Manipulation Can Be Hard in Tractable Voting Systems Even for Constant-Sized Coalitions

Curtis Menton and Preetjot Singh
Department of Computer Science
University of Rochester
Rochester, NY 14627 USA

April 28, 2013

Abstract

Voting theory has become increasingly integrated with computational social choice and multiagent systems. Computational complexity has been extensively used as a shield against manipulation of voting systems, however for several voting schemes this complexity may cause calculating the winner to be computationally difficult. Of the many voting systems that have been studied with regard to election manipulation, a few have been found to have an unweighted coalitional manipulation problem that is NP-hard for a constant number of manipulators despite having a winner problem that is in P. We survey this interesting class of voting systems and the work that has analyzed their complexity.

1 Introduction

Research in voting theory has become increasingly important due to the ubiquity of voting systems. While voting is most typically used for political or organizational elections, recently it has become relevant as a tool in multiagent systems and distributed artificial intelligence. From recommender systems [GMHS99, PHG00] such as those seen in Netflix which makes recommendations based on user activity, to consensus mechanisms for planning in artificial intelligence [ER91] and search engine and metasearch engine design [Li00, DKNS01], mechanisms that aggregate individual ‘votes’ have far-reaching applications. Many fields of study in computer science such as mechanism design and

*Supported in part by grants NSF-CCF-0915792 and NSF-CCF-1101479.
algorithmic game theory have become intertwined with research in voting theory. In this paper we concentrate on results in manipulation, that is, strategically changing one’s vote so as to change the result of the election.

Manipulation, as opposed to other ways of influencing election results, does not change the structure of the election (as in control [BTT92] and cloning [Tid87, EFS10]) or involve an external actor bribing voters to change their preferences (as in bribery [FHH09] and campaign management [EF10a, SFE11]). Manipulation does not require going outside of the election model but merely involves voters picking their optimal votes. Thus manipulation is the most immediate and frequently relevant of these problems.

The most representative subproblem of manipulation is unweighted coalitional manipulation (UCM). The model is simpler than one with weighted voters and so the focus is squarely on the voting rule itself, not on the interplay of differently weighted voters forming a coalition. As such, results in UCM are a purer test of a voting system’s vulnerability to manipulation.

We will first explore the history and key results of voting theory that imply significant issues with all voting systems, and subsequently we will show how we can cope with some of these difficulties through complexity theory.

1.1 Voting Theory

Voting has long been used as a tool for collaborative decision making, with democratic government known to have existed at least as far back as 6th century BCE in ancient Greece. For nearly as long people have studied voting in an attempt to find the best election methods and solve problems related to voting.

One milestone in the study of voting is the work of 13th century mystic Ramon Llull. Llull, a prominent Franciscan, was involved with the Catholic church and researched methods for electing church officials. Among his extensive body of work, encompassing at least 265 titles on subjects ranging from controversial theological viewpoints to romantic fiction, Llull’s contributions to voting theory stem from three works: Artifitium Electionis Personarum, En qual manera Natana fo eleta a abadessa (Chapter 24 of his novel Blaquerna) and De Arte Eleccionis, all featuring variants of a pairwise election system we now know as Condorcet voting [HP01, Szp10].

In the eighteenth century, the Marquis de Condorcet developed one of the first criteria for evaluating voting systems. Condorcet’s criterion is that, for a given election, if there exists a candidate that beats all other candidates in pairwise contests, then that candidate must be the winner of the election. It is a somewhat surprising result that such a candidate will not always exist due the possibility of cycles in the pairwise societal preferences. Condorcet proposed this criteria and showed that many popular voting systems do not meet it. Those that do are called Condorcet methods, or Condorcet-consistent
voting systems. Elections using Condorcet methods can be viewed as a number of pairwise majority-rule elections\footnote{A majority rule decides on the alternative that receives the majority of the votes.} or in the case of an election with two candidates, exactly equivalent to a majority-rule election. Hence the literature often refers to Condorcet methods as variants of the majority rule \cite{Ris05}.

Condorcet methods have often been contrasted with Borda voting, which takes complete preference orderings as the votes and gives points to each candidate based on the number of candidates they are ranked above in each vote. For instance a vote denoting $a$ is preferred to $b$ is preferred to $c$ would give two points to $a$, one to $b$, and none to $c$. There are fervent arguments between proponents of the two systems, debating the importance of the Condorcet criterion \cite{New92, Saa06, Ris05}, in a rivalry dating back to the Marquis de Condorcet’s criticism of Borda voting when it was first introduced \cite{Szp10}.

One persistent concern in elections is that some of the participants may be able to vote strategically thus unfairly gaining an advantage over honest voters. Pliny the Younger’s attempts to manipulate the Roman senate circa 105 CE is possibly the earliest recorded instance of strategic behavior in elections \cite{Szp10}. The senate, presiding over a murder trial, were divided into three blocs: those who favored acquittal, banishment, or death for the accused. The senators were more or less evenly distributed among the three positions, with the acquittal bloc (headed by Pliny) being slightly larger than the other two. The normal method of voting was similar to the current justice system in most countries: The senate would first vote on the guilt of the accused, followed by a vote on the punishment (banishment or death). Considering how the blocs were aligned, the probable outcome of the first election would be a decision of guilty, followed by banishment. To give his faction an edge, Pliny proposed the senators vote for acquittal, banishment or death in a single ternary-choice election. However, his strategy backfired. The death penalty faction, fearing an acquittal, voted for banishment.

Pliny’s story holds more than just strategy and counter-strategy: Pliny convinced the senate of the fairness of a single ternary-choice election by stating it aligned naturally with the principle of voting \textit{qua sentitits}, or according to your true preferences. His attempt proved unsuccessful but serves as an excellent example of the problem of getting people to vote honestly. While this problem was recognized by voting theoreticians through history, it was either dismissed or attempts to solve it were limited at best. For instance, Llull documents that voters were required to give an oath to vote sincerely, and Jean-Charles de Borda famously dismissed criticism of his system’s vulnerability to manipulation by saying “My scheme is only intended for honest men” \cite{Bla58}.

Later work drew from game theory to more formally model voter strategy and to analyze its possibilities, especially in the work of Allan Gibbard \cite{Gib73} and Mark Satterthwaite \cite{Sat75}. We will first explore the work of Kenneth Arrow, who initiated the
1.2 Arrow’s Impossibility Theorem

Arrow’s seminal work in modern social choice theory \cite{Arr50, Arr63} originated in an attempt to formalize an aggregate function for social opinion. Aggregate mechanisms existed previously in welfare economics: Called welfare functions, they attempted to measure societal welfare for a number of alternative possibilities by aggregating the utility or welfare of individuals (measuring, for instance, the impact of a change in fiscal policy or tax rates). These mechanisms all shared the assumption that as subjective a concept as individual utility could be compared or quantified. A breakthrough came with the Bergson-Samuelson social welfare function \cite{Ber38} which inspired Arrow’s own aggregate mechanism, also called a social welfare function \footnote{Differences between the two functions are elaborated on throughout Arrow’s paper \cite{Arr50}.}. Like the Bergson-Samuelson model, Arrow broke from previous economic models by considering an individual’s vote to be their ranked preferences rather than a collection of numerical utilities over the alternatives. Thus the output of the social welfare function is a ranked ordering of the alternatives as well. Arrow argues that this is a more appropriate model for aggregate functions due to the difficulty of interpersonal comparisons of utility.

Arrow’s key result, known as Arrow’s impossibility theorem, shows that no social welfare function can satisfy all of a set of five reasonable criteria whenever there are more than two alternatives. By reasonable criteria we mean these conditions “. . . accord with common sense and with our intuition about fairness and the democratic process” \cite{Szep10}. In formalizing his aggregate mechanism, Arrow laid down two postulates that directed the construction of individual preferences, and outlined the aforementioned five characteristics.

The first postulate states that for any pair of alternatives $a, b$, every individual will always have some opinion between them: individuals can be indifferent between them (generally represented as $aIb$ or $bIa$, denoting indifference between $a$ and $b$), or prefer one alternative to the other (generally represented as $aPb$ in the case that $a$ is preferred to $b$). The second postulate is that an individual’s preferences must be transitive, thus disallowing cycles in individual preference orderings. Note that this requirement is not universally held in voting theory, and intransitive preferences are sometimes considered reasonable when voters use different criteria to decide between different pairs of alternatives \cite{Hug80}. Consider the example of an individual Jeff who has to rent a car, and is willing to pay a little more for additional space. Between a compact and a midsize car, Jeff always chooses the midsize, since he has to pay just a little more for additional comfort. Similarly, between a midsize and a fullsize car, Jeff prefers a fullsize car. But
between a fullsize car and a compact, Jeff finds the price difference to be too great, and chooses the compact car instead.

Arrow defines five reasonable criteria for social welfare functions: unrestricted domain, monotonicity, nonimposition, independence of irrelevant alternatives and nondictatorship.

**Unrestricted Domain** By the two aforementioned postulates, an individual’s preferences are represented as an ordering complete over the set of alternatives. Any restriction on which sets of orderings are permitted as input to the function violates the democratic nature of the mechanism and would fail to satisfy this criterion. Unrestricted domain would be violated in the example of elections held in a despotic state where only votes with the current ruler ranked first are allowed.

**Nonimposition** The second criterion states that the function should not allow inclusion or preclusion of outcomes irrespective of the preferences of the electorate. This criterion implies the social outcome should depend entirely on the set of individual preference orderings. An example of imposition could be an election in a theocracy where only candidates from the state religion are permitted to be elected.

