QUOTIENTS OF CONIC BUNDLES

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Abstract. In this paper we study quotients of rational conic bundles over arbitrary fields of characteristic zero by finite groups of automorphisms. We construct smooth minimal models for such quotients and show that there are many examples of birationally non-trivial quotients.

1. Introduction

Let $k$ be a field of characteristic zero, $K = k(x_1, \ldots, x_n)$ be a pure transcendental extension of it and $G$ be a finite group acting on $k$. The main purpose of this paper is to study when the field of invariants $K^G$ is rational (i.e. pure transcendental) and classify non-rational fields of invariants. This is a special case of problem to classify subfields of $K$.

In the language of algebraic geometry this problem can be formulated in the following way. Let $X$ be a $k$-rational variety and $G$ be a finite subgroup of $\text{Aut}_k(X)$. When the quotient $X/G$ is $k$-rational? What is the $k$-birational classification of quotients $X/G$? This is a special case of problem to classify $k$-unirational varieties.

The answer for $n = 1$ is a classical result.

Theorem 1.1 (J. Lüroth [Lur76]). Any unirational curve is rational.

If $n = 2$ and the field $k$ is algebraically closed then rationality of any unirational surface follows from Castelnuovo’s rationality criterion [Cast94].

Theorem 1.2. Let $k$ be an algebraically closed field of characteristic zero. Then any $k$-unirational surface is $k$-rational.

If the field $k$ is not algebraically closed then this is not true. For example, any del Pezzo surface of degree 4 is $k$-unirational (see [Man74, Theorem 7.8]) but not all of them are $k$-rational (see [Isk96, Chapter 4]). Moreover, the unirationality degree of del Pezzo surface of degree
4 equals 1 or 2 thus this surface is either $k$-birationally trivial or $k$-birationally equivalent to a quotient of $k$-rational surface by group of order 2.

If a variety $X$ is $k$-birationally equivalent to a quotient of $k$-rational surface by a finite group of automorphisms then $X$ is called Galois $k$-unirational. Now there are not any known ways to prove that a geometrically rational (i.e. rational over algebraical closure of the field $k$) surface is not $k$-unirational. But Galois $k$-unirational surfaces can be studied in explicit way since groups which can minimally act on $k$-rational surfaces are described (see [DI09a]).

If $S$ is a $k$-rational surface and $G \subset \text{Aut}_k(S)$ by applying $G$-minimal model program one can obtain a $G$-minimal surface $X$ such that the quotients $S/G$ and $X/G$ are $k$-birationally equivalent. Any $G$-minimal surface is either a del Pezzo surface or a conic bundle (see [Isk79, Theorem 1]). Thus any quotient of a $k$-rational surface is birationally equivalent to a quotient of a del Pezzo surface or a conic bundle.

In this paper we study quotients of conic bundles. Its the biggest case since degree of del Pezzo surfaces is bounded and for a conic bundle $S$ the number $K_S^2$ can be arbitrary small. Also classification of quotients of conic bundles can be useful in studying $k$-unirationality of conic bundles.

The main results of this paper are the following.

**Theorem 1.3.** Let $k$ be a field of characteristic zero, $X$ be a $k$-rational surface and $G$ be a finite group acting on $X$. Then the quotient $X/G$ is $k$-birationally equivalent to either a quotient of a $k$-rational del Pezzo surface by a finite group of automorphisms or a quotient of $k$-rational conic bundle by cyclic or dihedral group of order $2^n$, alternating or symmetric group of degree 4 or alternating group of degree 5.

**Corollary 1.4.** Let $k$ be a field of characteristic zero, $S$ be a $k$-rational surface and $G$ be a finite group acting on $S$. Assume that $\text{ord } G > 10$, $\text{ord } G \neq 15$ and $\text{ord } G$ is odd. If $G$ is cyclic or $\text{ord } G$ is not divisible by 3 then $S/G$ is $k$-rational.

**Corollary 1.5.** Let $k$ be a field of characteristic zero, $K = k(x, y)$ and $G$ be a finite group acting on $K$. Assume that $\text{ord } G > 10$, $\text{ord } G \neq 15$ and $\text{ord } G$ is odd. If $G$ is cyclic or $\text{ord } G$ is not divisible by 3 then $K^G$ is pure transcendental extension of $k$.

**Proposition 1.6.** If $X$ is a $k$-rational $G$-equivariant conic bundle and $K_X^2 \geq 5$ then $X/G$ is $k$-rational.

**Proposition 1.7.** Let $k$ be a field of characteristic zero such that not all elements of $k$ are squares and $G$ be one of the following groups:
cyclic or dihedral group of order $2n$, alternating or symmetric group of degree 4 or alternating group of degree 5. Then the following three spaces are infinitely dimensional:

1. The space of Galois $k$-unirational surfaces up to $k$-birationally equivalence which is $k$-birationally equivalent to a quotient of $k$-rational surface $S$ by the group $G$;
2. The space of fields of invariants $k(x, y)^G$ up to isomorphism;
3. The space of subgroups $G \subset C_{r^2}(k)$ up to conjugation.

The plan of this paper is the following. In Section 2 we review main notions about minimal model program, conic bundles and rationality. In Section 3 we study how the geometric group $G$ acts on a $G$-minimal conic bundle $X$. As a by-product we prove Proposition 1.6. In Section 4 we prove Theorem 1.3 and construct an example showing that all possibilities listed in this theorem occurs. Also construction of this example proves Proposition 1.7.

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We use the following notation.

