Dissipative quantum dynamics

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Abstract. This article reviews recent research on dissipative macroscopic quantum bosonic and fermionic systems. These studies address the following issues. (i) The existence of static and dynamic phase transitions; order of the critical lines and type of phases. (ii) The dynamics of systems that are unable to reach equilibrium with their (equilibrium) environment due to their intrinsic slow dynamics or driven by an external drive in the form of a time-dependent magnetic field, coupling to source and drain leads, etc. (iii) The development of an effective temperature that controls the large-scale dynamics and a two-time dependent decoherence phenomenon. (iv) The role played by the environment in the behaviour of the systems.

1. Introduction

Interest in the out of equilibrium dynamics of (possibly macroscopic) quantum systems has been recently boosted by a number of experiments and applications.

On the experimental side spin-glass phases have been identified in many condensed matter systems at very low temperature. Among them we can cite the bi-layer Kagome system SrCr$_4$Ga$_4$O$_{19}$ [1], the polychlore structure Li$_2$Zn$_{1-x}$V$_2$O$_4$, [2], the dipolar magnet LiHo$_x$Y$_{1-x}$F$_4$ [3] and several high $T_c$ compounds [4]. Quantum glassy phases generated by other physical mechanisms appear also in electronic systems, an example being the Coulomb glass [5, 6], and ‘structural’ glasses as Mylar or BK7 [7]. As it is well-known, glasses are characterised by an extremely slow relaxation that occurs out of equilibrium.

The study of out of equilibrium steady states in small quantum systems has a practical interest due to their relevance to nano devices [8]. The external drive can be provided by time-dependent magnetic fields [9, 10] or electric currents flowing through the systems. It is only recently that some authors started to study the effect of a drive on quantum macroscopic systems undergoing phase transitions [11].

A rather satisfactory understanding of the non-equilibrium dynamics of coarsening phenomena, glasses, weakly driven complex liquids and granular matter has been achieved and it is based on the solution of mean-field simple models. This approach leads to a common picture for the dynamics of classical systems in the limit of small entropy production [12]. How much of the classical glassy phenomenology survives at very low temperatures where quantum effects are important is a question that deserves careful theoretical and experimental analysis. The impossibility of simulating the real-time evolution of quantum systems of moderate size enhances the importance of solving simple mean-field or low dimensional models with quantum fluctuations.
On the other hand, the rapidly developing field of quantum computation aims at solving hard optimisation problems with the help of quantum mechanics. Typically, hard problems are also glassy ones at least in their typical instances. Indeed, a mapping between dilute spin-glass models and the typical instances of several combinatorial optimisation problems has been established (K-satisfiability problems are related to p-spin glass models [13], etc.). Understanding the metastable state organisation and slow dynamics of quantum disordered systems appears then as a necessary step before attempting to develop a special quantum device to solve them.

Concretely, a number of questions we tried to give an answer to are:

(i) Find the (static and dynamic) critical surfaces in the $T, \Gamma, g, V$ parameter space, with $T$ temperature, $\Gamma$ strength of quantum fluctuations, $g$ coupling strength to the environment and $V$ drive strength. Establish the order of the all phase transitions. Characterise the static and dynamic phases. Determine the conditions under which equilibrium, steady state or glassy phases are reached.

(ii) Analyze the real-time dynamics after a quench to the different phases.

(iii) An important theoretical result in the field of classical glassy nonequilibrium dynamics is that an ‘effective temperature’, $T_{\text{eff}}$, controls the low-frequency linear response in the limit of small entropy production (long waiting-time, weak drive) [17, 18]. $T_{\text{eff}}$ is defined as the parameter replacing the environmental temperature, $T$, in the fluctuation-dissipation relation (FDR) evaluated at low-frequencies and it has the properties of a temperature in the sense that can be measured with a thermometer and controls heat flows and partial equilibration. Whether an effective temperature develops in quantum out of equilibrium problems and which are its consequences is a generic question that we also wished to address.

(iv) Study decoherence and localization in the different phases. The effect of dissipation on the phase transition, critical behaviour and ordered phase of macroscopic interacting systems is a relatively recent subject of study. Whether a localization transition [14, 15], known to exist in individual two-level system for sufficiently strong coupling strength to the bath, exists and which are the properties of quantum decoherence in a macroscopic ensemble of interacting two-level systems in contact with an environment is an interesting problem that might be of relevance in several areas of condensed-matter and, also, in establishing limitations to quantum computation (see, for example, [16] for a discussion in the context of Josephson junction realizations of qubits).

In short, we summarize here the results presented in [19]-[29]. The organisation of the paper is the following. In section 2 we recall the definition of the models that we studied. Section 3 is devoted to a very short review of the analytic and numeric techniques that we used. Section 4 summarizes the main results of our studies.

