The asymptotic behaviour of $F_L$ in the double scaling limit

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Abstract

In the kinematic region of small $x$ and large $Q^2$ in deep inelastic scattering, presently being explored by HERA, we present an analysis of the evolution of the longitudinal structure function $F_L^p(x,Q^2)$ in the double scaling limit of Ball and Forte. We fit the evolution to a $1/x^\lambda$ behaviour and extract the value of $\lambda$. We also study the behaviour of $R = F_2/2xF_1 - 1$. We present comparisons of both $F_L$ and $R$ with the corresponding MRS fits in this region of $x$ and $Q^2$. 

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The recent spurt of activity in the physics of low $x$ QCD has been guided by the electron-proton scattering measurements at HERA which have studied the proton structure function $F_2^p(x,Q^2)$ in the very low $x$ region, $x \leq 10^{-3}$. The steep rise of $F_2$ at these small values of $x$ have given rise to a flurry of theoretical activity in an attempt to understand the nature of this rise through 'standard' QCD.

There have essentially been two approaches to the study of the HERA data. In order to study the effects of summing $\alpha_s \ln \frac{1}{x}$ unaccompanied by $\ln Q^2$, the preferred approach has been to study the BFKL equation which generates a singular $x^{-\lambda}$ behaviour for the unintegrated gluon distribution $f(x,k^2_T)$, with $\lambda = \bar{\alpha}_s 4 \ln 2$ (for fixed, not running $\alpha_s$). For running $\alpha_s$, $\lambda \simeq 0.5$.\cite{1}. From this an asymptotic form for $F_i(i = 2, L)$ can be inferred.

The other distinct approach is to attempt to describe the data through a 'standard' Altarelli-Parisi (or DGLAP) evolution equation to the next-to-leading order approximation. Here again, the data imply a steep gluon distribution with the gluon density rising sharply as $x$ decreases, even for comparatively low values of $Q^2$. Ball and Forte \cite{2}, in a series of papers, have used this approach to exhibit the scaling properties at low $x$, generated by QCD effects and have shown that the HERA measurement of $F_2(x,Q^2)$ is well explained by this approach.

We describe their result very briefly. They show that at sufficiently small $x$ and large $t = \ln \frac{Q^2}{\Lambda^2}$, the structure function $F_2$ exhibits double scaling in the variables

$$\sigma \equiv \sqrt{\frac{\ln x}{x_0} \ln \frac{t}{t_0}}; \rho \equiv \sqrt{\frac{\ln x_0/x}{\ln t/t_0}}$$

where the starting scale for $Q^2_0$ in $t_0 \equiv \ln \frac{Q^2}{\Lambda^2}$ can be just a little more than $Q^2_0 = 1 GeV^2$, and $\Lambda = \Lambda_{QCD}$. Double asymptotic scaling results from the use of the operater product expansion and the renormalisation group at leading (and next-to-leading) order, and predicts the rise of $F_2$ on the basis of purely perturbative QCD evolution.

The asymptotic behaviour in this approach (to leading order) has the form

$$F_2^p(\sigma,\rho) \sim NF(\frac{\gamma}{\rho})\gamma \frac{1}{\rho \sqrt{\gamma \rho}} \exp[2 \gamma \rho - \delta(\sigma)]$$

$$\times [1 + O(\frac{1}{\sigma})]$$

where

$$\gamma \equiv 2 \sqrt{\frac{N_c}{\beta_0}} \beta_0 = 11 - \frac{2}{3} n_f$$

and

$$\delta \equiv (1 + \frac{2n_f}{11N_c^2})(1 - \frac{2n_f}{11N_c})$$

with $f$ an unknown distribution that depends on the starting value of the distribution.

In this paper, we concentrate on the Ball and Forte \cite{2} approach to QCD evolution to study the other independent proton structure function -- the longitudinal structure function for the proton, $F_L^p(x,Q^2)$. $F_L$ has not yet been measured at HERA and it will of course be very interesting to see how the data compares with theory.
Before we embark on the calculation, we would like to make a few comments about these different approaches. As is well known, in the traditional DGLAP or Ball-Forte analysis, there are two unknowns \( F_2(x, Q^2) \) and \( g(x, Q^2) \) (the gluon distribution) and these are determined by two measurements viz. a direct measurement of \( F_2 \) (from the cross section) and the \( Q^2 \) variation of \( F_2 \) which gives \( g(x, Q^2) \). These two measurements fix these two unknown quantities exactly since there are no free parameters in this approach that need to be fitted. In contrast, in the BFKL approach, due to the resummation of the \( \ln \frac{1}{x} \) leading to evolution also in \( x \) (by some exponent), there is a third parameter which needs to be fitted. Since there have been only two measurements till now, this parameter, in the BFKL approach, can only be fixed with a third measurement, for example, that of \( F_L \). This is in striking contrast to the Ball-Forte DGLAP approach where there are no parameters that can be tuned in order to fit any future measurement of \( F_L(x, Q^2) \). A theoretical analysis of the behaviour of \( F_L \) and subsequent comparison with experiment is, therefore, the need of the hour. We now present our analysis of \( F_L(x, Q^2) \).

