Dynamic Models of Satellite Relative Motion and their effects on Kalman Filter

Ali Imran*, Wang Xuechuan, Yue Xiaokui
School of Astronautics, Northwestern Polytechnical University, Xi’an, PRC

E-mail: aliimran900@mail.nwpu.edu.cn

Abstract. In this paper we will study the effects of an increasingly complex Kalman filter state transition matrix on the accuracy of the estimation of a non-linear relative navigation system. The propagation model will be constant in all the cases. The measurements are generated using Xu-Wang model and then noise is added. Whereas the process non-linear vector function/propagation matrix of the Kalman filter will be modified. In the first scenario the Clohessy-Wiltshire equations are considered as the basic propagation model of the Kalman Filter. In the second scenario the eccentricity of the chief satellite’s orbit is included in the model of the Kalman Filter. In the 3rd case we are using the model presented by Xu-Wang which incorporates the oblateness of Earth into the Tschauner-Hempel model. The divergence of the Kalman Filter is studied and provides a guideline to design engineers.

1. Introduction

Satellite formation flying provides solution to problems which cannot be solved by a single large satellite. This results in anticipation that a lot of future satellite missions will be formation flying. NASA measured the gravity of Earth accurately using GRACE. Similarly, work is being done to produce high resolution SAR images from formation flying satellites[1]. A large single satellite can be replaced by multiple small formation flying satellites. This increases the flexibility of the system design[2] and reduces the life cycle costs[3]. As a result, multiple formation flying missions have been proposed and launched, including: Terrestrial Planet Finder, Orion, Prisma, Darwin, Xeus, Earth Observing-1, Grace and Formation Flying Test Bed[4].

Formation flying satellites usually require the formation to maintain a specific shape or distance. However due to imperfections in the Earth’s gravitation and slight differences in perturbation forces like Lunar pull, solar radiation pressure, atmospheric drag etc the formation tends to drift from its intended shape. Important work has been done to get an understanding of these perturbation parameters and the most quoted and well-known work was done in the form of Clohessy-Wiltshire equations (C-W)[5]. These equations however had their own limitations as they assumed that the chief satellite’s orbit was circular and in equatorial plane. Further Earth is considered a perfect sphere. The closed form solution of C-W equations can easily be evaluated in terms of time[6]. Tschauner and Hempel (T-H) incorporated the eccentricity of the orbit into the C-W equations [7]. The state transition matrices of T-H equations both in terms of time[8][9] and true anomaly are available[10]. Further models were developed which included the J2 perturbation of Earth however, they assumed that the chief satellite’s reference orbit is unperturbed[11], this led to modelling errors. Xu and Wang developed the exact non-linear dynamic equations for satellite relative motion around an oblate (J2 perturbation) Earth [12].
2. Mathematical Models
There are two distinct parts of the simulation process

2.1. Propagation of Chief and Relative Orbit (Simulating the measurements)
For the sake of simplicity only perturbation included in this paper is J2. There are multiple models available including the unperturbed nonlinear model [13], Schweighard-Sedwick model[11], and the Xu-Wang model[12]. As stated earlier of these models Xu-Wang is most accurate. This model was developed keeping in view the very fact that the relative dynamics of the satellites are extremely dependent on any perturbations in the reference orbit. Any model that assumes a constant reference orbit for the chief satellite will have modelling errors. The ellipticity of the chief satellite’s orbit is defined in terms of five augmented first order differential equations. The Xu-Wang model does not have any modelling errors (apart from those introduced in the integration of said equations) and thus can be used to initialize the constellation in any initial conditions, the detail derivation of the model is available[14].

Same model is used in this paper with a few changes of variables to make the equations more process friendly. We will convert the Xu-Wang equations to first order differential equations as follows;

\[ \dot{x}_x = v_x \]  
\[ \dot{y}_r = v_y \]  
\[ \dot{z}_r = v_z \]

Thus, the second part of the equations simplifies to:

\[ v_x = 2v_y \omega_z - x_r (\eta^2 - \omega^2) + y_r \alpha_z - z_r \omega_x \omega_z - (\zeta_r - \zeta) s_i s_\theta - r (\eta^2 - \eta^2) + F_x \]  
\[ v_y = -2v_x \omega_z + 2v_z \omega_x - x_r \alpha_x + y_r (\eta^2 - \omega^2 - \omega^2) - z_r \alpha_x - (\zeta_r - \zeta) s_i c_\theta + F_y \]  
\[ v_z = -2v_y \omega_x - x_r \omega_x \omega_z - y_r \alpha_x + z_r (\eta^2 - \omega^2) - (\zeta_r - \zeta) c_i + F_z \]

Wherein \( \omega_x, \omega_z, \alpha_x, \alpha_z, r, \zeta, \eta, i, \theta \) are all periodic time varying parameters related to the chief orbit. The differential equations which define these parameters are not dependent on the geometry of the satellite constellation. It is necessary to evaluate these parameters of the chief satellite so that the relative dynamics can be evaluated, which in turn is a function of these parameters. They are given in reference [15].

