COUNTING ACYCLIC AND STRONG DIGRAPHS BY DESCENTS

KASSIE ARCHER, IRA M. GESSEL, CHRISTINA GRAVES, AND XUMING LIANG

Abstract. A descent of a labeled digraph is a directed edge \((s, t)\) with \(s > t\). We count strong tournaments, strong digraphs, and acyclic digraphs by descents and edges. To count strong tournaments we use Eulerian generating functions and to count strong and acyclic digraphs we use a new type of generating function that we call a graphic Eulerian generating function.

1. Introduction

A digraph \(D\) consists of a finite vertex set \(V\) together with a subset \(E\) of \(V \times V - \{(v, v) : v \in V\}\), the set of edges of \(D\). (We do not allow loops in our digraphs.) We call a digraph with vertex set \(V\) a digraph on \(V\). We assume that the vertices of our digraph are totally ordered, and for simplicity we take them to be integers.

A descent of a digraph is an edge \((s, t)\) with \(s > t\) and an ascent is an edge \((s, t)\) with \(s < t\). In this paper we count two important classes of digraphs, acyclic and strong, by edges and descents, generalizing the results of Robinson [16]. We also count strong tournaments and forests by descents.

A digraph is weakly connected (or simply weak) if its underlying graph is connected, and is strongly connected (or simply strong) if for every two vertices \(u\) and \(v\) there is a directed path from \(u\) to \(v\) (allowing the empty path if \(u = v\)). The weak and strong components of a digraph are the maximal weakly or strongly connected subgraphs. Note that every edge is contained in a weak component but there may be edges not contained in any strong component. A source strong component of a digraph is a strong component with no edges entering it from outside the component. (Robinson [16, 17] calls these out-components.)

We use the notation \([n]\) to denote the set \(\{1, 2, \ldots, n\}\). Given a digraph \(D\) on \([n]\), we denote by \(e(D)\) the total number of edges of \(D\), and by \(\text{des}(D)\) the number of descents of \(D\). We define the descent polynomial for a family of digraphs \(\mathcal{B}_n\) on \([n]\) to be

\[
b_n(u) = \sum_{D \in \mathcal{B}_n} u^{\text{des}(D)}.
\]
The coefficient of $u^k$ in $b_n(u)$ is the number of digraphs in $B_n$ with exactly $k$ descents. Similarly, the descent-edge polynomial $b_n(u, y)$ of $B_n$ is defined as

$$b_n(u, y) = \sum_{D \in B_n} u^{\text{des}(D)} y^{\text{e}(D)}.$$ 

The coefficient of $u^k y^m$ in $b_n(u, y)$ is the number of digraphs in $B_n$ with exactly $m$ edges, $k$ of which are descents. Note that $b_n(u) = b_n(u, 1)$.

This paper is organized as follows. In Section 2, we introduce several families of graphs and give known formulas for enumerating these families. In Section 3, we enumerate strong tournaments by the number of descents and enumerate strong digraphs by both the number of descents and edges. In Section 4, we enumerate acyclic digraphs by the number of edges and descents; we also derive a formula for rooted trees and forests with a given number of descents and leaves.

2. Families of digraphs

We are concerned primarily with four types of digraphs: strong tournaments, strong digraphs, acyclic digraphs, and trees.

2.1. Strong tournaments. A digraph is a tournament if there is exactly one directed edge between each pair of vertices. There are $2\binom{n}{2}$ tournaments on $[n]$ since for any two vertices $u$ and $v$, a tournament contains the edge $(u, v)$ or $(v, u)$ but not both.

In [12], Moon and Moser found a formula for the probability that a randomly chosen tournament is strongly connected. Equivalently, they showed that the number $t_n$ of strong tournaments on $n \geq 1$ labelled vertices is given by the recurrence

$$t_n = 2\binom{n}{2} - \sum_{k=1}^{n-1} \binom{n}{k} 2^{\binom{n-k}{2}} t_k. \quad (1)$$

This recurrence is equivalent to the generating function relation

$$\sum_{n=1}^{\infty} t_n \frac{x^n}{n!} = 1 - \left( \sum_{n=0}^{\infty} 2\binom{n}{2} \frac{x^n}{n!} \right)^{-1}.$$ 

In Section 3.1, we generalize these formulas to count strong tournaments by descents, replacing the exponential generating functions with Eulerian generating functions.
2.2. **Strong digraphs.** Strong digraphs were first counted by Liskovets [11], using a system of recurrences. Liskovets’s recurrences were simplified by Wright [22], who showed that the number $s_n$ of strong digraphs on $[n]$ is given by

$$s_n = \eta_n + \sum_{k=1}^{n-1} \binom{n-1}{k-1} s_k \eta_{n-k},$$

(2)

where

$$\eta_n = 2^{n(n-1)} - \sum_{k=1}^{n-1} \binom{n}{k} 2^{(n-1)(n-k)} \eta_k.$$  

(3)

A more direct approach to counting strong digraphs was given by Robinson [16] (see also [17]) which will be discussed in more detail in Section 3.3, and is the basis for our approach to counting strong digraphs by edges and descents. We use a new kind of generating function that we call an Eulerian graphic generating function whose properties are introduced in Section 3.2.

De Panafieu and Dovgal [1] have also counted acyclic and strong digraphs using an approach similar to Robinson’s. A different approach to counting strong digraphs has been given by Ostroff [13].

2.3. **Acyclic digraphs.** An acyclic digraph is a digraph with no directed cycles. That is, there is no nonempty directed path from any vertex to itself. Robinson [16] showed that the number $a_n$ of acyclic digraphs on $[n]$ is given by the generating function

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{n! 2^{(n)}} = \left( \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n! 2^{(n)}} \right)^{-1}.$$  

(Robinson had earlier [15] found a different recurrence for counting acyclic digraphs that we will not discuss here.) Using an approach similar to Robinson’s, in Section 4.1 we generalize Robinson’s formula to derive an Eulerian graphic generating function for counting acyclic digraphs by edges and descents.

Another proof of Robinson’s formula was given by Stanley [19], using properties of chromatic polynomials of graphs. In Section 4.2 we generalize Stanley’s proof to count acyclic digraphs by edges and descents, using a generalization of the chromatic polynomial related to the chromatic quasisymmetric function of Shareshian and Wachs [18].

2.4. **Trees.** We consider a rooted tree to be an acyclic digraph in which one vertex (the root) has outdegree 0 and every other vertex has outdegree 1. It is well known that there are $n^{n-1}$ rooted trees on $n$ vertices. More relevant to our results is that the exponential generating function $\sum_{n=1}^{\infty} n^{n-1} x^n / n!$ for rooted trees is the compositional inverse of $xe^{-x}$. In Section 4.3 we use a variation of our first approach to
counting acyclic digraphs to give a simple formula for the compositional inverse of the exponential generating function for counting rooted trees by descents.

