Can fusion coefficients be calculated from the depth rule?

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Abstract

The depth rule is a level truncation of tensor product coefficients expected to be sufficient for the evaluation of fusion coefficients. We reformulate the depth rule in a precise way, and show how, in principle, it can be used to calculate fusion coefficients. However, we argue that the computation of the depth itself, in terms of which the constraints on tensor product coefficients is formulated, is problematic. Indeed, the elements of the basis of states convenient for calculating tensor product coefficients do not have a well-defined depth! We proceed by showing how one can calculate the depth in an ‘approximate’ way and derive accurate lower bounds for the minimum level at which a coupling appears. It turns out that this method yields exact results for $\hat{su}(3)$ and constitutes an efficient and simple algorithm for computing $\hat{su}(3)$ fusion coefficients.

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1. Introduction

By now various methods have been proposed to calculate fusion rules in WZNW models [1-7]. The first method to be proposed was the depth rule of Gepner and Witten [1]. However it has not received much attention since it did not appear to be very practical, mainly because of the difficulties involved in calculating the depth. Therefore the depth rule has not been tested seriously as a calculation tool for fusion coefficients. Indeed, except for \( \widehat{su}(2) \), only low level examples of \( \widehat{su}(3) \) [1] and \( \widehat{E}_8 \) [8] have been worked out. In this work we examine carefully the depth rule as a computational tool for fusion rules. Section 2 contains a review of the original derivation of the rule (see also [9]).

Recall that a fusion rule tells us which primary fields can arise in the OPE of two given primary fields. More abstractly it defines a product \( \times \) as

\[
\lambda \times \mu = \sum_{\nu} N^{(k)}_{\lambda \mu} \nu \nu
\]

where \( \lambda, \mu \) and \( \nu \) are primary fields. The fusion coefficient \( N^{(k)}_{\lambda \mu} \nu \) gives the number of times \( \nu \) appears in the product \( \lambda \times \mu \). Notice further that \( N^{(k)}_{\lambda \mu} \nu = N^{(k)}_{\lambda \mu C_{\nu}} \), where \( C_{\nu} \) is the field conjugate to \( \nu \). For WZNW models the spectrum generating algebra is a Kač-Moody algebra \( \hat{g} \) at level \( k \) and the primary fields are in one-to-one correspondence with integrable representations of \( \hat{g} \) [1,10].

2. The depth rule of Gepner and Witten.

Let us start by writing explicitly the commutation relations for the untwisted Kač-Moody algebra \( \hat{g} \) at level \( k \) [11]:

\[
[H^i_m, H^j_n] = km \delta^{ij} \delta_{n+m,0}
\]

\[
[H^i_m, E^{\alpha}_n] = \alpha^i E^{\alpha}_{n+m}
\]

\[
[E^{\alpha}_m, E^{\beta}_n] = \frac{2}{\alpha^2} (\alpha \cdot H_{n+m} + km \delta_{n+m,0}) \quad \text{if} \quad \alpha = -\beta \quad (2.2)
\]

\[
= \epsilon(\alpha, \beta) E^{\alpha+\beta}_{n+m} \quad \text{if} \quad \alpha + \beta \quad \text{is a root}
\]

\[
= 0 \quad \text{otherwise}
\]

Here \( \alpha \) and \( \beta \) denote roots and \( \epsilon(\alpha, \beta) \) is a cocycle. To each node (i.e. each simple root) of the extended Dynkin diagram one can associate at least two distinct \( su(2) \) subalgebras.
In particular for the zeroth root, whose finite part is \(-\theta\) (\(\theta\) being the longest root), the explicit forms of the generators \(J^\pm\) and \(J_3\) of the two \(su(2)\) subalgebras we will use are

\[
(i) \quad E^\theta_0, \quad E^{-\theta}_0, \quad \theta \cdot H_0 = [E^\theta_0, E^{-\theta}_0]
\]

\[
(ii) \quad E^{-\theta}_1, \quad E^\theta_{-1}, \quad k - \theta \cdot H_0 = [E^{-\theta}_1, E^\theta_{-1}]
\]

The finite Lie algebra \(\hat{g}\) associated with \(\hat{g}\) is recovered from (2.2) by setting \(m = n = 0\) (this corresponds to the homogeneous gradation). Hence the first \(su(2)\) subalgebra in (2.3) is generated by finite Lie algebra operators while the second one has an intrinsic affine structure. Now let \(\lambda\) denote an affine weight of positive level \(k\), whose finite part \(\lambda\) is an integrable highest weight. There is a one-to-one correspondence between such affine weights and WZNW primary fields, and we denote them by the same symbol. To every field \(\lambda\) we associate a state \(\lambda(0)|0\rangle = |\lambda\rangle\). The degenerate nature of the WZNW primary fields translates into the following constraints for the singular vectors [1]:

\[
(E^{-\theta}_0)^{(\lambda, \theta)+1}|\lambda\rangle = 0 \quad (2.4)
\]

\[
(E^\theta_{-1})^{k-(\lambda, \theta)+1}|\lambda\rangle = 0 \quad (2.5)
\]

Of course there are also singular vectors with \(\theta\) replaced by an arbitrary simple root \(\alpha\).

These singular vectors lead to constraints on three-point functions. The constraints arising from the first type of singular vectors (2.4) being associated with finite Lie algebra generators, they are taken into account in ordinary tensor product coefficients of finite Lie algebras [12]. On the other hand, the other singular vectors yield new constraints on the ordinary tensor product coefficients which depend explicitly on the level \(k\). Since \((\lambda, \theta) \geq (\lambda, \alpha)\) for any root \(\alpha\) of \(\hat{g}\), the maximal constraint is obtained by considering the longest root, i.e., precisely from (2.5). Let us then derive the explicit form of this constraint. For this consider the three-point function

\[
\langle ((E^{-\theta}_1)^{p}\lambda)(z)\mu''(z_1)\nu''(z_2) \rangle \quad (2.6)
\]

where \(\mu''\) is associated to a weight (also at level \(k\)) whose finite part \(\bar{\mu}''\) belongs to the representation with highest weight \(\bar{\mu}\) (similarly for \(\nu''\)). This vanishes if \(p > k - (\lambda, \theta)\) as a result of (2.5). By considering the \(E^\theta_n(z)\) as the modes of the field \(E^\theta(\zeta)\) in a Laurent expansion about \(z\), i.e.

\[
E^\theta(\zeta) = \sum_n E^\theta_n(z)(\zeta - z)^{-n-1}
\]
one can write
\[ E_{-1}^\theta(z) = \oint \frac{d\zeta}{\zeta - z} E^\theta(\zeta) \]

(Factors of \((2\pi i)^{-1}\) in front of contour integrals are everywhere implicit). The three-point function (2.6) can thus be transformed into
\[ \oint \frac{d\zeta_1}{\zeta_1 - z} \cdots \oint \frac{d\zeta_p}{\zeta_p - z} (E^\theta(\zeta_1) \cdots E^\theta(\zeta_p) \lambda(z) \mu''(z_1) \nu''(z_2)) \]

Let us now deform the integration contour of every variable \(\zeta_i\) such that it circles around the singular points \(z_1\) and \(z_2\). Expanding the fields \(E^\theta(\zeta_i)\) in terms of their modes and performing the integrations (in which only the mode \(m = 0\) contributes) one finally gets
\[ \sum_{l=0}^{p} \frac{p!}{l!(p-l)!} \frac{1}{(z - z_1)^l (z - z_2)^{p-l}} \langle \lambda(z)[(E_0^\theta)^l \mu''](z_1)[(E_0^\theta)^{p-l} \nu''](z_2) \rangle \]

As already mentioned, this vanishes if \(p > k - (\lambda, \theta)\). Since all the terms in the sum are independent, they must all vanish separately, i.e.
\[ \langle \lambda(z)[(E_0^\theta)^l \mu''](z_1)[(E_0^\theta)^{p-l} \nu''](z_2) \rangle = 0 \quad \text{if} \quad p > k - (\lambda, \theta) \]  

This holds for any value \(0 \leq l \leq p\).

