A set of orthogonal polynomials, dual to alternative $q$-Charlier polynomials

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Abstract

The aim of this paper is to derive (by using two operators, representable by a Jacobi matrix) a family of $q$-orthogonal polynomials, which turn to be dual to alternative $q$-Charlier polynomials. A discrete orthogonality relation and a three-term recurrence relation for these dual polynomials are explicitly obtained. The completeness property of dual alternative $q$-Charlier polynomials is also established.

Mathematical Subject Classification (2000): 33D45, 47B36, 81Q10

1. Introduction

It is well known that orthogonal polynomials are closely connected with spectral properties of symmetric operators, which can be represented in some basis by a Jacobi matrix, and with the classical moment problem. Namely, a spectrum of an operator, represented by a Jacobi matrix, is determined by an orthogonality measure for corresponding orthogonal polynomials. If orthogonal polynomials admit many orthogonality relations, then the corresponding symmetric operator is not self-adjoint and it leaves room for infinitely many self-adjoint extensions. These extensions are determined by orthogonality measures for the appropriate orthogonal polynomials.

Contrary to orthogonal polynomials of the hypergeometric type (Wilson, Jacobi, Laguerre and so on), basic hypergeometric polynomials (or $q$-orthogonal polynomials) are not so deeply understood yet. These polynomials are collected in the $q$-analogue of the Askey–scheme of orthogonal polynomials (see, for example, [1]), which starts with Askey–Wilson polynomials and $q$-Racah polynomials, introduced in [2] and [3]. Importance of these polynomials is magnified by the fact that they are closely related to the theory of quantum groups. As an instance of such connection we refer to a paper [4], in which Al-Salam–Chihara $q$-orthogonal polynomials have been employed to construct locally compact quantum group $SU_q(1,1)$. Another application of $q$-orthogonal polynomials is related to the theory of $q$-difference equations, which often surface in contemporary theoretical and mathematical physics.

The purpose of present paper is to study completeness and duality properties of alternative $q$-Charlier polynomials. We shall show below that this originates a novel
type of q-orthogonal polynomials and a discrete orthogonality relation for them. To achieve this, we essentially use two operators $I_1$ and $q^{b_0}$, which are certain representation operators for the quantum algebra $U_q(su_{1,1})$ with a lowest weight (however, we do not use explicitly the theory of representations in what follows). The operator $I_1$ is related to the three-term recurrence relation for alternative $q$-Charlier polynomials. We diagonalize the trace class operator $I_1$ and obtain two bases in the Hilbert space: an initial basis and a basis of normalized eigenvectors of $I_1$. These bases are connected by an orthogonal matrix. The orthogonality relations for rows and columns of this matrix lead to orthogonality relations for alternative $q$-Charlier polynomials and for the functions, which are dual to these polynomials. We extract from the latter functions a dual set of polynomials and obtain a discrete orthogonality relation for them. As a result, one is led to the completeness property of dual alternative $q$-Charlier polynomials.

Observe that the present paper is a continuation of our research, initiated in [5] and [6].

Throughout the sequel we always assume that $q$ is a fixed positive number such that $q < 1$. We use (without additional explanation) notations of the theory of special functions and the standard $q$-analysis (see, for example, [7] or [8]).

2. Pair of operators $(I_1, J)$

Let $\mathcal{H}$ be a separable complex Hilbert space with an orthonormal basis $|n\rangle$, $n = 0, 1, 2, \ldots$. We define on $\mathcal{H}$ two operators. The first one, denoted as $q^{b_0}$, acts on the basis elements as $q^{b_0}|n\rangle = q^n|n\rangle$, and the second one, denoted as $I_1$, is given by the formula

$$I_1|n\rangle = a_n|n+1\rangle + a_{n-1}|n-1\rangle + b_n|n\rangle,$$

where $a_n$ and $b_n$ are the same as in (1). Collecting in this identity

$$a_n = -(aq^{2n+1})^{1/2} \sqrt{(1 - q^{n+1})(1 + aq^n)} \left(\frac{1}{1 + aq^{2n+1}}\right),$$

$$b_n = q^n \left(\frac{1}{1 + aq^{2n+1}} + aq^{n-1} \frac{1 - q^n}{1 + aq^{2n-1} + aq^{2n}}\right),$$

where $a$ is any fixed positive number. Clearly, $I_1$ is a symmetric operator.

