Curvature estimates for stable minimal surfaces with a common free boundary

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Abstract
The minimal surfaces meeting in triples with equal angles along a common boundary naturally arise from soap films and other physical phenomena. They are also the natural extension of the usual minimal surfaces. In this paper, we consider the multiple junction surface and show the Bernstein’s Theorem still holds for the stable multiple junction surfaces in some special cases. The key part is to derive the $L^p$ estimates of the curvature for multiple junction surfaces.

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1 Introduction
In the paper of Schoen, Simon, Yau [19], they showed the $L^p$ curvature estimates for the minimal hypersurfaces. As a corollary, they could get the generalized Bernstein’s theorem. That is, the only stable immersed hypersurface $\Sigma^n \hookrightarrow \mathbb{R}^{n+1}$ (dim $\Sigma^n = n$ and $n \leq 5$) with area growth condition (i.e. $\mathcal{H}^n(B^\Sigma_R) \leq CR^n$ for all the intrinsic ball $B^\Sigma$ with radius $R$) is a hyperplane. In [4], Colding, Minicozzi showed the stable and 2-sided, simply connected minimal surface in $\mathbb{R}^3$ has quadratic area growth. So the only stable, complete, 2-sided minimal surface in $\mathbb{R}^3$ is a plane. Thus, considering the triple junctions appearing in the natural phenomena such as the soap film, we have the following natural problem like the usual generalized Bernstein’s Theorem.

Problem 1 If three 2-sided minimal surfaces with boundary meet at the same boundary $\Gamma$ and each two of them meet at exactly 120 degrees along $\Gamma$. Suppose they are complete under the distance function and stable in some suitable variations, is it true that each of them is flat?
The surfaces with triple junctions have been studied extensively due to natural phenomena. J. Taylor [23] proved certain locally area-minimizing surfaces should have two types of singularities. The first one is the Y-type singularity, which we’re interested in. Furthermore, G. Lawlor and F. Morgan [14] have shown the triple junction surfaces are always locally minimizing area in any arbitrary dimension and codimension. In the result of C. Mese and S. Yamada [17], they have reproduced soap films with Y-type singularities studied by Taylor [23] by minimizing energy with a suitable boundary condition. So it’s natural to extend other properties of minimal surfaces to the case of minimal triple junction surfaces. Following from Schoen’s rigidity theorem for catenoids [20], J. Bernstein and F. Maggi [2] showed the rigidity of Y-shaped catenoid (the left figure in Fig. 1).

Following from Allard’s regularity [1], L. Simon [22] showed if a stationary integral 2-varifold has density $\frac{3}{2}$ at one point, then it looks like a $C^{1,\mu}$ triple junction minimal surface. Recently, B. Krummel [13] showed higher regularities along their triple junctions for stationary integral varifolds.

Besides considering the minimal triple junction surface, one can move the triple junction surface by mean curvature like the usual mean curvature flow for surfaces. For examples, A. Freire [9] (graph case) and D. Depner, H. Garcke, et al. [6, 7] (general case) have considered the mean curvature flow with triple junctions. F. Schulze and B. White [21] showed the local regularity for mean curvature flow with triple edges. Note that the mean curvature flow of curves with triple junctions in $\mathbb{R}^2$ is just the usual network flow. There are relatively more results in this direction, see for examples [3, 12, 15, 24].

In this paper, we will want to extend the curvature estimates for stable minimal surfaces to the case of minimal triple junction surfaces to see if we can get the Bernstein-type theorem like minimal surfaces. Instead of triple junctions, we can consider the arbitrary number of surfaces that meet at the same boundary.

**Theorem 1** Suppose $M = (\theta_1 \Sigma_1, \ldots, \theta_q \Sigma_q; \Gamma)$ is the (orientable) minimal multiple junction surface in $\mathbb{R}^3$. We assume $M$ is complete, stable and has quadratic area growth. Furthermore, we assume $\Gamma$ is compact and the angles between $\Sigma_i, \Sigma_j$ keep same along $\Gamma$ for $1 \leq i, j \leq q$. Then each $\Sigma_i$ is flat.

The terminology will be explained in Sect. 2.

The special case is the minimal triple junction surface, $M = (\Sigma_1, \Sigma_2, \Sigma_3; \Gamma)$ with $\Gamma$ compact. Note the angles between $\Sigma_i, \Sigma_j$ are $\frac{2\pi}{3}$ for all $i \neq j$. So from the above theorem, we know the Y-shaped catenoid is unstable in our sense.

Another case is the Y-shaped bent helicoid. Like the usual bent helicoid, one can construct Y-shaped bent helicoid by the classical Björling’s formula (see [16] for example). One can choose three unit normal vector fields making equal angles with each other instead of one

![Fig. 1](image.png) Two kinds of Y-shaped catenoid
along a unit circle in the construction. It has a circle as the triple junction and hence it is not stable.

For the case if $\Gamma$ is a straight line, we also have a similar result.

**Theorem 2** Suppose $M$ has the same condition with Theorem 1 except $\Gamma$ being a straight line instead of being compact. Then each $\Sigma_i$ is flat.

Note that this theorem is not enough to show the $Y$-shaped helicoid is unstable since it does not have quadratic area growth.

The key step proving the above theorems is the following curvature estimates for minimal multiple junction surfaces.

**Theorem 3** Suppose $M$ is a minimal multiple junction surface. We assume $M$ is stable. Then for the smooth function $\phi_i$ defined on $\Sigma_i$ with compact support and satisfying compatible condition along $\Gamma$, i.e. $\text{sign}(\phi_i) |A_i|^{p-1} |\phi_i|^p$ will be the projection of a smooth vector field along $\Gamma$ to the normal direction of $\Sigma_i$. Then we have

$$
\sum_{i=1}^{q} \int_{\Sigma_i} \theta_i |\phi_i|^{2p} |A_i|^{2p} \leq \sum_{i=1}^{q} \theta_i \left[ \int_{\Sigma_i} C |\nabla_{\Sigma_i} \phi_i|^{2} |A_i|^{2p-2} |\phi_i|^{2p-2} 
+ \int_{\Gamma} \left( \frac{p-1}{2} |\tau_i (\log |A_i|) - (H_\Gamma, \tau_i) | A_i|^{2p-2} |\phi_i|^{2p} \right) \right]
$$

for $p \in (1, \frac{5}{4})$, where $H_\Gamma$ is the curvature vector of $\Gamma$, $\tau_i$ is the outer conormal of $\Sigma_i$ along $\Gamma$. The constant $C = C(M)$ doesn’t rely on $p$.

The compatible condition will make sure the variation is well-defined on $M$, see (4) for details. Note that the integration of the term $|A_i|^{2p-2} |\phi_i|^{2p} \tau_i (\log |A_i|)$ is still well defined for $|A_i| = 0$ as we’ll explain later on.

For the case of triple junction, i.e. $q = 3$ and $\theta_1 = \theta_2 = \theta_3 = 1$, this condition is equivalent to the following identity.
\[ \sum_{i=1}^{3} \text{sign}(\phi_i) |A_i|^{p-1} |\phi_i|^p = 0 \]

The proof of (1) is essentially following Schoen, Simon, Yau’s proof [19]. Before that, we need to calculate the second variation formula to get the following stability operator,

\[ \sum_{i=1}^{q} \int_{\Sigma_i} \theta_i \left( |\nabla_{\Sigma_i} \phi_i|^2 - |A_i|^2 \phi_i^2 \right) - \int_{\Gamma} \theta_i \phi_i^2 \mathbf{H} \cdot \tau_i \] (2)

with \( \phi_i \) satisfying the compatible condition (4).

After getting curvature estimates, we can choose a suitable function. The trick part is we need our functions to satisfy the compatible conditions. So near \( \Gamma \), \( \phi_i \) should satisfy some compatible conditions and the gradient \( \phi_i \) cannot vanish near \( \Gamma \). This is why we need \( \Gamma \) to be compact or \( \Gamma \) to be a straight line. So we can choose \( p \) close to 1 to control the term near \( \Gamma \). For the part that is far from \( \Gamma \), we can choose \( \phi_i \) like the standard cutoff function in a large ball. After choosing a suitable function, we can deduce the curvature needs to vanish everywhere.

## 2 Minimal surfaces with multiple junctions

In this section, we will fix some notations and give the definition of the minimal multiple junction surface and several related concepts.

For \( q \in \mathbb{N} \), we suppose \( \Sigma_1, \cdots, \Sigma_q \) are all smooth 2-dimensional manifolds with boundary \( \partial \Sigma_i \) and \( \Gamma \) is a smooth 1-dimensional manifold. We only consider the case that each \( \Sigma_i \) is orientable so we have the well-defined unit normal vector field on \( \Sigma_i \) when immersing into \( \mathbb{R}^3 \). For each \( i \), we suppose there is a diffeomorphism \( p_i : \Gamma \rightarrow \partial \Sigma_i \). We will always immerse \( \Sigma_i \) into \( \mathbb{R}^3 \) when talking about extrinsic geometric quantities like normal vectors, second fundamental form, and so on. The Table 1 lists the notations used in this paper.

### Table 1 Notations

| Symbols       | Meaning                                      |
|---------------|----------------------------------------------|
| \( X \cdot Y \) | Standard inner product in \( \mathbb{R}^3 \). |
| \( D_X Y \)   | Standard coderivative in \( \mathbb{R}^3 \).  |
| \( B_r(x) \)  | Open ball centered at \( x \) with radius \( r \) in \( \mathbb{R}^3 \). |
| \( T \Sigma \) (resp. \( T \Gamma \)) | Tangent bundle of \( \Sigma \) (resp. \( \Gamma \)). |
| \( N \Sigma \) (resp. \( N \Gamma \)) | Normal bundle of \( \Sigma \) (resp. \( \Gamma \)) in \( \mathbb{R}^3 \). |
| \( v_i \in \Gamma(T \Sigma_i) \) | Unit normal vector filed on \( \Sigma_i \) in \( \mathbb{R}^3 \). |
| \( \tau_i \in \Gamma(N \partial \Sigma_i) \) | Unit outer conormal of \( \partial \Sigma_i \) on \( \Sigma_i \) pointing outside of \( \Sigma_i \). That is, \( \tau_i(p) \in T_p \Sigma_i \cap N_p \partial \Sigma_i \) for any \( p \in \partial \Sigma_i \). |
| \( A_i(X, Y) = D_X Y \cdot v_i \) | The second fundamental form on \( \Sigma_i \). |
| \( |A_i| \) | The norm of second fundamental form on \( \Sigma_i \). |
| \( H_i \) (resp. \( H_\Gamma \)) | The mean curvature vector of \( \Sigma_i \) (resp. \( \Gamma \)). |
| \( \text{sign}(x) \) | The sign function. |
2.1 Definition of multiple junction surfaces

**Definition 1** We say \( M = (\Sigma_1, \ldots, \Sigma_q; \Gamma) \) is an *intrinsic multiple junction surface* if it is a quotient space \( \bigcup_{i=1}^{q} \Sigma_i / \sim \) where the equivalent relation is defined as the following,
\[
x \sim y \text{ if and only if } x = y \text{ or } x \in \partial \Sigma_i, y \in \partial \Sigma_j \text{ for some } 1 \leq i, j \leq q \text{ and } x = p_i \circ p_j^{-1}(y).
\]

**Remark 1** We can define \( M \) as a topological space with coordinate charts like the definition of a smooth manifold.

