Dynamical dark energy or variable cosmological parameters?

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Abstract. One of the main aims in the next generation of precision cosmology experiments will be an accurate determination of the equation of state (EOS) for the dark energy (DE). If the latter is dynamical, the resulting barotropic index $\omega$ should exhibit a non-trivial evolution with the redshift. Usually this is interpreted as a sign that the mechanism responsible for the DE is related to some dynamical scalar field, and in some cases this field may behave non-canonically (phantom field). Present observations seem to favor an evolving DE with a potential phantom phase near our time. In the literature there is a plethora of dynamical models trying to describe this behavior. Here we show that the simplest option, namely a model with a variable cosmological term, $\Lambda = \Lambda(t)$, leads in general to a non-trivial effective EOS, with index $\omega_e$, which may naturally account for these data features. We prove that in this case there is always a “crossing” of the $\omega_e = -1$ barrier near our time. We also show how this effect is modulated (or even completely controlled) by a variable Newton’s gravitational coupling $G = G(t)$. 

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1. Introduction

Cosmology is becoming a science of precision and accuracy. In the last few years a flurry of experimental activity has been devoted to the measurement of the cosmological parameters. This effort has transformed observational cosmology into a fairly respectable branch of experimental physics [1, 2, 3]. As a result, evidence is piling up in favor of the existence of the dark energy (DE) component, $\rho_D$, pervading the Universe (or at least the known patch of it). Hopefully, this situation is bound to improve further with the advent (in the near future) of the promising SNAP and PLANCK projects [4], which will grant an experimental determination of the cosmological parameters to an unprecedented few percent level of accuracy. Quite in contrast, the nature of the DE remains at the moment a profound mystery. Historically, the DE was first identified with the cosmological constant (CC), $\Lambda$, and the vacuum energy contributions it receives from quantum field theory (QFT) [5]. This idea led to the famous cosmological constant problem [6, 7]. In more recent times the notion of DE has been detached from that of $\Lambda$ and has been extended to a variety of models leading to an accelerated expansion of the universe in which the DE itself is a time-evolving entity. These models include dynamical scalar fields (quintessence and the like) [8, 9], phantom fields [10], braneworld models [11], Chaplygin gas [12], and many other recent ideas like holographic dark energy, cosmic strings, domain walls etc (see e.g. [7] and references therein). Obviously the very notion of CC in such broader context becomes diminished, the CC could just be inexistent or simply relegated to the status of one among many other possible candidates.

A general DE model is described as being a sort of fluid characterized by a conserved DE energy density, $\rho_D$, and pressure $p_D$, related by an equation of state (EOS) $p_D = \omega_D \rho_D$. A particular and popular realization of the DE is the aforementioned quintessence idea [8, 9], where one has some scalar field $\chi$ which generates a non-vanishing $\rho_D$ from the sum of its potential and kinetic energy term at the present time: $\rho_D^0 = \{(1/2)\xi \dot{\chi}^2 + V(\chi)\}_{t=t_0}$. The sign of the coefficient $\xi$ determines whether the field can describe quintessence ($\xi > 0$) or phantom DE ($\xi < 0$). If the kinetic energy for $\chi$ is small enough, it is clear that $\rho_D^0$ looks as an effective cosmological constant. The scalar field $\chi$ is in principle unrelated to the Higgs boson or any other field of the Standard Model (SM) of particle physics, including all of its known extensions (e.g. the supersymmetric generalizations of the SM); in other words, the $\chi$ field is an entirely ad hoc construct just introduced to mimic the cosmological term. However, the field $\chi$ is usually thought of as a high energy field (unrelated to SM physics), i.e. $\chi \simeq M_X$ where $M_X$ is some high energy scale typically around the Planck mass $M_P \sim 10^{19}$ GeV. If one assumes the simplest form for its potential, namely $V(\chi) = (1/2) m_\chi^2 \chi^2$, its mass turns out to be $m_\chi \sim H_0 \sim 10^{-33} eV$ in order to approximately describe the present value of the energy density associated to $\Lambda$, namely $\rho_\Lambda^0 \equiv \Lambda_0/8\pi G \sim 10^{-47} \text{GeV}^4$. Obviously it is very difficult to understand the small mass $m_\chi$ in particle physics, and this is of course a serious problem underlying the quintessence models. On the other hand the possible observed transition from a quintessence into phantom-like behavior [13, 14] cannot be explained with only one scalar field (because the sign of the coefficient $\xi$ is fixed), and therefore one has to resort to at least two scalar fields, one with $\xi > 0$ and the other with another coefficient $\xi' < 0$ [3]. In spite of the many

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difficulties with the scalar field models they have the virtue that they may help to understand why the DE might be evolving with time as suggested by the analyses of the cosmological data \cite{13,14}, and in this sense one would like not to lose this useful aspect of the quintessence proposal.

