Notes for Quantum Gravitational Field

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Abstract

We discuss the problems of dynamics of the gravitational field and try to solve them according to quantum field theory by suggesting canonical states for the gravitational field and its conjugate field. To solve the problem of quantization of gravitational field, we assume that the quantum gravitational field $e^I$ changes the geometry of curved spacetime $x^\mu$, and relate this changing to quantization of the gravitational field. We introduce a field $\pi_I$ and consider it as a canonical momentum conjugates to a canonical gravitational field $\bar{e}^I$. We use
them in deriving the path integral of the gravitational field according to quantum field theory, we get Lagrangian with dependence only on the covariant derivative of the gravitational field $e^I$, similarly to Lagrangian of scalar field in curved spacetime. Then, we discuss the case of free gravitational field. We find that this case takes place only in background spacetime approximation of low matter density; weak gravity. Similarly, we study the Plebanski two form complex field $\Sigma^I$ and derive its Lagrangian with dependence only on the covariant derivative of $\Sigma^I$, which is represented in selfdual representation $|\Sigma^I\rangle$.

Then, We try to combine the gravitational and Plebanski fields into one field: $K^I_\mu$. Finally, we derive the static potential of exchanging gravitons between particles of scalar and spinor fields; the Newtonian gravitational potential.

Key words: Canonical gravitational field, Conjugate momentum, Path integral, Free gravitational propagator, Lagrangian of Plebanski field, Combination of Plebanski and gravitational fields.

**Introduction**

We search for conditions allow us to consider the gravitational field as dynamical field. The problem of dynamics in general relativity is that spacetime is itself dynamical. It interacts with matter, it is an operator $d\hat{x}^\mu$. So, we have to consider it as quantum field similarly to quantum fields. But to define dynamical gravitational field, we have to know that both dynamical curved spacetime $x^\mu$ and gravitational field $e^I$ have the same entity, it is the gravity. Therefore, we study only one of them as a gravitational field: $\hat{e}^I$. Then the dynamical spacetime $\hat{x}^\mu$ is implicitly included in the gravitational field $e^I$ via $\hat{e}^I = \hat{e}^I_\mu dx^\mu$, where $x^\mu$ are local coordinates defined on arbitrary curved spacetime manifold $M$. We will see that it is substantially different in background spacetime, where the gravitational field can be considered as a free field similarly to free quantum fields in flat spacetime.

To study dynamics of gravitational field which changes geometry of spacetime, we write the spin connection $\omega^{IJ}$ of local Lorentz frame as $\Omega^{IJ} + B^{IJ}$ by using a reference connection $B^{IJ}$ and a tensor $\Omega^{IJ}$. We relate this changing
in the geometry to fluctuation of the gravitational field. It is like to say that the quantum gravitational field consists of quanta (like $\delta e^I$), each of them contributes to spin connection changes $\delta \omega^{IJ}$, so dynamical changes in geometry of spacetime. Those changes are seen in the tensor $\Omega^{IJ}$. We consider the reference connection $B^{IJ}$ as spin connection in the vacuum, therefore when $\Omega^{IJ} = 0$, the GR equation becomes in the vacuum; $L(e, \omega) = 0$, with $\omega = B$.

To define dynamical variables and so canonical states, we introduce an anti-symmetric tensor $\pi^{IJK}$ by the formula $\Omega^{IJ} = \pi^{K IJ} e_K$ and a field $\pi_I$ to satisfy $\pi^{IJK} = \pi^L \epsilon^{LIJK}$. We consider $\pi_I$ as a canonical momentum conjugates to a canonical gravitational field $\tilde{e}^I = ee^I \mu n^\mu$, where $n^\mu$ is normal to 3D closed surface $\delta M$, which is embedded in arbitrary curved 4D spacetime manifold $M$. The closed surface $\delta M$ is parameterized by three parameters $X^1$, $X^2$ and $X^3$. We consider them in a certain gauge as spatial part of the local-Lorentz frame $X^I : I = 0, 1, 2, 3$. We will see that the path integral of the gravitational field is independent of this gauge. Therefore, the exterior derivative operator on the surface $\delta M(X^1, X^2, X^3)$ leads to a changing along the normal of that surface in direction of the time $dX^0$. That allows the 3D surface $\delta M(X^1, X^2, X^3)$ to extend, and so obtaining the local-Lorentz frame $(X^0, X^1, X^2, X^3)$. With that the gravitational field propagates from one surface to another by the extension of those surfaces.

So we can introduce canonical states $|\tilde{e}^I\rangle$ and $|\pi^I\rangle$ represented in local Lorentz frame and satisfy $\langle \tilde{e}^I | \pi_I \rangle_{\delta M} = \exp i \int_{\delta M} \tilde{e}^I(X) \pi_I(X) d^3X$. We use them in deriving the path integral of the gravitational field. We find that there is no propagation on the dynamical spacetime $x^\mu$. But in the background spacetime, the gravitational field propagates freely like the electromagnetic ans scalar fields.

We start with using our formula $\omega^{IJ} = \Omega^{IJ} + B^{IJ}$ in Einstein-Hilbert Lagrangian $L(e, \omega) = c\epsilon_{IJKL} e^I \wedge e^J \wedge R^{KL}(\omega)$, we get

$$L(e, \Omega, B) = c\epsilon_{IJKL} e^I \wedge e^J \wedge (\Omega \wedge \Omega + d\Omega + dB + \Omega \wedge B + B \wedge \Omega + B \wedge B)^{KL}. $$

Because $\Omega^{IJ}$ is tensor, the part $c\epsilon_{IJKL} e^I \wedge e^J \wedge (\Omega \wedge \Omega)^{KL}$ of this Lagrangian is scalar. If we use it in the path integral for the states $|\tilde{e}^I\rangle$ and $|\pi^I\rangle$, with
eq.(1.4), we get the Lagrangian
\[ L(e, \Omega, B) = c \varepsilon_{IJKL} e^I \wedge e^J \wedge (\Omega \wedge \Omega)^{KL} + \pi_I D\tilde{e}^I d^3 X. \]

Similarly to the path integral of one particle in one dimension for the states \( |x, t\rangle = e^{iHt} |x\rangle \), with using the formula \( \langle x | p \rangle = (2\pi)^{-1/2} \exp(ipx) \), it is
\[ \langle x_2, t_2 | x_1, t_1 \rangle = \int Dx dp \exp i \int_{t_1}^{t_2} dt (p\dot{x} - H). \]

The role of the term \( c \varepsilon_{IJKL} e^I \wedge e^J \wedge (\Omega \wedge \Omega)^{KL} \) in our Lagrangian is same role of the Hamiltonian \( H \) in this path integral, while the term \( \pi_I d\tilde{e}^I d^3 X \) is similar to the term \( p\dot{x} \). Therefore, we suggest the equality
\[ c \varepsilon_{IJKL} e^I \wedge e^J \wedge (\Omega \wedge \Omega)^{KL} = \pi_I D\tilde{e}^I d^3 X. \]

We try to show that there is at least one solution for this equation at end of section 1. Because the exterior derivative \( D\tilde{e}^I \) allows expanding of the 3D closed surface \( \delta M \), we consider the term \( \pi_I d\tilde{e}^I d^3 X \) as a kinetic energy which relates to the expansion of those surfaces, while the term \( c \varepsilon_{IJKL} e^I \wedge e^J \wedge (\Omega \wedge \Omega)^{KL} \) is gravitational energy of \( \tilde{e}^I \) on the surface \( \delta M \) on which the states \( |\tilde{e}^I\rangle \) and \( |\pi^I\rangle \) are defined, eq.(1.4). Finally, we get the Lagrangian:
\[ L(e, \omega) = \frac{1}{4\pi G} \frac{1}{4} (-D\mu e^\nu e^I_{\mu} + D\mu e^\nu D\nu e^I_{\mu}) ed^4 x, \]
where the covariant derivative \( D \) is defined in \( DV^I = dV^I + \omega^I_{J} \wedge V^J \), here we use the usual spin connection \( \omega^I_{J} \) not our \( \Omega^I_{J} + B^I_{J} \). Actually, we use \( \Omega^I_{J} + B^I_{J} \) just for getting this Lagrangian. Therefore, we consider the states \( |e^I_{\mu}\rangle \) and \( |\omega^I_{\mu}\rangle \) for the path integral of this Lagrangian.

We will write this Lagrangian using Riemann curvature tensor \( (R_{\mu\nu})_{IJ} \). For \( D\mu e^I_{\mu} = 0 \), we get
\[ Ld^4 x \rightarrow \frac{1}{48c^2} (e^I_{\mu} D^2 e^I_{\mu} + e^I_{\mu} e^I_{\nu} (R_{\mu\nu})_{IJ}) ed^4 x. \]

We use same method for Plebanski two form real field \( \Sigma_{\mu\nu}^{IJ} \) and get the Lagrangian
\[ L(e, \Sigma) = -4c' (D\mu \Sigma^{\nu\rho}_{IJ}) (D\mu \Sigma^{\nu\rho}_{IJ}) ed^4 x. \]
Then, we write this Lagrangian using the complex Plebanski selfdual two-form $\Sigma_{\mu\nu}^i$, and search for the reality conditions.

Notice: by parametrizing the 3D surface $\delta M$ by local Lorentz charts, the dynamical spacetime (the tangent vectors $dx^\mu$) becomes fibers, and the local Lorentz is basis space. Actually by considering the tangent vectors $dx^\mu$ as fibers, we get dynamics of general relativity.

1 Lagrangian and path integral of quantum gravitational field

We use the indices notation $\mu, \nu, ... = 0, 1, 2, 3$ for 4D spacetime tangent space on arbitrary spacetime manifold $M$, and the indices notation $I, J, ... = 0, 1, 2, 3$ for 4D Lorentz tangent space with the metric $[- + + +]$. At each point $x^\mu$ in the manifold $M$, we define gravitational field $e^I = e^I_\mu(x)dx^\mu$, and spin connection $\omega^{IJ}(x) = \omega^{IJ}_\mu(x)dx^\mu$ with values in Lie algebra of Lorentz group $SO(3,1)$. The spin connection defines covariant derivative $D_\mu$ on all fields that have Lorentz indices $(I, J, ...)$ [1, 2]:

$$D_\mu v^I = \partial_\mu v^I + \omega^{IJ}_\mu v^J.$$ 

The gravitational field determines compatible spin connection by

$$De^I = de^I + \omega^{IJ} \wedge e^J = 0.$$ 

Actually we consider this solution for the spin connection as a classical limit. So, the quantum fluctuation of gravitational field $e^I$ violates this formula.

The spin connection transforms under local Lorentz transformation $L(x)^I_J$, in a matrix notation, as [3]

$$\omega' = L\omega L^{-1} + LdL^{-1} \quad \text{or} \quad \omega'_\mu dx^\mu = L\omega_\mu L^{-1}dx^\mu + L\partial_\mu L^{-1}dx^\mu.$$ 

We write the connection $\omega$ as

$$\omega_\mu dx^\mu = \Omega_\mu dx^\mu + B_\mu dx^\mu,$$  \hspace{1cm} (1.1)
where $B^{IJ}$ is a reference connection (consider it as spin connection in the vacuum), and $\Omega^{IJ}_\mu dx^\mu$ is a tensor, so it transforms covariantly under local Lorentz transformation: $L\Omega L^{-1} = \Omega'$. Therefore,

$$L (\omega - B) L^{-1} = \omega' - B'$$

which yields

$$B' = LBL^{-1} + LdL^{-1} \text{ with } \omega' = L\omega L^{-1} + LdL^{-1}, \quad (1.2)$$

for $B = (B^{IJ})$ and $\omega = (\omega^{IJ})$,

and also yields

$$B' = LBL^{-1} - (dL)L^{-1} \text{ with } \omega' = L\omega L^{-1} - (dL)L^{-1}, \quad (1.3)$$

for $B = (B^I_J)$ and $\omega = (\omega^I_J)$.

