The local Langlands correspondence matches irreducible representations of a reductive \( p \)-adic group \( G(F) \) with enhanced \( L \)-parameters. It is conjectured by Hellmann and Zhu that it can be categorified. Then it should become a fully faithful functor from a derived category of representations to a derived category of equivariant sheaves on some variety of \( L \)-parameters.

We approach this conjecture in the case of finite length \( G(F) \)-representations. Then it runs via graded Hecke algebras associated to Bernstein components or to enhanced \( L \)-parameters. Here we work with graded Hecke algebras \( \mathbb{H} \) on the Galois side, those can be constructed entirely in terms of a complex reductive group, endowed with data from \( L \)-parameters.

We fix an arbitrary central character \((\sigma,r)\) of \( \mathbb{H} \) (which encodes the image of Frobenius by an \( L \)-parameter). That leads to a variety \( g^\sigma_N \) of nilpotent elements in the Lie algebra of \( G \) (possibilities for the monodromy operator from an \( L \)-parameter) and to a complex of equivariant constructible sheaves \( K^\sigma_N \) on \( g^\sigma_N \).

We relate the (derived) endomorphism algebra of \((g^\sigma_N, K^N)\) to a localization of \( \mathbb{H} \), which yields an equivalence between the appropriate categories of finite length modules of these algebras. From there we construct a fully faithful functor between:

- the bounded derived category of finite length \( \mathbb{H} \)-modules specified by the central character \((\sigma,r)\),
- the equivariant bounded derived category of constructible sheaves on \( g^\sigma_N \).

Also, we explicitly determine the images of standard modules under this functor. We expect that these results pave the way for more general instances of the aforementioned conjectural extension of the local Langlands correspondence.

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Introduction

The story behind this paper started with the seminal work of Kazhdan and Lusztig [KaLu]. They showed that an affine Hecke algebra $\mathcal{H}$ is naturally isomorphic with a $K$-group of equivariant coherent sheaves on the Steinberg variety of a complex reductive group. (Here $\mathcal{H}$ has a formal variable $q$ as single parameter and the reductive group must have simply connected derived group.) This isomorphism enables one to regard the category of equivariant coherent sheaves on that particular variety as a categorification of an affine Hecke algebra. Later that became quite an important theme in the geometric Langlands program, see for instance [Bez].

This paper is inspired by the quest for a generalization of such a categorification of $\mathcal{H}$ to affine Hecke algebras with more than one $q$-parameter. That is relevant because such algebras arise in abundance from reductive $p$-adic groups and types [ABPS2, §2.4]. However, up to today it is unclear how several independent $q$-parameters can be incorporated in a setup with equivariant $K$-theory or $K$-homology. The situation improves when one localizes an affine Hecke algebra with respect to (the kernel of) a central character, as in [Lus2]. Such a localization is Morita equivalent with a localization of a graded Hecke algebra with respect to a central character.

Graded Hecke algebras $\mathbb{H}$ with several parameters (now typically called $k$) do admit a geometric interpretation [Lus1, Lus3]. (Not all combinations of parameters occur though, there are conditions on the ratios between the different $k$-parameters.) For this reason graded Hecke algebras, instead of affine Hecke algebras, play the main role in this paper.

The appropriate geometric objects are still equivariant sheaves on a variety associated to a complex reductive group, but now the sheaves are constructible and one considers their equivariant cohomology instead of their $K$-theory. Results in [Lus3] strongly suggest that the module category of $\mathbb{H}$ is equivalent with some category of equivariant constructible sheaves. We will make this precise by involving derived categories. That can be regarded as a geometric categorification of $\mathbb{H}$, albeit of a different kind as that from [KaLu, Bez].

Motivation from the local Langlands program

Consider a reductive group $\mathcal{G}$ defined over a non-archimedean local field $F$. We denote its complex dual group by $\mathcal{G}^\vee$ and its Langlands dual group by $L\mathcal{G} = \mathcal{G}^\vee \rtimes W_F$. Let $\text{Rep}(\mathcal{G}(F))$ be the category of smooth complex $\mathcal{G}(F)$-representations and let $\text{Irr}(\mathcal{G}(F))$ be its set of irreducible objects. The local Langlands correspondence (LLC) predicts a nice bijection between $\text{Irr}(\mathcal{G}(F))$ and the set of $\mathcal{G}(F)$-relevant enhanced $L$-parameters with value in $L\mathcal{G}$. Over the years it gradually has become clear
that the LLC may admit a categorification, which

(1) relates $\text{Rep}(G(F))$ to coherent sheaves on a variety of $L$-parameters for $G(F)$.

It is expected that one should work with a stack of $L$-parameters and with derived categories of coherent sheaves on that. Then the (bounded) derived category of $\text{Rep}(G(F))$ should admit a fully faithful functor to a (bounded) derived category of such sheaves. We refer to [Hel, Zhu] for the technical conjectures and more background.

In [Hel] such an equivalence of derived categories is worked out for Iwahori-spherical representations of $GL_2(F)$, via an affine Hecke algebra. Of course, this picture calls for generalization. In fact, many of the pieces are already in places for arbitrary $L$-parameters and arbitrary $G(F)$-representations. Let us explain that, and at the same time sketch why Hecke algebras are so ubiquitous in these matters.

**Galois side**

First one defines the cuspidal support $\text{Sc}((\phi, \rho))$ of an enhanced $L$-parameter $(\psi, \rho)$ [AMS1, §7]. That gives rise to a partition of the space $\Phi_e(LG)$ of enhanced $L$-parameters (with values in $LG$) into Bernstein components $\Phi_e(LG)_s$ [AMS1, §8]. To every bounded element $(\phi, q\epsilon)$ of the cuspidal support of $\Phi_e(LG)_s$ one associates a (twisted) graded Hecke algebra $H_{\phi,q\epsilon}$, defined in terms of equivariant constructible sheaves, see [AMS2, §4] and [AMS3, §3.1]. The family of algebras

$$\{H_{\phi,q\epsilon} : (\phi, q\epsilon) \text{ as above}\}$$

can be glued to one (twisted) affine Hecke algebra $H^{\psi}$ [AMS3 §3.3]. The constructions are such that, upon specializing the parameters of $H^{\psi}$ to an array of positive real numbers $q$, there is a natural bijection

$$\text{Irr}(H^{\psi}(q)) \leftrightarrow \Phi_e(LG)^{\psi}.$$  

Here the subset of $\Phi_e(LG)^{\psi}$ with one fixed cuspidal support $(\phi', q\epsilon)$ corresponds to the set of irreducible representations of $H^{\psi}(q)$ with one fixed central character:

$$\text{Irr}_{\phi',q\epsilon}(H^{\psi}(q)) \leftrightarrow \text{Sc}^{-1}((\phi', q\epsilon)).$$

The choice of $q$ induces a specialization of the parameters of $H_{\phi,q\epsilon}$, which we denote briefly by $\log q$. Then the sets in (2) are also naturally in bijection with the subset $\text{Irr}_{\phi',q\epsilon}(H_{\phi,q\epsilon}(\log q))$ of $\text{Irr}(H_{\phi,q\epsilon}(\log q))$ picked out by one central character. (Here $\phi'$ determines $\phi$ as its “bounded part”.)

**$p$-adic side**

Consider an arbitrary Bernstein block $\text{Rep}(G(F)^{\psi}$ of $\text{Rep}(G(F)).$ Bernstein [BeRu, Ren] exhibited a progenerator $\Pi^{\psi}$ of $\text{Rep}(G)^{\psi}$. By a standard result from category theory, $\text{Rep}(G(F)^{\psi}$ is equivalent with $\text{End}_{G(F)}(\Pi^{\psi}) \cong \Mod$. Let $M$ be a Levi subgroup of $G$ such that the supercuspidal support of $\text{Rep}(G)^{\psi}$ can be realized in $\text{Irr}(M(F))$. Every tempered $(F)$-representation $\tau$ in that supercuspidal support of gives rise, via localizations of $\text{End}_{G(F)}(\Pi^{\psi})$, to a twisted graded Hecke algebra $H_{\tau}^{\psi}$ [Sol4, §7]. By construction $H_{\tau}^{\psi} \cong \Mod$ describes a well-defined part of $\text{Rep}(G(F)^{\psi}$. Under mild conditions, the family of algebras $H_{\tau}$ can be glued to one twisted affine Hecke algebra $H^{\psi}$, which is (almost) Morita equivalent with $\text{End}_{G(F)}(\Pi^{\psi})$ [Sol4 §10].
Local Langlands Correspondence

Suppose now that a LLC for $G$ matches a Bernstein component $\text{Irr}(G)^s$ with a Bernstein component $\Phi_e(LG)^s$. It can be expected that:

(i) each twisted graded Hecke algebra $H_{\phi,qe}(\log q)$ is isomorphic with $H_\tau$, where $	au$ has L-parameter $(\phi, qe) \in \Phi_e(LM)$;
(ii) for a suitable choice of $q$, $H^{s'}(\log q)$ is (almost) Morita equivalent with $H^s$, and hence with $\text{End}_{G(F)(\Pi^s)}$. In other words, $H^{s'}(\log q) - \text{Mod}$ should be equivalent with $\text{Rep}(G(F))^s$.

Indeed, many earlier results about Bernstein components, types and Hecke algebras, e.g. [Rec, Lus4, ScSt, ABPS1], can be interpreted as confirmations of cases of this expectation. In general all this remains conjectural, because we do not yet have a complete local Langlands correspondence. One may hope that requiring (i) and (ii) may help to determine a LLC in new cases, like for unipotent representations.

Independently, the parameters of the Hecke algebras $H^s$ and $H_\tau$ can be investigated. In many cases they can be determined [Sol6], and in all those instances they agree with the parameters of some Hecke algebras on the Galois side (or equivalently, with the parameters of a Hecke algebra for a Bernstein block of unipotent representations.)

To attack the conjecture (1) from [Hel, Zhu], the above suggests a strategy:

(a) match Hecke algebras on the $p$-adic and Galois sides of the LLC;
(b) relate $H^s$ to sheaves on a stack of L-parameters coming from $\Phi_e(LG)^{s'}$.

As a step in that direction, we consider the subcases of (b) obtained by restricting to one infinitesimal central character. In the end, these should account for all finite length $G(F)$-representations.

Let $\tau' \in \text{Irr}(M(F))$ be a twist of $\tau$ by an unramified character $\chi : M(F) \to \mathbb{R}_{>0}$. The pair $(M(F), \tau')$ defines a character of the Bernstein centre of $G(F)$ and a subcategory $\text{Rep}(G(F))^{(M(F), \tau')}$, consisting of those $G(F)$-representations all whose irreducible subquotients have supercuspidal support conjugate to $(M(F), \tau')$. Similarly $\tau'$ determines a central character of $H_\tau$, and a subcategory $H_\tau - \text{Mod}_{\tau'}$. The Langlands parameter of $\tau'$ should be a twist $(\hat{\chi}, qe)$ of $(\phi, qe)$ by the Langlands parameter $\hat{\chi}$ of $\chi$. Granting the above expectation (i), $H_\tau - \text{Mod}_{\tau'}$ will be equivalent with $H_{\phi,qe}(\log q) - \text{Mod}_{\phi',qe}$. In this way (1) motivates the main goal of the paper:

*find a fully faithful functor from $H_{\phi,qe}(\log q) - \text{Mod}_{\phi',qe}$ to some category of sheaves, maybe with derived categories.*

Geometric graded Hecke algebras

Let us describe the involved algebras a bit better, following [AMS2, AMS3]. Consider $\phi \in \Phi_e(LM)$ as a group homomorphism $W_F \times SL_2(\mathbb{C}) \to M^\vee \times W_F$. The group $Z_{G^\vee}(\phi(W_F))$ has a subgroup $Z_{M^\vee}(\phi(W_F))$, both reductive and possibly disconnected. The data $\phi|_{SL_2(\mathbb{C})}, qe$ give rise to an equivariant cuspidal local system $q\mathcal{E}$ on a unipotent orbit in $Z_{M^\vee}(\phi(W_F))$. We abbreviate

$$G = Z_{G^\vee}(\phi(W_F)) \quad \text{and} \quad M = Z_{M^\vee}(\phi(W_F)),$$

so that we are entirely in a context of complex reductive groups. To the data $(G,M,q\mathcal{E})$ we attach a twisted graded Hecke algebra $H = H(G,M,q\mathcal{E}) = H_{\phi,qe}$. That brings us, finally, to the actual setting of the paper. We have a family of
graded Hecke algebras which is defined purely in terms of complex geometry, and likewise our results and results will come in such terms. At the same time, all these algebras are solidly rooted in the representation theory of reductive p-adic groups. To all appearances, our algebras represent the general case of a graded Hecke algebra associated to a Bernstein block for a reductive p-adic group.

Main results
We will work in \( D_{G \times \mathbb{C}}(X) \), the equivariant (bounded) derived category of constructible sheaves on a complex variety \( X \) \cite{BeLu}. In \cite{Lus1, Lus3, AMS2} an important object \( K \in D_{G \times \mathbb{C}}(g) \) was constructed from \( qE \), by a process that bears some similarity with parabolic induction. Let \( g_N \) be the set of nilpotent elements in the Lie algebra \( g \) of \( G \) and let \( K_N \) be the pullback of \( K \) to \( g_N \). Up to degree shifts, both \( K \) and \( K_N \) are direct sums of simple perverse sheaves.

**Theorem A.** (see Theorem 3.2)
There exist natural isomorphisms of graded algebras
\[
\mathbb{H}(G, M, qE) \longrightarrow \text{End}_{D_{G \times \mathbb{C}}(g)}^*(K) \longrightarrow \text{End}_{D_{D_{G \times \mathbb{C}}(g)(N)}}^*(K_N).
\]

The object \( K_N \) generates (by the operations cones, degree shifts and taking direct summands) a full subcategory \( \langle K_N \rangle \) of \( D_{G \times \mathbb{C}}(g_N) \). Let \( \mathbb{H}(G, M, qE) - \text{Mod}_{fgp} \) be the category of finitely generated projective right \( \mathbb{H}(G, M, qE) \)-modules, and indicate its bounded derived category by a \( D \). It follows quickly from Theorem A that the functor \( \text{Hom}_{D_{G \times \mathbb{C}}(g)(N)}^*(K_N, ?) \) induces an equivalence of categories
\[
\langle K_N \rangle \rightarrow D(\mathbb{H}(G, M, qE) - \text{Mod}_{fgp}).
\]
However, this does not yet fit well with the LLC. The problem is that the variety \( g_N \) is too small: it sees only nilpotent elements of \( g \), and those only determine a part of an L-parameter. We bring in a semisimple element (related to the image of a Frobenius element under an L-parameter) by localizing \( \mathbb{H}(G, M, qE) \) with respect to a central character.

As a vector space, \( \mathbb{H}(G, M, qE) \) is the tensor product of \( \mathbb{C}[W_{qE}] \) (for a certain finite group \( W_{qE} \), generalizing a Weyl group) and \( O(t \oplus \mathbb{C}) \), where \( t = \text{Lie}(Z(M)) \) and \( O(\mathbb{C}) = \mathbb{C}[t] \) comes from the \( \mathbb{C}^\times \)-actions. The centre of \( \mathbb{H}(G, M, qE) \) can be described as \( O(t)^{W_{qE}} \otimes \mathbb{C}[t] \).

In these terms, the set of enhanced L-parameters \( \Phi_e(L_G) \) becomes a set of triples \( (\sigma, \rho) \), where \( \sigma \in g \) is semisimple, \( y \in g \) is nilpotent, \( \rho = 0 \) and \( \rho \) is a very specific kind of irreducible representation of the component group of \( (\sigma, y) \). One may also replace the condition \( [\sigma, y] = 0 \) by \( [\sigma, y] = 2ry \) for a fixed \( r \in \mathbb{C} \). We recall from \cite{AMS2} that the conjugacy classes of such triples parametrize both the irreducible and the standard modules of \( \mathbb{H}(G, M, qE)/(r-r) \), for any \( r \in \mathbb{C} \).

We fix \( (\sigma, r) \in t \oplus \mathbb{C} \) and consider it as a central character of \( \mathbb{H}(G, M, qE) \). We denote the corresponding completion of \( Z(\mathbb{H}(G, M, qE)) \) by \( \hat{Z}(\mathbb{H}(G, M, qE))_{\sigma, r} \). In the process of localization, \( G \times \mathbb{C}^\times \) will be replaced by \( Z_G(\sigma) \times \mathbb{C}^\times \) and \( g_N \) by
\[
\hat{g}_N^{\sigma, r} := \{ y \in g_N : [\sigma, y] = 2ry \}.
\]
A variation on the construction of \( K_N \) yields an object \( K_{N, \sigma, r} \in D_{Z_G(\sigma) \times \mathbb{C}^\times}(g_N^{\sigma, r}) \).

Since \( (\sigma, r) \in \text{Lie}(Z_G(\sigma) \times \mathbb{C}^\times) \), it defines a character of
\[
H_{Z_G(\sigma) \times \mathbb{C}^\times}^*(\text{pt}) \cong \mathbb{O}(\text{Lie}(Z_G(\sigma) \times \mathbb{C}^\times))^{Z_G(\sigma) \times \mathbb{C}^\times},
\]
and we can again complete with respect to \((\sigma, r)\).

**Theorem B.** (see Theorem 4.4)
There is a natural algebra isomorphism

\[
\tilde{\mathbb{H}}(G, M, q\mathcal{E})_{\sigma, r} \otimes_{\mathbb{H}(G, M, q\mathcal{E})} \mathbb{H}(G, M, q\mathcal{E}) \rightarrow \tilde{H}^*_Z G(\sigma) \otimes_{\mathbb{H}(G, M, q\mathcal{E})} \mathbb{H}(G, M, q\mathcal{E})
\]

This induces an equivalence of categories

\[
\mathbb{H}(G, M, q\mathcal{E}) - \text{Mod}_{\mathbb{H}, \sigma, r} \cong \text{End}^*_Z G(\sigma) \times \mathbb{C}^*(\mathfrak{g}^{\sigma, r}_N) (K, \sigma, r).
\]

When \(G\) is connected, Theorem B is due to Lusztig [Lus3]. Let \(\langle K, \sigma, r \rangle\) be the full subcategory of \(D_Z G(\sigma) \times \mathbb{C}^*(\mathfrak{g}^{\sigma, r}_N)\) generated by \(K, \sigma, r\). Generalizing (3), we obtain:

**Theorem C.** (see Theorem 5.1)
The functor \(\text{Hom}^*_Z G(\sigma) \times \mathbb{C}^*(\mathfrak{g}^{\sigma, r}_N) (K, \sigma, r, ?)\) gives an equivalence of categories

\[
\langle K, \sigma, r \rangle \rightarrow D(\text{End}^*_Z G(\sigma) \times \mathbb{C}^*(\mathfrak{g}^{\sigma, r}_N) (K, \sigma, r) - \text{Mod}_{\mathbb{H}}(\mathbb{g}^{\mathbb{H}, \sigma, r})).
\]

Now another problems arises: it is unclear whether all \(\text{End}^*_Z G(\sigma) \times \mathbb{C}^*(\mathfrak{g}^{\sigma, r}_N) (K, \sigma, r)\)-modules, or even those of finite length, admit a finite type projective resolution. Therefore the target in Theorem C could be much smaller than desired. With substantial effort, we show that at least the objects of \(\text{End}^*_Z G(\sigma) \times \mathbb{C}^*(\mathfrak{g}^{\sigma, r}_N) (K, \sigma, r) - \text{Mod}_{\mathbb{H}, \sigma, r}\) have such a resolution. That can be combined with Theorems B and C.

**Theorem D.** (see Proposition 8.4)
There exist derived functors

\[
\text{Hom}^*_Z G(\sigma) \times \mathbb{C}^*(\mathfrak{g}^{\sigma, r}_N) (K, \sigma, r, ?) : D_Z G(\sigma) \times \mathbb{C}^*(\mathfrak{g}^{\sigma, r}_N) \rightarrow D(\tilde{\mathbb{H}}(G, M, q\mathcal{E})_{\sigma, r} \otimes_{\mathbb{H}(G, M, q\mathcal{E})} \mathbb{H}(G, M, q\mathcal{E}) - \text{Mod})
\]

\[
\otimes K, \sigma, r : D(\tilde{\mathbb{H}}(G, M, q\mathcal{E}) - \text{Mod}_{\mathbb{H}, \sigma, r}) \rightarrow D_Z G(\sigma) \times \mathbb{C}^*(\mathfrak{g}^{\sigma, r}_N)
\]

such that \(\otimes K, \sigma, r\) is fully faithful and the composition

\[
\text{Hom}^*_Z G(\sigma) \times \mathbb{C}^*(\mathfrak{g}^{\sigma, r}_N) (K, \sigma, r) \otimes K, \sigma, r
\]

is naturally equivalent with the identity on \(D(\tilde{\mathbb{H}}(G, M, q\mathcal{E}) - \text{Mod}_{\mathbb{H}, \sigma, r})\).

In relation with the LLC it is preferable to compose both functors in Theorem D with the sign automorphism of \(\mathbb{H} = \mathbb{H}(G, M, q\mathcal{E})\), for only then tempered representations correspond to derived sheaves supported on bounded L-parameters. That yields functors

\[
\mathcal{F}_{\sigma, r} : D_Z G(\sigma) \times \mathbb{C}^*(\mathfrak{g}^{\sigma, r}_N) \rightarrow D(\tilde{\mathbb{H}}(G, M, q\mathcal{E}) - \text{Mod})
\]

\[
\mathcal{F}_{\sigma, r}^{-} : D(\mathbb{H} - \text{Mod}_{\mathbb{H}, \sigma, r}) \rightarrow D_Z G(\sigma) \times \mathbb{C}^*(\mathfrak{g}^{\sigma, r}_N)
\]

with properties as in Theorem D.

Recall that graded Hecke algebras coming from reductive \(p\)-adic groups have \(r\) specialized to a positive real number. Hence we need a version of the above results for \(\mathbb{H}(G, M, q\mathcal{E})/(r - r)\), with \(r \in \mathbb{C}\). As \(r\) came from the \(\mathbb{C}^\times\)-actions, a natural
attempt is to replace $G \times \mathbb{C}^\times$-equivariance by $G$-equivariance. That does not work well directly in Theorem A or in (2), only after localization.

**Theorem E.** (see Section 9)

Theorems B, C, D and (4) become valid for $\mathbb{H}(G, M, qE)/(r - r)$ once we forget the $\mathbb{C}^\times$-equivariant structure everywhere.

We note that for $r = \log(qF)/2$:

$$g_N^{\sigma, r} = \{ y \in g_N : [\sigma, y] = -\log(qF)y \} = \{ y \in g_N : \text{Ad}(\exp\sigma)y = qF^{-1}y \}.$$

Here $(\exp\sigma, y)$ defines an unramified $L$-parameter $\phi : W_F \rtimes \mathbb{C} \to G$, with $\exp(\sigma)$ the image of a Frobenius element of $W_F$. In this way the image of $F^{-\sigma, \log(qF)/2}$ can be translated to $Z_G(\sigma)$-equivariant derived sheaves on a variety of $L$-parameters $\phi$ with $\phi|_{W_F}$ fixed.

Although it may be superfluous in view of the title, we stress that these sheaves are constructible rather than coherent. Of course this is a consequence of the entire setup of the paper, starting already with the construction of $\mathbb{H}(G, M, qE)$. On the other hand, most of the equivariant sheaves we encountered are supported on only finitely many $G$-orbits, so large and by they are determined by their stalks at finitely many points. In such situations coherence or not does not make much of difference.

We hope that in subsequent work our results can be generalized to finitely generated modules of affine Hecke algebras, and that will most probably involve coherent sheaves.

**Structure of the paper**

We start with recalling (twisted) graded Hecke algebras in terms of generators and relations. We generalize a few results from [Sol5], which say that the set of irreducible representations of a a graded Hecke algebra is essentially independent of the parameters $k$ and $r$. In Section 3 we recall the construction and fundamental properties of graded Hecke algebras associated to complex reductive groups and cuspidal local systems.

Then (Section 4) we turn to localizing these algebras, generalizing [Lus3]. We show first how to replace the group $G \times \mathbb{C}^\times$ by $Z_G(\sigma) \times \mathbb{C}^\times$, and then how to replace $g$ by $g^{\sigma, r}$ and $K$ by $K_{\sigma, r}$. Our investigations revealed a technical problem in [Lus3], which is resolved in Appendix B.

The next three sections are mainly dedicated to the problem observed after Theorem C: the scarcity of finite type projective resolutions. We will approach this with standard $\mathbb{H}(G, M, qE)$-modules, which is reasonable because those generate the derived category of finite length $\mathbb{H}(G, M, qE)$-modules. After recalling some results about standard (left or right) modules from [AMS2], we discuss more convenient ways to realize them in Section 5. One involves the sign automorphism of $\mathbb{H}(G, M, E)$, the other is in terms close to Theorem B. For every standard right $\mathbb{H}(G, M, qE)$-module $E_{y, \sigma, r, \rho^\vee}$ with central character $(\sigma, r)$ and $y \in g_N^{\sigma, r}$, we construct (Section 7) an explicit object

$$j_N^*S_y(\Lambda_{y, \rho}^*) \in D_{Z_G(\sigma) \times \mathbb{C}^\times}(g_N^{\sigma, r})$$

that is mapped to $E_{y, \sigma, r, \rho^\vee}$ by the functor $\text{Hom}_{D_{Z_G(\sigma) \times \mathbb{C}^\times}(g_N^{\sigma, r})}^*(K_{N, \sigma, r}, ?)$. This involves a Koszul resolution for the algebra $H^*_Z(\mathbb{C}(y))(\text{pt})$. In our proof of Theorem D we need that $j_N^*S_y(\Lambda_{y, \rho})$ belongs to $\langle K_{N, \sigma, r} \rangle$. The purpose of Section 6 is to
gather a supply of objects of \(\langle K_{N,\sigma,r} \rangle\), sufficient to construct \(j_{N,*} \mathcal{S}_g(\tilde{\Lambda}^*_g)\) in that subcategory. With that in order we prove Theorem \([D]\) and \([A]\) in Section 8.

In the final section we specialize \(r\) to \(r \in \mathbb{C}\). We show that most of the earlier results remain valid if replace \(\mathbb{H}(G, M, q\mathcal{E})\) by \(\mathbb{H}(G, M, q\mathcal{E})/(r - r)\) and forget the \(\mathbb{C}^\times\)-equivariance of our sheaves.

Appendix \([A]\) is dedicated to the relation between standard modules and parabolic induction for graded Hecke algebras. The treatment of this topic in \([AMS2]\] omitted the most general case (both \(G\) and \(M\) disconnected) and contained a mistake. We provide the missing proofs. Although these results are not really needed here, we include them because they are used in \([AMS3, Sol3]\], both of which are important in the motivation of this work.

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We thank George Lusztig for some helpful email conversations.

1. Notations

Most of our notations can be found in Sections 2 and 3. Here we mention a few conventions that are not so common.

We denote the linear dual of a finite dimensional vector space \(V\) by \(V^\vee\) (we avoid \(V^*\) because * is already used heavily for degrees.) More generally we denote the dual of a local system \(L\) by \(L^\vee\).

We denote the identity component of a topological/algebraic group \(G\) by \(G^\circ\). If \(G\) acts on a set \(X\) and \(Y \subset X\), then we write \(Z^\circ G(Y)\) for \(Z G(Y)^\circ\).

For a commutative algebra \(A\) and a character \(\sigma\) of \(A\), we let \(\hat{A}_\sigma\) be the (formal) completion of \(A\) with respect to the powers of ker \(\sigma\).

For a differential complex \(C_*\) and \(m \in \mathbb{Z}\) the degree shift \(C_*[m]\) is the differential complex with \(C_n + m\) in degree \(n\).

2. Graded Hecke algebras

Let \(a\) be a finite dimensional Euclidean space and let \(W\) be a finite Coxeter group acting isometrically on \(a\) (and hence also on \(a^*\)). Let \(R \subset a^*\) be a reduced integral root system, stable under the action of \(W\), such that the reflections \(s_\alpha\) with \(\alpha \in R\) generate \(W\). These conditions imply that \(W\) acts trivially on the orthogonal complement of \(\mathbb{R}R\) in \(a^*\).

Write \(t = a \otimes_{\mathbb{R}} \mathbb{C}\) and let \(S(t^\vee) = \mathcal{O}(t)\) be the algebra of polynomial functions on \(t\). We also fix a base \(\Delta\) of \(R\). Let \(\Gamma\) be a finite group which acts faithfully and orthogonally on \(a\) and stabilizes \(R\) and \(\Delta\). Then \(\Gamma\) normalizes \(W\) and \(W \rtimes \Gamma\) is a group of automorphisms of \((a, R)\). We choose a \(W \rtimes \Gamma\)-invariant parameter function \(k : R \to \mathbb{C}\). Let \(r\) be a formal variable, identified with the coordinate function on \(\mathbb{C}\) (so \(\mathcal{O}(\mathbb{C}) = \mathbb{C}[r]\)).

Let \(\xi : \Gamma^2 \to \mathbb{C}^\times\) be a 2-cocycle and inflate it to a 2-cocycle of \(W \rtimes \Gamma\). Recall that the twisted group algebra \(\mathbb{C}[W \rtimes \Gamma, \xi]\) has a \(\mathbb{C}\)-basis \(\{N_w : w \in W \rtimes \Gamma\}\) and multiplication rules

\[N_w \cdot N_{w'} = \xi(w, w')N_{ww'}\]

In particular it contains the group algebra of \(W\).
Proposition 2.1. [AMS2 Proposition 2.2]
There exists a unique associative algebra structure on \( \mathbb{C}[W \rtimes \Gamma, z] \otimes \mathcal{O}(t) \otimes \mathbb{C}[r] \) such that:

- the twisted group algebra \( \mathbb{C}[W \rtimes \Gamma, z] \) is embedded as subalgebra;
- the algebra \( \mathcal{O}(t) \otimes \mathbb{C}[r] \) of polynomial functions on \( t \oplus \mathbb{C} \) is embedded as a subalgebra;
- \( \mathbb{C}[r] \) is central;
- the braid relation \( N_{s_\alpha} \xi - s_\alpha \xi N_{s_\alpha} = k(\alpha) r(\xi - s_\alpha \xi)/\alpha \) holds for all \( \xi \in \mathcal{O}(t) \) and all simple roots \( \alpha \);
- \( N_w \xi N_w^{-1} = w \xi \) for all \( \xi \in \mathcal{O}(t) \) and \( w \in \Gamma \).

We denote the algebra from Proposition 2.1 by \( \mathbb{H}(t, W \rtimes \Gamma, k, r, z) \) and call it a twisted graded Hecke algebra. It is graded by putting \( \mathbb{C}[W \rtimes \Gamma, z] \) in degree 0 and \( t^\vee \setminus \{0\} \) and \( r \) in degree 2. When \( \Gamma \) is trivial, we omit \( z \) from the notation, and we obtain the usual notion of a graded Hecke algebra \( \mathbb{H}(t, W, k, r) \).

