Claire Burrin · Samantha Fairchild · with an appendix by Jon Chaika

**Pairs in discrete lattice orbits with applications to Veech surfaces**

**Abstract.** Let $\Lambda_1, \Lambda_2$ be two discrete orbits under the linear action of a lattice $\Gamma < \text{SL}_2(\mathbb{R})$ on the Euclidean plane. We prove a Siegel–Veech-type integral formula for the averages

$$\sum_{x \in \Lambda_1} \sum_{y \in \Lambda_2} f(x, y)$$

from which we derive new results for the set $S_M$ of holonomy vectors of saddle connections of a Veech surface $M$. This includes an effective count for generic Borel sets with respect to linear transformations, and upper bounds on the number of pairs in $S_M$ with bounded determinant and on the number of pairs in $S_M$ with bounded distance. This last estimate is used in the appendix to prove that for almost every $((\theta, \varphi) \in S^1 \times S^1$ the translation flows $F^f_\theta$ and $F^f_\varphi$ on any Veech surface $M$ are disjoint.

**Keywords.** translation surfaces, saddle connections, Siegel–Veech transform, nonuniform lattice

---

**1. Introduction**

Let $\Gamma \subset G = \text{SL}_2(\mathbb{R})$ be a lattice acting linearly on the Euclidean plane $\mathbb{R}^2$. Any orbit of $\Gamma$ is either a dense or discrete subset of $\mathbb{R}^2$. When the orbit is dense, a limiting distribution was computed by Ledrappier [31] and extended to more general lattices in locally compact groups by Gorodnik and Weiss [19]. When the orbit is discrete, the lattice must be nonuniform and the problem of understanding the distribution is a venerable one that goes back to Gauss.

---

Claire Burrin: University of Zurich Institute of Mathematics, Winterthurerstrasse 190, 8057 Zürich Switzerland; claire.burrin@math.uzh.ch

Samantha Fairchild: Max Planck Institute for Mathematics in the Sciences, Inselstraße 22, 04103 Leipzig Germany; samantha.fairchild@mis.mpg.de

with an appendix by Jon Chaika: University of Utah Department of Mathematics 203, 155 S 1400 E RM 233, Salt Lake City, UT, 84112-0990, USA; chaika@math.utah.edu

*Mathematics Subject Classification (2020):* 22E40, 37E35, 11F72
A more recent incentive to understand the distribution of discrete lattice orbits is motivated by the study of rational polygonal billiards and of the linear flow $F^t_\theta$ (with direction $\theta \in S^1$) on translation surfaces. A complicating aspect in the study of the translation flow is the presence of the saddle points — for the billiard flow, the future trajectory of a ball hitting a corner of the polygonal table is ill-defined. Corner-to-corner trajectories correspond to finite geodesics on the translation surface called saddle connections. The set $S_M$ of holonomy vectors of saddle connections on a translation surface $M$ is a discrete planar set in $\mathbb{R}^2$ that records the length and direction of each saddle connection. For a typical translation surface, the number $|S_M \cap B_R|$ of saddle connections of length $||x|| < R$ grows quadratically, with a growth asymptotic of the form

$$|S_M \cap B_R| = c_M R^2 + O \left(R^{2-\delta}\right)$$

(1.1)

[13, 35], where $\delta > 0$ is a nontrivial but nonexplicit power saving.

In the case of Veech surfaces\footnote{We recall that a translation surface is a Veech surface if the image in $\text{PSL}_2(\mathbb{R})$ of its stabilizer group under the action of $\text{SL}_2(\mathbb{R})$ on the moduli space of all translation surfaces is a lattice. Veech surfaces are sometimes called lattice surfaces.} the set $S_M$ of holonomy vectors of saddle connections is a finite disjoint union of discrete lattice orbits [40], which may then be accessed using a combination of tools and ideas from dynamics, ergodic theory, spectral theory, or metric geometry. Hence one can hope to say more, and in particular to specify results to individual surfaces. This is of interest considering for instance that the set of surfaces constructed from rational billiard polygons have null measure in the stratum. As such, results for typical surfaces do not yield new information on billiards in rational polygons.

Our main theorem is an explicit mean value formula for the Siegel–Veech transform of pairs of discrete lattice orbits in the plane. We postpone the statement of the main theorem to Section 1.2 to first describe some applications to the study of holonomies for Veech surfaces. The main novelty of this paper is to obtain estimates towards counting pairs of vectors in $S_M$. For this we build on tools from the geometry of numbers, the spectral theory of automorphic forms, the weak mixing of the translation flow for nonarithmetic Veech surfaces established by Avila and Delecroix [4], and previous works of the authors [6, 11, 14].

1.1. Applications

Let $M$ be a Veech surface, $\Gamma_M$ its Veech group, and $S_M$ the set of holonomy vectors of its saddle connections. The results we present here for $S_M$ are proven in the main text in the more general setting of discrete lattice orbits in the plane. References to those more general statements are indicated in Section 1.3.

1.1.1. Counting in Borel sets. In the case of Veech surfaces, we have

$$|S_M \cap B_R| = c_M R^2 + O \left(R^{2-\delta}\right).$$

(1.2)
where this time the power-saving \( \delta = \delta(\Gamma_M) \) is explicit,\(^2\) in contrast to (1.1). In fact the same asymptotic holds when replacing \( S_M \) by \( gS_M \) for any \( g \in G \) — this amounts to counting points in \( S_M \) that lie in an ellipsoid centered at the origin. For more general shapes it is typical of analytic approaches that a lack of regularity of the shape’s boundary leads to weaker power-savings; see [6, Theorems 2.6, 2.7] or Section 10 for examples. The following theorem recovers a (nearly) optimal count for typical shapes with respect to linear transformations.

**Theorem 1.1.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded Borel set that contains the origin, and consider its dilates \( \Omega_R = R \cdot \Omega \). Then for almost every\(^3\) linear transformation \( A \) we have

\[
|A(S_M) \cap \Omega_R| \leq c_M |\Omega|R^2 + O \left( R^{2-\delta} \log^{3/2}(R) \right),
\]

where \( \delta = \delta(\Gamma_M) \) is as in (1.2).

**Remark 1.2.** The restriction to dilates is cosmetic; see Theorem 10.3 for a counting asymptotic that holds for every element of a linearly ordered family of Borel sets in the plane.

### 1.1.2. Weak uniform discreteness

Recall that a discrete planar set is \( \eta \)-uniformly discrete if \( |\Lambda \cap B_\eta(x)| \leq 1 \) for all \( x \in \mathbb{R}^2 \). Answering a question of Barak Weiss, Wu showed that \( S_M \) is not uniformly discrete when \( M \) is a nonarithmetic Veech surface [43]; in other words there exists a pair of vectors that are arbitrarily close. On the other hand, uniform discreteness is easy to prove when \( M \) is an arithmetic Veech surface; see Proposition 11.3. The next theorem quantifies the failure of uniform discreteness when \( M \) is nonarithmetic.

**Theorem 1.3.** Let \( M \) be a Veech surface. For each \( \varepsilon > 0 \) there is an \( \eta > 0 \) such that

\[
\limsup_{R \to \infty} \frac{|\{x \in S_M \cap B_R : |S_M \cap B_\eta(x)| \geq 2\}|}{|S_M \cap B_R|} < \varepsilon.
\]

The theorem shows that upon discarding an \( \varepsilon \)-dense set of saddle connections, \( S_M \) is \( \eta \)-uniformly discrete for some \( \eta > 0 \).

### 1.1.3. Disjointness of flows

The translation flow \( F_\theta^t \) is uniquely ergodic [30] for Lebesgue almost every direction \( \theta \). On a typical translation surface (with respect to the Lebesgue measure class on a given stratum) of genus \( g \geq 2 \), the flow \( F_\theta^t \) is weakly mixing for almost every \( \theta \) [5] (but never strongly mixing [26]). For a typical surface the flows \( F_\theta^t \) and \( F_\psi^t \) are disjoint (for almost every pair \( (\theta, \psi) \) of directions) [11]. An immediate consequence of disjointness is that \( F_\theta^t \) and \( F_\psi^t \) are not isomorphic.

Regarding individual surfaces, the work of [11] can be applied to establish disjointness for branched covers of the torus in typical directions. So far, covers of tori were the only

\(^2\)We refer the reader to Section 4 for an explicit description of \( \delta \), which is determined by the bottom of the residual spectrum of \( \Gamma_M \). It is worth noting that there are known constructions of lattices \( \Gamma \) for which \( \delta \) can be arbitrarily close to 0 [38].

\(^3\)With respect to the Euclidean metric induced by the matrix representation of \( A \).
examples of individual surfaces for which disjointness is shown. In Appendix A, the third author presents a direct proof of the following theorem based on Theorem 1.3.

**Theorem 1.4.** Let $M$ be any Veech surface. Then for Lebesgue almost every pair $(\theta, \psi) \in S^1 \times S^1$, the translations flows $F_\theta$ and $F_\psi$ are disjoint.

This provides the first family of surfaces other than branched covers of tori for which the flows in almost every pair of directions are not isomorphic. If Theorem 1.4 could be extended to all surfaces, we would even be able to recover the result of Chaika and Forni that there is a weakly mixing billiard in a polygon [9].

### 1.1.4. Counting pairs with bounded determinant.

Theorem 1.3 follows from the following stronger density theorem that also accounts for multiplicities. Here we denote $B_\eta^*(x) = B_\eta(x) \setminus \{x\}$.

**Theorem 1.5.** Let $M$ be a Veech surface. There exists a constant $\Gamma$ such that

$$\left| \{(x, y) \in S_M \times S_M : x \in B_R, y \in B_\eta^*(x)\} \right| / |S_M \cap B_R| < C\eta^2.$$

The same arguments also recover an upper bound on pairs of saddle connections with bounded determinant. For a vector $x \in \mathbb{R}^2$, set

$$\mathcal{D}_{D,1}(x) = \{y \in \mathbb{R}^2 : |y| \leq |x| \text{ and } |x \wedge y| \leq D\}.$$

For a typical surface $M$, the second author with Athreya and Masur [2] showed that for $D > 0$ there is a non-explicit constant $\Gamma_D > 0$ so that

$$\lim_{R \to \infty} \frac{\left| \{(x, y) \in S_M \times S_M : x \in B_R, y \in \mathcal{D}_{D,1}(x)\} \right|}{R^2} = \Gamma_D.$$

**Theorem 1.6.** Let $M$ be a Veech surface. For any $D > 0$, there are constants $C_M$ and $c$ depending only on $M$ so that

$$\limsup_{R \to \infty} \frac{\left| \{(x, y) \in S_M \times S_M : x \in B_R, y \in \mathcal{D}_{D,1}(x)\} \right|}{R^2} \leq C_M(D + c).$$

Note we have two terms in the upper bound. This comes from the fact that when $D$ is small, there are essentially only parallel pairs, and thus a constant multiple of the Siegel–Veech constant $c_\Gamma$ will be dominating (cf. [2, Theorem 1.2]). However for $D$ large, we have an upper bound which is asymptotically linear in $D$.

### 1.1.5. Pair correlations.

In view of these results it is natural to consider the two-dimensional pair correlation function

$$R_2(B_s, S_M, R) := \frac{\left| \{(x, y) \in S_M \times S_M : x \in B_R, y \in B_\eta^*(x)\} \right|}{|S_M \cap B_R|} (s > 0)$$
with \( s > 0 \) and a rescaling determined by the mean spacing \( \frac{|S_M \cap B_R|}{|B_R|} \sim c_M \). The pair correlation is said to be Poissonian if \( R_2(B_s, S_M, R) \to |B_s| \) as \( R \to \infty \). Poissonian expresses that we see the same asymptotic behavior for the pair correlation function as if we were looking at points in the plane generated by a two-dimensional Poisson process.

Our last application shows the pair correlation function to be Poissonian on average. To make this precise we need some notation. Let \( \mathcal{A} \) denote the probability Haar measure on \( \mathcal{A}/\Gamma \) and let \( \mathcal{A} = \{ A = g^{1/2} : g \in G/\Gamma, \det(A) = v \in (0, 1] \} \)
eq 0 \) and a rescaling determined by the mean spacing \( \frac{|S_M \cap B_R|}{|B_R|} \sim c_M \). The pair correlation is said to be Poissonian if \( R_2(B_s, S_M, R) \to |B_s| \) as \( R \to \infty \). Poissonian expresses that we see the same asymptotic behavior for the pair correlation function as if we were looking at points in the plane generated by a two-dimensional Poisson process.

Our last application shows the pair correlation function to be Poissonian on average. To make this precise we need some notation. Let \( \mu \) denote the probability Haar measure on \( G/\Gamma \) and let \( C \) be the cone

\[
C = \{ A = g^{1/2} : g \in G/\Gamma, \det(A) = v \in (0, 1] \}
\]
eq 0 \) and a rescaling determined by the mean spacing \( \frac{|S_M \cap B_R|}{|B_R|} \sim c_M \). The pair correlation is said to be Poissonian if \( R_2(B_s, S_M, R) \to |B_s| \) as \( R \to \infty \). Poissonian expresses that we see the same asymptotic behavior for the pair correlation function as if we were looking at points in the plane generated by a two-dimensional Poisson process.

Our last application shows the pair correlation function to be Poissonian on average. To make this precise we need some notation. Let \( \mu \) denote the probability Haar measure on \( G/\Gamma \) and let \( C \) be the cone

\[
C = \{ A = g^{1/2} : g \in G/\Gamma, \det(A) = v \in (0, 1] \}
\]
eq 0 \) and a rescaling determined by the mean spacing \( \frac{|S_M \cap B_R|}{|B_R|} \sim c_M \). The pair correlation is said to be Poissonian if \( R_2(B_s, S_M, R) \to |B_s| \) as \( R \to \infty \). Poissonian expresses that we see the same asymptotic behavior for the pair correlation function as if we were looking at points in the plane generated by a two-dimensional Poisson process.

Our last application shows the pair correlation function to be Poissonian on average. To make this precise we need some notation. Let \( \mu \) denote the probability Haar measure on \( G/\Gamma \) and let \( C \) be the cone

\[
C = \{ A = g^{1/2} : g \in G/\Gamma, \det(A) = v \in (0, 1] \}
\]
eq 0 \) and a rescaling determined by the mean spacing \( \frac{|S_M \cap B_R|}{|B_R|} \sim c_M \). The pair correlation is said to be Poissonian if \( R_2(B_s, S_M, R) \to |B_s| \) as \( R \to \infty \). Poissonian expresses that we see the same asymptotic behavior for the pair correlation function as if we were looking at points in the plane generated by a two-dimensional Poisson process.

Our last application shows the pair correlation function to be Poissonian on average. To make this precise we need some notation. Let \( \mu \) denote the probability Haar measure on \( G/\Gamma \) and let \( C \) be the cone

\[
C = \{ A = g^{1/2} : g \in G/\Gamma, \det(A) = v \in (0, 1] \}
\]
eq 0 \) and a rescaling determined by the mean spacing \( \frac{|S_M \cap B_R|}{|B_R|} \sim c_M \). The pair correlation is said to be Poissonian if \( R_2(B_s, S_M, R) \to |B_s| \) as \( R \to \infty \). Poissonian expresses that we see the same asymptotic behavior for the pair correlation function as if we were looking at points in the plane generated by a two-dimensional Poisson process.

Our last application shows the pair correlation function to be Poissonian on average. To make this precise we need some notation. Let \( \mu \) denote the probability Haar measure on \( G/\Gamma \) and let \( C \) be the cone

\[
C = \{ A = g^{1/2} : g \in G/\Gamma, \det(A) = v \in (0, 1] \}
\]
To simplify our formulas, we assume that \( \Lambda_1, \Lambda_2 \) are scaled so that \( c_{\Lambda_i} = c_{\Gamma} \). The general statement can be recovered from the observation that if \( \Lambda'_i = \lambda_i \Lambda_i \), then \( \Theta_{\Lambda'_1, \Lambda'_2; f} = \Theta_{\Lambda_1, \Lambda_2; f \circ A} \), with \( f \circ \lambda(x, y) := f(\lambda_1 x, \lambda_2 y) \).

Let \( \mathcal{N} = \{ \det(x | y) : (x, y) \in \Lambda_1 \times \Lambda_2 \} \). Two ordered pairs \((x_1, y_1), (x_2, y_2) \in \Lambda_1 \times \Lambda_2\) are considered equivalent if \((x_1, y_1) = (\gamma x_2, \gamma y_2)\) for some \( \gamma \in \Gamma \) and we set \( \varphi(c) \) to be the number of equivalence classes of pairs \((x, y) \in \Lambda_1 \times \Lambda_2\) with \( \det(x | y) = c \). Our main theorem is

**Theorem 1.8.** Let \( \Lambda_1, \Lambda_2 \) be scaled discrete \( \Gamma \)-orbits, and let \( f \) be a bounded semicontinuous function of compact support on \( \mathbb{R}^2 \times \mathbb{R}^2 \). Then \( \Theta_{\Lambda_1, \Lambda_2; f} \in L^\infty(G/\Gamma) \) and

\[
\int_{G/\Gamma} \Theta_{\Lambda_1, \Lambda_2; f}(g) \, d\mu(g) = c_{\Gamma} \sum_{c \in \mathcal{N}^{-1}} \frac{\varphi(c)}{|c|} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(x, tx + c\mathbf{x}^*) \, dx \, dt
\]

\[
+ \delta_{\Lambda_1, \Lambda_2} c_{\Gamma} \int_{\mathbb{R}^2} (f(x, -x) + f(x, x)) \, dx,
\]

where \( \mathbf{x}^* \) is such that \( \det(x | \mathbf{x}^*) = 1 \) and \( \delta_{\Lambda_1, \Lambda_2} = 1 \) if \( \Lambda_1 \) and \( \Lambda_2 \) are homothetic and is 0 otherwise.

The right-hand side provides a complete description of the arising Siegel–Veech measures, answering a question of Athreya, Cheung, and Masur [1, Sections 4.4, 4.5.3].

The range of admissible test functions \( f \) includes characteristic functions of open and closed measurable sets. However the evaluation of such integrals is not straightforward. Inspired by a trick used by Schmidt in the geometry of numbers, we rewrite the mean value formula above in terms of the cone measure \( m \) introduced earlier.

