ISOMETRIES BETWEEN WEIGHTED UNIFORM SPACES

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ABSTRACT. The linear isometries between weighted Banach spaces of continuous functions are considered. Some of well known theorems on isometries between spaces of continuous functions are proved and stated, but all they are in an appropriate form. In this paper, we present some new results, too, and in particular — an important equivalence relation is investigated.

CONTENTS

1. Introduction
2. Main Definitions and Notations
3. Main Theorem
4. Corollaries of the Main Theorem
5. The Equivalence Relation
6. Choquet Boundary
7. The spaces of continuous functions $C(X), C_0(Y)$
References

1. Introduction

We consider weighted uniform spaces and linear isometries between them. A weighted uniform space is a weighted Banach space of continuous functions on any completely regular topological space.

In this paper a systematic studying of some of basical results on linear isometries between weighted uniform spaces is presented. Our investigation is based on theorems concerning the geometry of dual spaces of Banach spaces.

It is necessary to note some of previous works on linear isometries between spaces classified as weighted uniform spaces: [1], [4, Ch. 5] — onto-isometries of the space $C(X)$ on a Hausdorff compact; [2] — onto-isometries between Banach spaces having appropriate function module representation; [7] — into-isometries of $C(X)$; [9] — into-isometries of the space $C_0(Y)$ of continuous functions on a locally compact Hausdorff topological space vanishing at infinity; [14] — linear isometries of the Bloch spaces of holomorphic functions in the unit disk; [12] and [13] —

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the onto-isometries of some weighted Banach spaces of holomorphic functions are characterized.

The main result of this paper is theorem 3.1. The basic idea of theorem 3.1 is demonstrated in \[4, \text{lemma 5.8.6}\].

In section 4, corollaries of theorem 3.1 are given. Some of them have analogous assertions in the particular case when the corresponding Banach spaces are uniform algebras or are concrete Banach spaces.

In section 5, an equivalence relation is defined. Some restriction of this relation is used earlier in \[12\]. In our investigation, it is better to use this equivalence relation instead of such a relation that is induced by a condition like that of \[8, \text{p.195, (2)}\].

In section 6, an extension of the classical notion Choquet boundary is given. The extension of this notion is based on the key property that such a boundary characterizes the extreme functionals.

In section 7, we apply the above developed theory to prove the well known classical results on isometries and on extreme functionals of spaces of continuous functions. We obtain the result \[9, \text{theorem 1}\], too. In this section we state some results that are important for the general theory of isometries between weighted uniform spaces, but not only for the special case under consideration in this section.

Moreover, it is possible to obtain theorems on isometries such as in \[1\], \[7\], \[9\] and theorems on extreme functionals as in \[4, \text{lemma 5.8.6, lemma 5.8.7}\], by using the presented here results.

We consider in this paper spaces of complex-valued functions. The proofs are given with the set \(S\) of all complex numbers of modulus one. Exactly the same results are valid for the spaces of real-valued functions, but in this case the set \(S = \{-1, 1\}\) is used instead of the set of all complex numbers of modulus one. The proofs are the same in both cases.

Remark. Some notations and some terminology explanations are placed just before their use. In each section it is used the terminology, notations and results of the previous sections.

2. Main Definitions and Notations

We use the standard terminology and notations of the theory of Banach spaces but it is necessary to give some explanations.

According to \[11, \S\ 3.20\], the definitions of extreme subsets and extreme points are next. Let \(L\) be any complex linear space. Suppose \(E\) is a non empty subset of \(L\). We recall that a subset \(D, \emptyset \neq D \subseteq E\), is said to be an extreme subset of \(E\) iff it is fulfilled the following condition:

\[
\text{for every } x, y \in E \text{ such that there is a number } t, 0 < t < 1, \text{ for which } tx + (1 - t)y \in D, \text{ in fact it is true that } x, y \in D.
\]

A point \(x_0\) of \(E\) is an extreme point of \(E\) iff the single point set \(\{x_0\}\) is an extreme subset of \(E\).

We make use of notations which are not in common use.

The set of all extreme points of \(E\) is denoted by \(\text{ex } E\). The convex hull of a set \(E \subseteq L\) is denoted by \(\text{conv } E\). If, in addition, \(L\) is a Hausdorff topological linear space, then the closed convex hull of \(E\) is \(\overline{\text{conv } E}\).
We recall that if $E$ is a compact subset of a Hausdorff topological linear space then:

- $\text{ex } E \neq \emptyset$;
- $E \subset \overline{\text{conv}}(\text{ex } E)$ and, in particular $E = \overline{\text{conv}}(\text{ex } E)$ when $E$ is convex;
- $\text{ex}(\overline{\text{conv}} E) \subset E$, when $\overline{\text{conv}} E$ is compact.

Note that the first two bullets represent the Krein-Milman theorem and the third bullet is a well-known assertion (see for instance one of the following two monographs [5, proposition 10.1.3] or [4, lemma 5.8.5]).

Let $(A, \| \cdot \|_A)$ be an abstract complex Banach space with the norm $\| \cdot \|_A$. Then $(A^*, \| \cdot \|_{A^*})$ is the corresponding dual Banach space. In this paper the following specific notation is used:

$$(A)_1 = \{ f : f \in A, \| f \| \leq 1 \},$$

$$\partial (A)_1 = \{ f : f \in A, \| f \| = 1 \},$$

and for brevity it is setted $\text{ext } A^* = \text{ex } (A^*)_1$.

We use some notations on isometries which are not in common use. Let for $i = 1, 2$, the abstract Banach space $(A_i, \| \cdot \|_{A_i})$ be with the norm $\| \cdot \|_{A_i}$. Then we write

$$T \in \text{Intois}(A_1, A_2)$$

iff the linear operator $T : A_1 \rightarrow A_2$ is such that $\| Tf \|_{A_2} = \| f \|_{A_1}, \forall f \in A_1$.

We set $\text{Is}(A_1, A_2)$ to be the family of all $T \in \text{Intois}(A_1, A_2)$ which maps $A_1$ onto $A_2$. It follows by the Krein-Milman theorem that for the dual map $T^*$ of $T$, $T \in \text{Intois}(A_1, A_2)$, we have

$$T^*(\text{ext } A^*_2) \supset \text{ext } A^*_1,$$

i.e., $\forall \ell \in \text{ext } A^*_1$ there exists $m \in \text{ext } A^*_2$ such that $T^*(m) = \ell$.

Note that if $F : \Omega_1 \rightarrow \Omega_2$ is an abstract map from $\Omega_1$ to $\Omega_2$ then as usual one set

$$F(Y) = \{ F(y) \mid y \in Y \},$$

where $\emptyset \neq Y \subset \Omega_1$, and $F(\emptyset) = \emptyset$.

The set of all scalars of modulus one is denoted by $S$, thus in the complex case

$$S = \{ \lambda : \lambda \in \mathbb{C}, |\lambda| = 1 \}$$

(and in the real case $S = \{ 1, -1 \}$).

**Definition 2.1.** We define that the triple $(Z, \Delta_A, A)$ has the property $(\ast)$ if and only if $Z$ is a set, $A$ is a non zero Banach space, $\Delta_A : S \times Z \rightarrow (A^*)_1$ is a single-valued map such that:

1. $\| f \|_A = \sup_{z \in Z} |\Delta_A(1, z)(f)|, \forall f \in A$;
2. $\Delta_A(\lambda, z) = \lambda \Delta_A(1, z), \forall \lambda \in S, \forall z \in Z$.

Let the triple $(Z, \Delta_A, A)$ have the property $(\ast)$. We shall say that $\Delta_A$ is continuous when $Z$ is a topological space, $(A^*)_1$ has the induced weak$^*$-topology of $A^*$ and the map $\Delta_A : S \times Z \rightarrow (A^*)_1$ is continuous.
Let the triple \((Z, \Delta_A, A)\) have the property \((\ast)\). We shall use the following version of the notion “boundary of the space \(A\): a set \(Y, Y \subseteq Z\), is said to be a boundary of \(Z\) with respect to the space \(A\) (or briefly, \(Y\) is a boundary of \(A\)) if

\[
\|f\|_A = \sup_{y \in Y} |\Delta_A(1, y)(f)|, \quad \forall f \in A.
\]

An example of such a triple \((Z, \Delta_A, A)\) that has the property \((\ast)\) is constructed as follows. Let \(Z\) be any completely regular topological space and let \(p : Z \to \mathbb{C} \setminus \{0\}\) be any continuous function. The Banach space \((C(Z; p), \| \cdot \|_p)\) is defined by

\[
C(Z; p) = \{ f : Z \to \mathbb{C} \text{ is continuous}, \| f \|_p := \sup_{z \in Z} |p(z)f(z)| < \infty \}.
\]

The notation \(U_n(Z; p)\) stands for the family of all non zero Banach subspaces of \((C(Z; p), \| \cdot \|_p)\). We say that the Banach space \(A\) is weighted uniform space iff \(A \in U_n(Z; p)\), where \(Z\) is any completely regular topological space and \(p : Z \to \mathbb{C} \setminus \{0\}\) is any continuous function. If \(A \in U_n(Z; p)\) then the map

\[
\Delta_A : S \times Z \to (A^*)_1
\]

is defined by \(\Delta_A(\lambda, z)(f) = \lambda p(z)f(z)\), where \(\lambda \in S, z \in Z, f \in A\). Then the triple \((Z, \Delta_A, A)\) has the property \((\ast)\) and moreover \(\Delta_A\) is continuous.

