LOCALIZED QUANTITATIVE ESTIMATES AND POTENTIAL BLOW-UP RATES FOR THE NAVIER-STOKES EQUATIONS

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IN MEMORY OF MY GRANDFATHER MAURICE (1930-2022)

ABSTRACT. We show that if \( v \) is a smooth suitable weak solution to the Navier-Stokes equations on \( B(0, 4) \times (0, T_*) \), that possesses a singular point \( (x_0, T_*) \in B(0, 4) \times \{T_*\} \), then for all \( \delta > 0 \) sufficiently small one necessarily has

\[
\limsup_{t \uparrow T_*} \frac{\|v(\cdot, t)\|_{L^3(B(x_0, \delta))}}{\log \log \log \left( \frac{1}{(T_* - t)^{\frac{1}{4}}} \right)^{\frac{1}{1129}}} = \infty.
\]

This local result improves upon the corresponding global result recently established by Tao \([34]\). The proof is based upon a quantification of Escauriaza, Sergeraert and Šverák’s qualitative local result \([11]\). In order to prove the required localized quantitative estimates, we show that in certain settings one can quantify a qualitative truncation/localization procedure introduced by Neustupa and Penel \([25]\). After performing the quantitative truncation procedure, the remainder of the proof hinges on a physical space analogue of Tao’s breakthrough strategy, established by Prange and the author \([7]\).

1. INTRODUCTION

In \([21]\), Leray showed that for any square integrable weakly divergence-free initial data, there exists an associated finite-energy solution\(^1\) \( v : \mathbb{R}^3 \times (0, \infty) \to \mathbb{R}^3 \) of the three-dimensional Navier-Stokes equations

\[
\begin{align*}
\partial_t v - \Delta v + v \cdot \nabla v + \nabla p &= 0, \\
\text{div } v &= 0, \\
v(x, 0) &= v_0(x) \quad \text{in } \mathbb{R}^3 \times (0, \infty).
\end{align*}
\]

The above equations are invariant under the Navier-Stokes rescaling

\[
\begin{align*}
(v_\lambda(x, t), p_\lambda(x, s), v_{0\lambda}(x)) := (\lambda v(\lambda x, \lambda^2 t), \lambda^2 p(\lambda x, \lambda^2 t), \lambda v_0(\lambda x)) \quad \text{with } \lambda > 0.
\end{align*}
\]

After almost 90 years, it remains unknown if finite energy solutions of (1), with sufficiently smooth initial data, remain smooth for all times.

This paper concerns localized quantitative behavior for potential blow-up solutions of (1) in terms of the \( L^3 \) norm, which is invariant with respect to the rescaling for the initial data (2) and is referred to as a critical norm.

The first quantitative blow-up rates for the Navier-Stokes equations were established by Leray. Leray showed that if a finite-energy solution of the Navier-Stokes equations first loses smoothness at \( T_* > 0 \) then for \( 3 < p \leq \infty \)

\[
\|v(\cdot, t)\|_{L^p(\mathbb{R}^3)} \geq \frac{C(p)}{(T_* - t)^{\frac{1}{2}(1 - \frac{3}{p})}} \quad \text{for every } 0 \leq t < T_*.
\]

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\( ^1\) For a definition of finite energy solutions see subsection 2.1.3.
This can be deduced from the fact that for \(3 < p \leq \infty\), a finite-energy solution of (1) (with sufficiently smooth initial data) stays smooth on \((0, T)\) with

\[
T \geq C_p \|v_0\|_{L^p(\mathbb{R}^3)}^{1 - \frac{3}{p}}.
\]

The behavior of the critical \(L^3\) norm of \(v\) is more subtle and the above argument cannot be used to deduce that it necessarily diverges near a potential blow-up time. As observed in [22], this obstruction also occurs for critical norms of other evolution equations with a scaling symmetry (see also [5]).

The first qualitative blow-up results for the critical \(L^3\) norm of the Navier-Stokes equations was shown in a celebrated paper by Escauriaza, Seregin and Šverák in [11]. In [11], a contradiction argument using backward uniqueness and unique continuation for parabolic operators was used to show that if a finite-energy solution on \((0, \infty)\) first loses smoothness at \(T_*\) then for any singular point \(2\) \((x_0, T_*)\) and any \(\delta > 0\) one necessarily has

\[
\limsup_{t \uparrow T_*} \|v(\cdot, t)\|_{L^3(B(x_0, \delta))} = \infty.
\]

This follows from a regularity criteria established in the same paper. Namely, for a suitable weak solution\(^3\) on \(B(0, 1) \times (-1, 0)\) we have the qualitative regularity criteria that

\[
v \in L^\infty(-1, 0; L^3(B(0, 1))) \Rightarrow v \in L^\infty(B(0, r) \times (-r^2, 0))\]

for all sufficiently small \(r > 0\).

Since [11], there have been many extensions regarding qualitative potential blow-up behavior of critical norms for the Navier-Stokes equations. See [30], [29], [14], [36], [1] and [2]. After Escauriaza, Seregin and Šverák’s result, the subtle question of the divergence of critical norms near a potential blow-up was examined for other supercritical partial differential equations. See [22], [24], [23] and [35], for example.

With the exception of [22], the aforementioned results regarding critical norms are based on contradiction arguments and are hence purely qualitative. To the best of the author’s knowledge, the first quantitative blow-up rates for critical norms of a supercritical partial differential equation was obtained by Merle and Raphaël in [22]. In [22], a logarithmic blow-up rate was obtained for the critical norms of radial solutions of the \(L^2\) supercritical nonlinear Schrödinger equation. In the breakthrough paper [34], Tao established the first quantitative blow-up rate for the critical \(L^3\) norm for finite energy solutions of the Navier-Stokes equations (1). In particular, Tao showed that if a finite-energy solution on \((0, \infty)\) first loses smoothness at \(T_*\) then one necessarily has

\[
\limsup_{t \uparrow T_*} \frac{\|v(\cdot, t)\|_{L^3(\mathbb{R}^3)}}{(\log \log \log(\frac{1}{T_* - t}))^c} = \infty.
\]

where \(c > 0\) is a universal constant. Tao’s result is a consequence of the following quantitative estimate. Namely, for a smooth finite-energy solution to the Navier-Stokes equations on \(\mathbb{R}^3 \times (-1, 0)\) we have that

\[
\|v\|_{L^\infty(-1, 0; L^3(\mathbb{R}^3))} \leq M \Rightarrow \|v\|_{L^\infty(\mathbb{R}^3)} \leq \exp \exp(\exp(M^C)),
\]

where \(C\) is a positive universal constant. The above quantitative blow-up rate and estimate have been subsequently extended and generalized in [7], [27]-[28] and [15].

In light of Escauriaza, Seregin and Šverák’s blow up criteria (5) and regularity criteria (6), it is natural to ask the following questions.

\(^2\)For the definition of a singular point, see subsection 2.1.4.

\(^3\)For a definition of suitable weak solutions see subsection 2.1.3.
1) Can Escauriaza, Seregin and Šverák’s regularity criteria (6) be quantified?

2) Can we obtain quantitative blow-up rates for the critical $L^3$ norm in any local neighborhood of a singular point?

Question 2) is also of potential interest in the numerical search for potentially singular solutions of the Navier-Stokes equations (1). In [16], the growth of the localized $L^3$ norm is investigated for certain numerical solutions for the three-dimensional axisymmetric solutions of the Navier-Stokes equation. In this paper we provide an answer to questions 1)-2) by means of the following results.

1.1. Main Results. Our first result quantifies Escauriaza, Seregin and Šverák’s regularity criteria (6), thus providing a positive answer to question 1) in the previous subsection.

**Theorem 1.** Suppose that $(v, p)$ is a smooth solution to the Navier-Stokes equations on $B(0, 4) \times (-16, 0)$ and that $(v, p)$ is a suitable weak solution on $Q(0, 4) := B(0, 4) \times (-16, 0)$.

Furthermore, suppose that $M \in (0, \infty)$ is such that

- $\|v\|_{L^\infty_t L^3_x(Q(0,4))} \leq M$ and
- $\max\{M_0, \|p\|_{L^{\frac{3}{2}}_x t(Q(0,4))}^{\frac{3}{2}} \} \leq M$.

Here, $M_0$ is a sufficiently large universal constant.

Then we conclude that

$$\|v\|_{L^\infty_{t,x}(B(0,\frac{1}{4}) \times (-e^{-e^{-e^{M^{128}}}},0))} \leq e^{e^{e^{M^{128}}}}.$$  \hspace{1cm} (9)

As a consequence of the quantitative estimate (9), we obtain quantitative blow-up rates of the critical $L^3$ norm in any local neighborhood of a singular point. This improves Tao’s necessary condition (7) and quantifies Escauriaza, Seregin and Šverák’s qualitative necessary condition (5).

**Theorem 2.** Suppose that $(v, p)$ is a smooth suitable weak solution to the Navier-Stokes equations on $B(0, 4) \times (0, T^*)$.

Furthermore suppose that

- $v \in L^\infty_{t,loc}(0, T^*); L^\infty(B(0,4))$ and
- there exists $x_0 \in B(0,4)$ such that for all $r \in (0, \text{dist}(x_0, \partial B(0,4)))$
  $$v \notin L^\infty_{x,t}(B(x_0, r) \times (T_* - r^2, T_*)).$$

Then, under the above assumptions we conclude that for all $\delta \in (0, \text{dist}(x_0, \partial B(0,4)))$ we have

$$\limsup_{t \uparrow T_*} \frac{\|v(\cdot, t)\|_{L^3(B(x_0,\delta))}}{(\log \log (\frac{1}{(T_*-t)^2}))^{\frac{1}{1129}}} = \infty.$$  \hspace{1cm} (10)

1.2. Comparison to Previous Literature and Novelty of Our Approach. As previously mentioned, Tao’s global quantitative potential blow-up rate (7) relies upon the quantitative estimate (8), which is achieved via a strategy\footnote{For a detailed description of Tao’s strategy, we refer the reader to [34] and [7].} using quantitative Carleman inequalities for parabolic differential inequalities. A key part of Tao’s strategy is using that a smooth solution $v : \mathbb{R}^3 \times [0, 1] \rightarrow \mathbb{R}^3$ with global bound

$$\|v\|_{L^\infty_{t,x} L^3_x(\mathbb{R}^3 \times (0,1))} \leq M$$  \hspace{1cm} (11)
possesses space-time regions of regularity, which can be quantified explicitly in terms of \( M \).

An elementary approach to obtaining quantitative regions of space-time regularity in this setting was provided in the subsequent paper \([7]\). In \([7]\), it was used that the global functions \((v, p)\) satisfying \((11)\) also satisfy

\[
\int_{\frac{1}{2}}^{1} \int_{B(0, \exp(M^C))} |v|^{\frac{10}{3}} + |p|^{\frac{5}{3}} \, dx \, dt \lesssim M^{O(1)}.
\]

Here, \( C \) is a positive universal constant. In particular, the estimate \((12)\) occurs for an integral over a region with exponentially more spatial scales compared to the right hand side bound. This then allows the use of the pigeonhole principle and Caffarelli, Kohn and Nirenberg’s \( \varepsilon \)-regularity criteria \([9]\) to obtain that \( v \) is quantitatively bounded in quantitative space-time annuli.

In the context of a smooth suitable weak solution \((v, p)\) on \( B(0, 4) \times (-16, 0) \) with

\[
\|v\|_{L^\infty_t L^3_x(B(0, 4) \times (-16, 0))} \leq M
\]

(\( M \) as in Theorem 1), then \((v, p)\) does not satisfy \((12)\) but instead obeys the Morrey bound

\[
\sup_{(x_0, t_0) \in B(0, 1) \times (-1, 0)} \int_{r_1}^{r_2} \int_{B(0, r)} |v|^{\frac{10}{3}} + |p|^{\frac{5}{3}} \, dx \, dt \lesssim M^{O(1)}.
\]

In contrast to the global setting \((12)\), \((13)\) is integrated over a region that is related to the right hand side bound by a power law. This prevents us applying the same arguments as in the global case \((34)\) and \([7]\) to obtain quantitative bounds for \( v \) on quantitative space-time annuli. This represents a major block when attempting to apply the previous strategies \((34)\) and \([7]\) for producing quantitative estimates to the local setting of suitable weak solutions.

A systematic bridge between global regularity results and local regularity results was established by Neustupa and Penel in \([25]\), in the context of one-component regularity criteria for the Navier-Stokes equations, which we will refer to as a ‘truncation procedure’. This bridge relies upon the following fact for suitable weak solutions. Namely, if \((v, p)\) is a suitable weak solution on \( B(0, 1) \times (-1, 0) \) then there exists \( 0 < r_1 < r_2 < 1 \) and \( 0 < \delta_1 < 1 \) such that

\[
v \in L^\infty_{x,t}(B(0, r_2) \setminus B(0, r_1) \times (-\delta_1, 0)).
\]

This relies upon the celebrated fact (proven in \([9]\)) that the parabolic one-dimensional Hausdorff dimension of the singular set is zero. Therefore, \((14)\) is purely qualitative.

Having established \((14)\), in \([25]\) an appropriate cut-off function \( \Phi \) is chosen and the Bogovskiǐ operator \([8]\) is utilized to ‘truncate’ the original suitable weak solution \( v \) to obtain a global function \( u : \mathbb{R}^3 \times (-\delta_1, 0) \rightarrow \mathbb{R}^3 \) possessing the following properties.

- \( u \) agrees with \( v \) on \( B(0, r_1) \times (-\delta_1, 0) \).
- \( u \) solves the Navier-Stokes equations with compactly supported forcing \( f : \mathbb{R}^3 \times (-\delta_1, 0) \rightarrow \mathbb{R}^3 \).
- \( f \) belongs to subcritical spaces, with respect to the Navier-Stokes rescaling for the forcing\(^5\).

\(^5\)For the notion of subcritical forcing, see subsection 2.1.4.
The fact that \( f \) belongs to subcritical spaces means that it does not infer with the regularity properties of \( u \) and one can then obtain apriori estimates for the global function \( u \). The main difficulty with applying Neustupa and Penel’s above truncation procedure to obtain the localized quantitative estimates in Theorem 1 is as follows. Neustupa and Penel’s truncation procedure cannot be quantified in the general case, since it relies upon the qualitative property (14) coming from Caffarelli, Kohn and Nirenberg’s qualitative result [9].

To overcome the aforementioned difficulties in producing the local quantitative estimates in Theorem 1, we show that under the assumptions of Theorem 1 Neustupa and Penel’s truncation procedure can be fully quantified (Proposition 4). A key part of quantifying the above truncation procedure is the application of local-in-space smoothing near the initial time for the Navier-Stokes equations, which was initiated in Jia and Šverák’s seminal work [17] and subsequently extended in [6], [18]-[19] and [4].

Having quantified Neustupa and Penel’s truncation procedure, we get a solution to the Navier-Stokes equations \( V : \mathbb{R}^3 \times (-2, 0) \to \mathbb{R}^3 \) with forcing \( F : \mathbb{R}^3 \times (-2, 0) \to \mathbb{R}^3 \) such that \( V \) has quantitatively controlled norms and \( F \) has quantitatively controlled (negligible) subcritical norms. This allows us to apply energy methods to \( V \) and then using \( \varepsilon \)-regularity in a similar way to [7], we can obtain quantitative space-time regions of space-time regularity for \( V \). This then allows us to apply the strategy for producing quantitative estimates in [7] to \( V \) instead of \( v \), which enables us to obtain Theorem 1. In order to apply the strategy in [7], some bookkeeping is required to ensure that the support in space of the forcing \( F \) does not intersect with the space-time regions where the quantitative Carleman inequalities are applied to \( \Omega = \nabla \times V \).

1.3. Extensions and Open Problems. Tao’s quantitative blow-up rate for the \( L^3 \) norm of the Navier-Stokes equations near a blow-up time has been improved in [7], [27]-[28] and [15]. In [7], it was shown that if \( u : \mathbb{R}^3 \times (0, T_*) \to \mathbb{R}^3 \) is a smooth solution to the Navier-Stokes equations on \( \mathbb{R}^3 \times (0, T_*) \), which satisfies the global bound

\[
\|v\|_{L^\infty(0, T_*; L^{3, \infty}(\mathbb{R}^3))} \leq M
\]

and possesses a singular point at \((x, t) = (0, T_*)\) then

\[
\int_{B(0, R)} |v(x, t)|^3 dx \geq \exp(-\exp(M^C)) \log \left( \frac{R^2 M^{-C}}{T_* - t} \right)
\]

for all \( t \) sufficiently close to \( T_* \). Here, \( C \) is a positive universal constant. In [27] it was shown that if \( v : \mathbb{R}^3 \times (0, \infty) \to \mathbb{R}^3 \) is a smooth finite-energy axisymmetric solution, which first loses smoothness at \( T_* \), then

\[
\lim_{t \to T_*} \sup_{t \leq T_*} \frac{\|v(\cdot, t)\|_{L^3(\mathbb{R}^3)}}{(\log \log \left( \frac{1}{T_* - t} \right))^{1/2}} = \infty.
\]

Here, \( c \) is a positive universal constant.

