Projective cofactor decompositions of Boolean functions and the satisfiability problem

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Abstract

Given a CNF formula $F$, we present a new algorithm for deciding the satisfiability (SAT) of $F$ and computing all solutions of assignments. The algorithm is based on the concept of cofactors known in the literature. This paper is a fallout of the previous work by authors on Boolean satisfiability [11, 12, 13], however the algorithm is essentially independent of the orthogonal expansion concept over which previous papers were based. The algorithm selects a single concrete cofactor recursively by projecting the search space to the set which satisfies a CNF in the formula. This cofactor is called projective cofactor. The advantage of such a computation is that it recursively decomposes the satisfiability problem into independent sub-problems at every selection of a projective cofactor. This leads to a parallel algorithm for deciding satisfiability and computing all solutions of a satisfiable formula.

1 Introduction

In this paper we consider the problem of deciding satisfiability of a Boolean equation

$$F(x_1, \ldots, x_n) = 1$$

and that of computing all satisfying assignments of variables $x_i$ in the Boolean algebra $B_0 = \{0, 1\}$. This is a celebrated problem of computer science with vast applications [2, 7, 5]. We shall denote by $B(n)$ the Boolean algebra of all Boolean functions $f : B^n \rightarrow B$ of $n$ variables over a Boolean algebra $B$. For all other notations on Boolean functions we refer to [3] and to [6] for satisfiability literature.

Although this paper is a sequel to previous papers [11, 12, 13] in which the satisfiability of the Boolean equation $F = 0$ was also considered, the central issue addressed in these papers was that of representation of Boolean functions $F$ in several variables in the Boole-Shannon expansion form and to express the satisfiability and all solutions of a satisfiable equation in terms of the expansion co-efficient functions with respect to orthonormal functions. In [13] it was shown that actually the orthonormality itself was not necessary for such an expansion and the expansion formula was vastly generalized. To recapitulate the ideas we shall re-state these results of [13] in a comprehensive form as follows [13, theorem 1,2, corollary 1].
Theorem 1. Let $G = \{g_1, g_2, \ldots, g_m\}$ be a set of non-zero Boolean functions and a Boolean function $f$ be such that

$$f \leq g_1 + g_2 + \ldots + g_m$$

Then,

1. $f$ can be expressed as

$$f = \alpha_1 g_1 + \alpha_2 g_2 + \ldots + \alpha_m g_m$$

where each $\alpha_i$, $i = 1, 2, \ldots, m$, is a Boolean function, called cofactor of $f$ wrt $g_i$, which can be chosen freely in the interval $[f, g_i, f + g_i']$.

2. $f = 0$ is consistent iff for some $i$ the system of equations $\alpha_i = 0$ and $g_i = 1$ is consistent, (i.e. $\alpha_i = 0$ has a solution on the set of satisfying assignments of $g_i$ for some $i$) for an arbitrary choice of cofactor $\alpha_i$ in its interval of existence.

3. All solutions of $f = 0$ arise as the union of all solutions of systems $\alpha_i = 0, g_i = 1$ whenever the later systems are consistent.

Hence this expansion of $f$ in terms of $g_i$ gives decomposition of the set of all satisfying assignments for $f = 0$ in terms of independent problems. Such a decomposition is thus of great importance for computation. In fact as the later result [13, corollary 2] shows an algorithm for satisfiability of a CNF formula $F$ and computation of all satisfying solutions does not require actual computation of the cofactors $\alpha_i$. The algorithm developed requires generation of independent reduced formulas $f_i$ at the satisfying assignments of $g_i$. This algorithm can thus be called (using terminology from logic) as the semantic decomposition algorithm.

1.1 Transformational algorithm for satisfiability

In this paper we also take up the problem of describing all satisfying assignments of a CNF formula $F$

$$F = \prod_{i=1}^{m} C_i$$

where $C_i$ are clauses, from the point of view of transformation of variables. The central idea of the algorithm proposed in this paper is to transform variables so that they successively map to the satisfying set of the partial clauses. This leads to yet another type of a decomposition of $F$ which gives an algorithm for satisfiability and computation of all solutions. This may be called a transformational decomposition. Using this central idea we propose in this paper a new algorithm for satisfiability of CNF formulas.

