Canonical Formulation of a New Action for a Non-relativistic Particle Coupled to Gravity

Rabin Banerjee $^{a,b}$, Pradip Mukherjee $^{c,d}$

$^a$ S. N. Bose National Centre for Basic Sciences, JD Block, Sector III, Salt Lake City, Kolkata - 700 098, India

$^b$ rabin@bose.res.in

$^d$ mukhpradip@gmail.com

Department of Physics, Barasat Government College, Barasat, India

Abstract

A detailed canonical treatment of a new action for a nonrelativistic particle coupled to background gravity, recently given by us, is performed both in the Lagrangian and Hamiltonian formulations. The equation of motion is shown to satisfy the geodesic equation in the Newton-Cartan background, thereby clearing certain confusions. The Hamiltonian analysis is done in the gauge independent as well as gauge fixed approaches, following Dirac’s analysis of constraints. The physical (canonical) variables are identified and the path to canonical quantisation is outlined by explicitly deriving the Schroedinger equation.

1 Introduction

Newton Cartan (NC) space time is a four dimensional differentiable manifold with two degenerate metrics. Just after Einstein formulated the general theory of relativity (GR) as a relativistic theory of gravity, Cartan demonstrated that Newtonian gravity can also be formulated as a geometric theory in NC manifold [1, 2]. Intense research on various aspects of the metric theory produced a rich literature [3, 4, 5, 6, 7]. The metric properties of the NC space time being so different from the Riemann or Riemann Cartan space time that great care had to be taken to derive Newtonian gravity from General relativity. In case of the Riemann space there is a unique non singular metric but in NC geometry there are two degenerate metrics. A direct outcome is the difficulties in coupling of matter theories with non relativistic gravity. In the ’classical’ age this issue was not very prominent.

Resurgence of this field in recent times, is due to the applications of the geometric approach to physical phenomena in varied topics including condensed matter physics, hydrodynamics and cosmology. So the question of coupling matter systems with non relativistic gravity occupied the centre stage. A number of different approaches to the problem have appeared in the recent past [8, 17], the most popular among these is based on the gauging of (extended) Galilean group algebra [10, 13]. However, there are several puzzling results reported in the literature. Some illustrative examples are discussed in the present paper taking two representative models, namely, non relativistic particle and Schroedinger field theory coupled to background gravity. There is a common thread in the problems that surface in both cases. This is the ad hoc introduction of a $U(1)$ gauge field. Surprisingly, the free Schroedinger theory could not be discussed, just as it was essential to include a gauge field in
the Newton-Cartan structure. This last feature is especially relevant for constructing an action for a nonrelativistic particle coupled to gravity that is the focus of this paper.

In this paper we develop a canonical formalism for a nonrelativistic particle coupled to background NC geometry. It is based on the action recently derived by us, following a systematic procedure of gauging the relevant Galilean symmetries, generically termed Galilean gauge theory (GGT) in analogy with Poincare gauge theory that is obtained by gauging the Poincare symmetries. A closer examination of the usual approaches indicates the necessity of a systematic algorithm that is provided by this approach and the action found by us was free of the various inconsistencies usually encountered. We derive the Lagrangian equation of motion from the reparametrisation invariant action and show that it yields the geodesic equation in the Newton-Cartan background, though not in the affine form. Then an affine parameter is identified and the affine geodesic is obtained. The obtention of the geodesic equation is nontrivial since earlier approaches had failed [10]. This was true both for particles without spin (the present case) [10] or with spin [14]. There is another approach that, however, assumes, rather than derives, the geodesic equation. [6].

The Lagrangian description is backed by a Hamiltonian analysis. There is a single first class constraint that is shown to generate the reparametrisation symmetry. The gauge independent analysis is supplemented by a gauge fixed formulation where the reparametrisation parameter is identified with the absolute time. We show that it is a good choice of gauge since it allows the abstraction of the physical (canonical) variables in a simple manner. The Dirac brackets of the canonical set are identical to the Poisson brackets which therefore allows us to elevate them to commutators, when quantising the theory. The operator version of the relevant variables are given in the coordinate representation. This allows us to write the Schroedinger equation, thereby paving the way for the quantisation of the model. In all cases the appropriate flat limit is reproduced.

In section 2 we take a critical look at the usual formulation, pointing out the inadequacies and inconsistencies. Section 3 reviews our construction of the action. Sections 4 and 5, respectively, deal with the Lagrangian and Hamiltonian formulations. Our concluding remarks are given in section 6.

