A Constructive Generalization of Nash Equilibrium for Better Payoffs and Stability

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Abstract. In a society of completely selfish individuals where everybody is only interested in maximizing his own payoff, does any equilibrium exist for the society? John Nash proved more than 50 years ago that an equilibrium always exists such that nobody would benefit from unilaterally changing his strategy. Nash Equilibrium is a central concept in game theory, which offers a mathematical foundation for social science and economy. However, it is important from both a theoretical and a practical point of view to understand game playing where individuals are less selfish. This paper offers a constructive generalization of Nash equilibrium to study $n$-person games where the selfishness of individuals can be defined at any level, including the extreme of complete selfishness. The generalization is constructive since it offers a protocol for individuals in a society to reach an equilibrium. Most importantly, this paper presents experimental results and theoretical investigation to show that the individuals in a society can reduce their selfishness level together to reach a new equilibrium where they can have better payoffs and the society is more stable at the same time. This study suggests that, for the benefit of everyone in a society (including the financial market), the pursuit of maximal payoff by each individual should be controlled at some level either by voluntary good citizenship or by imposed regulations.

1 Introduction

John Nash has proved in 1950 [1] using Kakutani fixed point theorem that any $n$-player normal-form game [2] has at least one equilibrium. In the game, each player has only a finite number of actions to take and takes one strategy at action playing. If a player takes one of the actions in a deterministic way, it is called a pure strategy. Otherwise, if a player takes anyone of the actions following some probability distribution defined on the actions, it is called a mixed strategy. At a Nash equilibrium, each player has chosen a strategy (pure or mixed) and no player can benefit by unilaterally changing his or her strategy while the other players keep theirs unchanged.

Nash Equilibrium is arguably the most important concept in game theory, which has significant impacts on many other fields like social science, economy,
and computer science. It is an important theory for understanding a common scenario in game playing.

In a Nash equilibrium, each player’s strategy is completely selfish because the player is only interested in maximizing his own payoff. Only the best action(s) to each player is accepted by the player, sub-optimal actions are not considered at all. The best action is defined as the one with the highest payoff. As a consequence of the selfishness, even if the payoff of a sub-optimal action is slightly less than the best one, the probability of picking this sub-optimal action by the player is still zero.

However, many cultures teach people to be less selfish in a society. Also, the scenario of less-selfish players may be closer to reality, such as individuals in human societies or animal kingdoms. Our conventional wisdom tells us that if each of us gives away a bit more in favor of others, we could end up with more gains as return. That is, reduced selfishness leads to better payoffs for the individuals in a society. For instance, if we, as drivers, respect other drivers sharing the same road and give considerations for each other either voluntarily and/or by following traffic laws, then each of us will end up with a faster, safer drive to his/her destination than the case when everyone is only interested in maximizing his own speed to his destination.

This paper presents both experimental results and theoretical investigation to show that, if the individuals in a society reduce their selfishness by simply accepting sub-optimal actions in some degree, a new equilibrium can be reached where better payoffs and social stability are obtained at the same time.

The first key observation of this paper is that, reducing selfishness can improve payoffs. When completely selfish players at a Nash equilibrium reduce their selfishness, they will shift to a new equilibrium with payoffs possibly better than the original one. The observations will range from the classic prisoner’s dilemma, a hard game used in other game theory literatures, to computer generated games with hundreds to thousands of players. It verifies the conventional wisdom that reducing selfishness could lead to better payoffs for everyone.

The second key observation is that, reducing selfishness can also improve social stability. A society of completely selfish individuals can be very sensitive to perturbations, the accuracy at representing individuals’ utility functions, and communication errors among the individuals in the society. The smallest change in utility function or the slightest communication error could knock the individuals out of their existing equilibrium. Furthermore, a society of completely selfish individuals can have an enormous number of equilibria. The number may increase exponentially with the population of the society. The society could end up with one Nash equilibrium or another, depending on the initial conditions and sensitive to perturbations. If the individuals reduce their selfishness together, they can reduce their sensitivity to perturbations, inaccuracy in utility functions, and communication errors. At the same time, the number of equilibria tends to drop significantly so that the outcome of the society can be more predictable. When the selfishness is below a certain level, the society tends to have only one equilibrium and converges to it with any initial conditions.
In particular, this paper gives a mathematical model for describing selfishness. The level of selfishness is controlled by one parameter of the model to cover the spectrum ranging from complete selfishness to complete selfishlessness. With the parameterized selfishness model, this paper offers a generalization of Nash equilibrium together with a proof of the existence of an equilibrium given any selfishness level using a fixed point theorem. It is a generalization because this paper offers a proof to show that a generalized equilibrium at the particular case of completely selfish players falls back to a Nash equilibrium. In other words, the definition of Nash equilibrium is a special case of the generalized one. It is important to note that the generalization is constructive because it defines a protocol for the players in a game to interact with each other so that an equilibrium can be reached with any selfishness level.

2 A Constructive Generalization

An $n$-player normal form game is defined as:

- $n$ players 1, 2, ..., $n$;
- Each player $i$ has a finite set of strategies $S_i = \{s_{i1}, s_{i2}, \ldots, s_{im_i}\}$. Strategies are also called actions. The Cartesian products of $S_i$, $S = S_1 \times S_2 \times \cdots \times S_n$, is called the set of the pure strategy profiles (the set of action tuples).
- Each player has a utility function defined as a real value function $u_i(x)$ defined on the set of the pure strategy profiles $S$, i.e., $u_i(x): S \rightarrow \mathbb{R}$ (a mapping from each action tuple to a real value).

If player $i$ takes one of the actions from $S_i$ in a deterministic way, it is called a pure strategy. Otherwise, if the player takes any of the actions following some probability distribution $p_i$ defined on the action set $S_i$, it is called a mixed strategy. That is, for each action $x_i \in S_i$, the player $i$ takes this action with a probability $p_i(x_i)$. A set of (mixed) strategies $\{p_1, p_2, \ldots, p_n\}$, one for each player, is called a (mixed) strategy profile $p$.

