TWO WEIGHT INEQUALITIES FOR RIESZ TRANSFORMS:
UNIFORMLY FULL DIMENSION WEIGHTS

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Abstract. Fix an integer $n$ and number $d$, $0 < d \neq n - 1 \leq n$, and two weights $w$ and $\sigma$ on $\mathbb{R}^n$. We impose an extra condition that the two weights separately are not concentrated on a set of codimension one, uniformly over locations and scales. (This condition holds for doubling weights.) Then, we characterize the two weight inequality for the $d$-dimensional Riesz transform on $\mathbb{R}^n$,

$$\left\| \int f(y) \frac{x - y}{|x - y|^{d+1}} \sigma(dy) \right\|_{L^2(\mathbb{R}^n; w)} \leq N \|f\|_{L^2(\mathbb{R}^n; \sigma)}$$

in terms of these two conditions, and their duals: For finite constants $A_2$ and $T$, uniformly over all cubes $Q \subset \mathbb{R}^n$,

$$\frac{w(Q)}{|Q|^{d/n}} \int_{\mathbb{R}^n} \frac{|Q|^{d/n}}{|Q|^{2d/n} + \text{dist}(x, Q)^{2d/n}} \sigma(dx) \leq A_2$$

$$\int_Q |R_\sigma 1_Q(x)|^2 w(dx) \leq T^2 \sigma(Q),$$

where $R_\sigma$ is the Riesz transform as above, and the dual conditions are obtained by interchanging the roles of the two weights. Examples show that a key step of the proof fails in absence of the extra geometric condition imposed on the weights.

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1. Introduction

We are interested in the two weight inequality for fractional Riesz transforms. Namely, for two weights, non-negative Borel measures \( w \) and \( \sigma \) on \( \mathbb{R}^n \), we consider the norm inequality

\[
\left\| \int_{\mathbb{R}^n} f(y) \frac{x - y}{|x - y|^{d+1}} \sigma(dy) \right\|_{L^2(\mathbb{R}^n, w)} \leq N \|f\|_{L^2(\mathbb{R}^n, \sigma)}.
\]

Here, \( 0 < d \neq n - 1 \leq n \), and the kernel above is vectorial, making it the standard \( d \)-dimensional Riesz transform on \( \mathbb{R}^n \). Throughout, we take \( N \) to be the best constant in the inequality above.

To avoid cumbersome notation, we shorten the inequality above to

\[
\|R_\sigma f\|_w \leq N \|f\|_\sigma.
\]

We also suppress the dimensionality of the Riesz transform, and since our results only hold in \( L^2 \), we write \( \|f\|_\sigma \equiv \|f\|_{L^2(\mathbb{R}^n, \sigma)} \). The norm inequality is written with the weight \( \sigma \) on both sides of the inequality since it then dualizes by interchanging the roles of the weights.

The desired characterization of the boundedness of the Riesz transform is in terms of a pair of testing inequalities and an \( A_2 \) type condition phrased in the language of this Poisson-like operator.

For a cube \( Q \subset \mathbb{R}^n \), set

\[
P^r(\sigma, Q) \equiv \int_{\mathbb{R}^n} \frac{|Q|^{d/n}}{|Q|^{2d/n} + \text{dist}(x, Q)^{2d/n}} \sigma(dx).
\]

The superscript ‘ denotes ‘reproducing Poisson’ due to the similarities with the analogous power in the setting of weighted estimates on reproducing kernel Hilbert spaces.

Below, we impose the condition that the dimension of the Riesz transform not be of codimension one. But, we also impose an extra geometric condition on each weight individually. Say that the weight \( w \) is uniformly of full dimension if there is a \( 0 < \eta < 1 \) such that for all cubes \( Q \subset \mathbb{R}^n \),

\[
\inf_H \int_Q \left( \frac{\text{dist}(x', H + x)}{|x - x'|} \right) w(dx) w(dx') \geq \eta w(Q)^2.
\]

The infimum is formed over all hyperplanes \( H \) through the origin of co-dimension one. Without this assumption, a crucial step in the proof of Lemma 3.7 fails. We thank Xavier Tolsa for pointing this out to us; elaborations of his example are given in §3.2.

A weight \( w \) is doubling if there is a constant \( C_d \) so that for all cubes \( Q \), \( w(2Q) \leq C_d w(Q) \), where \( 2Q \) is the cube concentric with \( Q \) but twice the side length. Such weights are uniformly of full dimension, so that in the case of both \( \sigma \) and \( w \) doubling, and \( n = d \geq 2 \), the Theorem below provides a characterization of the two weight inequality for the Riesz transforms.

**Theorem 1.4.** Assume that \( 0 < d \neq n - 1 \leq n \) and that \( \sigma \) and \( w \) are two weights on \( \mathbb{R}^n \), each of which are uniformly of full dimension, and do not share a common point mass. The norm inequality (1.1) holds if and only if for finite positive constants \( A_2, T \), these inequalities hold uniformly over cubes \( Q \), and over their dual formulations

\[
\frac{w(Q)}{|Q|^{d/n}} \cdot P^r(\sigma, Q) \leq A_2,
\]
\[ \int_Q |R_\sigma 1_Q(x)|^2 w(dx) \leq T^2 \sigma(Q), \]

(The dual statements are obtained by interchanging the roles of \( w \) and \( \sigma \).) Moreover, \( \mathcal{N} \simeq \mathcal{R} \equiv A_2^{1/2} + \mathcal{I} \).

The Theorem has two critical side conditions. The first is the uniformly full dimension assumption, which we need to derive a critical energy inequality. The second, of no common point masses, we can eliminate provided \( n - 1 < d \leq n \). We assume it in the general case in order to avoid some complications in the functional energy argument. A precise statement of a more general result is given in Theorem 3.6.

The difficult part of the theorem is to show the sufficiency of the \( A_2 \) and testing inequalities. The key property of Riesz transforms that we will exploit is the divergence condition

\[ \text{div} \left( \frac{y}{|y|^{d+1}} \right) = \frac{n-d-1}{|y|^{d+1}} \neq 0. \]

Note that in the case that the codimension is not one, i.e. \( d+1 \neq n \), the divergence of the kernel is signed, while in the case of codimension one, i.e. \( d+1 = n \), we gain no information since the divergence vanishes.

Concerning the additional condition that the weights be of uniformly full dimension, we have

**Proposition 1.8.** Sufficient conditions for a weight \( w \) to be of uniformly full dimension are either

1. the weight is doubling, or
2. the weight is Ahlfors-David regular of parameter \( n - 1 < d \leq n \).

The latter condition means that there are constants \( 0 < C_0 < C_1 < \infty \) so that for all points \( x \) in the support of \( w \), and balls \( B(x, r) \) centered at \( x \), of radius \( 0 < r < \infty \), there holds

\[ C_0 r^d < w(B(x, r)) < C_1 r^d. \]

We adopt key elements of the proof from the Hilbert transform \([1,2,4]\), with essential modifications. (1) An energy inequality, derived from the sufficient conditions for norm boundedness, is a decisive tool. This occurs in a very clean way for the Hilbert transform, but has additional complications herein, complications that are close to those of the Cauchy transform \([5]\). The divergence property and the side condition of being uniformly of full dimension are essential here. See Lemma 3.7. The divergence condition says that the trace of the tensor \( \nabla R_\sigma f \) is not zero, namely there is at least one non-negative eigenvector. But, by the example of Proposition 3.12, the Riesz transform can be nearly zero on a set as large as a square of co-dimension one. But, the weights cannot be concentrated on such sets by the uniformly full dimension assumption.

(2) The monotonicity inequality dominates certain off-diagonal terms. Estimates of this type are standard, but in the two weight setting they have an important modification to be consistent with the energy inequality. The inequality \((4.2)\) now has two components, arising from first and second gradient calculations. They are, by example, incomparable terms. Hence, both play a role in the off-diagonal considerations which dominate the proof. (3) In the global-to-local reduction, we use the argument of Hytönen \([1]\). A certain complication arises, in the case of \( 0 < d < n - 1 \), and in order to give a short argument, we impose the \( A_2 \) condition ‘without holes.’ (4) The local part of the proof, an argument invented in \([2]\), has additional complications that are not
present in other proofs of this type. (5) The \( A_2 \) condition, with its large exponent, if \( d > 1 \), requires new arguments to apply. Curiously, it is only needed in a couple of ‘standard’ lemmas, see Lemma 8.7. (6) The proofs for the Hilbert transform, likewise for the Cauchy operator \([5]\), employ Muckenhoupt’s two weight inequality for the Hardy inequality. This is of course no longer available to us, creating a new complication. We use the deep technique of surgery, invented by Nazarov-Treil-Volberg \([6]\), and a fundamental tool in local \( Tb \) theorems. In our setting, the application in Lemma 8.5 is new, but not very difficult.

2. Conventions

- Letters like \( P, Q, K, L \) will denote a cube, and it is convenient to denote the scale of cube \( Q \) by \( \ell(Q) \equiv |Q|^{1/n} \). The geometric center of the cube \( Q \) is denoted by \( x_Q \).
- We will introduce two distinct grids \( D_\sigma \) and \( D_w \), associated with the corresponding \( L^2 \) spaces. We will construct subgrids, for instance \( F \subset D_\sigma \). In this case, we set \( \pi_F^t Q \) to be the smallest element of \( F \) which contains \( Q \). This is well-defined for all \( Q \in D_\sigma \) which are contained in some element of \( F \). We then set \( \pi_{F}^1 Q = \pi_F Q \), and inductively set \( \pi_{F}^{t+1} Q \) to be the member of \( F \) that strictly contains \( \pi_{F}^t Q \).
- For two cubes \( P, Q \), and integer \( s \), we will write \( Q \subset^s P \), if \( Q \subset P \) and \( 2^s \ell(Q) \leq \ell(P) \). This notation will be used for \( s = r \) and \( s = 4r \), where \( r \) will be a fixed, large, integer, associated with the notion of goodness (defined below). This notation will be used when the cubes are in the same, or two distinct grids. (In the case of distinct grids, we will in addition be able to assume that \( 2^r \ell(Q) \leq \ell(P) \) and \( Q \cap P \neq \emptyset \) implies \( Q \subset P \).)
- Averages appear repeatedly in the analysis below. The average of \( f \) over cube \( Q \), with respect to the weight \( \sigma \) is

\[
[f]_Q^{\sigma} \equiv \frac{1}{\sigma(Q)} \int_Q f(x) \sigma(dx),
\]

this provided \( \sigma(Q) > 0 \). In particular, when we write

\[
[x]_Q^{\sigma} \equiv \frac{1}{\sigma(Q)} \int_Q x \sigma(dx) = \left( \frac{1}{\sigma(Q)} \int_Q x_1 \sigma(dx), \ldots, \frac{1}{\sigma(Q)} \int_Q x_n \sigma(dx) \right)
\]

we mean the \( \sigma \) center of mass of the cube \( Q \). Because of our hypotheses on the weights, it will always be the case that \( [x]_Q \in Q \). More commonly, the notation for averages is given by \( \langle f \rangle_Q^{\sigma} \), but we preserve angles for inner products.

2.1. Dyadic Grids. Let \( \hat{D} \) denote the standard dyadic grid in \( \mathbb{R} \). A random dyadic grid \( D^r \) is specified by \( \xi \in \{0,1\}^\mathbb{Z} \) and choice of \( 1 \leq \lambda \leq 2 \). The elements of \( D^r \) are given by

\[
I \equiv \hat{I} + \xi = \lambda \left\{ \hat{I} + \sum_{n:2^{-n}<|\xi|} 2^{-n} \xi_n \right\}.
\]

Place the uniform probability measure \( \mathbb{P} \) on \( \xi \in \{0,1\}^\mathbb{Z} \), and choose \( \lambda \) with respect to normalized measure on \( [1,2] \) with measure \( \frac{dx}{\lambda} \).
Let $D_\sigma$ be a $n$-fold tensor product of independent copies $D^1_\sigma \times \cdots \times D^m_\sigma$, which is used in $L^2(\mathbb{R}^n; \sigma)$. It is imperative to use a second random grid $D_w$ for $L^2(\mathbb{R}^n; w)$, but the subscripts on the two random grids are frequently suppressed, and largely irrelevant to the argument, except for the surgery argument in Lemma 8.5.

A choice of grid $D_\sigma$ is said to be \textit{admissible} if $\sigma$ does not assign positive mass to any lower dimensional face of a cube $Q \in D_\sigma$. By the construction of the random grid, in particular the use of dilations, a randomly selected grid is admissible with probability one. This is always assumed below, for both $D_\sigma$ and $D_w$. We assume that the dilation parameter is $\lambda = 1$, which should not cause any confusion.

For $Q \in D_\sigma$, we speak of $\text{ch}(Q)$, the \textit{children} of $Q$, the $2^n$ maximal elements of $D$ which are strictly contained in $Q$. If $\sigma(Q) > 0$ and $\sigma(Q') > 0$ for at least two children of $Q$, then we define the martingale difference

$$\Delta^\sigma_Q f \equiv -[f^\sigma]_Q \cdot Q + \sum_{Q' \in \text{ch}(Q) : \sigma(Q') > 0} [f^\sigma]_{Q'} \cdot Q'.$$

Otherwise, $\Delta^\sigma_Q f \equiv 0$. In this definition, we are identifying the cube $Q$ with $1_Q$, which we will do throughout. It is well-known that $f = \sum_{Q \in D_\sigma} \Delta^\sigma_Q f$, and that

$$\|f\|^2_\sigma = \sum_{Q \in D_\sigma} \|\Delta^\sigma_Q f\|^2_\sigma.$$

2.2. \textbf{Good and Bad Cubes.} Fix $0 < \epsilon < 1$ and $\tau \in \mathbb{N}$. An cube $I \in D_\sigma$ is said to be $(\epsilon, \tau)$-\textit{bad} if there is an cube $J \in D$ such that $\ell(J) > 2^{\tau}\ell(I)$ and $\text{dist}(I, \partial J) < \ell(I)^\epsilon \ell(J)^{1-\epsilon}$. Otherwise, an cube $I$ will be called $(\epsilon, \tau)$-\textit{good}. We have the following well-known properties associated to the random dyadic grid $D_\sigma$.

\textbf{Proposition 2.1.} The following properties hold:

1. The property of $I = \hat{I} + \xi$ being $(\epsilon, \tau)$-good depends only on $\xi$, and $|I|$;
2. $p_{\text{good}} \equiv \mathbb{P}(I \text{ is } (\epsilon, \tau) - \text{good})$ is independent of $I$;
3. $p_{\text{bad}} \equiv 1 - p_{\text{good}} \leq \epsilon^{-1} 2^{-\tau}.$

A similar result of course holds for the grid $D_w$. Now, write the identity operator in $L^2(\mathbb{R}^n; \sigma)$ as

$$f = p^\sigma_{\text{good}} f + p^\sigma_{\text{bad}} f \quad \text{where } p^\sigma_{\text{good}} f \equiv \sum_{Q \in D_\sigma: Q \text{ is } (\epsilon, \tau) - \text{good}} \Delta^\sigma_Q f.$$

Similar notation applies for the identity operator on $L^2(\mathbb{R}^n; w)$. Below, we will frequently impose the condition that the cubes are good, and will not explicitly point this out in the notation.

We have the following well-known proposition in this context.

\textbf{Proposition 2.2.} The following estimate holds:

$$\mathbb{E} \|p^\sigma_{\text{bad}} f\|_\sigma^2 \leq \epsilon^{-1} 2^{-\tau} \|f\|^2_\sigma.$$

An identical estimate is true for the weight $w$. 

