Degrees of Freedom for Instantaneous-Relay Aided Interference Channel: Bounds and Achievable Schemes

Amin Azari*, Farshad Lahouti*(Corresponding Author) and Tobias Weber*

*Center for Wireless Multimedia Communications, University of Tehran
+Institute of Communications Engineering, University of Rostock

Email: {aazari,lahouti}@ut.ac.ir, tobias.weber@uni-rostock.de, WEB: [http://wmc.ut.ac.ir](http://wmc.ut.ac.ir)

Abstract—The $K$-user flat fading MIMO interference channel with $J$ instantaneous relays ($K$IC/$JR$) is considered. In the $K$IC/$JR$, the effective channel between sources and destinations including the relays has certain structure and is non-generic. For non-generic channels, the achievable degrees of freedom (DoF) is still unknown. Lee and Wang showed that by using the aligned interference neutralization scheme $3/2$ degrees of freedom is achievable in a 2IC1R system, which is 50% more than the 2-user interference channel. But the DoF performance and achievable schemes for other $K$IC/$JR$ networks are not investigated in literature. In this paper we devise an achievable scheme called restricted interference alignment for instantaneous-relay aided interference channels. Also, to find insights to the maximum achievable degrees of freedom we develop linear beamforming based on the mean square error (MSE) minimization as an achievable scheme. Furthermore, we present upper-bounds on the maximum achievable degrees of freedom by investigating the properness of the interference alignment equation system. The numerical results show that the DoF performance of the proposed restricted interference alignment scheme and the MSE-based beamforming match the upper-bounds determined from the properness of the interference alignment equations.

Index Terms—Degrees of Freedom, Interference Alignment, Interference Channel, Instantaneous Relay.

I. INTRODUCTION

INTERFERENCE, a consequence of the broadcast nature of wireless channels is a limiting factor for the capacity of wireless networks. Interference alignment (IA) is shown to be a promising technique for combating interference and increasing the multiplexing gain in interference networks [1]-[3]. Interference alignment increases the achievable degrees of freedom by limiting the number of dimensions which are occupied by interferences. In wireless networks with generic channel matrices and by using infinite channel extensions, IA allocates approximately one half of the available dimensions to the desired signals and the other one to the interfering signals. Then, all the interfering sources are treated as a single interfering source and the degrees of freedom increase linearly with the number of users which means the channel is not interference limited. For interference channels with generic channel matrices and without time extensions, feasibility of the interference alignment is considered in [4] where the authors derived a system of equations and showed the relation between solvability of that equation system and the feasibility of the interference alignment. By investigating the properness of this system of equations, they developed upper bounds on the achievable DoF in the $K$-user interference channel. The feasibility of interference alignment was investigated later in [5]-[6] and the problem was settled by finding the general condition which must be satisfied by any feasible DoF in $K$-user interference channel. Using this condition, they showed the maximum achievable DoF in a $K$-user interference channel is $\frac{2K-2}{K+1}M$. The feasibility of interference alignment and the maximum achievable degrees of freedom for the $K$-user interference channel is well investigated in literature, however, the achievable degrees of freedom and the achievable schemes are still unknown for other interference networks. To this end, iterative algorithms for minimizing the mean square error (MSE) and interference leakage are investigated as linear interference alignment schemes in [7]-[9]. In these works, aiming at minimizing the per user interference subspace, sum-MSE or leakage functions are minimized by iteratively designing beamforming matrices at transmitters and receivers. The mentioned algorithms don’t necessarily converge to the optimal beamforming matrices for interference alignment, however, it is shown that they can provide a numerical insight into the feasibility of interference alignment in interference networks which is not known from theory. In [10], the interference alignment problem is converted to an interference rank minimization problem and it is shown that the iterative algorithm for solving this optimization problem converges to the interference-free beamforming matrices with less number of iterations than in [7]-[9].

As cooperation between users is proved to be useful for increasing the data rate of users in wireless communications [11], the effect of relaying on the achievable degrees of freedom in interference networks is investigated in [12] and it is shown that cooperation over fading links between the sources, the destinations, or both, cannot improve the degrees of freedom. In some special cases such as networks with cognitive relays (prior and non-causal knowledge of transmitting signals is available at the relays) [13], networks without power constraints at the relays (the power in the relays is some order of magnitudes higher than the case of the source nodes) [14], or when relays have full channel state information (CSI) and limited CSI is available at transmitters [15], relays can increase the achievable DoF. In networks where relays cannot increase the DoF, they can simplify the achievable
DoF scheme by enabling linear alignment schemes where nonlinear alignments are otherwise needed \cite{16}. Instantaneous relaying, in which the relay’s retransmitted signal depends on the past and presently received signal is investigated in \cite{17}. In \cite{18}, it is shown that $3M/2$ DoF in 2IC-1R network is achievable using a new scheme called aligned interference neutralization (AIN). In the AIN scheme, source nodes select their transmit beamforming matrices to align some of the received signals at the relays to each other. Immediately, the relays scale the received signals and retransmit them such that these signals neutralize some of the interferences which are received directly from the sources. In \cite{19}, the aligned interference neutralization scheme is applied to the $K$-user interference channel with 2 relays, when there is no direct links between the sources and destinations and it is shown that $2M − 1$ degrees of freedom is achievable. In \cite{20}, it is shown that by properly designing the processing matrix of the $K$-antennas instantaneous relay, one can neutralize all the interferences in the interference channel with $K$ single-antenna users. The $K$-user interference channel with $K$ relays and no direct links between sources and destinations is considered in \cite{21}, and it is shown that by nonlinear processing capability in relays, one can achieve all degrees of freedom.

**A. Contributions**

Our contribution in this work is threefold.

1) **Achievable scheme for instantaneous-relay aided interference channels:** We present restricted interference alignment (RIA) as an achievable scheme for interference channels aided by instantaneous relays. The RIA scheme benefits from instantaneous relays for decreasing the rank of the equivalent channels between undesired source-destination pairs.

