Stress-Energy-Momentum of Affine-Metric Gravity.
Generalized Komar Superpotential.

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Abstract
In case of the Einstein’s gravitation theory and its first order Palatini reformulation, the stress-energy-momentum of gravity has been proved to reduce to the Komar superpotential. We generalize this result to the affine-metric theory of gravity in case of general connections and arbitrary Lagrangian densities invariant under general covariant transformations. In this case, the stress-energy-momentum of gravity comes to the generalized Komar superpotential depending on a Lagrangian density in a precise way.

1

As is well-known, in the Einstein’s gravitation theory of metric fields \[6\], the stress-energy-momentum (SEM) conservation law corresponding to the invariance of the Hilbert-Einstein Lagrangian density under general covariant transformations takes the form

\[
\frac{d}{dx^\lambda} U(\tau)^\lambda = 0
\]

where

\[
U(\tau)^\lambda = \frac{d}{dx^\mu}[\sqrt{-g}(g^{\lambda\nu} r^\mu_{\nu} - g^{\mu\nu} r^\lambda_{\nu})]
\]

is the well-known Komar superpotential \[5\] associated with a vector field \(\tau\) on a world manifold \(X\). By the symbol \(\mu\) is meant the covariant derivative with respect to the Levi-Civita connection.

In a recent paper \[1\], it was shown that, in the Palatini model of metric fields and symmetric connections, the SEM complex corresponding to a Lagrangian density polynomial in a scalar curvature looks like the Komar superpotential \[2\].
We generalize this result to the affine-metric gravity in case of a general linear connection $K^\alpha_{\gamma\mu}$ and arbitrary Lagrangian density $L$ invariant under general covariant transformations. The corresponding SEM conservation law \[^3\] is brought into the form \(^{1}\) where

$$U(\tau)^\lambda = \frac{d}{dx^\mu}[\partial L/\partial K^\alpha_{\nu\mu,\lambda}(D_\nu \tau^\alpha + \Omega^\alpha_{\nu\sigma} \tau^\sigma)] \quad (3)$$

is the generalized Komar superpotential \(^2\). Here, by $D_\gamma$ is meant the covariant derivative with respect to the general linear connection $K$ and $\Omega$ is the torsion of this connection. In the particular case of the Hilbert-Einstein Lagrangian density and symmetric connections, we have

$$\frac{\partial L_{\text{HE}}}{\partial K^\alpha_{\nu\mu,\lambda}} = \frac{\sqrt{-g}}{2\kappa}(\delta^\mu_\nu g^{\nu\lambda} - \delta^\lambda_\nu g^{\nu\mu}),$$

so that the superpotential \(^3\) comes to the standard Komar superpotential \(^2\). If a Lagrangian density is a polynomial in the scalar curvature of a symmetric connection, the superpotential \(^3\) recovers the one in ref. \[^{1}\].

2

We follow the geometric approach to field theory when classical fields are described by global sections of a bundle $Y \to X$ over a world manifold $X$. Their dynamics is phrased in terms of jet manifolds \(^{4, 5, 6}\).

As a shorthand, one can say that the $k$-order jet manifold $J^kY$ of a bundle $Y \to X$ comprises the equivalence classes $j^k_s$, $x \in X$, of sections $s$ of $Y$ identified by the first $k + 1$ terms of their Taylor series at a point $x$. Recall that a $k$-order differential operator on sections of a bundle $Y \to X$ is defined to be a bundle morphism of the bundle $J^kY \to X$ to a vector bundle over $X$.

We restrict ourselves to the first order Lagrangian formalism, for most of contemporary field models are described by first order Lagrangian densities. This is not the case for the Einstein-Hilbert Lagrangian density of the Einstein’s gravitation theory which belongs to the special class of second order Lagrangian densities whose Euler-Lagrange equations are however of the order two as like as in the first order theory.