\[ \dot{r} = v_x \]  
\[ v_x = -\frac{\mu}{r^2} + \frac{h^2}{r^3} - \frac{k_{j2} s_i s_\theta}{r^3} (1 - 3s_i^2 s_\theta^2) \]  
\[ \dot{\theta} = \frac{h}{r^2} - \frac{2k_{j2} c_i s_\theta}{hr^3} \]  
\[ \dot{\omega} = \frac{k_{j2} s_2 i s_\theta}{2hr^3} \]

One thing should be noted here that if the value of J2 is set to zero these equations reduce to that of a simple propagation. We will use this property later in our paper. Here \( F_x, F_y, F_z \) can be either the control forces or the perturbations in the form of lunar pull, atmospheric drift or the differential solar
radiation pressure. However for simplicity’s sake these forces are assumed to be zero. The relative parameters $\eta, \varsigma, r, r_c, k_J$ can be evaluated from the expressions provided in Xu-Wang[12].

For propagation the initial conditions were set as follows: relative position (10km, 10km, 10km) and relative velocity was set to make the relative orbit periodic. The periodicity condition is[10]

$$\frac{\dot{y}(0)}{x(0)} = -\frac{n(2 + e)}{(1 + e)^{\frac{1}{2}}(1 - e)^{\frac{3}{2}}}$$

(12)

For all measurements the conditions are kept identical, integration time is set to 1 sec however the measurements are generated every 10 seconds, the chief satellite’s orbit is initialized at perigee, with a semi-major axis of 7368.4km, an arbitrarily high inclination of 70 degrees to have pronounced J2 perturbation effects.

The simulation was run for 20 orbits and the results are shown in Fig. 1 and Fig. 2. It can be seen how seriously the eccentricity of the orbit affects the relative position.

2.2. The prediction model for Kalman Filter

We explored three different prediction models for the Kalman Filter.

2.2.1. Clohessy Wiltshire Model. If n is the fundamental harmonic of the orbit then the solution to the Clohessy Wiltshire model at any time ‘t’ is given in [6]. The matrix for state transition is used as such in our state propagation for the filter. Estimates are generated at the same time stamps as of the measurements. It is given by:

$$\begin{pmatrix}
4 - 3 \cos nt & 0 & 0 & \frac{1}{n} \sin nt & \frac{2}{n} (1 - \cos nt) & 0 \\
6(\sin nt - nt) & 1 & 0 & -\frac{2}{n} (1 - \cos nt) & \frac{1}{n} (4 \sin nt - 3nt) & 0 \\
3n \sin nt & 0 & 0 & \cos nt & 2 \sin nt & 0 \\
-6n(1 - \cos nt) & 0 & 0 & -2 \sin nt & 4 \cos nt - 3 & 0 \\
0 & 0 & \cos nt & 0 & 0 & \frac{1}{n} \sin nt \\
0 & 0 & -n \sin nt & 0 & 0 & \cos nt
\end{pmatrix}$$

(13)

2.2.2. Taking Eccentricity into Account. As discussed already multiple models are available for taking into account the eccentricity of the reference orbit. Also as already pointed out if we set the value of J2 perturbation equal to 0 in the Xu-Wang model the effects of inclination and J2 perturbation are
eliminated and the model is equivalent to the Tschauner Hempel Model. It was noted that if we used
the Jacobian for the propagation period of 10 seconds the errors grew very large especially near
perigee, thus instead of using the Jacobian we integrated the estimates over 10 seconds with a time
step of 1 second. However, the Jacobian was used as is for the covariance propagation

2.2.3. Taking the J2 perturbation into Account. For the J2 perturbation we used the XuWang model,
as in the second model the Jacobian was used for the propagation of the Riccati equations however for
the state propagation we integrated the equations using RK4 integrator with a time step of 1 seconds.
As above the error in the systems grew very large, especially near perigee where the change in
accelerations is abrupt. The equations for the Xu-Wang model are already provided in Section 2.1.

3. Kalman Filter Initialization
The satellites were initialized in an inclined orbit with a reference orbit with semi-major axis of 7368.4
and an inclination of 70 degrees. The reference measurements were generated using the Xu-Wang
model as defined in Section 2.1. Once the measurements were generated a 3-dimensional noise with a
standard normal distribution was added to the measurement where the error amplitude was 300 m in
each axis. The P, Q and R matrices of the filter are defined as:

\[ P = 300^2 I_6; \ Q = Qbw \begin{bmatrix} \frac{d\ell T^2}{2} & I_3 \\ 0 & I_3 \end{bmatrix} \ ; \ R = 300^2 I_3 \]

(14)

Where d\ell T is the time step of 10 seconds and Qbw is the tuning factor for the Filter bandwidth. It
was further assumed that the filter had no prior knowledge of the relative position of the satellite and
the initial position and velocity estimates were set to 0. It was noted that it takes about 5-6
measurements for the filter to close in on the actual value however, this was dependent on the filter
bandwidth.