2 Properties of cofactors and construction

In [13] several properties of cofactors of Boolean functions $f$ relative to a base set $\{g_i\}$ were derived. For completeness these are summarized again here along with new properties. The cofactors $\alpha_i$ in the expansion shown in Eq. (1) can be thought of as kind of quotients and their interval of existence is given in the above theorem. We continue with the notation $\Xi(f, g)$ to denote the members of the set of cofactors of $f$ with respect to $g$. From the basic definition
of a cofactor [13, Definition 1] it follows that elements of $\Xi(f, g)$ are those functions which are restrictions of $f$ on $g_{ON}$ (i.e. match exactly with $f$ for all points on $g_{ON}$). Algebraically all such cofactors belong to the interval $[fg, f + g']$ and conversely every function in this interval is a cofactor. We call $fg$ the minimal cofactor and $f + g'$ as the maximal cofactor.

2.1 Algebra of co-factors

Some of the additional algebraic properties of cofactors are given below. For any two subsets of Boolean functions $A, B$, we define

$$A + B = \{u + v : u \in A, \ v \in B\}$$

$$A.B = \{u.v : u \in A, \ v \in B\}$$

$$A' = \{u' : u \in A\}$$

**Proposition 1.** Let $f, g, h$ be arbitrary Boolean functions. Then

1. $\Xi(f, g) + \Xi(f, h) = \Xi(f, g + h)$.
2. $\Xi(f, g).\Xi(f, h) = \Xi(f, g.h)$ as sets.
3. If $g \leq h$ then $\Xi(f, h) \subset \Xi(f, g)$.
4. $\Xi(f + h, g) = \Xi(f, g) + \Xi(h, g)$.
5. $\Xi(f, h, g) = \Xi(f, g).\Xi(h, g)$.
6. $\Xi(f', g) = \Xi(f, g)'$.

The following identity also holds

**Proposition 2.**

$$\Xi(f, g.h) = \{u.v | u \in \Xi(f, g), \ v \in \Xi(u, h)\}$$

*Proof:* By the definition of $u, v$, we see that $u.g = f.g$ and $v.h = u.h$. Thus,

$$ (u.v).(g.h) = (u.g).(v.h) $$

$$ = (f.g).(u.h) $$

$$ = (u.g).(f.h) $$

$$ = (f.g).(f.h) $$

$$ = f.(g.h) $$

so that $u.v \in \Xi(f, g.h)$.

The expansion shown in Equation 2 expresses $f$ in terms of cofactors wrt a base set $\{g_i\}$. We first show an example for computation of minimal and maximal cofactors.
Lemma 1. Let $P_h$ projects the entire space onto a subset of $P_g$ where $w$ is the minimal cofactor in $\Xi(f, C)$ is, the ON-set of $g$. We use $\zeta$ to denote the set of points on which $g$ define a particular kind of projection $P_g$. Suppose that $P_1$ is an arbitrary Boolean function.

Example 1. Consider $f = x_1 x_2 + x_2 x_3 + x_1 x_3$ and $C = (x_1 + x_3)$ then

$$fC = x_1 x_2 + x_2 x_3 = x_2 C$$

which is the minimal cofactor in $\Xi(f, C)$ while

$$f + C' = f$$

hence maximal $\Xi(f, C)$ is $f$ itself. A general cofactor in $\Xi(f, C)$ is

$$fC + pC' = x_2(x_1 + x_3) + p(x_1 x_3)$$

where $p$ is an arbitrary Boolean function.

3 Projective cofactors

Suppose that $g \neq 0, h \neq 1$ are Boolean functions in $B_0(n)$. For any Boolean function (say $g$), we use $g_{ON}$ to denote the set of points on which $g$ evaluates to 1 (the ON-set of $g$), and $g_{OFF}$ to be the set of points on which $g$ evaluates to 0 (the OFF-set of $g$).

A projection is a map $P : B_0^n \to B_0^n$. Given $g, h$ such that either $g = h = 1$ or $h \neq 1$, we define a particular kind of projection $P_{g,h} : B_0^n \to B_0^n$ as follows:

$$P_{g,h}(x) = \begin{cases} x & \text{if } x \in g_{ON} \\ y_x \in h_{OFF} & \text{if } x \in g_{OFF} \end{cases}$$

where $y_x$ is an arbitrary element in $h_{OFF}$ which may be chosen independently for each $x$). That is, the ON-set of $g$ is left unchanged, but the OFF-set of $g$ is mapped into the OFF-set of $h$.

