Infinite games with finite knowledge gaps

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Abstract

Inﬁnite games where several players seek to coordinate under imperfect information are believed to be intractable, unless the information is hierarchically ordered among the players.

We identify a class of games for which joint winning strategies can be constructed effectively without restricting the direction of information ﬂow. Instead, our condition requires that the players attain common knowledge about the actual state of the game over and over again along every play.

We show that it is decidable whether a given game satisﬁes the condition, and prove tight complexity bounds for the strategy synthesis problem under parity winning conditions.

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1. Introduction

Automated synthesis of systems that are correct by construction is a persistent ambition of computational engineering. One major challenge consists in controlling components that have only partial information about the global system state. Building on automata and game-theoretic foundations, signiﬁcant progress has been made towards synthesising ﬁnite-state components that interact with an uncontrollable environment either individually, or in coordination with other controllable components — provided the information they have about the global system is distributed hierarchically [1, 2]. For the general case, however, it was shown that the problem of coordinating two or more components of a distributed system with non-terminating executions are undecidable [3, 4].

The distributed synthesis problem can be formulated alternatively in terms of games between \(n\) players (the components) that move along the edges of a ﬁnite graph (the state-transitions of the global system) with imperfect information about the current position and the moves of the other players. The
outcome of a play is a possibly infinite path (system execution) determined by the joint actions of the players and moves of Nature (the uncontrollable environment). The players have a common winning condition: to form a path that corresponds to a correct execution with respect to the system specification, no matter how Nature moves. Thus, distributed synthesis under partial information corresponds to the problem of constructing a winning profile of finite-state strategies in a coordination game with imperfect information. This problem was shown to be undecidable by Peterson and Reif [5], already for the basic setting of two players with a reachability condition; infinitary winning conditions, which lead to higher degrees of undecidability, have been studied by Janin [6].

As in the case of distributed systems, decidable classes of coordination games rely on restrictions of the information flow according to an order among the players [7]. In their survey article on the complexity of multiplayer games [8], Azhar, Peterson, and Reif conclude that “[i]n general, multiplayer games of incomplete information can be undecidable, unless it the information is hierarchically arranged”.

The undecidability arguments cited above share a basic scenario: two players become uncertain about the current state of the game, due to moves of Nature. The structure of the game requires them to take into account not only their first-order uncertainty about the actual state, but also the higher-order uncertainty of one player about the knowledge of the other. Finally, the players can win only by attaining common knowledge about a property of the actual history, which requires them to maintain knowledge hierarchies of increasing height, as the play proceeds. The scenario is set up so that the uncertainty never vanishes and the knowledge hierarchies grow unboundedly, which leads to undecidability [9].

One systematic approach to characterising undecidable classes of problems that involve multiple players with imperfect information is the information fork criterion of Finkbeiner and Schewe [10]. Intuitively, an architecture has an information fork, if it allows for two players to reach a situation in which neither one can infer the observation received by the other player from his own observation. Under this condition, distributed architectures may allow the knowledge of players to diverge over an unbounded number of rounds, for certain specifications. Conversely, the authors show that all architectures that do not contain an information fork, admit an information ordering among the players and are hence decidable.

However, when applied to games, the information fork criterion yields only a coarse decidability classification, as the parametrisation over architectures has no natural correspondent in terms of games. Indeed, the set of game instances obtained by modelling a concrete family of distributed systems may be solvable, in spite of possible information forks in the architecture. This can occur, for instance, if the information flow affected by the fork is inessential to the players, or if the divergence between the knowledge of players vanishes after few rounds.

In this article, we identify a condition for the decidability of coordination games with imperfect information, which does not rely on the hierarchic arrangement of information. Similar to the information fork approach, our focus
is on situations in which the knowledge of players diverges. We use the term knowledge gap to describe an interval of rounds at which the players do not attain common knowledge about the actual state. Essentially, our condition requests that all knowledge gaps of a game are finite, or in other words, that the players attain common knowledge of the actual state of the game infinitely often, along every play. In this case we say that the game allows for recurring common knowledge of the state.

Questions about common knowledge in infinite runs are typically hard. In their study of the model checking complexity of epistemic temporal logics [11], van der Meyden and Shilov point out that already the problem of determining whether the players attain common knowledge about an atomic property is undecidable, even for synchronous models as we consider here. Indeed, it turns out that the question of whether the players attain common knowledge of the state at any history in a play is undecidable.

Surprisingly, the situation improves, when we look at the recurring formulation relevant for our characterisation: We are able to show that the question of whether the common-knowledge property holds infinitely often, on every play in a game is decidable with low complexity. This has several favourable consequences for solving infinite coordination games with imperfect information.

Our results are summarised as follows:

1. The question of whether a game for \( n \) players with imperfect information satisfies the condition of recurring common knowledge of the state is decidable in NLOGSPACE.

2. If a coordination game for \( n \) players with imperfect information satisfies the condition of recurring common knowledge of the state, then the problem of whether a joint winning strategy exists is decidable, and it is NEXPTIME-complete.

3. If there exists a joint winning strategy in a game with recurring common knowledge of the state, then there also exists a profile of finite-state strategies of exponential size, which can be synthesised in 2EXPSPACE.

The conclusions rely on three key arguments. Firstly, we show that under recurring common knowledge of the state, the intervals where the current state of the game is not common knowledge are bounded uniformly. This implies that the perfect-information tracking of such a game is finite, which yields decidability of the strategy synthesis problem as a consequence of a metatheorem from [12]. Secondly, we characterise recurring common knowledge in terms of recurring mutual knowledge. This allows us to establish tight complexity bounds. Finally, we prove that the problem of solving imperfect-information games with recurring common knowledge of the state can be reduced to solving parity games with perfect information, at a relatively low cost in terms of complexity.

The present article extends a previous report [13] presented at the Workshop on Strategic Reasoning.
2. Basic notions

2.1. Coordination games with imperfect information

Our game model is close to that of concurrent games [14]. There are \( n \) players, 1, \ldots, \( n \) and a distinguished agent called Nature. The grand coalition is the set \{1, \ldots, \( n \}\) of all players. We refer to a list of elements \( x = (x^i)_{1 \leq i \leq n} \), one for each player, as a profile.

For each player \( i \), we fix a set \( A^i \) of actions and a set \( B^i \) of observations, finite unless stated otherwise. The action space \( A \) consists of all action profiles. A game graph \( G = (V, \Delta, (\beta^i)_{1 \leq i \leq n}) \) consists of a finite set \( V \) of nodes called states, an edge relation \( \Delta \subseteq V \times A \times V \) representing simultaneous moves labelled by action profiles, and a profile of observation functions \( \beta^i : V \to B^i \) that label every state with an observation, for each player. We assume that every state has at least one outgoing move for every action profile, i.e., \( \Delta(v, a) \neq \emptyset \), for all \( v \in V \) and all \( a \in A \). For convenience, we include special sink states from which any outgoing move is a self loop.

Plays start at an initial state \( v_0 \in V \) known to all players and proceed in rounds where each player \( i \) chooses an action \( a^i \in A^i \), then Nature chooses a successor state \( v' \in \Delta(v, a) \) and each player \( i \) receives the observation \( \beta^i(v') \). Notice that the players are not informed about the action chosen by other players, nor about the state chosen by Nature.

Formally, a play is an infinite sequence \( \pi = v_0, a_1, v_1, a_2, v_2, \ldots \) alternating between positions and action profiles with \( (v_\ell, a_{\ell+1}, v_{\ell+1}) \in \Delta \), for all \( \ell \geq 0 \). A history is a prefix \( \pi = v_0, a_1, v_1, \ldots, a_\ell, v_\ell \) of a play; we refer to \( \ell \) as the length of the history. The observation function extends from states to histories and plays as \( \beta^i(\pi) = \beta^i(v_0), a_0^i, \beta^i(v_1), a_1^i, \ldots \). We say that two histories \( \pi, \pi' \) are indistinguishable to Player \( i \), and write \( \pi \sim^i \pi' \), if \( \beta^i(\pi) = \beta^i(\pi') \). This is an equivalence relation, and its classes are called the information sets of Player \( i \).

A game (graph) with perfect information is one where all information sets are singletons. In general, we do not assume that this is the case, so we speak about games with imperfect information.

When viewed as a distributed system in the taxonomy established by Halpern and Vardi [15], our game model corresponds to the class of synchronous systems with perfect recall. This is implicit in our definition of observation functions: the players are able to distinguish between histories of different length (synchronicity), and if two histories are indistinguishable for a player \( i \) at round \( \ell \), then so are they at any previous round \( r < \ell \) (perfect recall).

A strategy for Player \( i \) is a mapping \( s^i : V(AV)^* \to A^i \) from histories to actions such that \( s^i(\pi) = s^i(\pi') \), for any pair \( \pi \sim^i \pi' \) of indistinguishable histories. We denote the set of all strategies of Player \( i \) with \( S^i \) and the set of all strategy profiles by \( S \). A history or play \( \pi = v_0, a_1, v_1, \ldots \) follows the strategy \( s^i \in S^i \), if \( a_\ell = s^i(v_0, a_1, v_1, \ldots, a_\ell, v_\ell) \), for every \( \ell > 0 \). For the grand coalition, the play \( \pi \) follows a strategy profile \( s \in S \), if it follows all strategies \( s^i \). The set of possible outcomes of a strategy profile \( s \) is the set of plays that follow \( s \).

A general winning condition over a game graph \( G \) is a set \( W \subseteq (VA)^\omega \) of plays. A coordination game \( \mathcal{G} = (G, W) \) is described by a game graph and a
winning condition. We say that a play $\pi$ on $G$ is winning in $G$, if $\pi \in W$. A strategy profile $s$ is winning in $G$, if all its possible outcomes are so. In this case, we refer to $s$ as a joint winning strategy.

In this article, we restrict our attention to winning conditions represented by a colouring function $\gamma : V \to C$ with a finite range $C$ of colours, where the set of winning plays is represented by an $\omega$-regular set $W \subseteq C^\omega$ consisting of all plays $v_0, a_1, v_1, \ldots$ with $\gamma(v_0), \gamma(v_1), \cdots \in W$. Moreover, we assume that the colouring is observable to each player $i$, that is, $\beta^i(v) \neq \beta^i(v')$ whenever $\gamma(v) \neq \gamma(v')$.

The technical results are formulated in terms of parity winning conditions. A parity condition is described by a coloring function $\gamma : V \to C$ that maps every state to a number called priority; a play is winning if the least priority seen infinitely often along a play is even. Parity conditions provide a canonical form for $\omega$-regular winning conditions, in the sense that every automaton over $\omega$-words can be transformed into a parity automaton, over a larger state space, that recognises the same language. This allows to transfer results about games with parity condition to arbitrary regular conditions. For more background about such transformations, we refer the reader to the survey [16].

