Multiple peak aggregations for the Keller–Segel system

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Abstract

In this paper we derive matched asymptotic expansions for a solution of the Keller–Segel system in two space dimensions for which the amount of mass aggregation is $8\pi N$, where $N = 1, 2, 3, \ldots$ Previously available asymptotics had been computed only for the case in which $N = 1$.

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1. Introduction

The goal of this paper is to describe, using matched asymptotics, the asymptotic behaviour near blow-up points of a class of nonradially symmetric solutions of the following Keller–Segel system.

\begin{align*}
\frac{\partial u}{\partial t} &= \Delta u - \nabla (u \nabla v), & x \in \mathbb{R}^2, & t > 0, \\
0 &= \Delta v + u, & x \in \mathbb{R}^2, & t > 0.
\end{align*}

(1.1a) (1.1b)

The Keller–Segel system, which was introduced in [15], is a classical model of chemotactic aggregation. In this model $u$ is the density of a biological organism and $v$ is the concentration of a chemical substance produced by it having chemoattractant properties. It was conjectured in [5] and rigorously proven in [14], in the case of bounded domains, that solutions of (1.1a)–(1.1b) may blow up in finite time, showing the fact that is usually interpreted as the formation of a high density aggregate of cells.

The mathematical properties of the solutions of (1.1a)–(1.1b) have been extensively studied by many authors. One of the most peculiar features of (1.1a)–(1.1b) is the existence of a critical mass $m_0$ such that for solutions with initial total mass of organism $\int u_0$ larger than $m_0$, blow-up takes place, while solutions with smaller values of $\int u_0$ yield global existence.
of solutions (see [1, 17] for bounded domains, [6, 18] in the case of $\mathbb{R}^2$). The asymptotic behaviour of the solutions in $\mathbb{R}^2$ in the case of critical mass was studied in [4]. In this case mass aggregation in infinite time takes place for suitable initial data. It was proven that blow-up in finite time consists in the formation of a Dirac mass with an amount of mass larger than $4\pi$ in the case of Neumann boundary conditions and blow-up taking place at the boundary of the domain, and larger than $8\pi$ in the case of blow-up taking place at interior points (see [20]). The literature about the Keller–Segel system is huge and we will not attempt to summarize here all the existent research concerning singularity formation and global existence for (1.1a), (1.1b). Some of the main results in this area can be found in [1, 4, 6, 14, 17, 18].

In the case of radially symmetric solutions, the asymptotic behaviour of solutions of (1.1a)–(1.1b) near blow-up points was obtained in [11] using asymptotic methods, and a rigorous construction of such solutions was given in [12]. Actually the paper [11] also describes formally the asymptotics of the blow-up solutions in the parabolic–parabolic case in which (1.1b) is replaced by a parabolic equation. The rigorous construction of the corresponding solutions is given in [13]. The solutions constructed in [12] produce the aggregation of a Dirac mass with the mass $8\pi$. On the other hand, continuation of solutions after blow-up has been considered using formal arguments in [23, 24] and rigorous mathematical analysis in [7, 16]. We also remark that solutions of (1.1a), (1.1b) in function spaces which contain small measures were considered in [1, 8].

We will describe in this paper the asymptotics of solutions of (1.1a), (1.1b) yielding formation of Dirac masses whose amount of mass is $8\pi N$ with $N = 2, 3, 4, \ldots$. These solutions will be obtained by means of the coalescence at time $t = T$ of $N$ peaks of mass placed at distances of order $\sqrt{T - t}$, each of the peaks containing an amount of mass asymptotically close to $8\pi$. The behaviour of such solutions will be obtained using matched asymptotics. The peaks where most of the mass is concentrated near the blow-up time are placed at the vertices of some polygons to be described in detail later.

We summarize the main result of this paper in the following theorem. We emphasize that the results of this paper are obtained at the level of formal asymptotic expansions but not a rigorous theorem in the sense of mathematical analysis.

**Theorem 1.1.** It is possible to find formal asymptotic expansions for solutions of the Keller–Segel system (1.1a), (1.1b) that blow up at the time $t = T$ at the point $x = x_0$ and at each time $t < T$ the mass is concentrated around the points $x_j(t)$, $j = 1, 2$, where:

$$x_j(t) = x_0 + (-1)^{j+1}a\sqrt{T - t}, \quad j = 1, 2$$

with $a = (2, 0) \in \mathbb{R}^2$. More precisely, the formal solutions described by the asymptotics found in this paper have the following property. For any $\nu > 0$ arbitrarily small, there exists $R > 0$ sufficiently large such that:

$$\lim_{t \to T^-} \left| \int_{B_{\delta(t)}(x_j(t))} u(x, t) \, dx - 8\pi \right| \leq \nu$$

with

$$\delta(t) = \sqrt{T - t} e^{-\alpha \log(T - t)}$$

for some $\alpha > 0$.

Moreover, the total amount of mass concentrating at the point $x = x_0$ as $t \to T^-$ is $16\pi$. More precisely, for any function $\eta(t)$ such that $\lim_{t \to T^-} \eta(t)/\sqrt{T - t} = \infty$ and $\lim_{t \to T^-} \eta(t) = 0$ one has:

$$\lim_{t \to T^-} \int_{B_{\eta(t)}(x_0)} u(x, t) \, dx = 16\pi.$$
Remark 1.2. The argument used in the construction suggests that it would be possible to obtain solutions yielding the aggregation of an arbitrary number of multiples of $8\pi$. However, the feasibility of such a construction requires checking that a certain auxiliary elliptic problem, associated to suitable singular self-similar solutions of (1.1a), (1.1b) (see section 2), satisfy some sign condition that will be discussed in detail in section 4 for the case in which two peaks aggregate. We have checked that this sign condition holds in this particular case, solving numerically an elliptic equation. Analogous sign conditions should be checked for aggregations of multiple peaks, which we have not attempted in this paper. Precise asymptotic formulas for the solutions described in theorem 1.1 will be given in the rest of the paper. In particular, we will derive precise formulas for the width of the regions around the points $x_i(t)$, where the mass concentrates. The final profile of the solution at the blow-up time will be described in remark 5.2.

The results of this paper are of a local nature. For this reason we just restrict our analysis to the case in which the system is solved in the whole $\mathbb{R}^2$. Similar results could be derived for the Cauchy-Neumann problem in bounded domains with non-flux boundary conditions (see section 7).

We remark that numerical simulations showing aggregation of several peaks at the time of the singularity formation were obtained in [19].

2. Notation and preliminaries

As indicated in the introduction we will denote as $T$ the blow-up time. We will use repeatedly the following self-similar variables in the rest of the paper:

$$u(x, t) = \frac{1}{T - t} \Phi(y, \tau), \quad v(x, t) = W(y, \tau), \quad (2.1a)$$

$$y = \frac{x - x_0}{\sqrt{T - t}}, \quad \tau = -\log(T - t). \quad (2.1b)$$

The system (1.1a), (1.1b) becomes in these variables:

$$\Phi_{\tau} = \Delta \Phi - \frac{y \nabla \Phi}{2} - \nabla (\Phi \nabla W) - \Phi, \quad (2.2a)$$

$$0 = \Delta W + \Phi. \quad (2.2b)$$

It is natural to expect a self-similar behaviour for the solutions of (2.2a), (2.2b). Self-similar solutions of (1.1a), (1.1b) solve:

$$\Delta \Phi - \frac{y \nabla \Phi}{2} - \nabla (\Phi \nabla W) - \Phi = 0, \quad (2.3a)$$

$$\Delta W + \Phi = 0 \quad (2.3b)$$

in the variable (2.1a), (2.1b). The solutions that we construct in this paper approach asymptotically as $\tau \to \infty$ the singular steady states:

$$\Phi_j = 8\pi \sum_{\ell=1}^{N} \delta(y - y_j) \quad (2.4)$$

with the points $y_j$ satisfying:

$$\frac{y_j}{2} - 4 \sum_{\ell=1, \ell \neq j}^{N} \frac{y_j - y_\ell}{|y_j - y_\ell|^2} = 0, \quad j = 1, 2, ..., N. \quad (2.5)$$
The solutions (2.4), (2.5) solve (2.3a), (2.3b) in the sense that they can be obtained as a limit of bounded solutions \((\Phi_n, W_n)\) of (2.3a), (2.3b) in bounded domains \(B_{R_n}\) with \(R_n \to \infty\) as \(n \to \infty\). The reason for requiring the solutions to be obtained in such a way is because we want these solutions to appear as a limit of bounded solutions of (2.2a), (2.2b) as \(\tau \to \infty\). Seemingly this implies that the mass at each aggregation point must be \(8\pi\). We would not attempt to give a precise meaning to these solutions in this paper, although it is likely that they could be given a precise meaning using some of the methods used in [7, 16, 21] to define solutions of the two-dimensional Keller–Segel system for measures containing Dirac masses. Another alternative seems to be to use ideas analogous to the ones obtained in [10].

The solutions obtained in this paper will behave asymptotically as in (2.4), (2.5) as \(\tau \to \infty\). A particular case of these solutions corresponds to the case of radially symmetric solutions considered in [11, 12]. An alternative way of deriving the asymptotics of these solutions can be found in [22]. In this radially symmetric case, the corresponding solution of (2.3a), (2.3b) has the form:

\[
\Phi_{r,s}(y) = 8\pi \delta(y). \tag{2.6}
\]

As was seen in [11, 22], the solutions of (1.1a)–(1.1b) with the asymptotics near the blow-up characterized by (2.6) have the mass concentrated in a region of size:

\[
\varepsilon(\tau) = K e^{-\sqrt{\tau}}, \tag{2.7}
\]

where \(K = 2 e^{-(2+\gamma)/2}\) with classical Euler’s constant \(\gamma\). The region where the mass aggregates can be described by a rescaling with a factor \(\varepsilon(\tau)\) of the following stationary solution found in [5]:

\[
u_s(x) = -2 \log \left(1 + |x|^2\right). \tag{2.8}\]

This stationary solution plays an important role in describing the long time asymptotics of solutions of (1.1a)–(1.1b) in the case of critical mass (see [2, 3]).

In this paper we will give most of the details concerning the asymptotics of solutions of (1.1a)–(1.1b) which are bounded for \(t < T\) and blow up at \(t = T\) in the particular case of a limit function \(\Phi_s\), a solution of (2.3a), (2.3b) with the form (2.4) concentrated in two peaks, (i.e. \(N = 2\)). The reason is twofold. First, the computations become more cumbersome for an increasing number of peaks, but without requiring essentially different ideas. On the other hand, the construction requires checking a sign condition for a suitable elliptic problem, as indicated in remark 1.2, and this is what we have made numerically only in the case of two peaks. In any case, solutions of (2.3a), (2.3b) with the form (2.4) will be discussed in section 6.

Due to the symmetry of the problem under rotations we can restrict ourselves to the case in which \(\Phi_s\) is given by:

\[
\Phi_s(y) = 8\pi \left[\delta(y - y_1) + \delta(y - y_2)\right], \quad y_1 = a, \quad y_2 = -a, \quad a = (2, 0). \tag{2.9}
\]

The detailed structure near the points \(y_\ell, \ell = 1, 2\), can be computed by introducing boundary layers having many similarities to the ones described in [11, 22]. The rescaling factor \(\varepsilon(\tau)\) will have a form similar to the one given in (2.7), although the value of the constant \(K\) will differ in general from the one obtained for the radially symmetric case. Actually, in the case of the asymptotics given by (2.4), the value of this constant could be different for each of the aggregation points. This will not be the case if \(\Phi(y, \tau)\) approaches the singular stationary solution \(\Phi_s\) in (2.9) due to symmetry considerations.

A large portion of this paper consists of the detailed description of the boundary layers describing the regions of mass aggregation near the points \(y_1, y_2\). The computation of these
layers will be made using the methods developed in [22] because the validity of some of the arguments in [11] is restricted to the radially symmetric case.

We now briefly describe our strategy to compute the asymptotics of the solutions near the blow-up points. We will obtain outer and inner expansions for the solutions. The outer expansion is valid in the region where \( |y| \approx 1 \) and \( |y - y_\ell| \gg e^{-\alpha \sqrt{\tau}} \) as \( \tau \to \infty \), \( \ell = 1, 2 \), for some \( \alpha > 0 \) to be revealed later. The inner expansion is valid in the regions where \( |y - y_\ell| \approx e^{-\alpha \sqrt{\tau}} \), \( \ell = 1, 2 \). Both expansions are obtained under the assumption that the mass aggregating near the points \( y_\ell \) concentrates in a region with width \( \varepsilon_\ell (\tau) \ll 1 \), whose precise value will be computed later. Such assumption will be shown to be self-consistent with the derived asymptotics. There is a common region of validity where both outer and inner expansions make sense. The matching condition between both types of expansion in that intermediate region provides a set of differential equations for the functions \( \varepsilon_\ell (\tau) \) and these equations yield the asymptotics of such functions.

We make extensive use of the asymptotic notation. We write \( f \ll g \) as \( x \to x_0 \) to indicate \( \lim_{x \to x_0} f/g = 0 \), whereas \( f \sim g \) as \( x \to x_0 \) to denote \( \lim_{x \to x_0} f/g = 1 \). The notation \( f \approx g \) as \( x \to x_0 \) indicates that the terms \( f \) and \( g \) have a comparable order of magnitude, that is, the existence of \( C > 0 \) such that \( C^{-1} \leq \liminf_{x \to x_0} f/g \leq \limsup_{x \to x_0} f/g \leq C \).

3. Inner expansions

3.1. Expansion of the solutions

We compute the asymptotics of the functions \( \Phi(W) \) defined in (2.1a), (2.1b). In the case of radially symmetric solutions it is assumed that \( \nabla \Phi(y_\ell, \tau) = 0 \) with \( y_\ell = 0 \). However, due to the lack of symmetry, points where the maximum of \( \Phi \) are attained could change in time. We assume the existence of functions \( \{ \bar{y}_\ell (\tau) : \ell = 1, 2, ..., N \} \) such that:

\[
\nabla \Phi(\bar{y}_\ell (\tau), \tau) = 0, \quad \lim_{\tau \to \infty} \bar{y}_\ell (\tau) = y_\ell.
\]

It will be checked later that all these assumptions are self-consistent as usual in matched asymptotics. Let us introduce the following set of variables to describe the inner solutions near each point \( y_\ell \):

\[
\xi = \frac{y - \bar{y}_\ell (\tau)}{\varepsilon_\ell (\tau)}, \quad \Phi(y, \tau) = \frac{1}{(\varepsilon_\ell (\tau))^2} U(\xi, \tau), \quad W(y, \tau) = W(\xi, \tau).
\]

Notice that the variables \( \xi, U(\xi, \tau) \), and \( W(\xi, \tau) \) depend on \( \ell \), but these dependencies will not be explicitly written unless needed. Using (2.2a), (2.2b) and (3.3) we obtain:

\[
\varepsilon_\ell^2 \frac{\partial U}{\partial \tau} = \Delta_\xi U - \nabla_\xi \left( U \nabla_\xi W \right) + \left( 2\varepsilon_\ell \varepsilon_\ell, \tau - \varepsilon_\ell^2 \right) \left( U + \frac{\xi \nabla_\xi U}{2} \right) + \left( \varepsilon_\ell \bar{y}_\ell, \tau - \frac{\varepsilon_\ell \bar{y}_\ell}{2} \right) \nabla_\xi U,
\]

\[
0 = \Delta_\xi W + U.
\]

We will now assume that the function \( \varepsilon_\ell (\tau) \) satisfies:

\[
\varepsilon_\ell (\tau) \ll 1 \quad \text{as} \quad \tau \to \infty,
\]

\[
|\varepsilon_\ell, \tau| + |\varepsilon_\ell, \tau| \ll \varepsilon_\ell \quad \text{as} \quad \tau \to \infty.
\]

Assumptions similar to (3.5), (3.6) are made in [22]. In addition, we will also assume in this paper:

\[
|\bar{y}_\ell, \tau| \ll 1 \quad \text{as} \quad \tau \to \infty.
\]
We now define in a precise manner the functions \( \varepsilon_\ell (\tau) \). We expect \( U, W \) to behave like the stationary solution (2.8). The steady states of (1.1a), (1.1b) can be defined up to rescaling. Therefore the functions \( \varepsilon_\ell (\tau) \) could be computed up to a rescaling factor. The assumption \( U (\xi, \tau) \rightarrow u_\ell (\xi) \) as \( \tau \rightarrow \infty \) would prescribe uniquely the leading order asymptotics of \( \varepsilon_\ell (\tau) \). Moreover, we can prescribe uniquely the function \( U \), imposing the normalization:

\[
U (0, \tau) = 8.
\]  

(3.8)

We then look for solutions of the system (3.4a), (3.4b) with the form of the following expansions:

\[
U (\xi, \tau) = u_\ell (\xi) + U_1 (\xi, \tau) + U_2 (\xi, \tau) + U_3 (\xi, \tau) + U_4 (\xi, \tau) + \cdots,
\]

\[
W (\xi, \tau) = v_\ell (\xi) + W_1 (\xi, \tau) + W_2 (\xi, \tau) + W_3 (\xi, \tau) + W_4 (\xi, \tau) + \cdots,
\]

where \((u_\ell, v_\ell)\) are the stationary solution as in (2.8). Notice that the function \( v_\ell \) is prescribed up to the addition of an arbitrary constant, but this can be ignored due to the form of the system (1.1a)–(1.1b). On the other hand, it will be assumed, as in [22], the terms \( U_1, W_1 \) contain terms whose order of magnitude is \( \varepsilon_\ell \) and that the terms \( U_2, W_2 \) contain terms whose order of magnitude is \( (\varepsilon_\ell)^2 \) or \( \varepsilon_\ell \tilde{y}_{\ell, t} \), up to logarithmic corrections like \( \log |\varepsilon_\ell| \), \( \tau^b \) or similar ones. Such logarithmic corrections will arise from terms like \( \varepsilon_\ell \tilde{y}_{\ell, t}/\varepsilon_\ell \) or similar ones. The notation introduced in [22] and used also in this paper consists of writing all these terms as \( \varepsilon_\ell^a \) (w.l.a) (with logarithmic accuracy).

