Algebraic computation of some intersection D-modules

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Abstract

Let $X$ be a complex analytic manifold, $D \subset X$ a locally quasi-homogeneous free divisor, $\mathcal{E}$ an integrable logarithmic connection with respect to $D$ and $\mathcal{L}$ the local system of the horizontal sections of $\mathcal{E}$ on $X - D$. In this paper we give an algebraic description in terms of $\mathcal{E}$ of the regular holonomic $D_X$-module whose de Rham complex is the intersection complex associated with $\mathcal{L}$. As an application, we perform some effective computations in the case of quasi-homogeneous plane curves.

Introduction

On a complex analytic manifold, intersection complexes associated with irreducible local systems on a dense open regular subset of a closed analytic subspace are the simple pieces which form any perverse sheaf. The Riemann-Hilbert correspondence allows us to consider the regular holonomic $D$-modules which correspond to these intersection complexes, that we call “intersection $D$-modules”. They are the simple pieces which form any regular holonomic $D$-module. Whereas intersection complexes are topological objects, intersection $D$-modules are algebraic: they are given by a system of partial linear differential equations with holomorphic coefficients.

Intersection complexes can be constructed by an important operation: the intermediate direct image. Its description in terms of Verdier duality and usual derived direct images can be algebraically interpreted in the category of holonomic regular $D$-modules by using the deep properties of the de Rham functor. We need to compute localizations and $D$-duals.

This can be effectively done, in principle, by using the general available algorithms in [25, 27, 26], but in the case of integrable logarithmic connections along a locally quasi-homogeneous free divisor, we exploit the logarithmic point of view [2, 11, 12, 30, 31] to previously obtain a general algebraic description of their associated intersection $D$-modules, from which we can easily derive effective computations.

The main ingredients we use are the duality theorem proved in [5] and the logarithmic comparison theorem for arbitrary integrable logarithmic connections proved in [3], both with respect to locally quasi-homogeneous free divisors.

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The algorithmic treatment of the computations in this paper will be developed elsewhere.

Let us now comment on the content of this paper.

In section 1 we remind the reader of the basic notions and notations and we review our previous results on logarithmic \( \mathcal{D} \)-modules with respect to free divisors. We recall the logarithmic comparison theorem for arbitrary integrable logarithmic connections from [6], and we give the theorem describing the intersection \( \mathcal{D} \)-module associated with an integrable logarithmic connection along a locally quasi-homogeneous free divisor.

In section 2, given a locally quasi-homogeneous free divisor \( D \) with a reduced local equation \( f = 0 \) and a cyclic integrable logarithmic connection \( \mathcal{E} \) with respect to \( D \), we explicitly describe a presentation of \( \mathcal{D} \)[s]·(\( \mathcal{E}f^* \)) over \( \mathcal{D}[s] \) in terms of a presentation of \( \mathcal{E} \) over the ring of logarithmic differential operators. This description will be useful in order to compute the Bernstein-Sato polynomials associated with \( \mathcal{E} \).

In section 3 the general results of the previous section are explicitly written down in the case of a family of integrable logarithmic connections with respect to a quasi-homogeneous plane curves.

In section 4 we perform some explicit computations with respect to a cusp.

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1 Logarithmic connections with respect to a free divisor: theoretical set-up

Let \( X \) be a \( n \)-dimensional complex analytic manifold and \( D \subset X \) a hypersurface, and let us denote by \( j : U = X - D \to X \) the corresponding open inclusion.

We say that \( D \) is a free divisor \([28]\) if the \( \mathcal{O}_X \)-module \( \text{Der}(\log D) \) of logarithmic vector fields with respect to \( D \) is locally free (of rank \( n \)), or equivalently if the \( \mathcal{O}_X \)-module \( \Omega_X^1(\log D) \) of logarithmic 1-forms with respect to \( D \) is locally free (of rank \( n \)).

Normal crossing divisors, plane curves, free hyperplane arrangements (e.g. the union of reflecting hyperplanes of a complex reflection group), discriminant of stable mappings or bifurcation sets are examples of free divisors.

We say that \( D \) is quasi-homogeneous at \( p \in D \) if there is a system of local coordinates \( \mathfrak{p} \) centered at \( p \) such that the germ \((D,p)\) has a reduced weighted homogeneous defining equation (with strictly positive weights) with respect to \( \mathfrak{p} \). We say that \( D \) is locally quasi-homogeneous if it is so at each point \( p \in D \).

Let us denote by \( \mathcal{D}_X(\log D) \) the 0-term of the Malgrange-Kashiwara filtration with respect to \( D \) on the sheaf \( \mathcal{D}_X \) of linear differential operators on \( X \). When \( D \) is a free divisor, the first author has proved in [2] that \( \mathcal{D}_X(\log D) \) is the universal enveloping algebra of the Lie algebroid \( \text{Der}(\log D) \), and then it is coherent and has noetherian stalks of finite global homological dimension. Locally, if \( \{\delta_1, \ldots, \delta_n\} \) is a local basis of the logarithmic vector fields on an open set \( V \), any differential operator in \( \Gamma(V, \mathcal{D}_X(\log D)) \) can be written in a unique
way as a finite sum
\[ \sum_{\alpha \in \mathbb{N}^n, |\alpha| \leq d} a_\alpha \delta_1^{\alpha_1} \cdots \delta_n^{\alpha_n}, \]
where the \( a_\alpha \) are holomorphic functions on \( V \).

From now on, let us assume that \( D \) is a free divisor.

We say that \( D \) is a Koszul free divisor \([2]\) at a point \( p \in D \) if the symbols of any (some) local basis \( \{ \delta_1, \ldots, \delta_n \} \) of \( \text{Der}(\log D)_p \) form a regular sequence in \( \text{Gr}^n \mathcal{D}_X,p \). We say that \( D \) is a Koszul free divisor if it is so at any point \( p \in D \). Actually, as M. Schulze pointed out, Koszul freeness is equivalent to holonomicity in the sense of \([28]\).

Plane curves and locally quasi-homogeneous free divisors (e.g. free hyperplane arrangements or discriminant of stable mappings in Mather’s “nice dimensions”) are example of Koszul free divisors \([3]\).

A logarithmic connection with respect to \( D \) is a locally free \( \mathcal{O}_X \)-module \( \mathcal{E} \) endowed with:

1. a \( \mathbb{C} \)-linear morphism (connection) \( \nabla' : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X(\log D) \), satisfying \( \nabla'(ae) = a\nabla'(e) + e \otimes da \), for any section \( a \) of \( \mathcal{O}_X \) and any section \( e \) of \( \mathcal{E} \), or equivalently, with

2. a left \( \mathcal{O}_X \)-linear morphism \( \nabla : \text{Der}(\log D) \to \text{End}_{\mathcal{O}_X}(\mathcal{E}) \) satisfying the Leibniz rule \( \nabla(\delta)(ae) = a\nabla(\delta)(e) + \delta(a)e \), for any logarithmic vector field \( \delta \), any section \( a \) of \( \mathcal{O}_X \) and any section \( e \) of \( \mathcal{E} \).

The integrability of \( \nabla' \) is equivalent to the fact that \( \nabla \) preserve Lie brackets. Then, we know from \([2]\) that giving an integrable logarithmic connection on a locally free \( \mathcal{O}_X \)-module \( \mathcal{E} \) is equivalent to extending its original \( \mathcal{O}_X \)-module structure to a left \( \mathcal{D}_X(\log D) \)-module structure, and so integrable logarithmic connections are the same as left \( \mathcal{D}_X(\log D) \)-modules which are locally free of finite rank over \( \mathcal{O}_X \).

