Asymptotic stability of solutions to the Navier–Stokes–Fourier system driven by inhomogeneous Dirichlet boundary conditions

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ABSTRACT
We consider global in time solutions of the Navier–Stokes–Fourier system describing the motion of a general compressible, viscous and heat conducting fluid far from equilibrium. Using a new concept of weak solution suitable to accommodate the inhomogeneous Dirichlet time dependent data we find sufficient conditions for the global in time weak solutions to be ultimately bounded.

1. Introduction
The Navier–Stokes–Fourier system describing the time evolution of the mass density \(\rho = \rho(t,x)\), the velocity \(\mathbf{u} = \mathbf{u}(t,x)\), and the temperature \(\vartheta = \vartheta(t,x)\) of a general compressible, viscous, and heat-conducting fluid, endowed with inhomogeneous boundary conditions, is a prominent example of a dissipative system in the framework of continuum fluid mechanics. In general, a dissipative system is a thermodynamically open system confined to a physical space \(\Omega \subset \mathbb{R}^d\) and considered far from equilibrium, exchanging energy and matter with the outer world. The field equations of the Navier–Stokes–Fourier system describing the motion in the interior of the cavity \(\Omega\) read

\[
\partial_t \rho + \text{div}_x(\rho \mathbf{u}) = 0, 
\]

\[
\partial_t (\rho \mathbf{u}) + \text{div}_x(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\vartheta, \vartheta) = \text{div}_x \mathbf{S}(\vartheta, \nabla_x \mathbf{u}) + \rho \mathbf{g}, 
\]

\[
\partial_t (\rho \vartheta) + \text{div}_x(\rho \vartheta \mathbf{u}) + \text{div}_x \mathbf{q}(\vartheta, \nabla_x \vartheta) = \mathbf{S}(\vartheta, \nabla_x \mathbf{u} : \mathbf{D}_x \mathbf{u} - p(\vartheta, \vartheta) \text{div}_x \mathbf{u}), 
\]

where the viscous stress \(\mathbf{S}\) is given by Newton’s law

\[
\mathbf{S}(\vartheta, \nabla_x \mathbf{u}) = \mu(\vartheta) \left( \nabla_x \mathbf{u} + \nabla_x^T \mathbf{u} - \frac{2}{d} \text{div}_x \mathbf{u} \mathbf{I} \right) + \eta(\vartheta) \text{div}_x \mathbf{u} \mathbf{I}, \quad \nabla_x \mathbf{u} = \frac{1}{2} (\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u}),
\]

and the heat flux \(\mathbf{q}\) by Fourier’s law

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\( \mathbf{q} (\partial, \nabla_x \mathbf{u}) = -\kappa (\partial) \nabla_x \partial. \) \tag{1.5} 

The pressure \( p \) and the internal energy \( e \) are interrelated through Gibbs' equation

\[ \partial Ds = De + pD \left( \frac{1}{\partial} \right), \tag{1.6} \]

where \( s \) is the entropy. In view of (1.6), the internal energy balance (1.3) may be replaced by the entropy balance

\[ \partial_t (gs(\partial, \partial)) + \text{div}_x (gs(\partial, \partial) \mathbf{u}) + \text{div}_x \left( \frac{\mathbf{q}(\partial, \nabla_x \partial)}{\partial} \right) = \frac{1}{\partial} \left( S(\partial, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \partial}{\partial} \right). \] \tag{1.7}

As we shall see below, it is the entropy balance (1.7) that is more convenient for the weak formulation of the problem.

The fluid occupies a bounded domain \( \Omega \), whereas the velocity \( \mathbf{u} \) and the temperature \( \vartheta \) satisfy the inhomogeneous Dirichlet boundary conditions

\[ \mathbf{u} = \mathbf{u}_B \text{ on } (T, \infty) \times \partial \Omega, \]

\[ \vartheta = \vartheta_B \text{ on } (T, \infty) \times \partial \Omega. \] \tag{1.8}

Accordingly, the boundary of the space–time cylinder \( (T, \infty) \times \partial \Omega \) can be decomposed as

\[ \Gamma_{\text{in}} = \{(t, x) \in (T, \infty) \times \partial \Omega \mid \mathbf{u}_B(t, x) \cdot \mathbf{n}(x) < 0\}, \]

\[ \Gamma_{\text{wall}} = \{(t, x) \in (T, \infty) \times \partial \Omega \mid \mathbf{u}_B(t, x) \cdot \mathbf{n}(x) = 0\}, \]

\[ \Gamma_{\text{out}} = \{(t, x) \in (T, \infty) \times \partial \Omega \mid \mathbf{u}_B(t, x) \cdot \mathbf{n}(x) > 0\}, \]

where \( \mathbf{n} \) denotes the outer normal vector to \( \partial \Omega \). Finally, the density must be prescribed on the inflow component of the boundary,

\[ \vartheta = \vartheta_B \text{ on } \Gamma_{\text{in}}. \] \tag{1.10}

As shown in a series of papers by Matsumura and Nishida [1, 2], Valli [3, 4], Valli and Zajaczkowski [5], the Navier–Stokes–Fourier system is globally well posed in the class of classical solutions in a small neighborhood of a stable equilibrium. Unfortunately, this perturbation technique cannot be used in the far from equilibrium regime usually associated to turbulence. In view of the well known and so far unsurmountable problems concerning the existence of suitable \textit{a priori} bounds for large data global in time solutions to non–linear problems in fluid dynamics, the only available framework are the weak solutions in the spirit of the pioneering work by Leray [6], and later, in the context of compressible fluids, by Lions [7].

A mathematical theory based on weak solutions for the complete fluid systems was presented in [8]. Unfortunately, the concept of weak solutions developed in [8] applies to energetically closed systems with \( \mathbf{u}_B = 0 \) and with (1.9) replaced by the homogeneous Neumann boundary condition

\[ \nabla_x \vartheta \cdot \mathbf{n}|_{\partial \Omega} = 0. \]
Under these circumstances, the long time behavior of solutions is well understood, and, in the case of a time independent driving force \( g = g(x) \), obeys the following dichotomy:

- Either \( g = \nabla_x F \) and then all solutions tend an equilibrium;
- or \( g \neq \nabla_x F \) and then

\[
\int_\Omega \left[ \frac{1}{2} \rho |u|^2 + \rho e(\rho, \vartheta) \right] (t, \cdot) \, dx \to \infty \quad \text{as} \quad t \to \infty
\]

for any weak solution \((\rho, u, \vartheta)\);

see [9, 10]. It turns out that driving the system by means of a non–trivial volume force while keeping the conservative boundary conditions is not realistic and definitely not suitable for describing phenomena related to turbulence.

The mathematical theory of weak solutions has been extended to the **energetically open** Navier–Stokes–Fourier system only recently in [11], and, finally, in [12]. In particular, the inhomogeneous boundary condition for the temperature requires a new approach, developed in [12], based on the balance of ballistic energy in the sense of Ericksen [13]. Recently, the existence of weak solutions has also been proved for a bi-fluid model for a mixture of two compressible non interacting fluids with general boundary data in [14].

Note that this kind of boundary conditions is physically relevant, in particular for the Rayleigh–Bénard problem and Taylor experiment arising in models of turbulence, cf. Birnir and Svanstedt [15], Constantin et al. [16, 17], Davidson [18] among others.

To the best of our knowledge, this is the first attempt to describe the asymptotic behavior of the Navier–Stokes–Fourier system with the **inhomogeneous Dirichlet** boundary conditions. We focus on the problem of **global boundedness** of trajectories and the existence of bounded absorbing sets. This is the concept of **dissipativity in the sense of Levinson** extending the classical approach to closed systems via a Lyapunov function, see e.g. Haraux [19], Kuznetsov and Reitmann [20, Chapter 1, Section 1.2]. The crucial quantity is the **ballistic energy**

\[
\int_\Omega E_B(\rho, \vartheta, u) \, dx, \quad E_B(\rho, \vartheta, u) \equiv \frac{1}{2} \rho |u - u_B|^2 + \rho e(\rho, \vartheta) - \vartheta B \rho s(\rho, \vartheta)
\]

where \( u_B, \vartheta_B \) are suitable extensions of the boundary data inside \( \Omega \).