Notice that for \( k = 0 \) Proposition 2.1 yields the crossed product algebra
\[
\mathbb{H}(t, W \rtimes \Gamma, 0, r, z) = \mathbb{C}[r] \otimes_\mathbb{C} \mathcal{O}(t) \times \mathbb{C}[W \rtimes \Gamma, z],
\]
with multiplication rule
\[
N_w \xi N_w^{-1} = w \xi \quad w \in W \rtimes \Gamma, \xi \in \mathcal{O}(t).
\]
It is possible to scale all parameters \( k(\alpha) \) simultaneously. Namely, scalar multiplication with \( z \in \mathbb{C}^\times \) defines a bijection \( m_z : t^\vee \to t^\vee \), which clearly extends to an algebra automorphism of \( S(t^\vee) \). From Proposition 2.1 we see that it extends even further, to an algebra isomorphism
\[
m_z : \mathbb{H}(t, W \rtimes \Gamma, zk, r, z) \to \mathbb{H}(t, W \rtimes \Gamma, k, r, z)
\]
which is the identity on \( \mathbb{C}[W \rtimes \Gamma, z] \otimes_\mathbb{C} \mathbb{C}[r] \). Notice that for \( z = 0 \) the map \( m_z \) is well-defined, but no longer bijective. It is the canonical surjection
\[
\mathbb{H}(t, W \rtimes \Gamma, 0, r, z) \to \mathbb{C}[W \rtimes \Gamma, z] \otimes_\mathbb{C} \mathbb{C}[r].
\]
One also encounters versions of \( \mathbb{H}(t, W \rtimes \Gamma, k, r, z) \) with \( r \) specialized to a nonzero complex number. In view of (2.2) it hardly matters which specialization, so it suffices to look at \( r \mapsto 1 \). The resulting algebra \( \mathbb{H}(t, W \rtimes \Gamma, k, z) \) has underlying vector space \( \mathbb{C}[W \rtimes \Gamma, z] \otimes_\mathbb{C} \mathcal{O}(t) \) and cross relations
\[
\xi \cdot s_\alpha - s_\alpha \cdot s_\alpha(\xi) = k(\alpha)(\xi - s_\alpha(\xi))/\alpha \quad \alpha \in \Delta, \xi \in S(t^\vee).
\]
Since \( \Gamma \) acts faithfully on \( (a, \Delta) \), and \( W \) acts simply transitively on the collection of bases of \( R, W \rtimes \Gamma \) acts faithfully on \( a \). From (2.3) we see that the centre of \( \mathbb{H}(t, W \rtimes \Gamma, k, z) \) is
\[
Z(\mathbb{H}(t, W \rtimes \Gamma, k, z)) = S(t^\vee)^{W \rtimes \Gamma} = \mathcal{O}(t/W \rtimes \Gamma).
\]
As a vector space, \( \mathbb{H}(t, W \rtimes \Gamma, k, z) \) is still graded by \( \deg(w) = 0 \) for \( w \in W \rtimes \Gamma \) and \( \deg(x) = 2 \) for \( x \in t^\vee \setminus \{0\} \). However, it is not a graded algebra any more, because (2.3) is not homogeneous in the case \( \xi = \alpha \). Instead, the above grading merely makes \( \mathbb{H}(t, W \rtimes \Gamma, k, z) \) into a filtered algebra. The graded algebra associated to this filtration is obtained by setting the right hand side of (2.3) equal to 0. In other words, the associated graded of \( \mathbb{H}(t, W \rtimes \Gamma, k, z) \) is the crossed product algebra (2.1).
Graded Hecke algebras can be decomposed like root systems and reductive Lie algebras. Let $R_1, \ldots, R_d$ be the irreducible components of $R$. Write $a_i^\vee = \text{span}(R_i) \subset a^\vee$, $t_i = \text{Hom}_{\mathbb{R}}(a_i^\vee, \mathbb{C})$ and $\mathfrak{z} = R^k \subset t$. Then

$$t = t_1 \oplus \cdots \oplus t_d \oplus \mathfrak{z}. \quad (2.5)$$

The inclusions $W(R_i) \rightarrow W(R), t_i^\vee \rightarrow t^\vee$ and $\mathfrak{z}^\vee \rightarrow t^\vee$ induce an algebra isomorphism

$$\mathbb{H}(t_1, W(R_1), k) \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathbb{H}(t_d, W(R_d), k) \otimes_{\mathbb{C}} \mathcal{O}(\mathfrak{z}) \rightarrow \mathbb{H}(t, W, k). \quad (2.6)$$

The central subalgebra $\mathcal{O}(\mathfrak{z}) \cong S(\mathfrak{z}^\vee)$ is of course very simple, so the study of graded Hecke algebras can be reduced to the case where the root system $R$ is irreducible.

Now we list some isomorphisms of (twisted) graded Hecke algebras that will be useful later on. For any $z \in \mathbb{C}^\times$, $\mathbb{H}(t, W \rtimes \Gamma, k, r, z)$ admits a "scaling by degree" automorphism

$$x \mapsto z^n x \quad \text{if } x \in \mathbb{H}(t, W \rtimes \Gamma, k, r, z) \text{ has degree } 2n. \quad (2.7)$$

Extend the sign representation to a character $\text{sgn}$ of $W \rtimes \Gamma$, trivial on $\Gamma$. We have the Iwahori–Matsumoto and sign involutions

$$\text{IM} : \mathbb{H}(t, W \rtimes \Gamma, k, r, z) \rightarrow \mathbb{H}(t, W \rtimes \Gamma, k, r, z), \quad \text{IM}(N_w) = \text{sgn}(w)N_w, \quad \text{IM}(r) = r, \quad \text{IM}(\xi) = -\xi \quad (2.8)$$

$$\text{sgn} : \mathbb{H}(t, W \rtimes \Gamma, k, r, z) \rightarrow \mathbb{H}(t, W \rtimes \Gamma, k, r, z), \quad \text{sgn}(N_w) = \text{sgn}(w)N_w, \quad \text{sgn}(r) = -r, \quad \text{sgn}(\xi) = \xi \quad \text{if } w \in W \rtimes \Gamma, \xi \in t^\vee.$$

Upon specializing $r = 1$, these induce isomorphisms

$$\text{IM} : \mathbb{H}(t, W \rtimes \Gamma, k, z) \rightarrow \mathbb{H}(t, W \rtimes \Gamma, k, z), \quad \text{sgn} : \mathbb{H}(t, W \rtimes \Gamma, k, z) \rightarrow \mathbb{H}(t, W \rtimes \Gamma, -k, z).$$

More generally, we can pick a sign $\epsilon(s_\alpha)$ for every simple reflection $s_\alpha \in W$, such that $\epsilon(s_\alpha) = \epsilon(s_\beta)$ if $s_\alpha$ and $s_\beta$ are conjugate in $W \rtimes \Gamma$. Then $\epsilon$ extends uniquely to a character of $W \rtimes \Gamma$ trivial on $\Gamma$ (and every character of $W \rtimes \Gamma$ which is trivial on $\Gamma$ has this form). Define a new parameter function $\epsilon k$ by

$$\epsilon k(\alpha) = \epsilon(s_\alpha)k(\alpha).$$

Then there are algebra isomorphisms

$$\phi_\epsilon : \mathbb{H}(t, W \rtimes \Gamma, k, r, z) \rightarrow \mathbb{H}(t, W \rtimes \Gamma, \epsilon k, r, z), \quad (2.9)$$

$$\phi_\epsilon(N_w) = \epsilon(w)N_w, \quad \phi_\epsilon(r) = r, \quad \phi_\epsilon(\xi) = \xi, \quad w \in W \rtimes \Gamma, \xi \in \mathcal{O}(t).$$

Notice that for $\epsilon$ the sign character of $W$, $\phi_\epsilon$ agrees with $\text{sgn}$ from (2.8) on $\mathbb{H}(t, W \rtimes \Gamma, k, z)$ but not on $\mathbb{H}(t, W \rtimes \Gamma, k, r, z)$.

For $R$ irreducible of type $B_n, C_n, F_4$ or $G_2$, there are two further nontrivial possible $\epsilon$'s. Consider the characters $\epsilon_s, \epsilon_l$ of $W$ with

$$\epsilon_s(s_\alpha) = \begin{cases} 1 & \alpha \text{ long } \\ -1 & \alpha \text{ short } \end{cases}, \quad \epsilon_l(s_\alpha) = \begin{cases} 1 & \alpha \text{ short } \\ -1 & \alpha \text{ long } \end{cases}. \quad (2.10)$$

Since $\Gamma$ acts isometrically on $a$, $\epsilon_l$ and $\epsilon_s$ are $\Gamma$-invariant. Thus we obtain algebra isomorphisms

$$\phi_{\epsilon_s} : \mathbb{H}(t, W \rtimes \Gamma, k, z) \rightarrow \mathbb{H}(t, W \rtimes \Gamma, \epsilon_s k, z), \quad \phi_{\epsilon_l} : \mathbb{H}(t, W \rtimes \Gamma, k, z) \rightarrow \mathbb{H}(t, W \rtimes \Gamma, \epsilon_l k, z).$$
Lemma 2.2. Let \( \mathbb{H}(t, W \rtimes \Gamma, k, \lambda) \) be a twisted graded Hecke algebra with a real-valued parameter function \( k \). Then it is isomorphic to a twisted graded Hecke algebra \( \mathbb{H}(t, W \rtimes \Gamma, ek, \lambda) \) with \( ek : R \to \mathbb{R}_{\geq 0} \), via an isomorphism \( \phi_\epsilon \) that is the identity on \( \mathcal{O}(t) \).

Proof. Define
\[
\epsilon(s_\alpha) = \begin{cases} 
1 & k(\alpha) \geq 0 \\
-1 & k(\alpha) < 0 
\end{cases}.
\]
Since \( k \) is \( \Gamma \)-invariant, this extends to a \( \Gamma \)-invariant quadratic character of \( W \). Then \( \phi_\epsilon \) has the required properties. \( \square \)

With the above isomorphisms we will generalize the results of \cite{Sol5 §6.2}, from graded Hecke algebras with positive parameters to twisted graded Hecke algebras with real parameters.

For the moment, we let \( \mathbb{H} \) stand for either \( \mathbb{H}(t, W \rtimes \Gamma, k, \lambda) \) or \( \mathbb{H}(t, W \rtimes \Gamma, k, \lambda) \). Every finite dimensional \( \mathbb{H} \)-module \( V \) is the direct sum of its generalized \( \mathcal{O}(t) \)-weight spaces
\[
V_\lambda := \{ v \in V : (\xi - \xi(\lambda))^\text{dim} V v = 0 \; \forall \xi \in \mathcal{O}(t) \} \quad \lambda \in t.
\]
We denote the set of \( \mathcal{O}(t) \)-weights of \( V \) by
\[
\text{Wt}(V) = \{ \lambda \in t : V_\lambda \neq 0 \}.
\]
Let \( \mathfrak{a}^- \) be the obtuse negative cone in \( \mathbb{R}R \subset \mathfrak{a} \) determined by \( (R, \Delta) \). We denote the interior of \( \mathfrak{a}^- \) in \( \mathbb{R}R \) by \( \mathfrak{a}^{--} \). We recall that a finite dimensional \( \mathbb{H} \)-module \( V \) is tempered if
\[
\text{Wt}(V) \subset \mathfrak{a}^- \oplus i\mathfrak{a}
\]
and that \( V \) is essentially discrete series if, with \( \mathfrak{z} \) as in (2.5):
\[
\text{Wt}(V) \subset \mathfrak{a}^{--} \oplus (\mathfrak{z} \cap \mathfrak{a}) \oplus i\mathfrak{a}.
\]
For a subset \( U \) of \( t \) we let \( \text{Mod}_{\mathbb{H}, U}(\mathbb{H}) \) be the category of finite dimensional \( \mathbb{H} \)-modules \( V \) with \( \text{Wt}(V) \subset U \). For example, we have the category of \( \mathbb{H} \)-modules with "real" weights \( \text{Mod}_{\mathbb{H}, \mathfrak{r}}(\mathbb{H}) \). We indicate a subcategory/subset of tempered modules by a subscript "temp". In particular, we have the category of finite dimensional tempered \( \mathbb{H} \)-modules \( \text{Mod}_{\mathbb{H}}(\mathbb{H})_{\text{temp}} \).

We want to compare the irreducible representations of
\[
\mathbb{H}(t, W \rtimes \Gamma, k, \lambda) = \mathbb{H}(t, W \rtimes \Gamma, k, \lambda)/(r - 1)
\]
with those of
\[
\mathbb{H}(t, W \rtimes \Gamma, 0, \lambda) = \mathbb{H}(t, W \rtimes \Gamma, k, \lambda)/(r).
\]
The latter algebra has \( \text{Irr}(\mathbb{C}[W \rtimes \Gamma, \lambda]) \) as the set of irreducible representations on which \( \mathcal{O}(t) \) acts via evaluation at \( 0 \in t \). The correct analogue of this for \( \mathbb{H}(t, W \rtimes \Gamma, k, \lambda) \), at least with \( k \) real-valued, is
\[
\text{Irr}_{\mathfrak{a}}(\mathbb{H}(t, W \rtimes \Gamma, k, \lambda))_{\text{temp}} := \text{Irr}(\mathbb{H}(t, W \rtimes \Gamma, k, \lambda))_{\text{temp}} \cap \text{Mod}_{\mathbb{H}, \mathfrak{r}}(\mathbb{H}(t, W \rtimes \Gamma, k, \lambda)).
\]
As \( \mathbb{C}[W \rtimes \Gamma, \lambda] \) is a subalgebra of \( \mathbb{H}(t, W \rtimes \Gamma, k, \lambda) \), there is a natural restriction map
\[
\text{Res}_{W \rtimes \Gamma} : \text{Mod}_{\mathbb{H}}(\mathbb{H}(t, W \rtimes \Gamma, k, \lambda)) \to \text{Mod}_{\mathbb{H}}(\mathbb{C}[W \rtimes \Gamma, k, \lambda]).
\]
However, when \( k \neq 0 \) this map usually does not preserve irreducibility, not even on \( \text{Irr}_{\mathfrak{a}}(\mathbb{H}(t, W \rtimes \Gamma, k, \lambda))_{\text{temp}} \).
In the remainder of this section we assume that the parameter function \( k \) only takes real values. Let \( \epsilon \) be as in Lemma 2.2. Since \( \phi_\epsilon \) is the identity on \( \mathcal{O}(t \oplus \mathbb{C}) \), it induces equivalences of categories

\[
\begin{align*}
\text{Mod}_{R,U}(\mathbb{H}(t, W \times \Gamma, ek, \tilde{z})) & \rightarrow \text{Mod}_{R,U}(\mathbb{H}(t, W \times \Gamma, k, \tilde{z})) \quad U \subset t, \\
\text{Mod}_R(\mathbb{H}(t, W \times \Gamma, ek, \tilde{z}))(\text{temp}) & \rightarrow \text{Mod}_R(\mathbb{H}(t, W \times \Gamma, k, \tilde{z}))(\text{temp})
\end{align*}
\]

and a bijection

\[
\text{Irr}_a(\mathbb{H}(t, W \times \Gamma, ek, \tilde{z}))(\text{temp}) \rightarrow \text{Irr}_a(\mathbb{H}(t, W \times \Gamma, k, \tilde{z}))(\text{temp}).
\]

### Theorem 2.3

Let \( k : R \rightarrow \mathbb{R} \) be a \( \Gamma \)-invariant parameter function.

1. The set \( \text{Res}_{W \times \Gamma}(\text{Irr}_a(\mathbb{H}(t, W \times \Gamma, k, \tilde{z}))(\text{temp})) \) is a \( \mathbb{Z} \)-basis of \( \text{Z} \text{Irr}((C[W \times \Gamma, \tilde{z}])) \).

   Suppose that the restriction of \( k \) to any type \( F_4 \) component of \( R \) has \( k(\alpha) = 0 \) for a root \( \alpha \) in that component or is the form \( ek' \) for a character \( \epsilon : W(F_4) \rightarrow \{ \pm 1 \} \) and a geometric \( k' : F_4 \rightarrow \mathbb{R}_{>0} \).

2. There exist total orders on \( \text{Irr}_a(\mathbb{H}(t, W \times \Gamma, k, \tilde{z}))(\text{temp}) \) and on \( \text{Irr}((C[W \times \Gamma, \tilde{z}])) \), such that the matrix of the \( \mathbb{Z} \)-linear map

   \[
   \text{Res}_{W \times \Gamma} : \text{Z} \text{Irr}_a(\mathbb{H}(t, W \times \Gamma, k, \tilde{z}))(\text{temp}) \rightarrow \text{Z} \text{Irr}((C[W \times \Gamma, \tilde{z}]))
   \]

   is upper triangular and unipotent.

3. There exists a unique bijection

   \[
   \zeta_{\mathbb{H}(t, W \times \Gamma, k, \tilde{z})} : \text{Irr}_a(\mathbb{H}(t, W \times \Gamma, k, \tilde{z}))(\text{temp}) \rightarrow \text{Irr}((C[W \times \Gamma, \tilde{z}])
   \]

   such that \( \zeta_{\mathbb{H}(t, W \times \Gamma, k, \tilde{z})}(\pi) \) always occurs in \( \text{Res}_{W \times \Gamma}(\pi) \).

**Proof.** (a) is known from [Sol2, Proposition 1.7]. The proof of that shows we can reduce the entire theorem to the case where \( \tilde{z} \) is trivial. We assume that from now on, and omit \( \tilde{z} \) from the notations.

Parts (b) and (c) were shown in [Sol5, Theorem 6.2], provided that \( k(\alpha) \geq 0 \) for all \( \alpha \in R \). Choose \( \epsilon \) as in Lemma 2.2 so that \( ek : R \rightarrow \mathbb{R}_{\geq 0} \). For \( V \in \text{Mod}_R(\mathbb{H}(t, W, ek)) \) we have

\[
\text{Res}_{W}(\phi_\epsilon^* V) = \text{Res}_{W}(V) \otimes \epsilon,
\]

so we obtain a commutative diagram

\[
\begin{array}{ccc}
Z \text{Irr}_a(\mathbb{H}(t, W, ek))(\text{temp}) & \xrightarrow{\text{Res}_W} & Z \text{Irr}(W) \\
\downarrow \phi_\epsilon^* & & \downarrow \otimes \epsilon \\
Z \text{Irr}_a(\mathbb{H}(t, W, k))(\text{temp}) & \xrightarrow{\text{Res}_W} & Z \text{Irr}(W)
\end{array}
\] (2.10)

All the maps in this diagram are bijective and the vertical maps preserve irreducibility. Thus the theorem for \( \mathbb{H}(t, W, ek) \) implies it for \( \mathbb{H}(t, W, k) \).

The commutative diagram (2.10) also allows us to extend [Sol5, Lemma 6.5] from \( \mathbb{H}(t, W, ek) \) to \( \mathbb{H}(t, W, k) \). Then we can finish our proof for \( \mathbb{H}(t, W \times \Gamma, k) \) by applying [Sol5, Lemma 6.6].

**Remark.** Geometric parameter functions will appear in Section 3. We make the allowed parameter functions for a type \( F_4 \) root system explicit. Write \( k = (k(\alpha), k(\beta)) \) where \( \alpha \) is short root and \( \beta \) is a long root. The possibilities are

\[
(0, 0), (c, 0), (0, c), (c, c), (2c, c), (c/2, c), (4c, c), (-c, c), (-2c, c), (-c/2, c), (-4c, c),
\]

where \( c \in \mathbb{R}^\times \) is arbitrary. We expect that Theorem 2.3 also holds without extra conditions in type \( F_4 \).
Theorem 2.4. Let $\mathbb{H}(t, W \rtimes \Gamma, k, z)$ be as in Theorem 2.3.b. There exists a canonical bijection

$$\zeta_{\mathbb{H}(t, W \rtimes \Gamma, k, z)} : \text{Irr}(\mathbb{H}(t, W \rtimes \Gamma, k, z)) \to \text{Irr}(\mathbb{H}(t, W \rtimes \Gamma, 0, z))$$

which (as well as its inverse)

- respects temperedness,
- preserves the intersections with $\text{Mod}_{\mathbb{H}, \mathbb{M}}$,
- generalizes Theorem 2.3.c, via the identification

$$\text{Irr}_{\mathbb{M}}(\mathbb{H}(t, W \rtimes \Gamma, 0, z))_{\text{temp}} = \text{Irr}(\mathbb{C}[W \rtimes \Gamma, z]).$$

Proof. As discussed in the proof of Theorem 2.3.a, we can easily reduce to the case where $z$ is trivial. In [Sol5, Proposition 6.8], that case is derived from [Sol5, Theorem 6.2] (under more strict conditions on the parameters $k$). Using Theorem 2.3 instead of [Sol5, Theorem 6.2], this works for all parameters allowed in Theorem 2.3. Although [Sol5, Proposition 6.8] is only formulated for irreducible representations in $\text{Mod}_{\mathbb{H}, \mathbb{M}}(\mathbb{H}(t, W \rtimes \Gamma, k))$, the argument applies to all of $\text{Irr}(\mathbb{H}(t, W \rtimes \Gamma, k))$.  \qed

3. CUSPIDAL LOCAL SYSTEMS AND EQUIVARIANT COHOMOLOGY

We follow the setup from [Lus1, Lus3, AMS1, AMS2]. In these references a graded Hecke algebra was associated to a cuspidal local system on a nilpotent orbit for a complex reductive group. For applications to Langlands parameters we deal not only with connected groups, but also with disconnected reductive groups $G$.

Recall from [AMS1] that a quasi-Levi subgroup of $G$ is a group of the form $M = Z_G(Z(L)^\circ)$, where $L$ is a Levi subgroup of $G^\circ$. Thus $Z(M)^\circ = Z(L)^\circ$ and $M \hookrightarrow L = M^\circ$ is a bijection between the quasi-Levi subgroups of $G$ and the Levi subgroups of $G^\circ$.

Definition 3.1. A cuspidal quasi-support for $G$ is a triple $(M, C^M_v, qE)$ where:

- $M$ is a quasi-Levi subgroup of $G$;
- $C^M_v$ is the Ad($M$)-orbit of a nilpotent element $v \in m = \text{Lie}(M)$.
- $qE$ is a $M$-equivariant cuspidal local system on $C^M_v$, i.e. as $M^\circ$-equivariant local system it is a direct sum of cuspidal local systems.

We denote the $G$-conjugacy class of $(M, C^M_v, qE)$ by $[M, C^M_v, qE]_G$. With this cuspidal quasi-support we associate the groups

$$N_G(qE) = \text{Stab}_{N_G(M)}(qE) \quad \text{and} \quad W_{qE} = N_G(qE)/M.$$  \hspace{1cm} (3.1)

Such cuspidal quasi-supports are useful to partition the set of $G$-equivariant local systems on nilpotent orbits in $g = \text{Lie}(G)$. Let $E$ be an irreducible constituent of $qE$ as $M^\circ$-equivariant local system on $C^M_v$ (which by the cuspidality of $E$ equals the Ad($M^\circ$)-orbit of $v$). Then

$$W_E^\circ := N_G(M^\circ)/M^\circ \cong N_G(M^\circ)M/M$$

is a subgroup of $W_{qE}$. It is normal because $G^\circ$ is normal in $G$. Write $T = Z(L)^\circ$ and $t = \text{Lie}(T)$. It is known from [Lus1, Proposition 2.2] that $R(G^\circ, T) \subset t^\vee$ is a root system with Weyl group $W_E^\circ$.

Let $P^\circ$ be a parabolic subgroup of $G^\circ$ with Levi decomposition $P^\circ = M^\circ \ltimes U$. The definition of $M$ entails that it normalizes $U$, so

$$P := M \ltimes U$$
is a again a group, a "quasi-parabolic" subgroup of $G$. We put

$$N_G(P, q\mathcal{E}) = N_G(P, M) \cap N_G(q\mathcal{E}),$$

$$\Gamma_{q\mathcal{E}} = N_G(P, q\mathcal{E})/M.$$  

The same proof as for [AMS2, Lemma 2.1.b] shows that

\begin{equation}
W_{q\mathcal{E}} = W_\mathcal{E}^o \times \Gamma_{q\mathcal{E}}.
\end{equation}

The $W_{q\mathcal{E}}$-action on $T$ gives rise to an action of $W_{q\mathcal{E}}$ on $\mathcal{O}(t) = S(t^\vee)$.

We specify our parameters $c(\alpha)$. For $\alpha$ in the root system $R(G^\circ, T)$, let $\mathfrak{g}_\alpha \subset \mathfrak{g}$ be the associated eigenspace for the $T$-action. Let $\Delta_P$ be the set of roots in $R(G^\circ, T)$ which are simple with respect to $P$. For $\alpha \in \Delta_P$ we define $c(\alpha) \in \mathbb{Z}_{\geq 2}$ by

\begin{equation}
\begin{align*}
\text{ad}(v)^{c(\alpha) - 2} : & \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha} \to \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha} \text{ is nonzero,} \\
\text{ad}(v)^{c(\alpha) - 1} : & \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha} \to \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha} \text{ is zero.}
\end{align*}
\end{equation}

Then $(c(\alpha))_{\alpha \in \Delta_P}$ extends to a $W_{q\mathcal{E}}$-invariant function $c : R(G^\circ, T)_{\text{red}} \to \mathbb{C}$, where the subscript "red" indicates the set of indivisible roots. Let $\zeta : (W_{q\mathcal{E}}/W_\mathcal{E})^2 \to \mathbb{C}^\times$ be a 2-cocycle (to be specified later). To these data we associate the twisted graded Hecke algebra $\mathbb{H}(t, W_{q\mathcal{E}}, c, r, \zeta)$, as in Proposition 2.1.

To make the connection of the above twisted graded Hecke algebra with the cuspidal local system $q\mathcal{E}$ complete, we involve the geometry of $G$ and $\mathfrak{g}$. We work in the $G$-equivariant bounded derived category $\mathcal{D}_G(X)$, as in [BelLu], [Lus1] §1 and [Lus3] §1. Despite the terminology, this is not exactly the bounded derived category of the category of $G$-equivariant constructible sheaves on a $G$-variety $X$. Write

$$t_{\text{reg}} = \{ x \in t : Z_\mathfrak{g}(x) = I \} \quad \text{and} \quad \mathfrak{g}_{RS} = \text{Ad}(G)(\mathfrak{c}_v^M + t_{\text{reg}} + u).$$

Consider the varieties

$$\hat{\mathfrak{g}} = \{ (X, gP) \in \mathfrak{g} \times G/P : \text{Ad}(g^{-1})X \in \mathcal{C}_v^M + t + u \},$$

$$\hat{\mathfrak{g}}^o = \{ (X, gP) \in \mathfrak{g} \times G^o/P^o : \text{Ad}(g^{-1})X \in \mathfrak{c}_v^M + t + u \},$$

$$\hat{\mathfrak{g}}_{RS} = \hat{\mathfrak{g}} \cap (\mathfrak{g}_{RS} \times G/P).$$

We let $G \times \mathbb{C}^\times$ act on these varieties by

$$(g_1, \lambda) \cdot (x, gP) = (\lambda^{-2}\text{Ad}(g_1)x, g_1gP).$$

Equivariant cohomology for elements of $\mathcal{D}_{G \times \mathbb{C}^\times}(X)$ is defined via push-forward to a point. By [Lus1] Proposition 4.2 there is a natural isomorphism

\begin{equation}
H^*_c(G \times \mathbb{C}^\times)(\hat{\mathfrak{g}}) \cong \mathcal{O}(t) \otimes_{\mathbb{C}} \mathbb{C}[r].
\end{equation}

Consider the maps

\begin{equation}
\begin{align*}
f_1 : & \{ (x, g) \in \mathfrak{g} \times G : \text{Ad}(g^{-1})x \in \mathcal{C}_v^M + t + u \} \to \hat{\mathfrak{g}}, \\
& f_1(x, g) = \text{pr}_{\mathcal{C}_v^M}(\text{Ad}(g^{-1})x), \\
f_2(x, g) = (x, gP).
\end{align*}
\end{equation}

The group $G \times \mathbb{C}^\times \times P$ acts on $\{ (x, g) \in \mathfrak{g} \times G : \text{Ad}(g^{-1})x \in \mathcal{C}_v^M + t + u \}$ by

$$(g_1, \lambda, p) \cdot (x, g) = (\lambda^{-2}\text{Ad}(g_1)x, g_1gP).$$

Notice that $q\mathcal{E}$ is $M \times \mathbb{C}^\times$-equivariant, because $\mathbb{C}^\times$ is connected and stabilizes nilpotent $M$-orbits. Further $f_1$ is constant on $G$-orbits, so $f_1^*q\mathcal{E}$ is naturally a $G \times \mathbb{C}^\times$-equivariant local system. Let $q\mathcal{E}$ be the unique $G \times \mathbb{C}^\times$-equivariant local system on
\(\mathfrak{g}\) such that \(f^*_qq\mathcal{E} = f^*_1q\mathcal{E}\), and let \(q\mathcal{E}_{RS}\) be its pullback to \(\mathfrak{g}_{RS}\). Let \(\text{pr}_1: \mathfrak{g} \rightarrow \mathfrak{g}\) be the projection on the first coordinate. Its restriction

\[
\text{pr}_{1,RS}: \mathfrak{g}_{RS} \rightarrow \mathfrak{g}_{RS}
\]

is a fibration with fibre \(N_G(M)/M\), so \((\text{pr}_{1,RS})_!q\mathcal{E}_{RS}\) is a local system on \(\mathfrak{g}_{RS}\). Let \(K = IC_{G \times \mathbb{C}^*}(\mathfrak{g}, (\text{pr}_{1,RS})_!q\mathcal{E}_{RS})\) be the equivariant intersection cohomology complex on \(\mathfrak{g}\) defined by \((\text{pr}_{1,RS})_!q\mathcal{E}_{RS}\). It is shown in [Lus3, Proposition 7.12] that \((\mathfrak{g}, \mathfrak{g}_{RS})\) is a fibration with fibre \(\mathfrak{g}_{RS}\) and \(\mathfrak{g}\). Let \(\mathfrak{g}_{RS}\) be its pullback to \(\mathfrak{g}\). We can relate \(\dot{\mathfrak{g}}\) and \(K\) to their versions for \(G^o\), as follows. Write

\[
G = \bigsqcup_{\gamma \in N_G(P,M)/M} G^o \gamma M/M \quad \text{and} \quad G/P = \bigsqcup_{\gamma \in N_G(P,M)/M} G^o \gamma P/P.
\]

Then we can decompose

\[
\dot{\mathfrak{g}} = \bigcup_{\gamma \in N_G(P,M)/M} \{(X, g\gamma P) \in \dot{\mathfrak{g}} : g \in G^o\} = \bigcup_{\gamma \in N_G(P,M)/M} \{(X, g\gamma P^{-1}) : X \in \mathfrak{g}, g \in G^o/\gamma P^o \gamma^{-1}, \text{Ad}(g^{-1})X \in \text{Ad}(\gamma)(\mathfrak{g}_{RS}^o + t + u)\}
\]

Here each term \(\dot{\mathfrak{g}}^o_{\gamma}\) is a twisted version of \(\dot{\mathfrak{g}}^o\). Consequently \(K\) is a direct sum of \(G^o \times \mathbb{C}^*\)-equivariant subobjects, each of which is a twist of the \(K\) for \((G^oM, \mathfrak{c}_G^o, \mathfrak{q}\mathcal{E})\) by an element of \(N_G(M)/M\).

