TWO-TERM TILTING COMPLEXES OVER
BRAUER TREE ALGEBRAS

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Abstract
In this paper all two-term tilting complexes over a Brauer tree algebra
with multiplicity one are described using a classification of indecomposable
two-term partial tilting complexes obtained earlier in a joint paper with
M. Antipov. The endomorphism rings of such complexes are computed.

1 Introduction
Let \( A \) be a Brauer tree algebra corresponding to a Brauer tree \( \Gamma \) with mul-
tiplicity one. \( \text{TrPic}(A) \) is the derived Picard group of \( A \), that is the group of
standard autoequivalences of the derived category of \( A \) modulo the natural iso-
morphism. Let us consider the derived Picard groupoid, whose objects are the
Brauer tree algebras corresponding to the Brauer trees with \( n \) edges, and the
morphisms are the standard equivalences between them. \( \text{TrPic}(A) \) is the group
of endomorphisms of the object \( A \) in this category. The computation of the de-
rived Picard groupoid seems to be an easier problem than the computation of
\( \text{TrPic}(A) \). The derived Picard group is completely computed only in the case of
algebra with two simple modules [1]. In other cases only the action of different
braid groups on \( \text{TrPic}(A) \) is known [1], [2], [3]. On the other hand by the result
of Abe and Hoshino [4] the derived Picard groupoid corresponding to the class
of Brauer tree algebras with multiplicity of the exceptional vertex \( k \) and a fixed
number of simple modules is generated by one-term and two-term tilting com-
plexes. Thus if we describe all two-term tilting complexes over \( A \), we will obtain
the generating set of the derived Picard groupoid.

The computation of the derived Picard groupoid was the main motivation
while writing this paper. However, the two-term tilting complexes are also re-
lated to \( \tau \)-tilting theory [6] and to simple-minded systems [5].

This paper is a continuation of the joint paper with M. Antipov [7], in
which we classified all the indecomposable two-term partial tilting complexes
over Brauer tree algebras with multiplicity one. In section 3 all two-term tilting
complexes are described in combinatorial terms (theorem 1), in section 4 their
endomorphism rings are computed.

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2 Preliminaries

Let $K$ be an algebraically closed field, $A$ be a finite dimensional algebra over $K$. We will denote by $A$-mod the category of finitely generated left $A$-modules, by $K^b(A)$ – the bounded homotopy category and by $D^b(A)$ the bounded derived category of $A$-mod. The shift functor on the derived category will be denoted by $[1]$. Let us denote by $A$-perf the full subcategory of $D^b(A)$ consisting of perfect complexes, i.e. of bounded complexes of finitely generated projective $A$-modules.

In the path algebra of a quiver the product of arrows $a \rightarrow b \rightarrow$ will be denoted by $ab$.

**Definition 1.** A complex $T \in A$-perf is called tilting if

1. $\text{Hom}_{D^b(A)}(T, T[i]) = 0$, for $i \neq 0$;
2. $T$ generates $A$-perf as a triangulated category.

**Definition 2.** A complex $T \in A$-perf is called partial tilting if the condition 1 from definition 1 is satisfied.

**Definition 3.** A tilting complex $T \in A$-perf is called basic if it does not contain isomorphic direct summands or equally if $\text{End}_{D^b(A)}(T)$ is a basic algebra.

We will call a (partial) tilting complex a two-term (partial) tilting complex if it is concentrated in two neighboring degrees.

**Definition 4.** Let $\Gamma$ be a tree with $n$ edges and a distinguished vertex, which has an assigned multiplicity $k \in \mathbb{N}$ (this vertex is called exceptional, $k$ is called the multiplicity of the exceptional vertex). Let us fix a cyclic ordering of the edges adjacent to each vertex in $\Gamma$ (if $\Gamma$ is embedded into plane we will assume that the cyclic ordering is clockwise). In this case $\Gamma$ is called a Brauer tree of type $(n, k)$.

To a Brauer tree of type $(n, k)$ one can associate an algebra $A(n, k)$. The algebra $A(n, k)$ is a path algebra of a quiver with relations. Let us construct a Brauer quiver $Q_{\Gamma}$ using the Brauer tree $\Gamma$. The vertices of $Q_{\Gamma}$ are the edges of $\Gamma$, if two edges $i$ and $j$ are incident to the same vertex in $\Gamma$ and $j$ follows $i$ in the cyclic order of the edges incident to their common vertex, then there is an arrow from the vertex $i$ to the vertex $j$ in $Q_{\Gamma}$. $Q_{\Gamma}$ has the following property: $Q_{\Gamma}$ is the union of oriented cycles corresponding to the vertices of $\Gamma$, each vertex of $Q_{\Gamma}$ belongs to exactly two cycles. The cycle corresponding to the exceptional vertex is called exceptional. The arrows of $Q_{\Gamma}$ can be divided into two families $\alpha$ and $\beta$ in such a manner that the arrows belonging to intersecting cycles are in different families.

**Definition 5.** The basic Brauer tree algebra $A(n, k)$, corresponding to a tree $\Gamma$ of type $(n, k)$ is isomorphic to $Q_{\Gamma}/I$, where the ideal $I$ is generated by the relations:

1. $\alpha \beta = 0 = \beta \alpha$;
2. for any vertex $x$, not belonging to the exceptional cycle, $\alpha^x = \beta^x$, where $x_\alpha$, resp. $x_\beta$ is the length of the $\alpha$, resp. $\beta$-cycle, containing $x$;

3. for any vertex $x$, belonging to the exceptional $\alpha$-cycle (resp. $\beta$-cycle), $(\alpha^x)^k = \beta^x$ (resp. $\alpha^x = (\beta^x)^k$).

An algebra is called a Brauer tree algebra of type $(n,k)$, if it is Morita equivalent to the algebra $A(n,k)$.

Note that the ideal $I$ is not admissible. From now on for convenience all algebras are supposed to be basic.

Rickard showed that two Brauer tree algebras corresponding to the trees $\Gamma$ and $\Gamma'$ are derived equivalent if and only if their types $(n,k)$ and $(n',k')$ coincide [8] and it follows from the results of Gabriel and Riedtmann that this class is closed under derived equivalence [9].

**Definition 6.** Let $B$ be a Brauer tree algebra. $A$-cycle is a maximal ordered set of nonrepeating arrows of $Q$ such that the product of any two neighboring arrows is not equal to zero.

Note that the fact that the algebra is symmetric means that $A$-cycles are actually cycles.

In [7] we classified all indecomposable two-term partial tilting complexes over a Brauer tree algebra with multiplicity one. Note that any two-term indecomposable complex is either isomorphic to a stalk complex of a projective module concentrated in some degree (such complexes are obviously partial tilting), the minimal projective presentation of some indecomposable $A$-module.

**Theorem** Let $A$ be a Brauer tree algebra with multiplicity one. The minimal projective presentation of an indecomposable non-projective $A$-module $M$ is a partial tilting complex if and only if $M$ is not isomorphic to $P/soc(P)$ for any indecomposable projective module $P$.

### 3 Two-term tilting complexes over Brauer tree algebras with multiplicity one

To check the conditions from the definitions 2 it will be convenient to work with partial tilting complexes and not with the corresponding modules. Let us associate the following diagram on the quiver of $A$ to a two term complex consisting of projective modules.

**Definition 7.** Let $T = P_0 \xrightarrow{f} P_1 \in A – \text{perf}$ be the projective presentation of an indecomposable non-projective module $M$, $P_0 = \bigoplus_{i \in I} Ae_i$, $P_1 = \bigoplus_{i \in J} Ae_i$. Mark the vertices corresponding to the set $I \cup J$ on the quiver of $A$. Note that since $M$ does not have repeating composition factors and since $M$ is not isomorphic to $P/soc(P)$ for any indecomposable projective module $P$, each index can occur only once and only in one of the sets. Let us also mark the path from $i$ to
If \( f \) has a nonzero component between the corresponding summands of \( P_0 \) and \( P_1 \) on the quiver of algebra \( A \). The diagram obtained in such a manner will be called a diagram of a projective presentation \( T = P_0 \xrightarrow{f} P_1 \). We will call the vertices corresponding to the set \( I \cup J \) marked vertices of the diagram.

The obtained diagram is a connected path without self-intersections, it changes its orientation and \( A \)-cycle in every vertex from the set \( I \cup J \). This is satisfied since for any index \( j \in J \) there exists at most two indices from \( I \) such that the corresponding components of \( f \) are nonzero and visa versa: for any index \( i \in I \) there exists at most two indices from \( J \) such that the corresponding components of \( f \) are nonzero. It is clear that to any connected path \( \Theta \) without self-intersections, which changes its orientation every time it changes \( A \)-cycle and which consists of more than one vertex, one can associate a two-term partial tilting complex as follows: let \( I \) be the set of indices corresponding to the sources of \( \Theta \), let \( J \) be the set of indices corresponding to the sinks of \( \Theta \). Then 
\[
P_0 = \bigoplus_{i \in I} Ae_i, \quad P_1 = \bigoplus_{i \in J} Ae_i,
\]
\( f \) is induced by the morphisms corresponding to the directed subpaths from the sources to the sinks. (The projective modules corresponding to the neighboring sink and source belong to the same \( A \)-cycle and up to an invertible constant there is a unique morphism between them, the choice of the coefficient does not play any role, so we will assume that we always choose the multiplication by the corresponding path as a morphism.) So there is a one to one correspondence between minimal projective presentations of indecomposable non-projective modules non isomorphic to \( P / \text{soc}(P) \) for any indecomposable projective module \( P \) and connected paths \( \Theta \) on the quiver of \( A \) without self-intersections, which change their orientation every time they change \( A \)-cycle and which consist of more than one vertex.

For a diagram of a projective presentation and the projective presentation itself we will often use the same notation.

**Definition 8.** Let \( T_i \) and \( T_j \) be two diagrams of projective presentations which meet at more than one vertex so that the intersection has only one connected component. The restriction of \( T_i \) with respect to \( T_j \) is the intersection of the diagram \( T_i \) and the union of that \( A \)-cycles of the algebra which contain at least one marked vertex of \( T_j \).

A diagram which consists of more than one vertex and which is contained in one \( A \)-cycle will be called a string. As in [7] such a diagram will be denoted by \((k, \ldots, l)\), where \( k \) is a sink, and \( l \) is a source of the string.

**Remark 1.** Thus the restriction of \( T_i \) with respect to \( T_j \) is some subdiagram of \( T_i \) which contains the intersection of \( T_i \) and \( T_j \) and the completion of the intersection to the restriction can be defined independently at the ends of the intersection. If the intersection of \( T_i \) and \( T_j \) does not contain a marked vertex of \( T_i \) the restriction is a substring of \( T_i \) which contains the intersection; if the intersection ends at an unmarked vertex of \( T_i \) let us complete it to the smallest subdiagram (to the nearest marked vertex); if the intersection ends at a marked
vertex of $T_i$ which is not a marked vertex of $T_j$ we will not complete it, if the intersection ends at a marked vertex of $T_i$ which is a marked vertex of $T_j$, let us complete it to the next marked vertex of $T_i$, if it exists.