**Theorem 1.8'.** With the same notation as above, we have

\[
\int_{\mathcal{C}} \Theta_{\Lambda_1, \Lambda_2; f}(A) \, dm(A) = c_{\Gamma}^2 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x, y) \, dxdy + \delta_{\Lambda_1, \Lambda_2} c_{\Gamma} \int_{\mathbb{R}^2} (f(x, -x) + f(x, x)) \, dx
\]

\[
+ c_{\Gamma} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \Psi(|x \wedge y|) f(x, y) \, dxdy,
\]

where \( |x \wedge y| = |\det(x | y)| \), and \( \Psi(t) = t \sum_{t \leq c \in \mathcal{N}^{-1}} \varphi(c)c^{-3} - c_{\Gamma} \).

The evaluation of the last double integral therefore depends on counting determinants of pairs of linearly independent vectors in \( \Lambda_1 \times \Lambda_2 \). Using the spectral theory of automorphic forms, we show that

**Theorem 1.9.** As \( t \to \infty \), \( \Psi(t) = O\left(t^{-\delta}\right) \), where \( \delta \) is as in Eq. (1.2).

Specializing to \( \Lambda_1 = \Lambda_2 = \Lambda \) and \( f(x, y) = h(x)h(y) \), we recover the second moment formula (where \( \Lambda \) does not need to be scaled).
Corollary 1.10.

\[ \int_C \left( \sum_{x \in \Lambda} h(Ax) \right)^2 dm(A) = \left( c_\Lambda \int_{\mathbb{R}^2} h(x)dx \right)^2 + c_\Lambda \int_{\mathbb{R}^2} (h(x)h(-x) + h(x)h(x))dx + c_\Lambda \iint_{\mathbb{R}^2 \times \mathbb{R}^2} (\Psi(|x \land y|)) h(x)h(y)dxdy. \]

1.3. Discussion of proofs and structure of the paper

Section 2 collects notation and initial definitions. In Section 3 we show that nonuniform lattices yield, up to homothety, only finitely many discrete orbits, in one-to-one correspondence to the set of inequivalent cusps for \( \Gamma \).

In Section 4 we show that counting problems for discrete lattice orbits can be restricted to scaled discrete orbits. We also recall the state of the art for counting results of discrete lattice orbits, and include a full proof of

\[ N_R(\Lambda) = c_\Lambda |B_R| + o(|B_R|^{1/2}) \]

when \( \Lambda \) is a discrete orbit of a congruence subgroup of \( SL_2(\mathbb{Z}) \); see Theorem 4.3. (A stronger power-saving can be achieved if we assume the Riemann hypothesis.)

In Section 5 we give a representation theorem for all higher moments of the Siegel–Veech transform over the space of semi-continuous functions. Section 6 reviews Veech’s proof of (1.3), filling out some details left to the reader in [41] that help explain our restriction from measurable to semi-continuous functions when working with higher moments.

Concluding the background part of this paper, we give in Section 7 a direct proof of Theorem 1.9 (known to follow from a more general result of Good [17]) based on the meromorphic continuation of the scattering matrix.

Our main Theorem 1.8 (mean value formulas for pairs) and its Corollary 1.10 (second moment formulas) are proved in Section 8. The strategy of proof follows Veech’s ergodic approach [41] and its extension by the second author to the second moment of the Siegel–Veech transform for Hecke triangle groups [14, Theorem 1.2]. The key realization is that the geometric totient function introduced in [14] is a special instance of the counting function for double cosets of lattices with respect to maximal parabolic subgroups.

We are now in position to deduce the applications advertised above. Section 9 collects needed geometric estimates (using the spectral asymptotic provided by Theorem 1.9) and concludes with the proof of Theorem 1.7 (Poissonian pair correlation on average).

Theorem 1.1 (effective counting) follows directly from Theorem 10.4, which is itself the application of the more general Theorem 10.3 to dilated Borel sets containing the origin. The latter theorem should be seen as the discrete lattice orbit analogue of Schmidt’s bound on the discrepancy function for (primitive) lattices in \( \mathbb{R}^2 \) [37, Theorem 2]. In fact our proof mimicks Schmidt’s with input the second moment formula from Corollary 1.10 and a geometric estimate from Section 9. This argument of Schmidt has recently been revisited in various contexts; see [3], [29], [36].
In Section 11, Theorem 1.5 and Theorem 1.6 (counting pairs in balls or of bounded determinant) are proved using the full force of the mean value formula for pairs and the well-roundedness of Euclidean balls. The paper concludes with the proof of Theorem 1.4 (disjointness of flows) in Appendix A, building on a criterion for disjointness developed in [11].

2. Basic definitions

2.1. Notation

Given a countable set $A$, we denote by $|A|$ the number of its elements. If $A$ is a Borel set in $\mathbb{R}^2$, we denote its characteristic function by $1_A$ and its Lebesgue measure by $|A|$ or $\lambda(A)$. We use $\| \cdot \|$ to denote the Euclidean norm $\|x\| = \sqrt{\sum x_i^2}$. For the disk $B_R(x) = \{ y \in \mathbb{R}^2 : \|x - y\| \leq R \}$, we have $|B_R(x)| = \pi R^2$ and write $B_R := B_R(0)$.

For two functions $f(x)$ and $g(x)$, we write $f \ll g$ or $f = O(g)$ to say that there exists a constant $C > 0$ such that for $x$ sufficiently large, $|f(x)| \leq C |g(x)|$.

We set $G = \text{SL}_2(\mathbb{R})$ and distinguish the subgroups

$$K = \left\{ k_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} : \theta \in \mathbb{R}/2\pi \mathbb{Z} \right\},$$

$$A = \left\{ a_y = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} : y > 0 \right\},$$

$$N = \left\{ n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}.$$

The Haar measure $\eta$ on $G$ can be written in coordinates as the Lebesgue measure restricted to the set

$$\mathcal{S} = \{(a, b, s) : a \neq 0, b, s \in \mathbb{R}\} \quad (2.1)$$

with the coordinate transformation

$$(a, b, s) \mapsto g(a, b, s) = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} a & b \\ as & bs + a^{-1} \end{pmatrix}.$$

Let $\Gamma < G$ be a discrete subgroup, with respect to the topology induced from $G \subset \mathbb{R}^4$. We will assume that $\Gamma < G$ is a lattice, that is, the quotient $G/\Gamma$ carries a finite $G$-invariant Borel measure, which when normalized to a probability measure we denote by $\mu$. The image of $\Gamma$ in $\text{PSL}_2(\mathbb{R})$ via the canonical projection is a Fuchsian group, which we also denote by $\Gamma$.

The action of $\Gamma$ by fractional linear transformation on the hyperbolic plane $\mathbb{H}^2$, i.e.,

$$(\begin{pmatrix} a & b \\ c & d \end{pmatrix})z = \frac{az + b}{cz + d} \quad (c \neq 0, z \in \mathbb{C})$$

is properly discontinuous and admits a finite-area fundamental domain in $\mathbb{H}^2$ whose (hyperbolic) area we denote by $V = \text{area}(\Gamma \backslash \mathbb{H}^2)$. The area of a geometrically finite Fuchsian group is a numerical invariant that does not depend on the particular choice of fundamental domain; we refer the reader to [27] for more properties of Fuchsian groups.
2.2. Scaling transformations

A lattice \( \Gamma < G \) is nonuniform if and only if the action of \( \Gamma \) on \( \mathbb{H}^2 \) has parabolic fixed points, called cusps. We say that two cusps \( a \) and \( b \) are equivalent if their \( \Gamma \)-orbits coincide, i.e., if \( \Gamma . a = \Gamma . b \). Since \( \Gamma \) is a lattice, there are at most finitely many inequivalent cusps. For each cusp \( a \), we set \( \Gamma_a = \{ \gamma \in \Gamma \mid \gamma . a = a \} \) and note that if \( a \) and \( b \) are equivalent cusps with \( b = \gamma . a \), then \( \Gamma_b = \gamma \Gamma_a \gamma^{-1} \).

Computations are most convenient at the cusp at \( \infty \); up to conjugation by some \( \sigma_a \in G \) such that \( \sigma_a . \infty = a \), we may assume that \( \Gamma \) has a cusp at \( \infty \) (up to replacing \( \Gamma \) by \( \sigma_a^{-1} \Gamma \sigma_a \)). If \( -I \notin \Gamma \), then \( \Gamma_\infty \) is isomorphic to \( \mathbb{Z} \) and generated by a matrix of the form \( \pm \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} \), where we call \( \omega > 0 \) the **cusp width**, while if \( -I \in \Gamma \), \( \Gamma_\infty \) is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z} \). The image of \( \Gamma_\infty \) in \( \text{PSL}_2(\mathbb{R}) \) is always isomorphic to \( \mathbb{Z} \). The following criterion is well known; see [34, Lemma 1.7.3].

**Lemma 2.1.** Let \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \). If \( |c| \omega < 1 \), then \( \gamma \in \Gamma_\infty \).

Let \( a \) be a cusp for \( \Gamma \) and choose \( \sigma_a \in G \) such that

\[
\sigma_a . \infty = a \quad \text{and} \quad \sigma_a^{-1} \Gamma_a \sigma_a = \begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix},
\]

that is, we fix a cusp at \( \infty \) whose width \( \omega \) is scaled to 1. We call \( \sigma_a \) a **scaling transformation**. Notice that the above conditions only determine \( \sigma_a \) up to right translation by elements \( n \in N \). One has the following convenient double coset decomposition, which can be seen as a local expression of the Bruhat decomposition; see [25, Theorem 2.7].

**Theorem 2.2.** Let \( a, b \) be cusps for \( \Gamma < \text{PSL}_2(\mathbb{R}) \). We have a disjoint union

\[
\sigma_a^{-1} \Gamma \sigma_b = \delta_{ab} \begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix} \cup \bigcup_{0 < a < c} \begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & * \\ c & * \end{pmatrix} \begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix},
\]

(2.2)

where \( \delta_{ab} = 1 \) if \( a, b \) are \( \Gamma \)-equivalent and \( \delta_{ab} = 0 \) otherwise, and \( a, c \) run over real numbers such that \( \sigma_a^{-1} \Gamma \sigma_b \) contains \( \begin{pmatrix} a & * \\ c & * \end{pmatrix} \).

3. A dichotomy for lattice orbits in the plane

Given a nonzero vector \( \mathbf{x} = \left( \begin{smallmatrix} x \\ y \end{smallmatrix} \right) \) in the Euclidean plane, we say that the **direction of \( \mathbf{x} \) is fixed by a parabolic motion** in \( G = \text{SL}_2(\mathbb{R}) \) if there is a parabolic element \( g \in G \) such that \( g . \frac{x}{y} = \frac{x}{y} \). We have the following (now classical) dichotomy; see [12, Theorem 3.2].

**Theorem 3.1.** Let \( \mathbf{x} \in \mathbb{R}^2 \setminus \{0\} \) and let \( \Gamma < G \) be a lattice acting linearly on the plane and containing \(-I\). The (linear) orbit \( \Gamma \mathbf{x} \) is either discrete or dense in \( \mathbb{R}^2 \). More precisely, \( \Gamma \mathbf{x} \) is discrete if and only if the direction of \( \mathbf{x} \) is fixed by a parabolic motion in \( \Gamma \).

In particular, it follows that if the underlying lattice is uniform, each lattice orbit is dense in the plane. The distribution of dense lattice orbits was studied by many authors;
see [18–20,28,31,33]. We henceforth assume that $\Gamma$ is nonuniform. Note that the existence of discrete orbits for $\Gamma$ still holds if $-I \not\in \Gamma$; if $\Gamma^{(\pm)}$ denotes the degree 2 central extension generated by $-I$ and $\Gamma$ then $\Gamma^{(\pm)}x = \Gamma x \cup -\Gamma x$ is either a disjoint union or collapses to $\Gamma x$ and the previous result applies. The following proposition characterizes further the discrete orbits that arise.

**Proposition 3.2.** Let $\Gamma < G$ be a nonuniform lattice acting linearly on the plane. Up to homothety, there are only finitely many discrete $\Gamma$-orbits, which are pairwise disjoint and in one-to-one correspondence with the finitely many inequivalent cusps of $\Gamma$.

To see this, we record the following preliminary lemma.

**Lemma 3.3.** For each $t \in \mathbb{R} \cup \{\infty\}$, set

$$x_t = \begin{cases} e_1 & \text{if } t = \infty, \\ \left( \begin{array}{c} 1 \\ t \end{array} \right) & \text{if } t \in \mathbb{R}. \end{cases}$$

Let $x = \left( \begin{array}{c} x \\ y \end{array} \right) \in \mathbb{R}^2 \setminus \{0\}$ and let $\Gamma < \text{SL}_2(\mathbb{R})$ be a lattice acting linearly on $\mathbb{R}^2$. For $\gamma \in \Gamma$ we set $\gamma t$ to be the image of $\gamma$ under the extension of Möbius transformations to the extended real line. Then:

1. There is a unique $t \in \mathbb{R} \cup \{\infty\}$ such that $x$ and $x_t$ are collinear;
2. For every $\gamma \in \Gamma$ and $t \in \mathbb{R} \cup \{\infty\}$, the vectors $\gamma x_t$ and $x_{\gamma t}$ are collinear;
3. For every $s, t \in \mathbb{R} \cup \{\infty\}$ and $\gamma \in \Gamma$, $\gamma s = t$ if and only if $x_{\gamma s}$ and $x_t$ are collinear.

**Proof of Lemma 3.3.** (1) Clearly, each vector $x$ is collinear to $x_t$ for $t = \frac{x}{y}$ if $y \neq 0$ and to $e_1$ otherwise. Moreover, if $x_t$ and $x_{t'}$ are collinear, then $s = t$.

(2) Let $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$ and $t \neq \infty$. If $ct + d \neq 0$, then $\gamma x_t$ and $x_{\gamma t}$ are collinear. If $ct + d = 0$, then $\gamma t = \infty$, and $\gamma x_t$ is collinear to $e_1 = x_{\gamma t}$. Suppose now that $t = \infty$. Then $\gamma e_1 = \left( \begin{array}{c} a \\ c \end{array} \right)$ while $x_{\gamma, \infty} = x_{a/c}$.

The “only if” part of (3) is immediate. For the converse, we know by (2) that $x_t$ is collinear to $\gamma x_s$. Suppose first that $s, t \neq \infty$: we have a vector identity of the form $\left( \begin{array}{c} 1 \\ s \end{array} \right) = \lambda \left( \begin{array}{c} a s + b \\ c s + d \end{array} \right)$ for some $\lambda \neq 0$. We deduce that $\gamma_s = t$. If instead, $s = t = \infty$, we want to show that $\gamma \in \Gamma_{\infty}$. This then follows from the vector identity $e_1 = \lambda \left( \begin{array}{c} a \\ c \end{array} \right)$; since $c = 0$, we have $\gamma \in \Gamma_{\infty}$. The remaining cases follow along the same lines. ■

**Proof of Proposition 3.2.** If the direction of $x = \left( \begin{array}{c} x \\ y \end{array} \right)$ is fixed by a parabolic element $\gamma \in \Gamma$, we have that $x$ is collinear to $x_a$, whereby $a = \frac{x}{y}$ is a cusp for $\Gamma$ and thus the lattice orbits $\Gamma x$ and $\Gamma x_a$ are homothetic. Moreover, if $a$ and $b$ are equivalent cusps, then by Lemma 3.3, the orbits $\Gamma x_a$ and $\Gamma x_b$ are homothetic. Hence up to homothety, there are only finitely many discrete $\Gamma$-orbits, in one-to-one correspondence with the finitely many inequivalent cusps of $\Gamma$.

Suppose that $x \in \Gamma x_a \cap \Gamma x_b$; say $x = \gamma_1 x_a = \gamma_2 x_b$. Then $x_a = \gamma_1^{-1} \gamma_2 x_b$ is collinear to $x_{\gamma_1^{-1} \gamma_2 b}$ and this implies that $a$ and $b$ are equivalent. ■
4. Counting results for discrete lattice orbits

Given a discrete orbit \( \Lambda = \Gamma x \) as above, we wish to understand its distribution in the plane, starting with a precise estimate for

\[
N_R(\Lambda) = |\Lambda \cap B_R|
\]
as \( R \to \infty \), where \( B_R = \{ x \in \mathbb{R}^2 : \|x\| < R \} \).

We will first choose preferred representatives in each homothety class of discrete \( \Gamma \)-orbits. Then, up to computing a scaling factor, it suffices to restrict the counting problem to what we will call scaled discrete lattice orbits. Let \( a_1, \ldots, a_h \in \mathbb{R} \cup \{ \infty \} \) be a set of representatives for the inequivalent cusps of \( \Gamma \). Then for each cusp representative \( a_i \), we set

\[
\Lambda_{a_i} := \Gamma \sigma_{a_i} e_1, \tag{4.1}
\]

where \( \sigma_{a_i} \) is a scaling transformation for the cusp \( a_i \) as defined in Section 2.2. By Proposition 3.2, each discrete \( \Gamma \)-orbit is homothetic to one of \( \Lambda_{a_1}, \ldots, \Lambda_{a_h} \). We call \( \Lambda_a \) as in (4.1) the scaled \( \Gamma \)-orbit attached to \( a \). Observe that \( \Lambda_a \) does not depend on the particular choice of scaling \( \sigma_a \) or of cusp representative. Now if \( \Lambda = \lambda \Lambda_a \), the counting function scales by \( N_R(\Lambda) = N_{R/\lambda}(\Lambda_a) \). The next proposition provides an algebraic formula to compute the scaling \( \lambda \). (For a geometric interpretation of \( \lambda \), see [21, Theorem 6.1].)

**Proposition 4.1.** Given a discrete orbit \( \Gamma x \) associated to cusp \( a \) of \( \Gamma \), we have \( \Gamma x = \lambda \Lambda_a \) with

\[
\lambda = \frac{\|x\|}{\sqrt{\text{tr}(S\gamma_a)}},
\]

where \( S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) and \( \gamma_a \) denotes the cyclic generator of \( \Gamma_a \).

**Proof.** There exists some \( g \in G \) such that \( x = ge_1 \). Then \( g^{-1}\gamma_ag = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} \) for some \( \omega > 0 \). We may choose \( \sigma_a = ga\sqrt{\omega} \) as scaling transformation; it then follows that \( \Gamma x = \frac{1}{\sqrt{\omega}}\Lambda_a \).

