STABILITY MANIFOLD OF $\mathbb{P}^1$

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ABSTRACT. We describe the stability manifold of the bounded derived category $D(\mathbb{P}^1)$ of coherent sheaves on $\mathbb{P}^1$, denoted $\text{Stab}(D(\mathbb{P}^1))$. This is the first complete picture of a stability manifold for a non-Calabi-Yau manifold.

1. INTRODUCTION

T. Bridgeland defined the notion of the stability manifold of a triangulated category $\mathcal{T}$, motivated by M. Douglas’s work on $\Pi$-stability of $D$-branes [8], [9], and [10]. Some stability manifolds occur as an approximation of a part of moduli of $(2,2)$ superconformal field theories of interest in algebraic geometry. Another reason to be interested in the stability manifold $\text{Stab}(\mathcal{T})$ of a triangulated category $\mathcal{T}$ is that it can be viewed as a tool for navigation in $\mathcal{T}$. For instance, since the notion of a stability condition refines the notion of a heart, $\text{Stab}(\mathcal{T})$ decomposes into “cells” $\text{Stab}_C(\mathcal{T})$ corresponding to interesting hearts $C$. Moreover, the extra structure contained in a stability condition provides mechanisms of rotation (the action of $C$ in Definition 2.3) and wall crossing (Proposition 2.5 and Lemmas 5.4-5.5), that allow one to systematically construct new hearts from the known ones.

Throughout this paper, by a heart we mean the heart of a bounded $t$-structure of $\mathcal{T}$ (the heart actually determines the corresponding bounded $t$-structure [4, Section 3]).

In this paper, we study in some detail the stability manifold for the bounded derived category $D^b(\mathbb{P}^1)$ of coherent sheaves on $\mathbb{P}^1$. In particular:

**Theorem 1.1.** $\text{Stab}(D(\mathbb{P}^1))$ is isomorphic to $\mathbb{C}^2$ as a complex manifold.

The strategy is to show that the quotient of $\text{Stab}(D(\mathbb{P}^1))$ for a certain action of $\mathbb{C} \times \mathbb{Z}$ is isomorphic to $\mathbb{C}^*$. The main technical step is the following list of stability conditions of $D(\mathbb{P}^1)$ (the notions used in Theorem 1.2 are explained in Section 2).

**Theorem 1.2.** Up to the action of $\text{Aut}(D(\mathbb{P}^1))$, for any stability condition in $\text{Stab}(D(\mathbb{P}^1))$, there exists some $p > 0$ such that $O(-1)[p]$ and $O$ are semistable and $\phi(O(-1)[p]), \phi(O) \in (r, r + 1]$ for some $r \in \mathbb{R}$.

If $\phi(O(-1)[1]) < \phi(O)$, the multiples of the shifts of $O(-1)$ and $O$ are the only semistable objects. If $\phi(O(-1)[1]) \geq \phi(O)$, then all line bundles and torsion sheaves are semistable.

In Section 2, we briefly explain parts of [4]. The reader can consult [6] for basic notions of triangulated categories and hearts. In Section 3, we prove Theorem 1.2 and present all hearts that appear in $\text{Stab}(D(\mathbb{P}^1))$. In Section 4, we find a fundamental domain of $\text{Stab}(D(\mathbb{P}^1))/\mathbb{Z} \times \mathbb{C}$ in Lemma 4.3, and prove Theorem 1.1. In Section 5, we explicitly describe how $\text{Stab}(D(\mathbb{P}^1))$ is glued from “cells” corresponding to interesting hearts of $D(\mathbb{P}^1)$.
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2. Basic definitions and tools

2.1. Definition of stability conditions. Let us define the stability conditions using the following two “filtrations” for our model (we will loosely use the word filtration for sequences of exact triangles as below). First, a heart $A$ of a triangulated category $T$ gives a filtration of each object $E \in T$. For example, the standard heart $\text{Coh} \mathbb{P}^1$ of $\text{D}(\mathbb{P}^1)$ gives the following filtration for each object $E \in \text{D}(\mathbb{P}^1)$ by taking $E_k = \tau_{\leq k} E$.

$$
0 \to E_m \xrightarrow{\kappa} E_{m+1} \xrightarrow{\kappa} E_{m+2} \to \cdots \to E_{n-1} \xrightarrow{\kappa} E_n = E
$$

Each cone $A_j = H^j(E)[-j]$ lies in $\text{Coh}(\mathbb{P}^1)[-j]$.

We refine the filtration above. Now, $A_n[n] = I \oplus L$ for a torsion sheaf $I$ and $L = \mathcal{O}(s_2)^{\oplus t_2} \oplus \cdots \oplus \mathcal{O}(s_{u+1})^{\oplus t_{u+1}}$ for some $s_2 > \cdots > s_{u+1}$. Then

$$
E_{n-1} = E_n^0 \xrightarrow{\kappa} E_n^1 \xrightarrow{\kappa} E_n^2 \to \cdots \to E_n^u \xrightarrow{\kappa} E_n = E
$$

for $A_1^i = I[-n]$, $A_2^i = \mathcal{O}(s_2)^{\oplus t_2}[-n]$, ..., $A_{u+1}^i = \mathcal{O}(s_{u+1})^{\oplus t_{u+1}}[-n]$.

The crucial observation is the following: the property that the rightward $\text{Hom}$ is zero for cones in the first filtration remains true in the second filtration; i.e., $\text{Hom}_T(A_i, A_j) = 0$ for $i < j$, and $\text{Hom}_{\text{Coh} \mathbb{P}^1[-n]}(A_i^n, A_j^n) = 0$ for $i < j$. This situation is axiomatized in the following definition.

Definition 2.1. [4, Definition 1.1] A stability condition $(Z, \mathcal{P})$ on a triangulated category $T$ consists of a group homomorphism $Z : K(T) \to \mathbb{C}$ called the central charge, and a family of full additive subcategories $\mathcal{P}(\phi)$ of $T$ indexed by real numbers $\phi$, called the slicing, with the following properties:

(a) if $E \in \mathcal{P}(\phi)$ then $Z(E) = m(E) \exp(i\pi \phi)$ for some $m(E) \in \mathbb{R}_{>0}$;
(b) for all $\phi \in \mathbb{R}$, $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$;
(c) if $\phi_1 > \phi_2$ and $A_j \in \mathcal{P}(\phi_j)$ then $\text{Hom}_T(A_1, A_2) = 0$;
(d) for each nonzero object $E \in T$ there is a finite sequence of real numbers $\phi_1 > \phi_2 > \cdots > \phi_n$ and a collection of triangles

$$
0 \to E_0 \xrightarrow{\kappa} E_1 \xrightarrow{\kappa} E_2 \to \cdots \to E_{n-1} \xrightarrow{\kappa} E_n = E
$$

with $A_j \in \mathcal{P}(\phi_j)$ for all $j$. 

We call the filtration the Harder-Narasimhan filtration (or HN filtration for short) of \( E \) with respect to \( (Z, \mathcal{P}) \), and if \( E \in \mathcal{P}(\phi) \) then we call \( \phi \) the phase of \( E \). HN-filtrations are unique (when they exist).