Let us denote by \( \mathcal{O}_X(D) \) the sheaf of meromorphic functions with poles along \( D \). It is a holonomic left \( \mathcal{D}_X \)-module.

The first examples of integrable logarithmic connections (ILC for short) are the invertible \( \mathcal{O}_X \)-modules \( \mathcal{O}_X(mD) \subset \mathcal{O}_X(D) \), \( m \in \mathbb{Z} \), formed by the meromorphic functions \( h \) such that \( \text{div}(h) + mD \geq 0 \).

If \( f = 0 \) is a reduced local equation of \( D \) at \( p \in D \) and \( \delta_1, \ldots, \delta_n \) is a local basis of \( \text{Der}(\log D)_p \) with \( \delta_i(f) = \alpha_i f \), then \( f^{-m} \) is a local basis of \( \mathcal{O}_X(p(mD)) \) over \( \mathcal{O}_{X,p} \) and we have the following local presentation over \( \mathcal{D}_{X,p}(\log D) \) \((2, \text{th. } 2.1.4)\)

\[
\mathcal{O}_{X,p}(mD) \simeq \mathcal{D}_{X,p}(\log D) / \mathcal{D}_{X,p}(\log D)(\delta_1 + m\alpha_1, \ldots, \delta_n + m\alpha_n). \tag{1.1}
\]

For any ILC \( \mathcal{E} \) and any integer \( m \), the locally free \( \mathcal{O}_X \)-modules \( \mathcal{E}(mD) := \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X(mD) \) and \( \mathcal{E}^* := \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X) \) are endowed with a natural structure of left \( \mathcal{D}_X(\log D) \)-module, where the action of logarithmic vector fields is given by

\[
(\delta h)(e) = -h(\delta e) + \delta(h(e)), \quad \delta(e \otimes a) = (\delta e) \otimes a + e \otimes \delta(a) \tag{2}
\]
for any logarithmic vector field \( \delta \), any local section \( h \) of \( \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X) \), any local section \( e \) of \( \mathcal{E} \) and any local section \( a \) of \( \mathcal{O}_X(mD) \) (cf. \([5], \S 2\) ). Then \( \mathcal{E}(mD) \) and \( \mathcal{E}^* \) are ILC again, and the usual isomorphisms

\[
\mathcal{E}(mD)(m'D) \simeq \mathcal{E}((m + m')D), \quad \mathcal{E}(mD)^* \simeq \mathcal{E}^*(-mD)
\]
are $\mathcal{D}_X(\log D)$-linear.

(1.2) If $D$ is Koszul free and $\mathcal{E}$ is an ILC, then the complex $\mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} L \mathcal{E}$ is concentrated in degree 0 and its 0-cohomology $\mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} \mathcal{E}$ is a holonomic $\mathcal{D}_X$-module (see [3], prop. 1.2.3).

If $\mathcal{E}$ is an ILC, then $\mathcal{E}(\ast D)$ is a meromorphic connection (locally free of finite rank over $\mathcal{O}_X(\ast D)$) and then it is a holonomic $\mathcal{D}_X$-module (cf. [20], th. 4.1.3). Actually, $\mathcal{E}(\ast D)$ has regular singularities on the smooth part of $D$ (it has logarithmic poles! [10]) and then it is regular everywhere [19], cor. 4.3-14, which means that if $\mathcal{L}$ is the local system of horizontal sections of $\mathcal{E}$ on $U = X - D$, the canonical morphism

$$\Omega^\bullet_X(\mathcal{E}(\ast D)) \to Rj_* \mathcal{L}$$

is an isomorphism in the derived category.

For any ILC $\mathcal{E}$, or even for any left $\mathcal{D}_X(\log D)$-module (without any finiteness property over $\mathcal{O}_X$), one can define its logarithmic de Rham complex $\Omega^\bullet_X(\log D)(\mathcal{E})$ in the classical way (cf. [10, def. I.2.15]), which is a subcomplex of $\Omega^\bullet_X(\mathcal{E}(\ast D))$.

It is clear that both complexes coincide on $U$.

For any ILC $\mathcal{E}$ and any integer $m$, $\mathcal{E}(mD)$ is a sub-$\mathcal{D}_X(\log D)$-module of the regular holonomic $\mathcal{D}_X$-module $\mathcal{E}(\ast D)$, and then we have a canonical morphism

$$\rho_{\mathcal{E},m} : \mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} \mathcal{E}(mD) \to \mathcal{E}(\ast D),$$

given by $\rho_{\mathcal{E},m}(P \otimes e') = Pe'$.

Since $\mathcal{E}(m'D)(mD) = \mathcal{E}((m + m')D)$ and $\mathcal{E}(m'D)(\ast D) = \mathcal{E}(\ast D)$, we can identify morphisms $\rho_{\mathcal{E},m'D,m}$ and $\rho_{\mathcal{E},m,m'}$.

For any bounded complex $\mathcal{K}$ of sheaves of $\mathbb{C}$-vector spaces on $X$, let us denote by $\mathcal{K}^\vee = R\text{Hom}_{\mathcal{O}_X}(\mathcal{K}, \mathcal{C}_X)$ its Verdier dual.

The dual local system $\mathcal{L}^\vee$ appears as the local system of the horizontal sections of the dual ILC $\mathcal{E}^\ast$. We have the following theorem (see [3], th. 4.1 and [4] th. (2.1.1)):

(1.3) **Theorem.** Let $\mathcal{E}$ be an ILC (with respect to the divisor $D$) and let $\mathcal{L}$ be the local system of its horizontal sections on $U = X - D$. The following properties are equivalent:

1) The canonical morphism $\Omega^\bullet_X(\log D)(\mathcal{E}) \to Rj_* \mathcal{L}$ is an isomorphism in the derived category of complexes of sheaves of complex vector spaces.

2) The inclusion $\Omega^\bullet_X(\log D)(\mathcal{E}) \hookrightarrow \Omega^\bullet_X(\mathcal{E}(\ast D))$ is a quasi-isomorphism.

3) The morphism $\rho_{\mathcal{E},1} : \mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} L \mathcal{E}(D) \to \mathcal{E}(\ast D)$ is an isomorphism in the derived category of left $\mathcal{D}_X$-modules.

4) The complex $\mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} L \mathcal{E}(D)$ is concentrated in degree 0 and the $\mathcal{D}_X$-module $\mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} \mathcal{E}(D)$ is holonomic and isomorphic to its localization along $D$.

Moreover, if $D$ is a Koszul free divisor, the preceding properties are also equivalent to:
5) The canonical morphism \( j^! \mathcal{L} \to \Omega^*_X(\log D)(\mathcal{E}(-D)) \) is an isomorphism in the derived category of complexes of sheaves of complex vector spaces.

For \( D \) a locally quasi-homogeneous free divisor and \( \mathcal{E} = \mathcal{O}_X \), the equivalent properties in theorem \((1.3)\) hold: this is the so called “logarithmic comparison theorem” \([1]\) (see also \([4\text{ th. 4.4]} \) and \([5, \text{ cor. (2.1.3)}]\) for other proofs based on D-module theory).

