The reduced Hamiltonian for next-to-leading-order spin-squared dynamics of general compact binaries

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Abstract
Within the post-Newtonian framework the fully reduced Hamiltonian (i.e. with eliminated spin supplementary condition) for the next-to-leading-order spin-squared dynamics of general compact binaries is presented. The Hamiltonian is applicable to the spin dynamics of all kinds of binaries with self-gravitating components such as black holes and/or neutron stars taking into account spin-induced quadrupolar deformation effects in second post-Newtonian order perturbation theory of Einstein’s field equations. The corresponding equations of motion for spin, position and momentum variables are given in terms of canonical Poisson brackets. Comparison with a nonreduced potential calculated within the effective field theory approach is made.

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1. Introduction
A crucial prediction of Einstein’s theory of general relativity is the existence of gravitational waves (GWs), e.g. resulting from the inspiralling and merging process of two compact objects. Up till now those waves are purely theoretical predictions with lack of direct experimental verification, but their direct detection is under preparation by gravitational wave observatories on Earth, e.g. LIGO, VIRGO, GEO, or LISA—a future space mission [1].

A first indirect evidence for the existence of GWs was the observation of energy loss in the orbital motion of the Hulse–Taylor binary pulsar PSR B1913+16 being in full agreement with the predictions of Einstein’s theory using the quadrupole radiation formula. This discovery was awarded the Nobel prize in 1993. In the meantime, another strong indirect evidence occurred with the double pulsar system PSR J0737-3039A and B [2, 3]. For analysis of the measured GW patterns, one has to provide very accurate templates following from theory. This can be achieved by numerical calculations with the matching of functions to the results or directly using analytic tools. In the latter case, waveforms can be obtained only perturbatively due to missing analytic solutions of the Einstein field equations for two or more (spinning) compact...
objects (black holes (BHs), neutron stars (NSs)). One of the most successful approximation methods is the post-Newtonian one in which the metric remains close to the flat spacetime relying on the assumption that the typical velocity $v$ in a system divided by the speed of light $c$ is always small, $v/c \sim \epsilon \ll 1$. The deviations from the flat metric can be characterized by the Newtonian potential $\Phi$; so for a binary system, $\Phi/c^2 \sim v^2/c^2 \sim \epsilon^2$. In an appropriate limit (as $\epsilon \to 0$), the post-Newtonian (PN) approximation yields Newton’s equations.

The merging process of two compact objects is divided into four time scale sectors such as inspiral, plunge, merger and ring-down. Each sector delivers characteristic theoretical wave patterns, which are hoped to be matched against measured signals in the future. The PN approximation provides an excellent analytic handling for the inspiral phase. If the PN calculations are very accurate and thus high in order one can make very sensible predictions when comparing with measured signals. In this paper we focus on calculations of the next-to-leading-order (NLO) dynamics of spin-induced quadrupolar deformation effects. Surely, the most compact dynamical object is the Hamiltonian, generating the equations of motion (EoM), so we calculate in section 2 the NLO spin-squared one including a constant $C_Q$ parameterizing spin-induced quadrupolar deformation effects. $C_Q$ can be given definite values describing BHs or NSs. For NSs, $C_Q$ also depends on the model or equation of state (EoS). Thus our result, as it seems to be necessary to accurately measure $C_Q$ within future GW astronomy, could help to find the right EoS. The Hamiltonian in the present paper is calculated within the canonical formalism of Arnowitt, Deser and Misner (ADM) [4]. It should be noted that our Hamiltonian is fully reduced in the sense that the spin supplementary condition (SSC) is eliminated on the level of the Hamiltonian. Further we make a formal counting of the spin as $c^0$ and do not distinguish between fast and slowly spinning objects (see, e.g. [5] and also appendix A of [6]).