3. Main Theorem

In this section we state and prove our main theorem— theorem 3.1. In its proof it is used one of the ideas which are demonstrated in [4, lemma 5.8.6].

**Theorem 3.1.** Let \(A_1\) be a non zero Banach space and let the triple \((Z_2, \Delta_{A_2}, A_2)\) have the property \((\ast)\). Suppose \(\text{Int} \text{ois}(A_1, A_2) \neq \emptyset\). If \(T \in \text{Int} \text{ois}(A_1, A_2)\) then

\[
\text{ext} A_1^* \subset \text{conv} T^*(\overline{\Delta_{A_2}(S \times Y)}),
\]

where \(Y\) is any boundary of \(A_2\) and \(\overline{\Delta_{A_2}(S \times Y)}\) is the closure of \(\Delta_{A_2}(S \times Y)\) in \(A_2^*\) with respect to the weak* topology.

**Proof of theorem 3.1.** Note, ext \(A_1^* \neq \emptyset\) by the Krein-Milman theorem.

Let \(T \in \text{Int} \text{ois}(A_1, A_2)\) be any and let \(Y\) be any boundary of the space \(A_2\). Then \(\forall \ell \notin \text{conv} T^*(\Delta_{A_2}(S \times Y))\) there is an element \(f \in A_1\) such that

\[
\text{Re } \ell(f) > \sup_{m \in \text{conv} T^*(\Delta_{A_2}(S \times Y))} \text{Re } m(f).
\]

Hence

\[
\text{Re } \ell(f) > \sup_{m \in T^*(\Delta_{A_2}(S \times Y))} \text{Re } m(f) = \sup_{(\lambda, z) \in S \times Y} \text{Re } \Delta_{A_2}(\lambda, z)(Tf) = \|Tf\|_{A_2} = \|f\|_{A_1}.
\]

Therefore \(\ell \notin (A_1^*)_1\) and we obtain

\[(A_1^*)_1 \subset \text{conv} T^*(\Delta_{A_2}(S \times Y)).\]
The inclusion
\[(A^*_1)_1 \supset \text{conv} T^*(\Delta_{A_2}(S \times Y))\]
follows by an obvious way from \(T \in \text{Int}oI(A_1, A_2)\) and we omit the details. Thus
\[(A^*_1)_1 = \text{conv} T^*(\Delta_{A_2}(S \times Y))\]
Then by the weak*-compactness of \(T^*(\Delta_{A_2}(S \times Y))\) it follows
\[\text{ext } A^*_1 \subset T^*(\Delta_{A_2}(S \times Y))\]
So, theorem 3.1 is proved.

Remark. Suppose the triple \((Z, \Delta_A, A)\) has the property \((\ast)\) and in addition, let the map \(\Delta_A\) be continuous. If \(K\) is a compact subset of \(Z\) then, of course, the image \(\Delta_A(S \times K)\) is weak*-compact. So that
\[\Delta_A(S \times K) = \Delta_A(S \times K)\]
where \(\Delta_A(S \times K)\) is the weak*-closure of \(\Delta_A(S \times K)\) in the dual space \(A^*\).

4. Corollaries of the Main Theorem

In this section are placed corollaries of theorem 3.1 which are important for further results.

It is necessary to explain the following notations. Let \(A\) be any non zero Banach space. For every non empty family of its elements \(G, \emptyset \neq G \subset A \setminus \{0\}\), we set
\[\Sigma^A(G) = \{\ell | \ell \in \partial(A^*_1), |\ell(f)| = \|f\|_A \text{ for every } f \in G\}\]
We define that the family \(G\) is centered iff \(\Sigma^A(G) \neq \emptyset\).

Let the triple \((Z, \Delta_A, A)\) have the property \((\ast)\). We introduce the notation \(\text{suppmax}(f)\) as follows:
\[\text{suppmax}(f) = \{z | z \in Z, |\Delta_A(1, z)(f)| = \|f\|_A\}, \quad f \in A\]
We define that the family \(G, \emptyset \neq G \subset A \setminus \{0\}\), is placed over \(V\), where \(\emptyset \neq V \subset Z\), iff there is an element \(f \in G\) such that
\[\sup_{z \in Z \setminus V} |\Delta_A(1, z)(f)| < \|f\|_A\]

Lemma 4.1. Let \(E\) be any non empty subset of a complex linear space. If \(D\) is an extreme subset of \(E\), then
\[\text{ex } D = D \cap \text{ex } E\]
Instead of a detailed proof, we have to remark that lemma 4.1 is an immediate consequence of the definition of extreme subsets.

The inclusion \(\text{ex } D \subset D \cap \text{ex } E\) follows from the fact that \(D\) satisfies the condition characterizing extreme subsets (indeed, for such a \(D\) every one of its extreme points is an extreme point of \(E\), too); the case \(\text{ex } D = \emptyset\) is a trivial one. The opposite inclusion \(\text{ex } D \supset D \cap \text{ex } E\) is an obvious consequence of \(D \subset E\).
Lemma 4.2. Let the triple \((Z, \Delta_A, A)\) have the property \((*)\). If the family \(G\), \(\emptyset \neq G \subset A \setminus \{0\}\), is centered, then

1. \(\text{ex} \Sigma^A(G) \neq \emptyset\);
2. \(\Sigma^A(G) \subset \text{int}(\text{ex} \Sigma^A(G))\);
3. \(\text{ex} \Sigma^A(G) = \Sigma^A(G) \cap \text{ext} A^*\).

Proof of lemma 4.2. Let \(G\), \(\emptyset \neq G \subset A \setminus \{0\}\), be centered. Then, in particular,

\[\emptyset \neq \Sigma^A(G) \subset (A^*)_1\]

and \(\Sigma^A(G)\) is weak\(^*\)-closed. Therefore \(\Sigma^A(G)\) is weak\(^*\)-compact and by the Krein-Milman theorem we obtain assertions (1) and (2) of this lemma. In addition, by a direct computation, we verify that \(\Sigma^A(G)\) is an extreme subset of \((A^*)_1\). Then the assertion of lemma 4.2.(3) is the particular case of lemma 4.1 when \(E = (A^*)_1\), 

Thus lemma 4.2 is proved.

Theorem 4.1. Let \(A_1\) be a non zero Banach space, the triple \((Z_2, \Delta_{A_2}, A_2)\) have the property \((*)\) and let \(\Delta_{A_2}\) be a continuous map. Suppose, in addition, \(\text{Int}(A_1, A_2) \neq \emptyset\) and let \(T \in \text{Int}_{(A_1, A_2)}\) be any. If the centered family \(G\), \(\emptyset \neq G \subset A_1 \setminus \{0\}\), is such that \(TG\) is placed over some compact subset of \(Z_2\) then

\[\text{ex} \Sigma^{A_1}(G) \subset T^* \Delta_{A_2}(S \times (\bigcap_{f \in G} \text{suppmax}(T f))) \cap \text{ext} A_1^*,\]

and in particular, there exists \(z_0 \in \bigcap_{f \in G} \text{suppmax}(T f)\) for which we have

\[T^* \Delta_{A_2}(1, z_0) \in \text{ext} A_1^*\]

Proof of theorem 4.1. Suppose the family \(G\), \(\emptyset \neq G \subset A_1 \setminus \{0\}\), is centered and in addition, it is such that \(TG\) is placed over some compact subset of \(Z_2\). Let \(K\) stand for such a compact set. Then there is \(f_0 \in G\) such that

\[\sup_{z \in Z_2 \setminus K} |\Delta_{A_2}(1, z)(T f_0)| < \|T f_0\|_{p_2}.\]

Thus \(f_0\) represents a weak\(^*\)-continuous linear functional defined on \(A_1^*\) such that distinguishes \(\Sigma^{A_1}(G)\) and \(T^* (\bar{\Delta}_{A_2}(S \times (Z_2 \setminus K))\). (here, the overlined set stands for the weak\(^*\)-closure of \(\Delta_{A_2}(S \times (Z_2 \setminus K))\) in the space \(A_2^*\).) So,

\[(4.1) \quad \Sigma^{A_1}(G) \cap T^* (\bar{\Delta}_{A_2}(S \times (Z_2 \setminus K))\) = \emptyset.\]

Moreover, by lemma 4.2 when it is setted \(Z = Z_1\), \(p = p_1\), \(A = A_1\), we obtain for \(G\)

\[\emptyset \neq \text{ex} \Sigma^{A_1}(G) = \Sigma^{A_1}(G) \cap \text{ext} A_1^*\]

Further, it follows by theorem 3.1

\[\text{ex} \Sigma^{A_1}(G) \subset \Sigma^{A_1}(G) \cap T^* (\bar{\Delta}_{A_2}(S \times Z_2)) \cap \text{ext} A_1^*,\]
and hence
\[ \text{ex } \Sigma^{A_1}(G) \subset \Sigma^{A_1}(G) \cap (T^* \Delta_{A_2}(S \times K) \cup T^* (\Delta_{A_2}(S \times (Z_2 \setminus K)))) \cap \text{ext } A_1^*. \]

Then, by the equation (4.1)
\[ \text{ex } \Sigma^{A_1}(G) \subset \Sigma^{A_1}(G) \cap T^* (\Delta_{A_2}(S \times K)) \cap \text{ext } A_1^*. \]

and hence
\[ \text{ex } \Sigma^{A_1}(G) \subset T^* (\Delta_{A_2}(S \times (K \cap (\bigcap_{f \in G} \text{suppmax}(Tf)))))) \bigcap \text{ext } A_1^*. \]

In particular, the asserted inclusion in theorem 4.1 is proved.

