Intrinsic momentum in Poincaré gauge theory

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While it is generally accepted, in the framework of Poincaré gauge theory, that the Lorentz connection couples minimally to spinor fields, there is no general agreement on the coupling of the translational gauge field to fermions. We will show that the assumption that spinors carry a full Poincaré representation leads to inconsistencies, whose origins will be traced back by considering the Poincaré group both as the contraction of the de Sitter group, and as a subgroup of the conformal group. As a result, the translational fields do not minimally couple to fermions, and consequently, fermions do not possess an intrinsic momentum.

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I. INTRODUCTION

In Poincaré gauge theory, and especially in the more widely known Einstein-Cartan theory, a majority of authors follows the standard references [1, 2], considering scalar and gauge fields (other than the gravitational fields) as invariant under the Poincaré gauge group, while spinor fields are supposed to carry a Lorentz representation. This leads to a consistent theory, and one could justify the invariance under translations by the fact that the translational symmetry will be broken anyway, and that therefore the Lorentz group is the only physical gauge group.

On the other hand, it is certainly not unnatural to consider fermions carrying a full Poincaré representation. As a result, the covariant derivative of spinor fields will contain the full Poincaré connection, leading therefore to a minimal coupling to the translational gauge fields. Once the translational gauge freedom is broken, the fermions reduce to the usual Lorentz spinors, but the minimally coupled translational field (which becomes a simple Lorentz vector-valued one-form) remains in the Lagrangian. Both approaches are therefore not equivalent.

The first attempt to treat gravity as gauge theory goes back to Utiyama [3], with the gauge group taken to be the Lorentz group, while the tetrad fields had to be introduced ad hoc. The precise relation between tetrad fields and the translational gauge fields was clarified much later, which could explain why the possibility of fermions carrying a Poincaré representation is not even mentioned in most of the literature (see, e.g., [1, 2, 4]). Another good reason for this can also be seen from the experimental side. While it is a well established fact that fermions possess, apart from their orbital angular momentum, also an intrinsic spin momentum, there is no evidence in favor of a concept like intrinsic momentum.

Nevertheless, the alternative approach, with spinors carrying in addition a representation of the translational part of the Poincaré group, can be found in [3] and, more recently, in [2, 5].

In this paper, we will conclude that the coupling of the translational gauge fields to fermions faces problems, and that only the classical approach with Lorentz spinors is free of inconsistencies. This conclusion is further supported by the following. If the Poincaré gauge theory is treated as a subcase of the theory based on the conformal group $SO(4,2)$, it turns out that, on one hand, in the Lorentz invariant groundstate, no minimal coupling of the translational gauge field occurs, and moreover, no corresponding Poincaré invariant groundstate exists. In other words, the residual Lorentz theory cannot be interpreted as the result of a symmetry breakdown of the translational gauge freedom in the framework of a Poincaré gauge theory.

On the other hand, if we start from the de Sitter group $SO(4,1)$, and take the limit to the Poincaré group via a Wigner-Inönü contraction, we end up with a consistent Poincaré invariant theory, with, however, no direct coupling of the translational gauge fields to the spinor fields.

The article is organized as follows. In the next section, we briefly review the basic concepts of Poincaré gauge theory, focusing mainly on the translations and their relation to the tetrad fields. Then, in section III, we point out the inconsistencies arising from the minimal coupling of the translational field to fermions, and finally, in sections IV and V, we treat Poincaré theory as a limiting case of the de Sitter and the conformal gauge theories, respectively.

II. TRANSLATIONS IN POINCARÉ GAUGE THEORY

We start with a Poincaré connection one-form $(\Gamma^{ab}, \Gamma^a)$ (where $a = 0, 1, 2, 3$, $\Gamma^{ab} = -\Gamma^{ba}$) which transforms under an infinitesimal Poincaré transformation with coefficients $(\varepsilon^{ab}, \varepsilon^a)$ as

$$
\delta \Gamma^{ab} = -D \varepsilon^{ab}, \quad \delta \Gamma^a = -D \varepsilon^a + \varepsilon^a \Gamma^b.
$$

(1)

As is customary, we use the symbol $D$ to denote the Lorentz covariant derivative, i.e., $D \varepsilon^{ab} = d \varepsilon^{ab} + \Gamma^c \varepsilon^{cb} + \Gamma^c \varepsilon^{ac}$. It turns out that with those gauge fields alone, it is not possible to construct a consistent Lagrangian...
spinors, the generators are taken to be $\sigma$.

