The 3–string vertex and the AdS/CFT duality in the PP–wave limit

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Abstract

We pursue the study of string interactions in the PP-wave background and show that the proposal of hep-th/0211188 can be extended to a full supersymmetric vertex. Then we compute some string amplitudes in both the bosonic and fermionic sector, finding agreement with the field theory results at leading order in $\lambda$.
1 Introduction

The dynamics of type IIB strings on the maximally supersymmetric PP–wave background [1, 2] has been extensively investigated in the last year. Technically, this has been possible because in the Green–Schwarz formalism the light–cone string action contains only free fields [3, 4]. By using this basic result, various different aspects of string dynamics on PP–wave backgrounds have been analyzed.

At the conceptual level the motivation for studying strings on PP–waves is twofold. On one side, it is interesting by itself to have the possibility to test string theory on a curved background (which is even not asymptotically flat). On the other side, Berenstein, Maldacena and Nastase [5] showed that the dynamics of string theory in these backgrounds is directly connected to that of (Super) Yang–Mills theories. This relation appears to be a “corollary” of the AdS/CFT duality [6] and can be used both to derive new predictions on the gauge theory side and to understand better the original duality itself. We will focus only on the simplest case of the maximally supersymmetric PP–wave background. This background can be obtained [1, 5] by performing a Penrose limit [7, 8] of $\text{AdS}_5 \times S^5$, where one keeps only the perturbations with large energy and large spin along a chosen $SO(2)$ inside the five–sphere. This limit can be translated [5] to the $\mathcal{N} = 4$ SYM side by using the standard AdS/CFT dictionary. The result is a non–’t–Hooftian limit where $\lambda = g^2_{YM} N_c \to \infty$. Most of the gauge invariant operators should decouple in this limit since their conformal dimensions diverge. On the other hand, a particular set of operators [5] (which we call collectively BMN operators) have a well defined conformal dimension even in the limit and there is a one–to–one correspondence between this set and the type IIB string spectrum on the PP–wave background [5]. The BMN operators are characterized by having $J >> 1$, where $J$ is the R– symmetry charge under the generator singled out in the Penrose limit on the supergravity side. In formulae the limit is defined as $\lambda \to \infty$ with $g_{YM} \sim \text{const.}$, and the surviving operators are characterized by

$$\Delta, J \to \infty \text{, with } \lambda' = \frac{\lambda}{J^2} \sim \text{const.}, \quad g_2 = \frac{J^2}{N_c} \sim \text{const.}, \quad \Delta - J \sim \text{const.} \quad (1.1)$$

In this paper we focus on the study of the 3–string interaction on the PP–wave background and discuss the implications of our results for the PP–wave/CFT duality. A first proposal for the 3–string vertex was put forward in [11]. This vertex was subsequently corrected and completed in various papers [12], [13] and [14]. The basic idea of these papers is to apply the techniques used in the flat space case [15, 16, 17, 18] also to the PP–wave background. The explicit form of the string interaction is obtained in two steps. First one rewrites in terms of string oscillators the delta functional ensuring the smoothness of the string world–sheet. This yields a 3–vertex that is invariant under all the continuous kinematical symmetries$^2$. Then

$^1$The relevance of the double scaling limit has been first stressed in [9, 10].

$^2$In the light–cone quantization the symmetries are divided in two types: the symmetries that
following closely [18], this 3–string vertex has been supplemented by a prefactor so that it transforms correctly also under the dynamical symmetries. The final formulation of the prefactor has been given in [14]. However, it has been noticed in [19] that this approach does not constrain completely the form of the 3–string vertex and that a different starting point is possible. The basic idea is to use the same techniques as in [15, 16, 17, 18] but to build the PP–wave string vertex on the unique vacuum of the theory. In [20] this approach has been used to derive a 3–string vertex satisfying all the kinematical constraints.

In Section 2 we show that there is a simple supersymmetric completion for the vertex of [20] which seems very promising for two reasons. On one side, as discussed in the paragraph 2.3, this supersymmetric vertex shares some striking similarity with the behaviour of supergravity on \( AdS_5 \times S^5 \). This suggest that the 3–state interaction on the PP–wave should be compared with the results in \( AdS_5 \times S^5 \) rather than with those of flat–space. On the other side the vertex presented here represents a simple way to realize the proposal of [10] for comparing string and field theory quantities. In Section 3 we discuss some concrete examples. As in [10], we stay at leading order in \( \lambda' \), but we consider also BMN operators with vector and fermion impurities (and not only those with bosonic impurities).

The proposal of [10] has been subsequently criticized [21, 22] and some alternative prescriptions have been proposed [23, 24, 25, 26, 27, 28, 29, 30]. In the Conclusions we review the arguments usually presented against [10] and the various alternative proposals. In our opinion none of the proposals made so far can be surely discarded in the present state, not even the one of [10] that provides the simplest possible setup. The main point we want to stress is that the PP–wave/CFT duality has to be treated as close as possible to the usual AdS/CFT. In fact, the physical states of the PP–wave background are also present in the full \( AdS_5 \times S^5 \) geometry; working in the plane wave approximation means just keeping the leading order in the limits (1.1)\(^3\). Thus it would be highly desirable that the results obtained in the PP–wave geometry could be interpreted on the gauge theory side by means of the usual AdS/CFT dictionary. We are still far from this goal, mainly because one crucial ingredient [32, 33] of AdS/CFT duality is clearly spelled only in the supergravity limit (i.e. the boundary conditions to impose on the string partition function). However we think that the vertex presented in this paper is a step forward in this direction.

\(^3\)Also the subleading correction beyond the PP–wave limit has been considered in [31].
2 The supersymmetric 3–string vertex

2.1 Our conventions for the free string

The free motion of a IIB superstring in the maximally supersymmetric PP–wave background can be described by the sum of the following 2D actions\(^4\)

\[
S_b = \frac{1}{4\pi\alpha'} \int d\tau \int_0^{2\pi|\alpha|} d\sigma \left[ (\partial_\tau x)^2 - (\partial_\sigma x)^2 - \mu^2 x^2 \right],
\]

\[
S_f = \frac{1}{4\pi\alpha'} \int d\tau \int_0^{2\pi|\alpha|} d\sigma \left\{ i e(\alpha) \left[ \bar{\theta} \partial_\tau \theta + \theta \partial_\tau \bar{\theta} + \theta \partial_\sigma \theta + \bar{\theta} \partial_\sigma \bar{\theta} \right] - 2\mu \theta \bar{\theta} \right\}. \tag{2.2}
\]

Lorentz indices have been suppressed for sake of simplicity; the 8 bosons \(x^I\) and the 8 fermions \(\theta^a\) are always contracted with a Kronecker \(\delta\) except for the mass term in \(S_f\), where \(\Pi = \sigma_3 \otimes 1_{4 \times 4}\) appears. As usual, we indicate with \(\alpha'\) the Regge slope, while \(\alpha = \alpha' p^+\) is the rescaled light–cone momentum\(^5\) and \(e(\alpha) = 1\) if \(\alpha > 0\) and \(e(\alpha) = -1\) if \(\alpha < 0\). This form for \(S_b + S_f\) follows directly from specializing to light–cone gauge the covariant action written in \([3, 4]\)\(^6\). It is also useful to rewrite the fermionic action in terms of two real spinors

\[
S_f = \frac{1}{4\pi\alpha'} \int d\tau \int_0^{2\pi|\alpha|} d\sigma i \left\{ e(\alpha) \left[ \theta^1 \partial_\tau \theta^1 + \theta^2 \partial_\tau \theta^2 \right] - 2\mu \theta^1 \bar{\theta}^2 \right\}, \quad \partial_\pm = \partial_\tau \pm \partial_\sigma, \tag{2.3}
\]

where \(\theta^1\) and \(\theta^2\) are related to the complex \(\theta\) by

\[
\theta = \frac{\theta^1 + i\theta^2}{\sqrt{2}}, \quad \bar{\theta} = \frac{\theta^1 - i\theta^2}{\sqrt{2}}. \tag{2.4}
\]

From Eqs. (2.1) and (2.2) it is straightforward to derive the mode expansions, the commutation relations and the expressions for the free symmetry generators. Let us recall here some of these ingredients which we will use in what follows: \(\omega_n = \sqrt{n^2 + (\mu\alpha)^2}\) is the frequency of the \(n\)th string mode and

\[
c_n = \frac{1}{\sqrt{1 + \rho_n^2}}, \quad \rho_n = \frac{\omega_n - n}{\mu\alpha}. \tag{2.5}
\]

At \(\tau = 0\) the mode expansion for the fermionic fields\(^7\) is

\[
\theta = \theta_0 + \sqrt{2} \sum_{n=1}^{\infty} \left[ \theta_n \cos \frac{n\sigma}{|\alpha|} + \theta_{-n} \sin \frac{n\sigma}{|\alpha|} \right], \tag{2.6}
\]

\(^4\)We take \(\mu > 0\).

\(^5\)As usual in the study of string interaction in the light–cone, we work with dimensionful \((\tau, \sigma)\), so \(\alpha\) can be also seen as parameterizing the length of the string.

\(^6\)Often in the literature a redefinition of the spinor coordinates is adopted \((\theta \rightarrow e^{i\pi/4}\theta)\).