2 Closer Look at Coupling with Newton Cartan Space Time

As already discussed in the introduction Newtonian gravity was given a geometric formulation almost at the same time as Einsteins general theory of relativity. Recently new vistas of applications of this came up in strongly correlated electrons motion in lower dimensions, such as the fractional quantum Hall effect (FQHE). Naturally, the emphasis was laid on the coupling of matter fields with gravity. In absence of a single nondegenerate metric, the minimal coupling prescription is not obvious. In this situation different methods were suggested. However, as we show here, a close inspection reveals several pitfalls and discrepancies. These are primarily related to non-relativistic diffeomorphism invariance and the introduction of a gauge field in the Newton Cartan structure. We consider both these issues.

2.1 Non Relativistic Spatial Diffeomorphism

The idea of coupling the Schroedinger field with nonrelativistic (Newtonian) gravity, given in [3], is based on the action,

$$S = \int dtdx\sqrt{g} \left[ \frac{i}{2} \left( \psi^\dagger \partial_t \psi - \psi^\dagger \partial_t \psi \right) - A_0 \psi^\dagger \psi - \frac{g^{ij}}{2m} \left( \partial_i \psi^\dagger - iA_i \psi^\dagger \right) \left( \partial_j \psi + iA_j \psi \right) \right].$$

(1)
Requiring that it has both U(1) gauge symmetry,
\[ \psi \rightarrow e^{i\alpha} \psi, \; A_\mu \rightarrow A_\mu + \partial_\mu \alpha \]  
(2)

and spatial diffeomorphism,
\[ x^k \rightarrow x^k + \xi^k \]  
(3)

it was observed, by trial and error, that the symmetry of the action is ensured by the following
infinitesimal transformations,
\[ \delta \psi = i\alpha \psi - \xi^k \partial_k \psi, \]  
(4)
\[ \delta A_0 = -\dot{\alpha} - \xi^k \partial A_0 - A_k \dot{\xi}^k, \]  
(5)
\[ \delta A_i = -\partial_i \alpha - \xi^k \partial_k A_i - A_k \partial_i \xi^k + mg_{ik} \dot{\xi}^k, \]  
(6)
\[ \delta g_{ij} = -\xi^k \partial_k g_{ij} - g_{ik} \partial_j \xi^k - g_{kj} \partial_i \xi^k. \]  
(7)

The transformation of \( A_\mu \) is anomalous. Clearly the transformation laws are not canonical due to the
last terms of (5) and (6).

More problems surface when we take the flat limit, \( g_{ij} \rightarrow \delta_{ij} \) with corresponding changes to the
transformation parameters which now reduce to constant Galilean parameters. While the action (1)
passes over to its expected flat version, the same is not true for the transformations. Specifically the
result for \( \delta A_i \) does not agree for the boosts, once again due to the presence of the last term in the
relevant equation. To save the situation, one assumes the following rules,
\[ \alpha = mv^i x^i, \; \xi^i = v^i t. \]  
(8)

Apart from the abnormality of the equality of gauge parameters and boost parameters, more severe
puzzles persist. For instance, one independent symmetry has been lost in the process without any
reason or justification. On top of it, such an identification destroys the result for the transformation
of the \( \psi \) and \( A_0 \) fields.

A pertinent question can now be asked. The condition (8) states that Galilean covariance can only
be revived if the interaction is present. Obviously, this identification cannot be done for a free theory
where there is no gauge symmetry. In that case coupling of a free Schroedinger theory to gravity
becomes untenable in this approach.

### 2.2 Introduction of \( U(1) \) Gauge Field in Newton Cartan Geometry

That there are problems in the conventional approach mentioned above were also noticed by other
workers, notably in [9]. Indeed the paper correctly pointed out that ‘regrettably, this “general covari-
ance” suffers from the fact that it is not entirely covariant and the transformation laws of all of the
tensors can be formulated in a coordinate-independent way, with the exception of the transformation
of the gauge field \( A_\mu \).’ As we have shown, any attempt to rectify the situation by using an ansatz like
(8) would create other problems.

As we already mentioned a severe handicap is the lack of a viable method to couple a free
Schroedinger theory to gravity. In an attempt to justify this drawback a section of researchers (includ-
ing [9]) proposed to include a \( U(1) \) gauge field in the Newton Cartan geometric structure, thereby
implying that the $U(1)$ gauge field is an absolutely essential element in the coupling of nonrelativistic theories to a curved background. However, as is well known from the original work on Newton Cartan, no gauge field is necessary, it being purely defined from the two degenerate metrics. This gauge field is an extra baggage that is bound to lead to more problems. Indeed, a really tightrope walking was needed [9] to save the structure but the issue of connection could barely be managed, the arguments being rather contrived.

It should however be mentioned that the idea of including a $U(1)$ gauge field in the Newton Cartan structure was borrowed from earlier works. A particularly relevant example in the context of the present paper is its application in the study of coupling non relativistic point particle with NC geometry [10]. The approach was based on localising the transformation parameters of the Galilean group, obeying the peculiarities of the non relativistic idea of space and time. New gauge fields were introduced and their transformations were taken to be canonical. The Newton Cartan metrics were defined through this field. This has been readily adopted by many others. But problems remain as it were, as the following example will show.