**Definition 2** We say \( M = (\Sigma_1, \ldots, \Sigma_q; \Gamma) \) is a *multiple junction surface* in \( \mathbb{R}^3 \) if \( (\Sigma_1, \ldots, \Sigma_q; \Gamma) \) is an intrinsic multiple junction surface and there is a map \( \varphi : M \to \mathbb{R}^3 \) such that the restriction of \( \varphi \) on \( \Sigma_i \) is a smooth immersion for each \( 1 \leq i \leq q \).

We call the map \( \varphi \) smooth immersion for \( M \).

Note that we have a natural metric on each \( \Sigma_i \) for \( 1 \leq i \leq q \) by pulling back the metric on \( \mathbb{R}^3 \).

In general, we will consider the multiple junction surface \( M = (\Sigma_1, \ldots, \Sigma_q; \Gamma) \) with constant density \( \theta_1, \ldots, \theta_q > 0 \) such that we have constant density function \( \theta_i \) on the surface \( \Sigma_i \). We will write this surface as \( M = (\theta_1 \Sigma_1, \ldots, \theta_q \Sigma_q; \Gamma) \). So the associated 2-varifold of \( M \) has the form
\[
V_M := \sum_{i=1}^{q} \theta_i |\Sigma_i|.
\]

Here, \( |\Sigma_i| \) denotes the multiplicity one varifold associated with the surface \( \Sigma_i \). Note that we do not require \( \theta_i \) to be the integers.

**Definition 3** We say a multiple junction surface \( M = (\theta_1 \Sigma_1, \ldots, \theta_q \Sigma_q; \Gamma) \) is minimal if each \( \Sigma_i \) is a smooth minimal immersion and on \( \Gamma \), we have
\[
\sum_{i=1}^{q} \theta_i \tau_i = 0.
\]

**Remark 2** By the regularity of B. Krummel [13], suppose a stationary integral 2-varifold \( V \) has the form
\[
V = \sum_{i=1}^{q} \theta_i |\Sigma_i|
\]
for distinct \( C^{1,\mu} \) embedded hypersurfaces-with-boundary \( \Sigma_1, \ldots, \Sigma_q \) with a common boundary \( \Gamma \) for some \( 0 < \mu < 1 \). Then for any \( Z \in \Gamma \), if \( T_Z \Sigma_i \) are not the same plane in \( \mathbb{R}^3 \), then we can find a neighborhood \( O_Z \) of \( Z \) such that \( \Sigma_i \) is smooth and \( \Gamma \) is a smooth curve in \( O_Z \). Indeed, they are all analytic since \( \mathbb{R}^3 \) is a real analytic manifold.

Moreover, by the regularity of cylindrical tangent cones by L. Simon [22], if a stationary integral 2-varifold in \( U \) has density \( \frac{3}{2} \) at some point \( Z \in U \), then near \( Z \), \( M \) is the varifold associated with three \( C^{1,\mu} \) minimal surface with a common boundary \( \Gamma \) and \( \Gamma \) is still a \( C^{1,\mu} \) curve for some \( 0 < \mu < 1 \). So at least for the triple junction, we can assume a much weaker condition on the above definition.
For each $\Sigma_i$, we can define the intrinsic distance function $d_i(x, y)$ for $x, y \in \Sigma_i$, which is the length of the shortest geodesic jointing $x, y$ on $\Sigma_i$.

So we can define a global distance function $d(x, y)$ for $x \in \Sigma_i, y \in \Sigma_j$ by

$$d(x, y) := \inf \left\{ \sum_{k=0}^{l-1} d_i(x_k, x_{k+1}) : x_0 = x, x_{l+1} = y, x_1, \cdots, x_l \in \Gamma, \right.$$  

$$i_0 = i, i_{l-1} = j, 1 \leq i_1, \cdots, i_{l-2} \leq q, \text{ for } l \in \mathbb{N} \right\}.$$

Hence, we use $B^M_r(x) = \{ y \in M : d(x, y) < r \}$ to denote the intrinsic ball on $M$.

Now we can define the distance function with respect to $\Gamma$ as

$$d_\Gamma(x) = \inf_{y \in \Gamma} d(x, y) \text{ for } x \in M.$$

**Definition 4** We say a multiple junction surface $M$ is complete if it is complete in the distance function $d(\cdot, \cdot)$. That is, every Cauchy sequence converges to some point in $M$ under this distance function.

### 2.2 Definition of functional spaces on multiple junction surfaces

From now on, we will always assume $M = (\theta_1 \Sigma_1, \cdots, \theta_q \Sigma_q; \Gamma)$ is a complete minimal multiple junction surface in $\mathbb{R}^3$.

When we consider the variation on $M$, we will need to consider a kind of vector field on $M$. So we have the following definition.

**Definition 5** We say a map $X(x) : M \to T_x \mathbb{R}^3 \simeq \mathbb{R}^3$ is a $C^k$ vector field on $M$ if each $X|_{\Sigma_i}$ is a $C^k$ vector field on $\Sigma_i$ for $1 \leq i \leq q$ and they agree with each other along $\Gamma$. That is $X(p)|_{\Sigma_i} = X(p)|_{\Sigma_j}$ when $p \in \Gamma$ for any $1 \leq i, j \leq q$. We write this vector field space as $C^k(M; T \mathbb{R}^3)$.

Note that we do not require the vector field can be jointed smoothly across the junction. For example, let $M = (H_1, H_2; \Gamma)$ with $H_1, H_2$ the opposed two half-planes in $\mathbb{R}^3$ and $\Gamma$ the straight line in $\mathbb{R}^3$. Then as an immersion, $M$ can be regarded as a smooth plane in $\mathbb{R}^3$ but the smooth vector field on $M$ may not be smooth on this plane.

Let’s consider the space of functions on $M$. The natural definition is to consider the function on $M$, which write as $f : M \to \mathbb{R}$ and say it is $C^k$ if the restriction on each $\Sigma_i$ is $C^k$ up to the boundary.

Somehow this function space is not big enough to contain the function we are interested in. For example, give a vector field $V \in C^k(M, T \mathbb{R}^3)$, the function defined by $V \cdot v_i$ for $x \in \Sigma_i$ is not a $C^k$ function defined above. Actually, it isn’t well-defined on $M$ since on $\Gamma$, the value will depend on $i$. So we define some large function spaces as follows.

**Definition 6** We say a function $f(x) : \bigcup_{i=1}^q \Sigma_i \to \mathbb{R}$ is in a Sobolev space $W^{k, p}(M)$ for $1 \leq p \leq \infty$ if each restriction $f|_{\Sigma_i}$ is in $W^{k, p}(\Sigma_i)$ for each $i$.

Similarly, we can define the $L^p$ space as $L^p(M)$ and continuous function space $C^k(M)$.

Usually, we will write $H^k(M) = W^{k, 2}(M)$ to denote it as a Hilbert space.

By our definition, we do not impose any condition along $\Gamma$ for $f \in C^k(M)$. In general, we still wish our function can also be extended to a suitable vector field on $M$ at least. So we say $f \in C^k(M)$ satisfies the compatible condition if there exists a $C^k$ vector field $W$ along $\Gamma$ (i.e. $W \in C^k(\Gamma, T \mathbb{R}^3)$), such that
where \( f_i = f|_{\Sigma_i} \).

Note that by Trace Theorem, if \( f \in W^{1,p}(M) \) for some \( 1 \leq p \leq \infty \), then for any \( 1 \leq i \leq q \), the function \( f_i \) can be restricted to the boundary \( \partial \Sigma_i \) in the \( L^p(\partial \Sigma_i) \) sense.

So we can say \( f \in W^{1,p}(M) \) satisfies the compatible condition if there is a \( L^p_{loc}(\Gamma) \) vector field \( W \) along \( \Gamma \), such that

\[
f_i|_{\partial \Sigma_i} (x) = W(p_i^{-1}(x)) \cdot v_i(x) \text{ for } x \in \partial \Sigma_i, \quad 1 \leq i \leq q.
\]

(3)

Sometime we will write (3),(4) as \( f_i = W \cdot v_i \) for short.

Clearly, the function defined by \( V \cdot v_i \) is in \( C^k(M) \) for \( V \in C^k(M, T\mathbb{R}^3) \) and satisfies (3). Conversely, for any \( f \in C^k(M) \) satisfying (3), by definition we have \( W \in C^k(\Gamma, T\mathbb{R}^3) \), so \( f_i = W \cdot v_i \). For each \( 1 \leq i \leq q \), we can extend \( W^\top \) on the whole \( \Sigma_i \) to \( \tilde{V}_i \) such that \( \tilde{V}_i \) is a \( C^k \) tangential vector field on \( \Sigma_i \). This is because \( W^\top \) is \( C^k \) on \( \Gamma \) and \( \Gamma \) is smooth on \( \Sigma_i \). So we can define \( V_i = \tilde{V}_i + f_i v_i \), which is a \( C^k \) vector field on \( \Sigma_i \). So the vector field \( V \) defined by \( V = V_i \) on \( \Sigma_i \) is in \( C^k(M, T\mathbb{R}^3) \) and satisfies \( V \cdot v_i = f_i \).

**Remark 3** For the minimal triple junction surface, the compatible condition has a simple form. For \( f \in W^{1,p}(M) \), \( f \) satisfies the compatible condition if and only if

\[
f_1 + f_2 + f_3 = 0 \quad \mathcal{H}^1\text{-a.e. on } \Gamma.
\]

(4)

The “only if” part is trivial since if \( f \) has form \( f_i = W \cdot v_i \mathcal{H}^1\text{-a.e. along } \Gamma \) for some \( W \in L^p_{loc}(\Gamma) \), then

\[
f_1 + f_2 + f_3 = W \cdot \sum_{i=1}^{3} v_i = 0 \quad \mathcal{H}^1\text{-a.e. along } \Gamma.
\]

For the “if” part, we choose \( W \) in the following way

\[
W = \left( \frac{4}{3} f_1 + \frac{2}{3} f_2 \right) v_1 + \left( \frac{2}{3} f_1 + \frac{4}{3} f_2 \right) v_2 \quad \text{along } \Gamma.
\]

Since \( f_i \) is in \( L^p_{loc}(\Gamma) \) at least by trace theorem and \( v_i \) is smooth along \( \Gamma \), we know \( W \) is at least a \( L^p_{loc}(\Gamma) \) vector field along \( \Gamma \). Now we need to verify \( W \cdot v_i = f_i \) for \( i = 1, 2, 3 \).

When \( i = 1 \), we have

\[
W \cdot v_1 = \frac{4}{3} f_1 + \frac{2}{3} f_2 - \frac{1}{2} \left( \frac{2}{3} f_1 + \frac{4}{3} f_2 \right) = f_1.
\]

Similarly, we have \( W \cdot v_2 = f_2 \). For the last one, \( i = 3 \), we have

\[
W \cdot v_3 = -\frac{1}{2} \left[ \frac{4}{3} f_1 + \frac{2}{3} f_2 + \frac{2}{3} f_1 + \frac{4}{3} f_2 \right] = -f_1 - f_2 = f_3 \quad \mathcal{H}^1\text{-a.e. on } \Gamma.
\]

So based on our definition, \( f \) satisfies the compatible condition (4).

