Extremal sequences for the unit-weighted Gao constant of $\mathbb{Z}_n$

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Abstract

For $A \subseteq \mathbb{Z}_n$, the $A$-weighted Gao constant $E_A(n)$ is defined to be the smallest natural number $k$, such that any sequence of $k$ elements in $\mathbb{Z}_n$ has a subsequence of length $n$, whose $A$-weighted sum is zero. Sequences of length $E_A(n) - 1$ in $\mathbb{Z}_n$, which do not have any $A$-weighted zero-sum subsequence of length $n$ are called $A$-extremal sequences for the Gao constant. Such a sequence which has $n - 1$ zeroes is said to be of the standard type. When $A = U(n)$, is the set of units in $\mathbb{Z}_n$, where $n$ is odd, we characterize all such sequences and show that they are of the standard type. When $n$ is even, we give examples of such sequences which are not of the standard type. We also characterize the $U(n)$-extremal sequences for the Gao constant, when $n = 2^rp$, where $p$ is an odd prime.

Keywords: Gao constant, Davenport constant, Units in $\mathbb{Z}_n$, weighted zero-sum sequence

1 Introduction

Definition 1. Let $R$ be a ring and let $A \subseteq R$. A subsequence $T$ of a sequence $S : (x_1, x_2, \ldots, x_k)$ in $R$ is called an $A$-weighted zero-sum subsequence if the set $I := \{i : x_i \in T\}$ is non-empty and $\forall \ i \in I, \ \exists \ a_i \in A$ such that $\sum_{i \in I} a_i x_i = 0$.

Definition 2. Given a ring $R$ and a subset $A \subseteq R$, the $A$-weighted Davenport constant $D_A(R)$ is the least positive integer $k$ such that any sequence in $R$ of length $k$ has an $A$-weighted zero-sum subsequence.

Definition 3. Given a ring $R$ and a subset $A \subseteq R$, the $A$-weighted Gao constant $E_A(R)$ is the least positive integer $k$ such that any sequence in $R$ of length $k$ has an $A$-weighted zero-sum subsequence of length $|R|$.

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We denote the ring \( \mathbb{Z}/n\mathbb{Z} \) by \( \mathbb{Z}_n \). For a divisor \( m \) of \( n \), we define the natural map \( \mathbb{Z}_n \to \mathbb{Z}_m \) to be the map which sends \( x + n\mathbb{Z} \mapsto x + m\mathbb{Z} \). Let \( U(n) \) denote the group of units in \( \mathbb{Z}_n \).

When \( A \subseteq \mathbb{Z}_n \), we denote the constants \( D_A(\mathbb{Z}_n) \) and \( E_A(\mathbb{Z}_n) \) by \( D_A(n) \) and \( E_A(n) \) respectively. From Theorem 1.2 of \([7]\), we have \( E_A(n) = D_A(n) + n - 1 \).

**Definition 4.** Let \( A \subseteq \mathbb{Z}_n \). A sequence in \( \mathbb{Z}_n \) of length \( E_A(n) - 1 \) which does not have any \( A \)-weighted zero-sum subsequence of length \( n \), is called an \( A \)-extremal sequence for the Gao constant. A sequence in \( \mathbb{Z}_n \) of length \( D_A(n) - 1 \) which does not have any \( A \)-weighted zero-sum subsequence, is called an \( A \)-extremal sequence for the Davenport constant.

**Definition 5.** Let \( A \) be a subgroup of \( U(n) \) and let \( S : (x_1, \ldots, x_k) \) and \( T : (y_1, \ldots, y_k) \) be sequences in \( \mathbb{Z}_n \). We say that \( S \) and \( T \) are \( A \)-equivalent if there is a unit \( c \in U(n) \), a permutation \( \sigma \in S_k \) and we can find \( a_1, \ldots, a_k \in A \) such that for \( 1 \leq i \leq k \), we have \( c y_{\sigma(i)} = a_i x_i \).

**Remark:** If \( S \) is an \( A \)-extremal sequence for the Gao (resp. Davenport) constant and if \( S \) and \( T \) are \( A \)-equivalent, then \( T \) is also an \( A \)-extremal sequence for the Gao (resp. Davenport) constant.

From Theorem 1.3 of \([2]\) or from Theorem 1 of \([3]\), \( E_{U(n)}(n) = n + \Omega(n) \), for any \( n \). So, if \( S \) is a \( U(n) \)-extremal sequence for the Gao constant, then \( S \) has length \( n - 1 + \Omega(n) \).

For \( n \) odd, it was shown in Theorem 6 of \([1]\), that a sequence in \( \mathbb{Z}_n \) is a \( U(n) \)-extremal sequence for the Davenport constant if and only if it is \( U(n) \)-equivalent to a sequence of the following form:

\[
( b_1, p_1 b_2, p_1 p_2 b_3, \ldots, p_1 p_2 \cdots p_{k-1} b_k )
\]

where for \( 1 \leq i \leq k \), \( p_i \) is a prime and \( b_i \) is coprime to \( p_i \) and \( n = p_1 \cdots p_k \).

