Stochastic differential equations in a scale of Hilbert spaces 2. Global solutions.

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Abstract

A stochastic differential equation with coefficients defined in a scale of Hilbert spaces is considered. The existence, uniqueness and path-continuity of infinite-time solutions is proved by an extension of the Ovsyannikov method. This result is applied to a system of equations describing non-equilibrium stochastic dynamics of (real-valued) spins of an infinite particle system on a typical realization of a Poisson or Gibbs point process in $\mathbb{R}^n$. The paper improves the results of the work by the second named author "Stochastic differential equations in a scale of Hilbert spaces", Electron. J. Probab. 23, where finite-time solutions were constructed.

1 Introduction

The purpose of this work is to study an infinite-dimensional stochastic differential equation (SDE)

$$d\xi(t) = f(\xi(t))dt + B(\xi(t))dW(t),$$

with the coefficients $f$ and $B$ defined in a scale of densely embedded Hilbert spaces $(X_\alpha)_{\alpha \in A}$, where $A$ is a real interval, and $W$ is a cylinder Wiener process on a fixed Hilbert space $\mathcal{H}$. That is, $f$ and $B$ are Lipschitz continuous maps $X_\alpha \to X_\beta$ and $X_\alpha \to H_\beta := HS(\mathcal{H}, X_\beta)$, $\beta > \alpha$, respectively, but are not in general well-defined in any fixed $X_\alpha$, with the corresponding Lipschitz constants $L_{\alpha, \beta}$ becoming infinite as $|\alpha - \beta| \to 0$. Here $HS(\mathcal{H}, X_\beta)$ stands for the space of Hilbert-Schmidt operators $\mathcal{H} \to X_\beta$. 
Equation (1) cannot be treated by methods of the classical theory of SDEs in Banach spaces (see e.g. [11] and [15]), because its coefficients are singular in any fixed $X_\alpha$. Some progress can be achieved if

$$L_{\alpha\beta} \sim (\beta - \alpha)^{-1/q} \text{ as } |\alpha - \beta| \to 0,$$

with $q = 2$. Under this condition, a solution with initial value in $X_\alpha$ exists in $X_\beta$, with lifetime $T_{\alpha\beta} \sim (\beta - \alpha)^{1/2}$, see [8]. This result generalizes the Ovsyannikov method for ordinary differential equations, see e.g. [17], [4] and [9], in which setting it is sufficient to assume that $q = 1$.

It has been noticed in [9] that, in case of $q > 1$, a solution of the ODE

$$\frac{d}{dt}u(t) = f(u(t)), \ u(0) \in X_\alpha,$$

exists in any $X_\beta$, $\beta > \alpha$, with infinite lifetime. In the present paper, we build upon the ideas of [9], which enable us to generalize the results of [8] and prove the existence and uniqueness of a global solution $\xi(t)$ of equation (1) in any $X_\beta$, $\beta > \alpha$, with initial value $\xi(0) \in X_\alpha$, provided (2) holds with $q > 2$. Moreover, we show that $\xi(t)$ is $p$-integrable for any $p < q$ and has a continuous modification.

The structure of the paper is as follows. In Section 2 we introduce the framework and notations and formulate our main existence and uniqueness result. In Section 3 we obtain technical estimates, which play crucial role in what follows. Section 4 is devoted to the proof of our main existence and uniqueness result. In Sections 6 and 5 we derive an estimate of the growth of solutions and prove the existence of its continuous modification, respectively.

Section 7 is devoted to our main example, which is motivated by the study of countable systems of particles randomly distributed in a Euclidean space $\mathbb{R}^n(=:\mathcal{X})$. Each particle is characterized by its position $x$ and an internal parameter (spin) $\sigma_x \in S = \mathbb{R}^1$. For a given fixed (“quenched”) configuration $\gamma$ of particle positions, which is a locally finite subset of $\mathbb{R}^n$, we consider a system of stochastic differential equations describing (non-equilibrium) dynamics of spins $\sigma_x$, $x \in \gamma$. Two spins $\sigma_x$ and $\sigma_y$ are allowed to interact via a pair potential if the distance between $x$ and $y$ is no more than a fixed interaction radius $r$, that is, they are neighbors in the geometric graph defined by $\gamma$ and $r$. Vertex degrees of this graph are typically unbounded, which implies that the coefficients of the corresponding equations cannot be controlled in a single Hilbert or Banach space (in contrast to spin systems on a regular lattice, which have been well-studied, see e.g. [16] and more recent developments in [11], [2], [20], and references therein). However, under mild conditions on the density of $\gamma$ (holding for e.g. Poisson and Gibbs point
processes in $\mathbb{R}^n$), it is possible to apply the approach discussed above and construct a solution in the scale of Hilbert spaces $S^\gamma_\alpha$ of weighted sequences $(q_x)_{x \in \gamma} \in S^\gamma$ such that $\sum_{x \in \gamma} |q_x|^2 e^{-\alpha |x|} < \infty$, $\alpha > 0$. Local solutions of the above system were constructed in [8] by a somewhat different method.

Construction of non-equilibrium stochastic dynamics of infinite particle systems of the aforementioned type has been a long-standing problem, even in the case of linear drift and a single-particle diffusion coefficient. It has become important in the framework of analysis on spaces $\Gamma(\mathcal{X}, S)$ of configurations $\{(x, \sigma_x)\}_{x \in \gamma}$ with marks (see e.g. [14]), and is motivated by a variety of applications, in particular in modeling of non-crystalline (amorphous) substances, e.g. ferrofluids and amorphous magnets, see e.g. [27], [26 Section 11], [6] and [12, 13]. $\Gamma(\mathcal{X}, S)$ possesses a fibration-like structure over the space $\Gamma(\mathcal{X})$ of position configurations $\gamma$, with the fibres identified with $S^\gamma$, see [12]. Thus the construction of spin dynamics of a quenched system (in $S^\gamma$) is complementary to that of the dynamics in $\Gamma(\mathcal{X})$.

Various aspects of the study of deterministic (Hamiltonian) and stochastic evolution of configurations $\gamma \in \Gamma(\mathcal{X})$, in its deterministic (Hamiltonian) and stochastic form have been discussed by many authors, see e.g. [21, 22, 19, 5, 3, 18] and references given there. It is anticipated that (some of) these results can be combined with the approach proposed in the present paper allowing to build stochastic dynamics on the marked configuration space $\Gamma(\mathcal{X}, S)$. In particular, the results of Section 7 are used in a forthcoming paper [10] for the construction of a mixed-type jump diffusion dynamics in $\Gamma(\mathcal{X}, S)$.

Finally, in Section 8 we give two further examples of the maps satisfying condition (2).