In the first order Lagrangian formalism, the finite-dimensional configuration space of fields represented by sections $s$ of a bundle $Y \to X$ is the first order jet manifold $J^1Y$ of $Y$. Given fibered coordinates $(x^\mu, y^i)$ of $Y$, the jet manifold $J^1Y$ is endowed with the adapted coordinates $(x^\mu, y^i, y^i_\mu)$. In physical literature, the coordinates $y^i_\mu$ are usually called the velocity coordinates or the derivative coordinates because of the relation

$$y^i_\mu(j^1_x s) = \partial_\mu s^i(x).$$
The jet manifold $J^1Y$ is endowed with the natural bundle structures $J^1Y \to Y$ and $J^1Y \to X$. For the sake of convenience, we shall call $J^1Y \to X$ the configuration bundle and $J^1Y \to Y$ simply the jet bundle.

A first order Lagrangian density on $J^1Y$ is defined to be an exterior horizontal density

$$L : J^1Y \to \bigwedge^n \tau^*X, \quad n = \dim X,$$

$$L = \mathcal{L}(x^\mu, y^i, y^i_\mu) \omega, \quad \omega = dx^1 \wedge ... \wedge dx^n,$$

on the configuration bundle $J^1Y \to X$.

By a differential conservation law in first order field theories is meant a relation where the divergence of a current $T$ appears equal to zero, i.e.

$$dT = 0 \quad (4)$$

where $T$ is a horizontal $(n-1)$-form on the configuration bundle $J^1Y \to X$.

The relation (4) is called a strong conservation law if it is satisfied identically by all sections $s$ of the bundle $Y \to X$, while it is termed a weak conservation law if it takes place only on critical sections, i.e., on solutions of field equations. We shall use the symbol "$\approx$" for weak identities.

It may happen that the weakly conserved current $T$ takes the form

$$T = W + dU$$

where $W \approx 0$. In this case, one says that the current $T$ is reduced to the superpotential $U$ [1, 2]. For instance, the Nöther currents in gauge theory come to superpotentials which depend on parameters of gauge transformations that provide the gauge invariance of Nöther conservation laws.

Usually, one derives the differential conservation laws from invariance of a Lagrangian density under some group of transformations.

Let $G_t$ be a 1-parameter group of bundle isomorphisms of a bundle $Y \to X$ and

$$u = u^\lambda(x) \partial_\lambda + u^i(y) \partial_i$$

the corresponding vector field on $Y$. One can prove that a Lagrangian density $L$ on the configuration space $J^1Y$ is invariant under these transformations iff its Lie derivative by the jet lift

$$L_{j_0^1 u}u = u^\lambda \partial_\lambda + u^i \partial_i + (\hat{\partial}_\lambda u^i - y^i_\mu \partial_\lambda u^\mu) \partial_\lambda,$$

$$\hat{\partial}_\lambda = \partial_\lambda + y^j_\lambda \partial_j + y^j_\lambda \partial^\lambda_\mu + \cdots$$

of $u$ onto $J^1Y$ is equal to zero:

$$L_{j_0^1 u}L = 0. \quad (5)$$

3
The equality (5) gives rise to the weak differential conservation law as follows.

Let \( u \) be a projectable vector field on a bundle \( Y \to X \) and \( \pi \) its jet lift onto the configuration space \( J^1Y \to X \). Given a Lagrangian density \( L \), let us compute the Lie derivative \( L_u \). We get the canonical decomposition

\[
L_{\delta \nu}L = u_V \mathcal{E}_L + dT(u)
\]

where

\[
\mathcal{E}_L = [\partial_i - (\partial_\lambda + y^j_\lambda \partial_j + y^j_\mu \partial^\mu) \partial_i^\lambda) L dy^i \wedge \omega = \mathcal{E}_i dy^i \wedge \omega
\]

is the Euler-Lagrange operator,

\[
T(u) = T^\lambda(u) \omega_\lambda = \hat{\partial}_\lambda [\pi_i^\lambda(u^i - u^\mu y_i^\mu)] + u^\lambda L] \omega_\lambda,
\]

\[
\hat{\partial}_\lambda = \partial_\lambda + y^j_\lambda \partial_j + y^j_\mu \partial^\mu, \quad \omega_\lambda = \partial_\lambda \omega,
\]

is the corresponding current, and

\[
u_V = (u^i - y^i_\mu u^\mu) \partial_i
\]

is the vertical part of the vector field \( u \). We denote by \( \pi_i^\mu = \partial_i^\mu L \) the Lagrangian momenta. This is the well-known first variational formula of the calculus of variations.

