The Continuum Phase Diagram of the
2d Non-Commutative $\lambda \phi^4$ Model

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We present a non-perturbative study of the $\lambda \phi^4$ model on a non-commutative plane. The lattice regularised form can be mapped onto a Hermitian matrix model, which enables Monte Carlo simulations. Numerical data reveal the phase diagram; at large $\lambda$ it contains a “striped phase”, which is absent in the commutative case. We explore the question whether or not this phenomenon persists in a Double Scaling Limit (DSL), which extrapolates simultaneously to the continuum and to infinite volume, at a fixed non-commutativity parameter. To this end, we introduce a dimensional lattice spacing based on the decay of the correlation function. Our results provide evidence for the existence of a striped phase even in the DSL, which implies the spontaneous breaking of translation symmetry. Due to the non-locality of this model, this does not contradict the Mermin-Wagner Theorem.
1 Introduction

Since the dawn of the new millennium, field theory on non-commutative spaces ("NC field theory") has attracted a lot of interest, see Refs. [1, 2] for reviews. The idea as such is historic [3], but the observation that it can be related to low energy string theory [4] triggered an avalanche of thousands of papers on this subject.

The point of departure are space-time coordinates, which are given by Hermitian operators that do not commute. In the functional integral approach, however, NC field theory can be formulated with ordinary space-time coordinates $x$, if the field multiplications are carried out by a star product,

$$
\phi(x) \ast \psi(x) := \phi(x) \exp \left( \frac{i}{2} \partial_\mu \Theta_{\mu\nu} \partial_\nu \right) \psi(x).
$$

(1.1)

In particular the star commutator of the coordinates,

$$
x_\mu \ast x_\nu - x_\nu \ast x_\mu = i \Theta_{\mu\nu},
$$

(1.2)

yields the (anti-symmetric) non-commutativity "tensor" $\Theta$. Here we consider the simplest case, where $\Theta$ is constant.

The perturbative treatment of NC field theory is obstructed by the notorious problem of UV/IR mixing, i.e. the appearance of singularities at both ends of the energy scale in non-planar diagrams [5]. Therefore, renormalisation beyond one loop is mysterious. Effects due to this mixing also show up on the non-perturbative level, as this study is going to confirm.

In quadratic, integrated terms (with vanishing boundary contributions) the star product is equivalent to the ordinary product. Hence the Euclidean action of the NC $\lambda \phi^4$ model takes the form

$$
S[\phi] = \int d^d x \left[ \frac{1}{2} \partial_\mu \phi \partial_\mu \phi + \frac{m^2}{2} \phi^2 + \frac{\lambda}{4} \phi \ast \phi \ast \phi \ast \phi \right].
$$

(1.3)

The parameter $\lambda$ does not only control the strength of the coupling, but also the extent of the non-commutativity effects.

A 1-loop calculation in the framework of a Hartree-Fock approach led to the following conjecture about the phase diagram of this model in $d = 3$ and 4 [6]: there is a disordered phase, and — at strongly negative $m^2$ — an order regime. The latter splits into a phase of uniform magnetisation (at small $\lambda$, with an Ising-type transition to disorder, as in the commutative case), and
a “striped phase” (at large \( \lambda \)), where periodic non-uniform magnetisation patterns dominate. That phase, which does not exist in the commutative \( \lambda \phi^4 \) model, was also discussed based on renormalisation group methods [7] and on the Cornwall-Jackiw-Tomboulis effective action [8]. In \( d = 3 \), more precisely in the case of a NC plane and a commutative Euclidean time direction, the existence of such a striped phase was observed explicitly in a non-perturbative, numerical study on the lattice [9], in agreement with the qualitative prediction of Ref. [6].

The measurement of the dispersion relation (in lattice units), \( E(\vec{p}) \), \( (\vec{p} = (p_1, p_2)) \) revealed that for \( \lambda \) values above some critical \( \lambda_c \), the energy \( E \) grows not only at large momenta, but also at very small \( |\vec{p}| \), hence \( E \) takes its minimum at some finite momentum. A strongly negative parameter \( m^2 \) enforces ordering with the pattern corresponding to the modes of minimal energy. In a generalising sense, we denote the resulting order as a “stripe pattern”  

\footnote{Of course, this includes the possibility that a low mode condenses, which has several non-zero components. Then the resulting pattern corresponds to the interference of stripes in different directions, for instance with a checkerboard-type pattern.}

In fact, this minimal momentum in dimensional units, \( |\vec{p}|/a \) (where \( a \) is the lattice spacing), stabilises as one approaches a Double Scaling Limit (DSL) towards a continuous system of infinite extent, at a fixed non-commutativity tensor [9]. This shows that the striped phase does indeed exist in \( d = 3 \), which is a manifestation of the coexistence of UV and IR divergences. It further suggests non-perturbative renormalisability.

