THE SPACE $B_{\infty,\infty}^{-1}$, VOLUMETRIC SPARSENESS, AND 3D NSE

ASEEL FARHAT, ZORAN GRUJIĆ, AND KEITH LEITMEYER

Abstract. In the context of the $L^{\infty}$-theory of the 3D NSE, it is shown that smallness of a solution in Besov space $B_{\infty,\infty}^{-1}$ suffices to prevent a possible blow-up. In particular, it is revealed that the aforementioned condition implies a particular local spatial structure of the regions of intense velocity components, namely, the structure of local volumetric sparseness on the scale comparable to the radius of spatial analyticity measured in $L^{\infty}$.

1. Introduction

Motion of 3D incompressible, viscous fluid is modeled by 3D Navier-Stokes equations (NSE),

$$u_t + (u \cdot \nabla) u = -\nabla p + \Delta u,$$

supplemented with the incompressibility condition $\text{div} \, u = 0$, where $u$ is the velocity of the fluid and $p$ is the pressure (here, the viscosity is set to 1, and the external force to 0). Henceforth, the spatial domain will be the whole space $\mathbb{R}^3$.

A question of whether the 3D NSE allow a formation of singularities is an open problem; moreover, the problem is super-critical in the sense that there is a fixed ‘scaling distance’ between any presently known regularity criterion and the corresponding (presently known) a priori bound. A telling example is a highly nontrivial regularity criterion obtained in [ESS03], namely, $u \in L^{\infty}(0, T; L^3)$, to be contrasted to a priori boundedness of the kinetic energy, $u \in L^{\infty}(0, T; L^2)$, satisfied by any Leray weak solution. In particular, the regularity criteria are (at best) scaling-invariant with respect to the unique scaling leaving the equations invariant.

A (very) partial hierarchy of the scaling invariant spaces of interest $X$ is as follows,

$$L^3 \hookrightarrow L^{3, \infty} \hookrightarrow \text{BMO}^{-1} \hookrightarrow B_{\infty, \infty}^{-1}.$$

Looking at the corresponding (existing) regularity criteria in $L^{\infty}(0, T; X)$, the only criterion that does not require a smallness condition is the aforementioned result of Escauriaza, Seregin and Sverak [ESS03] in $L^{\infty}(0, T; L^3)$. On the other side of the spectrum, since $B_{\infty, \infty}^{-1}$ is the largest scaling-invariant space in play, obtaining even a smallness regularity criterion in $L^{\infty}(0, T; B_{\infty, \infty}^{-1})$ is of significant interest. A positive answer was given in [CP02] and [ChSh10], in the setting of mild solutions belonging to a suitable $p$-integrable ($p < \infty$) Besov space and Leray solutions, respectively.

Since $B_{\infty, \infty}^{-1}$ is an $\infty$-type space, a natural question becomes whether it is possible to derive a smallness regularity criterion in $L^{\infty}(0, T; B_{\infty, \infty}^{-1})$ without assuming any
global integrability of solutions, i.e., in the context of the $L^\infty$-theory. A class of weak/distributional solutions where this is relevant is the class of uniformly-local, non-decaying ‘local Leray solutions’ constructed by Lemarié-Rieusset (cf. [LR02]), where a spatial singular set at a (possible) singular time $T$ could consists of a sequence of points $\{x_j\}$ running off to infinity. More specifically, we could envision a criticality scenario where the singularity build up at each $(x_j, T)$ would feature a locally self-similar blow-up rate (such a solution could be a local Leray solution on a time-interval containing $T$). If, in addition, we suppose that $T$ is the first singular time, then the solution–up to $T$–could be in $L^\infty$, but not in any proper $L^p$. It is worth noting that, compared to the $L^p$-theory, the $L^\infty$-theory of the 3D NSE, as well as of the Stokes problem, is less established. A good illustration of this fact is that even such a fundamental question as whether the Stokes semigroup generates an analytic semigroup in an $L^\infty$-type space on domains with boundaries was addressed only recently (see [AG13] for the case of a bounded domain with no-slip boundary conditions, and [AG14] for the case of an exterior domain).

In this short note we give an affirmative answer to the above question; more precisely, we prove the following theorem.

