Abstract: In this paper, we investigate the problem for optimal control of a viscous generalized \( \theta \)-type dispersive equation (VG \( \theta \)-type DE) with weak dissipation. First, we prove the existence and uniqueness of weak solution to the equation. Then, we present the optimal control of a VG \( \theta \)-type DE with weak dissipation under boundary condition and prove the existence of optimal solution to the problem.

Keywords and phrases: optimal control, viscous generalized \( \theta \)-type dispersive equation, weak dissipation, existence and uniqueness, weak solution

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1 Introduction

In this paper, we investigate a viscous generalized \( \theta \)-type dispersive equation (VG \( \theta \)-type DE) with weak dissipation as follows:

\[
    u_t - \alpha^2 u_{xx} - \varepsilon (u - u_{xx})_x + ku_x + (a + b) uu_x + \gamma u_{xxx} + \lambda (u - u_{xx}) = \alpha^2 (au_x u_{xx} + buu_{xxx}),
\]

or

\[
    y_t - \varepsilon y_{xx} + au_x y + b uy_x + ku_x + \gamma y_{xxx} + \lambda y = 0,
\]

where \( u = G \ast y = \int_{-\infty}^{\infty} G(x - \xi) y(\xi) d\xi \), \( y = u - u_{xx} \) and \( G \) is chosen to be the Greens function for the Helmholtz operator \( 1 - \partial_x^2 \) on the line. Here, \( k \geq 0, \varepsilon > 0 \) (viscous effect), \( \lambda > 0 \) (weakly dissipative effect) \( a, b > 0 \) and \( u = u(x, t) \) denotes the fluid velocity. The constants \( \alpha^2 \) and \( \frac{\varepsilon}{\gamma} \) are squares scales, and \( c_0 = \sqrt{\frac{\gamma}{\varepsilon}} \) (\( c_0 = k \)) is the linear wave speed for undisturbed water at rest at spatial infinity. (1.1) describing the unidirectional propagation of surface waves in a shallow water region was derived by the method of asymptotic analysis and a near-identity normal form transformation from water wave theory, combining the linear dispersive of the Korteweg-de Vries (KdV) equation with the nonlinear dispersion of the Camassa-Holm (CH) equation, as well as viscous effect and weakly dissipative effect. For simplicity, we assume that \( \alpha = 1 \).

When \( \varepsilon = 0, \alpha \to 0 \) and \( \lambda = 0 \), (1.1) becomes the well-known KdV equation:

\[
    u_t + ku_x + (a + b) uu_x + \gamma u_{xxx} = 0.
\]
Equation (1.2) describes unidirectional propagation of waves at the free surface of shallow water under the influence of gravity. \( u(x, t) \) represents the wave height above a flat bottom, \( x \) is proportional to distance in the direction of propagation and \( t \) is proportional to the elapsed time, see, e.g., [1]. The Cauchy problem and long-time behavior of the KdV equation have been studied extensively, see, e.g., [2–4].

When \( k = 0, \gamma = 0, \alpha = 1 \) and \( \lambda = 0 \), (1.1) becomes the following family of evolutionary \( 1 + 1 \) partial differential equations, see, e.g., [5]:

\[
y_t + uu_y + b uy_x = \varepsilon y_{xx},
\]

which describes the balance between convection and stretching for small viscosity in the dynamics of 1-D nonlinear waves in fluids. In a recent study of soliton equations, it is found that (1.3) for \( \varepsilon = 0 \) and any \( b \neq -1 \) is included in the family of shallow water equations at quadratic order accuracy that are asymptotically equivalent under Kodama transformations [5].

In the absence of viscous effect, i.e., \( \varepsilon = 0 \), (1.3) reduces to the \( b \)-family of equations:

\[
y_t + uy_y + bu_y = 0,
\]

which describes a one-dimensional version of active fluid transport. It was shown by Degasperis and Procesi [6] that (1.4) cannot satisfy the asymptotic integrability condition unless \( b = 2 \) or \( b = 3 \), see, e.g., [5–7].

In the case of \( b = 2 \) in (1.4), (1.4) becomes the CH equation:

\[
y_t + u_y y + 2uy_x = 0 \iff u_t - u_{ext} + 3u_x = 2u_{xx} + u_{xxx},
\]

which was derived by Camassa and Holm in [7] by approximating directly the Hamiltonian for Euler equations in the shallow water regime. It turns out that it is also a model for the propagation of nonlinear waves in cylindrical hyperelastic rods, see [1]. Recently, the CH equation(s) has been investigated in [8–18]. For initial data \( u_0 \in H^s \) with \( s > \frac{3}{2} \), they have established the well-posedness (including local and global well-posedness), breaking-waves (i.e., the solutions that remain bounded while its slope becomes unbounded in finite time), blow up and blow-up rate. We note that the advantage of the CH equation in comparison with the KdV equation lies in the fact that the CH equation has peak solitons and models wave breaking.

In the case of \( b = 3 \) in (1.4), (1.4) becomes the Degasperis-Procesi (DP):

\[
y_t + u_y y + 3uy_x = 0 \iff u_t - u_{ext} + 4u_x = 3u_{xx} + u_{xxx},
\]

which has a similar form to CH Eq. (1.5). Degasperis, Holm and Hone [19] proved the formal integrability of DP Eq. (1.6) by constructing a Lax pair. They also showed that (1.6) has the bi-Hamiltonian structure and infinite sequence of conserved quantities to CH peakons. After (1.6) was derived, many contributions were devoted to its study, see [20–23]. For example, Yin [21] established local well-posedness to (1.6) with initial data \( u_0 \in H^s \) \( s > \frac{3}{2} \) on the line and precise blow-up scenario and blow-up result were derived. (1.6) can be regarded as a model for nonlinear shallow water dynamics and its asymptotic accuracy is the same for the CH equation. In [24], the authors showed that (1.5) can be obtained from the shallow water elevation by an approximate Kodama transformation.