3. Strong Tournaments and Strong Digraphs

3.1. Strong Tournaments. We first study strong tournaments, which are easier to enumerate than acyclic and strong digraphs. Since every tournament on \( n \) vertices has exactly \( \binom{n}{2} \) edges, we count strong tournaments by descents only. First, let us determine the descent polynomial for all tournaments on \([n]\). For each pair of vertices \( \{s,t\} \), exactly one of \((s,t)\) and \((t,s)\) is an edge. One of these edges is a descent and the other is an ascent, and thus the descent polynomial for all tournaments on \([n]\) is \((1 + u)^{\binom{n}{2}}\).

To count strong tournaments by descents, we will need some properties of the \( q \)-binomial coefficients (also called Gaussian binomial coefficients). We first define the \( q \)-factorial \( n!_q \) by

\[
n!_q = 1 \cdot (1 + q) \cdots (1 + q + \cdots + q^{n-2}) \cdot (1 + q + \cdots + q^{n-1}).
\]

The \( q \)-binomial coefficients, denoted \( \binom{n}{i}_q \), are defined by

\[
\binom{n}{i}_q = \frac{n!_q}{i!_q (n-i)!_q}.
\]

For \( q = 1 \) they reduce to ordinary binomial coefficients.

The \( q \)-binomial coefficients have several combinatorial interpretations, but the one that we need is given in Lemma 1 below. For disjoint sets of integers \( S \) and \( T \), we call an element \((s,t)\) of \( S \times T \) a descent if \( s > t \) and an ascent if \( s < t \). Let \( \text{des}(S,T) \) denote the number of descents in \( S \times T \).

Lemma 1. For nonnegative integers \( n \) and \( i \), with \( i \leq n \), we have

\[
\binom{n}{i}_q = \sum_{(S,T)} q^{\text{des}(S,T)},
\]

where the sum is over all ordered partitions \((S,T)\) of \([n]\) for which \( |S| = i \).

This lemma is proved in [4, Lemma 5.1] by showing that the right side satisfies the same recurrence as the left side, \( \binom{n}{i}_q = q^i \binom{n-1}{i}_q + \binom{n-1}{i-1}_q \). It can also be derived easily from other well-known combinatorial interpretations for the \( q \)-binomial coefficients such as [20, p. 56, Proposition 1.7.1].

We can now give a recurrence for the descent polynomial for strong tournaments on \([n]\). (Although the variable \( q \) is traditionally used in \( \binom{n}{i}_q \), we will replace \( q \) with \( u \), as we are using the variable \( u \) to weight descents.) Notice that setting \( u = 1 \) gives the recurrence [1] of Moon and Moser.
Theorem 2. Let \( t_n(u) \) be the descent polynomial for the set of strong tournaments on \([n]\). Then for \( n \geq 1 \) we have

\[
t_n(u) = (1 + u)^\binom{n}{2} - \sum_{k=1}^{n-1} \binom{n}{k} u^{\binom{n-k}{2}} t_k(u). \tag{4}
\]

Proof. Every nonempty tournament has a unique source strong component. Thus every tournament on \([n]\), for \( n \geq 1 \), can be constructed uniquely by choosing an ordered partition \((S, T)\) of \([n]\), with \( S \) nonempty, then constructing a strong tournament on \( S \) and an arbitrary tournament on \( T \), and adding all edges in \( S \times T \). By Lemma 1, the contribution to the descent polynomial for all tournaments on \([n]\) with \(|S| = k\) is \( \binom{n}{k} u^{\binom{n-k}{2}} t_k(u) \). Thus

\[
(1 + u)^\binom{n}{2} = \sum_{k=1}^{n} \binom{n}{k} u^{\binom{n-k}{2}} t_k(u). \tag{5}
\]

Solving for \( t_n(u) \) gives (4). \(\square\)

The first few values of the polynomials \( t_n(u) \) are

\[
t_1(u) = 1, \quad t_2(u) = 0, \quad t_3(u) = u + u^2, \quad \text{and} \quad t_4(u) = u + 6u^2 + 10u^3 + 6u^4 + u^5.
\]

Coefficients of \( t_n(u) \) for larger \( n \) can be easily computed from (4) and are given in Table 1. It is not difficult to show that \( t_n(u) \) is a polynomial of degree \( \binom{n}{2} - 1 \) for \( n \geq 3 \). Also, since reversing all the edges of a strong tournament gives another strong tournament, \( t_n(u) \) is symmetric; i.e., \( t_n(u) = u^{\binom{n}{2}} t_n(1/u) \).

The next result gives a divisibility property for \( t_n(u) \).

Proposition 3. The polynomial \( t_n(u) \) is divisible by \((1 + u)^{|n/2|}\).

Proof. Let \( v_n(u) = t_n(u)/(1 + u)^{|n/2|} \). Then from (4) we obtain the recurrence

\[
v_n(u) = (1 + u)^{\binom{n}{2} - |n/2|} - \sum_{k=1}^{n-1} \binom{n}{k} u^{\binom{n-k}{2} - |n/2| + |k/2|} v_k(u)
\]

for \( n \geq 1 \). It is easy to check that \( \binom{n}{2} - |n/2| \geq 0 \), so it suffices to show that the expression multiplied by \( v_k(u) \) in the sum on the right is a polynomial in \( u \).

Let

\[
E(n, k) = \left( \frac{n-k}{2} \right) - \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{k}{2} \right\rceil.
\]
We first show that $E(n, k) \geq 0$ for $k < n - 1$. Note that

$$E(n, k) \geq \frac{n - k}{2} - \frac{n}{2} + \frac{k - 1}{2}$$

$$= \frac{(n - k)(n - k - 2) - 1}{2}.$$ 

For $k \leq n - 2$, this gives $E(n, k) \geq -1/2$, so since $E(n, k)$ is an integer, we have $E(n, k) \geq 0$. In the case where $k = n - 1$, we must consider the parity of $n$. If $n$ is odd, then $E(n, n - 1) = 0$. However, if $n$ is even, $E(n, n - 1) = -1$. To complete the proof it suffices to show that if $n$ is even then \binom{n}{n-1}_u$ is divisible by $1 + u$. But if
n is even then
\[
\binom{n}{n-1}_u = 1 + u + u^2 + \cdots + u^{n-1} = (1 + u)(1 + u^2 + u^4 + \cdots + u^{n-2}),
\]
thus completing the proof. \(\square\)

The recurrence of Theorem 2 can also be expressed with generating functions. An Eulerian generating function is a generating function of the form
\[
\sum_{n=0}^{\infty} a_n \frac{x^n}{n!_q},
\]
Note that for \(q = 1\), this reduces to an exponential generating function.

