Largest Schmidt eigenvalue of random pure states and conductance distribution in chaotic cavities

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Abstract. A strategy to evaluate the distribution of the largest Schmidt eigenvalue for random pure states of bipartite systems is proposed. We point out that the multiple integral defining the sought quantity for a bipartition of sizes $N, M$ is formally identical (upon simple algebraic manipulations) to the one providing the probability density of Landauer conductance in open chaotic cavities supporting $N$ and $M$ electronic channels in the two leads. Known results about the latter can then be straightforwardly employed in the former problem for systems with both broken ($\beta = 2$) and preserved ($\beta = 1$) time-reversal symmetry. The analytical results, yielding a continuous but not everywhere analytic distribution, are in excellent agreement with numerical simulations.

Keywords: rigorous results in statistical mechanics, mesoscopic systems (theory), quantum dots (theory), random matrix theory and extensions
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1. Introduction

Consider two sets of $n$ correlated random variables in $[0, 1]$, $\{\lambda_i\}$ and $\{T_i\}$ ($i = 1, \ldots, n$), respectively distributed according to the following joint probability densities (jpd):

$$P_1(\lambda_1, \ldots, \lambda_n) = C_{n,\alpha}^{(\beta)} \delta \left( \sum_{i=1}^{n} \lambda_i - 1 \right) \prod_{i=1}^{n} \lambda_i^\alpha \prod_{j<k} |\lambda_j - \lambda_k|^\beta \quad (1)$$

$$P_2(T_1, \ldots, T_n) = K_{n,\alpha'}^{(\beta)} \prod_{i=1}^{n} T_i^\alpha' \prod_{j<k} |T_j - T_k|^\beta \quad (2)$$

where $C_{n,\alpha}^{(\beta)}$ and $K_{n,\alpha'}^{(\beta)}$ are known normalization constants and $\beta = 1, 2$. In sections 1.1 and 1.2 we will provide physical motivations for considering such sets, namely set 1 corresponds to the distribution of Schmidt eigenvalues for random pure states in bipartite systems (see section 1.1), while set 2 corresponds to the distribution of transmission eigenvalues of an open cavity in the chaotic regime (see section 1.2).

Consider now the following statistical quantities.

- The cumulative distribution $Q_n(x) = \text{Prob}[\lambda_{\text{max}} \leq x]$ of the largest member of set 1, $\lambda_{\text{max}} = \max_i \{\lambda_i\}$. By definition, this is given by the following $n$-fold integral:

$$Q_n(x) = \int_{[0,x]^n} d\lambda_1 \cdots d\lambda_n P_1(\lambda_1, \ldots, \lambda_n). \quad (3)$$

Differentiating $Q_n(x)$, one obtains the probability density of $\lambda_{\text{max}}$,

$$p_n(x) = \frac{d}{dx} Q_n(x). \quad (4)$$

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• The probability density \( P_G(y) = \text{Prob}[y \leq G \leq y + dy] \) of the quantity \( G = \sum_{i=1}^{n} T_i \), which is given by the following \( n \)-fold integral:

\[
P_G(y) = \int_{[0,1]^n} dT_1 \cdots dT_n P_2(T_1, \ldots, T_n) \delta \left( y - \sum_{i=1}^{n} T_i \right).
\]\( \tag{5} \)

Simple algebraic manipulations, summarized in appendix A, lead to the following relation between the two quantities above:

\[
Q_n(x) = \frac{C_n^{(\beta)}}{K_n^{(\beta)}} x^{n+\alpha n+\beta(n-1)\frac{1}{2}} P_G \left( \frac{1}{x} \right), \quad 1/n \leq x \leq 1, \quad \alpha = \alpha'.
\]\( \tag{6} \)

The identity in equation (6) is the main result of this paper. Notwithstanding its remarkable simplicity, equation (6) actually permits an exact evaluation of \( Q_n(x) \), the so far unavailable distribution of the largest Schmidt eigenvalue for finite \( n \) (see section 1.1), in terms of \( P_G(y) \) (the probability density of Landauer conductance, see section 1.2) about which much more is known.

The plan of the paper is as follows. In sections 1.1 and 1.2, we give a rather detailed introduction to the physics of random pure states in bipartite systems (related to the set 1 above) and to Landauer conductance in chaotic mesoscopic cavities supporting a finite number of electronic channels in the two attached leads (related to the set 2 above). In section 2 we exploit the identity (6) to derive analytically the cumulative distribution and the density of \( \lambda_{\text{max}} \) in a few illustrative cases. These results are then compared with numerical simulations with excellent agreement. Eventually we present concluding remarks in section 3 and technical developments in the three appendices.