### 2.3 Nonrelativistic Particle in Newton Cartan Background

The lagrangian of the nonrelativistic point particle coupled to gravity can be put in a form that is frequently used in the literature [10, 11, 12],

$$L = \frac{m}{2\tilde{\tau}_\mu x^\mu} \tilde{h}_{\mu\nu} x'^\mu x'^\nu - m\phi x'^\rho$$  \hspace{1cm} (9)

where,

$$\phi = -\frac{1}{\Theta^2} \left(A_0 - \frac{1}{2} h^{ij} A_i A_j\right)$$  \hspace{1cm} (10)

and $\tilde{h}_{\mu\nu}$ and $\tilde{\tau}_\mu$ are elements of the Newton Cartan geometry parametrised as [10],

$$\tilde{\tau}_\mu = [\Theta, 0, 0, 0] ; \tilde{h}_{\mu\nu} = \left(\begin{array}{cc} h^{rs} A_r A_s & A_j \\ A_i & h_{ij} \end{array}\right)$$  \hspace{1cm} (11)

The two other elements of the Newton Cartan geometry are given by [10],

$$\tilde{\tau}^\mu = \frac{1}{\Theta}[1; -h^{ij} A_j] ; \tilde{h}^{\mu\nu} = \left(\begin{array}{cc} 0 & 0 \\ 0 & h^{ij} \end{array}\right)$$  \hspace{1cm} (12)

One may verify that they satisfy the Newton Cartan algebra,

$$\tilde{h}^{\mu\nu} \tilde{\tau}_\nu = \tilde{h}_{\mu\nu} \tilde{\tau}^\nu = 0, \tilde{\tau}^{\mu} \tilde{\tau}_\mu = 1, \tilde{h}^{\mu\nu} \tilde{h}_{\nu\rho} = \delta^\mu_\rho - \tilde{\tau}^{\mu} \tilde{\tau}_\rho = P^\mu_\rho$$  \hspace{1cm} (13)

Two other useful relations that are valid in this parametrisation are,

$$\tilde{h}_{0i} = -\Theta \tilde{h}_{ij} \tilde{\tau}^j ; \tilde{h}_{00} = \Theta^2 \tilde{h}_{ij} \tilde{\tau}^i \tilde{\tau}^j$$  \hspace{1cm} (14)

The Lagrangian posited here is invariant under the local galilean transformations. Here, contrary to the usual global Galilean transformations,

$$x^\mu \rightarrow x'^\mu + \zeta^\mu ; \zeta^0 = -\epsilon, \zeta^k = \eta^k - v^k t; \eta^k = \omega^k_j x'^j + \epsilon^k$$  \hspace{1cm} (15)
where $\omega^k L$, the 3-dim rotation parameters, $-\epsilon, \epsilon^k$, parameters for time and space translations and $v^k$, the Galilean boosts, are taken to be space time dependent instead of constants. Recalling the universal role of time in nonrelativistic (NR) theory, time translation is a function of time only, while the others are space time dependent. The quasi invariance of the Lagrangian is achieved provided, along with the local Galilean transformations, the newly introduced fields are assumed to transform as,

\[
\begin{align*}
\delta \Theta &= \dot{\epsilon} \Theta \\
\delta h_{ij} &= -h_{kj} \partial_i \xi^k - h_{ki} \partial_j \xi^k \\
\delta A_0 &= 2\dot{\epsilon} A_0 - A_i \partial_i \xi^i - \Theta \partial_i (h_{ij} v^j x^i) \\
\delta A_i &= \dot{\epsilon} A_i - A_j \partial_i \xi^j - h_{ij} \partial_j \xi^i - \Theta \partial_i (h_{kj} v^k x^j)
\end{align*}
\]  

(16)

Identification of (9) as the Lagrangian for a NR particle coupled to gravity is, however, associated with many serious problems, like,

1. The equation of motion following from the above Lagrangian does not satisfy the geodesic equation. For more details including the difference from the geodesy we refer to eq. (4.2) of [10].

2. While the first term in (9) involves the appropriate Newton Cartan coupling, the second does not. Moreover the problems are further compounded by the fact that the Newton Cartan structure $\tilde{h}_{\mu \nu}$ appearing there does not even transform as a second rank tensor so that it cannot be regarded as an appropriate Newton Cartan metric, despite satisfying the algebra \[13\]. Hence the meaning and interpretation of the first term also is unclear. This is explicitly shown in \[18\].

3. There is a difficulty in taking the flat limit. In this case the new fields $(A)$ vanish, the field $\Theta$ goes to unity and the spatial metric $h_{ij}$ goes to the Kronecker delta. Then the Lagrangian in the form \[9\] reproduces the familiar parametrised invariant NR particle Lagrangian. But while this is essential for a proper flat limit, it is not sufficient. The point is that there is an anomaly in the transformation law for $A_0$ \[16\]. While the left side vanishes, the right side does not. There is a non-zero contribution from the last term containing the boosts.

