Abstract

After discussing the intrinsic ambiguity in determining the light quark mass ratio $m_u/m_d$, we reexamine the recent proposal that this ambiguity can be resolved by applying the QCD multipole expansion for the heavy quarkonium decays. It is observed that, due to instanton effects, some matrix elements which have been ignored in previous works can give a significant contribution to the decay amplitudes, which results in a large uncertainty in the value of $m_u/m_d$ deduced from quarkonium phenomenology. This uncertainty can be resolved only by a QCD calculation of some second order coefficients in the chiral expansion of the decay amplitudes.
It has been observed by a number of authors [1–5] that second order corrections in chiral perturbation theory can significantly affect the estimate of the light quark masses. In particular, it was pointed out that the determination of $m_u/m_d$ suffers from a large uncertainty due to the instanton-induced mass renormalization [1, 3]:

$$M \rightarrow \bar{M} (\omega) \equiv M + \omega M_I$$

(1)

where the real matrix $M = \text{diag}(m_u, m_d, m_s)$ denotes the light quark masses in the QCD lagrangian, $M_I \equiv \frac{1}{4\pi f} (\det M^\dagger)(M^\dagger)^{-1} = \frac{1}{4\pi f} (m_d m_s, m_u m_s, m_u m_d)$ is the instanton-induced second order mass with the pion decay constant $f \simeq 93 \text{ MeV}$, and $\omega$ is a dimensionless parameter of order unity. Most of the previous analyses on $M$ do not distinguish $M$ from $\bar{M}$, and thus the corresponding results can be interpreted as those on $\bar{M}$ for an arbitrary value of $\omega$, which leads to a large uncertainty in the extracted value of $m_u/m_d$.

Recently it was argued that the above mentioned difficulty can be overcome by noting that the instanton-induced mass $M_I$ is distinguished from the bare mass $M$ through its $\theta$-dependence where $\theta$ denotes the CP violating QCD vacuum angle [6]. If one keeps the $\theta$-dependence explicitly, $M_I$ always appears with the phase $e^{i\theta}$ due to the winding number of instantons. Note that $M$ and $e^{i\theta} M_I$ have the same transformation under the QCD chiral symmetry $SU(3)_L \times SU(3)_R \times U(1)_A$ under which

$$M \rightarrow e^{i\alpha} L M R^\dagger, \quad \theta \rightarrow \theta + 3\alpha,$$

(2)

where $L \in SU(3)_L$, $R \in SU(3)_R$, and $\alpha$ generates the anomalous $U(1)_A$ rotation. Then one may be able to measure $M$ directly, not $\bar{M}$ involving an arbitrary unknown parameter $\omega$, by probing the $\theta$-dependence of the QCD dynamics. Among quantities that probe the $\theta$-dependence, the matrix elements $\langle \phi | G \tilde{G} | 0 \rangle$ ($\phi = \pi^0$ or $\eta$) were considered in ref. [6]. It was then argued that the ratio $R_A \equiv \frac{\langle \phi | G \tilde{G} | 0 \rangle}{\langle \eta | G \tilde{G} | 0 \rangle}$ can be reliably determined by applying the QCD multipole expansion for the quarkonium decays $\psi' \rightarrow J/\psi + \phi$, which would allow us to precisely determine $m_u/m_d$. In
this paper, we wish to reexamine whether the QCD multipole expansion applied for the quarkonium decays can provide a way to determine $m_u/m_d$ without doing any nonperturbative QCD calculation. Our result then confirms the conclusion of ref. [7], viz. in order to precisely determine $m_u/m_d$, one needs to calculate a QCD matrix element which receives a potentially large instanton contribution. Since such a calculation is not available at present, a rather wide range of $m_u/m_d$, including zero, should be considered as being consistent with our present knowledge of QCD.

