An extension of Chaitin’s halting probability $\Omega$ to a measurement operator in an infinite dimensional quantum system

Kohtaro Tadaki

21st Century Center Of Excellence Program, Chuo University, 1-13-27 Kasuga, Bunkyo-ku, Tokyo 112-8551, Japan
E-mail: tadaki@kc.chuo-u.ac.jp

Abstract. This paper proposes an extension of Chaitin’s halting probability $\Omega$ to a measurement operator in an infinite dimensional quantum system. Chaitin’s $\Omega$ is defined as the probability that the universal self-delimiting Turing machine $U$ halts, and plays a central role in the development of algorithmic information theory. In the theory, there are two equivalent ways to define the program-size complexity $H(s)$ of a given finite binary string $s$. In the standard way, $H(s)$ is defined as the length of the shortest input string for $U$ to output $s$. In the other way, the so-called universal probability $m$ is introduced first, and then $H(s)$ is defined as $-\log_2 m(s)$ without reference to the concept of program-size.

Mathematically, the statistics of outcomes in a quantum measurement are described by a positive operator-valued measure (POVM) in the most general setting. Based on the theory of computability structures on a Banach space developed by Pour-El and Richards, we extend the universal probability to an analogue of POVM in an infinite dimensional quantum system, called a universal semi-POVM. We also give another characterization of Chaitin’s $\Omega$ numbers by universal probabilities. Then, based on this characterization, we propose to define an extension of $\Omega$ as a sum of the POVM elements of a universal semi-POVM. The validity of this definition is discussed.

In what follows, we introduce an operator version $\hat{H}(s)$ of $H(s)$ in a Hilbert space of infinite dimension using a universal semi-POVM, and study its properties.

Key words: algorithmic information theory, Chaitin’s $\Omega$, quantum measurement, computable analysis, POVM, universal probability

MSC (2000) 03F60, 68Q30, 81P15, 03D80, 47S30

1 Introduction

Algorithmic information theory is a framework to apply information-theoretic and probabilistic ideas to recursive function theory. One of the primary concepts of algorithmic information

*An extended abstract appeared in the Proceedings of the 6th Conference on Real Numbers and Computers (RNC’6), Schloß Dagstuhl, Germany, November 15–17, 2004, pp. 172–191.
theory is the program-size complexity (or Kolmogorov complexity) $H(s)$ of a finite binary string $s$, which is defined as the length of the shortest binary input for the universal self-delimiting Turing machine to output $s$. By the definition, $H(s)$ can be thought of as the information content of the individual finite binary string $s$. In fact, algorithmic information theory has precisely the formal properties of classical information theory (see [2]). The concept of program-size complexity plays a crucial role in characterizing the randomness of a finite or infinite binary string. In [2] Chaitin introduced the halting probability $Ω$ as an example of random infinite string. His $Ω$ is defined as the probability that the universal self-delimiting Turing machine halts, and plays a central role in the development of algorithmic information theory. The first $n$ bits of the base-two expansion of $Ω$ solves the halting problem for a program of size not greater than $n$. By this property, the base-two expansion of $Ω$ is shown to be an instance of a random infinite binary string. In [3] Chaitin encoded this random property of $Ω$ onto an exponential Diophantine equation in the manner that a certain property of the set of the solutions of the equation is indistinguishable from coin tosses. Moreover, based on this random property of the equation, Chaitin derived several quantitative versions of Gödel’s incompleteness theorems.

In [14] we generalized Chaitin’s halting probability $Ω$ to $Ω^D$ so that the degree of randomness of $Ω^D$ can be controlled by a real number $D$ with $0 < D ≤ 1$. As $D$ becomes larger, the degree of randomness of $Ω^D$ increases. When $D = 1$, $Ω^D$ becomes a random real number, i.e., $Ω^1 = Ω$. The properties of $Ω^D$ and its relations to self-similar sets were studied in [14]. In the present paper, however, we generalize Chaitin’s $Ω$ to a different direction from [14]. The aim of the present paper is to extend Chaitin’s halting probability $Ω$ to a measurement operator in an infinite dimensional quantum system (i.e., a quantum system whose state space has infinite dimension).

The program-size complexity $H(s)$ is originally defined using the concept of program-size, as stated above. However, it is possible to define $H(s)$ without referring to such a concept, i.e., we first introduce a universal probability $m$, and then define $H(s)$ as $− \log_2 m(s)$. A universal probability is defined through the following two definitions [16].

**Definition 1.1.** For any $r : Σ^* → [0, 1]$, we say that $r$ is a lower-computable semi-measure if $r$ satisfies the following two conditions:

(i) $\sum_{s ∈ Σ^*} r(s) ≤ 1$.

(ii) There exists a total recursive function $f : Σ^+ × Σ^* → Q$ such that, for each $s ∈ Σ^*$, $\lim_{n→∞} f(n, s) = r(s)$ and $∀ n ∈ Σ^+ \ 0 ≤ f(n, s) ≤ f(n + 1, s)$.

**Definition 1.2.** Let $m$ be a lower-computable semi-measure. We say that $m$ is a universal probability if for any lower-computable semi-measure $r$, there exists a real number $c > 0$ such that, for all $s ∈ Σ^*$, $cr(s) ≤ m(s)$.

In this paper we show that Chaitin’s $Ω$ can be defined using a universal probability without reference to the universal self-delimiting Turing machine, as in the case of $H(s)$.

In quantum mechanics, a positive operator-valued measure (POVM) is the mathematical tool which describes the statistics of outcomes in a quantum measurement in the most general setting. In this paper we extend the universal probability to an analogue of a POVM in an infinite dimensional quantum system, called a universal semi-POVM. Then, based on a universal semi-POVM, we introduce the extension $\hat{Ω}$ of Chaitin’s $Ω$ to a measurement operator in an infinite dimensional quantum system.
1.1 Quantum measurements

Let $X$ be a separable complex Hilbert space. We assume that the inner product $\langle u, v \rangle$ of $X$ is linear in the first variable $u$ and conjugate linear in the second variable $v$, and it is related to the norm by $\|u\| = \langle u, u \rangle^{1/2}$. $\mathcal{B}(X)$ is the set of bounded operators in $X$. We denote the identity operator in $X$ by $I$. For each $T \in \mathcal{B}(X)$, the adjoint operator of $T$ is denoted as $T^* \in \mathcal{B}(X)$. We say $T \in \mathcal{B}(X)$ is Hermitian if $T = T^*$. $\mathcal{B}_h(X)$ is the set of Hermitian operators in $X$. We say $T \in \mathcal{B}(X)$ is positive if $\langle T x, x \rangle \geq 0$ for all $x \in X$. $\mathcal{B}(X)_+$ is the set of positive operators in $X$. For each $S, T \in \mathcal{B}_h(X)$, we write $S \leq T$ if $T - S$ is positive. Let $\{A_n\}$ be a sequence of operators in $\mathcal{B}(X)$, and let $A \in \mathcal{B}(X)$). We say $\{A_n\}$ converges strongly to $A$ as $n \to \infty$ if $\lim_{n \to \infty} \|A_n x - Ax\| = 0$ for all $x \in X$.

With every quantum system there is associated a separable complex Hilbert space $X$. The states of the system are described by the nonzero elements in $X$. We assume that the inner product $\langle \cdot, \cdot \rangle$ of $X$ is linear in the first variable $u$ and conjugate linear in the second variable $v$, and it is related to the norm by $\|u\| = \langle u, u \rangle^{1/2}$. $\mathcal{B}(X)$ is the set of bounded operators in $X$. We denote the identity operator in $X$ by $I$. For each $T \in \mathcal{B}(X)$, the adjoint operator of $T$ is denoted as $T^* \in \mathcal{B}(X)$. We say $T \in \mathcal{B}(X)$ is Hermitian if $T = T^*$. $\mathcal{B}_h(X)$ is the set of Hermitian operators in $X$. We say $T \in \mathcal{B}(X)$ is positive if $\langle T x, x \rangle \geq 0$ for all $x \in X$. $\mathcal{B}(X)_+$ is the set of positive operators in $X$. For each $S, T \in \mathcal{B}_h(X)$, we write $S \leq T$ if $T - S$ is positive. Let $\{A_n\}$ be a sequence of operators in $\mathcal{B}(X)$, and let $A \in \mathcal{B}(X)$). We say $\{A_n\}$ converges strongly to $A$ as $n \to \infty$ if $\lim_{n \to \infty} \|A_n x - Ax\| = 0$ for all $x \in X$.

Let us consider a quantum measurement performed upon a quantum system. We first define a POVM on a $\sigma$-field as follows.

**Definition 1.3 (POVM on a $\sigma$-field).** Let $\mathcal{F}$ be a $\sigma$-field in a set $\Phi$. We say $M : \mathcal{F} \to \mathcal{B}(X)_+$ is a POVM on the $\sigma$-field $\mathcal{F}$ if the following holds for $M$: If $\{B_j\}$ is a countable partition of $\Phi$ into pairwise disjoint subsets in $\mathcal{F}$, then $\sum_j M(B_j) = I$ where the series converges strongly.$^1$

In the most general setting, the statistics of outcomes in a quantum measurement are described by a POVM $M$ on a $\sigma$-field in a set $\Phi$. The set $\Phi$ consists of all outcomes possible under the quantum measurement. If the state of the quantum system is described by an $x \in X$ with $\|x\| = 1$ immediately before the measurement, then the probability distribution of the measurement outcomes is given by $\langle M(B)x, x \rangle$. (See e.g. [8] for the treatment of the mathematical foundation of quantum mechanics.)

In this paper, we relate an argument $s$ of a universal probability $m(s)$ to an individual outcome which may occur in a quantum measurement. Thus, since $m(s)$ is defined for all finite binary strings $s$, we focus our thought on a POVM measurement with countably infinite measurement outcomes, such as the measurement of energy level of a harmonic oscillator. Since $\Phi$ is a countably infinite set for our purpose, we particularly define the notion of a POVM on a countably infinite set as follows.

**Definition 1.4 (POVM on a countably infinite set).** Let $S$ be a countably infinite set, and let $R : S \to \mathcal{B}(X)_+$. We say $R$ is a POVM on the countably infinite set $S$ if $R$ satisfies $\sum_{v \in S} R(v) = I$ where the series converges strongly.

Let $S$ be a countably infinite set, and let $\mathcal{F}$ be the set of all subsets of $S$. Assume that $R : S \to \mathcal{B}(X)_+$ is a POVM on the countably infinite set $S$ in Definition 1.4. Then, by setting $M(B) = \sum_{v \in B} R(v)$ for every $B \in \mathcal{F}$, we can show that $M : \mathcal{F} \to \mathcal{B}(X)$ is a POVM on the $\sigma$-field $\mathcal{F}$ in Definition 1.3. Thus Definition 1.4 is sufficient for our purpose. Consider the quantum measurement described by the $R$ performed upon a quantum system. We then see that if the state of the quantum system is described by an $x \in X$ with $\|x\| = 1$ immediately before the measurement then, for each $v \in S$, the probability that the result $v$ occurs is given by $\langle R(v)x, x \rangle$. Each operator $R(v) \in \mathcal{B}(X)_+$ is called a POVM operator associated with the measurement.

---

$^1$In Definition 1.3 and the subsequent Definition 1.4 and 1.6, we can equivalently replace the condition “the series converges strongly” by “the series converges weakly”, using Lemma 3.6 given below. Here, for any sequence $\{A_n\}$ of operators in $\mathcal{B}(X)$ and any $A \in \mathcal{B}(X)$, we say $\{A_n\}$ converges weakly to $A$ as $n \to \infty$ if $\lim_{n \to \infty} \langle A_n x, y \rangle = \langle Ax, y \rangle$ for all $x, y \in X$. 

---

3
In a POVM measurement with countably infinite measurement outcomes, we represent each measurement outcome by just a finite binary string in perfect register with the argument of a universal probability. Thus we consider the notion of a POVM on $\Sigma^*$ which is a special case of a POVM on a countably infinite set.

**Definition 1.5 (POVM on $\Sigma^*$).** We say $R: \Sigma^* \rightarrow \mathcal{B}(X)_+$ is a POVM on $\Sigma^*$ if $R$ is a POVM on the countably infinite set $\Sigma^*$.

In a quantum measurement described by a POVM on $\Sigma^*$, an experimenter gets a finite binary string as a measurement outcome.

Any universal probability $m$ satisfies $\sum_{s \in \Sigma^*} m(s) < 1$. This relation is incompatible with the relation $\sum_{s \in \Sigma^*} R(s) = I$ satisfied by a POVM $R$ on $\Sigma^*$. Hence we further introduce the notion of a semi-POVM on $\Sigma^*$, which is appropriate for an extension of universal probability.

**Definition 1.6 (semi-POVM on $\Sigma^*$).** We say $R: \Sigma^* \rightarrow \mathcal{B}(X)_+$ is a semi-POVM on $\Sigma^*$ if $R$ satisfies $\sum_{s \in \Sigma^*} R(s) \leq I$ where the series converges strongly.

Obviously, any POVM on $\Sigma^*$ is a semi-POVM on $\Sigma^*$. Let $R$ be a semi-POVM on $\Sigma^*$. It is easy to convert $R$ into a POVM on a countably infinite set by appending an appropriate positive operator to $R$ as follows. We fix any one object $w$ which is not in $\Sigma^*$. Let $\hat{\Omega}_R = \sum_{s \in \Sigma^*} R(s)$. Then $0 \leq \hat{\Omega}_R \leq I$ and $\sum_{s \in \Sigma^*} R(s) + (I - \hat{\Omega}_R) = I$. Thus, by setting $\overline{R}(s) = R(s)$ for every $s \in \Sigma^*$ and $\overline{R}(w) = I - \hat{\Omega}_R$, we see that $\overline{R}: \Sigma^* \cup \{w\} \rightarrow \mathcal{B}(X)_+$ is a POVM on the countably infinite set $\Sigma^* \cup \{w\}$ in Definition 1.4. Therefore a semi-POVM on $\Sigma^*$ has a physical meaning in the same way as a POVM on a countably infinite set. Hence, hereafter, we say that a POVM measurement $\mathcal{M}$ is described by a semi-POVM $R$ on $\Sigma^*$ if $\mathcal{M}$ is described by the POVM $\overline{R}$ on the countably infinite set $\Sigma^* \cup \{w\}$. Let us consider the quantum measurement described by the $R$ performed upon a quantum system. We then see that if the state of the quantum system is described by an $x \in X$ with $\|x\| = 1$ immediately before the measurement then, for each $s \in \Sigma^*$, the probability that the result $s$ occurs is given by $\langle R(s)x, x \rangle$.