**Remark 4** All the definitions in this section can be extended to arbitrary ambient manifolds with arbitrary dimension and codimension.
3 First and second variation of \( M \)

Now we can consider the variation of \( M \). We say \( M_t = (\theta_t \Sigma_{1t}, \ldots, \theta_q \Sigma_{qt}; \Gamma_t) \), \( t \in (-\varepsilon, \varepsilon) \) (considered as an immersion) is a \( C^k \) variation of \( M \) if each \( \Sigma_{it} \) is a \( C^k \) variation of \( \Sigma_i \) up to boundary and \( \Gamma_t \) is a \( C^k \) variation of \( \Gamma \). Of course, we can write this variation as one-parameter family of immersions \( \varphi_t(x) := \varphi(t, x) : (-\varepsilon, \varepsilon) \times M \to \mathbb{R}^3 \) such that for each \( t \), \( M_t = \varphi_t(M) \) is a multiple junction surface and restrict on each \( \Sigma_i \) the variation \( \varphi_t \) is \( C^k \).

For each \( C^k \) variation \( \varphi_t : M \to \mathbb{R}^3 \), there is an associated vector field \( V(x) : M \to T_x \mathbb{R}^3 \), which is \( C^k \) on \( \Sigma_i \) for each \( 1 \leq i \leq q \).

Let \( U \subseteq M \) be an open subset in \( M \) such that \( \overline{U} \) is compact. Suppose we have a \( C^k \) variation for \( M \) with associated vector field \( V(x) = \frac{d\varphi_t(x)}{dt} \) with compact support in \( U \). We can define the first variation of the area of \( M \) in \( U \), which is given by

\[
\frac{d}{dt} \bigg|_{t=0} |\varphi_t(U)| = \int_{M \cap U} \text{div}T_{\Gamma_t}V(x)d\|V_M\|(x)
\]

\[
= \sum_{i=1}^{q} \int_{\Sigma_i \cap U} \text{div}T_{\Gamma_t} \Sigma_i V(x) \theta_t d\mu_{\Sigma_i}(x)
\]

\[
= \sum_{i=1}^{q} - \int_{\Sigma_i \cap U} V \cdot H_i \theta_t d\mu_{\Sigma_i} + \sum_{i=1}^{q} \int_{\Gamma} V \cdot \tau_i \theta_t d\mu_{\Gamma}
\]

where \( \text{div}_P V = D_{e_1}V \cdot e_1 + D_{e_2}V \cdot e_2 \) for any orthonormal basis \( e_1, e_2 \) of the plane \( P \). The \( \mu_{\Sigma_i} \) is the area measure on \( \Sigma_i \). \( \|V_M\| \) is the weight measure of \( V_M \).

We say \( M \) is stationary in \( U \) if for any such variation, we have \( \frac{d}{dt} |\varphi_t(U)| = 0 \).

Note that every \( C^k \) vector field on \( M \) will give a \( C^k \) variation of \( M \). So \( M \) is stationary in \( U \) if and only if \( H_i = 0 \) on each \( \Sigma_i \cap U \) and on \( \Gamma \cap U \), we have

\[
\sum_{i=1}^{q} \theta_i \tau_i = 0.
\]

This is precisely the condition that we define the minimal multiple junction surfaces.

Now we can consider the second variation of area for minimal multiple junction surfaces.

**Definition 7** We say a minimal multiple junction surface \( M \) is stable in \( U \) whose closure is compact if for every variation \( \varphi_t \) of \( M \) in \( U \), we have

\[
\frac{d^2}{dt^2} \bigg|_{t=0} |\varphi_t(M \cap U)| \geq 0.
\]

So we say \( M \) is stable if for every \( U \) with compact closure, we always have \( \frac{d^2}{dt^2} \bigg|_{t=0} |\varphi_t(M \cap U)| \geq 0 \) for any variation \( \varphi_t \) in \( U \).

The remaining part of this section is to deduce the stability operator (2).

**Theorem 4** If \( M \) is a stable complete minimal multiple junction surface in \( \mathbb{R}^3 \). Then for any \( \phi \in C^k(M) \) satisfying (3) with compact support, we have

\[
\sum_{i=1}^{q} \int_{\Sigma_i} \left( |\nabla \Sigma_i \phi_i|^2 - |A_i|^2 \phi_i^2 \right) \theta_i d\mu_{\Sigma_i} - \int_{\Gamma} \phi_i^2 H_i \cdot \tau_i \theta_i d\mu_{\Gamma} \geq 0
\]
where, $\nabla_{\Sigma_i} \phi_i$ denotes the gradient on $\Sigma_i$ and $|A_i|$ denotes the norm of the second fundamental form on $\Sigma_i$. $H_\Gamma$ means the curvature vector of the curve $\Gamma$. Moreover, it holds even for $\phi \in H^1(M)$ satisfying (4) with compact support.

**Proof**

Let $\phi \in C^k(M)$. Since it satisfies the compatible condition, we can find $V \in C^k(M, T\mathbb{R}^3)$ such that $\phi_t = V \cdot \nu_t$. So there is a variation $\phi_t$ with compact support associated with vector field $V$, i.e. $V = \left. \frac{d}{dt} \right|_{t=0} \phi_t$.

Suppose $\phi_t$ is supported in $U$. So by the first variation formula, we have

$$
\left. \frac{d}{dt} \right|_{t=0} \| \phi_t(M \cap U) \| = \sum_{i=1}^q \int_{\Sigma_i} V_t \cdot \nu_{i t} H_{\Sigma_i} \theta_i d\mu_{\Sigma_i} + \sum_{i=1}^q \int_{\Gamma_t} V_t \cdot \tau_{i t} \theta_i d\mu_{\Gamma_t}
$$

where $H_{\Sigma_i} = H_{\Sigma_i} \cdot \nu_{i t}$ and $V_t = \left. \frac{d}{dt} \right|_{t=0} \phi_t$.

So after taking derivative with respect to $t$ on the first variation formula, we have

$$
\left. \frac{d^2}{dt^2} \right|_{t=0} \| \phi_t(M \cap U) \| = \sum_{i=1}^q \int_{\Sigma_i} V \cdot \nu_i \left( \left. \frac{d}{dt} \right|_{t=0} H_{\Sigma_i} \right) \theta_i d\mu_{\Sigma_i} + \sum_{i=1}^q \int_{\Gamma} V \cdot \theta_i d\mu_{\Gamma_t} + \sum_{i=1}^q \int_{\Gamma} \tau_{i t} \cdot \theta_i d\mu_{\Gamma_t}
$$

Note that by stationary condition, we know $H_{\Sigma_i} = 0$ and $\sum_{i=1}^q \theta_i \tau_i = 0$ along $\Gamma$, so the second and forth terms in (6) vanish. Moreover, we have the well known formula (cf. [18])

$$
\left. \frac{d}{dt} \right|_{t=0} H_{\Sigma_i} = \Delta_{\Sigma_i} \phi_i + |A_i|^2 \phi_i
$$

(7)

So actually, we only need to compute $V \cdot \left. \frac{d}{dt} \right|_{t=0} \tau_{i t}$. For simplicity, we use $(\cdot)'$ to denote $\left. \frac{d}{dt} \right|_{t=0} (\cdot)$.

Before computing $\tau_i'$, we need to get $v_i'$. Let $e_1, e_2$ be the orthonormal frame of $T_x \Sigma_i$ for some $x \in \Sigma_i$. Let $e_{i t} = d\phi_t(e_i)$.

We can decompose $V = \phi_t v_i + W_i$ on $\Sigma_i$ with $W_i$ tangential to $\Sigma_i$. Note that $[e_{i t}, V_t] = 0$, we have

$$
v_i' \cdot e_i = -v_i \cdot e_i' = -v_i \cdot D_{e_i} V = -v_i(\phi_t) - v_i \cdot D_{e_i} W = -e_i(\phi_t) - A_{\Sigma_i}(e_i, W)
$$

Suppose $\eta$ is the unit tangential vector field on $\Gamma$ and also write $\eta_t = d\phi_t(\eta)$. This time we decompose $V$ as $V = \phi_t v_i + f_i \tau_i + g \eta$ where $f_i = V \cdot \tau_i$, $g = V \cdot \eta$. Similarly with $v_i'$, we have

$$
\eta' \cdot \tau_i = \tau_i \cdot D_{\eta} V = \phi_t \tau_i + D_{\eta} f_i + g \tau_i \cdot D_{\eta} \eta
$$

$$
= -\phi_t A_{\Sigma_i}(\tau_i, \eta) + \eta(f_i) + g H_{\Gamma} \cdot \tau_i
$$

Hence,

$$
\tau_i' \cdot V = (\tau_i' \cdot v_i) \phi_i + (\tau_i' \cdot \eta) g = -(\tau_i \cdot v_i) \phi_i - (\tau_i \cdot \eta) g
$$
The proof of Theorem 4 can be extended to higher-dimensional multiple junction hypersurfaces in an arbitrary complete ambient manifold $N$ directly. The stability operator will have the form

$$
\sum_{i=1}^{q} \int_{\Sigma_i} \left[ \nabla_{\Sigma_i} \phi_i j^2 - \text{Ric}_N (\nu_i) \phi_i \right] d\mu_{\Sigma_i} - \int_{\Gamma} \phi_i \cdot (H \cdot \tau_i) \theta_i
$$

Remark 5 The proof of Theorem 4 can be extended to higher-dimensional multiple junction hypersurfaces in an arbitrary complete ambient manifold $N$ directly. The stability operator will have the form

$$
\sum_{i=1}^{q} \int_{\Sigma_i} \left[ |A_i|^2 \phi_i \right] d\mu_{\Sigma_i} - \int_{\Gamma} \phi_i \cdot (H \cdot \tau_i) \theta_i
$$
4 Functions with finite orders

Before giving the proof of \( L^p \) estimate, we need to consider some special function spaces on \( M \) containing \(|A_i|\) and test functions we’re interested in. Specifically, at least we want to show \( \tau_i (\log |A_i| |A_i|^{2p-2} \text{ are locally integrable for each } p > 1. \)

Let’s fix a surface \( \Sigma \subseteq \mathbb{R}^3 \) with smooth boundary \( \Gamma \). We will assume \( 0 < \alpha < \infty. \)

**Definition 8** We say a non-negative function \( g(x) \) on \( \Sigma \) has smooth order \( \alpha \) near \( x_0 \) if there is a conformal coordinate chart \( \varphi(z) : V \to U \subset \Sigma \) with \( 0 \in V \subset \mathbb{C} \) and \( \varphi(0) = x_0 \) such that \( g(z) := g(\varphi(z)) \) has form

\[
g(z) = h(z) |z|^{\alpha}
\]

where \( h(z) \) is positive and smooth in \( V \).

Here, the conformal coordinate chart is the coordinate chart that metric near \( x_0 \) has form \( \lambda^2(z) |dz|^2 \). We allow \( x_0 \) to be on \( \Gamma \) so that \( V \) is a domain with smooth boundary in \( \mathbb{C} \).

**Definition 9** We call a non-negative function \( g \) has smooth finite order on \( \Sigma \), if there exists a discrete subset \( P \subset \Sigma \) such that \( g \) is smooth and positive on \( \Sigma \setminus P \) and \( g \) has smooth order \( \alpha \) near \( x \) for each \( x \in P \). We write this function space as \( \tilde{C}_+ (\Sigma) \) for conveniences.