2. The models

We analyzed the statics and dynamics of macroscopic quantum systems coupled to equilibrium and out of equilibrium quantum environments. We mainly studied bosonic models coupled to bosonic (oscillator) baths but we also treated cases in which the system degrees of freedom and/or the environment ones are fermionic.

2.1. The system

Disordered quantum spin-$\frac{1}{2}$ models with two-body interactions are defined by

$$H_S = - \sum_{ij} J_{ij} \hat{\sigma}_i^z \hat{\sigma}_j^z + \sum_i \Gamma_i \hat{\sigma}_i^x + \sum_i h_i \hat{\sigma}_i^z.$$ (1)

The spins are represented by Pauli matrices satisfying the SU(2) algebra. $i = 1, \ldots, N$ labels the spins in the sample. In finite dimensions, the spins lie on the vertices of a cubic $d$ dimensional lattice and the interaction strengths $J_{ij}$ couple near-neighbours only. The coupling strengths
$J_{ij}$ are chosen from a probability distribution, $P(J)$. Mean values over $P(J)$ are indicated with square brackets and the average and variance are defined as $\langle J_{ij} \rangle = J_o$ and $\langle J_{ij}^2 \rangle = J^2/(2c)$, where $J_o$ and $J$ are order one and $c$ is the connectivity of the lattice. The next-to-last term is a coupling to a local “transverse” field $\Gamma_i$ that can be taken from another probability distribution, $\Phi(\Gamma)$.

The last term represents the coupling to a longitudinal field that one can include to compute local susceptibilities. Several generalizations that render the model easier to treat analytically, and become especially interesting in the context of optimization problems, have been considered.

2.1.1. Fully-connected limit

Allowing each spin to interact with all others in the sample, $c \to N - 1$, leads to the quantum extension of the Sherrington-Kirkpatrick mean-field spin-glass. This is the $d \to \infty$ limit in which the model is defined on the complete random graph. The scaling of the variance, $\langle J_{ij}^2 \rangle \approx J^2/(2N)$ ensures a good thermodynamic limit, $N \to \infty$.

2.1.2. Multi-spin interactions

In the fully-connected case one can consider $p$-spin interactions with Hamiltonian [21, 22, 25, 31, 32]

$$H_S = - \sum_{i_1, \ldots, i_p} J_{i_1 \ldots i_p} \hat{\sigma}_{i_1}^z \cdots \hat{\sigma}_{i_p}^z + \sum_i \Gamma_i \hat{\sigma}_i^x + \sum_i h_i \hat{\sigma}_i^z. \tag{2}$$

$p$ is an integer parameter that may take any integer value $p \geq 3$, and the sum runs over all $p$-uplets. The exchanges are Gaussian random independent variables with variance $p! J^2/(2N^{p-1})$. This model is particularly interesting since it provides a mean-field description of the structural glass transition and glassy physics that is also intimately related to the mode-coupling approach [12]. The $p$-spin model can also be defined on a dilute graph. The connection with the K-satisfiability optimization problem has been discussed in a number of papers, see, for example, [13].

2.1.3. Spherical case – a particle in a random potential

If, instead of working with spin-$\frac{1}{2}$ variables, one considers a ‘spherical limit’ $\sum_{i=1}^N \langle \hat{\sigma}_i^2 \rangle = N$ in which the $\hat{\sigma}_i$ play the role of coordinate on an $N$-dimensional space and one includes a kinetic term

$$K = \frac{1}{2M} \sum_{i=1}^N \dot{\hat{P}}_i^2, \quad \tag{3}$$

with $\hat{P}_i$ the conjugated momentum satisfying the commutation rules $[\hat{P}_i, \hat{P}_j] = 0$ and $[\hat{P}_i, \hat{\sigma}_j] = -i\hbar \delta_{ij}$, one obtains the model of a particle moving on an $N$-dimensional hypersphere under the effects of the random potential (1) [19, 20]. The spherical quadratic model ($p = 2$) is equivalent to a fully-connected model of rotors in the limit in which the number of components diverges, and to quantum coarsening in $d = 3$ as described by an $O(N)$ field theory in the large $N$ limit [29].