We will include in \( U_1, W_1 \) also the terms whose order of magnitude is \( \varepsilon_\ell \) (w.l.a). Therefore:

\[
(U_1, W_1) \approx \varepsilon_\ell (w.l.a) \quad \text{as } \tau \rightarrow \infty.
\]  

(3.11)

Including in \( U_2, W_2 \) also the terms whose order of magnitude is \( \varepsilon_\ell \tilde{y}_{\ell, t} \) (w.l.a), we have:

\[
(U_2, W_2) \approx \varepsilon_\ell^2 + \varepsilon_\ell \tilde{y}_{\ell, t} (w.l.a) \quad \text{as } \tau \rightarrow \infty.
\]  

(3.12)

In a similar manner, including in \((U_3, W_3)\) terms of order \( \varepsilon_\ell^3 \), \( \varepsilon_\ell^2 \tilde{y}_{\ell, t} \) and \( \varepsilon_\ell \tilde{y}_{\ell, t}^2 \) (w.l.a) and including in \((U_4, W_4)\) terms of order \( \varepsilon_\ell^4 \), \( \varepsilon_\ell^3 \tilde{y}_{\ell, t} \), \( \varepsilon_\ell^2 \tilde{y}_{\ell, t}^2 \) (w.l.a) we obtain:

\[
(U_3, W_3) \approx \varepsilon_\ell^3 + \varepsilon_\ell^2 \tilde{y}_{\ell, t} + \varepsilon_\ell \tilde{y}_{\ell, t}^2 (w.l.a) \quad \text{as } \tau \rightarrow \infty,
\]

\[
(U_4, W_4) \approx \varepsilon_\ell^4 + \varepsilon_\ell^3 \tilde{y}_{\ell, t} + \varepsilon_\ell^2 \tilde{y}_{\ell, t}^2 (w.l.a) \quad \text{as } \tau \rightarrow \infty.
\]

(3.13)

(3.14)

It follows from the assumptions (3.11)–(3.14) that the functions \((U_k, W_k)\), \( k = 1, 2, 3, 4 \), satisfy respectively the following systems:

\[
0 = \Delta_\varepsilon U_1 - \nabla_\varepsilon (u_\ell \nabla_\varepsilon W_1) - \nabla_\varepsilon \left( U_1 \nabla_\varepsilon v_\ell \right) - \frac{\varepsilon_\ell \tilde{y}_\ell}{2} \nabla_\varepsilon u_\ell,
\]

\[
0 = \Delta_\varepsilon W_1 + U_1,
\]

(3.15a)

(3.15b)

\[
0 = \Delta_\varepsilon U_2 - \nabla_\varepsilon \left( u_\ell \nabla_\varepsilon W_2 \right) - \nabla_\varepsilon \left( U_1 \nabla_\varepsilon W_1 \right) - \nabla_\varepsilon \left( U_2 \nabla_\varepsilon v_\ell \right) + \left( 2\varepsilon_\ell \tilde{y}_{\ell, t} - \varepsilon_\ell^2 \right) \left( u_\ell + \frac{\varepsilon_\ell \tilde{y}_\ell}{2} \right) + \varepsilon_\ell \tilde{y}_{\ell, t} \nabla_\varepsilon u_\ell - \frac{\varepsilon_\ell \tilde{y}_\ell}{2} \nabla_\varepsilon U_1,
\]

\[
0 = \Delta_\varepsilon W_2 + U_2,
\]

(3.16a)

(3.16b)

\[
0 = \Delta_\varepsilon U_3 - \nabla_\varepsilon \left( u_\ell \nabla_\varepsilon W_3 \right) - \nabla_\varepsilon \left( U_1 \nabla_\varepsilon W_2 \right) - \nabla_\varepsilon \left( U_2 \nabla_\varepsilon W_1 \right) - \nabla_\varepsilon \left( U_3 \nabla_\varepsilon v_\ell \right) + \left( 2\varepsilon_\ell \tilde{y}_{\ell, t} - \varepsilon_\ell^2 \right) \left( U_1 + \frac{\varepsilon_\ell \tilde{y}_\ell}{2} \right) - \varepsilon_\ell^2 \frac{\partial U_1}{\partial \tau} + \varepsilon_\ell \tilde{y}_{\ell, t} \nabla_\varepsilon U_1 - \frac{\varepsilon_\ell \tilde{y}_\ell}{2} \nabla_\varepsilon U_2,
\]

\[
0 = \Delta_\varepsilon W_3 + U_3,
\]

(3.17a)

(3.17b)
Multiple peak aggregations for the Keller–Segel system 325

We can easily obtain an exact solution of (3.15) of the outer expansion having the angular dependencies proportional to $a_0 = \bar{a}_0$. Due to (3.8) we must solve (3.15) with conditions:

$$U_k(0, \tau) = 0, \quad k = 1, 2, 3, 4. \quad (3.19)$$

3.2. Computation of $(U_1, W_1)$, $(U_2, W_2)$

We can easily obtain an exact solution of (3.15)–(3.16):

$$U_1(\xi, \tau) = 0, \quad W_1(\xi, \tau) = -\frac{\xi_i \bar{y}_j}{2} \xi. \quad (3.20)$$

In order to compute $(U_2, W_2)$ we split it as: $U_2 = U_{2,1} + U_{2,2} + U_{2,3}$, $W_2 = W_{2,1} + W_{2,2} + W_{2,3}$, where $(U_{2,j}, W_{2,j})$, $j = 1, 2, 3$, solve respectively:

$$0 = \Delta_\xi U_{2,1} - \nabla_\xi (u_x \nabla_\xi W_{2,1}) - \nabla_\xi \left( U_{2,1} \nabla_\xi v_x \right) + (2\epsilon_0 \epsilon_{\ell,\tau} - \epsilon^2_0) \left( u_x + \frac{\xi_i \bar{y}_j}{2} \xi \right), \quad (3.21a)$$

$$0 = \Delta_\xi W_{2,1} + U_{2,1}, \quad (3.21b)$$

$$0 = \Delta_\xi U_{2,2} - \nabla_\xi \left( u_x \nabla_\xi W_{2,2} \right) - \nabla_\xi \left( U_{2,2} \nabla_\xi v_x \right) + \epsilon_{\ell} \bar{y}_{\ell,\tau} \nabla_\xi u_x, \quad (3.21c)$$

$$0 = \Delta_\xi W_{2,2} + U_{2,2}, \quad (3.21d)$$

$$0 = \Delta_\xi U_{2,3} - \nabla_\xi \left( u_x \nabla_\xi W_{2,3} \right) - \nabla_\xi \left( U_{2,3} \nabla_\xi v_x \right), \quad (3.21e)$$

$$0 = \Delta_\xi W_{2,3} + U_{2,3}. \quad (3.21f)$$

We will check later that the term $\bar{y}_{\ell,\tau}$ is of order $(\epsilon_0)^2$ (w.l.o.g. Thus $(U_{2,2}, W_{2,2})$ will be of order $(\epsilon_0)^3$ (w.l.o.g. Notice that this means that the terms $(U_k, W_k)$ do not have a dependence $(\epsilon_0)^3$ (w.l.o.g.)

At first glance the system for $(U_{2,3}, W_{2,3})$ could seem a bit odd for the absence of source terms. Actually $(U_{2,3}, W_{2,3})$ will be chosen as a homogeneous solution of the system (3.21e), (3.21f) which are smooth for bounded values of $|\xi|$, but $W_{2,3}$ becomes unbounded as $|\xi| \to \infty$. The contribution of $(U_{2,3}, W_{2,3})$ will be required to obtain a matching with some quadratic terms of the outer expansion having the angular dependencies proportional to $\cos(2\theta)$, $\sin(2\theta)$) and giving corrections of order $\epsilon_0^2$ (w.l.o.g. A detailed analysis of the matching conditions for the terms with this order of magnitude shows that, after a suitable rotation of the coordinate system, we may assume that the angular dependencies of the term $(U_{2,3}, W_{2,3})$ are proportional to $\cos(2\theta)$.

Due to (3.19) we must have:

$$U_{2,k}(0, \tau) = 0, \quad k = 1, 2, 3. \quad (3.22)$$

The solution of (3.21a), (3.21b) satisfying (3.22) was obtained in [22] as:

$$U_{2,1}(\xi, \tau) = Q_{2,1}(r, \tau), \quad W_{2,1}(\xi, \tau) = V_{2,1}(r, \tau), \quad r = |\xi|; \quad (3.23a)$$

$$g_1(r, \tau) \equiv r \frac{\partial V_{2,1}}{\partial r}(r, \tau) = (2\epsilon_0 \epsilon_{\ell,\tau} - \epsilon^2_0) \frac{r^2}{(1 + r^2)} \int_0^{1 + r^2} \frac{(1 + t^2)^2}{t^2} \left[ \log (1 + t) - \frac{t}{1 + t} \right] dt; \quad (3.23b)$$

$$Q_{2,1} = -\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V_{2,1}}{\partial r} \right). \quad (3.23c)$$
According to the formulas (3.26) and (3.27) in [22], we have the following asymptotics:

\[ Q_{2,1}(r, \tau) = \left(2\xi_\ell \xi_{\ell, \tau} - \xi_\ell^2\right) \left[ \frac{2}{r^2} + O \left(\frac{(\log r)^2}{r^3}\right) \right] \quad \text{as } r \to \infty, \quad (3.24) \]

\[ \frac{\partial V_{2,1}}{\partial r}(r, \tau) = \left(2\xi_\ell \xi_{\ell, \tau} - \xi_\ell^2\right) \left[ \log \left(\frac{r^2}{r^3}\right) - \frac{2}{r^2} + O \left(\frac{(\log r)^2}{r^3}\right) \right] \quad \text{as } r \to \infty. \quad (3.25) \]

The solution of the system (3.21c), (3.21d) is given by the following simple formula:

\[ U_{2,2}(\xi, \tau) = 0, \quad W_{2,2}(\xi, \tau) = \xi_\ell \xi_{\ell, \tau} \xi. \quad (3.26) \]

We now consider the function \( U_{2,3}, W_{2,3} \). As explained before, this function, which is unbounded at infinity, is just a homogeneous solution of the linearized problem. It will be needed due to the effect of the other singular points at the point under consideration. More precisely, the function \( W \) due to the points placed near \( \bar{y}_k \) with \( k \neq \ell \) gives a contribution as \( |\xi| \to \infty \) that will be matched with the term \( W_{2,3} \). The angular dependence of this term is \( \cos(2\theta) \) and its size \( \epsilon_\ell^2 \) \((w.l.a.)\). Therefore we look for a solution \((U_{2,3}, W_{2,3})\) of (3.21e), (3.21f) with the form:

\[ U_{2,3}(\xi, \tau) = Q_{2,3}(r, \tau) \cos(2\theta), \quad W_{2,3}(\xi, \tau) = V_{2,3}(r, \tau) \cos(2\theta), \quad (3.27) \]

where \((r, \theta)\) is defined by \( \xi = (r \cos \theta, r \sin \theta) \). The system (3.21e), (3.21f) then reads:

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial Q_{2,3}}{\partial r} \right) - \frac{4}{r^2} Q_{2,3} - 2u_3 \frac{\partial V_{2,3}}{\partial r} + 2u_3 \frac{\partial Q_{2,3}}{\partial r} = 0, \quad (3.28a) \]

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V_{2,3}}{\partial r} \right) - \frac{4}{r^2} V_{2,3} + Q_{2,3} = 0. \quad (3.28b) \]

The smoothness of \((U_{2,3}, W_{2,3})\) at the origin (see (3.21e), (3.21f)) implies:

\[ Q_{2,3}(0, \tau) = V_{2,3}(0, \tau) = 0. \quad (3.29) \]

It was shown in [22] (see theorem 3.2) that the space of solutions of (3.28a), (3.28b) is a four dimensional linear space spanned by the set of functions \( \{(\psi_k, \omega_k) : k = 1, 2, 3, 4\} \). The condition (3.29) implies that: \( \{Q_{2,3}, V_{2,3}\} = K_1(\psi_1, \omega_1) + K_3(\psi_3, \omega_3) \) for some constants \( K_1, K_3 \in \mathbb{R} \). If \( K_3 \neq 0 \), the growth of \( (\psi_3, \omega_3) \) as \( |\xi| \to \infty \) would imply that \( (\Phi, W) \) are very large for \(|\psi|\) of order one, and this would contradict the hypothesis that \( \Phi \) approaches the steady state in (2.4) as \( \tau \to \infty \). Therefore \( K_3 = 0 \) and \((Q_{2,3}, V_{2,3})\) is given by:

\[ Q_{2,3}(r, \tau) = \frac{8B_{2,3}r^2}{(r^2 + 1)} \left( r^2 + 3 \right), \quad V_{2,3}(r, \tau) = \frac{B_{2,3}r^2}{r^2 + 1} \left( r^2 + 3 \right). \quad (3.30) \]

where \( B_{2,3} = B_{2,3}(\tau) \in \mathbb{R} \). Actually \( B_{2,3} \) can be expected to be a function of \( \tau \) changing slowly with respect to this variable. By this we mean that \( B_{2,3}(\tau) \) does not have a factor like \( e^{-\kappa \tau} \) with \( \kappa \neq 0 \). The precise value of \( B_{2,3} \) will be obtained later by matching the inner and the outer expansions. It will turn out to be of order \( (\epsilon_\ell)^2 \) \((w.l.a.)\). Finally, notice that the formulas (3.30) have been obtained for functions with angular dependence \( \cos(2\theta) \), but similar formulas could be obtained if the angular dependence is replaced by \( \sin(2\theta) \). The resulting coefficients \( B_{2,3} \) will be denoted for functions with such angular dependence as \( B_{2,3} \).

In the following arguments, more variables \( B_{1,2}, B_{1,2}, c_3(\infty), \ldots \) will appear. They have some dependence on \( \tau \), but we will not write this dependence explicitly unless needed. The solutions of (3.21e), (3.21f) cannot contain any radial contribution with angular dependence \( \cos \theta \). Indeed, arguing as in the derivation of (3.30) and using the fact that it is always possible to add a constant to \( V \), it follows that such a contribution would yield an additional term in \( U_{2,3} \) with the form \( K_1(\tau^2 - 1) \left( \tau^2 + 1 \right)^{-3} + K_2 \tau \left( \tau^2 + 1 \right)^{-3} \cos \theta \). However, if \( K_1 \neq 0 \) or \( K_2 \neq 0 \).
there would be a contradiction to (3.1), (3.8). Similar arguments exclude angular dependences \( \cos(\ell \theta) \) with \( \ell > 2 \), since they would imply large values for \( \Phi \), \( W \) in the outer region where \( |y| \) is of order one.

### 3.3. Computation of \((U_3, W_3)\)

Since \( U_1 = 0 \) and \(-\nabla_\ell \left( U_2 \nabla_\ell W_1 \right) - 2^{-1} \epsilon_3 \epsilon_3 \nabla_\ell U_2 = 0 \) by (3.20), the system (3.17a), (3.17b) reads:

\[
0 = \Delta_\ell U_3 - \nabla_\ell \left( u_3 \nabla_\ell W_3 \right) - \nabla_\ell \left( U_3 \nabla_\ell v_3 \right), \quad 0 = \Delta_\ell W_3 + U_3.
\]

This system is similar to (3.21e), (3.21f). Arguing as in the derivation of (3.30), we obtain:

\[
U_3(\xi, \tau) = \frac{8 \beta r^3}{(r^2 + 1)^3} (2r^2 + 4) \cos(3\theta), \quad V_3(\xi, \tau) = \frac{B_3 r^3}{r^2 + 1} (2r^2 + 4) \cos(3\theta) \tag{3.31}
\]

for some \( B_3 \in \mathbb{R} \), where \((r, \theta)\) is as before. As in the case of \( B_{2,3} \), \( B_3 \) could have some slow (meaning non-exponential in \( r \)) dependence on \( \tau \). More precisely, it will behave like \( \epsilon_3^2 (u.l.a.) \). We have just written terms with angular dependence \( \cos(3\theta) \), but there are similar terms with dependence \( \sin(3\theta) \) characterized by means of a coefficient \( B_3 \).

### 3.4. Computation of \((U_4, W_4)\)

Using (3.18a), (3.18b), (3.20) and (3.26) we have:

\[
0 = \Delta_\ell U_4 - \nabla_\ell \left( u_4 \nabla_\ell W_4 \right) - \nabla_\ell \left( U_2 \nabla_\ell W_{2,1} \right) - \nabla_\ell \left( U_2 \nabla_\ell W_{2,3} \right)
- \nabla_\ell \left( U_4 \nabla_\ell v_3 \right) + \left( 2 \epsilon_\ell \epsilon_\ell, e - \epsilon_\ell^2 \right) \left( U_2 + \frac{\xi \nabla_\ell U_2}{2} \right) - \epsilon_\ell^2 \frac{\partial U_2}{\partial \tau}, \tag{3.32a}
\]

\[
0 = \Delta_\ell W_4 + U_4. \tag{3.32b}
\]

It is now convenient to split \( U_4, W_4 \) as: \( U_4 = U_{4,1} + U_{4,2} \), \( W_4 = W_{4,1} + W_{4,2} \), where:

\[
0 = \Delta_\ell U_{4,1} - \nabla_\ell \left( u_4 \nabla_\ell W_{4,1} \right) - \nabla_\ell \left( U_{2,1} \nabla_\ell W_{2,1} \right)
- \nabla_\ell \left( U_{4,1} \nabla_\ell v_3 \right) + \left( 2 \epsilon_\ell \epsilon_\ell, e - \epsilon_\ell^2 \right) \left( U_{2,1} + \frac{\xi \nabla_\ell U_{2,1}}{2} \right) - \epsilon_\ell^2 \frac{\partial U_{2,1}}{\partial \tau}, \tag{3.33a}
\]

\[
0 = \Delta_\ell W_{4,1} + U_{4,1}, \tag{3.33b}
\]

\[
0 = \Delta_\ell U_{4,2} - \nabla_\ell \left( u_4 \nabla_\ell W_{4,2} \right) - \nabla_\ell \left( U_{4,2} \nabla_\ell v_3 \right) + S_{4,2}(\xi, \tau), \tag{3.34a}
\]

\[
0 = \Delta_\ell W_{4,2} + U_{4,2} \tag{3.34b}
\]

with

\[
S_{4,2}(\xi, \tau) = -\nabla_\ell \left( U_{2,3} \nabla_\ell W_{2,1} \right) - \nabla_\ell \left( U_2 \nabla_\ell W_{2,3} \right)
+ \left( 2 \epsilon_\ell \epsilon_\ell, e - \epsilon_\ell^2 \right) \left( U_{2,3} + \frac{\xi \nabla_\ell U_{2,3}}{2} \right) - \epsilon_\ell^2 \frac{\partial U_{2,3}}{\partial \tau}. \tag{3.34c}
\]

The system (3.33a), (3.33b) is the same as (3.16)–(3.18) in [22] and the solution can be obtained as indicated in that paper (although a slightly different notation for the functions has been used there). The relevant information that we will need in this paper is the asymptotics of the solutions for large values of \( |\xi| \), which can be computed as follows:

\[
U_{4,1} = -\frac{1}{r} \frac{\partial g_2}{\partial r}, \quad \frac{\partial W_{4,1}}{\partial r} = \frac{g_2}{r}, \tag{3.35a}
\]
\[ g_2 = e_3^2 \left( 2e_i \xi_{i \ell} - e_2^2 \right) r \left[ \frac{r^2 \log r}{4} - \frac{7r^2}{16} + O \left( \log r \right)^2 \right] + (2e_i \xi_{i \ell} - e_2^2)^2 \left[ \frac{r^2}{8} + O \left( \log r \right)^2 \right] \]  

(3.35b)

as \( r \to \infty \). Similar asymptotic formulas can be obtained for \( \frac{\partial g_2}{\partial r} \).