4. Kalman Filtering Results and Analysis
An analysis was performed for varying eccentricities. Inclination was kept constant. During the
evaluation of the measurements using the Xu-Wang model for simplicity the angular momentum of
chief orbit was estimated with that of a Keplerian orbit at initialization. It was noted that as the number
of orbits increased the error grew large, the filter converged back after some measurements but
eventually the errors grew large enough for the filter to diverge. Table 1 summarizes the error in each
axis along with the number of orbits at which the filter starts to diverge. A sample filter divergence
phenomenon is shown in fig 4, where it can be seen how the estimation error starts to diverge to errors
larger than the measurements error.

The eccentricity for each model was increased until the filter diverged. Figure 5 summarizes the
error in all 3 axes for the Clohessy-Wiltshire model. It was noted that radial error was most severely
affected followed by the in-plane error and then the out of plane error. However, the filter was only
able to handle eccentricities up to 0.7 but even at 0.7 the filter performance was below par. Figure 6
summarizes the performance for the T-H equivalent model. The filter also could not handle
eccentricities above 0.7, however, it offered far superior performance as compared to the Clohessy-
Wiltshire model. The radial error was in range of 300 m as compared to 2000 for the C-W model.
Similarly, the in-plane error was also around 300m as compared to 1000m in the C-W model. The out
of plane error also showed a very slight improvement. Figure 7 summarizes the performance of the X-
W model which incorporates the J-2 dynamics of Earth’s oblateness into the model. The filter was
able handle eccentricities up to 0.83 but the error exponentially increased with any increase in
eccentricity above 0.7. Figure 8 shows the no of orbits for each filter vs the eccentricity for the filter
divergence. It can be seen that for any long-term operation the performance of the T-H and the X-W
model are identical, whereas the performance of the C-W model is below par.
### Table 1. Error in estimation of all three filters

| Ecc | x    | y    | z    | n  | Ecc | x    | y    | z    | n  | Ecc | x    | y    | z    | n  |
|-----|------|------|------|----|-----|------|------|------|----|-----|------|------|------|----|
| 0   | 195  | 198  | 195  | 42 | 0   | 191  | 194  | 192  | 43 | 0   | 196  | 190  | 190  | 43 |
| 0.1 | 199  | 191  | 191  | 40 | 0.1 | 192  | 194  | 192  | 41 | 0.1 | 192  | 191  | 193  | 41 |
| 0.2 | 256  | 196  | 192  | 16 | 0.2 | 189  | 192  | 196  | 36 | 0.2 | 193  | 194  | 192  | 36 |
| 0.3 | 295  | 200  | 191  | 12 | 0.3 | 195  | 197  | 193  | 29 | 0.3 | 192  | 191  | 196  | 29 |
| 0.4 | 299  | 212  | 203  | 5.5| 0.4 | 197  | 196  | 193  | 21 | 0.4 | 193  | 192  | 189  | 21 |
| 0.5 | 318  | 232  | 200  | 2.5| 0.5 | 222  | 237  | 193  | 14 | 0.5 | 215  | 248  | 194  | 14 |
| 0.6 | 447  | 332  | 209  | 1  | 0.6 | 239  | 269  | 198  | 7.5| 0.6 | 215  | 211  | 196  | 7  |
| 0.7 | 2319 | 1169 | 242  | 0.6| 0.7 | 368  | 299  | 206  | 3 | 0.7 | 337  | 278  | 205  | 3 |
| 0.8 | -    | -    | -    | -  | 0.8 | -    | -    | -    | 0.2| 0.8 | 1789 | 2579 | 237  | 1 |
| 0.83| -    | -    | -    | -  | 0.83| -    | -    | -    | -  | 0.83| 1972 | 1972 | 254  | 0.6|

#### Figure 3. Measurement X-W e = 0.3, i = 70

#### Figure 4. Filter divergence for C-W e = 0.4 i = 70
Figure 5. Measurement error for C-W

Figure 6. Measurement error for T-H

Figure 7. Measurement error for X-W
5. Conclusion
In this paper the performance of the Kalman Filter based on various models (C-W, T-H, XW) was analyzed where the measurements were generated using the X-W model. A general consensus could be made here that the performance of the X-W and the T-H model were almost identical with only a slight performance improvement by the X-W model over very high eccentricities. However, the price here paid is in the form of added computational complexity, the T-H model and X-W have higher complexity as compared to the CW. So in case the eccentricity of the satellite constellation is low the C-W model should be preferred where the errors are similar as compared to the other two. However, for higher eccentricities the TH model should be preferred as the XW model does not provide sufficient measurement advantage unless the positional accuracy is of utmost importance.

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Figure 8. Number of orbits before filter diverges
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