Using the methodology of this paper, we expect that these results can be extended by being localized. For the blow-up rate (16), it seems plausible that the global assumption (15) can be replaced by the local assumption

\[
\|v\|_{L^\infty(T_* - \delta^2, T_*; L^{3, \infty}(B(0, \delta)))} \leq M \quad \text{for } \delta \in (0, T_*^{1/2}).
\]

\[ \text{This is a physical space analogue of Tao’s strategy in [34].} \]
We also expect that for axisymmetric solutions, the blow-up rate (17) can be refined to obtain
\[
\limsup_{t \to T^-} \frac{\|v(\cdot, t)\|_{L^3(B(x_0, \delta))}}{(\log \log \frac{1}{T-t})^3} = \infty \quad \text{for } \delta > 0.
\]
Here, \(x_0 = (0, 0, z_0)\) is any potential singular point at time \(T^-\). We also expect that similar localized refinements apply to the global results obtained in \([28]\) and \([15]\).

In \([3]\), Neustupa and Penel’s qualitative truncation procedure was used to show that if \(v : \mathbb{R}^3 \times (-1, 0) \to \mathbb{R}^3\) is a smooth finite-energy solution on \(\mathbb{R}^3 \times (-1, 0)\) then for any \(\delta > 0\)
\[
\sup_n \|v(\cdot, t(n))\|_{L^3(B(0, \delta))} = M \quad \text{with} \quad t(n) \uparrow 0
\]
implies that the space-time origin is not a singular point. It remains unclear how to quantify the regularity criteria (18). In particular, the following remains open.

**Open Problem:** Under the assumption (18), does there exist a function \(F : \{a(n)_{n \in \mathbb{N}} : a(n) \in (-1, 0) \times [0, \infty) \times [0, \infty) \to [0, \infty)\} \) such that
\[
|v(0, 0)| \leq F((t(n))_{n \in \mathbb{N}}, M, \delta)?
\]
The difficulty in obtaining (19) is that under the assumption (18), \(v\) does not necessarily belong to critical Morrey spaces, which is currently needed in this paper to quantify Neustupa and Penel’s truncation procedure.

2. Preliminaries

2.1. Notation.

2.1.1. Universal Constants, Vectors and Domains. We use the notation \(X \lesssim Y\), which means that there exists a positive universal constant \(C\) such that \(X \leq CY\). We will also use the notation \(X \lesssim_q Y\), which means there exists a positive constant \(C(q)\) depending on \(q\) only such that \(X \leq C(q)Y\). Sometimes in this paper, we will write \(X \lesssim Y \leq Z\). This means that there exists a positive universal constant \(C\) such that \(X \leq CY \leq Z\). In this paper we will sometimes refer to a quantity (for example \(\mathcal{M}\)) being ‘sufficiently large’, which should be understood as \(\mathcal{M}\) being larger than some universal constant that can (in principle) be specified.

For a vector \(a\), \(a_i\) denotes the \(i^{th}\) component of \(a\). For \((x, t) \in \mathbb{R}^4\) and \(r > 0\) we denote
\[
B(x, r) := \{y \in \mathbb{R}^3 : |y - x| < r\}
\]
and
\[
Q((x, t), r) := B(x, r) \times (t - r^2, t).
\]
When \((x, t) = (0, 0)\), we will sometimes write \(Q(0, r) := Q((0, 0), r)\). We denote the average of a function \(f\) over the ball \(B(x, r)\) by
\[
(f)_{B(x, r)} := \frac{1}{|B(0, r)|} \int_{B(x, r)} f dy.
\]
Here, \(|B(0, r)|\) denotes the Lebesgue measure of the ball \(B(0, r)\).

For \(a, b \in \mathbb{R}^3\), we write \((a \otimes b)_{\alpha\beta} = a_{\alpha}b_{\beta}\), and for \(A, B \in M_3(\mathbb{R})\), \(A : B = A_{\alpha\beta}B_{\alpha\beta}\). Here and in the whole paper we use Einstein’s convention on repeated indices. For \(F : \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}^3\), we define \(\nabla F \in M_3(\mathbb{R})\) by \((\nabla F(x))_{\alpha\beta} := \partial_\beta F_\alpha\).

For \(\lambda \in \mathbb{R}\), \([\lambda]\) denotes the greatest integer less than \(\lambda\). Furthermore, \([\lambda]\) denotes the smallest integer greater than \(\lambda\).
2.1.2. Sobolev Spaces, Bochner Spaces and Norms. For $O \subseteq \mathbb{R}^3$, $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, $W^{k,p}(O)$ denotes the Sobolev space with differentiability $k$ and integrability $p$.

If $X$ is a Banach space with norm $\| \cdot \|_X$, then $L^p(a, b; X)$, with $a < b$ and $s \in [1, \infty)$, will denote the usual Banach space of strongly measurable $X$-valued functions $f(t)$ on $(a, b)$ such that

$$\| f \|_{L^p(a, b; X)} := \left( \int_{a}^{b} \| f(t) \|_X^p \, dt \right)^{\frac{1}{p}} < +\infty.$$  

The usual modification is made if $s = \infty$. Let $C([a, b]; X)$ denote the space of continuous $X$ valued functions on $[a, b]$ with usual norm. In addition, let $C_w([a, b]; X)$ denote the space of $X$ valued functions, which are continuous from $[a, b]$ to the weak topology of $X$.

Let $O \subseteq \mathbb{R}^n$ and $a, b \in \mathbb{R}$. Sometimes we will denote $L^p(a, b; L^q(O))$ and $L^p(a, b; W^{k,a}(O))$ by $L^p_t(L^q_x) \cap (a, b))$ and $L^p_tW^{k,q}(O \times (a, b))$. For the case $p = q$ we will often write $L^p_t(O \times (a, b))$ to denote $L^p(a, b; L^p(O))$. For convenience, we will often use the following notation for energy type norms

$$\| f \|_{L^p_t(L^q_x) \cap (a, b))} := \left( \| f \|_{L^p_t(L^q_x(O))}^2 + \| \nabla f \|_{L^p_t(L^q_x(O))}^2 \right)^{\frac{1}{2}}.$$

2.1.3. Finite Energy Solutions and Suitable Weak Solutions. We say that $v$ is a finite energy solution on $(T_1, T)$ if $v \in C_w([T_1, T]; L^2(\mathbb{R}^3) \cap L^2(T_1, T; H^1(\mathbb{R}^3)))$ is a distributional solution to (1) and satisfies the global energy inequality

$$\| v(\cdot, t) \|_{L^2(\mathbb{R}^3)}^2 + \int_{T_1}^{T} \int |\nabla v(x, s)|^2 \, dxds \leq \| v(\cdot, T_1) \|_{L^2(\mathbb{R}^3)}^2 \quad \forall t \in [T_1, T].$$

Here, $L^2(\mathbb{R}^3) := \{ \phi \in C_c^\infty(\mathbb{R}^3) : \text{div} \phi = 0 \}$.

Let $O \subseteq \mathbb{R}^3$. We say that $(v, p)$ is a suitable weak solution to the Navier-Stokes equations in $O \times (T_1, T)$ if it fulfills the properties described in [31, Definition 6.1]. One such property we will refer to in this manuscript is the local energy inequality for such $(v, p)$. In particular, $(v, p) \in C_w([T_1, T]; L^2_x(\mathbb{R}^3) \cap L^2(T_1, T; H^1(\mathbb{R}^3))) \times L^2_x(\mathbb{R}^3)$ is said to satisfy the local energy inequality in $O \times (T_1, T)$ if the inequality

$$\int_O |v(x, t)|^2 \phi(x, t) \, dx + 2 \int_{-1}^{T} \int_{-1}^{T} |\nabla v|^2 \phi \, dxds \leq \int_{-1}^{T} \int_{-1}^{T} |v|^2 (\partial_t \phi + \Delta \phi) + (|v|^2 + 2p)v \cdot \nabla \phi \, dxds$$

holds for almost every $t \in (T_1, T)$ and all $0 \leq \phi \in C_0^\infty(O \times (T_1, \infty))$.

2.1.4. Singular Points and Criticality. Let $O \subseteq \mathbb{R}^3$. Let $(v, p)$ be a suitable weak solution to the Navier-Stokes equations on $O \times (T_1, T)$. We say that $z_0 = (x_0, t_0) \in \overline{O} \times (T_1, T)$ is a singular point if

$$v \notin L^\infty_x((B(x_0, r) \cap O) \times (t_0 - r^2, t_0))$$

for all $r > 0$ sufficiently small.

Consider the Navier-Stokes equations with forcing $f$

$$\partial_t v - \Delta v + v \cdot \nabla v + \nabla p = f, \quad \text{div} v = 0.$$
This is invariant with respect to the scaling

\[(v_\lambda(x,t), p_\lambda(x,s), v_{0\lambda}(x)) := (\lambda v(\lambda x, \lambda^2 t), \lambda^2 p(\lambda x, \lambda^2 t), \lambda^3 f(\lambda x, \lambda^2 t)) \quad \text{with} \quad \lambda > 0.\]

We say that the scalar quantity \(G(v, p, f)\) is critical or scale-invariant if \(G(v_\lambda, p_\lambda, f_\lambda) = G(v, p, f)\) for all \(\lambda > 0\). The quantity \(G(v, p, f)\) is said to be supercritical if there exists an \(\alpha > 0\) such that \(G(v_\lambda, p_\lambda, f_\lambda) = \lambda^{-\alpha} G(v, p, f)\) for all \(\lambda > 0\). Finally, \(G(v, p, f)\) is said to be subcritical if there exists a \(\beta > 0\) such that \(G(v_\lambda, p_\lambda, f_\lambda) = \lambda^{\beta} G(v, p, f)\) for all \(\lambda > 0\). We will sometimes use phrases such as ’\(v/f\) belongs to critical/supercritical/subcritical spaces’. This means that \(v/f\) belongs to a space-time Lebesgue space, whose norm is critical/supercritical/subcritical.

### 2.2. Bogovskiĭ Operator on an Annulus.

**Lemma 1.** (Bogovskiĭ operator on an annulus) Let \(R \geq 1\) and define the annulus \(A(R) := \{x \in \mathbb{R}^3 : R < |x| < 2R\}\).

There exists a linear operator \(B : C^\infty_{0, \text{avg}}(A(R)) \to C^\infty_0(A(R))\) satisfying (denote \(w = Bg\)) the equation

\[
\begin{cases}
\text{div} w = g & \text{in } A(R) \\
[w]_{\partial A(R)} = 0 & \text{on } \partial A(R).
\end{cases}
\]

Here and in the sequel, \(\text{avg}\) denotes zero spatial average.

Let \(k \in \mathbb{N}_0\) and \(1 < p < \infty.\) Then, for all \(g \in C^\infty_{0, \text{avg}}(A(R))\),

\[
\|\nabla^{k+1} w\|_{L^p(A(R))} \leq C(k, p) \|g\|_{W^{k+1,p}(A(R))}.
\]

Hence, \(B\) extends uniquely to a bounded linear operator \(B : \tilde{W}^{k+1,p}_{\text{avg}}(A(R)) \to \tilde{W}^{k+1,p}(A(R))\) solving (26), where \(\tilde{\cdot}\) denotes the closure of test functions.

Let \(I \subset \mathbb{R}\) be an open interval and \(g \in L_1(I; L^p_{\text{avg}}(A(R)))\). Consider the linear operator \(B\) defined by applying the above operator at almost every time.

If \(\partial_t g \in L_1(I; L^p_0(A(R)))\), then \(B\) commutes with the time derivative:

\[
\partial_t B(g) = B(\partial_t g).
\]

For the time-independent assertions, see [8] and [13, Theorem III.3.3]. The fact that the constant in (27) is independent of \(R \geq 1\) can be seen by rescaling \(w_R(x) = w(Rx)\) and \(g_R(x) = Rg(Rx)\) and then applying the Bogovskiĭ operator on \(A(1)\). To prove (28), one may use the finite difference operator \(D^h \varphi = (\varphi + \varphi(\cdot + h))/h\) and take \(h \to 0^+\). See [13, Exercise III.3.7].

### 3. Local-in-space smoothing and localized backward propagation of vorticity concentration

#### 3.1. Local-in-space smoothing for suitable weak solutions

In [7], Prange and the author used a strategy to produce quantitative estimates based on (i) local-in-space smoothing for a short time [7, Theorem 5.1], (ii) backward propagation of vorticity concentration [7, Lemma 3.1]. In [7], these concepts and statements are formulated in a global context (i.e. the velocity field is defined on \(\mathbb{R}^2\) for each time). In order to prove the localized quantitative estimate in Theorem 1, we require analogues of these concepts but in a local context rather than a global context.

It is already known that local-in-space short time smoothing is essentially a local phenomenon, that applies is a local context (see [7, Theorem 3] and [18, Theorem 1.1]). We use
Theorem 1.1, Remark 1.2 and Theorem 3.1 of [19], which are formulated in a convenient way for our purposes. Here is the statement below, which is essentially taken from [19].

**Proposition 1.** (Local-in-space short time smoothing, [19, Theorem 1.1, Remark 1.2 and Theorem 3.1])

There exists positive universal constants $c_0$ and $\epsilon_*$ such that the following holds true. Let $M \geq 1$ and define $T_0 = c_0 M^{-18}$. Suppose that $(v, p)$ is a suitable weak solution to the Navier-Stokes equations on $B(0,4) \times (0, T_0)$ such that

$$\lim_{t \to 0^+} \|v(\cdot, t) - v_0\|_{L^2(B(0,4))} = 0, \quad \text{with } v_0 \in L^2(B(0,4)).$$

Furthermore suppose that

$$\|v\|_{L^\infty_t L^2_x \cap L^2_t H^1_y(B(0,4) \times (0,T_0))} + \|p\|_{L^{\frac{3}{2}}_t x \cap L^\infty_t (B(0,4) \times (0,T_0))} \leq M$$

and

$$\|v_0\|_{L^3(B(0,3))} \leq \epsilon_*.$$

Then the above hypothesis imply that $v$ satisfies the quantitative bound

$$|\nabla^j v(x,t)| \lesssim t^{-\frac{j+1}{2}} \quad \text{where } j = 0, 1, \ldots \text{ and } (x, t) \in B(0,2) \times (0, T_0).$$

**Remark 1.** Note that in [19], (32) is stated and proven for $j = 0$ by showing that for certain $r > 0$ and $x_0$, we have

$$\int_0^r \int_{B(x_0, r)} |v|^3 + |p|^3 \, dx \, dt \leq \varepsilon_{CKN}$$

and then applying Caffarelli, Kohn and Nirenberg’s celebrated $\varepsilon$-regularity criteria [9]. It is well known that the $\varepsilon$-regularity criteria in [9] gives higher derivative estimates via parabolic smoothing. Thus the proof of [19, Theorem 1.1] also gives (32) for all $j \in \mathbb{N}$. 

**Remark 2.** From [19, Theorem 3.1] the hypothesis that $(v, p)$ is a suitable weak solution on $B(0,4) \times (0, T_0)$ can be replaced by the hypothesis that $v$ is a suitable weak solution on $B(0,4) \times (0, T)$ for any given $T > 0$. Under this change of hypothesis, the estimate (32) instead holds for $(x, t) \in B(0,2) \times (0, \min(T, T_0)).$

**Remark 3** (Removing the pressure). From [20, Theorem 16], a version of Proposition [19] is stated without any assumed bound on the pressure. Such a statement does not obviously improve our main results, due to the appearance of the pressure in the quantitative truncation procedure (Proposition 4).

Now we state a corollary, which is analogous to the global result of [7, Theorem 5.1] (which in turn is based upon ideas in [17]) but in the local setting.

**Corollary 1.** (Local-in-space short time smoothing, with initial data locally in $L^6$)

There exists a positive universal constant $c_1$ such that the following holds true. Let $M$ and $N$ be sufficiently large and define $T_1 = c_1 M^{-18} N^{-52}$. Suppose that $(v, p)$ is a suitable weak solution to the Navier-Stokes equations on $B(0,4) \times (0, T_1)$ such that

$$\lim_{t \to 0^+} \|v(\cdot, t) - v_0\|_{L^2(B(0,4))} = 0, \quad \text{with } v_0 \in L^2(B(0,4)).$$

Furthermore suppose that

$$\|v\|_{L^\infty_t L^2_x \cap L^2_t H^1_y(B(0,4) \times (0,T_1))} + \|p\|_{L^{\frac{3}{2}}_t x \cap L^\infty_t (B(0,4) \times (0,T_1))} \leq M$$

and

$$\|v_0\|_{L^6(B(0,3))} \leq N.$$
Then the above hypothesis imply that \( v \) satisfies the quantitative bound

\[
|\nabla^j v(x, t)| \lesssim_j t^{-\frac{j+1}{2}} \quad \text{where} \quad j = 0, 1, \ldots \quad \text{and} \quad (x, t) \in B(0, 2) \times (0, T_1).
\]

**Proof.** Take \( \mu \in (0, \frac{1}{3}) \), which will be fixed later. Throughout we fix \( x_0 \in B(0, 2\mu^{-1}) \).

