On non-singular inhomogeneous cosmological models

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In this talk we would like to review recent results on non-singular cosmological models. It has been recently shown that among stiff perfect fluid inhomogeneous spacetimes the absence of singularities is more common than it was expected in the literature. We would like to generalize these results and apply them to other matter sources.

1 Introduction

The interest for non-singular cosmologies has risen since it was shown that among inhomogeneous stiff fluid spacetimes there was a regular family depending on two nearly arbitrary functions [1].

Since the publication of the first known non-singular perfect fluid cosmological model by Senovilla [2], the number of regular spacetimes had not increased a lot. Senovilla and Ruiz enlarged the family [3] two years afterwards. All of them had the equation of state of an incoherent radiation fluid. The other regular models that were found either had no equation of state [4] or were for stiff fluids [5, 6, 7, 8].

As it has been said, the role of the stiff fluid equation of state in the absence of singularities is well known, since the gradient of pressure prevents the formation of singularities. In fact, it is a limiting case between causal and non-causal linear equations of state, since the velocity of sound for the stiff fluid is equal to the velocity of light. On the contrary, dust models, with vanishing pressure, are all singular.

Therefore, Senovilla’s original solution is intriguing for the fact that it contains incoherent radiation as a matter content, whereas every other barotropic regular model has been found for a stiff equation of state. Furthermore, in the generalization of the model by Senovilla and Ruiz incoherent radiation spacetimes are the only ones that are non-singular and they trace the separatrix between models which have Ricci and Weyl singularities.

In this talk we pursue to analyze the role of radiation in the absence of singularities in inhomogeneous cosmological models. To this aim we shall consider pure radiation cosmological models in the framework of orthogonally-transitive, mutually orthogonal Abelian $G_2$ spacetimes, where most regular perfect fluid models have been found so far. There are spherically symmetric non-singular models [9], but the matter content is an anisotropic fluid.

In the next section we shall derive the equations for pure radiation inhomogeneous cosmological models and in the last section we shall analyze the geodesic completeness of such models. The results will be summarized in a conclusions section.
2 Pure radiation inhomogeneous models

We start by defining the matter content of the models. The energy-momentum tensor for pure radiation,

$$ T^\mu{}_{\nu} = \Phi k^\mu k^\nu, $$

is written in terms of a null vector field describing the 4-velocity of the radiation field $k$ and a positive function $\Phi$. It is tempting to identify $\Phi$ with the energy density of the radiation field, but it must be noticed that it is not unambiguously defined, since it may be altered by rescaling the vector field $k$.

In full generality, conservation laws for this energy-momentum,

$$ T^\mu{}_{\nu} = \Phi k^\nu k^\mu + \Phi a^\mu + \Phi \Theta k^\mu = 0, $$

are written in terms of the expansion $\Theta = k^\nu$ and the acceleration, $a^\mu = k^\nu k^\mu$ of the vector field $k$.

If $k$ were timelike instead of null, we could split the equation in two components, parallel and orthogonal to $k$. But in the null case there is a single component if $\Phi$ is non-zero.

As we have already stated, we require that the spacetimes allow an orthogonally-transitive, mutually orthogonal $G_2$ group of isometries. The metric,

$$ ds^2 = e^{2K}(-dt^2 + dx^2) + \rho^2 e^{2U} dy^2 + e^{-2U} dz^2, $$

is written in coordinates $y$, $z$, adapted to the isometry generators, which do not appear in the metric functions $\rho$, $K$, $U$, and two isotropic coordinates $t$ and $x$. The function $K$ is just a conformal factor and $\rho$ is the surface element of the transitivity surfaces. It is much related to the absence of singularities, since all regular models have a spacelike gradient of $\rho$. The coordinates have the usual ranges.

We impose that $k$ has just components on $t$ and $x$,

$$ k = e^{-K} (\partial_t + \partial_x), $$

for an outgoing radiation flux. For ingoing radiation we have just to change the plus sign for a minus in the sum.

In these coordinates the conservation equation can be integrated,

$$ (\partial_t + \partial_x) \left\{ \Phi e^{2K} \rho \right\} = 0 \Rightarrow \Phi e^{2K} \rho = h(t - x), $$

in terms of one function $h$ of the outgoing null coordinate $t - x$.

