Sequential projective measurements

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We present a general theory for sequential projective measurement. Two bounds are provided for estimating the disturbance accumulation when the measurements are sequentially performed on a quantum state. As an application of the theory, we also show that the Holevo bound can be asymptotically achieved via sequential projective measurements.

I. INTRODUCTION

VG first proves that Holevo bound can be achieved by sequential measurements in [1][2]. Sen presents a probability theory for sequential projective measurement in [3]. Wilde generalizes Sen’s theory to sequential POVM measurement[4]. In this paper, we give a general theory for sequential projective which is stronger than Sen’s result. With our result, Wilde’s theory can also be improved.

This paper is organized as follows: In section2, we present two bounds for sequential projective measurement and give full proofs. In section3, we show that the Holevo bound can be achieved asymptotically via a sequence of projective measurements.

II. SEQUENTIAL PROJECTIVE MEASUREMENTS

In this section we present the main results of our work. We start with interpreting what is sequential measurement. Suppose we have a quantum state \( \rho \). The sequential measurement is a procedure like this: First we perform a measurement \( M_1 \) on \( \rho \) and get the result state \( \rho_1 \). Next we perform \( M_2 \) on \( \rho_1 \) and get the result state \( \rho_2 \). Next perform \( M_3 \) on \( \rho_2 \) and get the result \( \rho_3 \). And so carry on, each measurement is performed on the result state of previous one. After m times of measurement, we get \( \rho_m \). Certainly the finally state \( \rho_m \) may have various forms because each measurement may give different results. In many applications, we are only interested in some particular results. Suppose that \( M_i = \{ P_i, I - P_i \} \) for \( i = 1, \ldots, m \), where \( P_i \) are projectors. Now we are only interested in the case that all the measurements give the results corresponding to \( P_i \) rather than \( I - P_i \). In other word, the result states sequence is:

\[
\begin{align*}
\rho_1 &= \frac{P_1 \rho P_1}{tr(P_1 \rho)} \\
\rho_2 &= \frac{P_2 P_1 \rho P_2 P_1}{tr(P_2 P_1 \rho P_2 P_1)} \\
& \vdots \\
\rho_m &= \frac{P_m \ldots P_2 P_1 \rho P_1 P_2 \ldots P_m}{tr(P_m \ldots P_2 P_1 \rho P_1 P_2 \ldots P_m)}
\end{align*}
\]

Now we want to estimate the probability that \( \rho_m \) occurs and the distance between the initial state \( \rho \) and final state \( \rho_m \). To solve these problems, we give the following theorem:

Theorem1 Given a density operator \( \rho \) and projectors \( P_1, P_2, \ldots, P_m \) such that

\[
tr(P_i \rho) = 1 - \varepsilon_i \quad i = 1, 2, \ldots, m
\]

Let \( \rho_m = \frac{P_m \ldots P_2 P_1 \rho P_2 \ldots P_m}{tr(P_m \ldots P_2 P_1 \rho P_2 \ldots P_m)} \). If \( \sum \varepsilon_i < \frac{1}{2} \), then we have the following bounds:

(1-a) The trace distance between \( \rho \) and \( \rho_m \) obeys:

\[
D(\rho, \rho_m) \leq 2 \sqrt{\sum \varepsilon_i}
\]

where \( D(\rho, \rho_m) = tr(\sqrt{\rho - \rho_m})(\rho - \rho_m)^{\frac{1}{2}} \).

(1-b) The probability that \( \rho_m \) occurs obeys:

\[
tr(P_m \ldots P_2 P_1 \rho P_2 \ldots P_m) \geq \left( \frac{1 - \sum \varepsilon_i}{1 + \sum \varepsilon_i} \right)^2
\]

This bound can also be reduced to a weaker but simpler form,

\[
\left( \frac{1 - \sum \varepsilon_i}{1 + \sum \varepsilon_i} \right)^2 = \left( 1 - \frac{2 \sum \varepsilon_i}{1 + \sum \varepsilon_i} \right)^2 \geq 1 - 4 \sum \varepsilon_i
\]

Theorem1 gives the bounds for sequential projective measurements. In many occasions, \( \varepsilon_i \) is small. So the condition \( tr(P_i \rho) = 1 - \varepsilon_i \) implies that each measurement scarcely disturbs \( \rho \) if it is directly performed on \( \rho \). Theorem1 reveals how the disturbance increases when the measurements are sequentially performed. In the following, we present the proof of Theorem1. We first prove it holds for the case that \( \rho \) is pure state and then extend the proof to mixed state.

