ENERGY ESTIMATE FOR THE WAVE EQUATION DRIVEN BY 
A FRACTIONAL GAUSSIAN NOISE

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Abstract. We consider a general linear stochastic wave equation driven by fractional-in-time noise and study its energy. We provide a mild solution for the wave equation in terms its Fourier expansion. We calculate the expected energy and give asymptotic results for the expected energy for large and small times and as the Hurst parameter, \( H \), approaches \( 1/2 \). These results are phrased in terms of the norms of powers of the differential operator times powers of the spatial covariance operator.

1. Introduction. In this paper, which is an extension of [1], we consider for \( H \in (0, 1) \) the stochastic partial differential equation

\[
  u_{tt} = -Lu + \dot{G}^H(x,t),
\]

where

\[
  u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x),
\]

\[
  u(0,t) = 0 = u(1,t),
\]

\( u : [0,1] \times [0,\infty) \to \mathbb{R} \), \( L \) is a positive self-adjoint differential operator in the spatial variable, and \( \dot{G}^H(x,t) \) is a mean-zero Gaussian noise with covariance given by

\[
  E[G^H(x,t)G^H(x',t')] = \frac{1}{2}(|t|^H + |t'|^H - |t-t'|^H) f(x,x').
\]

(1a)

(1b)

(2)

(1) It is more convenient to express a covariance for \( \dot{G}^H \), than for \( G^H \) in this case.) Equivalently, we may consider a Banach-valued fBm \( B^H(t) := G^H(\cdot,t) \) with covariance given by

\[
  E[B^H(t)B^H(t')] = \frac{1}{2}(|t|^H + |t'|^H - |t-t'|^H) \cdot F,
\]

where \( F \) is the Hilbert-Schmidt operator induced by \( f(x,y) \). This paper, then, is an extension of [1] to the type of noise considered in [4].

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One physical motivation for (1) is the following. Consider (1) with
\[ L = (-\Delta)^{\gamma/2} \]
without an exterior force on \( \mathbb{R}^n \), but include a damping term
\[ u_{tt} = -(-\Delta)^{\gamma/2}u - (-\Delta)^{\sigma/2}u_t , \] (3)
and assume that the solution has the form of a plane wave,
\[ u(x, t) = e^{i(\omega t + \vec{k} \cdot \vec{x})} . \] (4)
Substituting (4) into (3) yields the so-called dispersion equation, i.e. the connection
between possible values of \( \omega \) and \( |\vec{k}| \),
\[ \omega^2 = |\vec{k}|^\gamma + i\omega|\vec{k}|^\sigma . \] (5)
Combining different powers of \( -\Delta \),
\[ u_{tt} = - \sum_{j=1}^J c_j (-\Delta)^{\gamma_j/2}u - \sum_{j_1=1}^{J_1} c_{j_1} (-\Delta)^{\sigma_{j_1}/2}u_t \] (6)
we may even get a rather arbitrary dispersion equation instead of (5),
\[ \omega^2 = \sum_{j=1}^J c_j |\vec{k}|^{\gamma_j} + i\omega \sum_{j_1=1}^{J_1} c_{j_1} |\vec{k}|^{\sigma_{j_1}} . \] (7)
This last relation is compatible with known dispersion phenomena in a complex
medium, at least at the qualitative level, see [3], [11]. We note that (6), without
the damping, is subsumed by (1). We consider the energy of (1), since the energy
is a main physical characteristic of a system governed by a hyperbolic pde, a major
component of the theory of which is to find conditions under which solutions pos-
sessing finite energy exist [8]. Typically only solutions possessing finite energy are
considered. Our goal is to find conditions which guarantee finite expected energy.

In Section 2 we address some background information, notation, and results
which will be used in following sections. In Section 3 we find an explicit formula for
the energy of the system governed by (1). We also provide an asymptotic analysis
of the expected energy for large and small times and as \( H \to 1/2 \).

