GEODESIC AND BILLIARD FLOWS ON QUADRICS IN PSEUDO–EUCLIDEAN SPACES: L–A PAIRS AND CHASLES THEOREM

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Abstract
In this article we construct L–A representations of geodesic flows on quadrics and of billiard problems within ellipsoids in the pseudo–Euclidean spaces. A geometric interpretation of the integrability analogous to the classical Chasles theorem for symmetric ellipsoids is given. We also consider a generalization of the billiard within arbitrary quadric allowing virtual billiard reflections.

1 Introduction

A pseudo–Euclidean space $E^{k,l}$ of signature $(k,l)$, $k, l \in \mathbb{N}$, $k + l = n$, is the space $\mathbb{R}^n$ endowed with the scalar product

$$\langle x, y \rangle = \sum_{i=1}^{k} x_i y_i - \sum_{i=k+1}^{n} x_i y_i \quad (x, y \in \mathbb{R}^n).$$

Two vectors $x, y$ are orthogonal, if $\langle x, y \rangle = 0$. A vector $x \in E^{k,l}$ is called space–like, time–like, light – like, if $\langle x, x \rangle$ is positive, negative, or $x$ is orthogonal to itself, respectively. Denote by $(\cdot, \cdot)$ the Euclidean inner product in $\mathbb{R}^n$ and let

$$E = \text{diag}(\tau_1, \ldots, \tau_n) = \text{diag}(1, \ldots, 1, -1, \ldots, -1),$$

where $k$ diagonal elements are equal to 1 and $l$ to $-1$. Then $\langle x, y \rangle = (Ex, y)$, for all $x, y \in \mathbb{R}^n$.

Let $M$ be a smooth hypersurface in $E^{k,l}$. A normal $\nu(x)$ at $x \in M$ is a vector orthogonal to the tangent plane $T_x M$. In particular, a normal to the hyperplane $(n, x) = 0$ is $En$. We say that $x \in M$ is singular point, if $\nu(x)$ is light–like, or equivalently, if the induced metric is degenerate at $x$. 

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Let \( A = \text{diag}(a_1, \ldots, a_n), \ a_i \neq 0, \ i = 1, \ldots, n. \)

Following Khesin and Tabachnikov [14] and Dragović and Radnović [7] we consider the geodesic flow and the billiard system (in the case when \( A \) is positive definite) on a \( n - 1 \)-dimensional quadric

\[
\mathbb{E}^{n-1} = \left\{ x \in E^{k,l} \mid (A^{-1}x, x) = 1 \right\}.
\]

Notice that \( EA^{-1}x \) is a normal at \( x \in \mathbb{E}^{n-1} \). Therefore, \( x \in \mathbb{E}^{n-1} \) is singular, if \((EA^{-2}x, x) = 0\).

Lax representations for geodesic lines and billiard trajectories outside of singular points are constructed (Theorems 2.1 and 3.1). For billiards, in a general non-symmetric case, the spectral curve is a non-singular hyperelliptic curve \( S \) of genus \( n - 1 \) for a space-like or time-like trajectory, while for a light-like trajectory its genus is \( n - 2 \). The billiard mapping transforms to a translation on the Jacobian variety of \( S \) by a constant vector (Theorem 3.2).

There is a nice geometric manifestation of the integrability. Consider the following "pseudo-confocal" family of quadrics in \( E^{k,l} \)

\[
Q_\lambda : \quad ((A - \lambda E)^{-1}x, x) = \sum_{i=1}^{n} \frac{x_i^2}{a_i - \tau_i\lambda} = 1, \quad \lambda \neq \tau_i a_i, \quad i = 1, \ldots, n.
\]

For a non-symmetric ellipsoid, the lines \( l_k, \ k \in \mathbb{Z} \) determined by a generic space-like or time-like (respectively light-like) billiard trajectory are tangent to \( n - 1 \) (respectively \( n - 2 \)) fixed quadrics from the pseudo-confocal family \( (1.2) \) (pseudo-Euclidean version of the Chasles theorem, see Theorem 4.9 in [14] and Theorem 5.1 in [7]). Also, tangent lines to a generic space-like or time-like (respectively light-like) geodesic are tangent to other \( n - 2 \) (respectively \( n - 3 \)) fixed quadrics from the pseudo-confocal family \( (1.2) \). A related geometric structure of the set of singular points for the pencil \( (1.2) \) is described in [7].

Here we consider the case of symmetric quadrics, when the systems are integrable in a noncommutative sense (Theorem 4.1) and prove the Chasles theorem for symmetric ellipsoids (Theorem 5.1). By combining Theorem 5.1 and a non-commutative version of Veselov’s discrete Arnold-Liouville theorem (see [21]), we formulate \textit{Poncelet theorem for a symmetric elliptic billiard} in the pseudo-Euclidean space \( E^{k,l} \) (Theorem 5.2).

Finally, in the last section, we define a natural generalization of the billiard within arbitrary quadric allowing the so called virtual reflections. The virtual billiard flow shows the same dynamical characteristics as the usual one: the Lax representation, integrability, and the Chasles theorem (Theorems 6.1, 6.2, and 6.3).

## 2 Geodesic flows

By the use of the scalar product we can identify tangent and cotangent spaces \( y \in T_x \mathbb{R}^n \leftrightarrow p = E_y \in T^*_x \mathbb{R}^n \). The canonical symplectic form \( dp \wedge dx \) on \( T^* \mathbb{R}^n (x, p) \) transforms to the form

\[
\sum_{i=1}^{n} dp_i \wedge dx_i = \sum_{i=1}^{k} dy_i \wedge dx_i - \sum_{i=k+1}^{n} dy_i \wedge dx_i.
\]
on $T\mathbb{R}^n(x, y)$. It induces the Poisson bracket
\[
\{f, g\} = \sum_{i=1}^k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \sum_{i=k+1}^n \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \sum_{i=1}^k \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i} + \sum_{i=k+1}^n \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i}. \tag{2.2}
\]

By a geodesic on $\mathbb{E}^{n-1}$ we mean a critical smooth curve $\gamma : t \mapsto x(t) \in \mathbb{E}^{n-1}$ of the action
\[
S[\gamma] = \int_\gamma L(x, \dot{x}) dt = \int_\gamma \frac{1}{2} \langle \dot{x}, \dot{x} \rangle dt.
\]
The Euler–Lagrange equation for the Lagrangian $L$ with the constraint $x(t) \in \mathbb{E}^{n-1}$ yields
\[
E \ddot{x} = \mu A^{-1} x, \tag{2.3}
\]
where the Lagrange multiplier is $\mu = -(A^{-1} \dot{x}, \dot{x})/(E A^{-2} x, x)$, provided that $x(t)$ is not a singular point.