2) **Numerical insights to the DoF performance of the KICJR:** As the maximum achievable DoF of the KICJR is not known from theory, we use a MSE-based beamforming to find a lower bound on the achievable degrees of freedom. Total MSE is chosen as the objective function to be minimized and an iterative algorithm is proposed for designing the interference-free beamforming matrices.

3) **Bounds on the maximum achievable DoF of the KICJR:** We derive upper bounds on the DoF performance of the KICJR. These bounds are validated by comparing against lower bounds from an MSE-based beamforming.

**B. Paper Organization**

The rest of this paper is organized as follows. In the next Section, the system model and problem formulation are introduced. In Section III, the restricted interference alignment scheme is presented. The iterative algorithm for the MSE-based beamforming is investigated in Section IV. In Section V, upper bounds on the DoF performance of the KICJR are presented. The performance of the AIN scheme in the KICJR network is investigated in Section VI. Numerical results are presented in Section VII. The concluding Remarks are given in Section VIII.

**II. SYSTEM MODEL AND PROBLEM FORMULATION**

**A. System Model**

The $K$-user interference channel aided by $J$ memory-less instantaneous relays is considered in this work (Fig. 1). Define the set of users as $\mathcal{K} = \{1, \ldots, K\}$. In this system, source $S_i, i \in \mathcal{K}$ sends its independent message to its paired destination $D_i$. The relays instantaneously amplify and forward the currently received data symbols from the sources to the destinations. All nodes in the system are equipped with $M$ antennas. The channel coefficients between the $n$th source node and $m$th destination node, the $n$th source node and $j$th relay and the $j$th relay and $m$th destination node are denoted by $H_{mn}, H_{jn}^s$ and $H_{mj}^r$, respectively. Furthermore, all elements of the channel matrices are drawn independently from a Gaussian distribution with unit variance and zero mean.

**B. Problem formulation**

At source $S_i, i \in \mathcal{K}$, message $W_i$ is split into $\delta_i$ submessages, where $\delta_i$ is the DoF for source $i$ and $\Delta = \sum_{i=1}^{K} \delta_i$ is the sum DoF. Submessage $W_{ij}, j \in \{1, \cdots, \delta_i\}$ is encoded using a Gaussian codebook with codewords of length $n$ denoted as $x_{i,j}$. Source $S_i$ sends symbol $x_{i,j}$ with the $M \times 1$ beamforming vector $v_{i,j}$. Then the transmit beamforming matrix, transmitted data stream, and transmitted signal for source $i$ are as follows:

$$V_i = [v_{i,1}, \cdots, v_{i,\delta_i}], \quad (1)$$

$$x_i = [x_{i,1}, \cdots, x_{i,\delta_i}]^T, \quad \text{and}$$

$$z_i = \sum_{j=1}^{\delta_i} v_{i,j} x_{i,j}. \quad (2)$$

Now, one can write the received signal at relay $j$ and destination $k$ as

$$y_j^r = \sum_{i=1}^{K} H_{ji}^r z_i + n_j^r \quad \text{and} \quad (3)$$

$$y_k = \sum_{i=1}^{K} H_{ik}^d z_i + \sum_{j=1}^{\delta_i} H_{kj}^r z_j^r + n_k, \quad (4)$$

respectively. In this expression, $z_j^r$ is the retransmitted signal from relay $j$, $n_k$ is the noise vector at the destination $k$, and $n_j^r$...
is the noise vector at the \( j \)th relay. The noise vector components are independent random variables which follow a Gaussian complex distribution with zero mean and unit variance. Then, one can write the received signal at destination \( k \) as follows:

\[
y_k = \sum_{i=1}^{K} H_{ki}^{ds} z_i + \sum_{j=1}^{J} H_{kj}^{ds} W_j \left[ \sum_{i=1}^{K} H_{ki}^{s} z_i + n^j_i \right] + n^k_k
\]

\[
= \sum_{i=1}^{K} \left[ H_{ki}^{ds} + \sum_{j=1}^{J} H_{kj}^{ds} W_j H_{ki}^{s} \right] z_i + n^k_k
\]

\[
= \sum_{i=1}^{K} \Phi_{ki} V_i x_i + n^k_k. \tag{5}
\]

In this expression, \( n^k_k \) is the equivalent noise at destination \( k \), \( W_j \) is the beamforming matrix at relay \( j \), and \( \Phi_{ki} \) is the equivalent channel matrix \( H_{ki}^{ds} \) between source \( i \) and destination \( k \).

Using a linear receive beamforming matrix \( R_k \) at destination \( k \), one can write the received data vector at the destination \( k \) as:

\[
x_k = R_k y_k = \sum_{i=1}^{K} R_k \Phi_{ki} V_i x_i + R_k n^k_k. \tag{6}
\]

The DoF tuple \((\delta_1, \ldots, \delta_K)\) is feasible if the beamforming matrices at users and relays can be designed such that the following conditions are satisfied:

\[
R_k \Phi_{ki} V_i = 0, \quad \forall k \neq i \tag{7}
\]

\[
\text{rank}(R_k \Phi_{ki} V_k) = \delta_k, \quad \forall k. \tag{8}
\]

This system of equations tries to align all the interferences at destination \( k \) in an \( M - \delta_k \) dimensional subspace and tries to guarantee \( \delta_k \) interference-free dimensions for the desired data streams. The solvability of this interference alignment system of equations and the achievable schemes investigated in this work.

III. INTERFERENCE ALIGNMENT AIDED BY INSTANTANEOUS-RELAYS: THE RIA SCHEME

Interference alignment is shown to be a promising technique for interference management through limiting the number of dimensions which are occupied by interfering streams \([1]-[3]\). By deploying instantaneous relays between sources and destinations in an interference channel, destination nodes receive multiple copies of the desired and the interfering signals. Then, one can design the beamforming matrices at the sources and the relays to avoid/align/neutralize as much interfering streams as possible. In this Section, we present an achievable scheme which benefits from instantaneous relays for decreasing the dimensions of the individual received interference subspaces at the destinations. In the following, the RIA scheme is presented in three steps: **Restricting**, **Aligning**, and **Zero forcing**.