We call the collection of such projections as the set of projections of $g$ into $h$, denoted by $\mathcal{P}(g, h)$. One possible choice is to select a fixed $y \in h_{OFF}$ and to set $y_x = y$ for all $x$. Thus $P_{g,h}$ projects the entire space onto a subset of $h_{OFF} \cup g_{ON}$.

The following important properties of such projections will be used in the sequel.

Lemma 1. Let $g_1, g_2$ and $h$ be Boolean functions with $h \neq 1$. Let $P_1$ be a projection in $\mathcal{P}_{g_1,h}$ and $P_2$ be a projection in $\mathcal{P}_{g_2,h}$. Then the compositions $(P_1 \circ P_2)$ and $(P_2 \circ P_1)$ are both projections in $\mathcal{P}_{g_1,g_2,h}$.

Proof: Suppose that $u$ is in the ON-set of $g_1, g_2$. Then $u$ is in $g_{1ON}$ and $g_{2ON}$, and thus $(P_1 \circ P_2)(u) = u$. If $u$ is not in the ON-set of $g_1, g_2$, then $u$ is either in $g_{2OFF}$ or in $g_{1OFF}$ or in both. Suppose that $u$ is in $g_{2OFF}$, then $w = P_2(u)$ is in $h_{OFF}$. Observe that $w$ may or may not be in $g_{1OFF}$. In either case, it is easy to check that $P_1(w)$ will be in $h_{OFF}$. Thus the composition $(P_1 \circ P_2)$ maps $u$ appropriately. The other cases can be handled similarly. \(\square\)

Given a projection $P$ and some function $h$, we can then obtain the function

$$(h \circ P)(x) = h(P(x))$$

where $\circ$ denotes composition. Now if $P$ is chosen to be in $\mathcal{P}_{g,h}$, we define

$$\zeta[h, g, P] = (h \circ P)$$
The function $\zeta[h, g, P]$ depends on $P \in \mathcal{P}_{g,h}$ which in turn depends on the elements $y_x$ in $h_{OFF}$ used in determining $P$. Further, every $\zeta$ defined in this manner matches with the restriction of $h$ on $g_{ON}$ and every such $\zeta$ is a cofactor in $\Xi(h, g)$ and also belongs to the interval $[hg, h + g']$. We call all such cofactors, the projective cofactors of $g$ in $h$. (Note that if $g \neq 1$, then such a projective co-factor is defined only if $h \neq 1$). The projective cofactors of $g$ in $h$ constitute a set of Boolean functions parametrized by the choice of elements $y_x \in h_{OFF}$ for each element $x \in g_{OFF}$. A similar (but not identical) concept of a projective cofactor has been used by Coudert and Madre to construct the restrict and constrain operators $[4]$.

Some properties of projective cofactors are easy to establish.

**Lemma 2.** Let $f, g, h_1$ and $h_2$ be Boolean functions with $f \neq 1$, $h_1 \neq 1$ and $h_2 \neq 1$. Then

1. If $f \leq g$, then $\zeta[f, g, P] = f$ for all $P \in \mathcal{P}_{g,f}$.
2. If $f.g = 0$, then $\zeta[f, g, P] = 0$ for any projection $P \in \mathcal{P}_{g,f}$.
3. If $h_1 \leq h_2$, then $\mathcal{P}_{g,h_2} \subset \mathcal{P}_{g,h_1}$.
4. If $h \neq 1$, then $\zeta[h, g, P] \leq h$ for all $P \in \mathcal{P}_{g,h}$.

Further, for any projection $P$, if $f_1, f_2$ are Boolean functions, then

$$\zeta[f_1, f_2, g, P] = \zeta[f_1, g, P] \cdot \zeta[f_2, g, P]$$

$$\zeta[f_1 + f_2, g, P] = \zeta[f_1, g, P] + \zeta[f_2, g, P]$$

$$\zeta[f', g, P] = \zeta[f, g, P]'$$

**Proof:** The conclusions follow from a routine application of the definition of $\mathcal{P}_{g,f}$ and $\zeta[]$. □

The importance of projective co-factors lies in the following result.

