Quasi-Locality for étale Groupoids

Baojie Jiang\textsuperscript{1}, Jiawen Zhang\textsuperscript{2}, Jianguo Zhang\textsuperscript{3}

\textsuperscript{1} College of Mathematics and Statistics, Chongqing University, Chongqing 401331, China  
E-mail: jiangbaojie@gmail.com  
\textsuperscript{2} School of Mathematical Sciences, Fudan University, 220 Handan Road, Shanghai 200433, China  
E-mail: jiawenzhang@fudan.edu.cn  
\textsuperscript{3} School of Mathematics and Statistics, Shaanxi Normal University, Xi’an 710119, China  
E-mail: jgzhang@snnu.edu.cn

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Abstract: Let $G$ be a locally compact étale groupoid and $\mathcal{L}(L^2(G))$ be the $C^*$-algebra of adjointable operators on the Hilbert $C^*$-module $L^2(G)$. In this paper, we discover a notion called quasi-locality for operators in $\mathcal{L}(L^2(G))$, generalising the metric space case introduced by Roe. Our main result shows that when $G$ is additionally $\sigma$-compact and amenable, an equivariant operator in $\mathcal{L}(L^2(G))$ belongs to the reduced groupoid $C^*$-algebra $C^*_r(G)$ if and only if it is quasi-local. This provides a practical approach to describe elements in $C^*_r(G)$ using coarse geometry. Our main tool is a description for operators in $\mathcal{L}(L^2(G))$ via their slices with the same philosophy to the computer tomography. As applications, we recover a result by Špakula and the second-named author in the metric space case, and deduce new characterisations for reduced crossed products and uniform Roe algebras for groupoids.

1. Introduction

In the last few decades, there has been increased interest in the study of $C^*$-algebras associated to groupoids. The story should be traced back to Renault [33], who introduced the notion of groupoid $C^*$-algebras as well as several counterparts for locally compact topological groupoids. These constructions are extremely powerful weapons to produce $C^*$-algebras, and provide a unified approach to a number of classic $C^*$-algebras including group $C^*$-algebras, crossed products, $C^*$-algebras associated to foliations, uniform Roe algebras, etc.

The groupoid $C^*$-algebra also provides a useful and concrete model for many classes of $C^*$-algebras, which is helpful to analyse their $C^*$-algebraic structures. Along this way, Renault [33, 34] introduced the notion of Cartan subalgebras in $C^*$-algebras, motivated by Feldman and Moore’s works [15] on von Neumann algebras. Since then, the work of

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Kumjian, Renault, Barlak, Li and others (e.g., [6,7,20,26,27,34]) showed that a large class of $C^*$-algebras can be realised as twisted groupoid $C^*$-algebras, based on the theory of Cartan subalgebras. Consequently, this allows people to use the groupoid models to study their ideal structure, nuclearity, the Universal Coefficient Theorem (UCT), etc.

On the other hand, the groupoid $C^*$-algebra plays a central role in (higher) index theory as a proxy for the algebra of continuous functions and its $K$-theory is the natural locus for generalised indices (see, e.g., [42]). For example, the equivariant index of a lifted elliptic differential operator on the universal cover of a closed manifold $M$ belongs to the $K$-theory of the group $C^*$-algebra of $\pi_1(M)$ [18,37]; while the index of an elliptic differential operator on an open manifold belongs to the $K$-theory of its Roe algebra [35], which is isomorphic to certain crossed product of the groupoid $C^*$-algebra for the associated coarse groupoid [38].

These indices provide geometric and topological information of the underlying manifold. Hence the computation for the $K$-theory of groupoid $C^*$-algebras lies at the heart of (higher) index theory, and the famous Baum-Connes conjecture provides an efficient and practical approach (see [44] for an excellent introduction, and [17] for a more recent survey). A number of excellent works have been made around this conjecture in the last few decades (see a comprehensive list of reference in [17]), while the general case is still widely open. This suggests that studying the structure of groupoid $C^*$-algebras will help to understand the indices of various operators, and hence to enhance our understanding of geometric properties of the underlying manifolds.

In spite of the importance of groupoid $C^*$-algebras, it is usually hard to determine whether a given element belongs to this algebra. According to the definition (see Sect. 2.3), we have to construct a sequence of compactly supported functions on the groupoid to approximate the given element in operator norm, while unfortunately there is no routine procedure for the construction.

The aim of this paper is to provide a practical approach to characterise elements in groupoid $C^*$-algebras using coarse geometry. For simplicity, we will only focus on locally compact étale groupoids (see Sect. 2.2 for a precise definition).

1.1. A motivating example: the coarse groupoid. We start with an illuminating example which is the motivation of this work. Let $(X, d)$ be a discrete metric space with bounded geometry. Regarding operators on $\ell^2(X)$ as $X$-by-$X$ matrices, we say that such an operator has finite propagation if the non-zero entries appear only in an entourage, i.e., a band of finite width (measured by the metric $d$ on $X$) around the main diagonal (see Sect. 2.5 for full details). The finite propagation operators form a $*$-subalgebra $C_u\[X\]$ of $B(\ell^2(X))$, and its closure $C^*_u(X)$ is called the uniform Roe algebra of $X$, introduced by Roe in [35].

To characterise operators in $C^*_u(X)$, Roe [35] also introduced a notion of quasi-locality for operators in $B(\ell^2(X))$. Roughly speaking, an operator is called quasi-local if for any $\varepsilon > 0$ we can find an entourage such that for any block sitting outside this entourage, its restriction on the block has norm less than $\varepsilon$ (see Sect. 2.5). The collection of all quasi-local operators on $\ell^2(X)$ forms a $C^*$-algebra, called the uniform quasi-local algebra of $X$ and denoted by $C^*_uq(X)$.

Clearly $C^*_u(X)$ is a subalgebra of $C^*_uq(X)$, and it is proved by Špakula and the second-named author [39] that they coincide when $X$ has Property A. Quasi-locality is a coarse geometric property in the sense that the algebra $C^*_uq(X)$ is preserved under coarse equivalence and moreover, this property is easier to verify than the case of the uniform Roe
algebra. Therefore, the quasi-local characterisation is crucial in the work of Engel [12, 13] on the index theory of pseudo-differential operators, and also in the work of White and Willett [45] on classifying Cartan subalgebras of Roe algebras, with some applications on the associated rigidity problem as well.

To build a bridge to the groupoid theory, Skandalis, Tu and Yu [38] introduced a locally compact étale groupoid for \((X, d)\), called the coarse groupoid \(G(X)\) (see Example 2.6.3).

Note that any compact subset in \(G(X)\) is contained in the closure of some entourage, and hence it is clear that \(\mathbb{C}_c(G(X))\) is \(\ast\)-isomorphic to \(\mathbb{C}_u[X]\). Moreover taking closures, the reduced groupoid \(C\ast\)-algebra \(C\ast_r(G(X))\) is \(\ast\)-isomorphic to the uniform Roe algebra \(C\ast_u(X)\). This suggests us to consider compactly supported operators in the place of finite propagation operators for general groupoids, and hence the reduced groupoid \(C\ast\)-algebra should correspond to certain version of the uniform Roe algebra. Also recall from [38] that \(X\) has Yu’s Property A if and only if \(G(X)\) is amenable.

Now a natural question is to ask how to recover the notion of quasi-locality for operators in \(\mathcal{B}(\ell^2(X))\) using the language of coarse groupoids. This will also be our starting point to introduce quasi-locality for general locally compact étale groupoids. Parallel to the situation of uniform Roe algebras, the practical approach we are looking for to describe elements in groupoid \(C\ast\)-algebras will be achieved by the coarse geometric property of quasi-locality.

1.2. Our strategy for quasi-locality. To introduce the notion of quasi-locality for general groupoids, we would like to mimic the approach for metric spaces. Recall from Sect. 1.1 that for a discrete metric space \((X, d)\) with bounded geometry, we have a natural ambient \(C\ast\)-algebra \(\mathcal{B}(\ell^2(X))\). For operators in \(\mathcal{B}(\ell^2(X))\), we have the notion of finite propagation and quasi-locality to form the associated \(C\ast\)-algebras. The main difficulty to generalise the notion of quasi-locality to general groupoids is that firstly we have to draw the border of a proper class of operators so that the notion of quasi-locality thereon has the chance to recover the reduced groupoid \(C\ast\)-algebra.

Let us explain our strategy in details. For a locally compact étale groupoid \(\mathcal{G}\), we follow the steps below:

Step I. We need to choose a suitable candidate for the ambient \(C\ast\)-algebra \(\mathcal{A}\). The criterion is that \(\mathcal{A}\) should be large enough to include the \(\ast\)-algebra \(\mathbb{C}_c(\mathcal{G})\), while not too large so that a well-defined notion of quasi-locality has the chance to characterise the reduced groupoid \(C\ast\)-algebra \(C\ast_r(\mathcal{G})\) when \(\mathcal{G}\) is “well-behaved”. Moreover in the case of coarse groupoid, the ambient \(C\ast\)-algebra \(\mathcal{A}\) should be (almost\(^1\)) the same as \(\mathcal{B}(\ell^2(X))\).

Step II. We would like to define the notion of compact support and quasi-locality for operators in the ambient \(C\ast\)-algebra \(\mathcal{A}\) chosen in Step I so that compactly supported operators are nothing but those from \(\mathbb{C}_c(\mathcal{G})\). Moreover in the case of the coarse groupoid, the notion of quasi-locality should recover the uniform quasi-local algebra \(C\ast_{uq}(X)\) introduced by Roe (see also [24]).

Step III. Finally we aim to show that when \(\mathcal{G}\) is amenable (which is equivalent to Property A in the case of the coarse groupoid), elements in the reduced groupoid \(C\ast\)-algebra \(C\ast_r(\mathcal{G})\) can be characterised by the property of quasi-locality, which fulfils our task.

\(^1\) As we shall see in Example 3.25, \(\mathcal{A}\) is smaller than \(\mathcal{B}(\ell^2(X))\). However this does not cause any trouble when considering quasi-local operators (see Example 4.9).
1.3. Our main contributions. Following the strategy from Sect. 1.2, now we introduce our main contributions to achieve each step above. Assume that $\mathcal{G}$ is a locally compact étale groupoid with unit space $\mathcal{G}^{(0)}$. For each $x \in \mathcal{G}^{(0)}$, denote $\mathcal{G}_x$ the set consisting of elements in $\mathcal{G}$ with source $x$.

For Step I, there are two natural candidates for the ambient $C^*$-algebras coming from the definition of the reduce groupoid $C^*$-algebra. Recall that the $C^*$-algebra $C_c(\mathcal{G})$ can be faithfully represented as adjointable operators on the Hilbert $C^*$-algebra. Denote the subalgebra might be a suitable candidate for the ambient $C^*$-algebra. By checking the case of the coarse groupoid, we realise that the algebra $\prod_{x \in \mathcal{G}^{(0)}} \mathcal{B}(\ell^2(\mathcal{G}_x))$ might be a suitable candidate for the ambient $C^*$-algebra.

By checking the issue, note that there is a natural embedding $\Phi : \mathcal{L}(L^2(\mathcal{G})) \to \prod_{x \in \mathcal{G}^{(0)}} \mathcal{B}(\ell^2(\mathcal{G}_x))$, called the slicing map (see (3.1)). We realise that operators in the image of $\Phi$ satisfy a condition which can be regarded as a vector-wise version of certain quasi-locality (Definition 3.12). Hence consulting the notion of operator fibre space introduced by Austin and the second-named author, we achieve the following (see Sect. 3.2 for precise definitions):

**Theorem A.** (Theorem 3.13) Let $\mathcal{G}$ be a locally compact étale groupoid with unit space $\mathcal{G}^{(0)}$. For $(T_x)_{x \in \mathcal{G}^{(0)}} \in \prod_{x \in \mathcal{G}^{(0)}} \mathcal{B}(\ell^2(\mathcal{G}_x))$, the following are equivalent:

1. $(T_x)_{x \in \mathcal{G}^{(0)}}$ belongs to $\Phi(\mathcal{L}(L^2(\mathcal{G})))$;
2. the map $x \mapsto T_x$ is a continuous section of the associated operator fibre space, and $(T_x)_{x \in \mathcal{G}^{(0)}}, (T_x^*)_{x \in \mathcal{G}^{(0)}}$ are vector-wise uniformly quasi-local.

**Theorem A** provides a characterisation for operators in $\mathcal{L}(L^2(\mathcal{G}))$ via their slices with the same philosophy to the computer toponomy, which describes an object using a family of its slices. This will be crucial in the proof of Theorem E below as well as in the computation of $\mathcal{L}(L^2(\mathcal{G}))$ for concrete examples. We also obtain a simplified version of Theorem A when the unit space contains a dense subset (see Proposition 3.18 and Corollary 3.19), which happens in a number of cases including coarse groupoids.

By checking coarse groupoids again, we realise that the algebra $\mathcal{L}(L^2(\mathcal{G}))$ might still be too large. Note that elements in $C^*_r(\mathcal{G})$ admit an extra equivariant property: an operator in $\mathcal{L}(L^2(\mathcal{G}))$ is called $\mathcal{G}$-equivariant if it commutes with elements in the image of the right regular representation (see Definition 3.21). Denote the subalgebra of $\mathcal{G}$-equivariant operators in $\mathcal{L}(L^2(\mathcal{G}))$ by $\mathcal{L}(L^2(\mathcal{G}))^G$. Thanks to Theorem A, we manage to compute $\mathcal{L}(L^2(\mathcal{G}))^G$ for the coarse groupoid (see Example 3.25). Moreover, we obtain the following realisation for equivariant operators:

**Proposition B.** (Proposition 3.31) Let $\mathcal{G}$ be a locally compact étale groupoid. For any $\mathcal{G}$-equivariant $T \in \mathcal{L}(L^2(\mathcal{G}))$, there exists a unique function $f_T \in C_b(\mathcal{G})$ such that $T$ is the convolution operator by $f_T$.

The clues above suggest that $\mathcal{L}(L^2(\mathcal{G}))^G$ should be a suitable candidate for the ambient $C^*$-algebra. We remark that Proposition B will also be crucial to prove Theorem E below since it transfers operators to functions.
Next, we move to Step II. Recall that we need to define the notion of compact support and quasi-locality for operators in the ambient $C^*$-algebra $\mathcal{L}(L^2(\mathcal{G}))$ so that those with compact support recover the algebra $C_c(G)$. Indeed, the definition we discover can be naturally defined on the larger algebra $\mathcal{L}(L^2(\mathcal{G}))$. Inspired by the case of coarse groupoids, we introduce the following key notion of this paper:

**Definition C.** [Definition 4.1 and Definition 4.2] Let $\mathcal{G}$ be a locally compact étale groupoid and $T \in \mathcal{L}(L^2(\mathcal{G}))$. We define the following:

1. For a subset $K \subseteq \mathcal{G}$, functions $f, g \in C_b(\mathcal{G})$ are called $K$-separated if we have $(K \cdot \text{supp}(f)) \cap \text{supp}(g) = \emptyset$ and $(K \cdot \text{supp}(g)) \cap \text{supp}(f) = \emptyset$.
2. $T$ is called compactly supported if there exists a compact subset $K \subseteq \mathcal{G}$ such that $gTf = 0$ for any $K$-separated functions $f, g \in C_b(\mathcal{G})$.
3. $T$ is called quasi-local if for any $\varepsilon > 0$, there exists a compact subset $K \subseteq \mathcal{G}$ such that for any $K$-separated functions $f, g \in C_b(\mathcal{G})$ we have $\|gTf\| < \varepsilon \|g\|_{\infty} \|f\|_{\infty}$.

Thanks to Theorem A, we obtain a characterisation for compactly supported and quasi-local operators in $\mathcal{L}(L^2(\mathcal{G}))$ using their slices (see Proposition 4.4 and Corollary 4.6). In particular for a quasi-local operator, each of its slice is an ordinary quasi-local operator on a discrete metric space. Hence our notion of quasi-locality reflects the coarse geometry of the underlying metric family. Applying this characterisation to the case of the coarse groupoid $G(X)$ for some metric space $X$, we verify that Definition C is compatible with the notion of finite propagation and quasi-locality for operators in $\mathcal{B}(l^2(X))$ as desired (see Example 4.9).

Moreover under the assumption of equivariance, we reach the following as a consequence of Proposition B:

**Proposition D.** (Proposition 4.12) Let $\mathcal{G}$ be a locally compact étale groupoid. A $\mathcal{G}$-equivariant operator $T \in \mathcal{L}(L^2(\mathcal{G}))$ is compactly supported if and only if $T$ belongs to the image of $C_r(G)$ under the left regular representation. Hence the reduced groupoid $C^*$-algebra $C^*_r(\mathcal{G})$ can be recovered as the norm closure of all $\mathcal{G}$-equivariant compactly supported operators in $\mathcal{L}(L^2(\mathcal{G}))$.

Hence we fulfil the task of Step II. For later use, we denote by $C_u(\mathcal{G})$ the $*$-subalgebra in $\mathcal{L}(L^2(\mathcal{G}))$ consisting of all compactly supported operators and define the uniform Roe algebra $C^*_u(\mathcal{G})$ of $\mathcal{G}$ to be its norm closure in $\mathcal{L}(L^2(\mathcal{G}))$. Also denote by $C_{uq}^*(\mathcal{G})$ the set of all quasi-local operators in $\mathcal{L}(L^2(\mathcal{G}))$, which forms a $C^*$-algebra and is called the uniform quasi-local algebra of $\mathcal{G}$. We also consider their equivariant counterparts and denote by $C_u(\mathcal{G})^G$, $C_u^*(\mathcal{G})^G$ and $C_{uq}^*(\mathcal{G})^G$ the subalgebras consisting of equivariant operators in $C_u(\mathcal{G})$, $C_u^*(\mathcal{G})$ and $C_{uq}^*(\mathcal{G})$, respectively.

Having established all necessary ingredients, finally we reach Step III and manage to prove the following main result of this paper:

**Theorem E.** (Theorem 5.1) Let $\mathcal{G}$ be a locally compact, $\sigma$-compact and étale groupoid. If $\mathcal{G}$ is amenable, then we have $C^*_r(\mathcal{G}) = C^*_u(\mathcal{G})^G = C_{uq}^*(\mathcal{G})^G$.

Theorem E shows that for a $\mathcal{G}$-equivariant operator in $\mathcal{L}(L^2(\mathcal{G}))$, it is quasi-local if and only if it belongs to the reduced groupoid $C^*$-algebra $C^*_r(\mathcal{G})$. Note that quasi-locality reflects the coarse geometry of the underlying object and is easier to verify than the situation of $C^*_r(\mathcal{G})$, hence Theorem E provides a practical and coarse geometric approach to characterise elements in the reduced groupoid $C^*$-algebra for amenable groupoids as desired.
As an example when $G$ is the coarse groupoid associated to a discrete metric space with bounded geometry, Theorem 5.1 recovers the main result of [39] in the Hilbert space case (see Sect. 6.3). Moreover when $G$ is the transformation groupoid associated to a discrete group acting on a compact Hausdorff space, we obtain a new characterisation for elements in the reduced crossed product (see Sect. 6.4).

The proof of Theorem E relies heavily on the coarse geometry of groupoids (see [28] and [30]). Roughly speaking, we assign a length function on the groupoid thanks to the $\sigma$-compactness, which induces a metric on each source fibre. Hence we can appeal to the weapon of coarse geometry for a family of metric spaces, and consult the idea of [39, Theorem 3.3]. Here we would like to clarify that the proof for Theorem E is not merely a family version of that for [39, Theorem 3.3] since we have certain continuity restriction from the topology of the groupoid, and technical analysis is also required especially when the unit space is no longer compact.

1.4. Beyond equivariance: uniform Roe algebras. Note that Theorem E only discusses the situation of equivariant operators, while our notion of quasi-locality (Definition C) is designed for general operators in $\mathcal{L}(L^2(G))$. Hence it is natural to ask whether we can obtain a similar result for the non-equivariant case.

As revealed by the work of Anantharaman-Delaroche [1], we can transfer the general case to the equivariant one by considering the semi-direct product. More precisely, for a locally compact étale groupoid $G$ we consider the fibrewise Stone-Čech compactification $\beta_r G$ and the associated semi-direct product $\beta_r G \rtimes G$ (see Sect. 7.1). Anantharaman-Delaroche showed in [1] that the reduced groupoid $C^*$-algebra $C^*_r(\beta_r G \rtimes G)$ is $C^*$-isomorphic$^2$ to the uniform Roe algebra $C^*_u(G)$.

Furthermore, we construct an embedding from $\mathcal{L}(L^2(\beta_r G \rtimes G))$ to $\mathcal{L}(L^2(G))$ in Sect. 7.3 (see Corollary 7.7), and apply Theorem A to show that it provides a $C^*$-isomorphism between the equivariant uniform quasi-local algebra of $\beta_r G \rtimes G$ and the uniform quasi-local algebra of $G$. Finally thanks to Theorem E, we obtain the following concrete description for elements in the uniform Roe algebra $C^*_u(G)$ (see Sect. 7.1 for precise definitions):

**Theorem F.** (Theorem 7.20) Let $G$ be a locally compact, $\sigma$-compact and étale groupoid. Suppose $G$ is either strongly amenable at infinity, or secondly countable weakly inner amenable and $C^*$-exact. Then we have $C^*_u(G) = C^*_uq(G)$.

1.5. Organisation. The paper is organised as follows. In Sect. 2, we recall necessary notions in groupoid theory and coarse geometry together with several examples. Section 3 is devoted to Step I where we provide the required characterisation for operators in the Hilbert $C^*$-module $\mathcal{L}(L^2(G))$ (Theorem A), discuss the notion of equivariance and finally prove Proposition B. In Sect. 4, we introduce the key notion of compact support and quasi-locality (Definition C) and recover the algebra $C_c(G)$ (i.e., Proposition D), which fulfils Step II. Section 5 is devoted to Step III where we prove Theorem 5.1, and as applications we provide several examples in Sect. 6. Finally in Sect. 7, we consider the general case and prove Theorem F.

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$^2$ More precisely, Anantharaman-Delaroche [1] defined the uniform Roe algebra $C^*_u(G)$ in a different way, while we show in Lemma 7.4 that it coincides with our definition.
2. Preliminaries

2.1. Standard notation. Here we collect the notation used throughout the paper.
To simplify the terminology, we always assume that locally compact spaces are Hausdorff. Given a locally compact space $X$, we denote by $C(X)$ the set of complex-valued continuous functions on $X$, and by $C_b(X)$ the subset of bounded continuous functions on $X$. Recall that the support of a function $f \in C(X)$ is the closure of $\{x \in X : f(x) \neq 0\}$, written as $\text{supp}(f)$, and denote by $C_c(X)$ the set of continuous functions with compact support. We also denote by $C_0(X)$ the set of continuous functions vanishing at infinity, which is the closure of $C_c(X)$ with respect to the supremum norm $\|f\|_\infty := \sup\{|f(x)| : x \in X\}$.

When $X$ is discrete, denote $\ell^\infty(X) := C_b(X)$ and $\ell^2(X)$ the Hilbert space of complex-valued square-summable functions on $X$. To be compatible with the setting of Hilbert $C^*$-modules, we always assume that the Hilbert space inner product $\langle \eta, \xi \rangle$ will be taken to be linear in the variable $\xi$ and conjugate-linear in $\eta$. Denote by $\mathcal{B}(\ell^2(X))$ the $C^*$-algebra of all bounded linear operators on $\ell^2(X)$, and $\mathcal{K}(\ell^2(X))$ the $C^*$-subalgebra of all compact operators.

2.2. Basic notions for groupoids. Let us start with some basic notions and terminology on groupoids. For details we refer to [33].

Recall that a groupoid is a small category, in which every morphism is invertible. Roughly speaking, a groupoid consists of a set $G$, a subset $G^{(0)}$ called the unit space, two maps $s, r : G \to G^{(0)}$ called the source and range maps respectively, a composition law:

$$G^{(2)} := \{(\gamma_1, \gamma_2) \in G \times G : s(\gamma_1) = r(\gamma_2)\} \ni (\gamma_1, \gamma_2) \mapsto \gamma_1 \gamma_2 \in G,$$

and an inverse map $\gamma \mapsto \gamma^{-1}$. These operations satisfy a couple of axioms, including the associativity law and the fact that elements in $G^{(0)}$ act as units. For $x \in G^{(0)}$, denote $G^x := r^{-1}(x)$ and $G_x := s^{-1}(x)$. A subset $Y \subseteq G^{(0)}$ is called invariant if $r^{-1}(Y) = s^{-1}(Y)$. For $A, B \subseteq G$, we denote

$$A^{-1} := \{\gamma \in G : \gamma^{-1} \in A\},$$

$$AB := \{\gamma \in G : \gamma = \gamma_1 \gamma_2 \text{ where } \gamma_1 \in A, \gamma_2 \in B \text{ and } s(\gamma_1) = r(\gamma_2)\}.$$

We say that $A \subseteq G$ is symmetric if $A = A^{-1}$.

A groupoid $G$ is principal if the map $(s, r)$ from $G$ into $G^{(0)} \times G^{(0)}$ is one-to-one, and $G$ is transitive if the map $(s, r)$ is onto.

A locally compact groupoid is a groupoid $G$ endowed with a locally compact topology for which composition, inversion, source and range maps are continuous with respect to the induced topologies.

We say that a locally compact groupoid $G$ is étale if the range (and hence the source) map is a local homeomorphism, i.e., for any $\gamma \in G$ there exists a neighbourhood $U$ of $\gamma$ such that $r(U)$ is open and $r|_U$ is a homeomorphism. In this case, fibers $G_x$ and $G^x$ with the induced topologies are discrete and $G^{(0)}$ is open in $G$. Throughout the paper, we shall limit ourselves to the étale case.

For a locally compact groupoid $G$, a subset $A \subseteq G$ is called a bisection if the restrictions of $s, r$ to $A$ are injective. It follows from definition that for a locally compact étale groupoid $G$, all open bisections form a basis for the topology of $G$. As a direct consequence, we obtain that for such a groupoid $G$, any function $f \in C_c(G)$ can be
written as a linear combination of continuous functions whose supports are contained in open precompact bisections.

2.3. Groupoid $C^*$-algebras and Hilbert $C^*$-modules. Here we recall the notion of reduced groupoid $C^*$-algebras in the étale case. Given a locally compact étale groupoid $G$, note that the space $C_c(G)$ can be formed into a $*$-algebra with the following operations: for $f, g \in C_c(G)$,

$$ (f * g)(\gamma) := \sum_{\alpha \in \mathcal{G}_{s(\gamma)}} f(\gamma \alpha^{-1}) g(\alpha), \quad (2.1) $$

$$ f^*(\gamma) := f(\gamma^{-1}). \quad (2.2) $$

The function $f * g \in C_c(G)$ is called the convolution of $f$ and $g$.

Recall that for each $x \in G^{(0)}$ the left regular representation at $x$, denoted by $\lambda_x : C_c(G) \to \mathcal{B}(\ell^2(G_x))$, is defined as follows:

$$ \lambda_x(f)(\xi)(\gamma) = \sum_{\alpha \in \mathcal{G}_x} f(\gamma \alpha^{-1}) \xi(\alpha), \quad \text{where } f \in C_c(G) \text{ and } \xi \in \ell^2(G_x). \quad (2.3) $$

It is routine to check that $\lambda_x$ is a well-defined $*$-homomorphism. The reduced norm on $C_c(G)$ is defined by

$$ \|f\|_r := \sup_{x \in G^{(0)}} \|\lambda_x(f)\|, $$

and the reduced groupoid $C^*$-algebra $C^*_r(G)$ is defined to be the completion of the $*$-algebra $C_c(G)$ with respect to the reduced norm $\| \cdot \|_r$. It is clear that each left regular representation $\lambda_x$ can be extended automatically to a $C^*$-homomorphism $\lambda_x : C^*_r(G) \to \mathcal{B}(\ell^2(G_x))$.