2.1.4. Disordered field-theories

Models of elastic manifolds in quenched random potentials are simple extensions of the previous case. For proper choices of the quenched random potential they model a large class of systems including charge density waves, Wigner crystals (upon generalization to a two component vector $\vec{u}$) and Luttinger liquids in $d = 1$ (see [28] and [34] for a list of references).
2.1.5. A metallic wire We also studied the dynamics of a metallic wire [26] driven out of equilibrium by a time-dependent magnetic flux and coupled to an electronic reservoir. We modelled the wire with an ideal system of non-interacting spin-less electrons that we described with a one dimensional periodic tight-binding chain with length \( L = Na \) (\( N \) is the number of sites and \( a \) the lattice spacing), hopping matrix element \( w = W/4 \) and bandwidth \( W \):

\[
H_S = -w \sum_{i=1}^{N} \left( e^{-i\phi t} c_i^\dagger c_{i+1} + e^{i\phi t} c_{i+1}^\dagger c_i \right).
\]

The time-dependent phase \( \phi t \) with \( \phi \equiv \Phi/\left(\Phi_0 N\right) \) and \( \Phi_0 = \hbar c/e \) accounts for the external magnetic flux that depends linearly on time, \( \Phi_M(t) = \Phi t \).

2.1.6. Other cases Of course, many other quantum situations deserve to be explored. A particular interesting case, which is actively being studied experimentally [5, 6], is the case of Coulomb glasses. Numerical [35] and analytic [36, 37] studies of different models intended to describe the physical problem appeared in the literature.

2.2. Environment

The dynamics of a classical system coupled to a classical environment are described by different kinds of stochastic evolution rules (Langevin, Glauber, Monte Carlo, etc.) depending on the symmetry of the dynamic variables. Quantum mechanically, the interaction of a system with a bath is highly non-trivial. We focused on two types of environments.

2.2.1. Harmonic oscillators in equilibrium A standard model of a (bosonic) bath is an ensemble of independent quantum harmonic oscillators with different masses and frequencies. The coupling between system variables and oscillators is typically chosen to be linear in the coordinates. The oscillator degrees of freedom appear quadratically and can be treated exactly. The interaction leads to dissipation, decoherence, and a localization transition in the case of a single two level system when the coupling is sufficiently strong (see below).

2.2.2. Electronic reservoirs (possibly out of equilibrium) By coupling a system to one or more fermionic reservoirs (whose properties are not affected by the coupling to the system) one also introduces dissipation. When using more than one ‘lead’ at, say, the same temperature but different chemical potential a voltage drop between the leads is induced and a current flows across the system driving it out of equilibrium.

In the ‘bosonic’ systems the coupling of the ‘coordinate’ or ‘spin’ to an electronic reservoir can be treated perturbatively as described in detail in [47, 29]. For free-electrons, at this level of the approximation, just the electron Green functions appear and then the environment is fully characterised by its density of states, temperature and chemical potential. At sufficiently low frequency (\( \omega \ll \beta \bar{\hbar} \)) the effect of the fermionic reservoir approaches the one of an Ohmic bosonic bath not necessarily in equilibrium. In the same limit a parameter playing a role similar to a classical temperature, \( T^* \), can be identified. Still, as we shall discuss below, \( T^* \), cannot be identified with the temperature of the (out of equilibrium) system.

In [26] we modelled the lead with a semi-infinite tight-binding chain in equilibrium at temperature \( T \) and chemical potential \( \mu \) and with a semicircular spectral density.

2.3. System-bath coupling

The coupling to the environment is modelled by three terms that are added to the Hamiltonian:

\[
H = H_S + H_B + H_I + H_{CT},
\]

(5)
where $H_B$ is the Hamiltonian of the bath, $H_I$ represents the interaction between the system and the bath and $H_{CT}$ is a counter term that is added to eliminate an undesired mass renormalization induced by the coupling to the oscillators [30, 19, 20].

In the case of a bosonic bath the full environment is usually taken to be a set of independent quantum harmonic oscillators [19, 20, 28]. For simplicity we considered a bilinear coupling with counter term given by

$$H_I = -\sum_{i=1}^{N} \xi_i \sum_{l=1}^{\tilde{N}} c_{il} \hat{x}_l$$

$$H_{CT} = \sum_{l=1}^{\tilde{N}} \frac{1}{2m_l \omega_l^2} \left( \sum_{i=1}^{N} c_{il} \sigma_z^i \right)^2.$$  \hspace{1cm} (6)

For $p = 2$ the fully-connected model reduces to a model for metallic spin-glasses [33].

In the problem of the metallic ring [26] the contact term between the lead and the ring was chosen to be $H_I = -w_{1\alpha} (c_1^\dagger c_\alpha + c_\alpha^\dagger c_1)$.

3. The techniques

The basic techniques used to study classical glassy models with or without quenched disorder are well documented in the literature (the replica trick, the cavity method, scaling arguments and droplet theories, the dynamic functional method used to derive macroscopic equations from the microscopic Langevin dynamics, functional renormalization, Monte Carlo and molecular dynamic numerical methods). On the contrary, the methods needed to deal with the statics and dynamics of quantum macroscopic systems are much less known in general. In this section we briefly mention the tools we used to study the statics and dynamics of the models listed above.