1.1. Random pure states in bipartite systems

Entanglement of pure bipartite systems is an interesting area of research nowadays, due to possible applications to quantum information and quantum computation problems [1,2]. It is also probably the simplest setting where well-behaved entanglement quantifiers can be defined, such as the von Neumann or Rényi entropies of either subsystem [2], the so-called concurrence for two-qubit systems [3] or other entanglement monotones [4,5].

Typical properties of such states are best addressed by considering random pure states (see e.g. [6] for an excellent review). More precisely, consider a bipartition of an \( NM \)-dimensional Hilbert space \( \mathcal{H}^{(NM)} \) as \( \mathcal{H}^{(NM)} = \mathcal{H}_A^{(N)} \otimes \mathcal{H}_B^{(M)} \), where we assume without loss of generality that \( N \leq M \). For example, \( A \) may be taken as a given system (say a set of spins) living in an external environment (e.g., a heat bath) \( B \). A quantum state \( |\psi\rangle \) of the composite system can be expanded as a linear combination

\[
|\psi\rangle = \sum_{i=1}^{N} \sum_{\alpha=1}^{M} x_{i,\alpha} |i^A\rangle \otimes |\alpha^B\rangle
\]\( \tag{7} \)

where \( |i^A\rangle \) and \( |\alpha^B\rangle \) are two complete bases of \( \mathcal{H}_A^{(N)} \) and \( \mathcal{H}_B^{(M)} \) respectively. The coefficients \( x_{i,\alpha} \) of this expansion form the entries of a rectangular \( (N \times M) \) matrix \( \mathbf{X} \).

We consider here random pure states \( |\psi\rangle \). This means that:

1. One has the following normalizations: \( \int_{1/n}^{1} dx p_n(x) = \int_{0}^{1} dy P_G(y) = 1 \).

2. The bound \( x \geq 1/n \) will be discussed in detail later on.

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(1) the expansion coefficients $x_{i,\alpha}$ are random variables drawn from a certain probability distribution,
(2) the density matrix of the composite system is simply given by $\rho = |\psi\rangle\langle \psi |$ with the constraint $\text{Tr}[\rho] = 1$, or equivalently $\langle \psi | \psi \rangle = 1$.

More precisely, the density matrix of $|\psi\rangle$ can then be straightforwardly expressed as
$$\rho = \sum_{i,\alpha} \sum_{j,\beta} x_{i,\alpha}^* x_{j,\beta} |i^A\rangle\langle j^A | \otimes |\alpha^B\rangle\langle \beta^B |$$
(8)
where the Roman indices $i$ and $j$ run from 1 to $N$ and the Greek indices $\alpha$ and $\beta$ run from 1 to $M$.

The reduced density matrix $\rho_A = \text{Tr}_B[\rho]$ is obtained by tracing out the environmental degrees of freedom (i.e. those of the subsystem $B$):
$$\rho_A = \text{Tr}_B[\rho] = \sum_{\alpha=1}^{M} \langle \alpha^B | \rho | \alpha^B \rangle.$$ (9)

Using the expansion in equation (8) one gets
$$\rho_A = \sum_{i,j=1}^{N} \sum_{\alpha=1}^{M} x_{i,\alpha} x_{j,\alpha}^* |i^A\rangle\langle j^A | = \sum_{i,j=1}^{N} W_{ij} |i^A\rangle\langle j^A |$$ (10)
where the $W_{ij}$s are the entries of the $N \times N$ matrix $W = \mathcal{X} \mathcal{X}^\dagger$, with $\mathcal{X} = (x_{i,\alpha})$. In a similar way, one could obtain the reduced density matrix $\rho_B = \text{Tr}_A[\rho]$ of the subsystem $B$ in terms of the $M \times M$ matrix $W' = \mathcal{X}' \mathcal{X}'$.

In the diagonal basis, one can express $\rho_A$ as
$$\rho_A = \sum_{i=1}^{N} \lambda_i |\lambda_i^A\rangle\langle \lambda_i^A |$$ (11)
where the $|\lambda_i^A\rangle$s are the normalized eigenvectors of $W = \mathcal{X} \mathcal{X}^\dagger$ and similarly for $\rho_B$, in terms of the same set of eigenvalues $\{ \lambda_i \}$ (the two matrices $W$ and $W'$ indeed share the same set of nonzero and positive real eigenvalues). The original composite state $|\psi\rangle$ in this diagonal basis reads
$$|\psi\rangle = \sum_{i=1}^{N} \sqrt{\lambda_i} |\lambda_i^A\rangle \otimes |\lambda_i^B\rangle.$$ (12)

Equation (12) is known as the Schmidt decomposition, and the normalization condition $\langle \psi | \psi \rangle = 1$, or equivalently $\text{Tr}[\rho] = 1$, imposes the constraint on the sum of eigenvalues, $\sum_{i=1}^{N} \lambda_i = 1$.