In the following, we would like to provide an alternative approach to define the reduced groupoid $C^*$-algebras using the language of Hilbert $C^*$-modules, which motivates our definition of quasi-locality for groupoids. Here we recall the necessary notions on Hilbert $C^*$-modules, and guide readers to [21] for details.

**Definition 2.1.** [21] Let $A$ be a $C^*$-algebra. An inner-product $A$-module is a linear space $E$ which is a right $A$-module (with compatible scalar multiplication: $\lambda(\xi a) = (\lambda \xi) a = \xi(\lambda a)$ for $\xi \in E$, $a \in A$ and $\lambda \in \mathbb{C}$), together with a map $(\xi, \eta) \mapsto \langle \xi, \eta \rangle$ from $E \times E$ to $A$ such that

1. $\langle \xi, \alpha \eta + \beta \zeta \rangle = \alpha \langle \xi, \eta \rangle + \beta \langle \xi, \zeta \rangle$ for $\xi, \eta, \zeta \in E$ and $\alpha, \beta \in \mathbb{C}$;
2. $\langle \xi, \eta a \rangle = \langle \xi, \eta \rangle a$ for $\xi, \eta \in E$ and $a \in A$;
3. $\langle \eta, \xi \rangle^* = \langle \xi, \eta \rangle$ for $\xi, \eta \in E$;
4. $\langle \xi, \xi \rangle \geq 0$ for $\xi \in E$, and $\langle \xi, \xi \rangle = 0$ if and only if $\xi = 0$.

For $\xi \in E$, we define its norm $\|\xi\|_E := \|\langle \xi, \xi \rangle\|^{1/2}$, also denoted by $\|\xi\|$ when there is no ambiguity. An inner-product $A$-module which is complete with respect to this norm is called a Hilbert $A$-module.

Recall that an inner-product $A$-module $E$ admits an $A$-valued norm $|\cdot|$ given by $|\xi| = \langle \xi, \xi \rangle^{1/2}$. Note that $|\langle \xi, \eta \rangle| \leq \|\xi\| \cdot |\eta|$ for $\xi, \eta \in E$ (see [21, Proposition 1.1]), and hence $|\xi a| \leq \|\xi\| \cdot |a|$ for $\xi \in E$ and $a \in A$. 


Definition 2.2. Let $A$ be a $C^*$-algebra and $E$ be a Hilbert $A$-module. A map $T : E \to E$ is called adjointable if there exists a map $T^* : E \to E$ such that

$$\langle \eta, T(\xi) \rangle = \langle T^*(\eta), \xi \rangle$$

for all $\xi, \eta \in E$.

It is easy to see that adjointable operators are $A$-linear and bounded with respect to the norm on $E$. Denote by $\mathcal{L}(E)$ the set of all adjointable operators on $E$, which forms a $C^*$-algebra.

The following standard result will be used in the sequel:

**Proposition 2.3.** [21, Proposition 1.2] Let $E$ be a Hilbert $A$-module. For $T \in \mathcal{L}(E)$ and $\xi \in E$, we have $|T(\xi)|^2 \leq \|T\|^2|\xi|^2$ and $|T(\xi)| \leq \|T\| \cdot |\xi|$.

Now we consider the Hilbert $C^*$-module associated to a locally compact étale groupoid $\mathcal{G}$. Let $L^2(\mathcal{G})$ be the Hilbert $C^*$-module over $C_0(\mathcal{G}^{(0)})$ obtained by taking completion of $C_c(\mathcal{G})$ with respect to the $C_0(\mathcal{G}^{(0)})$-valued inner product:

$$\langle \eta, \xi \rangle(x) := \sum_{\gamma \in \mathcal{G}_x} \overline{\eta(\gamma)} \xi(\gamma),$$

and the right $C_0(\mathcal{G}^{(0)})$-module structure is given by

$$(\xi f)(\gamma) := \xi(\gamma) f(s(\gamma)).$$

Due to étaleness, for any $\xi \in C_c(\mathcal{G})$ we have

$$\|\xi\|^2_{L^2(\mathcal{G})} = \sup_{x \in \mathcal{G}^{(0)}} \sum_{\gamma \in \mathcal{G}_x} |\xi(\gamma)|^2 \geq \sup_{\gamma \in \mathcal{G}} |\xi(\gamma)|^2 = \|\xi\|^2_{\infty}.$$ 

Hence we can regard $L^2(\mathcal{G})$ as a subspace in $C_0(\mathcal{G})$.

Note that all the regular representations $\lambda_x$ defined in (2.3) can be put together to form a single representation $\Lambda : C_c(\mathcal{G}) \to \mathcal{L}(L^2(\mathcal{G}))$ by the formula:

$$(\Lambda(f)\xi)(\gamma) := \sum_{\alpha \in \mathcal{G}_{s(\gamma)}} f(\gamma \alpha^{-1}) \xi(\alpha) = (\lambda_{s(\gamma)} f)\xi|_{\mathcal{G}_{s(\gamma)}}(\gamma). \quad (2.4)$$

And it is easy to check that for $f \in C_c(\mathcal{G})$, we have $\|f\|_r = \|\Lambda(f)\|_{\mathcal{L}(L^2(\mathcal{G}))}$. Therefore, $\Lambda$ can be extended to a faithful representation $\Lambda : C^*_c(\mathcal{G}) \to \mathcal{L}(L^2(\mathcal{G}))$, which is called the left regular representation.

It follows that the convolution operator $*: C_c(\mathcal{G}) \times C_c(\mathcal{G}) \to C_c(\mathcal{G})$ can be extended to an operator (using the same notation) $*: C_c(\mathcal{G}) \times L^2(\mathcal{G}) \to L^2(\mathcal{G})$ by (2.1), and by definition $\Lambda(f)\xi = f*\xi$ for $f \in C_c(\mathcal{G})$ and $\xi \in L^2(\mathcal{G})$.

There is a special class of operators in $\mathcal{L}(L^2(\mathcal{G}))$, called multiplication operators. For $g \in C_b(\mathcal{G})$, the multiplication operator $M(g) : L^2(\mathcal{G}) \to L^2(\mathcal{G})$ is defined by $(M(g)\xi)(\gamma) = g(\gamma)\xi(\gamma)$ for $\xi \in L^2(\mathcal{G})$ and $\gamma \in \mathcal{G}$. It is clear that $M(g) \in \mathcal{L}(L^2(\mathcal{G}))$, and we abbreviate $M(g)\xi$ by $g\xi$.

For later use, here we briefly recall the notion of interior tensor products for Hilbert $C^*$-modules. Readers might go to [21, Chapter 4] for more details.

Let $A, B$ be $C^*$-algebras, $E$ be a Hilbert $A$-module and $F$ be a Hilbert $B$-module. Let $\rho : A \to \mathcal{L}(F)$ be a $*$-homomorphism, which turns $F$ into a left $A$-module. Denote the algebraic tensor product of $E$ and $F$ over $A$ by $E \otimes_A F$, which is a right $B$-module.
According to [21, Proposition 4.5], $E \otimes_A F$ is an inner-product $B$-module under the inner product given on simple tensors by
\[ (x_1 \otimes y_1, x_2 \otimes y_2) = \langle y_1, \rho((x_1, x_2))(y_2) \rangle \] where $x_1, x_2 \in E$ and $y_1, y_2 \in F$.

The completion of the inner-product $B$-module $E \otimes_A F$ is a Hilbert $B$-module, denoted by $E \otimes_{\rho} F$, which is called the \textit{interior} (or \textit{internal}) tensor product of $E$ and $F$ with respect to $\rho$.

Furthermore, given $T \in \mathcal{L}(E)$, we define the tensor product $T \otimes 1$ on the algebraic tensor product $E \otimes_A F$ by setting $(T \otimes 1)(x \otimes y) := T(x) \otimes y$ for $x \in E$ and $y \in F$. One can easily check that this is well-defined and can be extended to an operator in $\mathcal{L}(E \otimes_{\rho} F)$, denoted by $T \otimes_{\rho} 1$.

\section{Amenable groupoids.}

Amenable groupoids comprise a large class of groupoids with relatively nice properties. Here we only focus on the case of étaleness, in which the notion of amenability behave quite well. A standard reference is [3] and another reference for the étale case is [9, Chapter 5.6].

\textbf{Definition 2.4.} [3, Chapter 2.2] A locally compact étale groupoid $G$ is said to be \textit{amenable} if for any $\varepsilon > 0$ and compact subset $K \subseteq G$, there exists a non-negative function $f \in C_c(G)$ such that for any $\gamma \in K$ we have:

- $\left| 1 - \sum_{\beta \in G_{r(\gamma)}} f(\beta) \right| < \varepsilon$;
- $\sum_{\beta \in G_{r(\gamma)}} |f(\beta) - f(\beta \gamma)| < \varepsilon$.

Using a standard normalisation argument, we have the following:

\textbf{Lemma 2.5.} A locally compact étale groupoid $G$ is amenable if and only if for any $\varepsilon > 0$ and compact subset $K \subseteq G$, there exists a non-negative function $f \in C_c(G)$ such that $\sum_{\beta \in G_x} f(\beta) \leq 1$ for any $x \in G^{(0)}$, and for any $\gamma \in K$ we have:

- $\sum_{\beta \in G_{r(\gamma)}} f(\beta) = 1$;
- $\sum_{\beta \in G_{r(\gamma)}} |f(\beta) - f(\beta \gamma)| < \varepsilon$.

We also need another description for amenability in terms of positive type functions. Recall that for a locally compact étale groupoid $G$, a function $h$ on $G$ is of \textit{positive type} if for any $x \in G^{(0)}$, $n \in \mathbb{N}$, $\gamma_1, \ldots, \gamma_n \in G_x$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, we have the following:

\[ \sum_{i,j=1}^n \overline{\lambda_i} \lambda_j h(\gamma_i \gamma_j^{-1}) \geq 0. \]

We remark that if $h$ is of positive type, then $h(\gamma^{-1}) = \overline{h(\gamma)}$ for $\gamma \in G$.

Combining Lemma 2.5 with [3, Proposition 2.2.13], we obtain the following:

\textbf{Lemma 2.6.} A locally compact étale groupoid $G$ is amenable if and only if for any $\varepsilon > 0$ and compact subset $K \subseteq G$, there exists a non-negative function $h \in C_c(G)$ of positive type such that

- $h(x) \leq 1$ for any $x \in G^{(0)}$, and $h(x) = 1$ for any $x \in K \cap G^{(0)}$;
- $|1 - h(\gamma)| < \varepsilon$ for any $\gamma \in K$. 

2.5. Notions from coarse geometry. Now we collect necessary notions from coarse geometry. For a discrete metric space \((X, d)\), denote the closed ball by \(B_X(x, r) := \{y \in X : d(x, y) \leq r\}\) for \(x \in X\) and \(r \geq 0\). We say that \(X\) has bounded geometry if for every \(r > 0\), the number \(\sup_{x \in X} \#B_X(x, r)\) is finite. Here we use the notation \(\#\) to denote the cardinality of a set \(S\). For a subset \(A \subseteq X\), we denote \(\chi_A\) the characteristic function of \(A\). For a point \(x \in X\), also denote \(\delta_x := \chi_{\{x\}}\).

Let us start with the notion of Property A, which was introduced by Yu in [47]. Property A plays a key role in Yu’s study on the coarse Baum-Connes conjecture, and it admits a number of equivalent definitions. Here we focus on the one in terms of kernels for later use.

For a set \(X\), a kernel on \(X\) is a function \(k : X \times X \to \mathbb{R}\). We say that a kernel \(k\) on \(X\) is of positive type if for any \(n \in \mathbb{N}, x_1, \ldots, x_n \in X\) and any \(\lambda_1, \ldots, \lambda_n \in \mathbb{R}\), we have the following:

\[
\sum^n_{i,j=1} \lambda_i \lambda_j k(x_i, x_j) \geq 0.
\]

We say that a kernel \(k\) has finite propagation if there is \(S > 0\) such that \(k(x, y) = 0\) whenever \(d(x, y) > S\), \(k\) is normalised if \(k(x, x) = 1\) for every \(x \in X\), and \(k\) is symmetric if \(k(x, y) = k(y, x)\) for any \(x, y \in X\). For \(R > 0\) and \(\varepsilon > 0\), we say that a kernel \(k\) has \((R, \varepsilon)\)-variation if \(|1 - k(x, y)| < \varepsilon\) whenever \(d(x, y) < R\).

Associated to a normalised, symmetric and positive type kernel \(k\) on \(X\), we have the so-called Schur multiplier \(m_k : \mathcal{B}(\ell^2(X)) \to \mathcal{B}(\ell^2(X))\). First note that every bounded linear operator \(T \in \mathcal{B}(\ell^2(X))\) can be written as an \(X\)-by-\(X\) matrix \(T = (T_{x,y})_{x,y \in X}\) with coefficient \(T_{x,y} := (\delta_x, T(\delta_y)) \in \mathbb{C}\). The Schur multiplier \(m_k\) is defined by

\[
m_k(T) := (k(x, y)T_{x,y})_{x,y \in X}
\]

in the matrix form. It is a standard result that \(m_k\) is well-defined linear bounded operator with norm 1 (see, e.g., [8, Theorem C.1.4] or [32, Theorem 5.1]), i.e., \(\|m_k(T)\| \leq \|T\|\) for \(T \in \mathcal{B}(\ell^2(X))\).

Now we recall the definition of Property A (see, e.g., [43, Proposition 3.2(v)] or [46, Proposition 1.2.4] for the equivalence to Yu’s original definition):

**Definition 2.7.** [47] Let \(X\) be a discrete metric space with bounded geometry. We say that \(X\) has Property A if for any \(R > 0\) and \(\varepsilon > 0\), there exists a normalised, finite propagation, symmetric kernel \(k\) on \(X\) of positive type and \((R, \varepsilon)\)-variation.

In the following, we would like to recall \(C^*\)-algebras associated to a metric space. Let \((X, d)\) be a discrete metric space. Recall that there is a \(*\)-representation \(M : \ell^\infty(X) \to \mathcal{B}(\ell^2(X))\) defined by point-wise multiplication, i.e., \((M(f)\xi)(x) := f(x)\xi(x)\) for \(f \in \ell^\infty(X), \xi \in \ell^2(X)\) and \(x \in X\). To simplify the notation, we will write \(f\xi\) instead of \(M(f)\xi\) in the sequel.

**Definition 2.8.** For a bounded linear operator \(T \in \mathcal{B}(\ell^2(X))\), we say that \(T\) has finite propagation if there exists \(R > 0\) such that for any \(f_1, f_2 \in \ell^\infty(X)\) with \(d(\text{supp}(f_1), \text{supp}(f_2)) > R\), then \(f_1 T f_2 = 0\). Equivalently, there exists \(R > 0\) such that \(T_{x,y} = 0\) for \(x, y \in X\) with \(d(x, y) > R\).

**Definition 2.9.** Let \((X, d)\) be a discrete metric space with bounded geometry. The set of all finite propagation operators in \(\mathcal{B}(\ell^2(X))\) forms a \(*\)-algebra, called the algebraic uniform Roe algebra of \(X\) and denoted by \(\mathcal{C}_u[X]\). The uniform Roe algebra of \(X\) is
defined to be the operator norm closure of $\mathbb{C}_u[X]$ in $\mathcal{B}(\ell^2(X))$, which is a $C^*$-algebra and denoted by $C^*_u(X)$.

Now we move to the case of quasi-locality, which was introduced by Roe in [35].

**Definition 2.10.** For a bounded linear operator $T \in \mathcal{B}(\ell^2(X))$ and $\varepsilon > 0$, we say that $T$ has finite $\varepsilon$-propagation if there exists $R > 0$ such that for any $f_1, f_2 \in \ell^\infty(X)$ with $d(\text{supp}(f_1), \text{supp}(f_2)) > R$, then $\|f_1Tf_2\| < \varepsilon \|f_1\|\|f_2\|$. We say that $T$ is quasi-local if $T$ has finite $\varepsilon$-propagation for any $\varepsilon > 0$.

Similar to the case of Roe algebras, we form the following:

**Definition 2.11.** [24] Let $(X, d)$ be a discrete metric space with bounded geometry. The set of all quasi-local operators in $\mathcal{B}(\ell^2(X))$ forms a $C^*$-algebra, called the uniform quasi-local algebra of $X$ and denoted by $C^*_{uq}(X)$.

It is clear that finite propagation operators are quasi-local, hence taking closures, we obtain that $C^*_u(X)$ is a $C^*$-subalgebra in $C^*_{uq}(X)$. For the converse, there have been a number of progresses mostly in the recent years [13, 22, 25, 39, 40]. Currently the most general answer is due to Ĺšpakula and the second-named author [39] as follows, while the general case is still open.

**Proposition 2.12.** [39, Theorem 3.3] Let $(X, d)$ be a metric space with bounded geometry and Property $A$, then the uniform Roe algebra $C^*_u(X)$ coincides with the uniform quasi-local algebra $C^*_{uq}(X)$.

2.6. Examples. We collect several classic examples of groupoids.

2.6.1. Discrete groups. Let $\Gamma$ be a discrete group. Then $\Gamma$ is a locally compact étale groupoid whose unit space consists of a single point $1_\Gamma$, i.e., the unit of $\Gamma$. Obviously, $\Gamma$ is amenable as a groupoid if and only if $\Gamma$ is amenable as a group.

In this case, the $*$-algebra $C_c(\Gamma)$ is just the group algebra $\mathbb{C}[\Gamma]$ and the left regular representation at $1_\Gamma$ defined in (2.3) coincides with the left regular representation of the group $\Gamma$. Taking completion, the reduced groupoid $C^*$-algebra of $\Gamma$ is the same as the reduced group $C^*$-algebra $C^*_r(\Gamma)$. We also record that the Hilbert $\mathbb{C}$-module $L^2(\Gamma)$ is nothing but the Hilbert space $\ell^2(\Gamma)$, and hence $\mathcal{L}(L^2(\Gamma)) = \mathcal{B}(\ell^2(\Gamma))$.

2.6.2. Pair groupoids. Let $X$ be a set. The pair groupoid of $X$ is $X \times X$ as a set, whose unit space is $\{(x, x) \in X \times X : x \in X\}$ and identified with $X$ for simplicity. The source map is the projection onto the second coordinate and the range map is the projection onto the first coordinate. The composition is given by $(x, y) \cdot (y, z) = (x, z)$ for $x, y, z \in X$. When $X$ is a discrete Hausdorff space, then $X \times X$ is a locally compact étale groupoid. It can be checked easily that the pair groupoid $X \times X$ is always amenable.

Each $f \in C_c(X \times X)$ can be regarded as an $X$-by-$X$ matrix with finitely many non-zero matrix entries, which induces a map

$$\theta : C_c(X \times X) \rightarrow \mathcal{B}(\ell^2(X))$$

by setting the matrix coefficients $\theta(f)_{x,y} := f(x, y)$ for $x, y \in X$. It is obvious that $\theta$ is a $*$-homomorphism, and induces a $C^*$-isomorphism

$$\Theta : C^*_r(X \times X) \cong \mathcal{K}(\ell^2(X)).$$
2.6.3. Discrete metric spaces  This example is the motivation for the entire work.

Let $(X, d)$ be a discrete metric space with bounded geometry. The coarse groupoid $G(X)$ on $X$ was introduced by Skandalis, Tu, and Yu in [38] (see also [36, Chapter 10]) to relate coarse geometry to the theory of groupoids. As a topological space, 

$$G(X) := \bigcup_{r > 0} E_r^{\beta(X \times X)} \subseteq \beta(X \times X),$$

where $E_r := \{(x, y) \in X \times X : d(x, y) \leq r\}$ and $\beta(X \times X)$ is the Stone-Čech compactification of $X \times X$. Note that $X \times X$ is the pair groupoid with source and range maps $s(x, y) = y$ and $r(x, y) = x$ (see Sect. 2.6.2). These maps extend to maps $\beta(X \times X) \to \beta X$, and hence to maps $G(X) \to \beta X$, still denoted by $r$ and $s$.

Consider the map $(r, s) : G(X) \to \beta X \times \beta X$. It was shown in [38, Lemma 2.7] that $(r, s)$ is injective, and hence $G(X)$ can be endowed with a groupoid structure induced by the pair groupoid $\beta X \times \beta X$, called the coarse groupoid of $X$. It was also shown in [38, Proposition 3.2] that the coarse groupoid $G(X)$ is étale, locally compact and principal. Clearly, the unit space of $G(X)$ is $\beta X$. We denote by $\partial_\beta X := \beta X \setminus X$ the Stone-Čech boundary of $X$. It was shown in [38, Theorem 5.3] that the coarse groupoid $G(X)$ is amenable if and only if $X$ has Property A.

Each $f \in C_c(G(X))$ is a continuous function supported in $\overline{E_r}$ for some $r > 0$; equivalently, we can interpret $f$ as a bounded continuous function on $E_r$. Define

$$\theta : C_c(G(X)) \to \mathcal{B}(\ell^2(X))$$

by setting the matrix coefficients $\theta(f)_{x,y} := f(x, y)$ for $x, y \in X$, which extends the map defined in (2.5). Hence we take the liberty of using the same notation. It is clear that $\theta$ provides a $\ast$-isomorphism from $C_c(G(X))$ to $\mathcal{B}(\ell^2(X))$. Moreover, it was shown in [36, Proposition 10.29] that $\theta$ extends to a $C^*$-isomorphism

$$\Theta : C^*_r(G(X)) \to C^*_u(X). \quad (2.7)$$

This map extends the isomorphism in (2.6), and hence again we abuse the notation.

2.6.4. Group actions  Let $X$ be a locally compact space and $\Gamma$ be a countable discrete group with an action $\Gamma \curvearrowright X$ by homeomorphisms. The transformation groupoid $X \rtimes \Gamma$ is $X \times \Gamma$ as a topological space, whose unit space is $X \times \{1\}$ and identified with $X$ for simplicity. The groupoid structure is given by $s((x, \gamma)) = \gamma^{-1}x$, $r((x, \gamma)) = x$ and $(x, \gamma)(\gamma^{-1}x, \gamma') = (x, \gamma \gamma')$. Then $X \rtimes \Gamma$ is a locally compact étale groupoid. It is not hard to check that $X \rtimes \Gamma$ is amenable if and only if the action is amenable (see, e.g., [9, Chapter 4]).

Denote by $C_c(\Gamma, C_0(X))$ the $\ast$-algebra consisting of all finitely supported functions on $\Gamma$ with values in $C_0(X)$ (see [9, Section 4.1] for details). The reduced crossed product $C_0(X) \rtimes_r \Gamma$ is its norm closure with respect to a regular representation of $C_c(\Gamma, C_0(X))$ (see [9, Definition 4.1.4]). We have the following $\ast$-isomorphism:

$$\theta : C_c(X \rtimes_r \Gamma) \to C_c(\Gamma, C_c(X))$$

given by

$$F \mapsto \sum_{\gamma \in \Gamma} f_\gamma \gamma, \quad \text{where } f_\gamma \in C_c(X) \text{ is defined by } f_\gamma(x) = F(x, \gamma).$$
It is known that $\theta$ can be extended to a $C^*$-isomorphism:
\[ \Theta : C_r^*(X \rtimes \Gamma) \to C_0(X) \rtimes_r \Gamma. \] (2.8)

Finally, we remark that $L^2(X \rtimes \Gamma) \cong \ell^2(\Gamma) \otimes C_0(X)$.

3. Characterisations for Operators in $\mathcal{L}(L^2(\mathcal{G}))$ and Equivariance

In this section, we aim to provide a detailed and practical approach to describe operators in $\mathcal{L}(L^2(\mathcal{G}))$ and discuss the notion of equivariance, which will play an important role in the sequel. We will start with characterisations for vectors in $L^2(\mathcal{G})$, and then move to operators.

Throughout this section, we assume that $\mathcal{G}$ is a locally compact étale groupoid with unit space $\mathcal{G}^{(0)}$.

3.1. Vectors in $L^2(\mathcal{G})$. Let us start by analysing vectors in the Hilbert $C_0(\mathcal{G}^{(0)})$-module $L^2(\mathcal{G})$ in terms of their “slices”.

Recall from Sect. 2.3 that vectors in $L^2(\mathcal{G})$ can be regarded as continuous functions on $\mathcal{G}$ vanishing at infinity. Hence for each $x \in \mathcal{G}^{(0)}$, we have the restriction map
\[ \phi_x : L^2(\mathcal{G}) \to \ell^2(\mathcal{G}_x) \text{ by } \phi_x(\xi) = \xi |_{\mathcal{G}_x}. \]

**Lemma 3.1.** With the notation as above, the map $\phi_x$ is surjective.

**Proof.** Given $\xi_0 \in C_c(\mathcal{G}_x)$, we assume that $\text{supp}(\xi_0) = \{\gamma_1, \ldots, \gamma_N\}$. Since $\mathcal{G}$ is étale, there exists an open subset $U$ in $\mathcal{G}^{(0)}$ containing $x$ and an open bisection $A_i$ containing $\gamma_i$ for each $i = 1, 2, \ldots, N$ such that $s(A_i) = U$.

Take a function $\rho \in C_c(U) \subseteq C_c(\mathcal{G})$ with range in $[0, 1]$ and value 1 at the given point $x$. Define a function $\xi : \mathcal{G} \to \mathbb{C}$ as follows: for $\gamma \in \mathcal{G}$ we set
\[ \xi(\gamma) := \begin{cases} \xi_0(\gamma_i) \rho(s(\gamma)), & \gamma \in A_i \text{ for } i = 1, 2, \ldots, N; \\ 0, & \text{otherwise}. \end{cases} \]

It is clear that $\xi \in C_c(\mathcal{G})$ and $\phi_x(\xi) = \xi_0$, which implies that $\phi_x(C_c(\mathcal{G})) = C_c(\mathcal{G}_x)$. Since $\phi_x$ is contractive and $C_c(\mathcal{G}_x)$ is dense in $\ell^2(\mathcal{G}_x)$, we conclude the proof. \(\square\)

Collecting $\phi_x$ together, we obtain the following *slicing map*:
\[ \phi : L^2(\mathcal{G}) \to \prod_{x \in \mathcal{G}^{(0)}} \ell^2(\mathcal{G}_x) \text{ by } \phi(\xi) := (\phi_x(\xi))_{x \in \mathcal{G}^{(0)}}, \]
which is clearly an isometric embedding. Here $\prod_{x \in \mathcal{G}^{(0)}} \ell^2(\mathcal{G}_x)$ denotes the Banach space consisting of uniformly norm-bounded families $(\xi_x)_{x \in \mathcal{G}^{(0)}}$ with norm $\|\xi_x\| := \sup_{x \in \mathcal{G}^{(0)}} \|\xi_x\|$.

The following result characterises elements in the image of $\phi$, which essentially comes from $[10, 11, 41]$. For completeness, here we provide a proof.

**Lemma 3.2.** With the same notation as above, let $(\xi_x)_{x \in \mathcal{G}^{(0)}} \in \prod_{x \in \mathcal{G}^{(0)}} \ell^2(\mathcal{G}_x)$ be a family of vectors. Then the following are equivalent:

1. $(\xi_x)_{x \in \mathcal{G}^{(0)}} \in \phi(L^2(\mathcal{G}))$: 


(2) the function $x \mapsto \|\xi_x - \eta|_{G_x}\|$ belongs to $C_0(G^{(0)})$ for all $\eta \in C_c(G)$;
(3) the function $x \mapsto \{\xi_x, \eta|_{G_x}\}$ belongs to $C_0(G^{(0)})$ for all $\eta \in C_c(G)$, and the function $x \mapsto \|\xi_x\|$ belongs to $C_0(G^{(0)})$. 

**Proof.** “(1) $\Rightarrow$ (2)”: Let $\xi \in L^2(G)$ and we consider $(\xi_x)_x = \phi(\xi)$. Note that condition (2) clearly holds when $\xi \in C_c(G)$. Since $C_c(G)$ is dense in $L^2(G)$ with respect to the norm $\|\xi\| = \sup_{x \in G^{(0)}} \|\xi|_{G_x}\|$, condition (2) holds for general $\xi$ by a standard approximating argument.