3.1. Matsubara replicas

The statics of a quantum model in the canonical ensemble can be analyzed by using the Matsubara approach to express the partition function as a path-integral. The quantum model can be the one of a system or a system+bath in which case the whole ensemble is assumed to be described by the canonical measure at a given temperature. In mean-field cases with quenched randomness one uses the replica approach to average over disorder and Parisi’s replica symmetry breaking prescription to study the ordered and disordered phases (see, e.g., [22, 24, 25] for more details).

In the freely relaxing problems a ‘trick’ within the replica method can be used to derive the dynamic phase diagram and some properties of the various phases: the $\text{Ansatz}$ of marginal stability (AMS). Originally developed for classical systems by Kirkpatrick and Thirumalai, this method was recently used to discuss the low-temperature properties of quantum glasses [22, 38]. Its main advantage is that it uses a formalism that is closely related to the imaginary-time approach to equilibrium quantum statistical mechanics. In Ref. [22] the AMS was extensively applied to the quantum spherical $p$-spin model in the absence of the bath. It was shown that the position of the dynamic transition line predicted by this method coincides precisely with that obtained using the real-time approach [19, 20]. It was also shown that the time dependent correlation function computed using the AMS in the absence of the bath is identical to the stationary part of the non-equilibrium correlation function $\langle C > q_{EA} \rangle$ when one takes the long-time limit first and the limit in which the coupling to the bath goes to zero next. The marginality condition imposed by the $\text{Ansatz}$ is intimately related to the fact that the correlation will further decay from $q_{EA}$ towards zero. (The details of this second decay as, for instance, the two-time scaling are not accessible with this method.) A localized solution with $C(\tau + t_w, t_w)$ approaching, and never leaving, the plateau at $q_{EA}$ corresponds, in replica terms, to a stable replica symmetry solution. In [24, 25] we extended the AMS to study the dynamics of models coupled to an environment.
3.2. Schwinger-Keldysh real-time
The usual methods of equilibrium quantum statistical mechanics are inappropriate to describe a system that is unable to relax to equilibrium or a system that is permanently driven. We studied the dynamics of the fully-connected models [19, 20, 24, 28, 29] using a method especially designed to treat systems out of equilibrium: the Schwinger-Keldysh real-time approach [30].

The typical initial condition we used is one in which system and bath are uncoupled until \( t = 0 \) when they set into contact. The initial density matrix is then \( \rho = \rho_S \otimes \rho_B \) with \( \rho_B \) in equilibrium at the working temperature (and chemical potentials in the case of fermions) and \( \rho_S \) representing a situation that is completely uncorrelated with the system Hamiltonian. Thus, the initial time corresponds to a rapid quench to a point in parameter space selected by the external parameters.

The analysis of initial conditions that ‘know’ about the equilibrium (or metastable) states of disordered systems is harder and we shall discuss it elsewhere [39].

3.3. Thouless-Anderson-Palmer method
In the classical case, the study of the Thouless-Anderson-Palmer (TAP) free energy landscape has been of much help to understand the behaviour of these systems [12]. A TAP approach can also be developed for the quantum problem [23]. It helps understanding the change in nature of the transition close to the quantum critical point.

3.4. Cavity method
The cavity method is specially suited to solve classical spin models on sparse random graphs (mean-field models with finite connectivity). An extension of this method to deal with quantum spin-\( \frac{1}{2} \) models in a transverse field was introduced in [42] using a discrete time Trotter-Suzuki representation of the path-integral and improved in [43] by taking the continuous imaginary-time limit analytically and presenting an explicit replica symmetry Ansatz for ferromagnetic model in a Bethe lattice.

4. The results
In this section we present a short survey of the main results found in the series of papers [19]-[29].

4.1. Phase transition
We start by summarizing the phase transitions found.

4.1.1. Pair interactions \((p = 2)\) In the \( p = 2 \) case without a drive \((V = 0)\) there is a static second order critical line in the \( T, \Gamma \) plane separating a disordered and an ordered phase. The former ‘continues’ the classical paramagnet while the latter is an ordered phase. In the real-time dynamics of with a bosonic [40] or fermionic [29] bath in equilibrium, taking the \( g \to 0 \) limit after the asymptotic long-times limit one finds the same second order critical line (in the sense that the Edwards-Anderson order parameter, \( q_{EA} = \lim_{\tau \to -\infty} \lim_{t_w \to -\infty} C(\tau + t_w, t_w) \) vanishes at criticality). The dynamics in the ordered phase are equivalent to 3d coarsening in the quantum \( O(N) \) model in the large \( N \) limit [29]. If \( g \) remains finite, the critical lines depend on \( g \) as discussed below.

