AN INDUCTION THEOREM FOR GROUPS ACTING ON TREES

MARTIN H. WEISSMAN

Abstract. If \( G \) is a group acting on a tree \( X \), and \( S \) is a \( G \)-equivariant sheaf of vector spaces on \( X \), then its compactly-supported cohomology is a representation of \( G \). Under a finiteness hypothesis, we prove that if \( H^0_{cc}(X, S) \) is an irreducible representation of \( G \), then \( H^0_{cc}(X, S) \) arises by induction from a vertex or edge stabilizing subgroup.

If \( G \) is a reductive group over a nonarchimedean local field \( F \), then Schneider and Stuhler realize every irreducible supercuspidal representation of \( G(F) \) in the degree-zero cohomology of a \( G(F) \)-equivariant sheaf on its reduced Bruhat-Tits building \( X \). When the derived subgroup of \( G \) has relative rank one, \( X \) is a tree. An immediate consequence is that every such irreducible supercuspidal representation arises by induction from a compact-mod-center open subgroup.

1. Introduction

According to a folklore conjecture, every irreducible supercuspidal representation of a reductive \( p \)-adic group arises by induction from a compact-mod-center open subgroup. This is proven for \( GL_n \) by Bushnell-Kutzko [BK93], for many classical groups by Stevens [Ste08], and for tame supercuspidals by Ju-Lee Kim [Kim07] (exhaustion) and Jiu-Kang Yu [Yu01] (construction). Outside of \( GL_n \), these results require some assumptions on characteristic and residue characteristic.

Here we prove the conjecture for all groups of relative rank one – those whose Bruhat-Tits building is a tree. Our method is less constructive, but follows directly from results of Schneider-Stuhler [SS97] and the geometry of equivariant sheaves on trees. No restrictions on residue characteristic (or characteristic!) are required, so the result seems new in many cases, e.g., for \( SU_3 \) in residue characteristic two and for quaternionic unitary groups.

There are nine classes of groups of relative rank one over a nonarchimedean local field \( F \), if one uses the Tits index to organize them [Tit79, §4]. These are conveniently tabulated and described in notes of Carbone [Car]. Their simply-connected forms are \( SL_2(F) \) and \( SL_2(D) \) (for \( D \) a division algebra of any degree over \( F \)), the quasisplit unitary groups \( SU_3^E \) and \( SU_4^E \) (\( E/F \) a separable quadratic field extension), and five types of quaternionic unitary groups \( SU_{2D, s} \), \( SU_{3D, s} \), \( SU_{3D, h} \), \( SU_{4D, h} \), \( SU_{5D, h} \) which have absolute types \( C_2 \), \( C_3 \), \( D_3 \), \( D_4 \), and \( D_5 \), respectively. For these, \( D \) denotes a quaternion division algebra over \( F \), \( h \) a nondegenerate Hermitian form, and \( s \) a nondegenerate skew-hermitian form. Previous results have addressed groups in three of these nine classes in a non-uniform manner, typically under restrictions on isogeny class, characteristic, and residue characteristic.

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2. Sheaves on Trees

Let $X$ be a tree with vertex set $V$ and edge set $E$. If $v \in V$ and $e \in E$, then we write $v < e$ to mean that $v$ is an endpoint of $e$. Fix a field $k$. A sheaf on $X$ will mean a cellular sheaf of $k$-vector spaces on $X$. Such a sheaf consists of $k$-vector spaces $S_v$ for every vertex $v \in V$, and $S_e$ for every edge $e \in E$, and linear maps $\gamma_{v,e} : S_v \to S_e$ for all $v < e$. The maps $\gamma_{v,e}$ are called restriction maps and the spaces $S_v$, $S_e$ are called the stalks of $S$. We write $(S, \gamma)$ or sometimes just $S$ for such a sheaf.