Considering \((\text{pr}_{1,RS})_!q\mathcal{E}_{RS}\) as a local system on \(\mathfrak{g}_{RS}\), [AMS1] Lemma 5.4] and [Lus3] Proposition 7.14] say that

\[
\mathcal{C}[W_{q\mathcal{E}}, \mathcal{Z}_{q\mathcal{E}}] \cong \text{End}_0^{G \times \mathbb{C}^*}(\mathfrak{g}_{RS})((\text{pr}_{1,RS})_!q\mathcal{E}_{RS}) \cong \text{End}_0^{G \times \mathbb{C}^*}(\mathfrak{g})(K),
\]

where \(\mathcal{Z}_{q\mathcal{E}} : (W_{q\mathcal{E}}/W_{\mathcal{E}})^2 \rightarrow \mathbb{C}^*\) is a suitable 2-cocycle. As in [AMS2] (8), we record the subalgebra of endomorphisms that stabilize \(\text{Lie}(P)\):

\[
\text{End}_{G}^{\mathcal{E}}((\text{pr}_{1,RS})_!q\mathcal{E}) \cong \mathcal{C}[\Gamma_{q\mathcal{E}}, \mathcal{Z}_{q\mathcal{E}}].
\]

Now we associate to \((M, \mathfrak{c}_G^o, \mathfrak{q}\mathcal{E})\) the twisted graded Hecke algebra

\[
\mathbb{H}(G, M, \mathfrak{q}\mathcal{E}) := \mathbb{H}(t, W_{q\mathcal{E}}, c, r, \mathcal{Z}_{q\mathcal{E}}),
\]

where the parameters \(c(\alpha)\) come from (3.3). As in [AMS2] Lemma 2.8], we can consider it as

\[
\mathbb{H}(G, M, \mathfrak{q}\mathcal{E}) = \mathbb{H}(t, W_{q\mathcal{E}}, c, r) \times \text{End}_{G}^{\mathcal{E}}((\text{pr}_{1,RS})_!q\mathcal{E}),
\]

and then it depends canonically on \((G, M, \mathfrak{q}\mathcal{E})\). We note that (3.2) implies

\[
\mathbb{H}(G^o N_G(P, \mathfrak{q}\mathcal{E}), M, \mathfrak{q}\mathcal{E}) = \mathbb{H}(G, M, \mathfrak{q}\mathcal{E}).
\]
There is also a purely geometric realization of this algebra. For \( \text{Ad}(G) \times \mathbb{C}^\times \)-stable subvarieties \( \mathcal{V} \) of \( \mathfrak{g} \), we define, as in \([\text{Lus1}] \S 3\),

\[
\mathcal{V} = \{ (X, gP) \in \mathfrak{g} : X \in \mathcal{V} \},
\]

(3.12) 
\[
\hat{\mathcal{V}} = \{ (X, gP, g'P) : (X, gP) \in \mathcal{V}, (X, g'P) \in \hat{\mathcal{V}} \}.
\]

The two projections \( \pi_{12}, \pi_{13} : \hat{\mathcal{V}} \to \mathcal{V} \) give rise to a \( G \times \mathbb{C}^\times \)-equivariant local system \( q\mathcal{E} = q\mathcal{E} \boxtimes \mathcal{E}^{\vee} \) on \( \mathcal{V} \), which carries a natural action of \( [3,4] \). As in \([\text{Lus1}] \), the action of \( \mathbb{C}[W_{q\mathcal{E}}, z_{q\mathcal{E}}^{-1}] \) on \( K' \) leads to

\[
(3.13) \quad \text{actions of } \mathbb{C}[W_{q\mathcal{E}}, z_{q\mathcal{E}}] \otimes \mathbb{C}[W_{q\mathcal{E}}, z_{q\mathcal{E}}^{-1}] \text{ on } q\mathcal{E} \text{ and on } H^G_{G \times \mathbb{C}^\times} (\hat{\mathcal{V}}, q\mathcal{E}).
\]

We will apply this with \( \mathcal{V} = \mathfrak{g} \) and with \( \mathcal{V} = \mathfrak{g}_N \), the variety of nilpotent elements in \( \mathfrak{g} \). In \([\text{Lus1}] \) and \([\text{AMS2}] \) §2 a left action \( \Delta \) and a right action \( \Delta' \) of \( \mathcal{H}(G, M, q\mathcal{E}) \) on \( H^G_{G \times \mathbb{C}^\times} (\mathfrak{g}_N, q\mathcal{E}) \) are constructed. Let

\[
pr_{1,N} : \mathfrak{g}_N = \mathfrak{g} \cap (\mathfrak{g}_N \times G/P) \to \mathfrak{g}_N
\]

be the restriction of \( pr_1 \). Let \( q\mathcal{E}_N \) be the pullback of \( q\mathcal{E} \) to \( \mathfrak{g}_N \) and define

\[
K_N = (pr_{1,N})_! q\mathcal{E}_N \in \mathcal{D}_{G \times \mathbb{C}^\times}(\mathfrak{g}_N).
\]

**Theorem 3.2.** (a) Let \( \mathcal{V} = \mathfrak{g} \) or \( \mathcal{V} = \mathfrak{g}_N \). The actions \( \Delta \) and \( \Delta' \) identify

\[
H^G_{G \times \mathbb{C}^\times} (\hat{\mathcal{V}}, q\mathcal{E}) \text{ with the biregular representation of } \mathcal{H}(G, M, q\mathcal{E}).
\]

(b) There exists a convolution product on \( H^G_{G \times \mathbb{C}^\times} (\hat{\mathfrak{g}}, q\mathcal{E}) \) which makes it isomorphic (as a graded algebra) to \( \text{End}_{\mathcal{D}_{G \times \mathbb{C}^\times}(\mathfrak{g})}^*(K) \). Similarly

\[
H^G_{G \times \mathbb{C}^\times} (\mathfrak{g}_N, q\mathcal{E}) \cong \text{End}_{\mathcal{D}_{G \times \mathbb{C}^\times}(\mathfrak{g}_N)}^*(K_N).
\]

(c) Parts (a) and (b) induce canonical isomorphisms of graded algebras

\[
\mathcal{H}(G, M, q\mathcal{E}) \to \text{End}_{\mathcal{D}_{G \times \mathbb{C}^\times}(\mathfrak{g})}^*(K) \to \text{End}_{\mathcal{D}_{G \times \mathbb{C}^\times}(\mathfrak{g}_N)}^*(K_N).
\]

**Proof.** (a) When \( G \) is connected, this is shown for \( \mathcal{V} = \mathfrak{g}_N \) in \([\text{Lus1}] \) Corollary 6.4 and for \( \mathcal{V} = \mathfrak{g} \) in the proof of \([\text{Lus3}] \) Theorem 8.11. In \([\text{AMS2}] \) Corollary 2.9 and §4 both are generalized to possibly disconnected \( G \).

(b) In \([\text{Lus3}] \) §2 it is shown that

\[
(3.14) \quad \text{End}_{\mathcal{D}_{G \times \mathbb{C}^\times}(\mathfrak{g})}^*(K) \cong H^G_{G \times \mathbb{C}^\times} (\hat{\mathfrak{g}}, i^! (q\mathcal{E} \boxtimes Dq\mathcal{E})),
\]

where \( i \) denotes the embedding \( \hat{\mathfrak{g}} \to \hat{\mathfrak{g}} \times \mathfrak{g} \). The Verdier duality operator \( D \) satisfies \( Dq\mathcal{E} = q\mathcal{E}^{\vee} \) and \( i^! = D i^* D \), so the right hand side of (3.14) is

\[
(3.15) \quad H^G_{G \times \mathbb{C}^\times} (\hat{\mathfrak{g}}, Di^! D(\mathcal{E} \boxtimes q\mathcal{E}^{\vee})) = H^G_{G \times \mathbb{C}^\times} (\mathfrak{g}, Di^* (q\mathcal{E}^{\vee} \boxtimes q\mathcal{E})).
\]

By definition of equivariant homology \([\text{Lus3}] \) §1.17, this equals

\[
(3.16) \quad H^G_{G \times \mathbb{C}^\times} (\mathfrak{g}, i^* (q\mathcal{E} \boxtimes q\mathcal{E}^{\vee})) = H^G_{G \times \mathbb{C}^\times} (\hat{\mathfrak{g}}, q\mathcal{E}).
\]

The same arguments apply with \( (\mathfrak{g}_N, K_N) \) instead of \( (\mathfrak{g}, K) \).

(c) In \([\text{Lus3}] \) Theorem 8.11 the first isomorphism is shown when \( G \) is connected. Using part (a) the same argument applies when \( G \) is disconnected. The second algebra isomorphism is a consequence of parts (a,b) and functoriality. \( \square \)
Let $\sigma \in \mathfrak{t}$, so that $M = Z_G(T) \subset Z_G(\sigma)$. We would like to compare Theorem 3.2 with its version for $(Z_G(\sigma), M, q\mathcal{E})$. First we analyse the variety 

$$(G/P)^\sigma := \{ gP \in G/P : \sigma \in \text{Lie}(gPg^{-1}) \}.$$ 

This is also the fixed point set of $\text{exp}(C\sigma)$ in $G/P$.

**Lemma 3.3.** For any $gP \in (G/P)^\sigma$, the subgroup $gP^\circ g^{-1} \cap Z_G^\sigma(\sigma)$ of $Z_G^\sigma(\sigma)$ is parabolic.

**Proof.** Consider the parabolic subgroup $P' := gP^\circ g^{-1}$ of $G^\circ$. Its Lie algebra $\mathfrak{p}'$ contains the semisimple element $\sigma$, so there exists a maximal torus $T'$ of $P'$ with $\sigma \in \mathfrak{t}'$. Let $M'$ be the unique Levi factor of $P'$ containing $T'$. The unipotent radical $U'$ of $P'$ and the opposite parabolic $M'\bar{U}'$ give rise to decompositions of $Z(\mathfrak{m}')$-modules

$$\mathfrak{g} = \mathfrak{u}' \oplus \mathfrak{p}', \quad \mathfrak{p}' = Z(\mathfrak{m}') \oplus \mathfrak{m}'_{\text{der}} \oplus \mathfrak{u}'.$$ 

Since $Z(\mathfrak{m}') \subset \mathfrak{t}' \subset Z_G(\sigma)$, these decompositions are preserved by intersecting with $Z_G(\sigma)$:

$$Z_G(\sigma) = Z_{\mathfrak{u}'}(\sigma) \oplus Z_{\mathfrak{p}'}(\sigma), \quad Z_{\mathfrak{p}'}(\sigma) = Z(\mathfrak{m}') \oplus Z_{\mathfrak{m}'_{\text{der}}}(\sigma) \oplus Z_{\mathfrak{u}'}(\sigma).$$ 

This shows that $Z_G(\sigma) \cap \mathfrak{p}'$ is a parabolic subalgebra of $Z_G(\sigma)$. Hence $Z_G^\sigma(\sigma) \cap \mathfrak{p}'$ is a parabolic subgroup of $Z_G^\sigma(\sigma)$.

The subgroup $Z_G(\sigma)$ of $G$ stabilizes $(G/P)^\sigma$, so the latter is a union of $Z_G(\sigma)$-orbits.

**Lemma 3.4.** The connected components of $(G/P)^\sigma$ are precisely its $Z_G^\sigma(\sigma)$-orbits.

**Proof.** Clearly every $Z_G^\sigma(\sigma)$-orbit is connected. From (3.17) we get an isomorphism of varieties

$$(3.17) \quad G/P = \bigsqcup_{T \in N_G(M)/M} \gamma G^\circ P/P \cong \bigsqcup_{\gamma} \gamma G^\circ /P^\circ.$$

Here $Z_G^\sigma(\sigma)$ acts on $\gamma G^\circ /P^\circ$ by

$$z \cdot \gamma gP^\circ = \gamma (\gamma^{-1} z \gamma) gP^\circ,$$

so via conjugation by $\gamma^{-1}$ and the natural action of $\gamma^{-1}Z_G^\sigma(\sigma)\gamma = Z_G^\sigma(\Ad(\gamma^{-1})\sigma)$ on $G^\circ/P^\circ$. Taking $\text{exp}(C\sigma)$-fixed points in (3.17) gives

$$(G/P)^\sigma \cong \bigsqcup_\gamma (\gamma G^\circ /P^\circ)^\sigma$$

$$= \bigsqcup_\gamma \{ gP^\circ : g \in G^\circ, \sigma \in \text{Lie}(\gamma gP^\circ g^{-1}\gamma^{-1}) \}$$

$$= \bigsqcup_\gamma \{ gP^\circ : g \in G^\circ, \Ad(\gamma^{-1})\sigma \in \text{Lie}(gP^\circ g^{-1}) \}$$

$$= \bigsqcup_\gamma (G^\circ /P^\circ)^{\Ad(\gamma^{-1})\sigma}.$$ 

This reduces the lemma to the case $G^\circ/P^\circ$, so to the connected group $G^\circ$. For that we refer to [ChGi, Proposition 8.8.7.ii]. That reference is written for Borel subgroups, but with Lemma 3.3 the proof also applies to other conjugacy classes of parabolic subgroups. \hfill \Box

It is also shown in [ChGi, Proposition 8.8.7.ii] that every $Z_G^\sigma(\sigma)$-orbit in $(G/P)^\sigma$ is a submanifold and an irreducible component.
Lemma 3.5. There are isomorphisms of \( Z_G(\sigma) \)-varieties
\[
\bigcup_{w \in N_{Z_G(\sigma)}(M) \setminus N_G(M)} Z_G(\sigma) / Z_{wPw^{-1}}(\sigma) \cong \bigcup_{w \in N_{Z_G(\sigma)}(M) \setminus N_G(M)} Z_G(\sigma) \cdot wP = (G/P)^\sigma.
\]

Proof. By Lemma 3.4 there exist finitely many \( \gamma \in G \) such that
\[
(G/P)^\sigma = \sqcup Z_G(\gamma) \cdot \gamma P.
\]
Then the same holds with \( Z_G(\sigma) \) instead of \( Z_G^0(\sigma) \), and fewer \( \gamma \)'s. The \( Z_G(\sigma) \)-stabilizer of \( \gamma P \) is
\[
\{ z \in Z_G(\sigma) : z\gamma P\gamma^{-1} = \gamma P\gamma^{-1} \} = Z_G(\sigma) \cap \gamma P\gamma^{-1} = \gamma P_{\gamma^{-1}}(\sigma).
\]
That proves the lemma, except for the precise index set.

Fix a maximal torus \( T' \) of \( Z_G^0(\sigma) \) with \( T \subset T' \). Every parabolic subgroup of \( G^0 \) or \( Z_G^0(\sigma) \) is conjugate to one containing \( T' \). The \( G^0 \)-conjugates of \( P^0 \) that contain \( T' \) are the \( wP^0w^{-1} \) with \( w \in N_{G^0}(T') \), or equivalently with
\[
w \in N_{G^0}(T') / N_{P^0}(T') = N_{G^0}(T') / N_{M^0}(T') \cong N_{G^0}(M^0) / M^0.
\]
For \( w, w' \in N_{G^0}(M^0) \), \( wP^0 \) and \( w'P^0 \) are in the same \( Z_G^0(\sigma) \)-orbit if and only if \( w'w^{-1} \in N_{Z_G^0(\sigma)}(M^0) / M^0 \). We find that
\[
(\text{3.19})
\]
\[
(G^0 / P^0)^\sigma = \bigcup_{w \in N_{Z_G^0(\sigma)}(M^0) \setminus N_{G^0}(M^0)} Z_G^0(\sigma) \cdot wP^0.
\]

We note that the group
\[
N_{G^0}(M^0) / M^0 = N_{G^0}(T) / Z_G^0(T) = N_{G^0}(M) / M^0 \cong N_{G^0}(M)M / M
\]
normalises \( P \). When we replace \( G^0 / P^0 \) by \( G/P \) in (3.19), the options for \( w \) need to be enlarged to \( N_G(M) / M \). Next we replace \( Z_G^0(\sigma) \) by \( Z_G(\sigma) \), so that \( wP \) and \( w'P \) are in the same \( Z_G(\sigma) \)-orbit if and only if \( w'w^{-1} \in N_{Z_G(\sigma)}(M) / M \). We conclude that
\[
(G/P)^\sigma = \bigcup_{w \in N_{Z_G^0(\sigma)}(M) \setminus N_G(M)} Z_G(\sigma) \cdot wP = \bigcup_{w \in N_{Z_G(\sigma)}(M) \setminus N_G(M)} Z_G(\sigma) \cdot wP. \quad \square
\]

The fixed point set of \( \exp(C \sigma) \) in \( \mathfrak{g}^\sigma \) is
\[
\mathfrak{g}^\sigma = \mathfrak{g} \cap (Z_G(\sigma) \times (G/P)^\sigma) = \{(X, gP) \in Z_G(\sigma) \times (G/P)^\sigma : \text{Ad}(g^{-1})X \in C_u^M + t + u \}.
\]
Clearly \( \mathfrak{g}^\sigma \) is related to \( Z_G(\sigma) \) and to \( Z_G^0(\sigma) \). With (3.19) and (3.8) we can make that precise:
\[
(\text{3.20})
\]
\[
\mathfrak{g}^\sigma = \bigcup_{w \in N_{Z_G^0(\sigma)}(M) \setminus N_G(M)} Z_G(\sigma)_{w} = \bigcup_{w \in N_{Z_G(\sigma)}(M) \setminus N_G(M)} Z_G(\sigma)_{w}
\]
\[
Z_G(\sigma)_{w} = \{(X, gZ_{wPw^{-1}}(\sigma)) \in Z_G(\sigma) \times Z_G(\sigma) / Z_{wPw^{-1}}(\sigma) : \text{Ad}(g^{-1})X \in \text{Ad}(w)(C_u^M + t + u) \}.
\]
Let \( j' : \mathfrak{g}^\sigma \to \mathfrak{g} \) be the inclusion and let \( \text{pr}_{1}^\sigma \) be the restriction of \( \text{pr}_{1} \) to \( \mathfrak{g}^\sigma \). We define
\[
K_\sigma = (\text{pr}_{1}^\sigma)_{j' \ast qE} \in \mathcal{D}_{G \times C^\times}(Z_G(\sigma)).
\]
From (3.20) we infer that \( K_\sigma \) is a direct sum of the parts \( K_{\sigma, w} \) (resp. \( K_{\sigma, w}^0 \)) coming from \( Z_G(\sigma)_{w} \) (resp. from \( Z_G^0(\sigma)_{w} \)), and each such part is a version of the \( K \) for \( Z_G(\sigma) \) (resp. for \( Z_G^0(\sigma) \)), twisted by \( w \in N_G(M) / M \).
These objects admit versions restricted to subvarieties of nilpotent elements, which we indicate by a subscript $N$. In particular

$$K_{N,\sigma} = (\text{pr}_{1,N})^* \mathcal{E}_N \in \mathcal{D}_G \times \mathbb{C}^\times (Z_0(\sigma)_N)$$

can be decomposed as a direct sum of subobjects $K_{N,\sigma,w}$ or $K_{N,\sigma,w}^\circ$.

**Lemma 3.6.** Let $w, w' \in N_G(M)/M$. The inclusion $Z_0(\sigma)_N \to Z_0(\sigma)$ induces an isomorphism of $H^*_{Z_0(\sigma)_X \times \mathbb{C}^\times (pt)}$-modules

$$\text{Hom}_{\mathcal{D}_{Z_0(\sigma)_X \times \mathbb{C}^\times (pt)}}(K_{N,\sigma,w}^\circ, K_{N,\sigma,w'}^\circ) \to \text{Hom}_{\mathcal{D}_{Z_0(\sigma)_X \times \mathbb{C}^\times (pt)}}(K_{N,\sigma,w}^\circ, K_{N,\sigma,w'}^\circ).$$

**Proof.** Decompose $q^* \mathcal{E}|_{Z_0(\sigma)_w}$ as direct sum of irreducible $Z_0^\circ(\sigma) \times \mathbb{C}^\times$-equivariant local systems. Each summand is of the form $\text{Ad}(w)_* \mathcal{E}$, for an irreducible summand $\mathcal{E}$ of $q^* \mathcal{E}$ as $M^\circ$-equivariant local system. Similarly we decompose $q^* \mathcal{E}|_{Z_0(\sigma)_w}$ as direct sum of terms $\text{Ad}(w')_* \mathcal{E}$. Then

$$K_{\sigma,w}^\circ = \bigoplus_\mathcal{E} (\text{pr}_{1,Z_0(\sigma)})_! \text{Ad}(w)_* \mathcal{E},$$

and similarly for $K_{N,\sigma,w}^\circ, K_{\sigma,w'}^\circ$ and $K_{N,\sigma,w'}^\circ$.

A computation like in (3.14)–(3.16) shows that

$$\text{Hom}_{\mathcal{D}_{Z_0(\sigma)_X \times \mathbb{C}^\times (pt)}}(K_{\sigma,w}^\circ, (\text{pr}_{1,Z_0(\sigma)})_! \text{Ad}(w)_* \mathcal{E}, (\text{pr}_{1,Z_0(\sigma)})_! \text{Ad}(w')_* \mathcal{E}'))$$

$$\cong \text{Hom}_{Z_0(\sigma)_X \times \mathbb{C}^\times} (Z_0(\sigma)_w^\circ, (\text{Ad}(w)_* \mathcal{E} \boxtimes \text{Ad}(w')_* \mathcal{E}')).$$

Here $Z_0(\sigma)_w^\circ = Z_0(\sigma)_w \times Z_0(\sigma)_w^\circ$ and

$$i_\sigma : Z_0(\sigma)_w^\circ \to Z_0(\sigma)_w^\circ \times Z_0(\sigma)_w^\circ$$

denotes the inclusion. The same applies with subscripts $N$:

$$\text{Hom}_{\mathcal{D}_{Z_0(\sigma)_X \times \mathbb{C}^\times (pt)}}(K_{N,\sigma,w}^\circ, (\text{pr}_{1,Z_0(\sigma)_N})_! \text{Ad}(w)_* \mathcal{E}_N, (\text{pr}_{1,Z_0(\sigma)_N})_! \text{Ad}(w')_* \mathcal{E}_N')$$

$$\cong \text{Hom}_{Z_0(\sigma)_X \times \mathbb{C}^\times} (Z_0(\sigma)_N^\circ, i_{\sigma,N}_* \text{Ad}(w)_* \mathcal{E} \boxtimes \text{Ad}(w')_* \mathcal{E}')).$$

When $w = w'$ and $\mathcal{E} = \mathcal{E}'$, (3.22) and (3.23) are computed in [Lus1, Proposition 4.7]. In fact [Lus1, Proposition 4.7] also applies in our more general setting, with different $\text{Ad}(w)_* \mathcal{E}$ and $\text{Ad}(w')_* \mathcal{E}'$. Namely, to handle those we add the argument from the proof of [AMS2, Proposition 2.6], especially [AMS2, (11)]. That works for both $Z_0(\sigma)_N$ and for $Z_0(\sigma)_N^\circ$, and entails that there are natural isomorphisms of graded $H^*_{Z_0(\sigma)_X \times \mathbb{C}^\times (pt)}$-modules

(3.24)

$$H^*_{Z_0(\sigma)_X \times \mathbb{C}^\times} (Z_0(\sigma)_N^\circ) \otimes \mathbb{C} H_0(Z_0(\sigma)_N^\circ, i_{\sigma,N}_* \text{Ad}(w)_* \mathcal{E} \boxtimes \text{Ad}(w')_* \mathcal{E}')) \cong (3.22),$$

$$H^*_{Z_0(\sigma)_X \times \mathbb{C}^\times} (Z_0(\sigma)_N^\circ) \otimes \mathbb{C} H_0(Z_0(\sigma)_N^\circ, i_{\sigma,N}_* \text{Ad}(w)_* \mathcal{E} \boxtimes \text{Ad}(w')_* \mathcal{E}')) \cong (3.23).$$

Moreover, the proof of [Lus1, Proposition 4.7] shows that the two lines of (3.24) are isomorphic via the inclusion $Z_0(\sigma)_N \to Z_0(\sigma)$.

Finally, we can generalize the second isomorphism in Theorem 3.2.c.

**Proposition 3.7.** The inclusion $Z_0(\sigma)_N \to Z_0(\sigma)$ induces an algebra isomorphism

$$\text{End}_{\mathcal{D}_{Z_0(\sigma)_X \times \mathbb{C}^\times (pt)}}(K_{\sigma}) \to \text{End}_{\mathcal{D}_{Z_0(\sigma)_X \times \mathbb{C}^\times (pt)}}(K_{N,\sigma}).$$
Proof. Take the direct sum of the instances of Lemma 3.6 over all \( w, w' \in N_{Z_G(\sigma)}(M) \setminus N_G(M) \). By (3.20), that yields a natural isomorphism
\[
\text{End}_D^*Z_{Z_G(\sigma) \times \mathbb{C}^\times}(K) \to \text{End}_D^*Z_{Z_G(\sigma) \times \mathbb{C}^\times}(K_N, \sigma).
\]
Now we take \( \pi_0(Z_G(\sigma)) \)-invariants on both sides, that replaces \( \text{End}_D^*Z_{Z_G(\sigma) \times \mathbb{C}^\times}(?) \) by \( \text{End}_D^*Z_{Z_G(\sigma) \times \mathbb{C}^\times}(?) \). \( \square \)

4. Localization at a Central Character

We want to localize \( \mathbb{H}(G, M, q\mathcal{E}) \cong \text{End}_{D_{G \times \mathbb{C}^\times}}^*(K) \) with respect to a central character. Recall from [AMS2, Lemma 2.3 and §4] that
\[
Z(\mathbb{H}(G, M, q\mathcal{E})) = \mathcal{O}(\mathfrak{t} \oplus \mathbb{C})^{W_{q\mathcal{E}}} = \mathcal{O}(\mathfrak{t}/W_{q\mathcal{E}}) \otimes \mathbb{C}[\mathfrak{r}].
\]
To localize in a geometric way, we need to interpret (4.1) in terms of equivariant homology. By [Lus1, §1.11] there are natural isomorphisms
\[
H^*_G(C\times pt) \cong \mathcal{O}(\mathfrak{g} \oplus \mathbb{C})^{G \times \mathbb{C}^\times} \cong \mathcal{O}(\mathfrak{g}/G) \otimes \mathbb{C}[\mathfrak{r}].
\]
The algebra \( H^*_G(C\times pt) \) acts naturally on
\[
H^*_G(C\times \mathbb{C}^\times)(\mathfrak{g}, \mathfrak{q}\mathcal{E}) \cong \text{End}_{D_{G \times \mathbb{C}^\times}}^*(K),
\]
by the product in equivariant homology. That determines a homomorphism
\[
H^*_G(C\times pt) \to Z(\text{End}_{D_{G \times \mathbb{C}^\times}}^*(K)) \cong \mathcal{O}(\mathfrak{t} \oplus \mathbb{C})^{W_{q\mathcal{E}}}.
\]
We take this opportunity to provide a proof that is missing in [AMS2, §4].

Lemma 4.1. The natural map \( t/W_{q\mathcal{E}} \to \mathfrak{g}/G \) is injective.

The support of \( \text{End}_{D_{G \times \mathbb{C}^\times}}^*(K) \) as \( H^*_G(C\times pt) \)-module is im(\( t/W_{q\mathcal{E}} \to \mathfrak{g}/G \times \mathbb{C} \)).

Proof. By [Lus3, 8.13.b] the natural map \( t/W_{q\mathcal{E}} \to \mathfrak{g}/G \) is injective. The group M equals \( Z_{G}(T) \), so it fixes \( t \) pointwise. Hence the equivalence class of \( X \in t \) in \( \mathfrak{g}/G \times \mathbb{C} \) equals its equivalence class in \( \mathfrak{g}/G^0 \).

By Condition 5.1
\[
W_{q\mathcal{E}}/W_{q\mathcal{E}}^0 \cong \Gamma_{q\mathcal{E}} \cong N_G(P, q\mathcal{E})/M \cong G/G^0M.
\]
Hence, the equivalence class of any \( X \in t \) in \( \mathfrak{g}/G \) is precisely \( \Gamma_{q\mathcal{E}} \) times its equivalence class in \( \mathfrak{g}/G^0M \), or equivalently in \( \mathfrak{g}/G^0 \). Similarly the \( W_{q\mathcal{E}} \)-orbit of \( X \) in \( t \) is \( \Gamma_{q\mathcal{E}} \times W_{q\mathcal{E}}^0 \)-orbit in \( t \), so the injectivity is preserved if we replace \( G^0 \) by \( G \).

The second statement is a direct consequence of (4.1), (4.2) and the first statement. \( \square \)

Lemma 4.1 entails that we can localize \( \text{End}_{D_{G \times \mathbb{C}^\times}}^*(K) \) with respect to elements of \( \mathfrak{g}/G \times \mathbb{C} \) that come from \( t/W_{q\mathcal{E}} \times \mathbb{C} \). With the techniques from [Lus3, §4], that can be done geometrically. In Appendix B we discuss why these techniques apply in the setting of [Lus3, §8], which is a special case of our current setting.

Fix \((\sigma, r) \in \text{Ad}(G)t \times \mathbb{C}\) and put
\[
C = Z_{G \times \mathbb{C}^\times}(\sigma, r) = Z_G(\sigma) \times \mathbb{C}^\times.
\]
The inclusion \( \mathfrak{c} = \text{Lie}(C) \subset \mathfrak{g} \) makes \( H^*_C(pt) \) into a module for \( H^*_{G \times C^\times}(pt) = \mathcal{O}(\mathfrak{g})^G \otimes \mathbb{C}[r] \). Further, any \( G \times C^\times \)-equivariant sheaf can be regarded as an \( C \)-equivariant sheaf. That induces an algebra homomorphism

\[
(4.4) \quad H^*_C(pt) \otimes H^*_{G \times C^\times}(pt) \to H^*_C(pt) \otimes H^*_{\mathcal{O}(\mathfrak{g})}(K).
\]

By the perversity of \( K \), all terms in negative degrees in \((4.4)\) are zero. Let

\[
\hat{H}^*_C(pt) = \hat{\mathcal{O}}(\mathfrak{c}/G \times C^\times)_{\sigma,r}
\]

be the completion of \( H^*_{G \times C^\times}(pt) \) with respect to the maximal ideal determined by \((\sigma,r)\). We define \( \hat{H}^*_C(pt)_{\sigma,r} = \hat{\mathcal{O}}(\mathfrak{c}/G)_{\sigma,r} \) analogously.

**Proposition 4.2.** (a) The natural map \( H^*_C\times K^* = H^*_C(pt) \) induces an algebra isomorphism \( \hat{H}^*_C(pt)_{\sigma,r} \to \hat{H}^*_C(pt)_{\sigma,r} \).