In this work, in contrast to the previous approaches, we wish to stick to the original idea that the primary cause of the DE is a “true” CC term in Einstein’s equations, but we assume it is a variable one: $\Lambda = \Lambda(t)$. Actually, a rich variety of variable $\Lambda$ models have been reported in the literature – some of them admittedly on purely phenomenological grounds \cite{15,17}. There are however models based on a more fundamental premise, even if ultimately phenomenological at the moment. For instance, there are models attempting to adjust the small value of the cosmological term with the help of a dynamical scalar field \cite{18}, as for example the cosmon model \cite{8,19}. More recently a variable $\Lambda$ has been proposed as a running quantity from the point of view of the renormalization group (RG) \cite{20-24}. Particularly interesting for the present work is the possibility that an effective equation of state can be associated to a variable cosmological model. This has been explored in detail in Ref.\cite{25} for a model of running cosmological term \cite{22}. In the present work we generalize this idea to arbitrary variable cosmological models (without specifying the underlying dynamics), and show that these models can effectively lead to a non-trivial EOS (therefore mimicking a dynamical DE model). We also show how to include in the effective EOS the possible effects from a variable Newton’s gravitational coupling. We provide a general algorithm to construct this effective EOS and prove some general results which describe the relationship between the variable $\Lambda$ models and the dynamical DE models. Finally we argue that this relationship may help to shed some light to understand the apparently observed transition from quintessence-like to phantom-like behavior as indicated by some analyses of the most recent cosmological data \cite{13,14}.

2. Two cosmological pictures: Variable $\Lambda$ and $G$ versus dynamical dark energy

We start from Einstein’s equations in the presence of the cosmological constant term,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} + g_{\mu\nu}\Lambda = 8\pi G (T_{\mu\nu} + g_{\mu\nu}\rho_\Lambda),$$

(1)

where $T_{\mu\nu}$ is the ordinary energy-momentum tensor associated to isotropic matter and radiation, and $\rho_\Lambda$ represents the energy density associated to the CC. Let us next contemplate the possibility that $G = G(t)$ and $\Lambda = \Lambda(t)$ can be both functions of the cosmic time within the context of the FRW (Friedmann-Robertson-Walker) cosmology. The possibility of a variable gravitational coupling and a variable cosmological term in cosmology (variable CC term, for short) has been considered by many authors, and in particular it has been discussed within the renormalization group approach to cosmology \cite{20-24}. It should be clear that the very precise measurements of $G$ existing in the literature refer only to distances within the solar system and astrophysical systems. In cosmology these scales are immersed into much larger scales (galaxies and clusters of galaxies) which are treated as point-like (and referred to as “fundamental observers”, co-moving with the cosmic fluid \cite{26}). Therefore, the variations of $G$ at the cosmological level could only be seen at much larger distances where we have never had the possibility to make direct experiments. To put it in another way, the potential variation of $G = G(t)$ and $\Lambda = \Lambda(t)$ is usually tested in terms of a
measurable redshift dependence of these functions, \( G = G(z) \) and \( \Lambda = \Lambda(z) \), say a variation for a redshift interval of at least \( \Delta z > 0.01 \), therefore implying scales of several hundred Mpc.

The only possible EOS for the CC term, whether strictly constant or variable, is \( p_\Lambda = -\rho_\Lambda \). However, we may still parametrize this variable \( \Lambda \) model as if it would be a dynamical field DE model with pressure and density \( (p_D, \rho_D) \). We will call this the “effective DE picture” of the fundamental \( \Lambda \) model. The matching of the DE picture with the fundamental CC picture generates a non-trivial “effective EOS” \[27\] for the latter, \( p_D = \omega_e \rho_D \), where the effective barotropic index \( \omega_e = \omega_e(z) \) is a function of the cosmological redshift to be determined. By exploring this alternative we can test the variable \( \Lambda \) models as a canonical source of effective dynamical DE models. In the following we will restrict our considerations to the flat space case \( (\Omega_K = 0) \). This is not only for the sake of simplicity, but also because it is the most favored scenario at present. Friedmann’s equation with non-vanishing \( \rho_\Lambda \) reads

\[
H_0^2 = \frac{8\pi G}{3} (\rho + \rho_\Lambda),
\]

where for convenience we have appended a subscript in the Hubble parameter. On the other hand the general Bianchi identity of the Einstein tensor in \[11\] leads to

\[
\nabla^\mu \left[ G(T_{\mu\nu} + g_{\mu\nu} \rho_\Lambda) \right] = 0.
\]

Using the FRW metric explicitly, the last equation results into the following “mixed” local conservation law:

\[
\frac{d}{dt} [G(\rho + \rho_\Lambda)] + 3 G H (\rho + p) = 0 .
\]

If \( \rho_\Lambda \equiv d\rho_\Lambda/dt \neq 0 \), \( \rho \) is not generally conserved as there may be transit of energy from matter-radiation into the variable \( \rho_\Lambda \) or vice versa (including a possible contribution from a variable \( G \), if \( \dot{G} \neq 0 \)). Thus this law indeed mixes the matter-radiation energy density with the vacuum energy \( (\rho_\Lambda) \). To be more precise, the following scenarios are possible: i) \( G = \text{const.} \) and \( \rho_\Lambda = \text{const.} \) This is the standard case of \( \Lambda \)CDM cosmology implying the canonical conservation law of matter-radiation:

\[
\dot{\rho} + 3H (\rho + p) = 0 ,
\]

