Safe Linear Thompson Sampling

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Abstract

The design and performance analysis of bandit algorithms in the presence of stage-wise safety or reliability constraints has recently garnered significant interest. In this work, we consider the linear stochastic bandit problem under additional linear safety constraints that need to be satisfied at each round. We provide a new safe algorithm based on linear Thompson Sampling (TS) for this problem and show a frequentist regret of order \( O(d^{3/2} \log^{1/2} d \cdot T^{1/2} \log^{3/2} T) \), which remarkably matches the results provided by [Abeille et al., 2017] for the standard linear TS algorithm in the absence of safety constraints. We compare the performance of our algorithm with a UCB-based safe algorithm and highlight how the inherently randomized nature of TS leads to a superior performance in expanding the set of safe actions the algorithm has access to at each round.

1 Introduction

The application of stochastic bandit optimization algorithms to safety-critical systems has received significant attention in the past few years. In such cases, the learner repeatedly interacts with a system with uncertain reward function and operational constraints. Yet, in spite of this uncertainty, the learner needs to ensure that her actions do not violate the operational constraints at any round of the learning process. As such, especially in the earlier rounds, there is a need to choose actions with caution, while at the same time learning more about the set of possible safe actions. Notably, the estimated safe set at each stage of the algorithm might not originally include the optimal action. This uncertainty about safety and the resulting conservative behavior means the learner could experience additional regret in such environments.

In this paper, we focus on a special class of stochastic bandit optimization problems where the reward is a linear function of the actions. This class of problems, referred to as linear stochastic bandits (LB), generalizes multi-armed bandit (MAB) problems to the setting where each action is associated with a feature vector \( x \), and the expected reward of playing each action is equal to the inner product of its feature vector and an unknown parameter vector \( \theta \). There exists several variants of linear stochastic bandits that study the finite [Auer et al., 2002] or infinite [Dani et al., 2008, Rusmevichientong and Tsitsiklis, 2010, Abbasi-Yadkori et al., 2011] set of actions, as well as the case where the set of feature vectors can change over time [Chu et al., 2011, Li et al., 2010].

Two efficient approaches have been developed for LB: linear UCB (LUCB) and linear Thompson Sampling (LTS). For LUCB, [Abbasi-Yadkori et al., 2011] provides a regret bound of order \( O(d \cdot T^{1/2} \log T) \). From a Bayesian point of view, [Agrawal and Goyal, 2012] show that LTS achieves an expected regret of order \( O(d^{3/2} \cdot T^{1/2} \log T) \), while [Agrawal and Goyal, 2013, Abeille et al., 2017] adopt a frequentist view and show a regret of order \( O(d^{3/2} \log^{1/2} d \cdot T^{1/2} \log^{3/2} T) \) for LTS.

Here we provide a LTS algorithm that respects linear safety constraints and study its performance. Let us formally define our problem setting before stating a summary of our contributions.

1.1 Safe Stochastic Linear Bandit Model

The setting. The learner is given a convex and compact set of actions \( X_0 \subset \mathbb{R}^d \). At each round \( t \), playing an action \( x_t \in X_0 \) results in observing a reward

\[ r_t := x_t^T \theta_* + \zeta_t, \]

where \( \theta_* \in \mathbb{R}^d \) is a fixed and unknown parameter and \( \zeta_t \) is a zero mean additive noise.

Safety constraint. We further assume that the envi-
The environment is subject to a linear side constraint (referred to as the safety constraint) which needs to be satisfied at every round $t$:

$$x_t^\top \mu_* \leq C.$$  

(2)

Here, $\mu_*$ is a fixed and unknown parameter and the positive constant $C$ is known to the learner. The set of “safe actions” denoted by $\mathcal{X}_0^*(\theta_*)$ that satisfy the safety constraint (2) is defined as

$$\mathcal{X}_0^*(\mu_*) := \{x \in \mathcal{X}_0 : x^\top \mu_* \leq C\}.$$  

(3)

The set of safe actions depends on the parameter $\mu_*$, which is unknown to the learner. Hence, the learner who is not able to identify the safe action set must play conservatively in order to not violate the safety constraint for all rounds $t$ (at least with high probability). When playing an action $x_t$, the learner does not observe $x_t^\top \mu_*$ but, rather, a noisy measurement:

$$w_t = x_t^\top \mu_* + \zeta_t,$$

where $\zeta_t$ is zero mean additive noise. For ease of notation, we will refer to the safe action set by $\mathcal{X}_0^*$ and drop the dependence on $\mu_*$. 

**Regret.** The cumulative pseudo-regret of the learner up to round $T$ is defined as $R(T) = \sum_{t=1}^T x_t^\top \theta_* - x_t^\top \theta_*$, where $x_*$ is the optimal safe action that maximizes the expected reward, i.e., $x_* = \arg\max_{x \in \mathcal{X}_0^*} x^\top \theta_*$. 

**Goal.** The learner’s objective is to control the growth of the pseudo-regret. Moreover, we require that the chosen actions $x_t$ for $t \in [T]$ are safe (i.e., belong to $\mathcal{X}_0^*$ (13)) with high probability. For simplicity, we use regret to refer to the pseudo-regret $R(T)$. 

### 1.3 Related Work

**Safety** - A diverse body of related works on stochastic optimization and control have considered the effect of safety constraints that need to be met during the run of the algorithm. For example, [Sui et al., 2015, Sui et al., 2018] study the problem of nonlinear bandit optimization with nonlinear constraints through a UCB approach using Gaussian processes (GPs) (as non-parametric models). The algorithms in [Sui et al., 2015, Sui et al., 2018] come with convergence guarantees but no regret bound. Such approaches for safety-constrained optimization using GPs have shown great promises in robotics applications [Ostafew et al., 2016, Akametalu et al., 2014]. Without the GP assumption, [Usmanova et al., 2019] proposes and analyzes a safe variant of the Frank-Wolfe algorithm to solve a smooth optimization problem with an unknown convex objective function and unknown linear constraints. Their main theoretical result provides convergence guarantees for their proposed algorithm. A large body of work has considered safety in the context of model-predictive control, see, e.g., [Aswani et al., 2013, Koller et al., 2018] and references therein. Focusing specifically on linear stochastic bandits, extensions of UCB-type algorithms to provide safety guarantees with provable regret bounds have been considered recently. For example, in [Kazerouni et al., 2017], safety refers to the requirement of ensuring that the cumulative reward up to each round of the algorithm stays above a given percentage of the performance of a known baseline approach. Recently, [Amani et al., 2019] considered the effect of safety constraints similar but not identical to ours on the regret of LUCB, where a problem-dependent upper bound on the regret is provided. The algorithm’s regret performance depends on the location of the optimal action in the safe action set, increasing significantly in problem instances for which the safety constraint is active.

**Thompson Sampling** - Even though Thompson sampling (TS)-based algorithms [Thompson, 1933] are computationally easier to implement than UCB-based algorithms and have shown great empirical performance, they were largely ignored by the academic community until a few years ago, when a series of papers (e.g., [Russo and Van Roy, 2014, Abeille et al., 2017, Agrawal and Goyal, 2012, Kaufmann et al., 2012]) showed that TS achieves optimal performance in both frequentist and Bayesian settings. Most
of the literature focused on the analysis of the Bayesian regret of TS for general settings such as linear bandits or reinforcement learning (see e.g., [Osband and Van Roy, 2015]). More recently, [Russo and Van Roy, 2016, Dong and Van Roy, 2018, Dong et al., 2019] provided an information-theoretic analysis of TS, where the key tool in their approach is the information ratio which quantifies the trade-off between exploration and exploitation. Additionally, [Gopalan and Mannor, 2015] provides regret guarantees for TS in the finite and infinite MDP setting. Another notable paper is [Gopalan et al., 2014], which studies the stochastic multi-armed bandit problem in complex action settings. They provide a regret bound that scales logarithmically in time with improved constants. However, none of the aforementioned papers study the performance of TS for linear bandits with safety constraints. Our proof for Safe-LTS is inspired by the proof technique in [Abeille et al., 2017].