2.2. Definition and local structure of \( \text{Stab}(\mathcal{T}) \). For an interval \( I \), let \( \mathcal{P}(I) \) be the full subcategory of \( \mathcal{T} \) generated under extensions by \( \mathcal{P}(\phi) \) for all \( \phi \in I \) (we say that \( B \) is an extension of \( A \) and \( C \) in \( \mathcal{T} \) if there exists an exact triangle \( A \rightarrow B \rightarrow C \) in \( \mathcal{T} \)). A slicing \( \mathcal{P} \) of a triangulated category \( \mathcal{T} \) is said to be locally-finite, if for any \( t \in \mathbb{R} \) we have an open interval \( I \) around \( t \), such that each object in \( \mathcal{P}(I) \) has a finite-length Jordan-H"{o}lder filtration. A stability condition \( (Z, \mathcal{P}) \) is locally-finite if the corresponding slicing \( \mathcal{P} \) is.

Definition 2.2. [4, Section 6] For a triangulated category \( \mathcal{T} \), \( \text{Stab}(\mathcal{T}) \) is the set of all locally-finite stability conditions on \( \mathcal{T} \). T. Bridgeland defines a topology on \( \text{Stab}(\mathcal{T}) \). Its characterizing property is

Theorem 2.1. [4, Theorem 1.2] Let \( \mathcal{T} \) be a triangulated category. For each connected component \( \Sigma \subset \text{Stab}(\mathcal{T}) \) there is a linear subspace \( V(\Sigma) \subset (K(\mathcal{T}) \otimes \mathbb{C})^* \) with a well-defined linear topology, and such that the map \( Z : \Sigma \rightarrow V(\Sigma) \), which maps a stability condition \( (Z, \mathcal{P}) \) to its central charge \( Z \in V(\Sigma) \), is a local homeomorphism.

Hence, when \( K(\mathcal{T}) \) has finite rank, \( \text{Stab}(\mathcal{T}) \) is a complex manifold, called the stability manifold of \( \mathcal{T} \).

2.3. Hearts and stability conditions. Let us see the relation between hearts and stability conditions. By a heart of \( \mathcal{T} \), we mean a heart of a bounded \( t \)-structure on \( \mathcal{T} \). For a given stability condition \( (Z, \mathcal{P}) \), \( \mathcal{P}((r, r+1]) \) is a heart for any \( r \in \mathbb{R} \), since a heart can be characterized as follows.

Lemma 2.2. [4, Lemma 3.2] Let \( \mathcal{A} \subset \mathcal{T} \) be a full additive subcategory of a triangulated category \( \mathcal{T} \). Then \( \mathcal{A} \) is a heart of \( \mathcal{T} \) if and only if the following two conditions hold:

(a) if \( k_1 > k_2 \) are integers and \( A, B \in \mathcal{A} \) then \( \text{Hom}_\mathcal{T}(A[k_1], B[k_2]) = 0 \);

(b) for every nonzero object \( E \in \mathcal{T} \) there is a finite sequence of integers \( k_1 > k_2 > \cdots > k_n \) and a collection of triangles

\[
\begin{array}{cccccc}
0 & \rightarrow & E_0 & \rightarrow & E_1 & \rightarrow & E_2 & \rightarrow & \cdots & \rightarrow & E_{n-1} & \rightarrow & E_n = E \\
& & \kappa & & \kappa & & \kappa & & \cdots & & \kappa & & \\
& & \downarrow A_1 & & \downarrow A_2 & & \downarrow A_n & & & & & & \\
& & \end{array}
\]

with \( A_j \in \mathcal{A}[k_j] \) for all \( j \).

The main relation between hearts and stability conditions is the following.

Proposition 2.3. [4, Proposition 5.3] To give a stability condition on a triangulated category \( \mathcal{T} \) is equivalent to giving a heart on \( \mathcal{T} \) and a centered slope-function on it with the Harder-Narasimhan property.

Here, a centered slope-function \( Z \) on a heart \( \mathcal{A} \) is a group homomorphism \( Z : K(\mathcal{A}) \rightarrow \mathbb{C} \), such that for \( 0 \neq E \in \mathcal{A} \), \( Z(E) \) lies in \( H \overset{\text{def}}{=} \{ r \exp(i\pi\phi) : r > 0 \text{ and } 0 < \phi < \pi \} \).
\( \phi \leq 1 \} \subset \mathbb{C} \). We define the \textit{slope} of \( E \neq 0 \), denoted by \( \phi(E) \), to be

\[ \phi(E) = \frac{1}{\pi} \arg Z(E) \in (0, 1]. \]

Object \( 0 \neq E \in \mathcal{A} \) is called \textit{semistable} if for any subobject \( 0 \neq A \subset E \) \( \phi(A) \leq \phi(E) \). Moreover, \( Z \) is said to have the \textit{Harder-Narasimhan property} (or \textit{HN-property} for short), if for every nonzero object \( E \in \mathcal{A} \) there is a finite short exact sequences (which we draw as triangles) in \( \mathcal{A} \)

\[
\begin{array}{cccccccc}
0 & \rightarrow & E_0 & \rightarrow & E_1 & \rightarrow & E_2 & \rightarrow & \cdots & \rightarrow & E_{n-1} & \rightarrow & E_n = E \\
& & \kappa & & \kappa & & \kappa & & \kappa & & \kappa & & \kappa \\
& & A_1 & & A_2 & & \cdots & & A_{n-1} & & A_n &
\end{array}
\]

with \( \phi(A_i) > \phi(A_j) \) for \( i < j \).

Proposition 2.3 says that a centered slope-function on a heart with the HN-property can be extended to a stability condition. Notice that for a heart \( A \) of \( \mathcal{T} \), we always have \( K(A) = K(\mathcal{T}) \). So, such an extension is achieved just by setting the central charge to be the same to the centered slope-function, and by setting the slice \( \mathcal{P}(\psi + k) \), for \( \psi \in (0, 1] \) and \( k \in \mathbb{Z} \), to be the full additive subcategory of \( \mathcal{T} \) consisting of \( E[k] \) with \( \phi(E) = \psi \). A non-zero object \( E \in \mathcal{P}(\phi) \) for each \( \phi \in \mathbb{R} \) is also called semistable.

Let us see how it works in our example \( \text{Coh} \mathbb{P}^1 \). Notice that we have to require \( Z(\mathcal{O}_x) \in \mathbb{R}_{<0} \) and \( Z(\mathcal{O}) \in H \setminus \mathbb{R}_{<0} \) in order to have \( Z(\text{Coh} \mathbb{P}^1) \subset H \). Let us graphically present a centered slope-function \( Z \) on the standard heart \( \text{Coh} \mathbb{P}^1 \) by the following figure.

![Figure 1. A centered slope-function \( Z \) on \( \text{Coh} \mathbb{P}^1 \)](image)

Then the second filtration in 2.1 proves that \( Z \) has the HN-property, with all the line bundles and the torsion sheaves semistable. Hence \( Z \) extends to a stability condition \( (Z, \mathcal{P}) \).

2.4. Wall crossing and rotation. Let \( K \) be the \textit{Kronecker quiver} (\( \cdot \rightarrow \cdot \)), and \( \text{Rep}(K) \) be the category of the representations of the quiver. We will construct Beilinson’s \textit{Kronecker heart} \( \mathcal{B} \) in \( D(\mathbb{P}^1) \) (see [1]), and some stability conditions with the heart \( \mathcal{B} \). We will use two operations on \( \text{Stab}(\mathcal{T}) \): \textit{rotation} affects the heart but preserves the semistable objects, while \textit{wall crossing} fixes the heart and affects semistable objects.