**Monotonicity** The third property, monotonicity, states that an individual cannot harm an alternative’s position by ranking it higher. In other words, if an aggregate preference ordering holds that alternative Ted is preferred to alternative Jeff, then an individual cannot harm Ted’s aggregate position by ranking him above Jeff in his or her ordering, all other individual orderings being constant. This property implies that the aggregate decision must be responsive to and representative of the individual’s preferences.

In the later version of his work, Arrow replaced monotonicity and nonimposition with the combined property of the Pareto criterion, or unanimity [Arr63]. The Pareto criterion is slightly stronger than monotonicity since it incorporates nonimposition. It states that for any two alternatives $a$ and $b$, if an individual preference ordering prefers $a$ to $b$ with all other individual preferences indifferent between these two alternatives, then the social outcome also prefers $a$ to $b$.

**Independence of Irrelevant Alternatives** The fourth property, independence of irrelevant alternatives (IIA), implies that individual preferences for any pair of alternatives should not be influenced by other alternatives. A famous anecdote attributed to Sidney Morgenbesser [Pon08] illustrates IIA: Morgenbesser, ordering dessert in a restaurant, was informed by the waitress that the dessert choices were apple pie and blueberry pie. Morgenbesser chose apple pie. A few minutes later, the waitress returned and informed him that cherry pie was also available. “In that case,” said Morgenbesser to the utter
confusion of the poor waitress, “I’ll have blueberry.”

IIA and the possibility of strategic behavior in a voting system are mutually exclusive: the presence of one in a voting model indicates the absence of the other. While any honest preference relation between a pair of alternatives would not be influenced by extraneous alternatives, strategic behavior often requires them. Consider a Borda election between two alternatives Jeff and Mike where a certain voter prefers Mike, the stronger candidate, to Jeff. A new candidate, Ted, is introduced that the voter prefers to all others. The voter, then, instead of his true preference ordering Ted > Mike > Jeff (where Ted > Mike implies that Ted is preferred to Mike), might misrepresent his preferences as Ted > Jeff > Mike, in order to weaken Ted’s strongest opponent, Mike. In other words, the introduction of Ted results in the voter switching his preferences for Jeff and Mike.

**Nondictatorship** The fifth property states the function should not permit an individual who is a dictator, i.e., for a given profile of individuals, the function cannot reflect any one individual’s preferences, irrespective of the preferences of all others in the profile.

Arrow proved that any thusly defined acceptable social welfare function, meeting all these criteria, cannot decisively aggregate the preferences of a profile of individuals if there are more than two alternatives. This result essentially meant that any social welfare function violated the most basic thresholds for acceptability, thus any such function would have to compromise on meeting at least one of these five criteria. Arrow, in discussing this problem opined that compromising on an unrestricted domain was the only reasonable alternative [Arr50].

One of the more notable approaches to this problem actually preceded Arrow’s work: In 1948 Scottish economist Duncan Black wrote about an intriguing property of societal preferences called single-peakedness [Bla48, Bla58]. Consider plotting a preference ordering with the horizontal axis representing a linear ordering of alternatives and the vertical axis representing their rank in the ordering. If the resulting curve (drawn from joining all alternative-representing points together) has a single peak (defined to be a point flanked by either lower-ranking points on both sides, or only on one side if the peak starts or ends the curve) then we can state that the preference ordering is single-peaked with respect to the linear ordering on the horizontal axis. For a given profile of preferences, if there exists at least one linear ordering such that all votes are single-peaked with reference to this linear ordering, then we pronounce the profile to be single-peaked, or admitting the property of single-peakedness.

Black showed that aggregate functions admitting single-peaked preference profiles (with respect to some linear ordering) meet all of Arrow’s criteria except for unrestricted domain. We lose this criterion since a linear ordering that induces single-peakedness does
Figure 1: A preference profile that is single peaked for the ordering abcd. The votes are $a > b > c > d$ (solid), $b > c > d > a$ (dashed), and $c > d > b > a$ (dotted).

not exist for every preference profile\footnote{The existence of a single-peaked profile would (partially) depend on the set of least-preferred alternatives across all given orderings. For example, in the case of exactly three alternatives $a, b, c$, any profile of two or more preference orderings having a total of two alternatives $a, b$ ranked last, linear orderings $acb$ and $bca$ exist relative to which the profile is single-peaked. For methods to determine single-peakedness, we refer the reader to the work of Ballester and Haeringer \cite{BH11} and Escoffier et al. \cite{ELO08}.}. Therefore, if an aggregate function is to admit only single-peaked input, it would have to exclude certain preference orderings, restricting the domain.

Another approach by Amartya Sen \cite{Sen69} showed the existence of aggregate mechanisms that have all of Arrow’s criteria but implement a relaxation of transitivity called quasi-transitivity. This permits the existence of certain preferences that violate standard transitivity—for example, for alternatives $a, b, c$ the following preference profile is acceptable: $aIb, bIc, aPc$. Sen also discusses an aggregate function where the input, instead of individual preference orderings, is individual utility functions.

Brams and Fishburn showed that approval voting similarly has very desirable properties when we restrict the preference domain \cite{Nie84}. In the case that voters have dichotomous preferences (that is, they can divide the candidates into two groups: one they they approve of and one they do not), approval voting has many positive properties, including immunity to strategic voting and the Condorcet criterion \cite{BF78}. In the general case with unrestricted preferences, the system no longer has these properties \cite{Nie84}.

The impact of Arrow’s work can be summed up in a quote attributed to Paul Samuelson \cite{Pou08}, “What Kenneth Arrow proved once and for all is that there cannot possibly be . . . an ideal voting scheme.” The Gibbard-Satterthwaite theorem held even more dismal news: In addition to being less-than-ideal, all voting schemes are also vulnerable to manipulation, unless they admit dictators.
1.3 Gibbard-Satterthwaite Theorem

In the 1970s Alan Gibbard [Gib73] and Mark Satterthwaite [Sat75] independently extended Arrow’s theorem for voting systems in a model that incorporated strategic misrepresentation of preferences. They proved that no strategy-proof voting system existed for elections with three or more alternatives, unless the voting system allowed dictators. A strategy-proof voting system is one where no manipulating strategy can exist in elections using this voting system. While this result revolutionized voting theory, it had been speculated previously: In 1960, William Vickrey, when discussing individuals strategically misrepresenting their preferences in Arrow’s model [Vic60], stated that “it is clear that social welfare functions that satisfy the non-perversity [monotonicity] and the independence [IIA] postulates and are limited to rankings as arguments, are also immune to strategy.” In addition, the Dummett-Farquharson conjecture of 1961 [DF61] parallels the Gibbard-Satterthwaite theorem.

The Gibbard-Satterthwaite theorem transformed voting theory for two reasons: one was the aforementioned result that nondictatorial voting rules were susceptible to strategic voting (manipulation) in cases with three or more outcomes, and the second was the adoption of the arcane science of game theory from Neumann and Morgenstern’s *Theory of Games and Economic Behavior* [vNM44], published just four years prior to Arrow’s work. While the influence of game theory was implied (and acknowledged) in Arrow’s social welfare function [Wil72], the Gibbard-Satterthwaite theorem proofs were more explicit in their treatment of voting functions as game-theoretic mechanisms. This provided the field of voting theory with a set of tools to examine a whole range of scenarios—for example, the motivations for the electorate to vote dishonestly, or their reaction to changes in the structure of the voting rule, or attempts to form coalitions to further their individual utility.

In Gibbard’s work, a voting scheme or social choice function is built upon a construct called a game form [Gib73]. A game form is similar to a game construct in game theory [vNM44] applied to a voting model: players are voters, and any player’s strategies are the set of all possible orderings of preferences (unless restricted for specific scenarios), over a set of alternatives or candidates. Game forms, unlike games, do not have functions assigning utilities for each player for a given action (or chosen strategy) or for a scenario of chosen strategies of all the players. Instead the social choice function has the concept of an honest or sincere strategy: Among the set of strategies for each player is a specially marked strategy denoting the honest representation of preferences for that player. This can be used to compare outcomes (which may be a preference ordering or a single alternative) for a voter’s different strategies, to see if one is less or more preferable to another in the honest ranking of alternatives.

This social choice function will be immune to manipulation only if each voter has a
*dominant strategy*, a strategy that will be at least as good as any other for that voter no matter what any other voter does. Otherwise, if a voter does not have a dominant strategy, then they might possibly be motivated to change their vote from their true preferences in order to obtain a better outcome. A social choice function where each voter has a dominant strategy (and hence is immune to manipulation) is said to be *straightforward*.

The proof of the Gibbard-Satterthwaite theorem relies on the Vickrey conjecture: a voting scheme (defined with the property of unrestricted domain) is strategy-proof if and only if it has the properties of IIA and unanimity. Since Arrow showed that no aggregate function with more than two outcomes can satisfy all of his model’s criteria, such a voting system then is necessarily a dictatorship. In other words, we have that any voting system with at least three outcomes will either be a dictatorship or it will be manipulable. A similar result, the Duggan-Schwartz theorem [DS00], exists for voting systems that elect multiple candidates.

The Gibbard-Satterthwaite theorem presents a problem and accepts a solution similar to Arrow’s theorem. By Black’s results, voting schemes that permit only single-peaked preferences restrict the domain of the function, but can have all other Arrow criteria, including unanimity and IIA, which together imply strategy-proofness. Thus, relaxing the condition of unrestricted domain is necessary for voting schemes that resist manipulation. Another example of this is Gibbard’s development of probabilistic mechanisms [Gib77, Gib78]. Procaccia took a similar approach by designing strategy-proof probabilistic voting systems that are similar to standard deterministic voting systems [Pro10].

