LOGARITHMIC BLOCH SPACES IN THE POLYDISC, ENDPOINT RESULTS FOR HANKEL OPERATORS AND POINTWISE MULTIPLIERS

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ABSTRACT. We define two notions of Logarithmic Bloch space in the polydisc for which we provide equivalent definitions in terms of symbols of bounded Hankel operators. We also provide a full characterization of the pointwise multipliers between two different Bloch spaces of the unit polydisc.

1. Introduction

For $0 < p < \infty$, the Bergman space $A^p(D)$ of the unit disc $D$ of the complex plane $\mathbb{C}$ consists of all holomorphic functions $f$ on $D$ such that

$$\|f\|_p := \left( \int_D |f(z)|^p d\nu(z) \right)^{1/p} < \infty$$

where $\nu$ is the normalized Lebesgue measure on $D$. The orthogonal projection from $L^2(D)$ onto its closed subspace $A^2(D)$ is called the Bergman projection and denoted $P$.

For a bounded holomorphic function on $D$, the Hankel operator with symbol $b$ is the operator defined for any integrable function $f$ on $D$ by

$$h_b(f) := P(bf).$$

Recall that a holomorphic function $f$ on $D$ is said to be a Bloch function if

$$\sup_{z \in D} (1 - |z|^2)|f'(z)| < \infty.$$ 

The set of all Bloch functions is called the Bloch space and denoted $\mathcal{B}(D)$. It is a Banach space when endowed with the following norm

$$\|f\|_{\mathcal{B}} := |f(0)| + \sup_{z \in D} (1 - |z|^2)|f'(z)|.$$

The Bloch space in one-parameter can be identified as the dual space of the Bergman space $A^1(D)$ (see [13]). Equivalent definitions of this space are also given in terms of symbols of bounded Hankel operators on the Bergman spaces $A^p$, $1 < p < \infty$ (see for example [4]) and image of $L^\infty(D)$ by the Bergman projection (see [13]). These equivalent characterizations of the Bloch space in one-parameter extend to higher-parameter [3, 9, 12].

Also in the unit disc, an analytic function $b$ is a multiplier of the Bloch space if and only if it is bounded and satisfies the following Bloch-type condition

$$\sup_{z \in D} (1 - |z|^2) \left( \log \left( \frac{2}{1 - |z|^2} \right) \right) |f'(z)| < \infty.$$
The space of all holomorphic functions satisfying the last condition is the
sometimes called logarithmic Bloch space and denoted $B_L(\mathbb{D})$ (more often
$B_{\log}(\mathbb{D})$). We endowed it with the norm

$$
\|f\|_{B_L} := |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2) \left( \log \left( \frac{2}{1 - |z|^2} \right) \right) |f'(z)|.
$$

It is also known that $B_L(\mathbb{D})$ is the exact range of symbols of bounded Hankel
operators on $A^1(\mathbb{D})$ (see for example [2]).

In the multi-parameter case, i.e. the polydisc $\mathbb{D}^n$, there is one Bloch space
that corresponds to the dual space of the Bergman space $A^1(\mathbb{D}^n)$ defined in
the next section, and another Bloch space which is a subspace of the first one
and can be obtained using the definition of a Bloch space in several complex
variables by R. Timoney [10]. For each of these Bloch spaces, we introduce
a corresponding logarithmic Bloch space for which we provide an equivalent
definition in terms of set of symbols of bounded Hankel operators. We also
characterize the multipliers algebra of the Bloch space corresponding to the
dual of $A^1(\mathbb{D}^n)$ and the pointwise multipliers from the smaller Bloch space
to the latter.

2. Function spaces

Recall that for $0 < p < \infty$, the Bergman space $A(\mathbb{D}^n)$ consists of analytic
functions $f$ in $\mathbb{D}^n$ such that

$$
\|f\|_p := \int_{\mathbb{D}^n} |f(z)|^p d\nu(z) < \infty,
$$

here $d\nu(z) = d\nu_1(z_1) \cdots d\nu_n(z_n)$ for $z = (z_1, \ldots, z_n)$, where $d\nu_j$ is the
normalized Lebesgue measure on the unit disc $\mathbb{D}$. That is $A^p(\mathbb{D}^n)$ is the
subspace of the Lebesgue space $L^p = L^p(\mathbb{D}^n, d\nu)$ consisting of analytic
functions. In particular the space $A^2(\mathbb{D}^n)$ is a reproducing kernel Hilbert space,
that is any $f \in A^2(\mathbb{D}^n)$ admits the representation

$$
f(z) = P(f)(z) = \langle f, B(\cdot, z) \rangle = \int_{\mathbb{D}^n} f(w) B(z, w) d\nu(z), \quad \text{for any } z \in \mathbb{D}^n,
$$

where the (weighted Bergman) kernel $B(\cdot, \cdot)$ is given by

$$
B(z, w) := \frac{1}{\prod_{j=1}^n (1 - \bar{w}_j z_j)^2}.
$$

$P$ is in fact the orthogonal projection from $L^2(\mathbb{D}^n)$ onto its closed subspace
$A^2(\mathbb{D}^n)$ and it is called the Bergman projection.

We denote by $H(\mathbb{D}^n)$ the space of all analytic functions in $\mathbb{D}^n$. $H^\infty(\mathbb{D}^n)$
is the set of all bounded analytic functions in $\mathbb{D}^n$ that is $f \in H^\infty(\mathbb{D}^n)$ if
$f \in H(\mathbb{D}^n)$ and

$$
\|f\|_\infty := \sup_{z \in \mathbb{D}^n} |f(z)| < \infty.
$$

For $j \in \{1, \ldots, n\}$, we consider the operator $D_j$ defined for $f \in H(\mathbb{D}^n)$ by

$$
D_j f(z) = 2f(z) + z_j \frac{\partial f}{\partial z_j}(z) = (2I + R_j)(f),
$$

where $I$ stands for the identity operator. We put $D = D_1 \cdots D_n$. 
The Bloch space of the polydisc $\mathbb{D}^n$ is denoted $\mathcal{B}(\mathbb{D}^n)$ and consists of all analytic functions $f$ such that

$$
\|f\|_B := |f(0)| + \sup_{z \in \mathbb{D}^n} \left[ \prod_{j=1}^n (1 - |z_j|^2) \right] |Df(z)| < \infty.
$$

The next space is called by several authors the Bloch space but here we name it the pointwise Bloch space to avoid any confusion with the Bloch space defined above. A function $f$ analytic in $\mathbb{D}^n$ belongs to the pointwise Bloch space $B_{wB}(\mathbb{D}^n)$ if

$$
\|f\|_{B_{wB}} := |f(0)| + \sup_{z \in \mathbb{D}^n} \left[ \prod_{j=1}^n (1 - |z_j|^2) \right] \left| \frac{\partial f}{\partial z_j}(z) \right| < \infty.
$$

Note that a function is in the pointwise Bloch space if and only if it is a one parameter Bloch function in each variable. This justifies our choice of the name for this space.

Let us introduce some other spaces of analytic functions in $\mathbb{D}^n$. We start with the logarithmic Bloch space $B_L(\mathbb{D}^n)$ which consists of analytic functions $f$ in $\mathbb{D}^n$ such that

$$
\|f\|_{B_L} := |f(0)| + \sup_{z \in \mathbb{D}^n} \left[ \prod_{j=1}^n (1 - |z_j|^2) \log \frac{2}{1 - |z_j|^2} \right] |Df(z)| < \infty.
$$

The above notion extends the notion of logarithmic Bloch space of the unit disc.

When it comes to the pointwise Bloch space, we also have a notion of logarithmic Bloch space, $B_{wB}(\mathbb{D}^n)$. An analytic function $f$ belongs to $B_{wB}(\mathbb{D}^n)$ if

$$
\|f\|_{B_{wB}} := |f(0)| + \sup_{z \in \mathbb{D}^n} \left[ \prod_{j=1}^n (1 - |z_j|^2) \log \frac{2}{1 - |z_j|^2} \right] \left| \frac{\partial f}{\partial z_j}(z) \right| < \infty.
$$

Our last space is the space $B_{Ld}(\mathbb{D}^n)$ defined as the space of all analytic functions $f$ in $\mathbb{D}^n$ such that there is a constant $C > 0$ such that for any $K = \{k_1, \ldots, k_l\} \subseteq \{1, \ldots, n\}$,

$$
|f(0)| + \sup_{z \in \mathbb{D}^n} \left[ \prod_{j=1}^n (1 - |z_j|^2) \log \frac{2}{1 - |z_j|^2} \right] \left| \frac{\partial^{k_1} \cdots \partial^{k_l} f}{\partial^{k_1} \cdots \partial^{k_l} z}(z) \right| \leq C.
$$

The smallest constant in the above definition is denoted $\|f\|_{B_{Ld}}$.

Note that a function $f$ belongs to $B_{Ld}(\mathbb{D}^n)$ if and only if $f \in B_{LL}(\mathbb{D}^n)$ and for any $w \in \mathbb{D}^k$ fixed ($1 \leq k < n$), the function $f(\cdot, w)$ which is a function of $n - k$ variables, is uniformly in $B_{LL}(\mathbb{D}^{n-k})$.

3. Statement of the results

We recall that given $b \in A^2(\mathbb{D}^n)$, the (small) Hankel operator with symbol $b$, $h_b$ is the operator defined for $f \in H^\infty(\mathbb{D}^n)$ by

$$
h_b(f) := P(b \overline{f}).
$$
Our first result says that the space $B_L(D^n)$ is the exact range of symbols of bounded Hankel operators on $A^1(D^n)$.