(b) Part (a) and \((4.4)\) induce an isomorphism of \( \hat{H}^*_C(pt)_{\sigma,r} \)-algebras

\[
(4.5) \quad \hat{H}^*_{G \times C^\times}(pt)_{\sigma,r} \otimes \text{End}^*_{\mathcal{O}(\mathfrak{g})}(K) \to \hat{H}^*_C(pt)_{\sigma,r} \otimes \text{End}^*_{\mathcal{O}(\mathfrak{g})}(K).
\]

(c) Part (b) also holds with \((\mathfrak{g}_N,K_N)\) instead of \((\mathfrak{g},K)\).

**Proof.** (a) According to [Lus3, 4.3.(a)] this holds for connected groups, so for \( G^0 \times C^\times \) and \( C^\times \) = \( Z_G(\sigma)^0 \times C^\times \). From \( H^*_G(pt) = H^*_{G^0}(pt)^{G/G^0} \) \([Lus1]\) \( \text{§1.9} \) we deduce that

\[
(4.5) \quad \hat{H}^*_{G \times C^\times}(pt)_{\sigma,r} = \left( \bigoplus_{g \in G/Z_G(\sigma)^0} \hat{H}^*_{G^0}(pt)_{\Ad(g)\sigma,r} \right)^{G/G^0}
\]

\[
\cong (\hat{H}^*_{G^0 \times C^\times}(pt)_{\sigma,r})_{Z_G(\sigma)^0} \cong (\hat{H}^*_C(pt)_{\sigma,r})_{Z_C(\sigma)}
\]

\[
= (\hat{H}^*_C(pt)_{\sigma,r})^{C/C^\times} = \hat{H}^*_C(pt)_{\sigma,r}.
\]

(b) Part (a) and Proposition \([B.1]\) show this for connected algebraic groups:

\[
(4.6) \quad \hat{H}^*_{G \times C^\times}(pt)_{\sigma,r} \otimes \text{End}^*_{\mathcal{O}(\mathfrak{g})}(K) \to \hat{H}^*_C(pt)_{\sigma,r} \otimes \text{End}^*_{\mathcal{O}(\mathfrak{g})}(K).
\]

This isomorphism is a tensor product of maps between the two respective tensor factors, so it can be restricted to \( Z_G(\sigma)\)-invariant elements in the first tensor factors on both sides. In view of \((4.5)\), this yields a natural isomorphism

\[
(4.7) \quad \hat{H}^*_C(pt)_{\sigma,r} \otimes \text{End}^*_{\mathcal{O}(\mathfrak{g})}(K) \cong (\bigoplus_{g \in G/Z_G(\sigma)^0} \hat{H}^*_{G^0 \times C^\times}(pt)_{\Ad(g)\sigma,r})^{G/G^0}
\]

\[
\cong (\bigoplus_{g \in G/Z_G(\sigma)^0} \hat{H}^*_G(pt)_{\Ad(g)\sigma,r})^{G/G^0} \otimes \text{End}^*_{\mathcal{O}(\mathfrak{g})}(K).
\]

Since all the elements of \( G/Z_G(\sigma)^0 \times C^\times \) correspond to different equivalence classes in \( \Ad(G) \subset \mathfrak{g}/Z_G(\sigma)^0 \), the \( Z_G(\sigma)^0 \times C^\times \) invariance of \( \text{End}^*_{\mathcal{O}(\mathfrak{g})}(K) \) is automatically promoted to \( G \)-invariance in the second line of \((4.7)\). Hence that
algebra can be identified with 
\[ \hat{H}^*_G \times \mathbb{C}^\times (\mathrm{pt})_{\sigma,r} \otimes \End^*_D (g)(K). \]

(c) This can be shown in the same way as part (b). \(\square\)

Having localized \(\End^*_D (g)(K)\) with respect to a central character, we want to see how this affects the underlying variety \(g\). Let \(T_{\sigma,r}\) be the smallest algebraic torus in \(G^0 \times \mathbb{C}^\times\) whose Lie algebra contains \((\sigma, r)\). Then \(g^\sigma_{\sigma,r} := g^{T_{\sigma,r}}\) is \(C\)-stable and

\[ g^\sigma_{\sigma,r} = g^{\exp(\mathbb{C}(\sigma, r))} = \{ X \in g : e^{-2\pi} \text{Ad}(\exp(z\sigma))X = X, \forall z \in \mathbb{C} \} \]

\[ = \{ X \in g : \text{Ad}(\exp(z\sigma))X = e^{2\pi}X, \forall z \in \mathbb{C} \} \]

\[ = \{ X \in g : \text{ad}(\sigma)X = 2rX \}. \]

Notice that \(g^\sigma_{\sigma,r}\) consists entirely of nilpotent elements (unless \(r = 0\)). Write

\[ \hat{g}^\sigma_{\sigma,r} = \hat{g}^{T_{\sigma,r}} = \hat{g} \cap (g^\sigma_{\sigma,r} \times (G/P)_{\exp(\mathbb{C}\sigma)}) \]

and consider the commutative diagram

\[ \begin{array}{ccc}
\hat{g}^\sigma_{\sigma,r} & \xrightarrow{j'} & \hat{g} \\
\downarrow \text{pr}_1 & & \downarrow \text{pr}_1 \\
\hat{g}^\sigma_{\sigma,r} & \xrightarrow{j} & \hat{g} \\
\end{array} \]

where the vertical maps are inclusions. We define

\[ K_{\sigma,r} = \text{pr}_{1,1}(j' \ast (q \mathcal{E})) \in \mathcal{D}_C (g^\sigma_{\sigma,r}). \]

Since \((4.8)\) is often not a pullback diagram, \(K_{\sigma,r}\) need not be isomorphic to \(j' \ast (K) = j' \ast \text{pr}_{1,1}(q \mathcal{E})\). Nevertheless \(K_{\sigma,r}\) can be regarded as some kind of restriction of \(K\) to \(g^\sigma_{\sigma,r}\). According to \([\text{Lus3}, \S 5.3]\) (where \(K\) is called \(B \) and pullbacks to \(T_{\sigma,r}\)-fixed subvarieties are indicated by a superscript tilde), \(K_{\sigma,r}\) is a direct sum of degree shifts of semisimple perverse sheaves on \(g^\sigma_{\sigma,r}\).

Notice that for \((\sigma, r) = (0, 0)\) we have

\[ g^{0,0} = g, \quad \hat{g}^{0,0} = \hat{g} \quad \text{and} \quad K_{0,0} = K. \]

Thus the objects in Theorem \([4.2]\) are special cases of their localized versions in this section. Let \(j_N : \hat{g}^{0,0}_{N} \rightarrow \hat{g}\) be the inclusion and define

\[ K_{N,\sigma,r} = (\text{pr}_{1,N} \ast j_N \ast (q \mathcal{E}_N)) \in \mathcal{D}_C (g^{\sigma}_{\sigma,r}). \]

We note that for \(r \neq 0\) we have \(g^{\sigma}_{\sigma,r} = g^\sigma_{\sigma,r}, \hat{g}^{\sigma}_{N} = \hat{g}^\sigma_{\sigma,r}\) and \(K_{N,\sigma,r} = K_{\sigma,r}\).

**Proposition 4.3.** (a) There exists a natural isomorphism of \(H^*_C (pt)_{\sigma,r}\)-algebras

\[ H^*_C (pt)_{\sigma,r} \otimes \End^*_D (g^\sigma_{\sigma,r}) (K_{\sigma,r}) \rightarrow \hat{H}^*_C (pt)_{\sigma,r} \otimes \End^*_D (g)(K). \]

(b) The algebras in part (a) are naturally isomorphic with

\[ H^*_C (pt)_{\sigma,r} \otimes \End^*_D (g^\sigma_{\sigma,r}) (K_{N,\sigma,r}) \cong \hat{H}^*_C (pt)_{\sigma,r} \otimes \End^*_D (g)(K). \]

**Proof.** (a) In \([\text{Lus3}, \S 4.9-4.10]\), which is applicable by Appendix \([3]\) this was proven under the assumption that \(G\) (and hence \(C\)) is connected. Explicitly, there exists an algebra homomorphism

\[ \Xi : \End^*_D (g^\sigma_{\sigma,r}) (K_{\sigma,r}) \rightarrow \End^*_D (g)(K) \]
such that \( \text{id} \otimes \Xi : \hat{H}^*_C(\sigma) \otimes \text{End}_{D_C(\sigma)}^*(g, r) K \to \hat{H}^*_C(\sigma) \otimes \text{End}_{D_C(\sigma)}^*(g)(K) \).

Next we take the \( C/C^\circ \)-invariants on both sides, which replaces \( \text{End}_{D_C(\sigma)}^*(g)(?) \) by \( \text{End}_{D_C(\sigma)}^*(g)(?) \).

(b) By Proposition 4.2.c and Theorem 3.2 there is a natural isomorphism

\[
\hat{H}^*_C(\sigma) \otimes \text{End}_{D_C(\sigma)}^*(g, r) \otimes \hat{H}^*_C(\sigma) \otimes \text{End}_{D_C(\sigma)}^*(g)(K) \cong \hat{H}^*_C(\sigma) \otimes \text{End}_{D_C(\sigma)}^*(g)(K).
\]

When \( r \neq 0 \), the left hand sides of parts (a) and (b) are the same. When \( r = 0 \), we assume (as we may by Lemma 4.1) that \( \sigma \in t \). Then \( g^{\sigma,0} = Z_\sigma(g) \), \( \hat{g}^{\sigma,0} = \hat{g}^\sigma \) and \( K_{\sigma,0} = K_\sigma \) (with the notations from page 18). Proposition 3.7 provides the final isomorphism

\[
\hat{H}^*_C(\sigma) \otimes \text{End}_{D_C(\sigma)}^*(g, 0) \otimes \hat{H}^*_C(\sigma) \otimes \text{End}_{D_C(\sigma)}^*(g, 0)(K).
\]

For \( \sigma \in \text{Ad}(G)t \) and \( r \in \mathbb{C} \), let \( Z_{\sigma, r} \) be the maximal ideal of

\[
Z(\mathbb{H}(G, M, q\mathcal{E})) \cong \mathcal{O}(t/W q\mathcal{E} \times \mathbb{C})
\]
determined by \( (\text{Ad}(G)\sigma \cap t, r) \). Every finite length module \( V \) of \( \mathbb{H}(G, M, q\mathcal{E}) \) can be decomposed as

\[
V = \bigoplus_{(\sigma, r) \in t/W q\mathcal{E} \times \mathbb{C}} V_{\sigma, r},
\]

\[
V_{\sigma, r} = \{ v \in V : Z_{\sigma, r}^n v = 0 \text{ for some } n \in \mathbb{N} \}.
\]

Hence the category of finite length left modules is a direct sum

\[
(4.11) \quad \text{Mod}_\mathbb{H}(\mathbb{H}(G, M, q\mathcal{E})) = \bigoplus_{(\sigma, r) \in t/W q\mathcal{E} \times \mathbb{C}} \text{Mod}_\mathbb{H},_{\sigma, r}(\mathbb{H}(G, M, q\mathcal{E})).
\]

The same holds for right modules:

\[
(4.12) \quad \mathbb{H}(G, M, q\mathcal{E}) - \text{Mod}_\mathbb{H} = \bigoplus_{(\sigma, r) \in t/W q\mathcal{E} \times \mathbb{C}} \mathbb{H}(G, M, q\mathcal{E}) - \text{Mod}_\mathbb{H},_{\sigma, r}.
\]

Let \( \hat{Z}(\mathbb{H}(G, M, q\mathcal{E}))_{\sigma, r} \) be the formal completion of \( Z(\mathbb{H}(G, M, q\mathcal{E})) \) with respect to \( Z_{\sigma, r} \). The category of finite length left modules, continuous with respect to the adic topology, of the localized algebra

\[
\hat{Z}(\mathbb{H}(G, M, q\mathcal{E}))_{\sigma, r} \otimes_{Z(\mathbb{H}(G, M, q\mathcal{E}))} \mathbb{H}(G, M, q\mathcal{E})
\]
can be identified with \( \text{Mod}_{\hat{Z},_{\sigma, r}}(\mathbb{H}(G, M, q\mathcal{E})) \). We use a similar notation for the algebras \( \text{End}_{D_{G \times \mathbb{C}^\times}}(\sigma)(K) \) and \( \text{End}_{D_{ZG(\sigma) \times \mathbb{C}^\times}}(\sigma)(K_{\sigma, r}) \), with respect to the maximal ideals determined by \( (\sigma, r) \) in \( H^*_G(\sigma) \times \mathbb{C}^\times(\sigma) \) and in \( H^*_ZG(\sigma) \times \mathbb{C}^\times(\sigma) \).
Theorem 4.4. (a) There are natural algebra isomorphisms
\[ \hat{Z}(\mathbb{H}(G, M, q\mathcal{E}))_{\sigma,r} \otimes_{\mathbb{H}(G, M, q\mathcal{E})} Z(\mathbb{H}(G, M, q\mathcal{E})) \xrightarrow{\sim} \hat{H}^*_{G \times \mathbb{C}^\times}(pt)_{\sigma,r} \otimes_{H^*_{G \times \mathbb{C}^\times}(pt)} \text{End}_{D_{G \times \mathbb{C}^\times}(\mathfrak{g})}^*(K) \xrightarrow{\sim} \hat{H}^*_{ZG(\sigma) \times \mathbb{C}^\times}(pt)_{\sigma,r} \otimes_{D_{ZG(\sigma) \times \mathbb{C}^\times}(\mathfrak{g}_{\sigma,r})} \text{End}_{D_{ZG(\sigma) \times \mathbb{C}^\times}(\mathfrak{g}_{\sigma,r})}^*(K_{\sigma,r}). \]

(b) Part (a) induces equivalences of categories
\[ \text{Mod}_{fl,\sigma,r}(\mathbb{H}(G, M, q\mathcal{E})) \cong \text{Mod}_{fl,\sigma,r}(\text{End}_{D_{G \times \mathbb{C}^\times}(\mathfrak{g})}^*(K)) \cong \text{Mod}_{fl,\sigma,r}(\text{End}_{D_{ZG(\sigma) \times \mathbb{C}^\times}(\mathfrak{g}_{\sigma,r})}^*(K_{\sigma,r})), \]
and analogously with right modules.

(c) Parts (a) and (b) also hold with \((\mathfrak{g}_N, K_N)\) instead of \((\mathfrak{g}, K)\).

Proof. (a) is a consequence of Theorem 3.2 and Propositions 4.2, 4.3.
(b) follows directly from (a).
(c) The proof is completely analogous to that of parts (a) and (b). \(\square\)

We point out that the data \((\sigma, r)\) in Theorem 4.4 can be scaled by an arbitrary \(z \in \mathbb{C}^\times\). Namely, 
\[ \mathfrak{g}^{z,\sigma, zr} = \mathfrak{g}^{\sigma, r}, \mathfrak{g}^{z,\sigma, zr} = \mathfrak{g}^{\sigma, r}, K_{z,\sigma, zr} = K_{\sigma, r} \]
and similarly with subscripts \(N\). The scaling by degree automorphism \((2.7)\) provides an isomorphism between of one the algebras associated to \((\sigma, r)\) and its analogue associated to \((z\sigma, zr)\).

5. Standard modules

In \([Lus1, AMS2]\) standard (left) modules for \(H(G, M, q\mathcal{E})\) were studied. We will quickly recall their construction and then we relate these standard modules to the previous section. From now on we suppose, as we may by \((3.11)\):

Condition 5.1. \(G = N_G(P, q\mathcal{E})G^\circ\).

By \([AMS2, (16)]\) this condition does not change the class of algebras that we consider. It is only needed to keep undesirable elements out of the varieties \(\mathcal{P}_y\), and hence for the construction of our standard modules. After developing some theory, we will be able to drop Condition 5.1.

Let \(y \in \mathfrak{g}\) be nilpotent and define
\[ \mathcal{P}_y = \{ gP \in G/P : \text{Ad}(g^{-1})y \in \mathcal{C}^M_v + u \}. \]

The group
\[ Z_{G \times \mathbb{C}^\times}(y) = \{(g_1, \lambda) \in G \times \mathbb{C}^\times : \text{Ad}(g_1)y = \lambda^2 y\} \]
acts on \(\mathcal{P}_y\) by \((g_1, \lambda) \cdot gP = g_1gP\). This puts \(\mathcal{P}_y\) in \(Z_{G \times \mathbb{C}^\times}(y)\)-equivariant bijection with \(\{y\} \times \mathcal{P}_y \subset \hat{\mathfrak{g}}\). The local system \(q\mathcal{E}\) on \(\hat{\mathfrak{g}}\) restricts to a local system on \(\{y\} \times \mathcal{P}_y \cong \mathcal{P}_y\), still called \(q\mathcal{E}\).

The action of \(\mathbb{C}[W_{q\mathcal{E}}, z_{q\mathcal{E}}]\) on \(K\) from \((3.9)\) induces an action on
\[ H^*_{G \times \mathbb{C}^\times}(\mathfrak{g}, K) \cong H^*_{G \times \mathbb{C}^\times}(\hat{\mathfrak{g}}, q\mathcal{E}) \]
From (3.3) and the product in equivariant cohomology, we get an action of $O(t \oplus \mathbb{C})$ on (5.2). These can be pulled back to actions on

$$H^*_Z(G, C, \gamma) \otimes (y, \rho, \sigma, \tau)$$

making that vector space into a left module over $\mathbb{H}(G, M, qE)$ and over $H^*_Z(G, C, \gamma) \otimes (pt)$.

Further, $Z_{G \times C} \otimes \gamma$ acts naturally on $H^*_Z(G, C, \gamma) \otimes (y, \rho, \sigma, \tau)$ and on $H^*_Z(G, C, \gamma) \otimes (pt)$, and those actions factor through the component group $\pi_0(Z_{G \times C} \otimes \gamma)$.

**Theorem 5.2.** (see [Lus1] Theorem 8.13) and [AMS2 Theorem 3.2 and §4])

(a) The actions of $\mathbb{H}(G, M, qE)$ and $H^*_Z(G, C, \gamma) \otimes (pt)$ on $H^*_Z(G, C, \gamma) \otimes (y, \rho, \sigma, \tau)$ commute.

(b) As $H^*_Z(G, C, \gamma) \otimes (pt)$-module, $H^*_Z(G, C, \gamma) \otimes (y, \rho, \sigma, \tau)$ is finitely generated and free.

(c) The action of $\pi_0(Z_{G \times C} \otimes \gamma)$ on $H^*_Z(G, C, \gamma) \otimes (y, \rho, \sigma, \tau)$ commutes with the action of $\mathbb{H}(G, M, qE)$ and is semilinear with respect to $H^*_Z(G, C, \gamma) \otimes (pt)$.

**Proof.** Comparing with the references, it only remains to see that in part (b) the module is free. This part ultimately relies on [Lus1 Proposition 7.2], where it is proven that the module is finitely generated and projective. However, that argument actually shows that the module is free. \qed

Recall from [Lus1] §1.11] that $H^*_Z(G, C, \gamma) \otimes (pt)$ is the ring of invariant polynomials on the maximal reductive quotient of $\text{Lie}(Z_{G \times C} \otimes \gamma)$. The characters of that ring are parametrized by the semisimple orbits in the reductive Lie algebra. We let $\mathfrak{g} \oplus \mathbb{C}$ act on $\mathfrak{g}$ by $(\sigma, r) \cdot X = [\sigma, X] - 2rX$ (that is the derivative of the $G \times C$-action).

Then we can write

$$(5.3) \quad \text{Lie}(Z_{G \times C} \otimes \gamma) = \{ (\sigma, r) \in \mathfrak{g} \oplus \mathbb{C} : [\sigma, y] = 2ry \} = Z_{\mathfrak{g} \oplus \mathbb{C}}(y).$$

Thus every semisimple $(\sigma, r) \in Z_{\mathfrak{g} \oplus \mathbb{C}}(y)$ defines a unique character of $H^*_Z(G, C, \gamma) \otimes (pt)$, which we denote $C_{\sigma, r}$. This gives us a family of $\mathbb{H}(G, M, qE)$-modules

$$E_{y, \sigma, r} := C_{\sigma, r} \otimes H^*_Z(G, C, \gamma) \otimes (y, \rho, \sigma, \tau)$$

for semisimple $(\sigma, r) \in Z_{\mathfrak{g} \oplus \mathbb{C}}(y)$. It is known from [AMS2 Proposition 3.5] that (when Condition 5.1 holds) $E_{y, \sigma, r}$ admits the central character $(\text{Ad}(G) \sigma \cap t, r)$. Let

$$C_y = Z_C(y) = Z_{G \times C} \otimes (y, \sigma, r)$$

be the intersection of $Z_{G \times C} \otimes (y)$ from (5.1) and $C = Z_{G \times C} \otimes (\sigma, r)$ (with respect to the adjoint action). The component group $\pi_0(C_y)$ acts naturally on $E_{y, \sigma, r}$ by $\mathbb{H}(G, M, qE)$-intertwiners. For $\rho \in \text{Irr}(\pi_0(C_y))$ we form the $\mathbb{H}(G, M, qE)$-module

$$E_{y, \sigma, r, \rho} = \text{Hom}_{\pi_0(C_y)}(\rho, E_{y, \sigma, r}).$$

Choose an algebraic homomorphism $\gamma_y : SL_2(\mathbb{C}) \rightarrow G^0$ with $d\gamma_y \frac{0 1}{1 0} = y$. It is often convenient to involve the element

$$(5.4) \quad \sigma_0 := \sigma + d\gamma_y \left( \begin{array}{cc} 1 & r \\ 0 & 1 \end{array} \right) \in Z_\rho(y).$$
For instance, by [AMS2] Lemma 3.6.a there are natural isomorphisms
\begin{equation}
\pi_0(C_y) \cong \pi_0(Z_G(y, \sigma)) \cong \pi_0(Z_G(y, \sigma_0))
\end{equation}
It was shown in [AMS2] Proposition 3.7 and §4 that \( E_{y,\sigma,r,\rho} \) is nonzero if and only if the cuspidal quasi-support \( q\Psi_{Z_G(\sigma_0)}(y, \rho) \), for the group \( Z_G(\sigma_0) \) and with \( \rho \) considered as representation of \( \pi_0(Z_G(y, \sigma_0)) \) via (5.a) is \( G \)-conjugate to \( (M, C^*_v M, qE) \). Equivalent conditions are described in Proposition 5.1. For such \( \rho \) we call \( E_{y,\sigma,r,\rho} \) a standard (geometric) \( \mathbb{H}(G, M, qE) \)-module.

**Theorem 5.3.** [AMS2] Theorem 4.6]
Fix \( r \in \mathbb{C} \).
(a) When \( r \neq 0 \), every standard \( \mathbb{H}(G, M, qE) \)-module \( E_{y,\sigma,r,\rho} \) has a unique irreducible quotient \( M_{y,\sigma,r,\rho} \).
(b) When \( r = 0 \), the standard module \( E_{y,\sigma,0,\rho} \) has a unique distinguished irreducible summand, called \( M_{y,\sigma,0,\rho} \).
(c) The correspondence \( M_{y,\sigma,r,\rho} \leftrightarrow (y, \sigma, \rho) \) provides a bijection between \( \text{Irr}_r(\mathbb{H}(G, M, qE)) \) and \( \text{Irr}(\mathbb{H}(G, M, qE)/(r - r)) \)
and the \( G \)-association classes of triples \( (y, \sigma, \rho) \) as above.

We would like to make the parametrization of \( \text{Irr}_r(\mathbb{H}(G, M, qE)) \) from Theorem 5.3 with \( r \in \mathbb{R}_{>0} \) compatible with the analytic properties temperedness and (essentially) discrete series. When \( G \) is connected, this is worked out in [Lus5]. Unfortunately the outcome is not exactly what we want, it rather produces "anti-tempered" representations where we would like temperedness.

To improve the temperedness properties of standard \( \mathbb{H}(G, M, qE) \)-modules, one can use the Iwahori–Matsumoto involution from (3.3). Notice that composing a \( \mathbb{H}(G, M, qE) \)-module with \( IM \) changes its \( O(t) \)-weights by a factor -1. To compensate for that, in [AMS2] [AMS3] the authors associate to \( (y, \sigma, \rho, r) \) and \( (y, \sigma_0, \rho, r) \) the modules
\[
IM^*E_{y,\sigma,r,\rho} \quad \text{and} \quad IM^*M_{y,\sigma,r,\rho}.
\]
We note that \( (d\gamma_y (\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}) - \sigma_0, r \sigma_0, -r \) = \(-\sigma, -r \) \( \in Z'_{\mathbb{R}^2}(y) \) and
\[
(d\gamma_y (\begin{smallmatrix} -r & 0 \\ 0 & r \end{smallmatrix}) - \sigma_0, -r \) = \(-\sigma, -r \) \( \in Z_{\mathbb{R}^2}(y) \).
\]
The Iwahori–Matsumoto involution commutes with the sign automorphism:
\[
\text{sgn} \circ IM = IM \circ \text{sgn} : \mathbb{H}(G, M, qE) \rightarrow \mathbb{H}(G, M, qE),
\]
\[
\text{sgn} \circ IM(N_w) = N_w, \quad \text{sgn} \circ IM(\xi) = -\xi \quad w \in W_{qE}, \xi \in t' \mathbb{C}.
\]
Composing with the involution \( \text{sgn} \circ IM \) has an easy effect on standard modules:

**Proposition 5.4.** Let \( E_{y,\sigma,r,\rho} \) be a geometric standard \( \mathbb{H}(G, M, qE) \)-module, as in Theorem 5.3. There are canonical isomorphisms of \( \mathbb{H}(G, M, qE) \)-modules
(a) \( \text{sgn}^*IM^*E_{y,\sigma,r} \cong E_{y, -\sigma, -r} \),
(b) \( \text{sgn}^*IM^*E_{y,\sigma,r,\rho} \cong E_{y, -\sigma, -r, \rho} \),
(c) \( \text{sgn}^*IM^*M_{y,\sigma,r,\rho} \cong M_{y, -\sigma, -r, \rho} \).

**Proof.** (a) By [Lus1] Proposition 7.2.d] there is a natural vector space isomorphism
\begin{equation}
E_{y,\sigma,r} = C_{\sigma,r} \otimes H^*_{G \times C^*(y)(pt)}(P_y, qE) \rightarrow H_*(P_y, qE).
\end{equation}
The construction of the $\mathbb{C}[W_{q,E}, z_{q,E}]$-action on $H^*_sZ_{G\times C^\times}^\mathbb{C}(y)(\mathcal{P}_y, q\mathcal{E})$ from [Lus1] p.193 comes from its action on $K$ and can also be performed without the $Z_{G\times C^\times}^\mathbb{C}(y)$-equivariance. That renders (5.6) $\mathbb{C}[W_{q,E}, z_{q,E}]$-equivariant. We can do the same with $(y, -\sigma, -r)$, that gives a $\mathbb{C}[W_{q,E}, z_{q,E}]$-linear bijection $E_{y,-\sigma,-r} \to H_*(\mathcal{P}_y, q\mathcal{E})$. Composing that with the inverse of (5.6), we obtain a natural isomorphism of $\mathbb{C}[W_{q,E}, z_{q,E}]$-modules

$$E_{y,-\sigma,-r} \to E_{y,\sigma, r}.$$  

Via (5.6) we transfer the $\mathbb{H}(G, M, q\mathcal{E})$-module structure of $E_{y,-\sigma,-r}$ to $E_{y,\sigma, r}$, and we call the new module $E'_{y,\sigma, r}$. The action of $\mathcal{O}(t \oplus \mathbb{C})$ on $E_{y,\sigma, r}$ comes from the natural maps

$$\mathcal{O}(t \oplus \mathbb{C}) \cong H^*_sH^*_sG\times C^\times \left(\hat{q}\right) \to H^*_sG\times C^\times \left(\hat{q}\right) \cong H^*_sZ^\mathbb{C}_G\times \left(\mathcal{P}_y, q\mathcal{E}\right).$$

and the product

$$H^*_sZ^\mathbb{C}_G\times \left(\mathcal{P}_y, q\mathcal{E}\right) \to H^*_sG\times C^\times \left(\mathcal{P}_y, q\mathcal{E}\right).$$

Similarly the $H^*_sZ^\mathbb{C}_G\times \left(\mathcal{P}_y, q\mathcal{E}\right)$-action comes from $H^*_sG\times C^\times \left(\mathcal{P}_y, q\mathcal{E}\right)$ and (5.7). It follows that (5.7) modifies the action of $\mathcal{O}(t \oplus \mathbb{C})$ by a factor $-1$ on $t^\vee$ and on $r$. In other words, $E'_{y,\sigma, r} = \text{sgn}^*\text{IM}^*E_{y,\sigma, r}$ as $\mathcal{O}(t \oplus \mathbb{C})$-modules. We already knew that for the $\mathbb{C}[W_{q,E}, z_{q,E}]$-action, so (5.7) induces the desired isomorphism of $\mathbb{H}(G, M, q\mathcal{E})$-modules.

(b) To enhance this picture with a $\rho$, we need more precise information. Recall from Theorem 5.2b that $H^*_sZ^\mathbb{C}_G\times \left(\mathcal{P}_y, q\mathcal{E}\right)$ is finitely generated and free over $H^*_sZ^\mathbb{C}_G\times \left(\mathcal{P}_y, \mathcal{E}\right)$. The action map

$$H^0_zG\times C^\times \left(\mathcal{P}_y, q\mathcal{E}\right) \to H^*_sZ^\mathbb{C}_G\times \left(\mathcal{P}_y, q\mathcal{E}\right)$$

respects degrees and $H^0_{Z^\mathbb{C}_G\times} \left(\mathcal{P}_y, q\mathcal{E}\right) = \mathbb{C}$ while $H^0_{Z^\mathbb{C}_G\times} \left(\mathcal{P}_y, q\mathcal{E}\right) = 0$ for $n < 0$. Let $E_y$ be the unique complement to

$$H^0_{Z^\mathbb{C}_G\times} \left(\mathcal{P}_y, q\mathcal{E}\right) \subset H^*_sZ^\mathbb{C}_G\times \left(\mathcal{P}_y, q\mathcal{E}\right)$$

which is spanned by homogeneous elements (i.e. elements that live in one degree). Then (5.9) restricts to a linear bijection

$$H^*_sG\times C^\times \left(\mathcal{P}_y, q\mathcal{E}\right) \to H^*_sZ^\mathbb{C}_G\times \left(\mathcal{P}_y, q\mathcal{E}\right).$$

Now (5.6) sends $\mathbb{C}[\sigma, r] \otimes \mathbb{C} E_y$ bijectively to $H_*(\mathcal{P}_y, q\mathcal{E})$ for any $(\sigma, r)$. In this way we can regard $E_y$ as a canonical copy of $H_*(\mathcal{P}_y, q\mathcal{E})$ in $H^*_sZ^\mathbb{C}_G\times \left(\mathcal{P}_y, q\mathcal{E}\right)$. The $C_y$-actions preserve the degrees in (5.9) and are the identity on $H^0_{Z^\mathbb{C}_G\times} \left(\mathcal{P}_y, q\mathcal{E}\right)$. Hence they restrict to an action of $C_y$ on $E_y$. Exactly the same action on $E_y$ is obtained if we start with $(-\sigma, -r)$ instead of $(\sigma, r)$. 


