The No-Boundary Wave Function and the Duration of the Inflationary Period

A. Lukas

Physik Department
Technische Universität München
D-85747 Garching, Germany

Abstract

For the simplest minisuperspace model based on a homogeneous, isotropic metric and a minimally coupled scalar field we derive analytic expressions for the caustic which separates Euclidean and Minkowskian region and its breakdown value $\phi_*$. This value represents the prediction of the no-boundary wave function for the scalar field at the beginning of inflation. We use our results to search for inflationary models which can render the no-boundary wave function consistent with the requirement of a sufficiently long inflationary period.
Quantum cosmology \[1\] can shed new light on the problem of initial conditions in the universe. To be predictive boundary conditions have to be imposed on the wave function of the universe. What quantum cosmology can do is therefore to replace the procedure of imposing specific boundary values by a “theory of boundary conditions”. Several proposals for such a theory have been made. The most prominent ones are the no-boundary proposal due to Hartle and Hawking \[2\] and the tunneling proposal due to Linde and Vilenkin \[3\]. Clearly, all these proposals are speculative and have to be confronted with reality i.e. with the main characteristics of our universe.

In the simplest minisuperspace model which is based on a homogeneous, isotropic metric with scale factor $a$ and a single, minimally coupled scalar field $\phi = \phi(t)$ both the no-boundary and the tunneling wave function predict a correlation between $a$, $\phi$ and their conjugate momenta if $a$ is sufficiently large. Assuming decoherence this correlation corresponds to a classical universe in a de-Sitter phase. They are therefore in agreement with the classical behaviour of the universe we live in and the inflationary paradigm of cosmology. In addition, a probability distribution $P(\phi)$ for the field $\phi$ at the beginning of the inflationary period can be extracted from the wave function. It can be used to compute a conditional probability $P_{\text{suff}}$ for sufficient inflation:

$$P_{\text{suff}} = \frac{\int_{\phi_{\text{suff}}}^{\phi_{2}} P(\phi) d\phi}{\int_{\phi_{1}}^{\phi_{2}} P(\phi) d\phi}. \quad (1)$$

Here $\phi_{\text{suff}}$ is the minimum value of $\phi$ which guarantees the $N_{\text{min}} \approx 60$ e-folds of inflation needed to solve the problems of standard cosmology. Since the wave function has been determined in a semiclassical approximation one should cut off the integral at the value $\phi_{2}$ corresponding to Planck energy. $\phi_{1}$ is the minimal $\phi$ which leads to an expanding universe.

For the tunneling wave function the probability distribution

$$P_{T}(\phi) = \exp \left( -\frac{2}{3V(\phi)} \right) \quad (2)$$

prefers large values of $\phi$ and therefore generically predicts sufficient inflation. However, the opposite is true for the no-boundary result

$$P_{\text{NB}}(\phi) = \exp \left( \frac{2}{3V(\phi)} \right). \quad (3)$$

Since conventionally the potential $V$ contains a tiny coupling in order to keep the density fluctuations small enough $P_{\text{NB}}$ is so strongly peaked on low values of $\phi$ that it essentially predicts the lower cutoff value $\phi_{1}$ which is much smaller than $\phi_{\text{suff}}$. Clearly, this argumentation can be doubted since the behaviour of $P(\phi)$ beyond Planck energy where the above expressions are not valid may change the picture completely. It has, however, been argued that the one loop contribution to the distribution $P(\phi)$ can suppress the constant tail at large $\phi$ \[4\]. This can render $P(\phi)$ normalizable and serve as a justification for the semiclassical approximation. Here we assume that such a suppression indeed takes place.
which, in addition, leaves the low energy behaviour of the probability distribution - as described by the above equations - essentially unchanged. Furthermore, we assume that we stay in a regime where the back reaction of quantum fluctuations can be neglected and the field $\phi$ follows a classical slow roll trajectory. Then $P_{\text{suff}}$ should indeed be large for a successful inflationary model and its computation provides a check for a certain boundary proposal.