Since $a_n \to 0$ and $b_n \to 0$ when $n \to \infty$, the operator $I_1$ is bounded. Therefore, we assume that it is defined on the whole Hilbert space $\mathcal{H}$. For this reason, $I_1$ is a self-adjoint operator. Let us show that $I_1$ is a trace class operator. For the coefficients $a_n$ and $b_n$ from (1), we have $a_{n+1}/a_n \to q^{3/2}$ and $b_{n+1}/b_n \to q$ when $n \to \infty$. Since $0 < q < 1$, for the sum of all matrix elements of the operator $I_1$ in the basis $|n\rangle$, $n = 0, 1, 2, \ldots$, we have $\sum_n (2a_n + b_n) < \infty$. This means that $I_1$ is a trace class operator. Thus, a spectrum of $I_1$ is discrete and has a single accumulation point at 0. Moreover, a spectrum of $I_1$ is simple, since $I_1$ is representable by a Jacobi matrix with $a_n \neq 0$ (see [9], Chapter VII).

To find eigenfunctions $\xi_\lambda$ of the operator $I_1$, $I_1\xi_\lambda = \lambda \xi_\lambda$, we set $\xi_\lambda = \sum_n \beta_n(\lambda)|n\rangle$, where $\beta_n(\lambda)$ are appropriate numerical coefficients. Acting by the operator $I_1$ upon both sides of this relation, one derives that $\sum_{n=0}^{\infty} \beta_n(\lambda) (a_n|n+1\rangle + a_{n-1}|n-1\rangle + b_n|n\rangle) = \lambda \sum_{n=0}^{\infty} \beta_n(\lambda)|n\rangle$, where $a_n$ and $b_n$ are the same as in (1). Collecting in this identity
all factors, which multiply $|n\rangle$ with fixed $n$, one derives the recurrence relation for the coefficients $\beta_n(\lambda)$:

$$\beta_{n+1}(\lambda)a_n + \beta_{n-1}(\lambda)a_{n-1} + \beta_n(\lambda)b_n = \lambda\beta_n(\lambda).$$

The substitution

$$\beta_n(\lambda) = \left(\frac{(-a;q)_n (1 +aq^{2n})}{(q;q)_n (1 + a)(a/q)^n}\right)^{1/2} q^{-n(n+3)/4} \beta'_n(\lambda)$$

reduces this relation to the following one

$$-A_n\beta'_{n+1}(\lambda) - C_n\beta'_{n-1}(\lambda) + (A_n + C_n)\beta'_n(\lambda) = \lambda\beta'_n(\lambda),$$

$$A_n = q^n \frac{1 +aq^n}{(1 +aq^{2n})(1 + aq^{2n+1})}, \quad C_n = qa^{2n-1} \frac{1 -q^n}{(1 +aq^{2n-1})(1 +aq^{2n})}.$$  

This is the recurrence relation for the alternative $q$-Charlier polynomials

$$K_n(\lambda; a; q) := \phi_1(q^{-n}, -aq^n; 0; q, q\lambda)$$

(see, formulas (3.22.1) and (3.22.2) in [1]). Therefore, $\beta'_n(\lambda) = K_n(\lambda; a; q)$ and

$$\beta_n(\lambda) = \left(\frac{(-a;q)_n (1 +aq^{2n})}{(q;q)_n (1 + a)a^n}\right)^{1/2} q^{-n(n+1)/4} K_n(\lambda; a; q).$$  

(2)

For the eigenvectors $\xi_\lambda$ we thus have the expression

$$\xi_\lambda = \sum_{n=0}^{\infty} \left(\frac{(-a;q)_n (1 +aq^{2n})}{(q;q)_n (1 + a)a^n}\right)^{1/2} q^{-n(n+1)/4} K_n(\lambda; a; q)|n\rangle.$$  

(3)

Since the spectrum of the operator $I_1$ is discrete, only for a discrete set of values of $\lambda$ these vectors belong to the Hilbert space $H$. This discrete set of eigenvectors determines a spectrum of $I_1$.

