Cosmological Implications of a Non-Separable 5D Solution of the Vacuum Einstein Field Equations

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An exact class of solutions of the 5D vacuum Einstein field equations (EFEs) is obtained. The metric coefficients are found to be non-separable functions of time and the extra coordinate $l$ and the induced metric on $l = \text{constant}$ hypersurfaces has the form of a Friedmann-Robertson-Walker cosmology. The 5D manifold and 3D and 4D submanifolds are in general curved, which distinguishes this solution from previous ones in the literature. The singularity structure of the manifold is explored: some models in the class do not exhibit a big bang, while other exhibit a big bang and a big crunch. For the models with an initial singularity, the equation of state of the induced matter evolves from radiation like at early epochs to Milne-like at late times and the big bang manifests itself as a singular hypersurface in 5D. The projection of comoving 5D null geodesics onto the 4D submanifold is shown to be compatible with standard 4D comoving trajectories, while the expansion of 5D null congruences is shown to be in line with conventional notions of the Hubble expansion.

I. INTRODUCTION

The vacuum EFEs for spacetime plus an extra dimension are given in terms of the Ricci tensor by $R_{AB} = 0$ ($A, B = 0 - 3, 4$). These contain the field equations of general relativity, which in terms of the Einstein tensor and an induced energy-momentum tensor are $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ ($\alpha, \beta = 0 - 3$). The latter is obtained by a well-known technique. Mathematically it can depend on $g_{4\alpha}, g_{44}$ and derivatives of $g_{\alpha\beta}$ with respect to $x^4 = l$, while physically it can describe a perfect fluid with density $\rho$ and pressure $p$. If there is no dependency on $l$, the equation of state is that of radiation or ultrarelativistic particles, $p = \rho/3$. If there is dependency on $l$, a wide range is available for the equation of state. What are often referred to as the standard 5D cosmological models were found as a class of solutions of $R_{AB} = 0$ by Ponce de Leon. These solutions are separable in the time ($t$), space ($r, \theta, \phi$) and the extra coordinate ($l$). On the hypersurfaces $l = \text{constant}$ which we label $\Sigma_l$, they reduce to the standard Friedmann-Robertson-Walker (FRW) models of 4D cosmology with flat space sections ($k = 0$). The class depends on one dimensionless parameter which fixes the scale-factor $a$ for the dynamics and the equation of state for the matter. It includes the $a = t^{2/3}, p = 0$ Einstein-de Sitter solution for the late universe, and the $a = t^{1/2}, p = \rho/3$ radiation solution for the early universe. However, the Ponce de Leon solutions are not unique as the 5D analysis of the standard 4D FRW solutions.

In Section II, we obtain a non-separable solution of the scale factor. Sections III and IV are devoted to cosmological implications, i.e. singularities and geodesics respectively. Section V is for comments.

II. NON-SEPARABLE SOLUTION

A class of solutions which is mathematically different and physically reasonable has a 5D line element given by

$$dS^2 = \sigma \left\{ dt^2 - a^2 \left[ \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right] \right\} + \epsilon b^2 dl^2. \tag{1}$$

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Here \( k = \pm 1,0 \) describes the 3D curvature, \( d\Omega^2 \equiv d\theta^2 + \sin^2 \theta d\phi^2 \) describes the 2D spherical geometry, and \( \epsilon = \pm 1 \) describes the nature of the extra dimension. The functions \( \sigma = \sigma(l) \), \( a = a(t,l) \), \( b = b(t,l) \) are to be fixed by the 5D field equations. We have studied the latter, particularly with regard to the 4D properties of matter. Hereafter we employ \( e = G = 1 \) unit system, that is, \( L = M = T \).