For random pure states, the expansion coefficients in equation (7) can be typically drawn from an unbiased (so-called Hilbert-Schmidt) distribution for the coefficient matrix $\mathcal{X}$, whose elements can be real or complex:
$$\text{Prob}[\mathcal{X}] \propto \delta \left( \text{Tr}(\mathcal{X} \mathcal{X}^\dagger) - 1 \right).$$ (13)
One can associate a Dyson index $\beta = 1, 2$ corresponding respectively to real and complex $X$ matrices\textsuperscript{3}. The meaning of equation (13) is clear: all normalized density matrices compatible with unitary invariance are sampled with equal probability, which corresponds to having minimal \textit{a priori} information about the quantum state under consideration. This in turn induces nontrivial correlations among the Schmidt eigenvalues (which are now real random variables between 0 and 1 whose sum is 1) and makes the investigation of several statistical quantities about such states quite interesting. Here we present a quick summary of known results:

- the joint probability density (jpd) of Schmidt eigenvalues, derived by Lloyd and Pagels [7], which is precisely given by $P_1(\lambda_1, \ldots, \lambda_N)$ in equation (1), with $n = N$ and $\alpha = (\beta/2)(N - M + 1) - 1$. Note that the delta function there guarantees that $\text{Tr}[\rho] = 1$. The normalization constant in this case reads [8]

$$C_{N, \alpha=(\beta/2)(N-M+1)-1}^{(\beta)} = \frac{\Gamma(MN\beta/2)(\Gamma(1+\beta/2))^N}{\prod_{j=0}^{N-1} \Gamma((M-j)\beta/2)\Gamma(1+(N-j)\beta/2)}; (14)$$

- the average von Neumann entropy for large $N, M$ (computed by Page [9] for $\beta = 2$ and extended in [10] to the case $\beta = 1$);
- the average von Neumann entropy for \textit{finite} $N, M$ and $\beta = 2$, conjectured by Page [9] and independently proven by many researchers soon after [11] also in a non-extensive setting [12];
- density of Schmidt eigenvalues (one-point function) for \textit{finite} $(N, M)$, derived independently in [13] and [14] for $\beta = 2$ and in [15] for $\beta = 1$;
- universality of eigenvalue correlations for $\beta = 2$ [16];
- distribution of so-called $G$-concurrence [4] for $\beta = 2$;
- distribution of so-called purity for \textit{small} $N$ [17], and phase transitions in its Laplace transform for \textit{large} $N$ [18];
- full distribution of Rényi entropies (including large deviation tails), computed in [19] for large $N = M$ and all $\beta$s using a Coulomb gas method. As a byproduct, the authors also obtain in [19] the average and variance of Rényi entropy valid for large $N = M$, and the density of Schmidt eigenvalues for all $\beta$s and $N = cM$ large;
- distribution of smallest eigenvalue (related to so-called \textit{Demmel condition number} [20]) for $\beta = 1, 2$ and finite $M = N$, derived independently in [21] and [22]. In the latter paper, a conjecture by Znidaric [23] was proven\textsuperscript{4} (see also [24] for an extension of these results to the case $N \neq M$);
- distribution of largest eigenvalue for \textit{large} $N = M$ and all $\beta$s [19], including small and large deviation laws. Typical fluctuations around the mean $\approx 4/N$ (once properly scaled) are found to follow the Tracy–Widom distribution (see also [25] for a related result). No results seem to be available for the case of \textit{finite} $N, M$.

\text{\textsuperscript{3} These two cases in turn correspond to quantum systems whose Hamiltonians preserve ($\beta = 1$) or break ($\beta = 2$) time-reversal symmetry.}

\text{\textsuperscript{4} For $\beta = 2$, $\langle \lambda_{\text{min}} \rangle = 1/N^3$ exactly for all $N = M$, while for $\beta = 1$, $\langle \lambda_{\text{min}} \rangle \sim c_1/N^3$ for large $N = M$, where the constant $c_1$ is precisely known [22].}
Given the current interest in the distribution of extreme Schmidt eigenvalues, the reader may on the one hand wonder whether they really encode useful information, and on the other why the largest eigenvalue distribution for finite \( N,M \) is much harder to obtain via the same strategy used for the smallest one [22].