\(^7\)Our conventions for the bosonic modes are equal to those of \([11, 13]\).
where the modes $\theta_n$ can be rewritten in terms of $b_n$’s that satisfy canonical commutation relations $\{b_n, b_m^\dagger\} = \delta_{nm}$, $\{b_n, b_m\} = \{b_n^\dagger, b_m^\dagger\} = 0$

$$\theta_0 = \frac{\sqrt{\alpha'}}{2 \sqrt{|\alpha|}} \left[ (1 + e(\alpha)\Pi) b_0 + e(\alpha)(1 - e(\alpha)\Pi) b_0^\dagger \right] , \quad (2.7)$$

$$\theta_n = \frac{\sqrt{\alpha'}}{\sqrt{2|\alpha|}} \sqrt{\frac{n}{\omega(n)}} \left[ P_n^{-1} b_n + e(\alpha) P_n b_n^\dagger \right] , \quad (2.8)$$

$$\theta_{-n} = i \frac{\sqrt{\alpha'}}{\sqrt{2|\alpha|}} \sqrt{\frac{n}{\omega(n)}} \left[ P_n^{-1} b_{-n} - e(\alpha) P_n b_{-n}^\dagger \right] . \quad (2.9)$$

where $P$ is a diagonal matrix in the spinor space

$$P_{\pm}^n = c_n \sqrt{\frac{\omega_n}{n}} (1 \mp \rho_n \Pi) = \frac{1}{\sqrt{1 - \rho_n^2}} (1 \mp \rho_n \Pi) . \quad (2.10)$$

The conjugate fermionic momentum is $\lambda_{\text{conj}} = -i \lambda = -i \frac{e(\alpha)}{2\pi \alpha'} \bar{\theta}$ and its mode expansion at $\tau = 0$ is

$$\lambda = \frac{1}{2\pi |\alpha|} \left\{ \lambda_0 + \sqrt{2} \sum_{n=1}^{\infty} \left[ \lambda_n \cos \frac{n\sigma}{|\alpha|} + \lambda_{-n} \sin \frac{n\sigma}{|\alpha|} \right] \right\} . \quad (2.11)$$

In terms of the $b_n$’s one has

$$\lambda_0 = \frac{\sqrt{|\alpha|}}{2\sqrt{\alpha'}} \left[ e(\alpha)(1 - e(\alpha)\Pi) b_0 + (1 + e(\alpha)\Pi) b_0^\dagger \right] , \quad (2.12)$$

$$\lambda_n = \frac{\sqrt{|\alpha|}}{\sqrt{2\alpha'}} \sqrt{\frac{n}{\omega(n)}} \left[ e(\alpha) P_n b_{-n} + P_n^{-1} b_{-n}^\dagger \right] , \quad (2.13)$$

$$\lambda_{-n} = -i \frac{\sqrt{|\alpha|}}{2\alpha'} \sqrt{\frac{n}{\omega(n)}} \left[ P_n^{-1} b_{-n}^\dagger - e(\alpha) P_n b_{-n} \right] . \quad (2.14)$$

The generator of the $x^+ \to x^+ + \text{const.}$ transformation is just the canonical Hamiltonian multiplied by $e(\alpha)$, since in the light–cone gauge $x^+$ is identified with $\tau$ ($x^+ = e(\alpha)\tau$). In the fermionic sector $H$ reads

$$H_f = e(\alpha) \int_0^{2\pi \alpha} d\sigma \left( \sum_{i=1}^{2} \partial_\tau \theta^i \lambda^i - L_f \right) = \frac{1}{\alpha} \sum_{n=-\infty}^{\infty} \frac{\omega_n}{2} (b_n^\dagger b_n - b_n b_n^\dagger) . \quad (2.15)$$

A similar expression holds also for the bosonic sector, so that the vacuum energy of the oscillators cancels and one can write

$$H = \frac{1}{\alpha} \sum_{n=-\infty}^{\infty} \omega_n (a_n^\dagger a_n + b_n^\dagger b_n) . \quad (2.16)$$
In the following we will need the explicit expression for the 16 dynamical supercharges

\[
Q^- = e(\alpha)\sqrt{\frac{\alpha'}{2\alpha}} \gamma \cdot \left[ a_0 (1 + e(\alpha)\Pi) + a_0^\dagger (1 - e(\alpha)\Pi) \right] \lambda_0 + \\
+ \frac{1}{\sqrt{|\alpha|}} \sum_{n=1}^{\infty} \sqrt{n} \gamma \cdot \left[ a_n^\dagger P_n b_{-n} + e(\alpha)a_n P_n^{-1} b_n^\dagger + ia_{-n} P_n b_n - ie(\alpha)a_{-n} P_n^{-1} b_{-n}^\dagger \right]
\]

\[
\bar{Q}^- = \sqrt{\frac{\mu}{2\alpha'}} \gamma \cdot \left[ a_0 (1 - e(\alpha)\Pi) + a_0^\dagger (1 + e(\alpha)\Pi) \right] \theta_0 + \\
+ \frac{1}{\sqrt{|\alpha|}} \sum_{n=1}^{\infty} \sqrt{n} \gamma \cdot \left[ a_n^\dagger P_n^{-1} b_n + e(\alpha)a_n P_n b_n^\dagger + ia_{-n}^\dagger P_n b_n - ie(\alpha)a_{-n} P_n b_n^\dagger \right].
\]

Instead of working directly with these expressions it is better to introduce the following linear combinations

\[
Q = \frac{1}{\sqrt{2}} \left( Q^- + \bar{Q}^- \right), \quad \bar{Q} = \frac{i}{\sqrt{2}} \left( Q^- - \bar{Q}^- \right).
\]

(2.18)

The reason is that in the \( \mu \to 0 \) limit \( Q \) contains just left moving oscillators, while \( \bar{Q} \) depends only on the right moving ones so that \( Q \) and \( \bar{Q} \) are the direct generalization of the supercharges usually considered in flat–space computations. These charges satisfy the algebra\(^8\)

\[
\{Q_\bar{a}, Q_\bar{b}\} = 2\delta_{\bar{a}\bar{b}}(H + T), \quad \{\bar{Q}_a, \bar{Q}_b\} = 2\delta_{ab}(H - T)
\]

(2.19)

\[
\{Q_\bar{a}, \bar{Q}_b\} = \mu \left[ -(\gamma_{ij}\Pi)_{\bar{a}\bar{b}} J^{ij} + (\gamma_{ij}'\Pi)_{\bar{a}\bar{b}'} J^{ij}' \right].
\]

(2.20)

Notice the appearance of \( T \) in (2.19):

\[
T = i e(\alpha) \int_0^{2\pi/|\alpha|} \left[ \partial_\sigma x(\sigma) p(\sigma) + \partial_\sigma \theta(\sigma) \lambda(\sigma) \right] d\sigma = \sum_{n=1}^{\infty} \frac{n}{\alpha} \left[ (b_n^\dagger b_{-n} + b_{-n}^\dagger b_n) - i (a_n^\dagger a_{-n} - a_{-n}^\dagger a_n) \right].
\]

(2.21)

At first sight the presence of \( T \) looks strange since it seems that the operator expressions of the supercharges (2.17) do not realize the PP–wave superalgebra [2, 3]. However, if one rewrites \( T \) in terms of the BMN–oscillators \( \hat{a}^i \) and \( \hat{b}^a \) (see (3.19) and (3.27) respectively), it is easy to see that \( T = 0 \) when the level matching condition is imposed. Thus the supercharges in (2.17) provide a representation of the superalgebra in the physical part of the whole Hilbert space; however for the subsequent manipulation it is important to remember that the anticommutation rules among the supercharges contain the operator \( T \).

\(^8\)The relative sign between \( J^{ij} \) and \( J^{ij}' \) has been noticed in [34]
2.2 The kinematical symmetries

As already mentioned in the Introduction, the standard procedure to construct the three-string interaction vertex consists of two steps. First, one looks for a state $|V\rangle$ in the three-string Hilbert space realising the kinematical symmetries of the PP–wave background. The invariance under all the kinematical symmetries translates into requiring the continuity of the bosonic and fermionic coordinates, and the conservation of the bosonic and fermionic momenta. In [20], it was shown that a solution to the above constraints (see (A.1) and (A.2) in Appendix A) is given by

$$|V\rangle = \delta \left( \sum_{r=1}^{3} \alpha_r \right) |E_a\rangle |E_b\rangle ,$$

(2.22)

where $\alpha_r$ is the light-cone momentum of the $r$-th string\(^9\) and $|E_a\rangle$ (resp. $|E_b\rangle$) is the contribution from the bosonic (resp. fermionic) oscillators. Both contributions can be represented as an exponential acting on the three-string vacuum.

The bosonic part is the same as for the vertex in [11, 12, 13, 14]

$$|E_a\rangle = \exp \left\{ \frac{3}{2} \sum_{r,s=1}^{3} \sum_{m,n} a_{m(r)}^\dagger N_{mn}^{rs} a_{n(s)}^\dagger \right\} |v\rangle_{123} ,$$

(2.23)

where $|v\rangle_{123} = |v\rangle_1 \otimes |v\rangle_2 \otimes |v\rangle_3$ is the tensor product of the three vacuum states (see Eq. (2.28) for the definition) and the matrix $N_{mn}^{rs}$ [11] is defined in the Appendix A.

