ON THE TUNNEL NUMBER AND THE MORSE-NOVIKOV
NUMBER OF KNOTS

A.V. PAJITNOV

ABSTRACT. Let $L$ be a link in $S^3$; denote by $\mathcal{MN}(L)$ the
Morse-Novikov number of $L$ and by $t(L)$ the tunnel number
of $L$. We prove that $\mathcal{MN}(L) \leq 2t(L)$ and deduce several
corollaries.

1. INTRODUCTION

1.1. Background. Let $L$ be a link in $S^3$, that is, an embedding
of several copies of $S^1$ to $S^3$. First off, we recall the definition
of three numerical invariants of $L$. In the sequel $N(L)$ denotes
a closed tubular neighbourhood of $L$.

A. Tunnel Number. An arc $\gamma$ in $S^3$ is called a tunnel for
$L$ if $\gamma \cap L$ consists of the two endpoints of $\gamma$. The tunnel
number $t(L)$ is the minimal number $m$ of disjoint tunnels
$\gamma_1, \ldots, \gamma_m$ such that the closure of $S^3 \setminus N(L \cup \gamma_1 \cup \ldots \cup \gamma_m)$
is a handlebody. The tunnel number was introduced by B.
Clark in [1]; this invariant was studied in the works of K. Mor-
rimoto, M. Sakuma, Y. Yokota, T. Kobayashi, M. Scharlemenn,
J. Schultens and others (see [11], [9], [10], [14]).

For any two knots $K_1, K_2$ we have $t(K_1 \# K_2) \leq t(K_1) +
t(K_2) + 1$. In the paper [8] T. Kobayashi and Y. Rieck defined
the growth rate for a knot $K$ by the formula

$$gr_t(K) = \lim \sup_{m \to \infty} \frac{t(mK) - mt(K)}{m - 1}$$

where $mK$ stands for the connected sum of $m$ copies of the
knot $K$. We have $1 \geq gr_t(K) \geq -t(K)$.

B. Bridge numbers. Let $S^3 = H_1 \cup H_2$ be a Heegaard de-
composition of $S^3$; put $\Sigma = H_1 \cap H_2$, and $g = g(\Sigma)$. We say
(following H. Doll [2]) that $L$ is in a $n$-bridge position with re-
spect to $\Sigma$ if $\Sigma$ intersects $L$ in $2n$ points and $\Sigma \cap H_i$ is a union
of $n$ trivial arcs in $H_i$ for $i = 1, 2$. The $g$-bridge number $b_g(L)$
of $L$ is defined as the minimal number $n$ such that $L$ can be
put in a $n$-bridge position with respect to a Heegaard decomposition of genus $g$ (thus $b_0(L)$ is the classical bridge number as defined in the paper [15] of H. Schubert). We have

$$t(L) \leq g + b_0(L) - 1.$$  

C. Morse-Novikov numbers. A framing of $L$ is a diffeomorphism $\phi : L \times D^2 \to N(L)$. Let $C_L$ denote the closure of $S^3 \setminus N(L)$. A Morse function $f : C_L \to S^1$ is called regular if its restriction to the boundary $\partial N(L)$ satisfies the following relation: $(f \circ \phi)(l, z) = \frac{z}{|z|}$. A regular Morse function has finite number of critical points; the number of the critical points of $f$ of index $i$ will be denoted by $m_i(f)$; the total number of critical points of $f$ will be denoted by $m(f)$. The minimal value of $m(f)$ over all possible framings $\phi$ and all possible Morse maps $f : C_L \to S^1$ is called the Morse-Novikov number of the link $L$ and denoted by $\mathcal{MN}(L)$ (see [12]). The Morse-Novikov theory implies that

$$\mathcal{MN}(L) \geq 2(b_1(L) + q_1(L))$$

where $b_1(L)$ and $q_1(L)$ are the Novikov numbers defined as follows. Let $\hat{C}_L$ be the infinite cyclic covering induced by $f$ from the covering $R \to S^1$. Denote the ring $\mathbb{Z}[t, t^{-1}]$ by $\Lambda$, and the ring $\mathbb{Z}((t))$ by $\hat{\Lambda}$. Then $b_1(L)$ and $q_1(L)$ are respectively the rank and torsion numbers of the module $H_1(\hat{C}_L) \otimes \hat{\Lambda}$. In case when the Novikov numbers are not sufficient to determine the $\mathcal{MN}(L)$ the twisted Novikov numbers (introduced by H. Goda and the author in [3]) are useful.

