A Grönewald Van Hove like formulation of the ordering problems of General Relativity

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A simple formal recasting of well known arguments concerning the ordering problems of General Relativity allows to obtain in such a context a Grönewald Van Hove like theorem.

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I. INTRODUCTION

Ordering problems occurs in two physically completely different contexts:

- in Quantum Mechanics \[1\] one has that in the canonical quantization of a monomial in \(q\) and \(p\) of the form \(q^n p^m\), \(n, m \in \mathbb{N}_+\) since the classical observables \(q\) and \(p\) commutes while \([\hat{q}, \hat{p}] = i\), to each of the classically equivalent orderings:

\[
q^n p^m = q^{n-1} p^m q = \cdots = qp^n q^{n-1} = p^m q^n = pq^n p^{m-1} = \cdots
\]  

(1.1)

corresponds a different quantum operator:

\[
\hat{q}^n \hat{p}^m \neq \hat{q}^{n-1} \hat{p}^m \hat{q} \neq \cdots \neq \hat{q}^m \hat{q}^{n-1} \neq \hat{p}^m \hat{q}^n \neq \hat{p}^m \hat{q}^{m-1} \neq \cdots
\]  

(1.2)

- in General Relativity the application of the the Minimum Substitution Prescription, following the terminology of \[2\], allows to generalize a non-gravitational law of Physics holding on Minkowski spacetime \((\mathbb{R}^4, \eta_{ab})\) (where in this paper the abstract index notation is adopted \[2\]) and containing a tensorial quantity \(\nabla(\eta) X^{a_1 \cdots a_r} b_1^{b_1} \cdots b_s \in T^r_s(\mathbb{R}^4) : (r, s) \neq (0, 0)\) to an arbitrary curved space-time \((M, g_{ab})\) through the ansatz:

\[
(\eta_{ab} \rightarrow g_{ab}, \nabla(\eta) \rightarrow \nabla^{(g)})
\]  

(1.3)

Ordering problems then occur owing to the fact that while covariant derivatives commute on Minkowski spacetime:

\[
\nabla(\eta) X^{a_1 \cdots a_r} b_1^{b_1} \cdots b_s = \nabla^{(g)} X^{a_1 \cdots a_r} b_1^{b_1} \cdots b_s \forall X^{a_1 \cdots a_r} b_1^{b_1} \cdots b_s \in T^r_s(\mathbb{R}^4), \forall r, s \in \mathbb{N}
\]  

(1.4)

covariant derivatives on a curved spacetime applied to something different from a scalar field don’t commute:

\[
\nabla(\eta) X^{a_1 \cdots a_r} b_1^{b_1} \cdots b_s \neq \nabla^{(g)} X^{a_1 \cdots a_r} b_1^{b_1} \cdots b_s \forall X^{a_1 \cdots a_r} b_1^{b_1} \cdots b_s \in T^r_s(M), \forall r, s \in \mathbb{N} : (r, s) \neq (0, 0)
\]  

(1.5)

Since in the former situation the Grönewald-Van Hove Theorem \[1, 3\] gives a conceptually satisfying characterization of the involved ordering problem stating the impossibility of making a consistent canonical quantization of any polynomial in \(q\) and \(p\), it appears then natural to investigate whether an analogous theorem holds in the latter situation, namely in General Relativity.

This is indeed the case as it may be appreciated as soon as one performs a simple formal recasting of well-known arguments concerning curvature terms related to the ordering problems.
II. THE GRÖENEWALD VAN HOVE THEOREM

Let us consider the symplectic manifold $(T^*\mathbb{R}, \omega)$ where of course $T^*\mathbb{R} = \mathbb{R}^2$ while $\omega := dp \wedge dq$ is the canonical symplectic form.

Let us denote by $\mathcal{U}$ the Lie algebra of the real-valued polynomials over $\mathbb{R}^2$ where the Lie brackets are the Poisson brackets associated to $\omega$.

