A $(4/3 + \epsilon)$-Approximation Algorithm for Arboricity From Pseudoforest Partitions

Markus Blumenstock
Johannes Gutenberg-Universität Mainz, Staudingerweg 9, 55128 Mainz, Germany
markusblumenstock@hotmail.com
https://orcid.org/0000-0003-3862-5922

Abstract

The arboricity of a graph is the minimum number of forests it can be partitioned into. Previous approximation schemes only approximated this value without computing a corresponding forest partition, as they operate on the related pseudoforest partitions or the dual problem.

We propose a linear-time algorithm for converting a partition of \( k \) pseudoforests into a partition of \( \lceil 4k/3 \rceil \) forests. For every fixed \( \epsilon > 0 \), this implies an \( O(m \log n) \)-time algorithm that constructively approximates the arboricity within a factor of \( (4/3 + \epsilon) \) plus a small additive constant. We also make several remarks on approximation algorithms for the pseudoarboricity and the equivalent graph orientations with smallest maximum indegree, and correct some mistakes made in the literature.

2012 ACM Subject Classification Mathematics of computing → Combinatorial optimization, Mathematics of computing → Approximation algorithms

Keywords and phrases Approximation Algorithms, Matroid Partitioning, Arboricity, Pseudoarboricity, Graph Orientations

Acknowledgements The author would like to thank Ernst Althaus and Łukasz Kowalik for discussions on the matter, as well as Frank Fischer for pointing out the use of Euler tours.

1 Introduction

Given a finite simple graph \( G = (V, E) \) with \( |V| = n \) and \( |E| = m \), the arboricity \( \Gamma(G) \) is the minimum number of forests the edge set \( E \) can be partitioned into. It is an important measure of the graph’s density, and has several applications, see e.g. [20, 4, 13].

A classic theorem by Nash-Williams [37] is the formula (for \( n \geq 2 \))

\[
\Gamma(G) = \max_{(V_H, E_H) \subseteq G} \left( \frac{|E_H|}{|V_H| - 1} \right),
\]

where \( (V_H, E_H) \subseteq G \) is the subgraph relation. The family of all subsets of \( E \) that are a forest on \( V \) is a matroid (the graphic or cyclic matroid), and \( \Gamma \) is its covering number. An algorithmic proof of (1) can be given via Edmonds’ matroid partitioning algorithm [14] [40]. It computes the corresponding forest partition in polynomial time.

A pseudoforest is a graph where every connected component contains at most one cycle. The family of all subsets of \( E \) that are a pseudoforest on \( V \) is also a matroid, the bicircular matroid. The pseudoarboricity \( p(G) \) is the covering number of the bicircular matroid. It can also be computed with Edmonds’ algorithm in polynomial time, and the formula

\[
p(G) = \max_{(V_H, E_H) \subseteq G} \left( \frac{|E_H|}{|V_H|} \right)
\]

can be proved similarly [10]. It is evident from (1) and (2) that \( \Gamma \) and \( p \) must be very close.
Theorem 1 ([38, 43]). For a simple graph \( G \), we have \( p(G) \leq \Gamma(G) \leq p(G) + 1 \).

This theorem can be proved from (1) and (2) via the inequality \( d^* < \gamma \leq d^* + 1/2 \) for the fractional arboricity \( \gamma \) (1) without the ceiling) and the maximum density \( d^* \) (2) without the ceiling) [38]. Despite the closeness of the two numbers, we face the following computational asymmetry:

If one approximates the arboricity and obtains a corresponding forest partition, it also is a pseudoforest partition that, by Theorem 1, approximates the pseudoarboricity. The reverse situation, however, is problematic: If we compute an approximate pseudoforest partition, the conversion into an approximate forest partition is not a trivial task, although we directly know an approximation of the value \( \Gamma \). This is not merely a hypothetical scenario: The approximation scheme by Kowalik [33] computes a partition into \( k \leq \lceil (1 + \epsilon) d^* \rceil \) pseudoforests in time \( O(m \log(n) \epsilon^{-1} \log p) \). However, converting it into a partition of \( k \) or \( (k + 1) \) forests takes \( O(n m \log k) \) [43]. Kowalik raised the question whether a faster (approximate) conversion from pseudoforests to forests exists. The main result of this paper is that this is the case, which implies the first near-linear time constructive algorithm for arboricity with an approximation factor smaller than two.

Theorem 2. For every fixed \( \epsilon > 0 \), a graph \( G = (V, E) \) can be partitioned into at most

\[
\left\lfloor \frac{4}{3} \cdot \lceil (1 + \epsilon) d^*(G) \rceil \right\rfloor
\]

forests in time \( O(m \log n) \).

2 Related Work

We list constructive algorithms for the arboricity in the Appendix in Table 1. For the pseudoarboricity, this is done in Table 2 and Table 3, respectively. We included algorithms that solve problems equivalent to the pseudoarboricity problem. We use \( \log \Gamma \) and \( \log p \) instead of \( \log n \) where this is easily possible by computing a 2-approximation in linear time first.

Arboricity

The arboricity can be computed with Edmonds’ matroid partitioning algorithm [14] in polynomial time. Picard and Queyranne [38] reduce the problem to a 0-1 fractional programming problem, which is solved with flow techniques. They also showed the strong relationship with the pseudoarboricity. Gabow and Westermann [20] give matroid partitioning algorithms specialized to the graphic matroid. Gabow’s algorithm [19] is the fastest known with a runtime of \( O(m^{3/2} \log(n^2/m)) \). This can be improved slightly to \( O(m^{3/2}) \) if \( \Gamma \) is asymptotically maximum by essentially the same argument as in [6] for pseudoarboricity.

To the best of our knowledge, no constructive algorithm with an approximation factor \( 1 < \alpha < 2 \) is known in general graphs. The well-known linear-time greedy algorithm [15, 2] is constructive. It computes an acyclic orientation that minimizes the maximum indegree among all acyclic orientations [7]. This indegree is at most \( \lfloor 2d^* \rfloor \). As every acyclic \( k \)-orientation can be converted into a forest \( k \)-partition (implicit in [15, 33]), this gives a partition of at most \( \lfloor 2d^* \rfloor \leq 2\Gamma - 1 \) forests. As cyclic orientations cannot be used in this manner directly, the approach is exhausted.

The approximation scheme of Worou and Galtier [44] computes for \( \epsilon > 0 \) a \( 1/(1 + \epsilon) \)-approximation of the fractional arboricity \( \gamma \) in time \( O(m \log^2(n) \log(n^2/m) \epsilon^{-2}) \). It constructs a subgraph of this density (in the sense of [11]), but apparently no forest partition is computed.
Barenboim and Elkin [4] propose a constructive distributed algorithm that computes a \((2 + \epsilon)\)-approximation of \(\Gamma\). Eden et al. [13] describe an algorithm that distinguishes with high constant probability between graphs that are \(\epsilon\)-close to and graphs that are \(c\epsilon\)-far from having arboricity \(\alpha\), for some constant \(c < 20\).

