MARTINGALE SOLUTION TO STOCHASTIC KORTEWEG - DE VRIES EQUATION DRIVEN BY LÉVY NOISE

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Abstract. We study stochastic Korteweg - de Vries equation driven by Lévy noise consisting of the compensated time homogeneous Poisson random measure and a cylindrical Wiener process. We prove the existence of a martingale solution to the equation studied. In proof of the existence theorem we use the Galerkin approximation and several auxiliary results suitable for the problem considered.

1. Introduction

In the paper we study the stochastic Korteweg - de Vries (for short KdV) equation with multiplicative noise of Lévy’s type

\[
\begin{cases}
    d u(t, x) + \left( u_3(t, x) + u(t, x) u_x(t, x) \right) \, dt = \int_Y F(t, u(t, x); y) \tilde{\eta}(dt, dy) + \Phi(t, u(t, x)) \, dW(t) \\
u(0, x) = u_0(x).
\end{cases}
\]

In the deterministic case, the assumption \( u(t, x) = 0 \) for "large" \( |x| \) leads to solitonic solutions, whereas the assumption in periodic form \( u(t, x) = u(t, x + l) \) leads to periodic solutions, so-called cnoidal waves \([6, 25]\), where \( l \) is the wavelength.

The deterministic Korteweg - de Vries equation \([17]\) (for short KdV) has been derived from the set of Eulerian shallow water and long wavelength equations. KdV can model the evolution in time, due to gravity force, of unidirectional weakly nonlinear waves appearing at the surface of the fluid. KdV corresponds to the case of a constant pressure on the surface of the fluid and an even bottom of the container. In more realistic physical cases small fluctuations of these quantities can be modelled by an additional random forcing term.

It is worth to note that KdV equation became a paradigm as weakly dispersive nonlinear wave equation, since it appears naturally as first order approximation in many fields, like fluid dynamics, ion-acoustic waves in plasma, electric currents, propagation of light in fibres and many others, see,
e.g. monographs [1][6][7][13][19][20][23][25]. Therefore it gained enormous interest among physicists, engineers, biologists and mathematicians.

The stochastic KdV equation has been studied extensively, see, e.g. [3–5, 8, 22] and [12][15]. The mentioned above papers deal with additive and/or multiplicative noise. Some discuss exact solutions to the stochastic KdV equation. However, to the best of our knowledge, there has been no result so far for the stochastic KdV equation driven by Lévy type noise.

In the paper we extend the results of the existence of martingale solution to that case. We apply and adapt for our purposes the approaches used in [3, 9] and [18].

2. Existence of martingale solution to KdV

Let \( \left( \Omega, \mathcal{F}, \{ \mathcal{F}_t \}_{t \geq 0}, \mathbb{P} \right) \) be a probability space with filtration and \( (Y, \mathcal{Y}) \) be a measurable space. Denote for \( T < \infty \) and \(-\infty < x_1 < x_2 < \infty\)

(i) \( \mathcal{Y} \) – the space of smooth functions \( f : [0, T] \times [x_1, x_2] \to \mathbb{R} \);

(ii) \( H \) – the closure of \( \mathcal{Y} \) in \( L^2([0, T] \times [x_1, x_2]; \mathbb{R}) \);

(iii) \( V_m, m \geq 1 \) – the closure of \( \mathcal{Y} \) in \( H^m([0, T] \times [x_1, x_2]; \mathbb{R}) \) [in particular \( V := V_1 \)].

Moreover, for arbitrary \( m > 1 \) by \( U \) we will denote a Hilbert space fulfilling the following conditions

(U1) \( U \subset V_m \);

(U2) \( U \) is dense in \( V_m \);

(U3) embedding \( U \hookrightarrow V_m \) is compact.

In [11], \( W(t), t \geq 0, \) is a cylindrical Wiener process adapted to filtration \( \{ \mathcal{F}_t \}_{t \geq 0}, \tilde{\eta} \) is a compensated time homogeneous Poisson measure on \((Y, \mathcal{Y})\) (see definition in Appendix) with \( \sigma \)-finite intensity measure \( \nu, u_0 \in H \) is a deterministic function, \( u(\omega, t, \cdot) : \mathbb{R}_+ \times \mathbb{R} \in \mathbb{R} \) is a càdlàg type function for any \( \omega \in \Omega \).

A measurable function \( F : [0, T] \times H \times Y \to H \) fulfils conditions

(F1) \( \int_Y \chi_{\{0\}}(F(t, x; y)) \nu(dy) = 0 \) for all \( x \in H \) and all \( t \in [0, T] \);

(F2) there exists a constant \( L > 0 \), such that for all \( u_1, u_2 \in H \) and all \( t \in [0, T] \)

\[
\int_Y |F(t, u_1; y) - F(t, u_2; y)|_H^2 \nu(dy) \leq L |u_1 - u_2|_H^2
\]

holds;

(F3) there exists \( \zeta > 0 \), such that for all \( p \in \{1, 2, 2 + \frac{1}{2} \zeta, 4 + \zeta\} \) there exists a constant \( C_p > 0 \), such that

\[
\int_Y |F(t, u; y)|_H^p \mu(dy) \leq C_p (1 + |u|^p_H), \quad \text{for } u \in H, \ t \in [0, T];
\]
(F4) for all $v \in \mathcal{V}$ the mapping $F_v : L^2(0, T; H) \to L^2([0, T] \times Y), dl \otimes \nu; \mathbb{R}$, where $dl \otimes \nu$ denotes the product of Lebesgue measure and the intensity $\nu$, defined by

\begin{equation}
(F_v(u))(t, y) := \langle F(t, u(t^-), y), v \rangle_H, \quad u \in L^2(0, T; H)
\end{equation}

is continuous if the space $L^2(0, T; H)$ is equipped with Fréchet topology from space $L^2(0, T; H_{loc})$.

We assume that a continuous mapping $\Phi[0, T] \times V \to L^2_0(L^2(\mathbb{R}))$ fulfills conditions (Φ1) there exists a constant $L_\Phi > 0$, such that for all $u_1, u_2 \in V$ and all $t \in [0, T]$

\begin{equation}
\|\Phi(t, u_1) - \Phi(t, u_2)\|_{L^2_0(L^2(\mathbb{R}))}^2 \leq L_\Phi \|u_1 - u_2\|_V^2 \quad \text{holds};
\end{equation}

(Φ2) there exist constants $\alpha, \beta, \kappa > 0$, such that for all $u \in V$ and all $t \in [0, T]$

\begin{equation}
\min \left\{ 2 \left( \langle \mathcal{X}_\lambda u, u \rangle_H - \|\Phi(t, u)\|_{L^2_0(L^2(\mathbb{R}))}^2, -\|\Phi(t, u)\|_{L^2_0(L^2(\mathbb{R}))}^2 \right) \right\}
\end{equation}

\begin{equation}
\geq \alpha |u(x)|_V^2 - \beta |u(x)|_H - \kappa,
\end{equation}

holds, where

\begin{equation}
\langle \mathcal{X}_\lambda u \rangle = u_{3x} + \lambda uu_x \quad \text{for} \quad \lambda \in [0, 1];
\end{equation}

(Φ3) there exists a constant $C_\Phi > 0$, such that

\begin{equation}
\|\Phi(t, u)\|_{L^2_0(L^2(\mathbb{R}))}^2 \leq C_\Phi \left( \max \left\{ |u|_V, |u|_H \right\} + 1 \right);
\end{equation}

(Φ4) for any $v \in \mathcal{V}$ the mapping $\Phi_v : L^2(0, T; H) \to L^2([0, T]; L^2_0(L^2(\mathbb{R})))$ given by

\begin{equation}
(\Phi_v(u))(t) := \langle \Phi(t, u(t)), v \rangle_H
\end{equation}

is continuous, if space $L^2(0, T; H)$ is equipped with Fréchet topology from $L^2(0, T; H_{loc})$.

**Definition 2.1.** We say that the problem [11] has a martingale solution on the interval $[0, T]$, $T < \infty$, if there exists a basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P, \bar{\eta}, \{\hat{W}_t\}_{t \geq 0})$, where

(i) $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ is a probability space with filtration;

(ii) $\bar{\eta}$ is a homogeneous Poisson random variable on measurable space $(Y, \mathcal{Y})$ with intensity measure $\nu$;

(iii) $\{\hat{W}_t\}_{t \geq 0}$ is cylindrical Wiener process adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$;

(iv) $\{u(t, x)\}_{t \geq 0}$ is a predictable process adapted to filtration $\{\mathcal{F}_t\}_{t \geq 0}$ with trajectories in

\begin{equation}
\mathbb{D}(0, T; H) \cap L^2(0, T; V) \cap L^\infty(0, T; H) \cap L^2(0, T; H_{loc}) \cap C(0, T; V_s_{loc}(\mathbb{R})),
\end{equation}
Lemma 2.5. The family of distributions (2.11) holds \( P \)-a.s.

Lemma 2.3. For all \( \theta \in \mathcal{L}(0, \infty) \), \( \Phi \) fulfills conditions

\[ \theta(\xi) = 1, \quad \xi \in [0, \frac{m}{2}]; \]

(2.9)

\[ \theta(\xi) \in [0, 1], \quad \xi \in \left[\frac{m}{2}, m]\right]; \]

\[ \theta(\xi) = 0, \quad \xi > m. \]

Lemma 2.4. For all \( p \in \left[\frac{1}{2}, 2 + \varsigma\right] \) there exist such constants \( \tilde{C}_1(p), \tilde{C}_2 \), that

(2.10)

\[ \sup_{m \geq 1} \mathbb{E} \left( \sup_{0 \leq s \leq T} \left| u^m(s) \right|^2_{H} \right)^p \leq \tilde{C}_1(p), \]

(2.11)

\[ \sup_{m \geq 1} \mathbb{E} \left( \int_0^T \left| u^m(s) \right|^2_{V} \, ds \right)^p \leq \tilde{C}_2. \]

Lemma 2.5. The family of distributions \( \mathcal{L}(u^m) \) is tight in \( \mathcal{F} := L^2(0, T; V) \cap L^2(0, T; H_{loc}) \cap \mathbb{D}(0, T; \mathbb{U}) \cap \mathbb{D}(0, T; H) \).