As for the upper bounds for $\mathcal{MN}(L)$ not much is known. M. Hirasawa proved that for every 2-bridge knot $K$ we have $\mathcal{MN}(K) \leq 2$ (unpublished). In the papers [13] and [6] of Lee Rudolph and M. Hirasawa it is proved that $\mathcal{MN}(K) \leq 4g_f(K)$ where $g_f(K)$ is the free genus of $K$, that is, the minimal possible genus of a Seifert surface $\Sigma$ bounding $K$ such that $S^3 \setminus \Sigma$ is an open handlebody.

1.2. Main results. The main result of this work is

**Theorem 1.1.** For every link $L$ in $S^3$ we have

$$(1) \quad \mathcal{MN}(L) \leq 2t(L).$$

The following corollaries are easily deduced.
Corollary 1.2. For every \( g \) we have
\[
\mathcal{MN}(L) \leq 2(g + b_g(L) - 1).
\]

Corollary 1.3. For every \((1,1)\)-knot \( K \) we have \( \mathcal{MN}(K) \leq 2 \).

Corollary 1.4. For every link \( L \) we have
\[
q_1(L) + b_1(L) \leq t(L).
\]

Corollary 1.5. For every knot \( K \)
\[
gr_t(K) \geq -t(K) + q_1(K).
\]

2. Proof of Theorem 1.1

Let \( m = t(L) \). Pick a framing \( \phi : L \times D^2 \to N(L) \). Then the manifold \( C_L = S^3 \setminus N(L) \) is obtained from \( \partial C_L \) by attaching \( m \) one-handles and then attaching a handlebody of genus \((m + 1)\) to the resulting cobordism. Thus we obtain a Morse function \( g : C_L \to \mathbb{R} \) which is constant on \( \partial C_L \) and has the following Morse numbers: \( m_0(g) = 0, \ m_1(g) = m, \ m_2(g) = m + 1, \ m_3(g) = 1 \). Pick any Morse map \( h : C_L \to S^1 \) such that \( h|\partial C_L \) is the canonical fibration: \( (h \circ \phi)(l, z) = \frac{z}{|z|} \). Consider a closed 1-form \( \omega_\varepsilon = dg + \varepsilon dh \). For \( \varepsilon > 0 \) sufficiently small \( \omega_\varepsilon \) is a Morse form with the same Morse numbers as \( dg \). Therefore the form
\[
\frac{1}{\varepsilon} \omega_\varepsilon = d\left(\frac{1}{\varepsilon} g + h\right)
\]
is the differential of a Morse map \( g_1 : C_L \to S^1 \) having the required behaviour on \( \partial C_L \). The map \( g_1 \) has one local maximum, and the standard elimination procedure (see for example [12] for details) gives us a Morse function \( f : C_L \to S^1 \) with \( m_0(f) = 0, \ m_1(f) = m, \ m_2(f) = m, \ m_3(f) = 0 \). Thus \( \mathcal{MN}(L) \leq 2m \).

3. Examples, and further remarks

A theorem of M. Hirasawa says that \( \mathcal{MN}(K) \leq 2 \) if \( K \) is a two-bridge knot. Since \( t(K) \leq b(K) - 1 \) our theorem implies this result. Observe that the proof of the M. Hirasawa’s theorem uses the H. Schubert’s classification of 2-bridge knots, and can not be generalized to the case of arbitrary bridge number.

The inequality (1) implies also the upper bound
\[
\mathcal{MN}(K) \leq 4g_f(K)
\]
obtained by Lee Rudolph and M. Hirasawa (see [13], [6]). Indeed Jung Hoon Lee [5] has shown that \( t(K) \leq 2g_f(K) \).

