EXISTENCE RESULTS IN THE LINEAR DYNAMICS OF QUASICRYSTALS WITH PHASON DIFFUSION AND NON-LINEAR GYROSCOPIC EFFECTS

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Abstract. Quasicrystals are characterized by quasi-periodic arrangements of atoms. The description of their mechanics involves deformation and a (so-called phason) vector field accounting at macroscopic scale of local phase changes, due to atomic flips necessary to match quasi-periodicity under the action of the external environment. Here we discuss the mechanics of quasicrystals, commenting the shift from its initial formulation, as standard elasticity in a space with dimension twice the ambient one, to a more elaborated setting avoiding physical inconveniences of the original proposal. In the new setting we tackle two problems. First we discuss the linear dynamics of quasicrystals including a phason diffusion. We prove existence of weak solutions and their uniqueness under rather general boundary and initial conditions. We then consider phason rotational inertia, non-linearly coupled with the curl of the macroscopic velocity, and prove once again existence of weak solutions to the pertinent balance equations.

1. Introduction

1.1. Quasi-periodic atomic arrangements. In a 1991 report of the International Union of Crystallography, under “terms of reference”, we find that “by ‘crystal’ we mean any solid having an essentially discrete diffraction diagram, and by ‘aperiodic crystal’ we mean any crystal in which three-dimensional lattice periodicity can be considered to be absent” [9]. The definition includes then the possibility of quasi-periodic crystals, also called quasicrystals to remind the circumstance.

Such a viewpoint overcame in the definition of crystals the previous mention of periodicity of the atomic distribution in space: a sliding in the basic paradigm of crystallography induced by the 1982 experimental discovery by D. Shechtman, published in 1984 [27], of the possibility of atomic arrangements with icosahedral symmetry in aluminium-based synthetic alloys, and pentagonal symmetry in thin films of the same materials, not determined by twinned atomic structures. Natural quasi-periodic alloys have been found since then in meteorites.

A typical example of a quasiperiodic function over the line is \( \sin x + \cos \alpha x \), with \( \alpha \) an irrational number. It can be intended as obtained from a periodic function in the plane, namely \( \sin x + \cos y \), with the additional constraint \( y = \alpha x \). Moreover, if we consider a quasi-periodic distribution of mass points in 3D space and compute the Fourier expansion of the mass distribution, we find that the emerging wave vector is six-dimensional. In general, a quasi-periodic atomic arrangement can be viewed as the orthogonal projection of a portion of a periodic lattice onto
an appropriate (incommensurate) subspace. The recurrent example includes a periodic lattice with square symmetry, filling a plane. If we select a strip around a straight line, inclined about $3/2$ with respect to the symmetry axes, and we project orthogonally the atoms in the strip over the line, we get a periodic one-dimensional lattice. In contrast, if we choose a straight line inclined by an irrational angle and reproduce the same process, what we find is a quasi-periodic one-dimensional lattice. Quasi-periodicity emerges even in the golden mean case (i.e. $3/2$) if we move orthogonally to the line the lattice in an appropriate way: some atoms go out the strip, others enter it. Such orthogonal shifts take the name of phason defects. The common terminology seems to recall that, by means of the orthogonal displacement of the lattice, we are changing the phase (at least with respect to the symmetry) of the lattice over the line. Such a construction, however, is just an ideal geometric representation of what is in nature. In the physical space, we can consider phasons as inner (low spatial scale) degrees of freedom exploited locally by the atoms to assure quasi-periodicity, compatibly with the boundary conditions imposed to a quasi-crystalline body by the interaction with the external environment. As inner degrees of freedom, they are invariant with respect to rigid translations of the whole body. This aspect has a key role in the developments presented in what follows.

1.2. Origins of and trends in the mechanics of quasicrystals. In building up continuum models of the mechanics of quasicrystals, the geometric constructions above have suggested at a first glance of proposing just a higher dimensional replica of crystal elasticity traditional setting (see, e.g., for the first proposals [14], [5] and also [10], [25], [8]). Researchers have then focused the attention primarily on linear elasticity, meeting this way the concrete advantage of having at disposal a format where we can easily reproduce systematically, without technical and conceptual difficulties, all the standard results of traditional linear elasticity. The approach may lead to conclusions with peculiar physical significance (a review of the production in this trend is in [6]; up-dated results are in [13], [12], [15], [16], [23]). However, in 2011, S. Colli and P. M. Mariano [4] showed the existence of at least two cases where such a format of quasicrystal linear elasticity produces non-physical results, i.e. the instantaneous propagation at infinity of the phason disturbances, described at a continuum level by a differentiable vector field (this way the representation is multi-scale) and they conjectured that a non-zero conservative phason self-action would avoid such a drawback, assuring phason decay in space. The analytical proof of the conjecture for one-dimensional quasicrystals is in [21]. Numerical simulations corroborating the conjecture in two-dimensional space appeared later in [22].

- In fact, a dissipative phason self-action has been already assumed to exist in [25] but just with dissipative nature, an action presumed to drive phason diffusion. The problems evidenced in [4] were, however, in purely conservative static setting.
- A first proof of the existence of a possibly nonzero phason self-action with both conservative and dissipative nature appeared first in [13]. Then it was rediscussed in [21], where an ancillary consequence shows the theoretical possibility of a rotational-type phason inertia induced by the local spin of the macroscopic velocity field, an aspect taken into account here.

1.3. What we discuss. Here we focus the attention on the dynamics of quasicrystals in small-strain regime, including a phason self-action with both conservative
and dissipative components, and gyroscopic type phason inertia effects. Before tackling the analysis of the pertinent balance equations, in Section 2 we rediscuss preliminarily their deduction from the invariance under rigid-body changes in observers of the power of external actions over a generic part of the body. We briefly reproduce the path followed in [21] by showing in addition a non-standard action-reaction principle for the phason traction and the existence of the phason stress (Cauchy-type theorem), which emerge as special occurrences of abstract results in the general model-building framework of the mechanics of complex materials, presented in [19]. In contrast with what is developed in [21], we restrict the treatment just to Euclidean frames, identifying covariant and contravariant components of the tensors considered, because the special cases tackled analytically in the subsequent sections refer to that frames.

In Section 3, we consider in small strain regime (1) linear constitutive structures for the Cauchy stress, the phason stress, and the conservative component of the phason self-action, (2) phason diffusion driven by a dissipative phason self-action, (3) macroscopic inertia. We provide existence and uniqueness theorems for the weak solution of the balance equations. Then, in Section 4, we consider non-linear rotational-type phason inertia and we provide a theorem of existence (and regularity) of the weak solutions of these modified balance equations. In the treatment we consider first regularization induced by viscous-type standard and phason stresses. Then we compute the limit when such regularizations vanish. The results in the linear case are a necessary prerequisite for the non-linear one.

2. Continuum mechanics of quasicrystals

2.1. Deformation and phason field. We write $B$ for the macroscopic reference shape of a quasicrystalline body (it is just a geometric setting where we may compare lengths, angles, surfaces and volumes to measure strain), assumed to be a bounded arcwise connected open region in the three-dimensional point space $E^3$, coinciding with the interior of its closure and endowed with surface-like boundary uniquely oriented everywhere to within a finite number of corners and edges. In another space, indicated by $\tilde{E}^3$ and distinguished by $E^3$ just by an isomorphism $i : E^3 \rightarrow \tilde{E}^3$, which we can choose as an orientation preserving isometry or even the simple identification, we record shapes of the body that we consider deformed with respect to $B$, reached by means of one-to-one, differentiable, orientation preserving maps $x \mapsto \tilde{y} := \tilde{\nu}(x) \in \tilde{E}^3$.

The distinction between the two spaces justifies the standard statement that two observers, i.e. two frames in the whole space, differing one another by a rigid-body motion, evaluate the same reference place. Moreover, such a distinction is crucial when we want to consider material mutations, which are naturally described by a non-unique choice of the reference place (see [19]).

A field taking values in a three-dimensional real vector space $V^3$, precisely $x \mapsto \nu := \tilde{\nu}(x) \in V^3$, assumed to be differentiable, accounts point-by-point at the continuum scale for the atomic flips, which allow to match quasi-periodicity. This is the so-called phason field in Lagrangian representation, i.e. considered as a field over the reference place. We then call $V^3$ the phason space.

From now on we endow $\tilde{E}^3$, $E^3$ and $V^3$ with Cartesian frames.

Motions are then (in generalized sense) pairs

$$(x, t) \mapsto y := \tilde{y}(x, t) \in \tilde{E}^3, \quad (x, t) \mapsto \nu := \tilde{\nu}(x, t) \in V^3,$$
assumed to be sufficiently differentiable in time.

We shall write \( F \) and \( N \) for the deformation gradient and the phason field gradient, evaluated at \( x \) and \( t \). The assumption that the deformation preserves the local orientation of triples of linearly independent vectors implies \( \det F > 0 \), a standard consequence, indeed. We define another vector field, the displacement, as

\[ (x, t) \mapsto u := \hat{u}(x, t) := \hat{y}(x, t) - \iota(x). \]

Consequently, we have \( \nabla u := \nabla \hat{u}(x, t) = F + I \), where \( I \) is the second-rank unit tensor.

