Morita equivalence of $w^*$-rigged modules

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Abstract

The $w^*$-rigged modules over dual operator algebras were introduced by Blecher and Kashyap as a generalization of $W^*$-modules. In this paper, we introduce two new types of Morita equivalence between right $w^*$-rigged modules over unital dual operator algebras and we examine whether these notions imply stable isomorphism between the corresponding modules. Furthermore, we investigate them in detail for the class of right $w^*$-rigged modules over nest algebras, a class which was characterized by G.K. Eleftherakis.

Keywords Operator algebras · Nest algebras · TRO · $w^*$-rigged modules · Stable isomorphism · Morita equivalence

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1 Introduction

The notion of a Hilbert $C^*$-module was developed in the early 1970’s by Rieffel and Paschke [15, 16]. These modules are useful tools in the Morita equivalence of $C^*$-algebras. The dual version is the notion of a $W^*$-module, that is a Hilbert $C^*$-module over a von Neumann algebra, which satisfies an analogue of the Riesz representation theorem for Hilbert spaces. Recently in [5], Blecher and Kashyap generalized the notion of $W^*$-module to the setting of dual operator algebras. The modules introduced there are called $w^*$-rigged modules. This work was continued by Blecher and Kraus in [4]. Unlike the $W^*$-module situation, $w^*$-rigged modules do not

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necessarily give rise to a w*-Morita equivalence of dual operator algebras in the sense of [2].

In this paper, we introduce two new types of Morita equivalence between right w*-rigged modules over unital dual operator algebras. We call them Morita equivalence and strongly projectively Morita equivalence. In particular, the notion of strongly projectively Morita equivalence which we introduce uses an interesting subclass of the w*-rigged modules, the strongly projectively w*-rigged modules which in turn define a subclass of the projectively w*-rigged modules, see [4, 5]. The notion of strongly projectively Morita equivalence also implies stable isomorphism between the corresponding w*-rigged modules in the sense of [14] as well as between the dual operator algebras on which these modules act, whereas the notion of Morita equivalence does not necessarily imply stable isomorphism between either the corresponding modules or between the algebras.

Furthermore, we study a special class of right w*-rigged modules, the right w*-rigged modules over nest algebras and we prove relevant theorems for Morita equivalence between them. The rest of this paper is organized as follows:

In Sect. 2, we recall some known definitions which are useful for the next sections. Also, we introduce the notion of a concrete strongly projectively w*-rigged module as well as the notion of a strongly projectively w*-rigged module. We provide a characterization for the class of concrete strongly projectively w*-rigged modules in the case the module is over a reflexive operator algebra, see Theorem 2.17.

Both new types of Morita equivalence between right w*-rigged modules over unital dual operator algebras are presented in Sect. 3. More specifically, if $A$ and $B$ are unital dual operator algebras and $E, F$ are right w*-rigged modules over $A$ and $B$ respectively, then we call $E$ and $F$ Morita equivalent if there exists a right w*-rigged module $Y$ over $B$ such that

(i) $A \cong Y \otimes_B^b \tilde{Y}$ as dual operator $A$–$A$-bimodules.

(ii) $B \cong \tilde{Y} \otimes_A^{\lhd} Y$ as dual operator $B$–$B$-bimodules.

(iii) $F \cong E \otimes_A^{\lhd} Y$ as right w*-rigged modules over $B$ and $E \cong F \otimes_B^b \tilde{Y}$ as right w*-rigged modules over $A$.

Here, $\tilde{Y}$ is the counterpart bimodule of $Y$ and $\otimes_A^{\lhd}$ (resp. $\otimes_B^b$) is the balanced normal Haagerup tensor product over $A$ (resp. $B$). For further details, see [2].

From the conditions (i) and (ii) above, we deduce that the dual operator algebras $A$ and $B$ are w*-Morita equivalent in the sense of [2].

We provide a counterexample to show that Morita equivalent right w*-rigged modules are not necessarily stably isomorphic.

The other type of Morita equivalence is strongly projectively Morita equivalence. If $E$ is a right w*-rigged module over a unital dual operator algebra $A$ and $F$ is a right w*-rigged module over a unital dual operator algebra $B$ then we call $E$ and $F$ strongly projectively Morita equivalent if there exists a strongly projectively right w*-rigged module $Y$ over $B$ such that

(i) $A \cong Y \otimes_B^b \tilde{Y}$ as dual operator $A$–$A$-bimodules.
(ii) $B \cong \bar{Y} \otimes_{A}^{\sigma_{B}} Y$ as dual operator $B$-$B$-bimodules.

(iii) $F \cong E \otimes_{A}^{\sigma_{B}} Y$ as right $w^{*}$-rigged modules over $B$ and $E \cong F \otimes_{B}^{\sigma_{B}} \bar{Y}$ as right $w^{*}$-rigged modules over $A$.

We will also prove that if $E$ and $F$ are strongly projectively Morita equivalent then $A, B$ are stably isomorphic (see Remark 3.6) and $E, F$ are also stably isomorphic (see Theorem 3.7).

In Sect. 4, we focus on right $w^{*}$-rigged modules over nest algebras. In [11], Eleftherakis characterized these modules. In particular, he proved the following theorem.

**Theorem 1.1** [11] Let $Y$ be a right dual operator module over a nest algebra $\text{Alg}(\mathcal{N}) \subseteq \mathcal{B}(H)$. Then $Y$ is a right $w^{*}$-rigged module over $\text{Alg}(\mathcal{N})$ if and only if there exist a Hilbert space $K$, a $w^{*}$-continuous and completely isometric right $\text{Alg}(\mathcal{N})$-module map $\Psi : Y \to \mathcal{B}(H, K)$, a nest $\mathcal{M}$ and a continuous onto nest homomorphism map $\phi : \mathcal{N} \to \mathcal{M}$ such that $\Psi(Y)$ is an $\text{Alg}(\mathcal{M})$-$\text{Alg}(\mathcal{N})$-bimodule and

$$\Psi(Y) = \text{Op}(\phi) := \{y \in \mathcal{B}(H, K) \mid \phi(p)^{\perp} yp = 0, \forall p \in \mathcal{N}\},$$

where, in general, for every projection $p$ of a Hilbert space $L$ we denote by $p^{\perp}$ the projection $\text{Id}_{L} - p$.

We prove that if $\phi : \mathcal{N}_{1} \to \mathcal{M}_{1}$ and $\psi : \mathcal{N}_{2} \to \mathcal{M}_{2}$ are continuous onto nest homomorphisms and $\chi : \mathcal{N}_{1} \to \mathcal{N}_{2}$ is a nest isomorphism such that $E = \text{Op}(\phi) \cong \text{Op}(\psi \circ \chi)$ as $\text{Alg}(\mathcal{N}_{1})$-right $w^{*}$-rigged modules, then $E$ and $F = \text{Op}(\psi)$ are Morita equivalent.

Finally, we prove that if $\phi : \mathcal{N}_{1} \to \mathcal{M}_{1}$ and $\psi : \mathcal{N}_{2} \to \mathcal{M}_{2}$ are continuous onto nest homomorphisms and $\chi : \mathcal{N}_{1} \to \mathcal{N}_{2}$ is a nest isomorphism which can be extended to a $*$-isomorphism $\chi' : \mathcal{N}_{1}' \to \mathcal{N}_{2}'$ such that $E = \text{Op}(\phi) \cong \text{Op}(\psi \circ \chi)$ as $\text{Alg}(\mathcal{N}_{1})$-right $w^{*}$-rigged modules, then $E$ and $F = \text{Op}(\psi)$ are strongly projectively Morita equivalent.

## 2 Preliminaries

Our notation is standard. If $H$ and $K$ are Hilbert spaces then $\mathcal{B}(H, K)$ is the space of all linear and bounded operators from $H$ to $K$. If $H = K$ we write $\mathbb{B}(H, H) = \mathbb{B}(H)$. A dual operator algebra is an operator algebra which is also a dual operator space. Every $w^{*}$-closed subalgebra of some $\mathbb{B}(H)$ is a dual operator algebra. Conversely, for any dual operator algebra $\mathcal{A}$ there exist a Hilbert space $H$ and a $w^{*}$-continuous and completely isometric homomorphism $\alpha : \mathcal{A} \to \mathbb{B}(H)$. In this case we identify the algebra $\mathcal{A}$ with the $w^{*}$-closed subalgebra $\mathcal{A} = \mathbb{B}(H)$. In the following, all dual operator algebras are unital, that is they possess an identity of norm 1. The diagonal of a dual operator algebra $\mathcal{A}$ is the $C^{*}$-algebra $\Delta(\mathcal{A}) = \mathcal{A} \cap \mathcal{A}^{*}$. 

