On Repdigits as Sums of Fibonacci and Tribonacci Numbers

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Abstract: In this paper, we use Baker’s theory for nonzero linear forms in logarithms of algebraic numbers and a Baker-Davenport reduction procedure to find all repdigits (i.e., numbers with only one distinct digit in its decimal expansion, thus they can be seen as the easiest case of palindromic numbers, which are a “symmetrical” type of numbers) that can be written in the form $F_n + T_n$, for some $n \geq 1$, where $(F_n)_{n \geq 0}$ and $(T_n)_{n \geq 0}$ are the sequences of Fibonacci and Tribonacci numbers, respectively.

Keywords: Diophantine equations; repdigits; Fibonacci; Tribonacci; Baker’s theory

MSC: 11B39; 11J86

1. Introduction

A palindromic number is a number that has the same form when written forwards or backwards, i.e., of the form $c_1c_2c_3\ldots c_3c_2c_1$ (thus it can be said that they are “symmetrical” with respect to an axis of symmetry). The first 19th palindromic numbers are

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 22, 33, 44, 55, 66, 77, 88, 99$$

and clearly they are a repdigits type. A number $n$ is called repdigit if it has only one repeated digit in its decimal expansion. More precisely, $n$ has the form

$$n = a \left( \frac{10^{\ell} - 1}{9} \right),$$

for some $\ell \geq 1$ and $a \in [1, 9]$ (as usual, we set $[a, b] = \{a, a+1, \ldots, b\}$, for integers $a < b$). An old open problem consists in proving the existence of infinitely many prime repunit numbers (sequence A002275 in OEIS [1]), where the $\ell$th repunit is defined as

$$R_\ell = \frac{10^{\ell} - 1}{9},$$

(it is an easy exercise that if $R_\ell$ is prime, then so is $\ell$).

There are many articles that address Diophantine equations concerning Fibonacci and Lucas numbers (see, e.g., [2–13]). In the last years, many authors have worked on Diophantine problems related to repdigits (e.g., their sums, concatenations) and linear recurrences (e.g., their product, sums). For more about this subject, we refer the reader to [14–24] and references therein.

Remark 1. We remark that the definition of repdigit is not restricted to decimal expansion. In fact, a repdigit in base $g \geq 2$, has the form
for some $\ell \geq 1$ and $a \in [1, g - 1]$.

In this work, we shall study two well-known recurrence sequences. The first one is the omnipresent sequence $(F_n)_n$. These numbers are defined by the second order linear recurrence

$$F_{n+2} = F_{n+1} + F_n,$$

for all $n \geq 0$ with initial values $F_0 = 0$ and $F_1 = 1$ (see, e.g., [25]). The sequence of Tribonacci numbers $(T_n)_n$ (generalizes the Fibonacci sequence) is defined by the third-order recurrence

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n,$$

for all $n \geq 0$ which begins with $T_0 = 0$ and $T_1 = T_2 = 1$ (see, e.g., [26,27]). We remark that Luca showed that $F_{10} = 55$ is the largest repdigit in the Fibonacci sequence, while Marques [28] proved that the largest repdigit in the Tribonacci sequence is $T_8 = 44$.

In this paper, we continue this program by searching for repdigits, which are the sum of a Fibonacci and a Tribonacci number (both with the same index). More specifically, our main result is the following:

**Theorem 1.** The only solutions of the Diophantine equation

$$F_n + T_n = a \left( \frac{10^\ell - 1}{9} \right),$$

in positive integers $(n, \ell, a)$, with $a \in [1, 9]$, are

$$(n, \ell, a) \in \{(1, 1, 2), (2, 1, 2), (3, 1, 4), (4, 1, 7)\}.$$

2. Auxiliary Results

First, we recall a very useful non-recursive formula for the $n$th Fibonacci numbers. The Binet’s formula is:

$$F_n = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}},$$

where $\phi = (1 + \sqrt{5})/2$. By a simple inductive argument, we can obtain that:

$$\phi^{n-2} \leq F_n \leq \phi^{n-1}, \text{ for all } n \geq 1.$$  \hspace{1cm} (3)

Also, we can write

$$F_n = \frac{\phi^n}{\sqrt{5}} + \nu,$$

where $|\nu| < 1/\sqrt{5}$ (actually, one has the asymptotic formula $F_n = (\phi^n / \sqrt{5})(1 + o(1))$).