In the following we will discuss how GGT may be applied to obtain a totally comprehensive analysis of all the problems.

3 Action for Non relativistic Particle in Curved Background

The parametrized action for a non relativistic particle in 3 dimensional Euclidean space and absolute time is given by,

\[
S = \int \frac{1}{2} m \frac{d x^\alpha}{d \lambda} \frac{d x^\alpha}{d \lambda} d \lambda
\]

(17)

The action is invariant under the global Galilean transformations \[15\]. This invariance is ensured by the transformations
$$\delta \frac{dx^0}{d\lambda} = \frac{d}{d\lambda}(\delta x^0) = -\frac{de}{d\lambda} = 0$$  \hspace{1cm} (18)$$
as $e$ is constant and,

$$\delta \frac{dx^k}{d\lambda} = w^j_k \frac{dx^j}{d\lambda} - v^k \frac{dx^0}{d\lambda}$$  \hspace{1cm} (19)$$
hold, which can be checked easily, The Lagrangian (17) changes by,

$$\delta L = -\frac{d}{d\lambda} \biggl( mv^k \frac{dx^k}{d\lambda} \biggr)$$  \hspace{1cm} (20)$$
due to (18) and (19). The change of the action (17) is then a boundary terms only, see (20). The same equations of motion follow from both the original and the transformed action. So the theory is invariant under the global Galilean transformations.

To localize the symmetry of the action (17) according to GGT, $\frac{dx^a}{d\lambda}$ is now substituted by the covariant derivatives $Dx^a$, where

$$\frac{Dx^a}{d\lambda} = \frac{dx^\nu}{d\lambda} \Lambda^a_{\nu} \partial^\nu x^a = \frac{dx^\nu}{d\lambda} \Lambda^a_{\nu}$$  \hspace{1cm} (21)$$
Here $\Lambda^a_{\nu}$ are a set of new compensating (gauge) fields, the transformations of which will ensure that the ‘covariant derivatives’ will transform in the form as the usual derivatives do under the global Galilean transformations. Then the new theory obtained by replacing the ordinary derivatives by the ‘covariant derivatives’ will be invariant under the local gauge transformations. This is the essence of GGT. The covariant derivatives indeed transform as expected [18],

$$\delta \frac{Dx^a}{d\lambda} = \frac{dx^\nu}{d\lambda} \Lambda^a_{\nu} \delta x^a = \frac{dx^\nu}{d\lambda} \delta \Lambda^a_{\nu}$$

Hence the cherished action is given by,

$$S = \int \frac{1}{2} m \frac{Dx^a}{d\lambda} \frac{Dx^a}{d\lambda} d\lambda$$  \hspace{1cm} (22)$$
obtained from (17), substituting $\frac{dx^\mu}{d\lambda}$ by $\frac{Dx^a}{d\lambda}$. This action has been explicitly demonstrated to be invariant under the local transformations [18].

### 3.1 Geometric Connection

The modified theory [22] has a geometrical interpretation. Following the tenets of galilean gauge theory, the new fields $\Lambda_{\mu}^a$ may be reinterpreted as inverse vielbeins in a general manifold charted by the coordinates $x^\mu$ connecting the spacetime coordinates with the local coordinates that represent the local Galilean symmetry. It has been proved that the 4-dim space time obtained in this way above is the Newton-Cartan manifold. This is done by showing that the metric formulation of our theory contains the same structures and satisfy the same structural relations as in NC space - time [19].
Indeed the various elements of the NC geometry introduced in section 2 are given in terms of the GGT variables by,

\[ h_{\mu\nu} = \Sigma_{\alpha}^{\mu} \Sigma_{\alpha}^{\nu}; \quad \tau_{\mu} = \Lambda_{\mu}^{\nu} \delta_{\nu}^{0} \]  

and,

\[ h_{\nu\rho} = \Lambda_{\nu}^{\alpha} \Lambda_{\rho}^{\beta}; \quad \tau_{\mu} = \Sigma^{\mu}_{\nu} \]  

where \( \Sigma_{\alpha}^{\nu} \) is the inverse of \( \Lambda_{\mu}^{\alpha} \),

\[ \Sigma_{\alpha}^{\nu} \Lambda_{\nu}^{\beta} = \delta_{\alpha}^{\beta}; \quad \Sigma_{\alpha}^{\nu} \Lambda_{\mu}^{\alpha} = \delta_{\nu}^{\mu} \]  

(25)

It is now easy to express (22) in a manifestly covariant form using the Newton Cartan elements. From (21) and (24) we obtain,

\[ \frac{Dx^\alpha}{d\lambda} \frac{Dx^\alpha}{d\lambda} = h_{\nu\rho} \frac{dx^\nu}{d\lambda} \frac{dx^\rho}{d\lambda} \]  

(26)

so that the action may be written as,

\[ S = \left( \frac{m}{2} \right) \int h_{\nu\rho} \frac{x'^\nu x'^\rho}{\Theta x'^0} d\lambda \]  

(27)

Clearly, this can be interpreted as the action of a non relativistic particle coupled with a Newton Cartan background.