We conclude that if
\[ (E_0^\theta)^{l_1} \mu'' \equiv \mu' \neq 0 \]
\[ (E_0^\theta)^{l_2} \nu'' \equiv \nu' \neq 0 \]  

for \(l_1 + l_2 = p > k - (\lambda, \theta)\), the nontrivial three-point function \(\langle \lambda \mu' \nu' \rangle\) must vanish. Here \(\mu'\) (reps. \(\nu'\)) is characterized by the fact that one can act \(l_1\) (reps. \(l_2\)) times on it with \(E_0^{-\theta}\) without leaving the representation, i.e., one obtains a non-zero field \(\mu''\) (resp. \(\nu''\)).

Let us now define the depth of the field \(\mu'\) as [1]
\[ d_{\mu'} = \max(l) \quad \text{such that} \quad (E_0^{-\theta})^l \mu' \neq 0 \]  

and similarly for \(\nu'\) (we will come back to this definition later and make it more precise). Since \(l_1 + l_2 < d_{\mu'} + d_{\nu'}\) we conclude that
\[ \langle \lambda(z)\mu'(z_1)\nu'(z_2) \rangle = 0 \quad \text{if} \quad k < d_{\mu'} + d_{\nu'} + (\lambda, \theta) \]
This is the explicit form of the constraints we were looking for. Of course, irrespective of the value of \( k \), it is always true that 

\[
\langle \lambda(z) \mu'(z_1) \nu'(z_2) \rangle = 0 \quad \text{if} \quad \bar{N}_{\bar{\lambda} \bar{\mu} \bar{\nu}} = 0
\]

where \( \bar{N}_{\bar{\lambda} \bar{\mu} \bar{\nu}} \) denotes the coefficient associated with the finite tensor product \( \bar{\lambda} \otimes \bar{\mu} \otimes \bar{\nu} \supset 0 \). Since 

\[
\bar{N}_{\bar{\lambda} \bar{\mu} \bar{\nu}} = \bar{N}_{\bar{\lambda} \bar{\mu} \bar{\nu}}^{C \bar{\nu}}
\]

where \( C \bar{\nu} \) is the conjugate of \( \bar{\nu} \), \( \bar{N}_{\bar{\lambda} \bar{\mu} \bar{\nu}} \) is equal to the number of times \( C \bar{\nu} \) appears in the tensor product \( \bar{\lambda} \otimes \bar{\mu} \). (Recall that a necessary condition for \( \bar{N}_{\bar{\lambda} \bar{\mu} \bar{\nu}} \neq 0 \) is the existence of a \( \bar{\mu}' \) (resp. \( \bar{\nu}' \)) in the highest weight representation \( \bar{\mu} \) (resp. \( \bar{\nu} \)) such that \( \bar{\lambda} + \bar{\mu}' + \bar{\nu}' = 0 \).

We summarize these results in the following depth rule [1]:

\[\text{The coupling } \langle \lambda \mu' \nu' \rangle \text{ vanishes if either } k < d_{\mu'} + d_{\nu'} + (\lambda, \theta) \text{ or } \bar{N}_{\bar{\lambda} \bar{\mu} \bar{\nu}} = 0.\]

Before proceeding to a detailed analysis of the depth rule, let us emphasize the following remarkable fact. As is clear from (2.9), the depth \( d_{\mu'} \) is a property of the finite part of the weight \( \mu' \) (i.e. it involves a generator of the finite Lie algebra). Hence, starting from a singular vector involving the action of a generator not in the finite Lie algebra, namely \( E_{-1}^\theta \), which amounts to a constraint on the affine weight, one ends up with a condition whose only residual affine characteristic is the level. Thus in applying this rule, all computations are done with finite weights.

Notice further that the constraint (2.5) is void when \( k \to \infty \). In that case we are left with purely finite Lie algebra constraints, so that

\[
\lim_{k \to \infty} N_{\lambda \mu \nu}^{(k)} = \bar{N}_{\bar{\lambda} \bar{\mu} \bar{\nu}} \quad \text{(2.11)}
\]

3. Deepening the depth rule

3.1. States vs. weights.

The derivation of the depth rule, reviewed in section 2, depends crucially upon considering correlation functions, which themselves involve states. We have pointed out that WZNW primary fields are in one-to-one correspondence with affine weights \( \lambda \) of positive
level $k$ whose finite parts are highest weights $\tilde{\lambda}$. These in turn are in one-to-one correspondence with states $|\lambda\rangle$. This one-to-one correspondence between weights and states no longer holds if $\lambda'$ has a finite part $\tilde{\lambda}'$ which is not a highest weight, but rather a degenerate weight with multiplicity $\text{mult}(\tilde{\lambda}') > 1$. To this weight correspond $\text{mult}(\tilde{\lambda}')$ states $|\lambda'(i)\rangle$, $i = 1, \ldots, \text{mult}(\tilde{\lambda}')$. This clarification is crucial since the depth is a property of states, not of weights, and it is the states which characterize a given coupling. The expression (2.9) for the depth should then be sharpened:

$$d_{\mu'} = \max(l) \text{ such that } (E_0^{-\theta})^l|\mu'(i)\rangle \neq 0$$

(3.2)

Similarly the notation $\langle \lambda\mu'\nu' \rangle$ should now stand for the correlation $\langle \lambda\mu'_{(j)}\nu'_{(\ell)} \rangle$ where the doublet $(j, \ell)$ can take $\bar{N}_{\tilde{\lambda}\tilde{\mu}\tilde{\nu}}$ values.

An aesthetic drawback of the original formulation of the depth rule is the lack of symmetry between the three states. This can be remedied by considering, instead of a three-point function, the following four-point function

$$\langle (E_0^{-\theta})^p I(0)\lambda''_{(i)}(z_1)\mu''_{(j)}(z_2)\nu''_{(\ell)}(z_3) \rangle$$

which vanishes when $p \geq k + 1$. Here $I$ denotes the identity field, associated with a null finite weight. Following the steps of the previous section, we obtain that the three-point function $\langle \lambda'_{(i)}(z_1)\mu'_{(j)}(z_2)\nu'_{(\ell)}(z_3) \rangle$ vanishes if:

$$k < d_{\mu''_{(j)}} + d_{\nu''_{(\ell)}} + d_{\lambda''_{(i)}}$$

(3.3)

where $\tilde{\mu}' + \tilde{\nu}' + \tilde{\lambda}' = 0$. Notice that this reduces to (2.10) when $\tilde{\lambda}' = \tilde{\lambda}$, since $d_{\lambda} = (\lambda, \theta)$. Eq. (3.3) is a symmetrized version of the depth constraint, which makes manifest the symmetry of the fusion coefficient $N^{(k)}_{\mu\nu\lambda}$ under the interchange of any two labels.