**Theorem 3.1.** Let $b \in H(D^n)$. Then the Hankel operator $h_b$ extends as a bounded operator on $A^1(D^n)$ if and only if $b \in B_L(D^n)$.

We also obtain that $B_L(D^n)$ is the exact range of symbols of bounded Hankel operators from $B(D^n)$ to $B(D^n)$.

**Theorem 3.2.** Let $b \in H(D^n)$. Then the Hankel operator $h_b$ extends as a bounded operator from $B(D^n)$ to $B(D^n)$ if and only if $b \in B_L(D^n)$.

Given two Banach spaces of analytic functions $X$ and $Y$, the set of pointwise multipliers from $X$ to $Y$ is defined by

$$\mathcal{M}(X, Y) := \{g \in H(D^n) : fg \in Y, \text{ for any } f \in X\}.$$ 

When $X = Y$, we just write $\mathcal{M}(X)$ for $\mathcal{M}(X, X)$. The norm of the multiplication operator by $\phi$ from $X$ to $Y$ is denoted $\|M_\phi\|_{X \to Y}$ or $\|\phi\|_{X \to Y}$.

Our first main result on pointwise multipliers is the following.

**Theorem 3.3.** Let $\phi \in H(D^n)$. Then $\phi$ is a multiplier from $B(D^n)$ to $B(D^n)$ if and only if $\phi \in H^\infty(D^n) \cap B_L(D^n)$. Moreover,

$$\|\phi\|_{B \to B} \leq \|\phi\|_\infty + \|\phi\|_{B_L}.$$ 

Here is our characterization of the pointwise multipliers of $B(D^n)$.

**Theorem 3.4.** Let $\phi \in H(D^n)$. Then $\phi$ is a multiplier of $B(D^n)$ if and only if

$$\phi \in H^\infty(D^n) \cap B_{LL}(D^n).$$ 

Moreover,

$$\|\phi\|_{B \to B} \leq \|\phi\|_\infty + \|\phi\|_{B_{LL}}.$$ 

In Section 4 we give some useful properties of $B(D^n)$ and their logarithmic counterparts. The proofs of Theorem 3.1 and Theorem 3.2 are given in Section 4.2. In Section 4.3 we prove our results on multipliers from $B(D^n)$ to $B(D^n)$ and the multipliers of $B(D^n)$. In the last section of this paper, we add some comments and remarks.

Finally, all over the text, $C$ will be a constant not necessarily the same at each occurrence. We will also use the notation $C_k$ to express the fact that the constant depends on the underlined parameter. Given two positive quantities $A$ and $B$, the notation $A \lesssim B$ means that $A \leq CB$ for some positive constant $C$. When $A \lesssim B$ and $B \lesssim A$, we write $A \simeq B$.

4. **Useful results on the Bloch spaces**

4.1. **The Bloch space of the unit disc.** On the unit disc $D$ of the complex plane $\mathbb{C}$, the Bloch space $B = B(D)$ consists of analytic functions $f$ such that

$$\sup_{z \in D}(1 - |z|^2)|f'(z)| < \infty. \quad (9)$$

The following norm makes $B(D)$ a Banach space:

$$\|f\|_B := |f(0)| + \sup_{z \in D}(1 - |z|^2)|f'(z)| < \infty. \quad (10)$$
The Bloch space strictly contains the space $H^\infty(D)$. A typical example of function in the Bloch space that does not belong to $H^\infty(D)$, is the function $f(z) = \log(1 - az)$ which is uniformly in $B(D)$. That is its $B$-norm is bounded by a constant that does not depend on the complex number $a$ (see [8]).

Note that in the above definition, $f'(z)$ can be replaced by $Rf(z) = zf'(z)$ and that equivalent norms are obtained by considering any derivative of higher order. That is for any integer $k \geq 1$,

$$
\|f\|_B \simeq |f(0)| + \sup_{z \in D} (1 - |z|^2)^k |f^{(k)}(z)|
$$

In general, higher order derivatives can be replaced by the so-called fractional derivatives (for a definition, see [13]). In particular, if for the integer $k \geq 1$ we define the operator

$$
D^k = [(1 + k)I + R] \cdots [2I + R]
$$

where $I$ is the identity, then

$$
\|f\|_B \simeq |f(0)| + \sup_{z \in D} (1 - |z|^2)^k |D^k f(z)|
$$

(see [2]). Let us still denote by $P$ the orthogonal projection from $L^2(D)$ onto its closed subspace $A^2(D)$. For $b \in A^2(D)$, we densely defined the (small) Hankel operator with symbol $b$ on $A^2(D)$ by

$$
h_b(f) := P(b\overline{f}).
$$

There are some other equivalent characterizations of the Bloch space (see [2, 13]).

**Proposition 4.1.** Let $b$ be an analytic function in the unit disc $D$. Then the following are equivalent.

(i) $b \in B(D)$;

(ii) $b = Pg$ for some $g \in L^\infty(D)$;

(iii) $b$ belongs to the dual space $(A^1(D))^*$ of $A^1(D)$ under the pairing

$$
\langle f, g \rangle := \lim_{r \to 1} \int_D f(rz)\overline{g(z)}d\nu(z);
$$

(iv) The Hankel operator $h_b$ is bounded on $A^2(D)$.

To deal with multipliers of $B(D)$, we recall that the logarithmic counterpart of the Bloch space called the logarithmic Bloch space and denoted $B_L(D)$, consists of all holomorphic functions $f$ in $D$ such that

$$
\|f\|_{B_L} \simeq |f(0)| + \sup_{z \in D} (1 - |z|^2) \left( \log \frac{2}{1 - |z|^2} \right) |f'(z)| < \infty.
$$

Remark that the same observations made on $B$ about higher order derivatives also work for the logarithmic Bloch space (see for example [2]). That
The proof is complete.

We test this inequality with the function \( f \) in the Bloch space. Above proposition, is the following pointwise estimate of functions in the Bloch space in the unit disc.

\[ \|f\|_{B_L} \simeq |f(0)| + \sup_{z \in \mathbb{D}}(1 - |z|^2)^k \left( \log \frac{2}{1 - |z|^2} \right) |f^{(k)}(z)| \]
\[ \simeq |f(0)| + \sup_{z \in \mathbb{D}}(1 - |z|^2)^k \left( \log \frac{2}{1 - |z|^2} \right) |R^k f(z)| \]
\[ \simeq |f(0)| + \sup_{z \in \mathbb{D}}(1 - |z|^2)^k \left( \log \frac{2}{1 - |z|^2} \right) |D^k f(z)|. \]

One has the following characterization of the multiplier algebra of the Bloch space.

**Proposition 4.2.** Let \( b \in \mathcal{H}(\mathbb{D}) \). Then \( b \in \mathcal{M}(\mathcal{B}(\mathbb{D})) \) if and only if
\[ b \in H^\infty(\mathbb{D}) \cap B_L(\mathbb{D}). \]

Moreover,
\[ \|M_b\|_{\mathcal{B}} \simeq \|b\|_{\infty} + \|b\|_{B_L}. \]

**Proof.** The proof is quite standard, we give it here as it will guide us along the text. One thing that one needs to know to prove the sufficiency in the above proposition, is the following pointwise estimate of functions in the Bloch space.
\[ |f(z)| \leq C \left( \log \frac{2}{1 - |z|^2} \right) \|f\|_{\mathcal{B}}. \]

The above pointwise estimate can be combined with the fact that the function \( \log(1 - \pi z) \) is uniformly in \( \mathcal{B}(\mathbb{D}) \) to prove that any element of \( \mathcal{M}(\mathcal{B}(\mathbb{D})) \) is bounded (see [3, 11]).

Suppose that \( b \) satisfies the condition in the proposition. Then for any \( f \in \mathcal{B}(\mathbb{D}) \) and any \( z \in \mathbb{D}, \)
\[ (1 - |z|^2) |(f b)'(z)| = (1 - |z|^2) |f(z)b'(z) + f'(z)b(z)| \leq (1 - |z|^2) |f(z)||b'(z)| + (1 - |z|^2) |f'(z)||b(z)| \leq C(1 - |z|^2) ||b'(z)| \left( \log \frac{2}{1 - |z|^2} \right) \|f\|_{\mathcal{B}} + \|b\|_{\infty}(1 - |z|^2) |f'(z)| \leq C \|f\|_{\mathcal{B}} (\|b\|_{\infty} + \|b\|_{B_L}). \]

Now suppose that \( b \) is multiplier of \( \mathcal{B}(\mathbb{D}) \). That is there exists a constant \( C > 0 \) such that for any \( f \in \mathcal{B}(\mathbb{D}) \) and \( z \in \mathbb{D}, \)
\[ (1 - |z|^2) |(f b)'(z)| = (1 - |z|^2) |f(z)b'(z) + f'(z)b(z)| \leq C \|f\|_{\mathcal{B}}. \]

We test this inequality with the function \( f(z) = f_a(z) = \log(1 - \pi z), a \in \mathbb{D} \) fixed. It comes that
\[ (1 - |z|^2) \log(1 - \pi z)b'(z) + \frac{\pi}{1 - \pi z}b(z) \leq C. \]

Putting \( z = a \), it comes since \( b \in H^\infty(\mathbb{D}) \) that for any \( z \in \mathbb{D}, \)
\[ (1 - |z|^2) \left( \log \frac{2}{1 - |z|^2} \right) |b'(z)| \leq C < \infty. \]

The proof is complete. \( \square \)
4.2. The product Bloch space and its logarithmic counterpart. For $K = (k_1, \ldots, k_n)$, $k_j \in \mathbb{N}$, we define on $H(D^n)$ the operator $D^K$ given by

$$D^K = D^{k_1} \times \cdots \times D^{k_n},$$

where $D_j^{k_j} f(z) = [(1 + k_j)I + R_j] \ldots [2I + R_j]$ with $R_j f(z) = z_j \frac{\partial f}{\partial z_j}(z)$.