3. ENERGY INEQUALITIES

We begin the discussion of energy inequalities with the case of codimension other than one, \(0 < d \neq n - 1 \leq n\).

Crucial to this discussion is the introduction of two different Poisson averages, with critically, a different power than that of the ‘reproducing Poisson’ average in (1.2). The Poisson-like average arise from gradient considerations, hence we write a superscript \(g\) on it, and \(g^+\) on the second.

\[
P^g([f], Q) \equiv \int_{\mathbb{R}^n} |f(x)| \frac{\ell(Q)}{\ell(Q)^{d+1} + \text{dist}(x, Q)^{d+1}} \sigma(dx),
\]

(3.1)

\[
P^{g^+}([f], Q) \equiv \int_{\mathbb{R}^n} |f(x)| \frac{\ell(Q)^2}{\ell(Q)^{d+2} + \text{dist}(x, Q)^{d+2}} \sigma(dx).
\]

(3.2)

It is important to note that the reproducing Poisson decay is \(2d\), as in (1.2), whereas the decay for \(P^g\) is \(d + 1\). These agree when \(d = 1\), e.g. the case of the Hilbert transform, which is included in our discussion, and the Cauchy transform, which is not. More generally, the reproducing decay is slower for \(0 < d < 1\), but is otherwise faster. Faster decay on the reproducing kernel creates additional technical problems for us. Fortunately, we will however not find it necessary to distinguish these three cases in the analysis below.

The energy of \(w\) over the cube \(K \in D_\sigma\) is taken to be

\[
E(w, K)^2 \equiv \frac{1}{w(K)^2} \int_K \int_K \frac{|x - x'|^2}{\ell(K)^2} w(dx) w(dx')
\]

\[
= \frac{2}{w(K)} \int_K \frac{|x - [x]_K|^2}{\ell(K)^2} w(dx)
\]

\[
= \frac{2}{w(K)} \sum_{Q \in D_\sigma : Q \subset K} \left\| \Delta_Q^w \frac{x}{\ell(K)} \right\|_w^2.
\]

(3.3)

The energy is the norm of \(\frac{x}{\ell(K)} \cdot K\) in \(L^2_0(\mathbb{R}^n; w)\), the subspace in \(L^2(\mathbb{R}^n; w)\) that is orthogonal to constants. In the middle line, we are subtracting off the mean value of \(x \cdot K\), in particular, \(E(w, K) \leq 1\), and is as small as zero if \(w \cdot Q\) is just a point mass. It is easy to check the equality of the three expressions above, and we will use all three. And, one should be careful to note that the last equality requires that \(K \in D_\sigma\), a condition we will not always have. Observe that

\[
\sum_{Q \in D_\omega : Q \subset K} \left\| \Delta_Q^w \frac{x}{\ell(K)} \right\|_w^2 \leq E(w, K)^2 w(K).
\]

The difference between this last display and (3.3) is that here is that we are summing over cubes in \(D_\omega\), while in (3.3), we are summing over cubes in \(D_\sigma\).

The energy constant is defined in terms of supplemental constants \(0 < C_0, C_1\), both of which are functions of \(n\) and \(d\). We will comment in more detail on the selection of these constants
below. The energy constant is the best constant \( E = \mathcal{E}(C_0, C_1) \) in the inequality

\[
(3.4) \quad \sum_{K \in \mathcal{K}} P_0^\sigma(Q_0 \setminus C_0 K, K)^2 \mathcal{E}(w, K)^2 w(K) \leq \varepsilon^2 \sigma(Q_0).
\]

Here, \( Q_0 \subset \mathbb{R}^n \) is a cube, and \( \mathcal{K} \subset \mathcal{D}_\sigma \) or \( \mathcal{K} \subset \mathcal{D}_w \) is any partition of \( Q_0 \) into dyadic cubes for which

\[
(3.5) \quad \sum_{K \in \mathcal{K}} (C_0 K)(x) \leq C_1 Q_0(x), \quad x \in \mathbb{R}^n.
\]

Importantly, we need the inequality (3.4) above with the roles of the weights reversed. And \( \mathcal{E} \) will denote the best constant in (3.4) and its dual statement.

With this notation the precise result we are proving in this paper is as follows. In particular, note that the condition that the pair of weights have no common point mass, and a stronger \( A_2 \) condition, are imposed in the case of \( 0 < d \leq n - 1 \).

**Theorem 3.6.** Let \( \sigma \) and \( w \) be two weights on \( \mathbb{R}^n \), and let \( 0 < d \leq n \). Assume that the energy constant \( \mathcal{E} \) as defined in (3.4) is finite.

1. For \( n - 1 < d \leq n \), assume the \( A_2 \) condition 'with holes', namely

\[
\sup_{Q \text{ a cube}} \frac{\sigma(Q)}{\ell(Q)} P_r^w(\mathbb{R}^n \setminus Q, Q) + \frac{w(Q)}{\ell(Q)} P_r^\sigma(\mathbb{R}^n \setminus Q, Q) = A_2 < \infty.
\]

Here, we are using the reproducing kernel Poisson average, as defined in (1.2).

2. For \( 0 < d \leq n - 1 \), assume that \( \sigma \) and \( w \) do not share a common point mass, and that the the \( A_2 \) condition 'with no holes' below holds.

\[
\sup_{Q \text{ a cube}} \frac{\sigma(Q)}{\ell(Q)} P_r^w(\mathbb{R}^n, Q) + \frac{w(Q)}{\ell(Q)} P_r^\sigma(\mathbb{R}^n, Q) = A_2 < \infty.
\]

Finally, assume that the two testing inequalities (1.6) and their duals hold. Then, the two weight norm inequality (1.1) holds, and \( N \lesssim A_2^{1/2} + \mathcal{E} + \mathcal{I} \).

Sawyer, Uriarte-Tuero, and Shen [7] have formulated a result along these lines. Our formulation is simpler, and proof much shorter.

This is the one place in which the side condition of \( \sigma \) and \( w \) being uniformly of full dimension is used.

**Lemma 3.7** (Energy Lemma). Assume \( 0 < d \neq n - 1 < n \) and both \( \sigma \) and \( w \) are uniformly of full dimension, in the sense of (1.3), with constant \( 0 < \eta < 1 \). In addition assume that they don’t share a common point mass. There is a constant \( C_0 = C_0(n, d, \eta) \), absolute, so that this holds. Let \( \sigma \) and \( w \) be a pair of weight for which the \( A_2 \) hypothesis (1.5) and the testing inequalities (1.6) and the dual to (1.6) holds. Then \( \mathcal{E}^2 \leq \mathcal{R}^2 \equiv A_2 + \mathcal{I}^2 \). The implied constant depends upon \( C_1 \), and the constants \( \eta \) that enter into (1.3).
Proof. The expression on the left in (3.4) is a sum of positive terms and so we can assume that the sum above is over just a finite number of terms. This is the main inequality:

\[
[P^g(\sigma \cdot Q_0 \setminus C_0 K)]^2 E(w, K)^2 w(K) \lesssim \int_K |R_\sigma(Q_0 \setminus C_0 K)(x)|^2 w(dx).
\] (3.8)

Using linearity in the argument of the Riesz transform, and the testing inequality (1.6), one sees that

\[
\sum_{K \in \mathcal{K}} \int_K |R_\sigma Q_0|^2 \, dw \leq \mathcal{T}^2 \sigma(Q_0),
\]

\[
\sum_{K \in \mathcal{K}} \int_K |R_\sigma C_0 K|^2 \, dw \leq \mathcal{T}^2 \sum_{K \in \mathcal{K}} \sigma(C_0 K) \leq C_1 \mathcal{T}^2 \sigma(Q_0).
\]

Note that the inequality (3.5) is used here in the second inequality, while for the first we use that \(K\) is a partition of \(Q_0\). These two estimates coupled with (3.8) prove (3.4), which would give the statement of the Lemma.

So, it remains to prove (3.8). Assume that it fails, namely for a constant \(0 < \kappa < 1\) that we will pick, as a function of \(0 < \eta < 1\) in (1.3), that there holds

\[
\int_K |R_\sigma(Q_0 \setminus C_0 K)(x)|^2 w(dx) \leq \kappa[P^g(\sigma \cdot Q_0 \setminus C_0 K)]^2 E(w, K)^2 w(K).
\]

We do not know how to use this condition directly, passing instead to its implication that

\[
\int_K \int_K |R_\sigma(Q_0 \setminus C_0 K)(x') - R_\sigma(Q_0 \setminus C_0 K)(x)|^2 w(dx') w(dx) \leq 2\kappa[P^g(\sigma \cdot Q_0 \setminus C_0 K)]^2 E(w, K)^2 w(K).
\] (3.9)

Fix \(x\) as in the integral on the right, and consider the symmetric tensor \(T_x = \nabla R_\sigma(Q_0 \setminus C_0 K)(x)\), which by the Spectral Theorem, has a diagonalization. By the divergence equality (1.7), the trace of this tensor is, up to a sign

\[
P_x \equiv \int_{Q_0 \setminus C_0 K} \frac{|n - d - 1|}{|x - y|^{d+1}} \sigma(dy).
\]

Thus, there is at least one eigenvalue of \(T_x\) of magnitude \(P_x\), and the maximal eigenvalue is \(C \cdot P_x\).

Observe that \(P_x\) and \(T_x\) are essentially constant on \(K\): For \(x', x \in K\):

\[
|P_{x'} - P_x| + |T_{x'} - T_x| \leq \frac{|x' - x|}{C_0 \ell(K)} P_x.
\] (3.10)

This depends only a second derivative calculation. We will choose \(C_0 \gtrsim \eta^{-d}\), and \(\kappa \approx \eta^d\), where \(\eta\) is as in (1.3). In particular, the right hand side above will be quite small.

Define

\[
L_x = \{y \in \mathbb{R}^n : |y| = 1, |T_x y| < \sqrt{\kappa P_x}\}.
\]
By our diagonalization observation, there must be a hyperplane $H_x$ with
\[ \sup_{y \in L_x} \text{dist}(y, H_x) \leq \sqrt{\kappa}. \]

Take $H_x$ to be orthogonal to the eigenvector with maximal eigenvalue. By (3.10), we can in addition take $H_x = H$, namely independent of $x \in K$.

For $x'$ as in (3.9), set $v$ to be the unit vector in the direction $x' - x$ and set $\delta = |x' - x|$. Then,
\[ R_{v,\sigma}(Q_0 \setminus C_0 K)(x') - R_{v,\sigma}(Q_0 \setminus C_0 K)(x) = \delta \cdot T_x v + O(C_0^{-1} P_x). \]

Indeed, the first term on the right is the first derivative approximation to the difference. By Taylor’s theorem, we should have a second derivative error term, but this is controlled by (3.10).

The inequality (3.9) can be rewritten as
\[ \frac{1}{w(K)} \int_K \int_K \delta^2 |T_x v|^2 \, dw \, dw \leq C_{\kappa} \frac{1}{w(K)} \int_K \int_K \delta^2 \cdot P_x^2 \, dw \, dw \]
where we are suppressing the dependence of $v$ and $\delta$ on $x$ and $x'$. We always have
\[ \delta^2 |v^T T_x v| \leq \delta^2 \cdot P_x, \]
therefore,
\[ \int_K \int_K \frac{\text{dist}(x', H + x)}{|x - x'|^2} \, w(dx) \, w(dx') \leq \sqrt{\kappa} w(K)^2. \]

The difference between this and (1.3) is the presence of the square above. But due to the normalization by $w(K)$, we have
\[ \int_K \int_K \frac{\text{dist}(x', H + x)}{|x - x'|^2} \, w(dx) \, w(dx') \leq \kappa^{1/4} w(K)^2. \]

Assuming the failure of (3.8) with $\kappa \approx \eta^4$ leads to a contradiction of the assumption (1.3). The proof is complete.

\[ \square \]

The proof above provides us with an absolute choice of $C_0$ in the definition of energy. The partitions that we use will be generated from examples of this type. For a cube $Q \in D_\sigma$, set $W_Q$ to be the maximal cubes $K \in D_w$ (or $D_\sigma$) such that $2^r \ell(K) \leq \ell(Q)$, and dist$(K, \partial Q) \geq \ell(K)^{\varepsilon} \ell(Q)^{1-\varepsilon}$ for all dyadic cubes $Q' \subset Q$, with $2^r \ell(K) \leq \ell(Q')$. These are the maximal cubes in the dual dyadic grid that are ‘good with respect to $Q$.’ They have this Whitney property, see [5, Prop. 3.18] or [3, Prop. 3.4].

Proposition 3.11. For any finite $C_0$, and $2^{r(1-\varepsilon)} > 4C_0$, there holds
\[ \sum_{K \in W_Q} (C_0 K)(x) \leq (2rC_0) Q(x). \]
Proof. The condition on $C_0$ and $r$, and the selection criteria for $W_Q$ imply that $C_0 K \subset Q$ for all $K \in W_Q$. So we need only control the overlaps. Suppose there are $K_1, K_2 \in W_Q$ with $2^r \ell(K_1) \leq \ell(K_2)$, and $C_0 K_1 \cap C_0 K_2 \neq \emptyset$. Then, we have

$$\text{dist}(K_1, \partial Q) \geq \ell(K_2)^c \ell(Q)^{1-\epsilon} - C_0 \ell(K_2) = (1 - C_0 2^{-r(1-\epsilon)}) \ell(K_2)^c \ell(Q)^{1-\epsilon}.$$  

That is, the cube $K_1$ with small side length is rather far from the boundary of $Q$. This contradicts the selection criteria for $K_1$. \hfill \Box

3.1. Sufficient Conditions for Uniformly Full Dimension. We give the proof of Proposition 1.8. First, if $w$ is doubling, then there is a constant $C_d$ so that for all cubes $Q \subset \mathbb{R}^n$, there holds $w(2Q) \leq C_d w(Q)$. To check the condition (1.3), take a cube $Q$, and partition it into $P$, cubes of side length $\frac{1}{2}\ell(Q)$. For each $x \in Q$, and hyperplane $H$ of co-dimension one, we can choose $Q', Q'' \in P$ so that $x \in Q''$ and for any $x' \in Q'$, the following holds

$$\text{dist}(x', H + x) \simeq \text{dist}(Q', Q'') \simeq \ell(Q).$$

Since for each $Q' \in P$, there holds $w(Q) \leq C_d^3 w(Q')$, we conclude that

$$\int_Q \int_Q \frac{\text{dist}(x', H + x)}{|x - x'|} w(dx) w(dx') \geq \eta(C_d) w(Q)^2,$$

as required.

Second, if $w$ is Ahlfors-David regular, namely satisfying (1.9), with $n - 1 < d \leq n$, we argue by contradiction that it is of uniformly full dimension. Namely we assume that the inequality (1.3) fails for some cube $Q$, for a sufficiently small $0 < \eta < 1$ specified below.

We can select an $x \in Q$ and a hyperplane $H$ so that

$$\int_Q \frac{\text{dist}(x', H + x)}{|x - x'|} w(dx') \leq 2\eta w(Q).$$

Let $\hat{H} = \{x' \in Q : \text{dist}(x', H + x) < 4\sqrt{n} \ell(Q)\}$ be a neighborhood of $H + x$. It follows that $w(Q \setminus \hat{H}) < \sqrt{n} \ell(Q)$. On the other hand, we can take a cover $\mathcal{C}$ of $\hat{H}$ by balls of radius $2 \sqrt{n} \ell(Q)$. Clearly, we can assume that $\mathcal{C} \leq c_n^{-\frac{n-1}{2}}$. Using both inequalities in the Ahlfors-David assumption (1.9), we then have

$$C_d \ell(Q)^d \leq w(Q)$$

$$\leq w(Q \setminus \hat{H}) + w(\hat{H})$$

$$\leq \sqrt{n} w(Q) + \sum_{B \in \mathcal{C}} w(B)$$

$$\leq \sqrt{n} w(Q) \sqrt{n} + c_n^{-\frac{n-1}{2}} \sup_{B \in \mathcal{C}} w(B)$$

$$\leq C_1 \left( \sqrt{n} \ell(Q) + c' \eta^{-\frac{n-1}{2} + \frac{d}{2}} \right) \ell(Q)^d.$$
By the assumption $n - 1 < d \leq n$, it follows that both exponents on $\eta$ above are positive. A contradiction is found for a sufficiently small $0 < \eta < 1$, as a function of $0 < C_0 < C_1 < \infty$. This completes the proof.

