When each continuous operator is regular, II *

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Abstract. The following theorem is essentially due to L. Kantorovich and B. Vulikh and it describes one of the most important classes of Banach lattices between which each continuous operator is regular.

**Theorem 1.1.** Let $E$ be an arbitrary $L$-space and $F$ be an arbitrary Banach lattice with Levi norm. Then $\mathcal{L}(E, F) = \mathcal{L}^r(E, F)$, (⋆) that is, every continuous operator from $E$ to $F$ is regular.

In spite of the importance of this theorem it has not yet been determined to what extent the Levi condition is essential for the validity of equality (⋆). Our main aim in this work is to prove a converse to this theorem by showing that for a Dedekind complete $F$ the Levi condition is necessary for the validity of (⋆).

As a sample of other results we mention the following. **Theorem 3.6.** For a Banach lattice $F$ the following are equivalent: (a) $F$ is Dedekind complete; (b) For all Banach lattices $E$, the space $\mathcal{L}^r(E, F)$ is a Dedekind complete vector lattice; (c) For all $L$-spaces $E$, the space $\mathcal{L}^r(E, F)$ is a vector lattice.

1. **Introduction.** As the title of this work indicates we will be concerned here with the study of Banach lattices $E$ and $F$ for which the space of all continuous operators, $\mathcal{L}(E, F)$, coincides with the space of all regular operators, $\mathcal{L}^r(E, F)$. Recall that a (linear) operator is said to be regular if it can be split into the difference of two positive operators. The following theorem, which is essentially due to L. Kantorovich and B. Vulikh

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[KV], describes one of the most important classes of Banach lattices between which each continuous operator is regular.

**Theorem 1.1.** Let $E$ be an arbitrary $L$-space and $F$ be an arbitrary Banach lattice with Levi norm. Then

$$\mathcal{L}(E,F) = \mathcal{L}'(E,F),$$

that is, every continuous operator from $E$ to $F$ is regular.

To be precise, it was assumed in [KV] that $F$ was a KB-space, and it was noticed in [S] that the original proof could be easily carried over from a KB-space $F$ to an arbitrary Banach lattice with a Levi norm. For a KB-space $F$ the proofs can be found in [AB, Theorem 15.3] and [V, Theorem 8.7.2]. Under the assumption, somewhat stronger than Levi property, that $F$ is positively complemented in $F^{**}$ a proof of Theorem 1.1 is presented in [MN, Theorem 1.5.11].

It is somewhat surprising that in spite of the importance of Theorem 1.1 it has not yet been determined to what extent the Levi condition is essential for the validity of equality ($\star$). Our main aim in this work is to prove a converse to Theorem 1.1 by showing that for a Dedekind complete $F$ the Levi condition is necessary for the validity of ($\star$).

**Definition 1.2.** A norm on a Banach lattice $E$ is said to be a Levi norm, if every norm-bounded upward directed set of positive elements has a supremum. If the previous property holds only for sequences, then we say that the norm is sequentially Levi.

Obviously, each Banach lattice with a Levi norm must be Dedekind complete, and each Banach lattice with a sequentially Levi norm must be Dedekind $\sigma$-complete. It is worth noticing that the above definition is, in fact, of an order-topological nature as it describes relationships between the topology and the order, rather than the properties of a particular (lattice) norm. These properties appear in the literature under many different names. It was D. Fremlin [F] who was the first to use Levi’s name in connection with this property. H. Nakano [N, pages 129-130] used the term monotone complete norm for a sequentially Levi norm and universally monotone complete norm for Levi. A. Zaanen [Z] considers sequentially Levi norms in two places under different names. First on page 305 he refers to them, like Nakano, as monotone complete norms, and then on page 421
as norms with the weak Fatou property for monotone sequences. The term weak Fatou property for directed sets (page 390) is used by Zaanen for what we refer to as a Levi norm. P. Meyer-Nieberg [MN, page 96] uses the term monotonically complete. Finally, the Soviet school on Banach lattices used symbols \( (B) \) and \( (B') \) to denote sequentially Levi and Levi properties respectively.

Our converse to Theorem 1.1, which we mentioned above, is somewhat partial since we assume \( F \) to be Dedekind complete. On the other hand, it is the best one can get since, as we will see below in Remark 3.2 and the comments after Theorem 3.5, there are non-Dedekind complete Banach lattices (hence, without a Levi norm) which, nevertheless, satisfy the equality (\( \ast \)).

This paper can be considered as a sequel to [A3]. As in [A3] we adhere in this work to an isomorphic point of view, i.e., we do not distinguish between equivalent norms. We use the standard terminology regarding Banach lattices and operators on them. Any notation or definition not mentioned explicitly in the text can be found in [AB], [V] or [A3].

2. Some Banach lattice preliminaries. In this section we present two new results of the so-called lateral analysis which will be needed later on. Lateral analysis is a convenient and powerful method of investigating Banach and vector lattices. The essence of this method can be described roughly as follows: instead of arbitrary nets or sequences (to be considered in a property or a definition) we try to deal with those of a much simpler structure by considering only the nets or sequences with mutually disjoint terms. We refer to [A2], [AB] and [MN] where this approach is used systematically. For an illustration we present two examples the former of which will be used later on.