Grishchuk and Rozhansky [5] have pointed out that the existence of a caustic in the $a$-$\phi$-plane is a necessary condition for a transition from the Euclidean region at small $a$ to the classical Minkowskian region at large $a$. For small $\phi$ this caustic breaks down so that below a certain value $\phi_* > \phi_1$ no classical universe develops. Therefore, in eq. (3) $\phi_1$ should be replaced by $\phi_*$. Then the no-boundary wave function predicts $\phi_*$ as the value of $\phi$ at the beginning of inflation. This value strongly depends on the potential and it has to be clarified whether $\phi_*$ can be larger than $\phi_{\text{suff}}$ for some choices.

In ref. [5] $\phi_*$ has been determined numerically for the potentials $V = \lambda \phi^2$ and $V = \lambda \phi^4$. In both cases it was found to be smaller than $\phi_{\text{suff}}$. However, it was left as an open problem whether or not a potential with $\phi_* > \phi_{\text{suff}}$ exists for the no-boundary wave function. This is the question we are going to address in this paper. As a first step we will derive an equation for the caustic and analytic expressions for $\phi_*$. These expressions allow us to discuss the general dependence of $\phi_*$ on the potential $V(\phi)$ and to analyze several classes of potentials in detail.

The Euclidean equations of motion for $a$ and $\phi$ which have to be solved in order to determine the semiclassical wave function are
\begin{align*}
\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} &= \frac{1}{2} \frac{dV}{d\phi} \\
\dot{a}^2 &= 1 - a^2 V + a^2 \dot{\phi}^2 \\
\ddot{a} &= -2\dot{\phi}^2 - V.
\end{align*}

According to the no-boundary proposal they have to be integrated by using the initial conditions
\begin{align*}
a(\tau = 0) &= 0, \\
\dot{\phi}(\tau = 0) &= 0, \\
\phi(\tau = 0) &= \phi_0.
\end{align*}

To illustrate the qualitative behaviour in fig. [4] we have plotted the trajectories for a number of starting values $\phi_0$ using the potential $V = \lambda \phi^4$. The caustic consists of the points of maximal $a$ and in this specific example it breaks down at $\phi_* \approx 3$. For large values of $\phi$ it behaves asymptotically like
\begin{equation}
a^2 V = 1
\end{equation}
(dashed line in fig. [4]) as it can be expected from the second eq. (4). Near the breakpoint, however, it deviates substantially from this asymptotic expression. As a first step we will now improve eq. (4) such that it also describes the caustic for small $\phi$.

\footnote{The value of $\phi_*$ does not depend on the coupling $\lambda$ which we have set to 1 for the numerical integration.}
Figure 1: Trajectories for the potential $V = \phi^4$, approximate caustic $a^2 V = 1$ (dashed line) and improved caustic (dotted line).

Obviously, informations about the time dependency of $a$ and $\phi$ are not needed for this. Instead, we can look at the equation

$$(1 - a^2 V)\phi'' + \left[\frac{1}{a}(3 - 4a^2 V) - a(2 - 3a^2 V)\phi'^2\right]\phi' = \frac{1 - a^2 \phi'^2}{2} \frac{dV}{d\phi}$$

(6)

for $\phi = \phi(a)$ ($\phi' = d\phi/da$) which can be derived from eq. (4). Surprisingly, this equation has two exact solutions

$$\phi' = \pm \frac{1}{a}$$

(7)

which are independent on the potential $V$. It turns out that the second of these solutions ($\phi' = -1/a$) asymptotically describes the “return trajectories” i. e. the trajectories after the turning point at maximal $a$. Since we are mainly interested in the behaviour near the approximate caustic $a^2 V = 1$ we linearize eq. (6) in $\epsilon = a^2 V_c - 1$, $V_c = V(\phi_c)$ to obtain

$$\epsilon \frac{d\xi}{de} \simeq 2 \left(\xi^2 - \frac{1}{4}\right) \left(\xi + \frac{1}{2y_c}\right), \quad \xi = \frac{d\phi}{de}.$$