As in the one parameter setting, we obtain equivalent norms on $B(D^n)$ by using higher order derivatives in each variable. More precisely, for a vector $K = (k_1, \cdot \cdot \cdot, k_n) \in \mathbb{N}^n$, we have

$$\|f\|_B \simeq |f(0)| + \sup_{z \in \mathbb{D}^n} \left[ \prod_{j=1}^n (1 - |z_j|^2)^{k_j} \right] |D^K f(z)|$$

Also,

$$\|f\|_{B_L} \simeq |f(0)| + \sup_{z \in \mathbb{D}^n} \left[ \prod_{j=1}^n (1 - |z_j|^2)^{k_j} \log \frac{2}{1 - |z_j|^2} \right] |D^K f(z)|$$

(see also [5, 6]).

Let us observe the following formula that can be proved as in the one parameter situation.

**Lemma 4.3.** Let $f$ and $g$ be two analytic polynomials in $\mathbb{D}^n$. Then for any $K = (k_1, \cdot \cdot \cdot, k_n) \in \mathbb{N}^n$, there exists a constant $C = C_{K,n}$ such that the following formula holds

$$\int_{\mathbb{D}^n} f(z) g(z) d\nu(z) = C \int_{\mathbb{D}^n} f(z) \left(1 - ||z||^2\right)^K D^K g(z) d\nu(z),$$

$$(1 - ||z||^2)^K := \prod_{j=1}^n (1 - |z_j|^2)^{k_j}.$$

The following first equivalent characterization of the Bloch space was obtained by K. Zhu in [12].

**Lemma 4.4.** Let $f$ be holomorphic in $\mathbb{D}^n$. Then the following assertions are equivalent.

(i) $f \in B(\mathbb{D}^n)$.

(ii) There exists a function $g \in L^\infty(\mathbb{D}^n)$ such that

$$f(z) = \int_{\mathbb{D}^n} \frac{g(w)}{\prod_{j=1}^n (1 - z_j \overline{w_j})^2} d\nu(z), \quad z \in \mathbb{D}^n.$$ 

Moreover, $\|f\|_B \simeq \|g\|_\infty$.

We refer also to [3] for the following duality result which provides another equivalent definition of $B(\mathbb{D}^n)$.

**Lemma 4.5.** The topological dual space $\left(A^1(\mathbb{D}^n)\right)^*$ of $A^1(\mathbb{D}^n)$ identifies with $B(\mathbb{D}^n)$ under the duality pairing

$$\langle f, g \rangle := \lim_{r \to 1} \int_{\mathbb{D}^n} f(rz) g(z) d\nu(z), f \in A^1(\mathbb{D}^n), g \in B(\mathbb{D}^n).$$
The proof of this result can be found in [3]. We give an alternative proof here using test functions. Let us first suppose that $b$ is analytic on $\mathbb{D}^n$ and such that for any $f \in A^p(\mathbb{D}^n)$, $pq = p + q$, using the duality in Lemma 4.5, we obtain
\[ |\langle b, f \rangle| = |\langle P(bf), m \rangle| = \|b\|_{B(\mathbb{D}^n)} \|fm\|_1 \leq \|b\|_{B(\mathbb{D}^n)} \|f\|_p \|m\|_q. \]
That is $h_b$ is bounded on $A^p(\mathbb{D}^n)$ as soon as $b$ is as in the statement of the proposition.

Now suppose that $b$ is analytic on $\mathbb{D}^n$ and such that $h_b$ extends as a bounded operator on $A^p(\mathbb{D}^n)$. Note that this means that there is a constant $C > 0$ such that for any $f \in A^p(\mathbb{D}^n)$ and any $g \in A^q(\mathbb{D}^n)$, $pq = p + q$,
\[ |\langle b, f \rangle| \leq C \|f\|_p \|g\|_q. \]

Let $a \in \mathbb{D}^n$ be fixed and put
\[ f(z) = f_a(z) = \prod_{j=1}^n \frac{(1 - |a_j|^2)^{k_j/p}}{(1 - |z_j|^2)^{2 + k_j}/p}, \]
and
\[ g(z) = g_a(z) = \prod_{j=1}^n \frac{(1 - |a_j|^2)^{k_j/q}}{(1 - |z_j|^2)^{(2 + k_j)/q}}, \quad k_j \in \mathbb{N}, \quad j = 1, 2, \ldots, n. \]

Observe that $f$ and $g$ are uniformly in $A^p(\mathbb{D}^n)$ and $A^q(\mathbb{D}^n)$ respectively. Taking $f = f_a$ and $g = g_a$ in (17), we obtain
\[ C \geq |\langle b, fg \rangle| = \left( \prod_{j=1}^n (1 - |a_j|^2)^{k_j} \right) \lim_{r \to 1} \int_{\mathbb{D}^n} \frac{b(rw)}{\prod_{j=1}^n (1 - a_j \overline{w_j})^{2 + k_j}} d\nu(w). \]
That is for any $a \in \mathbb{D}^n$ and any $K = (k_1, \ldots, k_n) \in \mathbb{N}^n$,
\[ \left( \prod_{j=1}^n (1 - |a_j|^2)^{k_j} \right) \left| D^K b(a) \right| \leq C \]
and consequently, $b \in B(\mathbb{D}^n)$. The conclusion then follows from Lemma 4.3. \qed
REMARK 4.8. As observed in [3], the above result is equivalent to saying that any \( f \in A^1(\mathbb{D}^n) \) admits a representation of the form

\[
f(z) = \sum_j f_j(z)g_j(z), \quad z \in \mathbb{D}^n, f_j \in A^p(\mathbb{D}), g_j \in A^q(\mathbb{D}),
\]

with

\[
\sum_{j \in \mathbb{N}_0} \|f_j\|_p \|g_j\|_q \leq C \|f\|_1, \quad pq = p + q, \quad 1 < p < \infty.
\]

4.3. The pointwise Bloch space. Let us start by considering some observations.

**Lemma 4.9.** The following assertions hold

1. The function \( f(z) = \sum_{j=1}^n f_j(z_j) \), with \( f_j \in B(\mathbb{D}) \) belongs to \( B(\mathbb{D}^n) \) and

\[
\|f\|_{B(\mathbb{D}^n)} \leq \sum_{j=1}^n \|f_j\|_{B(\mathbb{D})}.
\]

2. There is a constant \( C > 0 \) such that for any \( f \in B(\mathbb{D}^n) \) and any \( z \in \mathbb{D}^n \),

\[
|f(z)| \leq C \left[ \sum_{j=1}^n \log \frac{2}{1 - |z_j|^2} \right] \|f\|_B
\]

and this is sharp.

**Proof.** Assertion (1) is direct from the definition of \( B(\mathbb{D}^n) \). Let us prove (2).

We have for any \( z = (z_1, \ldots, z_n) \in \mathbb{D}^n \),

\[
f(z) - f(0) = \int_0^1 \frac{df(sz)}{ds} ds = \sum_{j=1}^n \int_0^1 z_j \frac{\partial f}{\partial z_j}(sz) ds.
\]

It easily follows using the definition of \( B(\mathbb{D}^n) \) that

\[
|f(z)| \leq |f(0)| + \sum_{j=1}^n \int_0^1 |z_j| \left| \frac{\partial f}{\partial z_j}(sz) \right| ds \\
\leq |f(0)| + \|f\|_{B(\mathbb{D}^n)} \sum_{j=1}^n \int_0^1 \frac{|z_j|}{1 - s^2 |z_j|^2} ds \\
\leq C \left( \sum_{j=1}^n \log \frac{2}{1 - |z_j|^2} \right) \|f\|_B.
\]

Sharpness follows by testing (18) with the function \( f(z) = f_\alpha(z) = \sum_{j=1}^n \log(1 - a_j z_j) \). \( \square \)

**Remark 4.10.** Let us observe that \( B(\mathbb{D}^n) \) is a strict subspace of \( B(\mathbb{D}^n) \). To see this, one only needs to observe that for \( f_j \in B(\mathbb{D}), j = 1, \cdots, n, \) the
tensor product

\[(f_1 \otimes f_2 \otimes \cdots \otimes f_n)(z_1, \cdots, z_n) = \prod_{j=1}^{n} f_j(z_j)\]

belongs to \(\mathcal{B}(\mathbb{D}^n)\) with

\[\|f_1 \otimes f_2 \otimes \cdots \otimes f_n\|_{\mathcal{B}(\mathbb{D}^n)} \leq \prod_{j=1}^{n} \|f_j\|_{\mathcal{B}(\mathbb{D})}\]

while \(f_1 \otimes f_2 \otimes \cdots \otimes f_n\) belongs to \(\mathcal{B}(\mathbb{D}^n)\) only if each \(f_j\) belongs to \(H^{\infty}(\mathbb{D})\), \(j = 1, 2, \cdots, n\).

To \(\tilde{j} = (j_1, \ldots, j_l)\), \(1 \leq l \leq n\), we associate the set \(J = \{j_1, \ldots, j_l\} \subseteq \{1, \ldots, n\}\). We denote by \(D_{\tilde{j}}\) the differential operator defined by

\[D_{\tilde{j}}f(z) = D_{j_1} \cdots D_{j_l}f(z)\]

We observe the following.

