A CAUCHY PROBLEM FOR MINIMAL SPACELIKE SURFACES IN $\mathbb{R}^4_2$

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Abstract. A definition of isoclinic parametric surfaces in $\mathbb{R}^4_2$ is given in this work. It has been proved that an isoclinic conformal immersion in $\mathbb{R}^4_2$ comes from two holomorphics functions. A Cauchy problem was proposed and solved, which consists of constructing an isoclinic and minimal positive (negative) spacelike surface in $\mathbb{R}^4_2$ containing a given positive (negative) real analytic curve. By last, it was studied the important and well-known problem called Björling problem and some examples was given in the last section.

1. Introduction

Basically, it has been done in this work a study about spacelike surfaces in $\mathbb{R}^4_2$ of well-known classical problems.

If we consider two planes $P^2$ and $\overline{P}^2$ in $\mathbb{R}^4$, we can take the angle that a unit vector $v$ in $P^2$ makes with its orthogonal projection $\bar{v}$ in $\overline{P}^2$. When $v$ describes a circumference of radius 1 centred at the origin, that angle varies between two extreme values, in general different, or equivalently, that unit circumference in $P^2$ projects as an ellipse in $\overline{P}^2$ and the axes correspond to the extreme values of the above angle. In [16], Wong developed a curvature theory for surfaces in $\mathbb{R}^4$ based on these two angles between two tangent planes of the surface. When the angle remains constant, i.e., the ellipse is also a circumference, it is said that the planes are isoclinic to each other. An interesting connection with functions of one complex variable is the well-known theorem that establishes: a 2-dimensional surface of class $C^2$ in $\mathbb{R}^4$ has the property that its tangent 2-planes are all mutually isoclinic if, and only if the surface is a $R$-surface, i.e., a surface given in suitable rectangular coordinates $(x, y, u, v)$ in $\mathbb{R}^4$ by $u = u(x, y)$, $v = v(x, y)$, where $u(x, y)$ and $v(x, y)$ are the real and imaginary parts of an analytic function $f(x + iy)$. In the higher dimensional case, Wong shows in [17] that the only $n$-dimensional surfaces of class $C^2$ in $\mathbb{R}^{2n}$ ($n > 2$) whose tangent $n$-planes are all mutually isoclinic are the $n$-planes.

In the same way, we can consider two planes positive (negative) planes in $\mathbb{R}_2^4$ and take the angle that a unit vector in one of those planes makes with its orthogonal projection in
the other plane, depending on causal character of the plane spanned by those vectors. In this way, in Definition 2.2 we define positive (negative) planes in $\mathbb{R}^4_2$ to be isoclinic to each other, and using the operator $L$ given by (4) we give a characterization of those planes. Based on it, we present a definition of isoclinic parametric surfaces in $\mathbb{R}^4_2$, which will be objects of study of our work.

In the section 3, we propose and solve a Cauchy problem which ask about the existence of an isoclinic and minimal positive (negative) spacelike surface in $\mathbb{R}^4_2$ containing a given positive (negative) real analytic curve.

In the section 4, we deal with the well-known problem called Björling problem, in this case for positive (negative) spacelike surface in $\mathbb{R}^4_2$. It consists of constructing a minimal positive (negative) spacelike surface in $\mathbb{R}^4_2$ containing a given real analytic strip. This problem was proposed in Euclidean space $\mathbb{R}^3$ by Björling in 1844 and its solution obtained by Schwarz in 1890 through an explicit formula in terms of initial data. Thereafter, the Björling problem has been considered in other ambient spaces, including in larger codimension or with indefinite metrics. Some works in the literature are [1, 2, 3, 4, 6, 7, 13].

2. SOME ALGEBRAIC AND ANALYTIC PRELIMINARY FACTS

The 4-dimensional pseudo-Euclidean space $\mathbb{R}^4_2$ is the 4-dimensional vector space $\mathbb{R}^4$ equipped with the pseudo-Riemannian metric

$$ds^2 = -(dx^1)^2 - (dx^2)^2 + (dx^3)^2 + (dx^4)^2,$$

and oriented by the volume form

$$\omega = dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4.$$

The inner product associated to the quadratic form $ds^2$ is given by

$$\langle v, w \rangle = -v^1 w^1 - v^2 w^2 + v^3 w^3 + v^4 w^4.$$

The elements of the canonical basis of the vector space $\mathbb{R}^4$ we will denote by $e_i, i = 1, 2, 3, 4$. The six coordinate planes $\pi_{ij} = \text{span}[e_i, e_j]$, for $i, j = 1, 2, 3, 4$, are such that $\pi_{12}$ is a negative Euclidean plane, this means that $-ds^2(\pi_{12}) = (dx^1)^2 + (dx^2)^2$ and thus it has a negative definite induced metric; $\pi_{34}$ is a positive Euclidean plane, since $ds^2(\pi_{34}) = (dx^3)^2 + (dx^4)^2$; whereas the other four coordinate planes $\pi_{13}, \pi_{14}, \pi_{23}$ and $\pi_{24}$ are Lorentzian planes, for example $ds^2(\pi_{13}) = -(dx^1)^2 + (dx^3)^2$.

A positive (negative) spacelike plane is a 2-dimensional subspace $V^2 \subset \mathbb{R}^4_2$ such that the induced metric $ds^2(V^2)$ is positive (respectively negative) definite. Given a positive
(negative) spacelike plane $V^2$, we say that a basis $\{v_1, v_2\}$ for $V^2$ is a $\epsilon \lambda$-isothermic basis if, and only if,

\[(3) \quad \langle v_i, v_j \rangle = \epsilon \lambda^2 \delta_{ij} \quad \text{for} \quad i, j = 1, 2; \quad \epsilon = 1 \ (\epsilon = -1).\]

If it is given a negative plane $V^2$ with an orthonormal basis $\{v_1, v_2\}$ and induced metric

\[-ds^2(v_i, v_j) = -\langle v_i, v_j \rangle = \delta_{ij},\]

then the geometry in this negative plane is the Euclidean geometry.

If $\{v, w\}$ is a $(-1)$-orthonormal set, we say that the negative plane $V^2 = \text{span}[v, w]$ has positive induced orientation if, and only if, the projection from $\mathbb{R}^4$ onto $\pi_{12}$ defined by

\[pr_{12}(v^1, v^2, v^3, v^4) = v^1e_1 + v^2e_2\]

give us a positive basis relative to $\{e_1, e_2\}$, in this order. Analogously, we define the positive induced orientation for positive planes with the projection $pr_{34}$. If it is given a negative plane $V^2 = \text{span}[v_1, v_2]$ and a positive plane $W^2 = \text{span}[w_1, w_2]$, both positively oriented, such that the set $\{v_1, v_2, w_1, w_2\}$ is an orthonormal set, then this set is a positive orthonormal basis of $\mathbb{R}^4$.

When necessary, we will use the following notation for the projections onto the planes $\pi_{12}$ and $\pi_{34}$:

\[pr_{12}(v) = \hat{v} \quad \text{and} \quad pr_{34}(v) = \tilde{v},\]

thus $v = \hat{v} + \tilde{v}$.

Two useful elements of the orthogonal group of $\mathbb{R}^4_2$ are given in matrix form by

\[(4) \quad L = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.\]

We have that: $\det L = \det N = 1$, $L^2 + I = 0$, $N^2 - I = 0$ and $LN + NL = 0$, where $I$ is the identity matrix.

If $v = (v^1, v^2, v^3, v^4)$ is a positive (negative) vector of $\mathbb{R}^4_2$, then $L(v) = (-v^2, v^1, -v^4, v^3)$ is another positive (negative) vector such that $\{v, L(v)\}$ is a $\epsilon |v|$-isothermic basis for the positive (negative) plane spanned by these vectors. We have an analogous statement for the transformation $N$. 
2.1. **Spheres in** \( \mathbb{R}^3_2 \). The positive (if \( \epsilon = 1 \)) sphere of radius \( r > 0 \), the negative (\( \epsilon = -1 \)) sphere of radius \( r > 0 \) and the null (\( \epsilon = 0 \)) null sphere in \( \mathbb{R}^4_2 \) are defined by

\[
S^3(\epsilon r) = \{ v \in \mathbb{R}^4_2 : \langle v, v \rangle = \epsilon r^2 \}.
\]

For \( r = 1 \), we give a parametrization for each of these spheres as follows:

If we considerer the negative sphere \( S^3_2(-1) \) and \( w = (a, b, x, y) \in S^3_2(-1) \), then we have

\[
a^2 + b^2 = 1 + x^2 + y^2 \quad \text{so we can take the parametric hypersurface}
\]

\[
F(\varphi, \theta, \eta) = (\cosh \varphi \cos \theta, \cosh \varphi \sin \theta, \sinh \varphi \cos \eta, \sinh \varphi \sin \eta)
\]

such that \( F(\mathbb{R} \times [0, 2\pi]^2) = S^3_2(-1) \).