2.2. Domino tiling problems

As a tool for proving lower complexity bounds, we use domino tiling problems, which allow a transparent representation of combinatorial problems as an alternative to encoding machine models. Our exposition follows the presentation of Börger, Grädel, and Gurevich [17] and the survey of Van Emde Boas [18].

A domino system $D = (D, H, V)$ is described by a finite set of dominoes together with a horizontal and a vertical compatibility relation $H, V \in D \times D$. The generic domino tiling problem is to determine, for a given system $D$, whether copies of the dominoes can be arranged to cover a given space $Z \subseteq \mathbb{Z} \times \mathbb{Z}$ in the discrete grid, such that any two vertically or horizontally adjacent dominoes are compatible.

Here, we consider finite rectangular grids, and it is convenient to view them as a torus $\mathbb{Z}_\ell \times \mathbb{Z}_m$ by identifying the upper and the lower edge, as well as the left and the right one. Then, the concrete question is, whether there exists a tiling $\tau : Z \to D$ that assigns to every point $(x, y)$ of the grid $Z$ a domino $\tau(x, y) \in D$ such that:

- if $\tau(x, y) = d$ and $\tau(x + 1, y) = d'$ then $(d, d') \in H$, and
- if $\tau(x, y) = d$ and $\tau(x, y + 1) = d'$ then $(d, d') \in V$.

We consider three variants with additional border constraints. The Corridor tiling problem takes as input a domino system, together with a top and a bottom constraint $\text{top}, \text{bot} \in D^\ell$, and asks whether there exists a height $m$ such that the torus $Z = \mathbb{Z}_\ell \times \mathbb{Z}_m$ allows a tiling $\tau$ that additionally satisfies

- $\tau(i, 0) = \text{bot}_i$, for all $i = 0, \ldots, \ell - 1$, and
- $\tau(i, m - 1) = \text{top}_i$, for all $i = 0, \ldots, \ell - 1$. 

5
The Exponential Square tiling problem takes as input a tiling system together with a bottom constraint $bot \in D^t$, and asks whether the torus $\text{Torus } \mathbb{Z}_2^t \times \mathbb{Z}_2^t$ allows a tiling $\tau$ that satisfies the $bot$ constraint as above. Finally, the Quadrant tiling problem asks whether there exists a tiling of the positive quadrant $\mathbb{N} \times \mathbb{N}$.

We use the following results from the literature.

**Theorem 2.1** ([17, 18, 19]).

1. The Corridor tiling problem is PSpace-complete.
2. The Exponential Square tiling problem is NExpTime-complete.
3. The Quadrant tiling problem is undecidable.

2.3. Common knowledge

We use the notion of knowledge in the sense of having information. That Player $i$ knows proposition $\varphi$ at history $\pi$ should mean that, from the structure of the game graph and the sequence $\beta^i(\pi)$ of observations, it can be inferred that $\varphi$ holds. Specifically, we are interested in propositions about play histories. To formalise knowledge and uncertainty, we rely on the standard semantic model introduced by Aumann [20] and follow the treatment of Osborne and Rubinstein [21, Ch. 5]. For an extensive account on distributed knowledge in computational systems, we refer the reader to the book of Fagin, Halpern, Moses, and Vardi [22, Ch. 10, 11] and, as a standard reference for common knowledge, to the handbook article of Geanakoplos [23]. The enlightening article of [24] addresses foundational issues about the formalisation of common-knowledge.

Let us fix a game graph $G$ and denote by $\Omega$ the set of histories. The **possibility** correspondence $P^i : \Omega \rightarrow \mathcal{P}(\Omega)$ associates to each history $\pi$ its information set:

$$P^i(\pi) := \{ \pi' \in \Omega \mid \pi' \sim^i \pi \}, \text{ for every player } i.$$  

Thus, at history $\pi$, Player $i$ knows that the actual history is in $P^i(\pi)$, but he may be uncertain which one it is. The sets $P^i(\pi)$ form a partition of $\Omega$. Observe that each information set $P^i(\pi)$ consists of histories of the same length as $\pi$, hence it is finite.

An **event** is a subset $E \subseteq \Omega$. We say that $E$ occurs at history $\pi$, if $\pi \in E$. The **knowledge** operator $K^i : \mathcal{P}(\Omega) \rightarrow \mathcal{P}(\Omega)$ associates to every event $E$ the set of histories at which Player $i$ knows that $E$ occurs:

$$K^i(E) := \{ \pi \in \Omega \mid P^i(\pi) \subseteq E \}, \text{ for every player } i.$$  

Note that $K^i(E)$ is itself an event. If $\pi \in K^i(E)$, then (the occurrence of) $E$ is private knowledge of Player $i$ at $\pi$: For any $\pi' \sim^i \pi$, it holds that $\pi' \in E$. If, moreover, $\pi \in K^i(E)$, for all players $i$, we say that $E$ is mutual knowledge among the players. In this case, every player knows that $E$ occurs, although one player may be uncertain about whether another player knows it.

An event $E \subseteq \Omega$ is common knowledge at $\pi$, if for every sequence of histories $\pi_1, \ldots, \pi_k$ and players $i_1, \ldots, i_k$ such that $\pi \sim i_1 \pi_1 \sim i_2 \pi_2 \sim \cdots \sim i_k \pi_k$, it is the case...
that $\pi_k \in E$. In other words, $\pi$ belongs to the image of $E$ under every iteration $K^{i_1}(K^{i_2}(\ldots K^{i_k}(E)\ldots))$ of knowledge operators.

We will use an alternative characterisation in terms of shared information. An event $E$ is self-evident, if it is mutual knowledge among the players at every history in $E$, that is, if $E \subseteq K^i(E)$, for all players $i$. As the converse inclusion $K^i(E) \subseteq E$ always holds, this amounts to saying that $E$ is a simultaneous fixed point of the player’s knowledge operators. Self-evident sets allow an interpretation of common knowledge that coincides with the iterated-knowledge interpretation if the range of the possibility correspondence is finite, which is the case in our setting.

Theorem 2.2 ([20, 24]). An event $C \subseteq \Omega$ is common knowledge at history $\pi$, if and only if, there exists a self-evident event $E \subseteq C$ with $\pi \in E$.

3. Uncertainty and coordination

Under perfect information, coordination games are trivial to solve: any winning play can be turned into a winning strategy profile by prescribing the unique action following the play, for every player, at every history. This is possible because the players know the current history. Unlike the case considered in [25], where players choose their strategy independently, in our setting there is no risk of discoordination as a consequence of strategic uncertainty: the joint strategy is centrally planned, it is common knowledge among the players.

Under imperfect information, the problem is more complex, because players may not know where they are in the game. Strategies need to adjust to the uncertainty surrounding the current history induced by moves of Nature. The prescribed actions should be suitable in any contingency of the unobservable state of nature. In the interaction between a single player and Nature (or, equivalently, between two players with strictly conflicting interests), the knowledge relevant to this task is of first order: it regards only the set of possible contingencies, that is, the information set.

Yet, in games among several players, whether an action of one player is suitable or not at a particular history may depend on whether another player chooses a matching action at the same history. He, the one player, should thus be certain that she, the other player, would play the matching action, according to her commonly known strategy, which, however, responds to the observations she received on her own side. In other words, to avoid discoordination, he needs to know about what she knows about the current history. In contrast to the one-player or the two-player conflict case, here it is relevant to consider higher-order knowledge, i.e., knowledge about the knowledge of other players.

The role of knowledge is particularly obvious in coordination games where the actions of players must agree at every history. Formally, we call consensus game a coordination game over a common action alphabet, where every move $(v, a, w) \in \Delta$ in which two players $i \neq j$ disagree on their actions $a^i \neq a^j$, leads to a special sink state $\ominus$ from which no play is winning.
Figure 1 and 2 show examples of consensus games for two players, Player 1 (he) and 2 (she), with actions in and out; unlabelled arcs represent moves where both players choose in. The observations •, ◦, and × are indicated in split states: he sees the left side, she the right side. Apart from the unsafe sink ⊖ that also collects the moves along any unrepresented action profiles, there is a safe sink ⊕, from which all plays are winning. The dots on the bottom stand for an arbitrary continuation. For each of these games, we consider the situation where the actual history corresponds to the rightmost path, marked by thicker arrows leading to the out state where playing (out, out) would lead to an immediate win. Along the examples, we will discuss the question of whether the information that players have at the marked history allows them to infer with certainty that (out, out) is a safe action.

In Figure 1(a), at the marked history, Player 2 knows about being in the out state which requires the action (out, out) to win. However, Player 1 cannot distinguish the current history from the one along the left path (•|•)(◦|◦)(×|×), where out would be losing. So, it occurs that at the current state (out, out) is the right move, but Player 1 lacks first-order knowledge about it, whereas Player 2 has the relevant first-order knowledge, but not the second-order knowledge to ascertain that Player 1 will out. At best, the players could coordinate on (in, in), based on their common knowledge that this leads to continuing the game, and not straight to the ⊖ sink.

In Figure 1(b), both players know that they are in the out state. Nevertheless, Player 2 is uncertain about whether Player 1 knows it, because according to her observation, the current history may be (●●)( ○● )( ×| ×), in which case, just as in Figure 1(a), Player 1 would consider the history (●●)( ○● )( ×| ×) possible, and thus not out. So, both players have first-order knowledge about being at the out state, Player 1 even has the second-order knowledge that Player 2 knows it, and still they cannot coordinate with certainty, because Player 2 does not know that Player 1 knows it. Moreover, in Figure 2(a), both players know that
the play is at the out state and each of them knows that they both know it. But Player 1 regards it as possible that Player 2 observed • • ∗ ×, again raising the uncertainty of Figure 1(b). Here, the reason why the players cannot coordinate is that Player 1 does not know that Player 2 knows that Player 1 knows about being in an out state.

The argument can be lifted to arbitrary levels of the knowledge hierarchy. This is illustrated in Figure 2(b), where the loop around the observation (● ●) may be unravelled n times to obtain an instance where coordination on the winning action fails in spite of the players having mutual knowledge of order n about being in a state where the action is safe.

Indeed, the examples incorporate the coordinated attack problem introduced in the distributed-systems literature by Gray [26], and which is at the basis of extensive investigations on communication and coordination from different angles in computing science following the seminal paper of Halpern and Moses [27], and in game theory starting with the analysis of Rubinstein [28]. In our setting, it is Nature that induces nondeterministically a possible loss of one message between the two generals who would attack if, and only if, they had common knowledge of being in the out-state. The paradigm put forward by these analyses is that, in certain types of systems, if common knowledge is not attainable, then coordination is impossible in spite of an arbitrarily high level of mutual knowledge.