The multiplication of Eulerian generating functions is similar to that of exponential generating functions. If
\[
a(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!_q} \quad \text{and} \quad b(x) = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!_q},
\]
then
\[
a(x)b(x) = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!_q},
\]
where the coefficient \(c_n\) is given by
\[
c_n = \sum_{i=0}^{n} \binom{n}{i}_q a_ib_{n-i}.
\]

The generating function for \(t_n(u)\) can be derived directly from the formula given in Theorem 2 and the multiplication property of Eulerian generating functions:

**Corollary 4.** Let \(T(x) = \sum_{n=1}^{\infty} t_n(u)x^n/n!_u\) be the Eulerian generating function for strong tournaments by descents and let \(U(x) = \sum_{n=0}^{\infty} (1+u)^\binom{n}{2} x^n/n!_u\) be the Eulerian generating function for all tournaments by descents. Then
\[
T(x) = 1 - U(x)^{-1}
\]
and
\[
U(x) = \frac{1}{1 - T(x)}. \tag{6}
\]

**Proof.** Equation (5) is equivalent to \(U(x) = 1 + T(x)U(x)\) from which the two formulas follow easily. \(\square\)
Equation (6) has a simple combinatorial interpretation obtained by iterating the decomposition described in the proof of Theorem 2: every tournament may be decomposed into a sequence of strong tournaments with all edges between the strong tournaments oriented from left to right.

3.2. Eulerian Graphic Generating Functions. In this section, we introduce a new type of generating function which will be useful in enumerating both acyclic and strong digraphs by descents and edges. This new generating function is a generalization of a graphic generating function (also called a special generating function), which is a generating function of the form

\[ \sum_{n=0}^{\infty} a_n x^n n! \left( 1 + y \right)^{(n)}_2 \],

often with \( y = 1 \). Graphic generating functions were first used by Robinson [16] and by Read [14] (in the case \( y = 1 \), with a slightly different normalization); further applications of graphic generating functions have been given by Gessel and Sagan [9], Gessel [6], and de Panafieu and Dovgal [1].

We define an Eulerian graphic generating function to be a generating function of the form

\[ \sum_{n=0}^{\infty} a_n x^n \frac{x^n}{n!q(1+y)^{(n)}_2} \].

(7)

Given two Eulerian graphic generating functions \( a(x) \) and \( b(x) \) defined by

\[ a(x) = \sum_{n=0}^{\infty} a_n x^n n!q(1+y)^{(n)}_2 \] and \( b(x) = \sum_{n=0}^{\infty} b_n x^n n!q(1+y)^{(n)}_2 \),

we multiply them to obtain

\[ a(x)b(x) = \sum_{n=0}^{\infty} c_n x^n n!q(1+y)^{(n)}_2 \]

where

\[ c_n = \sum_{i=0}^{n} \binom{n}{i}_q (1+y)^{(i-n)}_2 a_i b_{n-i}. \]

In all of our formulas from here on we will modify the Eulerian graphic generating functions by taking \( q = (1+uy)/(1+y) \). The combinatorial interpretation of these modified Eulerian graphic generating functions is explained by the following lemma.

Lemma 5. Let \( q = (1+uy)/(1+y) \). Then \( \binom{n}{i}_q (1+y)^{(i-n)}_2 \) is a polynomial in \( u \) and \( y \), and the coefficient of \( u^i y^m \) in \( \binom{n}{i}_q (1+y)^{(i-n)}_2 \) is the number of ordered pairs
(S, A) where S is an i-subset of [n] and A is an m-subset of S × ([n] − S) containing exactly j descents.

Proof. Let S be an i-subset of [n] such that S × ([n] − S) has k descents, and thus i(n − i) − k ordered pairs that are not descents. Define the weight of a subset A ⊆ S × ([n] − S) to be $u^{\text{des}(A)}y^{|A|}$. To count such weighted subsets of S × ([n] − S) we specify A by deciding which descents and ascents of S × ([n] − S) are included in A. Each descent in S × ([n] − S) can either be included in A, contributing a factor of uy to the weight of A, or excluded, contributing a factor of 1. Similarly, each ascent in S × ([n] − S) can either be included in A, contributing a factor of y, or excluded, contributing a factor of 1. Thus the sum of the weights of all A ⊆ S × ([n] − S) is $(1 + uy)^k(1 + y)^{i(n-i) - k}$.

Now define $Q_{n,i,k}$ by

$$\binom{n}{i} = \sum_{k=0}^{i(n-i)} Q_{n,i,k} q^k.$$ 

Then by Lemma 1, $Q_{n,i,k}$ is the number of i-subsets S of [n] such that S × ([n] − S) has k descents. Thus the sum over all i-subsets S ⊆ [n] of the weights of all A ⊆ S × ([n] − S) is

$$ \sum_{k=0}^{i(n-i)} Q_{n,i,k} (1 + uy)^k(1 + y)^{i(n-i) - k} = (1 + y)^i(1 + uy)^k \sum_{k=0}^{i(n-i)} Q_{n,i,k} \left( \frac{1 + uy}{1 + y} \right)^k$$

$$= (1 + y)^i \binom{n}{i}_q,$$

where $q = (1 + uy)/(1 + y)$.

The modified Eulerian graphic generating functions may be viewed another way. Note that

$$n!q(1 + y)^2 = \prod_{i=1}^{n} (1 + q + \cdots + q^{i-1})(1 + y)^{i-1}.$$ 

Setting $q = (1 + uy)/(1 + y)$ and letting $P(i)$ denote the ith factor in this product gives

$$P(i) := (1 + y)^{i-1} + (1 + uy)(1 + y)^{i-2} + (1 + uy)^2(1 + y)^{i-3} + \cdots + (1 + uy)^{i-1}$$

$$= \frac{(1 + y)^i - (1 + uy)^i}{y(1 - u)}.$$ 

So if we let $F(n) = P(1)P(2)\cdots P(n)$, then the modified Eulerian graphic generating functions are of the form $\sum_{n=0}^{\infty} a_n x^n / F(n)$. 