3.2. **An Example.** We discuss the proof of the energy lemma. The extra condition (1.3) is used to deduce (3.8), although that inequality is not directly proved rather, the variance inequality (3.9) is shown to fail for sufficiently small $\kappa$.

If $\sigma$ is Lebesgue measure, restricted to some affine hyperplane of dimension less or equal to $d$, at some distance from the hyperplane, the transform $R_\sigma 1$ is essentially constant in directions parallel to the hyperplane. We show here that there are worse examples: For certain choices of $d$, one can construct $R_\sigma$ so that it is essentially zero on a unit square of co-dimension 1.

**Proposition 3.12.** Let $n$ be an integer, and $\max(0, n - 2) < d \leq n$. Let $S \subset \mathbb{R}^n$ be a cube of co-dimension one. For all $\epsilon > 0$ there is a finite weight $\sigma$ so that

\[
\text{dist}(S, \text{supp}(\sigma)) \geq \epsilon^{-1}\ell(S)
\]

(3.13)

\[
\epsilon^{-1}\sup_{x \in S} |R_\sigma 1(x)| \leq \int_{\mathbb{R}^n} \frac{\ell(S)}{\ell(S)^{d+1} + \text{dist}(S, x)^{d+1}} \sigma(dx).
\]

The first condition says that $\sigma$ is not supported near $S$, and the second shows that the strongest norm we can place on $\sigma$, the $L^\infty$-norm, does not dominate the gradient Poisson average of $\sigma$. We thank Xavier Tolsa for a suggestion that leads to this example.

**Proof.** Write $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$, and write coordinates as $(y, z)$, for $y \in \mathbb{R}^{n-1}$ and $z \in \mathbb{R}$. A constant $\lambda > 1$ will be chosen. Take the cube $S$ to be $[-\lambda^{-1/2}, \lambda^{1/2}] \times \{0\}$. The weight $\sigma$ is supported on the union of the two hyperplanes $\Lambda = \bigcup_{\theta \in [-\epsilon, \epsilon]} \Lambda_\theta$, where $\Lambda_\theta = \mathbb{R}^{n-1} \times \{\theta \lambda\}$.

We take $\sigma$ to be translates of the same radial weight $\tilde{\sigma}$ on $\mathbb{R}^{n-1}$ to the hyperplanes $\Lambda_\pm$. Then, by oddness of the kernel for the Riesz transform, it follows that $R_\sigma 1(0) = 0$. Write the coordinates of the Riesz transform by $R_{j,\sigma}$, for $1 \leq j \leq n$. For $1 \leq j \leq n$, and $1 \leq j \neq k < n$, we have

\[
\frac{\partial}{\partial k} R_{j,\sigma} 1(0) = -(d + 1) \int_{\mathbb{R}^{n-1}} \frac{y_j y_k}{|z^2 + |y|^2|^{d+1}/2} \sigma(dy dz) = 0
\]

by the radial property of $\tilde{\sigma}$. For $1 \leq j < n$, we require

\[
\frac{\partial}{\partial i} R_{j,\sigma} 1(0) = \int_{\Lambda} \frac{z^2 + |y|^2 - (d + 1)|y_j|^2}{|z^2 + |y|^2|^{d+1}/2} \sigma(dy dz) = 0.
\]

By symmetry in $y_j$, we can recognize the integral involving $y_j$ as $\frac{1}{n-1}$ times the integral involving $|y|^2$. Thus, the equality above reduces to

\[
\int_{\mathbb{R}^{n-1}} \frac{\lambda^2}{\{\lambda^2 + |y|^2\}^{d+1}/2} \tilde{\sigma}(dy) = \frac{d + 2 - n}{n - 1} \int_{\mathbb{R}^{n-1}} \frac{|y|^2}{\{\lambda^2 + |y|^2\}^{d+1}/2} \tilde{\sigma}(dy).
\]

(3.14)
This is only possible for \( n - 2 < d \leq n \). The conditions we place on \( \sigma \) are the symmetry properties already described, the equality (3.14) above, and
\[
\int_{\mathbb{R}^n} \frac{1}{1 + \text{dist}(S, x)^{d+1}} \sigma(dx) = 1
\]  
(3.15)
And, one can check that under these assumptions
\[
\frac{\partial}{\partial_n} R_{n, \sigma} 1(0) = \int_{\Lambda} \frac{z^2 + |y|^2 - (d + 1)z^2}{(|z^2 + |y|^2|^{d+2})} \sigma(dy \, dz)
\]
\[
= (n - d - 1) \int_{\Lambda} \frac{1}{|z^2 + |y|^2|^{d+1}} \sigma(dy \, dz).
\]

To verify (3.13), namely that the Riesz transforms are uniformly small. Observe that
\[
|\nabla^2 R_{\sigma} 1| \lesssim 1,
\]
as follows by inspection, and the normalization (3.15). It follows then from Taylor’s theorem, that \( |R_{\sigma} 1(x)| \lesssim \lambda^{-1} \), for all \( x \in S \), since the side length of \( S \) is \( \lambda^{-1/2} \). On the other hand, by the selection of \( \Lambda \),
\[
\int_{\mathbb{R}^n} \frac{\lambda^{-1/2}}{\lambda^{-\frac{d+1}{2}} + \text{dist}(S, x)^{d+1}} \sigma(dx) \simeq \lambda^{-1/2}.
\]
and so for \( \lambda^{-1/2} \simeq \epsilon \), we have verified (3.13). \( \square \)

4. Monotonicity

We discuss how to use energy to dominate off-diagonal inner products, which turns on two points. First, we must impose smooth truncations on the Riesz transforms. Second, an analysis of the off-diagonal inner products will reveal a more complicated monotonicity principle than appears in the case of the Hilbert transform (\( d = n = 1 \)) or the Cauchy transform (\( d = 1, n = 2 \)).

4.1. Monotonicity. This is the lemma in which we dominate the off-diagonal terms by those which involve positive quantities.

Lemma 4.1. Let \( P \) be a cube and \( Q \) a cube with \( 10Q \subset P \). Then for all functions \( g \in L^2(\mathbb{R}^n; w) \), supported on \( Q \), and of \( w \)-mean zero, and functions \( f \in L^2(\mathbb{R}^n; \sigma) \) which are not supported on \( P \),
\[
|\langle R_{\sigma} f, g \rangle_w| \leq P_{\sigma}^g(|f|, Q) \left| \frac{x}{\ell(Q)} \cdot g \right|_w + P_{\sigma}^g(|f|, Q) E(w, Q) w(Q)^{1/2} \|g\|_w.
\]  
(4.2)

Note in particular that the first term on the right is quite economical, involving only a part of the energy term associated to the Haar support of \( g \) in \( L^2(\mathbb{R}^n; w) \). The second term however has the full energy term, but with a Poisson term with degree one more than \( P_{\sigma}^g \), as defined in (3.2). It will be the source of extra complications in the subsequent parts of the argument.
**Proof.** Since \( f \) is not supported on the cube \( 10Q \), it follows that \( R_\sigma f \) is a \( C^2 \) function on \( Q \). Thus, for \( x \in Q \) and \( [x]_Q^{\sigma} \), we have

\[
R_\sigma f(x) - R_\sigma f([x]_Q^{\sigma}) = \nabla R_\sigma f([x]_Q^{\sigma}) \cdot (x - [x]_Q^{\sigma}) + (x - [x]_Q^{\sigma})^t \cdot \nabla^2 R_\sigma f(x') \cdot (x - [x]_Q^{\sigma})
\]

for some point \( x' \) that lies on the line between \( x \) and \( [x]_Q^{\sigma} \). This just depends upon a Taylor's Theorem (with remainder) calculation.

By the mean zero property of \( g \),

\[
\langle R_\sigma f, g \rangle_w = \langle R_\sigma f - R_\sigma f([x]_Q^{\sigma}), g \rangle_w.
\]

Then, by Taylor's Theorem, the right hand side is dominated by two terms. The first of these is

\[
\left| \langle \nabla R_\sigma f([x]_Q^{\sigma}) \cdot (x - [x]_Q^{\sigma}), g \rangle_w \right| = \left| \nabla R_\sigma f([x]_Q^{\sigma}) \langle x - [x]_Q^{\sigma}, g \rangle_w \right| \lesssim P_g^\sigma(|f|, Q) \left| \left\langle \frac{x}{\ell(Q)}, g \right\rangle_w \right|.
\]

This is the first term on the right in (4.2).

The second term is at most

\[
\sup_{x \in Q} |\nabla^2 R_\sigma f(x)| \cdot \| x - [x]_Q^{\sigma} \cdot Q \|_w \| g \|_w.
\]

Above, we divide \( |x - [x]_Q^{\sigma}|^2 \) by \( \ell(Q)^2 \), and observe that

\[
\left\| \frac{|x - [x]_Q^{\sigma}|^2}{\ell(Q)^2} \cdot Q \right\|_w \lesssim \left\| \frac{|x - [x]_Q^{\sigma}|}{\ell(Q)} \cdot Q \right\|_w = E(w, Q)w(Q)^{1/2}.
\]

In addition, there holds

\[
\ell(Q)^2 \sup_{x \in Q} |\nabla^2 R_\sigma f(x)| \lesssim P^g_\sigma(|f|, Q).
\]

This completes the proof. \( \square \)

5. Functional Energy

Let \( \mathcal{F} \subset D_\sigma \) be a collection of intervals which is \( \sigma \)-Carleson. That is, for all \( F \in \mathcal{F} \), suppose that

\[
(5.1) \quad \sum_{F' \in \mathcal{F} : F' \subseteq F} \sigma(F') \leq \frac{1}{2} \sigma(F).
\]

To each \( F \in \mathcal{F} \), let \( \mathcal{V}_F \) be the maximal good cubes \( Q \in D_w \), such that \( Q \subset F \) and \( Q \in \pi F \). This definition is very similar to \( \bar{\pi}_F Q = F \), given in (6.5).
Theorem 5.2 (Functional Energy). The Poisson operator $P_\sigma^g$, as defined in (3.1), satisfies this inequality. For $\mathcal{F}$ satisfying (5.1), and non-negative $f \in L^2(\mathbb{R}^n, \sigma)$,

\begin{equation}
\sum_{F \in \mathcal{F}} \sum_{K \in \mathcal{V}_F} P_\sigma^g(f(\mathbb{R}^n \setminus F), K)^2 \left\| \sum_{Q \in D_\sigma': Q \subset K} \Delta^\sigma Q \frac{\chi}{\ell(K)} \right\|_w^2 \lesssim \{E^2 + A_2\} \|f\|_\sigma^2,
\end{equation}

where $E$ is as in (3.4). In particular, under the assumption that $w$ and $\sigma$ are of uniformly full dimension, $E \lesssim A_2^{1/2} + \mathcal{I}$.

5.1. Dyadic Approximate to the Poisson. It is more direct to work with this form of the inequality

\begin{equation}
\sum_{F \in \mathcal{F}} \sum_{K \in \mathcal{V}_F} \tilde{P}_\sigma(f(\mathbb{R}^n \setminus F), K)^2 \left\| \sum_{Q \in D_\sigma': Q \subset K} \Delta^\sigma Q \frac{\chi}{\ell(K)} \right\|_w^2 \lesssim \{E^2 + A_2\} \|f\|_\sigma^2,
\end{equation}

where $\tilde{P}_\sigma(h, R) \equiv \int_{\mathbb{R}^n} \frac{h(x)}{\ell(R)^{d+1} + \text{dist}(x, R)^{d+1}} \sigma(dx)$.

Namely, the side length of $R$ is canceled out.

Recall that $D_\sigma$ is our dyadic grid. The collection of cubes $\{3Q : Q \in D_\sigma\}$ is the union of collections $C_u$, $1 \leq u \leq 3^n$ each of which is like the dyadic grid with respect to covering and nested properties. This is straightforward in the case of dimension $n = 1$, and the general case follows from this.

It is also well-known in dimension one, that for any non-dyadic interval $I$, there are two choice of $1 \leq u \leq 3$ and intervals $J_u \in C_u$ such that $I \subset J$ and $|J| \leq 6|I|$. For each non-dyadic cube $Q$ let $\pi_u Q$ be the unique, if it exists, cube $L \in C_u$ such that $3Q \subset L$, and $9\ell(Q) \leq \ell(L) \leq 18\ell(Q)$, so that, in the dyadic case, $\ell(L) = 12\ell(Q)$. If no such cube exists, set $\pi_u Q = \emptyset$.

Then, for each $j \in \mathbb{N}$, there is a choice of $1 \leq u = u_{Q,j} \leq 3^n$ so that $2^j Q \subset \pi_{u_{Q,j}}(\pi_u Q) \equiv (\pi_{u_{Q,j}})^{(j)}$. From this, we have

Proposition 5.5. [1, Prop 6.6] For $R \in D_\sigma$, and function $\phi \geq 0$, there holds

\begin{equation}
\tilde{P}_\sigma(\phi, R) \simeq \sum_{u : \pi_u R \neq \emptyset} Q^g_{u}(\phi, \pi_u R)
\end{equation}

where $Q^g_{u,\sigma}(\phi, R) := \sum_{j \geq 0} \frac{1}{2^j \ell(R)^{d+1}} \int_{R^{(1)}} \phi \, d\sigma, \quad R \in C_u$.

Proof. It is clear that the Poisson term dominates the sum over the operators $Q^g_{u}$. In the reverse direction, we have

\begin{equation}
\tilde{P}_\sigma(\phi, R) \simeq \sum_{j \geq 0} \ell(2^j R)^{-d-1} \int_{2^j R} \phi \, d\sigma
\end{equation}

But then, the proposition follows immediately from our discussion above. □
Consider the two weight inequality with holes, for an operator $Q^u$, namely for non-negative constants $\eta$, the inequality is

\begin{equation}
\sum_{R \in C} Q_u(\phi \cdot R^c, R)^2 \eta_R \leq Q^2 \|\phi\|_\sigma^2.
\end{equation}

It is of course convenient to dualize this inequality. Thus, define $W(R) \equiv R \times [\ell(R)/2, \ell(R)]$, and define $\eta \equiv \sum_{R \in C} \eta_R \delta_{W(R)}$, and assume that $\eta$ is a weight, that is $\eta(R)$ is finite for all rectangles $R$. Then, (5.6) is equivalent to the boundedness of the bilinear form

$$
\sum_{R \in C} Q_u(\phi \cdot R^c, R) \int_{W(R)} \psi \, d\eta \leq Q \|\phi\|_\sigma \|\psi\|_\eta.
$$

Below, $Box_Q \equiv Q \times [0, \ell(Q)]$. Similar to the notation introduced in the introduction, we will let $\|\psi\|_\eta \equiv \|\psi\|_{L^2(R^{n+1}; \eta)}$.

