Global Complexity of an Output Dynamic Competition Model

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Abstract. In this paper, we use the critical curves theory to study the global complexity of an output dynamic competition model with adaptive adjustment. Two forms of contact bifurcations are investigated. As some parameters are variable, the global evolution of the output dynamic competition system’s feasible set can be regarded as the variation of players’ living space, which is used to explain the economic significance of every global bifurcation. The global complexity analysis can help players to take some measures and avoid the collapse of the output dynamic competition system.

1. Introduction
Recently, some scholars are interested in Global bifurcations in economic systems [1-4]. When there are several coexistent Nash equilibrium points, the study of the basins is useful in order to select a solution or strategy and to take actions. Also the exact definition of the basin of the attracting set at finite distance can help the decision maker to avoid the explosion of the economic system.

Currently, research on the complex dynamics of output dynamic competition model is mainly about local stability of equilibrium points and the creation of complex attractors through sequences of local bifurcation (see references [5-12]). The study of the global bifurcations that cause qualitative changes of the attractors and structure of their basin has been neglected. In this paper, the global bifurcations of an output dynamic competition model are analyzed by the use of critical curves. We also show how the global dynamics of the model can be analyzed through studying the structure of their basins of attraction.

The remainder of this paper is organized as follows. Section 2 introduces output dynamic competition model and gives the dynamics of the model. In Section 3 we discuss the global bifurcations of the resulting noninvertible map. The final section concludes the paper.

2. Output dynamic competition model

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In this paper, we generalize the duopoly model of Bowley [6] to the case of cost function with nonlinear term. We assume that the production cost function of producer \( i \) (\( i=1,2 \)) has the nonlinear form

\[
C_i(q_i) = c_i + d_i q_i + e_i q_i^2, i = 1,2,
\]

where the positive parameter \( c_i \) is fixed cost of producer \( i \), \( d_i \) and \( e_i \) are constants, \( q_i(t) \) is the output of producer \( i \) in period \( t \).

According to Agiza [6], the output dynamic competition model can be written in the form

\[
q_i(t + 1) = q_i(t) + \alpha_i q_i(t)[a - bQ(t) - (b + 2e_i)q_i(t) - d_i](1-r), i = 1,2,
\]

where \( Q(t) \) is the total supply of the two producers and \( Q(t) = q_1(t) + q_2(t) \), \( a \) and \( b \) are positive constants, and \( a \) is the highest price in the market, \( \alpha_i \) is positive parameter representing the speed of adjustment, \( r \) is the tax rate of business income tax, and \( 0 \leq r < 1 \).

The time evolution of the discrete dynamical system (2) is obtained by the iteration of the two-dimensional map

\[
M: \begin{cases} 
q'_1 = q_1 + \alpha_1 q_1[a - (2b + 2e_1)q_1 - bq_2 - d_1](1-r), \\
q'_2 = q_2 + \alpha_2 q_2[a - (2b + 2e_2)q_2 - bq_1 - d_2](1-r), 
\end{cases}
\]

where ‘′’ denotes the unit-time advancement operator.

The nonlinear map (3) has four equilibria

\[
E_0 = (0,0), \ E_1 = (\frac{a - d_1}{2b + 2e_1}, 0), \ E_2 = (0, \frac{a - d_2}{2b + 2e_2}), \ E^* = (q_1^*, q_2^*),
\]

where \( q_1^* = \frac{(a - d_1)(2b + 2e_2) - b(a - d_2)}{3b^2 + 4be_1 + 4be_2 + 4e_1e_2} \), \( q_2^* = \frac{(a - d_2)(2b + 2e_1) - b(a - d_1)}{3b^2 + 4be_1 + 4be_2 + 4e_1e_2} \).

\( E_0, E_1 \) and \( E_2 \) are called boundary equilibrium and are unstable. \( E^* \) is the unique Nash equilibrium. With given a set of parameters \( a, b, r, c_i, d_i \) and \( e_i \) (\( i=1,2 \)), \( E^* \) is local stable for the adjustment speeds \( \alpha_1 \) and \( \alpha_2 \). As usual in output dynamic model with adaptive adjustment, the Nash equilibrium point \( E^* \) will lose stability as one or both of the adjustment speeds are increased, and more complex attractors are created. These results can be obtained through a standard study of the local stability of the equilibrium points (see [5–12]). However, our attention will be mainly focused on the global properties of the map (3), which will be present in next section.