2 Safe Linear Thompson Sampling

Our proposed algorithm is a safe variant of Linear Thompson Sampling (LTS). At any round $t$, given a regularized least-squares (RLS) estimate $\hat{\theta}_t$, the algorithm samples a perturbed parameter $\tilde{\theta}_t$ that is appropriately distributed to guarantee sufficient exploration. Then, for the sampled $\tilde{\theta}_t$, the algorithm chooses the optimal action while making sure that safety constraint (2) holds. The presence of the safety constraint complicates the learner’s choice of actions. To ensure that actions remain safe at all rounds, the algorithm constructs a confidence region $C_t$, which contains the unknown parameter $\mu_s$ with high probability. It then ensures that the action $x_t$ chosen by the algorithm satisfies the safety constraint $\forall v \in C_t$. The summary is presented in Algorithm 1 and a detailed description follows in the rest of this section.

### 2.1 Model assumptions

#### Notation. For a positive integer $n$, $[n]$ denotes the set $\{1, 2, \ldots, n\}$. The Euclidean norm of a vector $x$ is denoted by $\|x\|_2$. The weighted 2-norm of the vector $x$ with respect to a positive semidefinite matrix $V$ is defined by $\|x\|_V = \sqrt{x^\top V x}$. We also use the standard $\tilde{\mathcal{O}}$ notation that ignores poly-logarithmic factors.

Let $\mathcal{F}_t = \{\mathcal{F}_t, \sigma(x_1, \ldots, x_t, \xi_t, \ldots, \xi_t, \zeta_t, \ldots, \zeta_t)\}$ denote the filtration that represents the accumulated information up to round $t$. In the following, we introduce standard assumptions on the problem.

**Assumption 1.** For all $t$, $\xi_t$ and $\zeta_t$ are conditionally zero mean and $R$-sub-Gaussian noises with a constant $\alpha$.

```plaintext
Algorithm 1: Safe Linear Thompson Sampling (Safe-LTS)
1 Input: $\delta, T, \lambda$
2 Set $\delta' = \frac{\delta}{m}$
3 for $t = 1, \ldots, T$ do
4 Sample $\eta_t \sim \mathcal{D}^\text{TS}$ (see Section 2.2.1)
5 Set $V_t = \lambda I + \sum_{s=1}^{t-1} x_s x_s^\top$
6 Compute RLS-estimate $\hat{\theta}_t$ and $\tilde{\mu}_t$ (see (6))
7 Set: $\tilde{\theta}_t = \hat{\theta}_t + \beta_t(\delta') V_t^{-\frac{1}{2}} \eta_t$
8 Build the confidence region:
9 $C_t(\delta') = \{v \in \mathbb{R} : \|v - \hat{\mu}_t\|_V \leq \beta_t(\delta')\}$
10 Compute the estimated safe set:
11 $X_t^s = \{x \in X_0 : x^\top v \leq C, \forall v \in C_t(\delta')\}$
12 Play the following safe action:
13 $x_t = \arg\max_{x \in X_t^s} x^\top \tilde{\theta}_t$
14 Observe reward $r_t$ and measurement $w_t$
12 end for
```

#### Assumption 2. There exists a positive constant $S$ such that $\|\theta_\star\|_2 \leq S$ and $\|\mu_s\|_2 \leq S$.

**Assumption 3.** The action set $X_0$ is a compact and convex subset of $\mathbb{R}^d$ that contains the origin. We assume $\|x\|_2 \leq L$, $\forall x \in X_0$ and $x_1^\top \theta_\star \leq 1$.

**Remark 2.1.** It is straightforward to generalize our results to the case where the noises $\xi_t$ and $\zeta_t$ are sub-Gaussian with different constants and the unknown parameters $\theta_\star$ and $\mu_s$ have different upper bounds. In this paper, for brevity of the notation we assume these are equal.

### 2.2 Algorithm description

Let $(x_1, \ldots, x_t)$ be the sequence of actions and $(r_1, \ldots, r_t)$ and $(w_1, \ldots, w_t)$ be the corresponding rewards and measurements, respectively. Then, $\theta_\star$ and $\mu_s$ can be estimated with $\lambda$-regularized least square. For any $\lambda > 0$, the Gram matrix $V_t$ and RLS-estimates $\hat{\theta}_t$ of $\theta_\star$ and $\mu_t$ of $\mu_s$ are as follows:

$$V_t = \lambda I + \sum_{s=1}^{t-1} x_s x_s^\top,$$

$$\hat{\theta}_t = V_t^{-1} \sum_{s=1}^{t-1} \xi_s x_s, \quad \tilde{\mu}_t = V_t^{-1} \sum_{s=1}^{t-1} w_s x_s.$$
Based on \( \hat{\theta}_t \) and \( \hat{\mu}_t \), Algorithm 1 constructs two confidence regions \( E_t(\delta') \) and \( C_t(\delta') \) as follows:

\[
E_t(\delta') := \{ \theta \in \mathbb{R}^d : \| \theta - \hat{\theta}_t \|_V < \beta_t(\delta') \},
\]
\[
C_t(\delta') := \{ v \in \mathbb{R}^d : \| v - \hat{\mu}_t \|_V < \beta_t(\delta') \}.
\]

The ellipsoid radius \( \beta_t(\delta') \) is chosen according to the Theorem 2.1 in [Abbasi-Yadkori et al., 2011] in order to guarantee that \( \theta_* \in E_t(\delta') \) and \( \mu_* \in C_t(\delta') \) with high probability.

**Theorem 2.1.** ([Abbasi-Yadkori et al., 2011]) Let Assumptions 1 and 2 hold. For a fixed \( \delta \in (0, 1) \), and

\[
\beta_t(\delta) = R \sqrt{d \log \left( \frac{1 + (t - 1) L^2}{\delta} \right) + \sqrt{d \log \frac{1}{\delta}}},
\]

with probability at least \( 1 - \delta \), it holds that \( \theta_* \in E_t(\delta) \) and \( \mu_* \in C_t(\delta) \), for all \( t \geq 1 \).

The algorithm proceeds in two steps which we describe next.

### 2.2.1 Sampling from the posterior: the safe setting

The LTS algorithm is a randomized algorithm constructed on the RLS-estimate of the unknown parameter \( \theta_* \). At any round \( t \), the LTS algorithm samples the parameter \( \hat{\theta}_t \) from the posterior as

\[
\hat{\theta}_t = \hat{\theta}_t + \beta_t(\delta') V_t^{-\frac{1}{2}} \eta_t.
\]

In the standard (non-safe) LTS setting, [Agrawal and Goyal, 2012] define TS as a Bayesian algorithm. As such, they consider a Gaussian Prior over the unknown parameter \( \theta_* \), which is updated according to reward observations. At each round, they play an optimal action corresponding to a random sample drawn from the posterior, and provide regret guarantees for this Bayesian approach. More recently, [Agrawal and Goyal, 2013, Abeille et al., 2017] showed that TS can be defined as a randomized algorithm over the RLS-estimate of the unknown parameter \( \theta_* \). They show that the same guarantees hold as long as the parameter \( \eta_t \) is sampled from a distribution \( D_{\text{TS}} \), which satisfies certain concentration and anti-concentration properties that hold for more distributions than the Gaussian prior.