**Lemma 4.11.** Let \(\tilde{j} = (j_1, \ldots, j_l)\), \(1 \leq l \leq n\), \(J = \{j_1, \ldots, j_l\} \subseteq \{1, \ldots, n\}\) be given. Then for any \(f \in \mathcal{B}(\mathbb{D}^n)\),

\[
\sup_{z \in \mathbb{D}^n} \left[ \prod_{j \in J} (1 - |z_j|^2) \right] |D_{\tilde{j}}f(z)| \lesssim \|f\|_{\mathcal{B}}.
\]

**Proof.** We can suppose without loss of generality that \(\tilde{j} = (1, \ldots, l)\), \(1 \leq l \leq n\), so that \(J = \{1, \ldots, l\}\). For \(z = (z_1, \ldots, z_n)\), we set \(w = (z_{l+1}, \ldots, z_n) \in \mathbb{D}^{n-l}\). We observe that for \(w \in \mathbb{D}^{n-l}\) fixed, the function \(g = f(\cdot, w)\) is uniformly in \(\mathcal{B}(\mathbb{D}^l)\) whenever \(f \in \mathcal{B}(\mathbb{D}^n)\) with \(\|g\|_{\mathcal{B}(\mathbb{D}^l)} \leq \|f\|_{\mathcal{B}(\mathbb{D}^n)}\). Hence, \(g = f(\cdot, w)\) is uniformly in \(\mathcal{B}(\mathbb{D}^l)\) with

\[
\sup_{a \in \mathbb{D}^l} \left[ \prod_{k=1}^{l} (1 - |a_k|^2) \right] |D_{\tilde{j}}f(a, w)| \leq \|g\|_{\mathcal{B}(\mathbb{D}^l)} \leq \|f\|_{\mathcal{B}(\mathbb{D}^n)}.
\]

This proves that (19) holds. \(\square\)

Let us observe that as for the space \(B_L(\mathbb{D}^n)\), we have the following equivalent definition for the logarithmic Bloch-type space that we have denoted \(B_L(\mathbb{D}^n)\).

\[\|f\|_{B_L} \asymp |f(0)| + \sup_{z \in \mathbb{D}^n} \left[ \prod_{j=1}^{n} (1 - |z_j|^2)^{k_j} \right] \left[ \sum_{j=1}^{n} \log \frac{2}{1 - |z_j|^2} \right] |D^K f(z)| < \infty,\]

\(K = (k_1, \cdots, k_n) \in \mathbb{N}^n\).

5. Endpoint results for Hankel operators

We now prove Theorem 3.1. This provides an equivalent definition of \(B_L(\mathbb{D}^n)\) in terms of symbols of bounded Hankel operators on \(A^1(\mathbb{D}^n)\) as in the one parameter case (see for example [2]). We will then be calling \(B_L(\mathbb{D}^n)\) the product logarithmic Bloch space.
Proof of Theorem 3.1. We start by the easy part which is the sufficiency. What we would like to prove is that giving any $f \in A^1(\mathbb{D}^n)$, the function $h_b(f)$ belongs to $A^1(\mathbb{D}^n)$ under the condition that $b \in B_L(\mathbb{D}^n)$ or equivalently that

$$\|h_b(f)\|_1 \leq C\|f\|_1$$

with $C$ not depending on $f$. To prove this, we observe with the help of Lemma 4.3 that we have in particular that

$$h_b(f)(z) = \int_{\mathbb{D}^n} \frac{b(w)f(w)}{\prod_{j=1}^n (1 - \frac{w_j}{z_j})^2} d\nu(w)$$

$$= C_{n,K} \int_{\mathbb{D}^n} \frac{|(1 - \|w\|^2)^K D^K b(w)f(w)|}{\prod_{j=1}^n (1 - \frac{w_j}{z_j})^2} d\nu(w),$$

for any $K \in \mathbb{N}^n$. It follows using this observation and [7, Proposition 1.4.10] that

$$\|h_b(f)\|_1 = \int_{\mathbb{D}^n} |h_b(f)(z)| d\nu(z)$$

$$\leq C \int_{\mathbb{D}^n} \left| \int_{\mathbb{D}^n} \frac{(1 - \|w\|^2)^K D^K b(w)f(w)}{\prod_{j=1}^n (1 - \frac{w_j}{z_j})^2} d\nu(w) \right| d\nu(z)$$

$$\leq C \int_{\mathbb{D}^n} (1 - \|w\|^2)^K |D^K b(w)||f(w)| \left( \int_{\mathbb{D}^n} \prod_{j=1}^n \frac{d\nu(z)}{|1 - \frac{w_j}{z_j}|^2} \right) d\nu(w)$$

$$\leq C \|b\|_{B_L} \int_{\mathbb{D}^n} |f(w)| d\nu(w)$$

$$= C \|b\|_{B_L} \|f\|_1.$$  

The converse is equivalent to saying that if $b$ is such that there exists a constant $C > 0$ so that for any $f \in A^1(\mathbb{D}^n)$ and any $g \in B(\mathbb{D}^n)$,

$$|\langle b, fg \rangle| \leq C\|f\|_1\|g\|_B,$$

then $b \in B_L(\mathbb{D}^n)$. We will need the following lemma.

**Lemma 5.1.** Let $w_j, z_j \in \mathbb{D}$, $j = 1, \ldots, n$ be given. The following formula holds.

$$\prod_{j=1}^n \log(1 - \frac{w_j}{z_j}) = \sum_{L \subseteq \{1, \ldots, n\}} (-1)^{|L|+1} T_L + \prod_{j=1}^n \log(1 - |w_j|^2)$$

where

$$T_L = \left( \prod_{j \in L} \log \frac{1 - \frac{w_j}{z_j}}{1 - |w_j|^2} \right) \left( \prod_{j \in \bar{L}} \log(1 - \frac{w_j}{z_j}) \right),$$

$L$ being the cardinality of the set $L$ and $\bar{L}$ its complementary in $\{1, \ldots, n\}$. 

Proof. One easily checks that for \( n = 2 \), we have
\[
\prod_{j=1}^{2} \log(1 - w_{j}z_{j}) = \log \left( \frac{1 - w_{1}z_{1}}{1 - |w_{1}|^2} \right) \log(1 - w_{2}z_{2}) + \log \left( \frac{1 - w_{2}z_{2}}{1 - |w_{2}|^2} \right) \log(1 - w_{1}z_{1})
\]
\[- \log \left( \frac{1 - w_{1}z_{1}}{1 - |w_{1}|^2} \right) \log \left( \frac{1 - w_{2}z_{2}}{1 - |w_{2}|^2} \right) + \log(1 - |w_{1}|^2) \log(1 - |w_{2}|^2).
\]

Next we suppose that (21) holds for \( n \geq 2 \) and prove that it then also holds for \( n + 1 \). Using our hypothesis, we obtain
\[
\prod_{j=1}^{n+1} \log(1 - w_{j}z_{j}) = \left( \prod_{j=1}^{n} \log(1 - w_{j}z_{j}) \right) \log(1 - w_{n+1}z_{n+1})
\]
\[- T_{1} + \prod_{j=1}^{n} \log(1 - |w_{j}|^2) \log(1 - w_{n+1}z_{n+1}).
\]

where
\[
T_{1} = \sum_{L \subseteq \{1, \ldots, n\}} (-1)^{|L|+1} \left( \prod_{l \in L} \log \left( \frac{1 - w_{l}z_{l}}{1 - |w_{l}|^2} \right) \right) \left( \prod_{l \in L} \log(1 - w_{l}z_{l}) \right) \log(1 - w_{n+1}z_{n+1})
\]

Before going ahead, let us deal with the second term in the sum on the right hand side of the above equality. We clearly have using our hypothesis again that
\[
M := \prod_{j=1}^{n} \log(1 - |w_{j}|^2) \log(1 - w_{n+1}z_{n+1})
\]
\[- \prod_{j=1}^{n} \log(1 - |w_{j}|^2) \log \left( \frac{1 - w_{n+1}z_{n+1}}{1 - |w_{n+1}|^2} \right) \log(1 - |w_{j}|^2) \]
\[- \prod_{j=1}^{n+1} \log(1 - |w_{j}|^2) + \prod_{j=1}^{n+1} \log(1 - |w_{j}|^2) \]
\[- T_{2} + T_{3} + \prod_{j=1}^{n+1} \log(1 - |w_{j}|^2),
\]

where
\[
T_{2} = \left( \prod_{j=1}^{n} \log(1 - w_{j}z_{j}) \right) \log \left( \frac{1 - w_{n+1}z_{n+1}}{1 - |w_{n+1}|^2} \right)
\]
and
\[
T_{3} = (-1)^{n} \left( \prod_{l=1}^{n} \log \left( \frac{1 - w_{l}z_{l}}{1 - |w_{l}|^2} \right) \log \left( \frac{1 - w_{n+1}z_{n+1}}{1 - |w_{n+1}|^2} \right)
\]

Taking this into the expansion of \( \prod_{j=1}^{n+1} \log(1 - w_j z_j) \), we obtain

\[
\prod_{j=1}^{n+1} \log(1 - w_j z_j) = T_1 + T_2 + T_3 + \prod_{j=1}^{n+1} \log(1 - |w_j|^2)
\]

\[
= \sum_{L \subseteq \{1, \ldots, n+1\}} (-1)^{|L|+1} \left( \prod_{l \in L} \frac{1 - w_l z_l}{1 - |w_l|^2} \right) \left( \prod_{l \in \mathcal{L}} \log(1 - |w_l|^2) \right)
\]

\[
+ \prod_{j=1}^{n+1} \log(1 - |w_j|^2).
\]

The proof of the lemma is complete.