(8)

We have introduced the variable $y$

$$y = \frac{2V}{V'}, \quad V' = \frac{dV}{d\phi}, \quad y_c = y(\phi_c).$$

(9)
which will be quite useful in the following. Eq. (8) can be integrated explicitly to give

\[
G(\xi) = \frac{|\epsilon|}{\epsilon_c(y_c)}
\]  

(10)

\[
G(\xi) = \left(\frac{\xi + 1}{2y_c}\right) \frac{2y_c^2}{\sqrt{1-y_c^2}} \left(\frac{\xi + 1}{2}\right)^{\frac{y_c^2}{\xi^2}} \left|\frac{\xi - 1}{2} \right|^{\frac{y_c^2}{\xi^2 + 1}}
\]  

(11)

with

\[
\epsilon_c(y_c) = \frac{2y_c^2}{y_c^2 - 1}
\]  

(12)

where we have imposed the boundary condition \( \xi = 0 \) at \( \epsilon = -1 \). At \( \epsilon = \epsilon_c(y_c) \) our solution shows a singularity \( \xi \to \infty \) which we identify with the turning point of the trajectory. From this observation we get for the caustic:

\[
a_c^2 V_c \simeq 1 + \epsilon_c(y_c).
\]  

(13)

In fig. 1 this curve is shown for our example \( V = \phi^4 \) (dotted line).

To determine \( \phi_* \) we have to distinguish two cases. Let us first assume that the potential in the range we are interested in is increasing rapidly. This is e.g. fulfilled for a polynomial potential like the one in fig. 1. Then the first part of the trajectories is approximately constant in \( \phi \) and because of the strong variation of the caustic with \( a \) the trajectories follow their asymptotic behaviour \( \phi' = -1/a \) even before they intersect the caustic for the second time. Then an expression for \( \phi_* \) can be computed by the following prescription. A return trajectory defined by \( \phi' = -1/a \) which starts at some point \((a_c, \phi_c)\) on the caustic (13) intersects the approximate caustic \( a^2 V = 1 \) at \((a, \phi)\). Using the eqs. (5), (7), (13) and (12) we find the relation

\[
\frac{e^\phi}{\sqrt{V(\phi)}} = \sqrt{1 + \epsilon_c(y_c)} \frac{e^{\phi_c}}{\sqrt{V(\phi_c)}}.
\]  

(14)

Now we minimize \( \phi \) as a function of \( \phi_c \) to identify the lowest return trajectory (cf. fig. 1). The value of \( \phi(\phi_c) \) at the minimum can be identified with \( \phi_* \). The explicit computation results in conditions on \( y_c = y(\phi_c) \) and \( \phi_* \):

\[
\frac{1}{y_c} - \frac{1}{2} \frac{d\epsilon_c}{dy}(y_c) y'(y_c) = 1
\]  

(15)

\[
\frac{e^{\phi_*}}{\sqrt{V(\phi_*)}} = \sqrt{1 + \epsilon_c(y_c)} \frac{e^{\phi_c}}{\sqrt{V(\phi_c)}}.
\]  

(16)

From eq. (12) it can be seen that \( \epsilon_c(y = 1) = e^{-1} \) and \( d\epsilon_c/dy(y = 1) = -e^{-1} \). Therefore, if

\[
y'(y) = 2 \left(1 - \frac{1}{4}y^2 \frac{V''}{V}(y)\right)
\]  

