PACKING AND COVERING DIRECTED TRIANGLES

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Abstract. We prove that if a directed multigraph $D$ has at most $t$ pairwise arc-disjoint directed triangles, then there exists a set of less than $2t$ arcs in $D$ which meets all directed triangles in $D$, except in the trivial case $t = 0$. This answers affirmatively a question of Tuza from 1990.

1. Introduction

In the 1980s, Tuza [9, 10] posed the following conjecture about packing and covering triangles in undirected simple graphs (hereafter called graphs). Given a graph $G$, let $\nu(G)$ be the maximum size of a family of pairwise edge-disjoint triangles in $G$, and let $\tau(G)$ be the minimum size of an edge set $X$ such that $G - X$ is triangle-free. Evidently $\tau(G) \geq \nu(G)$, since we are forced to delete at least one edge from each triangle in a family of edge-disjoint triangles (and these edges must be distinct), and on the other hand $\tau(G) \leq 3\nu(G)$, since it suffices to delete all edges from each triangle in a maximal family of edge-disjoint triangles. Tuza conjectured that in fact $\tau(G) \leq 2\nu(G)$ for every graph $G$. As Tuza observed, this upper bound is sharp if true, and in particular it is achieved by $K_4$ and $K_5$.

The best general result on Tuza’s conjecture is due to Haxell [3], who proved that $\tau(G) \leq 2.87\nu(G)$ for every graph $G$. Other authors have approached the conjecture by proving that $\tau(G) \leq 2\nu(G)$ for all graphs in some given family. Tuza [10] showed that his conjecture holds for all planar graphs, and Aparna Lakshmanan, Bujtás, and Tuza [8] showed that it holds for all 4-colorable graphs. The planar result has been generalized to graphs without $K_{3,3}$-subdivisions (Krivelevich [6]), and then to graphs with maximum average degree less than 7 (Puleo [7]). In the case where $G$ is a $K_4$-free planar graph, the stronger inequality $\tau(G) \leq \frac{4}{3}\nu(G)$ was proved by Haxell, Kostochka, and Thomassé [5].

Asymptotic, fractional, and multigraph versions of Tuza’s conjecture have also been considered. Yuster [12] proved that $\tau(G) \leq (2 + o(1))\nu(G)$ when $G$ is a dense graph, and this was shown to be asymptotically tight by Kahn and Baron [11]. Yuster [12] also noted that a combination of results by Krivelevich [6] and Haxell and Rödl [4] implies that for any graph $G$ with $n$ vertices, $\tau(G) < 2\nu(G) + o(n^2)$. Two fractional versions of Tuza’s Conjecture were proved by Krivelevich [6]. Chapuy, DeVos, McDonald, Mohar, and Scheide [2] tightened one of these fractional versions, and considered the natural extension of Tuza’s conjecture to multigraphs. Here by multigraph we mean that multiple edges are permitted, but not loops (they have no effect on our problem anyways); the definitions of $\mu$ and $\tau$ are identical to those given in the simple graph case. In [2], planar multigraphs were shown to satisfy Tuza’s conjecture, and $\tau(G) \leq 2.92\nu(G)$ was shown to hold for all multigraphs $G$.

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When posing his conjecture in [10], Tuza also discussed the problem of packing and covering directed triangles. Here by directed multigraph we shall mean any oriented multigraph; by directed graph we shall mean any directed multigraph without parallel arcs in the same direction (but we allow digons, i.e., a pair of arcs $u \to v$ and $v \to u$). Given a directed multigraph $D$, let $\nu_c(D)$ denote the maximum size of a family of pairwise arc-disjoint directed triangles, and let $\tau_c(D)$ denote the minimum size of an edge set $Y$ such that $D - Y$ has no directed triangles. Tuza asked: “Is $\tau_c(D) < 2 \nu_c(D)$ for every digraph $D$?” In this paper we answer this affirmatively with the following theorem.

**Theorem 1.1.** If $D$ is a directed multigraph with at least one directed triangle, then $\tau_c(D) < 2\nu_c(D)$.

Tuza [10] observed that the rotational 5-tournament $T_5$, pictured in Figure 1, satisfies $\tau_c(T_5)/\nu_c(T_5) = \frac{3}{2}$. Our computational efforts have not yielded any examples with a larger ratio for $\tau_c/\nu_c$, and in fact we find the following conjecture plausible.

**Conjecture 1.2.** If $D$ is a directed multigraph, then $\tau_c(D) \leq \frac{3}{2}\nu_c(D)$.

In [11], Tuza proved that if $D$ is a planar oriented graph, then $\tau_c(D) = \nu_c(D)$. This topic of packing and covering directed triangles appears not to have caught on in the literature however (in contrast to the undirected analogue), and we hope that Conjecture 1.2 and Theorem 1.1 may create interest.

### 2. Proof of Theorem 1.1

The main idea of our proof is based on the reducibility argument in Puleo [7]. We use induction on $|V(D)|$, with trivial base case when $|V(D)| = 1$.