2. Preliminaries.

2.1. Stochastic Calculus with respect to Fractional Brownian Motion. A
fractional Brownian motion (fBm), \( B^H(t) \), with Hurst parameter, \( H \in (0, 1) \), is a
mean-zero Gaussian stochastic process with covariance function given by
\[ E[B^H(t)B^H(t')] = \frac{1}{2}(|t|^H + |t'|^H - |t - t'|^H) . \] (8)
The increments of fBm are negatively correlated, independent, or positively corre-
lated for \( H < 1/2 \), \( H = 1/2 \), or \( H > 1/2 \), respectively. The case \( H = 1/2 \) corre-
sponds to the usual Brownian motion. Fractional Brownian motion is discussed in
detail in [9].

For \( H > 1/2 \) define as in [5]
\[ \theta_H(s, t) := H(2H - 1)|s - t|^{2H-2} = \frac{\partial^2}{\partial s \partial t} \text{Cov}(B^H(s), B^H(t)) . \]
Fot \( T \in (0, \infty] \), denote by \( L^2_H([0, T]) \) the set of nonrandom functions for which
\[ \int_0^T \int_0^T |f(s)f(t)|\theta_H(s, t)d\sigma dt < \infty . \]
It is proven in [7] that for \( f, g \in L^2_H([0,T]) \)
\[
E \left[ \int_0^T f(t) dB^H_t \int_0^T g(t) dB^H_t \right] = \int_0^T \int_0^T f(s)g(t) \theta_H(s,t) ds dt .
\] (9)

2.2. Decomposition of the Noise. Strictly speaking \( \dot{G}^H \) is not a real-valued stochastic process; it is the time-derivative (which exists in the sense of distributions) of the real-valued stochastic process \( G^H(x,t) \). The following Lemma is proven by comparing covariances.

Lemma 1. In equation (2), suppose \( f(x,x') \in L^2_H([0,1]^2) \) and let \( \{ \alpha_k \}_{k=1}^\infty \) be the eigenvalues (eigenvectors) of the Hilbert-Schmidt operator induced by \( f(x,x') \). Suppose also the trace of that operator, \( \sum_{k=1}^\infty \alpha_k \), is finite. Then
\[
G^H(x,t) = \sum_{k=1}^\infty \sqrt{\alpha_k} \varphi_k(x) B^H_k(t) ,
\]
where \( \{ B^H_k(t) \}_{k=1}^\infty \) is an independent and identically distributed (i.i.d.) sequence of real-valued fractional Brownian motions.

Proof. Since both \( G^H(x,t) \) and \( \sum_{k=1}^\infty \sqrt{\alpha_k} \varphi_k(x) B^H_k(t) \) are mean zero Gaussian processes we need only check that their covariances are the same. We compute
\[
E[G(x,t)G(x,t')] = E \left[ \left( \sum_{k=1}^\infty \sqrt{\alpha_k} \varphi_k(x) B^H_k(t) \right) \left( \sum_{k=1}^\infty \sqrt{\alpha_k} \varphi_k(x') B^H_k(t') \right) \right] = \frac{1}{2}(|t|^H + |t'|^H - |t - t'|^H) \sum_{k=1}^\infty \alpha_k \varphi_k(x) \varphi_k(x') .
\]

It only remains to observe that \( f(x,x') = \sum_{k=1}^\infty \alpha_k \varphi_k(x) \varphi_k(x') \), by checking the eigenvalues and eigenvectors of the integral operator with kernel given by each side of this equation. This completes the proof.

Formally then, we may write
\[
\dot{G}^H(x,t) = \sum_{k=1}^\infty \sqrt{\alpha_k} \varphi_k(x) \dot{B}^H_k(t) .
\] (10)

2.3. Asymptotic Expansion of the Fourier Integral. The following asymptotic expansion of the Fourier integral comes from [6].

