From modern observations of gravitational interactions, it can be inferred that there is much left to discover about the fundamental gravitational field. Since the advent of the General Theory of Relativity over a century ago, we have come to make exotic assumptions pertaining to the inner workings of an associated field theory. One of which is an elastic nature to spacetime and the behavior of gravity for strong and weak fields. In this work we investigate a more physical nature, expanding upon general relativity led by observations of strong sources. We introduce a candidate Lorentz-invariant field theory that employs an elastic and pseudoscalar nature to the field interpretation and it’s properties. A unique generation of the Euler-Lagrange equations of motion is presented; resulting in a longitudinal wave equation for the Dilation gravitational field. This provides a modern advancement of a relativistic gravitational field theory, supported by observation.

Keywords: longitudinal wave, elastic spacetime, gravitational waves, scalar tensor, dilation

I. INTRODUCTION

At the advent of the 20th century research conducted on fundamental physics generated a vast growth of the understanding of principal physical phenomena like that of Electromagnetism, Gravitation, Nuclear Physics and the genesis of Quantum Mechanics. Today, investigations into the mechanisms of the fundamental forces has taken a more robust approach contrasting older methods of research. Physics beyond Feynman’s introduction to modern unified field theory, in the form of QED, changed our treatment of investigating candidate field theories of fundamental forces. Historically, gravitational field theory has had few fundamentally profound breakthroughs when compared to the progress of other fundamental forces (strong and weak nuclear, and electromagnetism). Leaving a gaping hole in our understanding of the force in regards to a dynamic or time-dependant nature; and subsequently a quantum description of the gravitational field. Following with the communications of Einstein, Rosen and Infeld [1] from 1936-1938, Einstein himself was adamant that gravitational waves were not physically manifest and were a mathematical artifact of the linearized field equations per his publications in 1916-1918 [2, 3]. In which [he], Infeld and Hoffman arrived at the post-Newtonian approximation to the weak field equations [1] generating a perturbed linear set of plane wave equations in cylindrical coordinates; but not without apparent coordinate singularities propagated by these cylindrical plane waves. For many years the theoretical development of gravitational waves has eluded many scientists. Progressing from the beginning conversations exchanged between Einstein, Rosen and Infeld debating the mathematical artifacts arising from the implementation of a perturbation method[1], to the eventual discovery made by the LIGO/LSC community of scientists[4]. Experimental and observational studies have only tested for and observed gravitational waves to propagate transversely through space. These spatially transverse waves constitute the formal oscillations allowed by the well-known transverse-traceless gauge conditions[5]. Within these gauge conditions and under a linearization method, the reconstruction of the Riemann curvature and subsequently the Ricci curvature fails under this regime.

The current relativistic theory of gravitation, The General Theory of Relativity (GR), was not constructed to admit sufficient time-dependent solutions for the temporal variation of the gravitational field. Nor does it admit a substantial theory of gravitational field oscillations emanating as radiation in regards to localizing the energy-momentum carried. For gravitation proposed as a fundamental spin-2 self-interacting field, an appropriate non-linear representation of the field equations must be established. In order to recover the necessary curvature on a curved background, one must associate a source term in the field equations with gravitational mass-energy. Direct and indirect observations of strong sources (Black Holes, Neutron Stars, Binary Mergers, etc.) and Dark Matter-Energy elude to us that there is a missing facet to gravity not seen or theorized using GR under the standard model.
wide variety of solutions, in GR and its extensions, have been constructed to account for such dynamics introducing more complexities than simplifications. Looking at the detailed detections of LIGO and similar gravitational wave observatories, it is intuitive to infer that there is a fundamental “elastic” nature to spacetime and the interactions involving gravity. With this being said, in this paper we build upon the foundation of an elastic interpretation of gravity. With this in mind, we look into a formal description of volume deformations with respect to spacetime laid out in the previous work [6] and [7]. We look into an intuitive description of mass-energy distributions with respect to scalar-tensor gravitational strain field (Dilation, $D^{\mu\nu}$). When considering volume deformations of sourced mass distributions, this field (termed Dilation) constitutes deformations of spacetime geometry when considering the principle directions or local volumes within a local spacetime. Formulated from the observation of the aforementioned strong field sources, this proposed pseudoscalar-like gravitational field extends the idea of static curvature to a dynamic field that is scaled by the invariant curvature $\Lambda^{\alpha\beta}R_{\alpha\beta} = R$ of the local space. Where, $\Lambda^{\alpha\beta}$ is the strong-form metric of the local spacetime. Mass densities and scalar curvatures are considered the sources of this field. Under static conditions across $n = 4$ spacetime dimensions this field is defined by a finite volumetric gravitational strain tensor, $\varepsilon^{\sigma\tau}(\eta_{\mu}, \tau)$, for a constant local mass distribution. An expression for the dilation tensor is given in terms of the finite volumetric strain and the trace of the Ricci curvature tensor.

$$D^{\mu\nu}(\eta_{\mu}, \tau) = \frac{\beta}{2m_{vac}} \Lambda^{\mu\nu} (\Lambda_{\sigma\tau} \varepsilon^{\sigma\tau}(\eta_{\mu}, \tau))$$