Observe that the family $X_\alpha = S^\gamma_\alpha$, $\alpha > 0$, forms the dual to nuclear space $\Phi' = \cup_\alpha X_\alpha$. SDEs on such spaces were considered in [21, 22]. The existence of solutions to the corresponding martingale problem was proved under assumption of continuity of coefficients on $\Phi'$ and their linear growth (which, for the diffusion coefficient, is supposed to hold in each $\alpha$-norm). Moreover, the existence of strong solutions requires a dissipativity-type estimate in each $\alpha$-norm, too, which does not hold in our framework.

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2 Setting and main results

In this section we introduce the general framework we will be using. We start with the following general definition.

Let us consider a family $B$ of Banach spaces $B_\alpha$ indexed by $\alpha \in \mathcal{A} := [\alpha_*, \alpha^*]$ with fixed $0 \leq \alpha_* < \alpha^* < \infty$, and denote by $\|\cdot\|_{B_\alpha}$ the corresponding norms. When speaking of these spaces and related objects, we will always assume that the range of indices is $[\alpha_*, \alpha^*]$, unless stated otherwise. The interval $\mathcal{A}$ remains fixed for the rest of this work.

**Definition 1** The family $B$ is called a scale if

$$B_\alpha \subset B_\beta \text{ and } \|u\|_{B_\beta} \leq \|u\|_{B_\alpha} \text{ for any } \alpha < \beta, \ u \in X_\alpha,$$

where the embedding means that $B_\alpha$ is a vector subspace of $B_\beta$.

We will use the following notations:

$$\overline{B} := \bigcup_{\alpha \in [\alpha_*, \alpha^*)} B_\alpha, \ \underline{B} := \bigcap_{\alpha \in (\alpha_*, \alpha^*]} B_\alpha.$$

**Definition 2** For two scales $\mathcal{B}_1$, $\mathcal{B}_2$ (with the same index set) and a constant $q > 0$ we introduce the class $\mathcal{GL}_q(\mathcal{B}_1, \mathcal{B}_2)$ of (generalized Lipschitz) maps $f : \overline{B} \to \underline{B}$ such that

1. $f(B_\alpha) \subset B_\beta$ for any $\alpha < \beta$;

2. there exists constant $L > 0$ such that

$$\|f(u) - f(v)\|_{B_\beta} \leq \frac{L}{|\beta - \alpha|^{1/q}} \|u - v\|_{B_\alpha}$$

for any $\alpha < \beta$ and $u, v \in B_\alpha$.

We will write $\mathcal{GL}_q(\mathcal{B}) := \mathcal{GL}_q(\mathcal{B}_1, \mathcal{B}_2)$ if $\mathcal{B}_1 = \mathcal{B}_2 =: \mathcal{B}$.

**Remark 3** The constant $L$ may depend on $\alpha^*$ and $\alpha_*$, as usually happens in applications.

**Remark 4** Setting $v = 0$ in (4), we obtain the linear growth condition

$$\|f(u)\|_{B_\beta} \leq \frac{K}{|\beta - \alpha|^{1/q}} (1 + \|u\|_{B_\alpha}) , \ u \in B_\alpha,$$

for some constant $K$ and any $\alpha < \beta$. 

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Remark 5 Assume that $\phi$ is Lipschitz continuous in each $B_\alpha$ with a uniform Lipschitz constant $M$. Then $\phi \in GL_q(B)$ with $L = (\alpha^* - \alpha) M$.

Remark 6 Some authors have used the scale $B_\alpha$ such that $B_\alpha \subset B_\beta$ if $\alpha > \beta$. That framework can be transformed to our setting by an appropriate change of the parametrization, e.g. $\alpha \mapsto \alpha^* - \alpha$.

In what follows, we will use the following three main scales of spaces:

1. the scale $X$ of separable Hilbert spaces $X_\alpha$;
2. the scale $\mathcal{H}$ of spaces
   $$H_\alpha \equiv HS(\mathcal{H}, X_\alpha) := \{\text{Hilbert-Schmidt operators } \mathcal{H} \to X_\alpha\},$$
   for a fixed separable Hilbert space $\mathcal{H}$;
3. the scale $Z_{\alpha,T}$ of Banach spaces $Z_{\alpha,T}^p$ of progressively measurable random processes $u : [0, T) \to X_\alpha$ with finite norm
   $$\|u\|_{Z_{\alpha,T}^p} := \sup_{t \in [0, T)} \left( \mathbb{E} \|u(t)\|_{X_\alpha}^p \right)^{1/p},$$
   defined on a suitable filtered probability space $(\Omega, \mathcal{F}, P)$.

Our aim is to construct a strong solution of equation (7), that is, a solution of the stochastic integral equation

$$u(t) = u_0 + \int_0^t f(u(s))ds + \int_0^t B(u(s))dW(s), \; t \leq T, \; u_0 \in X_{\alpha^*},$$

where $W(t), \; t \leq T,$ is a fixed cylinder Wiener process in $\mathcal{H}$ (cf. (6)) defined on the probability space $(\Omega, \mathcal{F}, P)$, with coefficients acting in the scale $X$ for a fixed $p \geq 2$.

The following theorem states the main result of this paper.

Theorem 7 (Existence and uniqueness) Assume that $f \in GL_q(X)$ and $B \in GL_q(\mathcal{X}, \mathcal{N})$, $q > 2$ and $u_0 \in X_\alpha$, $\alpha \in A$. Then, for any $T > 0$, the following holds:

(1) equation (7) has a unique solution $u \in Z_{\alpha^*,T}^2$;
(2) $u \in Z_{\beta,T}^p$ for any $p \in [2, q)$ and $\beta > \alpha$;
(3) $u$ has continuous sample paths a.s.
The proof of the first two statements is given in Section 4 below. We will show that the map \( u \mapsto T(u) \), where
\[
T(u)(t) = u_0 + \int_0^t f(u(s))ds + \int_0^t B(u(s))dW(s), \quad t \in [0,T],
\]
has a unique fixed point in \( Z_{\beta,T}^p \) for any \( \beta > \alpha \), by Picard iterative process. The third statement is proved in Section 5 by Kolmogorov’s continuity theorem.

From now on, we keep \( u_0 \) fixed assume without loss of generality that \( u_0 \in X_\alpha \) (otherwise, we can always re-define the parameter set \( \mathcal{A} \)). We also fix an arbitrary \( T \) and write \( Z_{\beta}^p \) instead of \( Z_{\beta,T}^p \).

