GALOIS REPRESENTATIONS FOR GENERAL SYMPLECTIC GROUPS

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Abstract. We prove the existence of GSpin-valued Galois representations corresponding to cohomological cuspidal automorphic representations of general symplectic groups over totally real number fields under the local hypothesis that there is a Steinberg component. This confirms the Buzzard–Gee conjecture on the global Langlands correspondence in new cases. As an application we complete the argument by Gross and Savin to construct a rank seven motive whose Galois group is of type $G_2$ in the cohomology of Siegel modular varieties of genus three. Under some additional local hypotheses we also show automorphic multiplicity one as well as meromorphic continuation of the spin $L$-functions.

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Introduction

Let $G$ be a connected reductive group over a number field $F$. The conjectural global Langlands correspondence for $G$ predicts a correspondence between certain automorphic representations of $G(A_F)$ and certain $\ell$-adic Galois representations valued in the $L$-group of $G$. Let us recall from [BGII, §3.2] a rather precise conjecture on the existence of Galois representations for a connected reductive group $G$ over a number field $F$. Let $\pi$ be a cuspidal $L$-algebraic automorphic representation of $G(A_F)$. (We omit their conjecture in the $C$-normalization, cf. [BGII, Conj. 5.3.4], but see Theorem 10.1 below.) Denote by $\hat{G}(\mathbb{Q}_\ell)$ the Langlands dual group of $G$ over $\mathbb{Q}_\ell$, and by $L^G(\mathbb{Q}_\ell)$ the $L$-group of $G$ formed by the semi-direct product of $G(\mathbb{Q}_\ell)$ with $\text{Gal}(\overline{F}/F)$. According to their conjecture, for each prime $\ell$ and each field isomorphism $i : \mathbb{C} \cong \mathbb{Q}_\ell$, there should exist a continuous representation

$$\rho_{\pi,i} : \text{Gal}(\overline{F}/F) \to L^G(\mathbb{Q}_\ell),$$

which is a section of the projection $L^G(\mathbb{Q}_\ell) \to \text{Gal}(\overline{F}/F)$, such that the following holds: at each place $v$ of $F$ where $\pi_v$ is unramified, the restriction $\rho_{\pi,i,v} : \text{Gal}(\overline{F}_v/F_v) \to L^G(\mathbb{Q}_\ell)$ corresponds to $\pi_v$ via the unramified Langlands correspondence. Moreover $\rho_{\pi,i}$ should satisfy other desiderata, cf. Conjecture 3.2.2 of loc. cit. For instance at places $v$ of $F$ above $\ell$, the localizations $\rho_{\pi,i,v}$ are potentially semistable and have Hodge-Tate cocharacters determined by the infinite components of $\pi$. Note that if $G$ is a split group over $F$, we may as well take $\rho_{\pi,i}$ to have values in $\hat{G}(\mathbb{Q}_\ell)$. To simplify notation, we often fix $i$ and write $\rho_{\pi}$ and $\rho_{\pi,v}$ for $\rho_{\pi,i}$ and $\rho_{\pi,i,v}$, understanding that these representations do depend on the choice of $i$ in general.

Our main result confirms the conjecture for general symplectic groups over totally real fields in a number of cases. We find these groups interesting for two reasons. Firstly they naturally occur in the moduli spaces of polarized abelian varieties and their automorphic/Galois representations have been useful for arithmetic applications. Secondly new phenomena (as the semisimple rank grows) make the above conjecture sufficiently nontrivial, stemming from the nature of the dual group of a general symplectic group: e.g. faithful representations have large dimensions and locally conjugate representations may not be globally conjugate.

Let $F$ be a totally real number field. Let $n \geq 2$. Let $\text{GSpin}_{2n}$ denote the split general symplectic group over $F$ (with similitudes in $\mathfrak{g}_m$ over $F$). The dual group $G\text{Sp}_{2n}$ is the general spin group $G\text{Spin}_{2n+1}$, which we view over $\overline{\mathbb{Q}}_\ell$ (or over $\mathbb{C}$ via $i$), admitting the spin representation

$$\text{spin} : G\text{Spin}_{2n+1} \to \text{GL}_{2n}.$$ 

Consider the following hypotheses on $\pi$.

- **(St)** There is a finite $F$-place $v_{St}$ such that $\pi_{v_{St}}$ is the Steinberg representation of $G\text{Sp}_{2n}(F_{v_{St}})$ twisted by a character.
- **(L-coh)** $\pi_{\infty} : |\pi|^{(n+1)/4}$ is $\xi$-cohomological for an irreducible algebraic representation $\xi$ of the group $(\text{Res}_{F/Q}G\text{Sp}_{2n}) \otimes \mathbb{Q} \cong \mathbb{C}$ (Definition 1.8 below).
- **(spin-reg)** There is an infinite place $v_\infty$ of $F$ such that $\pi_{v_\infty}$ is spin-regular.

The last condition means that the Langlands parameter of $\pi_{v_0}$ maps to a regular parameter for $\text{GL}_{2n}$ by the spin representation, cf. Definitions 1.2 and 1.3 below. A trace formula argument shows that there are plenty of (in particular infinitely many) $\pi$ satisfying (St) and (L-coh), cf. [Clo86], whether or not (spin-reg) is imposed. Let $S_{bad}$ denote the finite set of rational primes $p$ such that either $p = 2$, $p$ ramifies in $F$, or $\pi_v$ ramifies at a place $v$ of $F$ above $p$.

**Theorem A** (Theorem 12.4). Suppose that $\pi$ satisfies hypotheses (St), (L-coh), and (spin-reg). Let $\ell$ be a prime number and $i : \mathbb{C} \to \overline{\mathbb{Q}}_\ell$ a field isomorphism. Then there exists a continuous representation

$$\rho_{\pi} = \rho_{\pi,i} : \text{Gal}(\overline{F}/F) \to G\text{Spin}_{2n+1}(\overline{\mathbb{Q}}_\ell),$$

unique up to $G\text{Spin}_{2n+1}(\overline{\mathbb{Q}}_\ell)$-conjugation, attached to $\pi$ and $i$ such that the following hold.

1. **(i)** The composition

$$\text{Gal}(\overline{F}/F) \to G\text{Spin}_{2n+1}(\overline{\mathbb{Q}}_\ell) \to \text{SO}_{2n+1}(\overline{\mathbb{Q}}_\ell) \subset \text{GL}_{2n+1}(\overline{\mathbb{Q}}_\ell)$$

is the Galois representation attached to a cuspidal automorphic $\text{Sp}_{2n}(A_F)$-subrepresentation $\pi^\flat$ contained in $\pi$. Further, the composition

$$\text{Gal}(\overline{F}/F) \to G\text{Spin}_{2n+1}(\overline{\mathbb{Q}}_\ell)/\text{Spin}_{2n+1}(\overline{\mathbb{Q}}_\ell) \cong \text{GL}_1(\overline{\mathbb{Q}}_\ell)$$

corresponds to the central character of $\pi$ via class field theory and $i$.

2. **(ii)** For every finite place $v$ which is not above $S_{bad} \cup \{\ell\}$, the semisimple part of $\rho_{\pi}(\text{Frob}_v)$ is conjugate to $i\phi_{\pi_v}(\text{Frob}_v)$ in $G\text{Spin}_{2n+1}(\overline{\mathbb{Q}}_\ell)$, where $\phi_{\pi_v}$ is the unramified Langlands parameter of $\pi_v$.
(iii) For every \( v|\ell \), the representation \( \rho_{n,v} \) is de Rham (in the sense that \( \pi \circ \rho_{n,v} \) is de Rham for all representations \( \pi \) of \( \text{GSpin}_{2n+1}(\mathbb{Q}_l) \)). Moreover,

(a) The Hodge-Tate cocharacter of \( \rho_{n,v} \) is explicitly determined by \( \xi \). More precisely, for all \( \varepsilon : F \to \mathbb{C} \) such that \( \varepsilon \) is tame, we have \( \mu_{HT}(\rho_{n,v}, \varepsilon) = \mu_{HT}(\xi, \varepsilon) \) (for \( \mu_{HT} \) and \( \mu_{H} \) see Definitions 1.6 and 1.9 below).

(b) If \( \rho_{n,v} \) has nonzero invariants under a hyperspecial (resp. Iwahori) subgroup of \( \text{GSpin}_{2n}(F_v) \), then either \( \rho_{n,v} \) or its quadratic character twist is crystalline (resp. semistable).

(c) If \( \ell \) is bad then \( \rho_{n,v} \) is crystalline.

(iv) For every \( v|\ell \), \( \rho_{n,v} \) is odd (see Definition 1.4 below).

(v) If \( n \geq 3 \) then the image of \( \rho_{n} \) is Zariski dense in \( \text{GSpin}_{2n+1}(\mathbb{Q}_l) \) modulo the center. For all \( n \geq 2 \), \( \text{spin} \circ \rho_{n} \) is an irreducible \( 2^n \)-dimensional representation, which remains irreducible when restricted to any open subgroup of \( \text{Gal}(\overline{F}/F) \).

(vi) If \( \rho' : \text{Gal}(\overline{F}/F) \to \text{GSpin}_{2n+1}(\mathbb{Q}_l) \) is any other continuous morphism such that, for almost all \( \ell \)-places \( v \) where \( \rho' \) and \( \rho_n \) are unramified, the semisimple part \( \rho'(\text{Frob}_v)_{\text{ss}} \) is conjugate to the semisimple part \( \rho_n(\text{Frob}_v)_{\text{ss}} \), then \( \rho_n \) and \( \rho_n' \) are conjugate.

Our theorem is new when \( n \geq 3 \). When \( n = 2 \), a better and fairly complete result without conditions (St) and (spin-reg) has been known by [Tay93, Lau05, Wei05, Urb05, Sor10, Jor12].

The above theorem in particular associates a weakly compatible system of \( \lambda \)-adic representations to \( \pi \). See also Proposition 17.1 below for precise statements on the weakly compatible system consisting of \( \pi \circ \rho_n \). It is worth noting that the uniqueness in (vi) would be false for general \( \text{GSpin}_{2n+1}(\mathbb{Q}_l) \)-valued Galois representations in view of Larsen’s example below Proposition 5.4 (as long as the finite group in that example can be realized as a Galois group). Our proof of (vi) relies heavily on the fact that \( \rho_n \) has large image.¹

Employing eigenvarieties, we can either replace condition (spin-reg) with weaker regularity conditions (HT) and (HT) as defined on page 13, or dispense with the condition at the expense of losing the Zariski-density and uniqueness of \( \rho_n \).

**Theorem B** (Theorem 14.3). Let \( \pi \) be a cuspidal automorphic representation of \( \text{GSpin}_{2n}(\mathbb{A}_F) \) satisfying (St) and (L-coh). Let \( \ell \) be a prime number and \( v : \mathbb{C} \to \overline{\mathbb{Q}}_l \) a field isomorphism. Assume that \( \psi_{\mathbb{Q}_l} \equiv \ell \). Then there exists a continuous representation

\[
\rho_n = \rho_{n,v} : \text{Gal}(\overline{F}/F) \to \text{GSpin}_{2n+1}(\mathbb{Q}_l),
\]

such that (i), (ii), (iii) and (iv) hold. Under conditions (HT) and (HT), the statements (v) and (vi) also hold, and \( \rho_n \) is unique up to conjugacy.²

Often Galois representations constructed via eigenvarieties are difficult to realize in the cohomology of algebraic varieties but our method shows that \( \pi \circ \rho_n \) for \( \pi \) as in Theorem B does appear in the cohomology of suitable Shimura varieties (up to normalization, without conditions (HT) and (HT)₂). When \( n = 3 \) and \( F = \mathbb{Q} \), we employ the strategy of Gross and Savin [GS98] to construct a rank 7 motive over \( \mathbb{Q} \) with Galois group of exceptional type \( G_2 \) in the cohomology of Siegel modular varieties of genus 3. The point is that \( \rho_n \) as in the above theorem factors through \( G_2(\overline{\mathbb{Q}}_l) \to \text{GSpin}_7(\overline{\mathbb{Q}}_l) \) if \( \pi \) comes from an automorphic representation on \( \mathbb{A} \) via theta correspondence. In particular we get yet another proof affirmatively answering a question of Serre in the case of \( G_2 \), cf. [KLS10, DR10, Yun14, Pat15] for other approaches to Serre’s question (none of which uses Siegel modular varieties). Along the way, we also obtain some new instances of the Buzzard–Gee conjecture for a group of type \( G_2 \). Our result also provides a solid foundation for investigating the suggestion of Gross–Savin that a certain Hilbert modular subvariety of the Siegel modular variety should give rise to the cohomology class for the complement of the rank 7 motive of type \( G_2 \) in the rank 8 motive cut out by \( \pi \), as predicted by the Tate conjecture. See Theorem 15.1, Corollary 15.2, and Remark 15.3 below for details.

As another consequence of our theorems, we deduce multiplicity one theorems for automorphic representations for inner forms of \( \text{GSpin}_{2n,F} \) under similar hypotheses from the multiplicity formula by Bin Xu [Xua]. As his formula suggests, multiplicity one is not always expected when all hypotheses are dropped.

**Theorem C** (Theorem 16.3). Let \( \pi \) be a cuspidal automorphic representation of \( \text{GSpin}_{2n}(\mathbb{A}_F) \) satisfying (St), (L-coh), (HT) and (HT)₂. The automorphic multiplicity of \( \pi \) is equal to 1.

¹In contrast, for cuspidal automorphic representations of the group \( \text{Sp}_{2n}(\mathbb{A}_F) \), the corresponding analogue of statement (vi) does hold (cf. Proposition B.1). So failure of (vi) is a new phenomenon for the cuspidal spectrum of the similitude group \( \text{GSpin}_{2n}(\mathbb{A}_F) \) that does not occur in the cuspidal spectrum of \( \text{Sp}_{2n}(\mathbb{A}_F) \).

²In the cases where conditions (HT) and (HT)₂ do not necessarily hold, Proposition 5.1 gives a description of the set of conjugacy classes of \( \rho_n \) satisfying (i) through (iv); in particular this set is finite by Corollary 5.3.
By the uniqueness in Theorem B, we have (a version of) strong multiplicity one for the $L$-packet of $\pi$. In Proposition 6.3 we prove a weak Jacquet-Langlands transfer for $\pi$ in Theorem C. This allows us to propagate multiplicity one from $\pi$ as above to the corresponding automorphic representations on a certain inner form. See Theorem 16.3 and Remark 16.6 below for details. Note that (weak and strong) multiplicity one theorem for globally generic cuspidal automorphic representations of $\text{GSp}_4(\mathbb{A}_F)$ has been known by Jiang and Soudry [Sou87, JS07].

Our results yield a potential version of the spin functoriality, thus also a meromorphic continuation of the spin $L$-function, for cuspidal automorphic representations of $\text{GSp}_{2n}(\mathbb{A}_F)$ satisfying (St), (L-coh), and the following strengthening of (spin-reg):

**Condition (L-coh)** is essential to our results. Remark 16.6 is necessary due to the current limitation of potential automorphy. Since Theorems A and C depend in turn on Theorem 16.3, we cannot control the poles. Before our work, some partial results have been obtained for $\text{GSp}_{2n}$ for $2 \leq n \leq 5$ by Andrianov, Novodvorsky, Piatetski-Shapiro, Vo, Bump–Ginzburg, and more recently by Pollack–Shah and Pollack, cf. [PS97, Vo97, BG00, PS, Pol]. See [Pol, 1.3] for remarks on spin $L$-functions with further references.

Finally we comment on the hypotheses of our theorems. The statements (ii) and (iii.c) are not optimal in that we exclude a little more than the finite places $v$ where $\pi_v$ is unramified. This is due to the fact that Shimura varieties are (generally) known to have good reduction at a $p$-adic place only when $p > 2$ and the defining data (including the level) are unramified at all $p$-adic places.\(^3\) Condition (L-coh) is essential to our method but it is perhaps possible to prove Theorem B under a slightly weaker condition that $\pi$ appears in the coherent cohomology of our Shimura varieties. We did get rid of (spin-reg) in Theorem B but the rather strong condition (spin-REG) in Theorem D is necessary due to the current limitation of potential automorphy theorems. Condition (St) is the most serious but believed to be superfluous. Unfortunately we have no clue how to work around it (although it should be possible to work with some other strong, if ad hoc, local hypotheses). We speculate that a level raising result for automorphic forms on $\text{GSp}_{2n}$ would be a big step forward, but such a result (for $n \geq 3$) seems out of reach at the moment. On the other hand, (St) is harmless to assume for local applications, which we intend to pursue in future work.

**Idea of proof.** Our proof of Theorem A relies crucially on Arthur’s book [Art13]. His results used to be conditional on the stabilization of the twisted trace formula, whose complete proof has been announced in a series of papers by Moeglin and Waldspurger; see [MW14] for the last one. Thus Arthur’s results are essentially unconditional.\(^4\) Since Theorems B, C, D depend in turn on Theorem A, all our results count on Arthur’s book.

Let us sketch our proof with a little more detail. We prove Theorem A by combining two approaches: (M1) Shimura varieties for inner forms of $\text{GSp}_{2n}$, and (M2) Lifting to $\text{GSpin}_{2n+1}(\overline{Q}_F)$ the $\text{SO}_{2n+1}(\overline{Q}_F)$-valued Galois representations as constructed by the works of Arthur and Harris-Taylor.

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\(^3\)When $F = \mathbb{Q}$ it is okay to allow $p = 2$ as the integral models for Siegel modular varieties are well known at all primes. The recent work [KMP] should make it unnecessary to assume $p > 2$ even when $F \neq \mathbb{Q}$.

\(^4\)Strictly speaking, Arthur postponed some technical details in harmonic analysis to future articles ([A25, A26, A27] in the bibliography of [Art13]), which have not appeared yet.
First Method. Consider the inner form $G$ over $F$ of $G^* := \text{GSp}_{2n,F}$, such that the local group $G_v$ is
\[
\begin{cases}
\text{non-split} & \text{if } v = v_\mathbb{S} \text{ and } [F = \mathbb{Q}] \text{ is even}, \\
\text{anisotropic modulo center} & \text{if } v|\infty, v \neq v_0, \\
\text{split} & \text{otherwise}.
\end{cases}
\]

We consider Shimura varieties arising from the group $\text{Res}_{F/Q} G$ (and the choice of $X$ as in §7 below). Note that (the $\mathbb{Q}$-points of) $\text{Res}_{F/Q} G$ has factor of similitudes in $F^*$, as opposed to $Q^*$, and that our Shimura varieties are not of PEL type (when $F \neq \mathbb{Q}$) but of abelian type. This should already be familiar for $n = 1$ (though we assume $n \geq 2$ for our main theorem to be interesting), where we obtain the usual Shimura curves, cf. [Car86a]. In case $F = \mathbb{Q}$ our Shimura varieties are the classical Siegel modular varieties.