**Theorem 1.1.** Let $u$ be a unique mild solution to the 3D NSE emanating from an initial datum $u_0$ in $L^\infty$, and $T > 0$ be the first possible blow-up time. There exists a positive (absolute) constant $m_0$ such that if the solution $u$ satisfies

$$\sup_{t \in (T-\epsilon, T)} \|u(t)\|_{B^{-1}_{\infty, \infty}} \leq m_0,$$

for some $0 < \epsilon < T$, then $T$ is not a blow-up time, and the solution can be continued past $T$.

The idea of the proof is to utilize a recent $L^\infty$-theory of formulating geometric regularity criteria for the 3D NSE based on local 1D sparseness of the super-level sets [Gr13] in conjunction with a technical lemma quantifying the amount of 3D/volumetric sparseness of a super-level set imposed by membership of the vector field in view in the space $B^{-1}_{\infty, \infty}$. It is worth mentioning that the argument reveals that the assumption on local sparseness of the super-level sets is in fact a weaker condition than the assumption on the smallness of the solution in $L^\infty(0, T; B^{-1}_{\infty, \infty})$ (see Remark 3.4).

A related avenue to understanding the role that scaling-invariant spaces play in the regularity theory of the 3D NSE is consideration of well-posedness for small initial data. The best result in this direction so far is the result of Koch and Tataru [KT01], taking the initial data in $BMO^{-1}$. The question of whether a small initial data result is possible in the largest scaling-invariant space $B^{-1}_{\infty, \infty}$ is still open; however, there are indications that the answer to this question might be negative (see, e.g., [BP08]).

The note is organized as follows. Section 2 contains the preliminaries regarding the role that local 1D sparseness of the super-level sets plays in controlling the $L^\infty$-norm of the solution. Section 3 presents a technical lemma connecting the space $B^{-1}_{\infty, \infty}$ to 3D sparseness of the super-level sets, and Section 4 contains the proof of the above theorem, and a remark on a scenario in which the smallness condition is not needed.
2. Sparseness

The concept of ‘local 1D sparseness’ of a set has recently emerged in the study of geometric conditions preventing possible formation of singularities in the 3D NSE (cf. [Gr13]).

Let $S \subseteq \mathbb{R}^3$ be an open set, $x_0$ a point in $\mathbb{R}^3$, $r > 0$, and $\delta \in (0, 1)$ ($m$ will denote the Lebesgue measure).

**Definition 2.1.** $S$ is $1D \delta$-sparse around $x_0$ at scale $r$ if there exists a unit vector $d$ in $\mathbb{S}^2$ such that

$$\frac{m(S \cap (x_0 - rd, x_0 + rd))}{2r} \leq \delta.$$ 

The volumetric version is as follows.

**Definition 2.2.** $S$ is $3D \delta$-sparse around $x_0$ at scale $r$ if

$$\frac{m(S \cap B(x_0, r))}{m(B(x_0, r))} \leq \delta.$$ 

**Remark 2.3.** It is plain that if $S$ is $3D \delta$-sparse around $x_0$ at scale $r$, then $S$ is automatically $1D (\delta)^{1/3}$-sparse around $x_0$ at scale $\rho$, for some $0 < \rho \leq r$. (This is easily seen by assuming the opposite; then, integrating the characteristic function of $S \cap B(x_0, \rho)$ in ‘polar’ coordinates–assuming the worst case scenario–yields the contradiction.)

The main idea of how the local sparseness of the super-level sets is used in conjunction with the spatial analyticity of solutions to obtain control of the $L^\infty$-norm is very simple (the super-level sets considered here are the regions in which the magnitude is above a fraction of the $L^\infty$-norm). Intuitively, a high degree of sparseness near a possible blow-up time indicates a high level of spatial complexity (e.g., rapid spatial oscillations) that eventually becomes incompatible with the uniform local-in-time spatial analyticity properties of solutions, leading to a contradiction (as in a typical blow-up argument). Technically, this is realized via the harmonic measure maximum principle (see [Gr13]).

One should mention that the morphology of the regions of intense velocity and the regions of intense vorticity in turbulent flows is quite different. On one hand, the velocity regions are (in the average) homogeneous and isotropic, while on the other hand, the vorticity regions are (in the average) locally anisotropic and dominated by vortex filaments [S81, SJ91, JWS93, VM94, CPS95]. However, in both cases, a geometric signature is the one of sparseness; more precisely, 3D sparseness and (local) 1D sparseness, respectively. A mathematical story about how the interplay between vortex stretching and locally anisotropic diffusion might lead to closing the ‘scaling gap’ in the 3D NS regularity problem—motivated by G.I. Taylor’s view on turbulent dissipation [Tay37]—was presented in [Gr13, DaGr12-3, BrGr13-2]; here, we address the scenario of the volumetric (3D) sparseness of the regions of intense velocity.