The remaining Einstein equations in these coordinates,

$$ U_{tt} - U_{xx} + \frac{1}{\rho}(U_t \rho_t - U_x \rho_x) = 0, $$

$$ \rho_{tt} - \rho_{xx} = 0, $$

$$ K_t \rho_x + K_x \rho_t = \rho_{tx} + U_t \rho_t + U_x \rho_x + 2 \rho U_t U_x - h, $$

$$ K_t \rho_t + K_x \rho_x = \frac{\rho_{tt} + \rho_{xx}}{2} + U_t \rho_t + U_x \rho_x + \rho (U_t^2 + U_x^2) + h, $$

may be reduced to a plane-wave equation for $U$ and a wave equation for $\rho$. The equations for $K$ are a quadrature. The integrability condition for these equations is a consequence of the other equations and therefore they can be integrated at the end.

If we take $\rho$ as a coordinate, the system is further simplified. There are three possibilities:
If $\rho$ is a null coordinate, $\rho = \rho(t - x)$, $U = U(t - x)$ is also null and the only equation is a wave equation,

$$K_{tt} - K_{xx} = 0 \Rightarrow K(u, v) = f(u) + g(v), \quad (7)$$

which may integrated in terms of null coordinates, $u = t - x$, $v = t + x$. Just $f$ is constrained by the quadrature,

$$f' = \frac{\rho''}{2\rho'} + U' + \frac{\rho U'^2}{\rho'} + \frac{h}{2\rho'}, \quad (8)$$

where the prima denotes derivation with respect to $u$.

After eliminating redundant functions by defining $\tilde{u} = \int du e^{2f}$, $\tilde{v} = \int dv e^{2g}$, the metric is written in a simple form,

$$ds^2 = -2dudv + F^2dy^2 + G^2dz^2, \quad (9)$$

in terms of two arbitrary functions, $F$ and $G$, of $u$. We have dropped the tilde for simplicity.

The function $\Phi$ must be positive,

$$\Phi = -\left(\frac{F''}{F} + \frac{G''}{G}\right) \geq 0. \quad (10)$$

These models have a five-dimensional group of isometries acting on null orbits. The vector field $\partial_\rho$ is covariantly constant and these spacetimes contain plane-fronted gravitational waves with parallel rays.

If $\rho$ is a spacelike coordinate, we choose $\rho = x$,

$$U_{tt} - U_{xx} - \frac{U_x}{x} = 0, \quad (11a)$$

$$K_t = U_t + 2xU_tU_x - h, \quad (11b)$$

$$K_x = U_x + x(U_t^2 + U_x^2) + h, \quad (11c)$$

and the equations are the same as for vacuum spacetimes except for the function $h$, which appears in the metric as a conformal factor $-H(t - x)$ added to $K$ [10].

Since according to equation 5 the function $h(t - x)$ must be positive for positive $x$ and negative for negative $x$ we have to interpret $x$ as a radial coordinate ranging from zero to infinity, although the axis is not flat. Therefore $H_t = h$ must be positive and $H$ is a growing function.

Finally, if $\rho$ is a timelike coordinate, we take $\rho = t$,

$$U_{tt} - U_{xx} + \frac{U_t}{t} = 0, \quad (12a)$$

$$K_x = U_x + 2tU_tU_x - h, \quad (12b)$$

$$K_t = U_t + t(U_t^2 + U_x^2) + h, \quad (12c)$$

and the results are quite similar to the spacelike case. We shall not develop this case since it is singular at $t = 0$.

3 Geodesic completeness

There are at least three different definitions for singularities [11]. One of them states that a spacetime is singular if any of the polynomial curvature scalars blow up at an event. This
definition has the drawback that singular events may be hidden by conveniently removing an open set around them.

Therefore a more reasonable definition would be considering causal observers on trajectories on the spacetime. For instance, restricting to observers in free fall, which move along timelike and null geodesics, a singularity is encountered if their proper time cannot be extended from minus infinity to infinity. This would mean that the observer has left the spacetime in a finite proper time. Spacetimes which do not suffer this inconvenience are called causally geodesically complete. We shall adopt this definition for regular spacetimes.

There is another definition of singularity which refers to general causal observers, not necessary in free fall and therefore not following causal geodesics.

In order to analyze the regularity of our spacetimes we are led to study their causal geodesics.