In order to answer the first question, first note that due to the constraint \( \sum_{i=1}^{N} \lambda_i = 1 \) and the fact that all eigenvalues are non-negative, it follows that\(^5\) \( 1/N \leq \lambda_{\text{max}} \leq 1 \) \( \text{and} \quad 0 \leq \lambda_{\text{min}} \leq 1/N \). Now consider the following limiting situations. Suppose that the largest eigenvalue \( \lambda_{\text{max}} = \max_i \{ \lambda_i \} \) takes its maximum allowed value 1. Then it follows immediately that all the remaining \((N-1)\) eigenvalues must be identically 0. In this situation equation (12) tells us that \( |\psi\rangle \) is fully unentangled (completely separable). On the other hand, if \( \lambda_{\text{max}} = 1/N \) (i.e., it takes its lowest allowed value), all the eigenvalues must have the same value, \( \lambda_i = 1/N \) for all \( i \). In this case, the pure state \( |\psi\rangle \) is maximally entangled, as this state maximizes the von Neumann entropy \( S_{\text{VN}} = -\sum_{i=1}^{N} \lambda_i \ln \lambda_i = \ln(N) \). In other words, the knowledge of the largest eigenvalue distribution provides at least some partial information about how entangled a random pure state is.

A discussion about the asymmetry in the treatment of smallest and largest Schmidt eigenvalues is included in appendix B.

1.2. Landauer conductance in open cavities

Consider a cavity of submicron dimensions etched in a semiconductor and connected to the external world by two leads supporting \( M \) and \( N \) electronic channels. It is well established that the electrical current flowing through such a cavity when brought out of equilibrium by an applied external voltage presents time-dependent fluctuations which persist down to zero temperature [26] and are thus associated with the granularity of the electron charge \( e \). Typical features observed in experiments include weak localization [27], universality in conductance fluctuations [28] and constant Fano factor [29]. The Landauer–Büttiker scattering approach [26,30,31] is rather successful in describing the statistics of quantum transport: it amounts to relating the wavefunction coefficients of the incoming and outgoing electrons through the unitary scattering matrix \( S \): for example, the dimensionless conductance and the shot noise are given respectively by \( G = \text{Tr}(T) \) [30] and \( P = \text{Tr}[T(1-T)] \) [32].

Random matrix theory (RMT) has been very successful in describing the statistics of universal fluctuations in such systems, when the corresponding classical dynamics is chaotic: the scattering matrix \( S \) is drawn from a suitable ensemble of random matrices,\(^5\) This means that both the smallest and the largest eigenvalue distributions have compact supports and justifies the bound \( x \geq 1/n \) in equation (6).

\(^6\) \((t,t')\) are respectively of size \( N \times M \) and \( M \times N \), while \((r,r')\) are of size \( M \times M \) and \( N \times N \).
with the overall constraint of unitarity \[33\]–\[35\]. A maximum entropy approach (under the assumption of ballistic point contacts \[26\]) forces the probability distribution of \(S\) to be uniform within the unitary group, i.e. \(S\) belongs to one of Dyson’s circular ensembles \[36,37\].

From the uniformity of the probability density of \(S\) within the unitary group, the jpd of the transmission eigenvalues \(\{T_i\}\) of the matrix \(T\), from which the statistics of interesting experimental quantities could be in principle derived, is precisely given by equation (2), with \(n = \min(N, M)\) and \(\alpha' = (\beta/2)(|N - M| + 1) - 1\) \[26,35,38\]. There, the Dyson index \(\beta\) characterizes different symmetry classes \((\beta = 1, 2\) according to the presence or absence of time-reversal symmetry). The eigenvalues \(T_i\) are thus correlated real random variables between 0 and 1. The normalization constant \(K^{(\beta)}_{n,\alpha'}\) is explicitly known from the celebrated Selberg integral as

\[
K^{(\beta)}_{n,\alpha'} = \prod_{j=0}^{n-1} \frac{\Gamma(1 + \beta/2)\Gamma(2 + \alpha' + (\beta/2)(n + j - 1))}{\Gamma(1 + (\beta/2)(1 + j))\Gamma(1 + \alpha' + \beta j/2)\Gamma(1 + \beta j/2)}.
\]

From (2), in principle the statistics of many observables of interest can be calculated. In particular, we focus here on the dimensionless Landauer conductance \(G = \text{Tr}(T) = \sum_{i=1}^{n} T_i\), which satisfies the bounds \(0 \leq G \leq n\). The mean and variance of \(G\) and other quantities have been known for a long time when \(N, M\) are large \[26,39,40\], and recently also for finite \(N, M\) \[40\]–\[43\].