### 1.2 Related works

There are precedent works which make an attempt to extend the universal probability to operators in quantum system [6, 15].

As we stated above, in quantum mechanics a POVM is the mathematical notion which describes the statistics of outcomes in a quantum measurement in the most general setting. Especially in quantum information processing such as quantum computation, quantum cryptography, and quantum teleportation and communication (see e.g. [10] for these subjects), prior to a real experiment we design an appropriate POVM in order to accomplish a certain purpose. Hence, in such applications of quantum mechanics, an experimenter has to be able to realize the quantum measurement described by a pre-designed POVM with any desired accuracy. Therefore the pre-designed POVM has to be computable. In the previous work [15], we investigated what appears in the framework of quantum mechanics if we take into account the computability of a POVM for a finite dimensional quantum system. We obtained a new kind of inequalities of quantum mechanics about the probability of each measurement outcome in a computable POVM measurement performed upon a finite dimensional quantum system. In order to derive these inequalities, we introduced the notion of a universal semi-POVM on a finite dimensional quantum system, as a generalization of the universal probability to a matrix-valued function. The present work is, in essence, an extension of the work [15] to infinite dimensional setting with respect to the form of the theory.

The first attempt to extend the universal probability to an operator is done by [6] for finite dimensional quantum system. The purpose of [6] is mainly to define the information content.
of an individual pure quantum state, i.e., to define the quantum Kolmogorov complexity of the quantum state, while such an attempt is not the purpose of both [15] and the present paper. [6] generalized the universal probability to a matrix-valued function $\mu$, called the quantum universal semi-density matrix. The function $\mu$ maps any positive integer $N$ to an $N \times N$ positive semi-definite Hermitian matrix $\mu(N)$ with its trace less than or equal to one. [6] proposed to regard $\mu(N)$ as an analogue of a density matrix of a quantum system whose state space has finite dimension $N$. Since the dependency of $\mu(N)$ on $N$ is crucial to the framework of [6], it would not seem clear how to extend the framework of [6] to an infinite dimensional quantum system. By comparison, the extension is clear to our framework.

In quantum mechanics, what is represented by an operator is either a quantum state or a measurement operator. In [15] and the present work we generalize the universal probability to an operator-valued function in different way from [6], and identify it with an analogue of a POVM. We do not stick to defining the information content of a quantum state. Instead, we focus our thoughts on properly extending algorithmic information theory to quantum region while keeping an appealing feature of the theory.

1.3 Organization of the paper

We begin in Section 2 with some basic notation and the results of algorithmic information theory. In Section 3 we introduce our definition of universal semi-POVM after considering mathematical constraints on it. We then propose our extension of $\Omega$ to an operator in infinite dimensional quantum system in Section 4. The introduction of universal semi-POVM also enables us to extend $H(s)$ to an operator in a Hilbert space of infinite dimension. In Section 5, we introduce the extension of $H(s)$ and study its properties. We conclude this paper with a discussion about the future direction of our work in Section 6.

2 Preliminaries

2.1 Notation

We start with some notation about numbers and matrices which will be used in this paper.

$\#S$ is the cardinality of $S$ for any set $S$. $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ is the set of natural numbers, and $\mathbb{N}^+$ is the set of positive integers. $\mathbb{Q}$ is the set of rational numbers. $\mathbb{R}$ is the set of real numbers, and $\mathbb{C}$ is the set of complex numbers. $\mathbb{C}_Q$ is the set of the complex numbers in the form of $a + ib$ with $a, b \in \mathbb{Q}$. For any matrix $A$, $A^\dagger$ is the adjoint of $A$. Let $N \in \mathbb{N}^+$. $\mathbb{C}^N$ is the set of column vectors consisting of $N$ complex numbers. $\text{Her}(N)$ is the set of $N \times N$ Hermitian matrices. For each $A \in \text{Her}(N)$, the norm of $A$ is denoted by $\|A\|$, i.e., $\|A\| = \max\{|\nu| \mid \nu$ is an eigenvalue of $A\}$. For each $A, B \in \text{Her}(N)$, we write $A \leq B$ if $B - A$ is positive semi-definite. $\text{Her}_Q(N)$ is the set of $N \times N$ Hermitian matrices whose elements are in $\mathbb{C}_Q$. $\text{diag}(x_1, \ldots, x_N)$ is the diagonal matrix whose $(j,j)$-element is $x_j$.

2.2 Algorithmic information theory

In the following we concisely review some definitions and results of algorithmic information theory [2, 3]. We assume that the reader is familiar with algorithmic information theory in addition to the theory of computable analysis. (See e.g. Chapter 0 of [11] for the treatment of the computability of complex numbers and complex functions on a discrete set.)

$\Sigma^* \equiv \{\lambda, 0, 1, 00, 01, 10, 11, 000, 001, 010, \ldots\}$ is the set of finite binary strings where $\lambda$ denotes the empty string, and $\Sigma^*$ is ordered as indicated. We identify any string in $\Sigma^*$ with a positive integer in this order, i.e., we consider $\varphi: \Sigma^* \to \mathbb{N}^+$ such that $\varphi(s) = 1s$ where the
concatenation 1s of strings 1 and s is regarded as a dyadic integer, and then we identify s with \( \varphi(s) \). For any \( s \in \Sigma^* \), \(|s|\) is the length of \( s \). A subset \( S \) of \( \Sigma^* \) is called a prefix-free set if no string in \( S \) is a prefix of another string in \( S \).

A computer is a partial recursive function \( C: \Sigma^* \to \Sigma^* \) whose domain of definition is a prefix-free set. For each computer \( C \) and each \( s \in \Sigma^* \), \( H_C(s) \) is defined by \( H_C(s) = \min \{ |p| \mid p \in \Sigma^* \& C(p) = s \} \). A computer \( U \) is said to be optimal if for each computer \( C \) there exists a constant \( \text{sim}(C) \) with the following property; if \( C(p) \) is defined, then there is a \( p' \) for which \( U(p') = C(p) \) and \( |p'| \leq |p| + \text{sim}(C) \). It is then shown that there exists an optimal computer. We choose any one optimal computer \( U \) as the standard one for use, and define \( H(s) = H_U(s) \), which is referred to as the program-size complexity of \( s \), the information content of \( s \), or the Kolmogorov complexity of \( s \) [5, 9, 2].

Let \( V \) be any optimal computer. For any \( s \in \Sigma^* \), \( P_V(s) \) is defined as \( \sum_{V(p)=s} 2^{-|p|} \). Chaitin’s halting probability \( \Omega_V \) of \( V \) is defined by

\[
\Omega_V \equiv \sum_{V(p) \text{ is defined}} 2^{-|p|}.
\] (1)

For any \( \alpha \in (0,1] \), we say that \( \alpha \) is random if there exists \( c \in \mathbb{N} \) such that, for any \( n \in \mathbb{N}^+ \), \( n - c \leq H(\alpha_n) \) where \( \alpha_n \) is the first \( n \) bits of the base-two expansion of \( \alpha \). Then [2] showed that, for any optimal computer \( V \), \( \Omega_V \) is random. It is shown that \( 0 < \Omega_V < 1 \) for any optimal computer \( V \).

The class of computers is equal to the class of functions which are computed by self-delimiting Turing machines. A self-delimiting Turing machine is a deterministic Turing machine which has two tapes, a program tape and a work tape. The program tape is infinite to the right, while the work tape is infinite in both directions. The program tape is read-only and the tape head of the program tape cannot move to the left. On the other hand, the work tape is read/write and the tape head of the work tape can move in both directions. A self-delimiting Turing machine computes a partial function \( f: \Sigma^* \to \Sigma^* \) as follows. The machine starts in the initial state with an input binary string \( s \) on its program tape and the work tape blank. The left-most cell of the program tape is blank and the tape head of the program tape initially scans this cell. The input string lies immediately to the right of this cell. If the machine eventually halts with the tape head of the program tape scanning the last bit of the input string \( s \), then \( f(s) \) is defined as the string extending to the right from the cell of the work tape which is being scanned to the first blank cell. Otherwise, \( f(s) \) is not defined. Since the computation must end with the tape head of the program tape scanning the last bit of the input string \( s \) whenever \( f(s) \) is defined, the domain of definition of \( f \) is a prefix-free set. A self-delimiting Turing machine is called universal if it computes an optimal computer. Let \( M_V \) be a universal self-delimiting Turing machine which computes an optimal computer \( V \). Then \( P_V(s) \) is the probability that \( M_V \) halts and outputs \( s \) when \( M_V \) starts on the program tape filled with an infinite binary string generated by infinitely repeated tosses of a fair coin. Therefore \( \Omega_V = \sum_{s \in \Sigma^*} P_V(s) \) is the probability that \( M_V \) just halts under the same setting. [2] showed the following theorem.

**Theorem 2.1.** For any optimal computer \( V \), both \( 2^{-H_V(s)} \) and \( P_V(s) \) are universal probabilities.

By Theorem 2.1 we see that, for any universal probability \( m \),

\[
H(s) = -\log_2 m(s) + O(1).
\] (2)

Thus it is possible to define \( H(s) \) as \( -\log_2 m(s) \) with any one universal probability \( m \) instead of as \( H_U(s) \). Note that the difference up to an additive constant is inessential to algorithmic
information theory. Any universal probability is not computable, as corresponds to the uncomputability of $H(s)$. As a result, we see that $0 < \sum_{s \in \Sigma^*} m(s) < 1$ for any universal probability $m$.

We can give another characterization of $\Omega_V$ using a universal probability, as seen in the following theorem. The proof of the theorem is based on Theorem 2.1 above and the result of [1].

**Theorem 2.2.** For any $\alpha \in \mathbb{R}$, $\alpha = \sum_{s \in \Sigma^*} m(s)$ for some universal probability $m$ if and only if $\alpha = \Omega_V$ for some optimal computer $V$.

**Proof.** The “if” part follows from Theorem 2.1 and $\Omega_V = \sum_{s \in \Sigma^*} P_V(s)$. The proof of the “only if” part is as follows. We say an increasing converging computable sequence $\{a_n\}$ of rational numbers is *universal* if for every increasing converging computable sequence $\{b_n\}$ of rational numbers, there exists a real number $c > 0$ such that, for all $n \in \mathbb{N}^+$, $c(\alpha - a_n) \geq \beta - b_n$ where $\alpha = \lim_{n \to \infty} a_n$ and $\beta = \lim_{n \to \infty} b_n$. Theorem 6.6 in [1] shows that, for any $\alpha \in (0, 1)$, $\alpha = \Omega_V$ for some optimal computer $V$ if and only if there exists a universal increasing computable sequence of rational numbers which converges to $\alpha$. Thus it is sufficient to show that there exists a universal increasing computable sequence of rational numbers converging to $\sum_{s \in \Sigma^*} m(s)$. Since $m$ is a lower-computable semi-measure, there exists a total recursive function $f : \mathbb{N}^+ \times \Sigma^* \to \mathbb{Q}$ such that, for each $s \in \Sigma^*$, $\lim_{n \to \infty} f(n, s) = m(s)$ and $\forall n \in \mathbb{N}^+ \ 0 \leq f(n, s) \leq f(n+1, s)$.

We define an increasing computable sequence $\{a_n\}$ of rational numbers by $a_n = \sum_{s=1}^n f(n, s)$. Then we have $|a_n - \sum_{s \in \Sigma^*} m(s)| \leq \sum_{s=1}^l |f(n, s) - m(s)| + \sum_{s=l+1}^\infty m(s)$ for any $l, n \in \mathbb{N}^+$ with $l < n$. Thus, by considering sufficiently large $n$ for each sufficiently large $l$, we see that $\lim_{n \to \infty} a_n = \sum_{s \in \Sigma^*} m(s)$. Let $\{b_n\}$ be an increasing computable sequence of rational numbers converging to $\beta$. We define $r : \Sigma^* \to \mathbb{Q} \cap [0, \infty)$ by $r(s) = (b_n - b_{n-1})/d$ for any $s > 1$ and $r(1) = 0$, where $d$ is any one positive integer with $\beta - b_1 \leq d$. Then we see that $\sum_{s \in \Sigma^*} r(s) = (\beta - b_1)/d \leq 1$ and $r$ is a total recursive function. Therefore $r$ is a lower-computable semi-measure. Thus there exists a $c > 0$ such that $cr(s) \leq m(s)$ for all $s \in \Sigma^*$. Hence we have $c(\beta - b_n)/d \leq \sum_{s=n+1}^\infty m(s) = \sum_{s=1}^\infty m(s) - \sum_{s=1}^n m(s)$ and therefore $\beta - b_n \leq d/c(\sum_{s \in \Sigma^*} m(s) - a_n)$. Thus the proof is completed.

In the present paper, we extend a universal probability to a semi-POVM on $\Sigma^*$. Thus, Theorem 2.2 suggests that an extension of $\Omega_V$ to an operator can be defined as the sum of the POVM elements of such a semi-POVM on $\Sigma^*$. Therefore the most important thing is how to extend a universal probability to a semi-POVM on $\Sigma^*$ on a Hilbert space of infinite dimension. We do this first in what follows.