Also in the 2d lattice model such a striped phase was observed numerically [9,10]. However, the fate of that phase in the DSL has never been clarified. One might suspect that it does not survive this limit, i.e. that the DSL removes the phase boundary inside to ordered regime, \( \lambda_c \to \infty \), because in such a phase translation and rotation symmetry are spontaneously broken. Indeed, there are phases in lattice field theory, which are regularisation artifacts, such as the confinement phase in lattice Quantum Electrodynamics at strong coupling. However, the Mermin-Wagner Theorem [11], which rules out the spontaneous breaking of continuous, global symmetries in \( d \leq 2 \), is based on assumptions like locality and a regular IR behaviour, which do not hold here. Hence this theorem is not automatically applicable in NC field theory.

Therefore Ref. [6] presented a refined consideration, and conjectured the absence of a striped phase in \( d = 2 \). The authors argued based on an effective
action of the Brazovskii form: in this formulation, the kinetic term is of fourth order in the momentum, which renders the statement of the Mermin-Wagner Theorem more powerful. This suggests that it might even capture the NC case. On the other hand, another effective action approach supported the existence of a striped phase in $d = 2$ \cite{12}.

Here we investigate this controversial question numerically; this has been reported before in a thesis and in a proceeding contribution \cite{13}. In Section 2 we review the matrix model formulation; in this form, the model is tractable by Monte Carlo simulations. Section 3 describes the phase diagram on the lattice. Section 4 introduces a physical scale in order to address the question if there exists a stable DSL in the vicinity of the striped phase. We add our conclusions and an appendix about numerical tricks, which are useful in the simulations of Hermitian matrix models.

## 2 Formulation on the lattice and as a matrix model

As relation (1.2) shows, the points in an NC plane are somewhat washed out, hence we cannot introduce a lattice with sharp sites. On the other hand, we assume the momentum components to commute. They can be restricted to a Brillouin zone, which does imply a (fuzzy) lattice structure \cite{2,17}. This is only consistent for discrete momenta, hence the NC lattice is automatically periodic. In particular, on a periodic $N \times N$ lattice of spacing $a$, the non-commutativity parameter $\theta$ is identified as

$$
\theta = \frac{1}{a^2}, \quad \text{with} \quad \Theta_{\mu\nu} = \theta \epsilon_{\mu\nu}.
$$

(2.1)

The DSL consists of the simultaneous limits $N \to \infty$ and $a \to 0$ at $Na^2 = \text{const.}$, \textit{i.e.} one extrapolates to the continuum and to infinite volume (since $Na$ also diverges), while keeping the non-commutativity parameter $\theta$ constant.

Still, such a formulation is not practical for numerical simulations, because the star product couples the field variables on \textit{any} pair of lattice sites. A computer-friendly formulation is obtained by mapping the system onto a twisted matrix model. Such a mapping was first suggested in the context of NC $U(N)$ gauge theory \cite{18} (twisted boundary conditions are required for
the algebra to close). Similarly, the matrix formulation of the NC $\lambda \phi^4$ model takes the form \[17\]

$$S[\Phi] = N \text{Tr} \left[ \frac{1}{2} \sum_{\mu=1}^{2} \left( \Gamma_{\mu} \Phi \Gamma_{\mu}^\dagger - \Phi \right)^2 + \frac{\tilde{m}^2}{2} \Phi^2 + \frac{\lambda}{4} \Phi^4 \right], \quad (2.2)$$

where $\Phi$ is a Hermitian $N \times N$ matrix, which captures in one point all the degrees of freedom of the lattice field. $\Gamma_{\mu}$ are unitary matrices denoted as twist eaters, which satisfy the 't Hooft-Weyl algebra

$$\Gamma_{\mu} \Gamma_{\nu} = Z_{\nu\mu} \Gamma_{\nu} \Gamma_{\mu}, \quad (2.3)$$

where $Z$ contains the twist factor in the boundary conditions. We choose $Z_{12} = Z_{21}^* = \exp(i\pi(N + 1)/N)$, where the matrix (or lattice) size $N$ has to be odd. This choice, and the form of the twist eaters,