Furthermore the isomorphism of $\mathbb{H}(G, M, q\mathcal{E})$-modules from part (a) factors as

$$E_{y, \sigma, r} \to E_y \to E_{y, -\sigma, -r},$$

so part (a) is also $C)\gamma$-equivariant. In particular, for any $\rho \in \text{Irr}(C_\gamma)$, part (a) induces isomorphisms of $\mathbb{H}(G, M, q\mathcal{E})$-modules

$$(5.11) \quad \text{sgn}^* \text{IM}^* E_{y, \sigma, r, \rho} = \text{Hom}_{C_\gamma}(\rho, \text{sgn}^* \text{IM}^* E_{y, \sigma, r}) \cong \text{Hom}_{C_\gamma}(\rho, E_{y, -\sigma, -r}) = E_{y, -\sigma, -r, \rho}.$$  

(c) When $r \neq 0$, $(5.11)$ sends the unique irreducible quotient $\text{sgn}^* \text{IM}^* M_{y, \sigma, r, \rho}$ on the left to the unique irreducible quotient $M_{y, -\sigma, -r, \rho}$ on the right. When $r = 0$, the distinguished irreducible summand $\text{sgn}^* \text{IM}^* M_{y, \sigma, 0, \rho}$ of $\text{sgn}^* \text{IM}^* E_{y, \sigma, r, \rho}$ is in the component of $E_y$ in one particular homological degree $[\text{AMS}2]$ Lemma 3.10 and Theorem 3.20. As part (a) factors via $E_y$, it preserves these homological degrees. Hence it sends $\text{sgn}^* \text{IM}^* M_{y, \sigma, 0, \rho}$ to the distinguished irreducible summand $M_{y, -\sigma, 0, \rho}$ of $E_{y, -\sigma, 0, \rho}$.

With Proposition 5.4 at hand, we can reformulate the results of $[\text{AMS}2]$ that use the Iwahori–Matsumoto involution in terms of the sign involution of $\mathbb{H}(G, M, q\mathcal{E})$. In particular we see that the module $\text{sgn}^* E_{y, \sigma, -r, \rho}$ is isomorphic to $\text{IM}^* E_{y, -\sigma, r, \rho}$. In $[\text{AMS}2]$ the latter module was associated to the data

$$(y, \sigma_0, r, \rho) \quad \text{and} \quad (y, \sigma_0 + \text{d}_\gamma y \left( \begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix} \right), r, \rho).$$

Similar statements holds without $\rho$ and with $E$ replaced by $M$. This is the class of modules that is standard in the analytic sense related to the Langlands classification, see $[\text{Sol}5]$ §3.5. To distinguish them from the earlier geometric standard modules, we refer to $\text{sgn}^* E_{y, \sigma, -r, \rho}$ as an analytic standard $\mathbb{H}(G, M, q\mathcal{E})$-module.

Using the sign automorphism we can vary on $[\text{AMS}2]$ Theorem 4.6]. Fix $r \in \mathbb{C}$ and consider triples $(y, \sigma, \rho)$ such that:

- $y \in \mathfrak{g}$ is nilpotent,
- $\sigma \in \mathfrak{g}$ is semisimple and $[\sigma, y] = -2ry,$
- $\rho \in \text{Irr}(\mathfrak{p}_0(\mathbb{Z}_G(\sigma, y)))$ and $q\Psi_{\mathbb{Z}_G(\sigma_0)}(y, \rho) = (M, \mathcal{C}_\nu^M, q\mathcal{E})$ up to $G$-conjugacy.

By Theorem 5.3 the map

$$(5.12) \quad (\sigma, y, \rho) \mapsto \text{sgn}^*(M_{y, \sigma, -r, \rho})$$

defines a bijection between the set of $G$-conjugacy classes of triples $(y, \sigma, \rho)$ as above and $\text{Irr}_r(\mathbb{H}(G, M, q\mathcal{E}))$. Notice that the central character of $\text{sgn}^*(M_{y, \sigma, -r, \rho})$ is $(\sigma, r)$.

This constitutes an improvement on $[\text{AMS}2]$ §3.5] because our new parametrization of $\text{Irr}_r(\mathbb{H}(G, M, q\mathcal{E}))$ has all the desired properties with respect to temperedness (see below) and is more natural – we do not have to involve $\text{d}_\gamma y \left( \begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix} \right)$ any more.

**Theorem 5.5.** $[\text{AMS}2]$ Theorem 3.25, Theorem 3.26 and §4]

Consider an analytic standard $\mathbb{H}(G, M, q\mathcal{E})$-module $\text{sgn}^* E_{y, \sigma, -r, \rho}$.

(a) Suppose that $R(r) \geq 0$. The $\mathbb{H}(G, M, q\mathcal{E})$-modules $\text{sgn}^*(E_{y, \sigma, -r, \rho})$ and $\text{sgn}^*(M_{y, \sigma, -r, \rho})$ are tempered if and only if $\sigma_0$ lies in $i\mathbb{R} = i\mathbb{R} \otimes \mathbb{Z} X_*(T)$.

Here $\sigma_0 = \sigma + \text{d}_\gamma y \left( \begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix} \right)$ is as in (5.3), but with $-r$ instead of $r$.

(b) Suppose that $R(r) > 0$. Then $\text{sgn}^*(E_{y, \sigma, -r, \rho})$ and $\text{sgn}^*(M_{y, \sigma, -r, \rho})$ are essentially discrete series if and only if $y$ is distinguished in $\mathfrak{g}$.

Moreover, when these conditions are fulfilled

$$\text{sgn}^*(E_{y, \sigma, -r, \rho}) = \text{sgn}^*(M_{y, \sigma, -r, \rho}) \in \text{Irr}_r(\mathbb{H}(G, M, q\mathcal{E})).$$
In terms of Theorems 5.3 and 5.5 the bijection from Theorem 2.4 becomes
\begin{equation}
\text{Irr}_\xi(\mathbb{H}(G, M, q\mathcal{E})) \quad \rightarrow \quad \text{Irr}_0(\mathbb{H}(G, M, q\mathcal{E}))
\end{equation}
\begin{equation}
\text{sgn}^\ast(M_{\gamma, \sigma} \rightarrow r, \rho) \quad \rightarrow \quad \text{sgn}^\ast(M_{\gamma, \sigma, 0, \rho})
\end{equation}
We would like to analyse the right $\mathbb{H}(G, M, q\mathcal{E})$-modules from Theorem 4.3 as left modules over the opposite algebra $\mathbb{H}(G, M, q\mathcal{E})^\text{op}$. This opposite algebra is easily identified via the isomorphism
\begin{equation}
\mathbb{H}(G, M, q\mathcal{E})^\text{op} \simeq \mathbb{H}(G, M, q\mathcal{E}^\vee)
\end{equation}
\begin{equation}
N_w \xi \quad \rightarrow \quad \xi(N_w)^{-1} \quad w \in W_q, \xi \in \mathcal{O}(t \oplus \mathbb{C}),
\end{equation}
see [AMS2 (14)]. That gives an equivalence of categories
\begin{equation}
\mathbb{H}(G, M, q\mathcal{E}) \quad \rightarrow \quad \mathbb{H}(G, M, q\mathcal{E}^\vee).
\end{equation}
The dual local system $q\mathcal{E}^\vee$ on $\mathcal{O}^M$ is also cuspidal, so all the previous results hold just as well for $\mathbb{H}(G, M, q\mathcal{E}^\vee)$. In particular we have a complete classification of its irreducible and its standard left modules.

Now we want to relate the standard modules of $\mathbb{H}(G, M, q\mathcal{E})$ (or its opposite) to Theorem 4.3. The vector spaces $H^\ast(\{y\}, i_y^\ast K_{\sigma, r})$ and $H^\ast(\{y\}, i_y^\ast K_{\sigma, r})$ become left $\text{End}_{D_\Sigma G(\sigma) \times \mathbb{C} \times (\mathfrak{g}^\sigma, r)}(K_{\sigma, r})$-modules via the natural algebra homomorphism
\begin{equation}
\text{End}_{D_\Sigma G(\sigma) \times \mathbb{C} \times (\mathfrak{g}^\sigma, r)}(K_{\sigma, r}) \rightarrow \text{End}_{D(\{y\})}(i_y^\ast K_{\sigma, r}),
\end{equation}
see [Lus3 §10.2]. Via Theorem 4.4, $H^\ast(i_y^\ast K_{\sigma, r})$ and $H^\ast(i_y^\ast K_{\sigma, r})$ also become left $\mathbb{H}(G, M, q\mathcal{E})$-modules. By [Lus3 §10.4], and as in Proposition 5.6, they carry natural actions of $\pi_0(C_y)$, which commute with the $\mathbb{H}(G, M, q\mathcal{E})$-actions.

Let $K_{\sigma, r}^\vee \in D_C(\mathfrak{g}^\sigma, r)$ be the analogue of $K_{\sigma, r}$, but constructed from $q\mathcal{E}^\vee$. From the definition of $K_{\sigma, r}$ and the properness of $\text{pr}_1 : \mathfrak{g}^\sigma, r \rightarrow \mathfrak{g}^\sigma, r$ we see that
\begin{equation}
DK_{\sigma, r} = K_{\sigma, r}^\vee [\dim_{\mathbb{R}}(\mathfrak{g}^\sigma, r)].
\end{equation}

**Proposition 5.6.** Assume Condition 5.1 and denote the subvariety of $\exp(\sigma)$-fixed points in $\mathcal{P}_y$ by $\mathcal{P}_y^\sigma$.

(a) There are natural isomorphism of $\mathbb{H}(G, M, q\mathcal{E}) \times \pi_0(C_y)$-representations
\begin{equation}
H^\ast(\{y\}, i_y^\ast K_{\sigma, r}) \simeq H^\ast(\mathcal{P}_y^\sigma, q\mathcal{E}) \simeq E_{\gamma, \sigma, r},
\end{equation}
\begin{equation}
H^\ast(\{y\}, i_y^\ast K_{\sigma, r}) \simeq \mathbb{C} \otimes H^\ast_C(\mathcal{P}_y^\sigma, q\mathcal{E}).
\end{equation}

(b) There are natural isomorphism of $\mathbb{H}(G, M, q\mathcal{E}^\vee) \times \pi_0(C_y)$-representations
\begin{equation}
H^\ast(\{y\}, i_y^\ast K_{\sigma, r})^\vee \simeq H^\dim_{\mathfrak{g}^\sigma, r} s_{\gamma, r}^{-1}(\{y\}, i_y^\ast K_{\sigma, r}^\vee) \simeq E_{\gamma, \sigma, r},
\end{equation}
\begin{equation}
H^\ast(\{y\}, i_y^\ast K_{\sigma, r})^\vee \simeq H^\dim_{\mathfrak{g}^\sigma, r} s_{\gamma, r}^{-1}(\{y\}, i_y^\ast K_{\sigma, r}^\vee) \simeq E_{\gamma, \sigma, r}.
\end{equation}

(c) Parts (a) and (b) are also valid with $(\mathfrak{g}_N^\sigma, K_{N, \sigma, r})$ instead of $(\mathfrak{g}^\sigma, r, K_{\sigma, r})$.

**Proof.** (a) When $G$ is connected, the isomorphisms with $H^\ast(\{y\}, i_y^\ast K_{\sigma, r})$ shown in [Lus3 Proposition 10.12]. We generalize those arguments to our setting. Consider the pullback diagram
\begin{equation}
\begin{array}{ccc}
\mathcal{P}_y^\sigma & \xrightarrow{k} & \mathfrak{g}^\sigma, r \\
\downarrow \pi & & \downarrow \text{pr}_1 \\
\{y\} & \xrightarrow{i_y} & \mathfrak{g}^\sigma, r,
\end{array}
\end{equation}
where \( k(gP) = (y, gP) \). From general results about derived sheaves, \( \text{[BeLu]} \) §1.4.6 and Theorem 1.8.ii, extended to the equivariant derived category in \( \text{[BeLu]} \) Theorem 3.4.3, it is known that

\[
(5.17) \quad i_y^* \text{pr}_{1, *} = \pi_* k^s \quad \text{and} \quad i_y^* \text{pr}_{1, *|y} = \pi_* k^l
\]
as functors \( \mathcal{D}_C(\mathfrak{g}^{σ-v}) \to \mathcal{D}_{C_y}(\{y\}) \). With that we compute

\[
H^*\left(\{y\}, i_y^! K_{σ,r}\right) \cong H^*\left(\{y\}, i_y^* \text{pr}_{1, *}, q\mathcal{E}\right) \cong H^*\left(\{y\}, \pi_* k^v q\mathcal{E}\right) \\
\cong H^*\left(\mathcal{P}_{y}^σ, k^\mathcal{E}\mathcal{C}_y\right) \cong H^*\left(\mathcal{P}_y^σ, Dk^d Dq\mathcal{E}\right) \\
\cong H^*\left(\mathcal{P}_y^σ, Dk^d q\mathcal{E}^∨\right) = H^\mathcal{H}_y^*\left(\mathcal{P}_y^σ, Dq\mathcal{E}^∨\right) = H_*(\mathcal{P}_y^σ, q\mathcal{E}).
\]
The last part of the proof of \( \text{[Lus3]} \) Proposition 10.12 shows that

\[
H_*(\mathcal{P}_y^σ, q\mathcal{E}) \cong \mathcal{C}_{σ,r} \otimes \mathcal{H}_{C_y}^*\left(\mathcal{P}_y^σ, q\mathcal{E}\right) = E_{γ_σ, γ_r}
\]
as \( \mathbb{H}(G, M, q\mathcal{E}^∨) \times π_0(\mathcal{C}_y) \)-representations.

Similarly we use \( \text{(5.17)} \) to compute

\[
H^*\left(\{y\}, i_y^! K_{σ,r}\right) \cong H^*\left(\{y\}, i_y^* \text{pr}_{1, *}, q\mathcal{E}\right) \cong H^*\left(\{y\}, \pi_* k^v q\mathcal{E}\right) \\
\cong H^*\left(\mathcal{P}_y^σ, q\mathcal{E}\right).
\]

Notice that \( \mathcal{P}_y^σ \) is compact, so cohomology coincides with compactly support cohomology here. The last part of the proof of \( \text{[Lus3]} \) Proposition 10.12 also shows that there is an isomorphism of \( \mathbb{H}(G, M, q\mathcal{E}^∨) \times π_0(\mathcal{C}_y) \)-representations

\[
H^*\left(\mathcal{P}_y^σ, q\mathcal{E}\right) \cong \mathcal{C}_{σ,r} \otimes \mathcal{H}_{C_y}^*\left(\mathcal{P}_y^σ, q\mathcal{E}\right).
\]

(b) From part (a) and \( \text{(5.16)} \) we see that

\[
H^*\left(\{y\}, i_y^! K_{σ,r}\right)^\mathcal{V} \cong \mathcal{H}_*\left(\{y\}, \mathcal{D}_{\mathfrak{g}^{σ-v}} K_{σ,r}\right) \cong \mathcal{H}_*\left(\{y\}, i_y^* \mathcal{D}_\mathcal{C}_y K_{σ,r}\right) \\
H_*\left(\{y\}, i_y^* \mathcal{D}_\mathcal{C}_y K_{σ,r}^\mathcal{V}\right) \cong \mathcal{H}_*\left(\{y\}, i_y^* K_{σ,r}^\mathcal{V}\right).
\]

Similarly there is a natural vector space isomorphism

\[
H^*\left(\{y\}, i_y^! K_{σ,r}\right)^\mathcal{V} \cong \mathcal{H}_*\left(\{y\}, i_y^* K_{σ,r}^\mathcal{V}\right).
\]
The \( \mathbb{H}(G, M, q\mathcal{E}) \times π_0(\mathcal{C}_y) \)-actions in part (a) become actions of \( \mathbb{H}(G, M, q\mathcal{E})^{op} \cong \mathbb{H}(G, M, q\mathcal{E}^∨) \) and of \( π_0(\mathcal{C}_y) \) upon taking vector space duals. We can reformulate the isomorphisms from part (b) as

\[
(5.18) \quad H_*(\mathcal{P}_y^σ, q\mathcal{E})^\mathcal{V} \cong \mathcal{H}_*\left(\mathcal{P}_y^σ, q\mathcal{E}\right)^\mathcal{V}, \\
H_*(\mathcal{P}_y^σ, q\mathcal{E})^\mathcal{V} \cong \mathcal{H}_*\left(\mathcal{P}_y^σ, q\mathcal{E}\right)^\mathcal{V}.
\]

Using the explicit description of the actions given in Section \( \mathfrak{g} \) and in \( \text{[AMS2]} \), one checks readily that in \( \text{(5.18)} \) we have isomorphisms of \( \mathbb{H}(G, M, q\mathcal{E}) \times π_0(\mathcal{C}_y) \)-representations.

(c) This can be shown in the same way as parts (a) and (b). \( \square \)

With Proposition \( \text{[5.6]} \) we can henceforth interpret the geometric standard \( \mathbb{H}(G, M, q\mathcal{E}) \)-module \( E_{γ_σ, γ_r, ρ} \) as

\[
(5.19) \quad \text{Hom}_{π_0(\mathcal{C}_y)}(ρ, H^*\left(\{y\}, i_y^! K_{σ,r}\right)) \cong \mathcal{C}_{σ,r} \otimes \mathcal{H}_{C_y}^*\left(\{y\}\right) \text{Hom}_{\mathcal{D}_\mathcal{C}_y}(ρ, i_y^! K_{σ,r}).
\]

In this way we can lift Condition \( \text{5.1} \).
6. Structure of the localized complexes \(K_{N,\sigma,r}\)

With \(K_{\sigma,r}\) and \(K_{\sigma,r}^\vee\) we can give an alternative interpretation of the cuspidal quasi-supports involved in standard modules (see in particular Theorem \[5.3\]c).

**Proposition 6.1.** Fix a nilpotent \(y \in \mathfrak{g}^{\sigma,r}\) and let \(i_y : \{y\} \to \mathfrak{g}^{\sigma,r}\) be the inclusion. For \(\rho \in \text{Irr}(\pi_0(Z_G(y,\sigma_0))) = \text{Irr}(\pi_0(C_y))\), the following are equivalent:

(i) the cuspidal quasi-support \(q\Psi_{Z_G(\sigma_0)}(y,\rho)\), with respect to the group \(Z_G(\sigma_0)\), is \(G\)-conjugate to \((M,\mathcal{C}_M^\vee,q\mathcal{E})\),

(ii) \(\text{Hom}_{\pi_0(C_y)}(\rho,H^*(\{y\},i_y^*\sigma_{\sigma,r})) \neq 0\),

(iii) \(\text{Hom}_{\pi_0(C_y)}(H^*(\{y\},i_y^*\sigma_{\sigma,r}),\rho) \neq 0\),

(iv) \(\text{Hom}_{\pi_0(C_y)}(\rho,H^*(\{y\},i_y^*\sigma_{\sigma,r})) \neq 0\),

(v) \(\text{Hom}_{\pi_0(C_y)}(H^*(\{y\},i_y^*\sigma_{\sigma,r}),\rho) \neq 0\).

**Proof.** The equivalence of (i) and (ii) is in the proof of [AMS2 Proposition 3.7], which can be extended to cuspidal quasi-supports with [AMS2 §4]. The equivalence of (ii) and (iii) follows from Proposition 6.1(b). By Proposition 5.6(a,c)

\[ H^*(\{y\},i_y^*\sigma_{\sigma,r}) \cong H^*(\{y\},i_y^*\sigma_{\sigma,r}) \quad \text{and} \quad H^*(\{y\},i_y^*\sigma_{\sigma,r}) \cong H^*(\{y\},i_y^*\sigma_{\sigma,r}) \]

as \(\mathbb{H}(G,M,q\mathcal{E}) \times \pi_0(C_y)\)-representations. That proves the equivalence of (ii) with (iv) and of (iii) with (v). \(\square\)

Cuspidal quasi-supports where defined [AMS1 §5], in relation with a Springer correspondence for disconnected reductive groups. In our context, it is more convenient to use the property (ii) or (iii) in Proposition 6.1 for a given triple \((y,\sigma,\rho)\) that determines \((M,\mathcal{C}_M^\vee,q\mathcal{E})\) up to \(G\)-conjugacy.

In the opposite direction, Proposition 6.1 almost determines the semisimple complex \(K_{\sigma,r}\). To work this out, let \(O_y = \text{Ad}(C)y \subset \mathfrak{g}^{\sigma,r}\) be the \(C\)-orbit of \(y\). Regarding \(\rho\) as a \(C_y\)-equivariant sheaf on \(\{y\}\) and invoking the equivalence of categories

\[ S_y : D_{C_y}(\{y\}) \sim D_C(O_y), \]

we obtain a \(C_y\)-equivariant local system \(S_y(\rho)\) on \(O_y\). We form the equivariant intersection cohomology complex \(IC_C(\mathfrak{g}^{\sigma,r},S_y(\rho)) \in D_C(\mathfrak{g}^{\sigma,r})\), which is supported on \(O_y\). This is the usual intersection cohomology complex \(IC(\mathfrak{g}^{\sigma,r},S_y(\rho))\), only now considered with its \(C\)-equivariant structure.

**Theorem 6.2.** (a) Fix \(r \in \mathbb{C}^\times\). Every simple direct summand of \(K_{\sigma,r}\) is isomorphic to \(IC_C(\mathfrak{g}^{\sigma,r},S_y(\rho))\), for data \((y,\sigma,\rho)\) that fulfill the conditions in Proposition 6.1.

Conversely, every such equivariant intersection cohomology complex is a direct summand of \(K_{\sigma,r}\) (with multiplicity \(\geq 1\)).

(b) For arbitrary \(r \in \mathbb{C}\), part (a) becomes valid when we replace all involved sheaves by their versions for \(\mathfrak{g}^{\sigma,r}\).

**Proof.** (a) From \([A.11]\) we see that, as \(M^\circ\)-equivariant local system on \(\mathcal{C}_M^\vee = C_M^\vee\), \(q\mathcal{E}\) is a direct sum of \(M\)-conjugates of \(E\) and that \((q\mathcal{E})_v \cong E_v \rtimes \rho_M\) for a suitable representation \(\rho_M\). Hence \(q\mathcal{E} \in D_{C^\circ}(\mathfrak{g})\) is a direct sum of \(G\)-conjugates of \(E \in D_{C^\circ}(\mathfrak{g})\).

Then the diagram \([4.3]\) shows that, as element of \(D_{Z_G(\sigma) \times \mathbb{C}^\times}(\mathfrak{g}^{\sigma,r})\), \(K_{\sigma,r}\) is a direct sum of \(Z_G(\sigma)\)-conjugates of \(K_{\sigma,r}^\circ\) (the version of \(K_{\sigma,r}\) for \((G^\circ,M^\circ,\mathcal{E}))\). By [Lus3 §5.3], \(K_{\sigma,r}^\circ\) is a semisimple perverse sheaf. Further [Lus3 Proposition 8.17] (for...
which we need \( r \neq 0 \) and Proposition \ref{Proposition6.1} entail that the simple direct summands of \( K_{\sigma,r}^z \) are the \( \mathbb{Z}^G(\sigma) \times \mathbb{C}^x \)-equivariant intersection cohomology complexes

\[
(6.2) \quad IC_{C_y(\mathfrak{g}^{\sigma,r}, S_y^z(\rho^o))} \quad \text{with} \quad Hom_{\pi_0(C_y)}(H^*(\{y\}, i^*_yK_{\sigma,r}, \rho^o) \neq 0.
\]

More precisely, every such summand appears with a multiplicity \( \geq 1 \). Then \( K_{\sigma,r} \) is a direct sum of terms \( IC_{C_y(\mathfrak{g}^{\sigma,r}, S_y^z(\rho'))} \), where \( \rho' \in \text{Irr}(\mathfrak{g}^0M) \) contains some \( \rho^o \) as before. That settles the geometric structure of \( K_{\sigma,r} \), it remains to identify exactly which \( \rho' \) occur.

The above works equally well with the group \( G^0M \) instead of \( G \). Let us assume that \( \sigma, \sigma_0 \in t \), as we may by \cite{AMS2} Proposition 3.5.c. Then \cite{AMS2} Lemma 4.4 says that every \( \rho^o \) as in \((6.2)\) corresponds to a unique \( \rho^o \times \rho_M \in \text{Irr}(\pi_0(Z_{G^0M}(\sigma_0,y))) \) with \( q\Psi_{Z_G(\sigma_0)}(y,\rho^o \times \rho_M) \) conjugate to \((M,C^M_y,q\mathcal{E})\) – see also \((A.12)\). A direct comparison of the constructions of \( K_{\sigma,r}^z \) and of \( K_{\sigma,r} \) for \( G^0M \) shows that the latter equals the direct sum of the complexes

\[
IC_{Z_G^0M(\sigma) \times \mathbb{C}^x}(\mathfrak{g}^{\sigma,r}, S_y(\rho^o \times \rho_M))
\]

with the same multiplicities as for \( K_{\sigma,r}^z \).

The step from \( K_{\sigma,r} \) for \( G^0M \) to \( K_{\sigma,r} \) for \( G \) is just induction, compare with \((A.10)\). This induction preserves the cuspidal quasi-supports (for \( G \)) from \cite{AMS1} §5, because those are based on what happens for objects coming from \( G^0M \) (when this support comes from \( M \)). We conclude that \( K_{\sigma,r} \) (for \( G \)) is a direct sum of terms

\[
IC_{C_y(\mathfrak{g}^{\sigma,r}, S_y(\text{ind}_{C_yG^0M}^G(\rho^o \times \rho_M))))
\]

with multiplicities coming from \((6.2)\). In particular \( K_{\sigma,r} \) is also a direct sum of simple perverse sheaves

\[
(6.3) \quad IC_{C_y(\mathfrak{g}^{\sigma,r}, S_y(\rho))} \quad \text{where} \quad q\Psi_{Z_G(\sigma_0)}(y, \rho) = [M,C^M_y,q\mathcal{E}]_G.
\]

By Frobenius reciprocity, applied to \( \text{ind}_{C_yG^0M}^G(\rho^o \times \rho_M) \), every term \((6.3)\) appears with a multiplicity \( \geq 1 \) in \( K_{\sigma,r} \).

(b) This can be shown in the same way, if we replace the crucial input from \cite{Lus3} Proposition 8.17 by \cite{Lus3} §9.5. \( \square \)

Let \( \langle K_{\sigma,r} \rangle \) be the subcategory of \( \mathcal{D}_C(\mathfrak{g}^{\sigma,r}) \) generated by \( K_{\sigma,r} \), via the operations degree shifts, cones and taking direct summands. We define the subcategory \( \langle K_{N,\sigma,r} \rangle \) of \( \mathcal{D}_C(\mathfrak{g}^{\sigma,r}_N) \) analogously.

**Proposition 6.3.** Assume that \( (y, \rho) \) fulfills the equivalent conditions in Proposition \ref{Proposition6.1} and let \( j : \mathcal{O}_y \rightarrow \mathfrak{g}^{\sigma,r} \) be the inclusion.

(a) When \( r \in \mathbb{C}^x \): \( j_*S_y(\rho) \in \langle K_{\sigma,r} \rangle \).

(b) Let \( j_N : \mathcal{O}_y \rightarrow \mathfrak{g}^{\sigma,r}_N \) be the inclusion. For any \( r \in \mathbb{C} \): \( j_{N*}S_y(\rho) \in \langle K_{N,\sigma,r} \rangle \).

**Proof.** (a) From Theorem \ref{Theorem6.2}a we know that \( IC_{C_y(\mathfrak{g}^{\sigma,r}, S_y(\rho))} \) is a direct summand of \( K_{\sigma,r} \), so in particular belongs to \( \langle K_{\sigma,r} \rangle \). That already shows that \( j_*S_y(\rho) \) lies in \( \langle K_{\sigma,r} \rangle \) whenever \( \mathcal{O}_y \) is closed in \( \mathfrak{g}^{\sigma,r} \).
In the remainder of the proof all relevant complexes sheaves will be supported on $\overline{\mathcal{O}_y}$, so we may take that as ambient space. That replaces $\langle K_{\sigma,r} \rangle$ by $\langle K_{\sigma,r} |_{\overline{\mathcal{O}_y}} \rangle$. Let
\[ j_1 : \mathcal{O}_y \to \overline{\mathcal{O}_y} \quad \text{and} \quad i_1 : \overline{\mathcal{O}_y} \setminus \mathcal{O}_y \to \overline{\mathcal{O}_y} \]
be the inclusions. Consider the exact triangle
\[
\begin{align*}
\text{(6.4)} & \quad i_1! i_1^! IC_C(\overline{\mathcal{O}_y}, S_y(\rho)) \longrightarrow IC_C(\overline{\mathcal{O}_y}, S_y(\rho)) \longrightarrow j_1_* j_1^* IC_C(\overline{\mathcal{O}_y}, S_y(\rho)) .
\end{align*}
\]
By construction
\[
\begin{align*}
\text{(6.5)} & \quad \mathcal{F}_1 := i_1! i_1^! IC_C(\overline{\mathcal{O}_y}, S_y(\rho)) = j_1_* S_y(\rho),
\end{align*}
\]
the object we want to belong to $\langle K_{\sigma,r} |_{\overline{\mathcal{O}_y}} \rangle$. We already have the middle term of (6.4) in there, so it suffices to show that
\[
\begin{align*}
\text{(6.5)} & \quad \mathcal{F}_1 \subseteq i_1! i_1^! IC_C(\overline{\mathcal{O}_y}, S_y(\rho)) \quad \text{belongs to} \quad \langle K_{\sigma,r} |_{\overline{\mathcal{O}_y}} \rangle.
\end{align*}
\]
As $\mathcal{F}_1$ is supported on a variety of dimension smaller than that of $\mathcal{O}_y$, it should be possible to reduce (6.5) to a statement analogous to the proposition, only in lower dimensions.

Let $\mathcal{O}_{y_2}$ be a $C$-orbit that is open in $\overline{\mathcal{O}_y} \setminus \mathcal{O}_y$. We claim that
\[
\begin{align*}
\text{(6.6)} & \quad \mathcal{F}_1|_{\mathcal{O}_{y_2}} \quad \text{and} \quad (i_1! i_1^! IC_C(\overline{\mathcal{O}_y}, S_y(\rho)))|_{\mathcal{O}_{y_2}}
\end{align*}
\]
have precisely the same $C$-equivariant irreducible local systems appearing in their cohomology sheaves.

By the properties of Verdier duality:
\[
\begin{align*}
\mathcal{F}_1 \cong D i_1! i_1^! D IC_C(\overline{\mathcal{O}_y}, S_y(\rho)) \cong Di_1! i_1^! IC_C(\overline{\mathcal{O}_y}, S_y(\rho^\vee))[\dim_\mathbb{R} \overline{\mathcal{O}_y}].
\end{align*}
\]
If a local system $\mathcal{L}_2$ on $\mathcal{O}_{y_2}$ appears in the cohomology sheaves here, then $D\mathcal{L}_2 \cong \mathcal{L}_2^\vee$ appears in the cohomology sheaf of $i_1! i_1^! IC_C(\overline{\mathcal{O}_y}, S_y(\rho^\vee))$, and conversely. The latter property is equivalent to $\mathcal{L}_2$ occurring in the cohomology sheaf of $i_1! i_1^! IC_C(\overline{\mathcal{O}_y}, S_y(\rho))$, which proves the claim (6.6).