Similarly, we say a non-negative function \( g(p) \) on \( \Gamma \) has smooth order \( \alpha \) near \( x_0 \) if \( g(p) \) can be written as \( g(t) := g(p(t)) = |t|^{\alpha} h(t) \) for some smooth positive function \( h(t) \) under the arc length parametrization such that \( g(0) = x_0 \). So we can define the smooth finite order function space \( \tilde{C}_+ (\Gamma) \) on \( \Gamma \) which contains the functions smooth outside a discrete set \( P \) and has smooth order near each point of \( P \).

We have the following lemma for the relation of these two spaces.

**Lemma 1** Let \( \Sigma \) be a two-dimensional analytic Riemannian manifold with smooth boundary \( \partial \Sigma \). For any \( \tilde{g} \in \tilde{C}_+ (\Gamma) \), there is an extension of \( \tilde{g} \) denoted by \( g \) such that \( g \in C_+ (\Sigma) \). Moreover, we can require \( g \) is positive on \( \Sigma \setminus \partial \Sigma \).

Conversely, for any \( g \in \tilde{C}_+ (\Sigma) \), the restriction of \( g \) on \( \Gamma \) lies in \( \tilde{C}_+ (\Gamma) \).

**Proof** Let’s write \( P = \{ x \in \Gamma : \tilde{g}(x) = 0 \} \). We can just focus on the extension near each \( x \in P \) since \( \partial \Sigma \) is smooth on \( \Sigma \) and positive smooth function can be easily extended from \( \partial \Sigma \) to \( \Sigma \) locally and keep positivity. Then we can use partition of unity to get a global extension.

Fix \( x_0 \in P \), we choose a conformal coordinate \( \varphi(z) : V \to U \) where \( V, U \) are all homeomorphic to a half disk such that \( \varphi(0) = x_0 \). WLOG, we assume the metric has form \( \lambda^2(z) d\bar{z}dz \) with \( \lambda(0) = 1 \) in this chart. Moreover, we can assume \( U \) is small enough such that \( \partial \Sigma \cap U \) has an arc length parametrization \( \gamma(t) : (a, b) \to \Gamma \) with \( \gamma(0) = x_0 \).

Let’s consider the function \( f(t) := \frac{|t|}{|\varphi^{-1}(\varphi(t))|} \) defined in \( (-\epsilon', \epsilon') \setminus 0 \) for some small \( \epsilon' \) where \( |z| \) is the usual absolute value in \( \mathbb{C} \) with respect to this coordinate chart. We want to show that, by define \( f(0) = 1 \), we can get a smooth function \( f \) on \( (-\epsilon'', \epsilon'') \).

Note that the map \( \psi(t) := \varphi^{-1} \circ \gamma(t) \) is smooth from \( (-\epsilon', \epsilon') \) to \( V \subset \mathbb{C} \) with \( \psi(0) = 0 \), so we can expand \( \psi(t) \) as \( \psi(t) = \psi'(0)t + \psi_1(t)t^2 \) for some smooth map \( \psi_1 \) near \( 0 \).

So we have

\[
f(t) = \frac{|t|}{|\psi(t)|} = \frac{1}{|\psi'(0) + \psi_1(t)t|} = \frac{1}{1 + t\frac{\psi_1(t)}{|\psi'(0)|}}
\]

is smooth in \( (-\epsilon'', \epsilon'') \) for some small \( \epsilon'' \) since \( |\psi'(0)| = 1 \).
Based on definition of $g$, we can write $g(t) = |t|^\alpha h(t)$ for some smooth function $h(t)$ near 0.

So since $h(t) f(t)^\alpha$ is smooth and positive near 0 in $\Gamma$, we can extend it smoothly to a neighborhood of $x_0 \in \Sigma$. We denote this extension function as $\tilde{h}(z)$. Then we define $\tilde{g} = |z|^\alpha \tilde{h}$ near $x$. This is a local extension of $g$ near $x$ which is positive except at the point $x$ since on $\partial \Sigma$, we have

$$\tilde{g}(\gamma(t)) = |\varphi^{-1} \circ \gamma(t)|^\alpha \left( \frac{|t|}{|\varphi^{-1} \circ \gamma(t)|} \right)^\alpha h(t) = h(t) |t|^\alpha.$$  

So by partition of unity, we can get an extension of $\tilde{g}$ as we want. Moreover, we can keep $\tilde{g}$ positive on $\Sigma \setminus \partial \Sigma$.

Another part is essentially similar to this case. \hfill \Box

So for our multiple junction surface $M = (\theta_1 \Sigma_1, \ldots, \theta_q \Sigma_q; \Gamma)$, the norm of second fundamental form $|A_i|$ will belong to $\tilde{C}_0(\Sigma_i)$ if $\Sigma_i$ is non-flat. This is because when $\Sigma_i$ is minimal, Gauss map $v_i(x) : \Sigma_i \to \mathbb{S}^2$ will be the holomorphic map. So $|A_i|^2 = |dv_i|^2$. Hence $|A_i|$ will have form $|z|^k f(z)$ near each zeros of $|A_i|$ with $f(z)$ positive and smooth for some $k \in \mathbb{Z}_+$ in some conformal coordinate. Hence,

$$|\tau(\log |A_i|)| |A_i|^{2p-2} \leq C_1 |z|^{2p-2} + C_2 |z|^{2p-3}$$

where $C_i$ only depends on $f$. Since $\int_{-\varepsilon}^{\varepsilon} |t|^{2p-2} dt$, $\int_{-\varepsilon}^{\varepsilon} |t|^{2p-3}$ are all finite, we know

$$\int (p-1) |\tau(\log |A_i|)| |A_i|^{2p-2} |\phi_i|^2 d\mu$$

is locally integrable for each $\phi_i \in L^\infty(\Gamma)$ and $p > 1$.

Moreover, we also have

$$|A|^{\alpha} \in L^\infty_{\text{loc}}(M) \cap H^1_{\text{loc}}(M)$$

by expanding the gradient of $|A_i|$ near its zeros for any $\alpha > 0$. To be more precise, $|A_i|$ has form $|A_i| = |z|^k f(z)$ near each zeros of $|A_i|$ with $f(z)$ positive and smooth for some $k \in \mathbb{Z}_+$ under some conformal coordinate. The fact $|A|^{\alpha} \in L^\infty_{\text{loc}}(M)$ is easy to see. For the gradient of $|A|$, we have

$$|\nabla_{\Sigma_i} (|A|^{\alpha})| \leq C \left( |z|^k |f|^{\alpha-1} |\nabla f| + k\alpha |z|^{k\alpha-1} |f|^{\alpha} \right).$$

Here $\nabla$ means the gradient under conformal coordinate and we’ve used the gradient on surfaces that can be controlled by the gradient under conformal coordinate. So the constant $C$ is related to the choice of conformal coordinate. No matter what, the right hand side of (10) is in $L^2_{\text{loc}}(\Sigma_i)$ for any $\alpha > 0$.

5 $L^p$ estimates for the multiple junction surfaces

In this section, we will prove the Theorem 3.

For convenience, we use the following notation. For any $\phi \in H^1(M)$ with compact support, we write

$$\int_{\mu} \phi := \sum_{i=1}^{q} \int_{\Sigma_i} \phi_i \theta_i d\mu_{\Sigma_i}$$
and
\[ \int_\Gamma \phi := \sum_{i=1}^q \int_\Gamma \phi_i \theta_i \, d\mu_\Gamma \]
for the integration on \( M = (\theta_1 \Sigma_1, \ldots, \theta_q \Sigma_q; \Gamma) \).

So the stability inequality can be written as
\[ \int_\Sigma | \nabla_\Sigma \phi |^2 - | A |^2 \phi^2 - \int_\Gamma \phi^2 \mathbf{H}_\Gamma \cdot \tau \geq 0 \]
for \( \phi \in H^1(M) \) with compact support satisfying (4).

**Theorem 5** Suppose \( M = (\theta_1 \Sigma_1, \ldots, \theta_q \Sigma_q; \Gamma) \) is a minimal multiple junction surface. Assume \( M \) is stable and complete. Let \( \phi \in H^1(M) \cap L^\infty(M) \) with compact support such that \( \text{sign}(\phi) | A |^{p-1} | \phi |^p \) satisfies the compatible condition (4). Then
\[ \int_\Sigma | A |^2 \phi |^2 \leq C_1 \int_\Sigma | A |^{2p-2} | \phi |^{2p-2} | \nabla_\Sigma \phi |^2 + \int_\Gamma \left[ \frac{p-1}{2} | \tau (\log | A |) - \mathbf{H}_\Gamma \cdot \tau \right] | A |^{2p-2} | \phi |^2. \quad (11) \]

Moreover, if \( \phi \in W^{1,2p}(M) \cap L^\infty(M) \), we also have
\[ \int_\Sigma | A |^2 \phi |^2 \leq C_1' \int_\Sigma | \nabla_\Sigma \phi |^2 + C_2' \int_\Gamma \left[ (p-1) | \tau (\log | A |) - \mathbf{H}_\Gamma \cdot \tau \right] | A |^{2p-2} | \phi |^2. \quad (12) \]

Here, we assume \( 1 < p < \frac{5}{4} \), and \( C_1, C_1', C_2' \) will only depend on \( M \), they do not depend on \( p \).

Note that if the \( \Sigma_i \) is flat, we can define
\[ \int_\Gamma | \tau_i (\log | A_i |) | A_i |^{2p-2} | \phi_i |^2 = 0 \]
So the right hand side of (11) will always be well-defined as we want.

**Remark 6** Although the \( L^p \) estimate (12) is the one appearing the original paper [19], we still need the slightly stronger version one like (11) in the later application since the condition \( \phi \in W^{1,2p}(M) \cap L^\infty(M) \) is not always satisfied based our choice of functions.

**Proof of Theorem 5** Let first consider the case that every \( \Sigma_i \) is non-flat. This means every \( | A_i | \) has only isolated zeros on \( \Sigma_i \).

We suppose \( \psi \in L^\infty(M) \cap H^1(M) \) with compact support. So by Hölder’s inequality, we know \( | A |^{p-1} \psi \in L^\infty(M) \cap H^1(M) \). We also suppose \( | A |^{p-1} \psi \) satisfies the compatible condition (4). This implies we can take \( \phi = | A |^{p-1} \psi \) in the stability inequality (5). Then we have
\[ \int_\Sigma | A |^2 \psi^2 \leq (p-1)^2 \int_\Sigma | A |^{2p-4} | \nabla_\Sigma | A | |^2 \psi^2 + \int_\Sigma | A |^{2p-2} | \nabla_\Sigma \psi |^2 + 2(p-1) \int_\Sigma | A |^{2p-3} \psi \nabla_\Sigma | A | \cdot \nabla_\Sigma \psi - \int_\Gamma | A |^{2p-2} \psi^2 \mathbf{H}_\Gamma \cdot \tau. \quad (13) \]

Note that we’ve used \( | A |^{p-1} \in L^\infty_{\text{loc}}(M) \cap H^1_{\text{loc}}(M) \) here (cf. (9)) to make sure the right hand side of (13) is finite.
On the minimal surface $\Sigma_i$, we have Simon’s identity (see [5] for example), we have

$$|A_i| \Delta_{\Sigma_i} |A_i| + |A_i|^4 = |\nabla_{\Sigma_i} |A_i||^2$$

where $\Delta_{\Sigma_i}$ is the Laplacian operator on $\Sigma_i$.