As our examples suggest, already in the simple case of consensus games with a safety condition (avoid the ◦-sink), coordination games with imperfect information are sensitive to common knowledge, and thus vulnerable to problems caused by its inapproximability through finite levels of mutual knowledge.

Still, one may argue that the problem of synthesising a joint winning strategy does not invoke the reasoning process of individual players; In the end, strategies only rely on first-order knowledge. Admittedly, this is true for the examples above. Nevertheless, in the remainder of this article we will show that the problem of attaining common knowledge of certain events – namely, the actual state of the play – is relevant for solving coordination games, in the following sense.

(i) There exists a family of games that admit a solution if, and only if, the players attain common knowledge about the state at a particular history.

(ii) Every game of infinite duration, where common knowledge of the actual state is attained infinitely often along every play, can be solved effectively.

4. Common knowledge of the state

As before, let Ω be the set of histories in a game graph. For a history π, we denote by State(π) ⊆ Ω the set of all histories that end at the same state as π. We say that the players have common knowledge of the state (CKS) at history π, if State(π) is common knowledge at π.
To develop familiarity with the notion, we first show that attaining cks at a particular history is equivalent to having a joint winning strategy in certain coordination games, more precisely, consensus games.

Towards this, let us consider a game graph $G$ and a history $\pi$ that starts at the initial state $v_0$ and ends at some state $z \in V$. For simplicity, we assume that all moves in $G$ are controlled by Nature i.e., the players have only one, trivial action. We construct a consensus game $G_{\pi}$ with a common action set $\{\text{in}, \text{out}\}$ as follows. The state space of $G_{\pi}$ consists of $G$ and a disjoint copy of the path $\pi$, which we denote by $\hat{\pi}$. All moves in $G$ and along $\hat{\pi}$ get inherited to $G_{\pi}$ and are assigned to the action $\text{in}$ for all players (in consensus). The initial state $\hat{v}_0$ of $\hat{\pi}$ is designated as initial state of $G_{\pi}$, and we allow a $\text{in}$ move to every successor of $v_0$ in $G$. There is an unsafe sink $\ominus$ and a safe sink $\oplus$; the winning condition requires to avoid $\ominus$. From both copies of the last state of $\pi$, i.e., $z$ in $G$ and $\hat{z}$ in $\hat{\pi}$, we allow $\text{out}$ moves to the safe sink $\oplus$. The action $\text{out}$ from any state except $z$ and $\hat{z}$, as well as the action $\text{in}$ from $\hat{z}$, lead to the unsafe sink $\ominus$.

**Proposition 4.1.** For a game graph $G$ and a history $\pi$, there exists a joint winning strategy in the game $G_{\pi}$ if, and only if, the players have common knowledge of the state at $\pi$ in $G$.

**Proof.** Consider a game graph $G$ and a history $\pi \in \Omega$ and let the game $G_{\pi}$ be constructed as above.

To prove that winning in $G_{\pi}$ implies common knowledge at $\pi$, assume that there exists a joint winning strategy $s$ in $G_{\pi}$ and let $C$ be the set of histories $\rho$ in $G$ such that $s(\rho) = \text{out}$. We argue that $C$ satisfies the conditions of Theorem 2.2 to witness that the players have cks at $\pi$:

- $C$ is a self-evident event, for each player $i$: for all histories $\rho \in C$, any
indistinguishable history \( \rho' \sim^i \rho \) must be assigned by \( s \) to the same action \( s(\rho') = s(\rho) = out \), which means \( \rho' \in C \).

- \( \pi \in C \): the action \( in \) is losing at \( \hat{\pi} \), hence we must have \( s(\hat{\pi}) = out \), and since \( \pi \) and \( \hat{\pi} \) are indistinguishable for every player, it follows that \( s(\pi) = out \).
- \( C \subseteq \text{State}(\pi) \): the action \( out \) is winning only from \( z \) or \( \hat{z} \). As we assumed that \( s \) is a winning strategy, all histories in \( C \) must therefore end at \( z \).

For the converse, assume that at the history \( \pi \) in \( G \) the players have common knowledge of \( \text{State}(\pi) \). We define a function that associates to any history \( \rho \) in \( G \) the action \( out \), if \( \text{State}(\pi) \) is common knowledge at \( \rho \), and \( in \) otherwise.

First, let us verify that \( s \) is a valid strategy on the game graph \( G \). Note that, if an event is common knowledge at a history \( \rho \), then it is also common knowledge at every indistinguishable history \( \rho' \sim^i \rho \), for every player \( i \). (Each history that is accessible from \( \rho' \) via a sequence of pairwise indistinguishable histories, is also accessible from \( \rho \) via the same sequence preceded by \( \rho \sim^i \rho' \)). In particular, whenever \( s(\rho) = out \), for a history \( \rho \), that is, when the event \( \text{State}(\pi) \) is common knowledge at \( \rho \), it is also common knowledge at every history \( \rho' \sim^i \rho \), hence \( s(\rho') = out = s(\rho) \). Consequently there is no pair \( \rho \sim^i \rho' \) with \( s(\rho) \neq s(\rho') \). To extend the strategy \( s \) on \( G \) to the game graph \( G_\pi \), we assign, to every history \( \rho \) in \( \hat{\pi} \), the unique action that \( s \) prescribes for the indistinguishable copy of \( \rho \) in \( G \).

Now, we claim that \( s \) is a winning strategy in \( G_\pi \). The unsafe sink \( \otimes \) can only be reached by taking a wrong move in one of the following two situations: either \( in \) at \( \hat{z} \), or \( out \) at a state different from \( z \) and \( \hat{z} \). The former situation is excluded, as \( s(\hat{\pi}) = s(\pi) = out \) must hold, because \( \hat{\pi} \) is indistinguishable from \( \pi \), and at \( \pi \) the players have common knowledge of \( \text{State}(\pi) \). The latter situation cannot occur either, because at all histories \( \rho \) with \( s(\rho) = out \), the players have common knowledge of \( \text{State}(\pi) \), so in particular, \( \rho \in \text{State}(\pi) \) (by definition of the knowledge operator, players can only know an event if it actually occurs). Thus, all plays that follow \( s \) are winning in \( G_\pi \).

Certainly, the complexity of coordination games with imperfect information cannot be explained entirely in terms of knowledge-related problems. For instance when we consider games of finite horizon, that is, games on acyclic graphs where every play reaches a sink after a bounded number of steps, we observe a complexity gap between the decision problems regarding winning strategies and those regarding common knowledge. As we will point out later, in Section 8.1, the former problem is \( \text{NExpTime} \)-complete. Here, we show that the latter problem is \( \text{PSPACE} \)-complete.

**Proposition 4.2.** Given a game graph and a history \( \pi \), the problem of deciding whether the players have common knowledge of the state at \( \pi \) is \( \text{PSPACE} \)-complete.
Proof. For membership, consider the procedure that takes a game graph and a history $\pi$ as input, and iterates the following loop: guess nondeterministically a player $i$ and a history $\rho \sim^i \pi$; reject, if $\pi$ and $\rho$ end at different states, otherwise, repeat with $\rho$ as the new value of $\pi$. As any pair of indistinguishable histories are of the same length, the procedure requires only linear space. It rejects if, and only if, there exists a sequence $i_1, \ldots, i_k$ of players and histories $\rho_1 \sim^{i_1} \rho_2 \sim^{i_2} \cdots \sim^{i_k} \rho_k$ with $\pi = \rho_1$ and $\rho_k \notin \text{State}(\pi)$, that is, if $\text{State}(\pi)$ is not common knowledge at $\pi$. Hence, we have a nondeterministic PSPACE procedure for deciding the complement problem, asking whether the players do not have CKS common knowledge of the state at the given history. Since nondeterministic and deterministic PSPACE are equivalent, it follows that the original problem can also be solved in PSPACE.

To prove hardness, we give a reduction from (the complement of) the corridor tiling problem. Given a domino system $D = (D, V, H)$ with a top and a bottom sequence $\text{top}, \text{bot} \in D^\ell$ we construct, in polynomial time, a game graph $G$ and a history $\pi$, such that the players have CKS at $\pi$ if, and only if, the domino-problem instance is negative, i.e., there does not exist a height $m \in \mathbb{N}$, such that the torus $Z(\ell, m)$ can be tiled correctly.

In $G$ there are two players with one, trivial action; Nature controls the moves. The players have a common observation set consisting of the dominoes in $D$. The states are arranged on $\ell$ levels. Each layer $k \in \{1, \ldots, \ell\}$ consists of three kinds of states:

(A) a state $d^k$, for every single domino $d \in D$,

(B) a state $(d, b)^k$ of each vertically compatible pair, $(d, b) \in V$, and

(C) a copy $\hat{d}^k$ of the domino $d$ at position $k$ in the bot sequence.

Additionally, there is an initial state $v^0$, and two sinks $\oplus$ and $\ominus$.

At the single-domino states $d^k$, and at the bot state $\hat{d}^k$, both players receive the observation $d$; at states $(d, b)^k$, the first player observes $d$ and the second player $b$; for the initial state and the two sinks, we fix one same observation arbitrarily, and assign it to all players.

From each layer $1 \leq k < \ell$, there are moves leading to the next layer as follows: for every single domino $d^k \to d^{k+1}$ whenever $(d, d') \in H$, for every pair $(d, b)^k \to (d', b')^{k+1}$ whenever both $(d, d')$ and $(b, b')$ are in $H$, and for the bot domino $\hat{d}^k \to \hat{d}^{k+1}$. From $v^0$ there are moves to all states on the first layer; every state on the last layer $\ell$ leads to $\oplus$, except for the copy of the bot domino $\hat{d}^\ell$, which leads to $\ominus$.

Note that any sequence $x = d_1, d_2, \ldots, d_\ell \in D^\ell$ that forms a horizontally consistent row in the tiling corresponds to a history $\pi_x = v^0, d_1^1, d_2^2, \ldots, d_\ell^\ell, \oplus$ in the game graph. Conversely, every history in $G$ that ends at a sink corresponds either to one possible row, in case Nature chooses a single domino or the bot copy in the first move, or to two rows, in case it chooses a pair. Moreover, a row $x$ can appear on top of a row $y = b_1, b_2, \ldots, b_\ell \in D^\ell$ in a tiling if,
and only if, there exists a history \( \rho \) in \( G \) such that \( \pi \sim^1 \rho \sim^2 \pi \), namely \( \rho = v^0, (d_1, b_1)^1, (d_2, b_2)^2, \ldots (d_\ell, b_\ell)^\ell, \oplus \).

Now, we claim that, at \( \pi_{\text{top}} \), the players have common knowledge of the state, if and only if there exists no correct corridor tiling for the given instance.