It remains to compute the cusp width \( \omega \). Write \( x = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \). Then \( g = \begin{pmatrix} \xi & \eta \\ -\xi & \eta \end{pmatrix} \) and a direct computation shows that \( \gamma_a = g \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} g^{-1} = \begin{pmatrix} \xi & \eta \\ -\eta \omega & \xi \eta \omega \end{pmatrix} \). \( S\gamma_a = \begin{pmatrix} \eta & \xi \omega \\ -\eta \omega & \xi \omega \end{pmatrix} \) and thus \( \omega = \frac{\text{tr}(S\gamma_a)}{\|x\|^2} \). \( \blacksquare \)

The current best counting result for \( N_R(\Lambda) \) is given by the following theorem.

**Theorem 4.2.** [6, Theorem 4.1] Let \( \Lambda \) be a scaled discrete \( \Gamma \)-orbit, and let \( g \in G \). There are numbers \( 1 = \rho_0 > \rho_1 > \cdots > \rho_k > 1/2 \) and constants \( c_1, \ldots, c_k \) such that we have the asymptotic expansion

\[
N_R(g\Lambda) = |g\Lambda \cap B_R| = c_1|B_R|^{\rho_1} + c_2|B_R|^{\rho_2} + \cdots + c_k|B_R|^{\rho_k} + O\left(|B_R|^{2/3}\right), \tag{4.2}
\]
where the leading constant is given by
\[
    c_\Gamma = \begin{cases} 
        \frac{2}{\pi V} & \text{if } -I \in \Gamma, \\
        \frac{1}{\pi V} & \text{if } -I \notin \Gamma.
    \end{cases} \tag{4.3}
\]

The constants \( \rho_i \in (\frac{1}{3}, 1) \) are eigenparameters of the finitely many possible small residual eigenvalues of the Laplacian, with \( \lambda_i = \rho_i(1 - \rho_i) \in (0, \frac{1}{3}) \), where \( \lambda_i \) belongs to the residual spectrum. Set
\[
    \delta = \min\{2(1 - \rho_1), 2/3\}. \tag{4.4}
\]

Then \( N_R(g\Lambda) = c_\Gamma \pi R^2 + O(R^{2-\delta}) \). There are known constructions of nonuniform lattices for which \( \rho_1 \) is arbitrarily close to 1 [38].

The small eigenvalues \( \lambda_i \), when they arise, are not explicit. It is known that \( SL_2(\mathbb{Z}) \) and its congruence subgroups have no small residual eigenvalues [25] and this is also true for triangle groups [22, Page 583]. In such cases (4.2) reduces to \( N_R(g\Lambda) = c_\Gamma |B_R| + O(|B_R|^{2/3}) \), where the exponent 2/3 is an artefact of the method of proof. For specific lattices, a better result is nonetheless possible. This is easy to see for congruence subgroups of \( SL_2(\mathbb{Z}) \). For the convenience of the reader, we include a full proof here.

**Theorem 4.3.** Let \( \Gamma \) be a congruence subgroup of \( SL_2(\mathbb{Z}) \), and let \( \Lambda \) be a scaled discrete \( \Gamma \)-orbit. Then
\[
    N_R(\Lambda) = c_\Gamma |B_R| + o(|B_R|^{1/2}).
\]
Assuming the Riemann hypothesis (RH), the error term \( o(|B_R|^{1/2}) \) can be replaced by \( O(|B_R|^{5/12+\varepsilon}) \).

**Proof.** Recall that \( \Gamma \) is a congruence subgroup of \( SL_2(\mathbb{Z}) \) if it contains a principal congruence subgroup \( \Gamma(N) = \ker(SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z})) \) for some \( N \geq 1 \). We have a finite disjoint decomposition \( \Gamma = \bigcup \Gamma(N) \) for a choice of coset representatives \( \tau \in \Gamma/\Gamma(N) \). This implies
\[
    N_R(\Gamma x) = \sum_{\tau} N_R(\tau \Gamma(N) x).
\]

We will consider each summand separately.

Since \( \Lambda \) is assumed to be scaled, we have \( x = \sigma_a e_1 \) for some choice of scaling transformation \( \sigma_a \). Each cusp of \( \Gamma \) is a cusp of \( \Gamma(1) = SL_2(\mathbb{Z}) \) and hence we may choose \( \sigma \in \Gamma(1) \) such that \( \sigma(\infty) = a \) and set \( \sigma_a = \sigma a_{\sqrt{\omega}} \), where \( \omega \) is the cusp width of \( \sigma^{-1} \Gamma \sigma \). Then
\[
    N_R(\tau \Gamma(N) x) = N_R(\tau \Gamma(N) \sigma a_{\sqrt{\omega}} e_1) = N_R(\tau \sigma \Gamma(N) a_{\sqrt{\omega}} e_1),
\]
where we used that \( \Gamma(N) \) is a normal subgroup of \( \Gamma(1) \). For readability we write \( A = \tau \sigma \).

Then
\[
    N_R(A \Gamma(N) a_{\sqrt{\omega}} e_1) = |\{ \gamma \in \Gamma(N)/a_{\sqrt{\omega}} \Gamma(N) a_{\sqrt{\omega}}^{-1} : \gamma e_1 \in \omega^{-1/2} A^{-1} B_R \}|, \tag{4.5}
\]

\[
    = \frac{\omega}{N} |\{ \xi \in Z_{\text{prim}}^2 : \xi \equiv e_1 \pmod{N}, \xi \in \omega^{-1/2} A^{-1} B_R \}|.
\]
Here $\mathbb{Z}_{\text{prim}}^2$ denotes the set of all vectors $\xi = \left( \frac{m}{n} \right)$ with coprime integer coordinates $(m, n) = 1$, and $\omega^{-1/2} A^{-1} B_R$ is an ellipsoid centered at the origin with area $\frac{\pi}{\omega} R^2$.

We have

$$
\frac{N}{\omega} \cdot (4.5) = \sum_{(m, n) \in \mathbb{Z}_{\text{prim}}^2, \frac{m}{n} (N)} \mathbf{1}_{\omega^{-1/2} A^{-1} B_R \left( \left( \frac{m}{n} \right) \right)} = \sum_{\substack{d \geq 1 \\ (d, N) = 1}} \mu(d) \sum_{(m, n) \in \mathbb{Z}_{\text{prim}}^2, \frac{m}{n} (N), d | m, d | m N | n} \mathbf{1}_{\omega^{-1/2} A^{-1} B_R \left( \left( \frac{m}{n} \right) \right)}.
$$

where $\mu(n)$ is the Möbius function and we used that $\sum_{d | (m, n)} \mu(d) = 1$ iff $(m, n) = 1$ (and is 0 otherwise). By the Chinese remainder theorem, the system of congruence equations for $m$ has a unique solution $m_0$ in $[0, dN)$ and thus

$$
\frac{N}{\omega} \cdot (4.5) = \sum_{\substack{d \geq 1 \\ (d, N) = 1}} \mu(d) \sum_{\substack{\xi \in \mathbb{Z}^2 \\ \xi \equiv \left( \frac{m_0}{0} \right) (dN)}} \mathbf{1}_{\omega^{-1/2} A^{-1} B_R \left( \left( \frac{m_0}{0} \right) \right)} = \sum_{\substack{d \geq 1 \\ (d, N) = 1}} \mu(d) \sum_{\xi \in \mathbb{Z}^2} \mathbf{1}_{\frac{1}{dN} \left( \omega^{-1/2} A^{-1} B_R - m_0 e_1 \right) \left( \frac{\xi}{dN} \right)}.
$$

For $R$ sufficiently large, $\Omega = \frac{1}{dN} \left( \omega^{-1/2} A^{-1} B_R - m_0 e_1 \right)$ is a compact convex planar set of area

$$
|\Omega| = \frac{|B_R|}{\omega (dN)^2}
$$

that contains the origin as an inner point. A simple geometric argument shows that

$$
\sum_{\xi \in \mathbb{Z}^2} \mathbf{1}_{\Omega} (\xi) = \frac{|B_R|}{\omega (dN)^2} + O_{\omega, N} \left( \frac{R}{d} \right) .
$$

(4.6)

Using available estimates towards the Gauss circle problem in convex planar domains, the error term can further be replaced by $O(R^{46/73} \log^{315/146} (R))$ [23, Theorem 5]. Let $\rho > 1/2$. To summarize we have, formally,

$$
(4.5) = \frac{1}{N^3} \sum_{(d, N) = 1} \frac{\mu(d)}{d^2} |B_R| + O \left( \sum_{d \geq 1} \frac{\mu(d)}{d^\rho} |B_R|^{\rho/2} \right)
$$

with convergence guaranteed when $\rho > 1$. Via the prime number theorem, the error term can be replaced with $o(|B_R|^{1/2})$ and under RH by $O(|B_R|^{5/12 + \varepsilon})$; see [24]. We conclude that

$$
N_R (\Lambda) = \left[ \Gamma : \Gamma (N) \right] \sum_{(d, N) = 1} \frac{\mu(d)}{d^2} |B_R| + o (|B_R|^{1/2})
$$

(unconditionally on RH or with $O(|B_R|^{5/12 + \varepsilon})$ under RH). The leading constant can be shown to equal $c_\Gamma$; see [34, Theorem 4.2.5].

\[ \square \]
5. Theta transforms

Let \( k > 0 \) and let \( \Lambda_1, \ldots, \Lambda_k \) be \( k \) discrete \( \Gamma \)-orbits. Let \( f \) be a Borel measurable function on \( (\mathbb{R}^2)^k = \mathbb{R}^2 \times \mathbb{R}^2 \times \cdots \times \mathbb{R}^2 \). We define the corresponding theta/Siegel–Veech transform to be

\[
\Theta = \Theta_{\Lambda_1, \ldots, \Lambda_k} : \mathbb{R}^k \to \mathbb{R}, \quad \Theta(g) = \sum_{(x_1, \ldots, x_k) \in \Lambda_1 \times \cdots \times \Lambda_k} f(gx_1, \ldots, gx_k).
\]

**Lemma 5.1.** If \( f \) is bounded with compact support, then \( \Theta \in L^\infty(G/\Gamma) \).

**Proof.** Since \( f \) is bounded and has compact support, for all \( (x_1, \ldots, x_k) \in (\mathbb{R}^2)^k \), and for some constant \( c_f, R > 0 \), \( f(x_1, \ldots, x_k) \leq c_f 1_{B_R}(x_1) \cdots 1_{B_R}(x_k) \). By [41, Lemma 16.10], since \( 1_{B_R} \) is bounded with compact support, \( \Theta_{\Lambda_i, 1_{B_R}} \) is uniformly bounded over \( G/\Gamma \). Thus since \( \mu \) is a probability measure, we conclude that \( \Theta_{\Lambda_i, 1_{B_R}} \in L^\infty(G/\Gamma) \). We then notice that \( \Theta_{\Lambda_1, \ldots, \Lambda_k} : f \leq c_f \prod_{i=1}^k \Theta_{\Lambda_i, 1_{B_R}} \in L^\infty(G/\Gamma) \) since \( G/\Gamma \) is a probability space. \( \blacksquare \)

Recall that a function \( f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\} \) is lower (resp. upper) semi-continuous at \( x_0 \) if \( \liminf_{x \to x_0} f(x) \geq f(x_0) \) respectively \( \limsup_{x \to x_0} f(x) \leq f(x_0) \). Characteristic functions of open (resp. closed) sets are lower (resp. upper) semi-continuous. Semi-continuous functions behave nicely with respect to monotone approximation; see Baire’s theorem or Remark 5.4 below.

**Definition 5.2.** We denote \( SC((\mathbb{R}^2)^k) \) to be the space of all lower semi-continuous functions bounded below, and all upper semi-continuous functions bounded above.

**Lemma 5.3.** There exists a unit regular \( G \)-invariant Borel measure \( \nu \) on \( (\mathbb{R}^2)^k \) such that for any \( f \in SC((\mathbb{R}^2)^k) \), we have

\[
\int_{G/\Gamma} \Theta_{\Lambda_1, \ldots, \Lambda_k} : f(g) d\mu(g) = \int_{(\mathbb{R}^2)^k} f(x) \, d\nu(x).
\]

**Proof.** By Lemma 5.1, for \( f \in C_c((\mathbb{R}^2)^k) \), we have \( \Theta_{\Lambda_1, \ldots, \Lambda_k} : f \in L^\infty(G/\Gamma) \). Thus the assignment \( f \mapsto \int_{G/\Gamma} \Theta_{\Lambda_1, \ldots, \Lambda_k} : f(g) d\mu(g) \) defines a \( G \)-invariant positive linear functional on \( C_c((\mathbb{R}^2)^k) \) where \( G \) acts diagonally on \( (\mathbb{R}^2)^k \). Hence by the Riesz–Markov–Kakutani representation theorem, there is a unit regular \( G \)-invariant Borel measure \( \nu \) on \( \mathbb{R}^2 \) such that Eq. (5.1) holds for \( f \in C_c((\mathbb{R}^2)^k) \).

Every lower semi-continuous function \( f \) bounded below can be approximated by a non-decreasing sequence \( (f_j) \subset C_c((\mathbb{R}^2)^k) \) that converges pointwise to \( f \). Since \( f \) is bounded from below, it suffices to consider non-negative semi-continuous functions. By Baire’s thereom, \( \bar{f} \) is the pointwise limit of continuous functions \( h_k \) which are monotonely non-increasing. Choosing a continuous bump function \( \tilde{h}_k \) to be 1 on \( B(0, k) \) and 0 outside \( B(0, k + 1) \), the sequence \( h_k \tilde{h}_k \) is the desired sequence of continuous functions of compact support. If \( f \) is upper-semicontinuous and bounded above, then \( -f \) is lower semi-continuous and bounded below, reducing to case of lower semi-continuous functions.
Moreover pointwise monotone convergence implies $\Theta_{\Lambda_1,\ldots,\Lambda_k;f}$ also monotonically increases to $\Theta_{\Lambda_1,\ldots,\Lambda_k;f}$ pointwise for each $g \in G/\Gamma$. We can then apply the monotone convergence theorem. We emphasize here that we use pointwise convergence (in $\mathbb{R}^2$) in order to automatically guarantee pointwise convergence (in $G/\Gamma$) under the transformation $f \mapsto \Theta_{\Lambda_1,\ldots,\Lambda_k;f}$.

Remark 5.4. The proof makes use of the fact that the dual of functions $f \in C_c((\mathbb{R}^2)^k)$ which are continuous with compact support are exactly the Radon measures on $(\mathbb{R}^2)^k$. It is not straightforward to extend this representation from $f \in C_c((\mathbb{R}^2)^k)$ to all integrable Borel functions. For example, if $A = \{x : 1 \leq |x| < 2\}$, the characteristic function $1_A$ cannot be pointwise approximated from above or below by sequences of continuous functions. But lower (or upper) semicontinuous functions precisely have this property.

6. Veech’s mean value theorem

For simplicity of exposition, we will from here on state our results for scaled discrete lattice orbits. Recall that each discrete $\Gamma$-orbit $\Lambda$ is homothetic to some scaled orbit $\Lambda_\alpha$; explicitly, we write $\Lambda_i = \alpha_i \Lambda_\alpha$ and set $\lambda = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k$. Then

$$\Theta_{\Lambda_1,\ldots,\Lambda_k;f} = \Theta_{\Lambda_\alpha_1,\ldots,\Lambda_\alpha_k;f \circ \lambda},$$

where $f \circ \lambda(x_1, \ldots, x_k) = f(\alpha_1 x_1, \ldots, \alpha_k x_k)$.

Let $\Lambda$ be a scaled discrete $\Gamma$-orbit. The following mean value theorem, extending the classical theorem of Siegel [39] in the geometry of numbers, is due to Veech [41].

Theorem 6.1. [41, Theorem 6.5] Let $\Lambda$ be a scaled discrete $\Gamma$-orbit. For each integrable Borel function $f : \mathbb{R}^2 \to \mathbb{R}$, the Siegel–Veech transform $\Theta_{\Lambda;f}$ satisfies

$$\int_{G/\Gamma} \Theta_{\Lambda;f}(g) \, d\mu(g) = c_\Gamma \int_{\mathbb{R}^2} f(x) \, dx,$$

with $c_\Gamma$ as defined in Eq. (4.3).

Remark 6.2. For a general (i.e., not scaled) discrete $\Gamma$-orbit $\Lambda$, the constant $c_\Gamma$ is simply replaced by $c_\Gamma/\lambda^2$ with $\lambda$ as defined in Proposition 4.1.

For completeness, we include a proof of Veech’s theorem. The statement is fairly immediate for semicontinuous functions, but the extension to integrable Borel functions stated by Veech actually requires the following technical lemma. Here $\lambda_n$ denotes the Lebesgue measure on $\mathbb{R}^n$.

Lemma 6.3. If $\{f_k\}_{k=1}^\infty$, $f$ are Borel functions in $L^1(\mathbb{R}^2, \lambda_2)$ with $f_k(y) \to f(y)$ for $\lambda$-a.e. $y$, then $\Theta_{\Lambda;f_k}(g) \to \Theta_{\Lambda;f}(g)$ for $\mu$-a.e. $g \in G/\Gamma$.

Proof. Recall the definition (2.1) of the set $\mathcal{S}$ and let $\mathcal{Y} \subset \mathcal{S}$ correspond to $\mathcal{Y} = \{g \in G : \Theta_{\Lambda;f_k}(g) \not\to \Theta_{\Lambda;f}(g)\}$ up to a set of measure zero. Let $Y = \{y \in \mathbb{R}^2 : f_k(y) \not\to f(y)\}$. The
set $\mathcal{X}^k$ is contained in

$$\bigcup_{x \in \Lambda} \{g(a, b, s)x \in Y\} = \bigcup_{x \in \Lambda} \bigcup_{k \geq 1} \{g(a, b, s)x \in Y : a \in [-2^k, -2^{-k}] \cup [2^{-k}, 2^k], (b, s) \in [-2^k, 2^k]^2\}.$$ 

We denote the latter sets by $\mathcal{X}_x^k \subseteq \mathbb{R}^3$. Fix $x \in \mathbb{R}^2 \setminus \{0\}$, and $k \in \mathbb{N}$. We will show that

$$\lambda_3(\mathcal{X}_x^k) = 0.$$ 

By the coarea formula for Lipschitz functions \cite{15} and our hypothesis $\lambda_2(Y) = 0$, for $J$ the Jacobian of the map $(a, b, c) \mapsto g(a, b, c)$ and $\mathcal{H}^1$ the Hausdorff 1-measure, we have

$$\int_{\mathcal{X}_x^k} J(g(a, b, s)x) d\lambda_3(a, b, s) = \int_Y \mathcal{H}^1(\{a, b, s) : g(a, b, s)x = y\}) d\lambda_2(y) = 0.$$

Hence $\lambda_3(\mathcal{X}_x^k) = 0$ if and only if $\{J(gx) = 0\} \cap \mathcal{X}_x^k$ has measure zero. In fact, we can directly compute that

$$D(g(\cdot, \cdot, \cdot)x)(a, b, s) = \begin{pmatrix} x_1 & x_2 & 0 \\ sx_1 - a^{-2}x_2 & sx_2 & ax_1 + bx_2 \end{pmatrix}. $$

Since $x \neq 0$, and $a \neq 0$, we in fact have $D(gx)$ has rank 2, so $D(gx)[D(gx)]^T$ has full rank and hence the Jacobian $(J(gx))^2 = \det(D(gx)[D(gx)]^T) \neq 0$.