To proceed, let us briefly review the points that were discussed in refs. [3, 4, 7]. In order to extract information on $M$, one usually considers measurable quantities whose $M$-dependence can be deduced from an effective lagrangian of hadron fields. This is not a severe limitation since it is hard to imagine a measurable quantity which may provide useful information on $M$ through its $M$-dependence but can not be described by any hadronic effective lagrangian. To satisfy the Ward identities of the QCD chiral symmetry, the effective lagrangian is required to be invariant under the chiral transformations of the involved hadrons and also those of $M$ and $\theta$ given in eq. (2). Since $M$ and $e^{i\theta}M_I$ have the same chiral transformation property, for any term in the effective lagrangian which is first order in $M$, e.g. $O_1 = a_i \langle M \Omega_i \rangle$, there exists also the second order term $O_2 = b_i e^{i\theta} \langle M_I \Omega_i \rangle$, where $\langle Z \rangle = \text{tr}(Z)$, $a_i$ and $b_i$ are chiral coefficients which are calculable within QCD, and $\Omega_i$ is a generic local functional of hadron fields which is transformed under the chiral symmetry as $M^{-1}$. This can be understood within QCD by noting that $e^{i\theta}M_I$ corresponds to the effective current mass induced by instantons [1, 3]. For any QCD diagram which contains an insertion of $M$ and thus gives a contribution to $O_1$, one can replace $M$ by the instanton-induced effective mass $e^{i\theta}M_I$ to make the new diagram contribute to $O_2$. This means that the $M$-dependent part of the effective lagrangian can always be written as

$$L_{eff}(\theta) \supset \sum_i \left[ a_i \langle M \Omega_i \rangle + b_i e^{i\theta} \langle M_I \Omega_i \rangle + ... \right],$$

(3)
where the ellipsis denotes other possible higher order terms.

To be invariant under the anomalous $U(1)_A$ symmetry, the effective lagrangian is $\theta$-dependent in general. One may then expand the $\theta$-dependent effective lagrangian around $\theta = 0$:

$$L_{\text{eff}}(\theta) = \sum_{n=0}^{\infty} \theta^n L_{\text{eff}}^{(n)}.$$  \hspace{1cm} (4)

Clearly $L_{\text{eff}}^{(0)} \equiv L_{\text{eff}}(\theta = 0)$ describes the normal CP conserving strong interactions while the terms of nonzero $n$ describe the $\theta$-dependence of the QCD dynamics, including the CP violation due to a nonzero $\theta$. In view of the arguments leading to eq. (3), for any $L_{\text{eff}}^{(0)}$ that includes the corrections of $O(M^2)$, one can define a transformation of the form:

$$M \rightarrow \bar{M} \equiv M + \omega M_I, \quad b_i \rightarrow b_i - \omega a_i,$$  \hspace{1cm} (5)

under which $L_{\text{eff}}^{(0)}$ is invariant\footnote{Throughout this paper, whenever we say about the IT, it is assumed that the corrections of $O(M^2)$ are included, while those of $O(M^3)$ are ignored.}. The above transformation mixes the instanton-induced mass $M_I$ with the bare mass $M$, and thus will be called as the instanton transformation (IT) in the following discussion. Obviously $L_{\text{eff}}^{(n)} (n \neq 0)$ in eq. (4) are not invariant under the IT. This is nothing but to mean that $L_{\text{eff}}^{(0)}$ does not distinguish the instanton-induced mass from the bare one, while $L_{\text{eff}}^{(n)} (n \neq 0)$ distinguish since they arise from the $\theta$-dependence. Based on this observation, it was stated in ref. [3] that the normal CP conserving strong interactions which are described by $L_{\text{eff}}^{(0)}$ always concern the effective mass $\bar{M}$ with an unspecified value of $\omega$, while the CP violating amplitudes or the nonderivative axion couplings probe directly the bare mass $M$. (Note that up to a small mixing with mesons, $\theta$ corresponds to the constant mode of axion in axion models.) For instance, if $\det(M) = 0$, then $\theta$ can be rotated away (even in the case without axion) and axion becomes massless although $\det(\bar{M}) \neq 0$.