**Corollary 4.1.** Let \( A_1 \) be a non zero Banach space and let the triple \((Z_2, \Delta_{A_2}, A_2)\) have the property \((*)\). Let \( Z_2 \) be a Hausdorff topological space and the map \( \Delta_{A_2} \) be continuous with respect to the weak*-topology of \( A_2^* \). Suppose, in addition, \( \text{Int}(A_1, A_2) \neq \emptyset \) and let \( T \in \text{Int}(A_1, A_2) \) be any. If the family \( G, \emptyset \neq G \subset A_1 \setminus \{0\} \), is such that

(i) \( \bigcap_{f \in G} \text{suppmax}(Tf) = \{z_0\} \), where \( z_0 \) is a point of \( Z_2 \),

(ii) the image \( TG = \{Tf | f \in G\} \) of \( G \), is placed over a compact subset of \( Z_2 \),

then
\[ \Sigma^{A_1}(G) = T^* \Delta_{A_2}(S \times \{z_0\}) \subset \text{ext } A_1^*. \]

**Proof of corollary 4.1.** Let for the family \( G, \emptyset \neq G \subset A_1 \setminus \{0\} \), be fulfilled the conditions (i) and (ii). In particular, from the condition (i) it follows \( T^* \Delta_{A_2}(1, z_0) \in \Sigma^{A_1}(G) \), so that \( \Sigma^{A_1}(G) \neq \emptyset \). Thus the family \( G \) is centered. Then by theorem 4.1 it follows \( \text{ex } \Sigma^{A_1}(G) \subset T^* (\Delta_{A_2}(S \times \{z_0\})) \cap \text{ext } A_1^* \), and hence
\[ \text{ex } \Sigma^{A_1}(G) = T^* (\Delta_{A_2}(S \times \{z_0\})) \subset \text{ext } A_1^*. \]

Further, from the Krein-Milman theorem
\[ \Sigma^{A_1}(G) \subset \text{conv } T^* (\Delta_{A_2}(S \times \{z_0\})) = \{rT^* (\Delta_{A_2}(\lambda, z_0)) | 0 \leq r \leq 1, \lambda \in S\}. \]

Moreover, according to the definition of \( \Sigma^{A_1}(G) \), the equation \( |\ell(f)| = ||f||_{A_1} \) holds for every \( \ell \in \Sigma^{A_1}(G) \) and \( \forall f \in G \). Then by the inclusion (4.2) we obtain
\[ \Sigma^{A_1}(G) = T^* (\Delta_{A_2}(S \times \{z_0\})). \]

Thus corollary 4.1 is proved.

**Proposition 4.1.** Let \( A_1 \) be a non zero Banach space and let the triple \((Z_2, \Delta_{A_2}, A_2)\) have the property \((*)\). Suppose, in addition, \( \text{Int}(A_1, A_2) \neq \emptyset \) and let \( T \in \text{Int}(A_1, A_2) \) be any. If the set of functionals \( L, \emptyset \neq L \subset (A_1^*)_1 \), is such that there are \( f, Q, f \in A_1 \) and \( Q \) is a subset of \( Z_2 \), such that at least one of the following two inequalities holds:

- \( \sup_{z \in Z_2 \setminus Q} |T^* (\Delta_{A_2}(1, z))(f)| < \inf_{\ell \in L} |\ell(f)| \),
- \( \inf_{z \in Z_2 \setminus Q} |T^* (\Delta_{A_2}(1, z))(f)| > \sup_{\ell \in L} |\ell(f)| \),

then \( L \cap T^* (\Delta_{A_2}(S \times (Z_2 \setminus Q))) = \emptyset \).

The assertion is an immediate consequence of the assumptions and we omit its proof.
Corollary 4.2. Let $A_1$ be a non zero Banach space and let the triple $(Z_2, \Delta_{A_2}, A_2)$ have the property (*). Let $Z_2$ be a Hausdorff topological space and the map $\Delta_{A_2}$ be continuous with respect to the weak* topology of $A_2^*$. Suppose, in addition, Int$(A_1, A_2) \neq \emptyset$ and let $T \in \text{Int}(A_1, A_2)$ be any. If the point $z_0 \in Z_1$ is such that $\Delta_{A_1}(1, z_0) \in \text{ext} A_1^*$ and if there are $f, K, f \in A_1 \setminus \{0\}$ and $K$ is a compact subset of $Z_2$, for which at least one of the following two inequalities holds:

- $\sup_{z \in Z_2 \setminus K} |T^*(\Delta_{A_2}(1, z)(f))| < |\Delta_{A_1}(1, z_0)(f)|$,
- $\inf_{z \in Z_2 \setminus K} |T^*(\Delta_{A_2}(1, z)(f))| > |\Delta_{A_1}(1, z_0)(f)|$,

then

1. $\Delta_{A_1}(S \times \{z_0\}) \cap T^*( (\Delta_{A_2}(S \times (Z_2 \setminus K)))) = \emptyset$
2. $\Delta_{A_1}(S \times \{z_0\}) \subset T^*(\Delta_{A_2}(S \times K))$.

Instead of a proof, note that corollary 4.2 is an immediate consequence of both theorem 3.1 and proposition 4.1.

5. The Equivalence Relation

In this section, it is defined the equivalence relation $\sim_A$ between the points of the set $Z$, where $(Z, \Delta_A, A)$ is any triple having the property (*). This relation is induced by the map $\Delta_A : S \times Z \rightarrow A^*$. It is important to note that there is triple $(Z, \Delta_A, A)$ such that between its points $x, y \in Z$ we have $x \sim_A y$ iff the functions of $A$ do not distinguish points $x$ and $y$. For instance, we have such an equivalence when $Z$ is any completely regular topological space and $A \in \text{Un}(Z; 1)$ is such that $A \ni 1$.

Definition 5.1. Let the triple $(Z, \Delta_A, A)$ has the property (*). The relation $\sim_A$ between points of $Z$ is defined by

$$x \sim_A y \iff |\Delta_A(1, x)(f)| = |\Delta_A(1, y)(f)|, \quad \forall f \in A.$$

Remark. A direct computation shows that the above defined relation is an equivalence relation.

Definition 5.2. Let the triple $(Z, \Delta_A, A)$ has the property (*). Suppose $V$ is any non empty subset of $Z$. We define that $|pA|$ distinguishes the points of $V$ iff for every couple $x, y \in V$, $x \neq y$, there is an element $f \in A$ such that

$$|\Delta_A(1, x)(f)| \neq |\Delta_A(1, y)(f)|.$$

Proposition 5.1. Let the triple $(Z, \Delta_A, A)$ has the property (*). Suppose $x, y \in Z$. Then

$$x \sim_A y \iff \text{there are } \lambda, \mu \in S \text{ such that } \Delta_A(\lambda, x) = \Delta_A(\mu, y).$$

Proof of proposition 5.1. Note, the assertion “$\iff$” is trivial and we omit its proof.

Proof of the part “$\Rightarrow$”. Thus we assume $x \sim_A y$. It is necessary to establish that there are scalars $\lambda, \mu \in S$ such that

$$(5.1) \quad \Delta_A(\lambda, x) = \Delta_A(\mu, y).$$
According to the definition 5.1, it follows from \( x \sim_A y \) that
\[
|\Delta_A(1, x)(f)| = |\Delta_A(1, y)(f)|, \quad \forall f \in A.
\]
Hence for the kernels \( \ker \Delta_A(1, x) \) and \( \ker \Delta_A(1, y) \) of functionals \( \Delta_A(1, x) \) and \( \Delta_A(1, y) \) we have
\[
\ker \Delta_A(1, x) = \ker \Delta_A(1, y).
\]
Therefore, there are two scalars \( z, w \in \mathbb{C} \), at least one of them is not zero, and such that
\[
z \Delta_A(1, x) + w \Delta_A(1, y) = 0.
\]
Further, we consider the following two hypotheses about the functional \( \Delta_A(1, y) \):
\( \Delta_A(1, y) = 0 \), and \( \Delta_A(1, y) \neq 0 \).