For dynamical gravitational field, let us suggest an anti-symmetry tensor $\pi^{IJK}$ to satisfy

$$\Omega^{IJ} = \pi^K e^{KIJ}.$$ 

We consider it as a conjugate momentum represented in the local Lorentz frame and acts on its vectors. Therefore, we consider it as dynamical operator. Let us introduce a field $\pi^I$ to satisfy

$$\pi^{IJK} = \pi_L e^{LJK}.$$ 

The element

$$e^{\varepsilon_{\mu\rho\sigma}}dx^\nu \wedge dx^\rho \wedge dx^\sigma / 3! = d^3x_\mu$$

is a co-vector, as $\partial_\mu$, therefore

$$\pi^K e^{K\mu} d^3x_\mu = \pi^K e^{K\mu} \varepsilon_{\mu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma / 3!$$

is invariant under local Lorentz transformation $V^I \rightarrow L^I_J(x) V^J$ and under arbitrary changing of the coordinates $dx^\mu \rightarrow \Lambda^\mu_\nu(x) dx^\nu$.

Let us integral it over 3D closed surface $\delta M$ in arbitrary curved spacetime manifold $M$, and introduce the phase

$$\exp i \oint_{\delta M} \pi^I e^{I\mu} \varepsilon_{\mu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma / 3!$$
in which we can define a canonical gravitational field $\tilde{e}'$ via

$$\tilde{e}' d^3X = \tilde{e}' dX^1 \wedge dX^2 \wedge dX^3 \equiv e'^\mu \varepsilon_{\mu\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma / 3!,$$

or

$$\tilde{e}' = e e'^\mu n_\mu(X^i),$$

where

$$n_\mu(X^i) = \frac{\varepsilon_{\mu\nu\rho\sigma} \partial x^\nu \partial x^\rho \partial x^\sigma \varepsilon^{ijk}}{3! \partial X^i \partial X^j \partial X^k} 3!$$

is the normal to the surface $\delta M(X^1, X^2, X^3)$. So, we get

$$\exp i \int_{\delta M} \pi_I \tilde{e}' d^3X,$$

with the parameters $X^I : I = i = 1, 2, 3$ parameterize the closed 3D surface $\delta M$. As mentioned before, in a certain gauge, we consider those parameters as spatial part of the local-Lorentz frame $X^I : I = 0, 1, 2, 3$. Therefore, the exterior derivative on the surface $\delta M$ is along the direction of the time $dX^0$, which is the direction of the normal to the surface $\delta M(X^1, X^2, X^3)$. We will see that the result of the path integral is independent of this gauge.

By comparing the previous formula with

$$\langle \phi | \pi \rangle = \exp i \int d^3X \phi(X)\pi(X)/\hbar,$$

which is a canonical formula in the scalar field theory on flat spacetime[4], for $\hbar = 1$, we suggest canonical states $|\tilde{e}'\rangle$ and $|\pi^I\rangle$ with

$$\langle \tilde{e}' | \pi_I \rangle_{\delta M} = \exp i \int_{\delta M} \tilde{e}'(X)\pi_I(X)d^3X,$$

(1.4)

where $\pi_I$ is canonical momentum conjugates to $\tilde{e}'$. Let us write this formula on the surface $\delta M(X^i)$ as

$$\langle \tilde{e}' | \pi_I \rangle_{\delta M} = \prod_{n,I} \langle \tilde{e}'(x_n + dx_n) | \pi_I(x_n) \rangle_{\delta M},$$

with

$$\langle \tilde{e}'(x_n + dx_n) | \pi_I(x_n) \rangle_{\delta M} = \exp i\tilde{e}'(x_n + dx_n)\pi_I(x_n) d^3X \rightarrow \exp i\tilde{e}'(x_n)\pi_I(x_n) d^3X.$$
In general, for two points in adjacent surfaces \( \delta M_1 \) and \( \delta M_2 \), let us rewrite it as

\[
\langle \tilde{e}^I (x_n + dx_n) \mid \pi_I (x_n) \rangle = \exp i \tilde{e}^I (x_n + dx_n) \pi_I (x_n) d^3 X.
\] (1.5)

Here the variation

\[
\tilde{e}^I (x_n + dx_n) - \tilde{e}^I (x_n)
\]

is exterior covariant derivative along the time \( dX^0 \) in the direction of the normal \( n^\mu (X^i) \) to the surface \( \delta M_1 \). It allows the extension of this surface: \( \delta M (X^1, X^2, X^3) \rightarrow M (X^0, X^1, X^2, X^3) \). This leads to the propagation of those surfaces.

We need to make \( \hat{e} d^4 \dot{x} \) commute with \( \hat{e} d^3 X \). For this purpose, we write

\[
- \hat{e} d^4 x = edx^\mu \wedge \varepsilon_{\mu \nu \rho \sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma / 4!
\]

\[
= edx^\mu \wedge \frac{\varepsilon_{\mu \nu \rho \sigma}}{4!} \frac{\partial x^\nu}{\partial X^I} \frac{\partial x^\rho}{\partial X^J} \frac{\partial x^\sigma}{\partial X^K} 3! d^3 X = \frac{1}{4} edx^\mu n_\mu d^3 X.
\]

The indices \( i, j \) and \( k \) in our gauge are the local-Lorentz frame indices for \( I = 1, 2, 3 \). We can rewrite it as

\[
- \hat{e} d^4 x = \frac{1}{4} edx^\mu n_\mu d^3 X = \frac{1}{4} e \frac{\partial x^\mu}{\partial X^0} n_\mu d^3 X dX^0 = \frac{1}{4} ee^\mu_0 n_\mu d^3 X dX^0.
\]

Comparing it with the term

\[
\tilde{e}^I d^3 X = e^{I \mu} \varepsilon_{\mu \nu \rho \sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma / 3! = ee^{I \mu} n_\mu d^3 X,
\]

we find that it commutes with it:

\[
[\hat{e} \tilde{e}^{I \mu} \hat{n}_\mu d^3 X, \hat{e} \tilde{e}^{I \nu} \hat{n}_\nu d^3 X dX^0] = 0 \rightarrow [\hat{e} d^3 X, \hat{e} d^4 \dot{x}] = 0,
\]

where \( [\hat{e}^{I \mu}, \hat{e}^{I \nu}] = 0 \). Thus, the operator \( \hat{e} d^4 \dot{x} \) takes eigenvalues when it acts on the states \( |\tilde{e}^I \rangle \).

The Einstein’s action for gravity written in the first order formalism is expressed by\(^5\)

\[
S(e, \omega) = \frac{1}{16 \pi G} \int \varepsilon_{IJKL} (e^I \wedge e^J \wedge R^{KL} (\omega) + \lambda e^I \wedge e^J \wedge e^K \wedge e^L).
\]
Let us consider only the first term:

\[ S(e, \omega) = c \int \varepsilon_{IKL} e^I \wedge e^J \wedge R^{KL}(\omega), \]

where \( C \) is constant. The Riemann curvature here is

\[ R^{KL}(\omega) = d\omega^{KL} + \omega^K_M \wedge \omega^M_L. \]

Inserting the formula we suggested before:

\[ (\omega)^{IJ} = \Omega^{IJ} + B^{IJ} : \Omega^{IJ} = \pi_K^{IJ} e^K, \]

the action becomes

\[ S(e, \pi) = c \int \varepsilon_{IKL} e^I \wedge e^J \wedge (d\Omega + dB + \Omega \wedge \Omega + B \wedge B + \Omega \wedge B \wedge B)^{KL}. \]

(1.6)

As we suggested before that \( \Omega^{IJ} \) transforms covariantly, so the remaining term \( d\Omega + dB + \Omega \wedge B + B \wedge \Omega + B \wedge B \) also transforms covariantly, this is because the Lagrangian transforms covariantly.

Let us test it by using the formulas eq.(1.1), eq.(1.2) and eq.(1.3), it becomes

\[
\begin{align*}
&d (L\Omega L^{-1}) + d (LBL^{-1} + LdL^{-1}) + L\Omega L^{-1} \wedge (LBL^{-1} + LdL^{-1}) \\
&+ (LBL^{-1} - (dL)L^{-1}) \wedge L\Omega L^{-1} + (LBL^{-1} - (dL)L^{-1}) \wedge (LBL^{-1} + LdL^{-1}).
\end{align*}
\]

Expanding it:

\[
\begin{align*}
&(dL) \wedge \Omega L^{-1} + L (d\Omega) L^{-1} - L\Omega \wedge (dL^{-1}) + (dL) \wedge BL^{-1} + L (dB) L^{-1} - LB \wedge dL^{-1} \\
&+ (dL) \wedge dL^{-1} + L\Omega L^{-1} \wedge LBL^{-1} + L\Omega L^{-1} \wedge LdL^{-1} + LBL^{-1} \wedge L\Omega L^{-1} \\
&- (dL)L^{-1} \wedge L\Omega L^{-1} + LBL^{-1} \wedge LBL^{-1} - (dL)L^{-1} \wedge LBL^{-1} + LBL^{-1} \wedge LdL^{-1} \\
&- (dL)L^{-1} \wedge LdL^{-1},
\end{align*}
\]

so

\[
\begin{align*}
&(dL) \wedge \Omega L^{-1} + L (d\Omega) L^{-1} - L\Omega \wedge (dL^{-1}) + (dL) \wedge BL^{-1} + L (dB) L^{-1} - LB \wedge dL^{-1} \\
&+ (dL) \wedge dL^{-1} + L\Omega L^{-1} \wedge BL^{-1} + L\Omega \wedge LdL^{-1} + LB \wedge \Omega L^{-1} - (dL) \wedge \Omega L^{-1} \\
&+ LB \wedge BL^{-1} - (dL) \wedge BL^{-1} + LB \wedge dL^{-1} - (dL) \wedge dL^{-1},
\end{align*}
\]
it becomes
\[ L (d\Omega) L^{-1} + L (dB) L^{-1} + L \Omega \wedge BL^{-1} + LB \wedge \Omega L^{-1} + LB \wedge BL^{-1}. \]

As expected, it transforms covariantly. Therefore, we can choose the equality
\[ c\varepsilon_{IJKL} e^I \wedge e^J \wedge (d\Omega + dB + \Omega \wedge B + B \wedge \Omega + B \wedge B)^{KL} = \pi_I D\tilde{e}^I d^3X, \quad (1.7) \]

where \( D\tilde{e}^I \) is the exterior covariant derivative of \( \tilde{e}^I \) along the normal of the surface \( \delta M(X^1, X^2, X^3) \), as mentioned in eq.(1.5). We try to show that there is at least one solution for this equation at end of this section. We postulated this equality because if we use the action \( c\varepsilon_{IJKL} e^I \wedge e^J \wedge (\Omega \wedge \Omega)^{KL} \) in the path integral for the states \( |\tilde{e}^I\rangle \) and \( |\pi^I\rangle \), we get the action
\[ S(e, \pi) = \int \left( c\varepsilon_{IJKL} e^I \wedge e^J \wedge (\Omega \wedge \Omega)^{KL} + \pi_I D\tilde{e}^I d^3X \right). \quad (1.8) \]

For \( \pi^I = 0 \), with the formulas \( \omega = \Omega + B \) and \( \Omega^{IJ} = \pi^K_{IJ} e^K \), eq.(1.7) becomes
\[ \varepsilon_{IJKL} e^I \wedge e^J \wedge (dB + B \wedge B)^{KL} = 0. \]

This is the Lagrangian of general relativity in the vacuum (for \( L = H = 0 \)), therefore we consider the reference connection \( B \) as spin connection in the vacuum.