The idea is to consider the compactly supported étale cohomology
\[
H^i_c(S_K, L_\xi) = \lim_{\longrightarrow} H^i_{c}(S_K, L_\xi), \quad i \geq 0,
\]
where $L_\xi$ is the $\ell$-adic local system attached to some irreducible complex representation $\xi$ of $G$. Then $H^i_c(S_K, L_\xi)$ has an action of $G(\mathbb{A}_F^\infty) \times \text{Gal}(\overline{F}/F)$; and one hopes to prove that through this action the module $H^i_c(S_K, L_\xi)$ realizes the Langlands correspondence. In particular, one tries to attach to a cuspidal automorphic representation $\pi$ of $\text{GSp}_{2n}(\mathbb{A}_F)$ the following virtual Galois representation
\[
\pi \mapsto \pi' \mapsto H_{\pi', n} \overset{\text{def}}{=} \sum_{i \geq 0} (-1)^i \text{Hom}(G(\mathbb{A}_F^\infty)(\pi', \overline{\xi}), H^i_{c}(S_K, L_\xi)),
\]
where the first mapping is a weak transfer of $\pi$ from $G^*$ to $G$. (We need to twist $\pi$ by $|\cdot|^{|n(n+1)|}$ to have $H_{\pi, n} \neq 0$ but this twist will be ignored in the introduction.) There are two issues: Firstly there are the usual issues coming from endoscopy. Our assumption (St) circumvents this difficulty (and helps us at other places). Secondly, even without endoscopy, one does not expect $H_{\pi, n}$ to realize the representation $\rho_{\pi}$ itself; rather it realizes (up to sign, twist, and multiplicity) the composition
\[
\text{Gal}(\overline{F}/F) \overset{\rho_{\pi}}{\rightarrow} \text{GSpin}_{2n+1}(\overline{\mathbb{Q}}_l) \overset{\text{spin}}{\rightarrow} \text{GL}_{2n}(\overline{\mathbb{Q}}_l).
\]

In particular if one wants to use $H_{\pi, n}$ to construct $\rho_{\pi}$, one has to show that $r: \text{Gal}(\overline{F}/F) \rightarrow \text{GL}_{2n}(H_{\pi, n}) \cong \text{GL}_{2n}(\overline{\mathbb{Q}}_l)$ has (up to conjugation) image in the group $\text{GSpin}_{2n+1}(\overline{\mathbb{Q}}_l) \subset \text{GL}_{2n}(\overline{\mathbb{Q}}_l)$. With point counting methods it can be shown that this is true for the Frobenius elements, but since these elements are only defined up to $\text{GL}_{2n}(\overline{\mathbb{Q}}_l)$-conjugation, we are unable to deduce directly that the entire representation $r$ has image in $\text{GSpin}_{2n+1}(\overline{\mathbb{Q}}_l)$. More information seems to be required.

Second Method. Consider a continuous morphism $\rho: \text{Gal}(\overline{F}/F) \rightarrow \text{SO}_{2n+1}(\overline{\mathbb{Q}}_l)$, and the exact sequence
\[
1 \rightarrow \overline{\mathbb{Q}}_l^\times \rightarrow \text{GSpin}_{2n+1}(\overline{\mathbb{Q}}_l) \rightarrow \text{SO}_{2n+1}(\overline{\mathbb{Q}}_l) \rightarrow 1.
\]

Using the theorem of Tate that $H^1(F, \mathbb{Q}/\mathbb{Z})$ vanishes, it is not hard to show that $\rho$ has a continuous lift $\tilde{\rho}: \text{Gal}(\overline{F}/F) \rightarrow \text{GSpin}_{2n+1}(\overline{\mathbb{Q}}_l)$. In particular one could try to attach to $\pi$ an (non-unique) automorphic $\text{Sp}_{2n}(\mathbb{A}_F)$-subrepresentation $\pi^b \subset \pi$, and then to $\pi^b$ the Galois representation $\rho_{\pi^b}: \text{Gal}(\overline{F}/F) \rightarrow \text{SO}_{2n+1}(\overline{\mathbb{Q}}_l)$, constructed by the combination of results of Arthur and many other people. One now hopes to construct $\rho_{\pi}: \text{Gal}(\overline{F}/F) \rightarrow \text{GSpin}_{2n+1}(\overline{\mathbb{Q}}_l)$ as a twist $\tilde{\rho}_{\pi^b} \otimes \chi$ for some continuous character $\chi: \text{Gal}(\overline{F}/F) \rightarrow \overline{\mathbb{Q}}_l^\times$. However it is a priori unclear where $\chi$ should come from. (The central character of $\pi$ only determines the square of $\chi$.)

Construction. We find the character $\chi$ by comparing $\tilde{\rho}_{\pi^b}$ with the representation $H_{\pi, n}$. Consider the diagram

\[
\begin{array}{ccc}
\text{Gal}(\overline{F}/F) & \overset{\rho_{\pi^b}}{\rightarrow} & \text{GSpin}_{2n+1}(\overline{\mathbb{Q}}_l) \\
& \rho_{\pi} & \text{spin} \\
\text{SO}_{2n+1}(\overline{\mathbb{Q}}_l) & \overset{\text{spin}}{\rightarrow} & \text{PGL}_{2n}(\overline{\mathbb{Q}}_l)
\end{array}
\]

where $\rho_{\pi^b} = \text{spin} \circ \tilde{\rho}_{\pi^b}$ for some choice of lift $\tilde{\rho}_{\pi^b}$ of $\rho_{\pi^b}$, and we construct $\rho_{\pi^b}$ from the representation $H_{\pi, n}$. We show in 3 steps that $\rho_{\pi^b}$ and $\rho_{\pi}$ are conjugate up to twisting by the sought-for character $\chi$. 

Step 1: Strong irreducibility. The representation $\rho_{\pi^b}$ is 'strongly irreducible'; that is, $\rho_{\pi^b}$ is irreducible, and stays irreducible upon restriction to all open subgroups of $\text{Gal}(\overline{F}/F)$. Since the spin representation is irreducible, it is enough to show that $\rho_{\pi^b}(\text{Gal}(\overline{F}/E)) \subset \text{SO}_{2n+1}(\overline{\mathbb{Q}}_l)$ is Zariski dense for all $E/F$ finite. Because $\pi$ is a twist
of the Steinberg representation at a finite place, we can assume that $\rho_n|_{\text{Gal}(\overline{F}/F)}$ has a regular-unipotent element $N$ in its image. Testerman–Zaleskii classified the semisimple subgroups of $SO_{2n+1}(\overline{Q})$ containing $N$. For $n \neq 3$, apart from $SO_{2n+1}(\overline{Q})$ only the principal $SL_2(\overline{Q})$ in $SO_{2n+1}(\overline{Q})$ is possible; but this group is easily ruled out using regularity of the Hodge-Tate numbers. In case $n = 3$, we have $G_2(\overline{Q}) \subset SO_7(\overline{Q})$, and this subgroup can be ruled out as well. Hence $\rho_3$ is strongly irreducible.

**Step 2**: Construct $\rho_2$. We compare the point counting formula for $(\text{Res}_{F/Q} G, f)$ with the Arthur-Selberg trace formula for $G/F$. Since the datum $(\text{Res}_{F/Q} G, f)$ is not of PEL type (unless $F = Q$), the classical work of Kottwitz [Kot92b, Kot90] does not apply. Instead we use the counting point formula as derived in [KSZ] from Kisin’s recent proof of the Langlands–Rapoport conjecture for Shimura data of abelian type, so in particular for $(\text{Res}_{F/Q} G, f)$. Consequently, at the unramified places we have $H^1(\text{Gal}(\overline{F}/F_v) = \mathfrak{a} \cdot \text{spin} \circ \phi_{\pi_v}$, where $\phi_{\pi_v}$ is the Satake parameter of $\pi_v$ and $\mathfrak{a} \in \mathbb{Z}_{>0}$ is essentially the automorphic multiplicity of $\pi$. The spin-regularity of $\pi$ and an argument due to Taylor show that $H^1_{\text{rig}}$ is the $\mathfrak{a}$-th power of a representation $\rho_2$. (Actually we show that $\mathfrak{a} = 1$ together with Theorem C but only after the construction of $\rho_\pi$ is done. See §17 below.)

**Step 3**: Produce $\chi$. To do this we prove:

**Lemma**. Let $r_1, r_2 : \text{Gal}(\overline{F}/F) \to \text{GL}_m(\overline{Q})$ two continuous representations, which are unramified almost everywhere, $r_1$ is strongly irreducible, and for almost all $F$-places

$$r_1(\text{Frob}_{F_v})_{\text{ss}}$$

is conjugate to

$$r_2(\text{Frob}_{F_v})_{\text{ss}}$$

in $\text{PGL}_m(\overline{Q})$.

Then $r_1 \simeq r_2 \otimes \chi$ for a continuous character $\chi : \text{Gal}(\overline{F}/F) \to \overline{\mathbb{Q}}^\times$.

The strong irreducibility condition is crucial. The lemma is false if $r_1$ is only assumed to be irreducible; Blasius constructs counter examples in the article [Bla94]. By counting points on Shimura varieties we control $\rho_2$ at the unramified places. By a different argument $\rho_1$ is, up to scalars, also controlled at the unramified places. Hence the lemma applies, and allows us to find a character $\chi$ such that $\rho_2 \simeq \rho_1 \chi$.

To prove Theorem A we define

$$\rho_\pi \begin{array}{c}
\text{def} \\
\chi \otimes \rho_n^\mathfrak{a} : \text{Gal}(\overline{F}/F) \to \text{GSpin}_{2n+1}(\overline{Q})
\end{array}$$

and check that $\rho_\pi$ satisfies the desired properties stated in the theorem.

The proof of Theorem B is an eigenvariety argument. Unlike in the usual case, we use two pseudorepresentations rather than one to overcome group-theoretic issues with general spin groups. Starting from $\rho_\pi$ that we have constructed in Theorem A for spin-regular automorphic representations $\pi$, we begin by interpolating the two representations

$$\text{Gal}(\overline{F}/F) \begin{array}{c}
\rho_\pi \\
\text{spin}
\end{array} \to \text{GSpin}_{2n+1}(\overline{Q}) \to \text{GL}_{2n}(\overline{Q})$$

and

$$\text{Gal}(\overline{F}/F) \begin{array}{c}
\rho_\pi \\
\text{std}
\end{array} \to \text{SO}_{2n+1}(\overline{Q}) \to \text{GL}_{2n+1}(\overline{Q})$$

as pseudorepresentations over the eigenvariety associated to a suitable inner form of $G$ which is compact at all infinite places. However it is unclear how to get the desired $\text{GSpin}_{2n+1}$-valued representation from the two pseudorepresentations if we naïvely specialize at a classical point with a non-spin-regular weight. Instead we adapt ideas from Steps 1-3 above to the family setting to construct $\rho_\pi$ as in Theorem B.

In fact we assume that the $\ell$-components of $\pi$ have nonzero Iwahori-invariants at $\ell$-adic places in the eigenvariety argument. The assumption is lifted by a patching argument along soluble extensions in Sorensen’s version [Sor08]. Some technical issues occur essentially because we do not have a precise control at $\ell$ over weak base change for $\text{GSp}_{2n}$ at $\ell$. We resolve them by using recent work of Bin Xu on $\text{GSp}_{2n}$ [Xua].

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5 It might also be possible to argue more directly with pseudorepresentations valued in reductive groups by Vincent Lafforgue but the authors do not see it.
We fix the following notation.

- \( n \geq 2 \) is an integer.
- \( F \) is a totally real number field, embedded into \( \mathbb{C} \).
- \( \mathcal{O}_F \) is the ring of integers of \( F \).
- \( \mathcal{A}_F \) is the ring of adeles of \( F \), \( \mathcal{A}_F := (F \otimes \mathbb{R}) \times (F \otimes \hat{\mathbb{Z}}) \).
- \( \Sigma \) is a finite set of \( F \)-places, then \( \mathcal{A}^0_\Sigma \subset \mathcal{A}_F \) is the ring of \( \Sigma \)-adeles with trivial components at the places in \( \Sigma \), and \( F_\Sigma := \prod_{\mathfrak{p} \nmid \Sigma} F_{\mathfrak{p}} ; \quad \mathcal{O}_\Sigma := F \otimes_\mathbb{Z} \mathbb{Z} \).
- \( p \) is a prime number, then \( F_p := F \otimes_\mathbb{Q} \mathbb{Q}_p \).
- \( \ell \) is a fixed prime number (different from \( p \)).
- \( \overline{Q}_F \) is a fixed algebraic closure of \( Q_F \), and \( i : \mathbb{C} \to \overline{Q}_F \) is an isomorphism.
- For each prime number \( p \) we fix the positive root \( \sqrt{p} \in \mathbb{R}_{>0} \subset \mathbb{C} \). From \( \ell \) we then obtain a choice for \( \sqrt{\ell} \in \overline{\mathbb{Q}}_F \).
- \( \Gamma := \text{Gal}(\overline{F}/F) \) is the absolute Galois group of \( F \).
- \( \Gamma_v := \text{Gal}(\overline{F_v}/F_v) \) is (one of) the local Galois group(s) of \( F \) at the place \( v \).
- \( \mathcal{V}_\infty := \text{Hom}(\overline{F}/F, \mathbb{C}) \) is the set of infinite places of \( F \).
- \( c_v \in \Gamma \) is the complex conjugation (well-defined as a conjugacy class) induced by any embedding \( \overline{F} \hookrightarrow \mathbb{C} \), extending \( v \in \mathcal{V}_\infty \).
- If \( G \) is a locally profinite group equipped with a Haar measure, then we write \( \mathcal{H}(G) \) for the Hecke algebra of locally constant, complex valued functions with compact support. We write \( \mathcal{H}(\overline{Q}) \) for the same algebra, but now consisting of \( \overline{\mathbb{Q}}_F \)-valued functions.
- We normalize parabolic induction by the half power of the modulus character as in [BZ77, 1.8], so as to preserve unitarity.

**The (general) symplectic group.** Write \( A_n \) for the \( n \times n \)-matrix with zeros everywhere, except on its antidiagonal, where we put ones. Write \( J_n := \begin{pmatrix} -A_n & \mathbb{I}_n \end{pmatrix} \in \text{GL}_{2n}(\mathbb{Z}) \). We define \( \text{GSp}_{2n} \) as the algebraic group over \( \mathbb{Z} \), such that for all rings \( R \),

\[
\text{GSp}_{2n}(R) = \{ g \in \text{GL}_{2n}(R) \mid \frac{1}{2} g \cdot J_n \cdot g = x \cdot J_n \text{ for some } x \in R^\times \}.
\]

The factor of similitude \( x \in R^\times \) induces a morphism \( \sim : \text{GSp}_{2n} \to \mathbb{G}_m \). Write \( T_{\text{GSp}} \subset \text{GSp}_{2n} \) for the diagonal maximal torus. Then \( X^*(T_{\text{GSp}}) = \bigoplus_{i=0}^{n} \mathbb{Z} e_i \) where

- \( e_i : \text{diag}(a_1, \ldots, a_n, ca_1^{-1}, \ldots, ca_i^{-1}) \mapsto a_i \) (\( i > 0 \)),
- \( e_0 : \text{diag}(a_1, \ldots, a_n, ca_1^{-1}, \ldots, ca_1^{-1}) \mapsto c \).

We let \( B_{\text{GSp}} \subset \text{GSp}_{2n} \) be the upper triangular Borel subgroup. We have the following corresponding simple roots and coroots,

- \( \alpha_1 = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = 2e_n - e_0 \in X^*(T_{\text{GSp}}) \),
- \( \alpha_\vee = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_\vee_n = e_n - e_0 \in X_*(T_{\text{GSp}}) \).

We define \( \text{Sp}_{2n} = \ker(\sim) \). Write \( B_{\text{Sp}} = B_{\text{GSp}} \cap \text{Sp}_{2n} \) and \( T_{\text{Sp}} = T_{\text{GSp}} \cap \text{Sp}_{2n} \).

**The (general) orthogonal group.** Let \( \text{GO}_{2n+1} \) be the algebraic group over \( \mathbb{Q} \) such that for all \( \mathbb{Q} \)-algebras \( R \),

\[
\text{GO}_{2n+1}(R) = \{ g \in \text{GL}_{2n+1}(R) \mid \frac{1}{2} g \cdot A_{2n+1} \cdot g = x \cdot A_{2n+1} \text{ for some } x \in R^\times \}.
\]

We have the factor of similitude \( \sim : \text{GO}_{2n+1} \to \mathbb{G}_m \) and put \( O_{2n+1} = \ker(\sim) \) and \( SO_{2n+1} = O_{2n+1} \cap \text{SL}_{2n+1} \). Let \( T_{\text{GO}} \subset \text{GO}_{2n+1} \), \( T_{\text{SO}} \subset \text{SO}_{2n+1} \) be the diagonal tori. We write \( \text{std} : \text{GO}_{2n+1} \to \text{GL}_{2n+1} \) for the standard representation. The root datum of \( SO_{2n+1} \) is dual to \( \text{Sp}_{2n} \). In particular, we identify \( X_*(T_{\text{GO}}) = X^*(T_{\text{Sp}}) \).
The (general) spin group. Consider the symmetric form

\[ (x, y) = x_1 y_{2n+1} + x_2 y_{2n} + \cdots + x_{2n+1} y_1 = \bar{t} x \cdot A_{2n+1} \cdot y \]
on \( \mathbb{Q}^{2n+1} \). The associated quadratic form is \( Q(x) = x_1 x_{2n+1} + x_2 x_{2n} + \cdots + x_{2n+1} x_1 \). Let \( C \) be the Clifford algebra associated to \( (\mathbb{Q}^{2n+1}, Q) \). It is equipped with an embedding \( \mathbb{Q}^{2n+1} \subset C \) which is universal for maps \( f : \mathbb{Q}^{2n+1} \rightarrow A \) into associative rings \( A \) satisfying \( f(x)^2 = Q(x) \) for all \( x \in \mathbb{Q}^{2n+1} \). Let \( e_1, \ldots, e_{2n+1} \) be the standard basis of \( \mathbb{Q}^{2n+1} \). The products \( E_I = \prod_{i \in I} e_i \) for \( I \subset \{1, 2, \ldots, n\} \) form a basis of \( C \). The algebra \( C \) is \( \mathbb{Z}/2\mathbb{Z} \)-graded, \( C = C^+ \oplus C^- \).