ii) \( G = \text{const.} \) and \( \dot{\rho}_\Lambda \neq 0 \), in which case Eq.\[4\] boils down to \( \dot{\rho}_\Lambda + \dot{\rho} + 3H (\rho + p) = 0 ; \) iii) \( \dot{G} \neq 0 \) and \( \rho_\Lambda = \text{const.} \), implying \( \dot{G}(\rho + \rho_\Lambda) + G[\dot{\rho} + 3H(\rho + p)] = 0 \); and finally iv) \( \dot{G} \neq 0 \) and \( \dot{\rho}_\Lambda \neq 0 \), which in the case of self-conservation of matter-radiation it leads to \( (\rho + \rho_\Lambda)\dot{G} + G\dot{\rho}_\Lambda = 0 \). Notice that in cases ii) and iii) the matter-radiation cannot be canonically conserved (if one of the two parameters \( \rho_\Lambda \) or \( G \) indeed is to be variable), whereas in case iv) it is assumed to be conserved. Explicit cosmological models with variable parameters as in cases ii) and iv) have been constructed within the context of RG cosmology in Ref.\[22\] and \[23\] respectively.

With only Eq.\[11\] and the FRW equations we cannot solve the cosmological model with variable parameters. One needs a fundamental model that informs us on the functional dependence of \( \rho_\Lambda \) and \( G \) on the remaining cosmological functions, like \( \rho, p \) and \( H \). However, in most of this work we will not commit ourselves to any specific model for the underlying fundamental dynamics, because the kind of results we are aiming at are to be valid for a large class of models with dynamical cosmological parameters. Our guiding paradigm will be the general expectation that
these parameters are variable because of the effective running that they should display according to the renormalization group. In this sense the possible variation of “fundamental constants” such as $\Lambda$ and $G$ could be an effective description of some deeper dynamics associated to QFT in curved space-time, or quantum gravity or even string theory, all of which share the powerful RG approach to quantum effects. For instance, in the specific framework of QFT in curved space-time the RG equations for $\rho_\Lambda$ and $G$ should entail definite laws $\rho_\Lambda = \rho_\Lambda(t), G = G(t)$ [23]. Given a fundamental model based on QFT, the parameter $\rho_\Lambda$ will primarily depend on some cosmological functions (matter density $\rho$, Hubble expansion rate $H$, etc) which evolve with time or redshift. Similarly for the Newton coupling $G$. Therefore, in general we will have two functions of the redshift

$$\rho_\Lambda(z) = \rho_\Lambda(\rho(z), H(z), ...), \quad G(z) = G(\rho(z), H(z), ...).$$

We understand that other fundamental parameters could also be variable. For example, the fine structure constant has long been speculated as being potentially variable with the cosmic time (and therefore with the redshift) including some recent experimental evidences – see e.g. [29]. This might also have an interpretation in terms of the RG at the level of the cosmological evolution. And similarly with other “constants” such as the ratio $m_e/m_p$ between the electron to the proton mass etc which could also have evolved throughout the cosmic history. However, in this paper we concentrate purely on the potential variability of the most genuine fundamental gravitational parameters of Einstein’s equations such as $\Lambda$ and $G$. The possibility that $G$ could be associated to a variable scalar field stems from the old Jordan and Brans-Dicke proposals [30] and has generated an abundant literature since then. Similarly the idea that the $\Lambda$ term could perhaps be variable and related to a dynamical field is also relatively old [18, 19, 13]. In short, whether related to dynamical fields or to the general phenomenon of RG running, the so-called “fundamental constants of Nature” are suspicious of being non-constant from the point of view of quantum theory. Adopting this general Ansatz, we wish to investigate whether general (model-independent) properties can be extracted if we take Einstein’s equations (1) in the FRW metric, with variable parameters (5), as the starting point for the study of cosmology.

Functions (5) will usually be monotonous; e.g. in the RG model of Ref. [22] $\rho_\Lambda$ inherits its time/redshift dependence through $\rho_\Lambda = c_1 + c_2 H^2$ and it satisfies $d\rho_\Lambda/dz \gtrless 0$ if $c_2 \gtrless 0$ respectively. Similarly, $G = G_0/(1 + \nu \ln H^2/H_0^2)$ in the RG model of [23], and therefore $dG/dz \gtrless 0$ if $\nu \lesssim 0$ respectively. Furthermore, in most phenomenological models in the literature (where the underlying dynamics is usually not specified) this monotonic character also applies [16]-[19]. Since the functions (5) are presumably known from the fundamental model, and the ordinary EOS for matter-radiation is also given, one may solve Eq. (11) and Friedmann’s equation to determine $\rho = \rho(z), \rho_\Lambda = \rho_\Lambda(z)$ and $G = G(z)$ as explicit functions of the redshift. These may then be substituted back in (2) to get the expansion rate of the variable $\Lambda$ model as a known function of $z$:

$$H^2_\Lambda(z) = H_0^2 \left[ \Omega_M f_M(z; r)(1 + z)^\alpha + \Omega_\Lambda f_\Lambda(z; r) \right],$$

where $\alpha = 3(1 + \omega_m)$ ($\omega_m = 1/3$ or $\omega_m = 0$ for the radiation or matter dominated epochs respectively). Functions $f_M$ and $f_\Lambda$ are completely determined at this stage, and may depend on some free parameters $r = r_1, r_2, ..$ – see [22, 23] for non-trivial examples with a single parameter.
Only for the standard $ΛCDM$ model we have $f_Λ = f_M = 1$. In general it is not so due to the mixed conservation law [11]. Whatever it be their form, these functions must satisfy $f_M(0;r) = f_Λ(0;r) = 1$ in order that the cosmic sum rule $Ω_0^M + Ω_0^Λ = 1$ is fulfilled.

