Projections in the Space $H^\infty$ and the Corona Theorem for Coverings of Bordered Riemann Surfaces

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Abstract

Let $M$ be a non-compact connected Riemann surface of finite type, and $R \subset M$ be a relatively compact domain such that $H_1(M, \mathbb{Z}) = H_1(R, \mathbb{Z})$. Let $\tilde{R} \to R$ be a covering. We study the algebra $H^\infty(U)$ of bounded holomorphic functions defined in some domains $U \subset \tilde{R}$. Our main result is a Forelli type theorem on projections in $H^\infty(D)$.

1. Introduction.

1.1. Let $X$ be a connected complex manifold and $H^\infty(X)$ be the algebra of bounded holomorphic functions on $X$ with pointwise multiplication and with norm

$$||f|| = \sup_{x \in X} |f(x)| .$$

Let $r: \tilde{X} \to X$ be the universal covering of $X$. The fundamental group $\pi_1(X)$ acts discretely on $\tilde{X}$ by biholomorphic maps. By $r^*(H^\infty(X)) \subset H^\infty(\tilde{X})$ we denote the Banach subspace of functions invariant with respect to the action of $\pi_1(X)$. In this paper we describe a class of manifolds $X$ for which there is a linear continuous projector $P: H^\infty(\tilde{X}) \to r^*(H^\infty(X))$ satisfying

$$P(fg) = P(f)g \text{ for any } f \in H^\infty(\tilde{X}), \ g \in r^*(H^\infty(X)). \quad (1.1)$$

For instance, according to Forelli [F], such $P$ exists in the case when $X$ is the interior of a compact bordered Riemann surface. (The universal covering of such $X$ is the
open unit disk $\mathbb{D} \subset \mathbb{C}$.) One of the possible applications of Forelli’s theorem is to the solution of the corona problem for $H^\infty(R)$ (for further results and references related to the corona problem we refer to Garnett [Ga1], Jones and Marshall [JM] and Slodkowski [S]). Extensions of Forelli’s theorem to some Riemann surfaces of Widom type were obtained by Carleson [Ca2] and Jones and Marshall [JM]. In this paper we consider another more general construction of $P$ satisfying (1.1). Let us formulate our result.

Let $N \subset M$ be a relatively compact domain (i.e. an open connected subset) in a connected Stein manifold $M$ such that

$$\pi_1(N) \cong \pi_1(M).$$

(1.2)

By $\mathcal{F}_c(N)$ we denote the class of unbranched coverings of $N$. Recall that any covering from $\mathcal{F}_c(N)$ corresponds to a subgroup of $\pi_1(N)$. Assume that the complex connected manifold $U$ admits a holomorphic embedding $i : U \hookrightarrow R$ into some $R \in \mathcal{F}_c(N)$. Let $i_* : \pi_1(U) \longrightarrow \pi_1(R)$ be the induced homomorphism of fundamental groups. We set $K(U) := \text{Ker}(i_*) \subset \pi_1(U)$. Consider the regular covering $p_U : \tilde{U} \longrightarrow U$ of $U$ corresponding to the group $K(U)$, that is, $\pi_1(\tilde{U}) = K(U)$ and $\pi_1(U)/K(U)$ acts on $\tilde{U}$ as the group of deck transformations. Further, by $p_U^*(H^\infty(U)) \subset H^\infty(\tilde{U})$ we denote the subspace of holomorphic functions invariant with respect to the action of $\pi_1(U)/K(U)$ (i.e. the pullback by $p_U$ of $H^\infty(U)$ to $\tilde{U}$). Let $F_z := p_U^{-1}(z)$, $z \in U$, and $l^\infty(F_z)$ be the Banach space of bounded complex-valued functions on $F_z$ with the supremum norm. By $C(F_z) \subset l^\infty(F_z)$ we denote the subspace of constant functions.

**Theorem 1.1** There is a linear continuous projector $P : H^\infty(\tilde{U}) \longrightarrow p_U^*(H^\infty(U))$ satisfying the properties:

1. There exists a family of linear continuous projectors $P_z : l^\infty(F_z) \longrightarrow C(F_z)$ holomorphically depending on $z \in U$ such that $P[f]_{p_U^{-1}(z)} := P_z[f]_{p_U^{-1}(z)}$ for any $f \in H^\infty(\tilde{U})$;
2. $P(fg) = P(f)g$ for any $f \in H^\infty(\tilde{U}), g \in p_U^*(H^\infty(U))$;
3. If $f \in H^\infty(\tilde{U})$ is such that $f|_{F_z}$ is constant, then $P(f)|_{F_z} = f|_{F_z}$;
4. Each $P_z$ is continuous in the weak * topology of $l^\infty(F_z)$;
5. The norm $||P|| \leq C < \infty$ where $C = C(N)$ depends on $N$ only.

As a simple corollary of Theorem 1.1 we obtain

**Corollary 1.2** Let $\tilde{R}$ be a covering of a bordered Riemann surface $R$. The fundamental group $\pi_1(\tilde{R})$ is a free group whose family of generators $J$ is finite or countable. Assume that $U \subset \tilde{R}$ is a domain such that $\pi_1(U)$ is generated by a subfamily of $J$. Let $r : \mathbb{D} \longrightarrow U$ be the universal covering map. Then there exists a linear continuous projector $P : H^\infty(\mathbb{D}) \longrightarrow r^*(H^\infty(U))$ satisfying the properties of Theorem 1.1.
**Example 1.3** Let \( r : \mathbb{D} \rightarrow X \) be the universal covering of a compact complex Riemann surface of genus \( g \geq 2 \). Let \( K \subset \mathbb{D} \) be the fundamental compact with respect to the action of the deck transformation group \( \pi_1(X) \). By definition, the boundary of \( K \) is the union of \( 2g \) analytic curves. Let \( D_1, \ldots, D_k \) be a family of mutually disjoint closed disks situated in the interior of \( K \). We set

\[
S := \bigcup_{i=1}^k D_i, \quad K' := K \setminus S, \quad \text{and} \quad \tilde{R} := \bigcup_{g \in \pi_1(X)} g(K').
\]

Then \( R := r(K') \subset X \) is a bordered Riemann surface, and \( r : \tilde{R} \rightarrow R \) is a regular covering corresponding to the quotient group \( \pi_1(X) \) of \( \pi_1(R) \). Here \( \pi_1(\tilde{R}) \) is generated by a family of simple closed curves in \( \tilde{R} \) with the origin at a fixed point \( x_0 \in \tilde{R} \) so that each such curve goes around only of one of \( g(D_i) \), \( g \in \pi_1(X) \), \( i = 1, \ldots, k \). Let \( Y \subset \mathbb{D} \) be a simply connected domain with the property: there is a subset \( L \subset \pi_1(X) \) so that

\[
Y \cap \left( \bigcup_{g \in \pi_1(X)} g(S) \right) = \bigcup_{g \in L} g(S).
\]

Clearly \( U := Y \setminus (\bigcup_{g \in L} g(S)) \) satisfies the conditions of Corollary 1.2. Therefore the projector \( P \), described above, exists for \( U \).

**Remark 1.4** In view of Example 1.3 it is natural to conjecture the following.

Let \( U \subset \mathbb{D} \) be a domain obtained by removing from \( \mathbb{D} \) a finite or countable family of pairwise disjoint closed disks \( D_i \). Let \( r_i \) be the radius of \( D_i \) with respect to the pseudohyperbolic metric \( \rho \),

\[
\rho(z, w) := \left| \frac{z - w}{1 - \overline{z}w} \right|, \quad z, w \in \mathbb{D}.
\]

Assume that \( \inf_i r_i = c > 0 \). Let \( r : \mathbb{D} \rightarrow U \) be the universal covering map.

**Conjecture.** There is a linear continuous projector \( P : H^\infty(\mathbb{D}) \rightarrow r^*(H^\infty(U)) \) satisfying \( P(fg) = P(f)g \) for \( f \in H^\infty(\mathbb{D}), g \in r^*(H^\infty(U)) \) whose norm depends on \( c \) only.

To formulate our next corollary we recall several definitions.

Let \( X \) be a Riemann surface such that \( H^\infty(X) \) separates points of \( X \). By \( M(H^\infty(X)) \) we denote the maximal ideal space of \( H^\infty(X) \), i.e. the set of non-trivial multiplicative linear functionals on \( H^\infty(X) \) with the weak \( * \) topology (which is called the Gelfand topology). It is a compact Hausdorff space. Each point \( x \in X \) corresponds in a natural way (point evaluation) to an element of \( M(H^\infty(X)) \). So \( X \) is naturally embedded into \( M(H^\infty(X)) \). Then the corona problem for \( H^\infty(X) \) asks: Is \( M(H^\infty(X)) \) the closure (in the Gelfand topology) of \( X \)?

Recall also that the corona problem has the following analytic reformulation.

A collection \( f_1, \ldots, f_n \) of functions from \( H^\infty(X) \) satisfies the corona condition if

\[
|f_1(x)| + |f_2(x)| + \ldots + |f_n(x)| \geq \delta > 0 \quad \text{for all} \ x \in X.
\]
The corona problem being solvable means that the Bezout equation
\[ f_1 g_1 + f_2 g_2 + \cdots + f_n g_n \equiv 1 \]
has a solution \( g_1, \ldots, g_n \in H^\infty(X) \) for any \( f_1, \ldots, f_n \) satisfying the corona condition. We refer to \( \max_j ||g_j|| \) as a “bound on the corona solutions”. Using Carleson’s solution \([Ca]\) of the corona problem for \( H^\infty(D) \) and property (2) for the projector constructed in Theorem 1.1 we immediately obtain.

**Corollary 1.5** Let \( N \subset\subset M, R \in F_c(N) \) and \( i : U \hookrightarrow R \) be open Riemann surfaces satisfying the conditions of Theorem 1.1. Assume also that \( K(U) := \text{Ker}(i_*) \) is trivial. Let \( f_1, \ldots, f_n \in H^\infty(U) \) satisfy the corona condition (1.4). Then the corona problem has a solution \( g_1, \ldots, g_n \in H^\infty(U) \) with the bound 
\[ \max_j ||g_j|| \leq C(N, n, \delta/ \max_j ||f_j||). \]

**Remark 1.6** In this case \( \pi_1(N) \) and \( \pi_1(M) \) are free groups. Therefore condition (1.2) is equivalent to \( H_1(M, \mathbb{Z}) = H_1(N, \mathbb{Z}) \) for the corresponding homology groups.

1.2. Another application of Theorem 1.1 is a result on the classification of interpolating sequences in \( U \) (cf. [St] and [JM]). Recall that a sequence \( \{z_j\} \subset U \) is an interpolating sequence for \( H^\infty(U) \) if for every bounded sequence of complex numbers \( \{a_j\} \), there is an \( f \in H^\infty(U) \) so that \( f(z_j) = a_j \).

**Theorem 1.7** Let \( N \subset\subset M, R \in F_c(N) \), \( i : U \hookrightarrow R \) and \( \tilde{U} \) be complex manifolds satisfying the conditions of Theorem 1.1. A sequence \( \{z_j\} \subset U \) is interpolating for \( H^\infty(U) \) if and only if \( r^{-1}(\{z_j\}) \) is interpolating for \( H^\infty(\tilde{U}) \).