For reader’s convenience proofs of Lemmas 2.3, 2.4 and 2.5 are given in section 3.
Lemma 2.6. ([18, p. 889]) Let \( \eta_m := \eta \) and \( W_m := W, m \in \mathbb{N} \). Then the following conditions hold

(i) the family \( \{ \mathcal{L}(\eta_m) \}_{m \in \mathbb{N}} \) is tight in \( M_\mathbb{N}([0, t] \times Y) \);

(ii) the family \( \{ \mathcal{L}(\eta) \}_{m \in \mathbb{N}} \) is tight in \( C(0, T; \mathbb{R}) \).

Due to Lemmas 2.5 and 2.6 the family of distributions \( \mathcal{L}(u^m, \eta, W_n) \) is tight in \( \widetilde{\mathcal{F}} \times M_\mathbb{N}([0, t] \times Y) \times C(0, T; \mathbb{R}) \). Then due to Corollary 7.3 in [18] there exists the subsequence \( \{ u^k \}_{k \in \mathbb{N}} \), probabilistic space \( (\Omega, \mathcal{F}, \{ F_t \}_{t \geq 0}, \mathbb{P}) \) and such random variables \( (\bar{u}, \bar{\eta}, W) \) and \( (\bar{u}^k, \bar{\eta}_k, W_k) \), \( k \in \mathbb{N} \) in this space with values in \( \mathcal{F} \), that

(i) \( \mathcal{L}(\bar{u}^k, \bar{\eta}_k, W_k) = \mathcal{L}(u^{m_k}, \eta_{m_k}, W_{m_k}) \), \( k \in \mathbb{N} \);

(ii) \( (\bar{u}^k, \bar{\eta}_k, W_k) \to (\bar{u}, \bar{\eta}, W) \) w \( \mathcal{F} \) a.s., when \( k \to \infty \);

(iii) \( (\bar{\eta}_k(\bar{\omega}), W_k(\bar{\omega})) = (\eta_{m_k}(\bar{\omega}), W_{m_k}(\bar{\omega})) \) for all \( \bar{\omega} \in \bar{\Omega} \).

Moreover, \( \bar{\eta}_k, k \in \mathbb{N} \) and \( \bar{\eta} \) are homogeneous Poisson random measures on \( (Y, \mathcal{Y}) \) with intensity measure \( \nu \) and \( W_k, k \in \mathbb{N} \), and \( W \) are cylindrical Wiener processes and \( \bar{u}^k \to \bar{u}, \mathbb{P} \)-a.s.

Since distributions \( \bar{u}^k \) and \( u^{m_k} \) are identical, then due to Lemma 2.4 for all \( p \in \left[ \frac{1}{2}, 2+\varsigma \right] \)

\[
\sup_{m \geq 1} \mathbb{E} \left( \sup_{0 \leq s \leq T} |\bar{u}^m(s)|^{2p} \right) \leq \tilde{C}_1(p)
\]

and

\[
\sup_{m \geq 1} \mathbb{E} \left( \int_0^T |\bar{u}^m(s)|^2 \, ds \right) \leq \tilde{C}_2.
\]

Denote

\[
M^{m_k}(t) := u^{m_k} - u^0 + \int_0^t \left[ \bar{u}^{m_k}(s) + \theta \left( \frac{|\bar{u}^{m_k}(s)|}{m_k} \right) \bar{u}^{m_k}(s)u^{m_k}(s) \right] \, ds - \int_0^t \int_Y P_{m_k} F(t, u(s^-, x); y) \eta_{m_k}(ds, dy);
\]

\[
M^k(t) := \bar{u}^k - \bar{u}^0 + \int_0^t \left[ \bar{u}^{k}(s) + \theta \left( \frac{|\bar{u}^{k}(s)|}{k} \right) \bar{u}^{k}(s)\bar{u}^{k}(s) \right] \, ds - \int_0^t \int_Y P_k F(t, u(s^-, x); y) \bar{\eta}_k(ds, dy).
\]

Note that

\[
M^{m_k}(t) = \int_0^t (\Phi(s, u^{m_k}(s))) \, dW^{m_k}(s),
\]

then it is a martingale with values in \( H \), square integrable, adapted to the filtration \( \sigma \{ u^{m_k}(s), 0 \leq s \leq t \} \) with variation

\[
[M^{m_k}] (t) := \int_0^t \Phi(s, u^{m_k}(s)) \left[ \Phi(s, u^{m_k}(s)) \right]^* \, ds.
\]
Substitute in the Doob inequality (e.g., see Theorem 2.2 in [10]) \( M_t := M^{m_k}(t) \) and \( p := 2p \). Then there exists \( K'_p \), such that

\[
\mathbb{E} \left[ \left( \sup_{t \in [0,T]} |M^{m_k}(t)|^p_H \right) \right] \leq K'_p.
\]

Let \( 0 \leq s \leq t \leq T \) and let \( \varphi \) be a bounded continuous function on \( L^2(0,s;H_{loc}) \) and \( a \in H \) be an arbitrary and fixed. Since \( M^{m_k} \) is a martingale and \( \mathcal{L}(\bar{u}^{m_k}) = \mathcal{L}(u^{m_k}) \), then

\[
\mathbb{E} \left( \left< M^{m_k}(t) - M^{m_k}(s); a \right>_H \varphi(u^{m_k}(s)) \right) = 0,
\]

\[
\mathbb{E} \left( \left< M^{m_k}(t) - M^{m_k}(s); a \right>_H \varphi(\bar{u}^{m_k}(s)) \right) = 0.
\]

Denote

\[
M(t) := \bar{u} - u_0 + \int_0^t \left[ \bar{u}_{3x}(s) + \bar{u}(s)\bar{u}_x(s) \right] \, ds - \int_0^t \int_Y F(t, \bar{u}(s^-, x); y)\eta(\, ds, dy).
\]

We will show that \( P_{m_k} \int_0^t \int_Y F(s, \bar{u}^{m_k}(s^-, x); y)\eta_{m_k}(\, ds, dy) \to \int_0^t \int_Y F(s, \bar{u}(s^-, x); y)\eta(\, ds, dy) \), when \( m_k \to \infty \). Let \( \nu \in U \) be arbitrary fixed. For all \( t \in [0,T] \) we have

\[
\int_0^t \int_Y \left| \left< \left[ F(s, \bar{u}^{m_k}(s^-); y) - F(s, \bar{u}(s^-); y) \right]; \nu \right>_H \right|^2 \, d\nu(y) \, ds
\]

\[
= \int_0^t \int_Y |F_{\nu}(\bar{u}^{m_k})(s,y) - F_{\nu}(\bar{u})(s,y)|^2 \, d\nu(y) \, ds
\]

\[
\leq \int_0^T \int_Y |F_{\nu}(\bar{u}^{m_k})(s,y) - F_{\nu}(\bar{u})(s,y)|^2 \, d\nu(y) \, ds
\]

\[
= \|F_{\nu}(\bar{u}^{m_k})(s,y) - F_{\nu}(\bar{u})(s,y)\|_{L^2([0,T] \times Y; \mathbb{R})}^2.
\]

In above equation \( F_{\nu} \) is the function defined by (2.3). Due to condition (2.1) there is

\[
\int_0^t \int_Y \left| \left< \left[ F(s, \bar{u}^{m_k}(s^-); y) - F(s, \bar{u}(s^-); y) \right]; \nu \right>_H \right|^2 \, d\nu(y) \, ds \leq \int_0^t \left| \left< \left[ \bar{u}^{m_k}(s^-) - \bar{u}(s^-) \right]; \nu \right>_H \right|^2 \, ds
\]

and since \( \bar{u}^{m_k} \to \bar{u} \) when \( m_k \to \infty \),

\[
\int_0^t \int_Y \left| \left< \left[ F(s, \bar{u}^{m_k}(s^-); y) - F(s, \bar{u}(s^-); y) \right]; \nu \right>_H \right|^2 \, d\nu(y) \, ds \to 0, \quad m_k \to \infty.
\]
Moreover, by inequalities (2.20) and (2.10), for arbitrary fixed $t \in [0, T], r \in (1, 2 + \frac{1}{2}], n \in \mathbb{N}$, there exist constants $C_1(r), C_2, C_3, C_4(r) > 0$, such that

$$
\mathbb{E} \left[ \left( \int_0^t \int_Y \left| \langle F(s, \bar{u}^{m_k}(s^-); y) - F(s, \bar{u}(s^-); y) \rangle_H \right|^2 \nu(y) \, ds \right)^\frac{r}{2} \right] \\
\leq 2^r |\nu|_{H}^{2r} \mathbb{E} \left[ \left( \int_0^t \int_Y \left\{ \left| F(s, \bar{u}^{m_k}(s^-); y) \right|^2 + \left| F(s, \bar{u}(s^-); y) \right|^2 \right\} \nu(y) \, ds \right)^\frac{r}{2} \right] \\
\leq 2^r C_2^r |\nu|_{H}^{2r} \mathbb{E} \left[ \left( \int_0^t \left\{ 2 + |\bar{u}^{m_k}(s)|^2_H + |\bar{u}(s)|^2_H \right\} \nu(y) \, ds \right)^\frac{r}{2} \right] \\
\leq C_3 \left( 1 + \mathbb{E} \left[ \sup_{s \in [0, T]} |\bar{u}^{m_k}(s)|^2_H \right] \right) \leq c(1 + C_1(r)) \leq C_4(r).
$$

(2.18)

Due to inequalities (2.17) and (2.18) we have

$$
\mathbb{E} \left[ \int_0^t \int_Y \left| \langle F(s, \bar{u}^{m_k}(s^-); y) - F(s, \bar{u}(s^-); y) \rangle_H \right|^2 \nu(y) \, ds \right] \to 0 \quad \text{as} \quad m_k \to \infty.
$$

(2.19)