“(2) $\Rightarrow$ (1)”: Taking $\eta = 0$ in condition (2), we obtain that for any $\varepsilon > 0$ there exists a compact subset $K \subseteq G^{(0)}$ such that $\sup_{x \in G^{(0)} \setminus K} \|\xi_x\| < \varepsilon$. For $x \in K$, it follows from Lemma 3.1 that there exists $\eta^x \in C_c(G)$ such that $\|\xi_x - \eta^x|_{G_x}\| < \varepsilon$. By condition (2), there exists an open neighbourhood $U_x$ of $x$ such that for $y \in U_x$ we have $\|\xi_y - \eta^x|_{G_y}\| < \varepsilon$.

Since $K$ is compact, there exist $x_1, \ldots, x_N \in K$ such that $K \subseteq \bigcup_{i=1}^N U_{x_i}$. Consider the open cover $\mathcal{U} := \{G^{(0)} \setminus K, U_{x_1}, \ldots, U_{x_N}\}$ of $G^{(0)}$ and take a partition of unity $\{\rho_0, \rho_1, \ldots, \rho_N\}$ subordinate to $\mathcal{U}$ such that $\text{supp}(\rho_0) \subseteq G^{(0)} \setminus K$ and $\text{supp}(\rho_i) \subseteq U_{x_i}$ for $i = 1, \ldots, N$. Define

\[ \eta := \sum_{i=1}^N \rho_i \cdot \eta^{x_i}. \]

It is easy to check that $\eta \in C_c(G)$ and $\sup_{x \in G^{(0)}} \|\xi_x - \eta|_{G_x}\| < \varepsilon$. Hence we conclude condition (1).

Finally, notice that “(2) $\Rightarrow$ (3)” follows directly from the polarization identity, and “(3) $\Rightarrow$ (2)” holds trivially. Hence we conclude the proof. \( \square \)

**Remark 3.3.** For those who are familiar with the notion of Hilbert bundles, we remark that the image of $\phi$ generates a continuous field of Hilbert spaces $\mathbb{G}$ over $G^{(0)}$ (see [10, Proposition 10.2.3]), and Lemma 3.2 is designed to show that $L^2(G)$ coincides with elements in $\mathbb{G}$ vanishing at infinity. Furthermore, it follows from [16, Theorem II.13.18] that there is a unique topology on $\bigsqcup_{x \in G^{(0)}} \ell^2(G_x)$ which makes it into a Hilbert bundle over $G^{(0)}$ such that $L^2(G)$ are continuous sections.

### 3.2. Operators in $\mathcal{L}(L^2(G))$.

Having established the characterisation for vectors in $L^2(G)$ (see Lemma 3.2), now we move to the case of operators in $\mathcal{L}(L^2(G))$.

First note that similar to the slicing map $\phi$, we can construct a slicing map on $\mathcal{L}(L^2(G))$. More precisely, for $T \in \mathcal{L}(L^2(G))$ and $x \in G^{(0)}$ we define an operator $\Phi_x(T)$ on $\ell^2(G_x)$ as follows:

\[ \Phi_x(T)(\xi|_{G_x}) := T(\xi)|_{G_x} \quad \text{for} \quad \xi \in L^2(G). \]

It follows from Proposition 2.3 and Lemma 3.1 that $\Phi_x(T)$ is well-defined and belongs to $\mathcal{B}(\ell^2(G_x))$, called the slice of $T$ at $x$. Collecting them together, we define the slicing map:

\[ \Phi : \mathcal{L}(L^2(G)) \longrightarrow \prod_{x \in G^{(0)}} \mathcal{B}(\ell^2(G_x)) \quad \text{by} \quad \Phi(T) := (\Phi_x(T))_{x \in G^{(0)}}, \quad (3.1) \]

where $\prod_{x \in G^{(0)}} \mathcal{B}(\ell^2(G_x))$ denotes the direct product of $C^*$-algebras. It is easy to see that $\Phi$ is a $C^*$-monomorphism, and hence an isometric embedding.
Remark 3.4. As pointed out by the anonymous referee, the maps $\phi_\mathcal{X}$ and $\Phi_\mathcal{X}$ constructed above can also be defined using the language of interior tensor products of Hilbert $C^*$-modules (recalled in Sect. 2.3). More precisely, for $x \in \mathcal{G}^{(0)}$ we consider the evaluation map $ev_x : C_0(\mathcal{G}^{(0)}) \to \mathbb{C}$ given by $ev_x(f) = f(x)$ for $f \in C_0(\mathcal{G}^{(0)})$. Then the interior tensor product $L^2(\mathcal{G}) \otimes_{ev_x} \mathbb{C}$ is a Hilbert space and isomorphic to $\ell^2(\mathcal{G}_\mathcal{X})$. It is not hard to verify that for $\xi \in L^2(\mathcal{G})$ and $\Phi \in \mathcal{L}(L^2(\mathcal{G}))$, the vector $\phi_\mathcal{X}(\xi)$ is nothing but the image of $\xi \otimes 1$ under this isomorphism, and the operator $\Phi_\mathcal{X}(T)$ can be identified with $T \otimes_{ev_x} 1$. However, we stick to the original definitions for $\phi_\mathcal{X}$ and $\Phi_\mathcal{X}$ since the tools of interior tensor products and related theories seem only responsible to simplify the definitions rather than the proofs.

The rest of this subsection is devoted to establishing characterisations for the image of $\Phi$. Our philosophy is that although operators in $\mathcal{L}(L^2(\mathcal{G}))$ are hard to describe, the family of their slices (i.e., the image of $\Phi$) are usually more achievable and easy to handle. This idea will play a crucial role when we study the quasi-locality for groupoids later.

Let us start with the following elementary result:

**Lemma 3.5.** Let $(T_x)_{x \in \mathcal{G}^{(0)}} \in \prod_{x \in \mathcal{G}^{(0)}} \mathcal{B}(\ell^2(\mathcal{G}_\mathcal{X}))$ be a family of operators. Then $(T_x)_{x \in \mathcal{G}^{(0)}}$ belongs to the image of $\Phi$ if and only if for any $\xi \in C_c(\mathcal{G})$, the families $(T_x(\xi |_{\mathcal{G}_\mathcal{X}}))_{x \in \mathcal{G}^{(0)}}$ and $(T^*_x(\xi |_{\mathcal{G}_\mathcal{X}}))_{x \in \mathcal{G}^{(0)}}$ belong to $\Phi(L^2(\mathcal{G}))$.

**Proof.** The necessity is clear, and hence we only focus on the sufficiency. First define a map $T : C_c(\mathcal{G}) \to \ell^2(\mathcal{G})$ by

$$T(\xi) := \phi^{-1}(T_x(\xi |_{\mathcal{G}_\mathcal{X}})) \quad \text{for} \quad \xi \in C_c(\mathcal{G}),$$

which is well-defined by assumption. It follows directly that $\|T(\xi)\| \leq \|T_x\| \cdot \|\xi\|$, which implies that $T$ can be extended to a bounded linear map $L^2(\mathcal{G}) \to L^2(\mathcal{G})$, still denoted by $T$. Similarly, we have a bounded linear map $S : \ell^2(\mathcal{G}) \to L^2(\mathcal{G})$ such that $S(\eta) = \phi^{-1}(T^*_x(\eta |_{\mathcal{G}_\mathcal{X}}))$ for $\eta \in C_c(\mathcal{G})$. For $\xi, \eta \in C_c(\mathcal{G})$, it is clear that $\langle \eta, T(\xi) \rangle = \langle S(\eta), \xi \rangle$. By continuity, we obtain that $S = T^*$ and hence conclude the proof. \qed

Combining Lemma 3.2 with Lemma 3.5, we obtain the following:

**Corollary 3.6.** For $(T_x)_{x \in \mathcal{G}^{(0)}} \in \prod_{x \in \mathcal{G}^{(0)}} \mathcal{B}(\ell^2(\mathcal{G}_\mathcal{X}))$, the following are equivalent:

1. $(T_x)_{x \in \mathcal{G}^{(0)}} \in \Phi(\mathcal{L}(L^2(\mathcal{G})))$;
2. the functions $x \mapsto \|T_x(\xi |_{\mathcal{G}_\mathcal{X}}) - \eta |_{\mathcal{G}_\mathcal{X}}\|$ and $x \mapsto \|T^*_x(\xi |_{\mathcal{G}_\mathcal{X}}) - \eta |_{\mathcal{G}_\mathcal{X}}\|$ belong to $C_0(\mathcal{G}^{(0)})$ for all $\xi, \eta \in C_c(\mathcal{G})$;
3. the functions $x \mapsto \langle \eta |_{\mathcal{G}_\mathcal{X}}, T_x(\xi |_{\mathcal{G}_\mathcal{X}}) \rangle$, $x \mapsto \|T_x(\xi |_{\mathcal{G}_\mathcal{X}})\|$ and $x \mapsto \|T^*_x(\xi |_{\mathcal{G}_\mathcal{X}})\|$ belong to $C_0(\mathcal{G}^{(0)})$ for all $\xi, \eta \in C_c(\mathcal{G})$.

Corollary 3.6 does provide a description for operators in $\mathcal{L}(L^2(\mathcal{G}))$. However, either condition (2) or (3) is still not easy to verify. We would like to explore a more delicate characterisation for elements in the image of $\Phi$. To fulfil the task, we would like to consult the machinery of operator fibre spaces introduced by Austin and the second-named author in [4, Section 3].

Consider the following disjoint union:

$$E := \bigsqcup_{x \in \mathcal{G}^{(0)}} \mathcal{B}(\ell^2(\mathcal{G}_\mathcal{X})). \quad (3.2)$$
We write $\sigma_x$ for an element in $\mathcal{B}(\ell^2(G_x)) \subseteq E$ to indicate the fibre in which it lives.

Following [4, Section 3], we endow a topology on $E$ as follows: a net $\{\sigma_{x_i}\}_{i \in I}$ converges to $\sigma_x$ in $E$ if and only if $x_i \to x$ in $G^{(0)}$ and for any $\gamma' \to \gamma'$, $\gamma'' \to \gamma''$ in $G$ with $s(\gamma'_i) = x_i = s(\gamma''_i)$, we have

$$\{\delta_{\gamma''_i}, \sigma_{x_i} (\delta_{\gamma'_i})\} \to \{\delta_{\gamma''}, \sigma_x (\delta_{\gamma'})\}.$$

**Definition 3.7.** [4, Definition 3.1] For a locally compact étale groupoid $G$, the space $E$ defined in (3.2) equipped with the above topology is called the operator fibre space associated to $G$.

A section of $E$ is a map $\sigma : G^{(0)} \to E$ such that $\sigma(x) \in \mathcal{B}(\ell^2(G_x))$ for $x \in G^{(0)}$. Denote by $\Gamma_b(E)$ the set of continuous sections $\sigma$ of $E$ with $\sup_{x \in G^{(0)}} \||\sigma(x)|| < \infty$. Equipped with pointwise operations and norm $\|\sigma\| := \sup_{x \in G^{(0)}} \|\sigma(x)\|$, $\Gamma_b(E)$ becomes a $C^*$-algebra. By definition, we have the inclusion map

$$\iota : \Gamma_b(E) \hookrightarrow \prod_{x \in G^{(0)}} \mathcal{B}(\ell^2(G_x)), \quad \sigma \mapsto (\sigma(x))_{x \in G^{(0)}}. \tag{3.3}$$

The following lemma characterises continuous sections in terms of their slices, which can be deduced easily from the definition, hence we omit the proof.

**Lemma 3.8.** For $(\sigma_x)_{x \in G^{(0)}} \in \prod_{x \in G^{(0)}} \mathcal{B}(\ell^2(G_x))$, the following are equivalent:

1. $(\sigma_x)_{x \in G^{(0)}} \in \iota(\Gamma_b(E));$
2. the function $x \mapsto \{\eta|_{G_x}, \sigma_x(\xi|_{G_x})\}$ belongs to $C_c(G^{(0)})$ for all $\xi, \eta \in C_c(G)$;
3. the function $\gamma \mapsto \bigl(\sigma_{s(\gamma)}(\xi|_{G_{s(\gamma)}})\bigr)$ is continuous on $G$ for all $\xi \in C_c(G)$;
4. the function $\gamma \mapsto \bigl(\sigma^*_s(\xi|_{G_{s(\gamma)}})\bigr)$ is continuous on $G$ for all $\xi \in C_c(G)$.

Combining Corollary 3.6 with Lemma 3.8, we obtain that $\Phi \bigl(\mathcal{L}(L^2(G))\bigr) \subseteq \iota(\Gamma_b(E))$. In other words, the homomorphism $\Phi$ factors through $\Gamma_b(E)$, i.e., there exists a $C^*$-homomorphism $\Phi : \mathcal{L}(L^2(G)) \to \Gamma_b(E)$ such that the following diagram commutes:

$$\xymatrix{ \mathcal{L}(L^2(G)) \ar[r]^\Phi \ar_{\Phi} [d] & \prod_{x \in G^{(0)}} \mathcal{B}(\ell^2(G_x)) \ar[l]_{\iota} \ar_{\iota}[d] \ar_{\iota}[d] }$$

**Remark 3.9.** We remark that for a general element $T \in \mathcal{L}(L^2(G))$, it was pointed out by Exel [14] that the map $x \mapsto \|\Phi_x(T)\|$ on $G^{(0)}$ might not be continuous. However, Corollary 3.6 and Lemma 3.8 indicate that the map $x \mapsto \Phi_x(T)$ is indeed continuous with respect to the topology in Definition 3.7. That is the reason for us to consult the notion of operator fibre spaces.

Thanks to Lemma 3.8, it remains to determine the image of $\widetilde{\Phi}$. The following result fulfils the task:

**Proposition 3.10.** Let $G$ be a locally compact étale groupoid, and $E$ be the associated operator fibre space. For $\sigma \in \Gamma_b(E)$, the following are equivalent:

1. $\sigma \in \widetilde{\Phi} \bigl(\mathcal{L}(L^2(G))\bigr);$
for any $\varepsilon > 0$ and $x \in C_c(\mathcal{G})$, there exists a compact subset $K \subseteq \mathcal{G}$ such that for any $x \in \mathcal{G} \subseteq \mathcal{G}$ and $A_x \subseteq \mathcal{G}$ with $A_x \cap K = \emptyset$, we have

$$\|\chi_{A_x} \cdot (\sigma(x)(\xi|g_x))\| \leq \varepsilon \quad \text{and} \quad \|\chi_{A_x} \cdot (\sigma(x)\ast(\xi|g_x))\| \leq \varepsilon; \quad (3.4)$$

(3) for any $\varepsilon > 0$ and $x \in C_c(\mathcal{G})$, there exists a compact subset $K \subseteq \mathcal{G}$ such that for any $x \in \mathcal{G}(0)$ and $A_x, B_x \subseteq \mathcal{G}$ with $A_x \cap (KB_x) = \emptyset$, we have

$$\|\chi_{A_x} \cdot (\sigma(x)(\xi|g_x))\| \leq \varepsilon \quad \text{and} \quad \|\chi_{A_x} \cdot (\sigma(x)\ast(\xi|g_x))\| \leq \varepsilon;$$

Proof. To save notation, we write $\sigma$, instead of $\sigma(x)$ where $x \in \mathcal{G}(0)$ for the given section $\sigma \in \Gamma_h(E)$.

“(1) \Rightarrow (2)” By assumption, there exists $T \in \mathcal{L}(L^2(\mathcal{G}))$ such that $T = \Phi(T).$ For any $x \in C_c(\mathcal{G}),$ we have $T(x) \in L^2(\mathcal{G})$ and $T_\ast(x) \in L^2(\mathcal{G}).$ Hence for any $\varepsilon > 0$, there exist $\eta, \zeta \in C_c(\mathcal{G})$ such that $\|T(x) - \eta\| < \varepsilon$ and $\|T_\ast(x) - \zeta\| < \varepsilon.$

Take $K := \text{supp}(\eta) \cup \text{supp}(\zeta).$ For any $x \in \mathcal{G}(0)$ and $A_x \subseteq \mathcal{G}$ with $A_x \cap K = \emptyset$, we obtain:

$$\|\chi_{A_x} \cdot (\sigma(x)(\xi|g_x))\| = \|\chi_{A_x} \cdot (T(x)|g_x)\| \leq \|\chi_{A_x} \cdot (\eta|g_x)\| + \varepsilon = \varepsilon.$$  

Similarly, we obtain $\|\chi_{A_x} \cdot (\sigma(x)(\xi|g_x))\| < \varepsilon.$

“(2) \Rightarrow (1)” By assumption, for any $\varepsilon > 0$ and $x \in C_c(\mathcal{G})$, there exists a compact subset $K \subseteq \mathcal{G}$ satisfying condition (2). Take a function $\rho \in C_c(\mathcal{G})$ such that $\rho|_K \equiv 1.$

Consider $\xi : \mathcal{G} \to \mathbb{C}$ defined by $\xi(\gamma) := \rho(\gamma) \cdot \left(\sigma_\mathcal{G}(\gamma)(\xi|g_{\mathcal{G}(\gamma)}))\right)(\gamma)$ for $\gamma \in \mathcal{G}.$ It follows from Lemma 3.8 that $\xi \in C_c(\mathcal{G}).$ Moreover, for $x \in \mathcal{G}(0)$ we have:

$$\|\sigma(x)(\xi|g_x) - \phi(x)\| = \|(1 - \rho)|g_x \cdot (\sigma(x)(\xi|g_x))\| = \|(1 - \rho)|g_x \cdot \chi_{g_x \setminus K} \cdot (\sigma(x)(\xi|g_x))\|,$$

which is less than $\varepsilon$ by the assumption (3.4). Letting $\varepsilon \to 0$, we obtain that $(\sigma(x)(\xi|g_x))_{x \in \mathcal{G}(0)}$ belongs to $\phi(L^2(\mathcal{G})).$ Similarly, $(\sigma(x)(\xi|g_x))_{x \in \mathcal{G}(0)}$ belongs to $\phi(L^2(\mathcal{G})).$ Therefore, we conclude condition (1) thanks to Lemma 3.5.

“(2) \Rightarrow (3)” As remarked in the last paragraph of Sect. 2.2, it suffices to prove condition (3) for $x \in C_c(\mathcal{G})$ whose support is contained in a compact bisection $K$. Given $\varepsilon > 0$, take a compact subset $K \subseteq \mathcal{G}$ satisfying condition (2). We set $\tilde{K} := KK^{-1}.$ Since supp($\xi$) is a bisection, it follows that for any $x \in \mathcal{G}(0)$ and $B_x \subseteq \mathcal{G}$ we have:

$$\chi_{B_x}(\xi|g_x) = \begin{cases} \xi|g_x, & \text{if } K \cap B_x \neq \emptyset; \\ 0, & \text{if } K \cap B_x = \emptyset. \end{cases}$$

Now for any $x \in \mathcal{G}(0)$ and $A_x, B_x \subseteq \mathcal{G}$ with $A_x \cap (\tilde{K}B_x) = \emptyset$, then $A_x \cap (KK^{-1}B_x) = \emptyset.$ If $K \cap B_x = \emptyset$, then:

$$\|\chi_{A_x}(x)(\xi|g_x)\| = \|\chi_{A_x}(\sigma(x)(\chi_{B_x}(\xi|g_x))\| = 0$$

and similarly, $\|\chi_{A_x}(\sigma_\mathcal{G}(\chi_{B_x}(\xi|g_x))\| = 0.$ If $K \cap B_x \neq \emptyset$, then

$$KK^{-1}B_x \supseteq K \cdot \{x\} = K \cap \mathcal{G}.$$
Hence $A_x \cap K = A_x \cap (K \cap \mathcal{G}_x) \subseteq A_x \cap (K K^{-1}_x) = \emptyset$. Applying condition (2), we obtain:

$$\| (\chi_{A_x} \sigma_x \chi_{B_x})(\xi |_{\mathcal{G}_x}) \| = \| (\chi_{A_x} \sigma) (\chi_{B_x} \xi |_{\mathcal{G}_x}) \| = \| \chi_{A_x} \cdot (\sigma_x (\xi |_{\mathcal{G}_x})) \| < \varepsilon.$$ 

Similarly, we have $\| (\chi_{A_x} \sigma_x^* \chi_{B_x})(\xi |_{\mathcal{G}_x}) \| < \varepsilon$. Therefore, we conclude condition (3).

"(3) \Rightarrow (2)". Given $\varepsilon > 0$ and $\xi \in C_c(\mathcal{G})$, let $K_\xi := \text{supp}(\xi)$. By condition (3), there exists a compact subset $\tilde{K} \subseteq \mathcal{G}$ satisfying the requirement therein. We set $K := \tilde{K} K_\xi$ and take $B_x = \text{supp}(\xi |_{\mathcal{G}_x})$. For any $x \in \mathcal{G}^{(0)}$ and $A_x \subseteq \mathcal{G}_x$ with $A_x \cap K = \emptyset$, we have

$$A_x \cap (\tilde{K} B_x) = A_x \cap (\tilde{K} \text{supp}(\xi |_{\mathcal{G}_x})) \subseteq A_x \cap (\tilde{K} K_\xi) = A_x \cap K = \emptyset.$$ 

Hence we obtain:

$$\| \chi_{A_x} \cdot (\sigma_x (\xi |_{\mathcal{G}_x})) \| = \| (\chi_{A_x} \sigma_x \chi_{B_x})(\xi |_{\mathcal{G}_x}) \| < \varepsilon$$

and similarly, $\| \chi_{A_x} \cdot (\sigma_x^* (\xi |_{\mathcal{G}_x})) \| < \varepsilon$. Therefore, we conclude the proof. \( \Box \)

**Remark 3.11.** Careful readers might already notice from Corollary 3.6(3) and Lemma 3.8(2) that in order to show that a continuous section $\sigma \in \Gamma_b(E)$ belongs to the image of $\Phi$, it suffices to show that the functions $x \mapsto \| \sigma(x)(\xi |_{\mathcal{G}_x}) \|$ and $x \mapsto \| \sigma(x)^*(\xi |_{\mathcal{G}_x}) \|$ belong to $C_0(\mathcal{G}^{(0)})$ for any $\xi \in C_c(\mathcal{G})$. In fact, one can verify these conditions directly using condition (2) in Proposition 3.10, which suggests an alternative proof for 

"(2) \Rightarrow (1)".

Comparing with the notion of quasi-locality recalled in Definition 2.10, we note that condition (3) in Proposition 3.10 can be regarded as a vector-wise uniform version of quasi-locality. Hence we introduce the following:

**Definition 3.12.** Let $\mathcal{G}$ be a locally compact étale groupoid with unit space $\mathcal{G}^{(0)}$. A family $(T_x)_{x \in \mathcal{G}^{(0)}} \in \prod_{x \in \mathcal{G}^{(0)}} \mathcal{B}(L^2(\mathcal{G}_x))$ is said to be vector-wise uniformly quasi-local if for any $\varepsilon > 0$ and $\xi \in C_c(\mathcal{G})$, there exists a compact subset $K \subseteq \mathcal{G}$ such that for any $x \in \mathcal{G}^{(0)}$ and $A_x, B_x \subseteq \mathcal{G}_x$ with $A_x \cap (K B_x) = \emptyset$, we have

$$\| (\chi_{A_x} T_x \chi_{B_x})(\xi |_{\mathcal{G}_x}) \| < \varepsilon.$$ 

Combining Lemma 3.8 with Proposition 3.10, we finally reach the following:

**Theorem 3.13.** Let $\mathcal{G}$ be a locally compact étale groupoid with unit space $\mathcal{G}^{(0)}$. For $(T_x)_{x \in \mathcal{G}^{(0)}} \in \prod_{x \in \mathcal{G}^{(0)}} \mathcal{B}(L^2(\mathcal{G}_x))$, the following are equivalent:

1. $(T_x)_{x \in \mathcal{G}^{(0)}}$ belongs to $\Phi(\mathcal{L}(L^2(\mathcal{G})))$;
2. the map $x \mapsto T_x$ is a continuous section of $E$, and $(T_x)_{x \in \mathcal{G}^{(0)}}$, $(T^*_x)_{x \in \mathcal{G}^{(0)}}$ are vector-wise uniformly quasi-local.

### 3.3. The case of dense subsets.

A lot of examples including the coarse groupoids (see Sect. 2.6.3) come with natural dense open subsets in their unit spaces. In these cases, we now show that the characterisations established in Sect. 3.2 can be further simplified.

Let $\mathcal{G}$ be a locally compact étale groupoid with unit space $\mathcal{G}^{(0)}$. Assume that $X$ is a dense subset of $\mathcal{G}^{(0)}$, and we set $\partial X := \mathcal{G}^{(0)} \setminus X$. Define

$$E_X := \bigcup_{x \in X} \mathcal{B}(L^2(\mathcal{G}_x)) \subseteq E.$$
with the induced topology from $E$, and denote by $\Gamma_b(\mathcal{E}_X)$ the $C^*$-algebra consisting of all continuous norm-bounded sections of $\mathcal{E}_X$. Consider the restriction map

$$\text{Res} : \Gamma_b(E) \to \Gamma_b(\mathcal{E}_X), \quad \sigma \mapsto \sigma|_X.$$ 

Since $X$ is dense and $\mathcal{G}$ is étale, it is easy to verify that the map $\text{Res}$ is injective. To determine its image, we introduce the following:

**Definition 3.14.** A section $\sigma \in \Gamma_b(\mathcal{E}_X)$ is called extendable if for any $\omega \in \partial X$ and $\gamma', \gamma'' \in \mathcal{G}_\omega$, there exists a constant $c_{\gamma', \gamma''} \in \mathbb{C}$ such that for any $x_i \in X$ with $x_i \to \omega$ and any $\gamma'_i, \gamma''_i \in \mathcal{G}_{x_i}$ with $\gamma'_i \to \gamma'$ and $\gamma''_i \to \gamma''$, then $(\delta_{\gamma''_i}, \sigma(x_i)(\delta_{\gamma'_i})) \to c_{\gamma', \gamma''}$.

The following lemma explains the terminology:

**Lemma 3.15.** The map $\text{Res} : \Gamma_b(E) \to \Gamma_b(\mathcal{E}_X)$ is a $C^*$-monomorphism with image consisting of all extendable sections in $\Gamma_b(\mathcal{E}_X)$.

**Proof.** It suffices to show that an extendable section $\sigma \in \Gamma_b(\mathcal{E}_X)$ can be extended to a section $\tilde{\sigma} \in \Gamma_b(E)$. For $\omega \in \partial X$, define an operator $\sigma_\omega : \ell^2(\mathcal{G}_\omega) \to \ell^2(\mathcal{G}_\omega)$ by

$$\langle \delta_{\gamma''}, \sigma_\omega(\delta_{\gamma'}) \rangle := \lim_i \langle \delta_{\gamma''_i}, \sigma(x_i)(\delta_{\gamma'_i}) \rangle \quad \text{for} \quad \gamma', \gamma'' \in \mathcal{G}_\omega,$$

where $x_i$ is a net in $X$ converging to $\omega$, and $\gamma'_i, \gamma''_i \in \mathcal{G}_{x_i}$ with $\gamma'_i \to \gamma'$ and $\gamma''_i \to \gamma''$. This is well-defined since $\mathcal{G}$ is étale and $\sigma$ is extendable. To show that $\sigma_\omega$ is a bounded operator, we note that

$$\|\sigma_\omega\| = \sup \left\{ \|\langle \eta_\omega, \sigma_\omega(\xi_\omega) \rangle\| : \xi_\omega, \eta_\omega \in C_c(\mathcal{G}_\omega) \text{ with } \|\xi_\omega\| = \|\eta_\omega\| = 1 \right\}.$$

Hence using the étaleness and $\sup_{x \in X} \|\sigma(x)\| < \infty$, we obtain that $\sigma_\omega$ is bounded (details are left to readers). Finally, we define a map $\tilde{\sigma} : \mathcal{G}^{(0)} \to E$ by

$$\tilde{\sigma} (z) := \begin{cases} \sigma(z), & \text{if } z \in X; \\ \sigma_\omega(z), & \text{if } z \in \partial X, \end{cases}$$

which clearly belongs to $\Gamma_b(E)$ and extends $\sigma$. Hence we conclude the proof. \hfill \qed

The following result shows that in some special cases, sections in $\Gamma_b(\mathcal{E}_X)$ are always extendable. First recall that for a locally compact Hausdorff space $X$, its Stone-Čech compactification is a compact space $\beta X$ together with a continuous map $i : X \to \beta X$ such that $i : X \to i(X)$ is a homeomorphism with dense image in $\beta X$ and satisfies the universal property: for any $f \in C_b(X)$ there exists a continuous map $f^\beta : \beta X \to \mathbb{C}$ such that $f^\beta \circ i = f$. Note that in this case, $i(X)$ is open in $\beta X$.