In this expression the scalar curvature is provided by the Einstein field equations of the strong-form background, in which the scalar holds values for the background vacuum ($\Lambda_{\mu\nu}$) and the source distribution. Considering the $^+\Lambda$-vacuum as a zero state configuration, the scalar curvature is always nonzero with a value of $R_{\text{vac}} \equiv R_{\text{zero}}$, where $R_{\text{zero}} = \Lambda^{\alpha\beta}R_{\alpha\beta} \equiv 4(\pm\Lambda)$ for dimension $(n = 4)$. For nonzero source distribution, $R_{\text{vac}} = R_{\text{zero}} + R_{\text{source}}$ where $R_{\text{source}}$ can take on values from any solution of the trace Einstein field equations for nonzero mass. This expression for the dilation field represents a way of dynamically describing the gravitational strain generated by a nonzero mass-energy distribution using the trace Einstein field equations; as first described in [7].

With this definition of the Dilation field, the development of a formal Lagrangian density is given in section [1] in which terms of $O(D^2)$ and higher are deemed valuable to the theory. This lagrangian density is thus a representation of the gravitational energy density involved in an isotropic compression and expansion of the local spacetime. In section [III] an action is then uniquely varied with respect to the field, $D^{\mu\nu}$, and then the background metric, $\Lambda^{\mu\nu}$. Resulting in the recovery of the tensorial einstein field equations and the corresponding field equation for dilation. Lastly generating a nonlinear longitudinal gravitational wave equation, describing the propagation of this proposed massive pseudoscalar field. In the succeeding sections, the curvature of the spacetime background are incorporated into the dilation field equations producing solutions dependent on the Einstein Field Equations when the curvature is not negligible; for the regime $\eta_{\mu\nu} \rightarrow \Lambda_{\mu\nu}$ incorporating strong gravitational fields.

II. LAGRANGIAN DENSITY OF THE DILATION FIELD

A. The Massive Phonon Basis

Looking into a fundamental basis of which we can formulate gravitational field equations from, we arrive at the generalized form of the Klein-Gordon equation representing massive free-scalar fields. Where the classical Klein-Gordon equation can be expressed in terms a massive field, $\phi(x_\mu, \tau)$, as

$$(\Box + M^2) \phi(x_\mu, \tau) = 0$$

Traditionally, the linear Klein-Gordon equation is utilized as a template relativistic wave equation. In which it is commonly used for the introduction of novel techniques and/or concepts pertaining to massive scalar or pseudoscalar free-fields. Here we explore the implications of beginning with a classically non-interacting massive scalar field. The description of this test field is similar to that of a phonon with nonzero mass. We are thus directed to begin our efforts with a generalized form of a lagrangian density describing the massive phonon field (our test field), $\phi(x_\mu, \tau)$. The Lagrangian density in a lattice space, $(L_{\text{phonon}})$, provides us with a statement of the associated energy density: $L_{\text{phonon}} = \sum_{k=1}^{n} \frac{m}{2} (\partial_\tau \phi)^2 - \frac{k_s}{2} (\phi_{k+1} - \phi_k)^2$.

$$L_{\text{phonon}} = \sum_{k=1}^{n} \frac{m}{2} (\partial_\tau \phi)^2 - \frac{k_s}{2} (\phi_{k+1} - \phi_k)^2$$

Here $k_s$ is an index associated with the number of neighboring parcels of spacetime, in which the field
at that point has mass $m$. Under this discrete approximation we make the assumption that the field is partitioned with respect to an inter-parcel spacing $A$. Indicating that there is a spatial discrete-ness about the test field at the $k^{th}$ point, $\phi_k(x)$, and its nearest neighbor $(\phi_{k+1}(x) - \phi_k(x))$. This makes for a valid approximation for a “spacetime phonon” in a discrete limit; behaving very much like the classical treatment of traditional phonons in a 1-dimensional crystal lattice. Continuing this analogy, in the continuum limit, where the sum is replaced by an integral over space $\sum_{k=1}^{n} \rightarrow \int dx$ and using a Taylor expansion on the field variable $\phi_k \rightarrow \sqrt{\Lambda} \phi (x_i)|_{x=k \cdot A}$ and $\phi_{k+1} - \phi_k \rightarrow \sqrt{\Lambda^2} \nabla_i \phi (x_i)|_{x=k \cdot A}$; the corresponding lagrangian in this limit is

$$\mathcal{L} (\phi, \partial_i \phi) = \frac{m}{2} (\partial_i \phi)^2 - \frac{k^2}{2} A^2 (\nabla_i \phi)^2$$

as $A \rightarrow 0$. We can see that the new behavior of this test field, again, resembles that of a “spacetime phonon” in a continuum limit. Conceptualizing a vibratory nature to spacetime parcels. In comparison this gives a continuous description of neighboring spacetime parcels as opposed to a discrete method pertaining to discrete infinitesimal sectors of spacetime. The linear behavior of this test field is analogous to that of the non-interacting longitudinal gravitational strain field (Dilation). The full interacting field description is explained in further detail following this section.

