Exact General Solutions to Extraordinary N-body Problems

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The N-body problem in which the force on the \( I \)th body is
\[
F_I = k \sum_J m_J (x_J - x_I)/r^3
\]
where \( r \) is the time dependent radius of the whole system is solved exactly for all initial conditions. Here, writing \( \mathbf{x} \) for the position of the barycentre, \( r^2 = M^{-1} \sum_I m_I (x_I - \mathbf{x})^2 \).

Also solved are all N-body problems for which the potential energy is of the form \( V = V(r) \) for any function \( V \) where \( r \) is given above. The first problem is the special case \( V \propto r^{-1} \). In all cases \( r \) vibrates for ever, just like the radius to a body which has angular momentum about a centre of attraction due to the potential \( V(r) \). Violent Relaxation does not occur in these systems. When \( r \) vibrates as in a highly eccentric orbit the exact solution should prove to be a useful test of N-body codes but a severer test follows.

We show further that when \( F_I \) is supplemented by an inverse cubic repulsion between each pair of bodies
\[
-k' \sum_{I \neq J} m_I m_J (x_J - x_I)/(x_J - x_I)^4
\]
there is still no damping of the fundamental breathing oscillation which performs its non-linear periodic motion for ever. This is a special case of continually breathing systems with potentials \( V_0(r) + r^{-2} V_2 ((x_I - x_J)/r) \).

A remarkable new statistical equilibrium is found for systems whose size is continually changing!

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1. Introduction

In 1995 one of us (Lynden-Bell, R.M. 1995, 1996) showed that the classical statistical mechanics of systems of N bodies interacting through mutual potential energies of the form
\[
V = NF \left( \sum q_i^2 / N \right)
\]
could be exactly calculated for any chosen function \( F \). By choice of a suitable function \( F \), she found a case with a simple phase transition which was calculated for any \( N \).

When the particles are of equal mass and when \( q_I \) measures the displacement of the \( I \)th particle from the barycentre \( \mathbf{x} \), \( \sum_I q_i^2 / N \) may be rewritten as a mutual
potential energy

\[ N^{-1} \sum \mathbf{q}_I^2 = N^{-1} \sum (\mathbf{x}_I - \mathbf{x})^2 = \sum_{I < J} (\mathbf{x}_I - \mathbf{x}_J)^2 / N^2. \]  

(1.2)

More than 300 years previously Newton showed that the N-body problem in which the force on body I due to body J was

\[ k m_I m_J (\mathbf{x}_J - \mathbf{x}_I) \]

could be exactly solved with all the interactions present (Newton (1687), see also Chandrasekhar (1995)). Newton’s case corresponds to a potential energy

\[ V = \frac{1}{2} k \sum_{I < J} m_I m_J (\mathbf{x}_I - \mathbf{x}_J)^2. \]  

(1.3)

When the masses are equal the expression (1.3) is the linear-F case of expression (1.1) by virtue of (1.2). As the statistical mechanics associated with expression (1.1) is still beautiful even when \( F \) is any non-linear function, we are led to wonder whether Newton’s exact solution to the N-body problem could be generalised to cover potential energies of the form \( V = V(r) \). Here \( V \) is any function and \( r \) is the mass weighted root mean square size of the system

\[ r^2 = M^{-2} \sum_{I < J} m_I m_J (\mathbf{x}_I - \mathbf{x}_J)^2 = M^{-1} \sum_{I} m_I (\mathbf{x}_I - \mathbf{x})^2 \]  

(1.4)

where

\[ M = \sum m_I \quad \text{and} \quad \mathbf{x} = M^{-1} \sum m_I \mathbf{x}_I. \]  

(1.5)

Following this idea we found a beautiful dynamics in \( 3N \) dimensional space which precisely mimics normal orbit theory in 3 dimensions.

2. Solutions to the N-body Problems

The equations of motion are

\[ m_I \ddot{\mathbf{x}}_I = -\partial V / \partial \mathbf{x}_I. \]  

(2.1)

Provided \( V \) is a mutual potential energy, i.e., a function of the \( \mathbf{x}_I - \mathbf{r}_J \) then \( \sum \partial V / \partial \mathbf{x}_I = 0 \) so \( \mathbf{X} = 0 \) and

\[ \mathbf{x} = \mathbf{x}_0 + \mathbf{u} t \]  

(2.2)

Writing \( \mathbf{q}_I = \mathbf{x}_I - \mathbf{x} \) we have \( r^2 = M^{-1} \sum m_I \mathbf{q}_I^2 \) so in centre of mass coordinates the equations of motion for the \( \mathbf{q}_I \) are

\[ m_I \ddot{\mathbf{q}}_I = -\partial V / \partial \mathbf{q}_I = -\partial V / \partial r \frac{m_I \mathbf{q}_I}{M r}. \]  

(2.3)

We now invent a \( 3N \) dimensional vector \( \mathbf{r} \) of length \( r \), whose components are

\[ \mathbf{r} = \left( \sqrt{\frac{m_1}{M}} \mathbf{q}_1, \sqrt{\frac{m_2}{M}} \mathbf{q}_2, \ldots, \sqrt{\frac{m_I}{M}} \mathbf{q}_I, \ldots \right). \]  

(2.4)

The first 3 components of \( \mathbf{r} \) involve the position of particle 1 and the next 3
particle 2, etc. Multiplying (2.3) by $\sqrt{M/m}$ we see that those equations for all $\mathbf{F}$ can be rewritten using unit vector $\mathbf{r}$

$$ M\ddot{\mathbf{r}} = -\frac{\partial V}{\partial r} \mathbf{r} = -V' \mathbf{r}. \tag{2.5} $$

For insight please consult here the note added in proof.