When \( n = p^r \), where \( p \) is an odd prime, in Theorem 3 of \([1]\), it was shown that a sequence in \( \mathbb{Z}_n \) is a \( U(n) \)-extremal sequence for the Gao constant if and only if it is \( U(n) \)-equivalent to the sequence

\[
( 1, p, p^2, \ldots, p^{r-1}, 0, \ldots, 0 )
\]

In this article, we have proved the following results:

- If \( n \) is odd, a sequence in \( \mathbb{Z}_n \) is a \( U(n) \)-extremal sequence for the Gao constant if and only if it is of the standard type, i.e., it has \( n - 1 \) zeroes.
• For any even $n$, we give examples of $U(n)$-extremal sequences for the Gao constant which are not of the standard type.

• For $n = 2^rp$, where $p$ is an odd prime, we characterize the $U(n)$-extremal sequences for the Gao constant.

2 When $n$ is any odd number

For the next theorem, we need the following ([2], Lemma 2.1 (ii)), which we restate here using our terminology:

**Lemma 1.** Let $A = U(p^r)$, where $p$ is an odd prime. If a sequence $S$ over $\mathbb{Z}_{p^r}$ has at least two terms coprime to $p$, then $S$ is an $A$-weighted zero-sum sequence.

The next result is Lemma 12 of [5].

**Lemma 2.** Let $S : (x_1, \ldots, x_k)$ be a sequence in $\mathbb{Z}_n$, let $d$ be a proper divisor of $n$ which divides every element of $S$ and let $n' = n/d$. For $1 \leq i \leq k$, let $x'_i$ denote the image of $x_i/d$ under the natural map $f : \mathbb{Z}_n \to \mathbb{Z}_{n'}$. Let $S' : (x'_1, \ldots, x'_k)$. Let $A \subseteq \mathbb{Z}_n$ and let $A' \subseteq f(A)$. Suppose $S'$ is an $A'$-weighted zero-sum sequence. Then $S$ is an $A$-weighted zero-sum sequence.

**Lemma 3.** Suppose $n = p_1 \cdots p_k$, where the $p_i$s are primes. Let

$$S : (b_1, p_1b_2, p_1p_2b_3, \ldots, p_1p_2 \cdots p_{k-1}b_k, 0, \ldots, 0) \quad (\star)$$

where for $1 \leq i \leq k$, $b_i$ is coprime to $p_i$. Then, $S$ is a $U(n)$-extremal sequence for the Gao constant.

**Proof.** Let $T$ be a $U(n)$-weighted zero-sum subsequence of $S$ of length $n$. Suppose the first term of $T$ is the $i$th term of $S$, where $1 \leq i \leq k$. Then, $p_1 \cdots p_i$ divides all the other terms of $T$. So, we get the contradiction that $b_i$ is divisible by $p_i$. Thus, the first term of $T$ cannot be any of the first $k$ terms of $S$. As $T$ has length $n$, so, the first term of $T$ cannot be any of the remaining $n - 1$ terms of $S$. Thus, $T$ cannot exist.

**Theorem 1.** Let $n$ be odd. Then a sequence in $\mathbb{Z}_n$ is a $U(n)$-extremal sequence for the Gao constant if and only if it is $U(n)$-equivalent to a sequence having the form $(\star)$.

**Proof.** From Lemma 3, a sequence which is $U(n)$-equivalent to a sequence having the form $(\star)$ is a $U(n)$-extremal sequence for the Gao constant.
Let \( S : (x_1, \ldots, x_l) \) be a \( U(n) \)-extremal sequence for the Gao constant. Suppose for each prime divisor \( p \) of \( n \), at least two terms of \( S \) are coprime to \( p \). As \( 2\omega(n) < n \), so we can find a subsequence \( T \) of \( S \) of length \( n \) such that for each prime divisor \( p \) of \( n \), at least two terms of \( T \) are coprime to \( p \). As \( n \) is odd, so, by Lemma 1 we get the contradiction that \( T \) is a \( U(n) \)-weighted zero-sum subsequence of \( S \) of length \( n \). Thus, there is a prime divisor \( p \) of \( n \), such that at most one term of \( S \) is coprime to \( p \).

Suppose \( p \) divides all the terms of \( S \). Let \( n' = n/p \), \( A' = U(n') \) and let \( f : \mathbb{Z}_n \to \mathbb{Z}_{n'} \) be the natural map. Let \( S' : (x'_1, \ldots, x'_l) \) denote the sequence in \( \mathbb{Z}_{n'} \) where \( x'_i = f(x_i/p) \), for \( 1 \leq i \leq l \). As \( E_{A'}(n') = n' + \Omega(n') \) and as the length of \( S' \) is \( n' - 1 + \Omega(n) = (p-1)n' + n' + \Omega(n') \), so, \( S' \) contains \( p \) disjoint subsequences \( S'_1, \ldots, S'_p \) which are \( A' \)-weighted zero-sum sequences of length \( n' \). So, their union \( T' \) is an \( A' \)-weighted zero-sum subsequence of \( S' \) of length \( n' \). Thus, by Lemma 2 we get the contradiction that \( S \) has an \( A \)-weighted zero-sum subsequence of length \( n \).