Suppose that \( \rho \) is a pure state \( | \psi \rangle \langle \psi | \) and the state \( \rho_m \) is \( | \psi_m \rangle \langle \psi_m | \). \( | \psi_m \rangle \) is generated from a sequence of measurements:

\[
| \psi_1 \rangle = \frac{P_1 | \psi \rangle}{\sqrt{\langle \psi | P_1 | \psi \rangle}} \\
| \psi_2 \rangle = \frac{P_2 | \psi \rangle}{\sqrt{\langle \psi | P_2 | \psi_1 \rangle}}
\]
Consider the $i$’th measurement, $|\psi_i\rangle = \frac{P_i|\psi_{i-1}\rangle}{\sqrt{\langle \psi_{i-1} | P_i | \psi_{i-1}\rangle}}$.

Let $|\psi_i^\perp\rangle = \frac{(I-P_i)|\psi_{i-1}\rangle}{\sqrt{\langle \psi_{i-1} | (I-P_i) | \psi_{i-1}\rangle}}$, we can write $|\psi_{i-1}\rangle$ in terms of $|\psi_i\rangle$ and $|\psi_i^\perp\rangle$ as follows:

$$|\psi_{i-1}\rangle = \cos\theta_i|\psi_i\rangle + \sin\theta_i|\psi_i^\perp\rangle$$

where $\theta_i = \arccos\left(\langle \psi_i | \psi_{i-1}\rangle\right)$. $\theta_i$ can be regarded as the angle between $|\psi_{i-1}\rangle$ and $|\psi_i\rangle$. The advantage of this representation lies that the trace distance and probability can be expressed in a simple form:

$$D(|\psi_{i-1}\rangle, |\psi_i\rangle) = 2\sin\theta_i$$

Likewise, if we perform the measurement $\{P_i, I-P_i\}$ on $\rho$ individually, then the result state will be $|\psi_i^\perp\rangle = \frac{(I-P_i)|\psi_i\rangle}{\sqrt{\langle \psi_i | (I-P_i) | \psi_i\rangle}}$.

The third equality uses the fact that $P_i(I-P_i) = 0$. From here, it is easy to get

$$\sin^2\beta_k = \cos^2\alpha_k \sin^2\gamma_k + \sin^2\alpha_k$$

Therefore, to prove the Lemma, we only need to prove the following inequality holds

$$\cos^2\alpha_k \sin^2\gamma_k \leq \sin^2\beta_{k-1}$$

By (13), we have

$$\cos\beta_{k-1} = \left|\langle \psi_{k-1} | \psi_i \rangle\right|$$

$$= \left|\cos\theta_k \cos\alpha_k \langle \psi_k | \psi_k'\rangle + \sin\theta_k \sin\alpha_k \langle \psi_k^\perp | \psi_k^\perp\rangle\right|$$

$$\leq \cos\theta_k \cos\alpha_k \langle \psi_k | \psi_k'\rangle + \sin\theta_k \sin\alpha_k$$

$$= \cos\theta_k \cos\alpha_k \cos\gamma_k + \sin\theta_k \sin\alpha_k$$

The inequality follows from the fact that $|a+b| \leq |a| + |b|$ and $\left|\langle \psi_k^\perp | \psi_k^\perp\rangle\right| \leq 1$.