The Euler-Lagrange operator \( \mathcal{E}_L \), by definition, vanishes on the critical sections of the bundle \( Y \to X \), and the equality (5) comes to the weak identity

\[
L_{\delta \nu}L \approx \hat{\partial}_\lambda [\pi_i^\lambda(u^i - u^\mu y_i^\mu)] + u^\lambda L] \omega.
\]

If the Lie derivative

\[
L_{\pi L} = [\partial_\lambda u^\lambda L + (u^\lambda \partial_\lambda + u^i \partial_i + (\partial_\lambda u^i + y^j_\lambda \partial_j u^i - y^j_\mu \partial_\lambda u^\mu) \partial_i^\lambda) L] \omega
\]

of a Lagrangian density \( L \) by a projectable vector field \( u \) satisfies the condition (5), then we have the weak conservation law

\[
0 \approx \hat{\partial}_\lambda [\pi_i^\lambda(u^i - u^\mu y_i^\mu)] + u^\lambda L].
\]

There are two theories where the condition (5) takes place: (i) the gauge theory of exact internal symmetries where \( u \) are vertical fields associated with gauge transformations and (ii) the gravitation theory on bundles of geometric objects \( Y \to X \) where \( u \) are canonical lifts of vector fields \( \tau \) on the base \( X \) onto \( Y \). In this case, we have the SEM conservation laws [3].
The bundles of geometric objects $Y \to X$ are exemplified by tensor bundles over $X$ and the bundle of linear connections. They admit the canonical lift of a vector field $\tau$ on $X$.

Let $\tau = \tau^\mu \partial_\mu$ be a vector field on the manifold $X$. There exists the canonical lift

$$\tilde{\tau} = T\tau = \tau^\mu \partial_\mu + \partial_\nu \tau^\alpha \dot{x}^\nu \frac{\partial}{\partial \dot{x}^\alpha}$$

of $\tau$ onto the tangent bundle $TX$ of $X$. This lift consists with the horizontal lift of $\tau$ by means the symmetric connection $K$ on the tangent bundle which has $\tau$ as the geodesic field:

$$\partial_\nu \tau^\alpha + K^\alpha{}_{\mu\nu} \tau^\mu = 0.$$ 

Generalizing the canonical lift (8), one can construct the lifts of a vector field $\tau$ on $X$ onto the following bundles over $X$ (for the sake of simplicity, we denote all these lifts by the same symbol $\tilde{\tau}$): We have:

- the canonical lift

$$\tilde{\tau} = \tau^\mu \partial_\mu - \partial_\beta \tau^\alpha \dot{x}_\beta \frac{\partial}{\partial \dot{x}_\alpha}$$

of $\tau$ onto the cotangent bundle $T^*X$;

- the canonical lift

$$\tilde{\tau} = \tau^\mu \partial_\mu + \left[ \partial_\nu \tau^\alpha \dot{x}_\beta \dot{x}_\gamma \dot{x}_\delta \ldots \right] - \partial_\beta \tau^\nu \dot{x}_\alpha \dot{x}_\gamma \dot{x}_\delta \ldots \partial_{\beta \mu} \tau^\alpha \dot{x}_\nu \dot{x}_\gamma \dot{x}_\delta \ldots \frac{\partial}{\partial \dot{x}_\alpha \dot{x}_\gamma \dot{x}_\delta \ldots}$$

of $\tau$ onto the tensor bundle

$$T^k_m X = (\otimes T X) \otimes (\otimes T^* X);$$

- the canonical lift

$$\tilde{\tau} = \tau^\mu \partial_\mu + \left[ \partial_\nu \tau^\alpha k^\beta_{\gamma \mu} - \partial_\beta \tau^\nu k^\alpha_{\gamma \mu} - \partial_\mu \tau^\nu k^\alpha_{\beta \nu} - \partial_\beta \tau^\nu k^\beta_{\nu \mu} - \partial_\mu \tau^\alpha \right] \frac{\partial}{\partial k^{\alpha \beta \mu}}$$

of $\tau$ onto the bundle $C$ of the linear connections on $TX$.