**Example 1.8** Let \( M \subset \mathbb{D} \) be a bounded domain, whose boundary \( B \) consists of \( k \) simple closed continuous curves \( B_1, \ldots, B_k \), with \( B_1 \) forming the outer boundary. Let \( D_i \) be the interior of \( B_i \), and \( D_2, \ldots, D_k \) the exteriors of \( B_2, \ldots, B_k \), including the point at infinity. Then each \( D_i \) is biholomorphic to \( \mathbb{D} \). Let \( \{z_{ji}\}_{j \in J} \) be an interpolating sequence for \( H^\infty(D_i) \), \( i = 1, \ldots, k \), such that the Euclidean distance between any two distinct sequences is bounded from below by a positive number. Then for any covering \( p : R \longrightarrow M \) the sequence \( p^{-1}(\{z_{ji}\}) \) is interpolating for \( H^\infty(R) \).

In Section 5 we also establish some results for interpolating sequences in \( U \) with \( U \) being a Riemann surface satisfying the assumptions of Corollary 1.5. These results have much in common with similar properties of interpolating sequences for \( H^\infty(D) \).

# 2. Construction of Bundles.

In this section we formulate and prove some preliminary results used in the proofs of our main theorems.

## 2.1. Definitions and Examples.

(For standard facts about bundles see e.g.
Hirzebruch’s book [Hi].) In what follows all topological spaces are assumed to be finite or infinite dimensional.

Let \( X \) be a complex analytic space and \( S \) be a complex analytic Lie group with the unit \( e \in S \). Consider an effective holomorphic action of \( S \) on a complex analytic space \( F \). Here holomorphic action means a holomorphic map \( \pi: X \times S \rightarrow X \) satisfying \( \pi \circ \alpha = \pi \circ \beta \) for any \( \alpha, \beta \in S \). Efficiency means that the condition \( \pi(\alpha u) = \pi(u) \) for some \( u \) and any \( f \) implies that \( \alpha = e \).

**Definition 2.1** A complex analytic space \( W \) together with a holomorphic map (projection) \( \pi: W \rightarrow X \) is called a holomorphic bundle over \( X \) with the structure group \( S \) and the fibre \( F \), if there exists a system of coordinate transformations, i.e., if

(1) there is an open cover \( U = \{ U_i \}_{i \in I} \) of \( X \) and a family of biholomorphisms \( \alpha_i : \pi^{-1}(U_i) \rightarrow U_i \times F \) such that \( \alpha_i \) and \( \alpha_j \) are compatible on \( U_i \cap U_j \); and the fibre \( \pi^{-1}(x) \) is a Banach space \( F \) and the structure group is \( GL(F) \) (the group of linear invertible transformations of \( F \)) is called a holomorphic Banach vector bundle.

(2) for any \( i, j \in I \) there are elements \( s_{ij} \in \mathcal{O}(U_i \cap U_j, S) \) such that

\[
(h_i h_j^{-1})(u \times f) = u \times s_{ij}(u)f \quad \text{for any } u \in U_i \cap U_j, \ f \in F.
\]

In particular, a holomorphic bundle \( \pi: W \rightarrow X \) whose fibre is a Banach space \( F \) and the structure group is \( GL(F) \) is called a holomorphic Banach vector bundle.

A holomorphic section of a holomorphic bundle \( \pi: W \rightarrow X \) is a holomorphic map \( s: X \rightarrow W \) satisfying \( \pi \circ s = \text{id} \). Let \( \pi_i: W_i \rightarrow X \), \( i = 1, 2 \), be holomorphic Banach vector bundles. A holomorphic map \( f: W_1 \rightarrow W_2 \) satisfying

(a) \( f(\pi_1^{-1}(x)) \subset \pi_2^{-1}(x) \) for any \( x \in X \);

(b) \( f|_{\pi_1^{-1}(x)} \) is a linear continuous map of the corresponding Banach spaces,

is called a homomorphism. If, in addition, \( f \) is a homeomorphism, then \( f \) is called an isomorphism.

We also use the following construction of holomorphic bundles (see, e.g. [Hi,Ch.1]):

Let \( S \) be a complex analytic Lie group and \( U = \{ U_i \}_{i \in I} \) be an open cover of \( X \). By \( Z_0^1(U, S) \) we denote the set of holomorphic \( S \)-valued \( U \)-cocycles. By definition, \( s = \{ s_{ij} \} \in Z_0^1(U, S) \), where \( s_{ij} \in \mathcal{O}(U_i \cap U_j, S) \) and \( s_{ij}s_{jk} = s_{ik} \) on \( U_i \cap U_j \cap U_k \). Consider disjoint union \( U_i \times F \) and for any \( u \in U_i \cap U_j \) identify point \( u \times f \in U_j \times F \) with \( u \times s_{ij}(u)f \). We obtain a holomorphic bundle \( W_s \) over \( X \) whose projection is induced by the projection \( U_i \times F \rightarrow U_i \). Moreover, any holomorphic bundle over \( X \) with the structure group \( S \) and the fibre \( F \) is isomorphic (in the category of holomorphic bundles) to a bundle \( W_s \).

**Example 2.2** (a) Let \( M \) be a complex manifold. For any subgroup \( G \subset \pi_1(M) \) consider the unbranched covering \( g: M_G \rightarrow M \) corresponding to \( G \). We will describe \( M_G \) as a holomorphic bundle over \( M \).

First, assume that \( G \subset \pi_1(M) \) is a normal subgroup. Then \( M_G \) is a regular covering of \( M \) and the quotient group \( Q := \pi_1(M)/G \) acts holomorphically on \( M_G \) by deck transformations. It is well known that \( M_G \) in this case can be thought of as a principle fibre bundle over \( M \) with fibre \( Q \) (here \( Q \) is equipped with discrete
topology). Namely, let us consider the map $R_Q(g) : Q \rightarrow Q$ defined by the formula

$$R_Q(g)(h) = h \cdot g^{-1}, \quad h \in Q.$$ 

Then there is an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of $M$ by sets biholomorphic to open Euclidean balls in some $\mathbb{C}^n$ and a locally constant cocycle $c = \{c_{ij}\} \in \mathbb{Z}_b(\mathcal{U}, Q)$ such that $M_G$ is biholomorphic to the quotient space of the disjoint union $V = \bigsqcup_{i \in I} U_i \times Q$ by the equivalence relation: $U_i \times Q \ni x \times R_Q(c_{ij})(h) \sim x \times h \in U_j \times Q$. The identification space is a holomorphic bundle with projection $p : M_G \rightarrow M$ induced by the projections $U_i \times Q \rightarrow U_i$. In particular, when $G = e$ we obtain the definition of the universal covering $M_e$ of $M$.

Assume now that $G \subset \pi_1(M)$ is not necessarily normal. Let $X_G = \pi_1(M)/G$ be the set of cosets with respect to the (left) action of $G$ on $\pi_1(M)$ defined by left multiplications. By $[Gq] \in X_G$ we denote the coset containing $q \in \pi_1(M)$. Let $H(X_G)$ be the group of all homeomorphisms of $X_G$ equipped with discrete topology. We define the homomorphism $\tau : \pi_1(M) \rightarrow H(X_G)$ by the formula:

$$\tau(g)([Gq]) := [Gqq^{-1}], \quad q \in \pi_1(M).$$

Set $Q(G) := \pi_1(M)/\text{Ker}(\tau)$ and let $\tilde{g}$ be the image of $g \in \pi_1(M)$ in $Q(G)$. By $\tau_{Q(G)} : Q(G) \rightarrow H(X_G)$ we denote the unique homomorphism whose pullback to $\pi_1(M)$ coincides with $\tau$. Consider the action of $G$ on $V = \bigsqcup_{i \in I} U_i \times \pi_1(M)$ induced by the left action of $G$ on $\pi_1(M)$ and let $V_G = \bigsqcup_{i \in I} U_i \times X_G$ be the corresponding quotient set. Define the equivalence relation $U_i \times X_G \ni x \times \tau_{Q(G)}(\tilde{c}_{ij})(h) \sim x \times h \in U_j \times X_G$ with the same $\{c_{ij}\}$ as in the definition of $M_e$. The corresponding quotient space is a holomorphic bundle with fibre $X_G$ biholomorphic to $M_G$.

(b) We retain the notation of example (a). Let $B$ be a complex Banach space with norm $| \cdot |$. Let $\text{Iso}(B) \subset GL(B)$ be the group of linear isometries of $B$. Consider a homomorphism $\rho : Q \rightarrow \text{Iso}(B)$. Without loss of generality we assume that $\text{Ker}(\rho) = e$, for otherwise we can pass to the corresponding quotient group. The holomorphic Banach vector bundle $E_\rho \rightarrow M$ associated with $\rho$ is defined as the quotient of $\bigsqcup_{i \in I} U_i \times B$ by the equivalence relation $U_i \times B \ni x \times \rho(c_{ij})(w) \sim x \times w \in U_j \times B$ for any $x \in U_i \cap U_j$. Further, we can define a function $E_\rho \rightarrow \mathbb{R}_+$ which will be called the norm on $E_\rho$ (and denoted by the same symbol $| \cdot |$). The construction is as follows. For any $x \times w \in U_i \times B$ we set $|x \times w| := |w|$. Since the image of $\rho$ belongs to $\text{Iso}(B)$, the above definition is invariant with respect to the equivalence relation determining $E_\rho$ and so it determines a “norm” on $E_\rho$. Let us consider some examples.

Let $l_1(Q)$ be the Banach space of complex-valued sequences on $Q$ with $l_1$-norm. The action $R_Q$ from (a) induces the homomorphism $\rho : Q \rightarrow \text{Iso}(l_1(Q))$,

$$\rho(g)(w)[x] := w(R_Q(g)(x)), \quad g, x \in Q, \ w \in l_1(Q).$$

By $E_1^M(Q)$ we denote the holomorphic Banach vector bundle associated with $\rho$.

Let $l_\infty(Q)$ be the Banach space of bounded complex-valued sequences on $Q$ with $l_\infty$-norm. The homomorphism $\rho^* : Q \rightarrow \text{Iso}(l_\infty(Q))$, dual to $\rho$ is defined as

$$\rho^*(g)(v)[x] := v(x \cdot g^{-1}), \quad g, x \in Q, \ v \in l_\infty(Q).$$
Further, recall that a $B$-valued function $f : U \to B$ defined in an open set $U \subset \mathbb{C}^n$ is said to be holomorphic if $f$ satisfies the $B$-valued Cauchy integral formula in any polydisk containing in $U$. Equivalently, locally $f$ can be represented as sum of absolutely convergent holomorphic power series with coefficients in $B$. Now any family $\{f_z \}_{z \in U}$, where $f_z$ is a $B$-valued holomorphic on $U$ function satisfying $|f_z(z)| < A$ for any $z \in U$ and $x \in X$, can be considered as a $B_\infty(X)$-valued holomorphic function on $U$. In fact, the local Taylor expansion in this case follows from the Cauchy estimates of the coefficients in the Taylor expansion of each $f_z$.