By introducing the variable $\dot{x} = y$, the system (2.3) takes the form
\[
\dot{x} = y, \quad \dot{y} = \mu E A^{-1} x = -\frac{(A^{-1} y, y)}{(E A^{-2} x, x)} E A^{-1} x, \tag{2.4}
\]
on the tangent bundle $T\mathbb{E}^{n-1} \setminus \Sigma$ described by the constraints
\[
F_1 = (A^{-1} x, x) - 1 = 0, \quad F_2 = (A^{-1} x, y) = 0, \tag{2.5}
\]
where
\[
\Sigma = \{(x, y) \in T\mathbb{R}^n \mid (E A^{-2} x, x) = 0\}. \tag{2.6}
\]
The system (2.4) is actually a Hamiltonian system with the Hamiltonian function
\[
H = \frac{1}{2} \langle y, y \rangle \tag{2.7}
\]
with respect to the Poisson–Dirac bracket
\[
\{f_1, f_2\}_D = \{f_1, f_2\} - \frac{\{F_1, f_1\}\{F_2, f_2\} - \{F_2, f_1\}\{F_1, f_2\}}{\{F_1, F_2\}}, \tag{2.8}
\]
where $\{\cdot, \cdot\}$ is the bracket (2.2) (cf. [16]). Note that
\[
\{F_1, F_2\} = 2(E A^{-2} x, x),
\]
and that the system (2.4) as well as the bracket (2.8), is well defined on $T\mathbb{R}^n \setminus \Sigma$. The functions $F_1$ and $F_2$ are Casimir functions of the Poisson–Dirac bracket considered on $T\mathbb{R}^n \setminus \Sigma$.

For arbitrary $\lambda \in \mathbb{R}$ let
\[
q_\lambda(x, y) = ((\lambda E - A)^{-1} x, y) = \sum_{i=1}^k \frac{x_i y_i}{\lambda - a_i} - \sum_{i=k+1}^n \frac{x_i y_i}{\lambda + a_i}. \tag{2.9}
\]
Similarly as in [12], we get
Theorem 2.1: Solutions of (2.4) on $T\mathbb{E}^{n-1} \setminus \Sigma$ satisfy the matrix equation

$$\mathcal{L}_{x,y}(\lambda) = [\mathcal{L}_{x,y}(\lambda), \mathcal{A}_{x,y}(\lambda)],$$

where the $2 \times 2$ matrices $\mathcal{L}_{x,y}(\lambda), \mathcal{A}_{x,y}(\lambda)$ are given by

$$\mathcal{L}_{x,y}(\lambda) = \begin{pmatrix} q_{\lambda}(x, y) & q_{\lambda}(y, y) \\ -1 - q_{\lambda}(x, x) & -q_{\lambda}(x, y) \end{pmatrix},$$

$$\mathcal{A}_{x,y}(\lambda) = \begin{pmatrix} 0 & \mu/\lambda \\ 1 & 0 \end{pmatrix}, \quad \mu = -(A^{-1}y, y)/(EA^{-2}x, x).$$

Corollary 2.1: The determinant $\det \mathcal{L}_{x,y}(\lambda)$ is an integral of the geodesic flow (2.4) for all $\lambda$.

We shall say that $\mathbb{E}^{n-1}$ is non-symmetric, if $\tau_i a_i \neq \tau_j a_j$ for $i \neq j$. Assuming that $\mathbb{E}^{n-1}$ is non-symmetric, the matrix representation described in Theorem 2.1 is equivalent to the system (2.4) up to the discrete group generated by the reflections

$$(x_i, y_i) \mapsto (-x_i, -y_i), \quad i = 1, \ldots, n. \quad (2.11)$$

Further, from the expression

$$\det \mathcal{L}_{x,y}(\lambda) = q_{\lambda}(y, y)(1 + q_{\lambda}(x, x)) - q_{\lambda}(x, y)^2 = \sum_{i=1}^{n} \frac{f_i(x, y)}{\lambda - \tau_i a_i}, \quad (2.12)$$

one can derive the integrals $f_i$ of the system (2.4) in the form

$$f_i(x, y) = \tau_i y_i^2 + \sum_{j \neq i} \frac{(x_i y_j - x_j y_i)^2}{\tau_j a_i - \tau_i a_j} \quad (i = 1, \ldots, n). \quad (2.13)$$

It is easy to check that they commute in the Poisson bracket (2.8), providing Liouville integrability of the geodesic flow. If $A$ is positive definite, $\mathbb{E}^{n-1}$ is an ellipsoid and the above integrals coincides with the ones given in [14].

It is also convenient to consider a polynomial L–matrix

$$\mathbb{L}_{x,y}(\lambda) = \prod_{i=1}^{n} (\lambda - \tau_i a_i) \mathcal{L}_{x,y}(\lambda).$$

The L–A pair $\hat{\mathbb{L}}_{x,y} = [\mathbb{L}_{x,y}, \mathcal{A}_{x,y}]$ belongs to a class of so called Jacobi–Mumford systems [13]. It has a spectral curve

$$S : \det (\mathbb{L}_{x,y}(\lambda) - \eta I) = 0 \iff \eta^2 + \det \mathbb{L}_{x,y}(\lambda) = 0, \quad (2.14)$$

$$\det \mathbb{L}_{x,y}(\lambda) = \prod_{i=1}^{n} (\lambda - \tau_i a_i)^2 \cdot \det \mathcal{L}_{x,y}(\lambda). \quad (2.15)$$

For a non-symmetric quadric, from (2.12), the polynomial $\det \mathbb{L}_{x,y}(\lambda)$ equals

$$\det \mathbb{L}_{x,y}(\lambda) = (\lambda - \tau_1 a_1) \cdots (\lambda - \tau_n a_n) \cdot (\lambda^{n-1} L_{n-1} + \lambda^{n-2} L_{n-2} + \cdots + L_0),$$

where the integrals $L_i$ depend on $f_i$, in particular $L_{n-1} = 2H = (y, y)$. The integrals $L_i$ are independent on $T\mathbb{R}^n$, while on $T\mathbb{E}^{n-1}$, due to $\det \mathbb{L}_{x,y}(0) = 0$, we have $L_0 \equiv 0$.

Therefore, for a space–like or time–like trajectory the degree of $\det \mathbb{L}_{x,y}(\lambda)$ is $2n - 1$, while for a light–like trajectory its degree is $2n - 2$. For a generic trajectory all zeros of $\det \mathbb{L}_{x,y}(\lambda)$ are different and $S$ is a non–singular hyperelliptic curve.
3 Billiards

Here we suppose that $A$ is positive definite and following [14], consider a billiard flow inside the ellipsoid $\{1\}$ in $E^{k,l}$. Between the impacts the motion is uniform along the straight lines. Suppose also that $x \in \mathbb{R}^{n-1}$ is non–singular. Then $\nu(x)$ is transverse to the quadric and the incoming vector $w$ can be decomposed as $w = t + n$, where $t$ is its tangential and $n$ the normal component in $x$. The billiard reflection is $w_1 = t - n$. If $x \in \mathbb{R}^{n-1}$ is singular, the flow stops.