First, suppose \( \Delta_A(1, y) = 0 \). Then \( \Delta_A(1, y)(f) = 0 \), \( \forall f \in A \). By the equation (5.2) it follows \( \Delta_A(1, x)(f) = 0 \), \( \forall f \in A \), i.e., \( \Delta_A(1, x) = 0 \). Hence \( \Delta_A(1, x) = \Delta_A(1, y) \) and the equation (5.1) is established with \( \lambda = \mu = 1 \).

Second, suppose \( \Delta_A(1, y) \neq 0 \). Then, in particular, there is an element \( f_0 \in A \) such that \( \Delta_A(1, y)(f_0) \neq 0 \). By the equation (5.2) it follows \( \Delta_A(1, x)(f_0) \neq 0 \). In addition, according to the equation (5.3) we obtain
\[
z \Delta_A(1, x)(f_0) + w \Delta_A(1, y)(f_0) = 0.
\]
Then from the equation (5.2) it follows \( |z| = |w| \) and in particular \( z \neq 0 \). Thus \( wz^{-1} \in S \). Then by (5.3)
\[
\Delta_A(1, x) = \Delta_A(-wz^{-1}, y),
\]
i.e., the equation (5.1) is established with \( \lambda = 1 \), \( \mu = -wz^{-1} \). So, the assertion “\( \Rightarrow \)” is proved and proposition 5.1 is proved, too.

**Proposition 5.2.** Let the triple \((Z, \Delta_A, A)\) has the property \((*)\). Suppose \( L \) and \( M \) are subsets of \( Z \) such that \( \emptyset \neq L \subset M \subset Z \). Then \( \forall x \in L \), \( \forall y \in M \), \( x \neq y \), we have \( x \not\sim_A y \) if and only if the following two conditions are fulfilled:

1. the restriction \( \Delta_A : S \times L \to (A^*)_1 \) is an injective map;
2. \( \Delta_A(S \times (M \setminus L)) \cap \Delta_A(S \times L) = \emptyset \).

Instead of a detailed proof we have to remark that proposition 5.2 is a direct consequence of proposition 5.1.

**Definition 5.3.** Let the triple \((Z, \Delta_A, A)\) has the property \((*)\). Then we define that the couple of sets \((M, L), \emptyset \neq L \subset M \subset Z\), is proper with respect to \((Z, \Delta_A, A)\) if the following three conditions are fulfilled:

1. the restriction \( \Delta_A : S \times L \to (A^*)_1 \) is an injective map;
2. the restriction \( \Delta_A : S \times M \to \Delta_A(S \times M) \) is closed, where \( \Delta_A(S \times M) \) has the induced weak*-topology of \( A^* \);
3. \( \Delta_A(S \times (M \setminus L)) \cap \Delta_A(S \times L) = \emptyset \).
Remark 5.1. Let the triple \((Z, \Delta_A, A)\) has the property \((*)\). Suppose the couple of sets \((M, L), \emptyset \neq L \subset M \subset Z\), is such that \(\forall x \in L, \forall y \in M, x \neq y\), we have \(x \not\sim_A y\) and in addition, let the restriction

\[
\Delta_A : S \times M \to \Delta_A(S \times M)
\]

be closed with respect to the induced weak*-topology of \(A^*\). Then the restriction

\[
\Delta_A : S \times L \to \Delta_A(S \times L)
\]

is a homeomorphism map. The following arguments prove this assertion. By proposition 5.2, the restriction \((5.5)\) is bijective. Moreover, by proposition 5.2, it is the restriction on the whole preimage of \(\Delta_A(S \times L)\) under the map \((5.4)\) and hence it is closed, too. Then it is a homeomorphic map.

6. Character Boundary

In this section, an extension of the classical notion Character boundary is given.

Definition 6.1. Let \(A_1\) be a non zero Banach space and let the triple \((Z_2, \Delta_{A_2}, A_2)\) have the property \((*)\). Suppose \(\text{Intois}(A_1, A_2) \neq \emptyset\) and let \(T \in \text{Intois}(A_1, A_2)\) be any. We define the following two notations:

- \(M_T(A_1) = \{ z \in Z_2, T^* \Delta_{A_2}(1, z) \in \text{ext } A_1^* \}\);
- \(\text{Ch}_T(A_1)\) is the family of all nonempty subsets \(Y\) of \(Z_2\) such that the image \(T^* \Delta_{A_2}(S \times Y) = \text{ext } A_1^*\).

Moreover, instead of \(M_T(A_1)\) and of \(\text{Ch}_T(A_1)\) it is written \(M(A)\) and \(\text{Ch}(A)\) resp., when \(A_1 = A_2 = A\), and \(T : A \to A\) is the identity. In the more special case when \(Z\) is a Hausdorff compact, \(A \in \text{Un}(Z; 1)\) and \(A \geq 1\), one define that a subset \(Y\) of \(Z\) is said to be Character boundary iff \(Y \in \text{Ch}(A)\).

Remark. If \(\text{Ch}_T(A_1) \neq \emptyset\) then it is obvious that \(M_T(A_1) \subset \text{Ch}_T(A_1)\).

Remark 6.1. Suppose \(A_1\) is a non zero Banach space and let the triple \((Z_2, \Delta_{A_2}, A_2)\) have the property \((*)\). If \(T \in \text{Intois}(A_1, A_2)\) is such that \(\text{Ch}_T(A_1) \neq \emptyset\). Then:

1. \(\text{Ch}_T(A_1) = \text{Ch}(TA_1)\);
2. \(M_T(A_1) = M(TA_1)\).

We sketch the proof by the following helping arguments. For brevity let’s put \(I : TA_1 \to A_2\) to be the identity embedding operator, let’s define \(\Delta_{TA_1} = I^* \Delta_{A_2}\), and let \(\tilde{T} : A_1 \to TA_1\) be the restriction of the isometry \(T \in \text{Intois}(A_1, A_2)\). In particular, \(\tilde{T} \in \text{Is}(A_1, TA_1)\) and by the Krein–Milman theorem \(\tilde{T}^* : \text{ext}(TA_1)^* \to \text{ext } A_1^*\) is bijective. Further by the factorization \(T^* = \tilde{T}^* I^*\) it follows

\[
T^* \Delta_{A_2} = \tilde{T}^* I^* \Delta_{A_2} = \tilde{T}^* \Delta_{TA_1}.
\]

A direct consequence of this equation are both (1) and (2), and we omit the details.

Proposition 6.1. Let \(A_1\) be a non zero Banach space and let the triple \((Z_2, \Delta_{A_2}, A_2)\) and \((M, L)\) have the property \((*)\). Then the following two assertions hold:

1. in the case when \(\Delta_{A_2}\) is continuous and \(A_2\) has a compact boundary \(K \subset Z_2\), if \(T \in \text{Intois}(A_1, A_2)\), then

\[
\text{ext } A_1^* \subset T^* \Delta_{A_2}(S \times K),
\]

\[
\text{ext } A_1^* \subset T^* \Delta_{A_2}(S \times K),
\]
and hence, in particular, \( M_T(A) \cap K \neq \emptyset \) and
\[ M_T(A) \cap K \subset \text{Ch}_T(A); \]

(2) in the case when \( \text{Ch}(A) \neq \emptyset \) and \( B \) is a non-zero Banach subspace of \( A \),
if \( T \in \text{Int}os(A,B) \), then for every \( Y \in \text{Ch}(A) \) (below, \( \Delta_B(S \times Y) \) stands
for the set of restrictions of functionals of \( \Delta_A(S \times Y) \) on the subspace \( B \)
of \( A \))
\[ \text{ext } A \subset T^* \Delta_B(S \times Y), \]
and hence, in particular, \( M_T(A) \cap Y \neq \emptyset \) and
\[ M_T(A) \cap Y \subset \text{Ch}_T(A). \]

Proof of proposition 6.1. Instead of an explicite proof of assertion (1), we have to remark that it is an immediate consequence of theorem 3.1 and we omit the details.

Proof of assertion (2). Let \( B \) be a non-zero Banach subspace of \( A \). Let the
operator \( I : B \to A \) stand for the identity embedding of \( B \) into \( A \). Then in
particular, the triple \( (Z, \Delta_B, B) \) has the property (\( \ast \)), where \( \Delta_B \) is defined by
\( \Delta_B = I^* \Delta_A \). Note \( I \in \text{Int}os(B, A) \) so, that \( \text{ext } B^* \subset I^*(\text{ext } A^*) \). Then
\[ \text{ext } A \subset T^*(\text{ext } B^*) \subset T^*(I^*(\text{ext } A^*)) \]
\[ \subset T^*(I^*(\Delta_A(S \times Y))) = T^*(\Delta_B(S \times Y)), \]
and so, assertion (2) is proved.

Thus proposition 6.1 is proved, too.