Let $t : X \to X$ be a bijection and $h : X \times X \to gl(B)$ be such that

$$h(t(x), x) \in GL(B) \quad \text{and} \quad \max\{\sup_{x \in X} ||h(t(x), x)||, \sup_{x \in X} ||h^{-1}(x, t(x))||\} < \infty.$$  

Then we can define $a(h, t) \in GL(B_\infty(X))$ by the formula

$$a(h, t)[(x, b(x))] := (t(x), h(t(x), x)[b(x)]), \quad b = \{(x, b(x))\}_{x \in X} \in B_\infty(X).$$

We retain the notation of Example 2.2. For the acyclic cover $U = \{U_i\}_{i \in I}$ of $M$ we have $g^{-1}(U_i) = \bigcup_{s \in X_G} V_{is} \subset M_G$ where $g|_{V_{is}} : V_{is} \to U_i$ is biholomorphic. Consider a holomorphic Banach vector bundle $\pi : E \to M_G$ with fibre $B$ defined by coordinate transformations subordinate to the cover $\{V_{is}\}_{i \in I, s \in X_G}$ of $M_G$, i.e. by a holomorphic cocycle $h = \{h_{is, jk}\} \in Z^1(\mathcal{O}(g^{-1}(U), GL(B)), h_{is, jk} \in \mathcal{O}(V_{is} \cap V_{jk}, GL(B))$, such that $E$ is biholomorphic to the quotient space of disjoint union $\sqcup_{i, s} V_{is} \times B$ by the equivalence relation $V_{is} \times B \ni (x, h_{is, jk}(x)[v]) \sim (x \times v) \in V_{jk} \times B$, $s := \tau_{Q(G)}(\tilde{c}_{ij})(k)$. The projection $\pi$ is induced by coordinate projections $V_{is} \times B \to V_{is}$. Assume also that for any $x$

$$\sup_{i, j, s, k} \max\{|h_{is, jk}(x)|, |h_{is, jk}^{-1}(x)|\} \leq A < \infty. \quad (2.1)$$

Further, define $\tilde{\pi} := g \circ \pi : E \to M$.

**Proposition 2.3** The triple $(E, M, \tilde{\pi})$ determines a holomorphic Banach vector bundle over $M$ with fibre $B_\infty(X_G)$. (We denote this bundle by $E_M$.)
Proof. Let $\phi_{ts} : U_t \to V_{is}$ be the map inverse to $g|_{V_{is}}$. We identify $V_{is} \times B$ with $U_t \times s \times B$ by $\phi_{ts}$, and $\{s \times B\}_{s \in X_G}$ with $B_\infty(X_G)$. Further, for any $x \in U_t \cap U_j$, we set $\tilde{h}_{is,jk}(x) := h_{is,jk}(\phi_{ts}(x))$. Then $E$ can be defined as the quotient space of $\sqcup_{i \in I} U_i \times B_\infty(X_G)$ by the equivalence relation $U_j \times B_\infty(X_G) \ni x \times \{(k, b(k))\}_{k \in X_G} \sim \{(\tau_{Q(G)}(\tilde{c}_{ij})(k), \tilde{h}_{i\tau_{Q(G)}(\tilde{c}_{ij})}(k)(x)[b(k)])\}_{k \in X_G} \in U_t \times B_\infty(X_G)$.

Define $\tilde{h}_{ij}(x) : X_G \times X_G \to GL(B_\infty(X_G))$, $x \in U_t \cap U_j$, and $d_{ij} \in \mathcal{O}(U_t \cap U_j, GL(B_\infty(X_G)))$ by the formulas

$$\tilde{h}_{ij}(x)(s, k) := \tilde{h}_{is,jk}(x), \quad \text{and}$$

$$d_{ij}(x)[b] := a(\tilde{h}_{ij}(x), \tau_{Q(G)}(\tilde{c}_{ij})[b], \ b \in B_\infty(X_G)).$$

Here holomorphy of $d_{ij}$ follows from (2.1). Clearly, $d = \{d_{ij}\}$ is a holomorphic cocycle with values in $GL(B_\infty(X_G))$, because $\{h_{is,jk}\}$ and $\{\tau_{Q(G)}(\tilde{c}_{ij})\}$ are cocycles. Now $E$ can be considered as a holomorphic Banach vector bundle over $M$ with fibre $B_\infty(X_G)$ obtained by identification in $\sqcup_{i \in I} U_t \times B_\infty(X_G)$ of $x \times d_{ij}(x)[b] \in U_t \times B_\infty(X_G)$ with $x \times b \in U_j \times B_\infty(X_G)$, $x \in U_t \cap U_j$. Moreover, according to our construction the projection $E \to M$ coincides with $\tilde{\pi}$. □

Let $W$ be a holomorphic Banach vector bundle over a complex analytic space $X$. In what follows by $\mathcal{O}(U, W)$ we denote the vector space of holomorphic sections of $W$ defined in an open set $U \subset X$.

We retain the notation of Proposition 2.3. According to the construction of Proposition 2.3, a fibre $\tilde{\pi}^{-1}(z)$, $z \in U_t$, of $E_M$ can be identified with $\prod_{s \in X_G} \pi^{-1}(\phi_{ts}(z))$ such that if also $z \in U_j$ then

$$\prod_{s \in X_G} \pi^{-1}(\phi_{ts}(z)) = \prod_{s \in X_G} \pi^{-1}(\phi_{j\tau_{Q(G)}(\tilde{c}_{ij})}(s)(z)).$$

(2.2)

We recall the following definitions.

Let $J_q$ be the set of sequences $(i_0 s_0, \ldots, i_q s_q)$ with $i_t \in I$, $s_t \in X_G$ for $t = 0, \ldots, q$. A family

$$f = \left\{ f_{i_0 s_0, \ldots, i_q s_q} : (i_0 s_0, \ldots, i_q s_q) \in J_q, \quad f_{i_0 s_0, \ldots, i_q s_q} \in \mathcal{O}(V_{i_0 s_0} \cap \cdots \cap V_{i_q s_q}, E), \right\}$$

is called a $q$-cochain on the cover $g^{-1}(U) := \sqcup_{i \in I, s \in X_G} V_{is}$ of $M_G$ with coefficients in the sheaf of germs of holomorphic sections of $E$. These cochains generate a complex vector space $C^q(g^{-1}(U), E)$. In the trivialization which identifies $\pi^{-1}(V_{i_0 s_0})$ with $V_{i_0 s_0} \times B$ any $f_{i_0 s_0, \ldots, i_q s_q}$ is represented by $b_{i_0 s_0, \ldots, i_q s_q} \in \mathcal{O}(V_{i_0 s_0} \cap \cdots \cap V_{i_q s_q}, B)$. Assume that for any $(i_0 s_0, \ldots, i_q s_q) \in J_q$ and any compact $K \subset U_{i_0} \cap \cdots \cap U_{i_q}$ there is a constant $C = C(K)$ such that

$$\sup_{s_0, \ldots, s_q, z \in K} |(b_{i_0 s_0, \ldots, i_q s_q} \circ \phi_{i_0 s_0})(z)| < C$$

(2.3)

The set of cochains $f$ satisfying (2.3) is a vector subspace of $C^q(g^{-1}(U), E)$ which will be denoted by $C^q_0(g^{-1}(U), E)$. Further, the formula

$$(\delta^q f)_{i_0 s_0, \ldots, i_q+1 s_{q+1}} = \sum_{k=0}^{q+1} (-1)^k r^W_k \left( f_{i_0 s_0, \ldots, i_{k+1} s_{k+1}}, \ldots, i_{q+1} s_{q+1} \right),$$

(2.4)
where \( f \in C^q(g^{-1}(U), E) \), determines a homomorphism

\[
\delta^q : C^q(g^{-1}(U), E) \rightarrow C^{q+1}(g^{-1}(U), E).
\]

Here \( \hat{\cdot} \) over a symbol means that this symbol must be omitted. Besides, we set \( W = V_{i_0 s_0} \cap \ldots \cap V_{i_q s_q+1} \), \( W_k = V_{i_0 s_0} \cap \ldots \cap \hat{V}_{i_k s_k} \cap \ldots \cap V_{i_q s_q+1} \) and \( r^W_k \) is restriction map from \( W \) to \( W_k \). Also condition (2.1) implies that \( \delta^q \) maps \( C^q_b(g^{-1}(U), E) \) into \( C^{q+1}_b(g^{-1}(U), E) \). We will denote \( \delta^q|_{C^q_b(g^{-1}(U), E)} \) by \( \delta^q_b \). As usual, \( \delta^{q+1}_b \circ \delta^q = 0 \) and \( \delta^{q+1}_b \circ \delta^q_b = 0 \). Thus one can define the cohomology groups on the cover \( g^{-1}(U) \) by

\[
H^q(g^{-1}(U), E) := \text{Ker}(\delta^q)/\text{Im}(\delta^{q-1}), \quad H^q_b(g^{-1}(U), E) := \text{Ker}(\delta^q_b)/\text{Im}(\delta^{q-1}_b).
\]

In what follows the cohomology group \( H^q(U, E_M) \) on the cover \( U \) of \( M \) with coefficients in the sheaf of germs of holomorphic sections of \( E_M \) is defined similarly to \( H^q(g^{-1}(U), E) \). Elements of \( \text{Ker}(\delta^q) \) and \( \text{Ker}(\delta^q_b) \) will be called \( q \)-cocycles and of \( \text{Im}(\delta^{q-1}) \) and \( \text{Im}(\delta^{q-1}_b) \) \( q \)-coboundaries.

**Proposition 2.4** There is a linear isomorphism \( \Phi^q : H^q_b(g^{-1}(U), E) \rightarrow H^q(U, E_M) \).

**Proof.** Let \( f = \{ f_{i_0 s_0, \ldots, i_q s_q} \} \in C^q_b(g^{-1}(U), E) \). Let \( b_{i_0 s_0, \ldots, i_q s_q} \in \mathcal{O}(V_{i_0 s_0} \cap \ldots \cap V_{i_q s_q}, B) \) be the representation of \( f_{i_0 s_0, \ldots, i_q s_q} \) in the trivialization which identifies \( \pi^{-1}(V_{i_0 s_0}) \) with \( V_{i_0 s_0} \times B \). If \( V_{i_0 s_0} \cap \ldots \cap V_{i_q s_q} \neq \emptyset \) then \( s_k = \tau_Q(\hat{c}_{i_k s_k})(s_0), k = 0, \ldots, q \), and \( U_{i_0} \cap \ldots \cap U_{i_q} \neq \emptyset \). For otherwise, \( b_{i_0 s_0, \ldots, i_q s_q} = 0 \). Thus for \( s_0, \ldots, s_q \) satisfying the above identities we can define

\[
\tilde{b}_{i_0, \ldots, i_q} := \{ b_{i_0 s_0, \ldots, i_q s_q} \circ \phi_{i_0 s_0} \}_{s_0 \in X_G}.
\]