Now we look for a spectrum of the operator $I_1$ and for a set of polynomials, dual to alternative $q$-Charlier polynomials. To this end we use the action of the operator

$$J := q^{-J_0} - a q^{J_0}$$

upon the eigenvectors $\xi_\lambda$, which belong to the Hilbert space $H$. In order to find how this operator acts upon these vectors, one can use the $q$-difference equation

$$(q^{-n} - aq^n)K_n(\lambda) = -aK_n(q\lambda) + \lambda^{-1}K_n(\lambda) - \lambda^{-1}(1 - \lambda)K_n(q^{-1}\lambda)$$  

(4)

for the alternative $q$-Charlier polynomials $K_n(\lambda) \equiv K_n(\lambda; a; q)$ (see formula (3.22.5) in [1]). Multiply both sides of (4) by $d_n |n\rangle$ and sum up over $n$, where $d_n$ are the coefficients of $K_n(\lambda; a; q)$ in the expression (2) for $\beta_n(\lambda)$. Taking into account the formula (3) and the fact that $J|n\rangle = (q^{-n} - aq^n)|n\rangle$, one obtains the relation

$$J \xi_\lambda = -a \xi_{q\lambda} + \lambda^{-1} \xi_\lambda - \lambda^{-1}(1 - \lambda) \xi_{q^{-1}\lambda}.\quad(5)$$
We shall see in the next section that the spectrum of the operator $I_1$ consists of the points $q^n$, $n = 0, 1, 2, \ldots$. This means that $J$ has the form of a Jacobi matrix in the basis of eigenvectors of $I_1$; that is, the pair of the operators $I_1$ and $J$ form a Leonard pair (see [9] for the corresponding definition).

3. Spectrum of $I_1$ and orthogonality of alternative $q$-Charlier polynomials

The aim of this section is to find, by using the operators $I_1$ and $J$, a basis in the Hilbert space $\mathcal{H}$, which consists of eigenvectors of the operator $I_1$ in a normalized form, and to derive explicitly the unitary matrix $U$, connecting this basis with the basis $|n\rangle$, $n = 0, 1, 2, \ldots$, in $\mathcal{H}$. This matrix leads directly to the orthogonality relation for alternative $q$-Charlier polynomials. For this purpose we first find a spectrum of $I_1$.

Let us analyze a form of the spectrum of $I_1$ from the point of view of the spectral theory of trace class operators. If $\lambda$ is a spectral point of the operator $I_1$, then (as it is easy to see from (5)) a successive action by the operator $J$ upon the vector (eigenvector of $I_1$) $\xi_\lambda$ leads to the eigenvectors $\xi_{q^m \lambda}$, $m = 0, \pm 1, \pm 2, \ldots$. However, since $I_1$ is a trace class operator, not all of these points may belong to the spectrum of $I_1$, since $q^{-m} \lambda \to \infty$ when $m \to +\infty$ if $\lambda \neq 0$. This means that the coefficient $1 - \lambda'$ of $\xi_{q^{-1}\lambda'}$ in (5) must vanish for some eigenvalue $\lambda'$. Clearly, it vanishes when $\lambda' = 1$. Moreover, this is the only possibility for the coefficient of $\xi_{q^{-1}\lambda'}$ in (5) to vanish, that is, the point $\lambda = 1$ is a spectral point for the operator $I_1$. Let us show that the corresponding eigenfunction $\xi_1 \equiv \xi_{q^0}$ belongs to the Hilbert space $\mathcal{H}$.

By formula (II.6) of Appendix II in [7], one has $K_n(1; a; q) = 2\phi_1(q^{-n}, -aq^n; 0; q, q) = (-a)^n q^{n^2}$. Therefore,

$$
\langle \xi_1, \xi_1 \rangle = \sum_{n=0}^{\infty} \frac{(-a; q)_n(1 + a q^{2n})}{(1 + a)(q; q)_n a^n q^{n(n+1)/2}} K_n^2(1; a; q) = \sum_{n=0}^{\infty} \frac{(-a; q)_n(1 + a q^{2n})a^n}{(1 + q)(q; q)_n q^{n(3n-1)/2}}.
$$