Proposition 6.2. Let for \( i = 1, 2 \) the triples \( (Z_i, \Delta_i, A_i) \) have the property (\( \ast \)).
Suppose that \( \text{Ch}(A) \neq \emptyset \). Suppose, in addition, \( \text{Int}os(A, A) \neq \emptyset \) and let \( T \in \text{Int}os(A, A) \) be any. If the set \( V, \emptyset \neq V \subset Z \), is such that there are \( f, W, f \in A \)
and \( W \) is a subset of \( Z \), such that at least one of the following two inequalities holds:
\[ \begin{align*}
\sup_{z \in W} |T^*(\Delta_A(S \times (Z \setminus V))(f))| &< \inf_{x \in Z \setminus V} |\Delta_A(1, x)(f)|, \\
\inf_{z \in W} |T^*(\Delta_A(S \times (Z \setminus V))(f))| &> \sup_{x \in Z \setminus V} |\Delta_A(1, x)(f)|,
\end{align*} \]
then
\[ \begin{align*}
(1) & \quad \Delta_A(S \times (Z \setminus V)) \cap T^*(\Delta_A(S \times W)) = \emptyset. \\
(2) & \quad \Delta_A(S \times (M(A) \cap V)) \supset T^*(\Delta_A(S \times (M_A(A_1) \cap W))).
\end{align*} \]

The assertion (1) of this proposition is a direct consequence of proposition 4.1
when we put \( Q = Z \setminus W \), \( L = \Delta_A(S \times (Z \setminus V)) \) and we omit its proof. Further,
(2) follows immediately from (1) and from the definition of \( M(A) \).

Proposition 6.3. Let \( Z \) be any compact Hausdorff topological space, \( A \in \text{Un}(Z; 1) \),
and \( 1 \in A \). Let’s put
\[ D = \{ \ell | \ell \in \partial(A^*1), \ell(1) = 1 \}. \]
Then \( M(A) = \{ z | z \in Z, \Delta_A(1, z) \in \text{ex } D \} \).

Proof of proposition 6.3. Note \( D \supset \{ \Delta_A(1, z) | z \in Z \} \) and moreover, \( D \) is an
extreme subset of \( (A^*) \). Then by lemma 4.1, \( \text{ex } D = D \cap \text{ext } A^* \). Hence
\[ \begin{align*}
M(A) & = \{ z | z \in Z, \Delta_A(1, z) \in \text{ext } A^* \} \\
& = \{ z | z \in Z, \Delta_A(1, z) \in D \cap \text{ext } A^* \} \\
& = \{ z | z \in Z, \Delta_A(1, z) \in \text{ex } D \}.
\end{align*} \]
and so, proposition 6.3 is proved.

**Remark 6.2.** It follows from both proposition 6.3 and [3, lemma 4.3] that in the special case when $Z$ is a Hausdorff compact, $A \in \text{Un}(Z; 1)$, $1 \in A$ our definition of $M(A)$ is equivalent to the classical definition of Choquet boundary (see for instance, [3, page 310]).

**Remark 6.3.** Note, in the general case under consideration in definition 6.1, $M(A)$ is a boundary of $A$ (see section 2 for the definition of boundary of $A$). If the family $G = \{f\} \subset A \setminus \{0\}$ is not placed over a compact subset of $Z$ (or briefly, “$f \in A, f \neq 0$ is not placed over a compact subset of $Z$”), then a well-known immediate consequence of the Krein-Milman theorem is

$$\sup_{z \in M(A)} |\Delta_A(1, z)(f)| = ||f||_A$$

and we omit the details. Note, it is possible that $M(A) \cap \text{suppmax}(f) = \emptyset$. But, if $f \in A \setminus \{0\}$ is placed over some of compact subsets of $Z$ then we have the following assertion — corollary 6.1.

**Corollary 6.1.** Let the triple $(Z, \Delta_A, A)$ have the property $(*)$, let $Z$ be a Hausdorff topological space, and $\Delta_A$ be continuous. Suppose, in addition, $\text{Ch}(A) \neq \emptyset$. If $f \in A$ is such that there exists a compact $K \subset Z$ for which

$$\sup_{z \in Z \setminus K} |\Delta_A(1, z)(f)| < ||f||_A,$$

i.e. $f$ is placed over the compact $K$, then

$$Y \cap \text{suppmax}(f) \neq \emptyset, \quad \forall Y \in \text{Ch}(A).$$

*Proof of corollary 6.1.* The one-dimentional space $< f >$ spaned over $f$ is such that $(Z, \Delta, < f >)$ has the property $(*)$, where it is setted $\Delta(\lambda, g) = \Delta_A(\lambda, g), \forall \lambda \in S$ and $\forall g \in < f >$. This space has a compact boundary and by proposition 6.1.(1) we obtain $\text{Ch}(< f >) \neq \emptyset$. Moreover, a direct computation shows $M(< f >) = \text{suppmax}(f)$. Further, $< f > \subset A$ and hence by proposition 6.1.(2), $M(< f >) \cap Y \neq \emptyset$, so that $Y \cap \text{suppmax}(f) \neq \emptyset, \forall Y \in \text{Ch}(A)$. Thus corollary 6.1 is proved.

**7. The Spaces of Continuous Functions $C(X), C_0(Y)$.

In this section it is demonstrated an application of the developed above theory. Let $X$ be any compact Hausdorff topological space. The notation $C(X)$ stands for the Banach space of all continuous complex valued functions on $X$, and this space is normed by the usual sup-norm. Let $Y$ be any locally compact Hausdorff topological space. The notation $C_0(Y)$ stands for the Banach space of all continuous complex valued functions on $Y$ that vanish at infinity, and this space is normed with the usual sup-norm. In this section we obtain all basical results on into-isometries between spaces such as $C(X), C_0(Y)$. The theorems stated below, all are simple consequences of the theory presented in the previous sections.

We define the abstract families $\mathcal{A}$ and $\mathcal{B}$ by

$\mathcal{A} = \{(C(X), X) | \text{X is a compact Hausdorff topological space} \},$

$\mathcal{B} = \{(C_0(Y), Y) | \text{Y is a locally compact Hausdorff topological space} \}.$
If \((A, Z) \in A \cup B\), then the map \(\Delta_A : S \times Z \to (A^*)_1\) is defined by

\[
\Delta_A(\lambda, z)(f) = \lambda f(z),
\]

where \(\lambda \in S\), \(z \in Z\), \(f \in A\). Then the triple \((Z, \Delta_A, A)\) has the property \((*)\) and \(\Delta_A\) is continuous. In particular, \(A \in \text{Un}(Z, 1)\) accordingly to the definitions of section 2.

**Lemma 7.1.** Let \(A\) be a non zero Banach space, \((B, Z) \in A \cup B\). Suppose that \(\text{Int}os(A, B) \neq \emptyset\) and let \(T \in \text{Int}os(A, B)\) be any. Then for every functional \(\ell \in \partial(A^*)_1\) there exists a compact \(K \subset Z\) such that

\[
\ell \notin T^*\left(\Delta_B(S \times (Z \setminus K))\right).
\]

**Proof of lemma 7.1.** Let \(\ell \in \partial(A^*)_1\) be any. Then, in particular, \(\ell \neq 0\) and hence, there is a function \(f \in A\) such that \(\ell(f) \neq 0\). Note \(Tf \in B\). So, that

\[
K = \{z \mid |Tf(z)| \geq |\ell(f)|/2\}
\]

is a compact subset of \(Z\). Then

\[
\sup_{z \in Z \setminus K} |T^*(\Delta_B(1, z))(f)| < |\ell(f)|.
\]

Further, in proposition 4.1 when \(A_1 = A\), \(Z_2 = Z\), \(A_2 = B\), \(Q = K\), \(L = \{\ell\}\), it is fulfilled the condition of the first bullet. Hence by proposition 4.1 it follows

\[
\{\ell\} \cap T^*\left(\Delta_B(S \times (Z \setminus K))\right) = \emptyset.
\]

Thus lemma 7.1 is proved.

**Lemma 7.2.** Let \(A\) be a non zero Banach space, \((B, Z) \in A \cup B\). Suppose that \(\text{Is}(A, B) \neq \emptyset\) and let \(T \in \text{Is}(A, B)\) be any. Then for every \(z_0 \in Z\) there exists a family of functions \(G, \emptyset \neq G \subset A \setminus \{0\}\) such that its image \(TG\) is placed over some compact subset of \(Z\) and in addition

\[
\bigcap_{f \in G} \text{supp} \text{max}(Tf) = \{z_0\}.
\]

**Proof of lemma 7.2.** There are two cases: \(Z = \{z_0\}\), and \(Z \setminus \{z_0\} \neq \emptyset\). In the first case lemma 7.2 is a trivial one and we omit the details of the proof. So, let’s consider the second case. Let \(z_0 \in Z\) be any. Then there exists an open neighbourhood \(U\) of \(z_0\) such that its closure \(\overline{U}\) is compact. Further, it follows by the Urysohn’s lemma that there is a function \(h_0 \in B\) such that \(z_0 \in \text{suppmax}(h_0)\), \(\sup_{z \in Z \setminus \overline{U}} |\Delta_B(1, z)(h_0)| = 0\). Moreover, by the Urysohn’s lemma, there exists a family of functions \(H \subset B\) such that \(\bigcap_{h \in H} \text{suppmax}(h) = \{z_0\}\). Thus the family \(\{h_0\} \cup H\) is placed over the compact \(\overline{U}\) and is such that

\[
\bigcap_{h \in \{h_0\} \cup H} \text{suppmax}(h) = \{z_0\}.
\]
Hence $G = \{ f \mid f \in A, Tf \in \{ h_0 \} \cup H \}$ has the desired properties.

