Simple homotopy type of the Hamiltonian Floer complex

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Abstract

For an aspherical symplectic manifold, closed or with convex contact boundary, and with vanishing first Chern class, a Floer chain complex is defined for Hamiltonians linear at infinity with coefficients in the group ring of the fundamental group. For two non-degenerate Hamiltonians of the same slope continuation maps are shown to be simple homotopy equivalences. As a corollary the number of contractible Hamiltonian orbits of period 1 can be bounded from below.

1 Introduction

Let \((M,\omega)\) be a symplectic manifold, closed or the completion of a compact manifold \(\hat{M}\) with convex boundary (see e.g. [6]). Assume that \(\omega|_{\pi_2(M)}\) and \(c_1(M)|_{\pi_2(M)}\) vanish. For a non-degenerate Hamiltonian \(H: \mathbb{R}/\mathbb{Z} \times M \to \mathbb{R}\), linear at infinity if \(M\) is open, and an almost complex structure satisfying a regularity assumption, let \(CF^*(H)\) denote the Hamiltonian Floer complex with coefficients twisted by the fundamental group of \(M\), reviewed in Section 3.

Theorem 1.1. Let \((G, J_G)\) and \((H, J_H)\) be regular pairs where the Hamiltonians \(G, H\) are linear at infinity with the same slope. Then a continuation map

\[
CF^*(G) \to CF^*(H),
\]

induced by any regular homotopy from \(G\) to \(H\) and from \(J_G\) to \(J_H\), is a simple homotopy equivalence.

Simple homotopy equivalence is an improvement compared to Floer’s result of homotopy equivalence [8]. Simple homotopy and the related notion of torsion has been studied for Floer complexes for instance in [16] [12] [11] [15] [1]. The complex \(CF^*(H)\) also appears in [14]. In particular for the case of closed \(M\), Theorem 1.1 and Corollary 1.2 appears in [2], which is based on [16]. The latter performs bifurcation analysis using a geometric stabilization while here we use action-energy arguments.

Theorem 1.1 is proved in Section 3 as Proposition 3.3.
The complex $CF^*(H)$ is generated by contractible 1-periodic orbits of the Hamiltonian flow of $H$. Thus their number is constrained by the minimum rank among complexes with the simple homotopy type of $CF^*(H)$. If the slope of $H$ is small, this is the simple homotopy type of a Morse complex on $\hat{M}$, since for $H$ a small Morse function with small slope Floer’s identification \cite{floer} of Floer and Morse complexes holds.

Damian \cite{damian} shows that any complex which is simple homotopic to a Morse complex on $\hat{M}$ may be realized as the Morse complex of a stable Morse function on $\hat{M}$: a Morse function $\hat{M} \times \mathbb{R}^k \rightarrow \mathbb{R}$ which is a compact perturbation of a Morse function on $\hat{M}$ plus a non-degenerate quadratic form on $\mathbb{R}^k$. See Theorem \ref{thm2.2}. The stable Morse number of $\hat{M}$ is the minimum number of critical points of such functions. Thus

\textbf{Corollary 1.2.} Suppose that $H : \mathbb{R}/\mathbb{Z} \times M \rightarrow \mathbb{R}$ is non-degenerate and linear at infinity with sufficiently small slope. Then the number of contractible 1-periodic orbits of the Hamiltonian vector field of $H$ is at least equal to the stable Morse number of $\hat{M}$.

The Arnold conjecture gives the Morse number of $\hat{M}$ as a lower bound (the minimum number of critical points among Morse functions respecting the boundary). Clearly this would be better than Corollary 1.2, and indeed there are examples due to Damian \cite{damian} where the Morse and stable Morse numbers differ. These examples depend only on the fundamental group; by work of Gompf \cite{gompf} any finitely presented group may be realized as the fundamental group of a symplectic manifold.

Examples of closed symplectic manifolds with $\pi_2 = 0$, hence satisfying the assumptions of Theorem 1.1, are given in \cite{seidel}, and there exists also examples of Gompf \cite{gompf} with $\pi_2 \neq 0$ but $\omega|\pi_2 = c_1|\pi_2 = 0$.