We have now shown $\lambda_3(\mathcal{X}_x^k) = 0$, which implies that $\eta(\mathcal{X}) = 0$. To conclude the proof, we note that since $\mu$ is the probability measure when $\eta$ is restricted to a finite volume fundamental domain associated to $G/\Gamma$, we in fact have convergence $\Theta_{\Lambda; f_k}(g) \to \Theta_{\Lambda; f}(g)$ for $\mu$-a.e. $g \in G/\Gamma$.

**Proof of Theorem 6.1.** Let $f \in SC(\mathbb{R}^2)$. Then Lemma 5.3 applies, and we seek to determine $\nu$. The plane $\mathbb{R}^2$ decomposes into the disjoint $G$-orbits $\{0\}$ and $(\mathbb{R}^2 \setminus \{0\})$ and hence the only $G$-invariant measures on $\mathbb{R}^2$ (up to scaling) are $\delta_0$, the point mass at the origin, and the Lebesgue measure $\lambda$. Since the Haar measure on $G$ is unique up to scaling, we conclude that $\nu = a\delta_0 + \beta\lambda$ for some real constants $a, \beta \geq 0$.

We first show that $a = 0$. Indeed since $f = \mathbf{1}_0$ is a characteristic function of a closed set, then $\Lambda \subset \mathbb{R}^2 \setminus \{0\}$ implies that $a = 0$.

Let $f$ be the characteristic function of the open disk of radius $R$ centered at the origin, and recall that characteristic functions of open sets are lower semi-continuous. Hence for $R > 0$, we are left with

$$\beta = \int_{G/\Gamma} \frac{|g\Lambda \cap B_R|}{|B_R|} d\mu(g).$$

Letting $R \to \infty$, we conclude with Theorem 4.2 that $\beta = c_\Gamma$.

To extend the statement from semicontinuous functions to integrable Borel functions, we first consider the class of bounded Borel functions with compact support. With Lemma 6.3
in hand, we can use the dominated convergence theorem (where pointwise almost everywhere convergence is needed on both the left and right hand side of (6.1)) to bootstrap our way to any integrable Borel function. By Lemma 5.1 and Lemma 6.3, we apply dominated convergence on both sides of Eq. (6.1) to a sequence of semi-continuous functions converging a.e. to any bounded Borel function with compact support.

We now extend to all nonnegative Borel functions. Consider \( A_k = \{ |x| \leq 2^k \} \cap \{ f(x) \leq 2^k \} \) and define \( f_k(x) := f(x)1_{A_k}(x) \). Monotone convergence implies that

\[
\lim_{k \to \infty} \int_{\mathbb{R}^2} f_k(x) \, dx = \int_{\mathbb{R}^2} f(x) \, dx
\]

(even when both sides are infinite). Each \( f_k(x) \) is a bounded Borel function with compact support, hence \( L^1 \) so that (6.1) holds for each \( k \). Clearly \( f_k \leq f_{k+1} \) implies that \( \Theta_{\Lambda; f_k} \leq \Theta_{\Lambda; f_{k+1}} \) for each \( k \). Hence, by Lemma 6.3 and monotone convergence, we have

\[
\int_{G/\Gamma} \Theta_{\Lambda; f}(g) \, d\mu(g) = \lim_{k \to \infty} \int_{G/\Gamma} \Theta_{\Lambda; f_k}(g) \, d\mu(g) = c_\Gamma \lim_{k \to \infty} \int_{\mathbb{R}^2} f_k(x) \, dx = c_\Gamma \int_{\mathbb{R}^2} f(x) \, dx.
\]

Finally for any integrable Borel function \( f \), we decompose \( f \) into its positive and negative parts to complete the proof of Theorem 6.1.

7. Admissible determinants

Our goal is to extend Theorem 6.1 to an integral formula for pairs of discrete lattice orbits. Similarly to the proof of Veech, our formula will build on the decomposition of the space \( \mathbb{R}^2 \times \mathbb{R}^2 \) into the union of disjoint \( G \)-orbits given by

\[
\mathbb{R}^2 \times \mathbb{R}^2 = \{0\} \cup (\{0\} \times \mathbb{R}^2) \cup (\mathbb{R}^2 \times \{0\}) \cup \{(x, y) : \det(x \mid y) = 0\} \cup \{(x, y) : \det(x \mid y) \neq 0\} = G(0, 0) \cup G(0, e_1) \cup G(e_1, 0) \cup \{G(e_1, te_1) : t \in \mathbb{R}\} \cup \{G(e_1, ce_2) : c \in \mathbb{R}^*\}.
\]

where we write \( \det(x \mid y) \) to be the determinant of the matrix with ordered columns \( x, y \). In particular, we will need to understand the set of determinants \( \det(x \mid y) \) arising from ordered pairs \( (x, y) \in \Lambda_1 \times \Lambda_2 \). We record necessary preliminary results in this section. The mean value formula for pairs of discrete lattice orbits will be derived in the next section.

In this section, we fix two (not necessarily distinct) scaled discrete lattice orbits \( \Lambda_a, \Lambda_b \). Let

\[
\mathcal{N}_{ab} = \{ c \in \mathbb{R} : \text{there are } x \in \Lambda_a, y \in \Lambda_b \text{ such that } \det(x \mid y) = c \}
\]

be the set of admissible determinants for vector pairs in \( \Lambda_a \times \Lambda_b \). Under the assumption that \(-I \in \Gamma\), we have \( c \in \mathcal{N}_{ab} \) if and only if \(-c \in \mathcal{N}_{ab} \).

**Lemma 7.1.** The following statements are equivalent:

1. \( 0 \in \mathcal{N}_{ab} \);
2. \( a \) and \( b \) are equivalent cusps;
(3) $\Lambda_a = \Lambda_b$.

**Proof.** If $0 \in \mathcal{N}_{ab}$, there is $\gamma \in \Gamma$ such that $\gamma x_a$ and $x_b$ are collinear, and hence by Lemma 3.3, the cusps $a$ and $b$ are equivalent. Then $\Lambda_a = \Lambda_b$ (we may take $\sigma_b = \gamma \sigma_a$) and this in turns implies immediately that $0 \in \mathcal{N}_{ab}$. ■

We introduce an equivalence relation on the set of ordered pairs $(x, y) \in \Lambda_a \times \Lambda_b$ by declaring $(x_1, y_1) \sim (x_2, y_2)$ iff $(x_1, y_1) = (\gamma x_1, \gamma y_2)$ for some $\gamma \in \Gamma$. The determinant $\det(x \mid y)$ is preserved by the equivalence relation. We set

$$\varphi_{ab}(c) = |\{(x, y) \in \Lambda_a \times \Lambda_b : \det(x \mid y) = c\}|$$

to be the counting function of equivalence pairs with determinant $c$.

**Lemma 7.2.** Suppose that $-I \in \Gamma$ and let $\mathcal{N}_{ab}^* := \mathcal{N}_{ab} \setminus \{0\}$. Then $c \in \mathcal{N}_{ab}^*$ if and only if there exists a vector $(a \ b) \in (\sigma_a^{-1} \Gamma \sigma_b)e_1$ with $0 < a \leq |c|$. In particular

$$\varphi_{ab}(c) = \left|\left\{0 < a \leq |c| : \begin{pmatrix} a \\ b \end{pmatrix} \in (\sigma_a^{-1} \Gamma \sigma_b)e_1\right\}\right|.$$

**Proof.** Let $(x, y)$ be an ordered pair in $\Lambda_a \times \Lambda_b$ such that $\det(x \mid y) = c$. We can choose $x'$ such that $(x \mid x') \in \Gamma \sigma_a$. Then

$$(x \mid x')^{-1}(x \mid y) = \begin{pmatrix} 1 & a \\ 0 & c \end{pmatrix}.$$

This shows that $(a \ b) \in \sigma_a^{-1} \Gamma y = (\sigma_a^{-1} \Gamma \sigma_b)e_1$. Under the assumption that $-I \in \Gamma$, we have $(1 \ 1) \in \sigma_a^{-1} \Gamma \sigma_a$. By matrix multiplication on the left with an appropriate power $(1 \ m)$, we have

$$\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a + mc \\ b \end{pmatrix}$$

with $0 < a + mc \leq |c|$. Hence we may always choose $a$ with this property. Moreover we recover the same vector for any equivalent ordered pair $(x_1, y_1)$. This assignment is easily seen to be surjective. Indeed, given $(a \ b) \in (\sigma_a^{-1} \Gamma \sigma_b)e_1$, choose $x = \sigma_a e_1$ and $y = \sigma_a (a \ b)$. To prove that it is also injective, suppose that we have $(x_1 \mid y_1) = (x_1 \mid x_1')(x_2 \mid x_2')^{-1}(x_2 \mid y_2)$. Then $(x_1 \mid y_1)$ and $(x_2 \mid y_2)$ are equivalent ordered pairs of vectors. ■

**Remark 7.3.** Observe that $\varphi_{ab}(c)$ counts the number of double coset representatives with lower left entry given by $c$; compare to Theorem 2.2.

When $\Gamma = \text{SL}_2(\mathbb{Z})$, we may take $\sigma_{\infty} = I$ and $\Lambda_{\infty} = \text{SL}_2(\mathbb{Z})e_1$ for the unique cusp at infinity. Then each integer $n \in \mathbb{Z}$ is realized as an admissible determinant, and $\varphi_{\infty}(n) = \varphi(|n|)$, Euler’s totient function, which is trivially bounded above by $\varphi(n) \leq n - 1$ for each $n \in \mathbb{N}$. For more general groups, and in particular groups that are not finite-index subgroups of $\text{SL}_2(\mathbb{Z})$, one only has the estimate $|\varphi_{ab}(c)| \ll |c|^2$ as consequence of the pigeonhole principle; see [25, Proposition 2.8]. However, we have stronger bounds on average via the
scattering matrix for $\Gamma$, which we quickly recall. If $h$ is the number of inequivalent cusps for $\Gamma$, the scattering matrix $(\phi_{ab}(s))_{a,b}$ for $\Gamma$ is the $h \times h$ matrix with entries given by

$$\phi_{ab}(s) = \sqrt{\pi} \frac{\Gamma(s - 1/2)}{\Gamma(s)} \sum_{c \in \mathcal{N}_{ab}} \frac{\varphi_{ab}(c)}{c^{2s}},$$

where $s \in \mathbb{C}$ and $\Gamma(s)$ is Euler’s gamma-function. Each entry converges absolutely and uniformly on compact subsets in the half-plane $\Re(s) > 1$ and admits a meromorphic continuation to the whole complex plane. (See [25] for these facts as well as the ones used in the proof below.)

**Theorem 7.4.** Let $\Gamma < G$ be a nonuniform lattice containing $-I$ with cusps $a, b$. There are constants $1 > \rho_1 > \cdots > \rho_k > \frac{1}{2}$ and $c_1, \ldots, c_k$ such that

$$\sum_{\substack{c \in \mathcal{N}_{ab} \\ 0 < c < T}} \varphi_{ab}(c) = \frac{c \Gamma}{2} T^2 + c_1 T^{2\rho_1} + \cdots + c_k T^{2\rho_k} + O(T^{4/3})$$

as $T \to \infty$.

**Proof.** The asymptotic is a special case of a more general theorem of Good (see [17, Theorem 4, p. 116]) on counting functions over double cosets of Fuchsian groups. We include a much simpler and direct proof for the convenience of the reader that recovers the weaker error growth rate $O(T^{4/3+\varepsilon})$.

We will need the following facts pertaining to the meromorphic continuation of the entries of the scattering matrix, which we also will denote by $\phi_{ab}(s)$. We record that $\phi_{ab}(s)$ is holomorphic for $\Re(s) \geq 1/2$ except for at most finitely many poles $\rho_0, \rho_1, \ldots, \rho_k$, all simple, and all lying on the real segment $(\frac{1}{2}, 1]$ with $\rho_0 = 1$ whose residue is

$$\text{Res}_{s=1} \phi_{ab}(s) = \frac{1}{V}.$$

We will also rely on the following standard complex analysis computation; uniformly for $y, \sigma > 0$ and $X \geq 1$ we have

$$\frac{1}{2\pi i} \int_{\sigma - iX}^{\sigma + iX} \frac{\Gamma(s)}{s} ds = \begin{cases} 1 + O\left(\frac{y^\sigma}{X \log y}\right) & \text{if } y > 1, \\ O\left(\frac{y^\sigma}{X \log y}\right) & \text{if } y < 1. \end{cases}$$

We may assume that $T \notin \mathcal{N}_{ab}$ and let $\sigma > 1$. Then

$$\frac{\pi^{-1/2}}{2\pi i} \int_{\sigma - iX}^{\sigma + iX} \frac{\Gamma(s)}{\Gamma(s - 1/2)} \phi_{ab}(s) \frac{T^{2s}}{s} ds = \sum_{\substack{c \in \mathcal{N}_{ab} \\ 0 < c < T}} \varphi_{ab}(c) + O\left(\frac{T^{2\sigma}}{X}\right).$$

Applying Stirling’s estimate for the gamma-function and the Phrágmen–Lindelöf principle we see that the integrand on the LHS is of order $O\left(\frac{T^{2\sigma}}{|T|^{1/2}}\right)$ for all $s \in \mathbb{C}$ with $\Re(s) \in [1/2, \sigma]$.
and \(|t| \geq 1\). To compute the integral on the LHS, we apply Cauchy’s residue theorem to the rectangle with sides \([\sigma - iX, \sigma + iX]\), \([\sigma + iX, 1/2 + iX]\), \([1/2 + iX, 1/2 - iX]\), \([1/2 - iX, \sigma - iX]\), and find that the LHS is

\[
\sum_{i=0}^{k} \frac{\Gamma(\rho_i) \pi^{-1/2}}{\rho_i \Gamma(\rho_i - 1/2)} \text{Res}_{s=\rho_i} \phi_{ab}(s) T^{2\rho_i} + O \left( TX^{1/2} + X^{-1/2} \frac{T^{2\sigma}}{\log T} \right).
\]

Combining these results, we have

\[
\sum_{c \in \mathbb{N}_{ab}} \varphi_{ab}(c) = \frac{T^2}{\pi V} + c_1 T^{2\rho_1} + \ldots + c_k T^{2\rho_k} + O \left( \frac{T^{2\sigma}}{X} + TX^{1/2} + X^{-1/2} \frac{T^{2\sigma}}{\log T} \right),
\]

for the appropriate choice of \(c_1, \ldots, c_k\). The error term is optimized by choosing \(X = T^{\sigma/3}\) and \(\sigma\) can be chosen as small as \(\sigma = 1 + \varepsilon\) for any \(\varepsilon > 0\).

We will later require asymptotics for the following modification of this average. For \(t > 0\), let

\[
\Phi_{ab}(t) := t \sum_{c \in \mathbb{N}_{ab}, 0 < c < T} \frac{\varphi_{ab}(c)}{c^3}.
\]

**Corollary 7.5.** Let \(\delta = \min\{2(1 - \rho_1), 2/3\} \in (0, 2/3]\) and \(c_\Gamma\) defined as in Eq. (4.3). Then

\[
\Phi_{ab}(t) = \begin{cases} 
  c_\Gamma + O \left( t^{-\delta} \right) & \text{as } t \to \infty, \\
  O(t) & \text{as } t \to 0.
\end{cases}
\]

**Proof.** The first statement follows by Abel summation on Theorem 7.4; indeed, we have for any \(T \gg t\),

\[
\sum_{c \in \mathbb{N}_{ab}, t < c \leq T} \frac{\varphi_{ab}(c)}{c^3} = \sum_{0 < c \leq T} \varphi_{ab}(c) T^{-3} - \sum_{0 < c \leq t} \varphi_{ab}(c) t^{-3} + 3 \int_t^T \sum_{0 < c \leq u} \varphi_{ab}(c) u^{-4} du.
\]

Letting \(T \to \infty\),

\[
\sum_{c \in \mathbb{N}_{ab}, t \leq c} \frac{\varphi_{ab}(c)}{c^3} = c_\Gamma t^{-1} + O \left( t^{-\delta} \right).
\]

For the second statement, we use Theorem 7.4 and positivity to see \(\varphi_{ab}(c) \ll c^{2-\varepsilon}\) for some \(\varepsilon > 0\). Hence the sum defining \(\Phi_{ab}(t)\) is absolutely convergent.

\[\square\]
8. Mean values formulas for pairs

A formula for the second moment of the Siegel–Veech transform is derived in [14] when the lattice \( \Gamma = H_q \) is a Hecke triangle group. In this setting there is only one cusp class, represented by the cusp at \( \infty \) and we have \( \Lambda := \Lambda_\infty = \Gamma \mathbf{e}_1 \). Then for \( f \in SC(\mathbb{R}^2 \times \mathbb{R}^2) \) we have

\[
\int_{G/H_q} \Theta_{\Lambda, \Lambda; f}(g) d\mu(g) = c_{H_q} \sum_{c \in N^*} \varphi(c) \int_{G} f(ge_1, cge_2) d\eta(g)
+ c_{H_q} \int_{\mathbb{R}^2} (f(x, -x) + f(x, x)) dx
\]

where \( N = N_{oooo}, \varphi = \varphi_{oooo} \), and \( \eta \) is the Haar measure on \( G \) normalized so that \( \eta_*(G/H_q) = c_{H_q} \) [14, Theorem 1.2].