Assume that the $n$ players take a strategy profile $p$. Then the payoff of player $i$ is defined as

$$u_i(p) = \sum_{x \in S} u_i(x) \prod_j p_j(x_j), \quad \text{for } i = 1, 2, \ldots, n.$$ 

The objective of each player is to maximize his payoff.

A strategy profile excluding the one for player $i$ is denoted as $p_{-i}$. A strategy profile $p^*$ is a Nash equilibrium if for all $i$ and for all $p_i$,

$$u_i(p_i, p^*_{-i}) \leq u_i(p^*_i, p^*_{-i}).$$

That is, no unilateral deviation in strategy by any player gives higher payoff for that player. Nash’s 1950 PNAS paper proves the existence for an equilibrium for any finite $n$-player game using Kakutani’s fixed point theorem.
In the following discussions, without loss of generality, we assume all utility functions are of positive function values, i.e., $u_i(x) > 0$, for any $x \in S$.

If player $i$ takes an action $x_i \in S_i$ in response to other players strategies $p_{-i}$, the payoff is $u_i(x_i,p_{-i})$. The optimal action $x_i^*$ for the player $i$ is defined as the one with the highest payoff, i.e.,

$$u_i(x_i^*,p_{-i}) = \max_{x_i \in S_i} u_i(x_i,p_{-i}).$$

Obviously,

$$u_i(p_i,p_{-i}) = \sum_{x_i \in S_i} p_i(x_i) u_i(x_i,p_{-i}).$$

One of the important properties of a Nash equilibrium $p^*$ is that only the optimal action(s) has non-zero probability, i.e., if $p_i^*(x_i) > 0$, then $x_i$ must be the optimal action for the player $i$. In other words, player $i$ is completely selfish because he only accepts the optimal action for himself.

Assume that, at a time instance $t$, the strategy profile excluding the one for player $i$ is $p_{-i}(t)$, the action payoff for player $i$ is $u_i(x_i,p_{-i}(t))$, for $x_i \in S_i$. Based on the above observation, we can define a mathematical model to formulate the construction of the next time strategy $p_i(x_i,t+1)$ for player $i$ based on his action payoff $u_i(x_i,p_{-i}(t))$ at the current time $t$. Specifically, we can construct $p_i(x_i,t+1)$ such that it is proportional to $u_i(x_i,p_{-i}(t))$, e.g.,

$$p_i(x_i,t+1) \propto (u_i(x_i,p_{-i}(t)))^\alpha,$$

where $\alpha$ is a parameter of a non-negative value. Since $p_i(x_i,t+1)$ should be normalized as a probability, the above formula can be rewritten as

$$p_i(x_i,t+1) = \frac{(u_i(x_i,p_{-i}(t)))^\alpha}{\sum_{x_i \in S_i} (u_i(x_i,p_{-i}(t)))^\alpha}, \quad \text{for } i = 1, 2, \ldots, n. \quad (1)$$

In $(1)$, when $\alpha \to \infty$, the best action has a non-zero probability while others have probability zero. That is, the player only accepts the best action, the one with the highest payoff $u_i(x_i,p_{-i})$. It is exactly same as the case of Nash equilibrium described before.

If the value of $\alpha$ is reduced from the above extreme case, the player $i$ starts to accept sub-optimal actions by assigning non-zero probability to them. The degree of the acceptance increases with further decrease of $\alpha$. At another extreme case, when $\alpha \to 0$, each action is assigned with the same probability and the player has no preference on any one of the actions. All of the actions are treated equally and they are sampled uniformly. In this case, the player is completely selfishless. In summary, the parameter $\alpha$ describes the selfishness level of player $i$. It covers the spectrum ranging from complete selfishness ($\alpha \to \infty$) to complete selfishlessness ($\alpha = 0$).

In the special case of $\alpha \to \infty$, the game playing defined by the constructive generalization $(1)$ is the same in principle as fictitious play introduced by G.W. Brown in 1951 [3]. In fictitious play, each player takes the optimal action(s) in respond to the strategies of other players.
Definition 1. Given a non-negative real value for the selfishness level \( \alpha \), i.e., \( \alpha \geq 0 \). If the iterative computation defined by (1) reaches an equilibrium, that is, there is a strategy profile \( p^* \) satisfying

\[
p^*_i(x_i) = \frac{(u_i(x_i, p^*_{-i}))^\alpha}{\sum_{x_i \in S_i} (u_i(x_i, p^*_{-i}))^\alpha} \quad \text{for } i = 1, 2, \ldots, n ,
\]  

then the strategy profile \( p^* \) is called a generalized equilibrium.

In parallel with Nash’s 1950 PNAS paper, the proof of the existence of a generalized equilibrium given any selfishness level is provided below. Furthermore, it will be shown that when the selfishness level is sufficiently high, a generalized equilibrium falls back to a Nash equilibrium.

Theorem 1. A generalized equilibrium \( p^* \) defined by (2) exists for any \( n \)-player normal form game with any selfishness level \( \alpha \) of a non-negative value (\( \alpha \geq 0 \)). It is still true even if each player \( i \) in the game has his own selfishness level \( \alpha_i \), possibly different from the rest.

Proof. The set of iterative equations (1) defines a mapping from the strategy profile set to itself. Because the set is compact and the mapping is continuous, so a fixed point exists based on Brouwer fixed point theorem.

The second part of this theorem tells us that, for any \( n \)-player normal form game, even if the selfishness level is different from player to player, a generalized equilibrium still exists for the game.

It is important to note that (2) defines a system of polynomials if \( \alpha \) is an integer. If it is also an even number, then any real value solution to this system, which must also be a positive solution, is also a generalized equilibrium for the game playing. Also, the game playing defined by (1) can be treated as an iterative, direct method to find an equilibrium of the game playing. It defines a protocol for the players in a game to interact with each other so that an equilibrium can be reached with any selfishness level. Following this protocol, each player only needs to know his own utility function and the strategies of other players at the current time to compute his strategy for the next time. The strategies of other players can be obtained through either statistical learning or message passing among the players.

