Optimizing Query Predicates with Disjunctions for Column Stores

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Abstract

Since its inception, database research has given limited attention to optimizing predicates with disjunctions. What little past work there is has focused on optimizations for traditional row-oriented databases. A key difference in predicate evaluation for row stores and column stores is that while row stores apply predicates to one record at a time, column stores apply predicates to sets of records. Not only must the execution engine decide the order in which to apply the predicates, but it must also decide how many times each predicate should be applied and on which sets of records it should be applied to. In our work, we tackle exactly this problem. We formulate, analyze, and solve the predicate evaluation problem for column stores. Our results include proofs about various properties of the problem, and in turn, these properties have allowed us to derive the first polynomial-time (i.e., \(O(n \log n)\)) algorithm \textsc{ShallowFish} which evaluates predicates optimally for all predicate expressions with a depth of 2 or less. We capture the exact property which makes the problem more difficult for predicate expressions of depth 3 or greater and propose an approximate algorithm \textsc{DeepFish} which outperforms \textsc{ShallowFish} in these situations. Finally, we show that both \textsc{ShallowFish} and \textsc{DeepFish} outperform the corresponding state of the art by two orders of magnitude.

1 Introduction

Query optimization has been a core topic of database research for decades. However, despite all the work done in the field, optimizing queries with disjunctive predicates is an area that has received limited attention. General boolean expressions with both conjunctions and disjunctions are notoriously difficult to minimize (it has been shown that the problem is \(\sum_2^P\)-complete \[3\]), and in comparison, conjunctive-only queries offer an easier target for optimization with more obvious performance benefits. Although works that focus on disjunctions do exist, the proposed optimizations were largely developed with traditional row-oriented databases in mind, and as modern databases have become increasingly column-oriented, new techniques are needed to adapt to the new setting.

Prior work on disjunctions typically optimize them through the following series of steps: 1. Break a complex predicate expression down to its predicate atoms (predicate subexpressions with no conjunctions or disjunctions). 2. Restructure the overall predicate expression into either conjunctive normal form (CNF) or disjunctive normal form (DNF) \[11\] \[21\]. 3. Apply the predicate atoms on each record in an order which maximizes the likelihood of “short-circuiting” (i.e., determining whether a record either satisfies or does not satisfy the overall predicate expression prematurely without having applied all predicate atoms) \[9\]. Because most storage devices retrieve data in blocks, row-oriented databases usually read entire records before applying the predicate atoms. Since every data value of every record is fetched regardless of the outcome of the predicate atoms, a lot of potential for optimization is missed.

In contrast, column-oriented databases store the values of a column together, so predicate atoms can retrieve and process only the required columns, without having to fetch entire records. Storing values in a columnar fashion has other benefits as well, such as being able to vectorize predicate atom applications \[22\]. Furthermore, with predicate atoms of sufficiently high selectivity, column stores may even skip retrieving large blocks of data for even relevant columns, potentially saving significant I/O and processing time. Column stores typically represent record indices in compact data structures such as bitmaps for which set operations (e.g., intersection for AND / union for OR) can be implemented efficiently as fast bit flipping operations.
Thus, evaluation results of predicate atom applications can quickly be combined using set operations, and only the still-valid records need be considered any further.

A crucial difference between how row-oriented and column-oriented databases evaluate predicates is that, in a row store, a group of predicate atoms is applied to each record; whereas in a column store, each predicate atom is applied on a set of records. Since row stores apply predicate atoms to only a single record at a time, they only need to concern themselves with ordering the predicate atom applications to get a single, overall true/false result for the record as soon as possible. On the other hand, when column stores apply a predicate atom, it is on multiple records, so a bitmap of true/false values (true for records which satisfy the predicate atom and false for the remaining) is returned. The query execution engine may combine this bitmap with the resultant bitmaps of previous predicate atom applications through set operations to further filter the records before the next predicate atom is applied. In addition, a curious case that is unique to column stores is that it may even be beneficial to apply the same predicate atom multiple times on separate disjoint sets instead of once on a larger superset. Thus, predicate evaluation for column stores has several additional dimensions to the problem which row stores do not. The execution engine must decide on not only the order of the predicate atom applications, but also how many times to apply each predicate atom, which set operations to perform between predicate atom applications, and the sets of records to apply these set operations and predicate atoms to.

1.1 Our Work

We wish to answer the question: “Given that our primitives are either predicate atom applications or any of the set operations (intersection, union, difference) on sets of records, in what order should we apply these primitives and on what sets of records should we apply them to to minimize query execution time in a column-oriented database?” We formulate our solution to the question as sequences of steps in which each step represents a predicate atom application, intersection, union, or difference on either a single or multiple sets of records. In our work, we show that for each optimal (minimum cost) sequence of steps in this formulation, there exists a corresponding optimal sequence of steps in an alternate formulation of only predicate atom application steps. Through this alternate formulation, we show that as long as the cost model obeys a “triangle inequality”-like property, the optimal sequence applies each predicate atom exactly once. We derive an algorithm, BestD, that calculates the optimal set of records on which to apply each predicate atom. For predicate atom ordering, we are able to reuse some of the prior work on disjunctions for row stores, such as Hanani’s predicate atom ordering algorithm [9] (which we refer to as ORDERP henceforth). When combined with BestD, we are able to come up with SHALLOWFISH, a provably optimal predicate evaluation algorithm for any predicate trees of depth 2 or less (i.e., ANDs of ORs / ORs of ANDs). Unfortunately, we show that due to the critical assumption of processing nodes in a depth-first manner, ORDERP is not optimal for predicate trees of depth 3 or greater. We capture the exact property which makes the problem more difficult for these cases, and we present our approximate algorithm DEEPFISH and show in our evaluation that it outperforms SHALLOWFISH on deeper predicate trees.

Our algorithms are general and are not constrained to just one specific cost model. Rather, we present multiple cost models which represent different real-world scenarios, and show that our proofs of optimality hold for all these cases. As a result, our BestD algorithm does not need to make any sort of independence assumption between predicate atoms, as has been done by much of the past work [15] [10]. In addition, our work does not require that the input predicate to be in CNF or DNF and is able to avoid the exponential blowup of terms that often plague these forms. Most importantly, we show that SHALLOWFISH has a runtime complexity of $O(n \log n)$ where $n$ is the number of predicate atoms; we believe this to be the first polynomial-time algorithm which generates optimal predicate evaluation plans for column stores.

Contributions. Our main contributions are:

- A formal formulation of how to optimally apply a predicate expression with both disjunctions and conjunctions in a column store.
- Our various theoretical results, which include reductions to predicate atom application-only sequences and proof that each predicate atom is applied exactly once in the optimal sequence.
- Our algorithms:
  1. **BestD**, which calculates the optimal sets of records to apply each predicate atom to.
2. **ShallowFish**, the first polynomial-time (i.e., $O(n \log n)$) algorithm to produce provably optimal predicate evaluation plans for predicate trees of depth 2 or less

3. **DeepFish**, an approximate algorithm specialized for predicate trees of depth 3 or more.

   • Evaluation of our algorithms, in which we outperform the current state-of-the-art by orders of magnitude.

The remainder of this paper is structured as follows. We formulate the problem in detail in Section 2 and formalize the problem in Section 3. Before solving the problem, we constrain the solution space to a tractable size in Section 4. Our algorithms are presented in Section 5 and proofs regarding correctness and optimality are detailed in Section 6. Evaluation of the algorithms are provided in Section 7. The paper wraps up with related work in Section 8 and conclusion in Section 9.

2 Problem Formulation

2.1 Setup

We first discuss the setting in which this work is applicable. We assume that our data is stored in a columnar fashion rather than in rows (i.e., that each attribute (column) is stored separately either in memory or externally on disk). When executing a query, we assume only the attributes referenced in the query need to be read, and lightweight data structures (e.g., bitmap indexes, record id lists) are used to manage indices of records which need to be filtered and joined from the applicable columns. Although no strong assumptions are made about the backend storage, we assume that the time to fetch and process the data is significantly greater than the time to manipulate the indices of records in memory. Backend storage such as hard disk drives (HDDs), flash drives, and network-attached storage all fit within this criteria. The primary resource we are concerned with is time. We assume we either have enough memory to hold the data needed to process the query or that the query can be executed off disk using a conventional buffer pool design. Either way, our algorithms do not require any additional data structures; we only use the data structures that would be constructed by a column store anyway to evaluate the query. If indexing structures to accelerate query execution are available, our algorithms can take advantage of them, but the absence of them does not affect the correctness of our system.

2.2 Query Properties

We assume that the queries given to us are selection queries with complex predicate expressions. The predicate expressions may be composed of any number and depth of conjunctions and disjunctions of boolean-result predicate atoms. The queries may have expensive (perhaps user-defined) predicate atoms, and different predicate atoms may take different lengths of time to finish, even on the same set of records. For clarity’s sake, the following analysis assumes that we are working with single-table queries, but this is not a necessary requirement. As long as each record in the source (single or joined) table has a global id (real or virtual), our algorithms are applicable. Even with pushdown-type optimizations, our algorithms still apply to the pushed-down predicate atoms at the individual table level.

2.3 Problem Model

Given a query with a complex boolean predicate expression with conjunctions and disjunctions, our problem is to determine how we should filter the records to minimize runtime. To answer this question, we frame the problem as follows. Suppose that a query’s predicate expression is composed of $n$ unique predicate atoms. For each record from the source table, we can evaluate the $n$ predicate atoms individually to build a tuple of $n$ binary (1 or 0) values based on whether the record satisfies the predicate atom or not. Each of these $n$-length tuples is called a vertex. Consider the following query:

\[
\text{SELECT color WHERE (length < 1.4 AND weight > 10) OR species ILIKE "wolffish" FROM fish}
\]

\[\text{1}^{1}\text{In the case of non-unique predicate atoms, techniques from boolean formula minimization or simple “lifting-up” strategies such as the one employed by Hyrise can be used to remove duplicates. If duplicate predicate atoms are treated as unique, BestD turns into an approximate algorithm, but the record sets it returns will still produce correct results.}

\[\text{2}^{2}\text{Each tuple is a vertex in an } n\text{-dimensional hypercube.}\]
If we label predicate atoms \( P_1 = (\text{length} < 1.4), P_2 = (\text{weight} > 10), \) and \( P_3 = (\text{species} \ 	ext{ILIKE} \ "\text{wolffish}"), \) a record of a wolffish\(^3\) with a length of 1.3 m and a weight of 8 kg would have the vertex \((1, 0, 1)\). Note that the vertex is independent of whether the predicates are combined using disjunctions or conjunctions.

We can create a logical mapping between each vertex and the records which produce that vertex value. Since each distinct vertex either satisfies or does not satisfy the predicate expression, we can operate on vertices to determine which records satisfy the predicate. In this way, we can simply maintain sets of distinct vertices (which we call vertex sets), and these vertex sets can become the basic units for our operations. Thus our problem is reduced to finding the vertex set that exactly matches the predicate expression.

Given that each predicate atom application splits a vertex set into two halves: the set which satisfies the predicate atom and the set which does not, we start with the set of all possible vertices (which we call vertex sets), and these vertex sets can become the basic units for our operations.

To continue with our example from before, if we started with the set of all possible vertices \(\{(0, 0, 0), \ldots, (1, 1, 1)\}\), we could apply predicate atom \(P_1\) to arrive at the set \(\{(1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}\). In addition, we could separately apply \(P_2\) to get \(\{(0, 1, 0), (0, 1, 1), (1, 1, 0), (1, 1, 1)\}\), and then we can take the intersection between these sets to produce the set of vertices which satisfy \(P_1 \land P_2\): \(\{(1, 0, 0), (1, 1, 0)\}\). Since our predicate expression is \((P_1 \land P_2) \lor P_3\), we can also apply \(P_3\) to find \(\{(0, 0, 1), (0, 1, 1), (1, 0, 1), (1, 1, 1)\}\) and take the union between this vertex set and \(P_1 \land P_2\) to get the result \(\{(1, 1, 0), (0, 0, 1), (0, 1, 1), (1, 0, 1), (1, 1, 1)\}\). This is the largest vertex set which satisfies the predicate expression and the corresponding records are the ones we want to fetch and operate on. Obviously, this is not the most efficient way to arrive at \((P_1 \land P_2) \lor P_3\).

To minimize the total cost by deciding which vertex sets to derive, the order they should be derived in, and what actions should be used to derive those vertex sets.

### 2.4 Cost Model

Since both predicate atom applications and set operations represent real actions that must be executed, we must have a cost model for them. The primary cost we are concerned with is time, so the basic cost model we propose is:

\[
C(O, D) = \begin{cases} 
\epsilon \cdot (\text{count}(D) + \kappa') & \text{if } O \in \{\cup, \cap, \}\ \\
\text{count}(D) + \kappa & \text{if } O \in \mathbf{P}
\end{cases}
\]

in which \(O\) is either the type of set operation or predicate atom application in the case of \(O \in \mathbf{P}\), and \(D\) is the sets of vertices we expect to operate on. Regardless of which action it is, the cost model says that we expect the cost to be linear in the number of records we process in addition to some constant overhead cost \(\kappa\) and \(\kappa’\). Note that “count” here refers to the number of underlying records represented by the vertices in \(D\) and not simply the number of vertices. Most critical is \(\epsilon\), which represents the ratio of costs between an in-memory set operation on record indices versus a predicate atom application on data that has to be physically fetched from storage. Depending on the environment, a predicate atom application could easily cost \(30 \times 100000 \times \text{more than a set operation, due to the difference in cost between a memory access and a disk seek, or the fact that a single record id can be represented with a single bit in a bitmap whereas a single int type element in a column likely takes up at least 4 bytes. To take a concrete example, in our experiment environment, just reading 10M 4-byte int values from a RAID5 setup took 274ms, while ANDing together two bitmaps of 10M elements in memory to produce another bitmap only took 8ms. Thus for our case, we set \(\epsilon \approx 0\) and use the following cost model instead:

\[
C(O, D) = \begin{cases} 
0 & \text{if } O \in \{\cup, \cap, \}\ \\
\text{count}(D) + \kappa & \text{if } O \in \mathbf{P}
\end{cases}
\]

\(^3\)The Atlantic wolffish generate their own antifreeze to survive in the cold deep sea [2]. Truly an animal optimized to its environment!
We could also change our cost model to more closely reflect a spinning HDD. Due to the differences in sequential access and random access, the access cost for HDDs generally goes up linearly for random access until a certain threshold, when it becomes cheaper to scan the entire column instead:

\[
C(O, D) = \begin{cases} 
0 & \text{if } O \in \{\cup, \cap, \setminus\} \\
\text{count}(D) + \kappa & \text{if } O \in P \land (\gamma < \vartheta) \\
|R| + \kappa & \text{if } O \in P \land (\gamma \geq \vartheta)
\end{cases}
\]

where \(|R|\) is the total number of records in the relation, \(\vartheta\) is the aforementioned threshold, and \(\gamma\) is the fraction of total records represented by \(D\) (i.e., \(\text{count}(D)/|R|\)).

Or, if we wanted to accommodate the fact that different predicate atoms have different processing factors:

\[
C(O, D) = \begin{cases} 
0 & \text{if } O \in \{\cup, \cap, \setminus\} \\
F_O \cdot \text{count}(D) + \kappa & \text{if } O \in \{P\}
\end{cases}
\]

where each predicate atom has a specific constant factor \(F_O\).

All of the above cost models are compatible with our analysis. In fact, our algorithms can be used with any other cost model, provided that they follow a "triangle inequality"-like property; for all vertex sets \(D\) and \(E\):

\[C(O, D \cup E) < C(O, D) + C(O, E)\]

3 Formalization

We assume we are given a selection query with a boolean predicate expression \(\phi^*\) which contains conjunctions and disjunctions of \(n\) unique predicate atoms \(P_1, \ldots, P_n\). We assume the boolean formula \(\phi^*\) is in negation normal form (NOT operators are pushed inwards until they appear only in front of literals); if not, this conversion can be done in linear time. Furthermore, all negative predicate atoms \(\neg P\) are replaced with positive predicate atoms \(P' = \neg P\). Given \(\phi^*\), we build a normalized predicate tree such that: (1) Each node type is one of AND, OR, Predicate Atom (2) All predicate atom nodes are leaf nodes (3) Parents of all AND nodes must be OR nodes and vice versa. Note that as a consequence of the last condition, there is an interleaving of ANDs and ORs across the levels of the predicate tree.

Our basic unit of operation is a vertex. Each vertex is an \(n\)-length tuple of 0s and 1s. The \(i\)th element indicates whether predicate atom \(P_i\) is satisfied (1) or not (0). Given that \(\mathcal{D}\) is the vertex set containing all possible \(n\)-length vertices (i.e., \(\{0, 1\}^n\)), our goal is to find a sequence of steps \([R_1, R_2, \ldots, R_m]\) that produce the largest vertex set whose vertices satisfy the boolean formula: \(\psi^*(\mathcal{D}) = \{v \in \mathcal{D} \mid \phi^*(v) = 1\}\). Each step \(R_i\) is defined as a \((O_i, D_i)\) pair, in which \(O_i\) can either be a set operation \((\cap, \cup, \setminus)\) or a predicate atom application \((P)\), and \(D_i\) is the vertex set(s) operation \(i\) is performed on. Applying predicate atom \(P\) on vertex set \(D\) is defined as: \(P(D) = \{v \in D \mid P(v) = 1\}\). The result of each step \(R_i\) is added to a growing universe of vertex sets \(U\), which contains the set of all vertex sets that can be derived using the steps so far. In particular, we let \(U_i\) be the snapshot of this universe after step \(i\): \(U_i = U_{i-1} \cup O_i(D_i)\). Future steps can use any vertex set in this universe as its input: \(D_{i+1} \in U_i\). Initially, we start with only \(\mathcal{D}\): \(U_0 = \{\mathcal{D}\}\). We are finished after step \(R_m\) if \(\psi^*(\mathcal{D}) \in U_m\).

Given these definitions, our problem can be defined as:

**Problem 1.** Find the sequence of steps \([R_1, R_2, \ldots, R_m]\), in which each step \(R_i\) is an \((O_i, D_i)\) pair representing either a set operation \((\cap, \cup, \setminus)\) or a predicate atom application \((P)\) over vertex set(s) \(D_i\) such that the sum of costs across all steps \(\sum_{i=1}^m C(R_i)\) is minimized, while ensuring that the largest vertex set satisfying \(\phi^*\) appears in the final universe of resulting sets: \(\psi^*(\mathcal{D}) \in U_m\).

For reference, Table 1 lists the notation used in the paper, including additional notation introduced later in Sections 5 and 6.

4 Solution Space

At first glance, the problem appears difficult due to its infinite solution space. However, thanks to certain properties of our cost model, we are able to constrain the solution space to a more tractable size.


| Symbol | Meaning |
|--------|---------|
| $\phi^*$ | Boolean formula given by query’s WHERE clause |
| $\psi^*$ | Set formula for the vertex set which matches $\phi^*$ |
| $R_i$ | $i$th step in sequence of steps to filter given query |
| $P_i$ | Predicate used in the $i$th step |
| $D_i$ | Vertex set(s) to operate on for the $i$th step |
| $U_i$ | Universe of possible output vertex sets after step $i$ |
| $D$ | Vertex set of all possible vertices (i.e., $\{0,1\}^v$) |
| $\lambda$ | A node in the predicate tree expressed by $\phi^*$ |
| $\Omega(i)$ | The index of the predicate tree nodes in $P_i$’s lineage |
| $\theta(\lambda)$ | Index of the predicate atom represented by node $\lambda$ |
| $\lambda(v)$ | Value of subtree $\lambda$ if $v$ were to be evaluated |
| $\Gamma(v,i)$ | Variations of vertex $v$ that are closely tied to $v$ |
| $\xi(\lambda,D)$ | Set of vertices in $D$ satisfying predicate subtree $\lambda$ |
| $L_\lambda$ | Level at which $\lambda$ is located in the predicate tree |
| $v_{i-1}$ | Vertex $v$ satisfying predicate subtree $\lambda$ |
| $v_i$ | $i$th element of vertex $v$ |
| $\Upsilon_P(\lambda)$ | Index of $\lambda$’s predicate atom descendant before $i$ |
| $\gamma_i$ | The selectivity of predicate atom $P_i$ |

Table 1: Notation for common symbols and expressions.

4.1 Predicate-Only Sequences

Since set operations are free under our cost model, any number of set operation steps may be added at any point to a predicate atom sequence while preserving the same overall cost. The addition of a set operation could potentially help derive a vertex set and remove the need for a costly predicate atom application later in the sequence. Rather than worrying about which set operations can be added at which points to create a more optimal solution, let us instead consider that after every predicate atom step, every possible combination of set operations among the already derived vertex sets (i.e., vertex sets in the universe) is available to us. This reduction is advantageous because we now only need to consider the order and appearances of predicate atom application steps, simplifying the problem. Once an optimal sequence of predicate atoms has been found, we can reconstruct the intermediate set operation steps by keeping track of which set operations produced which vertex sets.

More precisely, let predicate (atom) step $R'_i = (P_i, D_i)$ have the resultant vertex set $X = P_i(D_i)$. Before we would just update $U_i = U_{i-1} \cup X$, but now we add all possible set operation combinations to the universe after each step. More specifically, for each $Y \in U_{i-1}$, we update $U''_{i-1} = U_{i-1} \cup \{Y \cup X, Y\cap X, Y \setminus X, X \setminus Y\}$. Then, we update $U''_{i-1} = U''_{i-1} \cup \left(\bigcup_{Y' \in U''_{i-1}} \{Y' \cup X, Y' \cap X, Y' \setminus X, X \setminus Y'\}\right)$, and so on until no new sets are added. At the end of this process, $U_i$ becomes the exponential set of all possible interactions between $U_{i-1}$ and $X$. More formally, if $\Psi_{\{\cup,\cap,\setminus\}}$ is the space of all possible set formulas given its input sets and the set operators $\cup,\cap,\setminus$, then: $U_i = \bigcup_{\psi \in \Psi_{\{\cup,\cap,\setminus\}}} \psi(P_i(D_i), U_{i-1})$. Note that this extended universe is merely logical and will not be actually be realized in memory by the algorithm. In practice, only a few set operations are performed between predicate atom applications. Thus, our problem has been reduced to finding the sequence of steps $[R_1, R_2, ..., R_m]$, in which each step $R_i$ is a predicate atom application $P_i$ over a vertex set $D_i$.

**Theorem 1.** For any sequence $[R_1, ..., R_m]$ in which each step is either a set operation ($\cup, \cap, \setminus$) or predicate atom application, we can find a corresponding sequence $[R'_1, ..., R'_m]$ such that all steps are predicate atom application steps in the extended universe.

For the remainder of the paper, any mention of steps or $R_i$ refers to the steps of this predicate-only sequence.

4.2 Necessity of Predicates

It should be intuitive that to derive the final vertex set that exactly matches the given predicate expression, every predicate atom of that predicate expression must be applied at least once. Thus, the optimal sequence
must contain each predicate atom as a step at least once. While this may seem obvious, it helps us to constrain the space down further since we only need to search for sequences of length \( n \) or greater.

**Theorem 2.** Each predicate atom \( P \in \text{predicates}(\phi^*) \) must appear at least once as a step in the output sequence.

The full proof for this theorem is given in Appendix A.2.