In order to solve (3.34a)–(3.34e) we need to compute \( S_{4,2} (\xi, \tau) \). Using (3.23a) and (3.27) we obtain, after some elementary but tedious computations:

\[ S_{4,2} (\xi, \tau) = G_1 (r, \tau) + G_2 (r, \tau) \cos (2\theta) + G_3 (r, \tau) \cos (4\theta), \]  

(3.36)

where:

\[ G_1 (r, \tau) = -\frac{1}{2r} \frac{\partial}{\partial r} \left( r Q_{2,1} \frac{\partial V_{2,1}}{\partial r} \right), \]  

(3.37a)

\[ G_2 (r, \tau) = \frac{4Q_{2,1}V_{2,3}}{r^2} - \frac{1}{2r} \frac{\partial}{\partial r} \left( r Q_{2,1} \frac{\partial V_{2,3}}{\partial r} \right) - \frac{1}{r} \frac{\partial}{\partial r} \left( r Q_{2,3} \frac{\partial V_{2,1}}{\partial r} \right) - \xi_1 \frac{\partial Q_{2,3}}{\partial r} + (2e_i \xi_{i \ell} - e_2^2) \frac{48B_{2,3} r^2}{(r^2 + 1)^2}, \]  

(3.37b)

\[ G_3 (r, \tau) = \frac{4Q_{2,1}V_{2,3}}{r^2} - \frac{1}{2r} \frac{\partial}{\partial r} \left( r Q_{2,3} \frac{\partial V_{2,1}}{\partial r} \right). \]  

(3.37c)

The form of \( S_{4,2} (\xi, \tau) \) in (3.36) suggests to split \( \{ U_{4,2}, W_{4,2} \} \) as: \( U_{4,2} = U_{4,2,1} + U_{4,2,2} + U_{4,2,3}, \) \( W_{4,2} = W_{4,2,1} + W_{4,2,2} + W_{4,2,3} \) with \( \{ (U_{4,2,k}, W_{4,2,k}) : k = 1, 2, 3 \} \) having the angular dependencies \( \cos (2 \theta) \) and \( \cos (4 \theta) \).

\[ 0 = \Delta_\xi U_{4,2,k} - \nabla_\xi (u_\ell \nabla_\ell W_{4,2,k}) - \nabla_\xi (U_{4,2,k} v_\ell) + G_k \cos (2 (k - 1) \theta), \]  

(3.38a)

\[ 0 = \Delta_\xi W_{4,2,k} + U_{4,2,k} \]  

(3.38b)

with boundary conditions:

\[ U_{4,2,k} (0, \tau) = 0, \quad k = 1, 2, 3. \]  

(3.39)

The boundary conditions for \( U_{4,2,2}, \) \( U_{4,2,3} \) are just consequences of the angular dependence of these functions and their smoothness properties, whereas condition (3.39) for \( U_{4,2,1} \) is just a consequence of (3.8). On the other hand, the angular dependencies of the functions \( W_{4,2,2}, W_{4,2,3} \) yield:

\[ W_{4,2,2} (0, \tau) = W_{4,2,3} (0, \tau) = 0, \]  

(3.40)

whereas (3.1) implies:

\[ \frac{\partial W_{4,2,1}}{\partial r} (0, \tau) = 0. \]  

(3.41)

3.5. Computation of \( \{ U_{4,2,1}, W_{4,2,1} \} \)

**Lemma 3.1.** Under the conditions (3.39) and (3.41) the system (3.38a), (3.38b) with \( k = 1 \) has a unique exact solution:

\[ U_{4,2,1} = 2 \left( B_{2,3} \right)^2 r^4 \frac{r^4 + 4r^2 + 9}{(r^2 + 1)^4}, \quad \frac{\partial W_{4,2,1}}{\partial r} = - \left( B_{2,3} \right)^2 r^5 \frac{r^4 + 3}{(1 + r^2)^4}, \]  

(3.42)

where \( r = |\xi| \) and \( B_{2,3} \) is the parameter in (3.30).
Proof. Using (3.30), (3.37a), (3.38a) and (3.39) we obtain, after integration:

\[ \frac{r}{\partial r} \frac{\partial U_{4,2,1}}{\partial r} - r u_s \frac{\partial W_{4,2,1}}{\partial r} - r \frac{\partial V_{4,2,1}}{\partial r} = \frac{r}{\partial r} \frac{\partial Q_{2,3}}{\partial r} \frac{\partial V_{2,3}}{\partial r}. \]  

(3.43)

On the other hand, defining functions \( M_{4,2,1} \) and \( F_{4,2,1} \) by means of:

\[ M_{4,2,1} = \frac{r^2 F_{4,2,1}}{1 + r^2} = r \frac{\partial W_{4,2,1}}{\partial r} \]  

(3.44)

and using (3.38b), we obtain:

\[ U_{4,2,1} = \frac{1}{r} \frac{\partial M_{4,2,1}}{\partial r}. \]  

(3.45)

The smoothness of the function \( W_{4,2,1} \) implies \( M_{4,2,1} (0, \tau) = 0 \). Using (3.39) we then obtain

\[ M_{4,2,1} (r, \tau) = \mathcal{O}(r^2) \quad \text{as } r \to 0. \]  

(3.46)

Plugging (2.8), (3.30), (3.44) and (3.45) into (3.43) we have:

\[ \frac{r}{\partial r} \frac{\partial F_{4,2,1}}{\partial r^2} - \frac{r^2 - 3}{\partial r} \frac{\partial F_{4,2,1}}{\partial r} + \frac{r}{\partial r} \frac{\partial F_{4,2,1}}{\partial r} = -8 \left( B_{2,3} \right)^2 \frac{r^3 (r^2 + 3)}{(r^2 + 1)^3} \left( r^2 + 2r^2 + 3 \right). \]

This is a first order linear differential equation for \( \frac{\partial F_{4,2,1}}{\partial r} \), which may be integrated as:

\[ \frac{\partial F_{4,2,1}}{\partial r} = -\frac{4}{(r^2 + 1)^2} \frac{(B_{2,3})^2}{r^3} \frac{r^3 (r^2 + 3)}{(r^2 + 1)^3} \frac{(r^2 + 2r^2 + 3)}{(r^2 + 1)^3}. \]  

(3.47)

In the derivation of (3.47) we have used that, due to (3.46), the value of \( F_{4,2,1} (0, \tau) \) must be finite. Integrating now (3.47) and using (3.46) we obtain \( F_{4,2,1} (r, \tau) = -\left( B_{2,3} \right)^2 \frac{r^4 (r^2 + 3)}{(r^2 + 1)^2} \). Using now (3.44)–(3.46), we obtain the desired result. ■

3.6. Computation of \( \left( U_{4,2,2}, W_{4,2,2} \right) \)

3.6.1. Reduction of the problem to ODEs for \( Q_{4,2,2}, V_{4,2,2} \). The functions \( U_{4,2,2}, W_{4,2,2} \) satisfy the system (3.38a), (3.38b) with \( k = 2 \) together with conditions (3.39), (3.40). In order to remove angular dependence we look for solutions of these equations in the form:

\[ U_{4,2,2} = Q_{4,2,2} (r, \tau) \cos (2 \theta), \quad W_{4,2,2} = V_{4,2,2} (r, \tau) \cos (2 \theta), \]  

where \( (r, \theta) \) is as before. It then follows from (3.38a), (3.38b) with \( k = 2 \) as well as (2.8) that:

\[ 0 = \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r}{\partial r} \frac{\partial Q_{4,2,2}}{\partial r} \right) - \frac{4}{r^2} Q_{4,2,2} + \frac{32}{(r^2 + 1)^3} \frac{\partial V_{4,2,2}}{\partial r} + \frac{4}{r^2} \frac{\partial Q_{4,2,2}}{\partial r} + \frac{16}{(1 + r^2)^2} Q_{4,2,2} + G_2, \]  

(3.48a)

\[ 0 = \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r}{\partial r} \frac{\partial V_{4,2,2}}{\partial r} \right) - \frac{4}{r^2} V_{4,2,2} + Q_{4,2,2}. \]  

(3.48b)

The precise formula of \( G_2 \) may be computed, using (3.23b), (3.23c) and (3.37b), as:

\[ G_2 = -\frac{8B_{2,3}}{\left( r^2 + 1 \right)^3} \frac{r^4 + 4r^2 + 9}{\partial g_1} + \frac{32}{\left( r^2 + 1 \right)^4} \frac{B_{2,3} \left( r^2 - 3 \right)}{g_1} \]

\[ + \frac{8B_{2,3}}{\left( r^2 + 1 \right)^4} \frac{\left( 2B_{2,3} - \ell_1 \right) r^2}{g_1} \left( r^2 + 2r^2 + 9 \right) - \frac{8B_{2,3}}{\left( r^2 + 1 \right)^3} \frac{r^4 + 2r^2 + 9}{\left( B_{2,3} \right)^2} + \left( B_{2,3} \right)^3 \frac{8B_{2,3}^2 \left( r^2 + 3 \right)}{\left( r^2 + 1 \right)^3}, \]  

(3.49)
where \( g_1 \) is the function as in (3.23b). The derivation of (3.49) requires just a long but elementary computation. On the other hand, the conditions (3.39) and (3.40) imply:
\[
Q_{4,2,2} (0, \tau) = 0, \quad \tau = 0.
\]
(3.50)
The system (3.48a), (3.48b) is a nonhomogeneous linear system with source term \( G_2 \) as in (3.49). In order to study the asymptotics of their solutions we examine in detail the solutions of the homogeneous part of this system.

3.6.2. Study of the homogeneous system. The homogeneous part of the system (3.48a), (3.48b) can be written as the linear system:
\[
0 = \frac{1}{r} \frac{d}{dr} \left( r \frac{d\psi}{dr} \right) - \frac{L^2}{r^2} \psi + \frac{32r}{(r^2 + 1)^3} \frac{d\omega}{dr} + \frac{4r}{r^2 + 1} \frac{d\psi}{dr} + \frac{16}{(1 + r^2)^2} \psi,
\]
(3.51a)
\[
0 = \frac{1}{r} \frac{d}{dr} \left( r \frac{d\omega}{dr} \right) - \frac{L^2}{r^2} \omega + \psi
\]
(3.51b)
with \( L = 2 \). This system was studied in detail in [22]. Four linearly independent solutions \( (\psi_k, \omega_k), k = 1, 2, 3, 4 \), were obtained and their asymptotics for large and small \( r \) were computed there. We need to compute an error term in the asymptotics of \( \omega_k \) for \( L = 2, 3, 4 \), ... in a manner more detailed than in [22]. The following result is basically a reformulation of [22, theorem 4.3].

Theorem 3.2. Suppose that \( L = 2, 3, 4, \ldots \). A general solution of (3.51a), (3.51b) is a linear combination of four particular functions \( (\psi_k, \omega_k), k = 1, 2, 3, 4 \), whose asymptotics are given by:
\[
\psi_1 (r) = \frac{8r^L}{(r^2 + 1)^3} [(L - 1) r^2 + L + 1], \quad \omega_1 (r) = \frac{r^L}{r^2 + 1} [(L - 1) r^2 + L + 1],
\]
(3.52a)
\[
\psi_2 (r) = \frac{8r^L}{(r^2 + 1)^3} [(L + 1) r^2 + L - 1], \quad \omega_2 (r) = \frac{1}{r^L (r^2 + 1)} [(L + 1) r^2 + L - 1],
\]
(3.52b)
\[
\psi_3 (r) \sim 8r^L \text{ as } r \to 0^+, \quad \omega_3 (r) \sim -r^L \text{ as } r \to 0^+,
\]
(3.52c)
\[
\psi_3 (r) \sim 16K_L r^{-\sqrt{4r^2L^2 - 2}} \text{ as } r \to \infty, \quad \omega_3 (r) \sim -4K_L r^{-\sqrt{4r^2L^2}} \text{ as } r \to \infty,
\]
(3.52d)
\[
\psi_4 (r) \sim 8r^{-L} \text{ as } r \to 0^+, \quad \omega_4 (r) \sim -r^{-L} \text{ as } r \to 0^+,
\]
(3.52e)
\[
\psi_4 (r) \sim 16C_L r^{-\sqrt{4r^2L^2 - 2}} \text{ as } r \to \infty,
\]
\[
\omega_4 (r) \sim C_L r^{-L} - 4r^{-\sqrt{4r^2L^2}} + o (r^{-L-2}) \text{ as } r \to \infty
\]
(3.52f)
for some real numbers \( C_L, K_L \) and \( k_L \).

Remark 3.3. We have \( C_L, K_L > 0 \), whereas \( K_L \) could be zero for some \( L \).

Remark 3.4. The only difference between theorem 3.2 and [22, theorem 4.3] is that the last formula in (3.52f) is written as \( \omega_4 = o (r^{-L}) \) as \( r \to \infty \) in [22, theorem 4.3]. There is a typo there. The correct formula intended in that paper is \( \omega_4 = O (r^{-L}) \) as \( r \to \infty \). Formula (3.52f) provides a precise asymptotics for \( \omega_4 \).

Proof. It suffices to prove only (3.52f). To show it we define, given a solution \( (\psi, \omega) \) of (3.51a), (3.51b), two functions \( F, G \) by means of:
\[
\psi = \frac{8r^{-L}}{(r^2 + 1)^3} (F + G), \quad \omega = \frac{r^{-L}}{r^2 + 1} (F - G).
\]
(3.53)
Then:
\[
\frac{d^2 G}{dr^2} + \left( \frac{1 - 2L}{r} - \frac{8r}{r^2 + 1} \right) \frac{dG}{dr} + \left( \frac{8L + 12}{r^2 + 1} - \frac{16}{(r^2 + 1)^2} \right) G = 0,
\]
(3.54)
\[
\frac{d^2 F}{dr^2} + \left( \frac{1 - 2L}{r} - \frac{4r}{r^2 + 1} \right) \frac{dF}{dr} + \frac{4(L + 1)}{r^2 + 1} F = -\frac{4}{r^2 + 1} \left( r \frac{dG}{dr} - (L + 2) G \right) = S(r).
\]
(3.55)

It was proven in [22] that there exists a unique solution of (3.54) satisfying: \(G_\beta(0) = 1\), \(G_\beta(r) \sim C_L r^{\beta_L}\) as \(r \to \infty\) with \(\beta_L = 4 + L - \sqrt{4 + L^2}\). Moreover, since the point \(r = \infty\) is a regular singular point for (3.54), we can use the Frobenius theory to compute a power series expansion for \(G_\beta(r)\) as \(r \to \infty\) to observe:
\[
G_\beta(r) = C_L \left[ r^{\beta_L} + \frac{2\sqrt{L^2 + 4} - 1}{\sqrt{L^2 + 4} + 1} r^{\beta_L - 2} + O(r^{\beta_L - 4}) \right] \quad \text{as } r \to \infty.
\]
(3.56)

Two independent solutions of the homogeneous equation associated to (3.55) are:
\[
F_{1,h}(r) = (L + 1) r^3 + (L - 1), \quad F_{2,h}(r) = r^{2L} \left[ (L - 1) r^3 + (L + 1) \right].
\]
(3.57)

We then look for a solution of (3.55) with the form:
\[
F_\beta(r) = a_1(r) F_{1,h}(r) + a_2(r) F_{2,h}(r)
\]
(3.58)
under the constraint: \(a'_1(r) F_{1,h}(r) + a'_2(r) F_{2,h}(r) = 0\). Plugging (3.58) into (3.55) we obtain a system of algebraic equations for \(a'_1, a'_2\), whose solution may be expressed by the function \(S(r)\) in (3.55). Using the expression and the asymptotics of \(S(r)\) that may be computed with (3.56), we obtain, after integration, the asymptotics of \(a_1(r), a_2(r)\) as \(r \to \infty\). It then follows from (3.57) and (3.58) that
\[
F_\beta(r) = C_L \left[ r^{\beta_L} + K_L r^2 - \frac{2\sqrt{L^2 + 4} + 5}{\sqrt{L^2 + 4} + 1} r^{\beta_L - 2} + O(1) \right] \quad \text{as } r \to \infty.
\]
(3.59)

Using (3.56) and (3.59) in (3.53) we obtain (3.52f). This concludes the proof.

\[\Box\]