Here we note that the stable Morse number can be strictly larger than any bound coming from the cohomology of $M$. Firstly, base change using the augmentation $\varepsilon : \Lambda = \mathbb{Z}[\pi_1(M)] \rightarrow \mathbb{Z}$ which map all group elements to 1 sends the complex $CF^*(H)$ to the Floer complex $CF^*(H;\mathbb{Z})$ with integer coefficients, which for small slopes computes the cohomology of $M$ shifted down by $n = \frac{1}{2}\dim M$. Secondly, any complex in the simple homotopy type of $CF^*(H)$ has a module in degree $1 - n$ of rank at least equal to the minimal number $\delta$ of generators of the $\Lambda$-module $\ker \varepsilon$. Hence if for instance the fundamental group of $M$ is nontrivial, finite, and perfect ($\pi_1 = [\pi_1, \pi_1]$) then $\delta \geq 2$ \cite{damian}, and hence there must be at least two generators in degree $1 - n$ not seen in cohomology. For example we may find a Weinstein domain with 1- and 2-handles \cite{kuranishi} attached according to a presentation of the desired group.

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2 Simple-homotopic chain complexes and stable Morse numbers

General references for simple-homotopy theory are \cite{morse} or \cite{ganea}, but here the presentation is closer to \cite{ganea} and \cite{morse}.
Let $\Lambda$ be the ring $\mathbb{Z}[\pi_1(\hat{M})]$, and let $C^*$ be a free and finitely generated complex over $\Lambda$, with a basis $\underline{c}$ which is a union of bases $\underline{c}_i$ of $C^i$. The simple homotopy type of $(C^*, \underline{c})$ is the class of such based complexes under the equivalence relation generated by basis-preserving isomorphism and the following moves.

1. A trivial summand $0 \rightarrow \Lambda^r \xrightarrow{\text{id}} \Lambda^r \rightarrow 0$, with preferred basis at both positions the standard basis of $\Lambda^r$, can be added or removed.

2. The basis $\underline{c}$ can be replaced by a basis $\underline{c}'$ so that the change of basis matrix is either the identity matrix plus one off-diagonal element from $\Lambda$, or the identity matrix where one diagonal element is replaced by $\pm g \in \pm \pi_1(\hat{M})$.

Now let $C^*, D^*$ be free and finitely generated complexes with preferred bases $\underline{c}$ and $\underline{d}$. Write $C^*[1]$ for the shift $(C[1])^k = C^{k+1}$. A chain homotopy equivalence $f: C^* \rightarrow D^*$ is simple if the cone of $f$,

$$C_f = \left( C^*[1] \oplus D^*, d = \begin{pmatrix} -d_C & 0 \\ f & d_D \end{pmatrix} \right),$$

with the basis $\underline{c} \cup \underline{d}$ has the same simple homotopy type as the zero complex. If $f$ is simple, $C^*$ and $D^*$ have the same simple homotopy type [13].

Next suppose that $C^*$ carry an increasing filtration $F^p C^* \subset F^{p+1} C^*$ with associated quotient groups $G^p C^* = F^p C^*/F^{p-1} C^*$, and similarly for $D^*$. The following is essentially Corollary 2.4 in [1].

**Lemma 2.1.** Suppose that each filtration level $F^p C^*, F^p D^*$ is free on a subset of the bases $\underline{c}, \underline{d}$, and suppose that $f: C^* \rightarrow D^*$ is an equivalence respecting the filtration. Moreover suppose that each of the induced maps $f^p: G^p C \rightarrow G^p D$ are simple homotopy equivalences. Then $f$ is a simple homotopy equivalence.

**Proof.** The complex $B^* := C_f$ has a filtration $F^p C_f = (F^p C^*[1]) \oplus F^p D^*$ with associated quotients $G^p B^* = G^p C_f = C_f^p$. Let $k$ be the highest filtration level, so $F^{k-1} B^* \subset F^k B^* = B^*$. The complex $G^k B^*$ may be assumed to reduce to zero by a sequence of moves, which we will lift to reduce $F^k B^*$ to $F^{k-1} B^*$.

