ANALYSIS OF VELÁZQUEZ’S SOLUTION TO THE MEAN CURVATURE FLOW WITH A TYPE II SINGULARITY

SIAO-HAO GUO AND NATASA SESUM

Abstract. J.J.L. Velázquez in 1994 used the degree theory to show that there is a perturbation of Simons’ cone, starting from which the mean curvature flow develops a type II singularity at the origin. He also showed that under a proper time-dependent rescaling of the solution around the origin, the rescaled flow converges in the $C^0$ sense to a minimal hypersurface which is tangent to Simons’ cone at infinity. In this paper, we prove that the rescaled flow actually converges locally smoothly to the minimal hypersurface, which appears to be the singularity model of the type II singularity. In addition, we show that the mean curvature of the solution blows up near the origin at a rate which is smaller than that of the second fundamental form.

1. Introduction

J.J.L. Velázquez in [V] constructed a solution to the mean curvature flow which develops a type II singularity. Below is his result:

**Theorem 1.1.** Let $n \geq 4$ be a positive integer. If $t_0 < 0$ and $|t_0| \ll 1$ (depending on $n$), then there is a $O(n) \times O(n)$ symmetric mean curvature flow $\{\Sigma_t\}_{t_0 \leq t < 0}$ so that

1. $\{\Sigma_t\}_{t_0 \leq t < 0}$ develops a type II singularity at $O$ as $t \rightarrow 0$ in the sense that there is $0 < \sigma = \sigma(n) < \frac{1}{2}$ (see (3.2)) so that the second fundamental form of $\Sigma_t$ satisfies

$$\limsup_{t \rightarrow 0} \sup_{\Sigma_t \cap B(O; \sqrt{-t})} (-t)^{\frac{1}{2} + \sigma} |A_{\Sigma_t}| > 0$$

2. The type I rescaled hypersurfaces

$$\left\{ \Pi_s = \frac{1}{\sqrt{-t}} \Sigma_t \bigg|_{t = -e^{-s}} \right\}_{-\ln(-t_0) \leq s < \infty}$$

$C^2$-converge to Simons’ cone $C$ in any fixed annulus centered at $O$ (i.e. $B(O; R) \setminus B(O; r)$ with $0 < r < R < \infty$) as $s \rightarrow \infty$.

3. The type II rescaled hypersurfaces

$$\left\{ \Gamma_\tau = \frac{1}{(-t)^{\frac{1}{2} + \sigma}} \Sigma_t \bigg|_{t = -(2\sigma \tau)} \right\}_{2\sigma(-t_0)^{1/2} \leq \tau < \infty}$$

locally $C^0$-converges to a minimal hypersurface $M_k$ (see Section 2), which is tangent to Simons’ cone $C$ at infinity.

\footnote{Natasa Sesum thanks NSF for the support in DMS-1056387.}
Velázquez’s idea is to find a $O(n) \times O(n)$ symmetric solution to the “normalized mean curvature flow” $\{\Pi_s\}_{s_0 \leq s < \infty}$ which exists for a long time and converges (locally and away from $O$) to Simons’ cone $C$ as $s \to \infty$. Note that the minimal cone $C$ is a self-shrinker with a singularity at the origin and that this singularity of $C$ forces the normalized mean curvature flow $\{\Pi_s\}_{s_0 \leq s < \infty}$ to develop a singularity at $O$ as $s \to \infty$. Consequently, the corresponding mean curvature flow $\{\Sigma_t\}_{t_0 \leq t < 0}$ develop a type II singularity at $O$ in finite time (as $t \to 0$). In addition, he used the comparison principle to show that the type II singularity is a special case of the “translating mean curvature flow”. Velázquez’s solution converges to a “minimal hypersurface”, so it seems that there is a chance for the mean curvature of Velázquez’s solution to stay bounded up to the first singular time.

In this paper, we answer both of the above questions. More explicitly, we show the following:

**Theorem 1.2.** Let $\{\Sigma_t\}_{t_0 \leq t < 0}$ be Velázquez’s solution in Theorem 1.1 with $n \geq 5$. By choosing proper initial data outside a small ball centered at $O$, the origin is the only singularity of the solution at the first singular time $t = 0$. Moreover, the type II rescaled hypersurfaces $\{\Gamma_{\tau}\}_{2(|\tau| - 10) \leq \tau < \infty}$ converges locally smoothly to the minimal hypersurface $M_k$ as $\tau \to \infty$. It follows that the second fundamental form of $\Sigma_t$ satisfies

$$0 < \limsup_{t \to 0} \sup_{\sigma} (-t)^{\frac{1}{2} + \sigma} |A_{\Sigma_t}| < \infty$$

In addition, the mean curvature of $\Sigma_t$ blows up as $t \to 0$ at a rate which smaller than that of the second fundamental form. More precisely, there hold

$$\limsup_{t \to 0} \sup_{\sigma} (-t)^{\frac{1}{2} - \sigma} |H_{\Sigma_t}| > 0$$

$$\limsup_{t \to 0} \sup_{\sigma} (-t)^{\frac{1}{2} + (1 - 2\varrho)(-t)^{\frac{1}{2} + \sigma}} |H_{\Sigma_t}| < \infty$$

for some constant $0 < \varrho = \varrho(n) < 1$. 


Proof. The smooth convergence of the type II rescaled hypersurfaces \( \{\Gamma_\tau\} \to M_k \) as \( \tau \to \infty \) and the fact that the origin is the only singularity of \( \{\Sigma_t\} \) at \( t = 0 \) follow from Theorem 4.8 (see also Remark 4.9). The blow-up rates of the second fundamental form \( A_{\Sigma_t} \) and mean curvature \( H_{\Sigma_t} \) can be found in Proposition 5.1, Proposition 5.2, Proposition 5.3 and Proposition 5.4. \( \square \)

To improve the convergence of the type II rescaled flow, all we need is to derive some smooth estimates (see Proposition 4.4 and Proposition 4.5). One of the key ingredients to achieve that is to use the curvature estimates in [EH]. As for the blow-up of the mean curvature, it follows from the smooth convergence of type II rescaled flow and L’Hôpital’s rule. Moreover, by modifying Velázquez’s estimates, we show that the blow-up rate of the mean curvature is smaller than that of the second fundamental form.

The paper is organized as follows. In Section 2, we introduce the minimal hypersurface \( M_k \) found by Velázquez and then derive some smooth estimates for it. In Section 3, we specify the set up for constructing Velázquez’s solution and define various regions and rescalings for analyzing the solution. In Section 4, we state the key a priori estimates (Proposition 4.4 and Proposition 4.5) and explain how to use them to construct Velázquez’s solution (for the sake of completeness) and to see the behavior of the solution in different regions (see Theorem 4.8). In Section 5, we explain why the mean curvature blows up and why its blow-up rate is smaller than that of the second fundamental form. Lastly, in Section 6, Section 7 and Section 8 we prove Proposition 4.4 and Proposition 4.5 for completion of the argument.

2. Minimal hypersurfaces tangent to Simons’ cone at infinity
Let
\[ C = \{(r\nu, r\omega) \mid r > 0; \nu, \omega \in S^{n-1}\} \]
be Simons’ cone, where \( n \geq 4 \) is a positive integer and \( S^{n-1} \) is the unit sphere in \( \mathbb{R}^n \). It is shown in [V] that there is a smooth minimal hypersurface
\[ M = \left\{ (r\nu, \tilde{\psi}(r)\omega) \mid r \geq 0; \nu, \omega \in S^{n-1} \right\} \]
in \( \mathbb{R}^{2n} \) which is tangent to \( C \) at infinity, and that the function \( \tilde{\psi}(r) \) satisfies
\[ \frac{\partial^2 \tilde{\psi}}{1 + \left( \frac{\partial_r \tilde{\psi}}{1 + \left( \frac{\partial_r \tilde{\psi}}{r} \right)^2} \right)^2} + (n-1) \left( \frac{\partial_r \tilde{\psi}}{r} - \frac{1}{\tilde{\psi}} \right) = 0 \]
and
\[ \begin{cases} \partial_{rr} \tilde{\psi}(r) > 0 \\ \partial_r \tilde{\psi}(0) = 0, \lim_{r \to \infty} \frac{\partial_r \tilde{\psi}(r)-1}{r^{n-1}} = \alpha 2^{\frac{n+1}{2}} \\ \tilde{\psi}(r) > r, \lim_{r \to \infty} \frac{\tilde{\psi}(r)-r}{r^{n-1}} = 2^{\frac{n+1}{2}} \end{cases} \]
where
\[ \alpha = -\frac{(2n-3) + \sqrt{4n^2 - 20n + 17}}{2} \in [-2, -1] \]
is a root of the quadratic polynomial
\[ \alpha (\alpha - 1) + 2 (n-1) (\alpha + 1) = 0 \]
By symmetry, studying $\mathcal{M}$ is equivalent to analyzing the projected curves
\[
\tilde{\mathcal{M}} = \left\{ (r, \hat{\psi}(r)) \mid r \geq 0 \right\}
\]
(2.2)
\[
\bar{C} = \{ (r, r) \mid r > 0 \}
\]
Note that $\tilde{\mathcal{M}}$ is a convex curve which lies above $\bar{C}$ (i.e. $\hat{\psi}(r) > r$ for $r \geq 0$); moreover, $\tilde{\mathcal{M}}$ intersects orthogonally with the vertical ray $\{(0, r) \mid r > 0\}$ (i.e. $\partial_r \hat{\psi}(0) = 0$) and is asymptotic to $\bar{C}$ at infinity (i.e. $\hat{\psi}(r) = r + O(r^\alpha)$ as $r \to \infty$). Therefore, $\tilde{\mathcal{M}}$ is a graph over $\bar{C}$; more precisely,
\[
\tilde{\mathcal{M}} = \left\{ \left( r \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) + \psi(r) \left( \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \right) \mid r \geq \frac{\hat{\psi}(0)}{\sqrt{2}} \right\}
\]
Velázquez in [V] showed that the function $\psi(r)$ satisfies
\[
\frac{\partial^2_{rr} \psi}{1 + (\partial_r \psi)^2} + 2(n - 1) \frac{\partial_r \psi + \psi}{r^2 - \psi^2} = 0
\]
and
\[
\begin{cases}
\partial^2_{rr} \psi(r) > 0 \\
\partial_r \psi \left( \frac{\hat{\psi}(0)}{\sqrt{2}} \right) = -1, \quad \lim_{r \to \infty} \frac{\partial_r \psi(r)}{r^\alpha} = \alpha \\
\psi \left( \frac{\hat{\psi}(0)}{\sqrt{2}} \right) = \hat{\psi}(0) \sqrt{2}, \quad \lim_{r \to \infty} \frac{\psi(r)}{r^\alpha} = 1
\end{cases}
\]
More generally, for each $k > 0$, we can define
\[
\mathcal{M}_k = k^{1-\alpha} \mathcal{M}
\]
Then $\mathcal{M}_k$ is also a minimal hypersurface in $\mathbb{R}^{2n}$ which is tangent to $\bar{C}$ at infinity. Notice that
\[
\mathcal{M}_k = \left\{ (r, \nu, \hat{\psi}_k(r) \omega) \mid r \geq 0; \nu, \omega \in \mathbb{S}^{n-1} \right\}
\]
where
\[
\hat{\psi}_k(r) = k^{\frac{1}{1-\alpha}} \hat{\psi} \left( k^{\frac{1}{1-\alpha}} r \right)
\]
By rescaling, we deduce that
\[
\frac{\partial^2_{rr} \hat{\psi}_k}{1 + (\partial_r \hat{\psi}_k)^2} + (n - 1) \left( \frac{\partial_r \hat{\psi}_k}{r} - \frac{1}{\psi_k} \right) = 0
\]
(2.4)
\[
\begin{cases}
\partial^2_{rr} \hat{\psi}_k(r) > 0 \\
\partial_r \hat{\psi}_k(0) = 0, \quad \lim_{r \to \infty} \frac{\partial_r \hat{\psi}_k(r)}{r^{\frac{1}{\alpha}}} = k \alpha \ 2^{\frac{\alpha+1}{\alpha}} \\
\hat{\psi}_k(r) > r, \quad \lim_{r \to \infty} \frac{\hat{\psi}_k(r) - r}{r^{\alpha}} = k \ 2^{\frac{\alpha+1}{\alpha}}
\end{cases}
\]
Moreover, there holds a “monotonic” property of the rescaling family, i.e. \( \hat{\psi}_{k_1} (r) < \hat{\psi}_{k_2} (r) \) whenever \( 0 < k_1 < k_2 < \infty \). To see that, let’s first derive the following lemma.

**Lemma 2.1.** The function \( \hat{\psi}_k (r) \) satisfies
\[
(2.5) \quad \hat{\psi}_k (r) - r \partial_r \hat{\psi}_k (r) > 0
\]
for \( r \geq 0 \). In addition, there holds
\[
(2.6) \quad \lim_{r \to \infty} \frac{\hat{\psi} (r) - r \partial_r \hat{\psi} (r)}{r^\alpha} = (1 - \alpha) \frac{2^\frac{\alpha+1}{2}}{2}\]
Proof. Notice that
\[
\partial_r \left( \hat{\psi} (r) - r \partial_r \hat{\psi} (r) \right) = -r \partial^2_{rr} \hat{\psi} < 0
\]
which means the function \( \hat{\psi} (r) - r \partial_r \hat{\psi} \) is decreasing. Furthermore, we have
\[
\lim_{r \to \infty} \frac{\hat{\psi} (r) - r \partial_r \hat{\psi} (r)}{r^\alpha} = \lim_{r \to \infty} \frac{\hat{\psi} (r) - r \partial_r \hat{\psi} (r) + \frac{1 - \partial_r \hat{\psi} (r)}{r^{\alpha-1}}}{r^\alpha} = (1 - \alpha) \frac{2^\frac{\alpha+1}{2}}{2} > 0
\]
which implies
\[
\hat{\psi} (r) - r \partial_r \hat{\psi} (r) > 0
\]
for \( r \gg 1 \). The conclusions follow immediately. \( \square \)

Now we show the monotonic property of the rescaling family.

**Lemma 2.2.** There holds
\[
\partial_k \hat{\psi}_k > 0
\]
In other words, \( \hat{\psi}_k \) is monotonically increasing in \( k \).
Proof. By definition, we have
\[
\partial_k \hat{\psi}_k (z) = \partial_k \left( k^{\frac{1}{1-r}} \hat{\psi} \left( k^{\frac{1}{1-r}} z \right) \right)
= \partial_k k^{\frac{1}{1-r}} \left( \hat{\psi} (r) - r \partial_r \hat{\psi} (r) \right) \bigg|_{r=k^{\frac{1}{1-r}}} > 0
\]
\( \square \)

On the other hand, notice that the projected curve of \( \mathcal{M}_k \) is also a graph over \( \bar{C} \), i.e.
\[
(2.7) \quad \bar{\mathcal{M}}_k = \left\{ \left( r, \hat{\psi}_k (r) \right) \bigg| r \geq 0 \right\}
= \left\{ \left( (r - \psi_k (r)) \frac{1}{\sqrt{2}} \left( r + \psi_k (r) \right) \frac{1}{\sqrt{2}} \right) \bigg| r \geq \frac{\hat{\psi}_k (0)}{\sqrt{2}} \right\}
\]
where
\[
(2.8) \quad \psi_k (r) = k^{\frac{1}{1-r}} \psi \left( k^{\frac{1}{1-r}} r \right)
\]
By rescaling, the function \( \psi_k (r) \) satisfies
\[
(2.9) \quad \frac{\partial^2_{rr} \psi_k}{1 + (\partial_r \psi_k)^2} + 2 (n - 1) \frac{r \partial_r \psi_k + \psi_k}{r^2 - \psi_k^2} = 0
\]
Note that $\psi_k(r)$ decreases to 0 as $r \to \infty$. Below we have the decay estimates for $\psi_k(r)$.

**Lemma 2.3.** For any $m \in \mathbb{Z}_+$, there holds

$$|\partial^m_r \psi_k(r)| \leq C(n, m) kr^{\alpha-m}$$

for $r \geq \frac{\dot{\psi}(0)}{\sqrt{2}}$.

**Proof.** By rescaling, it is sufficient to check for $k = 1$.

From

$$\lim_{r \to \infty} \frac{\psi(r)}{r} = 1 = \lim_{r \to \infty} \frac{\partial_r \psi(r)}{\alpha r^{\alpha-1}}$$

we have

$$\max \left\{ \frac{\psi(r)}{r}, |\partial_r \psi(r)| \right\} \leq C(n) r^{\alpha-1}$$

for $r \geq \frac{\dot{\psi}(0)}{\sqrt{2}}$. In particular, there is $R \gg 1$ (depending on $n$) so that

$$\max \left\{ \frac{\psi(r)}{r}, |\partial_r \psi(r)| \right\} \leq \frac{1}{3}$$

for $r \geq R$. By (2.9), we have

$$\partial^2_{rr} \psi(r) = -2(n-1) \left( 1 + (\partial_r \psi(r))^2 \right) r \partial_r \psi(r) + \psi(r)$$

It follows that

$$|\partial^2_{rr} \psi(r)| \leq C(n) r^{\alpha-2}$$

for $r \geq R$. Continuing differentiating the equation of $\psi(r)$ and using induction yields

$$|\partial^m_r \psi(r)| \leq C(n, m) r^{\alpha-m}$$

for $r \geq R, m \in \mathbb{Z}_+$.

On the other hand, by the above choice of $R = R(n)$, we have

$$\sup_{\frac{\dot{\psi}(0)}{\sqrt{2}} \leq r \leq R} r^{m-\alpha} |\partial^m_r \psi(r)| \leq C(n, m)$$

for any $m \in \mathbb{Z}_+$. Therefore, we conclude that for any $m \in \mathbb{Z}_+$

$$|\partial^m_r \psi(r)| \leq C(n, m) r^{\alpha-m}$$

for $r \geq \frac{\dot{\psi}(0)}{\sqrt{2}}$. □

As a corollary, we have the following decay estimates for the higher order derivatives of $\psi_k(r)$.

**Lemma 2.4.** For any $m \geq 2$, there holds

$$|\partial^m_r \psi_k(r)| \leq C(n, m) kr^{\alpha-m}$$

for $r \geq 0$. 
Proof. By rescaling, it is sufficient to check for $k = 1$.

Let’s first parametrize the projected curve $\bar{M}$ by

$$Z = \left( r - \psi_k (r) \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \left( r + \psi_k (r) \right) \right)$$

In this parametrization, the normal curvature of $\bar{M}$ is given by

$$A_{\bar{M}} = \frac{\partial^2_{rr} \psi (r)}{\left( 1 + (\partial_r \psi (r))^2 \right)^{\frac{3}{2}}}$$

Let $\nabla_{\bar{M}}$ be the covariant derivative of $\bar{M}$, i.e.

$$\nabla_{\bar{M}} f = \frac{\partial_r f (r)}{\sqrt{1 + (\partial_r \psi (r))^2}}$$

for $f \in C^1 (\bar{M})$.

By Lemma 2.3 there is $R \gg 1$ (depending on $n$) so that

$$\max \left\{ \frac{\psi (r)}{r}, |\partial_r \psi (r)| \right\} \leq \frac{1}{3}$$

and

$$|Z|^m |\nabla_{\bar{M}}^m A_{\bar{M}}| \leq C (n, m) |Z|^{\alpha - 2}$$

for $r \geq R$, $m \in \mathbb{Z}_+$. Notice that

$$|Z| = \sqrt{r^2 + \psi^2 (r)}$$

is comparable with $r$ for $r \geq R$.

Next, let’s reparametrize $\bar{M}$ by

$$Z = \left( r, \hat{\psi} (r) \right)$$

In this parametrization, the normal curvature is given by

$$A_{\bar{M}} = \frac{\partial^2_{rr} \hat{\psi} (r)}{\left( 1 + (\partial_r \hat{\psi} (r))^2 \right)^{\frac{3}{2}}}$$

and the covariant derivative is defined by

$$\nabla_{\bar{M}} f = \frac{\partial_r f (r)}{\sqrt{1 + (\partial_r \hat{\psi} (r))^2}}$$

for $f \in C^1 (\bar{M})$.

Note also that by (2.3), we have

$$0 \leq \hat{\psi} (r) \leq C (n)$$

$$0 \leq \partial_r \hat{\psi} (r) \leq 1$$

for $r \geq R = R (n)$. Then by (2.10), (2.11), (2.12), (2.13) and (2.14), we infer that

$$|\partial_r^m \hat{\psi} (r)| \leq C (n, m) r^{\alpha - m}$$

for $r \geq 2R$, $m \geq 2$. 

On the other hand, by the above choice of \( R = R(n) \), there holds
\[
\sup_{0 \leq r \leq 2R} r^{m-\alpha} \left| \partial_r^m \hat{\psi}(r) \right| \leq (2R)^{m-\alpha} \sup_{0 \leq r \leq 2R} \left| \partial_r^m \hat{\psi}(r) \right| \leq C(n, m)
\]
for any \( m \geq 2 \). Consequently, we get
\[
\left| \partial_r^m \hat{\psi}(r) \right| \leq C(n, m) r^{\alpha-m}
\]
for \( r \geq 0, m \geq 2 \).

Lastly, we conclude this section by estimating the difference between \( \psi_k \) and its asymptotic function appeared in (2.9).

**Lemma 2.5.** The function \( \psi_k(r) \) satisfies
\[
|\psi_k(r) - kr^\alpha| \leq C(n) k^3 r^{3\alpha - 2}
\]
\[
|\partial_r \psi_k(r) - k\alpha r^{\alpha-1}| \leq C(n) k^3 r^{3\alpha - 3}
\]
for \( r \geq \hat{\psi}(0) \sqrt{2} \).

**Proof.** Without loss of generality, we may assume \( k = 1 \).

First, let’s rewrite the equation of \( \psi(r) \) as
\[
(2.15) \quad r \partial_r^2 \psi = -2(n-1) \frac{1 + (\partial_r \psi)^2}{1 - \left( \frac{\psi}{r} \right)^2} \left( \partial_r \psi + \frac{\psi}{r} \right)
\]
Let
\[
P = \partial_r \psi, \quad Q = \frac{\psi(r)}{r}
\]
and
\[
h = \ln(r)
\]
Then from (2.15), we deduce
\[
(2.16) \quad \left\{ \begin{array}{l}
\partial_h P = -2(n-1) \frac{1 + P^2}{1 - Q^2} (P + Q) \\
\partial_h Q = P - Q
\end{array} \right.
\]
On the other hand, by (2.1), we can also deduce that
\[
r \partial_r^2 r^\alpha = -2(n-1) \left( \partial_r r^\alpha + \frac{r^\alpha}{r} \right)
\]
Let
\[
P_* = \partial_r r^\alpha = \alpha r^{\alpha-1}, \quad Q_* = \frac{r^\alpha}{r} = r^{\alpha-1}
\]
and
\[
h = \ln(r)
\]
Similarly, there holds
\[
(2.17) \quad \left\{ \begin{array}{l}
\partial_h P_* = -2(n-1) (P_* + Q_*) \\
\partial_h Q_* = P_* - Q_*
\end{array} \right.
\]
Now subtract (2.17) from (2.16) to get
\[
\begin{align*}
\partial_h (P - P^*) &= -2(n - 1)((P - P^*) + (Q - Q^*)) - 2(n - 1) \frac{(P^2 + Q^2)(P + Q)}{1 - Q^2} \\
\partial_h (Q - Q^*) &= (P - P^*) - (Q - Q^*)
\end{align*}
\]
Note that by (2.9) we have
\[
\lim_{r \to \infty} \frac{\psi(r) - r^\alpha}{r^\alpha} = 0 = \lim_{r \to \infty} \frac{\partial_r \psi(r) - \alpha r^{\alpha - 1}}{r^{\alpha - 1}}
\]
which implies
\[
\begin{align*}
P - P^*_&= \partial_r \psi(r) - \alpha r^{\alpha - 1} = o \left(r^{\alpha - 1}\right) = o \left(e^{(\alpha - 1)h}\right) \\
Q - Q^*_&= \frac{\psi(r)}{r} - r^{\alpha - 1} = o \left(r^{\alpha - 1}\right) = o \left(e^{(\alpha - 1)h}\right)
\end{align*}
\]
as \(h \to \infty\). Now let
\[
\Theta = \begin{pmatrix} P - P^* \\ Q - Q^* \end{pmatrix}, \quad f(h) = \begin{pmatrix} -2(n - 1) \frac{(P^2 + Q^2)(P + Q)}{1 - Q^2} \\ 0 \end{pmatrix}
\]
and
\[
L = \begin{pmatrix} 2(n - 1) & 2(n - 1) \\ -1 & 1 \end{pmatrix}
\]
Then we have
\[
\begin{align*}
\partial_h \Theta + L \Theta &= f \\
\Theta(h) &= o \left(e^{(\alpha - 1)h}\right) \quad \text{as } h \to \infty
\end{align*}
\]
Notice that
\[
L = \begin{pmatrix} \alpha & \bar{\alpha} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -\alpha + 1 & 0 \\ 0 & -\bar{\alpha} + 1 \end{pmatrix} \begin{pmatrix} \alpha & \bar{\alpha} \end{pmatrix}^{-1}
\]
where
\[
\bar{\alpha} = \frac{-(2n - 3) - \sqrt{4n^2 - 20n + 17}}{2} < \alpha
\]
and
\[
|f(h)| \leq C(n) e^{3(\alpha - 1)h} \quad \text{for } h \geq \ln \left(\frac{\sqrt{\psi(0)}}{2}\right)
\]
It follows that for any \(R > h \geq \ln \left(\frac{\sqrt{\psi(0)}}{2}\right)\),
\[
|\Theta(h)| \leq e^{(R - h)(\alpha - 1)} |\Theta(R)| \quad + \quad \int_h^R e^{(R - h)(\alpha - 1)} |f(\xi)| d\xi
\]
\[
\leq \left(e^{(\alpha - 1)R} |\Theta(R)|\right) e^{(\alpha - 1)h} + C(n) e^{3(\alpha - 1)h}
\]
Note that
\[
\Theta(R) = o \left(e^{(\alpha - 1)R}\right)
\]
as $R \to \infty$ by (2.13). Let $R \to \infty$ to get
\[|\Theta (\mathfrak{h})| \leq C (n) e^{3(\alpha - 1)\mathfrak{h}} \quad \text{for} \quad \mathfrak{h} \geq \ln \left( \frac{\hat{\psi} (0)}{\sqrt{2}} \right)\]
which yields
\[|\nabla \hat{\psi} (r) - \alpha r^{\alpha - 1}| + \left| \frac{\psi (r)}{r} - r^{\alpha - 1} \right| \leq C (n) r^{3(\alpha - 1)} \quad \text{for} \quad r \geq \frac{\hat{\psi} (0)}{\sqrt{2}}\]
}\]

3. Admissible mean curvature flow

Let $n \geq 5$ be a positive integer and $\Lambda = \Lambda (n) \gg 1, \quad 0 < \rho \ll 1 \ll \beta$ (depending on $n$, $\Lambda$), $t_0 < 0$ with $|t_0| \ll 1$ (depending on $n$, $\Lambda$, $\rho$, $\beta$) be constants to be determined. Recall that an one-parameter family of smooth hypersurfaces $\{\Sigma_t\}_{t \leq t_0}$ in $\mathbb{R}^{2n}$, where $t < 0$ is a constant, is called a mean curvature flow (MCF) provided that
\[
\partial_t \Sigma_t \cdot N_{\Sigma_t} = H_{\Sigma_t}
\]
where $X_t$ is the position vector, $N_{\Sigma_t}$ and $H_{\Sigma_t}$ are the unit normal vector and mean curvature of $\Sigma_t$, respectively. We define the MCF $\{\Sigma_t\}_{t \leq t_0}$ to be admissible if every time-slice $\Sigma_t$ is a complete, embedded and smooth hypersurface which satisfies
\[
(1) \quad \Sigma_t \text{ is } O (n) \times O (n) \text{ symmetric and it can be parametrized as }
\]
\[
\Sigma_t = \{(x \nu, \hat{u} (x, t) \omega) \mid x \geq 0; \nu, \omega \in \mathbb{S}^{n - 1}\}
\]
where $\hat{u} (x, t)$ is a smooth function which satisfies
\[
\partial_t \hat{u} = \frac{\partial^2_{xx} \hat{u}}{1 + (\partial_x \hat{u})^2} + (n - 1) \left( \frac{\partial_x \hat{u}}{x} - \frac{1}{\hat{u}} \right)
\]
\[
\hat{u} (0, t) > 0, \quad \partial_x \hat{u} (0, t) = 0
\]
for $t_0 \leq t \leq \hat{t}$. Note that the above condition means that the projected curve
\[
\bar{\Sigma}_t = \{(x, \hat{u} (x, t)) \mid x \geq 0\}
\]
lives in the first quadrant and intersects orthogonally with the vertical ray $\{(0, x) \mid x > 0\}$.
(2) The projected curve $\bar{\Sigma}_t$ is a graph over $\mathcal{C}$ outside $B \left(O; \beta (-t)^{\frac{1}{2} + \sigma}\right)$, where
\[
\sigma = -\frac{1}{2} + \frac{2}{1 - \alpha} \in \left[ \frac{1}{6}, \frac{1}{2} \right]
\]
Equivalently, this is saying that $\Sigma_t$ is a normal graph over $\mathcal{C}$ outside $B \left(O; \beta (-t)^{\frac{1}{2} + \sigma}\right)$. In other words, we can reparametrize $\Sigma_t$ by
\[
X_t (x, \nu, \omega) = \left( x - u (x, t) - \frac{\nu}{\sqrt{2}} (x + u (x, t) \frac{\omega}{\sqrt{2}} \right)
\]
for $x \geq \beta (-t)^{\frac{1}{2} + \sigma}, \nu, \omega \in \mathbb{S}^{n - 1}$, where $u (x, t)$ is a smooth function satisfying
\[
\partial_t u = \frac{\partial^2_{xx} u}{1 + (\partial_x u)^2} + 2 (n - 1) \frac{x \partial_x u + u}{x^2 - u^2}
\]
(3) For the function \( u(x, t) \), there holds
\[
(3.8) \quad x^i |\partial_x^i u(x, t)| < \Lambda \left( (-t)^2 x^n + x^{2\lambda_2+1} \right), \quad i \in \{0, 1, 2\}
\]
for \( \beta (-t)^{\frac{1}{2}+\sigma} \leq x \leq \rho, \ t_0 \leq t \leq \tilde{t} \), where \( \lambda_2 = \frac{1}{2} (\alpha + 3) \) is a constant (see Proposition 3.1).

In order to analyze an admissible MCF, below we divide the space into three (time-dependent) regions and do proper rescaling for small regions.

- **The outer region** \( \Sigma_t \setminus B(O; \sqrt{-t}) \)
- **The intermediate region** \( \Sigma_t \cap \left( B(O; \sqrt{-t}) \setminus B(O; \beta (-t)^{\frac{1}{2}+\sigma}) \right) \): here we perform the "type I" rescaling

\[
(3.9) \quad \Pi_s = \frac{1}{\sqrt{-t}} \Sigma_t \bigg|_{t=-e^{-s}}
\]

By this rescaling, the intermediate region is then dilated to become
\[
\Pi_s \cap \left( B(O; 1) \setminus B(O; \beta e^{-\sigma s}) \right)
\]
for \( s_0 \leq s \leq \tilde{s} \), where \( s_0 = -\ln (-t_0) \) and \( \tilde{s} = -\ln (-\tilde{t}) \). Note that \( s_0 \gg 1 \) iff \( |t_0| \ll 1 \).

- **The tip region** \( \Sigma_t \cap B(O; \beta (-t)^{\frac{1}{2}+\sigma}) \): here we perform the "type II" rescaling

\[
(3.10) \quad \Gamma_s = \frac{1}{(-t)^{\frac{1}{2}+\sigma}} \Sigma_t \bigg|_{t=-(2\sigma \tau)^{-1}}
\]

By this rescaling, the intermediate region is dilated to become
\[
\Gamma_s \cap B(O; \beta)
\]
for \( \tau_0 \leq \tau \leq \tilde{\tau} \), where \( \tau_0 = \frac{1}{2\sigma(-t_0)} \), \( \tilde{\tau} = \frac{1}{2\sigma(-\tilde{t})} \). Note that \( \tau_0 \gg 1 \) iff \( |t_0| \ll 1 \).

In the outer region, we parametrize \( \Sigma_t \) by
\[
X_t(x, \nu, \omega) = \frac{1}{\sqrt{2}} \left( (x - u(x, t)) \nu \right) + \frac{1}{\sqrt{2}} \left( (x + u(x, t)) \frac{\omega}{\sqrt{2}} \right)
\]
and study the function \( u(x, t) \) via (3.7). In \( B(O; \rho) \setminus B(O; \sqrt{-t}) \), Velázquez showed that by choosing suitable initial data (see Section 4), there holds
\[
u(x, t) \sim x^{2\lambda_2+1}
\]
However, the behavior outside \( B(O; \rho) \) was not clear in [V]. In this paper we complete this part by providing smooth estimate for \( \Sigma_t \setminus B(O; \rho) \).

In the intermediate region, we first do the type I rescaling and parametrize the rescaled hypersurface \( \Pi_s \) by
\[
(3.11) \quad Y_s(y, \nu, \omega) = \frac{1}{\sqrt{2}} \left( (y - v(y, s)) \nu \right) + \frac{1}{\sqrt{2}} \left( (y + v(y, s)) \frac{\omega}{\sqrt{2}} \right)
\]
where
\[
(3.12) \quad v(y, s) = \frac{1}{\sqrt{-t}} u \left( \sqrt{-t} y, t \right) \bigg|_{t=-e^{-s}}
\]
From (3.7), we derive
\begin{equation}
\partial_x v = \frac{\partial_{yy} v}{1 + (\partial_y v)^2} + 2(n-1) \frac{y \partial_y v + v}{y^2 - v^2} + \frac{1}{2} \left(-y \partial_y v + v\right)
\end{equation}

Notice that (3.8) is equivalent to
\begin{equation}
y_1^1 \left| \partial_y v (y, s) \right| < \Lambda e^{-\lambda s} \left(y^\alpha + y^{2\lambda s+1}\right), \quad i \in \{0, 1, 2\}
\end{equation}
for \(\beta e^{-\alpha}s \leq y \leq \beta e^{s}, s_0 \leq s \leq s\). To study the function \(v(y, s)\), Velázquez linearized (3.13) and showed that
\[
v(y, s) \sim e^{-\lambda_2 s} \varphi_2(y)
\]
by (3.14) and the choice of initial data (see Section II). Where \(\lambda_2\) and \(\varphi_2(y)\) are the first positive eigenvalue and eigenfunction of the linearized operator (see Proposition 3.1). More precisely, (3.13) can be rewritten as
\begin{equation}
\partial_x v = -\mathcal{L}v + \mathcal{Q}v
\end{equation}
where
\begin{equation}
\mathcal{L}v = - \left(\frac{\partial_{yy} v + 2(n-1) y \partial_y v + v}{1 + (\partial_y v)^2} + \frac{1}{2} \left(-y \partial_y v + v\right)\right)
\end{equation}
\[
= - \left(y^{2(n-1)} e^{-\frac{y^2}{2}}\right)^{-1} \partial_y \left(y^{2(n-1)} e^{-\frac{y^2}{2}} \partial_y v\right) - \left(\frac{2(n-1)}{y^2} + \frac{1}{2}\right) v
\]
is the (negative) linearization of the RHS of (3.13), and
\begin{equation}
\mathcal{Q}v = - \frac{\left(\partial_y v\right)^2}{1 + (\partial_y v)^2} \partial_{yy} v + 2(n-1) \left(\frac{v}{y}\right)^2 \left(\frac{\partial_y v}{y} + \frac{v}{y^2}\right)
\end{equation}
is the remaining (quadratic) parts. Velázquez showed that the linear differential operator \(\mathcal{L}\) has the following properties (see [V]):

**Proposition 3.1.** Define an inner product
\[
\langle v_1, v_2 \rangle = \int_0^\infty v_1(y) v_2(y) y^{2(n-1)} e^{-\frac{y^2}{2}} dy
\]
and the associated norm
\[
\|v\| = \sqrt{\langle v, v \rangle}
\]
Let \(\mathcal{H}\) be the Hilbert space formed by the completion of \(C_c^\infty (\mathbb{R}_+)\) with respect to the following inner product:
\[
\langle v_1, v_2 \rangle \equiv \langle \partial_y v_1, \partial_y v_2 \rangle + \langle v_1, v_2 \rangle
\]
Then we have
\[
\left\| \frac{v}{y} \right\|^2 \leq \frac{4}{(2n-3)^2} \left\| \partial_y v \right\|^2 + \frac{1}{2n-3} \left\| v \right\|^2
\]
and \(\mathcal{L}\) is a bounded linear operator in \(\mathcal{H}\), which satisfies
\[
\langle \mathcal{L}v_1, v_2 \rangle = \langle \partial_y v_1, \partial_y v_2 \rangle - 2(n-1) \left(\frac{v_1}{y}, \frac{v_2}{y}\right) - \frac{1}{2} \langle v_1, v_2 \rangle
\]
\begin{equation}
\langle \mathcal{L}v, v \rangle \geq \frac{4n^2 - 20n + 17}{(2n-3)^2} \left\| \partial_y v \right\|^2 - \frac{6n - 7}{2(2n-3)} \left\| v \right\|^2
\end{equation}
Note that $4n^2 - 20n + 17 \geq 1$ if $n \geq 4$.

Moreover, the eigenvalues and eigenfunctions of $\mathcal{L}$ are given by

\[(3.19) \quad \lambda_i = -\frac{1}{2} (1 - \alpha) + i, \quad \text{for } i = 0, 1, 2, \cdots \]

and

\[\varphi_i (y) = c_i y^\alpha M \left( -i, n + \alpha - \frac{1}{2}; \frac{y^2}{4} \right)\]

respectively, where $c_i > 0$ is the normalized constant so that

\[\|\varphi_i\| = \sqrt{\langle \varphi_i, \varphi_i \rangle} = 1 \]

and $M (a, b; \xi)$ is the Kummer’s function defined by

\[M (a, b; \xi) = 1 + \sum_{j=1}^{\infty} \frac{a (a + 1) \cdots (a + j - 1)}{b (b + 1) \cdots (b + j - 1)} \xi^j j!\]

and satisfying

\[\xi \partial_{\xi}^2 M (a, b; \xi) + (b - \xi) \partial_{\xi} M (a, b; \xi) - a M (a, b; \xi) = 0\]

In addition, the family of eigenfunctions $\{\varphi_i\}_{i=0, 1, 2, \cdots}$ forms a complete orthonormal set in $\mathbf{H}$, and $\lambda_2$ is the first positive eigenvalue of $\mathcal{L}$, i.e.

\[\lambda_0, \lambda_1 < 0, \quad \lambda_2 > 0\]

Remark 3.2. The first three eigenfunctions of $\mathcal{L}$ are given by

\[\varphi_0 (y) = c_0 y^\alpha\]

\[\varphi_1 (y) = c_1 y^\alpha \left( 1 + \mathcal{Y}_1 y^2 \right)\]

\[\varphi_2 (y) = c_2 y^\alpha \left( 1 + 2 \mathcal{Y}_1 y^2 + \mathcal{Y}_2 y^4 \right)\]

where

\[\mathcal{Y}_1 = \frac{-1}{4 (n + \alpha - \frac{1}{2})}, \quad \mathcal{Y}_2 = \frac{1}{16 (n + \alpha - \frac{1}{2}) (n + \alpha + \frac{1}{2})}\]

Note that

\[\partial_{yy}^2 \varphi_2 (y) = c_2 y^{\alpha - 2} \left( \alpha (\alpha - 1) + 2 \mathcal{Y}_1 (\alpha + 2) y^2 + \mathcal{Y}_2 (\alpha + 4) (\alpha + 3) y^4 \right) > 0\]

for $y > 0$. In addition, for those constants, there hold

\[\alpha + 4 = 2 \lambda_2 + 1\]

\[\sigma = \frac{\lambda_2}{1 - \alpha}\]

Furthermore, when $n \gg 1$, we have

\[\alpha \approx -1 - \frac{1}{n}, \quad \sigma \approx \frac{1}{2} - \frac{1}{2n}\]

\[\lambda_0 \approx -1 - \frac{1}{2n}, \quad \lambda_1 \approx -\frac{1}{2n}, \quad \lambda_2 \approx 1 - \frac{1}{2n}\]
Lastly, in the tip region, we do the type II rescaling to get
\[
\Gamma_{\tau} = \{(z, w(z, \tau)) \mid z \geq 0; \nu, \omega \in S^{n-1}\}
\]
where
\[
\hat{w}(z, \tau) = \frac{1}{(-t)^{\frac{1}{2}+\sigma}} \hat{u}\left((-t)^{\frac{1}{2}+\sigma} z, t\right)_0 = -(2\sigma \tau)^{-\frac{1}{2}}\sigma
\]
From (3.3) we derive
\[
\partial_{\tau} \hat{w} = \frac{\partial_{z}^2 \hat{w}}{1 + (\partial_{z} w)^2} + (n-1) \left(\frac{\partial_{z} \hat{w}}{z} - \frac{1}{\hat{w}}\right) + \frac{1}{2\sigma\tau} (-z \partial_{z} \hat{w} + \hat{w})
\]
\[
\hat{w}(0, \tau) > 0, \quad \partial_{z} \hat{w}(0, \tau) = 0
\]
for \(\tau_0 \leq \tau \leq \hat{\tau}\). Velázquez showed that by choosing suitable initial data (see Section 4), there holds
\[
\hat{w}(z, \tau) = e^{\sigma z} v\left(e^{-\sigma z}, s\right)|_{s = \frac{\tau}{2\sigma\ln(2\sigma\tau)}}
\]
for some \(k \approx 1\), where \(\hat{\psi}_k\) is the function defined in Section 2. On the other hand, by the admissible condition and rescaling, we can regard the rescaled projected curve
\[
\bar{\Gamma}_{\tau} = \{(z, \hat{w}(z, \tau)) \mid z \geq 0\}
\]
as a graph over \(\bar{C}\) outside \(B(O; \beta)\). In other words, \(\Gamma_{\tau}\) can be reparametrized as a normal graph over \(C\) outside \(B(O; \beta)\), say
\[
Z_{\tau}(z, \nu, \omega) = \left((z - w(z, \tau)) \frac{\nu}{\sqrt{2}} (z + w(z, \tau)) \frac{\omega}{\sqrt{2}}\right)
\]
for \(z \geq \beta\), where
\[
w(z, \tau) = \frac{1}{(-t)^{\frac{1}{2}+\sigma}} u\left((-t)^{\frac{1}{2}+\sigma} z, t\right)_0 = -(2\sigma \tau)^{-\frac{1}{2}}\sigma
\]
From (3.7) we derive
\[
\partial_{\tau} w = \frac{\partial_{z}^2 w}{1 + (\partial_{z} w)^2} + 2(n-1) \frac{z \partial_{z} w + w}{z^2 - w^2} + \frac{1}{2\sigma\tau} (-z \partial_{z} w + w)
\]
Notice that (3.8) is equivalent to
\[
z^i |\partial_{z}^i w(z, \tau)| < \Lambda z^\alpha + \left(\frac{z^{2\lambda_2+1}}{(2\sigma\tau)^2}\right), \quad i \in \{0, 1, 2\}
\]
for \(\beta \leq z \leq \rho(2\sigma\tau)^{\frac{1}{2}+\frac{\sigma}{2\tau}}, \tau_0 \leq \tau \leq \hat{\tau}\).
ANALYSIS OF VELÁZQUEZ’S SOLUTION TO THE MCF

4. Construction of Velázquez’s solution

For readers’ convenience and also for the sake of the completeness of the argument, in this section we show how Velázquez’s solution is constructed. We basically follow Velázquez’s idea in [V] and modify his proofs and estimates. Also, our setting is slightly different from that in [V] since we assume more conditions in order to get better results. The key step is Proposition 4.4 and Proposition 4.5. The main theorem in this section is Theorem 4.8.

