LOCALLY UNMIXED MODULES AND LINEARLY EQUIVALENT
IDEAL TOPOLOGIES

MONA BAHADORIAN, MONIREH SEDGHI AND REZA NAGHIPOUR

Abstract. Let \( R \) be a commutative Noetherian ring, and let \( N \) be a non-zero finitely generated \( R \)-module. The purpose of this paper is to show that \( N \) is locally unmixed if and only if, for any \( N \)-proper ideal \( I \) of \( R \) generated by \( \text{ht}_N I \) elements, the topology defined by \( (IN)^{(n)} \), \( n \geq 0 \), is linearly equivalent to the \( I \)-adic topology.

1. Introduction

Let \( R \) denote a commutative Noetherian ring, \( I \) an ideal of \( R \) and \( N \) a non-zero finitely generated \( R \)-module. We denote by \( R[It] \) (resp. \( R[It, u] \)) the graded ordinary (resp. extended) Rees ring \( \oplus_{n \in \mathbb{N}_0} I^n t^n \) (resp. \( \oplus_{n \in \mathbb{Z}} I^n t^n \)) of \( R \) with respect to \( I \), where \( t \) is an indeterminate and \( u = t^{-1} \). Also, the graded ordinary Rees module \( \oplus_{n \in \mathbb{N}_0} I^n N \) over \( R[It] \) (resp. graded extended Rees module \( \oplus_{n \in \mathbb{Z}} I^n N \) over \( R[It, u] \)) is denoted by \( N[It] \) (resp. \( N[It, u] \)), which is finitely generated. For any multiplicatively closed subset \( S \) of \( R \), the \( n \)th \( (S) \)-symbolic power of \( I \) with respect to \( N \), denoted by \( S(I^n N) \), is defined to be the union of \( I^n N :_N s \) where \( s \) varies in \( S \). The \( I \)-adic filtration \( \{I^n N\}_{n \geq 0} \) and the \( (S) \)-symbolic filtration \( \{S(I^n N)\}_{n \geq 0} \) induce topologies on \( N \) which are called the \( I \)-adic topology and the \( (S) \)-symbolic topology, respectively. These two topologies are said to be linearly equivalent if, there is an integer \( k \geq 0 \) such that \( S(I^{n+k} N) \subseteq I^n N \) for all integers \( n \). In particular, if \( S = R \setminus \bigcup \{p \in \text{mAss}_R N/IN\} \), where \( \text{mAss}_R N/IN \) denotes the set of minimal prime ideals of \( \text{Ass}_R N/IN \), the \( n \)th \( (S) \)-symbolic power of \( I \) with respect to \( N \), is denoted by \( (IN)^{(n)} \), and the topology defined by the filtration \( \{(IN)^{(n)}\}_{n \geq 0} \) is called the symbolic topology. The purpose of this paper is to show that \( N \) is locally unmixed if and only if, for each \( N \)-proper ideal \( I \) that is generated by \( \text{ht}_N I \) elements, the \( I \)-adic and the symbolic topologies are linearly equivalent.

P. Schenzel has characterized unmixed local rings [19, Theorem 7] in terms of comparison of the topologies defined by certain filtrations. Also, D. Katz [9, Theorem 3.5] and J. Verma [21, Theorem 5.2] have proved a characterization of locally unmixed rings in terms of \( s \)-ideals. Equivalence of \( I \)-adic topology and \( (S) \)-symbolic topology has been studied, in the case \( N = R \), in [9 15 19 18 17], and has led to some interesting results.

Key words and phrases. Analytic spread, locally unmixed modules, ideal topologies, Rees ring.