Stimulated by some recent experimental progress \[44\], which made it eventually possible to explore the full distribution of the conductance (and not just its mean and variance), a lot of effort has been devoted to its theoretical characterization. An explicit expression was first obtained for \(n = 1, 2\) \[45\]–\[47\], while more results were available in the case of quasi-one-dimensional wires \[48\] and 3D insulators \[49\]. For the shot noise, the full distribution was known only for \(N = M = 1\) \[50\]. Very recently, Sommers et al \[51\] announced two formulae for the distribution of conductance and shot noise, valid at arbitrary number of open channels and for any \(\beta\), which are based on Fourier expansions. In \[52\], a complete and systematic approach to the full statistics of conductance and shot noise was brought forward. This is based on symmetric function expansions, and is valid for \(\beta = 1, 2\) and arbitrary \(M, N\). In addition, the authors of \[52\] were able to provide general formulae in terms of determinants or Pfaffians for the full probability distribution of conductance and shot noise at quantized \(\beta\) and general \(N, M\) (see below). In \[53\], the integrable theory of quantum transport in chaotic cavities (based on Painlevé transcendent) for \(\beta = 2\) was formulated, and recursion formulae for the efficient computation of conductance and shot noise cumulants have been derived. In two recent publications \[54\], the distributions of conductance, shot noise and integer moments were computed in the limit \(N = M \gg 1\) using a Coulomb gas method, and long power-law tails were detected in the distributions\(^7\), a result confirmed by extensive numerical simulations. In \[55\], exact results (basically equivalent to the general formulae in \[52\]) for the Laplace transform \(P_G(s)\) of the conductance distribution for any \(N, M\) and all \(\beta\)s are formally given as follows:

\(^7\) A careful asymptotic analysis of the formulae in \[53\] leads to the same findings. In particular, there are no long exponential tails in the distributions, as originally claimed in \[53\].

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- \( \beta = 1 \):
  \[
  \hat{P}_G(s) \propto \text{Pf}[\Psi_{j,k}(s)] \quad \text{for } N \text{ even}
  \]
  and
  \[
  \hat{P}_G(s) \propto \text{Pf}\left[ \begin{array}{cc} \Psi_{j,k}(s) & \Phi_j(s) \\ -\Phi_k(s) & 0 \end{array} \right] \quad \text{for } N \text{ odd}
  \]
  where \( \text{Pf}[A] \) is the Pfaffian of the even-dimensional antisymmetric matrix \( A \) [36] and the proportionality constants are known explicitly. The arrays \( \Psi_{j,k}(s) \) and \( \Phi_j(s) \) are given by
  \[
  \Psi_{j,k}(s) = \int_0^1 \int_0^1 \text{sgn}(x - y)e^{-sx} e^{-sy} x^{\alpha'+j} y^{\alpha'+k} \, dx \, dy
  \]  
  (17)
  \[
  \Phi_j(s) = \int_0^1 e^{-sx} x^{\alpha'+j} \, dx.
  \]  
  (18)

- \( \beta = 2 \): in this case, we have a representation in terms of a Hankel determinant
  \[
  \hat{P}_G(s) \propto \det[\Psi_{j,k}(s)] \quad \text{for } N \text{ odd}
  \]  
  (19)
  where
  \[
  \Psi_{j,k}(s) = \int_0^1 e^{-sx} x^{\alpha'+j+k} \, dx
  \]  
  (20)
  that was first derived in [53].

Explicit inversions of the Laplace transforms above are always possible on a case-by-case basis, and a catalogue of such evaluations for a few interesting cases is provided in [55]. In section 2 and in appendix C we will combine such explicit formulae with the identity in equation (6) to illustrate the validity of our approach to the distribution of largest Schmidt eigenvalue.

2. **Two applications of the main identity**

For illustrative purposes, we consider two applications of the main identity equation (6). A third one is discussed in great detail in appendix C.

- \( N = 3, M = 4, \beta = 1 \): in this case, the probability density of Landauer conductance has been derived explicitly in [55] as
  \[
  P_C(y) = \frac{3}{8} y^5 - (y - 1)^3(y^2 - 12y + 51)\theta(y - 1) - (y - 2)^3
  \]
  \[
  \times (y^2 + 6y + 24)\theta(y - 2)\theta(3 - y)
  \]  
  (21)
  where \( \theta(x) \) is the Heaviside step function.
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Figure 1. Cumulative distribution $Q_n(x)$ of the largest Schmidt eigenvalue via equation (6) for $n = N = 3, M = 4, \beta = 1$ (violet) and $n = N = 3, M = 5, \beta = 1$ (blue). This distribution is identically zero for $x \leq 1/n$ and $Q_n(1) = 1$ as it should be.