On the other hand, within the context of the effective DE picture the matter-radiation density satisfies by definition the standard (“unmixed”) conservation law

$$\dot{ρ}_s + α H_D ρ_s = 0.$$  \hfill (7)

Here we have denoted the matter-radiation density by $ρ_s$ to emphasize that in the DE picture it satisfies the standard self-conservation law, i.e. it corresponds to case i) mentioned above. Moreover $H_D$ stands for

$$H_D^2 = \frac{8πG_0}{3}(ρ_s + ρ_D)$$  \hfill (8)

with constant $G_0$. The subscript D here is to distinguish it from the Hubble function in the CC picture, Eq. (2). The DE density $ρ_D$ is also conserved, independently of matter:

$$\dot{ρ}_D + 3H_D(1 + ω_e) ρ_D = 0.$$  \hfill (9)

The solutions of the two conservation equations (7) and (9) read $ρ_s(z) = ρ_s(0)(1 + z)^α$ and

$$ρ_D(z) = ρ_D(0) ζ(z),$$  \hfill (10)

$$ζ(z) \equiv \exp\left\{3 \int_0^z dz' \frac{1 + ω_e(z')}{1 + z'}\right\}.$$  \hfill (11)

From (10) it is clear that if $ω_e(z) \gtrsim −1$, then $ρ_D$ increases with $z$ (hence decreases with the expansion). This is characteristic of standard quintessence [31]. In contrast, if at some point $z = z_p$ we hit $ω_e(z_p) < −1$, the DE density will decrease with $z$ and hence will increase with the expansion. This anomaly would signal the breakthrough of a phantom regime [10] in some interval of $z$ around $z_p$. The general solution for $ρ_s(z)$ and $ρ_D(z)$ given above can be substituted back to Friedmann’s equation in the DE picture and one gets the corresponding expansion rate:

$$H_D^2(z) = H_0^2 \left[Ω_M^0 (1 + z)^α + Ω_D^0 ζ(z)\right].$$  \hfill (12)

Here $Ω_M^0 + Ω_D^0 = 1$ because $ζ(0) = 1$. In general $ΔΩ_M ≡ Ω_M^0 - Ω_D^0$ is nonzero. In fact, suppose that one fits the high-z supernovae data using a variable $ρ_Λ$ model – cf. the second Ref. [22]. The fit crucially depends on the luminosity distance function, which is determined by the explicit structure of (6), so that the fitting parameters $Ω_M^0, Ω_D^0$ can be different from those obtained by substituting the alternate function (11) in the luminosity distance.

3. Effective EOS for a general model with variable $Λ$ and $G$

Let us now compute $ω_e$. With the help of (10) we have $ω_e(z) = −1 + (1/3)((1 + z)/ζ) dζ/dz$. Differentiating Eq. (11) on both sides we can eliminate $dζ/dz$. Next we apply the matching condition mentioned above, meaning that we fully identify $H_D^2$ with $H_Λ^2$ in the effective DE picture. The final result is

$$ω_e(z) = \frac{-1 + ((1 + z)/3) (1/H_Λ^2(z)) dH_Λ^2(z)/dz + (1 − α/3) Ω_M^0 (H_0^2/H_Λ^2(z)) (1 + z)^α}{1 − Ω_M^0 (H_0^2/H_Λ^2(z)) (1 + z)^α}.$$  \hfill (12)
This is the effective barotropic index for a variable $\rho_\Lambda(z)$ and/or $G(z)$ model. The procedure is formally similar to Ref. [32] but it is conceptually different. Here we do not try to reconstruct the scalar field dynamics as a fundamental model of the DE, not even a model-independent polynomial approximation to the DE [13]; rather, we start from a general (purportedly fundamental) $\rho_\Lambda$ model that can decay into matter and radiation and then we simulate it with a (generic) dynamical DE model in which the matter-radiation is strictly conserved, as usually assumed. The rationale for this is that fundamental physics (e.g. the RG in QFT) can offer us useful information on the possible functional forms [13], as shown in [20–24]. As a consequence, $H_\Lambda(z)$ can actually be computed after solving the coupled system of equations formed by Friedmann’s equation, the non-trivial conservation law [14] and the relations [15] provided by the fundamental model. Then one can check whether the effective EOS for the variable $\rho_\Lambda$ model does emulate quintessence or phantom energy depending on whether the calculation of the r.h.s. of (12) yields $-1 < \omega_e < -1/3$ or $\omega_e < -1$ respectively.