Since $\sum \varepsilon_i \leq \frac{1}{2}$, by (10) it holds that $\sin^2\alpha_k \leq \varepsilon_k \leq \frac{1}{2}$. Combined with $\sin^2\beta_{k-1} \leq \frac{1}{2}$ and (13), we can get that

$$\cos\theta_k \cos\alpha_k \cos\gamma_k \geq \cos\beta_{k-1} - \sin\theta_k \sin\alpha_k$$

Then we have

$$(\cos\theta_k \cos\alpha_k \cos\gamma_k)^2 \geq (\cos\beta_{k-1} - \sin\theta_k \sin\alpha_k)^2$$
This inequality can be reduced to
\[ \cos^2 \alpha_k \sin^2 \gamma_k \leq \frac{1}{\cos^2 \theta_k} \left( -\sin^2 \theta_k + 2 \sin \theta_k \sin \alpha_k \cos \beta_{k-1} + \sin^2 \beta_{k-1} - \sin^2 \alpha_k \right) \]

Let \( x = \sin \theta_k \), then the right side can be written as a function of \( x \),
\[ f(x) = \frac{-x^2 + 2x \sin \alpha_k \cos \beta_{k-1} + \sin^2 \beta_{k-1} - \sin^2 \alpha_k}{1 - x^2} \]

By \( f'(x) = 0 \), we can obtain that \( f(x) \) achieves the maximum value \( \sin^2 \beta_{k-1} \) when \( x = \frac{\sin \alpha_k}{\cos \beta_{k-1}} \). Thus, the inequality \( \cos^2 \alpha_k \sin^2 \gamma_k \leq \sin^2 \beta_{k-1} \) holds, then Lemma1 is proven.

Lemma1 gives us a recurrence relation between \( |\psi_{k-1}\rangle \) and \( |\psi_k\rangle \). Applying Lemma1, we immediately obtain that: if \( D^2 \left( |\psi\rangle, |\psi_{k-1}\rangle \right) \leq 4 \sum_{i=1}^{k-1} \varepsilon_i \), then
\[ D^2 \left( |\psi\rangle, |\psi_k\rangle \right) \leq D^2 \left( |\psi\rangle, |\psi_{k-1}\rangle \right) + D^2 \left( |\psi\rangle, |\psi'_k\rangle \right) \]
\[ \leq 4 \sum_{i=1}^{k-1} \varepsilon_i + 4 \varepsilon_k = \sum_{i=1}^{k} \varepsilon_i \]
(18)

By induction, the bound (1-a) is proven for pure state.

Next we show that (1-b) holds for pure state. The probability that \( |\psi_m\rangle \) occurs is
\[ tr \left( P_m \cdots P_1 |\psi\rangle \langle \psi | P_1 \cdots P_m \right) = tr \left( P_1 |\psi\rangle \langle \psi | P_2 |\psi_1\rangle \cdots tr \left( P_m |\psi_{m-1}\rangle \langle \psi_{m-1} | P_m \right) = \cos^2 \theta_1 \cos^2 \theta_2 \cdots \cos^2 \theta_m \]
(19)

From (14), we know that
\[ \cos \beta_{m-1} \leq \cos \theta_m \cos \alpha_m + \sin \theta_m \sin \alpha_m = \cos (\theta_m - \alpha_m) \]
(20)

So it holds that \( \theta_m \leq \beta_{m-1} + \alpha_m \). Then we have
\[ \cos \theta_1 \cdots \cos \theta_m \geq \cos \theta_1 \cdots \cos \theta_{m-1} \cos (\beta_{m-1} + \alpha_m) \]

To continue, we need the following lemma:

**Lemma2:** Define \( \{ a_k \} \) by
\[ a_k = \cos \alpha_m \cos \beta_m - \sqrt{\sum_{i=k+1}^{m} \sin^2 \alpha_i} \cdot \sqrt{\sum_{i=k+1}^{m-1} \sin^2 \beta_i + \sum_{i=k+1}^{m-1} \sin^2 \alpha_i} \]
\[ 1 + \sum_{i=k+1}^{m-1} \sin^2 \alpha_i \]

Then we have
\[ \cos \theta_k \cdot a_k \geq a_{k-1} \]
(21)

The proof of Lemma2 is in Appendix A.