The theorem of Veksler and Gejler [VG], characterizes Dedekind completeness of vector lattices by stating that a uniformly complete vector lattice is Dedekind complete if and only if every order bounded set of pairwise disjoint positive elements has a supremum. Many other completeness properties of vector or Banach lattices have been also characterized in the framework of lateral analysis. Meyer-Nieberg [MN1] and Fremlin [F, page 56] have shown that a Banach lattice has an order continuous norm if and only if every order bounded sequence of pairwise disjoint elements converges to zero in norm. (See also [AB,
Theorem 12.13] or [MN, Theorem 2.4.2] for alternative proofs of this theorem.)

Recall that a vector lattice is said to be universally complete if it is Dedekind complete and has the property that every set of pairwise disjoint positive elements has a supremum. It is well known [V, Chapter V] that every Dedekind complete vector lattice $E$ has a universal completion, $\hat{E}$, which, by definition, is a universally complete vector lattice containing $E$ as an order dense ideal.

**Definition 2.1.** If $E$ is a vector lattice then an upward directed set $A \subseteq E_+$ is called laterally increasing if for each $a, b \in A$ with $a \geq b$ we have $(a - b) \land b = 0$.

There is an important difference between laterally increasing sequences and nets which we would like to point out. If $(x_n)$ is a laterally increasing sequence, then one can easily produce a sequence $(u_n)$ with pairwise disjoint elements such that $x_n = u_1 + \ldots + u_n$ (take simply $u_1 = x_1$ and $u_n = x_{n+1} - x_n$ for $n \geq 2$). For laterally increasing nets, however, there is no convenient substitute for the previous representation, and this makes working with nets more complicated. Our next proposition and theorem deal with this problem.

**Proposition 2.2.** A Dedekind complete vector lattice $E$ is universally complete if and only if every laterally increasing subset of $E_+$ has a supremum.

**Proof.** If every laterally increasing subset of $E_+$ has a supremum and $A \subset E_+$ is a given pairwise disjoint set, then the set of all finite sums of elements from $A$ is laterally increasing so has a supremum. That supremum is clearly also the supremum of $A$ itself, so that $E$ is indeed universally complete.

Now suppose that $E$ is universally complete. We know that $E$ is isomorphic to a space $C_\infty(Q)$ for some Stonean space $Q$ [V, Chapter V]. Suppose that $A$ is a laterally increasing subset of $C_\infty(Q)$. Define a function $y$ on $Q_0 = \bigcup_{a \in A}\{s \in Q : a(s) > 0\}$ by $y(s) = a(s)$ if $a(s) > 0$. In order to show that this definition is unambiguous, it suffices to consider $a \geq b$ with $b(s) > 0$ and show that we obtain the same value for $y(s)$ using either $a$ or $b$, for then if we have any $b, c \in A$ we need only take $a \geq b, c$ to see that $b$ gives the same value as $a$, which in turn gives the same value as $c$. But if $a \geq b$ then $(a - b) \land b = 0$, and in particular $(a - b)(s) \land b(s) = 0$. As $b(s) > 0$ this means that $(a - b)(s) = 0$, i.e. $a(s) = b(s)$ and the definition is therefore unambiguous. This definition clearly makes $y$ continuous on the
open set $Q_0$, so it extends continuously to the closure of $Q_0$. If we now extend $y$ to the whole of $Q$ by making it zero on $Q \setminus Q_0$ then we now have an element of $C_\infty(Q)$ which is clearly the required supremum of the set $A$. ■

A characterization of Levi norms in terms of laterally increasing sets was obtained in [A1, Theorem 3'] or [A2, Theorem 2.5]. It is the equivalence of (a) and (b) in our next Theorem 2.3. However, for our further work we need slightly more, namely the equivalence of (a) and (c).

**Theorem 2.3.** For any Banach lattice $E$ the following three properties are equivalent.

(a) $E$ has a Levi norm.

(b) Every norm bounded laterally increasing subset of $E_+$ has a supremum.

(c) If $A \subset E_+$ is a set of pairwise disjoint elements such that the set

$$B = \{ \sum_{a \in \sigma} a : \sigma \text{ is a finite subset of } A \}$$

is norm bounded, then the set $A$ has a supremum (which will also be the supremum of $B$).

**Proof.** Implication (a)⇒(b) is obvious. Since the proof in [A1] of the implication (b)⇒(a) is not easily available we, answering the request of the referee, present here a rather complete sketch of this proof.

Note first that (b) certainly implies that every order bounded set of pairwise disjoint positive elements of $E$ has a supremum, and so the theorem of Veksler and Gejler cited above tells us that $E$ must be Dedekind complete. This allows us to embed $E$ as an order dense ideal in its universal completion $\hat{E} = C_\infty(Q)$, where $Q$ is the Stonean space of $E$.