By now there are a lot of results regarding spin effects at the conservative orders in the PN approximation. The leading order (LO) spin effects are well known for BHs, see, e.g. [7–11]. The LO $C_Q$-dependence is given in [9, 12]. The NLO spin effects were only tackled recently. The first derivation of the NLO spin–orbit (SO) EoM was given in [13] which was further developed in [14], both in harmonic gauge. Later, within the ADM canonical formalism, a Hamiltonian presentation was achieved [15] (see also [16]). The NLO spin(1)–spin(2) dynamics was found in [16, 17] and confirmed by [18, 19]. Higher PN orders linear in spin were tackled recently in [6, 20, 21]. In particular, [20] extended the point-mass ADM formalism to spinning objects, valid to any order linear in spin. Even Hamiltonians of cubic and higher order in spin were obtained for binary black holes (BBHs) [5, 22]. Besides quadrupolar deformations induced by proper rotation (spin) and treated in the present paper, tidal deformations induced through the gravitational field of the other object were also treated, see, e.g. [23–25].

A nonreduced potential (i.e. with SSC not eliminated on the level of the potential) corresponding to the result of the present paper was already calculated in [26]; however, a term relevant for the centre-of-mass motion was missing and only found recently [27]. A comparison with the result of the present paper is not trivial, if one wants to avoid comparing all (rather long) EoM; instead, it is more efficient to stay on the level of the (relatively short) potential. In section 3 we sketch how to transform the potential from [26, 27] into a reduced Hamiltonian where we will find full agreement with our result of the present paper. Considering the special case of BHs (or $C_Q = 1$), we already succeeded in calculating the Hamiltonian of the present paper in [5, 28], providing for the first time both the spin and correct centre-of-mass dynamics in this case. There we were only able to find agreement with [26] in the spin precession equation, see [29] (after identifying a sign typo in [26]). With the correction in [27] a full comparison can now be attempted. It will be provided in section 3.
More work needs to be done for an application of the result of the present paper to GW astronomy. In order to obtain the NLO radiation field (for the SO case see [30, 31]) the stress–energy tensor has to include spin-squared corrections. This stress–energy tensor arises from the one with a general quadrupole [32] by a spin-squared ansatz for the mass quadrupole, see [28]. Moreover, the NLO spin contribution should be of importance for data analysis. In a recent publication [33] it has been shown that for maximal spins (aligned with the total orbital angular momentum), the event rates are roughly 30 times larger than those matter systems with anti-aligned spins to the orbital angular momentum and 8 times as large as for non-spinning binaries. So especially considering such sources the event rate will increase with the inclusion of spin effects. Further, for the creation of templates, it is useful to find a parametrization of the orbits by solving the EoM. It is common to describe the conservative dynamics in terms of certain orbital elements. Spin precession and dissipative effects can then be formulated as secular EoM of the orbital elements. For explicit solutions including spin at LO SO see, e.g. [34, 35].

2. The NLO spin-squared Hamiltonian

We start with giving a short overview concerning the calculation of the Hamiltonian in question. This calculation is done within the ADM canonical formalism [4]. We use units in which \(16\pi G = c = 1\), where \(G\) is the Newtonian gravitational constant and \(c\) is the velocity of light. Greek indices will run over 0, 1, 2, 3, and Latin over 1, 2, 3. For the signature of spacetime we choose +2. We employ the following notations: \(x = (x^i)\) \((i = 1, 2, 3)\) denotes a point in the three-dimensional Euclidean space \(\mathbb{R}^3\) endowed with a standard Euclidean metric and a scalar product (denoted by a dot). The letters \(a\) and \(b\) are body labels (usually they are set to 1 or 2), so \(x_a \in \mathbb{R}^3\) denotes the position of the \(a\)th point mass. We also define \(r_a := x - x_a\), \(r_a := |r_a|\), and for \(a \neq b\), \(r_{ab} := x_a - x_b\). For a normal vector \(n\) we use \(|\cdot|\) stands here for the length of a vector. The linear momentum vector of the \(a\)th body is denoted by \(p_a := (p_{ai})\), and \(m_a\) denotes its mass parameter. The usual flat space spin vector of the \(a\)th body (in local coordinates) is denoted by \(S_a = (S_{ai}(\varphi))\) in correspondence with our paper [28], and its associated antisymmetric tensor by \(\epsilon_{a(i)jk} = \delta_{jk} S_{a(i)}\). For \(a\), \(\delta\)-symbol defined as \(\delta_{123} = 1\). We abbreviate \(\delta (x - x_a)\) by \(\delta_a\). The partial differentiation with respect to \(x^i\) is denoted by \(\partial_i\) or by a comma, i.e. \(\partial_i \phi \equiv \phi_{,i}\); the partial differentiation with respect to \(x^0\) we denote by \(\partial_{\const}\).