Note that \( \cos (\beta_{m-1} + \alpha_m) = a_{m-1} \), then by Lemma2, we have
\[ \cos \theta_1 \cos \theta_2 \cdots \cos \theta_{m-1} \cos (\beta_{m-1} + \alpha_m) \]
\[ = \cos \theta_1 \cos \theta_2 \cdots \cos \theta_{m-1} \cdot a_{m-1} \]
\[ \geq \cos \theta_1 \cos \theta_2 \cdots \cos \theta_{m-2} \cdot a_{m-2} \]
\[ \vdots \]
\[ \geq a_0 \]
\[ \cos \alpha_m = \sqrt{\sum_{i=1}^{m} \sin^2 \alpha_i} \cdot \sqrt{\sum_{i=1}^{m-1} \sin^2 \alpha_i} \]
\[ = \frac{1}{1 + \sum_{i=1}^{m-1} \sin^2 \alpha_i} \geq \frac{1 - \sum_{i=1}^{m-1} \varepsilon_i}{1 + \sum_{i=1}^{m-1} \varepsilon_i} \]
(22)

The second equality follows from the fact that \( \beta_0 = 0 \). The third inequality is proved in Appendix B. Combined with (14), we get
\[ tr (P_m \cdots P_2 P_1 |\psi\rangle \langle \psi | P_1 P_2 \cdots P_m) \geq \left( 1 - \sum_{i=1}^{m} \varepsilon_i \right)^2 \]
(23)

Now let us consider the case that \( \rho \) is mixed state. Suppose \( |\psi\rangle^{RA} \) and \( |\psi_m\rangle^{RA} \) are purifications of \( \rho \) and \( \rho_m \), where \( R \) denotes the reference system. Then the state \( |\psi_m\rangle^{RA} \) can be generated via the following projective measurements:
\[ Q_1 = I^R \otimes P_1, \quad |\psi_1\rangle^{RA} = \frac{Q_1 |\psi\rangle^{RA}}{\langle \psi | Q_1 |\psi\rangle^{RA}} \]
\[ Q_2 = I^R \otimes P_2, \quad |\psi_2\rangle^{RA} = \frac{Q_2 |\psi\rangle^{RA}}{\langle \psi | Q_2 |\psi\rangle^{RA}} \]
\[ \vdots \]
\[ Q_m = I^R \otimes P_m, \quad |\psi_m\rangle^{RA} = \frac{Q_m |\psi_{m-1}\rangle^{RA}}{\langle \psi_{m-1} | Q_m |\psi_{m-1}\rangle^{RA}} \]

Moreover, we have
\[ tr (Q_i |\psi\rangle \langle \psi |^{RA}) = tr (P_i \rho) \geq 1 - \varepsilon_i \]
(24)

Then we can apply the result for pure state and get that
\[ D \left( |\psi\rangle^{RA}, |\psi_m\rangle^{RA} \right) \leq 2 \sqrt{\sum \varepsilon_i} \]
(25)

By the monotonicity of trace distance, we obtain the bound on \( D (\rho, \rho_m) \),
\[ D (\rho, \rho_m) \leq D \left( |\psi\rangle^{RA}, |\psi_m\rangle^{RA} \right) \leq 2 \sqrt{\sum \varepsilon_i} \]
(26)
For the probability, we have

\[
tr(P_m \cdots P_1 \rho P_1 \cdots P_m) \geq tr\left(\frac{1}{1 - \sum \varepsilon_i} \right)^2
\]

The proof of Theorem 1 is complete.

III. ACHIEVING HOLOVE BOUND VIA SEQUENTIAL PROJECTIVE MEASUREMENTS

The Holevo bound sets a limit on the rates that can be achieved when transferring classical messages in a quantum channel. As an application of Theorem 1, we will show that the Holevo bound can be achieved via sequential projective measurement. A similar work has been done by VG [1, 2]. In this paper, we analyze a more simple and more natural decoding scheme.

Suppose \( j \) is a set of possible inputs to the quantum channel and let \( \sigma_j \) be the corresponding outputs. Let \( p_j \) be a probability distribution over the indices \( j \) and \( \sigma = \sum p_j \sigma_j \). Alice wants to send a message chosen from the set \( \{1, ..., 2^nR\} \) to Bob. The transmission of messages can be decomposed into three stages: the encoding, the transmission and the decoding. In the encoding stage, the standard random coding scheme is adopted. To the message \( i \), Alice associates a codeword \( \tilde{c}_i = c_1 c_2 \cdots c_n \), where \( c_1, c_2, ..., c_n \) are chosen from the index set \( \{j\} \) according to the distribution \( p_j \). She repeats this procedure for \( 2^nR \) times, creating a codebook \( C \) of \( 2^nR \) entries. The corresponding output states are simply denoted with \( \sigma_{\tilde{c}_i} \).