- $k$ denotes an arbitrary field of characteristic zero,
- $\overline{k}$ denotes the algebraic closure of a field $k$,
- $\overline{X} = X \otimes \overline{k}$,
- $\text{ord } G$ denotes the order of a group $G$,
- $\text{ord } g$ denotes the order of an element $g \in G$,
- $C_n$ denotes the cyclic group of order $n$,
- $D_{2n}$ denotes the dihedral group of order $2n$,
- $S_n$ denotes the symmetric group of degree $n$,
- $A_n$ denotes the alternating group of degree $n$,
- $(i_1i_2\ldots i_j)$ denotes a cyclic permutation of $i_1, \ldots, i_n$,
- $\text{diag}(a, b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$,
- $K_X$ denotes the canonical divisor of a variety $X$,
- $F_n$ denotes the rational ruled (Hirzebruch) surface $\mathbb{P}_1(\mathcal{O} \oplus \mathcal{O}(n))$,
- $X \approx Y$ means that $X$ and $Y$ are $k$-birationally equivalent.

2. Preliminaries

In this section we review main notions and results of $G$-equivariant minimal model program following the papers [Man67], [Isk79], [DI09a], [DI09b].
Definition 2.1. A \( k \)-rational surface \( X \) is a surface over \( k \) such that \( X \) is birationally equivalent to \( \mathbb{P}^2_k \).

We say that \( X \) is rational if \( \overline{X} = X \otimes \mathbb{F} \) is \( \overline{k} \)-rational.

A surface \( X \) over \( k \) is a \( k \)-unirational surface if there exists a \( k \)-rational variety \( Y \) and a dominant rational map \( \varphi : Y \dashrightarrow X \).

Definition 2.2. A \( G \)-surface is a pair \((X, G)\) where \( X \) is a projective surface over \( k \) and \( G \) is a subgroup of \( \text{Aut}_k(X) \). A morphism of surfaces \( f : X \to X' \) is called a \( G \)-morphism \((X, G) \to (X', G)\) if for each \( g \in G \) one has \( fg = gf \).

A smooth \( G \)-surface \((X, G)\) is called \( G \)-minimal if any birational morphism of smooth \( G \)-surfaces \((X, G) \to (X', G)\) is an isomorphism.

Let \((X, G)\) be a smooth \( G \)-surface. A \( G \)-minimal surface \((Y, G)\) is called a minimal model of \((X, G)\) if there exists a birational \( G \)-morphism \( X \to Y \).

The classification of \( G \)-minimal rational surfaces is well-known due to V. Iskovskikh and Yu. Manin (see [Isk79] and [Man67]). We recall some important notions before surveying it.

Definition 2.3. A smooth rational \( G \)-surface \((X, G)\) admits a conic bundle structure if there exists a \( G \)-morphism \( \varphi : X \to B \) such that any scheme fibre is isomorphic to a reduced conic in \( \mathbb{P}^2_k \) and \( B \) is a smooth curve.

The curve \( B \) is called the base of conic bundle.

A general fibre of a conic bundle \( \varphi : X \to B \) is isomorphic to \( \mathbb{P}^1_k \). The fibration \( \varphi \) has a finite number of singular fibres which are degenerate conics. Any irreducible component of a singular fibre is a \((-1)\)-curve. If \( n \) is the number of singular fibres then \( K^2_X + n = 8 \).

Definition 2.4. Let \( X \) be a \( G \)-surface that admits a conic bundle structure \( \varphi : X \to B \). The conic bundle is called relatively \( G \)-minimal over \( B \) if for any decomposition of \( \varphi \) into \( G \)-morphisms \( X \to X' \to B \) such that the first morphism is birational this morphism is isomorphism.

A conic bundle \( \varphi : X \to B \) is relatively \( G \)-minimal over \( B \) if and only if \( \text{rk} \text{Pic}(X)^G = 2 \).

Definition 2.5. A del Pezzo surface is a smooth projective surface \( X \) such that the anticanonical divisor \(-K_X\) is ample.

Theorem 2.6 ([Isk79, Theorem 1]). Let \( X \) be a \( G \)-minimal rational \( G \)-surface. Then either \( X \) admits a conic bundle structure over a rational curve with \( \text{Pic}(X)^G \cong \mathbb{Z}^2 \), or \( X \) is a del Pezzo surface with \( \text{Pic}(X)^G \cong \mathbb{Z} \).
Theorem 2.7 (cf. [Isk79, Theorem 4], [Isk79, Theorem 5]). Let $X$ admit a $G$-equivariant structure of a conic bundle. Then:

(i) If $K^2_X = 3, 5, 6, 7$ or $X \cong F_1$ then $X$ is not $G$-minimal.

(ii) If $K^2_X = 8$ then $X$ is isomorphic to $F_n$.

(iii) If $K^2_X \neq 3, 5, 6, 7$ or $X \neq F_1$ and $X$ is relatively $G$-minimal then $X$ is $G$-minimal.

(iv) If $K^2_X = 3, 5, 6$ and $X$ is relatively $G$-minimal, then $X$ is a del Pezzo surface.

(v) If $K^2_X = 7$, then $X$ is not relatively $G$-minimal.

The following theorem is an important criterion of $k$-rationality over an arbitrary perfect field $k$.

Theorem 2.8 ([Isk96, Chapter 4]). A minimal rational surface $X$ over a perfect field $k$ is $k$-rational if and only if the following two conditions are satisfied:

(i) $X(\mathbb{k}) \neq \emptyset$;

(ii) $K^2_X \geq 5$.

Corollary 2.9. Let $X$ admit a relatively minimal conic bundle structure over a perfect field $k$. The surface $X$ is $k$-rational if and only if the following two condition are satisfied:

(i) $X(\mathbb{k}) \neq \emptyset$;

(ii) $K^2_X \geq 5$.