The fermionic contribution reads

$$|E_b\rangle = \exp \left\{ 1 + \Pi \left[ \frac{3}{2} \sum_{r,s=1}^{3} \sum_{m,n=1}^{\infty} b_{m(r)}^\dagger Q_{mn}^{rs} b_{n(s)}^\dagger - \sqrt{\alpha'} \Lambda \sum_{r=1}^{3} \sum_{m=1}^{\infty} Q_m^{r} b_{-m(r)}^\dagger \right] \right\}$$

$$+ \frac{1}{2} \exp \left\{ -\sum_{i=1}^{2} \sqrt{\alpha_i} \frac{1}{\alpha_3} \left[ b_{0(i)}^\dagger b_{0(3)}^\dagger \right] \right\} |v\rangle_{123} ,$$

(2.24)

where

$$\Lambda \equiv \alpha_1 \lambda_{0(2)} - \alpha_2 \lambda_{0(1)} , \quad \Theta \equiv \frac{1}{\alpha_3} \left( \theta_{0(1)} - \theta_{0(2)} \right) , \quad \alpha \equiv \alpha_1 \alpha_2 \alpha_3 \, .$$

(2.25)

The matrices $Q$ are diagonal in the spinor space and are defined as (we use the notations of [13])

$$Q_{mn}^{rs} \equiv e(\alpha_r) \sqrt{\frac{|\alpha_s|}{|\alpha_r|}} \left[ U_{(r)}^{1/2} C_{(r)}^{1/2} N_{mn}^{rs} C_{(s)}^{-1/2} U_{(s)}^{1/2} \right]_{mn} ,$$

(2.26)

$$Q_m^{rs} \equiv \frac{e(\alpha_r)}{\sqrt{|\alpha_r|}} \left[ U_{(r)}^{1/2} C_{(r)}^{1/2} N^{rs} C_{(s)}^{-1/2} U_{(s)}^{1/2} \right]_m \, .$$

(2.27)

\(^9\)We choose $\alpha_i > 0$, $i = 1, 2$ and $\alpha_3 < 0$. 

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As vacuum state $|v\rangle$, we choose the state of minimal (zero) energy, which is annihilated by all annihilation modes

$$a_n(r)|v\rangle_r = 0, \quad b_n(r)|v\rangle_r = 0 \quad \forall n. \quad (2.28)$$

This choice of vacuum represents the main difference between the proposals in [19, 20] and [11, 12, 13, 14]\textsuperscript{10}. Notice that the last line of (2.24) is just a rewriting of the zero–mode structure proposed in [19]. In [19] the idea to construct the vertex on the vacuum (2.28) was suggested by the study of a $Z_2$ symmetry present in the supergravity background. The structure of (2.24) in the $b_0$ Hilbert space is the origin of the differences between the two vertices. This structure can also be understood in a different way by considering the path integral treatment of the fermionic “zero–modes” $b_0$. In Eq. (2.24) these are treated on the same footing as all the other modes. This is a very natural approach in the PP–wave background because $b_0$ is not really a zero–mode, but it is an harmonic oscillator exactly as the stringy modes (the only difference is that its energy is just $\mu$ and is independent of $\alpha'$). On the contrary in [11, 12, 13, 14] the $(\theta_0, \lambda_0)$ sector has been treated in the same way as in flat space. Thus in their case $|E_0\rangle$ has the same structure found in the flat space vertex [18]. In particular it contains an explicit $\delta$–function $(\sum_r \lambda_0(r))$ imposing a selection rule: string amplitudes are zero, unless the external states soak up the fermions present in the $\delta$–function. From the path integral point of view this result is natural only in flat space. In this case, in fact, the $\lambda_0$’s are genuine zero–modes appearing only in the measure and can not be saturated by the weight factor $\exp(iS)$. This yields a $\delta$–function enforcing a selection rule in the amplitudes. In the PP–wave case, on the contrary, all modes have a non–vanishing energy, thus one does not expect any selection rule. The vertex (2.24) exactly realizes this expectation, since it enforces the fermionic constraints with an exponential structure also in the $b_0$ sector, instead of imposing the constraint as in flat space: $\delta(\sum_r \lambda_0(r)) \to \sum_r \lambda_0(r)|0\rangle_{123}$.

As a final remark, we would like to mention that an alternative approach can be used to derive (2.23) and (2.24). We have verified that this form of the vertex follows uniquely from a path integral treatment [35]. We made use of the fact that the system may be considered as a sum of harmonic oscillators, in particular we treated the fermionic zero mode oscillators on the same footing as all others. Boundary conditions are conveniently specified in a coherent state basis, equivalent to the Bargmann-Fock formalism [36, 37]. We have also explicitly verified that this kinematical vertex in fact satisfies the complete set of kinematical constraints. In the path integral formalism the prefactor originates from the fact that the light cone gauge and $\kappa$ gauge conditions cannot be imposed at the point on the world sheet.

\textsuperscript{10}In [11, 12, 13, 14] the fermionic part of the vertex is built by acting on the ground state $|0\rangle$

$$a_n(\tau)|0\rangle = 0 \quad \forall n, \quad b_n|0\rangle = 0 \quad \forall n \neq 0, \quad \theta_0|0\rangle = 0. \quad (2.29)$$
where the 3 strings join. Ignoring this complication yields exactly the kinematical part of the vertex. Rather than attempting to derive the prefactor from first principles we follow the standard approach of seeking a supersymmetric completion as described below.

### 2.3 Supersymmetric completion

In order for the full supersymmetry algebra to be satisfied at the interacting level, the kinematical vertex has to be completed with a polynomial prefactor. An analogous construction also applies for the dynamical supersymmetry generators $Q$ and $\tilde{Q}$. Of course, the addition of the prefactor should not change the properties of the vertex under the kinematical symmetries. As for the dynamical symmetries, if we define the full hamiltonian and dynamical charges as $|H_3\rangle$, $|Q_{3a}\rangle$ and $|\tilde{Q}_{3\dot{a}}\rangle$,

$$
3 \sum_{r=1}^{3} Q_{(r)a}|Q_{3\dot{b}}\rangle + 3 \sum_{r=1}^{3} Q_{(r)b}|Q_{3a}\rangle = 2\delta_{\dot{a}b}|H_3\rangle ,
$$

$$
3 \sum_{r=1}^{3} \tilde{Q}_{(r)a}|\tilde{Q}_{3\dot{b}}\rangle + 3 \sum_{r=1}^{3} \tilde{Q}_{(r)b}|\tilde{Q}_{3\dot{a}}\rangle = 2\delta_{\dot{a}b}|H_3\rangle ,
$$

$$
3 \sum_{r=1}^{3} Q_{(r)a}|\tilde{Q}_{3\dot{b}}\rangle + 3 \sum_{r=1}^{3} \tilde{Q}_{(r)b}|Q_{3a}\rangle = 0 .
$$

To our knowledge there is no way to derive the prefactor from first principles, the standard approach being to write a suitable ansatz and then check that it is invariant under all symmetries [17, 18]. To proceed further, some physical inputs are then required. In [38] a supersymmetric completion of the kinematical vertex (2.22) has been obtained by requiring the continuity in the flat space $\mu \to 0$ limit. This forces to assign an even $Z_2$ parity to the state $|0\rangle$ (2.29). In this case the string vacuum has to be $Z_2$-odd because $|v\rangle$ and $|0\rangle$ have opposite parity [19]. Correspondingly, the prefactor proposed in [38] is also $Z_2$-odd, ensuring the $Z_2$-invariance of the interaction vertex. This vertex has been shown in [39] to be equivalent to that of [11, 12, 14].

Here we present a different approach: following the gauge theory intuition, the vacuum state of the string Fock space is defined to be even under the discrete $Z_2$ symmetry. Thus we are led to give up the continuity of the flat space limit $\mu \to 0$ for the string interaction. Also in this case it is possible to build a string vertex that is invariant under the $Z_2$ transformation. In this case, this symmetry is realized explicitly, i.e. both the interaction and the vacuum state are $Z_2$ invariant at the same time. A very simple way to realize the supersymmetry algebra is to act on the

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11For the relation between these states and the corresponding interaction operators, see for example Eq.(3.12) of [11]
The kinematical vertex with the free part of the Hamiltonian and the dynamical charges

\[ |H_3\rangle = \sum_{r=1}^{3} H_r |V\rangle, \quad |Q_{3a}\rangle = \sum_{r=1}^{3} Q_{r\bar{a}} |V\rangle, \quad |\bar{Q}_{3\bar{a}}\rangle = \sum_{r=1}^{3} \bar{Q}_{r\bar{a}} |V\rangle. \tag{2.33} \]

With this ansatz, the relations (2.30) and (2.32) are a direct consequence of the free-theory algebra (2.19)-(2.20)\textsuperscript{12}. It only remains to check that the full ansatz still satisfies the kinematical constraints. To this purpose, we need the explicit expressions for the prefactors in (2.33) and to prove that the various constituents commute with the constraints in (A.1), (A.2). Commuting the annihilation modes in \( H \) and \( Q \) through the kinematical vertex, we obtain

\[ |H_3\rangle = -\frac{\alpha'}{\alpha}(1-4\mu\alpha K) \left[ \frac{1}{4} (K^2 + \bar{K}^2) + \frac{1+\Pi}{2} \mathcal{W}_\Lambda \mathcal{Y}_\Lambda + \frac{1-\Pi}{2} \mathcal{W}_\Theta \mathcal{Y}_\Theta \right] |V\rangle, \]

\[ |Q_{3a}\rangle = -\frac{\alpha'}{\sqrt{2\alpha}}(1-4\mu\alpha K) \mathcal{K}^{ij} \gamma_{ab}^I (\mathcal{Y}_\Lambda - \mathcal{Y}_\Theta)_b |V\rangle, \tag{2.34} \]

\[ |\bar{Q}_{3\bar{a}}\rangle = -i\frac{\alpha'}{\sqrt{2\alpha}}(1-4\mu\alpha K) \bar{\mathcal{K}}^{ij} \gamma_{ab}^I (\mathcal{Y}_\Lambda + \mathcal{Y}_\Theta)_b |V\rangle, \]

where \( K = -(1/4) B \Gamma^{-1} B \) (see Appendix A for the definition of the matrix \( \Gamma \)).