Opening up the measure \( d\eta = da \, db \, ds \) under the coordinates in Section 2.1, the inner integral becomes

\[
\int_{a, b, s \in \mathbb{R}} f \left( \begin{pmatrix} a \\ as \\ \end{pmatrix}, c \begin{pmatrix} b \\ bs+a^{-1} \\ \end{pmatrix} \right) \, da \, db \, ds = \frac{1}{|c|} \int_{u, v, x \in \mathbb{R}} f \left( \begin{pmatrix} u \\ v \\ x(u^{-1}) + c \begin{pmatrix} 0 \\ u^{-1} \\ \end{pmatrix} \right) \, dx \, dv \, du
\]

under the change of coordinates \( a = u, s = vu^{-1}, ux = cb \) which has Jacobian \( 1/|c| \), and for \( x^* \) chosen such that \( \det(x | x^*) = 1 \).

We can extend this identity from Hecke triangle groups to all non-uniform lattices and pairs of not necessarily homothetic discrete lattice orbits. The mean value formula in Theorem 1.8 follows from

**Theorem 8.1.** Given two scaled discrete \( \Gamma \)-orbits \( \Lambda_\mathbf{a}, \Lambda_\mathbf{b}, f \in SC(\mathbb{R}^2 \times \mathbb{R}^2) \), we have

\[
\int_{G/\Gamma} \Theta_{\Lambda_\mathbf{a}, \Lambda_\mathbf{b}; f}(g) d\mu(g) = c_\Gamma \sum_{c \in N_{\mathbf{ab}}} \varphi_{\mathbf{ab}}(c) \int_{G} f(ge_1, cge_2) d\eta(g)
+ \delta_{\mathbf{ab}} c_\Gamma \int_{\mathbb{R}^2} (f(x, -x) + f(x, x)) dx,
\]

where \( \delta_{\mathbf{ab}} = 1 \) if and only if \( \Lambda_\mathbf{a} \) and \( \Lambda_\mathbf{b} \) are homothetic.

**Remark 8.2.** In order to obtain an extension from \( SC(\mathbb{R}^2 \times \mathbb{R}^2) \) to a larger class of functions, one can follow the techniques of Theorem 6.1, where the co-area formula is replaced by the area formula. This technique does not unconditionally yield the set of all integrable Borel functions.
Proof. Following the discussion in [14, Section 3.1], we decompose $\mathbb{R}^2 \times \mathbb{R}^2$ into $G$-invariant subsets on which $\mu$ must be supported; we write

$$
\int_{G/\Gamma} \Theta_{\Lambda_a, \Lambda_b}:f(g)d\mu(g) = a \cdot \delta_0(f) + b_\infty \int_{\mathbb{R}^2} f(0, x)d\mathbf{x} + \sum_{t \in \mathbb{R}} b_t \int_{\mathbb{R}^2} f(x, tx)d\mathbf{x} + \sum_{t \neq 0} c_t \int_{G} \langle g e_1, t g e_2 \rangle d\eta(g). \tag{8.1}
$$

In the following we use the fact that we can choose any $f \in SC(\mathbb{R}^2 \times \mathbb{R}^2)$. In each case we will define $f = 1_S$ for a specific closed set $S$, and use it to determine the coefficients for each term of Eq. (8.1).

Set $S = \{(0, 0)\}$. Then since no lattice orbit passes through the origin, we have $a = 0$.

Set $S = \{(0) \times \overline{B R} \}$. Since no discrete lattice orbit contains the origin, $\Theta_{\Lambda_a, \Lambda_b}:f = 0$, and we have $b_\infty = 0$. The same argument yields $b_0 = 0$.

Set $S = \{(x, tx) \in \overline{BR} \times \overline{BR}\}$. By Lemma 7.1, if $\delta_{ab} = 0$, then $S \cap (\Lambda_a \times \Lambda_b) = \emptyset$. Then (8.1) becomes

$$
0 = b_t \int_{\mathbb{R}^2} 1_S(x, tx)d\mathbf{x} = b_t |B R/ t|
$$

and hence $b_t = 0$ for all $t \in \mathbb{R}^*$ when $\delta_{ab} = 0$.

Suppose now $\delta_{ab} = 1$. Let $x \in \Lambda_a$, $tx \in \Lambda_a$. Choose $g \in \Gamma \sigma_a$ and $h \in \Gamma \sigma_a$ such that the first column vectors of $g$ and $h$ are respectively $x$ and $tx$. Then

$$
h^{-1}g = \begin{pmatrix} 1/t & * \\ 0 & t \end{pmatrix} \in \sigma_a^{-1} \Gamma \sigma_a.
$$

By Lemma 2.1, we conclude that $t \in \{\pm 1\}$. Set $f_+ = 1_S$ when $S = \{(x, x) \in \overline{BR} \times \overline{BR}\}$ and $f_- = 1_S$ when $S = \{(x, -x) \in \overline{BR} \times \overline{BR}\}$. Hence $\Theta_{\Lambda_a, \Lambda_b}:f \neq 0$ and (8.1) yields

$$
\int_{G/\Gamma} \Theta_{\Lambda_a, \Lambda_b}:f_+(g)d\mu(g) = b_{\pm 1} \int_{\mathbb{R}^2} f_\pm(x, \pm x)d\mathbf{x}.
$$

On the other hand, by Theorem 6.1, we have

$$
\int_{G/\Gamma} \Theta_{\Lambda_a, \Lambda_b}:f_+(g)d\mu(g) = c_{\Gamma} \int_{\mathbb{R}^2} f_\pm(x, \pm x)d\mathbf{x}.
$$

Since this is true for all $R$ we conclude that when $\delta_{ab} = 1$ we have $b_t = 0$ when $t \neq \pm 1$ and $b_{\pm 1} = c_{\Gamma}$.

Set $S = \{(x, y) : x, y \in \overline{BR}, \det(x | y) = c\}$. Then $\Theta_{\Lambda_a, \Lambda_b}:f(g) = \{|(x, y) \in (\Lambda_a \times \Lambda_b) \cap g^{-1}\overline{BR} : \det(xy) = c\}| \neq 0$ if and only if $c \in N_{ab}^*$. Now suppose $c \in N_{ab}^*$. By Lemma 7.2, there exists $0 < a \leq |c|$ so that $\begin{pmatrix} a \\ c \end{pmatrix} \in \sigma_a^{-1} \Gamma \sigma_b$, so that by definition $\varphi_{ab}(c) \neq 0$.

We want to show that (cf. [14, Lemma 3.6]) for all $R$, the support of $\Theta_{\Lambda_a, \Lambda_b}:f$ reduces to $\varphi_{ab}(c)$ total $\Gamma$-orbits. More precisely we show that

$$
D_{\mathbb{C}}^{ab} := \{(x_1, x_2) \in \Lambda_a \times \Lambda_b : \det(x_1 | x_2) = c\} = \bigcup_{0 < a \leq |c|} \Gamma \sigma_a \begin{pmatrix} a \\ 0 \end{pmatrix} \cdot \begin{pmatrix} b \\ c \end{pmatrix}.
$$
Indeed by the proof of Lemma 7.2, there is some \( h \in \Gamma \sigma_a \) so that \( h^{-1}(x_1|x_2) = \left( \begin{array}{cc} 1 & a \\ 0 & c \end{array} \right) \) for some \( 0 < a \leq |c| \) and \( \left( \begin{array}{c} a \\ c \end{array} \right) \in \sigma_a^{-1}\Gamma \sigma_b e_1 \). Moreover each of these orbits is distinct. Namely if \( |c| \geq a_2 > a_1 > 0 \) with

\[
\Gamma \sigma_a \left( \begin{array}{cc} 1 & a_1 \\ 0 & c \end{array} \right) = \Gamma \sigma_a \left( \begin{array}{cc} 1 & a_2 \\ 0 & c \end{array} \right),
\]

then since \( \sigma_a \) is a scaling transformation, we must have \( a_2 - a_1 \in c \mathbb{Z} \) which is impossible as \( 0 < a_2 - a_1 < c \).

Now that we know \( D_c^{ab} \) is decomposed into \( \varphi_{ab}(\|c\|) \) distinct orbits, we want to show we have the same contribution for each orbit. Namely (cf. [14, Lemma 3.7]) if we fix \( \left( \begin{array}{c} a \\ c \end{array} \right) \in \sigma_a^{-1}\Gamma \sigma_b e_1 \), with \( 0 < a \leq |c| \), then by our normalization of the Haar measure, we have

\[
\int_{G/\Gamma} \sum_{g \in \Gamma} f \left( g \gamma \sigma_a \left( \begin{array}{cc} 1 & a \\ 0 & c \end{array} \right) \right) d\mu(g) = c \int_{G/\Gamma} f \left( g \sigma_a \left( \begin{array}{cc} 1 & a \\ 0 & c \end{array} \right) \right) d\eta(g)
\]

\[
= c \int_{G/\Gamma} f \left( \left( \begin{array}{cc} 1 & 0 \\ 0 & c \end{array} \right) \right) d\eta(g)
\]

by multiplying \( g \) on the right by \( \left( \begin{array}{cc} 1 & -a \\ 0 & 1 \end{array} \right) \sigma_a^{-1} \) and using that \( \eta \) is invariant under the action of \( G \). Hence we get the same contribution from each of the \( \varphi_{ab}(\|c\|) \) orbits. Since this is true for all \( R \) we obtain the desired coefficients, thus concluding the proof of the first equality.

Let \( C \) be the cone of \( 2 \times 2 \) matrices \( A \) such that \( \lambda A \in G/\Gamma \) for some \( \lambda \geq 1 \), and let \( dA \) be the Euclidean volume element. There exists some constant \( c_0 \) so that

\[
\int_C f(A) dA = c_0 \int_0^1 \nu \int_{G/\Gamma} f(v^{1/2} g) d\mu(g) dv.
\]

Inspired by Schmidt [37], we introduce the equivalent\(^4\) cone measure

\[
\int_C f(A) dm(A) = \int_0^1 \int_{G/\Gamma} f(v^{1/2} g) d\mu(g) dv.
\]

Let \( \Lambda_a \), \( \Lambda_b \) be scaled discrete \( \Gamma \)-orbits. With respect to the cone measure \( dm \), we have \( m(C) = 1 \),

\[
\int_C \Theta_{\Lambda_a : f} (A) \det(A) dm(A) = \int_0^1 \nu \int_{G/\Gamma} \Theta_{\Lambda_a : f} (v^{1/2} g) d\mu(g) dv = c_\Gamma \int_{\mathbb{R}^2} f(x) \, dx
\]

(8.2)

\(^4\)I.e., the measures \( dm(A) \) and \( dA \) have the same null measure sets.
and
\[
\int_C \Theta_{\Lambda_a, \Lambda_b} f(A) \det(A)^2 \, dm(A) = \int_0^1 \int_{G/G} \Theta_{\Lambda_a, \Lambda_b} f(\nu^{1/2} g) \, d\mu(g) \, d\nu.
\]
The second mean value formula given by Theorem 1.8’ follows from the following theorem.

**Theorem 8.3.** Let \( \Gamma < G \) be a nonuniform lattice containing \(-I\), and let \( \Lambda_a, \Lambda_b \) be two scaled discrete \( \Gamma \)-orbits. For each \( f \in SC(\mathbb{R}^2 \times \mathbb{R}^2) \), we have
\[
\int_C \Theta_{\Lambda_a, \Lambda_b} f(A) \det(A)^2 \, dm(A) = c_\Gamma \int_{\mathbb{R}^2 \times \mathbb{R}^2} \Phi_{ab}(|x \wedge y|) f(x, y) \, dx \, dy
\]
\[
+ \frac{\delta_{ab} c_\Gamma}{2} \int_{\mathbb{R}^2} (f(x, x) + f(x, -x)) \, dx,
\]
where \( \Phi_{ab} \) is the function given by (7.1).

**Proof.** We will obtain this formula from Theorem 8.1 by a change of coordinates. Define \( f_\nu(x, y) = f(\nu^{1/2} x, \nu^{1/2} y) \), and apply Theorem 8.1 to \( \Theta_{\Lambda_a, \Lambda_b} f_\nu \). In the linearly dependent subsets, we have via substitution
\[
\delta_{ab} c_\Gamma \int_0^1 \int_{\mathbb{R}^2} f_\nu(x, x) + f_\nu(x, -x) \, dx \, d\nu = \delta_{ab} c_\Gamma \int_0^1 d\nu \int_{\mathbb{R}^2} f(x, x) + f(x, -x) \, dx.
\]

It remains to show that
\[
c_\Gamma \int_0^1 \nu^2 \sum_{c \in \mathcal{N}_{ab}} \varphi_{ab}(c) \int_{G} f(\nu^{1/2} g J_c) \, d\eta(g) \, d\nu = c_\Gamma \int_{\mathbb{R}^2} \Phi_{ab}(|x \wedge y|) f(x, y) \, dx \, dy.
\]
where we set \( J_c = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} \). We write \( t = \nu c \) and note that
(1) \( \mathcal{N}_{ab} \) and \( \varphi_{ab}(c) \) are symmetric about 0 if \(-I \in \Gamma\);
(2) and \( \mathcal{N}_{ab} \) is unbounded, so the parameter \( t \) ranges over \( \mathbb{R} \setminus \{0\} \).
Since $\Theta_{\Lambda_x, \Lambda_{\theta}; f}$ is integrable, Fubini applies and we may exchange the order of integration and summation in what follows. The LHS of (8.4) becomes

\[
c_G \sum_{c \in N^{\Lambda}_{ab}} \varphi_{ab}(c) \int_0^1 \nu^2 \int_G f(\nu^{1/2} g J_c) d\eta(g) d\nu
\]

(9.1) \[
= c_G \sum_{c \in N^{\Lambda}_{ab}} \int_0^1 \nu^2 \varphi_{ab}(c) f(g \left( \begin{smallmatrix} 1 & 0 \\ 0 & c \end{smallmatrix} \right)) d\nu d\eta(g)
\]

(9.2) \[
= c_G \int_G \left[ \int_0^\infty \nu^2 f(g \left( \begin{smallmatrix} 1 & 0 \\ 0 & c \end{smallmatrix} \right)) \sum_{c \in N^{\Lambda}_{ab}, c \geq t} \varphi_{ab}(c) \frac{dt d\eta(g)}{c^3} + \int_0^\infty \nu^2 f(g \left( \begin{smallmatrix} 1 & 0 \\ 0 & c \end{smallmatrix} \right)) \sum_{c \in N^{\Lambda}_{ab}, c \leq t} \varphi_{ab}(c) \frac{dt d\eta(g)}{c^3} \right]
\]

(9.3) \[
= c_G \int_\mathbb{R} |t| \Phi_{ab}(|t|) \int_G f(g \left( \begin{smallmatrix} 1 & 0 \\ 0 & t \end{smallmatrix} \right)) d\eta(g) dt.
\]

9. Geometric estimates

We may rewrite the mean value formula as

\[
\int_C \Theta_{\Lambda_x, \Lambda_{\theta}; f}(A) \det(A)^2 d\mu(A) = c_G^2 \int_{\mathbb{R}^2} f(x, y) dx dy + \frac{\delta_{ab}c_G}{2} \int_{\mathbb{R}^2} (f(x, x) + f(x, -x)) dx
\]

(9.4) \[
+ c_G \int_{\mathbb{R}^2} (\Phi_{ab}(|x \wedge y|) - c_G) f(x, y) dx dy.
\]

In this section, we use integration by parts to compute estimates for the last double integral,

\[
\int_{\mathbb{R}^2} (\Phi_{ab}(|x \wedge y|) - c_G) f(x, y) dx dy.
\]

(9.5) \[
\text{Namely, we will obtain an estimate in (9.4), and the results from this section to be used later are given in Example 9.1, Example 9.2, and Example 9.3.}
\]

For $t, u > 0$ consider the rectangle,

\[
P_{t, u}(x) = \{ y \in \mathbb{R}^2 : |x^1 \wedge y| \leq t, |x \wedge y| \leq u \}
\]

and set $P(x) = P_{1, 1}(x)$. Then $tP(x) = P_{t, t}(x)$. If we write $x = \|x\|k_\theta e_1$ and $x^1 = k_{\pi/2}\|x\|k_\theta e_1 = \|x\|k_\theta e_2$, we have

\[
P_{t, u}(x) = k_\theta P \frac{t \cdot \frac{u}{\|x\|}}{\|x\|}(e_1).
\]
In particular, \( P_{t,u}(x) \) is a rotated rectangle, centered at the origin, with sides of length \( 2t/\|x\| \) and \( 2u/\|x\| \). We consider the inner integral of (9.2). For \( t \) sufficiently large, writing \( x = \|x\| k_\theta e_1 \), we have

\[
\int_{\mathbb{R}^2} (\Phi_{ab}(\|x \land y\|) - c_\Gamma \delta_{ab}) f(x, y) dy = \int_{tP(x)} (\Phi_{ab}(\|e_1 \land \|x\| k_{-\theta} y\|) - c_\Gamma) f(x, y) dy
\]

\[
= \frac{1}{\|x\|^2} \int_{tP(e_1)} (\Phi_{ab}(\|e_1 \land y\|) - c_\Gamma) f(x, \frac{1}{\|x\|} k_\theta y) dy
\]

\[
= \frac{1}{\|x\|^2} \int_0^t \left( \Phi_{ab}(y_2) - c_\Gamma \right) \left( \int_0^t \sum f(x, \frac{1}{\|x\|} k_\theta (\frac{\|x\| y_1}{\|y_2\|}) \right) dy_1 dy_2,
\]

where the inner sum contains a summand for each of the four \( (\frac{\|x\| y_1}{\|y_2\|}) \). We call the inner integral \( g_r(y_2) \) and compute that its primitive is

\[
G_r(u) = \int_0^u \int_0^t \sum f(x, \frac{1}{\|x\|} k_\theta (\frac{\|x\| y_1}{\|y_2\|}) \right) dr ds
\]

\[
= \int_{P_{t,u}(e_1)} f(x, \frac{1}{\|x\|} k_\theta y) dy
\]

\[
= \|x\|^2 \int_{P_{t,u}(x)} f(x, y) dy.
\]

Recall that Corollary 7.5, there is some \( t_0 > 0 \) and constant \( M \) so that

\[
\Phi_{ab}(t) \leq \tilde{\Phi}_{ab}(t) := \begin{cases} 
  c_\Gamma + Mt^{-\delta} & t \geq t_0, \\
  Mt & 0 \leq t < t_0.
\end{cases}
\]

(9.3)

Applying integration by parts yields

\[
\int_{\mathbb{R}^2} (\Phi_{ab}(\|x \land y\|) - c_\Gamma) f(x, y) dy
\]

\[
= \frac{1}{\|x\|^2} \int_0^t \left( \Phi_{ab}(y_2) - c_\Gamma \right) \left( \int_0^t \sum f(x, \frac{1}{\|x\|} k_\theta (\frac{\|x\| y_1}{\|y_2\|}) \right) dy_1 dy_2
\]

\[
\ll \left( \Phi_{ab}(t) - c_\Gamma \right) \int_{tP(x)} f(x, y) dy - \int_0^t \tilde{\Phi}_{ab}^\prime(u) \int_{P_{t,u}(x)} f(x, y) dy du
\]

\[
\ll t^{-\delta} \int_{tP(x)} f(x, y) dy + \delta \int_{t_0}^\infty \left( \int_{P_{t,u}(x)} f(x, y) dy \right) u^{-1-\delta} du.
\]

(9.4)

**Example 9.1.** If \( f(x, y) = 1_{B_R}(x)1_{B_R}(y) \). Letting \( t \to \infty \) in (9.4), we have

\[
(9.2) \ll \int_0^\infty \int_{B_R} |P_{\infty,u}(x) \cap B_R| d\mathbf{x} u^{-1-\delta} du
\]

with \( |P_{\infty,u}(x) \cap B_R| = |B_R| \) if \( u \geq R/\|x\| \), and \( |P_{\infty,u}(x) \cap B_R| \ll \frac{R u}{\|x\|} \) otherwise. Hence

\[
(9.2) \ll \int_{B_R} \left( \int_0^{R/\|x\|} u R u^{-1-\delta} du + \int_{R/\|x\|}^\infty R^2 u^{-1-\delta} du \right) d\mathbf{x} \ll R^{2(2-\delta)}
\]
Example 9.2. Let \( f(x, y) = 1_{B_R}(x)1_{B_e^s}(y) \). Once again, \( |P_{\infty, u}(x) \cap B_e^s(x)| = |B_e| \) if \( u \geq \epsilon \|x\| \) and \( |P_{\infty, u}(x) \cap B_e^s(x)| \ll \frac{\epsilon u}{\|x\|} \) otherwise. Hence we have the estimate

\[
(9.2) \ll \int_{B_R} \left( \int_0^{\infty} \frac{e^\delta}{\|x\|} u^{-\delta} du + \int_0^\infty e^{2u}u^{-\delta} du \right) \, dx \ll (\epsilon R)^{2-\delta}.
\]

Proof of Theorem 1.7. Apply Eq. (9.1) to the characteristic function \( f \) supported on \( \{ x \in \mathbb{R}^2 : y \in B^s_{\sqrt{\nu}}(x) \} \) and use the estimate from Example 9.2 to bound the remaining term.