It is remarkable that the tracking of the $\rho_\Lambda(t)$ model by the DE picture (in particular a possible phantom behavior near our time) is actually expected for a wider class of variable $\rho_\Lambda$ models. It is not easy to see this from Eq.(12). Instead, let us return to Eq.(4). From the aforementioned matching condition we have $G(\rho + \rho_\Lambda) = G_0(\rho_s + \rho_D)$. Using also $H dt = -dz/(1 + z)$ we can transform the general Bianchi identity (4) into the following differential form

$$(1 + z) \, d(\rho_s + \rho_D) = \alpha \, (\rho_s + \rho_D - \xi_\Lambda) \, dz,$$

where we have defined

$$\xi_\Lambda(z) = \frac{G(z)}{G_0} \, \rho_\Lambda(z).$$

This new variable coincides with $\rho_\Lambda$ only if $G$ has the constant value $G_0$, but in general both $\rho_\Lambda$ and $G$ will develop a time/redshift evolution from the fundamental theory (for instance from given RG equations), so that $\xi_\Lambda$ will be a known function $\xi_\Lambda = \xi_\Lambda(\rho, H, ...)$). Using (17), rewritten in terms of the variable $z$, we can eliminate $\rho_s$ from (13) and we are left with a simple differential equation for $\rho_D$:

$$\frac{d\rho_D(z)}{dz} = \frac{\alpha \, \rho_D(z) - \xi_\Lambda(z)}{1 + z} \equiv \beta(\rho_D(z)).$$

(15)

This equation is particularly convenient to analyze the behavior of the effective DE picture. Consider the plane $(\rho_D, \beta(\rho_D))$. For constant $\xi_\Lambda$ (as in the ΛCDM model, where $G = G_0$ and $\xi_\Lambda = \rho_\Lambda^0$), Eq. (15) resembles a renormalization group equation for the “coupling” $\rho_D$, with “β-function” $\beta(\rho_D)$ and a fixed point (FP) at $\rho_D^* = \xi_\Lambda$, where $\beta(\rho_D^*) = 0$. Since $d\beta(\rho_D)/d\rho_D = \alpha/(1 + z) > 0$ (for all $z > -1$) it is clear that this FP is an infrared (IR) stable fixed point, hence $\rho_D(z) \to \rho_D^*$ for decreasing $z$. For variable $\xi_\Lambda = \xi_\Lambda(z)$ the conclusion is the same: the solution $\rho_D(z) \to \rho_D(z)$ of (15) is attracted to the given function $\xi_\Lambda = \xi_\Lambda(z)$ in the far IR. The quintessence regime occurs for $\beta(\rho_D) > 0$, i.e. for $\rho_D > \xi_\Lambda$, whereas the phantom regime is characterized by $\beta(\rho_D) < 0$. Whatever it be the sign of $\beta(\rho_D)$ at a particular instant of time in the history of the Universe, the function $\rho_D(z)$ will eventually be driven to $\xi_\Lambda(z)$ for decreasing $z$ (namely by the expansion of the Universe). Let us next study if the two functions $\rho_D(z)$ and $\xi_\Lambda(z)$ can be very close at some point near our time, and what are the phenomenological implications. The differential equation
can be integrated in closed form for any $\xi_\Lambda(z)$:

$$\rho_D(z) = (1 + z)^\alpha \left[ \rho_D(0) - \alpha \int_0^z \frac{dz' \xi_\Lambda(z')}{(1 + z')(\alpha + 1)} \right].$$  \hfill (16)

Expanding $\xi_\Lambda(z)$ around $z = -1$ one can check that $\rho_D(z) \rightarrow \xi_\Lambda(z)$ at sufficiently late time (i.e. $z \rightarrow -1$). Finally, from (10) and with the help of (15) we find

$$\omega_e(z) = -1 + \frac{\alpha}{3} \left( 1 - \frac{\xi_\Lambda(z)}{\rho_D(z)} \right) \equiv -1 + \epsilon(z),$$  \hfill (17)

where $\rho_D$ is given by (16). This formula is more useful than Eq. (12) in the present context. It provides an efficient recipe to compute $\omega_e$ directly from the sole knowledge of $\xi_\Lambda$. Using these general formulæ we can e.g. reproduce the effective EOS obtained for the particular model recently studied in Ref. [25], see Eq.(23-24) of the latter. At the present time (that is, near $z = 0$) we expect that $\epsilon(z)$ is a relatively small quantity because observations show that the effective barotropic index must be near $-1$ at present [11][2][3]. The sign of $\epsilon(z)$ at $z = 0$, however, is not known with certainty.

Model independent analyses of the most recent cosmological data seem to indicate [13][14] that $\omega_e$ has undergone a certain evolution from $\omega_e(0) = 0$ (at $z \approx 1.7$) reaching the phantom regime $\omega_e \lesssim -1$ at $z \gtrsim 0$. In [25] it was shown that a RG model for the $\rho_\Lambda$ evolution can exhibit this type of behavior. We will now prove that this feature actually holds for a large class of models with variable $\xi_\Lambda$.