As a matter of notation, we shall write \( u_t, u_{tt}, \) and \( \nu_t \) for the values \( \dot{\hat{y}} := \frac{d\hat{y}(x, t)}{dt}, \hat{y} := \frac{d^2\hat{y}(x, t)}{dt^2}, \) and \( \dot{\nu} := \frac{d\nu(x, t)}{dt} \), respectively, the latter chosen for the sake of notational uniformity. The velocity in the physical space and the phason time rate just listed are expressed in Lagrangian representation, i.e. as fields over the reference place and the time scale. We can have an Eulerian representation of such fields, i.e. we can consider them defined over the actual shape \( B_a = \hat{y}(B, t) \). In this case we write \( v(y, t) \) and \( \nu(y, t) \). Since at \( x \) and \( t \), the vector \( \hat{y}(x, t) \) is tangent to \( B_a \) at the point \( y \), we have the standard identity

\[ \hat{y}(x, t) = v(y, t). \]

An analogous relation does not hold between \( v(y, t) \) and \( \nu(x, t) \). The lack of identity depends on the circumstance that \( v \) is the time rate of the Eulerian representation of the phason field, which is a map \( \tilde{\nu}_a \) defined by

\[ \tilde{\nu}_a := \nu \circ \hat{y}^{-1}, \]

a definition possible for \( \hat{y} \) is one-to-one. The subscript \( a \) means actual, i.e. referred to the deformed configuration, here and in what follows.

The condition

\[ |\nabla u| < 1 \]

defines the small strain regime, in which we develop the analyses presented in Sections 3 and 4. In this setting we can ‘confuse’ \( B \) with \( B_a, u_t \) with \( v, \nu \) with \( \nu_a := \nu_a(y, t) \).

2.2. Changes in observers. According to the definition proposed explicitly in [18] and further refined in [19], we define observer frames of reference assigned on all spaces necessary to describe the shape of a body and its motion. In the setting discussed here, an observer is then (1) a frame in \( \mathcal{E}^3 \) or–it is the same–in the pertinent translation space containing \( u \), a space identified with \( \mathbb{R}^3 \), once we fix an origin, (2) a frame in \( \mathcal{E}^3 \), (3) a frame in \( \mathcal{V}^3 \)–all identified with copies of \( \mathbb{R}^3 \), which we can consider differing one another just by the identification–and (4) a time scale.

We consider time-varying synchronous changes in observers leaving invariant the reference space and changing the frame \( \mathcal{E}^3 \) by a rigid body motion. Precisely, let us write \( O \) and \( O' \) for these two observers. A place \( y \) for \( O \) becomes \( y' \) for \( O' \), with

\[ y' := w(t) + Q(t)(y - y_0), \]

where \( w(t) \) and \( Q(t) \) are the values at \( t \) of time-differentiable maps \( t \mapsto w(t) \in \mathbb{R}^3, t \mapsto Q(t) \in SO(3) \), with \( t \) running in the selected time interval, and \( y_0 \) an arbitrary point in space. The time rates are then \( \dot{y} \) for the first observer and \( \dot{y}' = \dot{w} + \dot{Q}(y - y_0) + \dot{y} \) for the second. By rotating back by \( Q^{-1} = Q^T \) the rate \( \dot{y}' \)
into the frame defining $\mathcal{O}$, and indicating by $\dot{y}^*$ the rotated velocity, which is $Q^T \dot{y}'$, we get

$$\dot{y}^* := c + q \times (y - y_0) + \dot{y},$$

where $c := Q^T w$, and $q$ is the axial vector of the skew-symmetric tensor $Q^T \dot{Q}$, both depending on time only. Since $\dot{y} = u_t$, for the displacement rate, under the change in observer considered here, we find

$$u^*_t = c(t) + q(t) \times (y - y_0) + u_t.$$

Since $\dot{y}(x,t) = v(y,t)$, we can also write

$$v^* := c(t) + q(t) \times (y - y_0) + v.$$

The distinction between the ambient space $\tilde{E}^3$ and the phason one $\mathcal{V}^3$ is just matter of modeling. Atomic flips, determining phason defects, occur in the physical space, indeed. We have also to remind that the notion of observer is just a formal representation of the concrete action of recording a phenomenon. When we rotate an observer in space we should perceive rotated the atomic flips. They are not affected by rigid translations in space for they are internal degrees of freedom. Consequently, with $\nu$ the value of the phason field for $\mathcal{O}$, the observer $\mathcal{O}'$ records a value $\nu' = Q\nu$. The relevant rates are then $\dot{\nu}$ and $\dot{\nu}' = Q\dot{\nu} + \dot{Q}\nu$ respectively. By writing $\dot{\nu}^* = \nu^*$ for $Q^T \dot{\nu}'$, we get

$$\dot{\nu}^* = \dot{\nu} + q \times \nu.$$

### 2.3. External power, invariance and balance

We have already mentioned in the Introduction that we derive balance equations from the invariance of power over a generic part of the body. The word part indicates here a subset $\mathcal{B}$ with non-null volume measure and the same regularity of $\mathcal{B}$ itself or a subset $\mathcal{B}_a$ of the current macroscopic shape $\mathcal{B}_a = \tilde{y}(\mathcal{B},t)$ of the body. Given a generic $\mathcal{B}_a$, we divide as usual all actions exerted on $\mathcal{B}_a$ by the environment and the rest of the body into bulk and contact families, the latter intended to be exerted through the boundary of the part considered. Each family is also subdivided into standard and phason components, all defined by the expression of the power that the external action must perform over $\mathcal{B}_a$ to change its state of motion with velocity $v$ in the physical space and phason rate $\nu$. For this reason we call such a power external, indicating it by $P^{ext}_b$ and defining it in Eulerian representation by

\begin{equation}
(2.1) \quad P^{ext}_{b_a}(v,\nu) := \int_{\mathcal{B}_a} \left( \beta^i \cdot v + \beta^i_\nu \cdot \nu \right) d\mu(y) + \int_{\partial \mathcal{B}_a} (t \cdot v + \tau \cdot \nu) d\mathcal{H}^2,
\end{equation}

where $d\mathcal{H}^2$ is the surface measure along $\partial \mathcal{B}_a$ and $d\mu(y)$ is the volume measure in $\mathcal{B}_a$.

At $y \in \partial \mathcal{B}_a$, where $\partial \mathcal{B}_a$ is oriented by the normal $n$, the standard traction $t$ depends on $y$ itself and $n$, besides the time $t$ (Cauchy’s assumption and Hamel-Noll theorem). Here we assume the same dependence for the phason traction $\tau$, i.e. we impose

$$\tau := \tilde{\tau}(y,n),$$

in addition to

$$t := \tilde{t}(y,n),$$

where we leave unexpressed the dependence on time for the sake of conciseness of some formulas below.

What we impose to $P^{ext}_{b_a}(v,\nu)$ is an axiom of invariance.
Axiom: $P_{ba}^{ext}(v, u)$ is invariant under rigid-body-based changes in observers, i.e.,

$$P_{ba}^{ext}(v^*, u^*) = P_{ba}^{ext}(v, u)$$

for any choice of $c$ and $q$.

**Theorem 2.1.** The axiom of invariance implies the following list of assertions:

(a) If the fields $y \mapsto b_a^1, y \mapsto \nu \times \beta_a^i, y \mapsto t$ and $y \mapsto \tau$ are integrable over $B_a$, the following integral balances hold for any part $b_a$ of $B_a$ and for $B_a$ itself:

\[
\int_{b_a} b_a^1 \, d\mu(y) + \int_{\partial b_a} t \, d\mathcal{H}^2 = 0,
\]

\[
\int_{b_a} ((y - y_0) \times b_a^1 + \nu \times \beta_a^i) \, d\mu(y) + \int_{\partial b_a} ((y - y_0) \times t + \nu \times \tau) \, d\mathcal{H}^2 = 0.
\]

(b) If the standard traction is continuous with respect to $y$ and the standard bulk action is bounded over $B_a$ at every instant, $t$ satisfies the action-reaction principle

\[
t(y, n) = -t(y, -n).
\]

(c) In the same continuity conditions, a second-rank tensor $\sigma$ independent of $n$ exists and is such that

\[
t(y, n) = \sigma(y)n(y).
\]

(d) If the phason traction is continuous with respect to $y$ and the field $y \mapsto \nu \times \beta_a^i$ is bounded over $B_a$ at every instant, $\tau$ satisfies a non-standard action-reaction principle

\[
\nu_a(y) \times (\tau(y, n) - \tau(y, -n)) = 0.
\]

(e) In the same regularity conditions above, a second-rank tensor $S_a$ independent of $n$ exists and is such that

\[
\tau(y, n) = S_a(y)n(y),
\]

a tensor that we call **phason stress**.

