A Topological Obstruction to Existence of Quaternionic Plücker Map

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Abstract

It is shown that there is no continuous map from the quaternionic Grassmannian $\mathbb{H}Gr_{k,n}(k \geq 2, \ n \geq k + 2)$ to the quaternionic projective space $\mathbb{H}P^{\infty}$ such that the pullback of the first Pontryagin class of the tautological bundle over $\mathbb{H}P^{\infty}$ is equal to the first Pontryagin class of the tautological bundle over $\mathbb{H}Gr_{k,n}$. In fact some more precise statement is proved.

1 Introduction

This note is a bi-product of an attempt to understand linear algebra over the (noncommutative) field of quaternions $\mathbb{H}$. For the basic material on linear algebra over noncommutative fields we refer to [Art], [GGRW]. For quaternionic linear algebra see [As],[Al1], [GRW], and for further applications of it to quaternionic analysis see [Al1], [Al2]. The main results of this note are Theorem 1 and its slight refinement Theorem 2 below. But first let us discuss the motivation of them.

Roughly speaking, it is shown that there is a topological obstruction to fill in the last row in the last column of the following table (compare with the table in [Ar]).

| $\mathbb{R}$ | $\mathbb{C}$ | $\mathbb{H}$ |
|----------|------------|------------|
| $w_1$    | $c_1$      | $p_1$      |
| $\mathbb{R}P^{\infty} = K(\mathbb{Z}/2\mathbb{Z}, 1)$ | $\mathbb{C}P^{\infty} = K(\mathbb{Z}, 2)$ | $\mathbb{H}P^{\infty} \to K(\mathbb{Z}, 4)$ |
| $w_1(V) = w_1(\wedge^{top} V)$ | $c_1(V) = c_1(\wedge^{top} V)$ | $p_1(V) = p_1(?)$ |
In this table $V$ denotes a vector bundle (respectively real, complex, or quaternionic) over some base, and $K(\pi, n)$ denotes the Eilenberg-Maclane space. Consider the category of (say) right $\mathbb{H}$-modules. It is not clear how to define in this category the usual notions of linear algebra like tensor products, symmetric products, and exterior powers. (However, it was discovered by D. Joyce [Jo] that one can define these notions in the category of right $\mathbb{H}$-modules with some additional structure, namely with some fixed real subspace generating the whole space as $\mathbb{H}$-module. See also [Qu] for further discussions). We show that in a sense there is a topological obstruction to the existence of the maximal exterior power of finite dimensional $\mathbb{H}$-modules.

We will use the following notation. For a field $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$ we will denote by $KGr_{k,n}$ the Grassmannian of $k$-dimensional subspaces in $K^n$, and by $KGr_{k,\infty}$ the inductive limit $\lim_{N \to \infty} KGr_{k,N}$. The homotopy classes of maps from a topological space $X$ to $Y$ will be denoted by $[X,Y]$. For the field $K = \mathbb{R}$ or $\mathbb{C}$ we will consider the Plücker map $\eta : KGr_{k,\infty} \to K\mathbb{P}^\infty$ given by $E \mapsto \bigwedge^k E$.

For $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$ the isomorphism classes of $K$- vector bundles of rank $k$ over $X$ correspond bijectively to the homotopy classes of maps $[X,KGr_{k,\infty}]$. In particular, the isomorphism classes of $K$- line bundles over $X$ are in bijection with the homotopy classes of maps to projective space $[X,K\mathbb{P}^\infty]$.

Recall that in the case of commutative field $K$ the $k$-th exterior power provides a canonical way to produce from a $k$-dimensional vector space a 1-dimensional one. In particular over the field $K = \mathbb{R}$ or $\mathbb{C}$ this gives a functorial construction of a line bundle from a vector bundle of rank $k$. Hence we get a map $\xi : [X,KGr_{k,\infty}] \to [X,K\mathbb{P}^\infty]$ which is just the composition of a map from $X$ to $KGr_{k,\infty}$ with the Plücker map $\eta$.