**Theorem 5.7.** *The inequality (5.6) holds if and only if these three constants are finite*

\begin{align}
\sup_{R \in C} \frac{\sigma(R^{(1)} \setminus R) \cdot \eta(Box_R)}{\ell(R)^{d+1}} & \equiv Q_1, \\
\sup_{R_0 \in C} \sigma(R_0)^{-1/2} \left\| \sum_{R \in C : R^{(1)} \subset R_0} \frac{\sigma(R^{(1)} \setminus R)}{\ell(R)^{d+1}} \cdot Box_R \right\|_\eta & \equiv Q_2, \\
\sup_{R_0 \in C} \eta(Box_{R_0})^{-1/2} \left\| \sum_{R \in C : R^{(1)} \subset R_0} \frac{(R^{(1)} \setminus R)}{\ell(R)^{d+1}} \cdot \eta(Box_R) \right\|_\sigma & \equiv Q_3.
\end{align}

Moreover, $Q \simeq \sum_{v=1}^3 Q_v$.

**Proof.** The three quantities are clearly necessary for the norm inequality (5.6). For the proof of sufficiency, we rewrite the

$$
Q_u(\phi, R) \simeq \sum_{S \subset R} \ell(S)^{-d-1} \int_{S^{(1)} \setminus S} \phi \, d\sigma
$$

so that if we consider the associated bilinear form, for non-negative $\psi \in L^2(R^{n+1}; \eta)$, it is

$$
\langle Q_u, \phi, \psi \rangle_\eta \simeq \sum_{R \in C} \sum_{S \subset R} \ell(S)^{-d-1} \int_{S^{(1)} \setminus S} \phi \, d\sigma \cdot \int_{W(R)} \psi \, d\eta = \sum_{S \subset R} \ell(S)^{-d-1} \int_{S^{(1)} \setminus S} \phi \, d\sigma \cdot \int_{Box_S} \psi \, d\eta.
$$

But this is the sum of $2^n$ operators of the following type: To the dyadic grid $D_u$, associate the grid $B_u = \{Box_Q : Q \in D_u\}$. To each $Q \in B_u$, associate two disjoint, distinguished subcubes
\( Q_+, Q_- \in \mathcal{B}_u \), and consider the bilinear form
\[
\Lambda(f, g) = \sum_{Q \in \mathcal{B}_u} \lambda_Q \int_{Q_+} f \, d\sigma \cdot \int_{Q_-} g \, dw.
\]

Hytönen in [1, Theorem 6.8], shows that the boundedness of operators of this type are characterized by three conditions similar to (5.8)—(5.10). The three conditions above imply the corresponding conditions for \( \Lambda(f, g) \), hence the Theorem follows. \( \square \)

The constants \( \eta_R \) in (5.6) are specified as follows. For \( F \in \mathcal{F}, \) and \( K \in \mathcal{V}_F \), set
\[
\tilde{\eta}_K \equiv \left\| \sum_{Q \in D_u^+ : Q \subset K \atop \pi_F \cap Q = \emptyset} \Delta^w_{Q} \right\|_w^2.
\]
Otherwise, set \( \tilde{\eta}_K = 0 \). Then, for \( 1 \leq u \leq 3^n \), and \( R \in D_u \), set \( \eta_R \equiv \tilde{\eta}_{\pi^{-1}_u R} \), provided \( \pi^{-1}_u R \) is defined, and if it is not defined, set \( \eta_R = 0 \). (We suppress the dependence of \( \eta \) on \( u \).) Recall that \( \eta \equiv \sum_{R \in C_u} \eta_R \delta_{\mathcal{W}(R)} \). With this choice of \( \eta \), the inequality (5.4) implies (5.3).

It remains to verify the three conditions (5.8)—(5.10), showing that each constant \( Q_y \leq R \). The first of these is clearly controlled by the \( A_2 \) constant, that is \( Q_1 \leq A_2^{1/2} \). Indeed, for this argument, one should use the bound \( \eta_R \leq \ell(R)^2 \mathcal{W}(R) \), which we will see again below.

5.2. **Forward Testing Condition.** We prove that \( Q_2 \leq \mathcal{E} \), where the former constant is as in (5.9), and we do so with a recursive argument along the stopping collection \( \mathcal{F} \).

The recursion is defined in terms of these definitions. Fix an cube \( R_0 \in C_u \) as in (5.9), and let \( \mathcal{R}_0(R_0) \) be those \( R \in C_u \) such that \( \eta_R \neq 0 \) and \( R \subset R_0 \), but \( R \) is not contained in any \( F \in \mathcal{F} \) which is strictly contained in \( R_0 \). This condition means in particular that there is a stopping interval \( F_R \in \mathcal{F} \), and a maximal good interval \( K_R \in \pi F \) with \( \pi_u K_R = R \). Also, set \( \mathcal{R}_1(R_0) \) to be cubes \( R \in C_u \) such that \( \eta_R \) is non-zero, and \( F_R \) is a maximal stopping cube strictly contained in \( R_0 \). We show that for \( k = 0, 1 \),
\[
N_k(R_0) = \left\| \sum_{R \in \mathcal{R}_k(R_0)} \sum_{S : R \supset S \supset \mathcal{R}} \frac{\sigma(S \setminus S)}{\ell(S)^{d+1}} \cdot R \right\|_\eta \leq \mathcal{E} \sigma(R_0)^{1/2}.
\]

The Carleson measure condition on \( \mathcal{F} \) will permit a recursion which completes the proof. Namely, let \( \mathcal{F}_j \) be the maximal cubes \( F \in \mathcal{F} \) which are contained in \( R_0 \), and inductively set \( \mathcal{F}_{j+1} \) to be the \( \mathcal{F} \)-children of the cubes in \( \mathcal{F}_j \), then using a standard Cauchy-Schwartz estimate in the summing index \( j \),
\[
\left\| \sum_{R : R \subset R_0} \frac{\sigma(R \setminus R)}{\ell(R)^{d+1}} \cdot \Box_R \right\|_\eta^2 \leq \sum_{k=0, 1} N_k(R_0)^2 + \sum_{j=1}^\infty j^2 \sum_{k=0, 1} N_k(F_j)^2 \\
\leq \mathcal{E}^2 \left\{ \sigma(R_0) + \sum_{j=1}^\infty j^2 \sum_{F \in \mathcal{F}_j} \sigma(F) \right\} \leq \mathcal{E}^2 \sigma(R_0).$
\]
The geometric decay in (5.1) clearly lets us sum this series.

To prove (5.11) argue first for the case of \( k = 1 \). The definition of \( \eta_n \) and the good property of intervals imply that the intervals \( \{ \pi_n K : K \in \mathcal{V}_F \} \) have bounded overlaps, hence

\[
N_1(R_0) \leq \sum_{R \in \mathcal{R}_1(R_0)} \left[ \sum_{S : R_0 \supseteq S \supseteq R} \frac{\sigma(S \setminus \overline{S})}{\ell(S)^{d+1}} \right]^2 \eta_R
\]

\[
\leq \sum_{R \in \mathcal{R}_1(R_0)} P^\varepsilon_R(\sigma \cdot (R_0 \setminus R), R)^2 E(w, R)^2 w(R) \lesssim E_2^2(R_0).
\]

Here, we have passed back from the discrete approximation to the Poisson, and then used the energy inequality.

Second, argue in the case of \( k = 0 \), the key point is that the collection \( \mathcal{R}_0(R_0) \) has bounded overlaps, in the sense of (3.5). Note that for \( R \in \mathcal{R}_0(R_0) \), we have \( R = \pi_n K_R \) where \( K_R \in \mathcal{V}_{F_R} \) for some \( F_R \in \mathcal{F} \), with \( F_R \) not contained in \( R_0 \). Now suppose, by way of contradiction of (3.5), that \( R_1, R_2 \in \mathcal{R}_0(R_0) \) satisfy

\[ C_0 R_1 \cap C_0 R_2 \neq \emptyset \quad \text{and} \quad 2^{2^r}|R_1| < 2^r|R_2| < |R_0|. \]

and \( F_{R_1} \subsetneq F_{R_2} \). (If \( F_{R_1} = F_{R_2} \), bounded overlaps is a consequence of goodness.) Then, we would have \( K_{R_2} \subsetneq K_{R_1} \), which is a contradiction. Thus, the cubes have bounded overlaps, and we can use the argument of the case of \( k = 1 \) to complete the proof.

5.3. **Backwards Testing Condition.** Let us first treat the case of \( n - 1 < d \leq n \). For an integer \( k \geq 0 \), let \( W^k_R = R \times [2^{-k-1}, 2^k]\ell(R) \), so that \( \Box_R = \bigcup_{k=0}^\infty W^k_R \). Then, using the \( A_2 \) condition with holes,

\[
\sum_{R \in \mathcal{C}_U : R^{(1)} \subseteq R_0} \frac{\eta(W^k_R)}{\ell(R)^{d+1}} \sum_{S \in \mathcal{C}_U : S \subseteq R} \frac{\eta(W^k_S)}{\ell(S)^{d+1}} \sigma(S)
\]

\[
\lesssim A_2 2^{-2k} \sum_{R \in \mathcal{C}_U : R^{(1)} \subseteq R_0} \frac{\eta(W^k_R)}{\ell(R)^{d+1}} \sum_{S \in \mathcal{C}_U : S \subseteq R} \ell(S)^{d+1}
\]

\[
\lesssim A_2 2^{-2k} \sum_{R \in \mathcal{C}_U : R^{(1)} \subseteq R_0} \eta(W^k_R) \lesssim A_2 2^{-2k} \eta(\Box_R).
\]

The bound on the sum of \( \ell(S)^{d+1} \) depends upon \( n - 1 < d \leq n \).

To argue along the same lines in the case of \( 0 < d \leq n - 1 \), the following Lemma will complete the proof, but this argument depends upon the \( A_2 \) condition with no holes, and it is only at this point that we need this stronger \( A_2 \) condition.

**Lemma 5.12.** Assuming the \( A_2 \) condition with no holes, the following estimate holds uniformly in \( R, \)

\[
\sum_{S \in \mathcal{C}_U : S \subseteq R} \frac{\eta(W^k_S)}{\ell(S)^{d+1}} \sigma(S) \lesssim 2^{-2k} \ell(R)^{d+1}.
\]
Proof. Let
\[ \alpha = \frac{w(R)}{\ell(R)^d} \cdot \frac{\sigma(R)}{\ell(R)^d} \leq \mathcal{A}_2. \]
For integers \( t \in \mathbb{Z} \) with \( 2^t \alpha \leq \mathcal{A}_2 \), and integers \( u \geq 1 \) let \( S_{t,u} \) be the intervals \( S \subset R \) such that
\[ 2^u \ell(S) = \ell(R), \]
and
\[ 2^t \alpha \leq \frac{w(S)}{\ell(S)^d} \cdot \frac{\sigma(S)}{\ell(S)^d} < 2^{t+1} \alpha. \]
Now, for each \( S \in S_{t,u} \), we have either
\[ \frac{w(S)}{\ell(S)^d} \geq \sqrt{\frac{w(R)}{\sigma(R)} 2^{t/2} \sqrt{\alpha}} \]
or
\[ \frac{\sigma(S)}{\ell(S)^d} \geq \sqrt{\frac{\sigma(R)}{w(R)} 2^{t/2} \sqrt{\alpha}}. \]
Assume that the former condition holds for at least half of the cubes in \( S_{t,u} \). From this, we deduce an upper bound on the cardinality of \( S_{t,u} \). Namely, we have
\[ \sqrt{w(R)/\sigma(R)} 2^{t/2} \sqrt{\alpha} 2^{-u \ell(R)^d} \cdot S_{t,u} \leq \sum_{S \in S_{t,u}} w(S) \leq w(R). \]
From this, it follows that \( S_{t,u} \leq 2^{-t/2+ud} \).
Hence, we have
\[ \sum_{S \in S_{t,u}} \eta(W_k) \frac{\ell(S)^d+1}{\sigma(S)} \leq 2^{-2k+t-u(d+1)} \alpha \ell(R)^{d+1} \cdot S_{t,u} \]
\[ \leq 2^{-2k+t/2-u} \alpha \ell(R)^{d+1}. \]
This is summable in the \( u \geq 1 \) and \( t \in \mathbb{Z} \) such that \( 2^t \alpha \leq 2 \mathcal{A}_2 \) to the estimate we need. \( \square \)

6. The Global to Local Reduction

6.1. Initial Steps. We begin the task of proving the norm boundedness of the Riesz transforms, assuming the \( \mathcal{A}_2 \), energy and testing hypotheses.

Lemma 6.1. Assume the a priori inequality (1.1). For all \( 0 < \vartheta < 1 \), and choices of \( 0 < \epsilon < (4d+4)^{-1} \), there is a choice of \( r \) sufficiently large so that,
\[ (6.2) \]
\[ |\mathbb{E} \langle R_\sigma(P_{g\text{good}}^g f), P_{g\text{good}}^w g \rangle_w | \leq [C_{\epsilon,r,a} R + \vartheta N] \|f\|_a \|g\|_w. \]
It follows that \( N \leq R \).

To prove (6.2), we can assume that \( f \) and \( g \) are supported on a fixed cube \( Q^0 \). After trivial application of the testing inequality, we can further assume that \( f \) and \( g \) are of mean zero in their respective spaces. With probability one, there is a cube \( Q^0_\sigma \in D_\sigma \) which contains \( Q^0 \). Then,
\[ f = \sum_{Q \in D_\sigma, Q \subset Q^0_\sigma} \Delta_Q^\sigma f. \]
The function \( g \) satisfies an analogous expansion.
Define the bilinear form
\[ B^{\text{above}}(f, g) \equiv \sum_{P \in D_f} \sum_{Q \in D_g, Q \supsetneq 4rP} \langle \Delta^P_{\sigma} f, \Delta^w_{\sigma} g \rangle_w, \]
and define \( B^{\text{below}}(f, g) \) similarly. Here \( Q \in D_w \) must be good, hence \( Q \subset R \), and is a relatively long way from the boundary of any child of \( R \). The cube \( P_Q \) is the child of \( P \) that contains \( Q \). We have also simply written \( R_{\sigma} \) above, suppressing the truncations.

A basic estimate is

**Lemma 6.3.** Under the hypotheses of Lemma 6.1, there holds

\[ \mathbb{E} \left| \langle R_{\sigma}(P^\sigma_{\text{good}} f, P^w_{\text{good}} g) \rangle_w - B^{\text{above}}(P^\sigma_{\text{good}} f, P^w_{\text{good}} g) \right| - B^{\text{below}}(P^\sigma_{\text{good}} f, P^w_{\text{good}} g) \right| \leq \{C_{\epsilon, r, \delta R + \delta N} \|f\|_\sigma \|g\|_w. \]

The proof of this lemma includes several elementary estimates, and critically, the surgery estimate, Lemma 8.5, which requires the expectation above. It remains to consider the form \( B^{\text{above}}(f, g) \), and its dual. This is indeed main point.

**Lemma 6.4.** For almost every choice of \( D_\sigma \) and \( D_w \), there holds

\[ |B^{\text{above}}(f, g)| \leq R \|f\|_\sigma \|g\|_w. \]

The same estimate holds for the dual form \( B^{\text{below}}(f, g) \).

In the proof, only the existence of the cube \( Q^0 \) is required of \( D_\sigma \). Hence, it suffices to assume that \( f \) and \( g \) are good, that is \( P^\sigma_{\text{good}} f = f \), and moreover that there is an integer \( 0 \leq i_f < 4r \), for which

\[ \Delta^P_{\sigma} f \neq 0 \quad \text{implies} \quad i_f = \log_2 \ell(Q) \mod 4r. \]

Impose the same assumptions on \( g \), with an integer \( 0 \leq i_g < 4r \). By passing to a larger cube, we can assume that \( \log_2 \ell(Q^0) = i_f - 1 \mod r \). Then, let

\[ D^r_{i_f} \equiv \{Q : \log_2 \ell(Q) = i_f \mod 4r\}, \]
\[ D^r_{i_f} \equiv \{Q : \log_2 \ell(Q) = i_f - 1 \mod r\}. \]

In particular, the grid \( D^r_{i_f} \) contains all the children of the cubes in \( D^4_{i_f} \). Let \( D^r_{g,s} \), for \( s = r, 4r \), have the corresponding definition.