(f) If the field $y \mapsto \sigma(y)$ is $C^1$ over $B_a$ and just continuous over its boundary, equation (2.8) implies the validity of the standard pointwise balance of forces

\[
b_a^1 + \text{div}\sigma = 0.
\]

(g) If, in addition, the field $y \mapsto S_a(y)$ is $C^1$ over $B_a$ and just continuous over its boundary, equation (2.9) implies the existence of a vector $z_a$ such that

\[
\text{div}S_a + \beta_a^i - z_a = 0.
\]

and

\[
\text{Skw}\sigma = \frac{1}{2}e(\nu_a \times z + (e\nabla_y (\nu_a))^T S_a),
\]

where the apex $T$ means transposition, $e$ is Ricci’s alternating symbol, and $\nabla_y$ is the gradient with respect to $y$. 
(h) The external power satisfies the following relation:

\[ P_{\text{ext}}^{\tau} (v, v) = \int_{b_a} (\mathbf{a} \cdot \nabla_y v + z_a \cdot v + \mathbf{S}_a \cdot \nabla_y v) \, d\mu(y), \]

for any choice of the rates involved. The right-hand side term takes the name of inner power.

**Proof.** The first item follows trivially by the arbitrariness of \( c \) and \( q \) for the axiom implies

\[ P_{\text{ext}}^{\tau} (c + q \times (y - y_0), q \times \nu) = 0. \]

The first integral balance (2.2) implies the boundedness of the absolute value of the traction average over the boundary of any part, once the bulk actions are bounded. Consequently, standard arguments (see [31]) allow us to obtain the action-reaction principle (2.4) for \( \mathbf{t} \) and the Cauchy theorem (2.5) about the existence of a stress independent of \( n \). Notice that the existence of the stress tensor can be obtained in less stringent regularity assumptions (see, e.g., [26] and [28]).

On defining \( r := (y - y_0) \times b_a^1 + \nu \times \beta_a^1 \) and \( p := (y - y_0) \times \mathbf{t} + \nu \times \mathbf{\tau} \), the integral equation (2.3) writes obviously as

\[ \int_{b_a} r \, d\mu(y) + \int_{\partial b_a} p \, d\mathcal{H}^2 = 0. \]

Since \( y_0 \) is arbitrary, once we choose \( y \), we can select \( y_0 \) such that the above boundedness assumption about \( |b_a^1| \) over \( B \) may imply the one of \( (y - y_0) \times b_a^1 \). Moreover, the above assumption of the boundedness \( |\nu \times \beta_a^1| \) over \( B \) implies the one of \( r \). The assumed boundedness of the first integral implies the one of the absolute value of the right-hand side term, so that we can apply the standard procedure adopted for the traction \( \mathbf{t} \) (see [31]), obtaining a non-standard action-reaction principle (2.6).

We find, in fact,

\[ p(x, n) = -p(x, -n), \]

i.e.

\[ (y - y_0) \times (\mathbf{t}(x, n) - \mathbf{t}(x, -n)) + \nu \times (\mathbf{\tau}(x, n) - \mathbf{\tau}(x, -n)) = 0, \]

so that from (2.4) we find (2.6).

Not writing explicitly time for the sake of conciseness, we can use a tetrahedron type argument to show the linearity of \( p \) with respect to \( n \), namely we show the existence of a second-rank tensor \( A(x) \) such that

\[ p(x, n) = A(x) n(x) . \]

Then we find

\[ (y(x) - y_0) \times \mathbf{t} + \nu(x) \times \mathbf{\tau} = (y(x) - y_0) \times P(x) n(x) + \nu(x) \times \mathbf{\tau} = A(x) n(x), \]

so that

\[ \nu(x) \times \mathbf{\tau}(x, n) = A(x) n(x) - (y(x) - y_0) \times P(x) n(x), \]

which implies the linearity of \( \mathbf{\tau}(x, n) \) with respect to \( n \).

The localization of the integral balance of forces, namely equation (2.2), which is possible due to the arbitrariness of \( b_a \), gives rise to the local balance (2.3). In contrast, the localization of the integral balance (2.3) implies

\[ \nu_a \times (\text{div} \mathbf{S}_a + \beta_a^1) = c_0 T - (\nabla_y \nu_a) \mathbf{T} \mathbf{S}_a, \]
which indicates the existence of a vector, say \( z_a \), satisfying the equation (2.10) with the constraint (2.9).

2.4. Inertia terms. A standard assumption is the additive decomposition of the bulk forces \( b^f \) into inertial, \( b^{in} \), and non-inertial, \( b \) components, the former identified by definition by equating their power to the negative of the time rate of the (canonical) kinetic energy. Since we do not have in mind any external bulk direct action over the phason flips, except perhaps the possible influence of magnetic fields in magnetizable quasicrystals, a material class not treated here, according to the proposal in [21], we consider \( \beta^f \) just with inertial nature.

Our inertia axiom is then the presumed validity of the integral equation

\[
\text{rate of the kinetic energy of } \mathfrak{b}_a = -\int_{\mathfrak{b}_a} (b^{in}_a \cdot \mathbf{v} + \beta^f_a \cdot \mathbf{v}) \, d\mu(y),
\]

for any choice of the rates involved and any part \( \mathfrak{b}_a \). The key point is then the expression of the kinetic energy. There is a debate about the possible existence of a peculiar phason kinetic energy. On one side, who is interested in having a structure duplicating the standard elasticity in a higher-dimensional space, with the obvious analytical advantages, would hope for it. However, just three sound-like branches seem to appear in dynamic spectra recorded in experiments (see, e.g., [29]) so that we should be inclined in not considering phason inertia. For this reason, we write explicitly the previous balance as

\[
\frac{d}{dt} \int_{\mathfrak{b}_a} \frac{1}{2} \rho |\mathbf{v}|^2 \, d\mu(y) = -\int_{\mathfrak{b}_a} (b^{in}_a \cdot \mathbf{v} + \beta^f_a \cdot \mathbf{v}) \, d\mu(y),
\]

where \( \rho := \tilde{\rho}(y,t) \) is the value at \( y \) and \( t \) of the mass density, assumed to be differentiable with respect to its entries, in the actual shape of the body. Here we presume that \( \rho \) is conserved along the motion, i.e. it satisfies the local mass balance

\[
\frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{v}) = 0.
\]

By taking into account the previous equation and applying a standard transport theorem to compute the time derivative of the first integral in the equation (2.13), in which the integration domain depends on time, the arbitrariness of \( \mathfrak{b}_a \)–or the one of the rate fields, which is the same–implies

\[
b^{in}_a = -\rho a,
\]

with \( a := \dot{\mathbf{v}} \) the acceleration in Euclidean representation, and

\[
\beta^f_a \cdot \mathbf{v} = 0.
\]

This last identity implies that \( \beta^f_a \) must be of the form \( \beta^f_a = h \times \mathbf{v} \), with \( h \) a generic vector. P. M. Mariano and J. Planas suggested [21] to identify the vector \( h \) with \(-\text{curl } \mathbf{v}\) so that we have

\[
\beta^f_a = -(\text{curl } \mathbf{v}) \times \mathbf{v}.
\]

Notice that these requirements are superabundant for we could just considered the integral extended to the whole body macroscopic shape \( \mathfrak{b}_a \), the arbitrariness of the rate fields allowing the selection of parts through the possibility of choosing compactly supported rate fields. However, we maintain the superabundant choice because it seems to us that it puts better in evidence the physical nature of the requirement, which would fail in the relativistic setting for the arbitrariness of \( \mathfrak{b}_a \) could be maintained, while the rate fields could not be selected at will.
Such a choice is motivated by the presumption that the local deformation spin tends to rotate the lattice structures influencing the way the atomic flips may develop. At continuum scale such circumstance should generate a coupled gyroscopic effect, as represented above.

2.5. Constitutive structures. Constitutive restriction on the dependence of the stresses and the phason self-action on the state variables characterizing a material class are prescribed by the need of not violating the second-law of thermodynamics. The statement remains vague till we specify an expression of the second law. In large strain regime, in which we distinguish between reference and actual shapes, it is natural to write such an expression in referential form. For it we write in isothermal setting

\[
\frac{d}{dt} \int_\mathcal{B} \psi \, d\mu(x) - \mathcal{P}_b^{\text{ext}}(\dot{y}, \dot{\nu}) \leq 0,
\]

adapting to the description of quasicrystals the traditional viewpoint in continuum mechanics on the constitutive matter (see [3] for it). In the previous inequality \( \psi \) is the free energy density and \( \mathcal{P}_b^{\text{ext}}(\dot{y}, \dot{\nu}) \) the referential description of the external power, obtained by changing variables in the integrals. It reads

\[
\mathcal{P}_b^{\text{ext}}(\dot{y}, \dot{\nu}) := \int_\mathcal{B} (b^\dagger \cdot \dot{y} + \beta^\dagger \cdot \dot{\nu}) \, d\mu(x) + \int_{\partial \mathcal{B}} (t \cdot \dot{y} + \tau \cdot \dot{\nu}) \, dH^2,
\]

where \( b^\dagger := (\det F) b_{a\nu}^\dagger, \beta^\dagger := (\det F) \beta_{a\nu}^\dagger, t \) and \( \tau \) are considered as the values of fields defined over \( \mathcal{B} \) through \( \tilde{t}(y(x,t),t) \) and \( \tilde{\tau}(y(x,t),t) \). With this version of the external power, the invariance axiom above requires the identity

\[
\mathcal{P}_b^{\text{ext}}(\dot{y}, \dot{\nu}) = \mathcal{P}_b^{\text{ext}}(\dot{y}^*, \dot{\nu}^*)
\]

for any choice of \( c, q \), involved in the definitions of \( \dot{y}^* \) and \( \dot{\nu}^* \), and the part considered. By exploiting such identity we can prove the referential version of Theorem 2.1, which includes, in particular, the relation

\[
\mathcal{P}_b^{\text{ext}}(\dot{y}, \dot{\nu}) := \int_\mathcal{B} (P \cdot \dot{F} + z \cdot \dot{\nu} + S \cdot \dot{N}) \, d\mu(x)
\]

to be substituted into the mechanical dissipation inequality. In the previous identity, \( P \) is the standard first Piola-Kirchhoff stress tensor defined by \( P := (\det F) \sigma F^{-T} \), \( z \) the referential phason self-action \( z := (\det F) z_a \) and \( S \) the referential phason stress (or microstress if you want to accept a nomenclature more common to the general model-building framework of the mechanics of complex materials, which includes the quasicrystal modeling) \( S := (\det F) S_a F^{-T} \).