Only when removing a trivial summand do we encounter any difficulty. Suppose basis elements $r \in F^k B^*, s \in F^k B^{i+1}$ span, in $G^k B^*$, a trivial summand with $dr = s$. However in $F^k B^*$ we may have that $dr = s + t$, where $t \in F^{k-1} B^*$. We replace the basis element $s \in F^k B^*$ with $s' = s + t \in F^k B^*$. Thus $dr = s'$ and this new basis still satisfies the assumptions. Moreover, let $r^\perp \subset F^k B^i$ be the span of the other basis elements in this degree and similarly $s^\perp \subset F^k B^{i+1}$. Since the differential of $G^k B^*$ takes $r^\perp / F^{k-1} B^i$ to $s^\perp / F^{k-1} B^{i+1}$, we have $dr^\perp \subset s^\perp$, too. Hence $r, s'$ span a trivial summand of $F^k B^*$, which we then may remove.

Thus $F^k B^*$ may be reduced to $F^{k-1} B^*$. Proceeding inductively we can reduce $F^k B^* = C_f$ to the zero complex.  

Let $(f, V)$ be a Morse-Smale pair on $\hat{M}$; if there is boundary, we assume that $V$ points out. By choosing a preimage in the universal cover of $\hat{M}$ for each
critical point and an orientation of the descending manifold, we may construct Morse complex $C^*(f, V)$ over the ring $\Lambda$. Namely if $\gamma$ is a rigid flow-line with $\gamma(-\infty) = x, \gamma(\infty) = y$, we lift $\gamma(\infty)$ to the chosen lift of $y$; the induced lift of $\gamma(-\infty)$ is then the chosen lift of $x$ acted on by some $g \in \pi_1(\tilde{M})$ and $\gamma$ contributes $\pm gx$ to $\partial y$. The simple homotopy type of the complex $C^*(f, V)$ is independent of the pair $(f, V)$.

When $\tilde{M}$ is closed, the arguments of M. Damian [3] apply to show that the minimum rank of complexes in the simple homotopy type of $C^*(f, V)$ is equal to the stable Morse number $\mu_{st}$ of $\tilde{M}$. This is the minimum number of critical points among Morse functions $h: \tilde{M} \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$ which, outside a compact set, agree with the quadratic form $Q(m, x, y) = |x|^2 - |y|^2$. If $W$ is a gradient-like vector field for $h$ which outside a compact set equals the gradient of $Q$ we say that $(h, W)$ is standard at infinity.

For $\tilde{M}$ with boundary, the argument of Damian can be adopted slightly as in Proposition 2.9 in [5] to conclude similarly. Here the stable Morse functions $h$ are required to be of the form $h = Q + f$ outside a compact set in the interior, where $f$ is a function on $\tilde{M}$ having $\partial \tilde{M}$ contained in a regular level set. Similarly a gradient-like vector field $W$ of $h$ is required to point out of along $\partial \tilde{M} \times \mathbb{R}^{2k}$ and outside a compact set be of the form $\nabla Q + V$, $V$ a vector field on $\tilde{M}$.

**Theorem 2.2** ([3], [4]). Let $D^*$ be a free finitely generated based $\Lambda$-complex in the simple homotopy type of $C^*(f, V)$. There is a $k \in \mathbb{N}$, a stable Morse function $h: \tilde{M} \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$, and a gradient-like vector field $W$ such that $(h, W)$ is Morse-Smale and standard at infinity, and such that

$$D^* = C^*(h, W)[k].$$

**Proof.** We merely emphasize the important points of the arguments of [3], [4]. Choose $k$ such that $D^*[-k]$ is supported in the degrees $3, \ldots, \dim \tilde{M} + 2k - 3$. One starts with a pair $(f, V)$ on $\tilde{M}$ and stabilize to $(f + Q, V + \nabla Q)$. Then, Morse moves are performed on the latter to mimic the algebraic manipulations to get from $C^*(f + Q, V + \nabla Q) = C^*(f, V)[-k]$ to $D^*[-k]$. These Morse moves are raising/lowering the value at a critical point, performing handle slides, add a pair of critical points which form a trivial subcomplex, cancel two trajectories which together form a nullhomotopic loop, and cancel a pair of critical points which form a trivial subcomplex and with only one trajectory between them.