**Lemma 3.16.** Let $\mathcal{G}$ be a locally compact étale groupoid with unit space $\mathcal{G}^{(0)}$, and $X$ be an open dense subset in $\mathcal{G}^{(0)}$ such that $\mathcal{G}^{(0)} = \beta X$ is the Stone-Čech compactification of $X$ (with the inclusion map). Then any section $\sigma \in \Gamma_b(\mathcal{E}_X)$ is extendable.

**Proof.** Fix a section $\sigma \in \Gamma_b(\mathcal{E}_X)$, $\omega \in \partial X$ and $\gamma', \gamma'' \in \mathcal{G}_\omega$. Using étaleness of $\mathcal{G}$, we can find two open neighborhoods $U_{\gamma'}$ and $U_{\gamma''}$ of $\gamma'$ and $\gamma''$, respectively, such that $s|_{U_{\gamma'}}$ and $s|_{U_{\gamma''}}$ are homeomorphisms. Let $U_\omega := s(U_{\gamma'}) \cap s(U_{\gamma''})$, and we take $\rho \in C_b(\mathcal{G}^{(0)})$ such that $\rho(\omega) = 1$ and $\rho|_{\mathcal{G}^{(0)} \setminus U_\omega} = 0$. Consider the function $f_{\gamma', \gamma''} : X \to \mathbb{C}$ defined by

$$f_{\gamma', \gamma''}(x) := \begin{cases} \rho(x)(\delta_{\gamma''}, \sigma(x)(\delta_{\gamma'})), & \text{if } x \in X \cap U_\omega; \\ 0, & \text{otherwise}, \end{cases}$$

where $\delta_{\gamma''} \in \mathcal{G}_\omega$. This function $f_{\gamma', \gamma''}$ is extendable by Lemma 3.15. Now, we define a map $\tilde{\sigma} : \mathcal{G}^{(0)} \to E$ by

$$\tilde{\sigma}(x) := \begin{cases} f_{\gamma'_i, \gamma''_i}(x), & \text{if } x \in X \cap U_\omega; \\ \sigma(x), & \text{if } x \in \partial X, \end{cases}$$

for some net $x_i \in X$ converging to $\omega$. Note that $\tilde{\sigma}$ is extendable since $\mathcal{G}$ is étale and $\sigma$ is extendable. Hence we conclude the proof. \hfill \qed
where $\gamma'_x \in U_{\gamma'} \cap G_x$ and $\gamma''_x \in U_{\gamma''} \cap G_x$ are uniquely determined. It is clear that $f_{\gamma', \gamma''} \in C_b(X)$. Thanks to the universal property of $\beta X = G^{(0)}$, $f_{\gamma', \gamma''}$ can be uniquely extended to a function $f_{\gamma', \gamma''}^\beta \in C(G^{(0)})$. Setting $c_{\gamma', \gamma''} := f_{\gamma', \gamma''}^\beta(\omega)$, we conclude that $\sigma$ is extendable.

Combining Lemma 3.15 and 3.16, we obtain:

**Corollary 3.17.** Let $G$ be a locally compact étale groupoid with unit space $G^{(0)}$, and $X$ be an open dense subset in $G^{(0)}$ such that $G^{(0)} = \beta X$ is the Stone-Čech compactification of $X$ (with the inclusion map). Then the map $\text{Res} : \Gamma_b(E) \to \Gamma_b(E_X)$ is a $C^\ast$-isomorphism.

Now we are in the position to simplify Proposition 3.10:

**Proposition 3.18.** Let $G$ be a locally compact étale groupoid with unit space $G^{(0)}$, and $X$ be a dense subset of $G^{(0)}$. For an extendable section $\sigma \in \Gamma_b(E_X)$, the following are equivalent:

1. $\sigma \in \text{Res} \circ \Phi \left( L^2(G) \right)$;
2. for any $\varepsilon > 0$ and $\xi \in C_c(G)$, there exists a compact subset $K \subseteq G$ such that for any $x \in X$ and $A_x \subseteq G_x$ with $A_x \cap K = \emptyset$, we have
   \[ \| \chi_{A_x} \cdot (\sigma(x)(\xi|\mathcal{G}_x)) \| < \varepsilon \quad \text{and} \quad \| \chi_{A_x} \cdot (\sigma(x)^\ast(\xi|\mathcal{G}_x)) \| < \varepsilon; \]
3. for any $\varepsilon > 0$ and $\xi \in C_c(G)$, there exists a compact subset $K \subseteq G$ such that for any $x \in X$ and $A_x$, $B_x \subseteq G_x$ with $A_x \cap (KB_x) = \emptyset$, we have
   \[ \| \chi_{A_x} \cdot (\sigma(x)(\xi|\mathcal{G}_x)) \| < \varepsilon \quad \text{and} \quad \| \chi_{A_x} \cdot (\sigma(x)^\ast(\xi|\mathcal{G}_x)) \| < \varepsilon; \]

**Proof.** From Lemma 3.15, there exists $\tilde{\sigma} \in \Gamma_b(E)$ such that $\text{Res}(\tilde{\sigma}) = \sigma$. Hence it suffices to show that condition (2) and (3) for $\sigma$ are equivalent to those in Proposition 3.18 for $\tilde{\sigma}$, respectively. Again to save the notation, we write $\tilde{\sigma}_x$ instead of $\tilde{\sigma}(x)$ for $x \in G^{(0)}$.

We only present the case for condition (2), while the other is similar. Assume that condition (2) holds for $\sigma$, we would like to verify Proposition 3.18(2) for $\tilde{\sigma}$. Given $\varepsilon > 0$ and $\xi \in C_c(G)$, let $K \subseteq G$ be a compact subset satisfying condition (2). For any $y \in G^{(0)}$ and any finite subset $A_y \subseteq \mathcal{G}_y$ with $A_y \cap K = \emptyset$, there exist $x \in X$ and a finite subset $A_x \subseteq G_x$ such that $A_x \cap K = \emptyset$ and

\[ \| \langle \tilde{\sigma}_y(\xi|\mathcal{G}_y), \eta|A_y \rangle - \langle \sigma_x(\xi|\mathcal{G}_x), \eta|A_x \rangle \| < \varepsilon \]

for any $\eta \in C_c(G)$ with $\| \eta \| \leq 1$, since $G$ is étale and $\tilde{\sigma}$ is the extension of $\sigma$. Note that

\[ \| \chi_{A_x}(\sigma_x(\xi|\mathcal{G}_x)) \| < \varepsilon \quad \text{and} \quad \| \chi_{A_y}(\tilde{\sigma}_y(\xi|\mathcal{G}_y)) \| \leq \sup_{\| \eta \|_G = 1} \| \langle \tilde{\sigma}_y(\xi|\mathcal{G}_y), \eta|A_y \rangle \|. \]

Hence we obtain $\| \chi_{A_y}(\tilde{\sigma}_y(\xi|\mathcal{G}_y)) \| < 2\varepsilon$, and similarly $\| \chi_{A_y}(\tilde{\sigma}_y^\ast(\xi|\mathcal{G}_y)) \| < 2\varepsilon$.

Finally using a standard approximating argument, the above estimates hold for arbitrary $A_y \subseteq \mathcal{G}_y$ with $A_y \cap K = \emptyset$, and therefore we conclude the proof.

 Analogous to Definition 3.12, for a dense subset $X$ of $G^{(0)}$ we say that a family $(T_x)_{x \in X} \in \prod_{x \in X} \mathcal{B}(\ell^2(\mathcal{G}_x))$ is vector-wise uniformly quasi-local if for any $\varepsilon > 0$ and $\xi \in C_c(G)$, there exists a compact subset $K \subseteq G$ such that for any $x \in X$ and $A_x$, $B_x \subseteq G_x$ with $A_x \cap (KB_x) = \emptyset$, we have $\| (\chi_{A_x}T_x\chi_{B_x})(\xi|\mathcal{G}_x) \| < \varepsilon$.

Finally, we conclude the following for the case of dense subsets:
Corollary 3.19. Let $G$ be a locally compact étale groupoid with unit space $G^{(0)}$, and $X$ be a dense subset of $G^{(0)}$. For $(T_x)_{x \in X} \in \prod_{x \in X} \mathcal{B}(\ell^2(G_x))$, the following are equivalent:

1. $(T_x)_{x \in X}$ belongs to $\text{Res} \circ \Phi(\mathcal{L}(L^2(G)))$;
2. the map $x \mapsto T_x$ is a continuous extendable section of $E_X$, and $(T_x)_{x \in X}$ are vector-wise uniformly quasi-local.

Moreover, we can omit the requirement of extendableness in condition (2) when $G^{(0)}$ is the Stone-Čech compactification of $X$ (with the inclusion map).

3.4. Equivariant operators. In this subsection, we study an extra condition for operators in $\mathcal{L}(L^2(G))$ called $G$-equivariance. It turns out that all operators in the reduced groupoid $C^*$-algebra $C_r^*(G)$ satisfy this condition, which suggests us to restrict ourselves to the equivariant case when studying quasi-locality in Sect. 4. This class of operators possesses a huge advantage since they can always be represented by certain continuous functions (see Sect. 3.5), which will play a key role in the proof of our main theorem.

Recall that the left regular representation $\Lambda : C_c(G) \rightarrow \mathcal{L}(L^2(G))$ is defined in Sect. 2.3. As for groups, we can also consider the right regular representation as follows. Given a function $g \in C_c(G)$, define the map $\rho(g) : C_c(G) \rightarrow C_c(G)$ by

$$(\rho(g)\xi)(\gamma) := (\xi \ast g)(\gamma) = \sum_{\beta \in G_\gamma} g(\beta)\xi(\gamma\beta^{-1}) = \sum_{\alpha \in G_\gamma} g(\alpha^{-1}\gamma)\xi(\alpha) \text{ for } \xi \in C_c(G).$$

Note that in general, $\rho(g)$ might not be a $C_0(G^{(0)})$-module homomorphism. However, we have the following:

Lemma 3.20. For $g \in C_c(G)$, the operator $\rho(g)$ can be extended to a bounded linear operator on $L^2(G)$ (regarded as a Banach space), still denoted by $\rho(g)$.

Proof. For any $\xi \in C_c(G)$, we have:

$$\|\rho(g)\xi\|^2 = \sup_{x \in G^{(0)}} \sum_{\gamma \in G_x} |\sum_{\beta \in G_x} g(\beta)\xi(\gamma\beta^{-1})|^2 \leq \sup_{x \in G^{(0)}} \sum_{\gamma \in G_x} (\sum_{\beta \in G_x} |g(\beta)|) \cdot (\sum_{\beta \in G_x} |g(\beta)| \cdot |\xi(\gamma\beta^{-1})|^2) = \sup_{x \in G^{(0)}} (\sum_{\beta \in G_x} |g(\beta)|) \cdot \sum_{\gamma \in G_x} (|g(\beta)| \cdot |\xi(\gamma\beta^{-1})|^2) \leq (\sup_{x \in G^{(0)}} \sum_{\beta \in G_x} |g(\beta)|)^2 \cdot \|\xi\|^2.$$

Note that $g \in C_c(G)$, hence the number $\sup_{x \in G^{(0)}} \sum_{\beta \in G_x} |g(\beta)|$ is finite. Therefore, $\rho(g)$ can be extended to a bounded operator on $L^2(G)$.

As a result, the convolution operator can also be extended to an operator (with the same notation) $\ast : L^2(G) \times C_c(G) \rightarrow L^2(G)$ by the same formula (2.1), and $\rho(g)\xi = \xi \ast g$ for $g \in C_c(G)$ and $\xi \in L^2(G)$.

Definition 3.21. Let $G$ be a locally compact étale groupoid. We say that $T \in \mathcal{L}(L^2(G))$ is $G$-equivariant, if $T\rho(g) = \rho(g)T$ (as bounded operators on $L^2(G)$) for any $g \in$

---

3 Warning: Note that our definition of $G$-equivariant operators is not the same as the standard one considered in the literature, e.g. [23, Definition 4.6].
$C_c(G)$. Denote by $\mathcal{L}(L^2(G))^G$ the set of all $G$-equivariant operators in $\mathcal{L}(L^2(G))$, which clearly forms a $C^*$-subalgebra in $\mathcal{L}(L^2(G))$.

The following result shows that operators in $C_c^*(G)$ are $G$-equivariant. The proof is straightforward, hence omitted.

**Lemma 3.22.** Given $f \in C_c(G)$, the operator $\Lambda(f)$ is $G$-equivariant, where $\Lambda$ is the left regular representation of $G$. Thus, we have $C_c^*(G) \subseteq \mathcal{L}(L^2(G))^G$.

Using the slicing map $\Phi$ defined in (3.1), we can characterise $G$-equivariance in terms of their slices. To start, for $\gamma \in G$ we consider the operator

$$V_\gamma : \ell^2(G s(\gamma)) \to \ell^2(G t(\gamma))$$

by

$$V_\gamma(\xi)(\gamma') = \xi(\gamma' \gamma)$$

for $\xi \in \ell^2(G s(\gamma))$. We have the following:

**Lemma 3.23.** Let $G$ be a locally compact étale groupoid. An operator $T \in \mathcal{L}(L^2(G))$ is $G$-equivariant if and only if $V_\gamma \Phi_{s(\gamma)}(T) = \Phi_{t(\gamma)}(T)V_\gamma$ for all $\gamma \in G$.

**Proof.** For the necessity, we fix $\gamma \in G$ and denote $x = s(\gamma)$, $y = r(\gamma)$. By étaleness, choose a pre-compact neighbourhood $U$ of $\gamma^{-1}$ such that $s|_U : U \to s(U)$ is a homeomorphism. Choose $g \in C_c(G)$ such that $g(\gamma^{-1}) = 1$ and $g(\alpha) = 0$ for $\alpha \in G \setminus U$. For any $\eta \in L^2(G)$ and $\gamma' \in G$, note that

$$(\rho(g)\eta)(\gamma') = \sum_{\alpha \in G t(\gamma')} g(\alpha^{-1} \gamma') \eta(\alpha) = \eta(\gamma' \gamma) = V_\gamma(\eta|_{G_x})(\gamma'),$$

which implies that $(\rho(g)\eta)|_{G_x} = V_\gamma(\eta|_{G_x})$. Hence for any $\xi \in L^2(G)$, we obtain

$$(V_\gamma \Phi_x(T))(\xi|_{G_x}) = V_\gamma(T(\xi)|_{G_x}) = (\rho(g)T\xi)|_{G_x} = (T\rho(g)\xi)|_{G_x} = \Phi_\gamma(T)((\rho(g)\xi)|_{G_x}) = (\Phi_\gamma(T)V_\gamma)(\xi|_{G_x}).$$

For the sufficiency, we need to show that $\rho(g)T = T\rho(g)$ for any $g \in C_c(G)$. Without loss of generality, we can assume that $\text{supp}(g)$ is a bisection (see Sect. 2.2). Given $\xi \in L^2(G)$ and $\gamma \in G$ with $s(\gamma) = x$ and $r(\gamma) = y$, the intersection $G_x \cap \text{supp}(g)$ contains at most one point. If $G_x \cap \text{supp}(g) = \{\gamma\}$, then

$$(\rho(g)T\xi)(\gamma) = \sum_{\beta \in G_x} g(\beta) \cdot (T\xi)(\gamma \beta^{-1}) = g(\gamma) \cdot (T\xi)(\gamma \gamma^{-1}) = g(\gamma) \cdot (V_{\gamma^{-1}} \Phi_{r(\gamma)}(T))(\xi|_{G_{r(\gamma)}})(\gamma)$$

$$= g(\gamma) \cdot ((\Phi_x(T)V_{\gamma^{-1}})(\xi|_{G_{r(\gamma)}}))(\gamma),$$

where we use $V_{\gamma^{-1}} \Phi_{r(\gamma)}(T) = \Phi_x(T)V_{\gamma^{-1}}$ in the last equality. On the other hand, note that for any $\gamma' \in G_x$ we have

$$(\rho(g)\xi)(\gamma') = \sum_{\beta \in G_x} g(\beta) \cdot \xi(\gamma' \beta^{-1}) = g(\gamma) \cdot \xi(\gamma' \gamma^{-1}) = g(\gamma) \cdot V_{\gamma^{-1}}(\xi|_{G_{r(\gamma)}})(\gamma').$$

Hence we obtain

$$(T\rho(g)\xi)(\gamma) = \Phi_x(T)((\rho(g)\xi)|_{G_x})(\gamma) = g(\gamma) \cdot (\Phi_x(T)V_{\gamma^{-1}})(\xi|_{G_{r(\gamma)}})(\gamma),$$

which shows that $\rho(g)T = T\rho(g)$. The case of $G_x \cap \text{supp}(g) = \emptyset$ is similar but much easier, hence omitted. □
In the case of dense subsets, Lemma 3.23 can be reduced to the following. The proof is straightforward, hence omitted.

**Corollary 3.24.** Let $\mathcal{G}$ be a locally compact étale groupoid and $X$ be a dense invariant subset of $\mathcal{G}^{(0)}$. Then an operator $T \in \mathcal{L}(L^2(\mathcal{G}))$ is $\mathcal{G}$-equivariant if and only if $V_\gamma T s(\gamma) = T r(\gamma) V_\gamma$ for any $\gamma \in s^{-1}(X) = r^{-1}(X)$.

Now we apply the tools developed above to calculate the $C^*$-algebra $\mathcal{L}(L^2(\mathcal{G}))^G$ for the coarse groupoid:

**Example 3.25.** Following the notation in Sect. 2.6.3, let $(X, d)$ be a discrete metric space with bounded geometry and $\mathcal{G} = G(X)$ be the associated coarse groupoid. The unit space of $G(X)$ is the Stone–Čech compactification $\beta X$ of $X$, with an open invariant dense subset $X$. Note that each $\mathcal{G}_x$ is bijective to $X$ via the map $(y, x) \mapsto y$, and hence Corollary 3.17 implies that the map

$$\text{Res} \circ \Phi : \mathcal{L}(L^2(\mathcal{G}))^G \to \prod_{x \in X} \mathcal{B}(\ell^2(\mathcal{G}_x)) \cong \prod_{x \in X} \mathcal{B}(\ell^2(X))$$

is a $C^*$-monomorphism. Moreover, it follows from Corollary 3.19 and 3.24 that $(T_x)_{x \in X}$ belongs to the image of $\text{Res} \circ \Phi$ if and only if there exists $T \in \mathcal{B}(\ell^2(X))$ such that the constant family $(T_x)_{x \in X}$ is vector-wise uniformly quasi-local.

To provide a more concrete picture, we claim that the family $(T_x)_{x \in X}$ is vector-wise uniformly quasi-local if and only if the following holds:

$$\lim_{S \to \infty} \sup_{x \in X} \sum_{y \notin B(x, S)} |T_{y,x}|^2 = 0 \quad \text{and} \quad \lim_{S \to \infty} \sup_{x \in X} \sum_{y \notin B(x, S)} |T_{x,y}|^2 = 0. \quad (3.5)$$

In fact, by definition we know that $(T_x)_{x \in X}$ is vector-wise uniformly quasi-local if and only if for any $\varepsilon > 0$ and $F \in \mathbb{C}_u[X]$, there exists $S > 0$ such that for any $A, B \subseteq X$ with $d(A, B) \geq S$ then $\|\chi_A T \chi_B(F_x)\| \leq \varepsilon$ and $\|\chi_A T^* \chi_B(F_x)\| \leq \varepsilon$, where $F_x(y) := F(y, x)$. It is easy to see that this is equivalent to the following: for any $\varepsilon > 0$ and $R > 0$, there exists $S > 0$ such that for any $A, B \subseteq X$ with $d(A, B) \geq S$ and any $\xi \in \ell^2(X)$ with $\text{diam}(\text{supp}(\xi)) \leq R$ and $\|\xi\|_\infty \leq 1$ we have $\|\chi_A T \chi_B \xi\| \leq \varepsilon$ and $\|\chi_A T^* \chi_B \xi\| \leq \varepsilon$. Taking $A$ to be $X \backslash B(x, S)$ and using bounded geometry, this is also equivalent to that for any $\varepsilon > 0$, there exists $S > 0$ such that for any $B \subseteq X$ and any $x \in X$ we have $\|\chi_{X \backslash N_\delta(B)} T \chi_B(\delta_x)\| \leq \varepsilon$ and $\|\chi_{X \backslash N_\delta(B)} T^* \chi_B(\delta_x)\| \leq \varepsilon$. Finally, note that it suffices to consider the case when $B = \{x\}$, and hence we conclude the claim.

In conclusion, we obtain the following:

**Lemma 3.26.** Let $(X, d)$ be a discrete metric space with bounded geometry and $G(X)$ be the associated coarse groupoid. Then for some fixed $x_0 \in X$, the map

$$\Theta : \mathcal{L}(L^2(G(X)))^{G(x_0)} \to \{T \in \mathcal{B}(\ell^2(X)) : T \text{ satisfies Equation (3.5)}\} \quad (3.6)$$

given by $T \mapsto \Phi_{x_0}(T)$ is a $C^*$-isomorphism. Note that $\Theta$ is independent of the choice of $x_0$.

As we will see in the next subsection, the map $\Theta$ in (3.6) extends the isomorphism $\Theta : C^*_\ell(G(X)) \to C^*_u(X)$ in (2.7). Hence we take the liberty of abusing the notation.
3.5. Realisations for equivariant operators and left convolvers. This subsection is devoted to providing a concrete description for equivariant operators. Let us start with the case of groups, i.e., the unit space consists of a single point.

Let \( \mathcal{G} = G \) be a discrete group, and hence \( \mathcal{L}(L^2(G)) = \mathcal{B}(\ell^2(G)) \). For \( f, g \in \ell^2(G) \), the convolution \( f * g \) is a well-defined bounded function on \( G \) thanks to the Cauchy-Schwarz inequality. Recall that \( f \in \ell^2(G) \) is a left convolver (see [2, Section 1.3]) if \( f * g \in \ell^2(G) \) whenever \( g \in \ell^2(G) \). In this case, the map \( g \mapsto f * g \) is a bounded operator on \( \ell^2(G) \) (by the closed graph theorem), denoted by \( \Lambda(f) \). It is shown in [2, Theorem 1.3.6] that an operator \( T \in \mathcal{B}(\ell^2(G)) \) is \( G \)-equivariant if and only if there exists a left convolver \( f_T \in \ell^2(G) \) such that \( T = \Lambda(f_T) \).

In the following, we would like to explore an analogous description for equivariant operators on groupoids. Let \( \mathcal{G} \) be a locally compact étale groupoid with unit space \( \mathcal{G}^{(0)} \). Similar to the assumption of square-summability in the group case, we consider functions \( f \in C_b(\mathcal{G}) \) satisfying the following:

\[
\begin{align*}
(\text{c.1}) & \ f|_{\mathcal{G}_x} \in \ell^2(\mathcal{G}_x) \text{ and } f^*|_{\mathcal{G}_x} \in \ell^2(\mathcal{G}_x) \text{ for every } x \in \mathcal{G}^{(0)}; \\
(\text{c.2}) & \ \sup_{x \in \mathcal{G}^{(0)}} \| f|_{\mathcal{G}_x} \|_{\ell^2(\mathcal{G}_x)} < \infty, \quad \text{and} \quad \sup_{x \in \mathcal{G}^{(0)}} \| f^*|_{\mathcal{G}_x} \|_{\ell^2(\mathcal{G}_x)} < \infty,
\end{align*}
\]

where \( f^* \) is defined by the same formula (2.2). Then for \( g \in L^2(\mathcal{G}) \), the Cauchy-Schwarz inequality implies that the convolution \( f * g \) and \( f^* * g \) by (2.1) are well-defined, and are bounded functions on \( \mathcal{G} \).

**Definition 3.27.** Let \( \mathcal{G} \) be a locally compact étale groupoid. A function \( f \in C_b(\mathcal{G}) \) is called a left convolver if \( f \) satisfies (c.1) and (c.2), and both \( f * g \) and \( f^* * g \) belong to \( L^2(\mathcal{G}) \) for every \( g \in L^2(\mathcal{G}) \).

For a left convolver \( f \in C_b(\mathcal{G}) \), it is routine to check that the map \( g \mapsto f * g \) is an adjointable operator on \( L^2(\mathcal{G}) \), still denoted by \( \Lambda(f) \). It is clear that \( \Lambda(f)^* = \Lambda(f^*) \).

**Remark 3.28.** Comparing with the case of groups, readers might wonder why we need requirements on both \( f \) and \( f^* \) in (c.1), (c.2) and Definition 3.27. The reason lies in the fact that on Hilbert \( C^* \)-modules, a bounded module morphism might not be adjointable in general. We provide some examples below.

**Example 3.29.** Let \( \mathcal{G} = \mathbb{N} \times \mathbb{N} \) be the pair groupoid from Sect. 2.6.2.

Consider the function \( f \in C_b(\mathcal{G}) \) given by \( f((m, n)) := m^{-1} \). Direct calculations show that \( f|_{\mathcal{G}_{(n,n)}} \in \ell^2(\mathbb{N}) \) for each \( n \in \mathbb{N} \), and \( \sup_{n \in \mathbb{N}} \| f|_{\mathcal{G}_{(n,n)}} \|_2 < \infty \). On the other hand, note that \( f^* \) is given by \( f^*((m, n)) = n^{-1} \), and hence \( f^*|_{\mathcal{G}_{(n,n)}} \notin \ell^2(\mathcal{G}_{(n,n)}) \) for any \( n \in \mathbb{N} \). Moreover, note that for the function \( g : \mathcal{G} \to \mathbb{R} \) defined by \( g((m, n)) = (nm)^{-1} \), it is clear that \( g \in L^2(\mathcal{G}) \) while

\[
 f * g((m, n)) = \sum_{m_1 \in \mathbb{N}} f((m, m_1))g((m_1, n)) = (nm)^{-1} \sum_{m_1 \in \mathbb{N}} m_1^{-1} = \infty.
\]

This illustrates that the requirements on \( f^* \) in (c.1) and (c.2) cannot be deduced from those on \( f \), and both of them are used to ensure that the convolutions by \( f \) and \( f^* \) are well-defined.

The following example shows that even if a function \( f \in C_b(\mathcal{G}) \) satisfies (c.1) and (c.2), it might occur that the convolution by \( f \) is a bounded operator on \( L^2(\mathcal{G}) \) while not adjointable. This example essentially comes from [29, Example 2.1.2], and explains that we need to consider both \( f \) and \( f^* \) in Definition 3.27.
Example 3.30. We consider the groupoid \( \mathcal{G} = \mathbb{N} \times \mathbb{N} \times [0, 1] \) with source and range maps given by \( s((m, n, x)) = (n, n, x) \) and \( r((m, n, x)) = (m, m, x) \) for \( (m, n, x) \in \mathcal{G} \), respectively. The composition map is given by \( (m, n, x) \cdot (n, k, x) = (m, k, x) \), and the inverse map is by \( (m, n, x) = (n, m, x) \) for \( m, n, k \in \mathbb{N} \) and \( x \in [0, 1] \).