Let $\phi : (x_j, y_j) \mapsto (x_{j+1}, y_{j+1})$ be the billiard mapping, where $x_j \in \mathbb{R}^{n-1}$ is a sequence of non–singular impact points and $y_j$ is the corresponding sequence of outgoing velocities (in the notation we follow [19] [21] [9], which slightly differs from the one given in [17], where $y_j$ is the incoming velocity).

It is evident from the definition that the Hamiltonian (2.7) is an invariant of the billiard mapping $\phi$. Therefore, the lines $l_k = \{x_k + sy_k | s \in \mathbb{R}\}$ containing segments $x_kx_k+1$ of a given billiard trajectory are of the same type: they are all either space–like ($H > 0$), time–like ($H < 0$), or light–like ($H = 0$).

As in the Euclidean case (see [19] [17] [9]), we have:

**Lemma 3.1:** (i) The billiard mapping $\phi$ is given by:

\[
x_{j+1} = x_j - 2 \frac{(A^{-1}x_j, y_j)}{(A^{-1}y_j, y_j)} y_j, \tag{3.1}
\]

\[
y_{j+1} = y_j + 2 \frac{(A^{-1}x_{j+1}, y_{j+1})}{(EA^{-2}x_{j+1}, x_{j+1})} EA^{-1}x_{j+1}. \tag{3.2}
\]

(ii) The function $J_j = (A^{-1}x_j, y_j)$ is an invariant of the billiard mapping.

**Proof.** (i) Since the normal component of $y_j$ and $y_{j+1}$ at $x_{j+1}$ is parallel to $EA^{-1}x_{j+1}$, we conclude that

\[
x_{j+1} - x_j = \mu_j y_j, \quad y_{j+1} - y_j = \nu_j EA^{-1}x_{j+1},
\]

for some $\mu_j, \nu_j \in \mathbb{R}$, $j \in \mathbb{Z}$, and the multipliers are determined from the conditions $(A^{-1}x_{j+1}, x_{j+1}) = 1$ and $(y_j, y_j) = (y_{j+1}, y_{j+1})$:

\[
\mu_j = -2 \frac{(A^{-1}x_j, y_j)}{(A^{-1}y_j, y_j)}, \quad \nu_j = 2 \frac{(A^{-1}x_{j+1}, y_{j+1})}{(EA^{-2}x_{j+1}, x_{j+1})}.
\]

(ii) From (3.2) we have

\[
(A^{-1}x_{j+1}, y_{j+1}) = (A^{-1}x_{j+1}, y_j) + 2(A^{-1}x_{j+1}, y_{j+1}),
\]

hence $(A^{-1}x_{j+1}, y_{j+1}) = -(A^{-1}x_{j+1}, y_j)$. Further, using (3.1), one obtains

\[
(A^{-1}x_{j+1}, y_{j+1}) = -(A^{-1}x_{j+1}, y_j) = -(A^{-1}x_j, y_j) + 2(A^{-1}x_j, y_j) = (A^{-1}x_j, y_j).
\]

The initial condition $(x_0, y_0)$ uniquely defines the billiard trajectory $x_k$. In the other direction, if the initial condition is given by the two successive non–singular
initial points $x_0, x_1 \in \mathbb{E}^{n-1}$ and $x_1 - x_0$ is space–like or time–like it is natural to take unit length $y_0 = (x_1 - x_0)/\sqrt{|\langle x_1 - x_0, x_1 - x_0 \rangle|}$. If $x_1 - x_0$ is light–like, we simply take $y_0 = x_1 - x_0$.

Note that in the limit, when $J_j$ tends to zero, the billiard flow transforms to the geodesic flow on $\mathbb{E}^{n-1}$. Conversely, when the smallest semi–axes of the ellipsoid $\mathbb{E}^{n-1}$ (say $a_n$) tends to zero, the geodesic flow on $\mathbb{E}^{n-1}$ transforms to the billiard flow within $(n-2)$–dimensional ellipsoid $\mathbb{E}^{n-1} \cap \{x_n = 0\}$.

Motivated by the L–A representation for the Euclidean elliptical billiard with the Hook potential given by Fedorov [9], we get:

**Theorem 3.1:** The trajectories $(x_j, y_j)$ of the billiard map (3.1), (3.2), outside the singular set (2.6), satisfy the matrix equation

$$L_{x_j+1} y_{j+1}(\lambda) = A_{x_j+1, y_j+1}(\lambda) L_{x_j, y_j}(\lambda) A_{x_j, y_j+1}(\lambda)^{-1},$$

with $2 \times 2$ matrices depending on the parameter $\lambda$

$$L_{x_j, y_j}(\lambda) = \begin{pmatrix} q_\lambda(x_j, y_j) & q_\lambda(y_j, y_j) \\ -1 - q_\lambda(x_j, x_j) & -q_\lambda(x_j, y_j) \end{pmatrix},$$

$$A_{x_j, y_j}(\lambda) = \begin{pmatrix} I_j \lambda + 2J_j \nu_j & -I_j \nu_j \\ -2J_j \lambda & I_j \lambda \end{pmatrix},$$

where $q_\lambda$ is given by (2.9), and

$$J_j = (A^{-1} x_j, y_j), \quad I_j = -(A^{-1} y_j, y_j), \quad \nu_j = 2J_j/(EA^{-2} x_{j+1}, x_{j+1}).$$

The theorem can be verified by direct calculations.

Analogous to the geodesic flow in Section 2, from Theorem 3.1 we arrive to the integrals (2.13) of the billiard flow (3.1), (3.2) associated to a non–symmetric ellipsoid (1.1).

Symplectic (for space–like and time–like trajectories) and contact properties (for light–like trajectories) of the mapping $\phi$ are studied in [14]. In particular, this is an example of a contact integrable system [15]. Recently, another integrable discrete contact system, the Heisenberg model in pseudo–Euclidean spaces, is given in [13].

By the use of Theorem 3.1 we have also an algebraic–geometrical interpretation of the integrability.

In a non–symmetric case and for generic initial conditions all zeros of (2.15) are real and different (see [7]). Thus, for a space–like or time–like trajectory, the spectral curve (2.14) is a hyperelliptic curve of genus $n - 1$, while for a light–like trajectory its genus is $n - 2$.

A generic complexified invariant manifold $L_0 = c_0, \ldots, L_{n-1} = c_{n-1}$ of the system factorized by the action of the discrete group generated by the reflections (2.11) is an open subsets of the Jacobian $J(S)$ of the spectral curve (2.14) (see [18] for the case of the Neumann system).