On the Clifford algebra \( C \) we have an unique anti-involution \( * \) that is determined by \( (v_1 \cdots v_r)^* = (-1)^r v_r \cdots v_1 \) for all \( v_1, \ldots, v_r \in V \). We define for all \( \mathbb{Q} \)-algebras \( R \),

\[ \text{GSpin}_{2n+1}(R) = \{ x \in (C^+ \otimes R)^n \mid g \cdot R^n \cdot \bar{g}^* = R^n \} \].

The spinor norm on \( C \) induces a character \( N : \text{GSpin}_{2n+1} \rightarrow \mathbb{C}^* \). The action of the group \( \text{GSpin}_{2n+1} \) stabilizes \( \mathbb{Q}^{2n+1} \subset C \) and we obtain a surjection \( q' : \text{GSpin}_{2n+1} \rightarrow \text{SO}_{2n+1} \). We write \( q \) for the surjection \( \text{GSpin}_{2n+1} \rightarrow \text{SO}_{2n+1} \) obtained from \( q' \).

We write \( T_{\text{GSpin}} \subset \text{GSpin}_{2n+1} \) for the torus \( (q')^{-1}(T_{\text{GSpin}}) \). We then have

\[ X'(T_{\text{GSpin}}) = \mathbb{Z} e_0^* \otimes \mathbb{Z} e_1^* \otimes \cdots \otimes \mathbb{Z} e_n^* = X(T_{\text{GSp}}) \],
\[ X(T_{\text{GSpin}}) = \mathbb{Z} e_0 \otimes \mathbb{Z} e_1 \otimes \cdots \otimes \mathbb{Z} e_n = X'(T_{\text{GSpin}}) \].

The group \( \text{GSpin}_{2n+1}(\mathbb{C}) \) is dual to \( \text{GSp}_{2n}(\mathbb{C}) \).

The spin representation. Let \( e_1, \ldots, e_{2n+1} \) be the standard basis of \( \mathbb{Q}^{2n+1} \). The products \( E_I = \prod_{i \in I} e_i \) for \( I \subset \{1, 2, \ldots, n\} \) form a basis of \( C^* \). The group \( \text{GSpin}_{2n+1} \) acts on \( C^* \), which induces an irreducible, \( 2^n \)-dimensional representation \( \text{spin} : \text{GSpin}_{2n+1} \rightarrow \text{GL}_{2^n} \), called the spin representation. The composition of spin with \( \text{GL}_{2^n} \rightarrow \text{PGL}_{2^n} \) induces a morphism \( \text{spin} : \text{SO}_{2n+1} \rightarrow \text{PGL}_{2^n} \).

Let \( k \) be an algebraically closed field of characteristic zero. Write \( W := \oplus_{i=1}^n k \cdot e_i \) and \( \wedge^* W \) for the exterior algebra of \( W \). We have a natural \( k \)-algebra morphism isomorphism from \( C^+_k := C^+ \otimes_k k \) onto \( \wedge^* W \) over \( \mathbb{C} \), cf. [FH91, (20.19)].

**Lemma 0.1.** When \( n \mod 4 = 0 \) or \( 3 \) (resp. \( 1 \) or \( 2 \)), there exists a symmetric (resp. symplectic) form on the \( 2^n \)-dimensional vector space underlying the spin representation such that the form is preserved under \( \text{GSpin}_{2n+1}(k) \) up to scalars. The resulting map \( \text{GSpin}_{2n+1} \rightarrow \text{GO}_{2^n} \) (resp. \( \text{GSpin}_{2n+1} \rightarrow \text{GSp}_{2n} \)) over \( k \) followed by the similitude character of \( \text{GO}_{2^n} \) (resp. \( \text{GSp}_{2n} \)) coincides with the spinor norm \( \beta \).

**Proof.** We may identify the \( 2^n \)-dimensional space with \( \wedge^* W \). Write \( * \) for the main involution on \( C^+_k \) as well as on \( \wedge^* W \). Given \( s, t \in \wedge^* W \), write \( \beta(s, t) \in C \) for the projection of \( s^* \wedge t \in \wedge^* W \) onto \( \wedge^* W = C \). It is elementary to check that \( \beta(s, \cdot) \) is symmetric if \( n \mod 4 = 0 \) or \( 3 \) and symplectic otherwise, cf. [FH91, Exercise 20.38]. Now let \( x \in \text{GSpin}_{2n+1}(k) \), also viewed as an element of \( C^*_k \). Then \( x^* x \in k^\times \) is the spinor norm of \( x \). Then \( \beta(xx, xt) = (xs)^* \wedge (xt) = (x^*x)s \wedge t = x^*x\beta(s, t) \), completing the proof.

\[
\square
\]

1. Conventions

Let \( G \) be a connected reductive group over \( \mathbb{Q} \). A \( \mathbb{G} \)-valued (\( \ell \)-adic) Galois representation is a continuous morphism \( \rho : \Gamma \rightarrow G(\overline{\mathbb{Q}}_\ell) \). If there is no danger of confusion, we write 'representation' instead of 'G-representable'. We call the representation \( G \)-irreducible if its image \( \rho(\Gamma) \) is not contained in any proper parabolic subgroup of \( G \). The representation \( \rho \) is semi-simple if for some, thus every (cf. [DMOS82, Prop. I.3.1]), faithful representation \( f \) of \( G \), the composition \( f \circ \rho \) is semisimple.

Let \( r_1, r_2 : \Gamma \rightarrow G(\overline{\mathbb{Q}}_\ell) \) be two semisimple continuous Galois representations, that are unramified almost everywhere. It is easily checked that the following statements are equivalent:

- There exists a dense subset \( \Sigma \subset \Gamma \) such that for all \( \sigma \in \Sigma \) we have \( r_1(\sigma)_{\text{ss}} \sim r_2(\sigma)_{\text{ss}} \in G(\overline{\mathbb{Q}}_\ell) \);
- for all \( \sigma \in \Gamma \) we have \( r_1(\sigma)_{\text{ss}} \sim r_2(\sigma)_{\text{ss}} \in G(\overline{\mathbb{Q}}_\ell) \);
- for (almost) all finite \( F \)-places \( v \) where \( r_1, r_2 \) are unramified, we have
  \[ r_1(\text{Frob}_v)_{\text{ss}} \sim r_2(\text{Frob}_v)_{\text{ss}} \in G(\overline{\mathbb{Q}}_\ell) ; \]
- for all linear representations \( \rho : G \rightarrow \text{GL}_N \), the representations \( \rho \circ r_1 \) and \( \rho \circ r_2 \) are isomorphic.

If one of the above conditions holds, we call \( r_1 \) and \( r_2 \) locally conjugate, and we write \( r_1 \approx r_2 \).

**Definition 1.1.** Let \( T \) be a maximal torus in a reductive group \( G \) over an algebraically closed field. A weight \( \nu \in X^*(T) \) is regular if \( (\alpha^\vee, \nu) \neq 0 \) for all coroots \( \alpha^\vee \) of \( T \) in \( G \).
**Definition 1.2.** Let $\phi : W_k \to \mathrm{GSpin}_{2n+1}(\mathbb{C})$ be a Langlands parameter. Denote by $T$ the diagonal maximal torus in $\mathrm{GL}_{2n}$ and by $\hat{T}$ its dual torus. We have $\mathbb{C}^\times \subset W_k$. The composition
\[
\mathbb{C}^\times \subset W_k \to \mathrm{GSpin}_{2n+1}(\mathbb{C}) \to \mathrm{GL}_{2n}(\mathbb{C})
\]
is conjugate to the character $z \mapsto \mu_1(z)\mu_2(z)$ given by some $\mu_1, \mu_2 \in X(\hat{T}) \otimes \mathbb{C} = X(T) \otimes \mathbb{C}$ such that $\mu_1 - \mu_2 \in X'(T)$. Then $\phi$ is spin-regular if $\mu_1$ is regular (equivalently if $\mu_2$ is regular; note that $\mu_1$ and $\mu_2$ are swapped if $\phi \circ \phi$ is conjugated by the image of the element $j \in W_k$ such that $j^2 = -1$ and $jw^{-1} = \varpi$ for $w \in W_\ell$).

**Definition 1.3.** An automorphic representation $\pi$ of $\mathrm{GSp}_{2n}(\mathbb{A}_f)$ is spin-regular at $v_\infty$ if the Langlands parameter of the component $\pi_{v_\infty}$ is spin-regular.

Let $H$ be a connected reductive group over $\overline{\mathbb{Q}}_\ell$ for the following two definitions (which could be extended to disconnected reductive groups). Let $\mathfrak{h}_{\text{der}}$ denote the Lie algebra of its derived subgroup. Write $c$ for the nontrivial element of $\text{Gal}(\mathbb{C}/\mathbb{R})$.

**Definition 1.4** (cf. [Gro]). A continuous representation $\rho : \text{Gal}(\mathbb{C}/\mathbb{R}) \to H(\overline{\mathbb{Q}}_\ell)$ is odd if the trace of $c$ on $\mathfrak{h}_{\text{der}}$ through the adjoint action of $\rho(c)$ is equal to $-\text{rank}(\mathfrak{h}_{\text{der}})$.

**Lemma 1.5.** Let $\rho^b : \text{Gal}(\mathbb{C}/\mathbb{R}) \to \text{GO}_{2n+1}(\overline{\mathbb{Q}}_\ell)$ be a continuous representation. Write $\rho^b : \Gamma \to \text{GL}_{2n+1}(\overline{\mathbb{Q}}_\ell)$ for the composition of $\rho^b$ with the standard embedding. If $\text{Tr} \rho^b(c) \in \{\pm 1\}$ then $\rho^b$ is odd.

**Proof.** We may choose a model for the Lie algebra of $\text{SO}_{2n+1}$ to consist of $A \in \text{GL}_{2n+1}$ such that $A + f^t A^{-1} = 0$, where $f$ is the matrix with $1$'s on the anti-diagonal and $0$'s everywhere else. Such an $A = (a_{ij})$ is characterized by the condition $a_{1,i} + a_{n+1-i,n+1-i} = 0$ for every $1 \leq i \leq n$. Let $t := \rho^b(c)$. By conjugation and multiplying $-1 \in \text{GL}_{2n+1}$ if necessary, we can assume that $t$ is in the diagonal maximal torus in $\text{SO}_{2n+1}$ (not only in $\text{GO}_{2n+1}$, using the fact that the latter is the product of $\text{SO}_{2n+1}$ with center) of the form $\text{diag}(1,-1,-1,1,\ldots,1)$, where $0 \leq a,b \leq n$ and $a + b = n$. Since $\rho^b(c) \in \{\pm 1\}$ we have $a = b$ if $n$ is even and $b - a = 1$ if $n$ is odd. Now an explicit computation shows that the trace of the adjoint action of $t$ on $\text{Lie}(\text{SO}_{2n+1})$ has trace $2(a-b)^2 + 2(a-b) - n$, which is equal to $-n - \text{rank}(\text{SO}_{2n+1})$ in all cases.

Let $K$ be a finite extension of $\mathbb{Q}_\ell$. Fix its algebraic closure $\overline{K}$ and write $\overline{\mathbb{Q}}_\ell$ for its completion.

**Definition 1.6** (cf. [BGIl, 2.4]). Let $\rho : \text{Gal}(\overline{K}/K) \to H(\overline{\mathbb{Q}}_\ell)$ be a continuous representation. We say that $\rho$ is crystalline/semistable/de Rham/Hodge-Tate if for some faithful (thus every) algebraic representation $\xi : H \to \text{GL}_N$ over $\overline{\mathbb{Q}}_\ell$, the composition $\xi \circ \rho$ is crystalline/semistable/de Rham/Hodge-Tate. Now suppose that $\rho$ is Hodge-Tate. For each field embedding $i : \overline{\mathbb{Q}}_\ell \to \overline{K}$, a cocharacter $\mu_{HT}(\rho,i) : \mathbb{G}_m \to H$ over $\overline{\mathbb{Q}}_\ell$ is called a Hodge-Tate cocharacter for $\rho$ and $i$ if for some (thus every) algebraic representation $\xi : H \to \text{GL}(V)$ on a finite dimensional $\overline{\mathbb{Q}}_\ell$-vector space $V$, the cocharacter $\xi \circ \rho$ induces the Hodge-Tate decomposition
\[
V \otimes_{\mathbb{Q}_\ell} \overline{K} \simeq \bigoplus_{k \in \mathbb{Z}} V_k
\]
into $\overline{K}$-vector spaces: namely $V_k$ is the weight $k$ space for the $\mathbb{G}_m$-action through $\xi \circ \mu_{HT}(\rho,i)$ which fixes each $V_j$ by the $(-k)$-th power of the cyclotomic character. (So our convention is that the Hodge-Tate number of the cyclotomic character is $-1$.)

In any of the above conditions on $\rho$, if it holds for one $\xi$ then it holds for all $\xi$ [use [DMOS82, Prop. I.3.1]]. Whenever $\rho$ is Hodge-Tate, a Hodge-Tate cocharacter exists by [Ser79, §1.4] and is shown to be unique (independent of $\xi$) by a standard Tannakian argument. Often we only care about the isomorphism class of $\rho$, in which case only the $H(\overline{\mathbb{Q}}_\ell)$-conjugacy class of a Hodge-Tate cocharacter matters.

**Lemma 1.7.** Let $f : H_1 \to H_2$ be a morphism of connected reductive groups over $\overline{\mathbb{Q}}_\ell$. If $\rho : \text{Gal}(\overline{K}/K) \to H_2(\overline{\mathbb{Q}}_\ell)$ is a Hodge-Tate representation then $f \circ \rho$ is also Hodge-Tate with $\mu_{HT}(f \circ \rho,i) = f \circ \mu_{HT}(\rho,i)$ for all $i : \overline{\mathbb{Q}}_\ell \to \overline{K}$.

**Proof.** This is obvious by considering $\xi \circ f \circ \rho$ for any faithful algebraic representation $\xi : H_2 \to \text{GL}_N$.

We go back to the global setting. A continuous representation $\rho : \Gamma \to H(\overline{\mathbb{Q}}_\ell)$ is said to be totally odd if $\rho|_{\text{Gal}(\overline{F}_v/F_v)}$ is odd for every $v \in V_\infty$. It is crystalline/semistable/de Rham/Hodge-Tate if $\rho|_{\text{Gal}(\overline{F}_v/F_v)}$ is crystalline/semistable/de Rham/Hodge-Tate for every place $v$ above $\ell$. 
Definition 1.8. Let $H$ be a real reductive group. Let $K_H$ be a maximal compact subgroup of $H(\mathbb{R})$. Put $\bar{K}_H := (K_H)^0Z(H)(\mathbb{R})$. Let $\xi$ be an irreducible algebraic representation of $H \otimes_{\mathbb{R}} \mathbb{C}$. An irreducible unitary representation $\pi$ of $H(\mathbb{R})$ is said to be cohomological for $\xi$ (or $\xi$-cohomological) if there exists $i \geq 0$ such that $H^i(\text{Lie } H(\mathbb{C}), \bar{K}_H, \pi \otimes_{\mathbb{C}} \xi) \neq 0$. (The definition is independent of the choice of $K_H$. The group $\bar{K}_H$ is consistent with $K_n$ in §7 below.)

Example 1. Let $\Pi_\xi$ be the set of (irreducible) discrete series representations which have the same infinitesimal and central characters as $\xi^\vee$. Then $\Pi_\xi$ is a discrete series $L$-packet, whose $L$-parameter is going to be denoted by $\phi_\xi : W_\mathbb{R} \rightarrow \mathcal{T}_H$. Every member of $\Pi_\xi$ is $\xi$-cohomological. More precisely $\mathcal{H}(\text{Lie } H(\mathbb{C}), \bar{K}_H, \pi \otimes \xi) \neq 0$ exactly when $i = \frac{1}{2} \dim_{\mathbb{R}} H(\mathbb{R})/K'_H$, in which case the cohomology is of dimension $[K_H Z(H)(\mathbb{R}) : \bar{K}_H]$, cf. Remark 7.2 below.

Definition 1.9. Consider a complex $L$-parameter $\phi : W_\mathbb{C} \rightarrow \mathcal{T}_H$. For a suitable maximal torus $\tilde{T} \subset \mathcal{T}_H$, one can describe $\phi$ as $z \mapsto z^\mu \mathcal{P}^\xi$ for $\mu_1, \mu_2 \in X_*(\tilde{T})_\mathbb{C}$ with $\mu_1 - \mu_2 \in X_*(\tilde{T})$. Write $\Omega_{\mathcal{T}_H}$ for the Weyl group of $\tilde{T}$ in $\mathcal{T}_H$.

We define $\mu_{\text{Hodge}}(\phi)$ to be $\mu_1$ viewed as an element of $X_*(\tilde{T})_\mathbb{C}/\Omega_{\mathcal{T}_H}$. When $\mu_1$ happens to be integral, i.e. in $X_*(\tilde{T})$, then we may also view $\mu_{\text{Hodge}}(\phi)$ as a conjugacy class of cocharacters $\mathcal{G}_m \rightarrow \mathcal{T}_H$ over $\mathbb{C}$.

Let $f : H_1 \rightarrow H_2$ be a morphism of connected reductive groups over $\mathbb{R}$ whose image is normal in $H_2$ such that $f$ has abelian kernel and cokernel. (Later we will consider the surjection from $\text{GSpin}_{2n+1}$ to $\text{GO}_{2n+1}$.) Denote by $\tilde{f} : \tilde{H}_2 \rightarrow \tilde{H}_1$ the dual morphism. We choose maximal tori $\tilde{T}_i \subset \tilde{H}_i$ for $i = 1, 2$ such that $f(\tilde{T}_2) \subset \tilde{T}_1$. If $\phi_2 : W_\mathbb{R} \rightarrow \tilde{H}_2$ is an $L$-parameter then obviously

$$ I. \quad \tilde{f}(\mu_{\text{Hodge}}(\phi_2|_{W_\mathbb{C}})) = \mu_{\text{Hodge}}(\tilde{f} \circ \phi_2|_{W_\mathbb{C}}). $$

Lemma 1.10. With the above notation, let $\tau_2$ be a member of the $L$-packet for $\phi_2$. Then the pullback of $\tau_2$ via $f$ decomposes as a finite direct sum of irreducible representations of $H_1(\mathbb{R})$, and all of them lie in the $L$-packet for $\tilde{f} \circ \phi_2$.

Proof. This is property (iv) of the Langlands correspondence for real groups on page 125 of [Lan89].

