. Let $\tau \in A$, and let $\sigma \in \Sigma$ be a cone containing it. Then, by construction, there exist cones $\delta_1, \ldots, \delta_k$ of $\Delta$ such that $\varphi(\tau) = \bigcup_i \delta_i$. Each $\delta_i$ is in the image of $\sigma$, hence $\varphi(\varphi^{-1}\delta_i \cap \sigma) = \delta_i$. With this, it follows that $\varphi(\tau) = \varphi(\bigcup_i (\varphi^{-1}\delta_i \cap \sigma))$, and hence that $\tau$ is a union of cones in $\Sigma'$ since $\varphi$ is injective on $\sigma$ by construction. \qed
Proposition 3.10 (Push-forwards). The assignment \( \Sigma \mapsto Z_*(\Sigma) \) is functorial. More precisely, there is a unique way to functorially associate to every morphism \( f : \Sigma \to \overline{\Sigma} \) a push-forward morphism \( f_* : Z_*(\Sigma) \to Z_*(\overline{\Sigma}) \) such that

(a) if \( f \) is a proper subdivision, then \( f \) restricts to the identity on \( Z_*(\Sigma) = Z_*(\Delta) \);
(b) if \( \Sigma = S_\delta(\Delta) \) for some \( \delta \in \Delta \), and \( f \) is the inclusion, then \( f_* \) is the inclusion;
(c) if \( f \) is finiteness-preserving, \( f(\sigma) \in \Delta \) for all \( \sigma \in \Sigma \), and \( c \in M_k(\Sigma) \), then \( f_*[c] \) is represented by the Minkowski weight \( f_*c \) defined by

\[
  f_*c(\delta) = \sum_{\sigma \mapsto \delta} |N^\delta : f(N^\sigma)|c(\sigma),
\]

where the sum is over all \( k \)-dimensional cones of \( \Sigma \) with image equal to \( \delta \).

Proof. To show the uniqueness, it suffices to prove that whenever \( f \) is finiteness-preserving and \( A \in Z_k(\Sigma) \), there exist proper subdivisions \( \Sigma' \) of \( \Sigma \) and \( \Delta' \) of \( \Delta \) such that \( A \) is represented by a Minkowski weight again. Say \( A \) is represented by a Minkowski weight \( c \) on a proper subdivision \( \overline{\Sigma} \) of \( \Sigma \). By Lemma 3.9, there is a proper subdivision \( \Delta' \) of \( \Delta \) such that \( f(\overline{\sigma}) \) is a union of cones of \( \Delta' \) for every \( \overline{\sigma} \in \overline{\Sigma} \). The cones of the form \( \overline{\sigma} \cap f^{-1}\delta' \) for \( \overline{\sigma} \in \overline{\Sigma} \) and \( \delta' \in \Delta' \) then form a proper subdivision \( \Sigma' \) of \( \overline{\Sigma} \), and standard arguments show that \( f(\sigma') \in \Delta' \) for all \( \sigma' \in \Sigma' \).

The proof of uniqueness above tells us how to define \( f_*A \) in terms of a representation \( A = [c] \) by a Minkowski weight \( c \) on a suitable refinement \( \Sigma' \) of \( \Sigma \). What is left to show is that

(i) \( f_*c \) is a Minkowski weight again,

(ii) \( [f_*c] \) is independent on the choice of \( \Sigma' \) and \( c \), and

(iii) the law \( A \mapsto [f_*c] \) defines a morphism and is functorial in \( f \).

Part (i) involves only a slight modification of the argument for the embedded case [24, Proposition 2.25], which we leave to the reader.

To see part (ii), we first note that the support \( [f_*c] \) is independent of \( c \). It is the union of the images of the \( k \)-dimensional cones of \( \Sigma' \) that are contained in \( |A| \) and on which \( f \) is injective. This does not change when we refine \( \Sigma' \) and does therefore not depend on any choices. Slightly abusing notation, we denote this set by \( [f_*A] \). Next, we observe that there is a dense open subset of \( [f_*A] \) over which \( |A| \to [f_*A] \) has finite fibers. For example, we can take \( [f_*A] \) minus the images of the cones of \( \Sigma' \) that are contained in \( |A| \) and on which \( f \) is not injective. Thus, locally, the number of summands occurring in the definition of \( f_*c \) does not depend on the chosen subdivisions. Since the occurring lattice indices are local in the same sense, the independence of \( f_*A \) of all choices follows. It is immediate from the construction that pushing forward is linear in \( A \).

Part (iii) is again left to the reader, as this works analogously to the embedded case [52, Remark 1.3.9]. \( \square \)

3.3. Intersecting with divisors

In this section, we construct two pairings of divisors and cycles. The first one will be the tropical analog of the intersection product of a boundary divisor on a toroidal embedding with a cycle, whereas the second will be the analog of the cup-product of cocycles on toroidal embeddings. The intersection of a boundary divisor with a cycle will only be well-defined without passing to rational equivalence if the cycle is not itself contained in the boundary. So tropically, we may expect to define a bilinear morphism

\[
  \cdot : \text{Div}(\Sigma) \times Z_*(\Sigma) \to Z_*(\overline{\Sigma}).
\]
Note that by what we just said, we cannot expect to extend the domain of the pairing to \( \text{Div}(\Sigma) \times \mathbb{Z}_+(\Sigma) \). Nevertheless, as the codomain, we need to take \( \mathbb{Z}_+(\Sigma) \) since intersections with boundary divisors will live in the boundary. To motivate the definition of the pairing, assume that \( \psi \in \text{Div}(\Sigma) \) is a divisor, and \( c \in \mathbb{M}_k(\Sigma) \) is a Minkowski weight. While technically not defined because we lack a good notion of fundamental classes, we should think about the associated Weil divisor of \( \psi \) as the linear combination of the boundary strata \( \mathbb{S}_\Sigma(\rho) \), with \( \rho \in \Sigma_{(1)} \), where the stratum \( \mathbb{S}_\Sigma(\rho) \) occurs with multiplicity \( \psi(\mu_\rho) \), where \( \mu_\rho \) is the primitive generator of \( \rho \). So, to define an intersection product \( \psi \cdot [c] \), it suffices to know what \( \mathbb{S}_\Sigma(\rho) \cdot [c] \) should be. The canonical choice here is \([i^*_\rho c]\), where \( i_\rho: \mathbb{S}_\Sigma(\rho) \to \Sigma \) is the inclusion and the pullback \( i^*_\rho \) is that of Proposition 3.5. It turns out that if we also want the projection formula to hold, for subdivision at the very least, then this uniquely defines the \( \cdot \)-pairing.

**Proposition 3.11.** There is a unique way to assign bilinear morphisms
\[
\cdot : \text{Div}(\Sigma) \times \mathbb{Z}_+(\Sigma) \to \mathbb{Z}_+(\Sigma)
\]
to weakly embedded cone complexes \( \Sigma \), such that

(a) for \( \psi \in \text{Div}(\Sigma) \) and \( c \in \mathbb{M}_k(\Sigma) \) we have
\[
\psi \cdot [c] = \sum_{\rho \in \Sigma_{(1)}} \psi(\mu_\rho)[i^*_\rho c] \in \mathbb{Z}_{k-1}(\Sigma),
\]
where \( \mu_\rho \) is the primitive generator of \( \rho \) and \( i_\rho: \mathbb{S}_\Sigma(\rho) \to \Sigma \) is the inclusion;

(b) the projection formula holds for proper subdivisions, that is, whenever \( f: \Sigma \to \Delta \) is a proper subdivision, \( A \in \mathbb{Z}_+(\Sigma) \) and \( \psi \in \text{Div}(\Delta) \), then
\[
f_*(f^*\psi \cdot A) = \psi \cdot f_*A.
\]

**Proof.** The uniqueness is clear, as every cycle \( A \) on a weakly embedded cone complex \( \Sigma \) is represented by a Minkowski weight \( c' \in \mathbb{M}_k(\Sigma') \) for some proper subdivision \( f: \Sigma' \to \Sigma \), which precisely means that \( A = f_*[c'] \).

To show existence, it suffices to show that if \( A = f_*[c'] \), then
\[
f_* \left( \sum_{\rho' \in \Sigma_{(1)}} \psi(\mu_\rho)[i^*_\rho c'] \right),
\]
where \( i^*_\rho : \mathbb{S}_{\Sigma'}(\rho') \to \Sigma' \) is the inclusion, does not depend on the choices of \( \Sigma' \) and \( c' \). As every two subdivisions have a common subdivision, we may assume that \( A \) is already represented by a Minkowski weight \( c \in \mathbb{M}_k(\Sigma) \) and it suffices to show that
\[
f_* \left( \sum_{\rho' \in \Sigma_{(1)}} \psi(\mu_\rho)[i^*_\rho c'] \right) = \sum_{\rho \in \Sigma_{(1)}} \psi(\mu_\rho)[i^*_\rho c].
\]
For a ray \( \rho \in \Sigma \), which is automatically a ray in \( \Sigma' \), it is not hard to see that the cone complex \( \mathbb{S}_{\Sigma'}(\rho') \) is a proper subdivision of \( \mathbb{S}_{\Sigma}(\rho) \). It immediately follows that \( f_*[i^*_\rho c] = [i^*_\rho c]. \) For a ray \( \rho' \in \Sigma' \setminus \Sigma \), there exists a minimal cone \( \tau \in \Sigma \) containing it which has dimension at least 2. Every \( k \)-dimensional cone \( \sigma' \in \Sigma' \) containing \( \rho' \) with \( c'(\sigma') \neq 0 \) is contained in a \( k \)-dimensional cone \( \sigma \in \Sigma \). But in this situation, \( \tau \) must be a face of \( \sigma \) and hence the image of \( \sigma' / \rho' \in \mathbb{S}_{\Sigma'}(\rho') \) in \( \mathbb{S}_{\Sigma}(\tau) \) has dimension at most \( \dim(\sigma / \tau) \), which is strictly less than \( k \). This shows that \( f_*[(i^*_\rho)^* c] = 0 \) and finishes the proof. \( \square \)
Remark 3.12. If $A$ is represented by a Minkowski weight $c$ on a proper subdivision $\Sigma'$, we can explicitly compute the weights on the cones of $\Sigma'$ to the intersection product $\psi \cdot A$. Assume that $\sigma'$ is a $k$-dimensional cone of $\Sigma'$ and we want to determine its contribution to the component of $\psi \cdot A$ in $S_{\Sigma}(\tau)$ for some $\tau \in \Sigma$. We can only have such a contribution if $\sigma'$ has a ray $\rho \in \sigma'_1$ intersecting the relative interior of $\tau$. In this case, let $\delta \in \Sigma$ be the smallest cone of $\Sigma$ containing $\sigma'$. As $\sigma' \cap \text{relint}(\tau) \neq \emptyset$, we must have $\tau \leq \delta$. The image of $\sigma'/\rho \subseteq S_{\Sigma}(\rho)$ in $\delta/\tau \subseteq S_{\Sigma}(\tau)$ is the image of $\sigma'$ under the canonical projection $\delta \to \delta/\tau$. This is $(k-1)$-dimensional if and only if the sublattice $N^{\sigma'} \cap N^\tau$ of $N^\delta$ has rank 1. In this case, we have $\sigma' \cap \tau = \rho$, and the contribution of $\sigma'$ to the component of $\psi \cdot A$ in $S_{\Sigma}(\tau)$ is the cone $(\sigma' + \tau)/\tau$ with weight

$$\psi(\rho)\text{ index } \left( N^\tau + N^{\sigma'} \right) c(\sigma'),$$

where $u_\rho$ is the primitive generator of $\rho$, and the occurring index is the index of $N^\tau + N^{\sigma'}$ in its saturation in $N^\delta$.

We see that $\sigma'$ contributes exactly to those boundary components of $\psi \cdot A$ which we would expect from topology. This is because ‘the part at infinity’ $\tilde{\Sigma}' \setminus \sigma'$ of $\Sigma'$ in $\Sigma$ intersects $S_{\Sigma}(\tau)$ if and only if $\text{relint}(\tau) \cap \sigma' \neq \emptyset$, and the intersection is $(k-1)$-dimensional if and only if $N^{\sigma'} \cap N^\tau$ has rank 1.

It is crucial for the definition of the ‘$\cdot$’-product that we work with extended cone complexes. However, the reader familiar with the intersection theory of Allermann and Rau [7] for embedded cone complexes will recall that, at least in the embedded case, there is also a pairing

$$\text{CP}(\Sigma) \times Z_*(\Sigma) \to Z_*(\Sigma),$$

which does not make use of the strata at infinity. As mentioned earlier, this second pairing has its algebraic analogy in the cup-product of cocycles (cf. Proposition 4.13). It turns out that Allermann and Rau’s construction for the embedded case carries over to the general situation with only minor adjustments.

Construction 3.13 (Tropical cup-products). Let $\Sigma$ be a weakly embedded cone complex, $c \in M_k(\Sigma)$ a Minkowski weight, and $\psi$ a cp-divisor on $\Sigma$. We construct a Minkowski weight $\psi \cup c \in M_{k-1}(\Sigma)$. For every $\sigma \in \Sigma$, choose an integral linear function $\psi_\sigma \in M^\Sigma$ such that $\psi_\sigma \circ \varphi_\Sigma|_\sigma = \psi|_\sigma$. Note that $\psi_\sigma$ is not uniquely defined by this property, but its restriction $\psi_\sigma|_{N^\Sigma_\tau}$ is. Whenever we have an inclusion $\tau \leq \sigma$ of cones of $\Sigma$, the function $\psi_\sigma - \psi_\tau$ vanishes on $N^\Sigma_\tau$ and hence defines a morphism $N^\Sigma_\tau \to \mathbb{Z}$. Therefore, we can define a weight on the $(k-1)$-dimensional cones of $\Sigma$ by

$$\psi \cup c: \tau \mapsto \sum_{\sigma: \tau \prec \sigma} (\psi_\sigma - \psi_\tau)(c(\sigma)u_{\sigma/\tau}),$$

where the sum is taken over all $k$-dimensional cones of $\Sigma$ containing $\tau$. The balancing condition for $c$ ensures that this weight is independent of the choices involved. It has been proven in [7, Proposition 3.7a] for the embedded case that $\psi \cup c$ is a Minkowski weight again, and a slight modification of the proof in loc. cit. works in our setting as well.

The cup-product can be extended to apply to tropical cycles as well. For $A \in Z_k(\Sigma)$, there is a subdivision $\Sigma'$ of $\Sigma$ such that $A$ is represented by a Minkowski weight $c \in M_k(\Sigma')$. It is easy to see that the tropical cycle $[\psi \cup c]$ does not depend on the choice of $\Sigma'$. Thus, we can define $\psi \cup A := [\psi \cup c]$. For details, we again refer to [7].

It is immediate from the definition that the cup-product is independent of the linear equivalence class of the divisor. Therefore, we obtain a pairing

$$\text{ClCP}(\Sigma) \times Z_*(\Sigma) \to Z_*(\Sigma),$$
which is easily seen to be bilinear. It can be extended to include tropical cycles in the boundary of \( \Sigma \). Let \( \tau \in \Sigma \), and denote the inclusion \( i : S(\tau) \to \overline{\Sigma} \) by \( i \). For a tropical cycle \( A \in S_{\Sigma}(\tau) \) and a divisor class \( \overline{\psi} \in \text{ClCP}(\Sigma) \), we define \( \overline{\psi} \cup A := i^* \overline{\psi} \cup A \). In this way, we obtain a bilinear map
\[
\text{ClCP}(\Sigma) \times Z_*(\Sigma) \to Z_*(\Sigma).
\] (\( \text{\circ} \))

In algebraic geometry, intersecting a cycle with multiple Cartier divisors does not depend on the order of the divisors. We would like to prove an analogous result for tropical intersection products on cone complexes. However, to define multiple intersections, we need rational equivalence, which we will only introduce in Definition 3.17. For cup-products, on the other hand, there is no problem in defining multiple intersections and they do, in fact, not depend on the order of the divisors.