Note that if we set \( u = 1 \), then \( q = (1 + uy)/(1 + y) \) becomes 1. So in this case the Eulerian graphic generating function (7) reduces to the “ordinary” graphic generating function
\[
\sum_{n=0}^{\infty} a_n \frac{x^n}{n! (1 + y)^{\binom{n}{2}}}.
\]

3.3. Strong Digraphs. We now find a generating function for the descent-edge polynomial for the set of strong digraphs. Before embarking on the proof, we need several preliminary definitions. Let \( \Delta \) be the transformation that converts an exponential generating function to an Eulerian graphic generating function. That is,
\[
\Delta \left( \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} \right) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n! (1 + y)^{\binom{n}{2}}}.
\]

Let \( D(x) \) be the Eulerian graphic generating function for the descent-edge polynomials of all digraphs. In any digraph on \([n]\), for each pair of vertices \( s < t \), the descent \((t, s)\) is either included as an edge or it is not, and the ascent \((s, t)\) is either included or it is not. Thus, the descent-edge polynomial of the set of all digraphs on \([n]\) is given by \((1 + uy)^{\binom{n}{2}}(1 + y)^{\binom{n}{2}}\) and \( D(x) \) is therefore
\[
D(x) = \sum_{n=0}^{\infty} (1 + uy)^{\binom{n}{2}}(1 + y)^{\binom{n}{2}} \frac{x^n}{n! q (1 + y)^{\binom{n}{2}}} = \sum_{n=0}^{\infty} (1 + uy)^{\binom{n}{2}} \frac{x^n}{n! q}.
\]

Let
\[
\Delta \left( \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} \right) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n! (1 + y)^{\binom{n}{2}}}.
\]

Then by the “exponential formula” [21, p. 5, Corollary 5.1.6], \( v_n(u, y; \alpha) \) is the descent-edge polynomial for digraphs on \( n \) in which every weak component is strong.
where each weak component is weighted $\alpha$. The Eulerian graphic generating function for the polynomials $v_n(u, y; \alpha)$ is thus $\Delta(e^{\alpha S(x)})$. We can now count strong digraphs by edges and descents, generalizing the result of Robinson who proved the case $u = 1$ (and thus $q = 1$) of the next result.

**Theorem 6.** Let $S(x)$ be the exponential generating function for the descent-edge polynomial for strong digraphs and let $D(x)$ be the Eulerian graphic generating function for all digraphs, given in (8). Then

$$S(x) = -\log \left( \Delta^{-1}(D(x)^{-1}) \right).$$

(11)

**Proof.** We will count in two ways ordered pairs $(D, C)$ where $D$ is a digraph on $[n]$ and $C$ is a subset of the set of source strong components of $D$. (We may identify $C$ with the digraph whose weakly connected components are the elements of the set $C$.) To such a pair we assign the weight $\alpha^{|C|} u^{\text{des}(D)} y^{\text{e}(D)}$. We compute the sum of the weights of these pairs in two ways.

First, we may choose $D$ as an arbitrary digraph on $[n]$ and then choose $C$ as an arbitrary subset of the source strong components of $D$. Thus the sum of the weights is $d_n(u, y; \alpha + 1)$.

Alternatively, we may count pairs $(D, C)$ by first choosing a subset $T$ of $[n]$, constructing a set of strong digraphs $C$ on $T$, choosing a digraph $D'$ on $[n] - T$ and choosing a subset $E$ of $T \times ([n] - T)$. We then construct $D$ by adding to $D'$ the digraphs in $C$ together with the elements of $E$ as edges. Then $\text{des}(D) = \text{des}(E) + \text{des}(C) + \text{des}(D')$ and $\text{e}(D) = |E| + \text{e}(C) + \text{e}(D')$. It follows from Lemma 5 that the sum of the weights of the pairs $(D, C)$ in which $C$ has a total of $i$ vertices is $(1 + y)^{i(n-i)} \binom{n}{i} q v_i(u, y; \alpha) d_{n-i}(u, y; 1)$. Summing over $i$ and using (9) gives

$$\sum_{i=0}^{n} (1 + y)^{i(n-i)} \binom{n}{i} v_i(u, y; \alpha) d_{n-i}(u, y; 1) = d_n(u, y; \alpha + 1),$$

(12)

which is equivalent by (10) to

$$\Delta(e^{\alpha S(x)}) D(x) = \sum_{n=0}^{\infty} d_n(u, y; \alpha + 1) \frac{x^n}{n! (1 + y)^{\binom{n}{2}}},$$

where $D(x)$ is given by (8). Now we set $\alpha = -1$. Since $d_n(u, y; 0) = 0$ for $n > 0$ we obtain

$$\Delta(e^{-S(x)}) D(x) = 1.$$

Solving for $S(x)$ yields (11).

We can now give extensions of Wright’s recurrences (2) and (3) for the polynomials $s_n(u, y)$. 


Corollary 7. The descent-edge polynomial for strong digraphs on \( n \) vertices \( s_n(u, y) \) satisfies the recurrence

\[
s_n(u, y) = \eta_n(u, y) + \sum_{k=1}^{n-1} \binom{n-1}{k-1} s_k(u, y) \eta_{n-k}(u, y), \quad n \geq 1, \tag{13}
\]

where the polynomials \( \eta_n(u, y) \) are determined by

\[
\eta_n(u, y) = (1 + y)\binom{n}{2}(1 + uy)\binom{n}{2} - \sum_{k=1}^{n-1} \binom{n}{k} q (1 + uy)^{(n-k)}(1 + y)^{(n-k)(n+k-1)/2} \eta_k(u, y), \tag{14}
\]

with \( q = (1 + uy)/(1 + y) \).

Proof. Let \( E(x) = 1 - e^{-S(x)} \), so

\[
S(x) = \log \frac{1}{1 - E(x)}, \tag{15}
\]

and define polynomials \( \eta_n(u, y) \) by \( E(x) = \sum_{n=1}^{\infty} \eta_n(u, y)x^n/n! \). Thus for \( n \geq 1 \), \( \eta_n(u, y) = -v_n(u, y; -1) \), where \( v_n(u, y; \alpha) \) is defined in (10). Then (14) is obtained by rearranging the case \( \alpha = -1 \) of (12), using (9) and \( d_n(u, y; 0) = 0 \) for \( n > 0 \).

Differentiating (15) with respect to \( x \) and simplifying gives

\[
S'(x) = E'(x) + S'(x)E(x).
\]

Equating coefficients of \( x^{n-1}/(n-1)! \) gives (13). \( \square \)

The first few values of the polynomials \( s_n(u, y) \) are \( s_1(u, y) = 1 \), \( s_2(u, y) = uy^2 \), and

\[
s_3(u, y) = uy^3 + u^2y^3 + uy^4 + 7u^2y^4 + u^3y^4 + 3u^2y^5 + 3u^3y^5 + u^3y^6.
\]

The values of \( s_n(u, y) \) for larger \( n \) can be easily computed from the recurrences of Corollary 7. We provide the values of \( s_n(u, 1) \) and \( s_n(1, y) \) for small values in Tables 2A and 2B.