2. Arthur parameters for symplectic groups

Assume $\pi^b$ is a cuspidal automorphic representation of $\text{Sp}_{2n}(A_F)$, such that

- $\pi^b$ is cohomological for an irreducible algebraic representation $\xi^b \cong \otimes_{v \in V_\mathbb{R}} \xi^b_v$ of $\text{Sp}_{2n,F} \otimes \mathbb{C}$,
- there is an auxiliary finite place $v_0$ such that the local representation $\pi_{v_0}^b$ is the Steinberg representation of $\text{Sp}_{2n}(F_{v_0})$.

In this section we apply the construction of Galois representations [Art13, Shill] to $\pi^b$ to obtain a morphism $\rho_{\pi^b} : \Gamma \rightarrow \text{GO}_{2n+1}(\mathbb{Q}_\ell)$ and then lift $\rho_{\pi^b}$ to a representation $\tilde{\rho}_{\pi^b} : \Gamma \rightarrow \text{GSpin}_{2n+1}(\mathbb{Q}_\ell)$. Let us briefly recall the notion of (formal) Arthur parameters as introduced in [Art13]. We will concentrate on the discrete and generic case as this is all we need (after Corollary 2.2 below); refer to loc. cit. for the general case. Here genericity means that no nontrivial representation of $\text{SU}_2(\mathbb{R})$ appears in the global parameter.

For any $N \in \mathbb{Z}_{\geq 2}$, let $\theta$ be the involution on [all the] general linear groups $\text{GL}_{N,F}$, defined by $\theta(x) = f_N x^t f_N$, where $f_N$ is the $N \times N$-matrix with 1’s on its anti-diagonal, and 0’s on all its other entries. A generic discrete Arthur parameter for the group $\text{Sp}_{2n,F}$ is a finite collection of unordered pairs $\{(m_i, \tau_i)\}_{i=1}^r$, where

- $m_i, r \geq 1$ are positive integers, such that $2n + 1 = \sum_{i=1}^r m_i$,
- for each $i$, $\tau_i$ is a unitary cuspidal automorphic representation of $\text{GL}_{m_i}(A_F)$ such that $\tau_i^\theta \cong \tau_i$,
- the $\tau_i$ are mutually non-isomorphic.

We write formally $\psi = \oplus_{i=1}^r \tau_i$ for the Arthur parameter $\{(m_i, \tau_i)\}_{i=1}^r$. The parameter $\psi$ is said to be simple if $r = 1$. The representation $\Pi_\psi$ is defined to be the isobaric sum $\oplus_{i=1}^r \tau_i$; it is a self-dual automorphic representation of $\text{GL}_{2n+1}(A_F)$.

Exploiting the fact that $\text{Sp}_{2n}$ is a twisted endoscopic group for $\text{GL}_{2n+1}$, Arthur attaches [Art13, Thm. 2.2.1] to $\pi^b$ a discrete Arthur parameter $\psi$. Let $\pi_\psi^G$ denote the corresponding isobaric automorphic representation of $\text{GL}_{2n+1}(A_F)$ as in [Art13, §1.3]. (If $\psi$ is generic, which will be verified soon, then $\psi$ has the form as in the preceding paragraph.) For each $F$-place $v$, the representation $\pi_\psi^G_v$ belongs to the local Arthur packet $\Pi(\psi_v)$ defined by $\psi$ localized at $v$. This packet $\Pi(\psi_v)$ satisfies the character relation ([Art13, Thm. 2.2.1])

$$ (2.1) \quad \text{Tr}(A_\theta \circ \pi_\psi^G(f_v)) = \sum_{\tau \in \Pi(\psi_v)} \text{Tr}(f_v^{\text{Sp}_{2n}}), $$

where $A_\theta = A_{\text{GO}_{2n+1}}$.
for all pairs of functions \( f_\nu \in \mathcal{H}(GL_{2n+1}(F_\nu)) \), \( f_\nu^{Sp_{2n,F}} \in \mathcal{H}(Sp_{2n}(F_\nu)) \) such that \( f_\nu^{Sp_{2n,F}} \) is a Langlands-Shelstad-Kottwitz transfer of \( f_\nu \). Here \( A_\theta \) is an intertwining operator from \( \pi^\sharp_\nu \) to its \( \theta \)-twist such that \( A^\sharp_\theta \) is the identity map. (The precise normalization is not recalled as it does not matter to us.)

**Lemma 2.1.** The component \( \pi^\sharp_{\nu_{St}} \) is the Steinberg representation of \( GL_{2n+1}(F_{\nu_{St}}) \).

**Proof.** Let \( f_{\nu_{St}}^{Sp_{2n}} \) be the Lefschetz function on the group \( Sp_{2n}(F_{\nu_{St}}) \) (Proposition A.1), and let \( f_{\nu_{St}}^{GL_{2n+1}'} \) be the twisted Lefschetz function on \( GL_{2n+1}(F_{\nu_{St}})_\theta \) (Proposition A.2). The functions \( (f_{\nu_{St}}^{Sp_{2n}}, C)_{f_{\nu_{St}}^{GL_{2n+1}'}}) \) are associated by Lemma A.3 for some constant \( C > 0 \). The endoscopic character relation of Arthur [Art13, Thm. 2.2.1] gives

\[
\text{Tr}(A_\theta \circ \pi^\sharp_{\nu_{St}}(f_{\nu_{St}}^{GL_{2n+1}'})) = C \cdot \sum_{\tau \in \Pi(\pi^\sharp_{\nu_{St}})} \text{Tr}(f_{\nu_{St}}^{Sp_{2n}}),
\]

where the right hand side defines the local packet \( \Pi(\pi^\sharp_{\nu_{St}}) \) attached to the parameter \( \pi^\sharp_{\nu_{St}} \). By Theorem 1.5.1 of [Art13], the representations in the packet \( \Pi(\pi^\sharp_{\nu_{St}}) \) are all tempered and in particular unitary. Thus, among the representations in \( \Pi(\pi^\sharp_{\nu_{St}}) \), the function \( f_{\nu_{St}}^{Sp_{2n}} \), can have non-zero trace only on the Steinberg representation (Proposition A.1(i)). The right hand side of Equation (2.2) is non-zero because the Steinberg representation \( \pi^\sharp_{\nu_{St}} \) appears in the packet \( \Pi(\pi^\sharp_{\nu_{St}}) \) (Proposition A.2(i)). Thus the left hand side of Equation (2.2) is non-zero. The representation \( \pi^\sharp \) is unitary, cuspidal automorphic and thus the component \( \pi^\sharp_{\nu_{St}} \) is unitary and infinite dimensional. Since \( f_{\nu_{St}}^{GL_{2n+1}'} \) has, among such representations, non-zero \( \theta \)-twisted trace only on the Steinberg representation (Proposition A.2), we conclude that \( \pi^\sharp_{\nu_{St}} \) must be the Steinberg representation. \( \square \)

**Corollary 2.2.** The Arthur parameter \( \psi \) of \( \pi^\sharp \) is simple (i.e. \( \psi = \tau_1 \) is cuspidal) and generic.

**Proof.** Lemma 2.1 implies in particular that \( \psi_{\nu_{St}} \) is a generic parameter which is irreducible as a representation of the local Langlands group \( W_{\nu_{St}} \times SU_2(\mathbb{R}) \). Hence the global parameter \( \psi \) is simple and generic. \( \square \)

Denote by \( \pi^\sharp \) the cuspidal automorphic representation \( \tau_1 = \Pi_\psi \). Let \( A_{\theta_0} : \Pi_\psi \rightarrow \Pi_\psi^{\theta_0} \) the canonical intertwining operator such that \( A^\sharp_{\theta_0} \) is the identity and \( A_{\theta_0} \) preserves the Whittaker model. Write \( \eta \) for the \( L \)-morphism \( \iota_{Sp_{2n},F_\psi} \rightarrow \iota_{GL_{2n+1},F_\psi} \) extending the standard representation \( \iota_{Sp_{2n}}: SO_{2n}(\mathbb{C}) \rightarrow GL_{2n+1}(\mathbb{C}) \) such that \( \eta|_{W_{\nu_{St}}} \) is the identity map onto \( W_{\nu_{St}} \).

**Lemma 2.3.** Let \( \nu \) be a finite \( F \)-place where \( \pi^\sharp_{\nu} \) is unramified. Then \( \pi^\sharp_{\nu} \) is unramified as well. Let \( \phi_{\pi^\sharp_{\nu}} : W_{\nu} \times SU_2(\mathbb{R}) \rightarrow GL_{2n+1}(\mathbb{C}) \) be the Langlands parameter of \( \pi^\sharp_{\nu} \). Let \( \phi_{\pi^\sharp_{\nu}} : W_{\nu} \times SU_2(\mathbb{R}) \rightarrow SO_{2n+1}(\mathbb{C}) \) be the Langlands parameter of \( \pi^\sharp_{\nu} \). Then \( \eta \circ \phi_{\pi^\sharp_{\nu}} = \phi_{\pi^\sharp_{\nu}}^{\eta} \).

**Proof.** The morphism \( \eta^*: \iota_{Sp_{2n+1}}^\text{unr}(GL_{2n+1}(F_\psi)) \rightarrow \iota_{Sp_{2n}}^\text{unr}(Sp_{2n}(F_\psi)) \) is surjective because the restriction of finite dimensional characters of \( GL_{2n+1} \) to \( SO_{2n+1} \) generate the space spanned by finite dimensional characters of \( SO_{2n+1} \). The lemma now follows from Equation (2.1). \( \square \)

The existence of the Galois representation \( \rho_{\pi^\sharp_\nu} \) attached to \( \pi^\sharp_\nu \) follows from [HT01, Thm. VII.1.9], which builds on earlier work by Clozel and Kottwitz. (The local hypothesis in that theorem is satisfied by Lemma 2.1. However this lemma is unnecessary for the existence of \( \rho_{\pi^\sharp_\nu} \) if we appeal to the main result of [Shill]). The theorem of [HT01] is stated over imaginary CM fields but can be easily adapted to the case over totally real fields, cf. [CHT08, Prop. 4.3.1] and its proof. (Also see [BLGGT14, Thm. 2.1.1]) for the general statement incorporating later developments such as the local-global compatibility at \( \nu|\nu_0 \) which we do not need.) We adopt the unitary normalization for \( L \)-algebraic representations as in [BGH, Conj. 3.2.3] unlike the references just mentioned, in which the arithmetic normalization for \( C \)-algebraic representations is used.

To state the Hodge-theoretic property at \( \ell \) precisely, we introduce some notation based on \( \mathcal{L} \). At each \( y \in \mathcal{V}_\infty \) we have a real \( L \)-parameter \( \phi_{\xi^y_\nu} : W_{\nu} \rightarrow \iota_{Sp_{2n}} \), arising from \( \xi^y_\nu \). The parameter is \( L \)-algebraic as well as \( C \)-algebraic. Via the embedding \( y : F \hookrightarrow C \) we may algebraic closure \( \overline{F}_y \) with \( C \) so that \( W_{\nu} \rightarrow \overline{W}_{\nu} = W_C \). The restriction \( \phi_{\xi^y_\nu}|_{W_{\nu}} : W_{\nu} \rightarrow SO_{2n+1}(\mathbb{C}) \) gives rise to \( \mu_{H^1}(\xi^y_\nu,y) := \mu_{H^1}(\phi_{\xi^y_\nu}|_{W_{\nu}}, y) \), a conjugacy class of cocharacters \( G_m \rightarrow SO_{2n+1}(\mathbb{C}) \).

**Theorem 2.4.** There exists an irreducible Galois representation

\[ \rho_{\pi^\sharp_\nu} = \rho_{\pi^\sharp_\nu}: \Gamma \rightarrow SO_{2n+1}(\mathbb{Q}_\ell), \]
unique up to $\text{SO}_{2n+1}(\mathbb{Q}_L)$-conjugation, attached to $\pi^b$ (and $i$) such that the following hold.

(i) Let $v$ be a finite place of $F$ not dividing $\ell$. If $\pi^b_v$ is unramified then
$$\left(\rho^b_v\big|_{W_F^v}\right)_{ss} = i\phi^b_v,$$
where $\phi^b_v$ is the unramified $L$-parameter of $\pi^b_v$, and $(\cdot)_{ss}$ is the semisimplification. For general $\pi^b_v$, the parameter $i\phi^b_v$ is isomorphic to the Frobenius-semisimplification of the Weil-Deligne representation associated to $\rho^b_v|_{F_F^v}$.6

(ii) Let $v$ be a finite $F$-place such that $v \nmid \ell$ where $\pi_v$ is unramified. Then $\rho^b_{\pi, v}$ is unramified at $v$, and for all eigenvalues $\alpha$ of $(\rho^b_{\pi, v}(Frob_v))_{ss}$ and all embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ we have $|\alpha| = 1$.

(iii) For every $v|\ell$, the representation $\rho^b_{\pi, v}$ is potentially semistable. For each $v : F \hookrightarrow \mathbb{C}$ such that $\imath v$ induces $v$, we have
$$\mu_{\text{HT}}(\rho^b_{\pi, v}, \imath v) = i\mu_{\text{Hodge}}(\xi^b_v, \imath v).$$

(iv) For every $v|\ell$, the Frobenius semisimplification of the Weil-Deligne representation attached to the de Rham representation $\rho^b_{\pi, v}$ is isomorphic to the Weil-Deligne representation attached to $\pi^b_v$ under the local Langlands correspondence.

(v) If $\pi_v$ is unramified at $v|\ell$, then $\rho^b_{\pi, v}$ is crystalline. If $\pi_v$ has a non-zero Iwahori fixed vector at $v|\ell$, then $\rho^b_{\pi, v}$ is semistable.

(vi) The representation $\rho^b_{\pi^b_v}$ is totally odd.

(vii) If $\pi^b_v$ is essentially square-integrable at $v \nmid \infty$ then $\text{std} \circ \rho^b_{\pi^b_v}$ is irreducible.

Remark 2.5. We need the ramified case of (ii) only when $\pi^b_v$ is the Steinberg representation.

Proof. By Theorem 1.2 of [Shill] (in view of Remark 7.6 in loc. cit.) there exists a Galois representation $\rho_{\pi^b_v} : \Gamma \to \text{GL}_{2n+1}(\mathbb{Q}_L)$ that satisfies properties (i)-(iii),(v) with $\pi^b$ in place of $\pi^b_v$; property (iv) is established by Carani [CarH, Thm. 1.1]. (The reader may also refer to the statement of [BLGGT14, Thm. 2.11].) Strictly speaking, the normalization there is different from here, so one has to twist the Galois representation there by the $\frac{1}{n}$-th power of the cyclotomic character. In particular $\pi^b_v$ is self-dual.

By Lemma 2.1, $\pi^b_v$ is the Steinberg representation and by Taylor–Yoshida [TY07] the representation $\rho_{\pi^b_v} : \Gamma \to \text{GL}_{2n+1}(\mathbb{Q}_L)$ is irreducible. Together with the self-duality of $\rho_{\pi^b_v}$, this implies that $\rho_{\pi^b_v}$ factors through a representation $\rho^b_{\pi^b_v} : \Gamma \to \text{SO}_{2n+1}(\mathbb{Q}_L)$ via the standard embedding $\text{SO}_{2n+1} \hookrightarrow \text{GL}_{2n+1}$ (after a conjugation by an element of $\text{GL}_{2n+1}(\mathbb{Q}_L)$). We know the uniqueness of $\rho^b_{\pi^b_v}$ from Proposition B.1 (and Chebotarev density). Properties (i)-(v) for $\rho^b_{\pi^b_v}$ follow from those for $\rho_{\pi^b_v}$. Part (vi) is deduced from Lemma 1.5 and the main theorem of [Tay12]. Lastly (vii) is [TY07, Cor. B].

For the rest of this section, let $\pi$ be a cuspidal $\xi^c$-cohomological automorphic representation of $\text{GSp}_{2n}(\mathbb{A}_F)$ for an irreducible algebraic representation $\xi$ of $\text{GSp}_{2n,F,\mathbb{Q}_C}$.

Lemma 2.6 (Labesse-Schwermer, Clozel). There exists a cuspidal automorphic $\text{Sp}_{2n}(\mathbb{A}_F)$-subrepresentation $\pi^b_v$ contained in $\pi$.

Proof. This follows from the arguments of Labesse-Schwermer [LS86]. See also Clozel [Clo91, p.137].

Lemma 2.7. Suppose that $\pi$ is a twist of the Steinberg representation at a finite place. Then $\pi_v$ is essentially tempered at all places $v$.

Proof. Let $\pi^b_v$ be as in the previous lemma. We know that $\pi^b_v$ is the Steinberg representation at a finite place and $\xi^b$-cohomological, where $\xi^b$ is the restriction of $\xi$ to $\text{Sp}_{2n,F,\mathbb{Q}_C}$ (which is still irreducible). Let $\pi^b$ be the self-dual cuspidal automorphic representation of $\text{GL}_{2n+1}(\mathbb{A}_F)$ as above. Note also that $\pi^b$ is $\mathbb{C}$-algebraic (and regular); this is checked using the explicit description of the archimedean $L$-parameters. Hence

- $\pi^b$ is tempered at all $v|\infty$ by Clozel's purity lemma [Clo90, Lem. 4.9],
- $\pi^b$ is tempered at all $v \nmid \infty$ by [Shill, Cor. 1.3] and (quadratic) automorphic base change.

(Since $\pi^b$ is self-dual, if a local component is tempered up to a character twist then it is already tempered.) Hence $\pi^b_v$ is a tempered representation of $\text{Sp}_{2n}(F_v)$, cf. [ArtL3, Thm. 1.5.1]. This implies that $\pi_v$ itself is essentially tempered. (Indeed, after twisting by a character, one can assume that $\pi_v$ restricts to a unitary tempered representation on $\text{Sp}_{2n}(F_v) \times Z(F_v)$, which is of finite index in $\text{GSp}_{2n}(F_v)$. Then temperedness is

\[\text{(This is equivalent to saying that } \rho_{\pi^b_v}\text{ is isomorphic to the Frobenius-semisimplification of the Weil-Deligne representation associated to } \text{std} \circ \rho_{\pi^b_v}\text{ in view of Proposition B.1 and the fact that self-dual representations into } \text{GL}_{2n+1}(\mathbb{Q}_L) \text{ factor through } \text{SO}_{2n+1}(\mathbb{Q}_L).\]
tested by whether the matrix coefficient (twisted by a character so as to be unitary on \(Z(F_v)\)) belongs to \(L^{2+}(\text{GSpin}_{2n+1}(F_v)/Z(F_v))\). This is straightforward to deduce from the same property of matrix coefficient for its restriction to \(\text{Sp}_{2n}(F_v) \times Z(F_v)\).

