SHARP SCHWARZ-TYPE LEMMAS FOR THE SPECTRAL UNIT BALL

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Abstract. We provide generalisations of two Schwarz-type lemmas — the first a result of Globevnik and the other due to Ransford and White — for holomorphic mappings into the spectral unit ball. The first concerns mappings of the unit disc in \( \mathbb{C} \) into the spectral unit ball, while the second concerns self-mappings. The aforementioned results apply to holomorphic mappings that map the origin to the origin. We extend these results to general holomorphic mappings into the spectral unit ball. We also show that our results are sharp.

1. Introduction and Statement of Results

The spectral unit ball (denoted by \( \Omega_n \)) is defined as

\[ \Omega_n := \{ W \in M_n(\mathbb{C}) : r(W) < 1 \}, \]

where \( r(W) \) denotes the spectral radius of the \( n \times n \) matrix \( W \). In this paper, \( D \) will denote the unit disc in \( \mathbb{C} \) and, if \( \Omega \) and \( G \) are complex domains, \( \mathcal{O}(\Omega; G) \) will denote the class of holomorphic mappings of \( \Omega \) into \( G \). We present two Schwarz-type lemmas for the spectral unit ball. These lemmas are inspired by the renewed interest in the function theory on the spectral unit ball — the reader is referred to \cite{2}, \cite{6}, \cite{1} and \cite{3}, to name just a few recent papers. This interest stems, to a large extent, from recent work on the spectral version of the Nevanlinna-Pick interpolation problem. We shall not address this problem directly in this paper; although Theorem 1.2 below might have some bearing on the two-point interpolation problem. We begin by considering the following result by Globevnik \cite{4}, which is perhaps the earliest Schwarz-type lemma for the spectral unit ball:

\[ (1.1) \quad F \in \mathcal{O}(D; \Omega_n) \text{ and } F(\zeta_j) = 0 \implies r(F(\zeta_2)) \leq \frac{|\zeta_1 - \zeta_2|}{|1 - \zeta_2\zeta_1|} =: \mathcal{M}(\zeta_1, \zeta_2). \]

One would like to generalise this result to the case when \( F(\zeta_j) \) is not necessarily 0, \( j = 1, 2 \).

The second Schwarz-type lemma that motivates this paper — this time a result on self-mappings of \( \Omega_n \) — is the following result of Ransford and White \cite{5}:

\[ (1.2) \quad G \in \mathcal{O}(\Omega_n; \Omega_n) \text{ and } G(0) = 0 \implies r(G(X)) \leq r(X) \quad \forall X \in \Omega_n. \]

Incidentally, we refer to the above results as “Schwarz-type lemmas” because they relate the growth of a holomorphic mapping to the growth of its argument(s). One would like to generalise \cite{12} in the similar manner that the Schwarz-Pick lemma generalises the Schwarz lemma for \( D \) — i.e. by formulating an inequality that is valid without assuming that the holomorphic mapping in question has a fixed point.

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What is the key idea needed to generalise (1.1) and (1.2) in appropriate ways? The following example shows how the conclusion of (1.1) can fail, even in the simple situation where \( r(F(\zeta_1)) = 0 \), if \( F(\zeta_1) \neq 0 \). However, it also suggests a way forward.

**Example 1.1.** For \( n \geq 3 \) and \( d = 2, \ldots, n - 1 \), define the holomorphic map \( F_d : D \to \Omega_n \) by

\[
F_d(\zeta) := \begin{bmatrix}
0 & \cdots & \zeta \\
1 & 0 & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{bmatrix}, \quad \zeta \in D,
\]

where \( \mathbb{I}_{n-d} \) denotes the identity matrix of dimension \( n - d \) for \( 1 < d < n \). One easily computes that \( r(F_d(\zeta)) = |\zeta|^{1/d} \). Hence

\[
(1.3) \quad r(F_d(\zeta))^d = |\zeta| = M(0, \zeta) \quad \forall \zeta \in D,
\]

but, for each \( q < d \), \( r(F_d(\zeta))^q > M(0, \zeta) \quad \forall \zeta \neq 0 \).