The idea is as follows. At the initial time \( t_0 \), we would choose a bunch of “initial hypersurfaces” \( \{ \Sigma_{(a_0, a_1)}^{t_0} \} \) (as candidates) and move each of them by the mean curvature vector. We then manage to show that for each \( \delta t \in [t_0, 0) \), there is an index \((a_0, a_1)\) for which the corresponding mean curvature flow \( \{ \Sigma_{(a_0, a_1)}^{t_0} \} \) exits and is admissible up to time \( \delta t \). In addition, we would establish uniform estimates for these solutions. Lastly, by the compactness theory, we then get a solution to the MCF which exists and is admissible for \( t_0 \leq t < 0 \) and also admits those uniform estimates.

Let’s start with choosing a proper family of initial hypersurfaces. Let

\[
\left\{ \Sigma_{(a_0, a_1)}^{t_0} \left| (a_0, a_1) \in B^2 \left( 0; \beta^{2(a-1)} \right) \right. \right\}
\]

be a continuous two-parameters family of complete, embedded and smooth hypersurfaces so that each element \( \Sigma_{(a_0, a_1)}^{t_0} \) is admissible at time \( t_0 \) and satisfies

1. The function \( v(y, s_0) = v^{(a_0, a_1)}(y, s_0) \) (defined in (3.11)) of the type I rescaled hypersurface

\[
\Pi_{(a_0, a_1)}^{s_0} = \frac{1}{\sqrt{-t_0}} \Sigma_{(a_0, a_1)}^{s_0}
\]

is given by

\[
v(y, s_0) = e^{-\lambda s_0} \left( \frac{1}{c_2} \varphi_2(y) + \frac{a_1}{c_1} \varphi_1(y) + \frac{a_0}{c_0} \varphi_0(y) \right)
\]

\[
e^{-\lambda s_0} y^\alpha \left( 1 + a_1 + a_0 + (2 + a_1) \gamma y^2 + \gamma y^4 \right)
\]

for \( \frac{1}{2} \beta e^{-\sigma s_0} \leq y \leq 2pe^{\frac{s_0}{\beta}} \) (see Proposition 3.1 and Remark 3.2).

2. The function \( u(x, t_0) = u^{(a_0, a_1)}(x, t_0) \) (defined in (3.6)) of \( \Sigma_{(a_0, a_1)}^{t_0} \) is chosen to be

\[
u(x, t_0) \approx \frac{\gamma x^{2\lambda_2+1}}{1 + x^4}
\]

for \( x \geq \rho \) so that

\[
|u(x, t_0)| \leq \frac{1}{3} \min \{ x, 1 \}
\]

\[
|\partial_x u(x, t_0)| \leq \frac{1}{5}
\]

\[
|\partial^2_{xx} u(x, t_0)| \leq C(n, \rho)
\]

for \( x \geq \frac{1}{3} \rho \).
The following remark shows that (4.1) fits in with the admissible condition and is compatible with (4.2).

\[
\Gamma^{(a_0, a_1)}_{\tau_0} = \frac{1}{(-t_0)^{\frac{a_0}{2} + \sigma}} \sum_{i}^{(a_0, a_1)}
\]

is chosen to be
\[
\hat{w}(z, \tau_0) \approx \psi_{1+a_1+a_0}(z)
\]

for \(0 \leq z \leq \beta\) so that
\[
\begin{align*}
\hat{\psi}_{1-\beta \frac{2a+1}{2}}(z) &< \hat{w}(z, \tau_0) < \hat{\psi}_{1+\beta \frac{2a-1}{2}}(z) \\
0 = \partial_z \hat{w}(0, \tau_0) &\leq \partial_z \hat{w}(z, \tau_0) < 1 \\
0 < \partial^2_z \hat{w}(z, \tau_0) &\leq C(n)
\end{align*}
\]

for \(0 \leq z \leq 5\beta\). Furthermore, if we reparametrize the projected curve \(\Gamma^{(a_0, a_1)}_{\tau_0}\) as a graph over \(\hat{C}\), the function \(w^{(a_0, a_1)}(z, \tau_0) = w(z, \tau_0)\) (defined in (3.24)) satisfies
\[
w(z, \tau_0) \approx \psi_{1+a_1+a_0}(z)
\]

for \(1 \leq z \leq \beta\) so that
\[
\begin{align*}
0 \leq w(z, \tau_0) &\leq C(n) z^\alpha \\
|\partial_z w(z, \tau_0)| &\leq C(n) z^{\alpha-1} \\
0 < \partial^2_z w(z, \tau_0) &\leq C(n) z^{\alpha-2}
\end{align*}
\]

for \(\hat{w}(0) \leq z \leq 5\beta\).

The following remark shows that (4.1) fits in with the admissible condition and is compatible with (4.2).

**Remark 4.1.** By (5.12) and Remark 3.2, (4.1) is equivalent to
\[
u(x, t_0) = (-t)^{\lambda_2 + \frac{1}{2}} \left( \frac{1}{c_2} \phi_2 \left( \frac{x}{\sqrt{-t}} \right) + \frac{a_1}{c_1} \phi_1 \left( \frac{x}{\sqrt{-t}} \right) + \frac{a_0}{c_0} \phi_0 \left( \frac{x}{\sqrt{-t}} \right) \right)
\]

\[
= (1 + a_1 + a_0) (-t_0)^{2 \lambda_2 + 2} x^\alpha + (2 + a_1) \gamma_1 (-t_0) x^{\alpha+2} + \gamma_2 x^{2 \lambda_2 + 1}
\]

(4.5)

\[
= x^{2 \lambda_2 + 1} \left( \gamma_2 + (2 + a_1) \gamma_1 \left( \frac{-t_0}{x^2} \right) + (1 + a_1 + a_0) \left( \frac{-t_0}{x^2} \right)^2 \right)
\]

for \(\frac{1}{\lambda_2} (-t_0)^{\frac{1}{2} + \sigma} \leq x \leq 2\rho\). In particular, there hold
\[
x^i |\partial_x^i u(x, t)| \leq C(n) (-t)^2 x^\alpha + x^{2 \lambda_2 + 1}, \quad i \in \{0, 1, 2\}
\]

(4.6)

\[
\left| \frac{u(x, t_0)}{x} \right| \leq C(n) (\beta^{\alpha-1} + \rho^{2 \lambda_2})
\]
for $\frac{1}{2} \beta (-t_0)^{1+\sigma} \leq x \leq 2\rho$. Thus, we may assume that

$$x^i \left| \partial^i_x u (x, t) \right| \leq \frac{A}{3} \left( (-t)^2 x^\alpha + x^{2\lambda_2 + 1} \right), \quad i \in \{0, 1, 2\}$$

for $\beta (-t_0)^{1+\sigma} \leq x \leq \rho$, provided that $\Lambda \gg 1$ (depending on $n$). Also by \eqref{1.2}, \eqref{4.3} and \eqref{4.0}, we may assume that

$$\dot{u} (x, t_0) > 0$$

for $x \geq 0$, provided that $0 < \rho \ll 1 \ll \beta$ (depending on $n$). Furthermore, by \eqref{4.5} we have

$$u (x, t_0) = x^{2\lambda_2 + 1} \left( T_2 + O \left( \frac{-t_0}{x^2} \right) \right)$$

for $\sqrt{-t_0} \lesssim x \lesssim 2\rho$, which is comparable with \eqref{4.2} provided that $0 < \rho \ll 1$ (depending on $n$ and $|t_0| \ll 1$ (depending on $n$, $\rho$).

The following remark shows that \eqref{4.1}, \eqref{4.3} and \eqref{4.4} are compatible.

Remark 4.2. By \eqref{4.3}, $\Gamma_t^{(a_0, a_1)}$ (see \eqref{5.23}) is a convex curve which lies between $M_{1-\beta}^{1-\frac{2}{n-\frac{1}{2}}}$ and $M_{1+\beta}^{1+\frac{2}{n+-\frac{1}{2}}}$ (see \eqref{2.7}) and intersects orthogonally with the vertical ray $\{0, z\}$ $z > 0$. Hence, if we reparametrize $\tilde{\Gamma}_t^{(a_0, a_1)}$ as a graph over $\tilde{C}$, it follows that

$$\psi_{1-\beta}^{\frac{2}{n-\frac{1}{2}}} (z) < w (z, \tau_0) < \psi_{1+\beta}^{\frac{2}{n-\frac{1}{2}}} (z)$$

Then \eqref{4.4} is compatible with \eqref{4.3} in view of Lemma \ref{2.3}

On the other hand, by \eqref{3.25} and Remark \ref{3.2} \eqref{4.1} is equivalent to

$$w (z, \tau_0) = (2\sigma \tau_0)^{\frac{2}{\nu}} \left( \frac{1}{c_2} \varphi_2 \left( \frac{z}{\sqrt{2\sigma \tau_0}} \right) + \sum_{i=0}^1 \frac{a_i}{c_1} \varphi_1 \left( \frac{z}{\sqrt{2\sigma \tau_0}} \right) \right)$$

(4.7)

$$= z^\alpha \left( 1 + a_1 + a_0 + (2 + a_1) \frac{z^2}{2\sigma \tau_0} + T_2 \left( \frac{z^2}{2\sigma \tau_0} \right) \right)$$

for $\frac{1}{2} \beta \leq z \leq 2 \rho (2 \sigma \tau_0)^{\frac{2}{\nu} + \frac{\nu}{2}}$, which means

$$w (z, \tau_0) = \left( 1 + a_1 + a_0 + O \left( \frac{z^2}{2\sigma \tau_0} \right) \right) z^\alpha$$

for $\frac{1}{2} \beta \leq z \leq \sqrt{2\sigma \tau_0}$. By Lemma \ref{2.3} we then get

$$|w (z, \tau_0) - \psi (z)| \leq |w (z, \tau_0) - z^\alpha| + |z^\alpha - \psi (z)|$$

$$\leq \left( |a_0| + |a_1| + C (n) \left( \frac{z^2}{2\sigma \tau_0} + z^{2(\alpha - 1)} \right) \right) \leq C (n) \beta^{2(\alpha - 1)} z^\alpha$$

for $\frac{1}{2} \beta \leq z \leq (2 \sigma \tau_0)^{\frac{\nu}{2}}$, provided that $\beta \gg 1$ (depending on $n$) and $\tau_0 \gg 1$ (depending on $n$, $\beta$). Note also that Lemma \ref{2.5} yields

$$\psi_{1 \pm \beta}^{\frac{2}{n-\frac{1}{2}}} (z) - \psi (z) = \left( \pm \beta^{\frac{2}{n-\frac{1}{2}}} + O \left( z^{2(\alpha - 1)} \right) \right) z^\alpha$$

in which we have

$$\frac{3}{2} \alpha - \frac{5}{2} > 2 (\alpha - 1)$$
Consequently, we get

$$\psi_{1-\beta^{n-\frac{3}{2}}} (z) < \psi_{1+\beta^{n-\frac{3}{2}}} (z)$$

for \( \frac{1}{2} \beta \leq z \leq (2\sigma \tau_0)^\frac{1}{\beta} \), provided that \( \beta \gg 1 \) (depending on \( n \)) and \( \tau_0 \gg 1 \) (depending on \( n, \beta \)).

Next, for each \((a_0, a_1) \in \overline{B^2 (O; \beta^{2(\alpha-1)})}\), by \([EH] \Sigma_t^{(a_0, a_1)} \) can be flowed by (4.1) for a short period of time. Let’s denote the corresponding solution by \( \Sigma_t^{(a_0, a_1)} \).

Given \( t \in [t_0, 0) \), let \( \mathcal{O}_t \) be a set consisting of all \((a_0, a_1) \in B^2 (O; \beta^{2(\alpha-1)})\) for which

- The corresponding mean curvature flow \( \{ \Sigma_t^{(a_0, a_1)} \} \) exists for \( t_0 \leq t \leq \hat{t} \)
- and can be extended beyond time \( \hat{t} \).
- \( \{ \Sigma_t^{(a_0, a_1)} \} \) is admissible for \( t_0 \leq t \leq \hat{t} \).

Clearly,

$$\mathcal{O}_{t_0} = B^2 (O; \beta^{2(\alpha-1)})$$

and \( \mathcal{O}_t \) is non-increasing in \( \hat{t} \).

Now let \( \zeta (r) \) be a smooth, non-decreasing function so that

$$\zeta (r) = \begin{cases} 0, & \text{for } r \leq 0 \\ 1, & \text{for } r \geq 1 \end{cases}$$

(4.8)

For each \( t \geq t_0 \), we define a map \( \Phi_t : \mathcal{O}_t \rightarrow \mathbb{R}^2 \) by

$$\Phi_t (a_0, a_1) = \left( \begin{array}{c} \langle \zeta (e^{\sigma s} y - \beta) \zeta (pe^{\sigma s} y) \psi (\cdot, s), c_0 \varphi_0 \rangle \\ \langle \zeta (e^{\sigma s} y - \beta) \zeta (pe^{\sigma s} y) \psi (\cdot, s), c_1 \varphi_1 \rangle \end{array} \right) \bigg|_{s = -\ln (-t)}$$

where the inner product \( \langle \cdot, \cdot \rangle \) is defined in Proposition 3.1 and \( v (y, s) = v^{(a_0, a_1)} (y, s) \) is the function of \( \Pi_t^{(a_0, a_1)} \) defined in (3.1) with \( s = -\ln (-t) \). Note that the localized function

$$\tilde{v} (y, s) = \zeta (e^{\sigma s} y - \beta) \zeta (pe^{\sigma s} y) v (y, s)$$

appeared in (4.9) is supported in \([\beta e^{-\sigma s}, pe^{\sigma s}]\) and would be studied carefully in Proposition 6.4. When \( t = t_0 \), we have the following lemma.

**Lemma 4.3.** If \( s_0 \gg 1 \) (depending on \( n, \rho, \beta \)), there hold

$$\left| \langle \zeta (e^{\sigma s_0} y - \beta) \zeta (pe^{\sigma s_0} y) \varphi_i, \varphi_j \rangle - \delta_{ij} \right| \leq C \left( n \right) e^{-2(n+\alpha-\frac{1}{2}) s_0}$$

$$\left\| (1 - \zeta (e^{\sigma s_0} y - \beta) \zeta (pe^{\sigma s_0} y) \varphi_i) \right\| \leq C \left( n \right) e^{-(n+\alpha-\frac{1}{2}) s_0}$$

for \( i, j \in \{0, 1, 2\} \), where \( s_0 = -\ln (-t_0) \) and \( \varphi_i \) is the \( i \)th eigenfunction of \( \mathcal{L} \) (see Proposition 5.7).

**Proof.** Notice that

$$\langle \varphi_i, \varphi_j \rangle = \delta_{ij}$$

and

$$\zeta (e^{\sigma s_0} y - \beta) \zeta (pe^{\sigma s_0} y) \rightarrow 1 \quad \text{as } s_0 \searrow \infty$$
Then we compute
\[
\left| \left\langle \zeta \left( e^{s \sigma_0} y - \beta \right) \zeta \left( pe^{\frac{\sigma_0}{\delta}} - y \right) \varphi_i, \varphi_j \right\rangle - \delta_{ij} \right|
\leq \int_0^{(\beta+1)e^{-\sigma_0}} |\varphi_i| y^{2(n-1)} e^{-\frac{y^2}{4}} dy + \int_{\rho e^{-1}}^{\infty} |\varphi_i| y^{2(n-1)} e^{-\frac{y^2}{4}} dy
\]
\[
\leq C(n) \left( \int_0^{(\beta+1)e^{-\sigma_0}} y^2 e^{\frac{\sigma_0}{\rho}} dy + \int_{\rho e^{-1}}^{\infty} y^{2(\lambda+2\lambda+2(n-1)) e^{-\frac{y^2}{4}}} dy \right)
\leq C(n) e^{-2(n+\frac{1}{2})\sigma_0}
\]

It follows that
\[
\left\| \left( 1 - \zeta \left( e^{s \sigma_0} y - \beta \right) \zeta \left( pe^{\frac{\sigma_0}{\delta}} - y \right) \right) \varphi_i \right\|^2
= \left\langle \left( 1 - \zeta \left( e^{s \sigma_0} y - \beta \right) \zeta \left( pe^{\frac{\sigma_0}{\delta}} - y \right) \right) \varphi_i, \left( 1 - \zeta \left( e^{s \sigma_0} y - \beta \right) \zeta \left( pe^{\frac{\sigma_0}{\delta}} - y \right) \right) \varphi_i \right\rangle
\leq \left\langle \left( 1 - \zeta \left( e^{s \sigma_0} y - \beta \right) \zeta \left( pe^{\frac{\sigma_0}{\delta}} - y \right) \right) \varphi_i, \varphi_i \right\rangle
\leq C(n) e^{-2(n+\frac{1}{2})\sigma_0}
\]

By (4.11) and Lemma 4.3, the function \( \Phi_{t_0} \) converges uniformly to the identity map in \( B^2 \left( O; \beta^{2(\alpha-1)} \right) \) as \( t_0 \to 0 \). Thus, if \( |t_0| \ll 1 \) (depending on \( n, \beta \)), we have
\[
(0, 0) \notin \Phi_{t_0} \left( \partial B^2 \left( O; \beta^{2(\alpha-1)} \right) \right)
\]
and
\[
1 = \deg \left( \text{Id}, B^2 \left( O; \beta^{2(\alpha-1)} \right), (0, 0) \right) = \deg \left( \Phi_{t_0}, B^2 \left( O; \beta^{2(\alpha-1)} \right), (0, 0) \right)
\]
(4.10)
\[
= \deg \left( \Phi_{t_0}, \mathcal{O}, (0, 0) \right)
\]
In addition, notice that \( \mathcal{O} \) is an open subset of \( B^2 \left( O; \beta^{2(\alpha-1)} \right) \) (by the continuous dependence on the initial data), and that \( \Phi_t \) is continuous in the parameter \( t \). Then we consider the following index set
\[
\mathcal{I} = \{ t \in [t_0, 0] \mid \deg \left( \Phi_t, \mathcal{O}, (0, 0) \right) = 1 \}
\]
Below are crucial a priori estimates of \( \left\{ \Sigma_{t \leq t_1} \right\} \) for which
\[
\Phi_{t_1} \left( a_0, a_1 \right) = (0, 0)
\]
We leave the proof in Section 5, Section 7, and Section 8.

**Proposition 4.4.** Let \( n \geq 5 \) be a positive integer and choose \( \varsigma = \varsigma(n) > 0 \), \( \vartheta = \vartheta(n) \in (0, 1) \) so that
\[
0 < \varsigma < \min \left\{ \frac{n+\alpha-\frac{5}{2}}{1-\alpha}, \frac{1}{\lambda_2} \right\}
\]
(4.11)
\[
\frac{-1-\alpha}{1-\alpha} < \vartheta < \min \left\{ \frac{(1-\alpha)\varsigma}{n+\alpha+\frac{1}{2}}, \frac{1-\alpha}{2-\alpha}, \frac{1}{2\sigma} \right\}
\]
(4.12)
Assume that \((a_0, a_1) \in \overline{O}_{t_1}\) for which
\[ \Phi_{t_1} (a_0, a_1) = (0, 0) \]
where \(t_1 \in [t_0, 0)\) is a constant. Suppose that
\[ (a_0, a_1) \in \overline{O}_t \]
for some \(t \in [t_1, e^{-1}t_1]\). Then if \(\Lambda > 1\) (depending on \(n\)), \(0 < \rho \ll 1 \ll \beta\) (depending on \(n, \Lambda\)) and \(|t_0| \ll 1\) (depending on \(n, \Lambda, \rho, \beta\)), we have the following estimates.

1. The function \(\dot{u}(x, t)\) defined in (3.3) satisfies
\[ \partial_{xx}^2 \dot{u}(x, t) \geq 0 \]
for \(0 \leq x \leq \rho, t_0 \leq t \leq \dot{t}\).

2. The function \(u(x, t)\) defined in (3.6) satisfies
\[ \left\{ \begin{array}{ll}
|u(x, t)| & \leq \frac{1}{\gamma} \min \{x, 1\} \\
|\partial_x u(x, t)| & \leq \frac{1}{\gamma} \\
|\partial_{xx}^2 u(x, t)| & \leq C(n, \rho) 
\end{array} \right. \]
for \(x \geq \frac{1}{\gamma} \rho, t_0 \leq t \leq \dot{t}\), and
\[ x^{|i|} \partial_{xx}^i u(x, t) \leq \frac{\Lambda}{2} \left((-t)^2 x^n + x^{2\lambda_2 + 1}\right), \quad i \in \{0, 1, 2\} \]
for \(\beta (-t)^{\frac{\beta - \sigma}{\sigma}} \leq x \leq \rho, t_0 \leq t \leq \dot{t}\).

3. In the tip region, if we do the type II rescaling, the rescaled function \(\hat{w}(z, \tau)\) defined in (3.21) satisfies
\[ \left\{ \begin{array}{l}
\hat{\psi}_{1-2\beta n-\lambda} (z) < \hat{w}(z, \tau) < \hat{\psi}_{1+2\beta n-\lambda} (z) \\
0 \leq \partial_z \hat{w}(z, \tau) \leq 1 + \beta^{-2} \\
|\partial_{zz}^2 \hat{w}(z, \tau)| \leq C(n) 
\end{array} \right. \]
for \(0 \leq z \leq 3\beta, \tau_0 \leq \tau \leq \dot{\tau}\), where \(\dot{\tau} = \frac{1}{2\sigma(-t)^{\sigma}}\).

Furthermore, we have the following asymptotic formulas and smooth estimates for the solution in Proposition (3.4)

**Proposition 4.5.** Under the hypothesis of Proposition (3.4), there is
\[ k \in \left(1 - C(n, \Lambda, \rho, \beta) (-t_0)^{\lambda_2}, 1 + C(n, \Lambda, \rho, \beta) (-t_0)^{\lambda_2}\right) \]
so that for any given \(0 < \delta \ll 1, m, l \in \mathbb{Z}_+\), the following smooth estimates hold.

1. In the outer region, the function \(u(x, t)\) of \(\Sigma_t^{(a_0, a_1)}\) defined in (3.6) satisfies
\[ |\partial_x^m \partial_t^l u(x, t)| \leq C(n, \rho, \delta, m, l) \]
for \(x \geq \frac{1}{\gamma} \rho, t_0 + \delta^2 \leq t \leq \dot{t}\), and
\[ x^{m+2l} \left| \partial_x^m \partial_t^l \left( u(x, t) - \frac{k}{c_2} (-t)^{\lambda_2 + \frac{1}{2}} \varphi \left( \frac{x}{\sqrt{1-t}} \right) \right) \right| \leq C(n, \Lambda, \rho, \delta, m, l) \rho^{4\lambda_2} x^{2\lambda_2 + 1} \]
for \((x, t)\) satisfying \(\frac{1}{2} \sqrt{-t} \leq x \leq \frac{4}{\delta} \rho, t_0 + \delta^2 x^2 \leq t \leq \tilde{t}\). Note that
\[
\frac{k}{c_2} (-t)^{\lambda_2 + \frac{1}{2}} \varphi_2 \left( \frac{x}{\sqrt{-t}} \right) = k x^{2 \lambda_2 + 1} \left( \frac{\tau_2}{2} \gamma_1 - t \frac{-t}{\gamma^2} \right)
\]
(see Proposition 3.1 and Remark 3.2).

2. In the intermediate region, if we rescale the hypersurface by the type I rescaling (see (3.9)), then the function \(v(y, s)\) of the rescaled hypersurface \(\Pi^\alpha_s(0, 1)\) defined in (3.10) satisfies
\[
y^{m+2l} \left| \nabla_y \nabla_s \left( v(y, s) - \frac{k}{c_2} e^{-\lambda_2} \varphi_2(y) \right) \right| \leq C (n, \Lambda, \delta, m, l) e^{-\lambda_2} s^{-2}
\]
for \((y, s)\) satisfying \(e^{-\theta_1} \leq y \leq 2, s_0 + \delta^2 y^2 \leq s \leq \tilde{s}\), and
\[
y^{m+2l} \left| \nabla_y \nabla_s \left( v(y, s) - e^{-\alpha} \psi_k(e^{-\gamma} y) \right) \right| \leq C (n, \Lambda, \delta, m, l) \beta_n e^{-\lambda_2} y^{\alpha}
\]
for \((y, s)\) satisfying \(\frac{2}{3} \beta e^{-\gamma} y \leq y \leq e^{-\gamma}, s_0 + \delta^2 y^2 \leq s \leq \tilde{s}\), where \(\tilde{s} = -\ln (-t)\) and
\[
\alpha = \min \left\{ \varsigma \lambda_2 - \theta \sigma \left( n + \alpha + \frac{3}{2} \right), \varsigma \lambda_2 + 2 (\lambda_2 + (\alpha - 2) \theta) \right\} > 0
\]
(4.19)
\[
\varrho = 1 - \frac{1}{2} (1 - \alpha) \left( 1 - \vartheta \right) \in (0, \vartheta)
\]
(4.22)
are constants. Note that
\[
k \frac{e^{-\lambda_2} \varphi_2(y)}{c_2} = k e^{-\lambda_2} y^{\alpha} \left( 1 + 2 \gamma_1 y^2 + \gamma_2 y^4 \right)
\]
(see Proposition 3.1 and (2.8)).

3. In the tip region, if we rescale the hypersurface by the type II rescaling (see (3.10)), then the function \(\hat{w}(z, \tau)\) of the rescaled hypersurface \(\Gamma^\alpha_s(0, 1)\) defined in (3.20) satisfies
\[
\delta^{m+2l} \left| \partial_z \partial_\tau \left( \hat{w}(z, \tau) - \hat{\psi}_k(z) \right) \right| \leq C (n, m, l) \beta_n \left[ \frac{\tau}{\tau_0} \right]^{-\varrho}
\]
for \(0 \leq z \leq 2\beta, \tau_0 + \delta^2 \leq \tau \leq \tilde{\tau}\), where \(\tilde{\tau} = \frac{1}{2\delta(t_0)}\).

Remark 4.6. By Proposition 4.4, Proposition 4.5 and [EH], we may infer that if \((a_0, a_1) \in \Sigma_{t_1}\) and
\[
\Phi_{t_1}(a_0, a_1) = (0, 0)
\]
then \((a_0, a_1) \in \Sigma_{t_1}^{(a_0, a_1)}\). In other words, \(\Sigma_{t_1}^{(a_0, a_1)}\) is a “good” candidate of initial hypersurfaces to flow.

We then have the following corollary.

Corollary 4.7. If \(|t_0| \ll 1\) (depending on \(n\)), then we have \(I = [t_0, 0]\).
Proof. Notice that by \((4.10)\) we have \(t_0 \in I\). Then we would like to prove the corollary by induction.

Assume that \(t_1 \in I\). The goal is to show that \(t_2 \in I\) for any \(t_2 \in [t_1, e^{-1}t_1]\). By definition, there holds

\[
\deg(\Phi_{t_1}, \mathcal{O}_{t_1}, (0, 0)) = 1
\]

It follows that there is \((a_0, a_1) \in \mathcal{O}_{t_2}\) for which

\[
\Phi_{t_1}(a_0, a_1) = (0, 0)
\]

By Remark 4.6, we then have \((a_0, a_1) \in \mathcal{O}_{t_2}\) and \((0, 0) \notin \Phi_t(\partial \mathcal{O}_{t_2})\) for all \(t_1 \leq t \leq t_2\). Consequently, \(\mathcal{O}_{t_2}\) is non-empty and the degree of \(\Phi_t\) at \((0, 0)\) is well defined in \(\mathcal{O}_{t_2}\) for each \(t_1 \leq t \leq t_2\). Since \(\Phi_t\) is continuous in \(t\), by the homotopy invariance of degree, there holds

\[
\deg(\Phi_{t_2}, \mathcal{O}_{t_2}, (0, 0)) = \deg(\Phi_{t_1}, \mathcal{O}_{t_2}, (0, 0))
\]

In addition, by Remark 4.6, \((0, 0) \notin \Phi_{t_1}(\mathcal{O}_{t_1} \setminus \mathcal{O}_{t_2})\), which, by the excision property of degree, implies that

\[
\deg(\Phi_{t_1}, \mathcal{O}_{t_2}, (0, 0)) = \deg(\Phi_{t_1}, \mathcal{O}_{t_1}, (0, 0)) = 1
\]

Therefore, we get \(t_2 \in I\).

Now we are ready to prove the existence theorem of Velázquez's solution.

**Theorem 4.8.** Let \(n \geq 5\) be a positive integer. If \(|t_0| < 1\) (depending on \(n\)), there is an **admissible** mean curvature flow \(\{\Sigma_t\}_{0 < t < 0}\) (see Section 3) for which the functions \(\hat{u}(x, t)\) and \(u(x, t)\) (defined in \((3.2)\) and \((3.6)\), respectively) satisfy \((4.13)\) and \((4.14)\). Besides, in the tip region, if we perform the type II rescaling, the rescaled function \(\hat{w}(\cdot, \tau)\) (defined in \((3.22)\)) satisfies \((4.17)\).

In addition, there is

\[
k \in \left(1 - C(n)(-t_0)^{\lambda_2}, 1 + C(n)(-t_0)^{\lambda_2}\right)
\]

so that for any given \(0 < \delta < 1\), \(m, l \in \mathbb{Z}_+\), there hold

1. In the **outer region**, the function \(u(x, t)\) of \(\Sigma_t\) defined in \((3.6)\) satisfies \((4.13)\) and \((4.14)\).
2. In the **intermediate region**, if we do the type I rescaling, the function \(v(y, s)\) of the rescaled hypersurface \(\Pi_s\) defined in \((3.11)\) satisfies \((4.11)\) and \((4.20)\).
3. In the **tip region**, if we do the type II rescaling, the function \(w(\cdot, \tau)\) of the rescaled hypersurface \(\Gamma_\tau\) defined in \((3.22)\) satisfies \((4.23)\).

**Proof.** Let \(t_i > t_0\) be a sequence so that \(t_i \not\to 0\). By Corollary 4.7, there is \((a_0^i, a_1^i) \in \mathcal{O}_{t_i}\) for which

\[
\Phi_{t_i}(a_0^i, a_1^i) = (0, 0)
\]

By the uniform estimates in Proposition 4.3 and Proposition 4.5 we may assume (by passing to a subsequence) that as \(i \to \infty\),

\[
k(a_0^i, a_1^i) \to k
\]

and the functions \(\hat{u}(a_0^i, a_1^i)(x, t)\) and \(u(a_0^i, a_1^i)(x, t)\) of \(\Sigma_t(a_0^i, a_1^i)\) (defined in \((3.2)\) and \((3.6)\)) converge locally smoothly to \(\hat{u}(x, t)\) and \(u(x, t)\), respectively. The conclusion follows immediately by passing the uniform estimates (in Proposition 4.4 and Proposition 4.5) to limit. \(\square\)
Remark 4.9. Let \( \{ \Sigma_t \}_{t_0 \leq t < 0} \) be Velázquez’s solution in Theorem 4.8. From (3.11), (3.12), (4.18) and (4.19), the type I rescaled hypersurfaces \( \Pi_s \) (see (3.9)) converges smoothly to \( \mathcal{C} \) on any fixed annulus centered at \( O \), i.e., for any \( 0 < r < R < \infty \),

\[
\Pi_s \xrightarrow{C^\infty} \mathcal{C} \quad \text{in } B(O; R) \setminus B(O; r)
\]
as \( s \to \infty \). Likewise, from (3.20), (3.24), (3.25), (4.20) and (4.23), the type II rescaled hypersurfaces \( \Gamma_{\tau} \) (see (3.10)) converges to \( \mathcal{M}_k \) locally smoothly, i.e.,

\[
\Gamma_{\tau} \xrightarrow{C^\infty_{\text{loc}}} \mathcal{M}_k
\]

In addition, by the admissible conditions, the projected curve \( \bar{\Sigma}_t \) (see (3.4)) is a graph over \( \bar{\mathcal{C}} \) outside \( B(O; \beta (-t)^{1/2 + \sigma}) \). By (4.13) and the admissible conditions, we know that inside \( B(O; \beta (-t)^{1/2 + \sigma}) \), \( \bar{\Sigma}_t \) is a convex curve which intersects orthogonally with the vertical ray \{ \((0, x) | x > 0\)\}; moreover, if we zoom in at \( O \) by the type II rescaling, by (2.4) and (6.8), the rescaled curve \( \bar{\Gamma}_\tau \) (see (3.23)) lies above \( \bar{\mathcal{C}} \) and tends to it for \( z \nearrow \beta \). Therefore, \( \bar{\Gamma}_\tau \) is a graph over \( \bar{\mathcal{C}} \) in \( B(O; \beta (-t)^{1/2 + \sigma}) \), which in turn implies that \( \bar{\Sigma}_t \) is also graph over \( \bar{\mathcal{C}} \) inside \( B(O; \beta (-t)^{1/2 + \sigma}) \). Hence, we get

\[
\bar{\Sigma}_t = \{(x, \hat{u}(x, t)) | x \geq 0\}
\]

\[
= \left\{ \left( (x - u(x, t)) \frac{1}{\sqrt{2}}, (x + u(x, t)) \frac{1}{\sqrt{2}} \right) \left| x \geq \hat{u}(0, t) \frac{1}{\sqrt{2}} \right\}
\]

5. Type II Singularity and Blow-up of the Mean Curvature

In this section we explain why Velázquez’s solution (see Theorem 4.8) develops a type II singularity at the origin and why its mean curvature blows up as \( t \nearrow 0 \). The lower bound for the blow-up rate of the second fundamental form is already shown in [V], while the upper bound (of the second fundamental form) and the blow-up of the mean curvature are new results.

To estimate the second fundamental form and mean curvature, we would use the asymptotic formulas in Theorem 4.8 to examine the solution in each region separately. Let’s start with analyzing the outer region by (3.6), (4.14) and (4.15).

**Proposition 5.1.** Let \( \{ \Sigma_t \}_{t_0 \leq t < 0} \) be Velázquez’s solution in Theorem 4.8. In the outer region, the second fundamental form of \( \Sigma_t \) is bounded by

\[
\sqrt{-t} |A_{\Sigma_t}| \leq C(n)
\]

for \( \frac{1}{2} t_0 \leq t < 0 \).

**Proof.** In the outer region, we parametrize \( \Sigma_t \) by (3.9). The second fundamental form is then given by

\[
A_{\Sigma_t} = \frac{1}{\sqrt{1 + (\partial_x u)^2}} \left( \begin{array}{cc}
\frac{\partial^2_x u}{1 + (\partial_x u)^2} & \frac{1 + \partial_x u}{x - u} I_{n-1} - \frac{1 + \partial_x u}{x + u} I_{n-1} \\
1 + (\partial_x u)^2 & \frac{1 + \partial_x u}{x - u} I_{n-1}
\end{array} \right)
\]
ANALYSIS OF VELÁZQUEZ’S SOLUTION TO THE MCF

By (4.14) and (4.15), we have
\[
\max \left\{ \left| \frac{u(x,t)}{x} \right|, |\partial_x u(x,t)| \right\} \leq \frac{1}{3}
\]
and
\[|\partial^2_{xx} u(x,t)| \leq C(n)
\]
for \(x \geq \sqrt{-t}, \frac{1}{2}t_0 \leq t < 0\). The conclusion follows immediately. \(\square\)

In the intermediate region, we first do the type I rescaling and study the rescaled hypersurface by (3.11), (3.12), (4.15), (4.19) and (4.20). Then we undo the rescaling to get the estimates for the solution.

Proposition 5.2. Let \(\{\Sigma_t\}_{t_0 \leq t < 0}\) be Velázquez’s solution in Theorem 4.8. In the intermediate region, the second fundamental form and the mean curvature of \(\Sigma_t\) are bounded by
\[
(-t)^{\frac{1}{2} + \sigma} |A_{\Sigma_t}| \leq C(n)
\]
\[
(-t)^{\frac{1}{2} + (1 - 2\rho)\sigma} |H_{\Sigma_t}| \leq C(n, t_0)
\]
for \(\frac{1}{2}t_0 \leq t < 0\), where \(0 < \sigma < \frac{1}{2}\) and \(0 < \rho < 1\) are constants defined in (3.5) and (4.22), respectively.

Proof. In the intermediate region, we rescale Velázquez’s solution by
\[
\Pi_s = \frac{1}{\sqrt{-t}} \Sigma_t \bigg|_{t = -e^{-s}}
\]
which can be parametrized by (3.11). The second fundamental form and the mean curvature of \(\Pi_s\) are then given by
\[
A_{\Pi_s} = \frac{1}{\sqrt{1 + (\partial_y v)^2}} \left( \frac{\partial^2_{yy} v}{1 + (\partial_y v)^2} \frac{1 + \partial_y v}{y + v} I_{n-1} - \frac{1 + \partial_y v}{y + v} I_{n-1} \right)
\]
\[
H_{\Pi_s} = \frac{1}{\sqrt{1 + (\partial_y v)^2}} \left( \frac{\partial^2_{yy} v}{1 + (\partial_y v)^2} + 2(n-1) \frac{y \partial_y v + v}{y^2 - v^2} \right)
\]
\[
= \frac{1}{\sqrt{1 + (\partial_y v)^2}} \left( \partial_y v - \frac{1}{2} (-y \partial_y v + v) \right)
\]
By (3.12) and (4.15), we have
\[
\max \left\{ \left| \frac{v(y,t)}{y} \right|, |\partial_y v(y,s)| \right\} \leq C(n) e^{-\lambda_2 s} y^{\alpha - 1} \leq \frac{1}{3}
\]
and
\[
|\partial^2_{yy} v(y,s)| \leq C(n) (e^{-\lambda_2 s} y^{\alpha - 1}) y^{-1} \leq C(n) e^{\sigma s}
\]
for \(\beta e^{-\sigma s} \leq y \leq 1, -\ln(-\frac{1}{2} t_0) \leq s < \infty\). Thus, we get
\[
|A_{\Pi_s}| \leq C(n) e^{\sigma s}
\]
in the intermediate region for \(-\ln(-\frac{1}{2} t_0) \leq s < \infty\).
As for the mean curvature, notice that

\[ v(y, s) \approx \begin{cases} \frac{k}{c_2} e^{-\lambda s} \varphi_2(y) & \text{for } e^{-\vartheta s} \leq y \leq 1 \\ e^{-s} \psi_k(e^{s}y) & \text{for } \beta e^{-s} \leq y \leq e^{-\vartheta s} \end{cases} \]

We then compute

\[
\left( \partial_s + \frac{y}{2} \partial_y - \frac{1}{2} \right) \left( \frac{k}{c_2} e^{-\lambda s} \varphi_2(y) \right)
= \left( \partial_s + \frac{y}{2} \partial_y - \frac{1}{2} \right) \left( ke^{-\lambda s} y^\alpha (1 + 2\gamma_1 y^2 + \gamma_2 y^4) \right)
= -2ke^{-\lambda s} y^\alpha (1 + \gamma_1 y^2)
\]

and

\[
\left( \partial_s + \frac{y}{2} \partial_y - \frac{1}{2} \right) (e^{-s} \psi_k(e^{s}y))
= -\left( \frac{1}{2} + \sigma \right) e^{-s} (\psi_k(z) - z \partial_z \psi_k(z)) \bigg|_{z=e^{s}y}
= -\left( \frac{1}{2} + \sigma \right) e^{-s} ((1 - \alpha) k (e^{s}y)^\alpha + O ((e^{s}y)^{3\alpha - 2}))
= -2ke^{-\lambda s} y^\alpha \left( 1 + O \left( (e^{s}y)^{-2(1 - \alpha)} \right) \right)
\]

It follows, by (4.19) and (4.20), that

\[
\left| \left( \partial_s + \frac{y}{2} \partial_y - \frac{1}{2} \right) v(y, s) \right|
\leq \left| \left( \partial_s + \frac{y}{2} \partial_y - \frac{1}{2} \right) \left( \frac{k}{c_2} e^{-\lambda s} \varphi_2(y) \right) \right| + C(n, t_0) e^{-\kappa s} \left( e^{-\lambda s} y^\alpha \right)
\leq |2ke^{-\lambda s} y^\alpha (1 + \gamma_1 y^2)| + C(n, t_0) e^{-\kappa s} \left( e^{-\lambda s} y^\alpha \right)
\leq C(n, t_0) e^{-\lambda s} y^\alpha \leq C(n, t_0)
\]

for \( e^{-\vartheta s} \leq y \leq 1, -\ln \left( -\frac{1}{2} t_0 \right) \leq s < \infty \), and

\[
\left| \left( \partial_s + \frac{y}{2} \partial_y - \frac{1}{2} \right) v(y, s) \right|
\leq \left| \left( \partial_s + \frac{y}{2} \partial_y - \frac{1}{2} \right) \left( e^{-s} \psi_k(e^{s}y) \right) \right| + C(n, t_0) e^{-2\kappa s} \left( e^{-\lambda s} y^\alpha - 2 \right)
\leq \left| 2ke^{-\lambda s} y^\alpha \left( 1 + O \left( (e^{s}y)^{-2(1 - \alpha)} \right) \right) \right| + C(n, t_0) e^{-\lambda s} y^\alpha \left( e^{-2\kappa s} y^{-1} \right)
\leq C(n, t_0) e^{-\lambda s} y^\alpha - 1 (y + e^{-2\kappa s} y^{-1}) \leq C(n, t_0) e^{(1 - 2\kappa) s}
\]

for \( \beta e^{-s} \leq y \leq e^{-\vartheta s}, -\ln \left( -\frac{1}{2} t_0 \right) \leq s < \infty \). Consequently,

\[
|H_{\Pi_s}| = \frac{|\partial_s v - \frac{1}{2} (-\vartheta \partial_y v + v)|}{\sqrt{1 + |\partial_y v|^2}} \leq C(n, t_0) e^{(1 - 2\kappa) s}
\]

Lastly, by the relation

\[
A_{\Pi_s}(y) = \sqrt{-t} A_{\Sigma_s} \left( \sqrt{-t} y \right) \bigg|_{t=-e^{-s}}
\]
\[
H_{\Pi_s}(y) = \sqrt{-t} H_{\Sigma_s} \left( \sqrt{-t} y \right) \bigg|_{t=-e^{-s}}
\]
the conclusion follow easily.