Let $E_\pm = (0, \pm \sqrt{-\det L(0)})$ and

$$T = A(E_+ - E_-),$$

where $A : Div^0(S) \to J(S)$ is the Abel mapping.

Repeating the arguments given for Theorem 3 in [9], we obtain
Theorem 3.2: The dynamics (3.1), (3.2) corresponds to the translation on the Jacobian variety of the spectral curve (2.14) by a vector $T$.

The Cayley–type conditions for periodic billiard trajectories within ellipsoids in the pseudo–Euclidean spaces are derived in [7]. Theorem 3.2 provides an alternative approach for the derivation of Cayley–type conditions modulo symmetries (2.11) (e.g., see Ch. 3, Section 8 and Ch. 7, Sections 2 and 3 in [6]).

4 Symmetric quadrics

In a more general situation, when the quadric is symmetric, we use the following notation (cf. [12]): the sets of indices $I_s \subset \{1, \ldots, n\}$ ($s = 1, \ldots, r$) are defined by the conditions,

$$
\begin{align*}
1^o & \quad \tau_ia_i = \tau Ja_j = \alpha_s \text{ for } i, j \in I_s \text{ and for all } s \in \{1, \ldots, r\}, \\
2^o & \quad \alpha_s \neq \alpha_t \text{ for } s \neq t.
\end{align*}
$$

One should observe the possibility that $a_i = a_j$ for $i \in I_s$, $j \in I_t$, $s \neq t$, but in this case it has to be $\tau_i \tau_j = -1$.

Owing to Corollary (2.1), the determinant $\det \mathcal{L}_{x,y}(\lambda)$ is an invariant of the flow (2.4), and by expanding it in terms of $1/(\lambda - \alpha_s), 1/(\lambda - \alpha_s)^2$, we get

$$
\det \mathcal{L}_{x,y}(\lambda) = (1 + q_\lambda(x, x))q_\lambda(y, y) - q_\lambda(x, y)^2 = \sum_{s=1}^r \frac{\tilde{f}_s}{\lambda - \alpha_s} + \frac{P_s}{(\lambda - \alpha_s)^2},
$$

where the integrals $\tilde{f}_s, P_s$ are given by

$$
\begin{align*}
\tilde{f}_s &= \sum_{i \in I_s} \left( \tau_iy_i^2 + \sum_{j \not\in I_s} \frac{(x_iy_j - x_jy_i)^2}{\tau_ja_i - \tau_i a_j} \right), \\
P_s &= \sum_{i,j \in I_s, i < j} (x_iy_j - x_jy_i)^2 \quad \text{for } |I_s| \geq 2 \quad (P_s \equiv 0, \text{ for } |I_s| = 1).
\end{align*}
$$

The Hamiltonian (2.7) is equal to the sum $H = \frac{1}{2} \sum_{s=1}^r \tilde{f}_s$. Also, the functions $\tilde{f}_s, P_s$ are independent on $T\mathbb{R}^n$, while restricted to $T\mathbb{E}^{n-1}$ they are related by

$$
\sum_{s=1}^r \frac{\tilde{f}_s}{\alpha_s} = \sum_{s=1}^r \frac{P_s}{\alpha_s^2},
$$

which is equivalent to $\det \mathcal{L}_{x,y}(0) = 0$.

An analog of Theorem 5.1 in [12] holds:

Theorem 4.1: In addition to (4.3), a non–singular geodesic $x(t)$ on a quadric $\mathbb{E}^{n-1}$ also has integrals

$$
\Phi_{s,ij} := y_i x_j - x_i y_j, \quad i, j \in I_s, \quad |I_s| \geq 2.
$$
The functions \( \hat{f}_s, P_s = \sum_{i<j} \Phi_{s,ij}^2 \) are central within the algebra of integrals generated by \( \hat{f}_s \) and \( \Phi_{s,ij} \):
\[
\{ \hat{f}_s, \hat{f}_t \}_D = 0, \quad \{ \hat{f}_s, P_t \}_D = 0, \quad \{ P_s, P_t \}_D = 0,
\{ \hat{f}_s, \Phi_{t,ij} \}_D = 0, \quad \{ P_s, \Phi_{t,ij} \}_D = 0.
\]

**Proof.** The functions (4.4) are components of the momentum mapping
\[ \Phi_s \colon T E^{n-1} \to so(k_s, l_s)^* \]
of the Hamiltonian \( SO(k_s, l_s) \)-action on \( T E^{n-1} \), where
\[ k_s = |\{ \tau_i : \tau_i = 1, i \in I_s \}|, \quad l_s = |\{ \tau_i : \tau_i = -1, i \in I_s \}|, \quad k_s + l_s = |I_s|, \]
Indeed, they are components of the momentum mapping of \( SO(k_s, l_s) \)-action on \( T \mathbb{R}^n(x, y) \) and since the action preserves the constraints (2.5), that is
\[ \{ \Phi_{s,ij}, F_1 \} = \{ \Phi_{s,ij}, F_2 \} = 0, \] (4.5)
they are also components of the momentum mapping of the Hamiltonian \( SO(k_s, l_s) \)-action on \( T E^{n-1} \). In particular, because \( P_s \) is a composition of the momentum mapping with a Casimir function on \( so(k_s, l_s)^* \), we have \( \{ P_s, \Phi_{s,ij} \} = \{ P_s, \Phi_{s,ij} \}_D = 0 \).

Since the Hamiltonian function (2.7), as well as of all its components \( \hat{f}_s \) are invariant with respect to the \( SO(k_s, l_s) \)-action, then the functions (4.4) are integrals of the system and commute with \( \hat{f}_s, s = 1, \ldots, r \) (the Noether theorem).

Next, since \( \Phi_{s,ij} \) and \( \Phi_{t,uv} \) for \( s \neq t \) depend on different sets of variables \((x, y)\), their canonical Poisson bracket vanishes. Thus, from (4.5) we also have \( \{ \Phi_{s,ij}, \Phi_{t,uv} \}_D = 0 \), implying that \( \{ P_s, P_t \}_D = 0, \{ P_s, \Phi_{t,ij} \}_D = 0 \).