Let $(x_\alpha)$ be an increasing norm bounded net in $E_+$. We need to consider the following two exclusive cases: either $(x_\alpha)$ is order bounded in $C_\infty(Q)$ or else $(x_\alpha)$ is not bounded. In the former case there exists $z = \sup_\alpha x_\alpha \in C_\infty(Q)$. Let $G_\alpha = \{ q \in Q : 2x_\alpha(q) > z(q) \}$. This is an open and closed set for each $\alpha$, and clearly $G_{\alpha_1} \subset G_{\alpha_2}$ whenever $\alpha_1 < \alpha_2$. Therefore, the net $(z\chi_{G_\alpha})_\alpha$ is laterally increasing. Since $x_\alpha \uparrow z$ we have that $\bigcup G_\alpha$ is dense in $Q$, and this implies that $z\chi_{G_\alpha} \uparrow z$. It remains to notice that each element $z\chi_{G_\alpha}$ belongs to $E$ since $z\chi_{G_\alpha} \leq 2x_\alpha$. Consequently (b) implies that $z \in E$, that is, indeed, the net $(x_\alpha)$ has its supremum in $E$. 5
Consider the second case when \((x_\alpha)\) is not order bounded in \(C_\infty(Q)\). Then there exists a nonempty open and closed subset \(Q_0\) of \(Q\) such that \(\sup_\alpha x_\alpha(q) = \infty\) for all \(q \in D\), where \(D\) is a dense subset of \(Q\). Take any \(0 \leq z \in C_\infty(Q)\) with its support in \(Q_0\). Consider the net \((z \wedge x_\alpha)\). This is an increasing norm bounded net in \(E\), and clearly its supremum \(\sup_\alpha z \wedge x_\alpha\) in \(C_\infty(Q)\) exists and equals \(z\), since \(\sup_\alpha(z \wedge x_\alpha)(q) = z(q)\) for each \(q \in D\).

By the previous part \(z \in E\). In other words, we have proved that the universally complete band \(C_\infty(Q_0)\) is normable. This is clearly impossible.

It is obvious that either of the conditions (a) or (b) implies (c), by considering the upward directed set \(B\). We will prove that \((c) \Rightarrow (b)\), which will complete the proof of the equivalence of the three statements.

Again notice that every order bounded set of pairwise disjoint positive elements of \(E\) will certainly have a supremum by (c), and so another application of the theorem of Veksler and Gejler tells us that \(E\) must be Dedekind complete. As before, we assume that \(E\) is embedded into its universal completion \(\hat{E} = C_\infty(Q)\).

Let \(A \subset E_+\) be laterally increasing and norm bounded, we must show that \(A\) has a supremum in \(E\). Note that by Proposition 2.2, \(A\) has a supremum in \(\hat{E}\), which we will denote by \(y\). We claim that \(y \in E\). Without loss of generality we may assume that \(y(q) = 1\) for each \(q \in Q\), otherwise we will consider an appropriate principal ideal generated in \(C_\infty(Q)\) by the element \(y\). As above, we do not distinguish between members of the ideal generated in \(\hat{E}\) by \(y\) and functions in \(C(Q)\).

The argument used in the proof of Proposition 2.2 shows that on a dense subset \(Q_0\) of \(Q\), we have \(1 = y(s) = a(s)\) whenever \(a \in A\) and \(a(s) > 0\). In particular, this shows that each \(a \in A\) is the characteristic function of some open and closed subset of \(Q\) and that the union of these open and closed sets is dense in \(Q\). Consider now the collection of all open and closed subsets \(G\) of \(Q\) each of which is contained in some set \(\{s \in Q : a(s) = 1\}\), where \(a\), of course, depends on \(G\). Let \(C\) be a maximal disjoint collection of such sets \(G\). If \(\bigcup\{G : G \in C\}\) is not dense in \(Q\) then there is \(t \in Q_0\) which does not meet its closure. But for some \(a \in A\) we have \(a(t) = 1\). Adding the set \(\{s : a(s) = 1\} \setminus \bigcup\{G : G \in C\}\) to \(C\) gives us a contradiction to the maximality of \(C\). Thus the family \(D = \{\chi_G : G \in C\}\) is a subset of \(C(Q)\) with supremum \(y\). Notice also that in actuality each function \(\chi_G\) from \(D\)
belongs to $E$, as $E$ is an ideal in $\hat{E}$ and $0 \leq \chi_G \leq a$ whenever $G \subseteq \{s : a(s) = 1\}$.

If $\chi_{G_k} \in D$ for $k = 1, 2, \ldots, n$ then there are $a_k \in A$ with $\chi_{G_k} \leq a_k$ for $1 \leq k \leq n$. As $A$ is upwards directed, there is $b \in A$ with $a_k \leq b$ for $1 \leq k \leq n$. Hence $\chi_{G_k} \leq b$ for each $k$. But the functions $\chi_{G_k}$ are disjoint, so we also have $\sum_{k=1}^n \chi_{G_k} \leq b$ and hence $\|\sum_{k=1}^n \chi_{G_k}\| \leq \|b\|$ so we may apply (c) to deduce that the family $D$ has a supremum, $z$, in $E$. Clearly $z \leq y$, so we may regard $z$ as an element of $C(Q)$. But in $C(Q)$ it is clear that $z$ must be at least 1 on $\bigcup\{G : G \in C\}$ which is dense in $Q$, so that $z \geq y$. That is, $z = y$, showing that $y \in E$ and, hence, that $A$ does indeed have a supremum in $E$ (and not just in $\hat{E}$). □

For the sequentially Levi property we have the following analogue of the previous theorem.

**Theorem 2.4.** For any Banach lattice $E$ the following three properties are equivalent.

(a) $E$ has a sequentially Levi norm.

(b) Every norm bounded laterally increasing sequence in $E_+$ has a supremum.

(c) Every disjoint positive sequence, for which the set of all possible finite sums is norm bounded, must have a supremum.