Fundamentally, a linear longitudinal oscillation is comprised of displacements $(\Delta S_n)$ parallel to the direction of propagation resulting in the relaxation (expansion) and rarefaction (compression) of a local volume. Notably, one might be inclined to suggest from this analogy that the Klein-Gordon equation alone provides a simple expression that may be used in the development of a resulting field equation. Suggesting that modifications could be made to the equation to condition or fashion it for use within the framework of general relativity directly. Unfortunately, the Klein-Gordon equation alone does not hold the necessary structure to cater to a scalar-tensor gravitational field. Lacking the necessary nonlinearity predicted by a generalization of interacting scalar-tensor fields. Thus, describing the tensor field of interest as a free-scalar is insufficient for the description of time-dependent variations in volume deformations of mass-energy distributions. We must move to consider an interacting field theory that includes nonlinear terms involving the field variable $D_{\mu \nu}$. With that being said, the Klein-Gordon equation provides us with a valid linear approxima-

### B. Lagrangian Density of the Dilation Field

Let the tensor field $D_{\mu \nu}$ be the object of interest. This tensor is expressed in terms of the components of the volumetric gravitational strain tensor $\varepsilon_{\mu \nu}$ in a 4-dimensional spacetime. Corresponding to the principle directions of strain, this presents the Dilation as a simple tensor with no off-diagonal elements. The trace $\Lambda^{\mu \nu} D_{\mu \nu} \equiv 4 \left| \mathcal{D}(\Lambda) \right|$ provides us with a description of volume deformations of mass-energy distributions $(\rho_0)$ in a local spacetime. Looking back at equation (1) this massive field effectively provides a means of transporting the scalar curvature associated with a local spacetime, scaled by the “resistance” or effective mass, $m_{vac} = \frac{\Lambda^{\mu \nu}}{8 \pi G} \int \sqrt{-|\Lambda|} dx^\nu$, of the background vacuum. In regards to a field equation further explained in the succeeding section, we consider the interaction of this field with the background of nonzero effective mass. This generates the sought after nonlinearities incorporating the Klein-Gordon form from the phonon propagation in resistive media analogy in the preceding section. A massive longitudinal wave equation can then be written from the nonlinear theory that would give the parallel displacements of constituent finite spacetime parcels with zero spacing between them. Analogous to the model of a lattice phonon as the vibratory motion of a particle continuum, the Dilation field can be described very much like the phononic model with appropriate modifications for a classical field\[3, 4]. With this, spacetime fluidity can be described by a collection of local unit volumes with mass $m_{vac}$.

In terms of the product of local 1-forms $(dx^n)$ and $\sqrt{|\Lambda|}$, a local volume form $(\omega^n)$ of the pseudo-Riemannian manifold is\[10, 11\]:

$$\omega^n (x^1, x^2, ..., x^n) = \sqrt{-|\Lambda|} dx^1 \wedge dx^2 \wedge ... \wedge dx^n$$

The effective mass of the local volume can be calculated as, $m_{vac} \equiv \frac{\Lambda^{\mu \nu}}{8 \pi G} V_n$. Because we can assume homogeneity of the background energy density characterized by the cosmological constant $(+\Lambda)$ this value of the effective mass holds for any number of continuum parcels $(k)$ with volumes equal to that of the n-ball $(V_n)$ for isotropic spaces. Moreover, for this discussion of the dilation field we limit ourselves to consider hyperbolic spaces of dimension $n = 4$, to restrict the complexity of cosmological implications on the field description. Now that an expression from which we can begin the formulation of the formal Dilation, Gravitational Strain field equations.
for the amount of mass per unit parcel is given, we continue our description of the dilation field as analogous to that of a traveling phonon in 4-dimensional hyperbolic spaces. Consider an initial position at \( \{x^0, x^1\} \), a 4-displacement \( S^\alpha \) can be introduced resulting in a dilated configuration of the continuum with a change in volume for \( \Delta V = V^* \). The displacement can be described as a continuous function of parameter \( \eta \) for \( S^\alpha(\eta) \), requiring that small gradients of \( S \) (strain) follow the linearized theory of elasticity conveyed in Hooke's law. While large deformations follow the nonlinear theory of continuum deformations. Sustaining the generality of the elastic theory, we state that gradients of the displacement are expressed in terms of the aforementioned spacetime strain tensor \( \varepsilon_{\mu\nu} \) as a function of the parameter \( \eta, \tau \). Similarly, from equation [1] the field can be written as a smooth function of \( (\eta, \tau) \),

\[
D^\mu\nu(\eta, \tau) = \frac{\beta R_{\text{vac}}}{2m_{\text{vac}}} \Lambda^{\mu\nu} [\Lambda_{\sigma\gamma} (E^{\sigma\gamma}(\eta, \tau) - \Lambda^{\sigma\gamma})]
\]

where \( D_{\mu\nu} = D^{\mu\nu} = 0 \) for \( \mu \neq \nu \) and the constant \( R_{\text{vac}} \) is the intrinsic scalar curvature of the background spacetime–\( \Lambda_{\mu\nu} \). The Lagrangian density, \( \mathcal{L} \), of the dilation field with a background spacetime can be expressed as a combination of the energy associated with the scalar curvature generated by the dilation field \( \mathcal{L}_R \), the kinetic energy density of the field \( \mathcal{L}_D \), and the mass of the field \( \mathcal{L}_m \):

\[
\mathcal{L}(\Lambda, D(\eta, \tau), \nabla D(\eta, \tau), \ldots) = \mathcal{L}_R - \mathcal{L}_D - \mathcal{L}_m \tag{7}
\]