\((1.4)\) Let \( \mathcal{E} \) be an ILC (with respect to \( D \)) and \( p \) a point in \( D \). Let \( f \in \mathcal{O} = \mathcal{O}_{X,p} \) be a reduced local equation of \( D \) and let us write \( D = D_{X,p}, V_0 = D_X(\log D)_p \) and \( E = \mathcal{E}_p \). We know from \([5, \text{ lemma (3.2.1)}]\) that the ideal of polynomials \( b(s) \in \mathbb{C}[s] \) such that \( b(s)Ef^* \subset D[s] \cdot (Ef^{s+1}) \subset E[f^{-1}, s]f^s \) is generated by a non constant polynomial \( b_{\mathcal{E},p}(s) \). By the coherence of the involved objects we deduce that \( b_{\mathcal{E},q}(s) \mid b_{\mathcal{E},p}(s) \) for \( q \in D \) close to \( p \).

If \( b_{\mathcal{E},p}(s) \) has some integer root, let us call \( \kappa(\mathcal{E}, p) \) the minimum of those roots. If not, let us write \( \kappa(\mathcal{E}, p) = +\infty \).

Let us call \( \kappa(\mathcal{E}) = \inf \{ \kappa(\mathcal{E}, p) \mid p \in D \} \in \mathbb{Z} \cup \{ \pm \infty \} \).

From now on let us suppose that \( D \) is a locally quasi-homogeneous free divisor.

\((1.5)\) \textbf{Theorem.} \( \text{Under the above hypothesis, if } \kappa(\mathcal{E}) > -\infty, \text{ then the morphism} \)

\[ \rho_{\mathcal{E}, k} : D_X \otimes_{D_X(\log D)} \mathcal{E}(kD) \to \mathcal{E}(\ast D) \] \[ (3) \]

is an isomorphism in the derived category of left \( D_X \)-modules, for all \( k \geq -\kappa(\mathcal{E}) \).

\textbf{Proof.} \( \text{It is a straightforward consequence of } \([3, \text{ th. 5.6]\) and theorem (3.2.6) of } \([3]\) \text{ and its proof.} \)

\( \text{Q.E.D.} \)

Let us note that the hypothesis \( \kappa(\mathcal{E}) > -\infty \) in theorem \((1.5)\) holds locally on \( X \).

In the situation of theorem \((1.5)\) if \( \mathcal{L} \) is the local system of the horizontal sections of \( \mathcal{E} \) on \( U = X - D \), then the derived direct image \( Rj_* \mathcal{L} \) is canonically isomorphic (in the derived category) to the de Rham complex of the holonomic \( D_X \)-module \( D_X \otimes_{D_X(\log D)} \mathcal{E}(kD) \):

\[ \text{DR} \left( D_X \otimes_{D_X(\log D)} \mathcal{E}(kD) \right) = \text{DR} \left( D_X \otimes_{D_X(\log D)} \mathcal{E}(\ast D) \right) \simeq \]

\[ \text{DR} \mathcal{E}(\ast D) \simeq \Omega^*_X(\mathcal{E}(\ast D)) \simeq Rj_* \mathcal{L}. \]

Proceeding as above for the dual ILC \( \mathcal{E}^* \), we find that if \( \kappa(\mathcal{E}^*) > -\infty \), then we have that the canonical morphism

\[ \text{DR} \left( D_X \otimes_{D_X(\log D)} \mathcal{E}^*(k'D) \right) \to Rj_* \mathcal{L}^\vee \]

is an isomorphism in the derived category for \( k' \geq -\kappa(\mathcal{E}^*) \).
Let us denote by
\[ g_{\xi,k,k'} : D_X \otimes_{D_X(\text{log }D)} E((1-k')D) \to D_X \otimes_{D_X(\text{log }D)} E(kD), \]
the \( D_X \)-linear morphism induced by the inclusion \( E((1-k')D) \subset E(kD) \), \( 1-k' \leq k \), and by \( IC_X(L) \) the intersection complex of Deligne-Goresky-MacPherson associated with \( L \), which is described as the intermediate direct image \( j_*L \), i.e. the image of \( j_*L \to Rj_*L \) in the category of perverse sheaves (cf. \[ 11 \], def. 1.4.22).

The following theorem describes the “intersection \( D_X \)-module” corresponding to \( IC_X(L) \) by the Riemann-Hilbert correspondence of Mebkhout-Kashiwara \[ 13 \, 16 \, 17 \].

1.6 THEOREM. Under the above hypothesis, we have a canonical isomorphism in the category of perverse sheaves on \( X \),
\[ IC_X(L) \simeq DR(\text{Im } g_{\xi,k,k'}), \]
for \( k \geq -\kappa(\xi) \), \( k' \geq -\kappa(\xi^*) \), and \( 1-k' \leq k \),

PROOF. Using our duality results in \[ 3 \, 8 \] the Local Duality Theorem for holonomic \( D_X \)-modules (\[ 8 \], ch. I, th. (4.3.1); see also \[ 22 \]) and theorem (1.5), we obtain
\[ DR(D_X \otimes_{D_X(\text{log }D)} E((1-k')D)) \simeq DR(D_X \otimes_{D_X(\text{log }D)} E^*(k'D)^*(D)) \simeq \]
\[ DR(D\ e(D_X \otimes_{D_X(\text{log }D)} E^*(k'D))) \simeq \left[ DR(D_X \otimes_{D_X(\text{log }D)} E^*(k'D)) \right]^\vee \simeq \]
\[ [Rj_*L]^\vee \simeq j_*L. \]

On the other hand, the canonical morphism \( j_*L \to Rj_*L \) corresponds, through the de Rham functor, to the \( D_X \)-linear morphism \( g_{\xi,k,k'} \), and the theorem is a consequence of the Riemann-Hilbert correspondence which says that the de Rham functor establishes an equivalence of abelian categories between the category of regular holonomic \( D_X \)-modules and the category of perverse sheaves on \( X \).

Q.E.D.

1.7 REMARK. For \( E = \mathcal{O}_X \), one has \( E^* = \mathcal{O}_X \) and there are examples where morphisms \( \rho_{\alpha,k} \) in \[ 3 \] are never isomorphisms (\[ 3 \], ex. 5.3). Nevertheless, for \( k = k' = 1 \) the image of the morphism
\[ g_{\mathcal{O}_X,1,1} : D_X \otimes_{D_X(\text{log }D)} \mathcal{O}_X \to D_X \otimes_{D_X(\text{log }D)} \mathcal{O}_X (D) \]
is always (canonically isomorphic to) \( \mathcal{O}_X \), which is the regular holonomic \( D_X \)-module corresponding by the Riemann-Hilbert correspondence to \( IC_X(\mathcal{O}_U) = \mathbb{C}_X \), where \( \mathbb{C}_U \) is the local system of horizontal sections of \( \mathcal{O}_X \) on \( U \). To see this, let us work locally as in \[ 11 \]. Then, morphism \( g_{\mathcal{O}_X,1,1} \) is given at point \( p \) by
\[ \mathcal{P} \in D_{X,p}/D_{X,p}(\delta_1, \ldots, \delta_n) \mapsto \mathcal{P}j \in D_{X,p}/D_{X,p}(\delta_1 + \alpha_1, \ldots, \delta_n + \alpha_n) \]
and the stalk at \( p \) of \( \text{Im } g_{\mathcal{O}_X,1,1} \) is given by \( D_{X,p}/J \) where \( J \) is the left ideal
\[ J = \{ P \in D_{X,p} \mid Pf \in D_{X,p}(\delta_1 + \alpha_1, \ldots, \delta_n + \alpha_n) \}. \]
By Saito’s criterion [28] we can suppose
\[
\begin{pmatrix}
\delta_1 \\
\vdots \\
\delta_n
\end{pmatrix} = A \begin{pmatrix}
\frac{\partial}{\partial x_1} \\
\vdots \\
\frac{\partial}{\partial x_n}
\end{pmatrix}
\]
where \(A\) is a \(n \times n\) matrix with entries in \(\mathcal{O}_{X,p}\) and \(\det A = f\). Writing \(B = \text{adj}(A)^t\) we obtain
\[
B \left( \begin{pmatrix}
\delta_1 \\
\vdots \\
\delta_n
\end{pmatrix} \right) = f \left( \begin{pmatrix}
\frac{\partial}{\partial x_1} \\
\vdots \\
\frac{\partial}{\partial x_n}
\end{pmatrix} \right) \text{ eval. on } f \rightarrow \begin{pmatrix}
\alpha_1 \\
\vdots \\
\alpha_n
\end{pmatrix} = \begin{pmatrix}
\frac{\partial f}{\partial x_1} \\
\vdots \\
\frac{\partial f}{\partial x_n}
\end{pmatrix}.
\]