2010 Mathematics Subject Classification: 13A30, 13E05.

*Corresponding author: e-mail: naghipour@ipm.ir (Reza Naghipour).
Let \( p \in \text{Supp}(N) \). Then \( N \)-height of \( p \), denoted by \( \text{ht}_N p \), is defined to be the supremum of lengths of chains of prime ideals of \( \text{Supp}(N) \) terminating with \( p \). We have \( \text{ht}_N p = \dim_{R_p} N_p \). We shall say an ideal \( I \) of \( R \) is \( N \)-proper if \( N/IN \neq 0 \), and, when this is the case, we define the \( N \)-height of \( I \) (written \( \text{ht}_N I \)) to be

\[
\inf \{ \text{ht}_N p : p \in \text{Supp}(N) \cap V(I) \}
\]

\( (= \inf \{ \text{ht}_N p : p \in \text{Ass}_R(N/IN) \}).\)

If \((R, m)\) is local, then \( \widehat{R} \) (resp. \( \widehat{N} \)) denotes the completion of \( R \) (resp. \( N \)) with respect to the \( m \)-adic topology. In particular, for any \( p \in \text{Spec}(R) \), we denote \( \widehat{R}_p \) and \( \widehat{N}_p \) the \( pR_p \)-adic completion of \( R_p \) and \( N_p \), respectively. Then \( N \) is said to be an \emph{unmixed module} if for any \( p \in \text{Ass}_R \widehat{N} \), \( \dim \widehat{R}/p = \dim N \). More generally, if \( R \) is not necessarily local and \( N \) is non-zero finitely generated, \( N \) is a \emph{locally unmixed module} if for any \( p \in \text{Supp}(N) \), \( N_p \) is an unmixed \( R_p \)-module.

As the main result of this paper we characterize the locally unmixed property of a non-zero finitely generated \( R \)-module \( N \) in terms of the linearly equivalence of the topologies defined by \( \{I^nN\}_{n \geq 0} \) and \( \{(IN)^{(n)}\}_{n \geq 0} \), for certain \( N \)-proper ideals \( I \) of \( R \). More precisely we shall show that:

**Theorem 1.1.** Let \( R \) be a Noetherian ring and \( N \) a non-zero finitely generated \( R \)-module. Then the following conditions are equivalent:

(i) \( N \) is locally unmixed.

(ii) For each \( N \)-proper ideal \( I \) of \( R \) that is generated by \( \text{ht}_N I \) elements, the topology given by \( \{(IN)^{(n)}\}_{n \geq 0} \) is linearly equivalent to the \( I \)-adic topology on \( N \).

One of our tools for proving Theorem 1.1 is the following, which plays a key role in this paper. Recall that a prime ideal \( p \) of \( R \) is called a \emph{quitessential prime ideal of} \( I \) with respect to \( N \) precisely when there exists \( q \in \text{Ass}_{R_p} \widehat{N}_p \) such that \( \text{Rad}(I\widehat{R}_p + q) = p\widehat{R}_p \).

The set of quitessential primes of \( I \) is denoted by \( Q(I, N) \). Then, the set of \emph{essential primes} of \( I \) with respect to \( N \), denoted by \( E(I, N) \), is defined to be the set \( \{q \cap R \mid q \in Q(uR[It, u], N[It, u])\} \).

**Theorem 1.2.** Let \( R \) denote a Noetherian ring, \( N \) a non-zero finitely generated \( R \)-module and \( I \) a \( N \)-proper ideal of \( R \) such that \( E(I, N) = \text{mAss}_R N/IN \). Then, the \( I \)-adic topology \( \{I^nN\}_{n \geq 0} \) and the topology defined by \( \{(IN)^{(n)}\}_{n \geq 0} \) are linearly equivalent.

The proof of Theorem 1.2 is given in 1.13.

Throughout this paper, \( R \) will always be a commutative Noetherian ring with non-zero identity, \( N \) will be a non-zero finitely generated \( R \)-module, and \( I \) will be an \( N \)-proper ideal of \( R \), i.e., \( N/IN \neq 0 \). For each \( R \)-module \( L \), we denote by \( \text{mAss}_p L \) the set of minimal primes of \( \text{Ass}_R L \). For any ideal \( J \) of \( R \), the \emph{radical of} \( J \), denoted by \( \text{Rad}(J) \), is defined to be the set \( \{x \in R : x^n \in J \text{ for some } n \in \mathbb{N}\} \). For any unexplained notation and terminology we refer the reader to [6] or [12].
2. The Results

The main result of this section is to show that a non-zero finitely generated module $N$ over a Noetherian ring $R$ is locally unmixed if and only if, for any $N$-proper ideal $I$ of $R$ that can be generated by $\text{ht}_N I$ elements, the topologies defined by $\{I^n N\}_{n \geq 0}$ and $\{(IN)^{(n)}\}_{n \geq 0}$, on $N$, are linearly equivalent. We begin with the following remark.