One can think of the vector fields $\tilde{\tau}$ on a bundle of geometric objects $Y$ as being the vector fields associated with the local 1-parameter groups of the holonomic isomorphisms of $Y$ induced by diffeomorphisms of its base $X$. In particular, if $Y = TX$ they are the tangent isomorphisms. We call these isomorphisms the general covariant transformations.

Let $Y$ be the bundle of geometric objects and $L$ a Lagrangian density on the configuration space $J^1Y$. Given a vector field $\tau$ on the base $X$ and its canonical lift $\tilde{\tau}$ onto $Y$, one may
utilize the first variational formula (6) in order to get the corresponding SEM transformation law. If the Lagrangian density $L$ is invariant under general covariant transformations, we have the equality

$$L_{\tilde{\tau}}L = 0$$

and get the weak conservation law (7). One can show that the conserved quantity is reduced to a superpotential term.

Let us verify this fact in case of a tensor bundle $Y \to X$. Let it be coordinatized by $(x, y^i)$ where the collective index $A$ is employed. Given a vector field $\tau$ on $X$, its canonical lift $\tilde{\tau}$ on $Y$ reads

$$\tilde{\tau} = \tau^\alpha \partial_\alpha + u^i_\beta \partial_\beta \tau^\alpha \partial_i.$$

Let a Lagrangian density $L$ on the configuration space $J^1Y$ be invariant under general covariant transformations. Then, it satisfies the equality (9) which takes the coordinate form

$$\partial_\alpha (\tau^\alpha \mathcal{L}) + u^i_\beta \partial_\beta \tau^\alpha \partial_i \mathcal{L} + \tilde{\partial}_\mu (u^i_\beta \partial_\beta \tau^\alpha) \partial_i^\mu \mathcal{L} - y^i_\alpha \partial_\beta \tau^\alpha \partial_i^\beta \mathcal{L} = 0.$$  \hspace{1cm} (10)

Due to the arbitrariness of the functions $\tau^\alpha$, the equality (10) is equivalent to the system of equalities

$$\partial_\lambda \mathcal{L} = 0,$$

$$u^i_\alpha \partial_\beta \tau^\alpha \partial_i \mathcal{L} + \tilde{\partial}_\mu (u^i_\beta \partial_\beta \tau^\alpha) \partial_i^\mu \mathcal{L} - y^i_\alpha \partial_\beta \tau^\alpha \partial_i^\beta \mathcal{L} = 0,$$

$$u^i_\alpha \partial_i^\mu \mathcal{L} + u^i_\alpha \partial_\lambda \mathcal{L} = 0.$$

(11a) \hspace{2cm} (11b) \hspace{2cm} (11c)

It is readily observed that the equality (11b) can be brought into the form

$$\delta_\alpha \mathcal{L} + u^i_\lambda \partial_\lambda \tau^\alpha + \tilde{\partial}_\mu (u^i_\beta \partial_\beta \tau^\alpha) \partial_i^\mu \mathcal{L} = y^i_\alpha \partial_\beta \tau^\alpha \mathcal{L}$$

(12)

where $\mathcal{L}_i$ are the variational derivatives of the Lagrangian density $L$. Substituting the relations (12) and (11c) into the weak identity

$$0 \approx \tilde{\partial}_\lambda [(u^i_\alpha \partial_\beta \tau^\alpha - y^i_\alpha \tau^\alpha) \partial_i^\lambda \mathcal{L} + \tau^\lambda \mathcal{L}]$$

(13), we get the conservation law

$$0 \approx \tilde{\partial}_\lambda [-u^i_\alpha \mathcal{L}_i \tau^\alpha - \tilde{\partial}_\mu (u^i_\alpha \partial_i^\mu \mathcal{L} \tau^\alpha)]$$

(13)

where the conserved current is reduced to the superpotential term

$$Q(\tau)^\lambda = -u^i_\alpha \mathcal{L}_i \tau^\alpha - \tilde{\partial}_\mu (u^i_\alpha \partial_i^\mu \mathcal{L} \tau^\alpha).$$

(14)

One can utilize the tensor fields described above as a matter source in the gravitation theory on bundles of geometric objects. Their Lagrangian densities are independent on a symmetric connection, and they contain the torsion of a general linear connection.
Proca fields which are described by sections of the cotangent bundle \( Y = T^*X \) exemplify a field model on bundles of geometric objects. Let us consider the SEM transformation law of Proca fields in the presence of a background world metric.

