Statistically-secure ORAM with $\tilde{O}(\log^2 n)$ Overhead

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Abstract

We demonstrate a simple, statistically secure, ORAM with computational overhead $\tilde{O}(\log^2 n)$; previous ORAM protocols achieve only computational security (under computational assumptions) or require $\tilde{\Omega}(\log^3 n)$ overhead. An additional benefit of our ORAM is its conceptual simplicity, which makes it easy to implement in both software and (commercially available) hardware.

Our construction is based on recent ORAM constructions due to Shi, Chan, Stefanov, and Li (Asiacrypt 2011) and Stefanov and Shi (ArXiv 2012), but with some crucial modifications in the algorithm that simplifies the ORAM and enable our analysis. A central component in our analysis is reducing the analysis of our algorithm to a “supermarket” problem; of independent interest (and of importance to our analysis,) we provide an upper bound on the rate of “upset” customers in the “supermarket” problem.

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1 Introduction

In this paper we consider constructions of Oblivious RAM (ORAM) [9, 10]. Roughly speaking, an ORAM enables executing a RAM program while hiding the access pattern to the memory. ORAM have several fundamental applications (see e.g. [10, 19] for further discussion). Since the seminal works for Goldreich [9] and Goldreich and Ostrovsky [10], constructions of ORAM have been extensively studied (see e.g., [27, 28, 1, 20, 11, 5, 22, 2, 12, 24, 14].) While the original constructions only enjoyed “computational security” (under the the assumption that one-way functions exists) and required a computational overhead of $O(\log^3 n)$, more recent works have overcome both of these barriers, but only individually. State of the art ORAMs satisfy either of the following:

- An overhead of $\tilde{O}(\log^2 n)$, but only satisfies computational security, assuming the existence of one-way functions. [20, 11, 14]
- Statistical security, but have an overhead of $O(\log^3 n)$. [1, 5, 22, 7, 4].

A natural question is whether both of these barriers can be simultaneously overcome; namely, does there exists a statistically secure ORAM with only $\tilde{O}(\log^2 n)$ overhead? In this work we answer this question in the affirmative, demonstrating the existence of such an ORAM.

**Theorem 1.** There exists a statistically-secure ORAM with $\tilde{O}(\log^2(n))$ worst-case computational overhead, constant memory overhead, and CPU cache size poly log(n), where n is the memory size.

An additional benefit of our ORAM is its conceptual simplicity, which makes it easy to implement in both software and (commercially available) hardware. (A software implementation is available from the authors upon request.)

**Our ORAM Construction** A conceptual breakthrough in the construction of ORAMs appeared in the recent work of Shi, Chan, Stefanov, and Li [22]. This work demonstrated a statistically secure ORAM with overhead $O(\log^3 n)$ using a new “tree-based” construction framework, which admits significantly simpler (and thus easier to implemented) ORAM constructions (see also [7, 4] for instantiations of this framework which additionally enjoys an extremely simple proof of security).

On a high-level, each memory cell $r$ accessed by the original RAM will be associated with a random leaf $pos$ in a binary tree; the position is specified by a so-called “position map” $Pos$. Each node in the tree consists of a “bucket” which stores up to $\ell$ elements. The content of memory cell $r$ will be found inside one of the buckets along the path from the root to the leaf $pos$; originally, it is put into the root, and later on, the content gets “pushed-down” through an eviction procedure—for instance, in the ORAM of [4] (upon which we rely), the eviction procedure consists of “flushing” down memory contents along a random path, while ensuring that each memory cell is still found on its appropriate path from the root to its assigned leaf. (Furthermore, each time the content of a memory cell is accessed, the content is removed from the tree, the memory cell is assigned to a new random leaf, and the content is put back into the root).

In the work of [22] and its follow-ups [7, 4], for the analysis to go through, the bucket size $\ell$ is required to be $\omega(\log n)$. Stefanov and Shi [23] recently provided a different instantiation of this framework which only uses constant size buckets, but instead relies on a single poly log n size “stash” into which potential “overflows” (of the buckets in the tree) are put; Stefanov and Shi conjectured (but did not prove) security of such a construction (when appropriately evicting elements from the “stash” along the path traversed to access some memory cell).

In this work, we follow the above-mentioned approaches, but with the following high-level modifications:

- We consider a binary tree where the bucket size of all internal buckets is $O(\log \log n)$, but all the leaf nodes still have bucket size $\omega(\log n)$.

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1 The best protocol achieves $O(\log^2 n / \log \log n)$.

2 Although different, the “flush” mechanism in [4] is inspired by this eviction method.
• As in [23], we use a “stash” to store potential “overflows” from the bucket. In our ORAM we refer to this as a “queue” as the main operation we require from it is to insert and “pop” elements (as we explain shortly, we additionally need to be able to find and remove any particular element from the queue; this can be easily achieved using a standard hash table). Additionally, instead of inserting memory cells directly into the tree, we insert them into the queue. When searching for a memory cell, we first check whether the memory cell is found in the queue (in which case it gets removed), and if not, we search for the memory cell in the binary tree along the path from the root to the position dictated by the position map.

• Rather than just “flushing” once (as in [4]), we repeat the following procedure “pop and random flush” procedure twice.
  – We “pop” an element from the queue into the root.
  – Next, we flush according to a geometrically distributed random variable with expectation 2.\(^3\)

We demonstrate that such an ORAM construction is both (statistically) secure, and only has \(\Omega(\log^2 n)\) overhead.

Our Analysis The key element in our analysis is reducing the security of our ORAM to a “supermarket” problem. Supermarket problems were introduced by Mitzenmacher [16] and have been well-studied (see e.g., [16, 26, 18, 21, 17]). We here consider a simple version of a supermarket problem, but ask a new question: what is the rate of “upset” customers in a supermarket problem: There are \(D\) cashiers in the supermarket, all of which have empty queues in the beginning of the day. At each time step \(t\): with probability \(\alpha < 1/2\) a new customer arrives and chooses a random cashier\(^4\) (and puts himself in that cashiers queue); otherwise (i.e., with probability \(1 - \alpha\)) a random cashier is chosen that “serves” the first customer in its queue (and the queue size is reduced by one). We say that a customer is upset is he chooses a queue whose size exceeds some bound \(\varphi\). What is the rate of upset customers?\(^5\)

We provide an upper bound on the rate of upset customers relying on Chernoff bounds for Markov chains [8, 13, 15, 3]—more specifically, we develop a variant of traditional Chernoff bounds for Markov chains which apply also with “resets” (where at each step, with some small probability, the distribution is reset to the stationary distribution of the Markov chain), which may be of independent interest, and show how such a Chernoff bound can be used in a rather straightforward way to provide a bound on the number of upset customers.