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If $X$ is a subset of $\mathcal{B}(H, K)$ and $Y$ is a subset of $\mathcal{B}(K, L)$ then we denote by $[YX]^{w^*}$ the w*-closure of the linear span of the set $\{yx \in \mathcal{B}(H, L) \mid x \in X, y \in Y\}$.

Furthermore, if $Z$ is a subset of $\mathcal{B}(L, R)$, then we denote by $[ZX]^{w^*}$ the w*-closure of the linear span of the set $\{zyx \in \mathcal{B}(H, R) \mid x \in X, y \in Y, z \in Z\}$.

A concrete right dual operator module $Y$ over a dual operator algebra $\mathcal{A}$ is a w*-closed subspace $Y \subseteq \mathcal{B}(H, K)$ such that $[Yx(A)]^{w^*} = Y$ for some w*-continuous and completely isometric unital homomorphism $x : \mathcal{A} \to \mathcal{B}(H)$.

An abstract right dual operator space $Y$ over a dual operator algebra $\mathcal{A}$ is defined to be a non-degenerate module over $\mathcal{A}$ which is also a dual operator space and the bilinear map $Y \times \mathcal{A} \to Y$ is separately w*-continuous with dense range. Similarly we define the two-sided bimodules which are considered to be non-degenerate both on the left and on the right. If $X$ is an operator space and $I, J$ are cardinals or sets, then we use the symbol $\mathcal{M}_{I,J}(X)$ for the operator space of $I \times J$ matrices over $X$ whose finite submatrices have uniformly bounded norm. We set $C_n^w(X) = \mathcal{M}_{I,1}(X)$ and $R_n^w(X) = \mathcal{M}_{1,J}(X)$ and these are written as $C_n(X)$ and $R_n(X)$ respectively if $J = n$ is finite. In the case where $X$ is a dual operator space, so is $\mathcal{M}_{I,J}(X)$ for any cardinals or sets $I, J$. For further details we refer the reader to [1, 2].

**Definition 2.1** [14] If $X, Y$ are dual operator spaces, then we say that $X$ and $Y$ are *stably isomorphic* if there exist a cardinal $J$ and a w*-continuous completely isometric map from $\mathcal{M}_J(X)$ onto $\mathcal{M}_J(Y)$. Here, $\mathcal{M}_J(X) = \mathcal{M}_{I,J}(X)$. Similarly $\mathcal{M}_J(Y) = \mathcal{M}_{I,J}(Y)$.

In [14], Eleftherakis et al. proved that two dual operator spaces $X$ and $Y$ are stably isomorphic if and only if they are $\Delta$-equivalent. The special case of dual operator algebras has been studied in [13, 14]. We remind the relevant definitions.

**Definition 2.2** [14] Let $X \subseteq \mathcal{B}(K_1, K_2)$ and $Y \subseteq \mathcal{B}(H_1, H_2)$ be dual operator spaces. We say that $X$ is *TRO-equivalent* to $Y$ if there exist w*-closed TRO’s $M_i \subseteq \mathcal{B}(H_i, K_i)$, $i = 1, 2$ (i.e w*-closed linear subspaces of $\mathcal{B}(H_i, K_i)$ satisfying $M_i M_i^* M_i \subseteq M_i$, $i = 1, 2$) such that

$$X = [M_2 Y M_1]^{w^*}, \quad Y = [M_2^* X M_1]^{w^*}.$$ 

**Definition 2.3** [14] Let $X$ and $Y$ be dual operator spaces. We say that $X$ is *$\Delta$-equivalent* to $Y$ if there exist w*-continuous and completely isometric maps $\Phi : X \to \mathcal{B}(K_1, K_2)$ and $\Psi : Y \to \mathcal{B}(H_1, H_2)$ for some Hilbert spaces $H_i, K_i$, $i = 1, 2$, such that $\Phi(X)$ is TRO-equivalent to $\Psi(Y)$. 

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Remark 2.4  Other notions of Morita equivalence of dual operator algebras or dual operator spaces exist in [3, 9, 10, 12].

2.1 w*-rigged modules

Definition 2.5  [5] Let $Y$ be a dual operator space which is a right operator module over a dual operator algebra $A$. Suppose that there exist a net of positive integers $(n(a))_{a \in J}$ and w*-continuous completely contractive right $A$-module maps

$$\phi_a : Y \to C_{n(a)}(A), \quad \psi_a : C_{n(a)}(A) \to Y$$

such that $\psi_a(\phi_a(y)) \to y$ in the w*-topology on $Y$, for all $y \in Y$. Then, we say that $Y$ is a right w*-rigged module over $A$.

We also give the definition of a left w*-rigged module over a dual operator algebra.

Definition 2.6  Let $Y$ be a dual operator space which is a left operator module over a dual operator algebra $A$. Suppose that there exist a net of positive integers $(n(a))_{a \in J}$ and w*-continuous completely contractive left $A$-module maps

$$\phi_a : Y \to R_{n(a)}(A), \quad \psi_a : R_{n(a)}(A) \to Y$$

such that $\psi_a(\phi_a(y)) \to y$ in the w*-topology on $Y$, for all $y \in Y$. Then, we say that $Y$ is a left w*-rigged module over $A$.

We identify two right (resp. left) w*-rigged modules $X, Y$ over a dual operator algebra $A$ as $A$-right (resp. left) dual operator modules, if there is a surjective w*-homeomorphic and completely isometric right (resp. left) $A$-module map between them. In this case we write $X \cong Y$.

Definition 2.7  [5] If $X$ and $Y$ are right w*-rigged modules over a dual operator algebra $A$, then we denote by $B(X, Y) = w^* CB_A(X, Y)$ the dual operator space of all w*-continuous and completely bounded right $A$-module maps from $X$ to $Y$. If $X = Y$, then we write $B(Y) = w^* CB_A(Y, Y) = w^* CB_A(Y)$.

Definition 2.8  [5] For a right w*-rigged module $Y$ over a dual operator algebra $A$ we denote by $\tilde{Y}$ the space $B(Y, A) = w^* CB_A(Y, A)$.

According to [5, Lemma 3.2] we have that $\tilde{Y}$ is a w*-closed subspace of $CB_A(Y, A)$ and a left w*-rigged module over $A$.

Remark 2.9  [2] If $A$ and $B$ are dual operator algebras, $Y$ is an $A$-$B$-dual operator bimodule and $X$ is a $B$-$A$-dual operator bimodule, then $Y \otimes_B^g X$ (resp. $X \otimes_A^g Y$) is a dual operator $A$-$A$-(resp. $B$) bimodule. In the case where $Y$ and $X$ are both right w*-rigged modules over $B$ and $A$, respectively, then $Y \otimes_B^g X$ (resp. $X \otimes_A^g Y$) is a right w*-rigged module over $A$ (resp. $B$) [5, Subsection 3.3].

We present now a subclass of right w*-rigged modules, the projectively right w*-rigged modules.
Definition 2.10 [5] If $\mathcal{A} \subseteq \mathcal{B}(H)$ is a dual operator algebra and $M \subseteq \mathcal{B}(H,K)$ is a w*-closed TRO such that $M^* M \subseteq \mathcal{A}$, then we call the space $Y = [M \mathcal{A}]^w$ a projectively right w*-rigged module over $\mathcal{A}$.

Remark 2.11 If $\mathcal{A}$, $M$ and $Y$ are as above, then by [5, Example (8)], $Y$ is a right w*-rigged module in the sense of Definition 2.5 and $Y \cong [\mathcal{A} M^*]^w$. The converse is not true. Not every w*-rigged module is a projectively w*-rigged module. For further details, we refer the reader to [4, Proposition 4.4] and [9].

We introduce the notion of a concrete strongly projectively right w*-rigged module as well as the notion of a strongly projectively right w*-rigged module over a unital dual operator algebra.

Definition 2.12 Let $\mathcal{A} \subseteq \mathcal{B}(H)$ be a unital dual operator algebra and let also $M \subseteq \mathcal{B}(H,K)$ be a w*-closed TRO such that $M^* M \subseteq \mathcal{A}$ and $\mathcal{A} = [M^* M]^w$. We call the dual operator space $Y = [M \mathcal{A}]^w$ a concrete strongly projectively right w*-rigged module over $\mathcal{A}$.

Definition 2.13 Let $\mathcal{A}$ be a unital dual operator algebra and let $Y$ be a dual operator space. We call $Y$ a strongly projectively right w*-rigged module over $\mathcal{A}$ if there exist a w*-continuous and completely isometric unital homomorphism $\alpha : \mathcal{A} \to \mathcal{B}(H)$ and a w*-continuous completely isometric right $\mathcal{A}$-module map $\Psi : Y \to \mathcal{B}(H,K)$ such that $\Psi(Y)$ is a concrete strongly projectively right w*-rigged module over $\alpha(\mathcal{A})$.