In order to be compatible with our dimension conventions effective Lorentz gauge theory. Thus, in some sense, the

while a Poincaré spinor would transform as

\[
\psi \rightarrow e^{-i/4} \epsilon^{ab} \sigma_{ab} \psi,
\]

which is invariant under translations ($\delta e^a = \epsilon^a e^b$). In order to avoid the introduction of an arbitrary length parameter, we suppose that $\Gamma$ is dimensionless (as opposed to $\Gamma^{ab}$ which has dimensions $L^{-1}$), while $e^a$ and $y^a$ have dimension $L$. The gravitational Lagrangian is now constructed from $e^a$, as well as from the curvature $R^{ab} = \partial \Gamma^{ab} + \Gamma^{ac} \Gamma^{cb}$ and the torsion $T^a = de^a + \Gamma^a_b e^b$.

The origin of the so-called Poincaré coordinates $y^a$ has been traced back in [4, 8, 10] to the non-linear realization of the translational part of the gauge group. There is a close connection between non-linear realizations and the Higgs symmetry breaking mechanism (see [4]). Indeed, it is possible to treat $y^a$ as a Higgs field, with groundstate $y^a = 0$. This groundstate breaks the translational invariance and leaves us with a residual Lorentz symmetry. The details of such an approach have been elaborated in [6].

A Lorentz spinor transforms under a Poincaré transformation as

\[
\psi \rightarrow e^{-i/4} (\epsilon^{ab} \sigma_{ab} + e^a P_a) \psi,
\]

while a Poincaré spinor would transform as

\[
\psi \rightarrow e^{-i/4} (\epsilon^{ab} \sigma_{ab} + e^a P_a) \psi,
\]

where $(\sigma_{ab}, P_a)$ are the generators of the Poincaré group. In order to be compatible with our dimension conventions $[e^a] = [L]$, we have to require $[P_a] = [L^{-1}]$.

Especially, in the classical approach with Lorentz spinors, the generators are taken to be $\sigma_{ab} = \frac{i}{2} [\gamma_a, \gamma_b]$, and the following Dirac Lagrangian

\[
\mathcal{L} = -\frac{i}{12} \epsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge (\overline{\psi} \gamma^d \nabla \psi - \nabla \overline{\psi} \gamma^d \psi),
\]

where $D \psi = d \psi - (i/4) \Gamma^{ab} \sigma_{ab}$ is easily shown to be Poincaré invariant. We omit the mass term, which is unrelated to our discussion.

It is clear that every Lagrangian constructed only out of $e^a, \Gamma^{ab}$, invariants (like the Maxwell field $A$ or scalar fields) as well as Lorentz spinors, will trivially be invariant under translations. Moreover, in view of the relation (3), no additional field equation will arise from variation with respect to $y^a$. The resulting equation will simply be the covariant derivative of the equation resulting from the $\Gamma^a$ variation and is thus identically satisfied. Further, instead of varying with respect to $\Gamma^a$, one can equivalently vary with respect to $e^a$. As a result, one can simply forget about $y^a$ and identify directly $\Gamma^a$ with $e^a$, resulting in an effective Lorentz gauge theory. Thus, in some sense, the

gauging of the translations and the introduction of the Higgs field $y^a$ is basically a way to explain the presence of the tetrad field, but for the rest, one considers only Lorentz transformations, just as Utiyama did right from the start.

Things changes, however, as soon as we introduce a Poincaré spinor (5). Then, clearly, the translational invariance of a Lagrangian is not a priori guaranteed, and moreover, the use of a Poincaré covariant derivative will break the equivalence of the $\Gamma^a$ and the $e^a$ variations. Let us introduce, for convenience, the following notations:

\[
\mathcal{P} = e^{-i/4} (\epsilon^{ab} \sigma_{ab} + e^a P_a), \quad \Lambda = e^{i/4} (\epsilon^{ab} \sigma_{ab}).
\]

Thus, we have for the Poincaré spinor $\psi \rightarrow \mathcal{P} \psi$. The covariant derivative $\nabla \psi$ transforming in the same way, $\nabla \psi \rightarrow \mathcal{P} \nabla \psi$, is given by

\[
\nabla \psi = d \psi - \frac{i}{4} (\Gamma^{ab} \sigma_{ab} + \Gamma^a P_a) = D \psi - \frac{i}{4} \Gamma^a P_a.
\]

We see that, if we have such a covariant derivative in our Lagrangian, the fields $\Gamma^a$ and $y^a$ do not only occur in the combination (3), and thus, the variation has to be carried out with respect to all independent fields $(\Gamma^{ab}, \Gamma^a, y^a, \psi)$. Together with the additional field equation, we get a new conservation law (because the translational gauge invariance is not trivially satisfied anymore), leading to the so-called intrinsic momentum conservation.