Several upper bounds of the type \(O(\sqrt{m})\) for the arboricity were given by Chiba and Nishizeki [9], Gabow and Westermann [20], Dean et al. [11] and Blumenstock [6]. The bound \(\Gamma \leq \lceil \sqrt{m/2} \rceil\) of Dean et al. is optimal.

**Pseudoarboricity**

The pseudoarboricity can be computed in polynomial time with Edmonds’ matroid partitioning algorithm [14]. Picard and Queyranne [38] reduce the problem to a 0-1 fractional programming problem. An algorithm by Gabow and Westermann [20] is specialized to the bicircular matroid. A pseudoforest \(k\)-partition can be converted into a \(k\)-orientation, and vice versa, in linear time [5, 33]. The smallest maximum indegree (or outdegree) problem can be solved with path-reversals [5, 12]. Flow algorithms can perform several path-reversals at the same time, and they operate on networks where almost all capacities are equal to one [17, 11, 5, 33, 3, 6] (see also [24, 23]). Dinitz’s algorithm, which has a runtime of \(O(m \min(\sqrt{m}, n^{2/3}))\) on unit capacity networks [29, 16], can be employed to find a \(k\)-orientation in the same runtime if it exists. A binary search for the minimum feasible \(k\) introduces a factor of \(O(\log p)\). This can be reduced to \(O(\sqrt{\log p})\) by the balanced binary search technique of Gabow and Westermann [20]. Blumenstock [6] improves this to \(O(\log \log p)\) by employing an approximation scheme [33] to shrink the search interval and using a balanced binary search on it. Recently, faster non-combinatorial flow algorithms for unit capacities were given by Mądry [35] and Lee and Sidford [34]. While they directly improve the runtime, the techniques for attacking the logarithmic factor carry over only for certain bounds on \(p\).

Kowalik’s approximation scheme [33] works by terminating Dinitz’s algorithm early. It computes an \([\lceil (1 + \epsilon)d^{*}\rceil\)-orientation in time \(O(m \log(n) \epsilon^{-1} \log p)\). The aforementioned greedy algorithm computes an acyclic \([2d^{*}]\)-orientation [15, 5] and a subgraph of density at least \([d^{*}/2]\) in linear time. It repeatedly removes the vertex of minimum degree and orients its unassigned edges towards it. Georgakopoulos and Politiopoulos [23] give a generalization to hypergraphs. Charikar [8] and Khuller and Saha [30] address directed graphs. The fractional orientation problem is dual to the densest subgraph problem [8].

Asahiro et al. [3] compute a \([\lceil 2d^{*} \rceil - 1]\)-orientation (assuming \(d^{*} > 1/2\)) with a variant of the greedy orientation algorithm in \(O(m^{2})\), but the orientation produced may contain cycles.

A partition of \(k\) pseudoforests can be converted into a partition of \(k + 1\) forests, and \(k\) if possible, in \(O(mn \log k)\). This is implicit in [43, 20]. (We claim in the appendix in Section B that the runtime bound of \(O(m^{2}/k \log k)\) is incorrect.)

For brevity, we exclude streaming algorithms for the densest subgraph problem, as well as NP-complete variants of it, from our literature review. The same holds for related concepts such as vertex arboricity.

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1 Let \(B\) denote the bounds, and write \(< \leq \) holds plus an example exists where the bounds differ. One can show \(B_{Dea} < B_{GW} \leq B < B_{CN}\). We do not know whether the second inequality is strict.

2 We note that this algorithm can be formulated entirely in terms of flows without any knowledge of matroid theory.
3 Contributions

Our contributions are as follows.

- In Sections 5 to 7, we propose three linear-time conversions from pseudoforest $k$-partitions into forest $\lceil ck \rceil$-partitions, the best of which has $c = 4/3$. This implies a $(4/3 + \epsilon)$-approximation algorithm for arboricity and is the main result. It also gives a way of finding a 4-forest partition of a planar graph in $O(n)$ without computing an embedding of the graph.

- For the pseudoarboricity approximation scheme [33], we show in Section 8 that for fixed $\epsilon > 0$ the factor $\log p$ arising from the binary search can be eliminated in the runtime analysis. We note that the central lemma for the approximation scheme in [33] is insufficiently stated (which was copied to [6]), but this can be fixed.

- In Section B we exhibit a (presumed) flaw in the (not explicitly stated) runtime analysis of Gabow and Westermann [43, 20] for a conversion of $k$ pseudoforests into $k + 1$ pseudoforests in time $O(m^2/k \cdot \log k)$.

- We note in Section C that the $(2 - 1/p)$-approximation algorithm of Asahiro et al. [3] for the smallest maximum indegree orientation problem (pseudoarboricity), whose runtime was analyzed to be $O(m^2)$, can be implemented in $O(m)$ time. The runtime in the weighted setting (where the problem is NP-complete [3]) is improved to $O(m + n \log n)$.

4 Notation and Preliminaries

We consider finite simple graphs $G = (V, E)$, i.e., $G$ is undirected and has no loops. We follow the standard graph-theoretic terminology. For technical reasons we assume $n \geq 2$ and $d^*(G) \geq 1$ for the input graphs. For a set $E' \subseteq E$, we sometimes write that $E'$ is acyclic etc. when we are talking about a subgraph of $G$ with edge set $E'$. Where appropriate, we may implicitly assume the subgraph to be vertex-minimal. If every vertex in the subgraph has degree zero or one, $E'$ is called a matching. The maximum degree of a graph is denoted by $\Delta(G)$. We sometimes write $\Delta$, $d^*$ etc. when the graph is clear from context.

We will consider directed graphs without loops in Sections 8 and C. In an orientation $\vec{G}$ of a simple graph $G$, every edge of $G$ is present once, directed in one of the two possible directions. Let indeg$\vec{G}(v)$ denote the indegree of $v$ in $\vec{G}$. If all indegrees are at most $k$, $\vec{G}$ is called a $k$-orientation.

In our definition, paths and cycles visit vertices only once. In simple graphs, a path of length $l \geq 1$ has two end vertices, its vertices of degree one. A path of length $l = 0$ is an isolated vertex, which is the sole end vertex.

An acyclic simple graph is called a forest, its connected components are called trees. A tree of $n$ vertices has exactly $n - 1$ edges. We call a forest linear if all its connected components are paths.

We denote the disjoint union of sets by $\dot{\cup}$. If $E$ is partitioned as $E = F_1 \dot{\cup} ... \dot{\cup} F_k$ where each $F_i$ is a forest, we call $(F_1, ..., F_k)$ a forest $k$-partition. The smallest $k$ such that a forest $k$-partition exists is called the arboricity $\Gamma(G)$.

If a graph has at most one cycle per connected component, it is called a pseudoforest. Its connected components are called pseudotrees. We define pseudoforest $k$-partitions $(P_1, ..., P_k)$ and the pseudoarboricity $p(G)$ analogously. We omit isolated vertices in each (pseudo-)forest in order to obtain linear runtimes.