It remains to prove \( \{ \hat{f}_s, \hat{f}_t \}_D = 0 \). Following [12], we introduce a family of deformed non–symmetric quadrics
\[ \mathbb{E}^n_{\epsilon} : (A_{\epsilon}^{-1} x, x) = 1, \quad A_{\epsilon} = \text{diag}(a_{1\epsilon}, \ldots, a_{n\epsilon}), \quad \tau_i a_{i\epsilon}^t \neq \tau_j a_{j\epsilon}^t, \quad i \neq j \text{ for } \epsilon \neq 0, \]
where \( \lim_{\epsilon \to 0} a_{i\epsilon}^t = a_i \), and \( a_i^t \) are smooth functions. The corresponding Poisson–Dirac bracket and integrals (2.13) are denoted by \( \{ \cdot, \cdot \}_D^{\epsilon} \) and \( f_i^{\epsilon} \), respectively. Define
\[ \hat{f}_s^{\epsilon} = \sum_{i \in I_s} f_i^{\epsilon} = \sum_{i \in I_s} \left( \frac{P_{ij}}{\tau_i a_i^t - \tau_j a_j^t} \right). \] (4.6)
Then \( \{ \hat{f}_s^{\epsilon}, \hat{f}_t^{\epsilon} \}_D^{\epsilon} = 0 \), and taking the limit \( \epsilon \to 0 \), we obtain \( \{ \hat{f}_s, \hat{f}_t \}_D = 0 \). \( \square \)

For a symmetric quadric (4.1), from (4.2), the polynomial (2.15) determining the spectral curve (2.11) equals
\[ \det L_{x,y}(\lambda) = (\lambda - \alpha_1)^{2|I_1|-\delta_1} \cdots (\lambda - \alpha_r)^{2|I_r|-\delta_r} \cdot P(\lambda), \]
where
\[ P(\lambda) = (\lambda - \alpha_1)^{\delta_1} \cdots (\lambda - \alpha_r)^{\delta_r} \det L_{x,y}(\lambda) \] (4.7)
\[ = \sum_{i=1}^{r} \left( (\lambda - \alpha_i)^{\delta_i-1} \prod_{i \neq s} (\lambda - \alpha_i)^{\delta_i} f_s + \prod_{i \neq s} (\lambda - \alpha_i)^{\delta_i} P_s \right) \]
\[ = \lambda^{N-1} K_{N-1} + \cdots + \lambda K_1 + K_0, \]
δ_1 = 2 for |I_s| ≥ 2, δ_1 = 1 for |I_s| = 1, N = δ_1 + ⋯ + δ_r.

In particular, K_{N-1} = 2H = \langle y, y \rangle. When considered on T\mathbb{R}^n, the functions K_i are independent, while on T\mathbb{E}^{n-1}, since P(0) = 0, we have K_0 ≡ 0.

Thus, the degree of P(λ) is N − 1 for a space-like or time-like vector y, or N − 2 for a light-like y. It can be proved that the geodesic flow (2.4) is integrable in a noncommutative sense by means of integrals described in Theorem 4.1 and that generic invariant isotropic manifolds are (N − 1)-dimensional. They are generated by the Hamiltonian flows of \tilde{f}_1, P_1, ⋯, \tilde{f}_r, P_r, that is, of the integrals K_1, ⋯, K_{N-1}.

5 The Chasles theorem for symmetric ellipsoids

In this section we assume that \mathbb{E}^{n-1} is an ellipsoid. Then the condition τ_i a_i = τ_j a_j can be satisfied only if

a_i = a_j, \quad τ_i τ_j = 1. \tag{5.1}

Therefore, a symmetric ellipsoid \mathbb{E}^{n-1} with conditions (4.1) has SO(|I_1|) × ⋯ × SO(|I_r|)-symmetry.

From the discrete L–A representation in Theorem 3.1 we get for billiards the integrals (4.3). Moreover, one can easily verify that the components (4.4) of the momentum mapping of SO(|I_s|)-action are also conserved by the billiard flow (3.1), (3.2), implying a noncommutative integrability of the mapping \phi both in the symplectic and in the contact setting (see [11]).

We now give a geometric interpretation of noncommutative integrability of the systems considered here analogous to the pseudo–Euclidean versions of the Chasles theorem stated in [14] (see Theorem 4.9) and in [7] (see Theorem 5.1) for the corresponding Liouville integrable non–symmetric systems. For the Euclidean case, see Lemma 6.2 in [12].

Consider the pencil of quadrics (1.2) in \mathbb{E}^{k,l}. The condition

\det L_{x,y}(λ) = q_λ(y, y)(1 + q_λ(x, x)) − q_λ(x, y)^2 = 0 \tag{5.2}

is equivalent to the geometrical property that the line

l_{x,y} = \{x + sy, s ∈ \mathbb{R}\}

is tangent to the quadric \mathcal{Q}_λ. This is proved in [16, 7] for \mathbb{E}^{n-1} being a non-symmetric ellipsoid, but the assertion holds for symmetric quadrics \mathbb{E}^{n-1} as well.

**Theorem 5.1:** (i) If a line \ell_k determined by the billiard segment x_k x_{k+1} (respectively a geodesic line x(t) at the moment t = t_0) is tangent to a quadric \mathcal{Q}_λ from the pseudo–confocal family (1.2), then it is tangent to \mathcal{Q}_λ for all k ∈ \mathbb{Z} (respectively for all t ∈ \mathbb{R}). In addition, \mathbf{R}(x_k) is a billiard trajectory (respectively \mathbf{R}(x(t)) is a geodesic line) tangent to the same quadric \mathcal{Q}_λ for all \mathbf{R} ∈ SO(|I_1|) × ⋯ × SO(|I_r|).

(ii) The lines \ell_k determined by a generic space–like or time–like (respectively light–like) billiard trajectory are tangent to N − 1 (respectively N − 2) fixed quadrics from the pseudo–confocal family (1.2), where, as above

N = r + |\{s ∈ \{1, ⋯, r\} : |I_s| ≥ 2\}|.
The tangent lines to a generic space–like or time–like (respectively light–like) geodesic on $E^{n-1}$ are tangent to other $N-2$ (respectively $N-3$) fixed quadrics from the pseudo–confocal family (1.2). Moreover, the billiard trajectories (geodesic lines) tangent to the same set of quadrics are of the same type: space–like, time–like or light–like.

**Proof.** (i) If the line $l_{x(t_0),y(t_0)}$ is tangent to $Q_{\lambda^*}$ then $\det L_{x(t_0),y(t_0)}(\lambda^*) = 0$, implying $\det L_{x(t),y(t)}(\lambda^*) = 0$ for all $t \in \mathbb{R}$ (Corollary 2.1). Therefore, the line $l_{x(t),y(t)}$ is tangent to the quadric $Q_{\lambda^*}$ for all $t \in \mathbb{R}$.

The second statement follows from the fact that $\det L_{x,y}(\lambda)$ is $SO(|I_1|) \times \cdots \times SO(|I_r|)$–invariant function.

(ii) From Lemma 5.1 below it follows that a space–like or time–like (respectively light–like) line $l_{x(t),y(t)}$ determined by a geodesic line $x(t)$ is tangent to $N-2$ (respectively, $N-3$) fixed quadrics from the pseudo–confocal family (1.2) different from $E^{n-1}$.