Since the local-in-time spatial analyticity properties of solutions play a key role in the theory, we recall a variant of the pertinent result obtained in [Gu10], inspired by the method of finding a lower bound on the uniform radius of spatial analyticity of solutions in $L^p$ spaces introduced in [GrKu98].
Theorem 2.4. [Gu10] Let $u_0$ be in $L^\infty$. Then, for any $M > 1$, there exist $C(M)$ and $\tilde{C}(M)$, such that setting $T = \frac{1}{C(M)^2\|u_0\|_\infty^2}$, a unique mild solution $u = u(t)$ on $[0, T]$ has the analytic extension $U = U(t)$ to the region 

$$\mathcal{R}_t = \{x + iy \in \mathbb{C}^3 : \sqrt{t} \leq \frac{1}{C(M)}\}$$

for any $t$ in $(0, T]$, and 

$$\|U(t)\|_{L^\infty(\mathcal{R}_t)} \leq M\|u_0\|_\infty$$

for all $t$ in $[0, T]$. 

(For the results on spatial analyticity of the 3D NSE in the critical Besov spaces, see [BBT12].) 

Then, a variant of the main result in [Gr13] reads as follows. 

Theorem 2.5. [Gr13] Suppose that a solution $u$ is regular on an interval $(0, T^*)$. (Recall that $u$ is then necessarily in $C\left((0, T^*); L^\infty\right)$.)

Let $M$ be the solution to the equation $\frac{1}{2}h + (1-h)M = 1$, where $h = \frac{3}{4}\arcsin\frac{1-\sqrt{2}}{1+\sqrt{2}}$, 

and let $C(M), \tilde{C}(M)$ be as in Theorem 2.4 (note that $M > 1$). Assume that there exists $\epsilon > 0$ such that for any $t$ in $(T^* - \epsilon, T^*)$, either 

(i) $t + \frac{1}{C(M)^2\|u(t)\|_\infty^2} \geq T^*$, or 

(ii) there exists $s = s(t)$ in $\left[t + \frac{1}{4C(M)^2\|u(t)\|_\infty^2}, t + \frac{C(M)^2\|u(t)\|_\infty^2}{1/2C(M)C(M)\|u(t)\|_\infty^2}\right]$ such that for any spatial point $x_0$, there exists a scale $r$, $0 < r \leq \frac{1}{2C(M)C(M)\|u(t)\|_\infty^2}$, with the property that the component super-level sets 

$$B_{\epsilon}^{i,\pm} = \{x \in \mathbb{R}^3 : u_i^\pm(x, s) > \frac{1}{2}\|u(t)\|_\infty\}$$

are 1D $(\frac{3}{4})$-sparse around $x_0$ at scale $r$, for $i = 1, 2, 3$ (here, as customary, for a real-valued function $g$, $g^+(x) = \max(g(x), 0)$ and $g^-(x) = -\min(g(x), 0)$). 

Then, there exists $\gamma > 0$ such that $u$ is in $L^\infty\left((T^* - \epsilon, T^* + \gamma); L^\infty\right)$, i.e., $T^*$ is not a singular time. 

This is a refinement of the theorem in [Gr13] in the sense that instead of postulating the sparseness of the full vectorial super-level set, only the sparseness of each of the six component super-level sets is required. The reason that the proof remains the same is that the argument is completely local, i.e., we are estimating $|u(x_0, s_0)|$ one spatial point at a time, and since (considering the maximum vector norm in $\mathbb{R}^3$) $|u(x_0, s_0)|$ is equal to one of the six $w_i^\pm(x_0, s_0)$, we can simply apply the harmonic measure maximum principle to the subharmonic function $w_i^\pm$.

Note that in the statement of the above theorem (as well as in the original theorem), the super-level sets are considered at a time $s(t)$, with respect to the level depending on a preceding time $t$. This is not an optimal setting for the argument that we wish to make in the final section. To alleviate this, we state a
different version of the theorem in which everything is evaluated at the same point in time; the trade-off is that this is possible only for suitably chosen times based off the concept of an ‘escape time’.