Although $L_{\text{eff}}^{(n)}$'s ($n \neq 0$) in eq. (4) might be useful also, one usually considers only
\( \mathcal{L}_{\text{eff}}^{(0)} \) in extracting information on \( M \) from experimental data. Of course the reason is that it is quite nontrivial to find a link between the available experimental data and the terms of nonzero \( n \). The IT was defined as acting on \( M \) and the second order chiral coefficients \( b_i \)'s. Since \( \mathcal{L}_{\text{eff}}^{(0)} \) is invariant, all measurable quantities described by \( \mathcal{L}_{\text{eff}}^{(0)} \), i.e. expressed in terms of \( M \) and other parameters that appear in \( \mathcal{L}_{\text{eff}}^{(0)} \), are also invariant under the IT. This can be understood also by observing that the IT can be considered as a kind of renormalization group (RG) transformation associated with the instanton-induced mass renormalization [7]. In taking this specific instanton effects into account, one can use the following scheme. For \( \theta \) which is fixed to be zero, the contribution to \( \mathcal{L}_{\text{eff}}^{(0)} \) from instantons with size \( \rho \leq \mu_I^{-1} \) is taken into account by redefining the chiral expansion parameter as \( M \rightarrow M + \omega M_I \), while that from larger instantons (\( \rho \geq \mu_I^{-1} \)) appears in the second order chiral coefficients \( b_i \). In this scheme, the transformation parameter \( \omega \) of the IT, being naturally of order unity, can be identified as \( \omega(\mu_I) = \int_{\mu_I}^{\infty} \frac{d \mu}{\mu} D(\mu) \), where \( D(\mu) \) denotes an appropriately normalized dimensionless instanton density. Then the IT corresponds to a renormalization group transformation changing \( \mu_I \), and all measurable quantities deduced from \( \mathcal{L}_{\text{eff}}^{(0)} \) should be independent of our choice of \( \mu_I \), i.e. are invariant under the IT.

As was argued previously [7], if we restrict ourselves to \( \mathcal{L}_{\text{eff}}^{(0)} \), the invariance of \( \mathcal{L}_{\text{eff}}^{(0)} \) under the IT prevents us from precisely determining \( m_u/m_d \). All equations deduced from \( \mathcal{L}_{\text{eff}}^{(0)} \) are covariant under the IT. As a result, any quantity which is sensitive to the IT, i.e. its variation under the IT for \( \omega = O(1) \) can be comparable to the expected central value, can not be fixed by the invariant experimental data. (Of course the invariant combination of such quantities can be fixed.) One can not avoid an uncertainty whose size is characterized by the variation under the IT. For \( m_u/m_d \),

\( \text{In considering the IT as a RG transformation, it must be noted that the scale } \mu_I \text{ is introduced only for the instanton-induced mass renormalization, but not for other kind of instanton effects. This isolation of the instanton-induced mass renormalization from other instanton effects can be easily achieved in the dilute instanton gas approximation. Then the conclusion that measurable quantities deduced from } \mathcal{L}_{\text{eff}}^{(0)} \text{ are invariant under the IT can be understood by restricting ourselves to the effects of relatively small instantons for which the dilute gas approximation is valid.} \)
its variation under the IT is roughly $\omega m_s/4\pi f$ which can be as large as about 1/2, implying a rather large uncertainty.

To be more specific, let us consider the case of using the pseudoscalar meson masses as experimental input. The relevant part of the effective lagrangian is

$$L_{\text{eff}}^{(0)} \supset \frac{1}{4} f^2 \langle \chi^\dagger U + \chi U^\dagger \rangle + L_6 \langle \chi^\dagger U + \chi U^\dagger \rangle^2 + L_7 \langle \chi^\dagger U - \chi U^\dagger \rangle^2 + L_8 \langle \chi^\dagger U \chi^\dagger U + \chi U^\dagger \chi U^\dagger \rangle,$$

where $\chi = 2 B_0 M$ for $B_0 = - \langle 0 | u u | 0 \rangle / f^2$ defined at chiral limit, and the $SU(3)$-valued $U$ denotes the pseudoscalar meson octet. For the above terms, the IT of eq. (5) can be written as [2, 5]:

$$m_u \rightarrow m_u + \frac{\omega m_d m_s}{4\pi f} \quad \text{(cyclic in } u, d, s),$$

$$L_i \rightarrow L_i - \frac{h_i \omega f}{128\pi B_0},$$

where $h_6 = h_7 = 1$ and $h_8 = -2$. Note that $m_s$ can be considered to be invariant since its variation is negligibly small compared to $m_s$ for $\omega$ of order unity. In refs. [2, 8], it was found that