Where the respective terms are,

\[
\begin{align*}
\mathcal{L}_R &= \frac{c^4}{16\pi G} \Lambda^{\mu\nu} \tilde{R}_{\mu\nu} \\
\mathcal{L}_D &= \frac{1}{2} \Lambda^{\alpha\beta} \nabla_\alpha D_{\mu\nu} \nabla_\beta D_{\mu\nu} \\
\mathcal{L}_m &= \frac{1}{2} m_{\text{vac}}^2 (D_{\mu\nu})^2.
\end{align*}
\]

Here, \( c \) is the proposed speed of propagation for the gravitational field, with \( \tilde{R}_{\mu\nu} \) and \( \tilde{R} \) as the resultant Ricci curvature tensor and scalar, respectively, produced by the field. Before combining terms a parametrization of the covariant derivative in terms of \( (\eta) \) and proper time \( (\tau) \) is needed to extract the dynamics produced by a sought after field equation. This parametrization essentially makes the covariant derivative a differential operator in terms of the coordinate-free parameter, proper time, giving us the following langrangian,

\[
\mathcal{L}(\Lambda, D(\eta, \tau), \nabla D(\eta, \tau), \ldots) = \frac{c^4}{16\pi G} \Lambda^{\mu\nu} \tilde{R}_{\mu\nu} - \frac{1}{2} \Lambda^{\alpha\beta} \nabla_\alpha D_{\mu\nu} \nabla_\beta D_{\mu\nu} - \frac{1}{2} m_{\text{vac}}^2 (D_{\mu\nu})^2 \tag{9}
\]

where the operators \( D(\tau) = T^\alpha \nabla_\alpha \), with timelike tangent vector \( T^\alpha = \frac{\partial}{\partial \tau} \) and \( D(\eta) = S^\alpha \nabla_\alpha \), with spacelike tangent vector \( S^\alpha = \frac{\partial}{\partial \eta} \). This separation of timelike and spacelike differential operators is key to producing evolutions of the tensor field and ultimately generating a parametrized wave operator.

This tensor field is also predicted to interact with the massive background spacetime and itself, thus we include nonlinear interaction terms in \( \mathcal{L}_m \) up to order-\( \sigma \) and coupled by a constant \( (g_\sigma) \) that accounts for the strength of the field-field and field-background coupling.

\[
\mathcal{L}_m \rightarrow \mathcal{L}_m = \frac{1}{2} m_{\text{vac}}^2 (D_{\mu\nu})^2 + \sum_{\sigma} \frac{1}{\sigma!} g_\sigma (D_{\mu\nu})^\sigma \tag{10}
\]

The addition of this \( \sigma \)-interaction gives a formal Lagrangian density as:

\[
\mathcal{L}(\Lambda_{\mu\nu}, D_{\mu\nu}, \nabla D_{\mu\nu}) = \frac{c^4}{16\pi G} \Lambda^{\mu\nu} \tilde{R}_{\mu\nu} - \frac{1}{2} [(D(\tau) D_{\mu\nu})^2 + (D(\eta) D_{\mu\nu})^2] - \frac{1}{2} m_{\text{vac}}^2 (D_{\mu\nu})^2
\]
III. THE DILATION FIELD EQUATION

From the lagrangian density outlined in the previous section we can continue further to find appropriate field equations for the field in question. We can proceed to write down a functional of the background metric and the field variable for some boundary $\Sigma$.

$$A[\mathcal{D}, \Lambda] = \int_\Sigma [\mathcal{L}_R - \mathcal{L}_D - \mathcal{L}_m] \sqrt{-\Lambda} d^m \eta$$

(12)

The extremization of the action ($A$) is taken with respect to the field variable ($D$) and also the background metric $\Lambda$. This allows a unique variation of the action, giving a new interdependendcy of the lagrangian density on two variables describing the nature of multiple species of the same gravitational field. Executing this variation, $A[\mathcal{D} + \delta \mathcal{D}]$ is

$$A[\mathcal{D} + \delta \mathcal{D}] = A[\mathcal{D}] + \delta A[\delta \mathcal{D}]$$

(13)

$$= \int_\Sigma [\mathcal{L}_D + \mathcal{L}_m] d^m \eta$$

$$+ \int_\Sigma \left[ \frac{\partial (\mathcal{L}_D + \mathcal{L}_m)}{\partial \Lambda_{\mu \nu}} \delta \Lambda_{\mu \nu} \right] d^m \eta$$

$$+ \int_\Sigma \left[ \frac{\partial (\mathcal{L}_D + \mathcal{L}_m)}{\partial (\partial_\alpha \mathcal{D}_{\mu \nu})} \delta (\partial_\alpha \mathcal{D}_{\mu \nu}) \right] \sqrt{-\Lambda} d^m \eta$$