**Proposition 3.14.** Let \( \Sigma \) be a weakly embedded cone complex, \( A \in Z_*(\Sigma) \) a tropical cycle, and \( \psi, \chi \in \text{CP}(\Sigma) \) two cp-divisors. Then we have
\[
\psi \cup (\chi \cup A) = \chi \cup (\psi \cup A).
\]

**Proof.** This can be proven similarly as in the embedded case \([7, \text{Proposition 3.7}]\). \( \square \)

Next, we will consider the projection formula for weakly embedded cone complexes. We will prove two versions, one for \( ' \cdot ' \)-products and one for \( ' \cup ' \)-products.

**Proposition 3.15.** Let \( f : \Sigma \to \overline{\Delta} \) be a morphism between weakly embedded cone complexes.

(a) If \( f \) is finiteness-preserving, \( A \in Z_*(\Sigma) \), and \( \psi \) is a divisor on \( \Delta \), then
\[
f_*(f^* \psi \cdot A) = \psi \cdot f_* A.
\]

(b) For \( \overline{\psi} \in \text{ClCP}(\Delta) \) and \( A \in Z_*(\Sigma) \), we have
\[
f_*(f^* \overline{\psi} \cup A) = \overline{\psi} \cup f_* A.
\]

**Proof.** In both cases, both sides of the equality we wish to prove are linear in \( A \). Therefore, we may assume that \( A \) is a \( k \)-dimensional tropical cycle. For part (a), choose proper subdivisions \( \Sigma' \) and \( \Delta' \) of \( \Sigma \) and \( \Delta \), respectively, such that \( A \) is defined by a Minkowski weight \( c \in M_k(\Sigma') \) and \( f \) maps cones of \( \Sigma' \) onto cones of \( \Delta' \). Consider the commutative diagram
\[
\begin{array}{ccc}
\Sigma' & \xrightarrow{f'} & \Delta' \\
\downarrow^p & & \downarrow^q \\
\Sigma & \xrightarrow{f} & \Delta
\end{array}
\]

By construction of the intersection product, we have \( f^* \psi \cdot A = p_*[f^* \psi \cdot c] \), where we identify \( f^* \psi \) with \( p^*(f^* \psi) \). Therefore, we have
\[
f_*(f^* \psi \cdot A) = (f \circ p)_*[f^* \psi \cdot c] = q_* \left( f'_*[f^* \psi \cdot c] \right).
\]

On the other hand, by construction of push-forward and intersection product, we have
\[
\psi \cdot f_* A = q_*[\psi \cdot f'_* c].
\]

This reduces the proof to the case where images of cones in \( \Sigma \) are cones in \( \Delta \), and \( A \) is represented by a Minkowski weight \( c \in M_k(\Sigma) \). Let \( \rho \in \Delta \) be a ray and \( \delta \in \Delta \) a \( k \)-dimensional cone containing it. Every cone \( \sigma \in \Sigma \) mapping injectively onto \( \delta \) contains exactly one ray \( \rho' \).
mapping onto $\rho$. The contribution of the cone $\sigma$ in $f_*(i^*_{\rho'}c)$, where $i_{\rho'}: S\Sigma(\rho') \to \Sigma$ is the inclusion, is a weight $[N^{\delta/\rho} : f(N^{\sigma/i'})]c(\sigma)$ on the cone $\delta/\rho$. The lattice index is equal to

$$[N^{\delta/\rho} : f(N^{\sigma/i'})] = [N^{\delta/N^\rho} : (f(N^\sigma) + N^\rho)/N^\rho] = [N^{\delta} : f(N^\sigma)/[N^\rho : f(N^\nu')]].$$

The index $[N^\rho : f(N^\nu')]$ is also defined by the fact that $f(u_\rho) = [N^\rho : f(N^\nu')]u_\rho$, where $u_\rho$ and $u_{\nu'}$ denote the primitive generators of $\rho$ and $\nu'$, respectively. Combined, we see that the weight of the component of $f_*(f^*\psi \cdot c)$ in $S_\Delta(\rho)$ at $\delta/\rho$ is equal to

$$\sum_{\rho' \to \rho} \sum_{\delta/\rho} \psi(f(u_{\rho'}))[N^{\delta} : f(N^\sigma)/[N^\rho : f(N^\nu')]c(\sigma) = \psi(u_\rho) \sum_{\sigma \to \delta} [N^{\delta} : f(N^\sigma)]c(\delta),$$

which is precisely the multiplicity of $\psi \cdot f_*c$ at $\delta/\rho$. Because the $S_\Delta(\delta)$-component is 0 for both sides of the projection formula if $\delta$ is not a ray, we have proven part (a).

For part (b), we may reduce to the case that $f$ is finiteness-preserving and $A \in Z_k(\Sigma)$. It also suffices to prove the equation for divisors instead of divisor classes. Analogous to part (a), we then reduce to the case where $A$ is given by a Minkowski weight on $\Sigma$, and images of cones of $\Sigma$ are cones of $\Delta$. In this case, a slight variation of the proof of the projection formula for embedded complexes [7, Proposition 4.8] applies. \hfill \Box

**Proposition 3.16.** Let $\Sigma$ be a weakly embedded cone complex.

(a) For every Minkowski weight $c \in M_k(\Sigma)$, cp-divisor $\psi \in CP(\Sigma)$, and cone $\tau \in \Sigma$, we have

$$i^*(\psi \cup c) = i^*\psi \cup i^*c,$$

where $i: S(\tau) \to \Sigma$ denotes the inclusion map.

(b) If $A \in Z_*(\Sigma)$, $\chi \in \text{Div}(\Sigma)$, and $\bar{\psi} \in \text{CICP}(\Sigma)$, then

$$\bar{\psi} \cup (\chi \cdot A) = \chi \cdot (\bar{\psi} \cup A).$$

**Proof.** We begin with part (a). If $\dim(\tau) \geq k$, then both sides of the equation are equal to 0. So, assume $\dim(\tau) < k$. Let $\sigma \in \Sigma$ be a $(k-1)$-dimensional cone containing $\tau$, and let $\gamma_1, \ldots, \gamma_l$ be the $k$-dimensional cones of $\Sigma$ containing $\sigma$. To compute the weights of the two sides of the equation at $\sigma/\tau$, we may assume that $\psi$ vanishes on $\sigma$. Let $\psi_i \in M_{\Sigma}^\tau$ be linear functions such that $\psi_i \circ \varphi_{\Sigma|\gamma_i} = \psi|\gamma_i$. Then if $v_1, \ldots, v_l$ are representatives in $N^\Sigma_\tau$ of the lattice normal vectors $u_{\gamma_1/\sigma}, \ldots, u_{\gamma_l/\sigma}$, the weight of $\psi \cup c$ at $\sigma$, and hence the weight of $i^*(\psi \cup c)$ at $\sigma/\tau$ is given by

$$\sum_{i=1}^l \psi_i(v_i)c(\gamma_i).$$

Since $\psi$ vanishes on $\sigma$, each $\psi_i$ vanishes on $N^\Sigma_\tau$. Thus, for every $i$, we have an induced map $\bar{\psi}_i \in \text{Hom}(N^\Sigma_\tau/N^\Sigma_\tau, \mathbb{Z}) = M^S(\tau)$. By construction of the pullbacks of divisors, these functions define $i^*\psi$ around $\sigma/\tau$. We saw in Construction 3.5 that the lattice normal vectors $u_{(\gamma_i/\tau)/(\sigma/\tau)}$ are the images of the lattice normal vectors $u_{\gamma_i/\sigma}$ under the quotient map $N^\Sigma_\tau/N^\Sigma_\tau \to N^S(\tau)/N^S(\tau)_{(\sigma/\tau)}$. This implies that the image $\bar{\psi}_i$ of $v_i$ under the quotient map $N^\Sigma_\tau \to N^S(\tau)$ represents $u_{(\gamma_i/\tau)/(\sigma/\tau)}$. Hence, the weight of $i^*\psi \cup i^*c$ at $\sigma/\tau$ is

$$\sum_{i=1}^l \bar{\psi}_i(\bar{\gamma}_i)i^*c(\gamma_i/\tau) = \sum_{i=1}^l \psi_i(v_i)c(\gamma_i),$$

proving the desired equality.
For part (b), we may assume that $A$ is pure-dimensional. We first treat the case in which it is given by a Minkowski weight $c$ on $\Sigma$. Using the definition of the intersection product and the bilinearity of the cup-product, we get

$$
\overline{\psi} \cup (\chi \cdot A) = \left[ \sum_{\rho \in \Sigma(1)} \chi(u_\rho) \left( \overline{\psi} \cup i_\rho^* c \right) \right] = \left[ \sum_{\rho \in \Sigma(1)} \chi(u_\rho) i_\rho^* (\overline{\psi} \cup c) \right] = \chi \cdot (\overline{\psi} \cup c),
$$

where $i_\rho : S(\rho) \to \Sigma$ denotes the inclusion map. The general case follows by considering a suitable proper subdivision of $\Sigma$ and using the projection formulas. \qed

3.4. Rational equivalence

We denote by $\Pi^1$ the weakly embedded cone complex associated to $\mathbb{P}^1$ equipped with the usual toric structure. It is the unique complete fan in $\mathbb{R}$, its maximal cones being $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{\leq 0}$. We identify the extended cone complex $\Pi^1$ with $\mathbb{R} \cup \{\infty, -\infty\}$, where the boundary points $\infty$ and $-\infty$ of $\Pi^1$ correspond to the boundary points 0 and $\infty$ of $\mathbb{P}^1$, respectively. The identity function on $\mathbb{R}$ defines a cp-divisor on $\Pi^1$ which we denote by $\psi_{\Pi^1}$. Its associated cycle $\psi_{\Pi^1} : [\Pi^1]$, where $[\Pi^1]$ is the 1-cycle on $\Pi^1$ with weight 1 on both of its cones, is $[\infty] - [-\infty]$. We use $\Pi^1$ to define rational equivalence on cone complexes in the same way as $\mathbb{P}^1$ is used to define rational equivalence on algebraic varieties.

**Definition 3.17.** Let $\Sigma$ be a weakly embedded cone complex, and let $p : \Sigma \times \Pi^1 \to \Sigma$ and $q : \Sigma \times \Pi^1 \to \Pi^1$ be the projections onto the first and second coordinates. For $k \in \mathbb{N}$, we define $R_k(\Sigma)$ as the subgroup of $Z_k(\Sigma)$ generated by the cycles $p_*(q^* \psi_{\Pi^1} \cdot A)$, where $A \in Z_{k+1}(S(\sigma \times 0))$ and $\sigma \in \Sigma$. We call two cycles in $Z_k(\Sigma)$ rationally equivalent if their difference lies in $R_k(\Sigma)$. The $k$th (tropical) Chow group $A_k(\Sigma)$ of $\Sigma$ is defined as the group of $k$-cycles modulo rational equivalence, that is, $A_k(\Sigma) = Z_k(\Sigma)/R_k(\Sigma)$. We refer to the graded group $A_*(\Sigma) = \bigoplus_{k \in \mathbb{N}} A_k(\Sigma)$ as the (total) Chow group of $\Sigma$.

**Remark 3.18.** Note that $q^*(\psi_{\Pi^1}) \cdot A$ is strictly speaking not defined if $A$ is not contained in $Z_*(\Sigma \times \Pi^1) = Z_*(S(0 \times 0))$. However, as $q^*(\psi_{\Pi^1})$ vanishes on $\sigma \times 0$ for $\sigma \in \Sigma$, there is a canonically defined pullback to $S(\sigma \times 0)$ which we can intersect with cycles in $Z_*(S(\sigma \times 0))$. Taking the canonical identification of $S(\sigma \times 0)$ with $S(\sigma) \times \Pi^1$, this linear function is nothing but the pullback of $\psi_{\Pi^1}$ via the projection onto $\Pi^1$. So, if we define $R_*(\Sigma)$ as the subgroup of $Z_*(\Sigma)$ generated by $p_*(q^* \psi_{\Pi^1} \cdot A)$ for $A \in Z_*(\Sigma \times \Pi^1)$, then $R_*(\Sigma)$ is generated by the push-forwards of $R_*(S(\sigma))$ for $\sigma \in \Sigma$. In particular, it follows immediately from the definition that the inclusions $i_s : S(\sigma) \to \Sigma$ induce push-forwards $i_* : A_*(S(\sigma)) \to A_*(\Sigma)$ on the level of Chow groups.

**Remark 3.19.** Our definition of tropical rational equivalence is analogous to the definition of rational equivalence in algebraic geometry given for example in [21, Section 1.6]: two cycles on an algebraic scheme $X$ are rationally equivalent if and only if they differ by a sum of elements of the form $p_*(\{q^{-1}\{0\} \cap V\} - \{q^{-1}\{\infty\} \cap V\})$, where $p$ is the projection from $X \times \mathbb{P}^1$ onto $X$, and $V$ is a subvariety of $X \times \mathbb{P}^1$ mapped dominantly to $\mathbb{P}^1$ by the projection $q : X \times \mathbb{P}^1 \to \mathbb{P}^1$. Rewriting $\{q^{-1}\{0\} \cap V\} - \{q^{-1}\{\infty\} \cap V\}$ as $q^* \text{div}(x) \cdot [V]$, where $x$ denotes the identity on $\mathbb{P}^1 \setminus \{0, \infty\} = \mathbb{G}_m$, we see that we obtain the tropical definition from the algebraic one by replacing $\mathbb{P}^1$ by its associated cone complex and $V$ by a cycle $A \in Z_*(S(\sigma \times 0))$. That we do not allow $A$ to be in $Z_*(S(\sigma \times \tau))$ for $\tau \neq 0$ corresponds to the dominance of $V$ over $\mathbb{P}^1$. 

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PROPOSITION 3.20. Let \( f : \Sigma \to \Delta \) be a morphism between weakly embedded extended cone complexes. Then taking push-forwards passes to rational equivalence, that is, \( f_* \) induces a graded morphism \( A_* (\Sigma) \to A_* (\Delta) \), which we again denote by \( f_* \).

Proof. First, assume that \( f \) is finiteness-preserving. Let \( p, p' \) be the projections to the first coordinates of \( \Sigma \times \Pi \) and \( \Delta \times \Pi \), respectively, and similarly, let \( q, q' \) be the projections to the second coordinates. For \( A \in Z_*(\Sigma \times \Pi (\sigma \times 0)) \), we have

\[
f_* p_* (q^* \psi \Pi^1 \cdot A) = p'_* (f \times \text{id})_* \left( (f \times \text{id})^* q^* \psi \Pi^1 \cdot (f \times \text{id})_* A \right),
\]

which is in \( R_{*}(\Delta) \). We see \( R_{*}(\Sigma) \) is mapped to 0 by the composite \( Z_*(\Sigma) \to Z_*(\Delta) \to A_*(\Delta) \), which proves the assertion.

If \( f \) is not finiteness-preserving, then \( f \) can be factored as a finiteness-preserving morphism followed by an inclusion \( \Sigma_\delta \to \Delta \) for some \( \delta \in \Delta \). For both of these morphisms, the push-forward respects rational equivalence, and hence it does the same for \( f \). \( \square \)

Let \( \Pi^0 \) denote the embedded cone complex associated to the trivial fan in the zero lattice 0. It is the tropical analog of the one point scheme \( \mathbb{P}^0 \) in the sense that it is the associated cone complex of \( \mathbb{P}^0 \), as well as in the sense of being the terminal object in the category of weakly embedded (extended) cone complexes. Note that \( \Pi^0 = \Pi^0 \). Since all 1-cycles on \( \Pi^1 \times \Pi^1 = \Pi^1 \) are of the form \( k \cdot [\Pi^1] \) for some \( k \in \mathbb{Z} \), there is a natural identification \( A_* (\Pi^0) = Z_*(\Pi^0) = \mathbb{Z} \).

DEFINITION 3.21. Let \( \Sigma \) be a weakly embedded cone complex and \( f : \Sigma \to \Pi^0 \) the unique morphism to \( \Pi^0 \). We define the degree map \( \text{deg} : Z_*(\Sigma) \to \mathbb{Z} \) as the composite of \( f_* \) and the identification \( Z_*(\Pi^0) = \mathbb{Z} \). By Proposition 3.20, the degree map respects rational equivalence, that is, there is an induced morphism \( A_* (\Sigma) \to \mathbb{Z} \), which we again denote by \( \text{deg} \).