When a drive is applied \((V \neq 0)\) there is a truly out of equilibrium phase transition between the ‘outer’ quantum nonequilibrium steady state and the ‘inner’ driven quantum coarsening regime. The form of the critical manifold depends on the details of the fermionic baths and it is discussed in [29]. The dynamic transitions are of second order.
4.1.2. **Multi-spin interactions ($p \geq 3$)** In the freely relaxing multi-spin models coupled to harmonic oscillator baths the static [31, 32, 21, 22] and dynamic [21, 21] transitions at the critical point ($T = 0, \Gamma_c$) are of first-order. The first-order line extends at small temperatures and ends in a tricritical point in which the transition becomes second order. Across the first order critical line the susceptibility is discontinuous and shows hysteresis. This is similar to what has been observed in the dipolar-coupled Ising magnet LiHo$_x$Y$_{1-x}$F$_4$ [3]. This behaviour appears to be generic of classical disordered models with random first order transitions at $\Gamma = 0$ (models whose statics is solved by a one-step replica symmetry Ansatz). It was shown in [41], using the random energy model as a simple example in this family of models, that at the first order transition the eigenstate suddenly projects onto the unperturbed ground state and the gap between the lowest states is exponentially small in the system size. This sets a clear limit into the performance of quantum annealing procedures to find the ground state of this class of systems.

4.2. **Dynamics**

Let us first discuss the freely relaxing (undriven) dynamics in the macroscopic disordered spin models [19, 20, 24] (see also [40, 46, 47] for similar studies of different models).

In the disordered phase the dynamics are fast and occur in equilibrium. The auto correlation and the linear response functions are invariant under time-translations. They both show oscillations, as is typical of a quantum problem. The frequency of the oscillations depends on $\Gamma$ [and the characteristics of the bath ($g, s$), see below]. Correlations and responses are linked by the quantum fluctuation dissipation theorem (QFDT). At high temperatures and after a short transient there is a decoherence effect and the dynamics become totally classical. For example, responses and correlations are related by the classical FDT.

In the ordered phase the dynamics show typical glassy features. There is separation of time scales controlled by $t_w$: for short $t - t_w$ with respect to $t_w$ the relaxation is stationary and the quantum FDT holds. For longer $t - t_w$ (comparable to $t_w$) the waiting-time dependence remains explicit and the relaxation is the slower the longer $t_w$. The quantum FDT is not verified but, instead, one observes that a classical one, with an effective temperature $T_{eff} > 0$ holds. The relaxation of the symmetric correlation and linear response show oscillations in the stationary regime but they are monotonic in the aging regime. In all respects, the relaxation in the aging ($t/t_w$, finite) regime looks classical at an effective temperature $T_{eff}$.

The effect of a drive on the $p = 2$ spherical model introduced in the form of a potential drop between the two fermionic reservoirs was analysed in [29]. The current flowing through the system drives it out of equilibrium in the full phase diagram. In the ‘outer’ phase a steady state is reached but the correlation and linear response do not satisfy the quantum FDT. In the ‘inner’ phase the system tends to order in the sense that one identifies a coarsening phenomenon as in the freely relaxing case with aging properties. Note the difference with the classical sheared problem [45] in which the external drive kills the aging relaxation (and renders the dynamics stationary) at all finite bath temperatures ($T > 0$). The effective temperature linking correlations and linear responses in the coarsening phase diverges, as in the classical limit.

Moreover, in the $p = 2$ driven problem we found an extension of the irrelevance of $T$ in classical ferromagnetic coarsening ($T = 0$ ‘fixed-point’ scenario): after a suitable normalization of the observables that takes into account all microscopic fluctuations (e.g. $q_{EA}$) the scaling functions are independent of all parameters including $V$ and $\Gamma$. We expect this result to hold in all instances with the same type of ordered phase, say ferromagnetic, and a long-time aging dynamics dominated by the slow motion of large domains. Thus, a large class of coarsening systems (classical, quantum, pure and disordered) should be characterized by the same scaling functions.

The effect of a drive on the $p \geq 3$ cases and, in particular, its effect on the (first-order) phase
transition lines has not been studied yet. We plan to explore this problem in the future.