Now, take arbitrary fixed $\tilde{\nu} \in H$ and $\varepsilon > 0$. Since $\mathcal{Y}$ is tight in $H$, then there exists $\nu_\varepsilon \in \mathcal{Y}$, such that $|\tilde{\nu} - \nu_\varepsilon| \leq \varepsilon$. By (2.2) there exists a constant $C_5 > 0$, such that

$$
\int_0^t \int_Y \left| \langle F(s, \bar{u}^{m_k}(s^-); y) - F(s, \bar{u}(s^-); y) \rangle_H \right|^2 d\tilde{\nu}(y) \, ds \\
\leq 2 \int_0^t \int_Y \left| \langle F(s, \bar{u}^{m_k}(s^-); y) - F(s, \bar{u}(s^-); y) \rangle_H \right|^2 d\tilde{\nu}(y) \, ds \\
+ 2 \int_0^t \int_Y \left| \langle F(s, \bar{u}^{m_k}(s^-); y) - F(s, \bar{u}(s^-); y) \rangle_H \right|^2 d\nu_\varepsilon(y) \, ds \\
\leq 4C_5 \varepsilon^2 \int_0^t \left\{ 2 + |\bar{u}^{m_k}(s)|^2_H + |\bar{u}(s)|^2_H \right\} \, ds \\
+ 2 \int_0^t \int_Y \left| \langle F(s, \bar{u}^{m_k}(s^-); y) - F(s, \bar{u}(s^-); y) \rangle_H \right|^2 d\nu_\varepsilon(y) \, ds,
$$

so, due to (2.10) there exist constants $C_6, C_7 > 0$, such that

$$
\mathbb{E} \left[ \int_0^t \int_Y \left| \langle F(s, \bar{u}^{m_k}(s^-); y) - F(s, \bar{u}(s^-); y) \rangle_H \right|^2 d\tilde{\nu}(y) \, ds \right] \\
\leq 4C_5 \varepsilon^2 \mathbb{E} \left[ \int_0^t \left\{ 2 + |\bar{u}^{m_k}(s)|^2_H + |\bar{u}(s)|^2_H \right\} \, ds \right] \\
+ 2 \mathbb{E} \left[ \int_0^t \int_Y \left| \langle F(s, \bar{u}^{m_k}(s^-); y) - F(s, \bar{u}(s^-); y) \rangle_H \right|^2 d\nu_\varepsilon(y) \, ds \right] \\
\leq C_6 \varepsilon^2 + 2 \mathbb{E} \left[ \int_0^t \int_Y \left| \langle F(s, \bar{u}^{m_k}(s^-); y) - F(s, \bar{u}(s^-); y) \rangle_H \right|^2 d\nu_\varepsilon(y) \, ds \right].
$$

(2.20)

Taking in (2.20) $m_k \to \infty$ and using (2.19) one obtains

$$
\limsup_{m_k \to \infty} \mathbb{E} \left[ \int_0^t \int_Y \left| \langle F(s, \bar{u}^{m_k}(s^-); y) - F(s, \bar{u}(s^-); y) \rangle_H \right|^2 d\tilde{\nu}(y) \, ds \right] \leq C_6 \varepsilon^2.
$$
Since \( \varepsilon > 0 \) was arbitrary, then

\[
\limsup_{m_k \to \infty} \mathbb{E} \left[ \int_0^t \int_Y \left| \left< F(s, \bar{u}^{m_k}(s^-); y) - F(s, \bar{u}(s^-); y) \right>; \tilde{\nu}_H \right|^2 d\tilde{\nu}(y) \, ds \right] = 0,
\]

so

\[
\mathbb{E} \left[ \int_0^t \int_Y \left| \left< P_{m_k} F(s, \bar{u}^{m_k}(s^-); y) - F(s, \bar{u}(s^-); y) \right>; \tilde{\nu}_H \right|^2 d\tilde{\nu}(y) \, ds \right] \to 0 \text{ as } m_k \to \infty,
\]

and since \( \tilde{\eta}_{m_k} = \tilde{\eta} \),

\[
(2.21) \quad \mathbb{E} \left[ \int_0^t \int_Y \left| \left< P_{m_k} F(s, \bar{u}^{m_k}(s^-); y) - F(s, \bar{u}(s^-); y) \right>; \tilde{\nu}_H \right|^2 d\tilde{\nu}(y) \, ds \right] \to 0 \text{ as } m_k \to \infty,
\]

where \( \tilde{\eta} \) denotes compensated Poisson random measure corresponding to \( \tilde{\eta} \). Using (2.2) and (2.10) one obtains

\[
\mathbb{E} \left[ \int_0^t \int_Y \left| \left< P_{m_k} F(s, \bar{u}^{m_k}(s^-); y) - F(s, \bar{u}(s^-); y) \right>; \tilde{\nu}_H \right|^2 d\tilde{\nu}(y) \, ds \right] \leq 2C_7 |\nu|^2_H \mathbb{E} \left[ \int_0^t \left\{ 2 + |\bar{u}^{m_k}(s)|_H^2 + |\bar{u}(s)|_H^2 \right\} ds \right] 
\]

\[
\leq C_8 \left( 1 + \mathbb{E} \left[ \sup_{s \in [0,T]} |\bar{u}^{m_k}_H|^2 \right] \right) \leq C_9 (1 + C_{10}(2)) \leq C_{11}
\]

for some \( C_7, C_8, C_9, C_{10}, C_{11} > 0 \). Due to (2.21) and (2.22) we have for all \( \nu \in H \)

\[
(2.23) \quad \int_0^T \mathbb{E} \left[ \int_0^t \int_Y \left| \left< P_{m_k} F(s, \bar{u}^{m_k}(s^-); y) - F(s, \bar{u}(s^-); y) \right>; \nu \right|_H \tilde{\eta}(ds, dy) \right]^2 dt \to 0 \text{ when } m_k \to \infty.
\]

This is true for all \( \nu \in U \), as well (since \( u \in H \)).

If \( k_m \to \infty \) then \( \bar{M}^{k_m}(t) \to \bar{M}(t) \) and \( \bar{M}^{k_m}(s) \to \bar{M}(s) \), \( \bar{P} \)-a.s. and since \( \varphi \) is continuous, \( \varphi(\bar{u}^{m_k}(s)) \to \varphi(\bar{u}(s)) \), \( \bar{P} \)-a.s. This and (2.23) gives for \( k_m \to \infty \)

\[
\mathbb{E} \left( \left< \left[ \bar{M}^{m_k}(t) - \bar{M}^{m_k}(s) \right]; a \right|_H \varphi(\bar{u}^{m_k}(s)) \right) \to \mathbb{E} \left( \left< \left[ \bar{M}(t) - \bar{M}(s) \right]; a \right|_H \varphi(\bar{u}(s)) \right).
\]

In particular

\[
(2.24) \quad \mathbb{E} \left( \left< \bar{M}^{m_k}(t); a \right|_H \varphi(\bar{u}^{m_k}(s)) \right) \to \mathbb{E} \left( \left< \bar{M}(t); a \right|_H \varphi(\bar{u}(s)) \right).
\]

Moreover, from (2.23) and (2.24) for all \( a \in U \)

\[
(2.25) \quad \int_0^t \left< \bar{u}^{m_k}_x(s) + \bar{u}^{m_k}(s)\bar{u}^{m_k}_x(s); a \right|_H ds \to \int_0^t \left< \bar{u}_x(s) + \bar{u}(s)\bar{u}_x(s); a \right|_H ds.
\]
By [IS p. 895-898], for all \( a \in U \)

\[
(2.26) \quad \int_0^T \mathbb{E} \left[ \left| \int_0^t [P_{mk} \Phi(s, \bar{u}^{mk}(s)) - \Phi(s, \bar{u}(s))] \, dW(s); a \right|_H^2 \right] \, dt \to 0 \text{ as } m_k \to \infty.
\]

Let \( a \in U \). Denote

\[
\bar{N}^{mk}(t) := \langle \bar{u}^{mk}(0); a \rangle_H + \int_0^t \langle [\bar{u}_{3x}^m(s) + \bar{u}^{mk}(s)\bar{u}_{x}^{mk}(s)]; a \rangle_H \, ds
\]
\[
+ \int_0^t \int_Y \langle P_{mk} F(s, \bar{u}^{mk}(s^-); y); a \rangle_H \bar{n}_{mk}(ds, dy)
\]
\[
+ \int_0^t \int_Y \langle P_{mk} \Phi(s, \bar{u}^{mk}(s)); a \rangle_H \, dW_{mk}(s); a \rangle_H \; ;
\]

\[
N^{mk}(t) := \langle u^{mk}(0); a \rangle_H + \int_0^t \langle [u_{3x}^m(s) + u^{mk}(s)u_x^{mk}(s)]; a \rangle_H \, ds
\]
\[
+ \int_0^t \int_Y \langle P_{mk} F(s, u^{mk}(s^-); y); a \rangle_H n_{mk}(ds, dy)
\]
\[
+ \int_0^t \int_Y \langle P_{mk} \Phi(s, u^{mk}(s)); a \rangle_H \, dW_{mk}(s); a \rangle_H \; ;
\]

\[
\bar{N}(t) := \langle \bar{u}(0); a \rangle_H + \int_0^t \langle [\bar{u}_{3x}(s) + \bar{u}(s)\bar{u}_x(s)]; a \rangle_H \, ds
\]
\[
+ \int_0^t \int_Y \langle F(s, \bar{u}(s^-); y); a \rangle_H \bar{n}(ds, dy) + \int_0^t \langle \Phi(s, \bar{u}(s)); a \rangle_H \, dW(s); a \rangle_H .
\]

Since \( u^{mk} \) is the solution of equation (2.8) for \( m := m_k \), then for all \( t \in [0, T] \) and \( a \in U \)

\[
\langle u^{mk}(t); a \rangle_H = N^{mk}(t), \quad \mathbb{P} - a.s.
\]

In particular

\[
\int_0^T \mathbb{E} \left[ \left| \langle u^{mk}(t); a \rangle_H - N^{mk}(t) \right|^2 \right] \, dt = 0.
\]

Because \( \mathcal{L}(u^{mk}, \eta_{mk}, W_{mk}) = \mathcal{L}(u, \eta, W) \), so

\[
\int_0^T \mathbb{E} \left[ \left| \langle \bar{u}^{mk}(t); a \rangle_H - \bar{N}^{mk}(t) \right|^2 \right] \, dt = 0
\]

and

\[
\int_0^T \mathbb{E} \left[ \left| \langle \bar{u}(t); a \rangle_H - \bar{N}(t) \right|^2 \right] \, dt = 0.
\]

This implies

\[
\langle \bar{u}(t); a \rangle_H = \bar{N}(t), \quad \mathbb{P} - a.s.,
\]

and also

\[
\langle \bar{u}^{mk}(t); a \rangle_H - \langle \bar{u}^{mk}(0); a \rangle_H + \int_0^t \langle [u_{3x}^m(s) + u^{mk}(s)u_x^{mk}(s)]; a \rangle_H \, ds
\]
\[
- \int_0^t \int_Y \langle F(s, \bar{u}(s^-); y); a \rangle_H \bar{n}(ds, dy) - \int_0^t \langle \Phi(s, \bar{u}(s)); a \rangle_H \, dW(s); a \rangle_H = 0,
\]

(2.27)
P-a.s. on Ω and l-a.s. on [0, T], where l is the Lebesgue measure. Since \( \bar{u} \) has values in \( \mathcal{Y} \), in particular \( \bar{u} \in \mathbb{D}(0, T; H) \), then the function on l.h.s. of the inequality (2.27) is càdlàg type with respect to t. Because two càdlàg type functions equal for almost all \( t \in [0, T] \) have to be equal for all \( t \in [0, T] \), so for all \( t \in [0, t] \) and all \( a \in U \)
\[
\langle \bar{u}^m_k(t); a \rangle_H - \langle \bar{u}^m_k(0); a \rangle_H + \int_0^t \langle [u^m_k(s) + u^m_k(s)u^m_k(s)]; a \rangle_H \, ds

- \int_0^t \int_Y \langle F(s, \bar{u}(s); y); a \rangle_H \tilde{\eta}(ds, dy) - \left( \int_0^t \Phi(s, \bar{u}(s)) \, d\bar{W}(s); a \right)_H = 0, \quad \bar{P}\text{-a.s.}
\]
Moreover, since \( U \) is dense in \( V \), then inequality (2.28) holds for all \( a \in V \). Then \( \bar{u} \) is the required martingale solution of the problem (1.1). What finishes the proof of Theorem 2.2.

\[\square\]

3. Proofs of Lemmas 2.3, 2.4 and 2.5

We start with the following auxiliary result.