3 Main estimates

In this section, we derive certain estimates of the map \( T \) defined by formula (8). We first observe that if \( \xi \in Z_{\alpha}^p \) and \( \alpha < \beta \) then \( \Phi(\xi) \) is a predictable \( H_{\beta} \)-valued process because \( \Phi \) is continuous by inequality (4). Now using inequality (5) we also see that
\[
\int_0^T \| \Phi(\xi(s)) \|^2_{H_{\beta}} ds \leq C_1 \int_0^T \left( 1 + \| \xi(s) \|^2_{B_{\alpha}} \right) ds
\]
\[
\leq C_2 \int_0^T \left( 1 + \| \xi(s) \|^p_{B_{\alpha}} \right) ds < \infty
\]
because \( \xi \in Z_{\alpha}^p \subset Z_{\alpha}^2 \). Thus the stochastic integral
\[
\int_0^t \Phi(\xi(s))dW(s), \quad t \leq T,
\]
is unambiguously defined as a square integrable \( X_{\beta} \)-valued martingale. Therefore \( T \) is a well-defined map \( \overline{Z} \to Z_{\alpha}^\ast \).

\textbf{Theorem 8} Assume that \( f \in \mathcal{G}\mathcal{L}_q(\mathfrak{X}) \) and \( B \in \mathcal{G}\mathcal{L}_q(\mathfrak{X},\mathfrak{H}) \). Then \( T \in \mathcal{G}\mathcal{L}_q(\mathfrak{Z}^p) \) for any \( p \geq 2 \).

\textbf{Proof.} Let us fix \( p \geq 2 \) and \( \alpha, \beta \in \mathcal{A}, \alpha < \beta \). For simplicity, we will use the shorthand notation \( Z_\alpha := Z_{\alpha,T}^p \).

We first show that \( T(Z_\alpha) \subset Z_\beta \). Observe that \( f(u(s)) \in X_\beta \) and \( B(u(s)) \in H_\beta \) for any \( s \in [0,T] \), and the integrals in the right-hand side of (8) are well-defined in \( X_\beta \). The inclusion in question immediately follows from the properties of those integrals.
Now we shall show that condition (4) of the Definition 2 also holds. Introduce notations
\[ \bar{F}(s) := F(\xi_1(s)) - F(\xi_2(s)) \quad \text{and} \quad \bar{\Phi}(s) := \Phi(\xi_1(s)) - \Phi(\xi_2(s)), \quad s \in [0, T], \]
we obtain
\[
E \left[ \left\| T(\xi_1)(t) - T(\xi_2)(t) \right\|_{X_\alpha}^p \right] = E \left[ \left\| \int_0^t \bar{F}(s)dW(s) \right\|_{X_\beta}^p \right] \\
\leq 2^{p-1}E \left[ \int_0^t \left\| \bar{F}(s) \right\|_{X_\beta}^p ds \right] + 2^{p-1}E \left[ \left\| \int_0^t \bar{\Phi}(s)dW(s) \right\|_{X_\beta}^p \right]. \quad (9)
\]
Applying the Hölder inequality, well-known formula for the moments of Itô integral (see e.g. [9]) and estimate (4) to inequality (9) above we obtain the estimate
\[
E \left[ \left\| T(\xi_1)(t) - T(\xi_2)(t) \right\|_{X_\alpha}^p \right] \leq 2^{p-1}t^{p-1} \int_0^t E \left[ \left\| \bar{F}(s) \right\|_{X_\beta}^p ds \right] \\
+ 2^{p-1} \left[ \frac{p}{2}(p-1) \right]^{p/2} t^{p/2-1} \int_0^t E \left[ \left\| \bar{\Phi}(s) \right\|_{H_\alpha}^p ds \right] \\
\leq \frac{\hat{L}(T)}{(\beta - \alpha)^{p/q}} \int_0^t E \left[ \| \xi_1(s) - \xi_2(s) \|_{X_\alpha}^p \right] ds, \quad (10)
\]
where \( \hat{L}(T) = (T^{p-1} + \left[ \frac{p}{2}(p-1) \right]^p T^{p/2-1})2^{p-1}L^p \). Consequently for all \( \alpha < \beta \in A \) and \( \xi_1, \xi_2 \in Z_\alpha \) the following general result holds:
\[
\left\| T(\xi_1) - T(\xi_2) \right\|_{Z_\beta} \leq \frac{\hat{L}(T)T}{(\beta - \alpha)^{p/q}} \sup_{s \in [0, T]} E \left[ \left\| \xi_1(s) - \xi_2(s) \right\|_{X_\alpha}^p \right] \\
= \sqrt{\hat{L}(T)T} (\beta - \alpha)^{1/q} \left\| \xi_1 - \xi_2 \right\|_{Z_\alpha}, \quad (11)
\]
and the proof is complete. \( \blacksquare \)

**Corollary 9** For any \( \alpha > \alpha_* \) and all \( n \in \mathbb{N} \)
\[ T^n : Z_{\alpha_*} \rightarrow Z_\alpha, \]
where \( T^n \) stands for the \( n \)-th composition power of \( T \).

Corollary 9 shows in particular that given \( \xi \in Z_{\alpha_*} \) the sequence of processes \{\( T^n(\xi) \)\}_{n=1}^\infty \) belongs to \( Z_\alpha \) for all \( \alpha > \alpha_* \).
Remark 10 Observe that we have
\[ \sqrt{\hat{L}(T)T} \leq a(T) = \begin{cases} \frac{a_pLT}{T}, & T \geq 1 \\ \frac{a_pLT^{1/2}}{T}, & T < 1 \end{cases}, \] (12)
where \( a_p = 2^{p-1} \left( \left( \frac{p}{2} \right)^{p/2} (p-1) + 1 \right) \).

Lemma 11 For any \( n \in \mathbb{N}, \alpha < \beta \) and \( \xi_1, \xi_2 \in Z_\alpha \) we have the estimate
\[ \|T^n(\xi_1) - T^n(\xi_2)\|_{Z_\beta}^p \leq \frac{n^{np/q}}{n!} \left( \frac{\hat{L}(T)^p}{(\beta - \alpha)^{p/q}} \right)^n \|\xi_1 - \xi_2\|_{Z_\alpha}^p. \] (13)