The configuration space \( J^1Y \) of Proca fields is coordinatized by \((x^\lambda, k_\mu, k_{\mu\lambda})\) where \( k_\mu = \dot{x}_\mu \) are the induced coordinates of \( T^*X \). The Lagrangian density of Proca fields is

\[
L_P = \left[ -\frac{1}{4\gamma} g^{\alpha\lambda} g^{\nu\beta} F_{\alpha\beta} F_{\mu\nu} - \frac{1}{2} m^2 g^{\mu\lambda} k_{\mu} k_{\lambda} \sqrt{| g |} \right] \omega \tag{15}
\]

where

\[
F_{\mu\nu} = k_{\nu\mu} - k_{\mu\nu}.
\]

Let \( \tau \) be a vector field on the base \( X \) and

\[
\tilde{\tau} = \tau^\mu \partial_\mu - \partial_\alpha \tau^\mu k_\nu \frac{\partial}{\partial k_\alpha}
\]

its canonical lift onto \( T^*X \).

The Lie derivative of the Lagrangian density (15) by the jet lift \( j^1_0 \tilde{\tau} \) of the field \( \tilde{\tau} \) is

\[
\mathbf{L}_{j^1_0 \tilde{\tau}} L_P = (\partial_\lambda \tau^\lambda \mathbf{L}_P + \tau^\lambda \partial_\lambda \mathbf{L}_P - \mathbf{F}_{\mu\nu} \partial_\lambda \tau^\mu \pi^{\nu\lambda} + m^2 g^{\nu\lambda} \partial_\lambda \tau^\mu k_\nu k_\mu \sqrt{| g |}) \omega
\]

where \( \pi^{\nu\lambda} = \partial^\nu \mathbf{L} \) are the Lagrangian momenta. Then, the corresponding SEM transformation law

\[
\mathbf{L}_{j^1_0 \tilde{\tau}} L_P \approx \mathbf{\hat{\partial}_\lambda} [\pi^{\nu\lambda} (-\partial_\nu \tau^\mu k_\mu - \tau^\mu k_{\nu\mu}) + \tau^\lambda \mathbf{L}]
\]

takes the form

\[
- \partial_\lambda \tau^\mu t^\lambda_\mu \sqrt{| g |} - \tau^\mu t^\lambda_{\nu\beta} \sqrt{| g |} \{^\beta_{\mu\alpha}\} \approx \mathbf{\hat{\partial}_\lambda} [-\tau^\mu t^\lambda_\mu \sqrt{| g |} + \tau^\nu k_\nu \mathbf{\mathcal{E}^\lambda} - \mathbf{\hat{\partial}_\mu} (\pi^{\mu\lambda} \tau^\nu k_\nu)] \tag{16}
\]

where

\[
t^\lambda_\mu \sqrt{| g |} = 2 g^{| \lambda \nu } \frac{\partial \mathbf{L}}{\partial g^{\nu \mu}}
\]

is the energy-momentum tensor.

A glance at the expression (16) shows that the SEM tensor of the Proca field

\[
\mathbf{T}(\mathbf{\tau})^\lambda = \tau^\mu t^\lambda_\mu \sqrt{| g |} - \tau^\nu k_\nu \mathbf{\mathcal{E}^\lambda} + \mathbf{\hat{\partial}_\mu} (\pi^{\mu\lambda} \tau^\nu k_\nu) \tag{17}
\]

is the sum of the familiar metric energy-momentum tensor and the superpotential term

\[
\mathbf{Q}(\mathbf{\tau})^\lambda = -\tau^\nu k_\nu \mathbf{\mathcal{E}^\lambda} + \mathbf{\hat{\partial}_\mu} (\pi^{\mu\lambda} \tau^\nu k_\nu) \tag{18}
\]
which is the particular case of the superpotential term (14). This term however does not make any contribution into the differential conservation law (16) which thus takes the standard form
\[ t_{\mu;\lambda}^\lambda \approx 0. \]

At the same time, the superpotential term reflects the fact that the symmetry of the Lagrangian density (15) under general covariant transformations is broken by the background metric field. In gravitation theory, when the general covariant transformations are exact, the total superpotential term contains the whole SEM tensor (17) of Proca fields. Thus, the Proca field model exemplifies the phenomenon of "hidden energy". Only the superpotential part of energy-momentum is displayed if the general covariant transformations are exact.