The two shortcomings of the lagrangian (9) do not occur here. The flat limit poses no problems. In this limit we recall that the vielbeins reduce to the Kronecker deltas. Then \( \Theta = 1 \) which implies \( T = t, h_{00} = h_{0i} = 0 \) and \( h_{ij} = \delta_{ij} \). The action (27) reduces to the standard NR action for a free particle in flat space. Contrary to (9) there are no gauge field terms and we do not have to worry about the consistency of their transformations. Also, as mentioned earlier, the Newton Cartan metric \( h_{\mu\nu} \) correctly transforms as a second rank covariant tensor.

4 Lagrangian Analysis

We start from the following action of the non relativistic particle in curved background,

\[ S = \left( \frac{m}{2} \right) \int h_{\nu\rho} \frac{x'^\nu x'^\rho}{\Theta x'^0} d\lambda = \left( \frac{m}{2} \right) \int h_{\nu\rho} \frac{x'^\nu x'^\rho}{\tau_{\sigma} x'^\sigma} d\lambda \]  

(28)

where, as earlier stated, a prime denotes a differentiation with respect to the parameter \( \lambda \). The passage from the first to the second equality follows on using the explicit representation of \( \tau_{\sigma} \). This action is manifestly invariant under the finite reparametrisations,

\[ \lambda \to \lambda', \quad x'^\mu(\lambda) \to x'^\mu(\lambda') \]  

(29)

The infinitesimal version is given by,

\[ \lambda' = \lambda + \delta \lambda, \quad \delta x^\mu(\lambda) = \delta \lambda \frac{dx^\mu}{d\lambda} \]  

(30)
The Euler-Lagrange equation following from (28) is,
\[ \frac{d}{d\lambda} \left( \frac{\partial L}{\partial x'^\mu} \right) - \frac{\partial L}{\partial x^\mu} = 0 \] (31)

We will give the calculations in some detail. The derivatives of \( L \) can be straightforwardly computed. Multiplying the overall equation by \( h^{\omega \mu} \) we get,
\[ x''_{\omega} + \left( \tau .x' \right) \tau^\omega - \left( \frac{\tau .x'}{\tau .x} \right) x^\omega = - \frac{h^{\omega \alpha}}{2} \frac{\tau .x' \tau .x^\beta}{\tau .x} + \frac{h^{\omega \alpha}}{2} \frac{h^{\mu \nu} \tau .x' x^\nu (\partial_\alpha \tau_\sigma) x^\sigma}{\tau .x} + \frac{1}{2} h^{\omega \alpha} (K_{\alpha \sigma} \tau_\beta + K_{\alpha \beta} \tau_\sigma) \] (32)

where the abbreviation,
\[ \tau .x = \tau_\sigma x^\sigma \] (33)
has been used.

We can now introduce the Dautcourt connection,
\[ \Gamma^\omega_{\sigma \beta} = \frac{1}{2} \frac{h^{\omega \alpha}}{2} (\partial_\sigma \tau_\beta + \partial_\beta \tau_\sigma) + \frac{h^{\omega \alpha}}{2} (\partial_\sigma h_{\alpha \beta} + \partial_\beta h_{\alpha \sigma} - \partial_\alpha h_{\sigma \beta}) + \frac{1}{2} h^{\omega \alpha} (K_{\alpha \sigma} \tau_\beta + K_{\alpha \beta} \tau_\sigma) \] (34)

where \( K \) is an arbitrary two form. Now from (34) we can write
\[ \frac{h^{\omega \alpha}}{2} (\partial_\sigma h_{\alpha \beta} + \partial_\beta h_{\alpha \sigma} - \partial_\alpha h_{\sigma \beta}) x^\sigma x^\beta = \Gamma^\omega_{\sigma \beta} x^\sigma x^\beta = \frac{1}{2} h^{\omega \alpha} (K_{\beta \sigma} \tau_\alpha + K_{\beta \alpha} \tau_\sigma) \] (35)