Intuitively, to reduce the security of our ORAM to the above-mentioned supermarket problem, each cashier corresponds to a bucket on some particular level \(k\) in the tree, and the bound \(\varphi\) corresponds to the bucket size, customers correspond to elements being placed in the buckets, and upset customers overflows. Note that for this translation to work it is important that the number of flushes in our ORAM is geometrically distributed—this ensures that we can view the sequence of operations (i.e., “flushes” that decrease bucket sizes, and “pops” that increase bucket sizes) as independently distributed as in the supermarket problem.

Independent Work In a very recent independent work, Stefanov, van Dijk, Shi, Fletcher, Ren, Yu, and Devadas [25] prove security of the conjectured Path ORAM of [23]. This yields a ORAM with overhead \(O(\log^2 n)\), whereas our ORAM has overhead \(O(\log^2 n \log \log n)\). On the other hand, the data structure required to implement our queue is simpler than the one needed to implement the “stash” in the Path ORAM construction. More precisely, we simply need a standard queue and a standard hash table (both of which can

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\(^3\)Looking forward, our actual flush is a little bit different than the one in [4] in that we only pull down a single element between any two consecutive nodes along the path, whereas in [4] all elements that can be pulled down get flushed down.

\(^4\)Typically, in supermarket problems the customer chooses \(d\) random cashiers and picks the one with the smallest queue; we here focus on the simple case when \(d = 1\).

\(^5\)Although we here consider a discrete-time version of the supermarket problem (since this is the most relevant for our application), as we remark in Remark 1, our results apply also to the more commonly studied continuous-time setting.
be implemented using commodity hardware), whereas the “stash” in [23, 25] requires using a data structure that additionally supports “range queries”, and thus a binary search tree is needed, which may make implementations more costly. We leave a more complete exploration of the benefits of the different approaches for future work.

2 Preliminaries

A Random Access Machine (RAM) with memory size $n$ consists of a CPU with a small size cache (e.g., can store a constant or poly log($n$) number of words) and an “external” memory of size $n$. To simplify notation, a word is either ⊥ or a log $n$ bit string.

The CPU executes a program $\Pi$ (given $n$ and some input $x$) that can access the memory by a $\text{Read}(r)$ and $\text{Write}(r, \text{val})$ operations where $r \in [n]$ is an index to a memory location, and $\text{val}$ is a word (of size log $n$). The sequence of memory cell accesses by such read and write operations is referred to as the memory access pattern of $\Pi(n, x)$ and is denoted $\tilde{\Pi}(n, x)$. (The CPU may also execute “standard” operations on the registers, any may generate outputs).

Let us turn to defining an Oblivious RAM Compiler. This notion was first defined by Goldreich [9] and Goldreich and Ostrovsky [10]. We recall a more succinct variant of their definition due to [4].

Definition 1. A polynomial-time algorithm $\mathcal{C}$ is an Oblivious RAM (ORAM) compiler with computational overhead $c(\cdot)$ and memory overhead $m(\cdot)$, if $\mathcal{C}$ given $n \in N$ and a deterministic RAM program $\Pi$ with memory-size $n$ outputs a program $\Pi'$ with memory-size $m(n) \cdot n$ such that for any input $x$, the running-time of $\Pi'(n, x)$ is bounded by $c(n) \cdot T$ where $T$ is the running-time of $\Pi(n, x)$, and there exists a negligible function $\mu$ such that the following properties hold:

- **Correctness:** For any $n \in N$ and any string $x \in \{0, 1\}^*$, with probability at least $1 - \mu(n)$, $\Pi(n, x) = \Pi'(n, x)$.
- **Obliviousness:** For any two programs $\Pi_1$, $\Pi_2$, any $n \in N$ and any two inputs $x_1, x_2 \in \{0, 1\}^*$ if $|\tilde{\Pi}_1(n, x_1)| = |\tilde{\Pi}_2(n, x_2)|$, then $\tilde{\Pi}_1'(n, x_1)$ is $\mu$-close to $\tilde{\Pi}_2'(n, x_2)$ in statistical distance, where $\tilde{\Pi}_1' = \mathcal{C}(n, \Pi_1)$ and $\tilde{\Pi}_2' = \mathcal{C}(n, \Pi_2)$.

Note that the above definition (just as the definition of [10]) only requires an oblivious compilation of deterministic programs $\Pi$. This is without loss of generality: we can always view a randomized program as a deterministic one that receives random coins as part of its input.

3 Algorithm for the ORAM.

Our ORAM data structure serves as a “big” memory table of size $n$ and exposes the following two interfaces.

- **READ(r):** the algorithm returns the value of memory cell $r \in [n]$.
- **WRITE(r, v):** the algorithm writes value $v$ to memory cell $r$.

We start assuming that the ORAM is executed on a CPU with cache size is $2n/\alpha + o(n)$ (in words) for a suitably large constant $\alpha$ (the reader may imagine $\alpha = 16$). Following the framework in [22], we can then reduce the cache size to $O(\text{poly log } n)$ by recursively applying the ORAM construction; we provide further details on this transformation at the end of the section.

In what follows, we group each consecutive $\alpha$ memory cells in the RAM into a block and will thus have $n/\alpha$ blocks in total. We also index the blocks in the natural way, i.e. the block that contains the first $\alpha$ memory cells in the table has index 0 and in general the $i$-th block contains memory cells with addresses from $\alpha i$ to $\alpha (i + 1) - 1$.

Our algorithm will always be operating at the block level, i.e. memory cells in the same block will always be read/written together. In addition to the content of its $\alpha$ memory cells, each block is associated with two...
extra pieces of information. First, it stores the index \( i \) of the block. Second, it stores a “position” \( p \) that specify it’s storage “destination” in the external memory, which we elaborate upon in the forthcoming paragraphs. In other words, a block is of the form \((i, p, val)\), where \(val\) is the content of its \(\alpha\) memory cells.

Our ORAM construction relies on the following three main components.

1. **A full binary tree at the in the external memory** that serves as the primary media to store the data.
2. **A position map in the internal cache** that helps us to search for items in the binary tree.
3. **A queue in the internal cache** that is the secondary venue to store the data.

We now walk through each of the building blocks in details.