A basic property of a cyclic pseudotree is that removing an arbitrary edge on its cycle leaves a tree. In reverse, adding an edge to a tree creates a unique cycle, hence it becomes a
cyclic pseudotree. Note that connecting two different trees by an edge results in a single tree. Let us now describe a basic operation that we will use extensively.

Let \((V,P)\) be a pseudoforest. For every cycle \(C \subseteq P\), select one edge \(e_C \in C\) arbitrarily. The set of all these selected edges is a matching, as every vertex can be in at most one cycle. We call this kind of matching a \(P\)-matching.

\[\text{Lemma 3. A pseudoforest} \ (V,P) \ \text{can be partitioned into a forest and a} \ P\text{-matching in linear time.}\]

\[\text{Proof. Determine all cycles in} \ (V,P) \ \text{in linear time, for example with depth-first search. Arbitrarily select an edge on each cycle to obtain a} \ P\text{-matching} \ M. \ P \setminus M \ \text{is a forest.} \quad \blacksquare\]

The lemma implies that a pseudoforest \(k\)-partition can be converted into a forest \(2k\)-partition in linear time. As a constructive 2-approximation algorithm for arboricity is known, this itself is not very useful. In the next section, we will see that the matching property can be exploited to obtain a factor of less than two.

5 Converting Three Pseudoforests Into Five Forests

We can employ a lemma by Duncan, Eppstein and Kobourov for a first result.

\[\text{Lemma 4 ([12]). Let} \ G \ \text{be a simple graph with} \ \Delta(G) \leq 3. \ \text{Then} \ G \ \text{can be partitioned into two linear forests in linear time.}\]

\[\text{Theorem 5. A pseudoforest partition} \ (P_1,P_2,P_3) \ \text{can be converted into a partition of five forests, two of which are linear forests, in linear time.}\]

\[\text{Proof. To convert} \ (P_1,P_2,P_3) \ \text{into five forests, first partition each} \ P_i \ \text{into a forest} \ F_i \ \text{and a} \ P_i\text{-matching} \ M_i \ \text{according to Lemma 3, which is possible in linear time.}\]

\[\text{Next, consider the graph on} \ V \ \text{with edge set} \ M_1 \cup M_2 \cup M_3. \ \text{Clearly, it has maximum degree three. Thus it can be partitioned into two linear forests by Lemma 4 in linear time.} \quad \blacksquare\]

This implies that a partition of \(k\) pseudoforests can be converted into \(\lceil 5k/3 \rceil\) forests in linear time. Better conversion algorithms will be developed in the following sections.

6 Converting Two Pseudoforests Into Three Forests

We now develop a conversion procedure by choosing the edges on the cycles more carefully. The following lemma is trivial, but crucial to all algorithms to follow.

\[\text{Lemma 6 ([21]). Let} \ G \ \text{be a simple graph with} \ \Delta(G) \leq 2. \ \text{Then every connected component of} \ G \ \text{is either a path or a cycle.}\]

\[\text{Theorem 7. A pseudoforest partition} \ (P_A,P_B) \ \text{can be converted into a partition of two forests} \ A,B \ \text{and a linear forest} \ L \ \text{whose edges are from} \ P_A \ \text{and} \ P_B \ \text{alternatingly.}\]

\[\text{Proof. Convert} \ P_A \ \text{and} \ P_B \ \text{into forests} \ A,B \ \text{and a} \ P_A\text{-matching} \ M_A \ \text{and a} \ P_B\text{-matching} \ M_B \ \text{as in Lemma 3. Let} \ L = M_A \cup M_B. \ \text{An example can be seen in Figure 1abc. Every vertex in} \ L \ \text{has a degree of at most two. By Lemma 6 \ (V,L) can only consist of paths and (even-length) cycles, their edges must be from} \ A \ \text{and} \ B \ \text{alternatingly. Determining all cycles is possible in linear time.}\]
We now modify $L$ by processing the cycles one after another in steps. Consider some cycle $Z \subseteq L$, and pick an arbitrary edge $e \in Z$ (Figure 1c). Without loss of generality, let this edge be from $P_A$. Adding it to $A$ would re-create the original cycle $C_e \subseteq P_A$ with $e \in C_e$. Let $e' \in C_e$ be incident to $e$ in $P_A$ (Figure 1a). Swap $e$ and $e'$. The modified set $A' = (A \cup \{e\}) \setminus \{e'\}$ is a forest, and the modified $L' = (L \cup \{e'\}) \setminus \{e\}$ still consists of paths and cycles by Lemma 6.

The edge $e' \neq e$ cannot link two vertices on $Z \setminus \{e\}$ because this would imply a vertex of degree three and thus a contradiction. Therefore, the path $Z \setminus \{e\}$ must have been joined at one of its endpoints to another component of $L$ upon insertion of $e'$ (Figure 1d). Again, as vertices have degree at most two, and the other components were not affected by the replacement, the vertex $e'$ links to must have been the end of a path before the replacement. Thus the number of cycles in $L'$ is one less than in $L$.

By breaking up cycles one after the other and joining them to paths at their ends, we postprocess $L$ to become a linear forest while maintaining the forest property for $A$ and $B$. The whole process takes linear time because we only determine cycles once in $L$.

Note that exchanging an edge $e$ for a non-incident edge $e'$ on the original cycle could link two end vertices of the same path in $L$ and thereby create a new cycle. An example is the squiggly edge in Figure 1a.

Theorem 7 implies a conversion into $\lceil 3k/2 \rceil$ forests in linear time. In the next section, we will exploit properties of $L$ to improve the conversion ratio further.

## 7 Converting Three Pseudoforests Into Four Forests

In this section, we will show how a pseudoforest 3-partition can be converted into a forest 4-partition in linear time. A first observation is that the linear forest $L$ constructed in
Theorem 7 is size-bounded: A pseudoforest can have at most \( n/3 \) cycles, as the smallest cyclic pseudotree is a triangle. Therefore, the linear forest \( L \) has at most \( 2n/3 \) edges.

If we tried to do combine matchings from three pseudoforests into a set \( S \), it would have \( |S| \leq n \). There are instances where exactly \( n \) edges are chosen, e.g., three pseudoforests on twelve vertices, each consisting of four triangles. As a forest has at most \( n - 1 \) edges, the set \( S \) cannot be a forest then, regardless of which edges we choose on the cycles! In terms of size, a surplus of one edge is not necessarily a problem, as inserting a single edge into a forest partition is possible in linear time [20] (see also Section [B]). However, \( S \) could have many interlocked cycles (see Figure 2 for an example). We note that we tried to utilize proven cases of the Strong Nine Dragon Tree Conjecture for \( d^* < 4/3 \) [36] and \( d^* < 3/2 \) [31], to no avail.

The intuition behind our approach is as follows. For three pseudoforests \( P_A, P_B, P_C \), we try to insert a \( P_C \)-matching \( M \) into \( L \), which is obtained from \( P_A \) and \( P_B \) as in Theorem 7.