Proof. If $K^2_X \geq 5$ and $X(\mathbb{k}) \neq \emptyset$ then any minimal model $Y$ of $X$ is $\mathbb{k}$-rational by Theorem 2.8. Thus $X$ is $\mathbb{k}$-rational.

If $K^2_X < 5$ and $X$ is relatively minimal then $X$ is minimal if $K^2_X \neq 3$. Therefore if $K^2_X \neq 3$ then $X$ is not $\mathbb{k}$-rational by Theorem 2.8. If $K^2_X = 3$ then there exists a minimal model $Y$ of $X$ such that $K^2_Y = 4$ (see the proof of [Isk79, Theorem 4]). Thus $Y$ and $X \approx Y$ are not $\mathbb{k}$-rational by Theorem 2.8.

For convenience of the reader we use the following definition.

Definition 2.10. Let $X$ be a $G$-surface, $\tilde{X} \rightarrow X$ be its ($G$-equivariant) minimal resolution of singularities, and $Y$ be a $G$-equivariant minimal model of $\tilde{X}$. We call the surface $Y$ a $G$-MMP-reduction of $X$.

Let $X$ be a $G$-surface and let $\varphi : X \rightarrow B \cong \mathbb{P}_k^1$ be a $G$-morphism such that the general fibre is a smooth rational curve. Let $\tilde{X} \rightarrow X$ be its ($G$-equivariant) minimal resolution of singularities, and $Y$ be a $G$-equivariant relatively minimal model of $\tilde{X}$. We call the surface $Y$ a relative $G$-MMP-reduction of $X$ over $B$. 

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3. Geometry of fibres and sections

In this section we prove the following theorem.

**Theorem 3.1.** Let $G$ be a finite group faithfully acting on a surface $X$ and $X$ admit a $G$-equivariant conic bundle structure $\varphi : X \to B$. Any relative MMP-reduction $Y$ over $B/G$ of the quotient $X/G$ admits a conic bundle structure. Denote by $G_B$ the image of $G$ under the natural map $\text{Aut}(X) \to \text{Aut}(B)$. Then

1. If $K^2_X \geq 5$ then $K^2_Y \geq 5$;
2. If $K^2_X < 5$ then $K^2_Y \geq K^2_Y$. If furthermore $K^2_X = K^2_Y$ then $K^2_X = K^2_Y = 4$ and $G_B \cong C_2$ or $G_B \cong D_4$.

Throughout this section we use the following notation.

Let a finite group $G$ faithfully act on a relatively $G$-minimal conic bundle $\varphi : X \to B$ and $n$ be the number of singular fibres of $X$. The morphism $\varphi$ is $G$-equivariant. It means that there exists an exact sequence:

$$1 \to G_F \to G \to G_B \to 1$$

where $G_F$ is a group of automorphisms of the generic fibre, acting trivially on the base $B \cong \mathbb{P}^1_k$, and $G_B$ is a group of automorphisms of $B$.

**Lemma 3.2.** Any relative $G_B$-MMP-reduction $Y$ of the quotient $X/G$ admits a conic bundle structure. The number of singular fibres of $Y$ is less or equal to $n$.

**Proof.** Let us consider a smooth fibre $F \cong \mathbb{P}^1_k$ of $\overline{X} \to \mathbb{P}^1_k$. The group $G_F$ acts on $F$ and the quotient $F/G_F$ is isomorphic to $\mathbb{P}^1_k$. Each non-trivial automorphism $g \in G_F$ has exactly two fixed points on $F$, but in a neighbourhood of these points $g$ acts as a reflection because it acts trivially on $B$. Therefore there are no singular points lying on $F/G_F$ in $\overline{X}/G_F$.

The $G$-morphism $\varphi : X \to B$ induces a $G_B$-morphism $\psi : X/G_F \to B$. A general fibre of $\psi$ is isomorphic to $\mathbb{P}^1_k$. Thus by applying $G_B$-equivariant Minimal Model Program over $B$ to $X/G_F$ one can obtain that a relative $G_B$-MMP-reduction $Y$ of $X/G_F$ is a conic bundle. Smooth fibres on $X$ correspond to smooth fibres on $Y$ therefore the number of singular fibres on $Y$ is less or equal to $n$. □

From now on we assume that $G_F$ is trivial and $G = G_B$. If $X$ is minimal then the components of each singular fibre of $\overline{X} \to \overline{B}$ are switched by some element of $G \times \text{Gal}(\overline{k}/k)$. 


Lemma 3.3. Any element $g \in G$ does not permute the components of any singular fibre.

Proof. Assume that there exists an element $g \in G$ permuting the components of a singular fibre $F$. Obviously, ord $g$ is even. The element $g$ faithfully acts on the base $B = \mathbb{P}^1_k$ and therefore there are exactly two fixed points $p_1$ and $p_2$ on $B$. If there exists a singular fibre $F$ over any other point $p$ and an integer $n$ such that $g^n$ permutes components of this fibre then $g^n$ has at least three fixed points $p, p_1$ and $p_2$ on the base. Thus the action of $g^n$ is trivial on the base and non-trivial on $X$. This contradicts the assumption that $G_F$ is trivial.

Apply $\langle g \rangle$-minimal model program to the surface $\overline{X}$. The obtained surface $\overline{X}$ has 1 or 2 singular fibres over points $p_1$ and $p_2$. Thus $K^2_{\overline{X}}$ equals 6 or 7. By Theorem 2.7 the surface $\overline{X}$ is a del Pezzo surface of degree 6.