3. Global analysis of the output dynamical system

3.1 Critical curve

An important feature of map (3) is that the two coordinate axes are invariant lines, since \( M(q_1,0) = (q_1',0) \) and \( M(0,q_2) = (0,q_2') \). The dynamics of (3) along the \( q_1 \)-axis are determined by the one-dimensional map \( q'_1 = M_1(q_1) \), where \( M_1 \) is the restriction of \( M \) to the \( q_1 \)-axis, given by

\[
M_1(q_1) = q_1 + \alpha_1 q_1[a - (2b + 2e_1)q_1 - d_1](1-r).
\]

Similarly, the dynamics of (3) along the \( q_2 \)-axis are governed by the one-dimensional map \( q'_2 = M_2(q_2) \), where \( M_2 \) is the restriction of \( M \) to the \( q_2 \)-axis, given by

\[
M_2(q_2) = q_2 + \alpha_2 q_2[a - (2b + 2e_2)q_2 - d_2](1-r).
\]
Supposed that $K$ is the set of points where the Jacobian determinant of $M$ vanishes, the critical curve of rank-0 $LC_{-1}$ is the subset of $K$. That is

$$LC_{-1} \subseteq K = \{(q_1, q_2) \in R^2 \mid \det DM = 0\}.$$ 

$LC_{-1}$ of map (3) is the restriction of $K$ to the positive quadrant $R^2_+$, which is given by the union of two branches, denoted by $LC^{(a)}_{-1}$ and $LC^{(b)}_{-1}$. The critical curve of rank-1 $LC$ is the rank-1 image of $LC_{-1}$ under $M$, i.e. $LC = M(LC_{-1})$. $LC$ is also the union of two branches, denoted by $LC^{(a)}$ and $LC^{(b)}$, where $LC^{(a)} = M(LC^{(a)}_{-1})$, $LC^{(b)} = M(LC^{(b)}_{-1})$. $LC$ and $LC_{-1}$ are displayed in Fig.1.

Note that the branches of critical curves $LC^{(b)}_{-1}$ and $LC^{(b)}$ intersect the coordinate axes $q_1$ and $q_2$ in the critical points of rank 0 and 1 of the restrictions $M_1$ and $M_2$, given by the points of coordinates

$$c^{(i)}_{-1} = \frac{\alpha_i(a - d_i)(1 - r) + 1}{\alpha_i(4b + 4e_i)(1 - r)} \quad \text{and} \quad c^i = M_i(c^{(i)}_{-1}) = \frac{[\alpha_i(a - d_i)(1 - r) + 1]^2}{\alpha_i(8b + 8e_i)(1 - r)}, \quad i = 1, 2;$$

As shown in Fig.1, $LC^{(b)}_{-1}$ separates the region $Z_{0b}$ whose points have no preimages, from the region $Z_2$, whose points have two distinct rank-1 preimages. $LC^{(a)}_{-1}$ separates the region $Z_2$ from $Z_4$, whose points have four distinct preimages. For example, we compute the preimages of the origin, by solving the system (3) with $q_1' = 0$ and $q_2' = 0$, we can obtain: $O^0_{-1} = (0, 0)$; $O^{(1)}_{-1} = \left(\frac{\alpha_1(a - d_1)(1 - r) + 1}{\alpha_1(2b + 2e_1)(1 - r)}, 0\right)$; $O^{(2)}_{-1} = \left(0, \frac{\alpha_2(a - d_2)(1 - r) + 1}{\alpha_2(2b + 2e_2)(1 - r)}\right)$ and $O^{(3)}_{-1}$ located at the intersection of preimages of two lines $oc^1$ and $oc^2$. For detailed account of critical curve theory, see Agliari [1], Bischi [2] and their reference.