$$\Gamma_1 = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & \ldots \\ \ldots & 1 & \ldots & \ldots & \ldots \\ \ldots & \ldots & 0 & \ldots & \ldots \\ 1 & \ldots & \ldots & 0 & 0 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 1 & 0 & \ldots & \ldots & 0 \\ 0 & Z_{21} & 0 & \ldots & \ldots \\ 0 & 0 & Z_{21}^2 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & Z_{21}^3 & \ldots \end{pmatrix}$$

follows Refs. \[9,17\], and it is similar to the formulation in Ref. \[10\].

This type of matrix model has also been studied in Ref. \[19\], which further supported the scenario of a striped phase phase in $d = 2$ and 4. Moreover, the formulation of the $\lambda \phi^4$ model on a “fuzzy sphere” also leads to NC coordinates on the regularised level, though they obey a different non-commutativity relation, where $\Theta(x)$ is not constant. Also that model can be translated into a Hermitian matrix formulation similar to eq. \(2.2\), which has been studied numerically in $d = 2$ \[20\] and in $d = 3$ \[21\]. In both cases, the phase diagram at finite $N$ involves a striped phase as well.

3 The phase diagram on the lattice

Figure 1 shows the phase diagram that we obtained based on Monte Carlo simulations of this model, in its matrix formulation \(2.2\). The qualitative
features agree with the phase diagram in $d = 3$ [6]: there is a disordered phase at positive or weakly negative $m^2$. When this parameter decreases below some critical value $m^2_c(\lambda) < 0$, we enter the order regime. At small $\lambda$ this order is uniform, but at larger $\lambda$ it is “striped”. The transition within the order regime is hard to identify, hence the dotted vertical line in Figure 1 is only suggestive.

\[
\text{Figure 1: The phase diagram of the 2d NC } \lambda \phi^4 \text{ model. The lattice size is } N \times N, \text{ and we observe a stable phase transition line between order and disorder for } N \geq 19. \text{ At small coupling } \lambda \text{ this order is uniform, but at large } \lambda \text{ it becomes striped, since the NC effects are amplified.}
\]

On the other hand, the boundary between order and disorder is identified well, as we will describe below. It stabilises for $N \geq 19$ to a good accuracy, if the axes are chosen as $N^{3/2}m^2$ and $N^2\lambda$. This condition differs from the 3d case, where the suitable axes were $N^2m^2$ and $N^2\lambda$ [9]. In $d = 2$, the limit of extremely large couplings $\lambda$ — where the kinetic term (in the matrix formulation) becomes negligible — can be solved by a 1-matrix consideration, which yields the edge of the disordered phase as $\bar{m}^2_c = -2\sqrt{\bar{\lambda}}$ [22]. This behaviour was confirmed numerically in the related fuzzy sphere model [20], but not in Figure 1 which only extends up to moderate coupling.

We can divide the phase diagram in Figure 1 into four sectors, depending
whether $N^2 \lambda$ and $-N^{3/2} m^2$ are small or large. For each of these sectors we illustrate a typical configuration in Figure 2. These plots are obtained by mapping the $N \times N$ matrices back to a scalar field configuration, following the instruction in Refs. [17], and converting the values of $\Phi_x$ into bright or dark spots.

These different phases were identified by measuring the momentum dependent order parameters

$$M(k) = \frac{1}{NT} \max_{\vec{p} \mid |\vec{p}| = k} \left| \sum_t \tilde{\phi} (\vec{p}, t) \right|.$$  

(3.1)
For $k = 0$ this is the standard order parameter for uniform magnetisation. Its generalisation to finite $k$ detects a possible dominance of striped patterns, after rotating the system such that this pattern becomes maximally manifest. Figure 3 shows examples how the order parameters $\langle M(0) \rangle$ or $\langle M(4) \rangle$ become significant for decreasing $m^2$.

![Figure 3: The order parameters $\langle M(0) \rangle$ and $\langle M(4) \rangle$, which are defined in eq. (3.1). We show the dependence on $m^2$ at fixed $N$ and $\lambda$. For small (large) $\lambda$ and decreasing $m^2$, the disorder turns into a uniform (4-stripe) pattern.](image)

Such pictures reveal the type of emerging order unambiguously, but for the identification of the critical value $m^2_c$ it is favourable to consider the connected correlation function of the relevant order parameter,

$$\langle M(k) \rangle_{\text{con}} = \langle M(k) \rangle^2 - \langle M(k) \rangle^2.$$  \hspace{1cm} (3.2)

Here the transition corresponds to a peak, which can be located well, as the examples in Figure 4 show. This property also shows that these order-disorder phase transitions are of second order.