Denote six $(-1)$-curves on $\overline{X}$ by $E_1, E_2, \ldots, E_6$, where $E_i^2 = -1$ and $E_i \cdot E_j = 1$ if $i = j \pm 1 \pmod{6}$, otherwise $E_i \cdot E_j = 0$. The element $g$ acts on these curves as follows:

$$gE_1 = E_2, gE_2 = E_1, gE_3 = E_6, gE_4 = E_5, gE_5 = E_4, gE_6 = E_3.$$  

The curves $E_3$ and $E_6$ are sections of the conic bundle $\overline{X} \to \mathbb{P}^1_k$, while $E_1 \cup E_2$ and $E_4 \cup E_5$ are singular fibres of this bundle.

Consider the element $h = g^{\text{ord}_g}$. Note that $h$ faithfully acts on the base $B$ and therefore all $h$-fixed points lie on the singular fibres of $\overline{X}$. The curves $E_i$ are not $h$-fixed since $\overline{X}$ is relatively $\langle g \rangle$-minimal. Thus by the Lefschetz fixed-formula the element $h$ has exactly 4 fixed points. But either all curves $E_i$ are $h$-invariant or $hE_1 = E_2$ and $hE_4 = E_5$. In the first case $h$ has six fixed points and in the second $h$ has two fixed points. This contradiction finishes the proof. □

One has

$$G = G_B \subset \text{Aut}(\mathbb{P}^1_k) \cong \text{PGL}_2(k) \subset \text{PGL}_2(\overline{k}).$$

Therefore $G$ is one of the following groups: $C_k, D_{2k}$ (including $D_4 = C_2^2$), $A_4, S_4, A_5$.

Lemma 3.4. Elements $g_1, g_2 \in \text{PGL}_2(\overline{k})$ such that the group $H = \langle g_1, g_2 \rangle$ is finite have the same pair of fixed points on $\mathbb{P}^1_k$ if and only if the group $H$ is cyclic. Otherwise the elements $g_1$ and $g_2$ do not have a common fixed point.

Proof. Let two elements $g_1, g_2 \in \text{PGL}_2(\overline{k})$ have a common fixed point $p \in \mathbb{P}^1_k$ and the group $H = \langle g_1, g_2 \rangle$ is finite. The natural representation $H \to \text{GL}(T_p \mathbb{P}^1_k) \cong \overline{k}^*$ is faithfull. Thus $H$ is cyclic. □
We use the following well-known lemma whose proof is a direct calculation.

**Lemma 3.5.** Let \( p \) be a smooth point of a surface \( S \) and \( g \in \text{Aut}(S) \) fix the point \( p \). Let \( g \) act on \( T_pS \) as \( \text{diag}(\lambda, \mu) \) and \( \tilde{S} \to S \) be a blowup of \( S \) at the point \( p \). Then \( g \) has on the exceptional divisor exactly two fixed points \( p_1 \) and \( p_2 \) and the actions on \( T_{p_1}\tilde{S} \) and \( T_{p_2}\tilde{S} \) have the form \( \text{diag}\left(\frac{a}{\mu}, \mu\right) \) and \( \text{diag}\left(\lambda, \frac{a}{\xi}\right) \) respectively.

**Lemma 3.6.** Let \( g \in G \) be an element of even order. Then \( g \)-invariant fibres are smooth.

**Proof.** Let \( p \) be a \( g \)-fixed point on \( \overline{\mathcal{B}} \) and \( F \) be a fibre over this point. We can assume that in the neighbourhood of \( p \) the element \( g \) acts on \( \overline{\mathcal{B}} \) as a multiplication by \( \xi_k \), where \( \xi_k \) is a primitive \( k \)-th root of unity.

Let \( F \) be a singular fibre so that \( F = E_1 + E_2 \). These curves \( E_1 \) and \( E_2 \) are \( g \)-invariant by Lemma 3.3. Let \( q_1 \in E_1 \) and \( q_2 \in E_2 \) be \( g \)-fixed points different from \( E_1 \cap E_2 \).

In the neighbourhood of \( q_1 \) the element \( g \) acts on \( E_1 \) as a multiplication by \( \xi_k^a \) for some \( a \). Applying Lemma 3.5 one can easily check that in the neighbourhood of \( q_2 \) the element \( g \) acts on \( E_2 \) as a multiplication by \( \xi_k^{-a-1} \). The point \( p \) is fixed only by elements of cyclic groups containing \( g \) by Lemma 3.4. Therefore if an element \( t \) of \( G \times \text{Gal}(\overline{k}/k) \) permutes \( E_1 \) and \( E_2 \) then \( tq_1 = q_2 \), \( tq_2 = q_1 \) and the groups \( G_{q_1} = \text{diag}(\xi_k, \xi_k^a) \) and \( G_{q_2} = \text{diag}(\xi_k, \xi_k^{-a-1}) \) coincide in \( \text{GL}_2(\overline{k}) \). It holds if and only if \( k \) is odd and \( a = \frac{k-1}{2} \). It contradicts assumption that \( k \) is even. \( \square \)

**Lemma 3.7.** Let \( Y \) be a relative MMP-reduction of the quotient \( X/G \) and \( f : X \dashrightarrow Y \) be the corresponding rational map. Let us consider an element \( g \in G \) of odd order such that the group \( \langle g \rangle \) is not contained in a bigger cyclic subgroup of \( G \). Then for a \( g \)-invariant smooth fibre \( F \) the fibre \( f(F) \) is smooth.

**Proof.** Let \( p \) be a \( g \)-fixed point on \( \overline{\mathcal{B}} \) and \( F \) be a fibre over this point. We can assume that in the neighbourhood of \( p \) the element \( g \) acts on \( \overline{\mathcal{B}} \) as a multiplication by \( \xi_k \), where \( \xi_k \) is a primitive \( k \)-th root of unity.