Note that

\[
B(x_0, 3) \subseteq B(0, 3\mu^{-1}) \quad \text{and} \quad B(x_0, 4) \subseteq B(0, 4\mu^{-1}).
\]

Now let us consider the rescaled functions \( v_\mu : B(0, 4\mu^{-1}) \times (0, T_1\mu^{-2}) \rightarrow \mathbb{R}^3 \) and \( p_\mu : B(0, 4\mu^{-1}) \times (0, T_1\mu^{-2}) \rightarrow \mathbb{R} \) defined by

\[
(v_\mu(x, t), p_\mu(x, t)) := (\mu v(\mu x, \mu^2 t), \mu^2 p(\mu x, \mu^2 t)).
\]

Let us also define \( v_{\mu 0} : B(0, 4\mu^{-1}) \rightarrow \mathbb{R}^3 \) by

\[
v_{\mu 0}(x) := \mu v_0(\mu x).
\]

Using the properties (34)-(35), we infer that

\[
\|v_\mu\|_{L^2_t L^2_x}^2 + \|\mu v_\mu\|_{L^2_t(B(0, 4\mu^{-1}) \times (0, T_1\mu^{-2}))}^2 + \|p_\mu\|_{L^2_t(B(0, 4\mu^{-1}) \times (0, T_1\mu^{-2}))} \leq \hat{M} := 3\mu^{-\frac{1}{2}} M
\]

and

\[
\|v_{\mu 0}\|_{L^3(B(x_0, 3))} \leq C_{\text{leb}} N \mu^\frac{1}{2}.
\]

Here, \( C_{\text{leb}} \in (0, \infty) \) is a universal constant coming from the continuous embedding \( L^6(B(x_0, 3)) \hookrightarrow L^3(B(x_0, 3)) \). In view of (41), we now take

\[
\mu := \min \left\{ \frac{\epsilon^2}{C_{\text{leb}}^2 N^2}, \frac{1}{3} \right\} = \frac{\epsilon^2}{C_{\text{leb}}^2 N^2}
\]

for \( N \) sufficiently large. This gives that

\[
\|v_{\mu 0}\|_{L^3(B(x_0, 3))} \leq \epsilon^* \quad \text{and} \quad \hat{M} = C_{\text{univ}} N^8 \mu^\frac{1}{2} M.
\]

Now we take \( c_1 \) in Corollary 1 to be an appropriate universal constant such that

\[
T_1 \mu^{-2} = c_0 \hat{M}^{-18}.
\]

Here, \( c_0 \) is as in Proposition 1. Then (43)-(44) and a translation in space allow us to apply Proposition 1 to get

\[
|\nabla^j v_\mu(x, t)| \lesssim_j t^{-\frac{j+1}{2}} \quad \text{where} \quad j = 0, 1, \ldots \quad \text{and} \quad (x, t) \in B(x_0, 2) \times (0, T_1\mu^{-2}).
\]

Recalling that \( x_0 \in B(0, 2\mu^{-1}) \) is arbitrary, we undo the Navier-Stokes rescaling (38) to finally infer that

\[
|\nabla^j v(x, t)| \lesssim_j t^{-\frac{j+1}{2}} \quad \text{where} \quad j = 0, 1, \ldots \quad \text{and} \quad (x, t) \in B(0, 2) \times (0, T_1).
\]

\( \square \)

**Remark 4.** It follows from Remark 2 and the above proof that the hypothesis that \( (v, p) \) is a suitable weak solution on \( B(0, 4) \times (0, T_1) \) can be replaced by the hypothesis that \( v \) is a suitable weak solution on \( B(0, 4) \times (0, T) \) for any given \( T > 0 \). Under this change of hypothesis, the estimate (36) instead holds for \( (x, t) \in B(0, 2) \times (0, \min(T, T_1)) \).

Let us now state another corollary to Proposition 1, which concerns local-in-space smoothing of solutions with locally square integrable vorticity.
Corollary 2. (Local-in-space short time smoothing, with initial vorticity locally in $L^2$)

There exists a positive universal constant $c_2$ such that the following holds true.

Let $M$ be sufficiently large and define $T_2 = c_2 M^{-44}$. Suppose that $(v, p)$ is a suitable weak solution to the Navier-Stokes equations on $B(0, 4) \times (0, T_2)$ such that

$$\lim_{t \to 0^+} \|v(\cdot, t) - v_0\|_{L^2(B(0, 4))} = 0, \quad \text{with } v_0 \in W^{1,2}(B(0, 4)).$$

Furthermore suppose that

$$\|v\|_{L^\infty L^2 \cap L^2 H^1(B(0, 4) \times (0, T_2))} + \|p\|_{L^\infty_{x,t}(B(0, 4) \times (0, T_2))} \leq M$$

and that for $\omega_0 = \nabla \times v_0$

$$\|\omega_0\|_{L^2(B(0, \frac{4}{M}))} \leq M.$$

Then the above hypothesis imply that $v$ satisfies the quantitative bound

$$|\nabla^j v(x, t)| \lesssim j^{-\frac{r+1}{r}} \quad \text{where} \quad j = 0, 1, \ldots \quad \text{and} \quad (x, t) \in B(0, 2) \times (0, T_2).$$

Proof. Recall that from the Biot-Savart law $-\Delta v_0 = \nabla \times \omega_0$. Using this and elliptic regularity theory gives

$$\|\nabla v_0\|_{L^2(B(0, 3))}^2 \lesssim \|\omega_0\|_{L^2(B(0, \frac{4}{M}))}^2 + \|v_0\|_{L^2(B(0, \frac{4}{M}))}^2.$$  

Applying the Sobolev embedding theorem then gives

$$\|v_0\|_{L^6(B(0, 3))} \lesssim \|\nabla v_0\|_{L^2(B(0, 3))} + \|v_0\|_{L^2(B(0, 3))} \lesssim \|\omega_0\|_{L^2(B(0, \frac{4}{M}))} + \|v_0\|_{L^2(B(0, \frac{4}{M}))}.$$  

Using this and (46)-(47) we get that

$$\|v_0\|_{L^6(B(0, 3))} \leq C_{univ} M^\frac{1}{2}.$$  

Here, $C_{univ}$ is a positive universal constant. Taking $N = C_{univ} M^\frac{1}{2}$ and taking $c_2$ to be an appropriate universal constant, we get the desired conclusion by directly applying Corollary 1. \qed

Remark 5. As was the case in Remark 4, the hypothesis that $(v, p)$ is a suitable weak solution on $B(0, 4) \times (0, T_2)$ can be replaced by the hypothesis that $v$ is a suitable weak solution on $B(0, 4) \times (0, T)$ for any given $T > 0$. Under this change of hypothesis, the estimate (48) instead holds for $(x, t) \in B(0, 2) \times (0, \min(T, T_2)).$

3.2. Local behavior of Vorticity Implies Localized Quantitative Estimates. The strategy to prove the local estimates in Theorem 1 is a localized version of the strategy in [7]. As in [7], we rely on connecting the behavior of the vorticity on spatial scales with quantitative estimates of the velocity field (see [7, Lemma 3.1-3.2]). However unlike in [7], in order to prove Theorem 1 we require such a connection to be in a purely local setting. This is given by the following proposition.

Proposition 2. Let $M$ be sufficiently large. Suppose that $(v, p)$ is a smooth solution to the Navier-Stokes equations on $B(0, 1) \times (-1, 0]$ satisfying

$$\sup_{0 < r \leq 1} \left\{ \frac{1}{r} \left( \|v\|_{L^\infty L^2 \cap L^2 H^1(Q(0,r))}^2 + \frac{1}{r^2} \|p\|_{L^\infty_{x,t}(Q(0,r))}^2 \right) \right\} \leq M.$$

Furthermore, suppose that there exists $t \in (-M^{-48}, 0)$ such that the vorticity $\omega = \nabla \times v$ satisfies

$$\int_{B(0,M^{24}(-t)^{\frac{1}{2}})} |\omega(x, t)|^2 \, dx \leq \frac{M^{-22}}{(-t)^{\frac{1}{2}}}.$$
Then the above assumptions imply that
\begin{equation}
|\nabla^j v(x,s)| \lesssim_j (s-t)^{-\frac{j+1}{2}} \quad \text{where} \quad j = 0,1 \ldots \quad \text{and} \quad (x,s) \in B(0,M^{23}(-t)^{\frac{1}{2}}) \times (t,0).
\end{equation}

\textbf{Proof.} First note that (50) implies that for \(M\) sufficiently large
\begin{equation}
\frac{1}{4M^{23}(-t)^{\frac{1}{2}}} \left\| v \right\|^2_{L_t^\infty L_x^2 \cap L_t^\infty H^1(Q(0,4M^{23}(-t)^{\frac{1}{2}}))} + \frac{1}{(4M^{23}(-t)^{\frac{1}{2}})^{\frac{3}{4}}} \left\| p \right\|^\frac{3}{4}_{L_{t,x}^2(Q(0,4M^{23}(-t)^{\frac{1}{2}}))} \lesssim M.
\end{equation}

Take \(\lambda := M^{23}(-t)^{\frac{1}{2}}\) and let us consider the rescaled functions \(\nu_\lambda : B(0,\lambda^{-1}) \times (0,M^{-46}) \rightarrow \mathbb{R}^3\) and \(p_\lambda : B(0,\lambda^{-1}) \times (0,M^{-46}) \rightarrow \mathbb{R}\) defined by
\begin{equation}
(v_\lambda(y,s), p_\lambda(y,s)) := (\lambda v(\lambda y, \lambda^2 s + t), \lambda^2 p(\lambda y, \lambda^2 s + t)).
\end{equation}

Under this rescaling \(B(0,\lambda^{-1}) \supset B(0,4)\) and furthermore (51) and (53) imply that for \(M\) sufficiently large we have
\begin{equation}
\left\| v_\lambda \right\|^2_{L_t^\infty L_x^2 \cap L_t^\infty H^1(B(0,4) \times (0,M^{-46}))} + \left\| p_\lambda \right\|^\frac{3}{4}_{L_{t,x}^2(B(0,4) \times (0,M^{-46}))} \lesssim M^{\frac{4}{7}}
\end{equation}
and
\begin{equation}
\int_{B(0,\frac{7}{4})} |\omega_\lambda(y,0)|^2 \, dy \leq M \leq M^{\frac{4}{7}}.
\end{equation}

Now, (55)-(56) allows us to apply Corollary 2 and Remark 5 with \(M\) replaced by \(\tilde{M} := M^{\frac{4}{7}}\). If \(\tilde{M}\) is sufficiently large this gives that
\begin{equation}
|\nabla^j v_\lambda(y,s)| \lesssim s^{-\frac{j+1}{2}} \quad \text{where} \quad j = 1 \ldots \quad \text{and} \quad (y,s) \in B(0,2) \times (0,M^{-46}).
\end{equation}

We then undo the Navier-Stokes rescaling (54) to finally infer the desired conclusion (52) for the original functions \((v,p)\).
\[\square\]

3.3. \textbf{Localized Backward Propagation of Vorticity Concentration.} As previously mentioned, backward propagation plays a central role in the strategy to produce quantitative estimates in [7]. The relevant statement is [7, Lemma 3.1], which is in a global context. Now, we state and prove a localized version of [7, Lemma 3.1], which will play a crucial role in proving Theorem 1.

\textbf{Proposition 3.} Let \(M\) be sufficiently large. Suppose that \((v,p)\) is a smooth solution to the Navier-Stokes equations on \(B(0,1) \times (-1,0]\) satisfying
\begin{equation}
\sup_{0 \leq r \leq 1} \left\{ \frac{1}{r} \left\| v \right\|^2_{L_t^\infty L_x^2 \cap L_t^\infty H^1(Q(0,r))} + \frac{1}{r^{\frac{3}{4}}} \left\| p \right\|^\frac{3}{4}_{L_{t,x}^2(Q(0,r))} \right\} \leq M.
\end{equation}

Furthermore, suppose that \(t\) and \(t' \in (-M^{-48},0)\) are such that
\begin{equation}
\frac{-t'}{-t} \leq M^{-50}
\end{equation}
and
\begin{equation}
\int_{B(0,M^{23}(-t')^{\frac{1}{2}})} |\omega(x,t')|^2 \, dx \geq \frac{M^{-22}}{(-t')^{\frac{3}{2}}}.
\end{equation}
Then the above assumptions imply
\begin{equation}
\int_{B(0,M^{24}(-t)^{\frac{1}{2}})} |\omega(x,t)|^2 \, dx > \frac{M^{-22}}{(-t)^{\frac{1}{2}}}.
\end{equation}

**Proof.** Assume that the contrapositive to (60) holds. Then by Proposition 2, we have that there exists a positive universal constant $C_{univ}$ such that
\begin{equation}
|\omega(x,s)| \leq \frac{C_{univ}}{t} \quad \text{where} \quad (x,s) \in B(0,M^{23}(-t)^{\frac{1}{2}}) \times (\frac{t}{2}, 0].
\end{equation}

Note that the assumption (58) implies that
\begin{equation*}
B(0,M^{24}(-t')^{\frac{1}{2}}) \times \{t'\} \subset B(0,M^{23}(-t)^{\frac{1}{2}}) \times (\frac{t}{2}, 0].
\end{equation*}

This together with (61) implies that for $M$ sufficiently large we have
\begin{equation}
\int_{B(0,M^{24}(-t')^{\frac{1}{2}})} |\omega(x,t')|^2 \, dx \leq \frac{M^{74}}{(-t')^{\frac{1}{2}}} \left(\frac{-t'}{-t}\right)^2.
\end{equation}

Now, using the assumption (58) gives that
\begin{equation*}
\int_{B(0,M^{24}(-t')^{\frac{1}{2}})} |\omega(x,t')|^2 \, dx \leq \frac{M^{-26}}{(-t')^{\frac{1}{2}}},
\end{equation*}

which contradicts (59). \[\square\]

4. Quantitative Truncation Procedure

4.1. Local Scale-Invariant Morrey Bounds. A key part of showing Theorem 1 is showing that the solution in Theorem 1 can be truncated to produce a solution of the Navier-Stokes equations with forcing. It is crucial that the forcing is compactly supported and that the subcritical norms of the forcing are quantified and sufficiently small. A central part of this quantitative truncation procedure involves using that the solution in Theorem 1 has a scale-invariant Morrey bound, which we demonstrate in this subsection. Before doing so, let us introduce some relevant notation.

If $(v, p)$ is a suitable weak solution on $Q(0, 4)$, then we define the following quantities for $\rho \in (0, 4]$. Namely,
\begin{equation}
A(v, \rho) := \frac{1}{\rho} \|v\|^2_{L_2^\infty L_2^2(Q(0,\rho))}, \quad E(v, \rho) := \frac{1}{\rho} \|
abla v\|^2_{L^2(Q(0,\rho))}, \quad C(v, \rho) = \frac{1}{\rho^2} \|v\|^2_{L^3(Q(0,\rho))},
\end{equation}

\begin{equation}
D(p, \rho) := \frac{1}{\rho^2} \|p\|^2_{L_{3,3}^\infty(Q(0,\rho))} \quad \text{and} \quad E(v, p, \rho) := A(v, \rho) + E(v, \rho) + D(p, \rho).
\end{equation}

**Lemma 2.** Suppose that $(v, p)$ is a suitable weak solution to the Navier-Stokes equations on $Q(0, 4)$. Furthermore, suppose that for some $M \geq 1$ we have
\begin{equation}
\|v\|_{L_2^\infty L_2^2(Q(0,4))} \leq M.
\end{equation}

Then the above assumptions imply that
\begin{equation}
\sup_{z_0 = (x_0, t_0) \in B(0,1) \times (-1,0], \ 0 < r \leq 1} \left\{ \frac{1}{r^2} \int_{Q(z_0, r)} |p|^2 \, dx dt + \frac{1}{r^2} \|v\|^2_{L_2^\infty L_2^2 \cap L_2^\infty H^1(Q(z_0, r))} \right\} \lesssim M^2 + \|p\|^2_{L_{3,3}^\infty(Q(0,4))}.
\end{equation}
Proof: First observe that by using spatial translations, it is sufficient to show that the following holds true. Namely, if \((v, p)\) is a suitable weak solution on \(Q(0, 2)\) with
\[
\|v\|_{L^\infty_t L^3_x(Q(0,2))} \leq \mathcal{M} \quad (\mathcal{M} \geq 1)
\]
then
\[
\sup_{0 < r \leq 1} \mathcal{E}(v, p, r) \lesssim \mathcal{M}^3 + \|p\|_{L^2_{x,t}(Q(0,2))}^{\frac{3}{4}}.
\]
From (67) and the local energy inequality (22), we observe that in order to show (68) it is sufficient to prove that
\[
\sup_{0 < r \leq 1} D(p, r) \lesssim \mathcal{M}^3 + \|p\|_{L^2_{x,t}(Q(0,2))}^{\frac{3}{4}}.
\]
From [32, pg. 847 (47)] we have that for any \(\theta \in (0, 1)\) and \(0 < r \leq 1\)
\[
D(p, \theta r) \leq C_{univ}(\theta D(p, r) + \theta^{-2}C(v, r)).
\]
Here, \(C_{univ}\) is a positive universal constant. Thus using (67) we obtain that for any \(\theta \in (0, 1)\) and \(0 < r \leq 1\)
\[
D(p, \theta r) \leq C_{univ}(\theta D(p, r) + \theta^{-2}\mathcal{M}^3).
\]
Fixing the universal constant \(\theta \in (0, 1)\) such that \(C_{univ}\theta \leq \frac{1}{2}\), one deduces that
\[
D(p, \theta r) \leq \frac{1}{2}D(p, r) + c(\theta)\mathcal{M}^3 \quad \forall r \in (0, 1].
\]
Iterating (70) gives that
\[
D(p, \theta^k) \leq \frac{1}{2^k}D(p, 1) + 2c(\theta)\mathcal{M}^3 \quad \text{for} \quad k = 0, 1, 2, \ldots.
\]
Take any \(r \in (0, 1]\). There exists \(k \in \{0, 1, 2, \ldots\}\) such that \(\theta^{k+1} < r \leq \theta^k\). From this, we see that
\[
D(p, r) \leq \frac{1}{\theta^2}D(p, \theta^k) \leq \frac{1}{\theta^2} \left( \frac{1}{2^k}D(p, 1) + 2c(\theta)\mathcal{M}^3 \right).
\]
Hence,
\[
\sup_{0 < r \leq 1} D(p, r) \lesssim D(p, 1) + \mathcal{M}^3
\]
as required. \(\square\)

4.2. Quantitative Truncation Procedure: Main Statement and Proof. A crucial part in proving Theorem 1 is the quantification of a qualitative localization procedure introduced in [25], when the solution has an assumed scale-invariant bound. This is given by the proposition below.