Note that setting \( u = y = 1 \) in the recurrences of Corollary 7 gives Wright’s recurrences (2) and (3), so Wright’s \( \eta_n \) is our \( \eta_n(1, 1) \). In fact, Wright also knew the corresponding recurrences for \( s_n(1, y) \) and \( \eta_n(1, y) \), counting strong digraphs by edges. He also stated that (2) and (3) look as if they should possess combinatorial interpretations, but he was not able to find one. He wrote, “We can show that \( \eta_n \) is non-negative, though \( \eta_2 = 0 \). But some of the coefficients in the polynomials \( \eta_n(y) \) are negative and this makes it seem somewhat unlikely that \( \eta_n \) has a simple combinatorial meaning.” (Wright’s \( \eta_n(y) \) is our \( \eta_n(1, y) \).)
**Table 2. Number of strong digraphs by descents and edges**

| \(d\) | \(n\) | 3  | 4  | 5  | 6       | \(e\) | 3  | 4  | 5       |
|---------|-------|----|----|----|---------|-------|----|----|---------|
| 1       | 2     | 10 | 122|    | 3,346   | 3     | 2  | 0  | 0       |
| 2       | 11    | 154| 3,418| 142,760 | 4     | 9  | 6  | 0       |
| 3       | 5     | 540| 27,304| 1,938,178| 5     | 6  | 84 | 24      |
| 4       | 0     | 581| 90,277| 12,186,976| 6     | 1  | 316| 720     |
| 5       | 0     | 272| 150,948| 42,696,630| 7     | 0  | 492| 6,440   |
| 6       | 0     | 49 | 150,519| 94,605,036| 8     | 0  | 417| 26,875  |
| 7       | 0     | 0  | 95,088| 145,009,210| 9     | 0  | 212| 65,280  |
| 8       | 0     | 0  | 37,797| 161,845,163| 10    | 0  | 66 | 105,566 |
| 9       | 0     | 0  | 8,714| 134,933,733| 11    | 0  | 12 | 122,580 |
| 10      | 0     | 0  | 893  | 84,656,743 | 12    | 0  | 1  | 106,825 |
| 11      | 0     | 0  | 0    | 39,632,149 | 13    | 0  | 0  | 71,700  |
| 12      | 0     | 0  | 0    | 13,481,441 | 14    | 0  | 0  | 37,540  |
| 13      | 0     | 0  | 0    | 3,156,845 | 15    | 0  | 0  | 15,344  |
| 14      | 0     | 0  | 0    | 455,917  | 16    | 0  | 0  | 4,835   |
| 15      | 0     | 0  | 0    | 30,649   | 17    | 0  | 0  | 1,140   |

**TOTAL** | 18 | 1,606 | 565,080 | 734,774,776 | 18 | 0  | 0  | 190     |

Despite Wright’s pessimism, \(\eta_n\) does have a simple combinatorial interpretation, which suggests a connection between the enumeration of strong tournaments and the enumeration of strong digraphs. If we multiply Moon and Moser’s recurrence (14) for strong tournaments by \(2^{(n)}\), we get

\[
2^{(n)}t_n = 2^{n(n-1)} - \sum_{k=1}^{n-1} \binom{n}{k} 2^{(n-1)(n-k)} \cdot 2^{(k)}t_k.
\]

Comparing with (13), we see that Wright’s \(\eta_n\) is equal to our \(2^{(n)}t_n\).

Thus (15) for \(u = y = 1\) may be written

\[
\sum_{n=1}^{\infty} s_n \frac{x^n}{n!} = -\log \left(1 - \sum_{n=1}^{\infty} 2^{(n)}t_n \frac{x^n}{n!} \right).
\]
Although the coefficients of \( \eta_n(u, y) \) are not in general nonnegative, we can derive a one-parameter refinement of the formula \( \eta_n = 2\binom{n}{2} t_n \) from Theorems 2 and 6. Note that \( \eta_2 = \eta_2(1, 1) = 0 \) and \( \eta_2(u, y) = -1 + uy^2 \). This suggests that if we want a specialization of \( \eta_n(u, y) \) with nonnegative coefficients, we might try setting \( u = y^{-2} \).

**Proposition 8.** The polynomials \( \eta_n(u, y) \) defined by (14) and the descent polynomials for strong tournaments by descents \( t_n(u) \), determined by (4), are related by

\[
\eta_n(y^{-2}, y) = (1 + y)^{\binom{n}{2}} (1 + y^{-1})^{\binom{n}{2}} 
- \sum_{k=1}^{n-1} \binom{n}{k} \frac{(1 + y^{-1})^{\binom{n-k}{2}}}{y^{-1}} \left( 1 + y \right)^{\binom{n-k}{2}} t_k(y^{-1}).
\]

Setting \( u = y^{-1} \) in (11), multiplying by \( (1 + y)^{\binom{n}{2}} \), and simplifying gives

\[
(1 + y)^{\binom{n}{2}} t_n(y^{-1}) = (1 + y)^{\binom{n}{2}} (1 + y^{-1})^{\binom{n}{2}} 
- \sum_{k=1}^{n-1} \binom{n}{k} (1 + y)^{\binom{n}{2}} (1 + y^{-1})^{\binom{n-k}{2}} \frac{k}{y^{-1}} \cdot (1 + y)^{\binom{k}{2}} t_k(y^{-1}).
\]

Then (17) follows from the fact that \( (1 + y)^{(n-k)(n+k-1)/2} = (1 + y)^{\binom{n}{2}} (1 + y)^{-\binom{k}{2}} \). □

Applying (17) to (15) gives

\[
\sum_{n=0}^{\infty} s_n(y^{-2}, y) \frac{x^n}{n!} = -\log \left( 1 - \sum_{n=1}^{\infty} (1 + y)^{\binom{n}{2}} t_n(y^{-1}) \frac{x^n}{n!} \right).
\]

If two exponential generating functions \( f \) and \( g \) are related by \( f = -\log(1 - g) = \sum_{n=1}^{\infty} (n-1)! g^n/n! \) then \( f \) may be interpreted as counting cycles of the objects counted by \( g \), so (16) and (18) might be explained combinatorially by a bijection from strong digraphs to cycles of strong tournaments with some additional structure. But we have not been able to find such a bijection.

### 4. Acyclic Digraphs and Trees

#### 4.1. Acyclic Digraphs

We begin this section by enumerating acyclic digraphs by their number of edges, descents, and sources. We again make use of Eulerian graphic
generating functions, and also follow closely the proof in [3] for enumeration of acyclic
digraphs by sources and edge (which is based on Robinson’s proof [16]).

Let
\[ a_n(u, y; \alpha) = \sum_D u^{\text{des}(D)} y^{e(D)} \alpha^{s(D)}, \]
where the sum is over all acyclic digraphs \( D \) on the vertex set \([n]\), \( e(D) \) is the number
of edges of \( D \), and \( s(D) \) is the number of sources of \( D \); that is, the number of vertices
of \( D \) of in-degree 0. Let \( a_n(u, y) = a_n(u, y; 1) \).