All the local systems $\mathcal{L}_2$ that satisfy the equivalent conditions of the above claim, occur in particular in the cohomology sheaf of $K_{\sigma,r} |_{\mathcal{O}_{y_2}}$. Then Theorem 6.2.a says that $IC_C(\overline{\mathcal{O}_{y_2}}, \mathcal{L}_2)$ is a direct summand of $K_{\sigma,r}$.

Now we can argue with induction to the dimension of $\mathcal{O}_y$. As $\dim \mathcal{O}_{y_2} \leq \dim \mathcal{O}_y$, the inductive hypothesis says that $j_{2*} \mathcal{L}_2 \in \langle K_{\sigma,r} \rangle$, where $j_2 : \mathcal{O}_{y_2} \to \mathfrak{g}^{\sigma,r}$ is the inclusion. Let $U \subset \overline{\mathcal{O}_y} \setminus \mathcal{O}_y$ be the union of the $C$-orbits that are open in $\overline{\mathcal{O}_y} \setminus \mathcal{O}_y$ and put $V := \overline{\mathcal{O}_y} \setminus (\mathcal{O}_y \cup U)$. Let
\[
\begin{align*}
\text{(6.7)} & \quad j_U : U \to \overline{\mathcal{O}_y} \setminus \mathcal{O}_y \quad \text{and} \quad i_V : V \to \overline{\mathcal{O}_y} \setminus \mathcal{O}_y
\end{align*}
\]
be the inclusions. Consider the exact triangle
\[
\begin{align*}
\text{(6.7)} & \quad i_V! i_V^! \mathcal{F}_1 \longrightarrow \mathcal{F}_1 \longrightarrow j_U! j_U^! \mathcal{F}_1 .
\end{align*}
\]
Here $U$ is a disjoint union of $C$-orbits, so $D_C(U)$ is equivalent with a finite product of categories of the $D_C(\mathcal{O}_{y_2}) \cong D_{C_{y_2}}(\{ y_2 \})$. The image of $j_U! j_U^! \mathcal{F}_1$ in $D_{C_{y_2}}(\{ y_2 \})$ is just a complex of $\pi_0(C_{y_2})$-representations, so the component $\mathcal{F}_{y_2}$ of $j_U! j_U^! \mathcal{F}_1$ in $D_C(\mathcal{O}_{y_2})$ is just a complex of $C$-equivariant local systems. Each of the irreducible local systems contained in the cohomology of $\mathcal{F}_{y_2}$ is of the form $\mathcal{L}_2$ above (it appears in $K_{\sigma,r}$), so by the induction hypothesis $\mathcal{F}_{y_2} \in \langle K_{\sigma,r} \rangle$. We deduce that
\[
\begin{align*}
\text{(6.7)} & \quad j_U! j_U^! \mathcal{F}_1 \in \langle K_{\sigma,r} \rangle.
\end{align*}
\]
To prove (6.5), now (6.2) entails that it suffices to show that \( \mathcal{F}_2 := i_{V!} i_V^! \mathcal{F}_1 \) belongs to \( (K_{\sigma,r}^{g})_0 \), or equivalently to \( (K_{\sigma,r})_0 \).

To \( \mathcal{F}_2 \) we can apply the same arguments as to \( \mathcal{F}_1 \), reducing the issue to a similar complex \( \mathcal{F}_3 \) that is supported on union of \( C \)-orbits of dimension smaller than \( \dim V \).

Continuing in this way, we finally arrive at a complex \( \mathcal{F}_0 \) of \( C \)-equivariant sheaves on a \( C \)-orbit which is closed in \( g^{\sigma,r} \), such that all the irreducible \( \pi_0(C) \)-representations occurring in the cohomology of \( \mathcal{F}_0 \) also occur in \( i_0^* K_{\sigma,r} \). Then Theorem 6.2.a ensures that \( \mathcal{F}_0 \) lies in \( (K_{\sigma,r})_0 \).

(b) This can be shown in the same way as part (a), using part (b) of Theorem 6.2. \( \square \)

We note that the homomorphisms of derived sheaves involved in the proof of Proposition 6.3 do not depend on the group \( C = Z_G(\sigma) \times \mathbb{C}^\times \). Therefore Proposition 6.3 is also valid if we consider all objects inside the category \( \mathcal{D}_C(g^{\sigma,r}) \) (or with variety \( g^{\sigma,r}_N \)), for any closed subgroup \( C' \subset Z_G(\sigma) \times \mathbb{C}^\times \).

7. A functor from sheaves to Hecke algebra modules

Let \( A = \text{Mod}_{g} \) be the category of finitely generated right modules of an algebra \( A \). We abbreviate the categories of finitely generated right modules of

\[ \tilde{Z}(\mathbb{H}(G,M,q\mathcal{E}))_{\sigma,r} \otimes_{\mathbb{H}(G,M,q\mathcal{E})} \mathbb{H}(G,M,q\mathcal{E}) \]

and

\[ \check{H}_C^*(pt)_{\sigma,r} \otimes_{H_C^*(pt)} \text{End}^*_{\mathcal{D}_C(g^r_{\sigma})}(K_{N,\sigma,r}) \]

to, respectively, \( \mathbb{H}(G,M,q\mathcal{E}) - \text{Mod}_{g,\sigma,r} \) and \( \text{End}^*_{\mathcal{D}_C(g_{\sigma,r}^r)}(K_{N,\sigma,r}) - \text{Mod}_{g,\sigma,r} \). Recall that \( C = Z_G(\sigma) \times \mathbb{C}^\times \). We indicate their bounded derived categories by a \( \mathcal{D}_C(g^r_{\sigma}) \) and \( \mathcal{D}_C(g_{\sigma,r}^r) \) and \( K_{N,\sigma,r} \) gives rise to a derived functor

\[ \mathcal{D}_C(g^r_{\sigma}) \quad \rightarrow \quad \mathcal{D}(\text{End}^*_{\mathcal{D}_C(g^r_{\sigma})}(K_{N,\sigma,r}) - \text{Mod}_{g}) \]

\[ S \quad \rightarrow \quad \text{Hom}^*_{\mathcal{D}_C(g^r_{\sigma})}(K_{N,\sigma,r}, S) \]

We can compose with \( \check{H}_C^*(pt)_{\sigma,r} \otimes_{H_C^*(pt)} \) to land in

\[ \mathcal{D}(\text{End}^*_{\mathcal{D}_C(g^r_{\sigma})}(K_{N,\sigma,r}) - \text{Mod}_{g,\sigma,r}) \cong \mathcal{D}(\mathbb{H}(G,M,q\mathcal{E}) - \text{Mod}_{g,\sigma,r}) \]

As in Theorem 5.5, we can improve by twisting \( \mathbb{H}(G,M,q\mathcal{E}) \)-modules with the sign automorphism. That yields a functor

\[ \mathcal{F}_{\sigma,r} : \mathcal{D}_C(g^{\sigma,r}_N) \quad \rightarrow \quad \mathcal{D}(\mathbb{H}(G,M,q\mathcal{E}) - \text{Mod}_{g,\sigma,r}) \]

\[ S \quad \rightarrow \quad \text{sgn}^*(\check{H}_C^*(pt)_{\sigma,-r} \otimes_{H_C^*(pt)} \text{Hom}^*_{\mathcal{D}_C(g^{\sigma,r}_N)}(K_{N,\sigma,r}, S)) \]

For any \( g \in G, \mathbb{H}(G,M,q\mathcal{E}) - \text{Mod}_{g,\sigma,r} = \mathbb{H}(G,M,q\mathcal{E}) - \text{Mod}_{g,\text{Ad}(g)\sigma,r} \) and \( \text{Ad}(g) : g^{\sigma,r}_N \rightarrow g^{\text{Ad}(g)\sigma,r}_N \) is an isomorphism of algebraic varieties.

**Lemma 7.1.** The family of functors \( \mathcal{F}_{\sigma,r} \) is \( G \)-equivariant, in the sense that

\[ \mathcal{F}_{\text{Ad}(g)\sigma,r}(S') \cong \mathcal{F}_{\sigma,r}(\text{Ad}(g)^*S') \quad \text{for all } S' \in \mathcal{D}_C(g^{\text{Ad}(g)\sigma,-r}_N) \].
By Proposition 5.6 that is isomorphic with \( \pi \). This can also be interpreted as \( \text{Hom} \) isomorphic with \( \pi \). By \[Lus3]\, (7.4) is isomorphic with \( \pi \). By the naturality in Proposition 4.3, (7.2) is compatible with the algebra isomorphisms in Theorem 4.4. □

It is easy to see that usually \( F_{\sigma,r} \) is far from injective; for instance it kills all sheaves on which \( Z(G) \) acts differently than on \( K \). We want to show that \( F_{\sigma,r} \) is close to surjective, in the sense that its essential image contains

\[ \mathbb{H}(G, M, q\mathcal{E}) – \text{Mod}_{\mathbb{H}_{\sigma,r}} \cong \text{Mod}_{\mathbb{H}_{\sigma,r}} – \mathbb{H}(G, M, q\mathcal{E}^\vee) \].

Given an analytic standard \( \mathbb{H}(G, M, q\mathcal{E}^\vee) \)-module, we look for an explicit element of its preimage under \( F_{\sigma,r} \). For \( \rho \in \text{Irr}(\pi(Z_G(\sigma, \rho))) \) appearing in \( i_y \pi K_{\sigma,r} \), Proposition 6.1 and (6.4) show that we can form the geometric standard \( \mathbb{H}(G, M, q\mathcal{E}^\vee) \)-module \( E_{y,\sigma,r,\rho} \).

Let \( j_y : \mathcal{O}_y \to \mathfrak{g}_N^{\sigma,r} \) be the inclusion.

**Lemma 7.2.** There are natural isomorphisms of \( \mathbb{H}(G, M, q\mathcal{E}^\vee) \)-modules

\[
\mathbb{C}_{\sigma,r} \otimes \text{Hom}^*_\mathcal{D}_C(\mathfrak{g}_N^{\sigma,r}) (K_{\sigma,r}, j_N, * \mathcal{S}_y(\rho)) \cong \mathbb{C}_{\sigma,r} \otimes \text{Hom}^*_\mathcal{D}_{\mathcal{C}_y}(\{y\}) (i_y^* K_{\sigma,r}, \rho) \cong E_{y,\sigma,r,\rho}.
\]

**Proof.** By adjunction and (6.1) there are natural isomorphisms

\[
\text{Hom}^*_\mathcal{D}_C(\mathfrak{g}_N^{\sigma,r}) (K_{\sigma,r}, j_N, * \mathcal{S}_y(\rho)) \cong \text{Hom}^*_\mathcal{D}_C(\mathcal{O}_y) (j_N^* K_{\sigma,r}, \mathcal{S}_y(\rho)) \cong \text{Hom}^*_\mathcal{D}_{\mathcal{C}_y}(\{y\}) (i_y^* K_{\sigma,r}, \rho).
\]

With [Lus3] §1.10] it can be rewritten as

\[
(\text{Hom}^*_\mathcal{D}_{\mathcal{C}_y}(\{y\}) (i_y^* K_{\sigma,r}, \rho))^{\pi_0(C_y)} \cong (H^*_\mathcal{C}_y(\{y\}, D i_y^* K_{\sigma,r} \otimes \mathbb{C} \rho))^{\pi_0(C_y)}
\]

(7.4)

By [Lus3] §1.21, (7.4) is isomorphic with

\[
(H^*_\mathcal{C}_y(\{y\}) \otimes \mathbb{C} H^*(\{y\}, D i_y^* K_{\sigma,r} \otimes \rho))^{\pi_0(C_y)}.
\]

(7.5)

Since \( C_y \) acts trivially on \( \mathbb{C}_{\sigma,r} \), we can tensor the isomorphisms (7.3)–(7.5) with \( \mathbb{C}_{\sigma,r} \) over \( H^*_\mathcal{C}_y(\{y\}) \), and it does not matter whether we do that before or after taking \( \pi_0(C_y) \)-invariants. That shows that the first two terms in the statement are naturally isomorphic with

\[
(H^*(\{y\}, D i_y^* K_{\sigma,r} \otimes \rho))^{\pi_0(C_y)} \cong (H^{-*}(\{y\}, i_y^* K_{\sigma,r})^{\vee} \otimes \rho)^{\pi_0(C_y)}.
\]

By Proposition 5.9] that is isomorphic with

\[
(H^*(\{y\}, i_y^* K_{\sigma,r}^{\vee})^{[\dim \mathfrak{g}_N^{\sigma,r}] \otimes \rho)^{\pi_0(C_y)} \cong (E_{y,\sigma,r} \otimes \rho)^{\pi_0(C_y)}.
\]

This can also be interpreted as \( \text{Hom}_{\pi_0(C_y)}(\rho^{\vee}, E_{y,\sigma,r}) = E_{y,\sigma,r,\rho^{\vee}}. \) □
Lemma [7.2] tells us that \( j_{N,*}S_y(\rho) \) is pretty close to a preimage of \( E_{y,\sigma,r,\rho^\vee} \) with respect to the functor \([7.4]\). To make that precise, we will transfer the functor \( C_{\sigma,r} \otimes {}_{H_C^C}(pt) \) to sheaves on \( C_y \). Recall that \( \gamma_y : SL_2(C) \rightarrow C^o \) is an algebraic group homomorphism with \( d\gamma_y( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} ) = y \). Then \( Z_C^o(\gamma_y) \) is a maximal reductive subgroup of \( C_y \), and isomorphic to the quotient of \( C^o_y \) by its unipotent radical, see [ChGi] Proposition 3.7.23. By [Lus1] §1.11

\[
H^*_{C_y^o}(\{y\}) = H^*_{C_y^o}(pt) \cong H^*_{Z_C^o(\gamma_y)}(pt) \cong \mathcal{O}(Z_C(\gamma_y)/Z_C^o(\gamma_y)).
\]

A classical theorem of Chevalley [Var §4.9] says that this is a polynomial algebra, generated by homogeneous polynomials. As \( (\sigma,r) \in Z(C_y) = Z(\text{Lie}(C_y^o)) \), the same holds if we regard \( (\sigma,r) \) as the origin of \( C_y \). Thus there are primitive invariant polynomials \( f \in \mathcal{O}(Z_C(\gamma_y)) \), vanishing at \( (\sigma,r) \), such that

\[
H^*_{C_y^o}(pt) \cong C[f_1, \ldots, f_d],
\]

(7.6)

\[
H^*_{C_y^o}(pt)_{\sigma,r} \cong \mathcal{O}(Z_C(\gamma_y)/Z_C^o(\gamma_y))_{\sigma,r} = C[[f_1, \ldots, f_d]].
\]

In particular (with \( \mathcal{T} \) for tangent space)

(7.7) \quad \text{Irr}(H^*_{C_y^o}(pt)) = Z_C(\gamma_y)/Z_C^o(\gamma_y) \cong \mathcal{F}_{\sigma,r}(Z_C(\gamma_y)/Z_C^o(\gamma_y))

has a natural vector space structure, with \( \{f_1, \ldots, f_d\} \) as basis of its linear dual. We define

\[
\Lambda^*_y = \Lambda^*(\text{Irr}(H^*_{C_y^o}(pt))) \in \mathcal{D}(\{y\}),
\]

so that \( (DA_y)^n = \Lambda^{-n}(\text{Irr}(H^*_{C_y^o}(pt)))^\vee \). We build the differential complex

\[
DA_y^* \otimes H^*_{C_y^o}(pt), d_i,
\]

where

\[
d_i(f_i \wedge \cdots \wedge f_i \otimes f) = \sum_{j=1}^n (-1)^{j+1} f_i \wedge \cdots \wedge f_{i+1} \wedge \cdots \wedge f_i \otimes f_i f.
\]

It possesses an augmentation

\[
d_0 = \text{ev}_{\sigma,r} : H^*_{C_y^o}(pt) \rightarrow C_{\sigma,r}.
\]

This is the Koszul resolution of \( C_{\sigma,r} \), graded so that the differentials raise the degrees by one. Notice that \( (DA_y^* \otimes H^*_{C_y^o}(pt), d_i) \) is also a complex of \( \pi_0(C_y) \)-representations.

With respect to the standard basis of \( DA_y^* \), the maps \( d_{-n} \) are given as linear combinations of multiplications with the elements \( f_i \in H^*_{C_y^o}(pt) \). From [Lus3] §1.21 we know that

(7.8) \quad \text{Hom}^*_{C_y^o}(\{y\})(DA_y^n, DA_y^{n+1}) \cong H^*_{C_y^o}(\{y\}) \otimes \text{Hom}_{C}(DA_y^n, DA_y^{n+1}).

Hence the differentials \( d_n \) can be considered as elements of (7.8).

We can tensor the free \( H^*_{C_y^o}(\{y\}) \)-module

\[
H^*_{C_y^o}(\{y\}) \otimes_C H^*({} \{y\}, Dl^*_y K_{N,\sigma,r}) \otimes \rho
\]

over \( H^*_{C_y^o}(\{y\}) \) with the above Koszul resolution, that gives a differential complex

(7.9) \quad DA_y^* \otimes_C H^*_{C_y^o}(\{y\}) \otimes_C H^*({} \{y\}, Dl^*_y K_{N,\sigma,r}) \otimes \rho

of \( H^*_{C_y^o}(\{y\}) \times \pi_0(C_y) \)-modules. By the exactness of the augmented Koszul resolution, the cohomology of (7.9) is naturally isomorphic with

\[
C_{\sigma,r} \otimes_C H^*({} \{y\}, Dl^*_y K_{N,\sigma,r}) \otimes \rho.
\]
Taking \( \pi_0(C_y) \)-invariants in (7.9), we obtain a differential complex of \( H^*_C(y) \)-modules

\[
(D\Lambda^*_y \otimes_C H^*_C(y)) \otimes_C H^*(\{y\}, D\Lambda^*_y K_N,\sigma,r) \otimes \rho)^{\pi_0(C_y)}
\]

(7.10)

Taking invariants for a linear action of a finite group is exact (in characteristic zero), so the cohomology of (7.10) is

\[
(C_{\sigma,r} \otimes_C H^*(\{y\}, D\Lambda^*_y K_N,\sigma,r) \otimes \rho)^{\pi_0(C_y)}.
\]

(7.11)

From the proof of Lemma 7.2 we know that (7.11) is naturally isomorphic with \( E_{y,\sigma,r,\rho^\vee} \). Using (7.8) we can build the complex

\[
(D\Lambda^*_y \otimes \rho, d_n \otimes \text{id}_\rho \in D_C(y)).
\]

The \( \pi_0(C_y) \)-actions on \( \rho \) and on (7.7) and the \( \pi_0(C_y) \)-equivariance of \( d_n \) make it into an element of \( D_C(y) \). This complex admits an augmentation

\[
ev_{\sigma,r} \otimes \text{id}_\rho : D\Lambda^*_y \otimes \rho \to \rho.
\]

**Lemma 7.3.** The augmentation \( \ev_{\sigma,r} \otimes \text{id}_\rho \) induces a quasi-isomorphism

\[
\text{Hom}_{D_C(y)}^*(i^*_y K_N,\sigma,r, D\Lambda^*_y \otimes \rho) \to E_{y,\sigma,r,\rho^\vee}
\]

in \( D(\text{End}_{D_C(y)}^*(i^*_y K_N,\sigma,r)) \). This complex admits a augmentation

\[
\ev_{\sigma,r} : D\Lambda^*_y \otimes \rho \to \rho.
\]

**Proof.** By construction

\[
\text{Hom}_{D_C(y)}^*(i^*_y K_N,\sigma,r, D\Lambda^*_y \otimes \rho)
\]

is naturally isomorphic with (7.10). We already identified its cohomology with \( E_{y,\sigma,r,\rho^\vee} \) after (7.11), and the argument shows that the isomorphism is induced by \( d_0 = \ev_{\sigma,r} \). □

Let \( \hat{\Lambda}^*_y,\rho \) be the sum of those irreducible \( \pi_0(C_y) \)-subrepresentations of \( D\Lambda^*_y \otimes \rho \) that appear in \( H^*(\{y\}, i^*_y K_N,\sigma,r) \), or equivalently (by Proposition 6.1) appear in \( H^*(\{y\}, i^*_y K_N,\sigma,r) \). By the \( \pi_0(C_y) \)-equivariance of \( d_n \), \( \hat{\Lambda}^*_y,\rho \in D_C(y) \). This object is independent of \( r \in \mathbb{C} \), because \( i^*_y K_N,\sigma,r \) is so as \( \pi_0(C_y) \)-representation. With (6.1) we obtain

\[
S_y(\hat{\Lambda}^*_y,\rho) \in D_C(O_y).
\]

**Proposition 7.4.** Let \( E_{y,\sigma,r,\rho^\vee} \) be a geometric standard \( \mathbb{H}(G,M,q\mathcal{E}) \)-module.

(a) \( \text{Hom}_{D_C(O_y)}^*(\{K_N,\sigma,r \}, j_N^* S_y(\hat{\Lambda}^*_y,\rho)) \) is isomorphic with \( E_{y,\sigma,r,\rho^\vee} \) in

\[
\mathcal{D}(\text{End}_{C}(K_N,\sigma,r)) \to \mathcal{D}(\text{Mod}_{\mathfrak{g}})
\]

(b) We replace \( r \) by \(-r\). Then

\[
\mathcal{F}_{\sigma,r}(j_N^* S_y(\hat{\Lambda}^*_y,\rho)) \cong \text{sgn}^* E_{y,\sigma,\sigma,\rho^\vee}
\]

in \( \mathcal{D}(\mathbb{H}(G,M,q\mathcal{E}) \to \mathcal{D}(\text{Mod}_{\mathfrak{g}}) \to \mathbb{H}(G,M,q\mathcal{E}^\vee)) \).

**Proof.** (a) By adjunction

\[
\text{Hom}_{D_C(O_y)}^*(\{K_N,\sigma,r \}, j_N^* S_y(\hat{\Lambda}^*_y,\rho)) \cong \text{Hom}_{D_C(O_y)}^*(\{K_N,\sigma,r \}, S_y(\hat{\Lambda}^*_y,\rho))
\]

\[
\cong \text{Hom}_{D_C(y)}^*(\{i^*_y K_N,\sigma,r \}, \hat{\Lambda}^*_y,\rho).
\]
By the definition of $\tilde{\Lambda}_{y,\rho}$ and by Lemma 7.3, the right hand side is naturally isomorphic with
\[ \text{Hom}_{D_C((y))}(i_y^*K, D\Lambda_y \otimes \rho) \cong E_{y,\sigma,\rho}^\vee. \]

(b) By part (a) and the exactness of localization,
\[ F_{\sigma,r}(j_*s(H(\tilde{\Lambda}_{y,\rho}))) \cong H^*_C(pt)_{\sigma,r} \otimes_{H^*_C(pt)} E_{y,\sigma,\rho}^\vee. \]

Since $E_{y,\sigma,\rho}^\vee$ has finite dimension and admits the central character $(\sigma,r)$, it is not changed by applying $H^*_C(pt)_{\sigma,r} \otimes_{H^*_C(pt)} \cdot$.

The complex $j_*s(H(\tilde{\Lambda}_{y,\rho}))$ is our preferred preimage of $s\vee E_{y,\sigma,-r,\rho}^\vee$ under $F_{\sigma,r}$.

**Proposition 7.5.** (a) The intersection of the essential image of $F_{\sigma,r}$ with
\[ \text{Mod}_{\mathrm{fg},\sigma,r}(F(G, M, qE^\vee)) \cong F(G, M, qE) - \text{Mod}_{\mathrm{fg},\sigma,r} \]
is closed under quotients, kernels and extensions.

(b) The essential image of $F_{\sigma,r}$ contains $\text{Mod}_{\mathrm{fg},\sigma,r} - F(G, M, qE^\vee)$.

**Proof.** (a) Consider an exact triangle
\[ F_1 \longrightarrow F_2 \longrightarrow F_3 \quad \text{in} \quad D(\text{Mod}_{\mathrm{fg},\sigma,r}(F(G, M, qE^\vee))). \]

Suppose that the essential image of $F_{\sigma,r}$ contains two out of these three complexes (e.g. $A$ and $B$). Then it also contains the third complex (for instance because $F_1 \to F_2$ is quasi-isomorphic with $F_3$). With degree shifts (which are allowed in $D_C(g_N^{\sigma,r})$ and hence in the image of $F_{\sigma,r}$) we can put every sheaf in the desired degree. Thus every short exact sequence in $D(\text{Mod}_{\mathrm{fg},\sigma,r}(F(G, M, qE^\vee)))$ gives rise to an exact triangle in the associated bounded derived category, and the two-out-of-three property applies to such exact sequences as well.

(b) In Proposition 7.4 we showed that the essential image contains all standard modules $s\vee E_{y,\sigma,-r,\rho}^\vee$. From Theorem 5.3 we know that this set of modules is in bijection with
\[ \text{Irr}(\tilde{Z}(F(G, M, qE^\vee))_{\sigma,r} \otimes_{Z(F(G, M, qE^\vee))} F(G, M, qE^\vee)) = \text{Irr}(F(G, M, qE^\vee)/(Z_{\sigma,r})), \]
by taking their irreducible quotients $s\vee M_{y,\sigma,-r,\rho}^\vee$. Moreover, by [Chu §3], extended to our generality in the proof of [AMS2 Theorem 3.20], every irreducible constituent of $s\vee E_{y,\sigma,-r,\rho}^\vee$ different from $s\vee M_{y,\sigma,-r,\rho}^\vee$ is isomorphic to a module $s\vee M_{y',\sigma',-r,\rho'}$ with $\text{dim} C^G_{y'}/\text{dim} C^G_y$ and $\sigma'$ $G$-conjugate to $\sigma$. In particular the essential image of $F_{\sigma,r}$ contains all
\[ s\vee E_{y',\sigma',-r,\rho'}^\vee = s\vee M_{y',\sigma',-r,\rho'}^\vee \]
for which $\text{dim} C^G_{y'}$ is maximal. For arbitrary $(y, \rho)$ with the correct cuspidal quasi-support, consider the quotient map
\[ q : s\vee E_{y',\sigma',-r,\rho'}^\vee \to s\vee M_{y',\sigma',-r,\rho'}^\vee. \]

By (downward) induction with respect to $\text{dim} C^G_y$ we may assume that all irreducible constituents of $\ker q$ belong to the essential image of $F_{\sigma,r}$. Then part (a) ensures that it also contains $\ker q$. Applying part (a) to $\ker q \to s\vee E_{y,\sigma,-r,\rho}^\vee$, we deduce that $s\vee M_{y,\sigma,-r,\rho}^\vee$ belongs to the essential image as well.
Now we know that the essential image of $F_{\sigma,r}$ contains the whole of (7.13). Every element of $\text{Mod}_{H,\sigma,r} = \mathbb{H}(G, M, qE')$ can be made from (7.13) by repeated extensions, so by part (a) such an element also belongs to the essential image of $F_{\sigma,r}$. □

Remark 7.6. We point out that the entire Section 7 is also valid with $(g^{\sigma,r}, K_{\sigma,r})$ instead of $(g^{\sigma_{N,r}, r}, K_{N,\sigma,r})$. Indeed the same arguments work, because earlier we also proved versions with and without restriction to nilpotent subvarieties. While for $r \in \mathbb{C}^*$ this modification would not make any difference (because $g^{\sigma,r} \subset g_N$), such results in the case $r = 0$ would combine less well with Section 6.

8. A functor from Hecke algebra modules to sheaves

The functor $\text{Hom}^*_{D_C(g^{\sigma,r}_{N})}(K_{N,\sigma,r}, ?)$ from (7.1) admits a partial inverse, whose construction requires a little preparation. A problem is that we cannot define it directly, because the involved derived categories are not abelian.

Instead, we consider the category $\text{End}^*_D(g^{\sigma,r}_{N}) (K_{N,\sigma,r}) - \text{Mod}_{fgp}$ of finitely generated projective right $\text{End}^*_D(g^{\sigma,r}_{N}) (K_{N,\sigma,r})$-modules. We claim that for any $P \in \text{End}^*_D(g^{\sigma,r}_{N}) (K_{N,\sigma,r}) - \text{Mod}_{fgp}$, the tensor product

$$P \otimes_{\text{End}^*_D(g^{\sigma,r}_{N}) (K_{N,\sigma,r})} K_{N,\sigma,r}$$

(8.1)

is a canonically an element of $D_C(g^{\sigma,r}_{N})$. Indeed, there exists another $P' \in \text{End}^*_D(g^{\sigma,r}_{N}) (K_{N,\sigma,r}) - \text{Mod}_{fgp}$ such that $P \oplus P'$ is free, say of rank $d$. Then (8.1) is naturally a direct summand of

$$(P \oplus P') \otimes_{\text{End}^*_D(g^{\sigma,r}_{N}) (K_{N,\sigma,r})} K_{N,\sigma,r} \cong K_{N,\sigma,r}^d.$$

Let $\langle K_{N,\sigma,r} \rangle$ be the subcategory of $D_C(g^{\sigma,r}_{N})$ generated by $K_{N,\sigma,r}$ (via degree shifts, forming cones and taking direct summands). The above provides a functor from $\text{End}^*_D(g^{\sigma,r}_{N}) (K_{N,\sigma,r}) - \text{Mod}_{fgp}$ to $\langle K_{N,\sigma,r} \rangle$. It extends to a derived functor

$$D(\text{End}^*_D(g^{\sigma,r}_{N}) (K_{N,\sigma,r}) - \text{Mod}_{fgp}) \rightarrow \langle K_{N,\sigma,r} \rangle$$

(8.2)

$$\mathcal{V} \ni \text{End}^*_D(g^{\sigma,r}_{N}) (K_{N,\sigma,r}) \otimes_{\text{End}^*_D(g^{\sigma,r}_{N}) (K_{N,\sigma,r})} K_{N,\sigma,r}.$$

Strictly speaking, on the right hand side we use the canonical functor from the derived category of $C$-equivariant constructible sheaves on $g^{\sigma,r}$ to $D_C(g^{\sigma,r}_{N})$ [BeLu]. We note that

$$D(\text{End}^*_D(g^{\sigma,r}_{N}) (K_{N,\sigma,r}) - \text{Mod}_{fgp})$$

is already generated by the single object $\text{End}^*_D(g^{\sigma,r}_{N}) (K_{N,\sigma,r})$, with respect to the operations degree shifts, cones and direct summands.