In particular, \( r(F_d(\zeta)) > M(0, \zeta) \quad \forall \zeta \neq 0 \), in contrast with (1.1). \( \square \)

The above example is rather suggestive when one notices that the exponent occurring in the left-hand side of (1.3) is the degree of the minimal polynomial of \( F_d(0) \). While \( r(F_d(\zeta_1)) = 0 \) for each \( d = 2, \ldots, n - 1 \) (take \( \zeta_1 = 0 \) in this discussion), what differs in each case is the degree of the minimal polynomial of \( F_d(\zeta_1) \). Presumably, this information should be encoded in any generalisation of (1.1). This idea is the key to establishing a result with the following features:

- It has a Schwarz-type structure: i.e. we get an expression in \( F(\zeta_1) \) and \( F(\zeta_2) \) — call it \( E(F(\zeta_1), F(\zeta_2)) \) — such that
  \[
  F \in \mathcal{O}(D; \Omega_n) \implies E(F(\zeta_1), F(\zeta_2)) \leq M(\zeta_1, \zeta_2).
  \]
- Globevnik’s result is recovered when we set \( F(\zeta_1) = 0 \) in the above inequality.
- The above Schwarz-type inequality is sharp in the sense that this inequality is the best one can achieve. In more precise terms: given \( z \neq w \in D \), we can find \( z^w, w^z \in \mathcal{O}(D; \Omega_n) \) such that \( E(z^w(z), w^z(w)) = M(z, w) \).

Before stating this result, let us, for any compact subset \( K \subseteq D \) and \( \zeta \in D \), define

\[
\text{dist}_M(\zeta; K) := \min_{z \in K} \left| \frac{\zeta - z}{1 - \overline{z_1}} \right|.
\]

Our first result is as follows.

**Theorem 1.2.** Let \( F \in \mathcal{O}(D; \Omega_n), \ n \geq 2 \), and let \( \zeta_1, \zeta_2 \in D \). Let \( W_j = F(\zeta_j), \ j = 1, 2 \), and define

\[
d_j := \text{the degree of the minimal polynomial of } W_j,
\]

for \( j = 1, 2 \). Then

\[
(1.4) \quad \max \left\{ \max_{\mu \in \sigma(W_2)} \left[ \text{dist}_M(\mu; \sigma(W_1))^{d_1} \right], \max_{\lambda \in \sigma(W_1)} \left[ \text{dist}_M(\lambda; \sigma(W_2))^{d_2} \right] \right\} \leq \left| \frac{\zeta_1 - \zeta_2}{1 - \overline{\zeta_2 \zeta_1}} \right|.
\]
Furthermore, \((1.4)\) is sharp in the sense that given any two points \(z, w \in D\), there exists a mapping \(F^{z,w} \in \mathcal{O}(D; \Omega_n)\) such that
\[
\max \left\{ \max_{\mu \in \sigma(F^{z,w}(w))} [\text{dist}_M(\mu; \sigma(F^{z,w}(z)))^d(z)] \right\} = \left| \frac{z - w}{1 - wz} \right|,
\]
where \(d(z)\) (resp. \(d(w)\)) is the degree of the minimal polynomial of \(F^{z,w}(z)\) (resp. \(F^{z,w}(w)\)).

Remark 1.3. Note that Globevnik’s result is recovered when we set \(F(\zeta_1) = 0\) in the above theorem. This is because if \(W_1 = 0\), then, in the notation of Theorem 1.2,
\[
d_1 = 1, \quad \max_{\lambda \in \sigma(W_1)} [\text{dist}_M(\lambda; \sigma(W_2))]^2 \leq \min_{\mu \in \sigma(W_2)} |\mu|^{d_2} = r(W_2).
\]
Hence \((1.4)\), in this case, is identical to the conclusion of Globevnik’s result.

The proof of Theorem 1.2 is presented in Section 2. The consideration of a pertinent minimal polynomial turns out to equally relevant to our next result. Essentially, the proofs of both results exploit the minimal polynomial of a key matrix lying in the ranges of \(F\), respectively \(G\), to transform the maps \(F\) and \(G\) to maps to which the results \((1.1)\) and \((1.2)\), respectively, are applicable. One is led to do this when examining the basic example of a mapping \(G \in \mathcal{O}(\Omega_n; \Omega_n)\) where \(G(0) \neq 0\) and \(G\) fails to satisfy the inequality in \((1.2)\), even though \(r(G(0)) = 0\). Since we do not wish to prolong this already protracted introduction, we refer the reader to the counterexample immediately following Theorem 2 in the Ransford-White paper \([5]\) (or to the end of Section 3 of this paper). The idea hinted at above leads to a new result that:

- Just like \((1.2)\), provides a bound on the growth of the spectral radius of \(G(X)\) in terms of \(r(X)\), and specialises precisely to \((1.2)\) when we set \(G(0)\);
- Is sharp in a manner analogous to our discussion of “sharpness” of our previous result.