Proof. We fix a partition of the interval \([\alpha, \beta]\) in \( n \) intervals \( [\psi_k, \psi_{k+1}], \) \( k = 0, \ldots, n - 1, \) \( \psi_0 = \alpha, \psi_n = \beta, \) of equal length \( \frac{\beta - \alpha}{n}. \) Then, iterating estimate (10), we obtain
\[ \mathbb{E} \left[ \|T(T^{n-1}(\xi_1))(t) - T(T^{n-1}(\xi_2))(t)\|_{X_\beta}^p \right] \leq \frac{\hat{L}(T)^{np/q}}{(\beta - \alpha)^{p/q}} \int_0^t \mathbb{E} \left[ \|T^{n-1}(\xi_1(s)) - T^{n-1}(\xi_2(s))\|_{X_\alpha}^p \right] ds \leq \cdots \leq \left[ \frac{\hat{L}(T)^{np/q}}{(\beta - \alpha)^{p/q}} \right]^{n} \int_0^{t_1} \cdots \int_0^{t_{n-1}} \mathbb{E} \|\xi_1(s) - \xi_2(s)\|_{X_\alpha}^p ds \, dt_{n-1} \cdots dt_1, \] (14)
and the result follows. ■

Corollary 12 For any \( n \in \mathbb{N}, \alpha < \beta \) and \( \xi \in Z_\alpha \) we have the estimates
\[ \mathbb{E} \left[ \|T^n(\xi)(t) - T^{n+1}(\xi)(t)\|_{X_\beta}^p \right] \leq \left[ \frac{\hat{L}(T)^{np/q}}{(\beta - \alpha)^{p/q}} \right]^{n} \int_0^{t_1} \cdots \int_0^{t_{n-1}} \mathbb{E} \|\xi(s) - T(\xi(s))\|_{X_\alpha}^p ds \, dt_{n-1} \cdots dt_1 \] (15)
and
\[ \|T^n(\xi) - T^{n+1}(\xi)\|_{Z_\beta}^p \leq \frac{n^{np/q}}{n!} \left( \frac{\hat{L}(T)^{np/q}}{(\beta - \alpha)^{p/q}} \right)^n \|\xi - T(\xi)\|_{Z_\alpha}^p. \] (16)

Lemma 13 Suppose \( \alpha < \beta \in A \) and \( \xi \in Z_\alpha. \) For all \( m, n \in \mathbb{N}, m > n, \) the following inequality holds
\[ \|T^n(\xi) - T^m(\xi)\|_{Z_\beta} \leq \|\xi - T(\xi)\|_{Z_\alpha} \sum_{k=n}^{m} \sqrt{\hat{L}(T)^{k}T^k} \frac{k^{\beta_k}}{(\beta - \alpha)^{k/q}} \sqrt{k!}. \] (17)
Proof. We have
\[ \|T^n(\xi) - T^m(\xi)\|_{Z_\beta} \leq \sum_{k=n}^{m-1} \|T^k(\xi) - T^{k+1}(\xi)\|_{Z_\beta} \]
\[ \leq \sum_{k=n}^{m-1} \frac{k^{n/q}}{(k!)^{1/p}} \left( \frac{\hat{L}(T)T}{(\beta - \alpha)^{p/q}} \right)^{n/p} \|\xi - \mathcal{T}(\xi)\|_{Z_\alpha}. \]

The result is proved. ■

Finally, we prove regularity of the right-hand side of (17). In what follows, we will use the notation
\[ E^{(p)}(t, \varepsilon, \theta) := 1 + \sum_{n=1}^{\infty} t^n \frac{n^{\theta n}}{(n!)^{1/p}} \varepsilon^{\theta n} \] (18)

Observe that for \( p = 1 \) and \( \theta = 0 \) the right-hand side of (18) reduces to an exponential series, so that \( E^{(1)}(c, \varepsilon, 0) = e^c \).

Lemma 14 For any \( t, p, \varepsilon > 0 \) and \( \theta \in [0, \frac{1}{p}) \) we have
\[ E^{(p)}(t, \varepsilon, \theta) < \infty. \]

Proof. By analyzing the ratio of terms of series (18) we get
\[ \lim_{n \to \infty} \frac{t(n+1)^{\theta n} \varepsilon^{\theta(n+1)} ((n+1)!)^{1/p}}{t^n \varepsilon^{\theta n} (n!)^{1/p}} = \lim_{n \to \infty} t \frac{(n+1)^{\theta n + \theta - \frac{1}{p}}}{n^{\theta n}} \frac{1}{n^{\theta n}}, \]
\[ = \lim_{n \to \infty} \frac{t}{\varepsilon^{\theta}} \left( 1 + \frac{1}{n} \right)^{\theta n} (n+1)^{\theta - \frac{1}{p}}, \]
\[ = \frac{t}{\varepsilon^{\theta}} e^{\theta} \lim_{n \to \infty} (n+1)^{\theta - \frac{1}{p}} = 0, \]
provided \( \theta - \frac{1}{p} < 0 \), which proves the result. ■

Corollary 15 For any \( t, p > 0 \) and \( \theta \in [0, \frac{1}{p}) \) we have
\[ \lim_{n \to \infty} t^n \frac{n^{\theta n}}{(n!)^{1/p}} = 0. \]

Corollary 16 We have
\[ \lim_{n \to \infty} \sum_{k=n}^{m} \frac{\sqrt[k]{\hat{L}(T)^k T^k}}{(\beta - \alpha)^{k/q} \sqrt[k]{k!}} = 0 \]
for any \( \alpha < \beta \) and \( q > p \).
4 Proof of the existence and uniqueness

We now prove an important result which will immediately allow us to establish Theorem 7.

**Theorem 17** There exists a unique element $\xi_0 \in \mathbb{Z}$ such that for all $t \in T$ we have $T(\xi_0)(t) = \xi_0(t)$ almost everywhere. Moreover and for all $\xi \in Z_{\alpha}$

$$\lim_{n \to \infty} T^n(\xi) = \xi_0,$$

is true in $Z_\alpha$ for all $\alpha \in (\alpha_*, \bar{\alpha})$.

**Proof.** Let us fix $\xi \in Z_{\alpha_*}$. Lemma 13 and Corollary 16 show that the sequence $\{T^n(\xi)\}_{n=1}^\infty$ is Cauchy in $Z_\beta$ and therefore converges in $Z_\beta$ for all $\beta > \alpha_*$. Thus there exists $\xi_0 \in \mathbb{Z} = \cap_{\beta > \alpha_*} Z_\beta$ such that

$$\lim_{n \to \infty} T^n(\xi) = \xi_0,$$

where the convergence takes place in $Z_\beta$ for all $\beta > \alpha_*$. We can now fix arbitrary $\delta < \beta$ and observe that $T : Z_\delta \to Z_\beta$ is continuous. Therefore, passing to the limit in both sides of the equality

$$T(T^n(\xi)) = T^{n+1}(\xi) \in Z_\beta$$

we can conclude that

$$T(\xi_0) = \xi_0$$

in $Z_\beta$ for any $\beta \in \mathcal{A}$, which implies that for all $t \in T$ we have $T(\xi_0)(t) = \xi_0(t)$ almost everywhere.