$N = 3, M = 5, \beta = 1$: again, the probability density of Landauer conductance in this case has been derived explicitly in [55] as

$$P_G(y) = \begin{cases} \frac{20}{143}y^{13/2} & \text{for } 0 \leq y \leq 1 \\ \frac{5}{2588} & \left[3003y^5 - 21021y^4 + 55770y^3 - 70070y^2 + 42315y - 9933 - 32(y-2)^{7/2}(2y^3 + 14y^2 + 63y + 231)\times \theta(y-2) \right] & \text{for } 1 \leq y \leq 3. \end{cases} \tag{22}$$

Note that:

- since the conductance density has always a compact support $[0, n]$ (where $n = \min(N, M)$), it follows immediately from equation (6) that the cumulative distribution of the largest Schmidt eigenvalue $Q_n(x)$ and its density $p_n(x)$ have compact support $1/n \leq x \leq 1$, as expected;
- since the conductance density is known to be continuous but not everywhere analytic [51, 52] (i.e. it displays ‘critical’ points at which higher derivatives are discontinuous), the cumulative distribution and the density of the largest Schmidt eigenvalue enjoy this property too.

In figure 1 we plot the cumulative distributions $Q_n(x)$ corresponding to the two cases above. The curves are obtained via equation (6), where $P_G(y)$ is respectively given by (21) and (22). The distributions are increasing functions of the argument $x$, as they should be, and such that $Q_n(1) = 1$. Differentiation of the analytical formulae provides the density of the largest eigenvalue, $p_n(x)$, which is plotted in figures 2 and 3 along with numerical simulations. These are obtained as follows [8, 56].

1. We generate $\kappa \simeq 10^4, 10^5$ real Gaussian $M \times N$ matrices $\mathcal{X}$.
2. For each instance we construct the Wishart matrix $\mathcal{W} = \mathcal{X}^T \mathcal{X}$.
3. We diagonalize $\mathcal{W}$ and collect its $N$ real and non-negative eigenvalues $\{\lambda_1, \ldots, \lambda_N\}$.

**Figure 1.** Cumulative distribution $Q_n(x)$ of the largest Schmidt eigenvalue via equation (6) for $n = N = 3, M = 4, \beta = 1$ (violet) and $n = N = 3, M = 5, \beta = 1$ (blue). This distribution is identically zero for $x \leq 1/n$ and $Q_n(1) = 1$ as it should be.
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Figure 2. Density of the largest Schmidt eigenvalue $p_n(x)$ via equations (4) and (6) for $n = N = 3, M = 4, \beta = 1$ (solid black curve), along with numerical diagonalization of $\kappa = 8 \times 10^4$ samples (red dots), see main text for the algorithm.

Figure 3. Density of the largest Schmidt eigenvalue $p_n(x)$ via equations (4) and (6) for $n = N = 3, M = 5, \beta = 1$ (solid black curve), along with numerical diagonalization of $\kappa = 10^5$ samples (red dots), see main text for the algorithm.

(4) We define a new set of variables $0 \leq \lambda_i \leq 1$ as $\lambda_i = \tilde{\lambda}_i / \sum_{i=1}^{N} \tilde{\lambda}_i$, for $i = 1, \ldots, N$. The set of variables $\lambda_i$ is guaranteed to be sampled according to the measure (1).

(5) We construct a normalized histogram of $\lambda_{\text{max}} = \max_i \{\lambda_i\}$.

The agreement between theory and simulations is excellent.

We also evaluated exactly the average of $\langle \lambda_{\text{max}} \rangle = \int_{1/n}^{1} dx \, p_n(x)$ for the two cases above and found

\begin{align}
\langle \lambda_{\text{max}} \rangle_1 &= \frac{25}{36} \approx 0.694444\ldots \\
\langle \lambda_{\text{max}} \rangle_2 &= \frac{1}{810} (378 + 89\sqrt{3}) \approx 0.656978\ldots
\end{align}

However, a general formula for $\langle \lambda_{\text{max}} \rangle$ is still elusive (unlike $\langle \lambda_{\text{min}} \rangle$ [22]) and may well attract further researches (see also appendix C).

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3. Conclusions

We have presented an exact identity relating two statistical quantities which arise in different contexts: the cumulative distribution of the largest Schmidt eigenvalue for random pure states in bipartite systems of sizes $(M,N)$ and the probability density of Landauer conductance in chaotic cavities supporting $N$ and $M$ electronic channels in the two external leads. Recent analytical results for the latter are exploited to derive (so far unavailable) exact formulae for the former quantity at finite $N, M$ (while large $N, M$ results are already available [19]), which is of interest in order to quantify the degree of entanglement of random pure states. A detailed introduction to the physics involved has been provided, along with a precise discussion of the asymmetry in the treatment of the smallest and largest Schmidt eigenvalue distributions. A general formula for $\langle \lambda_{\text{max}} \rangle$, the average of the largest eigenvalue, valid for arbitrary $N, M$ is unfortunately still lacking and certainly deserves further investigations.