### 1.4 Computational Difficulty of Manipulation

Another solution to the problem of inherent manipulability in voting was proposed by Bartholdi along with Tovey, Trick, and Orlin in a series of papers that started the field of computational social choice [BTT89a, BTT89b, BO91]. Their approach was to select voting schemes where manipulation is computationally difficult to carry out, i.e. where the manipulation problem is NP-hard. Our definition of the manipulation problem is that of constructive coalitional manipulation: i.e., does there exist a set of votes for the manipulating coalition that causes their preferred candidate to win the election? This subsumes the case of a single manipulator and contrasts with destructive manipulation, which is concerned with preventing a certain candidate from winning.

Bartholdi et al.’s initial work also highlighted the problems of selecting systems with complexity. Their impracticality theorem [BTT89b] is another instance of systems meeting seemingly reasonable criteria thereby inducing undesirable properties.

The theorem states that any *fair* voting system requires excessive computation to
determine the winner, making it impractical—a highly disturbing result. This theorem followed work by Kemeny [Kem59], Young and Levenglick [YL78] and Gardenfors [Gar76].

According to Bartholdi et al., a voting system is fair if it meets the Condorcet criterion, the condition of neutrality (symmetry in its treatment of candidates, implied by IIA [GPP09]), and the condition of consistency (if disjoint subsets of the voters voting separately arrive at the same preference ordering, then voting together always produces this same preference ordering as well). Their theorem states that computation of the winner in any such fair voting system is NP-hard.

The previously mentioned property of consistency (also called convexity [Woo94] and separability [Smi73]) has been proven to be present in ranked voting systems (those in which a vote is a ranking or ordering of preferences) only if they happen to be scoring protocols as well (where alternatives receive a certain number of points depending on their position in the ordering) [You75]. Scoring protocols are often incompatible with the Condorcet criterion (refer the aforementioned debate on Condorcet versus Borda) thus unsurprisingly so far we know of only one voting system, Kemeny scoring [Kem59], that meets all three conditions [YL78]. Kemeny scoring then, is NP-hard, but additionally it has been shown to be complete for parallel access to NP [HSV05].

However, this does not imply other voting systems are immune to having an intractable winner problem: systems such as Dodgson meet only two of the three fairness conditions—that of the Condorcet criterion and neutrality but not consistency—however computation of the winner in Dodgson is known to be not only NP-hard [BTT89b] but complete for parallel access to NP [HHR97], similar to the result for Kemeny scoring.

Research focusing on tractable voting systems4 was more promising: While Bartholdi et al. gave us a greedy algorithm that finds a manipulating vote for several tractable voting systems in polynomial time [BTT89a], two voting systems—second-order Copeland and single transferable vote [BTT89a, BO91]—proved to be resistant and were shown to have a manipulation problem that is NP-hard. Research in this field remained dormant for the next fifteen years until a revival starting in 2006 brought about results for most common tractable voting systems.

In this paper we are concerned with a restricted version of the manipulation problem. We survey tractable voting systems that resist manipulation in the unweighted coalitional manipulation (UCM) model with only a constant number of manipulators. This limited case subsumes hardness results in the weighted coalitional manipulation (WCM) model or with variably-sized coalitions, thus making a case for UCM being a stronger class of manipulation. We include both the initial work of Bartholdi, Tovey, and Trick [BTT89a] and Bartholdi and Orlin [BO91] that achieved the first results in this area and the recent resurgence of interest in this problem that has resulted in a number of new outcomes.

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4A voting system is tractable if calculating the winner takes at most polynomial time.
1.5 Terms Defined

An election is defined to be an instance of a voting system, comprising a voting rule $\nu r$, a set of candidates $C$ and a set of votes $V$. A voting rule is a function that takes as input a set of votes and a set of candidates and outputs a set of winners. Unless explicitly stated otherwise, references to $n$ and $m$ are defined as follows: $n = ||V||$ and $m = ||C||$. A vote is defined to be a linear ordering over the set of candidates. The advantage of a candidate $c_i$ over $c_j$ (hereafter referred to as $adv(c_i, c_j)$) is the number of votes that rank $c_i$ ahead of $c_j$, with values ranging from 0 to $n$. The net advantage of a candidate $c_i$ over $c_j$ is $adv(c_i, c_j) - adv(c_j, c_i)$, with values ranging from $-n$ to $n$. A netadv score between two candidates $c_i$ and $c_j$ is represented as $netadv(c_i, c_j)$. By definition we can see that $netadv(c_i, c_j) = -netadv(c_j, c_i)$, thus one netadv score can represent both directions.

UCM (unweighted coalitional manipulation) is defined to be a decision problem as follows.

Given An election, namely, a voting rule $\nu r$, a set of voters $V$ such that $V = V_{NM} \cup V_M$, where $V_M$ is the subset of voters that form the manipulating coalition and $V_{NM}$ is all other voters, and a set of candidates $C$ containing a distinguished candidate $c$.

Question Does there exist a set of votes for $V_M$ such that $\nu r$ over the complete set of votes yields $c$ as the winner?

We use the format $UCM_{2Cope}$ to refer to the UCM problem in second-order Copeland and likewise for other election systems.

2 UCM in Single Transferable Vote

Single transferable vote (henceforward STV) is a voting system with a long history. As esteemed a figure as John Stuart Mill said it was “among the greatest improvements yet made in the theory and practice of government.” It determines the winners with a simple multiround procedure that redistributes votes placed for less popular candidates. Also, unlike many of the esoteric voting systems studied in voting theory, STV has a history of being used for real-world political elections, in the United States and around the world.

The STV vote tallying procedure is as follows. Give a point to each candidate for each first-place vote it receives. If any candidate is the majority winner (i.e. with more than half the total points), that candidate will be the only winner of the election. If no majority winner exists, then select the candidates with the fewest number of points, remove them from consideration, and for the voters who currently give their support to these candidates, reallocate their support by giving their points to the next-highest ranked candidate on their ballots still under consideration. Repeat this procedure until a
winner is chosen or all candidates are removed. If the latter occurs, then all of candidates that were removed in the last round will be the winners.

The complex, shifting behavior of STV with multiple candidates is what gives it the resistance to manipulation we discuss here, but it also leads to STV failing to possess some very desirable voting system characteristics. Notably it does not possess the property of monotonicity: It is possible for a voter to increase his or her ranking of a candidate and for that candidate to subsequently do worse in the election. This was enough for Doron and Kronick [DK77] to refer to it as a “perverse social choice function,” and it certainly is a flaw of concern.

2.1 STV is Resistant to Manipulation

We show resistance to manipulation through a conventional, if difficult, reduction based on Bartholdi and Orlin’s work [BO91]. Their work was actually directed towards showing that the EFFECTIVE PREFERENCE problem is NP-complete. EFFECTIVE PREFERENCE is the problem of finding if a single voter can cause the preferred candidate to win an election. This is effectively the same as UCM with a single manipulator, and we will prove that this problem is NP-hard for STV with a proof based on the aforementioned work [BO91]. The proof is structured as a reduction from the exact cover by three-sets problem.

**Exact Cover by Three-Sets (X3C)**

**Given** A set \( D = \{d_1, \ldots, d_{3k}\} \) and a family \( S = \{S_1, \ldots, S_n\} \) of sets of size three of elements from \( D \).

**Question** Is it possible to select \( k \) sets from \( S \) such that their union is exactly \( D \)?

In other words, the goal of the problem is to find if there is an appropriately-sized set of subsets which covers each of the elements in \( D \). Since each \( S_i \) has exactly three elements and \( k \) such sets from \( S \) must be chosen, the set of subsets must be an exact cover with no repeated elements in all subsets chosen.

**Proof.** We will describe a reduction from an instance of X3C to a instance of the unweighted manipulation problem for STV. Note that since the reduction will only require a single manipulator, this shows that UCM_{STV} is NP-hard even for only a single manipulator.

Given an instance of X3C \((D, S)\) we construct the election as follows. Let the following comprise the candidate set \( C \):

- The possible winners \( c \) and \( w \);
• The set of “first losers” \( a_1, \ldots, a_n \) and \( \overline{a}_1, \ldots, \overline{a}_n \), one of each corresponding to each subset \( S_i \);

• The “second line” \( b_1, \ldots, b_n \) and \( \overline{b}_1, \ldots, \overline{b}_n \), one of each corresponding to each subset \( S_i \);

• The \( w \)-bloc \( d_0, \ldots, d_{3k} \), each of whose voters will just prefer them to \( w \);

• The “garbage collector” candidates \( g_1, \ldots, g_n \).

We will now describe the set of voters. Where we use ellipses, the remainder of a vote is arbitrary for our purposes and will not effect the result of the election.