$$\frac{m_s^2}{m_u^2 - m_u^2} \simeq (M_K^2/M_{\pi}^2)(M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2 - M_{\pi^0}^2)^{-1},$$

$$\frac{m_d + m_u}{m_d - m_u} \simeq (1 + \Delta_M)^2 M_{\pi}^2 (M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2 - M_{\pi^0}^2)^{-1},$$

where $\Delta_M = - 32 L_7 B_0 m_s / f^2 - c$ for $c \simeq 0.33 = O(m_s/4\pi f)$. These equations do not fix $m_u/m_d$, but give only a curve on the plane of $(m_u/m_d, \lambda)$ where $\lambda = m_s/m_d$ for the first equation and $\lambda = L_7$ for the second. Note that these curves are parametrized by $\omega$, and the uncertainties in $m_u/m_d$ and $\lambda$ are given by their variations under the IT for $\omega = O(1)$. We will encounter the same situation even when other kind of measurable quantities, e.g. the baryon masses and the decay amplitudes for $\eta \rightarrow 3\pi$, $\psi' \rightarrow J/\psi + \pi^0(\eta)$, are used in the context of an appropriate form of $L_{\text{eff}}^{(0)}$.

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3 Throughout this paper, we will use $m_d/m_s \ll m_s/4\pi f$, and thus ignore the corrections suppressed by either $m_d/m_s$ or $m_d/4\pi f$, while keeping only those of $O(m_s/4\pi f)$. 

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A definite value of $m_u/m_d$ would be obtained if one can choose a specific value of $\lambda$, but it is possible only through a QCD calculation of $\lambda$ since $\lambda$ is sensitive to the IT and thus can not be fixed by experimental data alone. Such an attempt was made recently by Leutwyler \cite{5} for $\lambda = L_7$. By invoking $\eta'$-dominance, it was argued that $L_7$ falls into a rather narrow range of negative values, thus ruling out $m_u = 0$ in view of the second equation in (8). However it has been pointed out later that instantons significantly suppress the negative $\eta'$-contribution to $L_7$, while enhancing the positive contribution from the pseudoscalar octet resonances \cite{7}. This would make $\eta'$-dominance for $L_7$ not valid, and thus results in a large uncertainty in the value of $L_7$, allowing $m_u = 0$. In fact, any quantity which is sensitive to the IT receives a potentially large contribution from instantons. This makes the QCD calculation of $\lambda$ and the precise determination of $m_u/m_d$ even more nontrivial.

So far, we have argued that there exists an intrinsic ambiguity in determining $m_u/m_d$ by usual manner. This ambiguity is due to the invariance of $L_{\text{eff}}^{(0)}$ under the IT, and as was pointed out in ref. \cite{3}, might be resolved by including the non-invariant terms $L_{\text{eff}}^{(n)} (n \neq 0)$ in the analysis. In QCD, one can probe the $\theta$-dependence through the matrix elements involving the insertion of the two gluon operator $G\tilde{G}$, e.g., $\langle \phi | G\tilde{G} | 0 \rangle$ ($\phi = \pi^0$ or $\eta$) and $\langle 0 | (G\tilde{G})(G\tilde{G}) | 0 \rangle$. The hadronic realizations of such matrix elements contain $L_{\text{eff}}^{(n)} (n \neq 0)$ in general. In axion models, $\langle \phi | G\tilde{G} | 0 \rangle$ describes the axion-meson mixing while the other matrix element is related to the axion mass. As a result, physical processes involving axion may provide information on these matrix elements, and thus on the bare quark mass $M$. However it is totally unclear whether the normal strong interaction data can also provide any information on these matrix elements. In ref. \cite{3}, it was observed that the matrix element $\langle \phi | G\tilde{G} | 0 \rangle$ appears in the QCD multipole expansion applied for the heavy quarkonium decays $\Phi' \to \Phi + \phi$ ($\Phi = J/\psi$ or $\Upsilon$). The major purpose of this paper is to examine whether this allows us to determine $m_u/m_d$ without doing any nonperturbative QCD calculation.
The quarkonium decays $\Phi' \to \Phi + \phi$ can be described by the effective lagrangian of eq. (3) with $\Omega_i$'s containing the $SU(3)$-singlet heavy quarkonium fields $\Phi$ and $\Phi'$ together with $\phi$. The resulting decay amplitudes $H(\phi)$ (for $\Phi'$ at rest) can be written as:

$$H(\phi) = \epsilon_{ijk} \Phi'_i \Phi_j p_k \left[ \hat{x} \langle M\phi \rangle + \hat{\gamma} \langle M_I\phi \rangle + \ldots \right],$$