Let $\alpha$ and $\beta$ be indices that run from 1 to 3$N$. Then taking the antisymmetric product of (2.5) with $\mathbf{r}$ we find

$$ M \frac{d}{dt} (r_\alpha \dot{r}_\beta - r_\beta \dot{r}_\alpha) = 0 $$

so the hyper-angular-momenta (per unit mass) are all conserved

$$ r_\alpha \dot{r}_\beta - r_\beta \dot{r}_\alpha = L_{\alpha\beta} = \text{const} = -L_{\beta\alpha}. \tag{2.6} $$

These quantities are not just normal angular momenta. They involve among others the $x$ component of particle 1 multiplied by the rate of change of the $x$ component of particle 2, i.e., $x_1 \dot{x}_2$ and also such terms as $x_1 \dot{y}_2$, etc. Only when both $\alpha$ and $\beta$ are in the range 1, 2, 3 does $L_{\alpha\beta}$ reduce to components of the normal angular momentum of particle 1 (in that case). The trace of the square of the antisymmetric tensor $L_{\alpha\beta}$ is

$$ L_{\alpha\beta} L_{\alpha\beta} = 2 \left[ r^2 \dot{r}^2 - (\mathbf{r} \cdot \dot{\mathbf{r}})^2 \right] = 2L^2. \tag{2.7} $$

Notice that if we were in 3 dimensions $L^2$ would be just $(\mathbf{r} \times \dot{\mathbf{r}})^2$ so it would then be the square of the specific angular momentum. We shall refer to $ML$ as the total hyper-angular-momentum and $ML_{\alpha\beta}$ as the hyper-angular-momentum. Taking the inner product of equation (2.5) with $\dot{\mathbf{r}}$ and integrating, we obtain the energy equation

$$ \frac{1}{2} M \dot{r}^2 + V(r) = E. \tag{2.8} $$

Now

$$ \dot{r} = \frac{d}{dt} (\mathbf{r} \cdot \mathbf{r})^{1/2} = \mathbf{r} \cdot \dot{\mathbf{r}} / r \tag{2.9} $$

so from (2.7)

$$ \dot{r}^2 = L^2 r^{-2} + \dot{r}^2 \tag{2.10} $$

and (2.8) may now be rewritten

$$ \frac{1}{2} M (\dot{r}^2 + L^2 r^{-2}) + V = E. \tag{2.11} $$

Differentiating (2.11) and dividing by $\dot{r}$ we get the equation of motion for scalar $r$

$$ M (\ddot{r} - L^2 r^{-3}) = -V'(r). \tag{2.12} $$

This we recognise as the equation of motion of a particle of mass $M$ and angular momentum $\dot{M}L$ in the central potential $V(r)$. Equation (2.11) is of course its energy equation and it may be integrated in the form

$$ t - t_0 = \int^r \frac{dr}{\sqrt{2M^{-1}(E - V(r)) - L^2 r^{-2}}}, \tag{2.13} $$

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which gives the standard relation for the periodic\footnote{Periodic when the system is bound.} function $r(t)$. Equation (2.13) integrates nicely for the standard cases $V \propto 1/r$, $V \propto r^2$ and the isochrone $V \propto (b + s)^{-1}$ where $s^2 = r^2 + b^2$. These potentials may be supplemented by an extra term in $r^{-2}$ giving a $r^{-3}$ supplementary force but they comprise the only central orbits that can be solved involving nothing more complicated than trigonometric functions (Eggen et al. 1962, Evans et al. 1990). There are many others that involve elliptic functions, etc., but for them the expression (2.13) is as simple as the function. We shall return to (2.13) later but for the present we merely note that it determines the periodic\footnote{Periodic when the system is bound.} function $r(t)$.

By analogy with normal central orbits the form of equation (2.12) strongly suggests that we invent an angle $\phi$ such that

$$r^2 \dot{\phi} = L .$$

(2.14)

Then from (2.11) we eliminate $t$ and integrate to find

$$\phi = \int_{r_{\min}}^{r} \frac{L}{r^2 \sqrt{2M^{-1}[E + V(r)] - L^2 r^{-2}}} dr ,$$

(2.15)

just as in ordinary orbit theory. We define $\phi$ by (2.15) and so there is no constant of integration. It gives $\phi$ as a function of $r$, or $r$ as a periodic function of $\phi$. Thus we have determined scalar $r$ but we need to find all components of the $3N$ vector $\mathbf{r}$ to determine the motion completely. Although $V'(r)/r$ may now be regarded as a known function of $t$ nevertheless the form of (2.5) does not at first look very encouraging, except for Newton’s case for which $V = \frac{1}{2} kM^2 r^2$ so $V'(r)/r$ is constant. However, a wonderful simplicity will emerge for the general case too. For a moment we discuss Newton’s case. Equations limited to this case will have the letter N appended to the equation number. Evidently (2.5) becomes

$$\ddot{\mathbf{r}} = -kM \mathbf{r}$$

(2.16N)

so each component of $\mathbf{r}$ vibrates harmonically, all with angular frequency $\sqrt{kM}$. Thus each body orbits in an ellipse centred on the barycentre and moving with it and the whole motion relative to the barycentre is periodic with period $2\pi/\sqrt{kM}$. Newton obtained this solution by realising that the linear law of attraction, when suitably mass weighted, gave a net total force on each particle due to all the others which was directed toward the barycentre and in constant proportion to distance from it.