In the tip region, we do the type II rescaling and study the rescaled hypersurface by (3.20), (4.16) and (4.23). Then we undo the rescaling to get estimates of the solution.

**Proposition 5.3.** Let \( \{ \Sigma_t \}_{t_0 \leq t < 0} \) be Velázquez’s solution in Theorem 4.8. In the tip region, the second fundamental form and the mean curvature of \( \Sigma_t \) satisfy

\[
\frac{1}{C(n)} \leq (-t)^{\frac{1}{2} + \sigma} |A_{\Sigma_t}| \leq C(n)
\]

\[
(-t)^{\frac{1}{2} + (1-2\sigma)\alpha} |H_{\Sigma_t}| \leq C(n, t_0)
\]

for \( \frac{1}{2} t_0 \leq t < 0 \), where \( 0 < \sigma < \frac{1}{2} \) and \( 0 < \alpha < 1 \) are constants defined in (3.3) and (4.22), respectively.

**Proof.** In the tip region, we first rescale Velázquez’s solution by

\[
\Gamma_\tau = \frac{1}{(-t)^{\frac{1}{2} + \sigma}} \Sigma_t \bigg|_{t = -\frac{1}{2\sigma} \tau}
\]

which can be parametrized by (3.20). Then the second fundamental form and the mean curvature of \( \Gamma_\tau \) are given by

\[
A_{\Gamma_\tau} = \frac{1}{\sqrt{1 + |\partial_\tau \hat{w}|^2}} \left( \frac{\partial^2 \hat{w}}{1 + |\partial_\tau \hat{w}|^2} \partial_\tau \hat{w} I_{n-1} - \frac{1}{\hat{w}} I_{n-1} \right)
\]

\[
H_{\Gamma_\tau} = \frac{1}{\sqrt{1 + (\partial_\tau \hat{w})^2}} \left( \frac{\partial^2 \hat{w}}{1 + (\partial_\tau \hat{w})^2} + (n-1) \left( \frac{\partial_\tau \hat{w}}{\hat{w}} - \frac{1}{\hat{w}} \right) \right)
\]

By (4.16), we have

\[
\frac{1}{C(n)} \leq \hat{w}(z, \tau) \leq C(n)
\]

\[
|\partial_\tau \hat{w}(z, \tau)| + |\partial^2_{zz} \hat{w}(z, \tau)| \leq C(n)
\]

for \( 0 \leq z \leq \beta, \frac{1}{2\sigma} (-\frac{1}{2} t_0)^{2\sigma} \leq \tau < \infty \). Thus, we get

\[
\frac{1}{C(n)} \leq |A_{\Gamma_\tau}| \leq C(n)
\]

As for the mean curvature, note, from (2.6), that

\[
\left| \left( \partial_\tau + \frac{1}{2\sigma} + \frac{\sigma}{2\sigma \tau} z \partial_z - \frac{1}{2\sigma} + \frac{\sigma}{2\sigma \tau} \right) \hat{w}(z, \tau) \right| \leq C(n) \frac{\tau}{2\sigma}
\]

By (4.23), we get

\[
\left\| \left( \partial_\tau + \frac{1}{2\sigma} + \frac{\sigma}{2\sigma \tau} z \partial_z - \frac{1}{2\sigma} + \frac{\sigma}{2\sigma \tau} \right) \hat{w}(z, \tau) \right\| \leq \left\| \left( \partial_\tau + \frac{1}{2\sigma} + \frac{\sigma}{2\sigma \tau} z \partial_z - \frac{1}{2\sigma} + \frac{\sigma}{2\sigma \tau} \right) \hat{w}(z, \tau) \right\| + C(n, t_0) (2\sigma)^{-\alpha}
\]
ANALYSIS OF VELÁZQUEZ’S SOLUTION TO THE MCF

\[ |H_{\Gamma_*} = \frac{\partial_x \hat{w} - \frac{1}{2\sigma \tau} (-z \partial_z \hat{w} + \hat{w})}{\sqrt{1 + (\partial_z \hat{w})^2}} \leq C(n, t_0)(2\sigma \tau)^{-\frac{\sigma}{2}} \]

Thus,

\[ |H_{\Gamma_*} = \frac{1}{C(n, t_0)(2\sigma \tau)^{-\frac{\sigma}{2}}} \]

The conclusion follows by noting that

\[ A_{\Gamma_*}(z) = (-t)^{\frac{1}{2} + \sigma} A_{\Sigma_t} (-t)^{\frac{1}{2} + \sigma} z \bigg|_{t=-(2\sigma \tau)^{-\frac{\sigma}{2}}} \]

\[ H_{\Gamma_*}(z) = (-t)^{\frac{1}{2} + \sigma} H_{\Sigma_t} (-t)^{\frac{1}{2} + \sigma} z \bigg|_{t=-(2\sigma \tau)^{-\frac{\sigma}{2}}} \]

\[ \square \]

Lastly, we would like to show that the mean curvature blows up in the tip region at a rate at least \( \frac{1}{-t} \) as \( t \nearrow 0 \).

**Proposition 5.4.** Let \( \{ \Sigma_t \}_{t_0 \leq t < 0} \) be Velázquez’s solution in Theorem 4.8. Let \( H_{\Sigma_t}(x) \) be the mean curvature of \( \Sigma_t \) at

\[ X_t(x, \nu, \omega) = (x \nu, \hat{u}(x, t) \omega) \]

(see (3.2)). Then for any \( z \geq 0 \), there holds

\[ \limsup_{t \nearrow 0} (-t)^{\frac{1}{2} + \sigma} H_{\Sigma_t} (-t)^{\frac{1}{2} + \sigma} z \bigg|_{t=-(2\sigma \tau)^{-\frac{\sigma}{2}}} > 0 \]

**Proof.** Note that

\[ H_{\Sigma_t} = \frac{1}{\sqrt{1 + (\partial_z \hat{u})^2}} \left( \frac{\partial^2_{xx} \hat{u}}{1 + (\partial_z \hat{u})^2} + (n - 1) \left( \frac{\partial \hat{u}}{x} - \frac{1}{\hat{u}} \right) \right) \]

(5.1)

We claim that for any \( z \geq 0 \), there holds

\[ \limsup_{t \nearrow 0} (-t)^{\frac{1}{2} + \sigma} \left( \frac{\partial \hat{u} \left( (-t)^{\frac{1}{2} + \sigma} z, t \right)}{(-t)^{-\frac{1}{2} + \sigma}} \right) > 0 \]

(5.2)

The conclusion follows immediately from (4.10), (5.1) and (5.2).

To prove (5.2), we use a contradiction argument. Suppose that there is \( z \geq 0 \) so that

\[ \limsup_{t \nearrow 0} \left( \frac{\partial \hat{u} \left( (-t)^{\frac{1}{2} + \sigma} z, t \right)}{(-t)^{-\frac{1}{2} + \sigma}} \right) = 0 \]

then obviously,

\[ \lim_{t \nearrow 0} \left( \frac{\partial \hat{u} \left( (-t)^{\frac{1}{2} + \sigma} z, t \right)}{(-t)^{-\frac{1}{2} + \sigma}} \right) = 0 \]

(5.3)
Recall that by (1.23), we have
\[
\frac{1}{(−t)^{\frac{1}{2} + \sigma}} \hat{u} \left( (−t)^{\frac{1}{2} + \sigma} z, t \right) = \hat{w} \left( z, \frac{1}{2\sigma} \right) \rightarrow \hat{\psi}_k (z) \quad \text{as } t \to 0
\]
It follows, by L'Hôpital's rule, that
\[
\hat{\psi}_k (z) = \lim_{t \to 0} \frac{\hat{u} \left( (−t)^{\frac{1}{2} + \sigma} z, t \right)}{(−t)^{\frac{1}{2} + \sigma}} = \lim_{t \to 0} \left( \frac{\partial_t \hat{u} \left( (−t)^{\frac{1}{2} + \sigma} z, t \right)}{z \partial_z \hat{w} \left( z, \frac{1}{2\sigma} \right)} + z \partial_z \hat{w} \left( z, \frac{1}{2\sigma} \right) \right)
\]
Notice that the limit on the RHS exists because of (1.23) and (5.3), so L'Hôpital's rule is applicable here. Thus, we get
\[
\lim_{t \to 0} \frac{\partial_t \hat{u} \left( (−t)^{\frac{1}{2} + \sigma} z, t \right)}{z \partial_z \hat{w} \left( z, \frac{1}{2\sigma} \right)} = \hat{\psi}_k (z) - z \partial_z \hat{\psi}_k (z) > 0
\]
by (2.3), which contradicts with (5.3).

6. \(C^0\) estimates in Proposition 4.4 and Proposition 4.5

Starting from this section, we are devoted to prove Proposition 4.4 and Proposition 4.5. From now on, we focus on the estimate of the admissible MCF \(\left\{ \Sigma_t^{(a_0, a_1)} \right\} \) for which
\[
(6.1) \quad \Phi_{t_1} (a_0, a_1) = (0, 0)
\]
where \(t_0 \leq t_1 \leq \hat{t} < 0\) are constants and \(\hat{t} \leq e^{-\hat{t}_1} t_1\). In this section, we would show that if \(0 < \rho \ll 1 \ll \beta (\text{depending on } n, \Lambda) \) and \(|t_0| \ll 1 (\text{depending on } n, \Lambda, \rho, \beta)\), there holds
\[
(6.2) \quad \sqrt{a_0^2 + a_1^2} \leq C (n, \Lambda, \rho, \beta) (-t_0)^{\varsigma \lambda_2}
\]
where \(\varsigma > 0\) is a constant defined in (4.11). Moreover, there is
\[
(6.3) \quad k \in \left\{ 1 - C (n, \Lambda, \rho, \beta) (-t_0)^{\varsigma \lambda_2}, 1 + C (n, \Lambda, \rho, \beta) (-t_0)^{\varsigma \lambda_2} \right\}
\]
so that the following hold. 

(1) In the outer region, the function \(u (x, t)\) of \(\Sigma_t^{(a_0, a_1)}\) defined in (3.6) satisfies
\[
(6.4) \quad |u (x, t) - u (x, t_0)| \leq C (n) \sqrt{t - t_0}
\]
for \(x \geq \frac{1}{3} \rho, t_0 \leq t \leq \hat{t}\), and
\[
(6.5) \quad \left| u (x, t) - \frac{k}{c_2} (-t)^{\lambda_2 + \frac{1}{2}} \varphi_2 \left( \frac{x}{\sqrt{-t}} \right) \right| \leq C (n, \Lambda, \rho, \beta) (-t_0)^{\varsigma} x^{2\lambda_2 + 1}
\]
for \(\frac{1}{3} \sqrt{-t} \leq x \leq \rho, t_0 \leq t \leq \hat{t}\), where \(\varsigma > 0\) is a constant defined in (4.21). Note that
\[
\frac{k}{c_2} (-t)^{\lambda_2 + \frac{1}{2}} \varphi_2 \left( \frac{x}{\sqrt{-t}} \right) = k x^{2\lambda_2 + 1} \left( \gamma_2 + 2 \gamma_1 \left( \frac{-t}{x^2} \right) + \left( \frac{-t}{x^2} \right)^2 \right)
\]
(2) In the intermediate region, if we do the type I rescaling, the function 
\( v(y, s) \) of the rescaled hypersurface \( \Pi_s^{(a_0, a_1)} \) defined in (3.11) satisfies
\[
(6.6) \quad \left| v(y, s) - \frac{k}{c_2} e^{-\lambda_2 s} \varphi_2(y) \right| \leq C (n, \Lambda, \rho, \beta) e^{-\kappa s} \left( e^{-\lambda_2 s} y^\alpha + 2 \right)
\]
for \( \frac{1}{3} e^{-\alpha s} \leq y \leq \sqrt[3]{\lambda_2 s} \), \( s_0 \leq s \leq \hat{s} \), and
\[
(6.7) \quad \left| \left( v(y, s) - e^{-\gamma s} \psi_k (e^{\sigma s} y) \right) \right| \leq C (n) \beta^{\alpha-3} e^{-2\sigma(s-s_0)} \left( e^{-\lambda_2 s} y^\alpha \right)
\]
for \( \frac{1}{3} e^{-\alpha s} \leq y \leq \frac{1}{2} e^{-\alpha s} \), \( s_0 \leq s \leq \hat{s} \), where \( \hat{s} = -\ln(-\hat{t}) \) and \( 0 < \rho < \hat{\vartheta} < 1 \) is a constant (see (4.12) and (4.22) for definition). Note that
\[
\frac{k}{c_2} e^{-\lambda_2 s} \varphi_2(y) = ke^{-\lambda_2 s} y^\alpha \left( 1 + 2 \gamma_1 y^2 + \gamma_2 y^4 \right)
\]
\[ e^{-\sigma s} \psi_k (e^{\sigma s} y) = ke^{-\lambda_2 s} y^\alpha \left( 1 + O \left( e^{\sigma s} y^{2(1-\alpha)} \right) \right) \]

(3) In the tip region, if we do the type II rescaling, the function \( \hat{w}(z, \tau) \) of the rescaled hypersurface \( \Gamma_s^{(a_0, a_1)} \) defined in (3.20) satisfies
\[
(6.8) \quad \hat{\psi} \left( \frac{1}{1-\beta^{\alpha-3} \left( \frac{z}{s_0} \right)^{-\vartheta}} \right)_{\kappa} (z) \leq \hat{w}(z, \tau) \leq \hat{\psi} \left( \frac{1}{1-\beta^{\alpha-3} \left( \frac{z}{s_0} \right)^{-\vartheta}} \right)_{\kappa} (z)
\]
for \( 0 \leq z \leq (2\sigma \tau)^{\frac{1}{\beta^{\alpha-3} \left( \frac{z}{s_0} \right)^{-\vartheta}}} \), \( \tau_0 \leq \tau \leq \hat{\tau} \), where \( \hat{\tau} = \frac{1}{2\sigma (-\hat{t})}\). To achieve that, we first establish (6.6) (see Proposition 6.4) by using the energy estimate and Sobolev inequality. Next, we use the comparison principle and the boundary values of (6.6) to show (6.3) (see Proposition 6.5) and (6.8) (see Proposition 6.6). Then we use (6.8) to deduce (6.7) by rescaling and analyzing the projected curves. Lastly, we use the gradient and curvature estimates in [EH] to prove (6.4) (see Proposition 6.4). The ideas of proving (6.5), (6.6) and (6.8) are due to Velázquez (see [V]). Here we improve his estimates to get better results.

**Remark 6.1.** By the above \( C^0 \) estimates, we deduce that
\[
-2 (\gamma_1^2 - \gamma_2) x^{2\lambda_2 + 1} \leq u(x, t) \leq 2 (1 + 2 \gamma_1 + \gamma_2) x^{2\lambda_2 + 1}
\]
for \( \sqrt{-\hat{t}} \leq x \leq \rho, t_0 \leq t \leq \hat{t} \), and
\[
2 (1 + 2 \gamma_1 + \gamma_2) e^{-\lambda_2 s} y^\alpha \leq v(y, s) \leq 2 e^{-\lambda_2 s} y^\alpha
\]
for \( \frac{1}{3} e^{-\alpha s} \leq y \leq 1, s_0 \leq s \leq \hat{s} \), provided that \( \beta \gg 1 \) (depending on \( n \)) and \( |t_0| \ll 1 \) (depending on \( n, \Lambda, \rho, \beta \)). In Section 8 we would use these estimates to choose the constant \( \Lambda = \Lambda(n) \).

In order to prove (6.10), we need the following two lemmas. The first lemma is the energy estimates for solutions to a parabolic equation associated with the linear operator \( L \) (see (3.16)). Recall that in Proposition 3.1 the eigenvalues of \( L \) satisfy \( \lambda_i \geq \lambda_3 > 1 \) for \( i \geq 3 \).

**Lemma 6.2.** Let \( H \) be the closed subspace of \( H \) (see Proposition 3.1) spanned by eigenfunctions \( \{ \varphi_i \}_{i \geq 3} \) of \( L \). Given
\[
f(\cdot, s) \in L^2 \left( [s_0, \hat{s}] ; L^2 \left( \mathbb{R}_+, y^{2(n-1)} e^{-\frac{y^2}{4}} dy \right) \right)
\]
and \( h \in H_s \), let \( \psi (\cdot , s) \in C ([s_0, \hat{s}] ; H_s) \) be the weak solution of

\[
(\partial_s + L) \psi (\cdot , s) = f (\cdot , s) \quad \text{for } s_0 \leq s \leq \hat{s}
\]

(6.9)

\[
\psi (\cdot , s_0) = h
\]

Then for any \( 0 < \delta < 1 \), there hold

\[
\| \psi (\cdot , s) \|^2 \\
\leq e^{-2(1-\delta)\lambda_3 (s-s_0)} \| \psi (\cdot , s_0) \|^2 + \frac{1}{2\delta \lambda_3} \int_{s_0}^{s} e^{-2(1-\delta)\lambda_3 (s-\xi)} \| f (\cdot , \xi) \|^2 \, d\xi
\]

and

\[
\langle L \psi (\cdot , s) , \psi (\cdot , s) \rangle \\
\leq e^{-2(1-\delta)\lambda_3 (s-s_0)} \langle Lh , h \rangle + \frac{1}{2\delta} \int_{s_0}^{s} e^{-2(1-\delta)\lambda_3 (s-\xi)} \| f (\cdot , \xi) \|^2 \, d\xi
\]

for \( s_0 \leq s \leq \hat{s} \), where the inner product \( \langle \cdot , \cdot \rangle \) and the corresponding norm \( \| \cdot \| \) are defined in Proposition 5.1.

Proof. Let \( \{ \varphi_m \}_{m \geq 3} \) be the Galerkin’s approximation of \( \psi \). Namely,

\[
\varphi_m (y , s) = \sum_{i=3}^{m} \left( e^{-\lambda_i (s-s_0)} \langle h , \varphi_i \rangle + \int_{s_0}^{s} e^{-\lambda_i (s-\xi)} \langle f (\cdot , \xi) , \varphi_i \rangle \, d\xi \right) \varphi_i (y)
\]

Then we have

\[
\begin{cases}
\partial_s \varphi_m (\cdot , s) + L \varphi_m (\cdot , s) = f_m (\cdot , s) \quad \text{for } s_0 \leq s \leq \hat{s} \\
\varphi_m (\cdot , s_0) = \sum_{i=3}^{m} \langle h , \varphi_i \rangle \varphi_i \to h \quad \text{in } H_s
\end{cases}
\]

where

\[
f_m (\cdot , s) = \sum_{i=3}^{m} \langle f (\cdot , s) , \varphi_i \rangle \varphi_i \to f (\cdot , s) \quad \text{in } L^2 \left( [s_0, \hat{s}] ; L^2 \left( \mathbb{R}_+ , y^{2(n-1)} e^{-\frac{y^2}{4}} \, dy \right) \right)
\]

It follows that

\[
\langle \partial_s \varphi_m (\cdot , s) , \varphi_m (\cdot , s) \rangle + \langle L \varphi_m (\cdot , s) , \varphi_m (\cdot , s) \rangle = \langle f_m (\cdot , s) , \varphi_m (\cdot , s) \rangle
\]

which, by Cauchy-Schwarz inequality, yields

\[
\frac{1}{2} \partial_s \| \varphi_m (\cdot , s) \|^2 + \lambda_3 \| \varphi_m (\cdot , s) \|^2 \leq \delta \lambda_3 \| \varphi_m (\cdot , s) \|^2 + \frac{1}{4 \delta \lambda_3} \| f_m (\cdot , s) \|^2
\]

for any \( 0 < \delta < 1 \). Thus, by integrating the inequality with respect to \( s \), we get

(6.10)

\[
\| \varphi_m (\cdot , s) \|^2 \\
\leq e^{-2(1-\delta)\lambda_3 (s-s_0)} \| \varphi_m (\cdot , s_0) \|^2 + \frac{1}{2\delta \lambda_3} \int_{s_0}^{s} e^{-2(1-\delta)\lambda_3 (s-\xi)} \| f_m (\cdot , \xi) \|^2 \, d\xi
\]

for \( s_0 \leq s \leq \hat{s} \).

Similarly, we have

\[
\langle \partial_s \varphi_m (\cdot , s) , \partial_s \varphi_m (\cdot , s) \rangle + \langle L \varphi_m (\cdot , s) , \partial_s \varphi_m (\cdot , s) \rangle = \langle f_m (\cdot , s) , \partial_s \varphi_m (\cdot , s) \rangle
\]

Substitute \( \partial_s \varphi_m (\cdot , s) = -L \varphi_m (\cdot , s) + f_m (\cdot , s) \) into the above equation to get

\[
\frac{1}{2} \partial_s \langle L \varphi_m (\cdot , s) , \varphi_m (\cdot , s) \rangle = - \langle L \varphi_m (\cdot , s) , L \varphi_m (\cdot , s) \rangle + \langle L \varphi_m (\cdot , s) , f_m (\cdot , s) \rangle
\]
By Cauchy-Schwarz inequality, we get
\[
\partial_s \langle L\nu_m (\cdot, s), \nu_m (\cdot, s) \rangle
\leq -2(1 - \delta) \langle L\nu_m (\cdot, s), L\nu_m (\cdot, s) \rangle + \frac{1}{2\delta} \| f_m (\cdot, s) \|^2
\]
\[
\leq -2(1 - \delta) \lambda_3 \langle L\nu_m (\cdot, s), \nu_m (\cdot, s) \rangle + \frac{1}{2\delta} \| f_m (\cdot, s) \|^2
\]
for any \(0 < \delta < 1\). Thus, we have
\[
(6.11) \quad \langle L\nu_m (\cdot, s), \nu_m (\cdot, s) \rangle
\leq e^{-2(1-\delta)\lambda_3(s-s_0)} \langle L\nu_m (\cdot, s_0), \nu_m (\cdot, s_0) \rangle + \frac{1}{2\delta} \int_{s_0}^{s_0} e^{-2(1-\delta)\lambda_3(s-s_\xi)} \| f_m (\cdot, \xi) \|^2 d\xi
\]
for \(s_0 \leq s \leq \hat{s}\).

On the other hand, for any \(m, l \geq 3\), there holds
\[
\partial_s (\nu_m (\cdot, s) - \nu_l (\cdot, s)) + L (\nu_m (\cdot, s) - \nu_l (\cdot, s)) = f_m (\cdot, s) - f_l (\cdot, s)
\]
By the same arguments as above, for any \(0 < \delta < 1\), we can deduce that
\[
||\nu_m (\cdot, s) - \nu_l (\cdot, s)\|^2
\]
\[
\leq e^{-2(1-\delta)\lambda_3(s-s_0)} ||\nu_m (\cdot, s_0) - \nu_l (\cdot, s_0)\|^2
\]
\[
+ \frac{1}{2\delta} \int_{s_0}^{s_0} e^{-2(1-\delta)\lambda_3(s-s_\xi)} || f_m (\cdot, \xi) - f_l (\cdot, \xi)\|^2 d\xi
\]
and
\[
(6.13) \quad \langle L (\nu_m (\cdot, s) - \nu_l (\cdot, s)), (\nu_m (\cdot, s) - \nu_l (\cdot, s)) \rangle
\]
\[
\leq e^{-2(1-\delta)\lambda_3(s-s_0)} \langle L (\nu_m (\cdot, s_0) - \nu_l (\cdot, s_0)), \nu_m (\cdot, s_0) - \nu_l (\cdot, s_0) \rangle
\]
\[
+ \frac{1}{2\delta} \int_{s_0}^{s_0} e^{-2(1-\delta)\lambda_3(s-s_\xi)} || f_m (\cdot, \xi) - f_l (\cdot, \xi)\|^2 d\xi
\]
for \(s_0 \leq s \leq \hat{s}\). Therefore, by (3.13), (6.12), (6.13) and the uniqueness of weak solutions, we get
\[
\nu_m \rightarrow \nu \quad \text{in} \quad C ([s_0, \hat{s}] ; H_\alpha)
\]
The conclusion follows by passing (6.10) and (6.11) to limit. \(\square\)

The second lemma is a Sobolev type inequality for functions in H, which is the Hilbert space defined in Proposition 3.1.

Lemma 6.3. Functions in H are actually continuous, i.e., \(H \subset C (\mathbb{R}_+)\). Moreover, for any \(v \in H\), there holds
\[
|v(y)| \leq C (n) \left( \frac{1}{y^{n-\gamma}} + e^{(\frac{\gamma y}{2})^2} \right) (\| \partial_y v \| + \| v \|)
\]
for \(y > 0\).
Proof. Let’s first assume that \( v \in C^1(\mathbb{R}_+) \cap H \).

For each \( 0 < y \leq 1 \), by the fundamental theorem of calculus, we have

\[
v(y) = v(z) + \int_z^y \partial_y v(\xi) \, d\xi \quad \forall \, \frac{y}{2} \leq z \leq y
\]

which, by Hölder’s inequality, implies

\[
|v(y)|^2 \leq C \left( |v(z)|^2 + y \int_{\frac{y}{2}}^y |\partial_y v(\xi)|^2 \, d\xi \right)
\]

\[
\leq C |v(z)|^2 + C(n) \frac{y}{y^{2(n-1)}} \left( \int_{\frac{y}{2}}^y |\partial_y v(\xi)|^2 \xi^{2(n-1)} e^{-\frac{\epsilon^2}{4}} \, d\xi \right)
\]

for \( \frac{y}{2} \leq z \leq y \). Integrate the above inequality against \( z^{2(n-1)} e^{-\frac{\epsilon^2}{4}} \, dz \) from \( \frac{y}{2} \) to \( y \) to get

\[
|v(y)|^2 \left( \int_{\frac{y}{2}}^y z^{2(n-1)} e^{-\frac{\epsilon^2}{4}} \, dz \right) \leq C \int_{\frac{y}{2}}^y |v(z)|^2 z^{2(n-1)} e^{-\frac{\epsilon^2}{4}} \, dz
\]

\[
+ C(n) \frac{1}{y^{2n-3}} \left( \int_{\frac{y}{2}}^y |\partial_y v(\xi)|^2 \xi^{2(n-1)} e^{-\frac{\epsilon^2}{4}} \, d\xi \right)
\]

which implies

\[
|v(y)|^2 \leq C(n) \frac{1}{y^{2n-3}} \left( \int_{\frac{y}{2}}^y |v(z)|^2 z^{2(n-1)} e^{-\frac{\epsilon^2}{4}} \, dz \right)
\]

\[
+ C(n) \frac{1}{y^{2n-3}} \left( \int_{\frac{y}{2}}^y |\partial_y v(\xi)|^2 \xi^{2(n-1)} e^{-\frac{\epsilon^2}{4}} \, d\xi \right)
\]

That is,

\[
|v(y)| \leq C(n) \left( \frac{1}{y^{n-\frac{1}{2}}} \|v\| + \frac{1}{y^{n-\frac{1}{2}}} \|\partial_y v\| \right)
\]

\[
\leq C(n) \frac{1}{y^{n-\frac{1}{2}}} (\|\partial_y v\| + \|v\|)
\]

for \( 0 < y \leq 1 \).

Likewise, for each \( y \geq 1 \), by the fundamental theorem of calculus, we have

\[
v(y) = v(z) - \int_y^z \partial_y v(\xi) \, d\xi \quad \forall \, y \leq z \leq y + 1
\]

which implies

\[
|v(y)|^2 \leq C \left( |v(z)|^2 + y \int_y^{y+1} |\partial_y v(\xi)|^2 \, d\xi \right)
\]

\[
\leq C |v(z)|^2 + Cy^{2(n-1)} e^{-\frac{\epsilon^2}{4}} \left( \int_y^{y+1} |\partial_y v(\xi)|^2 \xi^{2(n-1)} e^{-\frac{\epsilon^2}{4}} \, d\xi \right)
\]

for \( y \leq z \leq y + 1 \). Integrate both sides against \( z^{2(n-1)} e^{-\frac{\epsilon^2}{4}} \, dz \) from \( y \) to \( y + 1 \) to get

\[
|v(y)|^2 \left( \int_y^{y+1} z^{2(n-1)} e^{-\frac{\epsilon^2}{4}} \, dz \right) \leq C \int_y^{y+1} |v(z)|^2 z^{2(n-1)} e^{-\frac{\epsilon^2}{4}} \, dz
\]

\[
+ Cy^{2(n-1)} e^{-\frac{\epsilon^2}{4}} \left( \int_y^{y+1} |\partial_y v(\xi)|^2 \xi^{2(n-1)} e^{-\frac{\epsilon^2}{4}} \, d\xi \right)
\]
which yields
\[ |v(y)|^2 \leq C \left( \frac{1}{y^{n-2}} + e^{\frac{y+1}{2}} \right) \left( \|\partial_y v\| + \|v\| \right) \]
for \( y \geq 1 \).

More generally, given a function \( v \in H \), then choose a sequence \( \{v_i\} \subset C^1_c(\mathbb{R}_+) \cap H \) so that
\[ v_i \xrightarrow{C^1_{loc}} v \]
By the above arguments, we have
\[ |v_i(y) - v_j(y)| \leq C \left( \frac{1}{y^{n-2}} + e^{\frac{y+1}{2}} \right) (\|\partial_y v_i - \partial_y v_j\| + \|v_i - v_j\|) \]
for \( y > 0 \). It follows, by the second inequality, that
\[ v_i \xrightarrow{C^1_{loc}} v \]
Hence \( v \in C(\mathbb{R}_+) \). In addition, by passing the first inequality to limit, we get
\[ |v(y)| \leq C \left( \frac{1}{y^{n-2}} + e^{\frac{y+1}{2}} \right) (\|\partial_y v\| + \|v\|) \]
for \( y > 0 \). \( \square \)

Now we are ready to prove (6.6). The idea is to linearize (3.13) and do Fourier expansion. The condition (6.1) allow us to control the evolution of components in negative eigenvalue functions. For the remainder terms, we can use the energy estimate and Sobolev inequality to get a \( L^\infty \) estimate.

**Proposition 6.4.** If \( 0 < \rho \ll 1 \ll \beta \) (depending on \( n, \Lambda \)) and \( s_0 \gg 1 \) (depending on \( n, \Lambda, \rho, \beta \)), then (6.3) holds. Moreover, there is a constant \( k \) satisfying (6.3), for which the function \( v(y, s) \) of the type I rescaled hypersurface \( \Pi_s(\alpha_0, \alpha_1) \) (see (3.15)) satisfies (6.0).

**Proof.** Let
\[ \tilde{v}(y, s) = \zeta(e^{\alpha y} - \beta) \zeta(\rho e^{\tilde{\alpha}} - y) v(y, s) \]
then \( \tilde{v}(\cdot, s) \subset C([s_0, \hat{s}]; H) \). From (3.15), we have
\[ \left( \partial_s + \mathcal{L} \right)v(\cdot, s) = \mathcal{Q}v(\cdot, s) \]
which implies
\[ \left( \partial_s + \mathcal{L} \right)\tilde{v}(\cdot, s) = f(\cdot, s) \equiv f_1(\cdot, s) + f_{II}(\cdot, s) + f_{III}(\cdot, s) \]
where
\[ f_1(y, s) = \zeta(e^{\alpha y} - \beta) \zeta(\rho e^{\tilde{\alpha}} - y) \mathcal{Q}v(y, s) \]
\[ f_{II}(y, s) = \zeta'(e^{\alpha y} - \beta) e^{\alpha y} \left( -2 \partial_y v(y, s) + \frac{2(n-1)}{y} + \left( \frac{\sigma + 1}{2} \right) y v(y, s) \right) \]
\[ -\zeta''(e^{\alpha y} - \beta) e^{2\alpha y} v(y, s) \]
\( f_{\text{III}}(y, s) = \zeta' \left( \rho e^z - y \right) \left( \frac{\rho e^z - y}{2} + \frac{2(n - 1)}{y} \right) v(y, s) + 2 \partial_y v(y, s) \)

\[-\zeta'' \left( \rho e^z - y \right) v(y, s)\]

We claim that

\[
\|f(\cdot, s)\| \leq C(n, \Lambda, \rho, \beta) e^{-(1+\varsigma)\lambda_2 s}
\]

for \( s_0 \leq s \leq \hat{s} \), provided that \( 0 < \rho \ll 1 \ll \beta \) (depending on \( n, \Lambda \)) and \( s_0 \gg 1 \) (depending on \( n, \Lambda, \rho, \beta \)), where the norm \( \| \cdot \| \) is defined in Proposition 3.1. Notice that by (3.14), we have

\[
\max \left\{ \frac{v(y, s)}{y}, |\partial_y v(y, s)| \right\} \leq \Lambda e^{-\lambda_2 s} \left( y^{\alpha - 1} + y^{2\lambda_2} \right) \leq \Lambda (\beta^{-1} + \rho^{2\lambda_2})
\]

for \( \beta e^{-\sigma s} \leq y \leq \rho e^z \), so we have

\[
\max \left\{ \frac{v(y, s)}{y}, |\partial_y v(y, s)| \right\} \leq \frac{1}{3}
\]

for \( \beta e^{-\sigma s} \leq y \leq \rho e^z \) provided that \( 0 < \rho \ll 1 \ll \beta \) (depending on \( n, \Lambda \)). To prove (6.15), we use (3.14) to get

\[
\|f_1\| = \|\zeta (e^{\sigma s} y - \beta) \zeta (\rho e^z - y) Q v(y, s)\| 
\]

\[
\leq C(n) \Lambda^3 \left( e^{-\lambda_2 s} \left( y^{\alpha - 1} + y^{2\lambda_2} \right) \right)^2 e^{-\lambda_2 s} \left( y^{\alpha - 2} + y^{2\lambda_2 - 1} \right) \chi_{(\beta e^{-\sigma s}, \rho e^z)}
\]

\[
\leq C(n) \Lambda^3 e^{-(1+\varsigma)\lambda_2 s} \left( e^{-\lambda_2 s} \left( y^{\alpha - 1} + y^{2\lambda_2} \right) \right)^{2-\varsigma} \left( y^{\alpha - 2 + \varsigma(\alpha - 1)} + y^{2\lambda_2 - 1 + 2\varsigma \lambda_2} \right) \chi_{(\beta e^{-\sigma s}, \rho e^z)}
\]

\[
\leq C(n) \Lambda^3 e^{-(1+\varsigma)\lambda_2 s} \left( \int_0^\infty \left( y^{2(\alpha - 2 + \varsigma(\alpha - 1))} + y^{2(2\lambda_2 - 1 + 2\varsigma \lambda_2)} \right) y^{2(n - 1)} e^{-\frac{\rho^2}{4} dy} \right)^{\frac{1}{2}}
\]

\[
\leq C(n) \Lambda^3 e^{-(1+\varsigma)\lambda_2 s}
\]

since \( \zeta \leq \lambda_2^{-1} \leq 1 \) and \( 2(\alpha - 2 + \varsigma(\alpha - 1)) + 2(n - 1) > -1; \)

\[
\|f_{\text{II}}\| \leq C(n) \Lambda \left| e^{-\lambda_2 s} y^{\alpha - 2} \chi_{(\beta e^{-\sigma s}, (\beta + 1)e^{-\sigma s})} \left( \int_{\beta e^{-\sigma s}}^{(\beta + 1)e^{-\sigma s}} y^{2(\alpha - 2)} y^{2(n - 1)} dy \right)^{\frac{1}{2}} \right|
\]

\[
\leq C(n) \Lambda e^{-\lambda_2 s} \left( \int_{\beta e^{-\sigma s}}^{(\beta + 1)e^{-\sigma s}} y^{2(\alpha - 2)} y^{2(n - 1)} dy \right)^{\frac{1}{2}}
\]

\[
\leq C(n) \Lambda e^{-\lambda_2 s} \left( \beta e^{-\sigma s} \right)^{n + \alpha - \frac{1}{2}} \leq C(n) \Lambda \beta^{n + \alpha - \frac{1}{2}} e^{-(1+\varsigma)\lambda_2 s}
\]

and

\[
\|f_{\text{III}}\| \leq C(n) \Lambda \left| e^{-\lambda_2 s} \int_{\beta e^{-\sigma s}}^{(\beta + 1)e^{-\sigma s}} y^{2(\alpha - 2)} y^{2(n - 1)} e^{-\frac{\rho^2}{4} dy} \right|^{\frac{1}{2}}
\]

\[
= C(n) \Lambda e^{-\lambda_2 s} \left( \int_{\beta e^{-\sigma s}}^{(\beta + 1)e^{-\sigma s}} y^{2(\alpha - 2)} y^{2(n - 1)} e^{-\frac{\rho^2}{4} dy} \right)^{\frac{1}{2}}
\]

\[
\leq C(n) \Lambda e^{-\lambda_2 s} e^{-\sigma s} \leq C(n) \Lambda e^{-(1+\varsigma)\lambda_2 s}
\]

provided that \( s_0 \gg 1 \) (depending on \( n, \rho \)).
Next, we would like to estimate the components of negative eigenvalue functions in the Fourier expansion of $\tilde{v}(\cdot, s)$. For each $i \in \{0, 1\}$, by Proposition 5.1, (6.1) and (6.14), we have

$$
\begin{cases}
\partial_s \langle \tilde{v}(\cdot, s), \varphi_i \rangle + \lambda_i \langle \tilde{v}(\cdot, s), \varphi_i \rangle = \langle f(\cdot, s), \varphi_i \rangle \\
(\tilde{v}(\cdot, s_1), \varphi_i) = 0
\end{cases}
$$

Note that $\lambda_i = \lambda_2 - (2 - i) < 0$ and

$$
\hat{s} = -\ln (\hat{t}) \leq -\ln (-e^{-1} t_1) = s_1 + 1
$$

Therefore, for $s_1 \leq s \leq \hat{s}$, we have

$$
|\langle \tilde{v}(\cdot, s), \varphi_i \rangle| = \left| \int_{s_1}^{s} e^{-\lambda_i(s-\xi)} \langle f(\cdot, \xi), \varphi_i \rangle d\xi \right| \leq \int_{s_1}^{s} e^{-(\lambda_2 - 2)(s-\xi)} \|f(\cdot, \xi)\| d\xi \leq C(n, \Lambda, \rho, \beta) e^{-(\lambda_2 - 2)(s-s_1)} e^{-(1+\xi)\lambda_2 s_1} \leq C(n, \Lambda, \rho, \beta) e^{-(1+\xi)\lambda_2 s}
$$

and for $s_0 \leq s \leq s_1$, we have

$$
|\langle \tilde{v}(\cdot, s), \varphi_i \rangle| = \left| \int_{s_1}^{s} e^{\lambda_i(\xi-s)} \langle f(\cdot, \xi), \varphi_i \rangle d\xi \right| \leq \int_{s_1}^{s} e^{\lambda_2(\xi-s)} \|f(\cdot, \xi)\| d\xi \leq C(n, \Lambda, \rho, \beta) e^{-(1+\xi)\lambda_2 s}
$$

Thus, for $i \in \{0, 1\}$, there holds

$$
(6.16) \quad |\langle \tilde{v}(\cdot, s), \varphi_i \rangle| \leq C(n, \Lambda, \rho, \beta) e^{-(1+\xi)\lambda_2 s}
$$

for $s_0 \leq s \leq \hat{s}$. In addition, for $i \in \{0, 1\}$, by Lemma 4.3, we have

$$
|\langle \tilde{v}(\cdot, s_0), c_i \varphi_i \rangle - a_i e^{-\lambda_2 s_0}| = |\langle \zeta e^{\sigma_{s_0}} y - \beta \rangle \zeta (e^{\sigma_{s_0}} - y) v(\cdot, s_0), c_i \varphi_i \rangle - a_i e^{-\lambda_2 s_0}|
$$

$$
= e^{-\lambda_2 s_0} \left| \langle \zeta e^{\sigma_{s_0}} y - \beta \rangle \zeta (e^{\sigma_{s_0}} - y) \left( \frac{1}{c_2} \varphi_2 (y) + \frac{a_0}{c_0} \varphi_0 (y) + \frac{a_1}{c_1} \varphi_1 (y) \right), c_i \varphi_i \rangle - a_i \right|
$$

$$
\leq C(n, \Lambda, \rho, \beta) e^{-\xi \lambda_2 s_0}
$$

which, together with (6.16), implies

$$
|a_i| \leq \left| e^{\lambda_2 s_0} \langle \tilde{v}(\cdot, s_0), c_i \varphi_i \rangle \right| + \left| e^{\lambda_2 s_0} \langle \tilde{v}(\cdot, s_0), c_i \varphi_i \rangle - a_i \right| \leq C(n, \Lambda, \rho, \beta) e^{-\xi \lambda_2 s_0}
$$

We continue to estimate the components of the first positive eigenvalue functions in the Fourier expansion of $\tilde{v}(\cdot, s)$. By Proposition 5.1, Lemma 4.3, (6.1) and (6.14), we have

$$
\begin{cases}
\partial_s \langle e^{\lambda_2 s} \tilde{v}(\cdot, s), \varphi_2 \rangle = \langle e^{\lambda_2 s} f(\cdot, s), \varphi_2 \rangle \\
|e^{\lambda_2 s_0} \langle \tilde{v}(\cdot, s_0), c_2 \varphi_2 \rangle - 1| \leq C(n) e^{-2 \xi \lambda_2 s_0}
\end{cases}
$$

Now let

$$
k = e^{\lambda_2 s_1} \langle \tilde{v}(\cdot, s_1), c_2 \varphi_2 \rangle
$$

then for $s_1 \leq s \leq \hat{s}$, we have

$$
|e^{\lambda_2 s} \langle \tilde{v}(\cdot, s), c_2 \varphi_2 \rangle - k| = |e^{\lambda_2 s} \langle \tilde{v}(\cdot, s), \varphi_2 \rangle - e^{\lambda_2 s_1} \langle \tilde{v}(\cdot, s_1), c_2 \varphi_2 \rangle|
$$

$$
= \left| \int_{s_1}^{s} e^{\lambda_2 \xi} \langle f(\cdot, \xi), \varphi_2 \rangle d\xi \right| \leq \int_{s_1}^{s_1 + \hat{s}} e^{\lambda_2 \xi} \|f(\cdot, \xi)\| d\xi
$$
Thus, we get

\[ e^{\lambda_2 s} \langle \tilde{v} (\cdot, s), c_2 \varphi_2 \rangle - k = |e^{\lambda_2 s} \langle \tilde{v} (\cdot, s), c_2 \varphi_2 \rangle - e^{\lambda_2 s_1} \langle \tilde{v} (\cdot, s_1), c_2 \varphi_2 \rangle| \]

\[ = \left| \int_s^{s_1} e^{\lambda_2 \xi} \langle f (\cdot, \xi), \varphi_2 \rangle d\xi \right| \leq \int_s^{s_1} e^{\lambda_2 \xi} k \parallel f (\cdot, \xi) \parallel d\xi \]

\[ \leq C (n, \Lambda, \rho, \beta) e^{-c \lambda_2 s} \]

Thus, we get

\[ |k - 1| \leq |k - e^{\lambda_2 s_0} \langle \tilde{v} (\cdot, s_0), c_2 \varphi_2 \rangle| + |e^{\lambda_2 s_0} \langle \tilde{v} (\cdot, s_0), c_2 \varphi_2 \rangle - 1| \]

\[ \leq C (n, \Lambda, \rho, \beta) e^{-c \lambda_2 s} \]

for \( s_0 \leq s \leq s_1 \).