(9)

where $\Phi'_i$ and $\Phi_i$ denote the spin vectors of $\Phi'$ and $\Phi$ respectively, $p_i$ is the momentum of $\phi = \phi^a \lambda^a$. Clearly the above amplitudes must be invariant under the IT of eq. (5), acting on $M$ and also on $\hat{\gamma}$ which is proportional to an appropriate combination of $b_i$'s. As was discussed by Voloshin and Zakharov (VZ) [9], the QCD multipole expansion whose expansion parameter is the inverse of the heavy quark mass provides another expression for $H(\phi)$:

$$H(\phi) = \epsilon_{ijk} \Phi'_i \Phi_j X_k(\phi) W_0,$$

(10)

where $W_0$ depends on the quarkonium wavefunctions, and

$$X_i(\phi) \equiv \left\langle \phi \mid g^2 E^a_k D_k B_i^k \mid 0 \right\rangle \equiv p_i X(\phi).$$

(11)

At the leading order in the multipole expansion, the quarkonium wavefunctions can be considered to be independent of $M$, and thus $W_0$ is invariant under the IT. This means that $X(\phi)$ which is written as

$$X(\phi) = x \langle M\phi \rangle + \gamma \langle M_I\phi \rangle + \ldots,$$

(12)

is invariant under the IT of eq. (5), implying that $\gamma \equiv \hat{\gamma}/W_0$ is transformed as

$$\gamma \to \gamma - \omega x.$$  

(13)

In any case, one can define

$$A_i(\phi) \equiv \left\langle \phi \mid g^2 \partial_k (E^a_k B_i^a) \mid 0 \right\rangle = p_i A(\phi),$$

$$B_i(\phi) \equiv - \left\langle \phi \mid g^2 B_i^a D_k E_k^a \mid 0 \right\rangle = p_i B(\phi),$$

(14)
so that

\[ X(\phi) = A(\phi) + B(\phi). \]  

(15)

Note that \( A(\phi) \) can be written as

\[ A(\phi) = i \frac{1}{12} \langle \phi | g^2 G \tilde{G} | 0 \rangle \text{ where } G \tilde{G} \equiv C_{\mu \nu} \tilde{G}^{\mu \nu}. \]

In the attempt to determine \( m_u/m_d \) using the quarkonium decays, the quantities of interests are \( R_X \equiv X(\pi^0)/X(\eta) \) and \( R_A \equiv A(\pi^0)/A(\eta) \). Then \( R_X \) can be fixed by the quarkonium decay data, while it is necessary to fix \( R_A \) to determine \( m_u/m_d \). In fact \( R_A = R_X \) at the leading order in \( M \), however we need to include the corrections of \( O(M^2) \) for a meaningful determination of \( m_u/m_d \). In ref. [6], following ref. [9], it was simply assumed that \( |B(\phi)| \ll |A(\phi)| \), which would imply \( R_X \approx R_A \) even at \( O(M^2) \). Here we first argue that there is no reason for this assumption to be viable, and later show how the instanton-induced mass renormalization promotes \( R_A \) to be greater than \( R_X \).

The statement that \( B(\phi) \) may be significantly smaller than \( A(\phi) \) was first made by VZ [9] who observed that \( A(\phi) = O(\sqrt{N_c}) \), and thus is enhanced by one power of \( N_c \) with respect to the naive \( N_c \)-counting. This enhancement is due to the \( \eta_0 \)-pole where the \( SU(3) \)-singlet \( \eta_0 \) denotes \( \eta' \) at chiral limit. Roughly we have

\[ \langle \phi | g^2 G \tilde{G} | 0 \rangle \simeq \langle \eta_0 | g^2 G \tilde{G} | 0 \rangle \times M_{\eta_0}^2 \times \frac{1}{M_{\eta_0}^2}, \]  