Finally, suppose there exists another element $\eta_0 \in \mathbb{Z}$ such that for all $t \in T$ we have $T(\eta_0)(t) = \eta_0(t)$ almost everywhere. Then we see from the inequality (13) that,

$$\|\xi_0 - \eta_0\|_{Z_{\beta}} = \|T^n(\xi_0) - T^n(\eta_0)\|_{Z_{\beta}} \leq \frac{(a_p LT)^n}{(\alpha - \phi_1)^{\beta n}} \frac{n^{\beta n}}{\sqrt{n!}} \|\xi_0 - \eta_0\|_{Z_{\alpha}} \to 0, n \to \infty.$$

Thus $\|\xi_0 - \eta_0\|_{Z_{\beta}} = 0$ for any $\beta \in \mathcal{A}$. Hence $\xi_0$ is unique and the proof is complete.

The proof of the first two statements of Theorem 7 follows immediately from Theorem 17 above by letting $\xi \equiv u_0$. 


5 Continuity of the solution

Let \( \xi(t), t > 0 \), be the solution of equation (7) constructed in Theorems 17 and 7.

**Theorem 18** For any \( \alpha \in \mathcal{A} \), process \( \xi \) has a continuous modification \( \eta(t) \in X_\alpha, t \in [0, T] \), which solves equation (7).

**Proof.** Let us fix any \( \beta \in (\alpha_*, \alpha^*) \) and \( p \in [2, q) \). We can prove the existence of a continuous modification by an application of Kolmogorov’s continuity theorem in a rather standard way. Indeed, using (7) and the arguments similar to those used in the proof of (9) and (10) we obtain

\[
E||\xi(t) - \xi(s)||_{X_\beta}^p \leq \mathbb{E} \left[ \left| \int_s^t F(\xi(\tau))d\tau + \int_s^t \Phi(\xi(\tau))dW(\tau) \right|^p \right] \\
\leq \frac{C(t-s)}{(\beta - \alpha)^p} \|\xi\|_{Z_{a,T}^p}^p, \quad 0 \leq s < t \leq T,
\]

where, for \( \tau > 0 \),

\[
C(\tau) = (\tau^p + \left[ \frac{p}{2}(p-1) \right]^{p/2}) 2^{p-1}L_T \leq (T^{p/2} + \left[ \frac{p}{2}(p-1) \right]^{p/2}) 2^{p-1}L_T \tau^{p/2}.
\]

So we obtain the estimate

\[
E||\xi(t) - \xi(s)||_{X_\beta}^p = k(\xi, T) |t-s|^{p/2}
\]

with \( k(\xi, T) = (T^{p/2} + \left[ \frac{p}{2}(p-1) \right]^{p/2}) 2^{p-1}L_T \|\xi\|_{Z_{a,T}^p}^p \). The existence of a continuous modification \( \eta(t) \in X_\beta \) follows from Kolmogorov’s continuity theorem.

Since \( \xi \) satisfies (7) and \( \eta(t) = \xi(t) \) a.s. we have

\[
\eta(t) = \mathcal{T}(\xi)(t) \quad \text{a.s.}
\]

Observe now that by (11) we have

\[
\mathbb{E} \left[ ||\mathcal{T}(\xi)(t) - \mathcal{T}(\eta)(t)||_{X_\beta}^p \right] \leq \frac{C(t)}{(\beta - \alpha)^{p/2}} \|\xi - \eta\|_{Z_{a,T}^p}^p = 0, \quad 0 \leq t \leq T,
\]

which implies that \( \mathcal{T}(\xi)(t) = \mathcal{T}(\eta)(t) \) a.s. So we proved that

\[
\eta(t) = \mathcal{T}(\eta)(t) \quad \text{a.s.}
\]

This equality holds in \( X_\beta \) for any \( \beta \in (\alpha_*, \alpha^*) \). The proof is complete. ■

**Remark 19** Observe that \( \|\xi - \eta\|_{Z_{a,T}^p} = 0 \) for any \( \alpha \), so the processes \( \xi \) and \( \eta \) coincide as elements of \( Z_{a,T}^p \).
6 Estimate of the solution

In this section, we derive a norm estimate of the solution $\xi$ from Theorem 17.

Lemma 20 For any $\alpha < \beta \in A$ we have

$$\|\xi\|_{Z_\beta} \leq E^{(p)} \left( \sqrt[n]{L(T)T, \beta - \alpha, q^{-1}} \right) (1 + \|\xi_\alpha\|_{X_n})^p.$$ 

Proof. Consider the approximating sequence $\{\xi_n\} \subset Z$ defined by

$$\xi_n = T^n(\xi)$$

We can use inequality (15) and further estimate its right-hand side in the following way. We have

$$\|\xi_\alpha - T(\xi_\alpha)(s)\|_{X_\alpha}^p = E \left[ \| \int_0^t F(\xi_\alpha)ds + \int_0^t \Phi(\xi_\alpha)dW(s) \|_{X_\alpha}^p \right].$$

Remark III combined with the arguments similar to those used in the proof of (9) and (10) implies the estimate

$$E \left[ \|T^n(\xi_\alpha)(t) - T^{n+1}(\xi_\alpha)(t)\|_{X_\beta}^p \right] \leq \frac{n^{np/q} \tilde{L}(T)}{(\alpha - \alpha_*)^{p/q}} \left[ \frac{\tilde{L}(T)}{(\beta - \alpha_*)^{p/q}} \right]^{n+1} \frac{T^{n+1}}{(n+1)!} (1 + \|\xi_\alpha\|_{X_n})^p.$$ 

In particular, we can set $\alpha = \psi_n = \beta - \frac{\beta - \alpha_*}{n+1} = \alpha_* + \frac{n \beta - \alpha_*}{n+1}$. A direct calculation shows that the above inequality transforms into

$$E \left[ \|T^n(\xi_\alpha)(t) - T^{n+1}(\xi_\alpha)(t)\|_{X_\beta}^p \right] \leq \left[ \frac{\tilde{L}(T)(n+1)^{p/q}}{(\beta - \alpha_*)^{p/q}} \right]^{n+1} \frac{T^{n+1}}{(n+1)!} (1 + \|\xi_\alpha\|_{X_n})^p,$$

which implies that

$$\|T^n(\xi_\alpha) - T^{n+1}(\xi_\alpha)\|_{Z_{\beta}}^p \leq \frac{(n+1)^{n+1/p/q}}{(n+1)!} \left[ \frac{\tilde{L}(T)T}{(\beta - \alpha_*)^{p/q}} \right]^{n+1} (1 + \|\xi_\alpha\|_{X_n})^p.$$ 