This remarkably simple form has a very similar structure to what was proposed in [11]. As in [11], the contribution of the bosonic oscillators is contained in

\[ \mathcal{K} = \mathcal{K}_0 + \mathcal{K}_+ + \mathcal{K}_-, \quad \bar{\mathcal{K}} = \mathcal{K}_0 + \mathcal{K}_+ - \mathcal{K}_-. \tag{2.35} \]

where

\[ \mathcal{K}_0 = \mathbb{P} - i \mu \frac{\alpha}{\alpha'} \mathbb{R} = \sqrt{\frac{2}{\alpha'}} \sqrt{\mu a_1 a_2} (\sqrt{a_{1n(2)}} - \sqrt{a_{2n(1)}}), \tag{2.36} \]

\[ \mathcal{K}_+ = -\frac{1}{\sqrt{\alpha'} 1 - 4\mu\alpha K} \sum_{r=1}^{3} \sum_{n=1}^{\infty} \left[ \frac{1}{\alpha_t} \left( C^{1/2}(r) U^{-1/2}(r) \right)_n \right] a_{n(r)}^+, \tag{2.37} \]

\[ \mathcal{K}_- = -i \frac{1}{\sqrt{\alpha'} 1 - 4\mu\alpha K} \sum_{r=1}^{3} \sum_{n=1}^{\infty} \left[ \frac{1}{\alpha_t} \left( C^{1/2}(r) N^{1/2} \right)_n \right] a_{-n(r)}^+. \tag{2.38} \]

Similarly, for the fermionic oscillators we introduced the operators \( \mathcal{Y}_\Lambda \) and \( \mathcal{Y}_\Theta \)

\[ \mathcal{Y}_\Lambda = \frac{1+\Pi}{2} \left[ \Lambda - \frac{\alpha}{\sqrt{\alpha'}} \sum_{r=1}^{3} \frac{e(\alpha_t)}{\sqrt{|\alpha_t|}} \left( C^{1/2}(r) C^{1/2} U^{-1/2}(r) \right)_n b_{n(r)}^+ \right], \tag{2.39} \]

\[ \mathcal{Y}_\Theta = \frac{1-\Pi}{2} \left[ \frac{\alpha'}{\alpha} \Theta + \frac{\alpha}{\sqrt{\alpha'}} \sum_{r=1}^{3} \frac{e(\alpha_t)}{\sqrt{|\alpha_t|}} \left( C^{1/2}(r) C^{1/2} U^{-1/2}(r) \right)_n b_{-n(r)}^+ \right]. \]

\textsuperscript{12}One has to set \( T = 0 \) to get the usual commutation relations and use the fact that the kinematical vertex is annihilated by generators \( J^{ij} \) and \( J^{ij'} \), since it is \( SO(4) \times SO(4) \) invariant.
and

\[
W_\Lambda = \frac{1 + \Pi}{2} \left[ \frac{\alpha}{\sqrt{\alpha'}} \sum_{r=1}^{3} \frac{1}{\sqrt{|\alpha_r|^3}} \frac{(C^{1/2}_U^{1/2}U_{-1/2}^r N^r)_n b^\dagger_{-n(r)}}{1 - 4\mu K} \right]
\]

\[
W_\Theta = \frac{1 - \Pi}{2} \left[ \frac{\alpha}{\sqrt{\alpha'}} \sum_{r=1}^{3} \frac{1}{\sqrt{|\alpha_r|^3}} \frac{(C^{1/2}_U^{1/2}U_{-1/2}^r N^r)_n b^\dagger_{n(r)}}{1 - 4\mu K} \right].
\]

The bosonic constituents of the prefactor, \( K \) and \( \tilde{K} \), are the same as in [12, 13, 14]. So we refer to those papers for their commutation relations with the kinematical constraints. The fermionic constituents, (2.39) and (2.40), on the contrary, are new. Actually, \( Y_\Lambda \) coincides with the \( Y \) of [13, 14] in the \( \Pi = 1 \) sector. Since the kinematical constraints are diagonal in the spinor indices the results in [13, 14] still hold in our case. In Appendix B, we show that also \( Y_\Theta, W_\Lambda \) and \( W_\Theta \) commute with the kinematical constraints.

At first sight, it may appear surprising that the simple proposal (2.34) is fully consistent, because it is at most quadratic in the fermionic oscillators. In fact, in flat space it is crucial to have up to eight fermionic insertions in the prefactor in order to ensure the reality of the interacting Hamiltonian [43]. The point is that, when \( \mu = 0 \) the real and the imaginary part of \( \theta \) always appear quadratically and never mix. Thus the map \( \theta^2 \to -\theta^2 \) leaves the action invariant. The invariance of the flat space vertex under this change requires that the coefficient of \( \Lambda^k \) in the prefactor is related to the complex conjugate of the coefficient of \( \Lambda^{8-k} \). In the PP–wave case the situation is very different and already the free action (2.2) is not invariant under \( \theta^2 \to -\theta^2 \), as it is clear from the formulation given in Eq. (2.3). In order to restore this symmetry one has to send at the same time \( \Pi \) into \( -\Pi \). The functional form of the vertex presented here is unchanged under the transformation \( (\theta^2, \Pi) \to (-\theta^2, -\Pi) \). Notice that this difference has an important physical meaning. In flat space one knows that closed string amplitudes are factorized in two independent pieces involving left and right moving respectively. The requirement of having a real vertex according to the definition [43] is equivalent to the holomorphic factorization of physical amplitudes. When \( \mu \neq 0 \) this factorization breaks down and also the exchange \( \theta^2 \to -\theta^2 \) stops being a symmetry.

Let us now add some comments on the physical meaning of our proposal (2.34). Actually the simple possibility of supersymmetrizing the kinematical vertex just by dressing it with the free form of the dynamical supercharges also exists both in the flat space case and in the approach of [11, 12, 13, 14]. However in the flat space case the vertex ansatz (2.33) yields trivial on–shell amplitudes. In fact, in order to derive the \( S \)–matrix elements from \( |H_3\rangle \) one has first to take care of the \( x^+ \) dependence of the vertex. This dependence is not manifest in the formulae usually written, because it is standard to derive the string vertex at fixed interaction time \( \tau = 0 \). The general form for \( |H_3\rangle_{\tau} \) can be easily derived from \( |H_3\rangle_{\tau=0} \) by evolving the oscillators in the vertex with the free Hamiltonian \( (H^{(2)}) \). For on–shell external
states the usual recipe [15, 16] is that the \( S \)-matrix is derived by integrating over the interaction time
\[
A_{\text{phys.}} = \int_{-\infty}^{\infty} 123 \langle \text{state} | H_3 | \tau \rangle \, d\tau = \delta \left( \sum_{r=1}^{3} H_r^2 \right) 123 \langle \text{state} | H_3 | \tau = 0 \rangle. \tag{2.41}
\]
This shows that one has the possibility to redefine the off–shell vertices by adding terms with the structure of (2.33), but these terms will not contribute to the \( S \)-matrix. In the present case one is not interested in \( S \)-matrix elements but in matrix elements of the interaction hamiltonian, and these are picked out by the short time treatment. In any case the \( S \)-matrix energy conservation would correspond to a similar delta function on conformal dimensions on the field theory side, which is clearly unreasonable.