When Bob receives a particular state \( \sigma_{\tilde{c}_i} \), he tries to determine what the message was. To do this, he has two tools: the projector \( P \) onto the \( \delta \)-typical subspace of \( \sigma^{\otimes n} \) and the projectors \( \{P_{\tilde{c}_i}\} \) onto the \( \delta \)-typical subspace of corresponding \( \sigma_{\tilde{c}_i} \). They have the following properties [6]:

For any \( \varepsilon > 0 \) and for sufficiently large \( n \),

\[
tr(P \sigma_{\tilde{c}_i}) \geq 1 - \varepsilon
\]

(27)

\[
tr(P \sigma_{\tilde{c}_i} \sigma_{\tilde{c}_i}) \geq 1 - \varepsilon
\]

(28)

\[
tr(P \sigma_{\tilde{c}_i}) \leq 2^n(\sum p_j S(\sigma_j) + \delta)
\]

(29)

\[
P \sigma^{\otimes n} P \leq (1 - \varepsilon)^{-1} 2^{-n[S(\sigma) - \delta]} P
\]

(30)

Then Bob can perform sequential projective measurements to find the message: First, he performs the measurement \( \{P, I - P\} \) to determine whether or not the received state is in the typical subspace of \( \sigma^{\otimes n} \). If the answer is No, the process stops and he declares an error. If Yes, he performs the measurement \( \{P_{\tilde{c}_1}, I - P_{\tilde{c}_1}\} \) to test whether or not the received state is \( \sigma_{\tilde{c}_1} \). If No, he can declare that the message is \( \tilde{c}_1 \). If No, he performs the measurement \( \{P_{\tilde{c}_2}, I - P_{\tilde{c}_2}\} \) to test whether or not the received state is \( \sigma_{\tilde{c}_2} \). If Yes, then the message is \( \tilde{c}_2 \).

If No, he performs the measurement \( \{P_{\tilde{c}_2}, I - P_{\tilde{c}_2}\} \). The procedure goes on, until for some \( i \), he get a Yes, then he can say the message is \( \tilde{c}_i \).

Now we will calculate the error probability of this decoding procedure. Since every measurement may introduce small disturbance, the received state will gradually deviate from the original one and the error probability become larger. The last codeword \( c_m (m = 2^nR) \) is apparently in the worst location and its error probability is the largest over all codewords. So we only consider the error probability of the last codeword \( c_m \).

Suppose the received state is \( \sigma_{\tilde{c}_m} \). After the measurement \( \{P, I - P\} \), if the answer is Yes, then the state becomes \( \sigma'_{\tilde{c}_m} \),

\[
\sigma'_{\tilde{c}_m} = \frac{P \sigma_{\tilde{c}_m} P}{tr(P \sigma_{\tilde{c}_m})}
\]

(31)

The probability of getting \( \sigma'_{\tilde{c}_m} \) is

\[
p' = tr(P \sigma_{\tilde{c}_m}) \geq 1 - \varepsilon
\]

Next the measurements \( \{P_{\tilde{c}_1}, P_{\tilde{c}_2}, ..., P_{\tilde{c}_m}\} \), \( \{P_{\tilde{c}_1}, \overline{P_{\tilde{c}_2}}, ..., \overline{P_{\tilde{c}_m}}\} \) are sequentially performed on \( \sigma'_{\tilde{c}_m} \), where \( \overline{P_{\tilde{c}_i}} = I - P_{\tilde{c}_i} \). Then the probability that we get correct result is

\[
p'' = tr(P_{\tilde{c}_1} \overline{P_{\tilde{c}_2}} \cdots \overline{P_{\tilde{c}_m}}) \geq tr(P_{\tilde{c}_1} \overline{P_{\tilde{c}_2}} P_{\tilde{c}_3} \cdots P_{\tilde{c}_m})
\]

(32)

To use Theorem (1-b), we need to calculate \( tr(P_{\tilde{c}_1} \sigma'_{\tilde{c}_m}) \) and \( tr(P_{\tilde{c}_1} \sigma''_{\tilde{c}_m}) \). For the former, we have

\[
tr(P_{\tilde{c}_1} \sigma'_{\tilde{c}_m}) = tr\left[ P_{\tilde{c}_1} \frac{P \sigma_{\tilde{c}_m} P}{tr(P \sigma_{\tilde{c}_m})} \right] \geq tr(P_{\tilde{c}_1} P \sigma_{\tilde{c}_m} PP_{\tilde{c}_m}) \geq 1 - 8\varepsilon
\]

(33)

The first inequality holds because \( tr(P \sigma_{\tilde{c}_m}) \leq 1 \). The second inequality follows from (27) and Theorem (1-b).