Then
\[
\begin{pmatrix}
\frac{\partial}{\partial x_1} \\
\vdots \\
\frac{\partial}{\partial x_n}
\end{pmatrix} f = f \left( \begin{pmatrix}
\frac{\partial}{\partial x_1} \\
\vdots \\
\frac{\partial}{\partial x_n}
\end{pmatrix} \right) + \begin{pmatrix}
\frac{\partial f}{\partial x_1} \\
\vdots \\
\frac{\partial f}{\partial x_n}
\end{pmatrix} = \cdots = B \left( \begin{pmatrix}
\delta_1 + \alpha_1 \\
\vdots \\
\delta_n + \alpha_n
\end{pmatrix} \right)
\]
and \(\frac{\partial}{\partial x_i} \in J\) for \(i = 1, \ldots, n\). Since \(J\) is is not the total ideal, we deduce by maximality that \(J\) is the ideal generated by the \(\frac{\partial}{\partial x_i}\) and \(\mathcal{D}_{X,p}/J \simeq \mathcal{O}_{X,p}\). To conclude, one easily sees, from the fact that morphism \(\rho_{\mathcal{O}_{X,1,1}}\) factors through
\[
a \in \mathcal{O}_X \mapsto 1 \otimes a \in \mathcal{D}_X \otimes_{\mathcal{D}_X(log D)} \mathcal{O}_X(D)
\]
[it is \(\mathcal{D}_X\)-linear since, for any derivation \(\delta\) and any holomorphic function \(a\), \(\delta(1 \otimes a) = \delta \otimes a = \delta \otimes (f f^{-1} a) = (\delta f) \otimes (f^{-1} a) = 1 \otimes (\delta f)(f^{-1} a) = 1 \otimes (\delta a)\)] that the isomorphisms above at different \(p\) glue together and give a global isomorphism \(\text{Im} \rho_{\mathcal{O}_{X,1,1}} \simeq \mathcal{O}_X\).

This example suggests studying the comparison between \(\text{DR}(\text{Im} \rho_{\mathcal{E},k,k'})\), \(k,k' \gg 0\), and \(\text{IC}_X(\mathcal{L})\) in theorem [1.6], independent of the fact that \(\rho_{\mathcal{E},k}\) and \(\rho_{\mathcal{E}',k'}\) are isomorphisms or not.

## 2 Bernstein-Sato polynomials for cyclic integrable logarithmic connections

In the situation of [1.4] let us assume that \(E\) is a cyclic \(\mathcal{V}_0\)-module generated by an element \(e \in E\). The following result is proved in [10] prop. (3.2.3)).

(2.1) **Proposition.** Under the above conditions, the polynomial \(b_{\mathcal{E},p}(s)\) coincides with the Bernstein-Sato polynomial \(b_e(s)\) of \(e\) with respect to \(f\), where \(e\) is considered to be an element of the holonomic \(\mathcal{D}\)-module \(E[f^{-1}]\) (cf. [12]).

(2.2) Let \(\Theta_{f,s} \subset \mathcal{D}[s]\) be the set of operators in \(\text{ann}_{\mathcal{D}[s]} f^s\) of total order \((in s and in the derivatives) \leq 1\). The elements of \(\Theta_{f,s}\) are of the form \(\delta - \alpha s\) with \(\delta \in \text{Der}_C(\mathcal{O}), \alpha \in \mathcal{O}\) and \(\delta(f) = \alpha f\). In particular \(\Theta_{f,s} \subset \mathcal{V}_0[s]\).

The \(\mathcal{O}\)-linear map
\[
\delta \in \text{Der}(\text{log } D)_p \mapsto \delta - \frac{\delta(f)}{f} s \in \Theta_{f,s}
\]
is an isomorphism of Lie-Rinehart algebras over \((\mathbb{C}, \mathcal{O})\) and extends to a unique ring isomorphism \(\Phi : \mathcal{V}_0[s] \to \mathcal{V}_0[s]\) with \(\Phi(s) = s\) and \(\Phi(a) = a\) for all \(a \in \mathcal{O}\).

Let us note that \(\Phi^{-1}(\delta) = \delta + \frac{\delta(f)}{s}\) for each \(\delta \in \text{Der}(\log D)_{P}\).

It is clear that \(E[s]f^s\) is a sub-\(\mathcal{V}_0[s]\)-module of \(E[s, f^{-1}]f^s\) and that for any \(P \in \mathcal{V}_0[s]\) and any \(e' \in E[s]\), the following relation holds

\[
(Pe')f^s = \Phi(P)(e'f^s).
\]

(5)

(2.3) Proposition. Under the above conditions, the following relation holds

\[
\text{ann}_{\mathcal{V}_0[s]}(ef^s) = \mathcal{V}_0[s] \cdot \Phi(\text{ann}_{\mathcal{V}_0}e).
\]

Proof. The inclusion \(\supseteq\) comes from (5). For the other inclusion, let \(Q \in \text{ann}_{\mathcal{V}_0[s]}(ef^s)\) and let us write \(\Phi^{-1}(Q) = \sum_{i=1}^{d} P_is^i\) with \(P_i \in \mathcal{V}_0\). We have

\[
0 = Q(ef^s) = (\Phi^{-1}(Q)e)f^s = \left(\sum_{i=1}^{d} (P_is^i)f^s\right)
\]

and then \(P_i \in \text{ann}_{\mathcal{V}_0}e\). Therefore

\[
Q = \Phi\left(\sum_{i=1}^{d} P_is^i\right) = \sum_{i=1}^{d} \Phi(P_i)s^i \in \mathcal{V}_0[s] \cdot \Phi(\text{ann}_{\mathcal{V}_0}e).
\]

Q.E.D.