Theorem 2. When the selfishness level \( \alpha \) is sufficiently large, i.e., \( \alpha \to \infty \), any generalized equilibrium defined by (2) can be arbitrarily close to a Nash equilibrium and vice versa.

The proof is given in the subsection 5.1 in the Appendix.

As a consequence, any real value solution to the system of polynomials defined by (2) with a large even number for \( \alpha \) can be served as a good approximation to a Nash equilibrium. Alternatively, the game playing defined by (1) with a sufficiently large \( \alpha \) can be applied directly to reach an equilibrium which can also be served as a good approximation of a Nash equilibrium.
When \( \alpha \to \infty \), from (1) we can see that \( p_i(x_i, t + 1) \) is no longer a continuous function of \( u_i(x_i, p_{-i}(t)) \). In this case, any mixed strategy can be extremely unstable for the slightest change in \( u_i(x_i, p_{-i}(t)) \) caused by the inaccuracy at representing the utility functions, the variation of the utility functions, any perturbation and communication error among the players. For example, a small variation in the utility function could lead to a dramatic shift of the equilibrium from one point in the strategy profile space to another one. It is hard for an algorithmic method to converge to an unstable equilibrium purely based on iterations.

Even if a game of completely selfish players can reach an equilibrium, it may have an enormous number of equilibria, possibly growing exponentially with the number of the players of the game. The players could get stuck into one Nash equilibrium or another, depending on the initial conditions and sensitive to perturbations. How to reach an equilibrium which gives relatively good overall payoff for the game becomes a challenging problem (the overall payoff for a game is defined as the summation of the players’ payoffs, i.e., \( \sum_i u_i(x, t) \)).

As a summary, we can say that complete selfishness of the players in a game may lead to the difficulty for the players to reach an equilibrium. Even if an equilibrium is found, it could also be unstable, sensitive to perturbations, sensitive to inaccuracy or variations in utility functions, and vulnerable to communication errors. Furthermore, the overall payoff of the game may be ignored due to the fact that each player only tries to maximize his own payoff. It is desirable to improve the overall payoff for a society because it stands for improved individual payoff on average. Also, everyone in the society could benefit from the improved overall payoff if some social welfare system is implemented to redistribute the social wealth. Can we improve the overall payoff and the stability of a game playing by simply reducing the selfishness level of the players in the game?

Both experimental result in the following section and a theoretical investigation in Appendix (subsection 5.2) will affirm the above question. The theoretical investigation shows that the game playing defined by the constructive generalization (1) is a variation of a global optimization algorithm [8] defined by a multi-agent system. When the value of the parameter \( \alpha \) is reduced below a certain threshold, the global optimization algorithm has one and only one equilibrium and converges to it with an exponential rate. If the equilibrium is also a consensus one among all the agents, then it must be the global optimum, guaranteed by theory. Above the threshold, the number of equilibria of the global optimization algorithm may grow with the value of the parameter \( \alpha \). That is, the algorithm becomes less stable with the increase of \( \alpha \), but the chance of reaching a consensus increases, however.

The theory suggests that reducing the selfishness level from the extreme of complete selfishness can stabilize the game playing and possibly improve the overall payoff. The experiments in the next section verifies that the overall payoff for many games is best in a statistical sense at a certain level of selfishness, neither at the complete selfishness one nor at the complete selfishlessness one. Applying this to social situations, it suggests that a society should let some level
of selfishness remain in its individuals. Otherwise, nobody has any motivation to pursue better payoffs. Also, it is not recommended to take the other extreme where everyone is completely selfish. However, how to find the best selfish level for any game remains as an open question.

3 Experimental Results

The prisoner’s dilemma constitutes a basic problem in game theory. It is a typical non-zero-sum game in which two players can either “cooperate” or “defect” the other player. In this game, the only concern of each individual player (“prisoner”) is to maximize his/her own payoff. Regardless of what the opponent chooses, each prisoner always receives a higher payoff by defecting; i.e., defecting is the strictly dominant strategy. Therefore, the only possible Nash equilibrium for the game is for all prisoners to defect.

An example payoff matrix of the prisoner’s dilemma is given as follows:

|       | Cooperate | Defect |
|-------|-----------|--------|
| Cooperate | 3,3 | 1,4 |
| Defect   | 4,1 | 2,2 |

At the Nash equilibrium (the element in the matrix with a bold font), the payoffs of the two players are (2,2). It corresponds to the case when the selfishness level $\alpha = \infty$. When the two players reduce their selfishness level together, their payoffs at equilibria also increase together as shown in Figure 1. Those equilibria are found by the constructive generalization with different selfishness levels.

![Fig. 1. Payoffs of prisoner’s dilemma under different selfishness levels.](image)
From Fig. 1 we can see that when the two players have the same selfishness level and the level is of a high value ($\alpha = 30$), their payoffs are close to those of the Nash equilibrium. The moment that the both players reduce their selfishness level, both get better payoffs than those of the Nash equilibrium. When the selfishness level reduces to one ($\alpha = 1$), the payoffs are close to 2.4 for both, a 20% increase over the one of the Nash equilibrium.

The result seems counterintuitive because if one player could update his strategy to improve his payoff, he should go ahead to do it in order to receiving a better payoff. However, in many cases, all the players in a game are interconnected. The gain of one player often leads to the loss of other players. If everyone yields back a little bit of his payoff as a favor to others, everyone can end up with better payoff as a returned favor from others instead.