Example 9.3. For \( D, s > 0 \), let \( f(x, y) = 1_{B_R}(x)1_{D_{D,s}(x)}(y) \), where

\[
D_{D,s}(x) = \{ y \in \mathbb{R}^2 : |y| \leq s \|x\| \text{ and } |x \wedge y| \leq D \}.
\]

Then in this case \( |P_{\infty, u}(x) \cap D_{D,s}(x)| \leq \frac{\epsilon u}{\|x\|} \cdot s \|x\| = su \) if \( u \leq D \), and \( |P_{\infty, u}(x) \cap D_{D,s}(x)| \leq sD \) if \( u \geq D \). Therefore

\[
(9.2) \ll s \int_{B_R} \left( \int_0^D u^{-\delta} du + D \int_D^\infty u^{-1-\delta} du \right) \, dx \ll sD^2.
\]

10. Effective counting

10.1. Second moment estimate

We will derive Theorem 1.1 (see Theorem 10.3 for a full statement) by a combination of Borel–Cantelli and interpolation from the following ‘on average’ power saving. Our strategy is inspired by [37].

Proposition 10.1. Let \( \Lambda \) be a scaled discrete lattice orbit, and let \( B \) be a Lebesgue measurable set in the plane with finite volume \( |B| > C_\Lambda \). There is a constant \( M_\Lambda \) so that

\[
\int_C \left( \text{det}(A) \Theta_{\Lambda;1_B}(A) - c_\Gamma |B| \right)^2 \, dm(A) \leq M_\Lambda |B|^{2-\delta},
\]

where \( \delta \) is given by Eq. (4.4).

To prove Proposition 10.1, we use the following result of Schmidt for \( n = 2 \).

Lemma 10.2 ([37] Theorem 3). Let \( S \) be a Lebesgue measurable set in \( \mathbb{R}^2 \) with characteristic function \( 1_S \) and volume \( |S| \). Then

\[
\iint \chi(|x \wedge y|)1_S(x)1_S(y) \leq 8|S| \int_0^\infty \chi(t) \, dt
\]

for every nonnegative, nonincreasing function \( \chi(t) \) defined for \( t \geq 0 \) whose integral \( \int_0^\infty \chi(t) \, dt \) converges.
Proof of Proposition 10.1. Let \( \alpha(|B|) \) be a function increasing with \(|B|\) that we will choose later. Choose \(|B|\) large enough so that for \( t_0 < \alpha(|B|) \) we can use Corollary 7.5 (cf. Eq. (9.3)) to bound
\[
|\Phi(t) - c_T| \leq \chi(t) + \begin{cases} 
0 & t \leq \alpha(|B|) \\
M \alpha(|B|)^{-\delta} & t > \alpha(|B|)
\end{cases}
\]
where
\[
\chi(t) = \begin{cases} 
M t_0 & 0 \leq t < t_0 \\
M t^{-\delta} & t_0 \leq t < \alpha(|B|) \\
0 & t \geq \alpha(|B|).
\end{cases}
\]
Notice \( \chi(t) \) satisfies the assumptions of Lemma 10.2 with \( \int_0^\infty \chi(t) dt = M \left( t_0^2 - \frac{t_0^{1-\delta}}{1-\delta} \right) + \frac{M}{1-\delta} \alpha(|B|)^{1-\delta} \). Combining Eq. (8.2), Eq. (9.1), and Lemma 10.2 we compute
\[
\int_C (\det(A) \Theta_{\Lambda;1_B}(A) - c_T|B|)^2 dm(A) = \int_C \Theta_{\Lambda;1_B}(A) \det(A)^2 dm(A) - c_T^2|B|^2 \\
\leq c_T |B| + c_T \iint_{\mathbb{R}^2 \times \mathbb{R}^2} (\Phi_{ab}(|x \wedge y|) - c_T) 1_B(x) 1_B(y) dx dy,
\]
\[
\ll |B|(1 + \alpha(|B|)^{1-\delta}) + |B|^2 \alpha(|B|)^{-\delta}.
\]
Choosing \( \alpha(|B|) = |B| \), and noting that the implied constants only depend on \( \Lambda \) completes the proof.

10.2. Effective count via interpolation

We can now derive the main result of this section. The proof follows the same lines as that of [37, Theorem 2] and [29, Theorem 6.1]. We include it for the convenience of the reader.

Theorem 10.3. Let \( \Gamma < G \) be a lattice containing \(-I\) with scaled discrete orbit \( \Lambda \). Let \( \mathcal{B} \) be a linearly ordered family of Borel sets of finite volume in \( \mathbb{R}^2 \). Let \( \psi \) be a positive non-increasing function so that \( e^{(2-\delta)} \psi(t) \) is eventually non-decreasing and \( \int_1^\infty \psi(t) dt < \infty \). Then for almost every linear transformation \( A \)
\[
|A(\Lambda) \cap B| \det(A) = c_T|B| + O \left( \frac{|B|^{1-\delta/2} \log^{1/2}(|B|)}{\psi^{1/2}(|\log(|B|)|)} \right)
\]
Proof. Without loss of generality, we may assume \( \mathcal{B} \) has sets of arbitrarily larges volumes. By [37, Lemma 1] we may assume without loss of generality that \( \{|B| : B \in \mathcal{B}\} = \mathbb{R}^+ \). Thus for all \( N \in \mathbb{N}_{\geq 1} \) there exists \( B_N \in \mathcal{B} \) so that \( |B_N| = N \).

Define
\[
S_N(A) = |A\Lambda \cap B_N| - c_T N \det(A^{-1})|
\]
and for \( 0 \leq N_1 < N_2 \)
\[
N_1 S_{N_2}(A) = |A\Lambda \cap (B_{N_2} \setminus B_{N_1})| - c_T (N_2 - N_1) \det(A^{-1})|.
\]
For $T \geq 3$ set 
\[ \mathcal{K}_T = \{(N_1, N_2) \in \mathbb{Z}^2 \mid 0 \leq N_1 < N_2 \leq 2^T, \ N_1 = \ell 2^t \text{ and } N_2 = (\ell + 1)2^t \text{ for some } \ell, t \in \mathbb{N}_{\geq 0}\} . \]
By Proposition 10.1 with $B = B_{N_2} \setminus B_{N_1}$, notice that each value of $N_2 - N_1 = 2^t$ occurs $2^{T-t}$ times so we have 
\[ \sum_{(N_1, N_2) \in \mathcal{K}_T} \int_{\mathbb{C}} |N_1 S_{N_2}(A)|^2 \left| \det(A) \right|^2 \ dm(A) \leq M_A \sum_{t=0}^{T} 2^{T-t} 2^{t(2-\delta)} \leq 8M_A 2^{(2-\delta) T} , \tag{10.1} \]
where the last inequality follows from the geometric series and $\frac{1}{4} \leq 2^{1/3} \leq 2^{1-\delta} \leq 2$. 
Next let $\mathcal{E}_T \subseteq \mathcal{C}$ be the set of all $A \in \mathcal{C}$ so that 
\[ \sum_{(N_1, N_2) \in \mathcal{K}_T} |N_1 S_{N_2}(A)|^2 \left| \det(A) \right|^2 > \frac{2^{(2-\delta) T}}{48 \psi((T-1) \log(2))} . \tag{10.2} \]
Applying Chebyshev’s inequality and (10.1) implies 
\[ m(\mathcal{E}_T) < \frac{48 \psi((T-1) \log(2))}{2^{(2-\delta) T}} \int_{\mathcal{E}_T} \sum_{(N_1, N_2) \in \mathcal{K}_T} |N_1 S_{N_2}(A)|^2 \left| \det(A) \right|^2 \ dm(A) \]
\[ \leq 48 \cdot 8 M_A \psi((T-1) \log(2)). \tag{10.3} \]
By the Borel–Cantelli lemma and integrability of $\psi$, $m(\mathcal{E}_\infty) = 0$ for $\mathcal{E}_\infty = \limsup \mathcal{E}_T$. Define $C \setminus \mathcal{E}_\infty$ to be the full measure set (with respect to $dA$) of transformations where the discrepancy is small. 
For $N \leq 2^T$, we can write $[0, N) \subseteq \bigsqcup_{(N_1, N_2) \in I} [N_1, N_2)$ where $I \subset \mathcal{K}_T$ is given by 
\[ I = \{(0 \cdot 2^1, 1 \cdot 2^1)\} \cup \{(2^t, 2 \cdot 2^t)\}_{t=1}^{T-1} . \]
Thus we write 
\[ S_N(A) \leq \sum_{(N_1, N_2) \in I} N_1 S_{N_2}(A). \]
By (10.2), for any $A \notin \mathcal{E}_T$ and any $N < 2^T$, 
\[ |S_N(A)|^2 \left| \det(A) \right|^2 \leq \sum_{(N_1, N_2) \in I} |N_1 S_{N_2}(A)|^2 \left| \det(A) \right|^2 \]
\[ \leq |I| \sum_{(N_1, N_2) \in \mathcal{K}_T} |N_1 S_{N_2}(A)|^2 \left| \det(A) \right|^2 \]
\[ \leq T \frac{2^{(2-\delta) T}}{48 \psi((T-1) \log(2))} . \]
Consider $A \in C \setminus \mathcal{E}_\infty$. Choose $T_A \in \mathbb{N}$ so that for all $T \geq T_A$, $A \notin \mathcal{E}_T$. That is for all $T \geq T_A$ and for any $N < 2^T$, $|S_N(A)|^2 \left| \det(A) \right|^2 \leq T_A \frac{2^{(2-\delta) T}}{48 \psi((T-1) \log(2))}$. 

Pairs in discrete lattice orbits
In order to interpolate from \( B_N \) to any \( B \), we will set \( N_A = \max\{2^{T_A}, N_0\} \) where the assumption that \( e^{t(2-\delta)} \psi(t) \) is eventually non-decreasing ensures there exists some \( N_0 \) so that for all \( N \geq N_0 \),

\[
\frac{(N + 1)^{(1 - \frac{\delta}{2})} \log(N + 1)}{2 \psi^\frac{1}{2} (\log(N + 1))} + 1 \leq \frac{N^{(1 - \frac{\delta}{2})} \log(N)}{\psi^\frac{1}{2} (\log(N))}.
\]

(10.4)

Whenever \( N > N_A \), we have \( N > 2^{T_A} \) so we can choose an integer \( T \geq T_A \) so that \( 2^{T-1} \leq N \leq 2^T \). For such \( N \) we have

\[
|S_N(A)|^2 |\det(A)|^2 \leq \left( \frac{\log(N)}{\log(2)} + 1 \right) \frac{4N^2 - \delta}{48\psi(\log(N))} \leq \frac{N^{2 - \delta} \log(N)}{4\psi(\log(N))}
\]

where we used \( \psi \) in non-increasing, \( 2^{(2-\delta)T} = 4 \cdot 2^{2(T-1)} 2^{-\delta T} \), and \( \left( \frac{\log(N)}{\log(2)} + 1 \right) \leq 3 \log(N) \) for all \( N \geq 2 \).

We have now shown for all \( N > 2^{T_A} \) that

\[
|S_N(A)| |\det(A)| \leq \frac{N^{(1 - \frac{\delta}{2})} \log(N)}{2 \psi^\frac{1}{2} (\log(N))}
\]

For any \( B \in \mathcal{B} \) with \( |B| \geq N_A \) there exists an integer \( N \geq N_A - 1 \) so that \( N \geq 2^{T_A} \) and \( B_N \subseteq B \subseteq B_{N+1} \). We then interpolate and use (10.4) to obtain

\[
\left| |AA \cap B| - c_\Gamma |B| \det(A^{-1}) \right| \leq \max\{|S_N(A)|, |S_{N+1}(A)|\} + \frac{c_\Gamma}{|\det(A)|}
\]

\[
\leq \frac{|B|^{(1 - \frac{\delta}{2})} \log(|B|)}{\psi^\frac{1}{2} (\log(|B|)) |\det(A)|}.
\]

(10.5)

Since every \( \tilde{A} \) with \( 0 < |\det(\tilde{A})| \leq 1 \) is of the form \( \tilde{A} = A \gamma \) for \( A \in C \) and \( \gamma \in \Gamma \), and since \( \Gamma \) is enumerable, (10.5) holds for almost every linear transformation \( \tilde{A} \) with \( |\det(\tilde{A})| \leq 1 \). Now consider \( C \) a linear transformation with \( \det(C) = c \). Applying (10.5) to the family of sets \( \{C^{-1}B | B \in \mathcal{B}\} \),

\[
\left| |AA \cap C^{-1}B| - c_\Gamma |C^{-1}B| \det(A^{-1}) \right| = \left| |CAA \cap B| - c_\Gamma |B| \det((CA)^{-1}) \right|
\]

the fact that \( \det(C) \) can be taken arbitrarily large, we conclude (10.5) holds for almost every linear transformation. ■

10.3. Application to dilated sets

We obtain the discrete lattice orbit version of Theorem 1.1 by applying Theorem 10.3 to dilated sets.

**Theorem 10.4.** Let \( \Lambda \) be a scaled discrete \( \Gamma \)-orbit and let \( \Omega \subset \mathbb{R}^2 \) be a bounded Borel set that contains the origin. Consider its dilates \( \Omega_R = R \cdot \Omega \). Then for almost every linear transformation \( A \) we have

\[
|A(\Lambda) \cap \Omega_R| |\det(A)| = c_\Gamma |\Omega| |\mathbb{R}^2| + O \left( R^{2-\delta} \log^{3/2}(R) \right).
\]
where $\delta$ is as in (4.4).

**Proof.** Let $\psi(t) = t^{-2}$, which is integrable, non-increasing for $t > 1$, and satisfies that $e^{t(2-\delta)}\psi(t)$ is eventually non-decreasing. Now apply Theorem 10.3 to the family $\mathcal{B} = \{\Omega_R : R \in \mathbb{R}_{\geq 0}\}$ of dilates.

We recall the following known estimates from [6]. For simplicity, suppose that $\Gamma$ has trivial residual spectrum, i.e., $\delta = \frac{2}{3}$. By Theorem 10.4, for almost every linear transformation $A \in \text{GL}_2(\mathbb{R})$ we find

$$|\det(A)||A(\Lambda) \cap \Omega_R| = c_\Gamma |\Omega|R^2 + O \left( R^{\frac{3}{2}} \log^{3/2}(R) \right)$$

while in [6] it is shown that for every $A \in G$ (hence with $\det(A) = 1$), we have

$$|A(\Lambda) \cap \Omega_R| = c_\Gamma |\Omega|R^2 + O \left( R^\alpha \right) ,$$

with

$$\alpha = \begin{cases} 4/3 & \text{when } \Omega \text{ is the unit disk;} \\ 8/5 & \text{when } \Omega \text{ is a star-shaped domain with smooth boundary;} \\ 7/4 + \epsilon & \text{when } \Omega \text{ is a star-shaped domain with piecewise Lipschitz boundary;} \\ 12/7 + \epsilon & \text{when } \Omega \text{ is a sector.} \end{cases}$$

11. **Upper bounds on pair relationships**

In this section, we obtain asymptotic upper bounds for the number of ordered pairs $(x, y) \in \Lambda_1 \times \Lambda_2$, where

- $\Lambda_1, \Lambda_2$ are scaled discrete $\Gamma$-orbits;
- the first vector $x$ is restricted to $x \in B_R \cap \Lambda_1$;
- the second vector $y \in \Lambda_2$ has a prescribed metric relationship to $x$.

The first relationship we examine is given by those vectors $y \in \Lambda_2$ distinct from $x$ that are within distance $\epsilon$ of $x$. We say that two vectors $x, y$ are $\epsilon$-friends if $\|x - y\| < \epsilon$. The second relationship examined is that of a determinant, first studied in [2]. A key aspect of these results is that they hold for every Veech surface. This will be used in Appendix A to extend results previously only known for generic translation surface to every Veech surface.