Recently, Liu investigated the traveling wave solutions to a class of \( \theta \)-type dispersive models of the form [25]

\[
(1 - \partial_x^2)u_t + (1 - \theta \partial_x^2)\partial_x\left(\frac{u^2}{2}\right) = (1 - 4\theta)\partial_x\left(\frac{u_x^2}{2}\right) \iff u_t - u_{ext} + uu_x = (1 - \theta)u_x u_{xx} + \theta u_{xxx}, \quad 0 < \theta < 1,
\]

which were identified in his study of model equations for some dispersive schemes for approximating the Hopf equation. It is easy to see that (1.7) involves a convex combination of nonlinear terms \( uu_{xx} \) and \( u_x u_{xx} \).
Observing the coefficients of the peaked CH (1.5) and the DP (1.6), we note that in both models, the coefficient of \( uv \) is equal to the coefficient of \( u_{xx} \) plus the coefficient of \( uu_{xxx} \). That is, \( 3 = 2 + 1, 4 = 3 + 1 \). Recently, Lai and Wu [26] considered the following generalized equation:

\[
\begin{aligned}
    u_t - u_{xx} + (a + b)uv_t = au_{xx} + buu_{xxx},
\end{aligned}
\]

and established the global solutions and blow-up phenomena to (1.8).

In general, it is difficult to avoid energy dissipation mechanics in a real world. So it is reasonable to investigate the model with energy dissipation in propagation of nonlinear waves. For this reason, Zhang and co-workers [27–29] investigated how wave equation, Kein-Gordon equation and viscoelastic equation have to be modified to include the effect of dissipation and the influence of dissipation to the solution of the equation by using the perturbed energy method (see also [30–32]). When \( \varepsilon = 0, a = 2, b = 1 \), (1.1) becomes the weakly dissipative Dullin-Gottwald-Holm (DGH) equation as follows:

\[
\begin{aligned}
    u_t - u_{xxx} + ku_t + 3uu_x + yu_{xxxx} + \lambda(u - u_{xx}) = 2uu_{xx} + uu_{xxx},
\end{aligned}
\]

which describes that the unidirectional propagation of surface waves in a shallow water regime was derived by the method of asymptotic analysis and a near-identity normal form transformation from water wave theory, combining the linear dispersive of the KdV equation with the nonlinear dispersion of the CH equation, see, e.g., [1]. Weakly dissipative \( \theta \)-type Eq. (1.8) and DP Eq. (1.6) have been investigated in [26] and [33,34], respectively. Wu and Yin [26] discussed the blow-up, blow-up rate and decay of solution to the weakly dissipative \( \theta \)-type equation. Recently, Novruzov [35] considered blow-up phenomena the Cauchy problem for DGH Eq. (1.9) with weak dissipation and established certain conditions on the initial datum to guarantee that the corresponding positive strong solutions blow up in finite time. Later on, Zhang et al. [1] obtained the precise blow-up scenario and established some blow-up results for strong solutions, as well as the blow-up rate of the wave-breaking solutions to (1.9). This result complements the early one in the literature, such as [35].

To give a more comprehensive introduction, we shall present here some results concerning the control problems of nonlinear dispersive equations and other shallow water wave equations. For example, by adding a viscosity term to the equations, the optimal control problems with respect to the CH equation, the DP equation, the viscous weakly dispersive DP equation and the DGH equation are investigated in [36–39]. In [40], from the point of view of distributed control, Glass proved that the CH equation on the circle \( T \) is exact controllable, and he also obtained the global asymptotic stabilization result by means of an explicit stationary feedback law. In [41], Zhang investigated the optimal control of the Benjamin-Bona-Mahony (BBM) equation and showed that one can produce a water wave to match a desired water wave (called the desired state function) if the desired water wave is sufficiently small. By applying the analytical method, Sun [42,43] obtained the maximum principles of Pontryagin’s type for the viscous generalized CH equation and the viscous DGH equation, respectively. As for optimal control of other nonlinear dispersive equations, there are lots of contributions, such as [44–50].

In this paper, we present an optimal control problem for Eq. (1.1), which has never been studied until now. More precisely, we investigate the optimal control system

\[
\begin{aligned}
    \min \{ \| y(y, \omega) \| = \frac{1}{2} \| Cy - z \|_2^2 + \frac{\alpha}{2} \| \omega \|_{L^2(Q_0)}^2, \\
    y_t - \lambda_y + au_{xy} + bu \omega_t + k u_t + yu_{xxxx} + \lambda y = f + B \omega, \quad \text{in } (0, T) \times \Omega, \\
    u = u_0 = u_\delta = 0, \quad \text{on } (0, T) \times \partial \Omega, \\
    y(0, x) = y_0(x) \in H, \quad \text{in } \Omega,
\end{aligned}
\]

where \( f + \omega \in L^2(0; T; V^*) \), \( \Omega = (0, 1) \) and \( Q_0 = (0, T) \times \Omega \) in the \( L^2 \) sense. Our aim is to match the given desired state \( z \) by adjusting the body force \( f \), \( \omega \) is a control that belongs to the Hilbert space \( L^2(Q_0) \) with minimal energy, the first term in cost functional measures the physical objective, the second one is the size of the control, where \( \delta > 0 \) plays the role of a weight.

Throughout the paper, the letter \( C \) denotes positive constant, which may change from line to line. Before proceeding to our analysis, we present some notations which will be used throughout the paper.
Let $V = H^1_0(0, 1)$, $H = L^2(0, 1)$, $V^\prime = H^{-1}(0, 1)$ and $H^\prime = L^2(0, 1)$ be dual space, respectively. It is obvious that $V$ is dense in $H$, and $V \hookrightarrow H = H^\prime \hookrightarrow V^\prime$ with each embedding being dense. We supply $V$ with the inner product $(\varphi, \psi)_V = (\varphi_x, \psi_x)_H$, $\forall \varphi, \psi \in V$. The extended operator $B^\ast \in (L^2(Q_0), L^2(V^\prime))$ is given by

\[
B^\ast v = \begin{cases} v, & v \in Q_0, \\ 0, & v \in Q \setminus Q_0. \end{cases}
\]

We also supply $H$ with the inner product $(\cdot, \cdot)_H$ and the norm $\|\cdot\|_H$, and define $\|u\|_{H^m(\Omega)} = \|D^m u\|_H$, where $D^m = \frac{\partial^m}{\partial x^m}$ ($m = 0, 1, 2, \ldots$). A new space $W(0, T; V)$ is introduced as

\[
W(0, T; V) = \{ f : f \in L^2(0, T; V), f_t \in L^2(0, T; V^\prime) \},
\]

which is a Hilbert space endowed with a common inner product.