According to our observation, a correct tiling of the corridor from \( \text{top} \) to \( \text{bot} \) exists, if and only if, there exists a sequence of rows corresponding to histories \( \pi_1, \ldots, \pi_m \), and a sequence of witnessing histories \( \rho_1, \ldots, \rho_{m-1} \) such that

\[
\pi_{\text{top}} = \pi_1 \sim^1 \rho_1 \sim^2 \pi_2 \cdots \sim^1 \rho_{m-1} \sim^2 \pi_m = \pi_{\text{bot}}.
\]

However, the history along the copy of the \( \text{bot} \) sequence, \( \hat{\pi}_{\text{bot}} := v^0, d_1^1, \ldots, d_\ell^\ell, \ominus \) is indistinguishable from \( \pi_{\text{bot}} \), for both players. As these two histories end at different states, it follows that \( \text{State}(\pi_{\text{top}}) \) is not common knowledge at \( \pi_{\text{top}} \).

Conversely, if \( \text{State}(\pi_{\text{top}}) \) is not common knowledge at \( \pi_{\text{top}} \), there exists a sequence of pairwise indistinguishable histories that leads from \( \pi_{\text{top}} \) to \( \hat{\pi}_{\text{bot}} \), from which we can extract a correct corridor tiling.

This shows that the problem of deciding whether the players have cks at a given history is \text{Co-PSpace} hard, and as \text{PSPACE} is closed under complement, it is hard for \text{PSPACE}.

The argument above can be adapted to show that it is undecidable whether the players can ever attain cks in a given game.

**Proposition 4.3.** It is undecidable whether, for a given game graph, there exists a nontrivial history at which the players have common knowledge of the state.

**Proof.** We use a reduction from a variant of the Quadrant tiling problem, where the bottom row is initialised with one particular domino, that cannot appear in any other row and does not constrain the rest of the tiling.

Concretely, we consider domino systems \( D^w \) that contain one white domino \( w \) such that \( (w, w) \in H \) and, for every other domino \( d \in D \setminus \{w\} \), we have \( (w, d) \in V, (d, w) \notin V, \) and \( (w, d), (d, w) \notin H \). The Bottom Constrained Quadrant tiling problem takes as input a domino system \( D^w \) with a white tile \( w \) and asks, whether there exits a tiling \( \tau \) of \( \mathbb{N} \times \mathbb{N} \) with \( \tau(i, 0) = w \), for all \( i \in \mathbb{N} \). Clearly, a system \( D \) allows a Quadrant tiling if, and only if its extension \( D^w \) with a white tile allows a Bottom Constrained Quadrant tiling.

Given an instance \( D^w = (D, H, V) \) of the Bottom Constrained Quadrant tiling problem, we construct a game graph \( G \) for three players, such that there exists a history \( \pi \) in \( G \) at which the players attain cks if, and only if, there exists no correct tiling with \( D^w \) with only white tiles in the bottom row.

The construction is similar to the one in the proof of Proposition 4.2. To account for histories of arbitrary length, the layers are collapsed into one, with returning edges. Now, we have three players: the first two observe dominios in \( D \), as before, whereas the third one is mostly blind: He receives the same observation, say \( w \), in every round except at the sink nodes. Essentially, he can only count the length of the history.

In detail, the game graph is built as follows: There are inner states of three types: single-domino states \( d \) for each \( d \in D \), pair states \( (d, b) \) for each \( (d, b) \in V \),
and a copy \( \hat{w} \) of the white bottom domino. At single-domino states \( d \) the first two players receive the observation \( d \), in particular at \( d = w \); the also receive \( w \) at \( \hat{w} \). At states \((d, b)\), the first player observes \( d \) and the second player \( b \). The inner states are connected by moves \( d \rightarrow d' \) for every \((d, d') \in H\), and \((d, b) \rightarrow (d', b') \) whenever \((d, d') \) and \((b, b') \) are in \( H \). The bottom domino \( \hat{w} \) has a self loop. Again, there is an initial state \( v_0 \) and two sink nodes \( \oplus, \ominus \), all assigned to the same fresh observation, for the three players. To ensure that no player knows the current state unless a sink is reached, however, we duplicate the inner nodes, then add moves from \( v_0 \) to all inner states, and from every inner state to to \( \oplus \), except for (the two copies of) domino \( \hat{w} \), which lead to \( \ominus \).

This way, any sequence \( x = d_1, d_2, \ldots, d_\ell \in D_\ell \) that forms a horizontally consistent row in a tiling corresponds the duplicated history \( \pi_x = v_0, d_1, d_2, \ldots, d_\ell, \oplus \) in the game graph, and conversely, every history \( \pi \) of length \( \ell + 1 \) that ends at a sink corresponds to a horizontally consistent row of length \( \ell \). Moreover, by the same argument as in the proof of Proposition 4.2, the first two players have common knowledge of the state at \( \pi \), if the row corresponding to \( \pi \) does not appear (as prefix of an infinite row) in any correct tiling. Furthermore, the three players attain ckks at a history \( \pi \) of length \( \ell \), if and only if, the rectangle of size \( \ell \times 2^\ell \) at the origin of \( \mathbb{N} \times \mathbb{N} \) does not admit a correct tiling. Accordingly, there exists a history at which the players attain ckks, if, and only if, the Border Constrained Domino instance at the outset is negative.

5. Recurring common knowledge

Let us now turn to the use of common knowledge in infinite plays. We say that a play \( \pi \) allows for recurring common knowledge of the state (\( \omega \)-ckks), if there are infinitely many histories in \( \pi \), at which the players attain ckks. Likewise, we say that a game graph \( G \), or a game over \( G \), allows for \( \omega \)-ckks, if this is true for every play in \( G \).

A knowledge gap in a play \( \pi \) is an interval \( [\ell, t] \) with \( t \geq \ell > 0 \), such that the players do not attain ckks in \( \pi \) at any round in \( [\ell, t] \). The length of the gap is \( t - \ell + 1 \). Hence, a play allows for \( \omega \)-ckks, if the length of every knowledge gap in it is finite. The gap size (for ckks) of a play \( \pi \) is the least upper bound on the length of knowledge gaps in \( \pi \). Likewise, the gap size of a game (graph) is the least upper bound on the gap size of its plays.

Notice that for a play in an arbitrary game, the length of knowledge gaps may be unbounded, even if the play allows for \( \omega \)-ckks; its gap size is then infinite. In contrast, we show that, if a game allows for \( \omega \)-ckks, then there exists a uniform, finite bound on the length of the knowledge gaps in its plays.

**Proposition 5.1.** If a game graph allows for recurring common knowledge of the state, then its gap size is finite.

**Proof.** Let \( G \) be a game graph that allows for \( \omega \)-ckks. Without loss of generality, we assume that all states are reachable from the initial state \( v_0 \).

For each state \( v \in V \), we construct a tree \( T_v \) that may be understood as the unravelling of \( G \) from \( v \), up to common knowledge. The nodes of \( T_v \) correspond
to the histories in $G, v$ that have no strict, nontrivial prefix at which the players attain cks. The edges are labelled with action profiles and correspond to moves in $G$: for any history $\rho$ in the domain of $T_v$ at which the players do not attain cks, or for $\rho = v$, we have an edge $(\rho, a, paw)$ whenever $(u, a, w) \in \Delta$, for the last state $u$ of $\rho$. The leaves of $T_v$ thus correspond to the histories in $G, v$ at which the players attain cks for the first time (not counting the initial history). Finally, we associate to every history the observations of its last state.

Notice that each of the constructed trees has finite branching and all its paths are finite, according to our assumption that all plays allow for $\omega$-cks. Hence by König's lemma, every tree in the collection $(T_v)_{v \in V}$ is finite. We claim that the maximal height of a tree in this collection is an upper bound for the length of knowledge gaps in the plays of $G, v_0$.

To show this, we construct a game graph $G^{ck}$ over the disjoint union of all unravelling trees $T_v$, where we identify every leaf history with the root of the tree to its last state. Formally, in each tree $T_v$, we replace every edge $(\rho, a, \pi)$, where $\pi$ is a leaf history ending at $w$, with an edge $(\rho, a, w)$ to the root of the tree $T_w$. This induces a natural bijection $h$ between histories of $G, v_0$ and $G^{ck}, v_0$, which is also a bisimulation; clearly, the two game graphs have the same infinite unravelling. The bijection $h$ preserves cks: at a history $\pi$ in $G, v_0$, the players have cks, if, and only if, this holds for the image $h(\pi)$ in $G^{ck}, v_0$. Indeed, the only if part holds in an even stronger sense: if two histories $\pi$ and $\pi'$ end at the same state in $G, v_0$ but this is not common knowledge, then the (images of the) histories end at different states in $G^{ck}, v_0$. Observe that every history in $G^{ck}, v_0$ at which the players have cks ends at the root of some tree $T_v$. Accordingly, for every knowledge gap, i.e., every sequence of consecutive histories $\pi^1, \pi^2, \ldots, \pi^t$ in $G, v_0$ at which the players do not attain cks, the image $h(\pi^1), h(\pi^2), \ldots, h(\pi^t)$ describes a sequence of consecutive histories in $G^{ck}$ that never visit the root of any tree $T_v$. Hence, the length $t$ of the sequence is bounded by the maximal length of a path in any of the trees $(T_v)_{v \in V}$. This concludes the proof.

**Theorem 5.2.** For games that allow for recurring common knowledge of the state, with $\omega$-regular winning conditions,

(i) it is decidable whether there exists a joint winning strategy, and

(ii) if it is the case, there also exists a finite-state winning strategy, which can be constructed effectively.

**Proof.** Our argument relies on the tracking construction proposed in [12] that reduces the problem of solving coordination games with imperfect information for $n$ players against Nature to that of solving a zero-sum game for two players with perfect information. The construction proceeds via an unravelling process that generates epistemic models of the player’s information along the rounds of a play, and thus encapsulate their uncertainty.

An *epistemic model* for a game graph $G$ with the usual notation, is a Kripke structure $\mathcal{K} = (K, (Q_v)_{v \in V}, (\sim^i)_{1 \leq i \leq n})$ over a set $K$ of histories in $G$, equipped with predicates $Q_v$ designating the histories that end in state $v \in V$ and with
the players’ indistinguishability relations $\sim_i$. The construction keeps track of how the knowledge of players about the actual history is updated by generating, for each epistemic model $K$, a set of successor models along profiles $(a_k)_{k \in K}$ of actions $a_k \in A$ that are compatible with the players’ current knowledge, i.e. for every player $i$ and for all $k, k' \in K$ with $k \sim_i k'$, we have $a^i_k = a^i_{k'}$. This leads to a possibly disconnected epistemic model over the universe

$$K' = \{ka_kv \mid k \in K, k \in Q_v \text{ and } (w, a_k, v) \in \Delta\},$$

with $Q_v$ holding on all histories $ka_kv \in K'$, and $ka_kv \sim_i k'a_kv'$ if $k \sim_i k'$ in $K$ and $v \sim_i v'$ in $G$. By taking the connected components of this model under the coarsening $\sim := \bigcup_{i=1}^n \sim_i$, we obtain the set of epistemic successor models.