At low $\lambda$, this is an Ising-type transition, as in the commutative $\lambda \phi^4$ model, as argued in Ref. [6]. Regarding the transition to the striped phase we should note, however, that the correlation function does not decay exponentially, hence we cannot extract a correlation length in the usual sense. An example for the correlation function in the disordered phase, but close to striped ordering, is shown in Figure 5. We clearly see the trend to a 4-stripe pattern, which condenses at somewhat lower $m^2$.

Also in the 3d case non-exponential correlations were observed within the NC plane [9]. However, in that case the decay was exponential in the third
Figure 4: The connected correlation function of the order parameters $\langle M(0)^2 \rangle_{\text{con}}$ and $\langle M(4)^2 \rangle_{\text{con}}$. In both cases they exhibit a rather sharp peak, which allows us to identify the critical value $m_c^2$.

(commutative) direction, which did provide a correlation length, along with the aforementioned dispersion relation $E(p)$.

Figure 5: The correlation function $\langle \phi(0,0)\phi(x_1,0) \rangle$ in the point $(N^2\lambda, N^{3/2}m^2) = (272, -118)$, which is disordered, but close to the transition to the striped phase. We see a trend towards a 4-stripe pattern, which sets in at somewhat lower $m^2$. 

$N = 35$, $\lambda = 0.222$, $m^2 = -0.57$
4 The Double Scaling Limit

So far we have been dealing with lattice units. In order to discuss the DSL, we need to introduce a dimensional scale. In particular we are interested in the question if the DSL can be taken while staying in the proximity of the striped phase; if this is the case, it is evidence for that phase to persist.

Since the correlation function does not decay exponentially, we cannot resort to the standard procedure, which refers to the correlation length as the natural scale. Still, we extract the scale from the correlation decay, which slows down when $m^2$ approaches $m_c^2$. To suppress finite size effect, we consider the limit

$$\Delta m^2 := m^2 - m_c^2 \to 0 \quad (4.1)$$

from above, i.e. within the disordered phase. The goal is to increase $N$ at the same time, in such a way that the correlation function remains stable down to the first dip. Thus $\Delta m^2$ will be converted into the factor needed for the DSL, with a critical exponent that we denote as $\sigma$,

$$a^2 \propto (\Delta m^2)^\sigma, \quad (4.2)$$

and which remains to be determined.

As a normalisation, we choose the units such that $a = 1$ at $N = 35$, which means that distances take the form

$$ax = \sqrt{\frac{35}{N}} x, \quad (4.3)$$

if we follow the DSL rule $Na^2 = \text{const}$. We relate this term to $\Delta m^2$ as

$$Na^2 = N \frac{(\Delta m^2)^\sigma}{(m_c^2)^{1-\sigma}}. \quad (4.4)$$

This is the ansatz, which is consistent with the proportionality relation (4.3), and which adjusts the dimension in the spirit of a dimensionless temperature $\tau := (T - T_c)/T_c$, which is often used to parameterise the vicinity of a phase transition.

The practical method to determine $\sigma$ proceeds as follows: we consider two sizes $N_1$ and $N_2$, and we search the parameters which correspond to the same trajectory towards the DSL. Hence we fix the same coupling $\lambda$, so that the dimensionless product $\lambda \theta$ is kept constant. Now we adjust values of $\Delta m_1^2$
and $\Delta m_2^2$ in such a manner that the correlation decays coincide (as well as possible) until the first dip is reached. Figure 6 shows examples for such “matched correlations functions” at three values of $\lambda$, and various sizes in each case.

Having identified such pairs $\Delta m_1^2$, $\Delta m_2^2$ — and the corresponding critical values $m_{1,c}^2$, $m_{2,c}^2$ — we extract the critical exponent

$$\sigma = \frac{\ln(m_{1,c}^2/m_{2,c}^2)}{\ln(\Delta m_1^2/\Delta m_2^2) + \ln(m_{1,c}^2/m_{2,c}^2)}.$$  

(4.5)

This can be done for various pairs $N_1$, $N_2$, at fixed $\lambda$, always in the vicinity of the striped phase. In practice, the accessible values of $N^2\lambda$ are restricted by the feasibility of the simulation: for $N_1 < N_2$ the value of $\lambda$ has to be large enough for $N_1^2\lambda$ to be close to the striped phase, but $N_2^2\lambda$ should not become too large, to avoid a situation with a multitude of deep meta-stable minima, where the Monte Carlo history could get stuck. On the other hand, $N_1$ and $N_2$ should differ significantly — and both should be large enough to attain the asymptotic large volume regime — for the results for the $\sigma$ exponents to be sensible.