(2.4) Proposition. Under the above conditions, if \(D\) is a locally quasi-homogeneous free divisor, then

\[
\text{ann}_{D[s]}(ef^s) = D[s] \cdot \text{ann}_{\mathcal{V}_0[s]}(ef^s).
\]

Proof. From [4] we know that \(E[s]f^s = \mathcal{V}_0[s] \cdot (ef^s)\), and from [4] cor. (3.1.2)] we know that the morphism

\[
\rho_{E,s} : P \otimes (e'f^s) \in D[s] \otimes_{\mathcal{V}_0[s]} E[s]f^s \mapsto P(e'f^s) \in D[s] \cdot (E[s]f^s) = D[s] \cdot (ef^s)
\]

is an isomorphism of left \(D[s]\)-modules. Therefore

\[
\text{ann}_{D[s]}(ef^s) = D[s] \cdot \text{ann}_{\mathcal{V}_0[s]}(ef^s).
\]

Q.E.D.

(2.5) Corollary. Under the above conditions, if \(D\) is a locally quasi-homogeneous free divisor, then

\[
\text{ann}_{D[s]}(ef^s) = D[s] \cdot \Phi(\text{ann}_{\mathcal{V}_0}e).
\]

Proof. It follows from propositions [2.3] and [2.4] Q.E.D.
(2.6) Remark. Theorems (1.5) and (1.6), proposition (2.4) and corollary (2.5) remain true if we only assume that our divisor $D$ is Koszul free and of commutative linear type, i.e. its jacobian ideal is of linear type (see [3, §3]).

(2.7) Remark. As we shall see in sections 3 and 4, theorem (1.6), proposition (2.1) and corollary (2.5) provide an effective method of computing the intersection $\mathcal{D}_X$-module corresponding to $\text{IC}(L)$ in terms of the $\text{ILC}$ $\mathcal{E}$, at least if $D$ is a locally quasi-homogeneous free divisor, or more generally, if $D$ is Koszul free and of commutative linear type (see remark (2.6)).

(2.8) Remark. In the particular case of $E = \mathcal{O}_X$ and $E = \mathcal{O}$, corollary (2.5) says that

$$\text{ann}_{\mathcal{D}[s]}(f^s) = \mathcal{D}[s] \cdot (\delta_1 - \alpha_1 s, \ldots, \delta_n - \alpha_n s),$$

where $\delta_1, \ldots, \delta_n$ is a local basis of $\text{Der}(\log D)_p$ and $\delta_i(f) = \alpha_i f$ (see corollary 5.8, (b) in [4]).

(2.9) Example. Let us suppose that $D \subset X$ is a non-necessarily free divisor and let $f = 0$ be a reduced local equation of $D$ at a point $p \in D$. Let $\{\delta_1, \ldots, \delta_m\}$ a system of generators of $\text{Der}(\log D)_p$ and let us write $\delta_i(f) = \alpha_i f$.

Let us call $\text{ann}_{\mathcal{D}[s]}^{(1)}(f^s)$ the ideal of $\mathcal{D}[s]$ generated by $\Theta_{f,s}$ (see (2.2)):

$$\text{ann}_{\mathcal{D}[s]}^{(1)}(f^s) = \mathcal{D}[s] \cdot (\delta_1 - \alpha_1 s, \ldots, \delta_m - \alpha_m s) \subset \text{ann}_{\mathcal{D}[s]}(f^s).$$

The Bernstein functional equation for $f$

$$b(s)f^s = P(s)f^{s+1}$$

means that the operator $b(s) - P(s)f$ belongs to the annihilator of $f^s$ over $\mathcal{D}[s]$. Then, an explicit knowledge of the ideal $\text{ann}_{\mathcal{D}[s]}(f^s)$ allows us to find $b(s)$ by computing the ideal

$$\mathbb{C}[s] \cap (\mathcal{D}[s] \cdot f + \text{ann}_{\mathcal{D}[s]}(f^s)),
$$

(see [23]). However, the ideal $\text{ann}_{\mathcal{D}[s]}(f^s)$ is in general difficult to compute.

When $D$ is a locally quasi-homogeneous free divisor, or more generally, a divisor of differential linear type ([3, def. (1.4.5)]), $\text{ann}_{\mathcal{D}[s]}(f^s) = \text{ann}_{\mathcal{D}[s]}^{(1)}(f^s)$ and the computation of $b(s)$ is in principle easier.

But there are other examples where the Bernstein polynomial $b(s)$ belongs to

$$\mathbb{C}[s] \cap (\mathcal{D}[s] \cdot f + \text{ann}_{\mathcal{D}[s]}^{(1)}(f^s))$$

even if $\text{ann}_{\mathcal{D}[s]}(f^s) \neq \text{ann}_{\mathcal{D}[s]}^{(1)}(f^s)$. For instance, when $X = \mathbb{C}^3$ and $f = x_1x_2(x_1 + x_2)(x_1 + x_2x_3)$ (see example 6.2 in [4]) or in any of the examples in page 445 of [9]. In all these examples the divisor is free and satisfies the logarithmic comparison theorem.
3  Integrable logarithmic connections along quasi-homogeneous plane curves

Let $D \subset X = \mathbb{C}^2$ be a divisor defined by a reduced polynomial equation $h(x_1, x_2)$, which is quasi-homogeneous with respect to the strictly positive integer weights $\omega_1, \omega_2$ of the variables $x_1, x_2$. We denote by $\omega(f)$ the weight of a quasi-homogeneous polynomial $f(x_1, x_2)$. The divisor $D$ is free, a global basis of $\text{Der}(\log D)$ is $\{\delta_1, \delta_2\}$, where

\[
\begin{pmatrix}
\delta_1 \\
\delta_2
\end{pmatrix} = \begin{pmatrix}
\omega_1 x_1 & \omega_2 x_2 \\
-h_{x_2} & h_{x_1}
\end{pmatrix} \begin{pmatrix}
\frac{\partial}{\partial x_1} \\
\frac{\partial}{\partial x_2}
\end{pmatrix}.
\]

We have:
- $\delta_1(h) = \omega(h)h$, $\delta_2(h) = 0$,
- the determinant of the coefficient matrix is equal to $\omega(h)h$,
- $[\delta_1, \delta_2] = c\delta_2$, with $c = \omega(h) - \omega_1 - \omega_2$.

We consider a logarithmic connection $\mathcal{E} = \bigoplus_{i=1}^n \mathcal{O}_X e_i$ given by actions:

\[
\begin{align*}
\delta_1 \cdot \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} &= A_1 \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}, \\
\delta_2 \cdot \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} &= A_2 \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}.
\end{align*}
\]

For $\mathcal{E}$ to be integrable, the following integrability condition

\[
\delta_1(A_2) - \delta_2(A_1) + [A_2, A_1] = cA_2
\]

must hold.

(3.1) We shall focus on the case where $A_1, A_2$ are $n \times n$ matrices satisfying $\mathbb{F}$ and of the form:

\[
A_1 = \begin{pmatrix}
-a & 0 & 0 & \cdots & 0 & 0 \\
-\delta_2(a) & -a+c & 0 & \cdots & 0 & 0 \\
-\delta_2^2(a) & -\frac{n-2}{2} \delta_2(a) & -a+2c & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
-\delta_2^{n-2}(a) & -\binom{n-2}{1} \delta_2^{n-3}(a) & -\binom{n-2}{2} \delta_2^{n-4}(a) & \cdots & -a+(n-2)c & 0 \\
-\delta_2^{n-1}(a) & -\binom{n-1}{1} \delta_2^{n-2}(a) & -\binom{n-1}{2} \delta_2^{n-3}(a) & \cdots & -\binom{n-1}{n-2} \delta_2(a) & -a+(n-1)c
\end{pmatrix},
\]

\[
A_2 = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
-b_0 & -b_1 & -b_2 & \cdots & -b_{n-1}
\end{pmatrix},
\]

with $a, b_0, \ldots, b_{n-1}$ polynomials. Let us call $\mathcal{E}_{a,b}$ the corresponding ILC.