Remark 2.1. Let $R$ be a Noetherian ring and $N$ a finitely generated $R$-module. For a submodule $M$ of $N$ and an ideal $I$ of $R$, the increasing sequence of submodules

$$M \subseteq M :_N I \subseteq M :_N I^2 \subseteq \cdots \subseteq M :_N I^n \subseteq \cdots$$

becomes stationary. Denote its ultimate constant value by $M :_N \langle I \rangle$. Note that $M :_N \langle I \rangle = M :_N I^n$ for all large $n$. Let $M = Q_1 \cap \cdots \cap Q_r \cap Q_{r+1} \cap \cdots \cap Q_s$ be an irredundant primary decomposition of $M$, with $I \subseteq \text{Rad}(Q_i :_R N)$, exclusively for $r + 1 \leq i \leq s$. Then, from the definition, it easily follows that $M :_N \langle I \rangle = Q_1 \cap \cdots \cap Q_r$. Therefore

$$\text{Ass}_R N/(M :_N \langle I \rangle) = \{p \in \text{Ass}_R N/M : I \not\subseteq p\} = \text{Ass}_R (N/M) \setminus V(I).$$

Now we can state and prove the following lemma. Here $D_I(L)$ denotes the ideal transform of the $R$-module $L$ with respect to an ideal $I$ of $R$ (see \cite[2.2.1]{1}).

Lemma 2.2. Let $(R, \mathfrak{m})$ be local (Noetherian) ring, $I$ an ideal of $R$ and $N$ a non-zero finitely generated $R$-module such that depth $N > 0$. Then, for all integers $n \geq 0$, we have

$$I^n N :_N \langle \mathfrak{m} \rangle \subseteq D_{I^n}(N).$$

Proof. The assertion follows from \cite[Corollary 2.2.18]{1} and the fact that depth $I^n N > 0$ for all integers $n \geq 0$. \hfill $\square$

The next result concerns the associated prime ideals of the Rees module $N[It]$ for a non-zero finitely generated module $N$ over a Noetherian ring $R$ and an ideal $I$ in $R$.

Proposition 2.3. Let $R$ be a Noetherian ring, $I$ an ideal of $R$ and $N$ a non-zero finitely generated $R$-module. Then

$$\text{Ass}_{R[It]} N[It] = \left\{ \oplus_{n \geq 0} (I^n \cap p) : p \in \text{Ass}_R N \right\}.$$

Proof. Let $q \in \text{Ass}_{R[It]} N[It]$. Then in view of \cite[Lemma 1.5.6]{1} there exists a homogenous element $x$ of $N[It]$ such that $q = \text{Ann}_{R[It]} x$. Suppose that $x \in I^n N$ for some integer $v \geq 0$. Then we have

$$q = (0 :_{R[It]} x) = \oplus_{n \geq 0} (0 :_R x) \cap I^n.$$

Now, it is easy to see that $p := (0 :_R x)$ is a prime ideal of $R$ and so $p \in \text{Ass}_R N$. Hence $q = \oplus_{n \geq 0} (I^n \cap p)$ for some $p \in \text{Ass}_R N$. Conversely, let $p \in \text{Ass}_R N$ and $q = (0 :_R x)$ for an element $x \in N$. Then

$$q := (0 :_{R[It]} x) = \oplus_{n \geq 0} (I^n \cap p)$$

is a prime ideal of $\text{Ass}_{R[It]} N[It]$, because $R[It]/q \cong R/p((I + p/p)t)$ is a domain. \hfill $\square$
**Definition 2.4.** Let $R$ be a Noetherian ring and $N$ an $R$-module. A decreasing sequence $\{N_n\}_{n \geq 0}$ of submodules of $N$ is called a filtration of $N$. If $I$ is an ideal of $R$, then the filtration $\{N_n\}_{n \geq 0}$ is called $I$-filtration whenever $IN_n \subseteq N_{n+1}$ for all integers $n \geq 0$.