More precisely, we have the following:

**Theorem 1.4.** Let \(G \in \mathcal{O}(\Omega_n; \Omega_n), \ n \geq 2, \) and define \(d_G := \text{the degree of the minimal polynomial of } G(0)\). Then:
\[
(1.5) \quad r(G(X)) \leq \frac{r(X)^{1/d_G} + r(G(0))}{1 + r(G(0))r(X)^{1/d_G}} \quad \forall X \in \Omega_n.
\]

Furthermore, the inequality \((1.5)\) is sharp in the sense that there exists a non-empty set \(\mathcal{G}_n \subset \Omega_n\) such that given any \(A \in \mathcal{G}_n\) and \(d = 1, \ldots, n\), we can find a \(\mathcal{G}^{A,d} \in \mathcal{O}(\Omega_n; \Omega_n)\) such that
\[
d_{\mathcal{G}^{A,d}} = d, \quad \text{and}
\]
\[
r(\mathcal{G}^{A,d}(A)) = \frac{r(A)^{1/d} + r(\mathcal{G}^{A,d}(0))}{1 + r(\mathcal{G}^{A,d}(0))r(A)^{1/d}}.
\]
Remark 1.5. It is quite obvious why the Ransford-White bound is recovered when we set $G(0) = 0$ in the above theorem. When $G(0) = 0$, then $d_G = 1$ and $r(G(0)) = 0$, whence (1.5) is identical to the conclusion of (1.2).

2. The Proof of Theorem 1.2

The proofs in this section depend crucially on a theorem by Vesentini. The result is as follows:

Result 2.1 (Vesentini, [7]). Let $\mathcal{A}$ be a complex, unital Banach algebra and let $r(x)$ denote the spectral radius of any element $x \in \mathcal{A}$. Let $f \in \mathcal{O}(D; \mathcal{A})$. The the function $\zeta \mapsto r(f(\zeta))$ is subharmonic on $D$.

The following result is the key lemma of this section. The proof of Theorem 1.2 is reduced to a simple application of this lemma.

Lemma 2.2. Let $F \in \mathcal{O}(D; \Omega_n)$ and let $\lambda_1, \ldots, \lambda_s$ be the distinct eigenvalues of $F(0)$. Define $m(j) :=$ the multiplicity of the factor $(\lambda - \lambda_j)$ in the minimal polynomial of $F(0)$. Define the Blaschke product

$$B(\zeta) := \prod_{j=1}^{s} \left( \frac{\zeta - \lambda_j}{1 - \lambda_j \zeta} \right)^{m(j)}, \quad \zeta \in D.$$ 

Then $|B(\lambda)| \leq |\zeta| \ \forall \lambda \in \sigma(F(\zeta))$.

Proof. The Blaschke product $B$ induces a matrix function $\widetilde{B}$ on $\Omega_n$: for any matrix $A \in \Omega_n$, we set

$$\widetilde{B}(A) := \prod_{j=1}^{s} (I - \lambda_j A)^{-m(j)} (A - \lambda_j I)^{m(j)},$$

which is well-defined on $\Omega_n$ because whenever $\lambda_j \neq 0$,

$$(I - \lambda_j A) = \lambda_j (I/\lambda_j - A) \in GL(n, \mathbb{C}).$$

Furthermore, since $\zeta \mapsto (\zeta - \lambda_j)/(1 - \lambda_j \zeta)$ has a power-series expansion that converges uniformly on compact subsets of $D$, it follows from standard arguments that

$$\sigma(\widetilde{B}(A)) = \{B(\lambda) : \lambda \in \sigma(A)\} \quad \text{for any } A \in \Omega_n.$$ 

By the definition of the minimal polynomial, $\widetilde{B} \circ F(0) = 0$. At this point, we could apply Globevnik’s lemma — i.e. (1.1) above — to complete the proof. The actual argument, however, is very elementary, and we provide it here. Since $\widetilde{B} \circ F(0) = 0$, there exists a holomorphic map $\Phi \in \mathcal{O}(D; M_n(\mathbb{C}))$ such that $\widetilde{B} \circ F(\zeta) = \zeta \Phi(\zeta)$. Note that

$$\sigma(\widetilde{B} \circ F(\zeta)) = \sigma(\zeta \Phi(\zeta)) = \zeta \sigma(\Phi(\zeta)) \quad \forall \zeta \in D.$$ 