Following the ADM canonical formalism, the independent degrees of freedom of the gravitational field are described by \(h^{TT}_{ij}(\varphi)\), the transverse-traceless part of \(h_{ij} = g_{ij} - \delta_{ij}\) \((h^{TT}_{ii} = 0, h^{TT}_{ij, j} = 0\)), and by conjugate momenta \(\pi^{TT}_{ij}\). The needed energy and linear momentum density expressions are given by

\[
\gamma \frac{1}{2} T^{\mu\nu} n_\mu n_\nu = H^{\text{m(atter)}},
\]

\[
-\gamma \frac{1}{2} T^{\mu}_{\mu} n_\mu = H^{\text{m(atter)}}_{\text{const}},
\]

where \(\gamma = \det(g_{ij})\), \(\gamma^{ij}\) is inverse to \(g_{ij}\), \(n^\mu\) is a unit timelike normal to hypersurface \(x^0 = \text{const}\) and \(T^{\mu\nu}\) is the stress–energy tensor of the matter system. Hereafter, we call its constituents the ‘particles’, but they may well represent NSs or BHs. This is substantiated by ‘general relativity’s adherence to the strong equivalence principle’: BHs and other compact bodies, to some approximation, obey the same laws of motion as test bodies; see, e.g. [36]. Also, the analysis of the initial-value solutions for BHs shows that as in electromagnetism, where image charges are described by delta functions, BHs in full general relativity can be represented by
'image masses' with delta functions support [37]. It is convenient to choose the following four coordinate conditions \[ \pi^{ii} = 0, \quad g_{ij} = \psi^{4} \delta_{ij} + h_{ij}^{TT}, \quad \psi = \left(1 + \frac{1}{8} \phi\right). \] (3)

The standard ADM Hamiltonian (cf [4])
\[ H = \oint dS_{i} (g_{ij,i} - g_{jj,i}), \] (4)
then becomes, using the Gauss theorem,
\[ H = - \int d^{3}x \Delta \phi. \] (5)

The integrand \[ \Delta \phi = \partial_{i} \partial_{i} \phi \] can be expressed in terms of \[ x_{a}, p_{a}, S_{a}, h_{ij}^{TT} \] and \[ \pi_{ij}^{TT} \] using the constraint equations. By expansions of the field equations in powers of \[ G \] and after adopting suitable regularization procedures of integrals involved (see, e.g. [38] and the appendix in [39]), one can determine the Hamiltonian.

Following the procedure outlined in our previous papers [5, 28], the Hamiltonian and the other generators are constructed as to fulfil the Poincaré algebra up to 2PN order depending on standard canonical variables
\[ \{x_{a}, p_{b}\} = \delta_{ab}, \] (6)
\[ \{S^{(i)}_{a}, S^{(j)}_{a}\} = \epsilon_{ijk} S^{(k)}_{a}, \] (7)
with all other brackets being zero. The coefficient equations resulting from this procedure will change due to a modified Hamiltonian and CoM vector entering the crucial relation \[ \{G_{i}, H\} = P_{i}, \] see equation (2.4) in [5]. The modification leading to spin quadrupolar deformation effects of a general compact object has to be made in the LO spin-squared Hamiltonian labelled as \[ H_{S}^{2} \] in equation (2.8) in [5], which now has to include a general spin-quadrupole constant \[ C_{Q} \] see [12]; additionally an appropriate \[ S_{T}^{2} \]-CoM vector has to be found. Both can be accomplished by adopting the static source expression for \[ H^{m} \] from our paper [28] equation (4) and incorporating the \[ C_{Q} \] constant reading
\[ H_{S}^{2} \text{static} = \frac{c_{1}}{m_{1}} \left( I_{1}^{ij} \delta_{1}\right)_{ij} + \frac{1}{8m_{1}} g_{mn} \gamma^{pj} \gamma^{mi} \gamma^{pk} S_{ij} S_{kl} \delta_{1} \]
\[ + \frac{1}{4m_{1}} \left( \gamma^{ij} \gamma^{mn} \gamma^{kl} S_{1mn} S_{1kl} \delta_{1}\right)_{ij}, \] (8)
\[ c_{1} = - \frac{1}{2} C_{Q}. \] (9)