### 4.3 Minimizing Predicate Usage

Now that we have argued that every predicate atom must appear at least once in the sequence for correctness, we show a more surprising result: in the optimal sequence, each predicate atom appears exactly once. The extra overhead \( \kappa \) for each predicate atom application leads to the optimal sequence having as few predicate steps as possible.

**Theorem 3.** Given a boolean formula \( \phi^* \) with \( n \) unique predicate atoms, the optimal sequence will be of length \( n \) and have exactly one predicate step for each predicate atom \( P \in \text{predicates}(\phi^*) \).

**Proof.** Assume to the contrary that the optimal sequence is \( n + k \) steps long, for some \( k \geq 1 \). By the pigeonhole principle, at least one predicate atom must appear at least twice in the sequence. Let \( P \) be the first predicate atom to do so, and let indices \( i \) and \( j \), for some \( i < j \), be the first two steps that \( P \) appears in: \( R_i = (P, D_i) \) and \( R_j = (P, D_j) \). We show that we can always construct a new, less costly sequence which does not include both \( R_i \) and \( R_j \), leading to a contradiction. There are two major cases:

1. \( D_i \cap D_j \neq \emptyset \). We can replace step \( R_j \) with \( R_j' = (P, D_j \setminus D_i) \) and still construct every vertex set derived from \( P(D_j) \) since we can directly calculate \( P(D_j') = P(D_j \setminus D_i) \cup (D_i \cap P(D_i)) \). Therefore, replacing \( R_j \) with \( R_j' \) gives us a less costly sequence while maintaining equivalence, leading to a contradiction.

2. \( D_i \cap D_j = \emptyset \). In this case, instead of applying \( P \) separately, we can combine steps \( R_i \) and \( R_j \) and replace them with a single step \( R' = (P, D_i \cup D_j) \). With step \( R' \), we can directly calculate both \( P(D_i) = P(D_i \cup D_j) \cap D_i \) and \( P(D_j) = P(D_i \cup D_j) \cap D_j \). It is clear that \( C(R') = C(R_i) + C(R_j) \) due to the extra \( \kappa \) overhead. Thus, by replacing \( R_i \) with \( R' \) and removing \( R_j \), we have found a less costly sequence while maintaining equivalence, leading to a contradiction.

\( \square \)

### 4.4 Problem Restatement

With the properties given to us by Theorems 1, 2, and 3, we can restate the problem as:

**Problem 2.** Find the sequence of steps \( [R_1, R_2, ..., R_n] \) in which each step \( R_i \) is a \( (P_i, D_i) \) pair representing a unique predicate atom application of \( P_i \) over vertex set \( D_i \) such that the sum of costs across all steps \( \sum_{i=1}^n C(R_i) \) is minimized, while ensuring that the largest vertex set satisfying \( \phi^* \) appears in the universe of resulting sets: \( \psi^*(D) \in U_n \).

### 5 Algorithms

We now present the actual algorithms which determine the optimal sequence of predicate steps for a given query. In accordance with Problem 2 our solution must provide a \( n \)-length sequence of steps: \( R_i = (P_i, D_i) \). Thus, the problem can be divided into the following two distinct components.

**Problem 3.** Given a predicate atom sequence \( [P_1, ..., P_n] \), find the best corresponding sequence of vertex sets \( [D_1, ..., D_n] \) which minimizes the total cost for the given sequence of predicate atoms.

**Problem 4.** Find the best ordering of predicate atoms \( [P_1, ..., P_n] \) which minimizes the total cost given that we can find the corresponding best sequence of vertex sets \( [D_1, ..., D_n] \) that the predicate atoms will be applied to.

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4There is a caveat when \( D_i \cap D_j = \emptyset \) but \( D_j \) is somehow derived from \( P(D_i) \). Calculating \( D_i \cup D_j \) requires \( D_j \), which in turn requires \( P(D_i) \) which is not available before the \( i \)th step. Fortunately, we show that this situation cannot arise in Appendix A.3.
Note that for Problem 3, it is not a requirement for the given sequence \( [P_1, \ldots, P_n] \) to be in the optimal order. For any given sequence of predicate atoms, our solution \( \text{BESTD} \) returns the best corresponding sequence of vertex sets catered to that sequence of predicate atoms. When paired with an optimal predicate atom ordering algorithm like \text{ORDERP} suggested by Hanani [9] for Problem 1, our algorithms generate globally optimal predicate evaluation plans. In Section 5.2, we do precisely this and combine \( \text{BESTD} \) and \text{ORDERP} to present \( \text{SHALLOWFISH} \), which is provably optimal for all predicate trees of depth 2 or less. In Section 5.3 we show \text{ORDERP} is in fact not optimal for predicate trees of depth 3 or greater, and suggest our approximate algorithm \( \text{DEEPFISH} \) which is better suited to such situations.

5.1 \( \text{BESTD} \)

Before presenting \( \text{BESTD} \), we first provide some additional notation. If \( \lambda \) is a leaf node in the predicate tree of \( \phi^* \), given a sequence of predicate atoms \( [P_1, \ldots, P_n] \), let \( \theta(\lambda) \) return the index of the predicate atom referred to by \( \lambda \). In addition, let \( \Omega(i) \) return the sequence of nodes in \( P_i \)'s lineage (i.e., \( P_i \)'s ancestors), starting with the root first and ending with \( P_i \).

For any given node \( \lambda \) in the predicate tree, we say that \( \lambda \) is \textit{complete} on step \( i \) if all of its descendent predicate atoms have already been applied.

**Definition 1.** Given a predicate atom sequence \( [P_1, \ldots, P_n] \), a node \( \lambda \) of the predicate tree is \textit{complete} on step \( i \) if:

\[
\text{complete}(\lambda, i) = \begin{cases} 
\lambda \text{ is applied} \quad (\theta(\lambda) < i) & \text{if } \theta(\lambda) = P \\
\land_{c \in \text{children}(\lambda)} \text{complete}(c, i) & \text{otherwise}
\end{cases}
\]

On the other hand, determinability describes whether a node can already be determined to evaluate to 1/0 for some vertices without evaluating the remaining predicate atom descendants of that node. For example, if one child of an OR node evaluates to 1 for some vertex, the results of the other children on that vertex do not matter. Similarly, if one child of an AND node evaluates to 0 for some vertex, the overall value of the node on the vertex can be determined without evaluating the other children.

**Definition 2.** Given a predicate atom sequence \( [P_1, \ldots, P_n] \), a node \( \lambda \) of the boolean predicate tree is \textit{positively determinable} on step \( i \) if:

\[
\text{determ}^+(\lambda, i) = \begin{cases} 
\lambda \text{ is applied} \quad (\theta(\lambda) < i) & \text{if } \theta(\lambda) = P \\
\land_{c \in \text{children}(\lambda)} \text{determ}^+(c, i) & \text{if } \theta(\lambda) = \land \\
\lor_{c \in \text{children}(\lambda)} \text{determ}^+(c, i) & \text{if } \theta(\lambda) = \lor
\end{cases}
\]

**Definition 3.** Given a predicate atom sequence \( [P_1, \ldots, P_n] \), a node \( \lambda \) of the boolean predicate tree is \textit{negatively determinable} on step \( i \) if:

\[
\text{determ}^-(\lambda, i) = \begin{cases} 
\lambda \text{ is applied} \quad (\theta(\lambda) < i) & \text{if } \theta(\lambda) = P \\
\lor_{c \in \text{children}(\lambda)} \text{determ}^-(c, i) & \text{if } \theta(\lambda) = \land \\
\land_{c \in \text{children}(\lambda)} \text{determ}^-(c, i) & \text{if } \theta(\lambda) = \lor
\end{cases}
\]

We now present \( \text{BESTD} \), the algorithm which finds the best input vertex set \( D_i \) for the \( i \)th step given a sequence of predicate atoms \( [P_1, \ldots, P_n] \). Algorithm 1 gives the pseudocode. Here, \( \Xi[\hat{c}] \) is the vertex set which exactly matches the subtree represented by complete child \( \hat{c} \). The vertices in \( \Delta^+[c^+] \) are those guaranteed to make positively determinable child \( c^+ \) evaluate to 1, and the vertices in \( \Delta^-[c^-] \) guarantee that negatively determinable child \( c^- \) evaluates to 0. \( \text{BESTD} \) recursively climbs up the predicate tree until the root node, at which point \( D \) (the set of all vertices) is returned. Then, as the recursive calls return from top-to-bottom (i.e., root node to \( P_i \)), we filter from this vertex set, the vertices that can already be determined by previously applied predicate atoms. For example, if node \( \lambda \) is an OR node, \( c^+ \) is \( P_i \)'s ancestor and \( \lambda \)'s child, and \( c^+ \) is a positively determinable sibling of \( c^- \), we filter out the vertices in \( \Delta^+[c^+] \) before returning \( \text{BESTD} \) to \( c^+ \). The filtered vertices are already guaranteed to make \( \lambda \) evaluate to 1 and do not need to be considered further. This way, predicate atom \( P_i \) will only be applied to the vertices whose evaluation value on the overall predicate expression is still unknown. How \( \Xi, \Delta^+, \text{ and } \Delta^- \) are updated is discussed in Section 5.2.
Algorithm 1 BestD

**Input:** Predicate expression \( \phi^* \), Sequence of predicate atoms \( P = [P_1, ..., P_n] \), Vertex set maps \( \Xi, \Delta^+, \Delta^- \), Step index \( i \), The current level in the tree \( l \)

**Output:** Optimal vertex set \( D_i \)

1. if \( \text{type}(\Omega_l(i)) = \land \) then
2. \( \quad \text{return BestAndD}(\phi^*, P, \Xi, \Delta^+, \Delta^-, i, l) \)
3. else
4. \( \quad \text{return BestOrD}(\phi^*, P, \Xi, \Delta^+, \Delta^-, i, l) \)

5. function BestAndD(\( \phi^*, P, \Xi, \Delta^+, \Delta^-, i, l \))
6. \( \quad \text{if } l = 0 \text{ then return } D \)
7. \( \quad X \leftarrow \text{BestOrD}(\phi^*, P, \Xi, \Delta^+, \Delta^-, i, l - 1) \)
8. \( \quad \text{for } c \in \text{children}(\Omega_l(i)) \text{ do} \)
9. \( \quad \quad \quad \text{if complete}(c, i) \text{ then} \)
10. \( \quad \quad \quad \quad X \leftarrow X \cap \Xi[c] \)
11. \( \quad \quad \quad \quad \text{else if determ}^-(c, i) \text{ and } c \neq \Omega_{l+1}(i) \text{ then} \)
12. \( \quad \quad \quad \quad X \leftarrow X \setminus \Delta^-[c] \)
13. \quad \text{return } X \)

14. function BestOrD(\( \phi^*, P, \Xi, \Delta^+, \Delta^-, i, l \))
15. \( \quad \text{if } l = 0 \text{ then return } D \)
16. \( \quad X \leftarrow \{\} \)
17. \( \quad Y \leftarrow \text{BestAndD}(\phi^*, P, \Xi, \Delta^+, \Delta^-, i, l - 1) \)
18. \( \quad \text{for } c \in \text{children}(\Omega_l(i)) \text{ do} \)
19. \( \quad \quad \quad \text{if complete}(c, i) \text{ then} \)
20. \( \quad \quad \quad \quad X \leftarrow X \cup \Xi[c] \)
21. \( \quad \quad \quad \quad \text{else if determ}^+(c, i) \text{ and } c \neq \Omega_{l+1}(i) \text{ then} \)
22. \( \quad \quad \quad \quad X \leftarrow X \cup \Delta^+[c] \)
23. \quad \text{return } Y \setminus X \)
5.2 ShallowFish

Recall that ORDERP is the predicate atom ordering algorithm developed by Hanani [9]. The main idea is to evaluate predicate atoms of conjunctions in order of increasing selectivity while evaluating predicate atoms of disjunctions in decreasing selectivity. We can combine BESTD with ORDERP to construct SHALLOWFISH (Algorithm 2) which (1) computes the optimal ordering for predicate atoms $[P_1, \ldots, P_n]$ for predicate trees of depth 2 or less, (2) finds the best sequence of vertex sets $[D_1, \ldots, D_n]$ to apply those predicate atoms to, and (3) keeps track of which vertex sets satisfy which predicate tree nodes.

After calculating the best vertex $D_i$ for each step $i$, the predicate atom $P_i$ is “applied” using the UPDATE function. The UPDATE function calculates which vertices in $D_i$ satisfy $P_i$ and stores them in a mapping from predicate tree nodes to vertex sets (i.e., $\Xi$). UPDATE also calculates the satisfying vertex sets for each newly completed node among $P_i$’s lineage. For example, if an AND node is newly completed, its $\Xi$ value is updated as the intersection of its children’s $\Xi$ values. Positively and negatively determinable nodes are updated similarly according to Definitions 2 and 3.

The SHALLOWFISH presented in Algorithm 2 actually has a runtime complexity of $O(n^3)$ and is presented in this way to make the concepts and proofs of this paper clearer. However, an optimized SHALLOWFISH would combine the BESTD and UPDATE functions into one and have a runtime of complexity of $O(n \log n)$. We present the optimized algorithm in Appendix B.1. In addition, we give a brief overview on how to apply SHALLOWFISH on actual records instead of vertices in Appendix B.2.

5.3 DeepFish

In the case that the predicate tree has a depth of 3 or greater, ORDERP is no longer optimal. The reason is due to nodes which are positively/negatively determinable but not complete. To better demonstrate the idea, we present:

Example 1. Let predicate expression $\phi^* = P_A \land (P_B \lor (P_C \land P_D))$ in which $P_A$, $P_B$, $P_C$, and $P_D$ are predicate atoms with selectivities $\gamma_A = 0.820$, $\gamma_B = 0.313$, $\gamma_C = 0.469$, $\gamma_D = 0.984$ and constant cost factors $F_A = F_B = F_C = F_D = 1$ respectively.

According to ORDERP, we would apply predicate atoms in the order of $[P_C, P_D, P_B, P_A]$ for Example 1 which would result in a cost of 2.638. However, ordering $[P_B, P_C, P_A, P_D]$ would actually result in a lower cost of 2.586 (though this may seem like a small difference, more complex predicate expressions result in larger differences). In this situation, the root node is negatively determinable but not complete after predicate atoms $P_B$ and $P_C$ have been applied. The negative determinability removes enough vertices and the selectivity of $P_D$ is sufficiently high enough such that applying $P_A$ before $P_D$ becomes the optimal solution. However, ORDERP is inherently a depth-first processing algorithm, and is unable to consider this as a possible ordering. The reason SHALLOWFISH is optimal for predicate trees of depth 2 or less is due to Lemma 1 which states that if a child among BESTD’s path is positively/negatively determinable, it must also be complete.

Lemma 1. Given a predicate expression $\phi^*$ whose tree form has a depth of 2 or less and a sequence of predicate atoms $P = [P_1, \ldots, P_n]$, let $\Omega_l(i)$ refer to $P_i$’s $l$th level ancestor, and let $\lambda$ be $\Omega_l(i)$’s child. For any $l$, if $\Omega_l(i)$ is an AND (OR) node and $\lambda$ is negatively (positively) determinable respectively, then $\lambda$ must also be complete.

Determinability prevents us from dividing the problem into well-contained subproblems, and at each level, we must consider all possible determinable subsequences, of which there are exponentially many. Despite the authors’ best efforts, we were unable to come up with an optimal polynomial-time algorithm or a reduction from an NP-complete problem. Instead, for situations like this, we offer the approximate algorithm DEEPFISH, which correctly returns the sequence $[P_B, P_C, P_A, P_D]$ for Example 1. The pseudocode for DEEPFISH is detailed in Algorithm 3. Note that UPDATE is the function from Algorithm 2 and $C$ is the cost function introduced in our cost model. DEEPFISH calls ONELOOKAHEADP, which is a one predicate atom lookahead algorithm; it considers applying each unapplied predicate atom and observes the change in the overall cost of the remaining unapplied predicate atoms to decide the next predicate atom in the sequence. At each iteration, the predicate atom with the maximal change in remaining cost (benefit) to cost
BestD (for predicate trees of any depth). Note that since calls to
returns the best vertex set to operate on and minimizes the cost for any given sequence of predicate atoms
sequence of predicate atoms, if it is given an optimal ordering of predicate atoms, the overall sequences of
between the two, and returning the overall cheaper plan.

OneLookaheadP/BestD, comparing the estimated costs
between the two, and returning the overall cheaper plan.

6 Proofs

We now show the correctness and optimality of SHALLOWFISH for predicate trees of depth 2 or less. For
predicate trees of greater depth, SHALLOWFISH still remains correct, but it may no longer be optimal.
However, the problem also becomes much more difficult, which is why we have approximation algorithm
DEEPFISH rather than the optimal solution. The section is divided into: (1) Proving that the plan generated
by SHALLOWFISH correctly returns the filtered vertex set which satisfies \( \phi^* \), and (2) Proving that BESTD
returns the best vertex set to operate on and minimizes the cost for any given sequence of predicate atoms
(for predicate trees of any depth). Note that since calls to BESTD return the best vertex sets for any
sequence of predicate atoms, if it is given an optimal ordering of predicate atoms, the overall sequences of
Algorithm 3 DeepFish

Input: Boolean predicate $\phi^*$

Output: Ordered sequence of predicate atoms $[P_1, \ldots, P_n]$ and vertex sets $[D_1, \ldots, D_n]$

1: $P \leftarrow \emptyset$, $\Xi \leftarrow \emptyset$, $\Delta^+ \leftarrow \{\}$, $\Delta^- \leftarrow \{\}$
2: for $i \leftarrow 1, \ldots, n$ do
3: \hspace{1em} $P \leftarrow P + [\text{OneLookaheadP}(\phi^*, P, \Xi, \Delta^+, \Delta^-)]$
4: \hspace{1em} $D_i \leftarrow \text{BestD}(\phi^*, P, \Xi, \Delta^+, \Delta^-, i, |\Omega(i)| - 1)$
5: \hspace{1em} $\text{Update}(\phi^*, [P_1, \ldots, P_n], D_i, i, \Xi, \Delta^+, \Delta^-)$
6: \hspace{1em} $([P'_1, \ldots, P'_n], [D'_1, \ldots, D'_n]) \leftarrow \text{ShallowFish}(\phi^*)$
7: if $\sum_{i=1}^n C(P_i, D_i) < \sum_{i=1}^n C(P'_i, D'_i)$ then
8: \hspace{1em} return $([P_1, \ldots, P_n], [D_1, \ldots, D_n])$
9: else
10: \hspace{1em} return $([P'_1, \ldots, P'_n], [D'_1, \ldots, D'_n])$

11: function OneLookaheadP($\phi^*, P, \Xi, \Delta^+, \Delta^-$)
12: (P*, i, bestRatio) $\leftarrow$ (Nil, |P| + 1, 0)
13: for $P \in (\text{predicates}(\phi^*) \setminus P)$ do
14: \hspace{1em} $D \leftarrow \text{BestD}(\phi^*, P + [P], \Xi, \Delta^+, \Delta^-, i, LP - 1)$
15: \hspace{1em} $(\Xi', \Delta'^+, \Delta'^-) \leftarrow \text{copy}((\Xi, \Delta^+, \Delta^-))$
16: \hspace{1em} $\text{Update}(\phi^*, P + [P], D, i, \Xi', \Delta'^+, \Delta'^-)$
17: \hspace{1em} $\text{origCost} \leftarrow \text{RemainCost}(\phi^*, \phi^*, P, \Xi, \Delta^+, \Delta^-)$
18: \hspace{1em} $\text{newCost} \leftarrow \text{RemainCost}(\phi^*, \phi^*, P, \Xi', \Delta'^+, \Delta'^-) \leftarrow (\text{origCost} - \text{newCost}) / C(P, D)$
19: if newRatio $>$ bestRatio then
20: \hspace{1em} (P*, bestRatio) $\leftarrow$ (P, newRatio)
21: return $P^*$

23: function RemainCost($\phi^*, \lambda, P, \Xi, \Delta^+, \Delta^-$)
24: if type($\lambda$) = $P$ then
25: \hspace{1em} if $\lambda \in P$ then return 0
26: \hspace{1em} else return $C(\lambda, \text{BestD}(\phi^*, P, \Xi, \Delta^+, \Delta^-, |P|, L_\lambda - 1))$
27: \hspace{1em} else
28: \hspace{2em} $S \leftarrow 0$
29: \hspace{2em} for $c \in \text{children}(\lambda)$ do
30: \hspace{3em} $S \leftarrow S + \text{RemainCost}(\phi^*, c, P, \Xi, \Delta^+, \Delta^-)$
31: \hspace{2em} return $S$

steps generated by ShallowFish are globally optimal plans. This section only provides the proof ideas for the main theorems. Step-by-step derivations and proofs of lemmas are presented in Appendix A

6.1 Setup

Before we begin, we provide some notation, definitions, and properties that will be used throughout the proofs.

Notation. For a given vertex $v$, we let $v_i$ refer to the $i$th element of $v$. Furthermore, we let $v|_{i=1}$ refer to a copy of $v$ with the $i$th element set to 1 and $v|_{i=0}$ to a copy of $v$ with the $i$th element set to 0. More formally, let $v|_{i=1}$ be the vertex $u$ such that $u_i = 1$ and $\forall j \neq i, u_j = v_j$, while $v|_{i=0}$ be the vertex $u'$ such that $u'_i = 0$ and $\forall j \neq i, u'_j = v_j$. For any node $\lambda$ and vertex $v$, let $\lambda[v]$ be the result of evaluating the subtree $\lambda$ with respect to $v$. Specifically, if $\lambda$ is a leaf node, let $\lambda[v] = 1$ if $v|_{i=0}$ = 0. If $\lambda$ is an AND node, it is the conjunctive combination of its children’s evaluation results, and if $\lambda$ is an OR node, it is the disjunctive combination. As mentioned before $\Delta^+|\lambda|/\Delta^-|\lambda|$ keep track of the set of vertices which are determined to evaluate to true/false for $\lambda$ respectively. When denoted with a time step $\Delta^+_t|\lambda|/\Delta^-_t|\lambda|$, this refers to the
state of $\Delta^+ [\lambda] / \Delta^- [\lambda]$ at the beginning of step $i$. Note that from this point on, any mentions of the state of an object “at/on time step $i$” refer to its state at the beginning of step $i$. If $\lambda$ is not positively/negatively determinable at time step $i$, then we assume $\Delta^+ [\lambda] / \Delta^- [\lambda]$ evaluate to the empty set. On the other hand, $\Xi$ values are only ever updated once and are immutable, so they do not have time step denotations. For this section, we will use the following conventions for children of $\lambda$: $c$ for any child, $\hat{c}$ for a complete child, $c^+$ for a positively determinable child, $c^-$ for a negatively determinable child. In all cases, the children are assumed to be taken with respect to node $\lambda$. We use these conventions in conjunction with iterators to represent different sets of children. Near the top of the iterators is a time step enclosed in parentheses to instruct when to take an iteration. For example, $\bigcup_{i}^{j}$ would take the union of the children that are complete at time step $i$. Similarly, $\bigvee_{c}^{i}$ states “On time step $i$, for all negatively determinable children”. Operators without time step modifiers are time invariant (e.g., the intersection of all children: $\bigcap_{i}$). We denote $L_\lambda$ as the level that $\lambda$ occupies in the predicate tree representing $\phi^*$. Finally, when calling $\text{BESTD}$, we may omit any of the $\phi^*$, $P$, $\Xi$, $\Delta^+$, and $\Delta^-$ arguments since they refer to the same objects for every call (e.g., $\text{BESTD}(i, l) = \text{BESTD}(\phi^*, P, \Xi, \lambda, \Delta^+, \Delta^-, i, l)$).