Such Lagrangians will be investigated in the next section.

III. DIRAC FIELDS WITH INTRINSIC MOMENTUM

Recently, in [6, 8, 10], the following Dirac Lagrangian has been proposed

\[
\mathcal{L} = -\frac{i}{12} \epsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge (\overline{\psi} \gamma^d \nabla \psi - \nabla \overline{\psi} \gamma^d \psi),
\]

where

\[
\nabla \psi = d \psi - \frac{i}{4} (\Gamma^{ab} \sigma_{ab} + e^a P_a) = D \psi - \frac{i}{4} e^a P_a,
\]

and the Poincaré generators are taken to be

\[
\sigma_{ab} = \frac{i}{2} [\gamma_a, \gamma_b], \quad P_a = m \gamma_a (1 + \gamma_5).
\]

This Lagrangian is supposed to be invariant under the non-linear realization of the Poincaré group with Lorentz structure group. In other words, it is invariant after the reduction of the symmetry group to the Lorentz group (see [6, 8, 10] for the concept of non-linear realizations in the context of Poincaré gauge theory). That (9) is indeed Lorentz invariant is not hard to show, since the only difference to (6) comes from the coupling term $\sim \overline{\psi} e^a P_a \psi$ contained in the covariant derivatives, leading to a mass term for the Dirac particle (see [6]).
However, it is usually understood that different choices of the stability subgroup in the non-realization scheme should lead to physically equivalent results, since only the initial symmetry group (in our case the Poincaré group) is physically relevant. For instance, in $SO(3)$, the same Lagrangian (9) is also considered with stability group $SO(3)$ instead of the Lorentz group. Especially, one could also consider the stability subgroup to be the Poincaré group itself. In the more physical approach, using the Higgs field $y^a$, this simply means that the Lagrangian should be independent of any specific gauge we implement on $y^a$. This holds independently of how the specific gauge arises, be it simply a convenient choice, or be it the result of a symmetry breaking groundstate, like $\gamma^a = 0$. In any case, before we can choose a gauge, or be it the result of a symmetry breaking groundstate, like $y^a = 0$.

Thus, formally, we can be sure that the generators $\sigma^A$, where $A$ labels the ten Poincaré generators, are invariant (as they have too), if they transform as $\sigma^A \rightarrow P_\alpha^A P^{-1} P_A^\beta$, where $P_\alpha^A_B$ is the adjoint representation of the Poincaré group. This ensures, for instance, that $\sigma_{ab}$ is invariant, but still does not allow us to conclude on $\gamma^a$!

In view of the difficulties with the double indexed quantities, it is convenient to grab the problem from the other side, namely from (9). It is immediately clear that (9), with the covariant derivative (8), can only be Poincaré invariant if $\gamma^a$ transforms as follows

$$\gamma^a \rightarrow P \gamma^a P^{-1} \Lambda^a_a,$$  \hspace{1cm} (13)

where $\Lambda^a_a$ is the usual vector representation of the Lorentz group, $\Lambda^a_b = \delta^a_b + \epsilon^a_b$. This is simply because $\psi$ and $\nabla \psi$ transform under the fundamental representation of the Poincaré group, $\psi$ and $\nabla \psi$ under its inverse, and $\epsilon_{a b c d} \epsilon^{a b c d}$ as Lorentz vector (see (3)).

However, it can also be seen, using the explicit generators (11), that under (13), if $P$ contains a translation, $\gamma^a$ is not invariant. Moreover, neither is $[\gamma^a, \gamma^b]$ invariant. From this result, several problems arise. First, the generators (11), which have to be invariant, cannot be defined with the same $\gamma$-matrices that appear in the Lagrangian and are gauge dependent. This leads to a second problem, namely we will not be able to determine the commutation relations between $\gamma^a$ and the generators. Finally, only in specific gauges will $\gamma^a$ be constant. In general, $\gamma^a$ will depend on the spacetime coordinates, and consequently will have to be treated as dynamical field. This does certainly not look like a consistent theory.

We conclude that (9), with covariant derivative (8), is not the Poincaré invariant form of (9) with derivative (10). An alternative attempt can be found in [5]. They use the Lagrangian (9), again with the Poincaré covariant derivative (8), but replace $\gamma^a$ by

$$\tilde{\gamma}^a = \gamma^a + (i/4) g y^b [\gamma^a \gamma_b (1 + \gamma_5) - \gamma_b \gamma^a (1 - \gamma_5)] + (m^2/4) (g - 1/2) [y^2 \gamma^a] (1 + \gamma_5).$$  \hspace{1cm} (14)

In the gauge $y^a = 0$, this reduces to $\gamma^a$, and thus (14) too is a covariant generalization of (9). Unfortunately, the authors do not tell us how $\gamma^a$ and $\tilde{\gamma}^a$ are supposed to transform under gauge transformations, but they claim that the Lagrangian is invariant under the full Poincaré group.