If these results stabilise for increasing $N_1$, $N_2$ and decreasing $\Delta m_1^2$, $\Delta m_2^2$, we can conclude that it is indeed possible to take a DSL next to the striped phase, so that the latter persists — otherwise it would likely be removed in the DSL. Table 1 and Figure 7 show our results, which explore the window of sizes and couplings, which are numerically well tractable.

| $\lambda$ | $N_1$ | $N_2$ | $\sigma$    |
|----------|-------|-------|-------------|
| 0.222    | 35    | 45    | 0.152(7)    |
|          | 35    | 55    | 0.156(6)    |
|          | 45    | 55    | 0.161(11)   |
| 0.286    | 25    | 35    | 0.161(9)    |
|          | 25    | 45    | 0.167(7)    |
|          | 35    | 45    | 0.178(23)   |
| 0.4      | 25    | 35    | 0.147(13)   |

Table 1: The $\sigma$-values obtained for various pairs of sizes $N_1$, $N_2$ after tuning $\Delta m^2$ such that the short-distance decay of the correlation functions coincide.

We see a convincing trend towards a stable critical exponent of

$$\sigma = 0.16(1).$$  

(4.6)
Figure 6: Examples for “matched correlation functions”: at different system sizes $N$, but at fixed coupling $\lambda$, $\Delta m^2 = m^2 - m_c^2$ is tuned such that the short-distance correlations agree. Then the distance in physical units — as given in eq. (4.3) — agrees as well. Thus we identify the $\Delta m^2$ values to be inserted in eq. (4.5), which fixes the critical exponent $\sigma$. 
\[ \lambda = 0.4 \]
\[ \lambda = 0.286 \]
\[ \lambda = 0.222 \]
\[ (N_1 + N_2)/2 \]

**Figure 7:** An illustration of the values given in Table 1. There is a clear trend to a plateau value of \( \sigma = 0.16(1) \).

Therefore our results affirm the persistence of the striped phase in the DSL.

## 5 Conclusions

We have presented a non-perturbative, numerical study of the \( \lambda \phi^4 \) model on a NC plane. Monte Carlo simulations were possible in a matrix formulation of the lattice regularised system. We first explored the phase diagram as obtained on the lattice. The phase transition lines stabilise rapidly for increasing system size \( N \), if the axes are scaled suitably — that prescription differs from the 3d case.

In contrast to the commutative space, this phase diagram contains a striped phase, where patterns of non-uniform order dominate. We have introduced a dimensional scale through the decay behaviour of the correlation function (although this decay is not exponential). This allows us to identify trajectories in the parameter space, which can be extrapolated to the DSL, i.e. to the continuum and to infinite volume, at a constant NC parameter \( \theta \).

We have provided evidence that a DSL can be taken in the vicinity of the striped phase, which means that this exotic phase does persist in the continuum and in infinite volume. This implies in particular that the spontaneous breaking of translation and rotation symmetry occurs. The apparent
contradiction with the Mermin-Wagner theorem is avoided by the fact that this model is non-local, and that its IR behaviour is not smooth.

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A Numerical techniques

In the formulation that we simulated, a configuration is given by a Hermitian $N \times N$ matrix $\Phi$, where $N$ is odd, cf. Section 2. We applied the standard Metropolis algorithm, and proceeded with minimal updates of the matrix elements, $\Phi_{ij} \to \Phi'_{ij}$, $\Phi_{ji} \to \Phi'_{ij}^\ast$.

In this appendix we comment on a numerically efficient treatment of the action (2.2),

$$S[\Phi] = N \text{Tr} \left[ \frac{1}{2} \sum_{\mu=1}^{2} (\Gamma_\mu \Phi \Gamma_\mu^\dagger - \Phi)^2 + \frac{\bar{m}^2}{2} \Phi^2 + \frac{\bar{\lambda}}{4} \Phi^4 \right] := N \left[ \frac{1}{2} (s_{\text{kin},1} + s_{\text{kin},2}) + \frac{\bar{m}^2}{2} s_m + \frac{\bar{\lambda}}{4} s_\lambda \right]. \quad (A.1)$$

The twist eaters $\Gamma_\mu$ are given in Section 2; note that the diagonal elements of $\Gamma_2$ are powers of $z = -e^{-i\pi/N}$.