Now we would like to estimate the remaining parts in the Fourier expansion of \( \tilde{v} (\cdot, s) \). Let

\[ \tilde{v}_s (\cdot, s) = \tilde{v} (\cdot, s) - \sum_{i=0}^{2} \langle \tilde{v} (\cdot, s), \varphi_i \rangle \varphi_i \]

then \( \tilde{v}_s (\cdot, s) \in C ([s_0, s_1] : H_s) \), where \( H_s \) is defined in Lemma 6.2. By Proposition 3.1 and 6.14, we have

\[ (\partial_s + L) \tilde{v}_s (\cdot, s) = f (\cdot, s) - \sum_{i=0}^{2} \langle f (\cdot, s), \varphi_i \rangle \varphi_i \equiv f_s (\cdot, s) \]

Note that \( \parallel f_s (\cdot, s) \parallel \leq \parallel f (\cdot, s) \parallel \) and that \( \lambda_3 = \lambda_2 + 1 \). By Lemma 6.2 for any \( 0 < \delta < 1 \), we have

\[ \parallel \tilde{v}_s (\cdot, s) \parallel^2 \leq e^{-2(1-\delta)(\lambda_2+1)(s-s_0)} \parallel \tilde{v}_s (\cdot, s_0) \parallel^2 + \frac{1}{2 \delta \lambda_3} \int_{s_0}^{s} e^{-2(1-\delta)(\lambda_2+1)(s-\xi)} \parallel f (\cdot, \xi) \parallel^2 d\xi \]

\[ \parallel L \tilde{v}_s (\cdot, s) \parallel^2 \leq e^{-2(1-\delta)(\lambda_2+1)(s-s_0)} (L \tilde{v}_s (\cdot, s_0), \tilde{v}_s (\cdot, s_0)) + \frac{1}{2 \delta} \int_{s_0}^{s} e^{-2(1-\delta)(\lambda_2+1)(s-\xi)} \parallel f (\cdot, \xi) \parallel^2 d\xi \]

for \( s_0 \leq s \leq s_1 \). We claim that

\[ \parallel \tilde{v}_s (\cdot, s_0) \parallel + \parallel L \tilde{v}_s (\cdot, s_0) \parallel \leq C (n, \Lambda, \rho, \beta) e^{-c \lambda_2 s_0} \]

Note that since \( \varsigma < \lambda_2^{-1} \), there is \( \delta \in (0, 1) \) so that \( (1 - \delta) (\lambda_2 + 1) > (1 + \varsigma) \lambda_2 \).

Thus, we get

\[ \parallel \tilde{v}_s (\cdot, s) \parallel^2 + \parallel L \tilde{v}_s (\cdot, s) \parallel^2 \leq C (n, \Lambda, \rho, \beta) e^{-2(1+\varsigma)\lambda_2 s} \]

which, by (6.18), yields

\[ \parallel \tilde{v}_s (\cdot, s) \parallel^2 + \parallel \partial_y \tilde{v}_s (\cdot, s) \parallel^2 \leq C (n, \Lambda, \rho, \beta) e^{-2(1+\varsigma)\lambda_2 s} \]

By Lemma 6.3, we then get

\[ \parallel \tilde{v}_s (y, s) \parallel \leq C (n) (\parallel \partial_y \tilde{v}_s (\cdot, s) \parallel + \parallel \tilde{v}_s (\cdot, s) \parallel) \left( \frac{1}{y^{\varsigma \delta} + e^{(y+1)^2/4}} \right) \]
for $s_0 \leq s \leq \hat{s}$. To prove (6.18), we use Proposition 5.1, Lemma 4.3, (4.1) and previous computation for deriving (6.16) and (6.17) to get

$$\|\tilde{v}_* (\cdot, s_0)\| = \|\tilde{v} (\cdot, s_0) - \sum_{i=0}^{2} \langle \tilde{v} (\cdot, s_0), \varphi_i \rangle \varphi_i \|$$

\[
\leq \|\tilde{v} (\cdot, s_0) - e^{-\lambda_2 s_0} \sum_{i=0}^{2} \frac{a_i}{c_i} \varphi_i \| + \|e^{-\lambda_2 s_0} \sum_{i=0}^{2} \frac{a_i}{c_i} \varphi_i - \sum_{i=0}^{2} \langle \tilde{v} (\cdot, s_0), \varphi_i \rangle \varphi_i \| \\
\leq e^{-\lambda_2 s_0} \left( 1 - \zeta (e^{\sigma_0 y} - \beta) (\zeta (e^{\Phi\lambda} - y)) \sum_{i=0}^{2} \frac{a_i}{c_i} \varphi_i \right) + \sum_{i=0}^{2} \frac{1}{c_i} \langle \tilde{v} (\cdot, s_0), c_i \varphi_i \rangle - a_4 e^{-\lambda_2 s_0} \right) \\
\leq C (n, \Lambda, \rho, \beta) e^{-(1+\epsilon)\lambda_2 s_0}
\]

where $a_2 = 1$, and

\[
\|L\tilde{v}_* (\cdot, s_0)\| = \|L \left( \zeta (e^{\sigma_0 y} - \beta) (\zeta (e^{\Phi\lambda} - y)) v (\cdot, s_0) - \sum_{i=0}^{2} \langle \tilde{v} (\cdot, s), \varphi_i \rangle \lambda_i \varphi_i \right) \|
\]

\[
= \|L \left( \zeta (e^{\sigma_0 y} - \beta) (\zeta (e^{\Phi\lambda} - y)) e^{-\lambda_2 s_0} \sum_{i=0}^{2} \frac{a_i}{c_i} \varphi_i \right) - \sum_{i=0}^{2} \langle \tilde{v} (\cdot, s), \varphi_i \rangle \lambda_i \varphi_i \right) \|
\]

\[
\leq e^{-\lambda_2 s_0} \left| L \left( \zeta (e^{\sigma_0 y} - \beta) (\zeta (e^{\Phi\lambda} - y)) e^{-\lambda_2 s_0} \sum_{i=0}^{2} \frac{a_i}{c_i} \varphi_i \right) - \sum_{i=0}^{2} \frac{a_i}{c_i} \lambda_i \varphi_i \right| \\
+ \left| \sum_{i=0}^{2} \langle \tilde{v} (\cdot, s), \varphi_i \rangle \lambda_i \varphi_i - e^{-\lambda_2 s_0} \sum_{i=0}^{2} \frac{a_i}{c_i} \lambda_i \varphi_i \right| \\
\leq \|h\| + \sum_{i=0}^{2} \frac{\lambda_i}{c_i} \|\langle \tilde{v} (\cdot, s_0), c_i \varphi_i \rangle - a_4 e^{-\lambda_2 s_0} \|
\]

where

\[
h (y) = \zeta' (e^{\sigma_0 y} - \beta) e^{\sigma_0 y} \left( -2 \delta_v v (y, s_0) + \left( -\frac{2(n-1)}{y} + \frac{y}{2} \right) v (y, s_0) \right) \\
+ \zeta' (\zeta (e^{\Phi\lambda} - y)) \left( -\frac{y}{2} + \frac{2(n-1)}{y} \right) v (y, s_0) + 2 \partial_y v (y, s_0) \\
- \zeta'' (e^{\sigma_0 y} - \beta) e^{2\sigma_0 y} v (y, s_0) - \zeta'' (\zeta (e^{\Phi\lambda} - y)) v (y, s_0)
\]

Note that by similar computation as for $f_{11} (\cdot, s)$ and $f_{111} (\cdot, s)$, we have

\[
\|h\| \leq C (n, \Lambda, \rho, \beta) e^{-(1+\epsilon)\lambda_2 s_0}
\]

Hence,

\[
\|L\tilde{v}_* (\cdot, s_0)\| \leq C (n, \Lambda, \rho, \beta) e^{-(1+\epsilon)\lambda_2 s_0}
\]

Lastly, combining (6.16), (6.17), and (6.19), we conclude

\[
\|\tilde{v} (y, s) - \frac{k}{c_2} e^{-\lambda_2 s} \varphi_2 (y) \| = \| \sum_{i=0}^{2} \langle \tilde{v} (\cdot, s), \varphi_i \rangle \varphi_i (y) + \tilde{v}_* (y, s) - \frac{k}{c_2} e^{-\lambda_2 s} \varphi_2 (y) \|
\]
Thus, we get

\[ |\bar{v}(s, t)| + |(\bar{v}(s, t), \varphi_2(y) - \frac{k}{c_2} e^{-\lambda s} \varphi_2(y)| + |\bar{v}_s(s, t)| \leq C(n, \Lambda, \rho, \beta) e^{-(1+c)\lambda s} \left( \frac{1}{y^{n-1}} + e^{(\frac{1+n}{1}-\frac{1}{2})} \right) \]

for \( s_0 \leq s \leq \tilde{s} \). As a result, for \( \frac{1}{2} e^{-d \sigma s} \leq y \leq 1 \), we have

\[ \left| v(y, s) - \frac{k}{c_2} e^{-\lambda s} \varphi_2(y) \right| \leq C(n, \Lambda, \rho, \beta) e^{-\lambda s} y^{a+2} \]

and for \( 1 \leq y \leq \sqrt{\lambda} \), we have

\[ \left| v(y, s) - \frac{k}{c_2} e^{-\lambda s} \varphi_2(y) \right| \leq C(n, \Lambda, \rho, \beta) e^{-\lambda s} y^{a+2} \]

As a corollary, by \(3.12\), Proposition \(6.4\) and Remark \(3.2\) we get

\[ |u(x, t) - \frac{k}{c_2} (-t)^{\lambda s} \varphi_2 \left( \frac{x}{\sqrt{-t}} \right)| \leq C(n, \Lambda, \rho, \beta) (-t)^{a} (-t) x^{a+2} \]

(6.20)

for \( \frac{1}{3} \sqrt{-t} \leq x \leq \sqrt{\lambda} t \ln(-t), t_0 \leq t \leq \tilde{t} \). Below we use \(3.7\), \(4.5\), \(6.20\) and the comparison principle to prove \(6.5\).

**Proposition 6.5.** If \( 0 < \rho \ll 1 \) (depending on \( n, \Lambda \)) and \( |t_0| \ll 1 \) (depending on \( n, \Lambda, \rho \)), there holds \(6.7\).

**Proof.** First, by \(3.8\) we have

\[ \text{max} \left\{ \frac{|u(x, t)|}{x}, \frac{u(x, t)}{|x|} \right\} \leq \Lambda \left( (-t)^2 x^{a-1} + x^{2\lambda} \right) \leq \frac{1}{3} \]

for \( \sqrt{\lambda} t \ln(-t) \leq x \leq \rho, t_0 \leq t \leq \tilde{t} \), provided that \( 0 < \rho \ll 1 \) (depending on \( n, \Lambda \)) and \( |t_0| \ll 1 \) (depending on \( n, \Lambda, \rho \)).

By \(3.7\), \(3.8\) and Remark \(3.2\) there holds

\[ |\partial_{x} u(x, t)| \leq C(n) \left( |\partial_{x} u(x, t)| + \frac{|u(x, t)|}{x} \right) \leq C(n) x^{a+2} \]

for \( \sqrt{\lambda} t \ln(-t) \leq x \leq \rho, t_0 \leq t \leq \tilde{t} \). In addition, we have

\[ \partial_{t} \left( k(-t)^{\lambda s} \varphi_2 \left( \frac{x}{\sqrt{-t}} \right) \right) = k \partial_{t} \left( 2\lambda x^{2\lambda s+1} + 2\lambda (-t)^{a+2} + (-t)^2 x^{a} \right) \]

Thus, we get

\[ |\partial_{t} \left( u(x, t) - k(-t)^{\lambda s} \varphi_2 \left( \frac{x}{\sqrt{-t}} \right) \right)| \leq C(n, \Lambda) x^{a+2} \]
for $\sqrt{\lambda_2} t \ln (-t) \leq x \leq \rho$, $t_0 \leq t \leq \hat{t}$.

On the other hand, at time $t_0$, by (6.21), (6.22) and (6.23), there holds

$$
|u(x, t_0) - k(-t_0)^{\lambda_2 + \frac{1}{2}} \phi_2 \left( \frac{x}{\sqrt{-t}} \right)|
\leq (-t_0)^{\lambda_2 + \frac{1}{2}} \left( \frac{|k| - 1}{c_0} \phi_2 \left( \frac{x}{\sqrt{-t}} \right) + \sum_{i=0}^{1} \frac{|a_i|}{c_i} \phi_1 \left( \frac{x}{\sqrt{-t}} \right) \right)
$$

(6.22)

\[ \leq C(n, \Lambda, \rho, \beta) (-t_0)^{\kappa} x^{2\lambda_2 + 1} \]

for $\sqrt{\lambda_2} t \ln (-t) \leq x \leq \rho$. Moreover, by (6.20) we have

(6.23)

\[ |u(x, t) - k(-t)^{\lambda_2 + \frac{1}{2}} \phi_2 \left( \frac{x}{\sqrt{-t}} \right)| \leq C(n, \Lambda, \rho, \beta) (-t_0)^{\kappa} x^{2\lambda_2 + 1} \]

for $x = \sqrt{\lambda_2} t \ln (-t)$, $t_0 \leq t \leq \hat{t}$.

Combining (6.21), (6.22) and (6.23), we get

\[ |u(x, t) - k(-t)^{\lambda_2 + \frac{1}{2}} \phi_2 \left( \frac{x}{\sqrt{-t}} \right)| \leq C(n, \Lambda, \rho, \beta) (-t_0)^{\kappa} x^{2\lambda_2 + 1} + C(n, \Lambda) x^{\alpha + 2} (t - t_0) \]

\[ \leq C(n, \Lambda, \rho, \beta) (-t_0)^{\kappa} x^{2\lambda_2 + 1} \]

for $\sqrt{\lambda_2} t \ln (-t) \leq x \leq \rho$, $t_0 \leq t \leq \hat{t}$. The conclusion follows by (6.20) and the above.

Next, by (3.12) and Proposition 6.4, we have

\[ |w(z, \tau) - \frac{k}{c_2} (2\sigma \tau)^{\frac{1}{2}} \phi_2 \left( \frac{z}{\sqrt{2\sigma \tau}} \right)| \leq C(n, \Lambda, \rho, \beta) (2\sigma \tau)^{-\frac{1}{2}} \frac{z^2}{2\sigma \tau} z^{\alpha} \]

for $\frac{1}{2} (2\sigma \tau)^{\frac{1}{2}(1-\delta)} \leq z \leq \sqrt{2\sigma \tau}$, $\tau_0 \leq \tau \leq \hat{\tau}$. Notice that

\[ \frac{k}{c_2} (2\sigma \tau)^{\frac{1}{2}} \phi_2 \left( \frac{z}{\sqrt{2\sigma \tau}} \right) = k z^\alpha \left( 1 + 2 Y_1 \frac{z^2}{2\sigma \tau} + T_2 \left( \frac{z^2}{2\sigma \tau} \right)^2 \right) \]

Hence we get

\[ |w(z, \tau) - k z^\alpha| \leq C(n) \frac{z^2}{2\sigma \tau} z^{\alpha} \]

for $\frac{1}{2} (2\sigma \tau)^{\frac{1}{2}(1-\delta)} \leq z \leq \sqrt{2\sigma \tau}$, $\tau_0 \leq \tau \leq \hat{\tau}$, provided that $\tau_0 \gg 1$ (depending on $n$, $\Lambda$, $\rho$, $\beta$). On the other hand, by Lemma 2.5 and (0.3), we have

\[ |\psi_k(z) - k z^\alpha| \leq C(n) k^3 z^{3\alpha - 2} \leq C(n) z^{3\alpha - 2} \]

for $z \geq \frac{\tilde{w}(0)}{\sqrt{2}}$, provided that $\tau_0 \gg 1$ (depending on $n$, $\Lambda$, $\rho$, $\beta$). Therefore, we get

\[ |w(z, \tau) - \psi_k(z)| \leq |w(z, \tau) - k z^\alpha| + |k z^\alpha - \psi_k(z)| \]

(6.24)

\[ \leq C(n) \left( \frac{z^2}{2\sigma \tau} + z^{2(\alpha - 1)} \right) z^{\alpha} \]
for \( \frac{1}{2} (2 \sigma \tau)^{ \frac{1}{2} (1 - \theta) } \leq z \leq \sqrt{2 \sigma \tau} \), \( \tau_0 \leq \tau \leq \hat{\tau} \). Now consider the projected curves \( \tilde{M}_k \) and \( \tilde{\Gamma}^{(a_0, a_1)}_\tau \) (see (6.27) and (6.23)), which can be viewed as graphs of \( w(z, \tau) \) and \( \psi_k(z) \) over \( C \) (see (2.24)), respectively. Thus, (6.24) implies that

\[
|\hat{w}(z, \tau) - \hat{\psi}_k(z)| \leq C(n) \left( \frac{z^2}{2 \sigma \tau} + z^{2(\alpha - 1)} \right) z^\alpha
\]

for \( (2 \sigma \tau)^{ \frac{1}{2} (1 - \theta) } \leq z \leq \frac{1}{2} \sqrt{2 \sigma \tau} \), \( \tau_0 \leq \tau \leq \hat{\tau} \), provided that \( \tau_0 \gg 1 \) (depending on \( n, \Lambda, \rho, \beta \)). In particular, there holds

(6.25)

\[
|\hat{w}(z, \tau) - \hat{\psi}_k(z)| \leq C(n) (2 \sigma \tau)^{ -\theta } z^\alpha
\]

for \( z = (2 \sigma \tau)^{ \frac{1}{2} (1 - \theta) } \), \( \tau_0 \leq \tau \leq \hat{\tau} \), since \( 0 < \theta < \frac{\alpha - 1}{\alpha} \) (see (4.12)).

In addition, when \( \tau = \tau_0 \), by (4.17), (6.2) and (6.3), we have

\[
|w(z, \tau_0) - \psi_k(z)| \leq |w(z, \tau_0) - kz^\alpha| + |kz^\alpha - \psi_k(z)|
\]

\[
\leq \left( |k - 1| + |a_0| + |a_1| + C(n) \left( \frac{z^2}{2 \sigma \tau_0} + z^{2(\alpha - 1)} \right) \right) z^\alpha
\]

\[
\leq \left( C(n, \Lambda, \rho, \beta) (2 \sigma \tau_0)^{- \frac{\alpha - 1}{2}} + C(n) \left( (2 \sigma \tau_0)^{- \theta} + \beta z^{2(\alpha - 1)} \right) \right) z^\alpha
\]

\[
\leq C(n) \beta z^{2(\alpha - 1)} z^\alpha
\]

for \( \beta \leq z \leq 2 (2 \sigma \tau_0)^{ \frac{1}{2} (1 - \theta) } \), \( \tau_0 \leq \tau \leq \hat{\tau} \), provided that \( \tau_0 \gg 1 \) (depending on \( n, \Lambda, \rho, \beta \)). By reparametrizing \( \tilde{\Gamma}^{(a_0, a_1)}_\tau \) and \( \tilde{M}_k \), we deduce that

(6.26)

\[
|\hat{w}(z, \tau_0) - \hat{\psi}_k(z)| \leq C(n) \beta z^{2(\alpha - 1)} z^\alpha
\]

for \( \frac{3}{2} \beta \leq z \leq (2 \sigma \tau_0)^{ \frac{1}{2} (1 - \theta) } \), \( \tau_0 \leq \tau \leq \hat{\tau} \), provided that \( \tau_0 \gg 1 \) (depending on \( n, \Lambda, \rho, \beta \)).

Below we use (6.22), (6.27), (6.25) and the comparison principle to prove (6.8). We follow Velázquez’s idea of using the perturbation of \( \psi_k \) to construct barriers; moreover, we allow the perturbation to be time-dependent.

**Proposition 6.6.** If \( \beta \gg 1 \) (depending on \( n \)) and \( \tau_0 \gg 1 \) (depending on \( n, \Lambda, \rho, \beta \)), there holds (6.6). In particular, we have

(6.27)

\[
|\hat{w}(z, \tau) - \hat{\psi}_k(z)| \leq C(n) \beta z^{\alpha - 3} \left( \frac{\tau}{\tau_0} \right)^{-\theta} z^\alpha
\]

for \( \beta \leq z \leq (2 \sigma \tau)^{ \frac{1}{2} (1 - \theta) } \), \( \tau_0 \leq \tau \leq \hat{\tau} \), and

(6.28)

\[
|\hat{w}(z, \tau) - \hat{\psi}_k(z)| \leq C(n) \beta z^{\alpha - 3} \left( \frac{\tau}{\tau_0} \right)^{-\theta} z^\alpha
\]

for \( 0 \leq z \leq 5 \beta \), \( \tau_0 \leq \tau \leq \hat{\tau} \).

**Proof.** Given functions \( \lambda(\tau) \) and \( \mu(\tau) \), we define the perturbation of \( \hat{\psi}_k \) by

\[
\hat{\psi}_k^\lambda(\tau, \mu(z)) = \lambda^{\frac{1}{1 - \sigma}}(t) \hat{\psi}_k \left( \frac{z}{\lambda^\frac{1}{1 - \sigma}(t) \mu(\tau)} \right)
\]

(see also (2.3)). By (2.4), there holds

\[
\tilde{\psi}_k^\lambda(\tau, \mu(z)) = \left( \frac{\partial^2 \hat{\psi}_k^\lambda(\tau, \mu(z))}{1 + (\partial z \hat{\psi}_k^\lambda(\tau, \mu(z)))^2} + (n - 1) \left( \frac{\partial_z \hat{\psi}_k^\lambda(\tau, \mu(z))}{\hat{\psi}_k^\lambda(\tau, \mu(z))} - \frac{1}{\hat{\psi}_k^\lambda(\tau, \mu(z))} \right) + \frac{1}{2} \tilde{\psi}_k^\lambda(\tau, \mu(z)) \right)
\]
\[
\left( -\frac{1}{\alpha} + \frac{\sigma}{2\sigma^2} \right) \frac{\mu}{r^{\alpha}} \frac{1}{1-\alpha} \left( \partial_r \lambda \right) \left( \hat{\psi}_k (r) - r \partial_r \hat{\psi}_k (r) \right) - \frac{\lambda}{\mu} \left( \partial_r \mu \right) \left( r \partial_r \hat{\psi}_k (r) \right) \right|_{r = \frac{1}{\lambda^{1+\alpha}}} 
\]
\[
+ \mu^2 - 1 \frac{1}{\lambda^{1+\alpha} - \mu^2} \left( \frac{\partial^2_{rr} \hat{\psi}_k (r)}{1 + \left( \partial_r \hat{\psi}_k (r) \right)^2} \right) \left( 1 + \left( \partial_r \hat{\psi}_k (r) \right)^2 \right) + (n-1) \left( \mu \right) \right|_{r = \frac{1}{\lambda^{1+\alpha} - \mu^2}} 
\]
(6.29)

Notice that
\[
\left( \partial_\lambda \left( \hat{\psi}_k^{\lambda, \mu} (z) \right) \right) = \frac{1}{\lambda^{1+\alpha}} \left( \hat{\psi}_k (r) - r \partial_r \hat{\psi}_k (r) \right) \right|_{r = \frac{1}{\lambda^{1+\alpha}}} 
\]
(6.30)

Moreover, by (2.6), there holds
\[
\left( \partial_\mu \left( \hat{\psi}_k^{\lambda, \mu} (z) \right) \right) = - \frac{1}{\lambda^{1+\alpha}} \left( r \partial_r \hat{\psi}_k (r) \right) \right|_{r = \frac{1}{\lambda^{1+\alpha}}} 
\]

which implies
\[
\lim_{r \to \infty} \frac{\hat{\psi}_k (r) - r \partial_r \hat{\psi}_k}{r^\alpha} = k \lim_{r \to \infty} \frac{\psi (r) - r \partial_r \psi}{r^\alpha} = k (1 - \alpha) \, 2^{\frac{\alpha+1}{2}} 
\]
for \( r \geq \beta \), if \( \beta \gg 1 \) (depending on \( n \)) and \( \tau_0 \gg 1 \) (depending on \( n, \Lambda, \rho, \beta \)).

To get a lower barrier, we set
\[
\hat{w}_- (z, \tau) = \hat{\psi}_k^{\lambda_- (\tau)} (z, \tau) 
\]
with
\[
\lambda_- (\tau) = 1 - \beta^\alpha - 3 \left( \frac{\tau}{\tau_0} \right)^{\frac{\alpha}{2}}, \quad \mu_- (\tau) = 1 
\]
where \( \beta \gg 1 \) (depending on \( n \)). Firstly, for the initial value, by Lemma 2.2 and (4.3), we have
\[
\hat{w}_- (z, \tau_0) = \hat{\psi}_k (z) + (\lambda_- (\tau_0) - 1) \partial_\lambda \left( \hat{\psi}_k^{\lambda_- (\tau_0)} (z) \right) \right|_{\lambda = \lambda_- (\tau_0), z = z_0} = \frac{1}{\lambda^{1-\alpha}} 
\]
(6.32)

for \( 0 \leq z \leq \frac{3}{2} \beta \), provided that \( \beta \gg 1 \) (depending on \( n \)). Also, for each \( \frac{3}{2} \beta \leq z \leq \left( 2\sigma \tau_0 \right)^{\frac{1}{2}} (1 - \eta) \), by (6.30), (6.31), (6.29) and the mean value theorem, there is \( \lambda_- (\tau_0) \leq \lambda \leq 1 \) so that
\[
\hat{w}_- (z, \tau) = \hat{\psi}_k (z) + (\lambda_- (\tau_0) - 1) \partial_\lambda \left( \hat{\psi}_k^{\lambda_- (\tau_0)} (z) \right) \right|_{\lambda = \lambda_- (\tau_0), z = z_0} = \frac{1}{\lambda^{1-\alpha}} 
\]
(6.33)

\[
\leq \hat{\psi}_k (z) - (1 - o(1)) \beta^\alpha - 3 \frac{\alpha+1}{2} \, \hat{w}_- (z, \tau_0) < \hat{w}_- (z, \tau_0) 
\]
provided that $\beta \gg 1$ (depending on $n$). Secondly, for the boundary value, fix $\tau_0 \leq \tau \leq \hat{\tau}$ and let $z = (2\sigma \tau)^{\frac{1}{1-\theta}}$. By (6.29), (6.30), (6.31) and the mean value theorem, there is $\lambda_-(\tau_0) \leq \lambda_* \leq 1$ so that

$$\hat{w}_-(z, \tau_0) = \hat{\psi}_k(z) + (\lambda_-(\tau_0) - 1) \partial_\tau \left( \hat{\psi}_k(z) \right) \bigg|_{\lambda = \lambda_*}$$

$$\leq \hat{\psi}_k(z) - (1 - \theta(1)) \beta^{\alpha - 3} \frac{\alpha + 1}{\alpha} \left( \frac{\tau}{\tau_0} \right)^{1 - \alpha}$$

(6.34)

$$< \hat{\psi}_k(z) - C(n)(2\sigma \tau)^{-\theta} z^\alpha \leq \hat{w}(z, \tau)$$

provided that $\tau_0 \gg 1$ (depending on $n$, $\beta$), since $0 < \varrho < \vartheta$. Thirdly, for the equation, by (6.29), there holds

$$\partial_\tau \hat{w}_- - \left( \frac{\partial^2_{zz} \hat{w}_-}{1 + (\partial_z \hat{w}_-)^2} + (n - 1) \left( \frac{\partial_z \hat{w}_-}{z} - \frac{1}{\hat{w}_-} \right) + \frac{\frac{3}{2} + \sigma}{2\sigma \tau} \left( - z \partial_z \hat{w}_- + \hat{w}_- \right) \right) \right|_{r = \frac{z}{\lambda_1^{-\alpha} (\tau)}}$$

$$= \lambda_1^{-\alpha} (\tau) \left( - \left( \frac{1}{2} + \sigma \right) + \frac{2\sigma \beta^{\alpha - 3} \left( \frac{\tau}{\tau_0} \right)^{-\theta}}{(1 - \alpha) \lambda_1^{-\alpha} (\tau)} \right) \right|_{r = \frac{z}{\lambda_1^{-\alpha} (\tau)}} \leq 0$$

for $0 \leq z \leq (2\sigma \tau)^{\frac{1}{1-\theta}}$, $\tau_0 \leq \tau \leq \hat{\tau}$, provided that $\beta \gg 1$ (depending on $n$). Then we subtract the above equation from (3.22) to get

(6.35)

$$\partial_\tau (\hat{w} - \hat{w}_-) - \left( \frac{1}{1 + (\partial_z \hat{w}_-)^2} \partial^2_{zz} (\hat{w} - \hat{w}_-) + \frac{n - 1}{z} \partial_z (\hat{w} - \hat{w}_-) \right)$$

$$+ \left( \frac{\partial^2_{zz} (\hat{w} - \hat{w}_-)}{1 + (\partial_z \hat{w}_-)^2} \frac{\frac{3}{2} + \sigma}{(1 + (\partial_z \hat{w}_-)^2)} \right) \partial_z (\hat{w} - \hat{w}_-) - \left( \frac{n - 1}{\hat{w} - \hat{w}_-} + \frac{\frac{3}{2} + \sigma}{2\sigma \tau} \right) (\hat{w} - \hat{w}_-)$$

$$\geq 0$$

Now we are ready to show that $\hat{w}_-$ is a lower barrier. Let

$$\hat{w}_-(z, \tau) = \min_{0 \leq z \leq (2\sigma \tau)^{\frac{1}{1-\theta}}} (\hat{w} - \hat{w}_-) (z, \tau)$$

then by (6.32) and (6.33), we have

$$\hat{w}_-(\tau_0) > 0$$

We claim that

$$\hat{w}_-(\tau) \geq 0 \quad \forall \quad \tau_0 \leq \tau \leq \hat{\tau}$$

Suppose the contrary, then there is $\tau_0 < \tau^* \leq \hat{\tau}$ so that

(6.36)

$$\hat{w}_-(\tau^*) < 0$$
Let $\tau_0^* \in [\tau_0, \tau_*^1]$ be the first time after which $(\dot{w} - \bar{w}_-)_\min$ stays negative all the way up to $\tau_*^1$. By continuity, there holds

$$
(\dot{w} - \bar{w}_-)_\min (\tau_0^*) = 0
$$

On the other hand, by (6.34), the negative minimum of $\dot{w} - \bar{w}_-$ for each time-slice is achieved in $[0, (2\sigma \tau)^{2(1-\beta)}]$. Hence, applying the maximum principle to (6.35), we get

$$
\partial_\tau (\dot{w} - \bar{w}_-)_\min - \left(\frac{n-1}{\dot{w} \bar{w}_-} + \frac{1}{2} \frac{\dot{w}}{2\sigma \tau} \right) (\dot{w} - \bar{w}_-)_\min \geq 0
$$

Notice that

$$
\partial_\tau \dot{w} (0, \tau) = 0 = \partial_{\tau} \dot{w}_- (0, \tau) \quad \forall \tau_0 \leq \tau \leq \tau^*
$$

So at $z = 0$, by L'Hôpital's rule, the third term in (6.35) is interpreted as

$$
\lim_{z \to 0} \frac{n-1}{z} \partial_z (\dot{w} - \bar{w}_-) (z, \tau) = (n-1) \partial^2_{zz} (\dot{w} - \bar{w}_-) (0, \tau)
$$

It follows that

$$
\partial_\tau \left( e^{-\int_{\tau_0}^{\tau} \frac{n-1}{\dot{w} \bar{w}_-} \partial_\tau \frac{1}{2} \frac{\dot{w}}{2\sigma \tau} } \right) (\dot{w} - \bar{w}_-)_\min (\tau) \geq 0
$$

which, together with (6.36), contradicts with (6.37).

Next, for the upper barrier, we set

$$
\dot{w}_+ (z, \tau) = \dot{\psi}_k^{\lambda_+, \mu_+} (z, \tau)
$$

with

$$
\lambda_+ (\tau) = 1 + \beta^{\alpha-3} \left( \frac{\tau}{\tau_0} \right)^{-\delta}, \quad \mu_+ (\tau) = 1 + \delta \beta^{\alpha-3} (2\sigma \tau)^{1+\delta} \left( \frac{\tau}{\tau_0} \right)^{-\delta}
$$

where

$$
\delta = \delta (n, \beta) = \frac{1}{4(1-\alpha)} \inf_{0 \leq r \leq \frac{3}{2} \beta} \frac{\dot{\psi}_k (r) - r \partial_r \dot{\psi}_k (r)}{r \partial_r \dot{\psi}_k (r)} > 0
$$

by (2.34). Note that by (see (2.4)),

$$
0 \leq \partial_r \dot{\psi}_k (r) \leq 1, \quad \partial_{rr} \dot{\psi}_k (r) > 0
$$

for all $r \geq 0$. Firstly, for the initial value, given $0 \leq z \leq \frac{3}{2} \beta$, by Lemma 2.2, (4.3), (6.30), (6.31) and the mean value theorem, there are

$$
1 + \frac{1}{2} \beta^{\alpha-3} \leq \lambda_* \leq \lambda_+ (\tau_0), \quad 1 \leq \mu_* \leq \mu_+ (\tau_0)
$$

so that

$$
\dot{w}_+ (z, \tau_0) = \hat{\psi}_k^{1+\frac{1}{2} \beta^{\alpha-3}, 1} (z, \tau_0)
$$

$$
+ \left( \lambda_+ (\tau_0) - \left( 1 + \frac{1}{2} \beta^{\alpha-3} \right) \right) \partial_\lambda \left( \hat{\psi}_k^{\lambda, \mu} (z) \right) \bigg|_{\lambda = \lambda_+, \mu = \mu_+} + \left( \mu_+ (\tau_0) - 1 \right) \partial_\mu \left( \hat{\psi}_k^{\lambda, \mu} (z) \right) \bigg|_{\lambda = \lambda_+, \mu = \mu_+}
$$

$$
\quad \quad \quad + \frac{\beta^{\alpha-3} \lambda_*^{\alpha-3}}{2(1-\alpha)} \left( \hat{\psi}_k (z_*) - z_* \partial_z \hat{\psi}_k (z_*) \right)
$$

$$
= \hat{\psi}_k^{1+\frac{1}{2} \beta^{\alpha-3}, 1} (z, \tau_0) + \frac{\beta^{\alpha-3} \lambda_*^{\alpha-3}}{2(1-\alpha)} \left( \hat{\psi}_k (z_*) - z_* \partial_z \hat{\psi}_k (z_*) \right)
$$
\[-\delta \beta^{n-3}(2\sigma \tau_0)^{-1+\frac{1}{n}} \frac{\lambda^{\frac{1}{n}}}{\mu_*} \left(z_* \partial_z \hat{\psi}_k (z_*)\right)\]
\[\geq \hat{\psi}_k^{1+\frac{1}{2}\beta^{n-3},1} (z, \tau_0) + \frac{\beta^{n-3} \lambda^{\frac{1}{n}}}{2(1-\alpha)} \left(1 - \frac{\lambda}{2\mu_*} (2\sigma \tau_0)^{-1+\frac{1}{n}}\right) \left(\hat{\psi}_k (z_*) - z_* \partial_z \hat{\psi}_k (z_*)\right)\]
\[\geq \hat{\psi}_k^{1+\frac{1}{2}\beta^{n-3},1} (z, \tau_0) = \hat{\psi}_k (1+\frac{1}{2}\beta^{n-3}) (z, \tau_0) = \hat{\psi}_k (1+\frac{1}{2}\beta^{n-3})(1+o(1)) (z, \tau_0)\]
\[(6.40)\]
\[> w (z, \tau_0)\]

provided that \(\beta \gg 1\) (depending on \(n\)). Also, for each \(\frac{1}{2} \beta \leq z \leq (2\sigma \tau_0)^{\frac{1}{2}(1-\delta)}\), by \((4.22), (6.26), (6.30), (6.31), (6.38), (6.39)\) and the mean value theorem, there are
\[1 \leq \lambda_* \leq \lambda_+ (\tau_0), \quad 1 \leq \mu_* \leq \mu_+ (\tau_0)\]
so that
\[\hat{w}_+ (z, \tau_0) = \hat{\psi}_k (z, \tau_0) + (\lambda_+ (\tau_0) - 1) \partial_\lambda \left(\hat{\psi}_k^{\lambda_* \mu_+} (z)\right)\bigg|_{\lambda=\lambda_* \mu=\mu_* \ z=z_*} + \frac{1}{\lambda^{1-\delta}} \mu_*\]
\[= \hat{\psi}_k (z, \tau_0) + \frac{\beta^{n-3} \lambda^{\frac{1}{n}}}{1-\alpha} \left(\hat{\psi}_k (z_*) - z_* \partial_z \hat{\psi}_k (z_*)\right) - \delta \beta^{n-3}(2\sigma \tau_0)^{-1+\frac{1}{n}} \frac{\lambda^{\frac{1}{n}}}{\mu_*} \left(z_* \partial_z \hat{\psi}_k (z_*)\right)\]
\[\geq \hat{\psi}_k (z, \tau_0) + (1 + o(1)) \beta^{n-3} \mu_*^{-\alpha} 2^{\frac{n+1}{n}} z^\alpha - \delta \beta^{n-3}(2\sigma \tau_0)^{-1+\frac{1}{n}} \frac{\lambda^{\frac{1}{n}}}{\mu_*} \left(z_* \partial_z \hat{\psi}_k (z_*)\right)\]
\[= \hat{\psi}_k (z, \tau_0) + \frac{1}{2} (1 + o(1)) \beta^{n-3} \mu_*^{-\alpha} 2^{\frac{n+1}{n}} z^\alpha + \frac{1}{\lambda^{1-\delta}} \mu_*\]
\[(6.41)\]
\[\geq \hat{\psi}_k (z, \tau_0) + \frac{1}{2} (1 + o(1)) \beta^{n-3} \mu_*^{-\alpha} 2^{\frac{n+1}{n}} z^\alpha > w (z, \tau_0)\]

provided that \(\beta \gg 1\) (depending on \(n\)), since \(z \leq (2\sigma \tau_0)^{\frac{1}{2}(1-\delta)}\). Secondly, for the boundary value, fix \(\tau_0 \leq \tau \leq \hat{\tau}\) and let \(z = (2\sigma \tau)^{\frac{1}{2}(1-\delta)}\), by \((4.22), (6.26), (6.30), (6.31), (6.38), (6.39)\) and the mean value theorem, there are
\[1 \leq \lambda_* \leq \lambda_+ (\tau), \quad 1 \leq \mu_* \leq \mu_+ (\tau)\]
so that
\[\hat{w}_+ (z, \tau) = \hat{\psi}_k (z, \tau) + (\lambda_+ (\tau) - 1) \partial_\lambda \left(\hat{\psi}_k^{\lambda_* \mu_+} (z)\right)\bigg|_{\lambda=\lambda_* \mu=\mu_* \ z=z_*} + \frac{1}{\lambda^{1-\delta}} \mu_*\]
\[= \hat{\psi}_k (z) + \beta^{n-3}(\frac{\tau}{\tau_0})^{-\frac{\alpha}{n}} \frac{\lambda^{\frac{1}{n}}}{1-\alpha} \left(\hat{\psi}_k (z_*) - z_* \partial_z \hat{\psi}_k (z_*)\right)\]
\[-\delta \beta^{n-3}(2\sigma \tau)^{-1+\frac{1}{n}} \frac{\lambda^{\frac{1}{n}}}{\mu_*} \left(z_* \partial_z \hat{\psi}_k (z_*)\right)\]
provided that $\tau_0 \gg 1$ (depending on $n, \beta$), since $z = (2\sigma \tau_0)^{\frac{1}{2}(1-\beta)}$ and $0 < \varrho < \vartheta$.

Thirdly, by (6.21) and (6.33), there holds

$$
\begin{align*}
\partial_r \dot{w}_+ - \left( \frac{\partial^2_{zz} \dot{w}_+}{1 + (\partial_z \dot{w}_+)^2} \right) + (n-1) \left( \frac{\partial_z \dot{w}_+}{z} - \frac{1}{\dot{w}_+} \right) + \frac{1}{2} + \frac{\sigma}{2\sigma \tau} \left( -z \partial_z \dot{w}_+ + \dot{w}_+ \right)

&= \frac{\mu_+^2 - 1}{\lambda_+^{\alpha-\beta} \mu_+^2} \left( \frac{\partial^2_{rr} \dot{w}_+ (r)}{1 + (\partial_r \dot{w}_+ (r))^2} \right) \left( \frac{2}{1 + (\partial_r \dot{w}_+ (r))^2} \right) + (n-1) \frac{\partial_r \dot{w}_+ (r)}{r} \bigg|_{r = \frac{z}{\lambda_+^{\alpha-\beta} \mu_+}} \\
&+ \left( -\frac{1}{2} + \frac{\sigma}{2\sigma \tau} \frac{\lambda_+^{\beta}}{\lambda_+} + \frac{\lambda_+^{\beta-\alpha}}{1 - \alpha} (\partial_r \lambda_+) \right) \left( \dot{w}_+ (r) - r \partial_r \dot{w}_+ (r) \right) \bigg|_{r = \frac{z}{\lambda_+^{\alpha-\beta} \mu_+}} \\
&- \frac{\lambda_+^{\beta-\alpha}}{\mu_+} (\partial_r \mu_+ \partial_r \dot{w}_+ (r)) \bigg|_{r = \frac{z}{\lambda_+^{\alpha-\beta} \mu_+}} \\
&\geq 2 \left( 1 - O \left( \beta^{\alpha-3} \right) \right) \delta \beta^{\alpha-3} (2\sigma \tau_0)^{\vartheta} (2\sigma \tau)^{-1} \left( \frac{1}{4} \partial^2_{rr} \dot{w}_+ (r) + (n-1) \frac{\partial_r \dot{w}_+ (r)}{r} \right) \bigg|_{r = \frac{z}{\lambda_+^{\alpha-\beta} \mu_+}} \\
&- (1 + O \left( \beta^{\alpha-3} \right)) \left( \frac{1}{2} + \frac{2\sigma \varrho \beta^{\alpha-3}}{1 - \alpha} \left( \frac{\tau}{\tau_0} \right)^{-\vartheta} \right) (2\sigma \tau)^{-1} \left( \dot{w}_+ (r) - r \partial_r \dot{w}_+ (r) \right) \bigg|_{r = \frac{z}{\lambda_+^{\alpha-\beta} \mu_+}} \\
&\geq 0
\end{align*}
$$

provided that $\tau_0 \gg 1$ (depending on $n, \Lambda, \beta$), since

$$
\frac{\partial_r \dot{w}_+ (r)}{r} = (1 + o (1)) r^{-1} > (1 + o (1)) k (1 - \alpha) \frac{2^{\alpha+1}}{renos} = \dot{w}_+ (r) - r \partial_r \dot{w}_+ (r)
$$

for $r \gg 1$ (noting that $\alpha < -1$). Then we subtract the equation of $\dot{w}_+ (z, \tau)$ by (3.22) to get

$$
(6.43) \quad \partial_z (\dot{w}_+ - \dot{w}) - \left( \frac{1}{1 + (\partial_z \dot{w})^2} \partial^2_{zz} (\dot{w}_+ - \dot{w}) + \frac{n-1}{z} \partial_z (\dot{w}_+ - \dot{w}) \right)
$$
which, together with (6.44), contradicts with (6.45).