The equivalence of (a) and (b) was established in [A1, Theorem 3] or [A2, Theorem 2.4], while the equivalence of (c) and (b) is obvious in view of the comment made after Definition 2.1.

We conclude this section by a simple observation. If $X$ is a Dedekind $\sigma$-complete Banach lattice such that every disjoint family in $X$ is countable, then $X$ has a Levi norm if and only if $X$ has a sequentially Levi norm. Indeed, the disjoint families that need to be considered in Theorem 2.3 (c) will be countable and then we can use Theorem 2.4, taking into account the fact that $X$ must be Dedekind complete by [V, Theorem VI.2.1].

The condition that every disjoint family in $X$ is countable is stronger than the countable sup property, which also implies that $X$ is Dedekind complete. However, the countable sup property alone is not enough to imply the equivalence of Levi and sequentially Levi properties, as the Dedekind $\sigma$-complete Banach lattice, under the uniform norm, of all functions on $[0,1]$ with at most countable support shows.
3. Regularity of operators on L-spaces. As a first step in our proof of the converse of Theorem 1.1 we need an argument that involves only separable domains, so we isolate this as a separate result.

Theorem 3.1. The following conditions on a Dedekind σ-complete Banach lattice $F$ are equivalent.

(a) $F$ has a sequentially Levi norm.
(b) For every separable L-space $E$ the equality $\mathcal{L}(E, F) = \mathcal{L}^r(E, F)$ holds.
(c) For $E = L_1[0, 2\pi]$ the equality $\mathcal{L}(E, F) = \mathcal{L}^r(E, F)$ holds.

Proof. (a)⇒(b). Assume that $E$ is a separable L-space, $F$ has a sequentially Levi norm and $U : E \to F$ is norm bounded. As in the usual proof of Theorem 1.1 (see for example [V, Theorem 8.7.2]), for each $0 \leq x \in E$, we consider the set

$$A_x = \left\{ \sum_{k=1}^n |Ux_k| : x_k \geq 0, \sum_{k=1}^n x_k = x, n \in \mathbb{N} \right\}.$$

This set is upward directed and norm bounded. If $A_x$ has a supremum, then by the classical Riesz-Kantorovich formula this supremum is $|U|(x)$. In the familiar situation when $F$ has a Levi norm, this implies immediately that $A_x$ has a supremum and the existence of $|U|$ follows.

In the present situation, however, when we have only a sequentially Levi norm, the existence of sup $A_x$ is not obvious. To establish it we will use an argument first utilized in [W1, Theorem 5.2]. To prove the existence of sup $A_x$, it will suffice to show that there is a countable upward directed subset $B_x$ of $F_+$ such that $B_x$ is dense in $A_x$ and $B_x$ is dominated by $A_x$, i.e., for each $b \in B_x$ there is $a \in A_x$ with $b \leq a$. The latter condition implies that $B_x$ is norm bounded, and so in view of the sequentially Levi property this subset $B_x$ will have a supremum, which clearly will be also the supremum of $A_x$.

In order to find such a set $B_x$ it is sufficient to find a dense countable subset, $C_x$, of $A_x$. Indeed, the collection of all finite suprema from that set $C_x$ will be clearly upward directed, countable and dense in $A_x$. It will be also dominated by $A_x$ as if $y_1, y_2, \ldots, y_m \in C_x \subseteq A_x$, then there is $z \in A_x$ with $y_1, y_2, \ldots, y_m \leq z$ (since $A_x$ is upward directed) and hence $y_1 \lor y_2 \lor \cdots \lor y_m \leq z$. 


Finally, to prove the existence of a countable dense subset of $A_x$, it suffices (by standard arguments) to find a countable subset $D_x$ of $F$ with $A_x \subseteq \overline{D_x}$ (so we do not need this countable set to be contained in $A_x$). Since $E$ is separable, we can find a countable dense subset

$$E_x = \{z_k : k \in \mathbb{N}\}$$

of the order interval $[0, x]$. The set

$$D_x = \left\{ \sum_{k=1}^{n} |Uz_{m_k}| : n \in \mathbb{N} \right\}$$

will certainly be countable, we show that its closure contains $A_x$ (and will in general be much larger). Given a typical element $\sum_{k=1}^{n} |Ux_k|$ of $A_x$ and $\epsilon > 0$, for each $k$ we can find $z_{m_k} \in E_x$ with $\|z_{m_k} - x_k\| < \epsilon/(n\|U\|)$. It follows that

$$\left\| |Ux_k| - |Uz_{m_k}| \right\| \leq \|Uz_{m_k} - Ux_k\| \leq \|U\| \|z_{m_k} - x_k\| < \epsilon/n$$

so that

$$\left\| \sum_{k=1}^{n} |Uz_{m_k}| - \sum_{k=1}^{n} |Ux_k| \right\| < \epsilon$$

showing that $A_x \subseteq \overline{D_x}$ as claimed.

Implication (b)⇒(c) is obvious. In order to prove that (c)⇒(a), let us suppose, contrary to what we claim, that $(e_n)$ is a disjoint positive sequence in $F$ such that

(i) $\|\sum_{n \in \sigma} e_n\| \leq 1$ for all finite subsets $\sigma \subset \mathbb{N}$,

but with

(ii) $\{e_n : n \in \mathbb{N}\}$ not being bounded above

(if it were, then the set would have a supremum as we are assuming that $F$ is Dedekind $\sigma$-complete). We take $E = L_1[0, 2\pi]$ and define

$$b_n(f) = \int_0^{2\pi} f(t) \cos(nt) \, dt$$

for each $f \in E$ and $n \in \mathbb{N}$. Clearly

(iii) $|b_n(f)| \leq \|f\|_1$,

and by the Riemann-Lebesgue theorem we know that
(iv) the sequence \( (b_n(f)) \in c_0 \).