6.2. **Stopping Data.** Our next task is to make the global to local reduction, which is phrased in terms of this important stopping time construction. We construct \( \mathcal{F} \subset D^r_{i_f} \) in a recursive fashion. Initialize \( \mathcal{F} = \{Q_0^0\} \). Then, in the recursive step, if \( F \subset \mathcal{F} \) is minimal, we add to \( \mathcal{F} \) the maximal dyadic subcubes \( Q \subset F \), with \( Q \in D^r_{i_f} \) such that either

1. (A big average) \( ||f||^\sigma_Q \geq ||f||^\sigma_F \),
(2) (Energy is big) $\sum_{K \in \mathcal{W}_Q} P^\sigma_{\partial}(F \setminus \mathcal{C}_0 K, K)^2 E(K, w)^2 w(K) \geq 10^2 R^2 \sigma(Q)$.

In the second condition, recall that $E \leq R$. And, $\mathcal{W}_Q \subset \mathcal{D}^\sigma_\partial$ are the maximal cubes $Q' \subset Q$ such that $\operatorname{dist}(Q', \partial Q) \geq \ell(Q')^e \ell(Q)^{1-e}$, which have the bounded overlaps property of Proposition 3.11. Namely, the energy inequality (3.4) will hold.

It is elementary to see that the collection $\mathcal{F}$ is $\sigma$-Carleson:

$$\sum_{F' \in \mathcal{F}} |\sigma|_{\mathcal{F}} \leq 1, \quad F \in \mathcal{F}.$$ 

Define projections

$$P^\sigma_{\partial} f = \sum_{P: \pi_F P = F} \Delta^\sigma_{\partial} f, \quad P^w_{\partial} g = \sum_{Q: \pi_F Q = F} \Delta^\sigma_{\partial} f.$$ 

In the second line, by $\pi_F Q = F$, we mean that $F \in \mathcal{F}$ is the minimal element such that $Q \subset F$.

The important quasi-orthogonality bound is

$$\sum_{F \in \mathcal{F}} \{[|f|]^{\sigma}_{\mathcal{F}} \sigma(F)^{1/2} \} \|P^\sigma_{\partial} f\|_w \|P^w_{\partial} g\|_w \lesssim \|f\|_{\sigma} \|g\|_w,$$

as follows from orthogonality properties of the projections and the construction of the stopping cubes $\mathcal{F}$. This is the important reduction, which will follow from the functional energy inequalities.

Lemma 6.7 (Global to Local Reduction). With the construction above,

$$\left| B_{\text{above}}(f, g) - \sum_{F \in \mathcal{F}} B_{\text{above}}(P^\sigma_{\partial} f, P^w_{\partial} g) \right| \lesssim R \|f\|_{\sigma} \|g\|_w.$$ 

We prove this just below. Observe that this Lemma shows that the control of the form $B_{\text{above}}(f, g)$ is then reduced to a class of local estimates.

Lemma 6.8 (Local Estimate). Uniformly in $F \in \mathcal{F}$, we have

$$|B_{\text{above}}(P^\sigma_{\partial} f, P^w_{\partial} g)| \lesssim R \{[|f|]^{\sigma}_{\mathcal{F}} \sigma(F)^{1/2} + \|P^\sigma_{\partial} f\|_w \} \|P^w_{\partial} g\|_w.$$ 

In view of the quasi-orthogonality bound (6.6), this clearly completes the control of the form $B_{\text{above}}(f, g)$. The delicate proof is taken up in §7.

Proof of Lemma 6.7. We have

$$B_{\text{above}}(f, g) = \sum_{F \in \mathcal{F}} \sum_{F' \in \mathcal{F}} B_{\text{above}}(P^\sigma_{\partial} f, P^w_{\partial} g) = \sum_{F \in \mathcal{F}} \sum_{F' \in \mathcal{F}} B_{\text{above}}(P^\sigma_{\partial} f, P^w_{\partial} g).$$
Indeed, it is evident that \( f = \sum_{F \in \mathcal{F}} P^w F \). And, the form \( B^{\text{above}} \) is a sum over pairs of cubes \( Q, P \) with \( Q \subseteq 4rP \), so that we can assume that \( g = \sum_{F \in \mathcal{F}} P^w F \). But \( Q \subseteq 4rP \) also implies that \( \pi_F P \cap \pi_F Q \neq \emptyset \), hence one cube contains the other. The case \( \pi_F P \subsetneq \pi_F Q \neq \emptyset \) contradicts the definition of \( \pi_F Q \), see (6.5). The case of \( \pi_F P = \pi_F Q \) is contained in the form

\[
\sum_{F \in \mathcal{F}} B^{\text{above}}(P^w_F, P^w_P).
\]

And control of this is left to the local estimate.

Thus, we are concerned with \( \pi_F Q \subsetneq \pi_F P \), that is the stopping element \( \pi_F Q \) ‘separates’ \( Q \) and \( P_Q \). It remains to bound

\[
\left| \sum_{F \in \mathcal{F}} \sum_{F' \in \mathcal{F}} B^{\text{above}}(P^w_F, P^w_P) \right| \leq \mathcal{R} \left\| f \right\|_\sigma \left\| g \right\|_w.
\]

We invoke the exchange argument, which entails the following steps. (a) Use the stopping data to sum the martingale differences on \( f \). (b) Restrict the argument of the Riesz transforms to stopping cubes, invoke the testing inequality on these cubes, and quasi-orthogonality. (c) The remaining term has the argument of the Riesz transform and the martingale difference on \( f \) will be in off-diagonal position, letting the monotonicity principles come into play. (d) The remaining parts of the argument use either the energy inequality, or its functional variants, to control these terms.

For each \( F \in \mathcal{F} \),

\[
(6.9) \quad \left| \sum_{P, P' \supseteq F} [\Delta_F^\sigma]_{P_F} \right| \leq \left[ \left| f \right| \right] F^\sigma.
\]

This is relevant to (a) above. Concerning (b), for \( P \supseteq F \), we write the argument of the Riesz transform as \( P_F = F + (P_F \setminus F) \). With the argument of the Riesz transform being \( F \), we have

\[
\left| \sum_{F \in \mathcal{F}} \sum_{P, P' \supseteq F} [\Delta_F^\sigma]_{P_F} (R_\sigma P_F, P^w_P) \right| \leq \sum_{F \in \mathcal{F}} \left[ \left| f \right| \right] F^\sigma \left( R_\sigma P_F, P^w_P \right) \leq \mathcal{R} \left\| f \right\|_\sigma \left\| g \right\|_w.
\]

We first use (6.9), then the testing inequality, followed by the quasi-orthogonality bound (6.6).

It remains to bound the sum below, in which we are at point (c) of the exchange argument,

\[
\sum_{F \in \mathcal{F}} \sum_{P \supseteq F} [\Delta_F^\sigma]_{P_F} (R_\sigma (Q_F \setminus F), P^w_P) \leq \mathcal{R} \left\| f \right\|_\sigma \left\| g \right\|_w.
\]

Again, we appeal to the stopping data, estimating the argument of the Riesz transform by

\[
\left| \sum_{F \in \mathcal{F}} \sum_{P : P \supseteq F} [\Delta_P^\sigma]_{P_F} (Q_F \setminus F) \right| \leq \phi \equiv \sum_{F \in \mathcal{F}} \left[ \left| f \right| \right] F^\sigma \cdot F.
\]
By the $\sigma$-Carleson property of $\mathcal{F}$, we have that $\|\Phi\|_\sigma \lesssim \|f\|_\sigma$. Then, monotonicity applies to show that

$$\left| \sum_{P: P \supseteq F} [\Delta^\sigma_f]_P (R_\sigma(Q_F \setminus F), P^- g)_w \right| \lesssim \sum_{K \in W_F} P^\sigma_\sigma(\Phi \cdot F^c, K) \sum_{Q : \pi_F Q = F, Q \subseteq K} \| \Delta^w_Q \chi_{\ell(K)} \|_w \| \Delta^w_Q g \|_w$$

(6.10)

$$+ P^\sigma_\sigma(\Phi \cdot F^c, K) E(w, K)^{1/2} \left[ \sum_{Q : \pi_F Q = F, Q \subseteq K} \| \Delta^w_Q g \|_w \right]^{1/2} \equiv A_F + B_F.$$  

Recall that $f$ is supported on the cube $Q^0_\sigma \in D_\sigma$.

We are at point (d) of the exchange argument. Let us consider the first term on the right above. Summing over $F \in \mathcal{F}$, and applying Cauchy-Schwarz,

$$\sum_{F \in \mathcal{F}} A_F \leq \left[ \sum_{F \in \mathcal{F}} \sum_{K \in W_F} P^\sigma_\sigma(\Phi \cdot F^c, K)^2 \sum_{Q : \pi_F Q = F, Q \subseteq K} \| \Delta^w_Q \chi_{\ell(K)} \|_w^2 \times \sum_{P: P \supseteq F} \| P^- g \|_w^2 \right]^{1/2} \lesssim R \|f\|_\sigma \|g\|_w.$$

Here, appeal to the functional energy estimate of Theorem 5.2.

The second term in (6.10) only requires us to show that

$$\sum_{F \in \mathcal{F}} \sum_{K \in W_F} P^\sigma_\sigma(\Phi \cdot F^c, K)^2 E(w, K)^2 w(K) \lesssim R^2 \|f\|_\sigma^2.$$  

(6.11)

We will gain a geometric decay in the parameter $t \geq 1$. Fix $F \in \mathcal{F}$, and consider

$$\sum_{F' \in \mathcal{F}} \sum_{\pi_F^{-1} F' = F} P^\sigma_\sigma(\Phi(F \setminus \pi_F^{-1} F'), K)^2 E(w, K)^2 w(K)$$

\lesssim \left( \|\Phi\|_\sigma^2 \right)^2 \sum_{F' \in \mathcal{F}} \sum_{\pi_F^{-1} F' = F} P^\sigma_\sigma((F \setminus \pi_F^{-1} F'), K)^2 E(w, K)^2 w(K)$$

(6.12)

\lesssim 2^{-t} \|\Phi\|_\sigma^2 \sum_{F' \in \mathcal{F}} \sum_{\pi_F^{-1} F' = F} P^\sigma_\sigma((F \setminus \pi_F^{-1} F'), K)^2 E(w, K)^2 w(K)$$

\lesssim 2^{-t} R^2 \|\Phi\|_\sigma^2 \sigma(F).$

The point here is that we can use the faster decay of the gradient-plus operator and the goodness of the intervals $K$ to gain a geometric factor in $t$, at the cost of replacing $P^\sigma_\sigma$ by $P^\sigma$ in (6.12). Finally we use the energy inequality to bound the expression by $\sigma(F)$. By the $\sigma$-Carleson property of $\mathcal{F}$, there holds

$$\sum_{F \in \mathcal{F}} \left( \|\Phi\|_\sigma^2 \right)^2 \sigma(F) \lesssim \|\Phi\|_\sigma^2 \lesssim \|f\|_\sigma^2.$$  

A trivial summation over $t \geq 1$ will complete the proof of (6.11).
7. The Local Estimate

We prove Lemma 6.8. In so doing, we hold \( F \in \mathcal{F} \) fixed throughout the proof, and we assume that \( f = P^g_f g \) as well as \( g = P^w_f g \), to reduce notation. Thus, we should bound

\[
\sum_{P: \pi_P P = F} \sum_{Q: \pi_Q Q = F} [\Delta^g_f]_{PQ}^\sigma \langle R_{\sigma} P_Q, \Delta^w_Q g \rangle_w.
\]

The first step is a repetition of the exchange argument. The argument of the Riesz transform is \( P_Q \), where \( \pi_P P = F \) and \( Q \subseteq F \), \( P \). Write this as \( F + (P_Q - F) \). Define a real number \( \varepsilon_Q \) by

\[
\varepsilon_Q [\| f \| F] = \sum_{P: Q \subseteq F} [\Delta^g_f]_{PQ}^\sigma.
\]

It follows from the construction of the stopping data that we have \( |\varepsilon_Q| \leq 1 \). Thus, by reordering the sum below, an appeal to the testing condition can be made to see that

\[
\left| \sum_{P: \pi_P P = F} \sum_{Q: \pi_Q Q = F} [\Delta^g_f]_{PQ}^\sigma \langle R_{\sigma} F, \Delta^w_Q g \rangle_w \right| = [\| f \| F] \left| \sum_{Q: \pi_Q Q = F} \varepsilon_Q \Delta^w_Q g \right|_w.
\]

\[
\leq [\| f \| F] \sigma(F)^{1/2} \left| \sum_{Q: \pi_Q Q = F} \varepsilon_Q \Delta^w_Q g \right|_w.
\]

Thus, it remains to consider the sum when the argument of the Riesz transform is \( F \setminus P_Q \). This is the stopping form, and it will require a subtle, recursive analysis.

Lemma 7.1 (Stopping Form). *The following estimate holds*

\[
\left| \sum_{P: \pi_P P \subseteq F} \sum_{Q: \pi_Q Q = F} [\Delta^g_f]_{PQ}^\sigma \langle R_{\sigma} (F \setminus P_Q), \Delta^w_Q g \rangle_w \right| \leq R \| f \|_\sigma \| g \|_w.
\]

The analysis will combine on the one hand, a variant of an argument related to the so-called pivotal technique [8], and the other, the subtle recursion that was identified in [2]. In neither case is the bounded averages property of the function \( f \) relevant. (We used the bounded averages property in the exchange argument.) Rather, it is the fact that the energy stopping condition is incorporated into the stopping data that is the crucial point.

This definition is critical in our application of the monotonicity principle. For \( Q \) with \( \pi_P Q = F \), define \( Q^e \) to be the minimal element \( Q^e \in \mathcal{D}_f \) such that \( Q \subseteq F \), \( Q^e \), and

\[
P^g_{\sigma}(F \setminus Q^e, Q)^2 E(w, Q)w(Q) \leq P^g_{\sigma}(F \setminus Q^e, Q)^2 \| \Delta^w_Q \ell(Q) \|_w^2.
\]
Due to the nature of the two Poisson terms, if \(P_Q \supset Q^c\), then the inequality above continues to hold with \(Q^c\) replaced by \(P_Q\). This means that we will have a sharper form of the monotonicity principle to apply.

7.1. **Below** \(Q^e\). The Lemma to prove is

**Lemma 7.3.** There holds

\[
\sum_{Q : \pi_x P = 0} \sum_{Q : \pi_x P = \infty} [\Delta_1 Q^e F] P_Q \langle R_\sigma (F \setminus P_Q), \Delta_0 Q^e g \rangle_w \leq \mathcal{R}\|f\|_\sigma \|g\|_w.
\]

This preparation will be useful throughout the analysis of the local term.

**Lemma 7.4.** For \(P \in \mathcal{D}_r\), define \(K_P\) to be the maximal elements of \(\mathcal{D}_r\) such that \(10K \subset P\). Then, each good cube \(Q \in \mathcal{D}_r\) with \(Q \subset \pi P\), is satisfies \(Q \subset K\) for some \(K \in K_P\).