To perform these we need a little room. Let $F \subset \tilde{M}$ be the union of all critical points of $f$ and any trajectories between them, and let $U \subset \tilde{M} \times \mathbb{R}^{2k}$ be an open set containing $F \times 0$ and contained in a compact set $K \subset \text{int}(\tilde{M}) \times \mathbb{R}^{2k}$. All moves can be performed in $U$, so that the resulting pair $(h, W)$ equals $(f + Q, V + \nabla Q)$ outside $U$. We may choose $K$ such that any trajectory which exits $K$ in forwards or backwards time never enters it again, instead going to infinity or to the boundary $\partial \tilde{M} \times \mathbb{R}^{2k}$. From these considerations the arguments of Damian [3] produces $(h, W)$ as desired.

\[\square\]
3 Hamiltonian Floer setup and simple homotopy

For a linear at infinity function $H: \mathbb{R}/\mathbb{Z} \times M \rightarrow \mathbb{R}$ we define $X = X_H$, by $\omega(-, X) = dH_t$. The function $H$ is admissible if all 1-periodic orbits of $X$ are non-degenerate, in particular the slope at infinity is not equal to the length of any Reeb orbit. We choose a cylindrical almost complex structure $J = J_t$ compatible with $\omega$ and such that $(H, J)$ is a regular pair.

Let $P$ be the finite set of contractible 1-periodic orbits of $X$. For any filling disc $w_x$ of $x \in P$, we set the action of $x$ as

$$A_H(x) = \int_{D^2} w_x^* \omega - \int_0^1 H(t, x(t)) dt.$$  

which, since $\omega|_{\pi_2} = 0$, is independent of the filling. Grading $x$ by the Conley-Zehnder index $|x|$ is also independent of the filling as $c_1|_{\pi_2} = 0$. The Floer equation

$$\partial_s u + J(u) \partial_t u + \nabla H(u) = 0 \quad (1)$$

represents the positive gradient flow of $A_H$. Fix for each $x \in P$ a lift of $x$ to the cover $\tilde{M}$, and let $CF^*(H)$ be freely generated over $\Lambda$ by $P$ with grading $| \cdot |$. For $x \in P$, the differential $\partial x$ counts rigid solutions of (1) with input $x$ at $s = +\infty$: as in the Morse case lift the curve to $\tilde{M}$ with initial condition the chosen lift of $x$; then as $s \to -\infty$ it limits to a lift of some $y \in P$: we compare with the chosen lift for $y$ and record the curve’s contribution as $\pm g y$ for some $g \in \pi_1(M)$. The sign is determined using the coherent orientations of [9].

A regular path $(H^s, J^s)$, constantly equal to $(H^\pm, J^\pm)$ for $\pm s \gg 0$, induces a continuation map $\Phi: CF^*(H^+) \to CF^*(H^-)$ as long as the slope of $H^s$ is increasing as $s$ decreases. Whenever the slope is constant, or if $M$ is compact, $\Phi$ is a chain equivalence.

In the latter case $\Phi$ is a simple chain equivalence, which we will argue in this section. We may assume that the path $(H^s, J^s)$ is such that there is a dense set of $s' \in \mathbb{R}$ where the pairs $(H^s, J^{s'})$ are regular, and hence the complexes $CF^*(H^s)$ are defined; moreover we may assume that for all $s$ the sets $P(H^s)$ are finite. If $s' < s''$ are sufficiently close we will show that a continuation map $CF^*(H^{s''}) \to CF^*(H^{s'})$, induced by the path $H^s$, is a simple equivalence. The map $\Phi$ is homotopic to a composition of such maps and hence also simple [2].