**Corollary 2.8.** \(\pi_{\text{reg}}\) belongs to the discrete series \(L\)-packet \(\Pi_{\xi}\).

**Proof.** It follows from the Vogan-Zakharov classification [VZ84] that \(\Pi_{\xi}\) coincides with the set of essentially tempered \(\xi\)-cohomological representations.

### 3. Strong irreducibility

Let \(\rho_{n^b}\) be the Galois representation from Section 2. We first give a numerical criterion on the Hodge-Tate numbers of \(\rho_{n^b}\) to check that the restricted representation \(\rho_{n^b}|_{E^\infty}\) has Zariski dense image in \(SO_{2n+1}(\mathbb{Q}_L)\) for all finite extensions \(E/F\). We then lift the representation \(\rho_{n^b}\) to a \(\text{GSpin}_{2n+1}(\mathbb{Q}_L)\) representation \(\tilde{\rho}_{n^b}\), and check that \(\text{spin} \circ \tilde{\rho}_{n^b}|_{E^\infty}\) is irreducible for all finite extensions \(E/F\).

Fix an infinite place \(v_\infty\) of \(F\), and let \(\lambda|E\) be the \(F\)-place above \(E\) induced by \(iv_\infty\). Let \(\mu_{HT}(\rho_{n^b,\lambda,iv_\infty}) = (x_1, x_2, \ldots, x_n) \in X_*(T_{SO}) = \mathbb{Z}^n\) be the Hodge Tate cocharacter of \(\rho_{n^b,\lambda}\) attached to \(\rho_{n^b,\lambda,iv_\infty}\). We make the following assumptions:

- \((HT_1)\) \(\mu_{HT}(\rho_{n^b,\lambda,iv_\infty})\) is not of the form \(2(nx, (2n-2)x, \ldots, 2x)\) in \(X_*(T_{SO})/W_{SO}\) for some \(x \in \mathbb{Z}\);
- \((HT_2)\) if \(n = 3\), we assume additionally that \(\mu_{HT}(\rho_{n^b,\lambda,iv_\infty})\) is not of the form \((a, b, a+b) \in X_*(T_{SO})/W_{SO}\) for some \(a, b \in \mathbb{Z}\).

One checks easily that spin-regularity (spin-reg) implies the conditions \((HT_1)\) and \((HT_2)\).

**Theorem 3.1** (Testerman–Zalesski [TZ13, Thm. 1.4]). Let \(H\) be a semisimple subgroup of \(SO_{2n+1}(\mathbb{Q}_L)\) containing a regular unipotent element of \(H\). Then either \(H\) is the full group \(SO_{2n+1}(\mathbb{Q}_L)\), or \(n = 3\) and \(H\) is the simple exceptional group \(G_2\) over \(\mathbb{Q}_L\) (i.e. \(H\) is the automorphism group of the octonion algebra \(\mathbb{O} \otimes \mathbb{Q}_L\)).

**Lemma 3.2** (Liebeck–Testerman [LT04, Lem. 2.1]). Let \(G\) a semisimple connected algebraic group over an algebraically closed field. If \(F\) is a connected \(G\)-irreducible subgroup of \(G\), then \(F\) is semisimple, and the centralizer of \(X\) in \(G\) is finite.

**Lemma 3.3.** Assume \(n \geq 3\). The subset \(\rho_{n^b}(\Gamma) \subset SO_{2n+1}(\mathbb{Q}_L)\) is Zariski dense.

**Remark 3.4.** The lemma may fail when \(n = 2\) but if it fails, Theorem 3.1 tells us that \(\rho_{n^b}(\Gamma_E)\) is a Zariski dense subgroup of the principal \(SL_2(\mathbb{Q}_L)\). In case \(n = 3\), if we forgo condition \((HT_2)\) then it does happen that the Zariski closure of \(\rho_{n^b}(\Gamma_E)\) is \(G_2\), cf. §15 below.

**Proof.** Let \(E/F\) be a finite extension such that the group \(H = \rho_{n^b}(\Gamma_E)^{\text{Zar}}\) is Zariski connected. Let \(N_{v_\infty}\) be the unipotent operator of the Weil-Deligne representation attached to \(\rho_{n^b,v_\infty}\) so that it corresponds to the image of \((1, (1 1))\) under \(\phi_{n^b,v_\infty}\) by Theorem 2.4. Then \(N_{v_\infty}\) is regular unipotent since \(\pi_{v_\infty}\) is Steinberg. The attached Weil-Deligne representation has the property that a positive power of \(N_{v_\infty}\) lies in the image of \(\rho_{n^b,v_\infty}\). Therefore the image \(\rho_{n^b}(\Gamma_E)\) contains some (possibly higher) positive power of \(N_{v_\infty}\), which is again regular unipotent.

The representation \(\rho_{n^b}|_{E^\infty}\) is semisimple because it is the restriction to an open subgroup of an irreducible representation. Since \(\rho_{n^b}(\Gamma_E)\) contains the regular unipotent \(N_{v_\infty}\), the representation is indecomposable and semisimple, thus irreducible. It follows that \(H\) is an irreducible and connected subgroup of \(SL_{2n+1}(\mathbb{Q}_L)\). By Lemma 3.2, \(H\) is a semisimple group, and by Theorem 3.1, \(H\) is of the groups \(SO_{2n+1}, G_2\) or \(SL_2\) over \(\mathbb{Q}_L\).

Assume for a contradiction that \(\rho_{n^b}(\Gamma) \subset SL_2(\mathbb{Q}_L) \subset SO_{2n+1}(\mathbb{Q}_L)\). Let \(T\) be the diagonal torus of \(SL_2\). Then, after conjugating,

\[
\rho_{HT}(\rho_{n^b,\lambda,iv_\infty}) \in \text{Sym}^{2n} X_*(T) = (2n, 2n-2, \ldots, 2) \subset X_*(T_{SO}) = \mathbb{Z}^n,
\]

which contradicts \((HT_1)\). Similarly, in case \(n = 3\), assume that \(\rho_{n^b}(\Gamma_E) \subset G_2(\mathbb{Q}_L)\). Let \(T\) be the diagonal torus of \(G_2\). Then, after conjugating,

\[
\rho_{HT}(\rho_{n^b,\lambda,iv_\infty}) \in X_*(T) = \{(a, b, a+b) \mid a, b \in \mathbb{Z}\} \subset \mathbb{Z}^3 \subset X_*(T_{SO}),
\]

which contradicts \((HT_2)\). It follows that \(H = SO_{2n+1}(\mathbb{Q}_L)\) is the only possibility.

**Proposition 3.5.** (Tate, cf. [Con13, Prop. 5.3]) There exists a continuous representation \(\tilde{\rho}_{n^b}: \Gamma \to \text{GSpin}_{2n+1}(\mathbb{Q}_L)\) lifting \(\rho_{n^b}\). Moreover, we may choose the lift \(\tilde{\rho}_{n^b}\) such that the character \(N\circ \tilde{\rho}_{n^b}\) has finite order and such that \(\tilde{\rho}_{n^b}\) is unramified almost everywhere.
Proof. Consider for each $r \geq 1$ the group $\text{GSpin}_{2n+1}(\mathbb{Q}_L)$ of elements $g \in \text{GSpin}_{2n+1}(\mathbb{Q}_L)$ such that $N(g) = 1$. For each $n$ we have the exact sequence $[\pi_2^*(\mathbb{Q}_L) \to \text{GSpin}_{2n+1}(\mathbb{Q}_L) \to \text{GO}_{2n+1}(\mathbb{Q}_L)]$. The obstruction to lifting $\rho$ to the group $\text{GSpin}_{2n+1}(\mathbb{Q}_L)$ is measured by a continuous cohomology class $[\rho]_r$ in $H^2(\Gamma, \mathbb{Q}_L)$ for the trivial action of $\Gamma$ on $\mathbb{Q}_L$. The collection $\{[\rho]_r\}_{r \in \mathbb{Z}_p}$ defines an element of the inductive limit $\lim_{\rightarrow}\text{H}^2(\Gamma, \mathbb{Q}_L) \cong \text{H}^2(\Gamma, \mathbb{Q}/\mathbb{Z})$. By Tate's theorem $H^2(\Gamma, \mathbb{Q}/\mathbb{Z})$ vanishes and thus $[\rho]_r = 0$ for $r$ sufficiently divisible. Thus a continuous lift $\tilde{\rho}_{\mathbb{Q}_L}$ exists with image in $\text{GSpin}_{2n+1}(\mathbb{Q}_L)$ for such an $r$ so that $N' \circ \tilde{\rho}_{\mathbb{Q}_L}$ has order $r$. By [Conl3, Lem. 5.2.1], $\tilde{\rho}_{\mathbb{Q}_L}$ is unramified almost everywhere. 

Definition 3.6. Let $\mathbb{G}/\mathbb{Q}_L$ be a connected reductive group, and $\rho$ a $G$-valued Galois representation. We call $\rho$ strongly $G$-irreducible if, for all open subgroups $\Gamma \subset \Gamma$, the restricted representation $\rho|_{\Gamma}$ is $G$-irreducible (cf. Section I). We often omit $G$ if $G$ is a general linear group.

Remark 3.7. The Tate module of a CM elliptic curve over $\mathbb{Q}$ is an example of an irreducible Galois representation that is not strongly irreducible. In fact, this example is typical: Let $\rho : \Gamma \to \mathbb{G}(\mathbb{Q}_L)$ be an $\ell$-adic representation. Let $H_\rho$ be the Zariski closure of its image. If $F'/F$ is a finite extension with $\Gamma' := \text{Gal}(F'/F)$, write $H_\rho'$ for the Zariski closure of the image of $\rho|_{\Gamma'}$. Then $H_\rho'$ is a union of connected components of $H_\rho$ (it is closed and of finite index, hence open). We deduce

(i) If $H_\rho'$ is connected, then $\rho$ is irreducible if and only if it is strongly irreducible.

(ii) Assume $\rho$ is irreducible. Let $M_\rho \subset \mathbb{G}$ be the extension of $F$ corresponding to $\ker[\Gamma \to \pi_0\text{Zar}(\rho(\Gamma))]$.

Then for all finite extensions $F' \subset \mathbb{Q}$ of $F$ that are linearly disjoint to $M_\rho$, the representation $\rho|_{\Gamma'}$ is irreducible.

Consider a lift $\tilde{\rho}_{\mathbb{Q}_L}^\ast$ of $\rho_{\mathbb{Q}_L}^\ast$ to $\text{GSpin}_{2n+1}(\mathbb{Q}_L)$ and let $\rho_1 : \Gamma \to \text{GL}_{2n}(\mathbb{Q}_L)$ be the composition of $\tilde{\rho}_{\mathbb{Q}_L}^\ast$ with the spin representation $\text{GSpin}_{2n+1}(\mathbb{Q}_L) \to \text{GL}_{2n}(\mathbb{Q}_L)$.

Proposition 3.8. The representation $\rho_1$ is strongly irreducible.

Proof. By Proposition 3.3 the Zariski closure of the image of $\rho_1$ is connected. Thus, by Remark 3.7(i) it suffices to show that $\rho_1$ is irreducible. When $n = 2$, this is clear since $\rho_1$ contains a regular unipotent element. Assume $n \geq 3$. The image of the map $\text{spin} \circ \text{SO}_{2n+1} \to \text{GL}_{2n}$ is not contained in any parabolic subgroup. By Lemma 3.3, the projective representation $\text{spin} \circ \rho_{\mathbb{Q}_L}^\ast$ is thus irreducible, and consequently $\rho_1$ is irreducible as well.

4. Projective representations

A classical theorem for Galois representations states that, if $\rho_1, \rho_2 : \Gamma \to \text{GL}_n(\mathbb{Q}_L)$ are two continuous semisimple representations which are locally conjugate, then they are conjugate. This is a consequence of the Brauer-Nesbitt theorem combined with the Chebotarev density theorem. In this section we give an analogous statement for projective representations $\rho_1, \rho_2 : \Gamma \to \text{PGL}_n(\mathbb{Q}_L)$ (see Proposition 4.4).

Lemma 4.1. Let $m, t \geq 1$ be integers. Let $\tau_1, \tau_2$ be two $m$-dimensional $\ell$-adic representations of $\Gamma$ such that:

- $\tau_1, \tau_2$ are unramified at almost all places;
- $\tau_1$ is strongly irreducible;
- for almost all places $v$ where $\tau_1$ and $\tau_2$ are unramified, the semisimple elements $\tau_1(\text{Frob}_v)_s$ and $\tau_2(\text{Frob}_v)_s$ are conjugate in the group $\text{GL}_m(\mathbb{Q}_L)/\mu_1(\mathbb{Q}_L)$.

Then there exists a continuous character $\chi : \Gamma \to \mathbb{Q}_L^\ast$ such that $\tau_1 \equiv \chi \otimes \tau_2$.

Proof. Consider the quotient $\mathbb{Q}_L^\ast/\sim$ where for $x, y \in \mathbb{Q}_L$ we identify $x \sim y$ if and only if $y = \zeta x$ for some $\zeta \in \mu_1(\mathbb{Q}_L)$. The quotient $\mathbb{Q}_L^\ast/\sim$ is Hausdorff, and thus the loci in $\Gamma$ where $\text{Tr}\tau_1(\sigma) \sim \text{Tr}\tau_2(\sigma)$ is closed. Thus, by Chebotarev's density theorem we have $\text{Tr}\tau_1(\sigma) \sim \text{Tr}\tau_2(\sigma)$ for all $\sigma \in \Gamma$. We claim that there is an open neighborhood $U \subset \Gamma$ of the identity $e \in \Gamma$ such that $\text{Tr}\tau_1(\sigma) = \text{Tr}\tau_2(\sigma)$ for all $\sigma \in U$. To see this, pick some open ideal $I \subset \mathbb{Z}_p$ such that $(1 - \zeta)\ell \not\equiv 0$ for all $\zeta \in \mu_1(\mathbb{Q}_L)$ with $\zeta \not\equiv 1$. Take $U \subset \Gamma$ the set of $\sigma$ such that for $i = 1, 2$ we have for all $\sigma \in U$ that $\text{Tr}\tau_i(\sigma) \equiv m \mod I$. Then $U$ is open by continuity of the representations $\tau_i$. The traces $\text{Tr}\tau_1(\sigma)$ and $\text{Tr}\tau_2(\sigma)$ agree up to an element $\zeta_\sigma \in \mu_1(\mathbb{Q}_L)$. Reducing modulo $I$ we get $m \equiv \zeta_\sigma m$. Since $m \not\equiv \zeta_\sigma m$ modulo $I$, the element $\zeta_\sigma$ must be 1 for all $\sigma \in U$. For the unit element $e \in \Gamma$ we have $\text{Tr}\tau_e(\sigma) = m (i = 1, 2)$, consequently $U$ is an open neighborhood of $e \in \Gamma$. Thus the claim is true.

Let $E/F$ be a finite Galois extension so that $I_1 \subset U$. By construction $\text{Tr}\tau_1(\sigma) = \text{Tr}\tau_2(\sigma)$ for all $\sigma \in I_1$ and therefore the space $H := \text{Hom}_F(\tau_1|_{I_1}, \tau_2|_{I_1})$ is non-zero. Consider the $\Gamma$-action on elements $f$ of the space $H$ defined by $\sigma f := (\mathbb{Q}_L^\ast \ni v \mapsto \tau_2(\sigma) f(\tau_1^{-1}(\sigma) v))$. Let us check that the function $\sigma f$ is really an element of $H$, ...
i.e. is $\Gamma_E$-equivariant. Let $\sigma' \in \Gamma_E$, write $s$ for $\sigma^{-1} \alpha \sigma' \in \Gamma$ and compute: $\sigma'(\iota f) = \sigma' \sigma \sigma' = \sigma \sigma f = \sigma(\iota f) = \sigma f$. The last equality holds because $s \in \Gamma_E$ ($\Gamma_E \subset \Gamma$ is normal) and $f$ is $\Gamma_E$-equivariant. Thus $H$ is a $\Gamma$-representation. The space $H$ is one-dimensional by Schur’s lemma for $\Gamma_E$ (by assumption $r_{\Gamma_E}$ is irreducible). Consequently there exists a character $\chi: \Gamma \to \overline{\mathbb{Q}}_{\ell}$ such that $\sigma f = \chi(\sigma)f$ for all $\sigma \in \Gamma$ and all $f \in H$. Pick any non-zero element $f \in H$. Then for all $v \in \overline{\mathbb{Q}}_{\ell}$ and all $\sigma \in \Gamma$ we have $r_{\sigma} (\iota f)(r_{\sigma}(v)) = \chi(\sigma^{-1}) f(v)$, and therefore $f(r_{\sigma}(v)) = \chi(\sigma^{-1}) r_{\sigma} f(v)$ which means that $f$ is an intertwining operator $r_1 \to r_2 \otimes \chi^{-1}$.

In fact the character $\chi$ in the preceding lemma is unique by the following lemma.

**Lemma 4.2.** Let $r: \Gamma \to \text{GL}_m(\overline{\mathbb{Q}}_{\ell})$ be a strongly irreducible $\ell$-adic representation unramified almost everywhere. If $r \simeq r \otimes \chi$ for a continuous character $\chi: \Gamma \to \overline{\mathbb{Q}}_{\ell}$ then $\chi = 1$.

**Proof.** The order of $\chi$ is finite and divides $m$ since $\det r = \det(r \otimes \chi)$. Moreover $\chi$ has to be unramified at every place where $r$ is unramified, so $\chi$ factors through a faithful character on a finite quotient $\Gamma/G'$ If $\chi \neq 1$ then $r|_{G'}$ is reducible, contradicting the assumption.

**Lemma 4.3** (Tate). Let $\tilde{\sigma}: \Gamma \to \text{PGL}_m(\overline{\mathbb{Q}}_{\ell})$ be a continuous morphism which is unramified for almost all the $F$-places. There exists a lift $\tilde{\tau}: \Gamma \to \text{GL}_m(\overline{\mathbb{Q}}_{\ell})$ of $r$ which is unramified for almost all places, and such that the determinant $\det(\tilde{\tau})$ is a character of finite order.

**Proof.** Let $s \geq 1$. Consider the group scheme $\text{GL}_m^{(s)}/\mathbb{Z}$ such that $\text{GL}_m^{(s)}(R)$ is equal to the group of $g \in \text{GL}_m(R)$ such that $\det(g)^s = 1$ for all commutative rings $R$. By arguing as in the proof of Proposition 3.5 the representation $\tilde{\tau}$ lifts to $\text{GL}_m^{(s)}(\overline{\mathbb{Q}}_{\ell})$ for $s$ sufficiently large.