A similar simplicity emerges in the general case when we solve for $\ddot{\mathbf{r}} = \mathbf{r}/r$ as a function of $\phi$. To do this we write

$$\ddot{\mathbf{r}} = \frac{d}{dt} \left( \dot{\mathbf{r}} \dot{\mathbf{r}} + \ddot{\mathbf{r}} \ddot{\mathbf{r}} \right) = \frac{d}{dt} \left( \frac{L}{r} \frac{d \dot{\mathbf{r}}}{d\phi} + \ddot{\mathbf{r}} \ddot{\mathbf{r}} \right) = \frac{L^2}{r^3} \frac{d^2 \ddot{\mathbf{r}}}{d\phi^2} + \ddot{\mathbf{r}} \ddot{\mathbf{r}} ,$$

(2.17)

where two terms cancelled at the last step. Now scalar $\dot{r}$ is given by (2.12) so, inserting (2.17) into (2.5) and multiplying by $r^3 L^{-2}$, we find the lovely result

$$d^2 \ddot{r}/d\phi^2 + \ddot{r} = 0 .$$

(2.18)

Thus the unit vector $\hat{\mathbf{r}}$ vibrates harmonically in $\phi$ with period $2\pi$. For Newton's
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case, and for the hyper-Keplerian case \( V \propto 1/r \), the magnitude \( r \) is also periodic with period \( 2\pi \) in \( \phi \). But more generally, as in the isochrone, \( r \) is periodic but with a \( \phi \) period that is incommensurable with \( 2\pi \) (in general). The general solution of (2.18) is given by

\[
\hat{r} = A \sin \phi + B \cos \phi
\]  

(2.19)

where \( A \) and \( B \) are \( 3N \)-vector constants of integration. But \( \hat{r} \) is a unit vector so

\[
1 = A^2 \sin^2 \phi + 2A \cdot B \sin \phi \cos \phi + B^2 \cos^2 \phi = \frac{1}{2}(A^2 + B^2) - \frac{1}{2}(A^2 - B^2) \cos 2\phi + A \cdot B \sin 2\phi .
\]

Since this must be true for all \( \phi \) we find

\[
A^2 = B^2 = 1 \quad \text{and} \quad A \cdot B = 0,
\]

(2.20)

so \( A \) and \( B \) must be orthogonal unit \( 3N \)-vectors. There is a further restriction on these constants; because the \( q_I \) are the centre of mass coordinates they must obey

\[
0 = \left( \sum_I m_I q_I \right)_j = \left( \sum_I \sqrt{m_I} r_{3I-3+j} \right) \sqrt{M} r \quad \text{for} \quad j = 1, 2, 3
\]

therefore,

\[
0 = \sum_I \sqrt{m_I} A_{3I-3+j}
\]

and

\[
0 = \sum_I \sqrt{m_I} B_{3I-3+j}
\]

(2.21)

To construct constants \( A \) and \( B \) satisfying the constraints (2.20) and (2.21) we proceed as follows:

Take \( m_1 \) to be the largest mass; choose \( \overline{B}_\alpha \alpha > 3 \) and \( \overline{A}_\alpha \alpha > 3 \) arbitrarily, but set \( \overline{A}_j = \sum_{\alpha=1}^N \sqrt{m_\alpha} \overline{A}_{3J-3+j} \) for \( j = 1, 2, 3 \) and a similar relation for \( \overline{B}_j \). Then \( \overline{A}_j \) and \( \overline{B}_j \) satisfy (2.21) and so will \( \lambda \overline{A} \) and \( \mu(\overline{B} - \nu \overline{A}) \) where \( \lambda, \mu \) and \( \nu \) are any scalars, since (2.21) are linear.

Now choose \( \lambda = \frac{1}{\overline{A}} \) and set \( \overline{A} = \lambda \overline{A} \) so that \( \overline{A} \) is a unit vector; furthermore set \( \overline{B}' = \overline{B} - \overline{B} \cdot \overline{A} \overline{A} \) so that \( \overline{B}' \cdot \overline{A} = 0 \); finally normalise by writing \( B \equiv \mu B' \) with \( \mu = \frac{1}{|B'|} \). Then \( A \) and \( B \) so constructed are the general unit vectors satisfying the constraints (2.20) and (2.21). Thus our general solution for \( r \) is

\[
r = r\hat{r} = r(A \sin \phi + B \cos \phi)
\]  

(2.22)

Thus relative to the centre of mass every particle’s orbit becomes a centred ellipse after a universal scaling by \( r^{-1} \). This rescaling factor is time-dependent and \( r \) behaves like the radius to a particle in a central orbit which is an eccentric ellipse in the Keplerian case. This is our main result.

Differentiating (2.22) with respect to \( t \) one finds that (using \( r^2 \dot{\phi} = L \))

\[
L_{\alpha\beta} = r_\alpha \dot{r}_\beta - r_\beta \dot{r}_\alpha = (B_\alpha A_\beta - A_\alpha B_\beta)L .
\]

(2.23)

It is of interest here to count constants of integration. \( A \) and \( B \) together have \( 6N \) components but (2.20) and (2.21) give nine constraints, thus there are \( 6N-9 \) freedoms so far. In the solutions for \( r(t) \) and \( r(\phi) \) there are three further constants

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$L, E$ and $t_0$. Thus $r(t)$ depends on $6N$-6 constants. To these we may add the constants $\mathbf{x}_0$ and $\mathbf{u}$ involved in the motion of the centre of mass, and our final solution depends on $6N$ arbitrary constants. This equals the number of initial positions and velocities that determine the motion, which verifies that we have the general solution.

The solution is only fully explicit when we can carry out the integrations (2.13) and (2.15) which determine $r$ and $\phi$. We do not need this for Newton’s case because we already gave the solution. When $V \propto r\^{-1}$, the hyper-Keplerian case, it is actually simpler to follow Hamilton’s treatment of the problem with his eccentricity vector later used by Runge in 1919 and Lenz in 1924 (see Hamilton 1845, Laplace 1799, Goldstein 1976, or Chandrasekhar 1995). This was certainly known to Laplace and Bernoulli and can be traced to Newton. The method avoids the awkwardness of having to treat the elliptic, parabolic and hyperbolic cases separately. With $V = -kM^2/r$ equation (2.5) becomes

$$\ddot{r} = -kM\dot{r}/r^2 \quad (2.24K)$$

contracting with $L_{\alpha\beta}$ we find

$$\frac{d}{dt}(L_{\alpha\beta}\dot{r}_\beta) = -kM \left( \frac{\dot{r}_\alpha}{r} \dot{r}_\beta - \frac{\dot{r}_\beta}{r} \dot{r}_\alpha \right) = kM \frac{d}{dt} \left( \frac{r_\alpha}{r} \right).$$