For \( U_{i_0} \cap \ldots \cap U_{i_q} = \emptyset \) we set \( \tilde{b}_{i_0, \ldots, i_q} = 0 \). Further, according to (2.3), \( \tilde{b}_{i_0, \ldots, i_q} \in \mathcal{O}(U_{i_0} \cap \ldots \cap U_{i_q}, B_{\infty}(X_G)) \). This implies that

\[
\tilde{f}_{i_0, \ldots, i_q} := \{ f_{i_0 s_0, \ldots, i_q s_q} \circ \phi_{i_0 s_0} \}_{s_0 \in X_G}
\]

declared similarly to \( \tilde{b}_{i_0, \ldots, i_q} \) belongs to \( \mathcal{O}(U_{i_0} \cap \ldots, U_{i_q}, E_M) \), because \( \tilde{b}_{i_0, \ldots, i_q} \) is just another representation of \( \tilde{f}_{i_0, \ldots, i_q} \) in the trivialization which identifies \( \pi^{-1}(U_{i_0}) \) with \( U_{i_0} \times B_{\infty}(X_G) \). For \( \tilde{f} = \{ \tilde{f}_{i_0, \ldots, i_q} \} \) we set \( \tilde{\Phi}^q(f) = \tilde{f} \). Then, clearly, \( \tilde{\Phi}^q : C^q_b(g^{-1}(U), E) \rightarrow C^q(U, E_M) \) is linear and injective. Now for a cochain \( \tilde{f} \in C^q(U, E_M) \) we can convert the construction for \( \tilde{\Phi}^q \) to find a cochain \( f \in C^q_b(g^{-1}(U), E) \) such that \( \tilde{\Phi}^q(f) = \tilde{f} \). Thus \( \tilde{\Phi}^q \) is an isomorphism. Moreover, a simple calculation based on (2.2) shows that

\[
\delta^q \circ \tilde{\Phi}^q = \tilde{\Phi}^{q+1} \circ \delta_b^q,
\]

where \( \delta^q \) on the left means the operator for \( E_M \) defined similarly to (2.4). Hence \( \tilde{\Phi}^q \) determines a linear isomorphism \( \Phi^q : H^q_b(g^{-1}(U), E) \rightarrow H^q(U, E_M) \).

We complete this section by

**Proposition 2.5** Let \( \rho : G \rightarrow \text{Iso}(B) \) be a homomorphism and \( E_{\rho} \rightarrow M_G \) be the holomorphic Banach vector bundle associated with \( \rho \). Then \( E_{\rho} \) satisfies conditions of Proposition 2.4.
Proof. Let $M_e \to M_G$ be the universal covering (recall that $G = \pi_1(M_G)$). Since the open cover $g^{-1}(U) = \{V_{is}\}_{i \in I, s \in X_G}$ of $M_G$ is acyclic, $M_e$ can be defined with respect to $g^{-1}(U)$. Namely, there is a cocycle $h = \{h_{is,jk}\} \in Z^1_0(g^{-1}(U), G)$ such that $M_e$ is biholomorphic to the quotient space of $\sqcup_{is} V_{is} \times G$ by the equivalence relation $V_{is} \times G \ni x \times R_G(h_{is,jk})(f) \sim x \times f \in V_{jk} \times G$, $s = \tau_{q(l_j)}(\tilde{c}_{ij})(k)$; here $R_G(g)(f) := f \cdot g^{-1}$, $f, q \in G$. Now $E_{\rho}$ is biholomorphic to the quotient space of $\sqcup_{is} V_{is} \times B$ by the equivalence relation $V_{is} \times B \ni x \times \rho(h_{is,jk})(v) \sim x \times v \in V_{jk} \times B$. Clearly, the family $\{\rho(h_{is,jk})\}$ satisfies estimate (2.1). □

3. Proof of Theorem 1.1 and Corollaries 1.2, 1.5.

Proof of Theorem 1.1. Let $N \subset\subset M$ be an open connected subset of a connected Stein manifold $M$ satisfying (1.2). Let $G \subset \pi_1(M)$ be a subgroup. As before, by $M_G, N_G$ we denote the covering spaces of $M$ and $N$ corresponding to $G$. Then by the covering homotopy theorem (see e.g. [Hu,Ch.III,Sect.16]), there is a holomorphic embedding $N_G \hookrightarrow M_G$. Without loss of generality we regard $N_G$ as an open subset of $M_G$. Denote also by $g_{MG} : M_G \to M, g_{NG} : N_G \to N$ the corresponding projections such that $g_{MG}|N_G = g_{NG}$. Let $i : U \hookrightarrow N_G$ be a holomorphic embedding of a complex connected manifold $U$.

Lemma 3.1 It suffices to prove the theorem under the assumption that homomorphism $i_* : \pi_1(U) \to G (= \pi_1(N_G))$ is surjective.

Proof. Assume that $G' := Im(i_*)$ is a proper subgroup of $G$. By $t : N_{G'} \to N_G$ we denote the covering of $N_G$ corresponding to $G' \subset G$. By definition, $g_{NG} \circ t = g_{NG'} : N_{G'} \to N$ is the covering of $N$ corresponding to $G' \subset \pi_1(N)$. Further, by the covering homotopy theorem there is a holomorphic embedding $i' : U \hookrightarrow N_{G'}$ such that $t \circ i' = i$, $\text{Ker}(i') = \text{Ker}(i_*)$, and $i'_* : \pi_1(U) \to G' (= \pi_1(N_{G'}))$ is surjective. Clearly, it suffices to prove the theorem for $i'(U) \subset N_{G'}$. □

In what follows we assume that $i_*$ is surjective. By $p_U : \hat{U} \to U$ we denote the regular covering of $U$ corresponding to $K(U) := \text{Ker}(i_*)$, where $\pi_1(\hat{U}) = K(U)$. Consider the holomorphic Banach vector bundle $E^{MG}_1(G) \to M_G$ associated with homomorphism $\rho_G : G \to \text{Iso}(l_1(G)), [\rho_G(g)(v)](x) := v(x \cdot g^{-1})$, $v \in l_1(G), x, g \in G$ (see Example 2.2 (b)). Since $i_*$ is surjective, $E^{MG}_1(G)|_U = E^U_1(G)$.

Let $K_G \subset l_1(G)$ be the kernel of the linear functional $l_1(G) \ni \{v_g\}_{g \in G} \mapsto \sum_{g \in G} v_g$. Then $K_G$ is invariant with respect to any $\rho_G(g), g \in G$. In particular, $\rho_G$ determines a homomorphism $h_G : G \to \text{Iso}(K_G), h_G(g) = \rho_G(g)|_{K_G}$. Here we consider $K_G$ with the norm induced by the norm of $l_1(G)$. Let $F_G \to M_G$ be the holomorphic Banach vector bundle associated with $h_G$. Clearly, $F_G$ is a subbundle of $E^{MG}_1(G)$. Further, the quotient bundle $C_G := E^{MG}_1(G)/F_G \to M_G$ is the trivial flat vector bundle of complex rank 1. Indeed, it is associated with the quotient homomorphism: $h_G : G \to \mathbb{C}^*, h_G(g)(v + K_G) := \rho_G(g)(v) + K_G, g \in G, v \in l_1(G)$, where $w + K_G$ is the image of $w \in l_1(G)$ in the factor space $l_1(G)/K_G = \mathbb{C}$. This homomorphism is trivial because $\rho_G(g)(v) - v \in K_G$ by definition. Thus we have the short exact sequence

$$0 \to F_G \to E^{MG}_1(G) \xrightarrow{k_G} C_G \to 0$$

(3.1)
Our goal is to construct a holomorphic section \( I_G : C_G \rightarrow E_1^{MG}(G) \) (linear on the fibres) such that \( k_G \circ I_G = \text{id} \). Then we will obtain the bundle decomposition \( E_1^{MG}(G) = I(G)(C_G) \oplus F_G \).

Let \( \{t_s\}_{s \in B} \) be a standard basis of unit vectors in \( l_1(G) \), \( t_s(g) = \delta_{sg} \), \( s, g \in G \). Define \( A : \mathbb{C} \rightarrow l_1(G) \) by \( A(e) = c te \), where \( e \in G \) is the unit. Then \( A \) is a linear operator of norm 1. Now let us recall the construction of \( E_1^{MG}(G) \) given in Proposition 2.3.

Let \( M \rightarrow M_G \) be the universal covering. Consider an open cover \( \tilde{g}^{-1}_{MG}(U) = \{V_{G,is}\}_{i \in I, s \in X_G} \) of \( M_G \) where \( U = \{U_i\}_{i \in I} \) is an open cover of \( M \) by complex balls, and \( \cup_{s \in X_G} V_{G,is} = g^{-1}(U_i) \). Then there is a cocycle \( c_G = \{c_{G,is,jk}\} \in Z^1_B(\tilde{g}^{-1}_{MG}(U), G) \) such that \( E_1^{MG}(G) \) is biholomorphic to the quotient space of \( \cup_{i,s} V_{G,is} \times l_1(G) \) by the equivalence relation \( V_{G,is} \times l_1(G) \ni x \times \rho_G(c_{G,is,jk})(v) \sim x \times v \in V_{G,jk} \times l_1(G) \).

The construction of \( F_G \) is similar, the only difference is that in the above formula we take \( h_G \) instead of \( \rho_G \). The above constructions restricted to \( V_{G,is} \) determine isomorphisms of holomorphic Banach vector bundles: \( \epsilon_G, is : E_1^{MG}(G)|_{V_{G,is}} \rightarrow V_{G,is} \times l_1(G), \)

\( f_G, is : F_G|_{V_{G,is}} \rightarrow V_{G,is} \times K_G, \) \( c_G, is : c_G|_{V_{G,is}} \rightarrow V_{G,is} \times G \). Then we define \( A_G, is : C_G \rightarrow E_1^{MG}(G) \) on \( V_{G,is} \) as \( e_{G,is}^{-1} \circ \tilde{A} \circ c_{G,is} \), where \( \tilde{A}(x) = x \times A(e), x \in V_{G,is}, e \in \mathbb{C} \). Clearly, \( k_G \circ A_G, is = \text{id} \) on \( V_{G,is} \). Thus \( B_G, is,jk := A_G, is - A_G, jk : C_G|_{V_{G,is}} \rightarrow F_G|_{V_{G,is}} \) is a homomorphism of bundles of norm \( \leq 2 \) on each fibre (here norms on \( F_G, C_G \) and \( E_1^{MG}(G) \) are defined as in Example 2.2(b)). We also use the following identification \( \text{Hom}(C_G, F_G) \cong F_G \) (the last isomorphism is because \( C_G \) is trivial and \( \text{Hom}(\mathbb{C}, K_G) \cong \mathbb{C}^* \otimes K_G = K_G \)). Further, according to Proposition 2.3, the holomorphic Banach vector bundle \( \text{Hom}(C_G, F_G) \) associated with the homomorphism \( h_G \otimes h_G : G \rightarrow \text{Iso}(\text{Hom}(\mathbb{C}, K_G)) \) satisfies conditions of Proposition 2.3. Therefore, by definition, \( B_G = \{B_G, is,jk\} \) is a holomorphic 1-cocycle with respect to \( \delta^1 \) defined on the cover \( \tilde{g}^{-1}_{MG}(U) \). By \( \phi_G, is : U \rightarrow V_{G,is} \) we denote the map inverse to \( g_{MG}|_{V_{G,is}} \). Then we will prove

**Lemma 3.2** There is \( B_G = \{B_G, is\} \in C^0_b(\tilde{g}^{-1}_{MG}(U), F_G), \) \( B_G, is \in \mathcal{O}(V_{is}, F_G), \) such that \( \delta^0(B_G) = B_G \). Moreover, for any \( i \in I \) there is a continuous nonnegative function \( F_i : U_i \rightarrow \mathbb{R}_+ \) such that for any \( G \)

\[
\sup_{s \in X_G, z \in U_i} |(B_G, is \circ \phi_G, is)(z)| \leq F_i(z).
\]

(3.2)

Here \( | \cdot | \) denotes the norm on \( F_G \).