Next, we define $\Upsilon_P^{(i)}(\lambda)$ to be the index of the last applied predicate atom descendant of $\lambda$ before time step $i$:

**Definition 4.** For any node $\lambda$ and predicate atom sequence $P = [P_1, ..., P_n]$, let $\Upsilon_P^{(i)}(\lambda)$ be the largest index $j$ that is smaller that $i$ such that $P_j \in P$ is a descendant of $\lambda$. Let $\Upsilon_P(\lambda)$ be the shorthand to refer to the index of the last predicate atom descendant of $\lambda$ (i.e., $\Upsilon_P(\lambda) = \Upsilon_P^{(n+1)}(\lambda)$).

We define function $\xi(\lambda, D)$ to return the set of vertices in $D$ which would satisfy $\lambda$ (i.e., $\{v \in D \mid \lambda[v] = 1\}$):

**Definition 5.**

$$\xi(\lambda, D) = \begin{cases} P_{\phi(\lambda)}(D) & \text{if } \text{type}(\lambda) = P \\ \bigcap_{c \in \text{children}(\lambda)} \xi(c, D) & \text{if } \text{type}(\lambda) = \wedge \\ \bigcup_{c \in \text{children}(\lambda)} \xi(c, D) & \text{if } \text{type}(\lambda) = \vee \end{cases}$$

**Property 1.** For any node $\lambda$ and vertex sets $D$ and $Y$:

$$\xi(\lambda, D \cap Y) = \xi(\lambda, D) \cap Y$$

$$\xi(\lambda, D \setminus Y) = \xi(\lambda, D) \setminus Y$$

Since the result of $\xi(\lambda, D)$ is a vertex set which is a subset of $D$, $\xi$ acts like an intersection. Thus, the associative property of intersections gives us the above property.

**Property 2.** For any node $\lambda$ in the predicate tree for $\phi^*$, any vertex set $D$, and step index $i$: $\xi(\lambda, D) \cap \Delta^- [\lambda] = \emptyset$.

$\xi$ returns all the vertices for which the subtree $\lambda$ evaluates to true while $\Delta^-$ keeps track of all the vertices for which $\lambda$ evaluates to false. Thus, the intersection must be empty.

**Property 3.** For any node $\lambda$ and for all time steps $i$ in which $\lambda$ is complete, $\Xi[\lambda] = \Delta^+ [\lambda]$.

$\Delta^+$ keeps track of all vertices guaranteed to evaluate to true even if there are unapplied predicate atom descendants of $\lambda$, while $\Xi$ keeps track of all vertices for which $\lambda$ evaluates to true. If $\lambda$ is complete, the two sets are the same.

**Property 4.** For predicate expression $\phi^*$ and some sequence of predicate atoms $P = [P_1, ..., P_n]$:

$$\text{BESTD}(\phi^*, P, i, l) \subseteq \text{BESTD}(\phi^*, P, i, l')$$

for any $1 \leq i \leq n$ and all $l > l'$.

At each level, the $\text{BESTANDD}$ ($\text{BESTORD}$) function only takes intersections or removes some subset from the original $X$ ($Y$) vertex set given to it from a lower level, respectively. Thus, calls to $\text{BESTD}$ at higher levels will always be a subset of the calls to the lower levels for the same step $i$.  

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Property 5. If \( \lambda \) is a node in the predicate tree representing \( \phi^* \) and \( P \) is some arbitrary predicate atom sequence, then for all children \( c \) of \( \lambda \): \( \Upsilon_P(c) \leq \Upsilon_P(\lambda) \).

Since \( \Upsilon_P \) returns the index of the last descendent predicate atom of a node (in the sequence \( P \)), the index for the last descendant of any child of \( \lambda \) can be at most equal to the index of the last descendent of \( \lambda \).

6.2 Correctness of ShallowFish

We first show that the plan generated by ShallowFish results in a vertex set which satisfies the predicate expression \( \phi^* \); that is, it must be \( \psi^*(D) \). Based on Definition 5 if we let \( \lambda^* = \text{root}(\phi^*) \), then \( \xi(\lambda^*, D) = \psi^*(D) \). When the plan generated by ShallowFish is evaluated, the resulting vertex set is given by \( \Xi[\lambda^*] \). Therefore, it is sufficient to show that \( \Xi[\lambda^*] = \xi(\lambda^*, D) \).

To show that this equality holds for the root node \( \lambda^* \), we show that the equality actually holds more generally for any node \( \lambda \) when \( \xi \) is applied to the result of \( \text{BestD} \). Since \( \text{BestD}(\ast, 0) = D \), for the root node, we wish to show that \( \Xi[\lambda^*] = \xi(\lambda^*, \text{BestD}(\ast, 0)) \). More generally:

**Theorem 4.** Let node \( \lambda \) be a node in predicate tree given by \( \phi^* \). Given that vertex set maps \( \Xi, \Delta^+, \Delta^- \) are updated by \( \text{Update} \) and \( \text{BestD} \), for any sequence of predicate atoms \( P = [P_1, \ldots, P_n] \):

\[
\Xi[\lambda] = \xi(\lambda, \text{BestD}(\phi^*, P, \Xi, \Delta^+, \Delta^-, \Upsilon_P(\lambda), L_\lambda - 1))
\]

**Proof.** The proof is by strong induction on the height of the subtree referred to by node \( \lambda \). For the base case, \( \lambda \) refers to a subtree of height 1, which is just a single predicate atom \( P_i \). We see in Line 8 of Algorithm 2 that \( \Xi[P_i] = P_i(D_i) \) and in Line 4 that the value of \( D_i = \text{BestD}(i, L_{P_i} - 1) \). Thus, \( \Xi[P_i] = P_i(\text{BestD}(i, L_{P_i} - 1)) \).

According to Definition 5, \( \xi(\lambda, D) = P_{\theta(\lambda)}(D) \) for predicate atoms, so the theorem holds for all predicate atoms.

We assume the inductive hypothesis holds for all subtrees of height \( k \) or less. For the inductive step, we let \( \lambda \) be a predicate tree node with height \( k + 1 \). Since \( \lambda \)'s children cannot have height exceeding \( k \), the inductive hypothesis applies to all of \( \lambda \)'s children.

Before moving on to the inductive step, we introduce Lemma 2 which says that for any node \( \lambda \), the \( \text{BestD} \) values returned for \( \lambda \) at level \( L_{\lambda} \) are monotonically decreasing with respect to the time step. More specifically, for any two time steps \( i \) and \( j \) such that \( i < j \), if both \( P_i \) and \( P_j \) are descendants of \( \lambda \), then the value of \( \text{BestD}(i, L_\lambda) \) must be a superset of the value of \( \text{BestD}(j, L_\lambda) \). Formally stated:

**Lemma 2.** For any non-leaf node \( \lambda \) in the predicate tree for \( \phi^* \), let the sequence \([\eta_1, \ldots, \eta_J]\) be the descendent predicate atoms of \( \lambda \) ordered according to the given predicate atom sequence \( P = [P_1, \ldots, P_n] \). For all \( 1 \leq a < b \leq J \):

\[
\text{BestD}(\phi^*, P, \theta(\eta_a), L_\lambda) \supseteq \text{BestD}(\phi^*, P, \theta(\eta_b), L_\lambda)
\]

Note that as a direct corollary of Lemma 2, if the inequality holds for node \( \lambda \) and level \( L_\lambda \), it must also hold for \( \lambda \)'s parent and level \( L_{\lambda - 1} \), since all of \( \lambda \)'s descendants are \( \lambda \)'s parent’s descendants as well: \( \text{BestD}(\theta(\eta_b), L_{\lambda - 1}) \supseteq \text{BestD}(\theta(\eta_b), L_{\lambda - 1}) \).

We also present Lemma 3 which tells us that when intersected with the latest \( \text{BestD} \) value, \( \Delta^+[\lambda] \) is a subset of \( \Xi[\lambda] \) for any \( \lambda \).

**Lemma 3.** Let \( \lambda \) be a node in the predicate tree of \( \phi^* \), and let \( P \) be some predicate atom sequence. For any time step \( i \):

\[
\Delta^+[\lambda] \cap \text{BestD}(\phi^*, P, \Upsilon_P(\lambda), L_\lambda - 1) \subseteq \Xi[\lambda]
\]

Finally, Lemma 4 states that any two children of an OR node must have mutually exclusive \( \Delta^+ \) values.

**Lemma 4.** Given that \( \lambda \) is an OR node in the predicate tree for \( \phi^* \), if \( c \) and \( c' \) are children of \( \lambda \) such that \( c \neq c' \), then \( \Delta^+[c] \cap \Delta^+[c'] = \emptyset \) for all steps \( i \) and \( j \).
When we substitute in this $\Xi[c]$, apply Properties 1 and 3, we get:

$$\Xi[c] = \xi(c, \text{BESTD}(\Upsilon_{\mathbf{P}}(c), L_{\lambda} - 1))$$

which is the definition of $\xi$. By Property 5, we know that for any child $c$ of $\lambda$, $\Upsilon_{\mathbf{P}}(c) \leq \Upsilon_{\mathbf{P}}(\lambda)$. Thus, by Lemma 2, $\text{BESTD}(\Upsilon_{\mathbf{P}}(c), L_{\lambda} - 1) \supseteq \text{BESTD}(\Upsilon_{\mathbf{P}}(\lambda), L_{\lambda} - 1)$. If we apply Property 1 to this, get rid of duplicates, and let $Z = \text{BESTD}(\Upsilon_{\mathbf{P}}(\lambda), L_{\lambda} - 1)$ to derive:

$$\Xi[\lambda] = \bigcap_{c} \xi(c, \text{BESTD}(\Upsilon_{\mathbf{P}}(c), L_{\lambda} - 1)) \cap Z$$

$$\bigcap_{\hat{c} \neq c} \xi(\hat{c}, \text{BESTD}(\Upsilon_{\mathbf{P}}(\hat{c}), L_{\lambda} - 1)) \cap Z$$

\[ \setminus \left( \bigcup_{c^{-} \neq c} (\Upsilon_{\mathbf{P}}(c))_{c^{-}} \right) \]

When substituting $\Xi[c]$ into the expression for $\Xi[\lambda]$, we can use the equivalence $(A \cap B \setminus C) \cap D = (A \cap D) \setminus (B \cap D) \setminus C$ and let $Z = \text{BESTD}(\Upsilon_{\mathbf{P}}(\lambda), L_{\lambda} - 1)$ to derive:

$$\Xi[\lambda] = \bigcap_{c} \xi(c, \text{BESTD}(\Upsilon_{\mathbf{P}}(c), L_{\lambda} - 1)) \cap Z$$

By Property 5, we know that that for any child $c$ of $\lambda$, $\Upsilon_{\mathbf{P}}(c) \leq \Upsilon_{\mathbf{P}}(\lambda)$. Thus, by Lemma 2, $\text{BESTD}(\Upsilon_{\mathbf{P}}(c), L_{\lambda} - 1) \supseteq \text{BESTD}(\Upsilon_{\mathbf{P}}(\lambda), L_{\lambda} - 1)$. If we apply Property 1 to this, get rid of duplicates, and let $Z = \text{BESTD}(\Upsilon_{\mathbf{P}}(\lambda), L_{\lambda} - 1)$:

$$\Xi[\lambda] = \bigcap_{c} \xi(c, Z) \setminus \left( \bigcup_{c^{-} \neq c} (\Upsilon_{\mathbf{P}}(c))_{c^{-}} \right)$$

At this point, we can apply the equivalence $A \cap (B \setminus (C \cup D)) = (A \setminus C) \cap (B \setminus D)$ (shown in Appendix D) to rearrange the intersections and unions. Then, we get that for some set of indices $i$:

$$\Xi[\lambda] = \bigcap_{c} \xi(c, \text{BESTD}(\Upsilon_{\mathbf{P}}(\lambda), L_{\lambda} - 1)) \setminus \left( \bigcup_{i} \Delta^{-}_{\Upsilon_{\mathbf{P}}(c)}[c] \right)$$

Property 2 says that the $\xi$ value and the $\Delta^{-}$ value are mutually exclusive for any $c$ and index $i$, so this simplifies to:

$$\Xi[\lambda] = \bigcap_{c} \xi(c, \text{BESTD}(\Upsilon_{\mathbf{P}}(\lambda), L_{\lambda} - 1))$$

which is the definition of $\xi$ for AND nodes.

$\lambda$ is OR. Next, let us examine the case in which $\lambda$ is an OR node. Line 21 of Algorithm 2 tells us that $\Xi[\lambda] = \bigcup_{c} \Xi[c] \cap \text{BESTD}(\Upsilon_{\mathbf{P}}(\lambda), L_{\lambda} - 1)$. Since $c$ is a child of $\lambda$, the inductive hypothesis holds: $\Xi[c] = \xi(c, \text{BESTD}(\Upsilon_{\mathbf{P}}(c), L_{\lambda} - 1))$. If we expand out the BESTD value with respect to Algorithm 1 and apply Properties 1 and 3 we get:

$$\Xi[c] = \xi(c, \text{BESTD}(\Upsilon_{\mathbf{P}}(c), L_{\lambda} - 1))$$

$$\setminus \left( \bigcup_{c^{+} \neq c} (\Upsilon_{\mathbf{P}}(c))[c^{+}] \right)$$

When we substitute in this $\Xi[c]$ value into the expression for $\Xi[\lambda]$, apply the equivalence $(A \setminus B) \cap C =$
(A \cap C) \setminus (B \cap C), and let Z = \text{BestD}(\mathcal{T}_p(\lambda), L_\lambda - 1), we get:

\[\exists[\lambda] = \bigcup_c \xi(c, \text{BestD}(\mathcal{T}_p(c), L_\lambda - 1)) \cap Z\]

\[\setminus \left( \bigcup_{c^+ \neq c} (\mathcal{T}_p(c))^+ [c^+] \cap \text{BestD}(\mathcal{T}_p(\lambda), L_\lambda - 1) \right)\]

By Property 5, \(\mathcal{T}_p(c) \leq \mathcal{T}_p(\lambda)\) for any child \(c\). Thus, by Lemma 2, \(\text{BestD}(\mathcal{T}_p(c), L_\lambda - 1) \supseteq \text{BestD}(\mathcal{T}_p(\lambda), L_\lambda - 1)\) for any child \(c\). If we apply Property 1, we arrive at:

\[\exists[\lambda] = \bigcup_c \xi(c, \text{BestD}(\mathcal{T}_p(\lambda), L_\lambda - 1))\]

\[\setminus \left( \bigcup_{c^+ \neq c} (\mathcal{T}_p(c))^+ [c^+] \cap \text{BestD}(\mathcal{T}_p(\lambda), L_\lambda - 1) \right)\]

Equation 1 tells us that \(\exists[\lambda] \subseteq \xi(c, \text{BestD}(\mathcal{T}_p(c), L_\lambda - 1))\). Furthermore, recall that by Lemma 2 \(\text{BestD}(\mathcal{T}_p(\lambda), L_\lambda - 1) \supseteq \text{BestD}(\mathcal{T}_p(c), L_\lambda - 1)\) for any child \(c\). If we apply Lemma 3 and intersect both sides with \(\text{BestD}(\mathcal{T}_p(\lambda), L_\lambda - 1)\), we get that for all \(i\):

\[\Delta_i^+ [c] \cap \text{BestD}(\mathcal{T}_p(\lambda), L_\lambda - 1) \subseteq \xi(c, \text{BestD}(\mathcal{T}_p(\lambda), L_\lambda - 1))\]

Notice that the left-hand side is the same expression as the one in parentheses in Equation 2 while the right-hand side is the same as the expression outside of the parentheses. Thus, we apply the principle that if \(B \supseteq F\) and \(C \cap F = \emptyset\), then \(A \cup (B \setminus C) \cup (D \setminus (E \cup F)) = A \cup (B \setminus C) \cup (D \setminus E)\) for any sets \(A, B, C, D, E,\) and \(F\) (Appendix D). We focus specifically on child \(c^*\) and positively determinable child \(c^{++} \neq c^*\). Let:

\begin{align*}
B &= \xi(c^{++}, \text{BestD}(\mathcal{T}_p(\lambda), L_\lambda - 1)) \\
C &= \bigcup_{c^+ \neq c^{++}} (\mathcal{T}_p(c^{++}))^+ [c^+] \cap \text{BestD}(\mathcal{T}_p(\lambda), L_\lambda - 1) \\
D &= \xi(c^*, \text{BestD}(\mathcal{T}_p(\lambda), L_\lambda - 1)) \\
E &= \bigcup_{c^+ \neq c^{++}, c^*} (\mathcal{T}_p(c^*))^+ [c^+] \cap \text{BestD}(\mathcal{T}_p(\lambda), L_\lambda - 1) \\
F &= \Delta_i^+ [c^{++}] \cap \text{BestD}(\mathcal{T}_p(\lambda), L_\lambda - 1)
\end{align*}

and \(A\) be the remaining union operands in Equation 2. Based on Equation 3 it is clear that \(F \subseteq B\). Since \(c^+\) in \(C\) iterates over every child except \(c^{++}\), by Lemma 4 \(F \cap C = \emptyset\). Thus, we can remove \(F\) from Equation 2. If we apply this trick to every \(c^+\) in \(E\), then Equation 2 reduces to:

\[\exists[\lambda] = \xi(c^*, \text{BestD}(\mathcal{T}_p(\lambda), L_\lambda - 1))\]

\[\bigcup_{c \neq c^*} \xi(c, \text{BestD}(\mathcal{T}_p(\lambda), L_\lambda - 1))\]

\[\setminus \left( \bigcup_{c^+ \neq c} (\mathcal{T}_p(c))^+ [c^+] \cap \text{BestD}(\mathcal{T}_p(\lambda), L_\lambda - 1) \right)\]

After this reduction is done for every child \(c\) of \(\lambda\), we get:

\[\exists[\lambda] = \bigcup_c \xi(c, \text{BestD}(\mathcal{T}_p(\lambda), L_\lambda - 1))\]

which is the definition of \(\xi\) for OR nodes. \(\square\)
6.3 Optimality of BestD

Now that we have shown correctness, we show that there exists no better sequence of vertex sets \([D_1, ..., D_n]\) for predicate atom sequence \([P_1, ..., P_n]\) than the one given by our algorithm. This is true regardless of the depth of the predicate tree, and this allows us to reuse BestD even in DEEPFISH.

**Theorem 5.** For a predicate expression \(\phi^*\) and some ordering of predicate atoms \([P_1, ..., P_n]\), the sequence of vertex sets \([D_1, ..., D_n]\) generated by BestD leads to the lowest cost (i.e., \(\sum_{i=1}^{n} C(P_i, D_i)\)) among all sequences of vertex sets.

**Proof.** Let us assume to the contrary that there does exist a better sequence of \([D'_1, ..., D'_n]\). This must mean that there is at least one step \(i\) for which \(C(P_i, D'_i) < C(P_i, D_i)\). Let \(i\) be the index of the first such step. Since \(D'_i\) and \(D_i\) are different, to incur a cheaper cost, \(D'_i\) must be missing at least one vertex \(v\) that is present in \(D_i\). However, we show that if we apply \(P_i\) to the vertex set \(D'_i\) that is missing that \(v\), we are not able to satisfy the given predicate expression \(\phi^*\), leading to a contradiction.

Central to this argument is that before we apply predicate atom \(P_i\), we are unable to distinguish between \(v|_{i=1}\) and \(v|_{i=0}\) for any vertex \(v \in D\).Lemma 5 ensures that for every vertex set \(D \in U_{i-1}\) and every vertex \(v \in D\), either both or neither \(v|_{i=1}\) and \(v|_{i=0}\) must be in \(D\) since \(P_i\) has not yet been applied.

**Lemma 5.** Given a vertex set \(D\) and a predicate atom \(P_i\), if there exists a vertex \(v \in D\) such that \(v|_{i=1}\) and \(v|_{i=0}\) are also in \(D\), then every derivative vertex set of \(D\) which does not have \(P_i\) as a step will either have both or neither \(v|_{i=1}\) and \(v|_{i=0}\) in the vertex set. In other words, \(\forall \psi \in \Psi_{\{0,1\}}\), \(\psi(D) \land \psi(D) \lor (\psi(D) \land \psi(D))\), given that the set operations are on vertex sets that also either have both or neither \(v|_{i=1}\) and \(v|_{i=0}\).

Both \(D_i\) and \(D'_i\) come from \(U_{i-1}\), so they too must conform to this rule. Thus, for every \(v \in D_i \land v \notin D'_i\), it must be that \(v|_{i=1} \in D_i \land v|_{i=0} = D_i\) and \(v|_{i=0} \notin D_i \land v|_{i=0} = D_i\). We can apply this logic to all indices greater than \(i\) as well:

**Corollary 1.** Given a sequence of steps \([R_1, ..., R_n]\), after step \(R_{i-1}\) has been applied, for every vertex set \(D \in U_{i-1}\), if a vertex \(v \in D\), then for \(\Gamma(v, i) = \prod_{j=1}^{i-1}(v_i) \prod_{j=1}^{n}\{0, 1\}\), it must be that \(\Gamma(v, i) \subseteq D\). Similarly, if vertex \(v \notin D\), then \(\Gamma(v, i) \cap D = \emptyset\).

Since only steps up to \(R_{i-1}\) have been applied, predicate atoms \([P_1, ..., P_n]\) cannot have been applied yet. So, there is no way to distinguish the \([i, ..., n]\) elements for all vertex sets in \(U_i\). In other words, if vertex \(v \in D\) for some \(D \in U_{i-1}\), then all other variations of \(v\), in which the \([v_1, ..., v_n]\) elements can be either 0 or 1, must also be in \(D\). The vertex group, \(\Gamma(v, i)\), is the vertex set that constructs this set of variations by taking the Cartesian product between the first \(i - 1\) elements of \(v\) and \(\{0, 1\}\) for every index \(j \geq i\). If \(v \notin D\), then when removing \(v\) from vertex set \(D\), the vertex group \(\Gamma(v, i)\) must have also been removed since we can not differentiate between \(v\) and any vertex in \(\Gamma(v, i)\). Following Corollary 1 if \(v \in D_i \land v \notin D'_i\), then \(\Gamma(v, i) \subseteq D_i\), and \(\Gamma(v, i) \cap D'_i = \emptyset\).

**Lemma 6.** If vertex set \(D_i \in U_{i-1}\) is generated with BestD, for all vertices \(v \in D_i\), there exists a vertex \(u \in \Gamma(v, i)\) such that exactly one of \(u|_{i=1}\) and \(u|_{i=0}\) satisfies \(\phi^*\).