\[\begin{itemize}
   \item When we presume that the free energy \( \psi \), the stresses \( P \) and \( S \), and the self-action \( z \) depend all on \( F, N \) and \( \nu \), besides \( x \), with \( \psi \) a differentiable function of its entries, the arbitrariness of the rates in the mechanical dissipation inequality implies the constitutive restrictions
   \[
   P = \frac{\partial \psi}{\partial F}, \quad S = \frac{\partial \psi}{\partial N}, \quad z = \frac{\partial \psi}{\partial \nu},
   \]
   i.e.
   \[
   \sigma = (\det F)^{-1} \frac{\partial \psi}{\partial F} F^T, \quad S_a = (\det F)^{-1} \frac{\partial \psi}{\partial N} F^T, \quad z_a = (\det F)^{-1} \frac{\partial \psi}{\partial \nu}.
   \]
\end{itemize}\]

They characterize the elastic setting for quasicrystals.
By fixing $\nu$ and $N$, a standard argument shows that objectivity for $\psi$, i.e. invariance under the action of $SO(3)$ on the physical space, and convexity of $\psi$ with respect to $F$ are physically incompatible. Consequently, we commonly accept a polyconvex dependence of $\psi$ on $F$. With respect to $N$, the free energy can be quadratic in the so-called phason locked phase, and $\psi$ may admit a decomposed Ginzburg-Landau-type structure. In the so-called phason unlocked phase, $\psi$ depends on $|N|$. With reference to the phason locked phase, the existence of ground states (minimizers of the energy) has been found in [20] as a special case of a more general result presented there, further generalized in [7].

In small strain regime the dependence of the energy can be quadratic. With reference to the homogeneous and isotropic case, with $\varepsilon := \text{Sym} \nabla u$ the small strain tensor and $I$ the second-rank unit tensor, a rather general expression of the energy has been derived in [21]; it reads

$$\psi = \frac{1}{2} \lambda (\varepsilon \cdot I)^2 + \mu \varepsilon \cdot \varepsilon$$

from which we get

$$\sigma = \lambda (\text{tr} \varepsilon) I + 2\mu \varepsilon + k_3 (\text{tr} N) I + k_0 |\nu|^2$$

(2.14)

(2.15)

$$z_\alpha = k_0 \nu,$$

(2.16)

$$S_a = k_1 (\text{tr} N) I + 2k_2 \text{Sym} N + 2k_2' \text{Skw} N + k_3 (\text{tr} \varepsilon) I + k_3' \varepsilon.$$

(2.17)

$\lambda$ and $\mu$ are standard Lamé constants. The others are elastic constants related with the phason field. Experiments inform us about the values of $k_i$, $i = 1, 2, 3$, with some fluctuations in the literature (see, e.g., [2], [11], [24], [32]), but we do not know $k_2'$, $k_3'$ and $k_0$.

The quadratic expression of the energy written above is a special case of

$$\psi (\nabla u, \nu, \nabla \nu) = \frac{1}{2} \nabla u \cdot C \nabla u + \nabla u \cdot K' \nabla \nu + \frac{1}{2} \nabla \nu \cdot K \nabla \nu + \frac{1}{2} k_0 |\nu|^2,$$

with $C$, $K$ and $K'$ constitutive fourth-rank tensors, endowed at least with major symmetries, but it is a bit more general then the common choice

$$C_{ijhk} = \lambda \delta_{ij} \delta_{hk} + \mu (\delta_{ih} \delta_{jk} + \delta_{ik} \delta_{jh}),$$

$$K'_{ijhk} = k_1 \delta_{ih} \delta_{jk} + k_2 (\delta_{ij} \delta_{hk} - \delta_{ik} \delta_{jh}),$$

$$K_{ijhk} = k_3 (\delta_{i1} - \delta_{i2}) (\delta_{ij} \delta_{hk} - \delta_{ih} \delta_{jk} + \delta_{ik} \delta_{jh})$$

(see, e.g., [8]), where $i, j, h, k = 1, 2, 3, \delta_{ij}$ is the Kronecker symbol and the constants satisfy the inequalities

$$\mu > 0, \quad \lambda + \mu > 0, \quad k_1 > 0, \quad k_1 > |k_2|, \quad |k_3| < \sqrt{\frac{1}{2} \mu (k_1 + k_2)}, \quad k_0 \geq 0,$$

allowing nonnegative definition of the energy.
We could also imagine to have viscous effects represented through the dependence of the stresses $P$, $S$ and the self-action $z$ on $\dot{F}$, $\dot{N}$ and $\dot{\nu}$, besides $F$, $N$ and $\nu$. The mechanical dissipation inequality excludes the dependence of $\psi$ on $\dot{F}$, $\dot{N}$ and $\dot{\nu}$, provided that $F$, $N$ and $\nu$ are twice differentiable in time. Consequently, to be compatible with the second law of thermodynamics, $P$, $S$ and $z$ must admit a constitutive dependence on the rates of the state variables of the form

$$
P = \tilde{P}(F, N, \nu, \dot{F}, \dot{N}, \dot{\nu}) = \tilde{P}^e(F, N, \nu, \dot{F}, \dot{N}, \dot{\nu}) + \tilde{P}^d(F, N, \nu, \dot{F}, \dot{N}, \dot{\nu}),$$

$$
S = \tilde{S}(F, N, \nu, \dot{F}, \dot{N}, \dot{\nu}) = \tilde{S}^e(F, N, \nu) + \tilde{S}^d(F, N, \nu, \dot{F}, \dot{N}, \dot{\nu}),$$

$$
z = \tilde{z}(F, N, \nu, \dot{F}, \dot{N}, \dot{\nu}) = \tilde{z}^e(F, N, \nu) + \tilde{z}^d(F, N, \nu, \dot{F}, \dot{N}, \dot{\nu}),$$

where the superscripts $e$ and $d$ indicate the energetic and the dissipative (viscous) components. By inserting these choices in the mechanical dissipation inequality, we find the dependence of the energetic components of $P$, $S$ and $z$ form the derivatives of the free energy, as recalled above, with the consequent expressions of $\sigma^e$, $S_a^e$ and $z_a^e$, and the reduced dissipation inequality

$$
P^d \cdot \dot{F} + z^d \cdot \dot{\nu} + S^d \cdot \dot{N} \geq 0$$

valid for any choice of the velocity fields, which implies that $P^d$, $S^d$ and $z^d$ can be considered linear functions of $\dot{F}$, $\dot{N}$ and $\dot{\nu}$ as their actual counterparts $\sigma^d$, $S_a^d$ and $z_a^d$.

In this case and in small strain setting, for the energetic components of $\sigma$, $S_a$ and $z_a$ we shall consider the energy $(2.14)$ and dissipative components of the stresses and the self-action given by

$$
\sigma^d = \epsilon \nabla u_t, \quad S_a^d = \delta \nabla \nu_t, \quad z_a = \varsigma \nu_t,
$$

with $\epsilon$, $\delta$ and $\varsigma$ positive constants. Consequently, the constitutive equations $(2.15)$, $(2.16)$ and $(2.17)$ become

$$
\sigma = \lambda (\text{tr}\epsilon) I + 2\mu \epsilon + k_3 (\text{tr} N) I + k_3^0 \text{Sym} N + \epsilon \nabla u_t,
$$

(2.19)

$$
z_a = k_0 \nu + \varsigma \nu_t,
$$

(2.20)

$$
S_a = k_1 (\text{tr} N) I + 2k_2 \text{Sym} N + 2k_2^0 \text{Skw} N + k_3 (\text{tr} \epsilon) I + k_3^0 \epsilon + \delta \nabla \nu_t.
$$

We shall use the constitutive equations $(2.18)$ and $(2.20)$ just for technical purposes, due to the regularization induced by the gradients of the rate fields. We remark here only their mechanical motivation but we do not investigate further their experimental correspondence for we shall calculate the limits as $\epsilon$ and $\delta$ tend to zero. In contrast, there are estimates for $\varsigma$ (see [25]).