Let \( F \) be a smooth fibre. If \( g \) acts trivially on \( F \) then \( f(F) \) is a smooth fibre. Otherwise \( g \) has two fixed points \( q_1 \) and \( q_2 \) on \( F \). In the neighbourhood of \( q_1 \) the element \( g \) acts on \( F \) as a multiplication by \( \xi_k^a \) for some \( a \) and in the neighbourhood of \( q_2 \) the element \( g \) acts on \( F \) as a multiplication by \( \xi_k^{-a} \).

Let us consider the quotient map \( \pi : \overline{X} \to \overline{X}/G \). There are two singular points \( \pi(q_1) \) and \( \pi(q_2) \) on \( \pi(F) \). These singularities are toric.
and their resolutions are chains of negative curves. The selfintersection numbers of these curves \(-s_1, \ldots, -s_i\) and \(-r_1, \ldots, -r_j\) are determined by continued fractions

\[
\frac{k}{a} = s_1 - \frac{1}{s_2 - \frac{1}{\cdots - \frac{1}{s_i}}}, \quad \frac{k}{k-a} = r_1 - \frac{1}{r_2 - \frac{1}{\cdots - \frac{1}{r_j}}}.
\]

Let us consider a minimal resolution

\[
\mu : \widetilde{X}/G \to X/G.
\]

The transform \(\mu^{-1}\pi(F)\) is a chain of negative curves consisting of \(\mu^{-1}\pi(q_1), \mu^{-1}\pi(q_2)\) and \(\mu_*^{-1}\pi(F)\). The dual graph of \(\mu^{-1}\pi(F)\) admits at most one nontrivial automorphism.

If there is no non-trivial automorphism of the graph then the Galois group \(\text{Gal}(\overline{k}/k)\) can not permute irreducible components of \(\mu^{-1}\pi(F)\) and applying the relative minimal model program over \(k\) one can obtain that \(f(F)\) is a smooth fibre. If there is a non-trivial automorphism then

\[
i = j, \quad s_1 = r_1, \ldots, s_i = r_j.
\]

Therefore

\[
\frac{k}{a} = \frac{k}{k-a}, \quad a = k-a, \quad a = \frac{k}{2}
\]

and the number \(k\) is even. It contradicts assumption that \(k\) is odd. \(\square\)

**Lemma 3.8** (cf. [Spr77, 4.4.6]). Let a finite group \(G\) faithfully act on \(\mathbb{P}^1_k\) then orbits of points with non-trivial stabilizer have the following lengths:

- If \(G \cong C_k\), then there are two orbits of length 1.
- If \(G \cong D_{2k}\), then there are an orbit of length 2 and two orbits of length \(k\).
- If \(G \cong A_4\), then there are two orbits of length 4 and an orbit of length 6.
- If \(G \cong S_4\), then there are an orbit of length 6, an orbit of length 8 and an orbit of length 12.
- If \(G \cong A_5\), then there are an orbit of length 12, an orbit of length 20 and an orbit of length 30.

**Proposition 3.9.** Let \(X\) be a relatively \(G\)-minimal conic bundle with \(n\) singular fibres and \(G_F\) be trivial. Then any MMP-reduction \(Y\) of \(X/G\) admits a conic bundle structure. For \(n\) from the following table the number of singular fibres on \(Y\) is equal or less than the number \(m\) in the following table:
$$\begin{array}{|c|c|c|c|}
\hline
\text{Group} & n & m & \text{Conditions} \\
\hline
\mathcal{C}_{2k} & 2kb & a + b & a \leq 2 \\
\mathcal{C}_{2k+1} & a + (2k+1)b & a + b & a \leq 2 \\
\mathcal{D}_{4k} & 4kc & a + b + c & a \leq 1, b \leq 2 \\
\mathcal{D}_{4k+2} & 2a + (4k+2)c & a + b + c & a \leq 1, b \leq 2 \\
\mathcal{A}_4 & 4a + 12c & a + b + c & a \leq 2, b \leq 1 \\
\mathcal{S}_4 & 8a + 24d & a + b + c + d & a \leq 1, b \leq 1, c \leq 1 \\
\mathcal{A}_5 & 12a + 20b + 60d & a + b + c + d & a \leq 1, b \leq 1, c \leq 1 \\
\hline
\end{array}$$

**Proof.** Denote the map $X \rightarrow Y$ by $f$. By Lemmas 3.6 and 3.7 if $F$ is a singular fibre on $Y$ then:

- either $f^{-1}(F)$ is an orbit of $G$, consisting of singular fibres, and order of stabilizer of a fibre in $f^{-1}(F)$ is odd.
- or $f^{-1}(F)$ is an orbit of $G$, consisting of smooth fibres, and order of stabilizer of a fibre in $f^{-1}(F)$ is even.

Each orbit of fibres corresponds to an orbit of points on the base. Therefore we know numbers of orbits of fibres with non-trivial stabilizers from Lemma 3.3. Applying this, for each singular fibre $F$ we consider the orbit $f^{-1}(F)$. The numbers $a, b, c$ and $d$ in the table are numbers of orbits with given length.

The proof is the same for all cases listed in the table. Therefore we consider just one of them, for example, $G \cong \mathcal{D}_{4k+2}$. For any singular fibre $F$ on $Y$ the preimage $f^{-1}(F)$ consist of 2 singular fibres, $2k+1$ smooth fibres or $4k+2$ singular fibres. If we denote numbers of such fibres $F$ by $a$, $b$ and $c$ respectively then there are $a + b + c$ singular fibres on $Y$ and at least $2a + (4k+2)c$ singular fibres on $X$.

\[ \square \]

Now we prove Theorem 3.1

**Proof of Theorem 3.1.** Let $n$ and $m$ be numbers of singular fibres on $X$ and $Y$ respectively.

Let us consider a relative $G_B$-MMP-reduction $Z$ of $X/G_F$. By Lemma 3.2 the number of singular fibres of $Z$ is not greater than $n$. The surface $Y$ is a relative MMP-reduction of $Z/G_B$. Applying Proposition 3.9 to different cases of $G_B$ one can check the following:

- If $n \leq 3$ then $m \leq 3$;
- If $n > 3$ then $m \leq n$. If $m = n$ then $m = n = 4$ and $G_B = \mathcal{C}_2$ and $G_B = \mathcal{D}_4$.