Proposition 4. Suppose that \((v, p)\) is a smooth solution to the Navier-Stokes equations on \(B(0, 4) \times (-16, 0]\).
Furthermore, suppose that \(\mathcal{M} \in (0, \infty)\) is such that
\[
\|v\|_{L_t^\infty L_x^3(Q(0,4))} \leq \mathcal{M} \quad \text{and}
\]
\[
\max\{\mathcal{M}_0, \|p\|_{L^2_{x,t}(Q(0,4))}^{\frac{3}{4}}\} \leq \mathcal{M}.
\]
Here, \(\mathcal{M}_0\) is a sufficiently large universal constant. Then the above assumptions imply the following.
There exists functions $V : \mathbb{R}^3 \times [-2, 0] \to \mathbb{R}^3$, $P : \mathbb{R}^3 \times [-2, 0] \to \mathbb{R}$ and $R \geq 6e^{2M}$ such that
\begin{equation}
(V(x,t), P(x,t)) \equiv (\lambda v(\lambda x, \lambda^2 t), \lambda^2 p(\lambda x, \lambda^2 t)) \quad \text{on} \quad B(0,R) \times [-2,0] \quad (\lambda = e^{-2M} e^{-M^5})
\end{equation}
and
\begin{equation}
\sup_{0 < r \leq 1} \left\{ \frac{1}{r^2} \int_{Q(0,r)} |P|^2 dx dt + \frac{1}{r} \|V\|_{L^\infty_t L^2_x \cap L^2_t H^1(Q(0,r))}^2 \right\} \lesssim M^3.
\end{equation}
The functions $(V,P)$ are smooth on $\mathbb{R}^3 \times [-2,0]$ and $B(0,R) \times [-2,0]$. Furthermore, $(V,P)$ solves the Navier-Stokes equations with forcing $F$ on $\mathbb{R}^3 \times (-2,0)$ such that for every $t \in (-2,0)$
\begin{equation}
supp F(\cdot, t) \subseteq B(0,2R) \setminus B(0,R) \quad \text{and}
\end{equation}
\begin{equation}
supp V(\cdot, t) \subseteq B(0,2R).
\end{equation}
Additionally $V$ can be decomposed as $V = V^{(1)} + V^{(2)}$, where $V^{(i)} : \mathbb{R}^3 \times [-2,0] \to \mathbb{R}$ ($i \in \{1,2\}$) satisfy
\begin{equation}
\|V^{(1)}\|_{L^\infty_t L^2_x(\mathbb{R}^3 \times (-2,0))} = \|V\|_{L^\infty_t L^2_x(B(0,R) \times (-2,0))} \leq M \quad \text{and} \quad \|V^{(2)}\|_{L^\infty_t L^2_x(\mathbb{R}^3 \times (-2,0))} \leq e^{-eM}.
\end{equation}
The forcing term $F$ satisfies the estimates
\begin{equation}
\|F\|_{L^2_t L^\infty_x(\mathbb{R}^3 \times (-2,0))} \quad \text{and} \quad \|F\|_{L^2_t L^\infty_x(\mathbb{R}^3 \times (-2,0))} \leq e^{-eM}.
\end{equation}

**Proof.**  
**Step 1: localized quantitative regions of regularity via local-in-space smoothing**

First note that under the assumptions of Proposition 4, we can apply Lemma 2 to infer that for $M_0$ sufficiently large we have
\begin{equation}
\sup_{z_0 = (x_0,t_0) \in B(0,1) \times (-1,0)} \left\{ \frac{1}{r^2} \int_{Q(z_0,r)} |P|^2 dx dt + \frac{1}{r} \|v\|_{L^\infty_t L^2_x \cap L^2_t H^1(Q(z_0,r))}^2 \right\} \lesssim M^3.
\end{equation}
Next, we rescale the solution $(v,p)$:
\begin{equation}
(v_{\lambda(1)}(x,t),p_{\lambda(1)}(x,t)) := (\lambda(1)v(\lambda(1)x, \lambda(1)^2 t), \lambda(1)^2 p(\lambda(1)x, \lambda(1)^2 t)) \quad \text{with} \quad \lambda(1) := e^{-M^5}.
\end{equation}

Then $(v_{\lambda(1)}, p_{\lambda(1)})$ is a smooth solution to the Navier-Stokes equations on $B(0,4e^{M^5}) \times (-16e^{2M^5},0)$. From (72)-(73) we see that
\begin{equation}
\|v_{\lambda(1)}\|_{L^\infty_t L^2_x(Q(0,4e^{M^5}))} \leq M^3 \quad \text{and}
\end{equation}
\begin{equation}
\|p_{\lambda(1)}\|_{L^2_t L^\infty_x(Q(0,4e^{M^5}))} \leq M^2 e^{5/4 \cdot M^5}.
\end{equation}
Furthermore, from (80) we have
\begin{equation}
\sup_{z_0 = (x_0,t_0) \in B(0, e^{M^5}) \times (-e^{2M^5},0), \ 0 < r \leq e^{M^5}} \left\{ \frac{1}{r^2} \int_{Q(z_0,r)} |p_{\lambda(1)}|^2 dx dt + \frac{1}{r} \|v_{\lambda(1)}\|_{L^\infty_t L^2_x \cap L^2_t H^1(Q(z_0,r))}^2 \right\} \lesssim M^3.
\end{equation}
Note that for $\mathcal{M}_0$ sufficiently large, we have that $4^{\mathcal{M}_0^4+3} \leq e^{\mathcal{M}_0^5}$. Thus
\[
\sum_{k=2}^{[\mathcal{M}_0^4]+1} \int_{4^k < |x| < 4^{k+1}} |v_{\lambda(1)}(x, -\mathcal{M}^{-95})|^3 \, dx \leq \int_{B(0, e^{\mathcal{M}_0^5})} |v_{\lambda(1)}(x, -\mathcal{M}^{-95})|^3 \, dx \leq \mathcal{M}^3.
\]
Using the pigeonhole principle, there exists $k_0 \in \{2, \ldots, [\mathcal{M}_0^4] + 1\}$ such that
\[
\int_{A_{k_0}} |v_{\lambda(1)}(x, -\mathcal{M}^{-95})|^3 \, dx \leq \frac{1}{\mathcal{M}} \leq \varepsilon_* \quad \text{with} \quad A_{k_0} := \{ x : 4^{k_0} < |x| \leq 4^{k_0+1}\}.
\]
Here, $\mathcal{M}_0$ is sufficiently large and $\varepsilon_*$ is as in (31) (Proposition 1). Now we fix any $y_0 \in \{ x : 4^{k_0} + 4 \leq |x| \leq 4^{k_0+1} - 4 \} \subset A_{k_0} \subset B(0, e^{\mathcal{M}_0^5})$.

Then (84) implies that we have
\[
\frac{1}{16} \int_0^1 \int_{-\frac{1}{2}B(y_0, 4)} |p_{\lambda(1)}|^2 \, dx \, dt + \frac{1}{4} \| v_{\lambda(1)} \|_{L^\infty_t L^2_x \cap L^2_x H^1(B(y_0, 4))} \leq \mathcal{M}^3.
\]
This and (85) implies that for $\mathcal{M}_0$ sufficiently large we have
\[
\| v_{\lambda(1)} \|_{L^\infty_t L^2_x \cap L^2_x H^1(B(y_0, 4) \times (-\mathcal{M}^{-95}, 0))} + \| p_{\lambda(1)} \|_{L^2_x(B(y_0, 4) \times (-\mathcal{M}^{-95}, 0))} \leq \mathcal{M}^5
\]
and
\[
\int_{B(y_0, 3)} |v_{\lambda(1)}(x, -\mathcal{M}^{-95})|^3 \, dx \leq \varepsilon_*,
\]
This allows us to apply Proposition 1 and Remark 2 with $M$ replaced by $\mathcal{M}^5$. Thus, one deduces that for $\mathcal{M}_0$ sufficiently large
\[
\| \nabla^j v_{\lambda(1)} \|_{L^\infty_t L^2_x(B(y_0, 2) \times (-\mathcal{M}^{-96}, 0))} \leq \mathcal{M}_0^{50j+50} \quad \text{for} \quad j = 0, 1, 2.
\]
Since this holds for any $y_0$ such that $B(y_0, 4) \subset A_{k_0}$, we infer that
\[
\| \nabla^j v_{\lambda(1)} \|_{L^\infty_t L^2_x \{ x : 4^{k_0} + 2 < |x| < 4^{k_0+1} - 2 \} \times (-\mathcal{M}^{-96}, 0))} \leq \mathcal{M}_0^{50j+50} \quad \text{for} \quad j = 0, 1, 2.
\]

**Step 2: pressure and time derivative estimates inside the quantitative region of regularity**

Next, we want to use (83) and (87) to gain higher regularity of $p_{\lambda(1)}$ and $\partial_t v_{\lambda(1)}$ inside the region $\{ x : 4^{k_0} + 3 < |x| < 4^{k_0+1} - 3 \} \times (-\mathcal{M}^{-96}, 0)$.

Let the cut-off function $\varphi \in C^\infty_c(\mathbb{R}^3; [0, 1])$ be such that
\[
\varphi \equiv 1 \quad \text{on} \quad \{ x : 4^{k_0} + \frac{5}{2} < |x| < 4^{k_0+1} - \frac{5}{2} \},
\]
\[
\text{supp} \varphi \subset \{ x : 4^{k_0} + 2 < |x| < 4^{k_0+1} - 2 \} \quad \text{and} \quad \text{for} \quad j \in \mathbb{N} \quad \| \nabla^j \varphi \|_{L^\infty(\mathbb{R}^3)} \leq C_j.
\]
Here, $C_j$ is a positive constant depending only on $j$. Let us also recall from the previous step that $A_{k_0} := \{ x : 4^{k_0} < |x| \leq 4^{k_0+1}\} \subset B(0, e^{\mathcal{M}_0^5})$. Thus,
\[
|A_{k_0}| \lesssim e^{3\mathcal{M}_0^5}.
\]
Here, $|A_{k_0}|$ denotes the Lebesgue measure of $A_{k_0}$.

For every $s \in (-\mathcal{M}^{-96}, 0)$ we decompose the pressure
\[
p_{\lambda_1}(x, s) = p_{\text{Riesz}, \lambda(1)}(x, s) + p_{\text{h}, \lambda(1)}(x, s) \quad \text{with} \quad p_{\text{Riesz}, \lambda(1)}(x, s) = \mathcal{R}_i \mathcal{R}_j (\varphi v_i v_j(\cdot, s)).
\]
Here, \( R_t \) denote Riesz transforms and the Einstein summation convention has been adopted. Let us note that this decomposition implies that for every \( s \in (-M^{-96}, 0) \), \( p_{h, \lambda(1)}(\cdot, s) \) is harmonic in \( \{ x : 4^{-k_0} + \frac{5}{2} < |x| < 4^{-k_0+1} - \frac{5}{2} \} \).

First let us estimate \( p_{Riesz, \lambda(1)} \) and \( \nabla p_{Riesz} \). By the Calderón-Zygmund theory, Hölder’s inequality and (87)-(88), we infer the following for \( M_0 \) sufficiently large. Namely, for any \( q \in (1, \infty) \) and \( s \in (-M^{-96}, 0) \)

\[
\|p_{Riesz, \lambda(1)}(\cdot, s)\|_{L^2(R^3)} \lesssim_q \|v_{\lambda(1)}(\cdot, s)\|_{L^2(4^{-k_0+2} |x| < 4^{-k_0+1}-2)} \lesssim \|v_{\lambda(1)}(\cdot, s)\|_{L^\infty(4^{-k_0+2} |x| < 4^{-k_0+1}-2)} A_{k_0} \frac{1}{q} \lesssim M_0^{100} e^{3M^5} \leq e^{4M^5}.
\]

Similarly, for any \( q \in (1, \infty) \) and \( s \in (-M^{-96}, 0) \) we obtain

\[
\|\nabla p_{Riesz, \lambda(1)}(\cdot, s)\|_{L^q(R^3)} \lesssim_q \|v_{\lambda(1)}(\cdot, s)\|_{L^2(4^{-k_0+2} |x| < 4^{-k_0+1}-2)} + \|v_{\lambda(1)}(\nabla v_{\lambda(1)}(\cdot, s))\|_{L^q(4^{-k_0+2} |x| < 4^{-k_0+1}-2)} \lesssim_q \|v_{\lambda(1)}(\cdot, s)\|_{L^\infty(4^{-k_0+2} |x| < 4^{-k_0+1}-2)} A_{k_0} \frac{1}{q} \lesssim M_0^{150} e^{3M^5} \leq e^{4M^5}.
\]

From (89)-(90) and the Sobolev embedding theorem, we have that

\[
\|p_{Riesz, \lambda(1)}\|_{L^2 L^\infty(R^3 \times (-M^{-96}, 0))} \lesssim e^{4M^5}.
\]

Furthermore, for all \( q \in (1, \infty) \) we have

\[
\|p_{Riesz, \lambda(1)}\|_{L^q L^\infty(R^3 \times (-M^{-96}, 0))} + \|\nabla p_{Riesz, \lambda(1)}\|_{L^q L^2(R^3 \times (-M^{-96}, 0))} \lesssim_q e^{4M^5}.
\]

Next, we focus on estimating \( p_{h, \lambda(1)} \) and \( \nabla p_{h, \lambda(1)} \). Let us recall that for every \( s \in (-M^{-96}, 0) \), \( p_{h, \lambda(1)}(\cdot, s) \) is harmonic in \( \{ x : 4^{-k_0} + \frac{5}{2} < |x| < 4^{-k_0+1} - \frac{5}{2} \} \). Let \( x_0 \in \{ x : 4^{-k_0} + 3 < |x| < 4^{-k_0+1} - 3 \} \) and take \( s \in (-M^{-96}, 0) \). From properties of harmonic functions and the pressure decomposition, we infer that

\[
|p_{h, \lambda(1)}(x_0, s)| + \|\nabla p_{h, \lambda(1)}(x_0, s)\| \lesssim \|p_{h, \lambda(1)}(\cdot, s)\|_{L^2(B(x_0, 4^3))} \lesssim \|p_{Riesz, \lambda(1)}(\cdot, s)\|_{L^2(R^3)} + \|p_{\lambda(1)}(\cdot, s)\|_{L^2(B(2eM^5))}.
\]

Thus for every \( s \in (-M^{-96}, 0) \),

\[
\|p_{h, \lambda(1)}(\cdot, s)\|_{L^\infty(4^{-k_0+3} |x| < 4^{-k_0+1}-3)} + \|\nabla p_{h, \lambda(1)}(\cdot, s)\|_{L^\infty(4^{-k_0+3} |x| < 4^{-k_0+1}-3)} \lesssim \|p_{Riesz, \lambda(1)}(\cdot, s)\|_{L^2(R^3)} + \|p_{\lambda(1)}(\cdot, s)\|_{L^2(B(2eM^5))}.
\]

Using this, (83) and (92) we get that

\[
\|p_{h, \lambda(1)}\|_{L^\infty L^\infty(4^{-k_0+3} |x| < 4^{-k_0+1}-3) \times (-M^{-96}, 0)} + \|\nabla p_{h, \lambda(1)}\|_{L^\infty L^\infty(4^{-k_0+3} |x| < 4^{-k_0+1}-3) \times (-M^{-96}, 0)} \lesssim e^{4M^5}.
\]

This, Hölder’s inequality and (88) imply that for all \( q \in (1, \infty) \)

\[
\|p_{h, \lambda(1)}\|_{L^q L^\infty(4^{-k_0+3} |x| < 4^{-k_0+1}-3) \times (-M^{-96}, 0)} + \|\nabla p_{h, \lambda(1)}\|_{L^q L^\infty(4^{-k_0+3} |x| < 4^{-k_0+1}-3) \times (-M^{-96}, 0)} \lesssim q e^{7M^5}.
\]
Combining (91)-(94) yields that
\[ \|p_{\lambda(1)}\|_{L^q_t L^\infty_x(\{|x|<4^{k_0+1}-3\} \times (-M^{-96},0))} \leq e^{4M^5} \]
and that for all \( q \in (1, \infty) \)
\[ \|p_{\lambda(1)}\|_{L^q_t L^3_x(\{|x|<4^{k_0+1}-3\} \times (-M^{-96},0))} + \|\nabla p_{\lambda(1)}\|_{L^q_t L^\infty_x(\{|x|<4^{k_0+1}-3\} \times (-M^{-96},0))} \leq q e^{4M^5}. \]

Finally, let us estimate \( \partial_t v_{\lambda(1)} \). Using \( (v_{\lambda(1)}, p_{\lambda(1)}) \) is a smooth solution to the Navier-Stokes equations on \( B(0,4e^{M^5}) \times (-16e^{2M^5},0] \), we see that the estimates (87) and (96) (together with Hölder’s inequality) imply that for all \( q \in (1, \infty) \)
\[ \|\partial_t v_{\lambda(1)}\|_{L^q_t L^3_x(\{|x|<4^{k_0+1}-3\} \times (-M^{-96},0))} \leq q e^{4M^5}. \]

**Step 3: Bogovskiĭ operator estimates**

Let \( k_0 \in \{2, \ldots [M^4] + 1\} \) be as in the previous steps and define \( R_{k_0} := 4^{k_0} + 3 \). Note that
\[ 6 \leq R_{k_0} \leq e^{M^5}, \quad A(R_{k_0}) := \{ x : R_{k_0} < |x| < 2R_{k_0} \} \subset \{ x : 4^{k_0} + 3 < |x| < 4^{k_0+1} - 3 \} \]
and \( |A(R_{k_0})| \leq e^{3M^5} \).