Then

$$\|\xi_\alpha - T^n(\xi_\alpha)\|_{Z_\beta} \leq \sum_{n=1}^{m+1} \frac{n^{np/q}}{n!} \left[ \frac{\tilde{L}(T)T}{(\beta - \alpha_*)^{p/q}} \right]^{n} (1 + \|\xi_\alpha\|_{X_n})^p.$$
Passing to the limit as \( m \to \infty \) we obtain the bound

\[
||\zeta_{\alpha*} - \xi||_{Z_{\beta}} \leq \sum_{n=1}^{\infty} \frac{n^{p/q}}{n!} \left[ \frac{\hat{L}(T)T}{(\beta - \alpha_*)^{p/q}} \right]^n (1 + ||\zeta_{\alpha*}, x_{\alpha*}||)^p.
\]

Therefore

\[
||\xi||_{Z_{\beta}} \leq ||\zeta_{\alpha*}, x_{\alpha*}|| + \sum_{n=1}^{\infty} \frac{n^{p/q}}{n!} \left[ \frac{\hat{L}(T)T}{(\beta - \alpha_*)^{p/q}} \right]^n (1 + ||\zeta_{\alpha*}, x_{\alpha*}||)^p \\
\leq \left( 1 + \sum_{n=1}^{\infty} \frac{n^{p/q}}{n!} \left[ \frac{\hat{L}(T)T}{(\beta - \alpha_*)^{p/q}} \right]^n \right) (1 + ||\zeta_{\alpha*}, x_{\alpha*}||)^p \\
= E(p) \left( \sqrt[4]{\hat{L}(T)T}, \beta - \alpha, q^{-1} \right) (1 + ||\zeta_{\alpha*}, x_{\alpha*}||)^p,
\]

which completes the proof. ■

7 Stochastic spin dynamics of a quenched particle system

Our main example is motivated by the study of stochastic dynamics of interacting particle systems. We follow the scheme of paper [8], adapted to our present setting, which allows to show the existence of solutions with arbitrary large lifetime and their path-continuity.

Let \( \gamma \subset X = \mathbb{R}^d \) be a locally finite set (configuration) representing a collection of point particles. Each particle with position \( x \in X \) is characterized by an internal parameter (spin) \( \sigma_x \in S = \mathbb{R}^1 \).

We fix a configuration \( \gamma \) and look at the time evolution of spins \( \sigma_x(t), \ x \in \gamma \), which is described by a system of stochastic differential equations in \( S \) of the form

\[
d\sigma_x(t) = f_x(\bar{\sigma})dt + B_x(\bar{\sigma})dW_x(t), \ x \in \gamma,
\]

where \( \bar{\sigma} = (\sigma_x)_{x \in \gamma} \) and \( W = (W_x)_{x \in \gamma} \) is a collection of independent Wiener processes in \( S \). We assume that both drift and diffusion coefficients \( f_x \) and \( B_x \) depend only on spins \( \sigma_y \) with \( |y - x| < r \) for some fixed interaction radius \( r > 0 \) and have the form

\[
f_x(\bar{\sigma}) = \sum_{y \in \gamma} \varphi_{xy}(\sigma_x, \sigma_y), \quad B_x(\bar{\sigma}) = \sum_{y \in \gamma} \Psi_{xy}(\sigma_x, \sigma_y),
\]

which completes the proof. ■
where the mappings $\varphi_{xy} : S \times S \to S$ and $\Psi_{xy} : S \times S \to S$ satisfy finite range and uniform Lipschitz conditions, see Definition 23 and Condition 25 below.

Our aim is to realize system (19) as an equation in a suitable scale of Hilbert spaces and apply the results of previous sections in order to find its strong solutions.

We introduce the following notations:

- $S^\gamma := \prod_{x \in \gamma} S_x \ni \bar{\sigma} = (\sigma_x)_{x \in \gamma}, \sigma_x \in S_x = S$;
- $\gamma_{x,r} := \{ y \in \gamma : |x - y| < r \}, x \in \gamma$;
- $n_x \equiv n_{x,r}(\gamma) := \text{number of points in } \gamma_{x,r} ( = \text{number of particles interacting with particle in position } x)$.

Observe that, although the number $n_x$ is finite, it is in general unbounded function of $x$. We assume that it satisfies the following regularity condition.

From now on, we assume that the following condition holds.

**Condition 21** There exist constants $q > 2$ and $a(\gamma, r, q) > 0$ such that

$$n_{x,r}(\gamma) \leq a(\gamma, r, q) (1 + |x|)^{1/q} \tag{21}$$

for all $x \in X$.

**Remark 22** Condition (21) holds if $\gamma$ is a typical realization of a Poisson or Gibbs (Ruelle) point process in $X$. For such configurations, stronger (logarithmic) bound holds:

$$n_{x,r}(\gamma) \leq c(\gamma) [1 + \log(1 + |x|)] r^d,$$

see e.g. [28] and [23, p. 1047]. Thus (21) holds for any $q > 0$.

### 7.1 Existence of the dynamics

Our dynamics will live in the scale of Hilbert spaces

$$X_\alpha = S_\alpha^\gamma := \left\{ \bar{q} \in S^\gamma : \|\bar{q}\|_\alpha := \sqrt{\sum_{x \in \gamma} |q_x|^2 e^{-\alpha|x|}} < \infty \right\}, \quad 0 < \alpha_s < \alpha < \alpha^*.$$

We fix the parameters $\alpha_s$ and $\alpha^*$, which can be chosen in an arbitrary way.

We set

$$\mathcal{H} = S_0^\gamma := \left\{ \bar{q} \in S^\gamma : \|\bar{q}\|_0 := \sqrt{\sum_{x \in \gamma} |q_x|^2} < \infty \right\}$$
and define the corresponding spaces $GL_p(X)$ and $GL_p(X, \mathcal{F})$ (cf. Definition 2). Observe that $W(t) := (W_x(t))_{x \in \gamma}$ is a cylinder Wiener process in $\mathcal{H}$.

Let $V$ be a family of mappings $V_{xy} : S^2 \to S$, $x, y \in \gamma$.

**Definition 23** We call the family $V$ admissible if it satisfies the following two assumptions:

- finite range: there exists constant $r > 0$ such that $V_{xy} \equiv 0$ if $|x - y| \geq r$;
- uniform Lipschitz continuity: there exists constant $C > 0$ such that
  \[
  |V_{xy}(q'_1, q'_2) - V_{xy}(q''_1, q''_2)| \leq C (|q'_1 - q''_1| + |q'_2 - q''_2|) \]  
  (22)
  for all $x, y \in \gamma$ and $q'_1, q'_2, q''_1, q''_2 \in S$.