The last term in the above equation is the variation of the action with respect to the field denoted by $\delta A[\delta \mathcal{D}]$ and can be expanded as

$$\delta A[\delta \mathcal{D}] = \int_\Sigma \left[ \frac{\partial (\mathcal{L}_D + \mathcal{L}_m)}{\partial \mathcal{D}_{\mu \nu}} \delta \mathcal{D}_{\mu \nu} + \frac{\partial (\mathcal{L}_D + \mathcal{L}_m)}{\partial (\partial_\alpha \mathcal{D}_{\mu \nu})} \delta (\partial_\alpha \mathcal{D}_{\mu \nu}) \right] \sqrt{-\Lambda} d^m \eta$$

(14)

which includes a boundary term at the surface boundary $\partial \Sigma$, provided that the variation at the fixed boundary yields the following subset for $\mathcal{L}_D$ and $\mathcal{L}_m$ of the Euler-Lagrange equations of motion

$$0 = \int_\Sigma \left[ \frac{\partial (\mathcal{L}_D + \mathcal{L}_m)}{\partial \mathcal{D}_{\mu \nu}} \delta \mathcal{D}_{\mu \nu} + \frac{\partial (\mathcal{L}_D + \mathcal{L}_m)}{\partial (\partial_\alpha \mathcal{D}_{\mu \nu})} \delta (\partial_\alpha \mathcal{D}_{\mu \nu}) \right] \sqrt{-\Lambda} d^m \eta$$

(15)

Because the Lagrangian density is also dependent on the background metric $\Lambda^\mu_{\nu}$ through the interaction of the background, we vary the action once again but with respect to the inverse metric $\Lambda_{\mu \nu}$. This variation will give us the appropriate equation describing the curvature of space from the interaction with the dilation field. The variation in the action is now expressed as $\delta A[\delta \mathcal{D}, \delta \Lambda]$, where $\delta A[\delta \Lambda]$ is the variation of the action with respect to the inverse metric:

$$\delta \mathcal{A}[\delta \Lambda] = \int_\Sigma \frac{\delta \mathcal{L}_{\text{total}}}{\delta \Lambda^\mu_{\nu}} d^m \eta$$

(16)

$$= \int_\Sigma \left[ \frac{\delta (\mathcal{L}_R + \mathcal{L}_D + \mathcal{L}_m)}{\delta \Lambda^\mu_{\nu}} \cdot \left( \frac{\sqrt{-\Lambda}}{\delta \Lambda^\mu_{\nu}} \right) \right] d^m \eta$$

where the variation in $\Lambda^\mu_{\nu} \hat{R}_{\mu \nu}$ can be expanded in terms of $\frac{\delta \hat{R}}{\delta \Lambda^\mu_{\nu}}$ and $\frac{\delta \sqrt{-\Lambda}}{\delta \Lambda^\mu_{\nu}}$, such that

$$\frac{\delta \left( \Lambda^\mu_{\nu} \hat{R}_{\mu \nu} \right)}{\delta \Lambda^\mu_{\nu}} = \frac{\delta}{\delta \Lambda^\mu_{\nu}} \left[ (\delta \Lambda^\mu_{\nu}) \hat{R}_{\mu \nu} + \Lambda^\mu_{\nu} \left( \delta \hat{R}_{\mu \nu} \right) \right]$$

(17)

From the cancellation of $\delta \Lambda^\mu_{\nu} / \delta \Lambda^\mu_{\nu} = 1$, this simplifies to just the trace of the variation of the Ricci tensor with respect to the metric, $\frac{\Lambda^\mu_{\nu} \delta (\hat{R}_{\mu \nu})}{\delta \Lambda^\mu_{\nu}} \equiv \frac{\delta \hat{R}}{\delta \Lambda^\mu_{\nu}}$. In order to evaluate $\delta \hat{R}$, we first evaluate $\delta \hat{R}_{\mu \nu}$ as a contraction of the first and second indices in the respective Riemann curvature tensor;

$$\delta \hat{R}_{\mu \nu} = \delta \hat{R}^\rho_{\mu \nu} = \nabla_\nu (\delta \Gamma^\rho_{\mu \nu}) - \nabla_\mu (\delta \Gamma^\rho_{\nu \mu})$$

(18)

Thus, taking the trace results in obtaining the variation of the scalar curvature $\delta \hat{R}$

$$\Lambda^\mu_{\nu} \delta \hat{R}_{\mu \nu} = \delta \hat{R} = \hat{R}_{\mu \nu} \delta \Lambda^\mu_{\nu} + \nabla_\gamma (\Lambda^\mu_{\nu} \delta \Gamma^\gamma_{\nu \mu} - \Lambda^\gamma_{\mu \nu} \delta \Gamma^\rho_{\rho \mu})$$

(19)

The last term in the above equation becomes a total derivative with respect to the metric and does not contribute to the variation of the action at the boundary $\partial \Sigma$. Emulating the expected Einstein field equations, the resulting variation with respect to the
background metric becomes
\[
\frac{c^4}{16\pi G} \left( \tilde{R}_{\mu\nu} - \frac{1}{2} \Lambda_{\mu\nu} \tilde{R} \right) = \frac{c^4}{16\pi G} \tilde{G}_{\mu\nu} \tag{20}
\]

The result of this elegant variation is what we expect when considering the initial dilation field scalar curvature, \( \tilde{R} \), in the total Lagrangian density for the dilaton field and its interaction. This subset of the full equations of motion represents the curvature associated with the massive dilatation field. Accounting for a second source of curvature separate from the original background measure.