A. First step. Restricting the dimension of the received interference subspace from each source at each destination

Define the degrees of freedom for user \( i \) as \( \delta_i \). The received interference from source \( i \) at destination \( k \) is expressed as \( \Phi_{ki} V_i \), in which the rank of \( V_i \) is \( \delta_i \) to guarantee \( \delta_i \) DoF for user \( i \). Then, the maximum number of dimensions which can be occupied by interfering streams at destination \( i \) is \( M - \delta_i \).

In the first step of the RIA scheme we decrease the dimension of the received interference from source \( i \) at destination \( k \) from \( \delta_i \) to \( M - \delta_k \), where \( i, k \in K, i \neq k \). To this end, we need \([\delta_i + \delta_k - M]^+\) alignments for interfering streams from source \( i \) at destination \( k \) such that these \( \delta_i \) streams span only \( M - \delta_k \) dimensions, where \([x]^+ = \max(x, 0)\).

B. Second step. Aligning the restricted interferences at each destination

Up to now, the available space at destination \( k \) is spanned by \( K - 1 \) interference subspaces where each of them has \( M - \delta_k \) dimensions. As the maximum number of dimensions which can be spanned by interference subspaces at destination \( k \) is \( M - \delta_k \), these \( K - 1 \) interference subspaces must be aligned to each other. Define the \((M - \delta_k)\)-dimensional subspace for the interference at destination \( k \) as \( \Psi_k \), then the following must hold:

\[
\text{span}(\Phi_{ki} V_i) \subseteq \Psi_k, \quad i \in K, i \neq k \tag{9}
\]

This alignment can be done by adapting the transmit beamforming matrices at the sources, i.e. \( V_i \), or by adapting the beamforming matrices at the relays which construct the equivalent channel, \( \Phi_{ki} \).

C. Third step. Zero forcing at destinations

Applying step 1 and 2, the dimension of the received interference subspace at destination \( k \) is \( M - \delta_k \). As each destination is equipped with \( M \) antennas, one can use zero forcing beamforming at destination \( k \) to extract \( \delta_k \) desired streams from the \( M - \delta_k \) interfering streams.

One can see that the RIA scheme tries to convert the relay-aided interference channel to a conventional interference channel with low-rank cross channels. It decreases the rank of the the cross channels as much as possible while keeping the rank of the direct channels unchanged before applying a conventional interference alignment scheme.

D. Example of the RIA scheme in the 3IC2R network

Consider the 3-user interference channel aided by two instantaneous relays, where each node is equipped with 3 antennas. Here we describe how one can use the RIA scheme to achieve 6 degrees of freedom, i.e. the \((2, 2, 2)\) DoF tuple is achievable for the users.

1) Step 1. Restricting individual interference subspaces: The dimension of the spanned subspace by the interfering streams from source \( i \) at destination \( k \), where \( i \) and \( k \in \{1, 2, 3\} \), \( i \neq k \), is 2 and has to decrease to 1. Then for each interfering source at each destination, one alignment must be done. Without loss of generality, one can write the alignment equations at destination \( k \) for interfering streams from source
is one-dimensional. Then the received restricted interferences interfere with the desired streams and the associated subspace for interference at each destination, there are two interfering streams from two interfering sources, and the associated subspace for interference is one-dimensional. The solution to this system of equations can be found by decreasing the rank of the equivalent cross channels between the undesired users, i.e. \( \Phi_{ki} \), using beamforming matrices at the relays and then selecting the transmit beamforming matrices at the sources based on the null space in the equivalent cross channels.

2) Step 2. Aligning restricted interferences: Up to now in each destination, there are two interfering streams from two interfering sources, and the associated subspace for interference is one-dimensional. Then the received restricted interferences from undesired sources at each destination must be aligned as follows:

\[
\text{span}(\Phi_{12}v_{2,3}) = \text{span}(\Phi_{13}v_{3,3}), \\
\text{span}(\Phi_{21}v_{1,3}) = \text{span}(\Phi_{23}v_{4,1}), \\
\text{span}(\Phi_{31}v_{1,1}) = \text{span}(\Phi_{32}v_{2,1}).
\]

3) Step 3. Zero forcing at destinations: Each destination is equipped with 3 antennas. It receives the desired streams in a 2-dimensional subspace and the interfering streams in a 1-dimensional subspace. Then, one can use zero forcing beamforming at the destinations to extract the desired data from the interferences.

IV. MSE-BASED INTERFERENCE ALIGNMENT

In this Section we investigate an iterative algorithm for designing beamforming matrices at source, relay, and destination nodes such that the maximum degrees of freedom can be achieved. In this algorithm, the sum MSE of users is considered as the objective function to be minimized.

A. MSE-based beamforming algorithm

Using the definitions in Section II, one can define the MSE at destination \( k \) as presented at the top of the next page. Also, as the messages from different users are independent from each other and from the noises, one can simplify the expression in (9) as follows:

\[
\begin{align*}
\text{MSE}_k &= tr \left( R_k \left( \sum_{i=1}^{K} \Phi_{ki} V_i V_i^H \Phi_{ki}^H \right) R_k^H + I - R_k \Phi_{ki} V_k - V_k^H \Phi_{ki}^H R_k^H + R_k \sigma^2 R_k^H + R_k \left( \sum_{j=1}^{J} H_{kj}^{dr} W_j \sigma_j^2 W_j^H H_{kj}^{drH} \right) R_k^H \right),
\end{align*}
\]

where \( E \{ x_i x_i^H \} = I, \ E \{ u_i^r(n_i^r)^H \} = \sigma^2, \ E \{ u_i^r(n_i^r)^H \} = \sigma_r^2 \).