Let us note some properties of continuation maps like $\Phi$. If the $y$-coefficient of $\Phi(x)$ is nonzero, there is a curve $u$ which solves (1) with $J^s, H^s$ in place of $J, H$, and where $u(s, \cdot)$ has limits $x$ respectively $y$ as $s \to \infty$ or $-\infty$. The energy

$$E(u) = \frac{1}{2} \int_{\mathbb{R}} \int_0^1 |\partial_s u|^2 + |\partial_t u - X(u)|^2 dt ds \geq 0$$

of such a curve satisfies

$$E(u) = A_{H^+}(x) - A_{H^-}(y) + \int_{\mathbb{R}} \int_0^1 (\partial_s H^s)(u(s, t)) dt ds. \quad (2)$$
Hence if $\partial_s H^s$ is small, the action of $y$ is at most slightly greater than the action of $x$. Now fix $s_0 \in \mathbb{R}$. We may assume that $H^s$ is non-degenerate for all $s$ near $s_0$ except possibly for $H^{s_0}$ [3]. Let $b_1 < \cdots < b_k$ be the action values of $\mathcal{P}(H^{s_0})$, and pick $d > 0$ less than each $(b_{i+1} - b_i)/4$. Consider $s_\pm$ close to $s_0$ with $s_- < s_0 < s_+$ such that the complexes $\mathcal{C}F^*(H^{s_\pm})$ are defined. If they are sufficiently close, the action values of $\mathcal{P}(H^{s_\pm})$ lie in a $d$-neighbourhood of $\{b_1, \ldots, b_k\}$, and

$$
\int_{s_-}^{s_+} \int_{S^1} \max_{x \in M} |\partial_s H^s(x)| dt ds < d.
$$

For such $s_\pm$ we filter the complexes $\mathcal{C}F^*(H^{s_\pm})$ by setting $F^p \mathcal{C}F^*(H^{s_\pm})$ to be generated by those orbits in $\mathcal{P}(H^{s_\pm})$ of action $\leq (b_p + b_{p+1})/2$; $F^0$ is the zero complex, and $F^k$ is the full complex. Now, let $\eta(s)$ denote a function increasing from $s_-$ to $s_+$. From the path $(H^s, J^s)$ we get a path $(H^\eta, J^\eta)$ which induces a continuation map $\varphi: \mathcal{C}F^*(H^\eta^+) \to \mathcal{C}F^*(H^\eta^-)$. Due to the choice of $d$, $\varphi$ preserves the filtration and induces maps of associated groups

$$
\varphi^p: G^p(H^\eta^+) = F^p \mathcal{C}F^*(H^\eta^+)/F^{p-1} \mathcal{C}F^*(H^\eta^+) \to G^p(H^\eta^-)
$$

which are also equivalences. From Lemma 2.1 we need only show that each $\varphi^p$ is simple.

Now each $x \in \mathcal{P}(H^\eta^+)$ tend to a unique $z \in \mathcal{P}(H^{s_0})$ as $s_\pm \to s_0$; let $B^\pm_z$ be the collection of those tending to $z$, and which therefore have action close to $A_{H^{s_0}}(z)$. Thus we have a splitting of modules

$$
G^p(H^\eta^+) \cong \bigoplus_{z \in \mathcal{P}(H^{s_0}); A_{H^{s_0}}(z) = b_p} \Lambda B^\pm_z. \tag{3}
$$

A key point is to show that this is a splitting of subcomplexes which the continuation maps preserve, compare [1].

It is convenient to modify the chosen lifts for elements of $B^\pm_z$; namely, after choosing a lift of $z$ we prefer to choose the lift of $x \in B^\pm_z$ which is close to the lift of $z$. In the chain complex this is a change of basis $x \to gx$ which does not affect whether an equivalence is simple. This is assumed for the next proposition.