Applying equalities $K_X^2 = 8 - n$ and $K_Y^2 = 8 - m$ finishes the proof.

\[ \square \]

Now we prove Proposition 1.6.
Proof of Proposition 1.6. For an relative MMP-reduction $Y$ of $X/G$ one has $K_Y^2 \geq 5$ by Theorem 3.1. Thus $X/G \cong Y$ is $k$-rational by Corollary 2.9. □

4. The Galois group

In this section we prove the following proposition:

**Proposition 4.1.** Let $X \to B$ be a relatively $G$-minimal conic bundle, $G$ be a group faithfully acting on the base $B$ and $\Gamma = \text{Gal}(\mathbb{F}/k)$. Let $Y$ be a relative MMP-reduction over $B/G$ of the minimal resolution of $X/G$ and $f : X \dasharrow Y$ be the corresponding map. Let $F$ be a singular fibre over a point $p \in B$ such that $f(F)$ is a singular fibre on $Y$, $E_1$ and $E_2$ are irreducible components of $F$. Then there exist elements $g \in G$ and $\gamma \in \Gamma$ such that $g\gamma E_1 = E_2$ and $g\gamma E_2 = E_1$ and (up to conjugation) one of the following possibilities holds:

1. There exists an element $\delta \in \Gamma$ such that $\delta E_1 = E_2$.
2. The stabilizer of $p$ in $G$ is trivial, $\text{ord } g$ is even, $\gamma p = g^{-1} p$.
3. $G = D_{2k+2}$, $p$ is a point fixed by the normal cyclic subgroup $C_{2k+1}$, $g$ is any element of order 2, $\gamma p = gp$.
4. $G = S_4$, $p$ is a $(123)$-fixed point, $g = (12)$, $\gamma p = (12)p$.
5. $G = A_5$, $p$ is a $(12345)$-fixed point, $g = (25)(34)$, $\gamma p = (25)(34)p$.
6. $G = A_5$, $p$ is a $(123)$-fixed point, $g = (12)(45)$, $\gamma p = (12)(45)p$.

**Lemma 4.2.** In the previous notation if $\text{ord } g$ is odd then there exists an element $\delta \in \Gamma$ such that $\delta E_1 = E_2$.

**Proof.** One has $g\gamma E_1 = E_2$ and $\text{ord } g$ is odd, therefore $E_2 = (g\gamma)^{\text{ord } g} E_1 = \gamma^{\text{ord } g} g^{\text{ord } g} E_1 = \gamma^{\text{ord } g} E_1$ and $\delta = \gamma^{\text{ord } g}$ is an element of $\Gamma$ such that $\delta E_1 = E_2$. □

**Lemma 4.3.** In the previous notation all points in $\langle g \rangle$-orbit has the same stabilizer in $G$.

**Proof.** Let $H \subset G$ be the stabilizer of the point $p$. For any $h \in H$ one has $h g p = \gamma^{-1} h (g \gamma)^i p = \gamma^{-1} p = g^i (g \gamma)^{-1} p = g^i p$. Therefore all points in the $\langle g \rangle$-orbit of $p$ are fixed by the group $H$. □

**Proof of Proposition 4.1.** By Lemma 4.2 if $\text{ord } g$ is odd then there exists an element $\delta \in \Gamma$ such that $\delta E_1 = E_2$.

Assume that $\text{ord } g$ is even and stabilizer of $p$ in $G$ is non-trivial. By Lemma 4.3 any element $h$ of the stabilizer of the point $p$ also fixes the
points \(gp\) and \(g^2p\). If \(\text{ord} \ g > 2\) then \(h\) fixes more than 2 points on \(B\), so \(h\) fixes \(B\) pointwisely. It contradicts assumption that \(G\) acts faithfully on \(B\). Thus \(\text{ord} \ g = 2\).

By Lemma 3.6 the stabilizer of the point \(p\) is generated by an element \(h \in G\) of odd order. By Lemma 4.3 one has \(hgp = gp\) so by Lemma 3.4 the points \(p\) and \(gp\) are fixed only by the element \(h\) and its powers. Thus one has \(ghg^{-1} \in \langle h \rangle\). Applying Lemma 3.8 one can consider all possibilities (up to conjugation) of \(G\), \(h\) and \(g\) and check that they all are listed above:

- \(G = D_{4k+2}\), \(h\) is a generator of the normal subgroup \(C_{2k+1}\), \(g\) is any element of order 2.
- \(G = S_4\), \(h = (123)\), \(g = (12)\).
- \(G = A_5\), \(h = (12345)\), \(g = (25)(34)\).
- \(G = A_5\), \(h = (123)\), \(g = (12)(45)\).

\(\square\)

The following example shows that all listed above possibilities are reached for a suitable field \(k\):

**Example 4.4.** Consider a projective plane \(\mathbb{P}^2_k\) with coordinates \(x\), \(y\) and \(z\) and a projective line \(\mathbb{P}^1_k\) with coordinates \(t_1\) and \(t_0\). Let a finite group \(G\) acts on \(\mathbb{P}^1_k\) and \(g \in G\) is an element of even order \(l\) such that points \((0 : 1)\) and \((1 : 0)\) are fixed by \(g\). In particular, \(g(\lambda : 1) = (\xi \lambda : 1)\) where \(\xi = e^{2\pi i / l}\). The group \(G\) acts on homogeneous polynomials in variables \(t_1\) and \(t_0\) in the natural way.