For \( S \subset \mathcal{T} \), we let \( \langle S \rangle \) be the full subcategory generated under extension by objects in \( S \). \textit{Rotation} and \textit{rescaling} of a stability condition are the imaginary and the real parts of the \( \mathbb{C} \)-action below (it will be discussed in Proposition 4.1).

\[
\begin{array}{cccc}
\mathcal{O}(2) & \mathcal{O}(1) & \mathcal{O} \\
\mathcal{O}(n) & \mathcal{O}(n) & \mathcal{O}(n) \\
\mathcal{O}_x & \mathcal{O}_x & \mathcal{O}_x \\
\end{array}
\]
Definition 2.3. Let \((Z, P)\) be a stability condition and \(z = x + iy \in \mathbb{C}\). Then \(z \ast (Z, P)\) is defined to be \(z \ast Z = e^z Z\) and \((z \ast P)(\phi) = P(\phi - y/\pi)\).

Proposition 2.4. Let \((Z, P)\) be a stability condition as in the end of the Section 2.3. Then rotation by \(z = -i\phi(O)\) gives a stability condition \(z \ast (Z, P) = (Z, P)\) such that the heart \(\mathcal{P}((0, 1])\) is equivalent to \(\text{Rep}(\mathcal{K})\) and all line bundles and torsion sheaves are semistable.

Proof. We easily check that \(\mathcal{P}((0, 1]) = \langle O, O(-1)[1]\rangle\), since \(\mathcal{P}(O(-1)) = 0\). Let \(P = O \oplus O(1)\). The functor \(R\text{Hom}(P, -)\) sends \(O\) and \(O(-1)[1]\) to the irreducible generators of \(\text{Rep}(\mathcal{K})\), and gives the equivalence between the categories \(\mathcal{P}((0, 1])\) and \(\text{Rep}(\mathcal{K})\) (see [2, Section 6]).

Now we fix the heart \(\mathcal{B}\) and vary the centered slope-function \(\tilde{Z}\) on it. Since \(\mathcal{B}\) has finite-length objects, centered slope-functions with HN-property are defined by any choice of the values \(\tilde{Z}(O), \tilde{Z}(O(-1)[1]) \in H\), on irreducible objects \(O, O(-1)[1]\) in \(\mathcal{B}\).

Proposition 2.5. If \(\tilde{\phi}(O) < \tilde{\phi}(O(-1)[1])\), then we get \(\tilde{Z}\) above. If \(\tilde{\phi}(O) = \tilde{\phi}(O(-1)[1])\), then any non-zero object in \(\mathcal{P}((0, 1])\) is semistable. If \(\tilde{\phi}(O) > \tilde{\phi}(O(-1)[1])\), then the multiples of \(O\) and \(O(-1)[1]\) are the only semistable objects in \(\mathcal{P}((0, 1])\).

Proof. If \(\tilde{\phi}(O) = \tilde{\phi}(O(-1)[1])\), then all non-zero objects in \(\mathcal{B} = \langle O, O(-1)[1]\rangle\) lie in the same slope.

Consider the last case. We see that \(O(n)\) is not semistable for \(n > 0\), since we have the triangle \(O \rightarrow O(n) \rightarrow I\) for some torsion sheaf \(I\) and \(\tilde{\phi}(I) > \tilde{\phi}(O(n))\). Similarly, \(O(-n - 1)[1]\) for \(n > 0\) is not semistable because of the triangle \(I \rightarrow O(-n - 1)[1] \rightarrow O(-1)[1]\) for some torsion sheaf \(I\), and any torsion sheaf is not semistable because of the triangle \(O \rightarrow O_x \rightarrow O(-1)[1]\) for any \(x\).

T. Bridgeland defined wall, as a codimension-one submanifold of a stability manifold such that as one varies a stability condition, a semistable object can only become non-semistable if one crosses a wall (see [5, Section 8]). From Proposition 2.5, \(W = \{(Z, P) \in \text{Stab}(D(P^1)) \mid \phi(O) = \phi(O(-1)[1])\}\) is a wall.

3. Stability conditions for \(D(P^1)\)

The remark after [3, Theorem 3.1] says that \(\text{Aut}(D(P^1)) \cong \text{Aut} P^1 \times (\text{Pic}(P^1) \oplus \mathbb{Z})\), where \(\mathbb{Z}\) is generated by the shift \([1]\). We frequently use \(\text{Pic}(P^1) \oplus \mathbb{Z} \cong \mathbb{Z} \oplus \mathbb{Z}\), while \(\text{Aut} P^1\) acts trivially on \(\text{Stab}(D(P^1))\).

One way to obtain Theorem 1.2 is to classify the centered slope-functions with HN-property for each heart in the list provided by [11, Theorem 6.12]. However, for our purpose it is simpler to use the following lemma.

Lemma 3.1. [11, Lemma 6.6]
(a) For \(n \in \mathbb{Z}\), there exist exact triangles
\[
(1) \quad O(k + 1) \oplus_{n\neq k} \rightarrow O(n) \rightarrow O(k) \oplus_{n\neq k-1} [1] \quad \text{if } n > k + 1,
(2) \quad O(k + 1) \oplus_{n\neq k} [1] \rightarrow O(n) \rightarrow O(k) \oplus_{n\neq k+1} \quad \text{if } n < k.
\]
(b) For \(x \in P^1\) and \(k \in \mathbb{Z}\), there exists an exact triangle
\[
(3) \quad O(k + 1) \rightarrow O_x \rightarrow O(k)[1].
\]
(c) Any triangle $A \to M \to B$ with $\text{Ext}^\leq_0(A, B) = 0$ and $M$ either $\mathcal{O}(n)$ or $\mathcal{O}_x$ is in the form of $(a)$ or $(b)$.

(d) If some line bundle or torsion sheaf is not semistable, then there exist $k, n \in \mathbb{Z}$ such that the shifts of $\mathcal{O}(k)$ and $\mathcal{O}(k + 1)$ are semistable and the triangle found in $(a)$ or $(b)$ is the HN-filtration.

Proof. For (a)-(c), see the proof after [11, Remark 6.8]. For (d), see the proof of the semistability of $\mathcal{O}(k)$ and $\mathcal{O}(k + 1)$ after the statement of [11, Lemma 6.6]. □

**Corollary 3.2.** If there exists $k$ such that $\mathcal{O}(k)$ and $\mathcal{O}(k + 1)$ are semistable and $\phi(\mathcal{O}(k + 1)) > \phi(\mathcal{O}(k)[1])$, then no line bundle or torsion sheaf is semistable except $\mathcal{O}(k)$ and $\mathcal{O}(k + 1)$.

Proof. If $\mathcal{O}(n)$ for $n > k + 1$ is semistable, then $\mathcal{O}(n)$ itself is the HN-filtration, but the triangle (1) is also the HN-filtration, since $\mathcal{O}(k + 1)$ and $\mathcal{O}(k)[1]$ are semistable and the cone phases are decreasing by the assumption. So, by the uniqueness of the HN-filtrations, $\mathcal{O}(n)$ cannot be semistable. Likewise, $\mathcal{O}(n)$ for $n < k$ and torsion sheaf are not semistable, because of the triangle (2) and the triangle (3) respectively. □

**Proof of Theorem 1.2.** Consider the case when there exists some non-semistable line bundle. Lemma 3.1 (d) says that there is a line bundle $\mathcal{O}(n)$ with the HN-filtration

$$\mathcal{O}(k + 1)^{\oplus p}[j] \to \mathcal{O}(n)$$

for some $k \in \mathbb{Z}$, $j \in \{0, 1\}$, $p, s > 0$. The decreasing property of the phases the cones of the HN-filtration implies

$$\phi(\mathcal{O}(k + 1)) > \phi(\mathcal{O}(k)[1]).$$

Hence, Corollary 3.2 says that up to shifts $\mathcal{O}(k)$ and $\mathcal{O}(k + 1)$ are the only semistable objects in $D(\mathbb{P}^1)$; every object in $D(\mathbb{P}^1)$ can be decomposed into a direct sum of some shifts of line bundles and torsion sheaves, since the homological dimension of $\text{Coh} \mathbb{P}^1$ is one. After tensoring with $\mathcal{O}(-k - 1)$, the inequality (4) gives $\phi(\mathcal{O}) > \phi(\mathcal{O}(-1)[1])$ and hence

$$\phi(\mathcal{O}(-1)[p]), \phi(\mathcal{O}) \in (r, r + 1]$$

for some $r \in \mathbb{R}$ and $p > 0$.