**Remark 2.** The restriction is defined in such a manner that there are no nonzero morphisms between the projective summands of the components of $T_i$ and $T_j$ which correspond to the vertices not contained in the restrictions.

To classify all basic two-term tilting complexes it is necessary and sufficient to classify $n$-tuples of pairwise orthogonal nonisomorphic indecomposable two-term partial tilting complexes [4].

**Theorem 1.** Let $T_i$ and $T_j$ be indecomposable partial tilting complexes. The complex $T_i \oplus T_j$ is partial tilting iff one of the following conditions holds:

1) In the case when $T_i$ and $T_j$ are projective presentations of some modules, such that $T_i$ and $T_j$ are indecomposable two-term partial tilting complexes.
   a) The diagrams of $T_i$ and $T_j$ do not have vertices belonging to the same $A$-cycle.
   b) The diagrams of $T_i$ and $T_j$ have vertices belonging to the same $A$-cycle $\Upsilon$, but they do not intersect, and $\Upsilon$ does not contain a source of degree one of $T_j$ (resp. $T_i$) and a sink of degree one of $T_i$ (resp. $T_j$) such that these are the only vertices of $T_i$ and $T_j$ belonging to $\Upsilon$.
   c) The diagrams of $T_i$ and $T_j$ meet at one vertex $k$, and $k$ is neither a marked vertex of $T_i$ nor a marked vertex of $T_j$, or $k$ is a degree one sink of both $T_i$ and $T_j$, or $k$ is a degree one source of both $T_i$ and $T_j$.
   d) The intersection of $T_i$ and $T_j$ consists of one connected component, which contains more than one vertex. The diagrams of $T_i$ and $T_j$ intersect in such a way that one of the end points of the intersection is a sink and another is a source, the restriction of $T_i$ with respect to $T_j$ belongs to the restriction of $T_j$ with respect to $T_i$ or visa versa.
   e) The intersection of $T_i$ and $T_j$ consists of one connected component, which contains more than one vertex. The diagrams of $T_i$ and $T_j$ intersect in such a way that both of the end vertices of the intersection are sinks and neither the restriction of $T_i$ with respect to $T_j$ belongs to the restriction of $T_j$ with respect to $T_i$ nor the restriction of $T_j$ with respect to $T_i$ belongs to the restriction of $T_i$ with respect to $T_j$; or the diagrams of $T_i$ and $T_j$ have a coinciding vertex of degree one.
   f) The intersection of $T_i$ and $T_j$ consists of one connected component, which contains more than one vertex. The diagrams of $T_i$ and $T_j$ intersect in such a way that both of the end vertices of the intersection are sources and neither the restriction of $T_i$ with respect to $T_j$ belongs to the restriction of $T_j$ with respect to $T_i$ nor the restriction of $T_j$ with respect to $T_i$ belongs to the restriction of $T_i$ with respect to $T_j$; or the diagrams of $T_i$ and $T_j$ have a coinciding vertex of degree one.

2) In the case when $T_i$ is the projective presentation of some module, such that $T_i$ is an indecomposable two-term partial tilting complex and $T_j$ is an indecomposable stalk complex of a projective module $P$. 

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a) $P$ is concentrated in 0, the vertex corresponding to $P$ coincides with a source of degree one of $T_i$, or the vertex corresponding to $P$ does not belong to $T_i$ and there is no $A$-cycle which contains the vertex corresponding to $P$ and a sink of degree one of $T_i$ such that this is the only vertex of $T_i$ belonging to this $A$-cycle.

b) $P$ is concentrated in 1, the vertex corresponding to $P$ coincides with a sink of degree one of $T_i$, or the vertex corresponding to $P$ does not belong to $T_i$ and there is no $A$-cycle which contains the vertex corresponding to $P$ and a source of degree one of $T_i$ such that this is the only vertex of $T_i$ belonging to this $A$-cycle.

3) In the case when $T_i$ and $T_j$ are two indecomposable stalk complexes of projective modules and the vertices corresponding to $T_i$ and $T_j$ do not belong to the same $A$-cycle, or the vertices corresponding to $T_i$ and $T_j$ belong to the same $A$-cycle, $T_i$ and $T_j$ are concentrated in the same degree.

The rest of section 3 is devoted to the proof of this theorem.

As mentioned before, to classify all basic two-term tilting complexes it is necessary and sufficient to classify $n$-tuples of nonisomorphic indecomposable two-term partial tilting complexes \(\{T_1, ..., T_n\}\) such that $\text{Hom}_{D^b(A)}(T_i, T_j[1]) = 0 = \text{Hom}_{D^b(A)}(T_i, T_j[-1])$ [4]. By the following remark by Happel [10] it is sufficient to check only one of these conditions, for example, $\text{Hom}_{D^b(A)}(T_i, T_j[-1])$.

**Remark 3.** Let $A$ be a finite dimensional algebra over a field $K$, let $\text{proj} - A$, $\text{inj} - A$ be the categories of finitely generated projective and injective modules, $K^b(\text{proj} - A)$, $K^b(\text{inj} - A)$ the corresponding bounded homotopy categories, $D$ the duality with respect to $K$. Then the Nakayama functor $\nu$ induces an equivalence of triangulated categories $K^b(\text{proj} - A) \rightarrow K^b(\text{inj} - A)$ and there is a natural isomorphism $D\text{Hom}(P, -) \rightarrow \text{Hom}(-, \nu P)$, $P \in K^b(\text{proj} - A)$.

First, let us consider the case of two projective presentations of some modules.

**Lemma 1.** Let $T_i$ and $T_j$ not have vertices belonging to the same $A$-cycle then $\text{Hom}_{D^b(A)}(T_i, T_j[-1]) = 0$.

**Proof.** This is obvious since there are no nonzero morphisms between the summand of the component of $T_i$ and $T_j$. 

**Lemma 2.** Let the diagrams of $T_i$ and $T_j$ have vertices belonging to the same $A$-cycle $\Upsilon$, but not intersect, then the condition $\text{Hom}_{D^b(A)}(T_i, T_j[-1]) = 0 = \text{Hom}_{D^b(A)}(T_j, T_i[-1])$ holds in all the cases but the case when $\Upsilon$ contains a source of degree one of $T_j$ (resp. $T_i$) and a sink of degree one of $T_i$ (resp. $T_j$) such that these are the only vertices of $T_i$ and $T_j$ belonging to $\Upsilon$.

**Proof.** Note that if the diagrams of $T_i$ and $T_j$ have vertices belonging to the same $A$-cycle $\Upsilon$, but do not intersect, then since $\Gamma$ is a tree there is no other $A$-cycle they both meet. This situation can occur in the following cases: $\Upsilon$ contains a substring of $T_i$ and a substring of $T_j$, a substring of $T_i$ and a vertex of $T_j$, a vertex of $T_i$ and a vertex of $T_j$, namely:
1) \(\mathcal{Y}\) contains a substring \((i_1, ..., i_2)\) of \(T_i\) and a substring \((j_1, ..., j_2)\) of \(T_j\) but they do not intersect. Let us show that there is no nonzero morphism between \(T_i\) and \(T_j[-1]\). It is clear that there is a nonzero morphism only between the projective summands of the components of \(T_i\) and \(T_j\) with corresponding vertices belonging to \(\mathcal{Y}\) (i.e. between \( Ae_{i_1}, Ae_{i_2}, Ae_{j_1}, Ae_{j_2}\)), since all other projective summands of the components of \(T_i\) and \(T_j\) belong to different A-cycles. It is clear that \(\text{Hom}_{D^b(A)}(T_i, T_j[-1]) = 0\) iff \(\text{Hom}_{D^b(A)}(Ae_{i_2} \to Ae_{i_1}, Ae_{j_2} \to Ae_{j_1}, [-1]) = 0\), which is true since the strings \((i_1, ..., i_2)\) and \((j_1, ..., j_2)\) do not intersect.

2) \(\mathcal{Y}\) contains a substring \((i_1, ..., i_2)\) of \(T_i\) and a vertex of \(T_j\) which is not marked. \(\text{Hom}_{D^b(A)}(T_i, T_j[-1]) = 0\), since the projective summands of the components of \(T_i\) and \(T_j\) belong to different A-cycles. \(\text{Hom}_{D^b(A)}(T_j, T_i[-1]) = 0\) for the same reasons.

3) \(\mathcal{Y}\) contains a substring \((i_1, ..., i_2)\) of \(T_i\) and a vertex \((j_1)\) of \(T_j\) which is a sink of degree one. Then there is no nonzero morphism from \(Ae_{i_2}\) to the component of \(T_j[-1]\), concentrated in 1, i.e. \(\text{Hom}_{D^b(A)}(T_i, T_j[-1]) = 0\). It is clear that \(\text{Hom}_{D^b(A)}(T_j, T_i[-1]) = 0\) iff \(\text{Hom}_{D^b(A)}(0 \to Ae_{i_1}, Ae_{i_2} \to Ae_{i_1}, [-1]) = 0\), which is true since the string \((i_1, ..., i_2)\) does not contain \((j_1)\).

4) \(\mathcal{Y}\) contains a substring \((i_1, ..., i_2)\) of \(T_i\) and a vertex \((j_1)\) of \(T_j\) which is a source of degree one. This case is similar to the previous one.

5) \(\mathcal{Y}\) contains a vertex \((i_1)\) of \(T_i\) and a vertex \((j_1)\) of \(T_j\), both vertices are sources of degree one and \(\mathcal{Y}\) does not contain any other vertices of \(T_i\) and \(T_j\). The modules \(Ae_{i_1}\) and \(Ae_{j_1}\) are the only projective summands of the components of \(T_i\) and \(T_j\) with a nonzero hom-space, so after we apply a shift functor there will be no nonzero morphisms.

6) \(\mathcal{Y}\) contains a vertex \((i_1)\) of \(T_i\) and a vertex \((j_1)\) of \(T_j\), both vertices are sinks of degree one and \(\mathcal{Y}\) does not contain any other vertices of \(T_i\) and \(T_j\). This case is similar to the previous one.

7) \(\mathcal{Y}\) contains a degree one sink of \(T_i\) and an unmarked vertex of \(T_j\) and \(\mathcal{Y}\) does not contain any other vertices of \(T_i\) and \(T_j\). \(\text{Hom}_{D^b(A)}(T_i, T_j[-1]) = 0\), since the projective summands of the components of \(T_i\) and \(T_j\) belong to different A-cycles. \(\text{Hom}_{D^b(A)}(T_j, T_i[-1]) = 0\) for the same reasons.