### 3 Extension of universal probability

In order to extend a universal probability to a semi-POVM on $\Sigma^*$ which operates on an infinite dimensional Hilbert space, we have to develop a theory of computability for points and operators of such a space. We can construct the theory on any concrete Hilbert spaces such as $l^2$ and $L^2(\mathbb{R}^{3n})$ with $n \in \mathbb{N}^+$ (the latter represents the state space of $n$ quantum mechanical particles moving in three-dimensional space). For the purpose of generality, however, we here adopt an axiomatic approach which encompasses a variety of spaces. Thus we consider the notion of a *computability structure on a Banach space* which was introduced by [1] in the late 1980s.

#### 3.1 Computability structures on a Banach space

Let $X$ be a complex Banach space with a norm $\| \cdot \|$, and let $\varphi$ be a nonempty set of sequences in $X$. We say $\varphi$ is a *computability structure* on $X$ if the following three axioms; Axiom 3.1 3.2
and hold. A sequence in \( \varphi \) is regarded as a computable sequence in \( X \).

**Axiom 3.1 (Linear Forms).** Let \( \{x_n\} \) and \( \{y_n\} \) be in \( \varphi \), let \( \{\alpha_{nk}\} \) and \( \{\beta_{nk}\} \) be computable double sequences of complex numbers, and let \( d: \mathbb{N}^+ \to \mathbb{N}^+ \) be a total recursive function. Then the sequence

\[
s_n = \sum_{k=1}^{d(n)} (\alpha_{nk}x_k + \beta_{nk}y_k)
\]

is in \( \varphi \).

For any double sequence \( \{x_{nm}\} \) in \( X \), we say \( \{x_{nm}\} \) is computable with respect to \( \varphi \) if it is mapped to a sequence in \( \varphi \) by any one recursive bijection from \( \mathbb{N}^+ \) to \( \mathbb{N}^+ \times \mathbb{N}^+ \). An element \( x \in X \) is called computable with respect to \( \varphi \) if the sequence \( \{x, x, x, \ldots\} \) is in \( \varphi \).

**Axiom 3.2 (Limits).** Suppose that a double sequence \( \{x_{nm}\} \) in \( X \) is computable with respect to \( \varphi \), \( \{y_n\} \) is a sequence in \( X \), and there exists a total recursive function \( e: \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{N}^+ \) such that \( \|x_{ne(n,k)} - y_n\| \leq 2^{-k} \) for all \( n, k \in \mathbb{N}^+ \). Then \( \{y_n\} \) is in \( \varphi \).

**Axiom 3.3 (Norms).** If \( \{x_n\} \) is in \( \varphi \), then the norms \( \{\|x_n\|\} \) form a computable sequence of real numbers.

We say a sequence \( \{e_n\} \) in \( X \) is a generating set for \( X \) or a basis for \( X \) if the set of all finite linear combinations of the \( e_n \) is dense in \( X \).

**Definition 3.4.** Let \( X \) be a Banach space with a computability structure \( \varphi \). We say the pair \((X, \varphi)\) is effectively separable if there exists a sequence \( \{e_n\} \) in \( \varphi \) which is a generating set for \( X \). Such a sequence \( \{e_n\} \) is called an effective generating set for \((X, \varphi)\) or a computable basis for \((X, \varphi)\).

Throughout the rest of this paper, we assume that \( X \) is an arbitrary complex Hilbert space of infinite dimension with a computability structure \( \varphi \) such that \((X, \varphi)\) is effectively separable. We choose any one such a computability structure \( \varphi \) on \( X \) as the standard one throughout the rest of this paper, and we do not refer to \( \varphi \) hereafter. For example, we will simply say a sequence \( \{x_n\} \) is computable instead of saying \( \{x_n\} \) is in \( \varphi \).

We next define a notion of computability for a semi-POVM on \( \Sigma^* \) as a natural extension of the notion of an effectively determined bounded operator which is defined in [11].

**Definition 3.5 (computability of semi-POVM).** Let \( R \) be a semi-POVM on \( \Sigma^* \). We say \( R \) is computable if there exists an effective generating set \( \{e_n\} \) for \( X \) such that the mapping \((s, n) \mapsto (R(s))e_n\) is a computable double sequence in \( X \).

Recall that we identify \( \Sigma^* \) with \( \mathbb{N}^+ \) in this paper. For any semi-POVM \( R \) on \( \Sigma^* \), based on Axiom 3.1, 3.2, 3.3 and \( \|R(s)\| \leq 1 \) for all \( s \in \Sigma^* \), we can show that if \( R \) is computable then \( \{(R(s))e_n\} \) is a computable double sequence in \( X \) for every effective generating set \( \{e_n\} \) for \( X \).

The following two lemmas are frequently used throughout the rest of this paper.

**Lemma 3.6.** Let \( \{A_n\} \) be a sequence of operators in \( \mathcal{B}_h(X) \). Suppose that there exists a \( B \in \mathcal{B}_h(X) \) such that, for all \( n, A_n \leq A_{n+1} \leq B \). Then there exists an \( A \in \mathcal{B}_h(X) \) such that \( \{A_n\} \) converges strongly to \( A \) as \( n \to \infty \) and \( A \leq B \).

The proof of Lemma 3.6 is given at Section 104 of [13].

**Lemma 3.7.** Let \( \{A_n\} \) and \( \{B_n\} \) be sequences of operators in \( \mathcal{B}_h(X) \). Suppose that (i) \( A_n \leq B_n \leq A_{n+1} \) for all \( n \), and (ii) \( \{A_n\} \) converges strongly to some \( A \in \mathcal{B}_h(X) \) as \( n \to \infty \). Then \( \{B_n\} \) also converges strongly to \( A \) as \( n \to \infty \).
Proof. Since $A_n \leq A$ for all $n$, $B_n \leq B_{n+1} \leq A$ for all $n$. It follows from Lemma 3.6 that there exists a $B \in B_h(X)$ to which $\{B_n\}$ converges strongly as $n \to \infty$. Note that, for any $x \in X$, $\langle A_n x, x \rangle \leq \langle B_n x, x \rangle \leq \langle A_{n+1} x, x \rangle$. Thus $\langle B x, x \rangle = \langle A x, x \rangle$ for any $x \in X$, and therefore we have $B = A$. This completes the proof.

3.2 Universal semi-POVM

We first introduce the notion of a lower-computable semi-POVM on $\Sigma^*$, which is an extension of the notion of a lower-computable semi-measure over a semi-POVM on $\Sigma^*$. Our definition of a lower-computable semi-POVM premises the following lemma proved in [11]. We say a basis $\{e_n\}$ for $X$ is orthonormal if $\langle e_m, e_n \rangle = \delta_{mn}$ for any $m, n \in \mathbb{N}^+$. Let $Y$ be a Hilbert space with a computability structure $\phi$ such that $(Y, \phi)$ is effectively separable. Then there exists a computable orthonormal basis for $(Y, \phi)$.

By the above lemma, we are given free access to the use of a computable orthonormal basis for $X$ in what follows. The following definition is also needed to introduce the notion of a lower-computable semi-POVM on $\Sigma^*$.

Definition 3.9. Let $\{e_i\}$ be an orthonormal basis for $X$. For any $T \in B(X)$ and $m \in \mathbb{N}^+$, we say $T$ is an $m$-square operator on $\{e_i\}$ if for all $k, l \in \mathbb{N}^+$ if $k > m$ or $l > m$ then $\langle Te_k, e_l \rangle = 0$. Furthermore, we say $T$ is an $m$-square rational operator on $\{e_i\}$ if $T$ is an $m$-square operator on $\{e_i\}$ and for all $k, l \in \mathbb{N}^+$, $\langle Te_k, e_l \rangle \in \mathbb{C}_Q$.

The following Lemma 3.10 is suggestive to fix the definition of a lower-computable semi-POVM on $\Sigma^*$. By Lemma 3.10, we can effectively check whether $S \leq T$ holds or not, given $S, T \in B_h(X)$ and $m \in \mathbb{N}^+$ such that $S$ and $T$ are $m$-square operators on an orthonormal basis for $X$.

Lemma 3.10. Let $T \in B_h(X)$, and let $\{e_i\}$ be an orthonormal basis for $X$. Then, the following three conditions (i), (ii), and (iii) are equivalent to one another.

(i) $T$ is a positive operator.
(ii) For every $m \in \mathbb{N}^+$,

$$
\begin{pmatrix}
\langle Te_1, e_1 \rangle & \ldots & \langle Te_1, e_m \rangle \\
\vdots & \ddots & \vdots \\
\langle Te_m, e_1 \rangle & \ldots & \langle Te_m, e_m \rangle
\end{pmatrix} \geq 0.
$$

(iii) For every finite sequence $\nu_1, \ldots, \nu_m \in \mathbb{N}^+$ with $\nu_1 < \cdots < \nu_m$,

$$
\det\begin{pmatrix}
\langle Te_{\nu_1}, e_{\nu_1} \rangle & \ldots & \langle Te_{\nu_1}, e_{\nu_m} \rangle \\
\vdots & \ddots & \vdots \\
\langle Te_{\nu_m}, e_{\nu_1} \rangle & \ldots & \langle Te_{\nu_m}, e_{\nu_m} \rangle
\end{pmatrix} \geq 0.
$$

Proof. We note the elementary result of linear algebra that, for any $A \in \text{Her}(N)$, $0 \leq A$ if and only if all principal minors of $A$ are non-negative. Thus the conditions (ii) and (iii) are equivalent. We show the equivalence between the conditions (i) and (ii). For each $m \in \mathbb{N}^+$, let $V_m = \mathbb{C}e_1 + \cdots + \mathbb{C}e_m$. Then, for every $x \in V_m$, we see that $\langle Tx, x \rangle \geq 0$ if and only if $\sum_{i,j=1}^m c_i \langle Te_i, e_j \rangle \overline{c_j} \geq 0$ where $\{c_i\}$ satisfies that $x = \sum_{i=1}^m c_i e_i$. Thus, the condition (ii) is equivalent to the condition that, for any $m \in \mathbb{N}^+$ and any $x \in V_m$, $\langle Tx, x \rangle \geq 0$. Since $T \in B(X)$, the latter condition is further equivalent to the condition (i). Hence, the proof is completed.
We recall that, for any lower-computable semi-measure \( r \), there exists a total recursive function \( f: \mathbb{N}^+ \times \Sigma^* \to \mathbb{Q} \) such that, for each \( s \in \Sigma^* \), \( \lim_{n \to \infty} f(n, s) = r(s) \) and \( \forall n \in \mathbb{N}^+ \ 0 \leq f(n, s) \leq f(n + 1, s) \leq r(s) \). We here consider how to extend this \( f \) to an operator in order to define a lower-computable semi-POVM \( R \) on \( \Sigma^* \). Let \( \{e_i\} \) be an orthonormal basis for \( X \). When we prove the existence of a universal semi-POVM (i.e., Theorem 3.11) below, especially in the proof of Lemma 3.28, we have to be able to decide whether \( f(n, s) \leq f(n + 1, s) \) in the sequence \( \{f(n, s)\}_{n \in \mathbb{N}^+} \) of operators which converges to \( R(s) \). Thus, firstly, it is necessary for each \( f(s, n) \) to be an \( m \)-square rational operator on \( \{e_i\} \) for some \( m \in \mathbb{N}^+ \). If so we can use Lemma 3.10 to check \( f(n, s) \leq f(n + 1, s) \). On that basis, in order to complete the definition of a lower-computable semi-POVM, it seems at first glance that we have only to require that \( 0 \leq f(n, s) \leq f(n + 1, s) \leq R(s) \) and \( f(n, s) \) converges to \( R(s) \) in an appropriate sense. Note that each operator \( f(n, s) \) in the sequence has to be positive in order to guarantee that the limit \( R(s) \) is positive. However, this passing idea does not work properly as shown by the following consideration.

For simplicity, we consider matrices in \( \text{Her}(N) \) with \( N \geq 2 \) instead of operators in \( X \). We show that for some computable matrix \( A \geq 0 \) there does not exist a total recursive function \( F: \mathbb{N}^+ \to \text{Her}_\mathbb{Q}(N) \) such that

\[
\lim_{n \to \infty} F(n) = A \quad \text{and} \quad \forall n \in \mathbb{N}^+ \ 0 \leq F(n) \leq A. \tag{3}
\]

This follows from Example 3.12 below, which is based on the following result of linear algebra.

**Proposition 3.11.** Let \( A, B \in \text{Her}(N) \). Suppose that \( \text{rank } A = 1 \) and \( 0 \leq B \leq A \). Then \( B = \tau A \) for some \( \tau \in [0, 1] \).

**Proof.** Since \( A \in \text{Her}(N) \) and \( \text{rank } A = 1 \), there exist an \( N \times N \) unitary matrix \( U \) and a \( \lambda > 0 \) such that \( A = U \text{diag}(\lambda, 0, \ldots, 0) U^\dagger \). We write \( U = (u_1 \ u_2 \ \cdots \ u_N) \) with \( u_k \in \mathbb{C}^N \). For each \( k \geq 2 \), since \( u_k^\dagger A u_k = 0 \) and \( 0 \leq B \leq A \), we have \( u_k^\dagger B u_k = 0 \). It follows from \( 0 \leq B \) that \( B u_k = 0 \) for every \( k \geq 2 \). If \( B \) has a nonzero eigenvalue \( \nu \), then the eigenspace of \( B \) corresponding to \( \nu \) is \( \mathbb{C}u_1 \). Thus, we have \( B = U \text{diag}(\nu, 0, \ldots, 0) U^\dagger \) for some \( \nu \in \mathbb{R} \). Since \( 0 \leq \nu = u_1^\dagger B u_1 \leq u_1^\dagger A u_1 = \lambda \), by setting \( \tau = \lambda/\nu \), we have \( B = \tau A \) and \( \tau \in [0, 1] \).