Remark 1.1. According to the value of two parameters, it is not difficult to see that (1.8) is a generalization of (1.7) and (1.8) includes (1.5) and (1.6). In fact, let $a = 1 - \theta$, $b = \theta$, then $a + b = 1$. If $b = \theta = \frac{1}{3}$, $a = \frac{2}{3}$ ($a = 2b$), then by using transformation $t \rightarrow \theta t$, (1.8) becomes (1.5). If $b = \theta = \frac{1}{4}$, $a = \frac{3}{4}$ ($a = 3b$), then by using transformation $t \rightarrow \theta t$, (1.8) becomes (1.6).

2 Existence and uniqueness of weak solution to VG $\theta$-type DE with weak dissipation

In this section, we prove the existence of a weak solution for the following VG $\theta$-type DE with weak dissipation

\[
\begin{aligned}
\begin{cases}
\gamma_x - \gamma_{xx} + au_x y + bu_y + ku_x + yu_{xx} + \lambda y = f + B^\ast \omega, & \text{in } (0, T) \times \Omega, \\
u = u_x = u_{xx} = 0, & \text{on } (0, T) \times \partial \Omega, \\
y(0, x) = y_0(x) \in H, & \text{in } \Omega,
\end{cases}
\end{aligned}
\]

(2.1)

where $f + B^\ast \omega \in L^2(0, T; V^\prime)$. To facilitate further on our analysis, we introduce definition of the weak solution of (2.1) in the space $W(0, T; V)$ as follows:

Definition 2.1. A function $y(x, t) \in W(0, T; V)$ is called a weak solution to (2.1), if

\[
\frac{d}{dt}(y, \psi)_H + \varepsilon (y_x, \psi_x)_H + a(u_y y, \psi_H) + b(u_x, \psi_H) + k(u_x, \psi_H) + \gamma(u_{xx}, \psi_H) + \lambda(y, \psi_H) = (f + B^\ast \omega, \psi)_V, \forall \psi \in V, \text{ a.e. } t \in [0, T)
\]

and

$y_0(x) = \phi \in H$.

Next, by using the Galerkin method [27,51,52] and a priori estimates, one can obtain the following theorem, which ensures the existence of a unique weak solution to (2.1). Now, we are in position to state the main result in this section.

Theorem 2.1. Let $\phi \in H$, $f + B^\ast \omega \in L^2(0, T; V^\prime)$. Then there exists a unique weak solution to (1.1) in the interval $[0, T]$.

Proof. Let $\{ \varphi_j \}_{j=1}^\infty$ be an orthonormal basis in the space $H$ consisting of eigenfunctions of the operator $-\partial_x^2$, where $\varphi_j$ is the eigenfunction subject to the Dirichlet condition:

\[
\begin{cases}
-\partial_x^2 \varphi_j = \lambda_j \varphi_j, & \text{in } \Omega, \\
\varphi_j = 0, & \text{on } \partial \Omega.
\end{cases}
\]

We also normalize \( \varphi_j \) such that \( \|\varphi_j\|_{H^1} = 1 \) [53]. By the elliptic operator theory, \( \varphi_j \) forms base functions in \( V \). For \( n \in \mathbb{N} \), we define the discrete ansatz space by \( V_n = \text{span}\{\varphi_1, \varphi_2, \ldots, \varphi_n\} \subset V \). Let \( y_n(t) = \varphi_j(x, t) = \sum_{j=1}^n y_j^0 \varphi_j(x) \) with \( y_j^0, y_0 \in H^1 \).

Next, we prove the existence of a unique weak solution to (2.1) by analyzing the limiting behavior of sequences of smooth functions \( u_n \) and \( y_n \), where \( u_n \) and \( y_n \) are the solutions of the Cauchy problem as follows:

\[
\begin{align*}
    y_{n,t} - \nabla y_{n,xx} + au_{n,xx}y_n + bu_{n,x}y_n + cy_{n,xxx} + \lambda y_n &= f + B' \omega, \quad \text{in} \ (0, T) \times \Omega, \\
y_n &= u_{n,xx} = 0, \quad \text{on} \ (0, T) \times \partial \Omega, \\
y_n(0, x) &= y_n(0, x) = \phi \in H, \quad \text{in} \ \Omega.
\end{align*}
\]

(2.2)

By standard methods of differential equations [27], we prove the existence of a solution (2.2) on some interval \([0, t_0]\), then, this solution can be extended to the whole interval \([0, T]\) by using a priori estimates. That is, we will show that the solution is uniformly bounded as \( t \to T \). Thus, we divide our proof into four steps.

**Step 1.** We prove a uniform \( L^2(V) \) bound on a sequence \( \{u_n\} \). Taking the inner product of (2.2) with \( u_n \) in \( \Omega \), we have

\[
\frac{1}{2} \frac{d}{dt} (\|u_n\|^2 + \|u_{n,x}\|^2) + \varepsilon (\|u_{n,xx}\|^2 + \|u_{n,xxx}\|^2) + \frac{a - 2b}{2} (u_{n,xx}, u_{n,xx}^2) + A (\|u_n\|^2 + \|u_{n,x}\|^2) = (f + B' \omega, u_n)_{L^2(V^*)}.
\]

That is,

\[
\frac{1}{2} \frac{d}{dt} (\|u_n\|^2 + \|u_{n,x}\|^2) + \varepsilon (\|u_{n,xx}\|^2 + \|u_{n,xxx}\|^2) \leq \frac{a - 2b}{2} (u_{n,xx}, u_{n,xx}^2) + (f + B' \omega, u_n)_{L^2(V^*)} (\lambda > 0).
\]

(2.3)

First, we estimate the first term of the right hand side of (2.3) as follows:

\[
- \frac{a - 2b}{2} (u_{n,xx}, u_{n,xx}^2) \leq \frac{|a - 2b|}{2} \|u_{n,xx}\|_{L^2} \|u_{n,xx}\|^2.
\]

(2.4)

By the Sobolev embedding theorem \( (H^1 \hookrightarrow L^\alpha, \ H^2 \hookrightarrow H^1) \) and the Poincaré inequality, we get

\[
\|u_{n,xx}\|_{L^\alpha} \leq C \|u_{n,xx}\|_{H^1} \leq C \|u_n\|_{H^2} \leq C_1 \|u_{n,xx}\|,
\]

(2.5)

where \( C, C_1 \) are constants.