meaning \[ C_{Q} = 1 \] for BH. Symbolic abbreviations in this formula are taken unaltered from the original paper thus denoting the same mathematical objects. This means that \[ S_{0}(j) \] being given in an Euclidean basis can be related to a spin tensor \[ \tilde{S}_{ij} \] in a coordinate basis with the help of a triad (dreibein) \[ e_{i}(j) \] by \[ \tilde{S}_{ij} = e_{i(k)} e_{j(l)} S_{kl}. \] The dreibein as a function of the metric is just \[ e_{i}(j) = \psi^{2} \delta_{ij} \] because the metric can be taken as conformally flat, \[ g_{ij} = \psi^{4} \delta_{ij}, \] in our approximation. The mass quadrupole tensor of object 1, \[ I_{1}^{ij}, \] is given by
\[ I_{1}^{ij} = \gamma^{ik} \gamma^{jl} \gamma_{mn} \tilde{S}_{1mn} \tilde{S}_{1kl} + \frac{2}{S_{1}^{2}} \gamma^{ij}, \] (10)
\[ 2 S_{1}^{2} = \gamma^{ik} \gamma^{jl} \tilde{S}_{1ij} \tilde{S}_{1kl} = \text{const.} \] (11)
The relation to $Q_i^{ij}$ and $a_i^j$ is $I_1^{(i)(j)} = m_i^2 Q_i^{ij}$ and $S_i^j = m_i^2 a_i^j$, so in LO the related quadrupole-moment tensor $Q_i^{ij}$ is just given by

$$Q_i^{ij} = a_i^{(i)} a_i^{(j)} - \frac{1}{3} a_i^2 \delta^{ij}.$$  

(12)

This static source alone is also enough to determine all the $G^2$ terms (static, free of linear momenta) of the Hamiltonian in question. The LO spin-squared Hamiltonian and the $S_i^j$-CoM vector are calculated via the formulae $H = -\int d^3 x \phi$ and $G_i = -\int d^3 x \phi^2$, respectively, with a post-Newtonian perturbatively expanded $\phi$ and $\mathcal{H}^\text{matter}_{S_i^j, \text{static}}$ according to equations (4.14)–(4.16) in [5]. The results are

$$H_{S_i^j}^{CQ} = \frac{G m_1 m_2}{r_{12}^3} C_Q \left( \frac{3 (S_1 \cdot n_{12})^2}{m_i^2} - \frac{S_i^3}{m_i^2} \right),$$  

(13)

and

$$G_{S_i^j} = G m_2 m_1 \left[ v_1 \left( \frac{(S_1 \cdot n_{12}) S_1}{r_{12}^3} + \frac{(S_1 \cdot n_{12})^2}{r_{12}^3} \right) \left( v_2 x_1 + v_3 x_2 \right) + \frac{S_i^3}{r_{12}^3} \left( v_4 x_1 + v_5 x_2 \right) \right],$$  

(14)

with coefficients

$$v_1 = -1 + \frac{3}{2} C_Q, \quad v_2 = \frac{3}{2} C_Q, \quad v_3 = \frac{3}{2} C_Q, \quad v_4 = \frac{1}{2} + \frac{1}{2} C_Q, \quad v_5 = -1 + \frac{3}{2} C_Q.$$  

(15)