Given the Hilbert space $\mathcal{H} := L^2(\mathbb{R}, d\mu_{\text{Lebesgue}})$ let us introduce the following:

**Definition II.1**

Quantization:

- a map $\hat{\cdot} : \mathcal{U} \mapsto \mathcal{L}_{s.a.}(\mathcal{H})$ such that:

  1. for each finite subset $S \subset \mathcal{U}$ there is a dense subspace $\mathcal{D}_S \subset \mathcal{H}$ such that:

      $\mathcal{D}_S \subset \mathcal{D}_f$ and $\hat{f}\mathcal{D}_S \subset \mathcal{D}_S \ \forall f \in \mathcal{U}$

  2. $\hat{f} + \hat{g} = \hat{f} + \hat{g}$ pointwise on $\mathcal{D}_S \ \forall f \in S$

  3. $\hat{\lambda f} = \lambda \hat{f} \ \forall \lambda \in \mathbb{R}$

  4. $\{\hat{f}, \hat{g}\} = \frac{1}{i}[\hat{f}, \hat{g}]$ on $\mathcal{D}_S$

  5. $\hat{I} = I$

  6. $\hat{q} = \text{multiplication by } q$ and $\hat{p} = \frac{1}{i} \frac{d}{dq}$

**Theorem II.1**

GRÖENEWALD VAN-HOVE THEOREM

- It doesn’t exist a quantization map $\hat{\cdot}$.

**Remark II.1**

Indeed the proof of theorem II.1 (for which we demand to [1], [3]) shows that the quantization map $\hat{\cdot}$ can be defined only for the Lie subalgebra $\mathcal{U}_2$ of the polynomials of degree less or equal than 2 resulting in the operators:

\[
\hat{q}^2 = q^2
\]

\[
\hat{p}^2 = p^2
\]

\[
\hat{qp} = \frac{1}{2}(\hat{q}\hat{p} + \hat{p}\hat{q})
\]

but cannot be extended consistently to the set $\mathcal{U}_3$ of the polynomials of degree less or equal than 3 \(^1\).

**Remark II.2**

\(^1\) Let us observe that $\mathcal{U}_3$ is not a Lie subalgebra of $\mathcal{U}_2$ since, for instance, $\{q^3, p^3\} = 9q^2p^2 \notin \mathcal{U}_3$
One can of course define the Weyl ordering map $\hat{\cdot}^{\text{W.O.}} : \mathcal{U} \mapsto \mathcal{L}_{s.a.}(\mathcal{H})$ defined by symmetrizing the product of operators in all possible combinations with equal weight:

$$\hat{q^2p}^{\text{W.O.}} := \frac{1}{3}(\hat{q^2}\hat{p} + \hat{q}\hat{p}\hat{q} + \hat{p}\hat{q^2})$$ (2.4)

and so on.

As it is well-known, given a classical hamiltonian $H(q,p)$, this corresponds to the mid-point-prescription:

$$< q' | \hat{H}^{\text{W.O.}} | q > = \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \exp[-ip(q-q')/2] H(q+q'/2, p)$$ (2.5)

The key point is that the map $\hat{\cdot}^{\text{W.O.}}$ is not a quantization (i.e. it doesn’t respects all the conditions required by the definition II.1).

Alternatively one can introduce the normal ordering map $\hat{\cdot}^{\text{N.O.}} : \mathcal{U} \mapsto \mathcal{L}_{s.a.}(\mathcal{H})$ defined by the condition that the $\hat{p}$ operators are always put to the left of the $\hat{q}$ operators, i.e.:

$$\hat{q^2p}^{\text{N.O.}} := \hat{p}\hat{q}$$ (2.6)

$$\hat{p}\hat{q}$$

and so on.

As it is well-known, given a classical hamiltonian $H(q,p)$, this corresponds to the right-point-prescription:

$$< q' | \hat{H}^{\text{N.O.}} | q > = \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \exp[-ip(q-q')] H(q, p)$$ (2.8)

Let us observe that not only $\hat{\cdot}^{\text{N.O.}}$ is not a quantization but even its restriction to $\mathcal{U}_2$ doesn’t obey the conditions required by the definition II.1.
III. THE LIMITS IN THE APPLICATION OF THE MINIMUM SUBSTITUTION PRESCRIPTION

Given the Minkowski space-time \((\mathbb{R}^4, \eta_{ab})\) and a curved space-time \((M, g_{ab})\) let us introduce the following:

**Definition III.1**

*generalization map:*

a map \(\overrightarrow{\cdot} : \mathcal{T}(\mathbb{R}^4) \to \mathcal{T}(M)\):