Similarly there is a derived functor

$$D(\text{End}^*_D(g^{\sigma,r}_{N}) (K_{\sigma,r}) - \text{Mod}_{fgp}) \rightarrow \langle K_{\sigma,r} \rangle.$$ 

(8.4)

Theorem 8.1. (a) The functor (8.2) is an equivalence of categories, with inverse

$$\text{Hom}^*_D(g^{\sigma,r}_{N}) (K_{N,\sigma,r}, ?) : \langle K_{N,\sigma,r} \rangle \rightarrow D(\text{End}^*_D(g^{\sigma,r}_{N}) (K_{N,\sigma,r}) - \text{Mod}_{fgp}).$$
(b) The functor (8.1) is an equivalence of categories, with inverse
\[ \text{Hom}_{D_C(\mathfrak{g}^r)}(K_{\sigma,r}, \cdot) : (K_{\sigma,r}) \to D(\text{End}_{D_C(\mathfrak{g}^r)}^*(K_{\sigma,r}) - \text{Mod}_{fgp}). \]

Proof. (a) Clearly
\[ \text{Hom}_{D_C(\mathfrak{g}^r)}^*(K_{N,\sigma,r}, P) \cong \text{End}_{D_C(\mathfrak{g}^r)}^*(K_{N,\sigma,r}) \]
is naturally isomorphic whenever \( P \) is free of finite rank. Suppose \( P' \) is another free module, and say their ranks are \( d \) and \( d' \). We can naturally identify
\[ \text{Hom}_{\text{End}_{D_C(\mathfrak{g}^r)}^*(K_{N,\sigma,r})}(P, P') \cong \text{Mat}_{d \times d'}(\text{End}_{D_C(\mathfrak{g}^r)}^*(K_{N,\sigma,r})), \]
with action by multiplication from the left. But also
\[ \text{Hom}_{D_C(\mathfrak{g}^r)}^*(P, K_{N,\sigma,r}), P') \cong \text{End}_{D_C(\mathfrak{g}^r)}^*(K_{N,\sigma,r}) \]
\[ \cong \text{Hom}_{D_C(\mathfrak{g}^r)}^*(K_{N,\sigma,r}, K_{N,\sigma,r}) \cong \text{Mat}_{d' \times d}(\text{End}_{D_C(\mathfrak{g}^r)}^*(K_{N,\sigma,r})). \]

By the additivity of these functors, these properties hold for all \( P, P' \in \text{End}_{D_C(\mathfrak{g}^r)}^*(K_{N,\sigma,r}) - \text{Mod}_{fgp} \). That in turn extends automatically to the derived category of \( \text{End}_{D_C(\mathfrak{g}^r)}^*(K_{N,\sigma,r}) - \text{Mod}_{fgp} \), proving that (8.2) is fully faithful.

Its essential image of (8.2) contains \( K_{N,\sigma,r} \) and it is closed under the operations degree shifts, cones and direct summands (because its source is), so it is essentially surjective.

(b) The same arguments as for part (a) apply. \( \square \)

Combining Theorem 8.1 in the special case \((\sigma, r) = (0, 0)\) with Theorem 3.2 and (4.10) we find:

**Corollary 8.2.** The functors in Theorem 8.1 provide equivalences of categories
\[ (K_N) \cong D(\mathbb{H}(G, M, q\mathcal{E}) - \text{Mod}_{fgp}) \cong (K), \]
where the left (resp. right) hand side is a full subcategory of \( D_{G \times C^\times}(\mathfrak{g}_N) \) (resp. of \( D_{G \times C^\times}(\mathfrak{g}) \)).

Unfortunately, we do not know whether every finitely generated module over \( \text{End}_{D_C(\mathfrak{g}^r)}^*(K_{N,\sigma,r}) \) admits a finite type projective resolution. It is not hard to see that this algebra is Noetherian, but in general it is not clear how to show that its global dimension is finite. For \( \mathbb{H}(t, W, k) \) this was shown in [Sol1] Theorem 5.3, but that proof does not generalize to graded Hecke algebras with a formal variable \( r \).

Thus \( D(\text{End}_{D_C(\mathfrak{g}^r)}^*(K_{N,\sigma,r}) - \text{Mod}_{fgp}) \) can be a strict subcategory of \( D(\text{End}_{D_C(\mathfrak{g}^r)}^*(K_{N,\sigma,r}) - \text{Mod}_{fgp}) \). We will show that at least the objects of \( \text{End}_{D_C(\mathfrak{g}^r)}^*(K_{N,\sigma,r}) - \text{Mod}_{fgp} \) admit finite type projective resolutions, so that those belong to \( D(\text{End}_{D_C(\mathfrak{g}^r)}^*(K_{N,\sigma,r}) - \text{Mod}_{fgp}) \).

Recall the complex \( S_y(\hat{\Lambda}_{y,\rho}^*) \in D_C(\mathcal{O}_y) \) from (7.12).

**Corollary 8.3.** \( j_N^*S_y(\hat{\Lambda}_{y,\rho}^*) \in (K_{N,\sigma,r}) \), where \( j_N : \mathcal{O}_y \to \mathfrak{g}_N^r \) is the embedding.

Proof. By construction \( \hat{\Lambda}_{y,\rho}^* \) is a complex of \( \pi_0(C_y) \)-representations, all whose irreducible constituents satisfy Proposition 6.1. Thus Proposition 6.3 applies. \( \square \)
We note that for \( r \neq 0 \) the subscripts \( N \) are superfluous, and Corollary 8.3 says that \( j_N S_y (\Lambda^*_y, \rho) \in (K_{\sigma, r}) \).

**Proposition 8.4.** (a) \( \text{End}^*_{DC (g_N^r)} (K_{N, \sigma, r}) - \text{Mod}_{fgp} \) is contained in \( \text{D}(\text{End}^*_{DC (g_N^r)} (K_{N, \sigma, r}) - \text{Mod}_{fgp}) \), so all its objects admit finite type projective resolutions.

(b) For a standard \( \mathbb{H}(G, M, qE^\vee) \)-module \( E_{y, \sigma, r, \rho, v} \), regarded as an object of \( \text{End}^*_{DC (g_N^r)} (K_{N, \sigma, r}) - \text{Mod}_{fgp} \) via Theorem 4.4.d,

\[
E_{y, \sigma, r, \rho, v} \otimes_{\text{End}^*_{DC (g_N^r)} (K_{N, \sigma, r})} K_{N, \sigma, r}
\]

is quasi-isomorphic with \( j_N S_y (\Lambda^*_y, \rho) \).

**Proof.** (b) By Corollary 8.3 and Theorem 8.1.b,

\( \text{D}(\text{End}^*_{DC (g_N^r)} (K_{N, \sigma, r}) - \text{Mod}_{fgp}) \) contains \( \text{Hom}^*_{DC (g_N^r)} (K_{N, \sigma, r}, j_N S_y (\Lambda^*_y, \rho)) \).

Then Proposition 7.4 and Remark 7.6 say that this object is quasi-isomorphic with \( E_{y, \sigma, r, \rho, v} \). Next Theorem 8.1 and Corollary 8.3 say that \( j_N S_y (\Lambda^*_y, \rho) \) is the image of \( E_{y, \sigma, r, \rho, v} \) under \( DC \).

(a) We just saw that \( \text{D}(\text{End}^*_{DC (g_N^r)} (K_{N, \sigma, r}) - \text{Mod}_{fgp}) \) contains all geometric standard modules \( E_{y, \sigma, r, \rho, v} \). The proof of Proposition 7.5 shows that those generate a subcategory that contains \( \text{Mod}_{fgp} - \mathbb{H}(G, M, qE^\vee) \cong \text{End}^*_{DC (g_N^r)} (K_{N, \sigma, r}) - \text{Mod}_{fgp} \), inside the associated bounded derived categories.

Consider the functor

\[
F_{\sigma, r} : \text{D}(\text{Mod}_{fgp} - \mathbb{H}(G, M, qE^\vee)) \to \text{D}(\text{Mod}_{fgp} - \mathbb{H}(G, M, qE^\vee)) \to \text{D}(\mathbb{H}(G, M, qE) - \text{Mod}_{fgp} - \mathbb{H}(G, M, qE^\vee)) \to \text{D}(\text{End}^*_{DC (g_N^r)} (K_{N, \sigma, r}) - \text{Mod}_{fgp} - \mathbb{H}(G, M, qE^\vee)) \to \text{D}(g_N^{\sigma, -r})
\]

obtained by composing \( \text{sgn}^* \), (8.4), Theorem 4.4(b,d) and (8.4), where the last step is made possible by Proposition 8.4.a.

**Theorem 8.5.** (a) The functor \( F_{\sigma, r} \) is fully faithful and \( F_{\sigma, r} \circ F_{\sigma, r} \) is equivalent with the identity functor on \( \text{D}(\text{Mod}_{fgp} - \mathbb{H}(G, M, qE^\vee)) \).

(b) For an analytic standard \( \mathbb{H}(G, M, qE^\vee) \)-module \( \text{sgn}^* E_{y, \sigma, r, \rho, v} \):

\[
F_{\sigma, r} (\text{sgn}^* E_{y, \sigma, r, \rho, v}) = j_N S_y (\Lambda^*_y, \rho).
\]

**Proof.** (a) The last arrow in (8.5) is fully faithful by Theorem 8.1. All the other involved arrows are equivalences of categories, so certainly fully faithful. Notice that in the definition of \( F_{\sigma, r} \) the operation \( H_{E_C (pt)} (\sigma_{\sigma, r}) \otimes_{H_{E_C (pt)}} \) is just the identity on

\[
\text{D}(\text{End}^*_{DC (g_N^{\sigma, -r})} (K_{N, \sigma, r}) - \text{Mod}_{fgp} - \mathbb{H}(G, M, qE^\vee)).
\]

Thus every factor in the definition of \( F_{\sigma, r} \) has an inverse in (8.3), although the last arrow (8.5) is only a right inverse by Theorem 8.1. It follows that \( F_{\sigma, r} \) is a right inverse to \( F_{\sigma, r} \).

(b) This is a direct consequence of Proposition 8.4.b.

Again, we point out that for \( r \neq 0 \) the subscripts \( N \) are superfluous.
9. Twisted graded Hecke algebras with a fixed r

So far we mainly considered twisted graded Hecke algebras with a formal variable r. Often we localized r at a complex number r, but still we allowed modules on which r did not act as a scalar. In the Hecke algebras that arise from reductive p-adic groups, r is always specialized to some r ∈ ℜ, see [Sol4]. That prompts us to find versions of our main results for such algebras.

Fix r ∈ ℂ and write

\[ \mathbb{H}(G, M, q^E, r) = \mathbb{H}(G, M, q^E)/(r - r) = \mathbb{H}(t, W_{q^E}, c_r, z_{q^E}). \]

The centre of \( \mathbb{H}(G, M, q^E, r) \) is \( \mathcal{O}(t/W_{q^E}) \) (so it can be localized at any \( \sigma \in t \)). The irreducible and standard modules of \( \mathbb{H}(G, M, q^E, r) \) have already been classified in Theorems 5.3 and 5.5; they just come from \( \mathbb{H}(G, M, q^E) \) by imposing that r acts as r. However, the Ext-groups of two \( \mathbb{H}(G, M, q^E, r) \)-modules are usually not isomorphic to their Ext-groups as \( \mathbb{H}(G, M, q^E) \)-modules.

This can already be seen in the simple case \( W_{q^E} = \{1\}, \mathbb{H}(G, M, q^E) = \mathcal{O}(t ⊕ ℂ), \mathbb{H}(G, M, q^E, r) = \mathcal{O}(t) \). Then (with \( T \) for a tangent space)

\[
\begin{align*}
\text{Ext}^\ast_{\mathcal{O}(t ⊕ ℂ)}(\mathbb{C}_{σ, r}, \mathbb{C}_{σ, r}) & \cong \Lambda^\ast T_{(σ, r)}(t ⊕ ℂ) \cong \Lambda^\ast T_σ(t) \otimes \Lambda^\ast T_r(ℂ), \\
\text{Ext}^\ast_{\mathcal{O}(t)}(\mathbb{C}_{σ, r}, \mathbb{C}_{σ, r}) & \cong \Lambda^\ast T_σ(t).
\end{align*}
\]

One can also see this in terms of projective resolutions. The Koszul resolution of \( \mathbb{C}_{σ, r} \in \text{Mod} - \mathcal{O}(t ⊕ ℂ) \) does not become a resolution of \( \mathbb{C}_{σ, r} \in \text{Mod} - \mathcal{O}(t) \) by dividing out \( (r - r) \) everywhere, because that

\[
\begin{align*}
\mathcal{O}(t ⊕ ℂ) \xrightarrow{r - r} \mathcal{O}(t ⊕ ℂ) & \xrightarrow{0} \mathcal{O}(t).
\end{align*}
\]

For such reasons, we need a version of Theorem 4.4 for \( \mathcal{O}(t ⊕ ℂ) \) as \( \mathcal{H}_G^\ast(\text{pt}) \) comes in from the \( \mathbb{C}^\times \)-actions on our varieties and sheaves, it is natural to try to replace \( Z_G(σ) \times \mathbb{C}^\times \)-equivariant sheaves by \( Z_G(σ) \)-equivariant sheaves in Section 4. However, the \( \mathbb{C}^\times \)-actions are there for a reason. Without them, [Lus1] would just give

\[ \text{End}_{\mathcal{D}_G(\mathfrak{g})}(K) \cong \mathbb{H}(G, M, q^E)/(r) \cong \mathcal{O}(t) \times ℂ[W_{q^E}, z_{q^E}], \]

and from there one would never get any r in the picture. Therefore we proceed more subtly, localizing first at (σ, r) and only then forgetting the \( \mathbb{C}^\times \)-actions.

For σ ∈ t, Theorem 4.4.a implies

\[
\begin{align*}
\dot{Z}(\mathbb{H}(G, M, q^E, r))_σ \otimes_{Z(\mathbb{H}(G, M, q^E, r))} \mathbb{H}(G, M, q^E, r) & \cong \\
\dot{Z}(\mathbb{H}(G, M, q^E))/(r - r) \otimes_{Z(\mathbb{H}(G, M, q^E))} \mathbb{H}(G, M, q^E) & \cong \\
\dot{\mathcal{H}}_{G \times \mathbb{C}^\times}(\text{pt})_σ/(r - r) \otimes \text{End}_{\mathcal{D}_G(\mathfrak{g})}^\ast(K) & \cong \\
\dot{\mathcal{H}}_{Z_G(σ) \times \mathbb{C}^\times}(\text{pt})_σ/(r - r) \otimes \text{End}_{\mathcal{D}_{Z_G(σ) \times \mathbb{C}^\times}(\mathfrak{g}, r)}^\ast(K_σ, r).
\end{align*}
\]

It is much easier to analyse (9.3) when \( r = 0 \), so we settle that case first.
Lemma 9.1. There are natural algebra isomorphisms

\[
\hat{H}(\mathbb{H}(G, M, qE, 0))_{\sigma} \otimes \mathbb{H}(G, M, qE, 0) \cong \hat{H}_{ZG(\sigma)}(\text{pt})_{\sigma} \otimes \text{End}_{D_{ZG(\sigma)}(\mathfrak{g}^\sigma, 0)}(K_{\sigma, 0}) \cong \hat{H}_{ZG(\sigma)}(\text{pt})_{\sigma} \otimes \text{End}_{D_{ZG(\sigma)}(\mathfrak{g}^\sigma, 0)}(K_{N, \sigma, 0}).
\]

Proof. The final line of (9.3) simplifies because

\[
\hat{H}_{ZG(\sigma)}(\text{pt})_{\sigma} \cong \hat{H}_{ZG(\sigma)}(\text{pt})_{\sigma} \otimes \hat{H}_{C^\sigma}(\text{pt})_{0}/(r) \cong \hat{H}_{ZG(\sigma)}(\text{pt})_{\sigma}
\]

as \(H_{ZG(\sigma) \times C^\sigma}(\text{pt})\)-modules. Further, by \([\text{Lus3}] \S 4.11\) there is a natural isomorphism

\[
\hat{H}_{ZG(\sigma)}(\text{pt})_{\sigma} \cong \text{End}_{D_{ZG(\sigma)}(\mathfrak{g}^\sigma, 0)}(K_{\sigma, 0}).
\]

Taking \(\pi_0(ZG(\sigma))\)-invariants, as in the proof of Proposition 4.2 we obtain the analogue of (9.5) with \(ZG(\sigma)\) instead of \(Z^0_G(\sigma)\). Combining that with (9.3) and (9.4) proves the first isomorphism.

The isomorphism between the first and third terms in the statement can be shown in the same way, starting from part (c) instead of part (a) of Theorem 4.1. \(\square\)

Next we consider a nonzero \(r\). Let \(T'\) be a maximal torus of \(Z^0_G(\sigma)\) containing \(T\).

Lemma 9.2. Fix \((\sigma, r) \in \mathfrak{t} \times \mathbb{C}^\times\). There is a natural algebra isomorphism

\[
\hat{H}_{ZG(\sigma) \times C^\times}(\text{pt})_{\sigma, r} \cong \text{End}_{D_{ZG(\sigma) \times C^\times}(\mathfrak{g}^\sigma, r)}(K_{\sigma, r}) \cong \mathbb{C}[[r - r]] \otimes \hat{H}_{ZG(\sigma)}(\text{pt})_{\sigma} \otimes \text{End}_{D_{ZG(\sigma)}(\mathfrak{g}^\sigma, r)}(K_{\sigma, r}).
\]

Proof. From \([\text{Lus3}] \S 4.11\) and Proposition B.1 we get

\[
\hat{H}_{ZG(\sigma) \times C^\times}(\text{pt})_{\sigma, r} \cong \text{End}_{D_{ZG(\sigma) \times C^\times}(\mathfrak{g}^\sigma, r)}(K_{\sigma, r}) \cong \hat{H}_{T' \times C^\times}(\text{pt})_{\sigma, r} \otimes \text{End}_{D_{T' \times C^\times}(\mathfrak{g}^\sigma, r)}(K_{\sigma, r}).
\]

Let \(Z' := Z_{T' \times C^\times}(\mathfrak{g}^\sigma, r)\) be the pointwise stabilizer of \(\mathfrak{g}^\sigma, r\) in \(T' \times \mathbb{C}^\times\). It contains the subgroup \(\exp(\mathbb{C}(\sigma, r))\), which projects onto \(\mathbb{C}^\times\) because \(r \neq 0\). Consequently

\[
Z' = Z_{T' \times C^\times}(\mathfrak{g}^\sigma, r) = Z_{T'}(\mathfrak{g}^\sigma, r) \times \exp(\mathbb{C}(\sigma, r)).
\]

As \(Z'\) is a connected algebraic subgroup of \(T' \times \mathbb{C}^\times\) and it projects onto \(\mathbb{C}^\times\), we can find an algebraic subtorus \(Z_\perp \subset T'\) such that

\[
T' \times \mathbb{C}^\times = Z' \times Z_\perp.
\]
The group \( Z' \) acts trivially on \( g^{\sigma,r} \) and on \( K_{\sigma,r} \) (by connectedness), so we can further decompose according to (9.8):

\[
\text{End}^*_{D_{Z'\times C^\times}(g^{\sigma,r})}(K_{\sigma,r}) \cong \text{End}^*_{D_{Z'\times C}(pt)}(C) \otimes \text{End}^*_Z(g^{\sigma,r})(K_{\sigma,r}) \cong H^*_Z(pt) \otimes C \text{End}^*_Z(g^{\sigma,r})(K_{\sigma,r}).
\]  

(9.9)

From (9.7) we see that

\[ Z'\otimes C(g^{\sigma,r}) = Z'(g^{\sigma,r}) \otimes C(\sigma,r), \]

\[ H^*_Z(pt) \cong \mathcal{O}(Z'\otimes C(g^{\sigma,r})) \cong \mathcal{O}(Z'(g^{\sigma,r})) \otimes C[\sigma] \cong H^*_Z(g^{\sigma,r})(pt) \otimes C[\sigma]. \]

Plugging that back into (9.9), we obtain

\[
\text{End}^*_{D_{Z'\times C^\times}(g^{\sigma,r})}(K_{\sigma,r}) \cong C[\sigma] \otimes H^*_Z(g^{\sigma,r})(pt) \otimes \text{End}^*_Z(g^{\sigma,r})(K_{\sigma,r}) \cong C[\sigma] \otimes \text{End}^*_{D_{Z'(g^{\sigma,r})}}(K_{\sigma,r}).
\]

(9.10)

Then (9.6) and its analogue without \( C^\times \) yield

\[
H^*_{Z_G(\sigma)(g^{\sigma,r})}(pt)_{\sigma,r} \otimes \text{End}^*_{D_{Z_G(\sigma)(g^{\sigma,r})}}(K_{\sigma,r}) \cong C[\sigma][\sigma - r] \otimes \hat{H}^*_{Z_G(\sigma)}(pt)_{\sigma,r} \otimes \text{End}^*_{D_{Z_G(\sigma)(g^{\sigma,r})}}(K_{\sigma,r}) \cong C[\sigma][\sigma - r] \otimes \hat{H}^*_{Z_G(\sigma)}(pt)_{\sigma,r} \otimes \text{End}^*_{D_{Z_G(\sigma)(g^{\sigma,r})}}(K_{\sigma,r})
\]

(9.11)

As in the proof of Proposition 4.2, taking \( \pi_0(Z_G(\sigma)) \)-invariants in (9.11) replaces \( Z_G(\sigma) \) by \( Z_G(\sigma) \).

Now we can prove our desired variation on Theorem 4.4.

**Theorem 9.3.** Fix any \( (\sigma,r) \in t \times C \).

(a) There is a natural algebra isomorphism

\[
\hat{\mathbb{H}}(G,M,q\mathcal{E},r)_{\sigma} \otimes \mathbb{H}(G,M,q\mathcal{E},r) \cong H^*_{Z_G(\sigma)}(pt)_{\sigma,r} \otimes \text{End}^*_{D_{Z_G(\sigma)(g^{\sigma,r})}}(K_{\sigma,r}).
\]

(b) This induces an equivalence of categories

\[
\text{Mod}_{\mathbb{H},\sigma}(\mathbb{H}(G,M,q\mathcal{E},r)) \cong \text{Mod}_{\mathbb{H},\sigma}(\text{End}^*_{D_{Z_G(\sigma)(g^{\sigma,r})}}(K_{\sigma,r})).
\]

(c) Parts (a) and (b) also hold with \( (g^{\sigma,r}_N,K_N,\sigma,r) \) instead of \( (g^{\sigma,r},K_{\sigma,r}) \).

**Proof.** (b) follows directly from (a).

(a,c) For \( r = 0 \) this is Lemma 9.11 so we may assume \( r \neq 0 \). Then the subscripts \( N \) do not change anything. By (9.3) and Lemma 9.2

\[
\hat{\mathbb{H}}(G,M,q\mathcal{E},r)_{\sigma} \otimes \mathbb{H}(G,M,q\mathcal{E},r) \cong (C[\sigma][\sigma - r] \otimes \hat{H}^*_{Z_G(\sigma)}(pt)_{\sigma,r})/(r-r) \otimes \text{End}^*_{D_{Z_G(\sigma)(g^{\sigma,r})}}(K_{\sigma,r}) \cong \hat{H}^*_{Z_G(\sigma)}(pt)_{\sigma,r} \otimes \text{End}^*_{D_{Z_G(\sigma)(g^{\sigma,r})}}(K_{\sigma,r}).
\]

\[ \square \]
We note that for any $y \in \mathfrak{g}^{\sigma,r}_N$,
\[ \mathbb{C}^x y \subset \text{Ad}(Z_G^*(\sigma))y \]
because $y$ is part of a $\mathfrak{sl}_2$-triple in $Z_G^*(\sigma)$. In particular $Z_G^*(\sigma)$ and $C = Z_G^*(\sigma) \times \mathbb{C}^x$ have the same orbits on $\mathfrak{g}^{\sigma,r}_N$. Mainly for that reason, all the results in Sections 5–8 remain valid when we replace $C$ be $Z_G^*(\sigma)$ everywhere. There is only one thing that really changes, namely the Koszul resolution
\[ (9.12) \quad D\Lambda_y^* \otimes H^*_G(pt) \longrightarrow \mathbb{C}_{\sigma,}\]
With $Z_G(\sigma)$ instead of $C$, this resolution has one term less. More precisely, we can take $r - r$ as one of the primitive invariant polynomials $f_i \in H^*_G(pt)_{\sigma,r}$. Then the other $f_i$ can be taken in $H^*_G(pt)$, and the Koszul resolution with respect to $Z_G^*(\sigma)$ is made from those $f_i$. This explains how the problems (9.1) and (9.2) disappear.

We formulate a main conclusion, the version of Theorem 8.5 for $H(G, M, qE, r)$ and $Z_G(\sigma)$.

**Theorem 9.4.** Fix $(\sigma, r) \in \mathfrak{t} \oplus \mathbb{C}$. There exist derived functors
\[
F_\sigma : \mathcal{D}_{Z_G(\sigma)}(\mathfrak{g}^{\sigma,r}_N) \rightarrow \mathcal{D}(\mathbb{H}(G, M, qE, r) - \text{Mod}_{\mathfrak{g},\sigma}) \quad S \mapsto \text{sgn}^*(\tilde{H}^*_G(pt)_{\sigma} \otimes \text{Hom}^*_{\mathcal{D}_{Z_G(\sigma)}(\mathfrak{g}^{\sigma,r}_N)}(K_{\sigma,-r}, S))
\]
\[
F_\sigma : \mathcal{D}(\mathbb{H}(G, M, qE, r) - \text{Mod}_{\mathfrak{g},\sigma}) \rightarrow \mathcal{D}_{Z_G(\sigma)}(\mathfrak{g}^{\sigma,r}_N)
\]
such that:

(a) $F_\sigma$ is fully faithful and $F_\sigma \circ F_\sigma^-$ is equivalent with the identity functor on $\mathcal{D}(\mathbb{H}(G, M, qE, r) - \text{Mod}_{\mathfrak{g},\sigma})$.

(b) For any analytic standard left $\mathbb{H}(G, M, qE, r)$-module $\text{sgn}^*E_{y,\sigma,-r,\rho}$,
\[
F_\sigma(\text{sgn}^*E_{y,\sigma,-r,\rho}) \quad \text{is quasi-isomorphic with} \quad j_{\sigma}^*\tilde{S}_y(\Lambda_{y,\rho}^*)
\]

In part (b) $\Lambda_{y,\rho}^* \in \mathcal{D}_{Z_G(\sigma)}(\{y\})$ is built from the Koszul resolution (9.12) by first replacing $C$ with $Z_G(\sigma)$ and then applying the same operations as just before (7.12).

**Appendix A. Compatibility with parabolic induction**

The family of geometric standard modules $E_{y,\sigma,\rho}$ behaves well under parabolic induction. However, it does not behave as well as claimed in [AMS2 Theorem 3.4]: that result is slightly too optimistic. Here we repair [AMS2 Theorem 3.4] by adding an extra condition, and we extend it from graded Hecke algebras associated to a cuspidal support to graded Hecke algebras associated to a cuspidal quasi-support.

Let $L$ be a Levi subgroup of $G$, let $v \in \text{Lie}(L)$ be nilpotent and let $E$ be an $L$-equivariant cuspidal local system on $\mathcal{C}_v^L$. In [AMS2 §2], a twisted graded Hecke algebra $\mathbb{H}(G, L, E)$ is associated to the cuspidal support $(L, \mathcal{C}_v^L, E)$. Like in Condition 5.1, we assume without loss of generality that $G = G^0 N_G(P, E)$.

Let $Q$ be an algebraic subgroup of $G$ such that $Q^0 = Q \cap G^0$ is a Levi subgroup of $G$ and $L \subset Q^0$. Then $PQ^0$ is a parabolic subgroup of $G^0$ with $Q^0$ as Levi factor. The unipotent radical $\mathcal{R}_u(PQ^0)$ is normalized by $Q^0$, so its Lie algebra $\mathfrak{u}_Q = \text{Lie}(\mathcal{R}_u(PQ^0))$ is stable under the adjoint actions of $Q^0$ and $\mathfrak{q}$. In particular, for $Y \in \mathfrak{ad}(y)$ acts on $\mathfrak{u}_Q$. We denote the cokernel of $\text{ad}(y) : \mathfrak{u}_Q \rightarrow \mathfrak{u}_Q$ by $\mathfrak{u}_{Q^0}$. For $N \in \mathfrak{u}_Q$ and $(\sigma, r) \in Z_{\mathfrak{g}\mathfrak{n}\mathfrak{c}}(y)$ we have
\[ [\sigma, [y, N]] = [y, [\sigma, N]] + [[\sigma, y], N] = [y, [\sigma, N]] + [2ry, N] \in \text{ad}(y)\mathfrak{u}_Q. \]
Hence \( \text{ad}(y) \) descends to a linear map \( yu_Q \to yu_Q \). Following Lusztig \([\text{Lus5} \S 1.16]\), we define

\[
es : Z_{Q \otimes \mathbb{C}}(y) \rightarrow \mathbb{C} \\
(\sigma, r) \quad \mapsto \quad \det(\text{ad}(\sigma) - 2r : yu_Q ightarrow yu_Q)
\]

It is easily seen that \( \epsilon \) is invariant under the adjoint action of \( Z_{Q \otimes \mathbb{C}}(y) \), so it defines an element of \( H^*_Z Q \otimes \mathbb{C} (\{y\}) \). For a given nilpotent \( y \), all the parameters \((y, \sigma, r)\) for which parabolic induction from \( \mathbb{H}(Q, L, \mathcal{E}) \) to \( \mathbb{H}(G, L, \mathcal{E}) \) can behave problematically, are zeros of \( \epsilon \).

For any closed subgroup \( S \) of \( Z_{Q \otimes \mathbb{C}}(y) \), \( \epsilon \) yields an element \( \epsilon_S \) of \( H^*_S \{y\} \) (by restriction). We recall from \([\text{Lus1} \S 7.5]\) that for connected \( S \) there is a natural isomorphism

\[
(A.1) \quad H^*_S (\mathcal{P}_y, \mathcal{E}) \cong H^*_S (\{y\}) \otimes H^*_{Z_{Q \otimes \mathbb{C}}}(\mathcal{P}_y, \mathcal{E}).
\]

Here \( H^*_S (\{y\}) \) acts on the first tensor leg and \( \mathbb{H}(G, L, \mathcal{E}) \) acts on the second tensor leg. By \([\text{AMS2} \S 3.2.b]\) these actions commute, and \( H^*_S (\mathcal{P}_y, \mathcal{E}) \) becomes a module over \( H^*_S (\{y\}) \otimes \mathbb{H}(G, L, \mathcal{E}) \).

To indicate that an object is constructed with respect to the group \( Q \) (instead of \( G \)), we endow it with a superscript \( Q \). For instance, we have the variety \( \mathcal{P}^Q_y \), which admits a natural map

\[
(A.2) \quad \mathcal{P}^Q_y \rightarrow \mathcal{P}_y : g(P \cap Q) \mapsto gP.
\]

Now we can formulate an improved version of \([\text{AMS2} \S 3.4]\).