Since $\sigma(\widetilde{B} \circ F(\zeta)) \subset D$, the above equations give us:

$$r(\Phi(\zeta)) < 1/R \quad \forall \zeta : |\zeta| = R, \ R \in (0, 1).$$

Taking $A = M_n(\mathbb{C})$ in Vesentini’s theorem, we see that $\zeta \mapsto r(\Phi(\zeta))$ is subharmonic on the unit disc. Applying the Maximum Principle to (2.3) and taking limits as $R \rightarrow 1^-$, we get

$$r(\Phi(\zeta)) \leq 1 \quad \forall \zeta \in D.$$
In view of (2.1), (2.2) and (2.4), we get
\[ |B(\lambda)| \leq |\zeta| r(\Phi(\zeta)) \leq |\zeta| \quad \forall \lambda \in \sigma(F'(\zeta)). \]

We are now in a position to provide

**2.3. The proof of Theorem 1.2.** Define the disc automorphisms
\[ M_j(\zeta) := \frac{\zeta - \zeta_j}{1 - \zeta_j \zeta}, \quad j = 1, 2, \]
and write \( \Phi_j = F \circ M_j^{-1}, \quad j = 1, 2. \) Note that \( \Phi_1(0) = W_1. \) Let \( \lambda_1, \ldots, \lambda_{\tau} \) be the distinct eigenvalues of \( W_1 \) and define \( m_1(\lambda) := \) the multiplicity of the factor \( (\lambda - \lambda_j) \) in the minimal polynomial of \( W_1. \) Define
\[ B_1(\zeta) := \prod_{j=1}^{r} \left( \frac{\zeta - \lambda_j}{1 - \lambda_j \zeta} \right)^{m_1(\lambda_j)}, \quad \zeta \in D. \]

Applying Lemma 2.2 we get
\[ (2.5) \quad \left| \frac{\zeta_1 - \zeta_2}{1 - \zeta_2 \zeta_1} \right| = |M_1(\zeta_2)| \geq \prod_{j=1}^{r} \left| \frac{\mu - \lambda_j}{1 - \lambda_j \mu} \right|^{m_1(\lambda_j)} \geq \text{dist}_{\mathcal{M}}(\mu; \sigma(W_1))^{d_1} \quad \forall \mu \in \sigma(\Phi_1(M_1(\zeta_2))) = \sigma(W_2). \]

Now, swapping the roles of \( \zeta_1 \) and \( \zeta_2 \) and applying the same argument to
\[ B_2(\zeta) := \prod_{j=1}^{s} \left( \frac{\zeta - \mu_j}{1 - \mu_j \zeta} \right)^{m_2(\mu_j)}, \quad \zeta \in D, \]
where \( \mu_1, \ldots, \mu_s \) are the distinct eigenvalues of \( W_2 \) and \( m_2(\mu) := \) the multiplicity of the factor \( (\lambda - \mu_j) \) in the minimal polynomial of \( W_2, \) we get
\[ (2.6) \quad \left| \frac{\zeta_1 - \zeta_2}{1 - \zeta_2 \zeta_1} \right| \geq \text{dist}_{\mathcal{M}}(\lambda; \sigma(W_2))^{d_2} \quad \forall \lambda \in \sigma(W_1). \]

Combining (2.5) and (2.6), we get
\[ \max \left\{ \max_{\mu \in \sigma(W_1)} \text{dist}_{\mathcal{M}}(\mu; \sigma(W_1))^{d_1}, \max_{\lambda \in \sigma(W_1)} \text{dist}_{\mathcal{M}}(\lambda; \sigma(W_2))^{d_2} \right\} \leq \left| \frac{\zeta_1 - \zeta_2}{1 - \zeta_2 \zeta_1} \right|. \]