In the same way, if we considerer the positive sphere \( S^3_2(1) \) and \( w = (a, b, x, y) \in S^3_2(1) \), then we have

\[
a^2 + b^2 = 1 + x^2 + y^2 \quad \text{so we can take the parametric hypersurface}
\]

\[
G(\varphi, \theta, \eta) = (\sinh \varphi \cos \theta, \sinh \varphi \sin \theta, \cosh \varphi \cos \eta, \cosh \varphi \sin \eta)
\]

such that \( G(\mathbb{R} \times [0, 2\pi]^2) = S^3_2(1) \).

Now, if we take the null sphere \( S^3_2(0) \) and \( w = (a, b, x, y) \in S^3_2(0) \) then we have \( a^2 + b^2 = x^2 + y^2 \) so we can take the parametric hypersurface

\[
H(\varphi, \theta, \eta) = (e^\varphi \cos \theta, e^\varphi \sin \theta, e^\varphi \cos \eta, e^\varphi \sin \eta)
\]

such that \( H(\mathbb{R} \times [0, 2\pi]^2) = S^3_2(0) \).

Let \( pr_{12}(w) = \hat{w} \) and \( pr_{34}(w) = \tilde{w} \) be the orthogonal projections of the vector \( w \in \mathbb{R}^4_2 \) onto the coordinate planes \( \pi_{12} \) and \( \pi_{34} \), respectively. If \( w \in S^3_2(-1) \) then \( \langle \hat{w}, w \rangle = -\cosh^2 \varphi \), and if \( w \in S^3_2(1) \) then \( \langle \tilde{w}, w \rangle = \cosh^2 \varphi \). The following proposition is the basis for our definition of isoclinic planes.

**Proposition 2.1.** Let \( \{v, w\} \) be a \((-1)\)-isothermic basis of a negative spacelike plane \( V^2 \) in \( \mathbb{R}^4_2 \). Let's take the parametric curve \( x(\theta) = \cos \theta \ v + \sin \theta \ w \) in the sphere \( S^3_2(-1) \). Then, the orthogonal projection of this negative Euclidean circumference onto the plane \( \pi_{12} \) is a negative Euclidean circumference of radius \( r \) if, and only if the hyperbolic angle function \( \varphi(\theta) \) given by

\[
\varphi(\theta) = \langle \dot{x}(\theta), x(\theta) \rangle = -\cosh^2 \varphi(\theta),
\]

is the constant function \(-r^2\).

**Proof.** Taking \( E, F \) and \( G \) defined by \( E = -\langle \dot{v}, \dot{v} \rangle \), \( F = -\langle \dot{v}, \tilde{w} \rangle \) and \( G = -\langle \tilde{w}, \tilde{w} \rangle \), we obtain \( \cosh^2 \varphi(\theta) = E \cos^2 \theta + 2F \cos \theta \sin \theta + G \sin^2 \theta \), which is a circumference if, and only if \( E = G \) and \( F = 0 \), if, and only if \( E = \cosh^2 \varphi(\theta) = r^2 \). Moreover, \( \varphi \neq 0 \) imply that \( V^2 \cap \pi_{12} = \{0\} \) and, \( \varphi = 0 \) imply that \( V^2 = \pi_{12} \). \( \square \)
Note that we have an analogous proposition if we consider positive spacelike planes in $\mathbb{R}^4_2$ and the orthogonal projection onto the plane $\pi_{34}$. In this case, it is considered the Lorentzian spacelike angle between two positive vectors than span a timelike plane (see [15]).

**Definition 2.2.** Let $V^2$ be a negative spacelike plane in $\mathbb{R}^4_2$. Consider a $(-1)$-isothermic basis of $V^2$. We say that the negative planes $V^2$ and $\pi_{12}$ are isoclinic to each other if, and only if the hyperbolic angle $\varphi$ between $V^2$ and $\pi_{12}$ is constant. Analogously, we define the notion of isoclinic planes for a positive spacelike plane $W^2$ and $\pi_{34}$.

Let $f : M \to \mathbb{R}^4_2$ be a parametric surface. If the tangent plane at each point of this surface is a positive (negative) spacelike plane and is isoclinic to $\pi_{34}$ ($\pi_{12}$), we say that $f : M \to \mathbb{R}^4_2$ is a positive (negative) isoclinic surface.

The following proposition characterizes the isoclinic planes using the transformation $L$ given by (4).

**Proposition 2.3.** Let $L : \mathbb{R}^4_2 \to \mathbb{R}^4_2$ be the linear transformation given by (4). We have that:

(1) $x \in S^3_2(-1)$ if, and only if $L(x) \in S^3_2(-1)$ and, $x \in S^3_2(1)$ if, and only if $L(x) \in S^3_2(1)$.

(2) If $x \in S^3_2(-1)$, then span$[x, L(x)]$ is a negative plane isoclinic to $\pi_{12}$.

(3) If $x \in S^3_2(1)$, then span$[x, L(x)]$ is a positive plane isoclinic to $\pi_{34}$.

Reciprocally, if the negative planes $V^2 = \text{span}[x, y]$ and $\pi_{12}$ in $\mathbb{R}^4_2$ are isoclinic to each other, then $y = L(x)$. Analogously, if the positive planes $W^2 = \text{span}[x, y]$ and $\pi_{34}$ in $\mathbb{R}^4_2$ are isoclinic to each other, then $y = L(x)$.

**Proof.** We only need to show the reciprocal. We have that $\{\hat{x}, \hat{y}\}$ is a $(-\cosh \varphi)$-isothermic basis for $\pi_{12}$, where $\hat{x} = x^1e_1 + x^2e_2$ and $\hat{y} = y^1e_1 + y^2e_2$. Hence $y^1 = -x^2$ and $y^2 = x^1$. Thus, we also need to have $y^3 = -x^4$ and $y^4 = x^3$ to preserve orientation and ($-1$)-isothermality. □

Now, using the transformation $N$ given by (4), it follows:

**Corollary 2.4.** Let $V^2 = \text{span}[v_1, v_2]$ be a negative spacelike plane in $\mathbb{R}^4_2$ isoclinic to $\pi_{12}$. Then, the orthogonal complement of $V^2$ in $\mathbb{R}^4_2$, is the positive spacelike plane span$[N(v_1), N(v_2)]$, which is isoclinic to $\pi_{34}$ with Lorentzian spacelike angle between them equal to the hyperbolic angle between $V^2$ and $\pi_{12}$.

Our first example of an isoclinic surface in $\mathbb{R}^4_2$ is the following.
Example 2.5. Let \( f : \mathbb{R}^2 \to \mathbb{R}^4 \) be the parametric surface given by
\[
f(u, v) = (u, v, (u^2 - v^2)/2, uv).
\]
For \( U = \{w = u + iv \in \mathbb{C} : |w| < 1\}, \) the subset \( S_- = f(U) \) is a 2-dimensional negative submanifold of \( \mathbb{R}^4, \) with induced metric \(-ds^2 = (1 - u^2 - v^2)dudv.\)

For \( \Omega = \mathbb{C} \setminus \overline{U}, \) the subset \( S_+ = f(\Omega) \) is a 2-dimensional positive submanifold of \( \mathbb{R}^4, \) with induced metric \( ds^2 = (1 - u^2 - v^2)dudv.\)

These two surfaces are isoclinic surfaces and they are non flat (i.e., the Gauss curvature does not vanishes).