The last statement follows from the distribution of zeros of the polynomial $P(\lambda)$ described in the proof of Lemma 5.1.

The similar assertions hold for billiard trajectories as well. □

By combining Theorem 5.1 and a non-commutative version of Veselov’s discrete Arnold–Liouville theorem (see [21]) we can formulate Poncelet theorem for a symmetric elliptic billiard in the pseudo–Euclidean space $E^{k,l}$:

**Theorem 5.2:** If a billiard trajectory $(x_k)$ is periodic with a period $m$ and if the lines $l_k$ determined by the segments $x_k x_{k+1}$ are tangent to $N-1$ quadrics $Q_{\lambda_1}, \ldots, Q_{\lambda_{N-1}}$ (in the space–like or the time–like case) or to $N-2$ quadrics $Q_{\lambda_1}, \ldots, Q_{\lambda_{N-2}}$ (in the light–like case), then any other billiard trajectory within $E^{n-1}$ with the same caustics is also periodic with the same period $m$.

**Lemma 5.1:** If a point $x$ lies inside, or on the ellipsoid $E^{n-1}$, then the equation (5.2) generically has $N-1$ (respectively $N-2$) different real solutions for space–like and time–like (respectively light–like) vectors $y$. In particular, if the line $l_{x,y}$ is tangent to $E^{n-1}$, then (5.2) generically has $N-2$ (respectively $N-3$) different real non–zero solutions for space–like and time–like (respectively light–like) vector $y$.

**Proof.** Here we modify the idea used in [1, 7] for an analogous assertion in the case of non–symmetric ellipsoids.

We have

$$q_{\lambda}(y,y) = -\sum_{i=1}^{n} \alpha_i \tau_i \lambda - \sum_{s=1}^{r} \langle y, y \rangle_s = -\sum_{s=1}^{r} \frac{(y, y)_s}{\alpha_s - \lambda} = -\frac{R(\lambda)}{\prod_{s=1}^{r} (\alpha_s - \lambda)}, \quad (5.3)$$

where

$$\langle y, y \rangle_s = \sum_{i \in I_s} \tau_i y_i^2 \quad (5.4)$$
and

\[ R(\lambda) = \sum_{s=1}^{r} \langle y, y \rangle_s \prod_{t \neq s} (\alpha_t - \lambda) = (-1)^{r-1} \cdot \sum_{s=1}^{r} \langle y, y \rangle_s \prod_{t \neq s} (\lambda - \alpha_t). \]

We shall estimate the zeros of \( R(\lambda) \). Without losing a generality, we can assume that for (4.1) we have

\[ \alpha_1 > \alpha_2 > \cdots > \alpha_{\tilde{r}} > 0 > \alpha_{\tilde{r}+1} > \cdots > \alpha_r. \quad (5.5) \]

From the definition of \( R(\lambda) \) we obtain

\[ \text{sign } R(\alpha_s) = \epsilon_s (-1)^{s+r}, \quad \epsilon_s = \text{sign } \langle y, y \rangle_s, \quad s = 1, \ldots, r, \]

and for a space–like or a time–like vector \( y \):

\[ \text{sign } R(-\infty) = \text{sign } \langle y, y \rangle, \]

\[ \text{sign } R(\infty) = (-1)^{r-1} \text{sign } \langle y, y \rangle. \]

Then, since \((x, y)\) is generic, (5.1) and (5.5) yield

\[ \epsilon_1 = \cdots = \epsilon_{\tilde{r}} = +1, \quad \epsilon_{\tilde{r}+1} = \cdots = \epsilon_r = -1. \quad (5.6) \]

Therefore, the equation \( R(\lambda) = 0 \) has \( r - 2 \) solutions \( \zeta_s \in (\alpha_{s+1}, \alpha_s) \) for \( s \in \{1, \ldots, r - 1\} \setminus \{\tilde{r}\} \) and another solution \( \zeta_r \in (-\infty, \alpha_r) \) (if \( y \) is space–like) or \( \zeta_0 \in (\alpha_1, \infty) \) (if \( y \) is time–like).

Firstly, we consider the case when the line \( l_{x,y} \) is not tangent to \( E^{n-1} \). From the fact that the point \( x \) belongs to the interior of the ellipsoid, or to the ellipsoid itself, it follows that \( 1 + q_0(x, x) \geq 0 \). Furthermore, for a generic \((x, y)\) it is \( q_0(y, y) < 0 \), \( q_0(x, y) \neq 0 \). Whence,

\[ q_0(y, y)(1 + q_0(x, x)) - q_0(x, y)^2 < 0. \]

By the use of the polynomial (4.7), we can rewrite \( \det L_{x,y}(\lambda) \) in the form

\[ \det L_{x,y}(\lambda) = q_\lambda(y, y)(1 + q_\lambda(x, x)) - q_\lambda(x, y)^2 = \frac{P(\lambda)}{\prod_{s=1}^{r}(\lambda - \alpha_s)^{\delta_s}}. \quad (5.7) \]

Recall that the degree of \( P(\lambda) \) is \( N - 1 \) for a space–like or a time–like vector \( y \), while for a light–like vector \( y \) the degree is \( N - 2 \), \( N = \delta_1 + \cdots + \delta_r \). Thus, for a space–like or a time–like vector \( y \) we have:

\[ \det L_{x,y}(\lambda) \sim \langle y, y \rangle / \lambda, \quad \lambda \to \pm \infty. \quad (5.8) \]

Obviously, the left hand side of (5.7) takes negative values at the ends of each of the \( r - 2 \) intervals

\[ (\zeta_{\tilde{r}-1}, \zeta_{\tilde{r}-2}), \ldots, (\zeta_{\tilde{r}+2}, \zeta_{\tilde{r}+1}), (\zeta_{\tilde{r}+1}, 0), (0, \zeta_{\tilde{r}-1}), (\zeta_{\tilde{r}-1}, \zeta_{\tilde{r}-2}), \ldots, (\zeta_2, \zeta_1), \]

and in each of the indicated intervals lies \( \alpha_{\tilde{r}-1}, \ldots, \alpha_2 \), respectively. Since generically \( P_s > 0 \) for \( |I_s| \geq 2 \), i.e, \( \delta_s = 2 \), from

\[ \lim_{\lambda \to \alpha_s} \frac{\tilde{f}_s}{\lambda - \alpha_s} + \frac{P_s}{(\lambda - \alpha_s)^2} = \infty, \quad \lim_{\lambda \to \alpha_s} \frac{\tilde{f}_s}{\lambda - \alpha_s} + \frac{P_s}{(\lambda - \alpha_s)^2} = \infty, \quad (5.9) \]
and (4.2) follows that in the interval containing the corresponding $\alpha_s$ there are at least two zeros of the polynomial $P(\lambda)$.