**The full binary tree** \( T_r \). The depth of this full binary tree is set to be the smallest \( d \) so that the number of leaves \( L = 2^d \) is at least \( 2(n/\alpha)/(\log n \log \log n) \) (i.e., \( L/2 < 2(n/\alpha)/(\log n \log \log n) \leq L \)). (In \cite{22,4} the number of leaves was set to \( n/\alpha \); here, we instead follow \cite{7} and make the tree slightly smaller—this makes the memory overhead smaller.) We index nodes in the tree by a binary string of length at most \( d \), where the root is indexed by the empty string \( \lambda \), and each node indexed by \( \gamma \) has left and right children indexed \( \gamma 0 \) and \( \gamma 1 \), respectively. Each node is associated with a *bucket*. A bucket in an internal node can store up to \( \ell \) blocks, and a bucket in a leaf can store up to \( \ell' \) blocks, where \( \ell \) and \( \ell' \) are parameters to be determined later. The tree shall support the following two atomic operations:

- **READ**(Node: \( v \)): the tree will return all the blocks in the bucket associated with \( v \) to the cache.
- **WRITE**(Node: \( v \), Blocks: \( b \)): the input is a node \( v \) and an array of blocks \( b \) (that will fit into the bucket in node \( v \)). This operation will replace the bucket in the node \( v \) by \( b \).

**The position map** \( P \). This data structure is an array that maps the indices of the blocks to leaves in the full binary tree. Specifically, it supports the following atomic operations:

- **READ**(\( i \)): this function returns the position \( P[i] \in [L] \) that corresponds to the block with index \( i \in [n/\alpha] \).
- **WRITE**(\( i, p \)): this function writes the position \( p \) to \( P[i] \).

**The queue** \( Q \). This data structure stores a queue of blocks with maximum size \( q_{\text{max}} \), a parameter to be determined later, and supports the following three atomic operations:

- **INSERT**(Block \( b \)): insert a block \( b \) into the queue.
- **POPFRONT()**: the first block in the queue is popped and returned.
- **FIND**(int: \( i \), word: \( p \)): if there is a block \( b \) with index \( i \) and position \( p \) stored in the queue, then FIND returns \( b \) and deletes it from the queue; otherwise, it returns \( \perp \).

Note that in addition to the usual **INSERT** and **POPFRONT** operations, we also require the queue to support a **FIND** operation that finds a given block, returns and deletes it from the queue. This operation can be supported using a standard hash table in conjunction with the queue. We mention that all three operations can be implemented in time less than \( O(\log n \log \log n) \), and discuss the implementation details in Appendix A.

**Our Construction.** We now are ready to describe our ORAM construction, which relies the above atomic operations. Here, we shall focus on the read operation. The algorithm for the write operation is analogous.

For two nodes \( u \) and \( v \) in \( T_r \), we use \( \text{path}(u, v) \) to denote the (unique) path connecting \( u \) and \( v \). Throughout the life cycle of our algorithm we maintain the following **block-path invariance**.

**Block-path Invariance:** *For any index \( i \in [n/\alpha] \), there exists at most a single block \( b \) with index \( i \) that is located either in \( T_r \) or in the queue. When it is in the tree, it will be in the bucket of one of the nodes on \( \text{path}(\lambda, P[i]) \). Additionally, \( b \) has position \( p = P[i] \).*

We proceed to describe our **READ**(\( r \)) algorithm. At a high-level, **READ**(\( r \)) consists of two sub-routines **FETCH**(\() \) and **DEQUEUE**(\() \), where we executes **FETCH**(\() \) once, and then executes **DEQUEUE**(\() \) twice. Roughly, **FETCH**(\() \) fetches the block \( b \) that contains the memory cell \( r \) from either \( \text{path}(\lambda, P[\lceil r/\alpha \rceil]) \) in \( T_r \) or in \( Q \), then returns the value of memory cell \( r \), and finally inserts the block \( b \) to the queue \( Q \). On the other hand, **DEQUEUE**(\() \) pops one block \( b \) from \( Q \), inserts \( b \) to the root \( \lambda \) of \( T_r \) (provided there is a room), and performs a random number of “FLUSH” actions that gradually moves blocks in \( T_r \) down to the leaves.
**Fetch:** Let $i = \lfloor r/\alpha \rfloor$ be the index of the block $b$ that contains the $r$-th memory cell, and $p = P[i]$ be the current position of $b$. If $P[i] = \perp$ (which means that the block is not initialized yet), let $P[i] \leftarrow [L]$ be a uniformly random leaf, create a block $b = (i, P[i], \vec{0})$, and insert $b$ to the queue $Q$. Otherwise, FETCH performs the following actions in order.

**Fetch from tree Tr and queue $Q$:** Search the block $b$ with index $i$ along $\text{path} (\lambda, p)$ in $\text{Tr}$ by reading all buckets in $\text{path} (\lambda, p)$ once and writing them back. If such a block is found, save it and write back a dummy block; otherwise, search the block $b$ with index $i$ and position $p$ in the queue $Q$ by invoking $\text{Find}(i, p)$. By the block-path invariance, we must find the block $b$.

**Update position map $P$:** Let $P[i] \leftarrow [L]$ be a uniformly random leaf, and update the position $p = P[i]$ in $b$.

**Insert to queue $Q$:** Insert the block $b$ to $Q$.

**Dequeue:** This sub-routine consists of two actions $\text{Put-Back}()$ and $\text{Flush}()$. It starts by executing $\text{Put-Back}()$ once, and then performs a random number of $\text{Flush}()$ actions as follows: Let $C \in \{0, 1\}$ be a biased coin with $\text{Pr}[C = 1] = 2/3$. It samples $C$, and if the outcome is 1, then it continues to perform one $\text{Flush}()$ and sample another independent copy of $C$, until the outcome is 0. (In other words, the number of $\text{Flush}()$ is a geometric random variable with parameter $2/3$.)

**Put-Back:** This action moves a block from the queue, if any, to the root of $\text{Tr}$. Specifically, we first invoke a $\text{PopFront}()$. If $\text{PopFront}()$ returns a block $b$ then add it to $\lambda$.

**Flush:** This procedure selects a random path (namely, the path connecting the root to a random leaf $p^* \leftarrow \{0, 1\}^d$) on the tree and tries to move the blocks along the path down subject to the condition that the block always finds themselves on the appropriate path from the root to their assigned leaf node (see the block-path invariance condition). Let $p_0(= \lambda)p_1...p_d$ be the nodes along $\text{path} (\lambda, p^*)$. We traverse the path while carrying out the following operations for each node $p_i$ we visit: in node $p_i$, find the block that can be “pulled-down” as far as possible along the path $\text{path} (\lambda, p^*)$ (subject to the block-path invariance condition), and pull it down to $p_{i+1}$. For $i < d$, if there exists some $\eta \in \{0, 1\}$ such that $p_i$ contains more than $\ell/2$ blocks that are assigned to leaves of the form $p_i[\lceil \eta \rceil ..]$, then select an arbitrary such block $b$, remove it from the bucket $p_i$ and invokes an $\text{Overflow}(b)$ procedure, which re-samples a uniformly random position for the overflowed block $b$ and inserts it back to the queue $Q$. (See Figure 1 and 2 in Appendix for the pseudocode)

Finally, the algorithm aborts and terminates if one of the following two events happen throughout the execution.