**Proposition 3.1.** There is $\delta > 0$ such that if $|s_\pm - s_0| < \delta$, the splitting (3) is a splitting of subcomplexes, and $\varphi^p(B^\pm_z) \subset \Lambda B^\pm_z$ for each $z$. Furthermore with respect to the preferred basis, the matrices of the restrictions $\partial^\pm: \Lambda B^\pm_z \to \Lambda B^\pm_z$, $\varphi^p: \Lambda B^\pm_z \to \Lambda B^\pm_z$ have integer coefficients.

**Remark 3.2.** If the pairs $(H^\pm, J^\pm)$ or the path $(H^n, J^n)$ are non-regular, the conclusion still holds for regular pairs or paths sufficiently close.

The proof follows Lemma 3.3. This gives us

**Proposition 3.3.** The continuation map $\Phi: \mathcal{C}F^*(H^+) \to \mathcal{C}F^*(H^-)$ is a simple homotopy equivalence.
Proof. The $s$-dependence of $(H^*, J^*)$ is a compact set $A \subset \mathbb{R}$. We can choose finitely many $s_i$ and an $\delta_{\text{min}} > 0$ such that the intervals $(s_i - \delta_{\text{min}}, s_i + \delta_{\text{min}})$ cover $A$ and Proposition 3.1 applies whenever $\delta < \delta_{\text{min}}$.

Choose $r_i \in (s_{i+1} - \delta_{\text{min}}, s_i + \delta_{\text{min}})$. By Remark 5.2, we may assume that $(H^{r_i}, J^{r_i})$ respectively the induced paths $(H^{r_i}, J^{r_i})$ are regular, and there are thus equivalences

$$\varphi_i: CF^*(H^{r_i+1}) \rightarrow CF^*(H^{r_i}).$$

These are simple: from Proposition 3.1 we may regard the maps $\varphi_i$ of associated groups as maps of $\mathbb{Z}$-complexes with extended scalars; maps of $\mathbb{Z}$-complexes are always simple and the same holds for extensions of simple maps [2]. Thus by Lemma 2.1, so are the $\varphi_i$. Hence there is a map

$$\Phi': CF^*(H^+) \rightarrow CF^*(H^-)$$

which is formed by the composition of the simple $\varphi_i$. Being a composition of simple equivalences, $\Phi'$ is simple, too [2]. Finally note that $\Phi$ is homotopic to $\Phi'$ and thus also simple [2].

It remains to prove Proposition 3.1. The situation is as follows. We have the path $(H^*, J^*)$ and the chosen value $s_0$. Let $\sigma(s, b), b \in [0, 1]$, be a smooth homotopy from the constant function $s \mapsto s_0$ to a function increasing from $s_0 - 1$ to $s_0 + 1$ such that

$$\sigma(s_0, b) = s_0 \text{ for all } b$$
$$\sigma(s, b) = s_0 - b \text{ for } s < s_0 - 1 \text{ and all } b$$
$$\sigma(s, b) = s_0 + b \text{ for } s > s_0 + 1 \text{ and all } b$$
$$\partial_s \sigma \geq 0.$$

Consider the family $H^{\sigma(s,b)}$, suppressing the almost complex structure $J^{\sigma(s,b)}$

Let $\delta > 0$ be so small that if $\gamma$ is an orbit of $X^s$ with $|s - s_0| < \delta$ it makes sense to say that $\gamma$ lies close to some unique orbit of $X^{s_0}$. Let $\mathcal{P}(s) = \mathcal{P}(H^s)$. We will use the following notation for moduli spaces: $\mathcal{M}(gx, y; H)$ contains curves contributing $\pm gx$ to $\partial y$, whenever the dimension is correct, and similarly the spaces $\mathcal{M}_s(gx, y; H^s)$ defines the image of $y$ under the continuation map.

Lemma 3.4. Let $H^s$, $J^s$ be a given admissible path, fix $s_0 \in \mathbb{R}$, and let $\delta$ be as above. There is a $\beta \in (0, \delta)$, such that for any $b \in (0, \beta)$, any $x, y \in \mathcal{P}(s_0)$ with $\mathcal{A}_s(x) = \mathcal{A}_s(y)$, and any $x'$ close to $x$, $y'$ close to $y$, the following spaces are empty.