However, for practical calculations, it is preferable to fix two of the states, that is to set $\lambda'_{(i)} = \lambda$ and $\nu'_{(\ell)} = -C\nu$ and consider then the three-point function

$$\langle \lambda(z_1)\mu'_{(i)}(z_2)(-C\nu)(z_3) \rangle, \quad i = 1, \ldots, \bar{N}_{\tilde{\lambda}\tilde{\mu}\tilde{\nu}}$$

(3.4)

Here we stress that the $\tilde{\mu}'_{(i)}$'s are understood to be the states which, when coupled to $\tilde{\lambda}$ and $-C\tilde{\nu}$, give the scalar representation. The advantage of (3.4) is that only one out of
the three weights is neither a highest nor a lowest weight, which greatly simplifies the calculations. Since the depth of \(-C\nu\) is obviously zero, (3.4) vanishes whenever

\[ k < d_{\mu'_{ij}} + (\lambda, \theta) \]  

(3.5)

3.2. Threshold level \(k_0^{(i)}\) and the reformulation of the depth rule

When \(k \to \infty\), there are no affine constraints, so that \(\langle \lambda \mu'_{ij} (-C\nu) \rangle \neq 0\). This actually holds true for all values of \(k\) sufficiently large that one does not ‘hit’ the singular vector (2.5). For \(k < d_{\mu'_{ij}} + (\lambda, \theta)\), the coupling must vanish as a result of the depth constrain t. As a result, the smallest value of the level at which the coupling will be nonzero is

\[ k_0^{(i)} \equiv d_{\mu'_{ij}} + (\lambda, \theta) \]  

(3.6)

For each of the \(\bar{N}_{\bar{\lambda} \bar{\mu} \bar{\nu}}\) couplings associated to the tensor product \(\bar{\lambda} \otimes \bar{\mu} \otimes \bar{\nu} \supset 0\), there is then a threshold level, \(k_0^{(i)}\), such that for \(k \geq k_0^{(i)}\) the coupling is non zero, while for \(k < k_0^{(i)}\) it vanishes. Accordingly, the fusion coefficients are completely determined by the tensor product coefficients and by the possible values of \(k_0^{(i)}\) given above. Suppose now that the values of \(k_0^{(i)}\) are ordered such that \(k_0^{(i)} \leq k_0^{(i+1)}\). We can then reformulate the depth rule in a precise way as:

\[ N_{\lambda \mu \nu}^{(k)} = \max(i) \text{ such that } k \geq k_0^{(i)} \text{ and } \bar{N}_{\bar{\lambda} \bar{\mu} \bar{\nu}} \neq 0 \]
\[ = 0 \text{ if } k < k_0^{(1)} \text{ or } \bar{N}_{\bar{\lambda} \bar{\mu} \bar{\nu}} = 0 \]  

(3.7)

with \(k_0^{(i)}\) defined in (3.6) in terms of the depth.

It remains to see how to calculate the depth. But first we introduce an auxiliary concept, the depth charge.

3.3. Introducing the “depth charge” \(a_\theta\).

Suppose we have a basis of states such that \(E_0^{-\theta}\) applied to an element of this basis either vanishes or gives another single element. That is, one never gets linear combinations of basis elements by applying \(E_0^{-\theta}\). Then the depth is essentially the maximum number of times one can act on a given state with \(E_0^{-\theta}\) without leaving the representation. If
we denote by $|jm\rangle$ the projection of this state on the $su(2)$ subalgebra associated to $\theta$ ($J^2 = j(j+1)$ and $m$ is the $J_3$ eigenvalue) then the depth is simply $j + m$. We will write

$$a_\theta = 2j$$

and call $a_\theta$ the depth charge. For a state $|\mu'_{(i)}\rangle$, $m$ is the eigenvalue of $\theta \cdot H_0$, and is given by

$$(\mu', \theta) = 2m$$

Therefore, the depth can be reexpressed as

$$2d_{\mu'_{(i)}} = a_\theta(\mu'_{(i)}) + (\mu', \theta)$$

The calculation of $k_{0}^{(i)}$ then boils down to that of $a_\theta$. In terms of $a_\theta$, (3.6) reads

$$2k_0^{(i)} = a_\theta(\mu'_{(i)}) + (\lambda + \nu, \theta)$$

where we used $(\mu', \theta) = (C\nu - \lambda, \theta) = (\nu - \lambda, \theta)$.

3.4. Calculation of the depth.

In order to test the depth rule, one needs to know how to calculate the depth charge. For this one must answer the following three questions:

1. How can we describe states explicitly?
2. How can we characterize the three states appearing in a three-point function?
3. Given a state, how do we calculate the depth charge $a_\theta$?

These questions are looked at in the next section. From now on the discussion will be restricted to $su(N)$ for simplicity. However the results presented here can be generalized to other algebras.

4. The depth machinery

4.1. A basis for states: standard tableaux.

An $su(N)$ integrable highest weight $\bar{\lambda} = \sum_{i=1}^{N-1} \lambda_i \omega^i = (\lambda_1, \ldots, \lambda_{N-1})$, where $\omega^i$ is the $i$th fundamental weight, has a natural representation in terms of a Young tableau with
$\lambda_1 + \ldots + \lambda_{N-1}$ boxes on the first row, $\lambda_2 + \ldots + \lambda_{N-1}$ boxes on the second row, up to $\lambda_{N-1}$ boxes on the $(N - 1)$th row. Standard tableaux (also called semi-standard tableaux [13]) are Young tableaux in which each box is numbered according to the following rule. Let $c_{i,j}$ be the number appearing in the box on the $i$th row and the $j$th column. These numbers must satisfy the following constraints:

$$1 \leq c_{i,j} \leq N, \quad c_{i,j} \leq c_{i,j+1}, \quad c_{i,j} < c_{i+1,j}$$

In words, the numbers are non-decreasing from left to right and strictly increasing from top to bottom [14].

The allowed standard tableaux of shape $\bar{\lambda}$ are in one-to-one correspondence with the states in the highest weight representation $\bar{\lambda}$. Weights $\bar{\lambda}'$ with multiplicity $\text{mult}(\bar{\lambda}')$ are associated with $\text{mult}(\bar{\lambda}')$ different standard tableaux. Thus standard tableaux provide a convenient basis for states.

Notice that the weight of a standard tableau is obtained by adding the weights of all its boxes. The latter are easily obtained from the fundamental representations: $\omega^k$ is associated with a single column of $k$ boxes containing the numbers $1, 2, \ldots, k$.

**Example 1:**

For $su(3)$ the standard tableaux of the highest weight states of the fundamental representations are

$$(1, 0) \leftrightarrow \begin{array}{c} 1 \\ 2 \\ 3 \end{array}, \quad (0, 1) \leftrightarrow \begin{array}{c} 1 \\ 2 \\ 3 \end{array} , \quad (0, 0) \leftrightarrow \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

from which we conclude that $(-1, 1) \leftrightarrow \begin{array}{c} 2 \\ 3 \end{array}$ and $(0, -1) \leftrightarrow \begin{array}{c} 3 \end{array}$. The adjoint representation $(1, 1)$ contains 8 standard tableaux:

$\begin{array}{cccccccc}
11 & 12 & 13 & 11 & 12 & 13 & 22 & 23 \\
(1, 1) & (1, 2) & (0, 0) & (2, -1) & (0, 0) & (1, -2) & (-2, 1) & (-1, -1)
\end{array}$

Notice that to the doubly degenerate weight $(0, 0)$ one associates two distinct standard tableaux, i.e. two distinct states.