Further evidence supporting (2.33) and the interpretation just proposed comes from what is known of supergravity on \( \text{AdS}_5 \times S^5 \). The vertex derived here should play the same role as the cubic bulk couplings derived from the compactification of IIB theory on \( \text{AdS}_5 \times S^5 \) [44]. We should then be able to compare the results of our \( |H_3\rangle \) for supergravity states with the (leading order in \( J \)) results obtained in \( \text{AdS}_5 \times S^5 \). It is interesting to notice that the bulk vertices obtained in [44] for 3 scalars are indeed proportional to \( \Delta_1 + \Delta_2 - \Delta_3 \), exactly as the string 3–point functions derived from \( |H_3\rangle \) (2.33). It is also important to notice that in \( \text{AdS}/\text{CFT} \) this factor is necessary to have meaningful 3–point functions dual to extremal correlators \( (\Delta_1 + \Delta_2 = \Delta_3) \) [45]. Indeed, in the extremal case the integration over the \( \text{AdS}_5 \) position of the bulk vertex is divergent [46] and cancels exactly the factor \( \Delta_1 + \Delta_2 - \Delta_3 \) in the bulk vertex, leaving the expected field theory result. This pattern seems to be a general feature of extremal correlators and not just a peculiarity of correlators among chiral primaries\(^{13}\). Thus it is natural to expect that the 3–point vertices among supergravity BMN states both in AdS and in the PP–wave background are proportional to the \( \sum_r H_r^2 \). Notice that the vanishing of the energy conserving amplitudes is also one of the requirements of the holographic string field theory proposed in [48, 49]\(^{14}\).

As a final comment let us come back to the prescription for deriving the physically meaningful data from \( |H_3\rangle \). In the next section, we will check that the idea [10] of comparing field theory results with the interacting hamiltonian matrix element divided by \( \sum_r H_r^2 \) (see (3.1)) works very well at leading order in \( \lambda \). This proposal is very natural since it is strongly reminiscent of the formula obtained in quantum mechanics describing the first order transition probability induced by a perturbation acting for a very short time. It is then likely that this prescription has to be corrected in order to check the duality at higher orders in \( \lambda \). It would be interesting to see whether the analogy with quantum mechanics can be helpful in this generalization, following for instance the ideas in [50]. One can say that in the

\(^{13}\)See for instance [47], where the correlator between one scalar and two spinors was considered.

\(^{14}\)In [48, 49] it was also noticed that the fermionic action can be made \( SO(8) \) symmetric by the redefinition \((\theta_1, \theta_2) \rightarrow (\theta_1, \Pi \theta_2)\). This symmetry is satisfied by our vertex (2.34).
PP–wave/CFT duality we are in the opposite situation with respect to the usual AdS/CFT computations: we have an explicit and complete expressions for the 3–states coupling (while in AdS it is a challenging computation even to derive some of these couplings), but we do not have a completely clear prescription to relate them to the field theory side. One needs the analogue of the prescription in [32, 33]. Usually in AdS computations one uses the bulk–to–boundary propagator, but no simple analogue of this ingredient has been found in the PP–wave case so far.

3 Comparison between string and field theory results

The PP-wave vertex discussed in the previous section is in agreement with the proposal of [10] for comparing the string theory interaction with the three–point correlators of \( \mathcal{N} = 4 \) SYM theory. This proposal is motivated by the standard AdS/CFT dictionary between bulk and boundary correlation functions [32]: since the light-cone interaction vertex on the PP-wave in (2.33) can be understood as the generating functional of the correlation functions among string states, it is natural to put it in correspondence with the correlation functions of the dual field theory operators. A more general motivation for this proposal was indeed provided in [48, 49] by considering the Penrose limit of the AdS/CFT bulk–to–boundary formula of [32]. A specific prescription was proposed in [10] for the leading order in \( 1/\mu \)

\[
\frac{(\langle 1 \rangle \otimes \langle 2 \rangle \otimes \langle 3 \rangle | H_3)}{\sum_r (H_r^2)} = C_{ijk},
\]

where \( C_{ijk} \) is the coefficient appearing in the correlator among three BMN operators of R-charge \( J_i \)

\[
\langle \hat{O}_i(x_i)O_j(x_j)O_k(x_k) \rangle = \frac{C_{ijk}}{(x_{ij})^{\Delta_i^{(0)} + \Delta_j^{(0)} - \Delta_k^{(0)}}(x_{ik})^{\Delta_i^{(0)} + \Delta_k^{(0)} - \Delta_j^{(0)}}(x_{jk})^{\Delta_j^{(0)} + \Delta_k^{(0)} - \Delta_i^{(0)}}},
\]

and depends on the quantum numbers \( J_i \) and on the value of the BMN phase factors. Notice that in (3.2) we write the canonical dimensions \( \Delta^{(0)} \) of the BMN operators. In fact we will be concerned only with the comparison at the leading order in \( 1/\mu \), i.e. at the tree level in the field theory.

The proposal (3.1) has been questioned in the subsequent literature [21, 22], and other conjectures were put forward [23, 24, 25, 28]. The main argument which should invalidate (3.1) is the appearance of a mixing between single and multi-trace BMN operators at the genus-one order \( g_2 = J^2/N_c \) of the graph expansion [51, 52]. This mixing affects the tree–level evaluation of the field theory correlators (3.2). Thus, if taken into account in the dictionary (3.1), it would spoil the agreement with the string theory results. We notice however that similar issues appear already in the usual AdS/CFT correspondence for extremal correlators [45] (see also [53]).
For these correlators the mixing is enhanced and should in principle be taken into account in the comparison with the supergravity calculations. Actually, as was shown in [45], this is not the case, and one gets agreement between single–trace extremal correlators and three-point functions of supergravity on $AdS_5 \times S^5$, without invoking any mixing. Moreover, following the arguments of [54], the identification between the number of traces and the number of string states should be valid as long as the SYM operators are not too “big”. A simple quantitative characterization of big operators can be derived by realizing that overlap between single and double traces is of order $\sqrt{JJ'(J - J')}/N$. If this is not negligible in the planar limit, then we are dealing with big operators. This shows that, even if the BMN operators are made of an infinite number of fields, they are never big since $\sqrt{JJ'(J - J')}/N \sim g_2/\sqrt{J} \rightarrow 0$. Thus the usual rules of AdS/CFT should apply: the single trace operators should correspond to the elementary string states while the multi-trace operators should be bound states and so they are not present in the spectrum of the free string.

Further support to this picture comes from the fact that computations with multi–trace operators in the AdS/CFT correspondence seem to be related to string interactions which are non–local on the world sheet [55, 56]. Since the leading correlators in the BMN limit (1.1) are extremal correlators (for the supergravity modes) or non-BPS deformations of these (for the string modes), it is natural to expect that the same features are shared by the limiting PP-wave/CFT correspondence and that the local vertex (2.34) has to be compared with single–trace correlators.

In this section we will show that this is indeed the case for the leading order. We underline that we find agreement between string theory and field theory results for all kinds of BMN operator, including those containing vector and fermion impurities.

For the field theory computations, we adopt the formulation of $\mathcal{N} = 4$ SYM theory in terms of $\mathcal{N} = 2$ multiplets. This formulation has the advantage to realize explicitly the $R$-symmetry subgroup $SU(2)_V \times SU(2)_H \times U(1)_J \subset SU(4)$, where $SU(2)_V, SU(2)_H$ are respectively the internal symmetry groups of the $\mathcal{N} = 2$ vector multiplet and hypermultiplet. In this way we can naturally identify the charge under $U(1)_J$ with the $J$-charge of the BMN operators, and the $SU(2)_V \times SU(2)_H$ group with the $SO(4)$ rotations acting on the scalar impurities. The $Z$ field of the BMN operators is the complex scalar of the vector multiplet, while the scalar impurities are given by the four real scalars $\phi^i, i' = 1, \ldots, 4$ of the hypermultiplet. The fermionic excitations are associated with the Weyl fermions $\lambda_\alpha^u$ of the vector multiplet and $\bar{\psi}_\dot{\alpha}^{u'}$ of the hypermultiplet. These fields have in fact both charge $J = 1/2$. They transform in the fundamental representation $u = 1, 2$ of $SU(2)_V$ and $\dot{u} = 1, 2$ of $SU(2)_H$ respectively. The quadratic part of the $\mathcal{N} = 4$ SYM

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$^{15}$On the field theory side, BMN operators with scalar impurities were first analysed in [9, 10], while operators with vector impurities were first considered in [57, 58].
lagrangian is\(^{16}\)

\[
\mathcal{L} = \frac{1}{g_{YM}^2} \text{Tr} \left[ \frac{1}{4} F^{mn} F_{mn} + D_m \bar{Z} D^m Z + \frac{1}{2} D_m \bar{\phi}^I D^m \phi^I \right]
\]

(3.3)

where the covariant derivative is \(D_m \equiv \partial_m + [A_m, \cdot]\). The Green function for the complex field \(Z\) is then \(G(x - x') = g_{YM}^2 / 4\pi^2 (x - x')^2\). The BMN operator associated to the string theory vacuum is

\[
O_{\text{vac}}^J(x) = \frac{1}{\sqrt{JN^J}} \text{Tr} \left[ Z^J \right](x)
\]

(3.4)

with \(N = g_{YM}^2 N_c / 4\pi^2\). For single impurities we define

\[
O_i^J(x) = N_1 \text{Tr} \left[ D_i Z Z^J \right](x), \quad O_i^J(x) = N_1 \text{Tr} \left[ \bar{\phi}^I Z^J \right](x)
\]

(3.5)

\[
O_{\alpha}^J = N_1 \text{Tr} \left[ \lambda_\alpha Z^J \right](x), \quad O_{\alpha}^J = N_1 \text{Tr} \left[ \bar{\psi}^\alpha Z^J \right](x)
\]