Lemma 2. Let \( \varphi \in C^1[a,b] \), and \( p, q \in (0,1) \). Then
\[
\int_a^b e^{itu} (u-a)^{p-1}(b-u)^{q-1} \phi(u) du = \Gamma(q)e^{i(tu-ab)/2} t^{-q}(b-a)^{q-1} \phi(b)
- \Gamma(p)e^{i(tu-a(p-2)/2) t^{-p}(b-a)^{p-1} \phi(a)
+ O(t^{-1}) ,
\]
as \( t \to \infty \).
3. Solution and Energy. We develop here a solution to \([1]\) for \(H \in (0, 1)\), find its energy, and expected energy. Many results are proven rigorously only for \(H > 1/2\). A complete proof of the results for \(H < 1/2\) may be found in \([2]\). The first step is to insert \([10]\) into \([1]\), yielding
\[
u_{tt} = -Lu + \sum_{k=1}^{\infty} \sqrt{\alpha_k} \phi_k(x) \dot{B}^H_k(t) .
\] (11)

Next, we separate variables to decompose \(u(x, t)\) as
\[
u(x, t) = \sum_{n=1}^{\infty} c_n(t) e_n(x) ,
\] (12)

where \(e_n(x)\) are the eigenfunctions of the operator \(L\) with associated eigenvalues \(\lambda_n\). Combining \([11]\) and \([12]\) yields,
\[
\ddot{c}_n(t) = -\lambda_n c_n(t) + \sum_{k=1}^{\infty} \sqrt{\alpha_k} \phi_k(n) \dot{B}^H_k(t) ,
\] (13)

where \(\phi_k(n) = \langle \phi_k(x), e_n(x) \rangle\). The solution to \([13]\) is given by
\[
\begin{align*}
\dot{c}_n(t) &= -\lambda_n c_n(t) + \sum_{k=1}^{\infty} \sqrt{\alpha_k} \phi_k(n) \dot{B}^H_k(t) ,
\end{align*}
\]

This follows by writing \([13]\) as a system of first order SDE and treating the noise term as an ordinary forcing function, which may be done using the Stratonovich (ordinary) calculus. (It is known that the Itö and Stratonovich/symmetric integrals of non-random functions agree, since their difference involves a Malliavin derivative which is zero for non-random functions \([10]\).) Finally, we have the following theorem.

**Theorem 1.** The solution to the initial boundary value problem given by \([1]-(1b)\) is given by
\[
u(x, t) = \sum_{n=1}^{\infty} \left( \cos(\sqrt{\lambda_n} t)c_n(0) + \frac{\sin(\sqrt{\lambda_n} t)}{\sqrt{\lambda_n}} \dot{c}_n(0) \right)
+ \sum_{k=1}^{\infty} \int_{0}^{t} \sqrt{\alpha_k} \phi_k(n) \sin(\sqrt{\lambda_n}(t - s)) dB^H_k(s) \right) e_n(x),
\] (15)

and its derivative is given by
\[
u(t, x) = \sum_{n=1}^{\infty} \left( -\sqrt{\lambda_n} \sin(\sqrt{\lambda_n} t)c_n(0) + \cos(\sqrt{\lambda_n} t)\dot{c}_n(0) \right)
+ \sum_{k=1}^{\infty} \int_{0}^{t} \sqrt{\alpha_k} \phi_k(n) \sin(\sqrt{\lambda_n}(t - s)) dB^H_k(s) \right) e_n(x).
\] (16)
Note that it is important to obtain the time derivative (in the distributional sense) from the system of first order equations since it cannot be taken in the classical sense.

The energy of equation \((1)\) is defined as

\[
E^H (t) := \frac{1}{2} \left( \int_0^1 u_x^2 dx + \int_0^1 (L^{1/2} u)^2 dx \right). \tag{17}
\]

One may use \((15)\) to find \(u_x\) and apply Parseval’s identity to that and to \((16)\), yielding the following formula for the energy.