In the following we will use an equivalent representation of this basis, the so-called Gelfand-Tsetlin patterns [15]. To a given standard tableau we associate the following
triangular array of numbers:

\[
\begin{array}{cccc}
\beta_1^{(N)} & \beta_2^{(N)} & \cdots & \beta_N^{(N)} \\
\beta_1^{(N-1)} & \cdots & \beta_{N-1}^{(N-1)} \\
\cdots & \cdots & \cdots & \cdots \\
\beta_1^{(2)} & \beta_2^{(2)} & \cdots & \beta_1^{(1)}
\end{array}
\]  \hspace{1cm} (4.2)

such that \( \beta_i^{(j)} \) is the number of boxes containing numbers less or equal to \( j \) in the \( i \)th row of the standard tableau.

**Example 2:**

The following standard tableau and Gelfand-Tsetlin (GT) pattern corresponding to the weight \((-2, 1, 0)\) in the representation \((1, 2, 1)\) of \(su(4)\) are equivalent:

\[
\begin{array}{cccc}
1 & 2 & 2 & 4 \\
2 & 3 & 4 \\
3
\end{array} \approx
\begin{array}{cccc}
4 & 3 & 1 & 0 \\
3 & 2 & 1 \\
3 & 1 \\
1
\end{array}
\]

**4.2. Identifying states in tensor products.**

Given a triple product involving a highest weight state \( \bar{\lambda} \) and a lowest weight state \(-C\bar{\nu}\), what are the standard tableaux of shape \( \bar{\mu} \) and weight \( \bar{\mu}' = C\bar{\nu} - \bar{\lambda} \) that can appear in the product? To answer this question we first write down the \( \text{mult}(\bar{\mu}') \) standard tableaux of weight \( \bar{\mu}' \). Among these, the \( N_{\bar{\lambda}\bar{\mu}\bar{\nu}} \) standard tableaux contributing to the product \( \bar{\lambda} \otimes \bar{\mu} \otimes \bar{\nu} \) are those which, when inserted column by column from right to left into the Young tableau of \( \bar{\lambda} \), still give a regular tableau at every step. The insertion is made by adding to the \( i \)th row of the Young tableau the boxes of the standard tableau marked by \( i \) [16].

**Example 3:**

Consider the product \((1, 1) \otimes (2, 1) \supset (1, 0)\). We are looking for standard tableaux of shape \((2, 1)\) and weight \((0, -1)\). There are two of them

\[
\begin{array}{cc}
1 & 3 \\
2 & 3 \\
3
\end{array} \quad \text{and} \quad \begin{array}{cc}
1 & 2 \\
3 & 3
\end{array}
\]

Adding the first standard tableau to \( \begin{array}{cc} \hline 1 & 3 \\ \hline 2 & 3 \\ \hline \end{array} \) (the Young tableau of shape \((1, 1)\)) column by column from right to left, one ends with a non-regular tableau after 2 steps: the third
row has two boxes and the second row only one. Thus the first standard tableau does not contribute to the product, while the second one does (as is easily checked).

We now know how to characterize the states in triple products involving one highest and one lowest weight state. In general it is also possible to describe all the states in a given triple product, but this requires the introduction of further concepts which are not central in our argument, and consequently they are introduced in the appendix only.

4.3. Calculation of the depth charge for the standard tableaux.

A GT pattern indicates the chain of embeddings $u(2) \subset u(3) \subset \ldots \subset u(N)$ where the $su(2) \subset u(2)$ subalgebra is defined with respect to the root $\alpha_1$, the $su(3) \subset u(3)$ subalgebra is defined with respect to the roots $\alpha_1$ and $\alpha_2$, etc [15]. As a result the spin $j$ of a given state with respect to the $su(2)$ subalgebra associated to $\alpha_1$ can be read off directly from the GT pattern. Denoting by $a_{\alpha_1}$ twice this spin, one has $a_{\alpha_1} = \beta^{(2)}_1 - \beta^{(2)}_2$, where the $\beta$'s are defined just after Eq.(4.2). Thus the properties of the states with respect to the $su(2)$ subalgebra associated with $\alpha_1$ are easily extracted.

On the other hand the action of the operators $E_0^{\pm \theta}$ on standard tableaux or GT patterns is not so simple. Acting with $E_0^{\pm \theta}$ on a standard tableau generically yields a linear combination of standard tableaux with the same weight. Hence a GT pattern does not have a well-defined depth charge. A first consequence of this observation is that only lower and upper bounds for $a_\theta$ can be calculated for states whose weights have multiplicity $> 1$. These bounds are the minimal and maximal values that $a_\theta$ can take for all the states corresponding to a given weight (not just for the states contributing to the tensor product). As far as $k_0$ is concerned only the lower bound is relevant, but does not turn out to be very useful.

Thus one faces a problem with the depth rule approach for calculating fusion rules. In this scheme fusion coefficients are truncated tensor product coefficients. However the basis convenient for the calculation of tensor product coefficients (namely, standard tableaux) is not convenient when time comes to evaluate the degree of truncation appropriate to a certain level $k$ in fusion rules.

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3 Notice however that for a coupling $\langle \lambda \mu' \nu' \rangle$ such that $\nu' = -C \nu$ and $\text{mult}(\bar{\mu}') = \bar{N}_{\lambda \bar{\mu} \nu}$ we can say better. Indeed if we know all possible values of $a_\theta$ which corresponds to the weights, these values can be used in eq.(3.10) to calculate the possible values of $k_0$. 
4.4. Introducing the crystal depth charge $a^c_\theta$.

In spite of this situation, we will try to make further progress, motivated by the following observation. It is well known that fusion rules for $\widehat{su}(N)$ at level $k$ are equivalent to restricted tensor products in the quantum version of $su(N)$ with $q$ given by [17]:

$$q = \exp \left\{ \frac{2\pi i}{k + N} \right\}$$

On the other hand, at $q = 0$ the structure of the quantum group simplifies considerably [18] (the group ‘crystalizes’). For our purpose it is remarkable that an explicit realization of a crystal base is given by the standard tableaux [19]. Furthermore, it is a particularity of $q = 0$ that standard tableaux have a well-defined value of $a_\theta$. This prompts us to use these values of $a_\theta$ computed for the quantum group at $q = 0$, which we denote $a^c_\theta$ (the c stands for crystal), in the formula for $k_0$. Although the logical motivation for making such a naive extrapolation from $q = 0$ to $q = \exp\{2\pi i/(k + N)\}$ may look weak, it turns out that this procedure leads us to rather accurate results and, for $\widehat{su}(3)$, exact results!

One complication did arise, however. There is a one-to-one correspondence between the three couplings

$$\langle \lambda \mu'_{(i)}(-C\nu) \rangle, \langle \mu \nu'_{(i)}(-C\lambda) \rangle, \langle \nu \lambda'_{(i)}(-C\mu) \rangle \ .$$

(This correspondence can be seen for instance via the Berenstein-Zelevinsky triangles, to be introduced in section 6.) We found that the values of $k_0^{(i)}$ calculated in this way differed in general, for the 3 couplings above. It was therefore necessary to take the maximum of the values so obtained. We symbolise this by rewriting (3.6) as

$$k_0^{(i)} = \max\{d_{\mu'_{(i)}} + (\lambda, \theta)\} \ . \quad (4.3)$$

As already pointed out, standard tableaux or GT patterns naturally single out the $su(2)$ subalgebra associated with the root $\alpha_1$. Thus, in order to calculate $a_\theta$ we will make a suitable ‘rotation’ of the GT pattern such that $a_\theta$ will be the difference of the new entries sitting on the line second from the bottom. But since this operation is justified at $q = 0$ only, it is $a^c_\theta$ that we obtain in this way. Unfortunately, in order to define the explicit operation of ‘rotation’ some more technology must be introduced.
Let us define the operators $t_i$ which act on the GT patterns as follows [20]:

$$
t_i : \begin{cases} 
\beta_j^{(k)} \rightarrow \beta_j^{(k)} & \text{if } k \neq i \\
\beta_j^{(i)} \rightarrow \min(\beta_j^{(i+1)}, \beta_j^{(i-1)}) + \max(\beta_j^{(i+1)}, \beta_j^{(i-1)}) - \beta_j^{(i)} 
\end{cases}$$

If $\beta_{j-1}^{(i-1)}$ is absent it is set equal to $\infty$ and similarly if $\beta_j^{(i-1)}$ is absent it is set equal to zero. Then let us introduce the operator

$$\sigma = t_1 t_2 \ldots t_{N-1}$$

for $su(N)$. These operators have been introduced to describe the action of the Weyl group directly on GT patterns. The basic Weyl reflections $s_i$ (with respect to $\alpha_i$) are simply [20]

$$s_i = \sigma^{(i-1)} t_i \sigma^{-(i-1)}$$

This is equivalent to the action defined by Lascoux-Schützenberger directly on the standard tableaux [21].