**Lemma 2.5.** Let $R$ be a Noetherian ring, $I$ an ideal of $R$ and $N$ an $R$-module. Let $\{N_n\}_{n \geq 0}$ be an $I$-filtration of submodules of $N$ such that the ordinary Rees module $N/I[\mathfrak{m}]$ is finitely generated over $R[I]$. Then there exists an integer $k$ such that $N_{n+k} \supseteq I^nN_k$, for all integers $n \geq 0$.

*Proof.* The result follows easily from [4, Lemma 2.5.4]. □

**Corollary 2.6.** Let $(R, \mathfrak{m})$ be a local (Noetherian) ring and $I$ an ideal of $R$. Let $N$ be an $R$-module and set $N_n = I^nN : _N\langle \mathfrak{m}\rangle$ for each integer $n \geq 0$. Suppose that the module $\oplus_{n \geq 0}N_n$ is finitely generated over the ordinary Rees ring $R/I[\mathfrak{m}]$. Then there is an integer $k$ such that $I^{n+k}N : _N\langle \mathfrak{m}\rangle \subseteq I^nN$, for all integer $n \geq 0$.

*Proof.* As $I(I^nN : _N\langle \mathfrak{m}\rangle) \subseteq I^{n+1}N : _N\langle \mathfrak{m}\rangle$, for all integers $n \geq 0$, the claim follows from Lemma 2.5. □

**Definition 2.7.** Let $(R, \mathfrak{m})$ be a local (Noetherian) ring, $I$ an ideal of $R$ and $N$ an $R$-module. We define the $R$-module $D(I, N)$ as the following:

$$D(I, N) := \bigoplus_{n \geq 0}D_m(I^nN).$$

As $D_m(.)$ is an $R$-linear and left exact functor, it follows that $\{D_m(I^nN)\}_{n \geq 0}$ is a decreasing sequence and $ID_m(I^nN) \subseteq D_m(I^{n+1}N)$ for all integers $n \geq 0$. Hence $D(I, N)$ is an $R[I]$-module, by Lemma 2.5.

**Lemma 2.8.** Let $R$ be a Noetherian ring, $I$ an ideal of $R$ and $N$ a finitely generated $R$-module. Then the following conditions are equivalent:

(i) $D_I(N)$ is a finitely generated $R$-module.

(ii) For all $p \in \text{Ass}_R N$, the $R/p$-module $D_{R/p}(R/p)$ is finitely generated.

*Proof.* See [3, Lemma 3.3]. □

**Proposition 2.9.** Let $(R, \mathfrak{m})$ be a local (Noetherian) ring, $I$ an ideal of $R$ and $N$ a finitely generated $R$-module. Then the following conditions are equivalent:

(i) $D(I, N)$ is a finitely generated $R[I]$-module.

(ii) For all $p \in \text{Ass}_R N$, the module $\oplus_{n \geq 0}D_m(I^n + p/p)$ is finitely generated over the Rees ring $R/p[I + p/p][t]$.

*Proof.* In order to prove the implication (i) $\implies$ (ii), suppose that $p \in \text{Ass}_R N$. Then in view of Proposition 2.3, there exists $q \in \text{Ass}_{R/I[t]}(\oplus_{n \geq 0}I^nN)$ such that $q = \oplus_{n \geq 0}(I^n \cap p)$. Since

$$D(I, N) \cong D_m(\oplus_{n \geq 0}I^nN) \cong D_{m(R[I]/q)}(\oplus_{n \geq 0}I^nN),$$

is a finitely generated $R[I]$-module, it follows from Lemma 2.8 that the $R[I]/q$-module $D_{m(R[I]/q)}(R[I]/q)$ is finitely generated. Now, as

$$R[I]/q \cong R/p[(I + p/p)t],$$

and $D_{m(R[I]/q)}(R[I]/q) \cong \oplus_{n \geq 0}D_m(I^n + p/p)$,
we deduce that the $R/p[(I+p/p)t]$-module $\oplus_{n\geq 0} D_m(I^n + p/p)$ is finitely generated.