In order to calculate this sum, we take the limit $d, e \to \infty$ in the equality

$$
\sum_{n=0}^{\infty} \frac{(1 + a q^{2n})(-a; q)_n(d; q)_n(e; q)_n}{(1 + a)(-aq/d; q)_n(-aq/e; q)_n(q; q)_n q^{n(n-1)/2}} = \frac{(aq; q)_\infty(-aq/de; q)_\infty}{(-aq/d; q)_\infty(-aq/e; q)_\infty}
$$

(see formula in Exercise 2.12, Chapter 2 of [7]). Since

$$
\lim_{d,e \to \infty} (d; q)_n(e; q)_n(aq/de)^n = q^{n(n-1)}(aq)^n,
$$

we obtain from here that the sum in (6) is equal to $(-aq; q)_\infty$, that is, $\langle \xi_1, \xi_1 \rangle < \infty$ and $\xi_1$ belongs to the Hilbert space $\mathcal{H}$. Thus, the point $\lambda = 1$ does belong to the spectrum of the operator $I_1$.

Let us find other spectral points of the operator $I_1$ (recall that a spectrum of $I_1$ is discrete). Setting $\lambda = 1$ in (5), we see that the operator $J$ transforms $\xi_q$ into a linear combination of the vectors $\xi_q$ and $\xi_{q^0}$. Moreover, $\xi_q$ belongs to the Hilbert space $\mathcal{H}$, since the series

$$
\langle \xi_q, \xi_q \rangle = \sum_{n=0}^{\infty} \frac{(-a; q)_n(1 + a q^{2n})}{(1 + a)(q; q)_n a^n} K_n^2(q; a; q) q^{-n(n+1)/2}.
$$

4
is majorized by the corresponding series (6) for $\xi_q^n$. Therefore, $\xi_q$ belongs to the Hilbert space $\mathcal{H}$ and the point $q$ is an eigenvalue of the operator $I_1$. Similarly, setting $\lambda = q$ in (5), we find that $\xi_q^2$ is an eigenvector of $I_1$ and the point $q^2$ belongs to the spectrum of $I_1$. Repeating this procedure, we find that all $\xi_q^n$, $n = 0, 1, 2, \cdots$, are eigenvectors of $I_1$ and the set $q^n$, $n = 0, 1, 2, \cdots$, belongs to the spectrum of $I_1$. So far, we do not know yet whether other spectral points exist or not.

The vectors $\xi_q^n$, $n = 0, 1, 2, \cdots$, are linearly independent elements of the Hilbert space $\mathcal{H}$ (since they correspond to different eigenvalues of the self-adjoint operator $I_1$). Suppose that values $q^n$, $n = 0, 1, 2, \cdots$, constitute a whole spectrum of $I_1$. Then the set of vectors $\xi_q^n$, $n = 0, 1, 2, \cdots$, is a basis in the Hilbert space $\mathcal{H}$. Introducing the notation $\Xi_k := \xi_q^k$, $k = 0, 1, 2, \cdots$, we find from (5) that

$$J\Xi_k = -a\Xi_{k+1} + q^{-k}\Xi_k - q^{-k}(1 - q^k)\Xi_{k-1}.$$  

As we see, the matrix of the operator $J$ in the basis $\Xi_k$, $k = 0, 1, 2, \cdots$, is not symmetric, although in the initial basis $|n\rangle$, $n = 0, 1, 2, \cdots$, it was symmetric. The reason is that the matrix $(a_{mn})$ with entries $a_{mn} := \beta_m(q^n)$, $m, n = 0, 1, 2, \cdots$, where $\beta_m(q^n)$ are the coefficients (2) in the expansion $\xi_q^n = \sum_m \beta_m(q^n)|n\rangle$, is not unitary. This fact is equivalent to the statement that the basis $\Xi_n = \xi_q^n$, $n = 0, 1, 2, \cdots$, is not normalized. To normalize it, one has to multiply $\Xi_n$ by corresponding numbers $c_n$ (which are not known at this moment). Let $\hat{\Xi}_n = c_n \Xi_n$, $n = 0, 1, 2, \cdots$, be a normalized basis. Then the matrix of the operator $J$ is symmetric in this basis. Since $J$ has in the basis $\{\hat{\Xi}_n\}$ the form

$$J\hat{\Xi}_n = -c_{n+1}^{-1}c_n a \hat{\Xi}_{n+1} + q^{-n}\hat{\Xi}_n - c_{n-1}^{-1}c_n q^{-n}(1 - q^n)\hat{\Xi}_{n-1},$$

then its symmetricity means that $c_{n+1}^{-1}c_n a = c_{n-1}^{-1}c_{n+1} q^{-n-1}(1 - q^{n+1})$, that is, $c_n/c_{n-1} = \sqrt{aq^n/(1 - q^n)}$. Therefore,

$$c_n = c(a^n q^{(n+1)/2}/(q; q)_n)^{1/2},$$

where $c$ is a constant.