In perturbative QCD, the nucleon structure functions are determined in terms of the quark and gluon distribution functions which in turn follow the DGLAP equation,

\[
\frac{dG}{dt} = \frac{\alpha_s}{2\pi} P \otimes G(Q^2)
\]  

where,

\[
P = \begin{pmatrix} P_{qq} & 2n_f P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \quad \text{and} \quad G(x, Q^2) = \begin{pmatrix} q_s \\ g \end{pmatrix}
\]

are the splitting function and the parton distribution matrices respectively. The nonsinglet quark distribution follows a similar equation but we shall ignore it as the non singlet contribution to the longitudinal structure function is negligible.

The above equation (3) can be solved by using standard Mellin transformation techniques \[4\] and in the asymptotic limit \( \sigma \to \infty \) yields the result (2) for the structure function \( F_2 \).

Note however, that in the standard case, the quark distribution functions are defined in terms of the structure functions \( F_2 \), viz.,

\[
F_2(x, Q^2) = \left( \frac{1}{n_f} \sum_{i=1}^{n_f} e_i^2 \right) \left( q_s(x, t) + q_{NS}(x, t) \right). \tag{4}
\]

We could, however, start by using \( F_1 \) as the defining structure function instead of \( F_2 \) and then the definition of the former would yield

\[
F_1(x, Q^2) = \frac{1}{2} \sum_{i=1}^{n_f} e_i^2 \left( q_i^{(1)}(x, Q^2) + \bar{q}_i^{(1)}(x, Q^2) \right) \tag{5}
\]

which can be expressed in terms of the singlet and non singlet distributions as in the usual case, and where the superscript \( (1) \) denotes the fact that the distributions have been defined with respect to the structure function \( F_1 \). If we take the quark and gluon
distributions defined with respect to $F_2$ as the standard distribution, we can compute $q^{(1)}$ and $\bar{q}^{(1)}$ with respect to these and the result is

$$q^{(1)}(x, Q^2) = \int_x^1 \frac{dy}{y} \left\{ q(y, Q^2) \left[ \delta(1-z) + \alpha_s \Delta f_q^1(z) \right] + g(y, Q^2) \alpha_s \Delta f_g^q(z) \right\}$$

and a similar equation for $\bar{q}^{(1)}$, where we have

$$z = \frac{x}{y}; \quad \Delta f_q^1(z) = -\frac{4z}{3\pi}; \quad \Delta f_g^q = -\frac{z(1-z)}{\pi}$$

Now if we calculate the structure function $F_1$, in terms of these quark and gluon densities, using (5), the result, in terms of the standard parton distributions obtained from $F_2$, is given by

$$2F_1 = \int_x^1 \frac{dy}{y} \left\{ \sum_{i=q,\bar{q}} q_i(y)e_i^2 \left[ \delta(1-z) + \frac{\alpha_s}{2\pi} \sigma_{qg^*}^T(z) \right] + g(y) \left( \sum_{i=q} e_i^2 \right) \frac{\alpha_s}{2\pi} \sigma_{gq^*}^T(z) \right\}$$

where $q_i$ and $q$ are as usual the quark and gluon parton densities, $e_i$ is the charge of $q_i$, and in analogy with $F_2$, we have written the expression in terms of $\sigma_{qg^*}^T$ and $\sigma_{gq^*}^T$ which are the transverse virtual photon-quark (antiquark) and the virtual photon - gluon cross sections respectively. The longitudinal structure function $F_L = F_2 - 2xF_1$ can now be easily evaluated with the standard expression for $F_2$, in terms of the quark and gluon densities, and, rewritten in terms of $F_2$, yields the Altarelli-Martinelli formula

$$F_L(x, Q^2) = \frac{\alpha_s(Q^2)}{2\pi} x^2 \int_x^1 \frac{dy}{y^3} \left\{ \frac{8}{3} F_2(y, Q^2) + \frac{40}{9} y g(y, Q^2) \left( 1 - \frac{x}{y} \right) \right\}$$

In deriving the above formula, one uses the lowest order result for $F_1$, by introducing the running coupling constant, $\alpha_s(Q^2)$, and using the momentum dependent parton densities. This formula can be used to compute the value of $F_L$ for low $x$ and large $Q^2$, which combined with calculated values of $F_2$ will give an estimate of $R$.

In the present work, we work in the "double scaling" regime, i.e we consider the asymptotic solutions of $F_2$ and $g$ in the limit $\sigma \to \infty$ and fixed but large values of $\rho$ where $\sigma$ and $\rho$ are defined in (1).