Define a linear operator \( S : E \to F \) by \( Sf = \sum_{n=1}^{\infty} b_n(f)e_n \). It is routine to show, given (i) and (iv), that this series converges in \( F \), whilst the use of (i) and (iii) shows that \( \|S\| \leq 1 \).

We claim that \( S \) cannot be regular. If it were, let \( T : E \to F \) be a positive operator with \( T \geq S \). We denote by \( 1 \) the constantly one function on \([0, 2\pi]\), and let \( 2 = 2 \cdot 1 \).

For each \( n \in \mathbb{N} \) we have \( 2 \geq (1 + \cos(nt)) \geq 0 \), and so

\[
T2 \geq T(1 + \cos(nt)) \\
\geq S(1 + \cos(nt)) \\
= S1 + S(\cos(nt)) \\
= 0 + \pi \cdot e_n \geq 2e_n.
\]

Thus, \( T1 \geq e_n \) for all \( n \in \mathbb{N} \), which contradicts (ii). \( \blacksquare \)

**Remark 3.2.** The assumption that \( F \) be Dedekind \( \sigma \)-complete cannot be omitted from the hypotheses of Theorem 3.1.

Indeed it follows from [AG] that if \( K \) is a Stonean space and \( k_0 \in K \) is an arbitrary non-isolated point then, for an arbitrary Banach lattice \( E \), every bounded operator from \( E \) to \( C(\tilde{K}) \) is regular, where the Hausdorff compact space \( \tilde{K} \) is obtained from the Hausdorff compact space \( K \times \{1, 2\} \) by identifying \((k_0, 1)\) and \((k_0, 2)\). If we take \( K = \beta(\mathbb{N}) \) and \( k_0 \in \beta(\mathbb{N}) \setminus \mathbb{N} \), then \( \mathbb{N} \) is a dense open \( F_\sigma \)-set in \( K \) and clearly \( k_0 \in \overline{\mathbb{N}} \). Therefore, the two sets \( \mathbb{N} \times \{1\} \) and \( \mathbb{N} \times \{2\} \) are disjoint open \( F_\sigma \)-sets in \( \tilde{K} \) and the intersection of their closures is non-empty since it contains \((k_0, 1) = (k_0, 2)\). Hence \( \tilde{K} \) is not even an F-space, let alone a quasi-Stonean space, and consequently \( C(\tilde{K}) \) is not Dedekind \( \sigma \)-complete (in fact not even a Cantor space). \( \blacksquare \)

So far we discussed the implications of the equality \( (\star) \) on the properties of the target space \( F \). There is one more natural “parameter” which has a very important impact on \( F \), namely the order properties of the space \( \mathcal{L}^r(E,F) \). Following the lead given in [AG], we showed in [AW] that it is possible to characterize Dedekind \( \sigma \)-complete Banach lattices \( F \) by the fact that the space of regular operators from any separable Banach lattice into \( F \) forms a lattice. To parallel that result in the setting of the present paper we need to
consider operators on separable L-spaces. To do so we will generalize Theorem 3.10 in [AW], in which we considered operators defined on all separable Banach lattices. We are taking this opportunity to also mention that Theorem 3.10 of [AW] was rather carelessly worded (references to Banach lattices of operators should be replaced by references to vector lattices of operators; the former were not removed from an earlier draft which was formulated in a different setting), so we restate the result in its entirety, together with a new extra equivalence.

**Theorem 3.3.** For a Banach lattice $F$ the following are equivalent:

(a) $F$ is Dedekind $\sigma$-complete.

(b) For all separable Banach lattices $E$, $L^r(E, F)$ is a Dedekind $\sigma$-complete vector lattice.

(c) $L^r(c, F)$ is a vector lattice, where $c$ is the space of all convergent sequences.

(d) $L^r(L_1[0, 2\pi], F)$ is a vector lattice.

**Proof.** The equivalence of (a), (b) and (c) is proved in Theorem 3.10 of [AW], whilst (d) is an obvious consequence of (b).

To complete the proof we will deduce (a) from (d). Notice first that if $L^r(L_1[0, 2\pi], F)$ is a vector lattice, then so is $L^r(L_1[0, 2\pi], J)$, where $J$ is any principal ideal in $F$. If we can prove that each such $J$ is Dedekind $\sigma$-complete, then clearly, $F$ will also be. We may thus, using the Kakutani-Krein representation for unital M-spaces, reduce the problem to that of showing that if $K$ is a compact Hausdorff space and $L^r(L_1[0, 2\pi], C(K))$ is a vector lattice, then $K$ is quasi-Stonean.