11.1. $\epsilon$-friends

The main result of this section is
Theorem 11.1. Let $\Lambda_a, \Lambda_b$ be two scaled discrete $\Gamma$-orbits. There exists a constant $C$ (depending only on $\Gamma$) such that
\[
\limsup_{R \to \infty} \frac{|\{(x, y) \in \Lambda_a \times \Lambda_b : x \in B_R, y \in B_\eta^*(x)\}|}{|B_R|} < C|B_\eta|.
\]

Proof. Let $f = 1_S$ be the characteristic function supported on
\[
S := S(R, \eta) = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 : x \in B_R, y \in B_\eta^*(x)\},
\] (11.1)
and let
\[
\Theta(A) := \Theta_{\Lambda_a, \Lambda_b; f}(A) = \sum_{x \in \Lambda_a} \sum_{y \in \Lambda_b} f(Ax, Ay)
\]
for each $A \in \text{GL}_2(\mathbb{R})$. Fix $\varepsilon > 0$. We will show that we can choose $\eta > 0$ such that
\[
\limsup_{R \to \infty} \frac{\Theta(I)}{|B_R|} < \varepsilon.
\]

Fix $\alpha > 0$ and consider the open symmetric neighborhood $\mathcal{U}_a = K\left\{\left(\frac{e^{r/2}}{e^{-r/2}} : r \in (-\alpha, \alpha)\right) : K \subset G\right\}$ (in terms of the Cartan decomposition). If $f_\alpha$ denotes the characteristic function supported on $S(\sqrt{2}Re^{\alpha/2}, \sqrt{2}\eta e^{\alpha/2})$ and we set $\Theta_\alpha := \Theta_{\Lambda_a, \Lambda_b; f_\alpha}$ then for each $A \in \mathcal{U}_a$,
\[
\Theta(I) \leq \Theta_\alpha(A). \tag{11.2}
\]

Indeed, if $x \in B_R$, then $||Ax|| < \sqrt{2}e^{\alpha/2}R$, and if $y \in B_\eta^*(x)$, then $||Ay|| < \sqrt{2}\eta e^{\alpha/2}$.

Let $C_\alpha$ be the matrix cone over $\mathcal{U}_a \Gamma / \Gamma$. Then the above estimate together with the fact that $\Theta_\alpha \geq 0$ yield
\[
\Theta(I) \frac{\mu(\mathcal{U}_a \Gamma / \Gamma)}{3} = \int_{C_\alpha} \Theta(I) \det(A)^2 \, dm(A) \leq \int_{C_\alpha} \Theta_\alpha(A) \det(A)^2 \, dm(A). \tag{11.3}
\]
Expressing the Haar measure via the Cartan decomposition we find
\[
\mu(\mathcal{U}_a \Gamma / \Gamma) = \frac{1}{\pi \sqrt{V}} \int_{-\alpha}^{\alpha} |\sinh(r)|dr = \frac{1}{\pi \sqrt{V}} (\cosh(\alpha) - 1).
\]

By (9.1) and Example 9.2 the RHS of (11.3) is bounded above by
\[
c_1^2 |B_{\sqrt{2}Re^{\alpha/2}}| |B_{\sqrt{2}\eta e^{\alpha/2}}| + c_1^2 \int_{\mathbb{R}^2} (\Phi_{ab}(|x \wedge y|) - c_1^2) \, f_\alpha(x, y) \, dx \, dy
\]
\[
= 4c_1^2 e^{2\alpha} |B_R||B_\eta| + O\left(|B_{\sqrt{2}Re^{\alpha/2}}| |B_{\sqrt{2}\eta e^{\alpha/2}}|^{1-\delta/2}\right).
\]

Hence
\[
\limsup_{R \to \infty} \frac{\Theta(I)}{|B_R|} \leq 3c_1|B_\eta| \frac{e^{2\alpha}}{\cosh(\alpha) - 1}.
\]

The function in $\alpha$ has a local minimum at $\alpha = \ln(3)$ with value $\frac{2\pi}{2}$. \hfill \blacksquare

The proof of Theorem 11.1 extends to the set of holonomies of a Veech surface.
Theorem 11.2. Let $M$ be a Veech surface. Then there exists a constant $C > 0$, depending only on $M$, such that

$$
\limsup_{R \to \infty} \frac{|\{(x, y) \in S_M \times S_M : x \in B_R, y \in B^*_\eta(x)\}|}{|S_M \cap B_R|} < C|B_\eta|.
$$

Proof. Recall that if $M$ is a Veech surface, its Veech group $\Gamma_M$ is a nonuniform lattice in $G$ and the set $S_M$ is a disjoint finite union of discrete $\Gamma_M$-orbits, i.e., $S_M = \cup \Gamma_M x_i$. To each orbit corresponds a scaling factor $\lambda_i \neq 0$ given by $\Gamma_M x_i = \lambda_i \Lambda_i$ — here $\Lambda_i$ is the associated scaled discrete orbit — and we have

$$
|S_M \cap B^*_\eta(x)| = \sum \Lambda_i \cap B^*_{\eta/\lambda_i}(x/\lambda_i).
$$

Counting the number of pairs that are $\eta$-friends we find

$$
\sum_{x \in S_M \cap B_R} \sum_{y \in S_M \cap B^*_\eta(x)} 1 = \sum_{i,j} \sum_{x \in \Lambda_i \cap B_R, y \in \Lambda_j} |\Lambda_i \cap B^*_{\eta/\lambda_i}(x/\lambda_i)|
$$

Let $f_{ij}$ be the characteristic function supported on the set

$$
\{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 : x \in B_{\eta/\lambda_i}, y \in B^*_{\eta/\lambda_i}(x/\lambda_i)\}.
$$

Running through the proof of Theorem 11.1 we find that

$$
\limsup_{R \to \infty} \sum_{i,j} \frac{\Theta_{\Lambda_i, \Lambda_j; f_{ij}}(I)}{|S_M \cap B_R|} \ll \Gamma \sum_{i,j} \frac{c^2_i}{\lambda_i^2 \lambda_j^2} \frac{|B_\eta|}{c_M} = c_M |B_\eta|.
$$

Theorem 1.3 now follows immediately. We note that this result is trivially true for arithmetic lattice surfaces (e.g., square-tiled surfaces.)

Proposition 11.3. If $M$ is an arithmetic lattice surface, then $S_M$ is uniformly discrete.

Proof. Let $\Gamma_M$ be its arithmetic Veech lattice. As such, it is commensurable with $\text{SL}_2(\mathbb{Z})$, and there exists a lattice $L \subset \mathbb{R}^2$ such that any two points in $S_M$ differ by an element of $L$, see Lemma 5.4 and Theorem 5.5 in [21]. Let $\ell$ be the length of a shortest lattice vector in $L$. Then for any $x \in \mathbb{R}^2$, $|S_M \cap B_{\ell/2}(x)| \leq 1$.

11.2. Determinants

Theorem 11.4. Fix $D > 0$. Let $\Lambda_a$ and $\Lambda_b$ be scaled discrete lattice orbits. There is a constant $C = C_{\Gamma}$ so that

$$
\limsup_{R \to \infty} \frac{|\{(x, y) \in \Lambda_a \times \Lambda_b : x \in B_R, y \in \mathcal{D}_{D,1}(\mathbb{X})\}|}{|B_R|} \leq DC_\Gamma + 2\delta_{ab}C_\Gamma.
$$
Proof. We follow the proof of Theorem 11.1, where $f_\alpha$ is the characteristic function supported on
\[
\{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 : x \in B_{R_{e^\alpha}}, y \in D_{D_{e^2\alpha}}(x)\}.
\]
The conclusion comes from looking at Example 9.3, and noticing that the linearly dependent terms do indeed contribute to the upper bound since for small enough determinant, we are counting parallel pairs which is determined by the Siegel–Veech constant. 

The extension to holonomy vectors for Veech surfaces given by Theorem 1.6 is as straightforward as in the case of $\varepsilon$-friends.

Appendix A. An application to flows on translation surfaces

The first subsection introduces the notion of disjointness, and gives a criterion via spectral arguments for a family of flows to be disjoint. The second subsection verifies the criterion for non-arithmetic Veech surfaces. The main result of this section is Theorem A.7 below. Theorem A.7 implies Theorem 1.4 by Theorem 1.3; see the proof of Corollary A.8. The proof uses [4] and a largely geometric argument based on Theorem 1.3 and [42].

A.1. A criterion for disjointness

**Definition A.1.** Let $(X, \mathcal{B}, \mu, F'_1)$ and $(Y, \mathcal{A}, \nu, F'_2)$ be two ergodic probability measure preserving flows. A **joining** is an $(F'_1 \times F'_2)^t$-preserved measure with marginals $\mu$ and $\nu$. The product measure $\mu \times \nu$ is called the **trivial joining**. If the trivial joining is the only joining between two systems we say they are **disjoint**.

If $(X, \mathcal{B}, \mu, F'_1)$ and $(Y, \mathcal{A}, \nu, F'_2)$ are isomorphic with $\phi$ being the isomorphism then $(Id \times \phi)_* \mu$ is a joining. Thus if two systems are disjoint then they are not isomorphic (unless they are the one point system).

Let $(X, d)$ be a compact metric space and $(X, \mu)$ be a Borel probability measure space.

**Definition A.2.** Let $F^t$ be a measurable flow on $(X, d)$ that preserves a probability measure $\mu$. We say that $\{t_j\}_{j \in \mathbb{N}} \subset \mathbb{R}$ is a **c-partial rigidity sequence** if for all $j$ there exists $A_j \subset X$ with $\mu(A_j) \geq c$ so that
\[
\lim_{j \to \infty} \int_{A_j} d(F^{t_j}x, x)d\mu = 0.
\]
We say $\{t_j\}$ is a **partial rigidity sequence** if it is a c-partial rigidity sequence for some $c > 0$.

**Definition A.3.** Let $F^t$ be a measurable flow on $(X, d)$ that preserves a probability measure $\mu$. We say that $\{t_j\}_{j \in \mathbb{N}} \subset \mathbb{R}$ is a **mixing sequence** for $F^t$ if for all $f, g \in L^2(\mu)$ we have
\[
\lim_{j \to \infty} \langle f \circ F^{t_j}, g \rangle = \int f d\mu \int g d\mu.
\]
If \( L \subset \mathbb{R} \) has the property that any \( \{t_j\}_{j \in \mathbb{N}} \subset L \) with \( t_j \to \infty \) is a mixing sequence, then we call it a \textbf{mixing set}.

The following lemma is easier than results in the literature and we include a proof for the convenience of the reader.

\textbf{Lemma A.4.} Let \( F_1^t \) and \( F_2^t \) be two \( \mu \)-measure preserving and ergodic flows. If there is a mixing sequence for \( F_1^t \) which is a partial rigidity sequence for \( F_2^t \) then \( F_1^t \) and \( F_2^t \) are disjoint.

The proof uses the following notion: A continuous linear map \( P : L^2((X, \mu)) \to L^2((Y, \nu)) \) is called a \textbf{Markov operator} with adjoint operator \( P^* \) if it satisfies

1. \( P \geq 0 \) and \( P^* \geq 0 \);
2. \( P1_X = 1_Y \) and \( P^*1_Y = 1_X \);
3. \( PU_{F^t_1} = U_{F^t_2}P \) for \( s \in \mathbb{R} \) and where \( U_{F^t_j} \) is the unitary operator associated to \( F^t_j \).

Let \( \tau \) be a joining of \( (F_1^t, \mu) \) and \( (F_2^t, \nu) \). This defines a Markov operator \( P_{\tau} : L^2(\nu) \to L^2(\mu) \) by \( \int_B P_{\tau}(1_A)\mu = \tau(B \times A) \). Formally, this Markov operator \( P_{\tau} \) is the conditional expectation associated to the disintegration of \( \tau \) over \( \nu \). The set of Markov operators are in a one-to-one correspondence with joinings and given a joining \( \tau \) we will refer to the corresponding Markov operator \( P_{\tau} \) as the \textbf{associated Markov operator}. This identification respects the convex structure of preserved measures and so the extreme points come from ergodic joinings. For more on Markov operators, see [16].

\textbf{Proof of Lemma A.4.} Let \( \tau \) be a joining of \( F_1^t \) and \( F_2^t \) and \( P \) be the associated Markov operator. Choose \( c > 0 \) and \( \{t_j\}_{j \in \mathbb{N}} \) so that \( \{t_j\} \) is simultaneously a mixing sequence for \( F_1^t \) and a \( c \)-partial rigidity sequence for \( F_2^t \).

Since \( \{t_j\} \) is a mixing sequence for \( F_1^t \), we have \( PF_1^{t_j} = F_2^{t_j}P \) converge in the weak operator topology to the integral because the same is true for the unitary operators associated with \( F_1^{t_j} \). Recall the weak operator topology is compact on operators with uniformly bounded operator norm. Thus, there is a subsequence of \( t_{j_k} \) so that the sequence of unitary operators associated with \( F_2^{t_{j_k}} \) converges in the weak operator topology to an operator \( V \). Because \( t_j \) is a \( c \)-partial rigidity sequence for \( F_2^t \), \( V \) necessarily has the form \( c\mathrm{Id} + V' \) where \( V' \) is a positive operator. Set \( \int \) to be the bounded linear operator on \( L^2 \) given by integrating, so \( (c\mathrm{Id} + V')P = \int \). Since \( F_1^t \) has a mixing sequence, implying \( F_1^t \) is weakly mixing, we have \( (F_1 \times F_2)^t \) is ergodic. Thus \( \int \) (the Markov operator associated to the product measure) is an extreme point in the set of Markov operators. The equality \( (c\mathrm{Id} + V')P = \int \) combined with \( \int \) being an extreme point implies \( c\mathrm{Id}P = c\int \) and \( V'P = (1 - c)\int \). Thus \( P \) is in fact \( \int \), implying the trivial joining is the only joining.

The following disjointness criterion is motivated by the approach of [8] and is more or less easier than the criterion in [11]. Let \( \{F_\theta^t\}_{\theta \in \mathbb{S}^1} \) be the 1-parameter family of \( \mu \)-measure preserving flows on \( X \) coming from the straight line flow in each direction on \( X \). Let \( \lambda \) be the Lebesgue measure on \( \mathbb{S}^1 \).
Proposition A.5. If

- $F^t_\theta$ is weakly mixing for $\lambda$-almost every $\theta$, and
- for every $L \subset \mathbb{R}$, which is a union of intervals of length at least 1 and has density 1, we have that $F^t_\phi$ has a partial rigidity sequence in $L$ for $\lambda$-almost every $\phi$,

then for $\lambda^2$-almost every $(\theta, \phi) \in (S^1)^2$ we have that $F^t_\theta$ and $F^t_\phi$ are disjoint.

Lemma A.6. Let $F^t$ be a $\mu$-weakly mixing flow, and let $r < \infty$. We can find $L \subset \mathbb{R}$ which is a union of intervals of length at least $r$, has density 1, and any sequence in $L$ going to infinity is a mixing sequence for $F^t$.

The fact that there exists such an $L$ of density 1 is a well known equivalence of weak mixing. The fact that $L$ can be chosen as a union of intervals of length at least 1 is because the unitary operators for $F^t$ where $t \in [0, r]$ is a compact family in the strong operator topology. So, if $L'$ is a mixing set, its $r$-neighborhood is too.

Proof of Proposition A.5. It suffices to consider when $F^t_\theta$ is weakly mixing. By Lemma A.6 choose a set $L$ satisfying the assumptions of Lemma A.6 for $r = 1$. By assumption there is a full measure set $D$ of directions that have a partial rigidity sequence in $L$. By Lemma A.4, for all $\phi \in D$, $F^t_\theta$ and $F^t_\phi$ are disjoint, as desired.

A.2. Application to non-arithmetic Veech surfaces

We now set about adapting the criterion of Proposition A.5 to certain translation surfaces. The arguments have more or less been carried out in [7], [32] and [10]. For the sake of novelty, we present a slightly different, weaker (in particular avoiding ubiquity), argument that is sufficient for our purposes.

Theorem A.7. Let $M$ be a translation surface and $\mu$ denote the measure coming from the area. Let $F^t_\theta$ denote the flow in direction $\theta$ on $M$.

- Assume the flow in almost every direction is weakly mixing.
- Suppose for every $\varepsilon > 0$ there exists $\eta > 0$ so that all but at most an $\varepsilon$-proportion of holonomies of cylinders do not have a $\eta$-friend.

Then the family $\{F^t_\theta\}_{\theta \in S^1}$ with the measure $\mu$ satisfies the assumption of Proposition A.5.

Corollary A.8. Let $M$ be a non-arithmetic Veech surface. Then for almost every $\theta, \psi \in S^1$ we have $F_\theta$ and $F_\psi$ are disjoint. In particular, the flows are not isomorphic. Moreover, for almost every $\theta, \psi \in S^1$ the product flows are uniquely ergodic.

Proof of Corollary A.8 assuming Theorem A.7. First, the flow in almost every direction on any non-arithmetic Veech surface is weak mixing [4]. Theorem 1.3 implies that every non-arithmetic Veech surface has the property that for every $\varepsilon > 0$ there exists $\eta > 0$ so that for all but at most an $\varepsilon$ proportion of holonomies of cylinders do not have a $\eta$-friend. These two results are the assumptions of Theorem A.7 and so by Proposition A.5 the flows in almost every pair of directions on $M$ are disjoint.
We now establish the typical unique ergodicity of the product. Observe that the product of two flows is uniquely ergodic if and only if the flows are uniquely ergodic and disjoint. By [30], for every translation surface the flow in almost every direction is uniquely ergodic. Thus, for almost every \( \theta, \psi \) we have \( F_\theta \times F_\psi \) is uniquely ergodic.