Let $G$ be a group acting on $X$. A $G$-equivariant structure on a sheaf $(S, \gamma)$ consists of linear maps $\eta_{g,v} : S_v \to S_{gv}$, $\eta_{g,e} : S_e \to S_{ge}$ for all $g \in G$, $v \in V$, and $e \in E$, satisfying the following axioms.

- For all $v \in V$, $e \in E$, the linear maps $\eta_{1,v}$ and $\eta_{1,e}$ are the identity.
- For all $g, h \in G$, $v \in V$, and $e \in E$, $\eta_{g,hu} \circ \eta_{h,v} = \eta_{gh,v}$ and $\eta_{gh,e} \circ \eta_{h,e} = \eta_{gh,e}$.

A $G$-equivariant sheaf on $X$ will mean a sheaf $(S, \gamma)$ endowed with a $G$-equivariant structure.

2.1. Cohomology. For convenience, fix an orientation on every edge $e \in E$. This orders the endpoints of every edge $e$, and we write $x_e, y_e$ for the first and second endpoint, respectively. If $v < e$, then write $\text{or}(v, e) = 1$ if $v = x_e$ and $\text{or}(v, e) = -1$ if $v = y_e$.

Fix a sheaf $(S, \gamma)$ on $X$ in what follows. If $v \in V$, and $s \in S_v$, define

$$d_s = \sum_{e > v} \text{or}(v, e) \cdot \gamma_{v,e}(s) \in \bigoplus_{e > v} S_e.$$ 

The compactly-supported cohomology of $S$ is then computed by the complex

$$0 \to \bigoplus_{v \in V} S_v \xrightarrow{d} \bigoplus_{e \in E} S_e \to 0.$$ 

With reference to the complex above,

$$H^0_c(X, S) = \text{Ker } d, \quad H^1_c(X, S) = \text{Cok } d.$$ 

If $S$ is a $G$-equivariant sheaf, then the complex above and its cohomology inherit actions of $G$. In particular, $H^i_c(X, S)$ is a representation of $G$ on a $k$-vector space for $i = 0, 1$. 
2.2. The elliptic subsheaf. Fix a $G$-equivariant sheaf $(S, \gamma, \eta)$ on $X$ in what follows. If $v \in V$ and $s \in S_v$, we say that $s$ is elliptic if $d\, s = 0$, i.e., if
\[ \gamma_{v,e}(s) = 0 \text{ for all } e > v. \]
The elliptic elements of $S_v$ form a subspace $S_v^{\text{ell}} \subset S_v$. If $e$ is an edge, define $S_e^{\text{ell}} = 0$. This defines a $G$-equivariant subsheaf of $S$, which we call the elliptic subsheaf
\[ S^{\text{ell}} \subset S. \]

By construction, we have
\[ H^0_c(X, S^{\text{ell}}) = \bigoplus_{v \in V} S_v^{\text{ell}}, \quad \text{and} \quad H^1_c(X, S^{\text{ell}}) = 0. \]

For $x \in V$, let $G_x$ denote its stabilizer and $G \cdot x$ its orbit. Then $S_x$ is naturally (via $\eta$) a representation of $G_x$, and $S_x^{\text{ell}}$ is a $G_x$-subrepresentation. Algebraic induction gives a natural identification of $G$-representations,
\[ \bigoplus_{v \in G \cdot x} S_v^{\text{ell}} \cong \text{cInd}_{G_x}^G S_x^{\text{ell}}. \]

Above and in what follows, if $K$ is a subgroup of $G$ and $(\sigma, S)$ is a representation of $K$, $\text{cInd}_K^G(S)$ denotes “algebraic induction,” i.e., the space of functions $f : G \to S$ supported on a finite number of left $K$-cosets, satisfying $f(gk) = \sigma(k)^{-1} f(g)$ for all $g \in G$, $k \in K$.