The non-static (momenta based) terms of the Hamiltonian will be determined via the same ansätze for the source terms in $\mathcal{H}^m$ and $\mathcal{H}_i^m$ as in [5] with the LO quadrupole moment (12) reading

$$\mathcal{H}^m = \sum_{b=1}^{2} \left[ -\frac{m_b}{2} C_Q Q_b^{ij} \partial_i \partial_j - \frac{1}{2} p_b \cdot (a_b \times \partial) + \left( v^{ij} p_{b_i} p_{b_j} + m_b^2 \right)^{1/2} + \lambda_1 \frac{p_b^2}{2 m_b} Q_b^{ij} \partial_i \partial_j + \lambda_2 \frac{p_b \cdot (a_b \times \partial)}{m_b} \right],$$  

(16)

$$\mathcal{H}_i^m = -2 \sum_{b=1}^{2} \left[ Q_b^{ij} \left( \lambda_5 p_{b \delta} \partial_i \partial_j + \lambda_6 p_{b_i} \partial_j \partial_b + \lambda_7 (p_b \cdot \partial) \delta_i j \partial_b \right) + \lambda_4 a_i^2 (p_b \cdot \partial) \partial_i + \frac{m_b}{4} (a_b \times \partial)_i \left( 1 - \frac{1}{6} Q_b^{ij} \delta_i \partial_j \right) - \frac{1}{2} p_{b_i} \right] \delta_b.$$  

(17)

Note that the static term in $\mathcal{H}^m$ involves the $C_Q$ constant which follows from the expansion of (8) being the only modification of equation (4.11) in [5]. These sources allow the calculation of the NLO spin-squared Hamiltonian (with yet undetermined coefficients) leading to the same coefficient equations (4.50)–(4.62) in [5] except for the test particle terms $\beta_2$ and $\beta_3$ which are just multiplied by the quadrupole constant $C_Q$. These equations have to be matched to the ones resulting from the requirement of fulfilling the Poincaré algebra which now include the $C_Q$ constant and for that reason will slightly differ from equations (3.8)–(3.22) in [5]. The matching procedure then fixes all the coefficients left attributing to the source term coefficients the values

$$\lambda_1 = \frac{7}{4} - \frac{3}{2} C_Q, \quad \lambda_2 = \frac{5}{4} + \frac{3}{2} C_Q, \quad \lambda_3 = -\frac{1}{24}, \quad \lambda_4 = 0,$$  

(18)

$$\lambda_5 = \frac{1}{12} - \frac{C_Q}{6}, \quad \lambda_6 = -\frac{1}{8} + \frac{C_Q}{4}, \quad \lambda_7 = \frac{1}{8},$$  

(19)

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which agree with our results obtained in [5] when setting $C_Q = 1$. In view of section 3 from now on we label standard canonical variables with a ‘hat’ specifying its affiliation to the Newton–Wigner (NW) SSC except for the momentum which is chosen to be the same for NW and covariant SSC. The ‘hatted’ variables are then called NW variables in the sense that they are standard canonical, meaning

$$\{\hat{x}_a, \hat{p}_{bj}\} = \delta^j_i \delta_{ab},$$ (20)

$$\{\hat{S}_a^{(i)}, \hat{S}_a^{(j)}\} = \epsilon_{ijk} \hat{S}_a^{(k)},$$ (21)

$$\hat{p}_{ai} = p_{ai},$$ (22)

all other brackets being zero. Subtleties arising from that definition of NW variables are discussed in detail in our comment [29].