**Proof.** One should note that many cubes in \(K_P\) are of side length \(2^{-r} \ell(P)\), because the maximal side length of \(K \in K_P\) is \(2^{-r} \ell(P)\). But, the cube \(Q\) is much smaller than this bound: \(\ell(Q) \leq 2^{-4r+1} \ell(P)\). It follows from goodness that if \(Q\) and \(K \in K_P\) intersect, and \(2^r \ell(Q) \leq \ell(K)\), then we must have \(Q \subset K\). Thus, we can assume that \(\ell(K) \leq 2^{-3r+1} \ell(P)\) below, and this implies that \(\text{dist}(K, \partial P) \leq 10 \cdot 2^r \ell(K)\), by construction of \(K_P\).

Thus, we have (a) \(Q \cap K \neq \emptyset\); (b) \(\text{dist}(K, \partial P) \leq 10 \cdot 2^r \ell(K)\); (c) \(Q \not\subset K\) which implies \(2^r \ell(Q) \geq \ell(K)\), and (d) \(2^{4r} \ell(Q) \leq \ell(P)\). From goodness of \(Q\) it follows that

\[\ell(Q) \leq \ell(P)^{1-e} \leq \text{dist}(Q, \partial P)\]

This would contradict (b) if \(\ell(Q) \geq \ell(K)\), so \(\ell(Q) \leq \ell(K)\), in which case we derive

\[\ell(Q) \leq \ell(P)^{1-e} \leq (10 \cdot 2^r + 1) \ell(K) \leq (10 \cdot 2^r + 1) 2^r \ell(Q)\]

But this contradicts (d). So the proof is complete.

**Proof of Lemma 7.3.** Indeed, we hold the relative lengths of \(P\) and \(Q\) fixed, setting \(2^r \ell(Q) = \ell(P)\), and gain a geometric decay in \(s\). Hold \(P\), and \(P'\), a child of \(P\), fixed. We have for \(s \geq 0\), and \(K \in \mathcal{W}_P\),

\[
\sum_{Q : \subset K} P^g_{\sigma}(F \setminus P', Q)^2 E(w, Q)^2 w(Q)
\leq 2^{-s/2} \left[\frac{\ell(K)}{\ell(P)}\right] P^g_{\sigma}(F \setminus P', K)^2 E(w, K)^2 w(K).
\]

We use the extra decay on the gradient-plus Poisson to gain the geometric factor \(2^{-s/2}\).

It is essential to use construction of the stopping tree to deduce that the energy sum below is dominated by \(\sigma(P')\). Namely, the cube \(P'\) must fail the second rule in the stopping tree
construction, and so
\[ \sum_{K \in \mathcal{W}_p} P^g_{\sigma}(F \setminus P', K)^2E(w, K)^2w(K) \lesssim \mathcal{R}^2\sigma(P'). \]

Then observe that because we are ‘below’ $Q^c$, the monotonicity principle will be expressed solely in terms of the gradient-plus Poisson operator. Whence,
\[ \left| \sum_{K \in \mathcal{W}_p} \sum_{Q : \pi_F Q = \ell, 2^s \ell(Q) = \ell(K) \atop Q \subset P \cap K, P' \supseteq P_Q \subset Q^c} [\Delta^g_{\sigma}]_{P'} \langle R_{\sigma}(F \setminus P_Q), \Delta^w_{Q} g \rangle_w \right| \]
\[ \lesssim \left| [\Delta^g_{\sigma}]_{P'} \sum_{K \in \mathcal{W}_p} \sum_{Q : \pi_F Q = \ell, 2^s \ell(Q) = \ell(K) \atop Q \subset P \cap K, P' \supseteq P_Q \subset Q^c} P^g_{\sigma}(F \setminus P', Q)E(w, Q)w(Q)^{1/2}\|\Delta^w_{Q} g\|_w \right| \]
\[ \lesssim 2^{-s/2}\mathcal{R} \cdot |[\Delta^g_{\sigma}]_{P'}| \sigma(P')^{1/2} \left[ \sum_{K \in \mathcal{W}_p} \left( \frac{\ell(K)}{\ell(P)} \right)^2 \sum_{Q : \pi_F Q = \ell, 2^s \ell(Q) = \ell(K) \atop Q \subset P \cap K, P' \supseteq P_Q \subset Q^c} \|\Delta^w_{Q} g\|_w^2 \right]^{1/2}. \]

Apply Cauchy–Schwarz in $P$ and $P'$. This gives us two terms. The first is trivially estimated by
\[ \sum_{P \atop \pi_F P = F} \sum_{P' \atop \pi_F P' = F} \left| [\Delta^g_{\sigma}]_{P'} \right|^2 \sigma(P') \lesssim \|f\|_\sigma^2. \]

For the second, we have
\[ \sum_{P : \pi_F P = F} \sum_{P' : \pi_F P' = F} \sum_{K \in \mathcal{W}_p} \left( \frac{\ell(K)}{\ell(P)} \right)^2 \sum_{Q : \pi_F Q = \ell, 2^s \ell(Q) = \ell(K) \atop Q \subset P \cap K, P' \supseteq P_Q \subset Q^c} \|\Delta^w_{Q} g\|_w^2 \]
\[ \lesssim \sum_{Q'} \|\Delta^w_{Q'} g\|_w^2 \sum_{P : \pi_F P = F} \sum_{P' : \pi_F P' = F} \sum_{Q : \pi_F Q = \ell, 2^s \ell(Q) = \ell(K) \atop Q \subset P \cap K} \left( \frac{\ell(K)}{\ell(P)} \right)^2 \lesssim \|g\|_w^2. \]

We see that the resulting estimate is summable in $s \in \mathbb{N}$, due to the leading term in (7.5), so the proof is complete.

\[ \square \]

7.2. Above $Q^c$. The form that we have yet to control is
\[ \left| \sum_{P : \pi_F P = F} \sum_{Q : \pi_F Q = F} [\Delta^g_{\sigma}]_{P_Q} \langle R_{\sigma}(F \setminus P_Q), \Delta^w_{Q} g \rangle_w \right| \lesssim \mathcal{R} \|f\|_c \|g\|_w. \]

This case is far more subtle, requiring a delicate recursion, which in turn requires a more elaborate notation to explain. The recursion is expressed in the decomposition of the bilinear form, according to the pairs of cubes. For this, we need this definition.
Definition 7.7. We call a collection \( \mathcal{P} \subset \mathcal{D}_t \times \mathcal{D}_g \) of pairs of cubes \((P_1, P_2)\) \textit{admissible} if these conditions are met.

1. \( P_2 \in_4 P_1 \), with \( \pi_\mathcal{F} P_2 = \pi_\mathcal{F} P_1 = F \), \( P_1 \) and \( P_2 \) are good, and \((P_1)_{P_2} \supset (P_2)^c\).
2. (Convexity in \( \mathcal{P} \).) For each \( P_2 \), if \( (P_1, P_2), (P_1'', P_2) \in \mathcal{P} \), and \( P_1 \subset P_1' \subset P_1'' \), with \( P_1' \) good, then \((P_1', P_2) \in \mathcal{P}\).

We then set \( \tilde{P}_1 \equiv (P_1)_{P_2} \), and also set \( \mathcal{P}_1 \equiv \{P_1 : (P_1, P_2) \in \mathcal{P}, \text{ for some } P_2\} \), and we define \( \tilde{P}_1 \) and \( \mathcal{P}_2 \) similarly.

We then define
\[
B_\mathcal{P}(f, g) \equiv \sum_{(P_1, P_2) \in \mathcal{P}} [\Delta_f^\mathcal{P}]_{P_Q}^g \langle R_v(F \setminus P_Q), \Delta_Q g \rangle_{w}.
\]

Next, we define the \textit{size} of \( \mathcal{P} \), which must be formulated with some care. For a cube \( P \in \mathcal{D}_t \), set \( \mathcal{K}_P \) to be the maximal cubes \( K \in \mathcal{D}_t \) such that \( 10K \subset P \).

Proposition 7.8. A cube \( P_2 \in \mathcal{P}_2 \), with \( P_2 \in_4 P \) satisfies \( P_2 \in_r K \) for some \( K \in \mathcal{K}_P \).

Proof. Observe that the conclusion is obvious if \( K \) and \( P_2 \) intersect, and \( 2^r \ell(P_2) \leq \ell(K) \). But, also, many cubes \( K \in \mathcal{K}_P \) satisfy \( 2^r \ell(K) = \ell(P) \), due to the fact that \( K \in \mathcal{D}_t \). And, \( 2^{4r-1} \ell(P_2) \leq \ell(P) \), hence the conclusion is clear for \( 2^{3r-1} \ell(K) \geq \ell(P) \).

We have \( 2^{3r-1} \ell(K) \leq \ell(P) \), which implies \( \text{dist}(K, \partial P) \leq 20 \cdot 2^r \ell(K) \). Then, if \( \ell(K) \leq \ell(P_2) \), it follows that \( P_2 \subset K \) and
\[
20 \cdot 2^r \ell(K) \geq \text{dist}(K, \partial P) \geq \ell(K)^\epsilon \ell(P)^{1-\epsilon}
\]
by goodness of \( P_2 \). But this contradicts \( 2^{3r-1} \ell(K) \leq \ell(P) \).

Thus, \( P_2 \not\subset K \). And if \( P_2 \not\in_r K \), that means \( 2^r \ell(P_2) \geq \ell(K) \), whence, again by goodness,
\[
2^{-r} \ell(K)^\epsilon \ell(P)^{1-\epsilon} \leq 20 \cdot 2^r \ell(K).
\]

That means \( \ell(P) \leq [20 \cdot 2^{(1+\epsilon)}]^{1/(1-\epsilon)} \ell(K) \), which again is a contradiction. \( \square \)

Notice that we incorporate the previous proposition into the important definition of \textit{size}.

\[
\text{size}(\mathcal{P})^2 \equiv \sup_{K \in \mathcal{T}_\mathcal{P}} \frac{Pg(F \setminus K, K)^2}{\sigma(K)} \sum_{P_2 \in \mathcal{P}_2 : P_2 \subset K} \left\| \frac{x}{\ell(K)^w} \right\|_{w}^2,
\]

\[
\mathcal{T}_\mathcal{P} \equiv \bigcup \{K_{P_1} : \tilde{P}_1 \subset \tilde{P}_1\}.
\]

Note that we only ‘test’ the size of the collection by forming a supremum over the collection \( \mathcal{T}_\mathcal{P} \). Our care about \( \in_4 \) and \( \in_r \) has been designed for this proposition.

Proposition 7.9. There holds \( \text{size}(\mathcal{P}) \leq \mathcal{R} \).
Proof. Consider $P \in T_F$ for which $\pi_F P = F$. Then, the cube $P$ must fail the energy stopping condition of §6.2. Therefore, we can estimate first for the Poisson operator with a hole in the argument,

$$P_e^F(F \setminus P, P)^2 \sum_{P_2 \in P, \text{par in } P} \left\| \Delta^w_{P_2} \frac{x}{\ell(P)} \right\|_{L^2_w}^2 \leq \left[ \int_{F \setminus P} \frac{1}{\ell(P)^{d+1} + \text{dist}(x, P)^{d+1}} \sigma(dx) \right] \sum_{K \in W} \sum_{P_2 \in P, P_2 \subset K} \left\| \Delta^w_{P_2} x \right\|_{L^2_w}^2 \leq \sum_{K \in W} \left[ \int_{F \setminus P} \frac{\ell(K)^{d+1}}{\ell(K)^{d+1} + \text{dist}(x, K)^{d+1}} \sigma(dx) \right] \sum_{P_2 \in P, P_2 \subset K} \left\| \Delta^w_{P_2} x \right\|_{L^2_w}^2 \leq \text{size}(P)^2 \sigma(P).$$

If $\pi_F P \subset F$, then, by admissibility, there is no cube $P_2 \in P$ with $P_2 \subset K$. This is because otherwise we would have $\pi_F P_2 \neq F$. Then, the inequalities above are trivial.

Our task is then to show that

**Lemma 7.10.** For all admissible $P$,

$$|B_P(f, g)| \lesssim \text{size}(P) \|f\|_{\sigma} \|g\|_{w}. \quad (7.11)$$

The main step in the proof is phrased in terms of this variant of orthogonality. We say that an enumeration of admissible collections $\{P_j : j \in \mathbb{N}\}$ is orthogonal if and only if (a) the collections of cubes $P_j$ are pairwise disjoint, and (b) the collections of cubes $\tilde{P}_j$ are pairwise disjoint. One should note the asymmetry in the definition, which comes from a corresponding asymmetry in the roles of $P_1$ and $P_2$ in the definition of $B_P(f, g)$.

**Lemma 7.12.** Let $\{P_j : j \in \mathbb{N}\}$ be admissible and orthogonal. Then, there holds

$$|B_{\bigcup_j P_j}(f, g)| \leq \sqrt{2} \sup_j B_{P_j} \cdot \|f\|_{\sigma} \|g\|_{w}. \quad (7.12)$$

**Proof.** Notice that a give cube $P_1$ can be in two different collections $\tilde{P}_j$, which fact explains the $\sqrt{2}$ above. Let $\Pi^f \sigma = \sum_{P_j \in P} \Delta^w_{P_j} f$, and define $\Pi^g \sigma = \sum_{P_2 \in \tilde{P}} \Delta^w_{P_2} g$. Then, we have

$$\sum_{j \in \mathbb{N}} \|\Pi^f \sigma\|^2_{\sigma} \leq 2 \|f\|^2_{\sigma},$$

while the same inequality holds with constant one in $L^2(\mathbb{R}; \sigma)$. Then, there holds

$$|B_{\bigcup_j P_j}(f, g)| \leq \sum_{j \in \mathbb{N}} |B_{P_j}(f, g)| \leq B_{P_j} \sum_{j \in \mathbb{N}} \|\Pi^f \sigma\|_{\sigma} \|\Pi^g \sigma\|_{w}.$$
\[ \begin{align*}
\leq &\sup_j B_{P_j} \left[ \sum_{j \in \mathbb{N}} \|\Pi_j f\|_2^2 \times \sum_{j \in \mathbb{N}} \|\Pi_j g\|_w^2 \right]^{1/2} \\
\leq &\sqrt{2} \sup_j B_{P_j} \cdot \|f\|_\sigma \|g\|_w.
\end{align*} \]

With that preparation, our main lemma provides us with a decomposition of an arbitrary admissible collection into 'big' and 'small' collections. The big collections have a control on their operator norm, and we can recurse on the small collections.

**Lemma 7.13 (Size Lemma).** For any admissible \( \mathcal{P} \), there is a decomposition \( \mathcal{P} = \mathcal{P}_{\text{big}} \cup \mathcal{P}_{\text{small}} \) such that

\[ |B_{\mathcal{P}_{\text{big}}}(f, g)| \lesssim \text{size}(\mathcal{P}) \|f\|_\sigma \|g\|_w, \]

and, \( \mathcal{P}_{\text{small}} \) is the union of admissible collections \( \{ \mathcal{P}^j_{\text{small}} : j \in \mathbb{N} \} \), with

\[ \sup_{j \in \mathbb{N}} \text{size}(\mathcal{P}^j_{\text{small}}) \leq \frac{1}{4} \text{size}(\mathcal{P}). \]

Moreover, the collections \( \{ \mathcal{P}^j_{\text{small}} : j \in \mathbb{N} \setminus \{0\} \} \) are orthogonal.