Coming back to the proof of the necessity part of the theorem, we test \([20]\) with

\[
f(z) = f_w(z) = \prod_{j=1}^{n} \frac{(1 - |w_j|^2)^{k_j}}{(1 - w_j z_j)^{2+k_j}}, \quad k_j \in \mathbb{N}_0,
\]

and

\[
g(z) = g_w(z) = \prod_{j=1}^{n} \log(1 - w_j z_j).
\]

Clearly \(f\) and \(g\) are uniformly in \(A^1(\mathbb{D}^n)\) and \(B(\mathbb{D}^n)\) respectively.

Next, we take \(f_w\) and the expansion of \(g_w\) obtained in Lemma 5.1 into \([20]\) to obtain that

\[
C \geq |\langle b, f g \rangle| = \left| \left( \prod_{j=1}^{n} \log(1 - |w_j|^2) \right) \lim_{r \to 1} \int_{\mathbb{D}^n} \prod_{j=1}^{n} \frac{(1 - |w_j|^2)^{k_j}}{(1 - w_j z_j)^{2+k_j}} b(r z) d\nu(z) + T \right|
\]

where writing

\[
G^L_w(z) = \left( \prod_{l \in L} \frac{1 - w_l z_l}{1 - |w_l|^2} \right) \left( \prod_{l \in \mathcal{L}} \log(1 - |w_l|^2) \right),
\]

\[
T = \lim_{r \to 1} \int_{\mathbb{D}^n} \left( \sum_{L \subseteq \{1, \ldots, n\}} (-1)^{|L|+1} G^L_w(z) \right) \left( \prod_{j=1}^{n} \frac{(1 - |w_j|^2)^{k_j}}{(1 - w_j z_j)^{2+k_j}} \right) b(r z) d\nu(z)
\]

\[
= \sum_{L \subseteq \{1, \ldots, n\}} (-1)^{|L|+1} \lim_{r \to 1} \int_{\mathbb{D}^n} f_w^L(z) g_w^L(z) b(r z) d\nu(z)
\]

where

\[
f_w^L(z) := \left( \prod_{j=1}^{n} \frac{(1 - |w_j|^2)^{k_j}}{(1 - w_j z_j)^{2+k_j}} \right) \left( \prod_{l \in L} \log \frac{1 - w_l z_l}{1 - |w_l|^2} \right)
\]

\[
= \left( \prod_{j \in \mathcal{L}} \frac{(1 - |w_j|^2)^{k_j}}{(1 - w_j z_j)^{2+k_j}} \right) \left( \prod_{l \in L} \frac{1 - |w_l|^2}{1 - |w_l|^2} \right)
\]

and

\[
g_w^L(z) := \prod_{l \in \mathcal{L}} \log(1 - w_l z_l).
\]
Clearly, $g^L_w$ is uniformly in $B(\mathbb{D}^n)$. Observing that in the unit disc of $\mathbb{C}$, the function
\[ \frac{1 - |w_j|^2}{1 - \overline{w_j}z_j} \log \frac{1 - z_j}{1 - w_j} \]
is uniformly in $A^4(\mathbb{D})$ (see [2]), we conclude that $f^L_w$ is also uniformly in $A^1(\mathbb{D}^n)$. Hence, applying (20) to $f^L_w$ and $g^L_w$, we obtain that
\[ |T| \leq \sum_{L \subseteq \{1, \cdots, n\}} |\langle b, f^L_w g^L_w \rangle| \leq C. \]
We conclude that $B 
subseteq B_L(\mathbb{D}^n)$. The proof is complete. 

Let us now prove Theorem 3.2 that provides an equivalent definition of $B_L(\mathbb{D})$ in terms of symbols of bounded Hankel operators.

**Proof of Theorem 3.2** Let us start by the sufficiency. Assume $b \in B_L(\mathbb{D}^n)$. Then for any $f \in B(\mathbb{D}^n)$ and any $g \in A^1(\mathbb{D}^n)$, and for $K = (k_1, \cdots, k_n) \in \mathbb{N}^n$,
\[ |\langle h_b(f), g \rangle| = |\langle b, fg \rangle| \leq C \lim_{r \to 1} \int_{\mathbb{D}^n} \left| (1 - ||w||^2)^K D^K b(w) \overline{f(rw)g(rw)} \right| d\nu(z) \]
\[ \leq C \lim_{r \to 1} \int_{\mathbb{D}^n} (1 - ||w||^2)^K |D^K b(w)||f(rw)||g(rw)| d\nu(w) \]
\[ \leq C \|f\|_{B(\mathbb{D}^n)} \lim_{r \to 1} \int_{\mathbb{D}^n} (1 - ||w||^2)^K \left( \sum_{j=1}^{n} \log \frac{2}{1 - |w_j|^2} \right) |D^K b(w)||g(rw)| d\nu(w) \]
\[ \leq C \|f\|_{B(\mathbb{D}^n)} \|b\|_{B_L(\mathbb{D}^n)} \int_{\mathbb{D}^n} |g(w)| d\nu(w) \]
\[ = C \|f\|_{B(\mathbb{D}^n)} \|b\|_{B_L(\mathbb{D}^n)} \|g\|_1. \]
Thus
\[ \|h_b(f)\|_{B(\mathbb{D}^n)} = \sup_{g \in A^1(\mathbb{D}^n), \|g\|_1 \leq 1} |\langle h_b(f), g \rangle| \leq C \|f\|_{B(\mathbb{D}^n)} \|b\|_{B_L(\mathbb{D}^n)}. \]
That is $h_b$ is bounded from $B(\mathbb{D}^n)$ to $B(\mathbb{D}^n)$ for any $b \in B_L(\mathbb{D}^n)$.

For the converse, we have to prove that if $b$ is such that there exists a constant $C > 0$ so that for any $f \in B(\mathbb{D}^n)$, and any $g \in A^1(\mathbb{D}^n)$,
\[ |\langle b, fg \rangle| \leq C \|f\|_{B(\mathbb{D}^n)} \|g\|_1, \]
then $b \in B_L(\mathbb{D}^n)$. For this, we test (22) with
\[ f_a(z) = \sum_{j=1}^{n} \log (1 - a_j z_j), \ a = (a_1, a_2, \cdots, a_n) \in \mathbb{D}^n, \]
We would like to prove that

\[ C \geq |\langle b, fg \rangle| = \left| \sum_{r=1}^{n} \lim_{r \to 1} \int_{\mathbb{D}^n} \log\left(1 - \frac{1}{a_j z_j}\right) \prod_{j=1}^{n} \frac{(1 - |a_j|^2)^{k_j}}{(1 - a_j z_j)^{2+k_j}} b(r z) d\nu(z) \right| . \]

Next we observe that

\[
\int_{\mathbb{D}^n} \log\left(1 - \frac{1}{a_j z_j}\right) \prod_{j=1}^{n} \frac{(1 - |a_j|^2)^{k_j}}{(1 - a_j z_j)^{2+k_j}} b(r z) d\nu(z) = \int_{\mathbb{D}^n} \log\left(1 - \frac{1}{a_j z_j}\right)^{\prod_{j=1}^{n} \frac{(1 - |a_j|^2)^{k_j}}{(1 - a_j z_j)^{2+k_j}}} b(r z) d\nu(z) + \int_{\mathbb{D}^n} \log(1 - |a_j|^2) \prod_{j=1}^{n} \frac{(1 - |a_j|^2)^{k_j}}{(1 - a_j z_j)^{2+k_j}} b(r z) d\nu(z)
\]

and observing that as \( \log\left(\frac{1 - \frac{1}{a_j z_j}}{1 - |a_j|}\right) \in \mathbb{B}(\mathbb{D}^n) \) and \( \frac{(1 - |a_j|^2)^{k_j}}{(1 - a_j z_j)^{2+k_j}} \in A^1(\mathbb{D}) \) both uniformly, we have by (23) that

\[
\lim_{r \to 1} \int_{\mathbb{D}^n} \log\left(1 - \frac{1}{a_j z_j}\right) \prod_{j=1}^{n} \frac{(1 - |a_j|^2)^{k_j}}{(1 - a_j z_j)^{2+k_j}} b(r z) d\nu(z) \leq C,
\]

\( l = 1, \ldots, n \). It follows from the latter and (23) that

\[
\left( \sum_{j=1}^{n} \left| \log(1 - |a_j|^2) \right| \right) \left( \prod_{j=1}^{n} (1 - |a_j|^2) \right) \lim_{r \to 1} \int_{\mathbb{D}^n} \prod_{j=1}^{n} \frac{(1 - |a_j|^2)^{k_j}}{(1 - a_j z_j)^{2+k_j}} b(r z) d\nu(z) \leq C < \infty.
\]

That is for any \( K \in \mathbb{N}^n \),

\[
\sup_{z \in \mathbb{D}^n} \left( \sum_{j=1}^{n} \left( \log\left(\frac{2}{1 - |z_j|^2}\right) \right) \left( \prod_{j=1}^{n} (1 - |z_j|^2)^{k_j} \right) \right) \left| \mathcal{D}^K b(z) \right| < \infty.
\]

The proof is complete. \( \square \)

6. Pointwise multipliers of Bloch spaces

We prove Theorem 3.3 and Theorem 3.4 in this section.