It is again straightforward to show that for this to be the case, $\tilde{\gamma}^a$ has to transform as

$$\tilde{\gamma}^a \rightarrow P \tilde{\gamma}^a P^{-1} \Lambda^a_a.$$  \hspace{1cm} (15)
We were not able to find the transformation behavior for \( \gamma^a \) that leads to (15) for \( \tilde{\gamma}_a \). One can, however, easily verify that claiming \( \gamma^a \) to be invariant, does not lead to the above result. On the other hand, the generators (11) have to be invariant. Therefore, it would rather be a surprise if someone can come up with a non-trivial transformation behavior for \( \gamma^a \) leading both to (15) and to invariant generators. (Especially, it would be strange if the authors of [3] had known of such a transformation and did not find it necessary to write it down.) However, even if such a transformation exists, the fact that \( \gamma^a \) is not invariant (and will thus be coordinate dependent in certain gauges), is enough for the theory to be inconsistent, as outlined above.

We have to conclude that no consistent theory with translational gauge fields coupling to the Dirac field has yet been found. This does not mean that it is impossible to construct such a theory, but it shows at least that the straightforward approach does not lead to the expected success.

IV. THE DE SITTER GROUP

Part of the difficulties in constructing a consistent Poincaré invariant theory comes from the non semisimple nature of the group. It might therefore be instructive to start from rotational groups like \( SO(4,1) \) or \( SO(4,2) \) and consider the Poincaré theory as a limiting case of those theories.

The most promising candidate is probably the de Sitter group, since its algebra has the same dimension as that of the Poincaré group. The Poincaré group is recovered through the application of the so-called Wigner–Hilbni contraction. The de Sitter gauge theory has been discussed in full detail in [11]. The article contains a discussion of the non-linear realization scheme as well as of the possibility of a Higgs symmetry breaking mechanism. The latter has been further investigated in [5].

In this section, capital indices \( A, B \ldots \) take the values 0,1,2,3,5, while the four dimensional part is denoted by \( a, b \ldots \) as before, i.e., \( A = (a,5) \). We start with a de Sitter connection \( \Gamma^{AB} \), transforming as

\[
\delta \Gamma^{AB} = -\nabla \varepsilon^{AB},
\]

where \( \nabla \) denotes the de Sitter derivative. The de Sitter transformations (of the infinitesimal form \( G_B^A = \delta_B^A + \varepsilon_B^A \)) leave the de Sitter metric \( \eta_{AB} = \text{diag}(+1,-1,-1,-1,1) \) invariant. A de Sitter spinor transforms as

\[
\psi \to e^{-(i/4)e^{AB}\sigma_{AB}} \psi,
\]

where the generators are taken to be \( \sigma_{AB} = (i/2)[\gamma_A,\gamma_B] \), with \( \gamma_A = (\gamma_a, \gamma_5) \), where \( \gamma_5 = -(i/4)\varepsilon_{abcd}\gamma^a\gamma^b\gamma^c\gamma^d \). As in the Poincaré case, a Higgs field \( y^A \) is introduced, transforming as

\[
\delta y^A = \varepsilon^A_B y^B.
\]

We suppose that the complete theory contains a Higgs sector leading to the groundstate \( y^A = (0,0,0,0,l) \), invariant under the Lorentz group only (see [3]). The parameter \( l \) is an explicit ingredient of the theory. It has the dimensions of a length and in the limit \( l \to \infty \), the theory will reduce to a Poincaré gauge theory.

Let us introduce the following one-forms

\[
E^A = \nabla y^A,
\]

which reduce, in the groundstate, to \( E^A = (i\Gamma^5,0) \). In view of this, and its homogeneous transformation behavior under the residual Lorentz group, we identify \( i\Gamma_5 \) with the tetrad \( e^a \).