![Fig.1. (a) Critical Curves of rank-0. (b) Critical curves of rank-1.](image)

### 3.2 Boundaries of the feasible set

In the following, the feasible set of map (3) is denoted by $\mathbb{N}$, which is the set of points generating feasible trajectories. A feasible trajectory may converge to the positive steady state $E^*$, to more complex attractors inside $\mathbb{N}$ or to a one-dimensional invariant set embedded inside a coordinate axis [2]. The last occurrence means that one of the two producers drops out the market. Trajectories starting out of the set $\mathbb{N}$ stand for collapsing evolutions of the output dynamic competition system. In a sense, feasible set can be looked as the ‘living space’ of the players.

Let $\partial \mathbb{N}$ be the boundary of $\mathbb{N}$. Such a boundary can have a simple shape, or have a very complex structure which can be shown by using numerical simulation.
Using the method similar to that of Bischi [2], we can obtain the boundary of $\aleph$. In general, $\partial\aleph$ is the union of all the preimages of any rank of the segments $\xi_1$ and $\xi_2$:

$$\partial\aleph = \left( \bigcup_{n=0}^{\infty} M^{-n}(\xi_1) \right) \cup \left( \bigcup_{n=0}^{\infty} M^{-n}(\xi_2) \right),$$

(7)

where $\xi_1 = O^{(1)} O^{-1}$ and $\xi_2 = O^{(2)} O^{-1}$. As long as $\alpha_1(a-d_1)(1-r) \leq 3$ and $\alpha_2(a-d_2)(1-r) \leq 3$, the boundary of $\aleph$ has the simple shape shown in Fig.2. In this situation quadrilateral $O^{(1)}O^{(3)}O^{-1}$ constitutes the whole boundary $\partial\aleph$, as is shown in Fig.2. This is due to the fact that $\xi_1^{-1}$ and $\xi_2^{-1}$ are entirely included inside the region $Z_0$ whose points have no preimages. That is to say, no preimages of higher rank of $\xi_1$ and $\xi_2$ exist.

### 3.3 Global bifurcations

If $\alpha_1$ or $\alpha_2$ is increased, so that the bifurcation value $\alpha_1^B = 3/[(a-d_1)(1-r)]$ or $\alpha_2^B = 3/[(a-d_2)(1-r)]$ is crossed, then $\partial\aleph$ is changed from smooth to fractal. This transition between qualitatively different structures of the boundaries of the region $\aleph$ constitutes a global bifurcation. In order to display this bifurcation, we vary the speed of adjustment $\alpha_1$ and fix the other parameters.

As $\alpha_1$ is increased, the branch $LC^{(b)}$ of the critical curve that separates $Z_0$ and $Z_2$ moves upwards, and at $\alpha_1 = 3/[(a-d_1)(1-r)]$ it has a contact with $\xi_2^{-1}$ (i.e. $O^{(1)}O^{(3)}$) at point $O^{(1)}_1$. After this contact the sides $\partial\aleph$ are transformed from smooth to fractal. In fact, when $\alpha_1$ crosses $3/[(a-d_1)(1-r)]$, a segment of $\xi_2^{-1}$ enters the region $Z_2$, so that $G_0$, a portion of the complement of $\aleph$, which is bounded by $LC^{(b)}$ and $\xi_2^{-1}$ (see Fig.3), now has two preimages. These two preimages, say $G_{-1}$ and $G_{+1}$, merge in points of $LC^{(b)}$ and form a ‘gray stalagmite’ issuing from the $q_1$ axis (denoted by $G_{-1}$ in Fig.3, $G_{+1} = G_{-1} \cup G_{+1}$). Because the points of $G_{-1}$ are mapped into $G_0$, $G_{-1}$ belongs to the ‘gray set’ of points that generate non-feasible trajectories. This is only the rank-1 preimages of $G_0$. Preimages of $G_0$ of higher rank form a sequence of smaller and smaller gray stalagmites issuing from the $q_1$ axis. Only some of them are visible in Fig.3, but smaller stalagmites become numerically visible by enlargement. All of these stalagmites take on a fractal structure. Since $G_{-1}$ is in the region $Z_2$, it has two preimages $G_{1,1}$ and $G_{2,1}$, located at opposite sides with respect to $LC^{(b)}$. Because $G_{1,1}$ falls into the region $Z_4$ (shown in Fig.3), besides the two preimages along the $q_1$ axis, two more preimages exist issuing from $\xi_1^{-1}$ (i.e. $O^{(2)}O^{(3)}$) and located at opposite sides with respect to $LC^{(e)}$. Those stalagmites located in the region $Z_4$ will form into smaller stalagmites (some of them may become visible by enlargement) issuing from $\xi_1^{-1}$. The stalagmites located at $\xi_1^{-1}$ belong to $Z_0$, hence they do not give rise to new sequences of stalagmites.
Fig. 2. When $a_i(a-d_i)(1-r)=3$, critical curve $LC^{(b)}$ has a contact with $\frac{1}{b_2^2}$ at the point $O_2^{(2)}$. (The parameters are $a=10, \alpha_2=0.1, b=1, d_i=1, d_2=1, e_1=1, e_2=1.1$ and $r=0.3$). Fig. 3. After a contact between $LC^{(b)}$ and $\frac{1}{b_2^2}$, the boundary of the feasible set changes from smooth to fractal (Except that $a_i=3/[(a-d_i)(1-r)]+0.01$, The other parameters take the same values as in Fig. 2).