Proof of Theorem 3.3. We would like to prove that \( \phi \) is such that there is a constant \( C > 0 \) so that for any \( f \in \mathbb{B}(\mathbb{D}^n) \) and any \( z = (z_1, \ldots, z_n) \in \mathbb{D}^n \),

\[
\left| \prod_{j=1}^{n} (1 - |z_j|^2) \right| |D(\phi f)(z)| < C
\]

if and only if \( \phi \in H^\infty(\mathbb{D}^n) \cap B_L(\mathbb{D}^n) \).
We observe that
\[ D(\phi f) = \phi Df + f D\phi + \sum_{\emptyset \neq J \subset \{1, \ldots, n\}, \bar{K} = \bar{J}} D_J \phi D_{\bar{K}} f \]
where \( \bar{J} = (j_1, \ldots, j_l) \) is the vector associated to the set \( J = \{j_1, \ldots, j_l\} \) and the same for \( \bar{K} \) and the set \( K \) the complement of \( J \) in \( \{1, \ldots, n\} \). We also observe that if \( J \cap K = \emptyset, J \cup K = \{1, \ldots, n\} \) with \( J, K \neq \{1, \ldots, n\} \), we have using (19) that
\[ \sum_{j=1}^{n} (1 - |z_j|^2) |D_{\bar{J}} \phi(z) D_{\bar{K}} f(z)| \]
\[ \leq \left( \sup_{z \in B^n} \left| \prod_{j \in J} (1 - |z_j|^2) \right| \left| D_{\bar{J}} \phi(z) \right| \right) \left( \sup_{z \in B^n} \left| \prod_{k \in \bar{K}} (1 - |z_k|^2) \right| \left| D_{\bar{K}} f(z) \right| \right) \]
\[ \leq \|\phi\|_B \|f\|_B. \]
Remark that if \( \phi \in H^\infty(B^n) \), then for any \( z \in B^n \),
\[ \left| \prod_{j=1}^{n} (1 - |z_j|^2) \right| |\phi(z) Df(z)| \leq \|\phi\|_\infty \|f\|_{B(B^n)}. \]
All the above observations amount to saying that a bounded function \( \phi \) is a multiplier from \( B(B^n) \) to \( B(B^n) \) if and only if there is a constant \( C > 0 \) such that for any \( f \in B(B^n) \) and any \( z \in B^n \),
\[ \left| \prod_{j=1}^{n} (1 - |z_j|^2) \right| |f(z) D\phi(z)| \leq C. \]
Let us suppose that \( \phi \in B_L(B^n) \) and prove that in this case, (25) holds. Using the pointwise estimate of functions in \( B(B^n) \) given by Lemma 4.9, we obtain
\[ \left| \prod_{j=1}^{n} (1 - |z_j|^2) \right| |f(z) D\phi(z)| \leq \|f\|_B \left| \prod_{j=1}^{n} (1 - |z_j|^2) \right| \left| \sum_{j=1}^{n} \log \frac{2}{1 - |z_j|^2} \right| |D\phi(z)| \]
\[ \leq \|f\|_B \|\phi\|_{B_L}. \]
Conversely, if (25) holds, then testing with the function \( f(z) = f_a(z) = \sum_{j=1}^{n} \log(1 - z_j \bar{z}_j) \) with \( a \in B^n \) fixed, we obtain
\[ \left| \prod_{j=1}^{n} (1 - |z_j|^2) \right| \left| \sum_{j=1}^{n} \log(1 - z_j \bar{z}_j) \right| |D\phi(z)| \leq C. \]
Taking \( z_j = a_j, j = 1, \ldots, n \) in the latter inequality, we obtain that for any \( z \in B^n \),
\[ \left| \prod_{j=1}^{n} (1 - |z_j|^2) \right| \left| \sum_{j=1}^{n} \log \frac{2}{1 - |z_j|^2} \right| |D\phi(z)| < C \]
which proves that \( \phi \in B_L(B^n) \)
Let us now characterize the multiplier algebra of the product Bloch space. We start with the following elementary result.

**Lemma 6.1.** Let \( \vec{j} = (j_1, \ldots, j_l) \), \( 1 \leq l \leq n \), and the associated set \( J = \{ j_1, \ldots, j_l \} \subseteq \{ 1, \ldots, n \} \). Then there is a constant \( C > 0 \) such that for any \( f \in B(\mathbb{D}^n) \) and any \( z = (z_1, \ldots, z_n) \in \mathbb{D}^n \),

\[
\left| \prod_{j=1}^{n} (1 - |z_j|^2) \right| |D_{\vec{j}} f(z)| \leq C \left| \prod_{j \notin J} (1 - |z_j|^2) \right| \log \frac{2}{1 - |z_j|^2} \|f\|_B.
\]

**Proof.** The proof uses the representation formula of \( f \in B(\mathbb{D}^n) \) in Lemma 4.4 and [7, Proposition 1.4.10]. We obtain

\[
|D_{\vec{j}} f(z)| = \left| \int_{\mathbb{D}^n} \left[ \prod_{j \in J} \left( \frac{2}{1 - z_j w_j^*} \right)^3 \prod_{j \notin J} \left( \frac{1}{1 - z_j w_j^*} \right)^3 \right] g(w) d\nu(w) \right| \\
\leq C \|g\|_\infty \left[ \left| \prod_{j \in J} \left( \frac{2}{1 - z_j w_j^*} \right)^3 \prod_{j \notin J} \left( \frac{1}{1 - z_j w_j^*} \right)^3 \right| \right]_1 \\
\leq C \left[ \prod_{j \in J} (1 - |z_j|^2)^{-1} \prod_{j \notin J} \log \frac{2}{1 - |z_j|^2} \right] \|f\|_B.
\]

The proof is complete. \( \Box \)

As a consequence of the above lemma, we have the following result.

**Proposition 6.2.** Let \( \vec{j} \) and \( \vec{k} \) be two vectors such that their respective associated sets \( J \) and \( K \) are complementary in \( \{ 1, \ldots, n \} \), with none of them empty. Suppose that \( f \in B(\mathbb{D}^n) \) and \( \phi \in B_{LL}(\mathbb{D}^n) \). Then for any \( z \in \mathbb{D}^n \),

\[
\left| \prod_{j=1}^{n} (1 - |z_j|^2) \right| |D_{\vec{j}} f(z)||D_{\vec{k}} \phi(z)|, \lesssim \|f\|_B \|\phi\|_{B_{LL}}.
\]

**Proof.** For \( z \in \mathbb{D}^n \), following Lemma 6.1 and the definition of \( B_{LL}(\mathbb{D}^n) \) we obtain

\[
M := \left| \prod_{j=1}^{n} (1 - |z_j|^2) \right| |D_{\vec{j}} f(z)||D_{\vec{k}} \phi(z)| \\
\leq C \|f\|_B \left[ \prod_{j \in K} (1 - |z_j|^2)^2 \log \frac{2}{1 - |z_j|^2} \right] |D_{\vec{k}} \phi(z)| \\
\leq C \|f\|_B \|\phi\|_{B_{LL}}.
\]

Proof of Theorem [3.4]:

Let us start by proving the sufficiency. Let \( \phi \) be as in Theorem [3.4]. We would like to prove that for any \( f \in B(\mathbb{D}^n) \), the analytic function \( \phi f \) belongs
to $\mathcal{B}(\mathbb{D}^n)$. From the formula (24), one sees that we only have to estimate the following three terms.

$$M_1 := \left[ \prod_{j=1}^{n} (1 - |z_j|^2) \right] |\phi(z) Df(z)|;$$

$$M_2 := \left[ \prod_{j=1}^{n} (1 - |z_j|^2) \right] |D_j \phi(z) D_k f(z)|,$$

for $\emptyset \neq J, K \subset \{1, \ldots, n\} = J \cup K$, $J \cap K = \emptyset$;

$$M_3 := \left[ \prod_{j=1}^{n} (1 - |z_j|^2) \right] |f(z) D\phi(z)|.$$

From that $\phi \in H^\infty(\mathbb{D}^n)$, we obtain

$$M_1 := \left[ \prod_{j=1}^{n} (1 - |z_j|^2) \right] |\phi(z) Df(z)| \leq \|\phi\|_{\infty} \|f\|_{\mathcal{B}}.$$

We conclude that for $\phi \in H^\infty(\mathbb{D}^n) \cap \mathcal{B}_{LL}(\mathbb{D}^n)$, we use the pointwise estimate of $f \in \mathcal{B}(\mathbb{D}^n)$ to get

$$M_3 := \left[ \prod_{j=1}^{n} (1 - |z_j|^2) \right] |f(z) D\phi(z)|$$

$$\leq C \|f\|_{\mathcal{B}} \left[ \prod_{j=1}^{n} (1 - |z_j|^2) \log \frac{2}{1 - |z_j|^2} \right] |D\phi(z)|$$

$$\leq C \|f\|_{\mathcal{B}} \|\phi\|_{\mathcal{B}_{LL}}.$$

We now prove the necessity part in Theorem 3.3. We suppose that for any $\phi \in H^\infty(\mathbb{D}^n)$, the function $\phi f$ belongs to $\mathcal{B}(\mathbb{D}^n)$. That is there exists a constant $C > 0$ such that for any $z \in \mathbb{D}^n$,

$$|f(z)\phi(z)| \leq C \left( \prod_{j=1}^{n} \log \frac{2}{1 - |z_j|^2} \right) \|f\|_{\mathcal{B}} \leq C \left( \prod_{j=1}^{n} \log \frac{2}{1 - |z_j|^2} \right) \|f\|_{\mathcal{B}}.$$
We test (27) with the function

\[ f(z) = f_d(z) = \prod_{j=1}^{n} \log (1 - \overline{a_j} z_j), \]

\[ a = (a_j, \ldots, a_n) \text{ given in } \mathbb{D}^n. \]

We obtain that for any \( z \in \mathbb{D}^n \),

\[ \left| \left( \prod_{j=1}^{n} \log (1 - \overline{a_j} z_j) \right) \phi(z) \right| \leq C \left( \prod_{j=1}^{n} \log \frac{2}{1 - |z_j|^2} \right) \|f\|_B. \]

Taking in particular \( z_j = a_j \) (\( j = 1, \ldots, n \)) in the above inequality, we obtain that for any \( z \in \mathbb{D}^n \),

\[ |\phi(z)| \leq C < \infty, \]

that is \( \phi \in H^\infty(\mathbb{D}^n) \).