In fact, since \( f_u = (1, 0, u, v) \) and \( f_v = (0, 1, -v, u) \) it follows that \( E = \langle f_u, f_u \rangle = -1 + u^2 + v^2, \) \( G = \langle f_v, f_v \rangle = -1 + u^2 + v^2 \) and \( F = \langle f_u, f_v \rangle = 0.\)

Now, for \( w \in U, \) consider the \((-1)-\)isothermic negative basis \( \{a, b\} \) of \( T_{f(w)}S_-, \) where \( a = \frac{1}{\sqrt{-E(w)}} f_u(w) \) and \( b = \frac{1}{\sqrt{-G(w)}} f_v(w). \) The hyperbolic angle \( \varphi(w) \) is such that
\[
\cosh^2 \varphi(w) = \langle \cos \theta \hat{a} + \sin \theta \hat{b}, \cos \theta a + \sin \theta b \rangle = \frac{1}{1 - u^2 - v^2}.
\]

Note that \( \varphi(w) \to \infty \) when \( |w| \to 1.\)

On the other hand, we have that a basis for the normal bundle with positive induced orientation is given by
\[
N_1(w) = \frac{1}{\sqrt{-E(w)}}(u, -v, 1, 0) \quad \text{and} \quad N_2(w) = \frac{1}{\sqrt{-G(w)}}(v, u, 0, 1).
\]

We also see that the normal plane at each point is an positive plane isoclinic to \( \pi_{34}.\)

The second quadratic form of \( S_- \) is given by \( B_{ij} = \langle D_{ij}f, N_1 \rangle N_1 + \langle D_{ij}f, N_2 \rangle N_2 \) where
\[
B_{ij}^1 = \langle D_{ij}f, N_1 \rangle = \frac{1}{\sqrt{-E}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad B_{ij}^2 = \langle D_{ij}f, N_2 \rangle = \frac{1}{\sqrt{-G}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\]

Therefore, the Gauss curvature is:
\[
K_{S_-}(w) = K_{S_+}(w) = \frac{\det B_{ij}^1 + \det B_{ij}^2}{\det g_{ij}} = -\frac{2}{(1 - u^2 - v^2)^3}.
\]

2.2. On \( \mathbb{C}P^3.\) Let us extend the symmetric bilinear form \( \langle , \rangle \) of index 2 in \( \mathbb{R}^2 \) to one symmetric bilinear form in the 4-dimensional complex vector space \( \mathbb{C}^4 \equiv \mathbb{R}^4 \oplus i\mathbb{R}^4 \) by
\[
\langle x_1 + iy_1, x_2 + iy_2 \rangle = (\langle x_1, x_2 \rangle - \langle y_1, y_2 \rangle) + i(\langle x_1, y_2 \rangle + \langle y_1, x_2 \rangle).
\]

We can consider the complex projective space \( \mathbb{C}P^3 \) of complex dimension 3, which corresponds to the space of all the complex lines through the origin of \( \mathbb{C}^4.\)
Now, if it is given a positive plane $V^2 = \text{span}[v, w]$ where $\{v, w\}$ is a $\lambda$-isothermic basis, for each complex number $\mu \neq 0$ we have that $\{\Re(\mu z), \Im(\mu z)\}$, with $z = v + iw$, is a $|\mu|\lambda$-isothermic basis of the plane $V^2$. Therefore, we can identify the plane $V^2$ with the point $[v + iw]$ of $CP^3$, and define the Grassmannian of the positive planes as a quadric in $CP^3$ (See [9] for details).

So, we can define the Grassmannian of the positive planes in $\mathbb{R}^4_2$ as the complex subquadric of $CP^3$:

$$Q^2_{\text{pos}} = \{ [z] \in CP^3 : \langle z, z \rangle = 0 \text{ and } \langle z, \bar{z} \rangle > 0 \}.$$  

In same way, we can define the Grassmannian of the negative planes in $\mathbb{R}^4_2$ as the complex subquadric of $CP^3$:

$$Q^2_{\text{neg}} = \{ [z] \in CP^3 : \langle z, z \rangle = 0 \text{ and } \langle z, \bar{z} \rangle < 0 \}.$$  

Remark. In $\mathbb{R}^4_2$ there are pair of null vectors $x$ and $y$ such that $\langle x, y \rangle = 0$, but they are linearly independent. For example, $x = (1, 0, 1, 0)$ and $y = (0, 1, 0, 1)$. In this case, writing $z = x + iy$ we have

$$\langle z, z \rangle = \langle z, \bar{z} \rangle = 0.$$  

From the above, we have that $Q^2_{\text{null}} = \{ [z] \in CP^3 : \langle z, z \rangle = \langle z, \bar{z} \rangle = 0 \} \neq \emptyset$. Therefore,

$$Q^2 = \{ [z] \in CP^3 : \langle z, z \rangle = 0 \}$$

is the union of the pairwise disjoint subquadrics $Q^2_{\text{pos}}, Q^2_{\text{neg}}$ and $Q^2_{\text{null}}$.  

**Theorem 2.6.** Let $\mathbb{C} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere. The map

$$\Phi([z]) = \left( \frac{z^1 + iz^2}{z^3 - iz^4}, \frac{z^1 - iz^2}{z^3 - iz^4} \right)$$

is a homeomorphism from $Q^2$ onto $\mathbb{C} \times \mathbb{C}$.

**Proof.** We will give a homogeneous coordinate system on $Q^2$. To this end, we see that $[z] \in Q^2$ if, and only if $-(z^1)^2 - (z^2)^2 + (z^3)^2 + (z^4)^2 = 0$ if, and only if

$$(z^1 + iz^2)(z^1 - iz^2) = (z^3 + iz^4)(z^3 - iz^4).$$

For now, suppose that $z^3 - iz^4 \neq 0$, then we obtain

$$\frac{z^1 + iz^2}{z^3 - iz^4} = \frac{z^3 + iz^4}{z^3 - iz^4},$$
So, defining the complex numbers \( x = \frac{z^1 + iz^2}{z^1 - iz^2} \) and \( y = \frac{z^1 - iz^2}{z^1 - iz^2}, \) we get

\[
(10) \quad z = \mu(x + y, -i(x - y), 1 + xy, i(1 - xy)) \quad \text{with} \quad \mu = \frac{z^3 - iz^4}{2}.
\]

Now, if \( z^3 = \pm iz^4 \neq 0, \) then we must have \( z^1 = \pm iz^2. \) Hence, we can take

\[
z = \eta(x, \pm ix, 1, \pm i) \quad \text{for} \quad x = \frac{z^1}{z^3} \quad \text{and} \quad \eta = z^3.
\]

Finally, if \( z^3 = iz^4 = 0, \) then \( z = z^1(1, \pm i, 0, 0) \) is a complex representation of the plane \( \pi_{12} \in Q^2_{\text{neg}}. \)

On the other hand, since \( [z(x, y)] = [xy z(1/x, 1/y)] \) it follows the following.

**Claim.** For \( [z(x, y)] = [x + y, i(x - y), 1 + xy, i(1 - xy)], \) we have that

\[
\lim_{y \to \infty} [z(x, y)] = [1, -i, x, -ix], \quad \lim_{x \to \infty} [z(x, y)] = [1, i, y, -iy]
\]

and \( \lim_{(x, y) \to (\infty, \infty)} [z(x, y)] = [0, 0, 1, -i]. \)

To complete, we note that \( \langle z, \overline{z} \rangle = 2\mu \overline{\mu}(1-x\overline{y})(1-\overline{x}y) = 0 \) if, and only if \( [z] \in Q^2_{\text{null}}, \) that corresponds to \( ds^2|_{z} = 0 \) the induced metric of these null planes. \( \square \)

**Definition 2.7.** Let \( f : M \to \mathbb{R}^4_2 \) be a conformal immersion, \( M \subseteq \mathbb{C} \) connected open. We say that \( f \) is positive (negative) spacelike if it is a parametric surface for which

\[
f_w = \frac{\partial f}{\partial w} = \frac{1}{2} \left( \frac{\partial f}{\partial u} - i \frac{\partial f}{\partial v} \right)
\]

satisfy \( [f_w] \in Q^2_{\text{pos}} \left( Q^2_{\text{neg}} \right), \) where \( w = u + iv \) is a conformal parameter for \( M. \)

**Remark.** When the complex 1-form \( \beta = f_u dw \) has no real periods or, if \( M \) is a simple connected open subset of \( \mathbb{C}, \) it is well defined

\[
2\Re \int_{w_0}^{w} \beta = \int_{w_0}^{w} 2\Re(\beta) = \int_{(u_0, v_0)}^{(u, v)} f_u du + f_v dv.
\]

On the other hand, Let us note that each real valued exact 1-form \( d\phi = \phi_u du + \phi_v dv \) can be write in complex variable as \( d\phi = \phi_u dw + \phi_v d\overline{w} = 2\Re(\phi_u dw). \)

We have the following integral representation for conformal immersions:
Proposition 2.8 (Weierstrass integral formula). Let \( f : M \to \mathbb{R}^2_2 \) be a positive (negative) spacelike conformal immersion. We have that

\[
(11) \quad f(w) = f(w_0) + 2\Re \int_{w_0}^{w} \mu(\xi)W(x(\xi), y(\xi)) d\xi,
\]

where \([W(x(w), y(w))]] \in Q^{2}_{pos} \ (\left[ W(x(w), y(w)) \right]) \in Q^{2}_{neg} \), for all \( w \in M \). A reciprocal is also true.

Definition 2.9. Let \( f : M \to \mathbb{R}^4_2 \) be a parametric surface. We say that \( f \) is a minimal surface if the mean curvature vector \( H_f(w) = 0 \) for all \( w \in M \).