8) \(\mathcal{Y}\) contains a degree one source of \(T_i\) and an unmarked vertex of \(T_j\) and \(\mathcal{Y}\) does not contain any other vertices of \(T_i\) and \(T_j\). This case is similar to the previous one.

9) \(\mathcal{Y}\) contains an unmarked vertex of \(T_i\) and an unmarked vertex of \(T_j\) and \(\mathcal{Y}\) does not contain any other vertices of \(T_i\) and \(T_j\). This case is similar to the previous one.

10) \(\mathcal{Y}\) contains a degree one sink \((i_1)\) of \(T_i\) and a degree one source \((j_1)\) of \(T_j\). In this case \(Ae_{i_1}\) and \(Ae_{j_1}\) are projective summands of components of \(T_i\) and \(T_j\), which are concentrated in different degrees, after we apply the shift functor they will be concentrated in the same degree and so there will be a nonzero morphism between them, which induces a chain map. Hence \(\text{Hom}_{D^b(A)}(T_i, T_j[-1]) \neq 0\).

**Lemma 3.** Let diagrams of \(T_i\) and \(T_j\) have an intersection containing more than one connected component, then at least one of the spaces \(\text{Hom}_{D^b(A)}(T_i, T_j[-1])\), \(\text{Hom}_{D^b(A)}(T_j, T_i[-1])\) is nonzero.
Proof. Let diagrams of $T_i$ and $T_j$ have an intersection containing at least 3 connected components, then one of these components is an isolated vertex $k$ (to get from one $A$-cycle to another one should pass a unique vertex and since the diagrams have an intersection containing at least 3 connected components near this vertex they have an opposite orientation). Hence for one of the diagrams this vertex is a sink (say for $T_i$) and for the other this vertex is a source (for $T_j$). There is a nonzero morphism from $Ae_k$ to itself whose image is the socle of $Ae_k$. It induces a nonzero morphism from $\text{Hom}_{D^b(A)}(T_i, T_j[-1])$.

Let diagrams of $T_i$ and $T_j$ have an intersection containing two connected components, it is clear that both components belong to the same $A$-cycle. Let $(i_1, ..., i_2), (j_1, ..., j_2)$ be substrings of $T_i$ and $T_j$ corresponding to the restrictions of $T_i$ and $T_j$ to this $A$-cycle. The vertices have the following order on the $A$-cycle: $i_1, i_2, j_1, j_2$, where $i_1$ can coincide with $j_2$ and $j_1$ can coincide with $i_2$. There is a nonzero morphism from $Ae_{i_1}$ to $Ae_{j_2}$ (in the case when $i_1$ coincides with $j_2$ the morphism which has the socle as its image), it induces a chain map. □

Lemma 4. The diagrams of $T_i$ and $T_j$ have an intersection containing one vertex $k$, then $\text{Hom}_{D^b(A)}(T_i, T_j[-1]) = 0 = \text{Hom}_{D^b(A)}(T_j, T_i[-1])$ holds in the following cases: $k$ is neither a marked vertex of $T_i$ nor a marked vertex of $T_j$, or $k$ is a sink of degree one of both $T_i$ and $T_j$, or $k$ is a source of degree one of both $T_i$ and $T_j$.

Proof. Note that there is no $A$-cycle which intersects with $T_i$ and $T_j$ but does not contain $k$ (since $\Gamma$ is a tree). The vertex $k$ can be a degree one sink, a degree one source or an unmarked vertex of both $T_i$ and $T_j$, so we have to consider 6 cases.

1) Let $k$ be an unmarked vertex of both $T_i$ and $T_j$ then marked vertices of $T_i$ and marked vertices of $T_j$ belong to different $A$-cycles there are no nonzero morphisms between the corresponding projective modules.

2) Let $k$ be a degree one sink of both $T_i$ and $T_j$ then $\text{Hom}_{D^b(A)}(T_i, T_j[-1]) = 0$ iff $\text{Hom}_{D^b(A)}(Ae_{i_2} \to Ae_k, (Ae_{j_2} \to Ae_k)[-1]) = 0$ (where $i_2$, $j_2$ are the marked vertices of $T_i$ and $T_j$ which are next to $k$), which is true since the only nonzero morphism from $\text{Hom}_{A}(Ae_k, Ae_{j_2})$ does not induce a chain map. The space $\text{Hom}_{D^b(A)}(T_j, T_i[-1])$ is zero for the same reasons.

3) The case when $k$ is a degree one source of both $T_i$ and $T_j$ is similar to the previous one.

Let us check that in all other cases at least one of the spaces $\text{Hom}_{D^b(A)}(T_i, T_j[-1]), \text{Hom}_{D^b(A)}(T_j, T_i[-1])$ is not zero.

4) The case when $k$ is a source of $T_i$ and a sink of $T_j$: the morphism $Ae_k \to Ae_k$ which has a socle of $Ae_k$ as its image annihilates any noninvertible morphism between indecomposable projective modules, thus it induces a nonzero chain map from $\text{Hom}_{D^b(A)}(T_j, T_i[-1])$.

5) The case when $k$ is a source of $T_j$ but is not a marked vertex of $T_j$. Let $(j_1, ..., j_2)$ be a substring of $T_j$ containing $k$, $(i_1, ..., k)$ a substring of $T_i$, the corresponding vertices are ordered in the following way $j_1, j_2, k$ on the $A$-cycle. The morphism $Ae_{j_1} \to Ae_k$ induces a nonzero chain map from $\text{Hom}_{D^b(A)}(T_j, T_i[-1])$. 

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6) The case when \( k \) is a sink of \( T_i \) but is not a marked vertex of \( T_j \) is similar to the previous one.

If the intersection of \( T_i \) and \( T_j \) consists of more than one vertex then the following types of intersections can occur: one of the end points of the intersection is a sink and another is a source, both of the end points of the intersection are sinks, both of the end points of the intersection are sources.

**Lemma 5.** Let the intersection of \( T_i \) and \( T_j \) consist of one connected component, which contains more than one vertex. The diagrams of \( T_i \) and \( T_j \) intersect in such a way that one of the end points of the intersection is a sink and another is a source, then \( \text{Hom}_{D^n(A)}(T_i, T_j[-1]) = 0 = \text{Hom}_{D^n(A)}(T_j, T_i[-1]) \) iff the restriction of \( T_i \) with respect to \( T_j \) belongs to the restriction of \( T_j \) with respect to \( T_i \) or visa versa.

**Proof.** Let us first consider the cases when the end points of the intersection coincide with the marked vertices of only one of the diagrams (the cases 1 and 2), after that let us consider the cases when the end points of the intersection coincide with the marked vertices of both diagrams (the cases 3, 4, 5).

1) Assume that the restriction of \( T_i \) with respect to \( T_j \) belongs to the restriction of \( T_j \) with respect to \( T_i \) and the end points of the intersection of \( T_i \) and \( T_j \) do not coincide with the marked vertices of \( T_j \). The corresponding subdiagrams are arranged on the quiver of \( A \) as follows:

![Diagram](image)

where \( i_1, i_2, \ldots, i_n \) are the marked vertices of \( T_i \) (which belong to the restriction), the vertices with odd indices are sources, the vertices with even indices are sinks; \( j_1, j_2, \ldots, j_{n-1}, j_n \) are the marked vertices of \( T_j \) (which belong to the restriction), the vertices with odd indices are sources, the vertices with even indices are sinks (\( i_1 \neq j_1, i_n \neq j_n \)). The differentials of the restrictions of \( T_i \) and \( T_j \) are induced by the morphisms from \( Ae_{i_k} \) to \( Ae_{i_{k-1}} \), \( Ae_{i_{k+1}} \) for odd \( k = 3, \ldots, n - 3 \), and the morphisms from \( Ae_{i_1} \) to \( Ae_{i_2} \), from \( Ae_{j_1} \) to \( Ae_{j_2} \), from \( Ae_{i_{n-1}} \) to \( Ae_{i_{n-2}} \), and \( Ae_{i_n} \), from \( Ae_{i_{n-1}} \) to \( Ae_{i_{n-2}} \) and \( Ae_{j_n} \), the corresponding morphisms are given by the multiplication by the unique nonzero paths on the quiver of \( A \).

As it was already mentioned, there are no nonzero morphisms between the projective summands of the components of \( T_i \) and \( T_j \) which do not belong to the restrictions.

Assume that there is a nonzero chain map \( g \) from the restriction of \( T_j \) to the restriction of \( T_i \) shifted by \(-1\), it is nonzero on some projective summand of the component of \( T_j \) concentrated in 1. Let \( g \) restricted to \( Ae_{i_k} \) be nonzero, where \( k \) is even. If \( g \) has a nonzero component from \( Ae_{i_k} \) to \( Ae_{i_{k-1}} \), the composition of \( g \) and the differential should be equal to 0, but the composition \( Ae_{i_k} \to Ae_{i_{k-1}} \to \)
$Ae_{i_k}$ is not equal to zero, hence $g$ has a nonzero component from $Ae_{i_k}$ to $Ae_{i_{k+1}}$, but then $Ae_{i_{k+1}} \to Ae_{i_k} \to Ae_{i_{k+1}}$ is not equal to zero, hence $g$ has a nonzero component from $Ae_{i_{k+2}}$ to $Ae_{i_{k+1}}$, therefore $g$ has a nonzero component from $Ae_{j_n}$ to $Ae_{i_{n-1}}$, but $Ae_{j_n} \to Ae_{i_{n-1}} \to Ae_{i_n}$ is not equal to zero. Hence if $g$ has a nonzero component from $Ae_{i_k}$ to $Ae_{i_{k+1}}$, then $g$ can not be a chain map; if $g$ has a nonzero component from $Ae_{i_k}$ to $Ae_{i_{k+1}}$, then $g$ can not be a chain map for a similar reason.

Let us consider $\text{Hom}_{D^b(A)}(T_i, T_j[-1])$. Assume that there is a nonzero chain map $g$ from the restriction of $T_i$ to the restriction of $T_j$ shifted by $-1$, it is nonzero on some projective summand of the component of $T_i$ concentrated in 1. Let $g$ restricted to $Ae_{i_k}$ be nonzero, where $k$ is even. If $g$ has a nonzero component from $Ae_{i_k}$ to $Ae_{i_{k-1}}$, the composition of $g$ and the differential should be equal to 0, but the composition $Ae_{i_{k-1}} \to Ae_{i_k} \to Ae_{i_{k-1}}$ is not equal to zero, hence $g$ has a nonzero component from $Ae_{i_{k-2}}$ to $Ae_{i_{k-1}}$, but then $Ae_{i_{k-2}} \to Ae_{i_{k-1}} \to Ae_{i_{k-2}}$ is not equal to zero, hence $g$ has a nonzero component from $Ae_{i_{k-3}}$ to $Ae_{i_{k-2}}$, in such a manner we get that $g$ has a nonzero component from $Ae_{i_2}$ to $Ae_{i_1}$, but then $Ae_{i_2} \to Ae_{i_2} \to Ae_{i_1}$ is not equal to zero. Hence if $g$ has a nonzero component from $Ae_{i_k}$ to $Ae_{i_{k-1}}$, then $g$ can not be a chain map; if $g$ has a nonzero component from $Ae_{i_k}$ to $Ae_{i_{k+1}}$, then $g$ can not be a chain map for a similar reason. If $g$ has a nonzero component from $Ae_{i_2}$ to $Ae_{i_1}$, then it is straightforward that $g$ can not be a chain map.