2.2 Nest algebras

Let $\text{pr}(\mathcal{B}(H))$ be the set of all projections on $\mathcal{B}(H)$. A set $\mathcal{L} \subseteq \text{pr}(\mathcal{B}(H))$ which contains $0, I_H$ and arbitrary intersections and closed spans is called a lattice. We denote by $\text{Alg}(\mathcal{L})$ the unital dual operator algebra

$$\text{Alg}(\mathcal{L}) = \{ x \in \mathcal{B}(H) \mid p^* x p = 0, \forall p \in \mathcal{L} \}.$$ 

Dually, if $\mathcal{A}$ is a unital dual operator algebra acting on a Hilbert space $H$ the set

$$\text{Lat}(\mathcal{A}) = \{ p \in \text{pr}(\mathcal{B}(H)) \mid p^* a p = 0, \forall a \in \mathcal{A} \}$$

is a lattice. If $\mathcal{A} = \text{Alg}(\text{Lat}(\mathcal{A}))$ then the algebra $\mathcal{A}$ is called reflexive. It is obvious that $\text{Lat}(\mathcal{A}) \subseteq \Delta(\mathcal{A})'$, where $\Delta(\mathcal{A})'$ is the commutant of $\Delta(\mathcal{A}) = \mathcal{A} \cap \mathcal{A}^*$.

A nest $\mathcal{N}$ is a totally ordered set of projections of a Hilbert space $H$, which contains $0, I_H$ and is closed under arbitrary intersections and closed spans. The corresponding nest algebra is

$$\text{Alg}(\mathcal{N}) = \{ x \in \mathcal{B}(H) \mid p^* x p = 0, \forall p \in \mathcal{N} \}.$$
Remark 2.14 A nest algebra $\mathcal{A} = \text{Alg}(\mathcal{N})$ is a reflexive algebra and $\mathcal{N} = \text{Lat}(\mathcal{A})$, see [7]. Furthermore, by [7, Corollary 22.18] we have that $\mathcal{N}'' = \Delta(\mathcal{A})$ and as a consequence, $\mathcal{N}''' = \Delta(\mathcal{A})'$. Let $\mathcal{N}, \mathcal{M}$ be nests acting on the Hilbert spaces $H, K$ respectively. An order preserving map $\phi : \mathcal{N} \to \mathcal{M}$ is called a nest homomorphism. If this map is also surjective and injective, it is called a nest isomorphism. In the case where $\phi : \mathcal{N} \to \mathcal{M}$ is a continuous (with respect to WOT topologies on $\text{pr}(\mathcal{B}(H))$ and $\text{pr}(\mathcal{B}(K))$ respectively) onto nest homomorphism the space
\[
\text{Op}(\phi) = \left\{ y \in \mathcal{B}(H, K) \mid \phi(p)^\perp y p = 0, \forall p \in \mathcal{N} \right\}
\]
is an $\text{Alg}(\mathcal{N})$-right w*-rigged module which is also a left module over $\text{Alg}(\mathcal{M})$, see [11, Theorem 2.1].

Remark 2.15 [12, Theorem 2.9] Let $\mathcal{N}$ and $\mathcal{M}$ be nests acting on Hilbert spaces $H$ and $K$ respectively. The dual operator algebras $\mathcal{A} = \text{Alg}(\mathcal{N})$ and $\mathcal{B} = \text{Alg}(\mathcal{M})$ are w*-Morita equivalent in the sense of [2] if, and only if, there exists a nest isomorphism $\theta : \mathcal{N} \to \mathcal{M}$. In this case, if $Y = \text{Op}(\theta^{-1})$ and $X = \text{Op}(\theta)$, then

1. $\mathcal{A} \cong Y \otimes_B X$ as dual operator $\mathcal{A}$-$\mathcal{A}$-bimodules.
2. $\mathcal{B} \cong X \otimes_A Y$ as dual operator $\mathcal{B}$-$\mathcal{B}$-bimodules.
3. $\tilde{Y} \cong X$ as right dual operator modules over $\mathcal{B}$ [5, Definition 4.2].

Remark 2.16 Let $Y$ be a subspace of $\mathcal{B}(H, K)$. We denote by $\text{Map}(Y)$ the map which sends every $p \in \text{pr}(\mathcal{B}(H))$ to the projection of $K$ generated by the vectors of the form $y p(\xi)$ where $y \in Y$ and $\xi \in H$.

The following theorem provides a characterization for the class of concrete strongly projectively right w*-rigged modules over reflexive operator algebras.

Theorem 2.17 Let $Y$ be a dual operator space and let $\mathcal{A} \subseteq \mathcal{B}(H)$ be a reflexive operator algebra. The following are equivalent.

(i) $Y$ is a concrete strongly projectively right w*-rigged module over $\mathcal{A}$.
(ii) There exist a reflexive operator algebra $\mathcal{B} \subseteq \mathcal{B}(K)$ and a $*$-isomorphism $\theta : \Delta(\mathcal{A})' \to \Delta(\mathcal{B})'$ such that $\theta(\text{Lat}(\mathcal{A})) = \text{Lat}(\mathcal{B})$ and
\[
Y = \left\{ y \in \mathcal{B}(H, K) \mid \theta(p)^\perp y p = 0, \forall p \in \text{Lat}(\mathcal{A}) \right\}.
\]

Proof (i)⇒(ii) Let $M \subseteq \mathcal{B}(H, K)$ be a w*-closed TRO such that $M^* M \subseteq \mathcal{A}$, $1_{\frac{M^* M}{[M^* M]^{w^*}}} = 1_{\mathcal{A}}$ and $Y = \frac{[M \mathcal{A}]}{[M^* M]^{w^*}}$. We consider the dual operator algebra $\mathcal{B} = \frac{[M \mathcal{A} M^*]}{[M^* M]^{w^*}} \subseteq \mathcal{B}(K)$. Since $M^* M \subseteq \mathcal{A}$ and $1_{\frac{M^* M}{[M^* M]^{w^*}}} = 1_{\mathcal{A}}$ we have that
\[ [M^* BM]^w = [M^* MAM^* M]^w = \mathcal{A}. \]

Therefore, the algebras \( \mathcal{A} \) and \( \mathcal{B} \) are TRO-equivalent and by [8, Remark 2.7] it follows that \( \mathcal{B} \) is also a reflexive algebra. According to [8, Proposition 2.5] we also have that

\[ \Delta(\mathcal{A}) = [M^* \Delta(\mathcal{B}) M]^w, \quad \Delta(\mathcal{B}) = [M \Delta(\mathcal{A}) M^*]^w. \]

From [8, Lemma 2.6], the map \( \chi = \text{Map}(M) : \text{pr}(\Delta(\mathcal{A})^\prime) \to \text{pr}(\Delta(\mathcal{B})^\prime) \) can be extended to a \( * \)-isomorphism \( \theta : \Delta(\mathcal{A})^\prime \to \Delta(\mathcal{B})^\prime \) such that \( \theta(\text{Lat}(\mathcal{A})) = \text{Lat}(\mathcal{B}) \). If we define the \( w^* \)-closed TRO

\[ Z = \left\{ z \in \mathcal{B}(H, K) \mid zp = \theta(p) z, \quad \forall p \in \text{pr}(\Delta(\mathcal{A})^\prime) \right\} \]

then by [8, Proposition 2.8] we have that \( M \subseteq Z \) and

\[ \mathcal{A} = [Z^* BZ]^w, \quad \mathcal{B} = [ZAZ^*]^w, \quad \Delta(\mathcal{A}) = [Z^* Z]^w, \quad \Delta(\mathcal{B}) = [ZZ^*]^w. \]

We aim to show that

\[ Y = \left\{ y \in \mathcal{B}(H, K) \mid \theta(p) z y p = 0, \quad \forall p \in \text{Lat}(\mathcal{A}) \right\}. \]

We set \( Y_0 = \left\{ y \in \mathcal{B}(H, K) \mid \theta(p)^\perp z y p = 0, \quad \forall p \in \text{Lat}(\mathcal{A}) \right\} \) and we need to show that \( Y = Y_0 \). We observe that \( Y_0 = [ZA]^w \). Indeed, if \( za \in ZA \) then for all \( p \in \text{Lat}(\mathcal{A}) \) we have that

\[ \theta(p)^\perp z a p = z a p - \theta(p) z a p = z a p - z p a p = z p^\perp a p = 0 \]

so \( ZA \subseteq Y_0 \) and thus \( [ZA]^w \subseteq Y_0 \). On the other hand, since \( \mathcal{B} \) is unital we get that

\[ Y_0 \subseteq [BY_0]^w = [ZAZ^* Y_0]^w, \]

so it suffices to prove that \( Z^* Y_0 \subseteq \mathcal{A} \). If \( p \in \text{Lat}(\mathcal{A}), z \in Z \) and \( y \in Y_0 \) we have that

\[ p^\perp z^* y p = z^* y p - p z^* y p = z^* y p - z^* \theta(p) y p = z^* \theta(p)^\perp y p = 0 \]

which means that \( z^* y \in \text{Alg}(\text{Lat}(\mathcal{A})) = \mathcal{A} \). Therefore \( Z^* Y_0 \subseteq \mathcal{A} \) as desired. Now, by the fact that \( M \subseteq Z \) we have that \( M \mathcal{A} \subseteq Z \mathcal{A} \) which implies that

\[ Y \subseteq Y_0. \] \hspace{1cm} (2.1)

Moreover,

\[ Y_0 \subseteq [Y_0 M^* M]^w \subseteq [BZ M^* M]^w \subseteq [BM]^w = Y \] \hspace{1cm} (2.2)

since \( ZM^* \subseteq ZZ^* \subseteq \Delta(\mathcal{B}) \subseteq \mathcal{B} \).