We take the product topology on \( \mathcal{G} \), and then identify \( C_b(\mathcal{G}) \) with

\[
\{(f_{m,n})_{m,n \in \mathbb{N}} : f_{m,n} \in C([0, 1]) \text{ and } \sup_{(m,n) \in \mathbb{N} \times \mathbb{N}} \|f_{m,n}\|_\infty < \infty\},
\]

whose element can be regarded as an \( \mathbb{N} \)-by-\( \mathbb{N} \) matrix with entries in \( C([0, 1]) \). Elements in the subspace \( C_c(\mathcal{G}) \) can be written as \( (f_{m,n})_{m,n \in \mathbb{N}} \) such that \( f_{m,n} \neq 0 \) for only finitely many \( (m, n) \in \mathbb{N} \times \mathbb{N} \).

Consider the function \( f \in C_b(\mathcal{G}) \) given by

\[
f = \begin{pmatrix}
\varphi_1 & \varphi_2 & \cdots \\
0 & 0 & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix}
\]

where \( \varphi_i = \begin{cases} 0, & \text{on } [0, \frac{1}{i+1}] \text{ and } [\frac{1}{i}, 1]; \\
1, & \text{at the point } \frac{1}{i} (\frac{1}{i+1} + \frac{1}{i-1}); \\
\text{linear, on } [\frac{1}{i+1}, \frac{2i+1}{2i(i+1)}] \text{ and } [\frac{2i+1}{2i(i+1)}, \frac{1}{i}].
\end{cases}
\]

It is easy to see that \( f^* = \begin{pmatrix}
\varphi_1 & 0 & \cdots \\
\varphi_2 & 0 & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix} \) and \( f^* \notin L^2(\mathcal{G}) \) since the first coordinate of

\[
\langle f^*, f^* \rangle = \sum_{n \in \mathbb{N}} \varphi_{n,0}^2,
\]

which is not in \( C([0, 1]) \). It is routine to check that \( f \) satisfies (c.1) and (c.2).

Given \((\xi_{i,j}) \in L^2(\mathcal{G})\), we have:

\[
f * \xi = \begin{pmatrix}
\sum_{k \in \mathbb{N}} \varphi_k \xi_{k,1} & \sum_{k \in \mathbb{N}} \varphi_k \xi_{k,2} & \cdots \\
0 & 0 & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix},
\]

which belongs to \( L^2(\mathcal{G}) \) using a standard approximating argument. This implies that the convolution \( \Lambda(f) \) is a linear operator on \( L^2(\mathcal{G}) \), which is also bounded by the closed graph theorem. On the other hand, consider \( \eta = \begin{pmatrix}
\chi_{[0,1]} & 0 & \cdots \\
0 & 0 & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix} \in L^2(\mathcal{G}) \), and then

\[
f^* * \eta = f^* \notin L^2(\mathcal{G}).
\]

Now we are in the position to provide the realisation for equivariant operators:

**Proposition 3.31.** Let \( \mathcal{G} \) be a locally compact étale groupoid.

1. For any left convolver \( f \in C_b(\mathcal{G}) \), we have \( \Lambda(f) \in \mathcal{L}(L^2(\mathcal{G}))^\mathcal{G} \).
2. Conversely, for any \( T \in \mathcal{L}(L^2(\mathcal{G}))^\mathcal{G} \) there exists a unique left convolver \( f_T \in C_b(\mathcal{G}) \) such that \( T = \Lambda(f_T) \).

**Proof.** (1). It is clear that \( \Lambda(f) \in \mathcal{L}(L^2(\mathcal{G})) \) for any left convolver \( f \in C_b(\mathcal{G}) \). Moreover, note that for any \( g \in C_c(\mathcal{G}) \) and \( \xi \in L^2(\mathcal{G}) \), we have \( \Lambda(f) \rho(g) \xi = f * (\xi * g) = (f * \xi) * g = \rho(g) \Lambda(f) \xi \). Hence \( \Lambda(f) \) is \( \mathcal{G} \)-equivariant.

(2). Given \( T \in \mathcal{L}(L^2(\mathcal{G}))^\mathcal{G} \), we consider the function \( f_T : \mathcal{G} \to \mathbb{C} \) defined by

\[
f_T(\gamma) := (\Phi_{s(\gamma)}(T)(\delta_{s(\gamma)}))(\gamma) \quad \text{for } \gamma \in \mathcal{G}.
\]

(3.7)
Since $T$ is $\mathcal{G}$-equivariant, then for any $\gamma, \alpha \in \mathcal{G}$ with $r(\alpha) = s(\gamma)$ we have
\[
f_T(\gamma) = (\Phi_{s(\gamma)}(T)(\delta_{s(\gamma)}))(\gamma) = (\Phi_{s(\alpha)}(T)(\delta_\alpha))(\gamma \alpha).
\]
(3.8)

On the other hand, for $\gamma \in \mathcal{G}$ we have
\[
f_T^*(\gamma) = \Phi_{s(\gamma)}(T^*)(\delta_{s(\gamma)})(\gamma) = (V_{\gamma^{-1}}\Phi_{r(\gamma)}(T^*)V_\gamma)(\delta_{s(\gamma)})(\gamma)
\]
= \left(\Phi_{r(\gamma)}(T)^*V_\gamma\right)(\delta_{s(\gamma)})(\gamma \gamma^{-1}) = \left(\delta_{r(\gamma)}, \Phi_{r(\gamma)}(T)^*V_\gamma\right)(\delta_{s(\gamma)})
\]
= \left(\Phi_{r(\gamma)}(T)(\delta_{r(\gamma)}), V_\gamma(\delta_{s(\gamma)})\right) = \left(\Phi_{r(\gamma)}(T)(\delta_{r(\gamma)}), \delta_{\gamma^{-1}}\right)
\]
= \Phi_{r(\gamma)}(T)(\delta_{r(\gamma)})(\gamma^{-1}) = f_T(\gamma)(\gamma^{-1}) = \left(f_T^*\right)(\gamma^2),
\]
where $T$ being $\mathcal{G}$-equivariant is used in the second equality. This shows that $(f_T)^* = f_T^*$. Hence for any $x \in \mathcal{G}^{(0)}$, we have $f_T|_{\mathcal{G}_x} = \Phi_x(T)(\delta_x)$ and $(f_T)^*|_{\mathcal{G}_x} = \Phi_x(T^*)(\delta_x)$, which implies that (c.1) and (c.2) hold for $f_T$.

Since $\mathcal{G}$ is a locally compact étale groupoid, then for any $\xi \in L^2(\mathcal{G})$, it follows that $f_T$ is a bounded continuous function on $\mathcal{G}$. For any $\alpha \in L^2(\mathcal{G})$, note that $f_T \star \xi$ is well-defined since $f_T$ satisfies (c.1) and (c.2). Moreover, for any $\gamma \in \mathcal{G}$ with $s(\gamma) = x$ we have
\[
(f_T \star \xi)(\gamma) = \sum_{\alpha \in \mathcal{G}_x} f_T(\gamma \alpha^{-1})\xi(\alpha) = \sum_{\alpha \in \mathcal{G}_x} (\Phi_{r(\alpha)}(T)(\delta_{r(\alpha)}))(\gamma \alpha^{-1})\xi(\alpha)
\]
\[
= \sum_{\alpha \in \mathcal{G}_x} (\Phi_x(T)(\delta_x))(\gamma \alpha \xi)(\gamma) = (\Phi_x(T)(\xi|_{\mathcal{G}_x}))(\gamma) = (T\xi)(\gamma),
\]
where we use (3.8) in the third equality. This implies that $f_T \star \xi \in L^2(\mathcal{G})$ and $\Lambda(f_T) = T$.
Similarly, we have $(f_T)^* \star \xi \in L^2(\mathcal{G})$ and $\Lambda((f_T)^*) = T^*$. Therefore, $f_T$ is a left convolver and we conclude the proof.

Thanks to Proposition 3.31, we provide an alternative picture for the isomorphism $\Theta$ in (3.6):

**Example 3.32.** Following the notation in Example 3.25, for $T \in \mathcal{L}(L^2(\mathcal{G}(X)))^{G(X)}$ Proposition 3.31 implies that there exists $f_T \in C_b(\mathcal{G}(X))$ such that $T = \Lambda(f_T)$. Considering the restriction of $f_T$ on $X \times X$, it is easy to see that the isomorphism $\Theta$ in (3.6) can be written as follows:
\[
(\Theta(T)(\xi))(x) := \sum_{y \in X} f_T(x, y)\xi(y) \quad \text{for} \quad \xi \in \ell^2(X) \text{ and } x \in X.
\]

Therefore as claimed in Example 3.25, $\Theta$ in (3.6) extends the isomorphism $\Theta : C^*_r(\mathcal{G}(X)) \to C^*_u(X)$ in (2.7), which explains the notation.

We end this section with some discussions on multiplication operators. Recall that for $g \in C_b(\mathcal{G})$, the multiplication operator $M(g) : L^2(\mathcal{G}) \to L^2(\mathcal{G})$ is defined by
\[
(M(g)\xi)(\gamma) = g(\gamma)\xi(\gamma) \quad \text{for} \quad \xi \in L^2(\mathcal{G}) \text{ and } \gamma \in \mathcal{G}.
\]
To simplify the statement, we say that a function $g \in C_b(\mathcal{G})$ is $\mathcal{G}$-equivariant if $g(\beta) = g(\beta \gamma^{-1})$ for any $\beta, \gamma \in \mathcal{G}$ with $s(\beta) = s(\gamma)$.

**Lemma 3.33.** Let $\mathcal{G}$ be a locally compact étale groupoid and $g \in C_b(\mathcal{G})$. Then the multiplication operator $M(g)$ is $\mathcal{G}$-equivariant if and only if $g$ is $\mathcal{G}$-equivariant.
Proof. Assume that $g$ is $G$-equivariant, then for $h \in C_c(G)$, $\xi \in L^2(G)$ and $\gamma \in G$, we have:

\[
(\rho(h)M(g)\xi)(\gamma) = \sum_{\beta \in G_s(\gamma)} h(\beta)g(\gamma\beta^{-1})\xi(\gamma\beta^{-1}) = \sum_{\beta \in G_s(\gamma)} h(\beta)g(\gamma)\xi(\gamma\beta^{-1}) = g(\gamma) \cdot \sum_{\beta \in G_s(\gamma)} h(\beta)\xi(\gamma\beta^{-1}) = (M(g)\rho(h)\xi)(\gamma).
\]

Conversely, it follows from Lemma 3.23 that $V_{s(f)}\Phi_{s(y)}(M(g)) = \Phi_{r(y)}(M(g))V_{r}$ for all $\gamma \in G$. Hence for $\beta \in G_s(\gamma)$, we have

\[
g(\beta)\delta_{\beta^{-1}} = V_{s(f)}(g(\beta)\delta_{\beta}) = V_{s(f)}\Phi_{s(y)}(M(g))(\delta_{\beta}) = \Phi_{r(y)}(M(g))V_{r}(\delta_{\beta}) = g(\beta\gamma^{-1})\delta_{\beta^{-1}},
\]

which shows that $g$ is $G$-equivariant. \hfill \Box

On the other hand, the following lemma explores when an equivariant operator can be realised as a multiplication operator:

**Lemma 3.34.** Let $G$ be a locally compact étale groupoid. For $T \in \mathcal{L}(L^2(G))^G$, let $f_T$ be the associated left convolver from Proposition 3.31. Then $T = M(g)$ for some $g \in C_b(G)$ if and only if $\text{supp}(f_T) \subseteq G^{(0)}$. In this case, $f_T = g|_{G^{(0)}}$ and hence $M(g) = \Lambda(g|_{G^{(0)}})$.

**Proof.** If $T = M(g)$ for some $g \in C_b(G)$, from (3.7) we have:

\[
f_T(\gamma) = (\Phi_{s(y)}(M(g))(\delta_{s(\gamma)}))(\gamma) = g(s(y))\delta_{s(\gamma)}(\gamma) = \begin{cases} g(\gamma), & \text{if } \gamma \in G^{(0)}; \\ 0, & \text{otherwise}. \end{cases}
\]

Hence we obtain that $\text{supp}(f_T) \subseteq G^{(0)}$ and $f_T = g|_{G^{(0)}}$. Conversely, assume that $\text{supp}(f_T) \subseteq G^{(0)}$. Then for any $\xi \in L^2(G)$ and $\gamma \in G$, we have

\[
(f_T \ast \xi)(\gamma) = \sum_{\beta \in G_s(\gamma)} f_T(\gamma\beta^{-1})\xi(\beta) = f_T(r(\gamma))\xi(\gamma).
\]

Setting $g(\gamma) = f_T(r(\gamma))$ for $\gamma \in G$, it follows that $g \in C_b(G)$ is $G$-equivariant and $\Lambda(f_T) = M(g)$. Hence we conclude the proof. \hfill \Box

4. Compactly Supported and Quasi-Local Operators for Groupoids

In this section, we introduce the key notion of this paper, i.e., compactly supported and quasi-local operators for groupoids. Moreover, we show that the reduced groupoid C*-algebra can be recovered by equivariant and compactly supported operators.

**Definition 4.1.** Let $G$ be a locally compact étale groupoid and $K \subseteq G$. For $f, g \in C_b(G)$, we say that $f$ and $g$ are $K$-separated if $(K \cdot \text{supp}(f)) \cap \text{supp}(g) = \emptyset$ and supp$(f) \cap (K \cdot \text{supp}(g)) = \emptyset$, i.e., $\gamma_2\gamma_1^{-1} \notin K$ and $\gamma_1\gamma_2^{-1} \notin K$ for any $\gamma_1 \in \text{supp}(f)$ and $\gamma_2 \in \text{supp}(g)$ with $s(\gamma_1) = s(\gamma_2)$.

Recall that in the case of the coarse groupoids, compact subsets are always contained in closures of entourages. This inspires us to introduce the following:

**Definition 4.2.** Let $G$ be a locally compact étale groupoid and $T \in \mathcal{L}(L^2(G))$. We define the following:
(1) $T$ is supported in $K \subseteq \mathcal{G}$ if $gTf = 0$ for any $K$-separated functions $f, g \in C_b(\mathcal{G})$.
(2) $T$ is compactly supported if $T$ is supported in some compact subset of $\mathcal{G}$.
(3) $T$ is quasi-local if for any $\varepsilon > 0$, there exists a compact subset $K \subseteq \mathcal{G}$ such that for any $K$-separated functions $f, g \in C_c(\mathcal{G})$ we have $\|gTf\| < \varepsilon \|g\|_{\infty} \|f\|_{\infty}$.

It is clear that compactly supported operators are always quasi-local. Also note that $\mathcal{L}(L^2(\mathcal{G}))$ admits approximating units consisting of elements in $C_c(\mathcal{G})$, hence the following is straightforward:

**Lemma 4.3.** For an operator $T \in \mathcal{L}(L^2(\mathcal{G}))$, we have the following:

(1) $T$ is compactly supported if and only if there exists a compact subset $K \subseteq \mathcal{G}$ such that $gTf = 0$ for any $K$-separated functions $f, g \in C_c(\mathcal{G})$;
(2) $T$ is quasi-local if and only if for any $\varepsilon > 0$, there exists a compact subset $K \subseteq \mathcal{G}$ such that for any $K$-separated functions $f, g \in C_c(\mathcal{G})$ we have $\|gTf\| < \varepsilon \|g\|_{\infty} \|f\|_{\infty}$.

Using the slicing map $\Phi$ from (3.1), we provide the following “fibre-wise” characterisation for compactly supported and quasi-local operators:

**Proposition 4.4.** Let $\mathcal{G}$ be a locally compact étale groupoid and $T \in \mathcal{L}(L^2(\mathcal{G}))$. We have the following:

(1) $T$ is supported in a compact set $K \subseteq \mathcal{G}$ if and only if $\chi_{A_x} \Phi_x(T) \chi_{B_x} = 0$ for any $x \in \mathcal{G}^{(0)}$ and subsets $A_x, B_x \subset \mathcal{G}_x$ with $A_x \cap (K \cdot B_x) = \emptyset$ and $(K \cdot A_x) \cap B_x = \emptyset$;
(2) $T$ is quasi-local if and only if for any $\varepsilon > 0$ there exists a compact subset $K \subseteq \mathcal{G}$ such that $\|\chi_{A_x} \Phi_x(T) \chi_{B_x}\| < \varepsilon$ for any $x \in \mathcal{G}^{(0)}$ and subsets $A_x, B_x \subset \mathcal{G}_x$ with $A_x \cap (K \cdot B_x) = \emptyset$ and $(K \cdot A_x) \cap B_x = \emptyset$.

**Proof.** Here we only provide the proof for (2), since the other is similar and simpler.

**Necessity:** Given $\varepsilon > 0$, we take a compact subset $K \subseteq \mathcal{G}$ satisfying the condition in Definition 4.2(3). Fix an $x \in \mathcal{G}^{(0)}$. For any finite subsets $A_x, B_x \subset \mathcal{G}_x$ with $A_x \cap (K \cdot B_x) = \emptyset$ and $(K \cdot A_x) \cap B_x = \emptyset$, take open subsets $U$ and $V$ in $\mathcal{G}$ containing $A_x$ and $B_x$, respectively, such that $UV^{-1} \cap K = \emptyset$ and $VU^{-1} \cap K = \emptyset$. By Urysohn’s lemma, there exist $f, g \in C_b(\mathcal{G})$ with range in $[0, 1]$, $\text{supp}(g) \subset U$ and $\text{supp}(f) \subset V$ such that $g|_{\mathcal{G}_x} = \chi_{A_x}$ and $f|_{\mathcal{G}_x} = \chi_{B_x}$. It is then clear that $f$ and $g$ are $K$-separated. Hence we obtain:

$$\|\chi_{A_x} \Phi_x(T) \chi_{B_x}\| = \|\Phi_x(gTf)\| \leq \|gTf\| < \varepsilon.$$ 

Using an approximating argument, the estimate above holds for any subsets $A_x, B_x$ in $\mathcal{G}_x$ with $A_x \cap (K \cdot B_x) = \emptyset$ and $(K \cdot A_x) \cap B_x = \emptyset$.

**Sufficiency:** Given $\varepsilon > 0$, we take a compact subset $K \subseteq \mathcal{G}$ satisfying the assumption in (2). For any $K$-separated functions $f, g \in C_b(\mathcal{G})$, we have

$$\|gTf\| = \sup_{x \in \mathcal{G}^{(0)}} \|\Phi_x(gTf)\| = \sup_{x \in \mathcal{G}^{(0)}} \|g|_{\mathcal{G}_x} \cdot \Phi_x(T) \cdot f|_{\mathcal{G}_x}\|.$$ 

For $x \in \mathcal{G}^{(0)}$, denote $A_x := \text{supp}(g|_{\mathcal{G}_x})$ and $B_x := \text{supp}(f|_{\mathcal{G}_x})$. It is clear that $A_x \cap (K \cdot B_x) = \emptyset$ and $(K \cdot A_x) \cap B_x = \emptyset$. Therefore, the assumption implies that

$$\|g|_{\mathcal{G}_x} \cdot \Phi_x(T) \cdot f|_{\mathcal{G}_x}\| = \|g|_{\mathcal{G}_x} \cdot \chi_{A_x} \Phi_x(T) \chi_{B_x} \cdot f|_{\mathcal{G}_x}\| < \varepsilon \|g\|_{\infty} \|f\|_{\infty},$$

which shows that $\|gTf\| \leq \varepsilon \|g\|_{\infty} \|f\|_{\infty}$.

Inspired by Proposition 4.4, we introduce the following:
**Definition 4.5.** Let $\mathcal{G}$ be a locally compact étale groupoid with unit space $\mathcal{G}^{(0)}$. A family $(T_x)_{x \in \mathcal{G}^{(0)}} \in \prod_{x \in \mathcal{G}^{(0)}} \mathcal{B} (\ell^2 (\mathcal{G}_x))$ is said to be **compactly uniformly quasi-local** if for any $\varepsilon > 0$ there exists a compact subset $K \subseteq \mathcal{G}$ such that for any $x \in \mathcal{G}^{(0)}$ and $A_x, B_x \subseteq \mathcal{G}_x$ with $A_x \cap (K \cdot B_x) = \emptyset$ and $(K \cdot A_x) \cap B_x = \emptyset$, we have $\| \chi_{A_x} T_x \chi_{B_x} \| < \varepsilon$.

It is clear that the notion of compactly uniform quasi-locality implies vector-wise uniform quasi-locality introduced in Definition 3.12. Hence combining Theorem 3.13 with Proposition 4.4, we reach the following:

**Corollary 4.6.** Let $\mathcal{G}$ be a locally compact étale groupoid with unit space $\mathcal{G}^{(0)}$. For $(T_x)_{x \in \mathcal{G}^{(0)}} \in \prod_{x \in \mathcal{G}^{(0)}} \mathcal{B} (\ell^2 (\mathcal{G}_x))$, the following are equivalent:

1. there exists a quasi-local operator $T \in \mathcal{L} (L^2 (\mathcal{G}))$ such that $\Phi (T) = (T_x)_{x \in \mathcal{G}^{(0)}}$;
2. the map $x \mapsto T_x$ is a continuous section of $E$ introduced in (3.2) and $(T_x)_{x \in \mathcal{G}^{(0)}}$ is compactly uniformly quasi-local.

A similar result also holds for the case of compactly supported operators.

As revealed in Sect. 3.3, we can simplify Proposition 4.4 for dense subsets as follows. The proof is similar to that for Proposition 3.18, hence omitted.

**Lemma 4.7.** Let $\mathcal{G}$ be a locally compact étale groupoid with unit space $\mathcal{G}^{(0)}$, $X$ a dense subset of $\mathcal{G}^{(0)}$, and $T \in \mathcal{L} (L^2 (\mathcal{G}))$. Then we have the following:

1. $T$ is supported in a compact set $K \subseteq \mathcal{G}$ if and only if $\chi_{A_x} \Phi_x (T) \chi_{B_x} = 0$ for any $x \in X$ and subsets $A_x, B_x \subseteq \mathcal{G}_x$ with $A_x \cap (K \cdot B_x) = \emptyset$ and $(K \cdot A_x) \cap B_x = \emptyset$;
2. $T$ is quasi-local if and only if for any $\varepsilon > 0$ there exists a compact subset $K \subseteq \mathcal{G}$ such that $\| \chi_{A_x} \Phi_x (T) \chi_{B_x} \| < \varepsilon$ for any $x \in X$ and subsets $A_x, B_x \subseteq \mathcal{G}_x$ with $A_x \cap (K \cdot B_x) = \emptyset$ and $(K \cdot A_x) \cap B_x = \emptyset$.

Analogous to Definition 4.5, for a dense subset $X$ of $\mathcal{G}^{(0)}$ we say that a family $(T_x)_{x \in X} \in \prod_{x \in X} \mathcal{B} (\ell^2 (\mathcal{G}_x))$ is **compactly uniformly quasi-local** if for any $\varepsilon > 0$ there exists a compact subset $K \subseteq \mathcal{G}$ such that for any $x \in X$ and $A_x, B_x \subseteq \mathcal{G}_x$ with $A_x \cap (K \cdot B_x) = \emptyset$ and $(K \cdot A_x) \cap B_x = \emptyset$, we have $\| \chi_{A_x} T_x \chi_{B_x} \| < \varepsilon$.

It is clear that in the case of dense subsets, the notion of compactly uniform quasi-locality also implies vector-wise uniform quasi-locality. Hence combining Corollary 3.19 with Lemma 4.7, we reach the following:

**Corollary 4.8.** Let $\mathcal{G}$ be a locally compact étale groupoid with unit space $\mathcal{G}^{(0)}$, and $X$ be a dense subset of $\mathcal{G}^{(0)}$. For $(T_x)_{x \in X} \in \prod_{x \in X} \mathcal{B} (\ell^2 (\mathcal{G}_x))$, the following are equivalent:

1. there exists a quasi-local operator $T \in \mathcal{L} (L^2 (\mathcal{G}))$ such that $\text{Res} \circ \Phi (T) = (T_x)_{x \in X}$;
2. the map $x \mapsto T_x$ is a continuous extendable section of $E_X$ introduced in (3.2) and $(T_x)_{x \in X}$ is compactly uniformly quasi-local.

Moreover, we can omit the requirement of extendableness in condition (2) when $\mathcal{G}^{(0)}$ is the Stone-Čech compactification of $X$ (with the inclusion map).

A similar result also holds for the case of compactly supported operators.

The following example shows that our definition of quasi-locality for groupoids recovers Definition 2.10 in the case of metric spaces:

**Example 4.9.** Let $(X, d)$ be a discrete metric space with bounded geometry and $G(X)$ the associated coarse groupoid. Combining Example 3.25 and Corollary 4.8, we obtain that a $G(X)$-equivariant operator $T \in \mathcal{L} (L^2 (G(X)))$ is quasi-local if and only if there exists a quasi-local operator (in the sense of Definition 2.10) $T_0 \in \mathcal{B} (\ell^2 (X))$ such that $\text{Res} \circ \Phi (T) = (T_0)_{x \in X}$. 


Analogous to the uniform Roe and quasi-local algebras for metric spaces introduced in Definition 2.9 and 2.11, we define the following algebras for groupoids:

**Definition 4.10.** Let $\mathcal{G}$ be a locally compact étale groupoid.

1. Denote by $\mathbb{C}_u[\mathcal{G}]$ the set of all compactly supported operators in $\mathcal{L}(L^2(\mathcal{G}))$, and define the uniform Roe algebra $C_u^*(\mathcal{G})$ of $\mathcal{G}$ to be its norm closure in $\mathcal{L}(L^2(\mathcal{G}))$.
2. Denote by $C_{uq}^*(\mathcal{G})$ the set of all quasi-local operators in $\mathcal{L}(L^2(\mathcal{G}))$, which forms a $C^*$-algebra and is called the uniform quasi-local algebra of $\mathcal{G}$.

We also consider the equivariant counterpart. For a locally compact étale groupoid $\mathcal{G}$, denote by $\mathbb{C}_u[\mathcal{G}]^G$ the $*$-subalgebra in $\mathbb{C}_u[\mathcal{G}]$ consisting of $G$-equivariant operators. Also denote by $C_u^*(\mathcal{G})^G$ and $C_{uq}^*(\mathcal{G})^G$ the $C^*$-subalgebras in $C_u^*(\mathcal{G})$ and $C_{uq}^*(\mathcal{G})$, respectively, consisting of $\mathcal{G}$-equivariant operators.

**Remark 4.11.** Recall that Anantharaman-Delaroche introduced the notion of uniform $C^*$-algebra of a groupoid in [1, Definition 6.1] with the same notation $C_u^*(\mathcal{G})$. We will show in Sect. 7.2 that it coincides with Definition 4.10(1).

The following proposition connects the algebra $\mathbb{C}_u[\mathcal{G}]^G$ with $C_c(\mathcal{G})$, which parallels the result that the equivariant part of the algebraic uniform Roe algebra for a group coincides with the group algebra.

**Proposition 4.12.** Let $\mathcal{G}$ be a locally compact étale groupoid. Then $\mathbb{C}_u[\mathcal{G}]^G$ is $*$-isomorphic to $C_c(\mathcal{G})$. As a consequence, the norm closure of $\mathbb{C}_u[\mathcal{G}]^G$ in $\mathcal{L}(L^2(\mathcal{G}))$ is $C^*$-isomorphic to the reduced groupoid $C^*$-algebra $C_r^*(\mathcal{G})$.

**Proof.** Recall that we have the faithful representation $\Lambda : C_c(\mathcal{G}) \to \mathcal{L}(L^2(\mathcal{G}))$ defined in (2.4). Hence it suffices to show that the image $\Lambda(C_c(\mathcal{G}))$ coincides with the $*$-algebra $\mathbb{C}_u[\mathcal{G}]^G$.