PROPOSITION 3.22. Let \( \Sigma \) be a weakly embedded cone complex, let \( A \in Z_*(\Sigma) \), and let \( \psi \in \mathbb{CP}(\Sigma) \). Then

\[
\psi \cup A = \psi \cdot A \quad \text{in} \quad A_* (\Sigma).
\]

Proof. We may assume that \( A \) is purely \( k \)-dimensional, as both sides are linear in \( A \). We relate the intersection product and the cup-product via the ‘full graph’ of \( \psi \), which has been used in the embedded case, for example, in [7, 46]. Let \( \Sigma' \) be a proper subdivision of \( \Sigma \) such that \( A \) is represented by a Minkowski weight \( c \) on \( \Sigma' \), and \( \psi \) does not change signs on the cones of \( \Sigma' \). Let \( \Gamma_{\psi} : |\Sigma| \to |\Sigma \times \Pi^1| \) be the graph map of \( \psi \). Then the full graph \( \Gamma_{\psi}(A) \) of \( \psi \) on \( A \) is the tropical cycle in \( \Sigma \times \Pi^1 \) whose underlying subdivision consists of the subcones \( \Gamma_{\psi}(\sigma') \) for \( \sigma' \in \Sigma'_{(k)} \), and

\[
\Gamma_{\psi}^+ (\sigma)(\tau) = \{ (x,t) \mid x \in \tau, t \leq \Gamma_{\psi}(x) \} \cap (\tau \times \mathbb{R}_{\geq 0}) \quad \text{and} \quad \Gamma_{\psi}^- (\sigma)(\tau) = \{ (x,t) \mid x \in \tau, t \leq \Gamma_{\psi}(x) \} \cap (\tau \times \mathbb{R}_{\leq 0})
\]

for \( \tau \in \Sigma'_{(k-1)} \). The weights of these cones are \( c(\sigma') \) on \( \Gamma_{\psi}(\sigma') \) and \( (\psi \cup c)(\tau) \) on \( \Gamma_{\psi}^\pm (\tau) \). A slight modification of the argument given in [7, Construction 3.3] shows that this is a well-defined cycle, that is, satisfies the balancing condition.

Let \( p \) and \( q \) be the projections onto the first and second coordinates of \( \Sigma \times \Pi^1 \). We claim that

\[
p_* (q^* \psi \Pi^1 \cdot \Gamma_{\psi}(A)) = \psi \cdot A - \psi \cup A.
\]

To verify this let us analyze how the cones of \( \Gamma_{\psi}(A) \) contribute to the left-hand side of the equation using the computation of Remark 3.12. It is easily seen that the cones of the
form $\Gamma^+_{\psi}(\tau)$ do not contribute at all. The cones $\Gamma^-_{\psi}(\tau)$, on the other hand, contribute to the $S(0 \times \mathbb{R}_{\leq 0})$-component of $q^*\psi_{\Pi^1} \cdot \Gamma_{\psi}(A)$ with weight $-\psi \cup c(\tau)$ on the cone $\Gamma^-_{\psi}(\tau)/(0 \times \mathbb{R}_{\leq 0})$. This cone is mapped injectively onto $\tau$ by the isomorphism $S(0 \times \mathbb{R}_{\leq 0}) \rightarrow \Sigma$ induced by $p$. Hence, the total contribution of the cones considered so far is $-\psi \cup A$.

Now let $\sigma' \in \Sigma'_k$. Depending on whether $\psi$ is non-negative or non-positive on $\sigma'$, the cone $\Gamma^\psi(\sigma')$ can have contributions only in components corresponding to cones in $|\Sigma| \times \mathbb{R}_{\geq 0}$ or $|\Sigma| \times \mathbb{R}_{\leq 0}$. Without loss of generality, assume that $\psi$ is non-negative on $\sigma'$. For every $\tau \in \Sigma$, we have $\Gamma^\psi(\sigma') \cap (\tau \times \mathbb{R}_{\geq 0}) = \Gamma^\psi(\sigma' \cap \tau)$. In particular, this is 1-dimensional with relative interior contained in $\text{relint}((\tau \times \mathbb{R}_{\geq 0})$ if and only if $\rho := \sigma' \cap \tau$ is 1-dimensional, intersects the relative interior of $\tau$, and $|\psi|_\rho$ is nonzero. In this case, let $\delta \in \Sigma$ be a cone containing $\sigma'$, and denote by $\psi_\delta \in M^\delta$ the linear function defining $\psi_\delta$. We have the equality

$$\Gamma^\psi_\delta(\Delta_{\sigma'}) + (N^\tau \times \mathbb{Z}) = (N^\sigma' + N^\tau) \times \mathbb{Z}$$

of sublattices of $N^\delta \times \mathbb{R}_{\geq 0} = N^\delta \times \mathbb{Z}$. Therefore, $\Gamma^\psi(\sigma')$ has a contribution in the $S(\tau \times \mathbb{R}_{\geq 0})$-component of $q^*\psi_{\Pi^1} \cdot \Gamma_{\psi}(A)$ if and only $\sigma'$ has a contribution in the $S(\tau)$-component of $\psi \cdot A$.

To see that they even contribute with the same weight, we first notice the equality

$$\text{index } (N^\Gamma^\psi(\sigma') + N^\tau \times \mathbb{Z}) = \text{index } (\Gamma^\psi_\delta(\Delta_{\sigma'}) + (N^\tau \times \mathbb{Z})) = \text{index } (N^\tau + N^\sigma')$$

of indices. Since the primitive generator $u_{\Gamma^\psi(\rho)}$ of $\Gamma^\psi(\sigma') \cap (\tau \times \mathbb{R}_{\geq 0})$ is equal to the image $\Gamma^\psi(u_\rho)$ of the primitive generator $u_\rho$ of $\rho$, we also see that

$$q^*\psi_{\Pi^1}(u_{\Gamma^\psi(\rho)}) = q(u_\rho, \psi(u_\rho)) = \psi(u_\rho).$$

Finally, the weight of $\Gamma^\psi(A)$ on $\Gamma^\psi(\sigma')$ is equal to that of $A$ on $\sigma'$ by definition. Hence, the contributions are equal. Combining this with the fact that $S(\tau \times \mathbb{R}_{\geq 0})/(\tau \times \mathbb{R}_{\geq 0})$ is mapped injectively onto $(\sigma' + \tau)/\tau$ by the isomorphism $S(\tau \times \mathbb{R}_{\geq 0}) \rightarrow S(\tau)$ induced by $p$, we conclude that the total contribution of the cones of the form $\Gamma^\psi(\sigma')$ for $\sigma' \in \Sigma'_k$ to $p_*(q^*\psi_{\Pi^1} \cdot \Gamma^\psi(A))$ is $\psi \cdot A$, and with this, we have proven the desired equality.

**Proposition 3.23.** Let $\Sigma$ be a weakly embedded cone complex. Then there is a well-defined bilinear map

$$\text{ClCP}(\Sigma) \times A_*(\Sigma) \rightarrow A_*(\Sigma), \quad (\psi, [A]) \mapsto [\psi \cup A].$$

By abuse of notation, we denote this paring by $\cdot$.

**Proof.** We need to show that $\psi \cup A = 0$ in $A_*(\Sigma)$ for all $\psi \in \text{CP}(\Sigma)$ and $A \in R_*(\Sigma)$. It suffices to show this for $A = p_*(q^*\psi_{\Pi^1} \cdot B)$ for some $B \in Z_*(\Sigma \times \Pi^1)$, where $p$ and $q$ denote the projections to the first and second coordinates. In this case, we have

$$\psi \cup A = p_*(p^*\psi \cup (q^*\psi_{\Pi^1} \cdot A)) = p_*(q^*\psi_{\Pi^1} \cdot (p^*\psi \cup A)) \in R_*(\Sigma),$$

where the first equality uses the projection formula, and the second one uses Proposition 3.16.

**Remark 3.24.** Recall that the intersection product $\psi \cdot A$ of Proposition 3.11 is only defined when $A$ is contained in the finite part of $\Sigma$. That the bilinear pairing of the preceding proposition is its appropriate extension is justified by Proposition 3.22, which states that modulo rational equivalence the cup-product and the $\cdot$-product are essentially the same.
DEFINITION 3.25. Let $\Sigma$ be a weakly embedded cone complex, let $\psi_1, \ldots, \psi_k \in \text{ClCP}(\Sigma)$, and let $A \in A_*(\Sigma)$. Then we define the cycle class

$$\psi_1 \cdot \cdots \cdot \psi_k \cdot A = \prod_{i=1}^{k} \psi_i \cdot A$$

inductively by $\prod_{i=1}^{k} \psi_i \cdot A = \psi_1 \cdot (\prod_{i=2}^{k} \psi_i \cdot A)$, where the base case is the pairing of Proposition 3.23.

4. Tropicalization

In this section, we will tropicalize cocycles and cycles on toroidal embeddings. As already mentioned in Remark 3.3, cocycles will define Minkowski weights, whereas cycles will define tropical cycles on the associated fans by recording the degrees of the intersections with the boundary strata. The same method yields Minkowski weights associated to cocycles on complete toroidal embeddings.

PROPOSITION 4.1. Let $X$ be a complete $n$-dimensional toroidal embedding, and let $c \in A^k(X)$ be a cocycle. Then the $(n-k)$-dimensional weight

$$\omega: \Sigma(X)_{(n-k)} \to \mathbb{Z}, \quad \sigma \mapsto \deg(c \cap [V(\sigma)])$$

satisfies the balancing condition.

Proof. Let $\tau$ be an $(n-k-1)$-dimensional cone of $\Sigma(X)$, and let $\sigma_1, \ldots, \sigma_n$ be the $(n-k)$-dimensional cones containing it. To prove that the balancing condition is fulfilled at $\tau$, it suffices to show that

$$\sum_{i=1}^{n} \omega(\sigma_i)(f, u_{\sigma_i/\tau}) = \left< f, \sum_{i=1}^{n} \omega(\sigma_i)u_{\sigma_i/\tau} \right> = 0$$

for every $f \in (N^X/N^X)^* = (N^X)^\perp$. A rational function $f \in M^X$ is contained in $(N^X)^\perp$ if and only if it is an invertible regular function on $X(\tau)$. Therefore, for every $f \in (N^X)^\perp$, we have $\text{div}(f)|_{X(\sigma_i)} \in M^{\sigma_i} \cap \tau^\perp$. The lattice $M^{\sigma_i} \cap \tau^\perp$ is the group of integral linear functions of the ray $\sigma_i/\tau$, and $u_{\sigma_i/\tau}$ is by definition the image of the primitive generator of $\sigma_i/\tau$ in $N^X/N^X$. It follows that $(f, u_{\sigma_i/\tau})$ is the pairing of $\text{div}(f)|_{X(\sigma_i)}$ with the primitive generator of $\sigma_i/\tau$. By Lemma 2.3, this is equal to the pairing of $\text{div}(f)|_{V(\tau) \cap X(\sigma_i)}$ with the primitive generator of the ray $\sigma_{V(\tau)}^{\text{O}(\sigma_i)}$, which, in turn, is equal to the multiplicity of $\text{div}(f)|_{V(\tau)}$ at $V(\sigma_i)$. Using this, we obtain

$$\sum_{i=1}^{n} \omega(\sigma_i)(f, u_{\sigma_i/\tau}) = \sum_{i=1}^{n} \deg(c \cap [V(\sigma_i)])(f, u_{\sigma_i/\tau})$$

$$= \deg \left( c \cap \sum_{i=1}^{n} (f, u_{\sigma_i/\tau})[V(\sigma_i)] \right) = \deg \left( c \cap [\text{div}(f)|_{V(\tau)}] \right) = 0,$$
where the last equality follows from the fact that \( \text{div}(f)|_{V(\tau)} \) is principal. This finishes the proof. \( \square \)

**Definition 4.2.** Let \( X \) be a complete \( n \)-dimensional toroidal embedding with weakly embedded cone complex \( \Sigma \). We define the tropicalization map

\[
\text{Trop}_X : A^k(X) \to M_{n-k}(\Sigma),
\]

by sending a cocycle on \( X \) to the weight defined as in the preceding proposition. This is clearly a morphism of abelian groups. For a cone \( \sigma \in \Sigma \), and a cocycle \( c \in A^k(V(\sigma)) \), we define \( \text{Trop}_X(c) \in M_{n-\dim(\sigma)-k}(S_\Sigma(\sigma)) \) as the push-forward to \( M_*(S_\Sigma(\sigma)) \) of the Minkowski weight \( \text{Trop}_{V(\sigma)}(c) \in M_*(\Sigma(V(\sigma))). \) We will just write \( \text{Trop}(c) \) when no confusion arises.

**Proposition 4.3.** Let \( X \) be a complete toroidal embedding with weakly embedded cone complex \( \Sigma \), and let \( \tau \in \Sigma \). Furthermore, let \( c \in A^*(X) \) be a cocycle on \( X \), and let \( i : V(\tau) \to X \) and \( j : S(\tau) \to \Sigma \) be the inclusion maps. Then we have the equality

\[
\text{Trop}_X(i^*c) = j^* \text{Trop}_\tau(c)
\]

of Minkowski weights on \( S(\tau) \).

**Proof.** By Lemma 2.3, we have \( O(\sigma/\tau) = O(\sigma) \), hence \( V(\sigma/\tau) = V(\sigma) \). The projection formula implies that the weight of \( \text{Trop}(i^*c) \) at \( \sigma/\tau \) is equal to the weight of \( \text{Trop}(c) \) at \( \sigma \), which is equal to the weight of \( j^* \text{Trop}(c) \) at \( \sigma/\tau \) by Construction 3.5. \( \square \)

**Remark 4.4.** If the canonical morphism \( \Sigma(V(\tau)) \to S(\tau) \) is an isomorphism of weakly embedded cone complexes, the statement of Proposition 4.3 reads

\[
\text{Trop}(i^*c) = \text{Trop}(i)^* \text{Trop}(c).
\]

However, in general, there may be more invertible regular functions on \( O(\tau) \) than obtained as restrictions of rational functions in \( (N^X_\tau)_+ \subseteq M^X \). In this case, the pullback of a Minkowski weight on \( \Sigma \) to \( S(\tau) \) may not be a Minkowski weight on \( \Sigma(V(\tau)) \). In other words, the pullback \( \text{Trop}(i)^* : M_*(\Sigma(X)) \to M_*(\Sigma(V(\tau))) \) may be ill-defined. An example where this happens is when \( X \) is equal to the blowup of \( \mathbb{P}^2 \) in the singular point of a plane nodal cubic \( C \), and its boundary is the union of the exceptional divisor \( E \) and the strict transform \( \tilde{C} \). The cone complex of \( X \) consists of two rays \( \rho_C \) and \( \rho_E \), corresponding to the two boundary divisors, which span two distinct strictly simplicial cones, one for each point in \( E \cap \tilde{C} \). The lattice \( N^X \) is trivial, so that any two integers on the 2-dimensional cones define a Minkowski weight in \( M_2(\Sigma(X)) \). However, the pullback of such a weight to \( S_{\Sigma(X)}(\rho_E) \) defines a Minkowski weight in \( M_1(V(\rho_E)) \) only if the weights on its cones are equal. This is because the toroidal embedding \( O(\rho_E) \subseteq V(\rho_E) = E \) is isomorphic to \( \mathbb{P}^1 \) with two points in the boundary, the weakly embedded cone complex of which consists of two rays embedded into \( \mathbb{R} \).