By (2.4) and (2.5), we obtain

\[
- \frac{a - 2b}{2} (u_{n,xx}, u_{n,xx}^2) \leq \frac{|a - 2b|}{2} C_1 \|u_{n,xx}\| \|u_{n,xx}\|^2.
\]

(2.6)

Next, we estimate the second term of the right hand side of (2.3). Owing to \( f + B' \omega \in L^2(0, T; V^*) \) is a control item, there exists a constant \( M > 0 \) such that \( \|f + B' \omega\|_{L^2(0, T; V^*)} \leq M \). Thus, we deduce

\[
(f + B' \omega, u_n)_{L^2(V^*)} \leq \|f + B' \omega\|_{V^*} \|u_n\|_{V} \leq C_2 M \|u_{n,xx}\|.
\]

(2.7)

It follows from (2.3), (2.6), (2.7) and Young’s inequality that

\[
\frac{1}{2} \frac{d}{dt} (\|u_n\|^2 + \|u_{n,x}\|^2) + \varepsilon (\|u_{n,xx}\|^2 + \|u_{n,xxx}\|^2)
\leq \frac{|a - 2b|}{2} C_1 \|u_{n,xx}\| \|u_{n,xx}\|^2 + C_2 M \|u_{n,xx}\|
\leq \varepsilon \|u_{n,xx}\|^2 + \frac{|a - 2b|^2 C_1^2}{4 \varepsilon} \|u_{n,xx}\|^4 + \varepsilon \|u_n\|^2 + \frac{C_2^2 M^2}{\varepsilon}
\leq \varepsilon (\|u_{n,xx}\|^2 + \|u_n\|^2) + \frac{|a - 2b|^2 C_1^2}{4 \varepsilon} (\|u_{n,xx}\|^2 + \|u_n\|^2)^2 + \frac{C_2^2 M^2}{\varepsilon}.
\]
That is,
\[
\frac{d}{dt}(\|u_n\|^2 + \|u_{n,x}\|^2) \leq \frac{|a - 2b|^2 C^2}{2\varepsilon} (\|u_{n,xx}\|^2 + \|u_n\|^2) + \frac{2C^2 M^2}{\varepsilon}.
\] (2.8)
Therefore, by (2.8), we have
\[
\|u_n\|^2 + \|u_{n,x}\|^2 \leq \frac{2C_1 M}{|a - 2b| C_1} \tan \left\{ \frac{|a - 2b| C_2 M}{\varepsilon} t + C_3 \right\} \leq M^2,
\] (2.9)
where \( t \in [0, T], T < \frac{\varepsilon}{2 |a - 2b| C_1} \) and
\[ C_1 = \arctan \left( \frac{|a - 2b| C_2 (\|u_{n,0}\|^2 + \|u_{n,x}(0)\|^2)}{2C_2 M} \right). \]
Based on the above analysis, we obtain \( \|u_n\| \leq M_1 \) and \( \|u_{n,x}\| \leq M_1 \). By the Sobolev embedding theorem, we have
\[
\|u_n\|_{L^\infty} \leq \|u_n\|_{H^2} \leq (\|u_n\| + \|u_{n,x}\|) \leq C.
\] (2.10)

**Step 2.** Multiplying the first equation of (2.2) with \(-u_{n,xx}\) and integrating by parts with respect to \( x \) in \( \Omega \), we have
\[
\frac{1}{2} \frac{d}{dt}(\|u_{n,xx}\|^2 + \|u_{n,xxx}\|^2) + \varepsilon(\|u_{n,xx}\|^2 + \|u_{n,xxx}\|^2) + \lambda (\|u_{n,x}\|^2 + \|u_{n,xx}\|^2)
= -\frac{a + b}{2} \int_\Omega (u_{n,xx})^2 dx - \frac{2a - b}{2} \int_\Omega u_{n,x}(u_{n,xx})^2 dx + (f + B^*\omega, -u_{n,xx})_{V', V}.
\]
That is,
\[
\frac{1}{2} \frac{d}{dt}(\|u_{n,xx}\|^2 + \|u_{n,xxx}\|^2) + \varepsilon(\|u_{n,xx}\|^2 + \|u_{n,xxx}\|^2)
\leq -\frac{a + b}{2} \int_\Omega (u_{n,xx})^2 dx - \frac{2a - b}{2} \int_\Omega u_{n,x}(u_{n,xx})^2 dx + (f + B^*\omega, -u_{n,xx})_{V', V} (\lambda > 0).
\] (2.11)

Next, we estimate each term of the right hand side of (2.11). Owing to the Sobolev embedding theorem, we deduce that
\[
-\frac{a + b}{2} \int_\Omega (u_{n,xx})^2 dx \leq \frac{|a + b|}{2} \|u_{n,xx}\|_{L^\infty} \|u_{n,xx}\|_{L^2} \leq \frac{|a + b|}{2} K_1 \|u_{n}\|_{H^2} \|u_{n}\|_{V'},
\] (2.12)
\[
-\frac{2a - b}{2} \int_\Omega u_{n,x}(u_{n,xx})^2 dx \leq \frac{|2a - b|}{2} \|u_{n,x}\|_{L^\infty} \|u_{n,xx}\|_{L^2} \leq \frac{|2a - b|}{2} K_2 \|u_{n}\|_{H^2},
\] (2.13)
where \( K_1, K_2 > 0 \) are embedding constants.