2) Consider the case when neither the restriction of $T_i$ with respect to $T_j$ belongs to the restriction of $T_j$ with respect to $T_i$ nor the restriction of $T_j$ with respect to $T_i$ belongs to the restriction of $T_i$ with respect to $T_j$, and $i_1 \neq j_1, i_n \neq j_n$. This is satisfied if the degree one source of the restriction of $T_i$ does not belong to the intersection and the degree one sink of $T_i$ belongs to the intersection (or the same holds for $T_j$, but this case is analogous). The corresponding subdiagrams are arranged on the quiver of $A$ as follows:

Let us construct a nonzero morphism from $\text{Hom}_{D^b(A)}(T_i, T_j[-1])$. Let us describe how this morphism acts on the components: choose an arbitrary nonzero morphism $Ae_{i_2} \to Ae_{i_1}$, note that the composition $Ae_{i_1} \to Ae_{i_2} \to Ae_{i_1}$ is equal to zero; choose such a morphism $Ae_{i_2} \to Ae_{i_3}$ that the composition $Ae_{i_2} \to Ae_{i_3} \to Ae_{i_2}$ is equal to $Ae_{i_2} \to Ae_{i_3} \to Ae_{i_2}$ with the opposite sign; choose such a morphism $Ae_{i_4} \to Ae_{i_3}$ that the composition $Ae_{i_3} \to Ae_{i_2} \to Ae_{i_3}$ is equal to the composition $Ae_{i_3} \to Ae_{i_4} \to Ae_{i_3}$ with the opposite sign; continue to construct the morphism in this way until we reach $Ae_{i_n} \to Ae_{i_{n-1}}$, which was chosen such that $Ae_{i_{n-1}} \to Ae_{i_{n-2}} \to Ae_{i_{n-1}}$ is equal to the composition $Ae_{i_{n-1}} \to Ae_{i_n} \to Ae_{i_{n-1}}$ with the opposite sign, we can see that the
composition $Ae_{i_n} \to Ae_{i_{n-1}} \to Ae_{j_n}$ is equal to zero. Thus we have constructed a nonzero chain map from $\text{Hom}_{D^b(A)}(T_i, T_j[-1])$.

3) Let now $i_1 = j_1$, $i_n \neq j_n$, assume that $i_n$ belongs to the restriction of $T_j$. The following options are possible: a) the vertex $i_1$ is a degree one source of both $T_i$ and $T_j$; b) the vertex $i_1$ is a degree one source of $T_i$ but not $T_j$ c) the vertex $i_1$ is a degree one source of $T_j$ but not $T_i$.

a) The vertex $i_1$ is a degree one source of both $T_i$ and $T_j$. As before we can see that if there is a nonzero chain map $g$ from the restriction of $T_j$ to the restriction of $T_i$ shifted by $-1$, then it should have a nonzero component from $Ae_{i_2}$ to $Ae_{j_1} = Ae_{i_1}$, but $Ae_{i_2} \to Ae_{i_2} \to Ae_{j_1} = Ae_{i_1}$ is not equal to zero, hence $g$ cannot be a chain map, thus $\text{Hom}_{D^b(A)}(T_i, T_j[-1]) = 0$. It is clear that $\text{Hom}_{D^b(A)}(T_i, T_j[-1]) = 0$ for similar reasons.

b) The vertex $i_1$ is a degree one source of $T_i$ but not $T_j$. Let $j_0$ be the marked vertex of $T_j$ next to $j_1$. The restrictions of $T_i$ and $T_j$ are arranged on the quiver of $A$ as follows:

![Diagram](image)

As before we can see that if there is a nonzero chain map $g$ from the restriction of $T_i$ to the restriction of $T_j$ shifted by $-1$, then it should have a nonzero component from $Ae_{i_2}$ to $Ae_{i_1}$, but $Ae_{i_1} \to Ae_{i_2} \to Ae_{i_1}$ is not equal to zero, hence $g$ cannot be a chain map, thus $\text{Hom}_{D^b(A)}(T_i, T_j[-1]) = 0$. Let us consider $\text{Hom}_{D^b(A)}(T_i, T_j[-1]) = 0$. If there is a nonzero chain map $g$ from the restriction of $T_j$ to the restriction of $T_i$ shifted by $-1$, then it should have a nonzero component from $Ae_{j_n}$ to $Ae_{i_{n-1}}$, but $Ae_{j_n} \to Ae_{i_{n-1}} \to Ae_{i_n}$ is not equal to zero, hence $g$ cannot be a chain map.

c) The vertex $i_1$ is a degree one source of $T_j$ but not $T_i$. Let $i_0$ be the marked vertex of $T_i$ next to $i_1$. The restrictions of $T_i$ and $T_j$ are arranged on the quiver of $A$ as follows:

![Diagram](image)

Let us construct a nonzero morphism from $\text{Hom}_{D^b(A)}(T_i, T_j[-1])$. This morphism is constructed the same way as in the case when neither the restriction of $T_i$ with respect to $T_j$ belongs to the restriction of $T_j$ with respect to $T_i$ nor the restriction of $T_j$ with respect to $T_i$ belongs to the restriction of $T_i$ with respect
to $T_j$, and $i_1 \neq j_1, i_n \neq j_n$. Let us start from the morphism $Ae_{i_0} \to Ae_{i_1}$, and let us construct the morphism as described earlier until we reach $Ae_{i_n} \to Ae_{i_{n-1}}$, which was chosen such that the composition $Ae_{i_{n-1}} \to Ae_{i_{n-2}} \to Ae_{i_{n-1}}$ is equal to the composition $Ae_{i_{n-1}} \to Ae_{i_n} \to Ae_{i_{n-1}}$ with the opposite sign, we can see that the composition $Ae_{i_n} \to Ae_{i_{n-1}} \to Ae_{j_n}$ is equal to zero. Thus we have constructed a nonzero chain map from $Hom_{D^b(A)}(T_i, T_j[-1])$.

4) Let $i_n = j_n, i_1 \neq j_1$, assume that $i_1$ belongs to the restriction of $T_j$. The following options are possible: a) the vertex $i_n$ is a degree one sink of both $T_i$ and $T_j$; b) the vertex $i_n$ is a degree one sink of $T_i$ but not $T_j$ c) the vertex $i_n$ is a degree one sink of $T_j$ but not $T_i$.

a) The vertex $i_n$ is a degree one sink of both $T_i$ and $T_j$. A nonzero chain map from the restriction of $T_i$ to the restriction of $T_j$ shifted by $-1$, should have a nonzero component from $Ae_{i_n}$ to $Ae_{j_n}$, but $Ae_{i_n} \to Ae_{i_{n-1}} \to Ae_{j_n} = Ae_{i_n}$ is not equal to zero, hence $Hom_{D^b(A)}(T_i, T_j[-1]) = 0$. For similar reasons as before $Hom_{D^b(A)}(T_j, T_i[-1]) = 0$.

b) The vertex $i_n$ is a degree one sink of $T_i$ but not $T_j$. Let $j_{n+1}$ be the marked vertex of $T_j$ next to $i_n$. The restrictions of $T_i$ and $T_j$ are arranged on the quiver of $A$ as follows:

A nonzero chain map from the restriction of $T_i$ to the restriction of $T_j$ shifted by $-1$, should have a nonzero component from $Ae_{i_n}$ to $Ae_{j_1}$, but $Ae_{i_n} \to Ae_{i_2} \to Ae_{j_1}$ is not equal to zero, hence $Hom_{D^b(A)}(T_i, T_j[-1]) = 0$. If there is a nonzero chain map $g$ from the restriction of $T_j$ to the restriction of $T_i$ shifted by $-1$, it should have a nonzero component from $Ae_{i_n}$ to $Ae_{i_{n-1}}$, but $Ae_{i_n} \to Ae_{i_{n-1}} \to Ae_{i_n}$ is not equal to zero, hence $g$ can not be a chain map.

c) The vertex $i_n$ is a degree one sink of $T_j$ but not $T_i$. Let $i_{n+1}$ be the marked vertex of $T_i$ next to $i_n$. The restrictions of $T_i$ and $T_j$ are arranged on the quiver of $A$ as follows:

Let us construct a nonzero morphism from $Hom_{D^b(A)}(T_j, T_i[-1])$. This morphism is constructed the same way as in the case when neither the restriction of $T_i$ with respect to $T_j$ belongs to the restriction of $T_j$ with respect to $T_i$ nor the
restriction of $T_j$ with respect to $T_i$ belongs to the restriction of $T_i$ with respect to $T_j$, and $i_1 \neq j_1, i_n \neq j_n$. Let us start from the morphism $\text{Ae}_{i_2} \to \text{Ae}_{i_1}$ and let us construct the morphism as described earlier until we reach $\text{Ae}_{i_n} \to \text{Ae}_{i_{n-1}}$, which was chosen such that the composition $\text{Ae}_{i_{n-1}} \to \text{Ae}_{i_{n-2}} \to \text{Ae}_{i_{n-1}}$ is equal to the composition $\text{Ae}_{i_{n-1}} \to \text{Ae}_{i_n} \to \text{Ae}_{i_{n-1}}$ with the opposite sign, we can see that the composition $\text{Ae}_{i_n} \to \text{Ae}_{i_{n-1}} \to \text{Ae}_{i_n}$ is not equal to zero, choose $\text{Ae}_{i_n} \to \text{Ae}_{i_{n+1}}$ such that $\text{Ae}_{i_n} \to \text{Ae}_{i_{n+1}} \to \text{Ae}_{i_n}$ is equal to the composition $\text{Ae}_{i_n} \to \text{Ae}_{i_{n-1}} \to \text{Ae}_{i_n}$ with the opposite sign. Thus we have constructed a nonzero chain map from $\text{Hom}_{D^{b}(A)}(T_j, T_i[-1])$.