When starting from the trivial model that consists only of the initial state of the game $G$, and successively applying the update, one unravels a tree labelled with epistemic models, which corresponds to a two-player game $G'$ of perfect information where the strategies of one player translate into joint strategies of the grand coalition in $G$, and vice versa, such that a strategy in $G'$ is winning if and only if the corresponding joint strategy in $G$ is so.

The construction can be exploited algorithmically, if the perfect-information tracking of a game can be folded back into a finite game. A function $f$ is a homomorphism from an epistemic model $K$ to $K'$, if $Q_v(k) \Rightarrow Q_v(f(k))$ and $k \sim_i k' \Rightarrow f(k) \sim_i f(k')$. The main result of [12] shows that, whenever two nodes of the unravelling tree carry homomorphically equivalent labels, they can be identified without changing the (winning or losing) status of the game. This holds for all imperfect-information games with $\omega$-regular winning conditions that are observable. Consequently, the strategy synthesis problem is decidable for a class of such games, whenever the unravelling process of any game in the class is guaranteed to generate only finitely many epistemic models, up to homomorphic equivalence.

Let us now consider the tracking of a coordination game $G$ with observable $\omega$-regular winning condition that allows for $\omega$-cks. We claim that every history $\pi$ where the players have cks corresponds to an epistemic model that is homomorphically equivalent to a model over a single element labelled with the (commonly known) state at which the history ends. This is because, by our hypothesis, in the $\sim$-connected component of any epistemic model containing $\pi$, all histories end at the same state. On the other hand, when updating an epistemic model $K$, there are only finitely many successor models and each of them can be at most exponentially larger than $K$, for any fixed action space. Accordingly, the number of updating rounds in which the models can grow is bounded by the gap size of $G$, which is finite, according to Proposition 5.1.

Therefore, every game with $\omega$-cks has a finite tracking quotient under homomorphic equivalence. By [12], this implies that the winner determination problem is decidable for such games, and finite-state winning strategies can be effectively synthesised whenever the players have a joint winning strategy. \(\square\)
6. Characterisation via mutual knowledge

Theorem 4.3 leaves little hope for the approach of verifying whether a game allows for $\omega$-CKS by looking at individual plays. Fortunately, there is a way around this obstacle based on the weaker notion of mutual knowledge. Intuitively, the idea is as follows: A history at which the players do not attain CKS may be connected to long chains of indistinguishable histories that end at at the same state, but one chain will finally switch to a history that ends at a different state to witness the lack of common knowledge of the state. The history reached just before the state switched is a witness of special interest: it turns out that in a game that allows for $\omega$-CKS, we can always find a complete set of witnesses along one play. This will allows us to characterise games with recurring common knowledge of the state as those where mutual knowledge of the state is attained over and over again, along every play.

We say that the players attain mutual knowledge of the state ($\text{mks}$) at a history $\pi$ in a game, if $\text{State}(\pi)$ is mutual knowledge at $\pi$, that is, if all indistinguishable histories $\rho \sim^i \pi$ end at the same state as $\pi$, for all players $i$. A play $\pi$ allows for recurring mutual knowledge of the state ($\omega$-mks), if the players attain mks at infinitely many histories along $\pi$, and a game (graph) $G$ allows for $\omega$-mks, if all plays in $G$ do.

The link between common and mutual knowledge, is made by the notion of an ambiguous twin.

We say that two histories, or plays, $\pi$ and $\pi'$ are connected if there exists a sequence of histories $\pi_1, \ldots, \pi_k$ and a sequence of players $i_1, \ldots, i_{k+1}$ such that $\pi \sim^{i_1} \pi_1 \sim^{i_2} \cdots \sim^{i_k} \pi_k \sim^{i_{k+1}} \pi'$.

Two histories $\pi$ and $\pi'$ are twins, if they are connected and end at the same state. Clearly, the twin relation is an equivalence between histories.

A history $\pi$ is ambiguous, if the players do not attain mks at $\pi$, that is, if there exists an indistinguishable history $\rho \sim^i \pi$, for some player $i$, which ends at a different state. In this case, we call $\rho$ an ambiguity witness for $\pi$. Note that the players do not attain CKS at a history $\pi$ if, and only if, there exists an ambiguous twin of $\pi$.

Our goal is to show that every play in which the players do not attain recurring common knowledge of the state is witnessed by one where they do not attain mutual knowledge of state.

The following technical lemma asserts that ambiguity propagates backwards, via twins of preceding histories, as long as players do not have CKS.

**Lemma 6.1.** For any game, if a history $\pi$ at which the players do not attain common knowledge of the state has an ambiguous continuation $\pi_{av}$, then there exists an ambiguous twin $\pi'$ of $\pi$, such that $\pi'_{av}$ is an ambiguous twin of $\pi_{av}$.

**Proof.** For histories $\pi$ and $\pi_{av}$ as in the statement, consider an ambiguity witness $\rho_{cw} \sim^i \pi_{av}$ with $w \neq v$. Due to perfect recall, we have $\rho \sim^i \pi$. If $\pi$ and $\rho$ end at different states, $\pi$ already witnesses the claim (as an ambiguous twin of itself).
Else, if \( \pi \) and \( \rho \) end at the same state, since the players do not attain cks at \( \pi \), there exists an ambiguous twin \( \pi' \) of \( \pi \). As the histories \( \pi, \rho, \) and \( \pi' \) end at the same state, they allow the same continuations, and we have \( \piav \sim^i \text{pew} \sim^i \picw \) – the former by our assumption, the latter by definition of the observation function. From \( \piav \sim^i \picw \), it follows that \( \pi'av \sim^i \pi'cw \), again by definition of the observation function. Accordingly, \( \pi' \) is an ambiguous twin of \( \pi \) with an ambiguous successor \( \pi'av \), as required.

With the following lemma, we show that if the players never attain cks in a play, there exists a witnessing play in which all histories are ambiguous.

**Lemma 6.2.** For any game, if there exists a play \( \pi \) along which the players never attain common knowledge of the state (except for the initial state), then there also exists a play \( \pi' \) along which they never attain mutual knowledge of the state. Moreover, the plays \( \pi \) and \( \pi' \) are connected.

**Proof.** Let us fix a game \( G \) and consider an arbitrary play \( \pi \). We construct the tree \( T_\pi \) of hereditarily ambiguous twins of \( \pi \), that is, the subtree induced in the unravelling of \( G \) by the set of all histories \( \pi' \) that are connected to the history of the same length in \( \pi \), and in which every nontrivial history including \( \pi' \) is ambiguous.

By induction over the length \( \ell \) of histories \( \pi_\ell \) in \( \pi \) we show that, if the players do not attain cks along \( \pi_\ell \) (except for the trivial history), then every ambiguous twin \( \tau \) of \( \pi_\ell \) has a twin \( \tau' \) in \( T_\pi \).

For the base case with \( \ell = 1 \), if the players do not attain cks at the history \( \pi_1 = v_0, a_1, v_1 \), then there exist ambiguous twins, and all of them belong to \( T_\pi \), because the only preceding history is trivial.

For the induction step, suppose the statement holds for \( \ell \geq 1 \) and assume that players do not attain cks along \( \pi_\ell \) (except for the trivial history), then every ambiguous twin \( \tau \) of \( \pi_\ell \) has a twin \( \tau' \) in \( T_\pi \).

As \( \tau' \) ends at the same state as \( \tau' \) and \( \tau \), the continuation \( \tau''av \) is a valid history, and indeed a twin of \( \tauav \). Also \( \tau''av \sim^i \tau''cw \), because \( \tauav \sim^i \tau cw \). Accordingly, \( \tau''av \) is ambiguous and hence in \( T_\pi \), which completes the induction step.

In conclusion, for a play \( \pi \) in which the players do not attain cks at any round, there exist ambiguous twins in \( T_\pi \) for arbitrarily long histories of \( \pi \). As the tree \( T_\pi \) is finitely branching, it follows from König’s Lemma that it has an infinite path \( \pi' \). By construction, each nontrivial prefix of \( \pi' \) is an ambiguous history, and it is connected to the history of \( \pi \) of the same length. Hence, \( \pi' \) describes a play connected to \( \pi \) along which the players never attain mutual knowledge of the state.

We have now gathered all the arguments to prove our characterisation.
Theorem 6.3. A game allows for recurring common knowledge of the state if, and only if, it allows for recurring mutual knowledge of the state.

Proof. The only if direction is trivial: common knowledge of an event implies mutual knowledge.

For the converse, let us consider a game $G$ that does not allow for $\omega$-cks. Then, there exists a play $\pi$ in which the players attain $\text{cks}$ at some round $\ell$, but not at any later history. Accordingly, in the game $G, v$ starting from the (commonly known) state $v$ that is reached in round $\ell$ of $\pi$, there exists a play along which the player never attain $\text{cks}$, except for the initial state. Then, by Lemma 6.2, there exists a play $\pi'$ in $G, v$ along which the players never attain mutual knowledge of the state. Furthermore, in the play that follows $\pi$ for the first $\ell$ rounds and, upon reaching $v$, proceeds like $\pi'$, the players do not attain $\text{mks}$ at the infinitely many histories from round $\ell$ onwards. Hence, the game $G$ does not allow for $\omega$-mks, which concludes the proof.

Before turning to algorithmic questions, let us state the following corollary of arguments from the proofs of Lemma 6.2 and Theorem 6.3, which will be useful for bounding the gap size of games in Section 7.

Corollary 6.4. For any game $G$, if the players do not attain common knowledge of the state in a play $\pi$ along a sequence of rounds $\ell + 1, \ldots, \ell + t$, then there exists a play $\pi'$ in $G$ that is connected to $\pi$ and on which the players do not attain mutual knowledge of the state along the rounds $\ell + 1, \ldots, \ell + t$.

Proof. Let $G$ be a game graph and let $\pi$ be a play with the stated property, for some $\ell, t > 0$. We assume, without loss of generality, that the players attain $\text{cks}$ at round $\ell$ in $\pi$. For the game $G, v$ starting at the state $v$ reached in this round, we consider the continuation play $\tau$ of $\pi$ from round $\ell$ onwards, and construct the tree $T_\tau$ of hereditarily ambiguous twins of its histories, as in the proof of Lemma 6.2. The induction argument from the proof then shows that the history of length $t$ in $\tau$ has an ambiguous twin $\tau'$ in $T_\tau$. The histories of $\tau'$ from round 1 to $t$ are ambiguous and each of them is connected to the histories of the same length in $\tau$. Hence, the play $\pi'$ that follows $\pi$ for the first $\ell$ rounds, then proceeds like $\tau'$ for $t$ rounds, and then again follows $\pi$, satisfies the required properties: $\pi'$ is connected to $\pi$ and the players do not attain mutual knowledge of the state along the rounds $\ell + 1, \ldots, \ell + t$.