Using this and the identity
\[ \tau_\alpha' - \partial_\alpha \tau_\sigma x^\sigma = (\partial_\sigma \tau_\alpha - \partial_\alpha \tau_\sigma) x^\sigma \] (36)
in (32) we get the path of a particle falling freely in background gravity,
\[ x''_{\omega} + \Gamma^\omega_{\sigma \beta} x^\sigma x^\beta = \frac{(\tau .x')}{(\tau .x)} x^\omega \] (37)

where we have identified the arbitrary two form as,
\[ K_{\sigma \alpha} = \frac{1}{2} \frac{h^{\omega \alpha}}{(\tau .x)} (\partial_\sigma \tau_\alpha - \partial_\alpha \tau_\sigma) h_{\rho \beta} x^\rho x^\beta \] (38)

Note that in (37), \( \Gamma^\omega_{\sigma \beta} \) is completely specified. Also, the path is the equation of a geodesic in Newton Cartan geometry. Thus our action (22) produces the correct geodesic equation. Further, the arbitrariness of Dautcourt formula for the symmetric connection is eliminated. This does not mean that the arbitrariness in the NC affine connection is removed. It shows that such arbitrariness does not affect the NR particle dynamics.

In order to derive the affine form of the geodesic where the right side of (37) vanishes, the affine parameter has to be identified. This is easily done. The affine properties of the Newton Cartan
geometry is determined by the direction of flow of time. The Galilean frame assumed in this work has the time axis oriented along the direction of absolute time. Substituting $\lambda = t$ in (37) we get

$$\frac{d^2 x^\mu}{dt^2} + \Gamma^\mu_{\sigma\beta} \frac{dx^\sigma}{dt} \frac{dx^\beta}{dt} = \frac{\Theta}{\Theta^2} \dot{t}^\mu$$

where an overdot denotes time differentiation. Now to fix the scale of time define the affine parameter $T$ by

$$dT = \Theta dt$$

This leads to the affine geodesic equation

$$\frac{d^2 x^\mu}{dT^2} + \Gamma^\mu_{\sigma\beta} \frac{dx^\sigma}{dT} \frac{dx^\beta}{dT} = 0$$

We have successfully constructed the action for a NR particle coupled to Newtonian gravity following the systematic procedure provided by galilean gauge theory (GGT) [20, 19, 21]. The background space time is identified with the Newton Cartan space time. A Lagrangian analysis has shown that a freely falling particle follows a geodesic in this NC spacetime. This is a remarkable result because the usual analysis of non relativistic spinning particle predicts that the path will not be a geodesic in NC space time and follow a different trajectory given by the Papapetrou equation [14]. This puzzling result has remained unexplained since, one would expect, that the path for a free particle follows a geodesic, spin or no spin.

5 Hamiltonian Formulation

We follow Dirac’s method of constrained systems [15] to develop the Hamiltonian formulation. This can be done either in a gauge independent manner or by fixing a specific gauge [16]. We do in both ways.

5.1 Gauge independent analysis

The dynamical fields in the action are $x^\mu$. The vielbeins $\Lambda(x)$ are prescribed functions of $x$. The momentum, canonically conjugate to $x^\mu$ are,

$$p_\mu = \frac{\partial L}{\partial (\dot{x}^\mu)} = \frac{m h_{\mu\nu} x^\nu}{(x^\tau)} - \frac{m h_{\mu\nu} x^\tau x^\nu \tau_\mu}{2 (x^\tau)^2}$$

From here, after a couple of manipulations, we get a primary constraint,

$$\Omega_1 = \tau^\mu p_\mu + \frac{1}{2m} h^{\rho\sigma} p_\rho p_\sigma \approx 0$$

In flat space the only nonvanishing components are given by $\tau^0 = 1, h^{ij} = \delta^{ij}$, so that the constraint simplifies to the well known energy-momentum condition,

$$E = \frac{p^2}{2m}$$
where the energy is identified as, $E = -p_0$.

Expectedly, the canonical Hamiltonian vanishes,

$$H_c = p_\mu x'^\mu - L = 0$$

which is a manifestation of the reparametrization invariance of [22] [16]. The total Hamiltonian is then given by just the constraint,

$$H_T = \lambda \left( \tau^{\mu} p_\mu + \frac{1}{2m} h^{\rho\sigma} p_\rho p_\sigma \right)$$

where $\lambda$ is a Lagrange multiplier. Since the Poisson algebra of constraints is strongly involutive,

$$\{ \Omega_1, \Omega_1 \} = 0$$

there are no further constraints. We therefore have a single first class constraint which will subsequently be shown to generate the reparametrisation symmetry.

The Lagrange multiplier $\lambda$ can be fixed from the canonical equation of motion,

$$x'^\mu = \{ x^\mu, H \}_{PB}$$

Some calculation yields

$$\lambda = \tau x'$$

The total hamiltonian is then given by,

$$H_T = (\tau x') \left( \tau^{\mu} p_\mu + \frac{1}{2m} h^{\rho\sigma} p_\rho p_\sigma \right)$$

This is exactly equal to the super Hamiltonian of Kucher [6].