Proof of Lemma 2.3.

Lemma 3.1. \([14]\) Begin with initial value problem
\[
\begin{cases}
  du(t) = \sigma(u(t)) \, dW(t) + b(t, u(t)) \, dt + \int_Y F(t, u(t^-); y) \tilde{\eta}(dt, dy) \\
  u(0) = u_0,
\end{cases}
\]
where \( \sigma(u(t)) \) and \( b(t, u(t)) \) are continuous, \( \tilde{\eta} \) is a homogeneous compensated Poisson random measure on \((Y, \mathcal{Y})\) with \( \sigma \)-finite intensity measure \( \nu \) and \( \mathbb{E}u_0 < \infty \). Let \( \sigma(u) \), \( b(u) \) and \( F(t, u, y) \) fulfill the condition
\[
|\sigma(u) - \sigma(v)|^2 + \int_Y |F(t, u, y) - F(t, v, y)|^2 \tilde{\eta}(dt, dy) + \|b(u) - b(v)\|^2 \leq K |u - v|^2
\]
for some \( K > 0 \) and arbitrary \( u, v \). Then (3.1) has a martingale solution.

Let \( m \in \mathbb{N} \), \( u, v \in H \) be arbitrary fixed and
\[
b(u(t)) := \theta \left( \frac{u^m(t)}{m} \right) u^m(t)u^m_x(t); \\
\sigma(t, u(t)) := P_m \Phi(t, u^m(t)); \\
F(t, u(t), y) := P_m F(t, u^m(t); y).
\]
We have
\[
|\theta \left( \frac{u^m_x(t)}{m} \right) u^m_x(t) u^m_x(t) - \theta \left( \frac{v^m_x(t)}{m} \right) v^m(t) v^m_x(t)\|_H^2 = \int_{\mathbb{R}} \left( \theta \left( \frac{u^m_x(t)}{m} \right) u^m_x(t) u^m_x(t) - \theta \left( \frac{v^m_x(t)}{m} \right) v^m(t) v^m_x(t) \right)^2 \, dx
\]
(3.3)
\[
\leq m^4 \int_{\mathbb{R}} |u^m(t) - v^m(t)|^2 \, dx
\]
Moreover, due to conditions (2.1) and (2.4), there exist constants \( L_F, L_\Phi, C_1 > 0 \), such that
\[
\left| F(t, u^m(t); y) - F(t, v^m(t); y) \|_{L^2(\mathcal{H})} \right|^2 + \| (t, u^m(t)) - (t, v^m(t)) \|_{L^2(\mathcal{H})}^2 \leq L_F \| u^m(t) - v^m(t) \|_{L^2(\mathcal{H})}^2 + L_\Phi \| u^m(t) - v^m(t) \|_{L^2(\mathcal{H})}^2
\]
(3.4)
\[
\leq L_F \| u^m(t) - v^m(t) \|_{H} + L_\Phi \| u^m(t) - v^m(t) \|_{V} \leq L_F \| u^m(t) - v^m(t) \|_{H} + L_\Phi C_1 \| u^m(t) - v^m(t) \|_{H}.
\]
Addition inequalities (3.3) and (3.4) yield
\[
\left| \theta \left( \frac{u^m_x(t)}{m} \right) u^m_x(t) u^m_x(t) - \theta \left( \frac{v^m_x(t)}{m} \right) v^m(t) v^m_x(t) \right|^2 + \int_{\mathbb{R}} |F(t, u^m(t); y) - F(t, v^m(t); y)|^2 \, d\mathcal{H} + \| (t, u^m(t)) - (t, v^m(t)) \|_{L^2(\mathcal{H})}^2 \leq m^4 \| u^m(t) - v^m(t) \|_{H} + L_F \| u^m(t) - v^m(t) \|_{H} + L_\Phi C_1 \| u^m(t) - v^m(t) \|_{H}
\]
\[
\leq \max \left\{ m^4, L_F, L_\Phi C_1 \right\} \| u^m(t) - v^m(t) \|_{H}.
\]
This finishes the proof since it is enough to substitute \( K := \max \left\{ m^4, L_F, L_\Phi C_1 \right\} \) in (3.1).

\[\square\]

**Proof of the Lemma 2.4** Denote
\[
(3.5) \quad \tau_m := \inf \left\{ t \geq 0 : \| u^m(t) \|_{H} \geq R \right\}, \quad m \in \mathbb{N}, R > 0.
\]
Since every process \( \{ u^m(t) \}_{t \in [0, T]} \) is \( \mathcal{F}_t \)-adapted and right-continuous, then \( \tau_m(R) \) is its stopping time. Moreover, since \( \{ u^m(t) \}_{t \in [0, T]} \) is càdlàg type, so its trajectories \( t \mapsto u^m(t) \) are bounded on \( [0, T], \mathbb{P}\text{-a.s.} \) and \( \tau_m \uparrow T, \mathbb{P}\text{-a.s.} \) when \( R \uparrow \infty \).
Let $p = 1$ or $p = 2 + \frac{\zeta}{2}$ and let $\theta := \theta \left( \frac{|u_m(t)|}{m} \right) \leq 1$. Applying the Itô formula to function $A(u^m(t)) := \left| u^m(t) \right|_{H}^{2p}$ one obtains, similarly like in the proof of Lemma 2.4 in [16],

\[
\begin{align*}
|u^m(t \wedge \tau_m(R))|_{H}^{2p} &= |u^m(t \wedge \tau_m(R))|_{L^2(\mathbb{R})}^{2p} \\
&= |P_n u_0|_{H}^{2p} + \int_{0}^{t \wedge \tau_m(R)} 2p |u^m(s)|_{L^2(\mathbb{R})}^{2p-2} \left( u^m(s); \Phi^m(u^m(s)) \right) dW^m(s) \\
&\quad - \int_{0}^{t \wedge \tau_m(R)} 2p |u^m(s)|_{L^2(\mathbb{R})}^{2p-2} \left( u^m(s); \mathcal{K}_\theta(u^m(s)) \right) ds \\
&\quad + \int_{0}^{t \wedge \tau_m(R)} p^2 |u^m(s)|_{L^2(\mathbb{R})}^{2p-2} \left( \Phi^m(u^m(s)) \right)_{L^2_0(L^2(\mathbb{R}))}^2 ds \\
&\quad + \int_{0}^{t \wedge \tau_m(R)} \int_{Y} \left[ |u^m(s^-) + P_m F(s, u^m(s^-); y)|_{L^2_0(L^2(\mathbb{R}))}^{2p} - |u^m(s^-)|_{L^2_0(L^2(\mathbb{R}))}^{2p} \right] \tilde{\eta}(ds, dy) \\
&\quad + \int_{0}^{t \wedge \tau_m(R)} \int_{Y} \left[ |u^m(s^-) + P_m F(s, u^m(s^-); y)|_{L^2_0(L^2(\mathbb{R}))}^{2p} - |u^m(s^-)|_{L^2_0(L^2(\mathbb{R}))}^{2p} \right] \nu(dy) ds \\
&\quad - 2p |u^m(s^-)|_{L^2_0(L^2(\mathbb{R}))}^{2p-2} \left( u^m(s^-); P_m F(s, u^m(s^-); y) \right)_{H} \nu(dy) ds.
\end{align*}
\]

Denote

\[
\begin{align*}
K^m(t) := & \int_{0}^{t \wedge \tau_m(R)} 2p |u^m(s)|_{L^2(\mathbb{R})}^{2p-2} \left( u^m(s); \Phi^m(u^m(s)) \right) dW^m(s) \\
& - \int_{0}^{t \wedge \tau_m(R)} 2p |u^m(s)|_{L^2(\mathbb{R})}^{2p-2} \left( u^m(s); \mathcal{K}_\theta(u^m(s)) \right) ds \\
& + \int_{0}^{t \wedge \tau_m(R)} p^2 |u^m(s)|_{L^2(\mathbb{R})}^{2p-2} \left( \Phi^m(u^m(s)) \right)_{L^2_0(L^2(\mathbb{R}))}^2 ds; \\
M^m(t) := & \int_{0}^{t \wedge \tau_m(R)} \int_{Y} \left[ |u^m(s^-) + P_m F(s, u^m(s^-); y)|_{L^2_0(L^2(\mathbb{R}))}^{2p} - |u^m(s^-)|_{L^2_0(L^2(\mathbb{R}))}^{2p} \right] \tilde{\eta}(ds, dy); \\
I^m(t) := & \int_{0}^{t \wedge \tau_m(R)} \int_{Y} \left[ |u^m(s^-) + P_m F(s, u^m(s^-); y)|_{L^2_0(L^2(\mathbb{R}))}^{2p} - |u^m(s^-)|_{L^2_0(L^2(\mathbb{R}))}^{2p} \right] \nu(dy) ds \\
& - 2p |u^m(s^-)|_{L^2_0(L^2(\mathbb{R}))}^{2p-2} \left( u^m(s^-); P_m F(s, u^m(s^-); y) \right)_{H} \nu(dy) ds.
\end{align*}
\]