**Example 4:**

Here are various operations on a particular state of weight $(0, 1, -1, 0)$ in the highest weight representation $(0, 1, 1, 1)$ of $su(5)$ producing in the end a state of weight $(0, 0, 1, -1)$:

\[
\begin{array}{cccc}
1 & 1 & 3 & 3 3 2 1 0 \\
2 & 2 & 5 & 3 2 2 0 \\
4 & 4 & & 3 3 1 0 \\
5 & & & 3 3 1 0 \\
\end{array} = \begin{array}{cccc}
3 & 3 2 1 0 & 3 3 2 1 0 & 3 3 2 1 0 \\
3 2 0 & 3 3 1 0 & 3 3 1 0 & 3 3 1 0 \\
2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 \\
\end{array}
\]

\[
\begin{array}{cccc}
3 3 2 1 0 & 3 3 2 1 0 & 3 3 2 1 0 & 3 3 2 1 0 \\
3 3 1 0 & 3 3 1 0 & 3 3 1 0 & 3 3 1 0 \\
3 2 1 & 3 2 1 & 3 2 1 & 3 2 1 \\
3 1 & 3 1 & 3 1 & 3 1 \\
2 & 2 & 2 & 2 \\
\end{array} \cong \begin{array}{cccc}
1 & 1 & 2 & 1 1 2 \\
2 & 3 & 4 & 2 3 4 \\
3 & 5 & 5 & 3 5 5 \\
\end{array}
\]

This sequence of operations is nothing but the action of $\sigma$.

Now, $\sigma$ turns out to be the desired ‘rotation’ operator. Indeed, from the results of [20] one can show that

$$E_{0}^{-\theta} = \sigma^{-1} E_{0}^{\alpha_i} \sigma .$$
Example 5:

The action of $\sigma = t_1 t_2$ on the adjoint representation of $su(3)$ is shown on Fig.1. One sees that this action amounts to a rotation of the peripheral weights by $120^\circ$ and to an interchange of the two inner states. The $\theta$ direction has been rotated to the $\alpha_1$ direction except for a ‘sign’, which illustrates the fact that $E_0^{-\theta}$ (a $J_-$ operator) is actually related to $E_0^{\alpha_1}$ (a $J_+$ operator).

In terms of GT patterns this means that $a_\theta^c$ can be calculated as follows. Let us write

$$\tilde{GT} \equiv \sigma(GT)$$

where the entries of $\tilde{GT}$ are denoted $\tilde{\beta}_j^{(i)}$. Then

$$a_\theta^c = \tilde{\beta}_1^{(2)} - \tilde{\beta}_2^{(2)}$$

Example 6:

The following state corresponds to the weight $(0, -1, 0)$ in the $su(4)$ highest weight representation $(1, 1, 1)$:

$$\begin{array}{ccc}
1 & 3 & 4 \\
2 & 4 & \\
3 &
\end{array} \cong \begin{array}{ccc}
3 & 2 & 1 & 0 \\
2 & 1 & 1 \\
1 & 1 &
\end{array} \xrightarrow{\sigma} \begin{array}{ccc}
3 & 2 & 1 & 0 \\
3 & 1 & 0 \\
3 & 0 &
\end{array} \xrightarrow{} \begin{array}{ccc}
3 & 0 & \\
2 &
\end{array} : a_\theta^c = 3 - 0 = 3$$

Similarly the value of $a_\theta^c$ for the standard tableau considered in example 4 is 2.

4.5. A simple rule to calculate $a_\theta^c$ for $su(3)$.

In this section we derive a simple rule for computing the value of $a_\theta^c$ directly from standard tableaux in the case of $su(3)$. In this case $\sigma = t_1 t_2$, but since $t_1$ affects only the last row of the GT pattern, it is sufficient to consider only $t_2$. Hence one has

$$\begin{array}{ccc}
a & b & c \\
d & e & \end{array} \xrightarrow{t_2} \begin{array}{ccc}
a & b & c \\
\tilde{d} & \tilde{e} & f \\
f & f
\end{array}$$

where

$$\tilde{d} = a + \max(b, f) - d$$

$$\tilde{e} = \min(b, f) + c - e$$
Dropping from the outset columns of three boxes from the standard tableau, one can set $c = 0$ (this amounts to setting the irrelevant $u(1)$ charge of $u(1) \otimes su(N) \subset u(N)$ to zero). Thus

$$a_{\theta}^c = a - d + e + \max(b, f) - \min(b, f)$$

$$= a - d + e + |b - f|$$

This is easily checked to be equivalent to

$$a_{\theta}^c = \sum_{columns} |\text{number of 1's} - \text{number of 3's}|$$

(4.4)

In other words, the value of $a_{\theta}^c$ of a given standard tableau can be calculated by adding the values of $a_{\theta}^c$ corresponding to each of its columns. Furthermore, in each column the value of $a_{\theta}^c$ is just the number of 1’s minus the number of 3’s (in absolute value).

**Example 7:**

$$a_{\theta}^c \left( \begin{array} {c c c} 1 & 1 & 2 \\ 2 & 3 & 3 \end{array} \right) = 3$$

5. Testing the depth rule with $a_{\theta}^c$.

We are now in a position to test the depth rule with crystal depth charge through various examples.

5.1. The $\hat{su}(3)$ case.

**Example 8:**

Let us first consider a classic example, namely the triple product $(1, 1) \otimes (1, 1) \otimes (1, 1)$. It is well-known that $(1, 1) \otimes (1, 1) = 2(1, 1) \oplus \ldots$. In fusion rules one copy of $(1, 1)$ arises at level 2 while the other appears at level 3. Indicating the value of $k_0$ by a subscript one has

$$(1, 1) \otimes (1, 1) = (1, 1)_2 \oplus (1, 1)_3 \oplus \ldots$$

To the weight $(0, 0)$ correspond two standard tableaux of shape $(1, 1)$:

$$\begin{array} {c c} 1 & 2 \\ 3 \end{array} \quad \text{and} \quad \begin{array} {c c} 1 & 3 \\ 2 \end{array}$$
Their values of $a_\theta^c$ are respectively 0 and 2 (c.f. (4.4)). Thus here (3.10) gives directly $k_0 = 2, 3$, which is the correct result (permutations are not necessary since the three weights are equal).

Using a computer program we have tested the depth rule through (4.3) with the depth given by (3.9) and $a_\theta$ calculated at $q = 0$ by (4.4). We computed all fusion coefficients up to level 10 and found perfect agreement with the results obtained from the Kač-Walton formula [3]. This led us to suspect that the depth rule with crystal depth charge is correct for $\widehat{su}(3)$. Actually, this can be proved directly by relating our algorithm to other known algorithms for $\widehat{su}(3)$. We will return to this in the next section.