For the Tribonacci sequence, in 1982, Spickerman [29] found the following “Binet-like” formula:

$$T_n = \frac{a^n}{-a^2 + 4a - 1} + \frac{\beta^n}{-\beta^2 + 4\beta - 1} + \frac{\gamma^n}{-\gamma^2 + 4\gamma - 1}, \text{ for all } n \geq 1,$$

where $a, \beta, \gamma$ are the roots of polynomial $x^3 - x^2 - x - 1$. Numerically, if $w_1 := (19 + 3\sqrt{33})^{1/3}$ and $w_2 := (19 - 3\sqrt{33})^{1/3}$, then
\[\alpha = \frac{1}{4}(1 + w_1 + w_2), \quad \beta = \frac{1}{8}(2 - (w_1 + w_2) + i\sqrt{3}(w_1 - w_2)), \quad \text{and} \quad \gamma = \bar{\beta}.\]

Another very helpful formula provided by Spickermann is the following:

\[T_n = \left\lfloor \frac{\alpha}{(\alpha - \beta)(\alpha - \gamma)} \alpha^n \right\rfloor,\]

where, as usual, \([x]\) is the nearest integer to \(x\). In particular, it holds that

\[T_n = \alpha' \alpha^n + \eta, \quad (6)\]

where \(|\eta| < 1/2\) and \(\alpha' := \alpha / (\alpha - \beta)(\alpha - \gamma)\). Moreover, again by an inductive argument, we can deduce that

\[\alpha^{n-2} \leq T_n \leq \alpha^{n-1}, \quad \text{for all} \ n \geq 1. \quad (7)\]

The main approach to attack Theorem 1 is the Baker’s theory about lower bounds for linear forms in logarithms. The next result is due to Matveev [30] according to Bugeaud, Mignotte and Siksek [9]:

**Lemma 1.** Let \(\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\) be algebraic numbers and let \(b_1, b_2, b_3\) be nonzero integer numbers. Define

\[\Lambda = b_1 \log \alpha_1 + b_2 \log \alpha_2 + b_3 \log \alpha_3.\]

Let \(D = |\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3) : \mathbb{Q}|\) (degree of field extension) and let \(A_1, A_2, A_3\) be real numbers such that

\[A_j \geq \max\{Dh(\alpha_j), |\log \alpha_j|, 0.16\}, \quad \text{for} \ j \in \{1, 2, 3\}.\]

Take

\[B \geq \max\{1, \max\{|b_j|A_j / A_1; 1 \leq j \leq 3\}\}.\]

If \(\Lambda \neq 0, \) then

\[\log |\Lambda| \geq -C_1 D^2 A_1 A_2 A_3 \log(1.5eDB \log(eD)),\]

where

\[C_1 = 6750000 \cdot e^4(20.2 + \log(3^{5.5}D^2 \log(eD))).\]

In the previous statement, the logarithmic height of a \(t\)-degree algebraic number \(\alpha\) is defined by

\[h(\alpha) = \frac{1}{t} \left(\log |a| + \sum_{j=1}^{t} \log \max\{1, |\alpha(j)|\}\right),\]

where \(a\) is the leading coefficient of the minimal polynomial of \(\alpha\) (over \(\mathbb{Z}\)), \((\alpha(j))_{1 \leq j \leq t}\) are the algebraic conjugates of \(\alpha\). Some helpful properties of \(h(x)\) are in the following lemma (see Property 3.3 of [31]):

**Lemma 2.** Let \(x\) and \(y\) be algebraic numbers. Then

\[h(xy) \leq h(x) + h(y);\]
\[h(x + y) \leq h(x) + h(y) + \log 2;\]
\[h(\alpha^r) = |r| \cdot h(\alpha), \quad \text{for all} \ r \in \mathbb{Q}.\]

Our last ingredient is a reduction method provided by Dujella and Pethő [32], which is itself a variation of the result of Baker and Davenport [33]. For \(x \in \mathbb{R}\), set \(\|x\| = \min\{|x - n| : n \in \mathbb{Z}\} = |x - [x]|\) for the distance from \(x\) to the nearest integer.
Lemma 3. For a positive integer $M$, let $p/q$ be a convergent of the continued fraction of $\gamma \notin \mathbb{Q}$, such that $q > 6M$, and let $\mu, A$ and $B$ be real numbers, with $A > 0$ and $B > 1$. If the number $\epsilon = \|\mu q\| - M\|\gamma q\|$ is positive, then there is no solution to the Diophantine inequality

$$0 < m \gamma - n + \mu < A \cdot B^{-m}$$

in integers $m, n > 0$ with

$$\frac{\log(Aq/\epsilon)}{\log B} \leq m < M.$$

See Lemma 5 of [32].