The key property we want to exploit is that the number of edges removed from a pseudoforest is at most its number of connected components. Hence, if \( L \) is too full to insert an edge of \( M \), we can hope to insert it between two components of \( A \) or \( B \): If for an edge \((u,v) \in M\), there are two edges incident with \( u \) and at least one edge incident with \( v \) in \( L \), then \( M \) links two connected components in \( A \) or \( B \), or both, depending on which pseudoforest(s) the incident edges come from.

It is possible that connected components in \( A \) or \( B \) become linked in a cycle by several such \( M \)-edges. This will be resolved by moving a certain edge of \( L \) to \( C \), which allows inserting one carefully chosen \( M \)-edge that created the cycle in \( A \) or \( B \) into \( L \).

As an isolated vertex \( u \) can be linked to a tree without creating a cycle, an \( M \)-edge with such an endpoint \( u \) can always be inserted into \( L \).

The remaining case is where an \( M \)-edge links two vertices in \( L \) of degree one, i.e., end vertices of paths. The subcase where the incident \( L \)-edges are from different pseudoforests is problematic, because then the \( M \)-edge does not necessarily link different connected components in either \( A \) nor \( B \). In the following lemma, however, we will take care of all \( M \)-edges linking end vertices of paths.

**Lemma 8.** Given a pseudoforest partition \((P_A, P_B, P_C)\), let \( A, B, L \) be as in Theorem 7. Then a \( P_C \)-matching \( M \) can be computed in linear time such that \((V, L \cup M_1)\) is a linear forest for

\[
M_1 = \{(u,v) \in M \mid \deg_L(u) = 1 = \deg_L(v)\}.
\]

**Proof.** Choose an arbitrary \( P_C \)-matching \( M \) in linear time. Consider the set

\[
M_1 = \{(u,v) \in M \mid \deg_L(u) = 1 = \deg_L(v)\}.
\]
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As the degrees in \(L \cup M_1\) are bounded by two, its connected components are paths and cycles by Lemma 6. A cycle can only arise if \(M_1\)-edges link paths in a cycle at their end vertices (possibly a single path). An example can be seen in Figure 3a.

With respect to \(M_1\), the paths of \(L\) behave essentially like the vertices of \(L\) in Theorem 7. We modify the choice \(M\). It is possible to detect cycles in \(L \cup M_1\) in linear time. For each such cycle \(Z\), pick one edge \((u,v) \in M_1 \setminus Z\). This edge is from a cycle in \(P_C\). Exchange it with an incident edge on the original cycle in \(P_C\), say \((v,w)\) (Figure 3bc). This modified set \(M'\) is also a \(P_C\)-matching. Define \(M'_1\) analogously to \(M_1\). We now argue that \(L \cup M'_1\) is acyclic and hence a linear forest.

If one or several paths have been joined to form a cycle \(Z\) in \(L \cup M_1\), then one of these paths has one end vertex \(u\) that is not incident with any edge of \(M'_1\). Hence the cycle has been broken into a path of linked-together paths, which is attached at end vertex \(v\) to a vertex \(w\) of some path, while end vertex \(u\) now has no incident \(M\)-edge. If \(w\) is an end vertex of a path, this path was not part of a cycle, in particular \(Z\). Hence the paths of \(Z\) are linked end-to-end to a sequence of paths, i.e., no new cycle has been introduced. If \(w\) is an internal vertex of a path (\(w_1\) in Figure 3bc), then \((u,w) \notin M'_1\), hence it cannot be part of a cycle in \(L \cup M'_1\).

Equipped with Lemma 8, we can now attack the \(M\)-edges that link connected components in \(A\) and \(B\).

**Theorem 9.** A pseudoforest partition \((P_A, P_B, P_C)\) can be converted into a partition of four forests, one of which has maximum degree at most three, in linear time.

**Proof.** Turn \((P_A, P_B)\) into two forests \(A, B\) and a linear forest \(L\) according to Theorem 7.
Apply Lemma 8 to obtain the special $P_C$-matching $M$. Define $C = P_C \setminus M$ and

$$
M_0 = \{ (u,v) \in M \mid \deg_L(u) = 0 \},
$$

$$
M_1 = \{ (u,v) \in M \mid \deg_L(u) = 1 = \deg_L(v) \},
$$

$$
M_2 = \{ (u,v) \in M \mid \deg_L(u) = 2, \deg_L(v) \geq 1 \}.
$$

We have $M = M_0 \cup M_1 \cup M_2$. We know that $L \cup M_1$ is a linear forest.

Consider the set $M_2$ (see Figure 4a for a running example). As three or four $L$-edges are incident with each $(u,v) \in M_2$, at least two of them must be from the same pseudoforest. We can hence partition $M_2$ into $M_2 = M_A^2 \cup M_B^2$ such that for every $(u,v) \in M_A^2$, there exist $(t,u), (v,w) \in L \cap P_A$, and likewise for $M_B^2$. The following discussion is analogous for $P_B, B$ and $M_B^2$.

Every $(u,v) \in M_A^2$ links two different cyclic connected components in $P_A$ and hence two different components in $A$ (Figure 4b). Linking occurs only at the endpoints of edges $e \in P_A \setminus A = P_A \cap L$.

Therefore, components of $A$ behave like vertices that have at most two incident $M_A^2$-edges. In this contracted view, there are only paths and cycles according to Lemma 6. Inside the components, cycles of $A \cup M_A^2$ go through exactly the edges of $A$ that were part of a cycle in $P_A$. It is possible to determine all cycles of $A \cup M_A^2$ in linear time. For each such cycle $Z$, consider one arbitrary edge $e_Z \in P_A \cap L$ that ‘shortcuts the cycle’, i.e., it is an edge chosen from $P_A$ for $L$ that is incident to two $M_A^2$ edges (the squiggly line in Figure 4b). This implies that $e_Z$ links two cyclic connected components in $P_C$, and hence two components in $C$ (Figure 4c). Let $Y_A$ denote the set of all such edges $e_Z$ ($Y_B$ is analogously defined). The goal is to remove all edges $Y_A$ from $L$ (actually, $L \cup M_1$) to make room for one $M_A^2$-edge $m_Z$ on each cycle $Z$ in $A \cup M_A^2$ (Figure 4d). By removing one such edge per cycle of $A \cup M_A^2$, its forest property is restored. Let $X_A$ and $X_Y$ denote the sets of the $m_Z$ for $A$ and $B$, respectively. We will later carefully choose the $X$- and $Y$-sets such that $(L \cup M_1 \cup X_A \cup X_B) \setminus (Y_A \cup Y_B)$ is acyclic.
Add $Y_A \cup Y_B$ to $C$ and, only for the sake of argument, also to $P_C$. Thereby, cyclic connected components of $P_C$ are linked via $Y_A$-edges and $Y_B$-edges, and these must be incident to the endpoints of the $M^2_2$-edges.

\[ \textbf{Claim 1.} \] A $P_C$-component is linked via at most one $Y_A$-edge in $P_C \cup Y_A \cup Y_B$. Moreover, if it is linked via a $Y_A$-edge, then it is not linked via a $Y_B$-edge. The claim holds analogously with the roles of $Y_A$ and $Y_B$ reversed.

\[ \textbf{Proof of Claim 1.} \] If $e \in M^2_2$ is part of a cycle in $P_A \cup M^2_2$, then this is the only such cycle. As only one edge $e_Z$ on the cycle is selected and $M$ is a $P_C$-matching, the component of $P_C$ that contains $e$ is linked via at most one edge $e_Z \in Y_A$. The ‘moreover’-part of the claim follows from $M^2_2 \cap M^2_1 = \emptyset$.