**Abort-queue:** If the size of the queue $Q$ reaches $q_{\text{max}}$, then the algorithm aborts and outputs $\text{AbortQueue}$.

**Abort-leaf:** If the size of any leaf bucket reaches $\ell'$ (i.e., it becomes full), then the algorithm aborts and outputs $\text{AbortLeaf}$.

This completes the description of our $\text{Read}(r)$ algorithm; the $\text{Write}(r, v)$ algorithm is defined identically the same way, except that instead of inserting $b$ into the queue $Q$ (in the last step of $\text{Fetch}$), we insert a modified $b'$ where the content of the memory cell $r$ (inside $b$) has been updated to $v$.

It follows by inspection that the block-path invariance is preserved by our construction. Also, note that in the above algorithm, FETCH increases the size of the queue $Q$ by 1 and $\text{Put-Back}$ is executed twice which decreases the queue size by 2. On the other hand, the $\text{Flush}$ action may cause a few $\text{Overflow}$ events, and when an $\text{Overflow}$ occurs, one block will be removed from $\text{Tr}$ and inserted to $Q$. Therefore, the size of the queue changes by minus one plus the number of $\text{Overflow}$ for each $\text{Read}$ operation. The crux of our analysis is to show that the number of $\text{Overflow}$ is sufficiently small in any given (short) period of time, except with negligible probability.

We remark that throughout this algorithm’s life cycle, there will be at most $\ell - 2$ non-empty blocks in each internal node except when we invoke $\text{Flush}()$, in which case some intermediate states will have $\ell - 1$ blocks in a bucket (which causes an invocation of $\text{Overflow}$).
Reducing the cache’s size. We now briefly describe how the cache can be reduced to \(\text{poly log}(n)\). We will set the queue size \(q_{\text{max}} = \text{poly log}(n)\) (specifically, we can set \(q_{\text{max}} = O(\log^{2+\varepsilon} n)\) for an arbitrarily small constant \(\varepsilon\)). The key observation here is that the position map shares the same set of interfaces with our ORAM data structure. Thus, we may substitute the position map with a (smaller) ORAM of size \([n/\alpha]\). By recursively substituting the position map \(O(\log n)\) times, the size of the cache will reduce to \(\text{poly log } n\).

Efficiency and setting parameters. By inspection, it is not hard to see that the runtime of our READ and WRITE algorithms is \(O(\ell \log^2 n + \ell' \log n)\). Also, note that the position map of the base construction has size \(O((\ell + \ell') \cdot L) = O((\ell + \ell') \cdot (n/\alpha)/(\log n \log \log n))\), and each recursive level has a position map that is a constant factor smaller. Thus, the overall external memory required by our ORAM construction remains \(O((\ell + \ell') \cdot (n/\alpha)/(\log n \log \log n))\). To achieve the claims efficiency in Theorem 1, we set \(\ell = O(\log \log n)\) and \(\ell' = O(\log n \log \log n)\).

4 Security of our ORAM

The following observation is central to the security of our ORAM construction (and an appropriate analogue of it was central already to the constructions of [22, 4]):

Key observation: Let \(X\) denote the sum of two independent geometric random variables with mean 2. Each Read and Write operation traverses the tree along \(X + 1\) randomly chosen paths, independent of the history of operations so far.

The key observation follows from the facts that (1) just as in the schemes of [22, 4], each position in the position map is used exactly once in a traversal (and before this traversal, no information about the position is used in determining what nodes to traverse), and (2) we invokes the FLUSH action \(X\) times and the flushing, by definition, traverses a random path, independent of the history.

Armed with the key observation, the security of our construction reduces to show that our ORAM program does not aborts except with negligible probability, which follows by the following two lemmas.

Lemma 1. Given any program \(\Pi\), let \(\Pi'(n, x)\) be the compiled program using our ORAM construction. We have

\[
\Pr[\text{ABORTLEAF}] \leq \text{negl}(n).
\]

Lemma 2. Given any program \(\Pi\), let \(\Pi'(n, x)\) be the compiled program using our ORAM construction. We have

\[
\Pr[\text{ABORTQUEUE}] \leq \text{negl}(n).
\]

The proof of Lemma 1 is found in the Appendix and follows by a direct application of the (multiplicative) Chernoff bound. The proof of Lemma 2 is significantly more interesting. Towards proving it, in Section 5 we consider a simple variant of a “supermarket” problem (introduced by Mitzenmacher[16]) and show how to reduce Lemma 2 to an (in our eyes) basic and natural question that seems not to have been investigated before.

5 Proof of Lemma 2

We here prove Lemma 2: in Section 5.1 we consider a notion of “upset” customers in a supermarket problem [16, 26, 6]; in Section 5.2 we show how Lemma 2 reduced to obtaining a bound on the rate of upset customers, and in Section 5.3 we provide an upper bound on the rate of upset customers.
5.1 A Supermarket Problem

In a supermarket problem, there are $D$ cashiers in the supermarket, all of which have empty queues in the beginning of the day. At each time step $t$,

- With probability $\alpha < 1/2$, an arrival event happens, where a new customer arrives. The new customer chooses $d$ uniformly random cashiers and join the one with the shortest queue.
- Otherwise (i.e. with the remaining probability $1 - \alpha$), a serving event happens: a random cashier is chosen that “serves” the first customer in his queue and the queue size is reduced by one; if the queue is empty, then nothing happens.

We say that a customer is upset if he chooses a queue whose size exceeds some bound $\varphi$. We are interested in large deviation bounds on the number of upset customers for a given short time interval (say, of $O(D)$ or poly log($D$) time steps).

Supermarket problems are traditionally considered in the continuous time setting [16, 26, 6]. But there exists a standard connection between the continuous model and its discrete time counterpart: conditioned on the number of events is known, the continuous time model behaves in the same way as the discrete time counterpart (with parameters appropriately rescaled).

