Amplification of gravitational wave by a Kerr black hole

Yi Gong, Zhoujian Cao and Xian Chen

School of Physics and Technology, Wuhan University, Wuhan, Hubei 430072, China
Institute of Applied Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China
School of Fundamental Physics and Mathematical Sciences, Hangzhou Institute for Advanced Study, UCAS, Hangzhou 310024, China
Astronomy Department, School of Physics, Peking University, 100871 Beijing, China
Kavli Institute for Astronomy and Astrophysics at Peking University, 100871 Beijing, China

Binary black hole may form near a supermassive black hole. The background black hole (BH) will affect the gravitational wave (GW) generated by the binary black hole. It is well-known that the Penrose process can provide extra energy due to the ergosphere. In the present paper we investigate the energy amplification of the gravitational wave by a Kerr black hole background. In particular and different from the earlier studies, we compare the energies of the waves in the cases with and without a nearby Kerr BH.

I. INTRODUCTION

The Laser Interferometer Gravitational-wave Observatory (LIGO) and the Virgo detectors have detected dozens of stellar-mass binary black holes (BBHs) [1]. Recent astrophysical models [2] suggest that some of these binaries might form in the close vicinity of supermassive black holes (SMBHs). First, BBHs can be tidally captured by a SMBH to a distance as small as tens of gravitational radii of the SMBH without being disrupted [2]. However, the event rate is low, about four orders of magnitude below the rate inferred from the LIGO/Virgo events. Second, the formation and merger rates of BBHs would be enhanced in the accretion disks of active galactic nuclei (AGNs) as a result of the interaction with the gas in the disks [2]. More recent studies showed that the interaction also causes single stellar mass black holes (BHs) to migrate towards the central SMBHs [3] and eventually be trapped at a radius comparable to the innermost stable circular orbit (ISCO) [3]. The accretion of BHs at the ISCO also creates a condition favorable to the formation and merger of BBHs. The corresponding event rate is estimated to be 1% of the LIGO/Virgo rate [3].

If the BBHs in the above scenarios merge, they are doing so in a curved background induced by the nearby SMBH. In particular, when the SMBH is rotating, it is interesting to ask how the Kerr black hole background could affect the gravitational waves (GW) generated by the BBH. Especially due to the existence of the ergosphere, one may expect interesting energy amplification of the gravitational wave through the Penrose process.

The energy amplification problem has been extensively studied before for particles. Consider a particle with mass $M$ and energy $E_F$ which decays into two daughter particles. One daughter particle falls into a negative-energy orbit, and the other goes out with a mass $M_D$ and energy $E_D$. Wald [7, 8] concluded that the Penrose process does not work in this situation unless the initial particle moves with velocity larger than half of the speed of light. The energy amplification is limited by an inequality (Eq. (4) of [7])

$$\gamma_{FD} \frac{E_F}{M_F} - \gamma_{FD} v_{FD} \sqrt{1 - \frac{E_D^2}{M_D^2}} \leq \frac{E_F}{M_F} + \gamma_{FD} v_{FD} \sqrt{1 + \frac{E_D^2}{M_D^2}}$$

(1)

where $v_{FD}$ is the relative velocity between the initial particle and the outgoing daughter particle and $\gamma_{FD} = 1/\sqrt{1 - v_{FD}^2}$. More recently, however, it is found that if initially there are two particles and they are colliding near a Kerr SMBH, the total energy could be amplified due to the Penrose process by an arbitrary factor [9]. For this reason, Kerr black holes have been widely investigated as a particle accelerator [10–18].

For a BBH with a total mass of $M$, the GW energy radiated away during the merger is typically $\eta M$ with $\eta \approx 5\%$ [17]. For example, GW150914 is composed of two black holes with initial masses 36M$_\odot$ and 29M$_\odot$, and the GW carries a total amount of energy of 3M$_\odot$. If such a merger happens near a Kerr black hole, how much energy will be carried away by the gravitational wave? Especially if the final remnant black hole falls onto an orbit with a negative energy around the Kerr black hole, what will happen? We investigate this problem in the current paper.

We find that only when the BBH is moving relative to the Kerr BH can the GW energy be amplified. This
is similar to the finding by previous authors \cite{3, 8} for particles. Phenomenologically, our physical set up can be viewed as a particle, constituted by the initial binary, decaying into two other particles, corresponding to the final black hole and the gravitational wave which moves at the speed of light. Outside the ergosphere, astrophysical reality makes the relative speed between the BBH and the Kerr black hole small. In this case we find that the Kerr BH can amplify the energy of gravitational wave by at most 1.3 times. Inside the ergosphere, we find that the amplification is less than 5 times.