Note that \( tr(\overline{P_{\tilde{c}_i}} \sigma_{\tilde{c}_m} P) = 1 - tr(P_{\tilde{c}_i} \sigma_{\tilde{c}_m} P) \), we consider the expectation of \( tr(P_{\tilde{c}_i} \sigma_{\tilde{c}_m} P) \) over all random codes \( C \):

\[
E_C \left( tr(P_{\tilde{c}_i} \sigma'_{\tilde{c}_m}) \right) = E_C \left( tr\left[ P_{\tilde{c}_i} \frac{P \sigma_{\tilde{c}_m} P}{tr(P \sigma_{\tilde{c}_m})} \right] \right) \leq \frac{1}{1 - \varepsilon} E_C \left( tr(P_{\tilde{c}_i} P \sigma_{\tilde{c}_m} PP_{\tilde{c}_m}) \right) = \frac{1}{1 - \varepsilon} tr\left( E_C \left( P_{\tilde{c}_i} \right) P P \sigma_{\tilde{c}_m} \right)
\]

(34)

The last equality uses the fact that \( E_C (\sigma_{\tilde{c}_m}) = \sigma^{\otimes n} \). Continuing,

\[
tr\left( E_C \left( P_{\tilde{c}_i} \right) P \sigma^{\otimes n} P \right) \leq \frac{2^{-n[S(\sigma) - \delta]}}{1 - \varepsilon} tr\left( E_C \left( P_{\tilde{c}_i} \right) P \right)
\]

Continuing,
holds that

It then follows that there exists at least one code satisfies

\[ \min_{\chi} \geq \cos \theta - 1 \]

follows from (29). Thus, we have

\[ \frac{1}{1 - \varepsilon} 2^n (R - \chi + 2\delta + \frac{1}{n}) \]

\( m \leq \frac{1}{1 - \varepsilon} 2^n (R - \chi + 2\delta) \) (35)

where \( \chi = S(\sigma) - \sum p_j S(\sigma_j) \) be the Holevo quality. The first inequality follows from (33). The second inequality follows from the fact that \( P < I \). The third inequality follows from (28). Thus, we have

\[ \mathbb{E}_{\chi} \left\{ \text{tr} (P_{\chi} \sigma'_{\chi}) \right\} \leq \frac{2^n (R - \chi + 2\delta)}{(1 - \varepsilon)^2} \] (36)

Without generality, we can suppose \((1 - \varepsilon)^2 > \frac{1}{2}\), then it holds that

\[ \mathbb{E}_{\chi} \left\{ \sum_{i=1}^{m-1} \text{tr} (P_{\chi} \sigma'_{\chi}) \right\} \leq (2^n R - 1) \frac{2^n (R - \chi + 2\delta)}{(1 - \varepsilon)^2} < 2^n (R - \chi + 2\delta + \frac{1}{n}) \]

It then follows that there exists at least one code satisfies

\[ \sum_{i=1}^{m-1} \text{tr} (P_{\chi} \sigma'_{\chi}) < 2^n (R - \chi + 2\delta + \frac{1}{n}) \] (37)