Owing to \( f + B^*\omega \in L^2(0, T; V') \) is a control item, there exists a constant \( M > 0 \) such that
\( \|f + B^*\omega\|_{L^2(0, T; V')} \leq M \). Thus, by the Sobolev embedding theorem, we deduce
\[
|\langle f + B^*\omega, -u_{n,xx} \rangle_{V', V}| \leq \|f + B^*\omega\|_{V'} \|u_{n,xx}\|_{V} \leq K_3 \|u_{n}\|_{H^2},
\] (2.14)
where \( K_3 > 0 \) is the embedding constant. It follows from (2.9), (2.11)–(2.14) and Young’s inequality that
\[
\frac{1}{2} \frac{d}{dt}(\|u_{n}\|^2 + \|u_{n,xx}\|^2) + \varepsilon(\|u_{n}\|_{H^2}^2 + \|u_{n,x}\|_{H^2}^2)
\leq \frac{|a + b|}{2} K_1 \|u_{n}\|_{H^2}^2 + \frac{|2a - b|}{2} K_2 \|u_{n}\|_{H^2}^2 + K_3 M \|u_{n}\|_{H^2}
\leq \frac{\varepsilon}{2} \|u_{n}\|_{H^2}^2 + \frac{K_1^2 |a + b|^2}{4\varepsilon} \|u_{n}\|_{H^2}^2 + \frac{K_2^2 |2a - b|^2}{4\varepsilon} \|u_{n}\|_{H^2}^2 + \|u_{n,x}\|_{H^2}^2 + \frac{K_3^2 M^2}{\varepsilon}. \]
Then, we have
\[
\frac{d}{dt} \|u_n\|^2_{L^2} \leq \frac{K^1}{4\varepsilon} \|u_n\|^4_{L^4} + \frac{K^2}{\varepsilon} \|M_n\|^2_{L^2} + \frac{K^3}{\varepsilon} \|M_n\|^2_{L^2} + \frac{K^4}{\varepsilon} \|M_n\|^2_{L^2} + M_2^2 \|u_n\|^2_{L^2},
\]
where \( M_2^2 = \frac{K^1}{4\varepsilon} |2a - b|^2, \) \( M_2^2 = \frac{K^2}{\varepsilon} |a + b|^2 + \frac{K^3}{\varepsilon} |a + b|^2, \)

It follows from the aforementioned inequality that
\[
\|u_n\|^2_{L^2} \leq \frac{M_2}{M_2} \tan \arctan \left( \frac{M_3}{M_2} \|u_0\|^2_{L^2} \right) \leq M_2^2,
\]
where \( t \in [0, T] \) and \( T < \frac{n}{2M_2M_2}. \)

**Step 3.** We prove a uniform \( L^2(0, T; V) \) bound on a sequence \( \{y_n\} \). Taking the inner product of (2.2) with \( y_n \) in \( \Omega \), we have
\[
\frac{1}{2} \frac{d}{dt} \|y_n\|^2 + \|y_n\|^2 = (2a - b)(u_n, y_n) + (f + B^*\omega, y_n)v', v.
\]

First, we estimate the first term of the right hand side of (2.16). It follows from (2.10), the Poincaré inequality and the Hölder inequality that
\[
|(2a - b)(u_n, y_n)| \leq |2a - b| \|u_n\|_{L^2} \|y_n\|_{L^2} \leq \frac{C_1}{2}|2a - b| \|y_n\|^2_{L^2} \|y_n\|_{L^2} \leq \frac{C_1}{2}|2a - b| \|y_n\|^2_{L^2} + \|y_n\|^2_{L^2},
\]
where \( C_1 = \frac{C_1}{2}|2a - b|(C + 1). \)

Second, we estimate the second term of the right hand side of (2.16) in the following way:
\[
(f + B^*\omega, y_n)v' \leq \|f + B^*\omega\|_{L^2} \|y_n\|_{L^2} \leq C_2 \|y_n\|_{L^2},
\]

It follows from (2.16), (2.17) and (2.18) that
\[
\frac{1}{2} \frac{d}{dt} \|y_n\|^2 + \|y_n\|^2 \leq C_4 \|y_n\|^2 + C_3 M \|y_n\|_{L^2},
\]

By (2.19) and Young’s inequality, we have
\[
\frac{1}{2} \frac{d}{dt} \|y_n\|^2 + \|y_n\|^2 \leq C_4 \|y_n\|^2 + \left( \frac{e - C_4}{2} \|y_n\|^2 \right) + \frac{C_2^2 M^2}{2(e - C_4)}.
\]

That is,
\[
\frac{d}{dt} \|y_n\|^2 + \|y_n\|^2 \leq C_4 \|y_n\|^2 + \frac{C_2^2 M^2}{2(e - C_4)},
\]
where \( e - C_4 > 0 \). Integrating (2.20) with respect to \( t \) on \([0, T]\), we obtain
\[
\int_0^T \|y_n\|^2 dt \leq \frac{1}{e - C_4} \left[ \frac{C_2^2 M^2 T}{e - C_4} + \|y_{n, 0}\|^2 \right] \leq M^2_2.
\]

Moreover, we deduce
\[
\|y_n\|^2_{L^2(0, T; V)} \leq \int_0^T \|y_n\|^2 dt \leq C_3 \int_0^T \|y_n\|_{L^2} dt \leq M^2_2.
\]

On the other hand, it follows from (2.20) that
\[
\frac{d}{dt} \|y_n\|^2 \leq \frac{C_2^2 M^2}{e - C_4}.
\]
Integrating (2.23) with respect to \( t \) on \([0, T]\), we obtain

\[
\|y_{n,t}\|^2 \leq \frac{C_4^2 M_1^2 T}{\varepsilon - C_4} + \|y_{n,0}\|^2 = M^2.
\]

(2.24)

**Step 4.** We prove a uniform \( L^2(0, T; V^*) \) bound on a sequence \( \{y_{n,t}\} \). By (2.2), (2.9), (2.10), (2.15) and the Sobolev embedding theorem, we have

\[
\|y_{n,t}\|_{V^*} \leq \varepsilon \|y_n\|_{V} + a \|u_{n,v}\|_H \|y_n\|_H + b \|u_{n,v}\|_H \|y_n\|_H + k \|u_{n,v}\|_H + y \|u_{n,v}\|_H + \lambda \|y_n\|_H + \|f + B^* \omega\|_{V^*}
\]

\[
\leq \varepsilon \|y_n\|_V + a M_1 \|y_n\|_V + b c_1 \|y_n\|_H + k M_1 + y M_4 + \lambda \|y_n\|_H + C_3 M.
\]