The arrangement of the paper is as following. We describe the physical setup in the next section. In Sec. II and Sec. III, we discuss the energy amplification of the gravitational wave when the BBH merges, respectively, outside and inside the ergosphere of the Kerr black hole. A summary and discussion are given in the last section. Throughout the paper we use geometric units with \( c = G = 1 \).

II. PHYSICAL SETUP

We assume that the total mass of the BBH is \( M \), and the four velocity of the center of mass is \( U^a \) with respect to the background Kerr black hole. The mass of the merger remnant black hole is \( M' \). Due to the kick velocity, the four velocity of the final black hole, \( U'^a \), is different from \( U^a \). Using the 3+1 decomposition of \( U'^a \) relative to the observer \( U^a \), we have \cite{18}

\[
U'^a = \frac{\gamma (U^a + v^a)}{\gamma + u^a U_a},
\]

\[
\gamma \equiv -U'^a U_a,
\]

where \( v^a \) is perpendicular to \( U^a \) which means \( U^a v_a = 0 \). Since \( v^a \) is perpendicular to \( U^a \), \( v^a \) is spatial with respect to \( U^a \) and is called the three velocity of \( U^a \) with respect to \( U'^a \). Consequently, \( v^a \) corresponds to the kick velocity \cite{8}. Since four velocity \( U'^a \) satisfies \( U'^a U'_a = -1 \), we have \( \gamma = 1/\sqrt{1 - v^2} \) from the above relations. Here \( v^2 = g_{ab} v^a v^b \geq 0 \) because \( v^a \) is spatial. Eq. (2) is a four dimensional generalization of the classic formula of velocity superposition.

As a stationary spacetime, Kerr black hole admits a time-like Killing field \( \xi^a \equiv (\frac{\partial}{\partial t})^a \) where \( t \) is the time coordinate of the Boyer-Lindquist coordinate \cite{18}. Respect to \( \xi^a \) we denote

\[
\xi^a \sim -g_{ab} M U'^b \xi^a,
\]

\[
E' \equiv -g_{ab} M' U'^a \xi^b,
\]

\[
E_{GW} \equiv E - E' = -g_{ab} \xi^b (M U'^a - M' U'^a).
\]

The above equations are valid at the spacetime point where the BBH merger happens. \( MU'^a \) and \( M' U'^a \) correspond to the four momenta of, respectively, the initial binary and the final remnant black hole. According to the law of conservation of momentum, \( MU'^a - M' U'^a \) corresponds to the four momentum of the gravitational wave. Since the gravitational wave propagates to null infinity along a null geodesic, \( E_{GW} \) defined above is a constant along the geodesic because \( \xi^a \) is a Killing field. Moreover, \( E_{GW} \) corresponds to the gravitational wave energy measured by the asymptotic observer \( \omega' = (\frac{\partial}{\partial t})^{\omega'} \). These properties give our motivation to introduce the Killing field into our analysis. The quantities \( E \) and \( E' \) defined above significantly simplify our analysis in the following.

If a gravitational wave detector has a relative velocity respect to the background black hole, the energy seen by the detector will be \( E_{GW} \) multiplied by a Doppler factor. Usually the background Kerr black hole is the supermassive black hole locating at the center of a galaxy which moves together with the comoving frame of our Universe. For simplicity, we assume that our GW detector also moves with the comoving frame of our Universe. Consequently the energy detected will be \( E_{GW} \) itself.

If we denote \( \eta \equiv 1 - M' / M \), \( \Pi \equiv -\xi^a U_a \) and \( \zeta \equiv g_{ab} \xi^a \xi^b \equiv \xi^a \xi^a / v \), Eq. (6) results in

\[
\frac{E_{GW}}{M} = \Pi - \frac{1}{\sqrt{1 - v^2}} (1 - \eta)(\Pi - v \zeta) \tag{7}
\]