Direct calculations show that if $h \in C_c(\mathcal{G})$ has support in a compact set $K \subseteq \mathcal{G}$, then the operator $\Lambda(h)$ has support in $K$. Together with Proposition 3.31(1), this implies that $\Lambda(C_c(\mathcal{G})) \subseteq \mathbb{C}_u[\mathcal{G}]^G$. On the other hand, for $T \in \mathbb{C}_u[\mathcal{G}]^G$ with support in a compact subset $K \subseteq \mathcal{G}$, Proposition 3.31(2) implies that there exists a unique $f_T \in C_b(\mathcal{G})$ defined in (3.7) such that $T = \Lambda(f_T)$. Recall that

$$f_T(\gamma) = (\Phi_{s(\gamma)}(T)(\delta_{s(\gamma)}))(\gamma) = \langle \delta_{\gamma}, \Phi_{s(\gamma)}(T)(\delta_{s(\gamma)}) \rangle \quad \text{for} \quad \gamma \in \mathcal{G}.$$ 

Hence Proposition 4.4 implies that $f_T$ has support in $K \cup K^{-1}$, which concludes that $f_T \in C_c(\mathcal{G})$.

**Remark 4.13.** Note that $\mathbb{C}_u[\mathcal{G}]^G$ is contained in $C_u^*(\mathcal{G})^G$, hence doing completing we obtain that the norm closure of $\mathbb{C}_u[\mathcal{G}]^G$ in $\mathcal{L}(L^2(\mathcal{G}))$ is contained in $C_u^*(\mathcal{G})^G$. However in general, it is unclear whether these two algebras are identical. We provide a sufficient condition in the next section.

To conclude this section, we discuss a relative commutant picture for quasi-local operators on groupoids, which generalises the metric space case proved in [40, Theorem 2.8 “(i) $\iff$ (ii)’”] (see also [5, Proposition 3.3]). Recall the following notion:

**Definition 4.14.** Let $\mathcal{G}$ be a groupoid, $K$ a symmetric compact subset of $\mathcal{G}$ and $\delta > 0$. A function $h \in C_b(\mathcal{G})$ is said to have $(K, \delta)$--variation, if for any $\alpha, \beta \in \mathcal{G}$ with $\alpha\beta^{-1} \in K$, we have $|h(\alpha) - h(\beta)| < \delta$. 


Proposition 4.15. Let $\mathcal{G}$ be a locally compact étale groupoid and $T \in \mathcal{L}(L^2(\mathcal{G}))$. Then the following are equivalent:

1. $T$ is quasi-local;
2. for any $\varepsilon > 0$, there exist $\delta > 0$ and a symmetric compact $K \subseteq \mathcal{G}$ such that for any $h \in C_b(\mathcal{G})$ with $\|h\|_{\infty} = 1$ and $(K, \delta)$-variation, we have $\|T, h\| < \varepsilon$.

Proof. We follow the outline of the proof for [40, Theorem 2.8 “(i) $\Leftrightarrow$ (ii)”].

“(2) $\Rightarrow$ (1)”: Given $\varepsilon > 0$, choose $n \in \mathbb{N}$ such that $1/n \leq \delta$ and take $K' = K \cup K^2 \cup \cdots \cup K^n$. For any $K'$-separated $f, g \in C_c(\mathcal{G})$ with norm 1, there exists $h \in C_b(\mathcal{G})$ with $(K, \delta)$-variation and norm 1 such that $h$ equals 1 on $\text{supp}(f)$ and 0 on $\text{supp}(g)$. This can be achieved by Urysohn’s lemma. Then we have:

$$\|gTf\| = \|gThf\| = \|g[T, h]f\| < \varepsilon,$$

which implies that $T$ is a quasi-local operator by Lemma 4.3(2).

“(1) $\Rightarrow$ (2)”: For simplicity, we assume that $\|T\| = 1$. For any $0 < \varepsilon < 1$, take $\delta \leq \varepsilon/16$ and choose $N \in \mathbb{N}$ such that $\delta < 1/N < \varepsilon/8$. Due to quasi-locality, there exists a symmetric compact subset $K \subseteq \mathcal{G}$ such that for any $K$-separated $f, g \in C_b(\mathcal{G})$ with norm 1, we have $\|gTf\| \leq \varepsilon/(2N^2)$.

For $h \in C_b(\mathcal{G})$ with norm 1 and $(K, \delta)$-variation, we claim that $\|T, h\| < \varepsilon$. Without loss of generality, we assume that $h \geq 0$. Fixing $x \in \mathcal{G}^{(0)}$, denote by $h_x$ the restriction of $h$ on $\mathcal{G}_x$. We set

$$A_1 := h_x^{-1}([0, \frac{1}{N}]), \quad A_i := h_x^{-1}\left((\frac{i-1}{N}, \frac{i}{N})\right) \quad \text{for} \quad i = 2, \ldots, N.$$ 

It is clear that $(A_i A_j^{-1}) \cap K = \emptyset$ for $|i - j| > 1$. Applying Proposition 4.4(2), we have

$$\|\chi_{A_i} \Phi_x(T) \chi_{A_j}\| \leq \frac{\varepsilon}{2N^2}.$$ 

On the other hand, note that

$$\|h - \sum_{i=1}^{N} \frac{i}{N} \chi_{A_i}\|_{\infty} \leq 1/N.$$ 

Hence we obtain:

$$\|\Phi_x(T), h_x\| \leq \frac{2}{N} + \|\Phi_x(T), \sum_{i=1}^{N} \frac{i}{N} \chi_{A_i}\|$$

$$= \frac{2}{N} + \|\left(\sum_{j=1}^{N} \chi_{A_j}\right)\left(\sum_{i=1}^{N} \frac{i}{N} \Phi_x(T) \chi_{A_i}\right) - \sum_{j=1}^{N} \frac{j}{N} \chi_{A_j} \Phi_x(T) \sum_{i=1}^{N} \chi_{A_i}\|$$

$$= \frac{2}{N} + \|\sum_{i,j=1}^{N} \left(\frac{i-j}{N}\right) \chi_{A_j} \Phi_x(T) \chi_{A_i}\|$$

$$\leq \frac{2}{N} + \sum_{|i-j| > 1} \|\chi_{A_j} \Phi_x(T) \chi_{A_i}\| + \|\sum_{|i-j| \leq 1} \left(\frac{i-j}{N}\right) \chi_{A_j} \Phi_x(T) \chi_{A_i}\|.$$
Each item in the first sum is dominated by $\varepsilon/(2N^2)$, so the first item is less than $\varepsilon/2$.

The second sum can be divided into two parts for $j = i+1$ and $j = i-1$, each of which is dominated by $1/N$. Hence we have

$$\|[\Phi_x(T), h_x]\| \leq \frac{2}{N} + \frac{\varepsilon}{2} + \frac{2}{N} \leq \varepsilon,$$

which implies that

$$\|[T, h]\| = \sup_{x \in \mathcal{G}(\emptyset)} \|[\Phi_x(T), h_x]\| < \varepsilon.$$

Hence we conclude the proof. \qed

5. Main Theorem

In the previous section, we introduced the notion of quasi-locality for groupoids and constructed several associated $C^*$-algebras (see Definition 4.10). Now we are in the position to prove the following main result of this paper, showing that they coincide with each other under the assumption of amenability.

**Theorem 5.1.** Let $\mathcal{G}$ be a locally compact, $\sigma$-compact and étale groupoid. If $\mathcal{G}$ is amenable, then we have $C^*_r(\mathcal{G}) = C^*_u(\mathcal{G}) = C^*_uq(\mathcal{G})$.

Note that when $\mathcal{G}$ is the coarse groupoid associated to a discrete metric space with bounded geometry, Theorem 5.1 recovers the main result of [39] in the Hilbert space case (see Proposition 2.12). See Sect. 6.3 for more details.

The proof of Theorem 5.1 occupies the rest of this section. As mentioned in Sect. 1.3, it relies heavily on the coarse geometry of groupoids. Roughly speaking, we assign a length function on the groupoid thanks to the $\sigma$-compactness, which induces a metric on each source fibre. Hence we can appeal to the weapon of coarse geometry for a family of metric spaces, and consult the idea of [39, Theorem 3.3]. To start, let us recall some notion for metric families (comparing with Sect. 2.5).

A family of metric spaces $\{(X_i, d_i)\}_{i \in I}$ is said to have **uniformly bounded geometry** if for any $R > 0$, we have

$$\sup_{i \in I} \sup_{x \in X_i} \# B_{X_i}(x, r) < \infty.$$

For each $i \in I$, let $k_i$ be a kernel on $X_i$. We say that the family $(k_i)_{i \in I}$ has **uniformly finite propagation** if there exists $S > 0$ such that $k_i(x, y) = 0$ whenever $d_i(x, y) > S$ for $x, y \in X_i$ and $i \in I$.

**Definition 5.2.** Let $\{(X_i, d_i)\}_{i \in I}$ be a family of discrete metric spaces with uniformly bounded geometry. We say that $\{(X_i, d_i)\}_{i \in I}$ has **uniform Property A** if for any $R > 0, \varepsilon > 0$ and $i \in I$, there exists a normalised and symmetric kernel $k_i$ on $X_i$ of positive type and $(R, \varepsilon)$-variation such that the family $(k_i)_{i \in I}$ has uniformly finite propagation.

**Definition 5.3.** Let $\{(X_i, d_i)\}_{i \in I}$ be a family of metric spaces with uniformly bounded geometry, and $T_i$ be an operator in $\mathcal{B}(\ell^2(X_i))$ for each $i \in I$.

1. Say that $(T_i)_{i \in I}$ has **uniformly finite propagation** if there exists $R > 0$ such that for any $f, g \in \ell^\infty(X_i)$ with $d_i(\text{supp}(f), \text{supp}(g)) > R$, then $f T_i g = 0$. 


Given $\varepsilon > 0$, $(T_i)_{i \in I}$ is said to have uniformly finite $\varepsilon$-propagation if there exists $R > 0$ such that for any $f, g \in \ell^\infty(X_i)$ with $d_i(\text{supp}(f), \text{supp}(g)) > R$, then $\| f T_i g \| < \varepsilon \| f \|_\infty \| g \|_\infty$.

Say that $(T_i)_{i \in I}$ is uniformly quasi-local if for any $\varepsilon > 0$, the family $(T_i)_{i \in I}$ has uniformly finite $\varepsilon$-propagation.

Recall that we defined the notion of compactly uniform quasi-locality for operators on source fibres of a given groupoid in Definition 4.5. In fact, this has close relation with Definition 5.3(3) since we can endow each source fibre with a metric as follows. As explained above, this is the key ingredient to attack Theorem 5.1 where the coarse geometry of the underlying groupoid plays an important role.

Throughout the rest of this section, let $\mathcal{G}$ be a locally compact, $\sigma$-compact and étale groupoid with unit space $\mathcal{G}^{(0)}$. We also fix a sequence of subsets $\{ K_n \}_{n \in \mathbb{N}}$ in $\mathcal{G}$ satisfying the following:

- (M.1) $K_0 = \mathcal{G}^{(0)}$, and $K_n$ is compact and symmetric for $n \geq 1$.
- (M.2) $K_n \subset K_{n+1}^\circ$ for $n \geq 1$, where $K_{n+1}^\circ$ denotes the interior of $K_{n+1}$.
- (M.3) $K_n K_m \subset K_{n+m}$ for $n, m \geq 1$ and $\mathcal{G} = \bigcup_{n \geq 1} K_n$.

Note that we do not require $K_0 \subset K_1^\circ$ in general to deal with the case that $K_0 = \mathcal{G}^{(0)}$ might not be compact. However when $\mathcal{G}^{(0)}$ is compact, we further require that $K_0 \subset K_1^\circ$ for convenience. Using $\{ K_n \}_{n}$, we can endow a function $d_x$ on $\mathcal{G}_x \times \mathcal{G}_x$ for each $x \in \mathcal{G}^{(0)}$ as follows:

$$d_x(\gamma_1, \gamma_2) := \inf \{ n \in \mathbb{N} : \gamma_1 \gamma_2^{-1} \in K_n \} \quad \text{for} \quad \gamma_1, \gamma_2 \in \mathcal{G}_x. \quad (5.1)$$

It follows from (M.1)-(M.3) that $d_x$ is indeed a metric on $\mathcal{G}_x$.

**Lemma 5.4.** With the same notation as above, the family $\{ (\mathcal{G}_x, d_x) \}_{x \in \mathcal{G}^{(0)}}$ has uniformly bounded geometry.

**Proof.** Since $\mathcal{G}$ is étale, for each $n \in \mathbb{N}$ there exists a constant $C_n > 0$ such that $K_n$ can be covered by at most $C_n$-many open bisections, which implies that $\mathcal{Z}(\mathcal{G}^\gamma \cap K_n) \leq C_n$ for each $y \in \mathcal{G}^{(0)}$. Note that for each $n \in \mathbb{N}$, $x \in \mathcal{G}^{(0)}$, and $\alpha \in \mathcal{G}_x$, an element $\beta \in \mathcal{G}_x$ belongs to the ball $B_{\mathcal{G}_x}(\alpha, n)$ if and only if $\alpha \beta^{-1} \in K_n \cap \mathcal{G}^{(\alpha)}$. Hence the ball $B_{\mathcal{G}_x}(\alpha, n)$ contains at most $C_n$ points, which concludes the proof.

The following result indicates that the amenability of $\mathcal{G}$ affects the coarse geometry of the metric family $\{ (\mathcal{G}_x, d_x) \}_{x \in \mathcal{G}^{(0)}}$. This will be crucial in the proof of Theorem 5.1.

**Lemma 5.5.** With the same notation as above, the family $\{ (\mathcal{G}_x, d_x) \}_{x \in \mathcal{G}^{(0)}}$ has uniform Property A provided $\mathcal{G}$ is amenable.

**Proof.** Since $\mathcal{G}$ is amenable, Lemma 2.6 implies that for any $\varepsilon > 0$ and positive $n \in \mathbb{N}$, there exists a non-negative function $h \in C_r(\mathcal{G})$ of positive type such that:

- $h(x) \leq 1$ for any $x \in \mathcal{G}^{(0)}$, and $h(x) = 1$ for any $x \in r(K_n)$;
- $|1 - h(\gamma)| < \varepsilon$ for any $\gamma \in K_n$.

For each $x \in \mathcal{G}^{(0)}$, we define a function $k_x : \mathcal{G}_x \times \mathcal{G}_x \to [0, 1]$ by

$$k_x(\gamma, \beta) = h(\gamma \beta^{-1}) \quad \text{for} \quad \gamma, \beta \in \mathcal{G}_x.$$  

It follows that $k_x$ is a symmetric kernel on $\mathcal{G}_x$ of positive type such that:
the family \((k_x)_{x \in G^0}\) has uniformly finite propagation;
• for \(\gamma \in G_x\) with \(r(\gamma) \in r(K_n)\), then \(k_x(\gamma, \gamma) = 1\);
• for \(\gamma, \beta \in G_x\) with \(0 < d_x(\gamma, \beta) \leq n\), then \(|k_x(\gamma, \beta) - 1| < \varepsilon\).

Note that the kernel \(k_x\) might not have \((n, \varepsilon)\)-variation, hence we consider another kernel \(\tilde{k}_x\) on \(G_x\) defined by

\[
\tilde{k}_x(\gamma, \beta) = \begin{cases} 
k_x(\gamma, \beta), & \text{if } r(\gamma), r(\beta) \in r(K_n); \\
\delta_{\gamma, \beta}, & \text{otherwise}
\end{cases}
\]

Clearly \(\tilde{k}_x\) is a normalised symmetric kernel on \(G_x\) of positive type such that

• the family \((\tilde{k}_x)_{x \in G^0}\) has uniformly finite propagation;
• \(\tilde{k}_x\) has \((n, \varepsilon)\)-variation.

The last item comes from the fact that for any \(\gamma \in G_x\) with \(r(\gamma) / \in r(K_n)\) and \(\beta \in G_x\) with \(\beta \neq \gamma\), we have \(d_x(\gamma, \beta) > n\). Hence we conclude the proof.

\[\square\]

The following result is basic and the proof is straightforward, hence omitted.

**Lemma 5.6.** Given \(\alpha, \beta \in G_x\) for \(x \in G^0\) and an integer \(n > 0\), we have that \(\alpha \beta^{-1} \in K_n\) if and only if either \(d_x(\alpha, \beta) > n\), or \(\alpha = \beta \) with \(r(\alpha) \notin K_n\). If additionally \(G^0\) is compact, then \(\alpha \beta^{-1} \notin K_n\) if and only if \(d_x(\alpha, \beta) > n\).

By virtue of the metrics from (5.1) together with Proposition 4.4 and Lemma 5.6, we reach the following:

**Lemma 5.7.** With the same notation as above and given \(T \in \mathcal{L}(L^2(G))\), we have:

1. if \(T \in C_u[G]\), then \((\Phi_x(T))_{x \in G^0}\) has uniformly finite propagation.
2. if \(T \in C^*_uq(G)\), then \((\Phi_x(T))_{x \in G^0}\) is uniformly quasi-local.

If additionally \(G^0\) is compact, then both of the converse hold as well.

**Remark 5.8.** When \(G^0\) is compact, the above result shows that the notion of compactly uniform quasi-locality from Definition 4.5 coincides with that of uniform quasi-locality after endowing the metrics as above. However, note that this does not hold in general when \(G^0\) is non-compact. An easy example is to consider the identity operator, where each slice \(\Phi_x(T)\) is again the identity. However, it is direct to show that the identity does not belong to \(C^*_uq(G)\) when \(G^0\) is non-compact.

To provide a more concrete example, let us consider that \(X\) is a locally compact Hausdorff space which is not compact and regard it as a groupoid by setting \(\tilde{G} = G^0 = X\) with source and range maps being the identities on \(X\). It is clear that \(C^*_uq(G) = C^*_uq(G)G = C_0(X)\), and the constant function 1 on \(X\) is uniformly quasi-local, but not compactly uniformly quasi-local.

**Remark 5.9.** Up till now, we have introduced various figurations of quasi-locality and the terminologies bear a striking resemblance. As suggested by the anonymous referee, we provide the following diagram (Fig. 1) to conclude these notions and their relations. The right vertical implication can be checked directly using the properties of the metrics from (5.1). Note that the converse to each direction is incorrect.
To prove Theorem 5.1, we need another description for uniform quasi-locality in terms of relative commutants introduced in [40] (see also Proposition 4.15). Let us start with some extra notation. Let \((X, d)\) be a discrete metric space. A function \(f \in \ell^\infty(X)\) is called \(C\)-Lipschitz for some constant \(C > 0\) if \(|f(x) - f(y)| \leq Cd(x, y)\) for any \(x, y \in X\). Given \(L > 0\) and \(\varepsilon > 0\), we write \(T \in \text{Commut}_X(L, \varepsilon)\) if \(\|\left[\pi_0 T\right]\| < \varepsilon\) for any \(L\)-Lipschitz \(f \in \ell^\infty(X)\) with norm 1.

Tracking the parameters in the proof of [39, Lemma 5.2], we obtain the following version for a metric family:

**Lemma 5.10.** Let \(\{(X_i, d_i)\}_{i \in I}\) be a family of discrete metric spaces with uniformly bounded geometry and uniform Property A. For \(\varepsilon > 0\), \(L > 0\) and \(M > 0\), there exists \(s > 0\) such that for any \(i \in I\) and \(T_i \in \mathcal{B}(\ell^2(X_i))\) with \(T_i \in \text{Commut}_{X_i}(L, \varepsilon)\) and \(\|T_i\| \leq 2M\), there is a unit vector \(v_i \in \ell^2(X_i)\) with \(\text{diam}(\text{supp}(v_i)) \leq s\) such that

\[
\|T_i v_i\| \geq \|T_i\| - \frac{\varepsilon}{2}.
\]

Now we are in the position to prove Theorem 5.1. Before we dive into the details, let us explain the outline of the proof. The main goal is to approximate a given equivariant quasi-local operator \(T\) using elements in \(\Lambda(C_r(G))\). As indicated in the proof of Lemma 5.5, the amenability of \(G\) produces a family of kernels \((\tilde{k}_x)_{x \in G^{(0)}}\) which naturally produce a function \(g \in C_r(G)\). Moreover, these kernels can be modified to provide another family \((\tilde{k}'_x)_{x \in G^{(0)}}\) with nice behaviour. Conducting a “uniform” proof of [39, Theorem 3.3], the family of operators determined by \((\tilde{k}'_x)_{x \in G^{(0)}}\) indeed approximate the given operator \(T\). Finally thanks to the quasi-locality of \(T\), this approximating family of operators is simultaneously uniformly close to the slices of \(\Lambda(g)\), which concludes the proof.

**Proof of Theorem 5.1.** It is clear that \(C^*_r(G) \subseteq C^*_r(G) G^G \subseteq C^*_u(G) G^G\), hence it suffices to prove \(C^*_u(G) G^G \subseteq C^*_r(G)\). Fix an operator \(T \in C^*_u(G) G^G\) and \(\varepsilon > 0\), we aim to construct an operator \(S \in \Lambda(C_r(G))\) such that \(\|T - S\| \leq \varepsilon\). Throughout the proof, we also fix a sequence of subsets \(\{K_n\}_{n \in \mathbb{N}}\) of \(G\) satisfying (M.1)-(M.3).

Endow a metric \(d_x\) on \(G_x\) defined by (5.1) for each \(x \in G^{(0)}\).

By Proposition 4.4, there exists \(n_0 \in \mathbb{N}\) such that for any \(x \in G^{(0)}\) and \(A_x, B_x \subseteq G_x\) with \(A_x \cap (K_{n_0} \cdot B_x) = \emptyset\), then \(\|\chi_{A_x} \Phi_x(T) \chi_{B_x}\| \leq \frac{\varepsilon}{12}\). Moreover, it follows from Lemma 5.7(2) that the family \((\Phi_x(T))_{x \in G^{(0)}}\) is uniformly quasi-local. Hence following the proof of [40, Theorem 2.8, “(ii) \Rightarrow (i)"], there exists \(L > 0\) (which only depends on \(T\) and \(\varepsilon\)) such that \(\Phi_x(T) \in \text{Commut}_{G_x}(L, \frac{\varepsilon}{48})\) for any \(x \in G^{(0)}\).

Thanks to Lemma 5.4 and Lemma 5.5, we can apply Lemma 5.10 for \(\frac{\varepsilon}{2}\), \(L\) and \(M := \|T\|\) and obtain a constant \(s > 0\) satisfying the condition therein.

**Fig. 1.** A summary for different notions of quasi-locality.
We set \( N := \sup_{x \in G^{(0)}} \sup_{\gamma \in G_x} \| B_{G_x}(\gamma, s + \frac{1}{T}) \| \) and \( \widetilde{\varepsilon} = \min\left\{ \frac{\varepsilon}{8MN}, 1 \right\} \), and choose an integer \( \tilde{n} > s + \frac{1}{T} + n_0 \).

Since \( \mathcal{G} \) is amenable, it follows from Lemma 2.6 that there exists a non-negative function \( h \in C_c(\mathcal{G}) \) of positive type such that

- \( h(x) \leq 1 \) for any \( x \in G^{(0)} \), and \( h(x) = 1 \) for any \( x \in r(K_{\tilde{n}}) \);
- \( |1 - h(\gamma)| < \widetilde{\varepsilon} \) for any \( \gamma \in K_{\tilde{n}} \).

For each \( x \in G^{(0)} \) and \( \gamma, \beta \in G_x \), we define \( k_x(\gamma, \beta) = h(\gamma \beta^{-1}) \) and

\[
\tilde{k}_x(\gamma, \beta) = \begin{cases} 
h(\gamma \beta^{-1}), & \text{if } r(\gamma), r(\beta) \in r(K_{\tilde{n}}); \\
\delta_{\gamma, \beta}, & \text{otherwise.}
\end{cases}
\]

The proof of Lemma 5.5 shows that \( \tilde{k}_x \) is a normalised symmetric kernel on \( G_x \) of positive type and \( (\tilde{n}, \tilde{\varepsilon}) \)-variation such that the family \( \{\tilde{k}_x\}_{x \in G^{(0)}} \) has uniformly finite propagation.

For each \( x \in G^{(0)} \), we consider the operator \( T'_x = m_{\tilde{k}_x}(\Phi_x(T)) - \Phi_x(T) \) where \( m_{\tilde{k}_x} \) is the Schur multiplier.

It is clear that

\[
\sup_{x \in G^{(0)}} \| T'_x \| \leq 2M \quad \text{and} \quad T'_x \in \text{Commut}_{G_x}(L, \frac{\varepsilon}{24}).
\]

For the family \( \{T'_x\}_x \) and the parameters above, Lemma 5.10 implies that for each \( x \in G^{(0)} \) there exists a unit vector \( v_x \in \ell^2(G_x) \) with \( \text{diam}(\text{supp}(v_x)) \leq s \) such that

\[
\| T'_x v_x \| \geq \| T'_x \| - \frac{\varepsilon}{4}, \quad (5.2)
\]

Setting \( V_x = \{ \gamma \in G_x : d_x(\text{supp}(v_x), \gamma) < \frac{1}{T} \} \) for each \( x \in G^{(0)} \), we have

\[
\| T'_x v_x \| \leq \| \chi_{V_x}(T'_x v_x) \| + \| \chi_{G_x \setminus V_x}(T'_x v_x) \| \leq \| \chi_{V_x}(T'_x v_x) \| + \frac{\varepsilon}{24}, \quad (5.3)
\]

where the second inequality comes from \( T'_x \in \text{Commut}_{G_x}(L, \frac{\varepsilon}{24}) \).

For \( \gamma \in V_x \), we have

\[
| (T'_x v_x)(\gamma) | = \left| \sum_{\alpha \in \text{supp}(v_x)} (\tilde{k}_x(\gamma, \alpha) - 1) \Phi_x(T)_{\gamma, \alpha} v_x(\alpha) \right| \\
\leq \left( \sum_{\alpha \in \text{supp}(v_x)} |(\tilde{k}_x(\gamma, \alpha) - 1)|^2 \right)^{1/2} \| \Phi_x(T) v_x \| \leq \tilde{\varepsilon} \sqrt{N M} \leq \frac{\varepsilon}{8\sqrt{N}},
\]

where the second inequality comes from the Cauchy-Schwarz inequality and the third one is due to that \( \tilde{k}_x \) has \( (\tilde{n}, \tilde{\varepsilon}) \)-variation. Here \( \Phi_x(T)_{\gamma, \alpha} = \langle \delta_\gamma, \Phi_x(T)(\delta_\alpha) \rangle \) as defined in Sect. 2.5. Hence

\[
\| \chi_{V_x}(T'_x v_x) \|^2 = \sum_{\gamma \in V_x} | (T'_x v_x)(\gamma) |^2 \leq \frac{\varepsilon^2}{64}. \quad (5.4)
\]
Combining (5.2), (5.3) and (5.4) together, we obtain that for any $x \in \mathcal{G}^{(0)}$,

$$\|m_{k_x}(\Phi_x(T)) - \Phi_x(T)\| = \|T'\| \leq \|T'_x v_x\| + \frac{\varepsilon}{4} \leq \|\chi v_{\varepsilon}(T'_x v_x)\| + \frac{\varepsilon}{24} + \frac{\varepsilon}{4} < \frac{\varepsilon}{2} \quad (5.5)$$

On the other hand, it follows from Proposition 3.31 that $T = \Lambda(f_T)$ for some $f_T \in C_b(\mathcal{G})$. Hence for $x \in \mathcal{G}^{(0)}$ and $\gamma, \alpha \in \mathcal{G}_x$ we have

$$(m_{k_x}(\Phi_x(T)))_{\gamma,\alpha} = k_x(\gamma, \alpha) f_T(\gamma \alpha^{-1}) = h(\gamma \alpha^{-1}) f_T(\gamma \alpha^{-1}),$$

which implies that

$$m_{k_x}(\Phi_x(T)) = \Phi_x(\Lambda(h \cdot f_T)). \quad (5.6)$$

Set $C_x := \{ \gamma \in \mathcal{G}_x : r(\gamma) \in r(K_{\mathcal{G}}) \}$ and $D_x := \mathcal{G}_x \setminus C_x$, and decompose the operator $m_{k_x}(\Phi_x(T)) - m_{k_x}(\Phi_x(T))$ as follows:

$$m_{k_x}(\Phi_x(T)) - m_{k_x}(\Phi_x(T)) = (\chi c_x + \chi d_x) \left( m_{k_x - k_x}(\Phi_x(T)) \right) (\chi c_x + \chi d_x)$$

$$= m_{k_x - k_x} \left( \chi c_x \Phi_x(T) \chi c_x + \chi d_x \Phi_x(T) \chi d_x + \chi c_x \Phi_x(T) \chi d_x + \chi d_x \Phi_x(T) \chi c_x \right)$$

$$= m_{k_x - k_x} \left( \chi d_x \Phi_x(T) \chi d_x + \chi c_x \Phi_x(T) \chi d_x + \chi d_x \Phi_x(T) \chi c_x \right),$$

where we use the fact that $k_x$ and $\tilde{k}_x$ coincide on $C_x$ in the last equality.