A simple class of solutions of \( G_{AB} = 0 \) is obtained if we write \( \sigma = \text{constant} \equiv \sigma_0, \hat{a} \equiv \partial a/\partial l = f(l)b \) (here and henceforth, we use hats to denote \( \partial/\partial l \)). Then the scale factors for ordinary spacetime and the extra dimension are

\[
a = \sqrt{-F t^2 + gt + h},
\]

\[
b = \frac{-2\epsilon \sigma_0 f \hat{f} t^2 + \hat{g} t + \hat{h}}{2fa}.
\]

Here \( F \equiv \epsilon \sigma_0 f^2 + k \), \( g = g(l) \) and \( h = h(l) \) are functions, so including \( f = f(l) \) we have three such. There exists a relation between these, set by the field equations. It is

\[
h = -\frac{g^2 + \kappa}{4F},
\]

where \( \kappa \) is a constant with the physical dimensions of \((\text{length})^2\). In terms of this, the 3D scale factor (2) is given by

\[
a^2 = -\frac{[(2Ft - g)^2 + \kappa]}{4F}.
\]

The extra-dimensional scale-factor (3) is given by

\[
b = -\frac{[4\epsilon \sigma_0 f \hat{f} F t^2 - 2\hat{g} F^2 t + \hat{g} \hat{F} - \epsilon \sigma_0 f \hat{f}(g^2 + \kappa)]}{4\alpha_f F^2}.
\]

The new solutions have some properties in common with the Ponce de Leon cosmologies\(^4\) and some which are different. The Ponce de Leon solutions only exist for \( \epsilon = -1 \), but the new ones have \( \epsilon = \pm 1 \) so the extra dimension can be spacelike or timelike. Other (wave-like) solutions are known for \( \epsilon = \pm 1 \),\(^5\) but the new solutions open the way to testing the signature of the 5D manifold by observations of the 4D properties of matter as given by (8) and (9) below. The solution (6) is in general somewhat complicated, but we will not be much concerned with it because it is (5) which, via the Friedmann equations, determines the properties of matter. These for perfect fluid mean that Einstein’s equations read as usual

\[
G_{\alpha\beta} = 8\pi T_{\alpha\beta} = 8\pi[(\rho + p)u_\alpha u_\beta - pg_{\alpha\beta}].
\]

Here the 4-velocities \( u^\alpha = dx^\alpha/ds \) are defined in terms of the 4D interval \( s \), which is included in \( S \) of (1), and we have \( u^\alpha = (1,0,0,0) \). Then the density and pressure are given by

\[
\frac{8\pi \rho}{3} = -\frac{\epsilon f^2}{a^2} - \frac{\kappa}{4\sigma_0 a^4},
\]

\[
8\pi p = \frac{\epsilon f^2}{a^2} - \frac{\kappa}{4\sigma_0 a^4}.
\]

We recover FRW-like models on \( \Sigma_l \) that contain an exotic type of induced matter, with an equation of state which follows from (8) and (9),

\[
p = \frac{\rho}{3} \left[ 1 - \frac{8\epsilon \sigma_0 a^2 f^2}{(4\epsilon \sigma_0 a^2 f^2 + \kappa)} \right].
\]

This equation of state is manifestly dependent on time and completes the formal part of our analysis, which a fast computer package\(^6\) has confirmed. The form of these relations suggests a two-fluid model as used in solutions of straight general relativity. The first terms in (8), (9) by themselves imply \( (\rho + 3p) = 0 \), which is the 4D signature of matter with zero gravitational density.\(^1,7\) This kind of matter has been suggested as relevant to cosmic strings, i.e. open strings with sub-relativity transverse motions,\(^8\) and K-matter which existence mentions the possibility of a universe dominated by cosmic strings,\(^9\) to zero-point fields.
required by quantum theory\textsuperscript{10,11} and to extreme sources for the Reissner-Nordström metric which require that the material contents of the sphere has no effect on gravitational interactions at its centre.\textsuperscript{12} The second terms in (8), (9) by themselves imply \( p = \rho/3 \), which of course means photon-like matter. These identifications are supported by the behaviour of the scale-factor (2) or (5), where the latter may vary as \( t^{1/2} \) in the radiation model or as \( t \) in the empty (Milne) model. They become exact when in (8), (9) \( f \to 0, \kappa \to 0 \) respectively. Or on a given \( \Sigma_l \) hypersurface, we find in (2) that as \( t \to \infty \), \( a(t, l) \to \infty \) — provided that \( F < 0 \) — which implies in (10) that \( p \to -\rho/3 \). This is the equation of state of the Milne, or empty universe where the gravitational density of matter is zero. Hence, it might be understood that the 4D cosmologies on the \( \Sigma_l \) hypersurface are asymptotically empty as \( t \to \infty \). However, it should be noted that \( \rho, p \) of (8), (9) refer to the total density and pressure, and that any split is in general arbitrary in the absence of information about the particles which make up the matter. Some of the latter may, in principle, be of non-standard type, since \( g_{44} = \epsilon b^2(t, l) \) describes a 3D-homogeneous scalar-field.\textsuperscript{1}