**Definition 2.6.** Let $u_0$ be in $L^\infty$, $u$ a unique mild solution emanating form $u_0$, and $T > 0$ the first possible blow-up time. A time $t$ in $(0, T)$ is an escape time if $\|u(t)\|_\infty < \|u(\tau)\|_\infty$ for any $\tau$ in $(t, T)$.

**Remark 2.7.** Local-in-time well-posedness in $L^\infty$ implies that there are continuum many escape times.

Observing that for an escape time $t$ and any $s(t) = s(t)$ in $[t + \frac{1}{4C(M)^2}\|u(t)\|_\infty, t + \frac{1}{C(M)^2}\|u(t)\|_\infty]$, $\frac{1}{M}\|u(s(t))\|_\infty \leq \|u(t)\|_\infty < \|u(s(t))\|_\infty$, a slight modification of the proof of the theorem yields the desired version. More precisely, the utility of $t$ being an escape time is twofold. Firstly, since the $L^\infty$-norms at $t$ and $s(t)$ are now comparable, there is no need for the time-lag when setting the super-level set cut-off. Secondly, in the original argument ([Gr13]), the temporal points at which the super-level sets were considered were organized in a finite sequence, and the sparseness—via the harmonic measure maximum principle—guaranteed that the distance between the consecutive points did not shrink, causing the sequence to eventually surpass the possible singular time $T$, yielding the contradiction. In the current setting, the contradiction is obtained in a single temporal step as the sparseness-induced control on the $L^\infty$-norm contradicts the defining property of being an escape time.

**Theorem 2.8.** Let $u_0$ be in $L^\infty$, $u$ a unique mild solution emanating form $u_0$, $T > 0$ the first possible blow-up time, and $t$ an escape time.

Suppose that there exists $s = s(t)$ in $[t + \frac{1}{4C(M)^2}\|u(t)\|_\infty, t + \frac{1}{C(M)^2}\|u(t)\|_\infty]$ such that for any spatial point $x_0$, there exists a scale $\rho$, $0 < \rho \leq \frac{1}{2C(M)^2C(M)}\|u(s(t))\|_\infty$, with the property that the component super-level sets

$$A_{x_0}^{i, \pm} = \{x \in \mathbb{R}^3 : u_0^i(x, s(t)) > \frac{1}{2}\|u(s(t))\|_\infty\}$$

are 1D $(\frac{1}{3})$-sparse around $x_0$ at scale $\rho$, for $i = 1, 2, 3$.

Then $T$ is not a blow-up time.

### 3. Mixing

A closely related concepts of a set being $r$-mixed (or ‘mixed to scale $r$’) and $r$-semi-mixed appeared in the study of rearrangements and mixing properties of incompressible flows (see, e.g., [Br03] and [IKX14]).

**Definition 3.1.** Let $r > 0$. An open set $S$ is $r$-semi-mixed if

$$\frac{m(S \cap B(x, r))}{m(B(x, r))} \leq \delta$$
for every $x \in \mathbb{R}^3$, and for some $\delta \in (0, 1)$. If the complement, $S^c$, is $r$-semi-mixed as well, then $S$ is said to be $r$-mixed.

**Remark 3.2.** If the set $S$ is $r$-semi-mixed (with the ratio $\delta$), then it is $3D$ $\delta$-sparse around every point $x_0 \in \mathbb{R}^3$ at scale $r$.

The following lemma is a vector-valued, Besov space version of a scalar-valued, Sobolev space lemma in [IKX14]. All the norms to appear in the statement of the lemma are $\infty$-type norms.

**Lemma 3.3.** Let $\epsilon \in (0, 1]$, $r \in (0, 1]$ and $u$ a vector-valued function in $L^\infty$. Then, for any pair $\lambda, \delta$, $\lambda \in (0, 1)$ and $\delta \in \left(\frac{1}{1+\lambda}, 1\right)$, there exists an explicit constant $c = c(\lambda, \delta)$ such that if

$$\|u\|_{B_{\infty, \infty}^{-\epsilon}} \leq c(\lambda, \delta) r^\epsilon \|u\|_\infty,$$

then each of the six super-level sets $A_{\lambda}^{\pm} := \{x \in \mathbb{R}^3 : u_i^\pm > \lambda \|u\|_\infty\}$ is $r$-semi-mixed with the ratio $\delta$.