Now, we get the Lagrangian eq.(1.8) by using the path integral for the action \( S_1 \):
\[ S_1 = c \int \varepsilon_{IJKL} e^I \wedge e^J \wedge (\pi^K_{IJ} e^K)\wedge (\Omega \wedge \Omega)^{ML} = c \int \varepsilon_{IJKL} e^I \wedge e^J \wedge (\Omega^K_M \wedge \Omega^M_L). \quad (1.9) \]

We use our assumption \( \Omega^{IJ} = \pi^K_{IJ} e^K \) in this action, we get
\[ S_1 = c \int \varepsilon_{IJKL} e^I \wedge e^J \wedge (\pi^K_{1J} e^K_1 \wedge (\pi^K_{2L} e^K_2). \]

Making the replacement
\[ e^I \wedge e^J \wedge e^K_1 \wedge e^K_2 \rightarrow \varepsilon^{JJK_1K_2} e^0 \wedge e^1 \wedge e^2 \wedge e^3, \]
we get

\[ S_1 = c \int \varepsilon_{IJKL} \left( \pi_{K_1}^{K} M \right) \left( \pi_{K_2}^{ML} \right) \varepsilon^{IJK_1K_2} e^0 \wedge e^1 \wedge e^2 \wedge e^3. \]

Inserting the relation \( \pi_{IJL}^{I} = \pi_{K}^{K} \varepsilon_{IJL}^{K} \) we imposed before, and using \( \varepsilon_{IJKL} \varepsilon^{IJK_1K_2} = -2 \left( \delta_{K_1}^{K} \delta_{K_2}^{L} - \delta_{K_1}^{L} \delta_{K_2}^{K} \right) \), we obtain

\[ S_1 = c \int 2\pi^{I} \varepsilon_{ILMK} \pi^{J} \varepsilon_{JLMK} e^0 \wedge e^1 \wedge e^2 \wedge e^3, \]

and using \( \varepsilon_{ILMK} \varepsilon_{JLMK} = -6 \delta_{I}^{J} \), it becomes

\[ S_1 = -12c \int \pi^{I} \pi^{I} e^0 \wedge e^1 \wedge e^2 \wedge e^3, \]

so

\[ S_1 = -12c \int \pi^{2} e^4 x. \quad (1.10) \]

This action is scalar, so we can use it in the path integral for the states \( |\tilde{e}^{I}\rangle \).

We consider \(-12c \int \pi^{2} e^4 x\) as self-energy of \(\tilde{e}^{I}\) on the closed surface \(\delta M\) on which the states \(|\tilde{e}^{I}\rangle\) and \(|\pi^{I}\rangle\) are defined eq.(1.4). As we saw before, in our gauge, the operator \(\hat{e}^4 \hat{x}\) takes eigenvalues when it acts on the states \( |\tilde{e}^{I}\rangle \).

Using eq.(1.5), we get the amplitude

\[ \langle \tilde{e}^{I} (x + dx)| e^{iS}|\pi_{I}(x)\rangle \rightarrow \langle \tilde{e}^{I} (x + dx)| e^{-i12c\pi^{2} \hat{e}^4 \hat{x}}|\pi_{I}(x)\rangle \]

\[ = \exp \left( -i12c\pi^{2} (x) e (x + dx) d^4 x + i\tilde{e}^{I} (x + dx) \pi_{I}(x)d^{3}X \right) \]

\[ \rightarrow \exp \left( -i12c\pi^{2} (x) e (x) d^4 x + i\tilde{e}^{I} (x + dx) \pi_{I}(x)d^{3}X \right), \]

where we let the momentum \(\pi_{I}\) act on the left. The amplitude of the propagation between two points \(x\) and \(x + dx\) of adjacent surfaces \(\delta M_1\) and \(\delta M_2\)
is
\[
\langle \tilde{e}_I (x + dx) | e^{-ic\pi^2 \tilde{e} d^4 \tilde{x}} | \tilde{e}_I (x) \rangle_{\delta M_1 \rightarrow \delta M_2}
\]
\[
= \int \prod_I d\pi I \langle \tilde{e}_I (x + dx) | e^{-ic\pi^2 \tilde{e} d^4 \tilde{x}} | \pi_I (x) \rangle_{\delta M_1 \rightarrow \delta M_2} \langle \pi_I (x) | \tilde{e}_I (x) \rangle_{\delta M_1}
\]
\[
= \int \prod_I d\pi I \exp \left[ -i12c\pi^2 (x) e (x + dx) d^4 x + i\tilde{e}_I (x + dx) \pi_I (x) d^3 X \right] \exp \left( -i\tilde{e}_I (x) \pi_I (x) d^3 X \right)
\]
\[
\rightarrow \int \prod_I d\pi I \exp \left[ -i12c\pi^2 (x) e (x) d^4 x + i (\tilde{e}_I (x + dx) - \tilde{e}_I (x)) \pi_I (x) d^3 X \right].
\]
The exterior covariant derivative
\[
(\tilde{e}_I (x + dx) - \tilde{e}_I (x)) d^3 X = \frac{\partial \tilde{e}_I (x)}{\partial X^0} d^3 X dX^0 = D\tilde{e}_I (x) d^3 X
\]
is along the direction of time \(dX^0\), the direction of the normal to the surface \(\delta M(X^1, X^2, X^3)\). So, it leads to propagation from one surface to another.

Thus, we write the amplitude as
\[
\langle \tilde{e}_I (x + dx) | e^{-ic\pi^2 \tilde{e} d^4 \tilde{x}} | \tilde{e}_I (x) \rangle_{\delta M_1 \rightarrow \delta M_2}
\]
\[
= \int \prod_I d\pi I \exp \left[ -i12c\pi^2 (x) e (x) d^4 x + i\tilde{e}_I (x) \pi_I (x) D\tilde{e}_I (x) d^3 X \right].
\]
The path integral is the integral of ordered product of those amplitudes on all not intersected 3D closed surfaces, thus we write it as
\[
W = \int \prod_I D\tilde{e}_I D\pi_I \exp i \int \left( -12c\pi^2 e d^4 x + \pi_I D\tilde{e}_I d^3 X \right)
\]
\[
= \int \prod_I D\tilde{e}_I D\pi_I \exp i \int \left( -12c\pi^2 e^0 \wedge e^1 \wedge e^2 \wedge e^3 + \pi_I D\tilde{e}_I d^3 X \right).
\]
Thus, we obtained the same action eq.(1.8). There is no problem with Lorentz non-invariance in \(\frac{\partial \tilde{e}_I (x)}{\partial X^0} d^3 X dX^0\), because the equation of motion we get from this path integral is
\[
\frac{\partial \tilde{e}_I (x)}{\partial X^0} \propto -\pi^I,
\]
\[
\frac{\partial \tilde{e}^I(x)}{\partial X^0} \pi_I d^3X dX^0 \propto -\pi_I \tilde{e}^I d^3X dX^0.
\]

This is Lorentz invariant. It is similar to equation of motion of scalar field \(\phi\); \(\pi = \partial_0 \phi\), which solves the same problem.

In our gauge, we have

\[
\pi_I \pi^I d^3X dX^0 \rightarrow \pi^2 dX^0 \wedge dX^1 \wedge dX^2 \wedge dX^3 = \pi^2 \epsilon_{\mu}^{\nu} \epsilon_{\sigma}^{\rho} \epsilon_{\lambda}^{\sigma} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma
\]

\[
= \pi^2 \epsilon_{\mu}^{\nu} \epsilon_{\rho}^{\sigma} \epsilon_{\lambda}^{\sigma} \varepsilon^{\mu \nu \rho \sigma} d^4x = \pi^2 e d^4x,
\]

it is an invariant element.

The path integral

\[
W = \int \prod_I D\tilde{e}^I D\pi_I \exp i \int (-12\pi^2 e^0 \wedge e^1 \wedge e^2 \wedge e^3 + \pi_I D\tilde{e}^I d^3X) \quad (1.11)
\]

vanishes unless

\[
\frac{\delta S(\pi, e)}{\delta \pi_I} = \frac{\delta}{\delta \pi_I} (-12\pi^2 e^0 \wedge e^1 \wedge e^2 \wedge e^3 + \pi_I D\tilde{e}^I d^3X)
\]

\[
= -24\pi^I e^0 \wedge e^1 \wedge e^2 \wedge e^3 + D\tilde{e}^I d^3X = 0.
\]

Therefore we get the path (the equation of motion of \(\pi^I\)):

\[
\dot{\pi}^I = \frac{1}{24\epsilon} \left( \tilde{e}^0 \wedge \tilde{e}^1 \wedge \tilde{e}^2 \wedge \tilde{e}^3 \right)^{-1} D\tilde{e}^I d^3X,
\]

or

\[
\pi^I \pi^I = \frac{1}{(24\epsilon)^2} \left( e^0 \wedge e^1 \wedge e^2 \wedge e^3 \right)^{-2} D\tilde{e}^I d^3X D\tilde{e}^I d^3X. \quad (1.13)
\]

So,

\[
-12\pi^2 e^0 \wedge e^1 \wedge e^2 \wedge e^3 + \pi_I D\tilde{e}^I d^3X = -\frac{1}{48\epsilon} \left( e^0 \wedge e^1 \wedge e^2 \wedge e^3 \right)^{-1} (D\tilde{e}_I d^3X) (D\tilde{e}^I d^3X) + \frac{1}{24\epsilon} \left( e^0 \wedge e^1 \wedge e^2 \wedge e^3 \right)^{-1} (D\tilde{e}_I d^3X) (D\tilde{e}^I d^3X).
\]
Inserting it in the path integral eq. (1.11), we get

\[ W = \int \prod_l D\hat{e}^I \exp \frac{i}{48c} \int (e^0 \wedge e^1 \wedge e^2 \wedge e^3)^{-1} (D\hat{e}_I d^3 X) (D\hat{e}^I d^3 X). \] (1.14)

Now, we calculate it using the gravitational field \( e^I \). The canonical field \( \hat{e}^I \) is defined in

\[ \hat{e}^K d^3 X = e^K_\mu e_\mu \rho \sigma dx^\nu \wedge dx^\rho \wedge dx^\sigma / 3!. \]

Applying the covariant exterior derivative, we get

\[ \left( D\hat{e}^K \right) d^3 X = \left( D_{\mu_1} e^{K \mu} \right) \hat{e} e_\mu \rho \sigma d\hat{x}^{\mu_1} \wedge d\hat{x}^{\nu} \wedge d\hat{x}^{\rho} \wedge d\hat{x}^{\sigma} / 3!, \]

where the covariant derivative \( D \) is defined in

\[ DV^I = dV^I + \omega^I_{\ J} \wedge V^J. \]