1. \(\overrightarrow{\mathcal{T}}(\mathbb{R}^4) \subseteq \mathcal{T}'(M) \quad \forall r, s \in \mathbb{N}\)
2. \(X^{a_1\cdots a_r} + Y^{a_1\cdots a_r} = \overrightarrow{X^{a_1\cdots a_r}} + \overrightarrow{Y^{a_1\cdots a_r}} \quad \forall X^{a_1\cdots a_r}, Y^{a_1\cdots a_r} \in \mathcal{T}'(\mathbb{R}^4), \quad \forall r, s \in \mathbb{N}\)
3. \(\lambda \overrightarrow{X^{a_1\cdots a_r}} = \overrightarrow{\lambda X^{a_1\cdots a_r}} \quad \forall \lambda \in \mathbb{R}, \quad \forall X^{a_1\cdots a_r} \in \mathcal{T}'(\mathbb{R}^4), \quad \forall r, s \in \mathbb{N}\)
4. \(\overrightarrow{X Y} = \overrightarrow{X} \overrightarrow{Y} \quad \forall X, Y \in \mathcal{T}(\mathbb{R}^4)\)
5. \(\overrightarrow{\nabla_a X^{a_1\cdots a_r}} = \overrightarrow{\nabla_a X^{a_1\cdots a_r}} \quad \forall \nabla_a X^{a_1\cdots a_r} \in \mathcal{T}'(\mathbb{R}^4), \quad \forall r, s \in \mathbb{N}\)
6. \(\overrightarrow{\eta_{ab}} = g_{ab}\)

Following the terminology of [2] in General Relativity one would be tempted to assume that the generalization of a non-gravitational law of Special Relativity, that is a non-gravitational law holding in Minkowski-space-time \((\mathbb{R}^4, \eta_{ab})\), to a non-gravitational law of Physics holding on an arbitrary spacetime \((M, g_{ab})\) is ruled by the following:

**Principle III.1**

*Minimum Substitution Prescription*

if \(X^{a_1\cdots a_r} = 0\) is a non-gravitational Physics’ law on \((\mathbb{R}^4, \eta_{ab})\) then \(\overrightarrow{X^{a_1\cdots a_r}} = 0\) is a Physics’ law on \((M, g_{ab})\) where \(\overrightarrow{\cdot}\) is a suitable generalization map.

**Example III.1**

In the Minkowski spacetime \((\mathbb{R}^4, \eta_{ab})\) indices are raised and lowered by contraction with \(\eta_{ab}\). For instance:

\[
X_a = \eta_{ab} X^b
\]

Applying the Principle \(\text{III.1}\) it follows that on an arbitrary spacetime \((M, g_{ab})\) indices are raised and lowered by contraction with \(g_{ab}\). For instance:

\[
\overrightarrow{X_a} = \eta_{ab} \overrightarrow{X^b} = g_{ab} X^b
\]

**Example III.2**

The motion of a free particle of mass \(m > 0\) in the Minkowski spacetime is given by:

\[
u^a \nabla^{(\eta)}_a u^b = 0
\]

where \(u^a\) is the 4-velocity of the particle (defined as the tangent vector to its worldline parametrized through proper time \(\tau\)) and hence the worldline of such a particle is a time-like geodesic of \((\mathbb{R}^4, \eta_{ab})\).

Applying the Principle \(\text{III.1}\) it follows that the motion of a free-falling particle of mass \(m > 0\) on an arbitrary curved spacetime \((M, g_{ab})\) is given by:

\[
u^a \nabla^{(g)}_a u^b = 0
\]

i.e.:

\[
\overrightarrow{u^a} \nabla^{(g)}_a \overrightarrow{u^b} = 0
\]

It follows that the worldline of a massive free falling particle on an arbitrary curved spacetime \((M, g_{ab})\) is a time-like geodesic of \((M, g_{ab})\).
Example III.3

A perfect fluid on the Minkowski space-time \((\mathbb{R}^4, \eta_{ab})\) is defined as a continuous distribution of matter with stress-energy tensor \(T_{ab}\) of the form:

\[
T_{ab} = \rho u_a u_b + P(\eta_{ab} + u_a u_b)
\]

(3.6)

where \(u^a\) is the unit time-like vector field representing the 4-velocity of the fluid, and where \(\rho\) and \(P\) are, respectively, the mass-energy density and the pressure of the fluid measured in its rest frame.