**Proof.** Arguing by contradiction, assume that there exists $i \in \{1, 2, 3\}$ such that either $A_{\lambda}^{i+}$ or $A_{\lambda}^{i-}$ is not $r$-semi-mixed with the ratio $\delta$. Without loss of generality, assume that it is $A_{\lambda}^{i+}$ (if it were $A_{\lambda}^{i-}$, the only modification to the proof would be replacing the function $f$ below with $-f$).

Then, there exists $x_0 \in \mathbb{R}^3$ such that

$$\frac{m(A_{\lambda}^{i+} \cap B(x_0, r))}{m(B(x_0, r))} > \delta;$$

equivalently,

$$m(A_{\lambda}^{i+} \cap B(x_0, r)) > \delta \Pi(3) r^3,$$

where $\Pi(3)$ denotes the volume of the unit ball in $\mathbb{R}^3$.

Let $f \in B_{1,1}^\epsilon$ be a smooth radial cut-off equal to 1 in $B(x_0, r)$, and vanishing outside $B(x_0, (1 + \eta)r)$ for some $\eta > 0$ (the value to be determined at the end of the proof). Then,

$$\|u\|_{B_{\infty, \infty}^{-\epsilon}} \geq \frac{c}{\|f\|_{B_{1,1}^\epsilon}} \left| \int_{\mathbb{R}^3} u_i(x) f(x) \, dx \right|$$

for a positive constant $c$.

An explicit calculation of the $B_{1,1}^\epsilon$-norm of $f$ via the finite differences (using the finite differences of order two for the endpoint case $\epsilon = 1$; c.f. chapter II in [BCD11]) yields

$$\|f\|_{B_{1,1}^\epsilon} \leq c(\eta) r^{3-\epsilon}$$

for some $c(\eta) > 0$.

Next, write

$$\left| \int_{\mathbb{R}^3} u_i(x) f(x) \, dx \right| \geq \int_{\mathbb{R}^3} u_i(x) f(x) \, dx \geq I - |II| - |III|,$$

where

$$I = \int_{A_{\lambda}^{i+} \cap B(x_0, r)} u_i(x) f(x) \, dx,$$
\[ \begin{align*}
II &= \int_{B(x_0,r) \setminus A_{\lambda}^+} u_i(x) f(x) \, dx, \\
III &= \int_{(B(x_0,\eta) \setminus B(x_0,r))} u_i(x) f(x) \, dx.
\end{align*} \]

It is plain that
\[ I = \int_{A_{\lambda}^+ \cap B(x_0,r)} u_i(x) \, dx = \int_{A_{\lambda}^+ \cap B(x_0,r)} u_i^+(x) \, dx \]
\[ > \lambda \|u\|_\infty m(A_{\lambda}^+ \cap B(x_0,r)) \geq \lambda \delta \Pi(3) r^3 \|u\|_\infty, \quad (3.3) \]
\[ |II| = \left| \int_{B(x_0) \setminus A_{\lambda}^+} u_i(x) \, dx \right| \leq \|u\|_\infty \left( m(B(x_0,r) - m(A_{\lambda}^+ \cap B(x_0,r)) \right) \]
\[ \leq \|u\|_\infty (\Pi(3) r^3 - \delta \Pi(3) r^3) \]
\[ = (1 - \delta) \Pi(3) r^3 \|u\|_\infty, \quad (3.4) \]

and
\[ |III| \leq \int_{(B(x_0,\eta) \setminus B(x_0,r))} u_i(x) \, dx \]
\[ \leq \|u\|_\infty (m(B(x_0, (\eta) r) - m(B(x_0,r))) \]
\[ \leq ((1 + \eta)^3 - 1) \Pi(3) r^3 \|u\|_\infty. \quad (3.5) \]

It follows from (3.1), (3.2) and (3.3)-(3.5) that
\[ \|u\|_{B_{\infty,\infty}} > c^*(\eta) \Pi(3) r^\delta \|u\|_\infty (\lambda \delta + - (1 + \eta)^3). \]

Since \( \delta > \frac{1}{1 + \lambda} \), we can define \( \eta = \eta(\lambda, \delta) \) to be the solution of the equation
\[ (1 + \eta)^3 = \frac{\delta(1 + \lambda)^2}{2}; \]
this in turn yields
\[ \|u\|_{B_{\infty,\infty}} > c(\lambda, \delta) \Pi(3) r^\delta \|u\|_\infty \quad (3.6) \]
with \( c(\lambda, \delta) = c^*(\eta) \frac{\delta(1 + \lambda)^2 - 1}{1 + \lambda} \), which is positive since \( \delta > \frac{1}{1 + \lambda} \). This contradicts the statement in the lemma. \( \square \)