4.2. **Tropicalizations of cycles on toroidal embeddings**

Let \( X \) be a toroidal embedding with weakly embedded cone complex \( \Sigma \). In [63], Ulirsch constructs a map \( \text{trypo}_X : X^2 \to \Sigma \) as a special case of his tropicalization procedure for fine and saturated logarithmic schemes. We recall that, since \( X \) is separated, \( X^2 \) is the analytic domain of the Berkovich analytification \( X^\text{an} \) consisting of all points that can be represented by an \( R \)-integral point for some rank one valuation ring \( R \) extending \( k \). The notation is due to Thuillier [60]. The map \( \text{trypo}_X \) restricts to a map \( X_0^\text{an} \cap X^2 \to \Sigma \) generalizing the ‘ord’-map \( X_0(k((t))) \cap X(k[t]) \to \Sigma \) from [37]. Ulirsch’s tropicalization map is also compatible with Thuillier’s retraction map \( p_X : X^2 \to X^2 \) [60] in the sense that the retraction factors through
trop\(\mathbb{T}^n\). The tropicalization map allows to define the set-theoretic tropicalization of a subvariety \(Z\) of \(X_d\) as the subset trop\(X(Z) := \text{trop}\(\mathbb{T}^n(Z_{an} \cap X^2)\) of \(\Sigma\). This set has been studied in greater detail in [61], where several parallels to tropicalizations of subvarieties of tori are exposed: first of all, trop\(X(Z)\) can be given the structure of an at most dim\(Z\)-dimensional cone complex whose position in \(\Sigma\) reflects the position of \(Z\) in \(X\). Namely, there is a toroidal version of Tevelev’s lemma [59, Lemma 2.2], stating that trop\(X(Z)\) intersects the relative interior of a cone \(\sigma \in \Sigma\) if and only if \(\mathbb{T}\) intersects \(O(\sigma)\). Furthermore, if trop\(X(Z)\) is a union of cones of \(\Sigma\), then \(\mathbb{T}\) intersects all strata properly. When \(X\) is complete, we use this to give trop\(X(Z)\) the structure of a tropical cycle in a similar way as done in [58] in the toric case. First, we choose a simplicial proper subdivision \(\Sigma'\) of \(\Sigma\) such that trop\(X(Z)\) is a union of its cones. Since char \(k = 0\) by assumption and \(\Sigma'\) is simplicial, the toroidal modification \(X' = X' \times_\Sigma \Sigma'\) is the coarse moduli space of a smooth Deligne–Mumford stack [31, Theorem 3.3], and thus has an intersection product on its Chow group \(A_*(X)\) with rational coefficients [64].

**Definition 4.5.** Let \(d = \dim(Z)\) and \(\sigma \in \Sigma_{(d)}\). We define the multiplicity mult\(Z(\sigma)\) of the cone \(\sigma\) by \(\deg([\mathbb{T}] \cdot [V(\sigma)])\), where \(\mathbb{T}\) denotes the closure of \(Z\) in \(X'\).

**Remark 4.6.** The multiplicity mult\(Z(\sigma)\) is independent of the rest of the cone complex \(\Sigma'\). This is because \(\mathbb{T}\) intersects all strata properly so that \([\mathbb{T}] \cdot [V(\sigma)]\) is a well-defined 0-cycle supported on \(\mathbb{T} \cap O(\sigma)\) and only depending on \(X'(\sigma)\).

**Proposition 4.7.** Let \(\Delta\) be a simplicial proper subdivision of \(\Sigma'\), and let \(\delta \in \Delta_{(d)}\) be a \(d\)-dimensional cone contained in \(\sigma \in \Sigma_{(d)}\). Then mult\(Z(\delta) = \text{mult}_Z(\sigma)\). In particular, we have mult\(Z(\sigma) \in \mathbb{Z}\) for all \(\sigma \in \Sigma'\). Furthermore, the weight

\[
\Sigma'_{\Sigma(d)} \to \mathbb{Z}, \quad \sigma \mapsto \text{mult}_Z(\sigma)
\]

is balanced and the tropical cycle associated to it is independent of the choice of \(\Sigma'\).

**Proof.** To ease the notation, we replace \(X\) and \(X'\) by \(X \times_\Sigma \Sigma'\) and \(X \times_\Sigma \Delta\), respectively, and denote the closures of \(Z\) in \(X\) and \(X'\) by \(\mathbb{T}\) and \(\mathbb{T}'\). Furthermore, we denote the toroidal modification induced by the subdivision \(\Delta \to \Sigma'\) by \(f: X' \to X\). Since \(X\) is an Alexander scheme in the sense of [64], there is a pullback morphism \(f^*: A_*(X) \to A_*(X')\). The pullback \(f^*[\mathbb{T}']\) is represented by a cycle \([X']_f (\mathbb{T}) \in A_d(\text{f}^{-1}\mathbb{T}').\) Since \(\mathbb{T}\) intersects all strata properly, and \(f\) locally looks like a toric morphism, the preimage \(f^{-1}\mathbb{T}\) is \(d\)-dimensional and \(\mathbb{T}'\) is its only \(d\)-dimensional component. Hence, \(f^*[\mathbb{T}']\) is a multiple of \([\mathbb{T}']\). By the projection formula, we have

\[
f_* f^*[\mathbb{T}] = f_* (f^*[\mathbb{T}] \cdot [X']) = [\mathbb{T}] \cdot [X] = [\mathbb{T}],
\]

showing that we, in fact, have \(f^*[\mathbb{T}] = [\mathbb{T}']\). Again, using the projection formula, we obtain

\[
\text{mult}_Z(\delta) = \deg([\mathbb{T}] \cdot [V(\delta)]) = \deg(f_*(f^*[\mathbb{T}] \cdot [V(\delta)]) = \deg([\mathbb{T}] \cdot [V(\sigma)]) = \text{mult}_Z(\sigma).
\]

Since every complex has a strictly simplicial proper subdivision, we can choose \(\Delta\) to be strictly simplicial. In this case, the multiplicity mult\(Z(\delta)\) is defined by the ordinary intersection product on \(A_*(X')\), and hence we have mult\(Z(\sigma) = \text{mult}_Z(\delta) \in \mathbb{Z}\). Similarly, to prove the balancing condition for the weight \(\Sigma_{(d)} \ni \sigma \mapsto \text{mult}_Z(\sigma) \in \mathbb{Z}\), it suffices to show that the induced weight on a strictly simplicial proper subdivision \(\Delta\) is balanced. By what we just saw, this weight is equal to \(\Delta_{(d)} \ni \delta \mapsto \text{mult}_Z(\delta) \in \mathbb{Z}\). But this is nothing but the tropicalization of the cocycle corresponding to \([\mathbb{T}']\) by Poincaré duality, which is balanced by Proposition 4.1. That the
tropical cycle it defines is independent of all choices follows immediately from the fact that any two subdivisions have a common refinement.

The previous result allows us to assign a tropical cycle to the subvariety \( Z \) of \( X_0 \). Its support will be contained in the set-theoretic tropicalization of \( Z \).

\textbf{Definition 4.8.} Let \( X \) be a complete toroidal embedding with weakly embedded cone complex \( \Sigma \), and let \( Z \subseteq X_0 \) be a \( d \)-dimensional subvariety. We define the \textit{tropicalization} \( \trop_X(Z) \in \mathcal{Z}_d(\Sigma) \) as the tropical cycle represented by the tropicalization of the cocycle corresponding to the closure \( \overline{Z'} \) of \( Z \) in a smooth toroidal modification \( X' \) of \( X \) in which \( Z' \) intersects all strata properly. This is well defined by Proposition 4.7. Extending by linearity, we obtain a tropicalization morphism

\[
\trop_X: \mathcal{Z}_*(X) = \bigoplus_{\sigma \in \Sigma} \mathcal{Z}_*(O(\sigma)) \xrightarrow{\bigoplus_{\sigma \in \Sigma} \trop_{\Sigma}(\sigma)} \bigoplus_{\sigma \in \Sigma} \mathcal{Z}_*(S(\sigma)) = \mathcal{Z}_*(\Sigma).
\]

\textbf{Remark 4.9.} Strictly speaking the ‘\( \sigma \)th’ coordinate of \( \trop_X \) is not \( \trop_{\Sigma}(\sigma) \), but the composite \( \mathcal{Z}_*(O(\sigma)) \xrightarrow{\trop_{\Sigma}(\sigma)} \mathcal{Z}_*(\Sigma(V(\sigma))) \to \mathcal{Z}_*(S(\sigma)) \), where the second map is the push-forward induced by the identification of cone complexes of Lemma 2.3.

4.3. \textit{The Sturmfels–Tevelev multiplicity formula}

We now give a proof of the Sturmfels–Tevelev multiplicity formula [58, Theorem 1.1] in the toroidal setting. It has its origin in tropical implicitization and states that push-forward commutes with tropicalization. A version for the embedded case over fields with non-trivial valuation has been proven in [9, 29, 49].

\textbf{Theorem 4.10.} Let \( f: X \to Y \) be a subtoroidal morphism of complete toroidal embeddings, and let \( \alpha \in \mathcal{Z}_*(X) \). Then,

\[
\trop(f)_* \trop_X(\alpha) = \trop_Y(f_* \alpha).
\]

\textbf{Proof.} Let \( \Sigma \) and \( \Delta \) denote the weakly embedded cone complexes of \( X \) and \( Y \), respectively. Since both sides of the equation are linear in \( \alpha \), we may assume that \( \alpha = [Z] \), where \( Z \) is a closed subvariety of \( O(\sigma) \) for some \( \sigma \in \Sigma \), say of dimension \( \dim(Z) = d \). The subtoroidal morphism \( \Sigma(V(\sigma)) \to X \to Y \) factors through a toroidal morphism \( V(\sigma) \to V(\delta) \) for some \( \delta \in \Delta \), and since tropicalization and push-forward commute for closed immersions of closures of strata by definition, this allows us to reduce to the case where \( f \) is toroidal and \( O(\sigma) = X_0 \).

Assume that the dimension of \( f(Z) \) is strictly smaller than that of \( Z \). Since \( \trop(f)(\trop_X(Z)) \) is equal to \( \trop_Y(f(Z)) \) (this follows from the surjectivity of \( \trop_{\Sigma} \to f(\trop_{\Sigma}) \) [10, Proposition 3.4.6] and the functoriality of \( \trop_{\Sigma} \) [63, Proposition 6.2]), this implies that \( \trop(f) \) is not injective on any facet of \( \trop_X(Z) \), hence

\[
\trop(f)_* \trop_X([Z]) = 0 = \trop_Y(0) = \trop_Y(f_* [Z]).
\]

Now assume that \( \dim(f(Z)) = \dim(Z) = d \). We subdivide \( \Sigma \) and \( \Delta \) as follows. First, we take a proper subdivision \( \Sigma' \) of \( \Sigma \) such that \( \trop_{\Sigma'}(Z) \) is a union of cones of \( \Sigma' \). Then we take a strictly simplicial proper subdivision \( \Delta' \) of \( \Delta \) such that the images of cones in \( \Sigma' \) are unions of cones of \( \Delta' \). Pulling back the cones of \( \Delta' \), we obtain a proper subdivision \( \Sigma'' \) of \( \Sigma' \) whose cones map to cones in \( \Delta' \). By successively subdividing along rays (cf. [4, Remark 4.5]), we can achieve a proper subdivision of \( \Sigma'' \) whose cones are simplicial and mapped to cones of \( \Delta \) by \( \trop(f) \). After renaming, we see that there are proper subdivisions \( \Sigma' \) and \( \Delta' \) of \( \Sigma \) and \( \Delta \),
respectively, such that $\Sigma'$ is simplicial, $\Delta'$ is strictly simplicial, Trop($f$) maps cones of $\Sigma'$ onto cones of $\Delta'$, and trop$_X(Z)$ is a union of cones in $\Sigma'$.

Let $X' = X \times \Sigma \Sigma'$ and $Y' = Y \times \Delta \Delta'$ the corresponding toroidal modifications. Then the induced toroidal morphism $f' : X' \to Y'$ is flat by [4, Remark 4.6]. As trop$_Y(f(Z))$ is the image of trop$_X(Z)$, it is a union of cones of $\Delta'$. By construction of the tropicalization, the weight of Trop$_Y(f(Z))$ at a $d$-dimensional cone $\delta \in \Delta'$ is equal to

$$[K(\mathbb{Z}) : K(f(\mathbb{Z}))] \deg([f'(\mathbb{Z})] \cdot [V(\delta)]) = \deg(f'_*([\mathbb{Z}] \cdot [V(\delta)]) = \deg([\mathbb{Z}] \cdot f'^*[V(\delta)]),$$

where the second equality uses the projection formula. The irreducible components of $f'^{-1}V(\delta)$ are of the form $V(\sigma)$, where $\sigma$ is minimal among the cones of $\Sigma'$ mapping onto $\delta$. All of these cones $\sigma$ are $d$-dimensional, and the multiplicity with which $V(\sigma)$ occurs in $f'^{-1}V(\delta)$ is $[N^\delta : \text{Trop}(f)(V^\sigma)]$ as we see by comparing with the toric case using local toric charts. Combining this, we obtain

$$\deg([\mathbb{Z}] \cdot f'^*[V(\delta)]) = \sum_{\sigma \to \delta} [N^\delta : \text{Trop}(f)(V^\sigma)] \deg([\mathbb{Z}] \cdot [V(\sigma)]),$$

where the sum runs over all $d$-dimensional cones of $\Sigma'$ mapping onto $\delta$. Since $\deg([\mathbb{Z}] \cdot [V(\sigma)])$ is the weight of Trop$_X(Z)$ at $\sigma$, the right-hand side of this equation is precisely the multiplicity of Trop$_f \circ$ Trop$_X(Z)$ at $\delta$. \hfill \Box

### 4.4. Tropicalization and intersections with boundary divisors

Let $X$ be a toroidal embedding with weakly embedded cone complex $\Sigma$. By definition of $\Sigma$, the restriction of a Cartier divisor $D$ on $X$ that is supported away from $X_0$ to a combinatorial open subset $X(\sigma)$ for some $\sigma \in \Sigma$ is determined by an integral linear function on $\sigma$. Since the cones of $\Sigma$ and the combinatorial open subsets of $X$ are glued accordingly, $D$ defines a continuous function $\psi : [\Sigma] \to \mathbb{R}$ that is integral linear on all cones of $\Sigma$. Conversely, all such functions define a Cartier divisor on $X$ that is supported away from $X_0$. When $X$ is not toric, the Picard groups of its combinatorial opens need not be trivial. Hence, the restriction $D|_{X(\sigma)}$ is not necessarily of the form div$(f)|_{X(\sigma)}$ for some $f \in K(X)$. But if it is, the rational function $f$ must be regular and invertible on $X_0$, and by definition of the weak embedding, we then have $\psi|_{\sigma} = f \circ \varphi_X$. We see that $D$ is principal on the combinatorial opens of $X$ if and only if $\psi$ is combinatorially principal. By the same argument, we see that another such divisor $D'$ with corresponding tropical cp-divisor $\psi'$ is linearly equivalent to $D$ if and only if $\psi$ and $\psi'$ are linearly equivalent tropical cp-divisors on $\Sigma$.

**Definition 4.11.** Let $X$ be a toroidal embedding with weakly embedded cone complex $\Sigma$. We write Div$_{X_0}(X)$ for the subgroup of Div$(X)$ consisting of Cartier divisors that are supported on $X \setminus X_0$. We denote the tropical Cartier divisor on $X$ corresponding to $D \in$ Div$_{X_0}(X)$ by Trop$_X(D)$. If $D$ is principal on all combinatorial opens, or equivalently, if Trop$_X(D) \in$ CP$(\Sigma)$, we say that $D$ is combinatorially principal (cp). We write CP$(X)$ for the group of cp-divisors on $X$. Furthermore, we write ClCP$(X)$ for the image of CP$(X)$ in Pic$(X)$, and for $\mathcal{L} \in$ ClCP$(X)$, we write Trop$_X(\mathcal{L})$ for its corresponding tropical divisor class in ClCP$(\Sigma)$. If no confusion arises, we usually omit the reference to $X$.