Now, we show the conclusion (ii) $\implies$ (i). To do this end, let $q \in \text{Ass}_{R/I}[N[I]]$. Then, by virtue of Proposition 2.3, there exists $p \in \text{Ass}_R N$ such that $q = \oplus_{n\geq 0} (I^n \cap p)$. Since

$$R/I[q] \cong R/p[(I+p/p)t]$$

and

$$D_m(R/I[q]) \cong \oplus_{n\geq 0} D_m(I^n + p/p),$$

it follows from Lemma 2.8 that the $R/I$-module $D_m(R/I)(N[I])$ is finitely generated, and so the $R/I$-module $\oplus_{n\geq 0} D_m(I^n N)$ is finitely generated, as required. $\square$

The next proposition gives us a criterion for the finiteness of $R/I$-module $D_m(R/I)(N[I])$, whenever $(R, m)$ is a local ring and $N$ is a finitely generated module over $R$. To this end, let us, firstly, recall the important notion analytic spread of $I$ with respect to $N$, over a local ring $(R, m)$, introduced by Brodmann in [3]:

$$l(I, N) := \dim N[I]/(m, u) N[I],$$

in the case $N = R$, $l(I, N)$ is the classical analytic spread $l(I)$ of $I$, introduced by Northcott and Rees (see [14]).

**Proposition 2.10.** Let $(R, m)$ be a local (Noetherian) ring and $I$ an ideal of $R$. Let $N$ be a finitely generated $R$-module such that $l(I\hat{R} + p/p) < \dim R/p$ for all $p \in \text{Ass}_R \hat{N}$. Then the $R/I$-module $D_m(R/I)(N[I])$ is finitely generated, and depth $N > 0$.

**Proof.** It is easy to see that

$$D_m(R/I)(N[I]) \otimes_{R/I} \hat{R}[(I\hat{R})t] \cong D_m(R[I\hat{R}t])\oplus_{n\geq 0} I^n \hat{N},$$

and so by faithfully flatness of $R/I$ over $R/I$, it is enough for us to show that the $R/I$-module $\oplus_{n\geq 0} D_m(R/I)(I^n \hat{N})$ is finitely generated. In order to do this, in view of Proposition 2.9, it is enough to show that $\oplus_{n\geq 0} D_m(I^n \hat{R} + p/p)$ is finitely generated over $\hat{R}/p[(I\hat{R} + p/p)t]$ for all $p \in \text{Ass}_R \hat{N}$. But this follows easily from [19], Proposition and the assumption $l(I\hat{R} + p/p) < \dim \hat{R}/p$. $\square$

**Remark 2.11.** Before bringing the next result we fix a notation, which is employed by P. Schenzel in [13] in the case $N = R$. Let $S$ be a multiplicatively closed subset of a Noetherian ring $R$. For a submodule $M$ of a finitely generated $R$-module $N$, we use $S(M)$ to denote the submodule $\bigcup_{s \in S} (M : N s)$. Note that the primary decomposition of $S(M)$ consists of the intersection of all primary components of $M$ whose associated prime ideals do not meet $S$. In other words

$$\text{Ass}_R N/S(M) = \{ p \in \text{Ass}_R N/M : p \cap S = \emptyset \}.$$  

In particular, if $S = R \setminus \bigcup\{ p \in \text{mAss}_R N/IN \}$, then for any $n \in \mathbb{N}$, $S(I^n N)$ is denoted by $(IN)^{(n)}$, where $I$ is an ideal of $R$.

The following lemma is needed in the proof of Theorem 2.13.

**Lemma 2.12.** Let $R$ be a Noetherian ring and $N$ an $R$-module. Let $M$ and $L$ be two submodules of $R$ such that $M_p \subseteq L_p$ for all $p \in \text{Ass}_R N/L$. Then $M \subseteq L$.

**Proof.** The assertion follows from the fact that $\text{Ass}_R (M + L)/L \subseteq \text{Ass}_R N/L$. $\square$
Following, we investigate a fundamental characterization for linearly equivalence between the \( I \)-adic and symbolic topologies on a finitely generated \( R \)-module \( N \), for certain ideal \( I \) of \( R \). This result plays a key role in the proof of the main theorem.

