Local Antimagic Vertex Coloring of Gear Graph

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ABSTRACT
Let $G = (V, E)$ be a graph that consist of a vertex set $V$ and an edge set $E$. The local antimagic labeling $f$ of a graph $G$ with edge-set $E$ is a bijection map from $E$ to $\{1, 2, ..., |E|\}$ such that $w(u) \neq w(v)$, where $w(u) = \sum_{e \in E(u)} f(e)$ and $E(u)$ is the set of edges incident to $u$. In this labeling, every vertex $v$ is assigned $w(v)$ as its color. The minimum number of colors in a local antimagic labelling, is called a local antimagic chromatic number and denoted by $\chi_{la}(G)$. This paper contribution is to determine the local antimagic chromatic number $\chi_{la}(G_n)$ of a gear graph. A gear graph is a graph obtained by inserting additional vertex between each pair of adjacent vertices on the circumference of the wheel graph $W_n$. The gear graph $G_n$ has $2n + 1$ vertices and $3n$ edges.

Keywords: Antimagic labeling, Local antimagic labeling, Local antimagic chromatic number, Gear graph.

1. INTRODUCTION
Graphs are used in various fields, such as network topology modeling, database design, scheduling, traveling salesman problems, and so on [1]. Graph labeling is one of the interesting research topics in graph theory that assign an element of graph such as vertices and edges with the set of integers called labels [2].

Formally, the labeling of antimagic in a graph is defined as follows. Suppose $G = (V, E)$ be a simple undirected graph with a non-empty set of vertices $V$ and a set of edges $E$. The number of vertices, denoted by $|V(G)|$, referred to the order of $G$ and the number of edges, denoted by $|E(G)|$, called size of $G$ [3]. If vertex $v$ is an endpoint of edges $e$, then $v$ is said to be incident on $e$ and $e$ is incident on $v$. Two vertices are adjacent if they are the endvertices of the same edge [4]. The local antimagic labeling $f$ of a graph $G$ with edge-set $E$ is a bijection map from $E$ to $\{1, 2, ..., |E|\}$ such that $w(u) \neq w(v)$, where $w(u) = \sum_{e \in E(u)} f(e)$ and $E(u)$ is the set of edges incident to $u$, for $u, v \in V(G)$. A graph $G$ is called local antimagic if $G$ has a local antimagic labeling [5]. The local antimagic chromatic number of $G$, denoted by $\chi_{la}(G)$, is the minimum number of colors that is needed that is induced by local antimagic labelings of $G$ [6].

Arunumagam, et al., [7] presented local antimagic chromatic number of several classes of graphs such as complete graph $K_n$ for $n \geq 3, \chi_{la}(K_n) = n$; star graph $K_{1,n}$ for $n \geq 3, \chi_{la}(K_{1,n-1}) = n$; path graph $P_n$ for $n \geq 3, \chi_{la}(P_n) = 3$; cycle graph $C_n$ for $n \geq 3, \chi_{la}(C_n) = 3$; friendship graph $F_n$ for $n \geq 2, \chi_{la}(F_n) = 3$; complete bipartite graph $K_{m,n}$ for $m, n \geq 2, \chi_{la}(K_{m,n}) = 2$ if only if $m \equiv n (mod 2)$ and wheel graph $W_n$ with $n + 1$ order for $n \geq 3, \chi_{la}(W_n) = 4$, if $n \equiv 1, 3 (mod 4), \chi_{la}(W_n) = 3$, if $n \equiv 2 (mod 4)$, and $3 \leq \chi_{la}(W_n) \leq n$ if $n \equiv 0 (mod 4)$. Furthermore, Pratama, et al., [8] presented the local super antimagic total chromatic number of several classes of graph which is related with wheel graph, such as fan graph $F_n$ for $n \geq 3$, $3 \leq \chi_{la}(F_n) \leq 4$; even gear graph $G_n$ for $n \geq 4$, $2 \leq \chi_{la}(G_n) \leq 3$; odd sun flower graph $SF_n$ for $n \geq 5$, $4 \leq \chi_{la}(SF_n) \leq 5$.

We study and determine the local antimagic chromatic numbers of a gear graph in this paper.

2. MAIN RESULT
We begin this section by presenting the gear graph as follow. Gear graph, denoted by $G_n$ (Figure 1) is a graph obtained from wheel graph $W_n$ by adding a vertex between each pair of adjacent rim vertices [9]. Clearly $G_n$ has a set of vertices $V = \{c, v_i,u_i; 1 \leq i \leq n\}$ and
set of edges \( E = \{cv_i, viu_i; 1 \leq i \leq n \} \cup \{u_iv_{i+1}; 1 \leq i \leq n - 1 \} \cup \{u_nv_1\} \) such that \( |V| = 2n + 1 \) and \( |E| = 3n \).