Most of the existing works [16, 26, 6] study only the stationary behavior of the processes, such as the expected waiting time and the maximum load among the queues over the time. Here, we are interested in large deviation bounds on a statistics over a short time interval; the configurations of different cashiers across the time is highly correlated.

For our purpose, we analyze only the simple special case where the number of choice $d = 1$; i.e. each new customer is put in a random queue.

We provide a large deviation bound for the number of upset customers in this setting.\footnote{It is not hard to see that with $D$ cashiers, probability parameter $\alpha$, and “upset” threshold $\varphi$, the expected number of upset customers is at most $(\alpha/(1 - \alpha))^\varphi \cdot t$ for any $T$ steps time interval.}

**Proposition 1.** For the (discrete-time) supermarket problem with $D$ cashier, one choice (i.e., $d = 1$), probability parameter $\alpha \in (0, 1/2)$, and upset threshold $\varphi \in \mathbb{N}$, for any $T$ steps time interval $[t + 1, t + T]$, let $F$ be the number of upset customers in this time interval. We have

$$\Pr [F \geq (1 + \delta) (\alpha/(1 - \alpha))^\varphi T] \leq \begin{cases} \exp \left\{-\Omega \left(\frac{\delta^2 (\alpha/(1-\alpha))^\varphi T}{(1-\alpha)^2}\right)\right\} & \text{for } 0 \leq \delta \leq 1 \\ \exp \left\{-\Omega \left(\frac{\delta (\alpha/(1-\alpha))^\varphi T}{(1-\alpha)^2}\right)\right\} & \text{for } \delta \geq 1 \end{cases}$$

(1)

Note that Proposition 1 would trivially follow from the standard Chernoff bound if $T$ is sufficiently large (i.e., $T \gg O(D)$) to guarantee that we individually get concentration on each of the $D$ queue (and then relying on the union bound). What makes Proposition 1 interesting is that it applies also in a setting when $T$ is poly log $D$.

The proof of Proposition 1 is found in Section 5.3 and relies on a new variant Chernoff bounds for Markov chains with “resets,” which may be of independent interest.

**Remark 1.** One can readily translate the above result to an analogous deviation bound on the number of upset customers for (not-too-short) time intervals in the continuous time model. This follows by noting that the number of events that happen in a time interval is highly concentrated (provided that the expected number of events is not too small), and applying the above proposition after conditioning on the number of events happen in the time interval (since conditioned on the number of events, the discrete-time and continuous-time processes are identical).
5.2 From ORAM to Supermarkets

This section shows how we may apply Proposition 1 to prove Lemma 2. Central to our analysis is a simple reduction from the execution of our ORAM algorithm at level \( k \) in \( \text{Tr} \) to a supermarket process with \( D = 2^{k+1} \) cashiers. More precisely, we show there exists a coupling between two processes so that each bucket corresponds with two cashiers; the load in a bucket is always upper bounded by the total number of customers in the two cashiers it corresponds to.

To begin, we need the following Lemma.

**Lemma 3.** Let \( \{a_i\}_{i \geq 1} \) be the sequence of \text{PUT-BACK}/\text{FLUSH} operations defined by our algorithm, i.e. each \( a_i \in \{\text{PUT-BACK}, \text{FLUSH}\} \) and between any consecutive \text{PUT-BACK}s, the number of \text{FLUSH}es is a geometric r.v. with parameter \( 2/3 \). Then \( \{a_i\}_{i \geq 1} \) is a sequence of i.i.d. random variables so that \( \Pr[a_i = \text{PUT-BACK}] = \frac{1}{3} \).

To prove Lemma 3, we may view the generation of \( \{a_i\}_{i \geq 1} \) as generating a sequence of i.i.d. Bernoulli r.v. \( \{b_i\}_{i \geq 1} \) with parameter \( \frac{2}{3} \). We set \( a_i \) be a \text{FLUSH} if and only if \( b_i = 1 \). One can verify that the \( \{a_i\}_{i \geq 1} \) generated in this way is the same as those generated by the algorithm.

We are now ready to describe our coupling between the original process and the supermarket process. At a high-level, a block corresponds to a customer, and \( 2^{k+1} \) sub-trees in level \( k + 1 \) of \( \text{Tr} \) corresponds to \( D = 2^{k+1} \) cashiers. More specifically, we couple the configurations at the \( k \)-th level of \( \text{Tr} \) in the ORAM program with a supermarket process as follows.

- Initially, all cashiers have 0 customer.
- For each \text{PUT-BACK}(), a corresponding arrival event occurs: if a ball \( b \) with position \( p = (\gamma, |\eta|) \) (where \( \gamma \in \{0, 1\}^{k+1} \) is moved to \( \text{Tr} \), then a new customer arrives at the \( \gamma \)-th cashier; otherwise (e.g. when the queue is empty), a new customer arrives at a random cashier.
- For each \text{FLUSH}() along the path to leaf \( \rho^* = (\gamma, |\eta|) \) (where \( \gamma \in \{0, 1\}^{k+1} \)), a serving event occurs at the \( \gamma \)-th cashier.
- For each \text{FETCH}(), nothing happens in the experiment of the supermarket problem. (Intuitively, \text{FETCH}() translates to extra “deletion” events of customers in the supermarket problem, but we ignore it in the coupling since it only decreases the number of blocks in buckets in \( \text{Tr} \).)

**Correctness of the coupling.** We shall verify the above way of placing and serving customers exactly gives us a supermarket process. First recall that both \text{PUT-BACK} and \text{FLUSH} actions are associated with uniformly random leaves. Thus, this corresponds to that at each timestep a random cashier will be chosen.

Next by Lemma 3, the sequence of \text{PUT-BACK} and \text{FLUSH} actions in the execution of our ORAM algorithm is a sequence of i.i.d. variables with \( \Pr[\text{PUT-BACK}] = \frac{1}{3} \). Therefore, when a queue is chosen at a new timestep, an (independent) biased coin is tossed to decide whether an arrival or a service event will occur.

**Dominance.** Now, we claim that at any timestep, for every \( \gamma \in \{0, 1\}^{k+1} \), the number of customers at the \( \gamma \)-th cashier is at least the number of blocks stored at or above level \( k \) in \( \text{Tr} \) with position \( p = (\gamma, |\cdot|) \). This follows by observing that (i) whenever there is a block with position \( p = (\gamma, |\cdot|) \) moved to \( \text{Tr} \) (from \text{PUT-BACK}()), a corresponding new customer arrives at the \( \gamma \)-th cashier, i.e. when the number of blocks increase by one, so does the number of customers, and (ii) for every \text{FLUSH}() along the path to \( \rho^* = (\gamma, |\cdot|) \): if there is at least one block stored at or above level \( k \) in \( \text{Tr} \) with position \( p = (\gamma, |\cdot|) \), then one such block will be flushed down below level \( k \) (since we flush the blocks that can be pulled down the furthest)—that is, when the number of customers decreases by one, so does the number of blocks (if possible). This in particular implies that throughout the coupled experiments, for every \( \gamma \in \{0, 1\}^k \) the number of blocks in the bucket at node \( \gamma \) is always upper bounded by the sum of the number of customers at cashier \( \gamma 0 \) and \( \gamma 1 \).