Here \( \eta \) is the usual GW energy ratio used by current LIGO and other detectors. The value of \( \gamma \) depends only on the amplitude of the kick velocity \( v \). For a given Kerr black hole background, \( \zeta \) depends only on the direction of the kick velocity \( \tilde{v} \). Since \( \Pi \) depends on \( U^a \), which is independent of the kick velocity, \( \eta \), \( \gamma \), \( \zeta \) and \( \Pi \) are mutually independent. In other words we can take \( E_{GW} \) as a function of four independent arguments \( \eta \), \( v \), \( \zeta \) and \( \Pi \)

\[
\frac{E_{GW}}{M} = \Pi - \frac{1}{\sqrt{1 - v^2}} (1 - \eta)(\Pi - v \zeta) \tag{8}
\]

Now we analyze the dependence of \( E_{GW} \) on \( \zeta \), or equivalently, the dependence of \( E_{GW} \) on the direction of the kick velocity. We can decompose \( \xi^a \) as

\[
\xi^a = -\Pi U^a + \tilde{w}^a + \sqrt{\Xi + \Pi^2} \zeta \tilde{w}^a,
\]

\[
\Xi \equiv g_{ab} \xi^a \xi^b \tag{9},
\]

where \( \tilde{w} \) is spatial, normal, and perpendicular to \( U^a \) and \( v^a \), i.e., \( \tilde{w}^a U_a = \tilde{w}^a v_a = 0 \). So we have

\[
- \sqrt{\Xi + \Pi^2} \leq \zeta \leq \sqrt{\Xi + \Pi^2} \tag{10},
\]

Since \( \gamma (1 - \eta) = \gamma M' / M > 0 \), Eq. (7) indicates that \( E_{GW} \) increases with \( \zeta \). So the maximum energy of the gravitational wave is

\[
E_{GW1} \equiv \max_{\zeta} (E_{GW}/M) = \Pi - \gamma (1 - \eta) (\Pi - v \sqrt{\Xi + \Pi^2}) \tag{12},
\]

The values of \( \Pi \) and \( \Xi \) depend on the spacetime point where the merger happens. We note that this result is consistent with the Wald inequality given in \cite{21}.
Consequently we have units we have always 5% [17], we have (1 − Ξ) ≥ 0, Eq. (13) reduces to

\[ \Xi = -1 + \frac{2M_{Kerr}r}{r^2 + a_{Kerr}^2 \cos\theta} \]  (15)

Outside the ergosphere we have

\[ r > M_{Kerr} + \sqrt{M_{Kerr}^2 - a_{Kerr}^2 \cos^2\theta}, \]  (16)

\[ -1 < \Xi < 0. \]  (17)

Inside the ergosphere we have

\[ M_{Kerr} + \sqrt{M_{Kerr}^2 - a_{Kerr}^2 \cos^2\theta} < r < M_{Kerr} + \sqrt{M_{Kerr}^2 - a_{Kerr}^2 \cos^2\theta}, \]  (18)

\[ 0 < \Xi < \frac{a_{Kerr}^2 \sin^2\theta}{2M_{Kerr}(M_{Kerr} + \sqrt{M_{Kerr}^2 - a^2}) - a_{Kerr}^2 \sin^2\theta}. \]  (19)

Note that \( M_{Kerr} > a_{Kerr} \) due to the cosmic censorship, we have always

\[ -1 < \Xi < 1. \]  (20)

Consequently we have

\[ \Xi + \Pi^2 \leq 2 \max(1, \Pi^2). \]  (21)

The maximal kick velocity is about 5000 km/s [19]. With units \( c = 1, v_{max} \approx 10^{-2} \). Combining \( \mathcal{E}_{GW0} \approx \eta \approx 5\% \) [17], we have \( (1 - \mathcal{E}_{GW0})v \leq 0.2\mathcal{E}_{GW0}. \) Given these relations, the second term in Eq. (19) becomes

\[ (1 - \mathcal{E}_{GW0})v \sqrt{\Xi + \Pi^2} \leq \begin{cases} 0.3\mathcal{E}_{GW0}, & \text{if } \Pi^2 < 1 \\ 0.3\mathcal{E}_{GW0}|\Pi|, & \text{if } \Pi^2 > 1 \end{cases} \]  (22)

So Eq. (13) reduces to

\[ \mathcal{E}_{GW1} \leq \begin{cases} (\Pi + 0.3\mathcal{E}_{GW0}), & \text{if } \Pi^2 < 1 \\ (\Pi + 0.3|\Pi|)\mathcal{E}_{GW0}, & \text{if } \Pi^2 > 1 \end{cases} \]  (23)