Consider the case when all line bundles are semistable. Since there are more than two line bundles that are semistable, Corollary 3.2 says that all torsion sheaves have to be semistable, and in particular

$$\phi(\mathcal{O}(k + 1)) \leq \phi(\mathcal{O}(k)[1])$$

for any $k$.

Let us see that there exists $k$ such that

$$\phi(\mathcal{O}(k)[1]) - 1 < \phi(\mathcal{O}(k + 1)).$$

If not, $\phi(\mathcal{O}(k)[1]) - 1 \geq \phi(\mathcal{O}(k + 1))$ for all $k$; i.e., $\phi(\mathcal{O}(k)[1]) \geq \phi(\mathcal{O}(k + 1)[1])$ for all $k$. Since $\text{Hom}(\mathcal{O}(k)[1], \mathcal{O}(k + 1)[1]) \neq 0$, $\phi(\mathcal{O}(k)[1]) = \phi(\mathcal{O}(k + 1)[1])$ for all $k$ by Property (c) in Definition 2.1. However, the phases of the line bundles cannot
be the same, since \( Z(O(k+1)) - Z(O(k)) = Z(O_x) \neq 0 \) for all \( k \) and they cannot
be on the same ray. Therefore, the inequality (6) holds for some \( k \), and
\[
0 \leq \phi(O(-1)[1]) - \phi(O) < 1
\]
up to \( \text{Aut}(\mathbb{P}^1) \). \( \Box \)

Theorem 1.2 only tells us necessary conditions for the stability conditions in
\( \text{Stab}(\mathbb{D}(\mathbb{P}^1)) \), but Proposition 3.3 says all cases in the theorem actually exist and
reveals all hearts on which we can define a centered slope-function for each case.

**Proposition 3.3.** For any \( \alpha, \beta \in \mathbb{R} \) such that \( \alpha > \beta - 1 \), and any \( m_\alpha, m_\beta \in \mathbb{R}_{>0} \),
there exists a unique stability condition \( (Z, \mathcal{P}) \) such that \( \phi(O(-1)[1]) = \beta \)
and \( \phi(O) = \alpha \), and \( Z(O) = m_\alpha e^{i\pi \alpha} \) and \( Z(O(-1)[1]) = m_\beta e^{i\pi \beta} \).

Furthermore, we have the following cases:

1. if \( \alpha > \beta \), then for any \( r \in \mathbb{R} \), there exist \( p, q \in \mathbb{Z} \) such that \( \mathcal{P}((r-1, r]) = \langle O(-1)[p+1], O[q] \rangle \) and \( p - q \in (\alpha - \beta - 1, \alpha - \beta + 1) \);
2. if \( \alpha < \beta \), then for any \( r \), there exist \( i, j \in \mathbb{Z} \) such that \( \mathcal{P}((r-1, r]) = \langle O(i-1)[1+j], O(i)[j] \rangle \) and \( \phi(O(i-2)[1+j]) > r \geq \phi(O(i-1)[1+j]) \), or there exists \( j \in \mathbb{Z} \) such that \( \mathcal{P}((r-1, r]) = \text{Coh} \mathbb{P}^1[j] \) and \( r = \phi(O_x[j]) \);
3. if \( \alpha = \beta \), then for any \( r \), there exists \( j \) such that \( \langle O(-1)[1+j], O[j] \rangle = \mathcal{P}((r-1, r]) \).

**Proof.** In all cases, for any \( k \in \mathbb{Z} \), \( \mathcal{P} \) has to satisfy \( \mathcal{P}(\alpha + k) = \langle O[k] \rangle \), \( \mathcal{P}(\beta + k) = \langle O(-1)[1+k] \rangle \).

Consider the first case. By Theorem 1.2, the multiples of the shifts of \( O \) and
\( O(-1)[1] \) are the only semistable objects. So we have to put \( \mathcal{P}(\phi) = 0 \) for all \( \phi \in \mathbb{R} \)
except when \( \phi = \alpha + k \) or \( \beta + k \) for some \( k \in \mathbb{Z} \). Then \( (Z, \mathcal{P}) \in \text{Stab}(\mathbb{D}(\mathbb{P}^1)) \); (c)
in Lemma 3.1 implies (c) in Definition 2.1: the triangles (1), (2), and (3) are
the HN-filtrations for all summands of each object in \( \mathbb{D}(\mathbb{P}^1) \). If \( \alpha - \beta \in \mathbb{Z} \), for any \( r \in \mathbb{R} \),
there exists \( p, q \in \mathbb{Z} \) such that \( \mathcal{P}((r-1, r]) = \langle O(-1)[p+1], O[q] \rangle \) and \( p - q = \alpha - \beta \).
If \( n \not\in \mathbb{Z} \), then there exists \( p, q \in \mathbb{Z} \). \( \mathcal{P}((r-1, r]) = \langle O(-1)[1+p], O[q] \rangle \) and \( p - q \) is
either the smallest or the largest possible integer to \( \alpha - \beta \).

Consider the next case. By Theorem 1.2, all line bundles and torsion sheaves are
semistable. So for all \( \phi \in (0, 1) \) and \( k \in \mathbb{Z} \), we have to put \( \mathcal{P}(\phi + k) = \langle O(n)[k] \rangle \) when \( Z(O(n)) = m(O(n))e^{i\pi \phi} \) for some \( n > 0 \) and \( m(O(n)) \in \mathbb{R}_{>0} \), \( \mathcal{P}(\phi + k) = \langle O(-n-1)[1+k] \rangle \) when \( Z(O(-n-1)) = m(O(-n-1))e^{i\pi \phi} \) for some \( n > 0 \) and \( m(O(-n-1)) \in \mathbb{R}_{>0} \), \( \mathcal{P}(\phi + k) = \langle O_x[k] \mid x \in \mathbb{P}^1 \rangle \) when \( Z(O_x) = m(O_x)e^{i\pi \phi} \) for some \( x \in \mathbb{P}^1 \) and \( m(O_x) \in \mathbb{R}_{>0} \), and \( \mathcal{P}(\phi + k) = 0 \) for the other cases. Notice that \( Z(O_x) = m(O_x)e^{i\pi \psi} \) for some \( \beta > \psi > \alpha \) and \( m(O_x) \in \mathbb{R}_{>0} \) because of the triangle
\( O \to O_x \to O(-1)[1] \). So for \( z = e^{i\pi(1-\phi)} \), \( z \ast Z \) gives us a centered slope-
function with HN-property in 2.3. Hence we have \( z \ast (Z, \mathcal{P}) \in \text{Stab}(\mathbb{D}(\mathbb{P}^1)) \), and
\( (Z, \mathcal{P}) \in \text{Stab}(\mathbb{D}(\mathbb{P}^1)) \).