The expansions

$$\hat{\xi}_q^n(x) \equiv \hat{\Xi}_n(x) = \sum_m c_n \beta_m(q^n)|m\rangle \equiv \sum_m \hat{a}_{mn}|m\rangle$$  \hspace{1cm} \text{(7)}$$

connect two orthonormal bases in the Hilbert space $\mathcal{H}_t$. This means that the matrix $(\hat{a}_{mn})$, $m, n = 0, 1, 2, \cdots$, with entries

$$\hat{a}_{mn} = c_n \beta_m(q^n) = c \left( a^n q^{(n+1)/2}/(q; q)_n \right) \left( -a; q \right)_m (1 + a q^{2m})^{1/2} K_m(q^n; a; q)$$  \hspace{1cm} \text{(8)}$$

is unitary, provided that the constant $c$ is appropriately chosen. In order to calculate this constant, we use the relation $\sum_{m=0}^{\infty} |\hat{a}_{mn}|^2 = 1$ for $n = 0$. Then this sum is a multiple of the sum in (6) and, consequently, $c = (-aq;q)^{-1/2}$.

The matrix $(\hat{a}_{mn})$ is real and orthogonal, that is,

$$\sum_n \hat{a}_{mn} \hat{a}_{m'n'} = \delta_{mm'}, \quad \sum_m \hat{a}_{mn} \hat{a}_{m'n} = \delta_{nn'}.$$  \hspace{1cm} \text{(9)}$$
Substituting into the first sum over \( n \) in (9) the expressions for \( \hat{a}_{mn} \), we obtain the identity

\[
\sum_{n=0}^{\infty} \frac{a^n q^{n(n+1)/2}}{(q; q)_n} K_m(q^n; a; q) K_{m'}(q^n; a; q) = \frac{(-aq^m; q)_\infty a^n (q; q)_m}{(1 + aq^{2m})} q^{m(m+1)/2}\delta_{mm'},
\]

which must yield the orthogonality relation for alternative \( q \)-Charlier polynomials. An only gap, which remains to be clarified, is the following. We have assumed that the points \( q^n, n = 0, 1, 2, \cdots \), exhaust the whole spectrum of \( I_1 \). Let us show that this is the case.

Recall that the self-adjoint operator \( I_1 \) is represented by a Jacobi matrix in the basis \( |n\rangle, n = 0, 1, 2, \cdots \). According to the theory of operators of such type (see, for example, [9], Chapter VII), eigenvectors \( \xi_\lambda \) of \( I_1 \) are expanded into series in the basis \( |n\rangle, n = 0, 1, 2, \cdots \), with coefficients, which are polynomials in \( \lambda \). These polynomials are orthogonal with respect to some positive measure \( d\mu(\lambda) \) (moreover, for self-adjoint operators this measure is unique). The set (a subset of \( \mathbb{R} \)), on which these polynomials are orthogonal, coincides with the spectrum of the operator under consideration and the spectrum is simple.

We have found that the spectrum of \( I_1 \) contains the points \( q^n, n = 0, 1, 2, \cdots \). If the operator \( I_1 \) would have other spectral points \( x \), then on the left-hand side of (10) there would be other summands \( \mu_{x_k} K_m(x_k; a; q) K_{m'}(x_k; a; q) \), corresponding to these additional points. Let us show that these additional summands do not appear. To this end we set \( m = m' = 0 \) in the relation (10) with the additional summands. Since \( K_0(x; a; q) = 1 \), we have the equality

\[
\sum_{n=0}^{\infty} \frac{a^n q^{n(n+1)/2}}{(q; q)_n} + \sum_k \mu_{x_k} = (-aq; q)_\infty.
\]

According to the formula for the \( q \)-exponential function \( E_q(a) \) (see formula (II.2) of Appendix II in [7]), we have \( \sum_{n=0}^{\infty} a^n q^{n(n+1)/2}(q; q)_n^{-1} = (-aq; q)_\infty \). Hence, \( \sum_k \mu_{x_k} = 0 \) and all \( \mu_{x_k} \) disappear. This means that additional summands do not appear in (10) and it does represent the orthogonality relation for alternative \( q \)-Charlier polynomials.