However, the spaces $\mathbb{R}P^\infty$ and $\mathbb{C}P^\infty$ are Eilenberg-Maclane spaces $K(\mathbb{Z}/2\mathbb{Z},1)$ and $K(\mathbb{Z},2)$ respectively. Hence $[X,\mathbb{R}P^\infty] = H^1(X,\mathbb{Z}/2\mathbb{Z})$, and $[X,\mathbb{C}P^\infty] = H^2(X,\mathbb{Z})$. If $f \in [X,\mathbb{R}Gr_{k,\infty}]$ corresponds to a real vector bundle $V$ over $X$ then $\xi(f)$ corresponds to its first Stiefel-Whitney class $w_1(V) \in H^1(X,\mathbb{Z}/2\mathbb{Z})$. (Similarly if $f \in [X,\mathbb{C}Gr_{k,\infty}]$ corresponds to $V$ then $\xi(f)$ corresponds to its first Chern class $c_1(V) \in H^2(X,\mathbb{Z})$). Thus the Plücker map $\eta : \mathbb{R}Gr_{k,\infty} \to K(\mathbb{Z}/2\mathbb{Z},1) = \mathbb{R}P^\infty$ corresponds to the first Stiefel-Whitney class of the tautological bundle over $\mathbb{R}Gr_{k,\infty}$. Analogously the Plücker map $\eta : \mathbb{C}Gr_{k,\infty} \to K(\mathbb{Z},2) = \mathbb{C}P^\infty$ corresponds to the first Chern class of the tautological bundle over $\mathbb{C}Gr_{k,\infty}$. Thus the homotopy class of the Plücker map over $\mathbb{R}$ (resp. $\mathbb{C}$) is characterized uniquely by the property that the pull-back of the first Stiefel-Whitney (resp. Chern) class of the tautological bundle over
\( \mathbb{R}P^\infty \) (resp. \( \mathbb{C}P^\infty \)) is equal to the first Stiefel-Whitney (resp. Chern) class of the tautological bundle over \( \mathbb{R}Gr_{k,\infty} \) (resp. \( \mathbb{C}Gr_{k,\infty} \)).

One can try to obtain a quaternionic analogue of the last statement. Note first of all that the quaternionic projective space \( \mathbb{H}P^\infty \) has the rational homotopy type of the Eilenberg-MacLane space \( K(\mathbb{Z}, 4) \), but is not homotopically equivalent to it. More precisely, \( \pi_i(\mathbb{H}P^\infty) = 0 \) for \( i < 4 \), \( \pi_4(\mathbb{H}P^\infty) = \mathbb{Z} \), and \( \pi_i(\mathbb{H}P^\infty) \) is finite for \( i > 4 \). (This can be seen from the quaternionic Hopf fibration \( S^3 \to S^\infty \to \mathbb{H}P^\infty \) and the fact that \( \pi_3(S^3) = \mathbb{Z} \) and \( \pi_i(S^3) \) is finite for \( i > 3 \)). By gluing cells, one can construct an embedding \( \tau : \mathbb{H}P^\infty \to K(\mathbb{Z}, 4) \), which induces isomorphism of homotopy groups up to order 4. This map is unique up to homotopy.

One can easily see that this map corresponds to the first Pontryagin class of the tautological bundle over \( \mathbb{H}P^\infty \) (recall that \( [X, K(\mathbb{Z}, 4)] = H^4(X, \mathbb{Z}) \)). On the other hand, the first Pontryagin class of the tautological bundle over the quaternionic Grassmannian \( \mathbb{H}Gr_{k,\infty} \) defines a map \( \omega : \mathbb{H}Gr_{k,\infty} \to K(\mathbb{Z}, 4) \) which is unique up to homotopy. Our main result is

**Theorem 1** There is no map \( \eta : \mathbb{H}Gr_{k,\infty} \to \mathbb{H}P^\infty \), \( k \geq 2 \), which would make commutative the following diagram:

\[
\begin{array}{ccc}
\mathbb{H}P^\infty & \xrightarrow{\tau} & K(\mathbb{Z}, 4) \\
\downarrow{\eta} & & \\
\mathbb{H}Gr_{k,\infty} & \xrightarrow{\omega} & K(\mathbb{Z}, 4)
\end{array}
\]

In other words, there is no map \( \eta : \mathbb{H}Gr_{k,\infty} \to \mathbb{H}P^\infty \) such that the pull-back under \( \eta \) of the first Pontryagin class of the tautological bundle over \( \mathbb{H}P^\infty \) is equal to the first Pontryagin class of the tautological bundle over \( \mathbb{H}Gr_{k,\infty} \).

**Remark.** It will be clear from the proof of Theorem 1 that there is no map \( \eta : \mathbb{H}Gr_{k,n} \to \mathbb{H}P^\infty \) for \( k \geq 2, n \geq k + 2 \) with the same property (here \( \mathbb{H}Gr_{k,n} \) denotes the quaternionic Grassmannian of \( k \)-subspaces in \( \mathbb{H}^n \)).