Multiplying $|A_i|^{2p-4} \psi_i^2$ to the both side of Simon’s identity and integrating by part, we have

$$\int_{\Sigma} |A|^{2p-4} |\nabla_{\Sigma} |A||^2 \psi^2 = \int_{\Sigma} |A|^{2p} \psi^2 - (2p - 3) \int_{\Sigma} |A|^{2p-4} |\nabla_{\Sigma} |A||^2 \psi^2$$

$$-2 \int_{\Sigma} |A|^{2p-3} \psi \nabla_{\Sigma} \psi \cdot \nabla_{\Sigma} |A| + \int_{\Gamma} |A|^{2p-3} \tau (|A|) \psi^2. \quad (14)$$

Note that all the terms are finite in the above identity by the properties of $|A_i|$. Moving the second term in the right hand side of the above identity, we have

$$2(p - 1) \int_{\Sigma} |A|^{2p-4} |\nabla_{\Sigma} |A||^2 \psi^2 = \int_{\Sigma} |A|^{2p} \psi^2 - 2 \int_{\Sigma} |A|^{2p-3} \psi \nabla_{\Sigma} \psi \cdot \nabla_{\Sigma} |A|$$

$$+ \int_{\Gamma} |A|^{2p-3} \tau (|A|) \psi^2. \quad (15)$$

Substituting (15) in (13) and using Cauchy inequality, we get

$$\int_{\Sigma} |A|^{2p} \psi^2 \leq \frac{p-1}{2} \int_{\Sigma} |A|^{2p} \psi^2 - (p - 1) \int_{\Gamma} |A|^{2p-3} \psi \nabla_{\Sigma} \psi \cdot \nabla_{\Sigma} |A|$$

$$+ \frac{p-1}{2} \int_{\Gamma} |A|^{2p-3} \tau (|A|) \psi^2 + \int_{\Sigma} |A|^{2p-2} |\nabla_{\Sigma} \psi|^2$$

$$+ 2(p - 1) \int_{\Sigma} |A|^{2p-3} \psi \nabla_{\Sigma} \psi \cdot \nabla_{\Sigma} |A| - \int_{\Gamma} \psi^2 |A|^{2p-2} H_{\Gamma} \cdot \tau$$

$$= \frac{p-1}{2} \int_{\Sigma} |A|^{2p} \psi^2 + (p - 1) \int_{\Sigma} |A|^{2p-3} \psi \nabla_{\Sigma} \psi \cdot \nabla_{\Sigma} |A|$$

$$+ \int_{\Sigma} |A|^{2p-2} |\nabla_{\Sigma} \psi|^2 + \frac{p-1}{2} \int_{\Gamma} |A|^{2p-2} \tau (\log(|A|)) \psi^2$$

$$- \int_{\Gamma} \psi^2 |A|^{2p-2} H_{\Gamma} \cdot \tau$$

$$\leq (p - 1) \int_{\Sigma} |A|^{2p} \psi^2 + \left(1 + \frac{p-1}{2}\right) \int_{\Sigma} |A|^{2p-2} |\nabla_{\Sigma} \psi|^2$$

$$+ \frac{p-1}{2} \int_{\Gamma} |A|^{2p-2} \psi^2 (\log(|A|)) - \int_{\Gamma} |A|^{2p-2} \psi^2 H_{\Gamma} \cdot \tau. \quad (16)$$

So for $1 < p < \frac{3}{2}$, we have

$$\int_{\Sigma} |A|^{2p} \psi^2 \leq 3 \int_{\Sigma} |A|^{2p-2} |\nabla_{\Sigma} \psi|^2$$

$$+ \frac{p-1}{2} \int_{\Gamma} |A|^{2p-2} \psi^2 (\log(|A|)) - \int_{\Gamma} |A|^{2p-2} \psi^2 H_{\Gamma} \cdot \tau. \quad (17)$$

Now we suppose $\phi \in H^1(M) \cap L^\infty(M)$ with compact support is the function such that $\text{sign}(\phi) |A|^{p-1} |\phi|^p$ satisfies the compatible condition (4). We can check $\text{sign}(\phi) |\phi|^p \in L^\infty(M) \cap H^1(M)$ by following steps.

1. $\text{sign}(\phi) |\phi|^p \in L^\infty(M) \cap L^2(M)$ since $\phi$ has compact support and $\phi \in L^\infty(M)$.
2. $|\nabla_{\Sigma}(|\phi|^p)| = p |\phi|^{p-1} |\nabla_{\Sigma} \phi| \in L^2(M)$ since $\phi \in L^\infty(M)$ and $|\nabla_{\Sigma} \phi| \in L^2(M)$.

So we can take $\psi = \text{sign}(\phi) |\phi|^p$ in (17). This will give us

$$
\int_{\Sigma} |A|^{2p} |\phi|^{2p} \leq 3p^2 \int_{\Sigma} |A|^{2p-2} |\phi|^{2p-2} |\nabla_{\Sigma} \phi|^2 + \frac{p-1}{2} \int_{\Gamma} |A|^{2p-2} |\phi|^{2p} \tau (\log |A|) - \int_{\Gamma} |A|^{2p-2} |\phi|^{2p} H_{\Gamma} \cdot \tau.
$$

This is the first inequality (11) we want to prove since $p$ is bounded ($p < \frac{5}{4}$).

Now let’s change to the case $\phi \in W^{1,2p}(M) \cap L^\infty(M)$. Recall the Young’s inequality that for any $x, y > 0, a, b > 1$ with $\frac{1}{a} + \frac{1}{b} = 1$, we have

$$
xy \leq \frac{x^a}{a} + \frac{y^b}{b}.
$$

We choose $a = \frac{p}{p-1}, b = p$, then we have

$$
|A|^{2p-2} |\phi|^{2p-2} |\nabla_{\Sigma} \phi|^2 \leq \frac{p-1}{p} |A|^{2p} |\phi|^{2p} + \frac{1}{p} |\nabla_{\Sigma} \phi|^{2p}.
$$

So apply above inequality to (18), we have

$$(1 - 3p(p - 1)) \int_{\Sigma} |A|^{2p} |\phi|^{2p}$$

$$
\leq 3p \int_{\Sigma} |\nabla_{\Sigma} \phi|^2 + \frac{p-1}{2} \int_{\Gamma} |A|^{2p-2} |\phi|^{2p} \tau (\log |A|) - \int_{\Gamma} |A|^{2p-2} |\phi|^{2p} H_{\Gamma} \cdot \tau.
$$

Since we have condition $\phi \in W^{1,2p}(M)$, we know all the terms on the right hand side of (19) are finite. So for $1 < p < \frac{5}{4}$, we have

$$
\int_{\Sigma} |A|^{2p} |\phi|^{2p}$$

$$
\leq 64 \int_{\Sigma} |\nabla_{\Sigma} \phi|^2 + 8(p - 1) \int_{\Gamma} |A|^{2p-2} |\phi|^{2p} \tau (\log |A|) - 16 \int_{\Gamma} |A|^{2p-2} |\phi|^{2p} H_{\Gamma} \cdot \tau
$$

$$
\leq 64 \int_{\Sigma} |\nabla_{\Sigma} \phi|^2 + 16 \int_{\Gamma} [(p - 1) |\tau (\log |A|)| - H_{\Gamma} \cdot \tau] |A|^{2p-2} |\phi|^{2p}.
$$

This is exactly what we want.

If it happens that several minimal surfaces in $\{\Sigma_1, \cdots, \Sigma_q\}$ are flat, and we assume $|A_i|$ cannot equal to 0 on the support of $\phi_i$ on $\Sigma_i$ which is not flat, then after replacing $\phi$ by $|A|^{p-1} \psi, |A_i|^{p-1} \psi_i$ will vanish on $\Sigma_i$ which is flat. So all the integration will still make sense if we just drop the terms integrated on $\Sigma_i$ which is flat and all the formulas above are valid.

So this estimate still holds for the general case.

\[ \square \]

6 Proof of the main theorem

In this section, we will choose a suitable test function to get our main theorem.

Before that, let’s discuss the angle condition of $\Sigma_i$ along $\Gamma$ first.
Note that since \( \partial \Sigma_i \) is smooth in \( \Sigma_i \), the conormal vector fields \( \tau_i \) is a smooth unit normal vector field along \( \Gamma \). We say \( M = (\theta_1 \Sigma_1, \cdots, \theta_q \Sigma_q; \Gamma) \) has equilibrium angles along \( \Gamma \) if for all \( 1 \leq i, j \leq q \), the angles between \( \tau_i, \tau_j \) are constants along \( \Gamma \).

So if \( M \) has equilibrium angles along \( \Gamma \), we can choose a smooth normal vector fields \( W \) along \( \Gamma \) such that \( W \) has the constant angle with \( \tau_i \) and unit length along \( \Gamma \). After choosing a orientation on the normal vector field, we can write the angle between \( \alpha_i \) and unit length along \( \Gamma \). We will estimate these three terms one by one after choosing a suitable test function.

Proof First, let’s define a smooth cutoff function \( \eta(t) \) on \( \mathbb{R} \) by

\[
\eta(t) = \begin{cases} 
1, & t \leq 1 \\
0, & t \geq 2 
\end{cases}
\]

such that \( \eta(t) \) is a monotonically decreasing on \( \mathbb{R} \) and \( |\eta'(t)| < 2 \).

First Case: None of \( \Sigma_i \) is flat.

Let’s write the \( L^p \) estimate (11) with following notations,

\[
\int_\Sigma |A|^{2p} |\phi|^{2p} \leq C_1 I + II - III
\]

where

\[
I = \int_\Sigma |A|^{2p-2} |\phi|^{2p} |\nabla_\Sigma \phi|^2 \\
II = \int_\Gamma \frac{p-1}{2} |\tau (\log |A|)| |A|^{2p-2} |\phi|^{2p} \\
III = \int_\Gamma H_\Gamma \cdot \tau |A|^{2p-2} |\phi|^{2p}
\]

We will estimate these three terms one by one after choosing a suitable test function.

We use \( T_r(\Gamma) \) to denote the tubular neighborhood of \( \Gamma \), i.e. we define

\[
T_r(\Gamma) := \{ x \in M : d_\Gamma (x) < r \}.
\]

For simplicity, we assume \( |A| \) has no zeros in \( \overline{T_2(\Gamma) \setminus \Gamma} \). Otherwise, we can do a rescaling of \( M \) if necessary.

Define the cutoff function \( \rho_r \) on \( M \) by

\[
\rho_r(x) = \eta \left( \frac{d_\Gamma (x)}{r} \right).
\]

So \( \rho_r \) will have support in \( T_{2r}(\Gamma) \) and equal to 1 in \( T_r(\Gamma) \) and \( |\nabla_\Sigma \rho_r| < \frac{2}{r} \). We will write \( \rho(x) := \rho_1(x) \). Define \( \phi_i = W_0 \cdot v_i = \sin \alpha_i \).

Now let’s define a function \( g_i \) on \( \Gamma \) by

\[
g_i(x) = \prod_{j=1, \cdots, q, j \neq i} |A_j|.
\]
Since $|A_i| |_\Gamma \in \tilde{C}_+(\Gamma)$, $g_i(x) \in \tilde{C}_+(\Gamma)$, we can extend $g_i(x)$ to the whole $\Sigma_i$ such that $g_i \in \tilde{C}_+(\Sigma_i)$ and positive on $\Sigma_i \setminus \partial \Sigma_i$ by Lemma 1.