We say that the Navier–Stokes–Fourier system (1.1)–(1.10) is **Levinson dissipative** if there exists a universal constant \( \mathcal{E}_\infty \) such that

\[
\limsup_{t \to \infty} \int_\Omega E_B(\rho, \vartheta, u) \, dx \leq \mathcal{E}_\infty
\]

for any (weak) solution \((\rho, \vartheta, u)\) defined on a time interval \((T_0, \infty)\).

Our goal is to identify a class of constitutive relations (equations of state (EOS), viscosity and heat conductivity coefficients), for which the Navier–Stokes–Fourier system (1.1)–(1.10) is Levinson dissipative. We adopt the following strategy:

1. In **Section 2**, we recall the concept of weak solution introduced in [12], together with the associated ballistic energy inequality.
2. Inspired by [21], we introduce a class of equations of state penalizing the pressure if the density approaches a critical value \( \tilde{\rho} \), see Section 3.

3. The main results are stated in Section 4.

4. In Section 5, we show the uniform bounds of the ballistic energy proving Levinson dissipativity of the Navier–Stokes–Fourier system. We also study convergence to equilibrium solutions for a particular class of boundary data.

5. Possible applications including the existence of global attractors, statistical solutions, and the existence of time periodic solutions are briefly sketched in Section 6.

Although several ideas, in particular how to control the perturbation of the convective term by the hard sphere pressure already appeared in [22], the main contribution of the present paper is incorporating the temperature changes enforced through the Dirichlet boundary conditions. The resulting estimates are new and surprising to certain extent as they contrast the results for the energetically closed systems mentioned above. In particular, the energy of an open system may remain ultimately bounded though its flux through the physical boundary is a priori not controlled. Last but not least, the fluid motion driven by the boundary data is relevant in the models related to turbulence in contrast with the mathematically convenient but physically less convincing driving term in the form of a volume force amply used in many studies.

2. Weak solutions, ballistic energy

We recall the concept of weak solution and ballistic energy introduced in [12].

2.1. Weak solutions

We consider the Navier–Stokes–Fourier system (1.1)–(1.10) defined on the set \( (T, \infty) \times \Omega, \) where \( T < \infty. \) Although the existence theory is formulated in terms of the initial data

\[
\rho(T, \cdot) = \rho_0, \quad (\rho u)(T, \cdot) = \mathbf{m}_0, \quad \rho s(T, \cdot) = S_0, \quad S_0 = \rho_0 s(\rho_0, \vartheta_0),
\]

their specific form is irrelevant for the analysis of the present paper.

**Definition 2.1.** (Weak solution).

We say that a trio of functions \((\rho, \vartheta, \mathbf{u})\) is a weak solution of the Navier–Stokes–Fourier system (1.1)–(1.10) in \((T, \infty) \times \Omega\) if the following holds:

- Equation of continuity: \( \rho \geq 0 \) and the integral identity

\[
\int_T^\infty \int_\Omega [\rho \partial_t \varphi + \rho \mathbf{u} \cdot \nabla \varphi] \, dx \, dt = \int_T^\infty \int_{\partial \Omega} \varphi \rho_B [\mathbf{u}_B \cdot \mathbf{n}]^- d\sigma_x \, dt + \int_T^\infty \int_{\partial \Omega} \varphi \rho [\mathbf{u}_B \cdot \mathbf{n}]^+ d\sigma_x \, dt
\]  

(2.1)
holds for any $\varphi \in C^1_c((T, \infty) \times \Omega)$.

- Momentum equation:
  $$\mathbf{u} \in L^r_\text{loc}(T, \infty; W^{1, r}(\Omega; \mathbb{R}^d)) \text{ for some } r > 1,$$
  $$(\mathbf{u} - \mathbf{u}_B) \in L^r_\text{loc}(T, \infty; W^{1, r}_0(\Omega; \mathbb{R}^d)),$$
  (2.2)

and

\[ \int_T^\infty \int_\Omega \left[ \varrho \mathbf{u} \cdot \partial_t \mathbf{\varphi} + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla x \mathbf{\varphi} + p(\varrho, \vartheta) \text{div} x \mathbf{\varphi} \right] \, dx \, dt \]
\[ = \int_T^\infty \int_\Omega \left[ \mathbb{S}(\partial_\vartheta, \nabla x \mathbf{u}) : \nabla x \mathbf{\varphi} - \varrho \mathbf{g} \cdot \mathbf{\varphi} \right] \, dx \, dt \] (2.3)

for any $\mathbf{\varphi} \in C^1_c((T, \infty) \times \Omega; \mathbb{R}^d)$.

- Entropy inequality:
  \[ -\int_T^\infty \int_\Omega \left[ \varrho_s \partial_t \mathbf{\varphi} + \varrho s \mathbf{u} \cdot \nabla x \mathbf{\varphi} + \frac{q}{\vartheta} \cdot \nabla x \mathbf{\varphi} \right] \, dx \, dt \]
  \[ \geq \int_T^\infty \int_\Omega \frac{\varrho}{\vartheta} \left( \mathbb{S}(\partial_\vartheta, \nabla x \mathbf{u}) : \nabla x \mathbf{u} - \frac{q(\partial_\vartheta, \nabla x \vartheta)}{\vartheta} \cdot \nabla x \vartheta \right) \, dx \, dt \] (2.4)

for any $\varphi \in C^1_c((T, \infty) \times \Omega)$, $\vartheta \geq 0$.

- Ballistic energy inequality: For any
  \[ \tilde{\vartheta} \in C^1 \left( \left[ T, \infty \right) \times \bar{\Omega} \right), \tilde{\vartheta} > 0, \tilde{\vartheta}|_{\partial \Omega} = \vartheta_B \]

there holds

\[ -\int_T^\infty \partial_t \psi \int_\Omega \left( \frac{1}{2} \varrho |\mathbf{u} - \mathbf{u}_B|^2 + \varrho e - \vartheta \varrho_s \right) \, dx \, dt \]
\[ + \int_T^\infty \psi \int_{\partial \Omega} [\varrho e(\varrho_B, \vartheta_B) - \vartheta \varrho e(\varrho, \vartheta_B)] [\mathbf{u}_B \cdot \mathbf{n}]^{-} \, d\sigma_x \, dt \]
\[ + \int_T^\infty \psi \int_{\partial \Omega} [\varrho e(\varrho, \vartheta_B) - \vartheta \varrho e(\varrho_B, \vartheta_B)] [\mathbf{u}_B \cdot \mathbf{n}]^{+} \, d\sigma_x \, dt \]
\[ + \int_T^\infty \psi \int_\Omega \frac{\tilde{\vartheta}}{\vartheta} \left( \mathbb{S} : \nabla x \mathbf{u} - \frac{q \cdot \nabla x \vartheta}{\vartheta} \right) \, dx \, dt \] (2.5)

\[ \leq -\int_T^\infty \psi \int_\Omega \left[ \varrho (\mathbf{u} - \mathbf{u}_B) \otimes (\mathbf{u} - \mathbf{u}_B) + \rho \mathbf{I} - \mathbb{S} \right] : \nabla x \mathbf{u}_B \, dx \, dt \]
\[ + \int_T^\infty \psi \int_\Omega \varrho (\mathbf{u} - \mathbf{u}_B) \cdot (\mathbf{g} - \partial_t \mathbf{u}_B - \mathbf{u}_B \cdot \nabla x \mathbf{u}_B) \, dx \, dt \]
\[ - \int_T^\infty \psi \int_\Omega \left[ \varrho s (\partial_t \tilde{\vartheta} + \mathbf{u} \cdot \nabla x \tilde{\vartheta}) + \frac{q \vartheta}{\vartheta} \cdot \nabla x \tilde{\vartheta} \right] \, dx \, dt \]

for any $\psi \in C^1_c(T, \infty)$, $\psi \geq 0$.

The symbol $\mathbf{u}_B$ in (2.5) denotes any $C^1$ extension of the boundary velocity $\mathbf{u}_B$. It can be shown (cf. [22, Remark 2.2]) that the specific form of the ballistic energy inequality is independent of the extension $\mathbf{u}_B$. Specifically, if (2.5) holds for some $\mathbf{u}_B$, then it holds
for any $u_B$ attaining the same boundary value. We point out that such a statement may not hold for the temperature extension $\bar{\vartheta}$.