III. MERGERS OUTSIDE THE ERGOSPHERE

As we have analyzed in the above section, \(-1 < \Xi < 0\) outside the ergosphere. Denoting the normalized \( \xi^a \) as \( \xi \) we have

\[ \xi^a = \frac{\xi}{|\xi|}, \]  (24)

\[ |\xi| = \sqrt{|g_{ab}\xi^a\xi^b|}. \]  (25)

Outside the ergosphere, \( \xi \) is time-like. If the center of mass of the BBH is at rest with respect to the background Kerr black hole, then the four velocity of the center of mass of the BBH \( U^a = \xi^a \) in this situation. According to the definition of \( \Pi \) we have \( \Pi = \sqrt{-\Xi} \) which is always less than 1 because of Eq. (17). So Eq. (25) suggests that the energy amplification factor is less than 1.3 in this situation.

If the center of mass of the BBH has a relative velocity respect to the background Kerr black hole, we have an additional Doppler factor

\[ \Pi_D = -\xi^a U_a, \]  (26)

\[ \Pi = \Pi_D \sqrt{-\Xi} \]  (27)

where \( \Pi_D \) is the Lorentz factor between two observers whose four velocities are respectively \( \xi^a \) and \( U^a \).

If \( \Pi_D < 1/\sqrt{-\Xi} \), Eq. (26) suggests that the energy amplification factor is less than 1.3. If \( \Pi_D > 1/\sqrt{-\Xi} \) we have \( \mathcal{E}_{GW1} < 1.3\Pi \mathcal{E}_{GW0} \) and the amplification factor could be large. Since \( \Pi = \Pi_D \sqrt{-\Xi} \) corresponds to the specific energy \( E \) in Eq. (7) of [20], \( \Pi > 1 \) means that the BBH moves along an unbound, hyperbolic orbit of the Kerr black hole background. Such kind of binary is not realistic because astrophysical objects far from a SMBH are moving at velocities normally much smaller than the speed of light, which makes their orbits parabolic (\( \Pi \approx 1 \)). So we conclude that the GW energy will be amplified by the background Kerr black hole less than 1.3 times if the BBH merges outside the ergosphere.

IV. MERGERS INSIDE THE ERGOSPHERE

Inside the ergosphere, \( \xi \) is spatial and consequently \( \Pi \) may take any value in \((-\infty, +\infty)\) if there is no other constraint. If \( \Pi < -1 \), Eq. (26) suggests that \( \mathcal{E}_{GW1} \leq 0 \). If \( -1 < \Pi < 0 \), Eq. (25) tells us that \( \mathcal{E}_{GW1} \leq 0.3\mathcal{E}_{GW0} \). We discuss the situations with positive \( \Pi \) in the following.

If the BBH moves along a geodesic line of the background Kerr black hole, \( \Pi \) corresponds to the specific energy of the geodesic line and \( 0 < \Pi < 1 \) if a bound orbit is concerned. So if the BBH forms outside the ergosphere and falls into the ergosphere along a geodesic line, according to Eq. (25), the GW energy will be amplified by the background Kerr black hole less than 1.3 times.
In the following we consider another scenario. It involves an accretion disk extending to the ISCO of a Kerr SMBH, well inside the ergosphere. Astrophysical models predict that BBHs may form in such accretion disks so that some binaries may reside inside the ergosphere \[2\].

There is an unique future-pointing time-like normalized vector \(\hat{\xi}_T^a\) which is perpendicular to \(\xi^a\) and lies in the \(t,\phi\) subspace

\[
g_{ab}\hat{\xi}_T^a\hat{\xi}_T^b = -1,
\]

\[
g_{ab}\hat{\xi}_T^a\xi^b = 0.
\]

Then \((\hat{\xi}_T^a, \xi^a)\) form a convenient orthonormal basis for the \(t,\phi\) subspace. In general, the matter of the accretion disk moves along a circular orbit around the Kerr black hole. In this case the four velocity of the matter \(\psi^a\) locates in the \(t,\phi\) subspace. Consequently we can decompose it as

\[
\psi^a = \psi^0\hat{\xi}_T^a + \psi^1\xi^a,
\]

then we have the relations \[21\]

\[
-(\psi^0)^2 + (\psi^1)^2 = -1,
\]

\[
\psi^a\xi_a = \psi^1\xi = E_c,
\]

\[
E_c \equiv \frac{1 - 2u + \chi u^{3/2}}{\sqrt{1 - 3u + 2\chi u^{3/2}}}.
\]

\[
u \equiv \frac{M_{\text{Kerr}}}{r}, \chi \equiv \frac{a_{\text{Kerr}}}{M_{\text{Kerr}}}.
\]

Direct calculation results in

\[
\psi^0 = \sqrt{1 + \left(\frac{E_c}{\xi}\right)^2},
\]

\[
\psi^1 = \frac{E_c}{\xi}.
\]

If the BBH moves along with the accretion disk matter we have \(U^a = \psi^a\) and consequently

\[
\Pi = -\psi^a\xi_a = -E_c < 0.
\]

Then Eq. \(22\) tells us \(\mathcal{E}_{GW1} \leq 0.3\mathcal{E}_{GW0}\).