By (2.1) and (2.2) we have that \( Y = Y_0 \), as desired.
(ii)⇒(i) Let $B \subseteq \mathcal{B}(K)$ be a reflexive operator algebra and let $\theta : \Delta(A)' \to \Delta(B)'$ be a $*$-isomorphism such that $\theta(\text{Lat}(A)) = \text{Lat}(B)$ and

$$Y = \left\{ y \in \mathcal{B}(H, K) \mid \theta(p)^\perp yp = 0, \forall p \in \text{Lat}(A) \right\}.$$ 

Since $\Delta(A)'$ and $\Delta(B)'$ are von Neumann algebras, from [8, Theorem 3.2], we have that the space

$$M = \{ m \in \mathcal{B}(H, K) \mid mx = \theta(x)m, \forall x \in \Delta(A) '\}$$

is a $w^*$-closed TRO such that

$$B = \overline{[MAM]}^{w^*}, \quad A = \overline{[M^*BM]}^{w^*}.$$ 

It follows that $M^* M \subseteq A$ and $1_A = 1_{[M^*M]}^{w^*}$. Indeed, since the unit of $[M^*M]^{w^*}$ acts as a unit on the left and on the right on $A = [M^*BM]^{w^*}$ (as $M$ is a TRO) and since it is a subalgebra of $A$, then the unit of $[M^*M]^{w^*}$ is also a unit of $A$. We aim to show that $Y = \overline{[MA]}^{w^*}$. Indeed, for all $p \in \text{Lat}(A)$, $m \in M$ and $a \in A$, it holds that

$$\theta(p)^\perp m ap = m ap - \theta(p)m ap = m ap - mp ap = mp^\perp a p = 0$$

and thus

$$\overline{[MA]}^{w^*} \subseteq Y. \quad (2.3)$$

Since $B$ is unital we get that $Y \subseteq \overline{[BY]}^{w^*} = \overline{[MAM^*Y]}^{w^*}$. If $p \in \text{Lat}(A)$ and $m \in M$, $y \in Y$, then

$$p^\perp m^* yp = m^* \theta(p)^\perp yp = 0$$

which means that $M^* Y \subseteq \text{Alg}(\text{Lat}(A)) = A$. Therefore,

$$Y \subseteq \overline{[MAM^*Y]}^{w^*} \subseteq \overline{[MAA]}^{w^*} = \overline{[MA]}^{w^*}. \quad (2.4)$$

By (2.3) and (2.4) we deduce that $Y = \overline{[MA]}^{w^*}$ and as a consequence, $Y$ is a concrete strongly projectively right $w^*$-rigged module over $A$. $\square$

By combining Theorem 2.17 and Remark 2.14, we take the following.

**Corollary 2.18** Let $N$ be a nest acting on the Hilbert space $H$ with corresponding nest algebra $A = \text{Alg}(N) \subseteq \mathcal{B}(H)$ and let $Y$ be a dual operator space. The following are equivalent.

(i) $Y$ is a concrete strongly projectively right $w^*$-rigged module over $A$.

(ii) There exist a nest $M$ acting on some Hilbert space $K$ and a $*$-isomorphism $\theta : N'' \to M''$ such that $\theta(N) = M$ and
3 Morita equivalence

Throughout this section, we develop a theory of Morita equivalence for right w*-
rigged modules over unital dual operator algebras.

Let \( B \) be a unital dual operator algebra and let \( Y \) be a right w*-rigged module
over \( B \). By [5, Lemma 3.2] we get that \( \tilde{Y} \) is a left w*-rigged module over \( B \) and the
map \((\cdot, \cdot) : \tilde{Y} \times Y \to B\) which sends every \((f, y) \in \tilde{Y} \times Y\) to \(f(y) \in B\) is separately
w*-continuous and completely contractive. From the discussion above [5, Theorem 3.5], the space
\( A = Y \otimes_{B}^{\text{sh}} \tilde{Y} \) is a dual operator algebra with identity of
norm 1 and by [5, Definition 4.3], \( Y \) is a non-degenerate left module over \( A \) as well
as \( \tilde{Y} \) is a non-degenerate right module over \( A \).

We recall from Definition 2.7 that for a right w*-rigged module \( Y \) over a unital
dual operator algebra \( B \) we denote by \( B(Y) \) the dual operator space of all w*-continuous
and completely bounded right \( B \)-module maps from \( Y \) to \( Y \). In [5, Theorem 3.5] it is proven that
\( A \cong B(Y) \) as dual operator algebras and thus the
induced non-degenerate left action of \( A \) on \( Y \) is the following
\[
A \times Y \to Y, \ (a, y) \mapsto a y = \theta(a)(y),
\]
where \( \theta : Y \otimes_{B}^{\text{sh}} \tilde{Y} \to B(Y) \) is the isomorphism proved in the above theorem such that
\[
\theta(y \otimes_B f)(y') = yf(y'), \ y \in Y, f \in \tilde{Y}.
\]

Furthermore, \( B(Y) \) is an \( A\text{-}A \)-bimodule with actions
\[
a T : Y \to Y, \ (a T)(y) = a T(y), \quad \forall a \in A, \forall T \in B(Y)
\]
\[
T a : Y \to Y, \ (T a)(y) = T(a y), \quad \forall a \in A, \forall T \in B(Y).
\]

We claim that \( \theta \) is also a bimodule isomorphism in a canonical way. Since \( \theta \) is w*-continuous, it suffices to prove that
\[
\theta(a (y \otimes_B f)) = a \theta(y \otimes_B f) \quad \text{and} \quad \theta((y \otimes_B f) a) = \theta(y \otimes_B f) a,
\]
where \( a \in A, y \in Y, f \in \tilde{Y} \). Indeed, for every \( y' \in Y \) we have that
\[
\theta(a (y \otimes_B f))(y') = \theta(a y \otimes_B f)(y')
\]
\[
= ayf(y')
\]
\[
= a (yf(y'))
\]
\[
= (a \theta(y \otimes_B f))(y')
\]
and
\[\theta((y \otimes_B f) a)(y') = \theta(y \otimes_B f a)(y') = y(f(a))(y') = yf(ay') = \theta(y \otimes_B f)(ay') = (\theta(y \otimes_B f) a)(y').\]

**Remark 3.1** Let \( \mathcal{A} \) and \( \mathcal{B} \) be unital dual operator algebras and \( Y \) be a right \( \mathcal{B} \)-rigged module over \( \mathcal{B} \) such that \( \mathcal{A} \cong Y \otimes^\text{gh}_B \tilde{Y} \) as dual operator \( \mathcal{A} \)-bimodules (that is, via a completely isometric \( \mathcal{B} \)-homeomorphism which is also an \( \mathcal{A} \)-bimodule map) and also \( \mathcal{B} \cong \tilde{Y} \otimes^\text{gh}_A Y \) as dual operator \( \mathcal{B} \)-bimodules.

Therefore, the \( \mathcal{B} \)-\( \mathcal{A} \)-bimodule \( \tilde{Y} \) is a \( \mathcal{B} \)-Morita equivalence bimodule in the sense of \([2]\) and by the last paragraph above \([6, \text{Definition 1.3}]\) we deduce that \( \tilde{Y} \) is a right \( \mathcal{B} \)-rigged module over \( \mathcal{A} \).

Now, we can give the definition of Morita equivalence for right \( \mathcal{B} \)-rigged modules.

**Definition 3.2** Let \( \mathcal{A} \) and \( \mathcal{B} \) be dual operator algebras, \( E \) be a right \( \mathcal{B} \)-rigged module over \( \mathcal{A} \) and \( F \) be a right \( \mathcal{B} \)-rigged module over \( \mathcal{B} \). We call \( E, F \) Morita equivalent if there exists a right \( \mathcal{B} \)-rigged module \( Y \) over \( \mathcal{B} \) such that

(i) \( \mathcal{A} \cong Y \otimes^\text{gh}_B \tilde{Y} \) as dual operator \( \mathcal{A} \)-\( \mathcal{A} \)-bimodules.