To discuss the evaluation of these terms, we mention that any Hermitian matrix $H$ fulfils

$$\text{Tr} H^2 = \sum_{ij} |H_{ij}|^2 = \sum_i H_{ii}^2 + 2 \sum_{i>j} |H_{ij}|^2. \quad (A.2)$$

- We first address $s_{\text{kin},1}$, where it is useful to introduce the notation

$$\bar{i} := i \mod N \quad (A.3)$$
The matrix $\Gamma_1 \Phi \Gamma_1^\dagger - \Phi$ is also Hermitian, hence we apply the identity (A.2) to arrive at the computationally economic form

$$s_{\text{kin.1}} = \sum_i (\Phi_{i+1, i+1} - \Phi_{ii})^2 + 2 \sum_{i>j} |\Phi_{i+1, j+1} - \Phi_{ij}|^2 . \quad (A.4)$$

- In view of the term $s_{\text{kin.2}}$, we note that

$$(\Gamma_2 \Phi \Gamma_2^\dagger)_{ij} = \Phi_{ij} z^{i-j} \quad (A.5)$$

is still Hermitian, and $\Gamma_2 \Phi \Gamma_2^\dagger - \Phi$ as well. Inserting $z$ yields

$$s_{\text{kin.2}} = \sum_{ij} |\Phi_{ij}|^2 (z^{i-j} - 1) \quad (A.6)$$

- There are also simplifications of the action difference, which is needed in the Metropolis accept/reject step. For

$$(\Delta s_{\text{kin}} : = s_{\text{kin.1}}[\Phi'] + s_{\text{kin.2}}[\Phi'] - s_{\text{kin.1}}[\Phi] - s_{\text{kin.2}}[\Phi] . \quad (A.7)$$

we obtain

$$\Delta s_{\text{kin}}(i = j) = 2 \left[ \Phi_{ii}^2 - \Phi_{ii}'^2 + (\Phi_{ii} - \Phi_{ii}') (\Phi_{i+1, i+1} + \Phi_{i-1, i-1}) \right]$$

$$\Delta s_{\text{kin}}(i \neq j) = 4 \left[ (|\Phi_{ij}'|^2 - |\Phi_{ij}|^2) \left( 2 - \cos \frac{(N + 1)\pi(i-j)}{N} \right) + \text{Re} \left( (\Phi_{ij}' - \Phi_{ij}) (\Phi_{i+1, j+1} + \Phi_{i-1, j-1}) \right) \right] . \quad (A.8)$$

The cosine function in this formula, and in eq. (A.6), is only required for $N-1$ different arguments, which should be stored in a look-up array.

- The mass term $s_m = \text{Tr} \Phi^2$ is quick to evaluate, thanks to identity (A.2). The main computational challenge is the quartic term

$$s_\lambda = \text{Tr} \Phi^4 = \sum_r (\Phi^2)_{rr}^2 + 2 \sum_{r>s} |(\Phi^2)_{rs}|^2 . \quad (A.9)$$

If we update a matrix element $\Phi_{ij}, i \geq j$, the difference $\Delta s_\lambda$ is affected by the following elements of $\Phi^2$,

$$(\Phi^2)_{is} \text{ for } s = i \ldots N , \quad (\Phi^2)_{rj} \text{ for } r = 1 \ldots j . \quad (A.10)$$
These elements have to be computed explicitly for $\Phi^2$ and for $\Phi'^2$. A final detail is that the elements for the same indices $r, s$ — belonging to the set specified in (A.10) — can be obtained without doing the full summation twice, since most contributions are identical,

$$(\Phi'^2)_{rs} = (\Phi^2)_{rs} + \begin{cases} 
|\Phi'^{ij}|^2 - |\Phi^{ij}|^2 & \text{if } r = s \\
(\Phi'^{ij} - \Phi^{ij}) \Phi^js & \text{if } i = r \neq s \\
\Phi_{ri}(\Phi'^{ij} - \Phi^{ij}) & \text{if } j = s \neq r
\end{cases}. \quad (A.11)
$$

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