Thus, we get

\[ \tau + \hat{T} \]

To show that \( \hat{w}_+ \) is an upper barrier, let

\[ \hat{\omega}_+ - \hat{\omega} \]

Note that by (6.40) and (6.41), we have

\[ (\hat{\omega}_+ - \hat{\omega})_{\text{min}} (\tau) > 0 \]

We claim that

\[ (\hat{\omega}_+ - \hat{\omega})_{\text{min}} (\tau) \geq 0 \quad \text{for} \quad \tau_0 \leq \tau \leq \tilde{\tau} \]

Suppose the contrary, then there is \( \tau_0 < \tau^*_1 \leq \tilde{\tau} \) so that

(6.44)

\[ (\hat{\omega}_+ - \hat{\omega})_{\text{min}} (\tau^*_1) < 0 \]

Let \( \tau^*_0 \in [\tau_0, \tau^*_1) \) be the first time after which \( (\hat{\omega}_+ - \hat{\omega})_{\text{min}} \) is negative all the way up to \( \tau^*_1 \), then by the continuity, we must have

(6.45)

\[ (\hat{\omega}_+ - \hat{\omega})_{\text{min}} (\tau^*_0) = 0 \]

On the other hand, by (6.42), the minimum of \( \hat{\omega}_+ - \hat{\omega} \) for each time-slice is achieved in \([0, (2\sigma\tau)^{\frac{1}{4}}(1-\theta)]\). Applying the maximum principle to (6.43), we get

\[ \partial_{\tau} (\hat{\omega}_+ - \hat{\omega})_{\text{min}} - \left( \frac{n-1}{\hat{\omega}_+ \hat{\omega}} + \frac{\beta + \sigma}{2\sigma\tau} \right) (\hat{\omega}_+ - \hat{\omega})_{\text{min}} \geq 0 \]

Note that at \( z = 0 \), we always have

\[ \partial_z \hat{\omega}(0, \tau) = 0 = \partial_z \hat{\omega}_+(0, \tau) \quad \forall \tau_0 \leq \tau \leq \tilde{\tau} \]

so L'Hôpital's rule implies

\[ \lim_{z \to 0} \frac{n-1}{z} \partial_z (\hat{\omega}_+ - \hat{\omega}) (z, \tau) = \frac{n-1}{z} \partial^2_z (\hat{\omega}_+ - \hat{\omega}) (0, \tau) \]

It follows that

\[ \partial_{\tau} \left( e^{-f (\hat{\omega}_+ - \hat{\omega})_0} - \frac{1}{2} \frac{\beta + \sigma}{2\sigma\tau} (\hat{\omega}_+ - \hat{\omega})_{\text{min}} \right) \geq 0 \]

which, together with (6.44), contradicts with (6.45).

Lastly, by (6.30) and \( \mu_+ (\tau) \geq 1 \), we have

\[ \hat{\omega}_+ (z, \tau) = \hat{\psi}_{\lambda_+, \mu_+}^k (z, \tau) \leq \hat{\psi}_{\lambda_+, 1}^k (z, \tau) = \hat{\psi}_{\lambda_+ (\tau) k}^k (z) \]

Thus, we get

\[ \hat{\psi}_{\lambda_+ (\tau) k} (z) = \hat{\omega}_- (z, \tau) \leq \hat{\omega} (z, \tau) \leq \hat{\omega}_+ (z, \tau) \leq \hat{\psi}_{\lambda_+ (\tau) k} (z) \]

For (6.27), given \( \tau_0 \leq \tau \leq \tilde{\tau}, \beta \leq z \leq (2\sigma\tau)^{\frac{1}{4}}(1-\theta) \), by (6.30), (6.31) and the mean value theorem, there is \( 1 \leq \lambda_+ \leq \lambda_+ (\tau) \) so that

\[ \hat{\psi}_{\lambda_+, 1}^k (z, \tau) = \hat{\psi}_k (z, \tau) + (\lambda_+ (\tau) - 1) \partial_{\lambda_+} \left( \hat{\psi}_k^{\lambda_+ \mu_+} (z) \right) \big|_{\lambda = \lambda_+} = \hat{\psi}_k (z, \tau) - \lambda_+ \partial_{\lambda_+} \hat{\psi}_k (z, \tau) \]

\[ \hat{\psi}_k (z, \tau) + \beta^{\alpha-3} \left( \frac{\tau}{\tau_0} \right)^{-\theta} \frac{\lambda_+^{\alpha-1}}{1 - \alpha} \left( \hat{\psi}_k (z_*) - \partial_z \hat{\psi}_k (z_*) \right) \frac{z_0}{z} \]
ANALYSIS OF VELÁZQUEZ’S SOLUTION TO THE MCF

\[ \leq \hat{\psi}_k(z, \tau) + (1 + o(1)) 2^{\frac{\alpha + 1}{2}} \beta^{\alpha - 3} \left( \frac{\tau}{\tau_0} \right)^{-\theta} z^\alpha \]

Similarly,

\[ \hat{\psi}_{k-1}^+ (z, \tau) \geq \hat{\psi}_k(z, \tau) - (1 + o(1)) 2^{\frac{\alpha + 1}{2}} \beta^{\alpha - 3} \left( \frac{\tau}{\tau_0} \right)^{-\theta} z^\alpha \]

As for (6.28), given \( \tau_0 \leq \tau \leq \tilde{\tau} \), \( 0 \leq z \leq 5\beta \), by (6.30), (6.31) and the mean value theorem, there is \( 1 \leq \lambda_* \leq \lambda_+ (\tau) \) so that

\[ \hat{\psi}_{k+1}^+ (z, \tau) = \hat{\psi}_k(z, \tau) + (\lambda_+ (\tau) - 1) \partial_\lambda \left( \hat{\psi}_{k,\mu}^+ (z) \right) \bigg|_{\lambda = \lambda_*, z = z_*} \leq \hat{\psi}_k(z, \tau) + \beta^{\alpha - 3} \frac{c}{1 - \alpha} \left( \frac{\tau}{\tau_0} \right)^{-\theta} \]

where

\[ c = \sup_{r \geq 0} \left( \hat{\psi}_k(r) - r \partial_r \hat{\psi}_k(r) \right) \leq C(n) \]

(6.31). Similarly,

\[ \hat{\psi}_{k-1}^- (z, \tau) \geq \hat{\psi}_k(z, \tau) - \beta^{\alpha - 3} \frac{c}{1 - \alpha} \left( \frac{\tau}{\tau_0} \right)^{-\theta} \]

As a corollary, if we regard the projected curves \( \Pi_{x, y}^{(a_0, a_1)} \) and \( \mathcal{N}_k \) as graphs over \( \mathcal{C} \), (6.37) implies

\[ |w(z, \tau) - \hat{\psi}_k(z)| \leq C(n) \beta^{\alpha - 3} \left( \frac{\tau}{\tau_0} \right)^{-\theta} z^\alpha \]

for \( \frac{4}{3} \beta \leq z \leq \frac{1}{3} (2\sigma \tau)^{\frac{1}{2}(1-\theta)}, \tau_0 \leq \tau \leq \tilde{\tau} \). Then (6.7) follows immediately by (3.25).

Lastly, we prove (6.3) by using the gradient and curvature estimates in [EH].

**Proposition 6.7.** If \( 0 < \rho \ll 1 \) (depending on \( n, \Lambda \)) and \( |t_0| \ll \rho^2 \) (depending on \( n \)), there holds (6.4). Moreover, we have

\[ \left\{ \begin{array}{l} |\partial_x u(x, t)| \lesssim 1 \\ |\partial^2_x u(x, t)| \leq \frac{C(n)}{\sqrt{1-t_0}} \end{array} \right. \]

for \( x \geq \frac{1}{5} \rho, t_0 \leq t \leq \tilde{t} \).

**Proof.** For ease of notation, we denote \( \Sigma_t^{(a_0, a_1)} \) by \( \Sigma_t \). Let’s first parametrize \( \Sigma_{t_0} \) by (3.9), i.e.

\[ X_{t_0}(x, \nu, \omega) = \left( (x - u(x, t_0)) \frac{\nu}{\sqrt{2}}, (x + u(x, t_0)) \frac{\omega}{\sqrt{2}} \right) \]
for \( x \geq \frac{1}{4} \rho \), \( \nu, \omega \in S^{n-1} \). Then the (upward) unit normal vector of \( \Sigma_{t_0} \) at \( X_{t_0} \) is given by

\[
N_{\Sigma_{t_0}} (X_{t_0}) = \left( \frac{1 + \partial_x u(x, t_0)}{\sqrt{1 + (\partial_x u(x, t_0))^2}} \right) \frac{-\nu}{\sqrt{2}} \left( \frac{1 - \partial_x u(x, t_0)}{\sqrt{1 + (\partial_x u(x, t_0))^2}} \right) \frac{\omega}{\sqrt{2}}
\]

Note that by (4.2) we have

\[
\max \left\{ \left| \frac{u(x, t_0)}{x} \right|, \left| \partial_x u(x, t_0) \right| \right\} \leq \frac{1}{3}
\]

for \( x \geq \frac{1}{4} \rho \).

Now fix \( x_* \geq \frac{1}{4} \rho \) and let

\[
\nu_* = \omega_* = \left( \begin{array}{c} (n-1) \text{ copies} \\ 0, \cdots, 0, 1 \end{array} \right)
\]

\[
e = \left( -\frac{1}{\sqrt{2}} \nu_*, \frac{1}{\sqrt{2}} \omega_* \right)
\]

\[
X_* = X_{t_0} (x_*, \nu_*, \omega_*) = \left( (x_* - u(x_*, t_0)) \frac{\nu_*}{\sqrt{2}}, (x_* + u(x_*, t_0)) \frac{\omega_*}{\sqrt{2}} \right)
\]

Notice that

\[
|X_{t_0} - X_*|^2 \geq \frac{1}{2} (x - u(x, t_0))^2 \left( 1 - (\nu \cdot \nu_*)^2 \right) + \frac{1}{2} (x + u(x, t_0))^2 \left( 1 - (\omega \cdot \omega_*)^2 \right)
\]

\[
\geq \frac{x^2}{2} \left( 1 - \frac{|u(x, t_0)|}{x} \right)^2 \max \left\{ 1 - (\nu \cdot \nu_*)^2, 1 - (\omega \cdot \omega_*)^2 \right\}
\]

\[
\geq \frac{\rho^2}{9} \max \left\{ 1 - (\nu \cdot \nu_*)^2, 1 - (\omega \cdot \omega_*)^2 \right\}
\]

Thus, for \( X_{t_0} \in \Sigma_{t_0} \cap B(X_*; \frac{1}{30} \rho) \), there holds

\[
\min \{ \nu \cdot \nu_*, \omega \cdot \omega_* \} \geq \frac{\sqrt{91}}{10}
\]

which implies

\[
(N_{\Sigma_{t_0}} (X_{t_0}) \cdot e)^{-1} = \frac{2 \sqrt{1 + (\partial_x u(x, t_0))^2}}{(1 + \partial_x u(x, t_0)) (\nu \cdot \nu_*) + (1 - \partial_x u(x, t_0)) (\omega \cdot \omega_*)}
\]

\[
\leq \frac{\sqrt{10}}{\nu \cdot \nu_* + \omega \cdot \omega_*} \leq \frac{10 \sqrt{10}}{2 \sqrt{91}}
\]

(6.48)

By the gradient estimates in [EH], we then get

\[
(N_{\Sigma_t} (X_t) \cdot e)^{-1} \leq \left( 1 - \frac{|X_t - X_*|^2 + 2n (t - t_0)}{(1/30 \rho)^2} \right)^{-1} \sup_{\Sigma_{t_0} \cap B(X_*; 1/2 \rho)} (N_{\Sigma_{t_0}} \cdot e)^{-1}
\]
for $X_t \in \Sigma_t \cap B \left( X_x; \sqrt{\left( \frac{1}{30} \rho \right)^2 - 2n (t - t_0)} \right)$, where $N_{\Sigma_t} (X_t)$ is the unit normal vector of $\Sigma_t$ at $X_t$. Consequently,

\begin{equation}
(N (X_t) \cdot e)^{-1} \leq \left( 1 - \left( \frac{30}{31} \right) \right) \frac{10\sqrt{10}}{2\sqrt{91}}
\end{equation}

for $X_t \in \Sigma_t \cap B \left( X_x; \sqrt{\left( \frac{1}{30} \rho \right)^2 - 2n (t - t_0)} \right)$. It follows, by the curvature estimates in [EH], that

$$|A_{\Sigma_t} (X_t)| \leq C (n) \left( \frac{1}{\sqrt{t - t_0}} + \frac{1}{\rho} \right)$$

for $X_t \in \Sigma_t \cap B \left( X_x; \frac{\rho}{3}, \right)$, $t_0 \leq t \leq \hat{t}$.

Next, consider the “normal parametrization” for the MCF $\{\Sigma_t\}_{t_0 \leq t \leq \hat{t}}$, i.e. let $X_t (x, \nu, \omega) = X (x, \nu, \omega; t)$ so that

$$\left\{ \begin{array}{l}
\partial_t X (x, \nu, \omega; t) = H_{\Sigma_t} (X (x, \nu, \omega; t)) N_{\Sigma_t} (X (x, \nu, \omega; t)) \\
X (x, \nu, \omega; t_0) = X_{t_0} (x, \nu, \omega)
\end{array} \right.$$

For each $x \geq \rho$, $\nu, \omega \in S^{n-1}$, let $t(x, \nu, \omega) \in (t_0, \hat{t})$ be the maximal time so that

$$X_t (x, \nu, \omega) \in \Sigma_t \cap B \left( X_{t_0} (x, \nu, \omega); \frac{1}{33} \rho \right)$$

for all $t_0 \leq t \leq t(x, \nu, \omega)$. Then we have

$$|\partial_t X_t (x, \nu, \omega)| = |H_{\Sigma_t} (X_t (x, \nu, \omega))| \leq \frac{C (n)}{\sqrt{t - t_0}}$$

and hence

\begin{equation}
|X_t (x, \nu, \omega) - X_{t_0} (x, \nu, \omega)| \leq C (n) \sqrt{t - t_0}
\end{equation}

for all $t_0 \leq t \leq t(x, \nu, \omega)$. Thus, if $|t_0| \ll 1$ (depending on $n$), we may assume that $t(x, \nu, \omega) = \hat{t}$ and

\begin{equation}
d_H \left( \Sigma_t \setminus B \left( O; \frac{1}{5} \rho \right), \Sigma_{t_0} \setminus B \left( O; \frac{1}{5} \rho \right) \right) \leq C (n) \sqrt{t - t_0}
\end{equation}

for all $t_0 \leq t \leq \hat{t}$, where $d_H$ is the Hausdorff distance. It follows that

$$|u (x, t) - u (x, t_0)| \leq C (n) \sqrt{t - t_0}$$

for $x \geq \frac{1}{3} \rho$, $t_0 \leq t \leq \hat{t}$. 

Furthermore, by taking \( x = x_\ast, \nu = \nu_\ast, \omega = \omega_\ast \) in (6.48) and replace \( t_0 \) by \( t \), one could get
\[
(N_{\Sigma_t} (X_t (x_\ast, \nu_\ast, \omega_\ast)) \cdot e)^{-1} = \sqrt{1 + (\partial_x u (x_\ast, t))^2}
\]
So by (6.49) and (6.51), we have
\[
(\text{6.53})
\]
\[
|\partial_x u (x_\ast, t)| \lesssim 1
\]
for \( t_0 \leq t \leq \hat{t} \) (and any \( x_\ast \geq \frac{1}{5} \rho \)). For the second derivative, notice that
\[
|\partial_{xx} u (x_\ast, t)| \leq \frac{C(n)}{\sqrt{t - t_0}}
\]
for \( t_0 \leq t \leq \hat{t} \) (and any \( x_\ast \geq \frac{1}{5} \rho \)). □

7. Smooth estimates in Proposition 4.4 and Proposition 4.5

This section is a continuation of Section 6. For ease of notation, from now on, let’s denote \( \Sigma_t (a_0, a_1) \) by \( \Sigma_t \), \( \Gamma_t (a_0, a_1) \) by \( \Gamma_t \) and \( \Pi_s (a_0, a_1) \) by \( \Pi_s \). Here we would like to show that if \( 0 < \rho \ll 1 \ll \beta \) (depending on \( n, \Lambda \)) and \( |t_0| \ll 1 \) (depending on \( n, \Lambda, \rho, \beta \)), then

- In the outer region, the function \( u (x, t) \) of \( \Sigma_t (a_0, a_1) \) defined in (3.6) satisfies (4.14).
- In the tip region, if we do the type II rescaling, the function \( \hat{w} (z, \tau) \) of the rescaled hypersurface \( \Gamma_t (a_0, a_1) \) defined in (3.20) satisfies (4.16).

Moreover, for any \( 0 < \delta \ll 1, m, t \in \mathbb{Z}_+ \), there hold the following higher order derivatives estimates.

1. In the outer region, the function \( u (x, t) \) of \( \Sigma_t (a_0, a_1) \) defined in (3.6) satisfies (4.17) and (4.18) (see Proposition 7.4 and Proposition 7.5).
2. In the intermediate region, if we do the type I rescaling, the function \( v (y, s) \) of the rescaled hypersurface \( \Pi_s (a_0, a_1) \) defined in (3.11) satisfies (4.19) and (4.20) (see Proposition 7.6).
3. In the tip region, if we do the type II rescaling, the function \( \hat{w} (z, \tau) \) of the rescaled hypersurface \( \Gamma_t (a_0, a_1) \) defined in (3.20) satisfies (4.23) (see Proposition 7.12).

We establish (4.14) and (4.16) by using the maximum principle and curvature estimates in [EH]. Then we use Krylov-Safonov estimates and Schauder estimates, together with (3.8) (which is equivalent to (3.14) and (3.27)), (4.14) and (4.16), to derive (4.17), (4.18), (4.19), (4.20) and (4.23).

Let’s start with proving (4.14). The \( C^0 \) estimates has already been shown in Proposition 6.6 and Proposition 6.7, in which we also get the first and second derivative bounds for \( u (x, t) \) (see (6.47)). In the next lemma, we improve the first derivative bound in Proposition 6.7 by using the maximum principle, which turns out to be useful when we derive an improved second derivative estimate in Lemma 7.3.
Lemma 7.1. If \( 0 < \rho \ll 1 \) (depending on \( n, \Lambda \)) and \( |t_0| \ll 1 \) (depending on \( n, \rho \)), there holds

\[
\sup_{x \geq \frac{1}{\rho}} |\partial_x u (x, t)| \leq \sup_{x \geq \frac{1}{\rho}} |\partial_x u (x, t_0)| + C(n, \rho) \sqrt{t - t_0}
\]

for \( t_0 \leq t \leq \hat{t} \).

Proof. First, differentiate (3.7) with respect to \( x \) to get

\[
\partial_t (\partial_x u) - \frac{1}{1 + (\partial_x u)^2} \partial^2_{xx} (\partial_x u) - (a(x, t) \partial^2_{xx} u + b(x, t)) \partial_x (\partial_x u) = f(x, t)
\]

where

\[
a(x, t) = \frac{-2 \partial_x u (x, t)}{1 + (\partial_x u (x, t))^2}
\]

\[
b(x, t) = \frac{2(n-1)}{x \left(1 - \left(\frac{u(x, t)}{x}\right)^2\right)}
\]

\[
f(x, t) = \frac{-4(n-1) \left(\frac{u(x, t)}{x}\right) \left(1 - (\partial_x u (x, t))^2\right)}{x^2 \left(1 - \left(\frac{u(x, t)}{x}\right)^2\right)^2}
\]

For each \( R \geq 2 \), let \( \eta(x) \) be a smooth function so that

\[
\chi(\frac{1}{\rho}, R-1) \leq \eta \leq \chi(\frac{1}{\rho}, R)
\]

(7.1)

\[|\partial_x \eta (x)| + |\partial^2_{xx} \eta (x)| \leq C(\rho)\]

It follows that

\[
\partial_t (\eta \partial_x u) - \frac{1}{1 + (\partial_x u)^2} \partial^2_{xx} (\eta \partial_x u) - (a(x, t) \partial^2_{xx} u + b(x, t)) \partial_x (\eta \partial_x u)
\]

(7.2) \[= - \left( \frac{\partial^2_{xx} \eta}{1 + (\partial_x u)^2} + \left( a(x, t) \partial^2_{xx} u + b(x, t) \right) \partial_x \eta \right) (\partial_x u) + \eta(x) f(x, t) - \frac{2}{1 + (\partial_x u)^2} \partial_x \eta \left( \partial^2_{xx} u \right)\]

Now let

\[
(\eta \partial_x u)_{\max} (t) = \max (\eta(x) \partial_x u (x, t))
\]

By (3.8), (4.2) and (6.47), if \( 0 < \rho \ll 1 \) (depending on \( n, \Lambda \)), \( |t_0| \ll 1 \) (depending on \( n, \rho \)), we may assume that

(7.3)

\[
\begin{align*}
|\frac{u(x, t)}{x}| & \leq \frac{1}{3} \\
|\partial_x u (x, t)| & \lesssim 1 \\
|\partial^2_{xx} u (x, t)| & \leq \frac{C(n, \rho)}{\sqrt{t - t_0}}
\end{align*}
\]
for $x \geq \frac{1}{2} \rho$, $t_0 \leq t \leq \hat{t}$. Thus, by (7.1) and (7.3), applying the maximum principle to (7.2) yields
\[
\partial_t (\eta \partial_x u)_{\max} \leq \frac{C(n, \rho)}{\sqrt{t - t_0}}
\]
which implies
\[
(\eta \partial_x u)_{\max}(t) \leq (\eta \partial_x u)_{\max}(t_0) + C(n, \rho) \sqrt{t - t_0}
\]
Likewise, if we define
\[
(\eta \partial_x u)_{\min}(t) = \min_x (\eta(x) \partial_x u(x, t))
\]
by the same argument, we get
\[
(\eta \partial_x u)_{\min}(t) \geq (\eta \partial_x u)_{\min}(t_0) - C(n, \rho) \sqrt{t - t_0}
\]
□

Before moving on to the second derivative estimate, we derive the following lemma, which is about some properties of the cut-off functions to be used.

**Lemma 7.2.** Let $\eta(r)$ be a smooth, non-increasing function so that
\[
\chi_{(-\infty, 0)} \leq \eta \leq \chi_{(-\infty, 1)}
\]
and $\eta(r)$ vanishes at $r = 1$ to infinite order. Then
\[
\sup_r \frac{(\partial_r \eta(r))^2}{\eta(r)} < \infty
\]
for $r \leq 1$.

**Proof.** By L'Hôpital's rule, we have
\[
\lim_{r \to 1} \frac{(\partial_r \eta(r))^2}{\eta(r)} = 2 \lim_{r \to 1} \partial^2_r \eta(r) = 0
\]
Also, for $r \leq 0$ or $r > 1$, there holds
\[
\frac{(\partial_r \eta(r))^2}{\eta(r)} = 0
\]
Thus, the conclusion follows easily. □

Below is an improved estimate for the second derivative of $u(s, t)$ in the outer region. Note that the proof requires $|\partial_x u(x, t)| < \frac{1}{\sqrt{3}}$, which is guaranteed by (4.2) and Lemma 7.1.

**Lemma 7.3.** If $0 < \rho \ll 1$ (depending on $n$, $\Lambda$) and $|t_0| \ll 1$ (depending on $n$, $\rho$), there holds
\[
\sup_{x \geq \frac{1}{4} \rho} |\partial^2_{xx} u(x, t)| \leq \sup_{x \geq \frac{1}{4} \rho} |\partial^2_{xx} u(x, t_0)| + C(n, \rho)
\]
for $t_0 \leq t \leq \hat{t}$.

**Proof.** Differentiating (3.7) with respect to $x$ twice yields
\[
\partial_t (\partial^2_{xx} u) - \frac{1}{1 + (\partial_x u)^2} \partial^2_{xx} (\partial^2_{xx} u) - \left( \frac{-6 \partial_x u}{(1 + (\partial_x u)^2)^2} \partial^2_{xx} u + \frac{2(n - 1)}{x \left( 1 - \left( \frac{\rho}{4} \right)^2 \right)} \right) \partial_x (\partial^2_{xx} u)
\]
for $x \geq \frac{1}{2} \rho$, $t_0 \leq t \leq \hat{t}$. Thus, by (7.1) and (7.3), applying the maximum principle to (7.2) yields
\[
\partial_t (\eta \partial_x u)_{\max} \leq \frac{C(n, \rho)}{\sqrt{t - t_0}}
\]
which implies
\[
(\eta \partial_x u)_{\max}(t) \leq (\eta \partial_x u)_{\max}(t_0) + C(n, \rho) \sqrt{t - t_0}
\]
Likewise, if we define
\[
(\eta \partial_x u)_{\min}(t) = \min_x (\eta(x) \partial_x u(x, t))
\]
by the same argument, we get
\[
(\eta \partial_x u)_{\min}(t) \geq (\eta \partial_x u)_{\min}(t_0) - C(n, \rho) \sqrt{t - t_0}
\]
□

Before moving on to the second derivative estimate, we derive the following lemma, which is about some properties of the cut-off functions to be used.

**Lemma 7.2.** Let $\eta(r)$ be a smooth, non-increasing function so that
\[
\chi_{(-\infty, 0)} \leq \eta \leq \chi_{(-\infty, 1)}
\]
and $\eta(r)$ vanishes at $r = 1$ to infinite order. Then
\[
\sup_r \frac{(\partial_r \eta(r))^2}{\eta(r)} < \infty
\]
for $r \leq 1$.

**Proof.** By L'Hôpital's rule, we have
\[
\lim_{r \to 1} \frac{(\partial_r \eta(r))^2}{\eta(r)} = 2 \lim_{r \to 1} \partial^2_r \eta(r) = 0
\]
Also, for $r \leq 0$ or $r > 1$, there holds
\[
\frac{(\partial_r \eta(r))^2}{\eta(r)} = 0
\]
Thus, the conclusion follows easily. □

Below is an improved estimate for the second derivative of $u(s, t)$ in the outer region. Note that the proof requires $|\partial_x u(x, t)| < \frac{1}{\sqrt{3}}$, which is guaranteed by (4.2) and Lemma 7.1.

**Lemma 7.3.** If $0 < \rho \ll 1$ (depending on $n$, $\Lambda$) and $|t_0| \ll 1$ (depending on $n$, $\rho$), there holds
\[
\sup_{x \geq \frac{1}{4} \rho} |\partial^2_{xx} u(x, t)| \leq \sup_{x \geq \frac{1}{4} \rho} |\partial^2_{xx} u(x, t_0)| + C(n, \rho)
\]
for $t_0 \leq t \leq \hat{t}$.

**Proof.** Differentiating (3.7) with respect to $x$ twice yields
\[
\partial_t (\partial^2_{xx} u) - \frac{1}{1 + (\partial_x u)^2} \partial^2_{xx} (\partial^2_{xx} u) - \left( \frac{-6 \partial_x u}{(1 + (\partial_x u)^2)^2} \partial^2_{xx} u + \frac{2(n - 1)}{x \left( 1 - \left( \frac{\rho}{4} \right)^2 \right)} \right) \partial_x (\partial^2_{xx} u)
\]
For each \( R \geq 2 \), let \( \eta(x) \) be a smooth function so that

\[
\lambda\left(\frac{1}{R}, R^{-1}\right) \leq \eta \leq \lambda\left(\frac{1}{R}, R\right)
\]

and \( \eta(x) \) is increasing in \([\frac{1}{R}, \frac{1}{R}]\) and decreasing on \([R^{-1}, R]\); moreover, \( \eta(x) \) vanishes at \( x = \frac{1}{R} \) and \( x = R \) to infinite order. Notice that by Lemma 7.2, we may assume

\[
\eta \left( \frac{1}{R}, R^{-1} \right) \leq \eta \leq \lambda\left(\frac{1}{R}, R\right)
\]

and decreasing on \([R^{-1}, R]\). Now let

\[
\eta \left( \frac{1}{R}, R^{-1} \right) \leq \eta \leq \lambda\left(\frac{1}{R}, R\right)
\]

be a smooth function so that

\[
\lambda\left(\frac{1}{R}, R^{-1}\right) \leq \eta \leq \lambda\left(\frac{1}{R}, R\right)
\]

and decreasing on \([R^{-1}, R]\). Now let

\[
\eta \left( \frac{1}{R}, R^{-1} \right) \leq \eta \leq \lambda\left(\frac{1}{R}, R\right)
\]

be a smooth function so that

\[
\lambda\left(\frac{1}{R}, R^{-1}\right) \leq \eta \leq \lambda\left(\frac{1}{R}, R\right)
\]

and decreasing on \([R^{-1}, R]\). Now let

\[
\eta \left( \frac{1}{R}, R^{-1} \right) \leq \eta \leq \lambda\left(\frac{1}{R}, R\right)
\]

be a smooth function so that

\[
\lambda\left(\frac{1}{R}, R^{-1}\right) \leq \eta \leq \lambda\left(\frac{1}{R}, R\right)
\]

and decreasing on \([R^{-1}, R]\). Now let

\[
\eta \left( \frac{1}{R}, R^{-1} \right) \leq \eta \leq \lambda\left(\frac{1}{R}, R\right)
\]

be a smooth function so that

\[
\lambda\left(\frac{1}{R}, R^{-1}\right) \leq \eta \leq \lambda\left(\frac{1}{R}, R\right)
\]

and decreasing on \([R^{-1}, R]\). Now let

\[
\eta \left( \frac{1}{R}, R^{-1} \right) \leq \eta \leq \lambda\left(\frac{1}{R}, R\right)
\]

be a smooth function so that

\[
\lambda\left(\frac{1}{R}, R^{-1}\right) \leq \eta \leq \lambda\left(\frac{1}{R}, R\right)
\]

and decreasing on \([R^{-1}, R]\). Now let

\[
\eta \left( \frac{1}{R}, R^{-1} \right) \leq \eta \leq \lambda\left(\frac{1}{R}, R\right)
\]

be a smooth function so that

\[
\lambda\left(\frac{1}{R}, R^{-1}\right) \leq \eta \leq \lambda\left(\frac{1}{R}, R\right)
\]

and decreasing on \([R^{-1}, R]\). Now let

\[
\eta \left( \frac{1}{R}, R^{-1} \right) \leq \eta \leq \lambda\left(\frac{1}{R}, R\right)
\]

be a smooth function so that

\[
\lambda\left(\frac{1}{R}, R^{-1}\right) \leq \eta \leq \lambda\left(\frac{1}{R}, R\right)
\]

and decreasing on \([R^{-1}, R]\). Now let

\[
\eta \left( \frac{1}{R}, R^{-1} \right) \leq \eta \leq \lambda\left(\frac{1}{R}, R\right)
\]

be a smooth function so that

\[
\lambda\left(\frac{1}{R}, R^{-1}\right) \leq \eta \leq \lambda\left(\frac{1}{R}, R\right)
\]

and decreasing on \([R^{-1}, R]\). Now let

\[
\eta \left( \frac{1}{R}, R^{-1} \right) \leq \eta \leq \lambda\left(\frac{1}{R}, R\right)
\]

be a smooth function so that

\[
\lambda\left(\frac{1}{R}, R^{-1}\right) \leq \eta \leq \lambda\left(\frac{1}{R}, R\right)
\]

and decreasing on \([R^{-1}, R]\). Now let

\[
\eta \left( \frac{1}{R}, R^{-1} \right) \leq \eta \leq \lambda\left(\frac{1}{R}, R\right)
\]

be a smooth function so that

\[
\lambda\left(\frac{1}{R}, R^{-1}\right) \leq \eta \leq \lambda\left(\frac{1}{R}, R\right)
\]
where

\[
a(x, t) = \frac{2\left(1 - 3(\partial_x u)^2\right)}{\left(1 + (\partial_x u)^2\right)^3}
\]

\[
b(x, t) = \frac{6 \partial_x \eta \partial_x u}{\left(1 + (\partial_x u)^2\right)^2}
\]

\[
c(x, t) = -\frac{2(n - 1) \eta(x)\left(1 + \left(\frac{u}{x}\right)^2\right) - 6 \left(\frac{u}{x}\right) \partial_x u}{x^2\left(1 - \left(\frac{u}{x}\right)^2\right)^2}
\]

\[
f(x, t) = -\frac{4(n - 1)\left(1 - (\partial_x u)^2\right)}{x^3\left(1 - \left(\frac{u}{x}\right)^2\right)^3} \left(\left(1 + 3\left(\frac{u}{x}\right)^2\right) \partial_x u - \left(3 + \left(\frac{u}{x}\right)^2\right) \left(\frac{u}{x}\right)\right)
\]

By (3.8), (4.2), (6.47) and Lemma 7.1, if \(0 < \rho \ll 1\) (depending on \(n, \Lambda\)) and \(|t_0| \ll 1\) (depending on \(n, \rho\)), we have

\[
\max \left\{ \left|\frac{u(x, t)}{x}\right|, |\partial_x u(x, t)| \right\} \leq \frac{1}{3}
\]

for \(x \geq \frac{1}{4} \rho, t_0 \leq t \leq \hat{t}\), which, together with (7.4), implies

(7.6)

\[
\left\{ \begin{array}{c}
\frac{972}{1000} \leq a(x, t) \leq 2 \\
|b(x, t)| + |c(x, t)| + |f(x, t)| \leq C(n, \rho)
\end{array} \right.
\]

for \(x \geq \frac{1}{4} \rho, t_0 \leq t \leq \hat{t}\). Now let

\[
M = \max_{\frac{1}{4} \rho \leq x \leq R, t_0 \leq t \leq \hat{t}} \eta(x) \partial_{xx}^2 u(x, t)
\]

If

\[
M \leq \max_{\frac{1}{4} \rho \leq x \leq R} \left(\eta(x) \partial_{xx}^2 u(x, t_0)\right)_+
\]

then we are done; otherwise, we have

\[
M > \max_{\frac{1}{4} \rho \leq x \leq R} \left(\eta(x) \partial_{xx}^2 u(x, t_0)\right)_+
\]

In the later case, let \((x_*, t_*)\) be a maximum point of \(\eta \partial_{xx}^2 u\) in the spacetime, i.e.

\[
\eta(x_*) \partial_{xx}^2 u(x_*, t_*) = M
\]

then we have \(\frac{1}{4} \rho < x_* < R, t_0 < t \leq \hat{t}\). Applying the maximum principle to (7.5) yields

\[
0 \leq -a(x_*, t_*) \eta(x_*) \left(\partial_{xx}^2 u(x_*, t_*)\right)^3 + b((x_*, t_*)) \left(\partial_{xx}^2 u(x_*, t_*)\right)^2
\]

\[
+ c(x_*, t_*) \left(\partial_{xx}^2 u(x_*, t_*)\right) + \eta(x_*) f(x_*, t_*)
\]

\[
= \frac{1}{\eta^2(x_*)} (-a(x_*, t_*) M^3 + b(x_*, t_*) M^2 + \eta(x_*) c(x_*, t_*) M + \eta^3(x_*) f(x_*, t_*))
\]
It follows, by Young’s inequality and (7.6), that
\[ M^3 \leq \frac{8}{3} \left( \frac{b(x, t_*)}{a(x, t_*)} \right)^3 + \frac{4\sqrt{2}}{3} \left( \frac{c(x, t_*)}{a(x, t_*)} \right) + \frac{f(x, t_*)}{a(x, t_*)} \leq C(n, \rho) \]
Therefore, in either case, we have
\[ \max_{\frac{1}{4}\rho \leq x \leq R, t_0 \leq t \leq \bar{t}} \eta(x) \partial^2_{xx} u(x, t) \leq \max_{x \geq \frac{1}{4}\rho} \left( \eta(x) \partial^2_{xx} u(x, t_0) \right) + C(n, \rho) \]
Likewise, by the same argument, one could show that
\[ \min_{\frac{1}{4}\rho \leq x \leq R, t_0 \leq t \leq \bar{t}} \eta(x) \partial^2_{xx} u(x, t) \geq -\min_{x \geq \frac{1}{4}\rho} \left( \eta(x) \partial^2_{xx} u(x, t_0) \right) - C(n, \rho) \]
\[ \square \]

In the next proposition, we apply the standard regularity theory for parabolic equations to (3.7), together with (4.14), to derive (4.17).

**Proposition 7.4.** There holds (1.12).

**Proof:** Given \( 0 < \delta \ll 1 \), let’s fix \( x_* \geq \frac{1}{4}\rho, t_0 + \delta^2 \leq t_* \leq \bar{t} \). By (4.14) and Krylov-Safonov Hölder estimates (applying to (3.7)), there is
\[ \gamma = \gamma(n, \rho) \in (0, 1) \]
so that
\[ (7.7) \quad [u]_{\gamma; Q(x_*, t_*; \frac{\delta}{2})} \leq C(n, \rho, \delta) \|u\|_{L^\infty(Q(x_*, t_*; \delta))} \leq C(n, \rho, \delta) \]
Next, differentiate (3.7) with respect to \( x \) to get
\[
\partial_x (\partial_x u) - \frac{1}{1 + (\partial_x u)^2} \partial^2_{xx} (\partial_x u)
\]
\[
= \left( \frac{-2 \partial_x u \partial^2_{xx} u}{1 + (\partial_x u)^2} + \frac{2(n-1)}{x} \left( 1 - \frac{\rho}{x} \right)^2 \right) \partial_x (\partial_x u) - \left( \frac{4(n-1)(\frac{\rho}{x})}{x^2 \left( 1 - \frac{\rho}{x} \right)^2} \right) (\partial_x u)
\]
Then by (4.14) and Krylov-Safonov Hölder estimates (applying to the above equation of \( \partial_x u \)), we may assume that for the same exponent \( \gamma \), there holds
\[ [\partial_x u]_{\gamma; Q(x_*, t_*; \frac{\delta}{2})} \leq C(n, \rho, \delta) \left( \|\partial_x u\|_{L^\infty(Q(x_*, t_*; \delta))} + \|u\|_{L^\infty(Q(x_*, t_*; \delta))} \right) \]
\[ (7.8) \quad \leq C(n, \rho, \delta) \]
It follows, by (4.14), (7.7), (7.8) and Schauder \( C^{2, \gamma} \) estimates (applying to (3.7)), that
\[ [\partial^2_{xx} u]_{\gamma; Q(x_*, t_*; \frac{\delta}{2})} \leq C(n, \rho, \delta) \|u\|_{L^\infty(Q(x_*, t_*; \delta))} \leq C(n, \rho, \delta) \]
By the bootstrap argument, one could show that for any \( m \in \mathbb{Z}_+ \), there holds
\[ (7.9) \quad [\partial^m_x u]_{L^\infty(Q(x_*, t_*; \frac{\delta}{m+1}))} + [\partial^m_x u]_{\gamma; Q(x_*, t_*; \frac{\delta}{m+1})} \leq C(n, \rho, \delta, m) \]
Moreover, by (3.7) and (7.9), we immediately get
\[ [\partial^m_x \partial_x u]_{L^\infty(Q(x_*, t_*; \frac{\delta}{m+1}))} + [\partial^m_x \partial_x u]_{\gamma; Q(x_*, t_*; \frac{\delta}{m+1})} \leq C(n, \rho, \delta, m) \]
for any \( m \in \mathbb{Z}_+ \). Differentiating (3.7) with respect to \( t \) and using the above estimates gives
\[
\| \partial^m_x \partial^2_t u \|_{L^\infty(Q(x, t; \frac{r}{n}, \frac{\delta}{n+2+2r}))} + \| \partial^m_x \partial^2_t u \|_{L^\infty(Q(x, t; \frac{r}{n}, \frac{\delta}{n+2+2r}))} \leq C (n, \rho, \delta, m)
\]
for any \( m \in \mathbb{Z}_+ \). Continuing this process and using induction yields
\[
\| \partial^m_x \partial^l_t u \|_{L^\infty(Q(x, t; \frac{r}{n}, \frac{\delta}{n+2+2r}))} \leq C (n, \rho, \delta, m, l)
\]
for any \( m, l \in \mathbb{Z}_+ \).

In the following proposition, we prove (4.18) by using (3.7), (3.8), (6.5), (6.20) and the regularity theory for parabolic equations.

**Proposition 7.5.** If \( 0 < \rho \ll 1 \) (depending on \( n, \Lambda \)) and \( |t_0| \ll 1 \) (depending on \( n, \Lambda, \rho \)), there holds (4.18).

**Proof.** Notice that by (3.8), we have
\[
(7.10) \quad \max \left\{ \left| \frac{u(x, t)}{x} \right|, |\partial_z u(x, t)| \right\} \leq \frac{1}{3}
\]
(7.11) \( x^i |\partial^i_x u(x, t)| \leq \Lambda \left( -t \right)^2 x^n + x^{2\lambda_2 + 1} \leq C (n, \Lambda) x^{2\lambda_2 + 1} \quad \forall \ i \in \{0, 1, 2\} \)
for \( \frac{\sqrt{-t}}{4} \leq x \leq \rho, t_0 \leq t \leq \hat{t} \), provided that \( 0 < \rho \ll 1 \) (depending on \( n, \Lambda \)) and \( |t_0| \ll 1 \) (depending on \( n, \Lambda, \rho \)).