In the case $|I_s| = 1$, it is $P_s = 0$ and the interval contains at least one zero.

The analysis above shows that in $(\zeta_r, \zeta_{r-1})$ there are $\delta_2 + \cdots + \delta_{r-1}$ zeros. It remains to show that in $(-\infty, \zeta_{r-1}) \cup (\zeta_1, \infty)$ lie $\delta_1 + \delta_r - 2$ (if $y$ is light–like) or $\delta_1 + \delta_r - 1$ zeros (if $y$ is not light–like).

Indeed, note that when $y$ is not light–like, it also has negative value at the ends of one of the intervals $(\zeta_r, \zeta_{r-1})$ (if $y$ is space–like) or $(\zeta_1, \zeta_0)$ (if $y$ is time–like), containing $\delta_r$ and $\delta_1$ zeros, respectively (which is in agreement with (5.8)).

Consequently, in the case $\delta_1 = \delta_r = 1$ the assertion is clear.

If $\delta_1 = 2$, $\delta_r = 1$ and $y$ is time–like, the conclusion follows from the previous considerations. On the other hand, if $y$ is light–like or space–like, according to (5.9), the additional zero of $P(\lambda)$ lies within the interval $(\zeta_{r+1}, \zeta_{r-1})$.

The above analysis concerning the zeros of $P(\lambda)$ remains the same, except for the interval $(\zeta_{r+1}, \zeta_{r-1})$. However, owing to

$$\frac{dP}{d\lambda}|_{\lambda=0} = K_1$$

(see (4.7)) and the fact that the integral $K_1$ is generically different from zero, $P(\lambda)$ changes its sign at 0. Therefore, the number of zeros of $P(\lambda)$ lying in the interval $(\zeta_{r+1}, \zeta_{r-1})$ is the same as in the previous case.

6 Further generalization: virtual billiards within quadrics

Note that the billiard mapping (3.1), (3.2) is well defined for arbitrary quadric $\mathbb{E}^{n-1}$ given by (1.1) and not only for ellipsoids. Hence, segments $x_{k-1}x_k$ and $x_kx_{k+1}$ determined by 3 successive points of the mapping (3.1), (3.2) may be:

(i) on the same side of the tangent plane $T_{x_k}\mathbb{E}^{n-1}$;

(ii) on the opposite sides of the tangent plane $T_{x_k}\mathbb{E}^{n-1}$.

In the case (i) we have a part of the usual pseudo–Euclidean billiard trajectory, while in the case (ii) the billiard reflection corresponds to the points $x_{k-1}x_kx'_{k-1}$, where $x'_{k+1}$ is the symmetric image of $x_{k+1}$ with respect to $x_k$. In the three-dimensional Euclidean case, Darboux referred to such reflection as a virtual reflection (e.g., see [5] and [6], Ch. 5). In Euclidean spaces of arbitrary dimension, such configurations were introduced in [5]. It appears that a multidimensional variant of Darboux’s 4–periodic virtual trajectory with reflections on two quadrics, refereed as a double–reflection configuration [6], is fundamental in the construction of the double reflection nets in Euclidean and pseudo-Euclidean spaces (see [8]). They also played a role in a construction of the billiard algebra (see Ch. 8, [6]). The 4–periodic orbits of real and complex planar billiards with virtual reflections are also studied in [10].
Definition 6.1: Let $E^{n-1}$ be a quadric in the pseudo–Euclidean space $E^{k,l}$ defined by (1.1). We refer to (3.1), (3.2) as a virtual billiard mapping, and to the sequence of points $x_k$ determined by (3.1), (3.2) as a virtual billiard trajectory within $E^{n-1}$.

The virtual billiard dynamics is defined outside the singular set

$$\Sigma = \{(x, y) \in T\mathbb{R}^n \mid (EA^{-2}x, x) = 0 \lor (A^{-1}x, y) = 0 \lor (A^{-1}y, y) = 0\}. \quad (6.1)$$

The condition $(A^{-1}y_0, y_0) = 0$ implies that the line $l_0 = x_0 + sy_0, s \in \mathbb{R}$ does not intersect the quadric $E^{n-1}$ (for example, consider the light–like lines in the space $E^{1,1}$ and the quadric $x_1^2 - x_2^2 = 1$).

We can interpret (3.1), (3.2), in the case of non light–like billiard trajectories, as the equations of a discrete dynamical system (see [19, 17, 21]) on $E^{n-1}$ described by the discrete action functional:

$$S[x] = \sum_k L(x_k, x_{k+1}), \quad L(x_k, x_{k+1}) = \sqrt{|\langle x_{k+1} - x_k, x_{k+1} - x_k \rangle|},$$

where $x = (x_k), k \in \mathbb{Z}$ is a sequence of points on $E^{n-1}$. Note that a virtual billiard trajectory can have both virtual and real reflections.

Fig. 1: A segment of a virtual billiard trajectory within hyperbola ($a_1 > 0, a_2 < 0$) in the Euclidean space $E^{2,0}$. The caustic is an ellipse.

The Lax representation given in Theorem 3.1 applies for the virtual billiard dynamics as well.

Theorem 6.1: The trajectories $(x_j, y_j)$ of (3.1), (3.2), outside the singular set (6.1) satisfy the matrix equation (3.3).

Now, suppose that $E^{n-1}$ is a symmetric quadric defined by conditions (1.1). It has the $G = SO(k_1, l_1) \times SO(k_2, l_2) \times \cdots \times SO(k_r, l_r)$–symmetry (see Theorem 4.1). With the same proof as of the item (i) in Theorem 5.1, we have

Theorem 6.2: If a line $l_k$ determined by the segment $x_k x_{k+1}$ of a virtual billiard trajectory within $E^{n-1}$ (respectively a geodesic line $x(t)$ at the moment $t = t_0$) is tangent to a quadric $Q_{\lambda^*}$ from the pseudo–confocal family (1.2), then it is tangent to
\( Q_{\lambda^*} \) for all \( k \in \mathbb{Z} \) (respectively for all \( t \in \mathbb{R} \)). In addition, \( R(x_k) \) is a virtual billiard trajectory (respectively \( R(x(t)) \) is a geodesic line) tangent to the same quadric \( Q_{\lambda^*} \) for all \( R \in G \).

However, for a proof of the item (ii), in Lemma 5.1 we used the relations (5.1), which implied that, under the conditions (5.5), the signs of the restricted scalar products (5.4) satisfy (5.6).

Fig. 2: Families of pseudo-confocal quadrics for \( a_1 > 0, a_2 < 0 \) in \( E^{1,1} \) and \( E^{2,0} \), respectively.