**Lemma 6** tells us that there must exist at least one \(u \in \Gamma(v, i)\) such that only one of \(u|_{i=1}\) or \(u|_{i=0}\) satisfies \(\phi^*\). Since \(\Gamma(v, i) \cap D'_i = \emptyset\), neither can be in \(D'_i\), and based on Lemma 5, all vertex sets derived from \(P_i(D'_i)\) must have either both or neither \(u|_{i=1}\) and \(u|_{i=0}\). This is a contradiction since every vertex set in \(U_n\) will either be missing a vertex which satisfies \(\phi^*\) or have an extra vertex which does not satisfy \(\phi^*\). Thus, the sequence \([D_1, ..., D_n]\) produced by BestD is optimal. \(\square\)

7 Evaluation

We evaluated SHALLOWFISH and DEEPFISH against two predicate evaluation algorithms TDACB and NOOrOPT.

TDACB is the current state-of-the-art and is capable of generating the optimal predicate evaluation plan for general boolean expressions with conjunctions and disjunctions. However, it does so by searching the space of all possible evaluation plans. Although it has several clever optimizations, such as branch-and-
bound and memoization to constrain the overall runtime, it still has a runtime complexity of $O(n3^n)$ where $n$ is the number of predicate atoms.

NoOROpt is our straw man algorithm which does not have any optimizations for disjunctions. It evaluates conjunctions in increasing order of selectivity, but treats disjunctions as completely separate predicate expressions to be evaluated independently from one another. Although this may seem simplistic, this is the actual strategy employed by some real-world systems for disjunctions, e.g., Vertica [17].

We evaluated the algorithms with respect to two metrics:

- **Runtime** – The overall time it takes to generate and execute a predicate evaluation plan. For our evaluation, we measured only the predicate evaluation time; that is, we measured the time until a bitmap of indices which satisfies the predicate expression was returned.

- **Number of Evaluations** – To provide a system-agnostic measure of our algorithms, we also measured the sum total number records the predicate atoms evaluated.

As expected, ShallowFish & DeepFish outperformed both Tdacb and NoOrOpt for all depth-2 predicate expressions. Due to its exponential complexity, Tdacb quickly became unviable and resulted in runtimes orders of magnitude greater than the other algorithms. For predicate expressions of depth 3 or greater, although ShallowFish still performed quite well for many queries (coming close to the optimal), there were several situations in which DeepFish noticeably outperformed ShallowFish, achieving a speedup of up to $2.2 \times$.

### 7.1 Experimental Setup

**Dataset.** We used the Forest dataset [1] that was used in the majority of the experiments evaluating Tdacb [13]. The dataset has 10 quantitative attributes and 2 qualitative attributes of interest. Because we wanted to evaluate our algorithms on more than just 12 predicate atoms, we duplicated the original dataset 12 times and added the attributes of the duplicated datasets as additional attributes of the original dataset (no join), giving us a total of 144 attributes. To avoid having the exact same columns 12 times, we randomly shuffled the records of each duplicate dataset before adding them as extra attributes. Furthermore, since the original dataset only contains 581K records, we replicated it 10 times to for a total of 5.8M records.

**Queries.** Similar to the evaluation for Tdacb [13], we used randomly generated predicate expressions to evaluate our algorithms. Each of our predicate trees had a depth of 2, 3, or 4, and the root was randomly designated as either AND or OR. Each non-leaf node had a randomly chosen number of children between 2 and 5, and each child had some chance to be a leaf node, ensuring that we did not only test on balanced trees. For each of the qualitative attributes, we generated a simple $x < c$ comparison predicate atom where $c$ could one of 9 different constants which made the predicate atom have a selectivity of one of $[0.1, 0.2, ..., 0.9]$. For the two qualitative attributes, we had equality predicate atoms of the form $x = c$ where $c$ is one of the possible values of that attribute (one had 4 possible values and the other had 7). For the experiments with variable-cost predicate atoms, sleep times of 1-10 ns were added per evaluation of a record to emulate the

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For our algorithms, the time it took to generate each plan was on the order of a few milliseconds (less than 0.1% of the time it took to execute the plan), so we do not present the two times separately.

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Figure 1: Runtimes and number of evaluations for uniform-cost predicate atoms in a depth-2 predicate tree.
variable cost. Our queries were simple selection queries on a single table. 500 randomly generated predicate expressions were used for each experiment.

**System.** To evaluate our algorithms, we built a column-oriented database system capable of parsing and executing SQL selection queries with predicates, joins, and group-bys. In addition to simple inequality comparisons, our system is capable of handling predicates with regular expressions and commonly used functions, such as `avg` or `substr`. Generated plans are evaluated in single-threaded environment. Data is stored in columns on disk and intermediate filter results are stored as bitmaps. Our system is coded in Rust and contains around 6500 lines of code in total. All four algorithms were implemented in this system, and all of our experiments were conducted on a server with 64 Intel(R) Xeon(R) CPU E7-4830 @ 2.13GHz processors, 256GB of memory, and a RAID5 configuration of 7200rpm HDDs.

### 7.2 Depth-2 Predicate Expressions

For depth-2 predicate expressions, we first ran an experiment in which the costs of all the predicate atoms were the same. Figure 1 shows the mean runtimes and number of evaluations taken for each number of predicate atoms. In particular, Figure 1a shows the runtimes of all four algorithms with respect to the number of predicate atoms in the predicate expression. As we can see, although Tdacb and the other algorithms have similar runtimes for smaller numbers of predicate atoms, for predicate expressions with more than 12 predicate atoms, SHALLOWFish (and others) start to noticeably outperform Tdacb. By 16 predicate atoms, Tdacb has a runtime that is two orders of magnitude greater than the others, taking more than 100s to plan the query. This is to be expected since the runtime complexity of Tdacb is exponential, whereas it is $O(n \log n)$ for SHALLOWFish and $O(n^2)$ for DEEPFish. To Tdacb’s credit, it does eventually generate the optimal predicate evaluation plan, and the actual execution times of the generated plans were very similar to those of SHALLOWFish & DEEPFish. We can see evidence of this in Figure 1c, which shows that Tdacb had the same number of evaluations as SHALLOWFish & DEEPFish for every query. In Figure 1b, we remove Tdacb so that we can compare SHALLOWFish & DEEPFish with NoOrOpt. Though the difference between these algorithms is not as drastic, we still achieve about 1.5× speedup on average. Finally, note that Figure 1b and Figure 1c have almost the same shape, indicating that the runtimes closely follow the number of total evaluations.

We also ran the same experiment predicate atoms with varying costs as well. The trends for this experiment were very similar to the uniform-cost case. SHALLOWFish & DEEPFish outperformed the other algorithms, and Tdacb’s exponential growth led it to have runtimes that were orders of magnitude greater than the others.

### 7.3 Depth-3+ Predicate Expressions

In addition to the depth-2 experiments, we also evaluated our algorithms on deeper predicate trees (depths 3 and 4). Overall, the trends were similar to the depth-2 case. We present some of the results in Figure 2.

![Figure 2](image-url)

(a) Runtimes for Varying-Cost (b) CDF of speedups (c) CDF of extra number of evaluations

Figure 2: Number of evaluations for depth-3 predicate trees and CDF of the speedup between SHALLOWFish and DEEPFish.
Figure 2a shows the runtimes of the algorithms with respect to the number of varying-cost predicate atoms for depth-3 predicate trees (the results for the depth-4 predicates were very similar). Due to the extremely large runtimes of TDACB for predicate atoms greater than 15 predicate atoms, we did not include it in this experiment. We can see that although DeepFish is the best algorithm, ShallowFish comes very close in terms of performance. To compare the two, recall that DeepFish is a hybrid algorithm which tries both OrderP & BestD (ShallowFish) and OneLookaheadP & BestD and selects the cheaper plan. To compare the performance of the plans generated by OrderP & BestD and OneLookaheadP & BestD, we took the ratio of number of evaluations between these two schemes (i.e., speedup of OneLookaheadP over OrderP) and plot the CDF of this data. Figure 2b shows this plot. The x-axis is the amount of speedup, whereas the y-axis shows the percentage of queries that had at least that amount of speedup. The red dotted line shows the point at which speedup is 1.0. As we can see, for almost 90% of the queries, the plans generated by OrderP & BestD outperformed the plans generated by OneLookaheadP & BestD. However, for the remaining 10%, OneLookaheadP & BestD had results which were better than OrderP & BestD, with speedups reaching up to 2.2×. Regardless of which ordering scheme performs better in which case, DeepFish, as a hybrid algorithm, is able to quickly select the cheaper, better plan and execute it. Finally, we wanted to see how far off the plans generated by ShallowFish and DeepFish were from optimal. As mentioned earlier, in exchange for its exponential runtime, TDACB always generates the optimal plans, so we ran TDACB, ShallowFish, and DeepFish on depth-3 predicate trees with at most 16 predicate atoms. We measured the number of evaluations of ShallowFish and DeepFish compared to TDACB, and Figure 2c shows the CDF of the percentage of extra evaluations done by ShallowFish and DeepFish compared to TDACB. As we can see, ShallowFish and DeepFish actually come very close to the optimal for many queries, with 50% and 60% of the queries having less than 1% difference, respectively. Even in the cases in which ShallowFish and DeepFish performed extra evaluations, we can see that it is never too much. In fact, for 95% of the queries, the plans generated by DeepFish required only 20% more evaluations than the optimal plan. Given that DeepFish is a polynomial-time algorithm, we believe this to be a reasonable tradeoff.

8 Related Work

In this section, we present the works which focus on optimizing disjunctive query predicates. Note that few of these works provide the same formal rigor that we do in our work.

TDACB/Byp. As far as the authors are aware, there is only one line of work which has focused on optimizing query predicates with disjunctions for column stores: Kastrati and Moerkotte’s work on TDACB [13] and Byp [12]. TDACB is the algorithm that we introduced in the evaluation section. It does not require the input to be in CNF/DNF and produces the optimal predicate evaluation plan. Although it has many neat optimizations such as branch-and-bound and memoization, TDACB unfortunately has a runtime complexity of $O(n^3)$. As we showed in Section 7, the planning time can be quite expensive even for as few as 16 predicate atoms. In comparison, ShallowFish is able to find the optimal predicate evaluation plan for predicate trees of depth 2 or less in $O(n \log n)$ complexity and has a practical runtime that is shown to be orders of magnitude better than TDACB. Furthermore, TDACB is a fairly complex algorithm, and the implementation details have a large impact on the overall runtime. ShallowFish in comparison is much simpler and performs well even with minimal optimization. One key advantage of TDACB is that it does not make any independence assumption. ShallowFish, as it described in the paper, depends on OrderP, which makes the independence assumption; thus, ShallowFish also falls prey to the independence assumption. However, since BestD does not make the independence assumption, if OrderP is swapped out for a different ordering algorithm which does not make the independence assumption, then ShallowFish would also be free of the independence assumption. Byp is Kastrati and Moerkotte’s previous work which they improved upon with TDACB. Byp requires its input to be in DNF and searches the entire search space of predicate evaluation plans without restrictions. Thus, Byp is strictly worse than TDACB.

OrderP. In 1977, Hanani introduced OrderP [9], the predicate atom ordering algorithm. Although at the time, Hanani was trying to find the best order of evaluation to short-circuit predicate expressions, that same ordering is optimal for all predicate trees of depth 2 or less. Unfortunately, Hanani assumed a depth-first traversal of predicate trees, and as we have shown, due to determinability, OrderP is no longer guaranteed...
to be the optimal ordering for predicate trees of depth 3 or greater.

**Bypass.** Other than ORDERP, one of the other seminal works on optimizing disjunctions is the bypass technique [14]. When faced with a disjunction, the bypass technique stores the records which do not satisfy a predicate atom, and only those records are processed for the remaining predicate atoms. In the world of column stores, the bypass technique is akin to using the inverse of a predicate atom’s bitmap result. This is one of the fundamental operations allowed in our problem model, and our work can be considered a generalization of the bypass work to column stores.

**Boolean Difference Calculus.** Kemper et al. [15] propose a nice heuristic to ordering predicate atoms based on Boolean Difference Calculus, which measures the importance of each predicate atom. This ordering algorithm is actually quite similar to ONELOOKAHEADP since both measure the effect of evaluating a single predicate atom. However, the heuristic was developed for row stores, so the effect that they measure is different from DEEPFISH.

**CNF/DNF.** The remaining works on conjunction optimization all focus on first trying to convert the predicate expression into CNF/DNF and optimizing the execution from those forms [11] [21] [3] [6] [18]. However, it is well-known that the conversion process into CNF/DNF can result in an exponential number of terms [20]. Thus just transforming the input to be in the correct form can be quite expensive.

**Common Subexpression Elimination.** Chaudhuri et al. [5] present a work on factorizing predicate expressions to take advantage of existing indexes. This work is closely related since it also combines indexes for ANDs and ORs minimize query execution time. However, the work is also largely orthogonal because its emphasis is on reducing the number of reappearing subexpressions. Thus, their algorithm to reduce commonly reappearing predicate atoms may be applied before the execution of our algorithm, since our algorithm depends on uniqueness of predicate atoms for optimality. Other key differences are that Chaudhuri et al. do not consider set minus as a possible index operation, and they restrict the forms of the possible output expressions (e.g., DNF) to shrink their search space.

**Dewey Evaluation.** Fontoura et al. [7] present an algorithm similar to BESTD. However, it does not consider determinability or the bypass technique, so the same record may be processed redundantly several times.

9 Conclusion

In short, the predicate evaluation problem for complex predicates with conjunctions and disjunctions in column stores can be reduced to finding the optimal order to apply predicate atoms and the vertex sets to apply each predicate atom to. Our polynomial-time algorithms, SHALLOWFISH and DEEPFISH, are simple and have either optimal or close to optimal performance We hope that many systems can take advantage of our work and implement these optimizations to improve the performance of predicate evaluation.

References

[1] Forest dataset. [http://kdd.ics.uci.edu/databases/covertype/covertype.data.html](http://kdd.ics.uci.edu/databases/covertype/covertype.data.html).

[2] H. B. Bigelow and W. C. Schroeder. *Fishes of the Gulf of Maine.* Number 592. US Government Printing Office, 1953.

[3] D. Buchfuhrer and C. Umans. The complexity of Boolean formula minimization. *Journal of Computer and System Sciences*, 77(1):142–153, Jan. 2011.

[4] J.-y. Chang and S.-g. Lee. An optimization of disjunctive queries: union-pushdown. In *COMPSAC*, pages 356–361. IEEE Computer Society, 1997.

[5] S. Chaudhuri, P. Ganesan, and S. Sarawagi. Factorizing complex predicates in queries to exploit indexes. In *SIGMOD 2003*, 2003.

[6] P. Ciaccia and M. R. Scalas. Optimization Strategies for Relational Disjunctive Queries. *IEEE Trans. Software Eng.*, 15(10):1217–1235, 1989.

[7] M. Fontoura, S. Sadanandan, J. Shanmugasundaram, S. Vassilvitski, E. Vee, S. Venkatesan, and J. Zien. Efficiently evaluating complex boolean expressions. In *Proceedings of the 2010 international conference on Management of data - SIGMOD ’10*, page 3, Indianapolis, Indiana, USA, 2010. ACM Press.

[8] M. Grund, J. Krger, H. Plattner, A. Zeier, P. Cudre-Mauroux, and S. Madden. HYRISE: a main memory hybrid storage engine. *Proceedings of the VLDB Endowment*, 4(2):105–116, 2010.

[9] M. Z. Hanani. An Optimal Evaluation of Boolean Expressions in an Online Query System. *Commun. ACM*, 20(5):344–347, May 1977.
A Supplementary Proofs

A.1 Extra Notation

Here, we formalize the behavior of our algorithms in the following properties. Based on Algorithm 1, $\text{BestD}$ can be expanded out as follows:

Property 6. Let $\lambda$ be a node in the predicate tree representing $\phi^*$, and let $P_i$ (from predicate atom sequence $P$) be some descendant of $\lambda$. Furthermore, let $c^*$ refer to the node which is both $\lambda$’s child and $P_i$’s ancestor.

If $\lambda$ is an AND node:

$$\text{BestD}(\phi^*, P, i, L_\lambda) = \left( \text{BestD}(\phi^*, P, i, L_\lambda - 1) \cap \bigcap_{\hat{c} \neq c^*} (\Xi[\hat{c}]) \right) \cup \bigcup_{c^- \neq c^*} (\Delta^-_i[c^-])$$

If $\lambda$ is an OR node:

$$\text{BestD}(\phi^*, P, i, L_\lambda) = \text{BestD}(\phi^*, P, i, L_\lambda - 1) \setminus \bigcup_{c^+ \neq c^*} (\Delta^+_i[c^+])$$

Based on Algorithm 2, the values of $\Delta^+$ and $\Delta^-$ are updated as follows:
Property 7. Let λ be a node in the predicate tree representing ϕ*, and let P be the given predicate atom sequence.
If λ is an AND node:

\[ \Delta^+_λ[\lambda] = \text{BestD}(\phi^*, P, T^i_λ(\lambda), L_λ - 1) \cap (\bigcup_{c^+}^{(i)} \Delta^+_c[c^+] ) \]
\[ \Delta^-_λ[\lambda] = \text{BestD}(\phi^*, P, T^i_λ(\lambda), L_λ - 1) \cap (\bigcap_{c^-}^{(i)} \Delta^-_c[c^-] ) \]

If λ is an OR node:

\[ \Delta^+_λ[\lambda] = \text{BestD}(\phi^*, P, T^i_λ(\lambda), L_λ - 1) \cap (\bigcup_{c^+}^{(i)} \Delta^+_c[c^+] ) \]
\[ \Delta^-_λ[\lambda] = \text{BestD}(\phi^*, P, T^i_λ(\lambda), L_λ - 1) \cap (\bigcap_{c^-}^{(i)} \Delta^-_c[c^-] ) \]

A.2 Proof of Theorem 2

We first introduce Lemma 7 which says that for each predicate atom \(P_i\), there must exist some vertex for which only the result of \(P_i\) (ith element of vertex) can determine whether the vertex satisfies the overall predicate expression or not.

Lemma 7. For any predicate atom \(P_i\) from boolean formula \(\phi^*\), there exists a vertex \(v\) such that \(v\rangle_i = 1 \in \psi^*(D) \land v\rangle_i = 0 \notin \psi^*(D)\).

Proof. Assume to the contrary, sequence \([R_1, ..., R_m]\) does not contain any step with \(P_i\). Based on Lemma 7, let \(v\) be a vertex such that \(v\rangle_i = 1 \in \psi^*(D) \land v\rangle_i = 0 \notin \psi^*(D)\). For \([R_1, ..., R_m]\) to be a valid sequence, there must exist a set in the final universe \(U_m\) which contains \(v\rangle_i = 1\) but does not contain \(v\rangle_i = 0\). We show to that contrary that all sets in \(U_m\) either contain both \(v\rangle_i = 1\) and \(v\rangle_i = 0\) or neither \(v\rangle_i = 1\) nor \(v\rangle_i = 0\).

We prove this by induction on the index of the universes generated by our sequence. The base case is: \(U_0 = \{D\}\). By definition, \(D\) must contain both \(v\rangle_i = 1\) and \(v\rangle_i = 0\). Next, for the inductive step, recall from Lemma 6 that if there exists a vertex \(u\) for which both \(u\rangle_i = 1\) and \(u\rangle_i = 0\) are in vertex set \(D\), then any derivative vertex set of \(D\) created by a series of steps without \(P_i\) as a step must either include both \(u\rangle_i = 1\) and \(u\rangle_i = 0\) or neither. Here, we can more formally define derivative as:

Definition 6. For a vertex set \(D\), we say that \(D'\) is a derivative vertex set of \(D\) under operations \(\ast\) if there exists some set formula \(\psi \in \Psi_\ast\) such that \(D' = \psi(D)\).

By the inductive hypothesis, each vertex set in \(U_k\) must either contain both \(v\rangle_{i-1}\) and \(v\rangle_{i-0}\) or neither. If \(P_i\) is the predicate atom for the \(k\)th step, then every vertex set in \(U_{k+1}\) must be a derivative of the sets in \(U_k\) under the set formula space \(\Psi_{\{\cup, \cap \setminus \}, \Lambda}\). Since this formula space does not contain \(P_i\), by Lemma 6 every vertex set in \(U_{k+1}\) must either contain both \(v\rangle_{i-1}\) and \(v\rangle_{i-0}\) or neither. Thus, by induction, every vertex set in \(U_m\) must either contain both \(v\rangle_{i-1}\) and \(v\rangle_{i-0}\) or neither, and a contradiction is reached. So, \(\square\)

Proof of Lemma 7. We define \(\Omega(i)\) to be the sequence of nodes in \(P_i\)’s lineage (e.g., \(P_i\)’s parent and grand-parent) from \(\phi^*\)’s predicate tree, in order of decreasing distance to \(P_i\) (root is first and \(P_i\) is last). Given this, we can prove the lemma by construction with the following algorithm:

1. Start with an empty \(n\)-length vertex \(v = (\_ , \_ , ..., \_ )\).
2. Starting from \(P_i\)’s direct parent, for each ancestor \(a\) in \(\Omega(i)\):
   a. For each predicate atom descendant \(P_j\) of \(a\) where \(j \neq i\), if \(v_j\) is unset, set \(v_j = 1\) if \(a\) is an AND node and \(v_j = 0\) if \(a\) is an OR node.
3. Return the pair \((v\rangle_{i-1}, v\rangle_{i-0})\).
| Set Op | Both in $X$ | Neither in $X$ | Both in $Y$ | Neither in $Y$ |
|--------|-------------|----------------|-------------|----------------|
| $X \cup Y$ | Both | Both | Both | Neither |
| $X \cap Y$ | Both | Neither | Neither | Neither |
| $X \setminus Y$ | Neither | Both | Neither | Neither |
| $Y \setminus X$ | Neither | Neither | Both | Neither |

Table 2: Outcomes of set operations between vertex sets that have either both or neither $v_{i=1}$ and $v_{i=0}$.

To see why $v_{i=1}$ satisfies $\phi^*$ and $v_{i=0}$ does not, let $\Omega_l(i)$ be the $l$th ancestor node in $P_i$’s lineage $\Omega(i)$ (e.g., $\Omega_1(i)$ is root). Let all predicate atoms that are descendants of $\Omega_l(i)$ but not $\Omega_{l+1}(i)$ be called the other predicate atoms. If $\Omega_l(i)$ is an AND node, setting the elements corresponding to the other predicate atoms to 1 ensures that every child other than $\Omega_{l+1}(i)$ will evaluate to 1, thus the entire result of $\Omega(i)$ depends on whether $\Omega_{l+1}(i)$ is 0 or 1. In the case that $\Omega_l(i)$ is an OR node, setting all the elements corresponding to the other predicate atoms to 0 ensures that every other child other than $\Omega_{l+1}(i)$ will evaluate to 0, thus the entire result of $\Omega_l(i)$ depends on whether $\Omega_{l+1}(i)$ is 0 or 1. The above algorithm applies this logic recursively to construct $v_{i=1}$ and $v_{i=0}$, so the sole outcome of $P_i$ determines the outcome of the entire predicate expression.

Lemma 5. Given a vertex set $D$ and a predicate atom $P_i$, if there exists a vertex $v \in D$ such that $v_{i=1}$ and $v_{i=0}$ are also in $D$, then every derivative vertex set of $D$ which does not have $P_i$ as a step will either have both or neither $v_{i=1}$ and $v_{i=0}$ in the vertex set. In other words, let $\psi \in \Psi_{\{\cup, \cap, \lambda, \ldots, P_{i=1}, P_{i=2}, \ldots, P_n\}}$, $v_{i=1} \in \psi(D) \land v_{i=0} \in \psi(D)) \lor (v_{i=1} \notin \psi(D) \land v_{i=0} \notin \psi(D))$, given that the set operations are on vertex sets that also either have both or neither $v_{i=1}$ and $v_{i=0}$.