1. For $x \neq y$, $x' \in \mathcal{P}(s_0 - b), y' \in \mathcal{P}(s_0 + b), \mathcal{M}_s(gx', y'; H^{\sigma(s,b)}) = \emptyset$.
2. For $x \neq y$, $x', y' \in \mathcal{P}(s_0 \pm b), \mathcal{M}(gx', y'; H^{s_0 \pm b}) = \emptyset$.
3. For $x = y$, $x' \in \mathcal{P}(s_0 - b), y' \in \mathcal{P}(s_0 + b), \mathcal{M}_s(gx', y'; H^{\sigma(s,b)}) = \emptyset$, unless $g = 1$.
4. For $x = y$, $x', y' \in \mathcal{P}(s_0 \pm b), \mathcal{M}(gx', y'; H^{s_0 \pm b}) = \emptyset$, unless $g = 1$. 

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In the latter two cases the lifts of \( x', y' \) are chosen such that the lifts of \( x'(0), y'(0) \), and \( x(0) \) lie in the same sheet above \( x(0) \).

**Proof.** Consider part 1. Let \( x, y \in \mathcal{P}(s_0) \) be different and fix a small ball \( B \) at \( x(0) \) such that the closure of \( B \) does not contain \( z(0) \) for any \( z \in \mathcal{P}(s_0) \setminus x \). Consider a sequence \( b_n \to 0 \) and corresponding maps \( u_n \in \mathcal{M}_s(x_n, y_n; H^{s,b_n}) \), where the ends are close to \( x \) respectively \( y \) as in the Lemma. Thus \( E(u_n) \to 0 \).

Let \( s_n \) be the smallest number for which \( u_n(s_n, 0) \) hits \( \partial B \). The shifted sequence \( w_n = u_n(\cdot + s_n, \cdot) \) solves (1) for data \( H^{s(s+n,b_n)} \), which still converges to \( H^{s_0} \) as \( n \to \infty \).

We apply Gromov compactness to the sequence \( w_n \), so that after passing to a subsequence we find a limit \( v : \mathbb{R} \times S^1 \to M \) which solves (1) with data \( H^{s_0} \); since \( E(v) \leq \lim E(w_n) = 0 \) we must have \( v(s, t) = z(t) \) for some \( z \in \mathcal{P}(H^{s_0}) \). But \( w_n \) satisfies \( w_n(0, 0) \in \partial B \), hence \( z(0) = v(0, 0) \in \partial B \), which contradicts the choice of \( B \).

Hence there is no such sequence \( u_n \). We conclude that there is \( \beta(x, y) > 0 \) such that for \( b \in (0, \beta) \) there are no solutions of (1) between any \( x' \in \mathcal{P}(s_0 - b) \) near \( x \) and any \( y' \in \mathcal{P}(s_0 + b) \) near \( y \). The set \( \mathcal{P}(s_0) \) is finite, so we can take \( \beta \) as the minimum among \( \beta(x, y) \). This proves part 1. Part 2 is similar.

Assume that the situation is the same as above but that \( y = x \in \mathcal{P}(s_0) \). Consider a \( u_n \), and suppose that the path \( s \mapsto u_n(s, 0) \) is homotopic, relative ends, to a path contained in the neighbourhood \( B \) of \( x(0) \). In this case there is nothing to prove, so we may assume that it is not so for any \( u_n \). Then the hitting times \( s_n \) are well defined and the argument proceeds as above. This proves parts 3 and 4.

**Proof of proposition 3.1.** Take \( \delta < \beta \). By perturbing slightly, we may assume that the conclusions of Lemma 3.4 hold and that the pairs \( (H^\pm, J^\pm) \) and the path \( (H^u, J^u) = (H^{s(b)}, J^{s(b)}) \) are regular.

Then part 2 of Lemma 3.4 gives that the splitting (3) is a splitting of subcomplexes, and part 1 that \( \varphi \) preserves it. Parts 3 and 4 give that the matrices of the maps \( \partial^+, \partial^- \), and \( \varphi \) have integer coefficients.

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