**Lemma 4.12.** Let $f : X \to Y$ be a subtoroidal morphism of toroidal embeddings, and let $\mathcal{L} \in$ ClCP$(Y)$. Then $f^* \mathcal{L} \in$ ClCP$(X)$, and

$$\text{Trop}_X(f^* \mathcal{L}) = \text{Trop}(f)^*(\text{Trop}_Y(\mathcal{L})).$$

**Proof.** It suffices to prove the lemma for toroidal morphisms and closed immersions of closures of strata. The toroidal case follows immediately from the definition of Trop($f$). So,
we may assume that \( X = V(\delta) \) for some \( \delta \in \Delta = \Sigma(Y) \), and \( f: V(\delta) \to Y \) is the inclusion. Let \( D \in \text{CP}(Y) \) be a representative for \( \mathcal{L} \) and write \( \psi = \text{Trop}_Y(D) \). Since \( D \) is in \( \text{CP}(Y) \), there exists \( g \in M^\Delta \) such that \( \psi|_\delta = g \circ \varphi_\Delta|_\delta \). The tropicalization of the divisor \( D - \text{div}(g) \) is \( \psi - g \circ \varphi_\Delta \). This vanishes on \( \delta \) by construction, which means that \( D - \text{div}(g) \) is supported away from \( O(\delta) \). By Lemma 2.3, the tropicalization of the restriction of \( D - \text{div}(g) \) to \( V(\delta) \) is given by the tropical divisor in \( \text{Div}(S(\delta)) = \text{Div}(\Sigma(V(\delta))) \) induced by \( \psi - g \circ \varphi_\Delta \). This finishes the proof because \( (D - \text{div}(g))|_{V(\delta)} \) represents \( f^* \mathcal{L} \), and the divisor on \( S(\delta) \) induced by \( \psi - g \circ \varphi_\Delta \) represents \( \text{Trop}(f)^* \psi \) by Construction 3.8. \( \square \)

**Proposition 4.13.** Let \( X \) be a complete toroidal embedding, let \( \mathcal{L} \in \text{CI}_\text{CP}(X) \), and let \( c \in A^k(X) \). Then

\[
\text{Trop}_X(c_1(\mathcal{L}) \cup c) = \text{Trop}_X(\mathcal{L}) \cup \text{Trop}_X(c).
\]

**Proof.** Let \( n = \dim(X) \). Furthermore, let \( \tau \) be a \((n-k-1)\)-dimensional cone of \( \Sigma = \Sigma(X) \), and let \( \psi \in \text{CP}(\Sigma) \) and \( \psi_\tau \in \text{CP}(\Sigma(V(\tau))) \) be representatives for \( \text{Trop}(\mathcal{L}) \) and \( \text{Trop}(i)^* \text{Trop}(\mathcal{L}) \), respectively, where \( i: V(\tau) \to X \) is the inclusion map. By Lemma 4.12, the tropicalization of \( i^* \mathcal{L} \) is represented by \( \psi_\tau \), hence

\[
\sum_{\tau \prec \sigma} \psi_\tau(u_{\sigma/\tau})[V(\sigma)] = [i^* \mathcal{L}] = c_1(\mathcal{L}) \cap [V(\tau)] \quad \text{in } A_{n-k-1}(X).
\]

Therefore, the weight of \( \text{Trop}(c_1(\mathcal{L}) \cup c) \) at \( \tau \) is

\[
\deg((c_1(\mathcal{L}) \cup c) \cap [V(\tau)]) = \sum_{\tau \prec \sigma} \psi_\tau(u_{\sigma/\tau}) \deg(c \cap [V(\sigma)]).
\]

Since \( \deg(c \cap [V(\sigma)]) \) is the weight of \( \text{Trop}(c) \) at \( \sigma \), this is equal to the weight of \( \psi \cup \text{Trop}(c) \) at \( \tau \) by Construction 3.13. \( \square \)

**Theorem 4.14.** Let \( X \) be a complete toroidal embedding and let \( D \in \text{Div}_{X_0}(X) \). Then for every subvariety \( Z \subseteq X_0 \), we have

\[
\text{Trop}_X(D \cdot [\bar{Z}]) = \text{Trop}_X(D) \cdot \text{Trop}_X(Z).
\]

**Proof.** First, note that \( D \) is supported on \( X \setminus X_0 \) and hence \( D \cdot [\bar{Z}] \) is a well-defined cycle. Let \( \Sigma' \) be a strictly simplicial proper subdivision of \( \Sigma = \Sigma(X) \) such that \( \text{trop}(Z) \) is a union of its cones, and let \( f: X' := X \times_\Sigma \Sigma' \to X \) be the corresponding toroidal modification. The algebraic projection formula and Theorem 4.10 imply that

\[
\text{Trop}_X(D \cdot [\bar{Z}]) = \text{Trop}_X(f_* (f^* D \cdot [\bar{Z}'])) = \text{Trop}(f)_* \text{Trop}_X(f^* D \cdot [\bar{Z}']),
\]

whereas the tropical projection formula and Lemma 4.12 imply that

\[
\text{Trop}_X(D) \cdot \text{Trop}_X(Z) = \text{Trop}_X(D) \cdot (\text{Trop}(f)_* \text{Trop}_X(f^* D \cdot [\bar{Z}']))
\]

\[
= \text{Trop}(f)_* (\text{Trop}_X(f^* D) \cdot \text{Trop}_X(Z)).
\]

This reduces to the case where \( X = X' \) and we may assume that \( \Sigma \) is strictly simplicial and \( \bar{Z} \) intersects all boundary strata properly.

Denote \( \text{Trop}(D) \) by \( \psi \), and let \( \rho \in \Sigma(1) \) be a ray of \( \Sigma \). Since \( X \) is smooth, the boundary divisor \( V(\rho) \) is Cartier. Let \( V(\rho) \cap [\bar{Z}] = [V(\rho) \cap \bar{Z}] = \sum_{i=1}^k a_i [W_i] \). Because \( \bar{Z} \) intersects all strata of \( X \) properly, each \( W_i \) meets \( O(\rho) \). For the same reason, every \( W_i \) meets all strata of \( V(\rho) \) properly. It follows that \( \text{Trop}_X(W_i) \) is represented by the Minkowski weight on \( S(\rho) \) whose weight on a cone \( \sigma/\rho \) is equal to \( \deg([V(\sigma)] \cdot [W_i]) \), where the intersection product is taken in \( V(\rho) \). Denote
by \( i_\rho \) the inclusion of \( V(\rho) \) into \( X \). Then \( V(\rho) \cdot [Z] = i_\rho^*[Z] \) in \( A_*(V(\sigma)) \), and hence the weight of \( \text{Trop}_X(V(\rho) \cdot [Z]) \) at \( \sigma/\rho \) is equal to

\[
\sum_{i=1}^k a_i \deg([V(\sigma)] \cdot [W_i]) = \deg([V(\sigma)] \cdot i_\rho^*[Z]) = \deg([V(\sigma)] \cdot [Z]),
\]

where the first two intersection products are taken in \( V(\sigma) \), whereas the last one is taken in \( X \), and the last equality follows from the projection formula. Since \( \deg([V(\sigma)] \cdot [Z]) \) is the weight of \( \text{Trop}_X(Z) \) at \( \sigma \), this implies the equality

\[
\text{Trop}_X(V(\rho) \cdot [Z]) = j_\rho^* \text{Trop}_X(Z),
\]

where \( j_\rho : S(\rho) \to \Sigma \) is the inclusion. Together with the fact that the divisor \( V(\rho) \) occurs in \( D \) with multiplicity \( \psi(u_\rho) \), where \( u_\rho \) denotes the primitive generator of \( \rho \), we obtain

\[
\text{Trop}_X(D \cdot [Z]) = \sum_{\rho \in \Sigma(1)} \psi(u_\rho) \text{Trop}_X(V(\rho) \cdot [Z]) = \sum_{\rho \in \Sigma(1)} \psi(u_\rho) j_\rho^* \text{Trop}_X(Z),
\]

which is equal to \( \psi \cdot \text{Trop}_X(Z) \) by Proposition 3.11.

4.5. Tropicalizing cycle classes

As already pointed out in Remark 3.19, rational equivalence for cycles on weakly embedded extended cone complexes is defined very similarly as rational equivalence in algebraic geometry. In fact, now that we know that tropicalization respects push-forwards and intersections with boundary divisors, it is almost immediate that it respects rational equivalence as well. The only thing still missing is to relate the weakly embedded cone complex \( \Sigma(X \times Y) \) of the product of two toroidal embeddings \( X \) and \( Y \) to the product \( \Sigma(X) \times \Sigma(Y) \) of their weakly embedded cone complexes. First, note that by combining local toric charts for \( X \) and \( Y \), it is easy to see that \( X_0 \times Y_0 \subseteq X \times Y \) really is a toroidal embedding. With the same method, we see that the cone complexes \( \Sigma(X \times Y) \) and \( \Sigma(X) \times \Sigma(Y) \) are naturally isomorphic, the isomorphism being the product \( \text{Trop}(p) \times \text{Trop}(q) \) of the tropicalizations of the projections from \( X \times Y \). That this even is an isomorphism of weakly embedded cone complexes, that is, that \( N^{X \times Y} \to N^X \times N^Y \) is an isomorphism as well, follows from a result by Rosenlicht which states that the canonical map

\[
\Gamma(X_0, \mathcal{O}_X^*) \times \Gamma(Y_0, \mathcal{O}_Y^*) \to \Gamma(X_0 \times Y_0, \mathcal{O}_{X \times Y}^*)
\]

is surjective [39, Section 1].

**Proposition 4.15.** Let \( X \) be a complete toroidal embedding with weakly embedded cone complex \( \Sigma \). Then the tropicalization \( \text{Trop}_X : Z_*(X) \to Z_*(\Sigma) \) induces a morphism \( A_*(X) \to A_*(\Sigma) \) between the Chow groups, which we again denote by \( \text{Trop}_X \).

**Proof.** By definition, the Chow group \( A_k(X) \) is equal to the quotient of \( Z_k(X) \) by the subgroup \( R_k(X) \) generated by cycles of the form \( p_*([q^*([0] - [\infty])] \cdot [W]) \), where \( p \) and \( q \) are the first and second projections from the product \( X \times \mathbb{P}^1 \), and \( W \) is an irreducible subvariety of \( X \times \mathbb{P}^1 \) mapping dominantly to \( \mathbb{P}^1 \). Considering \( \mathbb{P}^1 \) with its standard toric structure, the projections are toroidal morphisms. The boundary divisor \( [0] - [\infty] \) is given by the identity on the cone complex \( \Pi^1 \) of \( \mathbb{P}^1 \) when considering its natural identification with \( \mathbb{R} \). This tropical divisor was denoted by \( \psi_{\Pi^1} \) in Subsection 3.4. It follows that the tropicalization of the Cartier divisor \( q^*([0] - [\infty]) \) is equal to \( \text{Trop}(q)^*(\psi_{\Pi^1}) \). Applying Theorems 4.10 and 4.14, we obtain that

\[
\text{Trop}_X(p_*([q^*([0] - [\infty])] \cdot [W])) = \text{Trop}(p)_* (\text{Trop}(q)^*(\psi_{\Pi^1} \cdot \text{Trop}_{X \times \mathbb{P}^1}(W))).
\]
The tropicalization $Trop_{X \times \mathbb{P}^1}(W)$ and its intersection with $\text{pr}_2^* \psi_{\Pi^1}$.

Noting that $\text{Trop}(p)$ and $\text{Trop}(q)$ are the projections from $\Sigma(X \times \mathbb{P}^1) = \Sigma \times \Pi^1$, and that the dominance of $W \to \mathbb{P}^1$ implies $\text{Trop}(W) \in \mathbb{Z}_k(\Sigma \times \mathbb{P}^1)(\sigma \times 0)$ for some $\sigma \in \Sigma$, we see that we have an expression exactly as given for the generators of $R_k(\Sigma)$ in Definition 3.17. Thus, the tropicalization $Trop_X$ maps $R_k(X)$ to zero in $A_k(\Sigma)$ and the assertion follows.

**Example 4.16.** Consider the toroidal embedding $X$ of Example 2.2(b), that is, $X = \mathbb{P}^2$, and the boundary is the union $H_1 \cup H_2$, where $H_i = V(x_i)$. Its weakly embedded cone complex $\Sigma$ is shown in Figure 1. Let $L_1 = V(x_1 + x_2)$, and $L_2 = V(x_1 + x_2 - x_0)$. The line $L_2$ intersects all strata properly as it does not pass through the intersection point $P = (1 : 0 : 0)$ of $H_1$ and $H_2$. Because $L_2$ intersects $H_1$ and $H_2$ transversally, this implies that $\text{Trop}(L_2)$ is represented by the Minkowski weight on $\Sigma$ having weight 1 on both of its rays. In contrast, $L_1$ passes through $P$. Its strict transform in the blowup of $X$ at $P$ intersects the exceptional divisor, and the intersection is, in fact, transversal, but none of the strict transforms of $H_1$ or $H_2$. The blowup at $P$ is the toroidal modification corresponding to the star subdivision of $\Sigma$. We see that $\text{Trop}(L_2)$ is the tropical cycle given by a ray in the direction of $e_1 + e_2$ with multiplicity 1, where the $e_i$ are as in Example 2.2(b). By Proposition 4.15, the two tropical cycles $\text{Trop}(L_1)$ and $\text{Trop}(L_2)$ are rationally equivalent. The proposition also tells us that the tropicalization of the closure of the graph of $(x_1 + x_2)/(x_1 + x_2 - x_0)$ in $X \times \mathbb{P}^1$ will give rise to the relation $\text{Trop}(L_1) - \text{Trop}(L_2) = 0$ in $A_*(\Sigma)$. Taking coordinates $x = x_1/x_0$ and $y = x_2/x_0$ on $X$, and a coordinate $z$ on $\mathbb{P}^1 \setminus \{\infty\}$, the graph $W$ is given by the equation $W = V(xz + yz - z - x - y)$. Using these coordinates, it is easy to compute the classical tropicalization of $W$ when considered as subvariety of $\mathbb{P}^2 \times \mathbb{P}^1$ with its standard toric structure. It is not hard to see that $\text{Trop}_{X \times \mathbb{P}^1}(W)$ can be obtained from the classical tropicalization by intersecting with $(\mathbb{R}_{\geq 0})^2 \times \mathbb{R}$. Its underlying set is depicted in Figure 2. The weights on its maximal faces are all 1.

Being able to tropicalize cycle classes, we can formulate the compatibility statements of Sections 4.3 and 4.4 modulo rational equivalence. That the tropicalization of cycle classes commutes with push-forwards follows immediately from Theorem 4.10. For intersections with cp-divisors, the compatibility is the subject of the following result.

**Proposition 4.17.** Let $X$ be a complete toroidal embedding, let $L \in \text{ClCP}(X)$, and let $\alpha \in A_*(X)$. Then we have

$$\text{Trop}(c_1(L) \cap \alpha) = \text{Trop}(L) \cdot \text{Trop}(\alpha).$$
**Proof.** Both sides are linear in $\alpha$, so we may assume that $\alpha = [Z]$ for some irreducible subvariety $Z$ of $X$. Let $\sigma \in \Sigma(X)$ be the cone such that $O(\sigma)$ contains the generic point of $Z$, and denote by $i : V(\sigma) \to X$ the inclusion map. Furthermore, let $D \in \text{CP}(V(\sigma))$ be a representative of $i^* \mathcal{L}$. By Theorem 4.10 and the projection formula, we have

$$\text{Trop}_X (c_1(L) \cap [Z]) = \text{Trop} (i)_* \text{Trop}_V (\sigma) (c_1(i^* \mathcal{L}) \cap [Z]) = \text{Trop} (i)_* \text{Trop}_V (\sigma) (D \cdot [Z]).$$

By Lemma 4.12 and Theorem 4.14, this is equal to

$$\text{Trop} (i)_* \left( (\text{Trop} (i)_* \text{Trop}_{V(\sigma)} (\mathcal{L}) \cdot \text{Trop}_{V(\sigma)} (Z \cap O(\sigma))) \right),$$

which is equal to $\text{Trop}_X (\mathcal{L}) \cdot \text{Trop}_X ([Z])$ by the tropical projection formula (Proposition 3.15) and the definition of the tropicalization. □

If $X$ is a smooth and complete toroidal embedding, we now have two possibilities to tropicalize cycle classes on $X$. Given $\alpha \in A^*(X)$, we can either take its tropicalization $\text{Trop}_X (\alpha) \in A^*(\Sigma)$, as introduced in Proposition 4.15, or we can first take the cocycle $c \in A^*(X)$ corresponding to $\alpha$ by Poincaré duality, then take the tropicalization $\text{Trop}_X (c) \in M_*(\Sigma)$ thereof (defined in Definition 4.2), and then take the class in $A_*(\Sigma)$ of the tropical cycle it represents. The following proposition shows that we obtain the same result in both cases — at least under the additional hypothesis of $X$ being projective that we need to be able to use the moving lemma.