(16)

where the \( \eta_0 \)-pole (\( \equiv 1/M_{\eta_0}^2 \)) is \( O(N_c) \). Note that the mass-squared mixing \( M_{\eta_0}^2 \) between \( \eta_0 \) and \( \phi \) is suppressed by the light quark mass \( M \), but is \( O(1) \) in the \( N_c \)-counting, and \( \langle \eta_0 | g^2 G \tilde{G} | 0 \rangle = O(1/\sqrt{N_c}) \) obeys the naive \( N_c \)-counting rule. In fact, the same enhancement can occur also for \( B(\phi) \) if \( \langle \eta_0 | B_i^a D_k E_k^a | 0 \rangle \) is nonvanishing. Then \( B(\phi) \) can be equally important as \( A(\phi) \) even in the large \( N_c \)-limit. If \( B(\phi) \) does not receive any contribution from the intermediate \( \eta_0 \) and thus is \( O(1/\sqrt{N_c}) \), we would have \( |B(\phi)/A(\phi)| \simeq M_{\eta_0}^2 / \Lambda^2 \), where \( \Lambda \) denotes a typical hadronic scale which is \( O(1) \) in the \( N_c \)-counting. Clearly then \( X(\phi) \) will be dominated by \( A(\phi) \) in the large \( N_c \) limit. However in the real case of \( N_c = 3 \), this is not necessarily true since \( M_{\eta_0}^2 \) is large enough to be comparable to \( \Lambda \). Again \( B(\phi) \) can be equally important as \( A(\phi) \).
VZ noted also that the equation of motion $D_k E^a_k = ig q^\dagger \lambda^a q$ gives $B(\phi)$ an additional power of the QCD coupling constant $g$. However it is hard to imagine that this extra $g$ means a real suppression of $B(\phi)$ compared to $A(\phi)$. First of all, $g$ is essentially of order one, even for the renormalization point above $m_b$, although the loop suppression factor $\alpha_s/4\pi = g^2/16\pi^2 \ll 1$. Note that the use of the equation of motion has nothing to do with the perturbative QCD loop expansion. Furthermore if we consider the valence quark contribution (in the sense of the parton model) to the vacuum to meson matrix element of an $n$-gluon operator, it would include a factor $g^n$. Before using the equation of motion, both $A(\phi)$ and $B(\phi)$ involve two-gluon operators. The equation of motion changes the two-gluon operator in $B(\phi)$ to $B^a_i q^\dagger q^a$ which includes only one gluon field. Then at least for the valence quark contribution, $B(\phi)$ is not higher order in $g$ compared to $A(\phi)$ since the matrix element of $B^a_i q^\dagger q^a$ will be enhanced by $g^{-1}$ compared to the matrix elements of two-gluon operators.

We have argued that there is no a priori reason to expect that $B(\phi)$ is significantly smaller than $A(\phi)$. One must include $B(\phi)$ in the analysis, and then $R_A$ can be significantly different from $R_X$ at $O(M^2)$. Among the $O(M^2)$ corrections, the instanton-induced mass renormalization is of particular importance for the determination of $m_u/m_d$. Thus from now on, we will concentrate on how the instanton-induced mass renormalization affects $A(\phi)$ and $B(\phi)$, so that promotes $R_A$ to be greater than $R_X$. For this purpose, we study the IT of the second order chiral coefficients $\alpha$ and $\beta$ that appear in the following chiral expansion\footnote{Here we ignore the electromagnetic effects which were shown to be negligibly small.}:

\begin{align}
A(\phi) &= a \langle M\phi \rangle + \alpha \langle M_I\phi \rangle + ..., \\
B(\phi) &= b \langle M\phi \rangle + \beta \langle M_I\phi \rangle + ...,
\end{align}