Finally, let \( \mathcal{P}(\alpha) = \langle O(-1)[1], O \rangle \). Then \( (Z, \mathcal{P}) \in \text{Stab}(\mathbb{D}(\mathbb{P}^1)) \) by Proposition
2.5. \( \Box \)

**Corollary 3.4.** The hearts on which we can impose a centered slope-function with
HN-property are \( \mathcal{C}_i \) def = \( \text{Coh} \mathbb{P}^1[j] \) and \( \mathcal{C}_{p,i,j} \) def = \( \langle O(i-1)[p+j], O(i)[j] \rangle \) for all \( i, j \in \mathbb{Z} \)
and \( p > 0 \).

**Proof.** The hearts found in Proposition 3.3 are \( \langle O(i-1)[p], O(i)[q] \rangle \) for \( i, j \in \mathbb{Z} \) and
any \( p, q \in \mathbb{Z} \) such that \( p - q > 0 \), and \( \text{Coh} \mathbb{P}^1[j] \) for any \( j \). Any heart on which we
can put a centered slope-function with HN-property lives in their orbits under the action of Aut(D(P^1)) by Theorem 1.2.

\begin{remark}
  The subcategory \( \mathcal{A} = \langle \mathcal{O}_x, x \in P^1; \mathcal{O}(n)[1], n \in \mathbb{Z} \rangle \) is a heart because it is the image of Coh \( P^1 \) under the Grothendieck duality functor \( \mathbb{D}(-) = R\text{Hom}(-, \omega_{P^1})[1] \). However, it carries no centered slope-function with HN-property by Corollary 3.4. This shows that the notion of stability is not invariant under passing from \( \mathcal{T} \) to \( \mathcal{T}^{opp} \).

For the one in Definition 2.1 right stability conditions, there is a notion of left stability conditions on \( \mathcal{T}^{opp} \), and we have such left stability conditions on \( \mathcal{A} \subset D(P^1)^{opp} \).

For any \( P \subset P^1 \), a subcategory \( \mathcal{A}(P) = \langle \mathcal{O}_x, x \in P; \mathcal{O}_y[1], y \notin P; \mathcal{O}(n)[1], n \in \mathbb{Z} \rangle \) is a heart (notice that \( \mathcal{A}(P) = \mathcal{A} \)). Up to \( \text{Aut}(D(P^1)) \), all hearts that do not bear any centered slope-function with HN-property are \( \mathcal{A}(P) \) by Corollary 3.4 and [11, Theorem 6.12].

\section{Stability Manifold}

\subsection{Quotient \( \widetilde{\text{Stab}}(\mathcal{T})/\mathbb{C} \)}

T. Bridgeland observes that \( \tilde{\text{GL}}^+(2, \mathbb{R}) \) (the universal covering of \( \text{GL}^+(2, \mathbb{R}) \)) acts on \( \text{Stab}(\mathcal{T}) \) for any triangulated category \( \mathcal{T} \) ([4, Lemma 8.2]). We notice that the \( \mathbb{C} \)-action in Definition 2.3 is a holomorphic part of this action.

\begin{proposition}
The \( \mathbb{C} \)-action is holomorphic, free, coincides with the action of a subgroup of \( \tilde{\text{GL}}^+(2, \mathbb{R}) \), and contains the shifts. The quotient \( \widetilde{\text{Stab}}(\mathcal{T})/\mathbb{C} \) is a complex manifold, modeled on a projective space of a topological vector space.
\end{proposition}

\begin{proof}
By Theorem 2.1, holomorphicity follows from the holomorphicity of the \( \mathbb{C} \)-action on a vector space via multiplication by \( e^z \). The stabilizers are trivial since \( z \cdot z = Z \) implies \( e^z = 1 \), so \( x = 0 \), while \( z \cdot \mathcal{P} = \mathcal{P} \) gives \( y = 0 \). Notice that the \( \mathbb{C} \)-action by \( z \) has the same effect as the action of \( (e^{-z}A, y) \in \tilde{\text{GL}}^+(2, \mathbb{R}) \) on a stability condition (notation from [4, Lemma 8.2]), where \( A \) is the rotation by the angle \( -\pi y \). The shift \([1]\) can be realized as the action of \( i\pi \in \mathbb{C} \). The action of \( \mathbb{C} \) on the manifold \( \text{Stab}(\mathcal{T}) \) is point-wise free, and locally isomorphic to the \( \mathbb{C}^* \)-action on \( V(z)\setminus \{0\} \) (notation from Theorem 2.1). This implies that \( \text{Stab}(\mathcal{T})/\mathbb{C} \) is a manifold as claimed.
\end{proof}

\subsection{Quotient \( \text{Stab}(D(P^1))/\mathbb{Z}\mathbb{C} \)}

We will denote by \( \mathbb{Z} \) the copy of \( \mathbb{Z} \) that acts on \( D(P^1) \) by the tensoring with line bundles.

In Lemma 4.2, we show that Theorem 1.2 gives a domain \( X \) that contains a fundamental domain of \( \text{Stab}(D(P^1))/\mathbb{Z}\mathbb{C} \), but we still need Lemma 4.3 to shrink \( X \) so that we avoid repetitions with respect to the action of \( (\mathbb{Z})\mathbb{C} \).

\begin{lemma}
Let \( X \) be the subset of \( \text{Stab}(D(P^1)) \) consisting of all stability conditions \( (Z, \mathcal{P}) \) with the following properties: (a) \( \mathcal{O}(-1)[1], \mathcal{O} \) are semistable; (b) \( \phi(\mathcal{O}(-1)[1]) = 1 \) and \( m(\mathcal{O}(-1)[1]) = 1 \); (c) \( \phi(\mathcal{O}) > 0 \). Then \( (\mathbb{Z})\mathbb{C} \cdot X = \text{Stab}(D(P^1)) \), and \( X \) is isomorphic to the open upper half-plane \( \mathbb{H} \), an isomorphism is given by \( \log(m(\mathcal{O})) + i\pi \phi(\mathcal{O}) : X \cong \mathbb{H} \).
\end{lemma}

\begin{proof}
Up to \( \text{Aut}(D(P^1)) \), Theorem 1.2 says that, for each stability condition \( (Z, \mathcal{P}) \), we have \( r \in \mathbb{R} \) such that \( \mathcal{O}(-1), \phi(\mathcal{O}) \) are semistable and the slope are in \( (r, r + 1] \). By the action of \( \mathbb{C} \), we can assume \( \phi(\mathcal{O}(-1)[1]) = 1 \) and \( m(\mathcal{O}(-1)[1]) = 1 \). So \( (r, r + 1] \ni \phi(\mathcal{O}(-1)[p]) = p \geq 1 \) forces \( r > 0 \), hence \( \phi(\mathcal{O}) > 0 \).
\end{proof}
The slope and the length of $\mathcal{O}(-1)[1]$ are fixed by (b). So each $(Z, \mathcal{P}) \in X$ in the stability manifold can be uniquely represented by $Z(\mathcal{O})$ on the $n$-th sheet of the Riemann surface of $\log z$, where $n$ is the greatest integer such that $\phi(\mathcal{O})/2 \geq n$. Here $m(\mathcal{O}), \phi(\mathcal{O}) > 0$ by (a) and (c).

**Lemma 4.3.** A fundamental domain of $\text{Stab}(\mathbb{P}^1)/\mathbb{Z}\mathbb{C}$ is isomorphic to $K \overset{\text{def}}{=} \{ x + iy \in \mathbb{C} \mid y > 0, \cos y \geq e^{-|x|} \}$ as in the shaded domain in the figure below. When passing to $\text{Stab}(\mathbb{P}^1)/\mathbb{Z}\mathbb{C}$ one identifies points on the boundary that have the same imaginary part.