By minimizing the MSE, one may find the optimal set of beamforming matrices, but solving this problem is very hard, if not impossible. We use the sum MSE at the destinations as the objective function to be minimized and develop the optimization problem as follows:

\[
\begin{align*}
& \min_{\{k \}} \{ r_k \} \{ w_{j} \} \sum_{k=1}^{K} \text{MSE}_k, \\
& \text{s.t.:} \quad tr(V_k V_k^H) \leq P_k^T, \quad tr(W_j W_j^H) \leq P_j^T,
\end{align*}
\]

in which, \( P_k^T \) and \( P_j^T \) are the power constraint at source \( k \) and relay \( j \) respectively. One can see that sum-MSE minimization over each set of beamforming matrices at the sources, relays, and destinations is convex, but it is not jointly convex. Then, an iterative algorithm for beamforming design is developed to minimize the sum MSE of the users. Structure is chosen when we develop sum-MSE minimization algorithm for the beamforming. The procedure of this algorithm is: (i) choose two sets of beamforming matrices as constants (e.g. beamforming matrices at the sources and the destinations); (ii) derive the optimal filters for the third set (e.g. beamforming matrices at the relays); and (iii) update the two constant sets. Then, we repeat this algorithm for the other combinations of beamforming matrices. One can write the Lagrangian function for the optimization problem in (11) as follows:

\[
L(V_k; R_k; W_j; \lambda_k; \gamma_j) = \sum_{k=1}^{K} \text{MSE}_k + \sum_{k=1}^{K} \lambda_k \left[ tr(V_k V_k^H) - P_k^T \right] + \sum_{j=1}^{J} \gamma_j \left[ tr(W_j W_j^H) - P_j^T \right],
\]

where \( \lambda_k \) and \( \gamma_j \) are Lagrange multipliers for satisfying the power constraint at source \( k \) and relay \( j \). Using the K.K.T conditions, the optimal transmit beamforming matrix (the other two set of beamforming matrices are treated as constants) is written as follows:

\[
V_k = \left( \sum_{i=1}^{K} \Phi_{ik}^H R_k^H \Phi_{ik} + \lambda_k I \right)^{-1} \Phi_{ik}^H R_k^H.
\]

By considering beamforming matrices at sources and relays as constants, one can optimize the beamforming at the destinations as

\[
R_k = V_k^H \Phi_{kk}^H \left( \sum_{i=1}^{K} \Phi_{ki} V_i V_i^H \Phi_{ki}^H + \sigma^2 I + \sum_{j=1}^{J} H_{kj}^{dr} W_j \sigma_j^2 W_j^H H_{kj}^{drH} \right)^{-1}.
\]

Finally the beamforming matrices at the relays are optimized as follows:

\[
\frac{\partial L}{\partial W_l} = \sum_{k=1}^{K} \left( \sum_{i=1}^{K} \left[ H^{drH} R_k^H \Phi_{ki}^{dr} V_i V_i^H H_{ki}^{drH} + \sum_{j=1}^{J} H_{kj}^{dr} R_k^H \Phi_{kj}^{drH} W_j W_j^H H_{kj}^{drH} \right] - H_{kl}^{drH} R_k^H V_k^H H_{kl}^{drH} + H_{kl}^{drH} \sum_{j=1}^{J} R_k^H \Phi_{kj}^{drH} W_l \sigma_j^2 \right) W_l,
\]

\[
\gamma_l W_l = 0.
\]

\[2\text{Karush-Kuhn-Tucker}\]
MSE\(_k\) = E\(|\mathbf{x}_k - \bar{\mathbf{x}}_k|^2\) = E\(\text{tr}[(\mathbf{x}_k - \bar{\mathbf{x}}_k)^H(\mathbf{x}_k - \bar{\mathbf{x}}_k)]\)

\[
(9)
\]

In table 1, the iterative algorithm for the MSE-based beamforming is presented.

**TABLE 1: Iterative algorithm for the MSE-based beamforming**

1) Randomly initialize beamforming matrices at sources \(\{\mathbf{V}_k\}_{k=1}^K\) and relays \(\{\mathbf{W}_j\}_{j=1}^J\).
2) Design receive beamforming matrices at destinations \(\{\mathbf{R}_k\}_{k=1}^K\) using (12), and update beamforming matrices at sources \(\{\mathbf{V}_k\}_{k=1}^K\) and relays \(\{\mathbf{W}_j\}_{j=1}^J\), using (12) and (13), respectively.
3) Design transmit beamforming matrices at sources \(\{\mathbf{V}_k\}_{k=1}^K\) using (12), and update beamforming matrices at the relays \(\{\mathbf{W}_j\}_{j=1}^J\) and destinations \(\{\mathbf{R}_k\}_{k=1}^K\), using (13) and (14), respectively.
4) Design beamforming matrices at relays \(\{\mathbf{W}_j\}_{j=1}^J\) using (14), and update beamforming matrices at sources \(\{\mathbf{V}_k\}_{k=1}^K\) and destinations \(\{\mathbf{R}_k\}_{k=1}^K\), using (12) and (13), respectively.
5) Repeat steps 2-4 up to the convergence or predetermined number of iterations.

**B. Convergence Issues**

This iterative algorithm is guaranteed to converge, because the iterative process between the beamforming matrices is a monotonically decreasing function of the sum-MSE. As the sum-MSE is lower bounded at least by zero, this iterative algorithm always converges. However, the convergence to the global minimum is not guaranteed because the sum-MSE objective function is jointly non-convex over the beamforming matrices at the sources, relays, and destinations. In numerical results, it is shown that the MSE-based beamforming provides a tight lower bound on the maximum achievable DoF in a KIC/R network.

**V. UPPER BOUND ON THE MAXIMUM ACHIEVABLE DOF**

In this Section, upper bounds on the maximum achievable DoF in the K-user interference channel aided by J instantaneous relays are presented. These bounds are derived by investigating the solvability of the interference alignment system of equations in (7)-(8). In the sequel, we present lemmas and facts that will be used in the subsequent Sections.

**Lemma 1:** Define \(\Phi_{ki}\) as the equivalent channel between source \(i\) and destination \(k\), where \(i, k \in \mathcal{K}, i \neq k\). Then \(r_{ki} \triangleq [\delta_k + \delta_i - M]^+\) singular values of \(\Phi_{ki}\) are zero, where \(\delta_i\) is the DoF for user \(i\).