7. Recognising recurring common knowledge

An automaton for recognising the plays that allow for $\omega$-mks could easily be constructed using the powerset construction described by Reif [29] for solving one-player games with imperfect information. This would yield a PSPACE-procedure for deciding whether a game allow for $\omega$-mks and thus for $\omega$-cks. For a sharper complexity bound, we will show that ambiguity witnesses along a play can be represented efficiently, by a very thin tree, which allows to reduce the complexity to NLOGSPACE.
Let us again fix a game graph $G$ with the usual notation. A fork tree for a play $\pi$ is a subtree of the unravelling of $G$ that contains $\pi$ as a central branch and one more history at every nontrivial level. Concretely, the root of a fork tree corresponds to the initial state, and every level $\ell > 0$ consists of two histories $\pi_\ell$ and $\rho_\ell$ such that:

(i) $\pi_\ell$ is the history of $\pi$ in round $\ell$, and

(ii) $\rho_\ell \sim^i \pi_\ell$, for some player $i$.

A fork tree is complete, if it additionally satisfies, for every level $\ell$:

(iii) if $\pi_\ell$ is ambiguous, then $\pi_\ell$ and $\rho_\ell$ end at different states.

Thus, a complete fork tree consists of the play $\pi$ and a family of witnesses $\rho_\ell$, one for each of its ambiguous histories $\pi_\ell$. In case $\pi_\ell$ and $\rho_\ell$ end at different states, we say that the level $\ell$ is a doubleton, else it is a singleton.

**Lemma 7.1.** For every play, there exists a complete fork tree, in any game.

**Proof.** It is convenient to extend the notion of an ambiguity witness to knowledge gaps of histories. For a history $\pi$ and an interval $[\ell, t]$, we say that a history $\pi'$ is an ambiguity witness along the gap $[\ell, t]$, if $\pi$ and $\pi'$ have length at least $t$, and $\pi'_r$ is an ambiguity witness for $\pi_r$, for every round $\ell \leq r \leq t$. Likewise, for a play $\pi$, we say that a play $\pi'$ is an ambiguity witness from round $\ell$ onwards, if $\pi'_r$ is an ambiguity witness for $\pi_r$ for every $r \geq \ell$.

Now, consider an arbitrary game $G$ and a play $\pi$. By induction on the number of rounds $\ell$, we construct a finite or infinite sequence of trees $T_\ell$ that satisfy the conditions (i), (ii), and (iii)* has been constructed. To extend it to $T_{\ell+1}$, we look at the set $R$ of histories $\tau$ that prolong either $\pi_\ell$ or $\rho_\ell$, and are ambiguity witnesses for $\pi$ along the gap $[\ell, t]$ up to the length $t$ of $\tau$. Now we distinguish three cases. (1) If $R$ is empty, we set $\rho_{\ell+1} = \pi_{\ell+1}$, that is, $\ell + 1$ is a singleton level. (2) If $R$ is nonempty, but finite, we pick a history $\tau \in R$ of maximal length,
and add $\rho_{t+1} = \tau_{t+1}$ together with $\pi_{t+1}$ as a new level to $T_t$. (3) Finally, if $R$ is infinite, there exists an infinite play $\tau$ in $G$ such that all its histories from round $\ell$ onwards are in $R$. This follows from König’s Lemma, since the histories in $R$ form an infinite tree that is finitely branching (indeed, a subtree of the unravelling of $G$). In this case, we add the histories $\pi_\ell$ and $\rho_\ell := \tau_\ell$, for all levels $r > \ell$ and terminate the sequence with this infinite tree $T_{t+1}$.

In any case, $\rho_{t+1}$ is a history in $G$ and indistinguishable from $\pi_{t+1}$ which is also contained on level $\ell + 1$. Condition (iii)* holds trivially in case (3), we shall verify that it is also maintained in case (1) and (2).

For case (1) assume, towards a contradiction, that $R$ is empty and there exists a history $\pi'$ of length $\ell + 1$ that is an ambiguity witness for $\pi_{t+1}$. Since $R$ is empty, $\pi'_\ell$ reaches a different state than $\pi_\ell$. Hence, by perfect recall, $\pi'_\ell$ is an ambiguity witness for $\pi_\ell$ along the gap $[[\ell, \ell + 1]]$, which, by induction hypothesis, implies that there also exists such a witness that prolongs $\rho_\ell$ and is thus contained in $R$, in contradiction to our assumption that $R = \emptyset$.

For case (2), consider an history $\pi'$ of length $t > \ell$ that is an ambiguity witness for $\pi$ along the gap $[[\ell + 1, t]]$. We claim that there also exists a prolongation of $\rho_{t+1}$ with this property. There are two situations to distinguish: If $\pi'_\ell$ reaches the same state as $\pi_\ell$, then the history $\pi''_\ell$ that follows $\pi$ until round $\ell$ and then continues like $\pi'$ belongs to $R$, and is at most as long as the witness $\tau$ chosen to construct $\rho_{t+1}$. Hence, $\tau$ prolongs $\rho_{t+1}$ and is an ambiguity witness for $\pi$ along the gap $[[\ell + 1, t]]$. Otherwise, if $\pi'_\ell$ reaches a different state than $\pi_\ell$, then, by perfect recall, we have $\pi'_\ell \sim^* \pi_\ell$, for some player $i$, and hence $\pi'$ is already an ambiguity witness for $\pi$ along the gap $[[\ell + 1, t]]$. By induction hypothesis, there exists an ambiguity witness $\pi''_\ell$ for $\pi$ along the gap $[[\ell, t]]$ that prolongs $\rho_\ell$. Hence, $\pi''_\ell \in R$ and, as the history $\tau \in R$ chosen to construct $\rho_{t+1}$ is of maximal length, $\tau$ prolongs $\rho_{t+1}$ and is also an ambiguity witness for $\pi$ along the gap $[[\ell + 1, t]]$.

Clearly, each tree $T_\ell$ constructed along the induction satisfies the conditions of a complete fork tree and agrees with its successor $T_{\ell+1}$, up to level $\ell$. In conclusion, the sequence converges and the infinite tree $T$ obtained at the limit is a complete fork tree for $\pi$.

As fork trees have only two nodes at every level, they can be viewed as $\omega$-words to be recognised by word automata. We say that a sequence $\tau \in V(AV)^\omega$ is a fork sequence for $\pi$, if $\tau_0 = v_0$ and there exists a fork tree $T$ for $\pi$ such that $\tau_\ell$ is the last action-state pair of $\rho_\ell$ in $T$, for every $\ell > 0$. In the following we construct a nondeterministic automaton for a game, such that every accepting run corresponds to a fork sequence.

**Proposition 7.2.** For any game with $m$ states, the set of plays that do not allow for recurring mutual knowledge of the state is recognisable by a nondeterministic co-Büchi automaton with $m^2$ states.

**Proof.** Let us fix an arbitrary game graph $G$. We construct an $\omega$-word automaton $A$ with co-Büchi acceptance condition that recognises the set of histories $\pi$ in $G$, for which there exists a fork tree with only finitely many singleton levels.
To witness this, the automaton guesses non-deterministically a fork sequence $\tau$ for $\pi$ and accepts if the states at $\tau_\ell$ and $\pi_\ell$ are different, for all but finitely many rounds $\ell$.

The states of the automaton are pairs of game states from $V$: the first component keeps track of the input play, the second one is used for guessing the fork sequence $\tau$. The transition function ensures that the two components evolve according to the moves available in the game graph and that the current input symbol yields the same observation as the second component to some player $i$.

Concretely, the co-B"uchi automaton $A$ is defined over the input alphabet $A \times V$, with set of states $V \times V$, initial state $(v_0, v_0)$, and transitions from state $(u, u')$ on input $(a, v) \in V$ to state $(v, v')$ if

- $(u, a, v) \in \Delta$, and
- $\beta^i(v') = \beta^i(v)$, for some player $i$, and
- either $(u', a', v') \in \Delta$ or $(u, a', v') \in \Delta$, for an action $a' \in A$ with $a'^i = a^i$.

The set of final states is $Q \setminus \{(v, v) \mid v \in V\}$. The automaton accepts an infinite input word, if all states that occur infinitely often in a run are final.

We claim that an input word $\pi \in V(AV)^\omega$ is accepted by $A$ if, and only if, $\pi$ corresponds to a play in $G$, and the players never attain mutual knowledge of the state along $\pi$, from some round onwards.

For the if direction, consider a play $\pi$ along which the players never attain mutual knowledge of the state from some round onwards. By Lemma 7.1, there exists a complete fork tree $T$ for $\pi$, in which all but finitely many levels are doubletons. Let $\tau$ be the fork sequence associated to $T$. Then, the sequence $((\pi_\ell, \tau_\ell))_{\ell < \omega}$ describes a run of $A$ on input $\pi$ in which non-final states $(v, v)$ occur only at the finitely many positions $\ell$ corresponding to a singleton level in $T$, thus witnessing that $\pi$ is accepted.

For the converse, inputs that do not correspond to histories in $G$ are rejected, by construction of $A$. Furthermore, if an input word $\pi$ corresponds to a play with infinitely many histories $\pi_\ell$ at which the players attain mutual knowledge of the state, then every run of the automaton visits a non-final state whenever such an input prefix $\pi_\ell$ is read. As this occurs infinitely often, the input $\pi$ is rejected. \qed

**Theorem 7.3.** The problem of whether a game graph allows for recurring common knowledge, or equivalently, recurring mutual knowledge of the state is $\text{NLogSpace}$-complete.

**Proof.** According to the Characterisation Theorem 6.3, a game graph $G$ allows for recurring common knowledge of the state if, and only if, it allows for recurring mutual knowledge of the state. Our problem thus reduces to checking whether the language recognised by the co-Buechi automaton $A$ constructed for $G$ in Proposition 7.2 is non-empty. The non-emptiness test for a co-Buchi automaton,
or equivalently, the emptiness test for a Büchi automaton, is in \( \text{NLogSpace} \), as shown by Vardi and Wolper [30].

Concretely, a nondeterministic procedure can guess a run of \( A \) that leads to a cycle included in the set of final states. This requires only pointers to three states of the automaton: two for the current transition and one for storing a state to verify that a cycle is formed. As each state of the automaton is formed by two states of the game, the overall space requirement is logarithmic in the size of the game graph \( G \). Accordingly, the problem of determining whether a game graph allows for common knowledge of state is in \( \text{NLogSpace} \).