**Theorem 2.** Let $f, g, h$ be Boolean functions with $f = g.h$ and $h \neq 1$. Then $f$ is satisfiable if and only if, for any projection $P \in \mathcal{P}_{g,h}$, $\zeta[h, g, P]$ is satisfiable. Further, the solution set to $f(z) = 1$ and $\zeta[h, g, P](z) = 1$ are the same, so that every solution of $f(z) = 1$ can be obtained by solving $\zeta[h, g, P](z) = 1$.

**Proof:** Since $h \neq 1$, the set of projections $\mathcal{P}_{g,h}$ contains at least one element.

Suppose that $f$ is satisfiable and $f(z) = 1$. Then $g(z) = 1$ and $h(z) = 1$. Observe that $z$ belongs to $g_{ON}$. Let $P$ be any projection in $\mathcal{P}_{g,h}$. Clearly $P(z) = z$ because $z \in g_{ON}$. Thus $\zeta[h, g, P](z) = h(P(z)) = h(z) = 1$. This proves necessity.

Conversely, assume that for some $z$ and $P \in \mathcal{P}_{g,h}$, $\zeta[h, g, P](z) = 1$. Let $u = P(z)$, so that $h(u) = 1$. Now consider $z$: if $z$ were to be in $g_{OFF}$ then by the definition of $P$, $u = P(z) \in h_{OFF}$ and thus we have $h(u) = 0$, a contradiction. Thus, $z \in g_{ON}$ and consequently $u = z$. Thus $g(z).h(z) = f(z) = 1$ and $f$ is satisfiable.

We have shown that $f(z) = 1$ if and only if $\zeta[h, g, P](z) = 1$ to complete the proof of the last claim.

□

From this result, we infer the following:
Corollary 1. Let $f = g.h$. Choose an arbitrary projection $P \in P_{g,h}$.

1. If $g$ and $\zeta[h,g,P]$ are independently satisfiable then $f$ is satisfiable.

2. The solution set of $\zeta[h,g,P](x) = 1$ is the same as the solution set of $f(x) = 1$.

3. If one of $g$, $\zeta[h,g,P]$ is identically 0, then $f$ is identically 0.

Thus, to check satisfiability of $f = g.h$, it is enough to independently check the satisfiability of $g$ and $\zeta[h,g,P_{g,h}]$ for any choice of $P_{g,h}$. This leads to a decomposition technique in which the satisfiability of $f$ can be determined in terms of simpler functions $g$ and $\zeta[h,g,P_{g,h}]$. In general, we may be asked to check the satisfiability of a Boolean function given as $f = h_1.h_2 \ldots h_k$, where each $h_i \neq 1$ is described by a small formula (a clause for example). Using Theorem 2, we can show the following.

Theorem 3. Let $f = h_1.h_2 \ldots h_k$, with $h_i \neq 0$ for each $i$. For $i = 1,2, \ldots k$, then $f$ is satisfiable, if and only if the following functions are satisfiable:

\[
\begin{align*}
    f_1 &= h_1, \text{ choose } P_1 \in P_{f_1,h_2} \\
    f_2 &= \zeta[h_2,f_1,P_1], \text{ choose } P_2 \in P_{f_2,h_3} \\
    f_3 &= \zeta[\zeta[h_3,f_1,P_1],f_2,P_2], \text{ choose } P_3 \in P_{f_3,h_4} \\
    & \quad \vdots \\
    f_k &= \zeta[\zeta[\ldots \zeta[h_k,f_1,P_1],f_2,P_2] \ldots],f_{k-1},P_{k-1}
\end{align*}
\]

where, for $j = 1,2, \ldots (k-1)$, $P_j \in P_{f_j,h_{j+1}}$. Further, if $f_k(x) = 1$, then $f(x) = 1$.