Unfortunately, it turns out that the presence of safety constraints leads to a more challenging problem, in which the distributional assumptions set forth in [Abeille et al., 2017] do not provide sufficient exploration needed to expand the safe set fast enough. As such, in order to obtain our regret guarantees for Algorithm 1 we need to impose appropriate modifications on the [Abeille et al., 2017] distributional properties. We state the new properties next and discuss them immediately after.

**Definition 2.1.** \( D_{\text{TS}} \) is a multivariate distribution on \( \mathbb{R}^d \), absolutely continuous with respect to the Lebesgue measure which satisfies the following properties:

- (anti-concentration) there exists a strictly positive probability \( p \) such that for all \( u \in \mathbb{R}^d \) with \( \| u \|_2 = 1 \),

\[
P_{u \sim D_{\text{TS}}} \left( u^\top \eta \geq 1 - \frac{2}{C} \right) \geq p,
\]

- (concentration) there exists positive constants \( c, c' \) such that \( \forall \delta \in (0, 1) \),

\[
P_{\eta \sim D_{\text{TS}}} \left( \| \eta \|_2 \leq \left( 1 + \frac{2}{C} \right) \sqrt{c d \log \left( \frac{c' d^2}{\delta} \right)} \right) \geq 1 - \delta.
\]

As observed in [Abeille et al., 2017], the algorithm explores far enough from \( \hat{\theta}_t \) (anti-concentration), but not too much (concentration). Compared to [Abeille et al., 2017], we have added an extra term \( (\frac{2}{C}) \) in (11) and (12). While inconspicuous, adding this term is critical for guaranteeing small regret in the Safe-LTS setting; see Section 3.1 for details. We note that, in the slightly more general setting where we do not assume \( x^\top \theta_* \leq 1 \) (Assumption 3), the constant \( 1 + \frac{2}{C} \) needs to be replaced by \( 1 + \frac{2}{C} \) LS (see Section D in Appendix for details).

### 2.2.2 Choosing the safe action

Since the learner does not know the safe action set \( \mathcal{X}_0^s \), she performs conservatively in order to satisfy the safety constraint (2). This is achieved by first building a confidence region \( C_t(\delta') \) around the RLS-estimate \( \hat{\mu}_t \) at each round \( t \). Based on \( C_t(\delta') \), the algorithm creates the so-called safe decision set denoted as \( \mathcal{X}_t^s \):

\[
\mathcal{X}_t^s = \{ x \in \mathcal{X}_0 : x^\top v \leq C, \forall v \in C_t(\delta') \},
\]

and chooses the safe action \( x_t \) from this set that maximizes the expected reward given the sampled parameter \( \hat{\theta}_t \). Note that \( \mathcal{X}_t^s \) contains actions which are safe with respect to all the parameters in the confidence region \( C_t(\delta') \), and not only \( \mu_* \). Therefore, this safe decision set \( \mathcal{X}_t^s \) is an inner approximation of safe action set \( \mathcal{X}_0 \), which may lead to extra regret for Safe-LTS that was otherwise absent from LTS.

### 3 Regret Analysis

In this section, we present our main result, a tight regret bound for Safe-LTS, and discuss key proof ideas.
For any $\theta \in \mathbb{R}^d$, denote the corresponding optimal value of playing the optimal action from the safe action set $X_0^s$ in (3) by

$$J(\theta) = \max_{x \in X_0^s} x^\top \theta.$$  \hfill (14)

At each round $t$, for the sampled parameter $\tilde{\theta}_t$, denote the optimal safe action that algorithm plays and its optimal value by

$$x_t = \arg \max_{x \in X_0^s} x^\top \tilde{\theta}_t,$$

$$J_t(\tilde{\theta}_t) = \max_{x \in X_0^s} x^\top \tilde{\theta}_t.$$  \hfill (15)

$$J_t(\tilde{\theta}_t) = \max_{x \in X_0^s} x^\top \tilde{\theta}_t.$$  \hfill (16)

Since, $\theta_*$ and $\mu_*$ are unknown, the learner does not know the safe action set $X_0^s$. Therefore, in order to satisfy the safety constraint (2), the learner chooses her actions from $X_0^s$, which is an inner approximation of $X_0^s$. Intuitively, the better this approximation, the more likely that safe linear Thompson sampling leads to a good regret, ideally the same regret as that of LTS in the original linear bandit setting. To see the changes due to the safety constraint, let us consider the following standard decomposition of the instantaneous regret:

$$R(T) \leq \sum_{t=1}^T (x_t^\top \theta_* - x_t^\top \tilde{\theta}_t) =$$

$$\sum_{t=1}^T \left( \frac{J(\theta_*) - J_t(\tilde{\theta}_t)}{\text{Term I}} \right) + \sum_{t=1}^T \left( \frac{J_t(\tilde{\theta}_t) - x_t^\top \theta_*}{\text{Term II}} \right).$$  \hfill (17)

Despite the addition of safety constraints, controlling Term II remains straightforward closely following previous results (e.g., [Abbasi-Yadkori et al., 2011]); see Appendix C.2 for more details. As such, here we focus our attention on bounding Term I. To see how the safety constraints affect the proofs let us review the treatment of Term I in the original setting. Specifically, in the absence of safety constraints, we have

$$J_t(\tilde{\theta}_t) = J_t(\tilde{\theta}_t),$$  \hfill (18)

as all actions are always “safe” to play. On the one hand, if an UCB algorithm is adopted, e.g., [Abbasi-Yadkori et al., 2011], Term I will be non-positive with high probability at all time steps since, by algorithm’s construction, $J_t(\tilde{\theta}_t) \geq J(\theta_*)$. As such, Term I no longer contributes to the growth of the regret. On the other hand, for LTS studied in [Agrawal and Goyal, 2013, Abeille et al., 2017], this term can be positive. As such, the authors in [Abeille et al., 2017] use the fact that

$$J(\theta_*) - J_t(\tilde{\theta}_t) \leq \|\nabla J(\theta_*)\|_{V_t} - 1 \|\theta_* - \tilde{\theta}_t\|_{V_t},$$  \hfill (19)

and they further show how to bound $\|\nabla J(\theta_*)\|_{V_t}$. With these, they obtain a regret bound of order $O(\sqrt{T \log T})$. Unfortunately, this approach cannot be adopted in the presence of safety constraints as (18) no longer holds. In the presence of safety constraints, $J_t(\tilde{\theta}_t)$ is in general less than or equal to $J(\theta_*)$ since $X_0^s \subseteq X_0^s$. Our main contribution towards establishing regret guarantees is upper bounding Term I given the existence of the safety constraint. Specifically, we are able to obtain the same regret bound as that of [Abeille et al., 2017] (i.e., $O(\sqrt{T \log T})$) in spite of the additional safety constraints imposed on the problem.