This can manifest itself in 4D as a time-dependent cosmological “constant”, as required to harmonize astrophysical data on the age of the universe.\textsuperscript{13,14} A time-dependent scalar field generalizes the constant vacuum of Einstein’s theory, whose density and pressure are given in terms of the cosmological constant by \( \rho = -p = \Lambda/8\pi \). The positive \( \Lambda \) is concluded by revisiting the Tinsley diagram with the recent determinations of the Hubble constants.\textsuperscript{15}

\section*{III. SINGULARITIES}

The age of the universe is defined in 4D models as the time elapsed since the big bang, but the latter concept has to be treated carefully in 5D models. In solutions of the 5D field equations which are flat in 5D but contain a curved space which is singular in 4D, the big bang has to be identified as a defect of the geometry.\textsuperscript{1,4}

To analyze the early behaviour of the 4D cosmologies embedded in (1), we need to establish whether or not these models contain a big bang singularities. Related to this issue is the nature of the singularity structure of 5D manifold as related to the singularity structure of the 4D sub-manifolds. It is well known that the 5D Ponce de Leon cosmological metrics contain 4D \( \Sigma_l \) hypersurfaces that precisely mimic standard FRW cosmologies complete with a big bang singularity. However, the 5D manifold is flat \( R^{A B C D} = 0 \), which suggests that the 4D big bang is merely a consequence of the way in which the \( \Sigma_l \) hypersurfaces are embedded in 5D Minkowski space, or, equivalently, the choice of 5D coordinates. The Ponce de Leon solutions are flat in 3D, curved in 4D and (perhaps surprisingly) flat in 5D.\textsuperscript{1,16} That is, the Riemann-Christoffel tensors for 5D and 4D are \( R^{A B C D} = 0 \) and \( R^\alpha_{\beta \gamma \delta} \neq 0 \), so a flat manifold smoothly embeds a curved one (locally), like 3D Euclidean space embeds the 2D surface of the Earth. We wish to address the issue of whether or not the present manifold (1) is curved or flat, and whether or not it contains a genuine singularity.

A direct attack on the problem could entail the calculation of the 5D Kretschmann curvature invariant \( K \equiv R^{ABCD}R_{ABCD} \) for (1). The divergence of this quantity is usually interpreted as being indicative of a curvature singularity in the manifold.\textsuperscript{1,17} However, the calculation of \( K \) for (1) is difficult, even by computer. We can make some headway by considering a special case defined by:

\[ \epsilon = -1, \quad k = 0, \quad f(l) = 1/\sqrt{2\sigma_0}, \quad g(l) = l. \]  \hfill (11)

In this model we have

\[ a(t, l) = \sqrt{\frac{(t + l)^2 + \kappa}{2}}. \]  \hfill (12)

The associated curvature invariant is

\[ K = \frac{72\kappa^2}{\sigma_0^4[(t + l)^2 + \kappa]^4}. \]  \hfill (13)

The scale factor \( a(t, l) \) goes to zero and the Kretschmann scalar \( K \) becomes infinity along the hypersurfaces

\[ 0 = t + l \pm \sqrt{-\kappa}. \]  \hfill (14)
Clearly, there will be no curvature singularity for $\kappa > 0$. Therefore, at least one specific case, we find that the manifold (1) is curved in 5D and singular where the scale factor $a(t,l)$ vanishes, in sharp contrast with the Ponce de Leon solutions. For the present model, the density and pressure of (8) and (9) will diverge in general as $a \rightarrow 0$ along the hypersurfaces of (14).