Let $U$ be an open $F_\sigma$-subset of $K$. By Proposition 2.1 of [W2] we can find two disjoint sequences of functions in $C(K)$, $(u_n)$ and $(v_n)$, vanishing on $K \setminus U$, such that $0 \leq u_n(k), v_n(k) \leq 1$ for all $k \in K$ and $U = \bigcup_{n=1}^{\infty} (u_n^{-1}(1) \cup v_n^{-1}(1))$. If $k_0 \in K \setminus \overline{U}$, use Urysohn’s lemma to find $w \in C(K)$ with $0 \leq w(k) \leq 1$ for all $k \in K$, $w|_U \equiv 1$ and $w(k_0) = 0$. With the same definition of $b_n$ as in Theorem 3.1, define two linear operators $S, T : L_1[0, 2\pi] \to C(K)$ by

$$Sf = \sum_{n=1}^{\infty} b_n(f)u_n,$$

$$Tf = Sf + b_0(f)w,$$

where $b_0(f) := \int_0^{2\pi} f(t) \, dt$. The convergence of the series defining $S$ may be proved in a
similar manner to the proof in Theorem 3.1. We notice for later use that the operator $S$ is independent of our choice of the point $k_0$ and of the function $w$.

It is clear that if $f \geq 0$ then $(T - S)f = b_0(f)w \geq 0$, so that $T \geq S$. Next we claim that the operator $T$ is positive. To this end first note that $b_0 + b_m$ is a positive functional for each $m \in \mathbb{N}$. Indeed $b_0(f) + b_m(f) = \int_0^{2\pi} f(t)(1 + \cos(mt)) dt \geq 0$, whenever $f \geq 0$. We have

$$Tf = \sum_{n=1}^{\infty} b_n(f) u_n + b_0(f)w = b_m(f) u_m + b_0(f)w + \sum_{n \neq m} b_n(f) u_n.$$ 

Since $w \geq u_n \geq 0$ and the functions $(u_n)$ are pairwise disjoint, the positivity of the functionals $b_0 + b_m$ implies that $Tf \geq 0$ on the closure of the set $\bigcup_n \{k \in K : u_n(k) > 0\}$. On the complement of the above set (i.e., on the interior of the set where all $u_n$ vanish), $Tf$ is simply $b_0(f)w$ which is certainly non-negative, so that $Tf \geq 0$. Thus we have established that $T \geq S$, which, among other things, tells us that $S$ is a regular operator.

By condition (d), $S^+$ exists in $\mathcal{L}^r(L_1[0, 2\pi], C(K))$.

Using again the inequality $2 \geq 1 + \cos(nt) \geq 0$ we see that

$$S^+ 2 \geq S^+ (1 + \cos(nt)) \geq S(1 + \cos(nt)) = \pi \cdot u_n,$$

whence $\pi^{-1} S^+ 2$ is an upper bound for the sequence $(u_n)$. We must also have $S^+ \leq T$, so that $0 \leq S^+ 2 \leq T 2 = 2w$ and in particular $(S^+ 2)(k_0) = 0$. Since our operator $S$ was independent of the choice of the point $k_0$, the previous argument is applicable to any point of $K \setminus \overline{U}$. Consequently, we must have $S^+ 2$ identically zero on $K \setminus \overline{U}$.

If we replace the sequence $(u_n)$ by $(v_n)$ to define an operator $S_1 f = \sum_{n=1}^{\infty} b_n(f)v_n$ and repeat the whole of the proof so far, then we will conclude that $\pi^{-1} S_1^+ 2$ vanishes on $K \setminus \overline{U}$ and is an upper bound for $(v_n)$.

Now $\pi^{-1} S^+ 2 \vee S_1^+ 2$ also vanishes on $K \setminus \overline{U}$ and is an upper bound for both sequences $(u_n)$ and $(v_n)$. It follows that $\pi^{-1} S^+ 2 \vee S_1^+ 2$ is at least 1 on $U$, whilst it vanishes on $K \setminus \overline{U}$. Since $S^+ 2 \vee S_1^+ 2$ is continuous, this certainly forces $\overline{U}$ to be open, and hence $K$ is indeed quasi-Stonean.  

Putting together Theorems 3.1 and 3.3 we now have:
Corollary 3.4. The following conditions on a Banach lattice $F$ are equivalent.

(a) $F$ has a sequentially Levi norm.

(b) For every separable $L$-space $E$, the space $L(E, F)$ is a vector lattice.

We turn now to the general case, when we allow the domain to be any $L$-space instead of considering only separable $L$-spaces. Our results are entirely analogous to those above and, in fact, depend on them.

Theorem 3.5. The following conditions on a Dedekind complete Banach lattice $F$ are equivalent.

(a) $F$ has a Levi norm.

(b) For every $L$-space $E$, the equality $L(E, F) = L^r(E, F)$ holds.

Proof. As mentioned earlier the implication (a)⇒(b) is true by Theorem 1.1. We need only prove that (b)⇒(a). Suppose that $A \subset F_+$ is a disjoint set such that for all finite sets $\sigma \subset A$, $\|\sum_{a \in \sigma} a\| \leq K$. By Theorem 2.3, it is enough to show that $A$ has a supremum in $F$. Let $\alpha$ denote the cardinality of $A$. We will work with operators whose domain is the space $L_1(\mu)$, where $\mu$ is the product of $\alpha$ copies of the probability measure on $\{0, 1\}$ which assigns measure $\frac{1}{2}$ to both $\{0\}$ and $\{1\}$. It is well known that integrable functions on $\{0, 1\}^\alpha$ depend on only countably many variables. Let $\phi_i$ denote the function in $L_\infty(\mu)$ which depends only on the $i$'th variable and takes the value 1 if this variable is 0, and takes the value $-1$ if this variable is 1. If $f \in L_1(\mu)$ and $f$ does not depend on the $i$'th variable, then $\int f \phi_i \, d\mu = 0$. We denote by $\Phi$ the collection of all these functions $\phi_i$ for $i \in \alpha$. Note that we may also write $A = \{a_i : i \in \alpha\}$ by indexing the members of $A$ by $\alpha$.