The resulting NLO spin-squared Hamiltonian for general compact binaries reads

$$H_{\text{NLO}}^{\text{ADM can}} = \frac{G}{r_{12}^3} \left[ \frac{m_2}{m_1} \left( \frac{21}{8} + \frac{9}{4} C_Q \right) p_1^2 (\hat{S}_1 \cdot \hat{n}_{12})^2 \right. + \left( \frac{15}{4} - \frac{9}{2} C_Q \right) (p_1 \cdot \hat{n}_{12}) (\hat{S}_1 \cdot \hat{n}_{12}) (\hat{S}_1 \cdot p_1) + \left( -\frac{5}{4} + \frac{3}{2} C_Q \right) (\hat{S}_1 \cdot p_1)^2 \right.$$

$$+ \left( \frac{9}{8} + \frac{3}{2} C_Q \right) (p_1 \cdot \hat{n}_{12})^2 \hat{S}_1^2 + \left( \frac{5}{4} - \frac{5}{4} C_Q \right) p_1^2 \hat{S}_1^2 \right.$$

$$+ \frac{1}{m_1} \left( -\frac{15}{4} C_Q (p_1 \cdot \hat{n}_{12}) (p_2 \cdot \hat{n}_{12}) (\hat{S}_1 \cdot \hat{n}_{12})^2 \right.$$

$$+ \left( 3 - \frac{21}{4} C_Q \right) (p_1 \cdot p_2) (\hat{S}_1 \cdot \hat{n}_{12})^2 \right.$$

$$+ \left( -\frac{3}{2} + \frac{9}{2} C_Q \right) (p_2 \cdot \hat{n}_{12}) (\hat{S}_1 \cdot \hat{n}_{12}) (\hat{S}_1 \cdot p_1) \right.$$ 

$$+ \left( -\frac{3}{2} + \frac{3}{2} C_Q \right) (p_1 \cdot \hat{n}_{12}) (\hat{S}_1 \cdot \hat{n}_{12})(\hat{S}_1 \cdot p_2) \right.$$ 

$$+ \left( \frac{3}{2} - \frac{3}{2} C_Q \right) (\hat{S}_1 \cdot p_1)(\hat{S}_1 \cdot p_2) \right.$$ 

$$+ \left( \frac{3}{2} - \frac{3}{4} C_Q \right) (p_1 \cdot \hat{n}_{12}) (p_2 \cdot \hat{n}_{12}) \hat{S}_1^2 \right.$$ 

$$+ \left( -\frac{3}{2} + \frac{9}{4} C_Q \right) (p_1 \cdot p_2) \hat{S}_1^2 + \frac{C_Q}{m_1 m_2} \left( \frac{9}{4} p_1^2 (\hat{S}_1 \cdot \hat{n}_{12})^2 - \frac{3}{4} p_1^2 \hat{S}_1^2 \right) \right] \right.$$ 

$$+ \frac{G^2 m_2}{r_{12}^4} \left[ \left( 2 + \frac{1}{2} C_Q + \frac{m_2}{m_1} (1 + 2 C_Q) \right) \hat{S}_1^2 \right.$$ 

$$+ \left( -3 - \frac{3}{2} C_Q + \frac{m_2}{m_1} (1 + 6 C_Q) \right) (\hat{S}_1 \cdot \hat{n}_{12})^2 \right],$$ (23)

for $C_Q = 1$ being in full agreement with the result for BH presented for the first time in [28].

3. Comparison with the NLO spin(1)–spin(1) potential

In order to transform the Routhian $R$ obtained within the effective field theory (EFT) approach [26, 27] to a nonreduced Hamiltonian $H$, we first have to eliminate the acceleration term [27]

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with the help of the Newtonian EoM (corresponding to a redefinition of the position variables, see [40]). This generates correction terms to the order $G^2$ in the NLO spin-squared potential. Next, one must replace velocities by canonical momenta $p_i = \frac{\partial H}{\partial v_i}$ to get the Hamiltonian by a Legendre transformation, i.e.

$$H = v_1 \cdot p_1 + v_2 \cdot p_2 - R.$$  

(24)

The canonical momentum $p_1$ necessary to cover all NLO spin effects explicitly reads

$$p_1 = \left(1 + \frac{1}{2} v_1^2\right) m_1 v_1 + \frac{G m_1 m_2}{2 r_{12}} \left[6 v_1 - 7 v_2 - (n_{12} \cdot v_2) n_{12}\right]$$

$$+ \frac{G}{r_{12}^2} \left[m_{12} (n_{12} \times S_1) + 2 m_1 (n_{12} \times S_2)\right],$$  