5) At last we can consider the case $i_1 = j_1, i_n = j_n$. The following options are possible: a) $i_1$ is a degree one source of $T_i$, but not $T_j$, $i_n$ is a degree one sink of $T_i$, but not $T_j$; b) $i_1$ is a degree one source of $T_i$, but not $T_j$, $i_n$ is a degree one sink of both $T_i$ and $T_j$; c) $i_1$ is a degree one source of both $T_i$ and $T_j$, $i_n$ is a degree one sink of $T_i$, but not $T_j$; d) $i_1$ is a degree one source of $T_i$, but not $T_j$, $i_n$ is a degree one sink of $T_j$, but not $T_j$.

a) The vertex $i_1$ is a degree one source of $T_i$, but not $T_j$, $i_n$ is a degree one sink of $T_i$, but not $T_j$. Let $j_0$ be the marked vertex of $T_j$ next to $i_1$, $j_{n-1}$ be the marked vertex of $T_j$ next to $i_n$. A nonzero chain map from the restriction of $T_i$ to the restriction of $T_j$ shifted by $-1$, should have a nonzero component from $\text{Ae}_{i_2} \to \text{Ae}_{i_1}$, but $\text{Ae}_{i_1} \to \text{Ae}_{i_2} \to \text{Ae}_{i_1}$ is not equal to zero, hence $\text{Hom}_{D^{b}(A)}(T_i, T_j[-1]) = 0$. A nonzero chain map from the restriction of $T_j$ to the restriction of $T_i$ shifted by $-1$, should have a nonzero component from $\text{Ae}_{i_1} \to \text{Ae}_{i_{n-1}}$, but $\text{Ae}_{i_n} \to \text{Ae}_{i_{n-1}} \to \text{Ae}_{i_n}$ is not equal to zero, hence $\text{Hom}_{D^{b}(A)}(T_j, T_i[-1]) = 0$.

b) The vertex $i_1$ is a degree one source of $T_i$, but not $T_j$, $i_n$ is a degree one sink of both $T_i$ and $T_j$. Let $j_0$ be the marked vertex of $T_j$ next to $i_1$. As in the previous case $\text{Hom}_{D^{b}(A)}(T_i, T_j[-1]) = 0$. A nonzero chain map from the restriction of $T_j$ to the restriction of $T_i$ shifted by $-1$, should have a nonzero component from $\text{Ae}_{i_1} \to \text{Ae}_{i_{n-1}}$, but $\text{Ae}_{i_n} \to \text{Ae}_{i_{n-1}} \to \text{Ae}_{i_n}$ is not equal to zero, hence $\text{Hom}_{D^{b}(A)}(T_j, T_i[-1]) = 0$.

c) The vertex $i_1$ is a degree one source of both $T_i$ and $T_j$, $i_n$ is a degree one sink of $T_i$, but not $T_j$. For reasons similar to the case (a) we have $\text{Hom}_{D^{b}(A)}(T_j, T_i[-1]) = 0$. A map from the restriction of $T_i$ to the restriction of $T_j$ shifted by $-1$, should have a nonzero component from $\text{Ae}_{i_1} \to \text{Ae}_{i_2}$, but $\text{Ae}_{i_1} \to \text{Ae}_{i_1} \to \text{Ae}_{i_1}$ is not equal to zero, hence $\text{Hom}_{D^{b}(A)}(T_i, T_j[-1]) = 0$.

d) The vertex $i_1$ is a degree one source of $T_i$, but not $T_j$, $i_n$ is a degree one sink of $T_j$, but not $T_i$. Let $j_0$ be the marked vertex of $T_j$ next to $i_1$, and $i_{n+1}$ be the marked vertex of $T_i$ next to $i_n$. Let us start the construction of a nonzero morphism from $T_j$ to $T_i[1]$ with an arbitrary nonzero morphism $\text{Ae}_{j_0} \to \text{Ae}_{i_1}$, choose a morphism $\text{Ae}_{i_2} \to \text{Ae}_{i_1}$ such that $\text{Ae}_{i_1} \to \text{Ae}_{j_0} \to \text{Ae}_{i_1}$ is equal to the composition $\text{Ae}_{i_1} \to \text{Ae}_{i_2} \to \text{Ae}_{i_1}$, with the opposite sign, as before let us construct a morphism from $T_j$ to $T_i[1]$ in such a manner that its composition with the differentials of $T_j$ and $T_i[1]$ is equal to zero, let us finish the construction choosing a morphism $\text{Ae}_{i_n} \to \text{Ae}_{i_{n+1}}$ such that the composition $\text{Ae}_{i_n} \to \text{Ae}_{i_{n-1}} \to \text{Ae}_{i_n}$ is equal to the composition $\text{Ae}_{i_n} \to \text{Ae}_{i_{n+1}} \to \text{Ae}_{i_n}$ with the opposite sign, thus $\text{Hom}_{D^{b}(A)}(T_j, T_i[-1]) \neq 0$. □
Lemma 6. Let the intersection of $T_i$ and $T_j$ consist of one connected component, which contains more than one vertex. The diagrams of $T_i$ and $T_j$ intersect in such a way that both of the end vertices of the intersection are sinks, then \( \text{Hom}_{D^b(A)}(T_i, T_j[-1]) = 0 = \text{Hom}_{D^b(A)}(T_j, T_i[-1]) \) iff one of the following conditions hold: 1) neither the restriction of $T_i$ with respect to $T_j$ nor the restriction of $T_j$ with respect to $T_i$ belongs to the restriction of $T_j$ with respect to $T_i$ nor the restriction of $T_j$ with respect to $T_i$; 2) the diagrams of $T_i$ and $T_j$ have a coinciding vertex of degree one.

Lemma 7. Let the intersection of $T_i$ and $T_j$ consist of one connected component, which contains more than one vertex. The diagrams of $T_i$ and $T_j$ intersect in such a way that both of the end vertices of the intersection are sources, then \( \text{Hom}_{D^b(A)}(T_i, T_j[-1]) = 0 = \text{Hom}_{D^b(A)}(T_j, T_i[-1]) \) iff one of the following conditions hold: 1) neither the restriction of $T_i$ with respect to $T_j$ nor the restriction of $T_j$ with respect to $T_i$ belongs to the restriction of $T_j$ with respect to $T_i$ nor the restriction of $T_j$ with respect to $T_i$; 2) the diagrams of $T_i$ and $T_j$ have a coinciding vertex of degree one.

Proof. The proof of lemmas 6 and 7 is completely analogous to that of lemma 5 and is left to the reader.

Remark 4. From lemmas 5, 6, 7 we can see that regardless of the type of the intersection if two diagrams have a common sink or source of degree one then the direct sum of the corresponding complexes is a partially tilting complex.

Let us now consider the case when at least one of the complexes is a stalk complex of a projective module. Note that if $P$ is an indecomposable stalk complex of a projective module concentrated in degree 0 and $T_i$ is concentrated in 0 and 1 then the condition \( \text{Hom}_{D^b(A)}(P, T_i[-1]) = 0 \) holds automatically.

Lemma 8. Let $T_i$ be the projective presentation of some module, such that $T_i$ is an indecomposable two-term partial tilting complex and let $P$ be an indecomposable stalk complex of a projective module concentrated in degree 0, then \( \text{Hom}_{D^b(A)}(T_i, P[-1]) = 0 \) iff one of the following conditions is satisfied: 1) the vertex corresponding to $P$ coincides with a source of degree one of $T_i$; 2) the vertex corresponding to $P$ does not belong to $T_i$ and there is no $A$-cycle which contains the vertex corresponding to $P$ and a sink of degree one of $T_i$ such that this is the only vertex of $T_i$ belonging to this $A$-cycle.

Proof. We have to consider the following cases: the vertex corresponding to $P$ does not belong to $T_i$, the vertex corresponding to $P$ coincides with a source of $T_i$, the vertex corresponding to $P$ coincides with a sink of $T_i$, the vertex corresponding to $P$ coincides with an unmarked vertex of $T_i$.

1) The vertex corresponding to $P$ does not belong to $T_i$. If the vertex corresponding to $P$ does not belong to any $A$-cycle containing vertices of $T_i$ then we clearly have \( \text{Hom}_{D^b(A)}(T_i, P[-1]) = 0 \); if the vertex corresponding to $P$ and some substring of $T_i$ belong to the same $A$-cycle, then the only morphism, from the sink of $T_i$ belonging to the same $A$-cycle to $P$ does not induce a chain map,
hence \( \text{Hom}_{D^b(A)}(T_i, P[-1]) = 0 \). If the vertex corresponding to \( P \) and just a source of \( T_i \) belong to the same \( A \)-cycle, then there is no nonzero maps from \( (T_i)^1 \) to \( P \). If the vertex corresponding to \( P \) and just a sink of \( T_i \) belong to the same \( A \)-cycle, then the morphism from this sink of \( T_i \) to \( P \) induces a nonzero chain map.

2) Let the vertex corresponding to \( P \) coincide with the degree one source of \( T_i \). Let \( i_2 \) be the marked vertex of \( T_i \) next to \( i_1 \), then \( \text{Hom}_{D^b(A)}(T_i, P[-1]) = 0 \) iff \( \text{Hom}_{D^b(A)}(Ae_{i_1} \rightarrow Ae_{i_2}, Ae_{i_1}[−1] = 0 \right. \). This is true since the only morphism \( Ae_{i_2} \rightarrow Ae_{i_1} \) does not induce a chain map. Let the vertex corresponding to \( P \) coincide with the degree two source \( i_2 \) of \( T_i \). Let \( i_1, i_3 \) be the neighboring marked vertices of \( T_i \), let us construct a nonzero morphism \( T_i \rightarrow P[-1] \) as follows. Choose an arbitrary nonzero restriction of this map \( Ae_{i_1} \rightarrow Ae_{i_2} \), the restriction to \( Ae_{i_2} \rightarrow Ae_{i_1} \) is chosen in such a manner that the composition \( Ae_{i_2} \rightarrow Ae_{i_1} \rightarrow Ae_{i_2} \) is equal to the composition \( Ae_{i_2} \rightarrow Ae_{i_3} \rightarrow Ae_{i_2} \) with the opposite sign, set all other components to be zero. Thus, \( \text{Hom}_{D^b(A)}(T_i, P[-1]) \neq 0 \).

3) If the vertex corresponding to \( P \) coincides with a sink \( i_1 \) of \( T_i \), then the map \( Ae_{i_1} \rightarrow Ae_{i_1} \) which has the socle of \( Ae_{i_1} \) as its image induces a nonzero chain map and \( \text{Hom}_{D^b(A)}(T_i, P[-1]) \neq 0 \).

4) Let us consider the case when the vertex \( k \) corresponding to \( P \) coincides with an unmarked vertex of \( T_i \). Let \( i_1, i_2 \) be the source and the sink of the substring of \( T_i \) containing \( k \). \( \text{Hom}_{D^b(A)}(T_i, P[-1]) = 0 \) iff \( \text{Hom}_{D^b(A)}(Ae_{i_1} \rightarrow Ae_{i_2}, Ae_{i_1}[−1] = 0 \). It is clear that a nonzero morphism \( Ae_{i_2} \rightarrow Ae_{k} \) induces a chain map since the composition \( Ae_{i_2} \rightarrow Ae_{i_2} \rightarrow Ae_{k} \) is equal to zero.