**Theorem A.1.** Let \( S \) be a maximal torus of \( Z_{Q \otimes \mathbb{C}}(y) \).

(a) The map \((A.2)\) induces an injection of \( \mathbb{H}(G, L, \mathcal{E})\)-modules

\[
\mathbb{H}(G, L, \mathcal{E}) \otimes \mathbb{H}(Q, L, \mathcal{E}) \rightarrow H^*_S (\mathcal{P}^Q_y, \mathcal{E}),
\]

It respects the actions of \( H^*_S (\{y\}) \) and its image contains \( \epsilon_S H^*_S (\mathcal{P}_y, \mathcal{E}) \).

(b) Let \((\sigma, r) \in Z_{Q \otimes \mathbb{C}}(y)\) be semisimple, such that \( \epsilon(\sigma, r) \neq 0 \) or \( r = 0 \). The map \((A.2)\) induces an isomorphism of \( \mathbb{H}(G, L, \mathcal{E})\)-modules

\[
\mathbb{H}(G, L, \mathcal{E}) \otimes E^Q_{\mathcal{P}^Q_y, \mathcal{E}} \rightarrow E_{\mathcal{P}_y, \mathcal{E}},
\]

which respects the actions of \( \pi_0(M^Q(y))_\sigma \cong \pi_0(Z_Q(\sigma, y)) \).

**Proof.** (a) The given proof of \([\text{AMS2} \S 3.4]\) is valid with only one modification. Namely, the diagram \([\text{AMS2} 25]\) does not commute. A careful consideration of \([\text{Lus5} \S 2]\) shows failure to do so stems from the difference between certain maps \( i_0 \) and \((p^*)^{-1}\), where \( p \) is the projection of a vector bundle on its base space and \( i_0 \) is the zero section of the same vector bundle. In \([\text{Lus5} \S 2.18]\) this difference is identified as multiplication by \( \epsilon_S \).

(b) For \((\sigma, r) \in \text{Lie}(S)\) with \( \epsilon(\sigma, r) \neq 0 \), the proof of \([\text{AMS2} \S 3.4.b]\) needs only one small adjustment. From \((A.1)\) we get

\[
\mathbb{C}_{\sigma, r} \otimes H^*_S (\mathcal{P}_y, \mathcal{E}) \cong \mathbb{C}_{\sigma, r} \otimes H^*_S (\{y\}) \otimes H^*_S (\{y\}) \otimes H^*_{M^Q(y)} (\mathcal{P}_y, \mathcal{E}) \]

\[
\cong \mathbb{C}_{\sigma, r} \otimes H^*_{M^Q(y)} (\mathcal{P}_y, \mathcal{E}) = E_{\mathcal{P}_y, \mathcal{E}}.
\]
The difference with before is the appearance of \( \epsilon_S \), with that and the above the proof of [AMS2] Theorem 3.4.b] goes through.

For \( r = 0 \), we know from [AMS2 (37)] that

(A.3) \[ \mathbb{H}(G^0, L, \mathcal{E}) \otimes \mathbb{H}(Z_{G^0}(\sigma), L, \mathcal{E}) E^{Z_{G^0}(\sigma)}_{y, \sigma, 0} = E^{G^0}_{y, \sigma, 0}. \]

Let \( Q_y \subset Z_{G^0}(\sigma) \) be a Levi subgroup which is minimal for the property that it contains \( L \) and \( \exp(y) \). Then \( \mathcal{O}(t \otimes \mathbb{C}) \) acts on both

(A.4) \[ E^{Z_{G^0}(\sigma)}_{y, \sigma, 0} \otimes \mathbb{H}(Z_{G^0}(\sigma), L, \mathcal{E}) \]

by evaluation at \((\sigma, 0)\). Hence the structure of these two \( \mathbb{H}(Z_{G^0}(\sigma), L, \mathcal{E})\)-modules is completely determined by the action of \( \mathbb{C}[W^{Z_{G^0}(\sigma)}_E] \). But by [AMS2] Theorem 3.2.c]

(A.5) \[ E^{Z_{G^0}(\sigma)}_{y, \sigma, r} \otimes \mathbb{H}(Z_{G^0}(\sigma), L, \mathcal{E}) \]

do not depend on \((\sigma, r)\) as \( \mathbb{C}[W^{Z_{G^0}(\sigma)}_E]\)-modules. From a case with \( \epsilon(\sigma, r) \neq 0 \) we see that these two \( W^{Z_{G^0}(\sigma)}_E\)-representations are naturally isomorphic. Together with [A.3] that gives a natural isomorphism of \( \mathbb{H}(G^0, L, \mathcal{E})\)-modules

(A.6) \[ \mathbb{H}(G^0, L, \mathcal{E}) \otimes \mathbb{H}(Q_{y, L, \mathcal{E}}) E^{Q^0}_{y, \sigma, 0} \rightarrow E^{G^0}_{y, \sigma, 0}. \]

By the transitivity of induction, (A.6) entails that

(A.7) \[ \mathbb{H}(G^0, L, \mathcal{E}) \otimes \mathbb{H}(Q_{y, L, \mathcal{E}}) E^{Q^0}_{y, \sigma, 0} \cong E^{G^0}_{y, \sigma, 0}. \]

It was shown in [AMS2] Lemma 3.3 that

(A.8) \[ H^*_\mathcal{G}_{G \times \mathbb{C}^\times}(\mathcal{P}_y, \mathcal{E}) \cong \text{ind}_{H(G^0, L, \mathcal{E})}^{H(G, L, \mathcal{E})} H^*_\mathcal{G}_{G \times \mathbb{C}^\times}(\mathcal{P}_y^{G^0}, \mathcal{E}). \]

From (A.7) and (A.8) we get natural isomorphisms of \( \mathbb{H}(G^0, L, \mathcal{E})\)-modules

\[
E^{G^0}_{y, \sigma, 0} \cong \mathbb{H}(G, L, \mathcal{E}) \otimes \mathbb{H}(Q, L, \mathcal{E}) \otimes \mathbb{H}(Q, L, \mathcal{E}) E^{Q^0}_{y, \sigma, 0},
\]

Here the composed isomorphism is still induced by (A.2), so just as in the case \( \epsilon(\sigma, r) \neq 0 \) it is \( \pi_0(Z_{Q}(\sigma, y))\)-equivariant.

There is just one result in [AMS2] that uses [AMS2 Theorem 3.4], namely [AMS2 Proposition 3.22]. It has to replaced by a version that involves only the cases of [AMS2 Theorem 3.4] covered by Theorem (A.1)b.

Now we set out to formulate and prove analogues of [AMS2 Theorem 3.4 and Proposition 3.22] in the setting of Section 3 with a cuspidal quasi-support \((M, \mathcal{C}_L^M, q\mathcal{E})\). Also, the condition \( G = G^0 N_G(P, q\mathcal{E}) \) is in force.

For comparison with Theorem (A.1) we assume that \( M^0 = L \) and that \( \mathcal{E} \) is contained in the restriction of \( q\mathcal{E} \) to \( \mathcal{C}_L^L \). It is known from [AMS2 (93)] that

(A.9) \[ \mathbb{H}(G^0 M, M, q\mathcal{E}) = \mathbb{H}(G^0, L, \mathcal{E}). \]
Taking this into account, [AMS2, (91)] provides a canonical isomorphism of
\[ H_\ast^{Z_0 G \times C}(y) \otimes (P_y, qE) \cong \text{ind}_{\Omega(G, M, qE)}^{H(G, M, qE)} H_\ast^{Z_0 G \times C}(y) \otimes (P_y^{G_0 M}, qE) \]
\[ \cong \text{ind}^{H(G, M, qE)}_{H(G, L, E)} H_\ast^{Z_0 G \times C}(y) \otimes (P_y^{G_0 M}, \hat{\mathcal{E}}). \]

According to [AMS2, (95)] we can write \((qE)_v = \mathcal{E}_v \rtimes \rho_M\) for a unique
\[ (A.11) \]
\[ \rho_M \in \text{Irr}(\mathbb{C}[M_\mathcal{E}/M^0; \mathcal{E}_v]) = \text{Irr}(\mathbb{C}[W_\mathcal{E}/W^0, \mathcal{E}_v]). \]

Next [AMS2, Lemma 4.4] says that, when \(\sigma_0 \in \mathfrak{t}\), the sets
\[ (A.12) \]
\[ \{ \rho^0 \in \text{Irr}(\pi_0(Z(\sigma_0, y))) : \Psi_{Z_0 G}(\rho^0)(y) = [L, C^L_{\mathcal{E}}]_{G^0} \}, \]
\[ \{ \tau^0 \in \text{Irr}(\pi_0(Z_{G^0 M}(\sigma_0, y))) : q\Psi_{Z^0 M}(\tau^0)(y, \tau^0) = [M, C^M_{\mathcal{E}}]_{G^0 M} \} \]
are in bijection via \(\rho^0 \mapsto \rho^0 \rtimes \rho_M\). Further, by [AMS2, Lemma 4.5] the identification
\[ (A.9) \]
turns a standard \(H(G, L, E)\)-module \(E_{y, \sigma, r, \rho^0}^{G_0 M, M, qE}\) into the standard \(H(G^0 M, M, qE)\)-module \(E_{y, \sigma, r, \rho^0} \rtimes \rho_M\).

**Theorem A.2.** Let \(Q\) be an algebraic subgroup of \(G\) such that \(Q^0 = G^0 \cap Q\) is a
Levi subgroup of \(G^0\) and \(M \subseteq Q\). Let \(y \in q\) be nilpotent and let \(S\) be a maximal torus
of \(Z_{Q \times C}(y)\). Further, let \((\sigma, r) \in Z_{Q \times \mathbb{C}}(y)\) be semisimple, such that \(e(\sigma, r) \neq 0\) or \(r = 0\).

(a) The map \((A.2)\) induces an injection of \(H(G, M, qE)\)-modules
\[ \text{Hom}_{\mathbb{H}(G, M, qE)} \otimes \mathbb{H}(G, M) \rtimes \mathcal{E}_v \rightarrow H_\ast^{S}(P_y \otimes qE). \]

(b) The map \((A.2)\) induces an isomorphism of \(H(G, M, qE)\)-modules
\[ \text{Hom}_{\mathbb{H}(G, M, qE)} \otimes E_{y, \sigma, r}^{Q} \rightarrow E_{y, \sigma, r}, \]
which respects the actions of \(\pi_0(Z_{Q}(\sigma, y))\).

Let \(\rho \in \text{Irr}(\pi_0(Z_{Q}(\sigma, y)))\) with \(q\Psi_{Z_0 Q}(\rho)(y, \rho) = [M, C^M_{\mathcal{E}}]_{Q}\) and let \(\rho^0 \in \text{Irr}(\pi_0(Z_{Q}(\sigma, y)))\) with \(q\Psi_{Z_0 Q}(\rho^0)(y, \tau^0) = [M, C^M_{\mathcal{E}}]_{Q}\).

(c) There is a natural isomorphism of \(H(G, M, qE)\)-modules
\[ \text{Hom}_{\mathbb{H}(G, M, qE)} \otimes E_{y, \sigma, r, \rho}^{Q} \cong \bigoplus_{\rho} \text{Hom}_{\pi_0(Z_{Q}(\sigma, y))}(\rho^0, \rho) \otimes E_{y, \sigma, r, \rho}, \]
where the sum runs over all \(\rho\) as above.

(d) For \(r = 0\) part (c) contains an isomorphism of \(\mathcal{O}(t \oplus \mathbb{C}) \rtimes \mathbb{C}[W_\mathcal{E}, \mathcal{E}_v]\)-modules
\[ \text{Hom}_{\mathbb{H}(G, M, qE)} \otimes M_{y, \sigma, 0, r}^{Q} \cong \bigoplus_{\rho} \text{Hom}_{\pi_0(Z_{Q}(\sigma, y))}(\rho^0, \rho) \otimes M_{y, \sigma, 0, r}. \]

(e) The multiplicity of \(M_{y, \sigma, r, \rho}\) in \(H(G, M, qE)\) is \([\rho^0 : \rho]_{\pi_0(Z_{Q}(\sigma, y))}\).

It already appears that many times as a quotient, via \(E_{y, \sigma, r, \rho}^{Q} \rightarrow M_{y, \sigma, r, \rho}^{Q}\). More
precisely, there is a natural isomorphism
\[ \text{Hom}_{\mathbb{H}(Q, L, E)}(M_{y, \sigma, r, \rho}^{Q}, M_{y, \sigma, r, \rho}) \cong \text{Hom}_{\pi_0(Z_{Q}(\sigma, y))}(\rho, \rho^0). \]
Proof. (a) By [A.1] and [AMS2] Lemma 3.3 and §4 the right hand side of the statement is canonically isomorphic with
\[
H_S^*(\{y\}) \otimes_{H_{Z^0_{G \times C}}^*(y)} (P_y, \hat{\epsilon}^* \{^*\}) \cong
\]
\[
H_S^*(\{y\}) \otimes_{H_{Z^0_{G \times C}}^*(y)} \mathbb{H}(G, M, q\mathcal{E}) \otimes_{H(G^0, L, \mathcal{E})} H_{S^0_{G \times C}}^*(y) (P_y^{G^0}, \hat{\epsilon}^* \{^*\}).
\]
Via [A.10] that is canonically isomorphic with
\[
H_S^*(\{y\}) \otimes_{H_{Z^0_{G \times C}}^*(y)} \mathbb{H}(G, M, q\mathcal{E}) \otimes_{H(G^0, L, \mathcal{E})} H_{S^0_{G \times C}}^*(y) (P_y^{G^0}, \hat{\epsilon}^* \{^*\}) \cong
\]
\[
\mathbb{H}(G, M, q\mathcal{E}) \otimes_{H(G^0, L, \mathcal{E})} H_{S^0_{G \times C}}^*(y) (P_y^{G^0}, \hat{\epsilon}^* \{^*\}).
\]
For similar reasons the left hand side of the statement is canonically isomorphic with
\[
\mathbb{H}(G, M, q\mathcal{E}) \otimes_{H(G^0, L, \mathcal{E})} H_{S^0_{G \times C}}^*(y) (P_y^{G^0}, \hat{\epsilon}^* \{^*\}) \cong
\]
\[
\mathbb{H}(G, M, q\mathcal{E}) \otimes_{H(G^0, L, \mathcal{E})} H_{S^0_{G \times C}}^*(y) (P_y^{G^0}, \hat{\epsilon}^* \{^*\}).
\]
Now we apply Theorem A.1a for \(G^0, Q^0, L, \mathcal{E}\) and use the exactness of \(\text{ind}_{\mathbb{H}(G^0, L, \mathcal{E})}^{\mathbb{H}(Q^0, L, \mathcal{E})}\).

(b) Like in part (a) there are canonical isomorphisms
\[
E_{y, \sigma, r} \cong \mathbb{H}(G, M, q\mathcal{E}) \otimes_{H(G^0, M, q\mathcal{E})} E_{y^*, \sigma, r}^{G^0} \cong \mathbb{H}(G, M, q\mathcal{E}) \otimes_{H(G^0, L, \mathcal{E})} E_{y, \sigma, r}^{G^0},
\]
\[
\mathbb{H}(G, M, q\mathcal{E}) \otimes_{H(G^0, M, q\mathcal{E})} E_{y, \sigma, r}^{Q^0} \cong \mathbb{H}(G, M, q\mathcal{E}) \otimes_{H(Q^0, M, q\mathcal{E})} E_{y, \sigma, r}^{Q^0}.
\]
It remains to apply Theorem A.1b.

(c,d,e) These can be shown in the same way as [AMS2] Proposition 3.22, with the following modifications:

- we use part (b) instead of [AMS2] Theorem 3.4.b],
- the references to [AMS1] §4 should be extended to the setting with cuspidal quasi-supports by means of [AMS1] §5,
- the references to [AMS2] §3 should be extended to the setting with cuspidal quasi-supports by invoking [AMS2] §4].

Since \(\epsilon\) is a polynomial function, its zero set is a subvariety of smaller dimension (say of \(V_y\)). Nevertheless, we also want to explicitly exhibit a large class of parameters \((y, \sigma, r)\) on which \(\epsilon\) does not vanish. By [AMS2] Proposition 3.5.c] we assume (via conjugation by an element of \(G^0\)) that \(\sigma, \sigma_0 \in \mathfrak{t}\).
Let us call \( x \in \mathfrak{t} \) (strictly) positive with respect to \( PQ^0 \) if \( \Re(\alpha(t)) \) is (strictly) positive for all \( \alpha \in R(\Re(\alpha(PQ^0)), T) \). We say that \( x \) is (strictly) negative with respect to \( PQ^0 \) if \( -x \) is (strictly) positive.

**Lemma A.3.** Let \( y \in \mathfrak{q} \) be nilpotent and let \((\sigma, r) \in \mathfrak{t} \oplus \mathbb{C} \) with \([\sigma, y] = 2ry\). Suppose that \( \sigma = \sigma_0 + d\gamma_y \begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix} \) as in [5.4], with \( \sigma_0 \in Z(y) \). Assume furthermore that one of the following holds:

- \( \Re(r) > 0 \) and \( \sigma_0 \) is negative with respect to \( PQ^0 \);
- \( \Re(r) < 0 \) and \( \sigma_0 \) is positive with respect to \( PQ^0 \);
- \( \Re(r) = 0 \) and \( \sigma_0 \) is strictly positive or strictly negative with respect to \( PQ^0 \).

Then \( \epsilon(\sigma, r) \neq 0 \).

**Proof.** Via \( d\gamma_y : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{q}, u_Q \) becomes a finite dimensional \( \mathfrak{sl}_2(\mathbb{C}) \)-module. Since \( \sigma_0 \in \mathfrak{t} \) commutes with \( y \), it commutes with \( d\gamma_y(\mathfrak{sl}_2(\mathbb{C})) \). For any eigenvalue \( \lambda \in \mathbb{C} \) of \( \sigma_0 \), let \( \chi_{u_Q} \) be the eigenspace in \( u_Q \).

For \( n \in \mathbb{Z}_{\geq 0} \) let \( \operatorname{Sym}^n(\mathbb{C}^2) \) be the unique irreducible \( \mathfrak{sl}_2(\mathbb{C}) \)-module of dimension \( n + 1 \). We decompose the \( \mathfrak{sl}_2(\mathbb{C}) \)-module \( \chi_{u_Q} \) as

\[
\chi_{u_Q} = \bigoplus_{n \geq 0} \operatorname{Sym}^n(\mathbb{C}^2)^{\mu(\lambda,n)} \quad \text{with} \quad \mu(\lambda,n) \in \mathbb{Z}_{\geq 0}.
\]

The cokernel of \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) on \( \operatorname{Sym}^n(\mathbb{C}^2) \) is the lowest weight space \( W_{-n} \) in that representation, on which \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) acts as \( -nr \). Hence \( \sigma \) acts on

\[
\operatorname{coker}(\operatorname{ad}(y) : \chi_{u_Q} \to \chi_{u_Q}) \cong \bigoplus_{n \geq 0} W_{-n}^{\mu(\lambda,r)} \quad \text{as} \quad \bigoplus_{n \geq 0} (\lambda - nr)\operatorname{Id}_{W_{-n}^{\mu(\lambda,r)}}.
\]

Consequently

\[
(\operatorname{ad}(\sigma) - 2r)|_{\chi_{u_Q}} = \bigoplus_{\lambda \in \mathbb{C}} \bigoplus_{n \geq 0} (\lambda - (n+2)r)\operatorname{Id}_{W_{-n}^{\mu(\lambda,r)}}.
\]

By definition then

\[
\epsilon(\sigma, r) = \prod_{\lambda \in \mathbb{C}} \prod_{n \geq 0} (\lambda - (n+2)r)^{\mu(\lambda,n)}.
\]

When \( \Re(r) > 0 \) and \( \sigma_0 \) is negative with respect to \( PQ^0 \), \( \Re(\lambda - (n+2)r) < 0 \) for all eigenvalues \( \lambda \) of \( \sigma_0 \) on \( u_Q \), and in particular \( \epsilon(\sigma, r) \neq 0 \).

Similarly, we see that \( \epsilon(\sigma, r) \neq 0 \) in the other two possible cases in the lemma. \( \square \)

As an application of Lemma A.3, we prove a result in the spirit of the Langlands classification for graded Hecke algebras [Lav]. It highlights a procedure to obtain irreducible \( \mathbb{H}(G, M, q\mathcal{E}) \)-modules from irreducible tempered modules of a parabolic subalgebra \( \mathbb{H}(Q, M, q\mathcal{E}) \): twist by a central character which is strictly positive with respect to \( PQ^0 \), induce parabolically and then take the unique irreducible quotient.

**Proposition A.4.** Let \( y \in \mathfrak{g} \) be nilpotent, \((\sigma, r) \in \mathbb{Z}_{\mathfrak{g} \oplus \mathbb{C}}(y) \) semisimple and let \( \rho \in \text{Irr}(\pi_0(ZG(\sigma, y))) \) with \( \Psi_{ZG(\sigma_0)}(y, \rho) = [M, C^M_{\nu}, q\mathcal{E}]_G \).

(a) If \( \Re(r) \neq 0 \) and \( \sigma_0 \in \mathfrak{t}_G + Z(\mathfrak{g}) \), then \( M_{y,\sigma,\rho} = E_{y,\sigma,\rho} \).

(b) Suppose that \( \Re(r) > 0 \) and \( \sigma, \sigma_0 \in \mathfrak{t} \) such that \( \Re(\sigma_0) \) is negative with respect to \( P \).

Then \( \Re(\sigma_0) \) is strictly negative with respect to \( PQ^0 \), where \( Q = Z_G(\Re(\sigma_0)) \). Further \( M_{y,\sigma,\rho} \) is the unique irreducible quotient of \( \mathbb{H}(G, M, q\mathcal{E}) \otimes M_{y,\sigma,\rho}^{Q,\sigma,\rho} \).
(c) In the setting of part (b), $\text{IM}^*(M_{y,\sigma,r,\rho}) \cong \text{sgn}^*(M_{y,-\sigma,-r,\rho})$ is the unique irreducible quotient of

$$\text{IM}^*(\mathbb{H}(G,M,q\mathcal{E}) \otimes M_{y,\sigma,r,\rho}^Q) \cong \mathbb{H}(G,M,q\mathcal{E}) \otimes \text{IM}^*(M_{y,\sigma,r,\rho}^Q).$$

(d) Let $(L,E)$ be related to $(M,q\mathcal{E})$ in (A.9). Then $\text{IM}^*(M_{y,\sigma,r,\rho}^Q) \cong \text{sgn}^*(M_{y,-\sigma,-r,\rho})$ comes from the twist of a tempered $\mathbb{H}(Q^0,L,\mathcal{E})$-module by a strictly positive character of $\mathcal{O}(Z(q^0))$.

**Remark.** By [AMS2] (82)] the extra condition in part (a) holds for instance when $\Re(r) > 0$ and $\text{IM}^*(M_{y,\sigma,r,\rho})$ is tempered. By [AMS2 Proposition 3.5] every parameter $(y,\sigma)$ is $G^o$-conjugate to one with the properties as in (b).

**Proof.** (a) Write $\sigma_0 = \sigma_{0,\text{der}} + z_0$ with $\sigma_{0,\text{der}} \in \mathfrak{g}_{\text{der}}$ and $z_0 \in Z(\mathfrak{g})$. Then, as in the proof of [AMS2 Corollary 3.28],

$$E_{y,\sigma,r,\rho}^0 = E_{y,\sigma,-z_0,r,\rho}^0 \otimes \mathbb{C}_{z_0} \quad \text{and} \quad M_{y,\sigma,r,\rho}^0 = M_{y,\sigma,-z_0,r,\rho}^0 \otimes \mathbb{C}_{z_0}.$$  

By [Lus9] Theorem 1.21] $E_{y,\sigma,-z_0,r,\rho}^0 = M_{y,\sigma,-z_0,r,\rho}^0$ as $\mathbb{H}(G_{\text{der}} \ltimes G_{\text{der}},\mathcal{E})$-modules, so $E_{y,\sigma,r,\rho}^0 = M_{y,\sigma,r,\rho}^0$ as $\mathbb{H}(G^o,L,\mathcal{E})$-modules. Together with [AMS2 Lemma 3.18 and (63)] this gives $E_{y,\sigma,r,\rho} = M_{y,\sigma,r,\rho}$.

(b) Notice that $Z_G(\sigma,y) = Z_Q(\sigma,y)$, so by [AMS1 Theorem 4.8.a] $\rho$ is a valid enhancement of the parameter $(\sigma,y)$ for $\mathbb{H}(Q,L,\mathcal{E})$.

By construction $\Re(\sigma_0)$ is strictly negative with respect to $PQ^0$. Now Lemma [A.3] says that we may apply [AMS2 Proposition 3.22]. That and part (a) yield

$$\mathbb{H}(G,L,\mathcal{E}) \otimes \mathbb{H}(G,L,\mathcal{E}) \otimes \mathbb{H}(G,L,\mathcal{E}) E_{y,\sigma,r,\rho}^0 = E_{y,\sigma,r,\rho}^0.$$  

Now apply [AMS2 Theorem 3.20.b].

(c) This follows from part (b) and the compatibility of $\text{IM}^*$ with parabolic induction, as is [AMS2 (81)].

(d) Write

$$M_{y,\sigma,r,\rho}^Q = M_{y,\sigma,-z_0,r,\rho}^Q \otimes \mathbb{C}_{z_0} = M_{y,\sigma,z_0-r,\rho}^Q \otimes \left(\mathbb{C}_{z_0} \otimes \mathbb{C}^{\Re(z_0)}\right)$$

as in the proof of part (a), with $Q$ in the role of $G$. By [AMS2 Theorem 3.25.b], $M_{y,\sigma,-z_0,r,\rho}^Q \otimes \mathbb{C}_{z_0}^{\Re(z_0)}$ is anti-tempered. The definition of $Q$ entails that $\Re(z_0) = \Re(\sigma_0)$, which we know is strictly negative. Hence

$$\text{IM}^*(M_{y,\sigma,r,\rho}^Q) = \text{IM}^*(M_{y,\sigma,z_0-r,\rho}^Q \otimes \mathbb{C}_{z_0}^{\Re(z_0)}) \otimes \mathbb{C}^{\Re(\sigma_0)},$$

where the right hand side is the twist of a tempered $\mathbb{H}(Q^0,L,\mathcal{E})$-module by the strictly positive character $-\Re(\sigma_0)$ of $S(Z(q)^*)$. By [AMS2 (68) and (80)]

(A.13) \quad $\text{IM}^*(M_{y,\sigma,r,\rho}^Q) = \text{IM}^*(\tau \times M_{y,\sigma,r,\rho}^Q) = \tau \times \text{IM}^*(M_{y,\sigma,r,\rho}^Q).$ \hfill$\Box$

We note that by [AMS2 Lemma 3.16] $\mathcal{O}(Z(q))$ acts on (A.13) by the characters $\gamma(-\Re(\sigma_0))$ with $\gamma \in \Gamma_{q^0}$. Since $\Gamma_{q^0}$ normalizes $PQ^0$, it preserves the strict positivity of $-\Re(\sigma_0)$. In this sense $\text{IM}^*(M_{y,\sigma,r,\rho}^Q)$ is essentially the twist of a tempered $\mathbb{H}(Q,L,\mathcal{E})$-module by a strictly positive central character.
Appendix B. Localization in equivariant cohomology

The fundamental localization theorem in equivariant (co)homology is [Lus3, Proposition 4.4]. It is analogous to theorems in equivariant K-theory [Seg, §4], [ChGi, §5.10] and in equivariant K-homology [KaLu, 1.3.k]. In [Lus3] it is proven for equivariant local systems \( \mathcal{L} \) on varieties \( X \) such that

\[ H_c^{\text{odd}}(X, \mathcal{L}^\vee) = 0. \tag{B.1} \]

During our investigations it transpired that the condition \( \text{(B.1)} \) is not always satisfied by \( (\hat{g}, \hat{L}) \) in the setting of [Lus3, §8.12] - on which important parts of [Lus3] and other papers rely. Fortunately, this can be repaired by relaxing the conditions of [Lus3, Proposition 4.4], as professor Lusztig kindly explained us.

**Proposition B.1.** Let \( G \) be a connected reductive complex group acting on an affine variety \( X \). Let \( M \) be a Levi subgroup of \( G \) (i.e. the connected centralizer of a semisimple element). Then the natural map

\[ H^*_M(\text{pt}) \otimes_{H^*_G(\text{pt})} H^*_G(X, \mathcal{L}) \longrightarrow H^*_M(X, \mathcal{L}) \]

is an isomorphism.

**Proof.** The proof of the analogous statement in equivariant K-homology [KaLu, 1.8.(a)] can be translated to equivariant cohomology. The crucial point is the Künneth formula in equivariant K-homology [KaLu, 1.3.(n3)], which is proven in [KaLu, §1.5–1.6].

The conditions about simple connectedness in [KaLu, §1] are only needed to ensure that the representation ring \( R(T) \) is a free module over \( R(G) \), for any maximal torus \( T \) of \( G \). In equivariant cohomology this translates to \( H^*_G(\text{pt}) \cong \mathcal{O}(g//G) \cong \mathcal{O}(t)^{W(G/T)} \), which is true for every connected reductive group \( G \). \( \square \)

For \( s \in G \) we consider the inclusion \( j: X^s \rightarrow X \).

**Proposition B.2.** In the setup of Proposition B.1, assume that \( s \in G \) is central (and hence semisimple). The map

\[ \text{id} \otimes j : \hat{H}^*_G(\text{pt}) \otimes_{\hat{H}^*_G(\text{pt})} \hat{H}^*_G(X^s, j_* \mathcal{L}) \longrightarrow \hat{H}^*_G(\text{pt}) \otimes \hat{H}^*_G(X, \mathcal{L}) \]

is an isomorphism.

**Proof.** Let \( T \subset G \) be a maximal torus, so \( s \in T \). The centrality of \( s \) and Chevalley’s theorem [Var, §4.9] entail that \( \hat{H}^*_T(\text{pt}) \) is a free module over \( \hat{H}^*_G(\text{pt}) \). By Proposition B.1

\[ \hat{H}^*_T(\text{pt}) \otimes_{\hat{H}^*_G(\text{pt})} \hat{H}^*_G(X, \mathcal{L}) \cong \hat{H}^*_T(\text{pt}) \otimes \hat{H}^*_G(X, \mathcal{L}) \]

and similarly with \( (X^s, j_* \mathcal{L}) \). In this way we reduce the issue from \( G \) to \( T \). That case is shown in [Lus3, Proposition 4.4.a]. \( \square \)
With Propositions 3.1 and 3.2 at hand, everything in Lusztig [Lus3, §4] can be carried out without assuming that certain odd cohomology groups vanish. In return, we have to assume that the involved groups are reductive, but that assumption can be lifted with Lusztig [Lus1, §1.h].

Problems with the condition (B.1) also entail that Lusztig [Lus3, 5.1.(b,c,d)] are not necessarily isomorphisms in the setting of Lusztig [Lus3 §8.12]. Professor Lusztig showed us that this can be overcome by rephrasing Lusztig [Lus3 Proposition 5.2] in the category $D_H \tilde{X}$ (instead of $D \tilde{X}$), see Lusztig [Lus7, # 121]. The upshot is that all the proofs about representations of graded Hecke algebras in Lusztig [Lus3 §8] can be fixed.

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