It follows that $H^0_c(X, S^{\text{ell}})$ is a direct sum of such induced representations.
\[ H^0_c(X, S^{\text{ell}}) = \bigoplus_{G \cdot x \in G / \! \! \perp V} \text{cInd}_{G_x}^G S_x^{\text{ell}}. \]

2.3. The unifacial subsheaf. Suppose now that $S^{\text{ell}} = 0$. If $v \in V$ and $s \in S_v$, we say that $s$ is unifacial if there exists a unique edge $e > v$ such that $\gamma_{v,e}(s) \neq 0$. Define
\[ S_{v,e}^{\text{uni}} = \text{Span}_k \{ s \in S_v : s \text{ is unifacial and } \gamma_{v,e}(s) \neq 0 \}. \]
Since we assume $S^{\text{ell}} = 0$, we find that
\[ S_{v,e}^{\text{uni}} = \{ s \in S_v : s \text{ is unifacial and } \gamma_{v,e}(s) \neq 0 \} \cup \{ 0 \}. \]
The spaces $S_{v,e}^{\text{uni}}$ for various $e$, are linearly independent in $S_v$. Hence putting these spaces together for various edges, we define
\[ S_v^{\text{uni}} = \text{Span}_k \{ s \in S_v : s \text{ is unifacial} \} = \bigoplus_{e > v} S_{v,e}^{\text{uni}}. \]

If $e$ is an edge with endpoints $x, y$, define
\[ S_e^{\text{uni}} = \gamma_{x,e}(S_{x,e}^{\text{uni}}) + \gamma_{y,e}(S_{y,e}^{\text{uni}}) \subset S_e. \]
The spaces $S_e^{\text{uni}}$ and $S_e^{\text{uni}}$ define a $G$-equivariant subsheaf $S^{\text{uni}} \subset S$, whose cohomology can be described explicitly.

**Proposition 2.1.** $H^1_c(X, S^{\text{uni}}) = 0$.

**Proof.** We demonstrate that $d$ is surjective as follows. Suppose $e \in E$ and $s \in S_e^{\text{uni}}$. Write $x = x_e$ and $y = y_e$. Then there exists $a \in S_{x,e}^{\text{uni}}$ and $b \in S_{y,e}^{\text{uni}}$ such that
\[ s = \gamma_{x,e}(a) + \gamma_{y,e}(b). \]
Since $\gamma_{x,e}(a) = 0$ and $\gamma_{y,e}(b) = 0$ for all $e' \neq e$, we find that $s = d(a - b)$. Hence $d$ is surjective and $H^1_c(X, S^{\text{uni}})$ vanishes. \qed
Proposition 2.2. \( H^0_c(X, S_{\text{uni}}) = \bigoplus_{e \in E} (\gamma_{x,e}(S_{x_e,c}^{\text{uni}}) \cap \gamma_{y,e}(S_{y_e,c}^{\text{uni}})) \).

Proof. We define auxiliary sheaves \((R, \rho)\) and \((\mathcal{I}, \tau)\) as follows. For every vertex \(v\), define \(R_v = S_{\text{uni}}^v\) and \(I_v = 0\). For every edge \(e\), with endpoints \(x, y\), define
\[
R_e = \gamma_{x,e}(S_{x_e,c}^{\text{uni}}) \oplus \gamma_{y,e}(S_{y_e,c}^{\text{uni}}), \quad I_e = \gamma_{x,e}(S_{x_e,c}^{\text{uni}}) \cap \gamma_{y,e}(S_{y_e,c}^{\text{uni}}).
\]
For \(v < e\), define \(\tau_{v,e}: I_v \rightarrow I_e\) to be the zero map. Define \(\rho_{v,e}: R_v \rightarrow R_e\) by \(\rho_{v,e}(s) = \gamma_{v,e}(s)\), where the latter is viewed in the summand \(\gamma_{v,e}(S_{v,e}^{\text{uni}})\) of \(R_e\).