Adding this back into the total Lagrangian density for the dilaton field we have,

\[
\delta A[\delta D_{\mu\nu}, \delta \Lambda_{\mu\nu}] = \int \sum \left[ \left[ \frac{c^4}{16\pi G} \left( \tilde{R}_{\mu\nu} - \frac{1}{2} \Lambda_{\mu\nu} \tilde{R} \right) + \frac{1}{2} \Lambda_{\mu\nu} (\mathcal{L}_D + \mathcal{L}_m) + \frac{\delta (\mathcal{L}_D + \mathcal{L}_m)}{\delta \Lambda_{\mu\nu}} \right] \delta \Lambda_{\mu\nu} 
+ \left( \partial \left( \mathcal{L}_D + \mathcal{L}_m \right) \right) \delta D_{\mu\nu} \right] \sqrt{-\Lambda} \eta_{\mu\nu} = 0 \tag{21}
\]

This statement holds true from the variational principle for stationary actions, stating that \( \delta A = 0 \). Since the variations in \( \delta D_{\mu\nu} \) and \( \delta \Lambda_{\mu\nu} \) are arbitrary, the interior terms for the varied action must vanish appropriately such that the full Euler-Lagrange equations of motion can be written as:

\[
\left[ \frac{c^4}{16\pi G} \left( \tilde{R}_{\mu\nu} - \frac{1}{2} \Lambda_{\mu\nu} \tilde{R} \right) + \frac{1}{2} \Lambda_{\mu\nu} (\mathcal{L}_D + \mathcal{L}_m) + \frac{\delta (\mathcal{L}_D + \mathcal{L}_m)}{\delta \Lambda_{\mu\nu}} \right] + \left[ \partial \left( \mathcal{L}_D + \mathcal{L}_m \right) \right] - \partial_{\alpha} \left( \partial \left( \mathcal{L}_D + \mathcal{L}_m \right) \right) = 0 \tag{22}
\]

Terms in this equation of motion can be compacted by replacing the differential terms of the Lagrangian density for their respective counterparts in terms of the field variable. In doing so, this allows us to restate this equation of motion in a form that is appropriate for a nonlinear wave equation. Identifying the terms as:

\[
\partial \left( \mathcal{L}_D + \mathcal{L}_m \right) \partial D_{\mu\nu} - \partial_{\alpha} \left( \partial \left( \mathcal{L}_D + \mathcal{L}_m \right) \right) = (\Lambda^{\alpha\beta} \nabla_\alpha \nabla_\beta + m_{\text{vac}}^2) D_{\mu\nu} + \sum \frac{(\sigma)}{\sigma!} g(D_{\mu\nu})^{(\sigma-1)} \tag{23}
\]

and

\[
\frac{c^4}{16\pi G} \tilde{R}_{\mu\nu} = -\frac{1}{2} \Lambda_{\mu\nu} \left( \tilde{R} + \mathcal{L}_D + \mathcal{L}_m \right) + \frac{\delta (\mathcal{L}_D + \mathcal{L}_m)}{\delta \Lambda_{\mu\nu}} \tag{24}
\]

Substituting in for the above terms gives a simplified equation of motion for the dilation field \( (D_{\mu\nu}) \):

\[
0 = (\Lambda^{\alpha\beta} \nabla_\alpha \nabla_\beta + m_{\text{vac}}^2) D_{\mu\nu} + \sum \frac{(\sigma)}{\sigma!} g(D_{\mu\nu})^{(\sigma-1)} + \frac{c^4}{16\pi G} \tilde{R}_{\mu\nu} \tag{25}
\]

The above equation of motion is very much close to the phononic approximation for a massive longitudinal wave. Replacing the second covariant derivatives with a modified wave operator \( \hat{\square} \) reminiscent of a gauge derivative, we arrive at the final form of the Longitudinal Wave equation for volume deformed spacetimes (where again, \( \tilde{R}^{\beta\nu} \) is the Ricci curvature tensor associated with the sourced background spacetime, and \( R_{\mu\nu} \) associated with the wave):

\[
\Lambda^{\alpha\beta} \nabla_\alpha \nabla_\beta \rightarrow \hat{\square} = \hat{\square} - R^{\beta\nu} \tag{26}
\]

Thus we have,

\[
0 = \left( \hat{\square} + m_{\text{vac}}^2 \right) D_{\mu\nu} - R^{\beta\nu} D_{\mu\beta} \tag{27}
\]
IV. DISCUSSION