**Proposition 3.5.** Let \(C_L\) and \(K_L\) with \(L \geq 2\) be the constants as in (3.52d) and (3.52f), respectively. Then the following identity holds:
\[
C_L K_L = \frac{L}{\sqrt{L^2 + 4}}.
\]
(3.60)

**Proof.** Set
\[
\mathcal{M}(r) = \begin{pmatrix}
\psi_1(r) & \psi_2(r) & \psi_3(r) & \psi_4(r) \\
\omega_1(r) & \omega_2(r) & \omega_3(r) & \omega_4(r) \\
\psi'_1(r) & \psi'_2(r) & \psi'_3(r) & \psi'_4(r) \\
\omega'_1(r) & \omega'_2(r) & \omega'_3(r) & \omega'_4(r)
\end{pmatrix}, \quad \Delta_L(r) = \det(\mathcal{M}(r)).
\]

We may derive a first order ODE for \(\Delta_L\), which may be solved explicitly as: \(\Delta_L(r) = E_L r^{-2} (r^2 + 1)^{-2}\) with a constant \(E_L\). Then: \(\Delta_L(r) \sim E_L r^{-2}\) as \(r \to 0\) and \(\Delta_L(r) \sim E_L r^{-6}\) as \(r \to \infty\). On the other hand the precise asymptotic formulas of \(\Delta_L(r)\) as \(r \to 0\) and \(r \to \infty\), respectively, may be derived from its definition and (3.52a)–(3.52f). The comparison of their coefficients yields: \(E_L = -2^{10} (L + 1) (L - 1) L^2 = -2^{10} L (L - 1) (L + 1) \sqrt{L^2 + 4 C_L K_L}\), whence (3.60) follows.

\[\Box\]
3.6.3. Asymptotics of \((Q_{4,2,2}, V_{4,2,2})\)

**Lemma 3.6.** For any fixed \(\tau\), the problem \((3.48a)-(3.48b)-(3.50)\) has a one-dimensional family of solutions parameterized by \(B_{4,2} = B_{4,2}(\tau)\). Its asymptotics as \(r \to \infty\) are given by:

\[
Q_{4,2,2}(r, \tau) = 16K_2B_{4,2}\varepsilon^2 - 2\varepsilon^2 \left(2\varepsilon_\ell \varepsilon_\ell,\tau - \varepsilon_\ell^2 + \frac{(B_{2,3})^2}{B_{2,3}}\right) + O\left(\frac{\varepsilon^4}{r^{4-2\sqrt{2}}}\right),
\]

\[
V_{4,2,2}(r, \tau) = -4K_2B_{4,2}\varepsilon^2 + \frac{C_2K_2B_{2,3}}{2} \left(2\varepsilon_\ell \varepsilon_\ell,\tau - \varepsilon_\ell^2 + \frac{(B_{2,3})^2}{B_{2,3}}\right) r^2 \log r + \cdots
\]

as \(r \to \infty\), where \(C_2\) and \(K_2\) are the constants as in theorem 3.2 and \(B_{2,3} = B_{2,3}(\tau)\) is the parameter in \((3.30)\).

**Proof.** Although the functions \(Q_{4,2,2}\) and \(V_{4,2,2}\) depend on \(\tau\) through the dependence on \(\tau\) of the source term \(G_2 = G_2(r, \tau)\) we do not write them explicitly in the proof, because the proof relies purely on standard ODE arguments and the dependence on \(\tau\) does not play any role.

We look for solutions with the form: \(Q_{4,2,2} = \sum_{i=1}^4 b_i(r) \psi_i(r)\), \(V_{4,2,2} = \sum_{i=1}^4 b_i(r) \omega_i(r)\), where the functions \((\psi_i, \omega_i)\) are as in theorem 3.2 with \(L = 2\), under the constraints: \(\sum_{i=1}^4 b_i^2(r) \psi_i^2(r) = 0, \sum_{i=1}^4 b_i^2(r) \omega_i^2(r) = 0\). Using \((3.48a)\) and \((3.48b)\) as well as the fact that the functions \((\psi_i, \omega_i)\) solve the homogeneous system \((3.51a), (3.51b)\), we then obtain:

\[
\mathcal{M}(r) \frac{d\mathcal{B}(r)}{dr} = \mathcal{S}(r),
\]

where

\[
\mathcal{B}(r) = \begin{pmatrix} b_1(r) \\ b_2(r) \\ b_3(r) \\ b_4(r) \end{pmatrix}, \quad \mathcal{S}(r) = \begin{pmatrix} 0 \\ 0 \\ -G_2(r) \\ 0 \end{pmatrix},
\]

and \(\mathcal{M}(r)\) is the matrix that appeared in the proof of lemma 3.5. Cramer’s formula then yields:

\[
b_m'(r) = (-1)^{m+1} 3^{-1} \cdot 2^{-12} G_2(r) D_{2,m}(r) r^2 (r^2 + 1)^2, \quad m = 1, 2, 3, 4,
\]

\[
D_{2,m}(r) = \begin{vmatrix} \psi_i(r) & \psi_j(r) & \psi_k(r) \\ \omega_i(r) & \omega_j(r) & \omega_k(r) \\ \omega'_i(r) & \omega'_j(r) & \omega'_k(r) \end{vmatrix}, \quad i, j, k \in \{1, 2, 3, 4\} \setminus \{m\}, \quad i < j < k.
\]

Our next goal is to compute the asymptotics of the functions \(b_m'(r)\) as \(r \to 0\). To this end we first compute the asymptotics of \(D_{2,m}(r)\) and \(G_2(r)\). Using \((3.52a)-(3.52c), (3.52e)\) in theorem 3.2 with \(L = 2\) we obtain:

\[
D_{2,1}(r) = -2r^{-3} (1 + O(r)), \quad D_{2,2}(r) = -3 \cdot 2^6 r (1 + O(r)), \quad D_{2,3}(r) = -3 \cdot 2^3 r^{-3} (1 + O(r)), \quad D_{2,4}(r) = -3 \cdot 2^6 r (1 + O(r))
\]

as \(r \to 0\). On the other hand, \((3.23b)\) and \((3.49)\) imply:

\[
G_2(r) = 3 \cdot 2^4 (3 (2\varepsilon_\ell \varepsilon_\ell,\tau - \varepsilon_\ell^2) B_{2,3} - (B_{2,3})^2) r^2 (1 + O(r))
\]
as $r \to 0$. Combining (3.63a)–(3.65) we obtain, after integration:

\[ b_1 (r) = \beta_1 - 2^{-4} \left( 3 \left( 2 \xi_t \xi_{t,t} - \xi_t^3 \right) B_{2,3}^2 - \left( B_{2,3} \xi_t \right)^2 \right) r^2 \left( 1 + O (r) \right), \]  

(3.66a)

\[ b_2 (r) = 2^{-4} \left( 3 \left( 2 \xi_t \xi_{t,t} - \xi_t^3 \right) B_{2,3}^2 - \left( B_{2,3} \xi_t \right)^2 \right) r^6 \left( 1 + O (r) \right), \]  

(3.66b)

\[ b_3 (r) = \beta_3 - 2^{-4} \cdot 3 \left( 2 \xi_t \xi_{t,t} - \xi_t^3 \right) B_{2,3}^2 - \left( B_{2,3} \xi_t \right)^2 \right) r^2 \left( 1 + O (r) \right), \]  

(3.66c)

\[ b_4 (r) = 2^{-4} \left( 3 \left( 2 \xi_t \xi_{t,t} - \xi_t^3 \right) B_{2,3}^2 - \left( B_{2,3} \xi_t \right)^2 \right) r^6 \left( 1 + O (r) \right) \]  

(3.66d)

as $r \to 0$ for some $\beta_1, \beta_3 \in \mathbb{R}$ to be determined, where we have used also (3.50). Notice that we can compute the functions $b_m (r)$ by means of:

\[ b_m (r) = \beta_m + \frac{(-1)^{m+1}}{3 \cdot 2^{12}} \int_0^r G_2 (\eta) \eta^2 (\eta^2 + 1)^2 D_{2,m} (\eta) \, d\eta, \quad m = 1, 2, 3, 4; \]  

(3.67a)

\[ \beta_2 = \beta_4 = 0. \]  

(3.67b)

We now proceed to compute the asymptotics of the terms $b_m (r)$ as $r \to \infty$. To this end we need to derive the asymptotics of the determinants $D_{2,m}$. It follows from theorem 3.2 that:

\[ D_{2,1} (r) = -2^8 \cdot 3 C_2 K_2 r^{5} (1 + o (1)) \]  

(3.68a)

\[ D_{2,2} (r) = 2^6 C_2 K_2 r^{2 \sqrt{7} - 3} (1 + o (1)) \]  

(3.68b)

\[ D_{2,3} (r) = -2^6 \cdot 3 C_2 r^{-2 \sqrt{7} - 3} (1 + o (1)) \]  

(3.68c)

\[ D_{2,4} (r) = -2^6 \cdot 3 C_2 r^{-2 \sqrt{7} - 3} (1 + o (1)) \]  

(3.68d)

as $r \to \infty$. Using (3.49) as well as (3.23b) we obtain:

\[ G_2 (r) \sim -\frac{8 B_{2,3}}{r^2} \left( 2 \xi_t \xi_{t,t} - \xi_t^3 \right) + \left( B_{2,3} \xi_t \right)^2 \xi_t^2 \]  

as $r \to \infty$.  

(3.69)

Using now (3.68a)–(3.68d) as well as (3.52a), (3.52b), (3.52d), (3.52f), (3.67a), and (3.67b) we obtain the following asymptotics:

\[ b_1 (r) \sim \frac{C_2 K_2 B_{2,3}}{2} \left( 2 \xi_t \xi_{t,t} - \xi_t^3 \right) + \left( B_{2,3} \xi_t \right)^2 \xi_t^2 \log r \]  

as $r \to \infty$,  

(3.70)

\[ b_2 (r) \sim \frac{C_2 K_2 B_{2,3}}{3 \cdot 2^4 \sqrt{2} + 1} \left( 2 \xi_t \xi_{t,t} - \xi_t^3 \right) + \left( B_{2,3} \xi_t \right)^2 \xi_t^2 r^{2 \sqrt{2} + 2} \]  

as $r \to \infty$.  

(3.71)

In the case of $b_2 (r)$ we have that $\int_0^\infty |G_2(\eta)| \eta^2 (\eta^2 + 1)^2 |D_{2,1} (\eta)| \, d\eta < \infty$. Assuming that $\beta_3 = O (\xi_t^3)$ (w.l.o.g.), as it corresponds to this class of terms, we would then obtain:

\[ b_2 (r) \to B_{4,2} + \beta_3 + \frac{1}{3 \cdot 2^{12}} \int_0^\infty G_2 (\eta) \eta^2 (\eta^2 + 1)^2 D_{2,3} (\eta) \, d\eta, \]  

(3.72)

where $B_{4,2} = O (\xi_t^3)$ (w.l.o.g.) uniformly for large $r$. Finally:

\[ b_4 (r) \sim \frac{C_2 K_2 B_{2,3}}{2^4 (\sqrt{2} + 1)} \left( 2 \xi_t \xi_{t,t} - \xi_t^3 \right) + \left( B_{2,3} \xi_t \right)^2 \xi_t^2 \xi_t^{2 \sqrt{2} + 2} \]  

as $r \to \infty$.  

(3.73)

We now compute the asymptotics of $Q_{4,2,2} (r)$, $V_{4,2,2} (r)$. We use (3.52a), (3.52b), (3.52d), (3.52f) combined with (3.70)–(3.73) to obtain the asymptotics of $V_{4,2,2}$ as in (3.61b) and

\[ Q_{4,2,2} (r) = 16 K_2 B_{4,2} r^{2 \sqrt{2} - 2} - \frac{C_2 K_2 B_{2,3}}{\sqrt{2} + 1} \left( 2 \xi_t \xi_{t,t} - \xi_t^3 \right) + \left( B_{2,3} \xi_t \right)^2 \xi_t^2 \]  

\[ - \frac{\eta^3 \eta^3}{3 \cdot 2^{12}} \int_r^\infty G_2 (\eta) \eta^2 (\eta^2 + 1)^2 D_{2,3} (\eta) \, d\eta + O \left( \frac{\xi_t^2}{r^{4 - 2 \sqrt{2}}} \right) \text{ (w.l.o.g.)} \]  

(3.74)
as \( r \to \infty \). It is important to remark that in the computation of the asymptotics of \( V_{4,2,2} \) there is a cancellation of the leading order of \( b_1 (r) a_0 (r) + b_3 (r) a_0 (r) \). We now estimate the integral term on the right-hand side of (3.74). The leading order of the integral is then computed as:

\[
\int_r^\infty G_2 (\eta) \eta^2 (\eta^2 + 1)^2 D_{2,3} (\eta) \, d\eta \sim 2^8 \cdot 3C_2 B_{2,3} \left( 2k_1 \epsilon \eta \epsilon, \tau - \epsilon^2 \right) + \left( \frac{B_{2,3}}{B_{2,3}} \right) \eta \left( \frac{r^2}{2} \right)^{1/2} - 1
\]

as \( r \to \infty \). Combining this formula with (3.74) as well as (3.52a) we obtain (3.61a).

**Remark 3.7.** It will be seen later in section 5.5 that in the outer region the terms \( K_2 B_{4,2} r^{2 \sqrt{2}} \) would give a contribution for \( \Phi \) of order \( B_{4,2} \epsilon^{4 + 2 \sqrt{2}} \) in the outer variables. In order for this term to be smaller than \( \epsilon^2 \) we would need \( B_{4,2} = \mathcal{O} (\epsilon^{4 + 2 \sqrt{2}}) \). This basically implies that \( B_{4,2} \) is very small in this region.

### 3.7. Computation of \((U_{4,2,3}, W_{4,2,3})\)

We now compute the functions \( U_{4,2,3}, W_{4,2,3} \) which satisfy (3.38a), (3.38b) with \( k = 3 \) together with boundary condition (3.39), (3.40). We can ignore the presence of homogeneous terms of the equations, because all such terms can be included in the parameters \( B_{2,3}, B_{3,3} \) (see (3.30)). We can assume also that the angular dependence of the functions \( U_{4,2,3}, W_{4,2,3} \) is \( \cos (4\theta) \): \( U_{4,2,3} = Q_{4,2,3} (r, \tau) \cos (4\theta) \), \( W_{4,2,3} = V_{4,2,3} (r, \tau) \cos (4\theta) \). The system is then reduced to:

\[
0 = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial Q_{4,2,3}}{\partial r} \right) - \frac{16}{r^2} Q_{4,2,3} - \frac{du_4}{dr} \frac{\partial V_{4,2,3}}{\partial r} + 2u_4 Q_{4,2,3} = \frac{\partial Q_{4,2,3}}{\partial r} + G_3 (r),
\]

\[
0 = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V_{4,2,3}}{\partial r} \right) - \frac{16}{r^2} V_{4,2,3} + Q_{4,2,3}.
\]

The conditions (3.39) and (3.40), respectively, imply:

\[
Q_{4,2,3} (0, \tau) = 0, \quad V_{4,2,3} (0, \tau) = 0.
\]

**Lemma 3.8.** For any fixed \( \tau \), the problem (3.75a)–(3.75b)–(3.75c) has a two-dimensional family of solutions parametrized by \( c_1 (\infty), c_3 (\infty) \). Moreover, its asymptotics as \( r \to \infty \) are given by:

\[
Q_{4,2,3} (r, \tau) = 16K_4 c_3 (\infty) r^{2 - \sqrt{2}} + \left( 24c_1 (\infty) + \sqrt{3} C_4 K_4 (B_{2,3})^2 \right) r + o (r),
\]

\[
V_{4,2,3} (r, \tau) = -4K_4 c_3 (\infty) r^{2 - \sqrt{2}} + 3c_1 (\infty) r^{4 + \left[ 2c_1 (\infty) - \frac{\sqrt{3} C_4 K_4 (B_{2,3})^2}{24} \right]} r^2 + o (r^2)
\]

as \( r \to \infty \), where \( C_4 \) and \( K_4 \) are the constants as in theorem 3.2 and \( B_{2,3} = B_{3,3} (\tau) \) is the parameter in (3.30).

**Proof.** Since the proof is similar to the one of lemma 3.6, we give only the main idea. Looking for solutions of (3.75a), (3.75b) with the form: \( Q_{4,2,3} = \sum_{i=1}^4 c_i (r, \epsilon) \psi_i (\tau), \quad V_{4,2,3} = \sum_{i=1}^4 c_i (r, \epsilon) \omega_i (\tau) \) under the constraint: \( \sum_{i=1}^4 c_i (r, \epsilon) \psi_i (\tau) - \sum_{i=1}^4 c_i (r, \epsilon) \omega_i (\tau) = 0 \), we obtain: \( c_i (r, \epsilon) = (-1)^{m+1} 2^{-14} \cdot 3^{-1} \cdot 5^{-1} r^2 (r^2 + 1)^2 G_3 (r) D_{4,m} (r) \) with determinants \( D_{4,m}, m = 1, 2, 3, 4, \) similarly defined to (3.63b). We then compute the asymptotics of \( D_{4,m} \) and \( G_3 \) both as \( r \to 0 \) and as \( r \to \infty \), using theorem 3.2 as well as (3.31) and (3.37c). Here
we should compute up to the second corrective terms of \( D_{4,2} \) and \( D_{4,4} \) as \( r \to \infty \) due to the cancellation of the leading term of the term \( c_3(r) \omega_2(r) + c_4(r) \omega_3(r) \), which contributes to the second corrective term of the asymptotics of \( V_{4,2,3} \) as \( r \to \infty \). The remaining modes may be readily computed.

**Remark 3.9.** To see the contribution due to \( c_3(r) \omega_3(r) \) we need to study the asymptotics of \( Q_{4,2,3} \). Using (3.52d) we obtain the matching condition:

\[
Q_{4,2,3}(r, \tau) \sim 16 c_3(\infty) K r^{2\sqrt{3} - 2} \quad \text{as} \quad r \to \infty.
\]

This contribution would give terms of order \( \varepsilon^{4-2\sqrt{3}/2} = \varepsilon^{6-2\sqrt{3}} \gg \varepsilon^2 \) in the self-similar region where \( |y - \bar{y}_i| \) is of order one. Then the contribution of this term to \( \Phi \) would be much larger than one unless \( c_3(\infty) \) is small as \( \tau \to \infty \). It will be seen in section 5.5 that \( c_3(\infty) = O(\varepsilon^{2\sqrt{3}/2}) \).