4. Effective quintessence/phantom behavior of models with variable $\Lambda$ and $G$

Let us rewrite the solution of the differential equation (15) in the following alternative way:

$$\rho_D(z) = \xi_\Lambda(z) - (1 + z)^\alpha \int_0^z \frac{dz' \xi_\Lambda(z')}{(1 + z')^\alpha},$$  \hfill (18)

where $z^*$ is a root of $\beta(\rho_D(z)) = 0$, i.e. a point where $\rho_D(z^*) = \xi_\Lambda(z^*)$. Remarkably, one can show that a root $z^*$ always exists near our present time, meaning in our recent past, in the immediate future or just at $z^* = 0$. The proof is based on establishing the following relation:

$$\frac{d\zeta(z)}{dz} = \frac{\alpha (1 + z)^{\alpha-1}}{1 - \tilde{\Omega}_M^0} \left( \Omega_M^0 f_M(z; r) - \tilde{\Omega}_M^0 \right).$$  \hfill (19)

It ensues after a straightforward calculation from: i) the matching condition of the two pictures, $H_\Lambda(z) = H_D(z)$, and ii) the constraint imposed on the functions $f_M$ and $f_\Lambda$ in [15] by the Bianchi identity [4]. Since $f_M(0; r) = 1$, as noted previously, it is clear that if the cosmological parameters of the two pictures coincide ($\Delta \Omega_M = 0$) the derivative (19) vanishes identically at $z^* = 0$, hence $\omega_e(0) = -1$. If, however, $\Delta \Omega_M \neq 0$ but is small (after all $\tilde{\Omega}_M^0$ and $\Omega_M^0$ in the two pictures should not be very different), then there exists a point $z^* \gtrsim 0$ not very far from $z = 0$ where $d\zeta/dz|_{z=z^*} = 0$ (and so $\omega_e(z^*) = -1$). This completes the proof. Therefore, $z^* \approx 0$ exists and defines a (local) divide between a quintessence phase and a phantom phase. The slope of the function $\rho_D$ reads

$$\frac{d\rho_D(z)}{dz} = -\alpha (1 + z)^{\alpha-1} \int_{z^*}^z \frac{dz' \xi_\Lambda(z')}{(1 + z')^\alpha} \frac{d\xi_\Lambda(z')}{dz'}.$$  \hfill (20)

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One could naively think that for increasing/decreasing $\xi$ with redshift, $\omega_e(z)$ should always be above/below $-1$. Let us assume that $\xi$ is monotonous (hence $d\xi_{\Lambda}/dz$ has a definite sign) in some interval containing $z^*$ and some reference point $z = z_1$. Eq. (20) shows that in this case $d\rho_D/dz$ is also monotonous, and if $z^* < z_1$ then $d\rho_D/dz|_{z=z_1}$ has opposite sign to $d\xi_{\Lambda}/dz|_{z=z_1}$. Thus e.g. if $d\xi_{\Lambda}/dz > 0$ (i.e. for $\xi_{\Lambda}$ decreasing with the expansion) then $d\rho_D/dz|_{z=z_1} < 0$, meaning that the dynamical DE picture will look as phantom energy at $z_1$ (contrary to naive expectations); if $z^* > z_1$ then $d\rho_D/dz|_{z=z_1} > 0$ and the DE will look as quintessence at $z_1$. Similarly, if $\xi_{\Lambda}(z)$ is monotonically increasing with the expansion ($d\xi_{\Lambda}/dz < 0$), and $z^* < z_1$, the observer at $z_1$ will see quintessence (again counterintuitive), but when $z_1 < z^*$ he/she will see phantom DE. The last situation is particularly worth noticing because if $z_1 = 0$ this case could just correspond to the present observational data (if the tilt into the phantom regime is finally confirmed [13, 14]). The exact position of $z^*$ depends on the peculiarities of the variable $\xi_{\Lambda}(z)$ model and also on the values of the parameter difference $\Delta \Omega_M$ in the two pictures (6) and (11). This is corroborated in concrete scenarios with monotonically decreasing $\rho_{\Lambda}(z)$ [25], where for parameter differences of a few percent one finds that the effective EOS develops a phantom phase at $z^* \gtrsim 0$ and persists phantom-like asymptotically until the remote future, where $\omega_e(z) \to -1$.

Finally, we note from (4) that if $G = G(t)$ and we assume that matter is conserved (no transfer of energy with a variable $\rho_{\Lambda}$), then $d\xi_{\Lambda}/dt = -\langle \rho/G_0 \rangle dG/dt$. In this case if $G(t)$ is monotonous, $\xi_{\Lambda}(t)$ is too. From (20) we get

$$
\frac{d\rho_D}{dz} = \alpha(1 + z)^{\alpha - 1} \frac{\rho(0)}{G_0} [G(z) - G(z^*)].
$$

(21)

This shows that if $G$ is asymptotically free – that is to say, if $G$ decreases with redshift – we should observe quintessence behavior for $0 \leq z \leq z^*$; whereas if $G$ is “IR free” (i.e. it increases with $z$ – hence decreases with the expansion), then we should have phantom behavior in that interval. In general, if $\rho_{\Lambda}$ is variable (and $G$ is fixed or variable) the monotonous property of $\xi_{\Lambda}$ is satisfied by most models in the literature [17], including the RG ones [20, 21, 22, 23, 24]. The monotonous variation of $\xi_{\Lambda}(z)$ around $z^*$ is important to insure a long quintessence-like regime preceding this point – as suggested by the cosmological data.