A geodesic of 4-velocity $u$, $$u^\mu = \dot{x}^\mu = \frac{dx^\mu(\tau)}{d\tau},$$ where $\tau$ is the proper time, satisfies the geodesic equations written in the form,

$$a^\mu = \ddot{x}^\mu + \Gamma^\mu_{\nu\rho} \dot{x}^\nu \dot{x}^\rho = 0, \quad -\infty < \tau < \infty,$$

in terms of the Christoffel symbols for the metric of the spacetime.

Since these models have isometries, the geodesic equations become simpler, because there are conserved quantities of geodesic motion assigned to them. If $\xi$ is a generator of an isometry, the quantity $X$,

$$X := g_{\mu\nu} \xi^\mu u^\nu,$$

is conserved along geodesics of 4-velocity $u$,

$$D_\sigma X = X_{,\sigma} u^\sigma = \left( g_{\mu\nu} \xi^\mu u^\nu + g_{\mu\nu} \xi^\mu u^\nu + g_{\mu\nu} \xi^\mu u^\nu \right) u^\sigma = 0,$$

since the metric $g$ is covariantly constant, the acceleration $a^\nu = u^\nu_{,\sigma} u^\sigma$ of the geodesics is zero and the generator satisfies the Killing equation,

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0.$$

Besides, the character of the geodesic is preserved,

$$\delta := -g_{\mu\nu} u^\nu u^\rho,$$

and equal to zero for null geodesics and to one for timelike geodesics. In fact, geodesic equations merely state this fact.

All these conserved quantities will be useful.

We have classified the spacetimes by the character of the coordinate function $\rho$:

- The function $\rho$ is a null coordinate: We have seen that the vector fields $\{\partial_\nu, \partial_\eta, \partial_\zeta\}$ are generators of isometries and therefore provide constants of geodesic motion,

$$V = \langle \partial_\nu, u \rangle = g_{\nu\mu} u^\mu = -\dot{u},$$

$$Y = \langle \partial_\eta, u \rangle = g_{\eta\mu} u^\mu = F^2 \dot{y},$$

$$Z = \langle \partial_\zeta, u \rangle = g_{\zeta\mu} u^\mu = G^2 \dot{z}.$$


All these equations are integrable provided that $F^{-2}$ and $G^{-2}$ are integrable functions. We are left just with one equation for $\dot{v}$, which we may derive directly from the expression for $\delta$, without using the Christoffel symbols,

$$\frac{dv}{du} = \frac{1}{2V^2} \{ \delta + Y^2 F^{-2} + Z^2 G^{-2} \}. \tag{22}$$

These results are comprised in a simple theorem:

**Theorem 1.** A spacetime endowed with a metric (9) and with a pure radiation field as matter content is causally geodesically complete if and only if the metric components $g^{\mu
u}$ and $g^{\nu
u}$ are integrable functions of the null coordinate $u$.

However, as it is explained in [12], the positiveness requirement for the function $\Phi$ implies that the metric functions are continuous but not differentiable.

- The function $\rho$ is a spacelike coordinate: In this case there are in general no additional isometries. However, there are theorems specifically derived for this case [13];

**Theorem 2.** A diagonal Abelian orthogonally transitive spacetime with spacelike orbits endowed with a metric in the form (3) with $C^2$ metric functions $K, U, \rho$, where $\rho$ has a spacelike gradient, is future causally geodesically complete provided that along causal geodesics:

1. For large values of $t$ and increasing $x$,
   
   a) $(K - U - \ln b)_t + (K - U - \ln \rho)_x \geq 0$, and either $(K - U - \ln \rho)_x \geq 0$ or $|(K - U - \ln \rho)_t + (K - U - \ln \rho)_x| \lesssim (K - U - \ln \rho)_t + (K - U - \ln \rho)_x$.
   
   b) $K_t + K_x \geq 0$, and either $K_x \geq 0$ or $|K_x| \lesssim K_t + K_x$.
   
   c) $(K + U)_t + (K + U)_x \geq 0$, and either $(K + U)_x \geq 0$ or $|(K + U)_t| \lesssim (K + U)_t + (K + U)_x$.

2. For large values of $t$, a constant $b$ exists such that
   
   $\frac{K(t, x) - U(t, x)}{2K(t, x)} \geq -\ln |t| + b.
   
   For geodesics pointing to the past the theorem imposes the same conditions but reversing the sign in time derivatives.