(3.2) Lemma. The $\mathcal{D}_X(\log D)$-module $\mathcal{E}_{a,b}$ is generated by $e_1$ (so it is cyclic) and the $\mathcal{D}_X(\log D)$-annihilator of $e_1$ is the left ideal $J_{a,b}$ generated by $\delta_1 + a$ and
\[ \delta_2^g + b_{n-1}\delta_2^{g-1} + \cdots + b_1\delta_2 + b_0. \] So, the \( D_X(\log D) \)-module \( \mathcal{E}_{a,b} \) is isomorphic to \( D_X(\log D)/J_{a,b} \).

**Proof.** The first part is clear since \( \delta_2 \cdot e_i = e_{i+1} \) for \( i = 1, \ldots, n - 1 \). For the second part, the inclusion \( J_{a,b} \subset \text{ann}_{D_X(\log D)}(e_1) \) is also clear. To prove the opposite inclusion, we use the fact that any germ of logarithmic differential operator \( P \) has a unique expression as a sum \( P = \sum_{i,j} a_{i,j}\delta_1^i\delta_2^j \), where the \( a_{i,j} \) are germs of holomorphic functions (see section (1.1)).

(3.3) **Remark.** Theorem 2.1.4 in [2] says that \( D_X(\log D) = \mathcal{O}_X[\delta_1, \delta_2] \) with relations:

\[ [\delta_1, f] = \delta_1(f), [\delta_2, f] = \delta_2(f), [\delta_1, \delta_2] = c\delta_2, \quad f \in \mathcal{O}_X. \]

In particular, we can define the support and the exponent of any germ of logarithmic differential operator \( P \) (or of any polynomial logarithmic differential operator in the Weyl algebra) by using the (unique) expression \( P = \sum_{i,j} a_{i,j}\delta_1^i\delta_2^j \), and we obtain a division theorem and a notion of Gröbner basis for ideals. Under this scope, the integrability condition reads out as the fact that the generators

\[ g_1 = \delta_1 + a, \quad g_2 = \delta_2^n + b_{n-1}\delta_2^{n-1} + \cdots + b_0 \]

of \( J_{a,b} \) satisfy Buchberger’s criterion, i.e. that \( \delta_2^2g_1 - \delta_1g_2 \) has a vanishing remainder with respect to the division by \( g_1, g_2 \), and then they form a Gröbner basis of \( J_{a,b} \).

(3.4) **Corollary.** The \( D_X \)-module \( D_X \otimes_{D_X(\log D)} \mathcal{E}_{a,b} \) is isomorphic to \( D_X/J_{a,b} \), where \( J_{a,b} = D_X(\delta_1 + a, \delta_2^n + b_{n-1}\delta_2^{n-1} + \cdots + b_0) \).

For any integer \( k \), we can consider the logarithmic connections \( \mathcal{E}_{a,b}(kD) \) and \( \mathcal{E}^*_{a,b} \) (see section (1.1)).

(3.5) **Lemma.** With the above notations, the ILC \( \mathcal{E}_{a,b}(kD) \) and \( \mathcal{E}_{a+\omega(h)k} \) are isomorphic.

**Proof.** An \( \mathcal{O}_X \)-basis of \( \mathcal{E}_{a,b}(kD) \) is \( \{ e_i^k = e_i \otimes h^{-k} \}_{i=1}^n \) and the action of \( \text{Der}(\log D) \) over this basis is given by (see [2]):

\[ \delta_1 \cdot e_i^k = (\delta_1 \cdot e_i) \otimes h^{-k} + e_i \otimes (-\omega(h)kh^{-k}), \quad \delta_2 \cdot e_i^k = (\delta_2 \cdot e_i) \otimes h^{-k}. \]

Then, the isomorphism of \( \mathcal{O}_X \)-modules

\[ \sum_{i=1}^n b_ie_i \in \mathcal{E}_{a+\omega(k)k} \rightarrow \sum_{i=1}^n b_ie_i^k \in \mathcal{E}_{a,b}(kD) \]

is clearly \( D_X(\log D) \)-linear. Q.E.D.

The proof of the following proposition is clear.

(3.6) **Proposition.** The morphism

\[ g_{\mathcal{E}_{a,b},k,k'} : D_X \otimes_{D_X(\log D)} \mathcal{E}_{a,b}(1-k')D \rightarrow D_X \otimes_{D_X(\log D)} \mathcal{E}_{a,b}(kD), \]
defined in (4), corresponds, through the isomorphisms in corollary (3.4) and lemma (3.5), to the morphism
\[ \varphi_{E,a,b,c,k,k'} : \mathcal{P} \in \mathcal{D}_{X/I_{\omega(h)^{1-k'}}} \mapsto \mathcal{P}h^{k-k'-1} \in \mathcal{D}_{X/I_{\omega(h)^{k}}} \]

For the dual connection \( E^*_{a,b,c} \), in order to simplify, let us concentrate on case \( n = 2 \), where the integrability condition (6) reduces to:
\[ (\delta_1 - c)(b_1) = 2\delta_2(a), \quad (\delta_1 - 2c)(b_0) = \delta_2^2(a) + b_1\delta_2(a). \]  
(7)

(3.7) Lemma. With the above notations, the ILC \( E^*_{a,b,c} \) and \( E_{c-a,b,c}^* \), with \( b = (b_1, b_0) \) and \( b' = (-b_1, b_0 - \delta_2(b_1)) \), are isomorphic.

Proof. The action of \( \text{Der}(\log D) \) over the dual basis \( \{ e_1^*, e_2^* \} \) in \( E^*_{a,b,c} \) is given by:
\[ (\delta_1 \cdot e_j^*)(e_k) = \delta_1(e_j^*(e_k)) - e_j^*(\delta_1 e_k) = -e_j^*(\delta_1 e_k), \]
for \( i = 1, 2 \) and \( j, k = 1, 2 \) (see (2)). Then
\[ \delta_1 \left( \begin{array}{c} e_1^* \\ e_2^* \end{array} \right) = -A_1^t \left( \begin{array}{c} e_1^* \\ e_2^* \end{array} \right), \quad \delta_2 \left( \begin{array}{c} e_1^* \\ e_2^* \end{array} \right) = -A_2^t \left( \begin{array}{c} e_1^* \\ e_2^* \end{array} \right). \]

Choosing the new basis \( \{ w_1 = e_2^*, w_2 = -e_1^* + b_1e_2^* \} \) of \( E^*_{a,b,c} \), we obtain
\[ \delta_1 \left( \begin{array}{c} w_1 \\ w_2 \end{array} \right) = \cdots = \left( \begin{array}{cc} a - c & 0 \\ \delta_2(a) & a \end{array} \right) \left( \begin{array}{c} w_1 \\ w_2 \end{array} \right), \]
\[ \delta_2 \left( \begin{array}{c} w_1 \\ w_2 \end{array} \right) = \cdots = \left( \begin{array}{cc} 0 & 1 \\ \delta_2(b_1) - b_0 & b_1 \end{array} \right) \left( \begin{array}{c} w_1 \\ w_2 \end{array} \right) \]
and the isomorphism of \( \mathcal{O}_X \)-modules
\[ \sum_{i=1}^{2} b_i w_i \in E^*_{a,b,c} \mapsto \sum_{i=1}^{2} b_i e_i \in E_{c-a,b,c}^* \]

is clearly \( \mathcal{D}_X(\log D) \)-linear. Q.E.D.