Now, we are in a position to prove our main theorem.

3. The Proof of Theorem 1

3.1. Finding an Upper Bound for $n$ and $\ell$

By using (4) and (6) in (1), we have

$$\left(\frac{\phi^n}{\sqrt{5}} + \nu \right) + (a' \alpha^n + \eta) = a \left(\frac{10^\ell - 1}{9}\right).$$

We can rewrite the previous equality as

$$\left|a' \alpha^n - a \frac{10^\ell}{9}\right| < 2.4 \phi^n,$$

where we used that $|\nu| \leq 1/\sqrt{5}$ and $|\eta| < 1/2$. After dividing by $a' \alpha^n$, we obtain

$$\left|1 - \frac{a}{9a'a'^{-n} 10^\ell}\right| < \frac{8}{(1.13)^n},$$

where we used that $a/\phi > 1.13$ and $a' > 0.3$. Let us define

$$\Lambda = \ell \log 10 - n \log a + \log \theta_a,$$

where $\theta_a := a/9a'$, with $a \in [1, 9]$. It follows from (9) that

$$|e^\Lambda - 1| < \frac{8}{(1.13)^n}.$$

Now, we claim that $\Lambda$ is nonzero. Indeed, on the contrary, we would have $10^\ell \theta_a = a^n$ and so $a \cdot 10^\ell = 9a'a^n$. However, the minimal polynomial of $a'$, namely $44x^3 - 2x - 1$, has all its roots inside the unit circle and also $|\beta| = |\gamma| < 1$. Thus, we can conjugate the relation $a \cdot 10^\ell = 9a'a^n$ by the Galois automorphism $a \rightarrow \beta$ in order to obtain $a \cdot 10^\ell = 9(\beta')\beta^n$. By applying absolute values in the previous expression, we get

$$10^\ell \leq |a| \cdot 10^\ell = 9|\beta'|^n < 9$$

which contradicts the fact that $\ell \geq 1$.

When $\Lambda > 0$, then $\Lambda < e^\Lambda - 1 < 8 \cdot (1.13)^{-n}$, while for $\Lambda < 0$, we can use that $1 - e^{-|\Lambda|} = |e^\Lambda - 1| < 8 \cdot (1.13)^{-n}$ to infer that

$$|\Lambda| < e^{\Lambda} - 1 < \frac{8 \cdot (1.13)^{-n}}{1 - 8 \cdot (1.13)^{-n}} < 8 \cdot (1.13)^{-n+1}.$$
Hence, we have $|\Lambda| < 8 \times (1.13)^{-n+1}$. Therefore

$$\log |\Lambda| < -(n-1) \log(1.13) + \log 8. \quad (11)$$

Now, we can apply Lemma 1 for the choice of

$$a_1 := 10, \ a_2 := a, \ a_3 = \theta_8, \ b_1 = \ell, \ b_2 = -n, \ b_3 = 1,$$

where $\theta_8 := a/(9a')$. Since $\mathbb{Q}(a_1, a_2, a_3) = \mathbb{Q}(a', a)$, then $D < 9$ and so $C_1 < 1.2 \times 10^{10}$.

By the definition of logarithm height, we deduce that $h(a_1) = h(10) = \log 10$ and $h(a_2) = h(a) = (\log a)/3 < (\log 2)/3$. Now, we use Lemma 2 to obtain

$$h(a_3) = h(\theta_8) \leq h(a) + h(a') + h(9) \leq 2 \log 9 + \frac{\log 44}{3} < 5.7,$$

where we used that the minimal polynomial of $a'$ is $44x^3 - 2x - 1$. Thus, we choose

$$A_1 := 20.8, \ A_2 := 2.1 \text{ and } A_3 := 51.3.$$

If $\ell \geq 3$, we have

$$\max\{1, \max\{|b_j|A_j/A_1; 1 \leq j \leq 3\}\} \leq \max\{\ell, 0.2n\}.$$

Now, we use the bound in (3) and (7) together with the main equation to get

$$\phi^{n-1} < \phi^{n-2} + \alpha^{n-2} \leq F_n + T_n = a(10^\ell - 1)/9 < 10^\ell$$

and so $n < 4.8\ell + 1$. On the other hand,

$$2^n > 2\alpha^{n-1} > \phi^{n-1} + \alpha^{n-1} \geq F_n + T_n = a(10^\ell - 1)/9 > 10^{\ell - 1} \implies 3.3(\ell - 1) < n.$$