In absence of external forces the equation of motion of a perfect fluid in Minkowski space-time is:

\[
\nabla^{(\eta)a} T_{ab} = 0
\]

(3.7)

i.e.:

\[
\nabla^{(\eta)a}[\rho u_a u_b + P(\eta_{ab} + u_a u_b)] = 0
\]

(3.8)

Projecting equation parallel and perpendicular to \(u_b\) one obtains that:

\[
u^a \nabla^{(\eta)a} \rho + (\rho + P) \nabla^{(\eta)a} u_a = 0
\]

(3.9)

\[
(P + \rho) u^a \nabla^{(\eta)a} u_b + (\eta_{ab} + u_a u_b) \nabla^{(\eta)a} P = 0
\]

(3.10)

Applying the Principle III.1 it follows that a perfect fluid on a \((M, g_{ab})\) is defined as a continuous distribution of matter with stress-energy tensor:

\[
\hat{T}_{ab} = \rho \hat{u}_a \hat{u}_b + P(\eta_{ab} + \hat{u}_a \hat{u}_b)
\]

(3.11)

If a perfect-fluid is free-falling on \((M, g_{ab})\) its equation of motion, applying the Principle III.1, is:

\[
\nabla^{(g)a} \hat{T}_{ab} = 0
\]

(3.12)

that leads to:

\[
u^a \nabla^{(g)a} \rho + (\rho + P) \nabla^{(g)a} u_a = 0
\]

(3.13)

\[
(P + \rho) u^a \nabla^{(g)a} u_b + (\eta_{ab} + u_a u_b) \nabla^{(g)a} P = 0
\]

(3.14)

Example III.4

Electromagnetism on the Minkowski space-time \((\mathbb{R}^4, \eta_{ab})\) is described by the Maxwell equations:

\[
\nabla^{(\eta)a} F_{ab} = -4\pi j_b
\]

(3.15)

\[
\nabla^{(\eta)a} F_{bc} = 0
\]

(3.16)

where the electromagnetic field \(F_{ab}\) is totally antisymmetric:

\[
F_{[ab]} = F_{ab}
\]

(3.17)

Equation 3.14 implies that:

\[
\nabla^{(g)a} j_b = -\frac{1}{4\pi} \nabla^{(g)b} \nabla^{(g)a} F_{ab} = 0
\]

(3.18)
so that the 4-current $j^b$ is conserved.

The stress-energy tensor of the electromagnetic field is:

$$T_{ab} = \frac{1}{4\pi}(F_{ac}F_c^b - \frac{1}{4}\eta_{ab}F_{de}F^{de})$$

(3.19)

The equation of motion of a particle of mass $m > 0$ and electric charge $q$ moving in the electromagnetic field $F_{ab}$ is:

$$u^a\nabla_a(u^b) = \frac{q}{m}F_{bc}u^c$$

(3.20)

Applying the Principle III.1 it follows that Electromagnetism on a spacetime $(M, g_{ab})$ is described by the generalized Maxwell equations:

$$\nabla_a(u^b)\nabla_a(u^b) = \nabla_b(u^a)\nabla_b(u^a) = 0$$

(3.21)

$$\nabla_a F_{bc} = \nabla_a F_{bc} = 0$$

(3.22)

where the electromagnetic tensor $\nabla_a F_{bc}$ is totally antisymmetric:

$$F_{[ab]} - F_{ab} = F_{[ab]} - F_{ab} = 0$$

(3.23)

Let us observe that:

$$\nabla_a j^b = \nabla_a j^b = 0$$

(3.24)

so that the 4-current $j^b$ is conserved.