So the term

\[ \left( e^0 \wedge e^1 \wedge e^2 \wedge e^3 \right)^{-1} \left( D\hat{e}_I d^3 X \right) \left( D\hat{e}^I d^3 X \right) = \frac{(D\hat{e}_I d^3 X) (D\hat{e}^I d^3 X)}{e^0 \wedge e^1 \wedge e^2 \wedge e^3}, \]

in the path integral eq. (1.14), becomes

\[ \frac{(D_{\mu_1} e^{\mu}) e e_\mu \rho \sigma \rho dx^{\mu_1} \wedge dx^{\nu} \wedge dx^{\rho} \wedge dx^{\sigma} (D_{\mu_2} e^{I \mu'}) e e_{\mu'} \rho \sigma' \sigma' dx^{\mu_2} \wedge dx^{\nu'} \wedge dx^{\rho'} \wedge dx^{\sigma'}}{3! 3! 3! e^0 e^1 e^2 e^3 e^{\mu} \rho \sigma dx^{\mu_1} \wedge dx^{\nu_3} \wedge dx^{\rho_3} \wedge dx^{\sigma_3}}. \] (1.15)

Let us define the inverse:

\[ \left( e^0 e^1 e^2 e^3 dx^{\mu} \wedge dx^{\nu} \wedge dx^{\rho} \wedge dx^{\sigma} \right)^{-1} = E^0_\mu E^1_\nu E^2_\rho E^3_\sigma \frac{\partial}{\partial x^\mu} \wedge \frac{\partial}{\partial x^\nu} \wedge \frac{\partial}{\partial x^\rho} \wedge \frac{\partial}{\partial x^\sigma}. \]

We can rewrite:

\[ e^0 e^1 e^2 e^3 dx^{\mu} \wedge dx^{\nu} \wedge dx^{\rho} \wedge dx^{\sigma} = \frac{1}{4} \epsilon^3 \mu dx^{\mu}. \]

(Actually, we have to rewrite the tensors \( \epsilon^{\mu \nu \rho \sigma} \) and \( e_{\mu \nu \rho \sigma} \) as \( \epsilon^{-1} \epsilon^{\mu \nu \rho \sigma} \) and \( e_{\mu \nu \rho \sigma} \), but here we neglect this because we get same results).
Also, we can rewrite:
\[ E_0^{\mu'} E_1^{\nu'} E_2^{\sigma'} E_3^{\rho'} \partial_{\sigma'} \wedge \partial_{\rho'} \wedge \partial_{\mu'} = E \partial_{\nu} \wedge \partial^{3\nu}, \]
with inner product like
\[ (E \partial_{\nu} \wedge \partial^{3\nu}) \left( \frac{1}{4} ed^3 x_\mu \wedge dx^\mu \right) = \frac{1}{4} E e \partial_{\nu} \wedge \partial^{3\nu} d^3 x_\mu \wedge dx^\mu = \frac{1}{4} E e (\delta^\nu_\mu) \partial_{\nu} dx^\mu = E e = 1. \]
In general, we can write it as
\[ (E \partial_{\nu} \wedge \partial^{3\nu}) \left( ed^3 x_\mu \wedge dx^\mu \right) = E e \partial_{\nu} \wedge \partial^{3\nu} d^3 x_\mu \wedge dx^\mu = E e \delta^\nu_\mu \partial_{\nu} dx^\mu = \delta^\mu_\mu. \]
In the path integral term eq.(1.15), let us make the replacements:
\[ (D_{\mu_1} e_I^\mu) e_{\epsilon_{\mu_\nu_\rho_\sigma}} dx^{\mu_1} \wedge dx^{\nu} \wedge dx^{\rho} \wedge dx^{\sigma} / 3! \rightarrow (D_{\mu_1} e_I^\mu) edx^{\mu_1} \wedge d^3 x_\mu = -(D_{\mu_1} e_I^\mu) ed^3 x_\mu \wedge dx^{\mu_1}, \]
and
\[ (D_{\mu_2} e_I^{\mu'}) e_{\epsilon_{\mu'\nu'\rho'\sigma'}} dx^{\mu_2} \wedge dx^{\nu'} \wedge dx^{\rho'} \wedge dx^{\sigma'} / 3! \rightarrow -(D_{\mu_2} e_I^{\mu'}) ed^3 x_{\mu'} \wedge dx^{\mu_2}. \]
Let us assume the following replacement:
\[ d^3 x_\mu \wedge dx^\mu = -dx_\mu \wedge d^3 x^\mu \rightarrow d^3 x_\mu \wedge dx^{\mu_1} = -dx_\mu \wedge d^3 x^{\mu_1}. \]
There is no problem with this trick because in any 4D spacetime we have the contraction \((d^3 x_\mu \wedge dx^\nu) = \delta^\nu_\mu d^4 x.\)
Therefore, we make the replacement:
\[ -(D_{\mu_1} e_I^\mu) ed^3 x_\mu \wedge dx^{\mu_1} \rightarrow (D_{\mu_1} e_I^\mu) edx_\mu \wedge d^3 x^{\mu_1}. \]
By that, the term
\[ (D_{\mu_1} e_I^\mu) e_{\epsilon_{\mu_\nu_\rho_\sigma}} dx^{\mu_1} \wedge dx^{\nu} \wedge dx^{\rho} \wedge dx^{\sigma} (D_{\mu_2} e_I^{\mu'}) e_{\epsilon_{\mu'\nu'\rho'\sigma'}} dx^{\mu_2} \wedge dx^{\nu'} \wedge dx^{\rho'} \wedge dx^{\sigma'} / 3! \]
\[ e^{\theta_1} e^{\theta_2} e^{\rho_3} e^{\sigma_3} dx^{\mu_1} \wedge dx^{\nu} \wedge dx^{\rho} \wedge dx^{\sigma} \]
in the path integral eq.(1.14), becomes
\[ -(E \partial_{\nu} \wedge \partial^{3\nu}) \left( (D_{\mu_1} e_I^\mu) edx_\mu \wedge d^3 x^{\mu_1} \right) \left( (D_{\mu_2} e_I^{\mu'}) ed^3 x_{\mu'} \wedge dx^{\mu_2} \right) \]
Thus we can write

\[(\mu, e_{\mu}^{I}) \left( D_{\mu_{2}} e_{\mu}^{I}\right) e \left( \partial_{\nu} \wedge \partial^{\mu} \right) \left( d^{3} x_{\mu_{1}} \wedge dx^{\mu} \right) \left( d^{3} x_{\mu'} \wedge dx^{\mu_{2}} \right),\]

where we used

\[-dx_{\mu} \wedge d^{3} x_{\mu_{1}} = d^{3} x_{\mu_{1}} \wedge dx_{\mu} \text{ then } d^{3} x_{\mu_{1}} \wedge dx_{\mu}.

Thus we can write

\[
\frac{(D\nu e_{I} d^{3} X) (D\nu e_{I} d^{3} X)}{e_{0} \wedge e_{1} \wedge e_{2} \wedge e_{3}} \rightarrow (D^{\mu_{1}} e_{I_{\mu}}) \left( D_{\mu_{2}} e_{\mu}^{I}\right) e \left( \partial_{\nu} \wedge \partial^{\mu} \right) \left( d^{3} x_{\mu_{1}} \wedge dx^{\mu} \right) \left( d^{3} x_{\mu'} \wedge dx^{\mu_{2}} \right).
\]

Let us choose the contraction:

\[
(\partial_{\nu} \wedge \partial^{\mu_{1}} \left( d^{3} x_{\mu_{1}} \wedge dx^{\mu} \right) \left( d^{3} x_{\mu'} \wedge dx^{\mu_{2}} \right) = (\partial_{\nu} \wedge \partial^{3} x_{\mu_{1}} \wedge dx^{\mu}) \left( d^{3} x_{\mu'} \wedge dx^{\mu_{2}} \right)
\]

\[
= \delta_{\mu_{1}}^{\nu} \left( \partial_{\nu} \wedge dx^{\mu} \right) \left( d^{3} x_{\mu'} \wedge dx^{\mu_{2}} \right) = \delta_{\mu_{1}}^{\nu} (-dx_{\mu} \wedge \partial_{\mu}) \left( -dx_{\mu_{2}} \wedge d^{3} x_{\mu'} \right)
\]

\[
= \delta_{\mu_{1}}^{\nu} d^{3} x_{\mu} \wedge d_{\nu} dx_{\mu_{2}} \wedge d^{3} x_{\mu'} = \delta_{\mu_{1}}^{\nu} \delta_{\nu_{2}} d^{3} x_{\mu} \wedge d^{3} x_{\mu'}. \]

So, in the path integral eq.(1.14), we make the replacement:

\[
\frac{(D\nu e_{I} d^{3} X) (D\nu e_{I} d^{3} X)}{e_{0} \wedge e_{1} \wedge e_{2} \wedge e_{3}} \rightarrow (D^{\mu_{1}} e_{I_{\mu}}) \left( D_{\mu_{2}} e_{\mu}^{I}\right) e^{\nu}_{\mu_{1}} \delta_{\mu_{2}}^{\nu_{1}} dx^{\mu} \wedge d^{3} x_{\mu'}
\]

\[
= (D_{\nu} e_{I_{\mu}}) \left( D_{\nu} e_{\mu}^{I}\right) edx^{\mu} \wedge d^{3} x_{\mu'} = - (D_{\nu} e_{I_{\mu}}) \left( D_{\nu} e_{\mu}^{I}\right) edx^{3} x_{\mu'} \wedge dx^{\mu}
\]

\[
= -(D_{\nu} e_{I_{\mu}}) \left( D_{\nu} e_{\mu}^{I}\right) ed^{4} x = -(D_{\nu} e_{I_{\mu}}) \left( D_{\nu} e_{\mu}^{I}\right) ed^{4} x.
\]

We can also choose another contraction:

\[
(D^{\mu_{1}} e_{I_{\mu}}) \left( D_{\mu_{2}} e_{\mu}^{I}\right) e \left( \partial_{\nu} \wedge \partial^{\mu} \right) \left( d^{3} x_{\mu_{1}} \wedge dx^{\mu} \right) \left( d^{3} x_{\mu'} \wedge dx^{\mu_{2}} \right)
\]

\[
= (D^{\mu_{1}} e_{I_{\mu}}) \left( D_{\mu_{2}} e_{\mu}^{I}\right) e \left( \partial_{\nu} \wedge \partial^{3} x_{\mu_{1}} \wedge dx^{\mu} \right) \left( d^{3} x_{\mu'} \wedge dx^{\mu_{2}} \right)
\]

\[
= (D^{\mu_{1}} e_{I_{\mu}}) \left( D_{\mu_{2}} e_{\mu}^{I}\right) e \left( \partial_{\nu} \wedge \partial^{3} x_{\mu_{1}} \wedge dx^{\mu} \right) \left( d^{3} x_{\mu'} \wedge dx^{\mu_{2}} \right)
\]

\[
= (D^{\mu_{1}} e_{I_{\mu}}) \left( D_{\mu_{2}} e_{\mu}^{I}\right) e \left( d^{3} x_{\mu'} \wedge dx^{\mu_{2}} \right)
\]

So we get

\[
\frac{(D\nu e_{I} d^{3} X) (D\nu e_{I} d^{3} X)}{e_{0} \wedge e_{1} \wedge e_{2} \wedge e_{3}} \rightarrow (D^{\mu_{1}} e_{I_{\mu}}) \left( D_{\mu_{2}} e_{\mu}^{I}\right) ed^{4} x.
\]
Considering the two possible contractions, we write final result as
\[- (e^0 \wedge e^1 \wedge e^2 \wedge e^3)^{-1} (D\tilde{e}_I d^3 X) (D\tilde{e}^I d^3 X) = \frac{1}{2} (D_\mu e^\nu_I D^\mu e^I_\nu - D_\mu e^\nu_I D_\nu e^I_\mu) e^4 x.\]

This Lagrangian depends only on covariant derivative of the gravitational field $e^I$, similarly to Lagrangian of scalar field in curved spacetime. It is also independent of the gauge we chose for the surface $\delta M$. This Lagrangian is invariant under local Lorentz transformation $V^I \rightarrow L^I J(x) V^J$ and under any coordinate transformation $V^\mu \rightarrow \frac{\partial x^\mu}{\partial x'^\mu} V^\mu$.