Proof of Lemma 5. We give a proof by case analysis. Let $X$ be a vertex set which has either both or neither $v_{i=1}$ and $v_{i=0}$. If we apply predicate atom $P_j$ for some $j \neq i$ on to $X$, the result $P_j(X)$ will have both vertices if $v_j = 1$ and neither vertices if $v_j = 0$. Let $Y$ be another vertex set which has either both or neither $v_{i=1}$ and $v_{i=0}$. We show for every set operation between the two vertex sets, the result will also have either both or neither vertices. Table 2 shows the results. Here, “Both” and “Neither” refer to $v_{i=1}$ and $v_{i=0}$. The table elements reflect whether both or neither vertices are present in the results of set operations between $X$ and $Y$. Since $v_{i=1}$ and $v_{i=0}$ are both in $D$ to begin with, any set operation or predicate atom application results in a vertex set with both or neither vertices, any derivative vertex set without $P_i$ as a step cannot have only one of $v_{i=1}$ and $v_{i=0}$.

A.3 Full Proof of Theorem 3

As a natural extension to Property 1, we introduce:

Property 1.B. For any node $\lambda$ and vertex sets $D$ and $Y$:

$$\xi(\lambda, D \cup Y) = \xi(\lambda, D) \cup \xi(\lambda, Y)$$

Since $\xi$ acts like an intersection, we can apply the distributive property of unions over intersections to arrive at the above property.

Proof. Assume to the contrary that the optimal sequence is $n + k$ steps long, for some $k \geq 1$. By the pigeonhole principle, at least one predicate atom must appear twice in the sequence. Let $P$ be the first predicate atom to do so, and let indices $i$ and $j$, for some $i < j$, be the first two steps that $P$ appears in: $R_i = (P, D_i)$ and $R_j = (P, D_j)$. We show by case analysis that we can always construct a new, less costly sequence which does not include both $R_i$ and $R_j$, leading to a contradiction. There are two major cases:

1. $D_i \cap D_j \neq \emptyset$ and 2. $D_i \cap D_j = \emptyset$.

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\[ D_i \cap D_j \neq \emptyset \rightarrow \text{For all sets } A \text{ and } B, \text{ it is true that } A = (A \setminus B) \cup (A \cap B) \text{ (Appendix D). With this equivalence, we can replace step } R_j \text{ with } R'_j = (P, D_j \setminus D_i). \text{ We can still construct every vertex set derived from } P(D_j) \text{ since we can directly calculate:}
\]
\[
P(D_j) = P((D_j \setminus D_i) \cup (D_j \cap D_i)) = P(D_j \setminus D_i) \cup P(D_j \cap D_i) = P(D_j \setminus D_i) \cup (D_j \cap P(D_i))
\]

Property 1.B allows Equation 4 to expand to 5, while Property 1 simplifies Equation 5 to 6. All of Equation 6's expressions can be obtained using just \( R_i \) and \( R'_j \). As long as \( D_i \cap D_j \neq \emptyset \), the vertex set \( D_j \setminus D_i \) contains at least one vertex and is a cheaper step \( C(R'_j) < C(R_j) \). Therefore, replacing \( R_j \) with \( R'_j \) gives us a less costly sequence while maintaining equivalence, leading to a contradiction.

1. \( D_i \cap D_j = \emptyset \rightarrow \) In this case, instead of applying \( P \) separately, we can combine steps \( R_i \) and \( R_j \) and replace them with a single step \( R'_j = (P, D_i \cup D_j) \). Since it is trivially true that for any sets \( A \) and \( B \), \( A = (A \cup B) \cap A \), we can directly calculate \( P(D_i) = P((D_i \cup D_j) \cap D_i) = P(D_i \cup D_j) \cap D_i \), once again simplifying by Property 1. Similarly, we can directly calculate \( P(D_j) = P(D_i \cup D_j) \cap D_j \). It is clear that \( C(R'_j) < C(R_i) + C(R_j) \) due to the extra \( \kappa \) overhead. Thus, by replacing \( R_i \) with \( R'_j \) and removing \( R_j \), we have found a less costly sequence while maintaining equivalence, leading to a contradiction.

**Caveat.** The trick from before does not work in the case that \( D_i \cap D_j = \emptyset \) but \( D_i \) is somehow derived from \( P(D_i) \) (i.e., \( D_j = \psi(P(D_i), \ldots) \)). Calculating \( D_i \cup D_j \) requires \( D_j \), which in turn requires \( P(D_i) \) which is not available before the \( i \)th step. However, this situation cannot arise. More specifically, we show that either \( D_j \) can be derived directly from \( D_i \) rather than from \( P(D_i) \), or \( D_j \) cannot be derived \( P(D_i) \) at all. If \( D_j \) can be derived directly from \( D_i \), \( R_i \) is no longer a requirement for \( D_j \) and we can replace steps \( R_i \) and \( R_j \) with \( R'_i = (P, D_i \cup D_j) \) just as before.

We prove this by induction. In the simplest case, \( D_j \equiv P(D_i) \). However, this is clearly a contradiction since \( D_i \cap D_j = \emptyset \) and \( D_i \cap P(D_i) = P(D_i) \) cannot be empty. Thus, \( D_j \equiv P(D_i) \) is not a valid derivation of \( D_i \) in this case. Next, for the inductive step, recall that any set formula can be constructed by taking an existing set formula and applying some operation to it. Let \( * \) be any set operation or predicate atom application other than \( P \). We show that for any set formula \( \psi \in \Psi_* \) and some arbitrary set \( Y \), the constructed formula \( Y * \psi \) can have two cases. Either 1. we can reproduce \( Y * \psi(P(D_i)) \) with \( \psi'(D_i, Y) \) for some other \( \psi' \in \Psi_* \) which does not include \( P \), or 2. the \( \psi(P(D_i)) \) derived from \( P(D_i) \) must contain a vertex \( v \) from \( D_i \). In the first case, we do not need \( P(D_i) \). In the second case, since \( D_j \) is also derived from \( P(D_i) \), it must also contain some vertex \( u \) which is from \( D_i \), but \( D_i \cap D_j = \emptyset \) was stated as an assumption. Either this way, this is a contradiction, and the situation from the caveat cannot arise. Note that it is sufficient to show this for \( \Psi_* \) since step \( j \) is the first time \( P \) appears after step \( i \), so \( * \) should not include \( P \). More formally:

**Lemma 8.** Let the \( [R_1, \ldots, R_m] \) be the optimal sequence for which predicate atom \( P \) appears at least twice, starting from steps \( i \) and \( j \). Let \( R_i = (P, D_i) \) and \( R_j = (P, D_j) \), and let \( Y \) refer to some arbitrary vertex set. If \( \Psi_* \) is the space of all set formulas computable from the input sets and the operations in \( * \) (where \( * \) can be one of \( \cup, \cap, \setminus \) or some predicate atom application which is not \( P \)), and \( \psi \) is some set formula from this space \( \Psi_* \), it must be that:

\[
(\exists \psi' \in \Psi_*, Y * \psi(P(D_i)) = \psi'(D_i, Y)) \lor (\exists u \in D_i, u \in Y * \psi(P(D_i)))
\]

**Proof of Lemma 8** First, we present an essential lemma that we will be using continuously throughout this proof. Lemma 9 says all vertex sets of the form \( \psi(P(D)) \setminus D \) can be calculated without using predicate atom \( P \).

**Lemma 9.** For all set formulas \( \psi \), predicate atoms \( P \), and vertex sets \( D \), it must be that \( \psi(P(D)) \setminus D = \psi(D) \setminus D \).

---

7 We assume here that every vertex contains at least one record. Even if this is not true we still have to apply the predicate atom to find out, so the step is an unavoidable cost.

8 If \( P(D_i) \) truly was empty, step \( i \) would be spent calculating the null set, which is clearly a waste of a step and cannot be in the optimal sequence.
Next, we reduce the equation in Lemma \[\textbf{8}\] to a form that is easier to reason about. For all sets $X$ and $Y$ the statement $\exists v \in X, v \in Y$ is equivalent to stating $X \cap Y \neq \emptyset$, so we transform our equation to:

\[
\begin{align*}
(\exists \psi' \in \Phi \ast Y \ast \psi(P(D_i))) &= \psi'(D_i, Y) \\
(D_i \cap (Y \ast \psi(P(D_i)))) &= \emptyset
\end{align*}
\tag{7}
\]

We now perform structural induction over the different values of $\ast$ to show that Equation \[\textbf{7}\] remains true regardless of the operation. The operations that $\ast$ can take on are $\{\cup, \cap, \setminus\}$ and any predicate atom other than \(P\). To avoid trivially fulfilling the second condition, we assume $D_i \cap (Y \ast \psi(P(D_i))) = \emptyset$ for most of these cases.

1. $Y \cup \psi(P(D_i)) \rightarrow$ It is trivially true that for any sets $A$ and $B$, $A \subseteq (B \cup A)$, so $\psi(P(D_i)) \subseteq Y \cup \psi(P(D_i))$ for any $Y$. Thus, $D_i \cap \psi(P(D_i)) \neq \emptyset$ must imply $D_i \cap (Y \cup \psi(P(D_i))) \neq \emptyset$, and the condition in Equation \[\textbf{7}\] is trivially satisfied.

2. $Y \cap \psi(P(D_i)) \rightarrow$ For any sets $A$ and $B$, it must be that $A = (A \setminus B) \cup (A \cap B)$ (shown in Appendix \[\textbf{D}\]). We use this equivalence to expand out:

\[
Y \cap \psi(P(D_i)) = (Y \cap \psi(P(D_i))) \setminus D_i \cup ((Y \cap \psi(P(D_i))) \cap D_i)
\]

If $((Y \cap \psi(P(D_i))) \cap D_i) \neq \emptyset$, the first implication of Equation \[\textbf{7}\] would be true (in which case we are done), so we assume otherwise. The second step is a direct application of Lemma \[\textbf{9}\]. Clearly we can construct a $\psi'(D_i, Y) = Y \cap (\psi(D_i) \setminus D_i)$, so the second condition of Equation \[\textbf{7}\] is true.

3. $Y \setminus \psi(P(D_i)) \rightarrow$ Once again we use $A = (A \setminus B) \cup (A \cap B)$:

\[
Y \setminus \psi(P(D_i)) = (Y \setminus \psi(P(D_i))) \setminus D_i \cup ((Y \setminus \psi(P(D_i))) \cap D_i)
\]

Same as before, we assume that $((Y \setminus \psi(P(D_i))) \cap D_i) = \emptyset$ to avoid trivially fulfilling the first condition. Then we use the equivalence, $(A \setminus B) \setminus C = (A \setminus (B \setminus C)) \setminus C$ (Appendix \[\textbf{D}\]), along with Lemma \[\textbf{9}\] to simplify. Finally, we can construct $\psi'(D_i, Y) = (Y \setminus (\psi(D_i) \setminus D_i)) \setminus D_i$, rendering the second implication true.

4. $\psi(P(D_i)) \setminus Y \rightarrow$ We use $A = (A \setminus B) \cup (A \cap B)$ again:

\[
\psi(P(D_i)) \setminus Y = ((\psi(P(D_i)) \setminus Y) \setminus D_i) \cup ((\psi(P(D_i)) \setminus Y) \cap D_i)
\]

In addition to the assumption that $((\psi(P(D_i)) \setminus Y) \cap D_i) = \emptyset$, we use the equivalence that $(A \setminus B) \setminus C = (A \setminus C) \setminus B$ to get the equation into a form where we can use Lemma \[\textbf{9}\]. We construct $\psi'(D_i, Y) = (\psi(D_i) \setminus D_i) \setminus Y$, proving Equation \[\textbf{7}\] true.

5. $P_j(\psi(P(D_i))) \rightarrow$ Even in the case that we apply some predicate atom $P_j$ for $j \neq i$, we use $A = (A \setminus B) \cup (A \cap B)$:

\[
P_j(\psi(P(D_i))) = (P_j(\psi(P(D_i))) \setminus D_i) \cup (P_j(\psi(P(D_i))) \cap D_i)
\]

After we use Property \[\textbf{1}\] to distribute, we can apply Lemma \[\textbf{9}\] to get $\psi'(D_i) = P_j(\psi(D_i) \setminus D_i)$.
**Proof of Lemma** We prove this by structural induction over the complexity of set formula \( \psi \). Here, we define complexity as the number of set operations/predicate atom applications in the formula:

**Definition 7.** We define \( \Psi^k_{\cap, \cup, \setminus, P_1, ..., P_n} \) as the set of all formulas which can be composed using \( k \) operations from \{\cap, \cup, \setminus, P_1, ..., P_n\} (operations may be reused). When we omit the domain \{\cap, \cup, \setminus, P_1, ..., P_n\} a la \( \Psi^k \), it is assumed that the domain consists of the set operations \{\cap, \cup, \setminus\} and any predicate atom application from the sequence \([P_1, ..., P_n]\).

The simplest set formula is the one with no operations (i.e., the identity function). Thus, for the base case, we must show \( P(D) \setminus D = D \setminus D \). By definition, \( P(D) \subseteq D \), so both sides reduce to \( \emptyset \). Next, by the inductive hypothesis, we assume that \( \forall \psi \in \Psi^k \), \( \psi(P(D)) \setminus D = \psi(D) \setminus D \). For the inductive step, we note that any set formula in \( \Psi^{k+1} \) can be constructed by combining two formulas from \( \bigcup_{i=0}^{k} \Psi^i \) with an operation from the domain. Thus, we perform case analysis over the operations in the domain. If \( \psi \in \Psi^{k+1} \), then using \( \psi_1, \psi_2 \in \bigcup_{i=0}^{k} \Psi^i \), we can construct it as:

1. \( \psi = \psi_1 \cap \psi_2 \). By the inductive hypothesis: \( \psi_1(P(D)) \setminus D = \psi_1(D) \setminus D \) and \( \psi_2(P(D)) \setminus D = \psi_2(D) \setminus D \). Since \( \psi(D) = \psi_1(D) \cap \psi_2(D) \)\(^9\) we have:

   \[
   \psi(P(D)) \setminus D = (\psi_1(P(D)) \cap \psi_2(P(D))) \setminus D \\
   = (\psi_1(P(D)) \setminus D) \cap (\psi_2(P(D)) \setminus D) \\
   = (\psi_1(D) \setminus D) \cap (\psi_2(D) \setminus D) \\
   = (\psi_1(D) \cap \psi_2(D)) \setminus D \\
   = \psi(D) \setminus D
   \]

2. \( \psi = \psi_1 \cup \psi_2 \).

   \[
   \psi(P(D)) \setminus D = (\psi_1(P(D)) \cup \psi_2(P(D))) \setminus D \\
   = (\psi_1(P(D)) \setminus D) \cup (\psi_2(P(D)) \setminus D) \\
   = (\psi_1(D) \setminus D) \cup (\psi_2(D) \setminus D) \\
   = (\psi_1(D) \cup \psi_2(D)) \setminus D \\
   = \psi(D) \setminus D
   \]

3. \( \psi = \psi_1 \setminus \psi_2 \).

   \[
   \psi(P(D)) \setminus D = (\psi_1(P(D)) \setminus \psi_2(P(D))) \setminus D \\
   = (\psi_1(P(D)) \setminus D) \setminus \psi_2(P(D)) \\
   = (\psi_1(D) \setminus D) \setminus \psi_2(P(D)) \\
   = \psi_1(D) \setminus (\psi_2(P(D)) \cup D) \\
   = \psi_1(D) \setminus ((\psi_2(P(D)) \setminus D) \cup D) \\
   = \psi_1(D) \setminus ((\psi_2(D) \setminus D) \cup D) \\
   = \psi_1(D) \setminus (\psi_2(D) \cup D) \\
   = (\psi_1(D) \setminus \psi_2(D)) \setminus D \\
   = \psi(D) \setminus D
   \]

\(^9\)Note that this is a little vaguely defined for \( k > 0 \) since each additional operation means a new possible input. However, here we assume that all the other inputs are constant with respect to the original input \( D \) and omit them.

\(^{10}\)In our formulation, we allow set formula \( \psi \) to take \( D \) as an input and not use it.
These steps are a combination of the equivalences \((A \setminus B) \setminus C = (A \setminus C) \setminus B\) and \((A \setminus B) \setminus C = A \setminus (B \cup C)\) from the algebra of sets, and the trivially true \(A \cup B = (A \setminus B) \cup B\). The inductive hypothesis is applied on steps two and give.

\[
\psi(P(D)) \setminus D = P'(\psi_1(P(D))) \setminus D
\]
\[
= P'(\psi_1(D)) \setminus D
\]
\[
= \psi(D) \setminus D
\]

These steps are a combination of Property \(1\) and the inductive hypothesis.

A.4 Proof of Lemma \([\text{I}]\)

Lemma 1. Given a predicate expression \(\phi^*\) whose tree form has a depth of 2 or less and a sequence of predicate atoms \(P = [P_1, ..., P_n]\), let \(\Omega_i(i)\) refer to \(P_i\)'s \(i\)th level ancestor, and let \(\lambda\) be \(\Omega_i(i)\)'s child. For any \(l\), if \(\Omega_i(i)\) is an AND (OR) node and \(\lambda\) is negatively (positively) determinable respectively, then \(\lambda\) must also be complete.

Proof. If \(\lambda\) is a predicate atom node, then the conditions for being positively or negatively determinable are equivalent to the condition for being complete, so the lemma is trivially true in this case.

If \(\lambda\) is an AND node, then \(\Omega_i(i)\) must be an OR node. \(\lambda\) can only be positively determinable if all of its children are positively determinable. However, since the predicate tree has a maximum depth of 2, all of \(\lambda\)'s children must be predicate atom nodes, for which the conditions for being positively determinable are the same as the conditions for being complete. If all of \(\lambda\)'s children are positively determinable, then they must all be complete, thereby causing \(\lambda\) to also be complete. The same reasoning in reverse can be applied for the case in which \(\lambda\) is an OR node and negative determinability. \(\square\)

A.5 Proofs for Correctness of ShallowFish

From here on out, to denote the time step of an iteration, we may denote it interchangeably either as \(\bigcup_{c}^{(i)}\) (same as before), or as: \(\bigcup_{c}^{(i)}\)

Lemma 2. For any non-leaf node \(\lambda\) in the predicate tree for \(\phi^*\), let the sequence \([\eta_1, ..., \eta_J]\) be the descendant predicate atoms of \(\lambda\) ordered according to the given predicate atom sequence \(P = [P_1, ..., P_n]\). For all \(1 \leq a < b \leq J\):

\[
\text{BESTD}(\phi^*, P, \theta(\eta_a), L_\lambda) \supseteq \text{BESTD}(\phi^*, P, \theta(\eta_b), L_\lambda)
\]

Proof of Lemma \([\text{I}]\) To prove Lemma \([\text{I}]\) we first present another lemma which shows that the value of \(\Delta^+\) grows monotonically when intersected with the \(\text{BESTD}\) value of the current step.

Lemma 10. Let \(\lambda\) be a node in the predicate tree of \(\phi^*\), and \(\mathbf{P} = [P_1, ..., P_n]\) be some predicate atom sequence. For any time step \(i\) in which \(P_i\) is a descendant of \(\lambda\), if it is true that:

\[
\text{BESTD}(\phi^*, P, \Psi_P^{(i)}(\lambda), L_\lambda - 1) \supseteq \text{BESTD}(\phi^*, P, i, L_\lambda - 1)
\]

If \(\lambda\) is positively determinable on step \(i + 1\):

\[
\Delta^+_{i+1}[\lambda] \supseteq \Delta^+_{i}[\lambda] \cap \text{BESTD}(\phi^*, P, i, L_\lambda - 1)
\]

If \(\lambda\) is negatively determinable on step \(i + 1\):

\[
\Delta^-_{i+1}[\lambda] \supseteq \Delta^-_{i}[\lambda] \cap \text{BESTD}(\phi^*, P, i, L_\lambda - 1)
\]

where \(\Delta^+_i[\lambda]\) is assumed to evaluate to the empty set if \(\lambda\) is not positively determinable before step \(i + 1\).
We first show that Lemma 2 holds for any consecutive descendants $\eta_j$ and $\eta_{j+1}$. Once the lemma holds for any $\eta_j$ and $\eta_{j+1}$, the monotonicity of $\supseteq$ ensures that the comparisons hold for all $1 \leq a < b \leq J$. To prove that the comparisons hold for $\eta_j$ and $\eta_{j+1}$, we use strong induction on the level of node $\lambda$ (i.e., $L_\lambda$). For the base case, let node $\lambda$ be the node on level $L_\lambda = 1$ (i.e., $\lambda$ is the root node). There are two major cases:

1. $\eta_j$ does not update any of the $\Xi$, $\Delta^+$, or $\Delta^-$ values for any of $\lambda$’s children. In this case, the values which affect $X$ in Algorithm 1 remain the same for both $\eta_j$ and $\eta_{j+1}$. Since $\text{BestD}(\ast, 0) = D$, there is no difference in the original $X/Y$ value either. As a result, $\text{BestD}(\theta(\eta_j), 1) = \text{BestD}(\theta(\eta_{j+1}), 1)$.

2. In the case that:

   (a) $\eta_j$ completes one of $\lambda$’s children, an additional intersection with the newly completed child’s $\Xi$ value makes $X$ smaller subset for $\text{BestAndD}$, and an additional union with the child’s $\Xi$ value makes $X$ larger for $\text{BestOrD}$. In both cases, the returned vertex set on time step $\theta(\eta_{j+1})$ is a subset of the vertex set on time step $\theta(\eta_j)$.

   (b) $\eta_j$ updates the $\Delta^+$ value of one of $\lambda$’s children and $\lambda$ is an OR node, we use Lemma 10 to assert that the $X$ value (when intersected with $\text{BestD}(\theta(\eta_j), L_\lambda)$ is a superset on time step $\theta(\eta_{j+1})$). Thus, the overall return value of $\text{BestD}$ is a subset of the vertex set returned on time step $\theta(\eta_j)$.

   (c) $\eta_j$ updates the $\Delta^-$ value of one of $\lambda$’s children and $\lambda$ is an AND node, we use Lemma 10 to assert that the $X$ value (when intersected with $\text{BestD}(\theta(\eta_j), L_\lambda)$ is a subset of the vertex set returned on time step $\theta(\eta_j)$.

   (d) $\eta_j$ updates $\Delta^+/\Delta^-$ and $\lambda$ is an AND/OR node respectively, the values affecting $X$ remain the same, and $\text{BestD}(\theta(\eta_j), 1) = \text{BestD}(\theta(\eta_{j+1}), 1)$.