It is interesting to note that $K = 0$ for $\kappa = 0$ in the model presented above. Indeed, an analysis of the general metric (1) via computer shows that $R^{A}_{BCD} = 0$ for $\kappa = 0$. In this eventuality, the scale factor (2) with (4) becomes

$$a(t, l) = \frac{g(l)t + 2h(l)}{2h^{1/2}(l)}, \quad (15)$$

which is linear in time, just like the scale factor for the Milne universe (indeed, the equation of state of the induced matter is $p = -\rho/3$). So the $\kappa = 0$ case entails induced matter with zero gravitational density, is curved in 4D and is flat in 5D. However, it bears repeating that when $\kappa \neq 0$, it is possible to have curved 5D solutions.

Now, by analogy with the standard FRW model, it is obvious that the 4D sub-manifolds on $\Sigma_1$ will be singular for times $t_*(l)$ such that $a(t_*,l) = 0$, an epoch commonly referred as the big bang. Solving $a(t_*,l) = 0$ in (5) for $t_*$ gives two solutions:

$$t_*^\pm(l) = \frac{g(l) \pm \sqrt{-\kappa}}{2F}. \quad (16)$$

Two things are apparent from this result. First, there can be no 4D big bang if $\kappa > 0$. This is in agreement with the special case presented above, where the scale factor can only vanish if $\kappa \leq 0$. Second, we see that if $\kappa < 0$ there are two separate singularities, or, in other words, two big bangs in $F < 0$.

These 4D events are located on the two 4D hypersurfaces defined by (16). In the neighborhood of the big bang, $a^2(t_*^\pm + \delta t, l) \approx a_*(l)\delta t$ where $a_*(l)$ is some function of $l$ at $t = t_*^\pm$. The equation of state of the cosmological matter is then

$$p = \frac{\rho}{3} \left[ \frac{\kappa - \dot{a}_*(l)}{\kappa + \dot{a}_*(l)} \right] = \frac{\rho}{3} - O(\delta t), \quad (17)$$

where $\dot{a}_*(l)$ is some other function of $l$. As $\delta t \to 0$, we recover $p = \rho/3$. So, near the big bang(s), the induced matter behaves like radiation while at late times it behaves like the matter in the Milne model.

Finally, we consider the case $F > 0$, which is guaranteed for $\epsilon = 1$ and $k = 0, 1$ since we must have $\sigma_0 > 0$. If we also have $\kappa < 0$, it is clear that on $\Sigma_1$, $a(t,l)$ is real only between $t_*^-$ and $t_*^+$. This is a cosmological model with both a big bang and a big crunch, similar to the standard $k = +1$ FRW metrics. There therefore exists a hierarchy of 4D cosmologies on the $\Sigma_1$ hypersurfaces. If $\kappa < 0$, then the issue of whether each of the cosmologies is forever expanding or destined to end in a big crunch is entirely determined by the value of $f(l)$ on that hypersurface. If $\kappa > 0$, the quantity inside the radical in the definition of $a(t,l)$ in (5) will be less than or equal to zero for all times, which makes $a(t,l)$ complex and has the effect of switching the signature of the $(x^1, x^2, x^3)$ coordinates from spacelike to timelike (this also happens for $t \not\in [t_*^-, t_*^+]$ if $\kappa < 0$). Table 1 summarizes all the case discussed in this section for both the global 5D geometry and the type of cosmology embedded on the $\Sigma_1$ hypersurfaces.

**Table 1**

**IV. GEODESICS**

The possibility the test particles move on higher dimensional geodesics has been studied by many authors. In particular, it has recently been demonstrated that particles that are massless 5D appear to be massive in 4D. In this section, we examine the 5D null and comoving geodesics of the metric (1) for the $\epsilon = -1$ case and identify the 4D GR limit.