It has already been observed—in the context of the Sobolev \( H^{-k} \)-spaces—that the converse of this type of result is not necessarily true (see [IKX14]). Here, we present a simple counterexample to the converse of the above lemma in the case \( \epsilon = 1 \). The function \( f \) will be a ‘dome with a lightning rod’ constructed as follows: let \( g = g(r) \) be a function on \( [0, \infty) \) obtained by smoothing out the edges of the polygonal line connecting the points \((0, 2), (1/n, 1), (1, 1), (2, 0)\) and \((\infty, 0)\), and set \( f(x) = g(|x|) \). (This can be done with the optimal bounds on the slopes of the secant lines to the graph of \( g \) analogous to the optimal bounds on the slopes of the tangent lines/derivatives when constructing standard cut-off functions.) On one hand, a simple geometric argument implies that the full vectorial super level set \( \{ |f| > \frac{3}{2} \|f\|_\infty \} \) is \( \frac{3}{2} \)-mixed with the ratio \( \delta = \frac{3}{2} \). On the other hand, the inequality
\[ \|f\|_{B_{\infty,\infty}} \leq c(\lambda, \delta) \ r \|f\|_\infty \]
is doomed. More precisely, since the scale $r$ of interest is now $r = \frac{2}{n}$, it is plain that the term $r \|f\|_\infty$ is equal to $\frac{2}{n}$, while the computation of $\|f\|_{B_{\infty,1}^{-1}}$ via ‘duality’ as in (3.1) (testing $f$ against itself) yields at least $O(1)$; namely, both the $L^2$-norm and the $B_{1,1}^1$-norm of $f$ are $O(1)$, the latter following from a calculation via the finite differences.

**Remark 3.4.** In the context of the study of the regularity theory of the 3D NSE, the above example is interesting as it indicates that the assumption on local sparseness of the super-level sets (Theorem 2.8) is a weaker condition than the smallness assumption in the Besov norm $B_{\infty,\infty}^{-1}$ (¢f. the proof of the main result below).

4. **Proof of the theorem and some thoughts on non-smallness**

In this section, we present a short proof of the main result, and a discussion about a blow-up scenario in which the smallness condition is not needed.

**Proof.** Let $t$ be an escape time in $(T - \epsilon, T)$, and $s(t)$ a time in the interval $[t + \frac{1}{4C(M)\|u(t)\|_\infty}, t + \frac{1}{C(M)\|u(t)\|_\infty}] \subset (T - \epsilon, T)$. Consider any of the component super-level sets

$$A_{i,s}^{t,\pm} = \{x \in \mathbb{R}^3 : u_i^{\pm}(x, s(t)) > \frac{1}{2}\|u(s(t))\|_\infty\}.$$ 

By Lemma 3.3, setting $\epsilon$ to be 1, $\lambda = \frac{1}{2}$, and $\delta = \frac{1}{4}$, there exists a constant $c_s > 0$ such that the condition

$$\|u(s(t))\|_{B_{\infty,\infty}^{-1}} \leq c_s r \|u(s(t))\|_\infty$$

implies that the super-level set $A_{i,s}^{t,\pm}$ is $r$-semi-mixed; in particular, this holds for $r = \frac{1}{2C(M)C(M)\|u(s(t))\|_\infty}$. Consequently, the condition

$$\|u(s(t))\|_{B_{\infty,\infty}^{-1}} \leq \frac{c_s}{2C(M)}$$

implies that $A_{i,s}^{t,\pm}$ is $1D (\frac{1}{4})^\#$-sparse at some scale $\rho$, $0 < \rho \leq \frac{1}{2C(M)C(M)\|u(s(t))\|_\infty}$, for any $x_0$ in $\mathbb{R}^3$. Setting $m_0 = \frac{c_s}{2C(M)C(M)}$, all the conditions in Theorem 2.8 are satisfied, and $T$ is not a blow-up time.

**Remark 4.1.** Note that in the statement of the theorem, the smallness condition is imposed over some interval $(T - \epsilon, T)$; however, the proof reveals that the condition is needed only at a single time $s(t)$.