(17)
is $\lesssim O(1)$ at $y = 1$ we can consistently neglect the second term in eq. (15). We will see that this is indeed justified for a number of potentials. The above equations then simplify to

$$y_c = \frac{2V}{VV'}(\phi_c) = 1 \quad (18)$$

$$\frac{e^{\phi_*}}{\sqrt{V(\phi_*)}} = k \frac{e^{\phi_c}}{\sqrt{V(\phi_c)}}. \quad (19)$$

with

$$k = \sqrt{1 + e^{-1}} \approx 1.17.$$  

These equations now allow an explicit calculation of $\phi_*$ for a given sufficiently increasing potential. Since the constant $k$ in eq. (19) is close to unity $\phi_c$ and $\phi_*$ are of the same order for almost all potentials. An order of magnitude estimate for $\phi_*$ is therefore given by the solution to the simple equation

$$y = \frac{2V}{VV'} = 1. \quad (20)$$

For potentials with a very flat region e. g. caused by a maximum or an asymptotic behaviour $V \to$ const the situation is different. Fig. 2 shows the example $V = (1 - \exp(-\phi/f))^2$ for $f = 0.1$. The trajectories increase substantially from $a = 0$ up to the caustic which is nearly vertical. The return up to the second intersection with the caustic is too short to be described by the asymptotic equation $\phi' = -1/a$. Instead, we follow a trajectory starting out at $y_0 = y(\phi_0)$, $a = 0$ and intersecting $a^2V = 1$ at $y_c = y(\phi_c)$ by using the equations

$$\xi = \frac{1}{6y_0} \quad a \text{ small}$$

$$\xi = \frac{1}{2} \sqrt{\frac{\epsilon_c(y_c)}{\epsilon_c(y_c) - \epsilon}} \quad \epsilon \text{ small}.$$  

They can be derived from eq. (3) and from eq. (10) by expanding $G(\xi)$ up to the first order in $1/\xi$. These approximate first integrals for small $a$ and the region around the caustic can be “glued” together. Their integration gives the second intersection value $\phi$ with the caustic $a^2V = 1$ as a function of $\phi_0$. As before the minimum of this function should be identified with $\phi_*$. Following this prescription we arrive at a set of equations

$$\phi_* = \phi_c + \frac{2}{y_c^2}$$

$$\phi_c = \phi_0 + \frac{1}{3y_0} - \frac{y_0}{y'(y_0)}$$

$$\frac{3}{2y_c^2} = \frac{1}{6y_0^2} - \frac{1}{y'(y_0)} \quad (21)$$

An exception to this is given by an exponential potential which cancels the exponentials in the numerators of eq. (19) to a high accuracy. We have discussed this special case in an example below.
where we have used that $\epsilon_c(y) \simeq 1/y^2$ since $y \gg 1$ in a flat region of the potential.

For the two types of potentials - the increasing “chaotic” type of potentials and the type with a flat region - the eqs. (15), (16) and eq. (21) represent the general answer for the value of $\phi_*$. Clearly, we have not proven that our approximations are justified for all potentials. The results should therefore be used as a guide to find promising potentials whose properties then have to be verified numerically. In this way large classes of potentials can be analyzed without a numerical case by case study. In fact, for all potentials we have checked so far our analytic formulae gave the correct answer.

We start by discussing the consequences of the previous results on the first type of potentials. The number of e-folds in the slow roll approximation is given by

$$N_{\text{e-folds}} = 3 \int_{\phi_c}^{\phi_f} y(\phi) \, d\phi$$  \hspace{1cm} (22)

where $\phi_f$ is the end point of the slow roll regime determined by the breakdown of the conditions

$$y \gg \frac{1}{3}, \quad \frac{V''}{V} \ll 18.$$  \hspace{1cm} (23)

Suppose we have found a potential $V$ of the first type which generates the desired number of e-folds. Since $y$ does not increase “irregularly” it is plausible that the slow roll conditions (23) are still fulfilled at $\phi_c$. Consequently, our approximation leading to (18) and (19)
is justified and we have \( y_c \simeq 1 \) at \( \phi_c \). This leads to the rough approximation

\[
N_{\text{e-folds}} \lesssim 3\phi_c. 
\]  

(24)