• 12\(n\) voters with preferences \((c, \ldots)\);

• 12\(n - 1\) voters with preferences \((w, c, \ldots)\);

• 10\(n + 2k\) voters with preferences \((d_0, w, c, \ldots)\);

• For each \( i \in \{1, \ldots, 3k\} \), 12\(n - 2\) voters with preferences \((d_i, w, c, \ldots)\);

• For each \( i \in \{1, \ldots, n\} \), 12\(n\) voters with preferences \((g_i, w, c, \ldots)\);

• For each \( i \in \{1, \ldots, n\} \), 6\(n + 4i - 5\) voters with preferences \((b_i, \overline{b}_i, w, c, \ldots)\) and for the three \( j \) such that \( d_j \in S_i \), 2 voters with preferences \((b_i, d_j, w, c, \ldots)\);

• For each \( i \in \{1, \ldots, n\} \), 6\(n + 4i - 1\) voters with preferences \((\overline{b}_i, b_i, w, c, \ldots)\) and 2 voters with preferences \((\overline{b}_i, d_0, w, c, \ldots)\);

• For each \( i \in \{1, \ldots, n\} \), 6\(n + 4i - 3\) voters with preferences \((a_i, g_i, w, c, \ldots)\), 1 voter with preferences \((a_i, b_i, w, c, \ldots)\), and 2 voters with preferences \((a_i, \overline{a}_i, w, c, \ldots)\);

• For each \( i \in \{1, \ldots, n\} \), 6\(n + 4i - 3\) voters with preferences \((\overline{a}_i, g_i, w, c, \ldots)\), 1 voter with preferences \((\overline{a}_i, \overline{b}_i, w, c, \ldots)\), and 2 voters with preferences \((\overline{a}_i, a_i, w, c, \ldots)\).

This reduction works by requiring the elimination order of a subset of the candidates to correspond to an exact cover over \( B \) in order for \( c \) to win the election. Namely, \( c \) will win the election if and only if \( I = \{ i \mid b_i \text{ is one of the first } 3n \text{ candidates to be eliminated} \} \) is an exact cover. Furthermore, there is a preference order for a single manipulator that will force \( I \) to be an exact cover if one exists. We will now consider the relevant properties of the election and show that this is the case.

Since this election is a single-winner STV election with more than two candidates, the scoring process will proceed for a number of rounds and a number of candidates will
be eliminated as the rounds progress. The first $3n$ candidates to be eliminated will be $a_1, \ldots, a_n, \overline{a}_1, \ldots, \overline{a}_n$, and exactly one of $b_i$ or $\overline{b}_i$ for every $i \in \{1, \ldots, n\}$.

Candidate $c$ initially has $12n$ votes, while $c$’s primary rival $w$ has $12n - 1$ votes. Every voter that does not have $c$ as their first choice ranks $c$ directly below $w$, and so $c$ can only gain more votes if $w$ is eliminated. In order to do so, the manipulator must ensure that $w$ does not gain additional votes before it is eliminated, as otherwise $w$ would have gained votes against $c$ and $c$ would have been eliminated first. The manipulator must consequently make sure that no candidate is eliminated such that any voter most prefers that candidate and prefers $w$ second-most. This is the case with every voter that prefers one of the $d_j$ candidates, so $w$ will gain a large number of votes if one of them is eliminated. Therefore if any $d_j$ candidate is eliminated before $w$, $c$ cannot possibly win. For every $b_i$ candidate that is eliminated, $\overline{b}_i$ gains a large number of votes, pushing it higher than $12n$ in score and preventing it from being eliminated early. Also, every $d_j \in S_i$ gains two votes, pushing them high enough to prevent their early elimination. Thus eliminating $b_i$ protects the $d_j$ candidates associated with the set $S_i$. Conversely, for every $\overline{b}_i$ that is eliminated, $d_0$ gains two votes and $b_i$ gains a large number of votes, preventing $b_i$ from being eliminated early. Thus for every $i$ only one of $b_i$ or $\overline{b}_i$ can be eliminated before $c$ or $w$.

The $a$ candidates are the other candidates that can be eliminated early. For every $a_i$ that is eliminated, $\overline{a}_i$ gains two votes, $b_i$ gains one vote, and $g_i$ gains the rest of $a_i$’s votes. The effect of this is that now $\overline{a}_i$ has been promoted above $b_i$ and $\overline{b}_i$ in the overall ranking, and since $b_i$ also gained a point over $\overline{b}_i$, $\overline{b}_i$ will be the next to be eliminated instead of $b_i$. Thus by controlling which of $a_i$ or $\overline{a}_i$ is eliminated first, we control which of $b_i$ or $\overline{b}_i$ is eliminated early.

Hence we can show that the candidate $c$ will win the election if and only if $I = \{i \mid b_i$ is one of the first $3n$ candidates to be eliminated$\}$ is an exact cover. We know that either $b_i$ or $\overline{b}_i$ will be among the first $3n$ eliminated candidates. If $b_i$ is the one eliminated, then every $d_j \in S_i$ will gain two votes and will each have at least $12n$ votes total, protecting them from early elimination. If $I$ is an exact cover, this will be true for every $d_j$ as each of them is covered by some selected $b_i$ and so each of them will win over $w$. Also, since $d_0$ gains two points for every one of the $\overline{b}_i$ eliminated, $d_0$ will gain at least $2(n - k)$ votes and will receive at least $12n$ votes overall, pushing it over the score of $w$ as well. Thus after the first $3n$ candidates have been eliminated, $w$ will have the least score with $12n - 1$, having gained no votes other than it’s initial first-place votes, and it will be the next candidate to be eliminated. The candidate $c$ will then gain a large number of votes from the elimination of $w$ and will go on to win the election.

If the set $I$ defined above does not correspond to an exact cover, $c$ cannot win the election. If $I$ is not an exact cover, some candidate $d_j$ will not gain the two points from
a corresponding $b_i$ being eliminated. Thus $d_j$ will only have $12n - 2$ votes after the first $3n$ candidates are eliminated while the other remaining candidates have at least $12n - 1$, leaving $d_j$ as the next candidate to be eliminated. The candidate $w$ then gains points from $d_j$’s elimination, preventing $c$ from gaining points against $w$ and winning the election.

If an exact cover exists, a single manipulator can construct it’s preference order as follows to ensure $c$ is a winner. For an exact cover $I$, if $i \in I$, let the $i$th candidate in the preference order be $a_i$ and otherwise $\overline{a}_i$. The rest of the preference order is arbitrary. This will result in the following order of elimination of the first $3n$ candidates: For $i \in I$, $\overline{a}_i, b_i, a_i$ will be eliminated in that order in positions $3i-2$, $3i-1$, and $3i$. For $i \notin I$, $a_i, b_i, a_i$ will instead be the candidates to be eliminated. For $i \in I$, since this preference ranks $a_i$ over $\overline{a}_i$, $a_i$ will gain one more vote and thus $\overline{a}_i$ will be eliminated first. This then gives two votes to $a_i$ and one more to $\overline{b}_i$, making $b_i$ the next least-preferred candidate. When $b_i$ is eliminated, $\overline{b}_i$ gains a large number of votes, so $a_i$ is now the least-preferred candidate and is eliminated next. The rounds proceed similarly in the case that $i \notin I$ and $\overline{a}_i$ is preferred instead. Thus the set $\{i \mid b_i$ is one of the first $3n$ candidates to be eliminated$\}$ will correspond to an exact cover and $c$ will win the election.

If no exact cover exists, no matter how a manipulator votes, at least one of the $d_j$ candidates will not receive the protective boost from the elimination of the corresponding $b$ candidate as previously described. This candidate will then be eliminated early, leading to $w$ being boosted past $c$ in votes and preventing $c$ from winning.

Thus even just setting the first $n$ rankings for the ballot of a single manipulator for STV is NP-hard, and the system is resistant even to this very limited case of manipulation.

3 UCM in Borda Voting

Borda voting is a classic voting system dating back at least to the eighteenth century. It was introduced by the French mathematician and engineer Jean-Charles de Borda to remedy the failure of plurality in reflecting the wishes of the electorate when used with more than two candidates: In plurality the candidate with the most votes is not necessarily preferred to all other candidates. Borda voting is very similar to a system introduced in the 15th century by Cardinal Nicolaus Cusanus [Szp10].

It has a rich and varied history of real-world use: In some form it has been used in political elections in Slovenia and the Micronesian countries of Kiribati and Nauru, in the Eurovision contest, the election of the board of directors of the X.Org foundation, and even in sports, in the election of the Most Valuable Player award in Major League Baseball. Borda is one of a class of systems known as scoring protocols, where each vote awards points to each candidate depending on their ranking in the vote. The winners are
candidates with the highest sum of points over all the votes. In the case of Borda voting, candidates receive linearly descending points for progressively less favorable positions in the votes, with the top position awarding \( m - 1 \) points, the next position awarding \( m - 2 \) points, and so on down to 0 points for the lowest position.

It was long an open problem whether \( \text{UCM}_{\text{Borda}} \) was hard in general, though manipulation with a single manipulator has long been known to be easy \[\text{BTT89a}\]. A greedy algorithm that can find a set of successful manipulating votes in polynomial time that is at most one larger than the optimum manipulative coalition size is known as well \[\text{ZPR08}\]. Recently, Betzler et al. \[\text{BNW11}\] and Davies et al. \[\text{DKNW11}\] proved Borda-manipulation to be NP-hard for instances with two or more manipulators.

3.1 Borda is Resistant to Manipulation

Both Betzler et al. \[\text{BNW11}\] and Davies et al. \[\text{DKNW11}\] prove their results by reduction from the problem of 2-numerical matching with target sums, a known NP-hard problem \[\text{YHL04}\] that closely corresponds to the problem of allocating points to the nonfavored candidates in the election.

2-Numerical Matching with Target Sums (2NMTS)

**Given** A sequence \( a_1, \ldots, a_k \) of positive integers with \( \sum_{i=1}^{k} a_i = k(k+1) \) and \( 1 \leq a_i \leq 2k \).

**Question** Are there permutations \( \psi_1 \) and \( \psi_2 \) of \( 1, \ldots, k \) such that \( \psi_1(i) + \psi_2(i) = a_i \) for \( 1 \leq i \leq k \)?