Then we choose our $\phi$ on $M$ as

$$\phi_i = \text{sign}(c_i) |c_i|^{\frac{1}{p}} \left( \rho g_i^{\frac{p-1}{p}} + \rho r - \rho \right)$$

for some $r > 2$.

At first, we note that $\phi$ will satisfy the compatible condition (4) since on $\Gamma$, $\phi_i = \text{sign}(c_i) |c_i|^{\frac{1}{p}} g_i^{\frac{p-1}{p}}$, and

$$\text{sign}(\phi_i) |A_i|^{p-1} |\phi_i|^p = c_i \prod_{i=1}^{q} |A_i|^{p-1} = \left( \prod_{i=1}^{q} |A_i|^{p-1} \right) W_0 \cdot v_i.$$

Note that $\prod_{i=1}^{q} |A_i| \in L^\infty_{\text{loc}}(\Gamma)$ by Trace Theorem (or just by the properties of functions in $\tilde{C}_+(\Gamma)$).

Now let’s check $\phi \in H^1(M)$. Note that $g_i^{\frac{p}{p-1}}$ is either smooth or has form $f(z) |z|^{\frac{k(p-1)}{p}}$ near an arbitrary point in $\Sigma_i$ for some smooth function $f$ in conformal coordinate, and $|z|^{\frac{k(p-1)}{p}} \in H^1_{\text{loc}}(M)$, so $g_i^{\frac{p}{p-1}} \in H^1_{\text{loc}}(M)$. Note that $\rho, \rho_r \in W^{1,\infty}(M)$ since they are Lipschitz functions with compact support, we get $\phi \in H^1(M)$ by Hölder’s inequality.

So by Theorem 5, we can put our $\phi$ in the estimate (11). The goal of the following proof is to make the terms, I, II, and III small enough with relatively large $r$ by choosing $p$ very close to 1 and some suitable $\phi$.

Now let’s fix some $\varepsilon > 0$ and some $r_0 > 2$ from now on.

**Estimation of III.**

Right now we do not know the sign of III. But if III < 0, we can rotate $W_0$ by 90 degrees in normal bundle to get a new vector field $\tilde{W}_0$ along $\Gamma$. So $\tilde{c}_i := \tilde{W}_0 \cdot v_i$. Then we can define the new test function $\tilde{\phi}$

$$\tilde{\phi}_i = \text{sign}(\tilde{c}_i) |\tilde{c}_i|^{\frac{1}{p}} \left( \rho g_i^{\frac{p-1}{p}} + \rho r - \rho \right).$$

Along with $\Gamma$, we have

$$|\phi_i| = |c_i|^{\frac{1}{p}} g_i^{\frac{p-1}{p}}, \quad |\tilde{\phi}_i| = |\tilde{c}_i|^{\frac{1}{p}} g_i^{\frac{p-1}{p}}.$$

If we define $g = \prod_{i=1}^{q} |A_i|$ along $\Gamma$, we have

$$\sum_{i=1}^{q} H_{\Gamma} \cdot \tau_i \theta_i |A_i|^{2p-2} |\tilde{\phi}_i|^{2p} = \sum_{i=1}^{q} H_{\Gamma} \cdot \tau_i \theta_i \tilde{c}_i^2 g^{2p-2}$$

$$= g^{2p-2} \sum_{i=1}^{q} H_{\Gamma} \cdot \tau_i \theta_i \left( \tilde{W}_0 \cdot v_i \right)^2$$

$$= g^{2p-2} \sum_{i=1}^{q} H_{\Gamma} \cdot \tau_i \theta_i (W_0 \cdot \tau_i)^2$$

$$= g^{2p-2} \sum_{i=1}^{q} H_{\Gamma} \cdot \tau_i \theta_i \left[ 1 - (W_0 \cdot v_i)^2 \right]$$
\[ = g^{2p-2} \sum_{i=1}^{q} -H_i \cdot \tau_i \theta_i (W_0 \cdot v_i)^2 \]

\[ = - \sum_{i=1}^{q} H_i \cdot \tau_i \theta_i |A_i|^{2p-2} |\phi_i|^{2p}. \]

Here we’ve used the fact that \( M \) is minimal along \( \Gamma \). Hence

\[ III := \int_{\Gamma} H_i \cdot \tau |A|^{2p-2} |\phi|^{2p} = -III. \]

So after replacing \( W_0 \) by \( \tilde{W}_0 \), we can get

\[ III > 0. \quad (22) \]

**Estimation of II.**

Note that \( \Gamma \) is compact, \( |A|^{2p-2} \) and \( |\phi|^{2p} \) are all bounded on \( \Gamma \) uniformly with respect to \( p \in (1, \frac{2}{3}) \). Since the integration \( \int_{\Gamma} |\tau(\log |A|)| \) is finite by the property of smooth finite order functions, the integration

\[ \int_{\Gamma} |\tau(\log |A|)| \|A|^{2p-2} |\phi|^{2p} < \infty. \]

Hence we can choose \( p_1 > 1 \) very close to 1 such that for every \( p \in (1, p_1) \), we have

\[ II < \varepsilon. \quad (23) \]

**Estimation of I.**

The trick part for estimating I is the points that \( \phi \) fails to be in \( W^{1,p}(M) \) nearby, which are the zeros of \( \phi \). Denote \( P_i = \{ x \in \Sigma_i : g_i(x) = 0 \} \). So by the definition of \( g_i \), we know \( P_i \subset \partial \Sigma_i \). Choose \( x \in P_i \), let’s consider the integration

\[ I_{x,\delta} := \int_{\Sigma_i \cap B^M_{\delta}(x)} |A_i|^{2p-2} |\phi_i|^{2p-2} |\nabla_{\Sigma_i} \phi_i|^2 \]

for \( \delta < 1 \).

In \( \Sigma_i \cap B^M_{\delta}(x) \), we have \( |\phi_i| = |c_i|^\frac{1}{p} g_i^{\frac{p-1}{p}} \). Still we work at conformal coordinate near \( x \) and we choose \( \delta \) small enough to make sure \( \Sigma_i \cap B^M_{\delta}(x) \) is in this coordinate chart. Then \( g_i \) has form \( g_i(z) = f(z) |z|^l \) for some \( f(z) \) positive and smooth near \( x \) and \( l \in \mathbb{Z}_+ \).

We compute

\[ |c_i|^{-\frac{2}{p}} |\nabla_{\Sigma_i} \phi_i|^2 = \left( \frac{p-1}{p} \right)^2 g_i^{-\frac{2}{p}} |\nabla_{\Sigma_i} g_i|^2 \]

\[ = \left( \frac{p-1}{p} \right)^2 f^{-\frac{2}{p}} |z|^{-\frac{2l}{p}} |z|^l \nabla_{\Sigma_i} f + lf |z|^{l-1} \nabla_{\Sigma_i} |z|^l \right|^2 \]

\[ \leq 2 \left( \frac{p-1}{p} \right)^2 f^{-\frac{2}{p}} \left( |z|^{2l-\frac{2l}{p}} |\nabla_{\Sigma_i} f|^2 + l^2 f^2 |z|^{2l-2-\frac{2l}{p}} |\nabla_{\Sigma_i} |z||^2 \right). \]

(24)

Note that \( |\nabla_{\Sigma_i} |z||^2 \) might not equal to 1. Nerveless, it is bounded near \( x \). Since \( f \) is smooth and positive so it has lower bound near \( x \), we know the term \( f^{-\frac{2}{p}} |z|^{2l-\frac{2l}{p}} |\nabla_{\Sigma_i} f|^2 \) is bounded near \( x \). Hence the integration on this term can be arbitrary small by choose \( \delta \) small enough.

\( \odot \) Springer
For the term \( l^2 f^{2-\frac{2}{p}} |z|^{2l-2-\frac{2l}{p}} \left| \nabla \Sigma_i |z| \right|^2 \), if we assume the metric in this coordinate is \( \lambda(z)^2 dz d\overline{z} \) and \( l^2 f^{2-\frac{2}{p}} \left| \nabla \Sigma_i |z| \right|^2 \lambda^2 \) is bounded by \( C \), then
\[
\int_{\Sigma_i \cap B^M(x)} l^2 f^{2-\frac{2}{p}} |z|^{2l-2-\frac{2l}{p}} \left| \nabla \Sigma_i |z| \right|^2 \leq \int_{|z|<\delta \text{ and } z \in \Sigma_i} C |z|^{2l-2-\frac{2l}{p}} \frac{\sqrt{-1}}{2} d\overline{z} \wedge dz
\leq \frac{2\pi C}{2l-\frac{2l}{p}} \delta^{2l-\frac{2l}{p}}
= \frac{\pi C}{l(p-1)} \delta^{2l-\frac{2l}{p}}.
\]
(25)

Combining (24), (25), and noting \( |A_i|^{2p-2} |\phi_i|^{2p-2} \) is bounded near \( x \), we get
\[
I_{x, \delta_x} \leq C \left( \frac{p-1}{p} \right)^2 \left( \varepsilon' + \frac{\pi C}{l(p-1)} \delta^{2l-\frac{2l}{p}} \right) \leq 2C \left( \frac{p-1}{p} \right)^2 \varepsilon'
\]
for some \( \delta_x \) small enough.

Here, the constant \( C \) does not depend on \( p \). So we can choose \( p_x \in (1, p_0) \) small enough such that if \( p \in (1, p_x) \), then
\[
I_{x, \delta_x} < \varepsilon'
\]
for any given \( \varepsilon' > 0 \). Note that the set \( P_i \) is a finite set for each \( 1 \leq i \leq q \), so we can choose \( p_1 = \min_{x \in \bigcup P_i} p_x \) and \( \delta_0 = \min_{x \in \bigcup P_i} \delta_x \). Denote \( B \in M \) as
\[
B := \bigcup_{i=1}^{q} \bigcup_{x \in P_i} (\Sigma_i \cap B_{\delta_0}(x)).
\]
Then
\[
\int_B |A|^{2p-2} |\phi|^{2p-2} |\nabla \Sigma \phi|^2 \leq \sum_{i=1}^{q} \theta_i z(P_i) \varepsilon'
\]
where \( z(P_i) \) denote the cardinality of the set \( P_i \). By requiring \( \varepsilon' \) small, we can find \( p_1 \in (1, p_0), \delta_0 > 0 \) such that
\[
I' = \int_B |A|^{2p-2} |\phi|^{2p-2} |\nabla \Sigma \phi|^2 \leq \frac{\varepsilon}{C_1}.
\]
(26)

Now we focus on the estimation of
\[
I'' = \int_{\Sigma \setminus B} |A|^{2p-2} |\phi|^{2p-2} |\nabla \Sigma \phi|^2.
\]

Clearly we have
\[
I = I' + I''.
\]
(27)