Thus, there is exactly one term of \( S \) which is coprime to \( p \), say \( x_1 \). By repeating the arguments in the above two paragraphs for the sequence \( (x'_2, \ldots, x'_l) \), we see that there is a prime divisor \( p' \) of \( n' \) such that exactly one term of this sequence, say \( x'_2 \) is coprime to \( p' \). Let us denote \( p \) by \( p_1 \) and \( p' \) by \( p_2 \). Let \( k = \Omega(n) \). For \( 2 \leq i \leq k \), we get prime divisors \( p_i \) of \( n \), such that \( x_i/(p_1 \cdots p_{i-1}) \) is coprime to \( p_i \) and \( p_1p_2 \cdots p_i \) divides \( x_j \) for \( j > i \). Thus, \( x_j = 0 \) for all \( j > k \). Hence, \( S \) is \( U(n) \)-equivalent to a sequence having the form \((*)\).

**Definition 6.** An \( A \)-extremal sequence for the Gao constant which has \( n - 1 \) zeroes, is said to be of the standard type.

Remark: For \( A \subseteq \mathbb{Z}_n \), \( E_A(n) = n - 1 + D_A(n) \). So, if \( S \) is an \( A \)-extremal sequence for the Gao constant which is of the standard type, then the subsequence consisting of the non-zero terms of \( S \) will be an \( A \)-extremal sequence for the Davenport constant. By Lemma 3 a sequence having the form \((*)\) is a \( U(n) \)-extremal sequence for the Gao constant. Such a sequence is of the standard type.

However, when \( n \) is even, there are \( U(n) \)-extremal sequences for the Gao constant which are not of the standard type.

### 3 When \( n \) is any even number

The next result is Theorem 10 of [6].
Theorem 2. Let \( n = 2^r \) and \( r \geq 2 \). A sequence in \( \mathbb{Z}_n \) is a \( U(n) \)-extremal sequence for the Gao constant if and only if it is \( U(n) \)-equivalent to a sequence \( S \) of length \( n + r - 1 \), such that, for each \( i \) with \( 0 \leq i \leq r - 2 \), \( 2^i \) occurs exactly once as a term of \( S \), \( 2^{r-1} \) occurs an odd number of times as a term of \( S \) and the remaining terms of \( S \) are zero.

For example, a sequence in \( \mathbb{Z}_8 \) is a \( U(8) \)-extremal sequence for the Gao constant if and only if it is \( U(8) \)-equivalent to one of the following sequences:

\[
(1, 2, 4, 0, 0, 0, 0, 0), (1, 2, 4, 4, 0, 0, 0, 0)
\]
\[
(1, 2, 4, 4, 4, 0, 0, 0), (1, 2, 4, 4, 4, 4, 0, 0)
\]

Lemma 4. Suppose \( n = 2p_1 \ldots p_k \), where the \( p_i \)s are primes.

If \( k \geq 2 \), let

\[
S : (b_1, p_1 b_2, p_1 p_2 b_3, \ldots, p_1 \cdots p_k-1 b_k, n/2, \ldots, n/2, 0, \ldots, 0)
\]

and if \( k = 1 \) and \( p_1 = p \), let

\[
S : (b_1, p, \ldots, p, 0, \ldots, 0)
\]

where \( m \) is odd and for \( 1 \leq i \leq k \), we have \( b_i \) is coprime to \( p_i \). Then, \( S \) is a \( U(n) \)-extremal sequence for the Gao constant.

Proof. Suppose \( T \) is a \( U(n) \)-weighted zero-sum subsequence of \( S \). Then, the first term of \( T \) cannot be any of the first \( k \) terms of \( S \). We claim that \( T \) cannot consist of the last \( n \) terms of \( S \).

Let \( f : \mathbb{Z}_n \to \mathbb{Z}_2 \) be the natural map and let \( T' \) be the sequence whose terms are obtained by dividing the terms of \( T \) by \( n/2 \) and then taking their images under \( f \). The sequence \( T' \) in \( \mathbb{Z}_2 \) has an odd number of non-zero terms which are all equal to one. So, we see that \( T' \) cannot be a zero-sum sequence. As the map \( f \) sends elements of \( U(n) \) to 1, hence, \( T \) cannot be a \( U(n) \)-weighted zero-sum sequence.

We now give an example of a \( U(n) \)-extremal sequence for the Gao constant whose all terms may be non-zero, when \( n \) is even.

Lemma 5. Suppose \( n = 2^{r+1} p_1 \ldots p_k \), where the \( p_i \)s are odd primes.
If \( k \geq 2 \), let \( S \) be the sequence

\[
(a_0, 2a_1, \ldots, 2^{r-1}a_{r-1}, c, 2^r b_1, 2^r p_1 b_2, \ldots, 2^r p_1 \cdots p_{k-1} b_k, n/2, \ldots, n/2)
\]

and if \( k = 1 \), let

\[
S : (a_0, 2a_1, \ldots, 2^{r-1}a_{r-1}, c, 2^r, n/2, \ldots, n/2)
\]

where for \( 0 \leq i \leq r - 1 \), \( a_i \) is odd, \( c \) is divisible by \( 2^{r+1} \) and for \( 1 \leq i \leq k \), we have \( b_i \) is odd and coprime to \( p_i \). Then, \( S \) is a \( U(n) \)-extremal sequence for the Gao constant.