Note that for $\gamma, \beta \in D_x$, we have $\gamma \beta^{-1} \notin K_{\mathcal{G}}$ (otherwise, we would have $r(\gamma) = r(\gamma \beta^{-1}) \notin r(K_{\mathcal{G}})$, which contradicts to that $\gamma \in D_x$).

Since $\tilde{n} > n_0$, we have $\|\chi d_x \Phi_x(T) \chi d_x\| < \frac{\varepsilon}{12}$. Similarly for $\gamma \in D_x$ and $\beta \in C_x$, we have $\gamma \beta^{-1} \notin K_{\mathcal{G}}$. This implies that $\|\chi d_x \Phi_x(T) \chi c_x\| < \frac{\varepsilon}{12}$ and $\|\chi c_x \Phi_x(T) \chi d_x\| < \frac{\varepsilon}{12}$. Hence

$$\|\chi d_x \Phi_x(T) \chi d_x + \chi c_x \Phi_x(T) \chi d_x + \chi d_x \Phi_x(T) \chi c_x\| < \frac{\varepsilon}{4},$$

which implies that

$$\|m_{k_x}(\Phi_x(T)) - m_{k_x}(\Phi_x(T))\| < \frac{\varepsilon}{2}. \quad (5.7)$$

Finally combining (5.5), (5.6) and (5.7), we obtain

$$\|\Lambda(h \cdot f_T) - T\| = \sup_{x \in \mathcal{G}^{(0)}} \|m_{k_x}(\Phi_x(T)) - \Phi_x(T)\| < \varepsilon.$$

Note that $h \cdot f_T \in C_c(\mathcal{G})$, hence we conclude that $T \in C_{r}^{\ast}(\mathcal{G})$. \qed

6. Examples

In this section, we apply Theorem 5.1 to the examples mentioned in Sect. 2.6.
6.1. Discrete groups. Let $\Gamma$ be a countable discrete group with unit $1_\Gamma$. As mentioned in Sect. 2.6.1, the reduced groupoid $C^*$-algebra of $\Gamma$ is the same as the reduced group $C^*$-algebra $C^*_r(\Gamma)$, and $\mathcal{L}(\ell^2(\Gamma)) = \mathcal{B}(\ell^2(\Gamma))$.

Writing in the matrix form, obviously an operator $T \in \mathcal{B}(\ell^2(\Gamma))$ is $\Gamma$-equivariant if and only if $T_{\alpha\gamma,\beta\gamma} = T_{\alpha,\beta}$ for any $\alpha, \beta, \gamma \in \Gamma$. Equipping $\Gamma$ with an arbitrary proper word length metric, it is clear that $T$ is compactly supported if and only if $T$ has finite propagation in the sense of Definition 2.8. Moreover, the notion of quasi-locality for $T$ from Definition 2.10 coincides with that from Definition 4.2(3). Hence in this case, Theorem 5.1 recovers the following known result:

**Theorem 6.1.** Let $\Gamma$ be a countable discrete amenable group. Then for an operator $T \in \mathcal{B}(\ell^2(\Gamma))$, the following are equivalent:

1. $T$ belongs to $C^*_r(\Gamma)$;
2. $T$ is $\Gamma$-invariant and belongs to $C^*_u(\Gamma)$;
3. $T$ is $\Gamma$-invariant and quasi-local.

**Remark 6.2.** The above theorem also holds for groups with the approximation property from [19] (see also [9]). In fact, it was shown in [48] that $C^*_u(\Gamma) = C^*_r(\Gamma)$ if $\Gamma$ has the approximation property. Moreover, it follows from [19,31] that amenable groups have the approximation property, and the approximation property implies Property A. Therefore combining with [39, Theorem 3.3], we obtain that if $\Gamma$ has the approximation property, then $C^*_r(\Gamma) = C^*_u(\Gamma) = C^*_{uq}(\Gamma)$.

6.2. Pair groupoids. Let $X$ be a set and we consider the pair groupoid $X \times X$ from Sect. 2.6.2. By Theorem 3.13, we know that the map

$$\Phi : \mathcal{L}(\ell^2(X))^{X \times X} \longrightarrow \prod_{x \in X} \mathcal{B}(\ell^2(X))$$

is a $C^*$-monomorphism with image consisting of constant families $(T')_{x \in X}$ which are vector-wise uniformly quasi-local. It is not hard (similar to, but much easier than, the analysis in Example 3.25) to see that $(T')_{x \in X}$ is vector-wise uniformly quasi-local for any $(T') \in \mathcal{B}(\ell^2(X))$. Hence similar to Lemma 3.26, we obtain the following:

**Lemma 6.3.** Let $X$ be a set and $X \times X$ be the pair groupoid. Then for some fixed $x_0 \in X$, the map

$$\Theta : \mathcal{L}(\ell^2(X))^{X \times X} \longrightarrow \mathcal{B}(\ell^2(X)), \quad T \mapsto \Phi_{x_0}(T)$$

is a $C^*$-isomorphism. Note that $\Theta$ is independent of the choice of $x_0$.

Analogous to the discussion in Example 3.32, the map $\Theta$ in (6.1) extends the map $\Theta : C^*_r(X \times X) \cong \mathcal{K}(\ell^2(X))$ in (2.6). Hence we abuse the notation.

Direct calculations show that an operator $T \in \mathcal{L}(\ell^2(X \times X))^{X \times X}$ is quasi-local in the sense of Definition 4.2(3) if and only if $\Phi_0(T) \in \mathcal{K}(\ell^2(X))$, which implies that $C^*_{uq}(X \times X)^{X \times X} = C^*_r(X \times X)$. Hence Theorem 5.1 holds trivially for the case of pair groupoids.
6.3. Coarse groupoids. This example is our motivation of the whole work, and details have already been spread in Sect. 3 and Sect. 4. Here we recall the whole picture again.

Let \((X, d)\) be a discrete metric space with bounded geometry, and \(G(X)\) be the associated coarse groupoid. Recall from Lemma 3.26 that for a fixed \(x_0 \in X\), the map
\[
\Theta : \mathcal{L}(L^2(G(X)))^{G(X)} \longrightarrow \{T \in \mathcal{B}(\ell^2(X)) : T \text{ satisfies Equation (3.5)}\}
\]
given by \(T \mapsto \Phi_{x_0}(T)\) is a \(C^*\)-isomorphism. Example 3.32 implies that \(\Theta\) extends the isomorphism
\[
C^*_r(G(X)) \cong C^*_u(X)
\]
from (2.7). Furthermore, Example 4.9 implies that the restriction \(\Theta_1 : C^*_uq(G(X)) \longrightarrow C^*_uq(X)\) is also an isomorphism. Therefore, our main result recovers [39, Theorem 3.3] in the Hilbert space case.

6.4. Transformation groupoids. Let \(X\) be a locally compact \(\sigma\)-compact space, and \(\Gamma\) be a countable discrete group acting on \(X\). We consider the transformation groupoid \(X \rtimes \Gamma\) in Sect. 2.6.4.

First note that \((X \rtimes \Gamma)_x\) is identified with \(\Gamma\) by \((\alpha x, \alpha) \mapsto \alpha\) for \(x \in X\) and \(\alpha \in \Gamma\). Under these identifications and equipping \(\Gamma\) with a proper word length metric, we have the following. The proof is straightforward, hence omitted.

**Lemma 6.4.** For a family \((T_x)_{x \in X} \in \prod_{x \in X} \mathcal{B}(\ell^2(\Gamma))\), we have:

1. Writing in the matrix form, \((T_x)_{x \in X}\) is \((X \rtimes \Gamma)\)-equivariant if and only if the following holds:
   \[
   (T_{\gamma x})_{\alpha, \beta} = (T_x)_{\alpha \gamma, \beta \gamma}\quad \text{for any } x \in X \text{ and } \alpha, \beta, \gamma \in \Gamma. \tag{6.2}
   \]
2. \((T_x)_{x \in X}\) belongs to the image of \(i\) from (3.3) if and only if the map \(X \rightarrow \mathcal{B}(\ell^2(\Gamma)), x \mapsto T_x\) is WOT-continuous.
3. When \(X\) is compact, \((T_x)_{x \in X}\) is compactly uniformly quasi-local if and only if \((T_x)_{x \in X}\) is uniformly quasi-local in the sense of Definition 5.3.(3).

Note that condition (3) above also holds for certain non-compact \(X\), while for simplicity we only focus on the compact case in the sequel. Combining Corollary 4.6 with Lemma 6.4, we reach the following:

**Corollary 6.5.** When \(X\) is compact, we have a \(C^*\)-monomorphism
\[
\Phi : C^*_uq(X \rtimes \Gamma)^{X \rtimes \Gamma} \longrightarrow \prod_{x \in X} \mathcal{B}(\ell^2(\Gamma)), \quad T \mapsto (\Phi_x(T))_{x \in X}
\]
with range consisting of elements \((T_x)_{x \in X}\) which is uniformly quasi-local and satisfies (6.2) such that the map \(x \mapsto T_x\) is WOT-continuous.

Hence applying Theorem 5.1, we reach the following characterisation for the reduced crossed product \(C(X) \rtimes_r \Gamma\):

**Theorem 6.6.** Let \(X\) be a compact space and \(\Gamma\) be a countable discrete group with an amenable action on \(X\). Then \(C(X) \rtimes_r \Gamma\) is \(C^*\)-isomorphic to the \(C^*\)-subalgebra in \(\prod_{x \in X} \mathcal{B}(\ell^2(\Gamma))\) consisting of elements \((T_x)_{x \in X}\) which is uniformly quasi-local and satisfies (6.2) such that the map \(x \mapsto T_x\) is WOT-continuous.
7. Beyond Equivariance

In Sect. 5, we study the equivariant parts in uniform Roe and quasi-local algebras for groupoids, and our main result (Theorem 5.1) provides a sufficient condition to ensure that they are the same. In this section, we will turn to these $C^*$-algebras themselves without the restriction of equivariance. Our approach is to consider the semi-direct product groupoids and transfer the general case to the equivariant one, inspired by the discussions in [1, Section 6].

7.1. Semi-direct product groupoids. Here we collect some preliminary knowledge on groupoid actions and semi-direct product groupoids, and guide readers to [1] for more details.

Let $X$ be a locally compact space. A fibre space over $X$ is a pair $(Y, p)$, where $Y$ is a locally compact space and $p : Y \to X$ is a continuous surjective map. For two fibre spaces $(Y_1, p_1)$ and $(Y_2, p_2)$ over $X$, we form their fibred product to be

$$Y_1 \times_{p_1} p_2 Y_2 := \{(y_1, y_2) \in Y_1 \times Y_2 : p_1(y_1) = p_2(y_2)\},$$

equipped with the topology induced by the product topology.

Definition 7.1. [1, Definition 1.2] Let $G$ be a locally compact groupoid. A left $G$-space is a fibre space $(Y, p)$ over $G(0)$, equipped with a continuous map $(y', y) \mapsto y y'$ from $G \times_p Y$ to $Y$ (called a $G$-action on $(Y, p)$) which satisfies the following:

- $p(y y') = r(y')$ for $(y', y) \in G \times_p Y$, and $p(y)y = y$;
- $y_2(y_1 y) = (y_2 y_1)y$ for $(y_1, y) \in G \times_p G$ and $s(y_2) = r(y_1)$.

Given a $G$-space $(Y, p)$, the associated semi-direct product groupoid $Y \rtimes G$ is defined to be $Y \rtimes_p G$ as a topological space. For $(y, \gamma) \in Y \rtimes G$, define its range to be $(y, r(\gamma))$ and its source to be $(\gamma^{-1} y, s(\gamma))$. The product and inverse are given by

$$(y, \gamma)(\gamma^{-1} y', \gamma') = (y, \gamma \gamma') \quad \text{and} \quad (y, \gamma)^{-1} = (\gamma^{-1} y, \gamma^{-1}).$$

Clearly $(y, p(y)) \mapsto y$ is a homeomorphism from the unit space of $Y \rtimes G$ onto $Y$. Hence from now on, we regard $Y$ as the unit space of the groupoid $Y \rtimes G$. As shown in [1, Proposition 1.4 and Proposition 1.6], the range map $r : Y \rtimes G \to Y$ is always open, and $Y \rtimes G$ is étale when the groupoid $G$ itself is étale.

The class of groupoid actions we are interested in come from certain compactification of the given groupoid. For completeness, let us recall the following:

Definition 7.2. Let $(Y, p)$ be a fibre space over a locally compact space $X$.

1. We say that $(Y, p)$ is fibrewise compact if $p$ is proper, i.e., $p^{-1}(K)$ is compact for any compact $K \subseteq X$.

2. A fibrewise compactification of $(Y, p)$ is a fibrewise compact fibre space $(Z, q)$ together with a continuous map $\varphi : Y \to Z$, which is a homeomorphism onto an open dense subset of $Z$ and satisfies $q \circ \varphi = p$. Usually we regard $Y$ as a subset of $Z$ in this case, and the morphism $\varphi$ is just the inclusion.

To introduce concrete fibrewise compactifications, we need refer to the Gelfand spectra. Let $(Y, p)$ be a fibre space over a locally compact space $X$. Denote

$$p^*(C_0(X)) := \{ f \circ p : f \in C_0(X) \}.$$
and \( C_0(Y, p) \) to be the closure of the following set in \( C_b(Y) \):

\[
C_c(Y, p) = \{ g \in C_b(Y) : \text{there is a compact } K \subseteq X, \text{ supp}(g) \subseteq p^{-1}(K) \}.
\]

It was shown in [1, Proposition A.2] that the fibrewise compactifications of \((Y, p)\) are in one-to-one correspondence with the \( C^* \)-subalgebras of \( C_0(Y, p) \) containing \( p^* (C_0(X)) \). The fibrewise compactification associated to \( C_0(Y, p) \) is called the fibrewise Stone-Čech compactification of \((Y, p)\), denoted by \((\beta_Y Y, \rho_Y)\).

As a special case, consider the following: Let \( G \) be a locally compact étale groupoid. Then the groupoid multiplication of \( G \) provides a \( G \)-action on the fibre space \((G, r)\), i.e., \( G \times_r G \rightarrow G \) defined by \((\gamma_1, \gamma_2) \mapsto \gamma_1 \gamma_2 \). Moreover, it was shown in [1, Proposition 2.5] that this \( G \)-action extends in a unique way to a \( G \)-action on the fibrewise Stone-Čech compactification \((\beta_Y Y, \rho_Y)\). Hence, we can consider the semi-direct product groupoid \( \beta_Y Y \ltimes G \).

Following the terminology in [1], a locally compact étale groupoid \( G \) is said to be strongly amenable at infinity if \( \beta_Y Y \ltimes G \) is amenable. As shown in [1], this has a close relation with the notion of exactness for groupoids. By definition, \( G \) is called \( C^* \)-exact if the reduced groupoid \( C^* \)-algebra \( C^*_r(G) \) is exact. Also, recall that \( G \) is said to be weakly inner amenable if for any compact subset \( K \subseteq G \) and any \( \varepsilon > 0 \), there exists a continuous bounded positive definite function \( f \) on the product groupoid \( G \times G \) which is properly supported such that \(|f(\gamma, \gamma) - 1| < \varepsilon \) for all \( \gamma \in K \).

**Proposition 7.3.** [1, Corollary 7.4 and Theorem 8.6] Let \( G \) be a locally compact, second countable and étale groupoid. If \( G \) is strongly amenable at infinity, then \( G \) is \( C^* \)-exact. Conversely, if \( G \) is weakly inner amenable and \( C^* \)-exact, then \( G \) is strongly amenable at infinity.

Throughout the rest of this section, let \( G \) be a locally compact étale groupoid and \((\beta_Y Y, \rho_Y)\) be the fibrewise Stone-Čech compactification of \((G, r)\). Consider the \( G \)-action on \((\beta_Y Y, \rho_Y)\) induced by the groupoid multiplication in \( G \), and the associated semi-direct product groupoid \( \beta_Y Y \ltimes G \). In the next few subsections, we will transfer the uniform Roe and quasi-local algebras for \( G \) to the equivariant part of the corresponding algebras for \( \beta_Y Y \ltimes G \).

### 7.2. Compactly supported case

Here we deal with the case of compactly supported operators and the uniform Roe algebra, which was originally studied in [1].

First, we show that our notion of the uniform Roe algebra \( C^*_u(G) \) from Definition 4.10(1) coincides with that of the uniform \( C^* \)-algebra from [1, Definition 6.1]. To start, let us recall an extra algebra \( C_t(G \ast_s G) \) from [1] (see also [4]). By definition,

\[
G \ast_s G := G \ast_s G = \{ (\gamma, \alpha) \in G \times G : s(\gamma) = s(\alpha) \}.
\]

A tube is a subset of \( G \ast_s G \) whose image by the map \((\gamma, \alpha) \mapsto \gamma \alpha^{-1}\) is relatively compact in \( G \). Denote by \( C_t(G \ast_s G) \) the space of continuous bounded functions on \( G \ast_s G \) with support in a tube. Recall from [1, Section 6] that there is a \( * \)-monomorphism \( T : C_t(G \ast_s G) \rightarrow \mathcal{L}(L^2(G)) \) given by:

\[
(T f)(\xi) := \sum_{\alpha \in G_0(\gamma)} f(\gamma, \alpha) \xi(\alpha) \quad \text{where} \quad \xi \in L^2(G) \text{ and } \gamma \in G,
\]

Note that the formula given in [1] is slightly different from the above, where the set \( G \ast_s G \) is considered instead of \( G \ast_s G \). Here we follow the formula from [4, Section 6.5] which is more convenient and compatible to our setting.
and the uniform \( C^*\)-algebra of \( \mathcal{G} \) is defined to be the norm closure of \( \mathcal{T}(C_r(\mathcal{G} \rtimes_s \mathcal{G})) \) in \( \mathcal{L}(L^2(\mathcal{G})) \).

The following lemma relates the algebra \( C_r(\mathcal{G} \rtimes_s \mathcal{G}) \) to our notion of compact supported operators:

**Lemma 7.4.** The image \( \mathcal{T}(C_r(\mathcal{G} \rtimes_s \mathcal{G})) \) consists of all compactly supported operators in the sense of Definition 4.2(2). Hence the uniform \( C^*\)-algebra of \( \mathcal{G} \) from [1, Definition 6.1] coincides with the uniform Roe algebra of \( \mathcal{G} \) from Definition 4.10(1).

**Proof.** First it is clear that given \( f \in C_r(\mathcal{G} \rtimes_s \mathcal{G}) \) with support in \( \{(\gamma, \alpha) \in \mathcal{G} \rtimes_s \mathcal{G} : \gamma \alpha^{-1} \in K\} \) for some compact \( K \subseteq \mathcal{G} \), then the operator \( \mathcal{T} f \) has support in \( \mathcal{K} \). Conversely, for an operator \( T \in \mathcal{L}(L^2(\mathcal{G})) \) with support in some compact \( K \subseteq \mathcal{G} \), we define a function \( f : \mathcal{G} \rtimes_s \mathcal{G} \to \mathbb{C} \) as follows: for \( (\gamma, \alpha) \in \mathcal{G} \rtimes_s \mathcal{G} \) with \( s(\gamma) = s(\alpha) = x \), set \( f(\gamma, \alpha) := (\delta_{\gamma}, \Phi_x(T)\delta_{\alpha})_{\ell^2(\mathcal{G})} \). Thanks to Corollary 4.6, it is easy to show that \( f \in C_r(\mathcal{G} \rtimes_s \mathcal{G}) \) and \( \mathcal{T}(f) = T \). Hence we conclude the proof. \( \Box \)

Next, we recall from [1, Lemma 6.3] that there is a \( *\)-algebraic isomorphism \( \vartheta : C_c(\beta_1 \mathcal{G} \rtimes \mathcal{G}) \to C_r(\mathcal{G} \rtimes_s \mathcal{G}) \) given by

\[
\theta(f)(\gamma, \alpha) := f(\gamma, \gamma \alpha^{-1}) \text{ for } f \in C_c(\beta_1 \mathcal{G} \rtimes \mathcal{G}).
\]

Hence we obtain a \( *\)-monomorphism

\[
\theta := \mathcal{T} \circ \vartheta : C_c(\beta_1 \mathcal{G} \rtimes \mathcal{G}) \to \mathcal{L}(L^2(\mathcal{G})).
\]

Direct calculations show that

\[
(f(\gamma))(\gamma) = \sum_{\alpha \in \mathcal{G}_{r(\gamma)}} f(\gamma, \gamma \alpha^{-1}) \xi(\alpha) \text{ for } f \in C_c(\beta_1 \mathcal{G} \rtimes \mathcal{G}) \text{ and } \xi \in L^2(\mathcal{G}). \quad (7.1)
\]

It is proved in [1, Theorem 6.4] that \( \theta \) extends to a \( C^*\)-isomorphism

\[
\Theta : C^*_r(\beta_1 \mathcal{G} \rtimes \mathcal{G}) \xrightarrow{\cong} C^*_u(\mathcal{G}).
\]

Combining with Proposition 4.12, we obtain a \( *\)-isomorphism (still denoted by \( \Theta \)):

\[
\Theta : \mathbb{C}_u[\beta_1 \mathcal{G} \rtimes \mathcal{G}] \rtimes \mathcal{G} \mathcal{L}(L^2(\beta_1 \mathcal{G} \rtimes \mathcal{G})) \xrightarrow{\cong} C^*_u(\mathcal{G}) \quad (7.2)
\]

such that \( \Theta(\Lambda(f)) = \theta(f) \) for \( f \in C_c(\beta_1 \mathcal{G} \rtimes \mathcal{G}) \).

7.3. From equivariance to non-equivariance. Now we aim to extend the map \( \Theta \) from (7.2) to all \( (\beta_1 \mathcal{G} \rtimes \mathcal{G})\)-equivariant operators in \( \mathcal{L}(L^2(\beta_1 \mathcal{G} \rtimes \mathcal{G})) \). The main observation is that the operator \( \theta(f) \) for \( f \in C_c(\beta_1 \mathcal{G} \rtimes \mathcal{G}) \) defined in (7.1) can be written as a composition of three maps, which will be explained in details.

First recall that the unit space of \( \beta_1 \mathcal{G} \rtimes \mathcal{G} \) is \( \beta_1 \mathcal{G} \) and note that the source fibre at \( z \in \beta_1 \mathcal{G} \) is

\[
(\beta_1 \mathcal{G} \rtimes \mathcal{G})_z = \{(\gamma z, \gamma) : \gamma \in \mathcal{G} \text{ such that } s(\gamma) = r_{\beta}(z)\},
\]

which is bijectively mapped onto \( \mathcal{G}_{r_{\beta}(z)} \) via the map

\[
p_{z} : (\beta_1 \mathcal{G} \rtimes \mathcal{G})_z \to \mathcal{G}_{r_{\beta}(z)}, \quad (\gamma z, \gamma) \mapsto \gamma.
\]
Hence in the following, we will write \((γz, γ) ∈ β_tG × G\) such that \(s(γ) = r_β(z)\) for a general element in \(β_tG × G\). Furthermore, \(p_z\) induces a unitary

\[
P_z : ℓ^2((β_tG × G)_z) → ℓ^2(G_{r_β(z)}) \quad \text{by} \quad P_z(δ_{(γz, γ)}) = δ_γ,
\]

which further induces a \(*\)-isomorphism

\[
\text{Ad}_{p_z} : B(ℓ^2((β_tG × G)_z)) → B(ℓ^2(G_{r_β(z)})), \quad T → p_zTP_z^*.
\]

(7.3)

Let us consider the map \(ι : C_c(G) → C_c(β_tG × G)\) defined by

\[
(ιξ)(γ, γ) := ξ(γ) \quad \text{for} \quad (γz, γ) ∈ β_tG × G \quad \text{and} \quad ξ ∈ C_c(G).
\]

This map is well-defined since for \(ξ ∈ C_c(G)\) with support in a compact set \(K ⊆ G\), then \(ιξ\) has support in \(r_β^{-1}(r(K)) × K\), which is compact in \(β_tG × G\). Moreover for \(ξ ∈ C_c(G)\), we have

\[
∥ιξ∥_{L^2(β_tG × G)}^2 = \sup_{z ∈ β_tG} ∥(ιξ)|_{(β_tG × G)_z}\|^2 = \sup_{z ∈ β_tG} \sum_{γ ∈ G_{r_β(z)}} |ξ(γ)|^2 = ∥ιξ∥_{L^2(G)}^2.
\]

Hence \(ι\) can be extended to an isometry (with the same notation):

\[
ι : L^2(G) → L^2(β_tG × G).
\]

On the other hand, we define another map

\[
κ : C_c(β_tG × G) → C_c(G) \quad \text{by} \quad (κη)(γ) := η(γ, γ)
\]

for \(η ∈ C_c(β_tG × G)\) and \(γ ∈ G\). It is clear that \(κ\) is well-defined. Moreover, we have:

\[
∥κη∥_{L^2(G)}^2 = \sup_{x ∈ G(0)} \sum_{γ ∈ G_x} |(κη)(γ)|^2 = \sup_{x ∈ G(0)} \sum_{γ ∈ G_x} |η(γ, γ)|^2
\]

\[
≤ \sup_{z ∈ β_tG} \sum_{γ ∈ G_{r_β(z)}} |η(γz, γ)|^2 = ∥η∥_{L^2(β_tG × G)}^2.
\]

Hence \(κ\) can be extended to a contractive map (with the same notation):

\[
κ : L^2(β_tG × G) → L^2(G).
\]

Now we are in the position to extend the map \(Θ\) from (7.2) to all \((β_tG × G)\)-equivariant operators in \(L(L^2(β_tG × G))\). More precisely, we define

\[
Θ : L(L^2(β_tG × G))^{β_tG × G} → L(L^2(G))
\]

by

\[
Θ(T) := κ ∘ T ∘ ι \quad \text{for} \quad T ∈ L(L^2(β_tG × G))^{β_tG × G}.
\]

(7.4)

In other words, the operator \(Θ(T)\) makes the following diagram commutes:

\[
\begin{array}{c}
L^2(G) \xrightarrow{ι} L^2(β_tG × G) \\
\downarrow Θ(T) \quad \downarrow T \\
L^2(G) \xrightarrow{κ} L^2(β_tG × G).
\end{array}
\]
We will show in Lemma 7.6 below that the image of Θ is indeed contained in \( \mathcal{L}(L^2(G)) \). First, note that Proposition 3.31 implies that \( T \in \mathcal{L}(L^2(\beta r G \rtimes G))^{\beta r G \rtimes G} \) can be written as \( T = \Lambda(f_T) \) where \( f_T \) is the associated left convolver. Direct calculations show that the \( \Theta(T) \) satisfies the following:

\[
(\Theta(T)\xi)(y) = \sum_{\alpha \in \mathcal{G}_{\mu(y)}} f_T(y, y\alpha^{-1})\xi(\alpha) \quad \text{for} \quad \xi \in L^2(G). \tag{7.5}
\]

Comparing (7.1) with (7.5), it is clear that \( \Theta \) in (7.4) indeed extends the isomorphism \( \Theta : C^*_r(\beta r G \rtimes G) \xrightarrow{\cong} \mathcal{C}^*_u(G) \) in (7.2). Hence again we use the same notation.