5.2. An $\widehat{su}(4)$ counterexample.

Here we present an example which shows that the depth rule with crystal depth charge is not correct in general. The simplest counterexample that we found is the following $\widehat{su}(4)$ product:

Example 9:

\[(1, 2, 1) \otimes (1, 2, 1) = 4(1, 2, 1)_5 \oplus (1, 2, 1)_6 \oplus \ldots \] (5.2)

To the weight $(0, 0, 0)$ in the representation $(1, 2, 1)$ correspond seven standard tableaux, among which five (obtained by the method of section 4.2) contribute to the product:

\[
\begin{array}{cccc}
1 & 1 & 2 & 3 \\
2 & 3 & 4
\end{array}, \quad
\begin{array}{cccc}
1 & 1 & 2 & 3 \\
2 & 4 & 4
\end{array}, \quad
\begin{array}{cccc}
1 & 1 & 2 & 4 \\
2 & 3 & 3
\end{array}, \quad
\begin{array}{cccc}
1 & 1 & 3 & 3 \\
2 & 2 & 4
\end{array}, \quad
\begin{array}{cccc}
1 & 1 & 3 & 4 \\
2 & 2 & 4
\end{array}, \quad
\begin{array}{cccc}
1 & 1 & 3 & 4 \\
2 & 2 & 4
\end{array}, \quad
\begin{array}{cccc}
1 & 1 & 3 & 4 \\
3
\end{array}
\]

The corresponding values of $a_\theta^c$, calculated according to the rule given in section 3.3, are respectively 0, 2, 2, 2, 4. Hence formula (3.10) yields the following values of $k_0$:

\[k_0 = 4, 5, 5, 5, 6\]

which disagrees with 5.2.
6. $\widehat{su}(3)$: relation with other methods.

In this section we show that, in the case of $\widehat{su}(3)$, the depth rule with crystal depth charge is equivalent to all other known algorithms for computing $\widehat{su}(3)$ fusion rules. This equivalence is obtained via the Berenstein-Zelevinsky (BZ) triangles, introduced in section 6.1. We then reexpress $k_0$ as given by eq. (3.10) in terms of the entries of these triangles. Next we recall that the generating function for $\widehat{su}(3)$ fusion rules obtained in [5] (and whose correctness was proven by Cummins [22]) can be translated into a simple formula for $k_0$. This in turn can also be reexpressed in terms of the parameters of the BZ triangles. The resulting formula for $k_0$ is equivalent to that obtained in [23] by other methods. We then show that it is also equivalent to the formula derived from the depth rule with $a_\theta$.

6.1. Berenstein-Zelevinsky triangles for tensor product coefficients.

Consider the set of three $su(3)$ highest weights $(\lambda_1, \lambda_2), (\mu_1, \mu_2)$ and $(\nu_1, \nu_2)$ (we give here the Dynkin labels). Berenstein and Zelevinsky [24] showed that the number of triangles one can construct according to the following rules:

\[
\begin{align*}
& a_4 \quad a_3 \quad a_5 \\
& a_2 \quad a_6 \\
& a_1 \quad a_9 \quad a_8 \quad a_7
\end{align*}
\quad \text{such that} \quad
\begin{align*}
a_1 + a_2 &= \lambda_1 \\
a_3 + a_4 &= \lambda_2 \\
a_4 + a_5 &= \mu_1 \\
a_6 + a_7 &= \mu_2 \\
a_7 + a_8 &= \nu_1 \\
a_9 + a_1 &= \nu_2 \\
a_2 + a_3 &= a_6 + a_8 \\
a_3 + a_5 &= a_9 + a_8 \\
a_5 + a_6 &= a_2 + a_9
\end{align*}
\] (6.2)

gives the value of $\tilde{N}_{\lambda\mu\nu}$. Such triangles make manifest most of the symmetries of the tensor product coefficients.

Example 10:

Corresponding to the coupling $(2, 2) \otimes (2, 2) \otimes (2, 2)$, three BZ triangles can be constructed:

\[
\begin{array}{cccc}
0 & 1 & 2 \\
2 & 2 & 1 & 1 & 0 & 0 \\
2 & 2 & ' & 1 & 1 & ' & 0 & 0 \\
0 & 2 & 2 & 0 & 1 & 1 & 1 & 2 & 0 & 0 & 2
\end{array}
\]

and accordingly the multiplicity of the coupling is 3.
The states involved in a specific coupling can be read off a triangle as follows. Consider the product $\bar{\lambda} \otimes \bar{\mu} = C\bar{\nu} \oplus \ldots$ associated with the BZ triangle (6.2). The state $|\bar{\mu}'(i)\rangle$ (of weight $C\bar{\nu} - \bar{\lambda}$) in this coupling is described by the GT pattern

$$
\begin{align*}
\mu_1 + \mu_2 & \quad \mu_2 & \quad 0 \\
\mu_1 + \mu_2 - a_4 & \quad \mu_2 - a_6 \\
\mu_1 + \mu_2 - a_4 - a_2
\end{align*}
$$

(6.3)

Example 11:

The GT patterns and corresponding standard tableaux (in the representation $\bar{\mu}$) associated with the three BZ triangles of the last example are (in the same order)

\begin{align*}
4 & 2 & 0 \\
4 & 0 & \approx & 1 & 1 & 2 & 2 \\
2 & & & 3 & 3
\end{align*}

\begin{align*}
4 & 2 & 0 \\
3 & 1 & \approx & 1 & 1 & 2 & 3 \\
2 & & & 2 & 3
\end{align*}

\begin{align*}
4 & 2 & 0 \\
2 & 2 & \approx & 1 & 1 & 3 & 3 \\
2 & & & 2 & 2
\end{align*}

6.2. $k_0$ from BZ triangles.

Consider the BZ triangle (6.2) describing the coupling between $\bar{\lambda}$, $\bar{\nu}' = -C\bar{\nu}$ and $\bar{\mu}'$ given by (6.3). The value of $a^c_\theta(\mu')$ can be obtained by acting on the GT pattern (6.3) with $\sigma = t_1t_2$ (actually $t_2$ alone is enough in this case) and by taking the difference of the entries on the second line. This gives

$$a^c_\theta(\mu') = a_4 + a_7 + |a_5 - a_2|$$

Therefore one has

$$a^c_\theta(\mu') + (\nu + \lambda, \theta) = 2 \max(a_4 + \nu_1 + \nu_2, a_7 + \lambda_1 + \lambda_2)$$

The permuted versions of the above are evaluated similarly. As a result, one can rewrite (3.10) in the form (There are only 3 permutations here because interchanging $\nu$ and $\lambda$ does not change the value of the crystal depth charge $a^c_\theta(\mu')$.) For $a_2 \leq a_5 \leq a_8$, this boils down to

$$k_0 = \max\{a_7 + \lambda_1 + \lambda_2, a_4 + \nu_1 + \nu_2\} \ .$$

But $(a_4 + \nu_1 + \nu_2) - (a_7 + \lambda_1 + \lambda_2) = a_5 - a_2 \geq 0$, by assumption, so we obtain

$$k_0 = a_4 + \nu_1 + \nu_2 \ .$$

(6.4)
It can similarly be shown that if we assume instead $a_2 \leq a_8 \leq a_5$, the formula (6.4) remains valid, and so holds for $a_2 \leq \min(a_5, a_8)$.