Given \( 0 < \delta \ll 1 \), let’s fix \((x_*, t_*)\) so that
\[
\frac{1}{2} \sqrt{-t_*} \leq x_* \leq \frac{3}{4} \rho, \quad t_0 + \delta x_*^2 \leq t_* \leq \hat{t}
\]
Define
\[
h(r, \iota) = u(rx_*, t_* + \iota x_*^2)
\]
for \( \frac{2}{3} \leq r \leq \frac{4}{3} \), \( -\delta^2 \leq \iota \leq 0 \). From (3.7), there holds
\[
(7.12) \quad \partial_t h - a(r, \iota) \partial^2_x h - b(r, \iota) \partial_x h - c(r, \iota) h = 0
\]
where
\[
a(r, \iota) = \frac{1}{1 + (\partial_x u(x, t))^2} \bigg|_{x=rx_*, t=t_* + \iota x_*^2} \quad b(r, \iota) = \frac{1}{r} \left( \frac{2(n-1)}{1 - \left( \frac{u(x, t)}{x} \right)^2} \right) \bigg|_{x=rx_*, t=t_* + \iota x_*^2}
\]
\[
c(r, \iota) = \frac{1}{r^2} \left( \frac{2(n-1)}{1 - \left( \frac{u(x, t)}{x} \right)^2} \right) \bigg|_{x=rx_*, t=t_* + \iota x_*^2}
\]

By (7.10), (7.11) and Krylov-Safonov Hölder estimates, there is
\[
\gamma = \gamma(n, \Lambda) \in (0, 1)
\]
so that
\[
[h]_{L^\infty(Q(1, 0; \frac{4}{3}))} \leq C(n, \delta) \| h \|_{L^\infty(Q(1, 0; \delta))} \leq C(n, \Lambda, \delta) x_*^{2\lambda_2 + 1}
\]
which implies

$$\partial_h (7.14)$$

then we have

$$\partial h \left( r, \iota \right) = \partial_x u \left( r x_s, t_s + \iota x_s^2 \right)$$

then we have

$$\partial \tilde{h} - \tilde{a} \left( r, \iota \right) \partial_x \tilde{h} - \tilde{b} \left( r, \iota \right) \partial_x \tilde{h} - \tilde{c} \left( r, \iota \right) \tilde{h} = \tilde{f} \left( r, \iota \right)$$

where

$$\tilde{a} \left( r, \iota \right) = \frac{1}{1 + (\partial_x u \left( x, t \right))^2} \bigg|_{x = r x_s, t = t_s + \iota x_s^2}$$

$$\tilde{b} \left( r, \iota \right) = \frac{1}{r^2} \left( \frac{-2 \partial_x u \left( x, t \right) \left( x \partial_x^2 u \left( x, t \right) \right)}{1 + (\partial_x u \left( x, t \right))^2} + \frac{2 (n - 1)}{1 - \left( \frac{u \left( x, t \right)}{x} \right)^2} \right) \bigg|_{x = r x_s, t = t_s + \iota x_s^2}$$

$$\tilde{c} \left( r, \iota \right) = \frac{1}{r^2} \left( \frac{4 (n - 1) \left( \frac{u \left( x, t \right)}{x} \right) \partial_x u \left( x, t \right)}{1 - \left( \frac{u \left( x, t \right)}{x} \right)^2} \right) \bigg|_{x = r x_s, t = t_s + \iota x_s^2}$$

$$\tilde{f} \left( r, \iota \right) = \frac{1}{r^2} \left( \frac{-4 (n - 1)}{1 - \left( \frac{u \left( x, t \right)}{x} \right)^2} \left( \frac{u \left( x, t \right)}{x} \right) \right) \bigg|_{x = r x_s, t = t_s + \iota x_s^2}$$

By (7.10), (7.11) and Krylov-Safonov Hölder estimates, we may assume that for the same exponent $\gamma$, there holds

$$\tilde{h} \left( 7, 8, Q \left( 1, 0, \frac{4}{x_s} \right) \right) \leq C \left( n, \Lambda, \delta \right) \left( \| \tilde{h} \|_{L^\infty \left( Q \left( 1, 0, \delta \right) \right)} + \| \tilde{f} \|_{L^\infty \left( Q \left( 1, 0, \delta \right) \right)} \right)$$

which implies

$$x^\gamma \left( \partial_x u \right) \left( 7, 8, Q \left( x_s, t_s, \frac{4}{x_s} \right) \right) \leq C \left( n, \Lambda, \delta \right) x^2 \lambda^2$$
Thus, by (7.10), (7.11), (7.13), (7.16), applying Schauder $C^{2,\gamma}$ estimates to (7.12) yields
\[ \left[ \partial_{xx}^2 h \right]_{\gamma; Q(1,0; \frac{4}{1}} \leq C(n, \Lambda, \delta) \| h \|_{L^\infty(Q(1,0; \frac{4}{1}}) \leq C(n, \Lambda, \delta) x_{*}^{2\lambda_2+1} \]
which implies
\[ x_{*}^{2+\gamma} \left[ \partial_{xx}^2 u \right]_{\gamma; Q(x_{*}, t_{*}; \frac{4}{1}} x_{*} \right) \leq C(n, \Lambda, \delta) x_{*}^{2\lambda_2+1} \]
By the bootstrap and rescaling argument, one could show that for any $m \in \mathbb{Z}_{+}$, there holds
\[ x_{*}^{m} \left[ \partial_{x}^{m} u \right]_{L^\infty(Q(x_{*}, t_{*}; \frac{4}{1}} x_{*} \right) \right) + x_{*}^{m+\gamma} \left[ \partial_{x}^{m} u \right]_{\gamma; Q(x_{*}, t_{*}; \frac{4}{1}} x_{*} \right) \right) \leq C(n, \Lambda, \delta, m) x_{*}^{2\lambda_2+1} \]
(7.18)
It follows, by (3.7) and (7.18), that
\[ x_{*}^{m+2} \left[ \partial_{x}^{m} \partial_{t} u \right]_{L^\infty(Q(x_{*}, t_{*}; \frac{4}{1}} x_{*} \right) \right) + x_{*}^{m+2+\gamma} \left[ \partial_{x}^{m} \partial_{t}^{2} u \right]_{\gamma; Q(x_{*}, t_{*}; \frac{4}{1}} x_{*} \right) \right) \leq C(n, \Lambda, \delta, m) x_{*}^{2\lambda_2+1} \]
for any $m \in \mathbb{Z}_{+}$. Then differentiate (3.7) with respect to $t$ and use the above estimates to get
\[ x_{*}^{m+4} \left[ \partial_{x}^{m} \partial_{t}^{2} u \right]_{L^\infty(Q(x_{*}, t_{*}; \frac{4}{1}} x_{*} \right) \right) + x_{*}^{m+4+\gamma} \left[ \partial_{x}^{m} \partial_{t}^{2} u \right]_{\gamma; Q(x_{*}, t_{*}; \frac{4}{1}} x_{*} \right) \right) \leq C(n, \Lambda, \delta, m) x_{*}^{2\lambda_2+1} \]
Continuing this process and using induction yields
\[ x_{*}^{m+2l} \left[ \partial_{x}^{m} \partial_{t}^{2} u \right]_{L^\infty(Q(x_{*}, t_{*}; \frac{4}{1}} x_{*} \right) \right) + x_{*}^{m+2l+\gamma} \left[ \partial_{x}^{m} \partial_{t}^{2l} u \right]_{\gamma; Q(x_{*}, t_{*}; \frac{4}{1}} x_{*} \right) \right) \leq C(n, \Lambda, \delta, m) x_{*}^{2\lambda_2+1} \]
(7.19)
for any $m, l \in \mathbb{Z}_{+}$.

On the other hand, by Proposition 3.1, there holds
\[ (\partial_{s} + \mathcal{L}) \left( k e^{-\lambda s} \varphi_{2}(y) \right) = 0 \]
By a rescaling argument, we get
\[ (\partial_{s} - \partial_{xx} - \frac{2(n-1)}{x} \partial_{x} - \frac{2(n-1)}{x^2}) \left( k (-t)^{\lambda_2 + \frac{t}{2}} \varphi_{2} \left( \frac{x}{\sqrt{-t}} \right) \right) = 0 \]
In addition, by (3.7) we have
\[ \left( \partial_{s} - \partial_{xx} - \frac{2(n-1)}{x} \partial_{x} - \frac{2(n-1)}{x^2} \right) u (x, t) = \frac{f(x, t)}{x^2} \]
where
\[ f(x, t) = -\frac{(\partial_{s} u)^2}{1 + (\partial_{s} u)^2} (x^{2} \partial_{xx} u) + \frac{2(n-1)}{1 - (\frac{w}{x})^2} (x \partial_{x} u) + \frac{2(n-1)}{1 - (\frac{w}{x})^2} u \]
Note that by (7.10) and (7.19) we have
\[ x_{*}^{m+2l} \left[ \partial_{x}^{m} \partial_{s} f (x, t) \right]_{L^\infty(Q(x_{*}, t_{*}; \frac{4}{1}} x_{*} \right) \right) + x_{*}^{m+2l+\gamma} \left[ \partial_{x}^{m} \partial_{s}^{l} f (x, t) \right]_{\gamma; Q(x_{*}, t_{*}; \frac{4}{1}} x_{*} \right) \right) \leq C(n, \Lambda, \delta, m, l) x_{*}^{4\lambda_2 x_{*}^{2\lambda_2+1}} \]
(7.22)
for any \( m, l \in \mathbb{Z}_+ \). Subtract (7.20) from (7.21) to get
\[
\left( \partial_t - \partial_{xx}^2 - \frac{2(n-1)}{x} \partial_x - \frac{2(n-1)}{x^2} \right) \left( u(x, t) - k(-t)^{\lambda_2 + \frac{1}{2}} \varphi_2 \left( \frac{x}{\sqrt{-t}} \right) \right) = \frac{f(x, t)}{x^2}
\]
Then by the rescaling argument, together with (7.22) and Schauder estimates, we get
\[
x_{s}^{m+2l} \left\| \partial_x^m \partial_t^l \left( u(x, t) - k(-t)^{\lambda_2 + \frac{1}{2}} \varphi_2 \left( \frac{x}{\sqrt{-t}} \right) \right) \right\|_{L^\infty(Q(x_*, t_*; \frac{x}{m+2l+2} x_*))} \\
\leq C(n, \Lambda, \delta, m, l) \left\| u(x, t) - k(-t)^{\lambda_2 + \frac{1}{2}} \varphi_2 \left( \frac{x}{\sqrt{-t}} \right) \right\|_{L^\infty(Q(x_*, t_*; \frac{x}{m+2l+2} x_*))} \\
+C(n, \Lambda, \delta, m, l) \sum_{i=0}^{m} \sum_{j=0}^{l} x_{s}^{i+j} \left\| \partial_x^i \partial_t^j f(x, t) \right\|_{L^\infty(Q(x_*, t_*; \frac{x}{m+2l+2} x_*))} \\
+C(n, \Lambda, \delta, m, l) \sum_{i=0}^{m} \sum_{j=0}^{l} x_{s}^{i+j+\gamma} \left\| \partial_x^i \partial_t^j f(x, t) \right\|_{L^\infty(Q(x_*, t_*; \frac{x}{m+2l+2} x_*))}
\]
for any \( m, l \in \mathbb{Z}_+ \).
\]

Below we use (3.13), (3.14), (6.6), (6.7) and the regularity theory to show (4.19) and (4.20).

**Proposition 7.6.** If \( \beta \gg 1 \) (depending on \( n, \Lambda \)), \( s_0 \gg 1 \) (depending on \( n, \Lambda, \beta \)), there hold (4.19) and (4.20).

**Proof.** By (3.14), we have
\[
(7.23) \quad y^{i} |\partial_y^i v(y, s)| \leq \Lambda \exp^{-\lambda_2 s} (y^\alpha + y^{2\lambda_2 + 1}) \leq C(n, \Lambda) \exp^{-\lambda_2 s} y^\alpha
\]
for \( \beta \exp^{-\alpha s} \leq y \leq 3 \), \( s_0 \leq s \leq \tilde{s} \). In particular, we may assume that
\[
\max \left\{ \left| \frac{v(y, s)}{y} \right|, \left| \partial_y v(y, s) \right| \right\} \leq C(n, \Lambda) \exp^{-\lambda_2 s} y^{\alpha-1} \leq \frac{1}{3}
\]
for \( \beta \exp^{-\alpha s} \leq y \leq 3 \), \( s_0 \leq s \leq \tilde{s} \), provided that \( \beta \gg 1 \) (depending on \( n, \Lambda \)).

Now given \( 0 < \delta \ll 1 \) and fix \( (y_*, s_*) \) so that
\[
\frac{3}{2} \beta \exp^{-\alpha s_*} \leq y_* \leq 2, \quad s_0 + \delta^2 y_*^2 \leq s_* \leq \tilde{s}
\]
From (3.13), we have
\[
\partial_{s} v - \frac{1}{1 + (\partial_y v)^2} \partial_{yy}^2 v = - \frac{1}{y^2} \left( \frac{2(n-1)}{1 - \left( \frac{v}{y} \right)^2} - \frac{y^2}{2} \right) \partial_y v - \frac{1}{y^2} \left( \frac{2(n-1)}{1 - \left( \frac{v}{y} \right)^2} + \frac{y^2}{2} \right) v = 0
\]
By (7.23) and Krylov-Safonov Hölder estimates, there is
\[
\gamma = \gamma(n, \Lambda) \in (0, 1)
\]
so that
\[
(7.24) \quad y^{\gamma} \left| v \right|_{\gamma; Q(y_*, s_*; \delta y_*)} \leq C(n, \delta) \left\| v \right\|_{L^\infty(Q(y_*, s_*; \delta y_*))} \leq C(n, \Lambda, \delta) \exp^{-\lambda_2 s_*} y^\alpha
\]
Differentiate (3.13) with respect to $y$ to get
\[
 \partial_s (\partial_y v) - \frac{1}{1 + (\partial_y v)^2} \partial_{yy}^2 (\partial_y v) \\
= - \frac{1}{y} \left( -2 \frac{\partial (\partial_y v) (y \partial_{yy}^2 v)}{(1 + (\partial_y v)^2)^2} + \frac{2 (n - 1)}{1 - \left( \frac{v}{y} \right)^2} \right) \partial_y (\partial_y v) - \frac{1}{y^2} \left( \frac{4 (n - 1)}{1 - \left( \frac{v}{y} \right)^2} \right) (\partial_y v) \\
= \frac{1}{y^2} \left( \frac{-4 (n - 1)}{1 - \left( \frac{v}{y} \right)^2} \right)
\]

By (7.23) and Krylov-Safonov Hölder estimates, we may assume that for the same $\gamma$, there holds
\[
y^*_\gamma [\partial_y v]_{\gamma; Q(y, s; \frac{4}{m+1} y_s)} \leq C (n, \Lambda, \delta) \left( \| \partial_y v \|_{L^\infty(Q(y, s; \frac{4}{m+1} y_s))} + \| v \|_{L^\infty(Q(y, s; \frac{4}{m+1} y_s))} \right) \leq C (n, \Lambda, \delta) e^{-\lambda^*_s} y^*_\alpha
\]
(7.25)

By (7.23), (7.24) and (7.25), applying Schauder $C^{2, \gamma}$ estimates to (3.13) yields
\[
y^{2+\gamma} [\partial_y v]_{\gamma; Q(y, s; \frac{4}{m+1} y_s)} \leq C (n, \Lambda, \delta) \| v \|_{L^\infty(Q(y, s; \frac{4}{m+1} y_s))} \leq C (n, \Lambda, \delta) e^{-\lambda^*_s} y^*_\alpha
\]
(7.26)

Then by the bootstrap argument, one could show that
\[
y^m_s \| \partial_y^m v (y, s) \|_{L^\infty(Q(y, s; \frac{4}{m+1} y_s))} + y^{m+\gamma} [\partial_y^m v (y, s)]_{\gamma; Q(y, s; \frac{4}{m+1} y_s)} \leq C (n, \Lambda, \delta, m) e^{-\lambda^*_s} y^*_\alpha
\]
(7.27)

for all $m \in \mathbb{Z}_+$. Furthermore, by (3.13) and (7.27), we get
\[
y^{m+2} \| \partial_y^m \partial_s v (y, s) \|_{L^\infty(Q(y, s; \frac{4}{m+2} y_s))} + y^{m+2+\gamma} [\partial_y^m \partial_s v (y, s)]_{\gamma; Q(y, s; \frac{4}{m+2} y_s)} \leq C (n, \Lambda, \delta, m) e^{-\lambda^*_s} y^*_\alpha
\]
for all $m \geq 0$. Differentiating (3.13) with respect to $s$ and using the above estimates gives
\[
y^{m+4} \| \partial_y^m \partial_s^2 v (y, s) \|_{L^\infty(Q(y, s; \frac{4}{m+2} y_s))} + y^{m+4+\gamma} [\partial_y^m \partial_s v (y, s)]_{\gamma; Q(y, s; \frac{4}{m+2} y_s)} \leq C (n, \Lambda, \delta, m) e^{-\lambda^*_s} y^*_\alpha
\]
(7.28)

Continuing this process and using induction yields
\[
y^{m+2l} \| \partial_y^m \partial_s^l v (y, s) \|_{L^\infty(Q(y, s; \frac{4}{m+2l} y_s))} + y^{m+2l+\gamma} [\partial_y^m \partial_s^l v (y, s)]_{\gamma; Q(y, s; \frac{4}{m+2l} y_s)} \leq C (n, \Lambda, \delta, m, l) e^{-\lambda^*_s} y^*_\alpha
\]

for any $m, l \in \mathbb{Z}_+$. If $e^{-\delta^*_s} y^* \leq 2$, recall that by Proposition 3.1, there holds
\[
(\partial_s + \mathcal{L}) \left( \frac{k}{c_2} e^{-\lambda^*_s} \varphi_2 (y) \right) = 0
\]
That is, (7.29)
\[
\left( \partial_s - \partial_{yy}^2 + \frac{1}{y} \left( 2(n-1) - \frac{y^2}{2} \right) \partial_y - \frac{1}{y^2} \left( 2(n-1) + \frac{y^2}{2} \right) \right) \left( \frac{k}{c_2} e^{-\lambda_2 s} \varphi_2(y) \right) = 0
\]
In addition, from (3.13) we have (7.30)
\[
\left( \partial_s - \partial_{yy}^2 + \frac{1}{y} \left( 2(n-1) - \frac{y^2}{2} \right) \partial_y - \frac{1}{y^2} \left( 2(n-1) + \frac{y^2}{2} \right) \right) v(y, s) = \frac{h(y, s)}{y^2}
\]
where
\[
h(y, s) = -\frac{\left( \partial_y v \right)^2}{1 + \left( \partial_y v \right)^2} \left( y^2 \partial_{yy}^2 v \right) + \frac{2(n-1) \left( \frac{v}{y} \right)^2}{1 - \left( \frac{v}{y} \right)^2} \left( y \partial_y v \right) + \frac{2(n-1) \left( \frac{v}{y} \right)^2}{1 - \left( \frac{v}{y} \right)^2} v
\]
Notice that by (7.28), the function \( h(y, s) \) satisfies
\[
y_s^{m+2l} \left\| \partial_y^m \partial_y^l h(y, s) \right\|_{L^\infty(\Omega(y, \delta, \lambda; \frac{m}{m+2l+1} y_2))} + y_s^{m+2l+\gamma} \left\| \partial_y^m \partial_y^l h(y, s) \right\|_{L^\infty(\Omega(y, \delta, \lambda; \frac{m}{m+2l+1} y_2))} \\
\leq C(n, \Lambda, \delta, m, l) \left( e^{-\lambda_2 s} y_s^{-1} \right)^2 \left( e^{-\lambda_2 s} y_s^{\alpha} \right) \\
= C(n, \Lambda, \delta, m, l) \left( e^{-\lambda_2 s} y_s^{\alpha-2} \right)^2 \left( e^{-\lambda_2 s} y_s^{\alpha+2} \right)
\]
(7.31)
for any \( m, l \in \mathbb{Z}_+ \). Then we substract (7.29) from (7.30) to get
\[
\left( \partial_s - \partial_{yy}^2 + \frac{1}{y} \left( 2(n-1) - \frac{y^2}{2} \right) \partial_y - \frac{1}{y^2} \left( 2(n-1) + \frac{y^2}{2} \right) \right) \left( v - \frac{k}{c_2} e^{-\lambda_2 s} \varphi_2(y) \right) = \frac{h(y, s)}{y^2}
\]
By (7.31) and Schauder estimates, we get
\[
y_s^{m+2l} \left\| \partial_y^m \partial_y^l v(y, s) - \frac{k}{c_2} e^{-\lambda_2 s} \varphi_2(y) \right\|_{L^\infty(\Omega(y, \delta, \lambda; \frac{m}{m+2l+1} y_2))} \\
\leq C(n, \Lambda, \delta, m, l) \left\| v(y, s) - \frac{k}{c_2} e^{-\lambda_2 s} \varphi_2(y) \right\|_{L^\infty(\Omega(y, \delta, \lambda; \frac{m}{m+2l+1} y_2))} \\
+ C(n, \Lambda, \delta, m, l) \sum_{i=0}^{m} \sum_{j=0}^{l} y_s^{i+2j} \left\| \partial_y^i \partial_y^j h \right\|_{L^\infty(\Omega(y, \delta, \lambda; \frac{m}{m+2l+1} y_2))} \\
+ C(n, \Lambda, \delta, m, l) \sum_{i=0}^{m} \sum_{j=0}^{l} y_s^{i+2j+\gamma} \left\| \partial_y^i \partial_y^j h \right\|_{L^\infty(\Omega(y, \delta, \lambda; \frac{m}{m+2l+1} y_2))} \\
\leq C(n, \Lambda, \delta, m, l) e^{-\lambda_2 s} \left( e^{-\lambda_2 s} y_s^{\alpha+2} \right)
\]
for any \( m, l \in \mathbb{Z}_+ \).
If \( \frac{\lambda}{\beta} e^{-\sigma y_s} \leq y_s \leq e^{-\sigma y_s} \), notice that
\[
\partial_s \psi_k(z) = 0 = \frac{1}{1 + (\partial_z \psi_k(z))^2} \partial_z^2 \psi_k(z) + 2(n-1) z \partial_z \psi_k(z) + \psi_k(z)
\]
Let
\[
\tilde{v}(y, s) = e^{-\sigma s} \psi_k(e^{\sigma s} y)
\]
then we have
\[ \partial_s \bar{v} + \sigma (-y \partial_y \bar{v} + \bar{v}) = \frac{1}{1 + (\partial_y \bar{v})^2} \partial^2_{yy} \bar{v} + 2(n - 1) \frac{y \partial_y \bar{v} + \bar{v}}{y^2 - \bar{v}^2}. \]

Then we subtract the above equation from (3.13) to get
(7.33)
\[ \partial_s (v - \bar{v}) - a(y, \sigma) \partial^2_{yy} (v - \bar{v}) - \frac{1}{y} b(y, \sigma) \partial_z (v - \bar{v}) - \frac{1}{y^3} c(y, \sigma) (v - \bar{v}) = \frac{1}{y^2} f(y, \sigma) \]
where
\[
\begin{align*}
a(z, \tau) &= \frac{1}{1 + (\partial_y v)^2} \\
b(z, \tau) &= \frac{-\left( y \partial^2_{yy} \bar{v} \right) \left( \partial_y v + \partial_y \bar{v} \right)}{1 + (\partial_y \bar{v})^2} + \frac{2(n - 1)}{1 - \left( \frac{\bar{v}}{\beta} \right)^2} - \frac{y^2}{2} \\
c(z, \tau) &= \frac{2(n - 1) \left( \partial_y \bar{v} + \bar{v} \right) \left( \frac{\bar{v}}{\beta} + \frac{1}{\beta} \right)}{1 - \left( \frac{\bar{v}}{\beta} \right)^2} + \frac{2(n - 1)}{1 - \left( \frac{\bar{v}}{\beta} \right)^2} + \frac{y^2}{2} \\
f(z, \tau) &= \left( \frac{1}{2} + \sigma \right) y^2 (-y \partial_y \bar{v} + \bar{v})
\end{align*}
\]

Note that by Lemma 2.3 and (7.32), we have
(7.34)
\[ y^m \left| \partial_y^m \bar{v} (y, \sigma) \right| \leq C(n, m) e^{-\lambda \sigma y^\alpha} \]
for \( y \geq \beta \), which yields
\[
y_s^{m+2l} \left\| \partial_y^m \partial_y^l f(y, \sigma) \right\|_{L^\infty(Q(y, s; \frac{4}{m+2l+1} y_s))} + y_s^{m+2l+\gamma} \left[ \partial_y^m \partial_y^l f(y, \sigma) \right]_{Q(y, s; \frac{4}{m+2l+1} y_s)} \\
\leq C(n, \delta, m, l) \left( e^{-\lambda \sigma y^\alpha} y_s^{\alpha+2} \right)
\]
(7.35)
\[ \leq C(n, \delta, m, l) e^{-\delta \sigma y_s^\alpha} \left( e^{-\lambda \sigma y_s^\alpha} y_s^\alpha \right) \]
since \( \delta \beta e^{-\sigma y_s} \leq y_s \leq e^{-\sigma y_s} \). Thus, by (7.25), (7.24), (7.26) and applying Schauder estimates to (7.33), we get
\[
y_s^{m+2l} \left\| \partial_y^m \partial_y^l \left( v(y, \sigma) - \bar{v}(y, \sigma) \right) \right\|_{L^\infty(Q(y, \sigma; \frac{4}{m+2l+1} y_s))} \\
\leq C(n, \Lambda, \delta, m, l) \left\| v(y, \bar{v}) - \bar{v}(y, \bar{v}) \right\|_{L^\infty(Q(y, \sigma; \frac{4}{m+2l+1} y_s))} \\
+ C(n, \Lambda, \delta, m, l) \sum_{i=0}^{l} \sum_{j=0}^{l} y_s^{i+2j} \left\| \partial_y^i \partial_y^j f \right\|_{L^\infty(Q(y, \sigma; \frac{4}{m+2l+1} y_s))} \\
+ C(n, \Lambda, \delta, m, l) \sum_{i=0}^{l} \sum_{j=0}^{l} y_s^{i+2j+\gamma} \left[ \partial_y^i \partial_y^j f \right]_{Q(y, \sigma; \frac{4}{m+2l+1} y_s)} \\
\leq C(n, \Lambda, \delta, m, l) \left( \beta^{\alpha-3} e^{-2\sigma y_s^{\alpha-\gamma}} e^{-\lambda \sigma y^\alpha} + e^{-2\sigma y_s^\alpha} \left( e^{-\lambda \sigma y^\alpha} y_s^\alpha \right) \right) \\
\leq C(n, \Lambda, \delta, m, l) \beta^{\alpha-3} e^{-2\sigma y_s^{\alpha-\gamma}} e^{-\lambda \sigma y^\alpha}
\]
provided that \( s_0 \gg 1 \) (depending on \( n, \beta \)). Notice that \( 0 < \varrho < \bar{v} \). \( \square \)
Next, we would like to prove (4.16). The $C^0$ estimate is already shown in Proposition 6.6. Below we would prove the first and second derivatives estimates in Lemma 7.9 and Lemma 7.11, respectively. Before that, notice that by (3.27) we have

\begin{equation}
|\partial_i^j w(z, \tau)| \leq \Lambda \left( z^\alpha + \frac{z^{2\lambda_2 + 1}}{(2\sigma \tau)^2} \right) \leq C(n, \Lambda) z^\alpha, \quad i \in \{0, 1, 2\}
\end{equation}

for $\beta \leq z \leq (2\sigma \tau)^{1/(1-\theta)}$, $\tau_0 \leq \tau \leq \tau_*$; in particular, we have

\begin{equation}
\max \left\{ \left| \frac{w(z, \tau)}{z} \right|, \left| \partial_z w(z, \tau) \right| \right\} \leq \frac{1}{3}
\end{equation}

for $\beta \leq z \leq (2\sigma \tau)^{1/(1-\theta)}$, $\tau_0 \leq \tau \leq \tau_*$, provided that $\beta \gg 1$ (depending on $n$, $\Lambda$). In the following lemma, we show how to transform the above estimates for $w(z, \tau)$ to $\hat{w}(z, \tau)$ via the projected curve $\bar{\Gamma}_\tau$ defined in (3.23). This lemma is useful since it provides the “boundary values” for estimating $\hat{w}(z, \tau)$ in the rescaled tip region.

**Lemma 7.7.** If $\beta \gg 1$ (depending on $n$, $\Lambda$) and $\tau_0 \gg 1$ (depending on $n$, $\Lambda$, $\rho$, $\beta$), there hold

\begin{equation}
|\partial_z \hat{w}(z, \tau) - 1| \leq C(n, \Lambda) z^{-1}
\end{equation}

\begin{equation}
|\partial^2_{zz} \hat{w}(z, \tau)| \leq C(n, \Lambda) z^{-2}
\end{equation}

for $2\beta \leq z \leq \frac{1}{2} (2\sigma \tau)^{1/(1-\theta)}$, $\tau_0 \leq \tau \leq \tau_*$.

**Proof.** Let’s first parametrize the projected curve $\bar{\Gamma}_\tau$ by

\[ Z_\tau = \left( (z - w(z, \tau)) \frac{1}{\sqrt{2}}, (z + w(z, \tau)) \frac{1}{\sqrt{2}} \right) \]

In this parametrization, there hold

\[ N_{\bar{\Gamma}_\tau} \cdot \mathbf{e} = \frac{-\partial_z w(z, \tau)}{\sqrt{1 + (\partial_z w(z, \tau))^2}} \]

\[ A_{\bar{\Gamma}_\tau} = \frac{\partial^2_{zz} w(z, \tau)}{\left(1 + (\partial_z w(z, \tau))^2\right)^{3/2}} \]

where $N_{\bar{\Gamma}_\tau}$ and $A_{\bar{\Gamma}_\tau}$ are the (upward) unit normal vector and normal curvature of $\bar{\Gamma}_\tau$ at $Z_\tau$, respectively, and

\[ \mathbf{e} = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \]

By (7.36) and (7.37), we get

\begin{equation}
|N_{\bar{\Gamma}_\tau} \cdot \mathbf{e}| \leq C(n, \Lambda) |Z_\tau|^{-1}
\end{equation}

\begin{equation}
|A_{\bar{\Gamma}_\tau}| \leq C(n, \Lambda) |Z_\tau|^{-2}
\end{equation}

Next, we would like to prove (4.16). The $C^0$ estimate is already shown in Proposition 6.6. Below we would prove the first and second derivatives estimates in Lemma 7.9 and Lemma 7.11, respectively. Before that, notice that by (3.27) we have

\begin{equation}
|\partial_i^j w(z, \tau)| \leq \Lambda \left( z^\alpha + \frac{z^{2\lambda_2 + 1}}{(2\sigma \tau)^2} \right) \leq C(n, \Lambda) z^\alpha, \quad i \in \{0, 1, 2\}
\end{equation}

for $\beta \leq z \leq (2\sigma \tau)^{1/(1-\theta)}$, $\tau_0 \leq \tau \leq \tau_*$; in particular, we have

\begin{equation}
\max \left\{ \left| \frac{w(z, \tau)}{z} \right|, \left| \partial_z w(z, \tau) \right| \right\} \leq \frac{1}{3}
\end{equation}

for $\beta \leq z \leq (2\sigma \tau)^{1/(1-\theta)}$, $\tau_0 \leq \tau \leq \tau_*$, provided that $\beta \gg 1$ (depending on $n$, $\Lambda$). In the following lemma, we show how to transform the above estimates for $w(z, \tau)$ to $\hat{w}(z, \tau)$ via the projected curve $\bar{\Gamma}_\tau$ defined in (3.23). This lemma is useful since it provides the “boundary values” for estimating $\hat{w}(z, \tau)$ in the rescaled tip region.

**Lemma 7.7.** If $\beta \gg 1$ (depending on $n$, $\Lambda$) and $\tau_0 \gg 1$ (depending on $n$, $\Lambda$, $\rho$, $\beta$), there hold

\begin{equation}
|\partial_z \hat{w}(z, \tau) - 1| \leq C(n, \Lambda) z^{-1}
\end{equation}

\begin{equation}
|\partial^2_{zz} \hat{w}(z, \tau)| \leq C(n, \Lambda) z^{-2}
\end{equation}

for $2\beta \leq z \leq \frac{1}{2} (2\sigma \tau)^{1/(1-\theta)}$, $\tau_0 \leq \tau \leq \tau_*$.

**Proof.** Let’s first parametrize the projected curve $\bar{\Gamma}_\tau$ by

\[ Z_\tau = \left( (z - w(z, \tau)) \frac{1}{\sqrt{2}}, (z + w(z, \tau)) \frac{1}{\sqrt{2}} \right) \]

In this parametrization, there hold

\[ N_{\bar{\Gamma}_\tau} \cdot \mathbf{e} = \frac{-\partial_z w(z, \tau)}{\sqrt{1 + (\partial_z w(z, \tau))^2}} \]

\[ A_{\bar{\Gamma}_\tau} = \frac{\partial^2_{zz} w(z, \tau)}{\left(1 + (\partial_z w(z, \tau))^2\right)^{3/2}} \]

where $N_{\bar{\Gamma}_\tau}$ and $A_{\bar{\Gamma}_\tau}$ are the (upward) unit normal vector and normal curvature of $\bar{\Gamma}_\tau$ at $Z_\tau$, respectively, and

\[ \mathbf{e} = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \]

By (7.36) and (7.37), we get

\begin{equation}
z \leq |Z_\tau| = \sqrt{z^2 + (w(z, \tau))^2} \leq \sqrt{\frac{10}{9} z}
\end{equation}

\begin{equation}
|N_{\bar{\Gamma}_\tau} \cdot \mathbf{e}| \leq C(n, \Lambda) |Z_\tau|^{-1}
\end{equation}

\begin{equation}
|A_{\bar{\Gamma}_\tau}| \leq C(n, \Lambda) |Z_\tau|^{-2}
\end{equation}
for $\beta \leq z \leq (2\sigma \tau)^{\frac{1}{2}(1-\theta)}$, $\tau_0 \leq \tau \leq \hat{\tau}$.

Now we reparametrize $\bar{\Gamma}_\tau$ as

$$Z_\tau = (z, \hat{w}(z, \tau))$$

In that case, we have

$$|Z_\tau| = \sqrt{z^2 + (\hat{w}(z, \tau))^2}$$

(7.42)

$$N_{\bar{\Gamma}_\tau} \cdot e = \frac{1 - \partial_z \hat{w}(z, \tau)}{\sqrt{2 \left(1 + (\partial_z \hat{w}(z, \tau))^2\right)}}$$

(7.43)

$$A_{\bar{\Gamma}_\tau} = \frac{\partial^2_z \hat{w}(z, \tau)}{\left(1 + (\partial_z \hat{w}(z, \tau))^2\right)^{\frac{3}{2}}}$$

Note that by (2.3), (6.3) and (6.8), there holds

(7.44)

$$1 \leq \frac{|N_{\bar{\Gamma}_\tau} \cdot e|}{100 \sqrt{2}}$$

for $2\beta \leq z \leq (2\sigma \tau)^{\frac{1}{2}(1-\theta)}$. Since

$$\lim_{p \to \pm \infty} \frac{1 - p}{\sqrt{2} (1 + p^2)} = \mp \frac{1}{\sqrt{2}}$$

it follows, by (7.42), that

(7.45)

$$|\partial_z \hat{w}(z, \tau)| \leq C$$

for $2\beta \leq z \leq (2\sqrt{2}\sigma \tau)^{\frac{1}{2}}$. The conclusion follows by (7.40), (7.41), (7.42), (7.43), (7.44) and (7.45).

Remark 7.8. Note that for the last lemma, when $\tau = \tau_0$, by (4.7) we have

$$N_{\bar{\Gamma}_{\tau_0}} \cdot e = \frac{-\partial_z \hat{w}(z, \tau_0)}{\sqrt{1 + (\partial_z \hat{w}(z, \tau_0))^2}} > 0$$

for $\frac{1}{2} \beta \leq z \leq (2\sigma \tau_0)^{\frac{1}{2}(1-\theta)}$, $\tau_0 \leq \tau \leq \hat{\tau}$. Consequently, by the same argument and (7.42), we can show that

(7.46)

$$0 \leq 1 - \partial_z \hat{w}(z, \tau_0) \leq C(n, \Lambda) z^{\alpha - 1}$$

for $\frac{1}{2} \beta \leq z \leq (2\sigma \tau_0)^{\frac{1}{2}(1-\theta)}$.

Below we use (3.22), (4.3), (7.38) and the maximum principle to show the first derivative estimate in (4.16).

Lemma 7.9. If $\beta \gg 1$ (depending on $n, \Lambda$), there holds

(7.47)

$$0 \leq \partial_z \hat{w}(z, \tau) \leq 1 + \beta^{\alpha - 2}$$

for $0 \leq z \leq \beta^2$, $\tau_0 \leq \tau \leq \hat{\tau}$.
Proof. By differentiating (3.22), we get

\[(7.48) \quad \partial_\tau (\partial_z \hat{w}) = \frac{1}{1 + (\partial_z \hat{w})^2} \partial_{zz}^2 (\partial_z \hat{w})
\]

\[+ \left( \frac{n-1}{z} - \frac{2 \partial_z \hat{w} \partial_{zz}^2 \hat{w}}{1 + (\partial_z \hat{w})^2} - \frac{\frac{1}{\sigma} + \sigma}{2\sigma^2} \right) \partial_z (\partial_z \hat{w}) + (n-1) \left( \frac{1}{\hat{w}^2} - \frac{1}{z^2} \right) (\partial_z \hat{w}) \]

Notice that for the last term on the RHS of (7.48), by (2.4) and (6.8), there holds

\[(7.49) \quad \hat{w} (z, \tau) > z \iff \frac{1}{\hat{w}^2 (z, \tau)} - \frac{1}{z^2} < 0\]

for \(0 \leq z \leq \beta^2\), \(\tau_0 \leq \tau \leq \tau^*\).

Let

\[(\partial_z \hat{w})_{\min} (\tau) = \min_{0 \leq z \leq \beta^2} \partial_z \hat{w} (z, \tau)\]

Then \((\partial_z \hat{w})_{\min} (\tau_0) \geq 0\) by (4.3). We claim that

\[(7.50) \quad (\partial_z \hat{w})_{\min} (\tau) \geq 0\]

for \(\tau_0 \leq \tau \leq \tau^*\). To prove that, we use a contradiction argument. Suppose that there is \(\tau_1^* > \tau_0\) so that

\[(\partial_z \hat{w})_{\min} (\tau_1^*) < 0\]

Let \(\tau_0^* > \tau_0\) be the first time after which \((\partial_z \hat{w})_{\min}\) stays negative all the way up to \(\tau_1^*\). By continuity, we have

\[(\partial_z \hat{w})_{\min} (\tau_1^*) \geq 0\]

Note that by (3.22) and (7.38), the negative minimum of \(\partial_z \hat{w} (z, \tau)\) for each time-slice must be attained in \((0, \beta^2)\), provided that \(\beta \gg 1\) (depending on \(n, \Lambda\)). Applying the maximum principle to (7.48) (and noting (7.49)) yields

\[\partial_\tau (\partial_z \hat{w})_{\min} \geq (n-1) \left( \frac{1}{\hat{w}^2} - \frac{1}{z^2} \right) (\partial_z \hat{w})_{\min} \geq 0\]

for \(\tau_0^* \leq \tau < \tau_1^*\). It follows that

\[(\partial_z \hat{w})_{\min} (\tau_0^*) \leq (\partial_z \hat{w})_{\min} (\tau_1^*) < 0\]

which is a contradiction.

Next, let

\[(\partial_z \hat{w})_{\max} (\tau) = \max_{0 \leq z \leq \beta^2} \partial_z \hat{w} (z, \tau)\]

Then

\[(\partial_z \hat{w})_{\max} (\tau_0) \leq 1\]

by (4.3) and (7.40). We claim that

\[(\partial_z \hat{w})_{\max} (\tau) \leq 1 + \beta^{\alpha-2}\]

for \(\tau_0 \leq \tau \leq \tau^*\). Suppose the contrary, then there is \(\tau_1^* > \tau_0\) so that

\[(\partial_z \hat{w})_{\max} (\tau_1^*) > 1 + \beta^{\alpha-2}\]

Let \(\tau_0^* > \tau_0\) be the first time after which \((\partial_z \hat{w})_{\max}\) is greater than \(1 + \beta^{\alpha-2}\) all the way up to \(\tau_1^*\). By continuity, we have

\[(\partial_z \hat{w})_{\max} (\tau_0^*) \leq 1 + \beta^{\alpha-2}\]
Notice that by (7.38), there holds
\[
\partial_z \hat{w} (\beta, \tau) \leq 1 + C (n, \Lambda) \beta^{2(\alpha - 1)} < 1 + \beta^{\alpha - 2}
\]
provided that \( \beta \gg 1 \) (depending on \( n, \Lambda \)). Thus, the maximum of \( \partial_z \hat{w} (z, \tau) \) for each time-slice which is greater than \( 1 + \beta^{\alpha - 2} \) must be attained in \((0, \beta^2)\), provided that \( \beta \gg 1 \) (depending on \( n, \Lambda \)). Applying the maximum principle to (7.38) (and using (7.49) and (7.50)) yields
\[
\partial_\tau (\partial_z \hat{w})_{\max} \leq 0
\]
for \( \tau_0^* \leq \tau < \tau_1^* \). It follows that
\[
(\partial_z \hat{w})_{\max} (\tau_0^*) \geq (\partial_z \hat{w})_{\max} (\tau_1^*) > 1 + \beta^{\alpha - 2}
\]
which is a contradiction. \( \Box \)

Then we start to show the second derivative estimate in (4.16). Note that the second fundamental form of \( \Gamma \) (in the parametrization of (3.20)) is given by
\[
A_{\Gamma} = \frac{1}{\sqrt{1 + |\partial_z \hat{w}|^2}} \left( \frac{\partial^2_z \hat{w}}{1 + |\partial_z \hat{w}|^2} \right) \frac{\partial_z \hat{w}}{\tau} I_{n-1} - \frac{1 + \beta \tau}{\tau} I_{n-1}
\]
By (6.8) and (7.47), to estimate \( \partial^2_z \hat{w} (z, \tau) \) is equivalent to estimate \( A_{\Gamma} \). In the following lemma, we derive an evolution equation of \( A_{\Gamma} \) and use that, together with (4.3), (7.39) and the maximum principle, to show that \( A_{\Gamma} \) can be estimated for a short period of time.