Let \(Z\) be a hypersurface in \(\mathbb{P}^2_k \times \mathbb{P}^1_k\) given by the equation

\[
Ax^2 \prod_{i=1}^{k} \prod_{h \in G} h(t_1 - \lambda_i t_0) + By^2 \left( \prod_{h \in G} h(t_0) \right)^k + Cz^2 \left( \prod_{h \in G} h(t_0) \right)^k = 0.
\]

The projection \(\pi : \mathbb{P}^2_k \times \mathbb{P}^1_k \to \mathbb{P}^1_k\) defines on \(Z\) a structure of a bundle over \(\mathbb{P}^1_k\) whose general fibre is a smooth conic. Let \(X\) be a relative \(G\)-MMP-reduction over \(\mathbb{P}^1_k\) of \(Z\). Note that the bundle \(Z\) has singular fibres exactly over points \(h(\lambda_i : 1)\) and \(h(1 : 0)\). Moreover, by Lemma 3.6 the fibres of \(X \to \mathbb{P}^1_k\) over points \(h(1 : 0)\) are smooth since these points are fixed by the elements of even order \(hgh^{-1}\).

Now assume that in the field \(k\) there is an element \(u\) such that

\[
\text{Gal} \left( \mathbb{K} \left( x^l - u \right) / k \right) \cong C_l.
\]

Let \(\lambda_i = \mu_i u^{\frac{1}{l}}\), \(\mu_i \in k\) and \(\frac{\mu_i}{\mu_j} = -1\). In this case the surface \(Z\) is correctly defined since \(G\)-orbit of the point \((\lambda_i : 1)\) includes all roots of
the equation \( t_1^l - \mu^l u t_0^l = 0 \) defined over \( k \). Let
\[
\gamma \in \text{Gal} \left( (k(x^l - u))/k \right)
\]
be the generator of the group acting as: \( \gamma (\lambda_i : 1) = (\xi_i^{-1} \lambda_i : 1) \).

Consider a singular fibre \( F \) over \( h (\lambda_i : 1) \). The image \( f(F) \) is a singular conic satisfying equation \( By^2 + Cz^2 = 0 \) over \( k \)-point of \( \mathbb{P}^1_k/G \). Therefore the element \( \gamma \) permutes components of \( f(F) \) since \( \frac{B}{C} = -u \). But all components of singular fibres of \( X \) are defined over \( k(x^l - u) \) and for any singular fibre \( F \) holds \( \gamma F \neq F \). So one can \( \text{Gal} \left( (k(x^l - u))/k \right) \)-equivariantly contract a component in each singular fibre of \( X \) and obtain a conic bundle without singular fibres.

Now fix a \( k \)-point \( q \) and find a coefficient \( A \) such that fibre over \( q \) contains a \( k \)-point. The obtained surface \( X \) is \( k \)-rational by Corollary 2.9 and the \( k \)-unirational relatively minimal conic bundle \( Y \) can contain any number of singular fibres.

To show that this example works for orbits consisting of points with non-trivial stabilizer (cases (3)–(6) of Proposition 4.1) it is sufficient to find a field \( k \) and a representation of the group \( G \) in \( \text{PGL}_2(k) \) such that the fixed points of the element \( g \) of order 2 are \( k \)-points and the fixed points of an element \( h \) of odd order, \( ghg^{-1} \in \langle h \rangle \), are not defined over \( k \).

For the group \( G = \mathfrak{D}_{4n+2} \) consider \( k = \mathbb{Q}(\cos \frac{2\pi}{2n+1}, \sin \frac{2\pi}{2n+1}) \). Then
\[
 h = \begin{pmatrix} \cos \frac{2\pi}{2n+1} & -\sin \frac{2\pi}{2n+1} \\ \sin \frac{2\pi}{2n+1} & \cos \frac{2\pi}{2n+1} \end{pmatrix}, \quad g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]
satisfy required conditions.

For the other three cases let
\[
\text{Id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.
\]

Then the group \( \mathfrak{A}_4 \) is generated by \( I \) and \( \frac{1}{2}(\text{Id} + I + J + K) \), the group \( \mathfrak{S}_4 \) is generated by \( \mathfrak{A}_4 \) and \( \frac{1}{\sqrt{2}}(\text{Id} + I) \), the group \( \mathfrak{A}_5 \) is generated by \( \mathfrak{A}_4 \) and \( \varphi \text{Id} + I + \varphi^{-1} J \), where \( \varphi = \frac{1 + \sqrt{5}}{2} \). Therefore these representations of \( \mathfrak{A}_4 \) and \( \mathfrak{S}_4 \) are defined over \( \mathbb{Q}(i) \), the representation of \( \mathfrak{A}_5 \) is defined over \( \mathbb{Q}(i, \sqrt{5}) \). The fixed points of an element conjugate to \( (12)(34) \) are defined over \( \mathbb{Q}(i) \), the fixed points of an element conjugate to \( (12) \) are defined over \( \mathbb{Q}(i, \sqrt{2}) \), the fixed points of an element conjugate to \( (123) \) are defined over \( \mathbb{Q}(i, \sqrt{3}) \), the fixed points of an element conjugate to \( (12345) \) are defined over \( \mathbb{Q}(i, \sqrt{5}, e^{\frac{2\pi i}{5}}) \).
Now we prove Proposition 1.7.