We have

\[
|u^m(t \wedge \tau_m(R))|_{H}^{2p} = |P_n u_0|_{H}^{2p} + K^m(t) + M^m(t) + I^m(t)
\]

\[
= |P_n u_0|_{H}^{2p} + \int_{0}^{t \wedge \tau_m(R)} 2p |u^m(s)|_{L^2(\mathbb{R})}^{2p-2} \left( u^m(s); \Phi^m(u^m(s)) \right) dW^m(s) \\
&\quad - \int_{0}^{t \wedge \tau_m(R)} 2p |u^m(s)|_{L^2(\mathbb{R})}^{2p-2} \left( u^m(s); \mathcal{K}_\theta(u^m(s)) \right) ds \\
&\quad + \int_{0}^{t \wedge \tau_m(R)} p^2 |u^m(s)|_{L^2(\mathbb{R})}^{2p-2} \left( \Phi^m(u^m(s)) \right)_{L^2_0(L^2(\mathbb{R}))}^2 ds + M^m(t) + I^m(t).
\]
Using condition (2.5) we obtain

\[|u^m(t \wedge \tau_m(R))|^{2p}_{L^2(\mathbb{R})} \leq |P_n u_0|^{2p}_H + \int_0^{t \wedge \tau_m(R)} 2p |u^m(s)|^{2p-2}_{L^2(\mathbb{R})} \left\langle u^m(s); \Phi(u^m(s)) \right\rangle dW^m(s)_H \]

\[+ \int_0^{t \wedge \tau_m(R)} p |u^m(s)|^{2p-2}_{L^2(\mathbb{R})} \left( -\alpha |u^m(s)|^2_{H^2(\mathbb{R})} + \beta |u^m(s)|^2_{L^2(\mathbb{R})} + \kappa \right) ds \]

\[+ M^m(t) + I^m(t),\]

where \(H^2(\mathbb{R})\) denotes Sobolev space. Then

\[|u^m(t \wedge \tau_m(R))|^{2p}_{L^2(\mathbb{R})} \leq |P_n u_0|^{2p}_H + \int_0^{t \wedge \tau_m(R)} 2p |u^m(s)|^{2p-2}_{L^2(\mathbb{R})} \left\langle u^m(s); \Phi(u^m(s)) \right\rangle dW^m(s)_H \]

\[+ \int_0^{t \wedge \tau_m(R)} p |u^m(s)|^{2p-2}_{L^2(\mathbb{R})} \left( (\delta - \alpha) |u^m(s)|^2_{H^2(\mathbb{R})} + \beta |u^m(s)|^2_{L^2(\mathbb{R})} + \kappa \right) ds \]

\[+ M^m(t) + I^m(t).\]

Moreover, for any \(\varepsilon > 0\),

\[|u^m(t \wedge \tau_m(R))|^{2p}_{L^2(\mathbb{R})} \leq |P_n u_0|^{2p}_H + \int_0^{t \wedge \tau_m(R)} \delta(p - p\varepsilon) |u^m(s)|^{2p-2}_{L^2(\mathbb{R})} |u^m(s)|^2_{H^2(\mathbb{R})} ds \]

\[\leq |P_n u_0|^{2p}_H + \int_0^{t \wedge \tau_m(R)} 2p |u^m(s)|^{2p-2}_{L^2(\mathbb{R})} \left\langle u^m(s); \Phi(u^m(s)) \right\rangle dW^m(s)_H \]

\[+ \int_0^{t \wedge \tau_m(R)} p |u^m(s)|^{2p-2}_{L^2(\mathbb{R})} \left( (\delta - \alpha) |u^m(s)|^2_{H^2(\mathbb{R})} + \beta |u^m(s)|^2_{L^2(\mathbb{R})} + \kappa \right) ds \]

\[+ M^m(t) + I^m(t);\]

\[|u^m(s)|^{2p}_{L^2(\mathbb{R})} \leq |P_n u_0|^{2p}_H + \int_0^{t \wedge \tau_m(R)} [p(\alpha - \varepsilon \delta)] |u^m(s)|^{2p-2}_{L^2(\mathbb{R})} |u^m(s)|^2_{H^2(\mathbb{R})} ds \]

\[\leq |P_n u_0|^{2p}_H + \int_0^{t \wedge \tau_m(R)} 2p |u^m(s)|^{2p-2}_{L^2(\mathbb{R})} \left\langle u^m(s); \Phi(u^m(s)) \right\rangle dW^m(s)_H \]

\[+ \int_0^{t \wedge \tau_m(R)} p |u^m(s)|^{2p-2}_{L^2(\mathbb{R})} \left( \beta |u^m(s)|^2_{L^2(\mathbb{R})} + \kappa \right) ds + M^m(t) + I^m(t).\]

Substitution in the Young inequality (e.g., see inequality 8.3 in [2]) \(a := |u^m(t, x)|^{2p-2}_{L^2(\mathbb{R})}\), \(b := \kappa p\), \(r := \frac{p}{p-1}\), \(r' = p\), gives

\[
\frac{\varepsilon}{r} = \frac{\varepsilon}{\frac{2p}{2p-2}} = \frac{\varepsilon(p - 1)}{p},
\]

\[
\frac{r'}{r} = \frac{p}{\frac{p}{p-1}} = p - 1,
\]

\[ab = \kappa p |u^m(t, x)|^{2p-2}_{L^2(\mathbb{R})}\]
\[ \kappa p |u^m(t, x)|^{2p-2}_{L^2(\mathbb{R})} \leq \frac{\varepsilon}{p-1} |u^m(t, x)|^{2p}_{L^2(\mathbb{R})} + \frac{1}{p(p-1)(\kappa p)^p} \]

Using (3.8) in (3.7) one gets

\[ |u^m(t \wedge \tau_m(R))|^{2p}_{L^2(\mathbb{R})} + \int_0^{t \wedge \tau_m(R)} |p(\alpha - \varepsilon \delta)| |u^m(s)|^{2p-2}_{L^2(\mathbb{R})} |u^m(s)|^2_{H^2(\mathbb{R})} \, ds \]

\[ \leq |P_m u_0|_{L^2}^{2p} + \int_0^{t \wedge \tau_m(R)} 2p |u^m(s)|^{2p-2}_{L^2(\mathbb{R})} \langle u^m(s), \Phi(u^m(s)) \rangle \, dW_s(s) \]

\[ + \int_0^{t \wedge \tau_m(R)} (p\beta + C_1(\varepsilon, p)) |u^m(s)|^{2p}_{L^2(\mathbb{R})} \, ds + \int_0^{t \wedge \tau_m(R)} C_2(\varepsilon, \kappa, p) \, ds \]

\[ = \int_0^{t \wedge \tau_m(R)} 2p |u^m(s)|^{2p-2}_{L^2(\mathbb{R})} \langle u^m(s), \Phi(u^m(s)) \rangle \, dW_s(s) \]

\[ + \int_0^{t \wedge \tau_m(R)} C_3(\varepsilon, \beta, p) |u^m(s)|^{2p}_{L^2(\mathbb{R})} \, ds + \int_0^{t \wedge \tau_m(R)} C_2(\varepsilon, \kappa, p) \, ds + M^m(t) + I^m(t) \]

Let \( \varepsilon < \frac{\alpha}{5} \) be arbitrary fixed. Then

\[ |u^m(t \wedge \tau_m(R))|^{2p}_{L^2(\mathbb{R})} \leq |P_m u_0(x)|_{L^2}^{2p} + \int_0^{t \wedge \tau_m(R)} p |u^m(s)|^{2p-2}_{L^2(\mathbb{R})} \langle u^m(s), \Phi(u^m(s)) \rangle \, dW_s(s) \]

\[ + \int_0^{t \wedge \tau_m(R)} C_3(\varepsilon, \beta, p) |u^m(s)|^{2p}_{L^2(\mathbb{R})} \, ds + \int_0^t C_2(\varepsilon, \kappa, p) \, ds + M^m(t) + I^m(t) \]

and

\[ \mathbb{E} |u^m(t \wedge \tau_m(R))|^{2p}_{L^2(\mathbb{R})} \leq C_4 + \mathbb{E} \int_0^{t \wedge \tau_m(R)} C_3(\varepsilon, \beta, p) |u^m(s)|^{2p}_{L^2(\mathbb{R})} \, ds \]

\[ + \mathbb{E} \int_0^{t \wedge \tau_m(R)} C_2(\varepsilon, \kappa, p) \, ds + \mathbb{E}(M^m(t)) + \mathbb{E}(I^m(t)). \]

In the following part we will use the following result from [18].

**Lemma 3.2.** [18 p. 882-883]) For any \( p \geq 1 \) there exist constants \( C_1(p), C_2(p), C_3(p) > 0 \), such that for arbitrary \( x, h \in H \) the following inequalities hold

\[ |x + h|^{2p}_{H} - |x|^{2p}_{H} - 2p |x|^{2p-2}_{H} \langle x, h \rangle_{H} \leq C_1(p) \left( |x|^{2p-2}_{H} + |h|^{2p-2}_{H} \right) |h|^{2p}_{H}; \]

\[ \left( |x + h|^{2p}_{H} - |x|^{2p}_{H} \right)^2 \leq 2 \left( 4p^2 |x|^{2p-2}_{H} |h|^{2p}_{H} + C_2(p) \left( |x|^{2p-2}_{H} + |h|^{2p-2}_{H} \right)^2 |h|^{4p}_{H} \right) \]

\[ \leq 4p^2 |x|^{4p-2}_{H} |h|^{2p}_{H} + C_3(p) |x|^{4p-4}_{H} |h|^{4p}_{H} + C_5(p) |h|^{4p}_{H}. \]
Using condition (2.2), Lemma 3.2 and (3.5) for the process \( I^n(t \land \tau_m(R)) \) one obtains

\[
|I^n(t)| := \left| \int_0^{t \land \tau_m(R)} \int_Y \left[ \left| u^m(s^-) + P_mF(s, u^m(s^-); y) \right|^{2p} - \left| u^m(s^-) \right|^{2p} \right. \\
- 2\left| r u^m(s^-) \right|^{2p-2} \left( \left| u^m(s^-) \right|; P_mF(s, u^m(s^-); y) \right) \right] \nu(dy) \, ds \right|
\]

\[
\leq \int_0^{t \land \tau_m(R)} \int_Y \left\{ C_4(p) \left( \left| u^m(s) + P_mF(s, u^m(s); y) \right|^{2p} + \left| P_mF(s, u^m(s); y) \right|^{2p-2} \right) \right. \\
\times \left. \left| P_mF(s, u^m(s); y) \right|^{2p} \right\} \nu(dy) \, ds
\]

\[
\leq C_4(p) \int_0^t \left\{ C_5 \left| u^m(s) \right|^{2p-2} \left( 1 + \left| u^m(s) \right|^2 \right) + C_6(p) \left( 1 + \left| u^m(s) \right|^{2p} \right) \right\} \, ds
\]

\[
\leq C_7(p) \int_0^t \left\{ 1 + \left| u^m(s) \right|^{2p} \right\} \, ds = C_7(p)t + C_7(p) \int_0^t \left| u^m(s) \right|^{2p} \, ds
\]

for some \( C_4(p), C_5, C_6(p), C_7(p) > 0 \). Then for any \( t \in [0, T] \)

\[
(3.11) \quad \mathbb{E} \left( |I^n(t)| \right) \leq C_7(p)t + C_7(p) \int_0^t \mathbb{E} \left( \left| u^m(s) \right|^{2p} \right) \, ds.
\]