We prove Theorem A.7 in the rest of this section. The key lemma is

**Lemma A.9.** Let \( M \) be a Veech surface. If \( L \subseteq R \) is a union of intervals of length at least 1 with density 1, then a density 1 set of holonomies of saddle connections have lengths contained in \( L \). That is, we have

\[
\lim_{R \to \infty} \frac{|\{(\theta, T) \in S_M : T \in L \cap [0, R)\}|}{|\{(\theta, T) \in S_M : T \in [0, R)\}|} = 1.
\]

**Proof of Lemma A.9.** Denote an element in \( S_M \) by its polar decomposition \((\theta, T)\). We want to find a bound for

\[
\limsup_{R \to \infty} \frac{|\{(\theta, T) \in S_M : T \in L^c \cap [0, R)\}|}{|\{(\theta, T) \in S_M : T \in [0, R)\}|}.
\]

Let \( \delta > 0 \). We split the set in the numerator into two cases, first those elements with \( \eta \)-friends, and then those without \( \eta \)-friends. Having fixed \( \delta \), choose \( \eta \) as given by Theorem 1.3. Then

\[
\limsup_{R \to \infty} \frac{|\{(\theta, T) \in S_M : T \in L^c \cap [0, R) \text{ and } (\theta, T) \text{ has an } \eta \text{-friend}\}|}{|\{(\theta, T) \in S_M : T \in [0, R)\}|} < \delta.
\]

We now consider the case of holonomy vectors with no \( \eta \)-friends, i.e.,

\[
\limsup_{R \to \infty} \frac{|\{(\theta, T) \in S_M : T \in L^c \cap [0, R) \text{ and } (\theta, T) \text{ has no } \eta \text{-friend}\}|}{|\{(\theta, T) \in S_M : T \in [0, R)\}|}.
\]

By quadratic growth, the denominator of (A.1) is bounded below by \( \rho R^2 \) for some \( \rho > 0 \) and \( R > 0 \). By the assumptions on \( L \) the complement \( L^c \cap [0, R) \) is a finite set of \( q \) points, where \( q < R \). We thicken \( L^c \) by placing an interval \( I_i \) of length \( |I_i| = \ell \) at each point. Making \( \ell \) small enough we define \( L^c_\ell \cap [0, R) \) to be the disjoint union of the intervals: \( L^c_\ell \cap [0, R) = \bigsqcup_{i=1}^{q} I_i \).

So by comparing areas, for any interval \( I_i \) we have

\[
|\{(\theta, T) \in S_M : T \in I_i \text{ and } (\theta, T) \text{ has no } \eta \text{-friend}\}| \leq \frac{\pi(\ell + \eta)^2}{\pi \eta^2} = \frac{(\ell + \eta)^2}{\eta^2}.
\]

Summing over the \( q < R \) intervals, we find

\[
|\{(\theta, T) \in S_M : T \in L^c_\ell \cap [0, R) \text{ and } (\theta, T) \text{ has no } \eta \text{-friend}\}| \leq R \frac{(\ell + \eta)^2}{\eta^2}.
\]

Since \( \ell \) can be taken to be non-increasing in \( R \) and \( \rho, \eta \) are independent of \( R \), (A.1) yields

\[
\limsup_{R \to \infty} \frac{|\{(\theta, T) \in S_M : T \in L^c \cap [0, R) \text{ and } (\theta, T) \text{ has no } \eta \text{-friend}\}|}{|\{(\theta, T) \in S_M : T \in [0, R)\}|} \leq \limsup_{R \to \infty} \frac{1}{R} \frac{(\ell + \eta)^2}{\rho \eta^2} = 0.
\]

We conclude by letting \( \delta \to 0 \).
Let $M$ be an area 1 translation surface of genus $g$ and $s$ be the length of the shortest saddle connection on $M$. Set $\sigma = \frac{1}{2g-2}$. We now fix $L \subset \mathbb{R}$, a union of intervals of length at least 1 with density 1. Let

$$\tilde{S}_M(\sigma, N) = \{ (\theta, T) : \theta \text{ is the direction of a periodic cylinder of length } T < N \text{ and volume at least } \sigma \}.\$$

Let

$$\tilde{S}'_M(\sigma, N) = \{ (\theta, T) \in \tilde{S}_M(\sigma, N) : T \in L \}.\$$

**Proposition A.10.** It suffices to show that there exists $\hat{c} > 0$ so that for any interval $J$, and for any $N$ large enough (depending on $J$) we have

$$\lambda \left( \bigcup_{(\theta, T) \in \tilde{S}'_M(\sigma, N)} B \left( \theta, \frac{1}{TN} \right) \cap J \right) > \hat{c} \lambda(J). \quad (A.3)$$

**Proof of Proposition A.10.** We first need a lemma connecting being close to the directions of cylinders which have area at least $c > 0$ to partial rigidity sequences.

**Lemma A.11.** If $v_i$ are the holonomies of the circumferences of cylinders on $Q$ with area at least $c$, $\theta_i$ are their directions and

$$\lim_{i \to \infty} |v_i|^2 |\theta_i - \theta| = 0$$

then $|v_i|$ is a $c$-partial rigidity sequence for $F_{r_o}Q$.

The proof of Lemma A.11 follows directly from the proof of [11, Lemma 12].

We now show that with Lemma A.11, Proposition A.10 follows from the Lebesgue density theorem because (A.3) for all intervals implies that for any $c' < \hat{c}$, for all large enough $N$ we have

$$\lambda \left( \bigcup_{(\theta, T) \in \tilde{S}'_M(\sigma, N)} B \left( \theta, \frac{\epsilon}{TN} \right) \cap J \right) > \epsilon c' \lambda(J). \quad (A.4)$$

To see how (A.4) follows from (A.3), if $B(\theta, \frac{1}{TN}) \subset J$ then $\lambda (B(\theta, \frac{\epsilon}{TN}) \cap J) = \epsilon \lambda (B(\theta, \frac{1}{TN}) \cap J)$.

Now

$$\lambda \left( J \cap \bigcup_{(\theta, T) \in \tilde{S}_M(\sigma, N) : B(\theta, \frac{1}{TN}) \notin J} B \left( \theta, \frac{1}{TN} \right) \right) \leq |\partial J| \frac{1}{sN} = \frac{2}{sN},$$

which clearly goes to zero as $N$ goes to infinity.

Now (A.4) (and the fact that $N \geq T$) establishes that for any $\epsilon > 0$, the complement of

$$\bigcup_{(\theta, T) \in \tilde{S}'_M(\sigma, N)} B \left( \theta, \frac{\epsilon}{TN} \right)$$
has no Lebesgue density points and so \( \cup_{(\theta, T) \in \tilde{S}_M'(\sigma, N)} B(\theta, \frac{c}{T}) \) has full measure. So for almost every \( \phi \in S^1 \) there exists \((\theta_j, T_j) \in \tilde{S}_M'(\sigma, \infty) \) so that
\[
\lim_{j \to \infty} T_j^2 |\theta_j - \phi| = 0.
\]

If \( \phi \) is not in the countable set of direction of elements of \( \tilde{S}_M'(\sigma, \infty) \) then the \( T_j \) necessarily tend to infinity. By Lemma A.11, the \( T_j \) are a \( \sigma \)-partial rigidity sequence for \( F^\phi \), and we have established the theorem. Thus we have shown that (A.3) implies Theorem A.7. \( \blacksquare \)

To prove (A.3), we use a result of Vorobets.

**Theorem A.12.** (Vorobets, [42, Page 16]) Let \( m \) be the sum of the multiplicities of singularities. Let \( c = 2^{2m} \) For large enough \( N \) we have
\[
\bigcup_{(\theta, T) \in \tilde{S}_M(\sigma, N)} B\left(\theta, \frac{c^2}{TN}\right) = S^1. \tag{A.5}
\]

**Derivation of Theorem A.12 from Vorobets.** Let \((\theta, T) \in S^1 \times \mathbb{R}^+ \). For \( \alpha > 0 \) let \( A_{(\theta, T)}(\alpha) \) be the set of directions \( \phi \) so that
\[
T |\sin(\theta - \phi)| \leq \lambda^{-1} c.
\]
Vorobets shows that for any \( N \geq c \) and \( \lambda = \frac{N}{c} \) we have
\[
S^1 = \bigcup_{(\theta, T) \in \tilde{S}_M(\sigma, N)} A_{(\theta, T)}(\lambda).
\]
If \( N \) is big enough, (depending on the shortest saddle connection in \( M \)) \( T |\sin(\theta - \phi)| \leq \left(\frac{N}{c}\right)^{-1} c \) implies \( |\sin(\theta - \phi)| \geq \frac{1}{2} |\theta - \phi| \). So for large enough \( N \) we have that for every \( \phi \) there exists \((\theta, T) \in \tilde{S}_M(\sigma, N) \) so that \( T \frac{1}{2} |\theta - \phi| \leq T |\sin(\theta - \phi)| \leq \left(\frac{N}{c}\right)^{-1} c \) or that \( |\theta - \phi| \leq \frac{c^2}{TN} \). \( \blacksquare \)

**Remark A.13.** For the reader’s convenience, we briefly connect our notation to the notation in [42]. There \( T_0 \) plays the role of \( c \). Because we are assuming our translation surface has area \( 1 \) the \( \sqrt{S} \) term (which denotes the area of the translation surface) does not appear. Also note that Vorobets allows for a smaller \( c \) when \( m = 1, 2 \). In [42], \( T \) plays the role of \( R \).

**Proof of Theorem A.7.** By Proposition A.10 it suffices to show that (A.5) implies (A.3). Let \( J \) be an interval. Observe that
\[
\lambda \left( \bigcup_{(\theta, T) \in \tilde{S}_M(\sigma, N)} B\left(\theta, \frac{1}{TN}\right) \cap J \right) \geq \frac{1}{c^2} \lambda \left( \bigcup_{(\theta, T) \in \tilde{S}_M(\sigma, N)} B\left(\theta, \frac{c^2}{TN}\right) \cap J \right) - \sum_{(\theta, T) \in \tilde{S}_M(\sigma, N) \setminus \tilde{S}_M'(\sigma, N)} 2 \frac{c^2}{TN}. \tag{A.6}
\]
By Lemma A.9 we may assume $|\tilde{S}_M(\sigma, N) \setminus \tilde{S}'_M(\sigma, N)| < \delta N^2$ for all $N$ large enough. Thus for large enough $N$, by Abel summation, 

$$\sum_{(\theta, \mathcal{T}) \in \tilde{S}_M(\sigma, N) \setminus \tilde{S}'_M(\sigma, N)} \frac{e^2}{TN} < 4c^2 \delta.$$ 

So for any $\delta > 0$ we can bound the right hand side of (A.6) by $\frac{1}{c^2} \lambda(J) - C\delta$. This establishes (A.3).

**Remark A.14.** In the previous referenced uses of similar disjointness arguments, [8], [11] one does not assume that the systems are weakly mixing. The author of the appendix considers one of the values of this appendix to be presenting a disjointness criterion of this flavor that is simplified by the presence of density one mixing sequences. However, similar to those earlier arguments, one can remove the condition that the flow in almost every direction on $M$ is weakly mixing by using [11, Proposition 2] in place of Lemma A.4. Lemma A.9 provides the necessary input for this criterion as well. Compare with [8, Corollary 5] which uses [8, Corollary 3]. By a similar argument to the one above, one can show that any Veech surface (including arithmetic Veech surfaces) has the property that the flow in almost every pair of directions is disjoint. However, even without the main result of the present paper, one can use similar spectral methods to prove that for any branched translation cover of a torus (a class of surfaces that includes all arithmetic Veech surfaces, which are branched translation covers of tori, branched over one point) the flow in almost every pair of directions is disjoint and so we omit it.

**Acknowledgments.** We would like to thank MSRI (now SLMath) where this project was started in the Fall of 2019 during the program on Holomorphic Differentials in Mathematics and Physics. The authors would like to thank Jayadev Athreya, Max Goering, Jiyoung Han, and Pedram Safaee for useful discussions, and the anonymous referees for their comments.

**Funding.** C. B. is supported by the Swiss National Science Foundation Grant No. 201557, and would like to thank the Hausdorff Institute for Mathematics in Bonn, where part of this work was completed in the Summer of 2021. S. F. was partially supported by the Deutsche Forschungsgemeinschaft (DFG)–Projektnummer 44546644. J. C. is supported by NSF grants DMS-2055354 and DMS-452762, the Sloan foundation, Poincaré chair, and Warnock chair.

**References**

[1] Athreya, J. S., Cheung, Y., Masur, H.: Siegel–Veech transforms are in $L^2$. J. Mod. Dyn. 14, 1–19 (2019) MR 3959354

[2] Athreya, J. S., Fairchild, S., Masur, H.: Counting pairs of saddle connections. Adv. Math. 431, 55 (2023)

[3] Athreya, J. S., Margulis, G. A.: Logarithm laws for unipotent flows. I. J. Mod. Dyn. 3, 359–378 (2009) MR 2538473

[4] Avila, A., Delecroix, V.: Weak mixing directions in non-arithmetic Veech surfaces. J. Amer. Math. Soc. 29, 1167–1208 (2016) MR 3522612

[5] Avila, A., Forni, G.: Weak mixing for interval exchange transformations and translation flows. Ann. of Math. (2) 165, 637–664 (2007) MR 2299743
[6] Burrin, C., Nevo, A., Rühr, R., Weiss, B.: Effective counting for discrete lattice orbits in the plane via Eisenstein series. Enseign. Math. 66, 259–304 (2020) MR 4254262

[7] Chaika, J.: Homogeneous approximation for flows on translation surfaces. arXiv:1110.6167 (2011)

[8] Chaika, J.: Every ergodic transformation is disjoint from almost every interval exchange transformation. Ann. of Math. (2) 175, 237–253 (2012) MR 2874642

[9] Chaika, J., Forni, G.: Weakly Mixing Polygonal Billiards. arxiv:2003.00890 (2020)

[10] Chaika, J., Frączek, K., Kanigowski, A., Ulcigrai, C.: Singularity of the spectrum for smooth area-preserving flows in genus two and translation surfaces well approximated by cylinders. Comm. Math. Phys. 381, 1369–1407 (2021) MR 4218685

[11] Chaika, J., Hubert, P.: Circle averages and disjointness in typical translation surfaces on every Teichmüller disc. Bull. Lond. Math. Soc. 49, 755–769 (2017) MR 3742443

[12] Dal’Bo, F.: Crossroads between hyperbolic geometry and number theory. In: Strasbourg master class on geometry, IRMA Lect. Math. Theor. Phys. 18, Eur. Math. Soc., Zürich, 183–232 (2012) MR 2931887

[13] Eskin, A., Masur, H.: Asymptotic formulas on flat surfaces. Ergodic Theory Dynam. Systems 21, 443–478 (2001) MR 1827113

[14] Fairchild, S.: A higher moment formula for the Siegel-Veech transform over quotients by Hecke triangle groups. Groups Geom. Dyn. 15, 57–81 (2021) MR 4235747

[15] Federer, H.: Geometric measure theory. Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York, Inc., New York (1969) MR 0257325

[16] Glasner, E.: Ergodic theory via joinings. Mathematical Surveys and Monographs 101, American Mathematical Society, Providence, RI (2003) MR 1958753

[17] Good, A.: Local analysis of Selberg’s trace formula. Lecture Notes in Mathematics 1040, Springer-Verlag, Berlin (1983) MR 727476

[18] Gorodnik, A.: Uniform distribution of orbits of lattices on spaces of frames. Duke Mathematical Journal 122, 549 – 589 (2004)

[19] Gorodnik, A., Weiss, B.: Distribution of lattice orbits on homogeneous varieties. Geom. Funct. Anal. 17, 58–115 (2007) MR 2306653

[20] Greenberg, L.: Discrete groups with dense orbits, 85–104. Princeton University Press (1963)

[21] Gutkin, E., Judge, C.: Affine mappings of translation surfaces: geometry and arithmetic. Duke Math. J. 103, 191–213 (2000) MR 1760625

[22] Hejhal, D. A.: The Selberg Trace Formula for PSL$(2, \mathbb{R})$: Volume 2, 429–506. Springer Berlin Heidelberg, Berlin, Heidelberg (1983)

[23] Huxley, M. N.: Exponential sums and lattice points. II. Proc. London Math. Soc. (3) 66, 279–301 (1993) MR 1199067

[24] Huxley, M. N., Nowak, W. G.: Primitive lattice points in convex planar domains. Acta Arith. 76, 271–283 (1996) MR 1397317

[25] Iwaniec, H.: Introduction to the spectral theory of automorphic forms. Revista Matemática Iberoamericana, Madrid (1995) MR 1325466

[26] Katok, A.: Interval exchange transformations and some special flows are not mixing. Israel J. Math. 35, 301–310 (1980) MR 594335

[27] Katok, S.: Fuchsian groups. Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL (1992) MR 1177168

[28] Kelmer, D.: Approximation of points in the plane by generic lattice orbits. Journal of Modern Dynamics 11, 143–153 (2017)

[29] Kelmer, D., Yu, S.: The second moment of the Siegel transform in the space of symplectic lattices. Int. Math. Res. Not. IMRN 5825–5859 (2021) MR 4251265

[30] Kerckhoff, S., Masur, H., Smillie, J.: Ergodicity of billiard flows and quadratic differentials. Ann. of Math. (2) 124, 293–311 (1986) MR 855297
[31] Ledrappier, F.: Distribution des orbites des réseaux sur le plan réel. C. R. Acad. Sci. Paris Sér. I Math. 329, 61–64 (1999) MR 1703338

[32] Marchese, L., Treviño, R., Weil, S.: Diophantine approximations for translation surfaces and planar resonant sets. Comment. Math. Helv. 93, 225–289 (2018) MR 3811752

[33] Maucourant, F., Weiss, B.: Lattice actions on the plane revisited. Geom. Dedicata 157, 1–21 (2012)

[34] Miyake, T.: Modular forms. English ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin (2006) MR 2194815

[35] Nevo, A., Rühr, R., Weiss, B.: Effective counting on translation surfaces. Adv. Math. 360, 106890 (2020) MR 4031118

[36] Rühr, R., Smilansky, Y., Weiss, B.: Classification and statistics of cut and project sets. J. Eur. Math. Soc. (2023)

[37] Schmidt, W. M.: A metrical theorem in geometry of numbers. Trans. Amer. Math. Soc. 95, 516–529 (1960) MR 117222

[38] Selberg, A.: On the estimation of Fourier coefficients of modular forms. In: Proc. Sympos. Pure Math., Vol. VIII, Amer. Math. Soc., Providence, R.I., 1–15 (1965) MR 0182610

[39] Siegel, C. L.: A mean value theorem in geometry of numbers. Ann. of Math. (2) 46, 340–347 (1945) MR 12093

[40] Veech, W. A.: Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards. Invent. Math. 97, 553–583 (1989) MR 1005006

[41] Veech, W. A.: Siegel measures. Ann. of Math. (2) 148, 895–944 (1998) MR 1670061

[42] Vorobets, Y.: Periodic geodesics on translation surfaces. arxiv:math/0307249 (2003)

[43] Wu, C.: Deloné property of the holonomy vectors of translation surfaces. Israel J. Math. 214, 733–740 (2016) MR 3544700