**Theorem 2.** The energy of the system governed by \((1)\) is given by

\[
2E^H(t) = \sum_{n=1}^{\infty} \left( \sqrt{\lambda_n} \cos(\sqrt{\lambda_n} t) c_n(0) + \sin(\sqrt{\lambda_n} t) \dot{c}_n(0) \right)
\]

\[
+ \sum_{k=1}^{\infty} \int_0^t \sqrt{\alpha_k} \varphi_{kn} \sin(\sqrt{\lambda_n}(t-s)) dB_k^H(s) \left( \sqrt{\lambda_n} \cos(\sqrt{\lambda_n} t) c_n(0) + \sin(\sqrt{\lambda_n} t) \dot{c}_n(0) \right)
\]

\[
+ \sum_{k'=1}^{\infty} \int_0^t \sqrt{\alpha_{k'}} \varphi_{k'n} \sin(\sqrt{\lambda_n}(t-s)) dB_{k'}^H(s') \right) \tag{18}
\]

Finding the expected energy requires expanding \((18)\). This yields terms involving no stochastic integrals, one stochastic integral, or a product of stochastic integrals. The terms involving no stochastic integrals give \(\lambda_n c_n^2(0) + \dot{c}_n^2(0)\). Upon summing over \(n\) and taking expectations we recognize this as \(E[E^H(0)]\). The terms involving one stochastic integral vanish upon taking expectations since the expected value of the stochastic integral is zero. Note that the expectation of the product of two stochastic integrals is zero if the fractional Brownian motions are independent (i.e. \(k \neq k'\)). We therefore see that

\[
E[E^H(t)] = E[E^H(0)] + \frac{1}{2} \sum_{n,k=1}^{\infty} \alpha_k \varphi_{kn}^2 E \left( \left( \int_0^t \sin(\sqrt{\lambda_n}(t-s)) dB_k^H(s) \right)^2 \right)
\]

\[
+ \left( \int_0^t \cos(\sqrt{\lambda_n}(t-s)) dB_k^H(s) \right)^2 \cdot. \tag{19}
\]

The following theorem gives formulae for the expected energy in the cases \(H > 1/2\) and \(H < 1/2\). It corrects the erroneous result concerning the expected energy for \(H > 1/2\) in \([4]\).
Theorem 3. The expected energy of the system governed by (1) is given for $H \in (1/2, 1)$ by

$$
E[\mathcal{E}^H(t)] - E[\mathcal{E}^H(0)] = \frac{1}{2} \sum_{n,k=1}^{\infty} \alpha_k \varphi_{kn}^2 \int_0^1 \int_0^t \cos(\sqrt{\lambda_n}(s - s')) H(2H - 1) |s - s'|^{2H-2} ds' ds
$$

$$
= t^{2H}H(2H - 1) \sum_{n,k=1}^{\infty} \alpha_k \varphi_{kn}^2 \int_0^1 (1 - u) \cos(\sqrt{\lambda_n}tu) u^{2H-2} du .
$$

For $H \in (0, 1/2)$ we have the following analytic continuation formula, which converges absolutely if

$$
\sum_{n,k} \alpha_k \varphi_{k,n}^2 \lambda_n = \|L^{1/2} F^{1/2}\|_2^2 < \infty ,
$$

$$
E[\mathcal{E}^H(t)] - E[\mathcal{E}^H(0)]
= \frac{t^{2H}}{2} Tr(F) + t^{2H}H(2H - 1) \cdot \sum_{n,k=1}^{\infty} \alpha_k \varphi_{kn}^2 \int_0^1 (1 - u) \cos(\sqrt{\lambda_n}tu) u^{2H-2} du .
$$

We note that the condition (21), which is more restrictive than finite trace of the spatial covariance, is a similar phenomena to that demonstrated in [12].