**Lemma 9.** Let \( T_i \) be the projective presentation of some module, such that \( T_i \) is an indecomposable two-term partial tilting complex and let \( P \) be an indecomposable stalk complex of a projective module concentrated in degree 1, then \( \text{Hom}_{D^b(A)}(T_i, P[1]) = 0 \) iff one of the following conditions is satisfied: 1) the vertex corresponding to \( P \) coincides with a sink of degree one of \( T_i \); 2) the vertex corresponding to \( P \) does not belong to \( T_i \) and there is no \( A \)-cycle which contains the vertex corresponding to \( P \) and a source of degree one of \( T_i \) such that this is the only vertex of \( T_i \) belonging to this \( A \)-cycle.

**Proof.** The proof is analogous to that of lemma 8.

**Lemma 10.** Let \( P_i, P_j \) be two indecomposable stalk complex of projective modules \( \text{Hom}_{D^b(A)}(P_i, P_i[-1]) = 0 = \text{Hom}_{D^b(A)}(P_j, P_j[-1]) \) iff one of the following conditions is satisfied: 1) the vertices corresponding to \( P_i \) and \( P_j \) do not belong to the same \( A \)-cycle 2) the vertices corresponding to \( P_i \) and \( P_j \) belong to the same \( A \)-cycle and \( P_i, P_j \) are concentrated in the same degree.

### 4 Endomorphism rings

In the previous section all two-term tilting complexes over a Brauer tree algebra with multiplicity one were described as direct sums of \( n \) indecomposable
partial tilting complexes which pairwisely satisfy some conditions. In this section the endomorphism rings of such complexes are described. It is well known that the endomorphism rings are isomorphic to some Brauer tree algebras with multiplicity one. Let $T$ be a tilting complex, to describe its endomorphism ring it is sufficient to determine the partition of the vertices of $\text{End}_{K^b(A)}(T)$ into $A$-cycles or equivalently which edges of the Brauer tree of $\text{End}_{K^b(A)}(T)$ are incident to the same vertex and the cyclic ordering of the edges incident to the same vertex in the Brauer tree. The vertices of the quiver of $\text{End}_{K^b(A)}(T)$ correspond to the indecomposable summands of $T$, two vertices $i, j$ belong to the same $A$-cycle iff $\text{Hom}_{K^b(A)}(T_i, T_j) \neq 0$ holds for the corresponding summands $T_i, T_j$. This condition is easy to verify by the well known formula due to Happel [11]: let $Q = (Q^r)_{r \in \mathbb{Z}}, R = (R^s)_{s \in \mathbb{Z}} \in A$-perf, then

$$\sum_i (-1)^i \text{dim}_K \text{Hom}_{K^b(A)}(Q, R[i]) = \sum_i (-1)^{r-s} \text{dim}_K \text{Hom}_A(Q^r, R^s).$$

Note that if $\text{Hom}_{K^b(A)}(Q, R[i]) = 0, i \neq 0$ (for example, in the case when $Q$ and $R$ are summands of a tilting complex) then the left hand side of the formula becomes $\text{dim}_K \text{Hom}_{K^b(A)}(Q, R)$.

Recall that it is convenient to assume that each vertex belongs to two $A$-cycles in the quiver of $A$, i.e. if some vertex has degree 2, then we assume that there is a formal loop, which is equal to the other $A$-cycle passing through this vertex and this loop is the second $A$-cycle containing this vertex.

**Proposition 1.** Let $T_i, T_j$ be projective presentations of nonisomorphic indecomposable nonprojective $A$-modules such that their direct sum is a partially tilting complex. $\text{Hom}_{K^b(A)}(T_i, T_j) \neq 0$ iff a degree one sinks of $T_i$ and $T_j$ belong to the same $A$-cycle and these are the only vertices of $T_i$ and $T_j$, belonging to this $A$-cycle, or a degree one sources of $T_i$ and $T_j$ belong to the same $A$-cycle and these are the only vertices of $T_i$ and $T_j$, belonging to this $A$-cycle.

**Proof.** Let a degree one sinks of $T_i$ and $T_j$ belong to the same $A$-cycle in such a manner that these are the only vertices of $T_i$ and $T_j$, belonging to this $A$-cycle. This holds in the following cases: the diagrams of $T_i$ and $T_j$ do not intersect, a degree one sinks of $T_i$ and $T_j$ belong to the same $A$-cycle and these are the only vertices of $T_i$ and $T_j$, belonging to this $A$-cycle and have a common sink of degree one. The case of the sources is analogous.

1) Let $T_i$ and $T_j$ not have vertices belonging to the same $A$-cycle, it is clear that $\text{dim}_K \text{Hom}_{K^b(A)}(T_i, T_j) = 0$.

2) Let $T_i$ and $T_j$ not intersect but have vertices belonging to the same $A$-cycle $\Upsilon$. Let a substring $(i_1, ..., i_2)$ of $T_i$ and a substring $(j_1, ..., j_2)$ of $T_j$ belong to $\Upsilon$, the substrings do not intersect. It is clear that $\text{dim}_K \text{Hom}_{K^b(A)}(T_i, T_j) = \text{dim}_K \text{Hom}_{K^b(A)}(Ae_{i_2} \rightarrow Ae_{i_1}, Ae_{j_2} \rightarrow Ae_{j_1}) = 1 - 1 + 1 = 0$. Now let a substring $(i_1, ..., i_2)$ of $T_i$ and an unmarked vertex of $T_j$ belong to $\Upsilon$, we get $\text{dim}_K \text{Hom}_{K^b(A)}(T_i, T_j) = 0$, since the projective summands of the components of $T_i$ and $T_j$ do not belong to the same $A$-cycle. If $\Upsilon$ contains a degree one sink of $T_i$ and an unmarked vertex of $T_j$; a degree one source of $T_i$ and an
we get that \( \dim K \) objective summands of the components of \( A \).

As it was already mentioned, there are no nonzero morphisms between the professional: \( \Hom A \) indices are sinks. Between the following modules the \( \Hom A \) is a sink and the other one is a source (the intersection contains more vertices; \( j \), then \( \dim K \Hom A(T_i, T_j) = 0 \) for the same reason. Let a substring \((i_1, \ldots, i_2)\) of \( T_i \) and a degree one sink \((j_1)\) of \( T_j \) belong to \( \Upsilon \). \( \dim K \Hom A(T_i, T_j) = \dim K \Hom A(Ae_{i_2} \to Ae_{i_1} \to 0 \to Ae_{j_1}) = -1 + 1 = 0 \). The case when a substring \((i_1, \ldots, i_2)\) of \( T_i \) and a degree one source \((j_1)\) of \( T_j \) belong to \( \Upsilon \) is similar to the previous one. Now let a vertex \((i_1)\) of \( T_i \) and a vertex \((j_1)\) of \( T_j \) belong to \( \Upsilon \), both vertices are degree one sources. \( \dim K \Hom A(T_i, T_j) = \dim K \Hom A(Ae_i \to 0, Ae_j \to 0) = 1 \). The case when \( \Upsilon \) contains a degree one sink of both \( T_i \) and \( T_j \) is analogous.

3) Let the diagrams of \( T_i \) and \( T_j \) meet at one vertex \( k \). Let \( k \) be an unmarked vertex of \( T_i \) and \( T_j \), then the marked vertices of \( T_i \) and \( T_j \) belong to different \( A \)-cycles and there are no nonzero morphisms between the corresponding projective modules, \( \dim K \Hom A(T_i, T_j) = 0 \). Let \( k \) be a degree one sink of both \( T_i \) and \( T_j \), then \( \dim K \Hom A(T_i, T_j) = \dim K \Hom A(Ae_{i_2} \to Ae_k, Ae_{j_2} \to Ae_k) = -1 + 1 = 0 \), where \( i_2 \) is the source of \( T_i \) next to \( k \), \( j_2 \) is the source of \( T_j \) next to \( k \) and \( i_2 \) and \( j_2 \) belong to different \( A \)-cycles. The case when \( k \) is a degree one source of both \( T_i \) and \( T_j \) is similar to the previous one.

Assume that the restriction of \( T_i \) with respect to \( T_j \) belongs to the restriction of \( T_i \) with respect to \( T_j \) and the end points of the intersection of \( T_i \) and \( T_j \) do not coincide with the marked vertices of \( T_j \). The corresponding subdiagrams are arranged on the quiver of \( A \) as follows:

4) Let \( T_i \) and \( T_j \) intersect in such a way that one of the end points of the intersection is a sink and the other one is a source (the intersection contains more than one vertex). Assume that the restriction of \( T_i \) with respect to \( T_j \) belongs to the restriction of \( T_j \) with respect to \( T_i \). The corresponding subdiagrams are arranged on the quiver of \( A \) as follows: (in the case when \( i_1 \neq j_1, i_n \neq j_n \)):

\[
\begin{array}{c}
\bullet & \bullet & \cdots & \bullet & \bullet & \bullet \\
\bullet & \bullet & \cdots & \bullet & \bullet & \bullet \\
\end{array}
\]

where \( i_1, i_2, \ldots, i_n \) are the marked vertices of \( T_i \) (which belong to the restriction), the vertices with odd indices are sources, the vertices with even indices are sinks; \( j_1, i_2, \ldots, j_n, i_n - 1 \) are the marked vertices of \( T_j \) (which belong to the restriction), the vertices with even indices are sources, the vertices with odd indices are sinks. Between the following modules the Hom-space is one dimensional: \( Ae_{i_k} \) and \( Ae_{i_{k+1}}, Ae_{i_{k+2}} \) for \( k = 2, 3, \ldots, n - 1 \) and between \( Ae_{i_1}, Ae_{i_2}, Ae_{j_1} \), and also between \( Ae_{i_{n-2}} \) and \( Ae_{i_{n-1}} \), and between \( Ae_{i_{n-1}} \) and \( Ae_{i_1} \), and between \( Ae_{i_{n-2}} \) and \( Ae_{i_{n-1}} \), and \( Ae_{j_n} \). As it was already mentioned, there are no nonzero morphisms between the projective summands of the components of \( T_i \) and \( T_j \) which do not belong to the restrictions. It is easy to see that \( \dim K \Hom A(T_i^n, T_j^n) = \dim K \Hom A(T_i^n, T_j^n) = \dim K \Hom A(T_i^n, T_j^n) = \dim K \Hom A(T_i^n, T_j^n) = n - 1 \), hence using the formula we get that \( \dim K \Hom A(T_i^n, T_j^n) = 0 \).