**Remark 2.2.** The regularity of a weak solution $(\varrho, \vartheta, u)$ is determined by the available a priori bounds based mostly on the ballistic energy inequality. In particular, the density $\varrho$ is only Lebesgue integrable while both (2.1) and (2.5) refer to its trace on $\Gamma_{out}$. The latter is understood in the following way. The velocity $u$ is a Sobolev function that admits a trace. Moreover, the vector field $[\varrho, 2\mathcal{O}u]$ having zero space–time divergence admits a normal trace on the space time cylinder $(T, \infty) \times \Omega$. Consequently, $\varrho|_{\Gamma_{out}}$ is determined through

$$
\varrho u \cdot n|_{\partial\Omega} = \varrho|_{\Gamma_{out}} u_B \cdot n.
$$

The reader may consult [11] for details.

The existence of global–in–time weak solutions under certain restrictions imposed on the constitutive relations was proved in [12, Theorem 4.2]. The weak solutions also comply with the weak–strong uniqueness principle, see [12, Theorem 3.1]. Specifically, any weak solution in the sense of Definition 2.1 coincides with the strong solution of the Navier–Stokes–Fourier system driven by the same initial/boundary data as long as the strong solution exists.

The ballistic energy inequality (2.5) can be written in a more concise form

$$
\frac{d}{dt} \int_{\Omega} \left( \frac{1}{2} \varrho |u - u_B|^2 + \varrho e - \vartheta \varrho s \right) \, dx
\quad + \int_{\partial\Omega} \left[ \varrho e(\varrho_B, \vartheta_B) - \vartheta_B \varrho_B s(\varrho_B, \vartheta_B) \right] [u_B \cdot n]^+ d\sigma_x
\quad + \int_{\partial\Omega} \left[ \varrho e(\varrho, \vartheta_B) - \vartheta_B \varrho_B s(\varrho, \vartheta_B) \right] [u_B \cdot n]^+ d\sigma_x
\quad + \int_{\Omega} \vartheta \left( \mathcal{D}u - \frac{q \cdot \nabla_x \vartheta}{\vartheta} \right) \, dx
\leq - \int_{\Omega} \left[ \varrho (u - u_B) \otimes (u - u_B) + pI - \mathcal{S} \right] : \mathcal{D}u_B \, dx
\quad + \int_{\Omega} \varrho (u - u_B) \cdot (g - \partial_t u_B - u_B \cdot \nabla_x u_B) \, dx
\quad - \int_{\Omega} \left[ \varrho s(\partial_t \vartheta + u \cdot \nabla_x \vartheta) + \frac{q \cdot \nabla \vartheta}{\vartheta} \right] \, dx \tag{2.6}
$$

understood in $\mathcal{D}'(T, \infty)$.

Our goal is to identify the class of boundary data $u_B, \vartheta_B$, for which (2.6) gives rise to a globally bounded ballistic energy,

$$
\limsup_{t \to \infty} \int_{\Omega} \left( \frac{1}{2} \varrho |u - u_B|^2 + \varrho e - \vartheta_B \varrho s \right) \, dx \leq \mathcal{E}_\infty, \tag{2.7}
$$

for a certain $\vartheta_B$. In other words, the Navier–Stokes–Fourier system is Levinson dissipative. Although the ballistic energy need not be non–negative, we show that its “entropy” component is dominated by the internal energy. More specifically, the bound (2.7) is equivalent to
\[ \limsup_{t \to \infty} \int_{\Omega} \left( \frac{1}{2} \rho |u - u_B|^2 + qe \right) \, dx \leq \mathcal{E}_{\infty}, \]

modulo a suitable modification of \( \mathcal{E}_{\infty} \).

Finally, note that if \( \vartheta_B \) is a positive constant, then (2.6) reduces to an energy inequality formally similar to that for the barotropic system studied in [22]. Thus, similarly to [22], a hard sphere pressure equation of state is necessary to keep the density bounded and to control the first integral on the right-hand side of (2.6). On the other hand, under the non-slip boundary conditions \( u_B = 0 \), inequality (2.6) gives rise to

\[ \frac{d}{dt} \int_{\Omega} \left( \frac{1}{2} \rho |u|^2 + qe - \vartheta \vartheta s \right) \, dx \]

\[ + \int_{\Omega} \vartheta \left( \mathbb{S} : \nabla u - \frac{q \cdot \nabla \vartheta}{\vartheta} \right) \, dx \]

\[ \leq \int_{\Omega} \left[ q \vartheta (\vartheta \vartheta) + u \cdot \nabla \vartheta + \frac{q \cdot \nabla \vartheta}{\vartheta} \right] \, dx; \]

whence the rightmost integral must be dominated by the dissipation

\[ \int_{\Omega} \vartheta \left( \mathbb{S} : \nabla u - \frac{q \cdot \nabla \vartheta}{\vartheta} \right) \, dx. \]

Again this does not seem realistic unless uniform bounds on the density \( \rho \) are a priori imposed. The above arguments justify the choice of the hard sphere pressure equation of state introduced in the section below.

### 3. Constitutive relations, equation of state

Before stating the main results, we introduce the structural hypotheses imposed on the equations of state motivated by [8]. We suppose

\[ \bar{p}(\rho, \vartheta) = \bar{\vartheta} \frac{P(\bar{\vartheta})}{\bar{\vartheta}} + a \frac{\bar{\vartheta}^4}{\bar{\vartheta}} + c(\rho, \vartheta) = \frac{3}{2} \bar{\vartheta} \frac{P(\bar{\vartheta})}{\bar{\vartheta}} + a \bar{\vartheta}^4, \quad a > 0, \]

where \( P \in C^1(0, \infty) \) satisfies

\[ P(0) = 0, \ P'(Z) > 0 \ \text{for} \ Z \geq 0, \ 0 < \frac{3}{2} \frac{P(Z) - P'(Z)Z}{Z} \leq c \ \text{for} \ Z > 0. \]

In particular, the function \( Z \mapsto P(Z)/Z^3 \) is decreasing, and we suppose

\[ \lim_{Z \to \infty} \frac{P(Z)}{Z^3} = p_\infty \geq 0. \]

The associated entropy \( s \) reads

\[ s(\rho, \vartheta) = S\left( \frac{\rho}{\vartheta^2} \right) + \frac{4a}{3} \frac{\vartheta^3}{\rho}, \]

where

\[ S'(Z) = \frac{3}{2} \frac{3}{2} \frac{P(Z) - P'(Z)Z}{Z^2}. \]
In addition, following [22], we introduce the hard sphere perturbation of the equation of state,

\[ p(q, \vartheta) = \tilde{p}(q, \vartheta) + p_{HS}(q), \]

\[ p_{HS} \in C^1(0, \tilde{q}), p_{HS}(0) = 0, p'_{HS} > 0 \text{ in } (0, \tilde{q}), \lim_{q \to \tilde{q}} p_{HS}(q) = \infty. \] (3.6)

The related internal energy reads

\[ e(q, \vartheta) = \tilde{e}(q, \vartheta) + \int_{\tilde{q}/2}^{q} \frac{p_{HS}(z)}{z} \, dz. \] (3.7)

**Remark 3.1.** The idea leading to the hard sphere perturbation is quite natural, namely, the density of any real fluid admits a natural bound enforced by the molecular theory: the volume of a real fluid cannot be made arbitrarily small, see e.g. the textbook by Kastler et al. [23]. We refer to Carnahan and Starling [24] or Kolafa et al. [25] for specific examples of equations of state that comply with this principle. The main effect enforced by the hard sphere pressure equation of state is, of course, the uniform bound imposed \textit{a priori} on the fluid density,

\[ 0 \leq q(t, x) \leq \tilde{q}, \] (3.8)

for any weak solution of the Navier–Stokes–Fourier system.

The transport coefficients \( \mu, \eta, \) and \( \kappa \) are continuously differentiable functions of the temperature \( \vartheta \) satisfying

\[ 0 < \mu (1 + \vartheta^{\Lambda}) \leq \mu(\vartheta) \leq \bar{\mu}(1 + \vartheta^{\Lambda}), |\mu'(\vartheta)| \leq c \text{ for all } \vartheta \geq 0, \frac{1}{2} \leq \Lambda \leq 1, \]

\[ 0 \leq \eta(\vartheta) \leq \bar{\eta}(1 + \vartheta^{\Lambda}), \]

\[ 0 < \kappa(1 + \vartheta^{\beta}) \leq \kappa(\vartheta) \leq \bar{\kappa}(1 + \vartheta^{\beta}), \beta \geq 0. \] (3.9)

In addition, we say that \( s \) is compatible with the Third law of thermodynamics, if

\[ \lim_{\vartheta \to 0} s(q, \vartheta) = 0 \text{ for any fixed } q > 0. \] (3.10)

The existence theory developed in [12] can be easily modified to accommodate the hard–sphere pressure, at least in the specific form

\[ p_{HS}(q) \approx (\tilde{q} - q)^{-\beta}, \beta > 3, \]

cf. also [26].