(2.25)

It follows from (2.25) that

\[
\|y_{n,t}\|^2 \leq 3(k M_1 + y M_4 + C_3 M)^2 + 3(\varepsilon + a M_1^2) \|y_n\|^2 + (3b^2 C^2 + \lambda) \|y_n\|^2.
\]

(2.26)

Integrating (2.26) with respect to \( t \) on \([0, T]\), we have

\[
\|y_{n,t}\|^2_{L^2(0, T; V^*)} \leq \left[3(k M_1 + y M_4 + C_3 M)^2 + (3b^2 C^2 + \lambda) M_2^2\right] T + 3(\varepsilon + a M_1^2) M_2^2 \geq M_2^2
\]

Observing the precious discussions, we have

(i) the sequence \( \{y_n\} \) is bounded in \( L^2(0, T; V) \) as well as \( L^2(0, T; V^*) \);

(ii) the sequence \( \{y_{n,t}\} \) is bounded in \( L^2(0, T, V^*) \).

Therefore, by (i) and (ii), we obtain the boundedness of \( \{y_n\} \) in \( W(0, T; V) \). Hence, by standard compactness arguments, we can know that there exists a weak limit \( y(t, x) \) to the subsequence of \( \{y_n\} \) in \( W(0, T; V) \). Moreover, one concludes convergence of the subsequence of, again denoted by \( \{y_n\} \) weak into \( W(0, T; V^*) \), weak-star in \( \infty L^2(0, T; V^*) \) and strong in \( L^2(0, T; V^*) \) to a function \( y(t, x) \in W(0, T; V) \). The uniqueness of weak solution can be easily derived by following ideas in [54].

This completes the proof of Theorem 2.1.

Our next result describes weak solution can be controlled by the initial value and control item.

**Theorem 2.2.** Let \( \phi \in H, f + B^* \omega \in L^2(V^*) \). Then there exist two constants \( L_1 > 0 \) and \( L_2 > 0 \), such that

\[
\|y\|^2_{W(V)} \leq 2 L_1 \left[ \left( \|\phi\|_H + \|f\|_{L^2(V^*)}\right)^2 + \|\omega\|^2_{L^2(0, T; \Omega)} \right] + L_2,
\]

where

\[
L_0 = 2 \max\{1, 2, \sqrt{C_2}, 2C_0M_4\},
\]

\[
L_1 = L_0 + 4 + 4(\varepsilon + a M_1^2) C_0 + 8(b C + \lambda) T L_0
\]

and

\[
L_2 = 4(k M_1 + y M_4)^2 T.
\]

**Proof.** Multiplying the first equation of (2.1) by \( y \) and integrating by parts with respect to \( x \) in \( \Omega \), we have

\[
\frac{1}{2} \frac{d}{dt} \|y\|^2_H + \varepsilon \int_{\Omega} y^2 dx + \lambda \|y\|^2_H = (2a - b) \int_{\Omega} u y y_x^2 dx + (f + B^* \omega, y)_{V^*}.
\]

(2.27)

By Poincaré’s inequality and Sobolev embedding theorem, we obtain

\[
(2a - b) \int_{\Omega} u y y_x^2 dx \leq |2a - b| \|u\|_{L^\infty} \|y\|^2_{L^2} \|y_x\|_H \leq \frac{|2a - b|}{2} \|u\|_{L^\infty} \|y\|^2_{L^2} + \|y_x\|^2_H
\]

\[
\leq \frac{|2a - b|}{2} K_a \|u\|_V (\lambda_1 \|y_x\|^2_H + \|y\|^2_H) = \frac{|2a - b|}{2} K_a (\lambda_1 + 1) \|u\|_V \|y\|^2_H,
\]

(2.28)
where $K_4 > 0$ is the embedding constant and $\lambda_4 > 0$ is the Poincaré coefficient.

By using the same argument as in the proof of Theorem 2.1, we have

$$
\|u\|_H \leq M_1, \quad \|u\|_V \leq M_4, \quad \|u\|_{H'} \leq M_4,
$$

(2.29)

where $M_1$ and $M_4$ are positive constants.

It follows from (2.28) and (2.29) that

$$
\left| (2a - b) \int_\Omega uyy_\lambda dx \right| \leq \frac{|2a - b|}{2}K_4(\lambda_4 + 1)M_4\|y\|_{L_2(H)}^2 \leq M_4\|y\|_{L_2(H)}^2,
$$

(2.30)

where $M_4 = \frac{|2a - b|}{2}K_4(\lambda_4 + 1)M_4$.

Combining (2.27) with (2.30), we get

$$
\frac{1}{2} \frac{d}{dt}\|y\|_H^2 + \varepsilon \int_\Omega y_t^2 dx \leq (f + B^\ast \omega, y)_V + M_8\|y\|_V^2.
$$

(2.31)

Integrating (2.31) with respect to $t$ on $[0, T]$, we have

$$
\frac{1}{2} \|y(T)\|_H^2 - \frac{1}{2} \|\phi\|_H^2 + \varepsilon \int_0^T \|y\|_{L_2(V)}^2 dt \leq \int_0^T (f + B^\ast \omega, y)_V dt + M_8\|y\|_{L_2(V)}^2.
$$

That is,

$$
\frac{1}{2} \|y(T)\|_H^2 - \frac{1}{2} \|\phi\|_H^2 + (\varepsilon - M_8)\|y\|_{L_2(V)}^2 \leq \int_0^T (f + B^\ast \omega, y)_V dt, \quad \varepsilon > M_8.
$$

(2.32)

From (2.32) and (2.33), we have

$$
\|y\|_{L_2(V)}^2 \leq \frac{1}{\varepsilon - M_8}\|\phi\|_H^2 + \frac{1}{(\varepsilon - M_8)^2}\|f + B^\ast \omega\|_{L_2(V')}^2 \leq \frac{1}{\varepsilon - M_8}\|\phi\|_H^2 + \frac{1}{(\varepsilon - M_8)^2}\left(\|\phi\|_H + \|f + B^\ast \omega\|_{L_2(V')}\right)^2
$$

(2.34)

$$
\leq C_0\left(\|\phi\|_H + \|f + B^\ast \omega\|_{L_2(V')}\right)^2,
$$

where $C_0 = \max \left\{ \frac{1}{\varepsilon - M_8}, \frac{1}{(\varepsilon - M_8)^2} \right\}$.