This implies that every component of $C$ is isolated or linked to a single other component in $C \cup Y_A \cup Y_B$. As the components are trees, the set $C \cup Y_A \cup Y_B$ is acyclic for any specific choice of $Y_A$ and $Y_B$.

We now choose which edges $X_A \subseteq M^2_2$ are inserted into $L \cup M_1$, and which edges $Y_A \subseteq A$ are removed from $L$.

For every cycle $Z$ of $A \cup M^2_1$, consider the endpoints of the $M^2_2$-edges. If we remove an edge $e_Z \in P_A \cap L$ from $L \cup M_1$, the path disconnects into two paths (trees). If the $e_Z$-edge has an endpoint $u_Z$ whose degree in $L$ is one, the $M^2_2$-edge incident to $u_Z$ can be inserted into $(L \setminus \{e_Z\}) \cup M_1$. Otherwise, it is possible that adding either of the two incident $M^2_2$-edges on $Z$ to $L \cup M_1$ creates a cycle. We would need to choose an $M^2_2$-edge on $Z$ that ‘bridges the gap’, i.e., that connects the two different trees.

We describe a simple general way of choosing an edge $e_Z$ together with an incident $M^2_2$-edge that also allows for a simple analysis of acyclicity: Number the vertices from 1 to $n$ such that every path of $L \cup M_1$ consists of a contiguous segment of the sequence $(1, ..., n)$. In other words, the paths are arranged in a sequence from left to right. This is possible in linear time. We view edges $(u, v)$ ordered as $u < v$.

Among the edges $e_Z = (u, v) \in P_A \cap L$ that shortcut a cycle $Z$ in $P_A \cup M^2_1$, we remove ‘the rightmost’ from $L$, i.e., the one that maximizes $v$. One of the two incident $M^2_2$-edges is $m_Z = (t, v)$ with $t < u$, which we add to $L \setminus \{e_Z\}$ (these are the choices in Figure 3 when the path is ordered from left to right). These edges can be determined in linear time in total by scanning each $Z$ once for the rightmost shortcut edge, and selecting the appropriate incident $M^2_2$-edge. We now prove that performing all these deletions and insertions does not create a cycle.

\[ \textbf{Claim 2.} \] For the above specific choices of $X_A, Y_A, X_B$ and $Y_B$, $(L \cup M_1 \cup X_A \cup X_B) \setminus (Y_A \cup Y_B)$ is a forest.

\[ \textbf{Proof of Claim 2.} \] We order the edges $e_Z = (u, v) \in Y_A \cup Y_B$ by their right endpoint $v$, and imagine the process of deleting them from $L \cup M_1$ and adding their incident edge $m_Z = (t, v) \in X_A \cup X_B$ in order of decreasing $v$ (‘from right to left’).

We prove by induction on $i \geq 0$ that after the $i$-th deletion of $e_Z = (u, v)$ and insertion of $m_Z = (t, v)$, the graph is a forest. Let $M^1_2 \subseteq M_2$ denote the edges of $X_A \cup X_B$ inserted in iterations $1, ..., i$, and let $L^i$ denote $L$ without the edges of $Y_A \cup Y_B$ removed in these iterations. Before the first insertion and deletion, $L \cup M_1$ is a linear forest ($i = 0$).

Let the induction hypothesis hold for some $i \geq 0$. After deleting the $(i + 1)$-th edge $e_Z = (u, v)$, the tree of $L^i \cup M_1 \cup M^1_2$ that $e_Z$ was a part of becomes disconnected into two different trees, one of which contains $u$ and the other $v$. The edge $m_Z = (t, v)$ has
t < u. We have to show that inserting \( m_Z \) does not create a cycle. This could only happen if \( t \) were in the same tree as \( u \). Assume this is the case. Then there is a unique path \( P \subseteq (L \setminus \{e_Z\}) \cup M_1 \cup M_2 \) from \( v \) to \( t \). (If \( v \) is now isolated, this is impossible.) Recall that we ordered the paths including the \( M_1 \)-edges. As we deleted \((u, v)\), \( P \) must pass through at least one edge \( e = (x, y) \in M_2^* \) with \( v < y \). Follow the path from \( v \) to \( t \) until the \( e \) with maximum \( y \) is visited. By construction, its left incident edge \((y - 1, y)\) was deleted. Hence there must be an \((x', y') \in M_2^* \) on \( P \) with \( v < y < y' \) in order to reach \( t < v \). This is a contradiction to \( y \) being maximum. ◼

Note that \((L \cup M_1 \cup X_A \cup X_B) \setminus (Y_A \cup Y_B)\) may have vertices of degree three.

Lastly, we consider the set \( M_0 \). Clearly, an isolated vertex \( u \) can be linked to a tree of \((L \cup M_1 \cup X_A \cup X_B) \setminus (Y_A \cup Y_B)\) via \((u, v) \in M_0 \) without creating a cycle. (This may also cause vertices of degree three.) As \( M = M_0 \cup M_1 \cup M_2 \), this concludes the proof. ◼

**Theorem 10.** Let \( G = (V, E) \) be a simple graph. A partition of \( G \) into \( k \) pseudoforests can be converted into a partition of \([4k/3]\) forests in linear time.

**Proof.** Make \([k/3]\) triplets of pseudoforests and convert each triplet into four forests as in Theorem 9. If \( k \) is divisible by three, the claim follows. If \( k \equiv 1 \mod 3 \), we convert the remaining pseudoforest into two forests. If \( k \equiv 2 \mod 3 \), we convert the two pseudoforests into three forests according to Theorem 4. The claim follows. ◼

Schnyder [11] and Chrobak and Eppstein [10] show that a planar graph can be partitioned into three forests in \( O(n) \) time from an embedding of the graph into the plane (which can also be computed in linear time, see e.g., [27]). The algorithm of Grossi and Lodi [25] finds, also using an embedding, a partition into three forests in time \( O(n \log n) \), and four forests in \( O(n) \). By using the second 3-orientation algorithm of [10] and converting it to a pseudoforest 3-partition (see Theorem 11), we can obtain four forests in linear time by applying Theorem 10 without computing an embedding first. Note that there are planar graphs with pseudoarboricity three.

**8 The Approximation Scheme for Pseudoforests**

The smallest maximum indegree problem is equivalent to the pseudoarboricity problem.