**4. Main results**

We are ready to present our main results. If not otherwise stated, we suppose that the boundary data enjoy the degree of smoothness necessary for the analysis, and that \( \partial \Omega \) is sufficiently smooth. In addition, without loss of generality, we suppose that the boundary data are restrictions of smooth functions defined on \((T, \infty) \times \mathbb{R}^d\).
4.1. General boundary conditions

We suppose that $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ is a bounded domain of class $C^\infty$ such that
\[
\partial\Omega = \bigcup_{i=0}^n \Gamma^i, \quad \Gamma^i \cap \Gamma^j = \emptyset \quad i \neq j,
\]
where $\Gamma^i$ are connected components of $\partial\Omega$ and $\Gamma^0$ is the boundary of the unbounded component of $\mathbb{R}^d \setminus \Omega$.

**Theorem 4.1.** Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a bounded domains of class $C^1$, the boundary of which admits the decomposition (4.1). In addition, suppose the following holds:

- The pressure $p$ and the internal energy $e$ are given by the hard sphere equations of state (3.5), the entropy is compatible with the Third law of thermodynamics (3.10).
- The transport coefficients $\mu, \eta, \kappa$ are continuously differentiable functions of $\nabla$ satisfying (3.9), with $\Lambda = 1, \beta > 6$.
- The boundary data $\varrho_B, \mathbf{u}_B, \vartheta_B$ are restrictions of continuously differentiable functions in $(T, \infty) \times \mathbb{R}^d$, and
\[
\begin{align*}
0 < \inf_{(T, \infty) \times \mathbb{R}^d} \varrho_B & \leq \sup_{(T, \infty) \times \mathbb{R}^d} \varrho_B \leq \bar{\varrho}, \\
0 < \vartheta & = \inf_{(T, \infty) \times \mathbb{R}^d} \vartheta_B \leq \sup_{(T, \infty) \times \mathbb{R}^d} \vartheta_B = \bar{\vartheta}, \\
\int_{\Gamma^i} \mathbf{u}_B \cdot \mathbf{n} \, d\sigma_x = 0, \quad i = 1, \ldots, n, \quad \inf_{(T, \infty) \times \mathbb{R}^d} \int_{\Gamma^0} \mathbf{u}_B \cdot \mathbf{n} \, d\sigma_x > 0, \\
|\partial_t \mathbf{u}_B(t, x)| + |\partial_t \vartheta_B(t, x)| + |\nabla \varrho_B(t, x)| + |\nabla^2 \mathbf{u}_B(t, x)| + |\nabla^2 \vartheta_B(t, x)| \leq D, \\
\vartheta = 0, 1, 2, \quad t \in (T, \infty), \quad x \in \mathbb{R}^d.
\end{align*}
\]
- The driving force $\mathbf{g}$ is a bounded measurable function,
\[
\|\mathbf{g}\|_{L^\infty((T, \infty) \times \Omega; \mathbb{R}^d)} \leq D. \tag{4.3}
\]

Then there exists a universal constant $\mathcal{E}_\infty$, depending solely on the norm of the boundary data and the driving force, such that
\[
\limsup_{t \to \infty} \int_\Omega \left( \frac{1}{2} |\mathbf{u} - \mathbf{u}_B|^2 + g(e(\varrho, \vartheta) - \vartheta \varrho s(\varrho, \vartheta)) \right) \, dx \leq \mathcal{E}_\infty \tag{4.4}
\]
for any weak solution $(\varrho, \vartheta, \mathbf{u})$ of the Navier–Stokes–Fourier system on $(T, \infty) \times \Omega$, where $\vartheta$ is the unique solutions of the Dirichlet problem
\[
\Delta_x \vartheta(t, \cdot) = 0 \quad \text{in } \Omega, \quad \vartheta(t, \cdot)|_{\partial\Omega} = \vartheta_B(t, \cdot). \tag{4.5}
\]

**Remark 4.2.** The condition
\[
\int_{\Gamma^0} \mathbf{u}_B \cdot \mathbf{n} \, d\sigma_x > 0
\]
required for any $t \in (T, \infty)$ is purely “compressible” as it excludes the possibility of $\mathbf{u}_B$ being solenoidal. Its stabilizing effect has been observed in [22]. It is worth noting that the same condition is imposed by Choe, Novotný and Yang [27, Theorem 2.5] to show
the existence of global in time weak solutions for the hard–sphere barotropic Navier–Stokes system. This condition is relaxed in the forthcoming section.

Note that the choice of parameters $\Lambda = 1$, $\beta > 6$ as well as (3.10) are also necessary to show the existence of global in time weak solutions in [12, Theorem 4.2]. The bound (4.4) can be equivalently stated as

$$\limsup_{t \to \infty} \int_{\Omega} \left( \frac{1}{2} \varrho |\mathbf{u} - \mathbf{u}_B|^2 + \varrho e(\varrho, \vartheta) \right) \, dx \leq \mathcal{E}_\infty,$$

(4.6)

without any reference to (4.5).

### 4.2. No-slip boundary conditions, Bénard problem

The reader will have noticed that the principal hypothesis (4.2) of Theorem 4.1 does not include the no-slip boundary conditions $\mathbf{u}_B|_{\partial \Omega} = 0$. The next result focuses on the Bénard problem, where the boundary temperature is prescribed, while the normal velocity vanishes on the boundary. In particular, the total mass

$$M = \int_{\Omega} \varrho \, dx$$

is a constant of motion.

**Theorem 4.3.** (Impermeable boundary). Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a bounded domains of class $C^\infty$, the boundary of which admits the decomposition (4.1). In addition, suppose the following holds:

- The pressure $p$ and the internal energy $e$ are given by the hard sphere equations of state (3, 5), the entropy is compatible with the Third law of thermodynamics (3.10).
- The transport coefficients $\mu, \eta, \kappa$ are continuously differentiable functions of $\vartheta$ satisfying (3.9), with $\Lambda = 1$, $\beta > 6$.
- The boundary data $\mathbf{u}_B, \vartheta_B$ are restrictions of continuously differentiable functions in $(T, \infty) \times \mathbb{R}^d$, and

$$0 < \vartheta = \inf_{(T, \infty) \times \mathbb{R}^d} \vartheta_B \leq \sup_{(T, \infty) \times \mathbb{R}^d} \vartheta_B = \bar{\vartheta},$$

$$\mathbf{u}_B \cdot \mathbf{n}|_{\partial \Omega} = 0,$$

$$|\partial_t \mathbf{u}_B(t, x)| + |\partial_x \vartheta_B(t, x)| + |\nabla_x \vartheta \mathbf{u}_B(t, x)| + |\nabla_x \vartheta \vartheta_B(t, x)| \leq D,$$

$$\alpha = 0, 1, 2, t \in (T, \infty), x \in \mathbb{R}^d.$$

(4.7)

- The driving force $\mathbf{g}$ is a bounded measurable function,

$$||\mathbf{g}||_{L^\infty((T, \infty) \times \Omega)} \leq D.$$

(4.8)

Then there exists a universal constant $\mathcal{E}_\infty$, depending solely on the norm of the boundary data, the total mass $M$, and the driving force, such that

$$\limsup_{t \to \infty} \int_{\Omega} \left( \frac{1}{2} \varrho |\mathbf{u} - \mathbf{u}_B|^2 + \varrho e_{HS}(\varrho, \vartheta) - \vartheta \varrho s(\varrho, \vartheta) \right) \, dx \leq \mathcal{E}_\infty.$$
for any weak solution \((\varrho, \vartheta, \mathbf{u})\) of the Navier–Stokes–Fourier system, where \(\tilde{\vartheta}\) is the unique solutions of the Dirichlet problem (4.5).

Note carefully that hypothesis (4.7) is not a special case of (4.2). In particular, (4.7) is compatible with the no-slip boundary condition \(\mathbf{u}_B|_{\partial \Omega} = 0\). As the proofs of Theorems 4.1, 4.3 reflect this diversity, we prefer to state the corresponding results separately rather than as a single statement.