### 4. Outer expansions

In the analysis of inner expansions we have derived the asymptotics of the solution for a general number of peaks, but we will compute outer expansions only for the case of two peaks for the reason stated in the previous sections. The notation of the singularities has been denoted by \( \{y_i\}_{i=1}^N \), which solves (2.5), in the derivation of inner expansions. In the particular case where \( N = 2 \) we will write \( y_1 = a \) and \( y_2 = -a \) with \( |a| = 2 \). We may assume that \( a = (2, 0) \) since the equation (2.5) is rotationally invariant.

In this section we derive outer expansions for the solution of (1.1a), (1.1b), i.e. for regions where \( |y| \) is of order one. To this end we argue as in the derivation of (3.40)–(3.48) in [22].

We look for expansions with the form:

\[
\Phi(y, \tau) = \varepsilon^2 \Omega(y) + \varepsilon^2 \nabla y \cdot Z(y) + \cdots, \quad (4.1a)
\]

\[
W(y, \tau) = W_0(y, \tau) + \varepsilon^2 W_1(y, \tau) + \cdots. \quad (4.1b)
\]

Then, to the leading order, we obtain: 
\( \Delta W_0 = 0 \), \( y \neq y_i \) with matching condition:

\[
\nabla_y W_0(y) \sim -4(y \mp a)/|y \mp a|^2 \quad \text{as} \quad y \to \pm a, \quad \text{where} \quad a = (2, 0).
\]

Then:

\[
\nabla_y W_0(y) = -\left( \frac{4(y - a)}{|y - a|^2} + \frac{4(y + a)}{|y + a|^2} \right). \quad (4.2)
\]

Neglecting the terms of order \( \varepsilon^2 \) we obtain:

\[
\Phi_\tau = \Delta \Phi - \frac{y \cdot \nabla \Phi}{2} + \left( \frac{4(y - a)}{|y - a|^2} + \frac{4(y + a)}{|y + a|^2} \right) \cdot \nabla \Phi - \Phi. \quad (4.3)
\]

Plugging the expansion (4.1a) into (4.3) we obtain the equations:

\[
L(\Omega) = 0, \quad y \neq \pm a, \quad (4.4)
\]

\[
L(Z) = 2\Omega, \quad y \neq \pm a, \quad (4.5)
\]

where

\[
L(\Omega) = \Delta \Omega - \frac{y \cdot \nabla \Omega}{2} + \left( \frac{4(y - a)}{|y - a|^2} + \frac{4(y + a)}{|y + a|^2} \right) \cdot \nabla \Omega - \Omega. \quad (4.6)
\]

Due to (2.8) and (3.9) we should impose the following matching condition for \( \Omega \):

\[
\Omega(y) \sim \frac{8}{|y \mp a|^4} \quad \text{as} \quad y \to \pm a. \quad (4.7)
\]

On the other hand, in order to obtain matchings in the regions where \( |x| \) is of order one, we must assume that \( \Omega(y) \) increases at most algebraically as \( |y| \to \infty \).
The function $\Omega$ cannot be computed in this case by means of a closed formula as in the radial case considered in [22]. Nevertheless, it is possible to prove that the problem (4.4)–(4.6)–(4.7) defines uniquely a function $\Omega$ with the properties required to describe the leading asymptotics of $\Phi$ in the outer region. More precisely, the following result holds.

**Lemma 4.1.** Assume that $|a| = 2$. Then for every $D_1, D_2 \in \mathbb{R}$ there exists a unique solution of (4.4), (4.6) satisfying:

$$\Omega(y) \sim \frac{D_1}{|y - a|^2} \quad \text{as} \quad y \to a, \quad \Omega(y) \sim \frac{D_2}{|y + a|^2} \quad \text{as} \quad y \to -a, \quad (4.8)$$

$$|\Omega(y)| \leq |y|^m \quad \text{for} \quad |y| \geq 5 \quad \text{for some} \quad m > 0. \quad (4.9)$$

Moreover, the asymptotics of $\Omega$ near the singular points $\pm a$ are given by:

$$\Omega(y) \sim D_1 \left[ \frac{1}{|y - a|^2} + \Psi_1(y - a) + \Psi_2(y - a) + \Psi_3(y - a) + A \right], \quad (4.10a)$$

$$\Omega(y) \sim D_2 \left[ \frac{1}{|y + a|^2} + \Psi_1(y + a) - \Psi_2(y + a) + \Psi_3(y + a) + A \right], \quad (4.10b)$$

where $A = A(D_1, D_2)$ is a constant and:

$$\Psi_1(Y) = \frac{1}{16} \left[ \frac{2}{|Y|^2} + \frac{(a \cdot Y)^2}{|Y|^4} \right], \quad \Psi_2(Y) = \frac{(a \cdot Y)}{96|Y|^2} \left[ 3 - \frac{(a \cdot Y)^2}{|Y|^2} \right],$$

$$\Psi_3(Y) = \frac{1}{256} \frac{(a \cdot Y)^4}{|Y|^2}. \quad (4.11)$$

**Remark 4.2.** Concerning the constant $A$ in the asymptotics (4.10a), (4.10b), we have computed its value using the PDE solver “PDE tool box” from the Matlab package. We have observed that the numerical value of $A$ is between $-1$ and $-0.9$ for $D_1 = D_2 = 8$. The crucial fact is that $A < 0$. This negativity is a sufficient condition to ensure that a certain differential equation satisfied by $\varepsilon \ell$ has solutions approaching zero as $\varepsilon \to \infty$ (See section 5.6).

**Remark 4.3.** A result analogous to lemma 4.1 could be shown using similar methods for singularities of $\Omega$ at arbitrary number of points, or more precisely for operators with the form:

$$L(\Omega) = \Delta \Omega - \frac{y \cdot \nabla \Omega}{2} + \sum_{j=1}^{N} \frac{4(y - a_j)}{|y - a_j|^2} \cdot \nabla \Omega - \Omega \quad (4.11)$$

with $a_j \in \mathbb{R}^2$, $j = 1, \ldots, N$, $a_j \neq a_k$ for $j \neq k$ satisfying:

$$\frac{a_k}{2} = \sum_{j=1, j \neq k}^{N} \frac{4(a_k - a_j)}{|a_k - a_j|^2}. \quad (4.12)$$

If (4.12) does not hold, the form of the asymptotics (4.10a), (4.10b) would contain additional terms with the homogeneity of $1/|y - a_j|^3$. Actually the condition $|a| = 2$ in lemma 4.1 is just the condition (4.12) in the case of two peaks, i.e. $N = 2$. The functions $\{\Psi_i\}_{i=1}^{N}$ that would appear in the study of the general case for (4.11) have similar homogeneity properties to the ones described in lemma 4.1, but slightly different functional forms.

**Remark 4.4.** A characteristic feature of the expansions (4.10a), (4.10b) is the absence of logarithmic terms in $\Psi_3$. Very likely this property holds in general under the assumption (4.12) for every integer $N \geq 2$. We prove, however, such absence of logarithmic terms just in the case of two peaks.
Proof. In what follows we denote by $Y_1 = y - \alpha$, $r_1 = |Y_1|$. $Y_2 = y + \alpha$, $r_2 = |Y_2|$ for notational simplicity. The key point of the proof is to define suitable sub- and supersolutions having the expected asymptotics near the singular points. To this end we define auxiliary functions $\hat{W}_j$ as:

$$
\hat{W}_j(y) = \left[ \frac{1}{r_j} + \Psi_1(Y_j) + (-1)^{j+1} \Psi_2(Y_j) + \Psi_3(Y_j) + \omega_j(Y_j) \right] \eta(Y_j), \quad j = 1, 2,
$$

where $\eta(\xi)$ is a $C^\infty$ cutoff function satisfying $\eta(\xi) = 1$ for $|\xi| \leq 1$, $\eta(\xi) = 0$ for $|\xi| \geq 2$, $0 \leq \eta \leq 1$ and where the functions $\omega_j(Y_j)$ will be defined later. We shall construct a supersolution $\Omega^+$ and a subsolution $\Omega^-$ of the form:

$$
\Omega^+(y) = D_1 \hat{W}_1(y) + D_2 \hat{W}_2(y) + K,
$$

$$
\Omega^-(y) = D_1 \hat{W}_1(y) + D_2 \hat{W}_2(y) - K
$$

with a constant $K > 0$ to be selected later. Some explicit but rather tedious computations yield:

$$
L \left( \frac{1}{r_j} + \Psi_1(Y_j) \right) = -\frac{(a \cdot Y_j)^2}{2r_j^2r_{j\tau(j)}} - \frac{2(a \cdot Y_j)^3}{r_j^2r_{j\tau(j)}^2} - \frac{1}{2} \Psi_1(Y_j) + \frac{(Y_1 \cdot Y_2)}{r_j^2r_{j\tau(j)}} + \frac{4(a \cdot Y_j)}{r_j^2r_{j\tau(j)}} + \frac{1}{64r_j^2} \left[ 1 - \frac{16}{r_{j\tau(j)}} \right] [a|^2(Y_1 \cdot Y_2) - 2(a \cdot Y_1)(a \cdot Y_2) + 4(a \cdot Y_j)^2 \frac{(Y_1 \cdot Y_2)}{r_j^2}] 
$$

for $j = 1, 2$, where $\tau(j) = 3 - j$ for $j = 1, 2$. In the derivation of these formulas we have used:

$$
-\frac{1}{4} Y_{(\tau(j))} \cdot \nabla \left( \frac{1}{r_j} \right) + \frac{4 Y_{(\tau(j))}}{r_j^2} \nabla \left( \frac{1}{r_j^2} \right) = O \left( \frac{1}{r_j^4} \right) \quad \text{as } r_j \to 0, \quad j = 1, 2.
$$

This formula holds due to the assumption that $|a| = 2$. In all these computations we often use: $r_{(\tau(j))}^2 - 16 = r_j^2 - 4(-1)^{j+1}(a \cdot Y_j)$, $j = 1, 2$. It follows from (4.15) that:

$$
L \left( \frac{1}{r_j} + \Psi_1(Y_j) \right) = \frac{(-1)^{j+1}(a \cdot Y_j)}{8} \frac{3r_j^2 - (a \cdot Y_j)^2}{r_j^2} + O \left( \frac{1}{r_j^4} \right) \quad \text{as } r_j \to 0, \quad j = 1, 2.
$$

We now use: $\Delta (\Psi_2)(Y) + \frac{4}{|Y|^2} (Y \cdot \nabla) (\Psi_2)(Y) = -\frac{(a \cdot Y)^2}{|Y|^2} \left[ 3|Y|^2 - (a \cdot Y)^2 \right]$ as well as the fact that the terms in $L$ that are not $\Delta$ and $\frac{a \cdot Y}{|Y|^2}(Y \cdot \nabla)$ yield only lower order contributions. Therefore, after some computations, it follows that:

$$
L \left( \frac{1}{r_j} + \Psi_1(Y_j) + (-1)^{j+1} \Psi_2(Y_j) \right) = -\frac{3(a \cdot Y_j)^2}{16r_j^2} + \frac{(a \cdot Y_j)^4}{16r_j^6} + O \left( \frac{1}{r_j^8} \right) \quad \text{as } r_j \to 0, \quad j = 1, 2.
$$

Using then that: $\Delta \Psi_3 + \frac{4(Y \cdot \nabla)\Psi_3}{|Y|^2} = \frac{3(a \cdot Y)^2}{16r_j^6} - \frac{(a \cdot Y_j)^4}{16r_j^6}$ as well as the fact that the remaining part of $L$ gives only lower order contributions, we obtain:

$$
L \left( \frac{1}{r_j} + \Psi_1(Y_j) + (-1)^{j+1} \Psi_2(Y_j) + \Psi_3(Y_j) \right) = \sum_{k=0}^{5} \beta_k \frac{(a \cdot Y_j)^k}{|Y_j|^{k+1}} + g_j(Y_j), \quad j = 1, 2
$$

(4.16)
with some \( g_j \in L^\infty(B_2((-1)^{j+1}a)) \) and suitable \( \beta_k \in \mathbb{R} \). Using a separation variables argument we may construct functions \( \omega_j, \ j = 1, 2, \) with the form: \( \omega_j(Y_j) = \sum_{k=0}^5 \kappa_k(a \cdot Y_j)^k / |Y_j|^{k+1}, \ j = 1, 2. \) The constant \( \kappa_k \) is selected in order that functions \( \omega_j \) may satisfy:

\[
\left( \Delta + 4\frac{(Y_j \cdot \nabla)}{|Y_j|^2} \right) \omega_j(Y_j) = -\sum_{k=0}^5 \beta_k(a \cdot Y_j)^k / |Y_j|^{k+1}, \ j = 1, 2. \tag{4.17}
\]

We then define the functions \( \hat{W}_j, \ j = 1, 2, \) as in (4.13). It follows from (4.16) and (4.17) that:

\[
L(\hat{W}_j) = f_j(y), \ j = 1, 2, \] 
with \( \|f_j\|_{L^\infty(\mathbb{R})} < \infty. \) Using the boundedness of \( f_j \) as well as the fact that the functions \( \hat{W}_j, \ j = 1, 2, \) are compactly supported, we observe that \( \Omega^- \), \( \Omega^+ \) in (4.14a), (4.14b) are respectively super- and subsolutions for (4.4) in \( \mathbb{R}^2 \setminus [-a, a] \) if \( K \) is chosen sufficiently large.

We now define a family of domains \( D_{5, R} \) as:

\[
D_{5, R} = B_R(0) \setminus [B_2(-a) \cup B_2(a)], \quad 0 < \delta < 1, \quad R > 8. \tag{4.18}
\]

Let us consider the following family of boundary value problems:

\[
L(\Omega_{5, R}) = 0 \quad \text{in } D_{5, R}, \tag{4.19a}
\]

\[
\Omega_{5, R} = D_j \left[ \frac{1}{r_j} + \Psi_1(Y_j) + (-1)^{j+1}\Psi_2(Y_j) + \Psi_3(Y_j) \right] \text{ on } \partial B_\delta \left((-1)^{j+1}a\right), \quad j = 1, 2, \tag{4.19b}
\]

\[
\Omega_{5, R} = 0 \quad \text{on } \partial B_R(0). \tag{4.19c}
\]

Classical results on elliptic equations (see [9, corollary 9.18] for instance) show that the functions \( \Omega_{5, R} \) are uniquely defined for any \( \delta \) and \( R \) in (4.18). Moreover, since \( \Omega^- < \Omega_{5, R} < \Omega^+ \) on \( \bigcup_{j=1}^2 \partial B_\delta \left((-1)^{j+1}a\right) \cup \partial B_R(0) \) for \( K > 0 \) sufficiently large independent of \( \delta \) and \( R \), it follows by comparison that:

\[
\Omega^- < \Omega_{5, R} < \Omega^+ \quad \text{in } D_{5, R}. \tag{4.20}
\]

Classical regularity theory for elliptic equations implies that \( |\nabla^k \Omega_{5, R}|, k = 1, 2, 3, \) are bounded in compact sets of \( D_{5, R}. \) A compactness argument then shows that there exists a smooth function \( \Omega \) satisfying \( \Omega^- < \Omega < \Omega^+ \) in \( \mathbb{R}^2 \setminus [-a, a] \) and a subsequence \( \{ (\delta_\ell, R_\ell) \} \) such that \( (\delta_\ell, R_\ell) \to (0, \infty) \) and: \( \Omega_{5, R} \to \Omega \) as \( \ell \to \infty \) in \( C^2(\Omega) \) for each compact \( \Omega \subset \mathbb{R}^2 \setminus [-a, a]. \)

Then \( L(\Omega) = 0 \) in \( \mathbb{R}^2 \setminus [-a, a]. \) Moreover, the functions

\[
\phi_j(y) := \Omega(y) - D_j \left[ \frac{1}{r_j} + \Psi_1(Y_j) + (-1)^{j+1}\Psi_2(Y_j) + \Psi_3(Y_j) \right], \quad j = 1, 2,
\]

are bounded in a neighbourhood of the points \( [-a, a] \) and satisfies: \( L(\phi) = Q_j(y) |Y_j|, \) where

\[
|Q_j(y)| \leq C \quad \text{in } 0 < |Y_j| \leq 1, \quad j = 1, 2. \tag{4.21}
\]

To conclude the proof it only remains to show that the limits \( \lim_{(\pm a + R)\xi} \phi_1(y) \) and \( \lim_{(\pm a + R)\xi} \phi_2(y) \) exist. To this end we estimate the derivatives of \( \phi \) as follows. For each \( 0 < R < 1 \) we define \( \varphi_R(\xi) = \phi(\pm a + R\xi). \) Then:

\[
\Delta \varphi_R + 4 |\xi|^{-2} \xi \cdot \nabla \varphi_R + a_R(\xi) \cdot \nabla \varphi_R = O(R), \quad 1/4 \leq |\xi| \leq 4, \quad \|a_R(\xi)\| \leq CR. \]