In fact we will prove a slightly more precise statement. Consider the Postnikov tower for \( X = \mathbb{H}P^\infty \):

\[
K(\mathbb{Z}, 4) = X_4 \leftarrow X_5 \leftarrow X_6 \leftarrow X_7 \leftarrow \cdots.
\]
Theorem 2 Let $k \geq 2$, $\infty \geq n \geq k + 2$. The map $\omega : \mathbb{H}Gr_{k,n} \to K(\mathbb{Z}, 4)$ corresponding to the first Pontryagin class of the tautological bundle over $\mathbb{H}Gr_{k,n}$ can be factorized through $X_6$, but cannot be factorized through $X_7$.

Clearly Theorem 2 implies Theorem 1.

2. Proof of Theorem 2. (1) Let us show that $\omega$ cannot be factorized through $X_7$. We first reduce the statement to the case $k = 2$, $n = 4$.

Let $k \geq 2$, $n \geq k + 2 \geq 4$. Fix a decomposition $\mathbb{H}^n = \mathbb{H}^4 \oplus \mathbb{H}^{n-4}$. Since $n - 4 \geq k - 2$ we can fix a subspace $\mathbb{H}^{k-2} \subset \mathbb{H}^{n-4}$. Consider the embedding $i : \mathbb{H}Gr_{2,4} \hookrightarrow \mathbb{H}Gr_{k,n}$ as follows: for every $E \subset \mathbb{H}^4$ let $i(E) = E \oplus \mathbb{H}^{k-2}$. Then the restriction of the topological bundle over $\mathbb{H}Gr_{k,n}$ to $\mathbb{H}Gr_{2,4}$ is equal to the sum of the topological bundle over $\mathbb{H}Gr_{2,4}$ and the trivial bundle $\mathbb{H}^{k-2}$. Hence the Pontryagin classes of these two bundles over $\mathbb{H}Gr_{2,4}$ are equal. Thus it is sufficient to prove the statement for $\mathbb{H}Gr_{2,4}$.

Claim 3. Consider the embedding $i : \mathbb{H}Gr_{2,4} \hookrightarrow \mathbb{H}Gr_{2,\infty}$. It induces isomorphism of homotopy groups of order $\leq 7$.

Let us prove Claim 3. It is well know (see e.g. [P-R]) the cell decomposition of quaternionic Grassmannians is given by quaternionic Schubert cell of real dimensions $0,4,8,12,\ldots$. Moreover $i$ induces isomorphism of 8-skeletons of both spaces. Hence Claim 3 follows.

Claim 4. $\pi_7(\mathbb{H}Gr_{2,4}) = 0$.

By Claim 3 it is sufficient to show that $\pi_7(\mathbb{H}Gr_{2,\infty}) = 0$. Consider the bundle $Sp_2 \to \mathbb{H}St_{2,\infty} \to \mathbb{H}Gr_{2,\infty}$, where $\mathbb{H}St_{2,\infty}$ is the quaternionic Stiefel manifold, i.e. pairs of orthogonal unit vectors in $\mathbb{H}^\infty$, and $Sp_2$ is the group of symplectic $2 \times 2$ matrices. Since $\mathbb{H}St_{2,\infty}$ is contractible $\pi_7(\mathbb{H}Gr_{2,\infty}) = \pi_6(Sp_2)$. But the last group vanishes by [M-T]. Claim 4 is proved.

Claim 5. $\pi_7(\mathbb{H}P\infty) \neq 0$.

Indeed using the quaternionic Hopf bundle $S^3 \to S^{\infty} \to \mathbb{H}P\infty$ one gets $\pi_7(\mathbb{H}P\infty) = \pi_6(S^3) = \mathbb{Z}/12\mathbb{Z}$ (see [Hu] Ch. XI.16 for the last equality).

Now let us assume the opposite to our statement, namely assume that there exists a map $\eta' : \mathbb{H}Gr_{2,4} \to X_7$ which makes the following diagram commutative:
where \( \rho_7 \) is the standard map for the Postnikov system. Let \( S \subset \mathbb{H}Gr_{2,4} \) denote the 4-skeleton of \( \mathbb{H}Gr_{2,4} \) consisting of Schubert cells. Then \( S \) can be described as follows. Fix a pair \( \mathbb{H}^1 \subset \mathbb{H}^3 \) inside \( \mathbb{H}^4 \). Then \( S = \{ E \in \mathbb{H}Gr_{2,4} \mid \mathbb{H}^2 \subset E \subset \mathbb{H}^3 \} \). Clearly, \( S \simeq \mathbb{H}P^1 \simeq S^4 \).