**Proposition 4.4.** Let $m \geq 1$ be an integer. Let $\tilde{\tau}_1, \tilde{\tau}_2: \Gamma \to \text{GL}_m(\overline{\mathbb{Q}}_{\ell})$ be two continuous morphisms such that

- $\tilde{\tau}_1, \tilde{\tau}_2$ are unramified at almost all places;
- $\tilde{\tau}_1$ is strongly irreducible, that is, for all finite extensions $E/F$ contained in $\overline{\mathbb{Q}}_{\ell}$, the group $\tilde{\tau}_1(\Gamma_E)$ is not contained in any proper parabolic subgroup of $\text{GL}_m(\overline{\mathbb{Q}}_{\ell})$;
- for almost all finite unramified $F$-places $v$, $\tilde{\tau}_1(\text{Frob}_v)$ is conjugate to $\tilde{\tau}_2(\text{Frob}_v)$ in $\text{GL}_m(\overline{\mathbb{Q}}_{\ell})$.

Then $\tilde{\tau}_1$ is conjugate to $\tilde{\tau}_2$.

**Remark 4.5.** The second condition of Proposition 4.4 cannot be removed (see Lemma 4.6 below). Blasius constructs in the article [Blu94] examples of pairs of irreducible representations $\tilde{\tau}_1, \tilde{\tau}_2$ with finite image that satisfy the first and third bullet, but not the conclusion of the proposition.

**Proof.** By Lemma 4.3 there exist lifts $\tilde{\tau}_i$ of the representations $\tilde{\tau}_i$ such that the determinants $\det(\tilde{\tau}_i)$ are characters of finite order, both of order dividing $s \geq 1$. Let $v$ be a finite $F$-place unramified in both representations and such that $\tilde{\tau}_1(\text{Frob}_v)$ and $\tilde{\tau}_2(\text{Frob}_v)$ are $\text{GL}_m(\overline{\mathbb{Q}}_{\ell})$-conjugate. Let $\text{GL}_m^{(s)}(\overline{\mathbb{Q}}_{\ell}) = \text{GL}_m(\overline{\mathbb{Q}}_{\ell})/\mu_{md}(\overline{\mathbb{Q}}_{\ell})$. By Lemma 4.1 the representations $\tilde{\tau}_1$ and $\tilde{\tau}_2$ are conjugate up to twist, which implies that $\tilde{\tau}_1, \tilde{\tau}_2$ are $\text{PGL}_m(\overline{\mathbb{Q}}_{\ell})$-conjugate.

Let $\tilde{\sigma}: \Gamma \to \text{PGL}_m(\overline{\mathbb{Q}}_{\ell})$ be a continuous morphism. Pick a lift $\tilde{\tau}$ of $\tilde{\sigma}$ to $\text{GL}_m(\overline{\mathbb{Q}}_{\ell})$ (Lemma 4.3), and consider the semisimplification $\tilde{\tau}_{ss}$. The representation $\tilde{\tau}_{ss} := \tilde{\tau}_{ss}: \Gamma \to \text{GL}_m(\overline{\mathbb{Q}}_{\ell})$ does not depend on the choice of lift $\tilde{\tau}$. We call $\tilde{\tau}_{ss}$ the semisimplification of $\tilde{\tau}$.

**Lemma 4.6.** Let $\tilde{\tau}_1, \tilde{\tau}_2: \Gamma \to \text{GL}_m(\overline{\mathbb{Q}}_{\ell})$ be two continuous morphisms such that

- $\tilde{\tau}_1, \tilde{\tau}_2$ are unramified at almost all $F$-places;
- for almost all finite unramified $F$-places $v$, the semisimple part $\tilde{\tau}_1(\text{Frob}_v)$ is conjugate to $\tilde{\tau}_2(\text{Frob}_v)$ in $\text{GL}_m(\overline{\mathbb{Q}}_{\ell})$.

Write $\tilde{\tau}_{1,ss}, \tilde{\tau}_{2,ss}$ for the semisimplification of $\tilde{\tau}_1, \tilde{\tau}_2$. Then there exists a finite extension $E/F$ such that $\tilde{\tau}_{1,ss}|_{\Gamma_E}$ is $\text{PGL}_m(\overline{\mathbb{Q}}_{\ell})$-conjugate to $\tilde{\tau}_{2,ss}|_{\Gamma_E}$.

**Proof.** By Lemma 4.3 there exists lifts $\tilde{\tau}_1, \tilde{\tau}_2$ of $\tilde{\tau}_1, \tilde{\tau}_2$ whose determinants have finite order. For almost all places $v$ we have $\text{Tr} \tilde{\tau}_1(\text{Frob}_v) = \zeta_{\text{Frob}_v} \text{Tr} \tilde{\tau}_2(\text{Frob}_v)$ for some element $\zeta_{\text{Frob}_v} \in \mu_{md}(\overline{\mathbb{Q}}_{\ell})$. By the Chebotarev density theorem, we get for all $\sigma \in \Gamma$ that $\text{Tr} \tilde{\tau}_1(\sigma) = \zeta_\sigma \text{Tr} \tilde{\tau}_2(\sigma)$ for some $\zeta_\sigma \in \mu_{md}(\overline{\mathbb{Q}}_{\ell})$. As in the proof of Lemma 4.1 we can then find a finite extension $E/F$ such that $(\tilde{\tau}_1)_{ss}|_{\Gamma_E}$ and $(\tilde{\tau}_2)_{ss}|_{\Gamma_E}$ are conjugate. The lemma follows.
5. GSpin-valued Galois Representations

In this section we study the notion of local conjugacy for the group $\text{GSpin}_{2n+1}(\mathbb{Q}_l)$. In general it is not expected that local conjugacy implies (global) conjugacy of Galois representations: In the paper [Lar94a, proof of Prop. 3.10] Larsen constructs a certain finite group $\Delta$ (in his text it is called $\Gamma$), which is a double cover of the (non-simple) Mathieu group $M_{10}$ in the 10-th alternating group $A_{10}$. More precisely, he realizes $M_{10} \subset \text{SO}_9(\mathbb{Q}_l)$ by looking at the standard representation of $A_{10} \subset \text{GL}_{10}(\mathbb{Q}_l)$. Then $\Delta$ is the inverse image of $M_{10}$ in $\text{Spin}_9(\mathbb{Q}_l)$. Let us just assume that $\Delta$ can be realized as a Galois group $s: \Gamma \to \Delta$. The group $\Delta$ comes with a map $\phi_1: \Delta \to \text{Spin}_9(\mathbb{Q}_l)$, and $\eta$ is the composition $\Delta \to M_{10} \to M_{10}/A_6 \cong \mathbb{Z}/2\mathbb{Z} \to \text{Spin}_9(\mathbb{Q}_l)$. He defines $\phi_2(x):=\eta(x)\phi_1(x)$. We may define $r_1:=\phi_1 \circ s$ and $r_2:=\phi_2 \circ s$. Then the argument of Larsen shows that $\phi_1(\sigma)$ and $\phi_2(\sigma)$ are $\text{Spin}_9(\mathbb{Q}_l)$-conjugate for every $\sigma \in \Gamma$, while $\phi_1$ and $\phi_2$ are not $\text{Spin}_9(\mathbb{Q}_l)$-conjugate. The maps $\phi_i$ cannot be $\text{GSpin}_{9}(\mathbb{Q}_l)$-conjugate: If we would have $\phi_2 = g\phi_1 g^{-1}$ for some $g \in \text{GSpin}_{9}(\mathbb{Q}_l)$ then we can find a $z \in \mathbb{Q}_l^\times$, such that $h = gz \in \text{Spin}_9(\mathbb{Q}_l)$ and $h\phi_1 h^{-1} = g\phi_1 g^{-1} = \phi_2$ which contradicts Larsen’s conclusion. Thus, assuming the inverse Galois problem holds over $F$ for $\Delta$, $(r_1, r_2)$ is a pair of locally conjugate, but non-conjugate Galois representations.

In [Lar94a] Larsen explains that counterexamples may be constructed for all $\text{Spin}_m(\mathbb{Q}_l)$ with $m \geq 8$ (for $m = 8$, see [Lar96, Prop. 2.5]). For $m \leq 7$, he says the group $\text{Spin}_m(\mathbb{Q}_l)$ is ‘acceptable’ (for $m = 7$, see [Lar96, Prop. 2.4]), which means that local conjugacy implies conjugacy for representations $r_i$ with finite image.

We say that a Galois representation $r: \Gamma \to \text{GSpin}_{2n+1}(\mathbb{Q}_l)$ is in bad position if there exists a quadratic extension $E/F$ such that every $\sigma \in \Gamma \backslash \Gamma_E$ maps via

$$\Gamma \to \text{GSpin}_{2n+1}(\mathbb{Q}_l) \to \text{SO}_{2n+1}(\mathbb{Q}_l) \to \text{SL}_{2n+1}(\mathbb{Q}_l)$$

to an element which has at least one eigenvalue equals to $-1$.

**Proposition 5.1.** Let $r_1,r_2: \Gamma \to \text{GSpin}_{2n+1}(\mathbb{Q}_l)$ be two Galois representations with $r_1$ semi-simple. Then $r_1, r_2$ are locally conjugate if and only if the following statement holds

- There exists an element $g \in \text{GSpin}_{2n+1}(\mathbb{Q}_l)$ and a character $\eta: \Gamma \to \{\pm 1\}$ such that $r_2 = \eta \cdot gr_1 g^{-1}$, and, if $\eta$ is non-trivial, $r_1$ is in bad position with respect to the quadratic extension $E/F$ corresponding to $\eta$.

Moreover, if $s|_E$ is irreducible, then for all quadratic extensions $E/F$, $r_1$ is not conjugate to $r_1 \cdot \eta|_E$, where $\eta|_E$ is the quadratic character corresponding to $E/F$.

**Remark 5.2.** Larsen shows that $\phi_1$ and $\phi_2: \Delta \to \text{GSpin}_{2n+1}(\mathbb{Q}_l)$ are two locally conjugate, but non conjugate representations, essentially by establishing the above for $\phi_1, \phi_2$ [Lar94b, Lem. 3.9, Prop. 3.10]. Our proof and statement are thus a simple extraction from his arguments.

**Proof.** Recall from Section 1 that we have the isomorphism

$$\mathbb{G}_m^{n+1} \to \text{T}_{\text{GO}} \left( \mathbb{G}_m \right) \cong \text{diag}(a_1, a_2, \ldots, a_n) \to \mathbb{G}_m \times \mathbb{G}_m \times \cdots \times \mathbb{G}_m$$

We also have the isomorphism

$$\text{T}_{\text{Spin}} \cong \{(z,a_1, \ldots, a_n) \in \mathbb{G}_m^{n+1} \mid z^2 = a_1 \cdots a_n\}.$$ 

The Weyl group $W = S_n \rtimes \{\pm 1\}^n$ acts on $\text{T}_{\text{Spin}}$ as follows: The group $S_n$ acts by permutation of the indices of $a_i$, and an element $\epsilon \in \{\pm 1\}^n$ acts via

$$(z,a_1, \ldots, a_n) \mapsto \left( z, a_1^{\epsilon(1)}, a_2^{\epsilon(2)}, \ldots, a_n^{\epsilon(n)} \right).$$

We have $\text{GSpin}_{2n+1} = (\mathbb{G}_m \times \text{Spin}_{2n+1})/\langle (-1, \tau) \rangle$ with $\tau := (-1, \ldots, -1)$, and hence

$$\text{T}_{\text{GO}} \cong (\mathbb{G}_m \times \text{T}_{\text{Spin}})/\langle (-1, \tau) \rangle \cong \mathbb{G}_m^{n+1}, \quad (x,z,a_1, \ldots, a_n) \mapsto (xz, a_1, \ldots, a_n).$$

We identify $\text{T}_{\text{GSpin}} = \mathbb{G}_m^{n+1}$ through the above isomorphism. Then an element of the Weyl group $w = \sigma \epsilon \in W$ with $\sigma \in S_n$ and $\epsilon \in \{\pm 1\}^n$ acts on $\mathbb{G}_m^{n+1}$ via the following recipes:

\begin{equation}
\sigma(c,a_1, a_2, \ldots, a_n) = (c, \sigma(a_1), \sigma(a_2), \ldots, \sigma(a_n)),
\end{equation}

\begin{equation}
\epsilon(c,a_1, a_2, \ldots, a_n) = \left( \frac{c}{\prod_{i=1}^{n} a_i}, a_1^{\epsilon(1)}, a_2^{\epsilon(2)}, \ldots, a_n^{\epsilon(n)} \right).
\end{equation}

We are now ready to prove “$\Rightarrow$”. Assume $r_1$ and $r_2$ are locally conjugate. Then $qr_1$ and $qr_2$ are locally conjugate $\text{GO}_{2n+1}(\mathbb{Q}_l)$-valued representations. By Appendix B, they are conjugate, and we may assume in fact...
equal: \( qr_1 = qr_2 \). Then \( \eta(\sigma) := r_1(\sigma)r_2(\sigma)^{-1} \) defines a character of \( \Gamma \). If it is trivial, the representations are conjugate and we are done. Assume \( \eta \) is non-trivial, then \( \eta \equiv \eta_{E/F} \) for some quadratic extension \( E/F \). Let \( \sigma \in \Gamma \setminus \Gamma_E \), and conjugate \( r_1(\sigma) \) to an element \((c, a_1, \ldots, a_n)\) of the maximal torus. Then \((c, a_1, \ldots, a_n)\) is conjugate to \((-c, a_1, \ldots, a_n)\). Since the Weyl group elements \( w \) that lie in \( S_n \subset W \) do not change the coordinate \( c \), we can find an \( \varepsilon \in \{\pm 1\}^n \) such that \( \varepsilon \cdot (c, a_1, \ldots, a_n) = (-c, a_1, \ldots, a_n) \). By Equation (5.2) we have \( \prod_i \varepsilon(i) = 1 \), and hence there is an odd number of indices \( i \) with \( \varepsilon(i) = -1 \). Consequently, there is at least one index \( i \) such that \( a_i = -1 \). Hence the implication \( \Rightarrow \). We prove \( \Leftarrow \). Let \( \sigma \in \Gamma \setminus \Gamma_E \), and conjugate \( r(\sigma) q_{\lambda} \) to an element \((c, a_1, \ldots, a_n)\) of \( \mathsf{T}_{\mathsf{GSpin}} \). Write \( \text{std}(q(c, a_1, \ldots, a_n)) = (a_1, \ldots, a_n, 1, a_1^{-1}, \ldots, a_n^{-1}) \) for some \( i \in \{1, 2, \ldots, n\} \), and hence also \( a_i = -1 \) for some \( i \in \{1, 2, \ldots, n\} \). Take \( \varepsilon \in \{\pm 1\}^n \) such that \( \varepsilon(k) = -1 \) if \( k = i \) and \( \varepsilon(k) = 1 \) otherwise. Then \( \varepsilon \in W \) conjugates \((c, a_1, \ldots, a_n)\) to \((-c, a_1, \ldots, a_n)\). This proves \( \Leftarrow \).

For the second statement of the Proposition, assume \( r \) is conjugate to \( r \cdot \eta \) with \( \eta \) a quadratic character of \( \Gamma \) and that \( \text{std} r \cdot \eta \) is irreducible. Let \( g \in \mathsf{GSpin}_{2n+1}(\mathbb{Q}_E) \) be such that \( g(r \cdot \eta)k^{-1} = r \). By Schur’s lemma \( \text{std}(g) \) is a scalar matrix and \( g \in \mathsf{GSpin}_{2n+1}(\mathbb{Q}_F) \) is central. Thus \( r = r \cdot \eta \) and hence \( \eta = 1 \).

\[ \boxed{\text{Corollary 5.3.} \quad \text{Let } r \in \mathsf{GSpin}_{2n+1}(\mathbb{Q}_F) \text{ be a Galois representation. Let } X(r) \text{ be the set of conjugacy classes of Galois representations } r' : \Gamma \to \mathsf{GSpin}_{2n+1}(\mathbb{Q}_F) \text{ that are locally conjugate to } r. \text{ The set } X(r) \text{ is finite.} } \]

Proof. In small neighborhoods around identity in \( \mathsf{SL}_{2n+1}(\mathbb{Q}_E) \) no element has an eigenvalue equal to \(-1\). Let \( M/F \) be a finite Galois extension such that \( \text{std} \circ q_{\lambda} \circ r(\Gamma_M) \) lies in such an open neighborhood. Then for any \( r' = \eta r \in F \) we must have \( E \subset M \).

\[ \boxed{\text{Corollary 5.4.} \quad \text{Let } r_1, r_2 : \Gamma \to \mathsf{GSpin}_{2n+1}(\mathbb{Q}_F) \text{ be two locally conjugate Galois representations, such that the representation } (\text{std} \circ r_1) |_{\mathbb{Q}_E} \text{ is irreducible for all quadratic extensions } E/F. \text{ Then } r_1 \text{ is } \mathsf{GSpin}_{2n+1}(\mathbb{Q}_F) \text{-conjugate to } r_2.} \]

Proof. After conjugating \( r_2 \), we may assume it is of the form \( r_1 \eta \) with \( \eta \) character with \( \eta^2 = 1 \). By local conjugacy we have \( \text{std} \circ r_1 \eta = \eta \circ \text{std} \circ r_1 \cong \text{std} \circ r_1 \). Let \( g \in \mathsf{GL}_2^+(\mathbb{Q}_E) \) be a matrix that conjugates \( \eta \circ \text{std} \circ r_1 \) to \( \text{std} \circ r_1 \). Let \( E/F \) be the field corresponding to the kernel of \( \eta \). Then, \( g \) defines a \( \Gamma_E \)-automorphism of \( \text{std} \circ r_1 \). Since \( \text{std} \circ r_1 \) is irreducible over \( \Gamma_E \), \( g \) is central by Schur’s lemma. Hence \( \eta \circ \text{std} \circ r_1 = \text{std} \circ r_1 \) and \( \eta = 1 \).

6. The trace formula with fixed central character

In this section we recall the general setup for the trace formula with fixed central character\(^7\) and prove some instances of the Langlands functoriality for \( \mathsf{GSp}_{2n} \), we will need later.

Let \( G \) be a connected reductive group over a number field \( F \) with center \( Z \). Write \( A_Z \) for the maximal \( Q \)-split torus in \( \mathsf{Res}_F \mathbb{Q}^\times \) and set \( A_{Z,\infty} := A_Z(\mathbb{R}) \). Consider a closed subgroup \( \mathcal{X} \subset Z(\mathbb{A}_F) \) which contains \( A_{Z,\infty} \) such that \( Z(F) \mathcal{X} \) is closed in \( Z(\mathbb{A}_F) \) (then \( Z(F) \mathcal{X} \) is always cocompact in \( Z(\mathbb{A}_F) \)) and a continuous character \( \chi : \mathcal{X} \mathbb{Q} \to \mathbb{C}^\times \). Such a pair \((\mathcal{X}, \chi)\) is called a central character datum.