This result imposes certain restrictions on the metric functions:

**Theorem 3.** A diagonal Abelian orthogonally-transitive spacetime with spacelike orbits with a metric in the form (3) and a spacelike $\rho$ and pure outgoing radiation as matter content is causally geodesically complete if:

1. $x^{2-\varepsilon}|U_x \pm U_t| \not\rightarrow 0$ for large values of $|t|$ and $x$.
2. $H(t) \leq \ln |t| + b$ and $U(t, 0) \geq b - \frac{1}{2} \ln |t| + H(t)$ for large values of $t$.
3. $U(t, 0) \geq b - \frac{1}{4} \ln |t| + H(t)/2$ for small values of $t$.

This result has obvious consequences. The amount of energy that can be radiated, which may be measured by $H$, must be bounded in order to prevent the formation of singularities. For ingoing radiation the result is very similar.

**Theorem 4.** A diagonal Abelian orthogonally transitive spacetime with spacelike orbits with a metric in the form (3) and a spacelike $\rho$ and pure ingoing radiation as matter content is causally geodesically complete if:

1. $x^{2-\varepsilon}|U_x \pm U_t| \not\rightarrow 0$ for large values of $|t|$ and $x$.
2. $U(t, 0) \geq b - \frac{1}{2} \ln |t| - H(t)/2$ for large values of $t$.
3. $H(t) \geq -\ln |t| + b$ and $U(t, 0) \geq b - \frac{1}{4} \ln |t| - H(t)$ for small values of $t$.

Therefore we may derive regular radiation cosmological models with a function $U$ which grows for large values of $|t|$ and $x$ and a function $H$ that does not grow or decrease too quickly.
4 Conclusions

We have derived sufficient conditions for an inhomogeneous spacetime filled with pure radiation to be singularity-free in the sense of causal geodesic completeness. The conditions are fairly easy to implement in terms of a growing function $H$ and a solution to the plane-wave equation with very few restrictions. This is much the same as it happened for stiff perfect fluids [14]. Regular models are pretty general among radiation spacetimes.

Pure radiation fields with positive energy density fulfill weak, strong and dominant energy conditions. Since these models have a function $t$ with timelike gradient everywhere, they are causally stable [11] and hence they satisfy every weaker causality condition. For instance, they do not contain closed causal curves. The only possibility to circumvent the powerful singularity theorems is the absence of closed trapped surfaces.

It has already been shown that singularity-free models are common among stiff perfect fluids and pure radiation inhomogeneous spacetimes. It remains to be seen if these results can be extended to other matter contents and symmetries.

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References

1. L. Fernández-Jambrina, L.M. González-Romero, Phys. Rev. D66 (2002) 024027 [arxiv: gr-qc/0402119].
2. J.M.M. Senovilla, Phys. Rev. Lett. 64 (1990) 2219.
   F.J. Chinea, L. Fernández-Jambrina, J.M.M. Senovilla, Phys. Rev D45 (1992) 481.
3. E. Ruiz, J.M.M. Senovilla, Phys. Rev D45 (1992) 1995.
4. M. Mars, Class. Quantum Grav. 12 (1995) 2831.
5. N. Dadhich: “On the uniqueness of the singularity free family of inhomogeneous cosmological models”. In: Proceedings of the Spanish Relativity Meeting on Inhomogeneous Cosmological Models, ed. by A. Molina, J.M.M. Senovilla, Singapore 1995.
6. M. Mars, Phys. Rev D51 (1995) R3989.
7. J.B. Griffiths, J. Bičák, Class. Quantum Grav. 12 (1995) L81.
8. L. Fernández-Jambrina, Class. Quantum Grav. 14 (1997) 3407 [arxiv: gr-qc/0404017].
9. J.M.M. Senovilla, Gen. Rel. Grav. 30 (1998) 701.
10. J. Krishna Rao, Proc. Nat. Inst. Sci. India A 30 (1964) 439.
11. S.W. Hawking, G.F.R. Ellis, The large scale structure of space-time, Cambridge University Press, Cambridge 1973.
12. L. Fernández-Jambrina, L.M. González-Romero, Mod. Phys. Lett. A19 (2004) 583. [arxiv: gr-qc/0402124]
13. L. Fernández-Jambrina, L.M. González-Romero, Class. Quantum Grav. 16 (1999) 953 [arxiv: gr-qc/9812039].
   L. Fernández-Jambrina, Journ. Math. Phys. 40 (1999) 4028 [arxiv: gr-qc/9906030].
14. L. Fernández-Jambrina, L.M. González-Romero, Journ. Math. Phys. 45 (2004) 2113 [arXiv: gr-qc/0405013].