**Proof.** According to Proposition 2.3, we can construct the holomorphic Banach vector bundle \( (F_G)_M \). It is defined on the cover \( U \) of \( M \) by a cocycle \( d_G := \{d_{G,ij}\} \in Z^1_B(U, \text{Iso}(K_G)_{\infty}(X_G)), \) where \( d_{G,ij} \in \mathcal{O}(U_i \cap U_j, \text{Iso}(K_G)_{\infty}(X_G)) \). Let \( \Phi_G^0 : C^0_b(\tilde{g}^{-1}_{MG}(U), F_G) \rightarrow C^0(U, (F_G)_M) \) be the isomorphism introduced in the proof of Proposition 2.4. Then \( \Phi_G(B_G) := b_G = \{b_{G,ij}\} \) is a holomorphic 1-cocycle with respect to \( \delta^1 \) defined on \( U \). Here \( b_{G,ij} \in \mathcal{O}(U_i \cap U_j, (F_G)_M) \), and

\[
\sup_{i,j \in I, z \in M} |b_{G,ij}(z)|_{(F_G)_M} \leq 2,
\]

where \( | \cdot |_{(F_G)_M} \) denotes the norm on \( (F_G)_M \).
Let \( \mathcal{G} \) be the set of all subgroups \( G \subset \pi_1(M) \). We define the Banach space \( K = \oplus_{G \in \mathcal{G}} (K_G)_{\infty}(X_G) \) such that \( x = \{ x_G \}_{G \in \mathcal{G}} \) belongs to \( K \) if \( x_G \in (K_G)_{\infty}(X_G) \) and

\[
|x| := \sup_{G \in \mathcal{G}} |x_G|(K_G)_{\infty}(X_G) < \infty,
\]

where \( | \cdot |(K_G)_{\infty}(X_G) \) is the norm on \( (K_G)_{\infty}(X_G) \). Further, let us define \( d := \{ d_{ij} \} \in Z_0^1(\mathcal{U}, \text{Iso}(K)) \) as \( d := \oplus_{G \in \mathcal{G}} d_G \). Here

\[
d_{ij} := \oplus_{G \in \mathcal{G}} d_{G,ij}, \quad [d_{ij}(z)](v_G)_{G \in \mathcal{G}} := \{ [d_{G,ij}(z)](v_G) \}_{G \in \mathcal{G}}, \quad z \in U_i \cap U_j.
\]

Clearly \( d_{ij} \in \mathcal{O}(U_i \cap U_j, \text{Iso}(K)) \). Now we define the holomorphic Banach vector bundle \( F \) over \( M \) by the identification \( U_i \times K \ni x \times d_{ij}(x)[v] \sim x \times v \in U_j \times K \) for any \( x \in U_i \cap U_j \). In fact, this bundle coincides with \( \oplus_{G \in \mathcal{G}} (F_G)_M \). A vector \( f \) of \( F \) over \( z \in M \) can be identify with a family \( \{ f_G \}_{G \in \mathcal{G}} \) so that \( f_G \in (F_G)_M \) is a vector over \( z \). Moreover, the norm \( |f|_F := \sup_{G \in \mathcal{G}} |f_G|_{(F_G)_M} \) of \( f \) is finite. Now we can define a holomorphic 1-cocycle \( b = \{ b_{ij} \} \) of \( F \) defined on the cover \( \mathcal{U} \) as

\[
b := \{ b_G \}_{G \in \mathcal{G}}, \quad b_{ij} := \{ b_{G,ij} \}_{G \in \mathcal{G}} \in \mathcal{O}(U_i \cap U_j, F).
\]

Here holomorphy of \( b_{ij} \) follows from the uniform estimate of norms of \( b_{G,ij} \).

Let us use the fact that \( M \) is a Stein manifold. According to a theorem of Bungart [B, Sect.4] (i.e. a version of the classical Cartan Theorem B for cohomology of sheaves of germs of holomorphic sections of holomorphic Banach vector bundles) cocycle \( b \) represents 0 in the corresponding cohomology group \( H^1(M, F) \). Further, the cover \( \{ U_i \}_{i \in I} \) of \( M \) consists of Stein manifolds (and so it is acyclic). Therefore by the classical Leré theorem (on calculation of cohomology groups on acyclic covers),

\[
H^1(M, F) = H^1(\mathcal{U}, F).
\]

Thus \( b \) represents 0 in \( H^1(\mathcal{U}, F) \), that is, \( b \) is a coboundary. In particular, there are holomorphic sections \( b_i \in \mathcal{O}(U_i, F) \) such that

\[
b_i(z) - b_j(z) = b_{ij}(z) \quad \text{for any } z \in U_i \cap U_j.
\]

We also set

\[
F_i(z) := |b_i(z)|_F.
\]

Then \( F_i \) is a continuous nonnegative function on \( U_i \). Further, by definition each \( b_i \) can be represented as a family \( \{ b_{G,i} \}_{G \in \mathcal{G}} \) where \( b_{G,i} \in \mathcal{O}(U_i, (F_G)_M) \). The family \( b_G = \{ b_{G,i} \}_{i \in I} \) belongs to \( C^0(\mathcal{U}, (F_G)_M) \). Using the isomorphism \( \Phi_G^0 \) from Proposition 2.4 we obtain a cochain \( \tilde{B}_G := [\Phi_G^0]^{-1}(\bar{b}_G) \in C^0_b(g_{M_0}^{-1}(\mathcal{U}), F_G) \). Now if \( \tilde{B}_G := \{ \tilde{B}_{G,is} \} \), \( \tilde{B}_{G,is} \in \mathcal{O}(V_{is}, F_G) \), from identity (2.3) it follows that

\[
\tilde{B}_{G,is}(z) - \tilde{B}_{G,jk}(z) = B_{G,is,jk}(z) \quad \text{for any } z \in V_{is} \cap V_{jk}.
\]

Finally, inequality (3.2) is the consequence of definitions of \( F_i \) and \( \Phi_G^0 \).

The lemma is proved. □

Let us consider now the family \( \{ A_{G,is} - B_{G,is} \}_{i,s} \). By definition, it determines a holomorphic linear section \( I_G : C_G \rightarrow E^1_{M_0}(G) \), \( k_G \circ I_G = id \). Thus we have \( E^1_{M_0}(G) = I_G(C_G) \oplus F_G \). In the next result the norm \( || \cdot || \) of \( I_G \) is defined with respect to the norms \( | \cdot |_{C_G} \) and \( | \cdot |_{E^1_{M_0}(G)} \).
Lemma 3.3  There is a constant $C = C(N)$ such that for any $G \in \mathcal{G}$

$$\sup_{z \in \tilde{N}} \|I_G(z)\| \leq C.$$  

**Proof.** Let $\mathcal{V} = \{V_i\}$ be a refinement of the cover $\mathcal{U}$ of $M$ such that each $V_i$ is relatively compact in some $U_{k(i)}$. Then from Lemma 3.1 it follows that

$$\sup_{s \in U_{G}, z \in V_i} |(\tilde{B}_{G,k(i)s} \circ \phi_{G,k(i)s})(z)| \leq \sup_{z \in V_i} F_{k(i)}(z) = C_i < \infty.$$  

Now for any $z \in g^{-1}_{MG}(V_i)$ we have

$$\|I_G(z)\| \leq \sup_{s \in X_{G}, y \in V_i} \{|(A_{G,k(i)s} \circ \phi_{G,k(i)s})(y)| + |(\tilde{B}_{G,k(i)s} \circ \phi_{G,k(i)s})(y)|\} \leq 1 + C_i.$$  

Since $\bar{\mathcal{N}} \subset M$ is a compact, we can find a finite number of sets $V_{i_1}, ..., V_{i_l}$ which cover $\bar{\mathcal{N}}$. Then

$$\sup_{z \in \mathcal{N}} \|I_G(z)\| \leq \max_{1 \leq i \leq l} \{1 + C_i\} := C < \infty. \quad \square$$

Consider now the restriction of exact sequence (3.1) to $U$. Using the identification $E_{1}^{MG}(G)\big|_{U} \cong E_{1}^{U}(G)$ we obtain

$$0 \longrightarrow (F_G)\big|_{U} \longrightarrow E_{1}^{U}(G) \longrightarrow (C_G)\big|_{U} \longrightarrow 0.$$  

Similarly, we have the dual sequence obtained by taken the dual bundles in the above sequence

$$0 \longrightarrow [(C_G)\big|_{U}]^{*} \longrightarrow E_{\infty}^{U}(G) \longrightarrow [(F_G)\big|_{U}]^{*} \longrightarrow 0.$$  

Let $C(G)$ be the space of constant functions in $l_{\infty}(G)$. By definition, $[(C_G)\big|_{U}]^{*}$ is a subbundle of $E_{\infty}^{U}(G)$ of complex rank 1 with fibre $C(G)$ associated with the trivial homomorphism $G \longrightarrow \text{Iso}(C(G))$. Let $P_U := [(I_G)\big|_{U}]^{*} : E_{\infty}^{U}(G) \longrightarrow [(C_G)\big|_{U}]^{*}$ be the homomorphism of bundles dual to $(I_G)\big|_{U}$. Then for any $z \in U$, $P_U(z)$ projects the fibre of $E_{\infty}^{U}(G)$ over $z$ onto the fibre of $[(C_G)\big|_{U}]^{*}$ over $z$. Moreover, we have

$$\sup_{z \in U} \|P_U(z)\| \leq C,$$  

where $\| \cdot \|$ is the dual norm defined with respect to $\| \cdot \|_{E_{\infty}^{U}(G)}$ and $\| \cdot \|_{[(C_G)\big|_{U}]^{*}}$. The operator $P_U$ induces also a linear map $P_{U}^{*} : \mathcal{O}(U, E_{\infty}^{U}(G)) \longrightarrow \mathcal{O}(U, [(C_G)\big|_{U}]^{*})$,

$$[P_{U}^{*}(f)](z) := [P_{U}(z)](f(z)), \quad f \in \mathcal{O}(U, E_{\infty}^{U}(G)).$$  

Further, any $f \in H^{\infty}(\tilde{U})$ can be considered in a natural way as a bounded holomorphic section of the trivial bundle $\tilde{U} \times \mathbb{C} \longrightarrow \tilde{U}$. This bundle satisfies assumptions of Proposition 2.3 (for $U$ instead of $M$). Furthermore, it easy to see that in this case the bundle $(\tilde{U} \times \mathbb{C})_{U}$ defined in Proposition 2.3 coincides with $E_{\infty}^{U}(G)$. Let $\Phi_{U}^{0} : H_{b}^{0}(g^{-1}(U), \tilde{U} \times \mathbb{C}) \longrightarrow H^{0}(U, E_{\infty}^{U}(G))$ be the isomorphism of Proposition 2.4.
(This is just the direct image map with respect to \( p_U : \tilde{U} \to U \).) We define the Banach subspace \( S_\infty(U) \subset H^0(U, E_U^*(G)) \) with norm \(| \cdot |_U\) by the formula

\[ f \in S_\infty(U) \iff |f|_U := \sup_{z \in U} |f(z)|_{E_U^*(G)} < \infty. \]

Clearly \( \Phi_U^0 \) maps \( H^\infty(\tilde{U}) \) isomorphically onto \( S_\infty(U) \). Moreover, \( s_U := \Phi_U^0|_{H^\infty(\tilde{U})} \) is a linear isometry of Banach spaces. By definition, the space \( s_U(p_U^*(H^\infty(U))) \) coincides with \( \mathcal{O}(U, [(C_U)|_U^*] \cap S_\infty(U) \). Then according to the definition of \( s_U \) and inequality (3.3) the linear operator \( P := s_U^{-1} \circ p_{U'}^* \circ s_U \) maps \( H^\infty(\tilde{U}) \) onto \( p_U^*(H^\infty(U)) \). According to our construction \( P \) is a bounded projector satisfying (1).