**Example 12:**

From the above formula one readily computes the values of $k_0$ for the three triangles of example 10 to be respectively 4, 5 and 6. (Notice that for these 3 triangles the inequality $a_2 \leq \min(a_5, a_8)$ is satisfied. For less obliging triangles, however, one simply uses their symmetry to rotate them until the inequality is obeyed.)

6.3. $k_0$ from the decomposition of BZ triangles into elementary couplings.

Recall that every coupling $\bar{\lambda} \otimes \bar{\mu} \otimes \bar{\nu}$ can be decomposed into a product of elementary couplings $E_i$ [25]. For $su(3)$ the 8 elementary couplings are

\[
E_1 = (1,0)(0,1)(0,0) \quad E_2 = (1,0)(0,0)(0,1) \quad E_3 = (0,0)(1,0)(0,1)
\]

\[
E_4 = (0,1)(1,0)(0,0) \quad E_5 = (0,1)(0,0)(1,0) \quad E_6 = (0,0)(0,1)(1,0)
\]

\[
E_7 = (1,0)(1,0)(1,0) \quad E_8 = (0,1)(0,1)(0,1)
\]

However there is some redundancy in such a decomposition since different products of elementary couplings can be equivalent. Since the BZ triangle of a product coupling is
simply the sum of the BZ triangles of the factors, one sees that \( E_1 E_3 E_5 = E_7 E_8 \), since they both have the same BZ triangle:

\[
E_1 E_3 E_5 = E_7 E_8 : \begin{array}{ccc}
0 & 1 & 1 \\
1 & 1 & 0 \\
0 & 1 & 1 \\
\end{array}
\]

For \( su(3) \) this is the only redundancy [25], and to obtain a unique decomposition for a general coupling one simply has to forbid one of the products \( E_7 E_8 \) or \( E_1 E_3 E_5 \) to appear. This leads directly to the construction of a generating function for tensor product coefficients.

It was conjectured in [5] that there is at least one choice of forbidden couplings which allows for the construction in a simple way of a generating function for fusion coefficients from that of tensor product coefficients. Given such a choice the conjecture boils down to a simple relation for the minimum level of a coupling in terms of the corresponding minimum level of the elementary couplings appearing in its decomposition. Writing the decomposition in the form

\[
\bar{\lambda} \otimes \bar{\mu} \otimes \bar{\nu} = \prod_i E^{b_i}_i
\]

the precise relation is

\[
k_0 = \sum_i b_i e_i \quad (6.5)
\]

where \( e_i \) is the minimum level for the elementary coupling \( E_i \).

The generating function for \( \widehat{su}(3) \) fusion coefficients obtained from this method has been proven to be equivalent to the combinatorial algorithm of Cummins [22]. This constitutes a proof of the relation (6.5) for \( \widehat{su}(3) \). The correct choice of forbidden coupling turns out to be \( E_1 E_3 E_5 \).

Conversely, granting (6.5), it is simple to find which coupling must be forbidden. Indeed, since \( E_1 E_3 E_5 = E_7 E_8 = (1,1)(1,1)(1,1) \), let us consider this specific fusion rule (c.f. example 8). We know that this coupling comes in two copies, with \( k_0 = 2, 3 \). The corresponding BZ triangles together with their decomposition into elementary couplings are

\[
E_2 E_4 E_6 \quad E_1 E_3 E_5 = E_7 E_8
\]

\[
\begin{array}{ccc}
0 & 1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 1 \\
1 & 0 & 0 \\
\end{array}
\quad
\begin{array}{ccc}
0 & 1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 1 \\
\end{array}
\quad
(6.6)
\]
For $\hat{s}u(3)$ $e_i = 1$ for all $i = 1, 2, \ldots, 8$, and therefore $E_1 E_3 E_5$ must be forbidden in order to obtain the correct result $k_0 = 2$ for the second triangle.

Having determined the forbidden coupling appropriate for the description of fusion rules, one can decompose uniquely any coupling into elementary couplings and read off $k_0$ from (6.5). By forbidding $E_1 E_3 E_5$ it is straightforward to obtain the general and unique decomposition of a BZ triangle into a product of the $E_i$’s. For instance, assuming that $a_2 \leq \min(a_5, a_8)$, one finds the following decomposition for (6.2):

$$E_2^{a_3} E_3^{a_5-a_2} E_4^{a_4} E_5^{a_3-a_6} E_6^{a_7} E_7^{a_2} E_8^{a_6}, \quad (a_2 \leq \min(a_5, a_8))$$

from which it follows that

$$k_0 = a_4 + \nu_1 + \nu_2, \quad (a_2 \leq \min(a_5, a_8))$$

(6.7)

in agreement with (6.4). This proves that the strong depth rule with $a_0^c$ is correct for $\hat{s}u(3)$.

**Example 13:**

The triangle $E_2 E_4 E_6$ $(k_0 = 3)$ in (6.6) is associated with the standard tableau $\begin{array}{c} 1 \\ 3 \\ 2 \end{array}$ (c.f. the relation between (6.2) and (6.3)) with $a_0^c = 2$ and thus $k_0 = 3$, while the triangle $E_7 E_8$ $(k_0 = 2)$ is associated with $\begin{array}{c} 1 \\ 2 \\ 3 \end{array}$, with $a_0^c = 0$ and thus $k_0 = 2$.

**7. Conclusion.**

We have reformulated in a precise way the depth rule of Gepner and Witten [1], and shown how it can be used in principle to calculate fusion coefficients. However, the

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4 An equivalent result has been obtained in [23]. However, their approach is different since they started from the Kač-Walton formula, which relates fusion coefficients to tensor product coefficients using the action of the affine Weyl group. Furthermore the expression for $k_0$ given by Lu is different from the one above. This is because his triangle is related to ours by a Schützenberger involution $(t_1 t_2 t_1)$ of the associated standard tableau. Explicitly, Lu’s result is

$$k_0(t_1 t_2 t_1 \text{BZ}) = \max[\min(a_3, a_8) + \mu_1 + \mu_2, \min(a_2, a_6) + \nu_1 + \nu_2, \min(a_5, a_9) + \lambda_1 + \lambda_2].$$
The cornerstone of the depth rule is the depth itself and we have argued that its calculation is problematic. More precisely, the depth rule is a constraint imposed on tensor product coefficients. But the basis convenient for calculating tensor product coefficients is not convenient for the calculation of the depth.

To carry on, we compute the depth in a special \( q = 0 \) limit of the quantum version of the finite part of the affine algebra under consideration, and freely extrapolate the result. Unfortunately, for \( \widehat{su}(N) \) \( (N > 3) \) it does not give the correct results. However we stress that counterexamples have been found only for moderately high levels and multiplicities. This indicates that the depth rule with crystal depth gives rather accurate lower bounds for the minimum levels at which couplings first appear. On the other hand, for \( \widehat{su}(3) \) this procedure gives exact results. It (namely, \( (6.3) \) with \( a_\theta \) given by \( (4.4) \)) turns out to be a rather efficient and simple algorithm for the calculation of fusion coefficients. A little puzzle remains, however: why does it work for \( \widehat{su}(3) \), and why is it so accurate in general?

As for \( \widehat{su}(3) \), it seems that the success of the present approach is linked to the fact that all elementary couplings have level one, which implies that the decomposition of a coupling into elementary couplings can be done column by column in Young tableaux. In this vein, the failure of the depth rule with crystal depth charge for \( \widehat{su}(N > 3) \) could be explained by the existence of elementary couplings with minimum level higher than 1. However, thanks to syzygies, they can most of the time be eliminated from the decomposition of a coupling into elementary couplings. This should account for the reasonable success of the crystalline version of the depth rule. Finally, for \( \widehat{su}(3) \), we noticed that BZ triangles provided a very convenient ground for the analysis of generating functions. We will report elsewhere on the extension of this approach to \( \widehat{su}(N > 3) \).