Moreover, due to Lemma 3.2, condition (2.2) and (3.5) the process \( \{M^m(t \land \tau_m(R))\}_{t \in [0, T]} \) is an integrable martingale, so

\[
(3.12) \quad \mathbb{E} \left( M^m(t \land \tau_m(R)) \right) = 0.
\]

Insertion of (3.11) and (3.12) into (3.10) yields

\[
\mathbb{E} \left| u^m(t \land \tau_m(R)) \right|^{2p} \leq C_4 + \mathbb{E} \int_0^{t \land \tau_m(R)} C_3(\varepsilon, \beta, p) \left| u^m(s) \right|^{2p} \, ds + \mathbb{E} \int_0^{t \land \tau_m(R)} C_2(\varepsilon, \kappa, p) \, ds
\]

\[
+ C_7(p)t + C_7(p) \int_0^t \mathbb{E} \left( \left| u^m(s) \right|^{2p} \right) \, ds
\]

\[
\leq C_4 + \mathbb{E} \int_0^{T \land \tau_m(R)} C_3(\varepsilon, \beta, p) \left| u^m(s) \right|^{2p} \, ds + \mathbb{E} \int_0^{t \land \tau_m(R)} C_2(\varepsilon, \kappa, p) \, ds
\]

\[
+ C_7(p)t + C_7(p) \int_0^T \mathbb{E} \left( \left| u^m(s) \right|^{2p} \right) \, ds
\]

\[
\leq C_4 + C_8 T + \int_0^{T \land \tau_m(R)} C_9(\varepsilon, \beta, p) \mathbb{E} \left( \left| u^m(s) \right|^{2p} \right) \, ds.
\]

Substitute in the Gronwall lemma (e.g., see Theorem 1.2 in [11]) \( u(t) := \mathbb{E} \left| u^m(t \land \tau_m(R)) \right|^{2p} \), \( \alpha(t) := C_4 + C_8 T \), \( \beta(t) := C_9(\varepsilon, \beta, p) \), \( a := 0 \), \( b := T \land \tau_m(R) \). Then for any \( t \in [0, T \land \tau_m(R)] \) and
\[ m \in \mathbb{N} \setminus \{0\} \]

\[ \mathbb{E} \left| u^m(t \wedge \tau_m(R)) \right|^{2p}_{H} \]

(3.14) \[ \leq C_4 + C_8 T + \left| \int_0^{t \wedge \tau_m(R)} [C_4 + C_8 s] C_9(\varepsilon, \beta, p) \exp \left\{ \int_s^{t \wedge \tau_m(R)} C_9(\varepsilon, \beta, p) \, d\xi \right\} \, ds \right| \]

\[ \leq C_{10}(\varepsilon, \beta, p, T) \]

for some constant \( C_{10}(\varepsilon, \beta, p, T) > 0 \). Moreover,

\[ \sup_{n \geq 1} \sup_{t \in [0, T]} \mathbb{E} \left| u^m(t \wedge \tau_m(R)) \right|^{2p}_{H} \leq C_{10}(\varepsilon, \beta, p, T) \]

and in particular

\[ \sup_{n \geq 1} \mathbb{E} \left[ \int_0^{T \wedge \tau_m(R)} \left| u^m(s) \right|^{2p}_{H} \, ds \right] \leq C_{10}(\varepsilon, \beta, p, T). \]

Due to this inequality, when \( R \uparrow \infty \) the following inequality holds

(3.15) \[ \sup_{n \geq 1} \mathbb{E} \left[ \int_0^{T} \left| u^m(s) \right|^{2p}_{H} \, ds \right] \leq C_{10}(\varepsilon, \beta, p, T). \]

Using (3.11), (3.12) and (3.15) in (3.9) one gets

\[ \sup_{m \geq 1} \mathbb{E} \left[ \left| u^m(t \wedge \tau_m(R)) \right|_{H}^{2p} \right] + \sup_{m \geq 1} \mathbb{E} \left[ \int_0^{t \wedge \tau_m(R)} \left| p(\alpha - \varepsilon \delta) \right| \left| u^m(s) \right|^{2p-2}_{H} \left| u^m(s) \right|_{V} \, ds \right] \]

\[ \leq \sup_{m \geq 1} \mathbb{E} \left[ \int_0^{t \wedge \tau_m(R)} 2p \left| u^m(s) \right|^{2p-2}_{H} \left( u^m(s), \Phi(u^m(s)) \right)_{H} \, ds \right] \]

+ \sup_{m \geq 1} \mathbb{E} \left[ \int_0^{t \wedge \tau_m(R)} C_3(\varepsilon, \beta, p) \left| u^m(s) \right|^{2p}_{H} \, ds + \int_0^{t \wedge \tau_m(R)} C_2(\varepsilon, \kappa, p) \, ds \right] \]

+ \sup_{m \geq 1} \mathbb{E} \left[ M^m(t) \right] + \sup_{m \geq 1} \mathbb{E} \left[ T^m(t) \right] \]

\[ \leq \sup_{m \geq 1} \mathbb{E} \left[ \int_0^{T} C_3(\varepsilon, \beta, p) \left| u^m(s) \right|^{2p}_{H} \, ds + \int_0^{T} C_2(\varepsilon, \kappa, p) \, ds \right] \]

+ \sup_{m \geq 1} \mathbb{E} \left[ M^m(t) \right] + \sup_{m \geq 1} \mathbb{E} \left[ T^m(t) \right] \]

\[ \leq C_{10}(\varepsilon, \beta, p, T)C_3(\varepsilon, \beta, p) + C_2(\varepsilon, \kappa, p)T + C_7(p)T + C_7(p) \sup_{m \geq 1} \int_0^{T} \mathbb{E} \left( \left| u^m(s) \right|^{2p}_{H} \right) \, ds \]

\[ \leq C_{11}(\varepsilon, \beta, \kappa, p, T) + C_7(p)C_{10}(\varepsilon, \beta, p, T) \leq C_{12}(\varepsilon, \beta, \kappa, p, T). \]
Substitution in the above inequality \( p := 1 \), give for any \( t \in [0, T] \)

\[
\sup_{m \geq 1} \mathbb{E} \left[ \left| u^m(t \wedge \tau_m(R)) \right|_H^2 \right] + \sup_{m \geq 1} \mathbb{E} \left[ \int_0^{t \wedge \tau_m(R)} C_{13}(\varepsilon, \alpha, \delta) |u^m(s)|_V^2 \, ds \right] \leq C_{12}(\varepsilon, \beta, \kappa, 1, T),
\]

From the Burkholder lemma (e.g., see Theorem 2.3 in [2]) for the process \( M^m(t) \) one obtains

\[
\mathbb{E} \left[ \sup_{r \in [0, t]} |M^m(r \wedge \tau_m(R))| \right] \leq C_{15}(p) \mathbb{E} \left[ \left( \int_0^{t \wedge \tau_m(R)} \int_Y \left( |u^m(s^-) + P_m F(s, u^m(s^-); y)|_H^{2p} - |u^m(s^-)|_H^{2p} \right) \nu(dy) \, ds \right]^{\frac{1}{2}}
\]

for some \( C_{15}(p) > 0 \). Moreover, due to condition (2.2), Lemma 3.2 for some \( C_{16}, C_{17}, C_{18}, C_{19} > 0 \) there holds

\[
\int_Y (u^m(s^-) + P_m F(s, u^m(s^-); y)|_H^{2p} - |u^m(s^-)|_H^{2p}) \nu(dy) \leq C_{16} + C_{17} |u^m(s^-)|_H^{4p-4} + C_{18} |u^m(s^-)|_H^{4p-2} + C_{19} |u^m(s^-)|_H^{4p}.
\]

Young’s inequality in (3.17) implies

\[
\int_Y (u^m(s^-) + P_m F(s, u^m(s^-); y)|_H^{2p} - |u^m(s^-)|_H^{2p}) \nu(dy) \leq C_{20} + C_{21} |u^m(s^-)|_H^{4p},
\]

for some constants \( C_{20}, C_{21} > 0 \). Therefore

\[
\left( \int_0^{t \wedge \tau_m(R)} \int_Y \left( |u^m(s^-) + P_m F(s, u^m(s^-); y)|_H^{2p} - |u^m(s^-)|_H^{2p} \right) \nu(dy) \, ds \right)^{\frac{1}{2}} \leq \sqrt{T} C_{20} + \sqrt{C_{21}} \left( \int_0^{t \wedge \tau_m(R)} |u^m(s^-)|_H^{4p} \, ds \right)^{\frac{1}{2}}.
\]
Using (3.15) and (3.19) in (3.16), one get for some constants $C_{22}, C_{23} > 0$

(3.20)

$$E \left[ \sup_{r \in [0,t]} |M^n(r \wedge \tau_m(R))| \right] \leq C_{15}(p) \sqrt{TC_2} + C_{15}(p) \sqrt{C_{21}E \left[ \left( \int_0^{t \wedge \tau_m(R)} |u_m(s^-)|^{4p} ds \right)^{\frac{1}{2}} \right]}$$

$$\leq C_{15}(p) \sqrt{TC_2} + C_{15}(p) \sqrt{C_{21}E \left[ \left( \sup_{s \in [0,t]} |u_m(s^-)|^{2p} \right) \left( \int_0^{t \wedge \tau_m(R)} |u_m(s^-)|^{2p} ds \right)^{\frac{1}{2}} \right]}$$

$$\leq C_{15}(p) \sqrt{TC_2} + C_{22}(p) \sqrt{C_{21}E \left[ \left( \int_0^{t \wedge \tau_m(R)} |u_m(s^-)|^{2p} ds \right)^{\frac{1}{2}} \right]}$$

$$\leq \frac{1}{4} E \left[ \left( \sup_{s \in [0,t]} |u_m(s^-)|^{2p} \right) \right] + C_{23}(p).$$