A. Interpretation of the Dilation Field Equations

This massive tensor field effectively generates a classical self-interaction that interacts with the background spacetime, as can be seen by the production of a second species of scalar curvature along with the quadratic and higher-order terms in the expanded potential. These accompanying dynamics provide a means of transporting the information on the scalar curvature associated with the source distribution. Here a “self-interaction” is defined as the coupling of the field with itself, generating a classical analog of the quantum field theoretic self-energy [8, 14–17]. A resistive factor, respective of the kinematic properties during propagation, included in this derivation of the dilation field is given by the effective mass \( m_{\text{vac}} = \frac{\Lambda c^2}{8\pi G} \int \sqrt{-\hat{\text{g}}} \, x_{\mu} \) of the \( \Lambda \)-vacuum background. In regards to the wave equation introduced from the dilation field equations, this generates the sought after Klein-Gordon like form for the massive longitudinal wave equation. Taking a closer look at the Lagrangian density of this field reveals that it admits a behavior similar to what is expected of a massive scalar field. The nonlinearity, and subsequently the self-interaction, of the field is evident in the terms that couple the field with itself (field-field interactions) and couplings with the background \( \Lambda \mu \nu \) spacetime (field-vacuum interactions), represented as:

\[
\Omega(D_{\mu \nu}) = \frac{c^4}{16\pi G} \hat{R}_{\mu \nu} + \sum_{\sigma} \frac{g_\sigma}{\sigma!} (D_{\mu \nu})^{(\sigma-1)} \quad (28)
\]

Involving a coupling index \( \sigma \) and strength constant in \( g \). Expanding this polynomial for \( n = 4 \) gives an expression for the coupling strength of the scalar-tensor gravitational field, with \( g_\sigma \) the coupling constants. Further constraints on the coupling constants will not be explored in this text, such that we limit ourselves to just approximating the form of the interaction polynomial.

\[
\sum_{\sigma} \frac{(\sigma)}{\sigma!} g_\sigma (D_{\mu \nu})^{(\sigma-1)} = g_1 + g_2 (D_{\mu \nu})^1 + \frac{1}{2} g_3 (D_{\mu \nu})^2 + \frac{1}{8} g_4 (D_{\mu \nu})^3 \quad (29)
\]

For an \( n=4 \) coupling we can see that only the third and fourth terms contribute to meaningful couplings of the field for which the proposed self-interactions are prominent. For the lagrangian density of the field, restated here for convenience,

\[
\mathcal{L}(\Lambda_{\mu \nu}, D_{\mu \nu}, \nabla D_{\mu \nu}) = \frac{1}{2}[(D_{\tau \rho} D_{\mu \nu})^2 - (D_{(\nu)} D_{\mu \rho})^2] - \frac{1}{2} m^2 D_{\mu \nu} + \Omega(D_{\mu \nu}) \quad (30)
\]

the following properties can be stated. The lagrangian is real if, \( m^2, D_{\mu \nu}, g_\sigma \in \mathbb{R} \); giving a stable field theory. The first two terms are quadratic in \( D_{\mu \nu} \), stating that if \( \hat{R} \), and \( g_\sigma \) are zero we have a free-field lagrangian density \( (\mathcal{L}_0) \) described by the Klein-Gordon equation of motion. Representing a free tensor field theory. This term represents the free-kinetic component of the lagrangian. Here \( m \) is the classical mass of the field. Foreshadowing a quantization of this lagrangian, \( g_\sigma \) is a coupling constant that is a measure of interaction strength with cross-section proportional to \( g^2 \). Renormalizability can remove all the infinities of the resulting quantum field theory, provided all coefficients have units of \( [M]^n, n \geq 0 \). This implies that there are no \( \frac{1}{4} g_6 (D_{\mu \rho})^6 \) terms appearing in the polynomial \( \Omega(D_{\mu \nu}) \). A cubic term, \( \frac{1}{4} g_3 (D_{\mu \nu})^3 \), is allowed if and only if the lagrangian is a Lorentz scalar and \( m \) is interpreted as mass. This requires that there be no linear terms of the field variable like that of \( \mu(m) D_{\mu \nu} \). We extend these properties to the kinetic term, \(-\frac{1}{2} \left( \nabla_\alpha D_{\mu \nu} \nabla^\alpha D_{\mu \nu} \right) \), for the propagation of the energy-momentum contained in the field. Requiring that this term be quadratic in the field variable.

Under the variation of the action, as seen in the previous section, this results in the generation of the wave operator of curved spacetimes, \( \Box - \hat{R}_\nu^\mu = (\Box)_\nu^\mu \).