4.3. Coupling to the bath: localization

The coupling of quantum two-level systems (TLS) to a dissipative environment has decisive effects on their dynamical properties. The dilute case, in which interactions between the TLS can be neglected, has been extensively investigated in the literature [14, 15, 44]. Under certain circumstances the model can be mapped onto the 1d Ising model with inverse squared interactions, the anisotropic Kondo model, or the resonant model [44]. Three different cases exist depending on the value of \(g\): in the Ohmic case, at zero temperature, there is a phase transition at \(g = 1\) [15]. For \(g < 1\) there is tunneling and two distinct regimes develop. If \(g < 1/2\) the system relaxes with damped coherent oscillations; in the intermediate region \(1/2 < g < 1\) the system relaxes incoherently. For \(g > 1\) quantum tunneling is suppressed and \(\langle \hat{\sigma}_z \rangle \neq 0\) signalling that the system remains localized in the state in which it was prepared. These results also hold for sub-Ohmic baths while weakly damped oscillations persist for super-Ohmic baths. At finite temperatures (but low enough such that thermal activation can be neglected), there is no localization but the probability of finding the system in the state it was prepared decreases slowly with time for \(g > g_{\text{crit}}\).

In thermodynamic equilibrium, in the absence of the bath, interactions between the TLS lead to the appearance of an ordered state at low enough temperature. If the interactions are of random sign, as in models with quenched randomness the latter will be a spin glass (SG) state. In this phase the symmetry between the states \(\sigma^z_i = \pm 1\) at any particular site is broken but there is no global magnetization, \(\sum_i \langle \hat{\sigma}_z^i \rangle = 0\). Since the coupling to the bath also tends to locally break the symmetry between the degenerate states of the TLS, both interactions compete with the tunneling term in the Hamiltonian. One thus expects the quantum noise to increase the stability of the SG state against quantum fluctuations. The consequences of this fact are particularly interesting when the coupling to the bath leads by itself to localization at some \(g = g_{\text{crit}}\). Consider a system of size \(N\) with \(g > g_{\text{crit}}\) at \(T = 0\) and suppose that we turn off the interactions between the TLS. The ground state of the system is then \(2^N\)-fold degenerate as each TLS can be in one of the states \(\langle \hat{\sigma}_z^i \rangle = \pm \sigma_0\) (say) independently. If we now turn on an infinitesimal random interaction between the TLS, this macroscopic degeneracy will be immediately lifted as the system will select among its \(2^N\) degenerate configurations the one (or one among the ones) that minimizes the interaction energy. If we denote by \(J\) the typical scale of the interactions and by \(g_{\text{crit}}\) the localization threshold, we thus expect a quantum critical point at \(J = 0\), \(g = g_{\text{crit}}\) between a quantum paramagnet and the ordered state such that, for \(g > g_{\text{crit}}\), the SG phase survives down to \(J = 0\).

A system of non-interacting localized TLS and a SG state in equilibrium are in some way similar: in both cases \(\sum_i \langle \hat{\sigma}_z^i \rangle = 0\) and the presence of order is reflected by a non-vanishing value of the long-time limit of the correlation function, \(q_{\text{EA}} = \lim_{t \to \infty} N^{-1} \sum_i \langle \sigma_z^i(t) \sigma_z^i(0) \rangle\) (since we assume equilibration the correlation is stationary and the reference time can be taken to be zero). However, this resemblance is only superficial since the dynamics are quite different, in particular, the way in which the correlation function reaches its asymptotic limit, \(q_{\text{EA}}\).

Further differences between the localized state and the SG state are seen from the study of the out of equilibrium relaxation of such states. As already mentioned, an important feature of glassy systems is that their low-temperature dynamics occur out of equilibrium and the dynamic correlation functions loose time translation invariance. If \(t_w\) denotes the time elapsed since a quench from the high temperature phase into the SG phase, \(C(\tau + t_w, t_w)\) depends on both \(\tau\) and \(t_w\). The order in which the limits \(t_w \to \infty\) and \(\tau \to \infty\) are taken is in this case very important. For sufficiently long \(\tau\) and \(t_w\) but in the regime \(\tau \ll t_w\), the dynamics are stationary and the correlation function reaches a plateau \(q_{\text{EA}}\). Much of what was said above for the equilibrium state also holds for this stationary regime. However, for times \(\tau \sim t_w\), the system enters an
aging regime where the correlation function depends on the waiting-time $t_w$ explicitly. In this regime, the dynamic correlation function vanishes at long times, \( \lim_{\tau \to \infty} C(\tau + t_w, t_w) = 0 \), at a rate that depends on $t_w$. This is to be confronted to the dynamics in the localized state, where $C(\tau + t_w, t_w)$ reaches, for any waiting-time $t_w$ and long enough $\tau$ a plateau that it never leaves. In the aging regime, even for $g > g_{\text{crit}}$, small interactions will result in the destruction of localization of the TLS at long enough times.