In what follows we need to choose Haar measures consistently for various groups, but we will suppress these choices as this is quite standard. For instance the same Haar measures on \( \mathbb{G}(\mathbb{A}_F) \) and \( \mathbb{X} \) have to be chosen for each term in the identity of Lemma 6.1 below.

Let \( F \) be a place of \( F \), and \( \mathbb{X}_F \) a closed subgroup of \( Z(F_v) \). Let \( \chi_{\mathbb{X}} : \mathbb{X}_F \to \mathbb{C}^\times \) be a smooth character. Write \( \mathcal{H}(\mathcal{G}(F_v), \chi_{\mathbb{X}}^{-1}) \) for the space of smooth functions on \( \mathcal{G}(F_v) \) which transform under \( \mathbb{X}_F \) via \( \chi_{\mathbb{X}} \). Given a semisimple element \( \gamma_{\mathbb{X}} \in \mathcal{G}(F_v) \) and an admissible representation \( \pi_{\mathbb{X}} \) of \( \mathcal{G}(F_v) \) with central character \( \chi_{\mathbb{X}} \) on \( \mathbb{X}_F \), the orbital integral and trace character for \( f_{\mathbb{X}} \in \mathcal{H}(\mathcal{G}(F_v), \chi_{\mathbb{X}}^{-1}) \) are defined as follows. Below \( I_{\gamma\mathbb{X}} \) denotes the connected centralizer of \( \gamma_{\mathbb{X}} \) in \( \mathcal{G} \).

\[
\begin{align*}
O_{\rho_{\mathbb{X}}}(f_{\mathbb{X}}) & := \int_{I_{\gamma\mathbb{X}}(F_v)/\mathcal{G}(F_v)} f_{\mathbb{X}}(x^{-1}\gamma_{\mathbb{X}}x)dx, \\
\text{Tr}(f_{\mathbb{X}}|\pi_{\mathbb{X}}) & = \text{Tr}\pi_{\mathbb{X}}(f_{\mathbb{X}}) := \text{Tr}\left( \int_{\mathcal{G}(F_v)/\mathbb{X}_F} f_{\mathbb{X}}(g)\pi_{\mathbb{X}}(g)dg \right).
\end{align*}
\]

Note that the trace is well defined since the operator \( \int_{\mathcal{G}(F_v)/\mathbb{X}_F} f_{\mathbb{X}}(g)\pi_{\mathbb{X}}(g) \) has finite rank. Likewise one can define the adelic Hecke algebra \( \mathcal{H}(\mathcal{G}(\mathbb{A}_F), \chi^{-1}) \) and orbital integrals and trace characters for its elements in the evident manner.

\(^7\)More details are available in [KSZ]. Compare with Section 1 of [Rog83], Sections 2 and 3 of [Art02], or Section 3.1 of [Art13].
For $f \in \mathcal{H}(G(\mathcal{A}_F), \chi^{-1})$ we define invariant distributions $T^G_{\text{ell}, \chi}$ and $T^G_{\text{disc}, \chi}$ by

\[
T^G_{\text{ell}, \chi}(f) := \sum_{\gamma \in \tilde{X}(G)} i(\gamma)^{-1} \text{vol}(I_{\gamma}(F) \backslash I_{\gamma}(G) \backslash \mathcal{X}) O_{\gamma}(f),
\]

\[
T^G_{\text{disc}, \chi}(f) := \text{Tr}(f \mid L^2_{\text{disc}, \chi}(G(F) \backslash G(\mathcal{A}_F))).
\]

Similarly $T^G_{\text{cusp}, \chi}$ is defined by taking trace on the space of square-integrable cuspidal forms. In general we do not expect that $T^G_{\text{ell}, \chi}(f) = T^G_{\text{disc}, \chi}(f)$ (unless $G/Z$ is anisotropic over $F$) but the equality should hold only after adding more terms on both sides. However we do have $T^G_{\text{ell}, \chi}(f) = T^G_{\text{disc}, \chi}(f)$ if $f$ satisfies some local hypotheses; this is often referred to as the simple trace formula. For our purpose, we henceforth assume the following:

- $X = \mathcal{X}\mathcal{X}_\infty$ for a open compact subgroup $\mathcal{X}\mathcal{X}_\infty \subset Z(\mathcal{A}_F^\infty)$ and $\mathcal{X}_\infty = Z(\mathcal{F}_\infty)$,
- $\chi = \prod_v \chi_v$ with $\chi_v = 1$ at every finite place $v$,
- $f = \prod_v f_v$ with $f_v \in \mathcal{H}(G(F_v), \chi_v^{-1})$.

We also need to consider the central character datum $(\mathcal{X}_0, \chi_0)$ with $\mathcal{X}_0 := A_{\mathcal{Z}_{\infty}}$ and $\chi_0 := \chi|_{A_{\mathcal{Z}_{\infty}}}$. There is a natural surjection $\mathcal{H}(G(\mathcal{A}_F), \chi_0^{-1}) \to \mathcal{H}(G(\mathcal{A}_F), \chi^{-1})$ given by

\[
f_0 \mapsto \left( g \mapsto f_0^\gamma(g) := \int_{\mathcal{X}_z \mathcal{Z}(\mathcal{Q}) \mathcal{X}_0} f_0(zg) \chi(z) \frac{dz}{z} \right).
\]

It is easy to check that $T^G_{\text{ell}, \chi_0}(f_0) = T^G_{\text{ell}, \chi}(f_G)$ and $T^G_{\text{disc}, \chi_0}(f_0) = T^G_{\text{disc}, \chi}(f_G)$.

Let $\xi$ be an irreducible algebraic representation of $G \times \mathbb{C}$. Denote by $\chi_{\xi} : Z(\mathcal{F}_\infty) \to \mathbb{C}^\times$ the restriction of $\xi^\vee$ to $Z(\mathcal{F}_\infty)$. Write $f_\xi = f^\xi_G \in \mathcal{H}(G(\mathcal{F}_\infty), \chi_\xi^{-1})$ for a Lefschetz function (a.k.a. Euler-Poincaré function) for $\xi$ such that $\text{Tr} \pi_\infty(f_\xi)$ computes the Euler-Poincaré characteristic for the Lie algebra cohomology of $\pi_\infty \otimes \xi$ for every irreducible admissible representation $\pi_\infty$ of $G(\mathcal{F}_\infty)$ with central character $\chi_\xi$. See Appendix A for details. Analogously we have the notion of Lefschetz functions at a finite place as recalled in Appendix A.

The following simple trace formula is standard for $\chi = A_{G_{\infty}}$ and some other choices of $\chi$ (such as $\chi = Z(G(\mathcal{A}_F)))$, but we want the result to be more flexible. Our proof reduces to the case that $\chi = A_{G_{\infty}}$.

**Lemma 6.1.** Consider the central character datum $(Z(\mathcal{F}_\infty), \chi)$. Assume that $f_\chi \in \mathcal{H}(G(\mathcal{F}_\chi))$ is a (truncated) Lefschetz function and that $f_0 \in \mathcal{H}(G(F, \chi^0))$ is a cuspidal function. Then

\[
T^G_{\text{ell}, \chi}(f_0) = T^G_{\text{disc}, \chi}(f) = T^G_{\text{cusp}, \chi}(f).
\]

**Proof.** Let $G(\mathcal{F}_\chi)^1$ denote the kernel of the restriction of the norm map $G(\mathcal{A}_F) \to \text{Hom}(X^*(G), \mathbb{R})$ to $G(\mathcal{F}_\chi)$. Consider $f_0 = \prod_v f_{0,v} \in \mathcal{H}(G(\mathcal{A}_F), \chi_0^{-1})$ given by $f_{0,v} := f_0$ at all finite places $v$ and $f_{0,\infty} := 1_{G^1(F_\infty)} f_0$. For semisimple elements $\gamma \in G(F_\infty)$ we see that $O_{\gamma}(f_{0,\infty})$ vanishes if $\gamma \not\in G^1(F_\infty)$ and equals $O_{\gamma}(f_0)$ if $\gamma \in G^1(F_\infty)$. Since $f_0$ is cuspidal (Lemma A.9), so is $f_{0,\infty}$.

We also know that $f_{0,\chi}$ is strongly cuspidal [Lemma A.7]. Hence the simple trace formula [Art68b, Cor. 7.3, 7.4] implies that

\[
T^G_{\text{ell}, \chi_0}(f_0) = T^G_{\text{disc}, \chi_0}(f_0) = T^G_{\text{cusp}, \chi_0}(f_0).
\]

We deduce the lemma by averaging over $Z(\mathcal{F}_\chi)/Z(\mathcal{F}_\chi) \cap G(\mathcal{F}_\chi)^1$ against $\chi$, noting that the average of $f_{0,\infty}$, namely the function $\gamma \mapsto \int_{Z(\mathcal{F}_\chi)/Z(\mathcal{F}_\chi) \cap G(\mathcal{F}_\chi)^1} f_{0,\infty}(\gamma) \chi_\xi(z) \frac{dz}{z}$, recovers $f_{0,\infty}$ up to a nonzero constant. \[\Box\]

Now we go back to the general central character data and discuss the stabilization for the trace formula with fixed central character under simplifying hypotheses. Assume that $G^\ast$ is quasi-split over $F$. Write $\Sigma_{\text{ell}, \chi}(G^\ast)$ for the set of $\chi$-orbits on the set of $F$-elliptic stable conjugacy classes in $G^\ast(F)$. Define

\[
ST^G_{\text{ell}, \chi}(f) := \tau(G^\ast) \sum_{\gamma \in \Sigma_{\text{ell}, \chi}(G^\ast)} i(\gamma)^{-1} \text{SO}_{\gamma, \chi}(f), \quad f \in \mathcal{H}(G^\ast(\mathcal{A}_F), \chi^{-1}),
\]

where $i(\gamma)$ is the number of $\Gamma$-fixed points in the group of connected components in the centralizer of $\gamma$ in $G^\ast$, and $\text{SO}_{\gamma, \chi}(f)$ denotes the stable orbital integral of $f$ at $\gamma$. If $G^\ast$ has simply connected derived subgroup (such as $\text{Sp}_{2n}$ or $\text{GSp}_{2n}$), we always have $i(\gamma) = 1$.

Returning to a general reductive group $G$, let $G^\ast$ denote its quasi-split inner form over $F$ (with a fixed inner twist $G^\ast \simeq G$ over $F$). Since $Z$ is canonically identified with the center of $G^\ast$, we may view $(\mathcal{X}_0, \chi_0)$ as a central character datum for $G^\ast$. Let $f^* \in \mathcal{H}(G^*(\mathcal{A}_F), \chi_0^{-1})$ denote a Langlands-Shelstad transfer of $f$ to $G^\ast$.\[8\] Such a transfer exists in this fixed-central-character setup: One can lift $f$ via the surjection $\mathcal{H}(G(\mathcal{A}_F)) \to$...
Remark 6.4. Corollary 2.8 tells us that the condition in (2) for the transfer from $G^*$ to $G$ is satisfied by $G^* = \text{GSp}_{2n}$. Later we will see in Corollary 8.5 that the same is also true for a certain inner form of $\text{GSp}_{2n}$.

Proof. We will only explain how to go from $\pi$ to $\pi^3$ as the opposite direction is proved by the exact same argument. Let $f = \prod v f_v$ be such that $f_v \in H^{\text{unr}}(G(F_v))$ for finite places $v \not\in S$, $f_{v_S}$ is a Lefschetz function at $v_S$, and $f_\infty$ is a Lefschetz function for $\xi$. Then we can choose the transfer $f^* = \prod v f^*_v$ such that $f_v^* = f_v$ for all $v \not\in S_\infty \cup \{v_S\}$, $f_\infty^*$ is a Lefschetz function, and $f_\infty^*$ is a Lefschetz function for $\xi$. We know from Lemmas A.4 and A.10 that $f_{v_S}$ (resp. $f_\infty$) and $f_{v_S}^*$ (resp. $f_\infty^*$) are associated up to a nonzero constant. Hence $cf$ and $f^*$ are associated for some $c \in \mathbb{C}^*$. The preceding two lemmas imply that

$$T_{\text{cusp}, \lambda}(f^*) = \text{ST}^G_{\text{ell}, \lambda}(f^*) = c \cdot T^G_{\text{cusp}, \lambda}(f).$$

By linear independence of characters, we have

$$\sum_{\tau \in \mathcal{A}_F} m(\tau) \text{Tr}(f^*_\tau | \tau_S) = c \cdot \sum_{\tau \in \mathcal{A}_F} m(\tau^3) \text{Tr}(f_\tau | \tau^3_S).$$

Let us prove (1). Choose $f_v = f_v^*$ at finite places $v$ in $S$ but outside $\{v_S\}$ such that $\text{Tr}(f_v^* | \tau_v) > 0$. This is vacuous if there is no such $v$. At infinite places, as soon as $\text{Tr}(f_\infty^* | \tau_\infty) \neq 0$ at $v_\infty$, the regularity condition on $\xi$ implies that $\tau_v$ is a discrete series representation and that $\text{Tr}(f_v^* | \tau_v) = (-1)^{\ell(\ell)}$ by Vogan–Zuckerman’s classification of unitary cohomological representations. At $v = v_S$, whenever $\text{Tr}(f_{v_S}^* | \tau_{v_S}) \neq 0$ (which is true for $v = v_S$), the unitary representation $\tau_{v_S}$ is either an unramified twist of the trivial or Steinberg function.
representation by Lemma A.1. If \( \tau_{v_{S}} \) were one-dimensional then the global representation \( \tau \) is one-dimensional by strong approximation for the derived subgroup of \( G \), implying that \( \tau_{\infty} \) cannot be tempered.\(^9\)

All in all, for all \( \tau \) as above such that \( \text{Tr}(f_{v}^{\tau} | \tau_{S}) \neq 0 \), we see that \( \tau_{v_{S}} \) is an unramified twist of the Steinberg representation and that \( \text{Tr}(f_{v}^{\tau} | \tau_{S}) \) has the same sign. Moreover \( \tau = \pi \) contributes nontrivially to the left sum by our assumption. Therefore the right hand side is nonzero, i.e. there exists \( \pi^{\pm} \in \mathcal{A}_{v}(G) \) such that \( \pi^{\pm} \neq \pi_{S}^{\pm} \) and \( m(\pi^{\pm}) \text{Tr}(f_{v} | \pi_{S}^{\pm}) = 0 \). The nonvanishing of trace confirms the conditions on \( \pi^{\pm} \) at \( v_{S} \) and \( \infty \) by the same argument as above.

Now we prove (2). Make the same choice of \( f_{v} = f_{v}^{\tau} \) at finite places \( v \) in \( S \). Since one-dimensional representations of \( \mathcal{G}(F_{v}) \) are not essentially tempered, the condition \( \text{Tr}(f_{v}^{\tau} | \tau_{S}) \neq 0 \) implies that \( \tau_{v_{S}} \) is an unramified twist of the Steinberg representation. By the assumption \( \tau_{\infty} \) is then a discrete series representation. Hence the nonzero contributions from \( \tau \) on the left hand side all has the same sign. Thereby we deduce the existence of \( \pi^{\pm} \) as in (1).

As before \( G^{*} \) is a quasi-split group over a totally real field \( F \) whose derived subgroup is simply connected. Let \( E/F \) be a finite cyclic extension of totally real fields such that \( [E : F] \) is a prime. For each place \( v \) of \( F \) put \( E_{v} := E \otimes_{F} F_{v} \). Let \( BC_{E/F} : \text{Irr}_{\text{unr}}^{\mathcal{G}^{*}(G_{E_{v}})}(\mathcal{G}(E_{v})) \rightarrow \text{Irr}_{\text{unr}}^{\mathcal{G}^{*}(G_{E_{v}})}(\mathcal{G}(E_{v})) \) denote the base change map for unramified representations. Writing \( BC_{E/F} : H^{\text{unr}}(G_{2n}(E_{v})) \rightarrow H^{\text{unr}}(G^{*}(F_{v})) \) for the base change morphism of unramified Hecke algebras, we have from Satake theory that

\[
(6.1) \quad \text{Tr}(BC_{E/F}(f_{v}) | \pi_{v}) = \text{Tr}(f_{v} | BC_{E/F}(\pi_{v})).
\]

**Proposition 6.5.** Let \( \pi \) be a cuspidal automorphic representation of \( \mathcal{G}^{*}(A_{E}) \) such that

- \( \pi \) is unramified at all finite places \( v \) apart from \( S \),
- \( \pi_{v_{S}} \) is an unramified character twist of the Steinberg representation,
- \( \pi_{\infty} = \xi \cdot \pi \)-cohomological.

Suppose that either \( \xi \) has regular highest weight or that the condition for \( \pi \) in Proposition 6.3.(2) is satisfied. (This is always true for \( G = \mathcal{G}_{2n}, \) cf. Remark 6.4.) Then there exists a cuspidal automorphic representation \( \pi_{E} \) of \( G_{2n}(A_{E}) \) such that

- \( \pi_{E} \) is unramified at all finite places \( v \) apart from \( S \),
- \( \pi_{E,S} \) is an unramified character twist of the Steinberg representation,
- \( \pi_{E,\infty} = \xi \cdot \pi \)-cohomological,

and moreover \( \pi_{E,v} \cong BC_{E/F}(\pi_{v}) \) at every finite place \( v \in S \).

**Proof.** We will be brief as our proposition and its proof are very similar to those in Labesse’s book \cite[§4.6]{Labesse99}, and also as the proof just mimics the argument for Proposition 6.3 in the twisted case. In particular we leave the reader to find further details about the twisted trace formula for base change in loc. cit. Strictly speaking one has to incorporate the central character datum \((\chi, \chi')\) above in Labesse’s argument, but this is done exactly as in the untwisted case.

We begin by setting up some notation. Take \( \tilde{K} \) to be a sufficiently small open compact subgroup of \( \text{GSp}_{2n}(A_{E}) \) such that \( N_{E/F} : Z(E) \rightarrow Z(F) \) maps \( Z(F) \cap \tilde{K} \) into \( Z(F) \cap K \). Let \( \tilde{\chi} := \chi \circ N_{E/F} \). Let \( \tilde{\xi} \) denote the representation \( \otimes_{E \rightarrow \mathbb{C}} \xi \) of \( \text{GSp}_{2n}(E \otimes_{\mathbb{Q}} \mathbb{C}) = \prod_{E \rightarrow \mathbb{C}} \text{GSp}_{2n}(F \otimes_{\mathbb{Q}} \mathbb{C}) \) (both indexed by \( F \)-algebra embedding \( E \hookrightarrow \mathbb{C} \)).