Thus lemma 7.2 is proved.

Recall that a map $F : \Omega_1 \to \Omega_2$ is said to be perfect [6, § 3.7] if and only if the following three conditions are fulfilled:

- $\Omega_1$ is a Hausdorff topological space and $\Omega_2$ is a topological space;
- $F$ is a continuous closed map;
- the whole preimage $F^{-1}(\{y\}) = \{x\mid x \in \Omega_1, F(x) = y\}$ is a compact subset of $\Omega_1$, for every point $y$ in the image $F(X) = \{F(x) \mid x \in \Omega_1\}$ of the map $F$.

**Lemma 7.3. A topological lemma.** Let for $i = 1, 2$ the set $\Omega_i$ be a Hausdorff topological space and let $F : \Omega_1 \to \Omega_2$ be a surjective continuous map. If for every $y \in \Omega_2$ there is a compact $K \subset \Omega_1$ such that $y \notin F(\Omega_1 \setminus K)$, then the map $F$ is perfect.

**Proof of lemma 7.3.** We prove this lemma directly by using the definition of perfect map.

First, we prove that $F$ is closed. Let the closed set $W \subset \Omega_1$ be any, and let $z$ be any point of the closure $\overline{F(W)}$ of $F(W)$. By the assumption there is a compact $K \subset \Omega_1$ such that $z \notin F(\Omega_1 \setminus K)$. Hence,

$$z \in \overline{F(W)} \subset \overline{F(W \cap K)} \cup \overline{F(W \setminus K)} \subset \overline{F(W \cap K)} \cup \overline{F(\Omega_1 \setminus K)},$$

so that $z \in \overline{F(W \cap K)}$. Further, it follows by continuity of $F$ and by compactness of $W \cap K$, that $\overline{F(W \cap K)} = F(W \cap K)$. Therefore, $z \in F(W \cap K)$ and in particular, $z \in F(W)$. Accordingly to the choice of $z \in \overline{F(W)}$, it follows that $F(W) \subset F(W)$. So, we obtain that the map $F$ is closed.

Second, we prove that the preimage $F^{-1}(y)$ is compact, $\forall y \in \Omega_2$. Let $y \in \Omega_2$ be any. Then from surjectivity of $F$ it follows $F^{-1}(y) \neq \emptyset$. Further, by the assumption, there is a compact $K \subset \Omega_1$ such that $y \notin F(\Omega_1 \setminus K)$. Then $F^{-1}(y) \cap (\Omega_1 \setminus K) = \emptyset$. So, $F^{-1}(y) \subset K$. Further, by continuity of $F$ it follows that $F^{-1}(y)$ is closed. Hence $F^{-1}(y)$ is compact.

Then by the definition, we obtain that $F$ is perfect. Thus lemma 7.3 is proved.

**Proposition 7.1.** Let $A$ be a non zero Banach space, the triple $(Z, \Delta_B, B)$ have the property $(\ast)$ and let $\Delta_B$ be a continuous map. If the isometry $T \in \text{Int}(\text{Iso}(A, B))$ is such that for every $\ell \in \partial(A^*)_1$ there is a compact $K \subset Z$ so that the following condition is satisfied

$$\ell \notin T^\ast(\overline{\Delta_B(S \times (Z \setminus K))})$$

then $\text{ext} A^* \subset T^\ast \Delta_B(S \times Z)$.

**Proof of proposition 7.1.** Let $\ell \in \text{ext} A^*$ be any. Then by theorem 3.1 it follows

$$\ell \in T^\ast(\overline{\Delta_B(S \times Z)})$$

Further, by the well-known inclusion $\text{ext} A^* \subset \partial(A^*)_1$ we obtain $\ell \in \partial(A^*)_1$. Hence there is a compact $K \subset Z$ such that the condition (7.1) is fulfilled. Therefore,

$$\ell \in T^\ast(\overline{\Delta_B(S \times K)}).$$
Note $\Delta_B(S \times K) = \Delta_B(S \times K)$ because of compactness of $K$ and of continuity of $\Delta_B$. So, that $$\ell \in T^*\Delta_B(S \times K).$$

Hence, accordingly to the choice of $\ell$, we obtain $\text{ext} \ A^* \subset T^*\Delta_B(S \times Z)$.

Thus proposition 7.1 is proved.

**Proposition 7.2.** Let $A$ be a non zero Banach space, the triple $(Z, \Delta_B, B)$ have the property $(\star)$ and let $\Delta_B$ be a continuous map. If the isometry $T \in \text{Intois}(A, B)$ is such that for every $z_0 \in Z$ there is a family of functions $G$, with the following properties $\emptyset \neq G \subset A \setminus \{0\}$, its image $\text{T}G$ is placed over some compact subset of $Z$, and

$$\bigcap_{f \in G} \text{suppmax}(\text{T}f) = \{z_0\},$$

then the map $T^*\Delta_B : S \times Z \to \text{ext} \ A^*$ is well defined, injective, and continuous.

We sketch the main arguments and omit the details. By corollary 4.1 it follows that this map is well defined. Further, by the assumptions on the isometry $T$ it follows that $|TA|$ distinguishes all points of $Z$ (see definition 5.2). Hence by proposition 5.2 it follows that the map $T^*\Delta_B$ is injective. And of course, it is continuous because $T^*$ and $\Delta_B$, both are continuous. These arguments are enough for the proof of proposition 7.2.

**Definition 7.1.** Let $A$ be a non zero Banach space, the triple $(Z, \Delta_B, B)$ have the property $(\star)$ and let $\Delta_B$ be a continuous map. Then we define the following two properties $(\alpha)$ and $(\beta)$ of an isometry $T \in \text{Intois}(A, B)$:

$(\alpha)$ for every $\ell \in \partial(A^*)_1$ there is a compact $K \subset Z$ so that the following condition is satisfied

$$\ell \notin T^*(\Delta_B(S \times (Z \setminus K))).$$

$(\beta)$ for every $z_0 \in Z$ there is a family of functions $G$, with the following properties $\emptyset \neq G \subset A \setminus \{0\}$, its image $\text{T}G$ is placed over some compact subset of $Z$, and

$$\bigcap_{f \in G} \text{suppmax}(\text{T}f) = \{z_0\},$$

**Theorem 7.1.** Let $A$ be a non zero Banach space, the triple $(Z, \Delta_B, B)$ have the property $(\star)$ and let $\Delta_B$ be a continuous map. Then we have the following assertions:

1. if $T \in \text{Intois}(A, B)$ has the property $(\alpha)$ then $\text{Ch}_T(A) \neq \emptyset$ and the map

$$T^*\Delta_B : S \times M_T(A) \to \text{ext} \ A^*$$

is well defined, surjective and perfect;

2. if $T \in \text{Intois}(A, B)$ has the two properties $(\alpha)$, $(\beta)$ then $M_T(A) = Z$ and the map

$$T^*\Delta_B : S \times Z \to \text{ext} \ A^*$$

is a homeomorphic map.
Proof of theorem 7.1. The assertion of (1) follows immediately from lemma 7.3 and from proposition 7.1. The assertion of (2) is a direct consequence of (1) and of proposition 7.2.

Thus theorem 7.1 is proved.

Corollary 7.1. Let for $i = 1, 2$, the couple $(A_i, Z_i) \in \mathcal{A} \cup \mathcal{B}$. Then we have the following assertions:

(1) if $T \in \text{Intois}(A_1, A_2)$ then $\text{Ch}_T(A) \neq \emptyset$ and the map

$$T^* \Delta_{A_2} : S \times M_T(A_1) \to \text{ext} A_1^*$$

is well defined, surjective and perfect;

(2) if $T \in \text{Is}(A_1, A_2)$ then $M_T(A_1) = Z_2$ and the map

$$T^* \Delta_{A_2} : S \times Z_2 \to \text{ext} A_1^*$$

is a homeomorphic map.

Proof of corollary 7.1. Note, that for $i = 1, 2$, the map $\Delta_{A_i} : S \times Z_i \to (A_i^*)_1$ is defined by $\Delta_{A_i}(\lambda, x)f(\lambda, x) = \lambda f(x), \forall (\lambda, x) \in S \times Z_i$ and $\forall f \in A_i$.

First we prove (1). We apply lemma 7.1 in the case $Z = Z_2$, $B = A_2$, $A = A_1$, $\Delta_b = \Delta_{A_2}$. Thus we obtain that the isometry $T \in \text{Intois}(A_1, A_2)$ has the property (a). Then by theorem 7.1(1), with $Z = Z_2$, $B = A_2$, $A = A_1$, $\Delta_b = \Delta_{A_2}$, it follows that the map $T^* \Delta_{A_2} : S \times Z_2 \to \text{ext} A_1^*$ is well defined, surjective and perfect. So, (1) is proved.