The path integral of the gravitational field
\[ W = \int \prod_I D\tilde{e}^I \exp \frac{i}{48c} \int (e^0 \wedge e^1 \wedge e^2 \wedge e^3)^{-1} (D\tilde{e}_I d^3 X) (D\tilde{e}^I d^3 X) \]
becomes
\[ W = \int \prod_I De^I \exp \frac{i}{48c} \frac{1}{2} (D_\mu e^\nu_I D^\mu e^I_\nu + D_\mu e^\nu_I D_\nu e^I_\mu) e^4 x, \]
with the gravitational field Lagrangian:
\[ L d^4 x = \frac{1}{48c} \frac{1}{2} (D_\mu e^\nu_I D^\mu e^I_\nu + D_\mu e^\nu_I D_\nu e^I_\mu) e^4 x. \quad (1.16) \]

The covariant derivative $D$ is defined as $DV^I = dV^I + \omega^I J V^J$. This path integral is now defined on the states $|e^I_\mu\rangle$ and $|\omega^I J_\mu\rangle$, but with considering that it is integrated over $\omega^I J_\mu$. So, the equation of motion of $\omega^I J_\mu$ must be satisfied in this Lagrangian. We determine the constant $c$ in the Newtonian gravitational potential $c > 0$.

In weak gravity, we can use the background spacetime approximation: $D_\mu \rightarrow \partial_\mu$ and $e \rightarrow 1 + \delta e$, so we get
\[ L \rightarrow \frac{1}{48c} \frac{1}{2} \left( -\partial_\mu e^\nu_I \partial^\mu e^I_\nu + \partial_\mu e^\nu_I \partial_\nu e^I_\mu \right) \]
or
\[ L_0 = \frac{1}{48c} \frac{1}{2} \eta_{IJ} e^I_\mu \left( g^{\mu \nu} \partial^2 - \partial^\mu \partial^\nu \right) e^J_\nu. \]
But, in strong gravity, we cannot use this approximation. So, we have a problem with the determinant $e$ in the path integral

$$W = \int \prod_I D e^I \exp \frac{i}{48c} \int \frac{1}{2} \left( -D_\mu e^I_\nu D^\mu e^I_\nu + D_\mu e^I_\nu D_\nu e^I_\mu \right) e^0_\mu e^1_\nu e^2_\rho e^3_\sigma e^{\mu_1 \nu_1 \rho_1 \sigma_1} d^4x.$$ 

Using $\eta_{0123} = -1$, we rewrite it as

$$\int \prod_I D e^I \exp \frac{i}{48c} \int \frac{1}{2} \left( -D_\mu e^I_\nu D^\mu e^I_\nu + D_\mu e^I_\nu D_\nu e^I_\mu \right) (-\eta_{1JKL}) e^{I_1}_\mu e^{I_2}_\nu e^{I_3}_\rho e^{I_4}_\sigma e^{\mu_1 \nu_1 \rho_1 \sigma_1} d^4x/4!.$$ 

The path integral is independent of arbitrary changing of the coordinates $x^\mu$:

consider $e^K_\rho \rightarrow e^K_\rho + \delta e^K_\rho$, then we have

$$\frac{\delta S(e)}{\delta e^K_\rho} = 0,$$

it yields

$$-D_\mu e^\nu_I D^\mu e^I_\nu + D_\mu e^\nu_I D_\nu e^I_\mu = 0 \rightarrow D_\mu e^I = 0.$$ 

Using eq.(1.13), we get

$$\pi^2 = \frac{1}{(24c)^2} \frac{1}{2} \left( -D_\mu e^\nu_I D^\mu e^I_\nu + D_\mu e^\nu_I D_\nu e^I_\mu \right) = 0.$$ 

So,

$$L(\omega, e) = 0, \quad \text{then} \quad H(\omega, e) = 0$$

This path integral is trivial; there is no propagation because there is no gravitational energy: $H(\omega, e) = 0$, similarly to the Wheeler-DeWitt equation $\hat{H}\psi = 0[6]$. The reason is that the gravitational field $e^I_\mu$ has the entity of spacetime. It is impossible for spacetime to be dynamical on itself, to propagate over itself.

But, if we use the approximation $e^I_\mu(x) \rightarrow \delta^I_\mu + h^I_\mu(x)$, the path integral exists. So, the propagation is possible. Thus, the gravitational field propagates freely only on background spacetime. This is case of weak gravity at low energy densities.
In background spacetime, we set \( g = \eta \) and \( k_\mu e^{\mu I} = 0 \), so the path integral of weak gravitational field becomes

\[
W = \int \prod_t D e^I \exp i \int \frac{1}{48c^2} \frac{1}{2} \eta_{IJ} g^{\mu\nu} \partial^2 \eta_{IJ} \partial^\mu \partial^\nu \right) e^I_\nu d^4x. \tag{1.17}
\]

Thus, the free gravitational field propagator becomes

\[
\Delta^{\mu\nu}_{IJ}(x_2 - x_1) = 48c \int \frac{d^4k}{(2\pi)^4} \frac{\eta_{IJ} g^{\mu\nu} e^{ik(x_2 - x_1)}}{k^2 - i\varepsilon},
\]

or

\[
\Delta^{\mu\nu}_{\rho\sigma}(x_2 - x_1) = 48c \int \frac{d^4k}{(2\pi)^4} \frac{g_{\rho\sigma} g^{\mu\nu} e^{ik(x_2 - x_1)}}{k^2 - i\varepsilon}. \tag{1.18}
\]

We will use this propagation in gravitational field interaction with scalar and spinor fields.

Let us write the Lagrangian eq.(1.16) using Riemann curvature. Integrate it by parts, we get

\[
L d^4x \rightarrow \frac{1}{48c^2} \left( e^\mu D^\mu D_\mu e^I_\nu - e_\nu D_\nu D_\mu e^I_\mu \right) d^4x,
\]

and using \( D_\mu D_\nu = [D_\mu, D_\nu] + D_\nu D_\mu \), we get

\[
L d^4x \rightarrow \frac{1}{48c^2} \left( e^\mu D^\mu D_\mu e^I_\nu - e_\nu [D_\mu, D_\nu] e^I_\mu \right) d^4x.
\]

But, \([D_\mu, D_\nu] e^I_\mu = (R_{\mu\nu})^I_J e^J_\mu\), where \((R_{\mu\nu})^I_J\) is Riemann curvature tensor. By choosing \( D_\mu e^I_\mu = 0 \), we obtain

\[
L d^4x \rightarrow \frac{1}{48c^2} \left( e^\mu D^2 e^I_\mu + e^I_\nu (R_{\mu\nu})^I_J \right) d^4x.
\]

The term \( e^I_\mu e^{J\nu} (R_{\mu\nu})^I_J \) is the usual Lagrangian of general relativity, while \(-D^\nu e^I_\mu D_\nu e^I_\mu\) is new term, it is similar to Lagrangian of real scalar field \( \phi \):

\[
L = 1/2 \left( -D^\mu \phi D_\mu \phi - m^2 \phi \phi \right).
\]

Now, we simplify the formula eq.(1.7):

\[
\pi e^I_1 e^J_K \pi \wedge e^J_1 \wedge (d\Omega + dB + \Omega \wedge B + B \wedge \Omega + B \wedge B)^{KL} = \pi_1 D e^I d^3X
\]
by using the equation eq.(1.12) of the momentum $\pi^I$:

$$\pi^I = \frac{1}{24c} \left( e^0 \wedge e^1 \wedge e^2 \wedge e^3 \right)^{-1} D e^I d^3 X.$$ 

Omitting $D e^I d^3 X$ from both equations, we get

$$\varepsilon_{IJKL} e^I \wedge e^J \wedge (d\Omega + dB + \Omega \wedge B + B \wedge \Omega + B \wedge B)^{KL} = 24\pi^2 \left( e^0 \wedge e^1 \wedge e^2 \wedge e^2 \right) = 24\pi^2 e d^4 x.$$ 

From eq.(1.9) and eq.(1.10), we get

$$\varepsilon_{IJKL} e^I \wedge e^J \wedge (\Omega \wedge \Omega)^{KL} = -12\pi^2 e d^4 x.$$ 

Using it in the previous formula, we get

$$\varepsilon_{IJKL} e^I \wedge e^J \wedge (d\Omega + dB + \Omega \wedge B + B \wedge \Omega + B \wedge B)^{KL} = -2\varepsilon_{IJKL} e^I \wedge e^J \wedge (\Omega \wedge \Omega)^{KL}.$$ 

Or

$$\varepsilon_{IJKL} e^I \wedge e^J \wedge (d(\Omega + B) + (\Omega + B) \wedge (\Omega + B))^{KL} = -\varepsilon_{IJKL} e^I \wedge e^J \wedge (\Omega \wedge \Omega)^{KL},$$

(1.19)

this formula determines the tensor $\Omega$ as a function of $e^I$ and $B^IJ$. As mentioned before, we fix $B^IJ$ as a spin connection in the vacuum. We need to show that there is at least one solution for this equation. For this, let us fix $B^IJ$ for arbitrary fluctuated gravitational field $e^I$ to satisfy

$$\varepsilon_{IJKL} e^I \wedge e^J \wedge (dB + B \wedge B)^{KL} = 0.$$ 

(1.20)

So eq.(1.19) becomes

$$\varepsilon_{IJKL} e^I \wedge e^J \wedge (d\Omega + \Omega \wedge B + B \wedge \Omega + 2\Omega \wedge \Omega)^{KL} = 0,$$ 

multiplying with 2, it becomes

$$\varepsilon_{IJKL} e^I \wedge e^J \wedge (2\Omega + 2\Omega \wedge B + 2\Omega \wedge 2\Omega)^{KL} = 0,$$ 

and adding eq.(1.20) again, we get

$$\varepsilon_{IJKL} e^I \wedge e^J \wedge (d(\Omega + B) + (2\Omega + B) \wedge (2\Omega + B))^{KL} = 0.$$
For $2\Omega + B = B'$, we have

$$\varepsilon_{IJKL}e^I \wedge e^J \wedge (dB' + B' \wedge B')^{KL} = 0,$$

it is the same eq.(1.20), so for two different solutions $B^{IJ}$ and $B'^{IJ}$ corresponding to two different fluctuated gravitational fields $e^I$ and $e'^I$, we get a tensor $\Omega = (B'(e') - B(e)) / 2$. Therefore there is at least one solution for the formula eq.(1.7). As we suggested in the beginning, the tensor $\Omega^{IJ}$ relates to fluctuation of the gravitational fields $e^I$, and this fluctuation $\delta e^I$ changes the geometry of spacetime.