We summarize the above in the following lemma.

**Lemma 4.** For every execution of our ORAM algorithm (i.e., any sequence of \text{READ} and \text{WRITE} operations), there is a coupled experiment of the supermarket problem such that throughout the coupled experiments, for
every $\gamma \in \{0, 1\}^k$, the number of blocks in the bucket at node $\gamma$ is always upper bounded by the sum of the number of customers at cashier $\gamma 0$ and $\gamma 1$.

**From Lemma 4 and Proposition 1 to Lemma 2.** Note that at any time step $t$, if the queue size is $\leq \frac{1}{3} \log^{2+\epsilon} n$, then by Proposition 1 with $\varphi = \ell/2 = O(\log \log n)$ and Lemma 4, except with negligible probability, at time step $t + \log^3 n$, there have been at most $\omega(\log n)$ overflows per level in the tree and thus at most $\frac{1}{2} \log^{2+\epsilon} n$ in total. Yet during this time “epoch”, $\log^3 n$ element have been “popped” from the queue, so, except with negligible probability, the queue size cannot exceed $\frac{1}{2} \log^{2+\epsilon} n$.

It follows by a union bound over $\log^3 n$ length time “epochs”, that except with negligible probability, the queue size never exceeds $\log^{2+\epsilon} n$.

### 5.3 Analysis of the Supermarket Problem

We now prove Proposition 1. We start with interpreting the dynamics in our process as evolutions of a Markov chain.

**A Markov Chain Interpretation.** In our problem, at each time step $t$, a random cashier is chosen and either an arrival or a serving event happens at that cashier (with probability $\alpha$ and $(1 - \alpha)$, respectively), which increases or decreases the queue size by one. Thus, the behavior of each queue is governed by a simple Markov chain $M$ with state space being the size of the queue (which can also be viewed as a drifted random walk on a one dimensional finite-length lattice). More precisely, each state $i > 0$ of $M$ transits to state $i + 1$ and $i - 1$ with probability $\alpha$ and $(1 - \alpha)$, respectively, and for state 0, it transits to state 1 and stay at state 0 with probability $\alpha$ and $(1 - \alpha)$, respectively. In other words, the supermarket process can be rephrased as having $D$ copies of Markov chains $M_i$, each of which starts from state 0, and at each time step, one random chain is selected and takes a move.

We shall use Chernoff bounds for Markov chains [8, 13, 15, 3] to derive a large deviation bound on the number of upset customers. Roughly speaking, Chernoff bounds for Markov chains assert that for a (sufficiently long) $T$-steps random walk on an ergodic finite state Markov chain $M$, the number of times that the walk visits a subset $V$ of states is highly concentrated at its expected value $\pi(V) \cdot T$, provided that the chain $M$ has spectral expansion $\lambda(M)$ bounded away from 1. However, there are a few technical issues, which we address in turn below.

**Overcounting.** The first issue is that counting the number of visits to a state set $V \subset S$ does not capture the number of upset customers exactly—the number of upset customers corresponds to the number of transits from state $i$ to $i + 1$ with $i + 1 \geq \varphi$. Unfortunately, we are not aware of Chernoff bounds for counting the number of transits (or visits to an edge set). Nevertheless, for our purpose, we can set $V_\varphi = \{i : i \geq \varphi\}$ and the number of visits to $V_\varphi$ provides an upper bound on the number of upset customers.

**Truncating the chain.** The second (standard) issue is that the chain $M$ for each queue of a cashier has infinite state space $\{0\} \cup \mathbb{N}$, whereas Chernoff bounds for Markov chains are only proven for finite-state Markov chains. However, since we are only interested in the supermarket process with finite time steps, we can simply truncate the chain $M$ at a sufficiently large $K$ (say, $K \gg t + T$) to obtain a chain $M_K$ with finite states $S_K = \{0, 1, \ldots, K\}$; that is, $M_K$ is identical to $M$, except that for state $K$, it stays at $K$ with probability $\alpha$ and transits to $K - 1$ with probability $1 - \alpha$. Clearly, as we only consider $t + T$ time steps, the truncated chain $M_K$ behaves identical to $M$. It’s also not hard to show that $M_K$ has stationary distribution $\pi_K$ with $\pi_K(i) = \alpha^i / (1 - \alpha^{K+1})$, and spectral gap $1 - \lambda(M_K) \geq \Omega(1/(1 - \alpha)^2)$.

**Correlation over a short time frame.** The main challenge, however, is to establish large deviation bounds for a short time interval $T$ (compared to the number $D$ of chains). For example, $T = O(D)$ or even

---

7. For an ergodic reversible Markov chain $M$, the spectral expansion $\lambda(M)$ of $M$ is simply the second largest eigenvalue (in absolute value) of the transition matrix of $M$. The quantity $1 - \lambda(M)$ is often referred to as the spectral gap of $M$.

8. One can see this by lower bounding the conductance of $M_K$ and applying Cheeger’s inequality.
poly log(D), and in these cases the expected number of steps each of the D chains take can be a small constant or even o(1). Therefore, we cannot hope to obtain meaningful concentration bounds individually for each single chain. Finally, the D chains are not completely independent: only one chain is selected at each time step. This further introduces correlation among the chains.

We address this issue by relying on a new variant of Chernoff bounds for Markov chains with “resets,” which allows us to “glue” walks on D separate chains together and yields a concentration bound that is as good as a T-step random walk on a single chain. We proceed in the following steps.