Here the required projector \( P_z : l^\infty(F_z) \to C(F_z) \) can be naturally identified with \( p_U(z) \). Let now \( f \in H^\infty(\tilde{U}) \) and \( g \in p_U^*(H^\infty(U)) \). Then by definition we have

\[ P[f \cdot g]|_{p_U^{-1}(z)} = P_z[(f \cdot g)|_{p_U^{-1}(z)}] = P_z[f|_{p_U^{-1}(z)}] \cdot g|_{p_U^{-1}(z)} = (P[f] \cdot g)|_{p_U^{-1}(z)}. \]

Here we used that \( g|_{p_U^{-1}(z)} \) is a constant and \( P_z \) is a linear operator. This implies (2).

Property (3) follows from the fact that \( p_U \) is a projector onto \( C(F_z) \). Further, (4) is a consequence of the fact that \( P_z(z) \) is dual to \( (I_C)(z) \) and so \( P_z \) is continuous in the weak * topology of \( l^\infty(F_z) \). Finally, the norm of \( P \) coincides with \( \sup_{z \in U} |P_U(z)| \).

Thus \( ||P|| \leq C \) for \( C \) as in (3.3). This completes the proof of (5).

The theorem is proved. \( \Box \)

**Proof of Corollary 1.2.** First, remark that any bordered Riemann surface \( N \) admits an embedding to a Riemann surface \( M \) such that the pair \( N \subset \subset M \) satisfies condition (1.2). Let \( \tilde{R} \) be a covering of \( N \) and \( i : U \leftrightarrow \tilde{R} \) be such that \( \pi_1(U) \) is generated by a subfamily of generators of the free group \( \pi_1(\tilde{R}) \). Then the homomorphism \( i_* : \pi_1(U) \to \pi_1(\tilde{R}) \) is injective. In particular, \( K(U) := Ker(i_*) = \{e\} \) and \( p_U : \tilde{U} \to U \) is the universal covering. Since \( \tilde{U} \) is biholomorphic to \( \mathbb{D} \), the existence of the projector \( P : H^\infty(\mathbb{D}) \to p_U^*(H^\infty(U)) \) follows from Theorem 1.1. \( \Box \)

**Proof of Corollary 1.3.** Let \( N \subset \subset M, R \subset F_z(N) \) and \( i : U \leftrightarrow R \) be open Riemann surfaces satisfying conditions of Theorem 1.1. Assume also that \( K(U) := Ker(i_*) = \{e\} \). Let \( p_U : \mathbb{D} \to U \) be the universal covering map. Then there is a projector \( P : H^\infty(\mathbb{D}) \to p_U^*(H^\infty(U)) \) with properties (1)-(5) of Theorem 1.1. Let \( f_1, \ldots, f_n \in H^\infty(U) \) satisfy the corona condition (1.4) with \( \delta > 0 \). Without loss of generality we will assume also that \( \max_i ||f_i||_{H^\infty(U)} \leq 1 \). For \( 1 \leq i \leq n \) we set \( h_i := p_U^*(f_i) \). Then \( h_1, \ldots, h_n \in H^\infty(\mathbb{D}) \) satisfy the corona condition in \( \mathbb{D} \) (with the same \( \delta \)). Also \( \max_i ||h_i||_{H^\infty(\mathbb{D})} \leq 1 \). Now according to the solution of Carleson’s Corona Theorem [Ca], there is a constant \( C(n, \delta) \) and some \( g_1, \ldots, g_n \in H^\infty(\mathbb{D}) \) satisfying \( \max_i ||g_i||_{H^\infty(\mathbb{D})} \leq C(n, \delta) \) such that \( \sum_{i=1}^n g_i h_i \equiv 1 \). Let us define \( d_i \in H^\infty(U) \) by the formula

\[ p_U^*(d_i) := P[g_i], \quad 1 \leq i \leq n. \]

Then property (2) for \( P \) implies that \( \sum_{i=1}^n d_i f_i \equiv 1 \). Moreover, \( \max_i ||d_i||_{H^\infty(U)} \leq C(N) \cdot C(n, \delta) \) where \( C(N) \) is the constant from Lemma 3.3.

The proof of the corollary is complete. \( \Box \)

**Remark 3.4** In a forthcoming paper we present the following generalization of Corollary 1.3.
Theorem. Let an open Riemann surface $U$ satisfy the conditions of Corollary 1.3. Let $b$ be an $n \times k$ matrix, $k < n$, with entries in $H^\infty(U)$. Assume that the corona condition (1.4) is valid for the family of minors of $b$ of order $k$. Then there is an $n \times n$ matrix $\hat{b}$ with entries in $H^\infty(U)$ which extends $b$ such that $\det(\hat{b}) \equiv 1$ on $U$.

The proof of the theorem is based on Theorem 1.1 and a Grauert type theorem for “holomorphic” vector bundles defined on maximal ideal spaces (which are not usual manifolds!) of certain Banach algebras.

4. Proof of Theorem 1.7.

Let $N \subset M$ be a relatively compact domain of a connected Stein manifold $M$ satisfying (1.2). For a subgroup $G \subset \pi_1(M)$ we denote by $g_{NG} : N_G \rightarrow N$ and $g_{MG} : M_G \rightarrow M$ the covering spaces of $M$ and $N$ corresponding to the group $G$ with $N_G \subset M_G$. Further, assume that $i : U \hookrightarrow N_G$ is a holomorphic embedding of a complex connected manifold $U$, $K(U) := Ker(i_*) \subset \pi_1(U)$, and $p_U : \tilde{U} \rightarrow U$ is the regular covering of $U$ corresponding to $K(U)$. As before, without loss of generality we may assume that homomorphism $i_* : \pi_1(U) \rightarrow G (= \pi_1(N_G))$ is surjective (see arguments of Lemma 3.1). Thus the deck transformation group of $\tilde{U}$ is $G$. We begin the proof of the theorem with the following

**Proposition 4.1** For any $z \in U$, the sequence $p_U^{-1}(z) := \{w_s\}_{s \in G} \subset \tilde{U}$ is interpolating with respect to $H^\infty(\tilde{U})$. Moreover, let

$$M(z) = \sup_{||a_s||_{H^\infty(\tilde{U})} \leq 1} \inf \{||g||_{H^\infty(\tilde{U})} : g \in H^\infty(\tilde{U}), g(w_s) = a_s, \ j = 1, 2, \ldots\}$$

be the constant of interpolation for $p_U^{-1}(z)$. Then there is a constant $C = C(N)$ such that

$$\sup_{z \in U} M(z) \leq C.$$

**Proof.** Consider the homomorphism $\rho_G^* : G \rightarrow Iso(l_\infty(G))$,

$$[\rho_G^*(g)](w)(x) := w(x \cdot g^{-1}), \ w \in l_\infty(G), x, g \in G.$$

Let $E^{\infty}_{MG}(G) \rightarrow M_G$ be the holomorphic Banach vector bundle associated with $\rho_G^*$. Then $E^{\infty}_{MG}(G)|_U = E^{U}_{\infty}(G)$ (see Example 2.2 (b)). According to Proposition 2.3, we can define the holomorphic Banach vector bundle $[E^{\infty}_{\infty}(G)]_M \rightarrow M$ with the fibre $[l_\infty(G)]_{\infty}(X_G)$. Let $G$ be the set of all subgroup $G \subset \pi_1(M)$. We define the Banach space $L = \bigoplus_{G \in G}[l_\infty(G)]_{\infty}(X_G)$ such that $x = \{x_G\}_{G \in G}$ belongs to $L$ if $x_G \in [l_\infty(G)]_{\infty}(X_G)$ and

$$|x|_L := \sup_{G \in G} |x_G|_{[l_\infty(G)]_{\infty}(X_G)} < \infty,$$

where $| \cdot |_{[l_\infty(G)]_{\infty}(X_G)}$ is the norm on $[l_\infty(G)]_{\infty}(X_G)$. Then similarly to the construction of Lemma 3.2, we can define the holomorphic Banach vector bundle $B$ over $M$ with the fibre $L$ by the formula

$$B := \bigoplus_{G \in G}[E^{\infty}_{\infty}(G)]_M.$$
Note that the structure group of $B$ is $Iso(L)$. Therefore the norm $|.|_L$ induces a norm $|.|_L$ on $B$ (see Example 2.2 (b)). Let $O \subset M$ be a relatively compact domain containing $\overline{N}$. Denote by $H^\infty(O, B)$ the Banach space of bounded holomorphic sections from $O(O, B)$, that is, 

$$f \in H^\infty(O, B) \iff ||f|| := \sup_{z \in O} |f(z)|_B < \infty.$$ 

For any $z \in O$ consider the restriction operator $r(z) : H^\infty(O, B) \rightarrow L,$

$$r(z)[f] := f(z), \quad f \in H^\infty(O, B).$$

Then $r(z)$ is a continuous linear operator with the norm $||r(z)|| \leq 1$. Moreover, by a theorem of Bungart (see [B, Sect.4]), for any $v \in L$ there is a section $f \in O(M, B)$ such that $f(z) = v$. Since $O$ is relatively compact in $M$, the restriction $f|_O$ belongs to $H^\infty(O, B)$. This shows that $r(z)$ is surjective. For any $v \in L$ we set $K_v(z) := (r(z))^{-1}(v) \subset H^\infty(O, B)$. The constant

$$h(z) := \sup_{|v|_L \leq 1} \inf_{t \in K_v(z)} ||t||$$

will be called the constant of interpolation for $r(z)$. We will show that

**Lemma 4.2**

$$\sup_{z \in \overline{N}} h(z) \leq C < \infty$$

where $C$ depends on $N$ only.

**Proof.** In fact it suffices to cover $\overline{N}$ by a finite number of open balls and prove the required inequality for $z$ varying in each of these balls. Moreover, since $\overline{N}$ is a compact, for any $w \in \overline{N}$ it suffices to find an open neighbourhood $U_w \subset O$ of $w$ such that $\{h(z)\}_{z \in U_w}$ is bounded from above by an absolute constant.