2 Lagrangian of the Plebanski two form field

The Plebanski two form complex field $\Sigma^i$, in selfdual representation $|\Sigma^i\rangle$, is defined by $\Sigma^i = P^i_{IJ}\Sigma^{IJ}$, where $\Sigma^{IJ} = e^I \wedge e^J$ is real anti-symmetric two form and $P^i$ is selfdual projector given in[7, 8]

$$(P^i)^{jk} = \frac{1}{2} \varepsilon^{ijk} : i = I \text{ for } I = 1, 2, 3.$$

That is

$$\Sigma^i = \frac{1}{2} \varepsilon^{ijk} e^j \wedge e^k + i e^0 \wedge e^i.$$

The complex field $\Sigma^i$ has spatial Lorentz index $i = I = 1, 2, 3$, so it transforms under $SO(3)$, the subgroup of Lorentz group $SO(3, 1)$. We derive its Lagrangian with dependence only on the covariant derivative of it, then we search for conditions which satisfy reality of the Lagrangian. We start with Lagrangian of the gravitational field eq.(1.6):

$$S(e, \omega) = c \int \varepsilon_{IJKL}e^I \wedge e^J \wedge (d\Omega + dB + \Omega \wedge \Omega + \Omega \wedge B + B \wedge \Omega + B \wedge B)^{KL}.$$

As we did before, we try to find Lagrangian that transforms covariantly under Local Lorentz transformations and contains only $\Sigma^i$, and cancel out the remaining terms. For that, we separate the action $S(e, \omega)$ to $S(e, \omega) + S(e, \Omega \wedge B, B)$. Because the actions $S(e, \omega)$ and $S(e, \Omega)$ transform covariantly, the action $S(e, \Omega \wedge B, B)$ also transforms covariantly. Therefore, we can
choose $S(e, \Omega \wedge B, B) = 0$. Let us write this action as

$$\int \varepsilon_{IJKL} e^I \wedge e^{J} \wedge (d\Omega + \Omega \wedge \Omega)^{KL} + I(e, \Omega \wedge B, B).$$

Using the assumption $\Omega^{IJ} = \pi^I_M e^M$ in the first part, we obtain

$$S(e, \pi) = c \int \left[ \varepsilon_{IJKL} e^I \wedge e^J \wedge d \left( \pi^K_M e^M \right) + \varepsilon_{IJKL} e^I \wedge e^J \wedge \left( \pi^K_1 e^K_1 \wedge (\pi^K_2^{ML} e^K_2) \right) \right] + I(e, \Omega \wedge B, B).$$

We assume that the integral of

$$\varepsilon_{IJKL} e^I \wedge e^J \wedge \left( \pi^K_M e^M \right)$$

is zero at infinities. Using

$$d\Sigma^{IJ} \wedge (\pi^K_M e^M) = - (\pi^K_M) e^M \wedge d\Sigma^{IJ} + e^I \wedge e^J \wedge d \left( \pi^K_M e^M \right),$$

the action becomes

$$S(e, \pi) = c \int \left[ \varepsilon_{IJKL} \left( \pi^K_M e^M \right) \wedge d\Sigma^{IJ} + \varepsilon_{IJKL} \Sigma^{IJ} \wedge \left( \pi^K_1 e^K_1 \wedge (\pi^K_2^{ML} e^K_2) \right) \right] + I(e, \Omega \wedge B, B),$$

or

$$S(e, \pi) = c \int \left[ \varepsilon_{IJKL} \left( \pi^K_M e^M \right) \wedge d\Sigma^{IJ} + \varepsilon_{IJKL} \left( \pi^K_1 M \right) \left( \pi^K_2^{ML} \right) \Sigma^{IJ} \wedge \Sigma^{K_1 K_2} \right] + I(e, \Omega \wedge B, B).$$

Using our assumption:

$$\pi^{IJK} = \pi_L \varepsilon^{LIJK},$$

we get

$$\varepsilon_{IJKL} \left( \pi^K_M e^M \right) = \varepsilon_{IJKL} \pi^{MKL} e_M = \varepsilon_{IJKL} \pi_N \varepsilon^{NML} e_M = -2 (\pi_I e_J - \pi_J e_I).$$
Making the replacement:

\[ \Sigma^{IJ} \wedge \Sigma^{K_1 K_2} \rightarrow \varepsilon^{IJK_1 K_2} \Sigma^{01} \wedge \Sigma^{23}, \]

we get

\[ \varepsilon_{IJKL} \left( \pi_{K_1}^{M} \right) \left( \pi_{K_2}^{ML} \right) \Sigma^{IJ} \wedge \Sigma^{K_1 K_2} = \varepsilon_{IJK_1 K_2} \Sigma^{01} \wedge \Sigma^{23} \]

\[ = 2 \left( \pi_{K}^{M} \right) \left( \pi_{K}^{ML} \right) \Sigma^{01} \wedge \Sigma^{23} = 2 \left( \pi_{LKM} \right) \left( \pi_{K}^{ML} \right) \Sigma^{01} \wedge \Sigma^{23} \]

\[ = 2 \left( \varepsilon_{IKML} \right) \left( \pi_{KML} \right) \Sigma^{01} \wedge \Sigma^{23} = 2 \left( \varepsilon_{IKML} \right) \left( \pi_{KML} \right) \Sigma^{01} \wedge \Sigma^{23} \]

\[ = -12 \pi^2 \Sigma^{01} \wedge \Sigma^{23}. \]

Therefore, the action becomes

\[ S(e, \pi, \Sigma) = c \int \left[ -2 (\pi_I e_J - \pi_J e_I) \wedge d\Sigma^{IJ} - 12 \pi_I \pi^I \Sigma^{01} \wedge \Sigma^{23} \right] + I(e, \Omega \wedge B, B). \]

Because the real Plebanski two form \( \Sigma^{IJ} = e^I \wedge e^J \) is anti-symmetric, we can rewrite:

\[ S(e, \pi, \Sigma) = c \int \left[ -4 \pi_I e_J \wedge d\Sigma^{IJ} - 12 \pi_I \pi^I \Sigma^{01} \wedge \Sigma^{23} \right] + I(e, \Omega \wedge B, B), \]

then using \( \varepsilon_{0123} = -1 \), we rewrite it as

\[ S(e, \pi, \Sigma) = c \int \left[ -4 \pi_I e_J \wedge d\Sigma^{IJ} + 12 \pi_I \pi^I \varepsilon_{IJKL} \Sigma^{IJ} \wedge \Sigma^{KL} / 4! \right] + I(e, \Omega \wedge B, B), \]

or

\[ S(e, \pi, \Sigma) = c \int \left[ -4 \pi_I e_J \wedge d\Sigma^{IJ} + \frac{1}{2} \pi^2 \varepsilon_{IJKL} \Sigma^{IJ} \wedge \Sigma^{KL} \right] + I(e, \Omega \wedge B, B). \]

The path integral over the momentum \( \pi^I \) vanishes unless

\[ \frac{\delta S(e, \pi, \Sigma)}{\delta \pi_I} = \frac{\delta}{\delta \pi_I} \int \left[ -4 \pi_I e_J \wedge d\Sigma^{IJ} + \frac{1}{2} \pi^2 \varepsilon_{IJKL} \Sigma^{IJ} \wedge \Sigma^{KL} \right] + \frac{\delta I(e, \Omega \wedge B, B)}{\delta \pi_I} = 0. \]

So we get the equation of motion of \( \Sigma^{IJ} \). But, it is not easy to separate \( \Sigma \) from \( e \). It is similar to the gravitational field, it is separable only in weak gravity in background spacetime. Thus, we solve it in background spacetime,
then we generate the solution to arbitrary spacetime. Therefore, we use the approximation:

\[
\int \left( -4\pi I e J \wedge d\Sigma^{I J} + \frac{1}{2} \pi^2 \varepsilon_{IJKL} \Sigma^{I J} \wedge \Sigma^{KL} \right)
\]

\[
\to \int \left( -4\pi I e_{\mu J} \partial_{\nu} \Sigma_{\rho \sigma}^{I J} \varepsilon^{\mu \nu \rho \sigma} + \frac{1}{2} \pi^2 \varepsilon_{IJKL} \Sigma_{\mu \nu}^{I J} \Sigma_{\rho \sigma}^{KL} \varepsilon^{\mu \nu \rho \sigma} \right) d^4 x.
\]

The background spacetime approximation is

\[
e^I_\mu (x) \to \delta^I_\mu (x), \quad e \to 1 + \delta e,
\]

thus we get

\[
\Sigma^{I J}_{\mu \nu} = \frac{1}{2} (e^I_\mu e^J_\nu - e^I_\nu e^J_\mu) \to \frac{1}{2} \left( \delta^I_\mu \delta^J_\nu - \delta^I_\nu \delta^J_\mu \right) + \frac{1}{2} \left( h^I_\mu \delta^J_\nu - h^I_\nu \delta^J_\mu \right) + \frac{1}{2} \left( \delta^I_\mu h^J_\nu - \delta^I_\nu h^J_\mu \right).
\]

Inserting it in the action

\[
S (e, \Sigma) = c \int \left( -4\pi I e_{\mu J} \partial_{\nu} \Sigma_{\rho \sigma}^{I J} \varepsilon^{\mu \nu \rho \sigma} + \frac{1}{2} \pi^2 \varepsilon_{IJKL} \Sigma_{\mu \nu}^{I J} \Sigma_{\rho \sigma}^{KL} \varepsilon^{\mu \nu \rho \sigma} \right) d^4 x + I (e, \Omega \wedge B, B),
\]

it becomes

\[
S (e, \Sigma) \to S (h, \delta \Sigma) = c \int \left( -4\pi I \partial_{\nu} \Sigma_{\rho \sigma}^{I J} \varepsilon_{J}^{\mu \nu \rho \sigma} + \frac{1}{2} \pi^2 (-24) + \ldots \right) d^4 x.
\]

Therefore, the condition

\[
\frac{\delta}{\delta \pi_I} \int \left[ -4\pi I e_{J} \wedge d\Sigma^{I J} + \frac{1}{2} \pi^2 \varepsilon_{IJKL} \Sigma^{I J} \wedge \Sigma^{KL} \right] + \frac{\delta I (e, \Omega \wedge B, B)}{\delta \pi_I} = 0
\]

approximates to

\[
\frac{\delta}{\delta \pi_I} \int \left( -4\pi I \partial_{\nu} \Sigma_{J \rho \sigma}^{I} \varepsilon_{J \nu \rho \sigma} + \frac{1}{2} \pi^2 (-24) + \ldots \right) d^4 x = 0.
\]

Its solution is

\[
\pi^I = -\frac{1}{6} \partial_{\nu} \Sigma_{J \rho \sigma}^{I} \varepsilon_{J \nu \rho \sigma} + \ldots = -\frac{1}{6} \partial_{\nu} \Sigma_{J \rho \sigma}^{I} \varepsilon_{J \nu \rho \sigma} + \ldots
\]
Thus, the action in the background spacetime approximates to

\[ S(\Sigma) \rightarrow c \int \left[ \frac{2}{3} \partial^{\rho_1} \Sigma^{I_1 J_1 \rho_1 \sigma_1} \varepsilon_{J_1 \rho_1 \sigma_1} \partial_{\rho} \Sigma^{I_1 J_1 \rho \sigma} \varepsilon^{\rho \sigma} + \ldots \right] d^4 x. \]