To count acyclic digraphs by sources we take an acyclic digraph and add some
new vertices as sources. The new vertices will be a subset of the set of sources of
the expanded digraph. This gives a formula expressing \( a_n(u, y; \alpha + 1) \) in terms of
\( a_j(u, y) \) for \( j \leq n \). Since every nonempty acyclic digraph has at least one source, the
formula for \( a_n(u, y; 0) \) gives a recurrence for \( a_n(u, y) \).

Lemma 9. For every nonnegative integer \( n \) we have
\[ \sum_{i=0}^{n} \binom{n}{i} (1 + y)^{i(n-i)} \alpha^i a_{n-i}(u, y) = a_n(u, y; \alpha + 1). \]

Proof. We count ordered pairs \((D, C)\), where \( D \) is an acyclic digraph on \([n]\) and \( C \) is a
subset of the set of sources of \( D \). To such a pair we assign the weight \( u^{\text{des}(D)} y^{e(D)} \alpha^{|C|} \).
We compute the sum of the weights of these pairs in two ways.

First, we may choose \( D \) as an acyclic digraph on \([n]\) and then choose \( C \) as an
arbitrary subset of the sources of \( D \). Thus the sum of the weights is \( a_n(u, y; \alpha + 1) \).

We may also count pairs \((D, C)\) by first choosing a subset \( C \) of \([n]\), choosing a
digraph \( D' \) on \([n] - C \) and choosing a subset \( E \) of \( C \times ([n] - C) \). We then
construct \( D \) by adding to \( D' \) the elements of \( C \) as vertices and the elements of \( E \) as
edges. Then \( \text{des}(D) = \text{des}(E) + \text{des}(D') \) and \( e(D) = |E| + e(D') \). Then it follows
from Lemma 5 that the sum of the weights of the pairs \((D, C)\) in which \(|C| = i \) is
\( \left(\binom{n}{i} (1 + y)^{i(n-i)} \alpha^i a_{n-i}(u, y)\right) \), and summing on \( i \) gives the left side of (19). \( \square \)

Theorem 10. Let \( a_n(u, y) \) be the descent-edge polynomial for the set of acyclic
digraphs on \( n \) vertices and let \( A(x) \) be the Eulerian graphic generating function for
\( a_n(u, y) \) where \( q = (1 + uy)/(1 + y) \). Then
\[ A(x) = \left( \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n! (1 + y)^{\binom{n}{2}}} \right)^{-1}. \]

More generally, the Eulerian graphic generating function for \( a_n(u, y; \alpha) \) is
\[ \left( \sum_{n=0}^{\infty} (\alpha - 1)^n \frac{x^n}{n! (1 + y)^{\binom{n}{2}}} \right) / \left( \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n! (1 + y)^{\binom{n}{2}}} \right). \]
Proof. Equation (19) is equivalent to
$$\sum_{n=0}^{\infty} a_n(u, y; \alpha + 1) \frac{x^n}{n!q(1 + y)^{\binom{n}{2}}} = \left( \sum_{n=0}^{\infty} \frac{\alpha^n}{n!q(1 + y)^{\binom{n}{2}}} \right) A(x).$$
(22)

Setting \(\alpha = -1\) in (22), and using the fact that \(a_n(u, y; 0) = 0\) for \(n > 0\) gives (20). Then replacing \(\alpha\) by \(\alpha - 1\) in (22) and applying (20) gives (21). \(\square\)

An interesting special case of (21) is obtained by setting \(u = 0\), so that we are counting (acyclic) digraphs with no descents by the number of sources. We find that
$$a_n(0, y; \alpha) = \prod_{i=0}^{n-1} (\alpha + (1 + y)^i - 1).$$
This is not difficult to prove directly.

From either (19) or (20) we obtain a recurrence for \(a_n(u, y)\):

Corollary 11. Let \(a_n(u, y)\) be the descent-edge polynomial for the set of acyclic digraphs on \(n\) vertices. Then
$$a_n(u, y) = \sum_{i=0}^{n-1} (-1)^{n-i-1} \binom{n}{i}_q (1 + y)^{i(n-i)} a_i(u, y)$$
where \(q = (1 + uy)/(1 + y)\). \(\square\)

The polynomials \(a_n(u, y)\) for the first few values of \(n\) are given by \(a_1(u, y) = 1, a_2(u, y) = 1 + y + uy,\) and
$$a_3(u, y) = 1 + (3 + 3u)y + (3 + 6u + 3u^2)y^2 + (1 + 2u + 2u^2 + u^3)y^3.$$
Coefficients of \(a_n(u, y)\) for larger \(n\) can be computed from the formula in Corollary 11, we provide the values of \(a_n(u, 1)\) for small values of \(n\) in Table 3.

Robinson [16, 17] gave a common generalization of the case \(u = 1\) of Theorems 10 and 6. Given a class \(\mathcal{S}\) of strong digraphs he found a relation between the generating function for digraphs in \(\mathcal{S}\) and the generating function for digraphs all of whose strong components are in \(\mathcal{S}\). If \(\mathcal{S}\) is the class of all strong digraphs we get the case \(u = 1\) of Theorem 6 and if \(\mathcal{S}\) is the class of 1-vertex graphs we get the case \(u = 1\) of Theorem 10. Our approach could be applied to extend Robinson’s generalization to include descents.

4.2. Chromatic polynomials. Stanley [19] derived the generating function for acyclic digraphs from his theorem [19, Corollary 1.3] that if \(\chi_G(\lambda)\) is the chromatic polynomial of a graph \(G\) with \(n\) vertices, then the number of acyclic orientations of \(G\) is \((-1)^n\chi_G(-1)\). We can use his approach to derive in another way the Eulerian
| \(n\) | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|------|---|---|---|---|---|---|---|
| 0    | 1 | 2 | 8 | 64 | 1,024 | 32,768 | 2,097,152 |
| 1    | 0 | 1 | 11 | 161 | 3,927 | 172,665 | 14,208,231 |
| 2    | 0 | 0 | 5 | 167 | 6,698 | 419,364 | 45,263,175 |
| 3    | 0 | 0 | 1 | 102 | 7,185 | 656,733 | 94,040,848 |
| 4    | 0 | 0 | 0 | 39 | 5,477 | 757,939 | 145,990,526 |
| 5    | 0 | 0 | 0 | 9 | 3,107 | 686,425 | 181,444,276 |
| 6    | 0 | 0 | 0 | 1 | 1,329 | 504,084 | 187,742,937 |
| 7    | 0 | 0 | 0 | 0 | 423 | 305,207 | 165,596,535 |
| 8    | 0 | 0 | 0 | 0 | 96 | 153,333 | 126,344,492 |
| 9    | 0 | 0 | 0 | 0 | 14 | 63,789 | 84,115,442 |
| 10   | 0 | 0 | 0 | 0 | 1 | 21,752 | 49,085,984 |
| 11   | 0 | 0 | 0 | 0 | 0 | 5,959 | 25,134,230 |
| 12   | 0 | 0 | 0 | 0 | 0 | 1,267 | 11,270,307 |
| 13   | 0 | 0 | 0 | 0 | 0 | 197 | 4,403,313 |
| 14   | 0 | 0 | 0 | 0 | 0 | 20 | 1,486,423 |
| 15   | 0 | 0 | 0 | 0 | 0 | 1 | 428,139 |
| 16   | 0 | 0 | 0 | 0 | 0 | 0 | 103,345 |
| 17   | 0 | 0 | 0 | 0 | 0 | 0 | 20,367 |
| 18   | 0 | 0 | 0 | 0 | 0 | 0 | 3,153 |
| 19   | 0 | 0 | 0 | 0 | 0 | 0 | 360 |
| 20   | 0 | 0 | 0 | 0 | 0 | 0 | 27 |
| 21   | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| TOTAL| 1 | 3 | 25 | 543 | 29,281 | 3,781,503 | 1,138,779,265 |