**Remark 7.5.** It is pointed out by the anonymous referee that the map \( \Theta \) from (7.4) can be defined in a more tidy way using interior tensor products of Hilbert \( C^* \)-modules as follows. Let \( \tau : G^{(0)} \to \beta r G \) be the inclusion map, which induces a \(*\)-homomorphism \( \tau^* : C_0(\beta r G) \to C_0(G^{(0)}) \). It is easy to see that \( L^2(\beta r G \rtimes G) \) is isomorphic to the interior tensor product \( L^2(G) \otimes_{\tau^*} C_0(\beta r G) \) as Hilbert \( C^*_r(\beta r G) \)-modules. Hence the interior tensor product \( L^2(\beta r G \rtimes G) \otimes_{\tau^*} C_0(G^{(0)}) \) is isomorphic to \( L^2(G) \) as Hilbert \( C_0(G^{(0)}) \)-modules due to that the composition \( G^{(0)} \xrightarrow{\tau} \beta r G \xrightarrow{\tau_\beta} G^{(0)} \) is the identity. Moreover, direct calculations show that \( \Theta(T) \) in (7.4) is identified with \( T \otimes_{\tau^*} 1 \) under the isomorphism above.

On the other hand, using (7.5) it is straightforward to prove:

**Lemma 7.6.** With the same notation as above, we have:

1. for \( T \in \mathcal{L}(L^2(\beta r G \rtimes G))^{\beta r G \rtimes G} \), the operator \( \Theta(T) \) is an adjointable operator on \( L^2(G) \) with adjoint \( \Theta(T^*) \);
2. for \( T, S \in \mathcal{L}(L^2(\beta r G \rtimes G))^{\beta r G \rtimes G} \), we have \( \Theta(TS) = \Theta(T) \Theta(S) \);
3. \( \Theta \) is injective.

Note that item (1) and (2) can be deduced directly from the alternative viewpoint of \( \Theta(T) \) given in Remark 7.5. However, we still need (7.5) to conclude item (3).

Combining the analysis above, we reach the following:

**Corollary 7.7.** The map \( \Theta : \mathcal{L}(L^2(\beta r G \rtimes G))^{\beta r G \rtimes G} \to \mathcal{L}(L^2(G)) \) given by (7.4) is a \( C^* \)-monomorphism, which extends the isomorphism in (7.2).

**Remark 7.8.** A careful reader might already notice that the map \( \Theta \) can be further extended to \( \mathcal{L}(L^2(\beta r G \rtimes G)) \) using the same formula (7.4). However in this case, it is unclear whether Lemma 7.6 still holds since (7.5) is no longer available. Due to the same reason, it is unclear whether \( \Theta(T) \) coincides with \( T \otimes_{\tau^*} 1 \) given in Remark 7.5 for a general \( T \in \mathcal{L}(L^2(\beta r G \rtimes G)) \).

Finally, we provide an extra viewpoint on \( \Theta \) in terms of the slicing maps from Sect. 3.2, which will be used later to characterise the quasi-local algebra. Recall from (3.1) that we have the slicing maps:

\[
\Phi^x : \mathcal{L}(L^2(\beta r G \rtimes G))^{\beta r G \rtimes G} \to \prod_{z \in \beta r G} \mathcal{B}(\ell^2((\beta r G \rtimes G)_{z})), \quad T \mapsto (\Phi^x_z(T))_{z \in \beta r G}
\]

and

\[
\Phi : \mathcal{L}(L^2(G)) \to \prod_{x \in G^{(0)}} \mathcal{B}(\ell^2(G_x)), \quad T' \mapsto (\Phi_x(T'))_{x \in G^{(0)}}. \tag{7.7}
\]
For $T \in \mathcal{L}(L^2(\beta_t \mathcal{G} \rtimes \mathcal{G}))^{\beta_t \mathcal{G} \rtimes \mathcal{G}}$ and $x \in \mathcal{G}^{(0)}$, we consider the operator $\text{Ad}_{P_x}(\Phi^X_x(T)) \in \mathcal{B}(l^2(\mathcal{G}_x))$ where $\text{Ad}_{P_x}$ is from (7.3). Using (7.5), it is straightforward to prove:

**Lemma 7.9.** With the same notation as above, for $T \in \mathcal{L}(L^2(\beta_t \mathcal{G} \rtimes \mathcal{G}))^{\beta_t \mathcal{G} \rtimes \mathcal{G}}$ and $x \in \mathcal{G}^{(0)}$ we have

$$\Phi_x(\Theta(T)) = \text{Ad}_{P_x}(\Phi^X_x(T)).$$

**Remark 7.10.** In fact, the map $\Theta$ can be alternatively defined using the slicing maps above. More precisely, consider the map

$$p : \bigoplus_{z \in \beta_t \mathcal{G}} \mathcal{B}(l^2((\beta_t \mathcal{G} \rtimes \mathcal{G})_z)) \to \bigoplus_{x \in \mathcal{G}^{(0)}} \mathcal{B}(l^2(\mathcal{G}_x)), \quad (T_z)_{z \in \beta_t \mathcal{G}} \mapsto (\text{Ad}_{P_x}(T_x))_{x \in \mathcal{G}^{(0)}}. \quad (7.8)$$

Given $T \in \mathcal{L}(L^2(\beta_t \mathcal{G} \rtimes \mathcal{G}))^{\beta_t \mathcal{G} \rtimes \mathcal{G}}$, we consider the family $p \circ \Phi^X_x(T)$. Then we can apply Theorem 3.13 to verify that $p \circ \Phi^X_x(T)$ belongs to $\Phi^X_x(L^2(\mathcal{G}))$, and hence it provides an operator $\Phi^{-1}(p \circ \Phi^X_x(T))$, which is defined to be $\Theta(T)$. More details will be provided in the next subsection, where Lemma 7.9 shows that these two definitions are the same.

### 7.4. Quasi-local case.

In this subsection, we consider the quasi-local algebras and the main result is the following:

**Proposition 7.11.** The restriction of the map $\Theta$ given by (7.4) provides a $C^*$-isomorphism:

$$\Theta : C^*_{uq}(\beta_t \mathcal{G} \rtimes \mathcal{G})^{\beta_t \mathcal{G} \rtimes \mathcal{G}} \cong C^*_{uq}(\mathcal{G}).$$

The proof is divided into several parts, making use of the slicing maps at the end of Sect. 7.3. First recall from Sect. 3.2 and 3.3, the operator fibre spaces are given by

$$E^X := \bigoplus_{z \in \beta_t \mathcal{G}} \mathcal{B}(l^2((\beta_t \mathcal{G} \rtimes \mathcal{G})_z)), \quad E^X_{\mathcal{G}} := \bigoplus_{z \in \mathcal{G}} \mathcal{B}(l^2((\beta_t \mathcal{G} \rtimes \mathcal{G})_z))$$

and

$$E := \bigoplus_{x \in \mathcal{G}^{(0)}} \mathcal{B}(l^2(\mathcal{G}_x)),$$

equipped with the topology from Definition 3.7 (note that $\mathcal{G}$ is a dense subset in $\beta_t \mathcal{G}$). Also denote by $\Gamma_b(E^X)$, $\Gamma_b(E^X_{\mathcal{G}})$ and $\Gamma_b(E)$ the associated algebras of continuous bounded sections. Similar to Lemma 3.16, we have the following:

**Lemma 7.12.** Every section $\sigma \in \Gamma_b(E^X_{\mathcal{G}})$ is extendable.

The proof is almost the same as the one for Lemma 3.16. In fact, borrowing the notation therein, we can shrink the neighbourhoods $U_{\gamma'}$ and $U_{\gamma''}$ to ensure that the function $f_{\gamma', \gamma''} \in C_0(\mathcal{G}, r)$ (see Sect. 7.1 to recall the definition). Hence we omit the details.
On the other hand, note that for the slicing maps (7.6) and (7.7), given \( T \in \mathcal{L}(L^2(\beta_rG \rtimes G))^{\beta_rG \rtimes G} \) the property of equivariance means the following:

\[
V_{(yz,y)}\Phi_z^\infty(T) = \Phi_{yz}^\infty(T)V_{(yz,y)} \quad \text{for} \quad (yz,y) \in \beta_rG \rtimes G,
\]

where \( V_{(yz,y)} : \ell^2((\beta_rG \rtimes G)_z) \to \ell^2((\beta_rG \rtimes G)_{yz}) \) is given by \( V_{(yz,y)}(\delta(\alpha z, \alpha)) = \delta(\alpha z, \alpha^{-1}) \) (see Sect. 3.4). In the following, we will consider families \( (T_z)_{z \in G} \) satisfying:

\[
V_{(yz,y)}T_z = T_{yz}V_{(yz,y)} \quad \text{for} \quad (yz,y) \in G \rtimes G.
\]  

Applying Corollary 4.8 together with Lemma 7.12, we obtain the following characterisations for the equivariant quasi-local algebras:

**Corollary 7.13.** The map

\[
\text{Res} \circ \Phi_z^\infty : C_{aq}^*(G) \to \bigoplus_{z \in G} \mathcal{B}(\ell^2(G_z^z)), \quad T \mapsto (\Phi_z^\infty(T))_{z \in G}
\]

is a \( C^* \)-isomorphism with image consisting of \( (T_z)_{z \in G} \) such that \( z \mapsto T_z \) is a continuous section of \( E^\infty_z \), \( (T_z)_{z \in G} \) is compactly uniformly quasi-local and satisfies (7.9).

**Corollary 7.14.** The map

\[
\Phi : C_{aq}^*(G) \to \bigoplus_{x \in G^{(0)}} \mathcal{B}(\ell^2(G_x)), \quad T \mapsto (\Phi_x(T))_{x \in G^{(0)}}
\]

is a \( C^* \)-isomorphism with image consisting of \( (T_x)_{x \in G^{(0)}} \) such that \( x \mapsto T_x \) is a continuous section of \( E \) and \( (T_x)_{x \in G^{(0)}} \) is compactly uniformly quasi-local.

Recall from Lemma 7.9 and Remark 7.10 that we have the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{L}(L^2(\beta_rG \rtimes G))^{\beta_rG \rtimes G} & \xrightarrow{\text{Res} \circ \Phi_z^\infty} & \bigoplus_{z \in G} \mathcal{B}(\ell^2((\beta_rG \rtimes G)_z)) \\
\Theta & \downarrow & \downarrow p \\
\mathcal{L}(L^2(G)) & \xrightarrow{\Phi} & \bigoplus_{x \in G^{(0)}} \mathcal{B}(\ell^2(G_x))
\end{array}
\]

Hence to prove Proposition 7.11, it suffices to verify whether the map \( p \) provides a bijection between the images of quasi-local algebras characterised by Corollary 7.13 and Corollary 7.14. First note that the equivariance condition (7.9) directly implies the following:

**Lemma 7.15.** The following restriction of \( p \) is a bijection:

\[
p : \{ (T_z)_{z \in G} \in \bigoplus_{z \in G} \mathcal{B}(\ell^2((\beta_rG \rtimes G)_z)) \text{ satisfying (7.9)} \} \to \bigoplus_{x \in G^{(0)}} \mathcal{B}(\ell^2(G_x)).
\]

Now we turn to the condition of continuity for sections:

**Lemma 7.16.** For a family \( (T_z)_{z \in G} \in \bigoplus_{z \in G} \mathcal{B}(\ell^2((\beta_rG \rtimes G)_z)) \) satisfying (7.9), the map \( z \mapsto T_z \) is a continuous section of \( E^\infty_z \) if and only if the map \( x \mapsto \text{Ad}_{P_x}(T_x) \) is a continuous section of \( E \).
Proof. By definition, the map \( z \mapsto T_z \) is continuous if and only if for any \( z_i \to z \), \( \gamma_i' \to \gamma' \) and \( \gamma_i'' \to \gamma'' \) in \( G \) with \( r(z_i) = s(\gamma_i') = s(\gamma_i'') \) and \( r(z) = s(\gamma') = s(\gamma'') \), then
\[
\langle \delta_{(\gamma''_i, \gamma'_i)}, T_{z_i} \rangle \to \langle \delta_{(\gamma''_i, \gamma'')} \rangle \to \langle \delta_{(\gamma'', \gamma')} \rangle
\] (7.10)

Setting \( s(z_i) = x_i \) and \( s(z) = x \), then (7.9) implies that
\[
V_{(z_i, z_i)} T_{x_i} V_{(z_i, z_i)^*} = T_{z_i} \quad \text{and} \quad V_{(z, z)} T_x V_{(z, z)^*} = T_z.
\]

Hence (7.10) can be rewritten as:
\[
\langle \delta_{(\gamma''_i, \gamma'_i)}, T_{x_i} \delta_{(\gamma''_i, \gamma'_i)} \rangle \to \langle \delta_{(\gamma'', \gamma')} \rangle.
\]

Applying the isomorphisms \( \text{Ad}_{P_x} \), the above is also equivalent to:
\[
\langle \delta_{\gamma''_i}, \text{Ad}_{P_x} (T_{x_i}) (\delta_{\gamma'_i}) \rangle \to \langle \delta_{\gamma''}, \text{Ad}_{P_x} (T_x) (\delta_{\gamma'}) \rangle,
\]
which is (by definition) nothing but the continuity of the map \( x \mapsto \text{Ad}_{P_x} (T_x). \) \( \square \)

Finally, we consider the condition of compactly uniform quasi-locality.

Lemma 7.17. A family \( (T_z)_{z \in G} \in \prod_{z \in G} \mathcal{B}(\ell^2 ((\beta_t G \times G)_z)) \) satisfying (7.9) is compactly uniformly quasi-local if and only if the family \( \langle \text{Ad}_{P_x} (T_x) \rangle_{x \in G^{(0)}} \) is compactly uniformly quasi-local.

Proof. Necessity: By definition, given \( \varepsilon > 0 \) there exists a compact subset \( \tilde{K} \subseteq \beta_t G \times G \) such that for any \( z \in G \) and \( A_z, B_z \subseteq (\beta_t G \times G)_z \) with \( \tilde{A}_z \cap (\tilde{K} \cdot \tilde{B}_z) = \emptyset \) and \( (\tilde{K} \cdot \tilde{A}_z) \cap \tilde{B}_z = \emptyset \), we have \( \| \chi_{\tilde{A}_z} \chi_{\tilde{B}_z} \| < \varepsilon \). Take
\[
K := \{ \gamma \in G : \text{there exists } \gamma' \in G \text{ such that } (\gamma', \gamma) \in \tilde{K} \},
\]
which is compact. Then for any \( x \in G^{(0)} \) and \( A_x, B_x \subseteq G_x \) with \( A_x \cap (K \cdot B_x) = \emptyset \) and \( (K \cdot A_x) \cap B_x = \emptyset \), then we have \( p_{x}^{-1} (A_x) \cap (\tilde{K} \cdot p_{x}^{-1} (B_x)) = \emptyset \) and \( (\tilde{K} \cdot p_{x}^{-1} (A_x)) \cap p_x^{-1} (B_x) = \emptyset \). Hence
\[
\varepsilon \geq \| \chi_{p_x^{-1}(A_x)} T_x \chi_{p_x^{-1}(B_x)} \| = \| \text{Ad}_{P_x} (\chi_{p_x^{-1}(A_x)} T_x \chi_{p_x^{-1}(B_x)}) \| = \| \chi_{A_x} \text{Ad}_{P_x} (T_x) \chi_{B_x} \|,
\]
which shows that the family \( \langle \text{Ad}_{P_x} (T_x) \rangle_{x \in G^{(0)}} \) is compactly uniformly quasi-local.

Sufficiency: Given \( \varepsilon > 0 \) there exists a compact subset \( K \subseteq G \) such that for any \( x \in G^{(0)} \) and \( A_x, B_x \subseteq G_x \) with \( A_x \cap (K \cdot B_x) = \emptyset \) and \( (K \cdot A_x) \cap B_x = \emptyset \), we have \( \| \chi_{A_x} \text{Ad}_{P_x} (T_x) \chi_{B_x} \| < \varepsilon \). Take
\[
\tilde{K} := \{ (\gamma', \gamma) \in \beta_t G \times G : \gamma \in K \},
\]
which is compact. Then for any \( z \in G \) and \( \tilde{A}_z, \tilde{B}_z \subseteq (\beta_t G \times G)_z \) with \( \tilde{A}_z \cap (\tilde{K} \cdot \tilde{B}_z) = \emptyset \) and \( (\tilde{K} \cdot \tilde{A}_z) \cap \tilde{B}_z = \emptyset \), we set \( x = s(z) \) and note that \( V_{(z, z)} T_x V_{(z, z)^*} = T_z \). Hence we have
\[
\| \chi_{\tilde{A}_z} T_z \chi_{\tilde{B}_z} \| = \| (V_{(z, z)} \chi_{\tilde{A}_z} V_{(z, z)}) \cdot T_x \cdot (V_{(z, z)} \chi_{\tilde{B}_z} V_{(z, z)}) \|
= \| \chi_{\tilde{A}_z(z, z)} T_x \chi_{\tilde{B}_z(z, z)} \|
= \| \text{Ad}_{P_x} (\chi_{\tilde{A}_z(z, z)}) \cdot \text{Ad}_{P_x} (T_x) \cdot \text{Ad}_{P_x} (\chi_{\tilde{B}_z(z, z)}) \|.
\] (7.11)
It is straightforward to check that \( \text{Ad}_{p_x}(\chi_{\tilde{A}_c \cdot (z, z)}) = \chi_{A_x} \) and \( \text{Ad}_{p_x}(\chi_{\tilde{B}_c \cdot (z, z)}) = \chi_{B_x} \), where

\[
A_x := \{ \alpha \in \mathcal{G}_x : (\alpha, \alpha) \in \tilde{A}_z \cdot (z, z) \} = \{ \alpha \in \mathcal{G}_x : (\alpha, \alpha z^{-1}) \in \tilde{A}_z \}
\]

and similarly, \( B_x := \{ \beta \in \mathcal{G}_x : (\beta, \beta z^{-1}) \in \tilde{A}_z \} \). If \( A_x \cap (K \cdot B_x) \neq \emptyset \), then there exists \( \alpha \in A_x \), \( \beta \in B_x \) and \( \gamma \in K \) such that \( \alpha = \gamma \beta \). Then we have

\[
(\alpha, \alpha z^{-1}) = (\alpha, \gamma) \cdot (\beta, \beta z^{-1}),
\]

which is a contradiction to that \( \tilde{A}_z \cap (K \cdot \tilde{B}_z) = \emptyset \). Similarly, we have \( (K \cdot A_x) \cap B_x = \emptyset \). Hence combining with (7.11), we obtain:

\[
\| \chi_{\tilde{A}_c} T_z \chi_{\tilde{B}_c} \| = \| \text{Ad}_{p_x}(\chi_{\tilde{A}_c \cdot (z, z)}) \cdot \text{Ad}_{p_x}(T_x) \cdot \text{Ad}_{p_x}(\chi_{\tilde{B}_c \cdot (z, z)}) \| = \| \chi_{A_x} \text{Ad}_{p_x}(T_x) \chi_{B_x} \| \leq \varepsilon,
\]

which concludes the proof. \( \square \)

**Proof of Proposition 7.11.** Combining Corollary 7.13, Corollary 7.14, Lemma 7.15, Lemma 7.16 and Lemma 7.17, we conclude the proof. \( \square \)

**Remark 7.18.** In fact, the above procedure is also available for the case of compactly supported operators. Hence combining with Lemma 7.4, we obtain an alternative proof for [1, Theorem 6.4].

**Remark 7.19.** Readers might wonder whether Lemma 7.17 still holds for vector-wise uniform quasi-locality. If fact using a similar idea in the proof of Lemma 7.17, we can show that if a family \( (T_z)_{z \in G} \in \prod_{z \in G} \mathcal{B}(l^2((\beta_z G \times G)_z)) \) satisfying (7.9) is vector-wise uniformly quasi-local, then the family \( (\text{Ad}_{p_x}(T_x))_{z \in G} \) is vector-wise uniformly quasi-local. Hence applying Theorem 3.13 together with Lemma 7.16, we obtain that \( p \circ \Phi^{\times T} \) belongs to \( \Phi(\mathcal{L}(L^2(G))) \) for \( T \in \mathcal{L}(L^2(\beta G \times G)) \). As already mentioned in Remark 7.10, this provides an alternative approach to define the map \( \Theta \).

However, it is unclear whether the opposite direction still holds for vector-wise uniform quasi-locality. If it is true, then \( \Theta \) will provide a \( C^* \)-isomorphism between \( \mathcal{L}(L^2(\beta G \times G))^{\beta G \times G} \) and \( \mathcal{L}(L^2(G)) \).

### 7.5. Conclusion

Combining the discussions in the previous subsections, we obtain that the \( C^* \)-monomorphism

\[
\Theta : \mathcal{L}(L^2(\beta G \times G))^{\beta G \times G} \longrightarrow \mathcal{L}(L^2(G))
\]

provides \( C^* \)-isomorphisms

\[
\overline{C}_u[\beta G \times G]^{\beta G \times G} \mathcal{L}(L^2(\beta G \times G)) (\cong C^*_r(\beta G \times G)) \cong C^*_u(G)
\]

and

\[
C^*_u(\beta G \times G)^{\beta G \times G} \cong C^*_u(G).
\]

Hence applying Theorem 5.1, we reach the following quasi-local characterisation for the uniform Roe algebra \( C^*_u(G) \):
Theorem 7.20. Let $G$ be a locally compact, $\sigma$-compact and étale groupoid. Suppose $G$ is either strongly amenable at infinity, or secondly countable weakly inner amenable and $C^*$-exact. Then we have $C_u^*(G) = C_uq^*(G)$.

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Declarations

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References

1. Anantharaman-Delaroche, C.: Exact groupoids. Preprint arXiv:1605.05117 (2016)
2. Anantharaman-Delaroche, C., Popa, S.: An Introduction to $\mathcal{I}_1$ factors (2019)
3. Anantharaman-Delaroche, C., Renault, J.: Amenable groupoids, volume 36 of Monographies de L’Enseignement Mathématique [Monographs of L’Enseignement Mathématique]. L’Enseignement Mathématique, Geneva, (2000). With a foreword by Georges Skandalis and Appendix B by E. Germain
4. Austin, K., Zhang, J.: Limit operator theory for groupoids. Trans. Am. Math. Soc. 373(4), 2861–2911 (2020)
5. Bao, H., Chen, X., Zhang, J.: Strongly quasi-local algebras and their $K$-theories. J. Noncommut. Geom. 17(1), 241–285 (2023)
6. Barlak, S., Li, X.: Cartan subalgebras and the UCT problem. Adv. Math. 316, 748–769 (2017)
7. Barlak, S., Li, X.: Cartan subalgebras and the UCT problem. II. Math. Ann. 378(1), 255–287 (2020)
8. Bachir, M., Bekka, B., de la Harpe, P., Valette, A.: Kazhdan’s property (T). New Mathematical Monographs, vol. 11. Cambridge University Press, Cambridge (2008)
9. Brown, N.P., Ozawa, N.: $C^*$-algebras and finite-dimensional approximations. Graduate Studies in Mathematics, vol. 88. American Mathematical Society, Providence, RI (2008)
10. Dixmier, J.: $C^*$-Algebras. North-Holland Publishing Co., Amsterdam-New York-Oxford (1977)
11. Dixmier, J., Douady, A.: Champs continus d’espaces hilbertiens et de $C^*$-algèbres. Bull. Soc. Math. France 91, 227–284 (1963)
12. Engel, A.: Index theory of uniform pseudodifferential operators. Preprint arXiv:1502.00494, (2015)
13. Engel, A.: Rough index theory on spaces of polynomial growth and contractibility. J. Noncommut. Geom. 13(2), 617–666 (2019)
14. Exel, R.: Invertibility in groupoid $C^*$-algebras. In: Operator Theory, Operator Algebras and Applications, volume 242 of Oper. Theory Adv. Appl., pp. 173–183. Birkhäuser/Springer, Basel (2014)
15. Feldman, J., Moore, C.C.: Ergodic equivalence relations, cohomology, and von Neumann algebras I, II. Trans. Am. Math. Soc. 234(289–324), 325–359 (1977)
16. Fell, J.M.G., Doran, R.S.: Representations of $*$-algebras, locally compact groups, and Banach $*$-algebraic bundles. Vol. 1, volume 125 of Pure and Applied Mathematics. Academic Press, Inc., Boston, MA (1988)
17. Gomez Aparicio, M.P., Julg, P., Valette, A.: The Baum-Connes conjecture: an extended survey. In: Advances in Noncommutative Geometry. Based on the noncommutative geometry conference, Shanghai, China, March 23 – April 7, 2017. On the occasion of Alain Connes’ 70th Birthday, pp. 127–244. Springer International Publishing, Cham (2019)
18. Gromov, M., Lawson, H.B.: Positive scalar curvature and the Dirac operator on complete Riemannian manifolds. Publ. Math. Inst. Hautes Étud. Sci. 58, 83–196 (1983)
19. Haagerup, U., Kraus, J.: Approximation properties for group $C^*$-algebras and group von Neumann algebras. Trans. Am. Math. Soc. 344(2), 667–699 (1994)
20. Kumjian, A.: On $C^*$-diagonals. Can. J. Math. 38(4), 969–1008 (1986)
21. Lance, E.C.: Hilbert $C^*$-modules, volume 210 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge (1995)
22. Lange, B.V., Rabinovich, V.S.: Noether property for multidimensional discrete convolution operators. Mat. Zametki 37(3), 407–421 (1985)
23. Le Gall, P.-Y.: Théorie de Kasparov équivariante et groupoïdes I. K-Theory 16(4), 361–390 (1999)
24. Li, K., Nowak, P., Špakula, J., Zhang, J.: Quasi-local algebras and asymptotic expanders. Groups Geom. Dyn. 15(2), 655–682 (2021)
25. Li, K., Wang, Z., Zhang, J.: Every classifiable simple $C^*$-algebra has a Cartan subalgebra. Invent. Math. 219(2), 653–699 (2020)
26. Li, X.: Every classifiable simple $C^*$-algebra has a Cartan subalgebra. Invent. Math. 219(2), 653–699 (2020)
27. Li, X., Renault, J.: Cartan subalgebras in $C^*$-algebras. Ir. Math. Soc. Bull. 61, 29–63 (2008)
28. Roe, J.: Lectures on coarse geometry. University Lecture Series, vol. 31. American Mathematical Society, Providence, RI (2003)
29. Rosenberg, J.: $C^*$-algebras, positive scalar curvature, and the Novikov conjecture. Publ. Math. Inst. Hautes Étud. Sci. 58, 409–424 (1983)
30. Skandalis, G., Tu, J.-L., Yu, G.: The coarse Baum-Connes conjecture and groupoids. Topology 41(4), 807–834 (2002)
31. Špakula, J., Zhang, J.: Quasi-locality and property A. J. Funct. Anal. 278(1), 108299 (2020)
32. Takahashi, A.: Hilbert modules and their representation. Rev. Colomb. Mat. 13, 1–38 (1979)
33. Tu, J.-L.: The Baum-Connes conjecture for groupoids. In: $C^*$-algebras. Proceedings of the SFB-workshop, Münster, Germany, March 8–12, 1999, pages 227–242. Berlin: Springer (2000)
34. Tu, J.-L.: Remarks on Yu’s “property A” for discrete metric spaces and groups. Bull. Soc. Math. France 129(1), 115–139 (2001)
35. Valette, A.: Introduction to the Baum-Connes conjecture. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, (2002). From notes taken by Indira Chatterji, with an appendix by Guido Mislin
36. White, S., Willett, R.: Cartan subalgebras in uniform Roe algebras. Groups Geom. Dyn. 14(3), 949–989 (2020)
37. Yu, G.: The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space. Invent. Math. 139(1), 201–240 (2000)
38. Zacharias, J.: On the invariant translation approximation property for discrete groups. Proc. Am. Math. Soc. 134(7), 1909–1916 (2006)

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