Define a map $\nabla : S^\gamma \to S^\gamma$ and a linear operator $\hat{\nabla} : S^\gamma \to S^\gamma, \bar{q} \in S^\gamma$, by the formula
\[
\nabla_x(\bar{q}) = \sum_{y \in \gamma} V_{xy}(q_x, q_y),
\]
and
\[
\left(\hat{\nabla}(\bar{q})\bar{\sigma}\right)_x := \nabla_x(\bar{q})\sigma_x, \ x \in \gamma, \ \bar{\sigma} \in S^\gamma,
\]
respectively.

**Lemma 24** Assume that $V$ is admissible. Then $\nabla \in GL_p(X)$ and $\hat{\nabla} \in GL_p(X, \mathcal{F})$.

The proof of this Lemma is quite tedious and will be given in Section 7.2 below.

Now we can return to the discussion of system (19). Assume that the following condition holds.

**Condition 25** The families of mappings $\{\varphi_{xy}\}_{x,y \in \gamma}$ and $\{\Psi_{xy}\}_{x,y \in \gamma}$ from (20) are admissible.

By Lemma 24 we have $\nabla \in GL_p(X)$ and $\hat{\Psi} \in GL_p(X, \mathcal{F})$. Thus we can write (19) in the form
\[
\bar{\sigma}(t) = \nabla(\bar{\sigma})dt + \hat{\Psi}(\bar{\sigma})dW(t),
\]
where $W(t) = (W_x(t))_{x \in \gamma}$, and apply the results of the previous sections to its integral counterpart. We summarize those results in the following theorem, which follows directly from Theorem 7.
This proof is a modification of the proof given in [8] for

\[ 7.2 \text{ Proof of Lemma 24.} \]

This result implies of course that, for each \( x \in \gamma \), equation (19) has a path-continuous strong solution, which is unique in the class of progressively measurable square-integrable processes.

\[ \text{Step 1.} \] We first show that \( V \) is a mapping \( S_{\alpha}^{\gamma} \to S_{\beta}^{\gamma} \) for any \( \alpha < \beta \). For any \( \bar{q} \in S_{\alpha}^{\gamma} \) we have

\[
||V(\bar{q})||_{\beta}^2 = \sum_{x \in \gamma} \left| \sum_{y \in \gamma} V_{xy}(q_{x}, q_{y}) \right|^2 e^{-\beta|x|} \\
\leq 3C^2 \sum_{x \in \gamma} \sum_{y \in \gamma, r} n_x \left( 1 + |q_x| + |q_y|^2 \right) e^{-\beta|x|}.
\]

The polynomial bound on the growth of \( n_x \) implies that

\[
\sum_{x \in \gamma} \sum_{y \in \gamma, r} n_x e^{-\beta|x|} = \sum_{x \in \gamma} n_x^2 e^{-\beta|x|} \leq \sum_{x \in \gamma} n_x^2 e^{-\alpha_*|x|} =: c(\gamma, \alpha_*) < \infty.
\]

Next, we estimate

\[
\sum_{x \in \gamma} \sum_{y \in \gamma, r} n_x |q_x|^2 e^{-\beta|x|} = \sum_{x \in \gamma} n_x^2 |q_x|^2 e^{-(\beta-\alpha)|x|} e^{-\alpha|x|} \\
\leq \sup_{x \in \gamma} (n_x^2 e^{-(\beta-\alpha)|x|}) ||\bar{q}||_{\alpha}^2.
\]

Observe that

\[
\sum_{x \in \gamma} \sum_{y \in \gamma, r} = \sum_{x, y \in \gamma, \gamma, y, y \in \gamma, y \in \gamma, r} = \sum_{y \in \gamma} \sum_{x \in \gamma, r} , \text{ and so}
\]

\[
\sum_{x \in \gamma} \sum_{y \in \gamma, r} n_x |q_y|^2 e^{-\beta|x|} \leq e^{\beta r} \sum_{y \in \gamma} N_y |q_y|^2 e^{-(\beta-\alpha)|y|} e^{-\alpha|y|} \\
\leq e^{\beta r} \sup_{y \in \gamma} (N_y e^{-(\beta-\alpha)|y|}) ||\bar{q}||_{\alpha}^2,
\]

where \( N_y := \sum_{x \in \gamma, y, r} n_x \). Here we used inequality \(|y| \leq |y - x| + |x| \leq r + |x|\) for \( y \in \gamma, r \), so that \( e^{-\beta|x|} \leq e^{|x|} e^{-\beta|y|} \). Condition (24) implies that

\[
N_x \leq a(\gamma, r, q^2 (1 + |x|)^{1/q} (1 + r + |x|)^{1/q} < c(\gamma, r, q) (1 + |x|)^{2/q},
\]

\[ 16 \]
for some constant $c(\gamma, r, q) > 0$, and

$$n_x^2 \leq a(\gamma, r, q)^2 (1 + |x|)^{2/q}$$

for any $x \in \gamma$. Eventually we obtain the bound

$$\|\nabla(q)\|_\beta^2 \leq L^2 \left\{ \sup_{s>0} (1 + s)e^{-(\beta - \alpha)s} \right\}^{2/q} \|q\|_\alpha^2 \leq L^2 (\beta - \alpha)^{-2/q} \|q\|_\alpha^2 < \infty, \quad L < \infty.$$

**Step 2.** Lipschitz condition (22) implies the estimate

$$\|\nabla(q') - \nabla(q'')\|_\beta^2 = \sum_{x \in \gamma} \left| \sum_{y \in \gamma} V_{xy}(q'_x, q'_y) - \sum_{y \in \gamma} V_{xy}(q''_x, q''_y) \right|^2 e^{-\beta|x|} \leq 2C^2 \sum_{x \in \gamma} \sum_{y \in \gamma} n_x \left( |q'_x - q''_x|^2 + |q'_y - q''_y|^2 \right) e^{-\beta|x|}$$

for any $q', q'' \in S^\gamma_\alpha$. Similar to Step 1, we obtain the bound

$$\|\nabla(q') - \nabla(q'')\|_\beta^2 \leq L^2 \left\{ \sup_{s>0} (1 + s)e^{-(\beta - \alpha)s} \right\}^{2/q} \|q' - q''\|_\alpha^2 \leq L^2 (\beta - \alpha)^{-2/q} \|q' - q''\|_\alpha^2 < \infty, \quad L < \infty.$$

**Step 3.** The inclusion $\nabla(q) \in S^\gamma_\beta$ implies that $\hat{V}(\hat{q})\hat{\sigma} \in S^\gamma_\beta$ for any $\hat{\sigma} \in H = S^\gamma_0$. A direct calculation shows that $\hat{V}(\hat{q}) : H \to S^\gamma_\beta$ is a Hilbert-Schmidt operator with the norm equal to $\|\hat{V}(\hat{q})\|_\beta$. Thus the inclusion $\nabla \in \mathcal{GL}^{(1)}$ implies that $\hat{V} \in \mathcal{GL}^{(2)}$. $\square$

### 8 Further examples

In this section we give two examples of linear maps of the class $\mathcal{GL}_q(\mathfrak{B})$.