### 4.3. Convergence to equilibrium

Finally, we discuss the situation, where the boundary temperature \(\vartheta_B\) is a positive constant, while the velocity field \(\mathbf{u}_B\) coincides with a rigid motion tangential to \(\partial \Omega\). As \(\Omega\) is bounded, this is possible only if:

- \(\mathbf{u}_B = 0\), or
- \(\mathbf{u}_B\) is a rigid rotation and \(\Omega\) is radially symmetric with respect to the axis of rotation.

If the driving force \(\mathbf{g} = \nabla_x G\) is potential, the global in time solutions are expected to converge to an equilibrium solution \((\varrho_E, \mathbf{u}_B, \vartheta_B)\), where

\[
\begin{align*}
div_x (\varrho_E \mathbf{u}_B) &= 0, \\
div_x (\varrho_E \mathbf{u}_B \otimes \mathbf{u}_B) + \nabla_x p(\varrho_E, \vartheta_B) &= \varrho_E \nabla_x G, \\
\int_{\Omega} \varrho_E \, dx &= M,
\end{align*}
\]  

where the total mass

\[
M = \int_{\Omega} \varrho(t, \cdot) \, dx
\]

is a constant of motion.

**Theorem 4.4.** (Convergence to equilibrium). Let \(\Omega \subset \mathbb{R}^d\), \(d = 2, 3\) be a bounded domains of class \(C^\infty\). In addition, suppose the following holds:

- The pressure \(p = \tilde{p}\) and the internal energy \(e = \tilde{e}\) are given by the constitutive equations (3.1)–(3.5), with \(p_\infty > 0\).
- The transport coefficients \(\mu, \eta,\) and \(\kappa\) are continuously differentiable functions of \(\vartheta\) satisfying (3.9), with \(\frac{1}{2} \leq \Lambda \leq 1, \beta = 3\).
- The boundary data satisfy

\[
\begin{align*}
\mathbf{u}_B &= \mathbf{u}_B(x), \quad \mathcal{D}_x \mathbf{u}_B = 0, \quad \mathbf{u}_B \cdot \mathbf{n}|_{\partial \Omega} = 0, \\
\vartheta_B &> 0 \text{ -- a positive constant}.
\end{align*}
\]

- The driving force \(\mathbf{g}\) is potential,

\[
\mathbf{g} = \nabla_x G, \quad G = G(x), \quad G \in W^{1, \infty}(\Omega), \quad \nabla_x G \cdot \mathbf{u}_B = 0.
\]
Then there exists a density profile \( \varrho_E \) solving the stationary problem (4.9) such that
\[
\begin{align*}
\varrho(t, \cdot) &\to \varrho_E \quad \text{in} \quad L^3(\Omega), \\
\varrho u(t, \cdot) &\to \varrho_E u_B \quad \text{in} \quad L^3(\Omega; \mathbb{R}^d), \\
\vartheta(t, \cdot) &\to \vartheta_B \quad \text{in} \quad L^4(\Omega)
\end{align*}
\] (4.12)
as \( t \to \infty \) for any weak solution \((\varrho, \vartheta, u, \vartheta_B)\) of the Navier–Stokes–Fourier system.

Unlike in Theorems 4.1, 4.3, the hard sphere pressure component is not necessary in Theorem 4.4.

5. Proof of the main results

Our goal is to prove the main results stated in Section 4. First, as \( \vartheta_B \) is a solution of the Dirichlet problem (4.5), we may apply the standard maximum principle together with the elliptic estimates to deduce
\[
0 < \vartheta_B \leq \vartheta(t, x) \leq \vartheta_b,
\]
\[
\inf_{(T, \infty) \times \partial \Omega} \vartheta_B \leq \vartheta_B(t, x) \leq \sup_{(T, \infty) \times \partial \Omega} \vartheta_B,
\]
\[
|\nabla_x \vartheta(t, x)| \leq c(D) \quad \text{for all} \quad t > T, \, x \in \Omega.
\] (5.1)

Next, we adapt the construction of a suitable extension of the velocity field \( u_B \) used in [22] to the time-dependent setting. It is easy to observe that the component \( \Gamma^0 \) of the boundary \( \partial \Omega \) contains at least one extremal point \( x_0 \in \Gamma_0 \) satisfying
\[
\overline{\Omega} \cap \tau_{x_0} = x_0, \quad \text{where} \quad \tau_{x_0} \quad \text{denotes the tangent plane to} \quad \partial \Omega \quad \text{at} \quad x_0.
\]

Without loss of generality, we may assume that
\[
\Omega \subseteq \{x : x^1 < x_0^1\} \quad \text{and} \quad x_0 = [x_0^1, 0, \ldots, 0], \quad \tau_{x_0} = x_0 + R^{d-1}.
\]

Now, consider a function
\[
\chi(z) = \begin{cases} 0 & \text{if} \quad z \leq 0, \\ \chi'(z) > 0 & \text{for} \quad z > 0 \end{cases}
\]
together with a vector field
\[
v_B^0(t, x) = \lambda(t) \left[ \chi(x^1 - x_0^1 + \delta), 0, \ldots, 0 \right].
\]

It is easy to check that
\[
\mathbb{D}_x v_B^0 = \begin{bmatrix} \lambda(t) \chi'(x^1 - x_0^1 + \delta) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{div}_x v_B^0 = \lambda(t) \chi'(x^1 - x_0^1 + \delta).
\]

Next, choose \( \delta > 0 \) small enough so that
\[
v_B^0|_{\Gamma_i} = 0, \quad i = 1, \ldots, n, \quad v_B^0|_{\Gamma_0} \neq 0,
\]
and then \( \lambda(t) > 0 \) large enough so that
\[
\int_{\Gamma^0} v_B^0 \cdot n \, d\sigma_x = \int_{\Gamma^0} u_B \cdot n \, d\sigma_x > 0
\]
in accordance with hypothesis (4.2). We decompose
\[ u_B = w_B + v_B^0, \]
where
\[
\int_{\Gamma^i} w_B \cdot n \, d\sigma_x = 0 \quad \text{for} \quad i = 0, 1, \ldots, n,
\]
\[
\mathbb{D}_x v_B^0 \geq 0, \quad \text{there is an open set} \quad B \subset \Omega, \quad |B| > 0,
\]
\[
\inf_{(T, \infty) \times B} (\text{div}_x v_B^0) \geq d > 0.
\]
(5.2)

Under the hypotheses of Theorem 4.3, where, obviously,
\[
d_{e}(x) = 0, \quad w_B = u_B:
\]
we simply set \( v_B^0 = 0, \quad w_B = u_B. \)

Now, exactly as in [22, Section 4.2], we use Galdi [28, Lemma IX.4.1], Kozono and Yanagisawa [29, Proposition 1] to write
\[
w_B(t, x) = \text{curl}_x(d_e(x)z_B(t, x)), \quad \text{curl}_x z_B = w_B,
\]
(5.3)
where
\[
|d_e| \leq 1, \quad d_e(x) \equiv 1 \quad \text{for all} \quad x \in \text{an open neighborhood of} \quad \partial \Omega,
\]
\[
d_e(x) \equiv 0 \quad \text{whenever} \quad \text{dist}[x, \partial \Omega] > \varepsilon,
\]
\[
|D^2 d_e(x)| \leq \frac{c}{\varepsilon} \frac{\varepsilon}{\text{dist}[x, \partial \Omega]}, \quad |x| = 1, 2, \quad 0 < \varepsilon < 1, \quad x \in \Omega.
\]
(5.4)

Remark 5.1. If \( d = 2 \), we adopt the convention that \( z_B \) is scalar and the operator \( \text{curl}_x \) is replaced by \( \nabla_x \) in (5.3).