**Proof.** We give the proof in the case \( k \geq 2 \). Let \( T \) be a \( U(n) \)-weighted zero-sum subsequence of \( S \) of length \( n \). Then, the first term of \( T \) cannot be any of the first \( r \) terms. Suppose the first term of \( T \) is \( c \). All the other terms of \( T \) are of the form \( 2^r a \), where \( a \) is odd. As \( n \) is even, \( T \) has an odd number (viz. \( n - 1 \)) of such terms. So, a \( U(n) \)-weighted sum of these terms is also of the same form and hence cannot be divisible by \( 2^{r+1} \). Thus, the first term of \( T \) cannot be \( c \).

It is easy to see that the first term of \( T \) cannot be any of the next \( k \) terms of \( S \). As there are only \( n - 1 \) terms remaining, so, \( S \) has no \( U(n) \)-weighted zero-sum subsequence of length \( n \).

4 When \( n = 2p \), where \( p \) is an odd prime

We begin this section by quoting Observation 2.2 in [2].

**Observation 1.** For every prime divisor \( p \) of \( n \), let \( v_p(n) \) denote the highest power of \( p \) which divides \( n \). Let \( S \) be a sequence in \( \mathbb{Z}_n \). For each prime divisor \( p \) of \( n \), let \( S^{(p)} \) denote the image of \( S \) under the natural map \( \mathbb{Z}_n \to \mathbb{Z}_{p^{v_p(n)}} \). Then, \( S \) is a \( U(n) \)-weighted zero-sum sequence in \( \mathbb{Z}_n \) if and only if for every prime divisor \( p \) of \( n \), \( S^{(p)} \) is a \( U(p^s) \)-weighted zero-sum sequence in \( \mathbb{Z}_{p^s} \), where \( s = v_p(n) \).

**Lemma 6.** Let \( n = 2p \), where \( p \) is an odd prime. If \( W \) is a sequence in \( \mathbb{Z}_n \) such that \( W^{(2)} \) has an even number of ones and \( W \) doesn’t have exactly one term which is coprime to \( p \). Then, \( W \) is a \( U(n) \)-weighted zero-sum sequence.

**Proof.** By Lemma 1 if \( W^{(p)} \) has at least two units, then it is a \( U(p) \)-weighted zero-sum sequence. If it has no non-zero term, then again it is a \( U(p) \)-weighted zero-sum sequence. As \( W^{(2)} \) has an even number of ones, so, it is a zero-sum sequence. Thus, by Observation 1 \( W \) is a \( U(n) \)-weighted zero-sum sequence. \( \square \)
Theorem 3. Let \( n = 2p \), where \( p \) is an odd prime. Then, \( S \) is a \( U(n) \)-extremal sequence for the Gao constant if and only if \( S \) is \( U(n) \)-equivalent to a sequence of length \( n + 1 \) having the form \((*)\), \((\ast)\) or \((\ast\ast)\).

Proof. From Lemmas 3, 4 and 5, we see that, when \( n \) is even, a sequence in \( Z_n \) which is \( U(n) \)-equivalent to a sequence having the form \((*)\), \((\ast)\) or \((\ast\ast)\), is a \( U(n) \)-extremal sequence for the Gao constant.

Let \( n = 2p \), where \( p \) is an odd prime. Suppose \( S \) is a \( U(n) \)-extremal sequence for the Gao constant. As \( E_{U(n)}(n) = n + \Omega(n) \), so, \( S \) has length \( n + 1 \).

Suppose all the terms of \( S^{(2)} \) are zero. If \( S^{(p)} \) has at most one non-zero term, then we get the contradiction that \( S \) has a subsequence of length \( n \) whose all terms are zero. So, \( S^{(p)} \) must have at least two non-zero terms. Let \( T \) denote a subsequence of \( S \) of length \( n \) such that \( T^{(p)} \) has at least two non-zero terms. Then, by Lemma 6, we get the contradiction that \( T \) is a \( U(n) \)-weighted zero-sum subsequence of \( S \) of length \( n \). So all terms of \( S^{(2)} \) cannot be zero. By a similar argument, all terms of \( S^{(2)} \) cannot be one.

Suppose \( S^{(2)} \) has exactly one non-zero term. Let \( T \) denote the subsequence of \( S \) of length \( n \) such that all terms of \( T^{(2)} \) are zero. If \( T^{(p)} \) has at least two units, then, by Lemma 6, we get the contradiction that \( T \) is a \( U(n) \)-weighted zero-sum subsequence of \( S \) of length \( n \). If \( T^{(p)} \) has no units, then all terms of \( T^{(p)} \) are zero and so, all terms of \( T \) are zero. So, we see that \( T^{(p)} \) has exactly one unit. Thus, \( S \) is \( U(n) \)-equivalent to the sequence \((a, 2b, 0, \ldots, 0)\), where there are \( n - 1 \) zeroes, \( a \) is odd and \( b \) is coprime to \( p \).