**Proof:** For guaranteeing \(\delta_k\) interference-free dimensions at destination \(k\), the interference space from source \(i\) at destination \(k\), \(\Phi_{ki}\), must span up to \(M - \delta_k\) dimensions. As the rank of the transmit beamforming matrix at source \(i\) is \(\delta_i\), the rank of \(\Phi_{ki}\) must not exceed \(M - (\delta_i - (M - \delta_k))\) = 2\(M - \delta_i - \delta_k\). Then, \([\delta_i + \delta_k - M]^+\) singular values of the equivalent channel matrix between source \(i\) and destination \(k\) have to be zero.

**Remark 1:** A system of polynomial equations which is improper (overdetermined) may have solutions if some equations are the consequence of the others. As the elements of the channel matrices are drawn from a continuous probability distribution, such a situation rarely occurs in a KIC/R network. In this work, we assume improperness implies infeasibility. By numerical evaluations, we show that the upper bound from this assumption matches the lower bounds from the MSE-based beamforming.

**Lemma 2:** Define \(C\) as an \(m \times n\) matrix where its elements are drawn randomly from a continuous probability distribution and its rank is \(\min(m, n)\) with probability one. To reduce the rank of \(C\) from \(\min(m, n)\) to \(d\), at least \((m - d)(n - d)\) elements of \(C\) are subject to change.

**Proof:** Define \(c_i\) as the row \(i\) of \(C\). Then one can derive the necessary conditions for \(C\) to have rank \(d\), as follows:

\[
c_i = \sum_{j=1}^{d} k_{j,i} c_{m-j+1}, \quad i \in \{1, \cdots, m-d\},
\]

in which \(k_{j,i}\) is a free variable. Using Remark 1, the properness of the polynomial system of equations in (15) is necessary for the solvability of (15). As each of the \(m - d\) equation sets in (15) has \(n\) equations (because each row of \(C\) consists of \(n\) elements), the total number of equations is \(n(m-d)\). Also, by counting the \(k_{j,i}\)s in (15), one can see that the total number of free variables is \(d(m-d)\). Then, for solvability of (15), at least \((n-d)(m-d)\) elements of \(C\) are subject to change for rank reduction.

**A. Necessary conditions on a DoF tuple to be feasible**

Now, we are able to develop necessary conditions which must be satisfied by any feasible DoF.

1) **First necessary condition:** The first necessary condition comes from the limited number of dimensions at each user, as follows:

\[
\delta_k \leq M \quad \text{for } k \in \mathcal{K}.
\]

2) **Second necessary condition:** Using Lemma 1, all interferences at destination \(k\) must be encapsulated in a subspace with \(M - \delta_k\) dimensions. The total number of received signals at destination \(k\) is \(\Delta\), where \(\Delta - \delta_k\) of them are interfering
signals. The interfering streams can be shown in an \( M \)-by-(\( \sum_{i \in K, i \neq k} \delta_k \)) matrix as follows:

\[
X_k = \begin{bmatrix} \Phi_{k1} V_1, \cdots, \Phi_{k(k-1)} V_{k-1}, \Phi_{k(k+1)} V_{k+1}, \cdots, \Phi_{kK} V_K \end{bmatrix}.
\]

Then, the rank of \( X_k \) must be less than or equal to \( M - \delta_k \). Using Lemma 2 for decreasing the rank of \( X_k \) to \( M - \delta_k \), one needs at least

\[
(\Delta - \delta_k - M + \delta_k)(M - \delta_k) = (\Delta - M)\delta_k
\]

free variables. Then for all destinations together, \( (\Delta - M)\Delta \) free variables are required. Now we count the number of free variables in the beamforming matrices at the sources and the relays, to check the properness of rank reduction problem. The transmit beamforming matrix at source \( i \), \( V_i \), is an \( M \times \delta_i \) matrix and its rank is \( \delta_i \). Then, \( \delta_i^2 \) elements of \( V_i \) are used for satisfying the rank constraint, and there are \( \delta_i(M - \delta_i) \) free variables at \( V_i \). Considering the transmit beamforming matrices at all sources, there are \( \sum_{i=1}^{K} \delta_i(M - \delta_i) \) free variables at the sources. Also, there are at most \( JM^2 \) free variables at the relays. Then the following condition has to hold for properness of rank reduction in \( X_k \):

\[
\sum_{k=1}^{K} \delta_k(M - \delta_k) + JM^2 \geq (\Delta - M)\Delta. \tag{17}
\]

3) Third necessary condition: As mentioned, the rank of \( X_k \) must be less than or equal to \( M - \delta_k \). Then the rank of \( \Phi_{ki} V_i \) has to be less than or equal to \( M - \delta_k \), where \( i, k \in K, i \neq k \). Furthermore, as the transmit beamforming matrix at source \( k \) has full rank, the channel matrix \( \Phi_{ki} \) must be rank deficient, and this is possible using appropriate beamforming at the relays. Using Lemma 2 for \( \Phi_{ki} \) to have \( r_{ki} = \lceil \delta_k + \delta_i - M \rceil \) zero singular values, we need at least \( r_{ki}^2 \) free variables at the relays. By considering all cross channels, \( \Phi_{ki}, i \neq k \); one can write the third necessary condition on feasible DoF tuples as follows:

\[
\sum_{k=1}^{K} \sum_{i=1, i \neq k}^{K} r_{ki}^2 \leq JM^2. \tag{18}
\]

4) Forth necessary condition: Here we present a strong necessary condition for the feasibility of a DoF tuple in the KIC1R, and verify its necessity without considering the assumption in Remark 1.

**Proposition 1:** Define \( A \) and \( B \) as two \( M \times M \) full rank matrices. A necessary condition on the existence of matrix \( W \) which satisfies

\[
\begin{align*}
\text{rank}(A + W) &= M - r_A \\
\text{rank}(B + W) &= M - r_B
\end{align*}
\]

is \( r_A + r_B \leq M \).