Hence

$$L_{\alpha\beta}\dot{r}_\beta = kM(\dot{r}_\alpha + e_\alpha), \quad (2.25K)$$

where $e$ is a 3N-vector eccentricity. Contracting with $r_\alpha/r$ and using the antisymmetry of $L_{\alpha\beta}$ we find,

$$\ell/r = 1 + e \cdot \dot{r}, \quad (2.26K)$$

where

$$\ell = L^2/(kM). \quad (2.27K)$$

If $e$ is the magnitude of $e$ and $\phi$ is the angle between $e$ and $\dot{r}$, then this equation becomes the equation of a conic section in the $r, \phi$ plane, i.e.,

$$\ell/r = 1 + e \cos \phi. \quad (2.28K)$$

To demonstrate that $\phi$ is indeed the angle called by that name previously, we first square (2.25K) writing out $L_{\alpha\beta}$ in full and using (2.7) on the left and eliminating $e \cdot \dot{r}$ via (2.26K) on the right. This gives just as for the 3 dimensional case

$$L^2\dot{r}^2 = k^2M^2(e^2 - 1) + 2L^2Mr^{-1},$$

which we rewrite in the form of the energy equation (2.8):

$$\frac{1}{2}M\dot{r}^2 - \frac{kM^2}{r} = \frac{1}{2}k^2M^3(e^2 - 1)/L^2 = E. \quad (2.29K)$$

Remembering (2.10) we have

$$\dot{r}^2 = 2 \left( \frac{E}{M} + k \frac{M}{r} \right) - \frac{L^2}{r^2} = \frac{k^2M^2}{L^2} \left[ e^2 - \left( \frac{\ell}{r} - 1 \right)^2 \right]. \quad (2.30K)$$

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Differentiating (2.28K)

\[ \frac{L^2 \dot{r}}{kM} = e \sin \phi (r^2 \dot{\phi}) , \]

(2.31K)

but by (2.30K) and (2.28K)

\[ \dot{r} = \frac{kM}{L} e \sin \phi . \]

(2.32K)

Thus \( r^2 \dot{\phi} = L \) and our new \( \phi \) can only differ from our old one by a constant \( \chi \). Such a constant is irrelevant as it merely makes a transformation

\[ A \rightarrow A \cos \chi - B \sin \chi \]

\[ B \rightarrow B \cos \chi + A \sin \chi \]

on our constants of the motion. The constraints (2.20) and (2.21) are invariant to such transformations. Thus our orbit in \( 3N \) space is given by

\[ r = \frac{\ell}{1 + e \cos \phi} (A \sin \phi + B \cos \phi) . \]

(2.33K)

This is our prime result and gives the \( x_I \) via (2.4) and (2.2). Relative to the centre of mass each particle’s orbit lies in a plane and eliminating \( \phi \) in favour of coordinates \( x \) and \( y \) in that plane we find it is a conic section. When \( e < 1 \) the ellipse has in general neither its focus nor its centre at the centre of mass. This is clear from the geometrical interpretation given in the note added in proof.

The relationships between \( r, \phi \) and \( t \) are given as usual by Kepler’s equation which one may derive from (2.13) and from (2.28K) and (2.31K):

\[ dt = \frac{dr}{\dot{r}} = \frac{d}{\frac{kM}{L} e \sin \phi} = \frac{L^3}{k^2 M^2} \frac{d\phi}{(1 + e \cos \phi)^2} . \]

(2.34K)

Defining \( a = GM/(-2e) \) these give, writing \( r = a(1 - e \cos \eta) \),

\[ t - t_{\text{min}} = GM (-2e)^{-3/2} (\eta - e \sin \eta) , \]

(2.35K)

and

\[ \phi = 2 \tan^{-1} \left( \sqrt{\frac{1 - e}{1 + e}} \tan \frac{\eta}{2} \right) , \]

(2.36K)

where \( \phi \) is measured from pericentre. From this one deduces \( \cos \phi = \frac{\cos \eta - e}{1 - e \cos \eta} \) and \( \ell = a(1 - e^2) \). The generalisations of these formulae for the isochrone potential are given in the Appendix.

3. Generalisation to Breathing Systems and their new Statistical Mechanics

When the potential is of the form \( V_0(r) + r^{-2} V_2(\mathbf{r}) \) equation (2.5) becomes

\[ M \ddot{\mathbf{r}} = -V_0'(r) \mathbf{r} + 2r^{-3} V_2 \mathbf{r} - r^{-3} \partial V_2 / \partial \mathbf{r} , \]

(3.1)

\( \ddot{r} \) is constrained by the barycentre constraint. It is understood that \( V_2 \) takes values dependent only on the subspace to which \( \mathbf{r} \) is constrained.