Let $w \in \overline{N}$. Without loss of generality we may identify a small open neighbourhood of $w$ in $O$ with the open unit ball $B_c(0, 1) \subset \mathbb{C}^n$, $n = dim O$, such that $w$ corresponds to 0 in this identification. It is easy to see that $r(z), z \in B_c(0, 1)$, is the family of linear continuous operators holomorphic in $z$. Let $R := 1/4h(w)$. Since $h(w) \geq 1$, $B_c(0, 1)$ contains $B_c(0, R)$. For a $y \in B_c(0, R)$ consider the one dimensional complex subspace $l_y$ of $\mathbb{C}^n$ containing $y$. Without loss of generality we may identify $l_y \cap B_c(0, 1)$ with the open unit disk $\mathbb{D} \subset \mathbb{C}$. With this identification, let $r(z) := \sum_{i=0}^\infty r_i z^i$ be the Taylor expansion of $r(z)$ in $\mathbb{D}$. Here $r_i : H^\infty(O, B) \rightarrow L$ is a linear operator with the norm $||r_i|| \leq 1$. The last estimate follows from the Cauchy estimates for derivatives of holomorphic functions. We also have $r_0 := r(0)$ (recall that $w = 0$). Let $v \in L, |v|_L \leq 1$. For $z < R$ we will construct $v(z) \in H^\infty(O, B)$ which depends holomorphically on $z$, such that $||v(z)|| \leq 8h(w)$ and $r(z)[v(z)] = v$. Let $v(z) = \sum_{i=0}^\infty v_i z^i$. Then we have the formal decomposition

$$v = r(z)[v(z)] = \sum_{i=0}^\infty z^i \cdot \sum_{j=0}^\infty r_i(v_j z^j) = \sum_{k=0}^\infty z^k \cdot \sum_{i+j=k}^{} r_i(v_j).$$
Let us define \( v_i \) from the equations
\[
  r_0(v_0) = v \quad \text{and} \quad \sum_{i+j=k} r_i(v_j) = 0, \quad \text{for} \quad k \geq 1.
\]

Since the constant of interpolation for \( r(0) \) is \( h(w) \), we can find \( v_0 \in H^\infty(O, B) \), \( ||v_0|| < 2h(w) \), satisfying the first equation. Substituting this \( v_0 \) into the second equation we obtain \( r_0(v_1) = -r_1(v_0) \). Here \( ||r_1(v_0)|| \leq 2h(w) \) because \( ||r_1|| \leq 1 \). Thus again we can find \( v_1 \in H^\infty(O, B) \) satisfying the second equation such that \( ||v_1|| \leq (2h(w))^2 \). Continuing step by step to solve the above equations we obtain \( v_n \in H^\infty(O, B) \) satisfying the \( n \)-th equation such that \( ||v_n|| \leq \sum_{i=1}^n (2h(w))^{i+1} < n(2h(v))^{n+1} \) (because \( h(w) \geq 1 \)). Thus we have
\[
  \|v(z)\| \leq \sum_{n=0}^\infty n(2h(w))^{n+1} R^n < \frac{2h(w)}{(1 - 2h(w)R)^2} = 8h(w).
\]

The above arguments show that \( h(z) \leq 8h(w) \) for any \( z \in B_c(0, 1/4h(w)) \).

This completes the proof of the lemma. \( \square \)

We proceed to prove Proposition 1.1. Consider the fibre \( p_U^{-1}(z) \subset \tilde{U} \) for \( z \in U \). Using the isometric isomorphism between \( H^\infty(\tilde{U}) \) and the space \( H^\infty(U, E^U_\infty(G)) \) of bounded holomorphic sections of \( E^U_\infty(G) \) defined by taking the direct image of each function from \( H^\infty(\tilde{U}) \) with respect to \( p_U \) (see the construction of Proposition 2.4), we can reformulate the required interpolation problem as follows:

**Given** \( h \in l_\infty(G) \) **find** \( v \in H^\infty(U, E^U_\infty(G)) \) **of the least norm** \( ||v|| \) such that \( v(z) = h \).

Let us consider \( y = g_{NG}(z) \in N \) and its preimage \( g_{NG}^{-1}(y) \subset N_G \). Further, consider the bundle \( E^{M_G}_\infty(G) \longrightarrow M_G \). We define a new function \( \tilde{h} \in [l_\infty(G)]_\infty(X_G) \) by the formula
\[
  \tilde{h}(z) = h \quad \text{and} \quad \tilde{h}(x) = 0 \quad \text{for any} \quad x \in g_{NG}^{-1}(y), \ x \neq z.
\]

Then \( |\tilde{h}|_{[l_\infty(G)]_\infty(X_G)} = |h|_{l_\infty(G)} \). Let us consider now the bundle \( [E^{M_G}_\infty(G)]_M \) over \( M \). Taking the direct image with respect to \( g_{M_G} \), we can identify \( \tilde{h} \) with a section of \( [E^{M_G}_\infty(G)]_M \) over \( y \). Since \( [E^{M_G}_\infty(G)]_M \) is a component of the bundle \( B \), we can extend \( \tilde{h} \) by 0 to obtain a section \( h' \) of \( B \) over \( y \) whose norm equals \( |h|_{l_\infty(G)} \). Therefore according to Lemma 1.2, there is a holomorphic section \( v' \in H^\infty(O, B) \) such that \( \sup_{w \in N} |v'(w)|_B \leq C|h|_{l_\infty(G)} \) and \( v'(y) = h' \). Now consider the natural projection \( \pi \) of \( B \) onto the component \( [E^{M_G}_\infty(G)]_M \) in the direct decomposition of \( B \). Then \( \tilde{v} := \pi(v') \) satisfies\n\[
  \sup_{w \in N} |\tilde{v}(w)|_{[E^{M_G}_\infty(G)]_M} \leq C|h|_{l_\infty(G)} \quad \text{and} \quad \tilde{v}(y) = \tilde{h}.
\]

Using identification of \( \tilde{v}|_N \) with a bounded holomorphic section \( v \) of \( E^{NG}_\infty(G) \) (see the construction of Proposition 2.4), we obtain that \( v(z) = h \) and \( \sup_{w \in U} |v|_{E^{NG}_\infty(G)} \leq C|h|_{l_\infty(G)} \). It remains to note that \( E^{NG}_\infty(G)|_U = E^U_\infty(G) \) and so \( v|_U \in H^\infty(U, E^U_\infty(G)) \). In particular, \( \sup_{z \in U} M(z) \leq C \).
This completes the proof of the proposition. \qed

**Proof of Theorem 1.7.** Assume that \( \{z_j\} \subset U \) is an interpolating sequence with the constant of interpolation

\[
M = \sup_{||a||_{\infty} \leq 1} \inf \{||g|| : g \in H^\infty(U), g(z_j) = a_j, \ j = 1, 2, \ldots \} .
\]

We will prove that \( p_U^{-1}(\{z_j\}) \subset \bar{U} \) is also interpolating. According to [Ga, Ch.VII, Th.2.2], there are functions \( f_n \in H^\infty(\bar{U}) \) such that

\[
f_n(z_n) = 1, \ f_n(z_k) = 0, \ k \neq n,
\]

\[
\sum_n |f_n(z)| \leq M^2 .
\]

Further, according to Proposition 4.1, for any \( x \in U \), \( p_U^{-1}(x) \) is an interpolating sequence with the constant of interpolation \( \leq C \). Let \( p_U^{-1}(z_n) = \{z_ng\}_{g \in G} \). Then [Ga, Ch.VII, Th.2.2] implies that there are functions \( f_{ng} \in H^\infty(\bar{U}) \) such that for any \( n \)

\[
f_{ng}(z_{ng}) = 1, \ f_{ng}(z_{kg}) = 0, \ k \neq n, \ g \neq s,
\]

\[
\sum_n |f_{ng}(z)| \leq C^2 .
\]

Define now \( b_{ng} \in H^\infty(\bar{U}) \) by the formula

\[
b_{ng}(z) := f_{ng}(z) \cdot (p_U^*(f_n))(z).
\]

Then we have

\[
b_{ng}(z_{ng}) = 1, \ b_{ng}(z_{ks}) = 0, \ k \neq n \text{ or } g \neq s,
\]

\[
\sum_{n,g} |b_{ng}(z)| = \sum_n \left(|(p_U^*(f_n))(z)| \cdot \sum_g |f_{ng}(z)|\right) \leq (MC)^2 .
\]

Now we have the linear interpolation operator \( S : l^\infty \rightarrow H^\infty(\bar{U}) \) defined by

\[
S(\{a_{ng}\}) = \sum_{n,g} a_{ng}b_{ng}(z) \text{ for any } \{a_{ng}\} \in l^\infty .
\]

This shows that \( \{p_U^{-1}(z_n)\} \) is interpolating.

Conversely, assume that \( \{z_n\} \subset U \) is such that \( \{p_U^{-1}(z_n)\} \) is interpolating for \( H^\infty(\bar{U}) \). Let \( \{a_n\} \in l^\infty \). Consider the function \( t \in l^\infty(\{p_U^{-1}(z_n)\}) \) defined by

\[
t|_{p_U^{-1}(z_n)} = a_n \text{ for } n = 1, 2, \ldots .
\]

Then there is \( f \in H^\infty(\bar{U}) \) such that \( f|_{p_U^{-1}(z_n)} = t \). Applying to \( f \) the projector \( P \) constructed in Theorem 1.1, we obtain a function \( k \in H^\infty(U) \) with \( p_U^*(k) = P(f) \) which solves the required interpolation problem.

The proof of the theorem is complete. \qed

5. Properties of Interpolating Sequences Defined on Riemann Surfaces.

In this section we establish some results for interpolating sequences in \( U \) where \( U \) is a Riemann surface satisfying conditions of Corollary 1.5.
Let \( r : \mathbb{D} \to U \) be the universal covering map. From Theorem 1.7 we know that for any \( z \in U \) the sequence \( r^{-1}(z) \subset \mathbb{D} \) is interpolating for \( H^\infty(\mathbb{D}) \). Then for any \( z \in U \), we can define a Blaschke product \( B_z \in H^\infty(\mathbb{D}) \) with simple zeros at all points of \( r^{-1}(z) \). If \( B'_z \) is another Blaschke product with the same property then we have \( B'_z = \alpha \cdot B_z \) for some \( \alpha \in \mathbb{C}, |\alpha| = 1 \). In particular, the subharmonic function \( |B_z| \) is invariant with respect to the action on \( \mathbb{D} \) of the deck transformation group \( \pi_1(U) \).

Thus there is a nonnegative subharmonic function \( P_z \) on \( U \) with the only zero at \( z \), such that \( r^*(P_z) = |B_z| \). It is also clear that \( P_z(y) = P_y(z) \) for any \( y, z \in U \), and \( \sup_U P_z = 1 \).

**Proposition 5.1** A sequence \( \{z_i\} \subset U \) is interpolating for \( H^\infty(U) \) if and only if
\[
\inf_j \left\{ \prod_{k: k \neq j} P_{z_k}(z_j) \right\} := \delta > 0 .
\] (5.1)

The number \( \delta \) will be called the characteristic of the interpolating sequence \( \{z_j\} \).