As we have shown, the orthogonality relation for the alternative \( q \)-Charlier polynomials is given by formula (10). Due to this orthogonality, we arrive at the following statement:

**Proposition.** The spectrum of the operator \( I_1 \) coincides with the set of points \( q^n, n = 0, 1, 2, \cdots \). The spectrum is simple and has one accumulation point at 0.

### 4. Dual alternative \( q \)-Charlier polynomials

Now we consider the second identity in (9), which gives the orthogonality relation for the matrix elements \( \hat{a}_{mn} \), considered as functions of \( m \). Up to multiplicative factors these functions coincide with the functions

\[
F_n(x; a|q) = 2\phi_1(x, -a/x; 0; q, q^{n+1}),
\]

(11)
considered on the set $x \in \{q^{-m} \mid m = 0, 1, 2, \cdots \}$. Consequently,

$$
\hat{a}_{mn} = \left( \frac{a^n q^{n(n+1)/2}}{(-aq^m; q)_n} \frac{(1 + a^2 q^{2m})}{(q; q)_n} a^m q^{m(m+1)/2} \right)^{1/2} F_n(q^{-m}; a|q)
$$

and the second identity in (9) gives the orthogonality relation for $F_n(q^{-m}; a|q)$:

$$
\sum_{m=0}^{\infty} \frac{(1 + a^2 q^{2m})}{a^m (-aq^m; q)_\infty (q; q)_m q^{m(m+1)/2}} F_n(q^{-m}; a|q) F_{n'}(q^{-m}; a|q) = \frac{(q; q)_n}{a^n q^{n(n+1)/2}} \delta_{nn'} \tag{12}
$$

The functions $F_n(x; a, b|q)$ can be represented in another form. Indeed, taking in the relation (III.8) of Appendix III in [7] the limit $c \to \infty$, one derives the relation

$$
2\phi_1(q^{-m}, -aq^m; 0; q, q^{n+1}) = (-a)^m q^{m^2} 3\phi_0(q^{-m}, -aq^m, q^{-n} - q, -q^n/a).
$$

Therefore, we have

$$
F_n(q^{-m}; a|q) = (-a)^m q^{m^2} 3\phi_0(q^{-m}, -aq^m, q^{-n} - q, -q^n/a) \tag{13}
$$

The basic hypergeometric function $3\phi_0$ in (13) is a polynomial of degree $n$ in the variable $\mu(m) := q^{-m} - a q^m$, which represents a $q$-quadratic lattice; we denote it by

$$
d_n(\mu(m); a|q) := 3\phi_0(q^{-m}, -a q^m, q^{-n} - q, -q^n/a) \tag{14}
$$

Then formula (12) yields the orthogonality relation

$$
\sum_{m=0}^{\infty} \frac{(1 + a^2 q^{2m}) a^m}{(-aq^m; q)_\infty (q; q)_m} q^{m(3m-1)/2} d_n(\mu(m)) d_n'(\mu(m)) = \frac{(q; q)_n}{a^n q^{n(n+1)/2}} \delta_{nn'} \tag{15}
$$

for the polynomials (14) when $a > 0$. As far as we know this orthogonality relation is new. We call the polynomials $d_n(\mu(m); a|q)$ dual alternative $q$-Charlier polynomials. Thus, we proved the following theorem.

**Theorem.** The polynomials $d_n(\mu(m); a|q)$, given by formula (14), are orthogonal on the set of points $\mu(m) := q^{-m} - a q^m$, $m = 0, 1, 2, \cdots$, and the orthogonality relation is given by formula (15).