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where the last term in the force is not hyperradial. Indeed since \( V_2 \) is independent of \( r \) it is constant along each radial line so its gradient is automatically perpendicular to \( r \). We therefore form a Virial theorem by taking the dot product of (3.1) with \( r \) and using

\[
\mathbf{r} \cdot \ddot{\mathbf{r}} = \frac{d}{dt}(\mathbf{r} \cdot \dot{\mathbf{r}}) - \dot{\mathbf{r}}^2 = \frac{1}{2} \frac{d^2}{dt^2}(r^2) - r^2 . \tag{3.2}
\]

We thus obtain writing \( T = \sum \frac{1}{2} M_i \)

\[
\frac{1}{2} M \frac{d^2}{dt^2}(r^2) = 2T - rV'_0 + 2r^{-2}V_2 . \tag{3.3}
\]

But by energy conservation

\[
T + V_0 + r^{-2}V_2 = E , \tag{3.4}
\]

so

\[
\frac{1}{2} M \frac{d^2}{dt^2}(r^2) = 2E - \frac{1}{r} \frac{d}{dr}(r^2V_0) . \tag{3.5}
\]

Notice that \( V_2 \) has disappeared from this equation which is now an equation for the scalar \( r \) only. This separation of the motion of the scaling coordinate \( r \) from the rest of the dynamics only occurs when the potential is of the special form we have chosen with the ‘angularly’ dependent part scaling as \( r^{-2} \). Multiplying (3.5) by \( dr^2/dt \) and integrating

\[
\frac{1}{4} M \left( \frac{dr^2}{dt} \right)^2 = 2Er^2 - 2r^2V_0 - ML^2 , \tag{3.6}
\]

where the final term is the constant of integration and \( L \) has the same dimensions as our former \( L \) whose part it plays. Dividing by \( 2r^2 \) we now have

\[
\frac{1}{2} M(\dot{r}^2 + L^2 r^{-2}) + V_0(r) = E . \tag{3.7}
\]

The equation of motion for \( r \) follows by differentiation

\[
M(\ddot{r} - L^2 r^{-3}) = -\partial V_0 / \partial r . \tag{3.8}
\]

Thus \( r \) vibrates as though it were in a central orbit of a particle of mass \( M \) and angular momentum \( ML \) with a central potential \( V_0(r) \). Integrating (3.7) we see that for bound orbits \( r \) vibrates for ever with a period

\[
P = \int \frac{dr}{\sqrt{2M^{-1}(E - V_0(r)) - L^2 r^{-2}}} . \tag{3.9}
\]

In the above we have assumed that \( L^2 \) is positive and that \( V_0(r) \) is not so strongly attractive that it beats the centrifugal repulsion. Then for bound states there are normally simple zeros of the expression in the surd above. When the constant \( L^2 \) is negative, one may change the nomenclature to make it positive by adding a constant \( K \) to \( V_2 \) and subtracting \( K/r^2 \) from \( V_0 \) so as to leave \( V \) unchanged. While this makes \( L^2 \) positive it may make \( V_0 \) so attractive that \( r \) can reach the origin. If we again define an angle \( \phi \) such that \( r^2 \dot{\phi} = L \) and plot an orbit for \( r \) in \( r, \phi \) space, then such orbits approach the origin in finite time following logarithmic spirals (for small \( r \)). They re-emerge with \( \phi \) still increasing

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but with \( r \) now increasing on logarithmic spirals. Although the time near the origin is finite, the increment in \( \phi \) on passing through the origin is infinite. The outside turning points of these orbits will be just as for others.

The form of the potential energy in (3.4), \( V_0(r) + r^{-2}V_2(\hat{r}) \) is the generalisation to \( 3N \) dimensions of the 3 dimensional potential \( V_0 + r^{-2}V_2(\theta, \phi) \) for which the radial motion separates from the transverse through the exact integral \( \frac{1}{2}m\mathbf{h}^2 + V_2(\theta, \phi) = \text{const} \) where \( \mathbf{h} = \mathbf{r} \times \mathbf{v} \). Further separations occur if \( V_2 = U(\theta) + W(\phi)/\sin^2 \theta \). Still further integrals can occur for special \( V_0 \) such as the simple harmonic and the Kepler potential, see Evans, 1990. In \( 3N-3 \) dimensions each separable potential gives an exactly soluble N-body problem. Marshall & Wojciechowski show that the general case occurs in hyperellipsoidal coordinates in \( 3N-3 \) dimensions and that other cases, such as the hyperspherically separable one we have used, can be found as degenerate cases of hyperellipsoidal coordinates.

We now apply the general theory of this section to the case mentioned in the abstract for which

\[
V_0 = \frac{1}{2}k \sum_{I < J} m_I m_J (\mathbf{x}_I - \mathbf{x}_J)^2 = \frac{1}{2}kM^2r^2, \tag{3.10}
\]

and

\[
r^{-2}V_2 = \frac{1}{2}k' \sum_{I < J} m_I m_J (\mathbf{x}_I - \mathbf{x}_J)^{-2}, \tag{3.11}
\]

with \( k \) and \( k' \) positive. Then we have the system of \( N \) bodies which attract each other linearly according to their separations and repel each other with an inverse cubic repulsion. Since the scale \( r \) of \( \mathbf{x}_I - \mathbf{x}_J \) cancels out in \( V_2 \) we see that indeed \( V_2 = V_2(\hat{r}) \). Thus the general theory applies and the period of \( r \) is given by (3.9) which gives

\[
P = \frac{\pi}{\sqrt{kM}}. \tag{3.12}
\]

This is only half the period in \( \phi \) because for a central ellipse \( r \) undergoes two oscillations as \( \phi \) increases by \( 2\pi \).

Thus the radius of such a system will continue to vibrate for ever and shows no Violent Relaxation (Lynden-Bell, 1967). With both attractive and repulsive forces present, it may be possible to get a giant lattice solution but the increased pressures near the centre will give interesting radial distortions to any such lattice as in a planet.

Even though the above hyper-radial motions are simple the \( \hat{r} \) motions are not, may show relaxation phenomena and we find below that they may even achieve a slightly modified form of equilibrium even though the scaling variable \( r \) continues to vibrate at large amplitude!