Note that right now we know \( \phi \) is positive and smooth with compact support in \( \Sigma \setminus B \), so \( \phi \in W^{1, \infty}(M) \). So we can apply Young’s inequality to get
\[
I'' \leq \frac{p-1}{p} \int_{\Sigma \setminus B} |A|^{2p} |\phi|^{2p} + \frac{1}{p} \int_{\Sigma \setminus B} |\nabla \Sigma \phi|^{2p} < \infty.
\]
(28)
So we choose $p_2 \in (1, p_1)$ small such that
\[
\frac{p_2 - 1}{p_2} < \frac{1}{4C_1}
\] (29)
so that the first term in the above inequality can be absorbed by left hand side of (11). For the second term in (28), we have
\[
\int_{\Sigma \setminus B} |\nabla \Sigma \phi|^2 \leq \int_{T_{2}(\Gamma) \setminus T_{r}(\Gamma)} |\nabla \Sigma \phi|^2
\]
(30)
In $T_{2}(\Gamma) \setminus B$, we know $|\phi_i| = |c_i|^\frac{p_2}{p} \rho \left( g_i^{-\frac{1}{p}} - 1 \right) + |c_i|^\frac{1}{p}$. Hence
\[
|\nabla \Sigma_i \phi_i|^2 = |c_i|^2 \left( p - 1 \right) g_i^{-\frac{1}{p}} \nabla \Sigma_i g_i \rho + \nabla \Sigma_i \rho \left( g_i^{-\frac{1}{p}} - 1 \right) \right|^2 \leq 2^2 c_i^2 \left( \frac{p - 1}{p} \right) g_i^{-2} \rho^2 |\nabla \Sigma_i g_i|^2
\]
(31)
Note that $g_i$ has positive upper and lower bound on $T_{2}(\Gamma) \setminus B$, $|\nabla \Sigma_i g_i|$ has an upper bound on $T_{2}(\Gamma) \setminus B$ since it is smooth. Hence, as $p \to 1^+$, $g_i^{-\frac{1}{p}} \to 1$ uniformly. So we can choose $p_3 \in (1, p_2)$ small enough to make
\[
I_1 < \frac{\varepsilon}{C_1}
\] (32)
for any $p \in (1, p_3)$. From now on, we will fix $p \in (1, p_3)$.
For the integration $I_2$, we have
\[
I_2 = \int_{T_{2}(\Gamma) \setminus T_{r}(\Gamma)} c^2 |\nabla \Sigma \rho r|^2 \leq C \int_{T_{2}(\Gamma) \setminus T_{r}(\Gamma)} \frac{1}{r^{2p}} \leq C r^{2-2p} \quad \text{By area growth condition.}
\] (33)
Here the constant $C$ does not depend on $p$ and $r$. Since we’ve fixed $p$, we can choose $r > r_0$ so large, such that
\[
I_2 \leq C r^{2-2p} < \frac{\varepsilon}{C_1}
\] (34)
Now, let’s combine the estimates (34), (32), (30), (28), (29), (27), (23), (22), (21) with (11) to get
\[
\int_{\Sigma} |A|^2 |\phi|^2 \leq \frac{1}{4} \int_{\Sigma \setminus B} |A|^2 |\phi|^2 + \varepsilon + \varepsilon + \varepsilon + \varepsilon.
\] (35)
So
\[
\int_\Sigma |A|^{2p} |\phi|^{2p} \leq 8\varepsilon.
\] (36)

By definition of \(\phi\), we have
\[
\int_{T_0(\Gamma) \setminus T_2(\Gamma)} \min\{1, |A|^4\} c^2 \leq \int_{T_1(\Gamma) \setminus T_2(\Gamma)} |A|^{2p} c^2 \leq \int_\Sigma |A|^{2p} |\phi|^{2p} \leq 8\varepsilon.
\] (37)

So the left hand side of the above inequality does not depend on \(r\) and \(p\). By the arbitrary choice of \(\varepsilon\) and noting \(c_i \neq 0\) on each \(\Sigma_i\), we know actually \(|A| = 0\) on \(T_0(\Gamma) \setminus T_2(\Gamma)\), and thus this implies each \(\Sigma_i\) is flat, which contradicts our assumption that each \(\Sigma_i\) is non-flat.

**Second case: One of \(\Sigma_i\) is flat.**

For the case that one of \(\Sigma_i\) being flat, we suppose \(\Sigma_1\) is lying the plane \(P\). Clearly, if there are another \(\Sigma_i\), which is flat and different from \(\Sigma_1\), \(\Sigma_i\) should be lying \(P\), too since we’ve assume \(\Gamma\) is compact. Again, we write it as \(\Sigma_2\) for simplicity. So the only possible choice of \(\Sigma_2\) is \(\Sigma_2 = P \setminus (\Sigma_1 \cup \partial \Sigma_1)\). Note that \(P\) is stable, so we can remove it from this multiple junction surface to get the remaining one \(((\theta_1 - \theta_2)\Sigma_1, \theta_3, \Sigma_3, \cdots, \theta_q, \Sigma_q; \Gamma)\) or \(((\theta_2 - \theta_1)\Sigma_2, \theta_3, \Sigma_3, \cdots, \theta_q, \Sigma_q; \Gamma)\) which is stable. So WLOG, we assume there are only one \(\Sigma_i\), which we call it \(\Sigma_1\), is flat.

This time we choose \(W_0 = \tau_1\). The trick park is the term III might not have a favorable sign. So we need a bit more precise estimation of the total curvature.

Let’s use \(K_i\) to denote the sectional curvature on \(\Sigma_i\). Following from B. White’s proof ([25]), we have the following lemma.

**Lemma 2** For each \(\Sigma_i\) which is non-compact with boundary \(\Gamma\), we have
\[
\int_\Gamma -H_{\Gamma} \cdot \tau_i \leq \int_{\Sigma_i} -K_{\Sigma_i}
\]
where the \(K_{\Sigma_i}\) is the sectional curvature of \(\Sigma_i\).

**Proof** Fix a point \(p_0 \in \Gamma\), and we define
\[
B_r := \{p \in \Sigma_i : d_i(p, p_0) < r\}.
\]

We write \(\Gamma_r = \partial B_r \setminus \Gamma\), the remaining boundary part of \(B_r\) except \(\Gamma\).

Note that we can choose a large \(r_0\) such that \(\Gamma \subset B_{r_0}\). So for \(r > r_0\), we know \(\Gamma\) and \(\Gamma_r\) do not connect with each other.

By the result of P. Hartman [10], we know \(\Gamma_r\) is, for almost all \(r\), a piecewise smooth, embedded closed curve in \(\Sigma_i\). So we can apply Gauss-Bonnet theorem to get
\[
\int_{B_r} K_{\Sigma_i} + \int_\Gamma H_{\Gamma} \cdot (-\tau_i) + \int_{\Gamma_r} \kappa_g + \sum(\text{exterior angles of } \Gamma_r)
= 2\pi \chi(B_r) = 2\pi (2 - 2h(r) - c(r))
\] (38)
where \(h(r)\) and \(c(r)\) are the number of handles and the number of boundary components, respectively, of \(B_r\) with \(r > r_0\). Here we also use \(\kappa_g\) to denote the curvature of \(\Gamma_r\) inside the surface \(\Sigma_i\) with respect to the inner conormal vector field.
Let $L(r)$ be the length of $\Gamma_r$. So by the first variation formula of a piecewise smooth curve, we have

$$L'(r) = \int_{\Gamma_r} \kappa_g + \sum \text{exterior angles of } \Gamma_r.$$

We also note that $c(r) \geq 2$ since $\Gamma$ has at least one component and $\Gamma_r$ is always non-empty since we’ve assume $\Sigma_i$ is complete and not compact for any $r > r_0$. So combining with the above Gauss-Bonnet formula we’ve got, we can get

$$- \int_{\Gamma} H_{\Gamma} \cdot \tau_i + L'(r) \leq - \int_{B_r} K_{\Sigma_i}.$$

Note that $L(r) > 0$ for all $r > r_0$, we have $\limsup_{r \to \infty} L'(r) \geq 0$. And since $K_{\Sigma_i} \leq 0$, we can take $r \to \infty$ to get

$$- \int_{\Gamma} H_{\Gamma} \cdot \tau_i \leq - \int_{\Sigma_i} K_{\Sigma_i}.$$

This is what we want. \qed

Let’s go back to the proof of the main theorem. Again, we choose the function $g_i$ on $\Gamma$ as

$$g_i(x) = \prod_{j=2, \cdots q, j \neq i} |A_j|$$

and choose our $\phi$ on $M$ as

$$\phi := \phi_{r, p} = \text{sign}(c) |c|^{\frac{1}{p}} \left( \rho g^{\frac{p-1}{p}} + \rho r - \rho \right)$$

where $c_j = W_0 \cdot v_i$. Here we use subscript to indicate $\phi$ depends on $r$ and $p$ if needed. Clearly, $\text{sign}(\phi_i) |A_i|^{p-1} |\phi_i|^p$ satisfies the compatible condition.

This time, we do not have a good sign for the term III, so we keep it in our estimates. Based on essentially same argument, we can get a similar estimate like (35) as

$$\int_{\Sigma} |A|^{2p} |\phi|^{2p} \leq \frac{1}{4} \int_{\Sigma \setminus B} |A|^{2p} |\phi|^{2p} + 4\varepsilon - \text{III} (39)$$

for $p$ small enough and $r$ large enough which might depend on $p$. If it happens that $\text{III} \geq 0$, the previous argument shows that each $\Sigma_i$ for $i = 2, \cdots, q$ is flat.

For simplicity, we write III in the form which depends on $p$ as

$$\text{III}_p := \int_{\Gamma} |A|^{2p-2} |\phi|^{2p} H_{\Gamma} \cdot \tau = \int_{\Gamma} g^{2p-2} c^2 H_{\Gamma} \cdot \tau$$

where $g = \prod_{i=2}^q |A_i|$. So if we write III0, we just mean

$$\text{III}_0 := \int_{\Gamma} c^2 H_{\Gamma} \cdot \tau.$$

We note $g^{2p-2} \to 1$ a.e. on $\Gamma$ since the zeros of $g$ are isolated, so by Dominated Convergence theorem, we have

$$\int_{\Gamma} g^{2p-2} c^2 H_{\Gamma} \cdot \tau \to \int_{\Gamma} c^2 H_{\Gamma} \cdot \tau$$
as $p \to 1$. This means we can choose $p$ large to make
\[ |\Pi_p - \Pi_0| \leq \frac{1}{8} |\Pi_0| \] (40)
since $\Pi_0 < 0$ as we’ve assumed.

Similarly, $g_i^{p-1/p} \to 1$ a.e. on $\Sigma_i$ and $|A_i|^{2p} \to |A_i|^2$ a.e. on $\Sigma_i$ as $p \to 1$, we have
\[ \int_{\Sigma} |A|^{2p} \phi^{2p} \to \int_{\Sigma} |A|^2 \rho^2 \] as $p \to 1$.

Hence, for some fixed $r$, we can always choose $p$ small enough to make sure
\[ \left| \int_{\Sigma} |A|^{2p} \phi^{2p} - |A|^2 \rho^2 \right| \leq \frac{1}{8} |\Pi_0| \].
(41)

By our previous lemma 2, we have
\[ \int_{\Sigma} |A|^2 \rho^2 = -\int_{\Sigma} 2K_{\Sigma} e^2 \geq -2 \int_{\Gamma} e^2 \mathbf{H} \cdot \tau = 2 |\Pi_0| . \]

Note $\int_{\Sigma} |A|^2 \rho^2 \to \int_{\Sigma} |A|^2 e^2$ as $r \to \infty$, so we can fix a $r_1$ large enough to get
\[ \int_{\Sigma} |A|^2 \rho^2_{r_1} \geq \frac{15}{8} |\Pi_0| . \] (42)

Combining with (41), we have
\[ \int_{\Sigma} |A|^{2p} |\phi_{r_0,p}|^{2p} \geq \frac{7}{4} |\Pi_0| \]
for $p$ sufficient small. Note $\phi_{r,p}$ is an increasing function with respect to variable $r$ as $r > 2$, so we actually have
\[ \int_{\Sigma} |A|^{2p} |\phi_{r,p}|^{2p} \geq \frac{7}{4} |\Pi_0| \] (43)
for all $r > r_0$, $1 < p < p_0$ for some $p_0$ sufficient closed to 1.