(ii) \( \mathcal{B} \cong \tilde{Y} \otimes^\text{gh}_A Y \) as dual operator \( \mathcal{B} \)-\( \mathcal{B} \)-bimodules.

(iii) \( F \cong E \otimes^\text{gh}_A Y \) as right \( \mathcal{B} \)-rigged modules over \( \mathcal{B} \) and \( E \cong F \otimes^\text{gh}_B \tilde{Y} \) as right \( \mathcal{B} \)-rigged modules over \( \mathcal{A} \).

**Remark 3.3** If \( E, F, \mathcal{A}, \mathcal{B} \) are as in the Definition above, then by (i), (ii), Remark 3.1 and the discussion above it, we get that \( \mathcal{A} \cong B(Y), \mathcal{B} \cong B(\tilde{Y}) \) and also \( \mathcal{A}, \mathcal{B} \) are \( \mathcal{B} \)-Morita equivalent in the sense of \([2]\) with equivalence bimodules \( Y \) and \( \tilde{Y} \).

**Remark 3.4** Let \( E, F, \mathcal{A}, \mathcal{B} \) and \( Y \) be as in the Definition 3.2. We observe that in the condition (iii), it suffices to prove one of the two isomorphisms since either of them implies the other one. Indeed, suppose that \( F \cong E \otimes^\text{gh}_A Y \) as right \( \mathcal{B} \)-rigged modules over \( \mathcal{B} \). Using the condition (i), the associativity of the normal Haagerup tensor product and the fact that \( E \) is a non-degenerate \( \mathcal{A} \)-right module, we have that

\[F \otimes^\text{gh}_B \tilde{Y} \cong (E \otimes^\text{gh}_A Y) \otimes^\text{gh}_B \tilde{Y} \cong E \otimes^\text{gh}_A (Y \otimes^\text{gh}_B \tilde{Y}) \cong E \otimes^\text{gh}_A \mathcal{A} \cong E.\]

Similarly, if \( E \cong F \otimes^\text{gh}_B \tilde{Y} \) as right \( \mathcal{B} \)-rigged modules over \( \mathcal{A} \) then by (ii), the associativity of the normal Haagerup tensor product and the fact that \( F \) is a non-degenerate \( \mathcal{B} \)-right module, we get that \( F \cong E \otimes^\text{gh}_A Y \) as right \( \mathcal{B} \)-rigged modules over \( \mathcal{B} \).

However, \( E \) and \( F \) are not always stably isomorphic. Indeed, by \([10, \text{Example 3.7}]\) there exist isomorphic nests \( \mathcal{N} \) and \( \mathcal{M} \) such that the nest algebras \( \mathcal{A} = \)
Alg(\mathcal{N}) and B = Alg(\mathcal{M}) are not stably isomorphic (since they are not \( \Delta \)-equivalent). Let \( \theta : \mathcal{N} \to \mathcal{M} \) be a nest isomorphism. If \( Y = \text{Op}(\theta) \) then by Remark 2.15 we have that \( \tilde{Y} \cong X = \text{Op}(\theta^{-1}) \) and also \( A \cong \tilde{Y} \otimes_B^\theta \tilde{Y} \), \( B \cong \tilde{Y} \otimes_A^\theta \tilde{Y} \). By defining \( E = A \) over \( A \) and \( F = Y \) over \( B \), we have that \( E \) and \( F \) are Morita equivalent in our sense but not stably isomorphic.

**Definition 3.5** Let \( A \) and \( B \) be dual operator algebras, \( E \) be a right \( w^* \)-rigged module over \( A \) and \( F \) be a right \( w^* \)-rigged module over \( B \). We call \( E, F \) strongly projectively Morita equivalent if there exists a strongly projectively right \( w^* \)-rigged module \( Y \) over \( B \) such that

(i) \( A \cong Y \otimes_B^\theta \tilde{Y} \) as dual operator \( A \text{-}\text{bimodules.} \)

(ii) \( B \cong \tilde{Y} \otimes_A^\theta Y \) as dual operator \( B \text{-}\text{bimodules.} \)

(iii) \( F \cong E \otimes_A^\theta Y \) as right \( w^* \)-rigged modules over \( B \) and \( E \cong F \otimes_B^\theta \tilde{Y} \) as right \( w^* \)-rigged modules over \( A \).

**Remark 3.6** If \( A, B, E, F \) and \( Y \) are as in the Definition 3.5, then since \( Y \) is a strongly projectively right \( w^* \)-rigged module over \( B \), there exist a \( w^* \)-continuous completely isometric unital homomorphism \( \beta : B \to \mathbb{B}(H) \) and a \( w^* \)-closed TRO \( M \subseteq \mathbb{B}(H, K) \) such that \( M^* M \subseteq \beta(B) \), \( 1_{\mathbb{M}^w M^w} = 1_{\beta(B)} \) and \( Y \cong [M \beta(B)]^w \) as right \( w^* \)-rigged modules over \( B \). Define the unital dual operator algebra \( C = [M \beta(B) M^*]^w \). Since \( M^* M \subseteq \beta(B) \) and \( [M^* M]^w \), \( \beta(B) \) share the same unit we have that

\[
[M^* M \beta(B)]^w = \beta(B) = [\beta(B) M^* M]^w
\]

which implies that

\[
[M^* C M]^w = [M^* M \beta(B) M^* M]^w = \beta(B).
\]

Therefore, the algebras \( \beta(B) \) and \( C \) are TRO-equivalent via \( M \). By [13, Theorem 2.4], for the \( M \)-generated \( B \text{-}\text{C} \) bimodules

\[
U = [\beta(B) M]^w \cong \tilde{Y}, \ V = [M \beta(B)]^w \cong Y
\]

it holds that

\[
\beta(B) \cong U \otimes_C^\theta V, \ C \cong V \otimes_B^\theta \beta(B) U \cong V \otimes_B^\theta U.
\]

Therefore, \( C \cong V \otimes_B^\theta U \cong Y \otimes_B^\theta \tilde{Y} \cong A \). Since \( \beta(B) \) and \( C \) are TRO-equivalent and \( C \cong A \), we deduce that \( A \) and \( B \) are \( \Delta \)-equivalent, i.e., \( A \) and \( B \) are stably isomorphic, [13, Theorem 3.2].

**Theorem 3.7** Let \( E \) and \( F \) be right \( w^* \)-rigged modules over the unital dual operator algebras \( A \) and \( B \), respectively. If \( E \) and \( F \) are strongly projectively Morita equivalent, then \( E \) and \( F \) are stably isomorphic.
Proof Let $E$ and $F$ be strongly projectively Morita equivalent. There exists a strongly projectively right $w^*$-rigged module $Y$ over $B$ such that

$$\mathcal{A} \cong Y \otimes_{B}^{\sigma} \tilde{Y}, \mathcal{B} \cong \tilde{Y} \otimes_{A}^{\sigma} Y, F \cong E \otimes_{A}^{\sigma} Y.$$  

Since $Y$ is a strongly projectively right $w^*$-rigged module over $B$, there exist a $w^*$-continuous completely isometric unital homomorphism $\beta : \mathcal{B} \to \mathcal{B}(H)$ and a $w^*$-closed TRO $M \subseteq \mathcal{B}(H, K)$ such that $M' M \subseteq \beta(B)$, $1_{[M' M]''} = 1_{\beta(B)}$ and $Y \cong [M \beta(B) M']^{w^*}$.

By Remark 3.6 it follows that the algebras $C = [M \beta(B) M']^{w^*}$ and $\beta(B)$ are TRO-equivalent via $M$. By [13, Lemma 3.1], for the $\beta(B)$-$C$-bimodule $\tilde{Y} \cong [\beta(B) M']^{w^*}$ we have that

$$R^w_I(\tilde{Y}) \cong R^w_I(\beta(B)) \cong R^w_I(B),$$

where $I$ is an infinite indexed set. Since $F \cong E \otimes_{A}^{\sigma} Y$, by Remark 3.4, we get that $E \cong F \otimes_{B}^{\sigma} \tilde{Y}$ as $\mathcal{A}$-right $w^*$-rigged modules and from [2, Lemma 2.7] we deduce that

$$\mathcal{M}_I(E) \cong \mathcal{M}_I(F \otimes_{B}^{\sigma} \tilde{Y}) \cong C^w_I(F) \otimes_{B}^{\sigma} R^w_I(\tilde{Y}) \cong C^w_I(F) \otimes_{B}^{\sigma} R^w_I(B) \cong \mathcal{M}_I(F \otimes_{B}^{\sigma} B) \cong \mathcal{M}_I(F).$$

Therefore, $E$ and $F$ are stably isomorphic.