**Proposition 4.18.** Let $X$ be a smooth projective toroidal embedding with weakly embedded cone complex $\Sigma$. Then the diagram

$$
\begin{array}{ccc}
A^*(X) & \xrightarrow{\text{Trop}_X} & M_*(\Sigma) \\
- \cap [X] & \downarrow & \downarrow \\
A_*(X) & \xrightarrow{\text{Trop}_X} & A_*(\Sigma),
\end{array}
$$

where the upper Tropicalization map is the one from Definition 4.2 and the lower Tropicalization map is the one from Proposition 4.15, commutes.

**Proof.** Let $c \in A^k(X)$. By Chow’s moving lemma (see [21, Section 11.4] and references therein), the cycles class $c \cap [X]$ can be represented by a sum $\sum_{i=1}^l a_i [V_i]$, where $a_i \in \mathbb{Z}$ and each $V_i$ is a $(\dim X - k)$-dimensional subvariety of $X$ that meets all boundary strata properly. Let $c_i \in A^k(X)$ denote the cocycle corresponding to $V_i$ by Poincaré duality. Then by definition, $\text{Trop}_X (c \cap [X])$ is represented by the cycle corresponding to the Minkowski weight

$$\sum_{i=1}^l a_i \text{Trop}_X (c_i),$$

which clearly equals $\text{Trop}_X (c)$. □

### 4.6. Comparison with classical tropicalization

Suppose we are given a complete toroidal embedding $X$, together with a closed immersion $\iota : X_0 \to T$ into an algebraic torus $T$ with lattice of 1-parameter subgroups $\Lambda$ and character lattice $\Lambda^\vee$. Then we have two different ways to tropicalize $X_0$, either by taking the tropicalization $\text{Trop}_X (X_0) = \text{Trop}_X ([X])$ in $\Sigma(X)$, or by taking the classical tropicalization in $\Lambda^\vee$. Every morphism $\iota : X_0 \to T$ induces a morphism

$$f : \Lambda^\vee \to M^X, \quad m \mapsto \iota^* \chi^m,$$
whose dualization gives rise to a weakly embedded cone complex $\Sigma_\iota(X)$ with cone complex $\Sigma(X)$ and weak embedding $\varphi_{X,\iota} = f_\iota^* \circ \varphi_X$. We define the tropicalization $\text{Trop}_\iota(X_0)$ of $X_0$ with respect to $\iota$ as the classical tropicalization of the cycle $\iota_*[X_0]$ in $T$. Here, by the classical tropicalization of a cycle $\alpha \in Z_\iota(T)$, we meant the cycle $\text{Trop}_Y(\alpha)$ in the sense of Definition 4.8, where $Y$ is any complete toric variety with big open torus $T$. This is an element of $\lim_{\alpha \to \Sigma} M_\iota(\Sigma)$, whose colimit is taken over all complete fans $\Sigma$ in $\Lambda_R$. It is independent of the choice of $Y$ \cite[Theorem 6.7.7]{42}. Combining \cite[Theorem 6.7.7]{42} with \cite[Theorem 3.2.3]{42}, we see that its underlying set $|\text{Trop}_\iota(X_0)|$ is the image of the composite map

$$X_0^\text{an} \xrightarrow{\iota^\text{an}} T^\text{an} \xrightarrow{\text{trop}^\text{an}_\iota} \Lambda_R,$$

where $\text{trop}^\text{an}_\iota$ takes coordinate-wise minus-log-absolute values. As the next lemma shows, this is completely determined by $\Sigma_\iota$:

**Lemma 4.19** (cf. \cite[Proposition 7.1]{63}). Let $X$ be a toroidal embedding, and let $\iota : X_0 \to T$ be a morphism into the algebraic torus $T$. Then the diagram

$$
\begin{array}{ccc}
X_0^\text{an} \cap X^\Sigma & \xrightarrow{\text{trop}^\Sigma_X} & \Sigma_\iota(X) \\
\downarrow \iota^\text{an} & & \downarrow \varphi_{X,\iota} \\
T^\text{an} & \xrightarrow{\text{trop}^\text{an}_\iota} & \Lambda_R
\end{array}
$$

is commutative. In particular, if $X$ is complete, we have $|\text{Trop}_\iota(X_0)| = \varphi_{X,\iota}(|\Sigma_\iota(X)|)$.

**Proof.** Let $x \in X_0^\text{an} \cap X^\Sigma$, and let $\sigma \in \Sigma_\iota(X)$ be the cone such that the reduction $r(x)$, that is, the image of the closed point of the spectrum of a rank-1 valuation ring $R$ under a representation $\text{Spec } R \to X$ of $x$, is contained in $O(\sigma)$. By construction of $\text{trop}^\Sigma_X$ (cf. \cite[p. 424; 63, Definition 6.1]{60}), the point $\text{trop}^\Sigma_X(x)$ is contained in $\sigma$ and its pairing with a divisor $D \in M^\sigma$ is equal to $-\log |f(x)|$ for an equation $f$ for $D$ around $r(x)$. It follows that the pairing of $\varphi_{X,\iota}(\text{trop}^\Sigma_X(x))$ with $m \in \Lambda^\vee$ is equal to $-\log |\xi^\text{an}_\iota(x^m(x))|$. This is clearly equal to $(m, \text{trop}^\Sigma_\iota(\iota^\text{an}(x)))$. The ‘in particular’ statement follows, since for complete $X$, we have $X^\Sigma = X^\text{an}$, and the tropicalization map $\text{trop}^\Sigma_X$ is always surjective \cite[Proposition 3.11]{60}. \qed

**Remark 4.20.** The set-theoretic equality of the ‘in particular’-part of the previous lemma is an instance of geometric tropicalization, that is, a situation where one can read off the tropicalization of a subvariety of an algebraic torus from the structure of the boundary of a suitable compactification. It has already been pointed out in \cite{41} how the methods for geometric tropicalization developed in \cite{30} can be used to obtain the equality $|\text{Trop}_\iota(X_0)| = \varphi_{X,\iota}(|\Sigma_\iota(X)|)$ for a complete toroidal embedding $X$.

The tropicalization $|\text{Trop}_\iota(X_0)|$ is a union of cones and hence the underlying set of arbitrarily fine embedded cone complexes, that is, fans, in $\Lambda_R$. Similarly as in Proposition 3.10, there exists such a fan $\Delta$ and a proper subdivision $\Sigma'$ of $\Sigma_\iota(X)$ such that $\varphi_{X,\iota}$ induces a morphism $\Sigma' \to \Delta$ of weakly embedded cone complexes. Therefore, there is a push-forward morphism

$$Z_\iota(\Sigma(X)) \to Z_\iota(\Sigma_\iota(X)) = Z_\iota(\Sigma') \to Z_\iota(\Delta) = Z_\iota(\text{Trop}_\iota(X_0)),$$

where $Z_\iota(\text{Trop}_\iota(X_0))$ denotes the group of affine tropical cycles in $\text{Trop}_\iota(X_0)$ \cite[Definition 2.15]{7}. Again, as in Proposition 3.10, we see that this morphism is independent of the choices of $\Sigma'$ and $\Delta$, and we denote it by $(\varphi_{X,\iota})_*$. 


THEOREM 4.21. Let $X$ be a complete toroidal embedding, and let $\iota : X_0 \to T$ be a morphism into the algebraic torus $T$. Then for every subvariety $Z$ of $X_0$, we have

$$\text{Trop}_X(Z) = \text{Trop}_{\iota}(Z),$$

(4.1)

where $\text{Trop}_{\iota}(Z)$ denotes the classical tropicalization of the cycle $\iota_*[Z]$ in $T$.

Proof. Since both sides of the equality are invariant under toroidal modifications, we may assume from the start that $\text{Trop}_X(Z)$ is a union of cones of $\Sigma(X)$. Let $d = \dim(Z)$. It follows from Lemma 4.19 that both sides of (4.1) vanish if the dimension of $\iota(Z)$ is strictly smaller than $d$. Hence, we may assume that $\dim(\iota(Z)) = d$. Let $\Delta$ be a complete strictly simplicial fan in $\Lambda_\mathbb{R}$ such that $|\text{Trop}_{\iota}(Z)|$, as well as $\varphi_{X,\iota}(\sigma)$ for every $\sigma \in \Sigma(X)$, is a union of cones of $\Delta$. Similarly as described in Proposition 3.10, we obtain a simplicial proper subdivision $\Sigma'$ of $\Sigma(X)$ such that images of cones of $\Sigma'$ under $\varphi_{X,\iota}$ are cones in $\Delta$. Namely, we can take suitable star-subdivisions of cones of the form $\varphi_{X,\iota}^{-1} \delta \cap \sigma$ for $\delta \in \Delta$ and $\sigma \in \Sigma(X)$ along rays. Let $X' = X \times_{\Sigma(X)} \Sigma'$, and let $Y$ denote the toric variety associated to $\Delta$. Whenever $\delta \in \Delta$ and $\sigma' \in \Sigma'$ such that $\varphi_{X,\iota}(\sigma') \subseteq \delta$, it follows from the definitions of $\text{div}(\iota_{\sigma'}^* \delta) |_{X'} \in M_{\mathbb{R}}^+$ for every $m \in \delta' \cap \Lambda'$. In particular, $\iota_{\sigma'}^* \delta$ is regular on $X'(\sigma')$, showing that $\iota$ extends to a morphism $X'(\sigma') \to U_\delta$, where $U_\delta$ is the affine toric variety associated to $U_\delta$. These morphisms glue and give rise to an extension $\iota : X' \rightarrow Y$ of $\iota$. If $\sigma' \in \Sigma'$, and $\delta \in \Delta$ is minimal among the cones of $\Delta$ containing $\varphi_{X,\iota}(\sigma')$, then $\iota(\text{Trop}_X(\sigma'))$ is contained in $\text{Trop}_{\iota}(\delta)$. Therefore, the preimages of torus orbits in $Y$ are unions of strata of $X$. Let $\delta \in \Delta(d)$. By [36, Lemma 2.3], the weight of $\text{Trop}_{\iota}(Z)$ at $\delta$ is equal to $\deg([V_Y(\delta)] \cdot \iota^*[\overline{Z}])$, where we denote by $\overline{Z}$ the closure of $Z$ in $X'$. Writing $D_1, \ldots, D_d$ for the boundary divisors associated to the rays $\rho_1, \ldots, \rho_d$ of $\delta$, this can be written as

$$\deg(D_1 \cdots D_d \cdot \iota^*[\overline{Z}]) = \deg(\iota_{\rho_1}^* D_1 \cdots \iota_{\rho_d}^* D_d \cdot [\overline{Z}])$$

by the projection formula. By the combinatorics of $\iota'$, the support of $\iota_{\rho_i}^* D_i$ is the union of all strata $\text{O}_{X'}(\sigma')$ associated to cones $\sigma' \in \Sigma'$ containing a ray that is mapped onto $\rho_i$. Hence, the intersection $|\iota_{\rho_1}^* D_1| \cap \cdots \cap |\iota_{\rho_d}^* D_d|$ has pure codimension $d$ with components $V_{X'}(\sigma')$ corresponding to the $d$-dimensional cones of $\Sigma'$ mapping onto $\delta$. Let $\sigma'$ be such a cone. The restriction $(\iota_{\rho_i}^* D_i)|_{X'(\sigma')}$ is equal to $[N^{\sigma'} : \varphi_{X,\iota}(N^{\rho_i})] \cdot V_{X'}(\rho_i)$, where $\rho_i$ is the ray of $\sigma'$ mapping onto $\rho_i$. By comparing with a local toric model, we see that $V_{X'}(\rho_1') \cdots V_{X'}(\rho_d') = \text{mult}(\sigma')^{-1} [V_{X'}(\sigma')]$ [16, Lemma 12.5.2], where the multiplicity $\text{mult}(\sigma')$ is the index of the sublattice of $N^{\sigma'}$ generated by the primitive generators of the rays $\rho_1', \ldots, \rho_d'$ of $\sigma'$. Thus, the multiplicity of $[V_{X'}(\sigma')]$ in $\iota_{\rho_1}^* D_1 \cdots \iota_{\rho_d}^* D_d$ is equal to

$$\frac{1}{\text{mult}(\sigma')} \prod_{i=1}^d [N^{\rho_i} : \varphi_{X,\iota}(N^{\rho_i})].$$

We easily convince ourselves that this is nothing but the index $[N^\delta : \varphi_{X,\iota}(N^{\sigma'})]$. It follows that the weight of $\text{Trop}_{\iota}(Z)$ at $\delta$ is equal to

$$\sum_{\sigma' \mapsto \delta} [N^\delta : \varphi_{X,\iota}(N^{\sigma'})] \deg([V_{X'}(\sigma')] \cdot [\overline{Z}]),$$

where the sum is taken over all $d$-dimensional cones $\sigma' \in \Sigma'$ that are mapped onto $\delta$ by $\varphi_{X,\iota}$. Since $\deg([V_{X'}(\sigma')] \cdot [\overline{Z}])$ is the weight of $\text{Trop}_X(Z)$ at $\sigma'$ by definition, the desired equality follows from the construction of the push-forward.

\square
Corollary 4.22. Let $X$ be a complete toroidal embedding, and let $\iota : X_0 \to T$ be a closed immersion into the algebraic torus $T$. Then the weight of $\text{Trop}_X(X_0)$ at a generic point $w$ of its support $\varphi_{X_0}(\Sigma(X))$ is equal to
\[
\sum_\sigma \text{index}(\varphi_{X_0}(N^\sigma)),
\]
where the sum is taken over all $\dim(X)$-dimensional cones of $\Sigma(X)$ whose image under $\varphi_{X_0}$ contains $w$.

Remark 4.23. Corollary 4.22 can also be proven using Cueto’s multiplicity formula for geometric tropicalization [17, Theorem 2.5].

4.7. Linear subvarieties of tori and the tropical Chow ring

While the tropical Chow group $A_*(\Sigma)$ of a weakly embedded extended cone complex $\Sigma$ allows the intersection-theoretic operations we desire, it can a priori be a very large group. In this section, we will show that if $\Sigma$ is a projective (that is, the associated toric variety is projective) strictly simplicial embedded cone complex, supported on the tropical linear space associated to a matroid $M$ that is realizable over $k$ (we refer to [42, Section 4.2] for an introduction to tropical linear spaces), then $A_*(\Sigma)$ is finitely generated. In fact, we will show that $A_*(\Sigma)$ is isomorphic to $M_*(\Sigma)$. To prove this, we will make use of the tropicalization map. Since the matroid $M$ is realizable, there exists an ideal $I \leq k[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$ generated by affine linear functions such that $M$ is realized by the vectors $1, x_1, \ldots, x_d$ in $k[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]/I$. Let $X_0 = \text{Spec} k[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]/I$ be the subvariety of the torus $T = \text{Spec} k[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$, and let $\iota : X_0 \to T$ denote the inclusion map. Then by [42, Proposition 4.1.6], we have $|\Sigma| = |\text{Trop}_X(X_0)|$. Let $X$ be the closure of $X_0$ in the smooth projective toric variety $Y_\Sigma$ with big open torus $T$ that corresponds to the fan $\Sigma$. Then by [59, Theorem 1.5] (reduce to the case where $M$ is connected), $X_0 \subseteq X$ is a smooth projective toroidal embedding. Since $X$ intersects each boundary stratum of $Y_\Sigma$ in precisely one component [26, Proposition 3.5], it follows that $\Sigma = \Sigma_i(X)$.