If we look the stalagmites as the ‘reef’ or ‘trap’ in the advance of enterprises, this bifurcation implies that player must have the sense of crisis in the business world. He should not be satisfied with previous or present comfortable surroundings, and frequently pay attention to the changes of interior condition (such as his adjustment speed) and exterior circumstance (such as market demand and information of his rival). Otherwise, he may step into the ‘trap’ or bump with the ‘reef’.

Fig 4. After a contact between $LC^{(b)}$ and a stalagmite located inside $Z_2$, the feasible set changes from simply connected to multiply connected (Except that $a_i=0.527$, the other parameters take the same values as in Fig. 3). Fig. 5. The stalagmite $G_{-1}$ crossed $LC^{(0)}$ forms into new islet $g_{-1}$ (Except that $a_i=0.532$, the other parameters take the same values as in Fig. 4).

As $\alpha_i$ is further increased, $LC^{(b)}$ moves upwards, the portion $G_0$ enlarges. Consequently, all its preimages (i.e. the infinitely many stalagmites) enlarge and become more protruding. When a stalagmite belonging to $Z_2$ has a contact with $LC^{(a)}$ and enters the region $Z_4$, the contact occurs out of
the $q_1$ axis, and causes the creation of a pair of new preimages, merging along $LC^{(a)}_{-1}$, whose union forms into a ‘grey crescent’. If we look the feasible set $\mathbb{N}$ as ‘gulf’ or ‘lake’, ‘grey crescent’ looks like ‘islet’ inside it. This contact causes the occurrence of another local bifurcation which makes the set $\mathbb{N}$ change from simply connected into multiply connected. This can be seen in Fig.4, where the islet $h_{-1}$ is the preimage of the portion $h_0$, inside $Z_4$, of a stalagmite that crossed $LC^{(a)}$.

As $\alpha_1$ is increased, other stalagmites cross $LC^{(a)}$ and, hence, new islets are created. The structure of $\mathbb{N}$ in this situation is shown in Fig.5, where $g_0$ inside $Z_4$, the portion of the stalagmite $G_{-1}$ crossed $LC^{(a)}$, forms into new islet $g_{-1}$.

This contact bifurcation may imply in a sense that with variation of circumstance, some production strategy looked as if it were safe previous may become no longer secure in business world.

4. Conclusion
In this paper we investigated the contact bifurcations of an output dynamic competition model with the method of critical curves. For the model analyzed in this paper the main qualitative changes of the global structure of the feasible set can be obtained by using the theory of critical curves, which allows us to learn more about the dynamical behavior of the output model than just focusing on local dynamics. Defining the non-feasible set can give some help for decision maker to avoid the explosion of the economic system. Global bifurcations occurring at the boundary of feasible set make the structure of basins of attraction become very complex.

We find that after rank-$1$ critical curve contact the boundary of feasible set, with the increase of the speed of adjustment, boundary of the feasible set changes from smooth to fractal, or the feasible set changes from simply connected to multiply connected. As if many obstacles appear in the advance of enterprises. These changes suggest that in business world producer should be prepared for danger in times of safety, and not puzzled by well situation at present, adjusting his policy according to the changes of circumstance.

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