Defining inner product via \( \Sigma^{I_1 J_1 \rho_1 \sigma_1} \Sigma_{I_1 J_1 \rho \sigma} = \Sigma^2 \delta_{I_1}^{J_1} \delta_{\rho_1}^{\rho} \delta_{\sigma_1}^{\sigma} \), we get

\[ S(\Sigma) \rightarrow c \int (-4 \partial_{\mu} \Sigma^{\nu \rho}_{I J} \partial_{\nu} \Sigma_{I J}^{\nu \rho} + \ldots) d^4 x \quad \text{with} \quad \partial_{\mu} \Sigma^{\nu \rho}_{I J} = 0. \]

This is action of real Plebanski two form in approximation of background spacetime. It is similar to scalar field. The corresponding Lagrangian is

\[ L_0(\Sigma) \rightarrow -4c (\partial_{\mu} \Sigma^{\nu \rho}_{I J}) (\partial_{\nu} \Sigma_{I J}^{\nu \rho}) \quad \text{with} \quad \partial_{\mu} \Sigma^{\nu \rho}_{I J} = 0. \]

In curved spacetime, we rewrite it as

\[ L_0(\Sigma) d^4 x \rightarrow -4c' (\partial_{\mu} \Sigma^{\nu \rho}_{I J}) (\partial_{\nu} \Sigma_{I J}^{\nu \rho}) \, ed^4 x. \tag{2.1} \]

It does not transform covariantly because the partial derivative \( \partial_{\mu} \) does not. But the total Lagrangian \( L(e, \omega) \) transforms covariantly, thus \( L(e, \omega) = L(e, d \Sigma, \Sigma) + L(e, \Sigma \wedge B, B) \) transforms covariantly. Let us rewrite:

\[ L(e, d \Sigma, \Sigma) + L(e, \Sigma \wedge B, B) = L(e, d \Sigma, \Sigma) + \Delta L - \Delta L + L(e, \Sigma \wedge B, B), \]

with \( L(e, d \Sigma, \Sigma) + \Delta L \) transforms covariantly, and choose \(-\Delta L + L(e, \Sigma \wedge B, B) = 0\), which determines the reference connection \( B \). Thus, we get

\[ L(e, d \Sigma, \Sigma) + \Delta L \rightarrow L(\Sigma) d^4 x = -4c' (D_{\mu} \Sigma^{\nu \rho}_{I J}) (D^{\mu} \Sigma_{I J}^{\nu \rho}) \, ed^4 x, \]

which transforms covariantly, where \( D \Sigma^{I J} = d \Sigma^{I J} + \omega^{I J} \wedge \Sigma^{K} + \omega^{J K} \wedge \Sigma^{I K} \).

We get Lagrangian of the complex Plebanski two form field \( \Sigma^i \) by using the selfdual projector \( P^i \), which projects the real Plebanski two form \( \Sigma^{I J} \) into two states: selfdual \( |\Sigma^i \rangle \) and anti-selfdual \( |\bar{\Sigma}^i \rangle \). Thus, the term

\[ D_{\mu} \Sigma^{\nu \rho}_{I J} D^{\mu} \Sigma_{I J}^{\nu \rho} \]

in the Lagrangian becomes

\[ D_{\mu} \Sigma^{\nu \rho}_{i} D^{\mu} \Sigma_{i}^{\nu \rho} + D_{\mu} \bar{\Sigma}^{\nu \rho}_{i} D^{\mu} \bar{\Sigma}_{i}^{\nu \rho}, \]

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where the hermitian conjugate $\bar{\Sigma}^\nu_\rho \Sigma^i_\nu\rho$ is represented in anti-selfdual $\bar{\Sigma}^i = \bar{P}^i_I J \Sigma^{IJ}$.

We search for conditions which allow us to rewrite the complex Plebanski two form field $\Sigma^i$ as a real field. To do this, let us choose $\bar{\Sigma}^i = 0$, which cancels out the terms of $\bar{\Sigma}^i$ and makes $\Sigma^i$ real field. So,

$$\bar{\Sigma}^i = \frac{1}{2} \varepsilon^{ijk} \Sigma_{jk} - i \Sigma^0_i = 0 \rightarrow \frac{1}{2} \varepsilon^{ijk} \Sigma_{jk} = i \Sigma^0_i,$$  

(2.2)

generally it becomes $\frac{1}{2} \varepsilon^{IJKL} \Sigma_{KL} = i \Sigma^{IJ}$. Therefore, $\Sigma^0_i$ and $e^0$ are pure imaginary, so we replace $X^0$ by $iX^0$ and the metric $\eta^{IJ} = (- + + +)$ by $(+ + + +)$.

Therefore, the Plebanski two form field in selfdual representation becomes

$$\Sigma^i = \frac{1}{2} \varepsilon^{ijk} \Sigma_{jk} + i \Sigma^0_i = \varepsilon^{ijk} \Sigma_{jk},$$

which is real as required for satisfying the reality condition.

Therefore, the Lagrangian of Plebanski two form field in selfdual representation becomes:

$$L_0(\Sigma) d^4x = -4c^I (D_\mu \Sigma^\nu_\rho) (D^\mu \Sigma^i_{\nu\rho}) e d^4x.$$  

(2.3)

Because $\Sigma_i$ is real, so the covariant derivative is $D \Sigma_i = d \Sigma_i + \omega^j_i \wedge \Sigma^j_i$. Let us combine the gravitational and Plebanski fields in one field $K^i_\mu$ via

$$K^i_\mu = \frac{1}{2} \left( e^i_\mu + \frac{i}{4} \varepsilon^{ijk} \varepsilon_{0 \mu \rho \sigma} \Sigma^{\rho \sigma}_{jk} \right),$$

its hermitian conjugate is

$$\bar{K}^i_\mu = \frac{1}{2} \left( e^i_\mu - \frac{i}{4} \varepsilon^{ijk} \varepsilon_{0 \mu \rho \sigma} \Sigma^{\rho \sigma}_{jk} \right),$$

where $i, j$ and $k$ are local-Lorentz frame indices for $I = i = 1, 2, 3$. And $K^0_\mu = e^0_\mu$ with $\bar{K}^0_\mu = -e^0_\mu$, here $e^0_\mu$ is pure imaginary as mentioned in eq.(2.2).

Therefore, we get

$$K^i_\mu + \bar{K}^i_\mu = e^i_\mu \text{ and } K^i_\mu - \bar{K}^i_\mu = \frac{i}{4} \varepsilon^{ijk} \varepsilon_{0 \mu \rho \sigma} \Sigma^{\rho \sigma}_{jk} = \frac{i}{2} \varepsilon_{0 \mu \rho \sigma} \Sigma^{i \rho \sigma},$$
then we get
\[ (D^\nu K^i_\mu + D^\nu \bar{K}^i_\mu) (D_\nu K^\mu_i + D_\nu \bar{K}^\mu_i) = D^\nu e^i_\mu D_\nu e^\mu_i, \]
and
\[ (D^\nu K^i_\mu - D^\nu \bar{K}^i_\mu) (D_\nu K^\mu_i - D_\nu \bar{K}^\mu_i) = -\frac{1}{4} \varepsilon_{0\mu\rho\sigma} \varepsilon^{0\mu'\sigma'} D^{\nu\Sigma} \Sigma_{\mu\rho\sigma} \Sigma_{\mu'\sigma'} \]
\[ = -D^{\nu\Sigma} \Sigma_{\mu\rho\sigma}. \]

Therefore, we have
\[ D^\nu e^i_\mu D^\nu e^\mu_i + D^\nu \Sigma_{\mu\rho\sigma} D^\nu \Sigma_{\mu\rho\sigma} \rightarrow 4D^\nu \bar{K}^i_\mu D^\mu K^i. \]

Using this in gravitational Lagrangian:
\[ L(e) d^4x = -\frac{1}{48c} \frac{1}{2} (D_\mu e^\nu_I) (D^{\nu} e_I) e d^4x, \]
and in Plebanski Lagrangian eq.(2.3):
\[ L(\bar{\Sigma}) d^4x = -8c' \frac{1}{2} (D^{\nu} \Sigma_{\mu\rho\sigma} D^\nu \Sigma_{\mu\rho\sigma}) e d^4x, \]
setting \(8c' = 1/(48c)\), we get
\[ L(e) d^4x + L(\bar{\Sigma}) d^4x \rightarrow \frac{1}{12c} \frac{1}{2} (D^\nu \bar{K}^i_\mu D_\nu K^\mu_i + D^\nu \bar{K}^\mu_0 D_\nu K^{0\mu}) e d^4x. \quad (2.4) \]

Using the metric (+ + ++), it becomes
\[ L(e) d^4x + L(\bar{\Sigma}) d^4x \rightarrow \frac{1}{12c} \frac{1}{2} (\delta_{IJ} D^\nu \bar{K}^I_\mu D_\nu K^{J\mu} + D^\nu \bar{K}^\mu_J D_\nu K^{J\mu}) e d^4x. \quad (2.5) \]

Generally, we write
\[ L(K) d^4x = \frac{1}{12c} \frac{1}{2} \delta_{IJ} (D^\nu \bar{K}^I_\mu D_\nu K^{J\mu} - D^\nu \bar{K}^\mu_J D_\nu K^J) e d^4x. \]

It satisfies the reality condition as required.
3 Static potential of weak gravity

We derive the static potential of scalar and spinor fields interactions with weak gravitational field in the static limit; the Newtonian gravitational potential. We find that this potential has same structure for both fields, it depends only on the distances between the particles and on their energies. By that, we determine the constant $c > 0$.

The action of scalar field in arbitrary curved spacetime is

$$S(e, \phi) = \int d^4x \left( \eta^{IJ} e_I^\mu e_J^\nu D_\mu \phi^+ D_\nu \phi - V(\phi) \right).$$

In weak gravity, we use approximation of background spacetime:

$$e_I^\mu(x) \to \delta_I^\mu + h_I^\mu(x), \quad e \to 1 + \delta e.$$ 

Thus, action approximates to

$$S(e, \phi) = \int d^4x \left( \partial_\mu \phi^+ \partial^\mu \phi + h^{\mu\nu}(x) \partial_\mu \phi^+ \partial_\nu \phi + h^{\mu\nu}(x) \partial_\mu \phi^+ \partial_\nu \phi - V(\phi) + ... \right).$$

The gravitational field is symmetric, so we get

$$S(e, \phi) = \int d^4x \left( \partial_\mu \phi^+ \partial^\mu \phi + 2h^{\mu\nu}(x) \partial_\mu \phi^+ \partial_\nu \phi - V(\phi) + ... \right).$$

The energy-momentum tensor of scalar field is

$$T_{\mu\nu} = \partial_\mu \phi^+ \partial^\nu \phi + g_{\mu\nu} L,$$

hence

$$\partial_\mu \phi^+ \partial^\nu \phi = T_{\mu\nu} - g_{\mu\nu} L.$$ 

Inserting it in the Lagrangian, it becomes

$$L = \partial_\mu \phi^+ \partial^\mu \phi + 2h^{\mu\nu}(x) (T_{\mu\nu} - g_{\mu\nu} L) - V(\phi) + ...$$

or

$$L = \partial_\mu \phi^+ \partial^\mu \phi + 2h^{\mu\nu} T_{\mu\nu} - V(\phi) - 2h^{\mu\nu} g_{\mu\nu} L + ....$$
Therefore, in the interaction term, we make the replacement:

\[ \partial_{\mu} \phi^+ \partial_{\nu} \phi \rightarrow T_{\mu \nu} \quad \text{and} \quad V \rightarrow V + 2h^{\mu \nu}g_{\mu \nu}L. \]

Because the gravitational field is weak, \( 2h^{\mu \nu}g_{\mu \nu}L \) is neglected compared with \( L \).