A gauge invariant Dirac-type Lagrangian for the de Sitter spinor is readily written down

\[
\mathcal{L} = -\frac{i}{12} \varepsilon_{ABCDE} \dot{E}^A \wedge E^B \wedge E^C \wedge (\bar{\psi} \gamma^D \nabla \psi - \nabla \bar{\psi} \gamma^D \psi) y^F / l.
\]

The spinor derivative is defined as usual, by \( \nabla \psi = d\psi - (i/4)\Gamma^{AB}\sigma_{AB} \). In the gauge \( y^A = (0,l) \), or, if you prefer, in the groundstate, \( \mathcal{L} \) takes the form (9), with the derivative

\[
\nabla \psi = d\psi - (i/4)\Gamma^{AB}\sigma_{AB} \\
= d\psi - (i/4)\Gamma^{ab}\sigma_{ab} + l^{-1}(1/2)e^a\sigma_{a5} \\
= D\psi - l^{-1}(1/2)e^a\gamma_a \gamma_5
\]

The Lagrangian therefore takes the form

\[
\mathcal{L} = -\frac{i}{12} \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge (\bar{\psi} \gamma^d D\psi - D\bar{\psi} \gamma^d \psi) \\
+ \frac{1}{12} \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge (e^d/2l)(\bar{\psi}(\gamma^a \gamma_b - \gamma^a \gamma_c) \gamma_5) \psi,
\]

where \( D \) the Lorentz covariant derivative. Clearly, the terms in the second line vanish, and thus, the Lagrangian reduces to the classical, Lorentz covariant Lagrangian (6), without a direct coupling of \( e^a \) to the spinor field, in contrast to the attempt (9).

What does that mean? Well, even without taking the Poincaré limit \( l \to \infty \), in the Lorentz invariant groundstate, there is no direct coupling of the pseudo-translational part \( \Gamma_5 = l^{-1}e^a \) of the connection to the spinor field. Therefore, if we take the limit \( l \to \infty \) in the de Sitter invariant form (20), this will still lead to a theory which, in the Lorentz invariant groundstate, will not present a minimal coupling of the translational fields to the spinor. In other words, whatever the coupling of the translational field looks like in the initial Poincaré Lagrangian, no coupling of the form (10) will remain in the Lorentz invariant gauge. But we know already the Poincaré invariant form of such a theory, it is the Lagrangian (6) itself, if we assume simply that \( \psi \) carries only a Lorentz representation, as we have argued in section II. Indeed, the same Lagrangian emerges by taking the limit \( l \to \infty \) in (20). This involves, however, a careful parameterization of the group generators and the fields in terms of \( l \), in a way that the corresponding Poincaré
structure results in the limit $l \to \infty$. The procedure is standard (see [12]), and is not needed for our purpose.

Summarizing, treating the Poincaré group as Wigner–Inönü contraction of the de Sitter group clearly suggests that in Poincaré gauge theory, the spinor fields do not carry a representation of the full group, but only of the Lorentz part, and consequently, couple minimally only to the Lorentz connection. In short, they do not possess intrinsic momentum.

Our result was achieved by analyzing the gauge theory of the group $SO(4,1)$. As far as the limit to the Poincaré group is concerned, one could expect that the $SO(3,2)$ theory leads to identical results. However, there are interesting differences between both theories. One might think that the $SO(3,2)$ spinor representation emerges from the $SO(4,1)$ representation simply through the replacement $\gamma_5 \to i\gamma_5$, changing in this way the signature in the metric $\gamma_{AB} = \gamma_A \gamma_B + \gamma^B \gamma^A$. It turns out that this is not correct for the following reason. In order to define the conjugate spinor $\bar{\psi}$, we write $\bar{\psi} = \psi^\dagger \gamma_5$. Then, $\gamma$ has to transform as $\gamma \to (G^\dagger)^{-1} \gamma G^{-1}$. Since we do not want $\gamma$ to be a dynamical field, it should be invariant. Thus, infinitesimally, for $G = \exp(i\sigma)$, we require

$$\sigma^\dagger \gamma = \gamma \sigma. \quad (22)$$

The only hermitian matrices, up to a global factor, satisfying this relation for all the Lorentz generators $\sigma_{ab}$ are $\gamma = \gamma_0$ and $\gamma = \gamma_5 \gamma_0$.

For the usual choice $\gamma = \gamma_0$, is easily verified that the only transformations $G = \exp(i\sigma)$ satisfying (22) are given by real linear combinations of

$$\sigma = (\sigma_{ab}, \gamma_a \gamma_5, i\gamma_5) \quad (23)$$

The largest possible gauge group with a four dimensional representation is therefore the conformal group. The generators $(\sigma_{ab}, \gamma_0 \gamma_5)$ span the algebra of the de Sitter group $SO(4,1)$, while $(\sigma_{ab}, \gamma_0)$ span the anti-de Sitter algebra $SO(3,2)$. However, that the latter, as opposed to case $SO(4,1)$, is not a spinor representation in the strict sense, i.e., we cannot write $2\sigma_{AB} = i[\gamma_A, \gamma_B]$ and $2\eta_{AB} = \{\gamma_A, \gamma_B\}$ for some $\gamma_A$. This, however, has the consequence there is no set of matrices such that $\gamma^A$ is a vector, i.e., we do not have the relations $\gamma^C, \sigma_{AB} = 2(i\delta^C_{[A} \gamma_{B]} - \delta^C_{[A} \gamma_{B]})(\text{meaning that } \gamma^A \text{ is invariant under gauge transformations, as in (12)}).