The problem of a single TLS being a difficult one, that of an infinite set of interacting TLS seems hardly solvable. Therefore, as a first step, we focused on the effect of the reservoir on the $p$-spin spherical model [24]. We found that the position of the static critical line separating the disordered and the ordered phases strongly depends on the strength of the coupling to the bath and the type of bath (ohmic, subOhmic, superOhmic). For a given type of bath, the ordered glassy phase is favored by a stronger coupling. Ohmic, subOhmic and superOhmic baths lead to different transition lines. The classical static transition temperature corresponding to $\Gamma \to 0$, with $\Gamma$ the parameter controlling the strength of the quantum fluctuations, remains unchanged by the coupling to the quantum heat reservoir. The spherical model localizes in the absence of interactions when coupled to a subOhmic bath. When interactions are switched on localization disappears and the system undergoes a phase transition towards a glassy phase.

The Ansatz of marginal stability allowed us to identify the dynamic critical line that is consistent with the one found studying the real-time dynamics of the coupled system with the Schwinger-Keldysh closed time-path formalism. The effect of the coupling to the environment on the dynamic critical line is similar. The classical dynamic transition temperature corresponding to $\Gamma \to 0$ also remains unchanged.

4.4. Effective temperature
The thermodynamic relevance of the parameter replacing the environmental temperature in the FDT in slow classical glassy relaxation was discussed in [17, 18]. A similar analysis in the quantum context has not been developed yet. In particular, questions such as what happens with heat flows when two quantum problems with different $T_{\text{eff}}$ are set in contact, can $T_{\text{eff}}$ be measured with a thermometer, is the violation of the quantum FDT – and hence the difference between the effective and the environmental temperature – bounded by a function of some kind entropy production time-variation as done in [48] for classical stochastic problems.

The environment plays, in a sense, a dual role: its quantum character basically determines the phase diagram but the coarsening process at long times and large length-scales is the one of a classical problem in a white noise at an effective temperature $T_{\text{eff}}$. This two-time dependent decoherence phenomenon (absence of oscillations, validity of a classical FDT when $t/t_w = O(1)$, etc.) is thus intimately related to the development of a non-zero (infinite in the $p = 2$ case) effective temperature, $T_{\text{eff}}$, of the system as defined from the deviation from the (quantum) FDT. $T_{\text{eff}}$ should be distinguished any temperature, say $T^*$, that could be identified from a low-frequency analysis of the noise kernels. $T_{\text{eff}}$ is generated not only by the environment but by the system interactions as well (e.g. cases in which $T_{\text{eff}} > 0$ even at $T^* = 0$ can be seen in [19, 20, 28, 29, 46, 47]).

We conclude that the non-vanishing character of the effective temperature is responsible for the decoherence effect observed at large time and length scales that renders the aging dynamics classical.