**Lemma 7.10.** If \( \beta \gg 1 \) (depending on \( n, \Lambda \)), then there is \( \delta > 0 \) (depending on \( n \)) so that the second fundamental form of \( \Gamma \) satisfies
\[
\max_{\Gamma \cap B(\sigma, 3\beta)} |A_{\Gamma, \tau}| \leq C (n)
\]
for \( \tau_0 \leq \tau \leq \min \{ \tau_0 + \delta, \tau^* \} \). In particular, there holds
\[
|\partial^2_{z \tau} \hat{w} (z, \tau)| \leq C (n)
\]
for \( 0 \leq z \leq 3\beta, \tau_0 \leq \tau \leq \min \{ \tau_0 + \delta, \tau^* \} \).

**Proof.** By (4.3), (6.8), (7.38), (7.39) and (7.51), the second fundamental form of \( \Gamma \) satisfies
\[
C \equiv |A_{\Gamma, \tau}|_{\max} (\tau_0) \geq \max_{\Gamma \cap B(\sigma, 3\beta)} |A_{\Gamma, \tau} (Z_\tau)| \leq C (n)
\]
provided that \( \beta \gg 1 \) (depending on \( n, \Lambda \)). By reparametrization of the flow, we may derive an evolution equation for \( A_{\Gamma, \tau} \), as follows:
\[
(\partial_\tau - \Delta_{\Gamma, \tau}) |A_{\Gamma, \tau}|^2 = -2 \left| \nabla_{\Gamma, \tau} A_{\Gamma, \tau} \right|^2 + 2 \left| A_{\Gamma, \tau} \right|^4 - \frac{1 + 2\sigma}{2\sigma \tau} \left| A_{\Gamma, \tau} \right|^2
\]
Let
\[
h (\tau) = \max_{\Gamma \cap B(\sigma, 3\beta)} |A_{\Gamma, \tau}|^2
\]
If \( h (\tau) \leq C \) for \( \tau_0 \leq \tau \leq \tau^* \), then we are done. Otherwise, there is \( \tau_1^* > \tau_0 \) so that
\[
h (\tau_1^*) > C
\]
Let $\tau_0^* > \tau_0$ be the first time after which $h$ is greater than $C$ all the way up to $\tau_1^*$. By continuity, we have

$$h(\tau_0^*) \leq C$$

(7.54)

Note that the maximum for each time-slice must be attained in the interior of $\Gamma_{\tau} \cap B(O; 3\beta)$. By applying the maximum principle to (7.53), we get

$$\partial_{\tau} h(\tau) \leq 2h(\tau)$$

for $\tau^*_0 \leq \tau \leq \tau^*_1$, which implies

$$h(\tau^*_1) \leq \frac{h(\tau_0^*)}{1 - 2(\tau_1^* - \tau_0^*)} h(\tau_0^*)$$

(7.55)

Thus, by (7.52), (7.54) and (7.55), there is $\delta = \delta(n)$ so that

$$h(\tau) \leq 2C$$

for $\tau_0^* \leq \tau \leq \tau^*_1$ for this choice of $\delta > 0$, we claim that $h(\tau) \leq 2C$ for $\tau_0^* \leq \tau \leq \tau^*_1 + \delta$ for $\tau_0^* \leq \tau \leq \tau^*_1 + \delta$. Otherwise, we may get a contradiction by the above argument. Then the conclusion follows immediately by (6.8), (7.47) and (7.51).

In the following lemma, we use Ecker-Huisken interior estimate for MCF to estimate $A_{\Gamma_{\tau}}$ for $\tau_0^* + \delta \leq \tau \leq \tau^*_1$. Combining with Lemma 7.10, we then get the second derivative estimate in (4.16).

**Lemma 7.11.** If $\beta \gg 1$ (depending on $n$, $\Lambda$), there holds

$$|\partial^2_{zz} \hat{w}(z, \tau)| \leq C(n)$$

for $0 \leq z \leq 3\beta$, $\tau_0^* \leq \tau \leq \tau^*_1$. Hence, to prove the lemma, we have to consider the case when $\tau^*_1 - \tau_0^* > \delta$.

Fix $\tau_0 + \delta \leq \tau_0^* \leq \tau_1^*$ and let

$$\Xi_\iota = (2\sigma_{\tau_0})^{\frac{n}{2} + \frac{1}{2}} \sum_{-(2\sigma_{\tau_0})}^{2(2\sigma_{\tau_0})} \left(1 - \frac{\iota}{2\sigma_{\tau_0}}\right)$$

where

$$\tilde{h}(r, \iota) = (2\sigma_{\tau_0})^{\frac{n}{2} + \frac{1}{2}} \hat{u} \left(\frac{r}{2\sigma_{\tau_0}} - (2\sigma_{\tau_0})^{\frac{1}{2}} \left(1 - \frac{\iota}{2\sigma_{\tau_0}}\right)\right)$$

Then $\{\Xi_\iota\}$ defines a MCF for $-(2\sigma_{\tau_0}) \left(\frac{\tau_0}{\tau_0^*} - 1\right) \leq \iota \leq 0$. Note that

$$\Xi_0 = (2\sigma_{\tau_0})^{\frac{n}{2} + \frac{1}{2}} \sum_{-(2\sigma_{\tau_0})}^{2(2\sigma_{\tau_0})} = \Gamma_{\tau_0}$$

and

$$(2\sigma_{\tau_0}) \left(\frac{\tau_0}{\tau_0^*} - 1\right) \geq \frac{\delta}{2}$$
provided that \( \tau_0 \gg 1 \) (depending on \( n \)). By \((3.21)\), we may rewrite \( \hat{h}(r, \iota) \) as

\[
\hat{h}(r, \iota) = \left(1 - \frac{r}{2\sigma \tau_*}\right)^{\frac{1}{2} + \sigma} \hat{w} \left(\frac{r}{1 - \frac{r}{2\sigma \tau_*}}, \frac{\tau_*}{1 - \frac{r}{2\sigma \tau_*}}\right)
\]

By \((6.8)\) and \((7.47)\), we have

\[
\hat{h}(r, \iota) \geq \frac{\hat{\psi}(0)}{2}
\]

\[
\left| \partial_r \hat{h}(r, \iota) \right| = \left| \partial_z \hat{w} \left(\frac{r}{1 - \frac{r}{2\sigma \tau_*}}, \frac{\tau_*}{1 - \frac{r}{2\sigma \tau_*}}\right)\right| \leq \frac{4}{3}
\]

for \( 0 \leq r \leq 4\beta, \ -\frac{\delta}{2} \leq \iota \leq 0 \), provided that \( \tau_0 \gg 1 \) (depending on \( n \)). Note that the unit normal vector of \( \Xi \) at \( X \) is given by

\[
N_{\Xi}(r, \nu, \omega) = \frac{(-\partial_r \hat{h}(r, \iota) \nu, \omega)}{\sqrt{1 + \left(\partial_r \hat{h}(r, \iota)\right)^2}}
\]

which satisfies

\[
(N_{\Xi}(r, \nu, \omega) \cdot e)^{-1} = \frac{\sqrt{1 + \left(\partial_r \hat{h}(r, \iota)\right)^2}}{(0, \omega) \cdot e}
\]

where

\[
e = \begin{pmatrix} (2n-1) \text{ copies} \\ 0, \ldots, 0 \end{pmatrix}, \quad \vec{0} = \begin{pmatrix} n \text{ copies} \\ 0, \ldots, 0 \end{pmatrix}
\]

Now fix \( 0 \leq z_* \leq 3\beta \) and let

\[
\mathcal{X}_* = \left(z_* \nu_*, \hat{h}(z_*, 0) \omega_*\right) = \left(z_* \nu_*, \hat{w}(z_*, \tau_*) \omega_*\right)
\]

where \( \nu_* = \omega_* = \begin{pmatrix} (n-1) \text{ copies} \\ 0, \ldots, 0 \end{pmatrix} \), we claim that

\[
(N_{\Xi}(r, \nu, \omega) \cdot e)^{-1} \leq \frac{5\sqrt{2}}{3}
\]

for \( X \in \Xi \cap B_n^2 \left(\mathcal{X}_*, \frac{\hat{\psi}(0)}{2\sqrt{2}}\right), \ -\frac{\delta}{2} \leq \iota \leq 0 \). Then by the curvature estimate in \([EH]\), the second fundamental form of \( \Gamma_{\tau_*} \) at \( \mathcal{X}_* \) satisfies

\[
|A_{\Gamma_{\tau_*}}(\mathcal{X}_*)| = |A_{\Xi_0}(\mathcal{X}_*)| \leq C(n) \left(\frac{2 \sqrt{2}}{\hat{\psi}(0)} + \sqrt{\frac{2}{\delta}}\right) = C(n)
\]
It follows that
\[
\left| \partial^2_{zz} \hat{w}(z_*, \tau_*) \right| \leq |A \Gamma, (X_*)| \leq C(n)
\]

Now let’s come back to (7.59). First notice that for each
\[
X_i (r, \nu, \omega) \in \Xi_i \cap B^{2n} \left( X_*, \frac{\hat{\psi}(0)}{2\sqrt{2}} \right), \quad -\frac{\delta}{2} \leq \iota \leq 0
\]
there holds
\[
\hat{h} (r, \iota) \sqrt{1 - \left( (\hat{0}, \omega) \cdot e \right)^2} \leq |X_i (r, \nu, \omega) - X_*| \leq \hat{\psi}(0) \frac{1}{\sqrt{2}}
\]
which, together with (7.56), implies
\[
(7.60)
\]
Then (7.59) follows by (7.57), (7.58) and (7.60).

Below we use (2.4), (3.22), (4.16), (6.8) and the standard regularity theory for parabolic equations to prove (4.23).

**Proposition 7.12.** If \( \beta \gg 1 \) (depending on \( n \)) and \( \tau_0 \gg 1 \) (depending on \( n, \beta \)), there holds (4.23).

**Proof.** Firstly, let \( \hat{w}(z, \tau) \) and \( \hat{\psi}_k(z) \) be radially symmetric functions so that
\[
\hat{w}(z, \tau) = \hat{w}(z, \tau) \bigg|_{z = |z|}, \quad \hat{\psi}_k(z) = \hat{\psi}_k(z) \bigg|_{z = |z|}
\]
where \( z = (z_1, \cdots, z_n) \). Note that
\[
\partial_{z_i} \hat{w} = \partial_{\hat{z}_i} \frac{z_i}{|z|}
\]
\[
\partial^2_{zz} \hat{w} = \partial^2_{\hat{z}_i \hat{z}_j} \frac{z_i z_j}{|z|^2} + \partial_{\hat{z}_i} \frac{|z|^2 \delta_{ij} - z_i z_j}{|z|^3}
\]
Then by (6.28), (7.47) and (4.16), there hold
\[
(7.61)
\]
for \( z \in B(O; 3\beta), \tau_0 \leq \tau \leq \hat{\tau}, m \in \mathbb{Z}_+ \), where
\[
\nabla = (\partial_{z_1}, \cdots, \partial_{z_n})
\]
Also, by (2.4) and Lemma 2.4, we get
\[
(7.62)
\]
Then we subtract (7.64) from (7.63) to get
\[
\partial_t \hat{w} = \sqrt{1 + |\hat{\nabla} \hat{w}|^2} \partial_z \left( \frac{z^{n-1}}{\sqrt{1 + (\partial_z \hat{w})^2}} \partial_z \hat{w} \right) - \frac{n-1}{\hat{w}} + \frac{1}{2} + \frac{\sigma}{2\sigma' \tau} \left(-z \partial_z \hat{w} + \hat{w}\right)
\]
and
\[
\partial_t \hat{v}_k = 0 = \sqrt{1 + \left(\frac{\partial_z \hat{v}_k}{\sqrt{1 + (\partial_z \hat{v}_k)^2}}\right)^2} \partial_z \left( \frac{z^{n-1}}{\sqrt{1 + (\partial_z \hat{v}_k)^2}} \partial_z \hat{v}_k \right) - \frac{n-1}{\hat{v}_k}
\]
which yield
\[
\partial_t \hat{w} = \sqrt{1 + |\hat{\nabla} \hat{w}|^2} \nabla \cdot \frac{\nabla \hat{w}}{\sqrt{1 + |\hat{\nabla} \hat{w}|^2}} - \frac{n-1}{\hat{w}} + \frac{1}{2} + \frac{\sigma}{2\sigma' \tau} \left(-z \cdot \nabla \hat{w} + \hat{w}\right)
\]
\[
(7.63) = \sum_{i,j=1}^{n} \left( \delta_{ij} - \frac{\partial_z \hat{w} \partial_{z_i} \hat{w}}{1 + |\hat{\nabla} \hat{w}|^2} \right) \partial^2_{z_i z_j} \hat{w} - \sum_{i=1}^{n} \left( \frac{z_i}{2\sigma' \tau} \hat{w} + \frac{1}{2} + \frac{\sigma}{2\sigma' \tau} \right) \partial_z \hat{w} + \left( \frac{1}{2} + \frac{\sigma}{2\sigma' \tau} \right) \hat{w} - \frac{n-1}{\hat{w}}
\]
and
\[
\partial_t \hat{v}_k = 0 = \sqrt{1 + |\hat{\nabla} \hat{v}_k|^2} \nabla \cdot \frac{\nabla \hat{v}_k}{\sqrt{1 + |\hat{\nabla} \hat{v}_k|^2}} - \frac{n-1}{\hat{v}_k}
\]
\[
(7.64) = \sum_{i,j=1}^{n} \left( \delta_{ij} - \frac{\partial_z \hat{v}_k \partial_{z_i} \hat{v}_k}{1 + |\hat{\nabla} \hat{v}_k|^2} \right) \partial^2_{z_i z_j} \hat{v}_k - \frac{n-1}{\hat{v}_k}
\]
Then we subtract (7.64) from (7.63) to get
\[
\partial_t \left( \hat{w} - \hat{v}_k \right) - \sum_{i,j=1}^{n} \left( \delta_{ij} - \frac{\partial_z \hat{w} \partial_{z_i} \hat{w}}{1 + |\hat{\nabla} \hat{w}|^2} \right) \partial^2_{z_i z_j} \left( \hat{w} - \hat{v}_k \right)
\]
\[
- \sum_{q=1}^{n} \left( \frac{\sum_{i,j=1}^{n} \partial_{z_i z_j} \hat{v}_k \partial_{z_q} \hat{v}_k \partial_z \hat{w} + \partial_{z_q} \hat{v}_k}{1 + |\hat{\nabla} \hat{v}_k|^2} \right) \partial_{z_q} \left( \hat{w} - \hat{v}_k \right)
\]
\[
+ \sum_{q=1}^{n} \left( \frac{\sum_{i,j=1}^{n} \partial^2_{z_i z_j} \hat{v}_k \partial_{z_q} \hat{w} + \partial_{z_q} \hat{v}_k}{1 + |\hat{\nabla} \hat{v}_k|^2} \right) \partial_{z_q} \left( \hat{w} - \hat{v}_k \right)
\]
\[
- \frac{n-1}{\hat{w} \hat{v}_k} + \frac{1}{2} + \frac{\sigma}{2\sigma' \tau} \left(-z \cdot \nabla \hat{v}_k (z) + \hat{v}_k (z)\right) \equiv f (z, \tau)
\]
(7.65)
\[
(7.66) \quad |\nabla^m \partial_t^l f (z, \tau)| \leq C (n, m, l) \tau^{-1}
\]

Note that by (7.66), we have
for \( z \in \mathbb{R}^n, \tau_0 \leq \tau \leq \hat{\tau}, m \geq 0 \).

Now fix \( 0 < \delta \ll 1 \) and \( z_\tau \in B(O; 2\beta), \tau_0 + \delta^2 \leq \tau_* \leq \hat{\tau} \). By (7.61), (7.62) and Krylov-Safonov H"older estimate (applying to (7.65)), there is

\[
\gamma = \gamma (n) \in (0, 1)
\]

so that

\[
(7.67) \quad \delta^\gamma \left[ \hat{w} - \hat{\psi}_k \right]_{\gamma; Q(z_\tau, \tau_*; \frac{1}{4}\delta)} 
\leq C(n) \left( \| \hat{w} - \hat{\psi}_k \|_{L^\infty(Q(z_\tau, \tau_*; \frac{1}{4}\delta))} + \delta^2 \| f \|_{L^\infty(Q(z_\tau, \tau_*; \frac{1}{4}\delta))} \right) \leq C(n)
\]

provided that \( \beta \gg 1 \) (depending on \( n \)) and \( \tau_0 \gg 1 \) (depending on \( n, \beta \)). Next, for each \( p \in \{ 1, \ldots, n \} \), differentiate (7.63) with respect to \( z_p \) to get

\[
\partial_r (\partial_{z_p} \hat{w}) = \Delta (\partial_{z_p} \hat{w}) - \frac{\nabla^2 (\partial_{z_p} \hat{w}) (\nabla \hat{w}, \nabla \hat{w})}{1 + |\nabla \hat{w}|^2}
\]

and

\[
\partial_{z_p} (\partial_{z_p} \hat{w}) = \partial_{z_p} (\partial_{z_p} \hat{w}) - \frac{\nabla^2 (\partial_{z_p} \hat{w}) (\nabla \hat{w}, \nabla \hat{w})}{1 + |\nabla \hat{w}|^2}
\]

Then by (7.61) and Krylov-Safonov H"older estimates, we may assume that for the same exponent \( \gamma \), there holds

\[
(7.68) \quad \delta^{1+\gamma} \left[ |\nabla \hat{w}| \right]_{\gamma; Q(z_\tau, \tau_*; \frac{1}{4}\delta)} \leq C(n) \delta \| \nabla \hat{w} \|_{L^\infty(Q(z_\tau, \tau_*; \frac{1}{4}\delta))} \leq C(n)
\]

Therefore, by (7.61), (7.62), (7.67) and (7.68), we can apply Schauder \( C^2, \gamma \) estimates to (7.65) to get

\[
\delta \left\| \nabla \left( \hat{w} - \hat{\psi}_k \right) \right\|_{L^\infty(Q(z_\tau, \tau_*; \frac{1}{4}\delta))} + \delta^2 \left\| \nabla^2 \left( \hat{w} - \hat{\psi}_k \right) \right\|_{L^\infty(Q(z_\tau, \tau_*; \frac{1}{4}\delta))}
\]

\[
+ \delta^{2+\gamma} \left[ \nabla^2 \left( \hat{w} - \hat{\psi}_k \right) \right]_{\gamma; Q(z_\tau, \tau_*; \frac{1}{4}\delta)} 
\leq C(n) \left( \| \hat{w} - \hat{\psi}_k \|_{L^\infty(Q(z_\tau, \tau_*; \frac{1}{4}\delta))} + \delta^2 \| f \|_{L^\infty(Q(z_\tau, \tau_*; \frac{1}{4}\delta))} + \delta^{2+\gamma} \left[ f \right]_{\gamma; Q(z_\tau, \tau_*; \frac{1}{4}\delta)} \right)
\]

\[
(7.69) \quad \leq C(n) \left( \beta^{\alpha-3} \left( \frac{\tau_*}{\tau_0} \right)^{-\varepsilon} + \tau_*^{-1} \right) \leq C(n) \beta^{2(\alpha-1)} \left( \frac{\tau_*}{\tau_0} \right)^{-\varepsilon}
\]
provided that $\tau_0 \gg 1$ (depending on $n$, $\beta$).

The conclusion follows by using the bootstrap argument on (7.65) and repeatedly differentiating equations with respect to $\tau$. \hfill $\square$

8. Determining the constant $\Lambda$

In this section, we would finish the proof of Proposition 4.4 and Proposition 4.5 What’s left is to show (4.13) and choose $\Lambda = \Lambda (\kappa) \gg 1$ so that (4.12) holds. To this end, it suffices to show that

(1) In the outer region, the function $u (x, t)$ defined in (3.6) satisfies

$$x^i \left| \partial^i_x u (x, t) \right| \leq C (n) x^2 \lambda^2 x^2 + 1 \quad \forall \ i \in \{0, 1, 2\}$$

(8.1)

$$\partial^2_{xx} u (x, t) \geq 0$$

for $\sqrt{-t} \leq x \leq \rho, t_0 \leq t \leq \hat{t}$

(2) In the intermediate region, if we perform the type I rescaling, the type I rescaled function $\bar{v} (y, s)$ defined in (3.11) satisfies

$$y^i \left| \partial^i_y \bar{v} (y, s) \right| \leq C (n) e^{-\lambda \sigma y^\alpha} \quad \forall \ i \in \{0, 1, 2\}$$

(8.3)

$$\partial^2_{yy} \bar{v} (y, s) \geq 0$$

for $2\beta e^{-\sigma s} \leq y \leq 1, s_0 < s \leq \hat{s}$

(3) Near the tip region, if we perform the type II rescaling, the type II rescaled function $\bar{w} (z, \tau)$ defined in (3.24) satisfies

$$z^i \left| \partial^i_z \bar{w} (z, \tau) \right| \leq C (n) z^\alpha \quad \forall \ i \in \{0, 1, 2\}$$

(8.5)

$$\partial^2_{zz} \bar{w} (z, \tau) \geq 0$$

for $0 \leq z \leq 5\beta, \tau_0 \leq \tau \leq \hat{\tau}$.

Note that (8.3) is equivalent to

$$x^i \left| \partial^i_x u (x, t) \right| \leq C (n) (-t)^{\frac{1}{2}} x^\alpha \quad \forall \ i \in \{0, 1, 2\}$$

for $2\beta (-t)^{\frac{1}{2} + \sigma} \leq x \leq \sqrt{-t}, t_0 \leq t \leq \hat{t}$ (see (3.12) and (3.14)). Also, (8.5) is equivalent to

$$x^i \left| \partial^i_x u (x, t) \right| \leq C (n) (-t)^{\frac{1}{2}} x^\alpha \quad \forall \ i \in \{0, 1, 2\}$$

for $\beta (-t)^{\frac{1}{2} + \sigma} \leq x \leq 2\beta (-t)^{\frac{1}{2} + \sigma}, t_0 \leq t \leq \hat{t}$ (see (3.5) and (3.25)). Moreover, by (8.2), (8.3), (8.4) and rescaling, we can show (4.13), i.e. the projected curve $\Sigma_t$ is convex in $B (O; \rho)$ for $t_0 \leq t \leq \hat{t}$.

Recall that in Remark 6.1, we already show the $C^0$ estimates in (8.1) and (8.3).

As for the derivatives, notice that the smooth estimates in Proposition 4.4 does not imply (8.1), (8.3) and (8.4), since those estimates donet not extend to the initial time. Therefore, in this section we compensate that by showing how to estimate the quantities in (8.1), (8.3) and (8.4) from the initial time to some extent. The idea is to derive evolution equations for these quantities and use the following lemma (see Lemma 8.1), together with (4.11), (4.5) and (4.7), to show that they can be bounded in terms of $n$ for a short period of time. Below is the lemma which we would use to prove the derivatives estimates in (8.1) and (8.3).
Lemma 8.1. Suppose that \( h(r, t) \) is a function which satisfies

\[
\partial_r h - a(r, t) \partial^2_{rr} h - b(r, t) \partial_t h = f(r, t)
\]

for \( \frac{1}{2} \leq r \leq \frac{3}{2}, \ 0 \leq t \leq T, \) with

\[
a(r, t) > 0
\]

\[
\max\{|a(r, t)|, |b(r, t)|\} \leq M
\]

for \( \frac{1}{2} \leq r \leq \frac{3}{2}, \ 0 \leq t \leq T, \) where \( T, M > 0 \) are constants. Then there hold

\[
h(r, t) \leq \max_{\frac{1}{2} \leq r \leq \frac{3}{2}} h(r, 0) + C(M) \epsilon \left( \|h\|_{L^\infty([\frac{1}{2}, \frac{3}{2}])} + \|f\|_{L^\infty([\frac{1}{2}, \frac{3}{2}] \times [0, T])} \right)
\]

\[
h(r, t) \geq \min_{\frac{1}{2} \leq r \leq \frac{3}{2}} h(r, 0) - C(M) \epsilon \left( \|h\|_{L^\infty([\frac{1}{2}, \frac{3}{2}])} + \|f\|_{L^\infty([\frac{1}{2}, \frac{3}{2}] \times [0, T])} \right)
\]

for \( \frac{3}{4} \leq r \leq \frac{5}{4}, \ 0 \leq t \leq T. \)

**Proof.** Let \( \eta(r) \) be a smooth function so that

\[
\chi\left[\frac{1}{4}, \frac{1}{2}\right] \leq \eta \leq \chi\left[\frac{1}{2}, \frac{3}{2}\right]
\]

and \( \eta(r) \) vanishes at \( \frac{1}{2} \) and \( \frac{3}{2} \) to infinite order. Note that by Lemma 7.2, we may assume that

\[
\left(\frac{\partial_r \eta(r)}{\eta(r)}\right)^2 + |\partial_t \eta(r)| + \left|\partial^2_{rr} \eta(r)\right| \lesssim 1
\]

It follows that

\[
\partial_r(\eta h) - a(r, t) \partial^2_{rr}(\eta h) - b(r, t) \partial_t(\eta h) = \eta f(r, t) - (a(r, t) \partial^2_{rr} \eta + b(r, t) \partial_t \eta) h - 2 a(r, t) \partial_r \eta \partial_r h
\]

For the last term on RHS of (8.7), if we evaluate it at any maximum point of \( \eta(r) h(r, t) \) for each time-slice, either \( \eta = 0 \) and hence

\[
\partial_r \eta = 0 \quad \Rightarrow \quad -2 a(r, t) \partial_r \eta \partial_r h = 0
\]

or \( 0 < \eta \leq 1 \), in which case we have

\[
\partial_r(\eta h) = 0 \quad \Leftrightarrow \quad \eta \partial_r h + h \partial_r \eta = 0
\]

which yields

\[
-2 a(r, t) \partial_r \eta \partial_r h = 2 a(r, t) \frac{(\partial_r \eta)^2}{\eta} h
\]

Now let

\[
(\eta h)_{\max}(t) = \max_r (\eta(r) h(r, t))
\]

By (8.8) and (8.9), if we apply the maximum principle to (8.7), we get

\[
\partial_r(\eta h)_{\max} \leq C(M) \left( \|h\|_{L^\infty([\frac{1}{2}, \frac{3}{2}] \times [0, T])} + \|f\|_{L^\infty([\frac{1}{2}, \frac{3}{2}] \times [0, T])} \right)
\]

which implies

\[
(\eta h)_{\max}(t) \leq (\eta h)_{\max}(0) + C(M) \epsilon \left( \|h\|_{L^\infty([\frac{1}{2}, \frac{3}{2}] \times [0, T])} + \|f\|_{L^\infty([\frac{1}{2}, \frac{3}{2}] \times [0, T])} \right)
\]

Similarly, if we define

\[
(\eta h)_{\min}(t) = \min_r (\eta(r) h(r, t))
\]

ANALYSIS OF VELÁZQUEZ’S SOLUTION TO THE MCF 73
then we have
\[
(\eta h)_{\min} (t) \geq (\eta h)_{\min} (0) - C (M) t \left( \|h\|_{L^\infty ([\frac{t}{2}, \frac{3}{2}] \times [0, T])} + \|f\|_{L^\infty ([\frac{t}{2}, \frac{3}{2}] \times [0, T])} \right)
\]

□

To prove the derivatives estimates in (8.1), we divide the region into two parts: 
\[
\frac{3}{4} \rho \leq x \leq \rho \quad \text{and} \quad \sqrt{\frac{1}{3}} \rho \leq x \leq \frac{3}{4} \rho.
\]
In the following proposition, we show (8.1) for \(\frac{3}{4} \rho \leq x \leq \rho\) by using (3.7), (4.5), (4.14) and Lemma 8.1.

**Proposition 8.2.** If \(|t_0| \ll 1\) (depending on \(n, \Lambda, \rho, \beta\)), then there hold

\[
(8.10) \quad \frac{1}{2} \mathcal{T}_2 (2 \lambda_2 + 1) x^{2 \lambda_2} \leq \partial_x u (x, t) \leq \frac{3}{2} \mathcal{T}_2 (2 \lambda_2 + 1) x^{2 \lambda_2}
\]

\[
(8.11) \quad \partial_{xx}^2 u (x, t) \leq \frac{3}{2} \mathcal{T}_2 (2 \lambda_2 + 1) (2 \lambda_2) x^{2 \lambda_2 - 1}
\]

\[
(8.12) \quad \partial_{xx}^2 u (x, t) \geq \frac{1}{2} \mathcal{T}_2 (2 \lambda_2 + 1) (2 \lambda_2) x^{2 \lambda_2 - 1} > 0
\]

for \(\frac{3}{4} \rho \leq x \leq \frac{7}{4} \rho\), \(0 \leq t \leq t_0\).

**Proof.** Let
\[
h (r, \iota) = x^{-2 \lambda_2} \partial_x u (x, t) \big|_{x=r \rho, t=t_0+\iota \rho^2}
\]

From (3.7), we derive
\[
\partial_x h - a (r, \iota) \partial_{xx}^2 h - b (r, \iota) \partial_x h = f (r, \iota)
\]
where
\[
a (r, \iota) = \frac{1}{1 + (\partial_x u (x, t))^2} \bigg|_{x=r \rho, t=t_0+\iota \rho^2}
\]

\[
b (r, \iota) = \frac{1}{r} \left( -2x (\partial_x u (x, t)) \partial_{xx}^2 u (x, t) \right) \bigg|_{x=r \rho, t=t_0+\iota \rho^2}
\]

\[
f (r, \iota) = \rho^{-2 \lambda_2+1} \left( \frac{2 \lambda_2}{r^{2 \lambda_2+1}} \right) \partial_{xx}^2 u (x, t) \bigg|_{x=r \rho, t=t_0+\iota \rho^2}
\]

\[
+ \rho^{-2 \lambda_2} \left( \frac{2 \lambda_2}{r^{2 \lambda_2+2}} \right) \left( -2x (\partial_x u (x, t)) \partial_{xx}^2 u (x, t) \right) \bigg|_{x=r \rho, t=t_0+\iota \rho^2}
\]

\[
+ \rho^{-2 \lambda_2} \left( \frac{2 \lambda_2}{r^{2 \lambda_2+2}} \right) \left( -2x (\partial_x u (x, t)) \partial_{xx}^2 u (x, t) \right) \bigg|_{x=r \rho, t=t_0+\iota \rho^2}
\]

\[
+ \rho^{-2 \lambda_2-1} \left( \frac{4 (n-1) \left( \partial_x u (x, t)^2 - 1 \right)}{r^{2 \lambda_2+3}} \right) \left( -2x (\partial_x u (x, t)) \partial_{xx}^2 u (x, t) \right) \bigg|_{x=r \rho, t=t_0+\iota \rho^2}
\]

\[
+ \rho^{-2 \lambda_2-1} \left( \frac{4 (n-1) \left( \partial_x u (x, t)^2 - 1 \right)}{r^{2 \lambda_2+3}} \right) \left( -2x (\partial_x u (x, t)) \partial_{xx}^2 u (x, t) \right) \bigg|_{x=r \rho, t=t_0+\iota \rho^2}
\]
It follows, by \([8.14]\) and Lemma \([8.1]\) that
\[
\min_{\frac{3}{4} \leq r \leq \frac{5}{4}} h(r, 0) - C(n, \rho) t \leq h(r, u) \leq \max_{\frac{3}{4} \leq r \leq \frac{5}{4}} h(r, 0) + C(n, \rho) t
\]
for \(\frac{3}{4} \leq r \leq \frac{5}{4}\). Undoing the change of variables, we get
\[
x^{-2\lambda_2} \partial_x u(x, t) \leq \max_{\frac{3}{4} \rho \leq x \leq \frac{3}{4} \rho} \left( x^{-2\lambda_2} \partial_x u(x, t_0) \right) + C(n, \rho) \frac{t - t_0}{\rho^2}
\]
\[
x^{-2\lambda_2} \partial_x u(x, t) \geq \min_{\frac{3}{4} \rho \leq x \leq \frac{3}{4} \rho} \left( x^{-2\lambda_2} \partial_x u(x, t_0) \right) - C(n, \rho) \frac{t - t_0}{\rho^2}
\]
for \(\frac{3}{4} \rho \leq x \leq \frac{5}{4} \rho, t_0 \leq t \leq \hat{t}\). Therefore, if \(|t_0| \ll 1\) (depending on \(n, \Lambda, \rho, \beta\)), then \((8.10)\) follows immediately from the above, \((4.9)\) and \((6.2)\).

For the second derivative, note that we have the following evolution equation:
\[
\partial_t \left( x^{-2\lambda_2 + 1} \partial^2_{xx} u \right) - \frac{1}{1 + (\partial_x u)^2} \partial_{xx} \left( x^{-2\lambda_2 + 1} \partial^2_{xx} u \right) = \frac{1}{x^{2\lambda_2 + 1}} \left( -6x (\partial_x u) \left( \partial^2_{xx} u \right) + \frac{2(n - 1)}{1 - \left( \frac{u}{x} \right)^2} + \frac{2(2\lambda_2 - 1)}{1 + (\partial_x u)^2} \right) \partial_x \left( x^{-2\lambda_2 + 1} \partial^2_{xx} u \right)
\]
\[
+ \frac{2\lambda_2 - 1}{x^{2\lambda_2 + 1}} \left( -6x (\partial_x u) \left( \partial^2_{xx} u \right) + \frac{2(n - 1)}{1 - \left( \frac{u}{x} \right)^2} + \frac{2(2\lambda_2 - 2)}{1 + (\partial_x u)^2} \right) (\partial^2_{xx} u)
\]
\[
+ \frac{1}{x^{2\lambda_2 + 2}} \left( \frac{4(n - 1) \left( \partial_x u \right)^2 - 1}{1 - \left( \frac{u}{x} \right)^2} \right) (\partial_x u)
\]
\[
+ \frac{1}{x^{2\lambda_2 + 3}} \left( \frac{4(n - 1) \left( \partial_x u \right)^2 - 1}{1 - \left( \frac{u}{x} \right)^2} + \frac{6}{3} \right) (u)
\]

By the same argument (as for the first derivative), we can show \((8.11)\) and \((8.12)\).

Now we show the derivatives estimates in \((8.1)\) for \(\sqrt{-t} \leq x \leq \frac{3}{2} \rho\) by using \((3.7)\), \((3.8)\), \((4.5)\), \((4.18)\) and Lemma \(8.1\).

**Proposition 8.3.** If \(0 < \rho \ll 1\) (depending on \(n, \Lambda\)) and \(|t_0| \ll 1\) (depending on \(n, \Lambda, \rho, \beta\)), then there hold
\[
(8.13) \quad 2(\alpha + 2\gamma_1 (\alpha + 2) + \gamma_2 (2\lambda_2 + 1)) x^{2\lambda_2} \leq \partial_x u(x, t) \leq 2\gamma_2 (2\lambda_2 + 1) x^{2\lambda_2}
\]
\[
(8.14) \quad \partial^2_{xx} u(x, t) \leq 2(\alpha - 1) + 2\gamma_1 (\alpha + 2) (\alpha + 1) + \gamma_2 (2\lambda_2 + 1) (2\lambda_2) x^{2\lambda_2 - 1}
\]
(8.15) \[ \partial_{xx}^2 u(x, t) \geq \frac{1}{2} T_2 (2\lambda_2 + 1) (2\lambda_2) x^{2\lambda_2 - 1} > 0 \]

for \( \sqrt{-1} \leq x \leq \frac{2}{3} \rho, \ t_0 \leq t \leq t \).

\textbf{Proof.} First, fix \( x_* \in \left[ \frac{2}{3} \sqrt{-t_0}, \frac{2}{3} \rho \right] \) and let

\[ h(r, \iota) = x^{-2\lambda_2} \partial_x u(x, t) \big|_{x = r \iota, t = t_0 + ix_*^2} \]

From (3.7), we derive

\[ \partial_r h - a(r, \iota) \partial_{xx}^2 h - b(r, \iota) \partial_r h = f(r, \iota) \]

where

\[ a(r, \iota) = \frac{1}{1 + (\partial_x u(x, t))^2} \bigg|_{x = r \iota, t = t_0 + ix_*^2} \]

\[ b(r, \iota) = \frac{1}{r^2} \left( \frac{-2 \partial_x u(x, t) \left( x \partial_{xx}^2 u(x, t) \right)}{1 + (\partial_x u(x, t))^2} + \frac{2(n - 1)}{1 - \left( \frac{u(x, t)}{x} \right)^2} \right) \bigg|_{x = r \iota, t = t_0 + ix_*^2} \]

\[ f(r, \iota) = \frac{1}{r^2} \left( \left( \frac{2\lambda_2}{1 + (\partial_x u(x, t))^2} \right) \left( x^{-2\lambda_2 + 1} \partial_{xx}^2 u(x, t) \right) \right) \bigg|_{x = r \iota, t = t_0 + ix_*^2} + \frac{1}{r^2} \left( \frac{2\lambda_2}{1 + (\partial_x u(x, t))^2} \right) \left( x^{-2\lambda_2} \partial_x u(x, t) \right) \bigg|_{x = r \iota, t = t_0 + ix_*^2} \]

\[ + \frac{1}{r^2} \left( \frac{4(n - 1) \left( (\partial_x u(x, t))^2 - 1 \right)}{1 - \left( \frac{u(x, t)}{x} \right)^2} \right) \left( x^{-2\lambda_2 - 1} u(x, t) \right) \bigg|_{x = r \iota, t = t_0 + ix_*^2} \]

Notice that by (3.3) we have

\[ \max \left\{ \left| \frac{u(x, t)}{x} \right|, \left| \partial_x u(x, t) \right|, \left| x \partial_{xx}^2 u(x, t) \right| \right\} \leq C(n, \Lambda) x^{2\lambda_2} \leq \frac{1}{3} \]

\[ x^{-2\lambda_2 - 1} + i \left| \partial_{xx}^2 u(x, t) \right| \leq C(n, \Lambda), \quad i \in \{ 0, 1, 2 \} \]

for \( \frac{1}{3} \sqrt{-t} \leq x \leq \rho, \ t_0 \leq t \leq t \), provided that \( 0 < \rho \ll 1 \) (depending on \( n, \Lambda \)). It follows, by Lemma 8.1, that

\[ \frac{1}{3} \sqrt{-t} \leq x \leq \rho, \ t_0 \leq t \leq t \]

which implies

\[ x_*^{-2\lambda_2} \partial_x u(x_*, t) \leq \max_{\frac{1}{3} \sqrt{-t_0} \leq x \leq \rho} \left( x^{-2\lambda_2} \partial_x u(x, t_0) \right) + C(n, \Lambda) \frac{t - t_0}{\rho^2} \]
\[ x_t \geq \frac{\min_{t_0 \leq x \leq \rho} (x_{-\lambda t^2}^2) \partial_x u (x, t)}{t - t_0} \]

for \( t_0 \leq t \leq t_0 + \delta^2 x^2 \). Thus, by (4.3) and (6.2), we can choose \( 0 < \delta \ll 1 \) (depending on \( n, \Lambda \)) so that

\[ (8.16) \quad 2 (\alpha + 2 \Gamma_1 (\alpha + 2) + \Gamma_2 (2 \lambda_2 + 1)) x^{2 \lambda_2} \leq \partial_x u (x, t) \leq 2 \Gamma_2 (2 \lambda_2 + 1) x^{2 \lambda_2} \]

for \( (x, t) \) satisfying \( \sqrt{-t} \leq x \leq \frac{\delta}{\sqrt{2}} \rho, t_0 \leq t \leq t_0 + \delta^2 x^2 \), provided that \( |t_0| \ll 1 \) (depending on \( n, \Lambda, \rho, \beta \)).

On the other hand, by this choice of \( \delta = \delta (n, \Lambda) \), (4.18) implies

\[ \left| \partial_x u (x, t) - \frac{k}{c_2} (-t)^{\lambda_2 + \frac{1}{2}} \varphi_2 \left( \frac{x}{\sqrt{-t}} \right) \right| \leq C (n, \Lambda) \rho^{4 \lambda_2} x^{2 \lambda_2} \]

for \( (x, t) \) satisfying \( \sqrt{-t} \leq x \leq \frac{3}{2} \rho, t_0 \leq t \leq t_0 + \delta^2 x^2 \leq \tilde{t} \), where

\[ \partial_x \left( \frac{k}{c_2} (-t)^{\lambda_2 + \frac{1}{2}} \varphi_2 \left( \frac{x}{\sqrt{-t}} \right) \right) = k x^{2 \lambda_2} \left[ \Gamma_2 (2 \lambda_2 + 1) + 2 \Gamma_1 (\alpha + 2) \left( \frac{-t}{x^2} \right) + \alpha \left( \frac{-t}{x^2} \right)^2 \right] \]

It follows, by (3.3), that

\[ (8.17) \quad 2 (\alpha + 2 \Gamma_1 (\alpha + 2) + \Gamma_2 (2 \lambda_2 + 1)) x^{2 \lambda_2} \leq \partial_x u (x, t) \leq 2 \Gamma_2 (2 \lambda_2 + 1) x^{2 \lambda_2} \]

for \( (x, t) \) satisfying \( \sqrt{-t} \leq x \leq \frac{3}{4} \rho, t_0 \leq t \leq t_0 + \delta^2 x^2 \), provided that \( |t_0| \ll 1 \) (depending on \( n, \Lambda, \rho, \beta \)). Then (8.17) follows immediately from (8.10) and (8.17).

As for the second derivatives, we have the evolution equation:

\[ \partial_t \left( x^{-2 \lambda_2 + 1} \partial_{xx}^2 u \right) - \frac{1}{1 + (\partial_x u)^2} \partial_{xx}^2 \left( x^{-2 \lambda_2 + 1} \partial_{xx}^2 u \right) \]

\[ = \frac{1}{x^2} \left[ \frac{-2 (2 \lambda_2 - 1) \partial_x u \left( \frac{1}{1 + (\partial_x u)^2} \right) + 2 (n - 1) \left( \frac{1}{1 + (\partial_x u)^2} \right)}{1 + (\partial_x u)^2} \right] \partial_x \left( x^{-2 \lambda_2 + 1} \partial_{xx}^2 u \right) \]

\[ + \frac{1}{x^2} \left[ \frac{4 (n - 1) \left( \frac{1}{1 + (\partial_x u)^2} \right)}{1 - \left( \frac{1}{2} \right)^2} \right] \partial_{xx} \left( x^{-2 \lambda_2} \partial_x u \right) \]

By a similar argument, we can deduce (8.14) and (8.15). \( \square \)

In the following proposition, we prove (8.3) by using (3.13), (3.14), (4.1), (4.19), (4.20) and Lemma 8.1.
Proposition 8.4. If $\beta \gg 1$ (depending on $n, \Lambda$) and $s_0 \gg 1$ (depending on $n, \Lambda$, $\rho$, $\beta$), then there hold

\begin{equation}
2(\alpha + 8T_1(\alpha + 2) + 16T_2(2\lambda_2 + 1)) e^{-\lambda_2 s y^\alpha - 1} \leq \partial_y v (y, s) \leq \frac{1}{2} e^{-\lambda_2 s y^\alpha - 1}
\end{equation}

\begin{equation}
\partial^2_{yy} v (y, s) \leq 2(\alpha - 1 + 8T_1(\alpha + 2)(\alpha + 1) + 16T_2(2\lambda_2 + 1)(2\lambda_2)) e^{-\lambda_2 s y^\alpha - 2}
\end{equation}

\begin{equation}
\partial^2_{yy} v (y, s) \geq \frac{1}{2} (\alpha (\alpha - 1)) e^{-\lambda_2 s y^\alpha - 2} > 0
\end{equation}

for $2\beta e^{-\sigma s} \leq y \leq 1$, $s_0 < s \leq \hat{s}$.