Moreover, using Burkholder’s inequality for the process

$$\int_0^{t \wedge \tau_m(R)} p |u_m(s^-)|^{2p-2} \langle \Phi(u_m(s)) dW^m(s), u_m(s) \rangle_H$$

one obtains for some constant $C_{23}(p) > 0$

(3.21)

$$E \left( \sup_{t \in [0,T]} \int_0^{t \wedge \tau_m(R)} p |u_m(s^-)|^{2p-2} \langle \Phi(u_m(s)) dW^m(s), u_m(s) \rangle_H \right) \leq C_{23}(p)E \left\{ \left[ \int_0^{t \wedge \tau_m(R)} p |u_m(s^-)|^{4p-2} \|\Phi(u_m(s))\|_{L_2^2(H)}^2 ds \right]^{\frac{1}{2}} \right\}$$

$$\leq C_{23}(p)pE \left\{ \left[ \sup_{0 \leq s \leq T} |u_m(s)|^{2p} \int_0^{t \wedge \tau_m(R)} |u_m(s)|^{2p-2} \|\Phi(u_m(s))\|_{L_2^2(H)}^2 ds \right]^{\frac{1}{2}} \right\}$$

$$\leq C_{23}(p)pE \left\{ \sup_{0 \leq s \leq T} |u_m(s)|^{2p} \int_0^{t \wedge \tau_m(R)} |u_m(s)|^{2p-2} \left( C \|u_m(s)\|_H^2 + 1 \right) ds \right\}^{\frac{1}{2}}$$

$$\leq \frac{1}{2} \left( \sup_{0 \leq s \leq T} |u_m(s)|^{2p} \right) + \frac{1}{2} C_{23}(p)^2 p^2 \left( \int_0^{t \wedge \tau_m(R)} C \sup_{0 \leq s \leq \xi} |u_m(s)|^{2p} d\xi \right)$$

$$+ \frac{1}{2} C_{23}(p)^2 p^2 \left( \int_0^{t \wedge \tau_m(R)} |u_m(s)|^{2p-2} ds \right)$$

$$\leq \frac{1}{2} \left( \sup_{0 \leq s \leq T} |u_m(s)|^{2p} \right) + \frac{1}{2} C_{23}(p)^2 p^2 \left( \int_0^{t \wedge \tau_m(R)} C \sup_{0 \leq s \leq \xi} |u_m(s)|^{2p} d\xi \right)$$

$$+ \frac{1}{2} C_{23}(p)^2 p^2 C_{16}(p, T)$$

$$\leq \frac{1}{2} \left( \sup_{0 \leq s \leq T} |u_m(s)|^{2p} \right) + C_{15}(p)E \left( \int_0^{t \wedge \tau_m(R)} \sup_{0 \leq s \leq \xi} |u_m(s)|^{2p} d\xi \right) + C_{16}(p, T)$$

$$\leq \frac{1}{2} \left( \sup_{0 \leq s \leq T} |u_m(s)|^{2p} \right) + C_4(p, T)E \left( \sup_{0 \leq s \leq T} |u_m(s)|^{2p} \right) + C_{16}(p, T)$$

$$\leq C_{25}(p, T)E \left[ \sup_{0 \leq s \leq T} |u_m(s)|^{2p} \right] + C_{16}(p, T).$$
Now, we have

\[
|u^m(t \land \tau_m(R))|^{2p}_{L^2(\mathbb{R})} \leq \int_0^{t \land \tau_m(R)} 2p |u^m(s)|^{2p-2}_{L^2(\mathbb{R})} \langle u^m(s), \Phi(u^m(s)) \rangle dW^m(s)_H \n + \int_0^{t \land \tau_m(R)} C_3(\varepsilon, \beta, p) |u^m(s)|^{2p}_{L^2(\mathbb{R})} ds + \int_0^{t \land \tau_m(R)} C_2(\varepsilon, \kappa, p) ds 
+ M^m(t) + I^m(t).
\]

Taking supremum from the r.h.s. of the above inequality, taking expectation values and using (3.20) and (3.21) one obtains

\[
E \sup_{t \in [0,T]} \left[ |u^m(t \land \tau_m(R))|^{2p}_{L^2(\mathbb{R})} \right] \leq E \sup_{t \in [0,T]} \left[ \int_0^{t \land \tau_m(R)} 2p |u^m(s)|^{2p-2}_{L^2(\mathbb{R})} \langle u^m(s), \Phi(u^m(s)) \rangle dW^m(s)_H \right] 
+ E \sup_{t \in [0,T]} \left[ \int_0^{t \land \tau_m(R)} C_3(\varepsilon, \beta, p) |u^m(s)|^{2p}_{L^2(\mathbb{R})} ds \right] 
+ E \sup_{t \in [0,T]} \left[ \int_0^{t \land \tau_m(R)} C_2(\varepsilon, \kappa, p) ds \right]
+ E \sup_{t \in [0,T]} [M^m(t)] + E \sup_{t \in [0,T]} [I^m(t)]
\]

\[
\leq + C_{26} + C_{25}(p, T)E \sup_{0 \leq s \leq T} |u^m(s)|^{2p}_H + C_{16}(p, T) + E \sup_{t \in [0,T]} \left[ \int_0^{t \land \tau_m(R)} C_3(\varepsilon, \beta, p) |u^m(s)|^{2p}_{L^2(\mathbb{R})} ds \right] 
+ \frac{1}{4} E \left[ \left( \sup_{s \in [0,t]} |u^m(s^-)|^{2p}_H \right) \right] + C_{23}(p) + E \sup_{t \in [0,T]} [I^m(t)]
\]

\[
\leq C_{27}(p, T) + C_{25}(p, T)E \sup_{0 \leq s \leq T} |u^m(s)|^{2p}_H + C_3(\varepsilon, \beta, p)E \sup_{t \in [0,T]} \left[ \int_0^{t \land \tau_m(R)} |u^m(s)|^{2p}_{L^2(\mathbb{R})} ds \right] 
+ E \sup_{t \in [0,T]} [I^m(t)].
\]

Inequalities (3.11), (3.15) imply

(3.22)

\[
E \sup_{t \in [0,T]} \left[ |u^m(t \land \tau_m(R))|^{2p}_{L^2(\mathbb{R})} \right] \leq C_{29}(p, T) + C_{30}(p, T) + C_{31}(p) \int_0^T E \left( |u^m(s)|^{2p}_H \right) ds + C_{32}(\varepsilon, \beta, p, T)
\]

\[
\leq C_{29}(p, T) + C_{30}(p, T) + C_{31}(p)C_{32}(\varepsilon, \beta, p, T) + C_{32}(\varepsilon, \beta, p, T) \leq C_{33}(\varepsilon, \beta, p, T).
\]

Taking the limit \(R \uparrow \infty\) yields (2.10).

Now, let \(p \in \left[ \frac{1}{2}, 2 + \frac{2}{3} \right] \setminus \{2\} \) and let \(m \in \mathbb{N}\) be arbitrary fixed. Then

\[
|u^m(s)|^{2p}_H = \left( |u^m(s)|^{2 + \frac{2}{3}}_H \right)^\frac{2p}{2 + \frac{2}{3}} \leq \left( \sup_{t \in [0,T]} |u^m(s)|^{2 + \frac{2}{3}}_H \right)^\frac{2p}{2 + \frac{2}{3}}
\]
and
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |u^m(t)|^{2p}_H \right] \leq \mathbb{E} \left[ \left( \sup_{t \in [0,T]} |u^m(s)|^{2+\frac{2p}{2+\frac{2p}{2}}} \right)^{\frac{2p}{2+\frac{2p}{2}}} \right] \leq \left( \mathbb{E} \left[ \sup_{t \in [0,T]} |u^m(s)|^{2+\frac{2p}{2}} \right] \right)^{\frac{2p}{2+\frac{2p}{2}}} \leq \left( \mathbb{E} \left[ \sup_{t \in [0,T]} |u^m(s)|^{2+\frac{2p}{2}} \right] \right)^{\frac{2p}{2+\frac{2p}{2}}} \leq \left[ C_{34} \left( 4 + \frac{\varsigma}{2} \right) \right]^{\frac{2p}{2+\frac{2p}{2}}}.
\]

Since \( m \in \mathbb{N} \) is fixed, so
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |u^m(t)|^{2p}_H \right] \leq C_{35}(p),
\]
what finishes the proof. \( \square \)

**Proof of Lemma 2.5.** For reader’s convenience we cite lemmas from [18] explicitly.

**Lemma 3.3.** ([18, Corollary 3.5, tightness criterium]) Let \( \{X_n\}_{n \in \mathbb{N}} \) be a sequence of processes of cádlág type, adapted to filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) with values in \( U' \), such that

(i) There exists a constant \( C_1 > 0 \), such that \( \sup_{n \in \mathbb{N}} \mathbb{E} \left[ \sup_{s \in [0,T]} |X_n(s)|_H \right] \leq C_1; \)

(ii) There exists a constant \( C_2 > 0 \), such that \( \sup_{n \in \mathbb{N}} \mathbb{E} \left[ \int_0^T |X_n|^2_V \, ds \right] \leq C_2; \)

(iii) \( \{X_n\} \) fulfils the Aldous condition in \( U' \).

Then the family of distributions \( \{\mathcal{L}(X_n)\} \) is tight in \( \mathcal{Z} \).

**Lemma 3.4.** ([18, Lemma 6.3]) Let \((E, |\cdot|_E)\) be a separable Banach space and let \( \{X_n\}_{n \in \mathbb{N}} \) be a sequence of random variables with values in \( E \). Let for any sequence of stopping times \( \{\tau_n\}_{n \in \mathbb{N}}, \tau_n < T, n \in \mathbb{N} \) and all \( n \in \mathbb{N} \) and \( \vartheta > 0 \)
\[
\mathbb{E} \left[ |X_n(\tau_n + \vartheta) - X_n(\tau_n)|^a_E \right] \leq C \vartheta^b
\]
holds for some \( a, b, C > 0 \). Then the sequence \( \{X_n\}_{n \in \mathbb{N}} \) fulfils the Aldous condition in \( E \).