On the other hand, in view of (2.31), we get

$$
\frac{1}{2} \frac{d}{dt}\|y\|_H^2 \leq (f + B^\ast \omega, y)_V + M_8\|y\|_V^2.
$$

Integrating the aforementioned inequality with respect to $t$ and observing (2.34) yields

$$
\|y\|_H^2 \leq \|\phi\|_H^2 + 2\|f + B^\ast \omega\|_{L_2(V')}\|y\|_{L_2(V)} + 2M_8\|y\|_{L_2(V')}^2 \leq 2\|f + B^\ast \omega\|_{L_2(V')} \sqrt{C_0}\left(\|\phi\|_H + \|f + B^\ast \omega\|_{L_2(V')}\right) + \|\phi\|_H^2 + 2C_0M_8\left(\|\phi\|_H + \|f + B^\ast \omega\|_{L_2(V')}\right)^2
$$

(2.35)
where $L_0 = 2\max\{1, 2\sqrt{C_0}, 2C_0M_0\}$.

It follows from (2.1) and (2.29) that
\[
\|y\|_{V'} \leq c\|y\|_V + a\|u\|_H\|y\|_V + b\|u\|_V\|y\|_H + k\|u\|_H + \gamma\|u\|_H^2 + \lambda\|u\|_H + \|f + B^*\omega\|_{V'}
\]
\[
\leq c\|y\|_V + aM_0\|y\|_V + bC\|y\|_H + kM_1 + \gamma M_3 + \lambda\|y\|_H + \|f + B^*\omega\|_{V'}
\]
\[
\leq (\varepsilon + aM_0)\|y\|_V + (bC + \lambda)\|y\|_H + (kM_1 + \gamma M_3) + \|f + B^*\omega\|_{V'}.
\]

It follows from (2.36) that
\[
\|y\|_{V'}^2 \leq 4(\varepsilon + aM_0)^2\|y\|_V^2 + 4(bC + \lambda)^2\|y\|_H^2 + 4(kM_1 + \gamma M_3)^2 + 4\|f + B^*\omega\|_{V'}^2.
\]

Integrating (2.37) with respect to $t$ on $[0, T]$, we have
\[
\left\{\begin{array}{l}
\|y\|_{L^2(V')}^2 \leq 4(\varepsilon + aM_0)^2\|y\|_V^2 + 4(bC + \lambda)^2\int_0^T \|y\|_H^2 dt + 4(kM_1 + \gamma M_3)^2 T + 4\|f + B^*\omega\|_{L^2(V')}^2 \\
\leq [4 + 4(\varepsilon + aM_0)^2C_0 + 8(bC + \lambda)^2 TL_0](\|\phi\|_H + \|f + B^*\omega\|_{L^2(V')})^2 + 4(kM_1 + \gamma M_3)^2 T.
\end{array}\right.
\]

Taking into account (2.34) and (2.38), we have
\[
\|y\|_{L^2(V')}^2 = \|y\|_{L^2(V')}^2 + \|y\|_{L^2(V')}^2
\]
\[
\leq L_1(\|\phi\|_H + \|f + B^*\omega\|_{L^2(V')})^2 + L_2
\]
\[
\leq 2L_1\left(\|\phi\|_H + \|f\|_{L^2(V')}\right)^2 + \|B^*\omega\|_{L^2(V')}^2 \right)^2 + L_2
\]
\[
\leq 2L_1\left(\|\phi\|_H + \|f\|_{L^2(V')}\right)^2 + \|\omega\|_{L^2(Q_0)}^2 \right)^2 + L_2,
\]

where
\[
L_1 = L_0 + 4 + 4(\varepsilon + aM_0)^2C_0 + 8(bC + \lambda)^2 TL_0
\]
and $L_2 = 4(kM_1 + \gamma M_3)^2 T$.

Thus, this completes the proof of Theorem 2.2.

\[\square\]

### 3 The distributed optima control of the VG $\theta$-type DE with weak dissipation

In this section, we investigate the distributed optimal control associated with the VG $\theta$-type DE with weak dissipation and prove the existence of optimal solution based on Lions’ theory [55].

Consider the following control system:
\[
\left\{\begin{array}{l}
\min\{J(y, \omega)\} = \frac{1}{2}\|\dot{y} - z\|_S^2 + \frac{\delta}{2}\|\omega\|_{L^2(Q_0)}^2,
\end{array}\right.
\]
\[
y_i = cy_{xx} + au_i y + bu_i + ku_i y_{xxx} + \delta y = f + B^*\omega, \quad \text{in } (0, T) \times \Omega,
\]
\[
u = u = u_{xx} = 0, \quad \text{on } (0, T) \times \partial\Omega,
\]
\[
y(0, x) = y_0(x) = \phi \in H, \quad \text{in } \Omega.
\]

where $f + B^*\omega \in L^2(0, T; V')$, $y = u - u_{xx}$ and $\omega$ is a control in $L^2(Q_0)$, $C \in L(W(0, T; V), S)$ is a given continuous observation operator, $S$ is a real Hilbert space and
\[
\min\{J(y, \omega)\} = \frac{1}{2}\|\dot{y} - z\|_S^2 + \frac{\delta}{2}\|\omega\|_{L^2(Q_0)}^2
\]
is the performance index of tracking type. Here, $z \in S$ is the desired state and $\delta > 0$ is fixed.
The optimal control problem for the VG θ-type DE with weak dissipation is \( \min \{ J_y(\omega) \} \), where the optimal control pair \((y, \omega)\) satisfies (2.1) with the given initial value and boundary condition.