**Proof:** Consider the following system of equations:

\[
A + W = F_1^T F_2 \tag{20}
\]

and

\[
B + W = G_1^T G_2, \tag{21}
\]

where \( F_1 \) and \( F_2 \) have \( M - r_A \) rows and \( M \) columns; \( G_1 \) and \( G_2 \) have \( M - r_B \) rows and \( M \) columns. The elements of these four matrices and \( W \) are to be designed such that the equations in (20)-(21) can be satisfied. Also, the rank of \( F_1 \) and \( G_1 \) matrices must be \( M - r_A \) and \( M - r_B \), respectively for \( i = 1, 2 \). One can write the difference of (21) and (20) as follows:

\[
A - B = F_1^T F_2 - G_1^T G_2. \tag{22}
\]

As \( A \) and \( B \) are full rank matrices, their difference is full rank with probability one. Then for consistency of (20)-(21), the right hand of (22) must be of full rank. But as the ranks of \( F_1^T F_2 \) and \( G_1^T G_2 \) are \( M - r_A \) and \( M - r_B \), respectively, the rank of their sum is less than or equal to \( \min(2M - r_A - r_B, M) \). Hence, one can write the necessary condition on the solvability of (20)-(21) as follows:

\[
2M - r_A - r_B \geq M \rightarrow r_A + r_B \leq M, \tag{23}
\]

and the proof is completed.

Now in a KIC1R network, for the channel matrices \( \Phi_{ki} \) and \( \Phi_{ik} \) have \( r_{ki} \) and \( r_{ik} \) zero singular values, the following equations must be satisfied:

\[
\Phi_{ki} = H_{ik}^{ds} + H_{ik}^{dr} W_i H_{ik}^{rs} = F^T G \tag{24}
\]

and

\[
\Phi_{ik} = H_{k}^{ds} + H_{ik}^{dr} W_i H_{ik}^{rs} = Y^T Z. \tag{25}
\]

in which \( W_i \) is the beamforming matrix at the relay and the channel matrices have the dimension \( M \times M \). The matrices \( F \) and \( G \) have \( M - r_k \) rows, \( M \) columns, and the matrices \( Y \) and \( Z \) have \( M - r_k \) rows and \( M \) columns, respectively. We design matrices \( W_i, F, G, Y, \) and \( Z \) such that the system of equations in (24)-(25) is satisfied, and the four latter matrices have full rank. As the elements of the channel matrices are drawn from continuous probability distributions, these matrices are full rank with probability one, and they are invertible. Then it is straightforward to rewrite (24)-(25) as follows:

\[
C + W_1 = Y_2^T Y_1 \tag{26}
\]

and

\[
D + W_1 = Z_2^T Z_1 \tag{27}
\]

in which,

\[
Y_1^T Y_2 = H_{k1}^{dr} F^T G H_{i}^{rs}, \tag{28}
\]

\[
Z_1^T Z_2 = H_{i1}^{dr} Y^T Z H_{i1}^{rs}, \tag{29}
\]

\[
C = H_{k1}^{dr} H_{ik}^{ds} H_{i1}^{rs}, \quad D = H_{i1}^{dr} H_{ik}^{ds} H_{ik}^{rs}. \tag{30}
\]

Using the condition in (23) for solvability of (26)-(27), we conclude that \( r_{ki} + r_{ik} \leq M \) must hold, which means:

\[
r_{ki} + r_{ik} = 2(\delta_i + \delta_j - M) \leq M \rightarrow \delta_i + \delta_k \leq 1.5M \text{ for } i \neq k; i, k \in K \tag{31}
\]

Now, we can summarize all investigated necessary conditions on feasible DoF tuples as follows:

**Proposition 2:** The \((\delta_1, \ldots, \delta_K)\) DoF tuple is feasible in the KICJR if it satisfies the following conditions:

a: \( 2M\Delta + JM^2 \geq \Delta^2 + \sum_{k=1}^{K} \delta_k^2 \),

b: \( \sum_{k=1}^{K} \sum_{i=1, i \neq k}^{K} (\delta_i + \delta_j - M)^2 \leq JM^2 \),

c: \( \delta_k \leq M \),

d: \( \delta_i + \delta_k \leq 3M/2 \) \( \forall i, k; i \neq k \) if \( J = 1 \).
B. Using upper bounds for DoF performance evaluation

One can use the necessary conditions in \((32)-(35)\) to analyze the DoF performance. In the following, we analyze the maximum achievable DoF in the 2IC1R, KIC0R, and KIC(K−1)R networks.

**Proposition 3:** The maximum achievable degrees of freedom in a 2IC1R network is \([1.5M]\).

**Proof:** Based on the necessary condition in \((35)\), the following bound is held in a 2IC1R network:

\[
\delta_1 + \delta_2 = \Delta \leq [1.5M].
\]

Also, in \([18]\) it is shown that \([1.5M]\) degrees of freedom is achievable in a 2IC1R network using the AIN scheme. Then, the maximum achievable DoF in this network is \([1.5M]\).

As the DoF performance of a K-user interference channel is known \([5]\), one can use the bounds for the KICJR in \((32)-(35)\) for the KIC0R and compare the results as follows.

**Proposition 4:** For the \((\delta_1, \ldots, \delta_K)\) DoF tuple to be feasible in the KIC0R, the following conditions must be met:

a: \(\delta_k \leq M\) for \(k \in \{1, \ldots, K\}\),

b: \(\sum_{k=1}^{K} \delta_k (M - \delta_k) + \geq (\Delta - M)\Delta\),

c: \(\delta_i + \delta_j \leq M \quad \forall i, k \in \{1, \ldots, K\}; \quad i \neq k\).