(17)

where the ellipses denote other possible second order terms. Note that eqs. (12) and (15) imply that $A(\phi)$ and $B(\phi)$ can be expanded as above although these matrix elements can not be evaluated by using $L^{(0)}_{eff}$ alone.
In order to derive the IT of $\alpha$, let us express the physical meson fields $\pi^0$ and $\eta$ in terms of $\phi_3$ and $\phi_8$: $\pi^0 = \phi_3 + \epsilon \phi_8$, $\eta = \phi_8 - \epsilon \phi_3$. The mixing parameter $\epsilon$ can be written as

$$\epsilon = \frac{\sqrt{3}}{4} \frac{m_d - m_u}{m_s} \left[ 1 + \frac{m_s}{4\pi f} \kappa \right],$$

where $\kappa$ is a dimensionless parameter of $O(1)$ which represents the size of the $O(M^2)$-corrections. Clearly $\epsilon$ should be invariant under the IT since it describes the physical meson fields in terms of the original wavefunctions $\phi_3$ and $\phi_8$ which are untouched by the IT. This then gives the following IT of $\kappa$:

$$\kappa \rightarrow \kappa + \omega.$$  \hfill (19)

In fact, using the chiral lagrangian, $\epsilon$ was evaluated as \[^5\] :

$$\kappa = -\frac{128\pi B_0}{f} (L_8 + 3L_7) + \text{chiral logs},$$

assuring the above IT of $\kappa$ (see the IT of $L_i$’s in eq. (7)).

Using eqs. (17) and (18), we can obtain

$$A(\pi^0) = \frac{3}{2} (m_u - m_d) a \left[ 1 + \frac{m_s}{4\pi f} \left( \frac{\kappa}{3} - \frac{2\alpha}{3a} + ... \right) \right],$$

while the anomalous Ward identity

$$\partial_{\mu} (\bar{q} \gamma^{\mu} \gamma_5 M^{-1} q) = \frac{g^2}{16\pi^2} \text{tr}(M^{-1}) G^{\mu}{}_{\nu} \tilde{G}_{\mu\nu} + 2\bar{q} i \gamma_5 q$$

gives

$$A(\pi^0) = \frac{4i\pi^2}{3} \frac{m_u - m_d}{m_u + m_d} f_\pi M_\pi^2 \left[ 1 + O\left( \frac{m_d}{4\pi f} \right) \right],$$

where $f_\pi$ and $M_\pi$ denote the decay constant and the mass of the physical $\pi^0$ respectively. Here and in what follows, ellipsis always denotes a dimensionless coefficient of $O(1)$ which is untouched by the IT. Then comparing (22) with (21) together with

\[^5\]In fact, due to the $SU(3)$ breaking, the $\pi^0$-$\phi_8$ mixing can be different from the $\eta$-$\phi_3$ mixing at $O(M^2)$. However this difference does not affect our analysis of the IT, and thus will be ignored.
the expressions of $f_\pi$ and $M_\pi$ given in ref. [8], we find

$$a = \frac{8i\pi^2 B_0 f}{9},$$

$$\frac{\alpha}{a} = -\frac{64\pi B_0}{f}(L_8 + 3L_7 + 3L_6) + \ldots,$$

which yield the following IT

$$\alpha \rightarrow \alpha + 2\omega a, \quad \beta \rightarrow \beta - \omega(b + 3a). \quad (23)$$

Here the IT of $\beta$ is derived from that of $\alpha$ and $\gamma$, using $x = a + b$ and $\gamma = \alpha + \beta$.

We are now ready to see how the instanton-induced mass renormalization affects $R_A$ and $R_X$, and what is still necessary to determine $m_u/m_d$ using the QCD multipole expansion applied for the quarknonium decays. Using eqs. (12), (17), and (18), we can obtain

$$R_X \equiv \frac{X(\pi^0)}{X(\eta)} = \frac{3\sqrt{3}}{4} \frac{m_d - m_u}{m_s} \left[ 1 + \frac{m_s}{4\pi f} \xi_X \right],$$

$$R_A \equiv \frac{A(\pi^0)}{A(\eta)} = \frac{3\sqrt{3}}{4} \frac{m_d - m_u}{m_s} \left[ 1 + \frac{m_s}{4\pi f} \xi_A \right],$$

where $\xi_X = \kappa/3 - 2\gamma/3x + \ldots$, $\xi_A = \kappa/3 - 2\alpha/3a + \ldots$ are dimensionless coefficients of $O(1)$. Then the IT of $\alpha$, $\gamma$, and $\kappa$ derived so far gives the following IT:

$$\xi_X \rightarrow \xi_X + \omega, \quad \xi_A \rightarrow \xi_A - \omega. \quad (25)$$