Lemma 2.10. Suppose that \( f : M \to \mathbb{R}^4_2 \) is given by (11). Then, \( f \) is a minimal surface if, and only if \( \mu(w), x(w) \) and \( y(w) \) are holomorphic functions from \( M \) into \( \mathbb{C} \).

Proof. The induced metric would be \( \pm ds^2(f) = \lambda^2 \delta_{ij} du^i du^j \) since \( f \) is a conformal parametric surface with \([f_w] \in Q^2_{pos} (\left[f_w \right] \in Q^2_{neg})\). The mean curvature vector is defined by Laplace-Beltrami equation (See [9] for details), so

\[
\pm H_f = \frac{2}{\lambda^2} \Delta M f = \frac{2}{\lambda^2} f_{w\overline{w}} = \frac{1}{2\lambda^2} (f_{uu} + f_{vv}) = 0.
\]

Hence, we will obtain \( H_f = 0 \) if, and only if \( (\mu W(x, y))_{\overline{w}} = 0 \). It follows from (10) that \( \mu_{\overline{w}} = 0, \ x_{\overline{w}} = 0 \) and \( y_{\overline{w}} = 0 \). \( \square \)

2.3. The cross product. Given three vectors \( x, y, z \in \mathbb{R}^4_2 \), by definition, the cross product of these vectors is the unique vector \( X(x, y, z) \) such that for each vector \( v \in \mathbb{R}^4_2 \) the following equation holds:

\[
(12) \quad \omega(x, y, z, v) = -\omega(v, x, y, z) = (X(x, y, z), v).
\]

Now, if we define the real numbers \( \Delta_{ijk} \), for \( i < j < k \) and \( i, j, k = 1, 2, 3, 4 \), by:

\[
\Delta_{ijk} = \begin{vmatrix}
  x^i & x^j & x^k \\
  y^i & y^j & y^k \\
  z^i & z^j & z^k
\end{vmatrix},
\]

then we obtain the formal determinant defined by Laplace expansion on the first line

\[
X(x, y, z) = -\begin{vmatrix}
  -e_1 & -e_2 & e_3 & e_4 \\
  x^1 & x^2 & x^3 & x^4 \\
  y^1 & y^2 & y^3 & y^4 \\
  z^1 & z^2 & z^3 & z^4
\end{vmatrix}.
\]
In coordinates, it takes the form
\[ \mathfrak{X}(x, y, z) = (\Delta_{234}, -\Delta_{134}, -\Delta_{124}, \Delta_{123}). \]
Moreover, we have that
\[ \omega(x, y, z, \mathfrak{X}(x, y, z)) = -\omega(\mathfrak{X}(x, y, z), x, y, z) = \sum_{i<j<k} (\Delta_{ijk})^2 \geq 0. \]

Now, let \( f : M \to \mathbb{R}^4 \) be a positive (negative) spacelike conformal immersion with normal bundle given by an orthonormal basis \( \{A, B\} \). Taking \( f_w = \frac{1}{2}(f_u - if_v) \), we have that \( \mathfrak{X}(f_u, A, B) = f_v \) and \( \mathfrak{X}(f_v, A, B) = -f_u \), assuming the positive orientation of the referential \( \{A, B, f_u, f_v\} \). Then,
\[ \mathfrak{X}(f_w, A, B) = \frac{1}{2}(\mathfrak{X}(f_u, A, B) - if_v(f_u, A, B)) = \frac{1}{2}(f_v + if_u) = if_w. \]

2.4. A type problem. The example \( \mathfrak{2.5} \) suggests the construction of spacelike parametric surfaces in \( \mathbb{R}^4 \) that are each one the graph of a holomorphic function \( Z(w) = \phi(w) + i\psi(w) \) defined in all complex plane. Therefore, the surface is given by
\[ f(w) = (u, v, \phi(u, v), \psi(u, v)). \]
In this case, the induced metric for those positive (negative) spacelike conformal surfaces is such that \( \lambda^2(f) = -1 + |Z'(w)|^2 \). By the small Picard Theorem of complex analysis, to obtain \( \lambda^2 > 0 \) in all \( \mathbb{C} \), the holomorphic function \( Z'(w) \) need omits more than two points, thus is a constant function. Note that the mean curvature vector \( H_f = 0 \), because \( f_w\mathfrak{m}(w) = 0 \) for each \( w \in \mathbb{C} \).

**Proposition 2.11.** If \( f : \mathbb{C} \to \mathbb{R}^4 \) is the graph of a holomorphic function \( Z(w) \), where result a positive (negative) spacelike parametric surface in \( \mathbb{R}^4 \), then \( Z(w) = aw + b \) for constant \( a, b \in \mathbb{C} \).

Now, we will give an example of conformal parametric surface non trivial in \( \mathbb{R}^4 \), which has mean curvature vector \( H_f = 0 \), its domain is all complex plane and it is free of singularities.

**Example 2.12.** Consider the formula (11) with \( x(w) = w \) and \( y(w) = iw \), \( w \in \mathbb{C} \). Then, we obtain
\[ f(w) = 2\Re \left( \frac{w^2 - iw^2}{2}, \frac{w^2 + iw^2}{2}, \frac{3w - iw^3}{3}, \frac{3w + iw^3}{3} \right). \]
It follows that \( f_w = (w - iw, i(w + iw), 1 - iw^2, i(1 + iw^2)) \). A simple calculation shows that \( \langle f_w, f_w \rangle = 0, \langle f_w, \overline{f_w} \rangle = 1 + |w|^4 \geq 1 \) and \( f_{w\overline{w}} = 0 \). Therefore, \([f_w] \in Q_{\text{pos}}^2\) and \(H_f = 0\).

2.5. The normal bundle. We have the followings result:

**Lemma 2.13.** Let \([v] = [v_1+iv_2] \in Q_{\text{neg}}^2\) be and \([u] = [u_1+iu_2] \in Q_{\text{pos}}^2\). Then, the set \(\{v_1, v_2, u_1, u_2\}\) is an orthonormal basis of \(\mathbb{R}_2^4\) if, and only if \(\langle v, u \rangle = \langle v, \overline{u} \rangle = 0\).

**Proof.** From 
\[
\langle v, u \rangle = (\langle v_1, u_1 \rangle - \langle v_2, u_2 \rangle) + i(\langle v_1, u_2 \rangle + \langle v_2, u_1 \rangle) = 0
\]
and
\[
\langle v, \overline{u} \rangle = (\langle v_1, u_1 \rangle + \langle v_2, u_2 \rangle) + i(\langle v_1, u_2 \rangle - \langle v_2, u_1 \rangle) = 0,
\]
it follows that
\[
\langle v_1, u_1 \rangle = 0, \quad \langle v_1, u_2 \rangle = 0, \quad \langle v_2, u_1 \rangle = 0 \quad \text{and} \quad \langle v_2, u_2 \rangle = 0.
\]
Therefore, each vector of \([v]\) is orthogonal to any vector of \([u]\). In symbols, \([v] \perp [u]\). □

**Lemma 2.14.** Let’s take in \(Q^2\) the following planes 
\([u] = [x + y, -i(x - y), 1 + xy, i(1 - xy)]\) and \([v] = [1 + ab, i(1 - ab), a + b, -i(a - b)]\). Then, they are mutually orthogonal if, and only if either
\[
a = x \text{ and } b = \overline{y}, \quad \text{or} \quad a = \frac{1}{x} \text{ and } b = \frac{1}{y}.
\]

**Proof.** In fact, it follows from
\[
\langle u, v \rangle = -(x + y + abx + aby) - (x - y - abx + aby)
\]
\[
+ (a + b + axy + bxy) + (a - b - axy + bxy)
\]
\[
= -2(x - a)(1 - by),
\]
and
\[
\langle u, \overline{v} \rangle = -(x + y + \overline{abx} + \overline{aby}) + (x - y - \overline{abx} + \overline{aby})
\]
\[
+ (\overline{a} + \overline{b} + \overline{axy} + \overline{bxy}) - (\overline{a} - \overline{b} - \overline{axy} + \overline{bxy})
\]
\[
= -2(y - \overline{b})(1 - \overline{ax}).
\]
□
Now, we consider the operator $\tilde{J} : Q^2 \rightarrow Q^2$ defined as follows:

$$\tilde{J}([x + y, -i(x - y), 1 + xy, i(1 - xy)]) = [1 + x\overline{y}, i(1 - x\overline{y}), x + \overline{y}, -i(x - \overline{y})].$$

Note that $\tilde{J}([z]) \perp [z]$.