We next prove that \( \phi \in \mathcal{B}_{LL}(\mathbb{D}^n) \). For this, we first observe the following fact.

**Lemma 6.3.** If \( \phi \in \mathcal{H}(\mathbb{D}^n) \) is a multiplier of \( \mathcal{B}(\mathbb{D}^n) \), then for any fixed \( a_1 \in \mathbb{D} \), the function \( \phi(a_1, \cdot) \) is a multiplier of \( \mathcal{B}(\mathbb{D}^{n-1}) \). Moreover,

\[ \|\phi(a_1, \cdot)\|_{\mathcal{B}(\mathbb{D}^{n-1}) \to \mathcal{B}(\mathbb{D}^{n-1})} \lesssim \|\phi\|_{\mathcal{B}(\mathbb{D}^n) \to \mathcal{B}(\mathbb{D}^n)}. \]

**Proof.** We first prove that for any fixed \( w_n \in \mathbb{D} \), for any \( b \in \mathcal{B}(\mathbb{D}^n) \), the function \( b(\cdot, w_n) \) which is a function of \( n - 1 \) variables, is in \( \mathcal{B}(\mathbb{D}^{n-1}) \) with

\[ \|b(\cdot, w_n)\|_{\mathcal{B}(\mathbb{D}^{n-1})} \lesssim \log \frac{4}{1 - |w_n|^2} \|b\|_{\mathcal{B}(\mathbb{D}^n)}. \]

Let \( z = (z_1, \ldots, z_{n-1}) \in \mathbb{D}^{n-1} \). From the integral representation of elements of \( \mathcal{B}(\mathbb{D}^n) \), we have that for some \( g \in L^\infty(\mathbb{D}^n) \),

\[ b(z, w_n) = \int_{\mathbb{D}^n} \frac{g(\xi) d\nu(\xi)}{(1 - z_1 \xi_1)^2 (1 - z_2 \xi_2)^2 \ldots (1 - z_{n-1} \xi_{n-1})^2 (1 - w_n \xi_n)^2}, \]

hence

\[ \left( \prod_{j=1}^{n-1} (1 - |z_j|^2) \right) |D_1 \ldots D_{n-1} b(z, w_n)| \]

\[ = \left( \prod_{j=1}^{n-1} (1 - |z_j|^2) \right) \left| \int_{\mathbb{D}^n} \frac{2^n g(\xi) d\nu(\xi)}{\left( \prod_{j=1}^{n-1} (1 - z_j \xi_j)^3 \right) (1 - w_n \xi_n)^2} \right| \]

\[ \leq 2^n \left( \prod_{j=1}^{n-1} (1 - |z_j|^2) \right) \left| \int_{\mathbb{D}^n} \frac{|g(\xi)| d\nu(\xi)}{\left( \prod_{j=1}^{n-1} (1 - z_j \xi_j)^3 \right) |1 - w_n \xi_n|^2} \right| \]

\[ \leq \left( \prod_{j=1}^{n-1} (1 - |z_j|^2) \right) \|g\|_{L^\infty(\mathbb{D}^n)} \int_{\mathbb{D}^n} \frac{d\nu(\xi)}{\left( \prod_{j=1}^{n-1} (1 - z_j \xi_j)^3 \right) |1 - w_n \xi_n|^2} \]

\[ \lesssim \|b\|_{\mathcal{B}(\mathbb{D}^n)} \log \frac{4}{1 - |w_n|^2}. \]
Now let $\phi$ be a multiplier of $\mathcal{B}(\mathbb{D}^n)$. Then from (28) we obtain that for any $b \in \mathcal{B}(\mathbb{D}^n)$ and any $w_n \in \mathbb{D}$ fixed,

$$
\left( \prod_{j=1}^{n-1} (1 - |z_j|^2) \right) |D_1 \ldots D_{n-1}(\phi b)(z, w_n)| \lesssim \left( \log \frac{4}{1 - |w_n|^2} \right) \|\phi b\|_{\mathcal{B}(\mathbb{D}^n)}
$$

and so

$$
\left( \prod_{j=1}^{n-1} (1 - |z_j|^2) \right) |D_1 \ldots D_{n-1}(\phi b)(z, w_n)| \lesssim \left( \log \frac{4}{1 - |w_n|^2} \right) \|\phi\|_{\mathcal{B}(\mathbb{D}^n) \to \mathcal{B}(\mathbb{D}^n)} \|b\|_{\mathcal{B}(\mathbb{D}^n)}
$$

Let us take in (29), $b(z, \xi) = g(z) \log(1 - \xi \bar{w}_n)$, $g \in \mathcal{B}(\mathbb{D}^{n-1})$, $z \in \mathbb{D}^{n-1}$ and $\xi \in \mathbb{D}$. We obtain

$$
S := \left( \prod_{j=1}^{n-1} (1 - |z_j|^2) \right) |\log(1 - \xi \bar{w}_n)| |D_1 \ldots D_{n-1}(\phi g)(z, w_n)|
$$

$$
= \left( \prod_{j=1}^{n-1} (1 - |z_j|^2) \right) |D_1 \ldots D_{n-1}(\phi b)(z, w_n)|
$$

$$
\lesssim \log \frac{4}{1 - |w_n|^2} \|\phi\|_{\mathcal{B}(\mathbb{D}^n) \to \mathcal{B}(\mathbb{D}^n)} \|g\|_{\mathcal{B}(\mathbb{D}^{n-1})}.
$$

Taking $\xi = w_n$ in the above inequalities, we obtain that for any $g \in \mathcal{B}(\mathbb{D}^{n-1})$, and any $z \in \mathbb{D}^{n-1}$,

$$
\left( \prod_{j=1}^{n-1} (1 - |z_j|^2) \right) |D_1 \ldots D_{n-1}(\phi g)| \lesssim \|M_\phi\|_{\mathcal{B}(\mathbb{D}^n) \to \mathcal{B}(\mathbb{D}^n)} \|g\|_{\mathcal{B}(\mathbb{D}^{n-1})}.
$$

Thus for any $w_n \in \mathbb{D}$ fixed, $\phi(\cdot, w_n)$ is a multiplier of $\mathcal{B}(\mathbb{D}^{n-1})$. The proof of the lemma is complete. 

We next proceed by induction on the number of parameters $n \geq 2$ to prove that if $\phi$ is a multiplier of $\mathcal{B}(\mathbb{D}^n)$, then $\phi \in \mathcal{B}_{LL}(\mathbb{D}^n)$. We start by the case $n = 2$. Let $\phi$ be a multiplier of $\mathcal{B}(\mathbb{D}^2)$. Then there exists a constant $C > 0$ such that for any $b \in \mathcal{B}(\mathbb{D}^2)$ and any $z = (z_1, z_2)$,

$$
(1 - |z_1|^2)(1 - |z_2|^2) |D\phi f(z)| \leq C \|f\|_{\mathcal{B}(\mathbb{D}^2)}.
$$

Recall that in this case, $D = D_1 D_2$ and

$$
D(\phi f)(z) = f(z) D_2 \phi(z) + \phi(z) D f(z) + D_1 \phi(z) D_2(z) + D_1 f(z) D_2 \phi(z).
$$

But by Lemma 6.3 and Proposition 4.2, $\phi(\cdot, 0)$ and $\phi(z_1, \cdot)$ are uniformly in $\mathcal{B}_{L}(\mathbb{D})$, that is there is a constant $C > 0$ such that for any $z \in (z_1, z_2) \in \mathbb{D}^2$,

$$
(1 - |z_1|^2) \left( \log \frac{4}{1 - |z_1|^2} \right) |D_1 \phi(z)| \leq C
$$

and

$$
(1 - |z_2|^2) \left( \log \frac{4}{1 - |z_2|^2} \right) |D_2 \phi(z)| \leq C.
$$
Hence for any \( z = (z_1, z_2) \in \mathbb{D}^2 \), we obtain using Lemma 6.1
\[
S := (1 - |z_1|^2)(1 - |z_2|^2)|D_1\phi(z)||D_2f(z)| \\
\lesssim (1 - |z_1|^2) \left( \log \frac{4}{1 - |z_1|^2} \right) |D_1\phi(z)||f|_{B(\mathbb{D}^2)}
\]
and consequently,
\[
(31) \quad (1 - |z_1|^2)(1 - |z_2|^2)|D_1\phi(z)||D_2f(z)| \leq C||f||_{B(\mathbb{D}^2)}.
\]
In the same way, we obtain for any \( z = (z_1, z_2) \in \mathbb{D}^2 \),
\[
(32) \quad (1 - |z_1|^2)(1 - |z_2|^2)|D_2\phi(z)||D_1f(z)| \leq C||f||_{B(\mathbb{D}^2)}.
\]
Also, note that as \( \phi \in H^\infty(\mathbb{D}^2) \), we have that for any \( z = (z_1, z_2) \in \mathbb{D}^2 \),
\[
(33) \quad (1 - |z_1|^2)(1 - |z_2|^2)|Df(z)| \leq \|\phi\|_\infty||f||_{B(\mathbb{D}^2)}.
\]
From (30), (31), (32) and (33), we deduce that there exists a constant \( C > 0 \) such that for any \( f \in B(\mathbb{D}^2) \) and any \( z = (z_1, z_2) \in \mathbb{D}^2 \),
\[
(34) \quad (1 - |z_1|^2)(1 - |z_2|^2)|D\phi(z)| \leq C||f||_{B(\mathbb{D}^2)}.
\]
For \( a = (a_1, a_2) \in \mathbb{D}^2 \) given, we test (34) with
\[
f(z) = f_\alpha(z) = \log(1 - z_1\bar{a}_1) \log(1 - z_2\bar{a}_2)
\]
which is uniformly in \( B(\mathbb{D}^2) \) and obtain for any \( z = (z_1, z_2) \in \mathbb{D}^2 \),
\[
(35) \quad (1 - |z_1|^2)(1 - |z_2|^2)\log(1 - z_1\bar{a}_1)\log(1 - z_2\bar{a}_2)||D\phi(z)| \leq C||f||_{B(\mathbb{D}^2)}.
\]
Taking in particular \( z_1 = a_1 \) and \( z_2 = a_2 \) in (35), we conclude that there is a constant \( C > 0 \) such that for any \( z = (z_1, z_2) \in \mathbb{D}^2 \),
\[
(1 - |z_1|^2)(1 - |z_2|^2)\left( \log \frac{4}{1 - |z_1|^2} \right) \left( \log \frac{4}{1 - |z_2|^2} \right) |D\phi(z)| \leq C,
\]
that is \( \phi \in B_{LL}(\mathbb{D}^2) \). This completes the proof for the case \( n = 2 \).