We thus certainly have

(i) For each $f \in L_1(\mu)$, $\int f \phi \, d\mu = 0$ for all but countably many $\phi \in \Phi$.

(ii) $\|\phi\|_\infty = 1$ for all $\phi \in \Phi$.

For notational convenience, we will write $\phi(f)$ in place of $\int f \phi \, d\mu$ and regard each $\phi$ as an element of $L_1(\mu)^*$. Note that

(iii) for each $\phi \in \Phi$ there is $f \in L_1(\mu)$ with $0 \leq f \leq 1$ and $|\phi(f)| \geq \frac{1}{2}$.

We refer the reader to [HS, §22] for the requisite details concerning infinite product measures and integration.
In view of Theorem 3.1 we already know that $F$ has a sequentially Levi norm. This implies, by a theorem of Amemiya [Am], that there is a constant $C > 0$ such that $0 \leq y_n \uparrow y \Rightarrow \|y\| \leq C \lim \|y_n\|$. In particular, for each countable subset $B \subset A$ we have that its supremum $\bigvee B$ exists in $F$ and that $\|\bigvee B\| \leq C$. Define a linear operator $S : L_1(\mu) \to F$ by

$$Sf = \sum_{i \in \alpha} \phi_i(f)a_i.$$ 

In order to see that this series is order convergent, recall that (i) guarantees that there is a countable subset $\beta \subset \alpha$ such that $\phi_i(f) = 0$ for all $i \in \alpha \setminus \beta$. The collection of all finite sums $\sum_{i \in \sigma} \phi_i(f)a_i$ is norm bounded in view of the inequality

$$\left| \sum_{i \in \sigma} \phi_i(f)a_i \right| \leq \sum_{i \in \sigma} \|f\|_1a_i = \|f\|_1 \sum_{i \in \sigma} a_i$$

which implies that $\|\sum_{i \in \sigma} \phi_i(f)a_i\| \leq K\|f\|_1$. The disjointness of the elements $a_i$ guarantees that we also have $\|\sum_{i \in \sigma} \phi_i(f)\pm a_i\| \leq K\|f\|_1$, so it follows from the sequentially Levi property of $F$ that the series $\sum_{i \in \beta} \phi_i(f)\pm a_i$ are both order convergent and hence $\sum_{i \in \beta} \phi_i(f)a_i$ is also order convergent. This implies that $Sf$ is indeed well-defined.

Notice that Amemiya’s result shows us that $\|\sum_{i \in \sigma} \phi_i(f)a_i\| \leq CK\|f\|_1$, so that $\|Sf\| = \|\sum_{i \in \beta} \phi_i(f)a_i\| \leq 2CK\|f\|_1$, so that $\|S\| \leq 2CK$ and, in particular, $S$ is norm bounded. By (b) there is $T : L_1(\mu) \to F$ with $T \geq S, -S$. We know that for each $i \in \alpha$ there is $f_i \in L_1(\mu)$ with $0 \leq f_i \leq 1$ and such that $|\phi_i(f_i)| \geq \frac{1}{2}$. Thus

$$T1 \geq Tf_i \geq |Sf_i| \geq |\phi_i(f_i)|a_i \geq \frac{1}{2}a_i,$$

so that $T2$ is an upper bound for $A$. As we are assuming that $F$ is Dedekind complete, this implies that the supremum of $A$ exists. ■

The example given in Remark 3.2 shows that we cannot omit the assumption of Dedekind completeness from the statement of Theorem 3.5. The L-space $E$ produced in the proof above is a nonseparable $L_1(\mu)$-space with a finite measure $\mu$. It is interesting to notice that one cannot avoid using a somewhat extravagant measure space to get the desired contradiction. For example, the classical L-spaces $\ell_1(\Gamma)$ will be definitely not
enough, since for any \( \Gamma \) and any Banach lattice \( F \) each continuous operator from \( \ell_1(\Gamma) \) into \( F \) is regular.

Similarly to what was done in Theorem 3.3, our next result characterizes the Dedekind completeness of \( F \) in terms of order properties of the space \( \mathcal{L}^r(E, F) \).

**Theorem 3.6.** For a Banach lattice \( F \) the following are equivalent:

(a) \( F \) is Dedekind complete.

(b) For all Banach lattices \( E \), the space \( \mathcal{L}^r(E, F) \) is a Dedekind complete vector lattice.

(c) For all \( L \)-spaces \( E \), the space \( \mathcal{L}^r(E, F) \) is a vector lattice.

**Proof.** Again, it is only (c) \( \Rightarrow \) (a) that we need to prove. As in the proof of Theorem 3.3, it suffices to consider the case that \( F = C(K) \). By that theorem, we already know that \( C(K) \) is Dedekind \( \sigma \)-complete, and so \( K \) is quasi-Stonean.