It shows that for such potentials one will need a quite large value \( \phi_* \) to generate the needed \( N_{\text{min}} \simeq 60 \) e-folds. From \( y(\phi) < 1 \) for \( \phi \lesssim \phi_* \) we conclude that

\[
\frac{V(\phi_2)}{V(\phi_1)} > e^{2(\phi_2 - \phi_1)}
\]

for any \( \phi_2, \phi_1 \lesssim \phi_* \). In particular we have

\[
V(\phi_*) \simeq \lambda e^{2\phi_*} \sim \lambda e^{2N_{\text{min}}} \sim \lambda e^{40}
\]

(25)

where \( \lambda \) is the coupling in front of \( V \). Therefore a potential which can provide a large enough \( \phi_* \) either exceeds Planck energy at this point - which makes the predicted \( \phi_* \) unreliable - or needs a very small coupling \( \lambda \). In the second case quantum fluctuations in de Sitter space are generically too small to generate a sufficiently large initial density spectrum. Then other sources of such density perturbations have to be considered.

Examples are provided by the standard chaotic potentials \( V = \lambda \phi^n \), \( n = 2, 4, 6, \ldots \). It is easy to show that \( y'(y = 1) = 2/n \) so that the eqs. (18) and (19) can be applied. We get

\[
\phi_c = \frac{n}{2}.
\]

(26)

Writing \( \phi_* = \phi_c + \delta \) for sufficiently large \( n \) we can approximate \( \sqrt{V(\phi_*)} \simeq \sqrt{V(\phi_c)}(e^{1 - \delta/n})^k \) to obtain an explicit solution of eq. (19):

\[
\phi_* \simeq \frac{n}{2} + \sqrt{1 - k^{-1}} \sqrt{n} \simeq \frac{n}{2} + 0.38 \sqrt{n}.
\]

(27)

By numerical integration of eq. (3) we have checked up to \( n = 70 \) that this formula indeed results in reasonable values for \( \phi_* \). The number of e-folds is given by

\[
N_{\text{e-folds}} \simeq \frac{3}{n} \phi_*^2 \simeq \frac{3}{4} n + 1.14 \sqrt{n}.
\]

In accordance with the general discussion a very large value of \( \phi_* \) and correspondingly of \( n \) is needed to generate sufficient inflation. In fact this is possible for \( V = \lambda \phi^n \) with \( n \gtrsim 70 \) but an enormously small \( \lambda \) is needed to keep \( V(\phi_*) \) below Planck energy.

Let us next discuss the potential

\[
V = \Lambda^4 \left( 1 - \exp \left( \frac{\phi}{f} \right) \right)^2
\]

(28)

which (for \( f = 1 \)) can be derived from higher order gravity theories [3]. We concentrate on the interesting range \( f \gtrsim 1 \) where the exponentials in eq. (19) are canceled. In that case
\( \phi_\ast \) may be substantially larger than \( \phi_c \) and the above general argument is somewhat weak. From the eqs. (18), (19) and (22) we obtain for \( 0 < d = f - 1 \ll 1 \):

\[
\begin{align*}
\phi_c & \simeq \ln \frac{1}{d} \\
\phi_\ast & \simeq \ln \frac{1}{d} + \frac{\ln k}{d} \\
N(\phi_\ast) & \simeq 3\phi_\ast.
\end{align*}
\]

(29)

These formulae have been confirmed by numerical integration. For \( d \lesssim 0.01 \) a sufficient number of e-folds may be generated, however, in accordance with our general conclusion a small coupling \( \Lambda^4 \simeq e^{-40} \) is needed to keep \( V(\phi_\ast) \) below the Planck scale.

Now we turn to the second type of potentials with a flat region. An interesting example is provided by the above potential (28) in the negative \( \phi \) direction where it approximates a constant value. Quantum cosmology of this potential has been studied in a number of papers including [7]. Only for a value of \( \phi_\ast \) in the asymptotically flat region we can hope for sufficient inflation. Such values are obtained for \( f \ll 1 \) where we can solve the system (21) explicitly. We find:

\[
\frac{\phi_\ast}{f} \simeq \ln \frac{1}{1.45f^2} + 4.20f.
\]

(30)

This has been confirmed numerically. Given the formula

\[
N_{\text{e-folds}}(\phi_\ast) \simeq 3f^2 \exp \left( \frac{\phi_\ast}{f} \right)
\]

we see that independent on \( f \) only a few e-folds are generated starting at \( \phi_\ast \).