**Proposition 2.15.** The normal operator $\tilde{J}$ is a smooth injective map from $Q^2$ onto $Q^2$ for which we have:

$$\tilde{J}(Q^2_{\text{neg}}) = Q^2_{\text{pos}}, \quad \tilde{J}(Q^2_{\text{pos}}) = Q^2_{\text{neg}} \quad \text{and} \quad \tilde{J}(Q^2_{\text{null}}) = Q^2_{\text{null}}.$$ 

Furthermore, if we take $[W(x, y)] = [x + y, -i(x - y), 1 + xy, i(1 - xy)]$ then

$$\tilde{J}([W(x, y)]) = [W(x, 1/\overline{y})].$$

As expected, it is an idempotent operator on $Q^2$, i.e., $\tilde{J} \circ \tilde{J} = \text{Id}$.

### 3. A Cauchy problem for isoclinic spacelike surfaces in $\mathbb{R}^4_2$

From now on, we will assume that $I = (-r, r)$ is a non-empty interval of the real line $\mathbb{R} \subset \mathbb{C}$ and, $M \subset \mathbb{C}$ is a simply connected and connected open subset such that $I \subset M$.

We would like to recall the following facts of complex analysis: if two holomorphic functions $x(w)$ and $y(w)$ from $M$ into $\mathbb{C}$ that coincide on $I$, then they coincide on the whole domain $M$, since $I$ has accumulation points.

Moreover, if it is given a real analytic function $x(t)$ from $I$ into $\mathbb{R}$, then the line integral

$$x(w) = x(0) + \int_0^w x'(t)dt$$

give us a holomorphic function $x : M \rightarrow \mathbb{C}$, which is the unique holomorphic extension of the real valued function $x(t)$.

Now, if $c(t) = (c_1(t), c_2(t), c_3(t), c_4(t))$ is a real analytic curve from $I$ into $\mathbb{R}^4_2$, we have that the holomorphic function $C(w) = (c_1(w), c_2(w), c_3(w), c_4(w))$ from $M$ into $\mathbb{C}^4$, where $c_i(w)$ is the holomorphic extension of the function $c_i(t)$, is the unique holomorphic extension of the curve $c(t)$.

We propose the following Cauchy problem which consists of constructing an isoclinic surface under certain conditions.

**Problem 3.1.** Given a positive (negative) real analytic curve $c : I \rightarrow \mathbb{R}^4_2$, thus $\langle c'(t), c'(t) \rangle > 0$ ($< 0$), to construct a conformal immersion $f : M \rightarrow \mathbb{R}^4_2$ satisfying the following conditions:

1. $f(t, 0) = c(t)$ for each $t \in I$ (extension condition),
2. $H_f(w) = 0$ for each $w \in M$ (minimal condition),
Then the following conditions:

(3) \([f_w(w)] \in Q^2_{pos} (|f_w(w)| \in Q^2_{neg})\) and is isoclinic to \(\pi_{34} (\pi_{12})\), for each \(w \in M\).

From condition (1) and taking a conformal parameter \(w = u + iv\) for \(M\), we will have \(f_u(t,0) = c'(t)\) and hence it would need to have that the vector field \(f_v(t,0)\) be pointwise orthogonal to \(c'(t)\) with \(\langle f_v(t,0), f_v(t,0) \rangle = \langle c'(t), c'(t) \rangle\).

Using the linear transformation given by (4), we define an isoclinic distribution, along our curve \(c(t)\), by \(D(t) = [c'(t) - iL(c'(t))]\), \(t \in I\). Now, consider the extension of the operator \(L\) to the complex vector space \(\mathbb{C}^4\) and also denote it by \(L\). We have the following map \(f : M \to \mathbb{R}^4_2\) defined by

\[
f(w) = c(0) + \Re \int_0^w (C'(\xi) - iL(C'(\xi)))d\xi,
\]

where \(C(w)\) is the unique holomorphic extension of \(c(t)\).

Since \(C(t,0) = c(t)\) we have that \(f(t,0) = c(t)\). From \(f_w(w) = \frac{1}{2}(C'(w) - iL(C'(w)))\) it follows \(f_{w\overline{w}} = 0\). To see that the conditions of our problem holds, we need of the following algebraic lemma.

**Lemma 3.2.** Let \(z = x + iy \in \mathbb{C}^4\) be with \(x\) and \(y\) positive (negative) vectors in \(\mathbb{R}^4_2\). Then

\[
w = z + iL(z) = x - L(y) + i(L(x) + y)
\]
satisfy the following conditions: \(\ll w, w \gg = 0\) and \(\ll w, \overline{w} \gg > 0\) (\(\ll w, \overline{w} \gg < 0\)).

**Proof.** Firstly,

\[
\ll w, w \gg = \langle x, x \rangle + \langle L(y), L(y) \rangle - 2\langle x, L(y) \rangle - (\langle L(x), L(x) \rangle + \langle y, y \rangle + 2\langle L(x), y \rangle).
\]

Since \(\langle L(y), L(y) \rangle = \langle y, y \rangle, \langle L(x), L(x) \rangle = \langle x, x \rangle\) and \(\langle x, L(y) \rangle = \langle L(x), y \rangle\), it follows that \(\ll w, w \gg = 0\).

Secondly,

\[
\ll w, \overline{w} \gg = \langle x, x \rangle + \langle y, y \rangle - 2\langle x, L(y) \rangle + (\langle x, x \rangle + \langle y, y \rangle + 2\langle x, L(y) \rangle)
\]

\[
= 2(\langle x, x \rangle + \langle y, y \rangle).
\]

Thus, being \(x\) and \(y\) positive (negative) vectors in \(\mathbb{R}^4_2\), we have that \(\ll w, \overline{w} \gg > 0\) (\(\ll w, \overline{w} \gg < 0\)).

**Remark.** We see from the example [2.3] that possibly we need to restrict the domain of the solutions of our problem, since we can obtain positive (negative) isoclinic surfaces with metric singularities when we extend the given functions.

We obtain the following result:
Theorem 3.3. Let $c : I \to \mathbb{R}^2$ be a positive (negative) real analytic curve. There exists a simply connected and connected open subset $M \subset \mathbb{C}$ with $I \subset M$ for what the map given by
\begin{equation}
    f(w) = c(0) + \Re \int_0^w (C'(\xi) - iL(C'(\xi)))d\xi,
\end{equation}
establishes a positive (negative) isoclinic and minimal conformal immersion (free of singularities in the induced metric) from $M$ into $\mathbb{R}^4$ such that $f(t,0) = c(t)$. Moreover, this is the unique solution of the Cauchy problem (3.1).

Proof. The previous Lemma guarantees that (14) is a conformal immersion satisfying the conditions (2) and (3) of our problem. Now, if we have another solution $g : M \to \mathbb{R}^4$, then $g$ satisfies the condition (1), thus $f$ and $g$ coincide on the interval $I$. Hence, they must coincide on the domain $M$. □

Example 3.4. Let $\Psi(z)$ and $\Phi(z)$ be two holomorphic functions from $M$ into $\mathbb{C}$. The map
\begin{equation}
    f(w) = \left( \frac{\Psi(w) + \overline{\Psi}(w)}{2}, \frac{\Psi(w) - \overline{\Psi}(w)}{2i}, \frac{\Phi(w) + \overline{\Phi}(w)}{2}, \frac{\Phi(w) - \overline{\Phi}(w)}{2i} \right),
\end{equation}
parametrizes an isoclinic minimal surface. When $|\Psi'| < |\Phi'|$ it is a positive parametric surface and when $|\Psi'| > |\Phi'|$ it is a negative parametric surface. The set of metric singularities is:
\begin{equation}
    \text{Sign} = \{ w \in M : |\Psi'| = |\Phi'| \} \subset S^4_2(0).
\end{equation}

In fact, a straightforward calculation shows that $\langle f_w, f_w \rangle = 0$ and the minimality of the surface. Also, since $2f_w = (\Psi', -i\Psi', \Phi', -i\Phi')$ it follows that $L(f_u) = -f_u$ and $L(f_v) = f_v$. Thus, by Proposition 2.3 the tangent plane at each point of the surface is isoclinic to $\pi_{34}$ ($\pi_{12}$).