To see this we first return to the general case with \( V_0(r) \) and \( V_2(\hat{r}) \) any functions and look for the equations of motion of the \( \hat{r} \). These follow from (3.1) if we remember that

\[
\hat{r} = \frac{d}{dt}(r \hat{r} + \hat{r} \hat{r}) = \hat{r} \hat{r} \hat{r} + \frac{1}{r} \frac{d}{dt}(r^2 \hat{r});
\]

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substituting this into (3.1) and eliminating \( \ddot{r} \) via (3.8) yields

\[
Mr^2 \frac{d}{dt} \left( \frac{dr}{dt} \right) = \left( M\mathcal{L}^2 + 2V_2 \right) \dot{r} - \frac{\partial V_2}{\partial \dot{r}} .
\]

Putting \( r^{-2}d/dt = d/d\tau \) this may be rewritten as an autonomous equation for \( \dot{r}(\tau) \)

\[
Md^2\dot{r}/d\tau^2 = -M\mathcal{L}^2\dot{r} + 2V_2\dot{r} - \frac{\partial V_2}{\partial \dot{r}} ,
\]

(3.13)

The radial component of this equation is irrelevant; it follows from the condition \( |\dot{r}| = 1 \) and the transverse components. We now write \( Q_I = \sqrt{m_I/M} q_{1I}/r \) so that the \( Q_I \) are the components of \( \hat{r} \) taken in triples. Then the centre of mass constraint becomes \( \sum_I \sqrt{m_I} Q_I = 0 \). When we expressed \( V_2 \) in terms of our original \( x_I \) it depended only on differences so \( V_2(x_1 + \Delta, \ldots x_I + \Delta, \ldots x_N + \Delta) \) was independent of \( \Delta \). This means that when we write \( V_2 \) as a function of our new coordinates \( Q_I \) it has the property that \( V_2(Q_1 + \sqrt{m_1/M} \Delta, \ldots Q_N + \sqrt{m_N/M} \Delta) \) is independent of \( \Delta \). Differentiating with respect of \( \Delta \) and then setting \( \Delta = 0 \) we deduce that

\[
\sum_I \sqrt{m_I} \partial V_2/\partial Q_I = 0 .
\]

(3.14)

Now consider the Lagrangian \( L(Q_I', Q_I) \) where \( Q_I' = dQ_I/d\tau \) and

\[
L = \sum_I \frac{1}{2} M Q_I'^2 - V_2(Q_1 \ldots Q_N) ,
\]

(3.15)

and the \( Q_I \) are subject to the constraints

\[
\sum_I Q_I^2 = 1 \quad \text{and} \quad \sum_I \sqrt{m_I} Q_I = 0 .
\]

(3.16)

Consider also the variational principle in \( \tau \)-time \( \delta \int L(Q_I', Q_I)d\tau = 0 . \)

Lagrange’s equations are

\[
M Q_I'' = -\partial V_2/\partial Q_I + \lambda(\tau) Q_I + \mu(\tau) \sqrt{m_I} ,
\]

(3.17)

where \( \lambda \) and \( \mu \) are Lagrange multipliers corresponding to the constraints (applied for all \( \tau \)). Multiplying (3.17) by \( \sqrt{m_I} \) and summing using (3.14), the second constraint equation (3.15) gives \( \mu \equiv 0 \). Evidently (3.17) and (3.13) are equivalent since \( \dot{r} = (Q_1, Q_2 \ldots Q_N) \). So (3.13) follows from the Lagrangian \( L \). The ‘energy’ equation for \( \dot{r} \) follows from \( r^2(3.4) - \frac{1}{2}(3.6) \) when we use \( \dot{r}^2 = r^2(\dot{r}/dt)^2 + \dot{r}^2 \) and we get

\[
\frac{1}{2} M \left( \frac{d\dot{r}}{d\tau} \right)^2 + V_2(\dot{r}) = \frac{1}{2} M \mathcal{L}^2 ,
\]

(3.18)

where by definition of \( Q_I \)

\[
\left( \frac{d\dot{r}}{d\tau} \right)^2 = \sum_I Q_I'^2 .
\]

In the statistical mechanics that follows we shall be concerned with the case in which \( N \) is large so that there are many terms in this sum.

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(a) **Statistical Equilibrium of a system in large amplitude oscillation**

In one sense it is surprising that a system may achieve a statistical equilibrium while its scale continues to oscillate (or in the unbound case evolve). However this only occurs for these special systems in which the large scale oscillations separate from the rest of the dynamics and only then when the potential \( V_2 \) is sufficiently complicated to give quasi-ergodic motion in \( \mathbf{r} \). Once the \( \mathbf{r} \) motion has been separated off, we have shown that the motion of \( \mathbf{Q}_I \) follows from a Lagrangian in \( \tau \)-time and that \( V_2 \) can be any function of the normalised relative coordinates, so it can achieve the necessary complications. Since the \( \mathbf{Q}_I \) motion pays no attention to the \( \mathbf{r} \) motion there is no reason why it should not achieve such an equilibrium. Whether such an equilibrium is a thermal equilibrium or not depends on semantics. The quantity shared by the different \( \mathbf{Q}_I \) is not energy but hyper-angular-momentum, \( M\mathcal{L}^2 \), and the time is replaced by \( \tau \) but, as we shall see, the equilibrium remains at all phases of the oscillation and does not depend on the timescale of the relaxation as compared with the period of oscillation. In many respects it is a true generalisation of the usual concept of statistical equilibrium albeit with sufficient differences to make it interesting. We have \( N, \mathbf{Q}_I \) subject to the 4 constraints

\[
\sum \sqrt{m_i} \mathbf{Q}_I = 0 \tag{3.19}
\]
\[
\sum \mathbf{Q}_I^2 = 1 \tag{3.20}
\]

and having equations of motion following the Lagrangian (3.15) and having ‘energy’ given by (3.18). When there are very many particles present the fixed centre of mass constraints (3.19) are unimportant. They only give changes of order \( 1/N \) in the results and are in any case statistically satisfied by the equilibrium found without imposing them. As they significantly complicate the arguments while adding very little to the result we shall now solve the problem when only the constraint (3.20) is imposed and we shall specialise to an interaction \( V_2 \) equivalent to a hard sphere gas. That is we shall take \( V_2 \) to be negligible at any one time but nevertheless to be present to perform its role of redistributing hyper-angular-momentum. Then by (3.18) our ‘energy’ equation takes the form

\[
\sum \mathbf{Q}_I^2 = \mathcal{L}^2
\]

where on the right we have a conserved quantity.