Hardness for \( \text{NLogSpace} \) follows via a straightforward reduction from directed graph acyclicity, shown to be \( \text{NLogSpace} \)-hard by Jones in [31]: For a directed graph \( G \), we construct a game graph \( G' \) for one player by taking two disjoint copies of \( G \) and assigning all non-terminal nodes with the same observation; each terminal node is assigned with a distinct observation and equipped with a self-loop. Finally, we add a fresh initial state to \( G' \), with moves to all other states. Clearly, the game graph \( G' \) can be constructed using logarithmic space, and the player has recurring (mutual, common) knowledge of the state in \( G' \) if, and only if, the directed graph \( G \) is acyclic.

This shows that the problem of determining whether a game graph allows for common knowledge of the state, or equivalently, for mutual knowledge of the state, is \( \text{NLogSpace} \)-complete.

**Theorem 7.4.** The gap size of any game with \( m \) states that allows for recurring common knowledge of the state is bounded by \( m^2 \).

**Proof.** Consider a game \( G \) with \( m \) states that allows for \( \omega \)-CKS. Towards a contradiction, suppose that in \( G \) there exists a play with gap size greater than \( m^2 \), that is, the players do not attain \( \text{CKS} \) along a sequence of consecutive rounds \( r, \ldots, r + m^2 \), for some \( r \). Due to Corollary 6.4, there also exists a play \( \pi \) in \( G \) such that the players do not attain \( \text{MKS} \) in \( \pi \) along these rounds. Let \( T \) be a complete fork tree for \( \pi \), according to Lemma 7.1, and let \( \tau \) be the associated fork sequence.

As \( G \) allows for \( \omega \)-MKS, the automaton \( A \) constructed in Proposition 7.2 is empty, in particular it rejects the run on \( \pi \) described by \( (\pi, \tau) \). But \( A \) has at most \( m^2 \) states, so there must be a cycle in the transition graph that is visited by run, say from position \( \ell \geq r \) to \( t \leq r + m^2 \). Along the interval \([\ell, t]\), the players do not attain \( \text{MKS} \) in \( \pi \), therefore the corresponding levels in the fork tree \( T \) are doubletons, and the states on the cycle visited in the run \( (\pi, \tau) \) from position \( \ell \) to \( t \) are final.

Consider now the sequences \( \pi' \) and \( \tau' \) that follow \( \pi \) and \( \tau \), respectively, until position \( t \) and then loop from \( \ell \) to \( t \) forever. Then, the pair \( (\pi', \tau') \) describes an run in \( A \) that eventually cycles through final states, hence, the input \( \pi' \) is accepted. But this means that \( \pi' \) is a play in \( G \) that does not allow for \( \omega \)-MKS, in contradiction to our assumption that all plays in \( G \) allow for \( \omega \)-CKS.

We observe that the quadratic bound on the gap size is tight. Consider, for instance, the one-player game graph \( G_m \) depicted in Figure 3, for an arbitrary.
number $m > 1$. There is only one bit of uncertainty induced by the choice of Nature at the initial state, where it can either move up, into the cycle with $m - 1$ white states followed by a black one, or down, to the path consisting of $m$ white states with selfloops, each followed by a black state, except for the last one which leads to the black state on the cycle. Consider the play $\pi$ where Nature moves into the cycle (and stays there forever). Along $\pi$, every nontrivial history up to round $m^2$ is indistinguishable from the one where Nature moves initially down to the path and loops on each white state precisely $m - 1$ times. For the first $m^2$ rounds in $\pi$, the player does therefore not know the current state, which means that the gap size of the game is at least $m^2$. On the other hand, notice that all histories that are distinguishable from $\pi$ are non-ambiguous, and that from round $m^2 + 1$ onwards, any history that is indistinguishable from $\pi$ leads to the same state as $\pi$ itself. Accordingly the game graph $G_m$ with $3m$ states allows for $\omega$-cks and its gap size is $m^2$.

One consequence of Theorem 7.4 is that the set of histories at which the players attain cks in a game that allows for $\omega$-cks is regular. Since the knowledge hierarchy for a game of size $m$ collapses at level $m^2$, common knowledge coincides with mutual knowledge of this level. An automaton that recognises the set of histories at which the players attain MKS of a fixed level for a game can be constructed by determinising and complementing the automaton constructed in Proposition 7.2.

However, for arbitrary games, there exists no automaton that recognises the histories at which the players attain cks. This follows immediately from the undecidability result in Theorem 4.3.

**Observation 7.5.** In arbitrary games, the set of histories along which the players attain cks is not regular.

### 8. Strategy synthesis

#### 8.1. Solving coordination games

We are now ready to establish complexity bounds for the basic algorithmic questions on games with recurring common knowledge of the state.

**Theorem 8.1.** For games that allow for recurring common knowledge of the state, with parity winning conditions,
(i) the problem of deciding whether there exists a joint winning strategy is
\text{NExpTime}-complete;

(ii) if joint winning strategies exist, there also exists a winning profile of finite-
state strategies of at most exponential size, which can be synthesised in
\text{2-ExpTime}.

The lower bound for the decision problem (i), follows from \text{NExpTime}-
hardness of the corresponding problem for two players over acyclic games, that
is games in which no state, except for the sinks, can be repeated in any play. A
detailed reduction from Exponential Square tiling problem to the decentralised
planning in partially observable Markov decision processes was presented by
Bernstein, Zilberstein, and Immerman in [32]. The same argument works for
our model of coordination games. The key ingredient is a construction that
allows Nature to send to each of the two players a pair of coordinates \((x,y)\)
and \((x',y')\) in binary encoding over \(\ell\) bits, such that the pairs are either equal,
or they differ by one one coordinate. Then, each player needs to produce one
domino, and the structure of the game verifies that the pair is compatible with
the relative position of the sent coordinates. If a tiling exists, the strategy
to produce the domino placed at the received coordinates guarantees a joint
win. Conversely, any winning strategy can be turned into a correct tiling. The
construction can be done in linear time in the size of the game.

For the upper bound and the strategy-construction procedure, it would be
inconvenient to rely on the tracking construction used to prove the decidability
Theorem 5.2, as the number of epistemic structures (over histories of quadratic
length that are relevant here) is already doubly exponential in the size of the
game graph. Instead, we introduce an auxiliary representation of the game that
also makes common-knowledge histories explicit, but is only simply exponential
in the size of the input game graph.

8.2. The abridged game

For the proof in the reminder of the section, let us fix a coordination game
\(G = (G, \gamma)\) for \(n\) players over a game graph \(G\) that allows for \(\omega\)-cks,
with a parity condition over a set of priorities \(C = \{1, \ldots, |C|\}\) described by the
colouring function \(\gamma : V \to C\). Recall that a play is winning under the parity
condition, if the least priority seen infinitely often along a play is even, and that
the priorities are assumed to be observable: if \(\gamma(v) \neq \gamma(v')\), then \(\beta_i(v) \neq \beta_i(v')\),
for all states \(v, v' \in V\) and all players \(i\).

The \textit{abridged} game \(\hat{G}\) of \(G\) is a game with perfect information for one player
against Nature. Intuitively, \(\hat{G}\) is obtained by contracting knowledge gaps and
recording only the most significant priority seen between two consecutive histo-
ries where the players attain cks.

Concretely, the states of the abridged game graph \(\hat{G}\) are pairs \((v, c)\) of states
\(v \in V\) and priorities \(c \in C\); for convenience, we also include a sink \(\odot\). We shall
refer to the the states of \(\hat{G}\) as \textit{positions}, to avoid confusion with the ones of \(G\).
The initial position \((v_0, |C|)\) corresponds to the initial state of \(G\) labelled with
the least significant priority. The set of actions consists of all nonempty subsets $U \subseteq V \times C$ of positions. The player has perfect information, so the observation function is the identity on $V \times C$.

To define the moves, we look at the unravelling $G^{ck}$ up to common knowledge of the game graph $G$, as constructed in the proof of Proposition 5.1. Recall that $G^{ck}$ is built from a disjoint collection of trees $(T_v)_{v \in V}$, which are then connected by identifying all leaves with the corresponding roots. For every state $v \in V$ and any strategy $t$ over the tree component $T_v$ of $G^{ck}$, we define the set $ouc_v(t)$ of pairs $(u, d) \in V \times C$, for which there exists a history $\tau$ in $T_v$ that follows $t$, such that $\tau$ ends at $u$, and the most significant priority that occurs along $\tau$ is $d$. Now, the set of available moves is defined as follows. For an action $U \subseteq V \times C$ there are moves from a position $(v, c)$ to every position $(u, d) \in U$, if there exists a joint strategy $t$ in $T_v$ with $ouc_v(t) = U$. Otherwise, the action leads to the $\ominus$-sink. Notice that the moves depend only on the first component of the position, that is, on the state and not on the priority.

At last, we define a parity condition on $G$, by assigning to every position $(v, c) \in V \times C$ the priority $c$.

The plays of $G$ and $\hat{G}$ are related via their summaries. Intuitively, this is the sequence of states reached when the players attain cks in a play, together with the most significant priority seen along the last knowledge gap. More precisely, for a play $\pi = v_0, a_1, v_1, \ldots$ in $G$, we look at the subsequence of rounds $t_0, t_1, t_2, \ldots$ such that, for all $\ell \geq 0$, the players attain cks at round $t_\ell$ in $\pi$, but not at any round $t$ in between $t_\ell < t < t_{\ell+1}$. Next, we associate to each index $t > 0$, the most significant colour that occurred in the gap between $t_\ell$ and $t_{\ell+1}$, setting $c_{t+1} := \min\{\gamma(v_t) : t_\ell < t < t_{\ell+1}\}$. Now, the summary of $\pi$ as the sequence $[\pi] := v_0, (v_1, c_1), (v_2, c_2), \ldots$. Notice that for every play $\pi$ in $G$, the summary $[\pi]$ corresponds to a sequence of states in $\hat{G}$, which is infinite, since we assume that $\pi$ allows for $\omega$-cks.

The notion of summary is defined analogously for histories, and it also applies to plays $\hat{\pi}$ in $\hat{G}$. Indeed, $[\pi]$ is obtained simply by dropping the actions in $\hat{\pi}$. We say that a play $\pi$ in $G$ matches a play $\hat{\pi}$ in $\hat{G}$, if they have the same summary: $[\hat{\pi}] = [\pi]$.

The winning or losing status is preserved among matching plays.

**Lemma 8.2.** If a play $\pi$ of $G$ matches a play $\hat{\pi}$ of $\hat{G}$, then $\pi$ is winning if, and only if, $\hat{\pi}$ is winning.