By a similar argument, if \( S^{(2)} \) has exactly one term which is zero and if \( T \) denotes the subsequence of \( S \) of length \( n \) such that all the terms of \( T^{(2)} \) are one, then \( T^{(p)} \) must have exactly one unit. Thus, \( S \) is \( U(n) \)-equivalent to the sequence \((1, c, p, \ldots, p)\), where \( p \) occurs \( n - 1 \) times and \( c \) is even.

Suppose \( S^{(2)} \) has at least two terms which are zero and at least two terms which are 1. Suppose \( S^{(p)} \) does not have exactly one unit. Then, by using Lemma 6, we will get a \( U(n) \)-weighted zero-sum subsequence of \( S \) of length \( n \). So, \( S^{(p)} \) has exactly one unit. Let \( T \) be the subsequence of \( S \) of length \( n \), such that \( T^{(p)} \) is the zero sequence.

If \( T^{(2)} \) has an even number of ones, then, \( T^{(2)} \) will be a zero-sum sequence and so, we get the contradiction that \( T \) is a zero-sum subsequence of \( S \) of length \( n \). Thus, \( T^{(2)} \) has an odd number of ones and hence \( S \) is \( U(n) \)-equivalent to the sequence \((b, p, \ldots, p, 0, \ldots, 0)\), where, \( b \) is coprime to \( p \) and \( p \) occurs an odd number of times.
Hence, when \( n = 2p \), where \( p \) is an odd prime, a sequence in \( \mathbb{Z}_n \) is a \( U(n) \)-extremal sequence for the Gao constant if and only if it is \( U(n) \)-equivalent to a sequence of length \( n + 1 \) having one of the following forms:

- \((a, 2b, 0, \ldots, 0)\), where \( a \) is odd and \( b \) is coprime to \( p \).
  This has the form \((\ast)\).

- \((b, p, \ldots, p, 0, \ldots, 0)\), where \( b \) is coprime to \( p \) and \( m \) is odd.
  This has the form \((\ast)\).

- \((1, c, p, \ldots, p)\), where \( c \) is even. This has the form \((\ast\ast)\).

\end{itemize}

5 When \( n = 2^r p \), where \( p \) is an odd prime

The next result is Lemma 1 (ii) of [3].

**Lemma 7.** Let \( n = 2^r \). If a sequence \( S \) in \( \mathbb{Z}_n \) has an even number (at least two) of units, then it is a \( U(n) \)-weighted zero-sum sequence.

**Lemma 8.** Let \( n = 2^r p \), where \( p \) is an odd prime. Let \( S \) be a sequence in \( \mathbb{Z}_n \) of length at least \( n + 2 \) such that \( S \) does not have any \( U(n) \)-weighted zero-sum subsequence of length \( n \). Then, \( S \) has at most two odd terms.

Also, if \( S \) has exactly two odd terms, then exactly one of the odd terms is a unit and all the other terms of \( S \) are divisible by \( p \).

**Proof.** Let \( S : (x_1, \ldots, x_k) \) be a sequence in \( \mathbb{Z}_n \) of length at least \( n + 2 \) such that \( S \) does not have any \( U(n) \)-weighted zero-sum subsequence of length \( n \). Suppose \( S \) has at least three odd terms. We consider the following two possibilities.

**Case:** Suppose \( S \) has at least two terms which are coprime to \( p \).

As the length of \( S \) is at least \( n + 2 \), so, (by removing a suitable \( x_i \) if there are an odd number of odd terms) we can get a subsequence \( T \) of \( S \) of length \( n \), which has an even number of odd terms and such that \( T \) has at least two terms which are coprime to \( p \). By Lemma [4] \( T^{(2)} \) is a \( U(2^r) \)-weighted zero-sum sequence and by Lemma [1] \( T^{(p)} \) is a \( U(p) \)-weighted zero-sum sequence. So, by Observation [1] we get the contradiction that \( T \) is a \( U(n) \)-weighted zero-sum subsequence of \( S \) having length \( n \).

**Case:** Suppose \( S \) has at most one term which is coprime to \( p \).

As the length of \( S \) is at least \( n + 2 \), so, (by removing at most two \( x_i \)'s if needed) we can get a subsequence \( T \) of \( S \) of length \( n \), which has an even number of odd terms.

\[ \square \]
terms and such that $T$ has no term which is coprime to $p$. By Lemma\textsuperscript{7} \(T^{(2)}\) is a \(U(2^r)\)-weighted zero-sum sequence. Also \(T^{(p)}\) is a \(U(p)\)-weighted zero-sum sequence, as all its terms are zero. So, by Observation\textsuperscript{11} we get the contradiction that $T$ is a \(U(n)\)-weighted zero-sum subsequence of $S$ having length $n$. Thus, $S$ has at most two odd terms.