![Figure 2. A fundamental domain of $\text{Stab}(\mathbb{P}^1)/\mathbb{Z}\mathbb{C}$](image_url)

**Proof.** Notice that the actions by line bundles and by $\mathbb{C}$ commute. Let $(Z, \mathcal{P}) \in X$. First, we will see that if $\phi(\mathcal{O}) > 1$, there are no repetitions; i.e., $\mathbb{C}(Z) \cdot (Z, \mathcal{P}) \cap X = \{(Z, \mathcal{P})\}$. By Theorem 1.2, all indecomposable semistable objects are shifts of $\mathcal{O}(-1)$ and $\mathcal{O}$. The action of $\mathcal{O}(i) \cdot (x + i\pi \psi) \in (\mathbb{Z}\mathbb{C})$ changes $(Z, \mathcal{P})$ into $(Z', \mathcal{P}')$ such that $\mathcal{P}'(\phi(\mathcal{O}) + \psi) = \mathcal{O}(i\mathcal{P})$ and $\mathcal{P}'(\phi(\mathcal{O}(-1)[1]) + \psi) = \mathcal{O}(-1 + i)[1]$. Hence, $\mathcal{O}$ and $\mathcal{O}(-1)[1]$ are not semistable unless $i = 0$, and even if $i = 0$ we have $\phi'(\mathcal{O}(-1)[1]) \neq 1$ or $m'(\mathcal{O}(-1)[1]) \neq 1$ unless $\phi = x = 0$.

In the remaining case $0 < \phi(\mathcal{O}) \leq \phi(\mathcal{O}(-1)[1]) = 1$; repetitions $\mathbb{C}(\mathcal{Z}) \cdot (Z, \mathcal{P}) \cap X$ are indexed by $Z \ni 1 \mapsto (Z_i, \mathcal{P}_i) = (z_i, \mathcal{O}(i)) \cdot (Z, \mathcal{P})$. Here, $z_i = \frac{1}{m(\mathcal{O}(-1)[1])} + i\pi(1 - \phi(\mathcal{O}(i - 1)[1]))$. Let us denote $(Z_i, \mathcal{P}_i) = \mathcal{O}(i) \cdot (Z, \mathcal{P})$; i.e.,

$$Z_i(\mathcal{O}(-1)[1]) = Z(\mathcal{O}(i - 1)[1]), \quad \phi_i(\mathcal{O}(-1)[1]) = \phi(\mathcal{O}(i - 1)[1]),$$

$$\hat{Z}_i(\mathcal{O}(i)) = \mathcal{O}(i), \quad \hat{\phi}_i(\mathcal{O}) = \phi(\mathcal{O}(i)).$$

We have $z_i \ast (Z_i, \mathcal{P}_i) \in X$, since $1 - (z_i \ast \phi_i)(\mathcal{O}) = \phi(\mathcal{O}(i - 1)[1]) - \phi(\mathcal{O}(i)) < 1$ implies $(z_i \ast \phi_i)(\mathcal{O}) \in (0, 1)$. Graphically we can explain actions above as follows.
Hence, the question is how to pick one \((Z_i, P_i)\). The first step is to require \((\dot{Z}_i, \dot{P}_i)\) to have the least possible \(\dot{\phi}_i\); it is easy to see from simple plane geometry that the minimality is achieved exactly when \(\dot{Z}_i(O(-1)[1])\) and \(\dot{Z}_i(O)\) are in the strip \(S_Z\) bounded by the two lines that are perpendicular to \(Z(O_x)\) and contain the initial or end point of \(Z(O_x)\). For example in Figures 3–5 \(S_Z\) is indicated by the dotted lines. Let us show the existence of such \((\dot{Z}_i, \dot{P}_i)\). Notice that \(\dot{Z}_i(O(-1)[1]) \in S_Z\) if and only if \(\dot{Z}_i(O) \in S_Z\), since \(\dot{Z}_i(O(-1)[1]) + \dot{Z}_i(O) = \dot{Z}_i(O_x) = Z(O_x)\). Moreover, we have some \(i\) such that \(\dot{Z}_i(O(-1)[1]) \in S_Z\), since \(\dot{Z}_i(O(-1)[1]) - \dot{Z}_{i-1}(O(-1)[1]) = Z(O_x)\).

Graphically, the minimality is achieved precisely when \(Z(O)\) is in the shaded region; \(Z(O(-1)[1]), Z(O) \in S_Z\) if and only if \(Z(O) = x + iy\) with \(x \leq 1\) and \((x - 1/2)^2 + y^2 \geq 1/4\).

The remaining repetitions occur precisely when one of \(Z(O(-1)[1])\) and \(Z(O)\) makes a right angle with \(Z(O_x)\); i.e., \((Z_{-1}, P_{-1}) \in X\) and \(Z_{-1}(O(-1)[1]) \in S_Z\), or \((Z_1, P_1) \in X\) and \(Z_1(O(-1)[1]) \in S_Z\). So two boundary points are identified when a ray from the origin connects them; when we have a right triangle between
STABILITY MANIFOLD OF $P^1$

$Z(O_x)$ and $Z(O)$ as $Z(O) = 1 + iy$ for some $y > 0$, then $Z_1(O) = \frac{1}{1+y^2}(1+iy)$, since $\dot{Z}_1(O(-1)[1]) = -1 + iy$ and $\dot{Z}_1(O) = 1$. This situation can be presented graphically in the following figures.

**Figure 7.** $Z$

**Figure 8.** $Z_1$

Let $X' \subset X$ consist of all $(Z, P) \in X$ such that $Z(O) \in S_Z$. Then $\log(m(O)) + i\pi\phi(O)$ gives us an isomorphism between $X'$ and $K$; we identify two boundary points with the same imaginary value in $K$, since in Figure 6 they correspond to the boundary points that can be connected by a ray from the origin.

**Lemma 4.4.** $\text{Stab}(P^1)/(Z)C$ is conformally equivalent to $C^*$.

**Proof.** This follows from Riemann mapping theorem and Reflection principle. We use notation from Figure 2; the origin and the infinite point are called $A$ and $B$, the boundary lines $e^{\pm x} \cos y = 1$ are called $L_{\pm}$, and the upper imaginary axis by $L$.

By $z \mapsto \frac{z}{1-z}$, $K$ is conformally equivalent to the subdomain in the unit disk ($|z| < 1$) bounded by $L_{\pm}$. Here, two boundary points of $K$ with the same imaginary value go to the boundary points with the same real value. Let us denote the upper-half and lower-half of $K$ by $K_u$ and $K_l$. See Figure 9.

Next, by the Riemann mapping theorem there is a bijective conformal mapping from $K_u$ to the unit disk. We can extend any isomorphism of bounded domains to a homeomorphism on their closures by [7, Theorem 11-1]. By a linear fractional transformation, we can rearrange three points on the boundary in arbitrary way as long as we keep their order. Hence $K_u$ is conformally equivalent to the unit disk where $A$ and $B$ correspond to $-1, 1$, and the upper-half circle and the lower-half circle correspond $L_-$ and $L$. See Figure 10.

**Figure 9.** $K$ in the unit disk

**Figure 10.** $K_u$ as the unit disk
Next, by the composition of \( z \mapsto -2i \left( \frac{z+1}{z-1} \right), z \mapsto \frac{z+\sqrt{z^2-1}}{2}, \) and \( z \mapsto \frac{1}{z} \), \( K_u \) as the unit disk is conformally equivalent to the lower-half disk, where \( B \) and \( A \) correspond \(-1\) and \(1\), and the lower-half circle corresponds to \( L_\ldots \). See Figure 11.