Proof: For $j = 1,2, \ldots k$, let $u_j = \Pi_{i=j}^k h_i$. Then $f = h_1.u_2$ and $u_i = h_i.u_{i+1}$ for $i = 1,2, \ldots, k-1$. By Theorem 2, since $h_i \neq 1$, $f$ is satisfiable if and only if $\zeta[u_2,h_1,P_1]$ is satisfiable. But $\zeta[u_2,h_1,P_1] = \zeta[h_2,h_1,P_1], \zeta[u_3,h_1,P_1]$. Thus $\zeta[u_2,h_1,P_1]$ is satisfiable if and only if $f_2 = \zeta[h_2,h_1,P_1]$ is satisfiable and $\zeta[\zeta[u_3,h_1,P_1],f_2,\ldots],f_{k-1},P_{k-1}$ is satisfiable for a projection $P \in P_{f_2,f_1}$. But since $f_2 \leq h_2$, it follows from Lemma 2 that $P_2 \in P_{f_2,f_1}$ and $P$ could be chosen to be $P_2$. Continuing in this fashion gives us the result. The conclusion that the solution set of $f_k(x) = 1$ is the same as the solution set of $f(x) = 1$ follows from Corollary 1. 

This result leads to a parallel algorithm to solve the satisfiability problem. This algorithm is described in Section 4.

4 An algorithm for SAT using projective cofactors

From Theorem 3, we obtain the algorithm shown in Figure 1. The algorithm starts with a function defined as a product $C_1.C_2 \ldots C_n$. At each step of the algorithm, we obtain a set of potentially simple active functions out of which we select one and check its satisfiability. In going from one step to the next, we form projective cofactors of the remaining elements in the set with the selected function (these can be performed in parallel). The algorithm stops when the set of elements has only one element whose satisfiability needs to be checked.

We observe the following:

- The inner loop can be parallelized.
Given $f = h_1.h_2\ldots h_k$, return SAT status, and assignment $A$

- status = true, $A \leftarrow$ empty
- Set $w_1 \leftarrow h_1, w_2 \leftarrow h_2, \ldots w_k \leftarrow h_k$
- for $i = 1; i \leq k; i = i + 1$ do
  - $f_i \leftarrow w_i$
  - if $f_i$ is not SAT then
    - status = false
    - break
  - end if
- if $i = k$ then
  - break
- end if
- select projection $P_i \in \mathcal{P}_{f_i,h_{i+1}}$
- for $j = i + 1; j \leq k; j = j + 1$ do
  - $w_j \leftarrow \zeta[w_j, f_i, P_i]$
- end for
- if status then
  - $A \leftarrow$ SAT-assignment for $f_k$
- end if
- return (status, $A$)

Figure 1: A simple parallel algorithm for SAT based on projective decomposition
Satisfiability checking is performed only on the functions $f_i$. This step can be very efficient if each $f_i$ depends on a small set of variables.

The main reduction step is the computation of $\zeta[w_j, f_i, P_i]$, which can be efficient if $w_j$ and $f_i$ depend on a small number of variables. Note that if each of the $h_i's$ is simple, then finding an element in the off-set of an $h_i$ is easy, and thus calculating a projection into $h_i$ is also easy.

The projections $P_1, P_2, \ldots P_{k-1}$ can be chosen arbitrarily. In practice, the choice of projections may have an impact on the effort involved in computing $\zeta[w_j, f_i, P_i]$.

4.1 Choosing a projection $P$ onto a function $g$, and the calculation of $\zeta[f, g, P]$

The critical step in the algorithm described in Figure 1 is the computation of $\zeta[h, g, P]$ given functions $h, g$ and a projection $P \in \mathcal{P}_{g,h}$. In general, $h$ and $g$ are available as Boolean formulas, and we are interested in obtaining a formula for $\zeta[h, g, P]$. There are many possible ways of choosing the projection $P \in \mathcal{P}_{g,h}$. We illustrate a simple technique to find a projection and express it as a set of boolean formulas. Once the projection is available as a set of boolean formulas, the actual calculation of a formula for $\zeta[h, g, P]$ can then be carried out easily.