In the field expression, the Ricci curvature again provides the Einstein field equations governing the local curvature of spacetime, in which the curvature scalar holds values for the background vacuum and the source distribution. When considering the \( \Lambda \)-vacuum as a zero state configuration, the scalar curvature takes on a unique role for this field. We consider the following relationships for all species of scalar curvatures

\[
R_{\text{vac}}[\Lambda^{\alpha \sigma}] = \Lambda^{\alpha \sigma} R_{\alpha \sigma \tau} \propto 4 \Lambda + R_{\text{source}} \quad (31a)
\]

\[
\hat{R}[e^{\alpha \sigma}] = \Lambda^{\alpha \sigma} \hat{R}_{\alpha \sigma \tau} \propto \Gamma(\varepsilon) \quad (31b)
\]

We can see that once the field equations are constructed, there exists two distinctly separate scalar curvatures that are unique to the background spacetime \( (\hat{R}_{\text{vac}}[\Lambda^{\alpha \sigma}]) \) and the dilation field \( (\hat{R}[e^{\alpha \sigma}]) \), respectively. With the explanation of a unique variation of the action given in the previous section, one
can use this explicit determination of scalar curvature species to make a statement on a proposed gravitational charge. One can see that in the expression for explicitly defining the field variable, $D^{\mu \nu}$, the scalar curvature behaves like that of the “charge” of the field. With this, one can make a substantial comparison to the behavior of a source charge generating a field. Considering the scalar curvature generated as a source term for the field we can state a massive longitudinal wave equation in terms of the field generated by a nonzero mass-energy distribution using the trace Einstein field equations. Relativistically, a linear longitudinal wave is comprised of oscillations parallel to the direction of propagation. For the 4-wave vector having a spacetime signature $(+,\tau,\tau,\tau)$, it can be defined as

$$K_\mu = \left( \frac{\omega}{c_g}, -K_i \right)$$

(33)

where $(c_g)$ is the group velocity of the longitudinal wave. We can make a substitution for the group velocity and restate the 4-wave vector for longitudinal waves as

$$K_\mu = \left( \sqrt{\frac{\rho_{\text{vac}}}{\beta}} \omega, -K_i \right)$$

(34)

Taking the inner product of the wave vector gives

$$K_\mu K^\mu = \left( \sqrt{\frac{\rho_{\text{vac}}}{\beta}} \omega \right)^2 - K_i K^i$$

(35)

$$= \frac{\rho_{\text{vac}}}{\beta} \omega^2 - K_i K^i$$

$$= \frac{\rho_{\text{vac}}}{\beta} \omega^2 - \left( \frac{\omega}{c_p} \right)^2 \hat{n}_j \hat{n}^j$$

The inner product of the spatial components are represented as the inner product of the spatial unit vector $(\hat{n}_i)$ multiplied by the square of the angular frequency over the phase velocity. In the massive vacuum the assumption that the phase velocity equal in magnitude to the frequency cannot be stated in this case for longitudinal gravitational waves. We can employ the dispersion relation for the longitudinal wave coming from a rotating compact dense object with the magnitude of the angular velocity equating to the angular frequency of the radiated wave. With this, the modulus for bulk gravity is given by the aforementioned derivation using the Friedmann equations[24], where $(\beta)$ is:

$$\beta = -\frac{\sqrt{-\det[\Lambda_{\mu \nu}]} }{\alpha^3 8 \pi G} \left( 3 H^2 c^2 - \Lambda c^4 \right)$$

(36)

This equation for the dilation field represents a way of dynamically describing the gravitational strain field generated by a nonzero mass-energy distribution using the trace Einstein field equations. Relativistically, a linear longitudinal wave is comprised of oscillations parallel to the direction of propagation resulting in the relaxation (expansion) and rarefaction (compression) of a local volume. For an
N-dimensional spacetime manifold, the dilation field tensor, $D_{\mu\nu}$. While the bulk modulus is characterized by a description of the deformed volume.1

The effective mass of the local volume can be calculated as, $m_{\text{vac}} \equiv \frac{\Lambda c^2}{8\pi G} V_n$. Because we can assume homogeneity of the background energy density characterized by the cosmological constant ($+\Lambda$) this value of the effective mass holds for any number of continuum parcels (k) with volumes equal to that of the unit n-ball, $(V_n)$, for isotropic spaces. Furthermore, for this discussion of the dilation field we limit ourselves to isotropic spaces of dimension $n = 4$, to restrict the complexity of the field description. Now that an expression for the amount of continuum parcels (k) with volumes equal to this value of the effective mass holds for any number characterized by the cosmological constant ($+\Lambda$) we can convert this energy density to an effective mass density with a proper handling of the speed of light (c) by way of the equivalence principle. Giving an approximate effective mass density of the vacuum to be: $\rho_{m-\text{vac}} = 6.0849 \times 10^{-10}$. Numerically integrating this expression bounded by the unit volume for a cartesian coordinate system gives an effective amount of mass contained in a unit cube in three spatial dimensions $m_{\text{vac}} = 6.0849 \times 10^{-10}$. A quick approximation shows that given the inner product above the dispersion relation for the longitudinal wave gives a non-trivial result. This tells us just a small sample of the full derived dispersion relation for the massive vacuum. Due to the nonlinearity of the longitudinal gravitational wave, after further investigation, we can expect a dispersion relation that would compliment this nonlinearity. Further discussion of this relationship will be given in subsequent work that further investigates this connection to the vacuum density.

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