**Proof of (7.11).** By recursive application of the Size Lemma, we can write \( \mathcal{P} = \bigcup_{t=1}^{\infty} \mathcal{P}_t \) where \( \mathcal{P}_t \) is the union of orthogonal collections \( \{ \mathcal{P}^t_{\text{small}} \} \), which satisfy \( B_{\mathcal{P}^t_{\text{small}}} \leq 4^{-t} \text{size}(\mathcal{P}) \). Thus,

\[ B_{\mathcal{P}} \leq \sum_{t=1}^{\infty} B_{\mathcal{P}_t} \lesssim \text{size}(\mathcal{P}) \sum_{t=1}^{\infty} 2 \cdot 4^{-t} \lesssim \text{size}(\mathcal{P}). \]

7.3. **Decomposition of \( \mathcal{P} \).** Define the measure on \( \mathbb{R}_+^{n+1} \) by

\[ \lambda = \lambda_\mathcal{P} = \sum_{P \in \mathcal{P}} \|\Delta^w_{P_2} x\|_w^2 \delta_{x_{P_2}} \]

where \( x_{P} = (x_P, \ell(\mathcal{P})) \). The main condition we have is this reformulation of the definition of size:

\[ \sup_{Q \in \mathcal{T}_\mathcal{P}} P^g_{\sigma}(F \setminus Q, Q)^2 \frac{\lambda(\text{Tent}_Q)}{\sigma(Q) \ell(Q)^2} = S^2 = \text{size}(\mathcal{P}), \]

\[ \text{Tent}_Q \equiv \bigcup_{K \in \mathcal{W}_Q} \text{Box}_K. \]

Here, we are using the notation \( \text{Box}_K \equiv K \times [0, \ell(K)) \), as it is used in the functional energy inequality.
This collection is used to make the decomposition of $\mathcal{P}$. Set $\mathcal{L}_0$ to be the minimal elements $Q \in \mathcal{T}_P$ such that
\[
P_{\sigma}^*(F \setminus Q, Q) \frac{\lambda(Tent_Q)}{\ell(Q)^2} \geq cS^2 \sigma(Q).
\]
Here $c = \frac{1}{32}$. Such $Q$ exist, by definition of size. Then, for $n \geq 1$, inductively define $\mathcal{L}_n$ to be the minimal elements $L \in \mathcal{T}_P$ such that
\[
\lambda(Tent_L) \geq (1 + c) \sum_{L' : L' \subset L}^* \lambda(Tent_{L'}).
\]
where the last sum is performed over the maximal elements $L' \in \bigcup_{m=0}^{n-1} \mathcal{L}_m$, with $L' \subset L$ (this is designated by the * appearing on the sum). Then set $\mathcal{L} = \bigcup_{n \geq 0} \mathcal{L}_n$.

The collection $\mathcal{P}_{\text{small}}$ is then defined this way. Set $\mathcal{P}_{\text{small}}^0$ to be the collection of pairs $(P_1, P_2) \in \mathcal{P}$ such that $P_1$ does not have a parent in $\mathcal{L}$. And, for each $L \in \mathcal{L}$, define
\[
\mathcal{P}_{\text{small}}^L \equiv \{(P_1, P_2) : P_2 = \pi_L P_1 = L, \ P_2 \subset L\}.
\]
Here and below, $\pi_L P_2$ is the minimal element of $L$ such that $P_2 \subset L$ and $P_2 \in \pi_L L$.

**Lemma 7.17.** The collections $\mathcal{P}_{\text{small}}^0$ and $\mathcal{P}_{\text{small}}^L$, for $L \in \mathcal{L}$ are admissible, have size at most $\frac{1}{4} \text{size}(\mathcal{P}) \leq \frac{1}{4} S$. Moreover, the collections $\{\mathcal{P}_{\text{small}}^L : L \in \mathcal{L}\}$ are orthogonal.

**Proof.** Admissibility is inherited from $\mathcal{P}$ and the construction of the collections. Orthogonality is also clear from the construction in terms of $\mathcal{L}$. Thus, it remains to check that the collections have small size. For $\mathcal{P}_{\text{small}}^0$, suppose there is a cube $Q \in \mathcal{T}_{\mathcal{P}_{\text{small}}^0}$ such that
\[
\frac{1}{16} S^2 \leq \frac{P_{\sigma}^*(F \setminus Q, Q)^2}{\sigma(Q) \ell(Q)^2} \times \lambda_{\mathcal{P}_{\text{small}}^0}(Tent_Q).
\]
If $Q$ does not contain any element of $\mathcal{L}$, we would contradict the construction of that collection.

Hence, it does contain elements of $\mathcal{L}$, and hence summing over the maximal such $L \in \mathcal{L}$ below, there holds
\[
\frac{1}{16} S^2 \leq c \frac{P_{\sigma}^*(F \setminus Q, Q)^2}{\sigma(Q) \ell(Q)^2} \times \sum_{L \in \mathcal{L} : L \subset Q}^* \lambda_{\mathcal{P}}(Tent) \leq cS^2.
\]
Notice that the constant $c$ enters in because of construction, see (7.15). We see a contradiction since $c = \frac{1}{32}$. Thus, $\mathcal{P}_{\text{small}}^0$ has small size.

Turn to the collections $\mathcal{P}_{\text{small}}^L$ as defined in (7.16). Again, if the size is more than $\frac{1}{4} S$, then there is a cube $Q \in \mathcal{T}_{\mathcal{P}_{\text{small}}^L}$ such that
\[
\frac{1}{16} S^2 \leq \frac{P_{\sigma}^*(F \setminus Q, Q)^2}{\sigma(Q) \ell(Q)^2} \lambda_{\mathcal{P}_{\text{small}}^L}(Tent_Q).
\]
Moreover, $Q$ cannot be contained in any $L' \in \mathcal{L}$ which is a descendant of $L$ in the $\mathcal{L}$-tree, otherwise the term on the right above is zero. It follows that the cube $Q$ must fail the inequality
7.4. Controlling the Big Collection. We have completed the proof of (7.14). By definition, the collection $\mathcal{P}_{\text{big}}$ is the (non-admissible) complementary collection. We decompose it into the union of two collections $\mathcal{P}_{\text{big}}^1$, for $j = 1, 2$, with the appropriate bound on the norm of the bilinear form $\mathcal{B}_{\mathcal{P}_{\text{big}}}$ in each case. The essential point is that in each of the big collections, the intricate relationship between $P_1$ and $P_2$ is moderated by a 'separating' collection of cubes, uniformly over pairs in the big collection. This permits the estimation of the operator norm.

Set $\mathcal{P}_{\text{big}}^{1,L} \equiv \bigcup_{L \in \mathcal{L}} \mathcal{P}_{\text{big}}^{1,L}$, where the latter collection is

$\mathcal{P}_{\text{big}}^{1,L} \equiv \{(P_1, P_2) \in \mathcal{P} : \tilde{P}_1 = L, \pi_L P_2 = L\}.$

Observe that these collections are admissible and orthogonal. Moreover, the structure of these collections is quite rigid, since $\tilde{P}_1$ is a fixed interval.

Lemma 7.18. There holds, uniformly over $L \in \mathcal{L}$, that $\mathcal{B}_{\mathcal{P}_{\text{big}}^{1,L}} \leq S$.

Proof. Set $Q \equiv \mathcal{P}_{\text{big}}^{1,L}$. We can estimate for each $K \in \mathcal{W}_L$,

$$\left| \sum_{(P_1, P_2) \in Q} \sum_{P_2 \in Q, K} [\Delta_{P_1}^w f]_{P_1}^w \langle R_\sigma(F \setminus \tilde{P}_1), \Delta_{P_2}^w g \rangle_w \right| \leq \left| \sum_{P_2 \in Q, K} \Delta_{P_2}^w \frac{x}{\ell(K)} \| \Delta_{P_2}^w g \|_w \right| \leq S \left| \sum_{P_2 \in Q, K} \Delta_{P_2}^w \frac{x}{\ell(K)} \| \Delta_{P_2}^w g \|_w \right|^{1/2} \leq S \| f \|_\sigma \| g \|_w.$$  

The monotonicity principle applies, as formulated in (4.2), which has both the gradient and gradient-plus Poisson terms. But with the condition that $(P_2)^c \subset P_1$, recall (7.2), we only have the term involving the gradient Poisson term. Then one appeals to Cauchy–Schwarz, and importantly, the definition of size($P$) to gain the term $S \sigma(K)^{1/2}$ above. The final inequality above is trivial. 

$\square$

The second collection is built of pairs that are 'separated' by $L$. Now, we set $\tilde{\pi}_L P_1 = \tilde{\pi}_L P_2$, and inductively define $\tilde{\pi}_L^{t+1} P_2$ to be the minimal element of $L$ that strictly contains $\tilde{\pi}_L^t P_2$. Then, $\mathcal{P}_{\text{big}}^{2,t} \equiv \bigcup_{t = 2}^\infty \mathcal{P}_{\text{big}}^{2,t}$, where

$$\mathcal{P}_{\text{big}}^{2,t} \equiv \bigcup_{L \in \mathcal{L}} \mathcal{P}_{\text{big}}^{2,t,L} \equiv \bigcup_{L \in \mathcal{L}} \{(P_1, P_2) \in \mathcal{P} : \pi_L \tilde{P}_1 = \tilde{\pi}_L^t P_2 = L\}.$$  

These collections are admissible, a property inherited from $\mathcal{P}$. For fixed $t \geq 2$, the collections $\mathcal{P}_{\text{big}}^{2,t,L} : L \in \mathcal{L}$ are orthogonal, as is very easy to see from the definition. We prove

(7.15). Hence, we can repeat the previous argument to deduce that the size of this collection is also small. 

$\square$
Lemma 7.19. For all \( t \geq 2 \) and \( L \in \mathcal{L} \), there holds

\[
B_{P_{big}^{2,t,L}} \leq (1 + c)^{-1/6} S.
\]  

(7.20)

In view of the orthogonality, and Lemma 7.12, this estimate is clearly summable in \( t \geq 2 \) and \( L \in \mathcal{L} \) to the estimate we need. Namely, we will have

\[
\sum_{t=2}^{\infty} \sum_{L \in \mathcal{L}} B_{P_{big}^{2,t,L}}(f, g) \leq \|f\|_s \|g\|_w \sum_{t=2}^{\infty} \sup_{L \in \mathcal{L}} B_{P_{big}^{2,t,L}} \leq S \|f\|_s \|g\|_w \sum_{t=2}^{\infty} (1 + c)^{-1/6} \leq S \|f\|_s \|g\|_w.
\]

Proof of Lemma 7.19. There are two parts of the proof. First, identifying a quantity that controls the norm of the bilinear form, and second that this quantity decreases geometrically in \( t \). Define \( C_L \) to be the \( \mathcal{L} \)-children of \( L \), and define

\[
T^2 \equiv \sup_{L' \in C_L} \sup_{K \in K_{L'}} P^g_\sigma(F \setminus K, K)^2 \frac{\lambda_Q(T_{ent_k})}{\sigma(K) \ell(K)} \frac{\lambda_Q}{\sigma(K) \ell(K)}.
\]

(7.21)

The first part is to show that \( B_Q \leq T \), where to ease notation, set \( Q = P_{big}^{2,t,L} \).

Let \( G \) be standard stopping data for \( f \). Since we can assume that the Haar support of \( f \) is contained in \( P_2 \), we can write \( G = \bigcup_{t \geq 0} G_t \), where \( G_0 = \{F\} \), and in the inductive stage, for \( G \in G_t \), add to \( G_t \) the maximal cubes \( Q \subset G \), \( Q \) a child of a \( P_1 \in P_1 \), such that \( [\|f\|_{Q}] \geq 10[\|f\|_Q] \).

With this version of the stopping data, a variant of the quasi-orthogonality bound holds, which we will use. In addition, the collapse of telescoping sums is given by this formula. For each \( P_2 \in P_2 \), and \( G \in G \), there holds

\[
\sum_{P_1 : (P_1 : P_2) \in P} [\Delta_{P_1} P_1 (F \setminus \tilde{P}_1)] \leq [\|f\|_{P_2}] \cdot F.
\]

(7.22)

This depends upon admissibility, namely the assumption of convexity in \( P_1 \), holding \( P_2 \) fixed.

For a cube \( G \) set \( \Pi_{G}^w \equiv \sum_{P_2 \in G} \Delta_{P_2}^w \). Observe that we can estimate as follows for \( G \in G \), and \( L' \in C_L \) such that \( \pi_G L' = G \),

\[
\left| \sum_{\tau_1 \in Q} \left[ \sum_{P_2 \in \tau_2} \left[ \Delta_{P_1} P_1 (R_{\sigma}(F \setminus \tilde{P}_1), \Delta_{P_2}^w) \right] \right] \right|
\]

\[
\leq \sum_{K \in K_{L'}} \sum_{P_2 \in \tau_2} \sum_{P_1 \in Q} \left| \sum_{\tau_1 \in Q} \left[ \Delta_{P_1} P_1 (R_{\sigma}(F \setminus \tilde{P}_1), \Delta_{P_2}^w) \right] \right|
\]

\[
\leq [\|f\|_{G}] \sum_{K \in K_{L'}} \sum_{P_2 \in \tau_2} \frac{\lambda_{\sigma}(F \setminus \tilde{P}_1, P_2)}{\ell(Q_2)} \left| \frac{x}{\ell(Q_2)} \right| \Delta_{P_2}^w \right|_w.
\]
\[
(7.23) \quad \lesssim T[f] G_0 \sum_{K \in K_L} \sigma(K)^{1/2} \|\Pi_K^w g\|_w \lesssim T[f] G \sigma(L')^{1/2} \|\Pi_L^w g\|_w.
\]

We are combining several observations: (a) for each \(P_2 \in Q_2\), we have \(P_2 \subset L'\) for some \(L' \in C_L\), and \(\pi G P_2 = \pi G L'\); (b) and then, \(P_2 \subset K\) for some \(K \in K_L\), thus the first inequality above holds; (c) we have convexity in \(P_1\), hence we can use (7.22), giving us the stopping value above; (d) we can also apply the monotonicity principle (4.2), which gives us two terms; (e) but in view of the definition of (7.2), and the fact that \(\tilde{P}_1 \cap G \supset (P_2)^c\), we only have the one term, the one using the gradient Poisson, after application of the monotonicity principle, giving us the second inequality above; (f) and a trivial use of Cauchy–Schwarz in \(P_2\), with the definition of \(T\) in (7.21) gives us the concluding inequality above.

Notice that an application of Cauchy–Schwarz, and quasi-orthogonality shows that
\[
\sum_{G \in \tilde{G}} \sum_{L' \in C : \pi_G L' = G} (7.23) \lesssim T[\|f\|_G \|g\|_w].
\]

We turn to the second half of the proof of (7.20), namely that \(T \lesssim (1 + c)^{-1/2}S\), where \(T\) is as in (7.21). It suffices to assume that \(t \geq 10\) say. Take a cube \(K \in K_{L'}\), for \(L' \in C_L\). It follows from Proposition 7.24 below, that \(\pi_G^1 K = L\), for \(u \leq 5\). Notice that by construction of \(L\), and simple induction, that for integer \(s = r - 5\),
\[
\lambda_Q(K) \leq \sum_{L'' \in L : \pi_G^1 L'' \subset K} \lambda(T_{L''}) \leq (1 + c)^{-1} \sum_{L'' \in L : \pi_G^1 L'' \subset K} \lambda(T_{L''}) \leq (1 + c)^{-s - 1} \sum_{L'' \in L : \pi_G^1 L'' \subset K} \lambda(T_{L''}) \leq (1 + c)^{-s} \lambda(T_{\text{tent}} K) \lesssim (1 + c)^{-1/2} \lambda(T_\text{tent} K).
\]

Note that we have used (7.15) a total of \(s - 1\) times to get the conclusion above. Therefore, by Proposition 7.24,
\[
(1 + c)^s \frac{\mathcal{P}^g(F \setminus K, K)^2}{\ell(K)^2} \lambda_Q(K) \lesssim \frac{\mathcal{P}^g(F \setminus K, K)^2}{\ell(K)^2} \lambda(T_\text{tent} K) \lesssim S^2 \sigma(K).
\]

This concludes our proof. \(\square\)

The Lemma above depends upon properties of the collection \(T_P\).