Let us note, that due to Lemma 2.4 the process \( u^m(t) \) fulfils conditions (i) and (ii) from Lemma 3.3 for any \( m \in \mathbb{N} \). Then it is sufficient to show that for any \( m \in \mathbb{N}, u^m(t) \) fulfils Aldous condition. We have
\[
u^m(t) = P_m u_0(t) - \int_0^t u^m_{3x}(s) \, ds - \int_0^t u^m(s) u^m_{m}(s) \, ds + \int_0^t \int_Y P_m F(s, u^m(s); y) \tilde{\eta}(ds, dy) + \int_0^t P_m \Phi(s, u^m(s)) \, dW(s).
\]
We will show that each of terms in the above equation fulfils assumptions of the Lemma 3.4. Let \( \vartheta > 0 \) and let \( \{\tau_m\}_{m \in \mathbb{N}} \) be a sequence of stopping times such that \( \tau_m < T, m \in \mathbb{N} \). Since
Definition A.6. \( \eta \) on \((Y, \mathcal{Y})\) over \((\Omega, \mathcal{F}, \mathbb{P})\) is a measurable function such that

\[ V \subset H \subset V_3 \subset V' \subset U', \text{ so} \]

(3.23)

\[
E \left[ \left| u^m_{3x}(\tau_m + \vartheta) - u^m_{3x}(\tau_m) \right|_{U_2} \right] = E \left[ \left| \int_{\tau_m}^{\tau_m+\vartheta} u^m_{3x}(s) \, ds \right|_{U_2} \right] \leq C_1 E \left[ \left| \int_{\tau_m}^{\tau_m+\vartheta} u^m(s) \, ds \right|_{H} \right] \leq C_1 C_2 E \left[ \left| \int_{\tau_m}^{\tau_m+\vartheta} u^m(s) \, ds \right|_{V_2} \right] \leq C_1 C_2 E \left[ \int_0^T \vartheta_i^{1/2} u^m(s) \, ds \right] \leq C_1 C_2 (C_2)^2 \vartheta_i^{1/2} = C_1 \vartheta_i^{1/2},
\]

then \( u^m_{3x}(t) \) fulfills assumptions of Lemma A.4 for \( a := 1 \) and \( b := \frac{1}{2} \) with the norm \( \cdot \mid_{U_2} \).

Similarly

(3.24)

\[
E \left[ \left| u^m(\tau_m + \vartheta)u^m_x(\tau_m + \vartheta) - u^m(\tau_m + \vartheta)u^m_x(\tau_m + \vartheta) \right|_{U_2} \right] = E \left[ \left| \int_{\tau_m}^{\tau_m+\vartheta} u^m(s)u^m_x(s) \, ds \right|_{U_2} \right] \leq E \left[ \left| \int_{\tau_m}^{\tau_m+\vartheta} u^m(s)u^m_x(s) \, ds \right|_{V_2} \right] \leq \frac{1}{2} C_4 E \left[ \int_{\tau_m}^{\tau_m+\vartheta} (u^m(s))^2 \, ds \right] \leq \frac{1}{2} C_4 C_5 C_6 \vartheta_i^{1/2} \leq \frac{1}{2} C_4 C_5 C_6 \vartheta_i^{1/2} \vartheta_i^{1/2} T = C_1 \vartheta_i^{1/2}.
\]

Therefore \( u^m(t)u^m_x(t) \) fulfills assumptions of Lemma A.4 for \( a := 1 \) and \( b := \frac{1}{2} \) with the norm \( \cdot \mid_{U_2} \). In the case of all other terms the result from [18] is used.

Lemma 3.5. ([18, p. 23])

(i) Let \( F : [0,T] \times H \times Y \to H \) fulfills conditions (F1)-(F4). Then the process \( P_m F(s, u^m, y) \) fulfills assumptions of Lemma A.4 for \( a := 2 \) and \( b := 1 \) with the norm \( \cdot \mid_{U_2} \).

(ii) Let \( \Phi : [0,T] \times V \to L_2^0(L^2(\mathbb{R})) \) fulfills conditions (P1)-(P4). Then the process \( P_m \Phi(s, u^m(s)) \) fulfills assumptions of Lemma A.4 for \( a := 1 \) and \( b := 1 \) with the norm \( \cdot \mid_{U_2} \).

Then due to Lemma A.4 the sequence \( \{u^m(t)\} \) fulfills the Aldous condition in the space \( U'_2 \), what finishes the proof.

Appendix A: Compensated time homogeneous Poisson random measure

The following definition is cited from [18] (see also [21]).

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space with filtration \( \mathbb{F} := (\mathcal{F}_t)_{t \geq 0} \).

Definition A.6. Let \((Y, \mathcal{Y})\) be a measurable space. A time homogeneous Poisson random measure \( \eta \) on \((Y, \mathcal{Y})\) over \((\Omega, \mathcal{F}, \mathbb{P})\) is a measurable function such that
(i) for all $B \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{Y}$, $\eta(B) := i_B \circ \eta : \Omega \to \bar{\mathbb{N}}$ is a Poisson random measure with parameter $\mathbb{E}[\eta(B)]$;

(ii) $\eta$ is independently scattered, i.e. if the sets $B_j \subset \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{Y}$, $j = 1, \ldots, n$, are disjoint then the random variables $\eta(B_j)$, $j = 1, \ldots, n$, are independent.

(iii) For all $U \in \mathcal{Y}$ the $\bar{\mathbb{N}}$-valued process $(N(t, U)_{t \geq 0}$ defined by

$$N(t, U) := \eta((0, t] \times U), \quad t \geq 0$$

is $\mathbb{F}$-adapted and its increments are independent of the past, i.e. if $t > s \geq 0$, then $N(t, U) - N(s, U) = \eta((s, t] \times U)$ is independent on $\mathcal{F}_s$.

If $\eta$ is a time homogeneous Poisson random measure then the formula

$$\nu(A) := \mathbb{E}[\eta(0, 1] \times A)], \quad A \in \mathcal{Y}$$

defines a measure on $(\mathcal{Y}, \mathcal{Y})$ called an intensity measure of $\eta$. Moreover, for all $T < \infty$ and all $A \in \mathcal{Y}$ such that $\mathbb{E}[\eta(0, 1] \times A)] < \infty$, the $\mathbb{R}$-valued process $\tilde{N}(t, A)_{t \in (0, T]}$ defined by

$$\tilde{N}(t, A) := \eta((0, T] \times A) - t \nu(A), \quad t \in (0, T]$$

is an integrable martingale on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. The random measure $l \otimes \nu$ on $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{Y}$, where $l$ stands for the Lebesgue measure, is called an compensator of $\eta$ and the difference between a time homogeneous Poisson random measure $\eta$ and its compensator, i.e.

$$\tilde{\eta} := \eta - l \otimes \nu,$$

is called a compensated time homogeneous Poisson random measure.

REFERENCES

[1] Ablowitz A., Nonlinear dispersive waves. Asymptotic Analysis and solitons, Cambridge University Press, Cambridge, 2001.

[2] Burkholder D. L., Davis B. J., Gundy R. F., Integral inequalities for convex functions of operators on martingales. Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability, Volume 2: Probability Theory, p. 223-240, University of California Press, Berkeley, California, 1972.

[3] de Bouard A., Debussche A., On the stochastic Korteweg - de Vries Equation. J. Funct. Anal., 154 (1998) 215-251.

[4] de Bouard A., Debussche A., On a stochastic Korteweg - de Vries with homogeneous noise. In Séminaire: Équations aux Dérivées Partielles. 2007-2008, École Polytech., Exp. No. V, 2009.

[5] Debussche A., Printems J., Effect of localized random forcing term on the Korteweg-de Vries equation. J. Comput. Anal. Appl., 3, (2001) 183-206.

[6] Dingemans, M. Water wave propagation over uneven bottoms, World Scientific, Singapore, 1997.
[7] Drazin P.G., Johnson R.S., *Solitons: An introduction*, Cambridge University Press, 1989.

[8] Gao W., Bao J., Exact solutions for a \((2+1)\)-dimensional stochastic KdV equation, J. Jilin Univ. Sci. 44 (2006) 46-49.

[9] Flandoli F., Gątarek D., *Martingale and stationary solutions for stochastic Navier-Stokes equations*, Probability Theory and Related Fields, 102, (1995) 367-391.

[10] Gawarecki L., Mandrekar V., *Stochastic differential equations in infinite dimensions*, Springer, New York, 2011.

[11] Hartman P., *Ordinary Differential Equations, 2nd ed.*, Society for Industrial and Applied Mathematics, Philadelphia, 2002.

[12] Herman R., Rose A., Numerical realizations of solutions of the stochastic KdV equation, Math. Comput. Simulation, 80 (2009) 164-172.

[13] Infeld E., Rowlands G., *Nonlinear Waves, Solitons and Chaos*, Cambridge University Press, 2nd Edition: UK, 2000.

[14] Ikeda N., Watanabe S., *Stochastic Differential Equations and Diffusion Processes, 2nd ed.*, North-Holland Publishing Company, Amsterdam, 1989.

[15] Karczewska A., Szczeciński M., Rozmej P., Boguniewicz B., Finite element method for stochastic extended KdV equations, Comput. Meth. Phys. Tech., 22 (2016), no 1, 19-29.

[16] Karczewska, A., Szczeciński, M. Martingale solution to stochastic extended Korteweg - de Vries equation. Submitted.

[17] Korteweg D.J., de Vries H., On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. Philosophical Magazine, 39 (1985) 422-443.

[18] Motyl E., Stochastic Navier - Stokes equations driven by Lévy noise in unbounded 3D domains, Potential Anal., 38 (2013), no. 3, 863-912.

[19] Newell A.C., *Solitons in Mathematics and Physics*, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1985.

[20] Osborne A., *Nonlinear Ocean Waves and the Inverse Scattering Transform*, Elsevier, 2010.

[21] Peszat S., Zabczyk, J., *Stochastic Partial Differential Equations with Lévy Noise*, Cambridge University Press, 2007.

[22] Printems J., The stochastic Korteweg - de Vries equation in \(L^2(\mathbb{R})\). J. Diff. Eq., 153 (1999) 338-383.

[23] Remoissenet M., *Waves Called Solitons*, Springer-Verlag, Berlin, 1994.

[24] Tao T., *Nonlinear Dispersive Equations, Local and Global Analysis*, CBMS Regional Conference Series, 106, American Mathematical Society: USA, 2006.

[25] Whitham G.B., *Linear and Nonlinear Waves*, Wiley, First Indian Reprint, 2014.
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