We take $\epsilon = -1$ and define $\Sigma^2 \equiv \sigma_0$. Then, the metric (1) may be written as

$$dS^2 = [\Sigma dt + b(t,l)dl][\Sigma dt + b(t,l)dl] - \Sigma^2 a^2(t,l)ds_5^2, \quad (18)$$
where $\Sigma^2 a^2(t, l) ds_3^2$ represents the spatial 3-metric, with

$$ds_3^2 = \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2.$$  \hfill (19)

Now, since the terms in square brackets represent a two dimensional manifold, which must be conformally flat, it is in principle possible to find a coordinate transformation

$$\eta = \eta(t, l), \quad \xi = \xi(t, l)$$  \hfill (20)

that will cast the metric in the form

$$dS^2 = C(\eta, \xi)d\eta d\xi - \Sigma^2 a^2(\eta, \xi) ds_3^2.$$  \hfill (21)

Then, the null geodesics of the manifold that are spatially comoving ($ds_3^2 = 0$) are just the $d\eta = 0$ or $d\xi = 0$ trajectories. However, the transformation (20) is not immediately obvious. Therefore, we must be content with an indirect analysis of the properties of the 5D null-comoving paths.

Consider the vectors

$$k^A \partial_A = \frac{1}{\sqrt{2\Sigma}} \partial_t + \frac{1}{\sqrt{2b(t, l)}} \partial_l,$$  \hfill (22)

$$N^A \partial_A = \frac{1}{\sqrt{2\Sigma}} \partial_t - \frac{1}{\sqrt{2b(t, l)}} \partial_l.$$  \hfill (23)

They satisfy

$$k^A k_A = 0, \quad N^A N_A = 0, \quad k^A N_A = 1.$$  \hfill (24)

Both $k^A$ and $N^A$ are tangent to null geodesics and can be thought of as the principle and auxiliary vectors of a null congruence. They both share the same parameterization, defined by

$$\frac{dt}{d\lambda} = \frac{1}{\sqrt{2\Sigma}},$$

$$\frac{dl}{d\lambda} = \frac{1}{\sqrt{2b(t, l)}} \quad \text{(for } k^A),$$

$$= -\frac{1}{\sqrt{2b(t, l)}} \quad \text{(for } N^A).$$  \hfill (25)

In order to determine if $\lambda$ is an 5D affine parameter, we calculate

$$k^A \nabla_A k^B = \left[ \frac{1}{\sqrt{2\Sigma}} \frac{\partial}{\partial t} \ln b(t, l) \right] k^B,$$  \hfill (26)

with a similar expression for $N^A$. $\nabla_A$ is the 5D covariant derivative operator. This shows that $\lambda$ is not a 5D affine parameter. However, it is easy to see that the 4D projections of $k^A$ and $N^A$ onto $\Sigma$ satisfy the 4D geodesic equation for timelike geodesics. Therefore, $\lambda$ is the 4D proper time $\tau = \Sigma t$ comoving geodesics (up to a constant prefactor). This is an excellent example of how a null path in 5D can appear to be the trajectory of a massive particle in 4D.

We now explore the possibility that galaxies travel along 5D trajectories described by (22) or (23). The tangent space to the null vectors $k^A$ and $N^A$ is necessarily three dimensional and has the metric

$$q_{AB} = g_{AB} - k_A N_B - k_B N_A,$$  \hfill (27)

which gives

$$q_{AB} dx^A dx^B = -\Sigma^2 a^2(t, l) ds_3^2.$$  \hfill (28)

It is clear that the tangent space of $k^A$ and $N^A$ is equivalent to the $t =$ constant, $l =$ constant 3-surface of the 5D manifold. Hence, observers traveling along 5D null-comoving geodesics 4D comoving geodesics
(confined to $\Sigma_l$) share the same tangent space, which suggests that they will observe the world around them in the same manner. In particular, we can calculate the expansion $\Theta$ of a congruence of 5D geodesics with tangent vector $k^A$. For non-affinity parameterized geodesics, $\Theta$ is given by:

$$\Theta = \nabla^A k_A - N_B k^A \nabla_A k^B.$$  \hspace{1cm} (29)