(3.6)

with \(N_1 = 1 / \sqrt{2N^{J+1}}\). The first operator in (3.6) corresponds to the insertion of a string fermionic oscillator of chirality \(\Pi = 1\), while the second to the chirality \(\Pi = -1\). For double impurities we define (in the dilute gas approximation)

\[
O_{ij,m}^J(x) = N_2 \sum_{l=0}^{J} q^l \text{Tr} \left[ D_j Z Z^J D_i Z Z^{J-l} \right](x),
\]

(3.7)

\[
O_{\alpha\beta,m}^J(x) = N_2 \sum_{l=0}^{J} q^l \text{Tr} \left[ \lambda_\alpha Z^J \lambda_\beta Z^{J-l} \right](x),
\]

(3.8)

with \(N_2 = (1/2) \sqrt{(J + 1)N^{J+2}}\). To simplify notations, in the above formulæ the BMN phase factor is \(q^l = \exp \left( 2\pi i m \frac{l+1}{J+2} \right)\), and the \(R\)-symmetry indices of the fermions are suppressed. The barred operators are defined according to the rules suggested by the radial quantization \([59]\). For example, for double–vector impurities we define\(^{17}\)

\[
\bar{O}_{ij,m}^J(x) \equiv N_2 \left( r^2 \right)^{J+2} \sum_{l=0}^{J} q^l \text{Tr} \left[ (C_{ik} \bar{D}_k r^2 \bar{Z}) \bar{Z}^l (C_{jh} \bar{D}_h r^2 \bar{Z}) \bar{Z}^{J-l} \right](x),
\]

(3.9)

where \(C_{ik}(x) = \delta_{ik} - 2x_i x_k / r^2\) is the tensor associated to the conformal inversion transformation \(x'_i = x_i / r^2\), with \(\partial x'_i / \partial x_k = C_{ik}(x) / r^2\). For fermionic impurities we define

\[
\bar{O}_i^J(x) \equiv N_1 \left( r^2 \right)^{J+3/2} \text{Tr} \left[ \bar{\lambda}^\alpha \bar{Z}^J \right](x),
\]

(3.10)

\[
\bar{O}_\alpha^J(x) \equiv N_1 \left( r^2 \right)^{J+3/2} \text{Tr} \left[ \bar{\psi}^\alpha \bar{Z}^J \right](x),
\]

(3.11)

\(^{16}\)We adopt the following convention for the sigma matrices: \(\sigma^m = (i\sigma^c, 1)\) and \(\bar{\sigma}^m = (-i\sigma^c, 1)\), where \(\sigma^c\) are the Pauli matrices.

\(^{17}\)M. Petrini, R. Russo and A. Tanzini are happy to acknowledge collaboration with C.S. Chu and V.V. Khoze about the definition of the barred operators in field theory. See also [30].
with \( \vec{f} = \vec{\sigma}^a x^a / r \), \( \vec{f} = \vec{\sigma}^a x^a / r \) and \( r \equiv |x| \).

The check of (3.1) for BMN operators containing only scalar impurities has been discussed in detail in [60, 61, 62]. We will focus on vector and spinor impurities. Inspired by the radial quantization analysis, we will evaluate the three-point correlator (3.2) in the \( x_i \rightarrow \infty \) limit. In this limit, the following identities hold for vector insertions

\[
\mathcal{N}^2 \lim_{r \rightarrow \infty} \langle 0 | (r^2)^{J+1} \text{Tr} \left[ C_{ik} \partial^k (r^2 Z) \bar{Z}^J \right] (x) \text{Tr} \left[ \partial^J Z \bar{Z}^J \right] (x') | 0 \rangle = \\
= \lim_{r \rightarrow \infty} C_{ik} \left( \delta_{kj} - \frac{2x_k x_j}{r^2} \right) = \delta_{ij} \tag{3.12}
\]

and

\[
\lim_{r \rightarrow \infty} \langle 0 | (r^2)^{J+1} \text{Tr} \left[ C_{ik} \partial^k (r^2 Z) \bar{Z}^J \right] (x) \text{Tr} \left[ Z^J \right] (x') | 0 \rangle = \\
= \lim_{r \rightarrow \infty} \partial_k \left( \frac{r^2}{(x-x')^2} \right) = 0 \tag{3.13}
\]

From (3.12) and (3.13) it immediately follows that the vector impurities in the BMN operators behave exactly as the scalar impurities. We are then left with the computation of the same combinatorial factors studied in [10]. Notice that (3.12) and (3.13) are consistent with the string state/operator correspondence.

The fermionic insertions display an analogous behaviour in the \( r \rightarrow \infty \) limit

\[
\mathcal{N}^2 \lim_{r \rightarrow \infty} \langle 0 | (r^2)^{J+3/2} \text{Tr} \left[ (\lambda \vec{f})^\alpha \bar{Z}^J \right] (x) \text{Tr} \left[ \lambda^J \bar{Z}^J \right] | 0 \rangle = \delta^\alpha_\beta , \tag{3.14}
\]

\[
\mathcal{N}^2 \lim_{r \rightarrow \infty} \langle 0 | (r^2)^{J+3/2} \text{Tr} \left[ (\bar{\psi} \vec{f})^\alpha \bar{Z}^J \right] (x) \text{Tr} \left[ \bar{\psi}^\beta \bar{Z}^J \right] | 0 \rangle = \delta_\alpha^\beta . \tag{3.15}
\]

This implies that also the tree-level evaluation of BMN correlators with fermion impurities can be reduced to the scalar impurity case, apart from some (important) signs due to the anticommuting nature of \( \lambda \) and \( \bar{\psi} \).

We now proceed to the comparison of some specific field theory correlators with the results derived from the three-string vertex (2.33). Let us start by considering BMN operators with one vector and one scalar impurity. From the results in the literature\(^\text{18}\) and using (3.12) and (3.13) one can find

\[
\langle \hat{O}_{i_3}^{J_3} O_{i_2}^{J_2} O_{i_1}^{J_1} \rangle \equiv \lim_{x_3 \rightarrow \infty} \langle \hat{O}_{i_3}^{J_3} (x_3) O_{i_2}^{J_2} (x_2) O_{i_1}^{J_1} (x_1) \rangle = \frac{J_2}{N} \sqrt{\frac{J_1 J_2}{J_3}} \frac{\sin^2(\pi ny)}{\pi^2 (m - ny)^2} , \tag{3.16}
\]

where \( y \equiv J_2 / J_3 \). For operators containing two vector impurities we can also consider the correlator

\[
\langle \hat{O}_{i_3}^{J_3} O_{i_2}^{J_2} O_{i_1}^{J_1} \rangle = \langle \hat{O}_{i_3}^{J_3} O_{i_2}^{J_2} O_{i_1}^{J_1} \rangle + \langle \hat{O}_{i_3}^{J_3} O_{i_2}^{J_2} O_{i_1}^{J_1} \rangle \tag{3.17}
\]

\(^\text{18}\)See for instance Eqs.(3.15) and (3.25) in [21]. At the level of planar free field theory there is no difference between the single/double–trace correlators in [21] and the single–trace 3-point functions considered here.
The last term in (3.17) comes from the fact that impurities with the same vector index can be contracted in two different ways, and the exchange of two impurities induces a change of sign in the BMN phase according to the definition (3.7).

Let us now see how these results are reproduced from the $\mu \to \infty$ limit of string theory. Due to the form (2.33) of the prefactor, the non-trivial part of the proposal (3.1) is simply given by the coefficient $C_{ijk}$. This justifies the approach of [60, 61, 62], and we can focus only on the kinematical part (2.22) of the 3-string vertex and rewrite (3.1) as

$$\langle i|\langle j|\langle k|V \rangle = \frac{C_{ijk}}{C_{ij(k)}} ,$$

(3.18)

where $C_{ij(k)}^{(0)} = \sqrt{J_1 J_2 J_3}/N$ is the combinatorial factor of the Green function among three vacuum operators (3.4). This factor ensures the same normalization for the two sides of (3.18), since the string overlap $\langle \bar{v}|V \rangle$ is set equal to one. For the comparison with field theory it is useful to recall the dictionary between the BMN oscillators $\hat{a}$ and those used in the previous section

$$\hat{a}_n = \frac{1}{\sqrt{2}}(a_n - ia_{-n}) \quad , \quad \hat{a}_{-n} = \frac{1}{\sqrt{2}}(a_n + ia_{-n}) .$$

(3.19)

According to the string state/operator mapping, the relevant amplitude is

$$A_{ij}^I = \langle \alpha_1, \alpha_2, \alpha_3|\hat{a}_{n(3)}^I \hat{a}_{-n(3)}^I \hat{a}_{m(2)}^I \hat{a}_{-m(2)}^I|V \rangle$$

$$= \frac{1}{4} \langle \alpha_1, \alpha_2, \alpha_3|\left( \hat{a}_{n(3)}^I \hat{a}_{m(2)}^I + \hat{a}_{m(2)}^I \hat{a}_{n(3)}^I + i \hat{a}_{n(3)}^I \hat{a}_{-m(2)}^I - i \hat{a}_{m(2)}^I \hat{a}_{-n(3)}^I \right)$$

$$\times \left( \hat{a}_{m(2)}^I \hat{a}_{m(2)}^I + \hat{a}_{-m(2)}^I \hat{a}_{-m(2)}^I + i \hat{a}_{m(2)}^I \hat{a}_{-m(2)}^I - i \hat{a}_{-m(2)}^I \hat{a}_{m(2)}^I \right)|V \rangle .$$