Let the cut-off function \( \Phi \in C^\infty(\mathbb{R}^3; [0,1]) \) be such that
\[ \Phi \equiv 1 \text{ on } B(0,R_{k_0}), \]
\[ \text{supp } \Phi \subset B(0,2R_{k_0}) \text{ and } \]
\[ \text{for } j \in \mathbb{N} \quad \|\nabla^j \Phi\|_{L^\infty(\mathbb{R}^3)} \lesssim j^M R_{k_0}^{-j}. \]

Then for every \( t \in (-M^{-96},0] \) we see that
\[ v_{\lambda(1)}(x,t) \cdot \nabla \Phi(x) \in C^\infty_0(A(R_{k_0})). \]

Furthermore, using that \( v_{\lambda(1)} \) is divergence-free, we see that for every \( t \in (-M^{-96},0] \)
\[ \int_{A(R_{k_0})} v_{\lambda(1)}(x,t) \cdot \nabla \Phi(x) dx = \int_{A(R_{k_0})} \text{div}(v_{\lambda(1)}(x,t)\Phi(x)) dx = \int_{\partial B(0,2R_{k_0})} v_{\lambda(1)} \Phi \cdot dS - \int_{\partial B(0,R_{k_0})} v_{\lambda(1)} \Phi \cdot dS \]
\[ = - \int_{\partial B(0,R_{k_0})} v_{\lambda(1)} \cdot dS = - \int_{B(0,R_{k_0})} \text{div} v_{\lambda(1)} dx = 0. \]

Using this, we can apply the Bogovskiĭ operator in Lemma 1 to \( -v_{\lambda(1)} \cdot \nabla \Phi \) on \( A(R_{k_0}) \) for each \( t \in (-M^{-96},0] \), which we denote by \( w_{\lambda(1)} := B(-v_{\lambda(1)} \cdot \nabla \Phi) \). Furthermore, by (99) we have that for every \( t \in (-M^{-96},0] \)
\[ w_{\lambda(1)}(\cdot, t) \in C^\infty_0(A(R_{k_0})). \]

Let us now estimate the spatial derivatives of \( w_{\lambda(1)} := B(-v_{\lambda(1)} \cdot \nabla \Phi) \). Using (87) and (88) allows us to apply (27) of Lemma 1 to get
\[ \|\nabla w_{\lambda(1)}\|_{L_t^\infty W^{2,4}_x(A(R_{k_0}) \times (-M^{-96},0))} \lesssim \|v_{\lambda(1)} \cdot \nabla \Phi\|_{L_t^\infty W^{2,4}_x(A(R_{k_0}) \times (-M^{-96},0))} \leq e^{M^5}. \]

Using this, (88)-(101) and Poincaré’s inequality gives that
\[ \|w_{\lambda(1)}\|_{L_t^\infty W^{2,4}_x(A(R_{k_0}) \times (-M^{-96},0))} \leq e^{3M^5}. \]
This and the Sobolev embedding theorem gives that
\begin{equation}
\|w_{\lambda(1)}\|_{L^\infty_t W^{2,\infty}_x(A(R_{k_0}) \times (-M^{-96},0))} \lesssim e^{3M^5}.
\end{equation}
Thus, by Hölder’s inequality we have
\begin{equation}
\|w_{\lambda(1)}\|_{L^\frac{2}{3}_t W^{2,\infty}_x(A(R_{k_0}) \times (-M^{-96},0))} \lesssim e^{3M^5}.
\end{equation}
By (103), (98) and Hölder’s inequality, we get
\begin{equation}
\|w_{\lambda(1)}\|_{L^\infty_t L^{\frac{4}{3}}_x(A(R_{k_0}) \times (-M^{-96},0))} \leq e^{4M^5}
\end{equation}
and
\begin{equation}
\|w_{\lambda(1)}\|_{L^\frac{4}{3}_t W^{2,\infty}_x(A(R_{k_0}) \times (-M^{-96},0))} \lesssim e^{\frac{51}{11}M^5}.
\end{equation}

Let us now estimate the time derivative of the Bogovskiï operator $w_{\lambda(1)} := B(-v_{\lambda(1)} \cdot \nabla \Phi)$. For every $t \in (-M^{-96},0)$ we see that
\begin{equation*}
\partial_t w_{\lambda(1)}(x,t) \cdot \nabla \Phi(x) \in C_0^\infty(A(R_{k_0})).
\end{equation*}
Using this, Lemma 1 (specifically, (28)) and (97), we see that for every $t \in (-M^{-96},0)$
\begin{equation}
\partial_t w_{\lambda(1)}(\cdot,t) \in C_0^\infty(A(R_{k_0}))
\end{equation}
and
\begin{equation}
\|\nabla \partial_t w_{\lambda(1)}\|_{L^\frac{2}{3}_t L^{2}_x(A(R_{k_0}) \times (-M^{-96},0))} \lesssim \|\partial_t v_{\lambda(1)} \cdot \nabla \Phi\|_{L^\frac{2}{3}_t L^1_x(A(R_{k_0}) \times (-M^{-96},0))} \lesssim e^{7M^5}.
\end{equation}
Noting (98) and (107)-(108), we apply Poincaré’s inequality to get
\begin{equation}
\|\partial_t w_{\lambda(1)}\|_{L^\frac{2}{3}_t W^{1,4}_x(A(R_{k_0}) \times (-M^{-96},0))} \lesssim e^{8M^5}.
\end{equation}
Hence, the Sobolev embedding theorem gives
\begin{equation}
\|\partial_t w_{\lambda(1)}\|_{L^\frac{4}{3}_t L^\infty_x(A(R_{k_0}) \times (-M^{-96},0))} \lesssim e^{8M^5}.
\end{equation}
Observing (98) and (110), we can apply Hölder’s inequality to conclude that
\begin{equation}
\|\partial_t w_{\lambda(1)}\|_{L^\infty_t L^\frac{11}{4}_x(A(R_{k_0}) \times (-M^{-96},0))} \lesssim e^{\frac{106}{11}M^5}.
\end{equation}

**Step 4: Truncating the solution and rescaling to conclude**

Let $\Phi$ and $w_{\lambda(1)}$ be as in the previous step. Let us now define the truncated solution
\begin{equation}
\begin{array}{ll}
u_{\lambda(1)}(x,t) := \Phi v_{\lambda(1)}(x,t) + w_{\lambda(1)}(x,t) & \text{and } \pi_{\lambda(1)}(x,t) := \Phi p_{\lambda(1)}(x,t) \quad \text{for } (x,t) \in \mathbb{R}^3 \times (-M^{-96},0].
\end{array}
\end{equation}
From Lemma 1, properties of the cut-off function $\Phi$ and (101), we deduce the following. Namely $u_{\lambda(1)}$ is divergence-free and that for every $t \in (-M^{-96},0)$, $u_{\lambda(1)}$ and $\pi_{\lambda(1)}$ are smooth with compact support in $B(0,2R_{k_0})$. Furthermore, we see that $u_{\lambda(1)}$ and $\pi_{\lambda(1)}$ are smooth on $\mathbb{R}^3 \times (-M^{-96},0)$ and $B(0,R_{k_0}) \times [-M^{-96},0]$. Furthermore, $u_{\lambda(1)}$ and $\pi_{\lambda(1)}$ solve the forced Navier-Stokes equations in $\mathbb{R}^3 \times (-M^{-96},0)$ with forcing term given by
\begin{equation}
f_{\lambda(1)} := -\Delta \Phi v_{\lambda(1)} - 2\nabla \Phi \cdot \nabla v_{\lambda(1)} + \Phi v_{\lambda(1)} \cdot \nabla \Phi v_{\lambda(1)} + (\Phi^2 - \Phi) v_{\lambda(1)} \cdot \nabla v_{\lambda(1)} + (\partial_t - \Delta) w_{\lambda(1)} + \Phi v_{\lambda(1)} \cdot \nabla w_{\lambda(1)} + w_{\lambda(1)} \cdot \nabla (\Phi v_{\lambda(1)}) + w_{\lambda(1)} \cdot \nabla w_{\lambda(1)} + \Phi p_{\lambda(1)}.
\end{equation}
Using properties of the cut-off function $\Phi$, (87), (95)-(96), (103)-(104), (106) and (110)-(111), we see that
\[
\left\|f_{\lambda(1)}\right\|_{L_t^\frac{4}{3}L_x^\infty(A(R_{k_0}) \times (-M^{-96}, 0))} + \left\|f_{\lambda(1)}\right\|_{L_t^\frac{11}{4}L_x^\infty(A(R_{k_0}) \times (-M^{-96}, 0))} \leq e^{12M^5}.
\]
From properties of the cut-off function $\Phi$ and (101), we see that for every $t \in (-M^{-96}, 0)$
\[
supp f_{\lambda(1)}(\cdot, t) \subseteq B(0, 2R_{k_0}) \setminus B(0, R_{k_0}) \quad \text{and} \quad supp u_{\lambda(1)}(\cdot, t) \subseteq B(0, 2R_{k_0}) \quad \text{with} \quad R_{k_0} \geq 6.
\]
From (82), properties of the cut-off function $\Phi$ and (105) one deduces that $u_{\lambda(1)} = \Phi v_{\lambda(1)} + w_{\lambda(1)}$ with
\[
\left\|\Phi v_{\lambda(1)}\right\|_{L_t^\infty L_x^2(\mathbb{R}^3 \times (-M^{-96}, 0))} = \left\|u_{\lambda(1)}\right\|_{L_t^\infty L_x^2(B(0,R_{k_0}) \times (-M^{-96}, 0))} \leq M \quad \text{and}
\]
\[
\left\|w_{\lambda(1)}\right\|_{L_t^\infty L_x^{10}(\mathbb{R}^3 \times (-M^{-96}, 0))} \leq e^{4M^5}.
\]
From properties of the cut-off function $\Phi$ and (101), we see that
\[
(u_{\lambda(1)}(x, t), \pi_{\lambda(1)}(x, t)) = (v_{\lambda(1)}(x, t), \rho_{\lambda(1)}(x, t)) = (\lambda_{(1)}v_{\lambda(1)}(x, \lambda_{(1)}^2 t), \lambda_{(1)}^2 \rho_{\lambda(1)}(x, \lambda_{(1)}^2 t))
\]
on $B(0, R_{k_0}) \times [-M^{-96}, 0] \supseteq B(0, 6) \times [-M^{-96}, 0].$

Using this, together with (84) and (98), gives that
\[
\sup_{0 < r \leq M^{-48}} \left\{ \frac{1}{r^2} \int_{Q(0, r)} |\pi_{\lambda(1)}|^2 dxdt + \frac{1}{r} \left\|u_{\lambda(1)}\right\|_{L_t^\infty L_x^2(Q(0, r))} \right\} \lesssim M^3.
\]
We now define the functions $V : \mathbb{R}^3 \times [-2, 0] \rightarrow \mathbb{R}^3$, $F : \mathbb{R}^3 \times [-2, 0] \rightarrow \mathbb{R}^3$ and $P : \mathbb{R}^3 \times [-2, 0] \rightarrow \mathbb{R}$ by
\[
(V(x, t), P(x, t), F(x, t)) := (\lambda_{(2)} u_{\lambda(1)}(x, \lambda_{(2)}^2 t), \lambda_{(2)}^2 \pi_{\lambda(1)}(x, \lambda_{(2)}^2 t), \lambda_{(2)}^3 f_{\lambda(1)}(\lambda_{(2)} x, \lambda_{(2)}^2 t))
\]
with $\lambda_{(2)} := e^{-2M}.$

We also define
\[
R := R_{k_0} \lambda_{(2)}^{-1}.
\]
With these definitions it is easy to see that $V$ and $P$ are smooth on $\mathbb{R}^3 \times [-2, 0]$. Additionally by (118), we see that $V$ and $P$ are smooth on $B(0, R) \times [-2, 0]$. Furthermore, the properties (114)-(119) for $(u_{\lambda(1)}, \pi_{\lambda(1)}, f_{\lambda(1)})$ and $R_{k_0}$ imply the desired properties for $(V, P, F)$ and $R$ stated in Proposition 4.

5. Quantitative Regions of Regularity for the Truncated Solution

5.1. Quantitative Epochs of Regularity for the Truncated Solution.

**Proposition 5.** There exists $M_0$ sufficiently large such that the following holds true. Let $(v, p)$, $(V, P)$, $F$ and $R$ be as in the statement of Proposition 4. Let $0 < T_1 \leq 1$ and define
\[
I := (-T_1, -\frac{T_1}{2}) \subset [-1, 0] \subset [-2, 0].
\]
Then the above assumptions imply that there exists an epoch of regularity $I' \subset I$, which is a closed interval such that
\[
|I'| = M^{-46}|I|
\]
and
Let us now estimate the kinetic energy of with any $\|F\|_{L^q(\mathbb{R}^3)} \lesssim \mathcal{M}^q$ for $q \in [\frac{10}{7}, \infty]$. 

Proof. Step 1: Preliminary estimates

The properties of $(V, P, F)$ and $R$ (as stated in Proposition 4) are also satisfied for the rescaled quantities

$$\left( V_{T_1}(x, t), P_{T_1}(x, t), F_{T_1}(x, t) \right) := \left( T_1^2 V(T_1^{-1} x, T_1 t), T_1^{-1} P(T_1^{-1} x, T_1 t), T_1^2 F(T_1^{-1} x, T_1 t) \right)$$

with any $0 < T_1 \leq 1$. Hence, without loss of generality we can take $T_1 = 1$ ($I := (-1, \frac{1}{2})$) in Proposition 5.