Table 3. Values of \(a_n(u,1)\), the number of acyclic digraphs on \(n\) vertices with \(u\) descents, for \(n \leq 8\). The total is the number of labelled acyclic digraphs on \(n\) vertices.

graphic generating function (20) for acyclic digraphs by edges and descents, using an interesting generalization of the chromatic polynomial.

Let us first sketch Stanley’s approach. By applying the combinatorial interpretation of multiplication of graphic generating functions, we can show that for \(\lambda\) a nonnegative integer, the coefficient of \(x^n / (1 + y)^\binom{n}{2} n!\) in

\[
\left( \sum_{n=0}^{\infty} \frac{x^n}{n! (1 + y)^\binom{n}{2}} \right)^\lambda
\]  

(23)
counts ordered partitions of \([n]\) into \(\lambda\) blocks (possibly empty), together with an arbitrary set of (undirected) edges between elements of different blocks, where each edge has weight \(y\). If we think of a vertex in the \(i\)th block as colored in color \(i\), then we may describe these objects as \(\lambda\)-colored graphs, i.e., graphs in which the vertices are colored using colors chosen from \(\{1, 2, \ldots, \lambda\}\) so that adjacent vertices have different colors, as shown (for \(y = 1\)) by Read \[14\]. Thus for \(\lambda\) a nonnegative integer, the coefficient of \(x^n/(1 + y)^{\binom{n}{2}}n!\) in (23) is the sum over all graphs \(G\) on \([n]\) of \(y^{e(G)}\chi_G(\lambda)\), where \(e(G)\) is the number of edges of \(G\). But if two polynomials in \(\lambda\) are equal whenever \(\lambda\) is a nonnegative integer, they are equal as polynomials in \(\lambda\), and in particular, they are equal for \(\lambda = -1\). Applying Stanley’s theorem on acyclic orientations, we find that setting \(\lambda = -1\) in (23), and replacing \(x\) with \(-x\), gives the graphic generating function for acyclic digraphs. (Stanley only considered the case \(y = 1\) but the extension to counting edges is straightforward.)

To include descents, we consider a generalization of the chromatic polynomial. Let \(G\) be a graph with totally ordered vertices. We define a descent of a proper coloring \(c\) of \(G\) to be an edge \(\{i, j\}\) of \(G\) with \(i < j\) and \(c(i) > c(j)\). We may define the refined chromatic polynomial \(X_G(\lambda)\) to be \(\sum_c u^{\text{des}(c)}\) where the sum is over all proper colorings \(c\) of \(G\) with colors chosen from \(\{1, 2, \ldots, \lambda\}\), and \(\text{des}(c)\) is the number of descents of the coloring \(c\). (It is not hard to show that \(X_G(\lambda)\) is indeed a polynomial in \(\lambda\); this follows from the proof of Theorem 12 below.) We note that \(X_G(\lambda)\) is a specialization of the chromatic quasisymmetric function introduced by Shareshian and Wachs \[18\].

Then we have the following analogue of Stanley’s theorem on acyclic orientations.

**Theorem 12.** Let \(G\) be a graph on a totally ordered \(n\)-element vertex set and let \(X_G(\lambda)\) be the refined chromatic polynomial of \(G\). Then

\[
X_G(-1) = (-1)^n \sum_O u^{\text{des}(O)}
\]

where the sum is over all acyclic orientations \(O\) of \(G\).

**Proof sketch.** We follow closely Stanley’s second proof in \[19\] of the case \(u = 1\) to which we refer for definitions not given here. To each proper coloring \(c\) of \(G\), we associate an acyclic orientation \(O_c\) of \(G\) in which each edge is directed from the lower-colored endpoint to the higher-colored endpoint. Then the coloring \(c\) and the acyclic orientation \(O_c\) have the same number of descents. As in Stanley’s proof, the number of proper colorings associated with a given acyclic orientation \(O\) is the strict order polynomial \(\Omega(O, \lambda)\), where \(\bar{O}\) is the transitive and reflexive closure of \(O\), regarded as a binary relation on the vertex set. Thus

\[
X_G(\lambda) = \sum_O u^{\text{des}(O)}\Omega(O, \lambda),
\]
where the sum is over all acyclic orientations of $G$. It is known, as a special case of Stanley’s reciprocity theorem for order polynomials, that for every acyclic orientation $O$, we have $\Omega(\bar{O}, -1) = (-1)^n$ and the result follows.

We can now give another proof of (20), counting acyclic digraphs by descents and edges. Using Lemma 5, we can show by induction on $\lambda$ a nonnegative integer, the coefficient of $x^n/n!q(1+y)(n/2)$ in

$$\left(\sum_{n=0}^{\infty} x^n \frac{n!q(1+y)^{\binom{n}{2}}}{n!q(1+y)^{\binom{n}{2}}}\right)^{\lambda}$$

counts $\lambda$-colored graphs on $[n]$, with edges weighted by $y$ and descents weighted by $u$. Thus this coefficient is the sum

$$\sum_G y^{\sigma(G)} X_G(\lambda)$$

over all graphs $G$ on $[n]$. Setting $\lambda = -1$ and using Theorem 12 gives (20).

4.3. Trees. We can use the basic idea of Lemma 9 to count rooted trees and forests by descents. Recall that we define a rooted tree (tree for short) to be an acyclic digraph in which one vertex (the root) has outdegree 0 and every other vertex has outdegree 1. The vertices of indegree 0 are called leaves but if the tree contains only one vertex, we do not consider this vertex to be a leaf. A (rooted) forest is a digraph in which every weak component is a tree. Let $t_n(u; \alpha)$ be the sum of the weights of all trees with vertex set $[n]$, where the weight of a tree with $i$ descents and $j$ leaves is $u^i \alpha^j$, and let $T(x, u; \alpha) = \sum_{n=0}^{\infty} t_n(u; \alpha)x^n/n!$.