For example, let us restrict ourselves to the Euclidean case. With the same notation as in Lemma 5.1 for a generic \( y \), we have

\[
\langle y, y \rangle_s > 0, \quad s = 1, \ldots, r,
\]

and, therefore,

\[
\begin{align*}
\text{sign } R(-\infty) &= 1, \\
\text{sign } R(\alpha_s) &= (-1)^{s+r}, \\
\text{sign } R(\infty) &= (-1)^{r-1}.
\end{align*}
\]

Consequently, the equation \( R(\lambda) = 0 \) has \( r - 1 \) solutions \( \zeta_s \in (\alpha_{s+1}, \alpha_s) \) for \( s \in \{1, \ldots, r - 1\} \) and the left hand side of (5.7) takes negative values at the ends of \( r - 2 \) intervals

\[
(\zeta_{r-1}, \zeta_{r-2}), \ldots, (\zeta_2, \zeta_1).
\]

Also, according to (5.8), the left hand side of (5.7) takes negative values at the ends of the interval

\[
(\zeta_{r}, \zeta_{r-1}),
\]

for a certain \( \zeta_r < \alpha_r \). We have \( \alpha_s \in (\zeta_s, \zeta_{s-1}) \), \( s = 2, \ldots, r \), and as in the proof of Lemma 5.1 this implies that the number of zeros of \( P(\lambda) \) is \( N - 1 \).

Thus, we get:

**Theorem 6.3:** The lines \( l_k \) determined by a generic virtual billiard trajectory within a quadric \( \mathbb{E}^{n-1} \) in the Euclidean space \( \mathbb{E}^{n,0} \) are tangent to \( N - 1 \) fixed quadrics from the confocal family (1.2). Also, the tangent lines to a generic geodesic on \( \mathbb{E}^{n-1} \) are tangent to other \( N - 2 \) fixed quadrics from the confocal family (1.2).
A sketch of the proof of Theorem 6.2 for a symmetric ellipsoid in the Euclidean space is given in Lemma 6.2 [12].

Finally, we mention that one can obtain similar results for geodesic flows and billiards on quadrics on a pseudo–sphere in $E^{k,l}$ (e.g., see [2] [20] [3]). Also, it would be interesting to describe the class of symmetric periodic (virtual) billiard trajectories (see [3] for a study of symmetric periodic elliptical billiard trajectories in the Euclidean space).

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References

[1] M. Audin, Courbes algébriques et systèmes intégrables: géodesiques des quadriques, Expo. Math. 12 (1994), 193–226.

[2] C. B. Bolotin, Интегрируемые бильярды на поверхностях постоянной кривизны, Мат. Заметки, 51 (1992), no. 2, 20–28, (Russian); English translation:
S. V. Bolotin, Integrable billiards on constant curvature surfaces, Math. Notes, 51 (1992), no. 1-2, 117-123.

[3] P. S. Casas, R. Ramírez-Ros, Classification of symmetric periodic trajectories in elliptical billiards, Chaos 22 (2012), 026110.

[4] V. Dragović, B. Jovanović and M. Radnović, On elliptical billiards in the Lobachevsky space and associated geodesic hierarchies, J. Geom. Phys. 47 (2003), 221–234, arXiv:math-ph/0210019.

[5] V. Dragović and M. Radnović, Geometry of integrable billiards and pencils of quadrics, J. Math. Pures Appl. 85 (2006), No. 6, 758–790, arXiv:math-ph/0512049.

[6] V. Dragović and M. Radnović, Poncelet porisms and beyond. Integrable billiards, hyperelliptic Jacobians and pencils of quadrics. Frontiers in Mathematics. Birkhauser/Springer Basel AG, Basel, 2011.

[7] V. Dragović and M. Radnović, Ellipsoidal billiards in pseudo-euclidean spaces and relativistic quadrics, Adv. Math. 231 (2012), 1173–1201, arXiv:1108.4552.

[8] V. Dragović and M. Radnović, Bicentennial of the Great Poncelet Theorem (1813-2013): Current Advances, Bulletin AMS 51 (2014) 373–445., arXiv:1212.6867.

[9] Yu. N. Fedorov, Задача об эллипсоидальном бильярде с квадратичным потенциалом, Функц. анализ и его приложения, 35 (2001), no. 3, 48–59, 95–96 (Russian); English translation:
Yu. N. Fedorov, An ellipsoidal billiard with quadratic potential, Funct. Anal. Appl. 35(3) (2001), 199–208.
Further generalization: virtual billiards within quadrics

[10] A. Glutsyuk, *On quadrilateral orbits in complex algebraic planar billiards*, Moscow Math. J., 14 (2014), No. 2, 239–289, [arXiv:1309.1843].

[11] B. Jovanović, *Noncommutative integrability and action angle variables in contact geometry*, Journal of Symplectic Geometry, 10 (2012), 535–562, [arXiv:1103.3611].

[12] B. Jovanović, *The Jacobi–Rosochatius problem on an ellipsoid: the Lax representations and billiards*, Arch. Rational Mech. Anal. 210 (2013), 101–131, [arXiv:1303.6204].

[13] B. Jovanović *Heisenberg model in pseudo–Euclidean spaces*, Regular and Chaotic Dynamics, 19 (2014), No. 2, 245–250, [arXiv:1405.0905 [nlin.SI]].

[14] B. Khesin and S. Tabachnikov, *Pseudo-Riemannian geodesics and billiards*, Adv. Math. 221 (2009), 1364–1396, [arXiv:math/0608620].

[15] B. Khesin and S. Tabachnikov, *Contact complete integrability*, Regular and Chaotic Dynamics, 15 (2010) 504-520, [arXiv:0910.0375 [math.SG]].

[16] J. Moser, *Geometry of quadric and spectral theory*. In: Chern Symposium 1979, Berlin–Heidelberg–New York, 147–188, 1980.

[17] J. Moser and A. P. Veselov, *Discrete versions of some classical integrable systems and factorization of matrix polynomials*, Comm. Math. Phys. 139 (1991) 217–243.

[18] D. Mumford, Tata lectures on theta, Birkhäuser, Boston 1984.

[19] А. П. Веселов, Интегрируемые системы с дискретным временем и разностные операторы, Функц. анализ и его прилож. 22(2) (1988), 1–13 (Russian); English translation:
A. P. Veselov, *Integrable discrete–time systems and and difference operators*, Funct. Anal. Appl. 22 (1988), 83–94

[20] A.P. Veselov, *Confocal surfaces and integrable billiards on the sphere and in the Lobachevsky space*, J. Geom. Phys. 7 (1990) 81-107.

[21] А. П. Веселов, Интегрируемые отображения, Успехи Мат. Наук, 46(5) (1991), 3–45 (Russian); English translation:
A. P. Veselov, *Integrable maps*, Russ. Math. Surv. 46 (5) (1991) 1–51.