Example 3.8 We provide an example of strongly projectively Morita equivalent $w^*$-rigged modules. Let $M \subseteq \mathcal{B}(H, K)$ be a $w^*$-closed TRO, where $H$ and $K$ are Hilbert spaces and let $E$ be a right $w^*$-rigged module over the unital dual operator algebra $\mathcal{A} = [M M^*]^{w^*} \subseteq \mathcal{B}(K)$. By defining $F = E \otimes_{A}^{\sigma} M$, we have that $F$ is a right $w^*$-rigged module over $B = [M' M]^{w^*} \subseteq \mathcal{B}(H)$ (see Remark 2.9) and $E, F$ are strongly projectively Morita equivalent with equivalence bimodules $Y = M = [M B]^{w^*}$ and $\tilde{Y} \cong M^* = [B M^*]^{w^*}$.

4 Morita equivalence of $w^*$-rigged modules over nest algebras

As mentioned in the introduction, in this section, we study the Morita equivalence and the strongly projectively Morita equivalence between right $w^*$-rigged modules over nest algebras. The main theorems of this section are Theorems 4.5 and 4.7. Before we state them, we prove some useful lemmas. For the first lemma, we recall the Remark 2.16, where for a subspace $Y \subseteq \mathcal{B}(H, K)$ we denote by Map($Y$) the map which sends every projection $p \in \text{pr}(\mathcal{B}(H))$ to the projection of $K$ generated by the vectors of the form $y p (\xi)$, where $y \in Y$ and $\xi \in H$.
Lemma 4.1  Let $\mathcal{M}$ be a nest acting on some Hilbert space $K$ and let $\text{Alg}(\mathcal{M})$ be the corresponding nest algebra. If $p \in \mathcal{M}$ and $q$ is the projection onto $\overline{\text{Alg}(\mathcal{M}) p(K)}$ then $p = q$.

**Proof**  Since $I_K \in \text{Alg}(\mathcal{M})$, it follows that $p \leq q$. On the other hand, it holds that $p^\perp \text{Alg}(\mathcal{M}) p = 0$ which implies that
\[
\text{Alg}(\mathcal{M}) p(K) = p \text{Alg}(\mathcal{M}) p(K) \subseteq p(K)
\]
that is $q \leq p$. We deduce that $p = q$. \hfill \Box

Lemma 4.2  Let $\mathcal{N}, \mathcal{M}$ be nests acting on the Hilbert spaces $H, K$, respectively. If $\phi : \mathcal{N} \to \mathcal{M}$ is a continuous onto nest homomorphism and
\[
U = \text{Op}(\phi) = \left\{ u \in \mathcal{B}(H, K) \mid \phi(p)^\perp u p = 0, \forall p \in \mathcal{N} \right\},
\]
then $\text{Map}(U)|_{\mathcal{N}} = \phi$.

**Proof**  Set $\phi' = \text{Map}(U)$. We will show that $\phi'(p) = \phi(p)$, $\forall p \in \mathcal{N}$. Denote by $V$ the space
\[
V = \left\{ v \in \mathcal{B}(K, H) \mid p^\perp v \phi(p) = 0, \forall p \in \mathcal{N} \right\}
\]
and let $\psi = \text{Map}(V)$. By [11, Theorem 2.1], we have that $\text{Alg}(\mathcal{M}) = [U V]^{w^*}$. Let $p \in \mathcal{N}$. Since $\phi(p)^\perp U p = 0$ it follows that $\phi(p)^\perp \phi'(p) = 0$ and thus
\[
\phi'(p) \leq \phi(p). \quad (4.1)
\]
On the other hand, $p^\perp V \phi(p) = 0$ which implies that $V \phi(p)(K) \subseteq p$ and as a consequence
\[
\psi(\phi(p)) \leq p. \quad (4.2)
\]
For all $u \in U, v \in V$, we have that
\[
\begin{align*}
u v \phi(p) &= u \psi(\phi(p)) v \phi(p) \\
&= \phi'(\psi(\phi(p))) u \psi(\phi(p)) v \phi(p) \\
&= \phi'(\psi(\phi(p))) u v \phi(p)
\end{align*}
\]
which means that
\[
[U V \phi(p)(K)]^{w^*} \subseteq \phi'(\psi(\phi(p)))(K). \quad (4.3)
\]
Since $\text{Alg}(\mathcal{M}) = [U V]^{w^*}$ and $\phi(p) \in \mathcal{M}$, from Lemma 4.1, $\phi(p)$ is the projection onto $[U V \phi(p)(K)]^{w^*}$ and thus by (4.3) we get that $\phi(p) \leq \phi'(\psi(\phi(p)))$. Using (4.2), we have that
By combining (4.1) and (4.4) we deduce that \( \phi(p) = \phi'(p) \).

**Lemma 4.3** Let \( N, M, L \) be nests acting on the Hilbert spaces \( H, K \) and \( R \) respectively. Let also \( \phi : N \to M, \zeta : N \to L \) and \( \theta : M \to L \) be continuous onto nest homomorphisms. The following are equivalent.

(i) \( \zeta = \theta \circ \phi \).

(ii) \( \text{Op}(\zeta) = \left[ \text{Op}(\theta) \text{Op}(\phi) \right]^{w^*} \).

**Proof** (i)\(\Rightarrow\)(ii) Let \( \zeta = \theta \circ \phi \). For all \( x \in \text{Op}(\theta), y \in \text{Op}(\phi) \) we have that

\[
\zeta(p)^{\perp} x y p = \theta(\phi(p))^{\perp} x y p \\
= \theta(\phi(p))^{\perp} x (\phi(p) + \phi(p)^{\perp}) y p \\
= \theta(\phi(p))^{\perp} x \phi(p) y p + \theta(\phi(p))^{\perp} x \phi(p)^{\perp} y p \\
= 0, \ \forall \ p \in N
\]

since \( \theta(\phi(p))^{\perp} x \phi(p) = 0 \) (\( \phi(p) \in M, x \in \text{Op}(\theta) \)) and \( \phi(p)^{\perp} y p = 0 \) (\( y \in \text{Op}(\phi) \)) which means that \( xy \in \text{Op}(\zeta) \). Thus, \( \text{Op}(\theta) \text{Op}(\phi) \subseteq \text{Op}(\zeta) \) which implies that

\[
\left[ \text{Op}(\theta) \text{Op}(\phi) \right]^{w^*} \subseteq \text{Op}(\zeta).
\] (4.5)

On the other hand, let \( z \in \text{Op}(\zeta) \). From [11, Theorem 2.1], we have that \( \text{Alg}(L) = \left[ \text{Op}(\theta) X \right]^{w^*} \) where \( X = \{ x \in \mathbb{B}(R, K) \ | \ q^{\perp} x 0(q) = 0, \ \forall \ q \in M \} \), so there exist nets \((u_j)_{j \in J}, (x_j)_{j \in J}\) such that \( u_j \in \text{Op}(\theta), x_j \in X, \ \forall \ j \in J \) and

\[
\sum_{r=1}^{k_j} u_j x_j, x_j \xrightarrow{w^*} I_R
\]

which implies that

\[
\sum_{r=1}^{k_j} u_j x_j, x_j \xrightarrow{w^*} z, z.
\] (4.6)

We observe that

\[
X = \{ x \in \mathbb{B}(R, K) \ | \ q^{\perp} x 0(q) = 0, \ \forall \ q \in M \} \\
= \{ x \in \mathbb{B}(R, K) \ | \ \phi(p)^{\perp} x 0(\phi(p)) = 0, \ \forall \ p \in N \} \\
= \{ x \in \mathbb{B}(R, K) \ | \ \phi(p)^{\perp} x \zeta(p) = 0, \ \forall \ p \in N \}.
\]

We claim that \( x_j, z \in \text{Op}(\phi), r = 1, \ldots, k_j, j \in J \). Indeed, for all \( p \in N \) it holds that
\[
\phi(p)^\perp x_j, z p = \phi(p)^\perp x_j, (\zeta(p) + \zeta(p)^\perp) z p \\
= \phi(p)^\perp x_j, \zeta(p) z p + \phi(p)^\perp x_j, \zeta(p)^\perp z p \\
= 0
\]
since \(\phi(p)^\perp x_j, \zeta(p) = 0\) (\(x_j, X\)) and \(\zeta(p)^\perp z p = 0\) (\(z \in \Op(\zeta)\)). Thus,

\[
z^{(4.6)} = \lim_{j} \sum_{r=1}^{k_j} u_j, x_j, z \in \overline{\Op(\theta) \Op(\phi)}^w^*
\]

where the above limit is on the w*-topology, that is

\[
\Op(\zeta) \subseteq \overline{\Op(\theta) \Op(\phi)}^w^*.
\]

By (4.5) and (4.7), we have that \(\Op(\zeta) = \overline{\Op(\theta) \Op(\phi)}^w^*\).