Proof. For $H > 1/2$ use (9) to evaluate (19), then apply the identity $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$. To prove the second equality substitute $tu = (s - s')$ in (a). For $H > 1/2$ equation (22) is a regularization of (20). For $H > 1/2$ we have

$$
E[\mathcal{E}^H(t)] - E[\mathcal{E}^H(0)]
= t^{2H}H(2H - 1) \sum_{n,k=1}^{\infty} \alpha_k \varphi_{kn}^2 \int_0^1 (1 - u) \cos(\sqrt{\lambda_n}tu) u^{2H-2} du
$$

$$
= t^{2H}H(2H - 1) \sum_{n,k=1}^{\infty} \alpha_k \varphi_{kn}^2 \left( \int_0^1 (1 - u) \cos(\sqrt{\lambda_n}tu) - 1) u^{2H-2} du \right)
$$

$$
+ \int_0^1 (1 - u) u^{2H-2} du
$$

$$
= t^{2H}H(2H - 1) \sum_{n,k=1}^{\infty} \alpha_k \varphi_{kn}^2 \left( \int_0^1 (1 - u) \cos(\sqrt{\lambda_n}tu) - 1) u^{2H-2} du \right)
$$

$$
+ \int_0^1 (1 - u) u^{2H-2} du
$$

$$
= t^{2H}H(2H - 1) \sum_{n,k=1}^{\infty} \alpha_k \varphi_{kn}^2 \left( \int_0^1 (1 - u) \cos(\sqrt{\lambda_n}tu) - 1) u^{2H-2} du \right)
$$

$$
+ \frac{1}{2H(2H - 1)}
$$

$$
= t^{2H}H(2H - 1) \sum_{n,k=1}^{\infty} \alpha_k \varphi_{kn}^2 \left( \int_0^1 (1 - u) \cos(\sqrt{\lambda_n}tu) - 1) u^{2H-2} du \right)
$$

$$
+ \frac{t^{2H}}{2} \sum_{k=1}^{\infty} \alpha_k ,
$$

For $H \in (0, 1/2)$ use (9) to evaluate (19), then apply the identity $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$. To prove the second equality substitute $tu = (s - s')$ in (a).
where the first series in the last equation converges absolutely if \( \sum_{n,k} \alpha_k \varphi_{k,n} \lambda_n = \| L^{1/2} F^{1/2} \|_2^2 < \infty \) since
\[
\left| \int_0^1 (1 - u) \left( \cos(\sqrt{\lambda_n tu}) - 1 \right) u^{2H-2} du \right| \\
\leq \int_0^1 (1 - u) 2 \sin^2 \left( \frac{\sqrt{\lambda_n tu}}{2} \right) u^{2H-2} du \leq \frac{\lambda_n t^2}{2} \int_0^1 (1 - u) u^{2H} du .
\]
This yields the analytic continuation formula and completes the proof.

The following corollary gives upper (resp. lower) bounds on the expected energy.

**Corollary 1.** Let \( H \in (1/2, 1) \). Then
\[
\frac{E[\mathcal{E}^H(t)] - E[\mathcal{E}^H(0)]}{t^{2H}} \leq \frac{\text{Tr}(F)}{2} .
\]

Let \( H \in (0, 1/2) \). Then according to the analytic continuation formula (22) we have
\[
\frac{E[\mathcal{E}^H(t)] - E[\mathcal{E}^H(0)]}{t^{2H}} \geq \frac{\text{Tr}(F)}{2} .
\]

**Proof.** Using the estimate \( \cos(\theta) \leq 1 \) and noting that
\[
\sum_{n,k} \alpha_k \varphi_{k,n} \lambda_n = \sum_k \alpha_k = \text{Tr}(F),
\]
completes the proof.

The next theorem, which considers the asymptotic behaviour of the energy as \( t \to 0 \), as \( t \to \infty \) and as \( H \to 1/2 \), shows that Corollary 1 is sharp as \( t \to 0 \).