Thus, on one hand, the generators $(\sigma_{ab}, i\gamma_a \gamma_5)$ span the algebra of the group $SO(3,2)$, but do not leave invariant the spinor metric $\gamma_0$. On the other hand, the generators $(\sigma_{ab}, \gamma_0)$ span the same algebra, leave the metric invariant, but do not allow for the construction of a Lagrangian in the form (20), because we do not have an invariant set of $\gamma$-matrices.

On the other hand, if we use instead $\gamma = i\gamma_5 \gamma_0$ as spinor metric, then the transformations allowed by (22) are generated by $\sigma = (\sigma_{ab}, i\gamma_a, i\gamma_a \gamma_5, i\gamma_5)$. Thus, in that case, the generators $(\sigma_{ab}, i\gamma_a \gamma_5)$ give rise to a true spinor representation of $SO(3,2)$, but now, the construction of an $SO(4,1)$ invariant Lagrangian is not possible. Both groups are thus simply interchanged and the conclusions concerning the translations are identical.

V. THE CONFORMAL GROUP

The conformal group contains the Poincaré group as a subgroup, and could thus reveal an alternative approach to the coupling of the translational part of the connection. In this section, capital indices take values $A = (0,1,2,3,5,6) = (a,5,6)$. The conformal group $SO(4,2)$ is defined to leave the metric $\eta_{AB} = \text{diag}(+1,-1,-1,-1,+1)$ invariant.

Although there exists a four dimensional representation of the conformal group, with the algebra spanned, e.g., by the generators (23), we face again the problem that there is no corresponding set of invariant $\gamma$-matrices for the construction of the Lagrangian. As outlined in the previous section, the largest possible transformation group is either given by $SO(3,2)$ or by $SO(4,1)$, depending on the choice of the spinor metric.

Therefore, we necessarily have to consider the eight dimensional spinor representation of $SO(4,2)$. The corresponding Dirac algebra can be found, e.g., in [12]. We define the 6 matrices $\beta^A$ by

$$\beta^a = \gamma^a \sigma_3, \beta^5 = i\sigma_1, \beta^b = \sigma_2, \quad (24)$$

where $\gamma^a$ and $\sigma_i$ ($i = 1,2,3$) are defined as

$$\gamma^a = \begin{pmatrix} \gamma^a & 0 \\ 0 & \gamma^a \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

composed of $4 \times 4$ block matrices. The $SO(4,2)$ generators are now given by

$$\sigma_{AB} = \frac{i}{2}[\gamma_A, \gamma_B]. \quad (25)$$

The generators $(\sigma_{ab}, P_a)$, with $P_a = (1/2)(\sigma_{a5} + \sigma_{ab})$, span the Poincaré subalgebra, which is what we are interested in.

The 8 component spinor $\psi$ transforms formally as in (17). Next, define the spinor metric $\gamma = \gamma_0 \sigma_1$, i.e., the adjoint spinor is given by

$$\tilde{\psi} = \psi^\dagger \gamma = \psi^\dagger \gamma_0 \sigma_1 = \tilde{\psi} \sigma_1. \quad (26)$$

The relation (22) for $\gamma$ is easily verified for the fifteen generators (25), and thus, $\tilde{\psi}$ transforms under the inverse of the fundamental representation. (We reserve the notation $\tilde{\psi}$ for $\psi^\dagger \gamma_0$, which will be useful later on.)

The gravitational gauge theory of the conformal group, in the fashion of Stelle and West [13], has been elaborated in [14]. The connection transforms as in (16). In contrast
to the Poincaré and de Sitter cases, we need two Higgs vectors $y^A$ and $z^A$ in order to break down the symmetry to the Lorentz group.

Let us introduce the quantities

$$E^A = \frac{\sqrt{ab}}{2} \nabla(z^A/b - y^A/a)$$

$$= \frac{\sqrt{ab}}{2} [d(z^A/b - y^A/a) + \Gamma^A_C(z^C/b - y^C/a)],$$

where $a, b$ are two length parameters of the theory. It is possible, alternatively, to define $E^A$ simply as $\nabla y^A$, but we prefer the more symmetric approach using both $y^A$ and $z^A$.