Hence, we can choose $p$ small enough again, and then choose $r > r_0$ large enough to make the estimate (39) holds for $\varepsilon = \frac{1}{12} |\Pi_0|$. So (39) implies
\[ \int_{\Sigma} |A|^{2p} |\phi|^{2p} \leq \frac{1}{6} |\Pi_0|- \frac{4}{3} |\Pi_p| \leq \frac{1}{6} |\Pi_0| + \frac{3}{2} |\Pi_0| = \frac{5}{3} |\Pi_0| \]
where we’ve used (40). This is a contradiction with the estimate (43). So it is impossible that only one of $\Sigma_i$ is flat. Hence, we finished our proof.\[ \square \]

**Theorem 7** Let $M = (\theta_1, \Sigma_1, \cdots, \theta_q \Sigma_q; \Gamma)$ be a minimal multiple junction surface in $\mathbb{R}^3$. We assume $M$ is complete, stable and has quadratic area growth. Furthermore, we assume $\Gamma$ is a straight line and $M$ has equilibrium angles along $\Gamma$, then each $\Sigma_i$ is flat.

**Proof** This case is much easier than the case of $\Gamma$ compact. Note that for any $i \neq j$, by rotating $\Sigma_i$ along $\Gamma$ for a suitable angle, we can make $\Sigma_i$ and $\Sigma_j$ share the same outward normal of boundary. Hence by Hopf’s boundary lemma, $\Sigma_i$ will be identical to $\Sigma_j$ after rotation.

\[ \diamond \] Springer
This says that each pieces of surface in \( M \) are all isometric to each other. Moreover, for each \( \Sigma_i \), we let \( \Sigma_i' \) to be the surface obtained by rotating \( \Sigma_i \) 180 degrees. Then the union of \( \Sigma_i \) and \( \Sigma_i' \) will form a global minimal surface by Hopf’s boundary lemma (Unique continuation of minimal surfaces shows we can extend \( \Sigma_i \) locally and Hopf’s boundary lemma shows \( \Sigma_i' \) is the only way to extend \( \Sigma_i \)). If we write \( \tau_i' \) as the unit outer normal of \( \Sigma_i' \) and \( A_i' \) as the second fundamental form of \( \Sigma_i' \), then we will have

\[
\tau_i(|A_i|) = \tau_i'(|A_i'|)
\]

By the definition of \( \Sigma_i' \)

\[
= -\tau_i(|A_i|)
\]

By the unique continuation of \( \Sigma_i \)

So we have \( \tau_i(|A_i|) = 0 \) for each \( 1 \leq i \leq q \). Another way to show \( \tau_i(|A_i|) = 0 \) is just directly compute it with the help of Codazzi equations.) Hence the stability operator for \( M \) is

\[
\int_{\Sigma} |A|^2 \phi^2 \leq \int_{\Sigma} |\nabla_{\Sigma} \phi|^2
\]

for some \( \phi = W \cdot \nu \). To apply the \( L_p \) estimate, we need \( \text{sign}(\phi) |A|^{p-1} |\phi|^2 |p \) to satisfy the compatible condition (4). Note that \( |A_i| = |A_j| \), we only need to require \( \text{sign}(\phi) |\phi|^p \) to satisfy (4).

Now, as usual we choose \( W_0 \) having constant angles with each \( \tau_i \) and make sure \( c_i := W_0 \cdot \nu_i \neq 0 \) for each \( i \). Now we choose an arbitrary point \( x_0 \) on \( \Gamma \). Define cutoff function \( \rho_r(x) \) support in \( B_{r/2}(x) \) which equals to 1 in \( B_r(x) \) and has gradient less than \( \frac{2}{r} \).

So we can choose our \( \phi \) as

\[
\phi = \text{sign}(c) |c|^\frac{1}{p} \rho_r.
\]

Hence we can apply \( L_p \) estimate to get

\[
\int_{\Sigma} |A|^2 |\phi|^2 \leq C_1 \int_{\Sigma} |\nabla_{\Sigma} \phi|^2 |p .
\]

Standard argument in [19] will imply each \( \Sigma_i \) will be flat. \( \square \)

**Remark 7** Indeed, we can remove the quadratic area growth condition in the case \( \Gamma \) a straight line. Actually, one can just use the result of Bernstein’s theorem for the stable minimal surface to get this result.

As a corollary, we can also get some results related to the stable capillary minimal surfaces.

**Corollary 1** Let \( P \) be a plane in \( \mathbb{R}^3 \). Then there is no (oriented) stable complete minimal surface \( \Sigma \) with boundary \( \partial \Sigma \) such that \( \partial \Sigma \in P \), \( \partial \Sigma \) compact, and \( \Sigma \) has constant angle with \( P \) along \( \partial \Sigma \).

Here the stability of the capillary minimal surface means this surface is stable of capillary energy under the variation fixing the plane \( P \).

This result is immediately following from the proof of Theorem 6. The variation we’ve taken in the proof of the second case is just the one fixing the plane \( P \).

**Remark 8** We note that H. Hong and A.B. Saturnino [11] gave more precise curvature estimates over capillary surfaces including capillary minimal surfaces. They’ve used different methods to give stronger results than Corollary 1.
7 Further questions

As we’ve seen, we still leave some questions related to minimal multiple junction surfaces.

The first one is, what if $\Gamma$ is neither compact nor a straight line?

**Question 1** Can we relax the condition of $\Gamma$ to be a compact or a straight line, so that we still get the similar Bernstein’s theorem for the stable minimal multiple junction surfaces?

In particular, we want to know the stability property of the universal cover of $Y$-shaped bent helicoid.

Another question is, we still need the help of area growth to get control of curvature. So we may still want to remove this condition in some sense.

**Question 2** Can one get the quadratic area growth for stable minimal multiple junction surfaces like the result in [4] in some sense?

In [4], one may need the simply connected condition to get the quadratic area growth. This is not a problem when talking about the usual smooth surface since we can always take a universal cover without affecting stability. But things get unusual, especially when requiring $\Gamma$ compact. Moreover, one may still need careful consideration when talking about the simply connected multiple junction surfaces.

The last question is related to the higher dimension case.

**Question 3** Can one get the similar Bernstein’s theorem for stable minimal multiple junction hypersurface with the hypersurface dimension greater than 2?

As we’ve seen, basically we can still get the similar $L^p$ estimation of curvature like Schoen, Simon, Yau’s result [19] for $2 \leq n \leq 5$. But we have an additional compatible condition along $\Gamma$, so we need $p \to 1$ in order to get the curvature estimation near $\Gamma$. This will force our surface dimension to be 2.

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References

1. Allard, W.K.: On the first variation of a varifold. Ann. Math. pp. 417–491 (1972)
2. Bernstein, J., Maggi, F.: Symmetry and rigidity of minimal surfaces with plateau-like singularities. Arch. Ration. Mech. Anal. 239(2), 1177–1210 (2021)
3. Bronsard, L., Reitich, F.: On three-phase boundary motion and the singular limit of a vector-valued ginzburg-landau equation. Arch. Ration. Mech. Anal. 124(4), 355–379 (1993)
4. Colding, T.H., Minicozzi, W.P.: Estimates for parametric elliptic integrands. Int. Math. Res. Not. 2002(6), 291–297 (2002)
5. Colding, T.H., Minicozzi, W.P.: A course in minimal surfaces. Am. Math. Soc. 121(2011)
6. Depner, D., Garcke, H., Kohsaka, Y.: Mean curvature flow with triple junctions in higher space dimensions. Arch. Ration. Mech. Anal. 211(1), 301–334 (2014)
7. Depner, D., Garcke, H., et al.: Linearized stability analysis of surface diffusion for hypersurfaces with triple lines. Hokkaido Math. J. 42(1), 11–52 (2013)
8. Fischer-Colbrie, D.: On complete minimal surfaces with finite Morse index in three manifolds. Inventiones Mathematicae 82(1), 121–132 (1985)

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1 It has been pointed out to me by Luen Fai Tam that we can deduce $\Sigma$ has quadratic area growth from stable condition and $\Gamma$ being compact by the argument due to D. Fischer-Colbrie [8].
9. Freire, A.: Mean curvature motion of triple junctions of graphs in two dimensions. Comm. Partial Differ. Equ. 35(2), 302–327 (2010)
10. Hartman, P.: Geodesic parallel coordinates in the large. Am. J. Math. 86(4), 705–727 (1964)
11. Hong, H., Saturnino, A.B.: Capillary surfaces: stability, index and curvature estimates. arXiv preprint arXiv:2105.12662 (2021)
12. Ilmanen, T., Neves, A., Schulze, F.: On short time existence for the planar network flow. J. Differ. Geom. 111(1) (2019)
13. Krummel, B.: Regularity of minimal hypersurfaces with a common free boundary. Calc. Var. Partial. Differ. Equ. 51(3–4), 525–537 (2014)
14. Lawlor, G., Morgan, F., et al.: Curvy slicing proves that triple junctions locally minimize area. J. Differ. Geom 44, 514–528 (1996)
15. Mantegazza, C., Novaga, M., Tortorelli, V.M.: Motion by curvature of planar networks. Annali della Scuola Normale Superiore di Pisa-Classe di Scienze 3(2), 235–324 (2004)
16. Meeks, W.H., Weber, M.: Bending the helicoid. Math. Ann. 339(4), 783–798 (2007)
17. Mese, C., Yamada, S.: The parameterized steiner problem and the singular plateau problem via energy. Trans. Am. Math. Soc. 358(7), 2875–2895 (2006)
18. Rosenberg, H.: Hypersurfaces of constant curvature in space forms. Bull. Sci. Math. 117(2), 211–239 (1993)
19. Schoen, R., Simon, L., Yau, S.T.: Curvature estimates for minimal hypersurfaces. Acta Math. 134(1), 275–288 (1975)
20. Schoen, R.M.: Uniqueness, symmetry, and embeddedness of minimal surfaces. J. Differ. Geomet. 18(4), 791–809 (1983)
21. Schulze, F., White, B.: A local regularity theorem for mean curvature flow with triple edges. J. für die reine und angewandte Mathematik 2020(758), 281–305 (2020)
22. Simon, L., et al.: Cylindrical tangent cones and the singular set of minimal submanifolds. J. Differ. Geomet. 38(3), 585–652 (1993)
23. Taylor, J.E.: The structure of singularities in soap-bubble-like and soap-film-like minimal surfaces. Ann. Math. pp. 489–539 (1976)
24. Tonegawa, Y., Wickramasekera, N.: The blow up method for brakke flows: networks near triple junctions. Arch. Ration. Mech. Anal. 221(3), 1161–1222 (2016)
25. White, B., et al.: Complete surfaces of finite total curvature. J. Differ. Geomet. 26(2), 315–326 (1987)

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