4 Some explicit examples

In this section we consider the case where \( D \subset X = \mathbb{C}^2 \) is defined by the reduced equation \( h = x_1^2 - x_2^3 \), and then \( \omega(x_1) = 3, \omega(x_2) = 2, \omega(h) = 6 \) and the basis of \( \text{Der}(\log D) \) is \( \{ \delta_1, \delta_2 \} \), with
\[ \left( \begin{array}{c} \delta_1 \\ \delta_2 \end{array} \right) = \left( \begin{array}{cc} 3x_1 & 2x_2 \\ 3x_2^2 & 2x_1 \end{array} \right) \left( \begin{array}{c} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{array} \right), \]
- \( \delta_1(h) = 6h, \quad \delta_2(h) = 0, \)
- the determinant of the coefficient matrix is equal to \( 6h, \)
- \( [\delta_1, \delta_2] = \delta_2 \quad (c = 1). \)
(4.1) Since the ILC $\mathcal{E}_{a,b}$ and the ideals $I_{a,b}$ in corollary [3.4] are defined globally by differential operators with polynomial coefficients and $D$ has a global polynomial equation, the study of morphism

$$\rho_{\mathcal{E}_{a,b},k} : \mathcal{D}_X^L \otimes_{\mathcal{D}_X(\log D)} \mathcal{E}_{a,b}(kD) \to \mathcal{E}_{a,b}(*D)$$

can be done globally at the level of the Weyl algebra $\mathcal{W}_2 = \mathbb{C}[x_1, x_2, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}]$.

The integrability conditions in $\mathcal{I}$ (for $n = 2$) become in our case

$$(\delta_1 - 1)(b_1) = 2\delta_2(a), \quad (\delta_1 - 2)(b_0) = \delta_2^2(a) + b_1\delta_2(a). \quad (8)$$

Once $a$ is fixed, it allows us to determine, uniquely, $b_1$ (the operator $\delta_1 - 1$ is injective), and to also determine $b_0$ up to a term $ex_2$, $e \in \mathbb{C}$ (the kernel of the operator $\delta_1 - 2$ is generated by $x_2$). In order to simplify, let us take

$$a = \lambda + mx_1 + nx_2,$$

where $\underline{\mu} = (\lambda, m, n)$ are complex parameters, and then

$$b_1 = 2mx_2^2 + 2nx_1$$

and

$$b_0 = ex_2 + 3nx_2^2 + 4mx_1x_2 + n^2x_1^2 + 2mnx_1x_2 + m^2x_2^2,$$

with $e$ another complex parameter. For convenience (see the rational factorization of $B(s)$ below), let us consider another complex parameter $\nu$ and make $e = \nu - \nu^2$.

Let us define the family of ILC of rank two, $\mathcal{F}_{\nu,\underline{\mu}} := \mathcal{E}_{a,b}$ (see [3.1]), with $a, b_0, b_1$ as above. We have $\mathcal{F}_{\nu,\underline{\mu}} = \mathcal{D}_X(\log D) \cdot e_1$ and $\text{ann}_{\mathcal{D}_X(\log D)} e_1 = \mathcal{D}_X(\log D)(g_1, g_2)$, with $g_1 = \delta_1 + a$ and $g_2 = \delta_2^2 + b_1\delta_2 + b_0$ (see lemma [3.2]). It is clear that $\mathcal{F}_{\nu,\underline{\mu}} = \mathcal{F}_{1-\nu,\underline{\mu}}$.

The conclusion of corollary [2.5] can be globalized and we obtain

$$\text{ann}_{\mathcal{D}_X[s]}(e_1h^* = \mathcal{D}_X[s](\Phi(g_1), \Phi(g_2)) = \mathcal{D}_X[s](\delta_1 + a - 6s, g_2)$$

and

$$\text{ann}_{\mathcal{W}_2[s]}(e_1h^*) = \mathcal{W}_2[s](\delta_1 + a - 6s, g_2).$$

Let us consider the Weyl algebra with parameters

$$\mathcal{W}' = \mathbb{C} \left[\lambda, m, n, \nu, x_1, x_2, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, s\right]$$

and the left ideal $I$ generated by

$$h, \quad \delta_1 + a - 6s, \quad \delta_2^2 + b_1\delta_2 + b_0.$$ 

By a Gröbner basis computation with an elimination order, for example, with the help of [11], we compute the generator $B(s)$ of the ideal $I \cap \mathbb{C}[s]$ and operators $P(s), C(s), D(s) \in \mathcal{W}'$ such that

$$B(s) = P(s)h + C(s)(\delta_1 + a - 6s) + D(s)(\delta_2^2 + b_1\delta_2 + b_0).$$
We find
\[
B(s) = \left(s - \frac{\lambda - 5}{6}\right) \left(s - \frac{\lambda - 8}{6}\right) \left(s - \frac{\lambda - \nu - 6}{6}\right) \left(s - \frac{\lambda + \nu - 7}{6}\right).
\]

For \(\lambda, \nu \in \mathbb{C}\), let us call \(B_{\lambda,\nu}(s) \in \mathbb{C}[s]\) the polynomial obtained from \(B(s)\) in the obvious way. We obtain then for each \(\nu, \lambda, m, n \in \mathbb{C}\) the global Bernstein-Sato functional equation
\[
B_{\lambda,\nu}(s) e_1 h^s = P(s) \left(e_1 h^{s+1}\right)
\]
in \(\mathcal{F}_{\nu,\mu}[h^{-1}, s]h^s\). Therefore, \(b_{\mathcal{F}_{\nu,\mu}}(s) | B_{\lambda,\nu}(s)\) (see prop. (2.1)) for any \(p \in D^1\) and
\[
\kappa(\mathcal{F}_{\nu,\mu}) \geq \tau(\lambda, \nu) := \min\{\text{integer roots of } B_{\lambda,\nu}(s)\} \in \mathbb{Z} \cup \{+\infty\}.
\]

We can apply theorem (1.5) to deduce that morphism
\[\rho_{\mathcal{F}_{\nu,\mu}} : \mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} \mathcal{F}_{\nu,\mu}(kD) \to \mathcal{F}_{\nu,\mu}(\ast D)\]
is an isomorphism for all \(k \geq -\tau(\lambda, \nu)\). On the other hand, from lemma (3.7) we know that \((\mathcal{F}_{\nu,\lambda,m,n})^* = \mathcal{F}_{\nu,1-\lambda,-m,-n}\) and then morphism
\[\rho_{\mathcal{F}_{\nu,\mu}}' : \mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} \mathcal{F}_{\nu,\mu}^*(k'D) \to \mathcal{F}_{\nu,\mu}^*(\ast D)\]
is an isomorphism for all \(k' \geq -\tau(1-\lambda, \nu)\).