**Example 3.12.** We consider the matrix \( A \in \text{Her}(2) \) given by

\[
A = \begin{pmatrix}
\frac{2}{3} & \frac{\sqrt{2}}{3} \\
\frac{\sqrt{2}}{3} & \frac{1}{3}
\end{pmatrix}.
\]

First, we see that all elements of \( A \) are computable real numbers, and therefore \( A \) itself is computable. We can check that \( \text{rank } A = 1 \). In fact, \( A \) has two eigenvalues 0 and 1. It can be shown that there does not exist any nonzero \( B \in \text{Her}_\mathbb{Q}(2) \) such that \( 0 \leq B \leq A \). Contrarily, assume that such a \( B \) exists. Then, by Proposition 3.11 we have \( B = \tau A \) for some \( \tau \in (0, 1] \), i.e.,

\[
B = \begin{pmatrix}
\frac{2}{3} \tau & \frac{\sqrt{2}}{3} \tau \\
\frac{\sqrt{2}}{3} \tau & \frac{1}{3} \tau
\end{pmatrix}.
\]

However, for any \( \tau > 0 \), it is impossible for all elements of \( B \) to be simultaneously in \( \mathbb{C}_Q \). \[ \square \]

Thus, even in a non-effective manner, we cannot get a sequence \( \{F(n)\} \subset \text{Her}_\mathbb{Q}(N) \) which satisfies the condition \( \boxed{[3]} \). On the other hand, for any positive semi-definite \( A \in \text{Her}(N) \) and any \( n \in \mathbb{N}^+ \), there exists a \( B \in \text{Her}_\mathbb{Q}(N) \) such that \( 0 \leq B \leq A + 2^{-n}E \), where \( E \) is the identity matrix. This is because, since \( \text{Her}_\mathbb{Q}(N) \) is dense in \( \text{Her}(N) \) with respect to the norm \( \| \cdot \| \), there exists a \( B \in \text{Her}_\mathbb{Q}(N) \) such that \( \|A + 2^{-n+1}/3E - B\| \leq 2^{-n}/3 \). Thus we have
0 \leq A + 2^{-n}/3E \leq B \leq A + 2^{-n}E. Furthermore we can show that, for any positive semi-definite \( A \in \text{Her}(N) \), if \( A \) is computable, then there exists a total recursive function \( F: \mathbb{N}^+ \rightarrow \text{Her}_2(N) \) such that (i) \( \lim_{n \rightarrow \infty} F(n) = A \), (ii) \( 0 \leq F(n) \), and (iii) \( F(n) - 2^{-n}E \leq F(n+1) - 2^{-(n+1)}E \leq A \). Note that a positive semi-definite matrix \( A \) with rank 1 as considered in Example 3.12 is not an atypical example as a POVM element in quantum measurements, since such a POVM element is common in a familiar projective measurement.

The foregoing consideration suggests the following definition of a lower-computable semi-POVM on an infinite dimensional Hilbert space.

**Definition 3.13.** Let \( \{e_i\} \) be a computable orthonormal basis for \( X \), and let \( R \) be a semi-POVM on \( \Sigma^* \). We say \( R \) is lower-computable with respect to \( \{e_i\} \) if there exist an \( f: \mathbb{N}^+ \times \Sigma^* \rightarrow \mathcal{B}(X)_+ \) and a total recursive function \( g: \mathbb{N}^+ \times \Sigma^* \rightarrow \mathbb{N}^+ \) such that

(i) for each \( s \in \Sigma^* \), \( f(n, s) \) converges strongly to \( R(s) \) as \( n \rightarrow \infty \),

(ii) for all \( n \) and \( s \), \( f(n, s) - 2^{-n}I \leq f(n+1, s) - 2^{-(n+1)}I \),

(iii) for all \( n \) and \( s \), \( f(n, s) \) is a \( g(n, s) \)-square rational operator on \( \{e_i\} \), and

(iv) the mapping \( \mathbb{N}^+ \times \Sigma^* \times \mathbb{N}^+ \times \mathbb{N}^+ \ni (n, s, i, j) \mapsto \langle f(n, s)e_i, e_j \rangle \) is a total recursive function.

In the above definition, we choose the sequence \( \{2^{-n}\} \) as the coefficients of \( I \) in the inequality of the condition (ii). However, by the following proposition, we can equivalently replace \( \{2^{-n}\} \) by a general nonincreasing computable sequence of non-negative rational numbers which converges to 0.

**Proposition 3.14.** Let \( \{e_i\} \) be a computable orthonormal basis for \( X \), and let \( R \) be a semi-POVM on \( \Sigma^* \). Then, \( R \) is lower-computable with respect to \( \{e_i\} \) if and only if there exist an \( f': \mathbb{N}^+ \times \Sigma^* \rightarrow \mathcal{B}(X)_+ \), a total recursive function \( g': \mathbb{N}^+ \times \Sigma^* \rightarrow \mathbb{N}^+ \), and a total recursive function \( h: \mathbb{N}^+ \times \Sigma^* \rightarrow \mathbb{Q} \) such that

(i) for each \( s \in \Sigma^* \), \( f'(n, s) \) converges strongly to \( R(s) \) as \( n \rightarrow \infty \),

(ii) for all \( n \) and \( s \), \( f'(n, s) - h(n, s)I \leq f'(n+1, s) - h(n+1, s)I \),

(iii) for all \( s \), \( \lim_{n \rightarrow \infty} h(n, s) = 0 \) and \( \forall n \in \mathbb{N}^+ \ h(n, s) \geq h(n+1, s) \geq 0 \),

(iv) for all \( n \) and \( s \), \( f'(n, s) \) is a \( g'(n, s) \)-square rational operator on \( \{e_i\} \), and

(v) the mapping \( \mathbb{N}^+ \times \Sigma^* \times \mathbb{N}^+ \times \mathbb{N}^+ \ni (n, s, i, j) \mapsto \langle f'(n, s)e_i, e_j \rangle \) is a total recursive function.

**Proof.** The “only if” part is obvious, and we show the “if” part. To begin with, we define \( h(n, s) \) as \( h(n, s) + 2^{-n} \). It follows that \( f'(n, s) - h(n, s)I \leq f'(n+1, s) - h(n+1, s)I \), \( \lim_{n \rightarrow \infty} h(n, s) = 0 \), and \( h(n, s) > h(n+1, s) \geq 0 \). Without loss of generality, we assume that \( h(1, s) > 1/2 \). In what follows, we assume the fact that, for any \( A, B \in \mathcal{B}_k(X) \) and any \( \alpha, \beta \in [0, 1] \), if \( A \leq B \) and \( \alpha \leq \beta \), then \( A \leq (1 - \alpha)A + \alpha B \leq (1 - \beta)A + \beta B \leq B \). In order to define \( f: \mathbb{N}^+ \times \Sigma^* \rightarrow \mathcal{B}(X)_+ \) and \( g: \mathbb{N}^+ \times \Sigma^* \rightarrow \mathbb{N}^+ \) which satisfy the conditions (i), (ii), (iii), and (iv) in Definition 3.13, we follow the procedure below for each \( s \). Initially we set \( m := 1 \) and \( n := 1 \).

Assume that \( f(k, s) \) and \( g(k, s) \) have so far been defined for all \( k \in \{1, \ldots, n-1\} \). We look for the least \( l > m \) with \( 2^{-n} \geq h(l, s) \). Since \( \lim_{l \rightarrow \infty} h(l, s) = 0 \), we can find such an \( l \). Once we get the \( l \), we calculate the finite set \( S = \{k \in \mathbb{N}^+ \mid k \geq n \land h(m, s) > 2^{-k} \geq h(l, s)\} \). For each \( k \in S \), we then define \( f(k, s) \) as \( (1 - \alpha_k)f'(m, s) + \alpha_kf'(l, s) \) where \( \alpha_k = (h(m, s) - 2^{-k})/(h(m, s) - h(l, s)) \), and we also define \( g(k, s) \) as \( \max\{g'(m, s), g'(l, s)\} \). It follows that, for every \( k \in S - \{n\} \),

\[
 f'(m, s) - h(m, s)I \leq f(k-1, s) - 2^{-(k-1)}I \leq f(k, s) - 2^{-k}I \leq f'(l, s) - h(l, s)I
\]
and \( f(k, s) \) is a \( g(k, s) \)-square rational operator on \( \{e_i\} \). We then set \( m := l \) and \( n := n + \#S \), and repeat this procedure.

It can be checked that the \( f \) and \( g \) defined by this procedure satisfy the desired properties. Especially, in a similar manner to the proof of Lemma 3.7 we can show that, for each \( s \in \Sigma^* \), \( f(n, s) \) converges strongly to \( R(s) \) as \( n \to \infty \). Thus the proof is completed.

In Proposition 3.16 below, we show that the lower-computability of a semi-POVM on \( \Sigma^* \) given in Definition 3.13 does not depend on the choice of a computable orthonormal basis used in the definition. The proof of Proposition 3.16 uses the following Lemma 3.15, which follows from the equivalence between the conditions (i) and (iii) in Lemma 3.10.

**Lemma 3.15.** Let \( T \in \mathcal{B}_h(X) \) be an \( m \)-square operator on an orthonormal basis \( \{e_i\} \) for \( X \). For any real number \( a > 0 \), \( 0 \leq T + aI \) if and only if \( 0 \leq T + aI_m \) where \( I_m \) is the operator in \( \mathcal{B}_h(X) \) such that \( I_me_i = e_i \) if \( i \leq m \) and \( I_me_i = 0 \) otherwise.

By Lemma 3.15 in order to check whether the condition (ii) of Definition 3.13 holds, we can equivalently check the condition that \( 0 \leq f(n + 1, s) - f(n, s) + 2^{-n-1}I_m \) if \( f(n, s) \) and \( f(n + 1, s) \) are \( m \)-square operators on an orthonormal basis \( \{e_i\} \) for \( X \).

For each \( T \in \mathcal{B}(X) \), the norm of \( T \) is denoted by \( \|T\| \). Throughout the rest of this paper, we will frequently use the property: For any \( \varepsilon \geq 0 \) and any \( T \in \mathcal{B}_h(X) \), \( \|T\| \leq \varepsilon \) if and only if \( -\varepsilon I \leq T \leq \varepsilon I \). For each \( T \in \mathcal{B}(X) \), we define \( \|T\|_2 \) as \( (\sum_{i=1}^{\infty} \|Te_i\|^2)^{1/2} \in [0, \infty] \), where \( \{e_n\} \) is an arbitrary orthonormal basis for \( X \). Note that \( \|T\|_2 \) is independent of the choice of an orthonormal basis \( \{e_n\} \) for \( X \), and \( \|T\| \leq \|T\|_2 \). These properties of \( \|\cdot\|_2 \) are used in the proof of Proposition 3.16.

**Proposition 3.16.** Let \( R \) be a semi-POVM on \( \Sigma^* \), and let \( \{e_i\} \) and \( \{e'_k\} \) be computable orthonormal bases for \( X \). Then, \( R \) is lower-computable with respect to \( \{e_i\} \) if and only if \( R \) is lower-computable with respect to \( \{e'_k\} \).

**Proof.** We first define \( u_{ki} = \langle e'_k, e_i \rangle \). Then \( \{u_{ki}\} \) is the computable double sequence of complex numbers which satisfies \( e'_k = \sum_{i=1}^{\infty} u_{ki}e_i \). Assume that \( R \) is lower-computable with respect to \( \{e_i\} \). Then there exist an \( f : \mathbb{N}^+ \times \Sigma^* \rightarrow \mathcal{B}(X)_+ \) and a total recursive function \( g : \mathbb{N}^+ \times \Sigma^* \rightarrow \mathbb{N}^+ \) which satisfy the conditions (i), (ii), (iii), and (iv) in Definition 3.13. In what follows, we show that \( R \) is lower-computable with respect to \( \{e'_k\} \). To begin with, we note that \( \sum_{i,j=1}^{g(n,s)} |\langle f(n, s)e_i, e_j \rangle|^2 = \|f(n, s)\|^2 = \sum_{k,l=1}^{g(n,s)} |\langle f(n, s)e'_k, e'_l \rangle|^2 \geq |\langle f(n, s)e'_k, e'_l \rangle|^2 \geq 2^{-2n-7} \). Here, since \( \langle f(n, s)e'_k, e'_l \rangle = \sum_{i,j=1}^{g(n,s)} u_{ki} \langle f(n, s)e_i, e_j \rangle w_{ij} \), \( \{\langle f(n, s)e'_k, e'_l \rangle\} \) is a computable fourfold sequence of complex numbers. Thus, there exists a total recursive function \( g' : \mathbb{N}^+ \times \Sigma^* \rightarrow \mathbb{N}^+ \) such that

\[
\left| \sum_{i,j=1}^{g(n,s)} |\langle f(n, s)e_i, e_j \rangle|^2 - \sum_{k,l=1}^{g(n,s)} |\langle f(n, s)e'_k, e'_l \rangle|^2 \right| \leq 2^{-2n-7}.
\]

On the other hand, it is easy to show that there exists \( \overline{T} : \mathbb{N}^+ \times \Sigma^* \rightarrow \mathcal{B}_h(X) \) such that (i) for every \( k, a \in \{1, \ldots, g(n, s)\} \), \( |\langle \overline{T}(n, s) - f(n, s), e'_k, e'_l \rangle|^2 \leq 1/g(n, s)^2 2^{-2n-7} \), (ii) \( \overline{T}(n, s) \) is a \( g'(n, s) \)-square rational operator on \( \{e'_k\} \), and (iii) the mapping \( (n, s, k, l) \mapsto \langle \overline{T}(n, s)e'_k, e'_l \rangle \) is a total recursive function. Therefore we have

\[
\|\overline{T}(n, s) - f(n, s)\|^2 = \sum_{k,l=1}^{g'(n,s)} |\langle \overline{T}(n, s) - f(n, s), e'_k, e'_l \rangle|^2 + \|f(n, s)\|^2 - \sum_{k,l=1}^{g'(n,s)} |\langle f(n, s)e'_k, e'_l \rangle|^2
\]

\[
\leq 2^{-2n-7} + 2^{-2n-7} \leq 2^{-2n-6}.
\]
Hence, \( |\mathcal{f}(n, s) - f(n, s)| \leq |\mathcal{f}(n, s) - f(n, s)|_2 \leq 2^{-n-3} \), and therefore \( 0 < f(n, s) \leq \mathcal{f}(n, s) + 2^{-n-3}I \). We then define \( f': \mathbb{N}^+ \times \Sigma^* \to B_h(X) \) by \( f'(n, s) = \mathcal{f}(n, s) + 2^{-n-3}I(n, s) \), where \( I(n, s) \in B_h(X) \) satisfies that \( I(n, s)e_k = e_k' \) if \( k \leq g'(n, s) \) and \( I(n, s)e_k = 0 \) otherwise. It follows that \( f'(n, s) = g'(n, s) \)|square rational operator on \( \{e_k'\} \) and the mapping \((n, s, k, l) \mapsto \langle f'(n, s)e_k', e_l'\rangle \) is a total recursive function. In particular, by Lemma 3.15 we have \( 0 < f'(n, s) \). Since \( \|f'(n, s) - f(n, s)\| \leq \|\mathcal{f}(n, s) - f(n, s)\| + 2^{-n-3}\|I(n, s)\| \leq 2^{-n-2}, f'(n, s) - 2^{-n-2}I \leq f(n, s) \leq f'(n, s) + 2^{-n-2}I. \) Using \( f(n, s) - 2^{-n}I \leq f(n + 1, s) - 2^{-n+1}I, \) we have

\[
f(n, s) - 2^{-(n-1)}I \leq f'(n, s) - (2^{-n-2} + 2^{-n-1} + 2^{-n})I \leq f(n + 1, s) - 2^{-n}I.
\]

From this inequality, it is shown that

\[
f'(n, s) - (2^{-n-2} + 2^{-n-1} + 2^{-n})I \leq f'(n + 1, s) - (2^{-n-3} + 2^{-n-2} + 2^{-n-1})I
\]

and, for each \( s \in \Sigma^* \), \( f'(n, s) \) converges strongly to \( R(s) \) as \( n \to \infty \). The latter follows from Lemma 3.17. Thus, by Proposition 3.14 \( R \) is lower-computable with respect to \( \{e_k'\} \). This completes the proof.