Proof. Firstly, for each $y_* \in \left[\frac{5}{4} \beta e^{-\sigma s_0}, 1\right]$, let

\[ h (r, i) = e^{\lambda_2 s y - \alpha + 1} \partial_y v (y, s) \Big|_{y = ry_*, s = s_0 + iy_*} \]

From \[3.43\], we derive

\[ \partial_i h - a (r, i) \partial^2_{rr} h - b (r, i) \partial_i h = f (r, i) \]

where

\[ a (r, i) = \frac{1}{1 + (\partial_y v (y, s))^2} \bigg|_{y = ry_*, s = s_0 + iy_*} \]

\[ b (r, i) = \frac{1}{r^2} \left( -2 (\partial_y v (y, s)) \left( y \partial^2_{yy} v (y, s) \right) \left( 1 + (\partial_y v (y, s))^2 \right)^2 \right) \bigg|_{y = ry_*, s = s_0 + iy_*} \]

\[ f (r, i) = \frac{1}{r^2} \left( -2 (\partial_y v (y, s)) \left( y \partial^2_{yy} v (y, s) \right) \left( 1 + (\partial_y v (y, s))^2 \right)^2 \right) \bigg|_{y = ry_*, s = s_0 + iy_*} \]

\[ + \frac{\alpha - 1}{r^2} \left( -2 (\partial_y v (y, s)) \left( y \partial^2_{yy} v (y, s) \right) \left( 1 + (\partial_y v (y, s))^2 \right)^2 \right) \bigg|_{y = ry_*, s = s_0 + iy_*} \]

\[ + \frac{1}{r^2} \left( -2 (\partial_y v (y, s)) \left( y \partial^2_{yy} v (y, s) \right) \left( 1 + (\partial_y v (y, s))^2 \right)^2 \right) \bigg|_{y = ry_*, s = s_0 + iy_*} \]

\[ + \frac{4 (n - 1) \left( \partial_y v (y, s) \right)^2 - 1}{1 - \left( \frac{v (y, s)}{y} \right)^2} e^{\lambda_2 s y - \alpha + 1} \partial_y v (y, s) \bigg|_{y = ry_*, s = s_0 + iy_*} \]

Notice that by \[3.43\] we have

\[ \max \left\{ \left| \frac{v (y, s)}{y} \right|, \left| \partial_y v (y, s) \right|, \left| y \partial^2_{yy} v (y, s) \right| \right\} \leq C (n, \Lambda) e^{-\lambda_2 s y^\alpha - 1} \leq \frac{1}{3} \]

\[ e^{\lambda_2 s y - \alpha + i} \left| \partial_y^i v (y, s) \right| \leq C (n, \Lambda) \quad \forall \ i \in \{0, 1, 2\} \]
for $\frac{3}{2}\beta e^{-\sigma s} \leq y \leq 2$, $s_0 \leq s \leq \dot{s}$, provided that $\beta \gg 1$ (depending on $n, \Lambda$). Then by Lemma 8.11 and 8.14, we get

$$\min_{\frac{t}{2} \leq \tau \leq \frac{t}{1}} h(r, 0) - C(n, \Lambda) \leq h(r, \tau) \leq \max_{\frac{t}{2} \leq \tau \leq \frac{t}{1}} h(r, 0) + C(n, \Lambda) \tau$$

which implies

$$e^{\lambda_2 s} y^{\alpha + 1} \partial_y v(y, s) \leq \max_{\beta e^{-\sigma y} \leq y \leq 2} \left( e^{\lambda_2 s_0} y^{\alpha + 1} \partial_y v(y, s_0) \right) + C(n, \Lambda) \frac{s - s_0}{y^2}$$

$$e^{\lambda_2 s} y^{\alpha + 1} \partial_y v(y, s) \geq \min_{\beta e^{-\sigma y} \leq y \leq 2} \left( e^{\lambda_2 s_0} y^{\alpha + 1} \partial_y v(y, s_0) \right) - C(n, \Lambda) \frac{s - s_0}{y^2}$$

for $s_0 \leq s \leq s_0 + \delta^2 y^2$. It follows, by (4.1) and (6.2), that we can choose $0 < \delta \ll 1$ (depending on $n, \Lambda$) so that

$$(8.21)$$

$$2(\alpha + 8 \gamma_1 (\alpha + 2) + 16 \gamma_2 (2\lambda_2 + 1)) e^{-\lambda_2 s} y^{\alpha - 1} \leq \partial_y v(y, s) \leq \frac{1}{2} \alpha e^{-\lambda_2 s} y^{\alpha - 1}$$

for $(y, s)$ satisfying $2\beta e^{-\sigma s} \leq y \leq 1$, $s_0 \leq s \leq s_0 + \delta^2 y^2$, provided that $s_0 \gg 1$ (depending on $n, \Lambda, \rho, \beta$).

On the other hand, by the above choice of $\delta = \delta(n, \Lambda)$, (4.19) and (4.20) yield

$$\left| \partial_y \left( v(y, s) - \frac{k}{c_2} e^{-\lambda_2 s} \varphi_2(y) \right) \right| \leq C(n, \Lambda) e^{-\kappa s} (e^{-\lambda_2 s} y^{\alpha + 1})$$

for $(y, s)$ satisfying $e^{-\sigma s} \leq y \leq 1$, $s_0 + \delta^2 y^2 \leq s \leq \dot{s}$, and

$$\left| \partial_y \left( v(y, s) - e^{-\sigma s} \psi_k(e^{\sigma s} y) \right) \right| \leq C(n, \Lambda) \beta^{\kappa y} e^{-2\sigma(e^{\sigma s} y - s_0)} (e^{-\lambda_2 s} y^{\alpha - 1})$$

for $(y, s)$ satisfying $2\beta e^{-\sigma s} \leq y \leq e^{-\sigma s}$, $s_0 + \delta^2 y^2 \leq s \leq \dot{s}$. Note that

$$\partial_y \left( \frac{k}{c_2} e^{-\lambda_2 s} \varphi_2(y) \right) = ke^{-\lambda_2 s} y^{\alpha - 1} (\alpha + 2 \gamma_1 (\alpha + 2) y^2 + \gamma_2 (2\lambda_2 + 1) y^4)$$

$$\partial_y \left( e^{-\sigma s} \psi_k(e^{\sigma s} y) \right) = ke^{-\lambda_2 s} y^{\alpha - 1} (\alpha + O((e^{\sigma s} y)^{-2(1 - \sigma)})$$

It follows, by (6.23), that

$$(8.22)$$

$$2(\alpha + 8 \gamma_1 (\alpha + 2) + 16 \gamma_2 (2\lambda_2 + 1)) e^{-\lambda_2 s} y^{\alpha - 1} \leq \partial_y v(y, s) \leq \frac{1}{2} \alpha e^{-\lambda_2 s} y^{\alpha - 1}$$

for $(y, s)$ satisfying $2\beta e^{-\sigma s} \leq y \leq 1$, $s_0 + \delta^2 y^2 \leq s \leq \dot{s}$, provided that $\beta \gg 1$ (depending on $n, \Lambda$) and $s_0 \gg 1$ (depending on $n, \Lambda$). Then (8.18) follows from (8.21) and (8.22).

As for the second derivative, we derive the following evolution equation:

$$\partial_s \left( e^{\lambda_2 s} y^{-\alpha + 2} \partial_y^2 \right) - \frac{1}{1 + (\partial_y)^2} \partial_y^2 \left( e^{\lambda_2 s} y^{-\alpha + 2} \partial_y^2 v \right)$$

$$- \frac{1}{y} \left( -6 (\partial_y v) (\partial_y^2 v) + 2 (n - 1) - \frac{y^2}{2} + \frac{y^2 + 2 (\alpha - 2)}{1 + (\partial_y v)^2} \right) \partial_y \left( e^{\lambda_2 s} y^{-\alpha + 2} \partial_y^2 v \right)$$

$$= \frac{1}{y^2} \left( -2 \left( \partial_y^2 v \right)^2 (1 - 3 (\partial_y v)^2) - \frac{y^2}{2} + \lambda_2 y^2 \right) \left( e^{\lambda_2 s} y^{-\alpha + 2} \partial_y^2 v \right)$$
for the projected curve $\tilde{\Gamma}$. More explicitly, we have
\begin{equation}
\frac{2(n-1)}{y^2} \left( \frac{4 \left( \frac{x}{y} \right) \partial y v - 1 - \left( \frac{x}{y} \right)^2}{1 - \left( \frac{x}{y} \right)^2} \right) \left( e^{\lambda_2 \rho y^{-\alpha+2} \partial y^2 v} \right)
+ \frac{\alpha - 2}{y^2} \left( \frac{-6 (\partial y v) (y \partial y^2 v)}{1 + (\partial y v)^2} + \frac{2(n-1)}{1 - \left( \frac{x}{y} \right)^2} - \frac{\alpha - 3}{1 + (\partial y v)^2} \right) \left( e^{\lambda_2 \rho y^{-\alpha+2} \partial y^2 v} \right)
+ \frac{1}{y^2} \left( \frac{4(n-1) \left( \frac{x}{y} \right) (y \partial y^2 v)}{1 - \left( \frac{x}{y} \right)^2} - \frac{4(n-1) (1 - (\partial y v)^2) \left( 3 + \left( \frac{x}{y} \right)^2 \right)}{1 - \left( \frac{x}{y} \right)^2} \right) \left( e^{\lambda_2 \rho y^{-\alpha+1} \partial y v} \right)
+ \frac{1}{y^2} \left( \frac{4(n-1) \left( \frac{x}{y} \right) \left( 3 + \left( \frac{x}{y} \right)^2 \right)}{1 - \left( \frac{x}{y} \right)^2} \right) \left( e^{\lambda_2 \rho y^{-\alpha} v} \right)
\end{equation}

Using the same argument as for the first derivative, (8.19) and (8.20) can be proved. \( \square \)

Note that by (3.24) and (8.3), we get
\begin{equation}
z^i \left| \partial_i \omega (z, \tau) \right| \leq C (n) z^\alpha \quad \forall \ i \in \{0, 1, 2\}
\end{equation}
for $2 \beta \leq z \leq \sqrt{2} \sigma \tau$, $\tau_0 < \tau \leq \hat{\tau}$. Also, by (8.12), (8.15), (8.20) and rescaling, the projected curve $\tilde{\Gamma}^\tau$ (see (3.25)) is convex in the corresponding rescaled region. More explicitly, we have
\begin{equation}
\partial_{zz}^2 \tilde{\omega} (z, \tau) \geq 0
\end{equation}
for $3 \beta \leq z \leq \rho (2 \sigma \tau)^{\frac{\beta}{\rho} + \frac{1}{\rho}}$, $\tau_0 \leq \tau \leq \hat{\tau}$. Below we prove (8.6) by using (2.4), (3.22), (8.8), (7.16) and (8.24).

Lemma 8.5. If $\beta \gg 1$ (depending on $n$, $\Lambda$) and $\tau_0 \gg 1$ (depending on $n$, $\Lambda$, $\rho$, $\beta$), there holds (8.6).

Proof. From (3.22), we deduce that
\begin{equation}
\partial_z \left( \partial_{zz}^2 \tilde{\omega} \right) = \frac{1}{1 + (\partial_z \tilde{\omega})^2} \partial_{zz}^2 \left( \partial_{zz} \tilde{\omega} \right)
+ \left( \frac{n-1}{z} - \frac{6 (\partial_z \tilde{\omega}) (\partial_{zz} \tilde{\omega})}{1 + (\partial_z \tilde{\omega})^2} \right) \partial_z \left( \partial_{zz}^2 \tilde{\omega} \right) - \frac{2 - 6 (\partial_z \tilde{\omega})^2}{1 + (\partial_z \tilde{\omega})^2} \left( \partial_{zz}^2 \tilde{\omega} \right)^3
+ \left( \frac{n-1}{z^3} - \frac{2}{z^2} \right) \left( \partial_{zz}^2 \tilde{\omega} \right) + 2 (n-1) \left( \frac{1}{z^3} - \frac{\partial_z \tilde{\omega}}{w^2} \right) \partial_z \tilde{\omega}
\end{equation}
Notice that the last term on the RHS is positive, i.e.
\begin{equation}
2 (n-1) \left( \frac{1}{z^3} - \frac{\partial_z \tilde{\omega} (z, \tau)}{w^2 (z, \tau)} \right) \partial_z \tilde{\omega} (z, \tau) > 0
\end{equation}
for $0 \leq z \leq 5\beta$, $\tau_0 \leq \tau \leq \tilde{\tau}$, since by [2.4], [6.3], [6.8] and (7.47), we have

$$
\left( \frac{\hat{\omega} (z, \tau)}{z} \right)^3 \geq \left( \frac{\hat{\psi}_{1-2\beta^\alpha - 3} (z)}{z} \right)^3 \geq \left( 1 + \frac{\alpha + 1}{2} (1 - 2\beta^{\alpha - 3}) (5\beta)^{\alpha - 1} \right)^3
$$

(8.27)

$$
> 1 + \beta^{\alpha - 2} \geq \partial_z \hat{\omega} (z, \tau)
$$

for $0 \leq z \leq 5\beta$, $\tau_0 \leq \tau \leq \tilde{\tau}$, provided that $\beta \gg 1$ (depending on $n$, $\Lambda$) and $\tau_0 \gg 1$ (depending on $n$, $\Lambda$, $\rho$, $\beta$).

Now let

$$
(\partial^2_{zz} \hat{\omega})_{\min} (\tau) = \min_{0 \leq z \leq 5\beta} \partial^2_{zz} \hat{\omega} (z, \tau)
$$

Note that by (4.3) we have

$$(\partial^2_{zz} \hat{\omega})_{\min} (\tau_0) > 0$$

Now we would like to prove

$$(\partial^2_{zz} \hat{\omega})_{\min} (\tau) \geq 0$$

for $\tau_0 \leq \tau \leq \tilde{\tau}$ by contradiction. Suppose that $(\partial^2_{zz} \hat{\omega})_{\min} (\tau)$ fails to be non-negative for all $\tau_0 \leq \tau \leq \tilde{\tau}$, there must be $\tau^*_1 > \tau_0$ so that

$$(\partial^2_{zz} \hat{\omega})_{\min} (\tau^*_1) < 0$$

Let $\tau^*_0 \geq \tau_0$ be the first time after which $(\partial^2_{zz} \hat{\omega})_{\min}$ is negative all the way up to $\tau^*_1$. By continuity, we have

$$(\partial^2_{zz} \hat{\omega})_{\min} (\tau^*_0) \geq 0$$

On the other hand, by (7.47) and (8.24), there hold

$$
\partial^2_{zz} \hat{\omega} (0, \tau) = \lim_{z \searrow 0} \partial_z \hat{\omega} (z, \tau) \geq 0
$$

$$
\partial^2_{zz} \hat{\omega} (5\beta, \tau) > 0
$$

for $\tau_0 \leq \tau \leq \tilde{\tau}$. As a result, the negative minimum of $\partial^2_{zz} \hat{\omega} (z, \tau)$ for each time-slice must be achieved in $(0, 5\beta)$. Then by the maximum principle (applying to (8.25)), (6.8), (8.26) and (8.27), we get

$$
\partial_\tau (\partial^2_{zz} \hat{\omega})_{\min} \geq \left( -\frac{2 - 6 (\partial_z \hat{\omega})^2}{1 + (\partial_z \hat{\omega})^2} (\partial^2_{zz} \hat{\omega})_{\min}^2 + \left( n - 1 \right) \left( \frac{1}{\hat{\omega}^2} - \frac{2}{\hat{\omega}^2} \right) \right) (\partial^2_{zz} \hat{\omega})_{\min}
$$

$$
\geq 6 (\partial_z \hat{\omega})^2 (\partial^2_{zz} \hat{\omega})_{\min}^2 \geq 6 (1 + \beta^{\alpha - 2})^2 (\partial^2_{zz} \hat{\omega})_{\min}^3
$$

for $\tau^*_0 < \tau \leq \tau^*_1$. It follows that $(\partial^2_{zz} \hat{\omega})_{\min} (\tau^*_0) < 0$, which is a contradiction. □

Recall that by the admissible conditions (see Section 3), the projected curve $\tilde{\Gamma}_\tau$ (see (8.23)) is a graph over $\tilde{\mathcal{C}}$ outside $B (O; \beta)$. By (8.6) and also the admissible conditions, we also know that inside $B (O; \beta)$, $\tilde{\Gamma}_\tau$ is a convex curve which intersects orthogonally with the vertical ray $\{ (0, z) | z > 0 \}$, i.e. $\partial_z \hat{\omega} (0, \tau) = 0$. Furthermore, by (2.4) and (6.8), $\tilde{\Gamma}_\tau$ lies above $\tilde{\mathcal{C}}$ and tends to it as $z \searrow \beta$. Therefore, we conclude that $\tilde{\Gamma}_\tau$ is “entirely” a graph over $\tilde{\mathcal{C}}$ and

(8.28)

$$
\tilde{\Gamma}_\tau = \{ (z, \hat{\omega} (z, \tau)) | z \geq 0 \}
$$

$$
= \left\{ \left( (z - w (z, \tau)) \frac{1}{\sqrt{2}}, (z + w (z, \tau)) \frac{1}{\sqrt{2}} \right) | z \geq \frac{\hat{\omega} (0, \tau)}{\sqrt{2}} \right\}
$$
Remark 8.6. For the admissible conditions in Section 3, we only require the function \( w (z, \tau) \) (see (8.24)) is defined for \( z \gtrsim \beta \). However, by the convexity (see (8.6)) and the above argument, we find the domain of definition for \( w (z, \tau) \) is given by

\[
\frac{\hat{w} (0, \tau)}{\sqrt{2}} \leq z < \infty
\]

On the other hand, by (6.8), inside \( \hat{w} (0, \tau) \) we may assume that inside \( B(\hat{O}; 5\beta) \), \( \hat{\Gamma} \) is bounded between \( \hat{M}_1 \) and \( \hat{M}_2 \), provided that \( \beta \gg 1 \) (depending on \( n \)) and \( \tau_0 \gg 1 \) (depending on \( n, \Lambda, \rho, \beta \)). In particular, we have

\[
\sup_{\tau_0 \leq \tau \leq \hat{\tau}} \frac{\hat{w} (0, \tau)}{\sqrt{2}} < \frac{\hat{w}_2 (0)}{\sqrt{2}}
\]

which means \( w (z, \tau) \) is defined for \( z \geq \frac{\hat{w}_2 (0)}{\sqrt{2}}, \tau_0 \leq \tau \leq \hat{\tau} \). In addition, since \( \hat{\Gamma} \) is a convex curve which lies below \( \hat{M}_2 \) and tends to \( \hat{C} \), we deduce that

\[
(8.29) \quad 0 \leq w (z, \tau) \leq \hat{\psi}_2 (z) \leq \frac{\hat{\psi}_2 (0)}{\sqrt{2}} z
\]

for \( \frac{\hat{w}_2 (0)}{\sqrt{2}} \leq z \leq 5\beta, \tau_0 \leq \tau \leq \hat{\tau} \). Note that the slope of the linear function on the RHS satisfies

\[
0 < \frac{\hat{\psi}_2 (0)}{\sqrt{2}} \leq \hat{w}_2 (0) \leq \frac{\hat{\psi}_2 (0)}{\sqrt{2}} = 1
\]

Lastly, in order to prove (8.30), we need the following two lemmas, which provide smooth estimates of the function \( w (z, \tau) \) in the rescaled tip region.

Lemma 8.7. If \( \beta \gg 1 \) (depending on \( n, \Lambda \)) and \( \tau_0 \gg 1 \) (depending on \( n, \Lambda, \rho, \beta \)), there holds

\[
(8.30)
\begin{align*}
|w (z, \tau) - \psi_k (z)| & \leq C (n \beta^\alpha - 3 \left( \frac{\tau}{\tau_0} \right)^{-\theta} \\
-1 & \leq \partial_z w (z, \tau) \leq \frac{1}{3} \\
0 & \leq \partial_{zz}^2 w (z, \tau) \leq C (n)
\end{align*}
\]

for \( \frac{\hat{w}_2 (0)}{\sqrt{2}} \leq z \leq 3\beta, \tau_0 \leq \tau \leq \hat{\tau} \).

Proof. By (8.24), inside \( B (O; 5\beta) \), the projected curve \( \hat{\Gamma} \) is bounded between \( \hat{M}_1 (1 - \beta^3 (\frac{z}{\tau_0})^{-\theta})^k \) and \( \hat{M}_2 (1 - \beta^3 (\frac{z}{\tau_0})^{-\theta})^k \), which implies

\[
\psi \left( 1 - \beta^3 (\frac{z}{\tau_0})^{-\theta} \right)^k (z) \leq w (z, \tau) \leq \psi \left( 1 + \beta^3 (\frac{z}{\tau_0})^{-\theta} \right)^k (z)
\]

for \( \frac{\hat{w}_2 (0)}{\sqrt{2}} \leq z \leq 3\beta, \tau_0 \leq \tau \leq \hat{\tau} \). Then by (2.9), (3.3) and using a similar argument as in the proof of Proposition 6.6, we can derive the \( C^0 \) estimate of (8.30).

As for the first derivative, note that by (4.27), (8.0), (8.24) and the admissible conditions in Section 3, \( \hat{\Gamma} \) is a convex curve which intersects orthogonally with the
vertical ray \( \{ (0, z) \mid z > 0 \} \). Thus, we have
\[
\partial^2_z w (z, \tau) \geq 0
\]
(8.31)
\[
\partial_z w (z, \tau) \geq \partial_z w \left( \frac{\hat{\psi}(0, \tau)}{\sqrt{2}}, \tau \right) = -1
\]
\[
\partial_z w (z, \tau) \leq \partial_z w (3 \beta, \tau) \leq C (n, \Lambda) \beta^{\alpha - 1} \leq 1
\]
for \( \frac{\hat{\psi}(0)}{\sqrt{2}} \leq z \leq 3 \beta, \tau_0 \leq \tau \leq \bar{\tau} \), provided that \( \beta \gg 1 \) (depending on \( n, \Lambda \)).

Lastly, for the second derivative, notice that by (4.16), the normal curvature of \( \hat{\Gamma}_\tau \) (in terms of \( \hat{w} (z, \tau) \)) satisfies
\[
\left| A_{\hat{\Gamma}_\tau} \right| = \frac{|\partial^2_z \hat{w} (z, \tau)|}{\left(1 + (\partial_z \hat{w} (z, \tau))^2\right)^{\frac{3}{2}}} \leq C (n)
\]
(8.32)
for \( 0 \leq z \leq 3 \beta, \tau_0 \leq \tau \leq \bar{\tau} \). Now if we reparametrize \( \hat{\Gamma}_\tau \) by means of \( w (z, \tau) \), the normal curvature is then given by
\[
A_{\hat{\Gamma}_\tau} = \frac{\partial^2_z w (z, \tau)}{\left(1 + (\partial_z w (z, \tau))^2\right)^{\frac{3}{2}}}
\]
(8.33)

The following lemma can be regarded as a counterpart of Proposition 7.11.

**Lemma 8.8.** If \( \beta \gg 1 \) (depending on \( n, \Lambda \)) and \( |\tau_0| \gg 1 \) (depending on \( n, \Lambda, \rho, \beta \)), then for any \( 0 < \delta \ll 1 \), \( m, l \in \mathbb{Z}_+ \), there holds
(8.34)
\[
\delta^{m+2l} \left| \partial_z^m \partial_{\tau}^l (w (z, \tau) - \psi_k (z)) \right| \leq C (n, m, l) \beta^{\alpha - 3} \left( \frac{\tau}{\tau_0} \right)^{-\delta}
\]
for \( (z, \tau) \) satisfying \( \hat{\psi}_2 (0) \leq z \leq 2 \beta, \tau_0 + \delta^2 \leq \tau \leq \bar{\tau} \).

**Proof.** By mimicking the proof of Proposition 7.12 and using (2.9), (3.26), (8.29), (8.30) and Lemma 2.3, we can deduce (8.34).

Below we show that the \( C^0 \) estimate of (8.31) follows directly from the \( C^0 \) estimate of (8.30).

**Proposition 8.9.** If \( \beta \gg 1 \) (depending on \( n \)) and \( \tau_0 \gg 1 \) (depending on \( n, \Lambda, \rho, \beta \)), there holds
(8.35)
\[
|w (z, \tau)| \leq C (n) z^\alpha
\]
for \( 2 \hat{\psi}_2 (0) \leq z \leq 2 \beta, \tau_0 \leq \tau \leq \bar{\tau} \).

**Proof.** By Lemma 2.3 (6.3) and (8.30), we have
\[
z^{-\alpha} |w (z, \tau)| \leq z^{-\alpha} |\psi_k (z)| + z^{-\alpha} |w (z, \tau) - \psi_k (z)|
\]
\[
\leq z^{-\alpha} |\psi_k (z)| + (2 \beta)^{-\alpha} |w (z, \tau) - \psi_k (z)|
\]
\[
\leq C (n) \left( 1 + \beta^{-3} \left( \frac{\tau}{\tau_0} \right)^{-\delta} \right) \leq C (n)
\]
for \( 2 \hat{\psi}_2 (0) \leq z \leq 2 \beta, \tau_0 \leq \tau \leq \bar{\tau} \), provided that \( \beta \gg 1 \) (depending on \( n \)).
In the following proposition, we show the first derivative estimate of (8.5) by using the maximum principle and (8.33).

**Proposition 8.10.** If $\beta \gg 1$ (depending on $n$) and $\tau_0 \gg 1$ (depending on $n$, $\Lambda$, $\rho$, $\beta$), there holds

(8.36) \[ |\partial_z w (z, \tau)| \leq C (n) z^{\alpha-1} \]

for $2 \psi_2 (0) \leq z \leq 2\beta$, $\tau_0 \leq \tau \leq \hat{\tau}$.

**Proof.** From (8.26), we derive

(8.37) \[
\partial_\tau \left( z^{-\alpha+1} \partial_z w \right) - \frac{1}{1 + (\partial_z w)^2} \partial_{zz} \left( z^{-\alpha+1} \partial_z w \right)
\]

\[
- \left( -2 \frac{\partial_z w \partial^2_{zz} w}{(1 + (\partial_z w)^2)^2} + \frac{2(n - 1)}{z \left( 1 - \left( \frac{w}{z} \right)^2 \right)} - \left( \frac{1}{2} + \sigma \right) \frac{z}{2\sigma\tau} \right) \partial_z \left( z^{-\alpha+1} \partial_z w \right)
\]

\[
= z^{-\alpha} \left( \frac{2(n - 1)}{1 + (\partial_z w)^2} \partial^2_{zz} w - \frac{4(n - 1)}{z^2 \left( 1 - \left( \frac{w}{z} \right)^2 \right)^2} w \right)
\]

\[
+ (\alpha - 1) z^{-\alpha} \left( \frac{-2(\partial_z w) \partial^2_{zz} w}{(1 + (\partial_z w)^2)^2} + \frac{2(n - 1)}{z \left( 1 - \left( \frac{w}{z} \right)^2 \right)} - \left( \frac{1}{2} + \sigma \right) \frac{z}{2\sigma\tau} - \sigma \right) \frac{z}{(1 + (\partial_z w)^2)} \right) \left( \partial_z w \right)
\]

Let

\[
M_{\text{boundary}} = \max_{\tau_0 \leq \tau \leq \tau^*} \left\{ z^{-\alpha+1} \partial_z w (z, \tau) \big|_{z = 2\psi_2 (0)}, z^{-\alpha+1} \partial_z w (z, \tau) \big|_{z = 2\beta} \right\}
\]

\[
M_{\text{initial}} = \max_{2 \psi_2 (0) \leq z \leq 2\beta} \left| z^{-\alpha+1} \partial_z w (z, \tau_0) \right|
\]

Then by (8.25) and (8.30), we have

\[
M_{\text{boundary}} \leq C (n)
\]

By (4.34), we have

\[
M_{\text{initial}} \leq C (n)
\]

Let

\[
h (\tau) = \max_{2 \psi_2 (0) \leq z \leq 2\beta} z^{-\alpha+1} \partial_z w (z, \tau)
\]

and

\[
M = \max \{ M_{\text{boundary}}, M_{\text{initial}} \}
\]

If $h (\tau) \leq M$ for $\tau_0 \leq \tau \leq \hat{\tau}$, then we are done. Otherwise, there is $\tau^*_1 > \tau_0$ for which

\[
h (\tau^*_1) > M
\]

Let $\tau^*_0$ be the first time after which $h$ is greater than $M$ all the way up to $\tau^*_1$.

By continuity, we have

\[
h (\tau^*_0) \leq M
\]

Applying the maximum principle to (8.37) (and using (8.29) and (8.30)) yields

\[
\partial_\tau h \leq C (n) \beta^{-\alpha}
\]

which implies that

\[
h (\tau) \leq M + C (n) \beta^{-\alpha} (\tau - \tau^*_0)
\]
for $\tau_0^* \leq \tau \leq \tau_1^*$. Now choose $0 < \varepsilon \ll 1$ (depending on $n$) so that

$$h(\tau) \leq M + 1$$

for $\tau_0^* \leq \tau \leq \tau_0^* + \varepsilon \beta^\alpha$. By the above argument, we claim that

$$(8.38) \quad \max_{2\hat{\psi}_2(0) \leq z \leq 2\beta} z^{-\alpha+1} \partial_\tau w(z, \tau) \leq M + 1$$

for $\tau_0 \leq \tau \leq \tau_0 + \varepsilon \beta^\alpha$; otherwise, we would get a contradiction by the above argument.

On the other hand, by (8.34) we have

$$\hat{\psi}_2(0) \leq 2\beta, \quad \tau_0 + \varepsilon \beta^\alpha \leq \tilde{\tau} \leq \hat{\tau}.$$  

It follows, by (2.1), (6.3) and Lemma 2.3, that

$$z^{-\alpha+1} \partial_\tau \hat{\psi}_k(z) \leq C(n) (\varepsilon \beta^\alpha)^{-\frac{1}{2}} (\beta^\alpha - 1) \left(\frac{\tau}{\tau_0}\right)^{-\varepsilon} z^{-\alpha+1}$$

(8.39)

$$\leq z^{-\alpha+1} \partial_\tau \hat{\psi}_k(z) + C(n) \beta^{-2 - \frac{\alpha}{n}} \leq C(n)$$

for $\hat{\psi}_2(0) \leq 2\beta, \tau_0 + \varepsilon \beta^\alpha \leq \tau \leq \hat{\tau}$, provided that $\beta \gg 1$ (depending on $n$) and $\tau_0 \gg 1$ (depending on $n, \Lambda, p, \beta$). Note that $\varepsilon = \varepsilon(n)$.

Combining (8.38) with (8.39) yields

$$\partial_\tau w(z, \tau) \leq C(n) z^{\alpha-1}$$

for $\hat{\psi}_2(0) \leq 2\beta, \tau_0 \leq \tau \leq \hat{\tau}$. By a similar argument, we can show that

$$\partial_\tau w(z, \tau) \geq -C(n) z^{\alpha-1}$$

$\square$

Next, given any constant $p$, from (8.30) we derive the following evolution equation in order to estimate the second derivative of (8.6).

$$(8.40) \quad \partial_\tau (z^{-p+2} \partial^2_{zz} w) = \frac{1}{1 + (\partial_\tau w)^2} \partial_x^2 (z^{-p+2} \partial^2_{zz} w) - \left(-\frac{6 (\partial_\tau w) (\partial^2_{zz} w)}{1 + (\partial_\tau w)^2} + \frac{2 (n - 1)}{z (1 - \frac{w}{z})^2} - \left(\frac{1}{2} + \sigma\right) \frac{z}{2 \sigma \tau} + \frac{2 (p - 2)}{z (1 + (\partial_\tau w)^2)} \right) \partial_\tau (z^{-p+2} \partial^2_{zz} w)$$

$$- \left(-\frac{2 (1 - 3 (\partial_\tau w)^2)}{1 + (\partial_\tau w)^2} \partial^2_{zz} w \right) + \frac{12 (n - 1)}{z^2 (1 - \frac{w}{z})^2} \partial_x w \right) \partial^2_{zz} (z^{-p+2} \partial^2_{zz} w)$$

$$\left(\frac{2 (n - 1)}{z^2 (1 - \frac{w}{z})^2} + \frac{2 (n - 1)}{z^2 (1 - \frac{w}{z})^2} \left(\frac{1}{2} + \sigma\right) \frac{1}{2 \sigma \tau} \right) \partial^2_{zz} (z^{-p+2} \partial^2_{zz} w)$$

$$+ (p - 2) \left(-\frac{6 (\partial_\tau w) (\partial^2_{zz} w)}{z (1 + (\partial_\tau w)^2)} + \frac{2 (n - 1)}{z^2 (1 - \frac{w}{z})^2} - \left(\frac{1}{2} + \sigma\right) \frac{1}{2 \sigma \tau} + \frac{p - 3}{z^2 (1 + (\partial_\tau w)^2)} \right) \partial^2_{zz} (z^{-p+2} \partial^2_{zz} w)$$
The following lemma is essential for the derivation of the second derivative estimates in (8.5), and its proof is very similar to the one in the previous lemma.

**Lemma 8.11.** If \( \tau_0 \gg 1 \) (depending on \( n, \Lambda, \rho, \beta \)), there holds

\[
|z \partial_{zz}^2 w(z, \tau)| \leq C(n)
\]

for \( 2 \psi_2(0) \leq z \leq 2 \beta, \tau_0 \leq \tau \leq \hat{\tau} \).

**Proof.** Let

\[
M_{\text{boundary}} = \max_{\tau_0 \leq \tau \leq \hat{\tau}} \left\{ |z \partial_{zz}^2 w(z, \tau)| \right\}_{z = 2 \psi_2(0)}, \quad M_{\text{initial}} = \max_{2 \psi_2(0) \leq z \leq 2 \beta} z \partial_{zz}^2 w(z, \tau_0)
\]

By (4.4), (8.23) and (8.30), we have

\[
M = \max \{ M_{\text{boundary}}, M_{\text{initial}} \} \leq C(n)
\]

Define

\[
h(\tau) = \max_{2 \psi_2(0) \leq z \leq 2 \beta} z \partial_{zz}^2 w(z, \tau)
\]

If \( h(\tau) \leq M \) for \( \tau_0 \leq \tau \leq \hat{\tau} \), then we are done. Otherwise, there is \( \tau_1^* > \tau_0 \) for which

\[
h(\tau_1^*) > M
\]

Let \( \tau_0^* \) be the first time after which \( h \) is greater than \( M \) all the way up to time \( \tau_1^* \). By continuity, we have

\[
h(\tau_0^*) \leq M
\]

Applying the maximum principle to (8.40) with \( p = 1 \) (and using (8.29) and (8.30)) yields

\[
\partial_\tau h(\tau) \leq C(n)(h(\tau) + 1)
\]

which implies that

\[
h(\tau) \leq C(n)^{\tau - \tau_0^*}(M + C(n)) \leq 2(M + C(n))
\]

for \( \tau_0^* \leq \tau \leq \tau_0^* + \varepsilon \), where \( 0 < \varepsilon = \varepsilon(n) \ll 1 \). Thus, we claim that

\[
(8.41) \max_{2 \psi_2(0) \leq z \leq 2 \beta} z \partial_{zz}^2 w(z, \tau) \leq 2(M + C(n))
\]

for \( \tau_0 \leq \tau \leq \tau_0 + \varepsilon \); otherwise, we would get a contradiction by the above argument.

On the other hand, by (3.34) we have

\[
\varepsilon |\partial_{zz}^2 w(z, \tau) - \psi_k(z)| \leq C(n) \beta^{\alpha - 3} \left( \frac{\tau}{\tau_0} \right)^{-\epsilon}
\]
for \(2 \dot{\psi}_2 (0) \leq z \leq 2 \beta, \tau_0 + \varepsilon \leq \tau \leq \tau_1^\ast\), which, together with (2.1), (6.3) and Lemma 2.3 implies

\[
(8.42) \quad z \partial_{zz}^2 w (z, \tau) \leq z \partial_{zz}^2 \psi (z) + C (n) \varepsilon^{-1} \beta^{n-3} \left( \frac{\tau}{\tau_0} \right)^{-\varepsilon} z \leq C (n)
\]

for \(2 \dot{\psi}_2 (0) \leq z \leq 2 \beta, \tau_0 + \varepsilon \leq \tau \leq \tau_1^\ast\) (since \(\varepsilon = \varepsilon (n)\)).

By (8.41) and (8.42), we get

\[
z \partial_{zz}^2 w (z, \tau) \leq C (n)
\]

for \(2 \dot{\psi}_2 (0) \leq z \leq 2 \beta, \tau_0 \leq \tau \leq \tau_1^\ast\). Similarly, by a similar argument, we can show that

\[
z \partial_{zz}^2 w (z, \tau) \geq -C (n)
\]

\(\square\)

Now we are ready to show the second derivative estimate of (8.5) with the help of the previous lemma.

**Proposition 8.12.** If \(\tau_0 \gg 1\) (depending on \(n\)), there holds

\[
|\partial_{zz}^2 w (z, \tau)| \leq C (n) z^{n-2}
\]

for \(2 \dot{\psi}_2 (0) \leq z \leq 2 \beta, \tau_0 \leq \tau \leq \tau_1^\ast\).

**Proof.** Let

\[
M_{\text{boundary}} = \max_{2 \dot{\psi}_2 (0) \leq z \leq 2 \beta} \left\{ z^{-\alpha+2} \partial_{zz}^2 w (z, \tau) \right\}
\]

\[
M_{\text{initial}} = \max_{z = 2 \dot{\psi}_2 (0)} z^{-\alpha+2} \partial_{zz}^2 w (z, \tau_0)
\]

By (4.4), (8.23) and (8.30), we have

\[
M = \max \{M_{\text{boundary}}, M_{\text{initial}}\} \leq C (n)
\]

Define

\[
h (\tau) = \max_{2 \dot{\psi}_2 (0) \leq z \leq 2 \beta} z^{-\alpha+2} \partial_{zz}^2 w (z, \tau)
\]

If \(h (\tau) \leq M\) for \(\tau_0 \leq \tau \leq \tau_1^\ast\), then we are done. Otherwise, there is \(\tau_1^\ast > \tau_0\) for which

\[
h (\tau_1^\ast) > M
\]

Let \(\tau_0^\ast\) be the first time after which \(h\) is greater than \(M\) all the way up to time \(\tau_1^\ast\). By continuity, we have

\[
h (\tau_0^\ast) \leq M
\]

By applying the maximum principle to (8.40) with \(p = \alpha\) and using (8.29), (8.30), (8.35) and (8.36), we get

\[
\partial_\tau h (\tau) \leq C (n) (h (\tau) + 1)
\]

which implies that

\[
h (\tau) \leq C (n) \tau^{-\tau_0} (M + C (n)) \leq 2 (M + C (n))
\]

for \(\tau_0^\ast \leq \tau \leq \tau_0^\ast + \varepsilon\), where \(0 < \varepsilon = \varepsilon (n) \ll 1\). Thus, we infer that

\[
(8.43) \quad \max_{2 \dot{\psi}_2 (0) \leq z \leq 2 \beta} z^{-\alpha+2} \partial_{zz}^2 w (z, \tau) \leq 2 (M + C (n))
\]
for $\tau_0 \leq \tau \leq \tau_0 + \varepsilon$, since otherwise, we would get a contradiction by the above argument.

On the other hand, by (8.34) we have

$$|\varepsilon \partial^2_{zz} w(z, \tau) - \psi_k(z)| \leq C(n) \beta^{\alpha-3} \left(\frac{\tau}{\tau_0}\right)^{-\theta}$$

for $2\hat{\psi}_2(0) \leq z \leq 2\beta$, $\tau_0 + \varepsilon \leq \tau \leq \check{\tau}$, which, together with (6.3) and Lemma 2.3 implies

$$z^{-\alpha+2} \partial^2_{zz} w(z, \tau) \leq z^{-\alpha+2} \partial^2_{zz} \psi_k(z) + C(n) \beta^{\alpha-3} \left(\frac{\tau}{\tau_0}\right)^{-\theta} z^{-\alpha+2}$$

(8.44)

for $2\hat{\psi}_2(0) \leq z \leq 2\beta$, $\tau_0 \leq \tau \leq \check{\tau}$, provided that $\beta \gg 1$ (depending on $n$).

Notice that $\varepsilon = \varepsilon(n)$.

Combining (8.43) with (8.44) yields

$$\partial^2_{zz} w(z, \tau) \leq C(n) z^{\alpha-2}$$

for $2\hat{\psi}_2(0) \leq z \leq 2\beta$, $\tau_0 \leq \tau \leq \check{\tau}$. Likewise, by a similar argument, we can show

$$\partial^2_{zz} w(z, \tau) \geq -C(n) z^{\alpha-2}$$

for $2\hat{\psi}_2(0) \leq z \leq 2\beta$, $\tau_0 \leq \tau \leq \check{\tau}$. □

References

[C] A. Cooper, A characterization of the singular time of the mean curvature flow. Proc. Amer. Math. Soc. 139 (2011), no. 8, 2933–2942.

[EH] K. Ecker, G. Huisken, Interior estimates for hypersurfaces moving by mean curvature. Invent. Math. 105 (1991), no. 3, 547–569.

[EMT] J. Enders, R. Müller, P. Topping, On type-I singularities in Ricci flow. Comm. Anal. Geom. 19 (2011), no. 5, 905–922.

[H] G. Huisken, Flow by mean curvature of convex surfaces into spheres. J. Differential Geom. 20 (1984), no. 1, 237–266.

[K] S. Kandanaarachchi, On the extension of axially symmetric volume flow and mean curvature flow. [arXiv:1301.1129]

[LS] N. Le, N. Sesum, The mean curvature at the first singular time of the mean curvature flow. Ann. Inst. H. Poincaré Anal. Non Linéaire 27 (2010), no. 6, 1441–1459.

[LW] H. Li, B. Wang, The extension problem of the mean curvature flow. [arXiv:1608.02832]

[V] J. J. L. Velázquez, Curvature blow-up in perturbations of minimal cones evolving by mean curvature flow. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 21 (1994), no. 4, 595–628.