The stress-energy tensor of the electromagnetic field is:

$$T_{ab} = \frac{1}{4\pi}(F_{ac}F_c^b - \frac{1}{4}\eta_{ab}F_{de}F^{de}) = \frac{1}{4\pi}(F_{ac}F_c^b - \frac{1}{4}\eta_{ab}F_{de}F^{de})$$

(3.25)

The equation of motion of a particle of mass $m > 0$ and electric charge $q$ moving in the electromagnetic field $F_{ab}$ is:

$$u^a\nabla_a(u^b) - \frac{q}{m}F_{bc}u^c = u^a\nabla_a(u^b) - \frac{q}{m}F_{bc}u^c = 0$$

(3.26)

That the Principle III.1 cannot be assumed in its whole generality is, anyway, a consequence of the following:

**Theorem III.1**

GRÖENWALD VAN HOVE LIKE THEOREM OF GENERAL RELATIVITY

it doesn’t exist a *generalization map* $\nabla^\alpha$.

**PROOF:**

Let us suppose, ad absurdum, that a *generalization map* $\nabla^\alpha$ exists.

One has then that:

$$\nabla^\alpha \nabla^\alpha T^c = \nabla^\alpha \nabla^\alpha T^c \quad \forall T^c \in \mathcal{T}^1_0(\mathbb{R}^4)$$

(3.27)

From the other side:

$$\nabla^\alpha \nabla^\alpha T^c = \nabla^\alpha \nabla^\alpha T^c = \nabla^\alpha \nabla^\alpha T^c \quad \forall T^c \in \mathcal{T}^1_0(\mathbb{R}^4)$$

(3.28)
Let us now observe that:
\[
\nabla^{(g)}_a \nabla^{(g)}_b \widehat{T}^c = \nabla^{(g)}_b \nabla^{(g)}_a \widehat{T}^c - R^{(g)c}_{\ abd} \widehat{T}^d \quad \forall T^c \in \mathcal{T}_0^1(\mathbb{R}^4)
\]
and hence:
\[
\nabla^{(g)}_a \nabla^{(g)}_b \widehat{T}^c = \nabla^{(g)}_b \nabla^{(g)}_a \widehat{T}^c - R^{(g)c}_{\ abd} \widehat{T}^d \quad \forall T^c \in \mathcal{T}_0^1(\mathbb{R}^4)
\]
that, being the Riemann curvature tensor $R^{(g)c}_{\ abd} \neq 0$, is absurd. ■

Example III.5

Let us describe Electromagnetism over Minkowski spacetime $(\mathbb{R}^4, \eta_{ab})$ in terms of the vector potential $A^a$ defined by:
\[
F_{ab} =: \nabla^{(g)}_a A_b - \nabla^{(g)}_b A_a
\]
Equation 3.15 becomes:
\[
\nabla^{(g)}_a (\nabla^{(g)}_b A_a - \nabla^{(g)}_a A_b) = -4\pi j_b
\]
The equation 3.31 individuates $A_a$ up to a gauge transformation:
\[
A_a \rightarrow A_a + \nabla^{(g)}_a f \quad f \in \mathcal{T}_0^0(\mathbb{R}^4)
\]
By solving the equation:
\[
\nabla^{(g)}_a \nabla^{(g)}_a f = -\nabla^{(g) b} A_b
\]
we can make a gauge transformation to impose the Lorentz gauge condition:
\[
\nabla^{(g)}_a A_a = 0
\]
In this gauge equation 3.32 reduced to:
\[
\nabla^{(g)}_a \nabla^{(g)}_b A_b = -4\pi j_b
\]
Applying the Principle III.1 it follows that Electromagnetism on a spacetime $(M, g_{ab})$ may be described in terms of the generalized vector potential $\widehat{A}^a$ defined by:
\[
\widehat{F}_{ab} =: \nabla^{(g)}_a \widehat{A}_b - \nabla^{(g)}_b \widehat{A}_a
\]
Equation 3.21 becomes:
\[
\nabla^{(g)}_a (\nabla^{(g)}_b \widehat{A}_a - \nabla^{(g)}_a \widehat{A}_b) + 4\pi j_b = \nabla^{(g)}_a (\nabla^{(g)}_b \widehat{A}_a - \nabla^{(g)}_b \widehat{A}_b) + 4\pi j_b = 0
\]
Let us now observe that applying the Principle III.1 it follow that in the Lorentz gauge:
\[
\nabla^{(g)}_a \widehat{A}_a = \nabla^{(g)}_a \widehat{A}_a = 0
\]
it should be:
\[
\nabla^{(g) a} \nabla^{(g) b} \widehat{A}_{ab} + 4\pi j_b = \nabla^{(g) a} \nabla^{(g) b} \widehat{A}_{ab} + 4\pi j_b = 0
\]
and hence:
\[
\widehat{j}_b = -\frac{1}{4\pi} \nabla^{(g) a} \nabla^{(g) b} \widehat{A}_{ab}
\]
But then:
\[
\nabla^{(g) b} \widehat{j}_b = -\frac{1}{4\pi} \nabla^{(g) b} \nabla^{(g) a} \nabla^{(g) b} \widehat{A}_{ab} = -\frac{1}{4\pi} \nabla^{(g) b} R^{(g)c}_{\ abd} A_d \neq 0
\]
(where $R^{(g)c}_{\ abd} := R^{(g)c}_{\ abd}$ is the Ricci tensor of the metric $g_{ac}$) in contradiction with equation 3.31.
Remark III.1