**Theorem 3.1.** Let $\lambda \geq 1$. Under Assumptions 1, 2, 3, the regret of safe linear Thompson Sampling is bounded with probability $1 - \delta$ as follows:

$$R(T) \leq \left( \beta T(\delta') + \gamma T(\delta')(1 + \frac{4}{p}) \right) \sqrt{2Td\log (1 + \frac{T L^2}{\lambda})} + \frac{4\gamma T(\delta')}{p} \sqrt{\frac{8TE^2}{\lambda} \log \frac{4}{\delta}},$$  \hfill (20)

where $\delta' = \frac{\delta}{gT}$, and,

$$\gamma_t(\delta) = \beta t(\delta') \left( 1 + \frac{2}{C} \right) \sqrt{cd \log \frac{e^c d}{\delta}}.$$  \hfill (21)

### 3.1 Sketch of the proof for bounding Term I

We provide the sketch of the proof for bounding Term I. The idea is inspired by [Abeille et al., 2017]: we wish to show that TS has a constant probability of being optimistic, but now in the presence of safety constraints. First, we define the set of the optimistic parameters as

$$\Theta_t^{opt}(\delta') = \{ \theta : J_t(\theta) \geq J(\theta_*) \}.$$  \hfill (22)

To see how $\Theta_t^{opt}$ is relevant note that if $\tilde{\theta}_t \in \Theta_t^{opt}$, then Term I at time $t$ is non-negative. Additionally, we define the ellipsoid $E_t^{TS}(\delta')$ such that

$$E_t^{TS}(\delta') := \{ \theta \in \mathbb{R}^d : \|\theta - \tilde{\theta}_t\|_{V_t} \leq \gamma_t(\delta') \}.$$  \hfill (22)

It is not hard to see by combining Theorem 2.1 and the concentration property that $\theta_t \in E_t^{TS}(\delta')$ with high probability. For the purpose of this proof sketch, we assume that at each round $t$, the safe decision set contains the previous safe action that the algorithm played, i.e., $x_{t-1} \in X_0^s$. However, for the formal proof in Appendix C.1, we do not need such an assumption.

Let $\tau$ be a time such that $\tilde{\theta}_\tau \in \Theta_t^{opt}$, i.e., $x^\top_t \tilde{\theta}_\tau \geq x^\top_t \theta_*$. Then, for any $t \geq \tau$ we have

$$\text{Term I} := R_t^{TS} = x^\top_t \theta_* - x^\top_t \tilde{\theta}_t \leq x^\top_t \tilde{\theta}_\tau - x^\top_t \tilde{\theta}_t \leq x^\top_t (\tilde{\theta}_\tau - \tilde{\theta}_t).$$  \hfill (23)
The last inequality comes from the assumption that at each round \( t \), the safe decision set contains the previous played safe actions for rounds \( s \leq t \); hence, \( x_t^\top \tilde{\theta}_t \leq x_{t-1}^\top \tilde{\theta}_t \). To continue from (23), we use Cauchy-Schwarz, and obtain

\[
R_t^{TS} \leq \|x_t\|_{V_t^{-1}} \|\tilde{\theta}_t - \tilde{\theta}_t\|_{V_t} \\
\leq \left(\|\tilde{\theta}_t - \theta_*\|_{V_t} + \|\theta_* - \tilde{\theta}_t\|_{V_t}\right) \|x_t\|_{V_t^{-1}}.
\]

Since the Gram matrix is increasing with time (i.e., \( V_t \leq V_t \)) and using \( \tilde{\theta}_t \in \Theta_t^{opt} \) and \( \theta_* \in \Theta_t(\delta) \), we can write

\[
R_t^{TS} \leq \left(\beta_t(\delta') + \gamma_t(\delta') + \beta_t(\delta') + \gamma_t(\delta')\right) \|x_t\|_{V_t^{-1}} \\
\leq 2 (\beta_t(\delta') + \gamma_t(\delta')) \|x_t\|_{V_t^{-1}}.
\]

Therefore, we can upper bound Term I with respect to the \( V_t \)-norm of the optimal safe action at time \( t \). Bounding the term \( \|x_t\|_{V_t^{-1}} \) is standard based on the analysis provided in [Abbasi-Yadkori et al., 2011] (see Proposition A.1 in the Appendix).

It only remains to show that TS samples a parameter \( \tilde{\theta}_t \) belonging to the optimistic set (i.e., \( \tilde{\theta}_t \in \Theta_t^{opt} \)) with some constant probability. The next lemma informally characterizes such claim (see the formal statement of the lemma and its proof in Section D of the Appendix).

**Lemma 3.2.** (informal) Let \( \Theta_t^{opt} = \{\theta \in \mathbb{R}^d : J_t(\theta) \geq J_t(\theta_*)\} \) be the set of optimistic parameters, \( \tilde{\theta}_t = \tilde{\theta} + \beta_t(\delta') V_t^{-\frac{1}{2}} \eta_t \) with \( \eta_t \sim \mathcal{D}^{TS} \), then \( \forall t \geq 1, \mathbb{P}\left(\tilde{\theta}_t \in \Theta_t^{opt}\right) \geq p \).

Simply stated, we need to show that

\[
q_t = \mathbb{P}\left(x_t^\top \tilde{\theta}_t \geq x_t^\top \theta_*\right) \geq p,
\]

where \( p \) denotes the strictly positive probability defined in (11). In order to do so, we introduce an enlarged confidence region centered on \( \mu_* \) as

\[
\tilde{C}_t(\delta') := \{v \in \mathbb{R}^d : \|v - \mu_*\|_{V_t} \leq 2\beta_t(\delta')\},
\]

and the shrunk safe decision set as

\[
\tilde{X}_t^s := \{x \in \tilde{X}_t : x^\top v \leq C, \forall v \in \tilde{C}_t\} \\
= \{x \in \tilde{X}_t : \max_{v \in \tilde{C}_t} x^\top v \leq C\} \\
= \{x \in \tilde{X}_t : x^\top \mu_* + 2\beta_t(\delta') \|x\|_{V_t^{-1}} \leq C\}.
\]

Define \( \alpha_t \in [0, 1] \) such that \( \alpha_t x_* \) belongs to the shrunk safe decision set \( \tilde{X}_t^s \), i.e.,

\[
\alpha_t \left(x_*^\top \mu_* + 2\beta_t(\delta') \|x_*\|_{V_t^{-1}}\right) \leq C.
\]

In particular, this is satisfied if we choose \( \alpha_t \) as follows (see Appendix D for more details)

\[
1 + \frac{2}{C} \beta_t(\delta') \|x_*\|_{V_t^{-1}} = \frac{1}{\alpha_t}.
\]

Then, by optimality of \( x_* \) and by feasibility of \( \alpha_t x_* \):

\[
x_t^\top \tilde{\theta}_t \geq \alpha_t x_*^\top \tilde{\theta}_t.
\]

Using (26), in order to show that (25) holds, it suffices to prove that

\[
q_t \geq \mathbb{P}\left(x_t^\top \tilde{\theta}_t + \mathbb{E}\left[\beta_t(\delta') \|x_*\|_{V_t^{-1}}\right] \geq x_t^\top \theta_* + \frac{2}{C} \beta_t(\delta') \|x_*\|_{V_t^{-1}}\right).
\]

Now, recall that \( \tilde{\theta}_t = \tilde{\theta} + \beta_t V_t^{-\frac{1}{2}} \eta_t \). Thus, we have

\[
q_t \geq \mathbb{P}\left(x_t^\top \beta_t(\delta') V_t^{-\frac{1}{2}} \eta_t \geq x_t^\top \theta_* + \frac{2}{C} \beta_t(\delta') \|x_*\|_{V_t^{-1}}\right).
\]

Further using the fact that \( x_t^\top \theta_* \leq 1 \) we need

\[
q_t \geq \mathbb{P}\left(\beta_t(\delta') V_t^{-\frac{1}{2}} \eta_t \geq x_t^\top \left(\theta_* - \tilde{\theta}_t\right) + \frac{2}{C} \beta_t(\delta') \|x_*\|_{V_t^{-1}}\right).
\]

Let us define a vector \( u \) such that \( u^\top = \frac{x_t^\top V_t^{-\frac{1}{2}}}{\|x_*\|_{V_t^{-1}}} \) (note that by definition, \( \|u\|_2 = 1 \)). Then, a direct application of Cauchy-Schwarz inequality and (7), gives:

\[
q_t \geq \mathbb{P}\left(u^\top \eta_t \geq 1 + \frac{2}{C}\right) \geq p.
\]

This holds due to the anti-concentration property of the \( \mathcal{D}^{TS} \) distribution in (11) completing the proof of Lemma 3.2.