Classical regularity theory for elliptic equations yields \( |\nabla \varphi_R| \leq C \) in \( 1/2 \leq |\xi| \leq 2. \) Therefore:

\[
|\phi_j(y)| + |Y_j| |\nabla \phi_j| \leq C \quad \text{in } 0 < |Y_j| \leq 1, \quad j = 1, 2 \tag{4.22}
\]
for some $C > 0$. In order to prove the existence of the limits $\lim_{y \to \pm \infty} \phi_j(y)$ we use a Fourier analysis argument. Using the polar coordinates defined as $Y_j = y \mp \alpha = \rho_j(\cos \theta_j, \sin \theta_j)$, $j = 1, 2$, we write: $\phi_j(y) = \Phi_j(\rho_j, \theta_j) = \sum_{n=-\infty}^{\infty} c_n(\rho_j) e^{i \alpha n \theta_j}$. The functions $c_n(\rho_j)$ solve a second order ODE, which can be solved explicitly:

$$c_n(\rho_j) = A_{1,n} \rho_j^{\alpha n} + A_{2,n} \rho_j^{-\alpha n} - \int_{\rho_j}^{1} \frac{\rho_j^{\alpha n}}{s^{\alpha n} - \alpha n} Q_{j,n}(s) \, ds - \int_{1}^{\rho_j} \frac{1}{s^{\alpha n} - \alpha n} Q_{j,n}(s) \, ds,$$

where:

$$Q_{j,n}(s) = \frac{1}{2\pi} \int_{0}^{2\pi} Q_j(\pm \alpha + s (\cos \theta_j, \sin \theta_j)) e^{-i \alpha n \theta_j} d\theta_j, \tag{4.23}$$

$$\alpha_n = -2 + \sqrt{4 + n^2}, \quad \alpha^-_n = -2 - \sqrt{4 + n^2}, \quad n \in \mathbb{Z} \tag{4.24}$$

and where $A_{1,n}, A_{2,n}$ are constants related to the Fourier coefficients of the functions $\Phi_j(1, \theta_j)$. These functions are in $C^\infty(S^1)$, for every $\beta > 0$, there exists a constant $C_\beta > 0$ such that

$$|A_{1,n} + |A_{2,n}| \leq C_\beta \frac{1}{1 + |n|^{\beta}}, \quad n \in \mathbb{Z}. \tag{4.25}$$

On the other hand, due to (4.21) and (4.22) the coefficients $c_n(\rho_j)$ and $Q_{j,n}(\rho_j)$ are bounded for $0 < \rho_j < 1$. This implies: $A_{2,n} = \int_{1}^{\rho_j} s^{-\alpha n} Q_{j,n}(s)/(\alpha^-_n - \alpha^+_n) \, ds$, whence:

$$c_n(\rho_j) = A_{1,n} \rho_j^{\alpha n} - \int_{\rho_j}^{1} \frac{\rho_j^{\alpha n}}{s^{\alpha n} - \alpha n} Q_{j,n}(s) \, ds + \int_{1}^{\rho_j} \frac{1}{s^{\alpha n} - \alpha n} Q_{j,n}(s) \, ds,$$

$j = 1, 2, \quad n \in \mathbb{Z}$

Using (4.21)–(4.25) we obtain:

$$|c_n(\rho_j) - A_{1,0} \delta_{n,0}| \leq \frac{C \rho_j^{\gamma_{\Omega}^2 - 2}}{1 + |n|^{\delta}} + \frac{C \rho_j^{\gamma_{\Omega}^2 - 2}}{1 + |n|^{\delta}},$$

where symbol $\delta_{n,0}$ stands for the Kronecker delta. Then:

$$|\phi_j(y) - A_{1,0} \delta_{n,0}| \leq \rho_j^{\gamma_{\Omega}^2 - 2} \sum_{n=-\infty}^{\infty} \frac{1}{1 + |n|^{\gamma_{\Omega}^2}} \leq C \rho_j^{\gamma_{\Omega}^2 - 2}$$

and therefore the limits $\lim_{y \to \pm \infty} \phi_j(y)$ exist. The fact that the value of $A$ is the same in (4.10a) and (4.10b) follows by a symmetry argument.

To prove the uniqueness result we construct a supersolution for (4.4). Consider a function $\Omega^* + \Omega$ with $K$ sufficiently large. We then modify the function $\Omega^*$ so that the constant $K$ becomes the polynomial $|y|^m$ for large values of $|y|$. Since the main terms in the operator $L$ for large values of $|y|$ are $2^{-1} |y|^2 \nabla \Omega$ and $-\Omega$, the modified function $\Omega^*$ satisfies $L(\Omega^*) \leq 0$. This modification is possible, because these leading terms yield positive contributions. The difference of two solutions of (4.4) satisfying (4.8) may be estimated by $\varepsilon \Omega^*$ for $y \to \pm \alpha$ and for $|y| = R$ with $\varepsilon > 0$ arbitrarily small and $R > 0$ large enough. A comparison argument then shows that the difference is bounded by $\varepsilon \Omega^*$ in the regions $B_R(0) \setminus B_\delta(\pm \alpha)$ for $\delta$ small. Taking the limit $\varepsilon \to 0$ we know that both functions are the same, whence the uniqueness follows.

Remark 4.5. Equation (4.4) suggests that $\Omega(y) \sim \varphi(\theta) / |y|^2$ as $|y| \to \infty$ for some function $\varphi(\theta)$ whose precise formula does not seem easy to derive. However, we will not attempt to compute this function in detail in this paper.
We also need to study the function $Z$, a solution of (4.5), satisfying:

$$Z (y) = o \left( \frac{1}{|y + a|^4} \right) \quad \text{as } y \to \pm a, \quad (4.26)$$

and

$$|Z (y)| \leq |y|^m \quad \text{for } |y| \geq 5 \quad \text{for some } m > 0. \quad (4.27)$$

**Lemma 4.6.** Suppose that $|a| = 2$. Let $\Omega$, $D_1$, and $D_2$ be as in proposition 4.1. Then there exists a unique solution of (4.5) satisfying (4.26) and (4.27). Its asymptotic behaviour near the singular points $\{-a, a\}$ is given by:

$$Z (y) \sim D_1 \left[ -\frac{1}{2} \frac{1}{|y - a|^2} + \frac{1}{8} \log |y - a| - \frac{1}{16} \frac{(a \cdot (y - a))^2}{|y - a|^2} + B \right] \quad \text{as } y \to a, \quad (4.28a)$$

$$Z (y) \sim D_2 \left[ -\frac{1}{2} \frac{1}{|y + a|^2} + \frac{1}{8} \log |y + a| - \frac{1}{16} \frac{(a \cdot (y + a))^2}{|y + a|^2} + B \right] \quad \text{as } y \to -a \quad (4.28b)$$

for some constant $B \in \mathbb{R}$.

**Proof.** The proof is similar to the one of lemma 4.1. Due to the linearity and the symmetry of the problem it is enough to consider the case $D_1 = 1$, $D_2 = 0$. We can obtain sub- and supersolutions of (4.5) with the help of the auxiliary function:

$$\tilde{W} (y) = \left[ -\frac{1}{2} \frac{1}{|Y_1|^2} + \frac{1}{8} \log |Y_1| + \frac{1}{8} - \frac{1}{16} \frac{(a \cdot Y_1)^2}{|Y_1|^2} \right] \eta (Y_1) \quad (4.29)$$

with the $C^\infty$ cutoff function $\eta (\xi)$ as in lemma 4.1. We then construct sub- and supersolutions in the form: $Z^\pm (y) = \tilde{W} (y) \pm K$. The terms between the brackets in (4.29) have been chosen in order to balance terms of the function $\Omega (y)$. This requires tedious but otherwise straightforward computations. Arguing then as in the proof of lemma 4.1 we obtain the desired result. 

We also need to study the asymptotics of the function $W_{1}$ in (4.1b), which solves the equation:

$$-\Delta W_1 = \Omega, \quad y \neq \pm a. \quad (4.30)$$

**Lemma 4.7.** Suppose that $|a| = 2$. Let $\Omega$ be as in lemma 4.1. Then for every $M^{(a)}_{1,W_1}, M^{(-a)}_{1,W_1} \in \mathbb{R}$ there exists at least one solution of (4.30) satisfying:

$$W_1 (y) - D_1 G (y - a) - D_1 M^{(a)}_{1,W_1} \log |y - a| = O (1) \quad \text{as } y \to a, \quad (4.31a)$$

$$W_1 (y) - D_1 G (y + a) - D_1 M^{(-a)}_{1,W_1} \log |y + a| = O (1) \quad \text{as } y \to -a, \quad (4.31b)$$

$$\lim_{|y| \to \infty} \frac{|W_1 (y)|}{|y|} = 0, \quad (4.31c)$$

where

$$G (Y) = -\frac{1}{4 |Y|^2} - \frac{1}{8} \left( \log |Y| \right)^2 + \frac{1}{32} \cos (2\theta) \quad (4.31d)$$

and where $\theta = 0 (Y)$ is the angle between the $y_1$-axis and $Y$. 

Moreover, two arbitrary solutions of (4.30) satisfying (4.31a)–(4.31c) differ by a constant. The asymptotics of \( \mathcal{W}_1(y) \) as \( y \to a \) and \( y \to -a \) are respectively given by:

\[
\mathcal{W}_1(y) = D_1 G(y - a) + D_1 M_1^{(a)} \log |y - a| - \frac{D_1}{27/3} |y - a| \cos (3\delta_{(a)}) + A_1^{(a)} + D_1 K_1^{(a)} \cdot (y - a) + O \left( |y - a|^2 \right),
\]

where \( \delta_{(a)}, \delta_{(-a)} \) are the angles between the horizontal axis and the vectors \( y - a \) and \( y + a \), respectively. The vectors \( K_1^{(a)}, K_1^{(-a)} \in \mathbb{R}^2 \) and the constants \( A_1^{(a)}, A_1^{(-a)} \in \mathbb{R} \) depend on an affine manner on the values of \( M_1^{(a)}, M_1^{(-a)} \).

**Proof.** The proof is similar to the one of lemma 4.1. It reduces just to compute explicitly the solutions of the Poisson equation having the terms in the asymptotics (4.10a), (4.10b) as sources. After removing the effect of these singular contributions, it only remains to obtain a solution of the Poisson equation with a source term bounded by \( C/(1 + |y|^2) \). This can be made using a supersolution behaving as \( C \left( \log |y| \right)^2 \) for large values of \( |y| \). The uniqueness result is a consequence of Liouville’s theorem for the Laplace equation. ■

5. Matching of the different terms

In this section we match the different terms in the inner and outer expansions, consequently deriving evolution equations for the functions \( \varepsilon_\ell (\tau) \) providing the width of the peaks. We will assume henceforth that, due to symmetry considerations, all the functions \( \varepsilon_\ell \) at the different peaks are the same. In general this does not need to be so. Moreover, there are non-symmetric singular self-similar solutions (see section 2) for which the corresponding values of the functions \( \varepsilon_\ell (\tau) \) cannot be expected to be the same. The question of determining the relative sizes of the functions \( \varepsilon_\ell (\tau) \) is interesting, but it will not be considered in this paper. Due to (3.3) this question is equivalent to determining the relative sizes of the maximum value of the function \( u \) at each of the different peaks. (Notice however that all of them have the same mass \( 8\pi \).)

We now describe how to match the different terms in the asymptotics as \( |\xi| \to \infty \) of the expansions (3.9), (3.10). We begin with the leading order terms. Restricting our analysis to the case of two peaks, we assume henceforth that \( \varepsilon_1 = \varepsilon_2 = \varepsilon \). We write for further reference the expansion of \( \nabla_y \mathcal{W}_0 \) near \( y = a \) (see (4.2)):

\[
\nabla_y \mathcal{W}_0 = -\frac{4Y}{|Y|^2} \frac{2a}{|a|^2} - \frac{Y}{|a|^2} + \frac{2a (a \cdot Y)}{|a|^4} + \frac{|Y|^2 a}{2 |a|^4} - \frac{2 (a \cdot Y)^2 a}{|a|^6} + \frac{(a \cdot Y) Y}{|a|^4} - \frac{(a \cdot Y) |Y|^2 a}{|a|^6} + \frac{2 (a \cdot Y)^3 a}{16 |a|^6} + \frac{|Y|^2 Y}{4 |a|^4} - \frac{(a \cdot Y)^4 Y}{|a|^6} - \ldots,
\]

where \( Y = y - a \). All the terms in this formula have been kept until the third order in \(|Y|\).

5.1. Leading terms

The leading order in (3.9), (3.10) is respectively given by the functions \( u_0 (\xi) \), \( v_0 (\xi) \). We will denote as \( \Phi_{0, \text{match}}, W_{0, \text{match}} \) the terms to be matched in the intermediate region.
|ξ| ≫ 1, |y − y_ξ| ≪ 1 due to these terms in the expansion. Keeping just terms of order ε_ξ^2 (w.l.a) in the region where |y − y_ξ| becomes of order one, we then obtain:

\[ \Phi_{0,\text{match}}(y, \tau) = \frac{8\varepsilon_ξ^2}{|y − y_ξ|^4}, \quad \nabla_y W_{0,\text{match}}(y, \tau) = -\frac{4(y − y_ξ)}{|y − y_ξ|^2}. \]  

(5.2)

The matching of the term ∇_y W_{0,\text{match}} has been already taken into account in the derivation of (4.2) that gives the asymptotics of the chemical field for |y| of order one up to corrections of order ε_ξ^2. On the other hand, due to (4.10a), (4.10b) we can match (5.2) with (4.1a), assuming D_1 = 8.

### 5.2. Terms coming from U_1, W_1

We now match the terms U_1, W_1 in (3.9), (3.10) with suitable terms in (4.1a), (4.1b) respectively. We denote as Φ_{1,\text{match}}, W_{1,\text{match}} the terms to be matched in the intermediate region |ξ| ≫ 1, |y − y_ξ| ≪ 1 due to these terms in the expansion. Notice that (3.20) shows: Φ_{1,\text{match}} = 0, ∇_y W_{1,\text{match}} = −y_ξ/2. We only need to match the term ∇_y W_{1,\text{match}} with some of the terms in (4.2). Let \lim_{y \to -\infty} y_ξ = a, since the case \lim_{y \to +\infty} y_ξ = −a can be treated in a symmetric way. The most singular term of (4.2) has been matched with ∇_y W_{1,\text{match}}. The next order in the expansion of ∇_y W_0 is −2a/|a|^2 (see (5.1)) and this matches with −y_ξ/2 if we impose |a| = 2. Therefore the matching of the terms of order ε_ξ (w.l.a) in the region where |ξ| is of order one is possible if we impose that the drift terms due to the change to the self-similar variables and the chemotactic terms balance with each other.

### 5.3. Terms coming from U_2, W_2

Let us denote as Φ_{2,\text{match}}, W_{2,\text{match}} the terms appearing in the matching condition arising from the terms U_2, W_2 in the inner expansion. Using (3.24), (3.25), (3.26) and (3.30) we obtain the following formulas in the intermediate region ε_ξ ≪ |y − y_ξ| ≪ 1:

\[ \Phi_{2,\text{match}}(y, \tau) \sim (2\varepsilon_ξ^2 \varepsilon_ξ \tau - \varepsilon_ξ^4) \left[ -\frac{2}{|y − y_ξ|^2} + O\left(\varepsilon_ξ^2 \frac{||y| - |y_ξ|^2|}{|y - y_ξ|^2}\right) \right] + \frac{8B_{2,3} \cos (2\theta)}{|y - y_ξ|^2} + \frac{8\bar{B}_{2,3} \sin (2\theta)}{|y - y_ξ|^2} + \cdots, \]  

(5.3a)

\[ W_{2,\text{match}}(y, \tau) \sim \tilde{y}_{ξ,τ} \cdot (y − y_ξ) + V_{2,1} + \frac{B_{2,3}}{\varepsilon_ξ^2} |y − y_ξ|^2 \cos (2\theta) \]  

\[ + \frac{\bar{B}_{2,3}}{\varepsilon_ξ^2} |y − y_ξ|^2 \sin (2\theta) + \cdots, \]  

(5.3b)

where V_{2,1} is a radial term. It is convenient to rewrite (5.3b) in Cartesian coordinates:

\[ W_{2,\text{match}}(y, \tau) \sim \tilde{y}_{ξ,τ} \cdot (y − y_ξ) + V_{2,1} + \frac{B_{2,3}}{\varepsilon_ξ^2} \left[ (y_1 - \tilde{y}_{ξ,1})^2 - (y_2 - \tilde{y}_{ξ,2})^2 \right] \]  

\[ + \frac{2\bar{B}_{2,3}}{\varepsilon_ξ^2} (y_1 - \tilde{y}_{ξ,1}) (y_2 - \tilde{y}_{ξ,2}) + \cdots \]  

(5.4)

The last two terms of this formula can be matched with the quadratic terms of the expansion of ∇_y W_0(y) near the points y_ξ. Using (5.1) it follows that ∇_y W_{2,\text{match}} matches with ∇_y W_0(y) if

\[ B_{2,3} = \frac{\varepsilon_ξ^4}{8}, \quad \bar{B}_{2,3} = 0, \]  

(5.5)
where we have used that |a| = 2. Using (5.5) in (5.3a) and transforming the resulting formula to Cartesian coordinates we obtain:

$$\Phi_{2, \text{match}}(y, \tau) \sim \varepsilon_{1}^{2} \left[ \frac{3(y_{1} - 2)^{2} + (y_{2})^{2}}{|y - y|^{4}} \right] - \frac{4 \varepsilon_{1} \varepsilon_{\ell, \tau}}{|y - y|^{2}} + O \left( \frac{\varepsilon_{1}^{2}}{|y - y|^{4}} \right) \ldots \quad (w.l.o)
$$

(5.6)

and the first term in (5.6) matches exactly with the term in the outer region multiplying $\Psi_{1}(y - a)$ in (4.10a) due to the fact that $D_{1} = 8$.

It is illuminating to compute $\overline{y}_{\ell, \tau}$ in the first term of (5.4), matching the first term on the right-hand side of this formula with one of the terms in the outer expansion (4.1b). We will examine the case in which $\lim_{y \to \infty} \overline{y}_{\ell} = a$, since the case in which $\lim_{y \to \infty} \overline{y}_{\ell} = -\alpha$ is similar. Using (4.1b), (4.32a) as well as the fact that $D_{1} = 8$ we obtain the following terms in the outer expansion of $V_{y} W$, which require to be matched with terms from the inner expansion:

$$\varepsilon_{1}^{2} \left[ \frac{4Y}{|Y|^{2}} - 2 \log |Y| \right] \frac{Y}{|Y|^{2}} - \frac{1}{2} \sin \left( \frac{\theta_{(a)}}{2} \right) \frac{Y^{\perp}}{|Y|^{2}} + \frac{8M_{1, W_{1}}^{01} Y}{|Y|^{2}} + \frac{\cos \left( \frac{3\theta_{(a)}}{2} \right) Y}{2^{2} \cdot 3} + \frac{\sin \left( \frac{3\theta_{(a)}}{2} \right) Y^{\perp}}{|Y|^{2}} + O \left( |Y| \right) \right],
$$

(5.7)

where $Y = y - a$ and we define $Y^{\perp} = (-y_{2}, y_{1})$ for $Y = (y_{1}, y_{2}) \in \mathbb{R}^{2}$.