We want to show that the tropicalization map
\[
A_*(X) \xrightarrow{\text{Trop}_X} A_*(\Sigma(X)) \to A_*(\Sigma(X)) = A_*(\Sigma),
\]
which, we will denote by $\text{Trop}_X$ again by abuse of notation, is an isomorphism. We first prove its surjectivity.

Proposition 4.24. The tropicalization $\text{Trop}_X : A_*(X) \to A_*(\Sigma)$ is surjective.

Proof. Let $A \in A_*(\Sigma)$. Since the boundary strata of $X$ are again given by affine linear equations in appropriate coordinates [26, Proposition 3.5], we may assume by induction on $\dim X$ that $A$ is represented by a Minkowski weight $c$ on a strictly simplicial projective subdivision $\Sigma'$ of $\Sigma$. Let $X'$ be the closure of $X_0$ in the toric variety $Y_{\Sigma'}$ corresponding to $\Sigma'$. Then by [26, Theorem 1.1], there exists a cocycle $d \in A^*(X')$ such that $\text{Trop}_{X'}(d) = c$. It now follows from Propositions 4.10 and 4.18 that
\[
\text{Trop}_X(f_*(d \cap [X'])) = A,
\]
where $f : X' \to X$ is the morphism induced by the toric modification $Y_{\Sigma'} \to Y_\Sigma$. \qed

To show injectivity, it is convenient to change our point of view and consider tropicalizations of cocycles instead of cycles. We can do this, as Proposition 4.18 guarantees that $\text{Trop}_X : A_*(X) \to A_*(\Sigma)$ is an isomorphism if and only if the composite
$A^*(X) \xrightarrow{Trop_X} M_*(\Sigma) \to A_*(\overline{\Sigma})$ is an isomorphism. It is already known that the tropicalization $Trop_X : A^*(X) \to M_*(\Sigma)$ of cocycles is an isomorphism [26, Theorem 1.1], so it suffices to show that the canonical morphism $M_*(\Sigma) \to A_*(\overline{\Sigma})$ is injective. To do so, we use the intersection product with divisors. Note that the pairings

$$\text{ClCP}(\Sigma) \times M_*(\Sigma) \to M_*(\Sigma), \ (\psi, c) \mapsto \psi \cup c$$

$$\text{ClCP}(\Sigma) \times A_*(\overline{\Sigma}) \to A_*(\overline{\Sigma}), \ (\psi, A) \mapsto \psi \cdot A$$

from Construction 3.13 and Proposition 3.23, respectively, give $M_*(\Sigma)$ and $A_*(\Sigma)$ natural $\text{Sym}(\text{ClCP}(\Sigma))$-module structures. By construction of the module structures and Proposition 4.24, the canonical morphism $M_*(\Sigma) \to A_*(\Sigma)$ is an epimorphism of $\text{Sym}(\text{ClCP}(\Sigma))$-modules.

For every ray $\rho \in \Sigma_{(1)}$, there is a distinguished divisor $\psi_\rho \in \text{Div}(\Sigma)$, characterized by the equalities

$$\psi_\rho(u_{\rho'}) = \begin{cases} 1 & \text{if } \rho = \rho', \\ 0 & \text{if } \rho \neq \rho'. \end{cases}$$

We denote by $\overline{\psi}_\rho$ the class of $\psi_\rho$ in $\text{ClCP}(\Sigma)$. It is not hard to see that if $\rho_1, \ldots, \rho_k$ are distinct rays that do not span a cone of $\Sigma$, then $\prod_{i=1}^k \overline{\psi}_{\rho_i}$ annihilates $M_*(\Sigma)$. Let $\mathcal{I}$ denote the ideal in $\text{Sym ClCP}(\Sigma)$ generated by the products of the form $\prod_{i=1}^k \overline{\psi}_{\rho_i}$, where the $\rho_i$'s are distinct rays of $\Sigma$ that do not span a cone of $\Sigma$. We define $A^*(\Sigma) := \text{Sym ClCP}(\Sigma)/\mathcal{I}$. This is precisely what is called the Chow ring of $\Sigma$ in [5]. Then $M_*(\Sigma)$ and hence $A_*(\overline{\Sigma})$ is an $A^*(\Sigma)$-module, and the canonical morphism $M_*(\Sigma) \to A_*(\overline{\Sigma})$ is a morphism of $A^*(\Sigma)$-modules. Note that the $A^*(\Sigma)$-action on $M_*(\Sigma)$ is precisely the one considered in [5].

**Remark 4.25.** The Chow ring $A^*(\Sigma)$ has a different description in terms of piecewise polynomials, which have been related to tropical intersection theory in [35] and [18]. If $\text{PP}(\Sigma)$ denotes the ring of continuous functions on $|\Sigma|$ that restrict to integer polynomials on each cone, then there is a natural morphism $\text{Sym}(\text{CP}(\Sigma)) \to \text{PP}(\Sigma)$ induced by the inclusion $\text{CP}(\Sigma) \to \text{PP}(\Sigma)$. A result by Brion [12] says that this map is a ring epimorphism, whose kernel is generated by the monomials $\prod_{i=1}^k \psi_{\rho_i}$, where $\rho_1, \ldots, \rho_k$ are distinct rays that do not span a cone of $\Sigma$ (Brion shows this for simplicial fans and polynomials with $\mathbb{Q}$-coefficients, but his proofs carry over to strictly simplicial fans and polynomials with integer coefficients). Therefore, $A^*(\Sigma)$ is naturally isomorphic to $\text{PP}(\Sigma)$ modulo the ideal generated by $M^Z$. We believe that this can be used to also define a Chow ring if $\Sigma$ is an arbitrary weakly embedded cone complex, namely, as the image of $\text{Sym}(\text{ClCP}(\Sigma))$ in $\text{PP}(\Sigma)/(M^Z)$. However, it is not clear if this acts on $M_*(\Sigma)$ or $A_*(\overline{\Sigma})$ in general, and we do not study this in this paper.

**Proposition 4.26** [5, Proposition 5.6]. The morphism

$$M_k(\Sigma) \to \text{Hom}(A^k(\Sigma), \mathbb{Z})$$

obtained by identifying $M_0(\Sigma)$ with $\mathbb{Z}$ is an isomorphism.

This can be used to prove the injectivity of $M_*(\Sigma) \to A_*(\overline{\Sigma})$:

**Proposition 4.27.** The morphism $M_*(\Sigma) \to A_*(\overline{\Sigma})$ is an isomorphism of $A^*(\Sigma)$-modules.

**Proof.** Denote the morphism $M_*(\Sigma) \to A_*(\overline{\Sigma})$ by $\pi$. Suppose $\pi(c) = 0$. Since $\pi$ is a graded morphism, we may assume that $c \in M_k(\Sigma)$ for some $k \in \mathbb{N}$. Then for all $\psi \in A^k(\Sigma)$, we have

$$0 = \deg(\psi \cdot \pi(c)) = \deg \pi(\psi \cdot c).$$
As the composite \( M_0(\Sigma) \xrightarrow{\pi} A_0(\Sigma) \xrightarrow{\text{deg}} \mathbb{Z} \) is an isomorphism, this shows that \( \psi \cdot c = 0 \) for all \( \psi \in A^k(\Sigma) \). But this implies that \( c = 0 \) by Proposition 4.26. Together with the observation from above that \( \pi \) is an epimorphism, the result follows. \( \square \)

**Remark 4.28.** The result of Proposition 4.26 proved by Adiprasito, Huh, and Katz does not rely on the fact that the support of \( \Sigma \) is the Bergman fan of a matroid, but merely uses that \( \Sigma \) is a strictly simplicial fan. Therefore, the argument in the preceding proof applies more generally and shows that \( M_*(\Sigma) \to A_*(\Sigma) \) is injective for all strictly simplicial embedded cone complexes \( \Sigma \).

Because \( \Sigma \) is supported on the tropical linear space of the matroid \( \mathcal{M} \), the group of cycles \( Z_*(\Sigma) \) has a natural graded ring structure given by the tropical intersection product \([19, 55]\). By \([55\text{, Proposition 3.13}]\), \( M_*(\Sigma) \) is a subring of \( Z_*(\Sigma) \). In fact, \( Z_*(\Sigma) \) is a graded \( A^*(\Sigma) \)-algebra by \([19\text{, Theorem 4.5}]\) and \( M_*(\Sigma) \) is a subalgebra. By Proposition 4.27, this induces an \( A^*(\Sigma) \)-algebra structure on \( A_*(\Sigma) \).

The natural morphism \( \text{ClCP}(\Sigma) \to A^1(X) \), which even is an isomorphism since \( \text{Pic}(X_0) = 0 \), makes \( A^*(X) \) a Sym \( \text{ClCP}(\Sigma) \)-algebra. Since \( \bigcap_{i=1}^l V(\rho_i) = \emptyset \) whenever \( \rho_1, \ldots, \rho_l \in \Sigma(1) \) are distinct rays that do not span a cone of \( \Sigma \), the morphism \( \text{Sym ClCP}(\Sigma) \to A^*(X) \) contains \( \mathcal{I} \) in its kernel. Therefore, \( A^*(X) \) is an \( A^*(\Sigma) \)-algebra.

**Remark 4.29.** It is an interesting question, whether the tropical intersection product on \( M_*(\Sigma) \), and hence on \( A_*(\Sigma) \), is fully determined by the product of \( A^*(\Sigma) \) in the sense that there is a Poincaré duality ring isomorphism \( A^*(\Sigma) \to M_*(\Sigma) \), \( \psi \mapsto \psi \cdot 1_{\Sigma} \), where \( 1_{\Sigma} \) denotes the unit of \( M_*(\Sigma) \), the Minkowski weight assigning 1 to every maximal cone of \( \Sigma \). By Proposition 4.26, this is true if and only if the pairings

\[ A^k(\Sigma) \times A^{\dim \Sigma - k}(\Sigma) \to \mathbb{Z} \]

induced by the degree morphism \( A^{\dim \Sigma}(\Sigma) \to M_0(\Sigma) \cong \mathbb{Z} \) are perfect for all \( k \). By \([5\text{, Proposition 5.10 and Theorem 6.19}]\), this is true at least if \( \Sigma \) is the fine subdivision in the sense of \([8\text{] on the tropical linear space of } \mathcal{M}\).

**Corollary 4.30.** The tropicalization map \( \text{Trop}_X : A^*(X) \to M_*(\Sigma) \) is an isomorphism of \( A^*(\Sigma) \)-algebras.

**Proof.** The map is a ring isomorphism by \([26\text{, Theorem 1.1}]\) and a morphism of \( A^*(\Sigma) \)-modules by Proposition 4.13. \( \square \)

By Poincaré duality, \( A_*(X) \) is an \( A^*(\Sigma) \)-algebra as well. As a corollary, we see that the tropicalization of cycle classes is an isomorphism of \( A^*(\Sigma) \)-algebras.

**Corollary 4.31.** The tropicalization map \( \text{Trop}_X : A_*(X) \to A_*(\Sigma) \) is an isomorphism of \( A^*(\Sigma) \)-algebras.

**Proof.** The assertion immediately follows from Proposition 4.18 and Corollary 4.30. \( \square \)
In this section, we show how to apply our methods to obtain classical/tropical correspondences for invariants on the moduli spaces of rational stable curves and genus 0 logarithmic stable maps to toric varieties. We begin by showing that the tropicalizations of $\psi$-classes on $M_{0,n}$ recover the tropical $\psi$-classes on the tropical moduli space $M_{0,n}^{\text{trop}}$. These have been defined in [48] as tropical cycles, and in [38] and [53] as tropical Cartier divisors on $M_{0,n}^{\text{trop}}$. We recall that the moduli space $M_{0,n}^{\text{trop}}$ of $n$-marked rational tropical curves consists of all (isomorphism classes of) metric trees with exactly $n$ unbounded edges with markings in $[n] := \{1, \ldots, n\}$ (cf. [24]).

To make it accessible to the intersection-theoretic methods of [7], it is usually considered as a tropical variety embedded in $\mathbb{R}^{(2)} / \Phi(\mathbb{R}^n)$. Here, $\Phi$ is the morphism sending the $i$th unit vector $e_i \in \mathbb{R}^n$ to $\sum_{j \neq i} e_{\{i,j\}}$, where we identify the coordinates of $\mathbb{R}^{(2)}$ with two-element subsets of $[n]$. Up to a factor, the embedding usually used is given by

$$M_{0,n}^{\text{trop}} \to \mathbb{R}^{(2)} / \Phi(\mathbb{R}^n), \quad \Gamma \mapsto -\sum_{i,j} \frac{\text{dist}_{ij}(\Gamma)}{2} e_{\{i,j\}},$$

where $\text{dist}_{ij}(\Gamma)$ denotes the distance of the two unbounded edges of $\Gamma$ marked by $i$ and $j$. We refer to [27, Section 3.1] for a justification of the factor $-1/2$. The image under this embedding has a canonical fan structure whose cones are the closures of the images of all tropical curves of fixed combinatorial type [57, Theorem 4.2]. From now on $M_{0,n}^{\text{trop}}$ will denote this fan. The dimension of a cone in $M_{0,n}^{\text{trop}}$ is equal to the number of bounded edges in the corresponding combinatorial type. In particular, its rays correspond to the combinatorial types with exactly one unbounded edge. These are determined by the markings on one of its vertices. For $I \subseteq [n]$ with $2 \leq |I| \leq n$, we denote by $v_I = v_f \in \mathbb{R}$ the primitive generator of the ray of $M_{0,n}^{\text{trop}}$ corresponding to $I$. It is equal to the image in $\mathbb{R}^{(2)} / \Phi(\mathbb{R}^n)$ of the tropical curve of the corresponding combinatorial type whose bounded edge has length one, that is,$$
v_I = -\frac{1}{2} \sum_{i \in I, j \notin I} e_{\{i,j\}}.
$$

The algebraic moduli space $\overline{M}_{0,n}$ parameterizes rational stable curves, that is, trees of $\mathbb{P}^1$'s with $n$ pairwise different non-singular marked points. The irreducible curves form an open subset $M_{0,n} \subseteq \overline{M}_{0,n}$, and since the boundary has simple normal crossings, this defines a toroidal embedding [60, Proposition 4.7]. The strata, and hence the cones in $\Sigma(\overline{M}_{0,n})$, are in natural bijection with the cones in $M_{0,n}^{\text{trop}}$: for two curves in $\overline{M}_{0,n}$, the dual graph construction, which replaces the components of a stable curve by nodes, their intersection points by edges, and marked points by unbounded edges with the respective marks, yields the same combinatorial type of tropical curves if and only if they belong to the same stratum. The bijection between the cones of $\Sigma(X)$ and those of $M_{0,n}^{\text{trop}}$ even preserves dimension. In particular, every boundary divisor of $\overline{M}_{0,n}$ is equal to the divisor $D_I$ corresponding to some subset $I \subseteq [n]$ with $2 \leq |I| \leq n$.

To further strengthen the connection between $\Sigma(\overline{M}_{0,n})$ and $M_{0,n}^{\text{trop}}$, we consider the embedding

$$\iota : M_{0,n} \to \mathbb{G}^{(2)}_m / \mathbb{G}^n_m,$$

given by the composite of the Gelfand–MacPherson correspondence $G^0(2,n)/\mathbb{G}^n_m \cong M_{0,n}$, where $G^0(2,n)$ is the open subset of the Grassmannian with non-vanishing Plücker coordinates and $\mathbb{G}^n_m$ acts on it by dilating the coordinates, with the Plücker embedding $G^0(2,n)/\mathbb{G}^n_m \to \mathbb{G}^{(2)}_m / \mathbb{G}^n_m$ (modulo $\mathbb{G}^n_m$). The weak embedding of $\Sigma(\overline{M}_{0,n})$ maps $|\Sigma(\overline{M}_{0,n})|$ into $\mathbb{R}^{(2)} / \Phi(\mathbb{R}^n)$. 