However, depending on the formation scenario of BBH, the BBH may have a relative velocity \(w\) with respect to the accretion disk matter \(\psi^a\). In order to get larger GW energy amplification, we need larger velocity \(w\). In the viewpoint of the observer whose four velocity coincides with \(\xi^a\), the BBH and the accretion matter should move in opposite directions to make \(w\) larger. That means the four velocity of the BBH can be expanded as

\[
U^a = U^0\hat{\xi}_T^a + U^1\xi^a,
\]

\[
-(U^0)^2 + (U^1)^2 = -1,
\]

\[
U^a\xi_a = U^1\xi = \Pi,
\]

\[
\psi^aU_a = -\frac{1}{\sqrt{1 - w^2}}.
\]

These equations give us

\[
\Pi = \frac{\chi}{\sqrt{1 - w^2}} \left[\psi^1 \pm w\psi^0\right],
\]

where the ‘+’ sign corresponds to the case where the relative velocity of \(U^a\) respect to \(\psi^a\) is along the \(\xi^a\) direction, while ‘-’ corresponds the \(-\xi^a\) direction. When the relative velocity is along the \(\xi^a\) direction the amplification effect is stronger because the frame-dragging effect enhances the relative motion. We denote this bigger one \(\Pi_+\). Apparently, \(\Pi_+\) depends on \(w\) and the location where the BBH merger happens. As an example we set \(\chi = 0.98\) and \(\theta = \pi/2\) to check the dependence of \(\Pi_+\) on \(w\) and \(r\). The result is shown in the left panel of Fig. 1. We can see that the dependence of \(\Pi_+\) on the location

FIG. 1: Left: The dependence of \(\Pi_+\) on \(w\) and \(r\). Right: \(\Pi_+\) as a function of \(w\) assuming that \(r = 1+r_{\text{ISCO}}/2\) which corresponds to the center of the \(r\) range in the left panel. We have set \(\chi = 0.98\) and \(\theta = \pi/2\) in the calculation.
the energy amplification could lead to a detectable signature in observation because the observable in GW astronomy is not energy, but the strain. As has been pointed out before [23], gravitational wave behaves like a scalar with respect to a boost transformation of a Minkowsky space. In the transformation, the gravitational wave behaves like a particle with ‘boost weight zero’ and ‘spin weight 2’. Correspondingly, two observers with a relative velocity will detect the same amplitude of the gravitational wave. The gravitational wave energy difference detected by these two observers only comes from the time shift. The presence of a SMBH near the GW source could change the above conclusion but in the current paper we did not calculate the GW strain directly. How the Kerr BH background affects GW strain is another fundamental problem. In order to study the GW strain, a Teukolsky equation sourced by a BBH should be used [24].

If the binary forms outside the ergosphere, astrophysical model requires the relative velocity of the binary with respect to the Kerr BH to be much smaller than the speed of light. We have shown that the quantity II, which is defined near Eq. 4, is, correspondingly, less than 1. So we conclude that in this case the background Kerr black hole amplifies the gravitational wave energy of the binary by at most 1.3 times.

Since the Killing vector becomes space-like inside the ergosphere, the relative velocity between the Kerr black hole and the binary has a different limit. If the binary forms outside the ergosphere and moves into the ergosphere along a geodesic line and merges there, the corresponding II is also limited by a bound orbit of the Kerr black hole. The GW energy amplification is also less than 1.3 times. If the binary forms inside the ergosphere, the most possible scenario is forming in the accretion disk, or to say the circular orbit, will present the energy amplification is at most 5 times. Ideally if two black holes fall into the ergosphere individually from outside and meet to form a binary in the ergosphere like the scenario considered in [9], the factor II can approach infinity in principle. But the possibility of forming such a binary is negligible in galactic nuclei. In summary we conclude that the Kerr black hole background amplifies the gravitational wave by at most a factor of 5.

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