At the end of this note, we would like to offer a possible scenario in which the smallness condition is not needed. Arguing the same way as in the proof of the theorem—but utilizing Lemma 3.3 with an $\epsilon$ in $(0, 1)$ instead—we see that the condition assuring the application of Theorem 2.8 can be formulated as

$$\|u(s(t))\|_{B_{\infty,\infty}^{-1}} \leq \frac{c_s}{2C(M)} \|u(s(t))\|_{\infty}^{1-\epsilon}.$$ 

Since the optimization in $\epsilon$ will not lead to a qualitatively different result, we set $\epsilon = \frac{1}{2}$, and rewrite the above as
Unraveling the characterization of the Besov norms in terms of the Littlewood-Paley decomposition yields that it is sufficient to require
\[
\|u(s(t))\|_{B^{-1}_{\infty,\infty}} \leq \frac{c_s}{2C(M)C(M)} \left( \frac{\|u(s(t))\|_{\infty}}{\|u(s(t))\|_{B^{0}_{\infty,\infty}}} \right)^{\frac{1}{2}} \|u(s(t))\|_{B^{0}_{\infty,\infty}}^{\frac{1}{2}}.
\]

This provides an (admittedly narrow) escape route from the smallness; namely, for certain classes of functions, the ratio
\[
\frac{\|f\|_{\infty}}{\|f\|_{B^{0}_{\infty,\infty}}}
\]
can become arbitrarily large. A typical example is given by the mollifications of the logarithm. More precisely, for \(\epsilon > 0\), set
\[
f_{\epsilon} = \rho_{\epsilon} \ast (\log^+(1/|x|)),
\]
where \(\rho_{\epsilon}\) is the standard mollifier. Then, on one hand, \(\|f_{\epsilon}\|_{\infty} = O(\log(1/\epsilon))\), while on the other hand, \(\|f_{\epsilon}\|_{B^{0}_{\infty,\infty}} = O(1)\), as \(\epsilon \to 0\). The first asymptotics is transparent, and to see that the second one is at most \(O(1)\) (which is what we need), recall that the inclusion of \(BMO\) in \(B^{0}_{\infty,\infty}\) is continuous, and observe that \(\|f_{\epsilon}\|_{BMO} = O(1)\); this follows from the scaling properties of the convolution and the \(BMO\)-norm similarly to the standard argument that \(\|\log |x|\|_{BMO}\) is finite (cf. \([St93]\), the first section in chapter IV). Consequently, the ratio
\[
\frac{\|f_{\epsilon}\|_{\infty}}{\|f_{\epsilon}\|_{B^{0}_{\infty,\infty}}}
\]
can indeed become arbitrarily large.

Of course, for this to be relevant, the question becomes whether it is realistic to expect that the local spatial structure of the flow around a possible singular point \(x^*\), at a time near a possible blow-up time \(T\), can exhibit a log-like profile. If we needed the condition to hold for a sequence of times converging to \(T\), the answer would be negative, as the spatial profile at \((x^*, T)\) has to be (essentially) at least as singular as \(\frac{1}{|x-x^*|}\) (see, e.g., \([Ko98]\) where it was shown that a spatial singularity to the 3D NSE that is \(o\left(\frac{1}{|x-x^*|}\right)\) is, in fact, removable). However, since it suffices that the condition holds at a single time \(s(t)\) (out of continuum many available), one could envision a scenario in which there is a log-like transition to the algebraic singularity that would include the spatial profile at \(s(t)\).

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References

[AG13] K. Abe and Y. Giga, Y, Analyticity of the Stokes semigroup in spaces of bounded functions, Acta Math. 211, 1 (2013).

[AG14] K. Abe and Y. Giga, The $L^\infty$-Stokes semigroup in exterior domains, J. Evol. Equ. 14, 1 (2014).

[BBT12] H. Bae, A. Biswas, and E. Tadmor, Analyticity and decay estimates of the Navier-Stokes equations in critical Besov spaces, Arch. Ration. Mech. Anal. 209, 963 (2012).

[BCD11] H. Bahouri, J.-Y. Chemin, and R. Danchin, Fourier analysis and nonlinear partial differential equations, volume 343 of Grundlehren der Mathematischen Wissenschaften. Springer, Heidelberg, 2011.

[BP08] J. Bourgain and N. Pavlović, Ill-posedness of the Navier-Stokes equations in a critical space in 3D. J. Funct. Anal. 255, 2233 (2008).