There is a natural short exact sequence of sheaves on \(X\),
\[
\mathcal{I} \hookrightarrow R \rightarrow S_{\text{uni}}^v.
\]

At vertices, the maps are obvious; for edges, the map \(I_e \rightarrow R_e\) sends an element
\[
t \in \gamma_{x,e}(S_{x_e,c}^{\text{uni}}) \cap \gamma_{y,e}(S_{y_e,c}^{\text{uni}})
\]
to the ordered pair \((t, -t)\). The map from \(R_e\) to \(S_{\text{uni}}^v\) is addition.

Essentially by construction, \(H^0_c(X, R) = H^1_c(X, I) = 0\). Indeed, we can decompose \(R\) as a product of sheaves,
\[
R = \prod_{v \in V} R^{(v)},
\]
where \(R^{(v)}\) is supported on the star-neighborhood of \(v\): \(R^{(v)}_v := R_v = S_{\text{uni}}^v\) and \(R^{(v)}_e = \gamma_{v,e}(S_{v,e}^{\text{uni}})\) for all \(e > v\). Since \(\gamma_{v,e}: S_{v,e}^{\text{uni}} \rightarrow \gamma_{v,e}(S_{v,e}^{\text{uni}})\) is an isomorphism for all \(v < e\) (recall \(S^{\text{uni}} = 0\)), we find that
\[
d: R^{(v)}_v \rightarrow \bigoplus_{e > v} R^{(v)}_e
\]
is an isomorphism (see (2.3)). Hence \(H^i_c(X, R^{(v)}) = 0\) for \(i = 0, 1\). The compactly supported cohomology of \(R\) is the direct sum of these, which vanishes.

From the short exact sequence of sheaves, \(\mathcal{I} \hookrightarrow R \rightarrow S_{\text{uni}}\), the long exact sequence in cohomology gives an identification,
\[
H^0_c(X, S_{\text{uni}}) \cong H^1_c(X, \mathcal{I}).
\]

Since \(I_v = 0\) for all vertices \(v\), we find that
\[
H^1_c(X, \mathcal{I}) = \bigoplus_{e \in E} I_e = \bigoplus_{e \in E} (\gamma_{x,e}(S_{x_e,c}^{\text{uni}}) \cap \gamma_{y,e}(S_{y_e,c}^{\text{uni}})).
\]

\Box

A \(G\)-equivariant structure on \(S\) transports to \(G\)-equivariant structures on \(\mathcal{I}\) and \(R\). It follows that, for any edge \(e \in E\), the space \(\mathcal{T}_e\) is naturally a representation of the stabilizer \(G_e\). Therefore, we find an identification of representations of \(G\),
\[
H^0_c(X, S_{\text{uni}}) \cong H^1_c(X, \mathcal{I}) \equiv \bigoplus_{G \in G \setminus E} \text{cInd}_{G_e}^G \mathcal{T}_e.
\]
(2.4)
2.4. Multifacial sheaves. Now, suppose that \((S, \gamma)\) is a sheaf on \(X\) and \(S^{\text{ell}} = 0\) and \(S^{\text{uni}} = 0\). Thus, for every \(v \in V\) and every nonzero \(s \in S_v\), there exist at least two edges \(e, f \in E\) such that \(v < e\) and \(v < f\) and
\[\gamma_{v,e}(s) \neq 0\] and \(\gamma_{v,f}(s) \neq 0\).
We call such a sheaf \textit{multifacial}.

**Proposition 2.3.** If \(S\) is a multifacial sheaf, then \(H^0_c(X, S) = 0\).

**Proof.** Suppose that \(s = (s_v : v \in V) \in H^0_c(X, S)\). Assume that \(s \neq 0\). Then there exists a nonempty finite set \(W \subset V\) of vertices such that \(s_v \neq 0\) if and only if \(v \in W\). Let \(\Omega\) be the convex hull of \(W\) in the tree \(X\). In other words, \(\Omega\) is the smallest connected subgraph of \(X\) containing every vertex from \(W\). In particular, \(\Omega\) is a \textit{finite tree}. Moreover, if \(\ell\) is a leaf of \(\Omega\), then \(\ell \in W\); otherwise one could prune the leaf while maintaining connectedness and containment of \(W\).