**Theorem 4.** The expected energy of the system governed by (1) satisfies
(a) \[
\lim_{t \to 0^+} \frac{E[\mathcal{E}^H(t)] - E[\mathcal{E}^H(0)]}{t^{2H}} = \frac{1}{2} \text{Tr}(F) ,
\]
(b) \[
\lim_{t \to \infty} \frac{E[\mathcal{E}^H(t)] - E[\mathcal{E}^H(0)]}{t} = \frac{\Gamma(2H + 1) \sin(\pi H)}{2} \| L^{1/4-H/2} F^{1/2} \|^2_2 ,
\]
(c) \[
\lim_{H \to 1/2} E[\mathcal{E}^H(t)] - E[\mathcal{E}^H(0)] = \frac{t}{2} \text{Tr}(F) = E[\mathcal{E}^{1/2}(t)] - E[\mathcal{E}^{1/2}(0)] .
\]
If \( H < 1/2 \) we require that
\[
\sum_{n,k} \alpha_k \varphi_{k,n} \lambda_n = \| L^{1/2} F^{1/2} \|_2^2 < \infty
\]
in order for the series representing the expected energy to converge.

We note that condition (24) while sufficient to guarantee finite energy is likely not necessary given that it does not depend on \( H \). The determination however of exact necessary and sufficient goes beyond the scope of this paper and the space allotted for its publication.
Proof. To prove part (a) for $H > 1/2$, we simply note that the integrals in (20) are bounded by 1. Since the series in (23) converges, the series in (20) converge uniformly. Take the limit as $t \to 0$.

To prove part (b) for $H > 1/2$, we first use a change of variables $v = \sqrt{n_u}$ on the integral in (20), yielding

$$
\int_0^1 (1 - u) \cos(\lambda_n tu) u^{2H-2} du = \lambda_n^{1/2-H} \int_0^{\sqrt{n_u}} (1 - \lambda_n^{-1/2} v) \cos(tv) v^{2H-2} dv .
$$

Splitting this last integral into two parts we apply Lemma 2 to each part with $a = 0$, $b = \sqrt{n_u}$, $p = 2H - 1$ (resp. $p = 2H$), $q = 1$, and $\phi(v) = 1$. Considering only the real parts yields

$$
\lambda_n^{1/2-H} \int_0^{\sqrt{n_u}} \cos(tv) v^{2H-2} dv = \lambda_n^{1/2-H} (-\Gamma(2H - 1) \cos(\pi(H - 3/2)) t^{1-2H} + O(t^{-1})) ,
$$

and

$$
- \lambda_n^{-H} \int_0^{\sqrt{n_u}} \cos(tv) v^{2H-1} dv = \lambda_n^{-H} (\Gamma(2H) \cos(\pi(H - 1)) t^{-2H} + O(t^{-1})) .
$$

Adding together (25) and (26) and noting that $- \cos(\pi(H - 3/2)) = \sin(\pi H)$, we see from (20) that (b) holds, provided that $\sum n,k \alpha_k \varphi_{k,n}^2$ in (23) converges absolutely.

To prove part (c) for $H > 1/2$, we again consider

$$
(2H - 1) \int_0^1 (1 - u) \cos(\lambda_n tu) u^{2H-2} du = \int_0^1 (\cos(\lambda_n tu) + (1 - u) \sqrt{\lambda_n t} \sin(\sqrt{\lambda_n tu})) u^{2H-1} du ,
$$

which follows after integrating by parts. Now the integrand on the right side of (27) is bounded by $1 + \sqrt{\lambda_n t}$ so it approaches

$$
\int_0^1 (\cos(\sqrt{\lambda_n tu}) + (1 - u) \sqrt{\lambda_n t} \sin(\sqrt{\lambda_n tu})) du = 1 ,
$$

as $H \to (1/2)+$. The result follows from noting (23).

For $H < 1/2$, part (a) is proven by simply taking a limit of (22) as $t \to 0$ since the series converges absolutely.