To show the power of the constructive generalization at finding Nash equilibria, a 2-player game is used with the following payoff matrix:

$$
\begin{pmatrix}
2 & 3 & -1 & 4 & 2 & 4 & 5 & 2 & 1 & -1 \\
2 & 2 & 3 & 0 & 4 & 1 & -2 & 4 & 1 & 3 \\
4 & 6 & 7 & 2 & 2 & 3 & 4 & 9 & 2 & 1 \\
9 & 0 & -2 & 6 & 6 & 3 & 7 & 0 & 0 & 5 \\
3 & 2 & 6 & 1 & 2 & 5 & 5 & 3 & 1 & 0
\end{pmatrix}
$$

This game has been used in other game theory literatures as a hard game because it has only one mixed Nash equilibrium. The strategy for the row player is $(0, 0, \frac{7}{3}, \frac{2}{3}, \frac{2}{3}, \frac{1}{3})$ with the payoff 4. The strategy for the column player is $(0, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{1}{3}, 0)$ with the payoff 3. This mixed Nash equilibrium is extremely unstable. Assume that the two players play the game by taking only the best action. Assume further that the column player couldn’t represent fraction numbers. Instead, the player uses real values to approximate them, just like the real values stored in most computers. Then, a very slight round-off error for the value $\frac{7}{3}$, say $0.4285714285714285714285714286$, could knock the row player off of his Nash equilibrium strategy to the new one $(0, 0, 0, 1, 0)$, which will in turn knock the column player off his Nash equilibrium strategy. As a consequence, both of them will immediately be knocked off the Nash equilibrium and get stuck into a chaotic situation. Therefore, this game is hard for an iteration-based direct method to reach the unique mixed Nash equilibrium.

Despite of its hardness, the constructive generalization as an iteration-based direct method can find a very good approximation to the Nash equilibrium. To improve the convergence property of the method, an additional step is added after computing $p_i(x_i, t)$ defined by (1) to smooth out its fluctuation. It is done by keeping some memory of the previous value of $p_i(x_i, t)$, i.e.,

$$
\lambda p_i(x_i, t + 1) + (1 - \lambda)p_i(x_i, t) \rightarrow p_i(x_i, t + 1)
$$

where $\lambda = 0.001$ was used in the experiment.

Furthermore, to increase the chance for the constructive generalization to reach an equilibrium at a high selfishness level $\alpha$, the value of $\alpha$ is progressively raised from a small value, say 1. When it reaches the value 1000, the payoff for
the row player is 4.0068, a difference around 0.17% to the payoff 4 of the Nash equilibrium. His strategy (left) is very close to the Nash equilibrium one (right) as shown below:

\[(0, 0, \frac{1.999}{11}, \frac{3.999}{11}, \frac{5.002}{11}) \approx (0, 0, \frac{2}{11}, \frac{4}{11}, \frac{5}{11})\].

The payoff for the column player is 3.0001097, a difference around 0.0037% to the payoff 3 of the Nash equilibrium. His strategy (left) is very close to the Nash equilibrium one (right) as shown below:

\[(0, \frac{1.997}{7}, \frac{2.981}{7}, \frac{2.023}{7}, 0) \approx (0, \frac{2}{7}, \frac{3}{7}, \frac{2}{7}, 0)\].

Fig. 2 shows the changes of the payoffs of the two players in relation to the selfishness level. The payoff of the row player is peaked around \(\alpha = 8\) with the value 4.77572. The payoff of the column player is peaked around \(\alpha = 5\) with the value 3.30341. At \(\alpha = 7\), the payoffs for both are \((4.7586, 3.2527)\), a 19% improvement for the row player and a 8.4% improvement for the column player over the payoffs (4, 3) of the Nash equilibrium.

Fig. 2. Payoffs of two players with 5 actions under different selfishness levels.

To illustrate the power of the constructive generalization (1) at stabilizing game playing, a 2-players game with 6 actions for each is constructed with the
following payoff matrix:

\[
\begin{pmatrix}
6 & 6 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 6.5 & 6.5 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 7 & 7 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 7.5 & 7.5 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 8 & 8 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 8 & 8.5
\end{pmatrix}
\]

Clearly, this game has six actions for each player and six pure Nash equilibria. Let us label the actions for each player as 1, 2, 3, 4, 5, 6. If any player picks the \(i\)th action in random, the other will take the same action as the best response. As a consequence, a Nash equilibrium is thus found.

With this best-response playing, the average payoff for each player is 7.725, the variance of the payoff is 35/48 \(\approx 0.729\). Three hundred generalized equilibria are found using the constructive generalization (1) with the selfishness levels \(\alpha = 100, 4, 2, 1\), respectively. The results for \(\alpha = 100, 4, 2\) are shown in Fig. 3, Fig. 4, and Fig. 5 respectively. From the figures we can see that the stability of the game playing defined by the constructive generalization improves progressively as the selfishness level \(\alpha\) decreases. Here, the stability is reversely proportional to the variance of the payoff. When \(\alpha = 1\), the game playing always converges to a unique equilibrium with the payoff=2.10373 after three hundred runs. That is, the game playing tends to have only one equilibrium when the selfishness level drops below a certain threshold. Also we can see from the three figures that the average payoff of each player is of the highest value when \(\alpha = 4\).

From this example, we can see that the constructive generalization yields the best payoffs for the players in a game at a certain selfishness level. The stability of the game playing continuously improves as the selfishness level reduces. That is, reducing the selfishness level can always improve the stability of the game playing. However, the players in a game can only get the highest payoffs at a statistical sense at a certain selfishness level (In the subsection 5.2 in the Appendix, a theoretical investigation is given to offer some explanation).

In the following set of experiments, computer-generated societies with a population ranging from hundreds to a thousand are used to demonstrate the improvement of payoffs and stability by reducing the selfishness level. In each society, each individual has a number of neighbors and his payoff function is defined by the summation of the pairwise joint actions of himself and his neighbors as follows

\[
u_i(x) = \sum_{j \in N(i)} f_{ij}(x_i, x_j), \tag{3}
\]

where \(N(i)\) is the set of the individual \(i\)'s neighbors. The overall payoff of the society is defined as

\[
\sum_i u_i(x) = \sum_i \sum_{j \in N(i)} f_{ij}(x_i, x_j).
\]

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Fig. 3. Overall payoffs of 300 generalized Nash equilibria with the selfishness level $\alpha = 100$. The average $\mu = 7.728$ and variance $\sigma^2 = 0.502$.