**Preliminaries** For manipulation to work, nonfavored candidates must be ranked low enough in manipulative votes such that the number of points they gain by said votes do not prevent the preferred candidate from winning. To that end we define the *gap* to be the maximum number of points nonfavored candidates can gain by all manipulative votes while still allowing the preferred candidate to win. In any Borda instance with a favored candidate \( c^* \), \( m \) other candidates, and \( t \) manipulators, the gap \( g_i \) for a candidate \( c_i \) is \( \text{score}(c^*) + t \cdot m - \text{score}(c_i) \). Here \( \text{score}(c) \) refers to the Borda score for the candidate \( c \) over the nonmanipulative votes. We assume these gap values \( g_1, \ldots, g_m \) to be ordered in a nondecreasing fashion.

**Result 1** Recall that the awarded points for the last \( j \) candidates in a vote will range from \( j - 1 \) to 0, hence the sum of their points equals \( j(j - 1)/2 \). Thus for any successful manipulation instance, we must have that \( \sum_{i=1}^{j} g_i \geq t \cdot j(j - 1)/2 \) for each \( j \in \{1, \ldots, m\} \). We define an instance to be *tight* if \( \sum_{i=1}^{j} g_i = t \cdot j(j - 1)/2 \). Thus, in a tight instance, for manipulation to be successful, the number of points a nonfavored candidate gains from manipulative votes must be exactly equal to its gap value.

**Proof.** Given any instance of 2NMTS we construct a \( \text{UCM}_{\text{Borda}} \) instance \((C,V,p)\) as
follows. The candidate set $C$ consists of candidates $c_1, \ldots, c_k$ and the preferred candidate $p$, and thus the range of Borda points is from 0 to $k$. The set of votes $V$ consists of a manipulating coalition of size two and a set of nonmanipulating votes of size three. The instance is constructed such that the gap $g_i$ for any nonfavored candidate $c_i$ is $2k - a_i$. In this context, the Borda problem can be considered as follows: for every nonfavored candidate $c_i$, can we assign a position in each of the manipulating votes such that the points $c_i$ gains from said votes is $\leq g_i$? This constructed instance of Borda manipulation will have a solution if and only if the 2NMTS instance has a solution.

**Direction 1** Given a solution to 2NMTS, we can obtain a solution for the Borda instance as follows: Preferred candidate $p$ is placed in the first position in both manipulating votes. A solution to 2NMTS exists so we have two orderings $\psi_1(i), \psi_2(i)$ such that $\psi_1(i) + \psi_2(i) = a_i$. For every candidate $c_i$, $(1 \leq i \leq k)$ set its position to $\psi_1(i) + 1$ in the first manipulative vote, and $\psi_2(i) + 1$ in the second manipulative vote. The corresponding Borda points are obtained from subtracting this position number from $||C|| = k + 1$. Therefore the points $c_i$ has gained from both manipulative votes is $(k + 1 - (\psi_1(i) + 1)) + (k + 1 - (\psi_2(i) + 1))$ which equals $2k - a_i = g_i$, permitting $p$ to win.

**Direction 2** Given a solution to the Borda instance, we have a solution for 2NMTS as follows: By construction, $\sum_{i=1}^{k} g_i = \sum_{i=1}^{k} (2k - a_i)$.

Since $\sum_{i=1}^{k} a_i = k(k + 1)$, $\sum_{i=1}^{k} (2k - a_i) = k(k - 1)$.

Hence, this Borda instance is tight for $j = k$ and $t = 2$ (from Result 1) and consequently each nonfavored candidate $c_i$ gains exactly $g_i$ points from the manipulative votes. If $pos_i(1)$ and $pos_i(2)$ are the positions for $c_i$ in the two manipulative votes, points gained from these positions total $(k + 1 - pos_i(1)) + (k + 1 - pos_i(2)) = g_i = 2k - a_i$, which yields $pos_i(1) + pos_i(2) = a_i + 2$. Therefore setting $\psi_1(i) = pos_i(1) - 1$ and $\psi_2(i) = pos_i(2) - 1$ gives us a solution for 2NMTS.

As mentioned earlier, we assume by construction $g_i = 2k - a_i$. This requires constructing the set of nonmanipulative votes such that the deficits for each candidate relative to the preferred candidate map precisely to the target sums in the original problem. Executing this involves complicated construction and the addition of a large number of “dummy” candidates to pad out the remaining positions and precisely set the required deficits for the primary candidate set $[BNW11]$ (in Davies et al., such padding is done with voters $[DKNW11]$). Thus the constructed instance has a much larger candidate set than a voter set (though it remains polynomially bounded), but it suffices to prove the desired hardness result.
4 UCM in Copeland Elections

Copeland voting is a voting system with a long history. One version of the system was discovered by the 13th century mystic Ramon Llull, and then another variation was discovered by A.H. Copeland in the 1950s. It is a Condorcet voting system.

Copeland voting is in fact a family of voting systems, parametrized on how ties are handled. In Copeland$^\alpha$, the score of a candidate $c$ in an election $E$ is $\text{wins}_E(c) + \alpha \cdot \text{ties}_E(c)$, where $\text{wins}_E(c)$ denotes the number of pairwise victories and $\text{ties}_E(c)$, the number of ties of the candidate $c$ in the election $E$. Llull’s system is Copeland$^1$, while “Copeland voting” has been used to describe Copeland$^{0.5}$ or to describe Copeland$^0$. Different parameter values subtly alter the behavior of the system and complicate the task of analyzing its computational properties, as we will explore.

4.1 Copeland is Resistant to Manipulation for Most Values of $\alpha$

UCM$_\text{Cope}^\alpha$ is in P when there is only one manipulator [BTT89a], but it is known to be resistant to manipulation even with two manipulators for $\alpha \in [0, 0.5) \cup (0.5, 1]$ [FHS10, FHS08]. The complexity of UCM$_\text{Cope}^\alpha$ for $\alpha = 0.5$ remains unknown as of date. Different proofs were required to show resistance to manipulation for different parameter ranges, as the behavior of the system changes in subtle but significant ways with different values of the parameter. We will describe the proof that was used to show hardness for Copeland$^\alpha$ for $\alpha \in \{0, 1\}$ [FHS10]. Both these cases were proved by Faliszewski et al. [FHS10] through similar reductions from X3C.

Exact Cover by Three-Sets (X3C)

Given A set $D = \{d_1, \ldots, d_{3k}\}$ and a family $S = \{S_1, \ldots, S_n\}$ of sets of size three of elements from $D$.

Question Is it possible to select $k$ sets from $S$ such that their union is exactly $D$?

Proof. The proof constructs a UCM$_\text{Cope}^\alpha$ instance from an X3C instance $(D, S)$ using graph representations to equate both problem instances. A election for the reduction can be constructed without an explicit collection of votes as such a collection can be elicited from the set of netadv or adv scores for all pairs of candidates$^5$. The graph representation of the election can be constructed from the set of netadv or adv scores as well$^6$. Both these constructions are polynomially bounded in the size of the set of candidates and the value of the netadv function. Thus, using these techniques we can construct the election given a partial set of significant candidates, the netadv scores for all pairs of candidates,

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$^5$Refer Appendix A.

$^6$Refer Appendix B.
and the lead in Copeland score that the distinguished candidate has over each specified candidate.

\[ c_{i,1} \quad c_{i,2} \quad c_{i,3} \]
\[ d_{i,1} \quad d_{i,2} \quad d_{i,3} \]
\[ -(n-k) \quad 2 \quad 2 \]

Figure 2: Gadget used in the Copeland manipulation NP-hardness proof \cite{FHST10}: The gadget is constructed for each \( S_i \) with numbers for each candidate showing the lead in Copeland score the preferred candidate has over them.

Given any instance of an X3C problem, the proof defines the set of candidates as the elements of sets \( D \) and \( S \), one preferred candidate \( p \), and one main adversary candidate \( c \) (the candidate holding the highest Copeland score prior to manipulation) as well as auxiliary candidates used for padding and constructing the mathematical gadgets. Such gadgets (also called widgets \cite{CLRS01}) are a common feature in reduction constructions: They are typically intricate subgraphs that are constructed for each element of the mapped-from problem. They align calculations and enforce constraints so as to ensure a tight mapping between instances of the two problems.

The numbers of the election are constructed in such a way that prior to manipulation, the elements of \( D \) each beat \( p \) by 1 unit of Copeland score and \( c \) beats \( p \) by \( n-k \) units of Copeland score. However, the elements of \( S \) each lose to \( p \) by 3 units. Hence any successful set of manipulating votes must cause the elements of \( D \) and \( c \) to lose just enough pairwise contests against the elements of \( S \) to erode their lead over \( p \) without making any \( S_i \) a possible winner. This involves selecting a subset of \( S \) that beats every candidate from \( D \) but still leaves enough elements of \( S \) to lower \( c \)’s score by \( n-k \) points. The construction of such a vote corresponds to selecting a \( k \)-sized subset of \( S \) that covers exactly the elements of \( D \), that is, a solution to the X3C problem. Hence, a solution to \( \text{UCM}_{\text{Cope}} \) gives us a solution to X3C.

\[ \square \]

5 UCM in Second-Order Copeland

Solutions for breaking ties in Copeland voting attempt to choose the more “powerful” candidate as the winner, which can be defined in a number of ways. One such method is second-order Copeland. It selects the candidate whose set of defeated opponents (hereafter referred to as \( DO_c \) for a candidate \( c \)) has the higher sum of Copeland scores. We
will let $\text{sum\_score}(S)$ for a set of candidates $S$ be the sum of the Copeland scores for candidates in $S$, and so $\text{sum\_score}(DO_c)$ gives the second-order Copeland score for a candidate $c$.