![Figure 11. \( K_u \) as the half disk](image)

![Figure 12. \( K \) as the unit disk](image)

Now, by the Reflection Principle we can extend the bijective conformal mapping from \( K \) to the unit disk, where the two points on \( L_\ldots \) with the same imaginary part are mapped to two points on the boundary on the unit disk with the same real part.

Finally, \( z \mapsto -i(\frac{z+1}{z-1}) \) sends \( K \) as the unit disk to the upper-half plane, where two points on the boundary with the same real value are mapped to two points on the boundary with the same absolute value. Then, \( z \mapsto z^2 \) sends \( K \) as the upper-half plane to \( \mathbb{C}^* \) and identifies on the real axis.

**Proof of Theorem 1.1.** By Lemma 4.4, we have \( \text{Stab}(D(\mathbb{P}^1))/\langle \mathbb{Z} \rangle \mathbb{C} \cong \mathbb{C}^* \). The action of \( \mathbb{Z} \) on \( \mathcal{X} = \text{Stab}(D(\mathbb{P}^1))/\mathbb{C} \) gives an exact sequence \( 0 \to \pi_1(\mathcal{X}) \to \pi_1(\mathcal{X}/\mathbb{Z}) \to \pi_0(\mathbb{Z}) \to \pi_0(\mathcal{X}) \). We will show that \( \mathcal{X} \) is connected. Recall that \( \text{Stab}(D(\mathbb{P}^1)) = (\mathbb{Z})\mathbb{C} \cdot X \) and, by Lemma 4.2, \( X \cong \mathbb{H} \) is connected, hence so is \( \mathbb{C} \cdot X \). It remains to check that \( \mathbb{Z} \) fixes some connected component of \( \text{Stab}(D(\mathbb{P}^1)) \), but it fixes \( \{ (Z, \mathcal{P}) \in \text{Stab}(D(\mathbb{P}^1)) \mid \mathcal{P}(\langle 0, 1 \rangle) = \text{Coh}\mathbb{P}^1 \} \), which lives in \( X \). So \( \mathcal{X} \) is connected and the map \( \alpha \) is a surjective map \( Z \to \mathbb{Z} \), therefore \( \alpha \) is injective and \( \pi_1(\mathcal{X}) = 0 \). Hence \( \mathcal{X} \) is the universal covering of \( \mathbb{C}^* \); i.e., \( \text{Stab}(D(\mathbb{P}^1))/\mathbb{C} \cong \mathbb{C} \). Moreover, since \( H^1(\mathcal{X}, \mathcal{O}) = 0 \), \( \text{Stab}(D(\mathbb{P}^1)) \cong \mathbb{C}^2 \).

5. **Walls and hearts of \( \text{Stab}(D(\mathbb{P}^1)) \)**

We define the “cell” \( \text{Stab}_C(\mathcal{T}) \) for a heart \( C \) by \( \text{Stab}_C(\mathcal{T}) = \{ (Z, \mathcal{P}) \in \text{Stab}(\mathcal{T}) \mid \mathcal{P}(\langle 0, 1 \rangle) = C \} \) \( S_C(\mathcal{T}) \) for short). In this section, we describe all cells \( S_C(D(\mathbb{P}^1)) \) and how they fit together. We describe a fundamental domain of \( \text{Stab}(D(\mathbb{P}^1))/\langle \mathbb{Z} \rangle\mathbb{R} \) as a real manifold, this shows that the quotient \( \text{Stab}(D(\mathbb{P}^1))/\langle \mathbb{Z} \rangle\mathbb{R} \) is an open torus.

For the hearts \( C_j \) and \( C_{p,i,j} \) from Corollary 3.4, let \( S_j = S_{C_j} \) and \( S_{p,i,j} = S_{C_{p,i,j}} \). Then they can be described as

\[
S_j = \{ (Z, \mathcal{P}) \in \text{Stab}(D(\mathbb{P}^1)) \mid \phi(O_x[j]) = 1, \ 0 < \phi(O[j]) < 1 \} \cong \mathbb{R}_{>0} \times \mathbb{H}; \\
S_{p,i,j} = \{ (Z, \mathcal{P}) \in \text{Stab}(D(\mathbb{P}^1)) \mid 0 < \phi(O(i-1)[p+j], \phi(O(i)[j]) \leq 1 \} \cong H^2.
\]
For $p \geq 1$ and $i, j \in \mathbb{Z}$, we have the following 4-dimensional manifolds with boundaries:

$$S_{p,i,j}^- = \{(Z, \mathcal{P}) \in S_{p,i,j} \mid \phi(\mathcal{O}(i-1)[p+j]) \geq \phi(\mathcal{O}(i)[j])\};$$

$$S_{p,i,j}^+ = \{(Z, \mathcal{P}) \in S_{p,i,j} \mid \phi(\mathcal{O}(i-1)[p+j]) \leq \phi(\mathcal{O}(i)[j])\};$$

$$P_{i,j} = S_{1,i,j}^- \cup (\cup_{p>0} S_{p,i,j}).$$

For each $S_{p,i,j}^\pm$, we will find which ones are neighbors in the sense that the intersection of their closures is a codimension-one submanifold. See the following graphic descriptions of $S_{p,i,j}$ and $S_{p,i,j}$, and see Figure 1 for $S_0$.

**Figure 13.** $S_{p,i,j}^-$

**Figure 14.** $S_{p,i,j}^+$

When $p = 1$, let us simplify notations as follows:

$$S_{i,j}^- = S_{1,i,j}^-; S_{i,j}^+ = S_{1,i,j}^+; W_{i,j} = S_{i,j}^- \cap S_{i,j}^+;$$

$$l_{i,j} = \{(Z, \mathcal{P}) \in W_{i,j} \mid \phi(\mathcal{O}(i-1)[j+1]) = \phi(\mathcal{O}(i)[j]) = 1\}.$$

**Proposition 5.1.** $W_{i,j}$ is a wall for any $i, j \in \mathbb{Z}$, and there is no other walls.

**Proof.** In $S_{i,j}^-$, all line bundles and torsion sheaves are semistable, but in $S_{i,j}^+$, only two line bundles are semistable. In the fundamental domain of $\text{Stab}(\mathcal{D}(\mathbb{P}^1))$ in Lemma 4.3, we have only one wall $y = \pi$ that is the quotient of the set consisting of $W_{i,j}$ for all $i, j$, \hfill $\square$

Before the neighbors lemma for $S_{i,j}^-$, we need the following technical corollaries from Proposition 3.3.

**Corollary 5.2.** Let $(Z, \mathcal{P})$ be a stability condition such that $\mathcal{O}(i), \mathcal{O}(i-1) \in \mathcal{P}(\alpha)$ for some $i$ and $\alpha$, then all line bundles and torsion sheaves are semistable.

**Proof.** It follows from the case 3 in Proposition 3.3. \hfill $\square$

**Corollary 5.3.** There is no stability condition $(Z, \mathcal{P})$ such that $m(\mathcal{O}(i)[j]) = m(\mathcal{O}(i-1)[j+1])(1+1/n)$ for some $n \in \mathbb{Z} \cup \{\infty\}$ and $\phi(\mathcal{O}(i)[j]) = \phi(\mathcal{O}(i-1)[j+1]) - 1$.