The conformally invariant Dirac-type Lagrangian is found in the form

$$\mathcal{L} = -\frac{1}{12} \varepsilon_{ABCDEF} E^A \wedge E^B \wedge E^C,$$

$$\wedge (\bar{\psi} \beta^D \nabla \psi - \nabla \bar{\psi} \beta^D \psi)(y^E/a)(z^E/b),$$

which is a straightforward generalization of (6) and (20). In [14], the Higgs fields were required, a priori, to satisfy the relations

$$y^A y_A = -a^2, \quad z^A z_A = b^2, \quad z^A y_A = 0.$$  

This is to be interpreted as groundstate configuration (see [3]). Let us first take a look at the Lorentz invariant groundstate, in order to see whether (28) reduces indeed to the Dirac Lagrangian.

The Lorentz invariant groundstate, compatible with (29), is given by $y^A = (0, 0, 0, 0, a, 0)$ and $z^A = (0, 0, 0, 0, b)$. In this gauge, (27) reduces to $E^A = (e^a, \sqrt{ab} A/2, -\sqrt{ab} A/2)$, where $e^a = \sqrt{ab}(\Gamma^{a5} + \Gamma^{a6})/2$, and $A = \Gamma^{a5}$. Let us also introduce the notation $B^a = \sqrt{ab}(\Gamma^{ab} - \Gamma^{a6})/2$. Note that if we choose, e.g., $z^A = (0, 0, 0, 0, -b)$, then, from (27), $E^a \sim B^a$. This is not a problem, since the corresponding generators, $Q_a = (\sigma_{a5} - \sigma_{a6})/2$, span, together with $\sigma_{ab}$, the Poincaré subalgebra too. Thus, the role of $(e^a, P_a)$ and $(B^a, Q_a)$ is simply inverted.

The residual Lorentz invariant Lagrangian takes the form

$$\mathcal{L} = -\frac{1}{12} \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge (\bar{\psi} \beta^d \nabla \psi - \nabla \bar{\psi} \beta^d \psi),$$

with

$$\nabla \psi = d\psi - (i/4) \Gamma^{ab} \sigma_{ab}$$

$$= -(i/4) \left[ \frac{4}{\sqrt{ab}} (e^a P_a + B^a Q_a) + 2 A \sigma_{56} \right] \psi, \quad (30)$$

It is straightforward to verify that $e^a$ and $B^a$ transform homogeneously under the residual Lorentz group, whereas $\Gamma^{ab}$ is a Lorentz connection and $A$ is an invariant. Thus, apart from the additional fields $B^a$ and $A$, corresponding to the parts of the conformal group that do not belong to the Poincaré subgroup (i.e., generated by $e^{a5} - e^{a6}$ and $e^{56}$), this Lagrangian is formally identical to the one proposed in [1], namely (9), with derivative (10). The role of the mass parameter $m$ is now played by the inverse length parameter $1/\sqrt{ab}$. Apart from this formal resemblance, we have to take into account that our particle interpretation is based on irreducible representations of the Lorentz group, and thus, in order to get an idea what particles are described by (30), we have to write the Lagrangian explicitly in terms of four component spinors. Introducing components $\psi = (\psi_1, \psi_2)$, we find from (31)

$$\nabla \psi = (\nabla \psi)^\dagger \gamma_0 \sigma_1,$$

$$\nabla \psi = (D \psi_1 + i \Gamma^{ab} \gamma_a \psi_2 - \frac{i}{2} A \psi_1, \quad D \psi_2 - i \Gamma^{ab} \gamma_a \psi_1 + \frac{i}{2} A \psi_2), \quad (32)$$

Using $\nabla \psi = (\nabla \psi)^\dagger \gamma_0 \gamma_1$, we get

$$\nabla \psi = (D \psi_2, \quad D \psi_1)$$

$$+ i \Gamma^{ab} \gamma_a \psi_2 - \frac{i}{2} A \psi_2 - \frac{i}{2} A \psi_1,$$

with $D$ denoting the Lorentz covariant derivative. Putting this into (30), it turns out that the diagonal terms all cancel out and we are left with

$$\mathcal{L} = -\frac{1}{12} \varepsilon_{abcd} e^a \wedge e^b \wedge e^c$$

$$\wedge (\bar{\psi}_2 \gamma^d D \psi_1 - D \psi_2 \gamma^d \psi_1 - \bar{\psi}_1 \gamma^d D \psi_2 + D \psi_1 \gamma^d \psi_2$$

$$- \bar{\psi}_2 \gamma^d A \psi_1 - \bar{\psi}_1 \gamma^d A \psi_2]. \quad (33)$$