**Proof of Proposition 1.7.** Let \( X \to B \cong \mathbb{P}^1_k \) and \( Y \to C \cong \mathbb{P}^1_k \) be two minimal conic bundles having singular fibres over points \( p_1, \ldots, p_n \) and \( q_1, \ldots, q_m \) respectively \((n \geq 8, m \geq 8)\). Applying Sarkisov program one can easily show that if there does not exist a map \( B \to C \) such that points \( p_1, \ldots, p_n \) map to \( q_1, \ldots, q_m \) then \( X \) and \( Y \) can not be \( k \)-birationally equivalent.

Note that the number \( k \) from Example 4.4 can be arbitrary large and the numbers \( \lambda_i \) are arbitrary. Thus the dimension of the space of quotients up to \( k \)-birational equivalence infinitely grows with increasing the number \( k \).

If two surface are not \( k \)-birationally equivalent then they have non-isomorphic function fields. The function field of quotient surface \( S/G \) is the invariant subfield \( k(S)^G \).

If for two subgroups \( G_1 \cong G_2 \) in \( \text{Cr}_2(k) \) birational quotients of \( \mathbb{P}^2_k \) by these groups are not \( k \)-birationally equivalent then \( G_1 \) and \( G_2 \) are not conjugate in \( \text{Cr}_2(k) \). \( \square \)

**Lemma 4.5.** Let a group \( G = \mathcal{C}_k \) or \( G = \mathcal{D}_{2k} \) act on a relatively \( G \)-minimal \( k \)-rational conic bundle \( X \). Let \( N \) be a maximal cyclic subgroup in \( G \) of odd order. Then \( X/N \) is \( k \)-rational.

**Proof.** Let \( \Gamma = \text{Gal}(\overline{k}/k) \). Let \( F \) be a singular fibre over a point \( p \in B \). By Lemma 1.2 if for an element \( g \in N \) there exists an element \( \gamma \in \Gamma \) such that \( g\gamma \) permutes components of \( F \), then there exists an element \( \delta \in \Gamma \) permuting the components of a singular fibre \( F \). Otherwise, there is no element \( \varepsilon \in \Gamma \) permuting the components of the image of \( F \) in \( X/N \). Therefore by Lemma 3.7, the number of singular fibres on relative \( G/N \)-MMP reduction \( Y \) of \( X/N \) whose components are permuted by an element of \( \Gamma \) is not greater than the number of singular fibres on \( X \) whose components are permuted by an element of \( \Gamma \). Thus \( Y \) is \( k \)-rational by Corollary 2.9. \( \square \)

**Proof of Theorem 1.3.** Let us consider a \( k \)-rational \( G \)-surface \( X \). Obviously, the quotient \( X/G \) is birationally equivalent to \( \widetilde{X}/G \) where \( \widetilde{X} \) is a \( G \)-equivariant minimal model of \( X \). If \( \widetilde{X} \) is a del Pezzo surface then \( X/G \) is \( k \)-birationalaly equivalent to the quotient \( \widetilde{X}/G \).

If \( \widetilde{X} \) is a conic bundle then there is a normal subgroup \( G_F \subset G \) acting trivially on \( B \). The quotient \( \widetilde{X}/G_F \) is \( k \)-rational by Lemma 3.2 and \( X/G \) is \( k \)-birationalaly equivalent to \( (\widetilde{X}/G_F)/G_B \) where \( G_B = G/G_F \) and \( X/G_F \) is the minimal resolution of singularities of \( X/G_F \).
Let $Y$ be a relative $G_B$-MMP reduction of $\widetilde{X}/G_F$. Then the group $G_B$ faithfully acts on the base $B$ thus $G_B$ is isomorphic to $\mathfrak{S}_4$, $\mathfrak{S}_5$, or $\mathfrak{A}_5$. In the first two cases we can consider a maximal cyclic group $N$ of odd order. By Lemma 4.5 the quotient $Y/N$ is a $k$-rational conic bundle. Thus the quotient $X/G$ is $k$-birationally equivalent to a quotient of the $k$-rational conic bundle $Y/N$ by a cyclic or dihedral group of order $2^n$.

To prove Corollary 1.4 we use the following well-known lemma.

**Lemma 4.6.** Let $X$ be an $n$-dimensional toric variety over a field $\mathbb{K}$ of arbitrary characteristic and let $G$ be a finite subgroup conjugate to a subgroup of an $n$-dimensional torus $\mathbb{T}^n \subset X$ acting on $X$. Then the quotient $X/G$ is a toric variety.

**Proof of Corollary 1.4.** Consider a $G$-minimal model $X$ of $S$. Then $X$ is either a conic bundle or a del Pezzo surface.

If $X$ is a conic bundle then $X/G$ is $k$-rational by Theorem 1.3. So we can assume that $X$ is a del Pezzo surface.

The classification of finite groups of automorphisms of del Pezzo surfaces is well-known (see [DI09a]). If $X$ is a del Pezzo surface of degree 5 or less and $G$ is a group of odd order, ord $G > 9$ and ord $G \neq 15$ then one can check that order of $G$ is divisible by 3 and $G$ is not cyclic. It contradicts conditions of this theorem.

Let $X$ be a del Pezzo surface of degree 6 or more. If ord $G$ is odd and $G$ is cyclic or ord $G$ is not divisible by 3 then $G$ is a subgroup of a torus acting on $X$. By Lemma 4.5 the quotient $X/G$ is a $k$-form of a toric surface. Thus an MMP-reduction $Y$ of $X$ is a $k$-form of a toric surface so $K_Y^2 \geq 6$. So both $X/G \approx Y$ and $Y$ are $k$-rational by Theorem 2.8.

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