To prove part (b) for $H < 1/2$ we start with (22). Dividing everything by $t$ we see that the first term clearly approaches zero as $t \to \infty$, so we focus our attention on

$$
t^{2H-1} (2H - 1) \sum_{n,k=1}^\infty \alpha_k \varphi_{k,n}^2 \int_0^1 (1 - u)(\cos(\lambda_n tu) - 1) u^{2H-2} du .
$$
Now the integral above equals

\[
\int_0^1 (\cos(\sqrt{\lambda_n}tu) - 1)u^{2H-2}du - \int_0^1 (\cos(\sqrt{\lambda_n}tu) - 1)u^{2H-1}du
\]

\[
= \left[ (\cos(\sqrt{\lambda_n}tu) - 1)u^{2H-1} \right]_0^1 - \sqrt{\lambda_n}t \int_0^1 \sin(\sqrt{\lambda_n}tu)u^{2H-1}du
\]

\[
- \int_0^1 (\cos(\sqrt{\lambda_n}tu) - 1)u^{2H-1}du
\]

\[
= \cos(\sqrt{\lambda_n}t) - 1 \frac{\lambda_n}{2H-1} t\lambda_n^{1/2-H} \int_0^\sqrt{\lambda_n} \sin(tu') (tu')^{2H-1} du' - \lambda_n^{-1} \int_0^\sqrt{\lambda_n} (\cos(tu') - 1)(tu')^{2H-1} du'
\]

\[
= O(1) + \frac{t\lambda_n^{1/2-H}}{2H-1} \left( \sin(\sqrt{\lambda_n}t - \pi/2)\lambda_n^{2H-1} - \frac{\Gamma(2H)\sin(\pi(2H-2)/2)}{t^{2H}} \right)
\]

\[
+ O(t^{-1}) - \frac{\lambda_n^{-H}}{t} \left( \frac{\cos(\sqrt{\lambda_n}t - \pi/2)\lambda_n^{2H-1}}{t} + \frac{\Gamma(2H)\cos(\pi(2H-2)/2)}{t^{2H}} + O(t^{-1}) \right)
\]

\[
= O(1) + \lambda_n^{H-1/2}O(1) - t^{1-2H}\lambda_n^{1/2-H} \Gamma(2H)\sin(\pi H - \pi)\left( \frac{1}{t} \right) + \lambda_n^{1/2-H}O(1)
\]

\[
- \lambda_n^{-1}O(t^{-1}) - \lambda_n^{-H}O(t^{-2H}) - \lambda_n^{-H}O(t^{-1})
\]

where the first equality is obtained integrating by parts, the second by substituting \( u' = \sqrt{\lambda_n}u \) in the integrals, and the third by applying Lemma 2 with \( a = 0, b = \sqrt{\lambda_n}, \varphi(u) = 1, q = 1, \) and \( p = 2H \). We see from (29) that the summability of the series \( \sum_{n,k} \delta_k \varphi_k^3 \lambda_n = \|L^{1/2}F^{1/2}\|_2^3 < \infty \) implies the summability of all other series involved in the expansion of (28) and that taking a limit of this expansion as \( t \to \infty \) yields the desired result since all terms but one approach zero and \( H\Gamma(2H) = \Gamma(2H + 1) \) and \( -\sin(\pi H - \pi) = \sin(\pi H) \).

To prove part (c) we start with (22) and consider again (28). Since this series converges uniformly we may take the limit as \( H \to \frac{1}{2}^- \) term by term. We have

\[
\lim_{H \to \frac{1}{2}^-} \frac{1}{2H-1} \left( \int_0^1 (\cos(\sqrt{\lambda_n}tu) - 1)u^{2H-2}du \right)
\]

\[
- \int_0^1 (\cos(\sqrt{\lambda_n}tu) - 1)u^{2H-1}du
\]

\[
= \lim_{H \to \frac{1}{2}^-} \left[ (\cos(\sqrt{\lambda_n}tu) - 1)u^{2H-1} \right]_0^1 - \sqrt{\lambda_n}t \int_0^1 \sin(\sqrt{\lambda_n}tu)u^{2H-1}du
\]

\[
- (2H - 1) \int_0^1 (\cos(\sqrt{\lambda_n}tu) - 1)u^{2H-1}du
\]

\[
= \lim_{H \to \frac{1}{2}^-} \left( \cos(\sqrt{\lambda_n}t) - 1 - \sqrt{\lambda_n}t \int_0^1 \sin(\sqrt{\lambda_n}tu)du - 0 \right) = 0
\]

This completes the proof.
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