Summarizing, we have

$$3.3(\ell - 1) < n < 4.8\ell + 2.$$

However, $3.3(\ell - 1) \geq \ell$, for $\ell > 1$ and then, we can take $B := n$. Thus, Lemma 1, yields

$$\log |\Lambda| > -2.2 \cdot 10^{15} \log(117.4n). \quad (12)$$

By combining the estimates (11) and (12), we obtain

$$n < 8.5 \cdot 10^{15} \log(117.4n).$$

From this inequality, we deduce that $n < 3.9 \cdot 10^{17}$ and by the estimate $3.3(\ell - 1) < n$, we infer that $\ell < 1.2 \cdot 10^{17}$.

3.2. Reducing the Bound

The next step is to use some reduction method in order to reduce the bounds for $n$ and $\ell$. For that, let us suppose, without loss of generality, that $\Lambda > 0$ (the other case can be handled in the same way by observing that $0 < \Lambda' = -\Lambda$).

The inequality $0 < \Lambda < 8\alpha^{-n+1}$ can be rewritten explicitly as

$$0 < \ell \log 10 - n \log \alpha + \log \theta_8 < 110.2 \cdot (7.3)^{-\ell},$$

where we used that $n > 3.3(\ell - 1)$. 

Now, dividing the previous inequality by $\log \alpha$, we get

$$0 < \ell \gamma - n + \mu_a < 182.4 \cdot (7.3)^{-\ell}, \quad (13)$$

with $\gamma := \log 10 / \log \alpha$ and $\mu_a := \log \theta_a / \log \alpha$.

Since $\gamma \notin \mathbb{Q}$ (in fact, on the contrary, $\alpha^p$ would be rational for some positive integer $n$, which is proved to be false by using Binomial Theorem). Also, we denote $p_n / q_n$ as the $n$th convergent of the continued fraction expansion of $\gamma$.

Now, we shall apply Lemma 3 to reduce the bound on $\ell$. For that, we take $M = 9 \cdot 10^{16}$ and so

$$\frac{p_{39}}{q_{39}} = \frac{490343570750402983}{1297691494959974192}$$

is the first approximant of $\gamma$, which satisfies all requirements of the lemma. In fact, $q_{39} = 1297691494959974192 > 6M$. Also, if

$$\epsilon_a := \|\mu_a q_{39} \|- \|\gamma q_{39} \|,$$

for $a \in [1, 9]$, then $\min_{a \in [1,9]} \epsilon_a = \epsilon_7 = 0.0086114 \ldots$ (here, we used a simple routine in Mathematica software).

Hence, we are in a position to apply Lemma 3 for the choice of $A = 182.4$ and $B = 7.3$. Thus, we conclude that there is no solution of the Diophantine inequality in (13) for $\ell$ in the range

$$\left[\left\lfloor \frac{\log(Aq_{39} / \epsilon_7)}{\log B} \right\rfloor + 1, M \right] = [27, 9 \cdot 10^{16}].$$

However $\ell < M$, and then $\ell \leq 26$ yielding $n < 4.8 \cdot 26 + 2 < 127$. Thus, we write a Mathematica routine, which returns that the solutions of $F_n + T_n = a(10^{\ell} - 1)/9$, in the range $n \in [1, 126], \ell \in [1, 26]$ and $a \in [1, 9]$, are

$$(n, \ell, a) \in \{(1, 1, 2), (2, 1, 2), (3, 1, 4), (4, 1, 7)\}.$$

This finishes the proof. $\Box$

4. Conclusions

In this paper, we solve the Diophantine equation $F_n + T_n = a(10^{\ell} - 1)/9$, where $(F_n)_n$ and $(T_n)_n$ are the Fibonacci and Tribonacci sequences, respectively. In other words, we found all repdigits (i.e., positive integers with only one distinct digit in its decimal expansion), which can be written as the sum of a Fibonacci number and a Tribonacci number with the same index. In particular, we proved that the only repdigits with the desired property are the trivial ones, i.e., with only one digit ($\ell = 1$). To prove that, we use Baker’s theory and a reduction method due to Dujella and Pethő.

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