5.1. Proof of Theorems 4.1 4.3

First, write the ballistic energy inequality (2.5) in the form
\[
\left[ \int_{\Omega} \left( \frac{1}{2} |u - u_B|^2 + qe - \tilde{q}s \right) \, dx \right]_{t = I + \tau}^{t = I + \tau}
\]
\[
+ \int_{I}^{I + \tau} \int_{\partial \Omega} \left[ qe(g, \tilde{q}_B) - \tilde{q}_B q s(g, \tilde{q}_B) \right] [u_B \cdot n]^{-} d\sigma_x \, dt
\]
\[
+ \int_{I}^{I + \tau} \int_{\partial \Omega} \left[ qe(g, \tilde{q}_B) - \tilde{q}_B q s(g, \tilde{q}_B) \right] [u_B \cdot n]^{+} d\sigma_x \, dt
\]
\[
+ \int_{I}^{I + \tau} \int_{\frac{\partial \Omega}{\partial \tilde{q}}} \left( S(\tilde{q}_B, \mathbb{D}_x u) : \mathbb{D}_x u - \frac{q \cdot \nabla_x \tilde{q}_B}{\tilde{q}_B} \right) \, dx \, dt
\]
\[
- \int_{I}^{I + \tau} \int_{\Omega} \left[ q g(\partial_t \tilde{q} + u \cdot \nabla_x \tilde{q}) + \frac{q}{\tilde{q}_B} \cdot \nabla_x \tilde{q} \right] \, dx \, dt, \quad I > T, \quad \tau \geq 0.
\]
(5.5)
As $A = 1$ in hypothesis (3.9), we may use Korn–Poincaré inequality to estimate

$$
\|u - u_B\|^2_{W^{1,2}_0(\Omega; \mathbb{R}^d)} \lesssim \|\nabla x (u - u_B) + \nabla x (u - u_B) - \frac{2}{d} \text{div}_x (u - u_B)^2\|_{L^2(\Omega; \mathbb{R}^{d+d})},
$$

$$
\lesssim \int_{\Omega} \left|\nabla x u + \nabla x u - \frac{2}{d} \text{div}_x u\right|^2 \; dx + c(\|\nabla x u_B\|_{L^2}),
$$

$$
\lesssim \int_{\Omega} \frac{1}{\varrho} S(\partial, D_x u) : D_x u \; dx + c(\|\nabla x u_B\|_{L^2}),
$$

where $A \lesssim B$ means $A \leq cB$ for a constant number $c \geq 0$. Consequently, as $q$ is bounded by $\tilde{\varrho}$ uniformly in (3.8),

$$
\int_{I}^{I+\tau} \int_{\Omega} q(u - u_B) : (g - \partial_t u_B - u_B \cdot \nabla x u_B) \; dx \; dt 
\leq \varepsilon \int_{I}^{I+\tau} \|u - u_B\|^2_{W^{1,2}_0(\Omega; \mathbb{R}^d)} \; dt + c(\tau, \varepsilon, D, \varrho, \|g\|_{L^\infty})
$$

for any $\varepsilon > 0$. Thus we deduce from (5, 7)

$$
\left[\int_{\Omega} \left(\frac{1}{2} q|u - u_B|^2 + qe - \tilde{\varrho}q s\right) \; dx\right]_{t=I}^{t=I+\tau}
+ \int_{I}^{I+\tau} \|u - u_B\|^2_{W^{1,2}_0(\Omega; \mathbb{R}^d)} \; dt + \int_{I}^{I+\tau} \int_{\Omega} \frac{\kappa(\varrho)|\nabla x \varrho|^2}{\varrho^2} \; dx \; dt
\lesssim - \int_{I}^{I+\tau} \int_{\Omega} \left[q(u - u_B) \odot (u - u_B) + p \mathbb{I} - S\right] : D_x u_B \; dx \; dt
- \int_{I}^{I+\tau} \int_{\Omega} \left[\kappa(\varrho) \nabla x \varrho \cdot \nabla x \varrho - \frac{\kappa(\varrho)}{\varrho} \nabla x \varrho \cdot \nabla x \varrho\right] \; dx + c(\tau, D, \varrho, \varrho, \partial, \|g\|_{L^\infty}).
$$

Next, as $\tilde{\varrho}$ solves (4.5), the Gauss–Green integration formula yields

$$
\int_{\Omega} \frac{\kappa(\varrho)}{\varrho} \nabla x \varrho \cdot \nabla x \varrho \; dx = - \int_{\Omega} \nabla x \mathcal{K}(\varrho) \cdot \nabla x \varrho \; dx = - \int_{\partial \Omega} \mathcal{K}(\varrho) \nabla x \varrho \cdot n \; d\sigma_x,
$$

where $\mathcal{K}^\prime(\varrho) = \frac{\kappa(\varrho)}{\varrho}$. Furthermore,

$$
\int_{\Omega} S(\partial, D_x u) : D_x u_B \; dx \lesssim \int_{\Omega} |\nabla x u|^2 |D_x u_B| \; dx + c(\varepsilon) \int_{\Omega} (1 + \varrho^2) |D_x u_B| \; dx
$$

for any $\varepsilon > 0$. Finally, as the entropy is compatible with the Third law of thermodynamics (3.10), we have

$$
|\kappa q(\varrho, \partial)| \lesssim \left(1 + \varrho |\log(\varrho)| + \varrho [\log(\varrho)]^+ + \varrho^3\right).
$$

In view of (5.8)-(5.10), the inequality (5.7) gives rise to

$$
\left[\int_{\Omega} \left(\frac{1}{2} q|u - u_B|^2 + qe - \tilde{\varrho}q s\right) \; dx\right]_{t=I}^{t=I+\tau}
+ \int_{I}^{I+\tau} \|u - u_B\|^2_{W^{1,2}_0(\Omega; \mathbb{R}^d)} \; dt + \int_{I}^{I+\tau} \int_{\Omega} \frac{\kappa(\varrho)|\nabla x \varrho|^2}{\varrho^2} \; dx \; dt
\lesssim - \int_{I}^{I+\tau} \int_{\Omega} \left[q(u - u_B) \odot (u - u_B) + p \mathbb{I} - S\right] : D_x u_B \; dx \; dt
- \int_{I}^{I+\tau} \int_{\Omega} \kappa(\varrho)|\nabla x \varrho|^2 \; dx \; dt
+ \int_{I}^{I+\tau} \int_{\Omega} \varrho^2 (1 + |u|) \; dx \; dt + c(\tau, D, \varrho, \varrho, \partial, \|g\|_{L^\infty}).
$$
Now, in accordance with hypothesis (3.9),
\[
\|\nabla_x \log (\vartheta)\|_{L^2(\Omega; \mathbb{R}^d)}^2 + \|\nabla_x \vartheta_x^e\|_{L^2(\Omega; \mathbb{R}^d)}^2 \lesssim \int_{\Omega} \frac{\kappa(\vartheta)|\nabla_x \vartheta|^2}{\vartheta^2} \, dx.
\]

Thus, as the boundary values of \(\vartheta\) are controlled,
\[
\|\log (\vartheta)\|_{W^{1,2}(\Omega)}^2 + \|\vartheta_x^e\|_{W^{1,2}(\Omega)}^2 \lesssim \int_{\Omega} \frac{\kappa(\vartheta)|\nabla_x \vartheta|^2}{\vartheta^2} \, dx + c(\vartheta, \bar{\vartheta})
\] (5.12)

Next, we have
\[
\int_{\Omega} \vartheta^3 |u| \, dx \leq \varepsilon \|u\|_{L^2(\Omega; \mathbb{R}^d)}^2 + c(\varepsilon)\|\vartheta^3\|_{L^2(\Omega)}^2
\]
and, since \(\beta > 6\),
\[
\|\vartheta^3\|_{L^2(\Omega)}^2 \leq \delta \|\vartheta_x^e\|_{L^2(\Omega)}^2 + c(\delta)
\] (5.13)
for any \(\varepsilon > 0, \delta > 0\). Consequently, going back to (5.11) we may infer that
\[
\begin{align*}
&\left[ \int_{\Omega} \left( \frac{1}{2} \vartheta|u - u_B|^2 + \varrho e - \bar{\vartheta} \delta s \right) \, dx \right]_{t = I + \tau} - I \\
&+ \int_{I}^{I + \tau} \|u - u_B\|_{W^{1,2}(\Omega; \mathbb{R}^d)}^2 \, dt + \int_{I}^{I + \tau} \left( \|\log (\vartheta)\|_{W^{1,2}(\Omega)}^2 + \|\vartheta_x^e\|_{W^{1,2}(\Omega)}^2 \right) \, dt \\
&+ \int_{I}^{I + \tau} \int_{\Omega} \rho \text{div}_x \vec{v}_B^0 \, dx \, dt \\
&\lesssim - \int_{I}^{I + \tau} \int_{\Omega} [\varrho(u - u_B) \otimes (u - u_B)] : \mathcal{D}_x u_B \, dx \, dt + c(\tau, D, \varrho, \bar{\varrho}, \vartheta, \bar{\vartheta}, \|g\|_{L^\infty})
\end{align*}
\] (5.14)