Denote this code by \( C^* \). Then for \( C^* \), we can apply theorem(1-b) and get that:

\[ p'' > 1 - 4 \left[ 8\varepsilon + 2^n (R - \chi + 2\delta + \frac{1}{n}) \right] \] (38)

\[ \cos \alpha_m \cos \beta_m \cos \theta_k \cos \alpha_k = \sum_{i=k+1}^{m} \sin^2 \alpha_i \cdot \frac{\cos^2 \theta_k - \cos^2 \beta_k \cos^2 \theta_k + \cos^2 \theta_k \sum_{i=k+1}^{m} \sin^2 \alpha_i}{1 + \sum_{i=k+1}^{m-1} \sin^2 \alpha_i} \]

\[ \cos \alpha_m (\cos \beta_{k-1} - \sin \theta_k \sin \alpha_k) - \sum_{i=k+1}^{m} \sin^2 \alpha_i \cdot \frac{\cos^2 \theta_k - (\cos \beta_{k-1} - \sin \theta_k \sin \alpha_k)^2 + \cos^2 \theta_k \sum_{i=k+1}^{m-1} \sin^2 \alpha_i}{1 + \sum_{i=k+1}^{m-1} \sin^2 \alpha_i} \]

\[ \cos \alpha_m \cos \beta_{k-1} - 2 \cos \alpha_m \sin \alpha_k - \sum_{i=k+1}^{m} \sin^2 \alpha_i \cdot \frac{\left( 1 + \sum_{i=k}^{m-1} \sin^2 \alpha_i \right) x^2 + 2x \cos \beta_{k-1} \sin \alpha_k + \sum_{i=k+1}^{m-1} \sin^2 \alpha_i + \sin^2 \beta_{k-1}}{1 + \sum_{i=k+1}^{m-1} \sin^2 \alpha_i} \] (A2)

Denote the right side by \( g(x) \). Then by \( g'(x) = 0 \), we can obtain the minimum value of \( g(x) \). It can be verified that \( g_{\text{min}}(x) = a_{k-1} \). Therefore, the inequality \( \cos \theta_k \cdot a_k \geq a_{k-1} \) holds.

The whole probability that we get the correct result is

\[ p_c = p'' > 1 - 33\varepsilon - 2^n (R - \chi + 2\delta + \frac{1}{n}) \]

The error probability \( p_e = 1 - p_c \), so

\[ p_e < 33\varepsilon + 2^n (R - \chi + 2\delta + \frac{1}{n}) \]

\( \varepsilon \) and \( \delta \) can be arbitrary small, so for any \( R > \chi \), \( p_e \to 0 \) when \( n \to \infty \). This actually complete our proof.

IV. CONCLUSION

In this paper, we focus on the sequential projective measurement and give a general theorem. This theorem shows that how the disturbance accumulates when a sequence of projective measurement are performed on a quantum state. As an application, we prove that the Holevo bound can be asymptotically achieved via a sequence of projective measurement.

Appendix A: Proof of Lemma2

From (13), (17), we get

\[ \cos \beta_k \cos \theta_k \geq \cos \beta_{k-1} - \sin \theta_k \sin \alpha_k \geq 0 \] (A1)

Let \( x = \sin \theta_k \), we have

\[ \cos \alpha_m \cos \beta_{k-1} - x \cos \alpha_m \sin \alpha_k = \sum_{i=k+1}^{m} \sin^2 \alpha_i \cdot \frac{\left( 1 + \sum_{i=k}^{m-1} \sin^2 \alpha_i \right) x^2 + 2x \cos \beta_{k-1} \sin \alpha_k + \sum_{i=k+1}^{m-1} \sin^2 \alpha_i + \sin^2 \beta_{k-1}}{1 + \sum_{i=k+1}^{m-1} \sin^2 \alpha_i} \] (A2)

Appendix B

Let \( W \) be defined by

\[ W = \cos \alpha_m - \sum_{i=1}^{m} \sin^2 \alpha_i \sum_{i=1}^{m-1} \sin^2 \alpha_i \left( 1 - \sum_{i=1}^{m} \sin^2 \alpha_i \right) \]
Then if $W \geq 0$, the inequality holds. So our target is to prove $W \geq 0$. It can be verified that

$$W = \frac{\sin^2 \alpha_m}{1 + \sqrt{1 - \frac{\sin^2 \alpha_m}{\sum \sin^2 \alpha_i}} - \frac{\sin^2 \alpha_m}{1 + \sqrt{1 - \sin^2 \alpha_m}}} \quad (\text{B1})$$

Since $\sum \sin^2 \alpha_i \leq \sum \varepsilon_i \leq \frac{1}{2}$, then from (B1) we have $W \geq 0$.

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