To this end, recall that, in \[16\], L.J. Ratliff, Jr., (resp. in \[2\] Brodman n) introduced the interesting set of associated primes \( A^*(I) := \text{Ass}_R R/(I^n)_a \) (resp. \( A^*(I, N) := \text{Ass}_R N/I^n(N) \), for large \( n \). Here \( I_a \) denotes the integral closure of \( I \) in \( R \), i.e., \( I_a \) is the ideal of \( R \) consisting of all elements \( x \in R \) which satisfy an equation \( x^n + r_1 x^{n-1} + \cdots + r_n = 0 \), where \( r_i \in I, i = 1, \ldots, n \).

Moreover, recall that a local ring \((R, \mathfrak{m})\) is said to be a quasi-unmixed ring if for every \( \mathfrak{p} \in \text{mAss} \hat{R} \), the condition \( \dim \hat{R}/\mathfrak{p} = \dim R \) is satisfied.

**Theorem 2.13.** Let \( R \) be a Noetherian ring, \( I \) an ideal of \( R \) and let \( N \) be a finitely generated \( R \)-module such that \( E(I, N) = \text{mAss}_R N/IN \). Then, the \( I \)-adic topology, \( \{I^n N\}_{n \geq 0} \) and the topology defined by the filtration \( \{(IN)^{(n)}\}_{n \geq 0} \) are linearly equivalent.

**Proof.** Let \( q \in A^*(I, N) \setminus \text{mAss}_R N/IN \) and let \( z \in \text{Ass}_R \hat{N}_q \). Then, by assumption, \( q \notin E(I, N) \). Hence, in view of \[11\] Lemma 3.2, \( qR_q \notin E(IR_q, N_q) \), and so it follows from \[11\] Proposition 3.6 that \( q\hat{R}_q/z \notin E(I\hat{R}_q + z/z) \). Thus by virtue of \[11\] Lemma 2.1, \( q\hat{R}_q/z \notin A^*(I\hat{R}_q + z/z) \). As \( \hat{R}_q/z \) is quasi-unmixed, it follows from McAdam’s result \[10\] Proposition 4.1 that

\[
\ell(I\hat{R}_q + z/z) < \dim \hat{R}_q/z. \tag{\dagger}
\]

Now, we show that there exists a non-negative integer \( k \) such that \( (IN)^{(n+k)} \subseteq I^n N \) for all integers \( n \geq 0 \). To do this, it is easy to see that, \( (IN)^{(s)}_p \subseteq (I^n N)_p \) for all \( p \in \text{mAss}_R N/IN \) and for all integers \( s \geq 0 \). Moreover, if for every \( q \in A^*(I, N) \setminus \text{mAss}_R N/IN \) there exists an integer \( k_q \) such that

\[
(IN)_{q}^{(n+k_q)} \subseteq (I^n N)_q,
\]

then by considering

\[
k := \max\{k_q : q \in A^*(I, N) \setminus \text{mAss}_R N/IN\},
\]

one easily sees that \( (IN)^{(n+k)} \subseteq I^n N \). Since both \( A^*(I, N) \) and \( \text{mAss}_R N/IN \) behave well under localization, we may assume by localizing at \( q \) that \((R, \mathfrak{m})\) is a local ring.

Now, we use induction on \( \dim N/IN := d \). It is clear that \( d \geq 1 \). Now, if \( d = 1 \), then, as \( \text{Ass}_R N/IN \subseteq \text{Supp} N/IN \) and \( \mathfrak{m} \in \text{Supp} N/IN \) it follows that the only possible embedded prime of \( \text{Ass}_R N/IN \) is \( \mathfrak{m} \), and so in view of Remark 2.1 we have

\[
I^n N :_N \langle \mathfrak{m} \rangle = (IN)^{(s)}
\]