Let \( U \) be an arbitrary open subset of \( K \). We need to prove that its closure \( \overline{U} \) is open. There obviously exists a maximal disjoint collection of closed and open subsets \( D_i \) (\( i \in I \)) of \( U \), and so \( \cup D_i \) is dense in \( U \). Fix an arbitrary point \( k_0 \in K \setminus \overline{U} \), and find a function \( w \in C(K) \) which lies between \( 0 \) and \( 1 \), is one on \( U \) and is zero at the point \( k_0 \).

Exactly as in the proof of Theorem 3.5, we construct now a measure \( \mu \) on \( \{0, 1\}^\alpha \), where \( \alpha \) is the cardinality of \( I \). Let \( \Phi \) have the same meaning as in that proof and let \( \psi \) be the linear functional \( f \mapsto \int f \, d\mu \), so that \( \psi - \phi_i \geq 0 \) for all \( i \in I \). Define \( S, T : L_1(\mu) \to C(K) \) by

\[
Sf = \sum_{i \in I} \phi_i(f) \chi_{D_i} \\
Tf = Sf + \psi(f)w.
\]

This time convergence of the series follows from the Dedekind \( \sigma \)-completeness of \( F \), using the facts that for each \( f \in L_1(\mu) \) only countably many terms are non-zero and that \( \psi \geq \phi_i \) for all \( i \in \alpha \), and showing that the finite partial sums all lie between \( \pm \psi(f)w \). It should be pointed out that the operator \( S \) above is independent of the function \( w \) and of the point \( k_0 \). As in the proof of Theorem 3.1, we can show that \( T \geq S, 0 \), so that \( S^+ \) exists by (c). For each \( i \in I \) there is \( f_i \in L_1(\mu) \) with \( 0 \leq f_i \leq 1 \) and \( \phi_i(f_i) \geq \frac{1}{2} \). It follows as before that \( S^+f_i \geq 2S^+x_i \). Noting that \( \phi_i(1) = 0 \) we also have \( 0 \leq S^+f_i \leq T2 = 2w \). In particular, \( S^+2(k_0) = 0 \). As \( k_0 \) was an arbitrary point of \( K \setminus \overline{U} \), this shows that \( S^+2 \) must
vanish on \( K \setminus \overline{U} \). On the other hand, the continuous function \( S^+ 2 \) dominates each \( \chi_{\alpha_i} \), so is at least one on a dense subset of \( U \). It follows that \( \overline{U} \) is indeed open and the proof is complete. ■

**Corollary 3.7.** The following conditions on a Banach lattice \( F \) are equivalent.

(a) \( F \) has a Levi norm.

(b) For every \( L \)-space \( E \), the space \( \mathcal{L}(E, F) \) is a vector lattice.

The results obtained in this section imply the following observation. If a Dedekind complete Banach lattice \( F \) does not have a Levi-norm then, by Theorem 3.5, for an appropriate \( L \)-space \( E \) the space \( \mathcal{L}(E, F) \) is bigger than \( \mathcal{L}^r(E, F) \), though the latter is a Dedekind complete vector lattice.

### 4. Regularity of operators with arbitrary domain.

There is only one known case of a Dedekind complete Banach lattice \( F \) such that all continuous operators, with any Banach lattice as domain and \( F \) as range, are regular—namely when \( F \) has a strong order unit. This result dates back to [K]. As we shall see next, it is in fact the only case.

**Theorem 4.1.** The following conditions on a Dedekind complete Banach lattice \( F \) are equivalent.

(a) \( F \) has a strong order unit.

(b) For every Banach lattice \( E \) the equality \( \mathcal{L}(E, F) = \mathcal{L}^r(E, F) \) holds.

**Proof.** As we said (a)\( \Rightarrow \) (b) is due to Kantorovich. We need only prove that (b)\( \Rightarrow \) (a). First notice that by Theorem 3.5, \( F \) has a Levi norm. Second note that (b) is “more” than enough to imply that \( F \) is isomorphic to an M-space. Indeed, either of the following two conditions is weaker than (b) and implies that \( F \) is isomorphic to an M-space.

1) There exists \( p \in (1, \infty] \) such that each continuous operator \( T : L_p[0, 1] \to F \) is regular.

2) There exists a Banach lattice \( E \) containing uniformly the subspaces \( \ell^n_\infty \) and such that each continuous operator \( T : E \to F \) is regular.

The sufficiency of the first condition was established by Cartwright and Lotz [CL] (see also [MN, Theorem 3.2.1], or [A3, Theorem 8.6]), whilst for sufficiency of the second and
several other conditions see [A3, §2]. The conjunction of the two properties of $F$ obtained so far immediately implies (a). Indeed, the collection of all finite suprema from the unit ball of $F$ is upward directed and norm bounded, so has a supremum. That supremum is a strong order unit for $F$. ■

Again, as Remark 3.2 shows, the hypothesis of Dedekind completeness may not be omitted. Combining Theorem 3.5 and Theorem 4.1, we obtain:

**Corollary 4.2.** The following conditions on a Banach lattice $F$ are equivalent.

(a) $F$ is Dedekind complete and has a strong order unit.

(b) For every Banach lattice $E$, the space $\mathcal{L}(E, F)$ is a vector lattice.

We conclude by noticing that in all our results starting with Corollary 3.4 the uniform operator norm and the regular norm on the space $\mathcal{L}^r$ are equivalent. The isometric version of these results, describing when these two norms are equal, will be discussed in a forthcoming paper by the authors and Z. L. Chen.
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