Second-order Copeland has been used by the National Football League and the United States Chess Federation to break ties, and has a special place in voting theory: it was the first tractable voting system for which the manipulation problem was shown to be NP-complete, even for just one manipulator [BTT89a].

### 5.1 Second-Order Copeland is Resistant to Manipulation

The problem of unweighted coalitional manipulation in second-order Copeland, hereafter referred to as UCM$_{2\text{Cope}}$, is NP-complete even for one manipulator [BTT89a]. Verifying a given solution is clearly in P as we simply have to calculate Copeland scores and second-order Copeland scores for each candidate. To prove UCM$_{2\text{Cope}}$ is NP-hard we show a polynomial-time reduction from 3,4-SAT, a known NP-hard problem [Tov84].

### 3,4-SAT

**Given** A set $U$ of Boolean variables, a collection of clauses $Cl$, each clause composed of disjunctions of exactly three literals, which may be a variable or its complement, and each variable occurs in exactly four clauses.

**Question** Does there exist a Boolean assignment over $U$ such that each clause in $Cl$ contains at least one literal set to true?

To facilitate the reduction we construct a graph representation of a second-order Copeland election that encodes the given 3,4-SAT instance. We will use an election graph representation with vertices representing candidates and directed edges representing the result of pairwise contests.

**Proof.** Given a 3,4-SAT instance, we create a second-order Copeland election graph as follows: Every clause ($C_1$ to $C_{|Cl|}$) and every literal is a candidate, represented as a vertex in our graph. The manipulating coalition’s chosen candidate is a separate candidate $C_0$. All pairs of vertices have directed edges between them, representing decided pairwise contests, except for any variable and its complement. The decided contests cannot be overturned by our manipulators, while undecided contests can be shifted in either direction according to the manipulating vote. Clauses beat (that is, have a directed edge to) literals they contain, and lose to all other literals.

In addition to these candidates derived from the 3,4-SAT instance, we pad the election with a number of auxiliary candidates in such a way to achieve the desired Copeland scores and second-order Copeland scores for each of the candidates. We will have that
Figure 3: A partial representation of the resultant election graph: Clauses represented are $C_1(x_1 \lor x_2 \lor x_3)$, $C_2(x_1 \lor \neg x_2 \lor x_3)$, $C_3(\neg x_1 \lor \neg x_2 \lor x_3)$ and $C_4(x_1 \lor x_2 \lor \neg x_3)$. Each clause vertex beats the variables (or their complements) that are its component literals. The dotted lines indicate undecided (second-order Copeland) contests between variables and their complements.

all the clause candidates and $C_0$ are tied with the highest Copeland score. We will also have that each clause candidate $C_i$ has $\text{sum\_score}(DO_{C_i}) = \text{sum\_score}(DO_{C_0}) - 3$.

The second-order Copeland score for $C_0$ will be independent of the variable-complement contests. For all possible outcomes of the variable-complement contests, $C_0$ still beats every candidate except for the clause candidates. Recall that the elements for every clause candidate’s defeated-opponent set are their component literals. Their second-order Copeland scores and the final result of the election will then depend on how each of the variable-complement contests are decided.

Consider if any clause candidate $C_i$’s literals win all their contests. The $\text{sum\_score}(DO_{C_i})$ increases by 3 points and $C_i$ is tied with $C_0$ for first place. Therefore, in order for $C_0$ to be the unique winner, at least one element of each clause candidate’s defeated-opponent set must lose one of their contests. For any variable $x$, we can interpret a directed edge from $x$ to $\neg x$ as setting $x$ to true. A vote that allows $C_0$ to win, then, would correspond exactly to each clause having at least one literal evaluating to true. In other words, we have a solution for UCM$_{2\text{Cope}}$ if and only if we have solution for 3,4-SAT. Therefore UCM$_{2\text{Cope}}$ is NP-hard with just a single manipulator.

6 UCM in Maximin

Maximin voting, also known as the Simpson-Kramer method, is a typical Condorcet voting system. As such it deals with contests between pairs of candidates, specifically their netadv scores. To find the winner under maximin, given a netadv function over the set of candidates, we first select the lowest netadv score for each candidate $k$ in $C$,
i.e., we select the minimum score for $\text{netadv}(k, k')$ for all $k'$ in $C$ such that $k' \neq k$. The winner is the candidate with the highest such score. We can trivially see that a candidate with a minimum $\text{netadv}$ score greater than 0 will be the Condorcet winner and there can only be one such candidate, thus maximin is a Condorcet voting system.

Calculating the maximin winner is easily seen to be polynomial in the size of the election, but Xia et al. [XCPR09] prove that for two or more manipulators the problem of $\text{UCM}_{\text{maximin}}$ is NP-complete.

6.1 Maximin is Resistant to Manipulation

$\text{UCM}_{\text{maximin}}$ is NP-complete for two or more manipulators [XCPR09]: Verifying an instance is easily seen to be polynomial in the size of the election as we can calculate the winner in polynomial time. To prove $\text{UCM}_{\text{maximin}}$ is NP-hard we construct a polynomial-time reduction from the vertex-disjoint-two-path problem, known to be NP-complete [LR78].

**Vertex-Disjoint-Two-Path problem (VDP$_2$)**

**Given** A directed graph $G$ and two sets of vertices $u, u'$ and $v, v'$ such that all four vertices are unique.

**Question** Do there exist two paths $u \to u_1 \to \ldots \to u_j \to u'$ and $v \to v_1 \to \ldots \to v_k \to v'$ in $G$ such that each path is a set of vertices disjoint from the other?

**Proof.** To facilitate our reduction construction from a graph problem such as the vertex-disjoint-two-path problem to maximin, we first construct a graph representation of maximin elections. A complete set of votes is not required to represent an election. We can construct the same given just a $\text{netadv}$ (or $\text{adv}$) function. Also, there exists a bijection between the $\text{netadv}$ function and directed edges of a complete antisymmetric graph such that given one, we can represent it in terms of the other.

$P_{\text{coal}}$ is the set of votes of the manipulating coalition and $P_{\text{noncoal}}$ is the set of all other votes. The term $\text{netadv}_{\text{noncoal}}$ indicates the $\text{netadv}$ score obtained by considering only the noncoalitional votes with $\text{netadv}_{\text{coal}}$ similarly defined. $M$ is the size of the coalition and $c$ is the candidate supported by the manipulating coalition.

Given a VDP$_2$ instance, that is a graph $G(V_G, E)$ and vertices $u, u', v, v' \in V$, we obtain a graph $G'$ using the following constructions and assumptions:

- Every vertex in our graph is reachable from $u$ or $v$.
- There are no directed edges $u \to v'$ or $v \to u'$.

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7Refer Appendix A.

8Refer Appendix B.
We add special edges \( u' \to v \) and \( v' \to u \) such that \( E_{G'} = E \cup \{(u', v), (v', u)\} \).

Our UCM\(_{\text{maximin}}\) instance then is as follows:

- Set of candidates \( C = V_G \).
- The set of \( P_{\text{noncoal}} \) preferences.
  - \( \forall c' \in C : c' \neq c, \text{netadv}(c, c') = -4M \);
  - \( \text{netadv}(u, v') = \text{netadv}(v, u') = -4M \);
  - For all other edges \((x, y)\) in \( E \), \( \text{netadv}(y, x) = -2M - 2 \);
  - For all other vertex pairs \( a, b \), \( \text{netadv}(a, b) = 0 \).

Regarding \( P_{\text{coal}} \) votes, we can freely assume that every coalition vote will rank \( c \) first, thus giving \( \text{netadv}_{\text{coal}}(c, c') = M \) and \( \text{netadv}_{\text{noncoal}\cup\text{coal}}(c, c') = -3M \). Thus, in our construction, scores for \( \text{netadv}(c, c') \) are fixed for all candidates \( c' \in C \). In order for \( c \) to be the winner, at least one \( \text{netadv} \) score must be less than \(-3M\) for every other candidate. We will see that in our construction this can occur if and only if there are two vertex-disjoint paths in \( G' \).

**Direction 1** The existence of vertex-disjoint paths \( u \to u_1 \to \ldots \to u_j \to u' \) and \( v \to v_1 \to \ldots \to v_k \to v' \) yields a \( P_{\text{coal}} \) that makes \( c \) the winner. In order to construct the manipulative preferences, we will make use of a connected subgraph over \( G' \) containing all the vertices, but with \( u \to \ldots \to u' \to v \to \ldots \to v' \to u \) as the only cycle in the graph.

We can construct \( P_{\text{coal}} \) votes in 3 parts as follows: Each manipulator vote will rank \( c \) the highest, followed by the vertex-disjoint-path vertices, followed by the other vertices.

*Other-vertex ordering:* These vertices will be ordered in the votes based on a linear order extracted from the single-cycle subgraph.