Consider the case that $g$ is a clause, and $h$ is available as a product of clauses. Suppose that

$$g = x + y + z$$

Now the projection must take the point $x = 0, y = 0, z = 0$ and map it to some element in the off-set of $h$. If $h$ is available as a product of sums, then such an element can be found easily. We assume that an element of the off-set of $h$ is known. Let this element be $(u_1, u_2, \ldots u_n)$. Note that if $h$ does not depend on variable $x_i$, then we may choose $u_i = x_i$. Then a projection $P \in \mathcal{P}_{g,h}$ can be determined as follows: map the point $(x_1, x_2, \ldots, x_n)$ to itself when it is in the on-set of $g$, and onto $(u_1, u_2, \ldots u_n)$ when it is not in the on-set of $g$. The projection then becomes

$$x_1 \rightarrow g.x_1 + g'.u_1$$
$$x_2 \rightarrow g.x_2 + g'.u_2$$
$$\vdots$$
$$x_n \rightarrow g.x_n + g'.u_n$$

This can be simplified further. For each $x_i$ which does not appear in $h$, we set $u_i = x_i$. If, for some $q, u_q = 0$, then $g.x_q + g'.u_q$ can be simplified to $g.x_q$, and when $u_q = 1$, then $g.x_q + g'.u_q$ can be simplified to $g' + x_q$. Thus, this projection takes

$$x_i \rightarrow \begin{cases} 
  x_i & \text{when } h \text{ does not depend on } x_i \\
  g.x_i & \text{when } u_i = 0 \\
  g' + x_i & \text{when } u_i = 1
\end{cases}$$

If $g$ itself can be written as a product $g_1.g_2.\ldots .g_m$, then we can find a projection in $\mathcal{P}_{g,h}$ by using Lemma 1. Choose $P_i \in \mathcal{P}_{g_i,h}$ for $i = 1, 2, \ldots m$, and set $P$ to be the composition

$$(P_1 \circ P_2 \circ \ldots P_m)$$
Instead of selecting a single element in $h_{OFF}$ to project all of $g_{OFF}$, a general procedure would be to select, for each element $x$ such that $g(x) = 0$, an element $u_x$ such that $h(u_x) = 0$. The resulting Boolean functions would then need to be simplified to obtain formulas describing the projection. Finding the projection with the simplest representation formula is something that needs to be studied further.

5 Some Examples

We will illustrate the use of Theorem 3 and the algorithm shown in Figure 1 on two simple examples.

5.1 A satisfiable function

Consider $f$ defined by

$$f = C_1C_2C_3 = (x' + y + w)(y' + z + w')(x + z + w')$$

This function is satisfiable. We use Theorem 3 to confirm that this is so.

- Let $f_1 = C_1$. We choose the point $y = 1, z = 0, w = 1$ in the off-set of $C_2$ to construct the projection of $f_1$ into $C_2$. The projection $P_1$ can then be worked out to be

  $$x \rightarrow x$$

  $$y \rightarrow y + C_1' = y + x w'$$

  $$z \rightarrow z.C_1 = z.(x' + y + w)$$

  $$w \rightarrow w + C_1' = w + x y'$$

- We compute

  $$f_2 = \zeta[C_2, f_1, P_1] = (y' x' + y' w + z x' + z y + z w + w' x' + w' y')$$

  Similarly,

  $$\zeta[C_3, f_1, P_1] = C_3$$

- For the projection $P_2$ of $f_2$ onto $C_3$, we choose the point $x = 0, z = 0, w = 1$ in the OFF-set $C_3$ and fix $P_2$ as:

  $$x \rightarrow x.f_2 = x y' w + x y w' + x w z$$

  $$y \rightarrow y$$

  $$z \rightarrow z.f_2 = z x' + z y + z w$$

  $$w \rightarrow w + f_2' = (w + x).(w + y).(x' + y' + w).(x + z' + w).(x + y + w)$$
Using $P_2$, we obtain

$$f_3 = \zeta[\zeta[C_3, f_1, P_1], f_2, P_2] = \zeta[C_3, f_2, P_2] = xy'w + xyw' + zx' + zy + zw + w'x + w'y'$$

Thus, by Theorem 3 we observe that since $f_3$ is satisfiable, $f$ is satisfiable. Further, note that the solutions of $f_3 = 1$ are exactly the solutions of $f = 1$.

### 5.2 A non-satisfiable example

Let

$$f = C_1C_2C_3C_4 = (x + y)(x + y')(x' + y)(x' + y')$$

In this case the function $f$ is not satisfiable. Let us proceed with the algorithm implied by Theorem 3 to confirm this.