The first term in (5.7) matches with a similar term coming from the function $v_{2}$ in (2.8) using the Taylor series as $|\xi| \to \infty$. The second term in (5.7) matches, to the leading order, with the first term on the right-hand side of (3.25). The third term in (5.7) matches with the first corrective term that results in the Taylor expansion of $V_{2, 3}$ in (3.30) as $|\xi| \to \infty$. Notice that we use also (5.5) in this matching. The matching of the term $8M_{1, W_{1}}^{01} Y / |Y|^{2}$ plays a relevant role in determining $\overline{y}_{\ell, \tau}$. Indeed, the contributions of similar order in the inner region are due to the terms $\log \left( r^{2} / r \right)$ and $-2 / r$ in (3.25). Due to the change of variables $r = |\xi| = |Y| / \varepsilon_{\ell}$ it follows that, to the leading order, $8M_{1, W_{1}}^{01} = \log \varepsilon_{\ell}$. Lemma 4.7 then yields that $K_{1}^{(a)} = B_{1}^{(a)} \log \varepsilon_{\ell}$ as $\tau \to \infty$ to the leading order. We can then match the term $8K_{1}^{(a)} \varepsilon_{1}^{2}$ in (5.7) with the first term in (5.4), whence:

$$\overline{y}_{\ell, \tau} \sim B_{1}^{(a)} \varepsilon_{1}^{2} (\tau) \log \varepsilon_{\ell} (\tau), \quad \overline{y}_{\ell} \sim a - B_{1}^{(a)} \int_{\tau}^{\infty} \varepsilon_{1}^{2} (s) \log \varepsilon_{\ell} (s) \, ds \quad \text{as} \quad \tau \to \infty.$$

This gives the desired asymptotic formula of the peaks stated in (3.2). The terms with the angular dependence $3\theta$ in (5.7) are matched with some of the high order corrections coming from (3.31). However, these terms give smaller contributions and we do not pursue this computation in detail.

5.4. Terms coming from $U_{3}, W_{3}$

We now match the terms coming from $U_{3}, W_{3}$, which can be computed by means of (3.31). We notice that, to the leading order, $U_{3}$ must match with the term $\varepsilon_{1}^{2} \Psi_{2}(Y)$ in (4.10a), (4.10b). Using that $D_{1} = 8$ and $D_{1, \Psi_{2}}(Y) = -6^{-1} \cos \left( \frac{3\theta}{|Y|} \right)$, we can match this term with the leading matching term coming from $U_{3}$, which can be written as (see (3.31)):

$$\Phi_{3, \text{match}}(y) = \frac{16B_{3} \cos \left( \frac{3\theta_{(a)}}{|y - a|} \right)}{\varepsilon_{1}}, \quad \text{whence:}
$$

$$B_{3} = -\frac{\varepsilon_{1}^{2}}{2^{2} \cdot 3}, \quad \overline{B}_{3} = 0.
$$

(5.8)
We can see that this gives also a matching for the terms in (5.1) with angular dependence cos \((3\theta_{(a)})\). Using that \(|\alpha| = 2\) we can write those terms as:

\[
\frac{|Y|^2}{2|\alpha|^2} - 2 \frac{(\alpha \cdot Y)^2 - (\alpha \cdot Y)^2}{|\alpha|^6} + \frac{Y^2}{2^4} Y [(-\cos (2\theta), \sin (2\theta))].
\]

(5.9)

On the other hand, we can compute two terms of the asymptotics of \(\nabla_{\xi} (V_{2}(r) \cos(3\theta))\) as \(|\xi| \to \infty\) using the Taylor series. Rewriting the resulting expansion using the variable \(y\), we obtain the following terms to be matched from the inner expansion:

\[
\frac{6B_{3}}{\varepsilon_{\ell}} r^2 \cos (2\theta), -\sin (2\theta)) + \frac{B_{3}}{\varepsilon_{\ell}} \left( 2 \cos (3\theta) \frac{Y}{|Y|} - 6 \sin (3\theta) \frac{Y^4}{|Y|^3} \right).
\]

(5.10)

Using (5.8) we observe that the first term in (5.10) matches with the term in (5.9) and the second one matches with the terms in (5.7) with angular dependence \(3\theta\).

5.5. Terms coming from \(U_{4}, W_{4}\)

We now match the asymptotics as \(|\xi| \to \infty\) in the terms \(U (\xi, \tau), V (\xi, \tau)\) with the terms in the outer expansions (4.1a), (4.1b) that are of order \(\varepsilon_{2}^2 \) \((w.l.a)\) as \(y \to \pm \alpha\). These are the terms in the outer expansion multiplying \(\Omega_{2} (Y)\) and \(A\) in (4.10a), (4.10b) as well as the terms multiplying \(16^{-1} (a \cdot (y - \alpha))^2 / |y - \alpha|^2\) and \(B\) in (4.28a), (4.28b). Therefore, using also that \(D_1 = 8\), we observe that the outer expansion for \(\Phi\) to be matched as \(y \to \alpha\):

\[
\frac{\varepsilon_{2}^2 (\alpha \cdot Y)^4}{2 Y^4} + 8A\varepsilon_{2}^2 \frac{\varepsilon_{2} \ell \xi_{\ell \tau} (\alpha \cdot Y)^2}{2 |Y|^2} + 8B \varepsilon_{2} \ell \xi_{\ell \tau} + \varepsilon_{2} \ell \xi_{\ell \tau} \log |Y|,
\]

(5.11)

where \(Y = y - \alpha\).

Concerning the inner expansion we notice that the only radial terms giving contributions of order \(\varepsilon_{2}^2 \) \((w.l.a)\) in the matching region are the terms \(U_{4,1} + U_{4,2,1}\). Using (3.35a), (3.35b), and (3.42) as well as the change of variables we observe the following radial terms for \(\Phi\) to be matched:

\[
- (2\varepsilon_{2} \ell \xi_{\ell \tau} - \varepsilon_{2}^2) \left[ \frac{\log |Y|}{2} - \frac{\log \varepsilon_{2}}{2} - \frac{5}{8} \right] + \frac{(2\varepsilon_{2} \ell \xi_{\ell \tau} - \varepsilon_{2}^2)^2}{4\varepsilon_{2}^2} + \varepsilon_{2}^2 / 2^5,
\]

(5.12)

where we have used (5.5). On the other hand, we can decompose the terms in (5.11) in radial terms and in terms with angular dependences \(\cos (2\theta)\) and \(\cos (4\theta)\). Using also that \(|\alpha| = 2\) we observe that the radial terms are: \(2^{-4} \cdot 3\varepsilon_{2}^2 + 8A\varepsilon_{2}^2 + (8B - 1) \varepsilon_{2} \ell \xi_{\ell \tau} + \varepsilon_{2} \ell \xi_{\ell \tau} \log |Y|\).

We notice that the term containing \(\log |Y|\) can be matched, to the leading order, with a similar term in (5.12). On the other hand, the matching of the remaining terms provides an equation for \(\varepsilon_{2} \ell\) in the same manner as in [22]:

\[
\frac{(2\varepsilon_{2} \ell \xi_{\ell \tau} - \varepsilon_{2}^2)}{2} \log \varepsilon_{2} + \frac{5}{8} (2\varepsilon_{2} \ell \xi_{\ell \tau} - \varepsilon_{2}^2) + \frac{(2\varepsilon_{2} \ell \xi_{\ell \tau} - \varepsilon_{2}^2)^2}{4\varepsilon_{2}^2} + \varepsilon_{2}^2 / 2^5
\]

\[
= \frac{3\varepsilon_{2}^2}{16} + 8A\varepsilon_{2}^2 + (8B - 1) \varepsilon_{2} \ell \xi_{\ell \tau}.
\]

(5.13)

We now consider the matching of the terms with angular dependence \(\cos (2\theta)\). The term in the outer region (see (5.11)) with such dependence is:

\[
(4^{-1} \varepsilon_{2}^2 - \varepsilon_{2} \ell \xi_{\ell \tau}) \cos (2\theta).
\]

(5.14)

This term must be matched with the contributions due to \(U_{4,2,2}\). Due to (3.61a) the corresponding outer part to be matched is:

\[
\left[ \frac{16K_{2} B_{4,2}}{\varepsilon_{2} \xi_{2}^{2} \xi_{2}^{2}} |Y|^{2} \xi_{2}^{2} - 2 + \sqrt{2} C_{2} K_{2} \left( \frac{\varepsilon_{2}^2}{4} - \varepsilon_{2} \ell \xi_{\ell \tau} \right) \right] \cos (2\theta).
\]

(5.15)
The matching of (5.14) and (5.15) requires:

\[ B_{4,2} = O \left( (\varepsilon_\ell)_{2^{\sqrt{2}-2}} \right) \quad \text{as} \quad \tau \to \infty. \]  

(5.16)

Computing higher order terms in the outer expansion it would be possible to derive more precise formulas for \( B_{4,2} \). Basically this would require computing higher order asymptotics of the function \( \Omega(y) \) as \( y \to \pm \alpha \). The next order correction to \( \Omega \) in (4.10a) is of order \( C \vert y \vert^{2\sqrt{2}-2} \cos(2\theta) \) for some \( C \in \mathbb{R} \). This would give exactly the behaviour (5.16). However, since the detailed form of these terms will not play any role in the following, we will not continue with this analysis. The matching of (5.14) and (5.15) requires also: \( \sqrt{2} C_2 K_2 = 1 \) and this is just a consequence of (3.60).

We now consider the matching of the terms with dependence \( \cos(4\theta) \). The term with this angular dependence in (5.11) is: \( 2^{-4} \varepsilon_\ell^2 \cos(4\theta) \). This term must be matched with the contributions due to \( U_{4,2,3} \). Due to (3.76a) the inner contribution to be matched is:

\[
\frac{16 K_4 \varepsilon_\ell^2 \left( (\varepsilon_\ell)_{32} \right) \varepsilon_\ell^2 \vert Y \vert^{2\sqrt{2}-2} + \frac{24 c_1 (\infty) + \sqrt{5} C_4 K_4 \left( B_{2,3} \right)^2}{\varepsilon_\ell^2} \cos(4\theta)}{\varepsilon_\ell^2}.
\]

Arguing as in the derivation of (5.16) we observe that \( c_1 (\infty) = O(\varepsilon_\ell^{2\sqrt{2}-2}) \), showing that these terms are very small in the inner region. Taking into account of (3.60) we have:

\[ c_1 (\infty) = \frac{\varepsilon_\ell^4}{2^8 \cdot 3}, \]  

(5.17)

which concludes the matching to this order of the functions \( \Phi \). We can also obtain matchings for the functions \( V \). We are just interested in the first term on the right-hand side of (3.76b) since it gives a term of order one for \( \vert V \vert \) of order one. The remaining terms give contributions of order \( \varepsilon_\ell^4 \) and we will ignore them. The term to be matched for \( V \) is \( 3 (c_1 (\infty) / \varepsilon_\ell^4) \vert Y \vert^4 \cos(4\theta) \). The gradient of this term with respect to \( y \) yields:

\[ \frac{12 c_1 (\infty)}{\varepsilon_\ell^4} \left( \vert Y \vert^3 \cos(4\theta) \frac{Y}{\vert Y \vert} - \frac{12 c_1 (\infty)}{\varepsilon_\ell^4} \vert Y \vert^3 \sin(4\theta) \frac{Y}{\vert Y \vert} \right). \]

In terms of the polar coordinates as well as (5.17), this becomes:

\[ \frac{\vert Y \vert^5}{2^{6\alpha}} (\cos(3\theta) - \sin(3\theta)). \]  

(5.18)

On the other hand the term in (5.1) containing cubic terms is:

\[ \left( \frac{[a_1 \varepsilon_\ell^2] Y Y}{4 a_1} + \left( \frac{[a_1 \varepsilon_\ell^2] Y Y}{4 a_1} - \frac{[a_2 \varepsilon_\ell^2] Y Y}{4 a_2} \right) \right), \]

which can be transformed, using polar coordinates in:

\[ \frac{\vert Y \vert^5}{2^{6\alpha}} (4 \cos^3 \theta - 3 \cos \theta, \sin \theta - 4 \cos^2 \theta \sin \theta). \]  

(5.19)

Standard trigonometric formulas show that (5.18) and (5.19) are the same.

5.6. Analysis of the ODE (5.13) and the derivation of the final profile

Neglecting the terms of order \( O((\varepsilon_\ell \varepsilon_\tau)^2) \) that will be seen to have a size of order \( O((\varepsilon_\ell)^2 / \tau) \) as \( \tau \to \infty \), we obtain:

\[ \varepsilon_\ell \varepsilon_\tau \log(\varepsilon_\ell) + M \varepsilon_\ell \varepsilon_\tau \varepsilon_\tau = L \varepsilon_\ell^2 \]  

with \( M := 5/4 + 8B \) and \( L := 3/32 - 8A \). This equation may be written as

\[ \frac{d}{d\tau} \left( (\log(\varepsilon_\ell))^2 + 2 M \frac{d}{d\tau} (\log(\varepsilon_\ell)) \right) = 2L + O ((\varepsilon_\ell \varepsilon_\tau)^2), \]

whence: \( (\log(\varepsilon_\ell))^2 + 2M (\log(\varepsilon_\ell)) = 2L \tau + O((\varepsilon_\ell \varepsilon_\tau)^2) \) as \( \tau \to \infty \), where we have used that \( \log(\varepsilon_\ell) \) is of order \( \sqrt{\tau} \) to the leading order. Therefore:

\[ \varepsilon_\ell (\tau) = \beta e^{-\alpha \sqrt{\tau}} \left( 1 + o(1) \right) \quad \text{as} \quad \tau \to \infty, \]  

(5.20)
where $\alpha \equiv \sqrt{2L} = \sqrt{3/16 - 16A}$, $\beta \equiv e^{-M} = e^{-5/4-B}$. In the original variables, the leading order corresponding to (5.20) is:

$$\beta \sqrt{T - t} e^{-\alpha \sqrt{|\log(T - t)|}}.$$  \hspace{1cm} (5.21)

Since $A < 0$, which we have checked numerically as was already mentioned in remark 4.2, the constant $L$ above is positive and thus $\alpha$ is a real positive number.

The asymptotics (5.21) characterizes the width of the peaks where the mass of $u$ is concentrated. The characteristic distance between these peaks is of order:

$$D = 4\sqrt{T - t}.$$  \hspace{1cm} (5.22)

**Remark 5.1.** It is interesting to notice that the formulas (5.22) provide information about the characteristic distance at which two peaks, with masses close to $8\pi$, and concentrated in a width of order $w$, must be, in order to obtain blow-up with two peaks aggregating together. Notice that, for $w$ small we have the following approximation for the critical distance required to have simultaneous blow-up and aggregation of the two peaks:

$$D = \frac{4e^{-\alpha^2}w}{\beta} \exp\left(\alpha \sqrt{2|\log(w)|}\right) \text{ as } w \to 0.$$

By critical distance we understand the distance at which two peaks containing a mass close to $8\pi$ in an area with radius $w$, should be localized in order to obtain singularity formation with an aggregating mass $16\pi$.

The numerical factor $4e^{-\alpha^2}/\beta$ cannot be expected to be really accurate if the concentrating masses in the initial peaks are not distributed exactly according to the stationary solutions (2.8).

**Remark 5.2.** Assuming that the asymptotics for $\Omega(y)$ stated in remark 4.5 hold, we can obtain an asymptotic formula for $u(x, T)$ as $x \to x_0$ using the methods in [22]. Indeed, using remark 4.5 as well as (2.1a) and (4.1a) we can approximate $u(x, \bar{t})$ for any $\bar{t} < T$, $\bar{t} \to T$ and $|x - x_0| = N\sqrt{T - \bar{t}}$, $N$ large. In such regions $u$ is basically constant in domains with a "parabolic size" $\sqrt{T - \bar{t}}$. Therefore the equation (1.1a) can be approximated as an ODE for times $\bar{t} \leq t < T$. This allows approximation of $u(x, T)$ as: $u(x, T) \sim \beta^2|x - x_0|^{-2} \exp(-2\alpha \sqrt{|\log|x - x_0|^2|}) \psi(\theta)$ as $x \to x_0$.

It is interesting to notice that the function $\psi(\theta)$ mentioned in remark 4.5 gives the angular dependence of $u$ at the blow-up point. Therefore a more detailed study of the asymptotics of the solutions of (4.4) as $|y| \to \infty$ would be in order.

### 6. Geometric configurations of singular self-similar solutions

In most of the previous computations we have assumed that $\Phi(y, \tau)$ approaches a very specific singular solution of (2.3a), (2.3b) with the form (2.9). However, there exist many other solutions of (2.3a), (2.3b) that could be taken as possible limits of $\Phi(y, \tau)$. The problem (2.3a), (2.3b) is meaningless if we assume that $\Phi$ is just a measure, or even a sum of Dirac masses. However, having in mind the matching arguments in the previous sections, it is natural to assume that $\Phi$ has the form (2.4) (i.e. all the masses of the peaks are $8\pi$) and also that the equation must be understood as (2.5) or, in an equivalent way, that a given peak does not interact with itself, which can be justified ‘a posteriori’ due to the local symmetry of the peaks during the process of aggregation.

In this section we obtain a few examples of solutions of (2.5). It is important to remark that the existence of these solutions does not guarantee the existence of solutions of the original problem (1.1a)–(1.1b). Indeed, although the formal arguments described in the previous
sections can be extended without much difficulty to more general self-similar solutions, a

crucial condition that must be satisfied in order to obtain a meaningful equation for the width

of the peaks \( \ell \) is the inequality \( 1/16 + 1/2^5 - 8A_\ell > 0 \) with \( A_\ell \) would be a constant defined

in a manner analogous to lemma 4.1 for the corresponding elliptic problem.