Based on the above proposition, we define the notion of a lower-computable semi-POVM on \( \Sigma^* \) independently of a choice of a computable orthonormal basis for \( X \).

**Definition 3.17 (lower-computable semi-POVM on \( \Sigma^* \)).** Let \( R \) be a semi-POVM on \( \Sigma^* \). We say \( R \) is lower-computable if there exists a computable orthonormal basis \( \{e_i\} \) for \( X \) such that \( R \) is lower-computable with respect to \( \{e_i\} \).

Thus, for any semi-POVM \( R \) on \( \Sigma^* \), based on Proposition 3.16 we see that if \( R \) is lower-computable then \( R \) is lower-computable with respect to every computable orthonormal basis for \( X \).

Any computable function \( r: \Sigma^* \to [0, 1] \) with \( \sum_{s \in \Sigma^*} r(s) \leq 1 \) is shown to be a lower-computable semi-measure. Corresponding to this fact we can show Theorem 3.18 below. In the theorem, however, together with the computability of a semi-POVM \( R \) on \( \Sigma^* \), we need an additional assumption that (i) each POVM element \( R(s) \) is Hilbert-Schmidt and (ii) given \( s \), \( \|R(s)\|_2 \) can be computed to any desired degree of precision. Here, for any \( T \in B(X) \), we say \( T \) is Hilbert-Schmidt if \( \|T\|_2 < \infty \). As an example, consider a POVM \( P \) on \( \Sigma^* \) with \( (P(s))e_i = \delta_{si}e_i \), where \( \{e_i\} \) is a computable orthonormal basis for \( X \). Then \( P \) is shown to be a computable POVM on \( \Sigma^* \) which satisfies this additional assumption (see the proof of Proposition 3.24). Note that the quantum measurement described by the \( P \) is a familiar projective measurement, such as the measurement of the number of photons in a specific mode of electromagnetic field.

**Theorem 3.18.** Suppose that (i) \( R: \Sigma^* \to B(X) \) is a computable semi-POVM on \( \Sigma^* \), (ii) \( R(s) \) is Hilbert-Schmidt for every \( s \in \Sigma^* \), and (iii) \( \{\|R(s)\|_2\}_{s \in \Sigma^*} \) is a computable sequence of real numbers. Then \( R \) is lower-computable.

**Proof.** Let \( \{e_i\} \) be any one computable orthonormal basis for \( X \). Since \( \{(R(s))e_i, e_j\} \) is a computable triple sequence of complex numbers and \( \{\|R(s)\|_2\} \) is a computable sequence of real numbers, it is easy to show that there exists a total recursive function \( g: \mathbb{N}^+ \times \Sigma^* \to \mathbb{N}^+ \) such that

\[
\left|\|R(s)\|_2^2 - \sum_{i,j=1}^{g(n, s)} |\langle R(s)e_i, e_j\rangle|^2 \right| \leq 2^{-2n-5}
\]

and \( g(n, s) \leq g(n + 1, s) \). Again, since \( \{(R(s))e_i, e_j\} \) is a computable triple sequence of complex numbers, we can show that there exists \( \mathcal{F}: \mathbb{N}^+ \times \Sigma^* \to B_h(X) \) such that (i) for every \( i, j \in \mathbb{N}^+ \times \Sigma^* \) and (ii) \( \mathcal{F}(n, s) \) converges strongly to \( R(s) \) as \( n \to \infty \). The latter follows from Lemma 3.17.
\{1, \ldots, g(n, s)\}, \quad \left\langle (R(s) - \mathcal{F}(n, s))e_i, e_j \right\rangle^2 \leq 1/g(n, s)^2 \, 2^{-2n-5}.
(iv) \quad \mathcal{F}(n, s)\) is a \(g(n, s)\)-square rational operator on \(\{e_i\}\), and (iii) the mapping \((n, s, i, j) \mapsto \left\langle \mathcal{F}(n, s)e_i, e_j \right\rangle\) is a total recursive function. Therefore we have

\[
\|R(s) - \mathcal{F}(n, s)\|_2^2 = \sum_{i,j=1}^{g(n, s)} \left| \left\langle (R(s) - \mathcal{F}(n, s))e_i, e_j \right\rangle \right|^2 + \|R(s)\|_2^2 - \sum_{i,j=1}^{g(n, s)} \left| \left\langle R(s)e_i, e_j \right\rangle \right|^2 \\
\leq 2^{-2n-5} + 2^{-2n-5} \leq 2^{-2n-4}.
\]

Hence, \(\|R(s) - \mathcal{F}(n, s)\| \leq \|R(s) - \mathcal{F}(n, s)\|_2 \leq 2^{-n-2}\), and therefore

\[
\mathcal{F}(n, s) - 2^{-n-2}I \leq R(s) \leq \mathcal{F}(n, s) + 2^{-n-2}I. \tag{4}
\]

We then define \(f : \mathbb{N}^+ \times \Sigma^* \to \mathcal{B}_h(X)\) by \(f(n, s) = \mathcal{F}(n, s) + 2^{-n-2}I(n, s)\), where \(I(n, s) \in \mathcal{B}_h(X)\) satisfies that \(I(n, s)e_i = e_i\) if \(i \leq g(n, s)\) and \(I(n, s)e_i = 0\) otherwise. It follows that \(f(n, s)\) is a \(g(n, s)\)-square rational operator on \(\{e_i\}\) and the mapping \((n, s, i, j) \mapsto (f(n, s)e_i, e_j)\) is a total recursive function. In particular, by \(0 \leq R(s)\), the inequality (4), and Lemma 3.15, we have \(0 \leq f(n, s)\). It follows also from the inequality (4) that \(\|R(s) - f(n, s)\| \leq \|R(s) - \mathcal{F}(n, s)\| + 2^{-n-2}\|I(n, s)\| \leq 2^{-n-1}\). Thus, for each \(s \in \Sigma^*\), \(f(n, s)\) converges strongly to \(R(s)\) as \(n \to \infty\). Finally, we show that \(f(n, s) - 2^{-n}I \leq f(n + 1, s) - 2^{-(n+1)}I\). For that purpose, we note that \(\mathcal{F}(n + 1, s) - \mathcal{F}(n, s) \geq -(2^{-n-3} + 2^{-n-2})I\) and \(I(n, s) \leq I(n + 1, s) \leq I\). The former follows from the inequality (4). Based on these inequalities, we have

\[
(f(n + 1, s) - 2^{-(n+1)}I) - (f(n, s) - 2^{-n}I) \geq 2^{-n-3}(I - I(n + 1, s)) \geq 0.
\]

This completes the proof.

**Remark 3.19.** It is open whether \(R\) can be proved to be lower-computable only under the assumption that \(R : \Sigma^* \to \mathcal{B}(X)\) is a computable semi-POVM on \(\Sigma^*\).

As a natural generalization of the notion of a universal probability, the notion of a universal semi-POVM is defined as follows.

**Definition 3.20 (universal semi-POVM).** Let \(M\) be a lower-computable semi-POVM on \(\Sigma^*\). We say that \(M\) is a universal semi-POVM if for each lower-computable semi-POVM \(R\) on \(\Sigma^*\), there exists a real number \(c > 0\) such that, for all \(s \in \Sigma^*\), \(cR(s) \leq M(s)\).

Most importantly we can show the existence of a universal semi-POVM.

**Theorem 3.21.** There exists a universal semi-POVM.

In order to prove Theorem 3.21, we need the following two lemmas.

**Lemma 3.22.** Let \(\{e_i\}\) be a computable orthonormal basis for \(X\), and let \(R\) be a semi-POVM on \(\Sigma^*\). If \(R\) is lower-computable, then there exist an \(f : \mathbb{N}^+ \times \Sigma^* \to \mathcal{B}(X)\) and a total recursive function \(g : \mathbb{N}^+ \times \Sigma^* \to \mathbb{N}^+\) such that

(i) the mapping \(\Sigma^* \ni s \mapsto \frac{1}{2}R(s) + \frac{1}{2^{s+1}}I\) is a lower-computable semi-POVM on \(\Sigma^*\),

(ii) for each \(s \in \Sigma^*\), \(f(n, s)\) converges strongly to \(\frac{1}{2}R(s) + \frac{1}{2^{s+1}}I\) as \(n \to \infty\),

(iii) for all \(n\) and \(s\), \(f'(n, s) \leq f'(n + 1, s)\),

\(14\)
We simulate the computations of \( h \). Initially we set \( k \) and update it accordingly. For each \( (n,s,i,j) \) we have \( e_i = e_j \) if \( i \leq g(n,s) \) and \( I(n,s)e_i = 0 \) otherwise. Then we have \( I(n,s) \leq I(n+1,s) \). It follows from Lemma 3.15 that \( f(n,s) \leq f(n+1,s) + 2^{-n-1}I(n+1,s) \) and \( f(n,s) = 1/2 f(n+s,s) + 2^{-n-1}(1-2^{-n})I(n+s,s) \), and define a total recursive function \( g : \mathbb{N}^+ \times \Sigma^* \to \mathbb{N}^+ \) by \( g(n,s) = g(n+s,s) \). Then we see that \( 0 \leq f(n,s) \leq f(n+1,s) \) and \( f(n,s) \) is the function \( h \).

Proof. Since \( R \) is lower-computable, there exist an \( f : \mathbb{N}^+ \times \Sigma^* \to \mathbb{B}(X)_+ \) and a total recursive function \( g : \mathbb{N}^+ \times \Sigma^* \to \mathbb{N}^+ \) which satisfy the conditions (i), (ii), (iii), and (iv) in Definition 3.13. Without loss of generality, we assume that \( g(n,s) < g(n+1,s) \). For each \( (n,s) \in \mathbb{N} \times \Sigma^* \), let \( I(n,s) \) be the operator in \( \mathcal{B}_h(X) \) such that \( I(n,s)e_i = e_i \) if \( i \leq g(n,s) \) and \( I(n,s)e_i = 0 \) otherwise. Then we have \( I(n,s) \leq I(n+1,s) \). It follows from f(n,s) \leq f(n+1,s) + 2^{-n-1}I(n+1,s) \) and Lemma 3.15 that \( f(n,s) \leq f(n+1,s) + 2^{-n-1}I(n+1,s) \) and \( f(n,s) = 1/2 f(n+s,s) + 2^{-n-1}(1-2^{-n})I(n+s,s) \), and define a total recursive function \( g : \mathbb{N}^+ \times \Sigma^* \to \mathbb{N}^+ \) by \( g(n,s) = g(n+s,s) \). Then we see that \( 0 \leq f(n,s) \leq f(n+1,s) \) and \( f(n,s) \) is the function \( h \).

It is easy to check that \( f(n,s) \) is a \( g(n,s) \)-square rational operator on \( \{e_i\} \) and the mapping \( \mathbb{N}^+ \times \Sigma^* \times \mathbb{N}^+ \times \mathbb{N}^+ \ni (n,s,i,j) \mapsto \langle f(n,s)e_i,e_j \rangle \) is a total recursive function. Since \( I(n,s) \) converges strongly to \( I \) as \( n \to \infty \), \( f(n,s) \) converges strongly to \( 1/2 R(s) + 1/2 s+1 I \). We have \( \sum_{s \in \Sigma^*} \frac{1}{2 R(s) + 1/2 s+1 I} \leq 1/2 \sum_{s \in \Sigma^*} R(s) + 1/2 I \leq I \). Thus, the mapping \( \Sigma^* \ni s \mapsto 1/2 R(s) + 1/2 s+1 I \) is a lower-computable semi-POVM on \( \Sigma^* \). This completes the proof.