## 4 Numerical Results

In this section, we present details of our numerical experiments on synthetic data. First, we show how the presence of safety constraints affects the performance of LTS in terms of regret. Next, we evaluate Safe-LTS by comparing it against Safe-LUCB presented in [Amani et al., 2019]. In all the implementations, we used the following parameters: \( T = 10000, \delta = 1/4T \) and \( R = 0.1 \). We considered a time independent decision set \( \tilde{X}_0 = [-1, 1]^d \) in \( \mathbb{R}^d \). The reward and constraint parameters \( \theta_t \) and \( \mu_* \) are drawn from \( \mathcal{N}(0, 1) \); \( C \) is drawn uniformly from \( [0, 1] \).
In view of the discussion in [Dani et al., 2008] regarding computational issues of LUCB algorithms with confidence regions specified with 2-norms, we implement a modified version of Safe-LUCB which uses 1-norms instead of 2-norms (see also [Amani et al., 2019]). This highlights a well-known benefit associated with TS-based algorithms, namely that they are easier to implement and more computationally-efficient than UCB-based algorithms, since action selection via the latter involves solving optimization problems with bilinear objective functions, whereas the former would lead to linear objectives by first sampling the unknown parameter vector (e.g. [Abeille et al., 2017]).

4.1 The effect of safety constraints on LTS

We compare the performance of our algorithm to an oracle that has access to set of safe actions $X_0^s$ and hence applies simply the LTS algorithm of [Abeille et al., 2017] to the problem. This experiment highlights the additional contribution of the safety constraint to the growth of regret. Specifically, Fig. 1 compares the average cumulative regret of Safe-LTS with the standard LTS algorithm with oracle access to the safe set over 20 problem realizations. As shown, even though the Safe-LTS requires that the chosen actions should belong to the more restricted set $X_t^s$ (i.e., inner approximation of the unknown safe set $X_0^s$), it achieves a regret of the same order as the oracle.

4.2 Comparison with a safe version of LUCB

We compare the performance of our algorithm with two safe versions of LUCB: 1) Naive Safe-LUCB: this is an extension of the LUCB algorithm of [Dani et al., 2008, Abbasi-Yadkori et al., 2011] to respect safety constraints by choosing actions from the estimated safe set defined in (13) instead. 2) Safe-LUCB: Inspired by a recent paper of [Amani et al., 2019] on safe LUCB in a similar but non-identical setting, we consider a modification of Naive-SLUCB that proceeds in two phases: starting with a so-called pure exploration phase, the algorithm randomly chooses actions from a seed safe set for a given period of time before switching to the same decision rule as Naive-SLUCB. The additional randomized exploration phase allows the algorithm to first learn a good representation of the safe action set before applying the LUCB action selection rule.

Fig. 2 compares average cumulative regret of Safe-LTS against those of Safe-LUCB and Naive Safe-LUCB over 30 problem realizations (see Section E in Appendix for plots with standard deviation). The reader can observe that Naive-SLUCB leads to very poor (almost linear) regret. For Safe-LUCB, a general regret bound of $O(T^{2/3} \log T)$ was provided by [Amani et al., 2019]. As our numerical experiment suggests, this is likely not a mere artifact of the proof provided in [Amani et al., 2019]. We observe that the LUCB action selection rule alone does provide sufficient exploration towards safe set expansion, thus requiring the algorithm to 1) have access to a seed safe set; 2) start with a pure exploration phase in order to guarantee safe set expansion. As pointed out in [Amani et al., 2019], this lack of exploration is especially costly for Safe-LUCB under problem instances where the safety constraint is active, i.e., $x^\top \mu = C$. We highlight this by comparing the regret of Safe-LUCB and Safe-LTS and their corresponding estimated safe sets at different rounds for a problem instance where $x^\top \theta = C$ in Figure 3. The left and middle figures depict the safe sets $X_t^s$ at different
rounds for a specific choice of parameters: $\theta^* = \begin{bmatrix} 1 \\ 0.23 \end{bmatrix}$, $\mu^* = \begin{bmatrix} 0.55 \\ 0.31 \end{bmatrix}$, and $C = 0.11$. Comparing the left and middle plots demonstrates that inherent randomness of TS leads to a natural exploration ability that is much faster at expanding the estimated safe set towards $x_*$ compared to Safe-LUCB. This results in an undesired regret performance for Safe-LUCB especially in instances where the safety constraint is active.

4.3 Sampling from a dynamic noise distribution

In order for Safe-LTS to be optimistic, i.e., for the event $\{x_i^\top \theta_t \geq x_i^\top \theta_*\}$ to happen with a constant probability, our theoretical results require the anti-concentration property assumed in (11). This requires that the noise $\eta_t$ is sampled from a distribution that satisfies:

$$\mathbb{P}_{\eta \sim D^{TS}} \left(u^\top \eta \geq 1 + \frac{2}{C}\right) \geq p. \quad (28)$$

The extra $\frac{2}{C}$ factor compared to the results of [Abeille et al., 2017] is needed in our theoretical results due to the presence of the safety constraint, restricting the choice of $x_i$’s to the set $X^t_0$, which is the inner approximation of the $X^t$ to which $x_i$ belongs.

However, here we highlight the performance of a heuristic modification of the algorithm in which the TS distribution $D^{TS}$ does not sample according to the above anti-concentration property for all rounds. We empirically observe that if the TS distribution satisfies (28), the TS algorithm explores more than what it needs to get a good approximation of the unknown parameter, which can cause the growth of the regret. Instead, Fig. 1 shows that if the TS distribution satisfies the following dynamic property

$$\mathbb{P}_{\eta \sim D^{TS}} \left(u^\top \eta \geq k(t)\right) \geq p, \quad (29)$$

where $k(t)$ is a linearly-decreasing function in time $k(t) = \left(1 + \frac{2}{C}\right)t + (1 + \frac{2}{C})$, the TS algorithm will have a smaller regret in comparison to when $k(t) = 1 + \frac{2}{C}$. This is intriguing to develop a theoretical understanding of this heuristic.

5 Conclusions

In this paper, we study a linear stochastic bandit (LB) problem in which the environment is subject to unknown linear safety constraints that need to be satisfied at each round. As such, the learner must make necessary modifications to ensure that the chosen actions belong to the unknown safe set. We propose Safe-LTS, which to the best of our knowledge, is the first safe linear TS algorithm with provable regret guarantees for this problem. We show that appropriate modifications of the [Abeille et al., 2017] distributional properties allow us to design an efficient algorithm for the more challenging LB problem with linear safety constraints. Moreover, we show that the Safe-LTS achieves the same frequentist regret of order $O(d^{3/2}\log^{1/2} t \cdot T^{1/2} \log^{3/2} T)$ as the original LTS problem studied in [Abeille et al., 2017]. We also compare Safe-LTS with Safe-LUCB of [Amani et al., 2019] a UCB-based safe algorithm for LB with linear safety constraints. We show that our algorithm has: better regret in the worst-case ($\tilde{O}(T^{1/2})$ vs. $\tilde{O}(T^{2/3})$), fewer parameters to tune and superior empirical performance. Interesting directions for future work include gaining a theoretical understanding of the regret of the algorithm when the TS distribution satisfies the dynamic property in (29), which empirically leads regret of smaller order.
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A Useful Results

The following result is standard and plays an important role in most proofs for linear bandits problems.