Figure 1 Gear Graph \( G_n \).

2.1. Local Antimagic Chromatic Number of \( G_n \) for \( n \) odd

Lemma 2.1.1. gives the upper bound of \( \chi_{la}(G_n) \) for \( n \equiv 1 \pmod{2} \).

Lemma 2.1.1. \( \chi_{la}(G_n) \leq 4 \) for \( n \equiv 1 \pmod{2} \) and \( n \geq 3 \).

Proof:

Case 1: for \( n = 3 \).

Figure 2 \( \chi_{la}(G_3) = 4 \).

Case 2: for \( n \equiv 1 \pmod{2} \) and \( n \geq 5 \).

Label the edges \( v_iu_i \), for \( i = 1, 2, \ldots, n \), as follows:

\[
f(v_iu_i) = \begin{cases} 
3n + i, & \text{for } i \text{ odd}, \\
6n - i, & \text{for } i \text{ even}.
\end{cases}
\]

Label the edges \( u_iu_{i+1} \), for \( i = 1, 2, \ldots, n - 1 \), as follows:

\[
f(u_iu_{i+1}) = \begin{cases} 
3n - i, & \text{for } i \text{ odd}, \\
4n + i, & \text{for } i \text{ even}.
\end{cases}
\]

Label the edges \( u_nv_1 \), as follows:

\[
f(u_nv_1) = 3n.
\]

In Figure 3, we have the labeling of the edges \( G_n \) for \( n \) odd.

Figure 3 Labeling Edges \( G_n \) for \( n \) odd.

This labelling provides different weights for any two adjacent vertices, namely:

\[
w(c) = \frac{1}{2}n(n + 1),
\]

\[
w(v_i) = \frac{1}{2}(9n + 3),
\]

\[
w(u_i) = \begin{cases} 
3n, & \text{for } 1 \leq i < n, i \text{ odd}., \\
5n, & \text{for } 1 < i < n, i \text{ even}, \\
i = n.
\end{cases}
\]

Thus, the labeling gives 4 different weights, that is, \( \chi_{la}(G_n) \leq 4 \), for \( n \equiv 1 \pmod{2} \) and \( n \geq 3 \). ■
In Figure 4, we have example of labeling \( G_7 \). We now prove the lower bound of the local antimagic chromatic number of \( G_n \) for \( n \) is odd.

**Figure 4** Labeling of Graph \( G_7 \).

**Lemma 2.1.2.** \( \chi_{la}(G_n) \geq 3 \) for \( n \equiv 1 \pmod{2} \) and \( n \geq 3 \).

**Proof:** Suppose that \( \chi_{la}(G_n) = 2 \) with \( w(v_i) \neq w(u_i) \). Then there is a labeling \( cv_i \) for \( 1 \leq i \leq n \) such that \( w(c) = w(u_i) \). The largest \( w(v_i) \) cannot be more than the sum of the edge labels labeled \( n \) and two largest edge labels, that is, \( w(v_i) = n + \frac{5n+1}{2} + (n+1) = \frac{9n+3}{2} \). While the weight of the central vertex must be at least the sum of the smallest \( n \) edge labels, that is, \( w(c) = 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \). It is clear that no \( n \in \mathbb{N} \) satisfies the equation \( 9n + 3 = n(n+1) \), this is a contradiction. So, it must be \( \chi_{la}(G_n) \geq 3 \) for \( n \equiv 1 \pmod{2} \). 

**Theorem 2.1.** The local antimagic chromatic number of \( G_n \) for \( n \) odd and \( n \geq 3 \) satisfies \( 3 \leq \chi_{la}(G_n) \leq 4 \).

**Proof:** Based on Lemma 2.1.1. and Lemma 2.1.2., it is proven that \( 3 \leq \chi_{la}(G_n) \leq 4 \).

### 2.2. Local Antimagic Chromatic Number of \( G_n \) for \( n \) even

Lemma 2.2.1. gives the upper bound of \( \chi_{la}(G_n) \) for \( n \equiv 0 \pmod{2} \).