(25)

and similarly for particle 2. The Poisson brackets at this stage are

$$\{x_i^a, p_j^b\} = \delta_i^b \delta_{ab},$$  

(26)

$$\{S_a^{(i)} , S_a^{(j)} \} = \epsilon_{ijk} S_a^{(k)},$$  

(27)

$$\{S_a^{(0)(i)} , S_a^{(0)(j)} \} = \epsilon_{ijk} S_a^{(0)(k)},$$  

(28)

$$\{S_a^{(0)(i)} , S_a^{(0)(j)} \} = -\epsilon_{ijk} S_a^{(0)(k)},$$  

(29)

and zero otherwise$^1$. Note that these are not yet the reduced or standard canonical brackets as $S_a^{(0)(i)}$ is still an independent degree of freedom and was not eliminated using the covariant SSC $S_a^{(m)} u_0 = 0$.

It is well known that one has to proceed to Dirac brackets (DBs) if $S_a^{(0)(i)}$ is going to be eliminated from the Hamiltonian $H$ using a SSC, see, e.g. [41, 42]. However, it is possible to find new variables $\hat{x}_i^a$, $\hat{p}_j^b$ and $\hat{S}_a^{(i)}$ for which the Dirac brackets take on the standard form

$$\{\hat{x}_i^a, \hat{p}_j^b\}_{\text{DB}} = \delta_i^b \delta_{ab},$$  

(30)

$$\{\hat{S}_a^{(i)} , \hat{S}_a^{(j)} \}_{\text{DB}} = \epsilon_{ijk} \hat{S}_a^{(k)},$$  

(31)

and zero otherwise. These new variables can only be unique up to canonical transformations. This freedom allows us to choose $p_{1i} = \hat{p}_{1i}$, as for the flat space case [41]. A possible transition to $\hat{x}_i^a$ and $\hat{S}_a^{(i)}$ then reads

$$S_{a1(i)(j)} = \hat{S}_{a1(i)(j)} - \left[\frac{p_{1i} \hat{S}_{11(i)(k)} p_{1k}}{m_1^2} \left(1 - \frac{3 p_1^2}{4 m_1^2}\right) - \frac{2 G m_2}{m_1^2 r_{12}} p_{1i} \hat{S}_{11(i)(k)} p_{1k}\right]$$

$$+ \frac{3 G}{m_1 r_{12}^2} p_{1i} \hat{S}_{11(i)(k)} p_{1k} + \frac{G}{m_1 r_{12}^2} p_{1i} \hat{S}_{11(i)(k)} \hat{p}_{12} (\hat{n}_{12} \cdot \hat{p}_2)$$

$$+ \frac{2 G m_2}{m_1 r_{12}^2} p_{1i} \hat{S}_{11(i)(k)} \hat{p}_{12} + \frac{2 G}{m_1 r_{12}^2} p_{1i} \hat{S}_{11(i)(k)} \hat{S}_{2(k)(l)} \hat{p}_{12},$$  

(32)

$$x_i^a = \hat{x}_i^a - \left[\frac{1}{2 m_1^2} p_{1i} \hat{S}_{11(i)(k)} \left(1 - \frac{p_1^2}{4 m_1^2}\right) - \frac{m_2}{m_1^2} \frac{p_{1k} \hat{S}_{11(i)(k)}}{r_{12}}\right]$$

$$+ \frac{3 G}{m_1 r_{12}^2} p_{1i} \hat{S}_{11(i)(k)} \hat{p}_{12} + \frac{G}{2 m_1 r_{12}^2} (\hat{n}_{12} \cdot \hat{p}_2) \hat{S}_{11(i)(k)}$$

$$+ \frac{G m_2}{m_1^2} \hat{S}_{11(i)(k)} \hat{p}_{12} \frac{1}{r_{12}} + \frac{G}{m_1^2 r_{12}^2} \hat{S}_{2(k)(l)} \hat{p}_{12}.$$  

(33)