It is now easy to check that $\mathcal{L}$ is hermitian. This justifies the choice of the omitted factor $i$ in (28), as compared to (20). For the special class of solutions $\psi_2 = -i \psi_1$, we recover exactly the conventional Dirac Lagrangian (6), coupling to the Lorentz connection only. More generally, (33) leads to two Dirac type equations for $\psi_1$ and $\psi_2$ with a minimal coupling to the Lorentz connection and to the field $A$. The latter coupling is identical to that of the Maxwell field, which justifies the identification made by Kerrick [14]. The addition of a mass term $m \bar{\psi} \psi \sim m(\bar{\psi}_1 \psi_2 + \bar{\psi}_2 \psi_1)$ is not possible, because it leads ultimately to an imaginary mass in the field equations. The correct way is to add a term $m \bar{\psi} \sigma_{AB} \psi y^{12} \gamma^b/(ab)$ which is hermitian in view of (22) and reduces to $\sim im(\bar{\psi}_1 \psi_2 - \bar{\psi}_2 \psi_1)$ in the groundstate. The Dirac limit is again obvious for $\psi_2 = -i \psi_1$, and in general, we get two Dirac particles with identical mass and opposite charge.

In any case, we see, just as in the case of the de Sitter group, that no minimal coupling of the translational gauge field occurs in the residual Lagrangian. Thus, we have to conclude that the coupling in the form (9) and (10) cannot be obtained from the reduction of a conformally invariant Dirac equation.

Apart from the fact that the Lorentz invariant groundstate does not lead to a coupling in the form (10), their is a more fundamental problem. Namely, the
Lagrangian (28) does not possess a Poincaré invariant groundstate. Indeed, under a Poincaré transformation $\varepsilon^{AB}$, with $\varepsilon^{a5} - \varepsilon^{a6} = \varepsilon^{56} = 0$, the Higgs field $y^A$ transforms as $\delta y^A = \varepsilon^A_B y^B$, thus, in particular

$$\delta y^a = \varepsilon^a_b y^b + \varepsilon^a_5 y^5 + \varepsilon^a_6 y^6 = \varepsilon^a_b y^b - \varepsilon^{a5}(y^5 - y^6).$$

(34)

The only possible Poincaré invariant groundstate therefore has to satisfy $y^a = 0$ and $y^5 = y^6$. The remaining conditions, $\delta y^5 = \delta y^6 = 0$ are then automatically satisfied. The same applies to $z^A$. Thus, the groundstate reads

$$y^A = (0, 0, 0, 0, a, a), \quad z^A = (0, 0, 0, 0, b, b),$$

(35)

which satisfies $y_A y^A = z_A z^A = z_A y^A = 0$, but not the constraints (29). However, it is not hard to see that, with such a groundstate, the Lagrangian (28) vanishes identically. (The same holds for the free Lagrangian of the gravitational fields themselves, as presented in [14].)

As a result, the only Poincaré invariant groundstate does not lead to a consistent theory. The other way around, in order to get a consistent theory, the Higgs fields have to satisfy the constraints (29). Those constraints, however, do not allow for a Poincaré invariant groundstate (apart from the trivial one, $y^A = z^A = 0$).

VI. CONCLUSIONS

We have argued that the assumption that spinor fields carry a Poincaré representation, and therefore couple minimally to the translational gauge fields, leads to inconsistencies. Although Lorentz invariant Lagrangians with a minimal coupling of the tetrad field to the spinor have been written down in the past, those theories cannot be considered as the result of a symmetry breakdown of the translational gauge freedom in the framework of a Poincaré gauge theory, and thus, neither as the result of a non-linear realization of the translational subgroup. We must therefore conclude that spinors, in the framework of Poincaré gauge theory, do not possess intrinsic momentum.

Apart from the direct analysis, we base our conclusion on the fact that, on one hand, no minimal coupling of translational fields arises from the theory obtained by applying a Wigner-Inönü contraction to the de Sitter gauge theory, while on the other hand, the gauge theory of the conformal group does not possess a Poincaré invariant groundstate.

As it seems, the only field in Poincaré gauge theory that carries a representation of the complete group is the Higgs field $y^a$, coupling (in a certain sense, minimally) to the full Poincaré connection via $e^a = dy^a + \Gamma^a_b y^b + \Gamma^a_a$. This underlines the special role played by the translational gauge field, serving ultimately in the construction of the spacetime metric, and the particular structure of gravitational theories in general.

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