Lemma 3.23. Let \( \{e_i\} \) be a computable orthonormal basis for \( X \). Then there exist an \( f : \mathbb{N}^+ \times \mathbb{N}^+ \times \Sigma^* \to \mathbb{B}(X)_+ \) and a total recursive function \( g : \mathbb{N}^+ \times \mathbb{N}^+ \times \Sigma^* \to \mathbb{N}^+ \) such that

(i) for all \( l \), \( n \), and \( s \), \( f(l,n,s) \leq f(l,n+1,s) \),

(ii) for all \( l \), \( n \), and \( s \), \( f(l,n,s) \) is a \( g(l,n,s) \)-square rational operator on \( \{e_i\} \),

(iii) for all \( (n,s,i,j) \), \( f(l,n,s) e_i = e_j \) for every \( s \in \Sigma^* \), \( f(l,n,s) \) converges strongly to \( R(s) \) as \( n \to \infty \), and

(iv) for each \( l \), there exists a lower-computable semi-POVM \( T_l \) on \( \Sigma^* \) such that, for every \( s \), \( f(l,n,s) \) converges strongly to \( R(s) / 2^{s+1} \) as \( n \to \infty \).

Proof. We first note that, for any \( A \in \text{Her}_Q(N) \), there exists a unique \( T_A \in \mathcal{B}_h(X) \) such that \( \langle T_A e_i,e_j \rangle = A_{ij} \) for every \( i,j \in \mathbb{N}^+ \) and \( T_A \) is an \( N \)-square rational operator on \( \{e_i\} \).

Given \( l \in \mathbb{N}^+ \), for all \( (n,s) \in \mathbb{N}^+ \times \Sigma^* \), \( f(l,n,s) \) and \( g(l,n,s) \) are defined through the following procedure.

We first build the \( l \)-th Turing machine \( M_l \). We make use of \( M_l \) as a machine which outputs a Hermitian matrix in \( \bigcup_{n=1}^{\infty} \text{Her}_Q(N) \) on an input \((n,s) \in \mathbb{N}^+ \times \Sigma^* \). Let \( f_l : \mathbb{N}^+ \times \Sigma^* \to \bigcup_{n=1}^{\infty} \text{Her}_Q(N) \) be a partial recursive function computed by \( M_l \) in this sense. For each \( n \in \mathbb{N}^+ \), let \( S_n = \{(n-s+1,1) \mid s \in \Sigma^* \& 1 \leq s \leq n\} \). In increasing order on \( n \), we simulate the computations of \( M_l \) on all inputs in \( S_n \). During the procedure, we keep the function \( h : \Sigma^* \to \bigcup_{n=1}^{\infty} \text{Her}_Q(N) \) and update it accordingly. For each \( (n,s) \in \mathbb{N}^+ \times \Sigma^* \), \( f(l,n,s) \) and \( g(l,n,s) \) are defined as \( T_{h(s)} \) and the order of the square matrix \( h(s) \), respectively. Here \( h(s) \) is one at the time step \( n \) in the simulations. Initially we set \( h(s) = 0 \) for all \( s \in \Sigma^* \) and \( n := 1 \).

Assume that the simulations of \( M_l \) on all inputs in \( \bigcup_{k=1}^{n-1} S_k \) have so far been completed. We simulate the computations of \( M_l \) on all inputs in \( S_n \). If all such computations halt then we check whether the following three conditions hold:

(i) \( f_l(k,s) \) is defined for all \( (k,s) \in S_n \),
(ii) \( T_{h(s)} \leq T_{f_l(k, s)} \) for all \((k, s) \in S_n\), and

(iii) \( \sum_{s=1}^{n} T_{f_l(n-s+1, s)} \leq I \).

Note that we can effectively check whether the above conditions (ii) and (iii) hold, based on the equivalence between the conditions (i) and (iii) in Lemma 3.10. If these three conditions hold then we set \( h(s) := f_l(n-s+1, s) \) for each \( s \in \{1, \ldots, n\} \) and \( n := n + 1 \). We then repeat this procedure.

We can show that the \( f \) and \( g \) defined by this procedure satisfy that (i) \( 0 \leq f(l, n, s) \leq f(l, n+1, s) \), (ii) \( f(l, n, s) \) is a \( g(l, n, s) \)-square rational operator on \( \{e_i\} \), and (iii) the mapping \( \mathbb{N}^+ \times \mathbb{N}^+ \times \Sigma^* \times \mathbb{N}^+ \times \mathbb{N}^+ \ni (l, n, i, j) \mapsto (f(l, n, s) e_i, e_j) \) and \( g \) are total recursive functions. We also see that \( \sum_{s=1}^{n} f(l, n, s) \leq I \) for any \( l, m, n \in \mathbb{N}^+ \). Thus we have \( f(l, n, s) \leq I \) and therefore, by Lemma 3.6, there exists an \( R_l : \Sigma^* \rightarrow \mathcal{B}(X)_+ \) such that \( f(l, n, s) \) converges strongly to \( R_l(s) \) as \( n \to \infty \). Hence we have \( \sum_{s=1}^{n} R_l(s) \leq I \). It follows from \( 0 \leq R_l(s) \) and Lemma 3.6 that \( \sum_{s=1}^{n} R_l(s) \) converges strongly to \( \sum_{s \in \Sigma^*} R_l(s) \in \mathcal{B}(X) \) as \( m \to \infty \) and \( \sum_{s \in \Sigma^*} R_l(s) \leq I \). Thus \( R_l \) is a lower-computable semi-POVM on \( \Sigma^* \) for all \( l \).

Now, let \( R \) be any lower-computable semi-POVM on \( \Sigma^* \). Then, by Lemma 3.22 there exist an \( f' : \mathbb{N}^+ \times \Sigma^* \rightarrow \mathcal{B}(X)_+ \) and a total recursive function \( g' : \mathbb{N}^+ \times \Sigma^* \rightarrow \mathbb{N}^+ \) which satisfy the conditions (i), (ii), (iii), (iv), and (v) in the lemma. Based on the above construction of \( f \), we see that there exists \( k \in \mathbb{N}^+ \) with the property that, for each \( s \in \Sigma^* \), the sequence \( \{f'(k, n, s)\}_{n \in \mathbb{N}^+} \) of operators is a subsequence of the sequence \( \{f(k, n, s)\}_{n \in \mathbb{N}^+} \). Thus \( f(k, n, s) \) converges strongly to \( 1/2R(s) + 1/2^{n+1}I \) as \( n \to \infty \). This completes the proof.

Based on the above lemmas, we can give the proof of Theorem 3.21 as follows.

**PROOF of Theorem 3.21**. Let \( \{e_i\} \) be a computable orthonormal basis for \( \langle X, \varphi \rangle \). Let \( f \) and \( g \) be the functions given by Lemma 3.23 and, for each \( l \in \mathbb{N}^+ \), let \( R_l \) be a lower-computable semi-POVM on \( \Sigma^* \) such that, for each \( s \in \Sigma^* \), \( f(l, n, s) \) converges strongly to \( R_l(s) \) as \( n \to \infty \). We first define an \( f_M : \mathbb{N}^+ \times \Sigma^* \rightarrow \mathcal{B}(X)_+ \) and a total recursive function \( g_M : \mathbb{N}^+ \times \Sigma^* \rightarrow \mathbb{N}^+ \) by

\[
 f_M(n, s) = \sum_{l=1}^{n} \frac{1}{2^l} f(l, n, s),
 g_M(n, s) = \max\{g(l, n, s) \mid 1 \leq l \leq n\}.
\]

Obviously, the mapping \( \mathbb{N}^+ \times \Sigma^* \times \mathbb{N}^+ \times \mathbb{N}^+ \ni (n, s, i, j) \mapsto (f_M(n, s) e_i, e_j) \) is a total recursive function and, for all \( n \) and \( s \), \( f_M(n, s) \) is a \( g_M(n, s) \)-square rational operator on \( \{e_i\} \). We also see that \( f_M(n, s) \leq f_M(n, s) + f(n+1, n+1, s) \leq f_M(n+1, s) \). Since \( f(l, n, s) \leq R_l(s) \leq I \), we have \( f_M(n, s) \leq (1 - 2^{-n})I \leq I \). Thus, by Lemma 3.6, there exists a \( M : \Sigma^* \rightarrow \mathcal{B}(X)_+ \) such that, for each \( s \in \Sigma^* \), \( f_M(n, s) \) converges strongly to \( M(s) \) as \( n \to \infty \). We show that this \( M \) is a universal semi-POVM.

To begin with, we note that, for any \( n, m \in \mathbb{N}^+ \), any \( s \in \Sigma^* \), and any \( x \in X \),

\[
 \left\| \left( \sum_{l=1}^{n} \frac{1}{2^l} R_l(s) \right) x - M(s)x \right\|
 \leq \left\| \sum_{l=1}^{n} \frac{1}{2^l} R_l(s)x - \sum_{l=1}^{n} \frac{1}{2^l} f(l, n + m, s)x \right\| + \left\| \sum_{l=n+1}^{n+m} \frac{1}{2^l} f(l, n + m, s)x \right\| + \left\| f_M(n + m, s)x - M(s)x \right\|
 \leq \sum_{l=1}^{n} \frac{1}{2^l} \left\| R_l(s)x - f(l, n + m, s)x \right\| + 2^{-n} \left\| x \right\| + \left\| f_M(n + m, s)x - M(s)x \right\|.
\]
Here we use \( \|f(l, n+m, s)x\| \leq \|f(l, n+m, s)\| \leq \|x\| \). Thus, by choosing any one sufficiently large \( m \) for each sufficiently large \( n \), we see that, for each \( s \in \Sigma^* \), \( \sum_{l=1}^{n} 1/2^l R_l(s) \) converges strongly to \( M(s) \) as \( n \to \infty \). For each \( m \in \mathbb{N}^+ \), since \( \sum_{l=1}^{n} 1/2^l \sum_{s=1}^{m} R_l(s) \) converges to \( M(s) \), we have \( \sum_{s=1}^{m} M(s) \leq I \). It follows from \( 0 \leq M(s) \) and Lemma 3.6 that \( \sum_{s=1}^{m} M(s) \) converges strongly to \( M(s) \in \mathcal{B}_h(X) \) as \( m \to \infty \) and \( 0 \leq \sum_{s \in \Sigma^*} M(s) \leq I \). Thus, since \( f_{M}(n, s) - 2^{-n}I \leq f_{M}(n+1, s) - 2^{-n-1}I \), \( M \) is a lower-computable semi-POVM on \( \Sigma^* \).

Now, let \( R \) be any lower-computable semi-POVM on \( \Sigma^* \). Then, by Lemma 3.23 there is a \( k \) with \( 1/2^k R(s) + (1/2)^{k+1}I = R_k(s) \). Since \( 1/2^k R_k(s) \leq \sum_{l=1}^{\infty} 1/2^l R_l(s) = M(s) \), we have \( 1/2^{k+1} R(s) \leq 1/2^{k+1} (R(s) + 2^{-s}I) \leq M(s) \). Hence, \( M \) is a universal semi-POVM.

In the previous work [15], we developed the theory of a universal semi-POVM for a finite dimensional quantum system, and we showed that, for every universal probability \( m \), the mapping \( \Sigma^* \ni s \mapsto m(s)E \) is a universal semi-POVM on a finite dimensional quantum system, where \( E \) is the identity matrix. On the other hand, as shown in the following proposition, the corresponding statement does not hold for the infinite dimensional setting on which we work at present.

**Proposition 3.24.** Let \( m \) be a universal probability. Then the mapping \( \Sigma^* \ni s \mapsto m(s)I \) is not a universal semi-POVM.

**Proof.** Let \( \{e_i\} \) be an orthonormal basis for \( X \), and let \( P: \Sigma^* \to \mathcal{B}(X)_+ \) with \( (P(s))(e_i) = \delta_{si} e_i \). Then \( P \) is shown to be a POVM on \( \Sigma^* \). By Axiom 3.3 we see that \( P \) is computable. Since \( \|P(s)\|_2 = 1 \) for every \( s \in \Sigma^* \), \( P(s) \) is Hilbert-Schmidt for every \( s \in \Sigma^* \) and \( \{\|P(s)\|_2\}_{s \in \Sigma^*} \) is a computable sequence of real numbers. It follows from Theorem 3.18 that \( P \) is a lower-computable semi-POVM on \( \Sigma^* \).

Now, let us assume contrarily that the mapping \( \Sigma^* \ni s \mapsto m(s)I \) is a universal semi-POVM. Then there exists a \( c > 0 \) such that, for all \( s \in \Sigma^* \), \( cP(s) \leq m(s)I \). Since \( \langle (P(s))e_s, e_s \rangle = 1 \), we have \( c \leq m(s) \) for all \( s \in \Sigma^* \). However, this contradicts the condition that \( \sum_{s \in \Sigma^*} m(s) \leq 1 \), and the proof is completed.

Thus, there is an essential difference between finite dimensional quantum systems and infinite dimensional quantum systems with respect to the properties of a universal semi-POVM.

### 4 Extension of Chaitin’s \( \Omega \)

In this section, we introduce an extension of Chaitin’s \( \Omega \) as a partial sum of the POVM elements of a POVM measurement performed upon an infinite dimensional quantum system. Before that, we give a relation between a universal semi-POVM and a universal probability. We first show a relation between a universal semi-POVM and a lower-computable semi-measure in Proposition 4.1.

**Proposition 4.1.** Let \( r \) be a lower-computable semi-measure, and let \( M \) be a universal semi-POVM. Then there exists a \( c > 0 \) such that, for all \( s \in \Sigma^* \),

(i) \( cr(s)I \leq M(s) \), and

(ii) for all \( x \in X \) with \( \|x\| = 1 \), \( cr(s) \leq \langle M(s)x, x \rangle \).