(ii)\(\Rightarrow\) (i) Suppose now that \(\Op(\zeta) = \overline{\Op(\theta) \Op(\phi)}^w^*\). From the direction (i)\(\Rightarrow\) (ii) we have that \(\overline{\Op(\theta) \Op(\phi)}^w^* = \Op(\theta \circ \phi)\), so

\[
\Op(\zeta) = \Op(\theta \circ \phi).
\]

Since \(\zeta\) and \(\theta \circ \phi\) are continuous onto nest homomorphism from \(\mathcal{N}\) to \(\mathcal{L}\), by Lemma 4.2 we deduce that

\[
\zeta = \Map(\Op(\zeta))|_{\mathcal{N}} = \Map(\Op(\theta \circ \phi))|_{\mathcal{N}} = \theta \circ \phi.
\]

\(\square\)

**Lemma 4.4** Let \(\mathcal{N}, \mathcal{M}, \mathcal{L}\) be nests with corresponding nest algebras

\[
\mathcal{A} = \Alg(\mathcal{N}), \mathcal{B} = \Alg(\mathcal{M}), \mathcal{C} = \Alg(\mathcal{L}).
\]

Let also \(\phi: \mathcal{N} \to \mathcal{M}, \theta: \mathcal{M} \to \mathcal{L}\) be continuous onto nest homomorphisms. Then

\[
\Op(\theta) \otimes_B^{\theta h} \Op(\phi) \cong \overline{\Op(\theta) \Op(\phi)}^w^*
\]
as \(\mathcal{A}\)-right w*-rigged modules.

**Proof** The map \(\Op(\theta) \times \Op(\phi) \to \Op(\theta) \Op(\phi), (x, u) \mapsto xu\) is a separately w*-continuous and completely contractive right \(\mathcal{B}\)-balanced map, so it induces a w*-continuous and completely contractive map

\[
\rho: \Op(\theta) \otimes_B^{\theta h} \Op(\phi) \to \overline{\Op(\theta) \Op(\phi)}^w^*
\]
satisfying \(\rho(x \otimes_B u) = xu, x \in \Op(\theta), u \in \Op(\phi)\). The map \(\rho\) is also a right \(\mathcal{A}\)-module map. Set \(Y = \{ y \mid q^\perp y \theta(q) = 0, \forall q \in \mathcal{M}\}\). From [11, Theorem 2.1] we have that \(\mathcal{C} = \overline{\Op(\theta) Y}^w^*\). Therefore, there exist contractive nets \((z_t)_{t \in T} \subseteq R^\infty_{\infty}(\Op(\theta))\) and \((y_t)_{t \in T} \subseteq C^\infty_{\infty}(Y)\) such that
$z_t y_t \rightarrow 1_C$, 

where, in general, if $W \subseteq \mathbb{B}(H, K)$ we denote by $R^\text{fin}_{\infty}(W)$ (resp. $C^\text{fin}_{\infty}(W)$) the space of operators $(w_1, w_2, \ldots) : H^\infty \rightarrow K$ (resp. $(w_1, w_2, \ldots)^t : H \rightarrow K^\infty$) such that $w_i \in W$, $\forall i \in \mathbb{N}$ and also there exists $i_0 \in \mathbb{N}$ such that $w_i = 0$, $\forall i \geq i_0$. Assume that $x_i \in \mathcal{F}$, $u_i \in \text{Op}(\phi)$, $i = 1, \ldots, n$, where $\mathcal{F}$ is the space of all finite rank operators in $\text{Op}(\theta)$. Since $x_i \in \mathcal{F}$ we have that $z_t y_t x_i \rightarrow x_i$. By the fact that the Haagerup tensor product is contractive as a bilinear map, for every $\epsilon > 0$ there exists $t \in T$ such that

$$\left\| \sum_{i=1}^n x_i \otimes_B u_i \right\| - \epsilon \leq \left\| \sum_{i=1}^n z_t y_t x_i \otimes_B u_i \right\|.$$  \hspace{1cm} (4.8)

Let $x \in \mathcal{F} \subseteq \text{Op}(\theta)$ and $y \in Y$. It holds that $\theta(q)^\perp x q = 0$, so it follows that $x q = \theta(q) x q$, $\forall q \in \mathcal{M}$ and as a consequence

$q^\perp y x q = q^\perp y \theta(q) x q = 0$, $\forall q \in \mathcal{M}$ (since $y \in Y$),

which means that $Y \mathcal{F} \subseteq \mathcal{B}$. Using (4.8) we get that

$$\left\| \sum_{i=1}^n x_i \otimes_B u_i \right\| - \epsilon \leq \left\| z_t y_t \sum_{i=1}^n x_i u_i \right\| \leq \left\| \sum_{i=1}^n x_i u_i \right\| = \rho \left( \sum_{i=1}^n x_i \otimes_B u_i \right).$$

But since $\epsilon$ is arbitrary, we have that

$$\left\| \sum_{i=1}^n x_i \otimes_B u_i \right\| \leq \rho \left( \sum_{i=1}^n x_i \otimes_B u_i \right)$$

which implies that the restriction of $\rho$ to

$$\mathcal{F} \otimes_B^h \text{Op}(\phi) = \left[ x \otimes_B u \mid x \in \mathcal{F}, u \in \text{Op}(\phi) \right]$$

is an isometry. We will prove that $\rho$ is an isometry in $\text{Op}(\theta) \otimes_B^h \text{Op}(\phi)$. To this end, let $z \in \text{Op}(\theta) \otimes_B^h \text{Op}(\phi)$. Thus, there exists a net $(z_\ell)_{\ell \in I}$ in $\mathcal{F} \otimes_B^h \text{Op}(\phi)$ such that $z_\ell \rightarrow^w z$. Let $f \in \text{Ball}(\mathcal{C})$, $g \in \text{Ball}(\mathcal{A})$ be finite rank operators. Then, $f z_\ell g \rightarrow^w f z g$ and since $\rho$ is $w^*$-continuous we have that $\rho(f z_\ell g) \rightarrow^w \rho(f z g)$ which implies that $\rho(f z_\ell g) \rightarrow^w \rho(f z g)$. Using the fact that $f$, $g$ are finite rank operators we deduce that

$$\| f z_\ell g \| = \| \rho(f z_\ell g) \| = \| \rho(f z_\ell g) \| \rightarrow \| \rho(f z g) \|.$$ 

For every $\epsilon > 0$ there exists $\ell_0 \in I$ such that $\| f z_\ell g \| < \| \rho(f z g) \| + \epsilon$, $\forall \ell \geq \ell_0$ so, $\| f z_\ell g \| < \| \rho(z) \| + \epsilon$, $\forall \ell \geq \ell_0$. We have that $f z_\ell g \rightarrow^w f z g$ which implies that $\| f z g \| \leq \sup \{ \| f z_\ell g \| : \ell \geq \ell_0 \}$. Thus, $\| f z g \| < \| \rho(z) \| + \epsilon$. Since $\epsilon$ is arbitrary we get
\[ \|f \cdot g\| \leq \|\rho(z)\|. \] (4.9)

By [7, Corollary 3.13] there exist contractive nets of finite rank operators \((f_j)_{j \in J} \subseteq \mathcal{C}\) and \((g_j)_{j \in J} \subseteq \mathcal{A}\) such that

\[ f_j \overset{w^*}{\to} 1_{\mathcal{C}}, \quad g_j \overset{w^*}{\to} 1_{\mathcal{A}}. \]

Thus, \(f_j g_j \overset{w^*}{\to} z\) and according to (4.9) it holds that

\[ \|z\| \leq \sup_{j \in J} \|f_j g_j\| \leq \|\rho(z)\| \]

which implies that \(\rho\) is an isometry. Similarly we can prove that \(\rho\) is a complete isometry and as a consequence (Krein–Smulian) \(\rho\) is onto \([\text{Op}(\theta) \text{Op}(\phi)]^{w^*}\).

The following theorem describes how to construct Morita equivalent right \(w^*\)-rigged modules.

**Theorem 4.5** Let \(\mathcal{N}_1, \mathcal{N}_2, \mathcal{M}_1, \mathcal{M}_2\) be nests, \(\mathcal{A} = \text{Alg}(\mathcal{N}_1) \subseteq \mathcal{B}(H_1), \mathcal{B} = \text{Alg}(\mathcal{N}_2) \subseteq \mathcal{B}(H_2)\), \(\phi : \mathcal{N}_1 \to \mathcal{M}_1, \psi : \mathcal{N}_2 \to \mathcal{M}_2\) be continuous onto nest homomorphisms and \(E = \text{Op}(\phi), F = \text{Op}(\psi)\). If \(\chi : \mathcal{N}_1 \to \mathcal{N}_2\) is a nest isomorphism such that \(E \cong \text{Op}(\psi \circ \chi)\) as \(\mathcal{A}\)-right \(w^*\)-rigged modules, then \(E\) and \(F\) are Morita equivalent.