Another way to obtain a flat region is to consider a potential with a maximum at some value \( \phi = \phi_m \). A sufficiently long period of inflation will be produced for these potentials if \( \phi \) starts its evolution in a small range around the maximum. Basically, we have to ask two kinds of questions: Can the shape of the potential be such that \( \phi_\ast \) falls into this range and is there a fine tuning involved in arranging such a situation?

Before we address these questions in general let us be more specific and discuss a prominent example for these potentials: The natural inflation potential \( V = \Lambda^4(1 - \cos(2\phi/f)) \) which can be interpreted as the effective potential of a pseudo-goldstone boson. Its inflationary properties have been studied extensively in ref. [8]. The quadratic maximum of this potential is still “curved” enough to apply the eqs. (18) and (19). From

\[
\cos \left( \frac{2\phi_c}{f} \right) = \frac{f^2 - 1}{f^2 + 1}
\]

(31)

we see that \( \phi_c \) and correspondingly \( \phi_\ast \) approach the maximum for decreasing \( f \). The latter is found from a numerical solution of eq. (19) and is in agreement with the results from numerical integration of (4). At \( f \simeq 0.6 \) the value \( \phi_\ast \) reaches the maximum. In principle,
one can therefore hope that a value of \( f \) close to 0.6 produces sufficient inflation. However, taking into account that

\[
\pi - \frac{2\phi_{\text{eff}}}{f} \sim \exp \left( -\frac{N_{\min}}{3f^2} \right)
\]

this seems almost impossible: For \( f \approx 0.6 \) the starting point of the field \( \phi \) has to be at the maximum to such an enormous precision that one has to fine tune \( f \) in order to obtain a \( \phi_* \) in this range. Even if such a tuning is accepted one would have to worry about the fluctuations of the higher, inhomogeneous modes and their back reaction on the field \( \phi \). This argument does not depend on the specific choice of the potential but just on the quadratic behaviour around the maximum. Our conclusion holds for all potentials with such a quadratic maximum provided that the slow roll near the maximum is the main source of inflation.

We can generalize this analysis to higher order maxima. If we assume that the relevant phenomena appear near the maximum we can expand the potential around \( \phi_m \) up to the first nontrivial order \( 2n \):

\[
V = \Lambda^4(1 - \Delta^{2n}) + O(\Delta^{2n+1}) , \quad \Delta = \frac{\phi_m - \phi}{f} .
\]

For the number of e-folds we get

\[
N_{\text{e-folds}}(\Delta_*) \sim \frac{3f^2}{2n(n-1)\Delta_*^{2n-2}} \quad \text{for } n > 1 .
\]

In contrast to the \( n = 1 \) case the number of e-folds does not show a logarithmic dependence on the initial value so that fine tuning problems are irrelevant. The equations (21) can be solved with the ansatz \( y_0 = d_n/f^{n/(n-1)} \), \( y_c = c_n/f^{n/(n-1)} \) leading to

\[
\Delta_* \approx (nc_n)^{\frac{1}{2n-1}} f^{\frac{1}{n-1}} \quad \text{and} \quad N_{\text{e-folds}}(\Delta_*) \approx \frac{3}{2n(n-1)(nc_n)^{\frac{2n-2}{2n-1}}} .
\]

As for the exponential potential the number of e-folds turns out to be independent on \( f \). The numerical constants \( c_n \) are small, e.g. \( c_2 \approx 2.6, c_3 \approx 5.0 \), showing that \( N_{\text{e-folds}}(\Delta_*) \) remains below the needed value. For \( n = 2,3 \) this result has been verified by numerical integration.