**Proposition A.1.** ([Abbasi-Yadkori et al., 2011]) Let \( \lambda \geq 1 \). For any arbitrary sequence of actions \((x_1, \ldots, x_t) \in \mathcal{X}^t\), let \( V_t \) be the corresponding Gram matrix \((5)\), then

\[
\sum_{s=1}^{t} \|x_s\|_{V_s^{-1}}^2 \leq 2 \log \left( \frac{\text{det}(V_{t+1})}{\text{det}(\lambda I)} \right) \leq 2d \log \left( 1 + \frac{t \lambda^2}{\lambda} \right).
\]

In particular, we have

\[
\sum_{s=1}^{T} \|x_s\|_{V_s^{-1}}^2 \leq \sqrt{T} \left( \sum_{s=1}^{T} \|x_s\|_{V_s^{-1}}^2 \right)^{\frac{1}{2}} \leq \sqrt{2T \log \left( 1 + \frac{t \lambda^2}{\lambda} \right)}.
\]

Also, we recall the Azuma’s concentration inequality for super-martingales.

**Proposition A.2.** (Azuma’s inequality [Boucheron et al., 2013]) If a super-martingale \((Y_t)_{t \geq 0}\) corresponding to a history \( \mathcal{F}_t \) satisfies \( |Y_t - Y_{t-1}| < c_t \) for some positive constant \( c_t \), for all \( t = 1, \ldots, T \) then for any \( \alpha > 0 \),

\[
P (Y_T - Y_0 \geq \alpha) \leq 2 e^{-\frac{\alpha^2}{2 \sum_{t=1}^{T} c_t^2}}.
\]

B Least-Square Confidence Regions

We start by constructing the following confidence regions for the RLS-estimates.

**Definition B.1.** Let \( \delta \in (0, 1) \), \( \delta' = \frac{\delta}{6T} \), and \( t \in [T] \). We define the following events:

- \( \hat{E}_t \) is the event that the RLS-estimate \( \hat{\theta} \) concentrates around \( \theta_* \) for all steps \( s \leq t \), i.e., \( \hat{E}_t = \{ \forall s \leq t, \| \hat{\theta}_s - \theta_* \|_{V_s} \leq \beta_s(\delta') \} \);

- \( \check{Z}_t \) is the event that the RLS-estimate \( \check{\mu} \) concentrates around \( \mu_* \), i.e., \( \check{Z}_t = \{ \forall s \leq t, \| \check{\mu}_s - \mu_* \|_{V_s} \leq \beta_s(\delta') \} \). Moreover, define \( Z_t \) such that \( Z_t = \hat{E}_t \cap \check{Z}_t \).

- \( \check{E}_t \) is the event that the sampled parameter \( \check{\theta}_t \) concentrates around \( \check{\theta}_t \) for all steps \( s \leq t \), i.e., \( \check{E}_t = \{ \forall s \leq t, \| \check{\theta}_s - \check{\theta}_t \|_{V_s} \leq \gamma_s(\delta') \} \). Let \( E_t \) be such that \( E_t = \check{E}_t \cap Z_t \).

**Lemma B.1.** Under Assumptions 1, 2, we have \( P(Z) = P(\hat{E} \cap \check{Z}) \geq 1 - \frac{\delta}{3} \) where \( \hat{E} = \hat{E}_T \subset \cdots \subset \hat{E}_1 \), and \( \check{Z} = \check{Z}_T \subset \cdots \subset \check{Z}_1 \).

**Proof.** The proof is similar to the one in [Abeille et al., 2017, Lemma 1] and is omitted for brevity.

**Lemma B.2.** Under Assumptions 1, 2, we have \( P(E) = P(\hat{E} \cap Z) \geq 1 - \frac{\delta}{2} \), where \( \hat{E} = \hat{E}_T \subset \cdots \subset \hat{E}_1 \).

**Proof.** We show that \( P(E) \geq 1 - \frac{\delta}{6} \). Then, from Lemma B.1 we know that \( P(Z) \geq 1 - \frac{\delta}{3} \), thus we can conclude that \( P(E) \geq 1 - \frac{\delta}{2} \). Bounding \( E \) comes directly from concentration inequality \((12)\).

Specifically,

\[
1 \leq t < T, \quad P \left( \| \hat{\theta}_t - \theta_* \|_{V_t} \leq \gamma_t(\delta') \right) = P \left( \| \eta_t \|_2 \leq \frac{\gamma_t(\delta')}{\beta_t(\delta')} \right) = P \left( \| \eta_t \|_2 \leq \left( 1 + \frac{2}{C} \right) \sqrt{cd \log \left( \frac{cd}{\delta'} \right)} \right) \geq 1 - \delta'.
\]

Applying union bound on this ensures that \( P(\hat{E}) \geq 1 - T \delta' = 1 - \frac{\delta}{6} \).
C Proof of Theorem 3.1

We use the following decomposition for bounding the regret:

\[
R(T) \leq \sum_{t=1}^{T} \left( x_t^\top \theta_t - x_t^\top \theta_* \right) \mathbb{1}\{E_t\}
\]

\[
= \sum_{t=1}^{T} \left( J(\theta_t) - J(\tilde{\theta}_t) \right) \mathbb{1}\{E_t\} + \sum_{t=1}^{T} \left( J(\tilde{\theta}_t) - x_t^\top \theta_* \right) \mathbb{1}\{E_t\},
\]

(33)

C.1 Bounding Term I.

Using the definition \( \frac{(\delta_t)}{BD} \) in (34), we can write \( \mathbb{E}\left[ J(\tilde{\theta}_t) - \inf_{\theta \in \Theta_{opt}^T} J_t(\theta) \right] \mathbb{1}\{Z_t\} \mathbb{I}\{F_t, \tilde{\theta} \in \Theta_{opt}^T\} \). We can bound (34) by the expectation over any random choice of \( \tilde{\theta} \in \Theta_{opt}^T \) that leads to

\[
R_{TS}^T \leq \mathbb{E}\left[ J(\tilde{\theta}_t) - \inf_{\theta \in \Theta_{opt}^T} J_t(\theta) \right] \mathbb{1}\{Z_t\} \mathbb{I}\{F_t, \tilde{\theta} \in \Theta_{opt}^T\},
\]

(35)

where \( x_t(\tilde{\theta}) = \arg \max_{x \in X_t} x^\top \tilde{\theta} \) and \( x_t(\theta) = \arg \max_{x \in X_t} x^\top \theta \). Then, using Cauchy–Schwarz and the definition of \( \gamma_t(\delta') \) in (21)

\[
\mathbb{E}\left[ \sup_{\theta \in \Theta_{opt}^T} \left( x_t(\tilde{\theta}) \right)^\top (\tilde{\theta} - \theta) \right] \mathbb{1}\{Z_t\} \mathbb{I}\{F_t, \tilde{\theta} \in \Theta_{opt}^T\}
\]

\[
\leq \mathbb{E}\left[ \left\| x_t(\tilde{\theta}) \right\|_{V_t^{-1}} \sup_{\theta \in \Theta_{opt}^T} \left\| \tilde{\theta} - \theta \right\|_{V_t} \right] \mathbb{I}\{F_t, \tilde{\theta} \in \Theta_{opt}^T, Z_t\} \mathbb{P}(Z_t)
\]

\[
\leq 2\gamma_t(\delta') \mathbb{E}\left[ \left\| x_t(\tilde{\theta}) \right\|_{V_t^{-1}} \right] \mathbb{I}\{F_t, \tilde{\theta} \in \Theta_{opt}^T, Z_t\} \mathbb{P}(Z_t).
\]