In all quantum models studied up to present a quench from the disordered state into the ordered one was considered and the effective temperature was found to be larger or equal than the ambient one. In classical problems the ‘reversed’ procedure in which one heats the system from the zero-temperature ground state to a finite temperature still within the ordered phase was considered and an effective temperature that is smaller than the working one was found. It would be extremely interesting to check whether a similar result is found quantum mechanically.
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References
[1] Uemura Y J et al 1994 Phys. Rev. Lett. 73 3306
[2] Urano C et al 2002 Phys. Rev. Lett. 85 1032
[3] Ancona-Torres C, Silevitch D M, Aeppli G and Rosenbaum T F 2008 Phys. Rev. Lett. 101 057201 and references therein.
[4] Julien M-H 2003 Physica B 329 693
[5] Ovadyahu Z 2006 Phys. Rev. B 73 214204
[6] Popovic D et al. 2003 Proceedings of SPIE 5112 99
[7] Rogge S, Natelson D and Osheroff D D 1996 Phys. Rev. Lett. 76 3136
[8] Imry Y 1997 Introduction to mesoscopic physics (Oxford: Oxford University Press)
[9] Arrachea L 2002 Phys. Rev. B 66 045315
[10] Arrachea L 2003 Eur. Phys. J. B 36 253
[11] Mitra A and Millis A J 2008 Current driven quantum criticality in itinerant electron ferromagnets Preprint arXiv:0804.3980 and references therein.
[12] Cugliandolo L F 2002 in Les Houches Session 77 eds J-L. Barrat et al (Berlin: Springer-EDP Sciences) (Preprint arXiv:cond-mat/0210312); Birolì G 2005 J. Stat. Mech. P05014
[13] Monasson R 2007 in Les Houches Session 85 eds J-P Bouchaud, M Mèzard and J Dalibard (Amsterdam: Elsevier) (Preprint arXiv:0704.2536)
[14] Chakravarty S 1982 Phys. Rev. Lett. 49 681
[15] Bray A J and Moore M 1982 Phys. Rev. Lett. 49 1545
[16] Faoro L and Ioffe L B 2007 Microscopic origin of low frequency flux noise in Josephson circuits Preprint arXiv:0712.2834; 2005 Quantum two level systems and Kondo-like traps as possible sources of decoherence in superconducting qubits Preprint arXiv:cond-mat/0510554
[17] Cugliandolo L F, Kurchan J and Peliti L 1997 Phys. Rev. E 55 3898
[18] Cugliandolo L F and Kurchan J 2000 J. Phys. Soc. Jpn. 69 247
[19] Cugliandolo L F and Lozano G 1998 Phys. Rev. Lett. 80 4979
[20] Cugliandolo L F and Lozano G 1999 Phys. Rev. B 59 915
[21] Cugliandolo L F, Grempel D R and da Silva Santos C A 2000 Phys. Rev. Lett. 85 2589
[22] Cugliandolo L F, Grempel D R and da Silva Santos C A 2001 Phys. Rev. B 64 014403
[23] Birolì G and Cugliandolo L F 2001 Phys. Rev. B 64 014206
[24] Cugliandolo L F, Grempel D R, Lozano G, Lozza H and da Silva Santos C A 2002 Phys. Rev. B 66 014444
[25] Cugliandolo L F, Grempel D R, Lozano G and Lozza H 2004 Phys. Rev. B 70 024422
[26] Arrachea L and Cugliandolo L F 2005 Europhys. Lett. 70 642
[27] Kennett M P, Chamon C and Cugliandolo L F 2005 Phys. Rev. B 72 024417; In this paper a model for a 2d doped classical antiferromagnet is studied with the scope of discussing super-conductor physics.
[28] Cugliandolo L F, Giamarchi T and Le Doussal P 2006 Phys. Rev. Lett. 96 217203
[29] Aron C, Birolì G and Cugliandolo L F 2008 Driven quantum coarsening Preprint arXiv:0809.0590
[30] Weiss U 1993 Quantum dissipative systems (Singapore: World Scientific); Kamenev A 2004 in Les Houches Session 81 eds H Bouchiat et al (Berlin: Springer-EDP Sciences) (Preprint arXiv:cond-mat/0412296)
[31] Goldschmidt Y 1990 Phys. Rev. B 41 4858
[32] Nieuwenhuizen T M and Ritort F 1998 Physica A 250 89
[33] Grempel D R and Rozenberg M J 1999 Phys. Rev. B 60 4702
[34] Giamarchi T 2002 in Lecture notes of the course given at the E. Fermi school on “Quantum Phenomena in Mesoscopic Systems” (Varenna) (Preprint arXiv.org:cond-mat/0403531)
[35] Grempel D R 2004 Europhys. Lett. 66 854
[36] Pankov S and Dobrosavljevic V 2005 Phys. Rev. Lett. 94 046402
[37] Mueller M and Ioffe L B 2007 Collective modes in quantum electron glasses and electron-assisted hopping Preprint arXiv:0711.2668
[38] Giamarchi T and Le Doussal P 1996 Phys. Rev. B 53 15206
[39] Cugliandolo L F, Grempel D R and da Silva Santos C A unpublished; Aron C, Biroli G and Cugliandolo L F in preparation
[40] Rokni M and Chandra P 2004 Phys. Rev. B 69 094403
[41] Jorg T, Krzakala F, Kurchan J and Maggs A J 2008 Simple Glass Models and their Quantum Annealing Preprint arXiv:cond-mat/0806414
[42] Laumann C, Scardicchio A and Sondhi S L 2007 Cavity method for quantum spin glasses on the Bethe lattice Preprint arXiv:cond-mat/0706.4391
[43] Krzakala F, Rosso A, Semerjian G and Zamponi F 2008 On the path integral representation for quantum spin models and its application to the quantum cavity method and to Monte Carlo simulations Preprint arXiv:cond-mat/0807:2553
[44] Leggett A J, Chakravarty S, Dorsey A T, Fisher M P A, Garg A and Zwerger W 1987 Rev. Mod. Phys. 59 1; 1995 67 725
[45] Cugliandolo L F, Kurchan J, Le Doussal P and Peliti L 1997 Phys. Rev. Lett. 78 350
[46] Kennett M P and Chamon C 2001 Phys. Rev. Lett. 86 1622
[47] Biroli G and Parcollet O 2002 Phys. Rev. B 65 094414
[48] Cugliandolo L F, Dean D S and Kurchan J 1997 Phys. Rev. Lett. 79 2168