Fig. 4. Overall payoffs of 300 generalized Nash equilibria with the selfishness level $\alpha = 4$. The average $\mu = 7.910$ and variance $\sigma^2 = 0.379$. 
Each function value $f_{ij}(x_i, x_j)$ is uniformly sampled from the interval $[0, 1]$. The neighbors of each individual are randomly picked from the entire population.

In the first experiment, an instance of a society of 121 individuals is generated where each one has 50 actions and 6 neighbors on average. 300 Nash equilibria are discovered by fictitious play and 300 generalized ones are discovered by the constructive generalization with the selfishness level $\alpha = 20$. Fig. 6 shows the overall payoffs of the first 300 ones versus the second 300 ones. From the figure we can see that, reducing the selfishness level can lead to remarkable improvement both in payoffs and stability.

In the second experiment, the population is increased to 601, the number of actions per person is reduced to 20, and the size of neighbors on average is increased to 30. Figure 7 shows the overall payoffs of 300 Nash equilibria versus the 300 generalized ones with the selfishness level $\alpha = 20$. From the figure we can see that, reducing the selfishness level can lead to remarkable improvement both in payoffs and stability with a larger population.

In the third experiment, the population is increased further to 1001, the number of actions per person is reduced to 10, and the size of neighbors on average is increased to 50. From Figure 8 we can make the same conclusions as above with an even larger population.

The last three experiments with societies of different population sizes are extended with more selfishness levels. The average overall payoff and the fluctuation of the overall payoff of a society with different selfishness levels $\alpha$ are shown in the following table. The fluctuation is indicated by the variance of the
Fig. 6. Overall payoffs of 300 Nash equilibria (bottom) versus 300 generalized ones (top, the selfishness level $\alpha = 20$) for a society of 121 individuals. For the former, the average $\mu = 600.67$ and variance $\sigma^2 = 17.5$. For the latter, the average $\mu = 622.60$ and variance $\sigma^2 = 10.7$.

Fig. 7. Overall payoffs of 300 Nash equilibria (bottom) versus 300 generalized ones (top, the selfishness level $\alpha = 20$) for a society of 601 individuals. For the former, the average $\mu = 11766$ and variance $\sigma^2 = 1009$. For the latter, the average $\mu = 11899$ and variance $\sigma^2 = 392$. 
Overall payoffs of 300 Nash equilibria versus 300 (bottom) generalized ones (top, the selfishness level $\alpha = 30$) for a society of 1001 individuals. For the former, the average $\mu = 30274$ and variance $\sigma^2 = 3335$. For the latter, the average $\mu = 30677$ and variance $\sigma^2 = 818$.

overall payoff given a selfishness level. The less fluctuation a society has, the more stable the society is.

| Selfishness Level | 121  | 601  | 1001 |
|-------------------|------|------|------|
| $\infty$          | 17.5 | 591  | 1371 |
| 100               | 16.9 | 516  | 1296 |
| 80                | 15.7 | 471  | 1151 |
| 60                | 13.2 | 461  | 1069 |
| 50                | 13.0 | 455  | 1029 |
| 40                | 10.1 | 411  | 818  |
| 30                | 10.1 | 411  | 818  |
| 20                | 8.68 | 392  | 681  |
| 10                | 458  | 0    | 0    |

From the above table, we can see that the overall payoffs of the three societies improve progressively with the reduction of the selfishness level $\alpha$ started from $\alpha = \infty$ (complete selfishness). Each society yields the highest overall payoff at a some selfishness level and degrades progressively with further reduction of the selfishness level. The stability of each society continuously improves as the selfishness level reduces. That is, reducing the selfishness level can always improve the stability of a society. This experiment shows us that a less selfish
society can be better in overall payoff and stability than a completely selfish society.

A less selfish society can also be more efficient than a completely selfish society. The efficiency of a society can be measured by the capability at finding a good equilibrium in terms of the overall payoff. To compare the efficiency, the same society of a population of 121 described before is used in the experiment. When the individuals in the society are less selfish (\(\alpha = 20\)), the average overall payoff of the 300 equilibria found by the society is 622.60 (see also Fig. [6]). When all the individuals become completely selfish, after exploring one million of equilibria by the society, the best overall payoff is of a value 621.5, less than the former one 622.60. This result says that the average payoff of the less selfish society in a generalized equilibrium is better than the best payoff out of those of one million Nash equilibria explored by the completely selfish society. The less selfish society spent seconds on average to find an equilibria while the completely selfish society took almost a whole day to find the one million equilibria using a laptop with a AMD Turion\textsuperscript{TM} X2 Dual-Core Mobile Processor and 3GB RAM. The less selfish society is several orders of magnitude more efficient than the completely selfish society.

Fig. [9] shows the improvement of the best overall payoff with the increase of the number of equilibria discovered by the completely selfish society mentioned above.

![Image](image.png)

**Fig. 9.** After exploring one million equilibria by a society of 121 completely selfish individuals, the best one in terms of overall payoff still couldn’t match the average one (dotted line) found by the same society when all the individuals are less selfish.
4 Conclusions

John Nash in his Nobel price-winning work defined an equilibrium and proved its existence for $n$ players games where all players are completely selfish. However, it is important from both a theoretical and a practical point of view to understand game playing where players are less selfish. The key contribution of this paper is a generalization of Nash equilibrium to cover the entire spectrum of selfishness ranging from complete selfishness to complete selfishlessness. It also gives the proof of the existence of an equilibrium for a game of $n$-players with any selfishness level. The definition of Nash equilibrium is a special case of this generalization where all players are completely selfish. The generalization is constructive since it offers a protocol for players in a game to reach an equilibrium. Most importantly, this paper presents experimental results and theoretical investigation to show that the players in a game can reduce their selfishness level together to reach a new equilibrium where they can have better payoffs and the game playing is more stable at the same time.