For the inductive step, we assume that Lemma 2 holds for all nodes of level $k$ or less. Let $\lambda$ be some node on level $k + 1$ with descendants $H = [\eta_1, \ldots, \eta_J]$ ordered according to $P$, and let $\lambda^{(p)}$ be $\lambda$’s parent on level $k$ with descendants $H^{(p)}$ also ordered according to $P$. Since $\eta_j$ and $\eta_{j+1}$ are descendants of $\lambda$, they must also be descendants of $\lambda^{(p)}$, but they are not required to be neighboring elements in $H^{(p)}$. There may exist a $\eta^{(p)}$ in $H^{(p)}$ such that $\theta(\eta_j) < \theta(\eta^{(p)}) < \theta(\eta_{j+1})$. However, as descendants of $\lambda^{(p)}$, they must obey the inductive hypothesis:

$$\text{BestD}(\theta(\eta_j), k) \supseteq \text{BestD}(\theta(\eta_{j+1}), k)$$

With this in mind, we can apply the same reasoning as we did for the base case. The only difference is that instead of $\text{BestD}(\ast, 0) = D$, now $\text{BestD}(\theta(\eta_j), k) \supseteq \text{BestD}(\theta(\eta_{j+1}), k)$. Thus, the original $X/Y$ values for $\text{BestAndD}/\text{BestOrD}$ respectively are updated so that their values on time step $\theta(\eta_{j+1})$ is a subset of the their values on time step $\theta(\eta_j)$. Since the reasoning from the base case shows that the final outcome of the $\text{BestD}$ becomes a subset even for the same $X/Y$ values, having a smaller starting point should yield the same result.

**Lemma 3.** Let $\lambda$ be a node in the predicate tree of $\phi^*$, and let $P$ be some predicate atom sequence. For any time step $i$:

$$\Delta^+_P(\lambda) \cap \text{BestD}(\phi^*, P, \Upsilon_P(\lambda), L_\lambda - 1) \subseteq \Xi[\lambda]$$

**Proof of Lemma 3.** Lemma 3 is actually a corollary of Lemma 10. For some node $\lambda$, if we let step $i = \Upsilon_P(\lambda)$, then Lemma 10 and Property 3 gives us that:

$$\Xi[\lambda] \supseteq \Delta^+_P(\lambda) \cap \text{BestD}(\Upsilon_P(\lambda), L_\lambda - 1) \quad (8)$$

---

11. In the case that the newly completed child was previously negatively determinable and $\lambda$ is an AND node, we use Property 2 to assert that intersecting with the $\Xi$ value instead of subtracting the $\Delta^-$ does not leave any extraneous vertices. If the child $c$ was positively determinable and $\lambda$ is an OR node, we use a combination of Property 2 and Lemma 10 to assert that $\Xi[\lambda] \supseteq \Delta^+_P(c) \cap \text{BestD}(\theta(\eta_j), L_\lambda)$.
12. Since $L_\lambda - 1 = 0$, the condition for Lemma 10 is trivially satisfied.
13. By the inductive hypothesis, we assumed that Lemma 2 holds for $\lambda$’s parent, thus the condition for Lemma 10 applies.
Let $i'$ be the index of the predicate atom descendant of $\lambda$ just before $P_{\mathcal{T}_P(\lambda)}$ in $\mathcal{P}$; that is, let $i' = \mathcal{Y}_P^{(\mathcal{T}_P(\lambda))}(\lambda)$.

By Lemma 10 we know that $\Delta^+_i[\lambda] \supseteq \Delta^+_i[\lambda] \cap \text{BESTD}(\lambda', L_\lambda - 1)$. Furthermore, for all $i' + 1 \leq j \leq \mathcal{T}_P(\lambda)$, we know that $\Delta^+_i[\lambda] = \Delta^+_i[\lambda] = \Delta^+_i[\lambda]$, since $\Delta^+_i[\lambda]$ cannot be updated if the step’s predicate atom is not a descendant of $\lambda$. Substituting, we get:

$$\Delta^+_i[\lambda] \supseteq \Delta^+_i[\lambda] \cap \text{BESTD}(\lambda', L_\lambda - 1)$$

By Lemma 2 $\text{BESTD}(\mathcal{T}_P(\lambda), L_\lambda - 1) \subseteq \text{BESTD}(\lambda', L_\lambda - 1)$, so we can take the intersection with respect to both sides in the above equation to get:

$$\Delta^+_i[\lambda] \cap \text{BESTD}(\mathcal{T}_P(\lambda), L_\lambda - 1) \supseteq \Delta^+_i[\lambda] \cap \text{BESTD}(\mathcal{T}_P(\lambda), L_\lambda - 1)$$

Combined with Equation 8 this gives us:

$$\Xi[\lambda] \supseteq \Delta^+_i[\lambda] \cap \text{BESTD}(\mathcal{T}_P(\lambda), L_\lambda - 1)$$

This reasoning can be made for $\mathcal{T}_{i'}(\lambda)$ and so on for all $i < \mathcal{T}_P(\lambda)$. If $i > \mathcal{T}_P(\lambda)$, then $\Delta^+_i[\lambda] = \Xi[\lambda]$ by Property 3 and the lemma is trivially true.

Note that as a direct corollary of this proof, we get that:

**Corollary 2.** Let $\lambda$ be a node in the predicate tree representing $\phi^*$. If $P_t$ is a descendant of $\lambda$, and $\lambda$ is positively determinable on step $i + 1$, then for all $j \leq i$:

$$\Delta^+_i[\lambda] \cap \text{BESTD}(i, L_\lambda - 1) \subseteq \Delta^+_i[\lambda]$$

**Lemma 4.** Given that $\lambda$ is an OR node in the predicate tree for $\phi^*$, if $c$ and $c'$ are children of $\lambda$ such that $c \neq c'$, then $\Delta^+_c[c] \cap \Delta^+_c[c'] = \emptyset$ for all steps $i$ and $j$.

**Proof of Lemma 4.** We prove this by induction over $k$, the maximum value that either $i$ or $j$ can take on. As the base case $\Delta^+_k[c] \cap \Delta^+_k[c'] = \emptyset$ is trivially true since no steps have been completed yet and $\Delta^+_k[c] = \emptyset$. Next, assume as the inductive hypothesis that given some step $k$, for all $i \leq k$ and $j \leq k$, $\Delta^+_k[c] \cap \Delta^+_k[c'] = \emptyset$. At this point, there are three cases: 1. $P_k$ is not a descendant of either $c$ or $c'$ 2. $P_k$ is a descendant of $c$. 3. $P_k$ is a descendant of $c'$.

1. If $P_k$ is not a descendant of either $c$ or $c'$, then $\Delta^+_c[c] \cap \Delta^+_c[c']$ must remain the same between steps $k$ and $k + 1$. In other words, $\Delta^+_k[c] = \Delta^+_k[c]$ and $\Delta^+_k[c'] = \Delta^+_k[c']$. Thus, with the inductive hypothesis, we can assert that $\Delta^+_k[c] \cap \Delta^+_k[c'] = \emptyset$ for all $j \leq k + 1$, and $\Delta^+_k[c] \cap \Delta^+_k[c'] = \emptyset$ for all $i \leq k + 1$.

2. If $P_k$ is a descendant of $c$, there is a chance that $\Delta^+_c[c]$ is updated according to the UPDATE function in Algorithm 2. If $\Delta^+_c[c]$ is not updated, we can use the same logic as case (1). However, if it is updated, since $\lambda$ is an OR node, it must be that $\Delta^+_k[c] \subseteq \text{BESTORD}(k, L_c - 1)$. From Algorithm 1 we know that $\Delta^+_k[c]$ is removed as part of calculating $\text{BESTORD}(k, L_c - 1)$, thus $\Delta^+_k[c] \cap \Delta^+_k[c'] = \emptyset$. Furthermore, since $P_k$ is not a descendant of $c'$, $\Delta^+_k[c'] = \Delta^+_k[c']$, therefore $\Delta^+_k[c] \cap \Delta^+_k[c'] = \emptyset$. Similarly, for all $\mathcal{Y}_P^{(k)}(c') < j < k$, we know $\Delta^+_k[c'] = \Delta^+_k[c']$, and $\Delta^+_k[c'] \cap \Delta^+_k[c'] = \emptyset$. Finally, Corollary 2 in conjunction with the previous equivalence tells us that for all $j \leq \mathcal{Y}_P^{(k)}(c')$, $\Delta^+_k[c'] \cap \text{BESTORD}(\mathcal{Y}_P^{(k)}(c'), L_c - 1) \subseteq \Delta^+_k[c']$. Lemma 2 tells us that $\text{BESTD}(\mathcal{Y}_P^{(k)}(c'), L_c - 1) \supseteq \text{BESTORD}(k, L_c - 1)$, so applying this, we get $\Delta^+_k[c'] \cap \text{BESTORD}(k, L_c - 1) \subseteq \Delta^+_k[c'] \cap \text{BESTORD}(k, L_c - 1)$. Given that $\Delta^+_k[c] \subseteq \text{BESTORD}(k, L_c - 1)$, we can simplify:

$$\Delta^+_k[c] \cap \Delta^+_k[c'] = (\Delta^+_k[c] \cap \text{BESTORD}(k, L_c - 1)) \cap \Delta^+_k[c']$$

$$= \Delta^+_k[c] \cap (\text{BESTORD}(k, L_c - 1) \cap \Delta^+_k[c'])$$

$$\subseteq \Delta^+_k[c] \cap \text{BESTORD}(k, L_c - 1) \cap \Delta^+_k[c']$$

$$= \Delta^+_k[c] \cap \Delta^+_k[c']$$

$$\implies \Delta^+_k[c] \cap \Delta^+_k[c'] = \emptyset$$
Since $\Delta^+_{k+1}[c] \cap \Delta^+_k[c'] = \Delta^+_{k+1}[c] \cap \Delta^+_k[c'] = \emptyset$ and $\Delta^+_{k+1}[c] \cap \Delta^+_k[c'] \subseteq \Delta^+_{k+1}[c] \cap \Delta^+_k[c']$, it must be that $\Delta^+_{k+1}[c] \cap \Delta^+_k[c'] = \emptyset$. The same reasoning can be made in reverse for all $i \leq k + 1.$

3. This is the same as Case (2) with simply the $c$ and $c'$ switched around, and the same reasoning applies.

\[\square\]

**Lemma 10.** Let $\lambda$ be a node in the predicate tree of $\phi^*$, and $P = [P_1, ..., P_n]$ be some predicate atom sequence. For any time step $i$ in which $P_i$ is a descendant of $\lambda$, if it is true that:

$$\text{BestD}(\phi^*, P, \Upsilon^{(i)}_P(\lambda), L_\lambda - 1) \supseteq \text{BestD}(\phi^*, P, i, L_\lambda - 1)$$

If $\lambda$ is positively determinable on step $i + 1$:

$$\Delta^+_{i+1}[\lambda] \supseteq \Delta^+_i[\lambda] \cap \text{BestD}(\phi^*, P, i, L_\lambda - 1)$$

If $\lambda$ is negatively determinable on step $i + 1$:

$$\Delta^-_{i+1}[\lambda] \supseteq \Delta^-_i[\lambda] \cap \text{BestD}(\phi^*, P, i, L_\lambda - 1)$$

where $\Delta^+_i[\lambda]$ is assumed to evaluate to the empty set if $\lambda$ is not positively determinable before step $i + 1$.

**Proof of Lemma 10.** We use strong induction on the height of the subtree referred to by $\lambda$. For the base case, let $\lambda = P_i$. $P_i$ will clearly be positively determinable on step $i + 1$, and $\Delta^+_i[\lambda]$ is assumed to evaluate the null set, so we arrive at: $\Delta^+_{i+1}[P_i] \supseteq \emptyset$, which is trivially true. The same is true for the negatively determinable case.

Next, assume $\lambda$ is an AND node. Based on Algorithm 2, we know that:

$$\Delta^+_{i+1}[\lambda] = \bigcap_c \Delta^+_{i+1}[c] \cap \text{BestD}(i, L_\lambda - 1)$$

$$\Delta^+_i[\lambda] = \bigcap_c \Delta^+_i[c] \cap \text{BestD}(\Upsilon^{(i)}_P(\lambda), L_\lambda - 1)$$

Based on the condition stated at the beginning of Lemma 10, $\text{BestD}(\Upsilon^{(i)}_P(\lambda), L_\lambda - 1) \supseteq \text{BestD}(i, L_\lambda - 1)$:

$$\Delta^+_i[\lambda] \cap \text{BestD}(i, L_\lambda - 1) = \bigcap_c \Delta^+_i[c] \cap \text{BestD}(i, L_\lambda - 1)$$

Let $c^*$ be the child of $\lambda$ which has $P_i$ as its descendant. By the inductive hypothesis, $\Delta^+_i[c^*] \cap \text{BestD}(i, L_{c^*} - 1) \subseteq \Delta^+_{i+1}[c^*] \cap \text{BestD}(i, L_{c^*} - 1)$, so if we expand this out we get:

$$\bigcap_{\hat{c}} \Xi[\hat{c}] \cap \Delta^+_i[c^*] \cap \text{BestD}(i, L_\lambda - 1) \subseteq \bigcap_{\hat{c}} \Xi[\hat{c}] \cap \Delta^+_{i+1}[c^*] \cap \text{BestD}(i, L_\lambda - 1)$$

Property 3 tells us that $\Xi[\hat{c}] = \Delta^+_i[\hat{c}]$ for all $i$ in which $\hat{c}$ is complete. So, we can replace with:

$$\bigcap_{\hat{c}} \Delta^+_i[\hat{c}] \cap \Delta^+_i[c^*] \cap \text{BestD}(i, L_\lambda - 1) \subseteq \bigcap_{\hat{c}} \Delta^+_i[\hat{c}] \cap \Delta^+_{i+1}[c^*] \cap \text{BestD}(i, L_\lambda - 1)$$

If we take the intersection of both sides with respect to $\bigcap_{c \neq c^*} \Delta^+_i[c]$, we get:

$$\bigcap_{c \neq c^*} \Delta^+_i[c] \cap \Delta^+_i[c^*] \cap \text{BestD}(i, L_\lambda - 1) \subseteq \bigcap_{c \neq c^*} \Delta^+_i[c] \cap \Delta^+_{i+1}[c^*] \cap \text{BestD}(i, L_\lambda - 1)$$

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Since $P_i$ is $c^+$’s descendant and predicate atoms are unique, for every other child $\Delta^+_i[c] = \Delta^+_{i+1}[c]$ since the values are not updated. Substituting, we get:

$$\bigcap_c \Delta^+_i[c] \cap \text{BESTD}(i, L_x - 1) \subseteq \bigcap_c \Delta^+_{i+1}[c] \cap \text{BESTD}(i, L_x - 1)$$
$$\bigcap_c \Delta^+_i[c] \cap \text{BESTD}(i, L_x - 1) \subseteq \Delta^+_{i+1}[\lambda]$$
$$\Delta^+_i[\lambda] \cap \text{BESTD}(i, L_x - 1) \subseteq \Delta^+_{i+1}[\lambda]$$

For the case when $\lambda$ is an OR node based on Algorithm 2, we know that:

$$\Delta^+_{i+1}[\lambda] = \bigcup_{c^+}^{(i+1)} \Delta^+_i[c^+] \cap \text{BESTD}(i, L_x - 1)$$
$$\Delta^+_i[\lambda] = \bigcup_{c^+}^{(i)} \Delta^+_i[c^+] \cap \text{BESTD}(\Psi_p^{(i)}(\lambda), L_x - 1)$$

Based on the premise of the lemma, this means:

$$\Delta^+_i[\lambda] \cap \text{BESTD}(i, L_x - 1) = \bigcup_{c^+}^{(i)} \Delta^+_i[c^+] \cap \text{BESTD}(i, L_x - 1)$$

If $c^{++}$ is $P_i$’s ancestor, then for all other $c^+ \neq c^{++}$, the $\Delta^+_i[c^+]$ values stay the same since they are not updated. For $c^{++}$, the inductive hypothesis applies: $\Delta^+_i[c^{++}] \supseteq \Delta^+_i[c^{++}] \cap \text{BESTD}(i, L_{c^{++}} - 1)$. If expand out the BESTD, we get:

$$\Delta^+_{i+1}[c^{++}] \cap \text{BESTD}(i, L_x - 1) \supseteq \bigcup_{c^+ \neq c^{++}} \Delta^+_i[c^+] \cap \text{BESTD}(i, L_x - 1) \cap \bigcup_{c^+ \neq c^{++}} \Delta^+_i[c^+]$$

By the principles that $(A \cap B) \setminus C = (A \cap B) \setminus (C \cap B)$ and $(A \setminus B) \cup B = A \cup B$, if we take the union of both sides with respect to $\bigcup_{c^+ \neq c^{++}} \Delta^+_i[c^+] \cap \text{BESTD}(i, L_x - 1)$, we get:

$$\bigcup_{c^+}^{(i+1)} \Delta^+_i[c^+] \cap \text{BESTD}(i, L_x - 1) \supseteq \bigcup_{c^+}^{(i)} \Delta^+_i[c^+] \cap \text{BESTD}(i, L_x - 1)$$
$$\Delta^+_{i+1}[\lambda] \supseteq \bigcup_{c^+}^{(i)} \Delta^+_i[\lambda] \cap \text{BESTD}(i, L_x - 1)$$

The reasoning is the same but reversed for the negatively determinable case and $\Delta^-$. \hfill \square

### A.6 Proofs for Optimality of $\text{BestD}$

**Proof of Lemma 6** Recall that $\lambda^*$ is the root node of the predicate tree for $\phi^*$, and $\lambda[u]$ evaluates the subtree $\lambda$ for vertex $u$. To prove Lemma 6, we now introduce the concept of static and dynamic nodes. Given a vertex set, static nodes are the nodes which evaluate to the same constant value for every vertex in that set, while dynamic nodes may evaluate to different values for different vertices in that set.

**Definition 8.** We call a predicate tree node $\lambda$ static with respect to vertex set $D$, if $\forall v \in D, \lambda[v] = 0$, or if $\forall v \in D, \lambda[v] = 1$. Otherwise, we say that the node is dynamic with respect to $D$.

**Property 8.** A non-leaf node can be static with respect to $D$ in the following five situations: 1. All its children are static. 2. The node is an AND node and one of its children has a static value of 0. 3. The node is an OR node and one of its child has a static value of 1. 4. The node is an AND node and $\forall v \in D$, at least one of its dynamic children must evaluate to 0. 5. The node is an OR node and $\forall v \in D$ at least one of its dynamic children must evaluate to 1.
Property 8 can be verified by iterating over the three states a child of λ can have: 1. static value of 1 2. static value of 0 3. dynamic, and exhaustively checking the situations which result in λ being static. Only the case of all children being static can lead to an AND node evaluating to a static value of 1 or an OR node evaluating to a static value of 0.

In addition, we introduce Lemma 11 and 12 which detail the behavior of the nodes in the predicate tree when evaluated against the vertex sets produced by BestD.

Lemma 11. For predicate expression \( \phi^\ast \) and sequence of predicate atoms \([P_1, ..., P_n]\), let \( D_i \) be the vertex set generated by BestD on the \( i \)th step, and let \( \Omega_i(i) \) refer to the \( i \)th level ancestor of \( P_i \). Then, for all \( v \in D_i \), there must exist a pair of vertices \( u \in \Gamma(v, i) \) and \( u' \in \Gamma(v, i) \) such that every one of \( \Omega_i(i) \)'s dynamic children (with respect to \( \Gamma(v, i) \)) has a value of 1 when evaluated against \( u \) and a value of 0 when evaluated against \( u' \).

Lemma 12. For predicate expression \( \phi^\ast \) and sequence of predicate atoms \([P_1, ..., P_n]\), let \( D_i \) be the vertex set generated by BestD on the \( i \)th step, and let \( \Omega_i(i) \) refer to the \( i \)th level ancestor of \( P_i \). Then, for each incomplete child \( \lambda \) of \( \Omega_i(i) \) for which \( \lambda \neq \Omega_{i+1}(i) \) and for all \( v \in D_i \), if \( \lambda \) is an AND/OR node, it cannot have a static value of 1/0 with respect to \( \Gamma(v, i) \) respectively.

Assume to the contrary of Lemma 6 that there exists a vertex \( v \in D_i \), such that \( \forall u \in \Gamma(v, i), \lambda^\ast[u]|_{i=1} = \lambda^\ast[u]|_{i=0} \). That is, \( \lambda^\ast \) is static with respect to \( \Gamma(v, i) \). By definition, predicate atom \( P_i \) must be dynamic with respect to \( \Gamma(v, i) \). Therefore, somewhere along \( P_i \)'s lineage, the nodes change from being dynamic to \( \Gamma(v, i) \) to being static to \( \Gamma(v, i) \). Let \( \Omega_i(i) \) be the \( i \)-th level ancestor in \( P_i \)'s lineage such that \( \Omega_i(i) \) is static to \( \Gamma(v, i) \) while \( \Omega_{i+1}(i) \) is dynamic. We show that the conditions for \( \Omega_i(i) \) do not fit any of the situations stated in Property 8 despite being static, leading to a contradiction. Already, we can see that case (1) cannot be true, since we assumed child \( \Omega_{i+1}(i) \) is dynamic.

Let us examine the situation in which \( \Omega_i(i) \) is an AND node. Either case (2) or (4) must hold. The BestANDD function prunes the \( D_i \) until it is the intersection of vertex sets that satisfy \( \Omega_i(i) \)'s completed children. This means that all completed children have a static value of 1 with respect to \( D_i \). Corollary 1 tells us that \( \Gamma(v, i) \subseteq D_i \), so all completed children must have a static value of 1 with respect to \( \Gamma(v, i) \) as well. According to Lemma 12 any incomplete children cannot have a static value of 0, thus removing the possibility of situation (2). This leaves situation (4) with the dynamic children. However, Lemma 11 states that there must exist \( u \in \Gamma(v, i) \) such that each dynamic child evaluates to 1, meaning situation (4) cannot hold either.

The same reasoning can be applied in reverse for the OR case. Therefore, none of the situations can hold and \( \Omega_i(i) \) cannot be static leading to a contradiction.

Proof of Lemma 12. We do a proof by construction. For any \( v \in D_i \), we can construct a vertex \( u \) such that \( u_j = v_j \) for all \( 1 \leq j < i \), and \( u_k = 1 \) for all \( i < k \leq n \). We claim that every one of \( \Omega_i(i) \)'s dynamic children (with respect to \( \Gamma(v, i) \)) must have a value of 1 when evaluated against \( u \). Since there are no negations in our predicate tree, our predicate expression \( \phi^\ast \) is a monotonically increasing function; thus, if for child \( c \), \( c[u] \) evaluates to 0, then \( \forall w \in \Gamma(v, i), c[w] = 0 \). Therefore, the child \( c \) cannot have been dynamic with respect to \( \Gamma(v, i) \) to begin with. The same reasoning can be applied with \( u' \) constructed such that \( u'_j = v_j \) for all \( 1 \leq j < i \) and \( u'_k = 0 \) for all \( i < k \leq n \), and \( c[u'] = 1 \).

Proof of Lemma 13. For this proof, we introduce Lemma 13 which shows the effect of determinability on node \( \lambda \)'s value, and Lemma 14 which gives us the analytical form of \( \Delta^+_i[\lambda]/\Delta^-_i[\lambda] \) on step \( i \).

Lemma 13. Let \( \lambda \) be a node in the predicate tree of \( \phi^\ast \) and \( v \) be some arbitrary vertex. Given a sequence of predicate atoms \([P_1, ..., P_n]\), on step \( i \), if \( \lambda \) is not positively determinable, then \( \exists u \in \Gamma(v, i), \lambda[u] = 0 \). If on step \( i \), \( \lambda \) is not negatively determinable, then \( \exists u \in \Gamma(v, i), \lambda[u] = 1 \).