**Theorem 11 ([5, 33]).** A pseudoforest \( k \)-partition can be converted into a \( k \)-orientation, and vice versa, in linear time.

One can determine the minimum feasible \( k \) with a binary search. Using the orientation view, a test for guess \( k \) can be performed by a maximum flow computation (see Related Work). Kowalik [33] turns such an exact algorithm into an approximation scheme by terminating the flow computation early. The central lemma that establishes the approximation was stated insufficiently, which was copied to [9]. It requires a given \( k \)-orientation, but this is exactly what a flow computation is supposed to compute for a guess \( k \), if it exists. Here, the corrected version for an arbitrary orientation is given. The proof is analogous, only one equality has to be replaced with an inequality. The lemma can also be generalized to fractional orientations as in [6].

**Lemma 12.** Let \( \bar{G} \) be an arbitrary orientation of a graph \( G \), and let \( k > d^* \). Then for every vertex \( v \) in \( \bar{G} \), the distance to a vertex with indegree smaller than \( k \) does not exceed \( \log_{k/d^*} n \).
If Dinitz’s algorithm is terminated after \(2 + \log_{1+\epsilon}\) phases, Lemma 12 guarantees that a \(k\)-orientation is found for guesses \(k \geq (1 + \epsilon)d^*\) despite the early termination, as the length of the shortest augmenting paths increases with every phase.

\[\text{Theorem 13 (33).} \quad \text{A partition into } k \leq \lceil (1 + \epsilon)d^* \rceil \text{ pseudoforests can be determined in time } O(m \log n \epsilon^{-1} \log p).\]

The factor \(O(\log p)\) comes from an exponential and a binary search\(^3\) for the minimum feasible \(k\). We can show that it can be made constant for every fixed \(\epsilon > 0\).

Let \(u_i\) and \(l_i\) denote the current upper and lower bounds in the \(i\)-th iteration of the binary search. We will keep \(u_i\) feasible at all times. Once the ratio \(u_i/l_i\) drops below \((1 + \epsilon)\), we can stop the algorithm and return the feasible upper bound.

\[\text{Theorem 14.} \quad \text{For any constant } \epsilon > 0, \text{we can compute a partition into } k \leq \lceil (1 + \epsilon)d^* \rceil \text{ pseudoforests in time } O(m \log n).\]

\[\text{Proof.} \quad \text{Compute an approximation } x \text{ satisfying } p \leq x \leq 2p \text{ in linear time with the greedy algorithm (see Section 2). Set } u_1 = x \text{ and } l_1 = x/2 \geq p/2. \text{ We have } u_1/l_1 \leq 4.\]

With every test \(t_i = \lfloor (u_i + l_i)/2 \rfloor\) of the binary search, either the lower or the upper bound is updated. It is straightforward to show that if \(u_i > \lceil (1 + \epsilon)l_i \rceil\), then

\[
\frac{u_{i+1}}{l_{i+1}} \leq \begin{cases} 
\frac{1}{2} \frac{u_i}{l_i} + \frac{1}{2} & \text{if the test is successful,} \\
\frac{1}{2} \frac{u_i}{l_i} & \text{otherwise.}
\end{cases}
\]

Since \(\epsilon\) is fixed, the bound ratio decays exponentially. Thus the initial ratio of four is reduced to \((1 + \epsilon)\) in a number of iterations that is constant.

The constants introduced in the proof in addition to \(\epsilon^{-1}\) are rather large. One can reduce them by repeated approximation, which lowers the initial ratios \(u_1/l_1\). This is similar to the iterated interval shrinking in [6]. We can now prove the main theorem.

\[\text{Proof of Theorem 2.} \quad \text{Use Theorem 14 to obtain the approximate pseudoforest partition for } \epsilon > 0. \quad \text{Then apply Theorem 10.}\]

\[9 \quad \text{Conclusion and Outlook}\]

We presented linear-time conversions from pseudoforest partitions to forest partitions, the best of which was from three pseudoforests to four forests, one of which has maximum degree at most three. An interesting question is whether a fast \(k\)-to-(\(k + 1\))-conversion can be obtained for general \(k\), say with a runtime of \(O(m \log n f(k))\). This would allow a fast constructive approximation scheme for arboricity. However, if we try to extend the 3-4-case to the 4-5-case, we don’t have the linearity property at our disposal, and some edges were inserted back into the forests \(A, B, C\). Hence new techniques would have to be developed.

Another interesting question is whether the approximation scheme for pseudoarboricity can be used to determine a \(1/(1 + \epsilon)\)-approximation to the densest subgraph (by [2], it approximates the value \(d^*\)), as the approximation scheme by Worou and Galtier (for \(\gamma\)) has worse runtime.

A linear-time algorithm for one of the three problems with an approximation ratio of less than two would also be of interest. We noted that \((2 - 1/p)\) can be obtained for the pseudoarboricity, but circumventing the degree sum formula appears to be hard.

\[3 \quad \text{We note that the binary search in [33, Algorithm 4.2] runs in an infinite loop for e.g., } d_1 = 1, d_2 = 2.\]
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A Tables with Runtimes

Table 1 Constructive exact algorithms for the arboricity $\Gamma$. $M(n,m)$ denotes the runtime of an arbitrary maximum flow algorithm, and $m' = m + n' \log n'$ where $n' = \min(n, m/n)$.

| Runtime | Note | Reference |
|---------|------|-----------|
| polynomial | $O(m^3 \log \Gamma)$ oracle calls | [14] |
| $O(M(n,m)n)$ | | [35] |
| $O(m^2)$ | | [28, 39] |
| $O(m (m/n) \log \Gamma)$ | | [20] |
| $O(m (m/m')^{1/3})$ | | [20] |
| $O(m^{3/2} \log(m'/m))$ | $O(m^{3/2})$ possible if $\Gamma \in \Omega(\sqrt{m})$ | [6] [19] |

Table 2 Constructive exact algorithms for the pseudoarboricity / smallest maximum indegree. $M(n,m)$ denotes the runtime of an arbitrary maximum flow algorithm, and $M_u(n,m)$ in the case of unit-capacity networks.

| Runtime | Note | Reference |
|---------|------|-----------|
| polynomial | $O(m^3 \log p)$ oracle calls | [14] |
| $O(M(n,m) \log p)$ | | [35, 24] |
| $O(m^2)$ | | [5, 3, 42] |
| $O(m (m/n) \log p)$ | sublogarithmic improvements for bounds on $p$ | [43, 6] |
| $O(m^{3/2} \log p)$ | sublogarithmic improvements for bounds on $p$ | [11, 5, 8] |
| $O(m^{3/2} \sqrt{\log p})$ | sublogarithmic improvements for bounds on $p$ | [20] |
| $O(M_u(n,m) \log p)$ | sublogarithmic improvements for bounds on $p$ | [6] |

Table 3 Approximation algorithms for the arboricity, they are constructive unless stated otherwise.