3. Existence results: the linear case

3.1. Dynamics with phason diffusion and absence of gyroscopic effects.

In small strain regime and under the validity of the linear constitutive structures $(2.15)$, $(2.17)$ and $(2.19)$, in absence of non-inertial body forces and gyroscopic-type phason inertia, by imposing $u$ and $\nu$ along $\partial B$ (Dirichlet boundary conditions) and
their values together with those of the velocity $u_t$ over $\mathcal{B}$ as initial conditions, the balance equations read

\begin{align}
\mu \Delta u + \xi \nabla \text{div} u + \kappa \Delta \nu + \bar{\xi} \nabla \text{div} \nu &= \rho u_{tt} \quad \text{in } (0, T) \times \mathcal{B}, \\
\zeta \Delta \nu + \gamma \nabla \text{div} \nu + \kappa \Delta u + \bar{\xi} \nabla \text{div} u - \kappa_0 \nu &= \zeta \bar{\nu}_t \quad \text{in } (0, T) \times \mathcal{B}, \\
u(t, x) &= \bar{u}(x), \quad \nu(t, x) = \bar{v}(x), \quad \text{on } (0, T) \times \partial \mathcal{B}, \\
 u|_{t=0} = u_0, \quad u_t|_{t=0} = \dot{u}_0, \quad \nu|_{t=0} = \nu_0, \quad \text{on } \mathcal{B},
\end{align}

(3.1)

where $u_0$, $\dot{u}_0$ and $\nu_0$ are the initial data, and the constitutive parameters are constants and satisfy the following relations: $\xi = \lambda + \mu$, $\bar{\xi} = k_3 + \frac{1}{2}k_3'$, $\zeta = k_2 + k_2'$, $\gamma = k_1 + k_2 - k_2'$, $\kappa = \frac{1}{2}k_3'$, and $\lambda$, $\mu$, $k_i$, $k_i'$, $i = 1, 2, 3$.

### 3.2. Preliminaries and notations

For $p \geq 1$, by $L^p(\mathcal{B})$ we indicate the usual Lebesgue space with norm $\| \cdot \|_p$. For $L^2$ we use the notation $\| \cdot \| = \| \cdot \|_2$. Moreover, by $W^{k,p}(\mathcal{B})$, $k$ a non-negative integer and $p$ as above, we denote the usual Sobolev space with norm $\| \cdot \|_{k,p}$. We write $W_0^{k,p}(\mathcal{B})$ for the closure of $C_0^\infty(\mathcal{B})$ in $W^{1,p}(\mathcal{B})$ and $W^{-1,p}(\mathcal{B})$, $p' = p/(p - 1)$, for the dual of $W^{1,p}(\mathcal{B})$ with norm $\| \cdot \|_{-1,p'}$. Let $X$ be a real Banach space with norm $\| \cdot \|_X$. We shall use the customary spaces $W^{k,p}(0, T; X)$, with norm denoted by $\| \cdot \|_{W^{k,p}(0, T; X)}$, recalling that $W^{0,p}(0, T; X) = L^p(0, T; X)$ are the standard Bochner spaces. The symbol $(\cdot, \cdot)$ indicates as usual the duality pairing. Here and in the sequel, we denote by $c$ or $\bar{c}$ positive constants that may assume different values, even in the same equation. We also define

$$\mathcal{H}^1 := \{ v \in W^{1,2}(\mathcal{B}) : v|_{\partial \mathcal{B}} = 0 \},$$

with dual space $\mathcal{H}^{-1}$. We denote by $\mathcal{B}_T$ the set product $(0, T) \times \mathcal{B}$ and, similarly, with $\partial \mathcal{B}_T$ we indicate $(0, T) \times \partial \mathcal{B}$.

### 3.3. Existence and uniqueness of weak solutions to (3.1)

**Definition 3.1 (Weak solution).** We affirm that a pair $(u, \nu)$ is a “weak solution” to the system (3.1) if, for a given $T > 0$, the following conditions hold true:

**Regularity:**

\begin{align}
&u \in L^\infty(0, T; \mathcal{H}^1) \cap C([0, T]; L^2(\mathcal{B})) \cap C_{weak}([0, T]; \mathcal{H}^1), \\
&\nu \in L^2(0, T; \mathcal{H}^1) \cap C([0, T]; L^2(\mathcal{B})), \\
&u_t \in L^\infty(0, T; L^2(\mathcal{B})) \cap C_{weak}([0, T]; L^2(\mathcal{B})), \quad u_{tt} \in L^2(0, T; \mathcal{H}^{-1}), \\
&\nu_t \in L^2(0, T; L^2(\mathcal{B})).
\end{align}

(3.2)
Weak formulation: For all \((w, h) \in C_0^\infty(0, T; \mathcal{H}^1) \times C_0^\infty(0, T; \mathcal{H}^1)\),
\begin{equation}
\rho \int_0^T \int_B u_t \cdot w + \mu \int_0^T \int_B \nabla u \cdot \nabla w + \kappa \int_0^T \int_B \nabla \nu \cdot \nabla w
- \int_0^T \int_{\partial B} w \cdot \left( \frac{\partial u}{\partial n} + \kappa \frac{\partial \nu}{\partial n} \right) + \xi \int_0^T \int_B \nabla (\nabla u) \cdot w + \bar{\xi} \int_0^T \int_B \nabla (\nabla \nu) \cdot w
\end{equation}
where, in order to keep the notation concise, we have erased all volume, surface
and time measures from the space-time integrals above, a choice that we adopt for
the remainder of the paper.

To prove our existence result, we use the Galerkin method to approximate a regular
weak solution to (3.1) with finite dimensional displacement and phason vector fields.

Let us consider the set \(\{\omega_k\}_{k \in \mathbb{N}}\) of eigenfunctions, with corresponding eigenvalues
\(\{\lambda_k\}\), of the problem
\[- \mu \Delta u = \lambda u \quad \text{in } B, \]
\[u = \bar{u} \quad \text{on } \partial B,\]
we define \(X_m := \text{span}\{\omega_1, \ldots, \omega_m\}\) and indicate by \(P_m\) the orthogonal projection
operator from \(\mathcal{H}^1\) over \(X_m\). Similarly, we also introduce the set \(\{\vartheta_r\}_{r \in \mathbb{N}}\) of the
eigenfunctions, with corresponding eigenvalues \(\{\varpi_r\}\), of
\[- \kappa \Delta \nu + \kappa_0 \nu = \varpi \nu \quad \text{in } B, \]
\[\nu = \bar{\nu} \quad \text{on } \partial B.\]

We define \(Y_n := \text{span}\{\vartheta_1, \ldots, \vartheta_n\}\) and indicate by \(\Pi_n\) the orthogonal projection
from \(\mathcal{H}^1\) over \(Y_n\). We are looking to approximate functions
\begin{equation}
u^m(t, x) = \sum_{j=1}^m \epsilon_j^m(t) \vartheta_j(x)\]
which are solutions of the system of ODEs below, for all \((\omega_k, \vartheta_r) \in X_m \times Y_m,\]
\(1 \leq k \leq m, 1 \leq r \leq m,\) and \(t \in [0, T]\):
\begin{align*}
\rho & \int_B u_t^m \cdot \omega_k + \mu \int_B \nabla u^m \cdot \nabla \omega_k + \kappa \int_B \nabla \nu^m \cdot \nabla \omega_k \\
& = \int_B \left( \mu \frac{\partial u^m}{\partial n} + \kappa \frac{\partial \nu^m}{\partial n} \right) \cdot \omega_k + \xi \int_B \nabla (\nabla u^m) \cdot \omega_k + \bar{\xi} \int_B \nabla (\nabla \nu^m) \cdot \omega_k, \\
\int_B (\kappa \nu^m) \cdot \vartheta_r + \xi \int_B \nabla \nu^m \cdot \nabla \vartheta_r + \kappa \int_B \nabla u^m \cdot \nabla \vartheta_r \\
& = \int_{\partial B} \vartheta_r \left( \kappa \frac{\partial u^m}{\partial n} + \kappa \frac{\partial \nu^m}{\partial n} \right) + \gamma \int_B \nabla (\nabla \nu^m) \cdot \vartheta_r + \bar{\gamma} \int_B \nabla (\nabla u^m) \cdot \vartheta_r.
\end{align*}
As a consequence, we have the following inclusions: \( u^m \in L^2(0,T;X_m) \), \( \nu^m \in L^2(0,T;Y_m) \), \( u^m_t \in L^2(0,T;X_m) \) and \( \nu^m_t \in L^2(0,T;Y_m) \). The Sobolev embedding theorem for functions (of a single variable \( t \)) implies \( u_m \in C([0,T];X_m) \) and \( \nu_m \in C([0,T];Y_m) \), so the initial conditions \( u_m(0) = P_m u_0 \) and \( \nu_m(0) = \Pi_m \nu_0 \) make sense.

The Galerkin approximation procedure, combined with a compactness argument (the Aubin-Lions lemma) and suitable a priori estimates, implies a first result.

**Theorem 3.1.** Assume \( \mu > -\lambda, \kappa > 0, \bar{\xi} > 0, \mu, \zeta > 2\kappa, \) and \( \xi, \gamma > 2\bar{\xi} \). Assume also \( u_0, \nu_0 \in W^{1,2}(B) \) so that \( \nabla u(0,x) = \nabla u_0 \) and \( \nabla \nu(0,x) = \nabla \nu_0 \) on \( B \) and \( \bar{u}, \bar{\nu} \in L^2(\partial B) \). Then, a unique regular weak solution to the problem \( (3.1) \) exists.