(3.20)

We will compute the string theory amplitudes in the $\mu \to \infty$ limit by using the relation $N_{nm}^{rs} = -(U(i)^N_{r(s)}U_{nm})$ and the expansions

$$N_{nm}^{32} \sim \frac{2 ny^{3/2} \sin(\pi ny)}{m^2 - n^2 y^2} , \text{ with } n, m > 0$$

(3.21)

$$U_{(i)} \sim \frac{n}{2 \mu \alpha_i} , \quad U_{(3)} \sim - \frac{2 \mu \alpha_3}{n} .$$

(3.22)

Let us first analyze the case $I = i$, $J = i'$. The amplitude (3.20) then becomes

$$A_{ii'}^I = \frac{1}{4} \left[ N_{nm}^{32} - N_{-n-m}^{32} \right]^2 \sim y \frac{\sin^2(\pi ny)}{\pi^2(m - ny)^2} ,$$

(3.23)

where in the last step we kept only the leading term in the $\mu \to \infty$ limit. By using (3.16) this provides a first check of (3.18). If we instead take $I = J = i$, (3.20) becomes

$$A_{ii}^I = \frac{1}{2} \left[ (N_{nm}^{32})^2 + (N_{-n-m}^{32})^2 + \frac{1}{2}(N_{nm}^{33} + N_{-n-m}^{33})(N_{nm}^{22} + N_{-n-m}^{22}) \right]$$

$$\sim \frac{y}{\pi^2} \frac{\sin^2(\pi ny)}{\left( m^2 - ny^2 \right)^2} + \frac{1}{\left( m^2 + ny^2 \right)^2} ,$$

(3.24)
where the two terms in the last parenthesis reproduce the r.h.s. of (3.17). Notice that the terms proportional to $N^{33}$ and $N^{22}$ do not contribute at the leading order.

We now pass to the analogous correlation functions for BMN operators with fermionic impurities. By using (3.14) and (3.15) their evaluation is reduced to the same combinatorics as for the scalar impurities case. Thus we have

$$\langle \bar{O}^{J_3}_{\alpha\beta,m} O^{J_2}_{\alpha\beta,n} O^{J_1}_{\alpha\beta,m} \rangle = - \frac{J_2}{N} \sqrt{\frac{J_1 J_2}{J_3}} \frac{\sin^2(\pi ny)}{\pi^2 (m - ny)^2},$$

(3.25)

which coincides with (3.16). For operators containing spinors of the same flavour we can also consider

$$\langle \bar{O}^{J_3}_{\alpha\alpha,n} O^{J_2}_{\alpha\alpha,m} O^{J_1}_{\alpha\alpha,\text{vac}} \rangle = \langle \bar{O}^{J_3}_{\alpha\beta,n} O^{J_2}_{\alpha\beta,m} O^{J_1}_{\alpha\beta,\text{vac}} \rangle - \langle \bar{O}^{J_3}_{\alpha\beta,-n} O^{J_2}_{\alpha\beta,m} O^{J_1}_{\alpha\beta,\text{vac}} \rangle.$$

(3.26)

Notice that due to the fermionic nature of the impurities one gets a relative minus sign in (3.26) with respect to (3.17).

The string amplitudes related to these correlators can be easily computed from (2.24) by using the dictionary

$$\hat{b}_n = \frac{1}{\sqrt{2}} (b_n + b_{-n}) \quad , \quad \hat{b}_{-n} = -\frac{i}{\sqrt{2}} (b_n - b_{-n}) .$$

(3.27)

In this case the relevant amplitude is

$$A^{ab}_f = \langle \alpha_1, \alpha_2, \alpha_3 | \hat{b}^{a}_{n(3)} \hat{b}^{b}_{-n(3)} \hat{b}^{a}_{m(2)} \hat{b}^{b}_{-m(2)} | V \rangle = -\frac{1}{4} \left[ (Q^{33})^2 + (Q^{32})^2 - 2 Q^{23} Q^{32} \right] .$$

(3.28)

By using now (2.26) we get at the leading order in $1/\mu$

$$Q_{mn}^{23} \sim -N_{mn}^{23} \quad , \quad Q_{mn}^{23} \sim -N_{m-n}^{23} .$$

(3.29)

This shows that in the large $\mu$ limit (3.28) takes the same form as (3.23), in agreement with the field theory result. Also the correlator (3.26) is reproduced by the corresponding string amplitude

$$A^{ab}_f = \langle \alpha_1, \alpha_2, \alpha_3 | \hat{b}^{a}_{n(3)} \hat{b}^{a}_{-n(3)} \hat{b}^{a}_{m(2)} \hat{b}^{a}_{-m(2)} | V \rangle = \left[ -Q_{mn}^{23} Q_{mn}^{22} + Q_{mn}^{23} Q_{mn}^{32} \right] \sim N^{23}_{m-n} N^{32}_{nm} .$$

(3.30)

This is again in agreement with the field theory result since

$$N_{mn}^{23} N_{nm}^{32} \sim -\frac{y}{\pi^2} \sin^2(\pi ny) \left[ \frac{1}{(m-ny)^2} - \frac{1}{(m+ny)^2} \right] .$$

(3.31)

Notice that the agreement in the fermion sector crucially depends on the form of our vertex (2.24). In fact, some fermionic amplitudes were derived also by using the proposal of [14] (see the last section of that paper). These results are quite different from ours because of the different zero mode structure of their vertex and have not been related to any field theory result.
4 Conclusions

In this paper we presented a supersymmetric string vertex that completes the construction of \[19, 20\]. We argued that this vertex has all the necessary properties to describe the local interaction of three strings in the maximally supersymmetric PP–wave background. We also observe that our vertex might be seen as an explicit realization of ”holographic string field theory” of \[48, 49\]. A different supersymmetric completion of the construction of \[19, 20\] was provided in \[38\] by requiring the continuity of the flat space limit $\mu \to 0$. This vertex has been shown in \[39\] to be equivalent to that in \[11, 12, 13, 14\].

Let us summarize here the main features of our proposal, since it is rather different from the one of \[11, 12, 13, 14\]. First, from the very beginning of our construction we gave up the idea of smoothly connecting our vertex to the one of flat space \[18\] when the parameter controlling the curvature of the background is sent to zero ($\mu \to 0$). One can argue that this limit is singular due to the enhancement of the isometry group that takes place exactly at the point $\mu = 0$. One consequence of this fact is that for all values of $\mu$ the modes $a_0$ and $b_0$ of the string expansion are harmonic oscillators and only for $\mu = 0$ one has to deal with genuine zero–modes $p_0$ and $\lambda_0$. For this reason we treated the modes $a_0$ and $b_0$ on the same footing as all the other modes in the string expansion. Moreover, since we do not expect any $\delta$–function on the energies in the matrix elements, there is no reason why the string 2–point functions should be diagonal at all (perturbative) orders. It is an open possibility that the identification between string states and single–trace field theory operators originally proposed in \[5\] is actually valid also at higher orders. We took this point of view, since it allows to keep the physical intuition of identifying the closed string with the “long” BMN trace made out of $Z$’s and the string excitations with the BMN impurities.

In the second part of the paper we used the proposal of \[10\] and compared the matrix elements of our 3–string vertex with the corresponding field theory correlators at leading order in $\lambda'$. The two results match in all cases. Our analysis concerns 3–point correlators of BMN operators with double insertions of scalar, vector and fermion impurities and provide a test for the various building blocks of our string vertex in the $\mu \to \infty$ limit. The proposal of \[10\] has been criticized in the subsequent literature also from the field theory point of view. It was argued that the presence of mixing between single–trace and multi–trace BMN operators had to be taken into account also at the level of planar computations. This affects the tree–level expression of the 3–point correlators and thus spoils the agreement with string theory results. However, the inclusion of this mixing in the comparison with bulk amplitudes seems a rather unnatural procedure from the point of view of the AdS/CFT correspondence. In fact, as we discussed in Section 3, a similar situation appears also in the standard AdS/CFT duality in the case of extremal correlators. There the results found in $AdS_5 \times S^5$ supergravity agree with field theory computations without invoking the mixing.
Let us finally comment on the other methods for comparing field and string theory that have been proposed [23, 24, 25, 28]. There are basically two distinct approaches. In the first one, the idea is to interpret the relation $\Delta - J = H$ as an operator equation and to identify the matrix elements of the string Hamiltonian with those of $\Delta - J$ between multi–trace operators. Technically this requires a modification of the dictionary between string states and field theory operators. However this new map contains several free parameters that are fixed by imposing the matching with the string theory results. In the more recent literature, this has been done by using the string vertex proposed by [11, 12, 13, 14]. However, it is possible to choose a different basis in field theory in order to get agreement with the results of the string vertex (2.33) presented here$^{19}$. In order to get a non–trivial check in this approach one is obliged to go to higher orders in the parameter controlling the genus expansion. At this level the computations are rather involved. Moreover, at subleading order only the field theory answer is known and is compared with results extrapolated from the tree–level string vertex by using the quantum mechanical perturbation theory. Genuine computations on the torus for the mass of string states remain a challenging open problem. Another approach advocated in [28] is to fix a new dictionary by taking, on the field theory side, the operators with definite conformal properties. The comparison with string theory has been done by using the kinematical vertex of [20] supplemented by a prefactor phenomenologically derived by the inputs coming from field theory. This prefactor is different from the one presented here, because it treats the $a_n$ modes with positive and negative frequency in a different way. Actually in string computations, like for instance that of the action of $Q$ on the kinematical vertex (2.34), both kinds of modes are treated on the same footing and appear always together in the combinations $K$ and $\tilde{K}$.