**Proof.** The assumption implies $Z(\mathcal{O}_x[j]) = (-1/n)Z(\mathcal{O}(i-1)[j+1])$ since $Z(\mathcal{O}(i)[j]) + Z(\mathcal{O}(i-1)[j+1]) = Z(\mathcal{O}_x[j])$. Then $Z(\mathcal{O}(i-1-n)[j+1]) = 0$ for $n \neq 0, \infty$, and $Z(\mathcal{O}_x) = 0$ or $\infty$ for otherwise. However, $\mathcal{O}(i)[j+1], \mathcal{O}(i-1)[j+1] \in \mathcal{P}(\phi(\mathcal{O}(i-1)[j+1]))$ implies all line bundles and torsion sheaves are semistable by Corollary 5.2. \hfill $\square$

**Lemma 5.4.** $S_{i,j}^-$ has neighbors: (1) $S_{i+1,j}^-; (2) S_{i-1,j}^-; (3) S_{i,j}^+$. Moreover, in each case we can describe the intersection of their closures as a submanifold with boundaries: (1) $\{(Z, \mathcal{P}) \in S_{i+1,j}^- \mid \phi(\mathcal{O}(i)[j]) = 0\}$ with boundaries $l_{i+n,j}$ for $n \geq 0$ and $l_{i-n,j+1}$ for $n > 0$; (2) $\{(Z, \mathcal{P}) \in S_{i,j}^- \mid \phi(\mathcal{O}(i-1)[j+1]) = 1\}$ with boundaries...
Let $\sigma = (Z, P) \in S_{i,j}^-$. If we rotate $Z(O(i)[j])$ counterclockwise, $\sigma$ will be in $W_{i,j}$ when $\phi(O(i)[j]) = \phi(O(i - 1)[j + 1])$.

If $\phi(O(i - 1)[j + 1]) \neq 1$, we see that $\sigma$ will be in $S_{i+1,j}^-$ when we rotate $Z(O(i)[j])$ clockwise, and $\sigma$ will be in $S_{i-1,j}^-$ when we rotate $Z(O(i - 1)[j + 1])$ counterclockwise.

Consider the case $\phi(O(i - 1)[j + 1]) = 1$. Let us see what happens when we have $\phi(O(j)[j]) = 0$ by rotating $Z(O(i)[j])$ clockwise. We can assume $m(O(i)[j]) \neq m(O(i - 1)[j + 1])(1 + 1/n)$ for any $n \in \mathbb{Z}$ by Corollary 5.3. Let us fix $m(O(i - 1)[j + 1]) = 1$, $i = j = 0$. If $1 - 1/(n - 1) < m(O) < 1 - 1/n$ for some $n > 1$, $m(O_x)$ will be $1 - m(O)$ and $\phi(O_x)$ will be one. Moreover, $\phi(O(-1 + (n - 1))[1])$ and $\phi(O(n - 1))$ will be one, hence $\sigma$ will be in $l_{n-1,0}$. For general $m(O(i - 1)[j + 1])$, $i$ and $j$, $\sigma$ will be in $l_{n-1+i,j}$. On the other hand, if $1 + 1/n < m(O) < 1 + 1/(n - 1)$ for some $n > 0$, then by the same manner, $\sigma$ will be in $l_{n-1,1}$. For general $m(O(i - 1)[j + 1])$, $i$ and $j$, $\sigma$ will be in $l_{n+i,j+1}$. Some examples when $\phi(O)$ becomes zero can be seen in the following figures.
Lemma 5.5. For \( p > 1 \), \( S_{i,j}^{-} \) has neighbors \( S_{p-i,j}^{+}, S_{p-i,j+1}^{+}, \) and \( S_{p-1,i,j}^{+} \). For \( p > 0 \), \( S_{p,i,j}^{+} \) has neighbors \( S_{p-1,i,j}^{-}, S_{p+1,i,j-1}^{-}, \) and \( S_{p+1,i,j}^{-} \). See the figure below for some fixed \( m(\mathcal{O}(i-1)[j]) \) and \( m(\mathcal{O}(i)[j+1]) \).

![Figure 20. Some components connected to \( S_{i,j}^{+} \)](image)

Proof. Let \( \sigma = (Z, P) \in S_{p,i,j}^{-} \). If we rotate \( Z(\mathcal{O}(i)[j]) \) clockwise, then \( \sigma \) will be in \( S_{p,i,j}^{+} \) when \( \phi(\mathcal{O}(i)[j]) = \phi(\mathcal{O}(i-1)[p+j]) \). If we rotate \( Z(\mathcal{O}(i)[j]) \) counterclockwise, \( \sigma \) will be in \( S_{p-1,i,j+1}^{+} \) when \( \phi(\mathcal{O}(i)[j]) = 0 \); i.e., \( \phi(\mathcal{O}(i)[j+1]) = 1 \). If we rotate \( Z(\mathcal{O}(i-1)[p+j]) \) counterclockwise, \( \sigma \) will be in \( S_{p-1,i,j}^{+} \) when \( \phi(\mathcal{O}(i-1)[p+j]) \) is greater than 1; i.e., \( 1 \geq \phi(\mathcal{O}(i-1)[p-1+j]) > 0 \).

Let \( \sigma = (Z, P) \in S_{p,i,j}^{+} \) for \( p > 0 \). By the same manner above, \( \sigma \) will be in \( S_{p+1,i,j-1}^{-} \) by rotating \( Z(\mathcal{O}(i)[j]) \) counterclockwise, and \( \sigma \) will be in \( S_{p+1,i,j}^{-} \) by rotating \( Z(\mathcal{O}(i-1)[p+j]) \) clockwise. □

Proposition 5.6. A fundamental domain of \( \text{Stab}(\mathcal{D}(\mathbb{P}^1))/\langle \mathbb{Z} \rangle \mathbb{R} \) is described by Figure 22. In particular \( \text{Stab}(\mathcal{D}(\mathbb{P}^1))/\langle \mathbb{Z} \rangle \mathbb{R} \) is isomorphic to an open torus.

Proof. Let us see how \( S_{n,j}^{-} \) and \( P(n, j) \) for all \( n \in \mathbb{Z} \), and \( S_{j} \) and \( S_{j+1} \) fit together. Let \( B_{i,j} \) be the union of them. We attach \( S_{i+1,j}^{-} \) to the front-side of \( S_{i,j}^{-} \), \( S_{i-1,j}^{-} \) to the backside of \( S_{i,j}^{-} \), and \( P(i,j) \) to \( W_{i,j} \). Then we get Figure 21, where we attach \( S_{j} \) to the front-side and \( S_{j+1} \) to the backside. Moreover, it is isomorphic to an open ball in Figure 22. Therefore, once we identify \( S_{j} \) and \( S_{j+1} \), we see that \( \text{Stab}(\mathcal{D}(\mathbb{P}^1))/\langle \mathbb{Z} \rangle \mathbb{R} \) is an open torus.
Remark 5.7. Heart $\text{Coh} \mathbb{P}^1 = C_0$ and its shifts $C_j$ are the most special hearts that appear in $\text{Stab}(D(\mathbb{P}^1))$, in the sense that the dimension of $S_j$ is the smallest. They are also the most symmetric ones; i.e., $C_j$ is fixed by $\text{Pic}(\mathbb{P}^1)$ and $\text{Aut}(\mathbb{P}^1)$, while any other heart is fixed only by $\text{Aut}(\mathbb{P}^1)$. The way all $S_j$ fit into $\text{Stab}(D(\mathbb{P}^1))$ gives a picture of degenerations of the hearts $C_j$ and $C_{i,j}$, that are of the homological dimension one, to the hearts $C_{p,i,j}$ for $p > 1$, that are of the homological dimension zero.

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