5. Applications

Applications
Its image can be computed explicitly. It follows from [42, Proposition 6.5.14] that the image of the primitive generator of the ray corresponding to \( n \not\in I \subseteq [n] \) is equal to

\[
\sum_{\{i,j\} \subseteq I} e_{\{i,j\}} = \sum_{\{i,j\} \subseteq I} e_{\{i,j\}} - \frac{1}{2} \Phi \left( \sum_{i \in I} e_i \right) = v_I.
\]

This shows that \( \varphi_{M_{0,n}} \) maps the rays of \( \Sigma_1(M_{0,n}) \) isomorphically onto those of \( M_{0,n}^{\text{trop}} \) in accordance with the bijection of cones from above. Since both cone complexes are strictly simplicial, and the rays corresponding to subsets \( I_1, \ldots, I_k \) of \([n] \) span a cone in \( \Sigma_1(M_{0,n}) \) if and only if they do so in \( M_{0,n}^{\text{trop}} \), it follows that \( \varphi_{M_{0,n}^{\text{trop}}} \) induces an isomorphism \( \Sigma_1(M_{0,n}) \cong M_{0,n}^{\text{trop}} \) of weakly embedded cone complexes. Note that by identifying \( M_{0,n} \) with the complement of the union of the hyperplanes \( \{ x_i = x_j \} \) in \( \mathbb{G}^{n-3} \), it is easy to see that we also have \( \Sigma(M_{0,n}) \cong \Sigma_1(M_{0,n}) \).

In addition to the boundary divisors, there are \( n \) more natural cohomology classes in \( A^1(M_{0,n}) \): the \( \psi \)-classes. The \( k \)th \( \psi \)-class \( \psi_k \) is defined as the first Chern class of the \( k \)th cotangent line bundle on \( M_{0,n} \). We refer to [40] for details. Mimicking this definition, Mikhalkin has defined tropical \( \psi \)-classes on \( M_{0,n}^{\text{trop}} \) as tropical cycles of codimension 1 [48]. They have been described by Kerber and Markwig [38] as tropical \( \mathbb{Q} \)-Cartier divisors in our sense. More precisely, they showed that Mikhalkin’s \( k \)th \( \psi \)-class is the associated cycle of the tropical Cartier divisor \( \psi_k^{\text{trop}} \) which they defined to be the unique divisor satisfying

\[
\psi_k^{\text{trop}}(v_I) = \frac{|I||I|-1}{(n-1)(n-2)}
\]

for all \( I \subseteq [n] \) with \( 2 \leq |I| \leq n-2 \) and \( k \not\in I \). They also computed the top-dimensional intersections of these tropical \( \psi \)-classes explicitly and obtained that they equal their classical counterparts. A more concrete proof of these equalities has been given by Katz [34] by considering \( M_{0,n} \) with its embedding into the toric variety \( Y(M_{0,n}^{\text{trop}}) \) associated to the fan \( M_{0,n}^{\text{trop}} \) and using the tropical description of the Chow cohomology ring of toric varieties. The following proposition strengthens this correspondence by showing that the tropical \( \psi \)-classes are, in fact, the tropicalizations of the classical ones. As a consequence, we will also obtain a correspondence for intersections of \( \psi \)-classes of lower codimension. For the statement to make sense, we consider the classical \( \psi \)-classes as elements in \( \text{Pic}(M_{0,n}) = A^1(M_{0,n}) \) and the tropical ones as classes in \( \text{ClCP}(M_{0,n}^{\text{trop}}) = \text{ClCP}(\Sigma(M_{0,n})) \).

**Proposition 5.1.** For every \( 1 \leq k \leq n \), we have \( \psi_k \in \text{ClCP}(M_{0,n}) \) and

\[
\text{Trop}_{M_{0,n}}(\psi_k) = \psi_k^{\text{trop}}.
\]

In particular, for arbitrary natural numbers \( a_1, \ldots, a_n \), we have

\[
\text{Trop}_{M_{0,n}} \left( \prod_{k=1}^n \psi_k^{a_k} \cdot [M_{0,n}] \right) = \prod_{k=1}^n \left( \psi_k^{\text{trop}} \right)^{a_k} \cdot [M_{0,n}^{\text{trop}}] \quad \text{in} \quad A_*(M_{0,n}^{\text{trop}}),
\]

where \( [M_{0,n}^{\text{trop}}] = \text{Trop}_{M_{0,n}}([M_{0,n}]) \) is the tropical cycle on \( M_{0,n}^{\text{trop}} \) with weight 1 on all maximal cones, and the tropical intersection product is that of Definition 3.25.

**Proof.** For the first part of the statement, we may assume \( k = 1 \), the other cases follow by symmetry. By [40, 1.5.2], the \( \psi \)-class \( \psi_1 \) is a sum of boundary divisors, namely,

\[
\psi_1 = \sum_{1 \leq i \leq 2; 3 \in J} D_I \quad \text{in} \quad \text{Pic}(M_{0,n}),
\]
where the sum runs over subsets \( I \subseteq [n] \) with \( 2 \leq |I| \leq n - 2 \), which satisfy the given condition. Hence, we have \( \psi_1 \in \text{ClCP}(\bar{M}_{0,n}) \), and its tropicalization is represented by

\[
\sum_{1 \in I; \ 2,3 \in J} \varphi_I,
\]

where \( \varphi_I \) is the Cartier divisor on \( M_{0,n}^{\text{trop}} \) that has value 1 at \( v_I \) and 0 at \( v_J \) for all \( I \neq J \neq J' \).

By [53, Lemma 2.24], this is equal to \( \psi_1^{\text{trop}} \) in \( \text{ClCP}(M_{0,n}^{\text{trop}}) \). The last part of the statement is an immediate consequence of Proposition 4.17.

In [44], Markwig and Rau defined genus 0 tropical descendant Gromov–Witten invariants of \( \mathbb{R}^r \) by substituting tropical for algebraic objects in the classical definition of genus 0 descendant Gromov–Witten invariants of toric varieties. In this definition, the tropical moduli space chosen to replace the moduli space of rational stable maps is the moduli space of tropical stable maps, which has been introduced in [24]. Tropical and classical invariants are known to be equal in special cases, but different in general. This reflects the fact that the moduli spaces of tropical stable maps are related, yet not completely analogous to the moduli spaces of (algebraic) stable maps. To make this more precise, let us recall their definition. Let \( n \) and \( m \) be positive natural numbers and \( \Delta \) an \( m \)-tuple of vectors in \( \mathbb{Z}^r \) summing to 0. Then the moduli space \( \text{TSM}^\circ(\mathbb{R}^r, \Delta) \), which is a priori only a set, consists of (isomorphism classes of) \((n+m)\)-marked tropical rational curves together with continuous maps into \( \mathbb{R}^r \), which have rational slopes on the edges, satisfy the balancing condition, contract the first \( n \) marked unbounded edges, and have slope \( \Delta_k \) on the \( k \)th of the remaining \( m \) unbounded edges (cf. [24] for details). The crucial differences to the moduli spaces of stable maps are the last \( m \) markings. Their algebraic meaning is the expected number of points on an algebraic curve of specified degree mapping to the toric boundary.

Let \( \Sigma \) be a complete fan in \( \mathbb{R}^r \) such that the elements of \( \Delta \) are contained in the rays of \( \Sigma \), and let \( Y \) be its associated toric variety. Denote by \( \text{LSM}^\circ(\Sigma, \Delta) \) (where \( \text{LSM} \) stands for rational stable maps) the set of (isomorphism classes of) \((n+m)\)-marked \( \mathbb{P}^1 \) to \( Y \) such that the points mapping to the boundary of \( Y \) are precisely the last \( m \) marked points. Moreover, we require that the \( k \)th of these \( m \) points is mapped to \( \mathcal{O}(\mathbb{R}_{\geq 0}\Delta_k) \), and the multiplicity with which it intersects the boundary is the index of \( \mathbb{Z}\Delta_k \) in its saturation. As explained in detail in [27, Section 5.2], every element in \( \text{LSM}^\circ(\Sigma, \Delta) \) is uniquely determined by its underlying marked curve together with the image of the first marked point. Hence, we can identify \( \text{LSM}^\circ(\Sigma, \Delta) \) with \( M_{0,n+m} \times \mathbb{G}_m^r \) (cf. [51, Proposition 3.3.3]). There is a modular compactification \( \text{LSM}(\Sigma, \Delta) \) of \( \text{LSM}^\circ(\Sigma, \Delta) \), whose points correspond to genus 0 logarithmic stable maps to \( Y \). We refer to [2, 15, 28] for details. The embedding \( \text{LSM}^\circ(\Sigma, \Delta) \to \text{LSM}(\Sigma, \Delta) \) is toroidal. In fact, if \( \mu: \text{LSM}(\Sigma, \Delta) \to \bar{M}_{0,n+m} \) denotes the morphism that forgets the map, and \( \text{ev}_i: \text{LSM}(\Sigma, \Delta) \to Y \) denotes the evaluation map at the \( i \)-th point, then \( \mu \times \text{ev}_i \) is a toroidal modification by [51, Theorem B] and proof thereof. Hence, the weakly embedded cone complex \( \Sigma(\text{LSM}(\Sigma, \Delta)) \) is a proper subdivision of \( M_{0,n+m} \times \Delta \). Since rational tropical stable maps are also determined by their underlying marked (tropical) curve and the image of the first marked point [24, Proposition 4.7], the tropical moduli space \( \text{TSM}^\circ(\mathbb{R}^r, \Delta) \) can be identified with \( \Sigma(\text{LSM}(\Sigma, \Delta)) \). For this reason, we will use \( \text{TSM}^\circ(\mathbb{R}^r, \Delta) \) to denote \( \Sigma(\text{LSM}(\Sigma, \Delta)) \). For a description of its cones, we refer to [51]. Also, we will write \( \text{TSM}(\mathbb{R}^r, \Delta) \) to denote the extended cone complex associated to \( \text{TSM}^\circ(\mathbb{R}^r, \Delta) \).

By construction, the tropicalization \( \text{Trop}(\mu) \) of the forgetful map is equal to the tropical forgetful map \( \mu^{\text{trop}}: \text{TSM}^\circ(\mathbb{R}^r, \Delta) \to M_{0,n+m}^{\text{trop}} \). Similarly, the tropicalization \( \text{Trop}(\text{ev}_i) \) of the \( i \)th evaluation map is equal to the \( i \)th tropical evaluation map \( \text{ev}^{\text{trop}}_i \) by [51, Proposition 5.1.2]. In accordance with [44], we define the \( k \)th \( \psi \)-class \( \hat{\psi}_k \) (respectively, \( \hat{\psi}^{\text{trop}}_k \) on \( \text{LSM}(\Sigma, \Delta) \) (respectively, \( \text{TSM}(\mathbb{R}^r, \Delta) \)) as the pullback via \( \mu \) (respectively, \( \mu^{\text{trop}} \)) of the \( k \)th \( \psi \)-class \( \psi_k \).
(respectively, $\psi_k^{(trop)}$) on $\overline{M}_{0,n+m}$ (respectively, $\overline{M}_{0,n+m}^{trop}$). As a direct consequence of our previous results, we obtain the following correspondence theorem.

**Corollary 5.2.** Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be natural numbers, and for $1 \leq i \leq n$ and $1 \leq j \leq b_i$ let $D_{ij} \in \text{Pic}(Y) = \text{ClCP}(\Sigma)$. Then we have the equality

$$\text{Trop}_{\text{LSM}(Y, \Delta)} \left( \prod_{i=1}^n \psi_i^{a_n} \prod_{j=1}^{b_i} \text{ev}_i^*(D_{ij}) \cdot [\text{LSM}(Y, \Delta)] \right)$$

$$= \prod_{i=1}^n (\psi_i^{trop})^{a_i} \prod_{j=1}^{b_i} (\text{ev}_i^{trop})^*(\text{Trop}_Y(D_{ij})) \cdot [\text{TSM}(\mathbb{R}^r, \Delta)]$$

of cycle classes in $A_*(\text{TSM}(\mathbb{R}^r, \Delta))$, where $[\text{TSM}(\mathbb{R}^r, \Delta)] = \text{Trop}_{\text{LSM}(Y, \Delta)}([\text{LSM}(Y, \Delta)])$ is the cycle on $\text{TSM}(\mathbb{R}^r, \Delta)$ with weight 1 on all maximal cones.

**Remark 5.3.** In case we take top-dimensional intersections with $Y = \mathbb{P}^r$, the tropical degree $\Delta$ containing each of the vectors $e_1, \ldots, e_r, -\sum e_i$ exactly $d$ times, and all the $D_{ij}$ equal to classes of lines, we obtain the tropical descendant Gromov–Witten invariants of $[44]$ and $[53]$ up to factor $(dl)^{r+1}$. In particular, slight variations of the recursion formulas of $[25, 44, 53]$ also hold for the corresponding algebraic invariants.

**Remark 5.4.** We defined the $\psi$-classes $\psi_k$ on $\text{LSM}(Y, \Delta)$ to be the analogs of the tropical $\psi$-classes on $\text{TSM}(\mathbb{R}^r, \Delta)$ of $[44]$, that is, as pullbacks of $\psi$-classes on $\overline{M}_{0,n+m}$. It has been shown in $[43]$ that these are equal to the classes on $\text{LSM}(Y, \Delta)$ obtained by taking Chern classes of cotangent line bundles.

**Index of notation**

The following notation has been introduced in Section 2.

$M^\sigma$ integral linear functions on $\sigma$

$M^\sigma_\geq$ nonnegative integral linear function on $\sigma$

$N^\sigma$ Hom$(N^\sigma, \mathbb{Z})$

$N^\sigma_\geq$ $N^\sigma \cap \sigma$

$|\Sigma|$ underlying space of the cone complex $\Sigma$

$N^{\Sigma}$ lattice in the target of the weak embedding of a weakly embedded cone complex $\Sigma$

$\varphi^{\Sigma}$ weak embedding of $\Sigma$

$N^{\Sigma}_\geq$ $N^{\Sigma} \cap \text{Span} \varphi^{\Sigma}(\sigma)$

$M^{\Sigma}$ Hom$(N^{\Sigma}, \mathbb{Z})$

$\Sigma_\sigma$ star of $\Sigma$ at $\sigma$

$\Sigma_\geq$ extended cone associated to a cone $\sigma$

$\Sigma_\Sigma$ extended cone complexes associated to cone complex $\Sigma$

$\Sigma_f(\sigma)$ morphism on stars induced by morphism $f$ of extended cone complexes

$X_0$ open stratum of a toroidal embedding $X$ (part of the data)

$X(Y)$ small open subset of toroidal embedding $X$ that is a union of strata and contains $Y$

$M^Y$ boundary divisors on $X(Y)$

$M^{Y}_\geq$ effective boundary divisors on $X(Y)$

$N^X$ Hom$(M^Y, \mathbb{Z})$

$\Sigma(X)$ cone complex associated to toroidal embedding $X$

$O(\sigma)$ stratum of $X$ corresponding to $\sigma \in \Sigma(X)$

$\Sigma(\sigma)$ closure of $O(\sigma)$

$X(\sigma)$ shorthand for $X(O(\sigma))$

$M^X$ $\Gamma(X_0, O_X^*)/k^*$, where $k$ is the base field

$N^X$ Hom$(M^X, \mathbb{Z})$; lattice in the target of the weak embedding of $\Sigma(X)$

$\varphi_X$ weak embedding on $\Sigma(X)$
Additional notation from Section 3.1

- $u_{\sigma}/\tau$: lattice normal vector of $\sigma$ relative to $\tau$
- $M_k(\Sigma)$: Minkowski (that is, balanced) weights on the $k$-dimensional cones of a weakly embedded cone complex $\Sigma$
- $|A|$: support of the tropical cycle $A$
- $[c]$: tropical cycle represented by the Minkowski weight $[c]$
- $\text{Div}(\Sigma)$: continuous functions on $\Sigma$ which are integer linear on the cones
- $\text{CP}(\Sigma)$: continuous functions on $\Sigma$ whose restriction to any cone of $\Sigma$ equals the pullback of a linear function via the weak embedding
- $\text{ClCP}(\Sigma)$: the quotient $\text{CP}(\Sigma)/M^\Sigma$

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