- Recall that we have D copies of truncated chains MK starting from state 0. At each time step, a random chain is selected and we take one step in this chain. We want to upper bound the total number of visits to Vφ during time steps [t + 1, t + T].
- We first note that, as we are interested in upper bounds, we can assume that the chains start at the stationary distribution πK instead of the 0 state (i.e., all queues have initial size drawn from πK instead of being empty). This follows by noting that starting from πK can only increase the queue size throughout the whole process for all of D queues, compared to starting from empty queues, and thus the number of visits to Vφ can only increase when starting from πK in compared to starting from state 0 (this can be formalized using a standard coupling argument).
- Since walks from the stationary distribution remain at the stationary distribution, we can assume w.l.o.g. that the time interval is [1, T]. Now, as a thought experiment, we can decompose the process as follows. We first determine the number of steps each of the D chains take during time interval [1, T]; let c_j denote the number of steps taken in the j-th chain. Then we take c_j steps of random walk from the stationary distribution πK for each copy of the chain MK, and count the total number of visit to Vφ.
- Finally, we can view the process as taking a T-step random walk on MK with “resets.” Namely, we start from the stationary distribution πK, take c_1 steps in MK, ”reset” the distribution to stationary distribution (by drawing an independent sample from πK) and take c_2 more steps, and so on. At the end, we count the number of visits to Vφ, denoted by X, as an upper bound on the number of upset customers.

Intuitively, taking a random walk with resets injects additional randomness to the walk and thus we should expect at least as good concentration results. We formalize this intuition as the following Chernoff bound for Markov chains with ”resets”—the proof of which follows relatively easy from recent Chernoff bounds for Markov chains [3] and is found in Appendix B.2—and use it to finish the proof of Proposition 1.

**Theorem 2** (Chernoff Bounds for Markov Chains with Resets). Let M be an ergodic finite Markov chain with state space S, stationary distribution π, and spectral expansion λ. Let V ⊂ S and µ = π(V). Let T, D ∈ N and 1 = T₀ ≤ T₁ ≤ · · · ≤ TD < TD+1 = T + 1. Let (W₁, . . . , WT) denote a T-step random walk on M from stationary with resets at steps T₁, . . . , TD: that is, for every j ∈ {0, . . . , D}, W₁ ← π and WTD+1,...,WD+1 are random walks from WD. Let Xi = 1 iff Wi ∈ V for every i ∈ [T] and X = ∑T i=1 Xi. We have

\[
Pr[X ≥ (1 + δ)µT] ≤ \begin{cases} \exp\{-Ω(δ²(1 − λ)µT)\} & \text{for } 0 ≤ δ ≤ 1 \\ \exp\{-Ω(δ(1 − λ)µT)\} & \text{for } δ > 1 \end{cases}
\]

Now, recall that 1 − λ(MK) = Ω(1/(1 − α)^2) and πK(φ) = βφ/(1 − βK+1) = (α/1 − α)φ/(1 − βK+1). Theorem 2 says that for every possible c₁, . . . , cD (corresponding to resetting time T_j = ∑j i=1 c_i + 1),

\[
Pr\left[X ≥ \frac{(1 + δ)(α/1 − α)^φT}{(1 − βK+1)} \mid c₁, . . . , cD\right] ≤ \begin{cases} \exp\{-Ω\left(\frac{δ²(α/1 − α)^φT}{(1 − α)^²(1 − βK+1)}\right)\} & \text{for } 0 ≤ δ ≤ 1 \\ \exp\{-Ω\left(\frac{δ(α/1 − α)^φT}{(1 − α)^²(1 − βK+1)}\right)\} & \text{for } δ ≥ 1 \end{cases}
\]

Since X is an upper bound on the number of upset customers, and the above bound holds for every c₁, . . . , cD and for every K ≥ t + T, Proposition 1 follows by taking K → ∞. □
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A Implementation details.

This section discusses a number of implementation details in our algorithm.

The queue at the cache. We now describe how we may use a hash table and a standard queue (that could be encapsulated in commodity chips) to implement our queue. Here, we only assume the hash table uses universal hash function and it resolves collisions by using a linked-list. To implement the \textsc{Insert}(Block : \( b \)) procedure, we simply insert \( b \) to both the hash table and the queue. The key we use is \( b \)'s value at the position \( \text{map} \). Doing so we may make sure the maximum load of the hash table is \( O(\log n) \) whp \cite{Mitzenmacher99}. To implement \textsc{Find}(int : \( i \), word : \( p \)), we find the block \( b \) from the hash table. If it exists, return the block and delete it. Notice that we do not delete \( b \) at the queue. This introduces inconsistencies between the hash table and the queue.

We now describe how we implement \textsc{PopFront}(). Here, we need to be careful with the inconsistencies. We first pop a block from the queue. Then we need to check whether the block is in hash table. If not, that means the block was already deleted earlier. In this case, \textsc{PopFront}() will not return anything (because we need a hard bound on the running time). One can see that \textsc{Insert}() takes \( O(1) \) time and the other two operations take \( \omega(\log n) \) time whp.

The \textsc{Flush()} and \textsc{Overflow()} procedures. Figure 1 and Figure 2 also give pseudocode for the \textsc{Flush}() and \textsc{Overflow}() procedures.
B Missing Proofs

B.1 Proof of Lemma 1

Proof. We turn to showing that the probability of overflow in any of the leaf nodes is small. Consider any leaf node $\gamma$ and some time $t$. For there to be an overflow in $\gamma$ at time $t$, there must be $\ell' + 1$ out of $n/\alpha$ elements in the position map that map to $\gamma$. Recall that all positions in the position map are uniformly and independently selected; thus, the expected number of elements mapping to $\gamma$ is $\mu = \log n \log \log n$ and by a standard multiplicative version of Chernoff bound, the probability that $\ell' + 1$ elements are mapped to $\gamma$ is upper bounded by $2^{-\ell'}$ when $\ell' \geq 6\mu$ (see Theorem 4.4 in [?]). By a union bound, we have that the probability of any node ever overflowing is bounded by $2^{-(\ell')} \cdot (n/\alpha) \cdot T$.

To analyze the full-fledged construction, we simply apply the union bound to the failure probabilities of the $\log_{\alpha} n$ different ORAM trees (due to the recursive calls). The final upper bound on the overflow probability is thus $2^{-(\ell')} \cdot (n/\alpha) \cdot T \cdot \log_{\alpha} n$, which is negligible as long as $\ell' = c \log n \log \log n$ for a suitably large constant $c$.

B.2 Proof of Theorem 2

We here prove Theorem 2. The high level idea is simple—we simulate the resets by taking a sufficiently long “dummy” walk, where we “turn off” the counter on the number of visits to the state set $V$. However, formalizing this idea requires a more general version of Chernoff bounds that handles “time-dependent weight functions,” which allows us to turn on/off the counter. Additionally, as we need to add long dummy walks, a multiplicative version (as opposed to an additive version) Chernoff bound is needed to derive meaningful bounds. We here rely on a recent generalized version of Chernoff bounds for Markov chains due to Chung, Lam, Liu and Mitzenmacher [3].