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Appendix A. Derivation of main identity

Consider the cumulative distribution $Q_n(x)$ of $\lambda_{\text{max}} = \max_i \{\lambda_i\}$ (equation (3)):

$$Q_n(x) = C_{n,\alpha}^{(\beta)} \int_{[0,x]^n} d\lambda_1 \cdots d\lambda_n \delta \left( \sum_{i=1}^n \lambda_i - 1 \right) \prod_{i=1}^n \lambda_i^\alpha \prod_{j<k} |\lambda_j - \lambda_k|^\beta. \quad (A.1)$$

A change of variables $\lambda_i = xT_i$ leads to

$$Q_n(x) = C_{n,\alpha}^{(\beta)} x^n + n\alpha + (\beta/2)n(n-1)$$

$$\times \int_{[0,1]^n} dT_1 \cdots dT_n \delta \left( x \sum_{i=1}^n T_i - 1 \right) \prod_{i=1}^n T_i^\alpha \prod_{j<k} |T_j - T_k|^\beta. \quad (A.2)$$

Using the property of delta functions $\delta(\gamma Y) = \delta(Y)/|\gamma|$, we obtain straightforwardly equation (6).

Appendix B. Asymmetry in the treatment of the smallest and largest Schmidt eigenvalues

Consider the cumulative distribution $\Theta_n(x)$ of the smallest member of set 1, $\lambda_{\text{min}} = \min_i \{\lambda_i\}$ (i.e. the distribution of the smallest Schmidt eigenvalue for random pure states). By definition it is given by

$$\Theta_n(x) = C_{n,\alpha}^{(\beta)} \int_{[x,\infty]^n} d\lambda_1 \cdots d\lambda_n \delta \left( \sum_{i=1}^n \lambda_i - 1 \right) \prod_{i=1}^n \lambda_i^\alpha \prod_{j<k} |\lambda_j - \lambda_k|^\beta. \quad (B.1)$$
where the upper limit of integration can be safely extended up to $\infty$ in view of the unit norm constraint. In the case $\alpha = 0$ and $\beta = 2$, which we focus on here for illustrative purposes, the evaluation of this multiple integral proceeds via the auxiliary function $\Theta_n(x, t)$ [22]:

$$\Theta_n(x, t) = C_{n,0}^{(2)} \int_{[x, \infty]^n} d\lambda_1 \cdots d\lambda_n \delta \left( \sum_{i=1}^n \lambda_i - t \right) \prod_{j<k} |\lambda_j - \lambda_k|^2$$

such that $\Theta_n(x) \equiv \Theta_n(x, 1)$. Next, one takes the Laplace transform of $\Theta_n(x, t)$:

$$\int_0^\infty dt \Theta_n(x, t)e^{-st} = C_{n,0}^{(2)} \int_{[x, \infty]^n} d\lambda_1 \cdots d\lambda_n e^{-s\sum_{i=1}^n \lambda_i} \prod_{j<k} |\lambda_j - \lambda_k|^2$$

and in the rhs performs a linear shift $y_i = s(\lambda_i - x)$ (this is the crucial technical step), to get

$$\int_0^\infty dt \Theta_n(x, t)e^{-st} = \frac{e^{-s N x}}{s^{N/2}} C_{n,0}^{(2)} \int_{[0, \infty]^n} dy_1 \cdots dy_n e^{-\sum_{i=1}^n y_i} \prod_{j<k} |y_j - y_k|^2.$$ 

Note that, thanks to the linear shift, the dependence on the Laplace variable $s$ has been entirely transferred outside the $n$-fold integral: this is now proportional to the partition function $Z_{WL}$ of an associated Wishart–Laguerre (WL) ensemble of random covariance matrices [36, 57] of the form $W = \mathcal{X}^\dagger \mathcal{X}$, where $\mathcal{X}$ is a Gaussian rectangular matrix with real or complex entries. The joint distribution of the $n$ non-negative eigenvalues of $W$ is known [58]:

$$P^{(WL)}(\lambda_1, \ldots, \lambda_n) = \mathcal{N}^{(\beta)}_{n,\alpha} e^{-\frac{s}{2} \sum_{i=1}^n \lambda_i} \prod_{i=1}^n \lambda_i^{\alpha} \prod_{j<k} |\lambda_j - \lambda_k|^\beta$$

where $\mathcal{N}^{(\beta)}_{n,\alpha}$ is a known normalization constant. Therefore, the jpd of Schmidt eigenvalues (1) can be seen as a fixed-trace (microcanonical) version of the Wishart–Laguerre (canonical) ensemble$^8$.