In conclusion, we have determined an analytic equation for the caustic separating the Euclidean and Minkowskian region in the simplest minisuperspace model. Moreover, we derived formulae for the breakdown value \( \phi_* \) of the caustic which is the value of the scalar field \( \phi \) predicted by the no-boundary wave function. Our results give some insight in the structure of minisuperspace models and its dependency on the scalar field potential. They allow to analyze inflationary properties of these models without relying on a numerical case by case study. We have applied these results to a wide range of models with a single scalar field. Unfortunately, for potentials with an extraordinary flat region, e.g. caused
by a maximum or an asymptotic behaviour \( V \to \text{const} \) we could not find an example which leads to a sufficiently long period of inflation. For potentials with a maximum and an inflationary phase which mostly arises due to the slow roll near the maximum this behaviour seems to be general.

On the other hand, some potentials of “chaotic” type like \( V = \lambda \phi^n \) with \( n \gtrsim 70 \) can cause sufficient inflation. However, to stay within the limits of the semiclassical approximation the coupling \( \lambda \) generically has to be very small. A positive answer within these limits therefore asks for an alternative explanation of primordial density perturbations. From our results, one might also argue that an understanding beyond the semiclassical limit is necessary to decide about the existence of a “successful” model. On the other hand, we have concentrated on the simplest class of models with a single scalar field which in the inflationary phase behaves essentially classical. Non negligible quantum fluctuations, in particular in the context of multidimensional models, might, however, change the picture. From the viewpoint of particle physics models with more than one scalar field are the more realistic anyway. Generalizations of the results presented in this paper might be helpful in analyzing the quantum cosmology of such models.

**Acknowledgment** I would like to thank H. van Elst, Z. Lalak and R. Poppe for helpful discussions. This work was partially supported by the Deutsche Forschungsgemeinschaft and the EC under contract no. SC1-CT92-0789 and the CEC Science Program no. SC1-CT91-0729.

**References**

[1] For a review see e. g.: J. J. Halliwell, “Introductory Lectures on Quantum Cosmology” in Quantum Cosmology and Baby Universes, Jerusalem Winter School on Theor. Physics, S. Coleman, J. B. Hartle, T. Piran and S. Weinberg eds. and references therein.

[2] J. B. Hartle and S. W. Hawking, *Phys. Rev. D* **28** (1983) 2960.

[3] A. Vilenkin, *Phys. Lett. B* **117** (1982) 25; *Phys. Rev. D* **27** (1983) 2848; **30** (1984) 509; **33** (1986) 3560; **37** (1988) 888; *Nucl. Phys. B* **252** (1985) 141; A. D. Linde, Zh. Eksp. Teor. Fiz. **87** (1984) 369 (Sov. Phys. JETP **60** (1984) 211); Nuov. Cim. **39** (1984) 401; Rep. Prog. Phys. **47** (1984) 925.

[4] A. O. Barvinsky, *Phys. Rep.* **230** (1993) 237.

[5] L. P. Grishchuk and L. V. Rozhansky, *Phys. Lett. B* **208** (1988) 369; **234** (1990) 9.

[6] K. Maeda, *Phys. Rev. D* **39** (1989) 3159; J. D. Barrow and S. Cotsakis, *Phys. Lett B* **214** (1988) 515.
[7] S. W. Hawking and J. Luttrell, *Nucl. Phys. B* **247** (1984) 250; G. T. Horowitz, *Phys. Rev. D* **31** (1985) 1169; A. Vilenkin, *Phys. Rev. D* **32** (1985) 2511; M. B. Mijić, M. S. Morris and W-M. Suen, *Phys. Rev. D* **39** (1989) 1496; S. R. Odintsov and I. L. Shapiro, *Class. Quantum Grav.* **9** (1992) 873; H. van Elst, J. E. Lidsey and R. Tavakol, FERMILAB-PUB-94-108-A, [gr-qc/9404041](https://arxiv.org/abs/gr-qc/9404041) (1994).

[8] F. C. Adams et al., *Phys. Rev. D* **47** (1993) 426.