This property shows that the regret \( R_{TS}^T \) is upper bounded by \( V_t^{-2} \)-norm of the optimal safe action corresponding to the any optimistic parameter \( \tilde{\theta} \). Hence, we need to show that TS samples from the optimistic set with high frequency. We prove in Lemma 3.2 that TS is optimistic with a fixed probability \( \left( \frac{K}{2} \right) \) which leads to bounding \( R_{TS}^T \) as follows:

\[
R_{TS}^T \leq 2\gamma_t(\delta') \mathbb{E}\left[ \left\| x_t(\tilde{\theta}) \right\|_{V_t^{-1}} \right] \mathbb{I}\{F_t, \tilde{\theta} \in \Theta_{opt}^T, Z_t\} \mathbb{P}(Z_t) \mathbb{P}(\tilde{\theta} \in \Theta_{opt}^T)
\]

\[
\leq 2\gamma_t(\delta') \mathbb{E}\left[ \left\| x_t(\tilde{\theta}) \right\|_{V_t^{-1}} \right] \mathbb{I}\{F_t, Z_t\} \mathbb{P}(Z_t)
\]

(37)
By reintegrating over the event $Z_t$ we get
\[ R_t^{TS} \leq \frac{4\gamma_t(\delta')}{p} E \left[ \left\| x_t(\hat{\theta}_t) \right\|_{V_t^{-1}} \mathbb{1}\{ Z_t \} \mid \mathcal{F}_t \right]. \] (38)

Recall that $E_t \subseteq Z_t$, hence
\[ R^{TS}(T) \leq \sum_{t=1}^{T} R_t^{TS} \mathbb{1}\{ E_t \} \leq \frac{4\gamma_T(\delta')}{p} \sum_{t=1}^{T} E \left[ \left\| x_t(\hat{\theta}_t) \right\|_{V_t^{-1}} \mid \mathcal{F}_t \right]. \]

For bounding this term, we rewrite the RHS above as:
\[ R^{TS}(T) \leq \left( \sum_{t=1}^{T} \left\| x_t \right\|_{V_t^{-1}} + \sum_{t=1}^{T} \left( E \left[ \left\| x_t(\hat{\theta}_t) \right\|_{V_t^{-1}} \mid \mathcal{F}_t \right] - \left\| x_t \right\|_{V_t^{-1}} \right) \right). \] (39)

We can now bound the first expression using Proposition A.1. For the second expression we proceed as follows:

- First, the sequence
  \[ Y_t = \sum_{s=1}^{t} \left( E \left[ \left\| x_s(\hat{\theta}_s) \right\|_{V_s^{-1}} \mid \mathcal{F}_s \right] - \left\| x_s \right\|_{V_s^{-1}} \right) \]
  is a martingale by construction.

- Second, under Assumption 3, $\left\| x_t \right\|_2 \leq L$, and since $V_t^{-1} \leq \frac{1}{L} I$, we can write
  \[ E \left[ \left\| x_s(\hat{\theta}_s) \right\|_{V_s^{-1}} \mid \mathcal{F}_s \right] - \left\| x_s \right\|_{V_s^{-1}} \leq \frac{2L}{\sqrt{\lambda}}, \forall t \geq 1. \] (40)

- Third, for bounding $Y_T$, we use Azuma’s inequality, and we have that with probability $1 - \frac{\delta}{T}$,
  \[ Y_T \leq \sqrt{\frac{8TL^2}{\lambda} \log \frac{4}{\delta}}. \] (41)

Putting this together, we conclude that with probability $1 - \frac{\delta}{T}$,
\[ R^{TS}(T) \leq \frac{4\gamma_T(\delta')}{p} \left( \sqrt{2Td \log \left( 1 + \frac{TL^2}{\lambda} \right)} + \sqrt{\frac{8TL^2}{\lambda} \log \frac{4}{\delta}} \right). \] (42)

### C.2 Bounding Term II

We can bound on Term II using the general result of [Abbasi-Yadkori et al., 2011]. In fact, we can use the following general decomposition:
\[ \sum_{t=1}^{T} \left( \text{Term II} \right) \mathbb{1}\{ E_t \} := R^{RLS}(T) \]
\[ = \sum_{t=1}^{T} \left( x_t^\top \hat{\theta}_t - x_t^\top \theta_s \right) \mathbb{1}\{ E_t \} \leq \sum_{t=1}^{T} \left| x_t^\top (\hat{\theta}_t - \hat{\theta}_s) \right| \mathbb{1}\{ E_t \} + \sum_{t=1}^{T} \left| x_t^\top (\hat{\theta}_t - \theta_s) \right| \mathbb{1}\{ E_t \}. \] (43)

By Definition B.1, we have $E_t \subseteq Z_t$ and $E_t \subseteq \hat{E}_t$, and hence
\[ \left| x_t^\top (\hat{\theta}_t - \hat{\theta}_s) \right| \mathbb{1}\{ E_t \} \leq \left\| x \right\|_{V_t^{-1}} \gamma_t(\delta') \]
\[ \left| x_t^\top (\hat{\theta}_t - \theta_s) \right| \mathbb{1}\{ E_t \} \leq \left\| x \right\|_{V_t^{-1}} \beta_t(\delta'). \]

Therefore, from Proposition A.1, we have with probability $1 - \frac{\delta}{T}$
\[ R^{RLS}(T) \leq (\beta_T(\delta') + \gamma_T(\delta')) \sqrt{2Td \log \left( 1 + \frac{TL^2}{\lambda} \right)}. \] (44)
C.3 Overall Regret Bound

Recall that from (17), \( R(T) \leq R^{TS}(T) + R^{RLS}(T) \). As shown previously, each term is bounded separately with probability \( 1 - \frac{\delta}{2} \). Using union bound over two terms, we get the following expression:

\[
R(T) \leq (\beta_T(\delta') + \gamma_T(\delta')(1 + \frac{4}{p}))\sqrt{2Td\log \left(1 + \frac{TL^2}{\lambda} \right) + \frac{4\gamma_T(\delta')}{p} \sqrt{\frac{8TL^2}{\lambda} \log 4 \delta}}.
\]  

(45)

holds with probability \( 1 - \delta \) where \( \delta' = \frac{\delta}{2T} \).

For completeness we show below that action \( x_1 \) is safe. Having established that, it follows that the rest of the actions \( x_t, t > 1 \) are also safe with probability at least \( 1 - \delta' \). This is by construction of the feasible sets \( X^*_t \) and by the fact that \( \mu_* \in C_t(\delta') \) with the same probability for each \( t \).

**Lemma C.1.** The first action that Safe-LTS chooses is safe, that is \( x_1^\top \mu_* \leq C \).

**Proof.** At round \( t = 1 \), the RLS-estimate \( \hat{\mu}_1 = 0 \) and \( V_1 = \lambda I \). Thus, Safe-LTS chooses the action which maximizes the expected reward while satisfying \( x_1^\top \hat{\mu}_1 + \beta_1(\delta') \| x_1 \|_{V^{-1}_1} \leq C \). Hence, \( x_1 \) satisfies:

\[
\beta_1(\delta') \| x_1 \|_{V^{-1}_1} \leq C.
\]

From (9) and \( V^{-1}_1 = (1/\lambda)I \) leads to \( S \| x_1 \|_2 \leq C \) which completes the proof. \( \square \)

D Proof of Lemma 3.2

In this section, we provide a formal statement and a detailed proof of Lemma 3.2. Note that our proof is significantly modified compared to [Abeille et al., 2017, Lemma 3]. This is required because in our setting, actions \( x_t \) belong to inner approximations of the true safe set \( X^*_0 \). Moreover, we follow an algebraic treatment that is perhaps simpler compared to the geometric viewpoint in [Abeille et al., 2017].