We have argued that the IT can be interpreted as a kind of RG transformation changing the scale $\mu_I$ that appears associated with the instanton-induced mass renormalization. With this interpretation, the IT of $\xi_{A,X}$ for a maximal value of the transformation parameter, i.e. $\omega = \omega_{\text{max}}$, represents the negative of the full instanton contribution to $\xi_{A,X}$. Then $\xi_A$ receives a positive contribution $\omega_{\text{max}}$ from instantons, while $\xi_X$ receives a negative one with the same size. Although its precise value is quite sensitive to the unknown nonperturbative QCD dynamics, it has been observed that $\omega_{\text{max}}$ can be as large as $2 \sim 3$ in the semiclassical instanton gas approximation.
Then although not very reliable, for such a large $\omega_{\text{max}}$, it is more likely that $R_A$ is significantly greater than $R_X$.

Applying the QCD multipole expansion for the measured quarkonium decay widths $\Gamma(\psi' \to J\psi + \phi)$ ($\phi = \pi^0$ or $\eta$), one finds $R_X \simeq 4.3 \times 10^{-2}$ \[6, 10, 11\]. Also in ref. \[6\], the size of $\hat{R}_A \equiv (m_d + m_u)R_A/(m_d - m_u)$ was estimated within the second order chiral perturbation theory of the light pseudoscalar mesons, which yields $\hat{R}_A \simeq 8 \times 10^{-2}$. Note that both $R_X$ and $\hat{R}_A$ are invariant under the IT, and thus can be fixed by experimental data. If we assume as in ref. \[6\] that $|B(\phi)| \ll |A(\phi)|$ which implies $R_A \simeq R_X$ or equivalently $\xi_A \simeq \xi_X$, the measured values of $R_X$ and $\hat{R}_A$ would give $m_u/m_d \simeq 0.3$. However we already argued that there is no reason for $|B(\phi)| \ll |A(\phi)|$. Furthermore once we include $B(\phi)$ in the analysis as it must be, instantons give a potentially large positive contribution $\omega_{\text{max}}$ to $\xi_A$, while $\xi_X$ receives a negative contribution with the same size. This is essentially due to the instanton-induced $O(M^2)$ piece in $B(\pi^0)$, i.e. the term with the coefficient $\beta$ arising from the instanton-induced mass renormalization. Then what we obtain from the entire analysis can be summarized by the equation:

$$\frac{m_d - m_u}{m_d + m_u} \simeq 0.54 \left(1 + \frac{m_s}{4\pi f} (\xi_A - \xi_X)\right),$$

where $\delta \xi \equiv (\xi_A - \xi_X)$ is a totally unknown coefficient of $O(1)$. If $\omega_{\text{max}} = 2 \sim 3$ whose possibility was assured within the semiclassical instanton gas approximation \[1, 3\], it is conceivable to assume that $\delta \xi$ is dominated by the positive instanton contribution $2\omega_{\text{max}}$, implying $\xi_A > \xi_X$. Then the measured values of $R_X$ and $\hat{R}_A$ would give $m_u/m_d < 0.3$. In any case, in order for $m_u/m_d$ to be precisely determined, we still need to compute (within QCD) the chiral coefficient $\delta \xi$ which is very sensitive to the IT.

To conclude, we have examined whether the QCD multipole expansion applied for the quarkonium decays $\Phi' \to \Phi \phi$ can be useful for the precise determination of $m_u/m_d$. The matrix element $B(\phi)$ which has been ignored in previous works...
can significantly affect the estimate of $m_u/m_d$, particularly through the $O(M^2)$-piece induced by instantons. As a result, a rather wide range of $m_u/m_d$ (including zero) can be consistent with the observed quarkonium decays. As was concluded in ref. [7], to determine $m_u/m_d$ precisely, we need a QCD calculation of the chiral coefficient $\delta \xi = (\xi_A - \xi_X)$ which is sensitive to the IT. If instanton contribution to $\delta \xi$ dominates over other contribution, which is conceivable for $\omega_{\text{max}} = 2 \sim 3$, and thus $\delta \xi > 0$, the observed quarkonium decays imply $m_u/m_d < 0.3$, allowing the massless up quark scenario [1, 2, 3] for the absence of CP violation in strong interactions.

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