Second, we prove (2). We apply both lemma 7.1 and lemma 7.2 in the case $Z = Z_2$, $B = A_2$, $A = A_1$, $\Delta_b = \Delta_{A_2}$. Thus we obtain that the isometry $T \in \text{Is}(A_1, A_2)$ has the properties (a), (b). Then by theorem 7.1(2), with $Z = Z_2$, $B = A_2$, $A = A_1$, $\Delta_b = \Delta_{A_2}$, it follows that the map $T^* \Delta_{A_2} : S \times Z_2 \to \text{ext} A_1^*$ is homeomorphic. Thus (2) is proved and the corollary 7.1 is proved, too.

Theorem 7.2. Let for $i = 1, 2$ $(A_i, Z_i) \in \mathcal{A} \cup \mathcal{B}$. Suppose $\text{Intois}(A_1, A_2) \neq \emptyset$. Then for every $T \in \text{Intois}(A_1, A_2)$ there exists a couple $(\phi, \tau)$ of maps such that:

(1) $\phi : M_T(A_1) \to S$ is continuous;

(2) $\tau : M_T(A_1) \to Z_1$ is surjective perfect map;

(3) $Tf(x) = \phi(x)f(\tau(x)), \forall f \in A_1, \forall x \in M_T(A_1)$;

This couple is unique in the following sence: if the couple $(\phi', \tau')$ is such that

$$\phi' : U \to S,$$

$$\tau' : U \to Z_1,$$

$$Tf(x) = \phi'(x)f(\tau'(x)), \forall x \in U, \forall f \in A_1$$

where $\emptyset \neq U \subset Z_2$, then $U \subset M_T(A_1)$ and $\phi'(x) = \phi(x)$, $\tau'(x) = \tau(x)$, for all $x \in U$.

Remark. We have to recall that in consequence of theorem 7.2(2) we obtain some of topological properties of $M_T(A_1)$. We use the topological results of [6, §3.7] to obtain the following:

(1) $M_T(A_1)$ is compact $\iff Z_1$ is compact;
We define the maps $\varphi, \tau$. Accordingly, $M_T(A_1)$ is surjective and perfect.

Remark. In the proof of theorem 7.2 we make use of the following projections:

$q: S \times Z_1 \to S,$ \hspace{1cm} $q(\lambda, z) = \lambda, \ \forall (\lambda, z) \in S \times Z_1$;

$p_1: S \times Z_1 \to Z_1,$ \hspace{1cm} $p_1(\lambda, z) = z, \ \forall (\lambda, z) \in S \times Z_1$;

$p_2: S \times M_T(A_1) \to M_T(A_1), \hspace{1cm} p_2(\mu, y) = y, \ \forall (\mu, y) \in S \times M_T(A_1),$

where for $i = 1, 2$ $(A_i, Z_i) \in \mathcal{A} \cup \mathcal{B}, T \in \text{Intos}(A_1, A_2)$. In particular, $p_1, p_2$, both are surjective perfect maps and $q$ is surjective continuous quotient map. Note that $q$ is perfect if and only if $Z_1$ is compact Hausdorff topological space.

Proof of theorem 7.2. Let $\text{ext} A_1^* \cap A_1^*$ has the induced weak$^*$-topology of $A_1^*$. By corollary 7.1.(1) we obtain $\text{Ch}_T(A_1) \neq \emptyset$ and

$$T^* \Delta_{A_2}: S \times M_T(A_1) \to \text{ext} A_1^*$$

is well defined, surjective and perfect. Further, by corollary 7.1.(2)

$$\Delta_{A_1}: S \times Z_1 \to \text{ext} A_1^*$$

is a homeomorphic map. Therefore the map

$$\Delta_{A_1}^{-1} T^* \Delta_{A_2}: S \times M_T(A_1) \to \text{ext} A_1^*$$

is surjective and perfect.

We set

$$\phi_0 = q \Delta_{A_1}^{-1} T^* \Delta_{A_2},$$

$$\tau_0 = p_1 \Delta_{A_1}^{-1} T^* \Delta_{A_2}.$$

Accordingly, $\tau_0$ is surjective and perfect. Further,

$$(\phi_0(\lambda, x), \tau_0(\lambda, x)) = \Delta_{A_1}^{-1} T^* \Delta_{A_2}(\lambda, x) = \Delta_{A_1}^{-1} T^*(\lambda \Delta_{A_2}(1, x))$$

$$= \Delta_{A_1}^{-1}(\lambda T^*_\Delta A_2(1, x)) = \Delta_{A_1}^{-1}(\lambda(\Delta_{A_1} \Delta_{A_1}^{-1} T^* \Delta_{A_2}(1, x)))$$

$$= \Delta_{A_1}^{-1}(\lambda(\Delta_{A_1}(\phi_0(1, x), \tau_0(1, x))))$$

$$= \Delta_{A_1}^{-1}(\lambda(\phi_0(1, x), \tau_0(1, x)))$$

$$= (\lambda \phi_0(1, x), \tau_0(1, x)), \hspace{1cm} \forall (\lambda, x) \in S \times M_T(A_1).$$

We define the maps $\phi: M_T(A_1) \to S$, $\tau: M_T(A_1) \to Z_1$ by

$$\phi(x) = \phi_0(1, x),$$

$$\tau(x) = \tau_0(1, x),$$
where \( x \in M_T(A_1) \). Then, \( \tau \) is surjective and perfect, \( \phi \) is continuous. We have to remark that \( \phi \) is perfect iff \( Z_1 \) is compact (in the case when \( Z_1 \) is compact then the diagonal map \( (\phi, \tau) : M_T(A_1) \to S \times Z_1 \) is perfect and is not necessary surjective).

Further, according to the definition of the maps \( \phi, \tau \)

\[
\Delta_{A_1}^{-1} T^* \Delta_{A_1}(1, x) = (\phi(x), \tau(x)), \quad \forall x \in M_T(A_1).
\]

Then \( T^* \Delta_{A_1}(1, x) = \Delta_{A_1}(\phi(x), \tau(x)) \) so, that

\[
Tf(X) = \phi(x)f(\tau(x)), \quad \forall x \in M_T(A_1), \forall f \in A_1.
\]

Thus it is proved the existence of maps \( \phi, \tau \) for which the assertions (1), (2), (3) of theorem 7.1 hold on. We claim that such a couple \((\phi, \tau)\) of maps is unique for the isometry \( T \). Indeed, for any couple \((\phi', \tau')\) with the properties:

\[
\phi' : U \to S, \\
\tau' : U \to Z_1, \\
Tf(x) = \phi'(\tau'(x)), \quad \forall x \in U, \forall f \in A_1,
\]

we have

\[
T^* \Delta_{A_2}(1, x) = \Delta_{A_1}(\phi'(x), \tau'(x)), \quad x \in U.
\]

Note, by corollary 7.1 \( \Delta_{A_1}(\phi(x), \tau(x)) \in \text{ext} A_1^*, x \in U \). So, \( T^* \Delta_{A_2}(1, x) \in \text{ext} A_1^* \) and hence \( x \in M_T(A_1) \). Thus \( U \subset M_T(A_1^*) \). Further

\[
(\phi'(x), \tau'(x)) = \Delta_{A_1}^{-1} T^* \Delta_{A_1}(1, x) = (\phi(x), \tau(x)), \quad \forall x \in U.
\]

Therefore the uniqueness is proved.

So, theorem 7.2 is proved.

**Lemma 7.4.** Let for \( i = 1, 2 \) \((A_i, Z_i) \in A \cup B\). Suppose there is a non closed subset \( U \) of \( Z_2 \) and there is a surjective perfect map \( \pi : U \to Z_1 \). Then for every \( x_0 \in \overline{U} \setminus U \) and for every \( f \in A_1 \)

\[
\lim_{x \to x_0, x \in U} f(\pi(x)) = 0.
\]

where \( \overline{U} \) stands for the closure of \( U \) in \( Z_2 \).

**Proof of lemma 7.4.** Let \( x_0 \in \overline{U} \setminus U, f \in A_1, \varepsilon > 0 \) all be chosen in an arbitrary manner. Then

\[
K = \{ y \mid y \in Z_1, |\Delta_{A_1}(1, x)(f)| \geq \varepsilon \}
\]

is compact. Hence its preimage \( \pi^{-1}(K) \) is a compact subset of \( U \), because of the perfectness of \( \pi \). So, that \( \pi^{-1}(K) \) is closed in \( \overline{U} \). Moreover, it follows from \( x_0 \notin U \) that \( x_0 \notin \pi^{-1}(K) \). Therefore \( \overline{U} \setminus \pi^{-1}(K) \) is an open neighbourhood of \( x_0 \) in \( U \) with respect to the subspace topology induced by \( Z_2 \). Then

\[
\lim_{x \to x_0, x \in U} |f(\pi(x))| \leq \sup_{x \in U \setminus \pi^{-1}(K)} |f(\pi(x))| = \sup_{z \in Z_1 \setminus K} |f(z)| \leq \varepsilon.
\]

Thus \( \lim_{x \to x_0, x \in U} |f(\pi(x))| = 0 \) and hence lemma 7.4 is proved.
Corollary 7.2. Let for \( i = 1, 2 \) the couple \((A_i, Z_i) \in \mathcal{A} \cup \mathcal{B}\). Suppose \( \text{Intois}(A_1, A_2) \neq \emptyset \). If \( T \in \text{Intois}(A_1, A_2) \) is such that \( M_T(A_1) \) is not closed then for every \( f \in A_1 \)
\[ Tf(x) = 0, \quad \forall x \in M_T(A_1) \setminus M_T(A_1). \]

Instead of an explicite proof we note that this corollary is an immediate consequence of both theorem 7.2 and lemma 7.4.