**Proof.** We proceed formally (since we lack the needed regularity to test directly against \( (u,\nu) \) or \( (u_t,\nu_t) \)), but the procedure actually goes through the use of the Galerkin approximation functions \( (u^m, \nu^m) \). Thus, to keep the notation compact we shall use \( (u,\nu) \) in place of \( (u^m, \nu^m) \), reminding however to adopt the sequence of the Galerkin approximations when we extract a suitable convergent subsequence.

By multiplying first and second equations in \( (3.1) \) respectively by \( u_t \) and \( \nu_t \) in \( L^2(B) \), by means of standard calculations we infer that

\[
\frac{\rho}{2} \frac{d}{dt} \|u_t\|^2 + \frac{\mu}{2} \frac{d}{dt} \|\nabla u\|^2 = \mu \int_{\partial B} u_t \cdot \frac{\partial u}{\partial n} + \kappa \int_B \nabla (\nabla u) \cdot u_t + \xi \int_B \nabla (\nabla \nu) \cdot u_t,
\]

\[
\xi \|\nu_t\|^2 + \frac{\kappa_0}{2} \frac{d}{dt} \|\nu_t\|^2 + \frac{\zeta}{2} \frac{d}{dt} \|\nabla \nu\|^2 = \zeta \int_{\partial B} \nu_t \cdot \frac{\partial \nu}{\partial n} + \gamma \int_B \nabla (\nabla \nu) \cdot \nu_t + \kappa \int_B \nabla u \cdot \nu_t + \xi \int_B \nabla (\nabla u) \cdot \nu_t
\]

Hence, by adding them and integrating by parts, we obtain

\[
(3.7) \quad \frac{\rho}{2} \frac{d}{dt} \|u_t\|^2 + \zeta \|\nu_t\|^2 + \frac{\kappa_0}{2} \frac{d}{dt} \|\nu\|^2
\]

\[
+ \frac{1}{2} \frac{d}{dt} (\mu \|\nabla u\|^2 + \zeta \|\nabla \nu\|^2) + \frac{1}{2} \frac{d}{dt} (\xi \|\nabla u\|^2 + \gamma \|\nabla \nu\|^2)
\]

\[
= \mu \int_{\partial B} u_t \cdot \frac{\partial u}{\partial n} + \zeta \int_{\partial B} \nu_t \cdot \frac{\partial \nu}{\partial n} - \kappa \frac{d}{dt} \int_B \nabla u \cdot \nabla \nu - \xi \frac{d}{dt} \int_B \nabla u (\nabla \nu)
\]

\[
+ \zeta \int_{\partial B} (\nabla u) u_t \cdot n + \gamma \int_{\partial B} (\nabla \nu) \nu_t \cdot n + \xi \int_{\partial B} (\nabla \nu) u_t \cdot n + \gamma \int_{\partial B} (\nabla \nu) \nu_t \cdot n.
\]

Due to the Dirichlet boundary conditions, we find \( u_t(t,x) = 0 \) on \( \partial B \) as well as \( u_t(t,x) = 0 \) on \( \partial B \). Consequently, from the equation \( (3.7) \) we get

\[
\frac{\rho}{2} \frac{d}{dt} \|u_t\|^2 + \zeta \|\nu_t\|^2 + \frac{\kappa_0}{2} \frac{d}{dt} \|\nu\|^2
\]

\[
+ \frac{1}{2} \frac{d}{dt} (\mu \|\nabla u\|^2 + \zeta \|\nabla \nu\|^2) + \frac{1}{2} \frac{d}{dt} (\xi \|\nabla u\|^2 + \gamma \|\nabla \nu\|^2)
\]

\[
= -\kappa \frac{d}{dt} \int_B \nabla u \cdot \nabla \nu - \xi \frac{d}{dt} \int_B \nabla u (\nabla \nu).
\]
and, by integrating in \((0, t), t \leq T\) and exploiting Hölder’s inequality, we compute
\[
\rho \|\dot{u}_t\|^2 + 2\kappa \int_0^t \|\nu_t\|^2 + \kappa_0 \|\nu\|^2 \\
+ (\mu \|\nabla u\|^2 + \zeta \|\nabla \nu\|^2) + (\xi \|\div u\|^2 + \gamma \|\div \nu\|^2)
\leq 2\kappa \|\nabla u\| \|\nabla \nu\| + 2\xi \|\div u\| \|\div \nu\| + \bar{c},
\]
where \(\bar{c} = \bar{c}(\|u_0\|_{1,2}, \|\dot{u}_0\|, \|\nu_0\|_{1,2}, \rho, \kappa, \zeta, \xi, \bar{c}, \gamma)\). Then, by using Young’s inequality and rearranging the terms in the expression above, we obtain
\[
(3.8) \quad \rho \|\dot{u}_t\|^2 + 2\kappa \int_0^t \|\nu_t\|^2 + \kappa_0 \|\nu\|^2 + \frac{1}{2} (\mu \|\nabla u\|^2 + \zeta \|\nabla \nu\|^2) \leq \bar{c},
\]
which implies the inclusions \(u_t, \nabla u \in L^\infty(0, T; L^2(B))\), \(\nu_t \in L^2(0, T; L^2(B))\) and \(\nu, \nabla \nu \in L^\infty(0, T; L^2(B))\).

By the first equation in (3.1), we also have
\[
\rho \left| \langle u_t(\tau) - u_t(s), \phi \rangle \right| \leq \mu \int_s^\tau |(\Delta u, \phi) + \xi \int_s^\tau |\nabla \div u, \phi| \\
+ \kappa \int_s^\tau |(\nabla^2 \phi)| + \bar{\xi} \int_s^\tau |(\nabla \div \phi)|
\]
for all \(\phi \in H^1\) and \(0 \leq s \leq \tau \leq T\). By the boundedness of \(\nabla u\) and \(\nabla \nu\), which belong to \(L^\infty(0, T; L^2(B))\), we realize that \(u_t(\tau) - u_t(s)\) is bounded in \(L^2(0, T; \mathcal{H}^{-1})\). Recalling that \((u, \nu)\) is actually the sequence \((u^m, \nu^m)\) (and that \((\hat{u}_t, \hat{\nu}_t)\) indicates \((u^m_t, \nu^m_t)\)), by using classical compactness arguments, we can extract a sub-sequence (still labeled by \((u^m, \nu^m)\)) such that
\[
\begin{align*}
\rho \quad u^m & \to \dot{u} \text{ in } L^2(0, T; L^2(B)) \text{–strong}, \\
& \quad L^\infty(0, T; W^{1,2}(B)) \text{–weak}^*, \\
& \quad L^2(0, T; W^{1,2}(B)) \text{–weak}, \\
\end{align*}
\]
\[
\begin{align*}
\rho \quad u^m_t & \to \dot{u}_t \text{ in } L^\infty(0, T; L^2(B)) \text{–weak}^*, \\
& \quad L^2(0, T; L^2(B)) \text{–weak}, \\
\end{align*}
\]
\[
\begin{align*}
\rho \quad u^m_{tt} & \to \dot{u}_{tt} \text{ in } L^2(0, T; \mathcal{H}^{-1}) \text{–weak}, \\
\end{align*}
\]
and
\[
\begin{align*}
\rho \quad \nu^m & \to \dot{\nu} \text{ in } L^\infty(0, T; L^2(B)) \text{–weak}^*, \\
& \quad L^2(0, T; W^{1,2}(B)) \text{–weak}, \\
\end{align*}
\]
\[
\begin{align*}
\rho \quad \nu^m_t & \to \dot{\nu}_t \text{ in } L^2(0, T; L^2(B)) \text{–weak}.
\end{align*}
\]

By exploiting these convergences, we can easily pass to the limit for the sequence \((u^m, \nu^m)\) in the weak formulation (3.4)–(3.5), proving that \((\dot{u}, \dot{\nu})\) is a regular weak solution to the problem (3.1). The continuity property of such a solution follows from the standard embedding of \(W^{1,2}(0, T; L^2(B)) \subset C^\beta([0, T]; L^2(B))\) of \(\beta\)-Hölder continuous functions on \([0, T]\) with values in \(L^2(B)\), for every \(\beta \in (0, 1)\) (see, e.g., [40]). Again, the circumstance that \(\hat{u}\) and \(\hat{u}_t\) are weakly continuous with values in \(H^1\), and \(L^2(B)\) respectively, is a direct consequence of the obtained regularity, i.e. \(\hat{u} \in L^\infty(0, T; H^1)\), \(u_t \in L^2(0, T; L^2(B))\), \(u_{tt} \in L^2(0, T; H^{-1})\), and the Sobolev embedding theorem.