**Proof** Let \(\chi : \mathcal{N}_1 \to \mathcal{N}_2\) be a nest isomorphism such that \(E \cong \text{Op}(\zeta)\) as \(\mathcal{A}\)-right \(w^*\)-rigged modules, where \(\zeta = \psi \circ \chi\). By Lemma 4.3 we get

\[ E \cong \text{Op}(\zeta) = [\text{Op}(\psi) \text{Op}(\chi)]^{w^*} = [F \text{Op}(\chi)]^{w^*}. \]

We define the \(\mathcal{B}\)-right \(w^*\)-rigged module \(Y = \text{Op}(\chi^{-1})\). Let

\[ X = \{ x \in \mathcal{B}(H_1, H_2) \mid q^\perp x \chi^{-1}(q) = 0, \forall q \in \mathcal{N}_2 \} = \text{Op}(\chi). \]

By [12, Theorem 2.9] we have that \(\mathcal{A} \cong Y \otimes_B^h X, \mathcal{B} \cong X \otimes_A^h Y\). According to Remark 2.15 we have that \(\tilde{Y} \cong X = \text{Op}(\chi)\) and thus

\[ \mathcal{A} \cong Y \otimes_B^h \tilde{Y}, \mathcal{B} \cong \tilde{Y} \otimes_A^h Y. \]

Furthermore by Lemma 4.4 we have that

\[ E \cong \text{Op}(\zeta) \cong [\text{Op}(\psi) \text{Op}(\chi)]^{w^*} \cong \text{Op}(\psi) \otimes_B^h \text{Op}(\chi) \cong F \otimes_B^h \tilde{Y}. \]

Therefore, \(E\) and \(F\) are Morita equivalent.

**Example 4.6** Let \(\mathcal{N}_1, \mathcal{N}_2\) be isomorphic nests acting on the Hilbert spaces \(H_1, H_2\) respectively. Let \(\mathcal{A} = \text{Alg}(\mathcal{N}_1) \subseteq \mathcal{B}(H_1)\) and \(\mathcal{B} = \text{Alg}(\mathcal{N}_2) \subseteq \mathcal{B}(H_2)\) be the corresponding nest algebras and let \(\chi : \mathcal{N}_1 \to \mathcal{N}_2\) be a nest isomorphism. We consider the nest
\[ \mathcal{N}_2 \oplus \mathcal{N}_1 = \{ \chi(p) \oplus p \mid p \in \mathcal{N}_1 \} \subseteq \mathbb{B}(H_2 \oplus H_1) \]

and we define the continuous onto nest homomorphism

\[ \psi : \mathcal{N}_2 \rightarrow \mathcal{N}_2 \oplus \mathcal{N}_1, \psi(q) = (q, \chi^{-1}(q)) \]  

If \( F = \operatorname{Op}(\psi) \) then by Theorem 4.5 the right \( \ast \)-rigged modules \( E = \operatorname{Op}(\psi \circ \chi) \) and \( F \) are Morita equivalent.

**Theorem 4.7** Let \( \mathcal{N}_1, \mathcal{N}_2, \mathcal{M}_1, \mathcal{M}_2 \) be nests and \( A = \operatorname{Alg}(\mathcal{N}_1), B = \operatorname{Alg}(\mathcal{N}_2) \). Let also \( \phi : \mathcal{N}_1 \rightarrow \mathcal{M}_1, \psi : \mathcal{N}_2 \rightarrow \mathcal{M}_2 \) be continuous onto nest homomorphisms and \( E = \operatorname{Op}(\phi), F = \operatorname{Op}(\psi) \). If \( \chi : \mathcal{N}_1 \rightarrow \mathcal{N}_2 \) is a nest isomorphism which can be extended to a \( \ast \)-isomorphism \( \chi' : \mathcal{N}'_1 \rightarrow \mathcal{N}'_2 \) such that \( E \cong \operatorname{Op}(\psi \circ \chi) \) as \( A \)-right \( \ast \)-rigged modules, then \( E \) and \( F \) are strongly projectively Morita equivalent.

**Proof** Let \( \chi : \mathcal{N}_1 \rightarrow \mathcal{N}_2 \) be a nest isomorphism which can be extended to a \( \ast \)-isomorphism \( \chi' : \mathcal{N}'_1 \rightarrow \mathcal{N}'_2 \) such that \( E \cong \operatorname{Op}(\psi \circ \chi) \) as \( A \)-right \( \ast \)-rigged modules. By the fact that \( (\chi')^{-1} : \mathcal{N}'_2 \rightarrow \mathcal{N}'_1 \) is a \( \ast \)-isomorphism such that \( (\chi')^{-1}(\mathcal{N}_2) = \mathcal{N}_1 \), we have that the space \( Y = \operatorname{Op}(\chi^{-1}) \) is a concrete strongly projectively right \( \ast \)-rigged module over \( B \), see Corollary 2.18. Then \( \tilde{Y} \cong \operatorname{Op}(\chi) \) (Remark 2.15) and \( Y \) is a bimodule of \( \ast \)-Morita equivalence between the algebras \( A \) and \( B \). By Lemma 4.3, we have that \( \operatorname{Op}(\psi \circ \chi) = \left[ \operatorname{Op}(\psi) \operatorname{Op}(\chi) \right]^{\ast \ast} \) and from Lemma 4.4 we deduce that

\[ E \cong \operatorname{Op}(\psi \circ \chi) = \left[ \operatorname{Op}(\psi) \operatorname{Op}(\chi) \right]^{\ast \ast} \cong \operatorname{Op}(\psi) \otimes_{B}^{\ast \ast} \operatorname{Op}(\chi) \cong F \otimes_{B}^{\ast \ast} \tilde{Y}. \]

Therefore, \( E \) and \( F \) are strongly projectively Morita equivalent.

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References

1. Blecher, D.P., Le Merdy, C.: Operator Algebras and Their Modules—An Operator Space Approach. London Mathematical Society Monographs. New Series, vol. 30. The Clarendon Press, Oxford (2004)
2. Blecher, D.P., Kashyap, U.: Morita equivalence of dual operator algebras. J. Pure Appl. Algebra 212(11), 2401–2412 (2008)
3. Blecher, D.P., Kashyap, U.: A Morita theorem for dual operator algebras. J. Funct. Anal. 256(11), 3545–3567 (2009)
4. Blecher, D.P., Kraus, J.E.: On a generalization of $W^*$ modules. Banach Cent. Publ. 91, 77–86 (2010)
5. Blecher, D.P., Kashyap, U.: A characterization and a generalization of $W^*$-modules. Trans. Am. Math. Soc. 363(1), 345–363 (2011)
6. Blecher, D.P., Kashyap, U.: Rigged modules I: modules over dual operator algebras and the Picard group. J. Pure Appl. Algebra 221(11), 2827–2837 (2017)
7. Davidson, K.R.: Nest Algebras: Triangular Forms for Operator Algebras on Hilbert Space. Longman Scientific & Technical, Harlow (1988)
8. Eleftherakis, G.K.: TRO equivalent algebras. Houston J. Math. 38(1), 153–176 (2012)
9. Eleftherakis, G.K.: A Morita type equivalence for dual operator algebras. J. Pure Appl. Algebra 212(5), 1060–1071 (2008)
10. Eleftherakis, G.K.: Morita type equivalences and reflexive algebras. J. Oper. Theory 64(1), 3–17 (2010)
11. Eleftherakis, G.K.: Applications of operator space theory to nest algebra bimodules. Integral Equ. Oper. Theory 72(4), 577–595 (2012)
12. Eleftherakis, G.K.: Morita equivalence of nest algebras. Math. Scand. 113(1), 83–107 (2013)
13. Eleftherakis, G.K., Paulsen, V.I.: Stably isomorphic dual operator algebras. Math. Ann. 341(1), 99–112 (2008)
14. Eleftherakis, G.K., Paulsen, V.I., Todorov, I.G.: Stable isomorphism of dual operator spaces. J. Funct. Anal. 258(1), 260–278 (2010)
15. Paschke, W.L.: Inner product modules over $B^*$-algebras. Trans. Am. Math. Soc. 182, 443–468 (1973)
16. Rieffel, M.A.: Morita equivalence for operator algebras. Operator Algebras and Applications, Part I (Kingston, ON, Canada, 1980), Proc. Sympos. Pure Math. 38, Amer. Math. Soc.,pp. 285–298. Providence, Rhode island (1982)