Theorem 3.5. Let $f : M \to \mathbb{R}^4$ be an isoclinic conformal immersion. Then, there exists two holomorphic functions $\Psi(z)$ and $\Phi(z)$ from $M$ into $\mathbb{C}$ such that
\begin{equation}
    f(w) = \left( \frac{\Psi(w) + \overline{\Psi}(w)}{2}, \frac{\Psi(w) - \overline{\Psi}(w)}{2i}, \frac{\Phi(w) + \overline{\Phi}(w)}{2}, \frac{\Phi(w) - \overline{\Phi}(w)}{2i} \right).
\end{equation}

Proof. Since the positive (negative) plane span $[\hat{f}_u, \hat{f}_v]$ is given by an isothermic basis, if we take $a = L(f_u)$ and $b = L(f_u)$, we have that $\hat{a} = -\hat{f}_v$ and $\hat{b} = \hat{f}_u$. The above implies $f_w^1 = -if_w^2$, here $f = (f^1, f^2, f^3, f^4)$. Hence, we take $\Psi = f^1 + if^2$. Analogously, we can take $\Phi = f^3 + if^4$, where the result is followed. □
Now, we know that for an abstract Riemannian surface the Gauss curvature is given by:

\[ K(w) = -\frac{1}{\lambda^2} \left( \frac{\lambda_u}{\lambda} \bigg|_u + \left( \frac{\lambda_v}{\lambda} \bigg|_v \right) \right) \text{ where } \pm ds^2 = \lambda^2(u,v)(du^2 + dv^2). \]

We have the following:

**Corollary 3.6.** If \( f : \mathbb{C} \to \mathbb{R}^4_2 \) is an isoclinic conformal immersion (free of metric singularities), then it is flat: \( K(w) = 0 \) for all \( w \in \mathbb{C} \).

**Proof.** If \( 0 \leq |\Psi'| < |\Phi'| \), then by Liouville Theorem there exists \( a \in \mathbb{C} \), with \( |a| < 1 \), such that \( |\Psi'/\Phi'| = |a| \). Hence, \( \varepsilon ds^2 = |\Phi'|^2(|a|^2 - 1)(du^2 + dv^2) \). Since, \( \Phi' \) is holomorphic and \( \Delta \ln |\Phi'| = 0 \), the surface is flat. \( \square \)

4. The Björling Problem

In this section we will deal with the following problem known as the Björling problem.

**Problem 4.1.** Assume that is given a triad \((c(t), A(t), B(t))_{t \in I}, \ I = (-r, r)\), composed by a positive (negative) real analytic curve \( c(t) \) in \( \mathbb{R}^4_2 \) and, a family of orthonormal sets \( \{A(t), B(t)\}_{t \in I} \) such that \( [A(t) + iB(t)] \in Q^2_{neg} \) \( [A(t) + iB(t)] \in Q^2_{pos} \) and defines a real analytic distribution along the curve \( c(t) \) pointwise orthogonal to the tangent vector \( T'(t) = c'(t)/\|c'(t)\| \).

We want to construct a positive (negative) spacelike conformal immersion \( f : M \to \mathbb{R}^4_2 \), with \( I \subset M \), satisfying the following conditions:

1. \( f(t, 0) = c(t) \) for all \( t \in I \),
2. Along the curve \( c(t) \), the normal bundle satisfies \( N_{c(t)}f(M) = [A(t) + iB(t)] \) for all \( t \in I \),
3. \( H_f(w) = 0 \) for all \( w \in M \).

To propose and solve this problem we need to assume good conditions for the curve \( c(t) \). We need to have that, pointwise, the orthogonal complement to the tangent vector \( c'(t) \) be isomorphic to \( \mathbb{R}^3_2 \) when the curve is positive, or be isomorphic to \( \mathbb{R}^3_2 \) when the curve is negative.

**Example 4.2.** Consider in \( \mathbb{R}^4_2 \) the curve \( c(t) = (\cos t, \sin t, 2 \cos \left( t/\sqrt{2} \right), 2 \sin \left( t/\sqrt{2} \right)) \). We have a positive curve such that \( \langle c', c' \rangle = 1 \), \( \langle c'', c' \rangle = 0 \) and \( \langle c'''', c' \rangle = \langle c'''', c'' \rangle = 0 \). Moreover, for \( k = 3, 4, ... \) it follows that \( \langle c^{(k)}, c^{(k)} \rangle < 0 \).

Therefore, we do not obtain a positive vector orthogonal to \( c' \), \( c'' \) and \( c''' \), to define a positive spacelike surface passing for the curve \( c \).
Definition 4.3. Let $c : I \to \mathbb{R}^2$ be a positive regular curve. We say that $c$ is a good curve, if there exists a referential

$$ \{T(s), N(s), B(s), R(s)\}_{s \in I} $$

which is an orthonormal positive basis of the space $\mathbb{R}^2$, satisfying the following conditions:

(i) $c'(s) = v(s)T(s)$ and $\langle c'(s), c'(s) \rangle = (v(s))^2 > 0$,

(ii) $N(s) \in \text{span}\{c'(s), c''(s)\}$ and $|\langle N(s), N(s) \rangle| = 1$,

(iii) $B(s) \in \text{span}\{c'(s), c''(s), c'''(s)\}$ and $|\langle B(s), B(s) \rangle| = 1$,

(iv) $R(s) = \mathfrak{X}(T(s), N(s), B(s))$.

In the same way, a definition for good negative curves can be given.

To solve the problem 4.1., we will use the technique called Schwarz integral equation.

Proposition 4.4. (Schwarz integral equation) Given the triad $(c(t), A(t), B(t))_{t \in I}$, $I = (-r, r)$, composed by a positive (negative) real analytic curve in $\mathbb{R}^2$ and, a family of orthonormal sets $\{A(t), B(t)\}_{t \in I}$ such that, $[A(t) + iB(t)] \in Q^2_{\text{neg}} \left([A(t) + iB(t)] \in Q^2_{\text{pos}}\right)$ and is pointwise orthogonal to the curve $c(t)$, we take the map

$$ f(w) = c(0) + \Re \int_0^w (C'(\xi) - i\mathfrak{X}(C'(\xi), A(\xi), B(\xi))) d\xi, $$

where $C(w)$, $A(w)$ and $B(w)$ are the (unique) analytic extensions of the curve $c(t)$ and, the vector fields $A(t)$ and $B(t)$, respectively.

It follows from (11), that $f$ define a minimal conformal immersion such that $f(t, 0) = c(t)$ with normal bundle along $c$, $N_{c(t)}f(M) = [A(t) + iB(t)]$.

Define $d'(w) = \mathfrak{X}(C'(w), A(w), B(w))$. From $f_w = \frac{1}{2}(C'(w) - id'(w))$ it follows that $f_w(t, 0) = \frac{1}{2}(c'(t) + id'(t))$ and, since $c'$ and $d'$ are vector fields in $\mathbb{R}^2$ along the curve, we obtain:

$$ \frac{\partial f}{\partial u}(t, 0) = c'(t) \quad \text{and} \quad \frac{\partial f}{\partial v}(t, 0) = d'(t). $$

We need the following result:

Lemma 4.5. Suppose that in the above data, $c$ is a good curve and the family of orthonormal sets is such that $\{A(t), B(t)\} \subset \text{span}[N, B, R]$. Define the complex vector field

$$ d'(w) = \mathfrak{X}(C'(w), A(w), B(w)). $$

Then, we have that $[C'(w) - id'(w)] \subset Q^2_{\text{pos}}$. 
functions define the unique holomorphic extension. Hence, (18) is a solution of our problem. □

Lemma 4.6 (Existence). Assume that is given a triad \((c(t), A(t), B(t))\)\(_{t \in I}\), \(I = (-r, r)\), composed by a positive (negative) real analytic good curve \(c(t)\) in \(\mathbb{R}^4\) and, a family of orthonormal sets \(\{A(t), B(t)\}_{t \in I}\) such that \([A(t) + iB(t)] \in Q^2_{\text{neg}}\) ([\(A(t) + iB(t)] \in Q^2_{\text{pos}}\) and defines a real analytic distribution along the curve \(c(t)\).
Then, the map, defined from a certain \(M\) into \(\mathbb{R}^4\), given by

\[
(18) \quad f(w) = c(0) + \Re \int_0^w (C'(\xi) - iX(C'(\xi), A(\xi), B(\xi))) d\xi,
\]

is a conformal immersion such that \(H_f = 0\) and

\[
f(t, 0) = c(t) \quad \text{and} \quad \frac{\partial f}{\partial w}(t, 0) = c'(t) - iX(c'(t), A(t), B(t)).
\]

Proof. Taking \(d'(w)\) given by (17) and using the operator given by Proposition 2.15 we have that \(\bar{J}(\{A(w) + iB(w)\}) = [C'(w) - id'(w)] = [z]\), we have the holomorphic functions from \(M\) into \(\mathbb{C}\) given by (11) of Theorem 2.8 \(\mu(w), x(w)\) and \(y(w)\). Therefore, these functions define the unique holomorphic extension

\[
f'(w) = C'(w) - id'(w) = C'(w) - iX(C'(w), A(w), B(w)) = 2\mu(w)W(x(w), y(w)).
\]

Hence, (18) is a solution of our problem. □

Lemma 4.7 (Unicity). Assume that \((M', g(z))\) and \((M, f(w))\) are two minimal immersion such that \(c(I) \subset f(M) \cap g(M')\), and

\[
[g_w(w(t))] = [f_z(z(s))] \quad \forall (s, t) \in I \times I' \quad \text{such that} \quad g(w(t)) = f(z(s)) = P \in C(I).
\]

Then, the subset \(f(M) \cap g(M') \supset c(I)\) is an open subset of the booth surfaces \(f(M)\) and \(g(M')\).