In many cases the estimate of Theorem 1.1 is better than the free genus estimate. For example, for a pretzel knot \( K = P(-2, m, n) \) where \( m, n \geq 3 \) are odd numbers, we have \( g(K) = \frac{m+n}{2} \) (see [4]), so that \( g_f(K) \geq \frac{m+n}{2} \). On the other hand \( t(K) = 1 \).

4. THE TUNNEL NUMBER AND THE HOMOLOGY WITH LOCAL COEFFICIENTS

Let \( L \) be a link in \( S^3 \), put \( m = t(L) \). As we have observed in the previous section there is a Morse function \( g : C_L \to \mathbb{R} \) such that \( g \) is constant on \( \partial C_L \) and takes there its minimal value, and with the Morse numbers as follows: \( m_0(g) = 0 \), \( m_1(g) = m \), \( m_2(g) = m + 1 \), \( m_3(g) = 1 \). The function \( -g \) provides a handle decomposition of the manifold \( C_L \), with \( (m + 1) \) one-handles, therefore we have the usual homological estimate \( m + 1 \geq \mu(Z(H_1(C_L, \mathbb{Z})) \). For the case of knots this estimate is trivial, however in some cases we can improve it using homology with local coefficients. Let \( \rho : \pi_1(C_L) \to GL(q, R) \) be a right representation (that is, \( \rho(ab) = \rho(b)\rho(a) \) for every \( a, b \)). Denote by \( \tilde{C}_L \) the universal covering of \( C_L \). The homology of the chain complex

\[
C_*(\tilde{C}_L, \rho) = R^q \otimes_{\rho} C_*(\tilde{C}_L)
\]

is called the homology with local coefficients \( \rho \) or \( \rho \)-twisted homology and denoted by \( H_*(C_L, \rho) \). If \( R \) is the principal ideal domain, then we have

\[
m + 1 \geq \frac{1}{q} \mu_R(H_1(C_L, \rho)) \tag{2}.
\]

In what follows we will concentrate on the case of knots. For a knot \( K \) consider a meridional embedding \( i : S^1 \to C_K \). Given a right representation \( \rho : \pi_1(C_K) \to GL(q, R) \) we can induce it to \( S^1 \) by \( i \) and obtain a local coefficient system \( i^* \rho \) on \( S^1 \).

The following proposition is an easy corollary of the main theorem of the paper of D. Silver and S. Williams [16].

\[\text{For a finitely generated } R\text{-module } T \text{ we denote by } \mu_R(T) \text{ the minimal number of generators of } T.\]
Proposition 4.1. Let \( K \) be any knot in \( S^3 \). Then there is a right representation \( \gamma : \pi_1(C_K) \to GL(q, R) \) with \( R \) a principal ideal domain such that

(i) \( H_1(C_K, \gamma) \neq 0 \).
(ii) \( H_1(S^1, i^*\gamma) = 0 \).

**Proof.** Let us first recall briefly the Silver-Williams theorem. Consider the meridional homomorphism \( \xi : \pi_1(C_K) \to \mathbb{Z} \) as a homomorphism of \( \pi_1(C_K) \) to \( \Lambda^\bullet = GL(1, \Lambda) \), where \( \Lambda = \mathbb{Z}[t, t^{-1}] \). For a right representation \( \theta : \pi_1(C_K) \to GL(q, \mathbb{Z}) \) form the tensor product \( \rho = \xi \otimes \theta : \pi_1(C_K) \to GL(q, \mathbb{Z}) \). Consider the \( \Lambda \)-module \( \mathcal{B} = H_1(C_K, \rho) \) and choose a free resolution for \( \mathcal{B} \):

\[
0 \leftarrow \mathcal{B} \leftarrow \Lambda^r \leftarrow \Lambda^k \leftarrow \cdots
\]

where \( k \geq r \). The GCD of the ideal of \( \Lambda \) generated by the \( r \times r \)-minors of \( \rho \) is called the *twisted Alexander polynomial* of \( k \) with respect to \( \rho \); we will denote it \( \Delta(K, \theta) \) (it is defined up to multiplication by \( \pm t^i \)). The Silver-Williams theorem says that for every \( K \) there is a representation \( \theta \) such that \( \Delta(K, \theta) \) is not a unit of \( \Lambda \). Pick such a representation \( \theta \) and consider two cases:

1) \( \Delta(K, \theta) \) is a monomial, that is \( \Delta(K, \theta) = at^n \) with \( a \in \mathbb{Z} \), \( a \neq \pm 1 \). In this case define the representation \( \gamma = \hat{\rho} \) to be the composition of \( \rho \) with the natural inclusion \( GL(q, \Lambda) \subset GL(q, \hat{\Lambda}) \). The \( \hat{\rho} \)-twisted homology is

\[
H_1(C_K, \hat{\rho}) = H_1(C_K, \rho) \otimes_{\Lambda} \hat{\Lambda}.
\]

We can obtain a free \( \hat{\Lambda} \)-resolution for this module by tensoring the resolution (3) by \( \hat{\Lambda} \) over \( \Lambda \). Since the GCD of elements of \( \Lambda \) remains the same when we extend the ring \( \Lambda \) to \( \hat{\Lambda} \) (see [12], Lemma 2.3), the GCD of the \( r \times r \)-minors of the matrix \( \hat{\rho} \) equals \( a \). Since \( \hat{\Lambda} \) is a principal ideal domain we deduce that \( H_1(C_K, \hat{\rho}) \) is non-zero and moreover it contains a cyclic direct summand. The property (ii) is easy to check.

2) \( \Delta(K, \theta) \) is a polynomial of non-zero degree. In this case consider the ring \( \Lambda_Q = \mathbb{Q}[t, t^{-1}] \). This ring is principal and \( \Delta(K, \theta) \) is not invertible in it; define the representation \( \gamma = \hat{\rho} \) to be the composition of \( \rho \) with the natural inclusion \( GL(q, \Lambda) \subset GL(q, \hat{\Lambda}) \). The property (ii) is easy to check.
Proposition 4.2. Let $K$ be any knot in $S^3$. Then there is $\lambda > 0$ such that for every $n \in \mathbb{N}$ we have $t(nK) \geq n\lambda - 1$.

Proof. Pick a representation $\gamma : \pi_1(C_K) \to GL(q, R)$ satisfying the conclusion of Proposition [4.1]. The module $H_1(C_K, \rho)$ contains then a cyclic $R$-submodule $T$.

Lemma 4.3. For any $n \geq 1$ there is a right representation $\gamma_n : \pi_1(C_{nK}) \to GL(q, R)$ such that the module $\mathcal{B}_n = H_1(C_{nK}, \gamma_n)$ contains a submodule isomorphic to $nT$.

Proof. We proceed by induction in $n$. Denote by $\mu \in \pi_1(C_K)$ the meridional element. Assume that we have constructed $\gamma_n : \pi_1(C_{nK}) \to GL(q, R)$ in such a way that $\gamma_n(\mu) = \gamma(\mu)$. The group $\pi_1(C_{(n+1)K})$ is isomorphic to the amalgamated product of the groups $\pi_1(C_K)$ and $\pi_1(C_{nK})$ over the subgroup $\mathbb{Z}$ included to both groups via the embedding of the meridian. Let $\gamma_{n+1} : \pi_1(C_{(n+1)K}) \to GL(q, R)$ be the product of the representations $\gamma$ and $\gamma_n$. Using the property (ii) from [4.1] and the Mayer-Vietoris exact sequence it is easy to deduce that the module $\mathcal{B}_{n+1} = H_1(C_{(n+1)K}, \gamma_{(n+1)})$ contains a submodule isomorphic to the direct sum of $T$ and $nT$.