For convenience, let \( X = W(0, T; V) \times L^2(Q_0), Y = L^2(0, T; V) \times H \) and we define an operator \( E = E(e_1, e_2) : X \to Y \) given by

\[
e_1 = (-\Delta)^{-1}[y_t - ey_{xx} + au_x + bu_x + ku + yu_{xxx} + \lambda y - (f + B'\omega)],
\]

\[
e_2 = y(x, 0) - \phi(x),
\]

where \( \Delta \) is an operator from \( V \) to \( V' \). Then we rewrite the optimal control problem in the following form:

\[
\min \{ J_y(\omega) \}, \quad \text{s. t.} \quad E(y, \omega) = 0.
\]

The following theorem is presented to demonstrate the existence of the optimal control to the VG θ-type DE with weak dissipation.

**Theorem 3.1.** There exists an optimal control solution \((y^*, \omega^*)\) to problem (2.39).

**Proof.** Let \((y, \omega) \in X\) satisfying \( E(y, \omega) = 0 \). Observing (2.39) and using Theorem 2.2, we get

\[
J(y, \omega) \geq \frac{\delta}{2} \| \omega \|^2_{L^2(Q_0)} \geq 0,
\]

\[
\| y \|_{W(V)} \to \infty, \quad \text{yields} \quad \| \omega \|^2_{L^2(Q_0)} \to \infty.
\]

Then, we have

\[
J(y, \omega) \to \infty \quad \text{as} \quad \| y(x, \omega) \|_{X} \to \infty. \tag{3.2}
\]

Note that the norm is weakly lower semicontinuous [56]; we deduce that \( J \) is weakly lower semicontinuous. Since \( J(y, \omega) \geq 0 \), for all \((y, \omega) \in X\), we conclude that there exists

\[
\kappa = \inf \{ J(y, \omega) \} \| (y, \omega) \in X, E(y, \omega) = 0 \}.
\]

This implies the existence of a minimizing sequence \((y_n, \omega_n)\) in \( X \) such that

\[
\kappa = \lim_{n \to \infty} J(y_n, \omega_n), \quad E(y_n, \omega_n) = 0, \quad n \in N. \tag{3.3}
\]

According to (3.2), there exists an element \((y^*, \omega^*) \in X\) such that, when \( n \to \infty \),

\[
y_n \to y^*, \quad \text{weakly} \quad y \in W(0, T; V), \tag{3.4}
\]

\[
\omega_n \to \omega^*, \quad \text{weakly} \quad \omega \in L^2(Q_0). \tag{3.5}
\]

From (3.4), we have

\[
\lim_{n \to \infty} \int_0^T \int_{\Omega} (y_n(t) - y^*(t), \varphi(t))_{H^1} \, dx \, dt = 0, \quad \forall \varphi(t) \in L^2(0, T; V). \tag{3.6}
\]

Since the fact that \( W(0, T; V) \) is compactly embedded into \( L^2(0, T; L^\infty) \) [57] and \( W(0, T; V) \) is continuously embedded \( C(0, T; H) \) [54], we obtain that \( y_n \to y^* \) strongly in \( L^2(0, T; L^\infty) \) and \( y_n \to y^* \) strongly in \( C(0, T; H) \). Furthermore, we obtain that \( u_n \to u^* \) strongly in \( C(0, T; H) \).

Owing to the sequence \( \{y_n\} \) converges weakly and \( \| y_n \|_{W(0, T; V)} \) is bounded [56], based on the embedding theorem, we obtain that \( \| y_n \|_{L^2(0, T; L^\infty)} \) is bounded. As a matter of fact, \( \| y_n \|_{L^2(0, T; L^\infty)} \) is also bounded, because \( y_n \to y^* \) strongly in \( L^2(0, T; L^\infty) \).

Thus, it follows from the Hölder inequality that

\[
\left| \int_0^T \int_{\Omega} (u_{n,x} y_n - u_{n,x} y^*) \varphi \, dx \, dt \right| \leq \left| \int_0^T \int_{\Omega} (u_{n,x} y_n - u_{n,x} y^* + u_{n,x} y^* - u_{n,x} y^*) \varphi \, dx \, dt \right|.
\]
Similarly, we have
\[
\left\| y_n - y* \right\|_{L^\infty(0,T;L^\infty)} \leq \left\| u_n - u^* \right\|_{L^1(0,T;H)} \|\varphi\|_{L^1(0,T;V)} + \left\| y_n - y^* \right\|_{L^1(0,T;L^\infty)} \|\varphi\|_{L^1(0,T;V)} \to 0,
\]
(as \( n \to \infty, \forall \varphi \in L^2(0,T;V) \)).

From (3.4), we obtain
\[
\left\| \int_0^T (u^* y - u_n y_n) \varphi \, dx \, dt \right\| \to 0, \quad n \to \infty, \forall \varphi \in L^2(0,T;V),
\]
\[
\left\| \int_0^T (y u_n,xxx - y u^*_xxx) \varphi \, dx \, dt \right\| \to 0, \quad n \to \infty, \forall \varphi \in L^2(0,T;V).
\]

From (3.5), we obtain
\[
\left\| \int_0^T (B^* y_n - B^* y^*_n) \varphi \, dx \, dt \right\| \to 0, \quad n \to \infty, \forall \varphi \in L^2(0,T;V).
\]

Based on the above discussion, we have \( e_1(y^*, \omega^*) = 0 \). Also, using the fact that \( y^* \in \mathcal{W}(0,T;V) \) and \( y_n \to y^* \) weakly in \( \mathcal{W}(0,T;V) \), we have \( y^*(0) \in H \) and \( y_n(0) \to y^*(0) \) weakly as \( n \to \infty \). Furthermore, we get
\[
(y_n(0) - y^*(0), \varphi) \to 0, \quad n \to \infty, \quad \forall \varphi \in H,
\]
which implies that \( e_2(y^*, \omega^*) = 0 \).

Thus, we conclude that
\[
E(y^*, \omega^*) = 0.
\]

Therefore, there exists an optimal solution \((y^*, \omega^*)\) to problem (2.39). In the meantime, we can infer that there exists an optimal solution \((u^*, \omega^*)\) to the VG \( \theta \)-type DE with weak dissipation owing to the relation \( u = (1 - \partial_x^2)^{-1} y \).

This completes the proof of Theorem 3.1.
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