Take any $v \in V(D)$, and define an auxiliary directed multigraph $N$ as follows: the vertex set of $N$ is the disjoint union of a set $\{s, t\}$ consisting of designated source and sink vertices, as well as two sets $W^+$ and $W^-$, where $W^+$ contains a copy $w^+$ of each vertex $w \in N^+(v)$, and $W^-$ contains a copy $w^-$ of each vertex $w \in N^-(v)$. (Note that if $w \in N^+(v) \cap N^-(v)$, then there is a copy of $w$ in each of $W^+$ and $W^-$.) Given vertices $u^+ \in W^+$ and $z^- \in W^-$, we include the arc $u^+ \to z^-$ in $E(N)$ with the same multiplicity as the arc $u \to z$ in $E(D)$. For each $w^+ \in W^+$, we include the arc $s \to w^+$ in $E(D)$ with the same multiplicity as the arc $v \to w$, and for each $w^- \in W^-$, we include the arc $w^- \to t$ in $E(N)$ with the same multiplicity as the arc $w \to v$ in $E(D)$.
Observe that there is a bijection between directed triangles in $D$ containing $v$, and directed $(s, t)$-paths in $N$; triangle $z \rightarrow v \rightarrow u \rightarrow z$ in $D$ corresponds to directed path $sv^+z^-t$ in $N$. Furthermore, two directed triangles in $D$ are arc-disjoint if and only if the corresponding paths in $N$ are arc-disjoint. (Whenever two triangles use different parallel arcs, the corresponding paths have parallel arcs as well.)

Let $\mathcal{P}$ be a maximum-size set of arc-disjoint $(s, t)$-paths in $N$, say with $|\mathcal{P}| = p$. Let $\mathcal{R}$ be the corresponding set of pairwise arc-disjoint triangles in $D$, all of which contain $v$. Each triangle in $\mathcal{R}$ has exactly one arc that is not incident to $v$; let $\mathcal{R}_v$ be the set consisting of these $p$ arcs. If possible, choose $\mathcal{P}$ so that $D' := D - v - \mathcal{R}_v$ has at least one directed triangle (subject to $\mathcal{P}$ being maximum-sized). Observe that triangles in $D'$ are precisely those triangles in $D$ that do not contain $v$ and do not share any edges with $\mathcal{R}$.

Let $X$ be a minimum-size set of arcs in $N$ so that $N - X$ has no $(s, t)$-paths. By Menger’s Theorem, $|X| = |\mathcal{P}| = p$. Note that in $D$, the set $X$ corresponds to a set $X_D$ of $p$ arcs, and every triangle incident to $v$ has at least one arc in $X_D$. Let $\tilde{C} = X_D \cup \mathcal{R}_v$, and observe that $\tilde{C}$ is a triangle arc cover of every triangle involving $v$ as well as every triangle sharing an edge with $\mathcal{R}$. We have $|\tilde{C}| \leq 2p$, with equality if and only if $X_D$ and $\mathcal{R}_v$ are disjoint.

Suppose that $D'$ has at least one directed triangle. By induction, $\tau_v(D') < 2\nu_v(D')$. Let $\mathcal{R}'$ be a maximum-size set of edge-disjoint directed triangles in $D'$ and let $\tilde{C}'$ be a minimum-size triangle arc cover in $D'$. By our observations above, note that $\mathcal{R} \cup \tilde{C}'$ is a triangle arc cover of $D$, and $\mathcal{R} \cup \mathcal{R}'$ is a set of edge-disjoint triangles in $D$. We get that

$$|\mathcal{R}' \cup \tilde{C}' < 2|\mathcal{R}'| + 2p = 2|\mathcal{R} \cup \mathcal{R}'|,$$

as desired.

We may now assume that $D'$ has no directed triangles. In this case, $\tilde{C}$ is a triangle arc cover for $D$ with size at most $2p$. Since $\mathcal{R}$ is set of $p$ arc-disjoint triangles in $D$, it suffices to show that $D$ has a triangle arc cover with size at most $2p - 1$. In particular, from our observations above, we may assume that $X_D$ and $\mathcal{R}_v$ are disjoint. Moreover, we may assume that every edge in $\mathcal{R}_v$ is in some directed triangle in $D - v$, otherwise this edge could be left out of the cover.

We claim that every maximum-size choice of paths $\mathcal{P}$ yields the same set $\mathcal{R}_v$ via the process defined above. Suppose to the contrary that some choice $\mathcal{P}'$ yields a different set $\mathcal{R}'_v$. Then there would be some arc in $\mathcal{R}_v \setminus \mathcal{R}'_v$ corresponding to a directed triangle in $D - v$ (by our assumption above). But then this triangle would still exist in $D - v - \mathcal{R}'_v$, contradicting our choice of $\mathcal{P}$. Hence every choice for $\mathcal{P}$ yields the same set $\mathcal{R}_v$.

Now let $e$ be any arc of $\mathcal{R}_v$, say from $w$ to $z$, and let $\tilde{N}$ be obtained from $N$ by deleting a copy of $w^+ \rightarrow z^-$. By the uniqueness of $\mathcal{R}_v$, the maximum number of edge-disjoint $(s, t)$-paths in $\tilde{N}$ is $p - 1$. Let $\tilde{X}$ be a minimum-size set of edges in $\tilde{N}$ so that $\tilde{N} - \tilde{X}$ has no $(s, t)$-paths. By Menger’s theorem $|\tilde{X}| = p - 1$. We claim that $\tilde{C} = \tilde{X} \cup \mathcal{R}_v$, which has size at most $2p - 1$, is a triangle-arc cover for $D$. To see this recall that, by assumption, all triangles in $D$ are either incident with $v$ or contain an edge of $\mathcal{R}_v$. The latter are covered by $\mathcal{R}_v$, and the former are covered by $\tilde{X}$, except for triangles containing the arc $e$ and the vertex $v$. Since $e \in \mathcal{R}_v$, we have our desired result.
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