Our requirement on \( \omega : \mathbb{H}Gr_{2,4} \to K(\mathbb{Z}, 4) \) that the pull-back under \( \omega \) of the fundamental class \( \kappa \in H^4(K(\mathbb{Z}, 4), 4) \) is equal to the first Pontryagin class of the topological bundle over \( \mathbb{H}Gr_{2,4} \), is equivalent to saying that \((\omega \circ j)^*(\kappa) \in H^4(S, \mathbb{Z}) \) is the canonical generator of \( H^4(S, \mathbb{Z}) = H^4(\mathbb{H}P^1, \mathbb{Z}) = \mathbb{Z} \) (here \( j : S \hookrightarrow \mathbb{H}Gr_{2,4} \) denotes the identity embedding). This is also equivalent to the fact that \( \omega \circ j : S \to K(\mathbb{Z}, 4) \) can be lifted to \( h : S \to \mathbb{H}P^\infty \) so that the diagram

\[
\begin{array}{ccc}
\mathbb{H}P^\infty & \overset{\rho_7}{\longrightarrow} & X_7 \\
\downarrow & & \downarrow \\
\mathbb{H}Gr_{2,4} \overset{\omega}{\longrightarrow} & K(\mathbb{Z}, 4)
\end{array}
\]

is commutative, and moreover \( h \) is homotopic to the composition \( S \overset{id}{\longrightarrow} \mathbb{H}P^1 \hookrightarrow \mathbb{H}P^\infty \), where \( \mathbb{H}P^1 \hookrightarrow \mathbb{H}P^\infty \) is the standard embedding. But the embedding \( \mathbb{H}P^1 \hookrightarrow \mathbb{H}P^\infty \) induces the \textit{surjective} map \( \pi_7(\mathbb{H}P^1) \to \pi_7(\mathbb{H}P^\infty) \) since \( \mathbb{H}P^1 \) is the 7-skeleton of \( \mathbb{H}P^\infty \) under the subdivision to Schubert cells. Let us choose any element \( \varphi \in \pi_7(S) \) that is under the composition \( S \overset{id}{\longrightarrow} \mathbb{H}P^1 \hookrightarrow \mathbb{H}P^\infty \) which is mapped to a nonzero element. (Recall that by Claim 4, \( \pi_7(\mathbb{H}P^\infty) \neq 0 \)). Consider the space \( Z := S \cup_\varphi D_8 \), where \( D_8 \) is the 8-dimensional disk with the boundary \( \partial D_8 = S^7 \). Since (by Claim 4) \( \pi_7(\mathbb{H}Gr_{2,4}) = 0 \) there is a map \( f : Z \to \mathbb{H}Gr_{2,4} \) such that \( f \big|_S \) is just the identity embedding \( S \hookrightarrow \mathbb{H}Gr_{2,4} \).

We have assumed that there exists a factorization as in Diag. 1. Consider \( \eta' \circ f : Z \to X_7 \). The restriction of this map to \( S \subset Z \) maps \( \varphi \in \pi_7(S) \) to an element of \( \pi_7(X_7) = \pi_7(\mathbb{H}P^\infty) \), which is not zero by our choice. But this element must vanish by construction of \( Z \). We get a contradiction.
(2) It remains to prove that there is a factorization of $\omega : \mathbb{H}Gr_{k,n} \to K(Z, 4)$ through $X_6$. First let us show that $\omega$ factorizes through $X_5$. We have the diagram

\[
\begin{array}{cccc}
\mathbb{H}Gr_{k,n} & \xrightarrow{\omega} & X_4 = K(Z, 4) & \xrightarrow{k_4(X)} & K(\pi_5, 6), \\
\downarrow & & \downarrow & & \\
X_5 & & X_4 & & \\
\end{array}
\]

where $X$ denotes for brevity $\mathbb{H}P^\infty$, $\pi_5 = \pi_5(X)$, $k_4(X)$ is the 4-th $k$-invariant of $X$. The composition $k_4(X) \circ \omega : \mathbb{H}Gr_{k,n} \to K(\pi_5, 6)$ is homotopic to the constant map. Indeed $H^6(\mathbb{H}Gr_{k,n}, \pi_5) = 0$ since all Schubert cells have dimensions divisible by 4. Hence the map $\omega$ can be lifted to a map $g : \mathbb{H}Gr_{k,n} \to X_5$ such that the following diagram is commutative (see [Mo-Ta], Ch.13):

\[
\begin{array}{cccc}
\mathbb{H}Gr_{k,n} & \xrightarrow{\omega} & X_4 = K(Z, 4) & \xrightarrow{k_4(X)} & K(\pi_5, 6), \\
\downarrow & & \downarrow & & \\
\mathbb{C}^n & \xrightarrow{g} & X_5 & & \\
\end{array}
\]

Since $H^7(\mathbb{H}Gr_{k,n}, \pi_5) = 0$ a similar argument shows that the constructed map $g$ can be lifted to a map $h : \mathbb{H}Gr_{k,n} \to X_6$. This map $h$ gives the necessary factorization. Q.E.D.

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