Let \(\ell\) be a leaf of \(\Omega\), so there is at most one edge of \(\Omega\) having \(\ell\) as an endpoint. Since \(\ell \in W\), we have \(s_\ell \neq 0\). Since \(S\) is multifacial, there exists an edge \(e\) such that \(\ell < e\), \(e\) does not belong to \(\Omega\), and \(\gamma_{\ell,e}(s_\ell) \neq 0\). Let \(v\) be the other endpoint of \(e\). Since \(\ell\) was a leaf of \(\Omega\), \(v \notin \Omega\).

![Figure 1](https://example.com/figure1.png)

\(\ell\) is a leaf of the finite tree \(\Omega\) (the subgraph contained in the shaded box), and \(e\) is an edge that protrudes outside of \(\Omega\).

Since \(d\) \(s = 0\), we have \((d \ s)_e = 0\). Equivalently,
\[\gamma_{v,e}(s_v) - \gamma_{\ell,e}(s_\ell) = 0.\]
Since \(\gamma_{\ell,e}(s_\ell) \neq 0\), this implies \(\gamma_{v,e}(s_v) \neq 0\), which implies \(s_v \neq 0\). But this contradicts the fact that \(v \notin \Omega\).

This contradiction proves that \(H^0_c(X, S) = 0\). \(\square\)

2.5. The Induction Theorem. Suppose that \(S\) is a \(G\)-equivariant sheaf on \(X\). We define its \textit{0-rank} to be the cardinal number
\[\text{Rank}^0(S) = \sum_{G \cdot v \in G \setminus V} \dim(S_v).\]
For example, if \(G \setminus V\) is finite and every stalk of \(S\) is finite-dimensional, then \(\text{Rank}^0(S)\) will be finite.

**Theorem 2.4.** Assume that \(\text{Rank}^0(S)\) is finite. If \(H^0_c(X, S) = 0\) or \(H^0_c(X, S)\) is an irreducible representation of \(G\), then \(H^0_c(X, S)\) is isomorphic to a representation induced from the stabilizer of a vertex or edge.
Proof. We proceed by induction on \( \text{Rank}^0(\mathcal{S}) \). If \( H^0_{c}(X, \mathcal{S}) = 0 \), then \( H^0_{c}(X, \mathcal{S}) \) is induced from the zero representation via any subgroup, and the result is trivial. This takes care of the \( \text{Rank}^0(\mathcal{S}) = 0 \) base case, in particular.

So assume that \( \text{Rank}^0(\mathcal{S}) > 0 \) and the result has been proven for lower 0-rank. If \( H^0_{c}(X, \mathcal{S}) = 0 \), then we are done. Otherwise, by Proposition 2.3 we find that \( \mathcal{S}^{\text{ell}} \neq 0 \) or \( (\mathcal{S}^{\text{uni}} = 0 \text{ and } \mathcal{S}^{\text{uni}} 
eq 0) \). We consider these two cases below.

If \( \mathcal{S}^{\text{ell}} \neq 0 \), then \( H^0_{c}(X, \mathcal{S}^{\text{ell}}) \) is a nonzero subrepresentation of \( H^0_{c}(X, \mathcal{S}) \). By irreducibility, we find that

\[
H^0_{c}(X, \mathcal{S}) \equiv H^0_{c}(X, \mathcal{S}^{\text{ell}}).
\]

By (2.2), this is a direct sum of representations induced from stabilizers of vertices. By irreducibility again, only one \( G \)-orbit of vertices can support \( \mathcal{S}^{\text{ell}} \) and

\[
H^0_{c}(X, \mathcal{S}) \equiv \text{cInd}_{G_x}^{G} \mathcal{S}^{\text{ell}}_x
\]

for some vertex \( x \in V \). Thus if \( \mathcal{S}^{\text{ell}} \neq 0 \), the result holds.