$^1$ Note that in [18, 26] a different sign convention was used for the Poisson brackets of the spin. As usual, we are not showing the canonical conjugate of the spin here; see [41].
where the antisymmetrization of indices pertaining to a tensor \( A_{ij} \) is defined as \( A_{[ij]} = 1/2(A_{ij} - A_{ji}) \). The rather complicated form of these variable transformations reflects the complicated structure of the DBs for self-interacting spinning objects. In [21] test-spinning objects are considered and [41] covers the flat space case only. We will elaborate on its specific calculation in another paper. Note that these results are applicable to all NLO spin effects (for the spin–orbit contributions the corrected form of the tetrad from [29] has to be inserted into the SSC). To the best of our knowledge this is the first time that DBs are applied to gravitationally self-interacting spinning objects. In [21] test-spinning objects are considered and [41] covers the flat space case only.

The Routhian from [26] now leads us to the reduced NLO spin-squared Hamiltonian in the form

\[
H_{\text{NLO}}^{\text{S}^{2}} = \frac{G}{r_{12}^{2}} \left[ \frac{m_{2}}{m_{1}} \left( \left( -\frac{21}{8} + \frac{9}{4} C_{Q} \right) \left( \hat{S}_{1} \cdot \hat{n}_{12} \right)^{2} + \left( 2 - \frac{3}{2} C_{Q} \right) \left( \hat{S}_{1} \cdot \hat{n}_{12} \right) \left( \hat{S}_{1} \cdot \hat{p}_{1} \right) + \left( 3 - \frac{15}{4} C_{Q} \right) \left( \hat{S}_{2} \cdot \hat{n}_{12} \right) \left( \hat{S}_{1} \cdot \hat{p}_{2} \right) \right] \right] + \frac{G^{2} m_{2}}{r_{12}^{4}} \left[ \left( \frac{1}{2} + \frac{3}{2} C_{Q} \right) \left( \hat{S}_{1} \cdot \hat{n}_{12} \right) \left( \hat{S}_{1} \cdot \hat{p}_{1} \right) - \frac{m_{2}}{m_{1}} \left( \frac{1}{2} + 6 C_{Q} \right) \left( \hat{S}_{1} \cdot \hat{n}_{12} \right) \left( \hat{S}_{1} \cdot \hat{p}_{1} \right) \right].
\]

This Hamiltonian and the one calculated in the last section should differ only up to a canonical transformation. It should thus be possible to generate the difference \( \Delta H_{\text{NLO}}^{\text{S}^{2}} = H_{\text{NLO}}^{\text{S}^{2}} - H_{\text{NLO}}^{\text{can}} \) by a canonical transformation of the form

\[
\Delta H_{\text{NLO}}^{\text{S}^{2}} = \left\{ H_{N}, g_{\text{NLO}}^{\text{can}} \right\},
\]

with \( H_{N} \) being the Newtonian Hamiltonian of a two-body system and \( g \) being an appropriate generator. It turns out that with the generator

\[
g_{\text{NLO}}^{\text{can}} = G \frac{m_{2}}{r_{12}^{2}} \left[ \left( \frac{1}{2} + C_{Q} \right) \left( \hat{S}_{1} \cdot \hat{n}_{12} \right) \left( \hat{S}_{1} \cdot \hat{p}_{1} \right) + \frac{1}{2} \left( \hat{S}_{1} \cdot \hat{n}_{12} \right) \left( \hat{S}_{1} \cdot \hat{p}_{1} \right) \right],
\]

equation (35) can be fulfilled and so the agreement is achieved. This means that the Hamiltonian \( H_{\text{NLO}}^{\text{can}} \) calculated with the aid of the ADM method, in terms of invariant physical quantities, agrees with the Routhian from above; hence, there is great confidence that
$H_{\text{ADMcan}}^{\text{NLO}}$ correctly describes a binary consisting of BHs and/or NSs or other kinds of compact objects in post-Newtonian Einsteinian theory. The new Hamiltonian may find immediate application in the problem of motion of orbiting binaries as investigated and solved in e.g. [35, 43].

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