In order to prove the sharpness of (1.3), fix an \( n \geq 2, \) and choose any \( z, w \in D. \) Next, define \( M(\zeta) := (\zeta - z)(1 - \pi \zeta)^{-1}. \) Pick any \( d = 1, \ldots, n, \) and define
\[ N_d(\zeta) := \begin{cases} [M(\zeta)], & \text{if } d = 1, \\ \begin{bmatrix} 0 & M(\zeta) \\ 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 \end{bmatrix}_{d \times d}, & \text{if } d \geq 2, \end{cases} \]
and, for the chosen \( d, \) define \( \mathcal{F}^{z, w} \) by the following block-diagonal matrix
\[ \mathcal{F}^{z, w}(\zeta) := \begin{bmatrix} N_d(\zeta) & M(\zeta) I_{n-d} \end{bmatrix} \quad \forall \zeta \in D. \]

Note that
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\( F(z,w) \) is nilpotent of degree \( d \), whence \( d(z) = d \) and

- Since \( |M(w)|^{1/d} > |M(w)| \),

\[
\max_{\mu \in \sigma(F^z,w(z))} [\text{dist}_M(\mu; \sigma(F^z,w(z))^d(z))] = |M(w)| = \frac{|z - w|}{1 - wz}.
\]

A similar argument yields

\[
\max_{\lambda \in \sigma(F^z,w(z))} [\text{dist}_M(\lambda; \sigma(F^z,w(z))^d(w))] = |M(w)|^{d+1} = \frac{|z - w|^{d+1}}{1 - wz}.
\]

Hence, we have the equality

\[
\max \left\{ \max_{\mu \in \sigma(F^z,w(z))} [\text{dist}_M(\mu; \sigma(F^z,w(z))^d(z))], \max_{\lambda \in \sigma(F^z,w(z))} [\text{dist}_M(\lambda; \sigma(F^z,w(z))^d(w))] \right\} = \frac{|z - w|}{1 - wz}.
\]

\( F^z,w \) is therefore the desired map that establishes the sharpness of (1.4). \( \square \)

3. The Proof of Theorem 1.4

In order to prove Theorem 1.4, we shall need the following elementary

**Lemma 3.1.** Given a Möbius transformation \( T(z) := (az + b)/(cz + d) \), if \( T(\partial D) \equiv \mathbb{C} \), then \( T(\partial D) \) is a circle with

\[
\text{centre}(T(\partial D)) = \frac{bd - ac}{|d|^2 - |c|^2}, \quad \text{radius}(T(\partial D)) = \frac{|ad - bc|}{|d|^2 - |c|^2}.
\]

We are now in a position to present

3.2. The proof of Theorem 1.4 Let \( G \in O(\Omega_n; \Omega_n) \) and let \( \lambda_1, \ldots, \lambda_s \) be the distinct eigenvalues of \( G(0) \). Define \( m(j) := \text{the multiplicity of the factor } (\lambda - \lambda_j) \) in the minimal polynomial of \( G(0) \). Define the Blaschke product

\[
B_G(\zeta) := \prod_{j=1}^s \left( \frac{\zeta - \lambda_j}{1 - \lambda_j \zeta} \right)^{m(j)}, \quad \zeta \in D.
\]

\( B_G \) induces the following matrix function which, by a mild abuse of notation, we shall also denote as \( B_G \)

\[
B_G(Y) := \prod_{j=1}^s (\mathbb{I} - \lambda_j Y)^{-m(j)} (Y - \lambda_j \mathbb{I})^{m(j)} \quad \forall Y \in \Omega_n,
\]

which is well-defined on \( \Omega_n \) precisely as explained in the proof of Lemma 2.2. Once again, owing to the analyticity of \( B_G \) on \( D \),

\[
\sigma(B_G(Y)) = \{ B_G(\lambda) : \lambda \in \sigma(Y) \} \quad \forall Y \in \Omega_n,
\]

whence \( B_G : \Omega_n \to \Omega_n \). Therefore, if we define

\[
H(X) := B_G \circ G(X) \quad \forall X \in \Omega_n,
\]

\[
\sigma(B_G(Y)) = \{ B_G(\lambda) : \lambda \in \sigma(Y) \} \quad \forall Y \in \Omega_n,
\]

where \( \sigma(B_G(Y)) \) is the spectrum of \( B_G(Y) \).
then \( H \in O(\Omega_n; \Omega_n) \) and, by construction, \( H(0) = 0 \). By the Ransford-White result, \( r(H(X)) \leq r(X) \), or, more precisely

\[
\max_{\mu \in G(X)} \left\{ \prod_{j=1}^{s} \left| \frac{\mu - \lambda_j}{1 - \lambda_j \mu} \right|^{m(j)} \right\} \leq r(X) \quad \forall X \in \Omega_n.
\]