We do not attempt to derive a complete classification of all the solutions of (2.5). However,

we will describe some particular classes of these solutions in order to illustrate what type of

geometries can arise in the aggregation of multiple peaks. The cases under consideration

will be the following ones: Points in a line, several regular polygons, several polygons with

different sizes, combined, complete classification of solutions for \( N = 2, 3 \), and particular

results for \( N = 4, 5 \).

We remark that the sum of the right-hand side of (2.5) vanishes for any \( N \geq 2 \) and for

any configuration of points \( \{y_j\} \) as it can be seen by symmetrization:

\[
\sum_{j=1}^{N} y_j = 0. \tag{6.1}
\]

### 6.1. Solutions where all the peaks are in a line

We begin with solutions of (2.5) where all the points \( \{y_j\} \) are placed in a line. We can assume

that this line is the horizontal coordinate axis: \( y_j = (x_j, 0) \) for some real numbers \( \{x_j\} \). Then

(2.5) becomes:

\[
\frac{x_j^2}{2} - 4 \sum_{\ell=1, \ell\neq j}^{N} \frac{x_j - x_\ell}{|x_j - x_\ell|^2} = 0, \quad j = 1, 2, \ldots, N, \quad N \geq 2. \tag{6.2}
\]

**Proposition 6.1.** For every integer \( N \geq 2 \) there exists a unique solution of (6.2). The solution

is invariant, up to the rearrangement of indexes, by the transformation \( x_j \mapsto -x_j \).

**Proof.** This problem can be reformulated in a variational form because the solutions of (6.2)

can be obtained as the minimizers of:

\[
E (x_1, x_2, \ldots, x_N) = \sum_{k=1}^{N} \frac{(x_k)^2}{4} - 2 \sum_{\ell=1}^{N} \sum_{k=1, k\neq \ell}^{N} \log |x_k - x_\ell|. \tag{6.3}
\]

The functional \( E (x_1, x_2, \ldots, x_N) \) is strictly convex and lower bounded in the convex set

\( \{-\infty < x_1 < x_2 < \ldots < x_N < \infty\} \). Therefore there exists a unique minimizer where (6.2)

holds. Moreover, symmetry considerations prove the invariance mentioned in the statement.

\[\blacksquare\]

**Remark 6.2.** The solutions of (2.5) can be characterized in general by means of the extremal

points of a functional similar to the one in (6.3) if the points \( \{y_j\} \) are not aligned. However,

in such general cases, the convexity properties of the functional are not satisfied and therefore

the functional does not allow obtaining information about the solutions in an easy manner.

### 6.2. Regular polygons

**Proposition 6.3.** For every integer \( N \geq 2 \) there exists a solution of (2.5) with the points

\( \{y_j\} \) placed at the vertices of a regular \( N \)-sided polygon centred at the origin. The solution

is unique up to rotation of coordinates. Moreover, the points lie on the circle with radius

\( 2\sqrt{N - 1} \) centred at the origin.
Proof. It is convenient to reformulate (2.5) using complex variables. Let us write \( y_j = (y_j,\Re, y_j,\Im) \) and \( z_j = y_j,\Re + iy_j,\Im \in \mathbb{C} \). Then (2.5) becomes:

\[
\bar{z}_j = 8\sum_{\ell=1,\ell\neq j}^N \frac{1}{z_j - z_\ell}, \quad j = 1, \ldots, N. \tag{6.4}
\]

We now look for solutions with the form:

\[
z_j = \rho e^{\frac{2\pi i}{N}j}, \quad j = 1, \ldots, N, \quad \rho > 0. \tag{6.5}
\]

Plugging (6.5) into (6.4) we obtain:

\[
\rho^2 = \frac{N-1}{2} + 1 + (-1)^N, \tag{6.6}
\]

where \([x]\) stands for the largest integer not greater than \(x \in \mathbb{R}\). We thus obtain \(\rho = 2\sqrt{N-1}\). This shows that there exists a solution of (2.5) constructing a regular \(N\)-sided polygon. The centre of the polygon is necessarily at the origin because of (6.1).

6.3. Classification of solutions for the cases \(N = 2\) and \(N = 3\)

In these particular cases we can characterize uniquely all the solutions of (2.5). The problem becomes more complicated if the number \(N\) increases, because, as it will be seen later, the number of geometrical configurations increases with \(N\).

6.3.1. The case \(N = 2\)

**Proposition 6.4.** Suppose that \(N = 2\). Then a solution of (2.5) is uniquely given by \(y_1 = (-2, 0), \ y_2 = (2, 0)\) up to rotation of coordinates.

**Proof.** Due to (6.1) we have \(y_2 = -y_1\). We can assume that \(y_1 = (x_1, 0)\) with \(x_1 > 0\). Then (2.5) is reduced to: \(x_1/2 = 2/x_1\), whence \(x_1 = 2\). This simultaneously proves the uniqueness of the obtained solution is in the class of solutions studied in sections 6.1 and 6.2 when \(N = 2\).

6.3.2. The case \(N = 3\) \n
This case is still sufficiently simple to obtain a complete classification of the solutions. There are just two solutions of (2.5) up to rotation: either the three points are in a line as in section 6.1 or in an equilateral triangle as in section 6.2.

**Proposition 6.5.** Suppose that \(N = 3\). Then for every solution of (2.5) the points \(\{y_1, y_2, y_3\}\) are placed, up to rotation, either at the ends and the intermediate point of a segment with length being \(4\sqrt{3}\) or at the vertices of the regular polygon with the length of the sides being \(2\sqrt{6}\).

**Proof.** Suppose first that the three points are in a line, i.e., \(y_j = (x_j, 0)\) for some \(x_j \in \mathbb{R}\). Then the line crosses the origin due to (6.1) and the resulting solution is, up to rotation, the one described by means of the minimizers of the functional \(E\) in (6.3). In this case, they can be computed explicitly. Indeed, the invariance of the solution under the transformation \(x_j \rightarrow -x_j\) implies that, under the assumption \(x_1 < x_2 < x_3\), we have \(x_2 = 0, x_1 = -x_3\). Then (6.2) reduces to: \(x_1/2 = 6/x_3\), whence \(x_3 = -x_1 = 2\sqrt{3}\).

Suppose now that the three points \(\{y_j\}\) are not in a line. We will prove that in this case the three points are placed at the vertices of an equilateral triangle. It is convenient to use the complex notation of section 6.2. We may assume, without loss of generality, that \(z_3 = \bar{z}_1\). On
the other hand, using also (6.1) we then observe that (2.5) becomes:

\[
\bar{z}_1 = \frac{24z_1}{(z_1 - z_2)(z_1 - z_3)}, \quad \bar{z}_2 = \frac{24z_2}{(z_2 - z_1)(z_2 - z_3)}.
\] (6.7)

Due to (6.1) we have \(z_1 z_2 \neq 0\) since otherwise the three points would be aligned against the assumption. Taking the absolute value of (6.7), we then have:

\[
|z_1 - z_2| |z_1 - z_3| = |z_2 - z_1| |z_2 - z_3| = 24.
\] (6.8)

Therefore \(|z_1 - z_3| = |z_2 - z_3| =: \sigma > 0\). On the other hand, there is nothing special about the point \(z_3\) and, using the rotational invariance of (2.5), we may replace \(z_3\) by \(z_1\) and prove in a similar way that \(|z_3 - z_1| = |z_2 - z_1| = \sigma\). Therefore the three points are placed at the vertices of an equilateral triangle and the obtained solution is the one considered in section 6.2.

The precise size of the triangle may be computed using (6.8) as \(\sigma = 2\sqrt{6}\). ■

6.4. The case \(N = 4\)

We have not obtained a complete classification of the solutions of (2.5) if \(N = 4\) but we have some partial results suggesting that there exist at least three types of solution (up to rotation).

Notice first that we can obtain two solutions as in sections 6.1 and 6.2. Actually they can be computed explicitly. In the case where the four peaks are in a line we write:

\[x_1 = -R, \quad x_2 = -\theta R, \quad x_3 = \theta R, \quad x_4 = R\]

with \(R > 0\) and \(0 < \theta < 1\). Then the equation (6.2) becomes:

\[
R^2 = 8 \left[ \frac{1}{1 - \theta} + \frac{1}{1 + \theta} + \frac{1}{2} \right], \quad \theta R^2 = 8 \left[ -\frac{1}{1 - \theta} + \frac{1}{2\theta} + \frac{1}{1 + \theta} \right].
\] (6.9)

whence we obtain: \(\theta = \sqrt{5 - 2\sqrt{6}}\), \(R = 2\sqrt{\sqrt{6} + 3}\), which concludes the characterization of the solution with \(N = 4\) and all the peaks aligned.

We can obtain a solution with all the peaks at the vertices of a square as indicated in section 6.2. Due to (6.5) and (6.6) the vertices are at the points: \(z_j = 2\sqrt{3}e^{i(j/2)\pi}, \quad j = 0, 1, 2, 3\).

We remark that there exists another solution in the case \(N = 4\) that is neither of the ones in sections 6.1 nor 6.2:

**Proposition 6.6.** Suppose that \(N = 4\). Then there exists a solution of (2.5) with one peak at the origin and three remaining peaks at the vertices of an equilateral triangle.

**Proof.** We look for a solution with the form: \(y_j = \rho e^{i(j/2)\pi}, \quad j = 1, 2, 3, 4 = 0\). Due to the symmetry of solutions under the rotation of an angle \(2\pi/3\), the equation (2.5) becomes:

\[
\frac{1}{\rho^2} (y_1 + y_2 + y_3) = 0, \quad \frac{\rho^2}{8} = 1 + \frac{1 - e^{i\pi/3}}{|1 - e^{i\pi/3}|^2} + \frac{1 - e^{-i\pi/3}}{|1 - e^{-i\pi/3}|^2}.
\]

The first equation is automatically satisfied by (6.1), whereas the second one gives \(\rho = 4\). ■

6.5. The case \(N = 5\)

In this case we do not attempt to obtain a complete classification of the solutions, but indicate some examples to illustrate what type of solution can arise. We can obtain solutions with all
the peaks in a line as in section 6.1. In this case we have, due to the symmetry of the problem: \( x_1 = -R, \; x_2 = -\theta R, \; x_3 = 0, \; x_4 = \theta R, \; x_5 = R \) with \( R > 0 \) and \( \theta \in (0, 1) \). The equations (2.5) are then reduced to:

\[
\frac{R^2}{8} = \frac{3}{2} + \frac{2}{1 - \theta^2}, \quad \frac{\theta R^2}{8} = \frac{3}{2\theta} - \frac{2\theta}{1 - \theta^2}.
\]

It then follows that \( \theta = \sqrt{7/3 - 2\sqrt{10}/3}, \; R = 3^{-1/3}\sqrt{10}\sqrt{\sqrt{10} - 2} \left( \sqrt{10} + 2 \right) \).

We can obtain also a solution where the peaks are placed at the vertices of a regular pentagon. Using (6.5) and (6.6) we obtain: \( z_j = 4e^{(2j/5)\pi}, \; j = 0, 1, 2, 3, 4 \).

There is also a solution that consists of one peak at the origin and the other four peaks at the vertices of a square centred at the origin. Assuming that the peaks are placed at the points \( z_j = \rho e^{(j/2)\pi}, \; j = 0, 1, 2, 3, \) solving the equations (2.5).

We finally remark that in the case \( N = 5 \) it is possible to obtain one configuration of peaks whose only symmetry is the reflection with respect to a line. More precisely, we have:

**Proposition 6.7.** There exists a solution of (2.5) with the points \( \{z_k\} \) placed, in terms of the complex notation of section 6.2, at the following positions:

\[
z_k = x_k \in \mathbb{R} \quad \text{for} \; k = 1, 2, 3, \quad z_4 = \alpha + i\beta, \quad z_5 = \alpha - i\beta \quad (6.10)
\]

with \( \alpha < 0, \beta > 0 \).

**Proof.** We prove the existence of a solution of (2.5) with the form (6.10) by means of a topological argument. Due to (6.1) we have:

\[
\alpha = -\frac{x_1 + x_2 + x_3}{2} \quad (6.11).
\]

We assume that \( \alpha \) is chosen as in (6.11). On the other hand, we can obtain an equation for \( \beta \) using the vertical component (or imaginary part in complex notation) of (2.5) with \( j = 4 \):

\[
1 = \frac{1}{8} \sum_{k=1}^{3} \frac{1}{(\alpha - x_k)^2 + \beta^2} + \frac{1}{2\beta^2} \quad (6.12).
\]

Since the right-hand side of (6.12) is a decreasing function of \( \beta \), we see that there exists a unique solution of (6.12) with \( \beta > 0 \) for any \( (x_1, x_2, x_3) \) in the set \( \{-\infty < x_1 < x_2 < x_3 < \infty\} \). We denote it as \( \beta(x_1, x_2, x_3) \). Notice that (6.12) implies

\[
\beta(x_1, x_2, x_3) > 2 \quad (6.13).
\]

Due to (6.10), equations (2.5) with \( j = 1, 2, 3 \) are reduced to:

\[
\frac{x_k}{8} = \sum_{j=1, j \neq k}^{3} \frac{x_k - x_j}{|x_k - x_j|^2} + \frac{2(x_k - \alpha)}{(\alpha - x_k)^2 + \beta^2}, \quad k = 1, 2, 3 \quad (6.14).
\]

To prove the existence of solutions of (6.14) in the cone: \( C = \{x = (x_1, x_2, x_3) : -\infty < x_1 < x_2 < x_3 < \infty\} \), we treat (6.14) as a perturbation of the equation: \( F_k(x) := x_k/8 - \sum_{j=1, j \neq k}^{3} (x_k - x_j)/|x_k - x_j|^2 = 0, \; k = 1, 2, 3 \), using topological degree. Since the function \( F(x) := (F_1, F_2, F_3)(x) \) becomes singular on the boundary of the cone \( C \), we will construct a subset \( \mathcal{U} \) so that \( |F(x)| \geq 100 \) on the boundary \( \partial \mathcal{U} \). The functions
$G_k(x) := 2(x_k - \alpha) / [(\alpha - x_k)^2 + \beta^2], k = 1, 2, 3,$ are bounded by 2 in $C$ as it can be easily checked considering separately the cases $|x_k - \alpha| \geq 1$ and $|x_k - \alpha| \leq 1$ and using (6.13). We thus obtain $|G(x)| < |F(x)|$ on $\partial U$. On the other hand, there is a unique nondegenerate solution of the equation $F(x) = 0$ in $C$ due to the results in section 6.1. Classical degree theory then shows that there exists at least one solution of $(F + G)(x) = 0$ in $U$, whence the existence of the desired solution of (6.14) follows.

We shall construct the subset $U$ with the form:

$$U = \{ x \in C : x_1 + \varepsilon < x_2, x_2 + \varepsilon < x_3, -R < x_k < R, k = 1, 2, 3 \},$$

where $\varepsilon$ and $R$ are positive constants to be determined. Notice that the boundary $\partial U$ is contained in the planes $\Pi_{1,2} = \{ x_2 - x_1 = \varepsilon \}, \Pi_{2,3} = \{ x_3 - x_2 = \varepsilon \}, \Pi_{-R} = \{ x_1 = -R \}, \Pi_R = \{ x_3 = R \}$. We will assume that $1/\varepsilon$ is much larger than $R$. Along the part of $\partial U$ contained in the planes $\Pi_{1,2}$ or $\Pi_{2,3}$ we have: $F_1(x) \geq 1/\varepsilon - R/8 \geq 1/(2\varepsilon)$.

We then proceed to consider the part of $\partial U$ contained in $\Pi_R$. Suppose first that $x_3 - x_1 \leq 1$. Then $x_1 \geq R - 1$ and $F_1(x) > x_1/8 > 100$ if $R > 801$. Suppose now that $x_3 - x_1 > 1$. We distinguish two cases. Suppose firstly that $x_3 - x_2 > 1$. Then: $F_2(x) = x_3/8 - 1/(x_3 - x_1) - 1/(x_3 - x_2) \geq R/16$ if $R$ is large, because the last two terms are bounded by one. Suppose secondly that $x_3 - x_2 \leq 1$. Let us consider firstly the case $x_2 - x_1 \leq 1/4$. We then have: $x_3 - x_2 \geq 3/4$ and $F_3(x) \geq R/16$. Let us consider secondly the case $x_2 - x_1 > 1/4$. Then: $F_2(x) = x_3/8 - 1/(x_3 - x_1) + 1/(x_3 - x_2) > x_3/8 - 4$. Since $x_3 - x_2 \leq 1$ we obtain $x_2 \geq R - 1$ and thus $F_2(x) \geq R/16$. Consequently, we have: $|F(x)| \geq 100$ for $x \in \partial U \cap \Pi_R$. The case of $x \in \partial U \cap \Pi_{-R}$ is similar.

We shall observe the existence of desired solutions of (6.14). It only remains to prove that the equation (2.5) with $j = 4$ is satisfied. The equation reads: $\alpha/8 = \sum_{k=1}^3 (\alpha - x_k)^2 / (\alpha - x_k)^2 + \beta^2$. Notice that, due to (6.11) and (6.14), this is equivalent to: $\sum_{k=1}^3 \sum_{j=1, j \neq k}^3 (x_k - x_j)^2 / |x_k - x_j|^2 = 0$. The last identity is trivially satisfied by symmetrization.

**Remark 6.8.** We have made some computations suggesting that in the case $N = 4$ the only trapezoidal solution is the square. The only rhombic solution is also the square. Increasing the value of $N$ it becomes possible to show that there are also solutions with nested squares, triangles, etc. However, we will not continue this discussion here. It would be interesting to determine the smallest number $N$ yielding solutions without any symmetry group.

### 7. Bounded domains

Solving the Keller–Segel model in the half circle, it is possible to obtain a wealth of shapes yielding aggregation at the boundary. The mass is, in all the cases, $4\pi m$ with positive integers $m$. It is possible to obtain for instance $8\pi$ instead of $4\pi$, just keeping one point at the interior of the domain. Notice that one must choose symmetric point configurations in order to ensure that the homogeneous Neumann boundary conditions are satisfied.

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