Since there is only one first class constraint, the gauge generator is just given by,

$$G = \epsilon \Omega_1 = \epsilon \left( \tau^{\mu} p_\mu + \frac{1}{2m} h^{\rho\sigma} p_\rho p_\sigma \right)$$

where $\epsilon$ is the gauge parameter.

The change in $x^\mu$ is given by,

$$\delta x^\mu = \epsilon \{ x^\mu, \Omega_1 \} = \epsilon \frac{\tau x'}{\tau x'} x'^\mu$$

As already shown, the model [22] has reparametrization invariance. Comparing the above result with [30] we find that the Hamiltonian gauge symmetry parameter is mapped to the reparametrisation parameter,

$$\delta \lambda = \frac{\epsilon}{\tau x'}$$

We now calculate the Hamilton’s equations of motion. These are given by bracketing with the total hamiltonian,
\[ x^\mu = \{ x^\mu, H_T \} = (\tau x') \left( \tau^\mu + \frac{1}{m} h^{\mu\sigma} p_\sigma \right) \] (54)

\[ p'_\mu = \{ p_\mu, H_T \} = (\tau x') \left( -\partial_{\tau^\mu} p_\alpha - \frac{1}{2m} \partial_{\mu} h^{\alpha\sigma} p_\rho p_\sigma \right) \] (55)

Taking derivative of (54) with respect to \( \lambda \) and using (54) and (55) we get back the geodesic equation (41). Both Lagrangian and Hamiltonian analysis show that the geodesic equation is obeyed by the freely falling particle.

5.2 Gauge fixed analysis

So far we were working in the gauge independent formalism. But to identify the physical variables and proceed with canonical quantization, gauge fixing is necessary.

A particularly suitable choice of gauge is to identify the parameter \( \lambda \) with the universal time,

\[ \Omega_2 = x^0 - \lambda = 0 \] (56)

As we shall see the Dirac brackets in this gauge are very simple and it is easy to identify the proper canonical variables of the theory.

The constraints \( \Omega_i \) and \( \Omega_2 \) now form a second class pair. The relevant matrix formed by the Poisson brackets of the two constraints is given by,

\[ C_{ij} = \{ \Omega_i, \Omega_j \} = -\Theta^{-1} \epsilon_{ij}, \text{ } \epsilon_{12} = 1 \] (57)

and its inverse is given by,

\[ C^{-1}_{ij} = \Theta \epsilon_{ij} \] (58)

The Dirac brackets (denoted by a star) between any two variables are defined by,

\[ \{ f, g \}^* = \{ f, g \} - \{ f, \Omega_i \} C^{-1}_{ij} \{ \Omega_j, g \} \] (59)

Then the only non-vanishing Dirac brackets are given by,

\[ \{ x^\mu, p_\nu \}^* = \delta^\mu_\nu - \delta^0_\nu \left( \tau^\mu + \frac{1}{m} h^{\mu\sigma} p_\sigma \right) \] (60)

It is now possible to identify the physical variables of the system. There are eight phase space variables, two of which are eliminated by the constraints. That leaves us with six physical (phase space) degrees of freedom. We identify these with the set \( (x^i, p_j) \). Moreover their Dirac brackets are identical to the Poisson brackets,

\[ \{ x^i, p_j \}^* = \delta^i_j \] (61)

so that these may be regarded as a canonical pair. The variable \( x^0 \) is just the time parameter and, expectedly, has vanishing brackets with all variables. The other variable \( p_0 \) is eliminated in favour of the canonical set by using (43). Realising that this constraint is now strongly implemented, we can solve for \( p_0 \) to get,
\[ p_0 = -\Theta \left( \tau^i p_i + \frac{1}{2m} h^{ij} p_ip_j \right) \]  

(62)

Finally, we have to identify the hamiltonian because the earlier expression (50), based solely on the constraint, is now strongly zero. The new hamiltonian is given by,

\[ H = -p_0 \]  

(63)

To prove this fact we reproduce the equations of motion by taking the relevant Dirac brackets with the canonical variables. For example,

\[ \dot{x}^i = \{x^i, H\}^* = \Theta \left( \tau^i + \frac{1}{m} h^{ij} p_j \right) \]  

(64)

where (61) is used. This matches exactly with \( x'^i \) (57) which has been obtained by gauge independent analysis, as \( \tau.x' = \Theta \), since now we can put \( \lambda = x^0 = t \), which is the gauge fixing constraint implemented strongly. Similarly, we can prove \( \dot{p}^i \) matches with \( p'^i \) in the gauge independent analysis. So \( H \) generates the equations of motion of the physical variables.