**Theorem 3** ([3]). Let $M$ be an ergodic finite Markov chain with state space $S$, stationary distribution $\pi$, and spectral expansion $\lambda$. Let $W = (W_1, \ldots, W_T)$ denote a $T$-step random walk on $M$ starting from stationary distribution $\pi$, that is, $W_1 \leftarrow \pi$. For every $i \in [T]$, let $f_i : S \rightarrow [0, 1]$ be a weight function at step $i$ with expected weight $\mathbb{E}_{v \sim \pi}[f_i(v)] = \mu_i$. Let $\mu = \sum_i \mu_i$. Define the total weight of the walk $(W_1, \ldots, W_t)$ by $X \triangleq \sum_{i=1}^t f_i(W_i)$. Then

$$\Pr [X \geq (1 + \delta)\mu] \leq \begin{cases} \exp \{-\Omega(\delta^2(1 - \lambda)\mu)\} & \text{for } 0 \leq \delta \leq 1 \\ \exp \{-\Omega(\delta(1 - \lambda)\mu)\} & \text{for } \delta > 1 \end{cases}$$

We now proceed to prove Theorem 2.

**Proof of Theorem 2.** We use Theorem 3 to prove the theorem. Let $f : S \rightarrow [0, 1]$ be an indicator function on $V \subset S$ (i.e., $f(s) = 1$ iff $s \in V$). The key component from Theorem 3 we need to leverage here is that the functions $f_i$ can change over the time. Here, we shall design a very long walk $W$ on $M$ so that the marginal distribution of a specific collections of “subwalks” from $W$ will be statistically close to $\pi$. Furthermore, we design $\{f_i\}_{i \geq 0}$ in such a way that those “unused” subwalks will have little impact to the statistics we are interested in. In this way, we can translate a deviation bound on $V$ to a deviation bound on $W$. Specifically, let $T(\epsilon)$ be the mixing time for $M$ (i.e. the number of steps needed for a walk to be $\epsilon$-close to the stationary distribution in statistical distance). Here, we let $\epsilon \triangleq \exp(-DT)$ (which is chosen in an arbitrary manner so long as it is sufficiently small). Given $1 = T_0 \preceq T_1 \preceq \cdots \preceq T_D < T_{D+1} = T + 1$, we define $W$ and $f_i$ as follows: $W$ will start from $\pi$ and take $T_1 - 2$ steps of walk. In the mean time, we shall set $f_i = f$ for all $i < T_1$. Then we “turn off” the function $f_i$ while letting $W$ keep walking for $T(\epsilon)$ more steps, i.e. we let $f_i = 0$ for all $T_1 \leq i \leq T_1 + T(\epsilon) - 1$. Intuitively, this means we let $W$ take a long walk until it becomes close to $\pi$ again. During this time, $f_i$ is turned off so that we do not keep track of any statistics. After that, we “turn on” the
function $f_i$ again for the next $T_2 - T_1$ steps (i.e. $f_i = f$ for all $T_1 + T(\epsilon) \leq i \leq T_2 + T(\epsilon) - 1$, followed by turning $f_i$ off for another $T(\epsilon)$ steps. We continue this “on-and-off” process until we walk through all $T_j$’s.

Let $\mathcal{V}'$ be the subwalks of $\mathcal{V}$ with non-zero $f_i$. One can see that the statistical distance between $\mathcal{V}'$ and $\mathcal{W}$ is $\text{poly}(D,T) \exp(-DT) \leq \exp(-T + o(T))$. Thus, for any $\theta$ we have

$$
\Pr\left[ \sum_{w \in \mathcal{W}} f(w) \geq \theta \right] \leq \Pr\left[ \sum_{v' \in \mathcal{V'}} f(v') \geq \theta \right] + \exp(-T + o(T)) = \Pr\left[ \sum_{v \in \mathcal{V}} f(v) \geq \theta \right] + \exp(-T + o(T)).
$$

(2)

By letting $\theta = (1 + \delta) \mu T$ and using Theorem 3 to the right hand side of (2), we finish our proof. \qed

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**Figure Description:**

- **Figure Content:**
  - **Title:** Side effect length
  - **Graph:** A simple line graph showing the relationship between side effect length and time.
  - **Legend:** The legend explains the color coding for different data sets.

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**Table Description:**

- **Table Content:**
  - **Table Title:** Comparison of Side Effect Rates
  - **Columns:** Side Effect Type, Frequency, Severity
  - **Rows:** Different categories of side effects with respective frequency and severity.
**FLUSH**(Tr)

1. Let $p^{*}$ be a uniformly random leaf.
2. Denote path($\lambda, p^{*}$) as $p_0p_1...p_d$ (with $p_0 = \lambda$ and $p_d = p^{*}$).
3. block ← null.
4. for $i ← 0$ to $d - 1$
   
   do
5. Read the buckets $\vec{b}$ at node $p_i$ to the client side.
6. if block $\neq$ null
    
    then Insert block to $\vec{b}$ by replacing it with a dummy block in $\vec{b}$.
8. Find a block $b[j] \in \vec{b}$ such that $p_{i+1} \in$ path($r, P[b[j]]$).
9. $\triangleright$ If there are more than one such blocks,
10. $\triangleright$ find the one that can travel furtherest.
11. if Can find such $b[j]$
    
    then block ← $b[j]$
    
    Replace $b[j]$ by a dummy block
12. else block ← null.
14. Let $S_L, S_R \subset \vec{b}$ be set of balls belong to left and right sub-trees.
16. if $|S_L| \geq \ell/2$
    
    then select any $b \in S_L$ and replace it by a dummy ball.
18. OVERFLOW($b$)
20. if $|S_R| \geq \ell/2$
    
    then select any $b \in S_R$ and replace it by a dummy ball.
22. OVERFLOW($b$)
24. Write back $\vec{b}$ to $p_i$.
26. Read the blocks $\vec{b}$ at $p_d$.
28. if block $\neq$ null
    
    then Insert block to $\vec{b}$ by replacing it with a dummy block in $\vec{b}$.
30. if $\vec{b}$ is full
    
    then abort the program.
32. Write back $\vec{b}$ to $p_d$.

**Figure 1:** Pseudocode for the FLUSH(·) action

**OVERFLOW**(block : b)

1. Update $P[i]$ to a uniformly random value from $[L]$, where $i$ is the index of $b$.
2. Update the position $p = P[i]$ in $b$.
3. Insert $b$ to the queue.

**Figure 2:** Pseudocode for the OVERFLOW procedure