Lemma 14. Let \( \lambda \) be a node in the predicate tree representing \( \phi^\ast \). Given a sequence of predicate atoms \( P = [P_1, ..., P_n] \), on step \( i \), if \( \lambda \) is positively determinable, then:

\[
\Delta^+_i[\lambda] = \{ v \in \text{BestD}(\phi^\ast, P, \Upsilon^+_P(i, \lambda), L_\lambda - 1) \mid \forall u \in \Gamma(v, i), \lambda[u] = 1 \}
\]

On step \( i \), if \( \lambda \) is negatively determinable, then:

\[
\Delta^-_i[\lambda] = \{ v \in \text{BestD}(\phi^\ast, P, \Upsilon^-_P(i, \lambda), L_\lambda - 1) \mid \forall u \in \Gamma(v, i), \lambda[u] = 0 \}
\]
Imagine first that $\lambda$ is an incomplete AND node, and $\lambda$’s parent $\Omega(i)$ is an OR node. Lemma 13 tells us that if $\lambda$ is not positively determinable, then there exists a $u \in \Gamma(v, i)$ for which $\lambda[u] = 0$, trivially satisfying the claim that $\lambda$ cannot have a static value of 1. So, let us assume that $\lambda$ is positively determinable. According to Algorithm 2, part of calculating $\text{BestD}(i, \Lambda - 1)$ is removing $\Delta_\lambda^+\lambda$ from $\text{BestD}(i, \Lambda - 2)$. More specifically, $\text{BestD}(i, \Lambda - 1) \subseteq \text{BestD}(i, \Lambda - 2 \setminus \Delta_\lambda^+\lambda)$. Property 4 tells us that $D_i \subseteq \text{BestD}(i, \Lambda - 1)$. Then, it must be that $D_i \cap \Delta_\lambda^+\lambda = \emptyset$. According to Lemma 14, the value of $\Delta_\lambda^+\lambda$ is $\{v \in \text{BestD}(\Gamma^\lambda\lambda \lambda, \Lambda - 1) | \forall u \in \Gamma(v, i), \lambda[u] = 1\}$. Lemma 2 tells us that $\text{BestD}(\Gamma^\lambda\lambda \lambda, \Lambda - 1) \supseteq \text{BestD}(i, \Lambda - 1)$, and Property 4 tells us that $\text{BestD}(i, \Lambda - 1) \supseteq D_i$. Since $D_i \cap \Delta_\lambda^+\lambda = \emptyset$, this implies $D_i \cap \{v \in D_i | \forall u \in \Gamma(v, i), \lambda[u] = 1\} = \emptyset$. In other words, $\lambda$ does not have a static value of 1 with respect to $\Gamma(v, i)$ for any $v \in D_i$. The same reasoning in reverse can be applied to when $\lambda$ is an OR node and negative determinability.

\[ \square \]

Proof of Lemma 13 We prove the contrapositive. First, if $\forall u \in \Gamma(v, i), \lambda[u] = 1$, then node $\lambda$ must be positively determinable. We prove this by strong induction based on the height of the subtree referred to by $\lambda$. For the base case $\lambda$ refers to a predicate atom, say $P_j$. Then, the only way for $\forall u \in \Gamma(v, i), \lambda[u] = 1$ is for $j < i$, otherwise $\Gamma(v, i)$ will include both 0 and 1 for the $j$th element if $j \geq i$. If $j < i$, then $P_j$ is positively determinable by definition.

Now let $\lambda$ be an AND node. The condition $\forall u \in \Gamma(v, i), \lambda[u] = 1$ is equivalent to $\forall c, \forall u \in \Gamma(v, i), c[u] = 1$. By the inductive hypothesis, for each child $c$, if $\forall u \in \Gamma(v, i), c[u] = 1$, the child must be positively determinable. Since all of $\lambda$’s children are positively determinable, $\lambda$ must be positively determinable as well.

If $\lambda$ is an OR node, then the condition $\forall u \in \Gamma(v, i), \lambda[u] = 1$ is equivalent to $\forall u \in \Gamma(v, i), \exists c, c[u] = 1$. However, this condition this implies $\exists u, \forall u \in \Gamma(v, i), c[u] = 1$. Why? Assume to the contrary that $\forall u \in \Gamma(v, i), \lambda[u] = 1$ and $c, \exists u \in \Gamma(v, i), c[u] = 0$. For each child $c$ of $\lambda$, there are three possibilities: $c$ is static with a value of 1 with respect to $\Gamma(v, i)$, $c$ is static with a value of 0 with respect to $\Gamma(v, i)$, and $c$ is dynamic. In the first case, a contradiction is already reached, so let us assume there are no children of this case. The second case is valid, but not all of $\lambda$’s children can be of the second case because then $\lambda[u] = 0$ for all $u \in \Gamma(v, i)$. Thus, there must be at least one child of the third case. Consider the vertex $u$ which is constructed with $u_j = v_j$ for all $j < i$ and $u_j = 0$ for all $j \geq i$. $u$ is certainly in $\Gamma(v, i)$, thus the assumption tells us that $\lambda[u] = 1$. This means there must exist a child $c$ such that $c[u] = 1$; specifically, this child $c$ has to be of the third case. For any child of the third case $c$, let $\{j_1, ..., j_H\}$ refer to the indices of the incomplete predicate atom descendants of $c$ (i.e., $j_h \geq i$). Note that there must be at least one incomplete predicate atom, otherwise $c$ would be static with respect to $\Gamma(v, i))$. Since it makes sense that evaluation child $c$ is a function of its dynamic predicate atom descendants, $c[u] = F(u_{j_1}, ..., u_{j_H})$ for some function $F$. However, if $c[u] = F(0, ..., 0) = 1$, then any variation $u'$ of $u$ which changes some subset of $\{u_{j_1}, ..., u_{j_H}\}$ to be 1 must also evaluate to $c[u'] = 1$, since there are no negations in our tree. Since $c$ does not depend on any other elements outside of those indexed by $\{j_1, ..., j_H\}$, this means that $\forall u \in \Gamma(v, i), c[u] = 1$. However, this is a contradiction to the statement that $\forall c, \exists u \in \Gamma(v, i), c[u] = 0$. Thus, $\forall u \in \Gamma(v, i), \lambda[u] = 1$ implies $\exists c, \forall u \in \Gamma(v, i), c[u] = 1$. Continuing on, if $\exists c, \forall u \in \Gamma(v, i), c[u] = 1$, then by the inductive hypothesis, child $c$ must be positively determinable. Based on the definition, if child $c$ is positively determinable, then $\lambda$ must also be positively determinable.

The same reasoning in reverse can be made for the case when $\lambda$ is not negatively determinable.

\[ \square \]

Proof of Lemma 14 Proof by strong induction on the height of the subtree referred to by node $\lambda$. The base case is when $\lambda = P_j$ for some $j < i$. Note that $P_j$ can only be positively determinable on step i if $j < i$. $P_j$ must be positively determinable since $j < i$ and we can calculate $\Delta_{j+1}^+P_j = \{v \in \text{BestD}(j, \Lambda - 1) | P_j[v] = 1\}$. For complete predicate atoms, the conditions $P_j[v] = 1$ and $\forall u \in \Gamma(v, i), P_j[u] = 1$ are the same for any vertex $v$ since it just means $v_j = u_j = 1$. Therefore, $\Delta_{j+1}^+P_j = \{v \in \text{BestD}(j, \Lambda - 1) | \forall u \in \Gamma(v, i), P_j[u] = 1\}$. Since predicate atom nodes become complete after the predicate atom has become evaluated, it must be that if predicate atom node $P$ is complete after step $j$, then $\Delta_{j+1}^+P = \Delta_{j+1}^+P$ for all $i > j$.

$\lambda$ is AND and determ+ If $\lambda$ is an AND node and positively determinable, then all of $\lambda$’s children must also be positively determinable by definition. By the inductive hypothesis, we know that for each child
c of λ, it must be that $\Delta^+_t[c] = \{v \in \text{BESTD}(\Upsilon^{(i)}_P(c), L_c - 1) \mid \forall u \in \Gamma(v, i), c[u] = 1\}$. Line 24 of Algorithm 2 tells us that $\Delta^+_t[\lambda] = \bigcap_c \Delta^+_t[c] \cap \text{BESTD}(\Upsilon^{(i)}_P(\lambda), L_\lambda - 1)$. If we substitute using the inductive hypothesis, we get:

$$\Delta^+_t[\lambda] = \bigcap_c \{v \in \text{BESTD}(\Upsilon^{(i)}_P(c), L_c - 1) \mid \forall u \in \Gamma(v, i), c[u] = 1\} \cap \text{BESTD}(\Upsilon^{(i)}_P(\lambda), L_\lambda - 1)$$

Line 10 of Algorithm 1 shows us that if $j = \Upsilon^{(i)}_P(c)$, then $\text{BESTD}(j, L_\lambda) = \bigcap_{c} \Xi[c] \cap \text{BESTD}(j, L_\lambda - 1) \setminus (\bigcup_{c^- \neq c^-} \Delta^+_j[c^-])$ where $c$ is a completed child of $\lambda$ and $c^-$ is a negatively determinable child of $\lambda$ at time step $j$, and $c^{-*}$ is $P_j$'s ancestor. Note that $\Xi$ values are only ever assigned once and immutable, and have the same value regardless of time as long as the node is complete. Furthermore, by Property 3, for any complete child $\hat{c}$, $\Delta^+_t[\hat{c}] = \Xi[\hat{c}]$ for all $i$ in which $\hat{c}$ is complete. Using the distributive property of intersections and Lemma 2 again, we get:

$$\Delta^+_t[\lambda] = \bigcap_c \{v \in \text{BESTD}(\Upsilon^{(i)}_P(c), L_c - 1) \mid \forall u \in \Gamma(v, i), c[u] = 1\} \cap \bigcap_{\hat{c}} \Xi[\hat{c}] \cap \text{BESTD}(\Upsilon^{(i)}_P(\lambda), L_\lambda - 1)$$

$$\setminus \left( \bigcup_{c^- \neq c^-} \Delta^-_{\Upsilon^{(i)}_P(c)}[c^-] \right)$$

Since $\Upsilon^{(i)}_P(c) \leq \Upsilon^{(i)}_P(\lambda)$ for any child $c$ of $\lambda$, Lemma 2 tells us that $\text{BESTD}(\Upsilon^{(i)}_P(c), L_\lambda - 1) \supseteq \text{BESTD}(\Upsilon^{(i)}_P(\lambda), L_\lambda - 1)$. Using the distributive property of intersections again:

$$\Delta^+_t[\lambda] = \bigcap_c \{v \in \text{BESTD}(\Upsilon^{(i)}_P(\lambda), L_\lambda - 1) \mid \forall u \in \Gamma(v, i), c[u] = 1\} \cap \bigcap_{\hat{c}} \Xi[\hat{c}] \cap \text{BESTD}(\Upsilon^{(i)}_P(\lambda), L_\lambda - 1)$$

$$\setminus \left( \bigcup_{c^- \neq c^-} \Delta^-_{\Upsilon^{(i)}_P(c)}[c^-] \right)$$

At this point, we note that $\Delta^-[c^-]$ is the set of vertices for which $c^-$ is guaranteed to evaluate to 0, while $\forall u \in \Gamma(v, i), c[u] = 1$ ensures that every vertex $v$ in that set, $c[v] = 1$ since $v \in \Gamma(v, i)$ by definition. Since the intersection is over all children, we know the $\Delta^-$ vertices must be mutually exclusive to $\bigcap_c \{v \in \text{BESTD}(\Upsilon^{(i)}_P(\lambda), L_\lambda - 1) \mid \forall u \in \Gamma(v, i), c[u] = 1\}$, and we can get rid of them:

$$\Delta^+_t[\lambda] = \bigcap_c \{v \in \text{BESTD}(\Upsilon^{(i)}_P(\lambda), L_\lambda - 1) \mid \forall u \in \Gamma(v, i), c[u] = 1\} \cap \left( \bigcap_{\hat{c}} \Xi[\hat{c}] \cap \text{BESTD}(\Upsilon^{(i)}_P(\lambda), L_\lambda - 1) \right)$$

The condition $\bigwedge_c c[u] = 1$ is the same as $\lambda[u] = 1$, so this can be reduced further to:

$$\Delta^+_t[\lambda] = \{v \in \text{BESTD}(\Upsilon^{(i)}_P(\lambda), L_\lambda - 1) \mid \forall u \in \Gamma(v, i), \lambda[u] = 1\} \cap \left( \bigcap_{\hat{c}} \Xi[\hat{c}] \cap \text{BESTD}(\Upsilon^{(i)}_P(\lambda), L_\lambda - 1) \right)$$

The set of complete children can only ever grow. Finally, there must exist a child $c$ of $\lambda$ such that $\Upsilon^{(i)}_P(c) = \Upsilon^{(i)}_P(\lambda)$.

$$\Delta^+_t[\lambda] = \{v \in \text{BESTD}(\Upsilon^{(i)}_P(\lambda), L_\lambda - 1) \mid \forall u \in \Gamma(v, i), \lambda[u] = 1\} \cap \left( \bigcap_{\hat{c}} \Xi[\hat{c}] \cap \text{BESTD}(\Upsilon^{(i)}_P(\lambda), L_\lambda - 1) \right)$$
By Property $[3]$ $\Xi[\lambda] = \Delta_i^+[\lambda]$ for all $i$ in which $\lambda$ is complete. We can apply the inductive hypothesis again to get:

$$\Delta_i^+[\lambda] = \{v \in \text{BestD}(\Upsilon_\mathcal{P}^{(i)} (\lambda), L_\lambda - 1) \mid \forall u \in \Gamma(v, i), \lambda[u] = 1\} \cap \left( \bigcap_{\hat{c}} \{v \in \text{BestD}(\Upsilon_\mathcal{P}^{(i)} (\hat{c}), L_{\hat{c}} - 1) \mid \forall u \in \Gamma(v, i), \hat{c}[u] = 1\} \right)$$

We can expand $\text{BestD}$ again:

$$\Delta_i^+[\lambda] = \{v \in \text{BestD}(\Upsilon_\mathcal{P}^{(i)} (\lambda), L_\lambda - 1) \mid \forall u \in \Gamma(v, i), \lambda[u] = 1\} \cap \left( \bigcap_{\hat{c}} \{v \in \text{BestD}(\Upsilon_\mathcal{P}^{(i)} (\hat{c}), L_{\lambda} - 1) \mid \forall u \in \Gamma(v, i), \hat{c}[u] = 1\} \right)$$

$$\left( \left( \bigcap_{\hat{c}} \Xi[\hat{c}] \setminus \bigcup_{\hat{c} \neq \hat{c}'} \Delta_i^{-\hat{c}'}[\hat{c}'] \right) \right)$$

We can continue to recursively expand the $\Xi$ and $\text{BestD}$ values, and remove the $\Delta_i^-$ values until in the end after applying Lemma $[2]$ and getting rid of duplicates, we are left with:

$$\Delta_i^+[\lambda] = \{v \in \text{BestD}(\Upsilon_\mathcal{P}^{(i)} (\lambda), L_\lambda - 1) \mid \forall u \in \Gamma(v, i), \lambda[u] = 1\} \cap \left( \bigcap_{\hat{c}} \{v \in \text{BestD}(\Upsilon_\mathcal{P}^{(i)} (\hat{c}), L_{\lambda} - 1) \mid \forall u \in \Gamma(v, i), \hat{c}[u] = 1\} \right)$$

By Lemma $[2]$, $\text{BestD}(\Upsilon_\mathcal{P}^{(i)} (\hat{c}), L_\lambda - 1) \supseteq \text{BestD}(\Upsilon_\mathcal{P}^{(i)} (\lambda), L_\lambda - 1)$ for any complete child $\hat{c}$ and since $\lambda[u] = \bigwedge c[u]$, it is a stricter constraint than $\bigwedge c[u]$, thus this reduces to:

$$\Delta_i^+[\lambda] = \{v \in \text{BestD}(\Upsilon_\mathcal{P}^{(i)} (\lambda), L_\lambda - 1) \mid \forall u \in \Gamma(v, i), \lambda[u] = 1\}$$

$\lambda$ is OR and $\text{determin}^+$. For OR nodes, we present Lemma $[13]$ which shows the effect of the children’s positive determinability on the parent.

**Lemma 15.** Let node $\lambda$ be an OR node in the predicate tree representing $\phi^*$, and let $c^+$ refer to a positively determinable child of $\lambda$. Given a sequence of predicate atoms $\mathcal{P} = \{P_1, ..., P_n\}$, if the following is true for each positively determinable child of $\lambda$ on step $i$:

$$\Delta_i^+[c^+] = \{v \in \text{BestD}(\phi^*, \mathcal{P}, \Upsilon_\mathcal{P}^{(i)} (c^+), L_{c^+} - 1) \mid \forall u \in \Gamma(v, i), c^+[u] = 1\}$$

Then, the following equivalence holds for $\lambda$:

$$\Delta_i^+\lambda = \{v \in \text{BestD}(\phi^*, \mathcal{P}, \Upsilon_\mathcal{P}^{(i)} (\lambda), L_\lambda - 1) \mid \forall u \in \Gamma(v, i), \lambda[u] = 1\}$$

Similarly if $\lambda$ is an AND node, and the following is true for each negatively determinable child $c^-$ of $\lambda$:

$$\Delta_i^-\lambda = \{v \in \text{BestD}(\phi^*, \mathcal{P}, \Upsilon_\mathcal{P}^{(i)} (c^-), L_{c^-} - 1) \mid \forall u \in \Gamma(v, i), c^-[u] = 0\}$$

Then, the following equivalence holds for $\lambda$:

$$\Delta_i^-\lambda = \{v \in \text{BestD}(\phi^*, \mathcal{P}, \Upsilon_\mathcal{P}^{(i)} (\lambda), L_\lambda - 1) \mid \forall u \in \Gamma(v, i), \lambda[u] = 0\}$$
If λ is an OR node, then we know from Line 26 that: $\Delta^+_i[\lambda] = \bigcup_{c^+} \Delta^+_{i+1}[c^+] \cap \text{BESTD}(\Upsilon^{(i)}_P(\lambda), L_\lambda - 1)$. For each positively determinable child $c^+$, the inductive hypothesis applies, so we can apply Lemma 15, substitute to get:

$$\Delta^+_i[\lambda] = \{ v \in \text{BESTD}(\Upsilon^{(i)}_P(\lambda), L_\lambda - 1) \mid \forall u \in \Gamma(v, i), \lambda[u] = 1 \}$$

**λ is AND and determ**+. If λ is an AND node and negatively determinable, Lemma 15 in conjunction with the inductive hypothesis gives us that:

$$\Delta^-_i[\lambda] = \{ v \in \text{BESTD}(\Upsilon^{(i)}_P(\lambda), L_\lambda - 1) \mid \forall u \in \Gamma(v, i), \lambda[u] = 0 \}$$

**λ is OR and determ**−. If λ is an OR node and negatively determinable, then by definition, all of λ’s children must be negatively determinable. Applying the inductive hypothesis to Property 7, we arrive at:

$$\Delta^-_i[\lambda] = \bigcap_c \{ v \in \text{BESTD}(\Upsilon^{(i)}_P(c), L_c - 1) \mid \forall u \in \Gamma(v, i), c[u] = 0 \} \cap \text{BESTD}(\Upsilon^{(i)}_P(\lambda), L_\lambda - 1)$$

If we expand out the value of $\text{BESTD}(\Upsilon^{(i)}_P(c), L_c - 1)$ by Property 6 and apply Property 1 to distribute, we get:

$$\Delta^-_i[\lambda] = \left( \bigcap_c \{ v \in \text{BESTD}(\Upsilon^{(i)}_P(c), L_c - 1) \mid \forall u \in \Gamma(v, i), c[u] = 0 \} \cap \text{BESTD}(\Upsilon^{(i)}_P(\lambda), L_\lambda - 1) \right)$$

$$\setminus \left( \bigcup_{c^+ \neq c} (\Upsilon^{(i)}_P(c)) \Delta^+_{i+1}[c^+] \right)$$

Since $\Upsilon^{(i)}_P(\lambda) \geq \Upsilon^{(i)}_P(c)$ for all children, by Lemma 2, $\text{BESTD}(\Upsilon^{(i)}_P(\lambda), L_\lambda - 1) \subseteq \text{BESTD}(\Upsilon^{(i)}_P(c), L_c - 1)$. If we apply Property 1, we can simplify to:

$$\Delta^-_i[\lambda] = \left( \bigcap_c \{ v \in \text{BESTD}(\Upsilon^{(i)}_P(c), L_c - 1) \mid \forall u \in \Gamma(v, i), c[u] = 0 \} \cap \text{BESTD}(\Upsilon^{(i)}_P(\lambda), L_\lambda - 1) \right)$$

$$\setminus \left( \bigcup_{c^+ \neq c} (\Upsilon^{(i)}_P(c)) \Delta^+_{i-1}[c^+] \right)$$

At this point, we note that $\Delta^+_{i+1}[c^+]$ is the set of vertices for which $c^+$ is guaranteed to evaluate to 1, while $\forall u \in \Gamma(v, i), c[u] = 0$ ensures that $c[v] = 0$, since $v \in \Gamma(v, i)$ by definition. Since the intersection is over all children, the $\Delta^+_{i+1}$ vertices must be mutually exclusive to $\bigcap_c \{ v \in \text{BESTD}(\Upsilon^{(i)}_P(c), L_c - 1) \mid \forall u \in \Gamma(v, i), c[u] = 0 \}$, and we can remove them to get:

$$\Delta^-_i[\lambda] = \bigcap_c \{ v \in \text{BESTD}(\Upsilon^{(i)}_P(c), L_c - 1) \mid \forall u \in \Gamma(v, i), c[u] = 0 \}$$

The intersection requires that $\forall u \in \Gamma(v, i), c[u] = 0$ hold for all children, and since OR nodes require all their children to be 0 to evaluate to 0, these are equivalent:

$$\Delta^-_i[\lambda] = \{ v \in \text{BESTD}(\Upsilon^{(i)}_P(\lambda), L_\lambda - 1) \mid \forall u \in \Gamma(v, i), \lambda[u] = 0 \}$$

**Proof of Lemma 15**. Proof by strong induction on the step number. Let $i$ be the first time step after which one of λ’s children becomes positively determinable (the child is considered positively determinable at step $i + 1$ but not $i$). We wish to show:

$$\Delta^+_{i+1}[\lambda] = \{ v \in \text{BESTD}(i, L_{\lambda + 1}) \mid \forall u \in \Gamma(v, i + 1), \lambda[u] = 1 \}$$

According to the basic assumption for the lemma, the following equivalence must hold for the first positively determinable child $c^+$:

$$\Delta^+_{i+1}[c^+] = \{ v \in \text{BESTD}(i, L_{c^+} - 1) \mid \forall u \in \Gamma(v, i + 1), c^+[u] = 1 \}$$
Based on Algorithm 1, $\text{BestD}(i, L_{c^*} - 1) = \text{BestD}(i, L_\lambda - 1) \setminus (\bigcup_{c^+ \neq c^*} \Delta^+_{i+1}[c^+])$. However, $c^+$ is the first child to become positively determinable, so $\text{BestD}(i, L_{c^*} - 1) = \text{BestD}(i, L_\lambda - 1)$. Given that $\Delta^+_{i+1}[\lambda] = \bigcup_{c^+} \Delta^+_{i+1}[c^+] \cap \text{BestD}(i, L_\lambda - 1)$, if we substitute for the current situation, we are left with: $\Delta^+_{i+1}[\lambda] = \Delta^+_{i+1}[c^+] \cap \text{BestD}(i, L_\lambda - 1)$. Now, we have:

$$\Delta^+_{i+1}[\lambda] = \{v \in \text{BestD}(i, L_\lambda - 1) \mid \forall u \in \Gamma(v, i), c^+[u] = 1\}$$

We now present Lemma 16 which shows that equivalence of expressions between the vertex group of a parent and the vertex groups of its positively determinable children.