**Proof.** The condition (ii) follows immediately from (i). Thus we show the condition (i). Since \( r \) is a lower-computable semi-measure, \( \sum_{s \in \Sigma^*} r(s) \leq 1 \) and there exists a total recursive function \( f': \mathbb{N}^+ \times \Sigma^* \to \mathbb{Q} \) such that, for each \( s \in \Sigma^* \), \( \lim_{n \to \infty} f'(n, s) = r(s) \) and \( \forall n \in \mathbb{N}^+ \ 0 \leq f'(n, s) \leq 1 \). Therefore, we have \( \sum_{s \in \Sigma^*} 1/2^k r(s) \) converges to \( M(s) \) as \( k \to \infty \). Since \( 0 \leq M(s) \) and Lemma 3.6, \( \sum_{s \in \Sigma^*} 1/2^k r(s) \) converges strongly to \( M(s) \in \mathcal{B}(X) \) as \( k \to \infty \). Thus, since \( f_{M}(n, s) - 2^{-n}I \leq f_{M}(n+1, s) - 2^{-n-1}I \), \( M \) is a lower-computable semi-POVM.
Proof. Since \( (i) \) follows. Since \( \Omega \) then Theorem 4.4. Let \( \{e_i\} \) be a computable orthonormal basis for \( X \) and, for each \( n \in \mathbb{N}^+ \), let \( I(n) \) be the operator in \( \mathcal{B}_h(X) \) such that \( I(n)e_i = e_i \) if \( i \leq n \) and \( I(n)e_i = 0 \) otherwise. We define \( f: \mathbb{N}^+ \times \Sigma^* \rightarrow \mathcal{B}_h(X) \) by \( f(n,s) = f'(n,s)I(n) \). Since \( 0 \leq I(n) \leq I(n+1) \), we have \( 0 \leq f(n,s) \leq f(n+1,s) \). Since \( I(n) \) converges strongly to \( I \), \( f(n,s) \) converges strongly to \( r(s)I \) as \( n \rightarrow \infty \). Obviously, \( f(n,s) \) is an \( n \)-square rational operator on \( \{e_i\} \), and the mapping \( \mathbb{N}^+ \times \Sigma^* \rightarrow \mathbb{N}^+ \times \mathbb{N}^+ \) \( \{(n,s,i,j) \mapsto \langle f(n,s)e_i,e_j \rangle \} \) is a total recursive function. It follows from \( \sum_{s \in \Sigma^*} \langle r(s)I \rangle \leq I \) that the mapping \( \Sigma^* \ni s \mapsto r(s)I \) is a lower-computable semi-POVM on \( \Sigma^* \). Thus, from the definition of a universal semi-POVM, the condition (i) follows. \( \square \)

Based on the above proposition, we can show the following.

**Theorem 4.2.** Let \( M \) be a universal semi-POVM, and let \( x \in X \) be computable with \( \|x\| = 1 \). Then the mapping \( \Sigma^* \ni s \mapsto \langle M(s)x,x \rangle \) is a universal probability.

Proof. Let \( \{e_i\} \) be a computable orthonormal basis for \( X \). We first define \( c_i = \langle x,e_i \rangle \). Then \( \{c_i\} \) is a computable sequence of complex numbers which satisfies \( x = \sum_{i=1}^\infty c_i e_i \). Since \( M \) is a lower computable semi-POVM on \( \Sigma^* \), there exist an \( f: \mathbb{N}^+ \times \Sigma^* \rightarrow \mathcal{B}(X) \) and a total recursive function \( g: \mathbb{N}^+ \times \Sigma^* \rightarrow \mathbb{N}^+ \) which satisfy the conditions (i), (ii), (iii), and (iv) in Definition 4.13. Since \( f(n,s) - 2^{-n}I \leq M(s) \), we have \( \langle f(n,s)x,x \rangle - 2^{-n} \leq \langle M(s)x,x \rangle \). It follows from \( \{\langle f(n,s)x,x \rangle\} \) is a computable sequence of real numbers. Therefore, since \( \lim_{n \rightarrow \infty} \langle f(n,s)x,x \rangle - 2^{-n} = \langle M(s)x,x \rangle \), there exists a total recursive function \( f': \mathbb{N}^+ \times \Sigma^* \rightarrow \mathbb{Q} \) such that, for each \( s \in \Sigma^* \), \( \lim_{n \rightarrow \infty} f'(n,s) = \langle M(s)x,x \rangle \) and \( \forall n \in \mathbb{N}^+ \ f'(n,s) \leq f'(n+1,s) \). We then define a total recursive function \( h: \mathbb{N}^+ \times \Sigma^* \rightarrow \mathbb{Q} \) by \( h(n,s) = \max\{f'(n,s),0\} \). Since \( \langle M(s)x,x \rangle \geq 0 \), we have \( \lim_{n \rightarrow \infty} h(n,s) = \langle M(s)x,x \rangle \) and \( \forall n \in \mathbb{N}^+ \ 0 \leq h(n,s) \leq h(n+1,s) \). We also have \( \sum_{s \in \Sigma^*} \langle M(s)x,x \rangle \leq \langleIx,x\rangle \leq 1 \). Thus the mapping \( \Sigma^* \ni s \mapsto \langle M(s)x,x \rangle \) is a lower-computable semi-measure. Finally, by Proposition 4.1 the theorem is obtained. \( \square \)

Since any universal probability is not computable, by Theorem 4.2, we can show that any universal semi-POVM is not a computable semi-POVM on \( \Sigma^* \).

Now, based on the intuition obtained from Theorem 2.2, we propose to define an extension \( \hat{\Omega} \) of Chaitin’s \( \Omega \) as follows.

**Definition 4.3 (extension of Chaitin’s \( \Omega \) to operator).** For each universal semi-POVM \( M \), \( \hat{\Omega}_M \) is defined by

\[
\hat{\Omega}_M \equiv \sum_{s \in \Sigma^*} M(s).
\]

Let \( M \) be a universal semi-POVM. Then, obviously, \( \hat{\Omega}_M \in \mathcal{B}(X)_+ \) and \( \hat{\Omega}_M \leq I \). We can further show that \( cI \leq \hat{\Omega}_M \) for some real number \( c > 0 \). This is because, by Proposition 4.1, there is a real number \( c > 0 \) with the property that \( c2^{-s}I \leq M(s) \) for all \( s \in \Sigma^* \).

The following theorem supports the above proposal.

**Theorem 4.4.** Let \( M \) be a universal semi-POVM. If \( x \) is a computable point in \( X \) with \( \|x\| = 1 \), then

(i) there exists an optimal computer \( V \) such that \( \langle \hat{\Omega}_M x,x \rangle = \Omega_V \), and

(ii) \( \langle \hat{\Omega}_M x,x \rangle \) is a random real number.

Proof. Since \( \langle \hat{\Omega}_M x,x \rangle = \sum_{s \in \Sigma^*} \langle M(s)x,x \rangle \), by Theorem 4.2 and Theorem 2.2, Theorem 4.4 (i) follows. Since \( \Omega_V \) is random for any optimal computer \( W \), Theorem 4.3 (ii) follows. \( \square \)
Let $M$ be any universal semi-POVM, and let $x$ be any point in $X$ with $\|x\| = 1$. Consider the POVM measurement $\mathcal{M}$ described by the $M$. This measurement produces one of countably many outcomes; elements in $\Sigma^*$ and one more something which corresponds to the POVM element $I - \Omega_M$. If the measurement $\mathcal{M}$ is performed upon the state described by the $x$ immediately before the measurement, then the probability that a result $s \in \Sigma^*$ occurs is given by $\langle M(s)x, x \rangle$. Therefore $\langle \hat{\Omega}_M x, x \rangle$ is the probability of getting some finite binary string as a measurement outcome in $\mathcal{M}$.

Now, assume that $x$ is computable. Recall that, for any optimal computer $V$, $\Omega_V$ is the probability that $V$ halts and outputs some finite string, which results from infinitely repeated tosses of a fair coin. Thus, by Theorem 4.4, $\langle \hat{\Omega}_M x, x \rangle$ has the meaning of classical probability that a universal self-delimiting Turing machine generates some finite string. Hence $\langle \hat{\Omega}_M x, x \rangle$ has the meaning of probability of producing some finite string in the contexts of both quantum mechanics and algorithmic information theory. Thus, in the case where $x$ is computable, algorithmic information theory is consistent with quantum mechanics in a certain sense. Note further that, even if $x$ is not computable, quantum mechanics still insists that $\langle \hat{\Omega}_M x, x \rangle$ has a meaning as probability, i.e., the probability of getting some finite binary string in the measurement $\mathcal{M}$.

5 Operator-valued algorithmic information theory

We choose any one universal semi-POVM $M$ as the standard one for use throughout the rest of this paper. The equation (2) suggests defining an operator-valued information content $\hat{H}(s)$ of $s \in \Sigma^*$ by

$$\hat{H}(s) \equiv -\log_2 M(s).$$

(5)

Here $\log_2 M(s)$ is defined based on the notion of continuous functional calculus (for the detail, see e.g. the section VII.1 of [12]). We here note the following properties for this notion.

**Proposition 5.1.** Let $S, T \in B_h(X)$. Suppose that $aI \leq S$ for some real number $a > 0$. Then $\log_2 S \in B_h(X)$ and the following hold.

(i) $\log_2 (cS) = \log_2 S + (\log_2 c)I$ for any real number $c > 0$.

(ii) If $S \leq T$ then $\log_2 S \leq \log_2 T$.

Proposition 5.1 follows the definition of the continuous functional calculus (especially, the proof of Proposition 5.1 (ii) is given at e.g. Chapter 5 of [7]). Since there is a real number $c > 0$ with the property that $c2^{-s}I \leq M(s)$ for all $s \in \Sigma^*$, by Proposition 5.1, we see that $\hat{H}(s) \in B_h(X)$ for all $s \in \Sigma^*$. The above definition of $\hat{H}(s)$ is also supported by the following Proposition 5.2. Let $S$ be any set, and let $f: S \rightarrow B_h(X)$ and $g: S \rightarrow B_h(X)$. Then we write $f(x) = g(x) + O(1)$ if there is a real number $c > 0$ such that, for all $x \in S$, $\|f(x) - g(x)\| \leq c$, which is equivalent to $-cI \leq f(x) - g(x) \leq cI$.

**Proposition 5.2.** Let $M$ and $M'$ be universal semi-POVMs. Then $\log_2 M(s) = \log_2 M'(s) + O(1)$.

**Proof.** This follows immediately from Proposition 5.1

By this proposition, the equation (5) is independent of the choice of a universal semi-POVM $M$ up to an additive constant. We show relations between $\hat{H}(s)$ and $H(s)$ in the following theorem.
Theorem 5.3. Let $x \in X$ with $\|x\| = 1$.

(i) There exists a real number $c > 0$ such that $\langle \hat{H}(s)x, x \rangle \leq H(s) + c$ for all $s \in \Sigma^*$.

(ii) If $x$ is computable then $\langle \hat{H}(s)x, x \rangle = H(s) + O(1)$.

Proof. Since $2^{-H(s)}$ is a lower-computable semi-measure, it follows from Proposition 4.1 that there is a $d > 0$ with the property that $d2^{-H(s)}I \leq M(s)$ for all $s \in \Sigma^*$. By Proposition 5.1 (i) and the equality $\log_2 I = 0$, we see that $\log_2 (d2^{-H(s)}I) = (-H(s) + \log_2 d)I$. Hence, by Proposition 5.1 (ii), we have $\hat{H}(s) \leq (H(s) - \log_2 d)I$ and therefore Theorem 5.3 (i) follows.

Using the concavity of the real function $\log_2 t$ and the spectral decomposition of the Hermitian operator $\log_2 M(s)$, we can show that $\log_2 \langle M(s)x, x \rangle \geq \langle \log_2 M(s)x, x \rangle$. In the case where $x$ is computable, by Theorem 4.2 the mapping $\Sigma^* \ni s \mapsto \langle M(s)x, x \rangle$ is a lower-computable semi-measure. By Theorem 2.1 there is a $c' > 0$ such that $c' \langle M(s)x, x \rangle \leq 2^{-H(s)}$ for all $s \in \Sigma^*$. Hence Theorem 5.3 (ii) follows.

In \[ \S \] Chaitin developed a version of algorithmic information theory where the notion of program-size is not used. That is, in the work he, in essence, defined $H(s)$ as $-\log_2 m(s)$ for a universal probability $m$, and showed several information-theoretic relations on $H(s)$. Thus we can develop the information-theoretic feature of algorithmic information theory to a certain extent even if we do not refer to the concept of program-size. On the lines of this Chaitin’s approach, we show in the following that an information-theoretic feature can be developed based on $\hat{H}(s)$. We first need the following theorem.

Theorem 5.4. Let $\psi: \Sigma^* \to \Sigma^*$ be a partial recursive function. Then the following hold.

(i) There exists a real number $c > 0$ such that, for all $s \in \Sigma^*$, if $\psi(s)$ is defined then $cM(s) \leq M(\psi(s))$.

(ii) There exists a real number $c > 0$ such that, for all $s \in \Sigma^*$, if $\psi(s)$ is defined then $\hat{H}(\psi(s)) \leq \hat{H}(s) + cI$.

Proof. Let $\{e_i\}$ be a computable orthonormal basis for $X$. Since $M$ is a universal semi-POVM, there exist an $f: \mathbb{N}^+ \times \Sigma^* \to \mathcal{B}(X)_+$ and a total recursive function $g: \mathbb{N}^+ \times \Sigma^* \to \mathbb{N}^+$ which satisfy the conditions (ii), (iii), and (iv) in Definition 3.13 and the condition that for each $s \in \Sigma^*$, $f(n, s)$ converges strongly to $M(s)$ as $n \to \infty$. We can define $R: \Sigma^* \to \mathcal{B}(X)$ by $R(s) = \sum_{t=1}^{\psi(t)=s} M(t)$, where the series converges strongly if $\psi^{-1}(s)$ is an infinite set. This limit exists by Lemma 3.6 since $\sum_{s=1}^{t} M(s) \leq I$ for any $l \in \mathbb{N}^+$ and $0 \leq M(s)$. In the case of $\psi^{-1}(s) = \emptyset$, we interpret $\sum_{t=1}^{\psi(t)=s} M(t)$ as 0. Obviously $0 \leq R(s)$ for any $s \in \Sigma^*$. Since $\langle R(s)x, x \rangle = \sum_{t=1}^{\psi(t)=s} \langle M(t)x, x \rangle$ and $\sum_{s \in \Sigma^*} \langle M(s)x, x \rangle \leq 1$, we see that $\sum_{s=1}^{t} R(s) \leq I$ for any $l \in \mathbb{N}^+$ and therefore, by Lemma 3.6, $R$ is a semi-POVM on $\Sigma^*$.