The previous Lemma implies that $\mu_R(\mathcal{B}_n) \geq n$. Our proposition follows, since

$$t(nK) + 1 \geq \frac{1}{q} \mu_R(\mathcal{B}_n) \geq \frac{n}{q}.$$

Corollary 4.4. For any knot $K$ we have

$$gr_t(K) > -t(K).$$

5. Generalizations and a question

The generalization of the results of the Section [1.2] to the case of knots and links in an arbitrary closed 3-manifold are straightforward. The same goes for the formula [2]. On the other hand it is not clear at all whether the Proposition [4.1] and [4.2] admit such generalizations, since the analogs of Silver-Williams theorem for arbitrary three-manifolds seem to be out of reach for the moment.
**Question.** Are the inequalities [2] sufficient to determine the tunnel number for every link? In other words is it true that

\[ t(L) + 1 = \max_{\rho} \left( \frac{1}{q} \mu_R(H_1(C_L, \rho)) \right). \]

where \( \rho \) ranges over all right representations \( \pi_1(C_L) \to GL(q, R) \)?

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REFERENCES

[1] B. Clark, The Heegaard genus of manifolds obtained by surgery on links and knots, Internat. J. Math. and Math. Sci. Vol.3 No.3 (1980), 583-589.

[2] H. Doll, A generalized bridge number, Math. Ann. 294. 1992, pp. 701–717.

[3] H.Goda, A.Pajitnov, Twisted Novikov homology and circle-valued Morse theory for knots and links, e-print: math.GT/0312374, Journal Publication: Osaka Journal of Mathematics, v.42 No. 3, 2005, p. 557 – 572.

[4] H.Goda, H. Matsuda, T. Morifuji, Knot Floer Homology of (1, 1)-Knots, Geometriae Dedicata, Volume 112, Number 1, April 2005 , pp. 197-214.

[5] Jung Hoon Lee, An upper bound for tunnel number of a knot using free genus, Talk at 4th East Asian School of knots, http://faculty.ms.u-tokyo.ac.jp/~topology/EAS4slides/JungHoonLee.pdf

[6] M. Hirasawa, Lee Rudolph, Constructions of Morse maps for knots and links, and upper bounds on the Morse-Novikov number, math.GT/0311134, to appear in J. Knot Theory Ramifications.

[7] T. Kobayashi, A construction of arbitrarily high degeneration of tunnel number of knots under connected sum, J. Knot Theory Ramifications 3, (1994) p. 179-186.

[8] T. Kobayashi, Y. Rieck, On the growth rate of tunnel number of knots math.GT/0402025, J. Reine Angew. Math. 592 (2006) 63 – 78.

[9] K. Morimoto, On the additivity of tunnel number of knots, Topology Appl., 53, No. 3 (1993), p. 37–66.

[10] K. Morimoto, M. Sakuma, Y. Yokota, Examples of tunnel number one knots which have the property "1+1=3", Math. Proc. Camb. Phil. Soc., 119 (1996), p. 113–118.

[11] K. Morimoto, On the super additivity of tunnel number of knots, Math. Ann., 317, No. 3 (2000), p. 489-508
[12] A.V. Pajitnov, C. Weber, L. Rudolph, *Morse-Novikov number for knots and links*, Algebra i Analiz, 13, no.3 (2001), (in Russian), English translation: Sankt-Petersbourg Mathematical Journal, 13, no.3 (2002), p. 417 – 426.

[13] Lee Rudolph, *Murasugi sums of Morse maps to the circle, Morse-Novikov numbers, and free genus of knots*, math.GT/0108006.

[14] M. Scharlemann, J. Schultens, *Annuli in generalized Heegaard splitting and degeneration of tunnel number*, Math. Ann 317 (2000) No. 4, 783–820.

[15] H. Schubert, *Über eine numerische Knoteninvariante*, Math. Z. 61 (1954), 245–288.

[16] D. Silver, S. Williams, *Twisted Alexander polynomials detect the unknot*, Algebraic and Geometric Topology, 6 (2006), 1893-1907. arXiv:math/0604084v3 [math.GT]

LABORATOIRE MATHÉMATIQUES JEAN LERAY UMR 6629, UNIVERSITÉ DE NANTES, FACULTÉ DES SCIENCES, 2, RUE DE LA HOUSSINIÈRE, 44072, NANTES, CEDEX

E-mail address: pajitnov@math.univ-nantes.fr