**Lemma 16.** Let node $\lambda$ be an OR node in the predicate tree representing $\phi^*$, and let $c^+$ refer to a positively determinable child of $\lambda$. The expression:

$$\{v \in \text{BestD}(\Upsilon_P^{(i)}(\lambda), L_\lambda - 1) \mid \forall u \in \Gamma(v, i), c^+[u] = 1\}$$

is equivalent to:

$$\bigcup_{c^+} \{v \in \text{BestD}(\Upsilon_P^{(i)}(\lambda), L_\lambda - 1) \mid \forall u \in \Gamma(v, i), c^+[u] = 1\}$$

Similarly if $\lambda$ is an AND node, and $c^-$ refers to a negatively determinable child of $\lambda$, the expression:

$$\{v \in \text{BestD}(\Upsilon_P^{(i)}(\lambda), L_\lambda - 1) \mid \forall u \in \Gamma(v, i), c^-[u] = 0\}$$

is equivalent to:

$$\bigcup_{c^-} \{v \in \text{BestD}(\Upsilon_P^{(i)}(\lambda), L_\lambda - 1) \mid \forall u \in \Gamma(v, i), c^-[u] = 0\}$$

If we apply Lemma 16, we can replace the $c^+[u] = 1$ condition with $\lambda[u] = 1$, proving the base case.

Now, assume that the Lemma 15 is true for any arbitrary time step $i$ in which there are a nonzero number of positively determinable children. That is:

$$\Delta^+_i[\lambda] = \{v \in \text{BestD}(\Upsilon_P^{(i)}(\lambda), L_\lambda - 1) \mid \forall u \in \Gamma(v, i), \lambda[u] = 1\}$$

If $P_i$ is not a descendant of $\lambda$, then the lemma must hold for the $(i+1)$th step since $\Delta^+_i[\lambda]$ is not updated ($\Delta^+_i[\lambda] = \Delta^+_i[\lambda]$), and $\Upsilon_P^{(i)}(\lambda) = \Upsilon_P^{(i+1)}(\lambda)$.

On the other hand, if $P_i$ is a descendant of $\lambda$ then $\Delta^+_i[\lambda]$ is updated as follows:

$$\Delta^+_{i+1}[\lambda] = \bigcup_{c^+} \Delta^+_{i+1}[c^+] \cap \text{BestD}(i, L_\lambda - 1)$$

If we apply the basic condition stated in the lemma, we can expand out to:

$$\Delta^+_{i+1}[\lambda] = \bigcup_{c^+} \text{BestD}(i, L_\lambda - 1) \cap \{v \in \text{BestD}(\Upsilon_P^{(i+1)}(c^+), L_{c^+} - 1) \mid \forall u \in \Gamma(v, i+1), c^+[u] = 1\}$$

If we expand out $\text{BestD}$ and apply Lemma 2, then we are left with:

$$\Delta^+_{i+1}[\lambda] = \text{BestD}(i, L_\lambda - 1) \cap \left( \bigcup_{c^+} \{v \in \text{BestD}(\Upsilon_P^{(i+1)}(c^+), L_\lambda - 1) \mid \forall u \in \Gamma(v, i+1), c^+[u] = 1\} \right) \setminus \bigcup_{c^+ \neq c^+} \left( \Upsilon_P^{(i+1)}(c^+) \Delta^+_{\Upsilon_P^{(i+1)}(c^+)}[c^+] \right)$$

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By the principle that $A \cap (B \setminus C) = A \cap (B \setminus (C \cap A))$, we can adjust this to:

$$\Delta_{i+1}[\lambda] = \text{BESTD}(i, L_\lambda - 1) \cap \left( \bigcup_{c^+} \{v \in \text{BESTD}(T_{P_\lambda}^{(i+1)}(c^+), L_\lambda - 1) \mid \forall u \in \Gamma(v, i + 1), c^+[u] = 1\} \setminus \bigcup_{c^+ \not\in c^*} \Delta_{T_{P_\lambda}^{(i+1)}(c^+)}^+[c^+] \cap \text{BESTD}(i, L_\lambda - 1) \right)$$

At this point, we introduce the following corollary to Corollary 2

**Corollary 3.** Let $\lambda$ be an OR node in the predicate tree representing $\phi^*$, and let $c^+$ refer to a child of $\lambda$. If predicate atom $P_i$ is a descendant of $c^+$, then for all $j \leq i$:

$$\Delta_j^+[c^+] \cap \text{BESTD}(i, L_\lambda - 1) \subseteq \Delta_{i+1}[c^+]$$

The premise of the lemma tells us that:

$$\Delta_{i+1}[c^+] = \{v \in \text{BESTD}(T_{P_\lambda}^{(i+1)}(c^+), L_\lambda - 1) \mid \forall u \in \Gamma(v, i + 1), c^+[u] = 1\} \setminus \bigcup_{c^+ \not\in c^*} \Delta_{T_{P_\lambda}^{(i+1)}(c^+)}^+[c^+] \cap \text{BESTD}(i, L_\lambda - 1)$$

By applying Corollary 3 and the principle that if $A \setminus B \supseteq C$, then $A \supseteq C$, we get that for any time step $j \leq i$: $\{v \in \text{BESTD}(T_{P_\lambda}^{(i+1)}(c^+), L_\lambda - 1) \mid \forall u \in \Gamma(v, i + 1), c^+[u] = 1\} \supseteq \Delta_j^+[c^+] \cap \text{BESTD}(i, L_\lambda - 1)$.

Similar to the proof for correctness presented in Section 6.2, we now remove the sets being subtracted one-by-one for each child. We assert once again that if $B \supseteq F$ and $C \cap F = \emptyset$, then $A \cup (B \setminus C) \cup (D \setminus (E \cup F)) = A \cup (B \setminus C) \cup (D \setminus E)$ for any sets $A, B, C, D, E,$ and $F$ (Appendix D). We focus on two specific positively determinable children $c^+$ and $c^*$, and let:

$$B = \{v \in \text{BESTD}(T_{P_\lambda}^{(i+1)}(c^*), L_\lambda - 1) \mid \forall u \in \Gamma(v, i + 1), c^*[u] = 1\}$$

$$C = \bigcup_{c^+ \not\in c^*} \Delta_{T_{P_\lambda}^{(i+1)}(c^+)}^+[c^+] \cap \text{BESTD}(i, L_\lambda - 1)$$

$$D = \{v \in \text{BESTD}(T_{P_\lambda}^{(i+1)}(c^*), L_\lambda - 1) \mid \forall u \in \Gamma(v, i + 1), c^*[u] = 1\}$$

$$E = \bigcup_{c^+ \not\in c^*, c^+} \Delta_{T_{P_\lambda}^{(i+1)}(c^+)}^+[c^+] \cap \text{BESTD}(i, L_\lambda - 1)$$

$$F = \Delta_{T_{P_\lambda}^{(i+1)}(c^*)}^+[c^*] \cap \text{BESTD}(i, L_\lambda - 1)$$

and $A$ be the remaining operands inside the intersection in Equation 13. We have just shown that $F \subseteq B$, and Lemma 4 tells us that $F \cap C = \emptyset$. By repeatedly applying this reduction, same as before, we can reduce Equation 13 to:

$$\Delta_{i+1}[\lambda] = \text{BESTD}(i, L_\lambda - 1) \cap \left( \bigcup_{c^+} \{v \in \text{BESTD}(T_{P_\lambda}^{(i+1)}(c^+), L_\lambda - 1) \mid \forall u \in \Gamma(v, i + 1), c^+[u] = 1\} \right)$$

We can apply Lemma 2 to simplify to:

$$\Delta_{i+1}[\lambda] = \bigcup_{c^+} \{v \in \text{BESTD}(i, L_\lambda - 1) \mid \forall u \in \Gamma(v, i + 1), c^+[u] = 1\}$$

By Lemma 16 this simplifies to:

$$\Delta_{i+1}[\lambda] = \{v \in \text{BESTD}(i, L_\lambda - 1) \mid \forall u \in \Gamma(v, i + 1), \lambda[u] = 1\}$$

The same reasoning in reverse can be made for the case in which $\lambda$ is an AND node and negatively determinable.
Proof of Lemma 10. Equation 10 is equivalent to:

\[ \{ v \in \text{BestD}(\Upsilon_{P}^{(i)}(\lambda), L_{\lambda} - 1) \mid \exists^{(i)} c_{\lambda}, \forall u \in \Gamma(v, i), c_{\lambda}^+[u] = 1 \} \]

To show equivalence between this expression and Equation 9, we must show that the conditions \( \forall u \in \Gamma(v, i), \lambda[u] = 1 \) and \( \exists^{(i)} c_{\lambda}, \forall u \in \Gamma(v, i), c_{\lambda}^+[u] = 1 \) are the same. We accomplish this by showing implication in both directions. Right off the bat, we see that the condition \( \exists^{(i)} c_{\lambda}, \forall u \in \Gamma(v, i), c_{\lambda}^+[u] = 1 \) trivially implies \( \forall u \in \Gamma(v, i), \lambda[u] = 1 \).

For the reverse direction, the contrapositive of Lemma 13 states that if \( \forall u \in \Gamma(v, i), \lambda[u] = 1 \), then \( \lambda \) must be positively determinable.

The same reasoning in reverse can be made for the case in \( \lambda \) is an AND node and negatively determinable. \( \square \)

Proof of Corollary 3. We first show that:

\[ \Delta_{i}^{+}[c_{\lambda}^{+}] \cap \text{BestD}(i, L_{\lambda} - 1) \subseteq \Delta_{i+1}^{+}[c_{\lambda}^{+}] \] (14)

By Corollary 2, we know that

\[ \Delta_{i}^{+}[c_{\lambda}^{+}] \cap \text{BestD}(i, L_{\lambda} - 1) \subseteq \Delta_{i+1}^{+}[c_{\lambda}^{+}] \]

If we expand out the BestD, we are left with:

\[ \Delta_{i}^{+}[c_{\lambda}^{+}] \cap \text{BestD}(i, L_{\lambda} - 1) \setminus \bigcup_{c_{\lambda}' \neq c_{\lambda}} \Delta_{i}^{+}[c_{\lambda}'] \subseteq \Delta_{i+1}^{+}[c_{\lambda}^{+}] \]

If we apply Lemma 4, which tells us \( \Delta^{+} \) values are mutually exclusive, this simplifies to Equation 14.

Now, let us look at the time step of \( c_{\lambda}^{+} \)'s descendent predicate atom before \( P, \Upsilon_{P}^{(i)}(c_{\lambda}^{+}) \). We can apply Equation 14 to this time step as well to get:

\[ \Delta_{\Upsilon_{P}^{(i)}}^{+}[c_{\lambda}^{+}] \cap \text{BestD}(\Upsilon_{P}^{(i)}(c_{\lambda}^{+}), L_{\lambda} - 1) \subseteq \Delta_{\Upsilon_{P}^{(i)}+1}^{+}[c_{\lambda}^{+}] \]

Since \( \Delta^{+}[c_{\lambda}^{+}] \) is not updated between steps \( \Upsilon_{P}^{(i)}(c_{\lambda}^{+}) \) and \( i \), this is equivalent to:

\[ \Delta_{\Upsilon_{P}^{(i)}}^{+}[c_{\lambda}^{+}] \cap \text{BestD}(\Upsilon_{P}^{(i)}(c_{\lambda}^{+}), L_{\lambda} - 1) \subseteq \Delta_{i}^{+}[c_{\lambda}^{+}] \]

Lemma 2 tells us that for all time steps \( j \leq i \), \( \text{BestD}(j, L_{\lambda} - 1) \supseteq \text{BestD}(i, L_{\lambda} - 1) \). So, if we take the intersection with respect to \( \text{BestD}(i, L_{\lambda} - 1) \) on both sides, we end up with:

\[ \Delta_{\Upsilon_{P}^{(i)}}^{+}[c_{\lambda}^{+}] \cap \text{BestD}(\Upsilon_{P}^{(i)}(c_{\lambda}^{+}), L_{\lambda} - 1) \subseteq \Delta_{i+1}^{+}[c_{\lambda}^{+}] \]

We can recursively apply this logic for all \( j \leq i \) until \( \Delta^{+}[c_{\lambda}^{+}] \) becomes the empty set, at which point it is trivially true. \( \square \)

B ShallowFish in Practice

B.1 Optimized ShallowFish

The ShallowFish presented in Algorithm 2 actually has a runtime complexity of \( O(n^{3}) \). BestAndD/BestOrD both have a runtime complexity of \( O(n) \) and may be called \( n \) times at worst for each newly completed ancestor in Lines 19 and 24. Since Update is called a total of \( n \) times, the overall complexity is \( O(n^{3}) \). However, a real world system would implement a merged BestD and Update function which would combine the \( X \) value from Algorithm 1 with the \( \Xi \) from Algorithm 2. The pseudocode for what this optimized ShallowFish might look like is presented in Algorithm 4. Here, orderTree traverses the predicate tree and orders the
children of each node according to the provided sequence of predicate atoms. By removing all the redundant
calculation, the optimized SHALLOWFish is able to limit itself to a single traversal of the tree, and the run-
time complexity is reduced to $O(n \log n)$ due to sorting. However, the Process function in the optimized
SHALLOWFish expects that the sequence of predicate atoms is already in the optimal order to work correctly,
whereas BestD will return the best vertex set for any sequence of predicate atoms. We present BestD
here because this is important for the proof in Section 6, but in practice the optimized SHALLOWFish
can be used.

Algorithm 4 Optimized SHALLOWFish

Input: Boolean predicate $\phi^*$
Output: Filtered vertex set $\psi^*(D)$

1: $[P_1, ..., P_n] \leftarrow \text{OrderP}(\phi^*)$
2: $\lambda \leftarrow \text{orderTree}($root$(\phi^*), [P_1, ..., P_n])$
3: return \text{Process}($[P_1, ..., P_n], \lambda, D$)

4: function \text{Process}($[P_1, ..., P_n], \lambda, D$)
5: if type($\lambda$) = $P$ then
6: return $P(\lambda)(D)$
7: if type($\lambda$) = $\land$ then
8: $X \leftarrow D$
9: for $c \in \text{children}(\lambda)$ do
10: $X \leftarrow \text{Process}([P_1, ..., P_n], c, X)$
11: return $X$
12: else
13: $X \leftarrow \emptyset$, $Y \leftarrow D$
14: for $c \in \text{children}(\lambda)$ do
15: $X \leftarrow X \cup \text{Process}([P_1, ..., P_n], c, Y \setminus X)$
16: return $X$

B.2 ShallowFish on Records

Evaluation on Records. Although everything we have talked about so far has been with vertices, modifying
the algorithm to work with data records in a real system is trivial. Instead of dealing with vertex sets, we
operate on lists/sets of records ids instead. We start by setting the best vertex for step 1 as: $D_1 = \{1, ..., m\}$,
where $m$ is the number of total records. We then apply the same SHALLOWFish algorithm as we did for
vertex sets to record id sets, and we get as output the set of indices for the records which satisfy $\phi^*$. Only
in Line 8 in the Update function, instead of logically determining the filtered vertices, we must actually
fetch data from the backend as specified by the records in $D_i$ and apply predicate atom $P_i$ to these fetched
records.

C OrderP

For the reader’s convenience, we provide a modern, updated version of the algorithm suggested Hanani [9].

For each predicate atom $P_i$ in $[P_1, ..., P_n]$, let $\gamma_i$ measure the selectivity of $P_i$; that is, $\gamma_i$ is the fraction
of records which satisfy $P_i$. Algorithm 5 presents ORDERP which returns the best sequence of predicate atoms
$[P_1, ..., P_n]$ for a given boolean predicate expression $\phi^*$ of depth 2 or less. ORDERNODE takes in a single
predicate tree node $\lambda$ and returns the estimated selectivity, cost, and the best ordering of its descendent
predicate atoms. In the case that $\lambda$ is a predicate atom node, we assume the selectivity is readily available.

The base cost is set as $F_\lambda$ in Line 5 which is the constant factor specific to each predicate atom from the

\footnote{Most databases calculate various statistics including cardinalities of columns for its data when the data first enters the database, so the selectivity of predicate atoms can be derived from that [?].}
cost model. The constant overhead $\kappa$ does not appear anywhere in the algorithm since it appears for all predicate atoms and cancels out.

In the case that $\lambda$ is not a predicate atom node, ORDERNode iterates over its children and stores their individual estimated selectivities, costs, and best orderings. The children are then sorted with a weight function which changes based on the type of $\lambda$. The sorted children and their information are then processed, and the combined estimated selectivity, cost, and ordering information of $\lambda$ are calculated. Note that the best ordering of descendent predicate atoms for a non-leaf node is simply the concatenation of the best orderings for its children.

Algorithm 5 ORDERP

Input: Predicate expression $\phi^*$
Output: Ordered sequence of predicate atoms $[P_1, ..., P_n]$

1: $(\gamma, \text{cost}, P) \leftarrow \text{ORDERNode(root(}$\phi^*$), $\{\gamma_1, ..., \gamma_n\}$)
2: return $P$

3: function ORDERNode($\lambda$)
4: if type($\lambda$) = $P$ then
5: return $(\gamma_{\theta(\lambda)}, F_\lambda, [\lambda])$
6: $S \leftarrow []$
7: for $c \in \text{children(}$\lambda$)$ do
8: $(\gamma, \text{cost}, P) \leftarrow \text{ORDERNode(c)}$
9: $S \leftarrow S + [(\gamma, \text{cost}, P)]$
10: if type($\lambda$) = $\land$ then
11: $S \leftarrow \text{increasingSort(S, GETANDWEIGHT)}$
12: else
13: $S \leftarrow \text{increasingSort(S, GETORWEIGHT)}$
14: return ORDERNodeHelper($S, \lambda$)

15: function ORDERNodeHelper($S, \lambda$)
16: totalCost $\leftarrow 0$
17: $\gamma_{\text{total}} \leftarrow 1$
18: $P_\text{all} \leftarrow []$
19: for $i \leftarrow 1, ..., |S|$ do
20: $(\gamma, \text{cost}, P) \leftarrow S[i]$
21: if type($\lambda$) = $\land$ then
22: totalCost $\leftarrow$ totalCost $+ \gamma_{\text{total}} \cdot \text{cost}$
23: $\gamma_{\text{total}} \leftarrow \gamma_{\text{total}} \cdot \gamma$
24: else
25: totalCost $\leftarrow$ totalCost $+ (1 - \gamma_{\text{total}}) \cdot \text{cost}$
26: $\gamma_{\text{total}} \leftarrow \gamma + \gamma_{\text{total}} \cdot (1 - \gamma)$
27: $P_\text{all} \leftarrow P_\text{all} + P$
28: return $(\gamma_{\text{total}}, \text{totalCost}, P_\text{all})$

29: function GETANDWEIGHT($\gamma, \text{cost}, P$)
30: return $\text{cost}/(1 - \gamma)$

31: function GETORWEIGHT($\gamma, \text{cost}, P$)
32: return $\text{cost}/\gamma$

The independence assumption is made for multi-attribute selectivities: $\gamma_{1 \land 2} = \gamma_1 \cdot \gamma_2$ and $\gamma_{1 \lor 2} = \gamma_1 + \gamma_2 - \gamma_1 \cdot \gamma_2$. If selectivities for multi-attribute indexes are present in the database, that information can be used in place of the independence assumption.
D Set Algebra

Here, we show the proofs for the various set equivalences we have used throughout the paper. Equivalences which are not proven come from related works [?].

1. \( A \cap (B \setminus (C \cup D)) = (A \setminus C) \cap (B \setminus D) \).

   \[
   A \cap (B \setminus (C \cup D)) = A \cap ((B \setminus C) \cap (B \setminus D)) \\
   = B \cap (A \setminus C) \cap (B \setminus D) \\
   = (A \setminus C) \cap (B \setminus D)
   \]

   The first step uses \( X \setminus (Y \cup Z) = (X \setminus Y) \cap (X \setminus Z) \). The second and third steps both use \( X \cap (Y \setminus Z) = (X \cap Y) \setminus Z = Y \setminus (X \setminus Z) \).

2. \( A = (A \setminus B) \cup (A \cap B) \).

   \[
   (A \setminus B) \cup (A \cap B) = (A \cup (A \cap B)) \setminus (B \setminus (A \cap B)) \\
   = (A) \setminus (\varnothing) \\
   = A
   \]

   The first step uses \( (A \setminus B) \cup C = (A \cup C) \setminus (B \setminus C) \). The remaining steps are trivially true.

3. \( (A \setminus B) \setminus C = (A \setminus (B \setminus C)) \setminus C \).

   \[
   (A \setminus (B \setminus C)) \setminus C = A \setminus ((B \setminus C) \cup C) \\
   = A \setminus (B \cup C) \\
   = (A \setminus B) \setminus C
   \]

   This uses the equivalence \( (A \setminus B) \setminus C = A \setminus (B \cup C) \) in the first and third steps. The second step is trivially true.

4. \( (A \setminus B) \setminus C = (A \setminus (B \setminus C)) \setminus C \).

   \[
   (A \setminus (B \setminus C)) \setminus C = A \setminus ((B \setminus C) \cup C) \\
   = A \setminus (B \cup C) \\
   = (A \setminus B) \setminus C
   \]

   This uses the equivalence \( (A \setminus B) \setminus C = A \setminus (B \cup C) \) in the first and third steps. The second step is trivially true.

5. If \( B \supseteq F \) and \( C \cap F = \varnothing \), then \( A \cup (B \setminus C) \cup (D \setminus (E \cup F)) = A \cup (B \setminus C) \cup (D \setminus E) \).

   Here, \( A \) is a common term on the outside, so it is sufficient to prove \( (B \setminus C) \cup (D \setminus (E \cup F)) = (B \setminus C) \cup (D \setminus E) \).

   \[
   (B \setminus C) \cup (D \setminus (E \cup F)) \\
   = ((B \setminus C) \cup D) \setminus ((E \cup F) \setminus (B \setminus C)) \\
   = ((B \setminus C) \cup D) \setminus ((E \setminus (B \setminus C)) \cup (F \setminus (B \setminus C))) \\
   = ((B \setminus C) \cup D) \setminus ((E \setminus (B \setminus C)) \cup (F \setminus C) \cup (F \setminus B)) \\
   = ((B \setminus C) \cup D) \setminus (E \setminus (B \setminus C)) \\
   = (B \setminus C) \cup (D \setminus E)
   \]

   The first and last step uses the equivalence: \( A \cup (B \setminus C) = (A \cup B) \setminus (C \setminus A) \). The second step uses the equivalence: \( (A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C) \). The third step uses the assumptions \( B \supseteq F \) and \( F \cap C = \varnothing \).