**Proof:** Here we prove that the results in \((36)-(38)\) are consistent with the results of \([5]\). Due to restricted number of antennas at each node, the condition in \((36)\) is presented which is the same as \((4)\) in \([5]\). The equation in \((37)\) is the extension of \((32)\) when there is no instantaneous relay, and counts the number of equations and unknowns as follows:

\[
\sum_{k=1}^{K} \delta_k (M - \delta_k) + M\Delta \geq \Delta^2.
\]

By subtracting \(\sum_{k=1}^{K} \delta_k^2\) from both sides of \((39)\), we have:

\[
\sum_{k=1}^{K} \delta_k (M - \delta_k) + \sum_{k=1}^{K} \delta_k (M - \delta_k) \geq \sum_{k=1}^{K} \sum_{i=1}^{K} \delta_k \delta_i,
\]

which is in accordance with \((6)\) in \([3]\). Also, in Lemma \([1]\) it is shown that the equivalent channel matrix \(\Phi_{k_i}\) between source \(i\) and destination \(k\), must have \([\delta_i + \delta_k - M] +\) zero singular values.

The beamforming matrices at the relays make the cross channels rank deficient and provide zero singular values. In a KIC0R network, as there is no instantaneous relay, the number of zero singular values must be zero. Then the following condition has to hold:

\[
\delta_i + \delta_j \leq M.
\]

One can see that \((41)\) is in accordance with \((5)\) in \([5]\), and the proof is completed.

In \([23]\) it is shown that using \(K(K - 1) + 1\) half-duplex relays, one can achieve \(KM\) degrees of freedom in a K-user interference channel. In the following, we investigate the same problem using instantaneous relays instead of the half-duplex relays.

**Proposition 5:** Consider the K-user interference channel. Using \(K(K - 1)\) instantaneous relays \(KM\) degrees of freedom is achievable.

**Proof:** As there are \(K(K - 1)\) interfering channels between undesired users, the relays must provide \(M\) zero singular values at each cross channel matrix. The channel matrices are \(M\)-by-\(M\) and the only rank zero \(M\)-by-\(M\) matrix is the all-zero matrix. Then, \(K(K - 1)\) linear system of equations must be satisfied where in each of them there are \(M^2\) zero forcing equations. The system of equations for removing the interfering channel between source \(i\) and destination \(k\) is written as

\[
H_{ki}^{dr} + \sum_{j=1}^{f} H_{kj}^{dr} W_j H_{ji}^{rs} = [0]_{M \times M},
\]

in which \(f = K(K - 1)\). Using Kronecker product, one can rewrite \((42)\) as:

\[
\left[ H_{1i}^{rsT} \otimes H_{ki}^{dr}, H_{2i}^{rsT} \otimes H_{ki}^{dr}, \ldots, H_{fi}^{rsT} \otimes H_{ki}^{dr} \right] \left[ \begin{array}{c} w_1 \\ w_2 \\ \vdots \\ w_f \end{array} \right] = \text{vec}(-H_{ki}^{dr}),
\]

where \(w_k = \text{vec}(W_k)\) and \(\text{vec}(A)\) is the column vector built from the columns of \(A\). Now we consider \(K(K - 1)\) system of equations for removing all interfering channels in matrix form, and rewrite the problem as shown on top of the next page, in which \(h_{ki}^{rs} = \text{vec}(H_{ki}^{rs})\). As the channel matrices are full rank with probability one, and the rank of the Kronecker product of two matrices is equal to the product of their ranks, the coefficient matrix in \((44)\) has full rank, and the system of equations in \((44)\) has a unique solution. This unique solution gives the beamforming matrices at the relays for removing interfering channels between undesired users and achieving \(KM\) degrees of freedom.

VI. ALIGNED INTERFERENCE NEUTRALIZATION IN THE KICJR

In \([18]\), the authors presented an achievable scheme for 2IC1R, called aligned interference neutralization. In this scheme, the sources transmit their streams such that some of them be aligned at the relay. Then, the relay chooses appropriate beamforming matrices, and retransmits the received streams to the destinations. The retransmitted streams from the relay neutralize some of the interferences which are received from the direct links. In \([18]\), it is shown that applying the AIN scheme, \(\frac{7}{3}\) degrees of freedom is achievable in a 2IC1R network. In this Section, we evaluate the performance of the AIN scheme when it is applied to a general KICJR network.

A. DoF performance evaluation for the AIN scheme

To find the maximum achievable DoF in the KICJR using the AIN scheme, we find the general condition which must be satisfied by any achievable DoF tuple for the AIN scheme. To this end, we investigate the role of the source, the relay, and the destination nodes in the AIN scheme.
1) Source nodes in the AIN: Source $S_i$, $i \in \mathcal{K}$, transmits $\delta_k$ independent data symbols to its paired destination, $D_k$. If $\Delta \geq M$, each source designs its transmit beamforming matrix such that $\Delta - M$ streams align on each other at each relay. The first necessary condition on feasibility of a DoF tuple comes from the limited number of dimensions in users, as in \[10\].

2) Relay nodes in the AIN: At relay $j$, $\Delta$ data streams arrive simultaneously, where $j \in \{1, \ldots, K\}$. The number of available dimensions at $j$th relay is $M$. Based on the AIN scheme, $\Delta - M$ alignments must be done at relay $j$. Then by considering all the relays together, $J(\Delta - M)$ alignment equations must be satisfied at the relays. The alignment equations at the relays construct a homogeneous linear system of equations as $Ax = [0]$. The homogeneous system of equations has a solution if the number of equations is less than the number of free variables. Then the alignment system of equations can be solved if the number of alignment equations is less than the number of transmit beamforming vectors, as follows:

$$\Delta - 1 \geq J(\Delta - M) \implies \Delta \leq \frac{JM - 1}{J - 1}, \text{ for } J \geq 1.$$  \hspace{1cm} (45)

3) Destination nodes in the AIN: At destination $k, k \in \mathcal{K}$, $\Delta$ streams arrive simultaneously, and $M$ dimensions are available for both desired and interfering streams. Destination $K$ is able to extract its $\delta_k$ desired streams, if we remove $\Delta - M$ data streams before arriving at the destination. This is possible by removing some of the received interferences from the direct links with received interferences from the relay links. The total number of interfering data streams that must be removed in destinations is $K(\Delta - M)$. Then, the same number of antennas must be available at the relays. Finally, one can write the third necessary condition on feasible DoF tuples via the AIN scheme in a $KICJR$ network as follows:

$$\Delta \leq (J + K)M/K.$$  \hspace{1cm} (46)

These conditions will be used in the next Section for evaluating the performance of the AIN scheme in a $KICJR$ network.