5. An example: renormalization group model with running $\rho_{\Lambda}$

As a concrete example of cosmological model with variable parameters, consider the case $G = \text{const.}$ and $\dot{\rho}_{\Lambda} \neq 0$. We have catalogued this possibility in Sect.2 as case ii), for which the general Bianchi identity (11) simplifies into

$$
\rho_{\Lambda} + \dot{\rho} + 3 H (\rho + p) = 0.
$$

(22)

This equation, in combination to Friedmann’s equation, has been used to solve the cosmological RG model of [22]. This model treats the CC semiclassically as a running parameter, therefore one evolving with time/redshift due to the quantum loop effects of the high energy fields (the only ones that can contribute significantly in this model, due to the “soft-decoupling” phenomenon associated to the CC in Ref. [22]). Interestingly enough this model can simulate an apparent
phantom behavior near our time [25], actually following very closely the polynomial data fits of Ref. [13]. In this particular model equations (5) read

$$\rho_\Lambda(z) = C_0 + C_1 H^2(z), \quad G = \text{const},$$

with

$$C_0 = \rho_{\Lambda,0} - \frac{3\nu}{8\pi} M_P^2 H_0^2, \quad C_1 = \frac{3\nu}{8\pi} M_P^2.$$

This model has a single parameter, $\nu$, defined essentially as the ratio (squared) of the masses of the high energy fields to the Planck mass ($M_P$) [22],

$$\nu = \frac{\sigma}{12\pi M^2}.$$

Here $\sigma = \pm 1$ depending on whether bosons or fermions dominate in their loop contributions to the running of $\rho_\Lambda$. The typical value for $\nu$ is obtained when $M = M_P$, namely

$$\nu_0 \equiv \frac{1}{12\pi} \simeq 0.026.$$

In general we expect $|\nu| \leq \nu_0$ because from the effective field theory point of view we assume $M \leq M_P$. This is also suggested from the bounds on $\nu$ obtained from nucleosynthesis [22] and also from the CMB [33].

Using equations (22) and (23) in combination with Friedmann’s equation one can solve this cosmological model explicitly and one can obtain the functions $H = H(z; \nu)$, $\rho = \rho(z; \nu)$ and $\rho_\Lambda = \rho_\Lambda(z; \nu)$ in close analytic form. We refer the reader to [22] for all the details concerning this running CC model. Once the model is solved we have an explicit expression for the function $\xi_\Lambda(z)$, Eq.(14), which in this particular case is obviously given by $\xi_\Lambda(z) = \rho_\Lambda(z; \nu)$. For the flat case one finds (in the matter-dominated epoch, $\alpha = 3$)

$$\xi_\Lambda(z; \nu) = \rho_\Lambda^0 + \rho_M^0 \frac{\nu}{1 - \nu} \left[(1 + z)^{3(1 - \nu)} - 1\right],$$

where $\rho_\Lambda^0$ and $\rho_M^0$ are the values of these parameters at $z = 0$. Notice that for $\nu = 0$ we recover the standard FRW case with constant cosmological term. Substituting (27) in the general formula (10), and integrating, one obtains the effective DE density associated to this model:

$$\rho_D(z; \nu) = \rho_\Lambda^0 + (\rho_D^0 - \rho_\Lambda^0) (1 + z)^3 + \frac{\rho_M^0}{1 - \nu} (1 + z)^3 \left[(1 + z)^{-3\nu} - 1\right] + \frac{\nu \rho_M^0}{1 - \nu} \left[(1 + z)^3 - 1\right].$$

As expected from the general discussion below Eq.(15) we confirm that $\rho_D(z; \nu) \to \xi_\Lambda(z; \nu)$ in the remote future; in other words $\omega_e(z \to -1) \to -1$, see Eq.(17). From (27) and (28) we have the asymptotic limit

$$\rho_D(z; \nu) \to \xi_\Lambda(z; \nu) \to \rho_\Lambda^0 - \frac{\nu \rho_M^0}{1 - \nu} \quad (\text{for } z \to -1).$$

Finally, the effective EOS parameter for this model is obtained by inserting (27) and (28) in Eq.(17). Upon some rearrangement it finally yields

$$\omega_e(z; \nu) = -1 + (1 - \nu) \frac{\Omega_M^0 (1 + z)^{3(1 - \nu)} - \Omega_M^0 (1 + z)^3}{\Omega_M^0 [(1 + z)^{3(1 - \nu)} - 1] - (1 - \nu) [\Omega_M^0 (1 + z)^3 - 1]},$$

$$10.$$
Figure 1: (a) Numerical analysis of the effective EOS parameter $\omega_e$, Eq. (17), as a function of the redshift for fixed $\nu = -0.8 \nu_0 < 0$ and for $\Delta \Omega_M = -0.01$. The Universe is assumed to be spatially flat ($\Omega_K^0 = 0$) with the standard parameter choice $\Omega_0^M = 0.3$, $\Omega_0^\Lambda = 0.7$. In the figure, $z^*$ is the crossing point of the “barrier” $\omega_e = -1$; (b) The corresponding evolution and crossing (at $z = z^*$) of the density functions $\xi_\Lambda(z)$ and $\rho_D(z)$ in the two pictures, Eqs. (27)-(28), in units of the critical density $\rho_{c,0}$ at present.

where we have defined $\Omega^0_0 = \rho^0_M/\rho_{c,0}$ and $\bar{\Omega}^0_0 = \rho^0_/\rho_{c,0}$ corresponding to the matter densities in the two pictures. Recall that in the CC picture $\rho_M$ is non-conserved in this model – see Eq.(22) – whereas in the DE picture $\rho_s$ is conserved – cf Eq (7) –, so the two mass density parameters $\rho^0_M$ and $\rho^0_s$ involved in the two fits (in each picture) of the same data need not to coincide. In our general discussion below Eq. (11) we expressed this fact by asserting that the parameter difference $\Delta \Omega_M \equiv \Omega_0^M - \bar{\Omega}^0_0$ is in general expected to be nonzero \footnote{We note that the result \footnote{\cite{25}}, obtained from the general method devised in this paper, does perfectly agree with the particular calculation performed in \cite{25}.}