As far as the statistical mechanics is concerned the \( \mathbf{Q}_I \) are equivalent, so we invent a 6 dimensional phase space \( \mathbf{Q}, \mathbf{Q}' \) and divide it into cells of equal volume. The number of ways \( W \) of putting \( n_a \) distinguishable particles in the \( a^{th} \) cell centred on \( \mathbf{Q}_a, \mathbf{Q}'_a \) is

\[
W = N! \left( \prod_a n_a! \right)
\]

We maximise \( W \) subject to the constraints

\[
\sum n_a = N
\]
\[
\sum n_a \mathbf{Q}_a^2 = \mathcal{L}^2 \tag{3.21}
\]
\[
\sum n_a \mathbf{Q}'_a^2 = 1
\]
and using Lagrange multipliers $\tilde{\alpha}, \frac{1}{2}\tilde{\beta}, \frac{1}{2}\tilde{\gamma}$ for the constraints we obtain
\[
\delta \ln W - \sum \delta n_a (\tilde{\alpha} + \frac{1}{2} \tilde{\beta} Q^2_a + \frac{1}{2} \tilde{\gamma} Q^2_a) = 0 .
\]
The normal use of Stirling’s theorem then leads to
\[
n_a = \exp \left[ \tilde{\alpha} - 1 + \frac{1}{2} \tilde{\beta} Q^2_a + \frac{1}{2} \tilde{\gamma} Q^2_a \right] . \tag{3.22}
\]

Returning to the constraint equations (3.21) we see that the $\gamma$ are exactly constant however slowly the interactions have led to the equilibrium. Hence when we solve them for $\tilde{\beta} L^2$ and $\tilde{\gamma}$ we inevitably find those to be equal. If we now pass to a continuous distribution function $f(Q, Q')$ in our six-dimensional phase space by replacing $n_a$ by $f(Q, Q') d\beta Q d\gamma Q$ we find
\[
f(Q, Q') = A \exp \left[ -\frac{1}{2} \tilde{\beta} (Q'^2 + L^2 Q^2) \right] . \tag{3.23}
\]
Where $Q'^2$ stands for any one of the equivalent $Q^2_i$ and $Q^2$ stands for one of the equivalent $Q^2_i = \frac{m_i q_i^2}{r^2}$. We shall now re-express the $Q'_i$ in terms of the $q_i$ and the $\dot{q}_i$
\[
Q'^2_i = \frac{m_i}{M} \left( \frac{q_i}{r} \right)^2 = \frac{r^4}{M^2} m_i \left[ \frac{d}{dt} \left( \frac{q_i}{r} \right) \right]^2 = \frac{r^2}{M} m_i \left( \frac{\dot{q}_i - \dot{r} q_i}{r} \right)^2 .
\]

Inserting these expressions into (3.23) and writing $H$ for the ‘Hubble’ expansion rate $\dot{r}/r$, which is of course time dependent in our problem, the expression becomes simplest written in terms of the peculiar velocity $v_i$ relative to the Hubble flow
\[
v_i = \dot{q}_i - H q_i . \tag{3.24}
\]
The distribution of $v_i$ and $q_i$ at given $r$ is then
\[
f(v_i, q_i|r) = A \exp \left[ -\frac{1}{2} \frac{r^2}{M} (\frac{1}{2} m_i v_i^2) - \frac{\tilde{\beta} L^2}{r^2} \left( \frac{1}{2} \frac{m_i}{M} q_i^2 \right) \right] . \tag{3.25}
\]
$r$ is continually changing in time and by no means slowly, nevertheless $\tilde{\beta}$ and $L^2$ are exactly constant however slowly the interactions have led to the equilibrium distribution. We notice that at each $r$ the peculiar velocity $v_i$ with respect to the ‘Hubble’ flow $H q_i$ is Maxwellianly distributed with a temperature proportional to $r^{-2}$ and the heavier particles move more slowly with respect to that flow. Likewise $m_i^{1/2} \dot{q}_i$ are Gaussianly distributed with a dispersion proportional to $r$.

The value of $\tilde{\beta}$ is readily deduced from the condition $\sum \frac{m_i}{M} q_i^2 = r^2$ which gives $\tilde{\beta} = (3N-1)/L^2$. By (3.23) $f$ is unchanging so there is no entropy change. Here the $-1$ follows from the fact that there are $3N-1$ independent $Q_i$ components when the constraint (3.20) is imposed. Had we imposed also the barycentre constraints we would have found $3N-4$. The sharing of $L^2$ without the sharing of energy has arisen before in Cometary Theory (see Rauch & Tremaine, 1996).

4. Conclusions

It is natural to ask whether the new form of equilibrium (3.25) can be generalised to the Hubble flow of relativistic cosmology and to look for connections.
with the interesting fact that the cosmic background radiation remains in equilibrium during cosmic expansion and creates no entropy while a classical gas creates entropy during uniform expansion due to its bulk viscosity. These are beyond the scope of this paper but we hope they show that exploration of somewhat esoteric N-body problems can have considerable interest outside classical dynamics. Readers will have appreciated the beauty of these N-body problems treated exactly in $3N$ dimensions as well as the novelty of the unusual statistical mechanics of systems with isotropic expansions as seen in the distribution (3.25) where the velocities are those relative to the time-dependent flow as given by (3.24).