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5 Appendix

5.1 Proof for Theorem 2

Definitions and Notations At time instance $t$, let $u_i(x_i, p_{-i}(t))$ be the payoff of player $i$ by taking action $x_i$ in response to other players' strategies $p_{-i}(t)$. It
is a function of \( x_i \) and \( t \), called the action payoff function, denoted as \( \Psi_i(x_i, t) \). Obviously, we have

\[
\Psi_i(x_i, t) = u_i(x_i, p_{-i}(t)) = \sum_{x_i} \left( u_i(x) \prod_{j \neq i} p_j(x_j, t) \right), \quad \text{for any } i .
\]

Using the notation, the constructive generalization (1) can be rewritten as

\[
p_i(x_i, t + 1) = \frac{(\Psi_i(x_i, t))^\alpha}{\sum_{x_i \in S_i} (\Psi_i(x_i, t))^\alpha}, \quad \text{for } i = 1, 2, \ldots, n .
\]

That is, \( p_i(x_i, t + 1) \) equals to the normalized \((\Psi_i(x_i, t))^\alpha\). To show the relationship, \( p_i(x_i, t) \) can be expressed as \((\bar{\Psi}_i(x_i, t))^\alpha\) with the bar standing for the normalization. That is,

\[
p_i(x_i, t) = \left( \bar{\Psi}_i(x_i, t) \right)^\alpha, \quad \text{for } i = 1, 2, \ldots, n .
\]

Substituting (6) into (4), we have an iterative update function for \( \Psi_i(x_i, t) \) as follows

\[
\Psi_i(x_i, t + 1) = \sum_{x_i} \left( u_i(x) \prod_{j \neq i} (\bar{\Psi}_j(x_j))^{\alpha} \right), \quad \text{for } i = 1, 2, \ldots, n .
\]

If a strategy profile \( p^* \) is a generalized equilibrium satisfying (2), then there is a corresponding set of action payoff functions \( \{\Psi^*_1(x_1), \Psi^*_2(x_2), \ldots, \Psi^*_n(x_n)\} \) defined by (1), or simply \( \Psi^* \), such that (7) is satisfied. That is,

\[
\Psi^*_i(x_i) = \sum_{x_i} \left( u_i(x) \prod_{j \neq i} (\bar{\Psi}^*_j(x_j))^{\alpha} \right), \quad \text{for } i = 1, 2, \ldots, n .
\]

Both a strategy profile \( p^* \) satisfying (2) and an action payoff function set \( \Psi^* \) satisfying (8) can be used to represent a generalized equilibrium. Based on (4), we have

\[
\Psi^*_i(x_i) = \sum_{x_i} \left( u_i(x) \prod_{j \neq i} p^*_j(x_j) \right), \quad \text{for } i = 1, 2, \ldots, n .
\]

Based on (6), we have

\[
p^*_i(x_i) = \left( \bar{\Psi}^*_i(x_i) \right)^\alpha, \quad \text{for } i = 1, 2, \ldots, n .
\]

**The Proof** The best action of player \( i \) at time \( t \) is defined as the one with the highest payoff, i.e., the \( x_i \) that maximizes the action payoff function \( \Psi_i(x_i, t) \). Assume that the total number of actions of player \( i \) is \( m_i \). Assume further that
α ≥ 1. At a generalized equilibrium with a strategy profile \( p^* \) and its corresponding action payoff function set as \( \Psi^* \), based on (8), we can find out the difference between the best payoff \( \max_x \Psi^*_i(x_i) \) and the expected payoff \( \sum_{x_i} \Psi^*_i(x_i)p^*_i(x_i) \). It is straightforward to verify that the difference should satisfy the following inequality:

\[
0 \leq \max_x \Psi^*_i(x_i) - \sum_{x_i} \Psi^*_i(x_i)p^*_i(x_i) < \left( \frac{m_i - 1}{\max_x \Psi^*_i(x_i)} \right) \alpha^{-1} .
\]

Obviously, the difference can be arbitrarily small when the parameter \( \alpha \) is sufficiently large. That is, the difference is reduced to zero when \( \alpha \to \infty \),

\[
\lim_{\alpha \to \infty} \left( \max_x \Psi^*_i(x_i) - \sum_{x_i} \Psi^*_i(x_i)p^*_i(x_i) \right) = 0, \quad \text{for any } i . \quad (9)
\]

Given a strategy profile \( p^* \), it is a Nash equilibrium if and only if, given any player, its best payoff is equal to its expected payoff \( \sum_{x_i} \Psi^*_i(x_i)p^*_i(x_i) \). That is, for any \( i \),

\[
\max_x \Psi^*_i(x_i) - \sum_{x_i} \Psi^*_i(x_i)p^*_i(x_i) = 0 . \quad (10)
\]

Compare the statement (9) with the statement (10), we can conclude that any generalized equilibrium (8) can be arbitrarily close to a Nash equilibrium if the parameter \( \alpha \) is sufficiently large.

The other way around is also true. That is, for any Nash equilibrium, there exists a generalized equilibrium defined as (8) which is arbitrarily close to the Nash equilibrium if the parameter \( \alpha \) is sufficiently large. To prove this statement, recall that the action payoff function \( \Psi_i(x_i) \) computed by (7) is the payoff of player \( i \) taking the action \( x_i \) while other players taking the strategies \( p_j \) (\( j \neq i \)). Assume that a strategy profile \( p^* \) is a Nash equilibrium. Then the payoff \( \Psi^*_i(x_i) \) at the Nash equilibrium should satisfy the following condition,

\[
\max_x \Psi^*_i(x_i) = \Psi^*_i(x_i), \quad \text{if } p^*_i(x_i) > 0 ;
\]
\[
\max_x \Psi^*_i(x_i) \leq \Psi^*_i(x_i), \quad \text{if } p^*_i(x_i) = 0 .
\]

Let \( \epsilon \) is a positive infinitesimal. Note that for any probability \( p_i \), if \( 0 < p_i \leq 1 \), then

\[
\lim_{\epsilon \to 0^+} (1 + \epsilon \ln p_i)^{1/\epsilon} = p_i .
\]