**Lemma D.1.** Let \( \Theta^{opt}_t = \{ \theta \in \mathbb{R}^d : J_t(\theta) \geq J_t(\theta_0) \} \cap \mathcal{E}^{TS}_t \) be the set of optimistic parameters, \( \hat{\theta}_t = \hat{\theta} + \beta_t(\delta') V_t^{-\frac{1}{2}} \eta_t \) with \( \eta_t \sim D^{TS} \), then \( \forall t \geq 1, P \left( \hat{\theta}_t \in \Theta^{opt}_t | \mathcal{F}_t, Z_t \right) \geq \frac{5}{6} \).

**Proof.** First, we provide the shrunk version \( \tilde{X}^*_t \) of \( X^*_t \) as follows:

**A shrunk safe decision set \( \tilde{X}^*_t \).** Consider the enlarged confidence region \( \tilde{C}_t \) centered at \( \mu_* \) as

\[
\tilde{C}_t := \{ v \in \mathbb{R}^d : \| v - \mu_* \|_{V_t} \leq 2 \beta_t(\delta') \}.
\]

We know that \( C_t \subseteq \tilde{C}_t \), since \( \forall v \in C_t \), we know that \( \| v - \mu_* \|_{V_t} \leq \| v - \hat{\mu}_t \|_{V_t} + \| \hat{\mu}_t - \mu_* \|_{V_t} \leq 2 \beta(t) \). From the definition of enlarged confidence region, we can get the following definition for shrunk safe decision set:

\[
\tilde{X}^*_t := \{ x \in X_0 : x^\top \gamma_v \leq C, \forall v \in \tilde{C}_t \} = \{ x \in X_0 : \max_{v \in \tilde{C}_t} x^\top v \leq C \}
\]

\[
= \{ x \in X_0 : x^\top \mu_* + 2 \beta_t(\delta') \| x \|_{V^{-1}_t} \leq C \},
\]

(47)

and note that \( \tilde{X}^*_t \subseteq X^*_t \), and they are not empty, since they include zero due to Assumption 3.

Then, we define the parameter \( \alpha_t \) such that the vector \( z_t = \alpha_t x_* \) in direction \( x_* \) belongs to \( \tilde{X}^*_t \) and is closest to \( x_* \). Hence, we have:

\[
\alpha_t := \{ \alpha \in [0, 1] : z_t = \alpha x_* \in \tilde{X}^*_t \}.
\]

(48)

Since \( X_0 \) is convex by Assumption 3 and \( 0, x_* \in X_0 \), we have

\[
\alpha_t = \max \left\{ \alpha \in [0, 1] : \alpha \left( x_*^\top \mu_* + 2 \beta_t(\delta') \| x_* \|_{V^{-1}_t} \right) \leq C \right\}.
\]

(49)
Figure 4: Comparison of mean per-step regret for Safe-LTS, Safe-LUCB, and Naive Safe-LUCB. The shaded regions show one standard deviation around the mean. The results are averages over 30 problem realizations.

From constraint (2), we know that $x^\top_\star \mu_\star \leq C$. We choose $\alpha_t$ such that

$$1 + \frac{2}{C} \beta_t(\delta') \|x_\star\|_{V_t^{-1}} = \frac{1}{\alpha_t}. \quad (50)$$

We need to study the probability that a sampled $\tilde{\theta}_t$ drawn from $D^{TS}$ distribution at round $t$ is optimistic, i.e.,

$$p_t = \mathbb{P} \left( J_t(\tilde{\theta}_t) \geq J(\theta_\star) \mid F_t, Z_t \right).$$

Using the definition of $\alpha_t$ (49), we have

$$J_t(\tilde{\theta}_t) = \max_{x \in X_t} x^\top \tilde{\theta}_t \geq \alpha_t x^\top_\star \tilde{\theta}_t. \quad (51)$$

Hence, we can write

$$p_t \geq \mathbb{P} \left( \alpha_t x^\top_\star \tilde{\theta}_t \geq x^\top_\star \theta_\star = J(\theta_\star) \mid F_t, Z_t \right)$$

$$= \mathbb{P} \left( x^\top_\star (\tilde{\theta}_t + \beta_t(\delta') V_t^{-\frac{1}{2}} \eta_t) \geq \frac{x^\top_\star \theta_\star}{\alpha_t} \mid F_t, Z_t \right)$$

Then, we use the value that we chose for $\alpha_t$ in (50), and we have

$$= \mathbb{P} \left( x^\top_\star \tilde{\theta}_t + \beta_t(\delta') x^\top_\star V_t^{-\frac{1}{2}} \eta_t \geq x^\top_\star \theta_\star + \frac{2}{C} \beta_t(\delta') \|x_\star\|_{V_t^{-1}} x^\top_\star \theta_\star \mid F_t, Z_t \right)$$

From Assumption 3, we know that $x^\top_\star \theta_\star \leq 1$. Hence,

$$p_t \geq \mathbb{P} \left( \beta_t(\delta') x^\top_\star V_t^{-\frac{1}{2}} \eta_t \geq x^\top_\star (\theta_\star - \tilde{\theta}_t) + \frac{2}{C} \beta_t(\delta') \|x_\star\|_{V_t} \mid F_t, Z_t \right)$$

From Cauchy-Schwarz inequality and (7), we have

$$\| x^\top_\star (\theta_\star - \tilde{\theta}_t) \| \leq \|x_\star\|_{V_t^{-1}} \|x_\star\|_{V_t} \leq \beta_t(\delta') \|x_\star\|_{V_t^{-1}} \|x_\star\|_{V_t^{-1}}.$$ 

Therefore, we can write

$$p_t \geq \mathbb{P} \left( x^\top_\star V_t^{-\frac{1}{2}} \eta_t \geq \|x_\star\|_{V_t^{-1}} + \frac{2}{C} \|x_\star\|_{V_t^{-1}} \mid F_t, Z_t \right) \quad (52)$$

We define $u^\top = \frac{x^\top_\star V_t^{-\frac{1}{2}}}{\|x_\star\|_{V_t^{-1}}}$, and hence $\|u\|_2 = 1$. It follows from (52) that

$$p_t \geq \mathbb{P} \left( u^\top \eta_t \geq 1 + \frac{2}{C} \right) \geq p, \quad (53)$$
where the last inequality follows the concentration inequality (12) of the TS distribution. We also need to show that the high probability concentration inequality event does not affect the TS of being optimistic. This is because the chosen confidence bound $\delta' = \frac{1}{3p}$ is small enough compared to the anti-concentration property (11). Moreover, we assume that $T \geq \frac{1}{3p}$ which implies that $\delta' \leq \frac{p}{2}$. We know that for any events $A$ and $B$, we have

$$\mathbb{P}(A \cap B) = 1 - \mathbb{P}(A^c \cup B^c) \geq \mathbb{P}(A) - \mathbb{P}(B^c).$$

(54)

We apply (54) with $A = \{J_t(\tilde{\theta}_t) \geq J(\theta_\star)\}$ and $B = \{\tilde{\theta}_t \in \mathcal{E}_t^{TS}\}$ that leads to

$$\mathbb{P} \left( \tilde{\theta}_t \in \Theta_t^{\text{opt}} \middle| \mathcal{F}_t, Z_t \right) \geq p - \delta' \geq \frac{p}{2}.$$

E More on experimental results

In this section, we provide Figure 4 which highlights the sample standard deviation of regret around the average per-step regret for each curve depicted in Figure 2. We remark the strong dependency of the performance of LUCB-based algorithms on the specific problem instance, whereas the performance of Safe-LTS does not vary significantly under the same instances.