We find the potential \( V(r) \) of exchanged virtual gravitons by two particles \( k_1 \) and \( k_2 \) using \( M (k_1 + k_2 \rightarrow k_1' + k_2') \) matrix element (like Born approximation to the scattering amplitude in non-relativistic quantum mechanics [10]).

For one of Feynman diagrams, we have

\[ iM (k_1 + k_2 \rightarrow k_1' + k_2') = \frac{i}{48c} \eta_{IJ} (g^{\mu \nu} \partial_\rho - \partial_\mu \partial_\nu) e^I_{\rho \sigma} \Delta^{IJ} (q) \frac{i}{q^2} \eta_{IJ} (g^{\mu \nu} \partial_\rho - \partial_\mu \partial_\nu) e^J_{\sigma \nu}, \]

with

\[ q = k_1' - k_1 = k_2 - k_2'. \]

The propagator \( \Delta^{\mu \nu \rho \sigma} (x_2 - x_1) \) is the gravitons propagator eq.(1.18) that we get from Lagrangian of free gravitational field in background spacetime:

\[ L_0 = \frac{1}{48c^2} \eta_{IJ} e^I_{\mu} (g^{\mu \nu} \partial_2^2 - \partial_\mu \partial_\nu) e^J_{\nu} \rightarrow \frac{1}{48c^2} \eta_{IJ} h^I_{\mu} (g^{\mu \nu} \partial_2^2 - \partial_\mu \partial_\nu) h^J_{\nu}. \]

With the gauge \( \partial_\mu e^I_{\mu} = 0 \), we get

\[ \Delta^{IJ}_{\mu \nu} (y - x) = \int \frac{d^4q}{(2\pi)^4} \Delta^{IJ}_{\mu \nu} (q^2) e^{iq(y-x)} : \Delta^{IJ}_{\mu \nu} (q^2) = 48c^2 \frac{g_{\mu \nu} \eta_{IJ}}{q^2 - i\varepsilon}. \]

Therefore, the \( M \) matrix element becomes

\[ iM (k_1 + k_2 \rightarrow k_1' + k_2') = i48c \frac{g_{\mu \nu} g^{\rho \sigma}}{q^2} (-ik_1')_{\rho} (ik_1)_{\nu}, \]

where

\[ g = \eta \quad \text{and} \quad q = k_1' - k_1 = k_2 - k_2'. \]

Comparing it with [10]

\[ iM (k_1 + k_2 \rightarrow k_1' + k_2') = -i \bar{V} (q) \delta^4 (k_{\text{out}} - k_{\text{in}}), \]
we get

\[ \bar{V} (q^2) = -48c (-i k'_2)_\mu (i k_2)_\mu \frac{g^{\mu \nu} g^{\rho \sigma}}{q^2} (-i k'_1)_\sigma (i k_1)_\nu . \]

Then, comparing this formula with the replacement:

\[ \partial_\mu \phi^+ \partial_\nu \phi \rightarrow T_{\mu \nu}, \]

and evaluating inverse Fourier transform, we get

\[ V (y - x) = -48c T_{\mu \nu} (y) g^{\mu \nu} g^{\rho \sigma} T_{\nu \sigma} (x) \frac{1}{4\pi |y - x|} = -48c \frac{T_{\mu \nu} (y) T^{\mu \nu} (x)}{4\pi |y - x|}, \]

where \( T^{\mu \nu} \) is transferred energy-momentum tensor. It is anti-symmetric, so the summation over the indices \( \mu \) and \( \nu \) is repeated twice. Therefore, we divide the right side by 2:

\[ V (y - x) = -\frac{48c T_{\mu \nu} (y) T^{\mu \nu} (x)}{2} \cdot \frac{1}{4\pi |y - x|}. \]

In static limit, for one particle, we approximate \( T^{00} \) to \( m \), where \( m \) is the mass of interacted particles.

Thus, we get Newtonian gravitational potential:

\[ V (y - x) = -\frac{48c m^2}{2} \cdot \frac{1}{4\pi |y - x|} = -\frac{G m^2}{|y - x|} \rightarrow 48c = 8\pi G. \]

Therefore, the weak gravitational Lagrangian becomes

\[ L_0 = \frac{1}{4\pi G} \frac{1}{4} \eta_{IJ} e^I_\mu \left( g^{\mu \nu} \partial^2 - \partial^\mu \partial^\nu \right) e^J_\nu. \]

We do the same thing for spinor fields’ interactions with gravitational field. The action is[1]

\[ S(e, \psi) = \int d^4x e\left( i e^\mu \bar{\psi} \gamma^\mu D_\mu \psi - m \bar{\psi} \psi \right), \]

where the covariant derivative \( D_\mu \) is

\[ D_\mu = \partial_\mu + (\omega_\mu)_J^I L^J_I + A_\mu T^a. \]
In background spacetime, it becomes

\[ S(e, \psi) = \int d^4x \left( i \bar{\psi} \gamma^\mu D_\mu \psi + i h_\mu^I \bar{\psi} \gamma^I D_\mu \psi - m \bar{\psi} \psi + \ldots \right). \]

Let us consider only the terms:

\[ \int d^4x \left( i \bar{\psi} \gamma^\mu \partial_\mu \psi + i h_\mu^\nu \bar{\psi} \gamma^\nu \partial_\mu \psi - m \bar{\psi} \psi \right) : \quad g = \eta. \]

The energy-momentum tensor of spinor field is\([9]\]

\[ T^{\mu\nu} = -i \bar{\psi} \gamma^\mu \partial^\nu \psi + g^{\mu\nu} L. \]

Therefore, in the interaction term, we have the replacements

\[ i \bar{\psi} \gamma^\mu \partial^\nu \psi \rightarrow -T^{\mu\nu} \quad \text{and} \quad L \rightarrow L + h^{\mu\nu} g_{\mu\nu} L. \]

The term \( h^{\mu\nu} g_{\mu\nu} L \) is neglected compared with the Lagrangian \( L \). We find \( M \) matrix element of exchanged virtual gravitons \( p_1 + p_2 \rightarrow p'_1 + p'_2 \). For one of Feynman diagrams\([10]\):

\[ i M (p_1 + p_2 \rightarrow p'_1 + p'_2) = i 48 c \bar{u} (p'_1) \gamma^\mu (-i p_1)_\nu u (p_1) \frac{g_{\mu\sigma} g^{\nu\rho}}{q^2} \bar{u} (p'_2) \gamma^\sigma (-i p_2)_\rho u (p_2), \]

with

\[ q = p'_1 - p_1 = p_2 - p'_2 \quad \text{and} \quad g = \eta, \]

we get

\[ \bar{V} (q^2) = -48 c \bar{u} (p'_1) \gamma^\mu (-i p_1)_\nu u (p_1) \frac{g_{\mu\sigma} g^{\nu\rho}}{q^2} \bar{u} (p'_2) \gamma^\sigma (-i p_2)_\rho u (p_2). \]

Comparing this formula with the replacement:

\[ i \bar{\psi} \gamma^\mu \partial^\nu \psi \rightarrow -T^{\mu\nu}, \]

and evaluating inverse Fourier transform, we get

\[ V (y - x) = -48 c \left( -T^{\mu\rho} (y) \right) g^{\mu\sigma} g^{\nu\rho} \left( -T_{\nu\sigma} (x) \right) \frac{1}{4\pi |y - x|} = -48 c \frac{T^{\mu\nu} (y) T^{\mu\nu} (x)}{4\pi |y - x|}, \]

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where $T^{\mu\nu}$ is transferred energy-momentum tensor. Dividing the right side by 2:

$$V(y - x) = -\frac{48c}{2} \frac{T^{\mu\nu}(y) T^{\mu\nu}(x)}{4\pi |y - x|}.$$  

In the static limit, for one particle, we approximate $T^{00}$ to $m$, where $m$ is the mass of interacted particles. Thus, we get Newtonian gravitational potential:

$$V(y - x) = -\frac{48c}{2} \frac{m^2}{4\pi |y - x|} = -G \frac{m^2}{|y - x|} \rightarrow 48c = 8\pi G.$$  

It is the same potential we found for the interaction of a scalar field with the gravitational field.

4 Summary

We have derived Lagrangian of gravitational field with dependence only on the second covariant derivative, like electromagnetic and scalar fields, that means it has the same symmetries. So it makes it easier for unification the gravity with other fields. We postulated the gravity propagation as expansion of closed 3D surfaces in 4D arbitrary spacetime manifold, so this propagation relates to changing in geometry of those surfaces due to that expansion. This is dynamics of gravity; changing the geometry. We suggested spin connection splitting, $\omega \rightarrow \Omega + B$, and canonical states $|\tilde{e}^I\rangle$ and $|\pi^I\rangle$ just for using them in path integral to get eq.(1.11):

$$W = \int \prod_I d\tilde{e}^I D\pi_I \exp i \int (-12c\pi^2 e_0 \wedge e^1 \wedge e^2 \wedge e^3 + \pi_I D\tilde{e}^I d^3X).$$  

Comparing the Lagrangian $-12c\pi^2 e_0 \wedge e^1 \wedge e^2 \wedge e^3 + \pi_I D\tilde{e}^I d^3X$ with Lagrangian eq.(1.6):

$$c\varepsilon_{IJKL} e^J \wedge e^I \wedge (d\Omega + dB + \Omega \wedge \Omega + \Omega \wedge B + B \wedge \Omega + B \wedge B)^{KL},$$

we postulate the equality eq.(1.7):

$$c\varepsilon_{IJKL} e^J \wedge e^I \wedge (d\Omega + dB + \Omega \wedge B + B \wedge \Omega + B \wedge B)^{KL} = \pi_I D\tilde{e}^I d^3X.$$
We wrote this formula in simpler form; eq.(1.19).
In that path integral we considered $c\varepsilon_{IJKL}e^I \wedge e^J \wedge (\Omega \wedge \Omega)^{KL}$ as self-energy of $\tilde{e}^I$ on the surface $\delta M$, and $\pi_I D\tilde{e}^I d^3X$ as kinetic energy which relates to expansion of those surfaces. Using the original states $|e^I_\mu\rangle$ and $|\omega^I_\mu\rangle$, we get

$$16\pi GL(e, \omega) = (-D_\mu e^\nu_I D^\mu e^I_\nu + D_\mu e^\nu_I D_\nu e^I_\mu) e d^4x.$$ 

We use same methods for Plebanski field to get Lagrangian like

$$L(\Sigma) d^4x = -4c^I (D_\mu \Sigma^\nu_i) (D^\mu \Sigma^i_\nu) e d^4x,$$

with the gauge $(1/2)\varepsilon^{IJKL}\Sigma_{KL} = i\Sigma^{IJ}$. Therefore, $\Sigma^\nu_i$ and $e^0$ are pure imaginary, so we replace $X^0$ by $iX^0$ and the metric $\eta^{IJ} = (-+++)$ by $(++++)$. This metric allows us to combine the gravititional and Plebanski fields in one field $K^I_\mu$, we get the Lagrangian

$$24cL(K) d^4x = -\delta_{IJ} (D^\nu K^I_\mu D_\nu K^J_\mu - D^\nu \bar{K}^I_\mu D^\mu K^J_\nu) e d^4x.$$ 

Finally we derived the interaction potential of spinor and scalar fields with the gravity in static limit "Newtonian gravitational potential" which allows us to determine the constant $c$.

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