We saw in the remark II.1 that a quantization map could be consistently defined on the algebra $U_2$ of polynomials in $q$ and $p$ of order less or equal than two, the inconsistencies stated by Theorem II.1 arising as soon as one tries to extend such a map to the set $U_3$ of polynomials in $q$ and $p$ of order less or equal than 3.

In a similar way the proof of theorem III.1 shows that defined the set of tensors of order $n$ over a differentiable manifold $M$ as:

$$T_n(M) := \bigcup_{r,s \in \mathbb{N}: r+s \leq n} T^{r,s}_n(M)$$  \hspace{1cm} (3.43)

a generalization map could be consistently defined on the set $T_2(\mathbb{R}^4)$ (it is essential, with this regard, that $\nabla^a_a \nabla^b_b f = \nabla^a_b \nabla^b_a f \ \forall f \in T^0_0(M)$), the inconsistencies arising as soon as one tries to extend such a map to the set $T_3(\mathbb{R}^4)$.

Remark III.2

One could think, with analogy to the Weyl-ordering of the remark II.2 to define a map $\overset{\text{symmetric}}{\cdot} : \mathcal{T}(\mathbb{R}^4) \ni T(M)$ defined by symmetrizing the double product of covariant derivatives in all possible combinations with equal weight, i.e.:

$$\overset{\text{symmetric}}{\nabla^a_a \nabla^b_b T^c} := \frac{1}{2} (\nabla^a_a \nabla^b_b T^c + \nabla^b_b \nabla^a_a T^c) \ \forall T^c \in T^1_0(\mathbb{R}^4)$$ \hspace{1cm} (3.44)

But then:

$$\overset{\text{symmetric}}{\nabla^a_a \nabla^b_b T^c} = \frac{1}{2} (2\nabla^a_a \nabla^b_b T^c - R^a_b c d T^d) \ \forall T^c \in T^1_0(\mathbb{R}^4)$$ \hspace{1cm} (3.45)

and hence $\overset{\text{symmetric}}{\cdot}$ is not a generalization map.
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Following once more we adopt Planck units defined by the condition $\hbar = G = c = 1$. In Planck units all the quantities having International System’s dimensions expressible in terms of $L$, $T$ and $M$ are dimensionless. In particular all lengths are expressed as dimensionless multiples of the Planck length $l_p := \left(\frac{G\hbar}{c^3}\right)^\frac{1}{2}$. 

APPENDIX A: UNITS AND DIMENSIONS
## APPENDIX B: NOTATION

| Symbol | Description |
|--------|-------------|
| $L_{s.a.}(\mathcal{H})$ | self-adjoint operators over the Hilbert space $\mathcal{H}$ |
| $\nabla^g_a$ | Levi Civita covariant derivative of the metric $g_{bc}$ |
| $T(M)$ | tensor fields over $M$ |
| $T^q(M)$ | tensor fields of type $(q,r)$ over $M$ |
| $T^n(M)$ | tensor fields of order $n$ over $M$ |
| $R^g_{(g)abcd}$ | Riemann curvature tensor of the metric $g_{ef}$ |
| $R^g_{ab}$ | Ricci tensor of the metric $g_{cd}$ |
| $T_{(a_1\cdots a_n)}$ | totally symmetric tensor of type $(0,n)$ |
| $T_{[a_1\cdots a_n]}$ | totally antisymmetric tensor of type $(0,n)$ |
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