| Runtime | Approximation | Note | Reference |
|---------|--------------|------|-----------|
| $O(n)$ | $k \leq 4$ | planar graphs | [25], this paper |
| $O(n \log n)$ | $k \leq 3$ | planar graphs | [25] |
| $O(n)$ | $k \leq 3$ | planar graphs | [41, 10] |
| $O(m)$ | $k \leq [2d^*] \leq 2\Gamma - 1$ | | |
| $O(m \log n)$ | $k \leq [4/3 \lceil (1+\epsilon)d^* \rceil]$ for fixed $\epsilon > 0$ | non-constructive / dual | [14] |
| $O(m \log^2(n) \log \left( \frac{m}{n} \right) \epsilon^{-2})$ | $k \leq (1+\epsilon)\gamma$ | | |
### Table 4 Constructive approximation algorithms for pseudoarboricity.

| Runtime            | Approximation | Note                        | Reference |
|--------------------|---------------|-----------------------------|-----------|
| \( O(n) \)        | \( k \leq 3 \) | planar graphs               | \[11\]    |
| \( O(m) \)        | \( k \leq \lfloor 2d \rfloor \) | \[13\] |\[15\] |
| \( O(m^2) \)      | \( k \leq \lfloor 2d \rfloor - 1 \) | \[3\]       |           |
| \( O(m) \)        | \( k \leq \lfloor 2d \rfloor - 1 \) | this paper |           |
| \( O(m \log n \epsilon^{-1} \log p) \) | \( k \leq \lceil (1 + \epsilon)d \rceil \) | \[34\]   |
| \( O(m \log n) \) | \( k \leq \lfloor (1 + \epsilon)d \rfloor \) for fixed \( \epsilon > 0 \) | this paper |           |

### B Runtime Analysis of the Conversion by Gabow and Westermann

Gabow and Westermann \[43\] \[20\] show that an edge \( e \) can be inserted into a forest partition \( (F_1, \ldots, F_k) \) in \( O(m) \) time (which involves a pre- and a postprocessing), if possible. In order to insert it, the algorithm must possibly move other edges between the forests to obtain a forest \( k \)-partition of \( F_1 \cup \ldots \cup F_k \cup \{ e \} \). If this is impossible, it outputs a forest \( (k + 1) \)-partition. We will use this algorithm ("cyclic scanning") as a black box without further explanation.

In order to convert \( k \) pseudoforests into \( k + 1 \) forests, and \( k \) if possible, the authors proposed Algorithm \[1\]. In a nutshell, the algorithm divides the \( k \) pseudoforests into two groups of \( \lfloor k/2 \rfloor \) and \( \lceil k/2 \rceil \) pseudoforests, recursively converts them to \( \lfloor k/2 \rfloor + 1 + \lceil k/2 \rceil + 1 \) forests, and then inserts the edges of smallest forest into the \( k + 1 \) others, which is always feasible by Theorem \[1\]. Once the recursion is done, one tries to insert the edges of the smallest forest into the \( k \) others, which may be feasible or infeasible. It is possible to show that the time of the insertions is bounded by \( O(m^2/k) \) in both functions of Algorithm \[1\].

Gabow and Westermann pick an arbitrary forest for insertion. They use the fact that it has at most \( m/k \) edges due to their preprocessing, but it easier to argue with the forest of minimum cardinality\[^4\].

In a partition of \( k + 1 \) forests, there is at least one forest that has less than \( m/k \) edges. This proves the total insertion runtime as each insertion takes linear time. Thus, for some \( c > 0 \), the following recurrence for runtime \( T \) holds:

\[
T(m, k) \leq T(m_1, \lfloor k/2 \rfloor) + T(m_2, \lceil k/2 \rceil) + cm^2/k, \quad T(1) = cn.
\]

Gabow and Westermann claim that it satifies \( T(k) \in O(m^2/k \log k) \), without giving a proof. If we try to prove this by induction with the ansatz

\[
T(m, k) \leq c' m^2/k \log k
\]

for some \( c' > 0 \), we see that we obtain \( 2c' m^2/k \log(k/2) \) from each recursive call \( T(m_i, k/2) \) by the induction hypothesis, as \( 1/(k/2) = 2/k \). While \( c' \) can be made arbitrarily large, it has to be a constant that is valid on all levels of the recursion. Hence we think that the proof the authors had in mind is incorrect. Fortunately, the runtimes stated in \[20\], Table 1 are unaffected.

The analysis of \( O(n^2k \log k) \) in \[43\] \[20\] is correct: We obtain \( c'n^2k/2 \log(k/2) \) from each call \( T(m_i, k/2) \) by the induction hypothesis. However, it is unclear why this estimate was used at all: As \( k \geq m/n \), the runtime \( O(mn \log k) \) that is immediate from Westermann’s

\[^4\] This is done in \[43\] for Convert, but not Divide.
Algorithm 1: Converting \(k\) pseudoforests to \(k + 1\) forests.

**Input:** A pseudoforest partition \(E = P_1 \cup \ldots \cup P_k\) of a simple graph \(G = (V,E)\).

**Output:** A forest partition \(E = F_1 \cup \ldots \cup F_l\) with \(l = k\) if possible and \(l = k + 1\) otherwise.

function Convert\((P_1, \ldots, P_k)\):

if \(k = 1\) then
  if \(P_1\) is a forest then
    return \((P_1)\)
  else
    Determine \(P_1\)-matching \(M\) return \((P_1 \setminus M, M)\)

\((F_1, \ldots, F_{k+1}) \leftarrow\) Divide\((P_1, \ldots, P_k)\)

W.l.o.g. let \(F_{k+1}\) be the forest of smallest cardinality

foreach \(e \in F_{k+1}\) do
  if \(e\) can be inserted into \((F_1, \ldots, F_k)\) using cyclic scanning then
    insert \(e\) into \((F_1, \ldots, F_k)\)
    \(F_{k+1} \leftarrow F_{k+1} \setminus \{e\}\)

if \(F_{k+1} = \emptyset\) then
  return \((F_1, \ldots, F_k)\)
else
  return \((F_1, \ldots, F_{k+1})\)

function Divide\((P_1, \ldots, P_k)\):

\((F_1, \ldots, F_{\lfloor k/2 \rfloor + 1}) \leftarrow\) Divide\((P_1, \ldots, P_{\lfloor k/2 \rfloor})\) \hspace{1cm} // \(m_1 = |F_1 \cup \ldots \cup F_{\lfloor k/2 \rfloor + 1}|\)

\((F_{\lfloor k/2 \rfloor + 2}, \ldots, F_{k+2}) \leftarrow\) Divide\((P_{\lfloor k/2 \rfloor + 1}, \ldots, P_k)\) \hspace{1cm} // \(m_2 = |F_{\lfloor k/2 \rfloor + 2} \cup \ldots \cup F_{k+2}|\)