We have argued that in the absence of an appropriate basis, generic fusion coefficients cannot be easily computed from the depth rule (which is based on the consideration of three-point functions). But we should mention another way by which they could be extracted solely from the analysis of singular vectors. Indeed, among the Kac-Moody singular vectors are those that may be interpreted as mixed Virasoro-Kač-Moody singular vectors, given the Sugawara construction. These mixed singular vectors lead to the Knizhnik-Zamolodchikov equation [10]. From this equation, one can evaluate the four-point functions and extract from these the three-point functions, which then yield the

\[ \text{We have no proof that it is actually a lower bound, but the statement is motivated by the very large number of examples we have worked out.} \]
fusion coefficients. However, to our knowledge, no such calculations have been performed for three-point functions such that \( \bar{N}_{\bar{\lambda} \bar{\mu} \bar{\nu}} > 1 \).

On the other hand, the depth rule has been used in [6] to compute arbitrary fusion coefficients using a Schubert type calculus. This does not contradict our conclusion since the depth rule is used there only to calculate fusion coefficients involving at least one fundamental representation (i.e. the truncated Pieri’s formula). For a fundamental representation, the depth charge is easily evaluated and it agrees with its crystal version.

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Appendix A. Tensor products with standard tableaux: words, bumping and the Robinson-Schensted correspondence.

In this appendix we describe a more general way of representing states in a given coupling and of calculating tensor product coefficients. To a standard tableau one may associate a word, which is the sequence of numbers appearing in the tableau, read column after column from bottom to top and from left to right.

Example 14:

To the standard tableau

\[
\begin{array}{cccc}
1 & 1 & 2 & 3 \\
3 & 4 & 4 & \\
5 & & & \\
\end{array}
\]

It should be stressed that this approach is conceptually very different from the method of generating functions [5]. Starting from the fact that any irreducible representation can be obtained from an appropriate multiple product of fundamental representations, it is shown in [6] that this description can be extended to the affine case provided we impose some level dependent constraints, called syzygies, on the ordinary product of the fundamental representations. On the other hand, for the method of generating functions, the basic variables are the elementary couplings and the syzygies there refer to relations between products of elementary couplings, already present for the calculation of tensor products.
we associate the word \([531414235]\).

On the other hand, given a word one can reconstruct the corresponding standard tableau by the \textit{bumping} method. One places the leftmost number in a box, the upper left box of the tableau being constructed. For the next number one proceeds as follows: If it is greater than or equal to the number already placed in the first box, it is put in the second box of the first row; otherwise it takes the place of the first number in the first box, the latter being \textit{bumped} in the first box of the second row. Proceeding in this way for all numbers of the word and allowing bumping in every row, one easily reconstructs the corresponding standard tableau. This relation between standard tableaux and words is known as the Robinson-Schensted correspondence \cite{26}.

\textbf{Example 15:}

Here is the step-by-step reconstruction of the tableau associated with the word \([3121]\):

\[
\begin{array}{c}
  3 \leftarrow [121] = 1 \leftarrow [21] = 1 \leftarrow [1] = 1 \leftarrow 2 \leftarrow 3
\end{array}
\]

(This example illustrates the fact that words are invariant with respect to some rearrangements in the order of the entries: the words \([3121]\) and \([3211]\) are manifestly equivalent.)

Any intermediate step in the reconstruction of a standard tableau from a word can be viewed as the insertion of a word into a standard tableau. The representation of a standard tableau by a word, followed by the insertion of this word into another standard tableau defines a ‘fusion’ procedure for two standard tableaux which exactly reproduces the Littlewood-Richardson rule for calculating tensor products coefficients. This construction, due to Thomas \cite{27}, provided the first combinatorial proof of the Littlewood-Richardson rule. More explicitly, in order to obtain all possible \(\tilde{\nu}\) which could appear in the product \(\tilde{\lambda} \otimes \tilde{\mu}\) one proceeds as follows. Consider all the states \(T(\tilde{\lambda}'_{(i)})\) (resp. \(T(\tilde{\mu}'_{(j)})\)) in the \(\tilde{\lambda}\) (resp. \(\tilde{\mu}\)) representation. By fusing all possible pairs of standard tableaux like \(T(\tilde{\lambda}'_{(i)}) \leftarrow W(T(\tilde{\mu}'_{(j)}))\), one fills up all the highest weight representations \(\tilde{\nu}\) which appear in the product \(\tilde{\lambda} \otimes \tilde{\mu}\). Notice that the symmetry of the tensor product with respect to the

\[\text{In our notation } T(\lambda) \text{ is the standard tableau associated with the state } |\lambda\rangle, \text{ and } W(T(\lambda)) \text{ is the corresponding word.}\]
interchange of $\bar{\lambda}$ and $\bar{\mu}$ is reflected by the existence of a dual word $W^*$ and a right insertion operation such that [27]

$$W^*(T(\bar{\lambda}')) \to T(\bar{\mu}') = T(\bar{\lambda}') \leftarrow W(T(\bar{\mu}'))$$

**Example 16:**

Fusion of the ST  

```
1 2 3
2 4
```

and

```
1 4
3
4
```

is done as follows:

```
1 2 3
2 4
```

$\leftarrow [4314] =

```
1 1 3 3 4
2 2 4
4
```

which shows that the representation $(2, 2, 1)$ appears in the $su(4)$ tensor product $(1, 2, 0) \otimes (1, 0, 1)$.

The generalization of the Thomas construction to the case of a triple product $\bar{\lambda} \otimes \bar{\mu} \otimes \bar{\nu}$ is straightforward: one simply has a chain of insertions of the type $[T \leftarrow W(T')] \leftarrow W(T'')$.

**Example 17:**

Let us identify the two triplets of states contributing to the $su(3)$ tensor product $(1, 1) \otimes (1, 1) \otimes (1, 1) \supset (0, 0)$. These are

$$\left\{ \begin{array}{c} 2 3 \\ 3 \end{array}, \begin{array}{c} 1 2 \\ 3 \end{array}, \begin{array}{c} 1 1 \\ 2 \end{array} \right\} \quad \text{and} \quad \left\{ \begin{array}{c} 2 3 \\ 3 \end{array}, \begin{array}{c} 1 3 \\ 2 \end{array}, \begin{array}{c} 1 1 \\ 2 \end{array} \right\}$$

Indeed, in the first case one has

$$\begin{pmatrix} 2 & 3 \\ 3 \end{pmatrix} \leftarrow [312] = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 3 \end{pmatrix}$$

followed by

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 3 \end{pmatrix} \leftarrow [211] = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix} = (0, 0)$$

In this example, associativity is easily verified, i.e.

$$[T \leftarrow W(T')] \leftarrow W(T'') = T \leftarrow W[T' \leftarrow W(T'')]$$
Example 18:

With this method one can redo example 9. In order appropriate to left insertion, the contributing tableaux are

\[
\begin{array}{cccc}
2 & 3 & 3 & 4 \\
3 & 3 & 4 & 4 \\
4 & & & \\
\end{array}, \quad X, \quad \begin{array}{cccc}
1 & 1 & 1 & 1 \\
2 & 2 & 2 & \\
3 & & & \\
\end{array}
\]

where \(X\) stands for five tableaux given in Ex. 9. Since \(a_\theta^c\) for the two other tableaux above is easily checked to be 4, one reobtains the set \(\{4, 5, 5, 5, 6\}\) for \(k_0\).

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