VII. Performance Evaluation

In this section, the DoF performance of the $KICJR$ network under the RIA scheme, MSE-based beamforming, and the AIN scheme is investigated. For the benchmark, we also consider interference alignment and time division multiple access (TDMA) schemes in a $KICOR$ network. In the following figures, $UB$ means the upper bounds on the maximum achievable DoF in the $KICJR$, which are investigated in \([32, 35]\). For $3IC1R$ network, one can see in Fig. [2] that the lower bounds from the MSE-based beamforming and the RIA scheme match quite well the upper bounds on degrees of freedom. For a $3IC2R$ network, we see that the lower bounds from the RIA scheme and the MSE-based beamforming are close to the DoF upper bounds, but there is a gap between them, when the number of antennas at each node is higher than 5. In both cases, there is a big gap between the DoF performance of the RIA and AIN schemes. The DoF performances of the $2IC1R$ and $2IC2R$ networks are investigated in Fig. [4]. One can see that the DoF performances of the RIA, MSE, and AIN schemes in a $2IC1R$ network are the same and match the upper bounds on the DoF performance. Also, the achieved DoFs are in accordance with the Proposition [3] that implies the maximum achievable DoF in $2IC1R$ is $[3M/2]$. For a $2IC2R$ network, one can see that the DoF performances from the RIA and the MSE schemes are the same and match quite well the DoF upper bounds, while the DoF performance of the AIN scheme is worse than the maximum achievable DoF in this network. Also, the DoF performance in a $2IC2R$ network is in accordance with the Proposition [5] that implies the maximum achievable DoF in the 2-user interference channel is achieved using two instantaneous relays.

VIII. Conclusions

The DoF performance of the $K$-user interference channel with instantaneous relays is considered in this work. In the $KICJR$, the effective channels between the sources and destinations including the relays are non-generic, and for non-generic channels, the maximum achievable DoF is still unknown. Previously, it is shown that one instantaneous relay can increase the DoF of a 2-user interference channel by fifty percent, applying a new scheme named aligned interference neutralization. In this work, restricted interference alignment for the $KICJR$ is proposed. In this scheme, the interferences from each source are restricted at each destination, and the received restricted interferences at each destination are aligned. As the exact bounds on the DoF performance of the non-generic channels, including the $KICJR$ network, is not known from theory, an iterative algorithm for the MSE-based beamforming is investigated in this work. Also, we present upper bounds on the DoF performance of the $KICJR$ network, by investigating the properness of the system of equations resulted from the interference alignment equations. Numerical results show that the DoF performance of the proposed RIA scheme and the MSE-based beamforming are close to the upper bounds determined from the properness of the interference alignment equations.

References

[1] M. Maddah-Ali, A. Motahari, and A. Khandani, “Signaling over MIMO multi-base systems: Combination of multi-access and broadcast schemes,” in Information Theory, IEEE International Symposium on, 2006, pp. 2104–2108.

[2] S. Jafar and S. Shamai, “Degrees of freedom region of the MIMO X channel,” Information Theory, IEEE Transactions on, vol. 54, no. 1, pp. 151–170, 2008.
number of antennas at each node
Δ: sum degrees of freedom

TDMA−3IC0R
IA−3IC0R
AIN−−3IC1R
AIN−−3IC2R
UB−−3IC1R
UB−−3IC2R
MSE−−3IC1R
MSE−−3IC2R
RIA−−3IC1R
RIA−−3IC2R

Fig. 2. DoF performance of a 3IC/JR network under different transmission schemes

[3] V. Cadambe and S. Jafar, “Interference alignment and degrees of freedom of the K-user interference channel,” *Information Theory, IEEE Transactions on*, vol. 54, no. 8, pp. 3425–3441, 2008.

[4] C. M. Yetis, T. Gou, S. A. Jafar, and A. H. Kayran, “On feasibility of interference alignment in MIMO interference networks,” *Signal Processing, IEEE Transactions on*, vol. 58, no. 9, pp. 4771–4782, 2010.

[5] M. Razaviyayn, G. Lyubeznik, and Z.-Q. Luo, “On the degrees of freedom achievable through interference alignment in a MIMO interference channel,” *Signal Processing, IEEE Transactions on*, vol. 60, no. 2, pp. 812–821, 2012.

[6] G. Bresler, D. Cartwright, and D. Tse, “Settling the feasibility of interference alignment for the MIMO interference channel: the symmetric square case,” *arXiv preprint arXiv:1104.0888*, 2011.

[7] H. Shen, B. Li, M. Tao, and X. Wang, “MSE-based transceiver designs for the MIMO interference channel,” *Wireless Communications, IEEE Transactions on*, vol. 9, no. 11, pp. 3480–3489, 2010.

[8] M. Razaviyayn, M. Sanjabi, and Z.-Q. Luo, “A distributed numerical approach to interference alignment and applications to wireless interference networks,” *Information Theory, IEEE Transactions on*, vol. 57, no. 6, pp. 3309–3322, 2011.

[9] D. S. Papailiopoulos and A. G. Dimakis, “Interference alignment as a rank constrained rank minimization,” *Signal Processing, IEEE Transactions on*, vol. 60, no. 8, pp. 4278–4288, 2012.

[10] A. Nosratinia, T. Hunter, and A. Hedayat, “Cooperative communication in wireless networks,” *Communications Magazine, IEEE*, vol. 42, no. 10, pp. 74–80, Oct 2004.

[11] V. R. Cadambe and S. A. Jafar, “Can feedback, cooperation, relays and full duplex operation increase the degrees of freedom of wireless networks?” in *Information Theory, IEEE International Symposium on*, 2008, pp. 1263–1267.

[12] S. Sridharan, S. Vishwanath, S. Jafar, and S. Shamai, “On the capacity of cognitive relay assisted gaussian interference channel,” in *Information Theory, IEEE International Symposium on*, 2008, pp. 549–553.
Fig. 3. DoF performance of a 2ICJR network under different transmission schemes