5

Let us consider the affine-metric gravitational model where dynamic variables are pseudo-Riemannian metrics and general linear connections on \( X \). They are called the world metrics and the world connections respectively.

The 4-dimensional base manifold \( X \) is assumed to satisfy the well-known topological condition in order that it can be provided with a pseudo-Riemannian metric. We call it the world manifold.

Let \( LX \to X \) be the principal bundle of linear frames in the tangent spaces to \( X \). Its structure group is \( GL^+(4, \mathbb{R}) \).

World metrics are represented by sections of the bundle \( \Sigma_g \to X \). It is the 2-fold covering of the bundle
\[ \Sigma = LX/SO(3, 1) \]
where \( SO(3, 1) \) is the connected Lorentz group. Hereafter, we shall identify \( \Sigma_g \) with the open subbundle of the tensor bundle
\[ \mathcal{O} T^* X \to X. \]

The bundle \( \Sigma_g \) is coordinatized by \( (x^\lambda, g_{\alpha\beta}) \).

The world connections are thought as principal connections on the principal bundle \( LX \to X^4 \). Indeed, there is the 1:1 correspondence between the world connections and the global sections of the bundle
\[ C = J^1 LX/GL^+(4, \mathbb{R}). \]

With respect to a holonomic atlas, the bundle \( C \) is coordinatized by \( (x^\lambda, k^{\alpha}_{\beta\lambda}) \) so that, for any section \( K \) of \( C \),
\[ K^\alpha_{\beta\lambda} = k^{\alpha}_{\beta\lambda} \circ K \]
are the coefficients of the linear connection
\[ K = dx^\lambda \otimes \left( \frac{\partial}{\partial x^\lambda} + K^\alpha_\beta \lambda \frac{\partial}{\partial \dot{x}^\beta} \right) \]
on T^*X.

Note that, since the world connections are the principal connections, one may apply the standard procedure of gauge theory, but in this case the nonholonomic gauge isomorphisms of the linear frame bundle \( LX \) and the associated bundles to be considered [4]. The canonical lift \( \tau \) of a vector field \( \tau \) on the base \( X \) onto the bundles \( \Sigma_g \) and \( C \) does not correspond to these isomorphisms. One must use a horizontal lift of \( \tau \) by means of some connections on these bundles. Here, we limit our consideration to the framework of field theory on the bundles of geometric objects.

The total configuration space of the affine-metric gravity is
\[ J^1Y = J^1(\Sigma_g \times C) \]coordinatized by \( (x^\lambda, g^{\alpha\beta}, k^\alpha_\beta\lambda, g^{\alpha_\beta\mu}, k^\alpha_\beta\lambda\mu) \).

We assume that a Lagrangian density \( L \) of the affine-metric gravitation theory on the configuration space (19) depends on a metric \( g^{\alpha\beta} \) and the curvature
\[ R^\alpha_\beta\nu\lambda = k^{\alpha}_\beta\lambda\nu - k^{\alpha}_\beta\nu\lambda + k^{\alpha}_\nu\epsilon\lambda k^\epsilon_\beta\nu - k^{\alpha}_\epsilon\lambda k^\epsilon_\beta\nu. \]

In this case, we have the relations
\[ \frac{\partial L}{k^{\alpha}_\beta\nu} = \pi^{\beta\nu\lambda\sigma} k^{\alpha}_\sigma\lambda - \pi^{\alpha\sigma\nu\lambda} k^{\beta}_\sigma\lambda, \]
\[ \pi^{\alpha\beta\nu\lambda} = \partial^{\alpha\beta\nu\lambda} L = -\pi^{\alpha\beta\nu}. \]