Now, in the case of the cumulative distribution of the largest eigenvalue $Q_n(x)$, the integrals on the rhs run over $[0, x]$ instead of $[x, \infty]$, making the aforementioned linear shift less useful. One could keep pursuing the Laplace transform route (introducing an auxiliary function $Q_n(x, t)$) with the change of variables $s\lambda_i = y_i$, obtaining

$$\int_0^\infty dt Q_n(x, t)e^{-st} = \frac{C_{n,0}^{(2)}}{s^{N/2}} \int_{[0, x]^n} dy_1 \cdots dy_n e^{-\sum_{i=1}^n y_i} \prod_{j<k} |y_j - y_k|^2$$

but the integral on the rhs does not permit this time a friendly Laplace inversion (see however appendix C). Indeed, this integral is readily recognized as proportional to $E_{WL}([sx, \infty])$, where $E_{WL}([a, b])$ is the gap probability for a Wishart–Laguerre ensemble,

$^8$ Note that the presence of a fixed-trace constraint has crucial consequences on the spectral properties of random matrix ensembles [59, 60].
i.e. the probability that the interval \((a, b)\) on the real axis is free of eigenvalues. This quantity is exactly known in terms of Painlevé V \([61]\), with the consequence that an explicit Laplace inversion formula is not available to date.

In summary, the asymmetry in the treatment of the smallest and largest Schmidt eigenvalues arises because the linear shift that works in the former case fails in the latter, and this fact calls for the alternative approach developed in this paper.

**Appendix C. A third application of main identity for \(\beta = 2 \text{ and } N = M\)**

In this appendix we discuss in more detail a third application of the main identity equation (6). In the case \(\beta = 2 \text{ and } M = N\), the Hankel determinant representation in equation (19) actually allows a more systematic (and more easily automatized) treatment of the cumulative distribution of the largest Schmidt eigenvalue. Following the Laplace transform route outlined in appendix B, one can easily write down the following equation:

\[
\int_0^\infty dt Q_n(x, t)e^{-st/x} = C_n(2)x^n \int_{[0,1]^n} dy_1 \cdots dy_n e^{-s\sum_{j<k}y_j} \prod_{j<k} |y_j - y_k|^2
\]

where we have used the Hankel determinant representation for the case \(n = 0\) provided in \([53]\) (fully equivalent to (19)), with \(F_1(s) = (1 - e^{-s})/s\).

A change of variable \(t = xt\) on the lhs, followed by a (formal) Laplace inversion, leads to the final formula for \(Q_n(x)\) in this case\(^9\):

\[
Q_n(x) = C_n(2)n!x^{n-1} L^{-1} \det[(-\partial_s)^{j+k} F_1(s)]_{j,k=0, \ldots, n-1} (1/x)
\]

where \(L^{-1}[f(s)](t)\) is the inverse Laplace transform of \(f(s)\) with parameter \(t\). The rhs of (C.2) can be systematically evaluated in Mathematica\(^\text{®}\) and its derivative \((p_n(x))\) has been plotted in figure C.1 for \(n = N = M = 3, 4\). The computational complexity of the determinant evaluation does not permit us, however, to go too high in \(n\) (\(n \simeq 20\) is likely to require already a few hours of computation on a standard laptop).

Also the average of the largest eigenvalue \(\langle \lambda_{\text{max}} \rangle = \int_0^1 dx x p_n(x)\) can be evaluated exactly in Mathematica\(^\text{®}\) for a given \(n\), and we provide a few evaluations in the following table.

| \(N = M\) | \(\langle \lambda_{\text{max}} \rangle \) (exact) | \(\langle \lambda_{\text{max}} \rangle \) (approx.) |
|---|---|---|
| 2 | \(\frac{5}{8}\) | 0.875 |
| 3 | \(\frac{313}{432}\) | 0.724537 |
| 4 | 1.387907 | 0.610768 |
| 5 | 4581882694877 | 0.526222 |
| 6 | 8707129344000 | 0.461814 |

Unlike the average of the smallest eigenvalue, which in the same circumstances \((\beta = 2 \text{ and } M = N)\) has the attractively simple form \(\langle \lambda_{\text{min}} \rangle = 1/N^3\) exactly for all \(N\) \([22]\), the

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\(^9\) One could have started directly from equation (6) and would have been led to the very same equation (C.2) upon noticing that \(P_1(1/x) = (n!K_n^{(2)}(x))L^{-1}[\det[(-\partial_s)^{j+k} F_1(s)]_{j,k=0, \ldots, n-1}](1/x)\) \([53]\).
situation for the largest eigenvalue seems more complicated and it is much harder to even conjecture a possible formula for the sought average valid for all $N = M$. We leave this as a challenging open problem, noticing en passant that the asymptotic value $\langle \lambda_{\text{max}} \rangle \sim 4/N$ is approached very slowly as $N$ increases [15]. This fact makes the investigation of finite (small) $N$ results very much called for.

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