Suppose $S$ has exactly two odd terms. By using a similar argument as in the second paragraph, we see that $S^{(p)}$ cannot have at least two units. By using a similar argument as in the third paragraph, we see that $S^{(p)}$ must have at least one unit. Thus, $S^{(p)}$ has exactly one unit and so, by a similar argument as in the third paragraph, both the odd terms cannot be divisible by $p$. Hence, exactly one of the two odd terms must be coprime to $p$, and so, it will be a unit.

\[\square\]

**Theorem 4.** Let $n = 2^r p$, where $p$ is an odd prime and $r \geq 2$. Let $S$ be a sequence in \(\mathbb{Z}_n\). Suppose $S$ is a \(U(n)\)-extremal sequence for the Gao constant. Then, one of the following two cases will occur:

An odd multiple of $2^i$ occurs exactly once in $S$ for each $i$, such that $0 \leq i \leq r-2$.

There is a unique $j$ such that $0 \leq j \leq r-2$ and an odd multiple of $2^j$ occurs exactly two times in $S$. Also, for any $i \neq j$ such that $0 \leq i \leq r-2$, an odd multiple of $2^i$ occurs exactly once in $S$.

**Proof.** Let $S$ be a \(U(n)\)-extremal sequence for the Gao constant. As $E_{U(n)}(n) = n+\Omega(n)$, so, $S$ has length $k = n+r$. Suppose there exists $i$ such that $0 \leq i \leq r-2$ and an odd multiple of $2^i$ occurs at least two times in $S$.

Let $j$ be the smallest value of $i$, such that $0 \leq i \leq r-2$ and an odd multiple of $2^i$ occurs at least two times in $S$. Suppose $S$ has at least three terms which are an odd multiple of $2^i$. As $S$ has at most one term which is an odd multiple of $2^i$, where $0 \leq i \leq j-1$, so at most $j$ terms of $S$ will not be divisible by $2^j$. As $j \leq r-2$, so at least $k - (r-2) \geq n + r - (r-2) = n + 2$ terms of $S$ are divisible by $2^j$.

Let $U$ be the subsequence of $S$ consisting of all the terms of $S$ which are divisible by $2^j$. If $U$ has a \(U(n)\)-weighted zero-sum subsequence of length $n$, then $S$ will also have such a subsequence and this is not possible. As $U$ has length at least $n + 2$, so, by Lemma\textsuperscript{8} at most two terms of $U$ are an odd multiple of $2^j$. This contradicts the assumption that $S$ has at least three terms which are an odd multiple of $2^j$. Thus, exactly two terms of $S$ are an odd multiple of $2^j$.
By Lemma 8 exactly one of the two terms of $U$ which are an odd multiple of $2^j$, is coprime to $p$ and all the other terms of $U$ are divisible by $p$. Suppose $j \leq r - 3$ and at least two terms of $S$ are odd multiples of $2^i$ for some $i$, where $j + 1 \leq i \leq r - 2$. Let us assume that $i'$ is the smallest such value of $i$. Let $T$ denote the subsequence of $S$ consisting of all the terms which are divisible by $2^{i'}$. Then, $T$ has length at least $k - (i' + 1)$. As $i' \leq r - 2$, so $k - (i' + 1) \geq n + r - (r - 1) = n + 1$.

By using Lemma 7 we can find a $U(2^r)$-weighted zero-sum subsequence of length $n$ of $T(2)$. Also, $T(p)$ is the zero sequence, as all the terms of $U$ which are even multiples of $2^j$, are divisible by $p$, and so, all the terms of $T$ are divisible by $p$. Thus, by Observation 1 we get the contradiction that $T$ (and hence $S$) has a $U(n)$-weighted zero-sum subsequence of length $n$. Hence, if $j \leq r - 3$, at most one term of $S$ is an odd multiple of $2^i$, for any $j + 1 \leq i \leq r - 2$.

Suppose for some $i$, such that $0 \leq i \leq r - 2$ and $i \neq j$, there is no term of $S$ which is an odd multiple of $2^i$. As any non-zero term of $\mathbb{Z}_{2^r}$ is a unit multiple of a power of 2 which is between 0 and $r - 1$, so, $S^{(2)}$ will have at least $k - (r - 1) = n + 1$ terms which are either zero or a unit multiple of $2^{r-1}$. So, we can find a subsequence $V$ of $S$ having length $n$, such that $V^{(2)}$ has an even number of terms which are a unit multiple of $2^{r-1}$.

So, $V^{(2)}$ is a zero-sum sequence in $\mathbb{Z}_{2^r}$. As all the even terms of $U$ are divisible by $p$ and as $V$ is a subsequence of $U$, so, $V(p)$ is the zero sequence. Thus, by Observation 1 we get the contradiction that $V$ is a $U(n)$-weighted zero-sum subsequence of $S$ having length $n$. Hence, if $0 \leq i \leq r - 2$ and $i \neq j$, then there is at least one term (and hence exactly one term) of $S$ which is an odd multiple of $2^i$.

Corollary 1. Let $n = 2^r p$, where $p$ is an odd prime and $r \geq 2$. Let $S$ be a sequence in $\mathbb{Z}_n$. Suppose $S$ is a $U(n)$-extremal sequence for the Gao constant. Then, $S$ has at least one and at most two odd terms.