Proof. Defining the real valued function \(s = \rho(t)\) such that \(w(t) = z(\rho(t))\) that it is equivalent to say that \(g(w(t)) = f(z(\rho(t)))\), since \(g\) and \(f\) are holomorphic fields in \(\mathbb{C}^4\), the function \(\rho\) is a real analytic function. Let \(\eta(w)\) be a local holomorphic extension of the function \(\rho(t)\). This means that \(\eta(t, 0) = \rho(t)\).

Taking \(h(w) = f(z(w))\) we have that \(h(U) \subset f(M)\), and \(h_w(w) = z'(w)f_z(z(w))\). Since, by hypothesis, \([g_w(w(t))] = [f_z(z(\rho(t)))] = [h_w(z(\rho(t)))] = [W(x(t), y(t))]\) imply that along
the curve \( c(I) \) we have:

\[
g_w(t, 0) = \mu(t)W(x(t), y(t)) = \chi(t)z' (\rho(t))W(x(t), y(t)) = \beta(t)W(x(t), y(t)) = h_w(t, 0).
\]

Therefore,

\[
\mu(t) = \chi(t)z'(\rho(t)) = \beta(t)
\]

and then, we obtain by unicity of the holomorphic extensions

\[
h(w) = P + 2\Re \int_{w_0}^{w} \beta(\xi)W(x(\xi), y(\xi))d\xi = P + 2\Re \int_{w_0}^{w} \mu(\xi)W(x(\xi), y(\xi))d\xi = g(w),
\]

locally around \( c(I) \).

**Theorem 4.8.** Assume that is given the analytic triad \((c(t), A(t), B(t))\) composed by a positive (negative) regular curve and a family of orthonormal set such that \([A, B] \subset [N, B, R]\) and \([A + iB] \in Q^2\) is a negative (positive) distribution along this curve.

The integral Schwarz formula

\[
f(w) = c(0) + 2\Re \int_{w_0}^{w} \left(c_w(\xi) - i\mathcal{X}(c_w(\xi), A(\xi), B(\xi))\right) d\xi,
\]

is a maximal solution of the problem 3.3, free of metric singularities, where, the complex triad

\[
(c_w(w), A(w), B(w))
\]

is the (unique) holomorphic extension of the triad \((c'(t), A(t), B(t))\).

5. Constructions and examples

**Example 5.1.** As a first example, consider the regular curve \( c(t) = (t, 0, t^2/2, 0), t \in I = (-1, 1) \). This is a negative curve with \( c'(t) = (1, 0, t, 0) \). Moreover, take

\[
A(t) = \frac{1}{\sqrt{1-t^2}}(t, 0, 1, 0) \quad \text{and} \quad B(t) = -\frac{1}{\sqrt{1-t^2}}(0, t, 0, 1), t \in I.
\]

After a simple calculation, we obtain that \( d'(t) = \mathcal{X}(c'(t), A(t), B(t)) = (0, 1, 0, t) \). The holomorphic extension of \( d'(t) \) is

\[
d'(w) = \mathcal{X}(c'(w), A(w), B(w)) = (0, 1, 0, w).
\]

Hence,

\[
f(w) = \Re \int_{0}^{w} ((1, 0, \xi, 0) - i(0, 1, 0, \xi))d\xi = \Re \left((w, 0, w^2/2, 0) - i(0, w, 0, w^2/2)\right).
\]
Therefore, the solution of the Björling problem for the given data is
\[ f(u, v) = (u, v, (u^2 - v^2)/2, uv) \quad \text{with} \quad w \in \mathbb{C}, \ |w| < 1, \]
which is the surface of example 2.3.

5.1. Periodic curves in the Lorentzian Torus of the null sphere. Let \( T_1^2 \) be the Torus of the null sphere in \( \mathbb{R}^4_2 \), parametrically given by
\[ T(u, v) = (\cos u, \sin u, \cos v, \sin v), \]
for \((u, v) \in \mathbb{R}^2\). Since the induced metric on surface is \( ds^2(T_1^2) = -du^2 + dv^2 \), the Torus \( T_1^2 \) is a compact Lorentz surface in \( \mathbb{R}^4_2 \) isometric to the Lorentz plane \( \mathbb{R}^2_1 \).

Now, for a real number \( k > 0 \), we take the parametric curve \( c(t) = T(t, kt) \). For this curve, \( \langle c', c' \rangle = -1 + k^2 \) and \( \langle c'', c'' \rangle = -1 + k^4 \). Therefore, if \( k > 1 \) we obtain a positive parametric curve.

Claim. For \( n \in \{2, 3, \ldots, k, \ldots\} \), we have that
\[ c_n(t) = T(t, nt) = (\cos t, \sin t, \cos nt, \sin nt) = a(t) + b(nt), \]
where \( a(u) = T(u, 0) \) and \( b(v) = T(0, v) \), and it is a periodic positive good curve.

Now, we take the parametric surface given by
\[ X_n(r, \theta) = (r \cos \theta, r \sin \theta, r^n \cos n\theta, r^n \sin n\theta), \]
for which its induced metric is
\[ ds^2 = (-1 + n^2 r^{2(n-1)}) dr^2 + r^2 (-1 + n^2 r^{2(n-1)}) d\theta^2. \]
This parametric surface \( X_n(r, \theta) \), for \( r > 1/n \) and \( n > 1 \), is a positive surface for what we have \( X(1, t) = c_n(t) \).

Claim. The periodic positive parametric curve \( c_n(t) \) traces the cycle of \( T_1^2 \)
\[ c([0, 2\pi]) = T_1^2 \cap S, \]
where, \( S \subset \mathbb{R}^4_2 \) is the graphic of the holomorphic function \( \phi(z) = z^n \), for \( |z| > 1/n \).

Let us define the family of unitary and mutually orthogonal functions
\[
\begin{align*}
    a(t) &= (\cos t, \sin t, 0, 0), & x(t) &= (-\sin t, \cos t, 0, 0), \\
    b(t) &= (0, 0, \cos t, \sin t), & y(t) &= (0, 0, -\sin t, \cos t).
\end{align*}
\]
We have:
\[ c'(t) = x(t) + ny(nt) \quad c''(t) = -a(t) - n^2 b(nt) \quad c'''(t) = -x(t) - n^3 y(nt), \]
and then, the negative vector fields \( V(t) = n^2 a(t) + b(t) \) is orthogonal to \( c', c'' \) and \( c''' \). Taking \( V'(t) = n^2 x(t) + ny(t) \) we obtain a negative plane \([V'(t), V(t)]\) orthogonal to the plane \([c'(t), c''(t)]\).

Since, by construction \( V = \alpha \mathcal{X}(c', c'', c''') \) for a some \( \alpha \in \mathbb{R} \) we have that the Frenet referential of the curve \( c(t) \), can be taking real numbers \( \eta, \xi \), such that

\[
T(t) = c'(t), \quad N_1(t) = \eta c''(t), \quad N_3(t) = \xi V(t) \quad \text{and} \quad N_2(t) = \mathcal{X}(T(t), N(t), V(t)).
\]

**Claim.** The normal plane of the surface \( S = \text{graph}(z^n) \) is given by \([A(u, v) + iB(u, v)]\). Since \( X(r, \theta) = (r \cos \theta, r \sin \theta, r^n \cos n \theta, r^n \sin n \theta) \), we obtain

\[
A(r, \theta) = \frac{1}{\sqrt{-1 + n^2 r^{2n-2}}} (nr^{n-1} \cos n \theta, nr^{n-1} \sin n \theta, \cos \theta, \sin \theta)
\]

\[
B(r, \theta) = \frac{1}{r \sqrt{-1 + n^2 r^{2n-2}}} (-nr^n \sin n \theta, nr^n \cos n \theta, -r \sin \theta, r \cos \theta).
\]

Therefore, along the curve \( r = 1 \) and \( \theta = t \) we obtain:

\[
A(t) = \frac{1}{\sqrt{n^2 - 1}} (na(nt) + b(t)) \quad \text{and} \quad B(t) = \frac{1}{\sqrt{n^2 - 1}} (nx(nt) + y(t)),
\]

For the triad \( (c(t) = X(1, t), A(t), B(t)) \) we obtain

\[
c'(t) = X_\theta(1, t) = x(t) + ny(nt) \quad \text{and} \quad d'(t) = X_r(1, t) = \mathcal{X}(c', A, B) = -a(t) - nb(nt).
\]

**Example 5.2.** For the triad \( (c(t), A(t), B(t)) \) where \( c(t) = a(t) + b(nt) \) and

\[
A(t) = \frac{1}{\sqrt{n^2 - 1}} (na(nt) + b(t)) \quad \text{and} \quad B(t) = \frac{1}{\sqrt{n^2 - 1}} (nx(nt) + y(t)),
\]

the unique solution of problem 3.3 is given by

\[
f(w) = \Re(e^{i w}, -i e^{i w}, e^{i n w}, -i e^{i n w}) = \Re(z, -iz, z^n, -iz^n) \quad \text{for} \quad z = e^{i w}.
\]

**Proof.** First: \( f(t, 0) = \Re(e^{i t}, -i e^{i t}, e^{i n t}, -i e^{i n t}) = (\cos t, \sin t, \cos nt, \sin nt) = c(t) \).