**Proof.** Let $c$ be the least priority that appears infinitely often in $\pi$. As each knowledge gap in $\pi$ is finite, $c$ appears in infinitely many knowledge gaps in $\pi$, hence it is recorded infinitely often in the summary $[\pi]$. Conversely, all priorities that appear infinitely often in the summary $[\pi]$, also appear infinitely often in $\pi$, so $c$ is minimal among them. In conclusion, the least priority appearing infinitely in the summaries $[\pi] = [\hat{\pi}]$ is the same as in the plays $\pi$ and $\hat{\pi}$. \qed

### 8.3. Reduction to parity games with perfect information

To use results from the standard literature on parity games, it is convenient to view the abridged game $\hat{G}$ formally as a turn-based game between two players,
Coordinator and Nature. In contrast to before, we shall hence regard Nature as an actual player with proper positions, moves, and strategies.

Towards this, we view the game graph \( \hat{G} \) as a bipartite graph, with one partition \( V \times C \) controlled by Coordinator, and a second one formed by the nonempty subsets of \( V \times C \), controlled by Nature. The initial position \((v_0, |C|)\) is unchanged. Coordinator can move from every position \((v, c) \in V \times C\) to a position \(U \subseteq V \times C\), if \(U = \text{outcome}_v(t)\) for some joint strategy \(t\) on \(T_v\), whereas Nature can move from every position \(U \subseteq V \times C\) to any element \((u, d) \in U\). The new positions from \(U \subseteq V \times C\) receive the least significant priority \(|C|\), whereas position \((v, c) \in V \times C\) have priority \(c\), as before.

A fundamental result about parity games is that they enjoy positional determinacy. A strategy is positional, if the choice prescribed at a history \(\pi\) depends only on the last position in \(\pi\). The following theorem was first proved by Emerson and Jutla [33], a comprehensive exposition can be found in the survey of Zielonka in [34].

**Theorem 8.3** ([33]). For every parity game with perfect information, one of the two players has a positional winning strategy.

For our setting, positional determinacy means that in the abridged game \( \hat{G} \), either Coordinator or Nature has a winning strategy defined on the set of positions. This yields witnesses of manageable size for determining which player the abridged game.

In the following, we argue that positional strategies for the abridged game \( \hat{G} \) can be translated effectively into strategies on \( G \), such that the resulting plays match in the sense of Lemma 8.2.

**Proposition 8.4.** Let \( G \) be a game graph that allows for \( \omega \)-cks, and let \( \hat{G} \) be the graph of the abridged game.

(i) For every positional Coordinator strategy \( \hat{s} \) in \( \hat{G} \), we can effectively construct a strategy profile \( s \) for the coalition in \( G \) such that, for every play \( \pi \) that follows \( s \), there exists a matching play \( \hat{\pi} \) that follows \( \hat{s} \).

(ii) For every positional Nature strategy \( \hat{r} \) in \( \hat{G} \), and every strategy profile \( s \) for the coalition in \( G \), there exists a play \( \pi \) in \( G \) that follows \( s \) with a matching play \( \hat{\pi} \) that follows \( \hat{r} \).

**Proof.** (i) Let \( \hat{s} : V \times C \to 2^{V \times C} \) be a positional strategy for Coordinator in the abridged game \( \hat{G} \). We construct a strategy \( s \) for the unravelling \( G^{\text{ck}} \) of \( G \) up to common knowledge. As the two game graphs have the same unravelling, \( s \) is also a strategy for \( \hat{G} \).

Towards this, we consider, for each state \( v \in V \), the tree component \( T_v \) of \( G^{\text{ck}} \) separately. Here, we look at the set \( U := \hat{s}(v) \) and pick a joint strategy \( t_v \) on \( T_v \) with \( \text{outcome}_v(t_v) = U \). Now, for every history \( \pi \) that ends in \( T_v \), we take the suffix \( \pi_v \) contained in \( T_v \), that is, we forget the prefix history up to entering the tree, and set \( s(\pi) = t_v(\pi_v) \).
With $s$ constructed this way, every following play $\pi$ in $G$ has the same summary $[\pi] = v_0,(v_1,c_1),(v_2,c_2),\ldots$ as the play $\hat{\pi} = v_0,a_1,(v_1,c_1),a_2,(v_2,c_2),\ldots$ in $\hat{G}$ with actions $a_\ell = \hat{s}(v_\ell,c_\ell)$. Hence, $\hat{\pi}$ follows $\hat{s}$ and matches $\pi$, as required.

(ii) For the converse, let $\hat{r} : 2^V \times C \rightarrow V \times C$ be a positional strategy for Nature in $\hat{G}$ and let $s$ be an arbitrary strategy for Coordinator in $G$. We construct a pair of plays $\pi$ in $G$, and $\hat{\pi}$ in $\hat{G}$ with the desired properties.

The construction is by induction along the number of knowledge gaps in $\pi$:

For every $\ell$, we construct a history $\pi_\ell$ in $G$ with $\ell$ knowledge gaps that follows $s$ and ends at some state $v$, where the players attain cks. At the same time, we construct a matching history $\hat{\pi}_\ell$ that follows $\hat{r}$ and ends at a position $(v,c)$ in $\hat{G}$, associated with the same state $v$.

For the base case, both histories $\pi_0$ and $\hat{\pi}_0$ are set to $v_0$. For the induction step, suppose that the two histories $\pi_\ell$ and $\hat{\pi}_\ell$ satisfy the hypothesis, and that they end at state $v$ and position $(v,c)$, respectively. We construct a prolongation $\pi_{\ell+1}$ that follows $s$ over the $\ell + 1$st knowledge gap and matches a one-round prolongation $\hat{\pi}_{\ell+1}$ of $\hat{\pi}_\ell$.

Towards this, we consider the strategy $t_v$ induced by $s$ in the set of histories $\pi_\ell T_v$, that is, the prolongations of $\pi_\ell$ into the tree component $T_v$ of $G^{ck}$. For $U := outcome_v(t_v)$ and $(u,d) := \hat{r}(U)$, there exists a history $\tau$ in $T_v$ that ends at $u$ and has $d$ as most significant priority after the initial state $v$. Now, we update $\pi_{\ell+1} := \pi_\ell \tau$ and $\hat{\pi}_{\ell+1} := \hat{\pi}_\ell U(u,d)$. This way, $\pi_{\ell+1}$ follows $s$ and the players attain cks, whereas $\hat{\pi}_{\ell+1}$ follows $\hat{r}$. Moreover, the two plays have the same summary, and $\hat{\pi}_{\ell+1}$ ends at a position corresponding to the last state of $\pi_{\ell+1}$.

For the infinite plays $\pi$ and $\hat{\pi}$ obtained at the limit, we have: $\pi$ follows $s$ and matches $\hat{\pi}$ which follows $\hat{r}$, as required.

The correspondence between strategies in the abridged game and in the original game, allows us to draw the following conclusion.

**Proposition 8.5.** Let $G$ be a coordination game that allows for recurring common knowledge of the state, with $m$ states and a parity winning condition over $d$ priorities.

(i) The coalition has a joint winning strategy for $G$ if, and only if, Coordinator has a positional winning strategy for the abridged game $\hat{G}$, that is a perfect-information parity game with $md + 2^{md}$ positions and $d$ priorities.

(ii) If the coalition has a joint winning strategy in $G$, then there exists a winning profile of finite-state strategies with $2^{O(m^2 \log m)}$ states.

**Proof.** (i) If Coordinator has a positional winning strategy $\hat{s}$ in $\hat{G}$, then the corresponding profile $s$ according to Proposition 8.4(i) is winning in $G$, because every play $\pi$ that follows $s$ has a matching play in $G$ that follows $\hat{s}$ and is hence winning, which implies that $\pi$ is also winning, by Lemma 8.2.
Conversely, assume that there exists a joint winning strategy $s$ in $G$. By Proposition 8.4(i), for any arbitrary positional strategy $\tilde{r}$ of Nature, there exists a play $\hat{\pi}$ that follows $\tilde{r}$ and matches some play $\pi$ in $G$ which follows $s$ and thus wins. Hence, $\hat{\pi}$ is also winning for Coordinator, by Lemma 8.2 which means that $\tilde{r}$ is not a winning strategy. By positional determinacy, it follows that Coordinator has a positional winning strategy in $\tilde{G}$.

The state space of the abridged game is $V \times C \cup 2^{V \times C}$, it has $md + 2^{md}$ positions; the number $d$ of priorities is as in $G$.

(ii) Let $\hat{s} : V \times C \to 2^{V \times C}$ be a winning strategy for Coordinator in the abridged game $\tilde{G}$. Since for any fixed state $v$, all positions $(v, c)$ have the same set of successors, we can assume, without loss, that the strategy prescribes the same move $U = \hat{s}(v, c)$, at all positions corresponding to $v$, independently of the priority $c$. To construct a joint winning strategy $s$ for the coalition, according to Proposition 8.4(i), for each of the $m$ states $v \in V$, the move $\hat{s}(v, c)$ of the perfect-information strategy is translated into an imperfect-information strategy $t_v$ over the tree component $T_v$ of $G^{ck}$. For each player, such a local strategy can be implemented by an automaton that records the sequence of his observations along the history suffixes contained in the tree. As these histories correspond to knowledge gaps, their length is at most $m^2$, by Theorem 7.4. Since there are no more observations than states, $m^{m^2}$ states are sufficient to store these sequences. Globally, the strategy $s^i$ of each player $i$ combines $m$ local strategies for the trees $T_v$, hence we need at most $m \cdot m^{m^2} = 2^{O(m^2 \log m)}$ many states to represent each component of the profile $s$ by a strategy automaton. 

8.4. Complexity

A nondeterministic procedure for deciding whether there exists a joint winning strategy in a game $G$ with $\omega$-cks can guess the abridged game $\tilde{G}$ and determine whether Coordinator has a winning strategy in the obtained parity game with perfect information. The complexity is dominated by the verification of the transition relations between Coordinator positions $(v, c)$ and Nature positions $U \subseteq V \times C$, which involves guessing a witnessing strategy profile $t_v$ over the tree $T_v$ such that outcome$_\nu(T_v) = U$. According to our argument in Proposition 8.5, for a game $G$ of size $m$, such a strategy $t_v$ can be represented by a collection of $n$ trees of size $2^{O(m^2 \log m)}$, one for every player; once the local strategy trees $t^i$ are guessed, the verification that outcome$_\nu(t) = U$ is done in time linear in their size. Given the abridged game, a winning strategy for Coordinator can be guessed and verified in nondeterministic linear time, in the size $md + 2^{md}$ of $\tilde{G}$, with $d$ priorities. Overall, the procedure runs in $NTime(2^{O(m^2 \log m)})$, that is nondeterministic exponential time.

With a deterministic procedure, the abridged game can be constructed by exhaustive search over witnessing strategies over the component trees in $G^{ck}$ in time $2^{O(m^2 \log m)}$. Once this is done, winning strategies for the obtained parity game $\tilde{G}$ can be constructed in time $O(2^{md^2})$ using the basic iterative algorithm presented by Zielonka in [34]. This concludes the proof of Theorem 8.1.
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