Proof. Let $S$ be a $U(n)$-extremal sequence for the Gao constant. As $E_{U(n)}(n) = n + \Omega(n)$, so, $S$ has length $k = n + r$. Then, $k \geq n + 2$ as $r \geq 2$. So, by Lemma 8 $S$ has at most two odd terms. By Theorem 4 $S$ has at least one odd term.

Theorem 5. Let $n = 2^r p$, where $p$ is an odd prime and $r \geq 2$. Let $S$ be a sequence in $\mathbb{Z}_n$ which has exactly two odd terms. Then, $S$ is a $U(n)$-extremal sequence for the Gao constant if and only if $S$ is $U(n)$-equivalent to a sequence having the form $(\ast)$. 

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Proof. From Lemma 4, we see that if a sequence \( S \) is \( U(n) \)-equivalent to a sequence having the form \((*)\), then \( S \) is a \( U(n) \)-extremal sequence for the Gao constant.

Let \( S \) be a sequence in \( \mathbb{Z}_n \) which is a \( U(n) \)-extremal sequence for the Gao constant, where \( n = 2^r p \), where \( p \) is an odd prime and \( r \geq 2 \). Then, \( S \) has length \( k = n + r \). If \( S \) has exactly two odd terms, then, by Lemma 8 exactly one of the odd terms, is a unit and all the other terms of \( S \) are divisible by \( p \). By Theorem 4 for each \( i \) such that \( 1 \leq i \leq r - 2 \), an odd multiple of \( 2^i \) occurs exactly once in \( S \).

Thus, there are \( k - r = n \) terms of \( S \) which are either zero or an odd multiple of \( 2^{r-1} \). Let \( T \) denote the subsequence of \( S \) consisting of these \( n \) terms. As the terms of \( T \) are divisible by \( p \), so, the non-zero terms of \( T \) must be equal to \( n/2 \). As \(-1\) is a unit and as \( T \) cannot be a \( U(n) \)-weighted zero-sum sequence, so, the number of terms of \( T \) which are equal to \( n/2 \) must be odd.

Hence, \( S \) is \( U(n) \)-equivalent to the sequence

\[
(1, pa_0, 2pa_1, 2^2pa_2, \ldots, 2^{r-2}pa_{r-2}, n/2, \ldots, n/2, 0, \ldots, 0)
\]

where \( m \) is odd and for \( 0 \leq i \leq r - 2 \), \( a_i \) is odd. This has the form \((*)\). \( \square \)

**Theorem 6.** Let \( n = 2^r p \), where \( p \) is an odd prime and \( r \geq 2 \). Let \( S \) be a sequence in \( \mathbb{Z}_n \) which has exactly one odd term. Then, \( S \) is a \( U(n) \)-extremal sequence for the Gao constant if and only if \( S \) is \( U(n) \)-equivalent to a sequence having the form \((*)\), \((*)\) or \((***)\).

**Proof.** From Lemmas 3, 4 and 5 we see that a sequence which is \( U(n) \)-equivalent to a sequence having the form \((*)\), \((*)\) or \((***)\), is a \( U(n) \)-extremal sequence for the Gao constant.

Let \( S \) be a \( U(n) \)-extremal sequence for the Gao constant, where \( n = 2^r p \), such that \( p \) is an odd prime and \( r \geq 2 \). Then, \( S \) has length \( k = n + r \). By Theorem 4 two cases can occur.

**Case:** An odd multiple of \( 2^i \) occurs exactly once in \( S \), for each \( i \), such that \( 0 \leq i \leq r - 2 \).

Let \( S_* \) denote the subsequence of \( S \) consisting of the remaining \( k - (r - 1) = n + 1 \) terms. Then, every non-zero term of \( S_* \) is an odd multiple of \( 2^{r-1} \). The arguments for the remaining part of this case of the proof, are similar to the arguments in Theorem 3. So, we do not repeat those arguments and just state the conclusions directly. We see that \( S_*^{(2)} \) cannot have all terms zero or all terms
non-zero. We use the fact that the sum of an even number of terms, all of which are odd multiples of $2^{r-1}$, will be zero in $\mathbb{Z}_{2r}$.

If $S_k^{(2)}$ has exactly one non-zero term, then $S_k$ is $U(n)$-equivalent to the sequence $(2^{r-1}a, 2^{r-2}b, 0, \ldots, 0)$, where there are $n - 1$ zeroes, $a$ is odd and $b$ is coprime to $p$. If $S_k^{(2)}$ has exactly one term which is zero, then $S_k$ is $U(n)$-equivalent to the sequence $(c, 2^{r-1}, 2^{r-2}, p a_2, \ldots, 2^{r-1}p a_n)$, where $c$ is divisible by $2^r$ and $a_i$ is odd, for $2 \leq i \leq n$. If $S_k^{(2)}$ has at least two terms which are zero and at least two terms which are non-zero, then $S_k$ is $U(n)$-equivalent to the sequence $(2^{r-1}b, 2^{r-2}, p a_1, \ldots, 2^{r-1}p a_m, 0, \ldots, 0)$, where $b$ is coprime to $p$, $m$ is odd and $a_i$ are odd, for $1 \leq i \leq m$.