The evaluation of $F_2$ and the gluon distribution function $g$ and the fit of $F_2$ to experimental values have been carried out in [2] and we have used the same best fit normalizations as in [2]. In the absence of any existing data for $F_L$, in this kinematical regime, we have fixed an overall normalization for $F_L$ as 0.14 in order to closely match the various parametrizations for the same. Indeed, it would be possible to carry out a detailed analysis of the possible boundary conditions along the lines of [3], once experimental values for $F_L$ are obtained. We must mention here that we are interested in only the asymptotic form of the evolution of the various functions and the DGLAP evolution equations cannot fix the numbers absolutely and these must be uniquely determined by the experimental data. In particular, in a recent paper, De Roeck et al. [3] have fitted the expression (2) for $F_2$ to the latest measurement of the proton structure function by the H1 collaboration.
at HERA. We have assumed the result of their fit, which gives $\Lambda = \Lambda_{QCD} = 248\,\text{MeV}$ and $Q_0^2 = 1.12\,\text{GeV}^2$.

We have numerically evaluated equations (7) and (8) and subsequently $F_L$ and $R$, in the double scaling limit. Since HERA has not measured either $F_L$ or $R$, we have given a comparison to the MRSA-fit \cite{4}. The data for $F_L$ is shown in figure (1a) and (1b), while that for $R$ is shown in fig (2). The fits are shown on the same graph. As is clear, apart from overall normalization, about the arbitrariness of which we have already commented before, the behaviour of $F_L$ and $R$ closely follows the fit.

In order to get a somewhat more quantitative handle on the data, we have done a fit of the graph for $F_L$ to the form $a(Q^2)x^{-\lambda(Q^2)}$. The result of the fit is shown in the Table. It is strikingly clear from the table that $a(Q^2)$ reaches an asymptotic value of $0.0034$ and, more interestingly, $\lambda(Q^2)$ reaches a asymptotic value of $0.55$ We will have more to say on this later.

A few words about our results are now in order. In this paper, we have considered the form of the longitudinal structure function at very low values of $x$ ($10^{-5} < x < 10^{-2}$) for a wide range of $Q^2$. We have also calculated $R$ and compared it with the present MRSA-fit.

We see clearly that the steep rise of the longitudinal structure function as predicted by hard QCD processes gives a behaviour of the form $x^{-0.5}$. This behaviour is similar to that predicted by the BFKL \cite{1} analysis. In the BFKL analysis, summing the $\alpha_s log(1/x)$ terms unaccompanied by $logQ^2$ generates a singular $x^{-\lambda}$ behaviour for the unintegrated gluon distribution. For a running coupling, the BFKL equation has to be solved numerically under some reasonable assumptions, and it yields $\lambda \simeq 0.5$. (For a somewhat detailed discussion, see, for example, \cite{5}). This behaviour subsequently gives a similar behaviour for $F_2$ and $F_L$.

The rise of $F_L$ is very close to that predicted by the MRSA-fit even though the overall normalization is presumably different and will be a function of the input distribution.

At this level of analysis, it is impossible to distinguish, purely from measurements of $F_L$, the behaviour as predicted by BFKL and that of the Ball-Forte analysis. Thus the role of the hard pomeron in deciding the nature of the rise of the structure function is not at all clear, at this level of analysis. It is possible that very precise measurements of $F_L$ coupled with the non-leading contribution to the Ball-Forte result will allow us to make a clear statement on the role of the hard pomeron. However, if indeed, $F_L$ is seen to go as $x^{-0.5}$ asymptotically, while experimentally $F_2$ goes as $x^{-0.3}$, then, even though the absolute value of $F_2$ at these values of $x$ and $Q^2$ is much larger that $F_L$, it is possible to find some $x$ value at which $F_L > F_2$, if indeed this behaviour persists. If that happens then since $2xF_1 = F_2 - F_L$, this would imply a negative LHS. Since the LHS is the momentum density and hence positive, some new physics must take over at this stage to reduce the rise of $F_L$ to keep the RHS positive.

A measurement of the longitudinal structure function and $R$ will shed light on the importance of hard processes in the kinematic region being explored at HERA and will consequently give us more information about the gluon.

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Table

| $Q^2$ | $a(Q^2)$ | $\lambda(Q^2)$ |
|-------|----------|----------------|
| 20    | 0.0183   | -0.316         |
| 50    | 0.00777  | -0.443         |
| 100   | 0.00652  | -0.467         |
| 400   | 0.00483  | -0.505         |
| 800   | 0.00424  | -0.521         |
| 1200  | 0.00395  | -0.529         |
| 1600  | 0.00376  | -0.535         |
| 2000  | 0.00362  | -0.539         |
| 2400  | 0.00351  | -0.542         |
| 2800  | 0.00342  | -0.545         |
Figure Captions

1a and 1b: Plot of $F_L$ vs. $x$ for different $Q^2$. The dashed lines are the MRS-fit shown for comparison.
2: Plot of $R$ vs $x$ for different $Q^2$. The dashed lines are the MRS-fits shown for comparison.

Table Caption

Values of $a(Q^2)$ and $\lambda(Q^2)$ as a function of $Q^2$ (in the fit for $F_L \sim a(Q^2)x^\lambda$).
Figure 1a
Figure 1b
Figure 2