Next, suppose that \( \mathcal{S}^{\text{ell}} = 0 \) and \( \mathcal{S}^{\text{uni}} \neq 0 \). Consider the short exact sequence,

\[
\mathcal{S}^{\text{uni}} \rightarrow \mathcal{S} \rightarrow \mathcal{S}/\mathcal{S}^{\text{uni}}.
\]

Since \( \mathcal{S}^{\text{uni}} \neq 0 \), \( \mathcal{S}^{\text{uni}} \neq 0 \) for some vertex \( v \), and so \( \text{Rank}^0(\mathcal{S}/\mathcal{S}^{\text{uni}}) < \text{Rank}^0(\mathcal{S}) \).

By Proposition 2.4 the long exact sequence in cohomology yields

\[
0 \rightarrow H^0_{c}(X, \mathcal{S}^{\text{uni}}) \rightarrow H^0_{c}(X, \mathcal{S}) \rightarrow H^0_{c}(X, \mathcal{S}/\mathcal{S}^{\text{uni}}) \rightarrow 0.
\]

This is a short exact sequence of \( G \)-representations, so irreducibility of the middle term yields

\[
H^0_{c}(X, \mathcal{S}) \equiv H^0_{c}(X, \mathcal{S}^{\text{uni}}) \text{ or } H^0_{c}(X, \mathcal{S}) \equiv H^0_{c}(X, \mathcal{S}/\mathcal{S}^{\text{uni}}).
\]

In the first case, (2.4) and irreducibility yields

\[
H^0_{c}(X, \mathcal{S}) \equiv \text{cInd}_{G_x}^{G} \mathcal{S}
\]

for some edge \( e \in E \) and the result holds.

In the second case, \( H^0_{c}(X, \mathcal{S}) \equiv H^0_{c}(X, \mathcal{S}/\mathcal{S}^{\text{uni}}) \). But \( \mathcal{S}/\mathcal{S}^{\text{uni}} \) has lower 0-rank. Hence the second case follows from the theorem for sheaves of lower 0-rank. \( \square \)

**Corollary 2.5.** Let \( G \) be a reductive group over a nonarchimedean local field \( F \), whose derived subgroup has relative rank one. Let \( G = G(F) \). Then every irreducible supercuspidal representation of \( G \) on a complex vector space is isomorphic to \( \text{cInd}^G_K \sigma \) for some compact-mod-center open subgroup \( K \subset G \) and some irreducible representation \( \sigma \) of \( K \).

**Proof.** Let \( X \) be the Bruhat-Tits tree of \( G \) over \( F \). Then \( G \) acts on \( X \) with finitely many orbits on vertices and edges. The stabilizer of any vertex or any edge is a compact-mod-center open subgroup of \( G \); every compact-mod-center open subgroup is contained in such a stabilizer.

Let \((\pi, V)\) be an irreducible supercuspidal representation of \( G \). In [SS97] §IV.1 and Theorem IV.4.17, Schneider and Stuhler construct a \( G \)-equivariant sheaf \( \mathcal{S} \) on \( X \), with finite-dimensional stalks, such that \( H^0_{c}(X, \mathcal{S}) \equiv V \). The result now follows from the previous theorem. \( \square \)

Note that Vigneras [Vig97, Theorem 4.6] has proven that the main results of Schneider and Stuhler adapt to representations of \( G \) on \( R \)-vector spaces, when \( R \) is a field whose characteristic is coprime to the residue characteristic of \( F \).
Hence the previous result holds for cuspidal (see [Vig97, §3.7] for the definition) \( R \)-representations as well as for supercuspidal complex representations.

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Dept. of Mathematics, University of California, Santa Cruz, CA 95064

E-mail address: weissman@ucsc.edu