Finally, exactly as in [22, Section 5], it can be shown that the integral
\[
\int_{\Omega} [\varrho(u - u_B) \otimes (u - u_B)] : \mathcal{D}_x u_B \, dx
\]
can be absorbed by the left–hand side of (5.14) thanks to the decomposition (5.3) as long as \(\varepsilon > 0\) in (5.3) is fixed small enough. Thus, by virtue of (5.2), we conclude
\[
\begin{align*}
&\left[ \int_{\Omega} \left( \frac{1}{2} \vartheta|u - u_B|^2 + \varrho e - \bar{\vartheta} \delta s \right) \, dx \right]_{t = I + \tau} - I \\
&+ \int_{I}^{I + \tau} \|u - u_B\|_{W^{1,2}(\Omega; \mathbb{R}^d)}^2 \, dt + \int_{I}^{I + \tau} \left( \|\log (\vartheta)\|_{W^{1,2}(\Omega)}^2 + \|\vartheta_x^e\|_{W^{1,2}(\Omega)}^2 \right) \, dt \\
&+ d \int_{I}^{I + \tau} \int_{B} p \, dx \, dt \leq c(\tau, D, \varrho, \bar{\varrho}, \vartheta, \bar{\vartheta}, \|g\|_{L^\infty})
\end{align*}
\] (5.15)

Note that the same inequality with \(d = 0\) is obtained under the hypotheses of Theorem 4.3. Our ultimate goal is therefore to derive the estimate
\[
\int_{\Omega} \varrho e \, dx \lesssim \|u - u_B\|_{W^{1,2}(\Omega; \mathbb{R}^d)}^2 + \|\log (\vartheta)\|_{W^{1,2}(\Omega)}^2 + \|\vartheta_x^e\|_{W^{1,2}(\Omega)}^2 + \int_{B} p(\varrho, \bar{\varrho}) \, dx.
\] (5.16)
Similarly to [22], the desired bound (5.16) follows from the pressure estimates. As the present setting is slightly different due to the temperature depending terms, we reproduce some details of the proof for reader's convenience. First, we recall the Bogovskii operator:

\[ B : L^q_0(\Omega) \equiv \{ f \in L^q(\Omega) \mid \int_{\Omega} f \, dx = 0 \} \rightarrow W^{1,q}_0(\Omega, R^d), \quad 1 < q < \infty, \]

\[ \text{div}_x B[f] = f, \]

see e.g. Galdi [28, Chapter 3], Geissert, Heck, and Hieber [30].

Using \( \psi(t)B[\Phi], \Phi = \Phi(x) \) as a test function in the momentum equation (2.3), we deduce

\[
\int_1^{t + \tau} \int_{\Omega} p(\varrho, \vartheta) \Phi \, dx \, dt = \left[ \int_{\Omega} \varrho \cdot B[\Phi] \, dx \right]_{t = 1}^{t = t + \tau} \\
- \int_1^{t + \tau} \int_{\Omega} \varrho u \otimes u : \nabla_x B[\Phi] \, dx \, dt + \int_1^{t + \tau} \int_{\Omega} \mathcal{S}(\partial_t, D_x \mathcal{A}) : \nabla_x B[\Phi] \, dx \, dt \\
- \int_1^{t + \tau} \int_{\Omega} \varrho g \cdot B[\Phi] \, dx \, dt.
\]

All integrals on the right-hand side can be estimated exactly as in [22, Section 5].

\[
\int_{\Omega} \mathcal{S}(\vartheta, D_x \mathcal{A}) : \nabla_x B[\Phi] \, dx \nabla_x B[\Phi] \, dx
\]

that can be treated as

\[
\left| \int_{\Omega} \mathcal{S}(\vartheta, D_x \mathcal{A}) : \nabla_x B[\Phi] \, dx \right| \lesssim \| \nabla_x B[\Phi] \|_{L^4(\Omega)} \left( \| \mathcal{S}(\vartheta, D_x \mathcal{A}) \|_{L^4(\Omega)} + \| \vartheta \|_{L^4(\Omega)}^2 \right)
\]

\[
\lesssim \| \nabla_x B[\Phi] \|_{L^4(\Omega)} \left( \| \mathcal{S}(\vartheta, D_x \mathcal{A}) \|_{L^4(\Omega)} + \| \vartheta \|_{L^4(\Omega)} + c(D, q, \vartheta, \vartheta, \vartheta) \right).
\]

With the estimate (5.18) at hand, we can repeat step by step the arguments of [22, Section 5] to deduce from (5.16) the final bound

\[
\left[ \int_{\Omega} \left( \frac{1}{2} \varrho |u - u_0|^2 + \varrho e - \vartheta q s - \varrho u \cdot B[\Phi] \right) \, dx \right]_{t = 1}^{t = t + \tau} \\
+ \int_1^{t + \tau} \| u - u_0 \|_{W^{1,2}_0(\Omega, R^d)}^2 \, dt + \int_1^{t + \tau} \left( \| \log (\vartheta) \|_{W^{1,2}_0(\Omega)}^2 + \| \vartheta \|_{W^{1,2}_0(\Omega)}^2 \right) \, dt
\]

\[
+ \int_1^{t + \tau} \int_{\Omega} p \, dx \, dt \leq c(\tau, D, q, \vartheta, \vartheta, \vartheta, \vartheta, \| g \|_{L^\infty}, \Phi),
\]

for a suitable \( \Phi \) and \( \delta > 0 \).

Similarly to [22, Section 5], relation (5.19) yields the conclusion of Theorem 4.1. First observe that
\( \psi e^{\sim p(\rho, \vartheta)} \)  

(5.20)

see [22, Equation (2.4)].

Denote

\[
\mathcal{E}(t) = \int_{\Omega} \left( \frac{1}{2} \rho |u - u_\delta|^2 + \psi e - \vartheta \varphi s - \rho \mathbf{u} \cdot \mathbf{B} [\varphi] \right) (t, \cdot) \, dx
\]

(5.21)

As the entropy \( s \) satisfies (5.10),

\[
\mathcal{E} \geq \lambda \int_{\Omega} \left( \frac{1}{2} \rho |u - u_\delta|^2 + \psi e \right) \, dx - \frac{1}{\lambda}
\]

(5.22)

for some \( \lambda > 0 \). Consequently, inequality (5.19) yields

\[
[\mathcal{E}(t)]_{t=I+\tau} + \delta \int_{I}^{I+\tau} \mathcal{E}(s) \, ds \leq C(\tau, \delta) \quad \text{for some } \delta > 0.
\]

(5.23)

For \( \tau = 1 \) we get the following dichotomy:

1. Either there exists \( t \in [I, I+1] \) such that

\[
\mathcal{E}(t) \leq \frac{2C(1, \delta)}{\delta};
\]

(5.24)

2. or

\[
\mathcal{E}(I + 1) \leq \mathcal{E}(I) - C(1, \delta).
\]

(5.25)

In virtue of (5.22), \( \mathcal{E} \) is bounded below and thus it follows that there exists \( x_t \in [t-1, t] \subset (T, \infty) \) such that

\[
\mathcal{E}(x_t) \leq \frac{2C(1, \delta)}{\delta}.
\]

From (5.23),

\[
\mathcal{E}(t) \leq \mathcal{E}(x_t) + C(1, \delta) \leq \frac{2C(1, \delta)}{\delta} + C(1, \delta).
\]

So we get

\[
\limsup_{t \to \infty} \mathcal{E}(t) \leq C(1, \delta) \left( 1 + \frac{2}{\delta} \right).
\]

Consequently, we prove that

\[
\limsup_{t \to \infty} \int_{\Omega} \left( \frac{1}{2} \rho |u - u_\delta|^2 + \psi e \right) \, dx \leq \frac{C(1, \delta)}{\lambda} \left( 1 + \frac{2}{\delta} \right) + \frac{1}{\lambda^2} := \mathcal{E}_\infty,
\]

which completes to prove Theorem 4.1.
5.1.2. **Pressure estimates, proof of Theorem 4.3**

Under the hypotheses of Theorem 4.3, the total mass of the fluid is conserved,

\[ \int_{\Omega} \rho(t, \cdot) \, dx = M \text{ for any } t. \]

Keeping in mind (5.19), the proof of Theorem 4.3 can be done by the same arguments as in [22, Section 6.1]. Specifically, we repeat the pressure estimates with the test function

\[ \phi = \psi(t)B \left[ \rho - \frac{1}{|\Omega|} \int_{\Omega} \rho \, dx \right]. \]

**Remark 5.2.** It is worth noting that the above proof does not use any structural properties of \( p \) and \( e \) as soon as the uniform bound on the density is established. The only piece of information to be retained being \( e \). In particular, the radiation component is irrelevant and possibly a more realistic equation of state similar to [21] can be used.