Now let \( i_1 = j_1, i_n \neq j_n \), assume that \( i_n \) belongs to the restriction of \( T_j \), the
vertex \( i_1 \) is a degree one source of both \( T_i \) and \( T_j \), then \( \dim_K \text{Hom}_A(T_i^0, T_j^0) = n \), 
\( \dim_K \text{Hom}_A(T_i^0, T_j^1) = \dim_K \text{Hom}_A(T_i^1, T_j^0) = \dim_K \text{Hom}_A(T_i^1, T_j^1) = n-1 \), hence by the formula \( \dim_K \text{Hom}_{K^0(A)}(T_i, T_j) = 1 \).

Let the vertex \( i_1 \) be a degree one source of \( T_i \) but not \( T_j \). Let \( j_0 \) be the marked vertex of \( T_j \) next to \( j_1 \). The restrictions of \( T_i \) and \( T_j \) are arranged on the quiver of \( A \) as follows:

![Quiver of A](image)

And we have \( \dim_K \text{Hom}_A(T_i^0, T_j^0) = \dim_K \text{Hom}_A(T_i^0, T_j^1) = n \), 
\( \dim_K \text{Hom}_A(T_i^1, T_j^0) = \dim_K \text{Hom}_A(T_i^1, T_j^1) = n-1 \), hence by the formula we get \( \dim_K \text{Hom}_{K^0(A)}(T_i, T_j) = 0 \).

Now let \( i_n = j_n \), \( i_1 \neq j_1 \), assume that \( i_1 \) belongs to the restriction of \( T_j \), the vertex \( i_n \) is a degree one sink of both \( T_i \) and \( T_j \). Then \( \dim_K \text{Hom}_A(T_i^0, T_j^0) = \dim_K \text{Hom}_A(T_i^0, T_j^1) = \dim_K \text{Hom}_A(T_i^1, T_j^0) = \dim_K \text{Hom}_A(T_i^1, T_j^1) = n-1 \), \( \dim_K \text{Hom}_A(T_i^1, T_j^1) = n \) hence by the formula \( \dim_K \text{Hom}_{K^0(A)}(T_i, T_j) = 1 \).

Let the vertex \( i_n \) be a degree one sink of \( T_i \) but not \( T_j \). Let \( j_{n+1} \) be the marked vertex of \( T_j \) next to \( i_n \). The restrictions of \( T_i \) and \( T_j \) are arranged on the quiver of \( A \) as follows:

![Quiver of A](image)

Then, \( \dim_K \text{Hom}_A(T_i^0, T_j^0) = \dim_K \text{Hom}_A(T_i^0, T_j^1) = n-1 \), 
\( \dim_K \text{Hom}_A(T_i^1, T_j^0) = \dim_K \text{Hom}_A(T_i^1, T_j^1) = n \) hence by the formula we get \( \dim_K \text{Hom}_{K^0(A)}(T_i, T_j) = 0 \).

At last, let us consider the case \( i_1 = j_1 \), \( i_n = j_n \). Let the vertex \( i_1 \) be a degree one source of \( T_i \), but not \( T_j \), \( i_n \) a degree one sink of \( T_i \), but not \( T_j \). Let \( j_0 \) be the marked vertex of \( T_j \) next to \( i_1 \), and \( j_{n+1} \) be the marked vertex of \( T_j \) next to \( i_n \). Then \( \dim_K \text{Hom}_A(T_i^0, T_j^0) = \dim_K \text{Hom}_A(T_i^0, T_j^1) = \dim_K \text{Hom}_A(T_i^1, T_j^0) = \dim_K \text{Hom}_A(T_i^1, T_j^1) = n \) hence by the formula we get \( \dim_K \text{Hom}_{K^0(A)}(T_i, T_j) = 0 \).

Now let \( i_1 \) be a degree one source of \( T_i \), but not \( T_j \), \( i_n \) a degree one sink of both \( T_i \) and \( T_j \). Let \( j_0 \) be the marked vertex of \( T_j \) next to \( i_1 \). Then, \( \dim_K \text{Hom}_A(T_i^0, T_j^0) = \dim_K \text{Hom}_A(T_i^0, T_j^1) = \dim_K \text{Hom}_A(T_i^1, T_j^0) = \dim_K \text{Hom}_A(T_i^1, T_j^1) = n \), \( \dim_K \text{Hom}_A(T_i^1, T_j^0) = n-1 \), hence by the formula we get \( \dim_K \text{Hom}_{K^0(A)}(T_i, T_j) = 1 \).
Let the vertex $i_1$ be a degree one sink of both $T_i$ and $T_j$, $i_n$ a degree one sink of $T_i$ but not $T_j$. Then $\dim_k \text{Hom}_A(T_i^0, T_j^0) = \dim_k \text{Hom}_A(T_i^1, T_j^1) = \dim_k \text{Hom}_A(T_i^1, T_j^1) = n$, $\dim_k \text{Hom}_A(T_i^0, T_j^1) = n - 1$, hence by the formula we get $\dim_k \text{Hom}_{K^b(A)}(T_i, T_j) = 1$.

As it was mentioned before the following types of intersections can occur: one of the end points of the intersection is a sink and another is a source (this case we have just analysed), both of the end points of the intersection are sinks, both of the end points of the intersection are sources. The cases when both of the end points of the intersection are sinks or both of the end points of the intersection are sources can be analysed similar to the case 4 and is left to the reader.

**Proposition 2.** Let $T_i$ be the projective presentation of an indecomposable nonprojective $A$-modules, $P$ a stalk complex of an indecomposable projective module concentrated in degree 0 and let $T_i \bigoplus P$ be a partial tilting complex. $\text{Hom}_{K^b(A)}(T_i, P) \neq 0$ iff a degree one source of $T_i$ and the vertex corresponding to $P$ belong to the same $A$-cycle and this source is the only vertex of $T_i$ belonging to this $A$-cycle.

**Proof.** Let $j$ denote the vertex corresponding to $P$. It is clear that if $j$ and vertices of $T_i$ do not belong to the same $A$-cycle, then $\text{Hom}_{K^b(A)}(T_i, P) = 0$. Let $j$ and some vertices of $T_i$ belong to the same $A$-cycle $\Upsilon$, $j$ does not belong to $T_i$, there are the following possibilities: 1) there is a substring $(i_1, \ldots, i_2)$ of $T_i$ which belongs to $\Upsilon$ 2) there is just a degree one source $i$ of $T_i$ which belongs to $\Upsilon$. In the first case $\dim_k \text{Hom}_{K^b(A)}(T_i, P) = \dim_k \text{Hom}_{K^b(A)}(Ae_{i_2} \to Ae_{i_1}, Ae_j) = 0$. In the second case $\dim_k \text{Hom}_{K^b(A)}(T_i, P) = \dim_k \text{Hom}_{K^b(A)}(Ae_i, Ae_j) = 1$.

Let now $j$ belong to the diagram of $T_i$, i.e. $j$ coincides with a degree one source of $T_i$. Denote by $i$ the marked sink of $T_i$ next to $j$. Then $\dim_k \text{Hom}_{K^b(A)}(T_i, P) = \dim_k \text{Hom}_{K^b(A)}(Ae_j \to Ae_i, Ae_j) = 2 - 1 = 1$. 

**Proposition 3.** Let $T_i$ be the projective presentation of an indecomposable nonprojective $A$-modules, $P$ a stalk complex of an indecomposable projective module concentrated in degree 1 and let $T_i \bigoplus P$ be a partial tilting complex. $\text{Hom}_{K^b(A)}(T_i, P) \neq 0$ iff a degree one sink of $T_i$ and the vertex corresponding to $P$ belong to the same $A$-cycle and this sink is the only vertex of $T_i$ belonging to this $A$-cycle.

**Proof.** The proof is similar to the proof of proposition 2.

**Proposition 4.** Let $P_i$, $P_j$ be stalk complexes of indecomposable projective modules and let $P_i \bigoplus P_j$ be a partial tilting complex. $\text{Hom}_{K^b(A)}(P_i, P_j) \neq 0$ iff $P_i$ and $P_j$ are concentrated in the same degree and the vertices corresponding to $P_i$ and $P_j$ belong to the same $A$-cycle.

From the description of the Cartan matrix we see that there are two types of $A$-cycles in $\text{End}_{K^b(A)}(T)$: $A$-cycles corresponding to sources, to such an $A$-cycle belong all the indecomposable partial tilting complexes with diagrams having a degree one source on some fixed $A$-cycle $\Upsilon$ of algebra $A$ such that no other vertices of these diagrams belong to $\Upsilon$ and all the indecomposable stalk complexes
of projective modules concentrated in degree 0 such that the corresponding vertices belong to $\Upsilon$; $A$-cycles corresponding to sinks, to such an $A$-cycle belong all the indecomposable partial tilting complexes with diagrams having a degree one sink on some fixed $A$-cycle $\Upsilon$ of algebra $A$ such that no other vertices of these diagrams belong to $\Upsilon$ and all the indecomposable stalk complexes of projective modules concentrated in degree 1 such that the corresponding vertices belong to $\Upsilon$.

Let us describe the cyclic ordering of the indecomposable summands of $T$ belonging to the $A$-cycle corresponding to sources. With this aim in view let us introduce a sort of cyclic lexicographic order. Fix a vertex on $\Upsilon$ and set it to be the greatest (the vertex with an arrow coming from the fixed vertex set to be less than the fixed one, the next vertex even less and so on), order the other vertices linearly. To a diagram with a degree one source on $\Upsilon$ let us associate an ordered set of vertices as follows: the first vertex is the degree one source on $\Upsilon$, after that take all the marked vertices according to their order on the diagram. To an indecomposable stalk complex of a projective module associate a set consisting of the vertex which corresponds to that module. Let us consider a usual lexicographic order on the sets of vertices (except for we set the empty spot on an even position to be the least and on an odd position the greatest): if the first vertex of the set corresponding to $T_i$ is less than the first vertex of the set corresponding to $T_j$, then $T_i < T_j$, if these vertices coincide, then the second vertices of the sets corresponding to $T_i$ and $T_j$ belong to the same $A$-cycle, the first vertex of the sets belong to the same $A$-cycle, set the first vertex of the sets to be the greatest among the vertices (the empty spot can be greater) we can compare the $i+1$-st vertices. The cyclic ordering is glued from the linear one.

Recall that we identify the indecomposable summands of $T$ and the edges in the Brauer tree of $\text{End}_{K^b(A)}(T)$.

**Proposition 5.** In the Brauer tree of $\text{End}_{K^b(A)}(T)$ the cyclic ordering of the edges incident to a vertex corresponding to an $A$-cycle of sources coincides with the order introduced above.