**Appendix A.**

Motion in Hénon's (1959) isochrone potential $V = -GM^2/(b+s)$ where $s^2 = r^2 + b^2$. We write $\epsilon = E/M$ for the specific energy. We need to evaluate the integral (2.13) for $t$ and (2.15) for $\phi$. If we write

$$a = GM/(-2\epsilon) ,$$

and put

$$(1 - \epsilon^2) = L^2/[-2\epsilon(a - b)^2] ,$$

then the surd in (2.13) $\times r$ is

$$S = [2\epsilon(s^2 - b^2) + 2GM(s - b) - L^2]^{1/2} = (-2\epsilon)^{1/2}[(a - b)^2e^2 - (s - a)^2]^{1/2} ;$$

so we write

$$(s - a) = -(a - b)e \cos \eta ,$$

and (2.13) becomes

$$t - t_0 = \int (s/S)ds = a(-2\epsilon)^{-1/2}(\eta - (1 - b/a)e \sin \eta) .$$

Putting

$$\kappa = (-2\epsilon)^{1/2}/a = (GM/a^3)^{1/2} = (-2\epsilon)^{3/2}/GM ,$$

then $2\pi/\kappa$ is the radial period as for Kepler’s case and

$$\kappa(t - t_0) = \eta - (1 - b/a)e \sin \eta ,$$

which we recognise as the generalisation of Kepler’s equation ($b = 0$). Both in Kepler’s case and for the isochrone, direct evaluation of (2.15) in terms of $\eta$ gives untidy formulae (cf. (A17)). In the Keplerian case one uses $1/r$ as the integration variable to get $\ell/r = 1 + e \cos \phi$. For the isochrone one has from (2.15)

$$\phi = \frac{1}{2}L \int \left( \frac{1}{s - b} + \frac{1}{s + b} \right) \frac{ds}{S} ,$$

and we need to use $\frac{1}{s - b}$ and $\frac{1}{s + b}$ instead of $1/r$. We therefore use two new angles $\chi$ and $\chi_+$ and rewrite (A4) in terms of $a_p$, the pericentric value of $s - b$.

$$\sqrt{r^2 + b^2} - b = s - b = \frac{a_p}{1 - e} (1 - e \cos \eta) = \frac{a_p(1 + e)}{1 + e \cos \chi} = -2b + \frac{(a_p + 2b)(1 + e_+)}{(1 + e_+ \cos \chi_+)} ,$$

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where

\[ a_p = (a - b)(1 - e) \]  \hspace{1cm} (A10)

and, taking apocentre values and writing

\[ f = \frac{2b}{a_p} \]  \hspace{1cm} (A11)

we find

\[ \frac{1 + f}{1 - e} = \frac{1}{2} (1 + f) + \frac{1}{1 - e} \]  \hspace{1cm} (A12)

The two parts of the integral (A8) then give

\[ \phi = \frac{1}{2} \left[ \chi + \frac{L}{\sqrt{L^2 + 4GMb}} \chi_+ \right] \]  \hspace{1cm} (A13)

Since both \( \chi \) and \( \chi_+ \) (and \( \eta \)) increase by \( 2\pi \) in one radial period we find that \( \phi \) increases by

\[ \Phi = \pi \left[ 1 + \frac{L}{\sqrt{L^2 + 4GMb}} \right] < 2\pi \]  \hspace{1cm} (A14)

Only when \( b = 0 \) is \( \Phi = 2\pi \) and then both \( \chi \) and \( \chi_+ \) reduce to \( \phi \). Thus the orbits do not close (unless \( \frac{L}{\sqrt{L^2 + 4GMb}} \) is rational) and they form rosettes with inner and outer radii of

\[ r_p = \sqrt{a_p(a_p + 2b)} \quad \text{and} \quad r_a = \sqrt{a_a(a_a + 2b)} \]  \hspace{1cm} (A15)

where

\[ a_a = a_p(1 + e)/(1 - e) = (a - b)(1 + e) \]  \hspace{1cm} (A16)

If in place of inventing \( \chi \) and \( \chi_+ \) we proceed with (A4), we obtain the ugly formula (cf. (2.36K))

\[ \phi = \tan^{-1} \left( \sqrt{1 - e} \tan \eta/2 \right) + \frac{L}{\sqrt{L^2 + 4GMb}} \tan^{-1} \left( \frac{1 - e_+}{1 + e_+} \tan \eta/2 \right) \]  \hspace{1cm} (A17)

which is term for term the same as (A13).

One may show from the above formulae that

\[ \frac{L}{\sqrt{L^2 + 4GMb}} = \sqrt{\frac{1 - e^2}{1 - e_+^2}} \]  \hspace{1cm} (A18)

To get the general solution to the N-body problem one procedure is to take a value of \( \eta \), from that determine \( t \) from (A5), \( s \) from (A4) and \( r = \sqrt{s^2 - b^2} \) and finally \( \phi \) from (A17). We finally have

\[ r = r(A \sin \phi + B \cos \phi) \]

as before. (A9) gives the basic relationship between \( r, s, \eta, \chi \) and \( \chi_+ \) whereas (A13) relates \( \phi \) to \( \chi \) and \( \chi_+ \) and (A5) relates \( t \) to \( \eta \).
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Note Added in Proof

Geometrical Insight gives greater clarity. From (2.5), the problem is a central one both in $3N$ dimensions and for each particle. Together the initial $\mathbf{r}$ and $\dot{\mathbf{r}}$ define a plane in $3N$ dimensions through the centre. The central force lies in that plane so all the motion continues in it. The motion is just planar motion under the potential $V(\mathbf{r})$ so we get the usual orbit. The orbit of each particle is the projection of this $3N$ planar orbit onto the orbital plane of that particle (that has most of the $3N$ coordinates fixed at zero). Thus in all cases the particle orbits are projections of the $3N$ planar orbit which is a familiar one. If this is an ellipse with the barycentre as its focus, then the particle orbit will be the projected ellipse but the projection does not preserve the barycentre as its focus. Similarly if the $3N$ orbit is a rosette, the individual particle orbits will be projected rosettes, i.e., rosettes between similar ellipses. That will happen for most potentials, e.g., the isochrone.