5.2. **Proof of Theorem 4.4**

First we claim that for a given velocity field \( u_B \), the temperature \( \vartheta_B \), and the total mass \( M > 0 \), there exists a unique density profile \( \rho_E \) solving the stationary problem (4.9). Indeed, as \( \mathbb{D}_x u_B = 0 \), the convective term in (4.9) reads

\[ \text{div}_x (\rho_E \mathbf{u}_B \otimes \mathbf{u}_B) = \text{div}_x (\rho_E \mathbf{u}_B) \mathbf{u}_B + \rho_E \mathbf{u}_B \cdot \nabla_x \mathbf{u}_B = -\frac{1}{2} \rho_E \nabla_x |\mathbf{u}_B|^2. \]

Accordingly, the problem (4.9) can be rewritten as

\[ \begin{align*}
\nabla_x \rho_E \cdot \mathbf{u}_B &= 0, \\
\nabla_x P(\rho_E, \vartheta_B) &= \rho_E \left( \nabla_x G + \frac{1}{2} |\mathbf{u}_B|^2 \right), \\
\int_{\Omega} \rho_E \, dx &= M. 
\end{align*} \]

As \( P'(0) > 0 \) in (3.2) and \( \vartheta_B \) is given, the desired uniqueness result follows from [31, Theorem 2.1].

Under the hypotheses of Theorem 4.4, the ballistic energy inequality (2.6) simplifies considerably

\[ \begin{align*}
\frac{d}{dt} \int_{\Omega} \left( \frac{1}{2} \rho |\mathbf{u} - \mathbf{u}_B|^2 + \rho e - \vartheta_B Q \right) \, dx \\
+ \int_{\Omega} \frac{\partial}{\partial \vartheta} \left( \mathbb{S} : \mathbb{D}_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right) \, dx \\
\leq \int_{\Omega} \rho (\mathbf{u} - \mathbf{u}_B) \cdot (\nabla_x G - \mathbf{u}_B \cdot \nabla_x \mathbf{u}_B) \, dx. 
\end{align*} \]

Moreover, by virtue of hypothesis (4.11),

\[ \int_{\Omega} \rho (\mathbf{u} - \mathbf{u}_B) \cdot (\nabla_x G - \mathbf{u}_B \cdot \nabla_x \mathbf{u}_B) \, dx = \int_{\Omega} \rho \mathbf{u} \cdot \left( \nabla_x G + \frac{1}{2} \nabla_x |\mathbf{u}_B|^2 \right) \, dx; \]

whence (5.27) reduces to
\[
\frac{d}{dt} \int_{\Omega} \left( \frac{1}{2} \rho |u - u_B|^2 + \rho e - \rho \left( G + \frac{1}{2} |u_B|^2 \right) - \varrho_B G \right) \, dx \\
+ \int_{\Omega} \frac{\partial}{\partial \varrho} \left( S(\varrho, \mathbb{D}_x u) : \mathbb{D}_x u + \frac{\kappa(\varrho)}{\varrho} |\nabla_x \varrho|^2 \right) \, dx \leq 0.
\]

(5.28)

It turns out that the modified ballistic energy
\[
\int_{\Omega} \left( \frac{1}{2} \rho |u - u_B|^2 + \rho e - \rho \left( G + \frac{1}{2} |u_B|^2 \right) - \varrho_B G \right) \, dx
\]
is a Lyapunov function decreasing along trajectories for which
\[
\int_{\Omega} \frac{\partial}{\partial \varrho} \left( S(\varrho, \mathbb{D}_x u) : \mathbb{D}_x u + \frac{\kappa(\varrho)}{\varrho} |\nabla_x \varrho|^2 \right) \, dx > 0.
\]

In particular, there holds
\[
\int_T^\infty \int_{\Omega} \frac{\partial}{\partial \varrho} \left( S(\varrho, \mathbb{D}_x u) : \mathbb{D}_x u + \frac{\kappa(\varrho)}{\varrho} |\nabla_x \varrho|^2 \right) \, dx \, dt
\]
\[
= \int_T^\infty \int_{\Omega} \frac{\partial}{\partial \varrho} \left( S(\varrho, \mathbb{D}_x u - \mathbb{D}_x u_B) : (\mathbb{D}_x u - \mathbb{D}_x u_B) + \frac{\kappa(\varrho)}{\varrho} |\nabla_x \varrho - \nabla_x \varrho_B|^2 \right) \, dx \, dt < \infty
\]
(5.29)
for any weak solution of the Navier–Stokes–Fourier system defined on the time interval \((T, \infty)\).

Let \(T_n \to \infty\) be a sequence of time. Let
\[
\varrho_n(t, x) = \varrho(T_n + t, x), \quad \varrho_n(t, x) = \varrho(T_n + t, x), \quad u_n(t, x) = u(T_n + t, x)
\]
be the associated time shifts of a global in time weak solution to the Navier–Stokes–Fourier system. It follows from (5.29) that
\[
\varrho_n \to \varrho_B \quad \text{in} \quad L^{1+\varphi}((0, T) \times \Omega),
\]
\[
u_n \to \nu_B \quad \text{in} \quad L^{\varphi}(0, T; W^{1, \varphi} (\Omega; \mathbb{R}^2))
\]
as \(n \to \infty\) for some \(\varphi > 1\). In particular,
\[
\text{div} \, u_n \to 0 \quad \text{in} \quad L^\infty((0, T) \times \Omega),
\]
which yields (cf. [10, Chapter 4, Theorem 4.2])
\[
\varrho_n \to \varrho \quad \text{in} \quad L^{\frac{\varphi}{\varphi - 1}}((0, T) \times \Omega),
\]
(5.30)

passing to a suitable subsequence as the case may be.

Our final claim is that \(\varrho = \varrho_B\), in particular, the convergence in (5.30) is unconditional, which completes the proof of Theorem 4.4. Seeing that the limit is again a weak solution of the Navier–Stokes–Fourier system we get
\[
\partial_t \varrho + \text{div} \, (\varrho \nu_B) = 0,
\]
\[
\partial_t (\varrho \nu_B) + \text{div} \, (\varrho \nu_B \otimes \nu_B) + \nabla_x \rho (\varrho, \nu_B) = \varrho \nabla_x G
\]
in \(\mathcal{D}'((0, T) \times \Omega)\). Consequently,
$\nabla_x p(\varrho, \varrho_B) = \varrho \nabla_x \left( G + \frac{1}{2} |\mathbf{u}_B|^2 \right)$ for any $t \in (0, T)$.

Since $\int_\Omega \varrho(t, \cdot) \, dx = M$, the uniqueness result [31] yields $\varrho(t, \cdot) = \varrho_E$ for a.a. $t \in (0, T)$.

6. Concluding remarks

The hypotheses of Theorems 4.1, 4.3 can be slightly relaxed. In the presence of the singular hard–sphere pressure component, the structural hypotheses (3.1)–(3.5) are not necessary. We may consider $p, e$ in the form (3, 5), where $p, e$ are related to the entropy $s$ through general Gibbs’ equation

$$\partial Ds = De + pD \left( \frac{1}{\varrho} \right)$$

and satisfy the hypothesis of thermodynamic stability

$$\frac{\partial p}{\partial \varrho} > 0, \frac{\partial e}{\partial \varrho} > 0.$$

Strictly speaking, the presence of the radiation pressure is not necessary for the results of this paper, however, it is essential for the existence of weak solutions.

The structural hypotheses that guarantee the existence of bounded absorbing sets are expected to provide positive results concerning the qualitative behavior of solutions in the long run. In particular:

- The existence of time periodic solutions for problems driven by time periodic boundary data.
- The existence of global attractors.
- Convergence of the ergodic averages and the existence of statistical stationary solutions in the spirit of Constantin and Wu [32], Foias, Rosa, Temam [33, 34], Vishik and Fursikov [35].

These issues will be addressed in the future work.

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