for all integers \( s \geq 0 \). Next, it follows from (\dagger) and Proposition 2.10 that the \( R[I^t] \)-module \( \oplus_{n \geq 0} D_{\mathfrak{m}}(I^n N) \) is finitely generated and depth \( N > 0 \). Hence in view of Lemma 2.2, the module \( \oplus_{n \geq 0} (I^n N :_N \langle \mathfrak{m} \rangle) \) is finitely generated over the Rees ring \( R[I^t] \), and so by virtue of Corollary 2.6, there exists an integer \( t \) such that \( I^{n+t} N :_N \langle \mathfrak{m} \rangle \subseteq I^n N \) for all integers \( n \geq 0 \). Therefore \( (IN)^{(n+k)} \subseteq I^n N \), and so the result holds for \( d = 1 \).
We therefore assume, inductively, that $d > 1$ and the result has been proved for smaller values of $d$. If $q \neq m$ and $q \in A^s(I, N)$, then
\[ \dim N_q/IN_q = \text{ht}_{N/IN} q < \text{ht}_{N/IN} m = \dim N/IN = d. \]
Hence by induction hypothesis, there exists a non-negative integer $k_q$ such that
\[ (IN)_q^{(n+k_q)} \subseteq (I^N)_q, \]
for all integers $n \geq 0$. Now, in view of Remark 2.1,
\[ \text{Ass}_R N/(I^n N :_N (m)) = \text{Ass}_R N/IN \setminus V(m), \]
it follows that for all $q \in \text{Ass}_R N/(I^n N :_N (m))$, there exists a non-negative integer $k_q$ such that
\[ (IN)_q^{(n+k_q)} \subseteq (I^N)_q \subseteq ((I^N)_q :_{N_q} (mR_q)), \]
for all integers $n \geq 0$. Hence by considering
\[ k := \max\{k_q : q \in \text{Ass}_R N/(I^n N :_N (m))\}, \]
we get
\[ (IN)_q^{(n+k)} \subseteq (I^N)_q :_{N_q} (mR_q), \]
for all $q \in \text{Ass}_R N/(I^n N :_N (m))$ and all integers $n \geq 0$. Therefore, by virtue of the Lemma 2.12, we have
\[ (IN)^{(n+k)} \subseteq (I^N) :_N (m). \]
On the other hand, in view of Corollary 2.6, there exists an integer $s \geq 0$ such that
\[ I^{n+s} :_N (m) \subseteq I^n N \]
for all integers $n \geq 0$. Consequently
\[ (IN)^{(n+k+s)} \subseteq I^{n+s} :_N (m) \subseteq I^n N, \]
for all integers $n \geq 0$, and thus the topologies defined by the filtrations $\{I^N\}_{n \geq 0}$ and $\{(I^N)^{(n)}\}_{n \geq 0}$ are linearly equivalent. \qed

We are now ready to state and prove the main theorem of this paper, which is a new characterization of locally unmixed modules in terms of comparison of the topologies defined by certain decreasing families of submodules of finitely generated modules over a commutative Noetherian ring. One of the implications in the proof of this theorem follows from [13, Theorem 3.2].

**Theorem 2.14.** Let $R$ be a Noetherian ring and $N$ a non-zero finitely generated $R$-module. Then the following conditions are equivalent:

(i) $N$ is locally unmixed.

(ii) For any $N$-proper ideal $I$ of $R$ generated by $\text{ht}_N I$ elements, the $I$-adic topology is linearly equivalent to the symbolic topology.
Proof. The implication (ii) ⇒ (i) follows easily from [13, Theorem 3.2]. In order to prove the conclusion (i) ⇒ (ii), let \( I \) be an \( N \)-proper ideal of \( R \) which is generated by \( h_N I \) elements. Then, in view of Theorem 2.13 it is enough for us to show that \( E(I, N) = \text{mAss}_R N/I N \). Suppose that \( p \in E(I, N) \), and we show that \( p \in \text{mAss}_R N/I N \).