**Remark.** The duality of polynomials is the well-known notion (see [7] and [11]) and, in particular, in the case of polynomials, orthogonal with respect to a finite number of discrete points, it reflects the simple fact that a finite-dimensional matrix, orthogonal by rows, is also orthogonal by its columns (cf (9)). There is also an analytical way of deriving a dual orthogonality for polynomials, whose weight functions are supported on an infinite number of discrete points (see, for example, [12]-[14]). But this derivation is given in terms of a dual set of functions (for instance, their explicit form (11) for the case of alternative $q$-Charlier polynomials is given above) and one needs to make one step further in order to extract an appropriate family of dual polynomials from these functions. Observe that in our approach to the duality of $q$-polynomials it is not
assumed that an initial family is orthogonal, this orthogonality property is straightforwardly derived. Besides, one naturally extracts from a dual set of orthogonal functions an appropriate dual family of $q$-polynomials.

Let $L^2$ be the Hilbert space of functions on the set $m = 0, 1, 2, \cdots$ with the scalar product

$$\langle f_1, f_2 \rangle = \sum_{m=0}^{\infty} \frac{(1 +aq^{2m})a^m}{(-aq^m; q)_\infty(q; q)_m} q^{m(3m-1)/2} f_1(m) \overline{f_2(m)},$$  \hspace{1cm} (16)

where the weight function is taken from (15). The polynomials (14) are in one-to-one correspondence with the columns of the orthogonal matrix $(a_{mn})$ and the orthogonality relation (15) is equivalent to the orthogonality of these columns. Due to (9) the columns of the matrix $(\hat{a}_{mn})$ form an orthonormal basis in the Hilbert space of sequences $a = \{a_n \mid n = 0, 1, 2, \cdots\}$ with the scalar product $\langle a, a' \rangle = \sum_n a_n \overline{a'_n}$. This scalar product is equivalent to the scalar product (16) for the polynomials $d_n(\mu(m); a; q)$. For this reason, the set of polynomials $d_n(\mu(m); a; q)$, $n = 0, 1, 2, \cdots$, form an orthogonal basis in the Hilbert space $L^2$. This means that the point measure in (15) is extremal for the dual alternative $q$-Charlier polynomials $d_n(\mu(m); a; q)$ (for the definition of an extremal orthogonality measure see, for example, in [11]). This means that the dual alternative $q$-Charlier polynomials (14) form a complete system in the $L^2$-space with respect to the point measure in (15). Observe that the completeness of alternative $q$-Charlier polynomials in the $L^2$-space with respect to the point measure in (10) is a consequence of the fact that the operator (1) is bounded.

A recurrence relation for the polynomials $d_n(\mu(m); a; q)$ is derived from (4). It has the form

$$(q^{-m} - aq^m)d_n(\mu(m)) = -ad_{n+1}(\mu(m)) + q^{-n}d_n(\mu(m)) - q^{-n}(1 - q^n)d_{n-1}(\mu(m)),$$ \hspace{1cm} (17)

where $d_n(\mu(m)) \equiv d_n(\mu(m); a; q)$. A $q$-difference equation for $d_n(\mu(m); a; q)$ can be obtained from the three-term recurrence relation for alternative $q$-Charlier polynomials.

Note that for the polynomials $d_n(\mu(m); a; q^{-1})$ with $q < 1$ we have the expression

$$d_n(\mu(m); a; q^{-1}) = 3\phi_2(q^{-m}, -aq^m, q^{-n}; 0, 0; q, q).$$ \hspace{1cm} (18)

However, the recurrence relation for these polynomials (which can be obtained from the relation (17), does not satisfy the positivity condition $A_nC_{n+1} > 0$, that is, they are not orthogonal polynomials for $a > 0$ (as it is the case for alternative $q$-Charlier polynomials). This positivity condition holds only if we demand $a < 0$. In this case, the polynomials (18) are the continuous big $q$-Hermite polynomials $H_n(x; a|q)$ (for an explicit form of these polynomials see, for example, [1], formula (3.18.1)), which are orthogonal on a certain continuous set of points.

**Acknowledgments**

This research has been supported in part by the SEP-CONACYT project 41051-F and the DGAPA-UNAM project IN112300 “Óptica Matemática”. A. U. Klimyk acknowledges the Consejo Nacional de Ciencia y Tecnología (México) for a Cátedra Patrimonial Nivel II.
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