Otherwise, if \( p_i = 0 \), then

\[
\lim_{\epsilon \to 0^+} (1 + \epsilon \ln \epsilon)^{1/\epsilon} = p_i(= 0) .
\]

Given each player \( i, \ i = 1, 2, \ldots, n \), define its action payoff function \( \Psi'_i(x_i) \) as

\[
\Psi'_i(x_i) = \begin{cases} 
(1 + \epsilon \ln p^*_i(x_i)) \max_x \Psi^*_i(x_i), & \text{if } p^*_i(x_i) > 0 ; \\
(1 + \epsilon \ln \epsilon) \max_x \Psi^*_i(x_i), & \text{if } p^*_i(x_i) = 0 \text{ and } \max_x \Psi^*_i(x_i) = \Psi^*_i(x_i) ; \\
\Psi^*_i(x_i), & \text{if } \Psi^*_i(x_i) < \max_x \Psi^*_i(x_i) .
\end{cases}
\]
Obviously,
\[ \lim_{\epsilon \to 0^+} \Psi'_i(x_i) = \Psi^*_i(x_i). \]

Let \( \alpha = 1/\epsilon \), from (5) used for computing the strategy \( p_i(x_i, t) \), we have
\[
\lim_{\epsilon \to 0^+} \frac{\left( \Psi'_i(x_i) \right)^{1/\epsilon}}{\sum_{x_i} \left( \Psi'_i(x_i) \right)^{1/\epsilon}} = p_i^*(x_i), \quad \text{for } i = 1, 2, \ldots, n.
\]

Hence, the set of action payoff functions \( \{ \Psi'_1(x_1), \Psi'_2(x_2), \ldots, \Psi'_n(x_n) \} \) is a generalized equilibrium satisfying (8) when the parameter \( \alpha \) is sufficiently large. Its corresponding strategy profile \( \{ p_1^*(x_1), p_2^*(x_2), \ldots, p_n^*(x_n) \} \) is the strategy profile \( p^* \) of the Nash equilibrium in the assumption. In other words, for any Nash equilibrium with a strategy profile \( p^* \), there always exists a generalized equilibrium satisfying (2) which is arbitrarily close to the Nash equilibrium when the selfishness level \( \alpha \) is sufficiently large.

5.2 Theoretical Investigation

From Cooperative Optimization to the Constructive Generalization

The constructive generalization can be derived from a recently discovered general global optimization method, called cooperative optimization [8]. Cooperation is an ubiquitous phenomenon in nature. The cooperative optimization theory is a mathematical theory for understanding cooperative behaviors and translating it into optimization algorithms. The major theoretical results can be found in [8].

Let \( E(x_1, x_2, \ldots, x_n) \), or simply \( E(x) \), be a multivariate objective function of \( n \) variables. Assume that \( E(x) \) can be decomposed into \( n \) sub-objective functions \( E_i(x) \), one for each variable, such that those sub-objective functions satisfying
\[ E_1(x) + E_2(x) + \ldots + E_n(x) = E(x). \]

In terms of a multi-agent system, let us assign \( E_i(x) \) as the objective function for agent \( i \), for \( i = 1, 2, \ldots, n \). There are \( n \) agents in the system in total. The objective of each agent \( i \) is to maximize \( E_i(x) \). The objective of the system is to maximize \( E(x) \), called the global objective function.

There is a simple form of cooperative optimization where each agent \( i \) is associated with a function \( \Psi_i(x_i, t) \) defined on the variable \( x_i \) and time \( t \). The function is called the assignment function for the agent. Each agent updates its assignment function iteratively as follows:
\[ \Psi_i(x_i, t) = \sum_{\sim x_i} \left( e^{E_i(x_i)/h} \prod_{j \neq i} p_j(x_j, t - 1) \right), \quad \text{for } i = 1, 2, \ldots, n, \quad (11) \]
where \( \sum_{\sim x_i} \) stands for the summation over all variables except \( x_i \) and \( h \) is a constant of a small positive value. \( p_i(x_i, t) \) is defined as
\[ p_i(x_i, t) = \frac{(\Psi_i(x_i, t))^\alpha}{\sum_{x_i} (\Psi_i(x_i, t))^\alpha}, \quad (12) \]
where $\alpha$ is a parameter of a non-negative real value.

By the definition, $p_i(x_i, t)$ is a probability-like function satisfying

$$\sum_{x_i} p_i(x_i, t) = 1.$$ 

It is, therefore, called the assignment probability function. It defines the soft decisions for assigning variable $x_i$ at the time instance $t$. If a variable value $x_i$ is of a higher function value $p_i(x_i, t)$, then it is more likely to be assigned to the $i$-th variable than any other value of a lower function value.

The assignment function $\Psi_i(x_i, t)$ is also called the assignment state function, representing the state of agent $i$ at the time instance $t$. From (12) we can see that the assignment probability function $p_i(x_i, t)$ is defined as the assignment state function $\Psi_i(x_i)$ to the power $\alpha$ with normalization.

With the bar notation for normalization introduced in the subsection 5.1, the iterative update function (11) can be rewritten as

$$\Psi_i(x_i, t) = \sum_{\sim x_i} \left( e^{E_i(x_i)/\hbar} \prod_{j \neq i} (\bar{\Psi}_j(x_j, t-1))^{\alpha} \right), \quad \text{for } i = 1, 2, \ldots, n. \quad (13)$$

Without loss of generality, let the utility function $u_i(x)$ for the agent $i$ be

$$u_i(x) = e^{E_i(x)/\hbar}.$$ 

In this case, the agent $i$ tries to maximize the utility function $u_i(x)$ instead of maximizing the objective function $E_i(x)$ where the former task is fully equivalent to the latter. Accordingly, the simple form (13) of cooperative optimization becomes exactly same as the iterative update function (7) for the action payoff function $\Psi_i(x_i, t)$. The assignment probability function $p_i(x_i, t)$ of agent $i$ in (13) is called the strategy of player $i$ in (7).