Let \( h_N I := n \). Then by [13, Theorem 2.1], there exist the elements \( x_1, \ldots, x_n \) in \( I \) such that \( h_N (x_1, \ldots, x_i) = i \) for all \( 1 \leq i \leq n \). As, in view of [13, Corollary 3.11], \( x_1, \ldots, x_n \) is an essential sequence on \( N \), and the fact that \( \text{egrade}(I, N) \leq h_N I \), it follows that \( \text{egrade}(I, N) = n \). Now, analogous to the proof of [8, Theorem 125], it is easy to see that \( I \) can be generated by an essential sequence of length \( n \). Therefore by [13, Lemma 3.8], we have \( p \in \text{mAss}_R N/I N \), and so \( E(I, N) \subseteq \text{mAss}_R N/I N \). As the opposite inclusion is obvious, the result follows.

\[ \square \]

Acknowledgments

The authors are deeply grateful to the referee for his/her careful reading of the paper and valuable suggestions. Also, we would like to thank Professors M.P. Brodmann and S. Goto for their useful comments on Theorem 2.13.

References

[1] S.H. Ahn, Asymptotic primes and asymptotic grade on modules, J. Algebra 174 (1995), 980-998.
[2] M. P. Brodmann, Asymptotic stability of \( \text{Ass}_R(M/I^n M) \), Proc. Amer. Math. Soc. 74 (1979), 16-18.
[3] M.P. Brodmann, The asymptotic nature of the analytic spread, Math. Proc. Cambridge Philos. Soc. 86 (1979), 35-39.
[4] M.P. Brodmann, Finiteness of ideal transforms, J. Algebra 63 (1980), 162-185.
[5] M.P. Brodmann and R.Y. Sharp, Local Cohomology; an Algebraic Introduction with Geometric Applications, Cambridge University Press, Cambridge, 1998.
[6] W. Bruns and J. Herzog, Cohen-Macaulay Rings, Cambridge University Press, Cambridge, UK, 1998.
[7] E.E. Enochs and M.G. Jenda, Relative Homological Algebra, Walter de Gruyter, Berlin, New York, 2000.
[8] I. Kaplansky, Commutative Rings, Univ. of Chicago Press, Chicago, 1974.
[9] D. Katz, Prime divisors, asymptotic \( R \)-sequences and unmixed local rings, J. Algebra 95 (1985), 59-71.
[10] S. McAdam, Asymptotic Prime Divisors, Lecture Notes in Math. 1023, Springer-Verlag, New York, 1983.
[11] S. McAdam, Quintasymptotic primes and four results of Schenzel, J. Pure Appl. Algebra 47 (1987), 283-298.
[12] M. Nagata, Local Rings, Interscience, New York, 1961.
[13] R. Naghipour, Locally unmixed modules and ideal topologies, J. Algebra 236 (2001), 768-777.
[14] D.G. Northcott and D. Rees, Reductions of ideals in local rings, Proc. Cambridge Philos. Soc. 50 (1954), 145-158.
[15] L. J. Ratliff, The topology determined by the symbolic powers of primary ideals, Comm. Algebra 13 (1985), 2073-2104.
[16] L.J. Ratliff, Jr., On asymptotic prime divisors, Pacific J. Math. 111 (1984), 395-413.
[17] P. Schenzel, Finiteness of relative Rees ring and asymptotic prime divisors, Math. Nachr. 129 (1986), 123-148.
[18] P. Schenzel, On the use of local cohomology in algebra and geometry, Six lectures on commutative algebra (Bellaterra, 1996), 241-292.
[19] P. Schenzel, *Independent elements, unmixedness theorems and asymptotic prime divisors*, J. Algebra 92 (1985), 157-170.

[20] P. Schenzel, *Symbolic powers of prime ideals and their topology*, Proc. Amer. Math. Soc. 93 (1985), 15-20.

[21] J. K. Verma, *On ideals whose adic and symbolic topologies are linearly equivalent*, J. Pure Appl. Algebra 47 (1987), 205-212.

Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran.
E-mail address: mona.bahadorian@gmail.com

Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran.
E-mail address: m.sedghi@tabrizu.ac.ir
E-mail address: sedghi@azaruniv.ac.ir

Department of Mathematics, University of Tabriz, Tabriz, Iran.
E-mail address: naghipour@ipm.ir
E-mail address: naghipour@tabrizu.ac.ir