Second: \( \frac{\partial f}{\partial w}(t, 0) = i(e^{i w}, -i e^{i w}, n e^{i n w}, -i n e^{i n w})|_{(t,0)} = c'(t) - id'(t). \)

Now, we need to check the signal of \( d'(t) \). To this end,

\[
(n^2 - 1) \mathcal{X}(c'(0), A(0), B(0)) = \begin{vmatrix} e_1 & e_2 & -e_3 & -e_4 \\ 0 & 1 & 0 & n \\ n & 0 & 1 & 0 \\ 0 & n & 0 & 1 \end{vmatrix} = (1 - n^2)e_1 + n(1 - n^2)e_3.
\]

Thus, this computation shows that \( d'(t) = -(a(t) + nb(nt)) \). In ded, \( d'(t), A(t) = 0 \), because, \( (a(t) + nb(nt), na(nt) + b(t)) = n(- \cos t \cos nt + \cos nt \cos t) = 0 \). □
It is well known that an arc-length parametric regular and **good** curve $c(t) = f(u(t), v(t))$ of a positive (negative) parametric surface $(M, f)$ is a (pre) geodesic line if, and only if, the normal $N_c(t)$ belongs to the normal bundle $N_c f(M)$ along this curve. For asymptotic lines, we also need that the normal $N_c$ is positive (negative) to have $N_c(t)$ belongs to the normal bundle $T_c f(M)$.

On Example 2.4, the negative curve $\alpha(t) = \frac{1}{2}(\cos t, \sin t, \frac{1}{4}\cos 2t, \frac{1}{4}\sin 2t)$ is such that $\langle c''(t), c''(t) \rangle = 0$, for all $t \in \mathbb{R}$. This curve corresponds to $u^2 + v^2 = 1/4$.

**Theorem 5.3.** Let $c(t)$ be a positive arc-length parametric good curve with Frenet reference $\{T(t), N_1(t), N_2(t), N_3(t)\}$. If the first normal $N_1(t)$ is a negative fields along this curve, and $V(t) \in \{N_2(t), N_3\}$ is a negative vector fields along this curve, then, the solution $M, f$ of the Problem 3.3 for the triad $(c(t), N_1(t), V(t))$ is such that, the curve $c(t) = f(t, 0)$ is a geodesic line.

Now, assuming that $N_1(t)$ is a positive vector fields along this curve, then the solution $M, f$ of the Problem 3.3 for the triad $(c(t), N_2(t), V(3))$ is such that, the curve $c(t) = f(t, 0)$ is an asymptotic line.

Now, we take the cycle of Claim 4.1, $c(t) = a(t) + b(nt)$, and the distribution $D(t) = [N_2(t), N_3(t)]$.

**Example 5.4.** The parametric surface given by

$$f(w) = c(0) + \Re \int_0^w \left(c'(\xi) - i \frac{1}{\sqrt{1 + n^2}} c''(\xi) \right) d\xi$$

is a solution of the Problem 3.3 such that the curve $c(t) = f(t, 0)$ is an asymptotic line of this new surface $f(M)$.

### 5.2. A type of Helices.

For positive real numbers $0 < b < a < \sqrt{b}$, we take in $\mathbb{R}^4_2$ the curve

$$\alpha(s) = \left( b \cos \frac{s}{\sqrt{a^2 - b^2}}, b \sin \frac{s}{\sqrt{a^2 - b^2}}, \cos \frac{as}{\sqrt{a^2 - b^2}}, \sin \frac{as}{\sqrt{a^2 - b^2}} \right).$$

Since $\langle \alpha', \alpha' \rangle = 1$ and $\langle \alpha'', \alpha'' \rangle = \frac{a^4 - b^2}{(a^2 - b^2)^2} < 0$, this is a positive curve with negative normal vector field along $\alpha(s)$. 
We have that,
\[ T(s) = \frac{1}{\sqrt{a^2 - b^2}} \left( -b \sin \frac{s}{\sqrt{a^2 - b^2}}, b \cos \frac{s}{\sqrt{a^2 - b^2}}, -a \sin \frac{s}{\sqrt{a^2 - b^2}}, a \cos \frac{s}{\sqrt{a^2 - b^2}} \right), \]
\[ N(s) = \frac{1}{\sqrt{b^2 - a^2}} \left( b \cos \frac{s}{\sqrt{a^2 - b^2}}, b \sin \frac{s}{\sqrt{a^2 - b^2}}, a^2 \cos \frac{s}{\sqrt{a^2 - b^2}}, a^2 \sin \frac{s}{\sqrt{a^2 - b^2}} \right), \]
\[ N'(s) = \frac{1}{\sqrt{b^2 - a^2}} \left( -b \sin \frac{s}{\sqrt{a^2 - b^2}}, b \cos \frac{s}{\sqrt{a^2 - b^2}}, -a^3 \sin \frac{s}{\sqrt{a^2 - b^2}}, a^3 \cos \frac{s}{\sqrt{a^2 - b^2}} \right). \]

Let us define the following set of unitary and mutually orthogonal functions where \( t = s/\sqrt{a^2 - b^2} \):
\[
\begin{aligned}
&\{ x(t) = (\cos t, \sin t, 0, 0),
\quad y(t) = (-\sin t, \cos t, 0, 0),
\quad \eta(t) = (0, 0, \cos at, \sin at), \\
&\quad \xi(t) = (0, 0, -\sin at, \cos at).
\end{aligned}
\]

Now, we have that
\[ \alpha' = by(t) + a\xi(t), \quad \alpha'' = -bx(t) - a^2\eta(t) \quad \text{and} \quad \alpha''' = -by(t) - a^3\xi(t). \]

The exterior product of this vectors give us
\[
V = \begin{vmatrix}
  x(t) & y(t) & -\eta(t) & -\xi(t) \\
  0 & -b & 0 & -a \\
  b & 0 & a^2 & 0 \\
  0 & b & 0 & a^3 \\
\end{vmatrix} = (ba^3(1 - a^2)x(t) + (b^2a(1 - a^2))\eta(t).
\]

Therefore, we obtain the third normal vector \( V = \sqrt{b^2 - a^2}R = a^2x(t) + b\xi(t). \)

**Example 5.5.** For \( 0 < b < a < \sqrt{b} \). Given the positive Helix
\[ \alpha(s) = \left( b \cos \frac{s}{\sqrt{a^2 - b^2}}, b \sin \frac{s}{\sqrt{a^2 - b^2}}, \cos \frac{as}{\sqrt{a^2 - b^2}}, \sin \frac{as}{\sqrt{a^2 - b^2}} \right), \]
and taking the positive fields given by its third normal
\[ R(s) = \frac{1}{\sqrt{b^2 - a^2}} \left( -a^2 \sin \frac{s}{\sqrt{a^2 - b^2}}, a^2 \cos \frac{s}{\sqrt{a^2 - b^2}}, -b \sin \frac{as}{\sqrt{a^2 - b^2}}, b \cos \frac{as}{\sqrt{a^2 - b^2}} \right), \]
the parametric surface given by
\[
\begin{aligned}
f(w) &= \Re \int_0^w (T(\xi) - iR(\xi))d\xi
\end{aligned}
\]
is a solution of the Problem 3.3 such that the curve \( \alpha(s) = f(s, 0) \) is a geodesic line of this surface \( f(M) \).
An example of a isoclinic "bi-Helix."

**Example 5.6.** Let \( b > 0 \) be and, let \( m \) and \( n \) be two distinct natural numbers. Consider the negative curve

\[
c(t) = \frac{\sqrt{1 + b^2}}{m} (\cos mt, \sin mt, 0, 0) + \frac{b}{n} (0, 0, \cos nt, \sin nt).
\]

The parametric isoclinic surface is given by

\[
f(w) = c(0) + \int_0^w (c'(\xi) - iM(c'(\xi)))d\xi.
\]

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