On $\mathbb{R}^3 \times (-2, 0)$ we have

$$V(\cdot, t) = L(F)(\cdot, t) + B(V, V)(\cdot, t).$$

Here,

$$L(F)(\cdot, t) := e^{(t+2)\Delta} V(\cdot, -2) + \int_{-2}^{t} e^{(t-s)\Delta} F(\cdot, s) ds$$

and

$$B(A, B)(\cdot, t) := - \int_{-2}^{t} e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (A \otimes B)(\cdot, s) ds.$$

Here, $e^{t\Delta}$ denotes the heat semigroup on $\mathbb{R}^3$ and $\mathbb{P}$ denotes the projection onto divergence-free vector fields in $\mathbb{R}^3$. Using $V = V(1) + V(2)$ on $\mathbb{R}^3 \times [-2, 0]$, (78) and standard estimates of the heat semigroup yields

$$\sup_{t \in [-1, 0]} \| e^{(t-s)\Delta} V(\cdot, -2) \|_{L^q(\mathbb{R}^3)} \lesssim q \mathcal{M} \quad \forall q \in [\frac{10}{7}, \infty].$$

Using standard estimates of the heat semigroup and Leray projector, together with Hölder’s inequality, yields that for $t \in [-2, 0]$

$$\left\| \int_{-2}^{t} e^{(t-s)\Delta} \mathbb{P} F(\cdot, s) ds \right\|_{L^q(\mathbb{R}^3)} \lesssim \| F \|_{L_t^2 L_s^{\frac{14}{7}}(\mathbb{R}^3 \times (-2, 0))}$$

and

$$\left\| \int_{-2}^{t} e^{(t-s)\Delta} \mathbb{P} F(\cdot, s) ds \right\|_{L^\infty(\mathbb{R}^3)} \lesssim \int_{-2}^{t} \frac{\| F(\cdot, s) \|_{L^q(\mathbb{R}^3)}}{(t-s)^{\frac{q}{2}}} ds \lesssim \| F \|_{L_t^2 L_s^{\infty}(\mathbb{R}^3 \times (-2, 0))}.$$

Using (126)-(127) and (79) gives that for $\mathcal{M}_0$ sufficiently large

$$\sup_{t \in [-1, 0]} \| L(F)(\cdot, t) \|_{L^q(\mathbb{R}^3)} \lesssim q \mathcal{M} \quad \forall q \in [\frac{10}{7}, \infty].$$

Let us now estimate the kinetic energy of $B(V, V)(\cdot, t)$, which we will now denote by

$$W(\cdot, t) := B(V(1), V(1))(\cdot, t) + B(V(1), V(2))(\cdot, t) + B(V(2), V(1))(\cdot, t) + B(V(2), V(2))(\cdot, t).$$

for convenience. From [26], it is known that $e^{(t-s)\Delta} \mathbb{P} \nabla$ has an associated convolution kernel $K_{ijk}$ ($i, j, k \in \{1, 2, 3\}$). Furthermore, from [33] we have

$$\| K(x, t) \| \lesssim \frac{1}{(|x|^2 + t)^2} \quad \text{for } (x, t) \in \mathbb{R}^3 \times (0, \infty).$$

Using (130), (78) and Young’s convolution inequality gives that for $t \in (-2, 0)$

$$\| B(V(1), V(1))(\cdot, t) \|_{L^2(\mathbb{R}^3)} \lesssim \int_{-2}^{t} \frac{1}{(t-s)^2} \| V(1)(\cdot, s) \|_{L^3(\mathbb{R}^3)}^2 ds \lesssim \mathcal{M}^2.$$
Performing a standard energy estimate on \((135)\)\textsuperscript{1}

Observe that the smooth function \(W\) satisfies the following equation on \(\mathbb{R}^3 \times (-2, 0)\):

\[
\partial_t W - \Delta W + W \cdot \nabla W + W \cdot \nabla L(F) + L(F) \cdot \nabla W + L(F) \cdot \nabla L(F) + \nabla \Pi = 0,
\]

\(\text{div } W = 0\) and \(W(\cdot, -2) = 0\).

Performing a standard energy estimate on \((135)\) yields that for \(t \in [-1, 0)\)

\[
\|W(\cdot, t)\|_{L^2(\mathbb{R}^3)}^2 + 2 \int_{-1}^{t} \int_{\mathbb{R}^3} |\nabla W(x, s)|^2 ds = \|W(\cdot, -1)\|_{L^2(\mathbb{R}^3)}^2 + 2 \int_{-1}^{t} \int_{\mathbb{R}^3} (W+L(F)) \otimes L(F) : \nabla W dxds.
\]

Using this, Hölder’s inequality and Young’s algebraic inequality yields that for \(t \in [-1, 0)\)

\[
\|W(\cdot, t)\|_{L^2(\mathbb{R}^3)}^2 + \int_{-1}^{t} \int_{\mathbb{R}^3} |\nabla W(x, s)|^2 ds \lesssim \|W(\cdot, -1)\|_{L^2(\mathbb{R}^3)}^2 + \int_{-1}^{t} \int_{\mathbb{R}^3} |W|^2 |L(F)|^2 + |L(F)|^4 dxds.
\]

Using this, the Sobolev embedding theorem and the pigeonhole principle, we see that there exists \(t_1 \in (-1, -\frac{3}{2})\) such that

\[
\|W(\cdot, t)\|_{L^6(\mathbb{R}^3)} \lesssim \mathcal{M}^3.
\]

Combining this with \((128)\) gives that

\[
\|V(\cdot, t_1)\|_{L^6(\mathbb{R}^3)} \lesssim \mathcal{M}^3.
\]

Using this estimate for \(V(\cdot, t_1)\) and \((79)\), one deduces that

\[
\sup_{t \in [t_1, 0]} \left\|e^{(t-t_1)\Delta} V(\cdot, t_1) + \int_{t_1}^{t} e^{(t-s)\Delta} \mathbb{P} F(\cdot, s) ds \right\|_{L^6(\mathbb{R}^3)} \lesssim \mathcal{M}^3.
\]
Using this and applying [12, Theorem (3.2)], we see that for \( M \) sufficiently large there exists a solution \( \tilde{V} : \mathbb{R}^3 \times [t_1, t_1 + C_{\text{univ}}M^{-12}] \to \mathbb{R}^3 \) to

\[
(140) \quad \tilde{V}(\cdot, t) = e^{(t-t_1)\Delta} V(\cdot, t_1) + \int_{t_1}^{t} e^{(t-s)\Delta} \mathbb{P} F(\cdot, s) ds - \int_{t_1}^{t} e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (\tilde{V} \otimes \tilde{V})(\cdot, s) ds,
\]

with

\[
(141) \quad \sup_{s \in [t_1, t_1 + C_{\text{univ}}M^{-12}]} \| \tilde{V}(\cdot, s) \|_{L^6(\mathbb{R}^3)} \lesssim M^3.
\]

Here, \( C_{\text{univ}} \) is a positive universal constant. Since \( \tilde{V} \) is smooth on \( \mathbb{R}^3 \times [-t_1, 0) \) and with compact support (77), we have that for \( M \) sufficiently large that

\[
V \in L^\infty_t L^3_x(\mathbb{R}^3 \times (t_1, t_1 + C_{\text{univ}}M^{-12})).
\]

Furthermore, \( V \) also satisfies the same equation (140) as \( \tilde{V} \). So by [12, Theorem (3.3)], we infer that \( \tilde{V} \equiv V \) on \( \mathbb{R}^3 \times [t_1, t_1 + C_{\text{univ}}M^{-12}] \). Hence, from (141) we obtain

\[
(142) \quad \sup_{s \in [t_1, t_1 + C_{\text{univ}}M^{-12}]} \| V(\cdot, s) \|_{L^6(\mathbb{R}^3)} \lesssim M^3.
\]

**Step 3: Higher differentiability for \( V \) via \( \varepsilon \)-regularity**

The pressure \( P \) can be decomposed as

\[
(143) \quad P = P_{V \otimes V} + P_F \quad \text{with} \quad P_{V \otimes V} := \mathcal{R}_i \mathcal{R}_j (V_i V_j) \quad \text{and} \quad P_F := (\Delta)^{-1}(\text{div} \, F).
\]

Using (79), (142) and Calderón-Zygmund bounds gives that

\[
(144) \quad \| P_{V \otimes V} \|_{L^6_t L^3_x(\mathbb{R}^3 \times (t_1, t_1 + C_{\text{univ}}M^{-12}))} \lesssim M^6 \quad \text{and} \quad \| \nabla P_F \|_{L^6_t L^6_x(\mathbb{R}^3 \times (t_1, t_1 + C_{\text{univ}}M^{-12}))} \lesssim e^{-e^M}.
\]

Next, we fix \( x \in B(0, \frac{1}{2}) \), \( r := M^{-\frac{12}{2}} \) and \( t \in [t_1 + \frac{1}{2} M^{-13}, t_1 + 2 M^{-13}] \). Using the properties of \( R \) in Proposition 4 and (74), we see that

\[
(145) \quad B(x, r) \times (t-r^2, t) \subseteq B(0, R) \times (t_1, t_1 + C_{\text{univ}}M^{-12}), \quad (V, P) \text{ solve the Navier-Stokes equations in } B(x, r) \times (t-r^2, t).
\]

Using (142), (145) and Hölder’s inequality, one infers that

\[
(146) \quad \frac{1}{r^2} \int_{t-r^2}^{t} \int_{B(x, r)} |V(y, s)|^2 dy ds \lesssim r^\frac{2}{3} \sup_{s \in [t_1, t_1 + C_{\text{univ}}M^{-12}]} \| V(\cdot, s) \|_{L^6(\mathbb{R}^3)}^3 \lesssim M^{-\frac{4}{3}}.
\]

The same arguments applied to \( P_{V \otimes V} \) yield

\[
(147) \quad \frac{1}{r^2} \int_{t-r^2}^{t} \int_{B(x, r)} |P_{V \otimes V}(y, s) - (P_{V \otimes V})_{B(x, r)}(s)|^\frac{2}{3} dy ds \lesssim M^{-\frac{4}{3}}.
\]

Next, using (144), (145), Poincaré’s inequality and Hölder’s inequality gives

\[
\frac{1}{r^2} \int_{t-r^2}^{t} \int_{B(x, r)} |P_F(y, s) - (P_F)_{B(x, r)}(s)|^\frac{2}{3} dy ds \lesssim \frac{1}{r^2} \int_{t-r^2}^{t} \int_{B(x, r)} |\nabla P_F|^\frac{2}{3} dy ds
\]

\[
\lesssim r^{\frac{22}{3}} \int_{t-r^2}^{t} \left( \int_{B(x, r)} |\nabla P_F|^\frac{1}{6} \, dy \right)^{\frac{9}{10}} ds \lesssim M^{-\frac{13}{11}} e^{-\frac{2}{5} e^M}.
\]
Combining (146)-(148) gives
\[
\left(149\right) \frac{1}{r^2} \int_{t-r^2}^t \int_{B(x,r)} |V(y,s)|^3 + |P(y,s) - (P)_{B(x,r)}(s)|^2 \, dy \, ds \lesssim \mathcal{M}^{-\frac{3}{4}} + \mathcal{M}^{-\frac{11}{4}} e^{-\frac{3}{2} \varepsilon \mathcal{M}}.
\]
Using (145) and Caffarelli, Kohn and Nirenberg’s $\varepsilon$-regularity criteria [9] and [31, Lemma 6.1], we infer from (149) that for $\mathcal{M}$ sufficiently large
\[
|\nabla^j V(x,t)| \lesssim_j r^{-(j+1)} \lesssim \mathcal{M}^{\frac{39}{2j}} \quad \text{for} \quad j = 0, 1, 2.
\]
Thus, we have
\[
\left(150\right) \sup_{B(0,\frac{r}{2}) \times \left[|t_1| + \frac{3}{4} \mathcal{M}^{-13}, t_1 + 2 \mathcal{M}^{-13}\right]} |\nabla^j V| \lesssim \mathcal{M}^{\frac{39}{2j}} \quad \text{for} \quad j = 0, 1, 2.
\]
Let
\[
\left(151\right) I' := \left[t_1 + \frac{3}{4} \mathcal{M}^{-13}, t_1 + 3 \mathcal{M}^{-13} + \frac{1}{\mathcal{M}^{46}}\right] \subset \left[t_1 + \frac{3}{2} \mathcal{M}^{-13}, t_1 + 2 \mathcal{M}^{-13}\right].
\]
Then $|I'| = \frac{1}{2} \mathcal{M}^{-46} = |I| \mathcal{M}^{-46}$. From this and (150), we get
\[
\sup_{B(0,\frac{r}{2}) \times I'} |\nabla^j V| \lesssim \mathcal{M}^{\frac{39}{2j}} \lesssim \frac{1}{\mathcal{M}} |I'|^{-\frac{j+1}{2j}} \quad \text{for} \quad j = 0, 1, 2
\]
as required. \hfill \square

5.2. Quantitative Annuli of Regularity for the Truncated Solution. In order to show the existence of quantitative annuli of regularity for the truncated solution $V$, we will make use of the following proposition taken from [7].

**Proposition 6.** [7, Proposition 6.4]

There exists a universal constant $\varepsilon_1^* \in (0, 1)$, such that if $(v, p)$ is a suitable weak solution to the Navier-Stokes equations on $Q(0, 1)$ and for some $\varepsilon_1 \leq \varepsilon_1^*$
\[
\left(152\right) \int_{Q(0,1)} |v|^3 + |p|^{\frac{1}{4}} \, dx \, dt \leq \varepsilon_1
\]
then one concludes that for $j = 0, 1$
\[
\left(153\right) \nabla^j v \in L^\infty(Q(0,\frac{1}{2})) \quad \text{with} \quad \|\nabla^j v\|_{L^\infty(Q(0,\frac{1}{2}))} \lesssim \frac{1}{\varepsilon_1^*}
\]
and
\[
\left(154\right) \nabla \omega \in L^\infty(Q(0,\frac{1}{2})) \quad \text{with} \quad \|\nabla \omega\|_{L^\infty(Q(0,\frac{1}{2}))} \lesssim \frac{1}{\varepsilon_1^*}.
\]

**Proposition 7.** Fix any $\mu \geq 1$ and let $\mathcal{M}$, $(v, p)$, $(V, P)$, $F$ and $R$ be as in Proposition 4. Let $\Omega := \nabla \times V$. There exists a positive constant $\lambda_0(\mu)$ such that if $\mathcal{M} \geq \lambda_0(\mu)$ then the following holds true.

In particular, for any $T_1 \in (0, 1]$ and
\[
\left(155\right) T_1^{\frac{1}{2}} \leq R_0 \leq T_1^{\frac{1}{4}} e^{2\mathcal{M}} e^{-4\mu \mathcal{M}^{11\mu+22}}
\]
there exists $R_2$ such that
\[
\left(156\right) 2R_0 \leq R_2 \leq 2R_0 e^{\mu \mathcal{M}^{11\mu+22}} \leq R
\]
such that
\[
\left(157\right) \|\nabla^j V\|_{L^\infty_r(2R_2 < |x| < \frac{R_2 \mathcal{M}^{11\mu+22}}{4}, (-\frac{T_1}{4}, 0))} \lesssim \mathcal{M}^{-\lambda_0(\mu) - \frac{j+1}{2}} T_1^{-\frac{j+1}{2}} \quad \text{for} \quad j = 0, 1.
\]
and
\begin{equation}
\|\nabla \Omega\|_{L^\infty_x(B_2 < |x| < R_{\alpha,\mu}^{11\mu} \times (-\frac{T_1}{4}, 0))} \lesssim M^{-3\mu} T_1^{-\frac{3}{2}}.
\end{equation}

**Proof.** **Step 1: Preliminary estimates**

From now on we take $M \geq M(\mu)$ such that
\[ 1 \leq 6e^{2M} e^{-4\mu M^{11\mu+22}} \quad \forall M \geq M(\mu). \]

For identical reasons as described in the proof of Proposition 5, we may assume without loss of generality that $T_1 = 1$ by a scaling argument. As in the proof of Proposition 5, we decompose $V : \mathbb{R}^3 \times [-2, 0] \to \mathbb{R}^3$ as
\[ V(\cdot, t) = W(\cdot, t) + L(F)(\cdot, t) \quad \text{for } \quad t \in (-2, 0). \]

Here, $L(F)$ is defined in (125) and $W$ is defined in (129). We decompose the pressure $P : \mathbb{R}^3 \times [-2, 0] \to \mathbb{R}$ as
\begin{equation}
P(\cdot, t) = P_{V\otimes V}(\cdot, t) + P_F(\cdot, t) \quad \text{for } \quad t \in (-2, 0).
\end{equation}

Here, $P_{V\otimes V}$ and $P_F$ are as in (143).