In particular:

\[
\max_{\mu \in G(X)} \left[ \text{dist}_M(\mu; \sigma(G(0))) \right] \leq r(X) \quad \forall X \in \Omega_n.
\]

For the moment, let us fix \( X \in \Omega_n \). For each \( \mu \in \sigma(G(X)) \), let \( \lambda_\mu \) be an eigenvalue of \( G(0) \) such that \(|\mu - \lambda_\mu| (1 - \lambda_\mu)^{-1}| = \text{dist}_M(\mu; \sigma(G(0)))\). Now fix \( \mu \in \sigma(G(X)) \). The above inequality leads to

\[
(3.1) \quad \left| \frac{\mu - \lambda_\mu}{1 - \lambda_\mu \mu} \right| \leq r(X)^{1/d_G}.
\]

Applying Lemma 3.1 to the Möbius transformation

\[
T(z) = \frac{|\mu| z - \lambda_\mu}{1 - \lambda_\mu |\mu| z},
\]

we deduce that

\[
\left| \frac{\zeta - \lambda_\mu}{1 - \lambda_\mu \zeta} \right| \geq \left| \frac{|\mu| - |\lambda_\mu|}{1 - |\mu||\lambda_\mu|} \right| \quad \forall \zeta : |\zeta| = |\mu|.
\]

Applying the above fact to (3.1), we get

\[
(3.2) \quad \frac{|\mu| - |\lambda_\mu|}{1 - |\mu||\lambda_\mu|} \leq r(X)^{1/d_G}
\]

\[
\Rightarrow |\mu| \leq \frac{r(X)^{1/d_G} + r(G(0))}{1 + |\lambda_\mu| r(X)^{1/d_G}}, \quad \mu \in \sigma(G(X)).
\]

Note that the function

\[
t \mapsto \frac{r(X)^{1/d_G} + t}{1 + r(X)^{1/d_G} t}, \quad t \geq 0,
\]

is an increasing function on \([0, \infty)\). Combining this fact with (3.2), we get

\[
|\mu| \leq \frac{r(X)^{1/d_G} + r(G(0))}{1 + r(G(0)) r(X)^{1/d_G}},
\]

which holds \( \forall \mu \in \sigma(G(X)) \), while the right-hand side is independent of \( \mu \). Since this is true for any arbitrary \( X \in \Omega_n \), we conclude that

\[
r(G(X)) \leq \frac{r(X)^{1/d_G} + r(G(0))}{1 + r(G(0)) r(X)^{1/d_G}} \quad \forall X \in \Omega_n.
\]

In order to prove the sharpness of (1.4), let us fix an \( n \geq 2 \), and define

\[
\mathcal{S}_n := \{ A \in \Omega_n : A \text{ has a single eigenvalue of multiplicity } n \}.
\]
Pick any $d = 1, \ldots, n$, and define
\[
M_d(X) := \begin{cases}
[\frac{\text{tr}(X)}{n},]
\begin{bmatrix}
0 & \frac{\text{tr}(X)}{n} \\
1 & 0 \\
\vdots & \vdots \\
1 & 0
\end{bmatrix}, & \text{if } d = 1,
\end{cases}
\]
and, for the chosen $d$, define $\mathfrak{S}^{(d)}$ by the following block-diagonal matrix
\[
\mathfrak{S}^{(d)}(X) := \left[ M_d(X) \frac{\text{tr}(X)}{n} I_{n-d} \right] \quad \forall X \in \Omega_n.
\]
For our purposes $\mathfrak{S}^{A,d} = \mathfrak{S}^{(d)}$ for each $A \in \mathfrak{S}_n$, i.e., the equality (1.6) will hold with the same function for each $A \in \mathfrak{S}_n$. To see this, note that
- $r(\mathfrak{S}^{(d)}(X)) = |\text{tr}(X)/n|^{1/d}$; and
- $\mathfrak{S}^{(d)}(0)$ is nilpotent of degree $d$, whence $d_{\mathfrak{S}^{(d)}} = d$.

Therefore,
\[
\frac{r(A)^{1/d} + r(\mathfrak{S}^{(d)}(0))}{1 + r(\mathfrak{S}^{(d)}(0))} = r(A)^{1/d} = r(\mathfrak{S}^{(d)}(A)) \quad \forall A \in \mathfrak{S}_n,
\]
which establishes (1.6). \hfill \Box

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