**Lemma 2.2.1.** \( \chi_{la}(G_n) \leq 4 \) for \( n \equiv 0 \pmod{2} \) and \( n \geq 4 \).

**Proof:** For \( n \equiv 0 \pmod{2} \) and \( n \geq 4 \),

Label the edges \( cv_i \), for \( i = 1,2,\ldots,n \), as follows:

\[
\begin{align*}
  f(cv_i) &= \begin{cases} 
  i, & 1 \leq i \leq \frac{n}{2}, \text{odd}, \\
  n - i + 1, & 1 \leq i \leq \frac{n}{2}, \text{even}, \\
  n, & i = \frac{n}{2} + 1, \\
  i - 1, & \frac{n}{2} + 1 < i \leq n, \text{odd}, \\
  n - i + 2, & \frac{n}{2} + 1 < i \leq n, \text{even}.
  \end{cases}
\end{align*}
\]

Label the edges \( v_iu_i \), for \( i = 1,2,\ldots,n \), as follows:

\[
\begin{align*}
  f(v_iu_i) &= \begin{cases} 
  \frac{4n - i + 1}{2}, & 1 \leq i \leq \frac{n}{2}, \text{odd}, \\
  \frac{3n + i}{2}, & 1 \leq i \leq \frac{n}{2}, \text{even}, \\
  \frac{3n - i + 1}{2}, & \frac{n}{2} < i \leq n, \text{odd}, \\
  \frac{2n + i}{2}, & \frac{n}{2} < i \leq n, \text{even}.
  \end{cases}
\end{align*}
\]

Label the edges \( u_iv_{i+1} \), for \( i = 1,2,\ldots,n-1 \), as follows:

\[
\begin{align*}
  f(u_iv_{i+1}) &= \begin{cases} 
  \frac{4n + i + 1}{2}, & 1 \leq i \leq \frac{n}{2}, \text{odd}, \\
  \frac{5n - i + 2}{2}, & 1 \leq i \leq \frac{n}{2}, \text{even}, \\
  \frac{5n + i + 1}{2}, & \frac{n}{2} < i \leq n, \text{odd}, \\
  \frac{6n - i + 2}{2}, & \frac{n}{2} < i \leq n, \text{even}.
  \end{cases}
\end{align*}
\]

Label the edges \( u_nv_1 \), as follows:

\[
\begin{align*}
  f(u_nv_1) &= \frac{5n + 2}{2}.
\end{align*}
\]

In Figure 5, we have labeling the edges \( G_n \) for \( n \) even.

This labelling formula gives us four different values of vertex weight as follows:

\[
w(c) = \frac{1}{2}n(n + 1),
\]
Lemma 2.2.2. \( \chi_{la}(G_n) \geq 4 \) for \( n \equiv 0 \pmod{2} \) and \( n \geq 4 \).

Proof: From Lemma 2.1.2, it has been proved that \( \chi_{la}(G_n) \geq 3 \) for \( n \) is odd. Suppose for \( n \) is even \( \chi_{la}(G_n) = 3 \). Similar to the proof of Lemma 2.1.2, the weights are \( w(c) \neq w(v_j) \). Since vertices \( u_i \) and \( v_j \) are neighbors, they must be \( w(u_i) \neq w(v_j) \). Suppose the weight of each vertex \( v_j \) is the same. Because every vertex \( v_j \) is adjacent to edge \( cv_j, v_ju_i \), and \( u_i-v_j \) (or \( u_nv_1 \) if \( i = 1 \)), a labeling of all edges \( G_n \) will be constructed so that each vertex \( v_j \) has the same weight. It is known that the sum of all edge labels \( G_n \) is \( \sum_{i=1}^{3n} i = \frac{3n(n+1)}{2} \), in this case there is no integer for even \( n \) that can evenly divide the sum of all edge labels \( G_n \). This results in at least two vertices \( v_j \) has different weights. This is a contradiction, so it must be \( \chi_{la}(G_n) \geq 4 \) for \( n \equiv 0 \pmod{2} \).

Theorem 2.2. The local antimagic chromatic number of \( G_n \) for \( n \) even and \( n \geq 4 \) is \( \chi_{la}(G_n) = 4 \).

Proof: Based on Lemma 2.2.1, and Lemma 2.2.2, it is proven that \( \chi_{la}(G_n) = 4 \).

3. CONCLUSION

We proved the local antimagic chromatic number of gear graph \( \chi_{la}(G_n) \). We propose an open problem: What is the local antimagic chromatic number for other graph that is also related to wheel graph?

AUTHORS’ CONTRIBUTIONS

Conceptualization, M.N.S., K.A.S.; proof methodology, M.N.S., K.A.S.; writing-original draft preparation, M.N.S, K.A.S.; writing-review and editing, M.N.S., K.A.S.; funding acquisition, K.A.S. All authors have read and agreed to the published version of the manuscript.

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