The advantage of the setup presented here, beyond its simplicity, is that there are no “free parameters” that can be modified and each computation represents a test for the proposal. It would be very interesting to see whether the agreement is preserved when operators with more than two impurities are considered. The main limitation of our approach is that it only considers the leading order in $\lambda'$. Generalizing it beyond this approximation is an important open problem. On the other hand, since there are two different proposals for the 3–string vertex both satisfying the same super–algebra, it is also important to analyze again the derivation of $|H_3\rangle$. Clearly to understand the origin of the differences it is necessary to use in the derivation a dynamical principle (like the path integral) instead of symmetry arguments. Work is in progress along these directions.

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$^{19}$See for example the first version of [23].
Appendix A: Definitions

In this Appendix we recall some of the definitions we use in the paper (we refer to [13] for a more detailed list of formulae and identities). Let us start with the mode expansion of the kinematical constraints

\begin{align}
x_{m(3)} &= - \sum_{n=-\infty}^{\infty} \sum_{i=1}^{2} \frac{\alpha_i}{\alpha_3} X_{mn}^{(i)} x_{n(i)} , \quad p_{m(3)} &= - \sum_{n=-\infty}^{\infty} \sum_{i=1}^{2} X_{mn}^{(i)} p_{n(i)} ,
\end{align}

where the matrices \(X\) are

\[
X_{mn}^{(r)} \equiv \begin{cases} 
(C^{1/2}A^{(r)}C^{-1/2})_{mn} , & m > 0 , n > 0 , \\
\frac{\alpha_1}{\alpha_3} (C^{-1/2}A^{(r)}C^{1/2})_{-m,-n} , & m < 0 , n < 0 , \\
-\frac{1}{\sqrt{2}} \epsilon^{rs} \alpha_s (C^{1/2}B)_m , & m > 0 , n = 0 , \\
1 , & m = 0 = n , \\
0 , & \text{otherwise} .
\end{cases} \tag{A.3}
\]

All the quantities in the equations below are defined for \(m,n > 0\)

\[
C_{mn} = m \delta_{mn} , \\
A_{mn}^{(1)} = (-1)^n \frac{2\sqrt{mn} \beta \sin m \pi \beta}{\pi} \frac{m^2 \beta^2 - n^2}, \\
A_{mn}^{(2)} = \frac{2\sqrt{mn}(\beta + 1) \sin m \pi \beta}{\pi} \frac{m^2(\beta + 1)^2 - n^2}, \\
A_{mn}^{(3)} = \delta_{mn} , \\
B_m = -\frac{2}{\pi} \frac{\alpha_3}{\alpha_1 \alpha_2} m^{-3/2} \sin m \pi \beta . \tag{A.4}
\]

The matrices \(N_{rs}^{(t)}\) and \(N_{m}^{T}\) in the definition of the \(Q\)’s are given by

\[
N_{mn}^{(rs)} = \delta_{mn} \delta_{rs} - 2 \left( C^{1/2} A^{(r)} T \Gamma^{-1} A^{(s)} C^{-1/2} C^{1/2} \right)_{mn} , \tag{A.5}
\]

where \(C_{(s)} = \delta_{mn} \omega_{(r)m} = \delta_{mn} \sqrt{m^2 + (\mu \alpha_{(r)})^2} , \quad \Gamma = \sum_{r=1}^{3} A^{(r)} U_{(r)} A^{(r) T} \) and finally \(U_{(s)} = C^{-1} \left(C^{(r)} - \mu \alpha_{(r)} 1 \right)\).
Appendix B: Prefactor

In this Appendix we show that the fermionic constituents of the prefactor, $Y_\Theta$, $W_\Lambda$, and $W_\Theta$, commute with the kinematical constraints (A.2). The fermionic constituents of the prefactor have to satisfy (the bosonic commutators are trivially zero)

$$\left\{ \sum_{r=1}^{3} \sum_{m \in \mathbb{Z}} \alpha_r X_{mn}^{(r)} \theta_{n(r)} , Y_\Theta \right\} = 0, \quad \left\{ \sum_{r=1}^{3} \sum_{m \in \mathbb{Z}} X_{mn}^{(r)} \lambda_{n(r)} , Y_\Theta \right\} = 0,$$  \hspace{1cm} (B.1)

$$\left\{ \sum_{r=1}^{3} \sum_{m \in \mathbb{Z}} \alpha_r X_{mn}^{(r)} \theta_{n(r)} , W_\Lambda(\Theta) \right\} = 0, \quad \left\{ \sum_{r=1}^{3} \sum_{m \in \mathbb{Z}} X_{mn}^{(r)} \lambda_{n(r)} , W_\Lambda(\Theta) \right\} = 0,$$  \hspace{1cm} (B.2)

where we expanded the fermionic constraints in oscillation modes. As usual [13, 14, 20] it is convenient to split the above equations for $m > 0$, $m < 0$ and $m = 0$.

Consider first the anticommutators of $W_\Lambda$. Since $W_\Lambda$ only contains $b^+_n$ oscillators the non-trivial anticommutators are those with the constraints containing $b_n$.

By using the mode expansions of $\theta$ and $\lambda$, one obtains for $m > 0$

$$\left\{ \sum_{r=1}^{3} \sum_{n=1}^{\infty} X_{mn}^{(r)} \lambda_{n(r)} , W_\Lambda \right\} = -\frac{1}{2\sqrt{\alpha'}} \frac{\alpha}{1 - 4\mu\alpha K} \sum_{r=1}^{3} \frac{1}{\alpha_r} \left(C^{1/2} A^r C^{3/2} N^r\right)_m = 0 \quad (B.3)$$

because of Eq. (5.12) of [13]. For $m < 0$

$$\left\{ \sum_{r=1}^{3} \sum_{n=1}^{\infty} \alpha_r X_{-m-n}^{(r)} \theta_{n(r)} , W_\Lambda \right\} = -\frac{i}{\sqrt{\alpha'}} \frac{\alpha}{1 - 4\mu\alpha K} \sum_{r=1}^{3} \frac{1}{\alpha_r} \left(C^{1/2} A^r C^{3/2} C_{(r)} N^r\right)_m \quad (B.4)$$

The two terms in this equation vanish because of Eq. (5.14) and (5.12) of [13].

For $W_\Theta$, the situation is very similar. $W_\Theta$ only contains $b^+_n$ oscillators, so the non-trivial anticommutators are

$$\left\{ \sum_{r=1}^{3} \sum_{n=1}^{\infty} \alpha_r X_{mn}^{(r)} \theta_{n(r)} , W_\Theta \right\} \quad \text{for } m > 0, \quad \left\{ \sum_{r=1}^{3} \sum_{n=1}^{\infty} X_{-m-n}^{(r)} \lambda_{n(r)} , W_\Lambda \right\} \quad \text{for } m < 0,$$  \hspace{1cm} (B.5)

which give respectively Eq. (B.3) and (B.4).

Finally we have to compute the anticommutator of $Y_\Theta$. In this case also the contribution of the zero modes has to be taken into account. For $m > 0$ there is only one non-trivial anticommutator

$$\left\{ \sum_{r=1}^{3} \sum_{n=0}^{\infty} X_{mn}^{(r)} \lambda_{n(r)} , Y_\Theta \right\} = \frac{\alpha}{2\sqrt{\alpha'}} C^{1/2} C^{1/2} \frac{1}{1 - 4\mu\alpha K} \sum_{r=1}^{3} \left(A^r C^{1/2} U_{(r)}^{-1} N^r\right)_m + B_m = 0.$$  \hspace{1cm} (B.6)
To prove Eq. (B.6), it is convenient to rewrite \( U_{m(r)}^{-1} = U_{m(r)} + 2\mu\alpha_r C_m^{-1} \). Then by using Eq. (B.5) of [13] in the first term and Eq. (B.9) in the second, it is easy to see that (B.6) vanishes. For \( m < 0 \) the non-trivial anticommutator reads

\[
\left\{ \sum_{i=1}^{3} \sum_{n=1}^{\infty} \alpha_r X_{-m-n}^{(r)} , \mathcal{Y}_\Theta \right\} = i \sqrt{\alpha'} \frac{\alpha \alpha_3}{1 - 4\mu \alpha K} \sum_{i=1}^{3} \frac{1}{\alpha_r} (C^{-1/2} A^r C^{3/2} N^r)_m = 0,
\]

which is zero by Eq. (B.12) of [13]. For \( m = 0 \) the anticommutators for the zero modes are easily checked to be zero.

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