**Proof.** Let $\Upsilon$ be some $A$-cycle of algebra $A$, denote the $A$-cycle of $\text{End}_{K^b(A)}(T)$ corresponding to the sources belonging to $\Upsilon$ by $\Psi$. Assume that $r$ vertices belong to $\Psi$, then to determine the cyclic ordering of the vertices belonging to $\Psi$ it is sufficient to construct $r$ morphisms between the corresponding summands of $T$ such that their consecutive composition is not homotopic to zero. First let us construct morphisms from $T_j$ to $T_i$, where $T_i < T_j$ then from the least summand to the greatest. Let us construct $\alpha_{2,1}$, assume that in the sets of the vertices corresponding to $T_2, T_1$ the first $i$ vertices coincide and
the \( i + 1 \)-st vertices are different, \( T_2 > T_1 \). Denote by \( P_1, \ldots, P_i \) the projective modules corresponding to the first \( i \) vertices of the sets of \( T_1, T_2 \), denote by \( P_{i+1} \) the projective module corresponding to the \( i + 1 \)-st vertex of \( T_1 \), by \( P_{i+2} \) the projective module corresponding to the \( i + 1 \)-st vertex of \( T_2 \).

Let us set the morphism \( \alpha_{2,1} : T_2 \to T_1 \) on the projective summands of the components of \( T_1 \) and \( T_2 \). On the coinciding modules \( P_1, \ldots, P_i \) it is the identity morphism and \( \alpha_{2,1}|_{P_{i+2} \to P_{i+1}} \) is the multiplication by the unique path in the quiver of \( A \) between the corresponding vertices. All other components are zero. Let us check that the map obtained is a chain map. It is clear that the commutativity of the square

\[
\begin{array}{c}
T_2^0 \longrightarrow T_2^1 \\
\downarrow \alpha_{1,2} \downarrow \alpha_{1,2} \\
T_1^0 \longrightarrow T_2^1
\end{array}
\]

follows from the commutativity of the squares consisting of direct summands of \( T_2^0, T_2^1, T_1^0, T_2^1 \). The squares with only identity or only zero morphisms are commutative. Let \( i \) be odd, the square

\[
\begin{array}{c}
P_i \longrightarrow P_{i+2} \\
\downarrow 1 \downarrow \alpha_{1,2} \\
P_i \longrightarrow P_{i+1}
\end{array}
\]

is commutative, since the path from the vertex corresponding to \( P_i \) to the vertex corresponding to \( P_{i+1} \) passes the vertex corresponding to \( P_{i+2} \). If \( P_{i+1} = 0 \), the square remains commutative.

Let \( i \) be even, the square

\[
\begin{array}{c}
P_{i+2} \longrightarrow P_i \\
\alpha_{1,2} \downarrow 1 \\
P_{i+1} \longrightarrow P_i
\end{array}
\]

is commutative, since the path from the vertex corresponding to \( P_{i+2} \) to the vertex corresponding to \( P_i \) passes the vertex corresponding to \( P_{i+1} \). If \( P_{i+2} = 0 \), the square remains commutative. The second square containing \( \alpha_{1,2}|_{P_{i+2} \to P_{i+1}} \) is commutative since the modules it contains belong to different \( A \)-cycles and the compositions are equal to zero.

Let as construct the morphism from the least summand of \( T \) to the greatest \( \alpha_{\min,\max} : T_{\min} \to T_{\max} \). The projective modules corresponding to the first elements of the sets of \( T_{\min}, T_{\max} \) are denoted by \( P_{\min}, P_{\max} \) respectively. If \( P_{\min} \neq P_{\max} \) then the component \( \alpha_{\min,\max}|_{P_{\min} \to P_{\max}} \) is set to be the multiplication by the unique nonzero path, all other components are set to be zero. If \( P_{\min} = P_{\max} \), then the component \( \alpha_{\min,\max}|_{P_{\min} \to P_{\max}} \) is set to be
the multiplication by the long nonzero path, i.e. the morphism with the socle of $P_{\text{max}}$ as its image, all other components are set to be zero. The commutativity of

$$
\begin{array}{ccc}
T^0_{\text{min}} & \to & T^1_{\text{min}} \\
\downarrow{\alpha^0_{1,2}} & & \downarrow{\alpha^1_{1,2}} \\
T^0_{\text{max}} & \to & T^1_{\text{max}}
\end{array}
$$

follows from the commutativity of

$$
\begin{array}{ccc}
P_{\text{min}} & \to & P_1 \\
\downarrow{\alpha^0_{1,2}} & & \downarrow{0} \\
P_{\text{max}} & \to & P_2
\end{array}
$$

Where $P_1, P_2$ are the modules corresponding to the second elements of the sets of $T_{\text{min}}, T_{\text{max}}$ respectively (note that $P_1, P_2$ can be zero). If $P_{\text{min}} \neq P_{\text{max}}$, then $P_{\text{min}}$ and $P_2$ belong to different $A$-cycles, hence the composition $P_{\text{min}} \to P_{\text{max}} \to P_2$ is equal to 0. If $P_{\text{min}} = P_{\text{max}}$, then the composition $P_{\text{min}} \to P_{\text{max}} \to P_2$ is equal to 0, since the image of $P_{\text{min}} \to P_{\text{max}}$ is the socle of $P_{\text{max}}$.

An $A$-cycle $\Psi$ consists of $r$ vertices, let us check that the consecutive composition of $r$ constructed morphisms is not homotopic to 0. That is for any indecomposable summand of $T$ (say $T_i$) the composition $\alpha_{i,i} : T_i \to T_i$ is not homotopic to 0. Let $P_i$ be the module corresponding to the first vertex of the set $T_i$, it’s easy to see that $\alpha_{i,i} |_{P_i \to P_i}$ is the multiplication by the long nonzero path, i.e. the morphism with the socle of $P_i$ as its image, all other components of $\alpha_{i,i}$ are zero. This morphism is not homotopic to 0.

Let us describe the cyclic ordering of the indecomposable summands of $T$ belonging to the $A$-cycle corresponding to sinks. It differs from the ordering of the indecomposable summands of $T$ belonging to the $A$-cycle of sources only by the fact that the sets corresponding to the diagrams start with a degree one sink on $\Upsilon$ and that the empty spot on the odd position is set to be least and on the even position the greatest. Namely fix a vertex on $\Upsilon$ and set it to be the greatest, order the other vertices linearly. To a diagram with a degree one sink on $\Upsilon$ let us associate an ordered set of vertices as follows: the first vertex is the degree one sink on $\Upsilon$, after that take all the marked vertices according to their order on the diagram. To an indecomposable stalk complex of a projective module associate a set consisting of the vertex which corresponds to that module. Let us consider a lexicographic order on the sets of vertices (except for we set the empty spot on an odd position to be the least and on an even position the greatest) as before, glue the cyclic ordering from this linear one. The proof of the next proposition is similar to the proof of proposition 5.

**Proposition 6.** In the Brauer tree of $\text{End}_{K^A}(A)(T)$ the cyclic ordering of the edges incident to a vertex corresponding to an $A$-cycle of sinks coincides with the order introduced above.
Proposition 7. Over any Brauer tree algebra $A$ there exists a two-term tilting complex $T$ such that the algebra $\text{End}_{K^b(A)}(T)$ is isomorphic to the Brauer star algebra.

Proof. Fix some $A$-cycle $\Upsilon$ of algebra $A$. To any vertex $x$ of algebra $A$ let us associate an indecomposable two-term partial tilting complex $T_x$ such that the sum over all the vertices is the desired $T$. All the summands of $T$ will belong to an $A$-cycle of $\text{End}_{K^b(A)}(T)$, corresponding to sources. If the vertex $x$ belong to $\Upsilon$, then $T_x$ is the stalk complex of the projective module corresponding to $x$, concentrated in 0. If the vertex $x$ does not belong to $\Upsilon$, consider a diagram such that one of its end points is $x$, and the other end point is some vertex $y$ belonging to $\Upsilon$ such that $y$ is the only vertex of the diagram belonging to $\Upsilon$ and $y$ is a source. It is clear that since the Brauer graph of $A$ is a tree, this diagram exists and is unique, $T_x$ is the indecomposable complex corresponding to this diagram. From the construction we get that $T := \bigoplus_{x \in A} T_x$ is a tilting complex: firstly, $T$ contains the required number of nonisomorphic direct summands, secondly, $T$ is partially tilting. If the diagrams corresponding to different vertices $x$ and $y$ do not intersect, use lemma 2, if they intersect then remark 2, if one of the vertices belong to $\Upsilon$, use lemma 8, if both belong to $\Upsilon$, use lemma 10. □

Remark 5. It is clear that similarly to the construction from proposition 7 we could construct a tilting complex $T$ such that $\text{End}_{K^b(A)}(T)$ is isomorphic to the Brauer star algebra and all summands of $T$ belong to an $A$-cycle $\text{End}_{K^b(A)}(T)$ corresponding to sinks.

Remark 6. Over a Brauer tree algebra associated to a Brauer tree with $n$ edges there are exactly $2(n+1)$ nonisomorphic basic two-term tilting complexes $T$ such that $\text{End}_{K^b(A)}(T)$ is isomorphic to a Brauer star algebra. Each of $n+1$ cycles of $A$ can generate a complex $T$ such that all its summands belong to $A$-cycle of $\text{End}_{K^b(A)}(T)$ corresponding to sinks or sources.

References

[1] R. Rouquier, A. Zimmermann, Picard groups for derived module categories, Proc. London Math. Soc. (3) 87 (2003), no. 1, 197–225.

[2] I. Muchtadi-Alamsyah, Braid action on derived category of Nakayama algebras, Comm. Algebra 36 (2008), no. 7, 2544–2569.

[3] M.Schaps and E. Zakay-Illouz, Braid group action on the refolded tilting complex of the Brauer star algebra, Proceedings ICRA IX (Beijing) vol.2 (2002), 434–449.

[4] H. Abe, M. Hoshino, On derived equivalences for selfinjective algebras, Comm. in Algebra 34 (2006), no. 12, 4441–4452.

[5] A.Chan, Two-term tilting complexes of Brauer star algebra and simple-minded systems, arXiv:1304.5223 (2013).
[6] T. Adachi, O. Iyama, I. Reiten, τ-tilting theory, to appear in Compos. Math., arXiv:1210.1036 (2012).

[7] M. Antipov, A. Zvonareva. Two-term partial tilting complexes over Brauer tree algebras, Problems in the theory of representations of algebras and groups, Part 24, Zap. Nauchn. Sem. POMI, 413 (2013), 5–25.

[8] J. Rickard, Derived categories and stable equivalence, J. Pure Appl. Algebra, 61 (1989), 303–317.

[9] P. Gabriel, C. Riedtmann, Group representations without groups, Comment. Math. Helv. 54 (1979), 240–287.

[10] D. Happel, Auslander-Reiten triangles in derived categories of finite-dimensional algebras, Proc. Amer. Math. Soc. 112 (1991), 641–648.

[11] D. Happel, Triangulated Categories in the Representation of Finite Dimensional Algebras, Cambridge University Press (1988).