Now, we enumerate the domain of definition of $\psi$. Let $t(k, s)$ be the $k$-th element in $\psi^{-1}(s)$ generated in the enumeration, and let $h(n, s)$ be the number of elements in $\psi^{-1}(s)$ which are generated until the time step $n$ in the enumeration (possibly $h(n, s) = 0$). We define $f': \mathbb{N}^+ \times \Sigma^* \to \mathcal{B}(X)_+$ by $f'(n, s) = \sum_{k=1}^{h(n,s)} f(n+k, t(k, s))$. It is then shown that $f'(n, s) - 2^{-n} I \leq f'(n+1, s) - 2^{-n-1} I$. We also define the total recursive function $g': \mathbb{N}^+ \times \Sigma^* \to \mathbb{N}^+$ by $g'(n, s) = \max\{g(n+k, t(k, s)) \mid 1 \leq k \leq h(n, s)\}$. Then $f'(n, s)$ is a $g'(n, s)$-square rational operator on $\{e_i\}$. Since $f(n, s) - 2^{-n} I \leq M(s)$, we have $f'(n, s) - 2^{-n} I \leq R(s) - 2^{-n-h(n, s)} \leq I$. Thus, by Lemma 3.6, again, for each $s \in \Sigma^*$, there exists an $R'(s) \in \mathcal{B}_h(X)$ such that $f'(n, s) - 2^{-n} I$ converges strongly to $R'(s)$ as $n \to \infty$. We show that $R(s) = R'(s)$ for all $s$. Since $f(n, s) -$
\[ 2^{-n} I \leq M(s), \text{ we have } |\langle M(s)x, x \rangle - \langle f(n, s)x, x \rangle| \leq \langle M(s)x, x \rangle + 2^{-n} \|x\|^2. \] Hence we see that if \( 1 \leq l < h(n, s) \) then
\[
|\langle R(s)x, x \rangle - \langle f'(n, s)x, x \rangle| \\
\leq |\langle R(s)x, x \rangle - \sum_{k=1}^{h(n,s)} \langle M(t(k,s))x, x \rangle| + \sum_{k=1}^{h(n,s)} |\langle M(t(k,s))x, x \rangle - \langle f(n+k, t(k,s))x, x \rangle| \\
\leq |\langle R(s)x, x \rangle - \langle \sum_{k=1}^{l} M(t(k,s))x, x \rangle| \\
+ \sum_{k=l+1}^{h(n,s)} |\langle M(t(k,s))x, x \rangle - \langle f(n+k, t(k,s))x, x \rangle| + \sum_{k=l+1}^{h(n,s)} \langle M(t(k,s))x, x \rangle + 2^{-n} \|x\|^2.
\]

In the case where \( \psi^{-1}(s) \) is an infinite set, since \( \sum_{s \in \Sigma^*} \langle M(s)x, x \rangle < \infty \), by considering sufficiently large \( n \) for each sufficiently large \( l \), we have \( \lim_{n \to \infty} \langle f'(n, s)x, x \rangle = \langle R(s)x, x \rangle \). In the case where \( \psi^{-1}(s) \) is a finite set, obviously the same holds. It follows that \( \langle R(s)x, x \rangle = \langle R'(s)x, x \rangle \) for all \( x \in X \) and \( s \in \Sigma^* \), and therefore \( R(s) = R'(s) \) for all \( s \in \Sigma^* \). Hence \( R \) is a lower computable semi-POVM on \( \Sigma^* \), and there is a real number \( c > 0 \) such that, for all \( s \in \Sigma^* \), \( cI \leq M(s) \).

From the definition of \( R \), if \( \psi(s) \) is defined then \( M(s) \leq R(\psi(s)) \). Thus Theorem 5.4 (i) follows.

By Proposition 5.1 we see that Theorem 5.4 (ii) holds. \( \Box \)

We choose any one computable bijection \( s, t > \) from \( (s, t) \in \Sigma^* \times \Sigma^* \) to \( \Sigma^* \). Let \( s, t \in \Sigma^* \). The joint information content \( \hat{H}(s, t) \) of \( s \) and \( t \) is defined as \( \hat{H}(s, t) \equiv \hat{H}(< s, t >) \). We then define the conditional information content \( \hat{H}(s|t) \) of \( s \) given \( t \) by the equation \( \hat{H}(s|t) \equiv \hat{H}(t, s) - \hat{H}(t) \). Finally we define the mutual information content \( \hat{H}(s, t) \) of \( s \) and \( t \) by the equation \( \hat{H}(s, t) \equiv \hat{H}(s) + \hat{H}(t) - \hat{H}(s|t) \). Thus \( \hat{H}(s; t) \equiv \hat{H}(t) - \hat{H}(s|t) \). We can then show the following theorem using Theorem 5.3 (ii). In particular, by Theorem 5.3 (i), we have \( \hat{H}(s; t) = \hat{H}(s) - \hat{H}(s|t) + O(1) \).

**Theorem 5.5.**

(i) \( \hat{H}(s, t) = \hat{H}(t, s) + O(1) \) and \( \hat{H}(s|t) = \hat{H}(t; s) + O(1) \).

(ii) \( \hat{H}(s, s) = \hat{H}(s) + O(1) \) and \( \hat{H}(s; s) = \hat{H}(s) + O(1) \).

(iii) \( \hat{H}(s, \lambda) = \hat{H}(s) + O(1) \) and \( \hat{H}(s; \lambda) = O(1) \).

(iv) \( \exists c \in \mathbb{R} \forall s, t \in \Sigma^* \ cI \leq \hat{H}(s|t) \).

**Proof.** Consider the total recursive function \( \psi: \Sigma^* \to \Sigma^* \) with \( \psi(< s, t >) =< t, s > \). By Theorem 5.4 (ii), there is a \( c > 0 \) such that, for all \( s, t \in \Sigma^* \), \( \hat{H}(< s, t >) \leq \hat{H}(< s, t >) + cI \). Thus Theorem 5.3 (i) follows. Next consider the function \( \psi \) with \( \psi(< s, s >) = s \). By Theorem 5.4 (ii), there is a \( c > 0 \) such that, for all \( s \in \Sigma^* \), \( \hat{H}(s) \leq \hat{H}(< s, s >) + cI \). On the other hand, by considering the function \( \phi \) with \( \phi(s) = < s, s > \), we see that there is a \( c' > 0 \) such that, for all \( s \in \Sigma^* \), \( \hat{H}(s; s) \leq \hat{H}(s) + c'I \). Thus Theorem 5.3 (ii) follows. Similarly, by considering the functions \( \psi \) with \( \psi(< s, \lambda >) = s \) and \( \phi \) with \( \phi(s) = < s, \lambda > \), we have Theorem 5.3 (iii). Finally, by considering the function \( \psi \) with \( \psi(< s, t >) = s \), we have Theorem 5.3 (iv). \( \Box \)
The above relations can be compared with the following relations in information theory except for the relation (v) (see the discussion in Section 6 for this exception).

Theorem 5.6.

(i) \( H(X, Y) = H(Y, X) \) and \( I(X; Y) = I(Y; X) \).

(ii) \( H(X, X) = H(X) \) and \( I(X; X) = H(X) \).

(iii) \( H(X, Y) = H(X) \) and \( I(X; Y) = 0 \) if \( Y \) takes any one fixed value with probability 1, i.e., \( H(Y) = 0 \).

(iv) \( 0 \leq H(X|Y) \).

(v) \( H(X, Y) \leq H(X) + H(Y) \) and \( 0 \leq I(X; Y) \).

Here \( X \) and \( Y \) are discrete random variables, and \( H(X) \), \( H(X, Y) \), \( H(X|Y) \), and \( I(X; Y) \) denote the entropy, joint entropy, conditional entropy, and mutual information, respectively (see e.g. [4] for the detail of these quantities). Thus, our theory built on \( \hat{H}(s) \) has the formal properties of information theory to a certain extent.

6 Discussion

Based on a universal semi-POVM, we have introduced \( \hat{\Omega}_M \) which is an extension of Chaitin’s halting probability \( \Omega_U \) to a measurement operator in an infinite dimensional quantum system, and also we have introduced the operator \( \hat{H}(s) \) which is an extension of the program-size complexity \( H(s) \). In algorithmic information theory, however, \( \Omega_U \) is originally defined through (1) based on the behavior of an optimal computer \( U \), i.e., \( \Omega_U \) is defined as the probability that the universal self-delimiting Turing machine which computes \( U \) halts. Likewise \( H(s) \) is originally defined as the length of the shortest input for a universal self-delimiting Turing machine to output \( s \). Thus \( \Omega_U \) and \( H(s) \) are directly related to a behavior of a computing machine. Therefore, in order to develop our operator version of algorithmic information theory further, it is necessary to find more concrete definitions of \( \hat{\Omega}_M \) and \( \hat{H}(s) \) which are immediately based on a behavior of some sort of computing machine.

In general, a POVM measurement can be realized by first interacting the quantum system on which we make the POVM measurement with an ancilla system, and then making a projective measurement upon the composite system, which consists of the original quantum system and the ancilla system. This interaction is described by a unitary operator. Let \( U_M \) be such a unitary operator in the POVM measurement described by an arbitrary universal semi-POVM \( M \). If we can identify a computing machine \( \mathcal{M} \) of some sort which performs the unitary transformation \( U_M \) in a natural way in the POVM measurement, then we might be able to give a machine interpretation to \( \hat{\Omega}_M \) and \( \hat{H}(s) \). Note that the machine \( \mathcal{M} \) might be different kind of computing machine from the so-called quantum Turing machine. This is because the unitary time evolution operator defined by a quantum Turing machine makes local changes on a quantum system, whereas \( U_M \) makes global changes in general. We leave the development of this line to a future study.

Now, by defining \( H(s) \) as \( -\log_2 m(s) \) for any one universal probability \( m \), [3] proved the following theorem, which corresponds to the inequality in information theory called subadditivity, i.e., Theorem 5.6(v).

Theorem 6.1 (subadditivity). \( \exists c \in \mathbb{R} \ \forall s, t \in \Sigma^* \ c \leq H(s; t) \).
Here $H(s,t)$ was defined as $H(s) + H(t) - H(s,t)$ in $[3]$. Because of the non-commutativity of operators in $X$, however, it is open to prove the corresponding formula for our $\hat{H}(s,t)$. In the proof of Theorem 6.1 given in $[3]$, the product $m(s)m(t)$ is considered. In general, a product of two POVM elements has no physical meaning unless they commute. For a universal semi-POVM $M$, it would seem difficult to prove the commutativity of $M(s)$ and $M(t)$ for distinct $s$ and $t$. Thus $M(s)M(t)$ seems to have no physical meaning as a product of two POVM elements. Hence the difficulty in proving the subadditivity for our $\hat{H}(s,t)$ seems to justify our interpretation of a universal semi-POVM as measurement operators which describe a quantum measurement performed upon a quantum system. Note that, as is shown in $[13]$, we have the subadditivity in finite dimensional setting. This is because $m(s)E$ is a universal semi-POVM in a finite dimensional linear space for any universal probability $m$, where $E$ is the identity matrix. Obviously, $m(s)E$ and $m(t)E$ commute in this case.

**Acknowledgments**

The author is grateful to the 21st Century COE Program of Chuo University for the financial support.

**References**

[1] C. S. Calude, P. H. Hertling, B. Khoussainov, and Y. Wang, Recursively enumerable reals and Chaitin $\Omega$ numbers. Theoret. Comput. Sci. **255**, 125–149 (2001).

[2] G. J. Chaitin, A theory of program size formally identical to information theory. J. Assoc. Comput. Mach. **22**, 329–340 (1975).

[3] G. J. Chaitin, Incompleteness theorems for random reals. Adv. in Appl. Math. **8**, 119–146 (1987).

[4] T. M. Cover and J. A. Thomas, Elements of Information Theory (John Wiley & Sons, Inc., New York 1991).

[5] P. Gács, On the symmetry of algorithmic information. Soviet Math. Dokl. **15**, 1477–1480 (1974); correction, ibid. **15**, 1480 (1974).

[6] P. Gács, Quantum algorithmic entropy. J. Phys. A: Math. Gen. **34**, 6859–6880 (2001).

[7] F. Hiai and K. Yanagi, Hilbert Spaces and Linear Operators (Makino-Shoten, Tokyo 1995). In Japanese.

[8] A. S. Holevo, Statistical Structure of Quantum Theory (Springer-Verlag, Berlin 2001).

[9] L. A. Levin, Laws of information conservation (non-growth) and aspects of the foundations of probability theory. Problems of Inform. Transmission **10**, 206–210 (1974).

[10] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge 2000).

[11] M. B. Pour-El and J. I. Richards, Computability in Analysis and Physics. Perspectives in Mathematical Logic (Springer-Verlag, Berlin 1989).

[12] M. Reed and B. Simon, Methods of Modern Mathematical Physics I: Functional Analysis. Revised and Enlarged Edition (Academic Press, New York 1980).
[13] F. Riesz and B. Sz.-Nagy, Functional Analysis (Dover Publications, Inc., New York 1990).

[14] K. Tadaki, A generalization of Chaitin’s halting probability Ω and halting self-similar sets. Hokkaido Math. J. 31, 219–253 (2002). Electronic version available at URL: http://arxiv.org/abs/nlin/0212001

[15] K. Tadaki, Upper bound by Kolmogorov complexity for the probability in computable quantum measurement. In: Proceedings 5th Conference on Real Numbers and Computers, Lyon, France, September 3–5, 2003, pp. 193–214.

[16] A. K. Zvonkin and L. A. Levin, The complexity of finite objects and the development of the concepts of information and randomness by means of the theory of algorithms. Russian Math. Surveys 25, no. 6, 83–124 (1970).