**Proof.** Assume that \( \{z_j\} \) is an interpolating sequence. Then by Theorem 1.7, \( r^{-1}(\{z_j\}) \) is interpolating for \( H^\infty(\mathbb{D}) \). Let \( r^{-1}(z_j) = \{z_jg \in \pi_1(U)\} \). Then by the Carleson theorem [Ca1] on the characterization of interpolating sequences we have (for any \( j, g \))
\[
\left( \prod_{k: k \neq j} |B_{z_k}(z_{jg})| \right) \cdot \left( \prod_{h: h \neq g} \frac{|z_{jh} - z_{jg}|}{1 - z_{jh}\bar{z}_{jg}} \right) \geq c > 0 .
\]

Further, since
\[
\prod_{h: h \neq g} \left| \frac{z_{jh} - z_{jg}}{1 - z_{jh}\bar{z}_{jg}} \right| \leq 1,
\]
from the above inequality it follows that for any \( j \)
\[
\prod_{k: k \neq j} P_{z_k}(z_j) := \prod_{k: k \neq j} |B_{z_k}(z_{jg})| \geq c > 0 .
\]

Conversely, assume that for any \( j \) we have
\[
\prod_{k: k \neq j} P_{z_k}(z_j) \geq c > 0 .
\]

From the proof of Theorem 1.7 we know that the constant of interpolation for \( r^{-1}(z) \) with an arbitrary \( z \in U \) is bounded from above by some \( C = C(N) < \infty \). Thus according to the inequality which connects the constant of interpolation with the characteristic of an interpolating sequence (see [Ca1]) we obtain for any \( j \) and any \( g \in \pi_1(U) \) : 
\[
\prod_{h: h \neq g} \left| \frac{z_{jh} - z_{jg}}{1 - z_{jh}\bar{z}_{jg}} \right| \geq 1/C > 0 .
\]
Combining these two inequalities we have (for any $j, g$)

$$
\left( \prod_{k: k \neq j} P_{zk}(z_j) \right) \cdot \left( \prod_{h: h \neq g} \left| \frac{z_{jh} - z_{jg}}{1 - z_{jh}z_{jg}} \right| \right) = \\
\left( \prod_{k: k \neq j} |B_{zk}(z_{jg})| \right) \cdot \left( \prod_{h: h \neq g} \left| \frac{z_{jh} - z_{jg}}{1 - z_{jh}z_{jg}} \right| \right) \geq \frac{c}{C} > 0 .
$$

This inequality implies that the sequence $r^{-1}(\{z_j\})$ is interpolating (see [Ca1]). Hence by Theorem 1.7, $\{z_j\}$ is interpolating for $H^\infty(U)$.

The proof of the proposition is complete. $\square$

**Corollary 5.2** Let $\{z_j\} \subset U$ be an interpolating sequence with characteristic $\delta$. Let $K$ be the constant of interpolation for $\{z_j\}$. Then there is a constant $A$ depending only on the original Riemann surface $N$ (and not of the choice of $U$) such that

$$K \leq \frac{A}{\delta} \left( 1 + \log \frac{1}{\delta} \right) .$$

**Proof.** From the proof of Proposition 5.1 and Theorem 1.7 it follows that the characteristic $\delta'$ of the interpolating sequence $r^{-1}(\{z_j\})$ is $\geq \delta/C$, where $C \geq 1$ depends on $N$ only. Then according to the Carleson theorem [Ca1], the constant of interpolation $K'$ of $r^{-1}(\{z_j\})$ is $\leq \frac{C_2}{\delta}(1 + \log \frac{\delta}{2}) < \frac{C_1}{\delta}(1 + \log \frac{1}{\delta})$. Here $c$ is an absolute constant and $C_1 = C_1(N)$. Thus applying the projector $P$ of Theorem 1.7 to functions $f \in H^\infty(D)$ which are constant on each fibre $r^{-1}(z_j)$, $j = 1, 2, \ldots$, and using that $||P|| \leq C_2 = C_2(N) < \infty$ we obtain that

$$K \leq C_2K' \leq \frac{C_1C_2}{\delta} \left( 1 + \log \frac{1}{\delta} \right) .$$

$\square$

The next result states that a small perturbation of an interpolating sequence in $U$ is also an interpolating sequence. Let $\rho$ be the pseudometric on $D$ (see definition (1.3)). Let $x, y \in U$ and $x_0 \in D$ be such that $r(x_0) = x$. We define the distance $\rho^*(x, y)$ by the formula:

$$\rho^*(x, y) := \inf_{w \in r^{-1}(y)} \rho(x_0, w) .$$

It is easy to see that this definition does not depend on the choice of $x_0$ and determines a metric on $U$ compatible with its topology.

**Proposition 5.3** Let $\{z_j\} \subset U$ be an interpolating sequence with characteristic $\delta$. Assume that $0 < \lambda < 2\lambda/(1 + \lambda^2) < \delta < 1$. If $\{\xi_j\} \subset U$ satisfies $\rho^*(\xi_j, z_j) \leq \lambda$, $j = 1, 2, \ldots$, then for any $k$

$$\prod_{j: j \neq k} P_{\xi_j}(\xi_k) \geq \frac{\delta - 2\lambda/(1 + \lambda^2)}{1 - 2\lambda\delta/(1 + \lambda^2)} .$$

In fact, this proposition is similar to [Ga, Ch.VII, Lemma 5.3] which is used in the proof of Earl’s theorem on interpolation. We will show how to modify the proof of this lemma to obtain our result.

**Proof.** Let \( r^{-1}(z_j) = \{z_{jg}\}_{g \in \pi_1(U)} \) and \( r^{-1}(\xi_j) = \{\xi_{jg}\}_{g \in \pi_1(U)} \). According to the definition of \( \rho^* \) and because \( \pi_1(U) \) acts discretely on \( \mathbb{D} \), we can choose the above indices such that \( \rho(\xi_{jg}, z_{jg}) \leq \lambda \) for any \( g \). Let us fix some \( h \in \pi_1(U) \). Then by definition for \( j \neq k \) we have

\[
P_{\xi_j}(\xi_k) = \prod_{g \in \pi_1(U)} \rho(\xi_k, \xi_{jg}) .
\]

Using an inequality from the proof of Lemma 5.3 in [Ga, Ch.VII] gives

\[
\rho(\xi_{jg}, \xi_{kh}) \geq \frac{\rho(z_{jg}, z_{kh}) - \alpha}{1 - \alpha \rho(z_{jg}, z_{kh})}
\]

for \( \alpha := 2\lambda/(1 + \lambda^2) \). According to our assumption we have

\[
\prod_{j: j \neq k} P_{z_j}(z_k) := \prod_{j: j \neq k} \prod_{g \in \pi_1(U)} \rho(z_{kh}, z_{jg}) \geq \delta .
\]

Therefore \( \rho(z_{kh}, z_{jg}) \geq \delta \) for any \( j \neq k \) and any \( g \in \pi_1(U) \). Hence we can apply the inequality of [Ga, Ch.VII, Lemma 5.2] to obtain

\[
\prod_{j: j \neq k} P_{\xi_j}(\xi_k) := \prod_{j: j \neq k} \prod_{g \in \pi_1(U)} \rho(\xi_{jg}, \xi_{kh}) \geq \prod_{j: j \neq k} \prod_{g \in \pi_1(U)} \frac{\rho(z_{jg}, z_{kh}) - \alpha}{1 - \alpha \rho(z_{jg}, z_{kh})} \geq \frac{(\prod_{j: j \neq k} \prod_{g \in \pi_1(U)} \rho(z_{jg}, z_{kh})) - \alpha}{1 - \alpha \left( \prod_{j: j \neq k} \prod_{g \in \pi_1(U)} \rho(z_{jg}, z_{kh}) \right)} \geq \frac{\delta - \alpha}{1 - \alpha \delta} .
\]

This gives the required inequality. \( \square \)

**Proposition 5.4** Let \( \{z_i\} \) and \( \{y_i\} \) be interpolating sequences in \( U \). Assume that there is a constant \( c > 0 \) such that for any \( i, j \)

\[
\rho^*(z_j, y_i) \geq c .
\]

Then the sequence \( \{z_i\} \cup \{y_i\} \subset U \) is interpolating.

**Proof.** From the condition of the proposition it follows that the distance in the pseudo-hyperbolic metric on \( \mathbb{D} \) between interpolating sequences \( r^{-1}(\{z_i\}) \) and \( r^{-1}(\{y_i\}) \) is \( \geq c \). This implies that \( r^{-1}(\{z_i\}) \cup r^{-1}(\{y_i\}) \) is interpolating for \( H^\infty(\mathbb{D}) \) (see e.g. [Ga, Ch.VII, Problem 2]). Therefore by Theorem 1.7 \( \{z_i\} \cup \{y_i\} \subset U \) is interpolating for \( H^\infty(U) \). \( \square \)

Finally we formulate an analog of Corollary 1.6 from [Ga, Ch.X].

**Proposition 5.5** Let \( \{z_j\} \subset U \) be an interpolating sequence with characteristic \( \delta \). Then \( \{z_j\} \) can be represented as a disjoint union \( \{z_{1j}\} \cup \{z_{2j}\} \) of two subsequences such that the characteristic of \( \{z_{sj}\} \), is \( \geq \sqrt{\delta} \), \( s = 1, 2 \).
Proof. Consider the function \( F(z) := \prod_j P_{z_j}(z) \). Then we have a decomposition \( F(z) = F_1(z) \cdot F_2(z) \) with \( F_s(z) := \prod_j P_{z_{s,j}}(z), \ s = 1, 2 \). It suffices to choose the required decomposition such that

\[
\text{if } F_1(z_n) = 0, \text{ then } \prod_{j: j \neq n} P_{z_{1,j}}(z_n) \geq F_2(z_n)
\]

\[
\text{if } F_2(z_n) = 0, \text{ then } \prod_{j: j \neq n} P_{z_{2,j}}(z_n) \geq F_1(z_n).
\]

The proof of the above inequalities repeats word-for-word the combinatorial proof of Lemma 1.5 in [Ga,Ch.X] given by Mills, where we must define the matrix \( [a_{kn}] \) by the formula

\[
a_{kn} = \log P_{z_k}(z_n), \quad k \neq n; \quad a_{nn} = 0.
\]

We leave the details to the reader. Now from the above inequalities for \( F_1(z_n) = 0 \) we have

\[
\delta \leq \prod_{j: j \neq n} P_{z_{j}}(z_n) = \left( \prod_{j: j \neq n} P_{z_{1,j}}(z_n) \right) F_2(z_n) \leq \left( \prod_{j: j \neq n} P_{z_{2,j}}(z_n) \right)^2
\]

which gives the required estimate of the characteristic for \( \{z_{1,j}\} \). The same is valid for \( \{z_{2,j}\} \). \( \square \)

Remark 5.6 Using the above properties of interpolating sequences in \( U \) it is possible to define non-trivial analytic maps of \( \mathbb{D} \) to the maximal ideal space of \( H^\infty(U) \) related to limit points of an interpolating sequence. The construction is similar to the construction given in the case of \( H^\infty(\mathbb{D}) \) (see [Br]).

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