Corollary 7.3. Let for \( i = 1, 2 \) \((A_i, Z_i) \in \mathcal{A} \cup \mathcal{B}\). If there exist both an open subset \( U \) of \( Z_2 \) and a surjective perfect map \( \pi : U \to Z_1 \), then \( \text{Intois}(A_1, A_2) \neq \emptyset \).

Instead of a detailed proof we remark the following arguments. By lemma 7.4, for every \( f \in A_1 \) the function
\[
(\pi^0 f)(x) = \begin{cases} f(\pi(x)), & x \in U \\ 0, & x \notin U \end{cases}
\]
is continuous on \( Z_2 \). Hence the linear operator \( \pi^0 : f \mapsto \pi^0 f \) belongs to \( \text{Intois}(A_1, A_2) \).

These arguments prove corollary 7.3.

Remark 7.1. The problem when \( \text{Intois}(A_1, A_2) \neq \emptyset \) under consideration in corollary 7.3 is not new. There are results in [10] when \( U \) is closed.

Corollary 7.4. Let for \( i = 1, 2, 3 \) the couple \((A_i, Z_i) \in \mathcal{A} \cup \mathcal{B}\). Suppose \( \text{Intois}(A_1, A_2) \neq \emptyset, \text{Intois}(A_2, A_3) \neq \emptyset \). Let \( T_1 \in \text{Intois}(A_1, A_2), T_2 \in \text{Intois}(A_2, A_3) \) both be any and let’s put \( T_3 = T_2 T_1 \) (so, that \( T_3 \in \text{Intois}(A_1, A_3) \)). Let for \( i = 1, 2, 3 \), the couple of maps \((\phi_i, \tau_i)\) be the corresponding couple to \( T_i \) which existence is established in theorem 7.2. Then:

1. \( \emptyset \neq \tau_2^{-1}(M_{T_1}(A_1)) \subset M_{T_2}(A_2) \cap M_{T_3}(A_1) \);
2. \( \tau_2^{-1}(M_{T_1}(A_1)) \subset \text{Ch}_{T_2}(A_1), \text{ and } M_{T_2}(A_2) \cap M_{T_3}(A_1) \subset \text{Ch}_{T_2}(A_1) \);
3. \( \phi_3(x) = \phi_2(x) \phi_1(\tau_2(x)), \text{ for } x \in \tau_2^{-1}(M_{T_1}(A_1)) \);
4. \( \tau_3(x) = \tau_1(\tau_2(x)), \text{ for } x \in \tau_2^{-1}(M_{T_1}(A_1)) \);
5. \( \tau_3 : \tau_2^{-1}(M_{T_1}(A_1)) \to Z_1 \) is surjective and perfect;

Proof of corollary 7.4. By theorem 7.2

- \( \tau_1 : M_{T_1}(A_1) \to Z_1, \tau_1 \) is surjective and perfect;
- \( \tau_2 : M_{T_2}(A_2) \to Z_2, \tau_2 \) is surjective and perfect;
- \( \tau_3 : M_{T_3}(A_1) \to Z_1, \tau_3 \) is surjective,

where \( M_{T_1}(A_1) \subset Z_2, M_{T_2}(A_2) \subset Z_3, \text{ and } M_{T_3}(A_1) \subset Z_4 \). In particular, \( \emptyset \neq \tau_2^{-1}(M_{T_1}(A_1)) \subset M_{T_2}(A_2) \), and by the definition \( T_3 = T_2 T_1 \) it follows
\[
(T_3 f)(x) = (T_2(T_1 f))(x) = \phi_2(x)(T_1 f)(\tau_2(x)) = \phi_2(x)\phi_1(\tau_2(x))f(\tau_1(\tau_2(x))).
\]

where \( x \in \tau_2^{-1}(M_{T_1}(A_1)) \), and \( f \in A_1 \) both are arbitrary. Then by theorem 7.2 (the part about uniqueness) we obtain \( \tau_2^{-1}(M_{T_1}(A_1)) \subset M_{T_2}(A_2) \) and hence corollary 7.4.(1) is proved. Note, in addition, the assertions of corollary 7.4.(3) and of corollary 7.4.(4), are proved, too. Further, from (4) we obtain the part of (5) on surjectivity. The assertion on perfectness we obtain by using the general properties of perfect maps [6, § 3.7]: the restriction of the perfect map \( \tau_2 : M_{T_2}(A_2) \to Z_2 \) on
the whole preimage $\tau_2 : \tau_2^{-1}(M_{T_1}(A_1)) \to M_{T_1}(A_1)$ is perfect, too; the composition
$\tau_3 = \tau_1 \circ \tau_2$ of perfect maps is perfect, too. So, (5) is proved. Note, in particular,
$\tau_3^{-1}(M_{T_1}(A_1)) \to Z_1$ is surjective. Recall, by corollary 7.1.(2), we have
$\Delta_{A_1} : S \times Z_1 \to \text{ext} A_1$ is bijective. Therefore, $\tau_2^{-1}(M_{T_1}(A_1)) \in \text{Ch}_{T_1}(A_1)$ and so,
the first part of (2) is proved. The second one follows from both, from this one and
from (1). Thus corollary 7.4 is proved.

**Corollary 7.5.** Let for $i = 1, 2$ the couple $(A_i, Z_i) \in \mathcal{A} \cup \mathcal{B}$. Suppose $\text{Is}(A_1, A_2) \neq \emptyset$
and $T \in \text{Is}(A_1, A_2)$ be any. Then

1. $\tau_T : Z_2 \to Z_1$ is a homeomorphic map;
2. $\tau_{T^{-1}} = \tau_T^{-1}, \phi_T^{-1} = \phi_T \circ \tau_T^{-1},$

where the two couples of maps $(\phi_T, \tau_T)$ and $(\phi_T^{-1}, \tau_T^{-1})$ are the corresponding
 couples to the isometries $T$ and $T^{-1}$, resp. (see theorem 7.2).

Instead of a detailed proof we note that corollary 7.5 is a direct consequence of
corollary 7.4.

**References**

1. R. F. Arens and J. L. Kelley, *Characterizations of the space of continuous functions over a compact Hausdorff space*, Trans. Amer. Math. Soc. **62** (1947), 499–508.
2. E. Behrends, *M-structure and the Banach-Stone theorem*, Lecture Notes in Mathematics, vol. 736, Springer-Verlag, [MR81h:46002], Berlin, Heidelberg, New York, 1979.
3. E. Bishop and K. de Leeuw, *The representations of linear functionals by measures on set of extreme points*, Ann. Inst. Fourier **9** (1959), 305–331, [MR22#4945].
4. N. Dunford and J. T. Schwartz, *Linear operators, Part I*, Interscience, New York, 1958.
5. R. E. Edwards, *Functional analysis- theory and applications*, Holt, Rinehart and Winston, N.Y. etc.; translated in russian by G. Berman, I. Raskin, ed. V. Lin, M., Mir, 1969.
6. R. Engelking, *General topology*, Polish SP, Warszawa; translated in russian by M. Antonovski and A. Archangeliski, M., Mir, 1986 from the manuscript of the second edition, 1985.
7. W. Holsztyński, *Continuous mappings induced by isometries of spaces of continuous function*, Studia Mathematica **26** (1966), 133–136, [MR33#1711].
8. K. Jarosz and V. Pathak, *Isometries between function spaces*, Trans. Amer. Math. Soc. **305** (1988), 193–200p., [MR89e:46020].
9. Jeang, J.-Sh. and Wong, N.-Ch., *Weighted composition operators of $C_0(X)$’s*, J. of Math. Ann. and Appl. **201** (1996), 981–993p., [MR97f:47029].
10. McDonald, John N., *Isometries of function algebras*, Illinois J. Math. **17**, 579-583p., [ ... ].
11. W. Rudin, *Functional analysis*, MacGraw Hill, N.Y. etc.; translated in russian by V. Lin, ed. E. Gorin, M., Mir, 1975.
12. M. A. Stanev, *Surjective isometries in Banach spaces of slowly growing holomorphic functions on a ball (Russian original)*, Dokl. Akad. Nauk, Ross. Akad. Nauk **334**, 702–704; translated in Russ. Acad. Sci. Dokl. Math. vol. **49** (1994), no.1, 206–209p., [MR95e:46032].
13. M. A. Stanev, *Isometrical automorphisms for function spaces of Lipschitz type (English original)*, Complex analysis and generalized functions. Papers from the fifth intern. conf. on complex analysis and applic. on generaliz detions held in Varna, Bulgaria, Sept. 15-21, 1991 (Dimovski, Iv. et al., eds.), Publ. House of the Bulgar. Acad. Sci. 1993, Sofia, pp. 283–288p., [MR95h:30044].