Let the Lagrangian density \( L \) be invariant under general covariant transformations. Given a vector field \( \tau \) on \( X \), its canonical lift onto the bundle \( \Sigma_g \times C \) reads
\[ \tilde{\tau} = \tau^\lambda \partial_\lambda + (g^{\nu\beta} \partial_\nu \tau^\alpha + g^{\alpha\nu} \partial_\nu \tau^\beta) \frac{\partial}{\partial g^{\alpha\beta}} \]
\[ + \left[ \partial_\nu \tau^\alpha k^{\nu}_\beta\mu - \partial_\beta \tau^{\nu} k^{\alpha}_\nu\mu - \partial_\mu \tau^{\nu} k^{\alpha}_\beta\nu - \partial_\beta \tau^{\mu} k^{\alpha}_\beta\nu \right] \frac{\partial}{\partial k^{\alpha}_\beta\mu}. \]

For the sake of simplicity, the compact notation
\[ \tilde{\tau} = \tau^\lambda \partial_\lambda + (g^{\nu\beta} \partial_\nu \tau^\alpha + g^{\alpha\nu} \partial_\nu \tau^\beta) \partial_{\alpha\beta} + (u^{A\beta}_\alpha \partial_\beta \tau^\alpha - u^{A\epsilon\beta}_\alpha \partial_\epsilon \tau^\alpha) \partial_A \]
is employed.
Since the Lie derivative of $L$ by the jet lift $j^1_0 \bar{\tau}$ of the field $\bar{\tau}$ (20) is equal to zero:

$$L_{j^1_0 \bar{\tau}}L = 0,$$

we have the weak conservation law

$$0 \approx \frac{\partial \lambda}{\partial \lambda} L(u^{A\beta}_\alpha \partial_{\beta} \tau^\alpha - u^{A\beta}_\alpha \partial_{\epsilon \beta} \tau^\alpha - y^{A\gamma}_\alpha \tau^\lambda) + \tau^\lambda L,$$

where

$$\partial \lambda L u^{A\beta}_\alpha = \pi^{e\beta}_\lambda,$$

$$\partial \lambda L u^{A\beta}_\alpha = \pi^{\gamma \mu \epsilon \beta \lambda}_\alpha - \pi^{\beta \epsilon \lambda}_{\alpha \mu} - \pi^{\gamma \beta \epsilon \lambda}_{\alpha \mu} = \partial \lambda \beta \epsilon L - \partial \lambda \gamma \beta \epsilon k \gamma \lambda.\

Due to the arbitrariness of the functions $\tau^\alpha$, (21) implies the following equality

$$\delta^\lambda \alpha L + \sqrt{-g} T^\alpha_\beta + u^{A\beta}_\alpha \partial_{\lambda} L + \delta_{\mu} (u^{A\beta}_\alpha) \partial^\mu L - y^{A\gamma}_\alpha \partial^\beta \alpha L = 0.$$ (23)

One can think of

$$\sqrt{-g} T^\alpha_\beta = 2g^{\alpha \nu} \partial_{\nu} \beta L$$

as being the metric energy-momentum tensor of general linear connections.

Substituting the term $y^{A\gamma}_\alpha \partial^\beta \alpha L$ from the expression (23) into the conservation law (22), we bring it into the form

$$0 \approx \frac{\partial \lambda}{\partial \lambda} L(u^{A\beta}_\alpha \partial_{\beta} \tau^\alpha - u^{A\beta}_\alpha \partial_{\epsilon \beta} \tau^\alpha) - \partial \lambda L u^{A\lambda}_\alpha \tau^\alpha - \partial \lambda \mu L \delta_{\mu} (u^{A\lambda}_\alpha) \tau^\alpha.$$