Thus, in the case when an odd multiple of $2^t$ occurs exactly once in $S$, for each $i$, such that $0 \leq i \leq r - 2$, we see that $S$ is $U(n)$-equivalent to a sequence having one of the following forms:

- $(a_0, 2a_1, \ldots, 2^{r-2}a_{r-2}, 2^{r-1}a_{r-1}, 2^r b, 0, \ldots, 0)$, where $a_i$ is odd, for $0 \leq i \leq r - 1$ and $b$ is coprime to $p$. This has the form $(\ast)$.  
- $(a_0, 2a_1, \ldots, 2^{r-2}a_{r-2}, 2^{r-1}b, n/2, \ldots, n/2, 0, \ldots, 0 )$, where $a_i$ is odd, for $0 \leq i \leq r - 2$, $b$ is coprime to $p$ and $m$ is odd. This has the form $(\ast)$.  
- $(a_0, 2a_1, \ldots, 2^{r-2}a_{r-2}, c, 2^{r-1}, n/2, \ldots, n/2)$, where $a_i$ is odd, for $0 \leq i \leq r - 2$ and $c$ is divisible by $2^r$. This has the form $(\ast\ast)$.  

Case: There is a unique $j$ such that $1 \leq j \leq r - 2$ and an odd multiple of $2^j$ occurs exactly two times in $S$. Also, for any $i \neq j$ such that $0 \leq i \leq r - 2$, an odd multiple of $2^i$ occurs exactly once in $S$.

Let $S_k$ denote the subsequence of $S$ consisting of the remaining $k - r = n$ terms. We claim that all the terms of $S_k$ are divisible by $p$. Consider the subsequence $T$ of $S$ consisting of all the terms of $S$ which are divisible by $2^j$. Then $T$ has length $k - j = n + r - j \geq n + 2$. By Lemma 8, exactly one of the two terms which are odd multiples of $2^j$, is coprime to $p$ and all the other terms of $T$ are divisible by $p$. This proves our claim. Then, every non-zero term of $S_k$ is an odd multiple of $2^{r-1}$. If the number of non-zero terms in $S_k$ is even, then $S_k$ will be a zero-sum subsequence of $S$ of length $n$. So, the number of non-zero terms in $S_k$ must be odd. Thus, $S$ is $U(n)$-equivalent to

$$(a_0, 2a_1, \ldots, 2^{r-1}a_{j-1}, 2^j, 2^i p a_j, \ldots, 2^{r-2}p a_{r-2}, n/2, \ldots, n/2, 0, \ldots, 0 ),$$

where $a_i$ is odd, for $0 \leq i \leq r - 2$ and $m$ is odd. This has the form $(\ast)$.  

\[\square\]
6 Concluding remarks

From Theorem 1, we see that when $n$ is odd, there is only one type of $U(n)$-extremal sequence for the Gao constant (and hence also for the Davenport constant). However, in this paper, we have seen that for any even $n$, there are at least three types of $U(n)$-extremal sequences for the Gao constant. Sequences of the type $(\ast)$ which have a term equal to $n/2$, are also of the type $(\ast)$.

When $n$ is even and $\omega(n) \leq 2$, it can be shown that there is only one type of $U(n)$-extremal sequence for the Davenport constant. When $n$ is even, with $\omega(n) \geq 3$, there are other types of $U(n)$-extremal sequences for the Davenport constant. So, we will get other types of $U(n)$-extremal sequences for the Gao constant (which are of the standard type).

Lemma 9. Let $n = 2p_1p_2 \ldots p_r$ be squarefree where the $p_i$s are primes. Then, the following sequence is a $U(n)$-extremal sequence for the Davenport constant; $S : (a, \hat{p}_1, \hat{p}_2, \ldots, \hat{p}_r)$, where $\hat{p}_i = n/(2p_i)$, for $1 \leq i \leq r$ and $a$ is chosen to be either 1 or 2, so that $S$ has an odd number of odd terms.

Proof. Let $n$ and $S$ be as in the statement of the lemma. Suppose $T$ is a $U(n)$-weighted zero-sum subsequence of $S$. Then, the first term of $T$ cannot be $\hat{p}_i$, for any $i$ with $1 \leq i \leq r$, as all the other terms of $T$ are divisible by $p_i$, if $i \leq r - 1$. Suppose the first term of $T$ is $a$. If $T$ does not contain $\hat{p}_i$, for some $i$ with $1 \leq i \leq r$, then, we get the contradiction that $a$ is divisible by $p_i$. Finally, as $S$ has an odd number of odd terms, so, any unit-weighted sum of these terms cannot be even. Thus, $T$ cannot be $S$.

The constant $C_A(n)$ has been defined, and the $U(n)$-extremal sequences for that constant have been characterized in [4], when $n$ is odd. When $n$ is a power of 2, such sequences have been characterized in [6]. It will be interesting to characterize such sequences, for other even values of $n$.

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