*Vertex-disjoint-path orderings:* We have two vertex-disjoint-path orderings: \( u \to \ldots \to u' \to v \to \ldots \to v' \) and \( v \to \ldots \to v' \to u \to \ldots \to u' \) and thus two possible vote constructions for \( P_{\text{coal}} \). We construct \( M - 1 \) votes as per the first ordering and 1 vote as per the second\(^9\). Thus \( \text{netadv}(c, c') \) increases by \( M \) points but every other \( \text{netadv} \) score increases by less than \( M \) points, making \( c \) the winner. The calculations are as follows for the complete (coalitional and noncoalitional) set of votes:

- \( \text{netadv}(u, v') = -4M + (M - 1) - 1 = -3M - 2 \)
- \( \text{netadv}(v, u') = -4M + 1 - (M - 1) = -5M + 2 \)

\(^9\)Switching these orderings results in the same outcome.
For any other candidate $c' \not\in \{c, u, v\}$, we can see that there exists some candidate $d$ in every vote of $P_{coal}$ that beats $c'$, i.e., the lowest netadv score for $c'$ is $netadv_{coal}(c', d) = -M$, and thus for the complete set of votes the (lowest) netadv for any such candidate $c'$ is no more than $-2M - 2 - M = -3M - 2$. All the above netadv scores are less than $-3M$ for all values of $M \geq 2$, thus $c$ is the winner.

**Direction 2** The existence of a $P_{coal}$ that makes $c$ a winner yields a positive VDP$_2$ instance in the graph $G'$:

Since $c$ is the winner, we know that for any other candidate $c'$ in $C$, there exists a candidate $d$ that beats $c'$ such that:

- $netadv(c', d) < -3M$;
- There exists an edge $(d, c')$ in $G'$;
- $d$ is ranked higher than $c'$ in a majority of the total votes and in at least one vote in $P_{coal}$—the proof of this is as follows.

Consider such an edge $(d, c')$:

10: either $(d, c')$ is one of the special edges $(v', u)$, $(u', v')$ or $(d, c') \in E$. If $(d, c')$ is a special edge, then at least one vote in $P_{coal}$ must prefer $d$ to $c'$ (since $netadv(c', d) < -3M$). If $(d, c') \in E$, then all $M$ votes in $P_{coal}$ must prefer $d$ to $c'$.

For this $d$, we can choose a candidate that beats it with sufficient margin, and continue to find such a candidate for the previous choice of $d$.

11 Choosing such a $d$ is formalized in the proof of Xia et al. [XCPR09] as a composite function $f$.

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7 **UCM in Tideman Ranked Pairs**

Tideman ranked pairs (TRP) was conceived by Nicolaus Tideman in 1987 when attempting to define a voting system that “almost always” has the property of independence of clones [Tid87]. It is defined as follows: given a netadv function over the set of candidates, create a list by ranking the pairs in descending order of their scores. In the case of a tie between two netadv pairs, e.g., $netadv(a, b) = netadv(x, y)$, we break ties by ordering

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10 If there exists more than one such $d$ we choose one arbitrarily.

11 Choosing such a $d$ is formalized in the proof of Xia et al. [XCPR09] as a composite function $f$.
the pairs lexicographically according to an arbitrary ordering of the candidates. We add the top-scoring pair to an election graph $G$ if the resultant graph does not contain a cycle. Otherwise we skip this pair and move on to the next one in the list. We continue until we have considered all pairs. Since we now have a directed acyclic graph, there must exist a source vertex, which we state to be the TRP winner.

7.1 TRP is Resistant to Manipulation

Xia et al. [XCPR09] found that UCM$_{TRP}$ is NP-complete even for one manipulator. We can easily see that verifying a given solution to UCM$_{TRP}$ is in P. UCM$_{TRP}$ was proven to be NP-hard by a polynomial-time reduction from 3SAT.

Three-Conjunctive-Normal-Form Satisfiability (3SAT)

**Given** A set $U$ of Boolean variables, a collection of clauses $Cl$, each clause composed of disjunctions of exactly three literals, which may be a variable or its complement.

**Question** Does there exist a Boolean assignment over $U$ such that each clause in $Cl$ contains at least one literal set to true?

**Proof.** Given a 3SAT instance, we construct a UCM$_{TRP}$ election graph as follows. Clauses $C_1 \ldots C_{|Cl|}$ are vertices as is the coalition’s preferred candidate $c$. For each clause $C_i$, we construct six other special clause candidates—three for the literals it contains and three for their complements. $C_i$ beats (has a directed edge to) the special clause candidates corresponding to the literals it contains, which in turn beat the literals they correspond to. We also have a $C'_i$ such that it is beaten by the complement of the literals in $C_i$. Candidate $c$ starts out beating the $C_i$ candidates but gets defeated (though by a smaller margin) by the $C'_i$ candidates. The intuition of this correspondence is that the edges beating $c$ are so weak and so far down the ordering that they are not added, leaving $c$ to be the source vertex (and TRP winner) if and only if there exists a solution to 3SAT. Thus the UCM problem is NP-complete even in the case of a single manipulator. The proof of Xia et al. [XCPR09] relies on mathematical gadgets to achieve this correspondence.

8 Conclusion

Our survey of UCM results can be seen as qualifying election systems by a single metric. Determining which election system is superior is an ongoing debate often reflecting differing philosophies. Pierre-Simon Laplace in his lectures at the Ecole Normale Superieure in 1795 attacked the éléction par ordre de mérite (election by ranking of merit) system of his contemporary Jean-Charles de Borda [Szp10], later proposing a variation of the majority rule in its place. Much of the modern literature on voting theory is still devoted

\[ \text{As usual, } V_G = C \text{ and directed edges represent netadv scores.} \]
Voting rule | Coalition size = 1 | Coalition size ≥ 2
--- | --- | ---
Copeland\(^\alpha\)(0 < \(\alpha\) < 0.5)(0.5 < \(\alpha\) < 1) | P [BTT89a] | NP-complete [FHS08]
Copeland\(^\alpha\)(\(\alpha\) = \{0, 1\}) | P [BTT89a] | NP-complete [FHS10]
Copeland\(^\alpha\)(\(\alpha\) = 0.5) | P [BTT89a] | ?
Second-order Copeland | NP-complete [BTT89a] | NP-complete [BTT89a]
Single Transferable Vote | NP-complete [BO91] | NP-complete [BO91]
Maximin | P [BTT89a] | NP-complete [XCP09]
Tideman Ranked Pairs | NP-complete [XCP09] | NP-complete [XCP09]
Borda | P [BTT89a] | NP-complete [BNW11, DKNW11]
Bucklin | P [XCP09] | P [XCP09]
Plurality with Runoff | P [ZPR08] | P [ZPR08]
Veto | P [BTT89a] | P [ZPR08]
Cup | P [CSL07] | P [CSL07]

*Table 1: Table of UCM results for common tractable voting systems.*

To advocacy for particular voting systems, arguing their superiority by one metric or another [New92, Saa06, Ris05].

In our survey we showcase manipulation results for a particular class of voting systems, namely those with a tractable winner problem but where unweighted coalitional manipulation is hard for a constant coalition size. The complexity of this case of UCM has been determined for most common voting rules, though a few remain: Copeland\(^0.5\) remains unsolved even as results for all other parameter values have been found [FHS10].

Several related areas of research, however, remain more or less uncharted. The most significant simplification in the literature is that most of the hardness results achieved are just worst-case. Several papers have studied whether voting systems are difficult to manipulate in a large fraction of instances, finding that manipulation can be easy in the average case while being hard in the worst case [CS06, PR07b, PR07a]. Additionally, approximation algorithms exist for several worst-case hardness results. Brelsford et al. [BFH+08] formalized manipulation as an optimization problem and then studied whether this version of the problem is approximable. Zuckerman et al. [ZPR08] discovered an approximation algorithm for Borda manipulation before it was known to be NP-hard, and Davies et al. [DKNW11] gave several other approximation algorithms for this problem. In other results, approximation algorithms for manipulation of maximin as well as families of scoring protocols exist [ZLR10, XCP10]. Other techniques include the use of relatively efficient algorithms for the NP-complete integer partitioning problem to solve manipulation instances [Lin11].

Conitzer et al. [CSL07] qualified the manipulation problem with an additional metric: the minimum number of candidates that must be present for manipulation to be NP-hard. Additionally, their work breaks from the standard model and studies whether manipulation is hard for cases where manipulators do not have complete information of
all of the votes. Slinko explored how often elections will be manipulable based on the size of the manipulative coalition [Sli04].

Research into how often elections can be manipulated [FKN08], and more general areas such as parametrization of NP-hard problems [Nie10] and phase transitions [CKT91, KS94, Zha01] lead to a more nuanced approach to problem classification. Phase transitions have been examined in the manipulation problem for the veto rule [Wal09].

Another issue is that votes are most commonly represented in the literature as transitive linear preference orderings over the set of candidates and the concept of irrational votes has only been sparsely dealt with. Irrational (by which we mean intransitive) votes, may be more apt for any number of real-world scenarios where voters tend to rank candidates according to multiple criteria. Irrational votes are not represented as a linear ordering but as a preference table which holds the voter’s choice for any pair of candidates. For Copeland\(\alpha\) for \(\alpha \in \{0, 0.5, 1\}\), manipulation is in \(P\) in the irrational voter model, while it is known to be \(NP\)-hard for \(\alpha \in \{0, 1\}\) in the standard voter model [FHS10]. Thus voting systems may have different behavior with regard to manipulation in the irrational voter model and it deserves more study. Another convention is that the default definition of UCM is constructive—i.e., efforts are directed to making a preferred candidate a winner, rather than preventing a certain candidate from winning. Variations of UCM with a destructive approach is another area rich with possibilities.

UCM instances presented in this paper typically have a large number of candidates and a smaller constant-sized coalition of manipulators. In contrast, cases with a small number of candidates and a relatively large manipulating coalition might be considered more natural. Betzler et al. [BNW11] mention a specific open problem in this area: whether there exists a combinatorial algorithm to solve Borda efficiently with few candidates and an unbounded coalition size. Another open problem is solving a UCM\(Borda\) instance having a coalition of size 2 in less than \(O(||C||!))\).