In Fig. 1a we show the numerical analysis of Eq. (30) for a typical choice of the parameters. In general we do not expect $\Delta \Omega_M$ to be very large because the two pictures are supposed to give a similar representation of the same data. Therefore we have chosen $\Delta \Omega_M$ of order one percent. Similarly, the parameter $\nu$ should be close to (26). For $\Delta \Omega_M < 0$ and $\nu < 0$ as given in Fig. 1a we find a transition point from quintessence-like behavior into phantom-like behavior very near our recent past, namely around $z^* \simeq 0.7$. – marked explicitly in the figure – i.e. around the time when the Universe deceleration changed into acceleration. In Fig. 1b we plot the density functions (27) and (28) normalized to the critical density at present, $\rho_{c,0}$. This plot also exhibits the crossing point $z = z^*$ of the two curves, where $\xi_\Lambda(z^*) = \rho_D(z^*)$. As predicted by the general differential equation (15), in this figure the quintessence-like behavior is seen to be characterized by $\rho_D(z) > \xi_\Lambda(z)$ whereas the phantom-like behavior by $\rho_D(z) < \xi_\Lambda(z)$, with $z = z^*$ acting as a transition point between the two. We can also check in Fig. 1b the accomplishment of the asymptotic condition given by Eq. (29).

The interesting features displayed by this running cosmological model are only a particular illustration of the general expectations for cosmological models with variable cosmological param-
eters. The alternate representation of the effective DE density $\rho_D(z)$ and its derivative $d\rho_D/dz$ – see equations (18) and (20) – allowed us to describe these features in a completely general way without committing to any particular model. If we change the model (namely the kind of evolution of the redshift functions (5)) similar features will arise depending on the parameters of the model and on the value of $\Delta \Omega_M$, but in all cases a transition point $z^*$ near our present time (lying either in our recent past or in our immediate future) will be found. The existence of the crossing point $z^*$ $(\omega_e(z^*) = -1)$ where the effective EOS may change from quintessence-like to phantom-like behavior near our present time is, as we have proven in this paper, a general result for all the models with variable cosmological parameters.

6. Conclusions

To summarize, we have shown that a model with variable $\rho_\Lambda$ and/or $G$ generally leads to an effective EOS with a non-trivial barotropic index, with the property $\omega_e(z) \to -1$ in the far IR. Such model can mimic a dynamical DE model and effectively appear as quintessence and even as phantom energy. The eventual determination of an empirical EOS for the DE in the next generation of precision cosmology experiments should keep in mind this possibility. The nature of the dynamics behind $\rho_\Lambda$ is not known in principle, but various works in the literature suggest that a fundamental $\rho_\Lambda$ can display a RG running which can be translated into redshift evolution. Here we have generalized these ideas and have shown that in any model with variable $\xi_\Lambda(z) = (G(z)/G_0) \rho_\Lambda(z)$ there is an effective DE density, $\rho_D(z)$, that tracks the evolution of $\xi_\Lambda(z)$ and converges to it in the far IR. Moreover, we have proven that there always exists a point $z^*$ near $z = 0$ where $\rho_D(z^*) = \xi_\Lambda(z^*)$, hence $\omega_e(z^*) = -1$. If this point lies in our recent past ($z^* \geq 0$) and $\xi_\Lambda$ is a decreasing function of $z$ around $z^*$, then there must necessarily be a recent transition into an (effective) phantom regime $\omega_e(z) \lesssim -1$. This would explain in a natural way another "cosmic coincidence": why the crossing of $\omega_e = -1$ just occurs near our present time? Our results are model-independent, they only assume the formal structure of Einstein’s equations of General Relativity and FRW cosmology in the presence of a variable cosmological term, $\rho_\Lambda = \rho_\Lambda(z)$, and possibly (though not necessarily) a variable gravitational coupling $G = G(z)$. We conclude that there is a large class of variable ($\rho_\Lambda, G$) models that could account for the observed evolution of the DE, without need of invoking any combination of fundamental quintessence and phantom fields. Last, but not least, we wish to point out that PLANCK and SNAP data can constrain the underlying model with varying parameters (e.g. the RG model of Sect. 5) – see also [22]. Since, in principle, the underlying model may also have implications at astrophysical scales [23], one could even check if the constraints from SNAP and PLANCK can yield the expected behavior at these scales. Most important, if the SNAP and PLANCK data should confirm the crossing of the CC boundary, $\omega_e(z^*) = -1$, it would strengthen this approach substantially since the CC boundary crossing is generic in the entire class of cosmological models with variable cosmological parameters studied in this paper. Notice, however, that if observations would prove that $\omega_e > -1$ this would not invalidate the model because the crossing point could be in our immediate future ($z^* \lesssim 0$). What we have proven, indeed, is that in this kind of models there always exists a crossing point $z^*$ around our present ($z = 0$), whether in our recent past or in our near future.
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