Plugging in expression for $k^A$ and $N^A$, we get

$$\Theta = \frac{3}{a} \left( \frac{1}{\sqrt{2} \Sigma} \frac{\partial a}{\partial t} + \frac{1}{\sqrt{2} b(t,l)} \frac{\partial a}{\partial l} \right) = \frac{3}{a} \frac{da}{d\lambda}.$$  \hspace{1cm} (30)

Since the expansion scalar represents the fractional rate of change in the 3-volume $\delta V$ of the congruence, we see that $\delta V \sim a^3(\lambda)$ as expected. That is, an observer in a galaxy traveling along a 5D null-comoving geodesic will see the other galaxies receding away in a Hubble-like expansion with scale factor $a(\lambda)$ (recall that $\lambda = \sqrt{2} \tau$).

We are now in a position to identify the 4D limit of the theory. The geodesic paths described by $k^A$ and $N^A$ would be identical to 4D geodesics if $dl/d\tau = 0$, that is, they are confined to $\Sigma_l$ hypersurfaces. This will be true if $|dl/d\tau| = b^{-1}(t,l) \ll 1$, which can be physically interpreted as demanding that large changes in the 4D proper time $\tau$ be accompanied by small changes in the extra coordinate $l$. This can be accomplished in the $\epsilon = -1$ case by choosing $f(l)$ and $\sigma_0$ such that $|f(l)| \ll 1$ and $|f^2(l)\sigma_0 - k| > 0$, which ensures that $a(t,l)$ is real as $t \to \infty$. In terms of the toy model (11), we recover the 4D limit for $\Sigma \to \infty$.

V. COMMENTS

The new non-separable solution (2) gives a physical meaning of the coefficients $K_1$ and $K_2$ which appear in the solution of the scale factor. $^{22}$ That is, these constants and the coefficient of $t^2$ as well are originated in $f(l)$, $g(l)$ and $h(l)$ on the hypersurface $\Sigma_l$. This helps us to consider the special case (11).

The physical big bang in 4D is a hypersurface in 5D. This kind of behaviour has been observed in other 5D solutions, e.g. in a “wave-like” class of exact cosmological solutions which look like waves propagating in the fifth dimension. $^{23}$ For the example just considered, the 4D big bang is akin to a 5D shock wave. $^{24}$ It is different to what happens in the Ponce de Leon cosmologies, $^{4}$ because of the non-separable nature of the solutions. In general, 5D cosmologies where the 4D big bang is a geometrical effect have major implications for the early universe, notably in regard to particle masses during the inflationary epoch $^{25}$ and the thermalization of photons by the particle mass varying quadratically with the time during the subsequent radiation epoch. $^{26}$ 5D cosmologies imply the variability of the masses of all particles. $^{5}$ The precise nature of the transformation (20) for general choices of $f(l)$, $g(l)$ and $h(l)$ will naturally give us closed form solutions for the comoving null geodesic equation and enable us to study those cosmological phenomena in the early universe. These studies and a study of timelike 5D geodesics ($\epsilon = 1$) are left for the future work.

ACKNOWLEDGEMENTS

S.S.S. would like to thank NSERC of Canada for financial support. T.F. is grateful for the hospitality of the Department of Physics, University of Waterloo while staying on the research program of Dokkyo University.

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\[ \kappa > 0 \text{ and } R_{ABCD} \neq 0 \text{ in general} \]

no big bang

\[ \kappa = 0, \ R_{ABCD} = 0 \]

one big bang

\[ \kappa < 0 \text{ and } R_{ABCD} \neq 0 \text{ in general} \]

two big bangs

\[ \kappa > 0 \text{ and } R_{ABCD} \neq 0 \text{ in general} \]

\[ a(t, l) \text{ is complex} \]

\[ \kappa = 0, \ R_{ABCD} = 0 \]

\[ a(t, l) \text{ is complex} \]

\[ \kappa < 0 \text{ and } R_{ABCD} \neq 0 \text{ in general} \]

\[ \text{big bang and big crunch} \]

Table 1: Characteristics of the 5D Manifold and the 4D Cosmologies embedded on the \( \Sigma_l \) hypersurfaces