**Some Computational Properties of Cooperative Optimization** In the simple form (13) of cooperative optimization, we can replace the constant $\alpha$ by $\lambda(t)w_{ij}$, where both $\lambda(t)$ and $w_{ij}$ are parameters, i.e.,

$$\Psi_i(x_i, t) = \sum_{\sim x_i} \left( e^{E_i(x)/\hbar} \prod_{j \neq i} (\bar{\Psi}_j(x_j, t-1))^{\lambda(t)w_{ij}} \right). \quad (14)$$

Note that a summation operator can be approximated by a maximization operator as follows:

$$\max_x e^{f(x)/\hbar} \approx \sum_x e^{f(x)/\hbar}.$$ 

(Under the assumption that the function $f(x)$ has a unique global maximum.)
Such an approximation becomes accurate when \( \hbar \to 0^+ \), i.e.,

\[
\lim_{\hbar \to 0^+} \left( \max_x e^{f(x)/\hbar} - \sum_x e^{f(x)/\hbar} \right) = 0 .
\]

With this approximation, the iterative update function \((14)\) becomes

\[
\Psi_i(x_i, t) = \max_{\sim x_i} \left( \frac{e^{E_i(x_i) / \hbar}}{\prod_{j \neq i} (\Psi_j(x_j, t - 1))^{\lambda(t) w_{ij}}} \right) .
\]

Taking the logarithm of both sides, we have

\[
\Psi_i(x_i, t) = \max_{\sim x_i} \left( E_i(x_i) + \lambda(t) \sum_{j \neq i} w_{ij} \Psi_j(x_j, t - 1) \right) . \tag{15}
\]

This is the original general form of cooperative optimization.

In this form, each agent optimizes an objective function defined at the right side of the above equation. It is called the compromised objective function in the sense that it is the linear combination of the original objective function \( E_i(x) \) for agent \( i \) and the assignment state functions \( \Psi_j(x_j, t - 1) \) of other agents \( j \) at the previous time instance \( t - 1 \). Given a variable value \( x_i \), the function value \( \Psi_i(x_i, t) \) stores the maximal value of the compromised objective function with the \( i \)-th variable fixed to the value.

Let \( \tilde{x}_i(t) \) be the value of \( x_i \) with the highest function value \( \Psi_i(x_i, t) \), i.e.,

\[
\tilde{x}_i(t) = \arg \max_{x_i} \Psi_i(x_i, t) . \tag{16}
\]

That value represents the best value of \( x_i \) at iteration time instance \( t \) for maximizing the compromised objective function defined at the right side of \((15)\). The solution of the system at iteration time instance \( t \) is the collection of those best values as follows

\[
(\tilde{x}_1(t), \tilde{x}_2(t), \ldots, \tilde{x}_n(t)) , \quad \text{simply } \tilde{x}(t) .
\]

All of the parameters \( w_{ij} \)'s together form a \( n \times n \) matrix called the propagation matrix \( W \). To have \( \sum_i E_i(x) \) as the global utility function to be maximized, it is required that the propagation matrix \( W = (w_{ij})_{n \times n} \) is non-negative, irreducible, aperiodic, and satisfying

\[
\sum_{i=1}^n w_{ij} = 1, \quad \text{for } j = 1, 2, \ldots, n .
\]

**Theorem 3.** Given a constant cooperation strength \( \lambda \) of a non-negative value less than 1 \( (0 \leq \lambda < 1) \), the general form \((15)\) of cooperative optimization has one and only one equilibrium. It always converges to the unique equilibrium with an exponential rate regardless of initial conditions.
To be more general, assume that the objective function $E_i(x)$ for agent $i$ is defined on variable set $X_i$. Recall that the solution at iteration $t$ is $\tilde{x}(t)$ (see (16)). Let $\tilde{x}(t)(X_i)$ denote the restriction of the solution on $X_i$.

**Definition 2.** The solution $\tilde{x}(t)$ is called a consensus solution if it is the optimal solution for each optimization problem defined by (14). That is,

$$\tilde{x}(t)(X_i) = \arg \max_{x_i} \left( E_i(x) + \lambda(t) \sum_{j \neq i} w_{ij} \Psi_j(x_j, t-1) \right), \quad \text{for } i = 1, 2, \ldots, n.$$

**Theorem 4.** If the general form (15) of cooperative optimization converges to a consensus equilibrium with a constant $\lambda$ satisfying $0 \leq \lambda < 1$, then it must be the global optimum of the global objective function $E_1(x) + E_2(x) + \cdots + E_n(x)$.

From (15), we can see that the agents can increase the chance of reaching a consensus when the value of the parameter $\lambda$ is increased. However, when $\lambda \geq 1$, it is no longer guaranteed that any consensus equilibrium is the global optimum. Also, the uniqueness of equilibrium is no longer guaranteed. Assume that the maximization of $E_i(x)$, for any $i$, also leads to the maximization of the global objective function $E(x)$. Then, when $\lambda \to \infty$, the cooperative optimization (15) falls back to local search, a classic optimization method (see Section 3.5 in [9]). A local search algorithm can have many local optimal solutions and the number of them may grow exponentially with the problem size.

In summary, the cooperative optimization algorithm (15) is absolutely stable when the cooperation strength $\lambda$ is less than one ($\lambda < 1$). Above that value, the number of equilibria may grow with the value. As a consequence, the algorithm may become less stable because it can get stuck into one equilibrium or another. On the other hand, the chance of reaching a consensus equilibrium increases. A consensus equilibrium is guaranteed to be the global optimal one only when $\lambda < 1$. Hence, the performance of the algorithm usually peaks at some positive value for the cooperation strength $\lambda$. It deteriorates when the value is moved away from the best performing value, either further up or further down towards the value zero.

The above investigations are not on a rigorous basis. The exact performance of the cooperative optimization algorithm (15) in relationship with the cooperation strength $\lambda$ is an open question.