Uniqueness emerges from direct computations: Let \((u_1, \nu_1)\) and \((u_2, \nu_2)\) be two solutions of (3.1). We take differences \(U := u_1 - u_2\) and \(V := \nu_1 - \nu_2\) and consider
the related equations. We use $U_t$ and $V_t$ as test functions for the equations satisfied by $U$ and $V$, respectively. Thus, by taking the $L^2$-inner products and integrating in time on $(0,T)$, as in the procedure above, we obtain

$$
\rho \|U_t\|^2 + 2\epsilon \int_0^t \|V_t\|^2 + \kappa_0 \|V\|^2
+ (\mu \|\nabla U\|^2 + \zeta \|\nabla V\|^2) + (\xi \|\text{div } U\|^2 + \gamma \|\text{div } V\|^2)
\leq 2\kappa \|\nabla U\| \|\nabla V\| + 2\tilde{\epsilon} \|\text{div } U\| \|\text{div } V\|, \quad t \in (0,T),
$$

from which the conclusion follows by applying Young’s inequality on the right-hand side terms and reabsorbing the emerging integrals. \qed

**Remark 3.1.** Let $(u, \nu)$ be a weak solution to (3.1) constructed in Theorem 3.1. By using standard arguments it is possible to show that, actually, it is such that $u \in C(0,T; \mathcal{H}^1)$ and $u_t \in C(0,T; L^2(B))$. Indeed, this may be proved by taking a regularization (convolution in time) $(u^\epsilon, \nu^\delta)$ of $(u, \nu)$, given by

$$
u^\delta = \eta_\delta * \nu \in C^\infty_0(0,T; \mathcal{H}^1), \quad \nu^\delta = \eta_\delta * \nu \in C^\infty_0(0,T; \mathcal{H}^1),$$

where the smooth function $\eta_\epsilon$ is even, positive, supported in $(-\epsilon, \epsilon)$ and so is $\eta_\delta$ in $(-\delta, \delta)$, with $\int_{-\epsilon}^\infty \eta_\epsilon(s) ds = 1$ (similarly $\int_{-\delta}^\infty \eta_\delta(s) ds = 1$), and using subsequently the properties of the considered system of equations along with the convergence $u^\epsilon \to u$ ($\nu^\delta \to \nu$) in $L^2(0,T; \mathcal{H}^1)$ as $\epsilon \to 0$ (as $\delta \to 0$, respectively).

Alternatively, as we do in analyzing the non-linear system (4.1) below, we can use a parabolic regularization of the equations in (3.1) by consider viscous components of the standard and phason stresses determining the terms $-\epsilon \Delta u_t$ and $-\delta \Delta \nu_t$, which appear respectively to the right-hand side of the first and second equation in (3.1). In this case, to prove the strong continuity of the weak solution $(u, \nu)$, we can exploit the convergence of the regularized solution $(u^\epsilon, \nu^\delta)$ to $(u, \nu)$, as $(\epsilon, \delta) \to (0,0)$, together with the intrinsic properties of the equations (3.1) (see the next section for additional details).

4. Existence results: dynamics with phason diffusion and non-linear gyroscopic phason inertia

In presence of gyroscopic-type phason inertia, the system (3.1) becomes

$$
\begin{align*}
\mu \Delta u + \xi \nabla \text{div } u + \kappa \Delta \nu + \xi \nabla \text{div } \nu &= \rho u_{tt} \quad &\text{in } B_T, \\
\zeta \Delta \nu + \gamma \nabla \text{div } \nu + \kappa \Delta u + \xi \nabla \text{div } u - \kappa_0 \nu &= \zeta \nu_t + \ell (\text{curl } u_t) \times \nu_t \quad &\text{in } B_T, \\
u(t,x) &= \bar{u}(x), \quad (x), \\
u(t,x) &= \bar{v}(x), &\text{on } \partial B_T, \\
|t=0 &= u_0, \quad u_t|_{t=0} = \bar{u}_0, \quad \nu|_{t=0} = \nu_0, &\text{on } B,
\end{align*}
$$

(4.1)

$\ell$ is a positive constant and $\ell (\text{curl } u_t) : (\text{curl } u_t) := \ell e_{ij} e_{rkh} u_{rk} \nu_{ij}$, leaving understood the sum over repeated indexes, as usual. $e_{rkh}$ is the $rhk$-th component of the Ricci alternating symbol $e$, recalled in Section 2.

**Definition 4.1** (Weak solution). We say that a pair $(u, \nu)$ is a “weak solution” to the system (4.1) if, for a given $T > 0$, the following conditions hold true:
Weak formulation: For all \( \epsilon > 0 \) and \( \delta > 0 \):

\[
\begin{align*}
\int_B (\zeta \nabla \nu_t + \kappa \nabla u_t) \cdot h + (\zeta \nabla \nu + \kappa \nabla u) \cdot \nabla h & \Rightarrow 0, \\
\int_B (\zeta \nabla \nu_t + \kappa \nabla u_t) \cdot h + \ell \nabla \nu_t \cdot h & \Rightarrow 0.
\end{align*}
\]

To determine existence of a weak solution to (4.1), we analyze first its regularized counterpart, obtained by introducing dissipative components of the stresses, fixing the parameters \( \epsilon > 0 \) and \( \delta > 0 \):

\[
\begin{align*}
\mu \Delta u + \xi \nabla \nabla u + \kappa \nabla u + \xi \nabla \nabla \nu & = -\epsilon \Delta u_t + \rho u_t, \\
\zeta \Delta \nu + \gamma \nabla \nabla \nu + \kappa \nabla u + \xi \nabla \nabla \nu & = -\delta \Delta \nu_t + \varsigma \nu_t + \ell (\text{curl } u_t) \times \nu_t.
\end{align*}
\]

**Theorem 4.1.** Consider problem (4.1). Assume \( \mu > -\lambda, \kappa > 0, \xi > 0, \mu, \zeta > 2\kappa, \) and \( \xi, \gamma > 2\xi \). Assume also \( u_0, \nu_0 \in W^{1,2}(B) \), \( \dot{u}_0 \in W^{1,2}(B) \), such that \( \ell \| \dot{u}_0 \| _{1,2} < \xi/2 \) and that \( \tilde{u}, \tilde{\nu} \in \mathcal{L}(\partial B) \). Then, the system (4.1) admits a weak solution.

**Proof.** First we consider the regularized model (4.1). In order to prove the existence of pertinent weak solutions \( (u^\epsilon, \nu^\delta) \), we follow the same path leading to Theorem 3.1. We apply the Galerkin method by using the approximating functions \( (u^\epsilon, \nu^\delta) \); thus we proceed by testing the equations in (4.6) by \( u^\epsilon_t, \nu^\delta_t \) and \( \nu^\delta_t \), respectively. Due to the identity

\[
\int_B (\text{curl } u^\epsilon_t) \times \nu^\delta_t \cdot \nu^\delta_t = 0,
\]

a priori estimates for the equations (4.6) are nearly the same made for the system (3.1).
Thus, in the case of the system (4.6), the following estimate holds true, provided that $\|\nabla u^\varepsilon_{t,m}(0)\|$ and $\|\nabla \nu^\varepsilon_{t,m}(0)\|$ are bounded:

$$\rho \|u^\varepsilon_{t,m}\|^2 + 2\zeta \int_0^t \|\nu^\varepsilon_{t,m}\|^2 \leq \kappa_0 \|\nabla \nu^\varepsilon_{t,m}\|^2 + 1 \int_0^t (\epsilon \|\nabla u^\varepsilon_{t,m}\|^2 + \delta \|\nabla \nu^\varepsilon_{t,m}\|^2)$$

(4.7)

Consequently, we get an improved regularity: $\nabla u_t \in L^2(0,T;L^2(B))$ and $\nabla \nu_t \in L^2(0,T;L^2(B))$. Here, we have $\nabla u^\varepsilon_{t,m}(0) = P_m \nabla u_0$ and the constant in the inequality is $\bar{c} = c(\|u_0\|_{L^2}, \|\nabla u_0\|_{L^2}, \|\nabla \nu_0\|_{L^2}, \|\nabla \nu^\varepsilon_{t,m}(0)\|, \epsilon, \delta, \rho, \kappa_0, \mu, \zeta, \xi, \bar{\xi}, \gamma)$. In order to guarantee the validity of the estimate (4.7) we have to ensure uniform a priori estimates for the initial datum $(\nabla \nu^\varepsilon_{t,m}(0))(0)$.