- Let $f_1 = C_1 = (x + y)$. Fix the point $x = 0$, $y = 1$ in the offset of $C_2$ so that $P_1 \in \mathcal{P}_{C_1,C_2}$ is the projection:

$$
\begin{align*}
  x & \rightarrow x \\
  y & \rightarrow x' + y
\end{align*}
$$

For this projection we find

$$
\begin{align*}
  f_2 &= \zeta[C_2, f_1, P_1] = x \\
  w_{31} &= \zeta[C_3, f_1, P_1] = x' + y \\
  w_{41} &= \zeta[C_4, f_1, P_1] = x' + y'
\end{align*}
$$

- Choose (using the point $x = 1$, $y = 0$ in the set of $C_3$) $P_2 \in \mathcal{P}_{f_2,C_3}$ to be the projection:

$$
\begin{align*}
  x & \rightarrow 1 \\
  y & \rightarrow x.y
\end{align*}
$$

Using this, we find

$$
\begin{align*}
  f_3 &= \zeta[w_{31}, f_2, P_2] = x.y \\
  w_{42} &= \zeta[w_{41}, f_2, P_2] = x' + y'
\end{align*}
$$

- Choose the element $x = 1$, $y = 1$ in the offset of $C_4$ to obtain $P_3 \in \mathcal{P}_{f_3,C_4}$ as:

$$
\begin{align*}
  x & \rightarrow 1 \\
  y & \rightarrow 1
\end{align*}
$$

We then find that

$$
\begin{align*}
  f_4 &= \zeta[w_{42}, f_3, P_3] \\
       &= 0
\end{align*}
$$

Thus, by Theorem 3 the function $f$ is not satisfiable.
6 Conclusions

Given a Boolean function $f(x_1, x_2, \ldots, x_n)$ of $n$ variables, the satisfiability question asks if there is a point at which $f$ evaluates to 1. An analogous consistency problem is to determine whether $f = 0$ has a solution. Decompositions of $f$ can help in solving such problems.

If $f$ satisfies

$$f \leq g_1 + g_2 + \ldots + g_m$$

where $g_1, g_2, \ldots, g_m$ are non-zero functions, then $f$ can be written as

$$f = \alpha_1 g_1 + \alpha_2 g_2 + \ldots + \alpha_m g_m$$

where $\alpha_i$ is an arbitrary Boolean function in the interval $[f.g_i, f + g'_i]$ \textsuperscript{13}. Such an $\alpha_i$ is termed a co-factor of $g_i$ in $f$. For this decomposition, $f$ is satisfiable if and only if $\alpha_i g_i$ is satisfiable for some $i \in \{1, 2, \ldots, m\}$. Thus, the original satisfiability problem can be solved as $m$ parallel and independent problems. If the functions $\alpha_i g_i$ are simpler than $f$, there can be substantial reduction in computational effort.

We have introduced the notion of a projective co-factor: If $g$ and $h$ are two Boolean functions ($h \neq 1$), then we define a projection $P_{g,h} : B_0^n \rightarrow B_0^n$ which maps the ON-set of $g$ to itself and the OFF-set of $g$ into the OFF-set of $h$. With respect to such a projection, we can define, for any function $f$, a projective co-factor of $g$ in $f$ (denoted by $\zeta[f,g,P_{g,h}]$ which maps $x$ to $f(P_{g,h}(x))$). We have shown that if $f = g.h$ with $h \neq 1$, then $f$ is satisfiable if and only if $g$, and $\zeta[h,g,P_{g,h}]$ are independently satisfiable for any projection $P_{g,h}$ chosen as above. This result enables us to check the satisfiability of $f$ by separately (hence, in parallel) checking the satisfiability of two potentially simpler functions $g$ and $\zeta[h,g,P_{g,h}]$.

From the projective decomposition result, we obtain a new easily parallelizable transformational algorithm for the solution of a general satisfiability problem: check the satisfiability of $f = g_1.g_2.\ldots.g_m$ where each $g_i$ is a clause (or in the case of XOR-sat, an XOR-clause). The decomposition property can in principle be used to devise a variety of decomposition schemes for solving the satisfiability problem. Further algorithmic development and heuristics need to be investigated in order to determine the practical feasibility of this approach.

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