W.l.o.g. let \(F_{k+2}\) be the forest of smallest cardinality

foreach \(e \in F_{k+2}\) do
  insert \(e\) into \((F_1, \ldots, F_{k+1})\) using cyclic scanning \hspace{1cm} // feasible

return \((F_1, \ldots, F_{k+1})\)

thesis [43, Equation (1) on page 46] is better and in turn, the alleged runtime \(O(m^2/k \log k)\) would have been even better. Let us finish the discussion by stating the recurrence for \(O(nm \log k)\):

\[ T(m,k) \leq T(m_1,\lfloor k/2 \rfloor) + T(m_2,\lceil k/2 \rceil) + cnm, \quad T(1) = cn. \]

The fact that every forest has less than \(n\) edges simplifies the discussion, and the runtime analysis by induction is straightforward.

\[ \textbf{Theorem 15 (43,20).} \text{ A pseudoforest } k\text{-partition can be converted into a forest } (k + 1)\text{-partition, and a forest } k\text{-partition if possible, in } O(nm \log p) \text{ time.} \]

We note that instead of inserting the edges one-by-one, one could try using the batch routine of 43,20. Moreover, from the knowledge of the existence of a pseudoforest \(k\)-partition, one can solve the \(k\)-forests and \((k + 1)\)-forests problem from scratch using the algorithms in 43,20, which can be faster or slower than \(O(nm \log k)\) depending on \(m/n\) and \(k\).

\[ \textbf{C} \quad \text{A New Runtime Analysis of the Algorithm of Asahiro et al.} \]

Asahiro et al. [3] propose Algorithm 2 for approximating \(p\) via orientations (see Theorem 11). By [2], the average density \(l\) of a subgraph is a lower bound on \(p\). At most \(\lfloor 2l \rfloor - 1 \leq 2p - 1\)
Algorithm 2: The $(2 - \frac{1}{p})$-orientation algorithm of Asahiro et al.

Input: A simple graph $G = (V, E)$.
Output: A $(2 - \frac{1}{p})$-orientation $\vec{G}$ of $G$ (if $p \geq 1$).

function orient($V, E$):
  Let $l \leftarrow \frac{m}{n}$
  Let $V_{\text{orig}} \leftarrow V$
  Let $\vec{E} \leftarrow \emptyset$
  repeat
    while $\exists v \in V$ with $\deg_G(v) \leq \lceil 2l \rceil - 1$ do
      foreach $uv \in E$ do
        orient $u \rightarrow v$ in $\vec{E}$
        $E \leftarrow E \setminus \{uv\}$
        $V \leftarrow V \setminus \{v\}$
      if $V = \emptyset$ then
        return $(V_{\text{orig}}, \vec{E})$
    $l \leftarrow \frac{m}{n}$ /* current sizes of the sets $V, E$ */
  until $\forall v \in V : \deg_G(v) = \lceil 2l \rceil$
  while there is cycle $(v_1, ..., v_k)$ in $G$ do
    orient $v_1 \rightarrow ... \rightarrow v_k \rightarrow v_1$ in $\vec{E}$
    $E \leftarrow E \setminus \{v_1 v_2, ..., v_k v_1\}$
  /* A forest remains, orient towards the leaves */
  while $\exists v \in V$ with $\deg_G(v) \leq 1$ do
    orient $u \rightarrow v$ in $\vec{E}$ for the unique $uv \in E$
    $E \leftarrow E \setminus \{uv\}$
    $V \leftarrow V \setminus \{v\}$
  return $(V_{\text{orig}}, \vec{E})$

edges are oriented towards a vertex in the executions of the repeat-until loop (unless $d^* \leq 1/2$). When the while loop for cycles starts, every remaining vertex $v$ has $\deg(v) = \lceil 2l \rceil$. At most half its degree is assigned to a $v$ when we orient along the cycles. At the beginning of the last while loop, the graph is a forest and the algorithm strips the trees from their leaves repeatedly. Hence in this loop every vertex is assigned at most one edge, and the loop terminates.

Asahiro et al. correctly claim that the threshold $\lceil 2l \rceil$ is non-decreasing with every execution of the repeat-until loop. However, we need a strictly increasing sequence for termination of the repeat-until loop. This is guaranteed as there is at least one vertex of degree $\lceil 2l \rceil + 1$, and thus

$$\lceil 2l_{i+1} \rceil = 2 \left[ \frac{|E_{i+1}|}{|V_{i+1}|} \right] \geq 2 \left[ \frac{|V_{i+1} - 1}{2|V_{i+1}|} \right] = \lceil 2l_i \rceil + 1,$$

where $l_i, V_i, E_i$ are the lower bounds and sets after the $i$-th execution of the repeat-until loop. One may wonder whether a constant $1 < c < 2$ could be used for a threshold of $\lceil cl \rceil - 1$ in

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5. In the fractional orientation problem, where a value of 1 is to be divided among its two endpoints for every edge, this can be done trivially by orienting all edges one-half to each of their end vertices.

6. We note that $\ell$ may well decrease between two vertex deletions of the inner while loop.
the repeat-until loop in order to obtain smaller indegrees, because the orientation loop could cope with indegrees of about $[2c]$. However, the degree sum formula does not guarantee termination of the repeat-until loop for such $c$.

Asahiro et al. give a straightforward runtime analysis. The repeat-until loop can be implemented in time $O(mn)$, the cycle loop in $O(m^2)$ by performing depth-first searches, and the forest loop takes $O(n)$. They showed this for edge-weighted graphs, where the maximum weighted indegree is to be minimized and $l$ is defined to be the weighted density. They also showed the weighted orientation problem to be NP-complete even for bipartite planar graphs.

**Theorem 16.** Algorithm 2 can be implemented in linear time. For edge-weighted graphs, it can be implemented in $O(m + n \log n)$ time.

**Proof.** All iterations of the repeat-until loop constitute a partial run of the greedy algorithm, which can be implemented in linear time with linked lists that keep track of the degrees of the vertices (see e.g., [8]). The forest loop is also a run of the greedy algorithm on a forest.

It remains to show linear runtime of the cycle loop. It is possible to perform a single modified depth-first search instead of one search per cycle. We omit the details because we can use known results instead:

As the sum of all degrees is $2m$, the number of vertices with odd degree must be even. Add a vertex $v^*$ to the remaining graph and add edges $(u, v^*)$ for every vertex $u$ of odd degree. Now all vertices have even degree. Therefore an Euler tour exists in every connected component of the graph. The Euler tours can be determined in $O(m)$ total time with Hierholzer’s algorithm [26]. We orient the edges as they are traversed in an Euler tour. After removing $v^*$ and its incident edges, every vertex has at most $\deg(v)/2 + 1$ ingoing edges. The forest loop is now not necessary. Hierholzer’s algorithm is essentially identical to the modified depth-first search that appeared in the earlier version of this paper.

In the case of edge weights, the repeat-until loop can be implemented in $O(m + n \log n)$ time: The vertex of minimum weighted degree can be extracted from a priority queue in $O(\log n)$ amortized time, and the weighted degree of each neighboring vertex can be updated in constant amortized time [18].

\[\]