To bound $\|\nabla (\nu^\varepsilon_{t,m}(0))\|$, let $\varphi \in H^1$ with $\|\varphi\|_{L^2} \leq 1$. From the second equation in (4.6) we get

$$\zeta |\nabla \nu^\varepsilon_{t,m}(0), \varphi| + \delta |\nabla \nu^\varepsilon_{t,m}(0), \varphi| = \zeta |\nabla \nu^\varepsilon_{t,m}(0), \varphi| + \delta |\nabla \nu^\varepsilon_{t,m}(0), \varphi|$$

$$= \zeta |\nabla \nu^\varepsilon_{t,m}(0), \varphi| + \delta |\nabla \nu^\varepsilon_{t,m}(0), \varphi|$$

$$\leq \left(\epsilon |\nabla u_0| + \delta |\nabla \nu_0| + \gamma \nabla \nu^\varepsilon_{t,m}(0), \varphi\right)$$

$$+ \kappa_0 |\nabla \nu^\varepsilon_{t,m}(0), \varphi| + \ell |\nabla \nu^\varepsilon_{t,m}(0), \varphi|$$

$$\leq \left(\epsilon |\nabla u_0| + \delta |\nabla \nu_0| + \gamma \nabla \nu^\varepsilon_{t,m}(0), \varphi\right)$$

$$+ \kappa_0 |\nabla \nu^\varepsilon_{t,m}(0), \varphi| + \ell |\nabla \nu^\varepsilon_{t,m}(0), \varphi|$$

$$\leq c(\|u_0\|_{L^2}, \|\nu_0\|_{L^2}, \ell |\nabla \nu^\varepsilon_{t,m}(0), \varphi|)$$

Since $\ell |\nabla \nu_0| < \zeta/2$, we obtain

$$\min \left\{ \frac{\zeta}{2}, \delta \right\} \|\nabla \nu^\varepsilon_{t,m}(0)\|_{L^2} \leq c(\|u_0\|_{L^2}, \|\nu_0\|_{L^2}),$$

that is $|\nabla \nu^\varepsilon_{t,m}(0)|_{L^2} \leq c(\|u_0\|_{L^2}, \|\nu_0\|_{L^2})$. As a consequence, the bound (4.7) depends just on $\|u_0\|_{L^2}, \|\nu_0\|_{L^2}, \epsilon, \delta, \rho, \kappa_0, \mu, \zeta, \xi, \bar{\xi}, \gamma$.

Consider the first equation in (4.6). From it we get

$$\rho \|u^\varepsilon_{t,m}(\tau) - u^\varepsilon_{t,m}(s), \phi\| \leq \epsilon \int_0^{\tau} \|\nabla u^\varepsilon_{t,m}, \nabla \phi\|$$

$$+ \mu \int_0^{\tau} \|\nabla u^\varepsilon_{t,m}, \nabla \phi\| + \kappa \int_0^{\tau} \|\nabla \nu^\varepsilon_{t,m}, \nabla \phi\|$$

(4.8)

for all $\phi \in H^1$ and $0 \leq s \leq \tau \leq T$. By the boundedness of $\nabla u$ and $\nabla \nu$, which belong to $L^\infty(0,T;L^2(B))$, we find that $u^\varepsilon_{t,m}(\tau) - u^\varepsilon_{t,m}(s)$ is bounded in $L^2(0,T;H^{-1})$. By exploiting the Aubin-Lions compactness argument, we obtain the same kind of convergences in the proof of Theorem 3.1 and, in addition, we realize that

$$u^\varepsilon_{t,m} \to u^\varepsilon_{t} \quad \text{in} \quad L^2(0,T;W^{1,2}(B)) - \text{weak},$$

(4.9)

$$L^2([0,T] \times B) - \text{strong},$$

and

$$\nu^\varepsilon_{t,m} \to \nu^\varepsilon_{t} \quad \text{in} \quad L^2(0,T;W^{1,2}(B)) - \text{weak},$$

(4.10)

$$L^2([0,T] \times B) - \text{weak}.$$
As a consequence of the obtained regularity of $u_t^\epsilon$ and the interpolation theorem (see, e.g., [30]), we find $u_t^\epsilon \in C([0,T];L^2(\mathcal{B}))$. Moreover, due to the inclusion $W^{1,2}(0,T;H^1) \subset C^\beta([0,T];H^1)$, $\beta \in (0,1)$, we have in particular that $u^\epsilon, \nu^\delta \in C([0,T];H^1)$.

To pass to the limit in the weak formulation, the only relevant point to be proved is the following: For every $\phi \in C^\infty_c([0,T[ \times \mathcal{B})$, and $0 \leq t \leq T$, the limit
\begin{equation}
\int_0^t \int_{\mathcal{B}} \left| [\text{curl}(u_t^{\epsilon,m}) \times \nu_t^{\delta,m} - (\text{curl} u_t^\epsilon) \times \nu_t^\delta] \cdot \phi \right| \to 0, \quad m \to +\infty
\end{equation}
exists. In fact, we get
\begin{align}
&\int_0^t \int_{\mathcal{B}} \left| [\text{curl}(u_t^{\epsilon,m}) \times \nu_t^{\delta,m} - (\text{curl} u_t^\epsilon) \times \nu_t^\delta] \cdot \phi \right| \\
&= \int_0^t \int_{\mathcal{B}} \nu_t^{\delta,m} \cdot \text{curl}(u_t^{\epsilon,m} - u_t^\epsilon) + \int_{\mathcal{B}} (\nu_t^{\delta,m} - \nu_t^\delta) \times \phi \cdot (\text{curl} u_t^\epsilon) \\
&= \int_0^t \int_{\mathcal{B}} \text{curl}(\nu_t^{\delta,m} \times \phi) \cdot (u_t^{\epsilon,m} - u_t^\epsilon) + \int_0^t \int_{\mathcal{B}} \phi \times (\text{curl} u_t^\epsilon) \cdot (\nu_t^{\delta,m} - \nu_t^\delta) \\
&=: I_1^m + I_2^m.
\end{align}

Let us consider $I_1^m$. We have
\begin{align}
\int_0^t \|\text{curl}(\nu_t^{\delta,m} \times \phi)\|^2 &= \int_0^t \|\nu_t^{\delta,m} \cdot \nabla \phi - (\phi \cdot \nabla)\nu_t^{\delta,m} + (\text{div} \nu_t^{\delta,m})\phi - (\text{div} \phi)\nu_t^{\delta,m}\|^2 \\
&\leq c\|\nu_t^{\delta,m}\|^2_{L^2(0,T;H^1)} \|\phi\|^2_{L^\infty(0,T;W^{1,\infty}(\mathcal{B}))}.
\end{align}

Since $\nu_t^{\delta,m}$ and $\nu_t^\delta$ are uniformly bounded (with respect to $m$ and $\delta$) in $L^2([0,T[ \times \mathcal{B})$, in view of the inequality above, it follows that $\text{curl}(\nu_t^{\delta,m} \times \phi)$ is uniformly bounded in $L^2([0,T[ \times \mathcal{B})$.

Hence, the inequalities
\begin{align}
\left| \int_0^t \int_{\mathcal{B}} \text{curl}(\nu_t^{\delta,m} \times \phi) \cdot (u_t^{\epsilon,m} - u_t^\epsilon) \right| &\leq \int_0^t \|\text{curl}(\nu_t^{\delta,m} \times \phi)\| \|u_t^{\epsilon,m} - u_t^\epsilon\| \\
&\leq c\|u_t^{\epsilon,m} - u_t^\epsilon\|^2_{L^2(0,T;L^2)}
\end{align}
hold and, by the strong convergence of $u_t^{\epsilon,m}$ to $u_t^\epsilon$, we compute $I_1^m \to 0$ as $m \to +\infty$.

As regards the integral $I_2^m$, for every $\phi \in L^\infty(0,T;H^1)$ we find
\begin{align}
\int_0^t \|\phi \times (\text{curl} u_t^\epsilon)\|^2 &\leq c\|u_t^\epsilon\|^2_{L^2(0,T;H^1)} \|\phi\|^2_{L^\infty(0,T;L^\infty(\mathcal{B}))},
\end{align}
and hence $\phi \times (\text{curl} u_t^\epsilon) \in L^2([0,T[ \times \mathcal{B})$. Recalling that
\begin{align}
I_2^m &= \int_0^t \int_{\mathcal{B}} \phi \times (\text{curl} u_t^\epsilon) \cdot (\nu_t^{\delta,m} - \nu_t^\delta)
\end{align}
and $\nu_t^{\delta,m}$ converges to $\nu_t^\delta$ weakly in $L^2([0,T[ \times \mathcal{B})$, it follows that $I_2^m \to 0$ as $m \to +\infty$. Hence, we can pass to the limit in (4.11), obtaining the conclusion.

By the same arguments used above (essentially, by exploiting again the inequalities (4.7) and (4.8) for the weak solution $(u^\epsilon, \nu^\delta)$), we can deduce that $(u^\epsilon, \nu^\delta)$ is uniformly bounded in $W^{1,2}(0,T;H^1)$ and that $u_t^\epsilon$ is bounded in $L^2(0,T;H^{-1})$. Hence,
we can pass to the limit as \((\epsilon, \delta) \to (0, 0)\) and we have
\[
u_t^\delta \to \nu_t \quad \text{in} \quad L^2([0, T] \times B) - \text{weak},
\]
and
\[
u_t^\delta \to \nu_t \quad \text{in} \quad L^2([0, T] \times B) - \text{weak}.
\]
These convergences types are enough to pass to the limit as \((\epsilon, \delta) \to (0, 0)\) in the weak formulation for \((u^\epsilon, \nu^\delta)\), and hence the pair \((u, \nu)\) is a weak solution of (4.6) for every \(T \geq 0\).

Remark 4.1. The lack of uniqueness for the weak solutions to (4.6) is mainly related to the need of having \(\nu_t\) uniformly bounded in the \(L^\infty(B)\)-norm in \((0, T)\). Such bound seems to be essential in estimating the difference of two possible solutions to (4.6). Such requirement is a bit more than what is implied by the boundedness of the bulk actions required for proving the non-standard action-reaction principle satisfied by the phason tractions and the existence of the phason stress. The uniform boundedness of \(\nu_t\) could be in principle reached with initial data more regular than those we have presumed here.

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