Now for \( n > 2 \), we suppose that \( \phi \) is a multiplier of \( B(\mathbb{D}^n) \) implies that \( \phi \in B_{LL}(\mathbb{D}^n) \). We prove that this implies that if \( \phi \) is a multiplier of \( B(\mathbb{D}^{n+1}) \), then \( \phi \in B_{LL}(\mathbb{D}^{n+1}) \).

Let \( \phi \) be a multiplier of \( B(\mathbb{D}^{n+1}) \). Then by Lemma 6.3, for any \( w_{n+1} \in \mathbb{D} \) fixed, \( \phi(\cdot, w_{n+1}) \) is a multiplier of \( B(\mathbb{D}^n) \) with uniformly bounded multiplier norm. Hence by our hypothesis, \( \phi(\cdot, w_{n+1}) \in B_{LL}(\mathbb{D}^n) \) uniformly. It follows in particular that there is a constant \( C > 0 \) such that for any \( \vec{j} = (j_1, \ldots, j_t) \) with associated set \( J = \{j_1, \ldots, j_t\} \subset \{1, 2, \ldots, n + 1\} \), and any \( z = (z_1, \ldots, z_{n+1}) \in \mathbb{D}^{n+1} \),
\[
(36) \quad \left( \prod_{j \in J}(1 - |z_j|^2)\log \frac{4}{1 - |z_j|^2} \right) |D_J\phi(z)| \leq C.
\]
Denoting by \( K \) the complement set of \( J \) in \( \{1, 2, \ldots, n+1\} \) with associated vector \( \vec{k} \), we obtain using Lemma 6.1 that for any \( f \in B(\mathbb{D}^{n+1}) \), and any
$z = (z_1, \ldots, z_{n+1}) \in \mathbb{D}^{n+1}$, 

$$Q := \left( \prod_{j=1}^{n+1} (1 - |z_j|^2) \right) |D_j \phi(z)||D_k f(z)|$$

$$\leq C \|f\|_{B(\mathbb{D}^{n+1})} \left( \prod_{j \in J} (1 - |z_j|^2) \log \frac{4}{1 - |z_1|^2} \right) |D_j \phi(z)|.$$ 

Hence applying (36) to the above, we obtain

$$\left( \prod_{j=1}^{n+1} (1 - |z_j|^2) \right) |D_j \phi(z)||D_k f(z)| \leq C \|f\|_{B(\mathbb{D}^{n+1})}.$$ 

Also we have since $\phi \in H^\infty(\mathbb{D}^{n+1})$, that for any $z = (z_1, \ldots, z_{n+1}) \in \mathbb{D}^{n+1}$,

$$\prod_{j=1}^{n+1} (1 - |z_j|^2) |\phi(z)||D f(z)| \leq \|\phi\|_{H^\infty} \|f\|_{B(\mathbb{D}^{n+1})}.$$ 

We recall that in this case,

$$D(\phi f) = fD\phi + \phi Df + \sum_{\emptyset \neq J \subset \{1, \ldots, n+1\}} D_J \phi D_k f.$$ 

From (37), (38) and the fact that we have a constant $C > 0$ such that for any $f \in B(\mathbb{D}^{n+1})$ and any $z = (z_1, \ldots, z_{n+1}) \in \mathbb{D}^{n+1}$,

$$\left( \prod_{j=1}^{n+1} (1 - |z_j|^2) \right) |D(\phi f)(z)| \leq C \|f\|_{B(\mathbb{D}^{n+1})},$$

we obtain that there exists a constant $C > 0$ such that for any $f \in B(\mathbb{D}^{n+1})$ and for any $z = (z_1, \ldots, z_{n+1}) \in \mathbb{D}^{n+1}$,

$$\prod_{j=1}^{n+1} (1 - |z_j|^2) |f(z)||D\phi(z)| \leq C \|f\|_{B(\mathbb{D}^{n+1})}.$$ 

For $a = (a_1, a_2, \ldots, a_{n+1}) \in \mathbb{D}^{n+1}$ given, we test (41) with

$$f(z) = f_a(z) = \log(1 - z_1 a_1) \ldots \log(1 - z_{n+1} a_{n+1})$$

which is uniformly in $B(\mathbb{D}^{n+1})$ and obtain for any $z = (z_1, \ldots, z_{n+1}) \in \mathbb{D}^{n+1}$,

$$\prod_{j=1}^{n+1} (1 - |z_j|^2) \left| \log(1 - z_j a_j) \right| |D\phi(z)| \leq C.$$ 

Taking in particular $z_j = a_j$, $j = 1, 2, \ldots, n+1$ (41), we obtain that there is a constant $C > 0$ such that for any $z = (z_1, \ldots, z_{n+1}) \in \mathbb{D}^{n+1}$,

$$\prod_{j=1}^{n+1} (1 - |z_j|^2) \left( \log \frac{4}{1 - |z_j|^2} \right) |D\phi(z)| \leq C.$$ 

That is $\phi \in B_L(\mathbb{D}^{n+1})$. The latter and (36) allow us to conclude that $\phi \in B_{LL}(\mathbb{D}^{n+1})$. The proof is complete.
7. Remarks on the pointwise Bloch space

The multiplier algebra of $\mathcal{B}(\mathbb{D}^n)$ has been found by F. Colonna and R. F. Allen in [1]. They proved exactly the following.

**Proposition 7.1.** The only multipliers of $\mathcal{B}(\mathbb{D}^n)$ are the constants.

We have the following consequence of the above proposition.

**Corollary 7.2.** Let $X$ be a Banach space of analytic functions strictly containing $\mathcal{B}(\mathbb{D}^n)$. Then $\mathcal{M}(X, \mathcal{B}(\mathbb{D}^n)) = \{0\}$.

In particular, we obtain the following.

**Corollary 7.3.** $\mathcal{M}(\mathcal{B}(\mathbb{D}^n), \mathcal{B}(\mathbb{D}^n)) = \{0\}$.

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**Conflict of interest statement**

The author declares that there is no conflict of interests regarding the publication of this paper.

**References**

[1] R. F. Allen, F. Colonna, *Multiplication operators on the Bloch space of bounded homogeneous domains*. Comput. Methods Funct. Theory 9 (2009), no. 2, 679–693.

[2] A. Bonami, Luo Luo, *On Hankel operators between Bergman spaces on the unit ball*. Houston J. Math. 31 (2005), no. 3, 815–827.

[3] O. Constantin, *Weak product decompositions and Hankel operators on vector-valued Bergman spaces*. J. Oper. Theor. 59 (2008), no. 1, 157–178.

[4] S. Janson, J. Peetre, R. Rochberg, *Hankel forms and the Fock space*. Revista Math. Ibero-Amer. 3 (1987) 61–138.

[5] A. Harutyunyan, *Bloch spaces of holomorphic functions in the polydisk*. J. Funct. Spaces Appl. 5 (2007), no. 3, 213–230.

[6] A. Harutyunyan, W. Lusky, *Weighted holomorphic Besov spaces on the polydisk*. J. Funct. Spaces Appl. 9 (2011), no. 1, 1–16.

[7] W. Rudin, *Function theory in the unit ball of $\mathbb{C}^n$*. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], 241. Springer-Verlag, New York-Berlin (1980).

[8] B. F. Sehba, *On some equivalent definitions of $p$-Carleson measures on the unit ball*. Acta Sci. Math. (Szeged) 75 (2009), no. 3-4, 499–525.

[9] S. Stevic, *A note on a Theorem of Zhu on weighted Bergman projections on the polydisk*. Houston J. Math. 34 (2004), no. 4, 511–521.

[10] R. M. Timoney, *Bloch functions in several complex variables I*. Bull. London Math. Soc. 12 (1980), 241–267.

[11] R. Zhao, *On logarithmic Carleson measures*, Acta Sci. Math (Szeged) 69 (2003), no 3-4, 605–618

[12] K. Zhu, *Weighted Bergman projections on the polydisk*. Houston J. Math. 20 (1994), no. 2, 275–292.

[13] K. Zhu, *Spaces of holomorphic functions in the unit ball*. Graduate Texts in Mathematics 226, Springer Verlag (2004).
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