We first illustrate the approach with $u = 1$. The same approach to counting trees was taken in [5]. The result was stated there as a recurrence, but here we use exponential generating functions directly since the analogue for general $u$, discussed in Proposition 13 below, would be more complicated as a recurrence.

Writing $T(x; \alpha)$ for $T(x, 1; \alpha)$ (counting trees by leaves) and $T(x)$ for $T(x, 1; 1)$ (just counting trees), we will show that

$$T(x; \alpha + 1) = T(xe^{\alpha x}).$$

The left side of (24) counts trees in which some subset of the leaves are marked, where each marked leaf is weighted $\alpha$. To interpret the right side of (24), we assume that the reader is familiar with the combinatorics of exponential generating functions, as described, for example, in [21 Chapter 5]. The exponential generating function $xe^{\alpha x}$ counts “short trees”": trees in which every vertex other than the root is a leaf, where the leaves are weighted by $\alpha$. Then $T(xe^{\alpha x}; 1)$ is the exponential generating function for structures obtained from rooted trees by replacing each vertex with a short tree.
It is clear that these structures are essentially the same as the trees counted by the left side; the marked leaves corresponding to the leaves of the short trees.

Setting $\alpha = -1$ in (24) gives $T(xe^{-x}) = x$. In other words $T(x)$ is the compositional inverse of $xe^{-x}$, so $T(x)e^{-T(x)} = x$, or $T(x) = xe^{T(x)}$, the more common form of the functional equation for $T(x)$. These equations can be solved by Lagrange inversion or other methods to obtain the well-known formula

$$T(x) = \sum_{n=1}^{\infty} n^{n-1} \frac{x^n}{n!},$$

and more generally,

$$e^{zT(x)} = \sum_{n=0}^{\infty} z(n + 1)^{n-1} \frac{x^n}{n!},$$

which counts forests of rooted trees by the number of trees. There is also a simple functional equation for $T(x; \alpha) = T(xe^{(\alpha-1)x})$, which counts trees by leaves. From the functional equation $T(x) = xe^{T(x)}$ we can easily obtain the functional equation for $T(x; \alpha) = T(xe^{(\alpha-1)x})$:

$$T(x; \alpha) = xe^{T(x; \alpha) + (\alpha-1)x} \quad (25)$$

Equation (25) is easy to see combinatorially, interpreting $T(x; \alpha) + (\alpha - 1)x$ as counting trees by leaves, but now considering the root of a one-vertex tree to be a leaf.

Next, we can generalize (24) to keep track of descents. The argument is essentially the same as for (24) but we need to replace $xe^{\alpha x}$ with something a little more complicated.

**Proposition 13.** The exponential generating function $T(x, u; \alpha)$ for trees by descents and leaves satisfies

$$T(x, u; \alpha + 1) = T\left(\frac{e^{\alpha x} - e^{\alpha ux}}{\alpha (1 - u)}, u\right), \quad (26)$$

where $T(x, u) = T(x, u; 1)$ is the exponential generating function for trees by descents. Moreover, $T(x, u)$ is compositional inverse (as a power series in $x$) of

$$\frac{e^{-x} - e^{-ux}}{u - 1} = \sum_{n=1}^{\infty} (-1)^{n-1}(1 + u + \cdots + u^{n-1}) \frac{x^n}{n!} \quad (27)$$

**Proof.** A short tree on $[n]$ with root $i + 1$ has $i$ descents. Thus the exponential generating function for short trees, with descents weighted by $u$ and leaves weighted by $\alpha$, is

$$\sum_{n=1}^{\infty} (1 + u + \cdots + u^{n-1}) \frac{\alpha^{n-1} x^n}{n!} = \sum_{n=1}^{\infty} \frac{1 - u^n}{1 - u} \frac{\alpha^{n-1} x^n}{n!} = \frac{e^{\alpha x} - e^{\alpha ux}}{\alpha (1 - u)}.$$
Then we obtain (26) in the same way that we obtained (24). As before, \( T(x, u; 0) = x \), so setting \( \alpha = -1 \) in (26) gives

\[
T \left( \frac{e^{-x} - e^{-ux}}{u - 1} \right) = x.
\]

\[\square\]

Another combinatorial proof that \( T(x, u) \) is the compositional inverse of \( \frac{e^{-x} - e^{-ux}}{u - 1} \) was given by Drake [2, Example 1.7.2].

There is a simple formula for the coefficients of \( T(x, u) \) that can be derived from our results and known formulas.

**Proposition 14.** For the exponential generating function \( T(x, u) \) for trees by descents, we have the formulas

\[
T(x, u) = \sum_{n=1}^{\infty} \frac{n-1}{n!} \prod_{i=1}^{n-1} (iu + n - i) \frac{x^n}{n!}.
\]

and

\[
e^{zT(x,u)} = 1 + \sum_{n=1}^{\infty} z \prod_{i=1}^{n-1} (iu + n - i + z) \frac{x^n}{n!}.
\]

**Proof.** Since \( T(x, u) \) is the compositional inverse of \( \frac{e^{-x} - e^{-ux}}{u - 1} \), we have

\[
\frac{e^{-T(x,u)} - e^{-uT(x,u)}}{u - 1} = x.
\]

Multiplying both sides by \((1 - u)e^{T(x,u)}\) gives

\[
e^{(1-u)T(x,u)} - 1 = (1 - u)xe^{T(x,u)}.
\]

Now set \( G = e^{(1-u)T(x,u)} \). Then (27) may be written

\[
G = 1 + (1 - u)xe^{T(x,u)}.
\]

and the desired formulas follow from the results of [10, Section 5] or by Lagrange inversion (see, e.g., [8, Section 3.3]). \[\square\]

We note that the formulas of Proposition 14 are proved by a different method in [10, Section 9], and more general enumerative results for trees have been proved bijectively by Egecioğlu and Remmel [3].

Forests have been counted by leaves and descents of a different kind in [7] but there does not seem to be any connection between the results described here and the results of [7].
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(K. Archer) University of Texas at Tyler, Tyler, TX 75799 USA
E-mail address: karcher@uttyler.edu

(I. Gessel) Brandeis University, Waltham, MA 02254 USA
E-mail address: gessel@brandeis.edu

(C. Graves) University of Texas at Tyler, Tyler, TX 75799 USA
E-mail address: cgraves@uttyler.edu

(X. Liang) Harvey Mudd College, Claremont, CA 91711 USA
E-mail address: eliang@hmc.edu