Using (128), (138) and the Sobolev embedding theorem gives
\begin{equation}
0 \int_{-1}^0 \int_{\mathbb{R}^3} |V(y, s)|^\frac{10}{3} \, dy \, ds \lesssim M^{10}.
\end{equation}

Using this and Calderón-Zygmund estimates, we have
\begin{equation}
0 \int_{-1}^0 \int_{\mathbb{R}^3} \left| P_{V\otimes V}(y, s) \right|^\frac{5}{3} \, dy \, ds \lesssim M^{10}.
\end{equation}

Thus, for $M$ sufficiently large we have
\begin{equation}
0 \int_{-1}^0 \int_{\mathbb{R}^3} \left| V(y, s) \right|^\frac{10}{3} \, dy \, ds + \left| P_{V\otimes V}(y, s) \right|^\frac{5}{3} \, dy \, ds \leq \lambda \equiv M^{11}.
\end{equation}

Using (79) and Calderón-Zygmund estimates also give that
\begin{equation}
\|\nabla P_F\|_{L^\frac{11}{2} L^\frac{11}{6} (\mathbb{R}^3 \times [-1, 0])} \lesssim e^{-e^M}.
\end{equation}

**Step 2: Finding a good scale with pigeonholing arguments**

Now we fix any $\mu \geq 1$ and $R_0$ such that
\begin{equation}
1 \leq R_0 \leq 6e^{2M} e^{-4\mu M^{11\mu+22}}.
\end{equation}

From (162), we see that
\[ \sum_{k=0}^{\infty} \int_0^1 \int_{\{x: \lambda^{k+1} R_0 < |x| < \lambda^k R_0\}} |V(y, s)|^\frac{10}{3} + \left| P_{V\otimes V}(y, s) \right|^\frac{5}{3} \, dy \, ds \leq \lambda \equiv M^{11}. \]

By the pigeonhole principle, there exists
\begin{equation}
k_0 \in \{0, 1, \ldots, \lfloor \lambda^{k+1} \rfloor \}
\end{equation}
such that
\[
(166) \quad \int_0^1 \int_{-1 \{x : \lambda^{k_0} R_0 < |x| < \lambda^{k_0+1} R_0 \}} |V(y, s)|^{10} + |P_V \otimes V(y, s)|^{5} dyds \leq \lambda^{-\mu} = \mathcal{M}^{-11\mu}.
\]

**Step 3: Concluding with $\varepsilon$-regularity**

Next, we define
\[
(167) \quad R_1 := \lambda^{k_0} R_0
\]

From now on we impose the restriction $\mathcal{M} \geq \max(\mathcal{M}_0, \mathcal{M}(\mu), 5^{11\mu})$ and we define
\[
(168) \quad A := \{x : R_1 + 1 < |x| < \lambda^\mu R_1 - 1\}.
\]

This, (166) and Hölder’s inequality gives that
\[
(169) \quad \sup_{x \in A} \int_{-1 B(x, 1)} |V(y, s)|^{3} + |P_V \otimes V(y, s) - (P_V \otimes V)_{B(x, 1)}(s)|^{\frac{3}{2}} dyds \lesssim \lambda^{-\frac{6}{11}} \leq \lambda^{-\frac{9\mu}{11}}.
\]

Combining this with (163) and Poincaré’s inequality gives that for $\mathcal{M}$ sufficiently large, there exists a positive universal constant $C_{univ}$ such that
\[
(170) \quad \sup_{x \in A} \int_{-1 B(x, 1)} |V(y, s)|^{3} + |P(y, s) - (P)_{B(x, 1)}(s)|^{\frac{3}{2}} dyds \lesssim \mathcal{M}^{-9\mu} + e^{-\frac{3}{11} \mathcal{M}} \leq C_{univ} \mathcal{M}^{-9\mu}.
\]

Thus for
\[
(171) \quad \mathcal{M} \geq \lambda_0(\mu) := \max \left( \mathcal{M}_0, \mathcal{M}(\mu), 5^{11\mu}, \left( \frac{C_{univ}}{\varepsilon_1^*} \right)^{\frac{1}{11}} \right)
\]

we have
\[
(172) \quad \sup_{x \in A} \int_{-1 B(x, 1)} |V(y, s)|^{3} + |P(y, s) - (P)_{B(x, 1)}(s)|^{\frac{3}{2}} dyds \leq C_{univ} \mathcal{M}^{-9\mu} \leq \varepsilon_1^*.
\]

Here, $\mathcal{M}_0$ is a sufficiently large universal constant and $\varepsilon_1^*$ is as in Proposition 6. Using (165) and (167), we see that
\[
R_0 \leq R_1 \leq R_0 \lambda^{\mu \lambda^{\mu+1}} \leq R_0 e^{\lambda^{\mu \lambda^{\mu+2}}} = R_0 e^{\lambda^{\mu \lambda^{\mu+2}} + 22} \quad \text{and} \quad \lambda^\mu R_1 \leq R_0 e^{2\mu \lambda^{\mu+2}} = R_0 e^{2\mu \lambda^{\mu+2}}.
\]

Using this and (164) gives that
\[
(174) \quad \{x : R_1 < |x| < \lambda^\mu R_1\} \subset B(0, 6e^{2M}) \subset B(0, R).
\]

Here, $R$ is as in Proposition 4. Combining (174) and (76), we deduce that $(V, P)$ solves the Navier-Stokes equations on $\{x : R_1 < |x| < \lambda^\mu R_1\} \times (-1, 0)$. This and (172) allows us to apply Proposition 6 to get that for $j = 0, 1$
\[
(175) \quad \|\nabla^j V\|_{L^\infty(A \times (-\frac{1}{16}, 0))} \lesssim \mathcal{M}^{-3\mu} \quad \text{and} \quad \|\nabla^j \Omega\|_{L^\infty(A \times (-\frac{1}{16}, 0))} \lesssim \mathcal{M}^{-3\mu}.
\]

From (164), (171) and (173), we see that for $R_2 := 2R_1$
\[
(176) \quad R_1 + 1 \leq R_2 < \frac{\lambda^\mu R_2}{4} \leq \lambda^\mu R_1 - 1 \quad \text{and} \quad 2R_0 \leq R_2 \leq 2R_0 e^{\lambda^{\mu \lambda^{\mu+2}}}.\]
We then see that (175) implies that for $j = 0, 1$

\[(177) \quad \| \nabla^j V \|_{L^\infty_t(L^2_x(R_2 < |x| < R_x M^{111/4} \times (-\frac{1}{10}, 0)))} \lesssim M^{-3\mu} \quad \text{and} \quad \| \nabla \Omega \|_{L^\infty_t(L^2_x(R_2 < |x| < R_x M^{111/4} \times (-\frac{1}{10}, 0)))} \lesssim M^{-3\mu}
\]
as required.

6. Proof of Theorem 1

We will show that the assumptions of Theorem 1 imply that

\[(178) \quad \sup_{t \in [-e^{-eM^{1126}}, 0]} |v(0, t)| \leq e^{e^{eM^{1126}}}.
\]

A translation and scaling argument (replacing $\mathcal{M}$ with $\mathcal{M}' = \frac{64}{\sqrt{5}} \mathcal{M}$) then allows us to deduce the desired estimate (9) in Theorem 1.

From now on we will consider $(V, P)$, $F$ and $R$ as in Proposition 4, which are derived from $(v, p)$. Recall from Proposition 4 that

\[(179) \quad V(x, t) = \lambda v(\lambda x, \lambda^2 t) \quad \text{on} \quad B(0, R) \times [-2, 0] \quad \text{and} \quad \| V \|_{L^\infty_t L^3_x(B(0, R) \times (-2, 0))} \leq \mathcal{M}
\]

with

\[(180) \quad R \geq 6e^{eM} \quad \text{and} \quad \lambda = e^{-eM} e^{-eM^5}.
\]

Recall that $(V, P)$ are smooth on $B(0, R) \times [-2, 0]$, solve the Navier-Stokes equations on $B(0, R) \times [-2, 0]$ and satisfy (75). Hence, we see that $(V, P)$ obey the hypothesis (50) of Proposition 2 with $M$ replaced by $M^4$. Using (179)-(180) and Proposition 2, we see that in order to prove (178) it suffices to show the following for $\Omega := \nabla \times V$. Namely,

\[(181) \quad \int_{B(0, M^{96}(-t_M)^{1/2})} |\Omega(x, t_M)|^2 dx \leq \frac{M^{-88}}{(-t_M)^{1/2}} \quad \text{with} \quad -t_M := \frac{M^{-401}}{16} \exp\{-2M^{1126} \exp \exp(M^{1125})\}.
\]

In order to show this, we assume that there exists

\[(182) \quad 0 < -t'_0 < \frac{M^{-401}}{8}
\]
such that

\[(183) \quad \int_{B(0, M^{96}(-t'_0)^{1/2})} |\Omega(x, t'_0)|^2 dx > \frac{M^{-88}}{(-t'_0)^{1/2}}.
\]

We then show that this necessarily implies that

\[(184) \quad -t'_0 \geq \frac{M^{-401}}{8} \exp\{-2M^{1126} \exp \exp(M^{1125})\},
\]

which then gives (181).

Assuming (183), we see that Proposition 3 then implies that

\[(185) \quad \int_{B(0, M^{96}(-t''^0)^{1/2})} |\Omega(x, t'')|^2 dx > \frac{M^{-88}}{(-t'')^{1/2}}.
\]
at any well-separated backward time

\[(186) \quad t'' \in [-M^{-192}, t'_0] \quad \text{such that} \quad M^{200}t'_0 > t''.
\]
The rest of the proof relies on quantitative Carleman inequalities [34, Proposition 4.2-4.3].\footnote{For reasons relating to notation we will refer to [7, Proposition B.1-B.2], which are directly taken from Tao’s paper [34, Proposition 4.2-4.3] but adopt different notation.} These are the tools used to transfer the concentration information (185) from the time $t''$ to time 0 and from the small scales $B(0, \mathcal{M}^{06}(-t''))$ to larger scales, which are smaller than $R$ in Proposition 4.

The proof of Theorem 1 using quantitative Carleman inequalities closely follows [7], which is a physical space analogue of Tao’s strategy [34]. In order to not to fully repeat parts of the proof in [7], we provide a detailed sketch. This allows the reader with [7] in hand to recover all the details. Throughout the detailed sketch, $C$ denotes a positive universal constant, which may vary from line to line unless otherwise specified.

**Step 1: quantitative unique continuation.** The purpose of this step is to prove the following estimate:

\begin{equation}
T_1^\frac{1}{2} e^{-\mathcal{C} M^{147} R^2_{1, s}} \lesssim \int_{-T_1} T_1 B(0,2R_s) \setminus B(0,R_s/2) \frac{d|\Omega(x,t)|^2}{dxdt},
\end{equation}

for all $T_1$ and $R_s$ such that

\begin{equation}
2 \mathcal{M}^{200}(-t'_0) < T_1 < \mathcal{M}^{-192} \quad \text{and} \quad R \geq e^{c_K} \geq R_s \geq \mathcal{M}^{100} \left( \frac{T_1}{2} \right)^{\frac{1}{2}}.
\end{equation}

Denote $I_1 := (-T_1, -T_1)$ and $T_1 := t'_0$. We see that (188) implies that (186) is satisfied for every $t \in (-T_1, -T_1)$. Hence, (185) is satisfied for every $t \in (-T_1, -T_1)$. Using that $(-T_1, -T_1) \subset [-1, 1] \subset [-2, 0]$, we can apply Proposition 5. This implies that there exists an epoch of regularity $I''_1 = [t''_1 \setminus T''_1, t'_1] \subset I_1$ such that

\begin{equation}
T''_1 = |I''_1| = \mathcal{M}^{-46} \quad \text{and} \quad \mathcal{M}^{-46} \frac{T}{2}
\end{equation}

and for $j = 0, 1, 2$,

\begin{equation}
\|\nabla^j V\|_{L^\infty_B(\mathcal{M}^{03}|x| \geq \mathcal{M}^{51} T''_1)} \lesssim \frac{1}{\mathcal{M}^j}\frac{T''_1^{(j+1)}}{2} = \frac{1}{\mathcal{M}^j}\frac{T''_1^{(j+1)}}{2}.
\end{equation}

Let $T''_1 := \frac{3}{2} T''_1$ and $s'^{(''} := [t''_1 \setminus T''_1, t''_1]$. Let $x_1 \in \mathbb{R}^3$ be such that $e^{c_K} \geq |x_1| \geq \mathcal{M}^{100} \left( \frac{T_1}{2} \right)^{\frac{1}{2}}$ and let $r_1 := \mathcal{M}^{50} |x_1| \geq \mathcal{M}^{150} \left( \frac{T_1}{2} \right)^{\frac{1}{2}}$. Notice that for $\mathcal{M}$ large enough

\begin{equation}
r_1 \geq \mathcal{M}^{50} |x_1| \geq \mathcal{M}^{150} \left( \frac{T_1}{2} \right)^{\frac{1}{2}} \geq \mathcal{M}^{53} (\mathcal{M}^{06}(-t'')) \quad \text{and} \quad r_1^2 \geq 4000 T''_1.
\end{equation}

Notice that

\begin{equation}
B(x_1, r_1) \subset B(0, \mathcal{M}^{51} |x_1|) \subset B(0, 3 e^{c_K}).
\end{equation}

Using the above facts we can apply the Carleman inequality corresponding to quantitative unique continuation [7, Proposition B.2] on the cylinder $C_1 = \{(x, t) \in \mathbb{R}^3 \times \mathbb{R} : t \in [0, T''_1] \text{, } |x| \leq r_1\}$. We apply this to the function $w : C_1 \rightarrow \mathbb{R}^3$, defined for all $(x, t) \in C_1$ by

\[w(x, t) := \Omega(x, x, s'' - t),\]

with

\[r = r_1, \quad S = S_1 := T''_1, \quad C_{\text{Carl}} = \frac{16}{\mathcal{M}, \quad \mathcal{S}_1 = \frac{T''_1}{20000} \quad \text{and} \quad \mathcal{S}_1 = \mathcal{M}^{150} T''_1 \leq \frac{T''_1}{10000}.\]
This gives

\[(193) \quad Z_1 \lesssim e^{-\frac{r_1^2}{20000}} X_1 + \left(\frac{\xi_1}{\xi_1}\right)^{\frac{3}{2}} \left(\frac{C_1^2}{\xi_1}\right) Y_1,\]

where

\[
X_1 := \int_{s''-\frac{T''}{20000}}^{s''} \int_{B(x_1, \mathcal{M}^{50}|x_1|)} ((T''')^{-1}|\Omega|^2 + |\nabla \Omega|^2) \, dx \, ds, \quad Y_1 := \int_{B(x_1, \mathcal{M}^{50}|x_1|)} \left| \nabla (x, s''\big|2(\xi_1)\right) - \frac{1}{2} e^{-\frac{|x-x_1|^2}{\xi_1}} \, dx,
\]

\[
Z_1 := \int_{s''-\frac{T''}{20000}}^{s''} \int_{B(x_1, \mathcal{M}^{50}|x_1|)} ((T''')^{-1}|\Omega|^2 + |\nabla \Omega|^2) e^{-\frac{|x-x_1|^2}{\xi_1}} \, dx \, ds.
\]

By (191), we have

\[
B(0, \mathcal{M}^{96}(-s)\frac{1}{2}) \subset B(0, \mathcal{M}^{96}(-s'')\frac{1}{2}) \subset B(0, |x_1|) \subset B(x_1, 2|x_1|) \subset B(x_1, \mathcal{M}^{50}|x_1|)
\]

for all \( s \in [s'' - \frac{T''}{20000}, s'' - \frac{T''}{20000}] \) and for \( \mathcal{M} \) sufficiently large. Combining this with (185) gives

\[
Z_1 \gtrsim \int_{s''-\frac{T''}{20000}}^{s''} \int_{B(0, \mathcal{M}^{96}(-s)\frac{1}{2})} (T''')^{-1}|\Omega(x, s)|^2 \, dx \, ds e^{-\frac{C|x_1|^2}{s''}} \gtrsim \mathcal{M}^{-111}(T'')^{-\frac{1}{2}} e^{-\frac{C|x_1|^2}{s''}}.
\]

Using (192), we use the quantitative regularity (190) we obtain

\[
e^{-\frac{r_1^2}{50000}} X_1 \lesssim (T''')^{-2} \mathcal{M}^{150}|x_1|^3 e^{-\frac{C\mathcal{M}^{100}|x_1|^2}{s''}} \lesssim (T'')^{-\frac{1}{2}} e^{-\frac{C\mathcal{M}^{100}|x_1|^2}{s''}}
\]

and

\[
\left(\frac{\xi_1}{\xi_1}\right)^{\frac{3}{2}} \left(\frac{C_1^2}{\xi_1}\right) Y_1 \lesssim \mathcal{M}^{225} e^{-\frac{C\mathcal{M}^{101}|x_1|^2}{s''}} \int_{B(x_1, \mathcal{M}^{50}|x_1|)} |\nabla (x, s'\big|2(\xi_1)|^2 \, dx + \mathcal{M}^{225}(T'')^{-\frac{1}{2}} e^{-\frac{C\mathcal{M}^{150}|x_1|^2}{s''}}.
\]

Gathering these bounds and combining with (193) yields

\[
\mathcal{M}^{-111}(T'')^{-\frac{1}{2}} e^{-\frac{C|x_1|^2}{s''}} \lesssim (T''')^{-\frac{1}{2}} e^{-\frac{C\mathcal{M}^{100}|x_1|^2}{s''}}
\]

\[\quad + \mathcal{M}^{225} e^{-\frac{C\mathcal{M}^{101}|x_1|^2}{s''}} \int_{B(x_1, \mathcal{M}^{50}|x_1|)} |\nabla (x, s'\big|2(\xi_1)|^2 \, dx + \mathcal{M}^{225}(T'')^{-\frac{1}{2}} e^{-\frac{C\mathcal{M}^{150}|x_1|^2}{s''}}.
\]

Using (189) and \( e^{c\mathcal{M}} \geq |x_1| \geq \mathcal{M}^{100}(T'')^{\frac{1}{2}} \), we see that for \( \mathcal{M} \) sufficiently large

\[
\mathcal{M}^{-336}(T''')^{-\frac{1}{2}} e^{-\frac{C\mathcal{M}^{101}|x_1|^2}{s''}} \lesssim \int_{B(x_1, \mathcal{M}^{50}|x_1|)} |\nabla (x, s'\big|2(\xi_1)|^2 \, dx.
\]
Hence, for all $s'' \in [t''_1 - \frac{T''}{4}, t''_1]$, for all $e^{CM}\geq |x_1| \geq \mathcal{M}^{100}(\frac{T}{2})^{\frac{1}{2}}$, \[
 \int_{B(x_1, \frac{1}{2}T)} |\Omega(x, s'')|^2 \, dx \geq \mathcal{M}^{-336}(T''_1)^{-\frac{1}{2}} e^{\frac{C_{\mathcal{M}^{100}} |x_1|^2}{T''_1}} = \mathcal{M}^{-336}(T''_1)^{-\frac{1}{2}} e^{\frac{2C_{\mathcal{M}^{147}} |x_1|^2}{T''_1}}. \]

Let $e^{CM}\geq R_0 \geq \mathcal{M}^{100}(\frac{T}{2})^{\frac{1}{2}}$ and $x_1 \in \mathbb{R}^3$ be such that $|x_1| = R_0$. Integrating in time $[t''_1 - \frac{T''}{4}, t''_1]$ yields the claim (187) of Step 1.