Another approach to the manipulation problem taken by Conitzer and Sandholm [CS03] and Elkind and Lipmaa [EL05] is modifying voting systems to give them greater resistance to manipulation. Both add an extra initial round of subelections between subsets of the candidates. Conitzer and Sandholm [CS03] describe techniques that can make manipulation \(NP\)-hard or even \(PSPACE\)-hard for these modified voting systems. Elkind and Lipmaa [EL05] present a version of this technique that uses one-way functions to construct the initial-round schedule from the set of votes. Reversing the one-way function is computationally hard, preventing election organizers from gaming the initial round and forcing their desired result in polynomial time. These techniques essentially construct new voting systems by structurally augmenting standard systems to imbue them with complexity.

Other related work includes the study of electoral control, which encompasses attempts
by an election organizer to change the result by modifying the election structure in various ways [BTT92, FHHR09, EF10b, HHR09], which also encompasses cloning, or adding candidates very similar to existing candidates in an attempt to split their support [Tid87, EFS10]. Other ways to influence elections include bribery and campaign management, where in both cases a briber attempts to sway the result of an election by paying off a set of voters to change their votes [FHH09, FHHR09, Fal08, EFS09, EF10a, SF11]. These, too, are problems endemic to many voting systems to which complexity can serve as a defense.

Another possible response to the problems presented by Arrow’s theorem and the Gibbard-Satterthwaite theorem is to reconsider the standard model of the aggregate function. Balinski and Laraki [BL07] introduce a model where voters give candidates independent grades, such as the letter grades F to A or {good, average, bad}, similar to approval voting or range voting, rather than ranking them in a linear order. In a sense this represents a reversion to the pre-Bergson-Samuelson model of welfare functions. Balinski and Laraki’s method defines the aggregate grade of each candidate to be the median grade over all votes, unlike range voting where the aggregate grade is the average. We can obtain a complete aggregate preference ordering of the candidates with this method provided that ties can be broken. Balinski and Laraki give a tie-breaking mechanism that successively removes one of the median-score-awarding voters from the votes for each tied candidate and recomputes the median grades until they are no longer tied [BL07]. Their approach does not rely on complexity but instead redesigns the election model to become strategy-proof in a limited case defined by the authors.

Faliszewski et al. [FHHR11] showed that with a restriction to single-peaked preferences, a wide range of manipulation and control instances that are NP-hard in the general case turn out to be easy (though not any of the results we describe here). For some voting systems these problems remain easy even with a partial relaxation of the single-peaked model that allows for a small number of “mavericks”, whose votes are not aligned with the single-peaked ordering [FHH11]. The single-peaked model is considered “the canonical setting for models of political institutions” [GPP09], so this work calls the significance of a number of hardness results into question.

After the birth of research in the manipulation problem with the work of Bartholdi et al. [BTT89a], most research moved towards the weighted voter model and many results for the weighted coalitional manipulation problem (WCM) were achieved [CSL07, HH07], until the resurgence of interest in the UCM problem [BNW11, DKNW11, FHS10, FHS08, XCP09]. It can be argued that as compared to WCM, UCM is a better test of a voting system’s vulnerability to manipulation. UCM serves as a special case of WCM and hence subsumes its hardness results. In other words, if an election system is resistant to manipulation in the UCM case, it will resist manipulation in the WCM case, but the other direction does not necessarily follow. With this problem solved for most common
voting systems, we look forward to the resolution of the remaining open problems as well as new avenues of research into the manipulability of voting systems.

A Constructing an Election Given a netadv Function: the McGarvey Method

While the traditional representation of an election requires a set of votes, we can construct these votes given a pairwise relation denoting preference over the set of candidates. This method was given by McGarvey [McG53], and can be applied with very little modification to a netadv function.

A.1 From a Preference Pattern to a Set of Votes

Theorem A.1. (McGarvey’s Theorem) Given a preference pattern we can elicit a set of votes (defined to be strict and complete preference orderings over the set of candidates) such that (the ordering derived from) the preference pattern is the result of the election.

Definition 1. A preference pattern is a set of relations over the set of candidates. The relations are a preference relation (expressed as $aPb$ viz. $a$ is preferred to $b$) and an indifference relation ($aIb$ viz. $a$ is neither preferred to $b$ nor is $b$ preferred to $a$).

Both relations are distinct for any pair of candidates - i.e., $aPb$ implies $\neg bPa$, and $aIb$ implies $bIa$. Thus we will have $m(m - 1)/2$ pairs over both relations where $m$ is the number of candidates. McGarvey’s method constructs a set of votes as follows:

For each pair $aPb$ with remaining candidates $c_1, \ldots, c_{m-2}$, we construct two preference orderings $abc_1 \ldots c_{m-2}$ and $c_{m-2} \ldots c_1ab$. For each pair $aIb$, we construct $abc_1 \ldots c_{m-2}$ and $c_{m-2} \ldots c_1ba$. The idea is that on evaluation for these preferences, rankings of all candidates besides $a, b$ from these orderings will be equal, and the rankings of $a, b$ reflect the preference relation under consideration. Consider the example $C = a, b, c, d$. The six pairs we consider are $aPb, aPc, aPd, bPc, bPd, cPd$.

To represent $aPb$ we construct two votes $abcd$ and $dcab$. Evaluating these two votes in the context of pairwise rankings leads to two votes for $aPb$ and no votes for any other pair over the set of candidates. Similarly, for $aPc$, we construct $acbd$ and $dbac$ and so on. The idea is that for all candidates besides the ones under consideration, preferences for and against them cancel each other out. Hence the need for two votes for each pair. The total number of thus-constructed votes is twice the cardinality of the preference pattern. Thus in our example, we obtain a set of votes which yield exactly the relations in the given preference pattern.

\footnote{Where candidates are listed in order of decreasing preference.}
A.2 From a netadv Function to a Set of Votes

We consider the following useful property of the netadv function when constructing the corresponding election:

**Theorem A.2.** For all pairs of candidates $c_i, c_j$ where $c_i \neq c_j$, the values of $\text{netadv}(c_i, c_j)$ are either all even or all odd.

**Proof.** Consider two blocs of votes where each bloc takes one side in a pairwise election. Let the sizes of the blocs be $x$ and $y$ such that $x + y = n$.

If $n$ is even:

- Then $x, y$ are either both odd or either both even since the sum components of an even number are either both even, or both odd.
- The difference between two even numbers or two odd numbers is always even.

If $n$ is odd:

- Then $x, y$ are either odd and even, or even and odd, respectively, since the sum components of an odd number are always a combination of even and odd.
- The difference between an even and odd number is always odd.

Thus, all the netadv values are either all even or all odd. \qed

We can see that the netadv function corresponds to elements of a preference pattern: 
netadv($c_i, c_j$) $> 0$ corresponds to $c_i Pc_j$, netadv($c_i, c_j$) $= 0$ corresponds to $c_i Ic_j$, and netadv($c_i, c_j$) $< 0$ corresponds to $c_j Pc_i$. However, the key difference between the netadv functions and preference-pattern elements is that netadv has scores, which we must factor into our construction.

Since we construct two votes for each pair of candidates, the problem of applying McGarvey’s method to a netadv relation with an even score is trivial—for each of the two preference orderings constructed we simply have $n/2$ such votes. In fact, given a netadv function, if one netadv score is even, then (1) the number of votes will be even, and (2) every netadv score in that set will be even. The converse also applies: if one netadv score in a given set is odd, then the number of votes will be odd, and every netadv score in the given set will be odd.

Applying McGarvey’s method to an odd set of netadv scores requires a slight tweak. We first must select some arbitrary ordering of the candidates. Then for any netadv score $s = \text{netadv}(a, b)$ where $a$ precedes $b$ in the ordering, we construct $s - 1$ (net) votes
(or units of score) exactly as in the case where the scores are all even (i.e., two preference orderings, with 
\((s - 1)/2\) such votes for each one). In the case of \(s = \text{netadv}(b,a)\) where 
a precedes \(b\), we will instead create \((s + 1)/2\) pairs of votes to give a score of \(s + 1\) for 
b over \(a\). To obtain the last points, we construct one preference ordering corresponding 
to the previously chosen ordering of the candidates. This will give one net vote to every 
pair \(a,b\) where \(a\) precedes \(b\), and minus one vote where \(b\) precedes \(a\). Thus we achieve 
the desired odd numbers for each \(\text{netadv}\) input.

**Number of Constructed Votes** In the above two cases (\(\text{netadv}\) scores being even or 
odd) we can see the upper bound on the number of votes constructed is one for every 
unit of score across all \(\text{netadv}\) values. The size of the set of votes will be bounded 
by \(\sum_{c_i,c_j \in C} |\text{netadv}(c_i,c_j)|\). Thus given a \(\text{netadv}\) function with bounded value, we can 
construct a reasonably small set of votes.

**B Graph Representation of the \(\text{netadv}\) Function**

Consider a complete directed antisymmetric graph \(G(V_G,E)\) where \(V_G\) is the set of ver-
tices and \(E\) is the set of directed weighted edges. We can trivially see that there will 
be \(m(m - 1)/2\) directed edges, where \(m = ||V_G||\). Recall that this is the same as the 
number of distinct scores required to represent a \(\text{netadv}\) function (where \(m\) is the number 
of candidates). Thus we can represent a \(\text{netadv}\) function with such a graph \(G\) where \(V_G\) 
is the set of candidates \(C\) and \(E\) encodes the \(\text{netadv}\) function. Similarly, given such a 
graph, we can obtain an equivalent \(\text{netadv}\) function.

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