Let us separate the components of the Euler-Lagrange operator

$$E_L = (\varepsilon_{\alpha \beta} dg^{\alpha \beta} + \varepsilon_{\alpha \gamma} \partial_{\gamma} \alpha k^{\alpha \gamma}) \land \omega$$

in the expression (24). We get

$$0 \approx \frac{\partial \lambda}{\partial \lambda} L[-2g^{\lambda \mu} \tau^\alpha \varepsilon_{\alpha \mu} - u^{A\lambda}_\alpha \tau^\alpha \varepsilon_{\lambda} A] +\frac{\partial \lambda}{\partial \lambda} L u^{A\mu}_\alpha \partial_{\mu} \tau^\alpha - \delta_{\mu} (u^{A\lambda}_\alpha) \tau^\alpha + \delta_{\mu} (\varepsilon^{\mu \lambda}) \partial_{\mu} \tau^\alpha + \delta_{\mu} (\varepsilon^{\mu \lambda}) \partial_{\mu} \tau^\alpha$$

and then

$$0 \approx \frac{\partial \lambda}{\partial \lambda} L[-2g^{\lambda \mu} \tau^\alpha \varepsilon_{\alpha \mu} - (k^{\lambda \gamma}_\mu \varepsilon^{\mu \gamma} k^{\sigma}_{\alpha \mu} \varepsilon_{\sigma} \lambda \mu - k^{\sigma}_{\gamma \alpha} E_{\sigma} \gamma \lambda) \tau^\alpha + \varepsilon_{\alpha \gamma} \partial_{\gamma} \tau^\alpha] + \delta_{\mu} (k^{\lambda \mu}_\alpha \varepsilon_{\lambda \mu} \tau^\alpha) + \delta_{\mu} (\varepsilon^{\mu \lambda}) \partial_{\mu} \tau^\alpha + \delta_{\mu} (\varepsilon^{\mu \lambda} (D_\nu \tau^\alpha + \Omega^{\alpha \nu}_{\nu \sigma} \tau^\sigma)).$$
The final form of the conservation law (22) is

\[
0 \approx \hat{\partial}_\lambda \left[ -2g^{\lambda \mu} \tau^\alpha \mathcal{E}_{\alpha \mu} - (k^\lambda \gamma_\mu \mathcal{E}_\alpha^{\gamma \mu} + k^\sigma \gamma_\alpha \mathcal{E}_\sigma^{\gamma \lambda} + \mathcal{E}_\alpha^{\varepsilon \lambda} \partial_\varepsilon \tau^\alpha - \hat{\partial}_\mu (\mathcal{E}^{\lambda \mu} \tau^\alpha) + \mathcal{E}_\alpha^{\varepsilon \lambda} \partial_\varepsilon \tau^\alpha - \hat{\partial}_\mu (\mathcal{E}^{\lambda \mu} \tau^\alpha) \right] + \\
\hat{\partial}_\lambda \left[ - \hat{\partial}_\mu (\pi_\alpha^{\nu \mu \lambda} (D_\nu \tau^\alpha + \Omega_\alpha^{\nu \sigma} \tau^\sigma) \right].
\] (25)

It follows that the SEM conservation law in the affine-metric gravity is reduced to the form \( (1) \) where \( U \) is the generalized Komar superpotential \( (2) \).

Note that, as like as in gauge theory, the fact that this superpotential depends on the components of a vector field \( \tau \) provides invariance of the SEM conservation law \( (1) \) under general covariant transformations.

Let us now consider the total system consisting of the affine-metric gravity and tensor fields described above, e.g., a Proca field. In the presence of a general linear connection, their Lagrangian density \( L_m \) is naturally generalized through covariant derivatives and depends on the torsion. The total SEM conservation law is the sum of the expression \( (13) \) and \( (25) \) plus the additional contribution

\[ \hat{\partial}_\mu (\partial_\alpha^{\lambda \mu} L_m \tau^\alpha) \]

in the superpotential term.

One can consider general linear connections in the presence of a background world metric \( g \) when the general covariant transformations are not exact. In this case, the SEM complex of affine-metric gravity takes the form

\[ T^\lambda = \sqrt{-g} T_\alpha^{\lambda \tau^\alpha} + \hat{\partial}_\mu (\pi^{\nu \mu \lambda}_\alpha (D_\nu \tau^\alpha + \Omega^{\alpha \nu \sigma} \tau^\sigma) \]

and the conservation law comes to the form of the familiar covariant conservation law

\[ T^{\lambda \mu}_{\mu \lambda} \approx 0. \]

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