**Step 2: quantitative backward uniqueness.** The goal of this step and Step 3 below is to prove the following claim:

\[
(194) \quad T_2^{-\frac{1}{2}} \exp \left( - \exp(\mathcal{M}^{1123}) \right) \lesssim \int_{B(0, \frac{3}{16}\mathcal{M}^{1100}R_2) \setminus B(0, 2R_2)} |\Omega(x, 0)|^2 \, dx,
\]

for all $8\mathcal{M}^{401}(-t''_0) < T_2 \leq 1$ and $\mathcal{M}$ sufficiently large. Here, $R_0$ and $R_2$ are as in (196)-(198). This is the key estimate for Step 4 below and the proof of Theorem 1.

Let $T_1$ and $T_2$ be such that

\[
(195) \quad 8\mathcal{M}^{401}(-t''_0) < T_2 \leq 1 \quad \text{and} \quad T_1 := \frac{T_2}{4\mathcal{M}^{201}}.
\]

Let

\[
(196) \quad R_0 := R^2(T_2)^{\frac{1}{2}},
\]

for a universal constant $K^2 \geq 1$ to be chosen sufficiently large below. In particular it is chosen in Step 3 such that (214) holds, which makes it possible to absorb the upper bound (213) of $X_3$ in the left hand side of (212). From (196), we see that for $\mathcal{M}$ sufficiently large we have that

\[
T_2^{\frac{1}{2}} \leq R_0 \leq T_2^{\frac{1}{2}} 6 e^{2\mathcal{M}} e^{-400\mathcal{M}^{1122}}.
\]

By Proposition 7, for $\mathcal{M} \geq \lambda_0(100)$ there exists a scale

\[
(197) \quad 2R_0 \leq R_2 \leq 2R_0 e^{100\mathcal{M}^{1122}}
\]

and a good cylindrical annulus

\[
(198) \quad \mathcal{A}_2 := \{R_2 < |x| < \frac{\mathcal{M}^{1100}R_2}{16} \} \times \left( -\frac{T_2}{16}, 0 \right)
\]

such that for $j = 0, 1$,

\[
(199) \quad \|\nabla^j V\|_{L^\infty(\mathcal{A}_2)} \leq \tilde{C}_j \mathcal{M}^{-300} T_2^{\frac{j+1}{2}}, \quad \|\nabla \Omega\|_{L^\infty(\mathcal{A}_2)} \leq \tilde{C}_2 \mathcal{M}^{-300} T_2^{-\frac{3}{2}}.
\]

Let

\[
(200) \quad \mathcal{A}_2 := \left\{4R_2 \leq |x| \leq \frac{\mathcal{M}^{1100}R_2}{16} \right\} \times \left[ -\frac{T_2}{\mathcal{M}^{201}}, 0 \right] \subseteq B(0, R) \times \left[ -\frac{T_2}{\mathcal{M}^{201}}, 0 \right]
\]

and choose $\mathcal{M}$ sufficiently large such that $\tilde{C}_j \mathcal{M}^{-300} \leq 1$ and $\tilde{C}_2 \mathcal{M}^{-300} \leq 1$. Using (200) we see that $(V, P)$ solve the Navier-Stokes equations on $\mathcal{A}_2$.

The above facts allow us to apply the Carleman inequality corresponding to quantitative backward uniqueness [7, Proposition B.1] to the function $w : \mathcal{A}_2 \to \mathbb{R}^3$ defined for all $(x, t) \in \mathcal{A}_2$ by

\[
(210) \quad w(x, t) = \Omega(x, -t),
\]

with

\[
(211) \quad r_- := 4R_2, \quad r_+ := \frac{1}{16} \mathcal{M}^{1100} R_2, \quad S = S_2 := \frac{T_2}{\mathcal{M}^{201}} \quad \text{and} \quad C_{\text{Carl}} = \mathcal{M}^{201}.
\]
Step 3: a final application of quantitative unique continuation. Assume that the bound (203) holds. The same reasoning as in [34] and the subsequent paper [7] (involving the pigeonhole principle three times) gives the following. There exists

\[ 8R_2 \leq R_3 \leq \frac{1}{8} M^{1100} R_2, \]

and \( x_3 \in B(0, R_3) \setminus B(0, \frac{R_2}{2}) \) such that

\[ T_2^{-\frac{3}{2}} \exp\left(-\frac{8R_3^2}{T_2}\right) T_2 \leq -t_3 \leq \frac{T_2}{M^{201}} \]

and the subsequent paper [7] (involving the pigeonhole principle three times) gives the following. There exists

\[ 8R_2 \leq R_3 \leq \frac{1}{8} M^{1100} R_2, \]

where

\[ \frac{1}{2} \exp\left(-\frac{8R_3^2}{T_2}\right) T_2 \leq -t_3 \leq \frac{T_2}{M^{201}} \]

and the subsequent paper [7] (involving the pigeonhole principle three times) gives the following. There exists

\[ 8R_2 \leq R_3 \leq \frac{1}{8} M^{1100} R_2, \]

where

\[ \frac{1}{2} \exp\left(-\frac{8R_3^2}{T_2}\right) T_2 \leq -t_3 \leq \frac{T_2}{M^{201}} \]

and the subsequent paper [7] (involving the pigeonhole principle three times) gives the following. There exists

\[ 8R_2 \leq R_3 \leq \frac{1}{8} M^{1100} R_2, \]

where

\[ \frac{1}{2} \exp\left(-\frac{8R_3^2}{T_2}\right) T_2 \leq -t_3 \leq \frac{T_2}{M^{201}} \]

and the subsequent paper [7] (involving the pigeonhole principle three times) gives the following. There exists

\[ 8R_2 \leq R_3 \leq \frac{1}{8} M^{1100} R_2, \]

where

\[ \frac{1}{2} \exp\left(-\frac{8R_3^2}{T_2}\right) T_2 \leq -t_3 \leq \frac{T_2}{M^{201}} \]
We apply now the Carleman inequality corresponding to quantitative unique continuation [7, Proposition B.2]. We apply this to the function $w : B(0, r) \times [0, -20000t_3] \to \mathbb{R}^3$ defined for all $(x, t) \in B(0, r) \times [0, -20000t_3]$ by
\[
w(x, t) = \Omega(x + x_3, -t),
\]
with
\[
S = S_3 := -20000t_3, \quad r = r_3 := 1000R_3\left(-\frac{t_3}{T_2}\right)^\frac{1}{4}, \quad \pi_3 = z_3 = -t_3 \quad \text{and} \quad C_{\text{Carle}} = 1.
\]
This gives
\[
Z_3 \lesssim e^{\frac{r_3^2}{5000}X_3} + (-t_3)^\frac{3}{2}e^{-\frac{Cr_3^2}{16}Y_3},
\]
where
\[
X_3 := \int_{-S_3}^0 \int_{B(x_3, r_3)} (S_3^{-1}|\Omega|^2 + |\nabla \Omega|^2) \, dx \, dt, \quad Y_3 := \int_{B(x_3, r_3)} |\Omega(x, 0)|^2 (-t_3)^{-\frac{3}{2}} e^{-\frac{|x-x_3|^2}{4t_3}} \, dx,
\]
\[
Z_3 := \int_{2S_3}^{t_3} \int_{B(x_3, \frac{r_3}{2})} (S_3^{-1}|\Omega|^2 + |\nabla \Omega|^2)e^{\frac{|x-x_3|^2}{4t_3}} \, dx \, dt.
\]
Using (196)-(197) and (205)-(206), we have that
\[
B(x_3, (-t_3)^\frac{1}{4}) \subset B(x_3, \frac{r_3}{2}) \subset B(x, r) \subset B\left(x_3, \frac{|x_3|^2}{2} \right)
\]
\[
\subset \{ \frac{R_3}{4} < |y| < \frac{R_3}{2} \} \subset \{ 2R_2 < |y| < \frac{1}{10}M^{1100}R_2 \}.
\]
Using this and (207) and $T_2^{-1} \leq S_3^{-1}$ we have
\[
T_{2}^{\frac{1}{4}} \exp\left(-\frac{18R_3^2}{T_2}\right) \lesssim \int_{2S_3}^{t_3} \int_{B(x_3, (-t_3)^\frac{1}{4})} (T_2^{-1}|\Omega|^2 + |\nabla \Omega|^2)e^{-\frac{|x-x_3|^2}{4t_3}} \, dx \, dt \leq Z_3
\]
Hence, we have
\[
T_{2}^{-\frac{1}{4}} \exp\left(-\frac{18R_3^2}{T_2}\right) \leq C_{\text{univ}}e^{\frac{r_3^2}{5000}X_3} + C_{\text{univ}}(-t_3)^\frac{3}{2}e^{-\frac{Cr_3^2}{16}Y_3}.
\]
Where $C_{\text{univ}}$ is a positive universal constant. Using the bounds (199) along with (206), we find that
\[
C_{\text{univ}}e^{\frac{r_3^2}{5000}X_3} \lesssim S_3^{-2}r_3^3 e^{\frac{r_3^2}{5000}} \lesssim (-t_3)^{-\frac{1}{2}} e^{\frac{r_3^2}{1000t_3}} \lesssim T_2^{-\frac{1}{2}} e^{\frac{4R_3^2}{T_2} e^{\frac{r_3^2}{1000t_3}}} \lesssim T_2^{-\frac{1}{2}} e^{\frac{596R_2^2}{T_2}} \lesssim T_2^{-\frac{1}{4}} e^{-\frac{18R_2^2}{T_2} e^{\frac{978R_2^2}{T_2}}} \leq C'_{\text{univ}}T_2^{-\frac{1}{4}} e^{-\frac{18R_2^2}{T_2} e^{-978\cdot 256(K^2)^2}}.
\]
We choose $K^2$ sufficiently large so that
\[
C'_{\text{univ}}e^{-978\cdot 256(K^2)^2} \leq \frac{1}{2},
\]
where $C'_{\text{univ}} \in (0, \infty)$ is the universal constant appearing in the last inequality of (213). Therefore, the term in the right hand side of (213) can be absorbed into the left hand side of
(212). From this and (212), we obtain
\[ T_2^{-\frac{1}{2}} \exp \left(- C \frac{R_3^2}{T_2} \right) \lesssim \int_{B(x, r_3)} |\Omega(x, 0)|^2 \, dx. \]

Using (196), the upper bound
\[ R_3 \leq \frac{1}{5} \mathcal{M}^{1100} R_2 \leq \frac{1}{4} \mathcal{M}^{1100} \exp(100 \mathcal{M}^{1122}) R_0 \]
and (210), it follows that
\[ T_2^{-\frac{1}{2}} \exp(- \exp(\mathcal{M}^{1123})) \lesssim \int_{B(0, \frac{3}{10} \mathcal{M}^{1100} R_2) \setminus B(0, 2R_2)} |\Omega(x, 0)|^2 \, dx. \]

**Step 4, conclusion: summing the scales and lower bound for the global \( L^3 \) norm.** Next
we use (215) and argue in a similar way to [7]\(^8\). In particular, the pigeonhole principle, the
quantitative estimate (199), integration by parts and Hölder’s inequality gives that
\[ \int_{B(0, \exp(\mathcal{M}^{1123})) \setminus B(0, T_2^{\frac{1}{2}})} |V(x, 0)|^3 \, dx \geq \exp(- \exp(\mathcal{M}^{1123})), \]
for all \( 8 \mathcal{M}^{401}(-t_0') \leq T_2 \leq 1 \). We note that this holds for \( T_2 = 8 \mathcal{M}^{401}(-t_0') \) using that \( V \)
is smooth.

We then sum (216) over \( k \) scales starting with \( T_2 = 8 \mathcal{M}^{401}(-t_0') \), where
\[ k := \lfloor \frac{1}{2} \mathcal{M}^{-1123} \log \left( \frac{\mathcal{M}^{-401}}{8(-t_0')} \right) + 1 \rfloor. \]

This gives that
\[ \exp(- \exp(\mathcal{M}^{1125})) \frac{1}{2} \mathcal{M}^{-1123} \log \left( \frac{\mathcal{M}^{-401}}{8(-t_0')} \right) \leq \int_{B(0, \exp(\mathcal{M}^{1123})) \setminus B(0, 8 \mathcal{M}^{401}(-t_0')^{\frac{1}{2}})} |V(x, 0)|^3 \, dx \leq \mathcal{M}^3. \]

Rearranging this gives the lower bound (184), as required.

### 7. Proof of Theorem 2

Without loss of generality, assume \( 0 < \delta < \min(\text{dist}(x_0, \partial B(4)), (\frac{1}{4} T_*)^{\frac{2}{3}}) \). Assume
the hypothesis of Theorem 2 holds with the contrapositive of (10). Then there exists \( N \in (0, \infty) \) such that
\[ \sup_{0 < s < t} \|v(\cdot, s)\|_{L^3(B(x_0, \delta))} \leq N \left( \log \log \left( \frac{1}{(T_* - t)^{\frac{1}{3}}} \right) \right)^{\frac{1}{1+2\delta}} \quad \forall t \in (0, T_*). \]

Next, we fix \( \varepsilon_0 > 0 \) such that
\[ \left( \frac{N}{\log \log \left( (\varepsilon_0)^{-\frac{1}{3}} \right) \frac{1+2\delta}{1+2\delta}} \right)^{\frac{1}{1+2\delta}} \leq 1 \quad \text{and} \]
\[ \left( \log \log \left( (\varepsilon_0)^{-\frac{1}{3}} \right) \right)^{\frac{1}{1+2\delta}} \geq \max\{\mathcal{M}_0, 16\delta^{-2} \|p\|_{L^2_\varepsilon,t(B(0,4)\times(0,T_*))}^3 \}. \]

\(^8\)These arguments are in turn based on Tao’s arguments in [34].
Here, $\mathcal{M}_0$ is as in Theorem 1. Then (218)-(219) gives that
\[
\sup_{0 < s < t} \left\| v(\cdot, s) \right\|_{L^3(B(x_0, \delta))} \leq \left( \log \log \left( \frac{1}{(T_* - t)^{\frac{4}{7}}} \right) \right)^{\frac{1128}{7}} \forall t \in (T_* - \varepsilon_0, T_*).
\]
From now on, we take any $t \in (\max(\frac{1}{2}T_*, T_* - \varepsilon_0), T_*)$. Next we rescale
\[
(V_{x_0, \delta}(y, s), P_{x_0, \delta}(y, s)) = \left( \frac{\delta}{4}(\frac{\delta y}{x_0} + x_0, \frac{\delta^2}{16}s + t), \frac{\delta^2}{16}p(\frac{\delta y}{x_0} + x_0, \frac{\delta^2}{16}s + t) \right).
\]
Then, $(V_{x_0, \delta}, P_{x_0, \delta})$ is a smooth solution to the Navier-Stokes equations in $B(0, 4) \times [-16, 0]$. By (221) we have
\[
\sup_{0 < s < t} \left\| v(\cdot, s) \right\|_{L^3(B(0, 4))} \leq M := \left( \log \log \left( \frac{1}{(T_* - t)^{\frac{4}{7}}} \right) \right)^{\frac{1128}{7}}.
\]
By (220), we have that
\[
\max\{\mathcal{M}_0, \sup_{0 < \delta \leq 1} \left\| P_{x_0, \delta} \right\|_{L^6_t(\bar{Q}(0, 4))} \} \leq M.
\]
From (222)-(223), we see that we can apply Theorem 1. This gives
\[
\frac{\delta}{4} \left\| v(\cdot, t) \right\|_{L^\infty(B(x_0, \delta))} = \left\| V_{x_0, \delta}(\cdot, 0) \right\|_{L^\infty(B(0, \frac{4}{\delta}))} \leq \exp(\exp(M^{1128})) = \frac{1}{(T_* - t)^{\frac{4}{7}}},
\]
So (218) implies that
\[
\left\| v(\cdot, t) \right\|_{L^\infty(B(x_0, \delta))} \leq \frac{4}{\delta(T_* - t)^{\frac{4}{7}}} \forall t \in (\max(\frac{1}{2}T_*, T_* - \varepsilon_0), T_*).
\]
This implies that
\[
v \in L^2_t L^\infty_x(B(x_0, \delta) \times (\max(\frac{1}{2}T_*, T_* - \varepsilon_0), T_*)).
\]
By [32, Theorem 7], this implies that
\[
v \in L^\infty_{2,t}(B(x_0, r) \times (T_* - r^2, T_*))
\]
for all sufficiently small $r > 0$. This violates the hypothesis of Theorem 2. Hence the contrapositive (218) cannot hold and we get the desired conclusion. □

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