**Proposition 7.24.** Let \(Q \in \tilde{P}_1\), with \(\pi_C Q = L\), and that \(K \in K_Q\) is a cube on which we are testing the size of \(P_{\text{big}}^{2,1} \). Then \(Q \subset \pi_C^1 K\).
Proof. Now, \( \ell(\pi^3 K) \geq 2^{3r} \ell(K) \), by our construction, so the conclusion is obvious if \( \ell(K) \geq 2^{-3r} \ell(Q) \).

Continuing under the hypothesis that \( 2^{4r} \ell(K) \leq \ell(Q) \), it follows that we must have \( \text{dist}(K, \partial Q) \leq 20 \cdot 2^r \ell(K) \). By way of contradiction, suppose that

\[
(7.25) \quad K \subseteq L_1 \subseteq L_2 \subseteq L_3 \subseteq Q, \quad L_1, L_2, L_3 \in \mathcal{L}.
\]

Suppose that \( L_s \in \mathcal{P}_1 \), for either \( s = 1, 2 \). Since all cubes are from a grid with scales separated by \( r \), we then have \( L_s \in 4r \), and then goodness of the parent of \( L_s \) implies

\[
|L_s|^c |Q|^{1-c} \leq \text{dist}(L_s, \partial Q) \leq \text{dist}(K, \partial Q) \leq 20 \cdot 2^r|L_s|
\]

which is a contradiction.

Thus, we must have \( L_s \notin \mathcal{P}_1 \), for \( 1 \leq s \leq 2 \). Let \( Q_s \in \mathcal{P}_1 \) be such that \( L_s \in \mathcal{K}_{Q_s} \). If \( Q_1 \subseteq Q \), then \( Q_1 \in 4r Q \), and we again see a contradiction to \( K \in \mathcal{K}_Q \), namely

\[
|Q_1|^c |Q|^{1-c} \leq \text{dist}(Q_1, \partial Q) \leq \text{dist}(K, \partial Q) \leq 10 \cdot 2^{r+1}|Q_1|.
\]

Assume that \( Q_1 \notin 4r Q \). Equality \( Q_1 = Q_2 \) cannot hold, since \( K \subsetneq L_1 \). We must have \( Q \in 4r Q_1 \in 4r Q_2 \), and that means

\[
|Q|^c |Q_2|^{1-c} \leq \text{dist}(Q, \partial Q_2) \leq \text{dist}(L_{r+1}, \partial Q_2) \leq 10 2^r \ell(Q).
\]

And this final contradiction proves that (7.25) cannot hold, and this prove the proposition. \( \square \)

8. Proof of Lemma 6.3

The proof of Lemma 6.3 depends upon a well-known case analysis, however the very weak form of the \( A_2 \) condition\(^1\) complicates our analysis, and indeed, every case below requires a new analysis.

The expression to be dominated is a sum over pairs of cubes \( 2^{-4r} \ell(Q) \leq \ell(P) \leq 2^{4r} \ell(Q) \).

Throughout, we assume that \( \ell(Q) \leq \ell(P) \), with the other case being handled by duality. The cases are

**Far Apart:** \( \ell(P) \leq 2^{4r} \ell(Q) \) and the cubes \( 3P \) and \( 3Q \) do not intersect.

**Surgery:** \( \ell(P) \leq 2^{4r} \ell(Q) \) and \( 3P \cap 3Q \neq \emptyset \). This is a delicate surgery argument, one in which we use the \textit{a priori} bound, and take advantage of the presence of random grids.

**Nearby:** \( \ell(P) \leq 2^{4r} \ell(Q) \) and \( Q \subsetneq 3P \setminus P \).

**Inside:** \( \ell(P) \leq 2^{4r} \ell(Q) \) and \( Q \subset P \). In this case we bound the sum of terms

\[
\langle R(\Delta^w_p f \cdot (P \setminus P_Q)), \Delta^w_Q g \rangle_w.
\]

Lemma 8.1 (Far Apart). \textit{The following estimate is true:}

\[
\sum_{P} \sum_{Q : \ell(Q) \leq \ell(P)} \langle R(\Delta^w_p f, \Delta^w_Q g) \rangle_w \leq \mathcal{R} \|f\|_\sigma \|g\|_w.
\]

\(^1\)Our \( A_2 \) condition is weak, but also in the case of \( d > 1 \), the necessary \( A_2 \) condition comes with a power decay of \( 2d \), which is too weak to directly apply in some of the cases below.
The dual estimate also holds.

Proof. Hold the relative side lengths of $P$ and $Q$ fixed, thus for an integer $s \geq 0$, $2^s \ell(Q) = \ell(P)$. For an integer $t \geq 0$, and dyadic cube $R$, consider the two projections

$$\Pi_{R,s,t}^g f \equiv \sum_{P \subset 3^{t+2}R \setminus 3^{t+1}R \ell(P) = \ell(R)} \Delta_p^g f,$$

$$\Pi_{R,s,t}^w g \equiv \sum_{Q \subset 3^{t+2}R \setminus 2^t \ell(Q) = \ell(R)} \Delta_Q^w g.$$

Observe that it suffices to bound

$$\sum_{s,t=0}^{\infty} \sum_{R \in D} \left| \langle R, \Pi_{R,s,t}^g f, \Pi_{R,s,t}^w g \rangle \right|.$$  \hfill (8.2)

Moreover, we have

$$\sum_{R \in D} \| \Pi_{R,s,t}^g f \|_\sigma^2 \lesssim \| f \|_\sigma^2$$  \hfill (8.3)

and likewise for the projections $\Pi_{R,s,t}^w f$.

Then, estimate as follows.

$$\sum_{Q : Q \subset 3^t R \ell(Q) = \ell(R)} \left| \langle R, \Pi_{R,s,t}^g f, \Delta_Q^w g \rangle \right|_w \lesssim \frac{2^{-s} \ell(R)}{3^{t \ell(R)} d + t} \sum_{Q : Q \subset 3^t R \ell(Q) = \ell(R)} \| \Pi_{R,s,t}^g f \|_w \| \Delta_Q^w g \|_{L^1(\mathbb{R}^n, w)}$$

$$\lesssim 2^{-s} \sqrt{3^{-3t} \langle 3^{t+2} R \setminus 3^{t+1} R \rangle \Delta_Q^w g \|_{L^1(\mathbb{R}^n, w)}}$$

$$\lesssim 2^{-s} \Delta_2^1 \| \Pi_{R,s,t}^g f \|_w \| \Pi_{R,s,t}^w g \|_w. $$  \hfill (8.4)

The first inequality is just the standard off-diagonal kernel bound, which can be used since $\Delta_Q^w g$ has $w$ integral zero, giving us the constant in front, which is the side length of $Q$ times the $L^\infty$ norm of the gradient of the Riesz transform on the complement of $3^{t+1}R$. After that, apply Cauchy–Schwarz, and the $A_2$ condition.

By (8.3), the sum over $R$ of the terms in (8.4) is bounded. We have gained a geometric factor in $s$ and $t$, so we have the required bound for (8.2). \hfill \Box

The delicate surgery estimate is contained in the following lemma. It is phrased differently as the a priori norm estimate is needed to complete the proof.

Lemma 8.5 (Surgery). For all $0 < \theta < 1$, and choices of $0 < \epsilon < (4d + 4)^{-1}$, there is a choice of $r$ sufficiently large so that, uniformly in $0 < a_0 < b_0 < \infty$,

$$\sup_{a_0 < a < b < b_0} \mathbb{E} \left| \sum_{P : 2^{-4r} \ell(Q) \leq \ell(P) \leq 2^{4r} \ell(Q)} \langle R, \sigma \Delta_P^g f, \Delta_Q^w g \rangle_w \right| \lesssim \{ C_{\epsilon, r, a} R + \delta N_0 \| f \|_\sigma \| g \|_w. $$
Proof. It is important that the expectation over the random choice of grids appears above. We fix the relative lengths of $P$ and $Q$, setting $2^s \ell(Q) = \ell(P)$, where $0 \leq s \leq r$ is fixed. The case of $-r \leq s < 0$ is handled by duality. There are only a bounded number of cubes $Q$ with length as above, such that $3P \cap 3Q \neq \emptyset$, and so we can assume that $Q$ is a function of $P$, but this is suppressed in the notation. Further, enumerating the children $P_i, Q_j$, $1 \leq i, j \leq n$ of $P$, and $Q$ respectively, we fix $i, j$, and only consider

$$E|\Delta^\sigma f \cdot \langle R_\sigma P_i, Q_j \rangle_{w} \cdot \Delta^\sigma g|.$$  

Here and below, we are suppressing the truncation parameter $s$.

Now, we have by the testing hypothesis, uniformly over the probability space

$$|\langle R_\sigma P_i, Q_j \cap P_i \rangle_{w}| \leq \mathcal{R} \sqrt{\sigma(P_i)w(Q_j)}.$$  

In this case, Cauchy–Schwarz completes the proof, so we need only consider

$$E|\Delta^\sigma f \cdot \langle R_\sigma P_i, (Q_j \setminus P_i) \rangle_{w} \Delta^\sigma g|.$$  

The set $Q_j \setminus P_i$ is decomposed into the sets $Q_\partial \cup Q_{sep}$, where

$$Q_\partial \equiv \{x \in Q_j \setminus P_i : \text{dist}(x, P_i) < \vartheta \ell(Q)\}, \quad 0 < \vartheta < 1.$$  

This latter set depends upon $P_i$, which is a function of the dyadic grid $D_{\sigma}$, holding $D_{w}$ fixed. Observe that we can estimate, using the \textit{a priori} norm inequality (1.1),

$$E_{D_{\sigma}} \sum_{P} |\Delta^\sigma f \cdot \langle R_\sigma P_i, Q_\partial \rangle_{w} \Delta^\sigma g| \leq N \mathbb{E}_{D_{\sigma}} \sum_{P} |\Delta^\sigma f \sqrt{\sigma(P_i)w(Q_\partial)\Delta^\sigma g}| \leq N\|f\|_{\sigma} \left[ \mathbb{E}_{D_{\sigma}} \sum_{P} |\Delta^\sigma g|^{2}w(Q_\partial) \right]^{1/2} \leq \tilde{N}_{\sigma} \vartheta^{1/2} \|f\|_{\sigma} \|g\|_{w}.$$  

We gain the factor of $\vartheta^{1/2}$ since $\mathbb{E}_{D_{\sigma}} w(Q_\partial) \leq \vartheta w(Q_j)$.

It therefore remains to bound the term below, with cube $P_i$ and set $Q_{sep}$. But these sets are separated by distance $\vartheta \ell(P)$, so that using just the kernel bound, we have

$$|\langle R_\sigma P_i, Q_{sep} \rangle_{w}| \leq C_{\sigma, r} \frac{\sigma(P_i)w(Q_j)}{\ell(P)_{d}} \leq C_{\sigma} A_{1/2}^{1/2} \sqrt{\sigma(P_i)w(Q_j)},$$  

and this is clearly enough to complete the proof.

\[\square\]

Lemma 8.6 (Inside Term). There holds

$$\sum_{P} \sum_{Q : Q \preceq s_{r}P_Q} |\langle R_\sigma (\Delta^\sigma f \cdot (P \setminus P_Q)) \rangle_{w} \Delta^\sigma g| \leq \mathcal{R} \|f\|_{\sigma} \|g\|_{w}.$$
Lemma 8.7 (Nearby Term). There holds

$$\sum_{P} \sum_{Q : 2^s \ell(Q) \leq \ell(P)} |\langle R_\sigma \Delta^q_p f, \Delta_Q^w g \rangle| \leq \mathcal{R}_s \|f\| \|g\|_w.$$  

Both are argued by a similar method, and we present the proof of the nearby term. The key points are the goodness of cubes and the $A_2$ condition. These two proofs are the only place in which the full strength of the $A_2$ condition is used. Moreover, the $A_2$ bound has to be arranged correctly to overcome certain dimensional obstructions not present in the Hilbert transform case.

Proof. We hold the relative lengths of $Q$ and $P$ to be fixed by an integer $s \geq 4\epsilon$. By a crude application of the monotonicity principle, there holds for each child $P'$ of $P$,

$$|\langle R_\sigma (\Delta^q_p f \cdot P'), \Delta_Q^w g \rangle| \leq [\|\Delta^q_p f\|]_{p^o} P^o(Q, P) w(Q)^{1/2} \|\Delta_Q^w g\|_w.$$  

Clearly, we will use Cauchy-Schwarz in $Q$. To organize the sum of the Poisson related terms, we first estimate

$$P^o(P', Q)^w(Q) \leq \sigma(P') \int_{P} \frac{\ell(Q)^2}{\ell(Q)^{2d+2} + |x - x_Q|^{2d+2}} \sigma(dx) \cdot w(Q).$$  

Now, $Q$ is good, and $2^s \ell(Q) = \ell(P)$, hence for each $x \in P'$, $|x - x_Q| \geq 2^{-\epsilon} \ell(P)$. Let $\mathcal{R}_p^s$ be a collection of cubes $R \subset 3P \setminus P$ such that (1) each has side length $\ell(R) \simeq 2^{-\epsilon} \ell(P)$, (2) $\text{dist}(R, P) \geq 2^{-\epsilon} \ell(P)$, (3) each good $Q$ with $2^s \ell(Q)$ is contained in some $R \in \mathcal{R}_p^s$, and (4) the cardinality of $\mathcal{R}_p^s$ is at most $C_\epsilon N^{2^{\epsilon s}}$.

Then, for each $R \in \mathcal{R}_p^s$, we can use the $A_2$ bound.

$$\sum_{Q : 2^s \ell(Q) = \ell(P)} \sum_{Q \subset R} \int_{P} \frac{\ell(Q)^2}{\ell(Q)^{2d+2} + |x - x_Q|^{2d+2}} \sigma(dx) \cdot w(Q) \leq 2^{-2(1-\epsilon)s} \int_{P} \frac{1}{\ell(R)^{2d} + |x - x_R|^{2d}} \sigma(dx) \cdot w(R) \leq 2^{-2(1-\epsilon)s} A_2.$$  

Combining these observations, we have

$$\sum_{R \in \mathcal{R}_p^s} \sum_{Q : 2^s \ell(Q) = \ell(P)} |\langle R_\sigma (\Delta^q_p f \cdot P'), \Delta_Q^w g \rangle| \leq 2^{-2(1-\epsilon)s} A_2^{1/2} [\|\Delta^q_p f\|]_{p^o} \|P'\|^{1/2} \sum_{R \in \mathcal{R}_p^s} \left[ \sum_{Q \subset R} \|\Delta_Q^w g\|_w^2 \right]^{1/2}.$$  

$$\leq 2^{-2(1-\epsilon-s/2)A_2} A_2^{1/2} [\|\Delta^q_p f\|]_{p^o} \|P'\|^{1/2} \left[ \sum_{Q \subset R} \|\Delta_Q^w g\|_w^2 \right]^{1/2}.$$
Notice that we have gained a factor comparable to the square root of the cardinality of $\mathcal{R}_\delta$. We take $0 < \epsilon < 1$ so small that the exponent on $s$ above is positive. A further sum over $P$, children $P'$, and $s \geq 4r$ are very easy to complete. This finishes the proof.

\[ \square \]

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