[BrGr13-2] Z. Bradshaw and Z. Grujić, A spatially localized $L \log L$ estimate on the vorticity in the 3D NSE, Indiana Univ. Math. J. 64, 433 (2015).

[Br03] A. Bressan, A lemma and a conjecture on the cost of rearrangements, Rend. Sem. Mat. Univ. Padova 110, 97 (2003).

[CP02] M. Cannone and F. Planchon, More Lyapunov functions for the Navier-Stokes equations, The Navier-Stokes equations: theory and numerical methods (Varenna, 2000), Lecture Notes in Pure and Appl. Math. 223, Dekker, New York, 19 (2002).

[ChSh10] A. Cheskidov and R. Shvydkoy, The regularity of weak solutions of the 3D Navier-Stokes equations in $B_{-1}^{-1},\infty$, Arch. Ration. Mech. Anal. 195, 159 (2010).

[CPS95] P. Constantin, I. Procaccia and D. Segel, Creation and dynamics of vortex tubes in three dimensional turbulence, Phys. Rev E 51, 3207 (1995).

[DaGr12-3] R. Dascaliuc and Z. Grujić, Vortex stretching and criticality for the 3D NSE, J. Math. Phys. 53, 115613 (2012).

[ESS03] L. Escauriaza, G. Seregin and V. Šverák, $L_{3,\infty}$-solutions of Navier-Stokes equations and backward uniqueness, Uspekhi Mat. Nauk. 58, 3 (2003).

[GrKu98] Z. Grujić and I. Kukavica, Space analyticity for the Navier-Stokes and related equations with initial data in $L^p$, J. Funct. Anal. 152, 447 (1998).

[Gr13] Z. Grujić, A geometric measure-type regularity criterion for solutions to the 3D Navier-Stokes equations, Nonlinearity 26, 289 (2013).

[Gu10] R. Guberović, Smoothness of Koch-Tataru solutions to the Navier-Stokes equations revisited, Discrete Cont. Dynamical Systems 27, 231 (2010).

[IKX14] G. Iyer, A. Kiselev and X. Xu, Lower bound on the mix norm of passive scalars advected by incompressible enstrophy-constrained flows, Nonlinearity 27, 973 (2014).

[JWSR93] J. Jimenez, A.A. Wray, P.G. Saffman and R.S. Rogallo, The structure of intense vorticity in isotropic turbulence, J. Fluid Mech. 255, 65 (1993).

[KT01] H. Koch and D. Tataru, Well posedness for the Navier-Stokes equations, Adv. Math. 157, 22 (2001).
[Ko98] H. Kozono, *Removable singularities of weak solutions to the Navier-Stokes equations*, Commun. Part. Diff. Eq. **23**, 949 (1998).

[LR02] P. G. Lemarié-Rieusset, Recent developments in the Navier-Stokes problem, volume 431 of Chapman & Hall/CRC Research Notes in Mathematics. Chapman & Hall/CRC, Boca Raton, FL, 2002.

[Le34] J. Leray, *Sur le mouvement d’un liquide visqueux emplissant l’espace*, Acta Math. **63**, 193 (1934).

[St93] E.M. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals, Princeton University Press, 1993.

[SJO91] Z.-S. She, E, Jackson and S. Orszag, *Structure and dynamics of homogeneous turbulence: Models and simulations*, Proc. R. Soc. Lond. A **434**, 101 (1991).

[S81] E. Siggia, *Numerical Study of Small Scale Intermittency in Three-Dimensional Turbulence*, J. Fluid Mech. **107**, 375 (1981).

[Tay37] G. I. Taylor, *Production and dissipation of vorticity in a turbulent fluid*, Proc. Roy. Soc., A**164**, 15 (1937).

[VM94] A. Vincent and M. Meneguzzi, *The dynamics of vorticity tubes in homogeneous turbulence*, J. Fluid Mech. **225**, 245 (1994).

(Aseel Farhat) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF VIRGINIA, CHARLOTTESVILLE, VA 22904, USA
E-mail address, Aseel Farhat: af7py@virginia.edu

(Zoran Grujić) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF VIRGINIA, CHARLOTTESVILLE, VA 22904, USA
E-mail address, Zoran Grujić: zg7c@virginia.edu

(Keith Leitmeyer) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF VIRGINIA, CHARLOTTESVILLE, VA 22904, USA
E-mail address, Keith Leitmeyer: kl2ju@virginia.edu