The above results can be rephrased in the following way:

1) Morphism
\[\rho_{\mathcal{F}_{\nu,\mu}} : \mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} \mathcal{F}_{\nu,\mu}(kD) \to \mathcal{F}_{\nu,\mu}(\ast D)\]
is an isomorphism if the four following conditions hold:
- \(\lambda + 6k \neq -1, -7, -13, -19, \ldots\)
- \(\lambda + 6k \neq -2, -4, -10, -16, \ldots\)
- \(\lambda + 6k - \nu \neq 0, -6, -12, -18, \ldots\)
- \(\lambda + 6k + \nu \neq -1, -15, -11, -17, \ldots\)

2) Morphism
\[\rho_{\mathcal{F}_{\nu,\mu}}' : \mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} \mathcal{F}_{\nu,\mu}^*(k'D) \to \mathcal{F}_{\nu,\mu}^*(\ast D)\]
is an isomorphism if the four following conditions hold:
- \(1 - \lambda + 6k' \neq -1, -7, -13, -19, \ldots\)
- \(1 - \lambda + 6k' \neq -2, -4, -10, -16, \ldots\)
- \(1 - \lambda + 6k' - \nu \neq 0, -6, -12, -18, \ldots\)
- \(1 - \lambda + 6k' + \nu \neq -1, -5, -11, -17, \ldots\)

or equivalently, if the four following conditions hold:
- \(\lambda - 6k' \neq 2, 8, 14, 20, \ldots\)
- \(\lambda - 6k' \neq -1, 5, 11, 17, \ldots\)
- \(\lambda + \nu - 6k' \neq 1, 7, 13, 19, \ldots\)
- \(\lambda - \nu - 6k' \neq 1, -5, -11, -17, \ldots\)

In particular, if the four following conditions:

\[\text{In fact it is possible to show that } b_{\mathcal{F}_{\nu,\mu,0}}(s) = B_{\lambda,\nu}(s).\]
(i) $\lambda \not\equiv 2 \pmod{6}$ or $\lambda = 2$

(ii) $\lambda \not\equiv 5 \pmod{6}$ or $\lambda = -1$

(iii) $\lambda + \nu \not\equiv 1 \pmod{6}$ or $\lambda + \nu = 1$

(iv) $\lambda - \nu \not\equiv 0 \pmod{6}$ or $\lambda - \nu = 0$

hold, both morphisms

$\rho \iota_{\nu,\mu}^{-1} : \mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} \mathcal{F}_{\nu,\mu}(D) \to \mathcal{F}_{\nu,\mu}^*(\ast D),$  

$\rho \iota_{\nu,\mu}^{-1} : \mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} \mathcal{F}_{\nu,\mu}^*(D) \to \mathcal{F}_{\nu,\mu}^*(\ast D)$

are isomorphisms.

Let us denote by $\mathcal{L}_{\nu,\mu}$ the local system over $X - D$ of the horizontal sections of $\mathcal{F}_{\nu,\mu}$. By theorem (1.6), we have

$\text{IC}_X(\mathcal{L}_{\nu,\mu}) \simeq \text{DR}(\text{Im} \, \varrho_{\nu,\mu}, 1),$  

provided that conditions (i)-(iv) are satisfied.

Proposition (3.6) and (4.1) reduce the computation of $\text{Im} \, \varrho_{\nu,\mu}, 1$ to the computation of the image of the map

$\theta_{\nu,\mu} : \mathcal{I} \in \mathbb{W}_2/\mathbb{W}_2(g_1, g_2) \mapsto \mathcal{I} h \in \mathbb{W}_2/\mathbb{W}_2(g_1 + 6, g_2),$

but $\text{Im} \, \theta_{\nu,\mu} = \mathbb{W}_2/K_{\nu,\mu}$ where

$K_{\nu,\mu} = \{ R \in \mathbb{W}_2 \mid Rh \in \mathbb{W}_2(g_1 + 6, g_2) \}.$

Now, in order to compute generators of $K_{\nu,\mu}$, we proceed as follows. Since $[g_1, g_2] = 2g_2$ (for any $\nu, \mu$) and the symbols $\sigma(g_1) = \sigma(\delta_1)$, $\sigma(g_2) = \sigma(\delta_2)^2$ form a regular sequence ($D$ is Koszul free!), we deduce that

$\sigma \left( \mathbb{W}_2(g_1 + 6, g_2) \right) = (\sigma(\delta_1), \sigma(\delta_2)^2)$

and consequently $\sigma \left( K_{\nu,\mu} \right) \subset (\sigma(\delta_1), \sigma(\delta_2)^2) : h$. A straightforward (commutative) computation shows that

$\left( \sigma(\delta_1), \sigma(\delta_2)^2 \right) : h = (\sigma(\delta_1), \sigma(Q_0))$

with $Q_0 = 9x_2 \frac{\partial^2}{\partial x_1^2} - 4 \frac{\partial^2}{\partial x_2^2}$, and

$\sigma(Q_0)h = x_2 \sigma(\delta_1)^2 - \sigma(\delta_2)^2 = x_2 \sigma(\delta_1)\sigma(g_1 + 6) - \sigma(g_2).$  \hfill (10)

Searching to lift the relation (10) to $\mathbb{W}_2$, we find

$Qh = x_2(\delta_1 + mx_1 + nx_2 + 7 - \lambda)(g_1 + 6) - g_2 + (\lambda^2 - \lambda + \nu - \nu^2)x_2,$

with $Q = Q_0 + 6mx_2 \frac{\partial}{\partial x_1} - 4nx_2 \frac{\partial}{\partial x_2} + m^2x_2 - n^2$. In particular, if condition

$\lambda^2 - \lambda + \nu - \nu^2 = 0 \quad (\iff \lambda - \nu = 0 \; \text{or} \; \lambda + \nu = 1) \hfill (11)$
holds, then $Q \in K_{\nu,\mu}$.

Actually, by using the equality $[Q, g_1] = 4Q$ and the fact that $\sigma(Q) = \sigma(Q_0)$ and $\sigma(g_1) = \sigma(\delta_1)$ also form a regular sequence in $\text{Gr } \mathbb{W}_2$, condition (11) implies that

$$K_{\nu,\mu} = \mathbb{W}_2(g_1, Q), \quad \sigma(K_{\nu,\mu}) = (\sigma(\delta_1), \sigma(Q_0)).$$

On the other hand, since $\sigma(Q_0)$ is not contained in the ideal $(x_1, x_2)$, we finally deduce the following result:

If parameters $\nu, \mu = (\lambda, m, n)$ satisfy conditions (i)-(iv) and (11), then the conormal of the origin $T_0^*(X)$ does not appear as an irreducible component of the characteristic variety of $\text{Im } \theta_{\nu,\mu} = \mathbb{W}_2/K_{\nu,\mu}$, and consequently

$$\text{Ch}(\text{IC}_X(\mathcal{L}_{\nu,\mu})) = \text{Ch} \left( \mathbb{W}_2/K_{\nu,\mu} \right) = \{ \sigma(\delta_1) = \sigma(Q_0) = 0 \} = T_X^* (X) \cup T_0^*(X).$$

The existence of such an example has been suggested by [21], example (3.4), but the question on the values of the parameters $\nu, \mu$ for which the local system $\mathcal{L}_{\nu,\mu}$ is irreducible will be treated elsewhere.

If condition (11) does not hold, it is not clear that there exists a general expression for a system of generators of $K_{\nu,\mu}$ as before.

(4.2) REMARK. The relationship between the preceding results and examples and the hypergeometric local systems (cf. [23, 24, 29]) is interesting and possibly deserves further work.

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