5.2.1 Canonical Quantization and Schroedinger Equation

Canonical quantization is now possible. We elevate the Dirac algebra (61) to commutators, replacing the \( x^i, p_j \) by operators. Then,

\[ [\hat{x}^i, \hat{p}_j] = i \delta^i_j \]  

(65)

In the coordinate representation, therefore,

\[ \dot{x}^i = x^i, \quad \dot{p}_i = -i \frac{\partial}{\partial x^i} \]  

(66)

Using (66) along with (62) and (63), we can write down the following equation,

\[ i \frac{\partial \psi}{\partial t} = \Theta \left( \tau^i (-i \frac{\partial}{\partial x^i}) + \frac{1}{2m} h^{ij} (-\partial_i \partial_j) \right) \psi \]  

(67)

This is the Schroedinger equation for a nonrelativistic particle in a Newton Cartan background. As a consistency check we study its flat limit. In this limit \( \Theta = 1, \quad \tau^i = 0, \quad h^{ij} = \delta^{ij} \) and the above equation reduces to,

\[ i \frac{\partial \psi}{\partial t} = -\frac{1}{2m} \nabla^2 \psi \]  

(68)

Similarly, from (63) we find that \( H \) goes to its flat limit \( \frac{p^2}{2m} \). These agreements are really wonderful.
6 Conclusions

The coupling of nonrelativistic matter to gravity is quite nontrivial when compared to the relativistic case. A prime reason is the lack of a single nondegenerate metric. While such a metric occurs naturally for relativistic theories, the nonrelativistic theory is saddled with a pair of degenerate metrics [3, 4, 5, 6, 7]. Thus it becomes necessary to adopt different techniques than those used conventionally.

Among the various approaches, a large part is devoted to gauging the Galilean (or its centrally extended) algebra [17]. While this does reproduce the elements of the Newton Cartan geometry, it cannot illuminate the dynamics. Thus one takes recourse to a trial and error approach to write the corresponding action [8, 9]. But, as already discussed in section 2, it runs into inconsistencies.

For the specific problem of writing an action for a nonrelativistic free particle coupled to gravity, two methods were used in the literature. By a combination of an algebraic approach and ad hoc assumptions of introducing a $U(1)$ gauge field in the Newton Cartan structure, an action was written [10, 9]. But, as shown in section 2.3, this is untenable. It lacks algebraic consistency, does not yield the proper flat limit and, finally, its equation of motion was not a geodesic [10]. This last point brings us to the other approach [6]. Here the geodesic equation was assumed, from which an action was guessed. Incidentally, there are mistaken claims in the literature that the actions following from these two approaches are same [14]. There is only a structural resemblance but otherwise they are very different.

Recently we have given a new form for the action of a nonrelativistic particle [18] which was based on a systematic algorithm developed by us over the last few years to couple nonrelativistic theories to gravity [20, 19, 21]. This action had a proper flat limit. No assumptions were used and the coupling to Newton Cartan geometry was simply an outcome of the method.

In this paper we have made a detailed canonical analysis of that action. Both lagrangian and hamiltonian formulations were discussed. We have shown that the Euler-Lagrange equation of motion is just the geodesic equation in the Newton Cartan background. This is a nontrivial check on the validity of our action, particularly since the issue of geodesy has been a recurring theme. Even for the nonrelativistic spinning particle model, the equation of motion did not turn out to the geodesic equation [14]. However, no explanation was provided for this bizarre result, since one expects that the equation of motion for a free particle should be the geodesic, spin or no spin. We also like to mention that, contrary to [6], we have not assumed the geodesic equation. Rather, we have derived it from our action.

The hamiltonian analysis of our model is based on Dirac’s theory of constraints [15]. This has been done using both gauge independent and gauge fixed versions. We find the appearance of a single first class constraint. The canonical hamiltonian vanishes, revealing the reparametrisation invariance possessed by the action. This invariance is shown to be generated by the first class constraint. The time evolution of the system is given by the total hamiltonian, which is proportional to the first class constraint. Fixing this arbitrariness appropriately, we are able to reproduce the Euler Lagrange equations of motion. Thus the geodesic equation is also obtained in the hamiltonian formulation.

In the gauge fixed approach, we choose a gauge where the reparametrisation parameter is taken to be the absolute time. The reparametrisation freedom is thus removed and the original first class constraint gets converted to second class. The Dirac brackets were computed. From these brackets the physical (canonical) variables were abstracted. The new hamiltonian was identified from which the original equations of motion were reproduced using the Dirac brackets. Thus the consistency of
the gauge independent and gauge fixed formulations was established. Since the canonical pair had been abstracted it was possible to quantise the theory, say in the Schrödinger representation. The Schrödinger equation was explicitly written. All these results reproduced the expected flat limit. For instance, the Schrödinger equation found here smoothly goes over to the normal Schrödinger equation for a free nonrelativistic particle in flat spacetime.

As future possibilities, an immediate application would be to extend the analysis for a spinning particle. Apart from illuminating the confusion surrounding the geodesic equation in this case, such a model can be the starting point for more involved theories like the superparticle or even superstrings in a Newton Cartan background.

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