**Example 1.** Let $B_\alpha := L^p[\mathbb{R}^1, e^{-\alpha|x|}dx], \quad p > 1, \quad f(u) = Au$, where $A$ is the integral operator with kernel $K$:

$$Au(x) = \int K(x, y)u(y)dy, \quad x, y \in \mathbb{R}^1.$$

**Condition 27** There exist $\beta^* > \alpha^*$ and $a > 0$ such that

$$|K(x, y)| \leq ae^{-\alpha^*|x-y|} (1 + |y|)^{\delta}, \quad \delta > 0,$$

for a.a. $x \in \mathbb{R}$.
Assume that Condition 27 holds. Then Proposition 29 we need \( q = \frac{p-1}{p} \).

Remark 30 For an implementation of any version of Ovsyannikov-type method, we need \( q \geq 1 \), which implies \( \delta \leq \frac{p-1}{p} < 1 \).

Proof. We start with the following estimate of the norm of operator \( A \):

\[
\|Au\|_{B^\alpha}^p = \int \left[ \int K(x,y)u(y)dy \right]^p e^{-\beta|x|} dx
\]

\[
\leq a^p \int \left[ \int e^{-\beta|x-y|} (1 + |y|)^{\delta} |u(y)| dy \right]^p e^{-\beta|x|} dx
\]

\[
= a^p \int \left[ \int e^{-\varepsilon|x-y|} (1 + |y|)^{\delta} |u(y)| e^{-\varepsilon|x-y|} dy \right]^p e^{-\beta|x|} dx,
\]

where \( \varepsilon = \frac{\beta - \varepsilon}{p} \). Observe that

\[
e^{-\varepsilon|x-y|} e^{-\varepsilon|x-y|} \leq e^{-\varepsilon|x|},
\]

so that

\[
\|Au\|_{B^\alpha}^p \leq a^p \int \left[ \int e^{-\varepsilon|x-y|} (1 + |y|)^{\delta} |u(y)| e^{-\varepsilon|x-y|} dy \right]^p dx.
\]

For \( \theta \) such that \( \theta^{-1} + p^{-1} = 1 \) we have

\[
e^{-\varepsilon|x-y|} (1 + |y|)^{\delta} |u(y)| e^{-\varepsilon|x-y|} = \left[ e^{-\varepsilon|x-y|} (1 + |y|)^{\delta} e^{-\varepsilon|x-y|} \right] \times \left[ e^{-\varepsilon|x-y|} |u(y)| e^{-\varepsilon|x-y|} \right].
\]

Then, by Holder’s inequality,

\[
\|Au\|_{B^\alpha}^p \leq a^p \int \left[ \int e^{-\varepsilon|x-y|} (1 + |y|)^{\delta} e^{-\varepsilon|x-y|} \right]^{\theta} \times \left[ \int e^{-\varepsilon|x-y|} |u(y)| e^{-\varepsilon|x-y|} \right]^{p/\theta} dx
\]

\[
= a^p \int \left[ \int e^{-\varepsilon|x-y|} (1 + |y|)^{\delta} e^{-\varepsilon|x-y|} \right]^{\theta} \times \left[ \int e^{-\varepsilon|x-y|} |u(y)| \right]^{p/\theta} e^{-\alpha|y|} dy dx
\]

\[
\leq a^p b \left[ \int e^{-\varepsilon|x-y|} dy \right]^{p/\theta} \times \int e^{-\varepsilon|x-y|} |u(y)|^{p} e^{-\alpha|y|} dy dx
\]

\[
\leq a^p b^{p/\theta} \int \int e^{-\varepsilon|x-y|} |u(y)|^{p} e^{-\alpha|y|} dy dx,
\]
where
\[ b = \sup_{s \geq 0} (1 + s)^{\theta \delta} e^{-\frac{\theta}{p}(\beta - \alpha)s} \]
and
\[ c = \int e^{-\frac{p}{p'}|x-y|} dy > \int e^{-\varepsilon|y|} dy. \]

Observe that
\[ \int \int e^{-\varepsilon|x-y|} |u(y)|^p e^{-\alpha|y|} dy dx = \int e^{-\varepsilon|x-y|} dy \|u\|_{B_\alpha}^p = c \|u\|_{B_\alpha}^p, \]

which leads to the bound
\[ \|Au\|_{B_\alpha}^p \leq d^p bc^{p/\theta + 1} \|u\|_{B_\alpha}^p. \]

It remains to compute constant \( b = \left[ \sup_{s \geq 0} (1 + s) e^{-\frac{1}{\theta \delta}(\beta - \alpha)s} \right]^{\theta \delta} \).

Equating to 0 the derivative \( \frac{\partial}{\partial s} (1 + s) e^{-\frac{1}{\theta \delta}(\beta - \alpha)s} \) we obtain
\[ b = \frac{C}{(\beta - \alpha)^{\theta \delta}}, \]
for some constant \( C > 0 \).

It is clear that estimate (1) holds with \( q = \frac{1}{\theta \delta} = \frac{p-1}{p\delta} \). □

**Example 2.** A somewhat similar example is given by the spaces of sequences
\[
B_\alpha := \left\{ (u_k)_{k \in \mathbb{Z}} : \sum_{k \in \mathbb{Z}} |u_k|^p e^{-\alpha|k|} < \infty \right\}, p > 1,
\]

and the linear map given an infinite matrix \( A = (A_{kj})_{k,j \in \mathbb{Z}} \) is with elements satisfying the bound
\[ |A_{kj}| \leq a e^{-\frac{\theta}{p'}|k-j|} (1 + |j|)^{\delta} \]
for some \( \beta^* > \alpha^*, a > 0 \) and all \( k \in \mathbb{Z} \). The proof of the inclusion \( A \in \mathcal{G}L_q(\mathcal{B}), q = \frac{p-1}{p\delta} \), is similar to that of Proposition 29. Similar to the previous example, we have in general
\[ |A_{kk}| \to \infty, \ k \to \infty, \]
so that operator \( A \) is unbounded in any weighted \( L^p \).
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