We choose the test functions

\[
\tilde{f} = \sum_{w} f_{w} \in \mathcal{H}(\text{GSp}_{2n}(A_{E}), \tilde{\chi}^{-1}) \quad \text{and} \quad f = \prod_{v} f_{v} \in \mathcal{H}(\text{GSp}_{2n}(A_{F}), \chi^{-1})
\]

as follows. If \( v = v_{S} \) then \( f_{v} = \prod_{w/v} f_{w} \) and \( f_{v} \) are set to be Lefschetz functions as in Appendix A. So \( f_{v} \) and \( f_{v} \) are associated up to a nonzero constant by Lemma A.8. If \( v \) is a finite place of \( F \) contained in \( S \) then choose \( f_{v} \) to be the characteristic function on a sufficiently small open compact subgroup \( K_{v} \) of \( \text{GSp}_{2n}(F_{v}) \) such that \( \pi_{E}^{K_{v}} \neq 0 \). By \cite[Prop. 3.3.2]{Labesse99} \( f_{v} \) is a base change transfer of some function \( f_{E} = \prod_{w/\mathfrak{p}} f_{w} \in \mathcal{H}(\text{GSp}_{2n}(E_{v}), \tilde{\chi}^{-1}) \). Given a finite place \( v \in S \) and each place \( w \) of \( E \) above it, let \( f_{v} \) be an arbitrary function in \( \mathcal{H}^{\text{unr}}(\text{GSp}_{2n}(E_{v}, \tilde{\chi}^{-1})) \). The image of \( f_{E} \) in \( H^{\text{unr}}(\text{GSp}_{2n}(F_{v})) \) under the base change map is denoted by \( f_{v} \). At infinite places let \( f_{\infty} = \prod_{v/\mathfrak{p}} f_{w} \) be the twisted Lefschetz function determined by \( \tilde{\xi} \) and \( f_{\infty} = \prod_{v/\mathfrak{p}} f_{w} \) the usual Lefschetz function for \( \xi \). Again \( f_{\infty} \) and \( f_{\infty} \) are associated up to a nonzero constant by Lemma A.11. By construction \( \tilde{f} \) and \( \tilde{c} \) are associated for some \( c \in \mathbb{C}^{x} \).

\(^9\)The strong approximation is true since the derived subgroup of our \( G \) is simply connected but this is inessential; one can always reduce to this case via \( z \)-extensions.
We write \( T^G_{\text{cusp}, \tilde{x}} \) and \( T^G_{\text{ell}, \tilde{x}} \) for the cuspidal and elliptic expansions in the base-change twisted trace formula, which are defined analogously as their untwisted analogues. Just like the trace formula for \( G \) and \( f \), the twisted trace formula for \( \tilde{G} \) and \( \tilde{f} \) as well as its stabilization simplifies greatly exactly as in Lemmas 6.1 and 6.2 in light of Lemmas A.8 and A.11. So we have

\[
T^G_{\text{cusp}, \tilde{x}}(\tilde{f}) = T^G_{\text{ell}, \tilde{x}}(\tilde{f}) = e \cdot ST^G_{\text{ell}, \tilde{x}}(f) = e \cdot T^G_{\text{cusp}, \tilde{x}}(f).
\]

By linear independence of characters and the character identity (6.1), we have

\[
\sum_{\tilde{x} \in A}(S_{\text{ell}, \tilde{x}}(\tilde{A}_\ell)) \sum_{\tilde{x} \in A}(S_{\text{ell}, \tilde{x}}(\tilde{A}_\ell)) = e \sum_{\tilde{x} \in A}(S_{\text{ell}, \tilde{x}}(\tilde{A}_\ell))
\]

where \( S_{\text{ell}, \tilde{x}}(\tilde{A}_\ell) \) denotes the \( \theta \)-twisted trace (for a suitable intertwining operator for the \( \theta \)-twist). The right hand side is nonzero as in the proof of Proposition 6.3. Therefore there exists \( \pi_E \approx \bar{\tau} \in A_\ell(G_{\text{Sp}_{2n}(A_E)}) \) contributing nontrivially to the left hand side. By construction of \( \tilde{f} \) such a \( \pi_E \) satisfies all the desired properties. \( \square \)

7. Cohomology of certain Shimura varieties of abelian type

In this section we construct a Shimura datum and then state the outcome of the Langlands-Kottwitz method on the formula computing the trace of the Frobenius and Hecke operators in the case of good reduction.

We first construct our Shimura datum \( (\text{Res}_{E/F}G,X) \). The group \( G/F \) is a certain inner form of the quasi-split group \( G^* := G_{\text{Sp}_{2n,F}} \). We recall the classification of such inner forms, and then define our \( G \) in terms of this classification. The inner twists of \( G_{\text{Sp}_{2n}} \) are parametrized by the cohomology group \( H^1(F,\text{PSp}_{2n}) \). Kottwitz defines in [Kot86, Thm. 1.2] for each \( F \)-place \( v \) a morphism of pointed sets

\[
\alpha_v : H^1(F_v,\text{PSp}_{2n}) \to \pi_0(Z(\text{Spin}_{2n+1}(C))^{\overline{v}})
\]

\((D)\) denotes the Pontryagin dual). If \( v \) is finite, then \( \alpha_v \) is an isomorphism. If \( v \) is infinite, then [Kot86, (I.2.2)] tells us that

\[
\ker(\alpha_v) = \text{im}[H^1(F_v,\text{Sp}_{2n}) \to H^1(F_v,\text{PSp}_{2n})],
\]

\[
\text{im}(\alpha_v) = \ker[\pi_0(Z(\text{Spin}_{2n+1}(C))^{\overline{v}})]D \to \pi_0(Z(\text{Spin}_{2n+1}(C))^{\overline{v}}).\]

Thus \( \alpha_v \) is surjective, with trivial kernel as \( H^1(R,\text{Sp}_{2n}) \) vanishes by [PR94, Chap. 2]. However \( \alpha_v \) is not a bijection (when \( n \geq 2 \)). In fact \( \alpha_v^{-1}(1) \) classifies unitary groups associated to Hermitian forms over the Hamiltonian quaternion algebra over \( F_v \) with signature \( (a,b) \) with \( a + b = n \) modulo the identification as inner twists between signatures \( (a,b) \) and \( (b,a) \). (See [Tail5, 3.1.1] for an explicit computation of \( H^1(F_v,\text{PSp}_{2n}) \). So \( \alpha_v^{-1}(1) \) has cardinality \( \frac{n}{2} + 1 \). There is a unique nontrivial inner twist of \( G_{\text{Sp}_{2n,F}} \) (up to isomorphism), to be denoted by \( G_{\text{Sp}_{2n,F}}^{\text{cmt}} \), such that \( G_{\text{Sp}_{2n,F}}^{\text{cmt}} \) is compact modulo center. It comes from a definite Hermitian form. By [Kot86, Prop. 2.6] we have an exact sequence

\[
\ker^1(F,\text{PSp}_{2n}) \to H^1(F,\text{PSp}_{2n}) \to \bigoplus_v H^1(F_v,\text{PSp}_{2n}) \to Z/2Z
\]

where \( \alpha \) sends \( (c_v) \in H^1(F,G(A_F \otimes F_v)) \) to \( \sum_v \alpha_v(c_v) \). By [Kot84b, Lem. 4.3.1] we have \( \ker^1(F,\text{PSp}_{2n}) = 1 \). We conclude

\[
H^1(F,\text{PSp}_{2n}) = \left\{ (x_v) \in \bigoplus_v H^1(F_v,\text{PSp}_{2n}) \bigg| \sum_v \alpha_v(x_v) = 0 \in Z/2Z \right\}.
\]

In particular there exists an inner twist \( G \) of \( G_{\text{Sp}_{2n,F}} \) such that:

- For the infinite places \( y \in \mathcal{V}_\infty \setminus \{v_\infty\} \) the group \( G_{F_y} \) is isomorphic to \( G_{\text{Sp}_{2n,F_y}}^{\text{cmt}} \). For \( y = v_\infty \) we have \( G_{F_{v_\infty}} = G_{\text{Sp}_{2n,F_{v_\infty}}} \). (Recall: \( v_\infty \) is the finite \( F \)-place corresponding to the embedding of \( F \) into \( C \) that we fixed in the Notation section.)
- If \( [F : Q] \) is odd, we take \( G_{\text{Sp}_{2n,A}^{\text{cmt}}} \).
- If \( [F : Q] \) is even we fix a finite \( F \)-place \( v_{St} \), and take \( G_{\text{Sp}_{2n,A}^{\text{cmt}}} \). The form \( G_{F_{v_{St}}} \) then has to be the unique nontrivial inner form of \( G_{\text{Sp}_{2n,F_{v_{St}}}} \).

More concretely \( G \) can be defined itself as a similitude group. See the first two paragraphs of §II below.
Let be the Deligne torus \( \text{Res}_{G/K_{Gm}} \). Over the real numbers the group \((\text{Res}_{F/Q}G_{R})\) decomposes into the product \( \prod_{v \in V_{\infty}} G \otimes F_{v} \). Let \( I_{n} \) be the \( n \times n \)-identity matrix and \( A_{n} \) be the \( n \times n \)-matrix with all entries 0, except those on the anti-diagonal, where we put 1. Let \( h_{0} : \rightarrow (\text{Res}_{F/Q}G_{R}) \) be the morphism given by

\[
(7.2) \quad (R) \rightarrow G(F \otimes Q, R), \quad a + bi \mapsto \left( \begin{array}{cc} a_{11} & b_{12} \\ -b_{11} & a_{12} \end{array} \right)_{v_{\infty}}, \quad 1, \ldots, 1 \in \prod_{v \in V_{\infty}} (G \otimes F_{v})(R),
\]

for all \( R \)-algebras \( R \) (the non-trivial component corresponds to the non-compact place \( v_{\infty} \in V_{\infty} \)). We let \( X \) be the \((\text{Res}_{F/Q}G_{R})(R)\)-conjugacy class of \( h_{0} \). This set \( X \) can more familiarly be described as Siegel double half space \( \mathcal{F}_{n} \), i.e. the \( n(n+1)/2 \)-dimensional space consisting of complex symmetric \( n \times n \)-matrices with definite (positive or negative) imaginary part. Let us explain how the bijection \( X \cong \mathcal{F}_{n} \) is obtained. The group \( \text{GSp}_{2n}(\mathbb{R}) \) acts transitively on \( \mathcal{F}_{n} \) via fractional linear transformations:

\[ (\begin{array}{cc} A & B \\ C & D \end{array}) \in \text{GSp}_{2n}(\mathbb{R}), \quad Z \in \mathcal{F}_{n}, \quad \text{Sie64} \]

So the bijection \( X \cong \mathcal{F}_{n} \) is obtained. This is an isomorphism \( X \cong \mathcal{F}_{n} \) under which \( h_{0} \) corresponds to \( I_{n} \). It is a routine verification that Deligne's axioms (2.1.1.1), (2.1.1.2), and (2.1.3) for Shimura data [Del79] are satisfied for \((\text{Res}_{F/Q}G_{R}, h_{0})\). Since moreover the Dynkin diagram of the group \( G_{\mathbb{R}} \) is of type \( C \), it follows from Deligne [Del79, Prop. 2.3.10] that \((\text{Res}_{F/Q}G_{R}, h_{0})\) is of abelian type.\textsuperscript{10}

We write \( \mu = (\mu_{y})_{y \in V_{\infty}} \in X(\mathbb{Q})^{\infty} \) for the cocharacter of \((\text{Res}_{F/Q}G_{R}) \) such that \( \mu_{y} = 1 \) if \( y \neq v_{\infty} \) and \( \mu_{v_{\infty}}(x) = (\frac{v_{\infty}}{y})_{0} \). Then \( \mu \) is the holomorphic part of \( h_{0} \otimes \mathbb{C} \). An element \( \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) sends \( \mu = (\mu_{y}) \) to \( \sigma(\mu) = (\mu_{\sigma y}) \). Thus the conjugacy class of \( \mu \) is fixed by \( \sigma \) if and only if \( \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{F}) \). Therefore the reflex field of \((\text{Res}_{F/Q}G_{R}, h_{0})\) is \( F \). For \( K \subset G(\mathbb{A}^{\infty}_{\mathbb{F}}) \) a sufficiently small compact open subgroup, write \( S_{K}/F \) for the corresponding Shimura variety. In case \( F = \mathbb{Q} \) the datum \((\text{Res}_{F/Q}G_{R}, h_{0})\) is the classical Siegel datum (of PEL type) and \( S_{K} \) are the usual non-compact Siegel modular varieties. If \( [F : \mathbb{Q}] > 1 \) so that \( G \) is anisotropic modulo center, then it follows from the Baily-Borel compactification [BB64, Thm. 1] that \( S_{K} \) is projective. Whenever \([F : \mathbb{Q}] > 1\) the datum \((\text{Res}_{F/Q}G_{R}, h_{0})\) is not of PEL type. If moreover \( n = 1 \) then the \( S_{K} \) have dimension 1 and they are usually referred to as Shimura curves, which have been extensively studied in the literature.

Let \( \xi = \otimes_{v \mid p} \xi_{v} \) be an irreducible algebraic representation of \((\text{Res}_{F/Q}G_{R}) \otimes \mathbb{Q} = \prod_{v \mid \infty} G(F_{v}) \) with each \( F_{v} \) canonically identified with \( \mathbb{C} \). The central character \( \omega_{\xi} \) of \( \xi_{v} \) has the form \( z \mapsto z^{\nu_{y}} \) for some integer \( \nu_{y} \in \mathbb{Z} \).

**Lemma 7.1.** If there exists a \( \xi \)-cohomological discrete automorphic representation \( \pi \) of \( G_{\mathbb{F}} \) then \( \omega_{\pi} \) has the same value for every infinite place \( y \) of \( F \).

**Proof.** Under the assumption, the central character \( \omega_{\pi} : F_{\mathbb{F}} \rightarrow \mathbb{C}^{\times} \) of \( \pi \) is an \( (\mathbb{L}) \)-algebraic Hecke character. Hence \( \omega_{\pi} = \omega_{\{0\}} \mid \nu \) for a finite Hecke character \( \omega_{\{0\}} \) and an integer \( \nu \) by Weil [Wei56]. It follows that \( \nu = \nu_{y} \) for every infinite place \( v \).

In light of the lemma we henceforth make the hypothesis as follows, implying that \( \xi \) restricted to the center \( Z(F) = F^{\times} \) of \((\text{Res}_{F/Q}G_{R})(\mathbb{Q}) = \text{GSp}_{2n}(F) \) is the \( w^{th} \) power of the norm character \( N_{F/Q} \).

\textbf{(cent)} \( \nu_{y} \) is independent of the infinite place \( y \) of \( F \) (and denoted by \( w \)).

Following [Car86b, 2.1] (especially paragraph 2.1.4) we construct an \( \ell \)-adic sheaf on \( S_{K} \) for each sufficiently small open compact subgroup \( K \) of \( G(\mathbb{A}^{\infty}_{\mathbb{F}}) \) from the \( \ell \)-adic representation \( \xi_{\mathbb{Q}} = \xi \otimes \mathbb{Q} \). For simplicity we write \( \mathcal{L}_{\xi} \) for the \( \ell \)-adic sheaf (omitting \( K \)). It is worth pointing out that the construction relies on the fact that \( \xi \) is trivial on \( Z(F) \cap K \) for small enough \( K \), cf. (12.4) below where we discuss \( \ell \)-adic sheaves further. (For a fixed open compact subgroup \( K_{0} \), we see that \( Z(F) \cap K_{0} \subset \mathcal{O}_{F}^{\infty} \) is mapped to \{\pm 1\} under \( N_{F/Q} \). So \( \xi \) is trivial on \( Z(F) \cap K \) for every \( K \) contained in some subgroup of \( K_{0} \) of index at most 2.)

Without loss of generality we assume throughout that \( K \) decomposes into a product \( K = \prod_{v \mid \infty} K_{v} \) where \( K_{v} \subset G(F_{v}) \) is a compact open subgroup. We call \((G,K)\) unramified at a finite \( F \)-place \( v \) if the group \( K_{v} \) is hyperspecial in \( G(F_{v}) \). If so, fix a smooth reductive model \( G_{F} \) of \( G_{\mathbb{F}} \) over \( \mathcal{O}_{F} \), such that \( G(\mathcal{O}_{F_{v}}) = K_{v} \). We say that \((G,K)\) is unramified at a rational prime \( p \) if it is unramified at all \( F \)-places \( v \) above \( p \).

Our primary interest lies in the \( \ell \)-adic étale cohomology with compact support

\[
H_{\ell}(S_{K}, \mathcal{L}_{\xi}) : = \sum_{i=0}^{n(n+1)} (-1)^{i} H_{\ell}^{i}(S_{K}, \mathcal{L}_{\xi}^{\ell})
\]

as a virtual \( \mathcal{H}(G)(\mathbb{A}^{\infty}_{\mathbb{F}})/K) \times \Gamma \)-module. We are going to apply the Langlands-Kottwitz method to relate the action of Frobenius elements of \( \Gamma \) at primes of good reduction to the Hecke action. In the PEL case of type A\textsuperscript{10}

\textsuperscript{10}This fact can also be deduced in a more explicit way from the constructions in Section 12.
or C. Kottwitz worked it out in [Kot90,Kot92b] including the stabilization. As we are dealing with Shimura varieties of abelian type, we import the result from [KSZ]. In fact the stabilization step is very simple under the hypothesis ([St]).

Suppose that \((G,K)\) is unramified at \(p\). Let \(\rho\) be a finite \(F\)-place above \(p\) with residue field \(k(\rho)\). Let us introduce some more notation.

- For \(j \in \mathbb{Z}_{\geq 1}\) denote by \(Q_{p,j}\) the unramified extension of \(Q_p\) of degree \(j\), and \(Z_{p,j}\) its integer ring.
- Write \(F_p := F \otimes Q_{p,j}\), \(F_{p,j} := F \otimes Q_{p,j}\), and \(O_{F_{p,j}} := Q_{p,j} \otimes Z_{p,j}\).
- Let \(\sigma\) be the automorphism of \(F_{p,j}\) induced by the (arithmetic) Frobenius automorphism on \(Q_{p,j}\) and the trivial automorphism on \(F\).
- Let \(H(G(A^\infty_{\sigma}))\) (resp. \(H(G(F_p)), H(G(F_{p,j}))\)) be the convolution algebra of compactly supported smooth complex valued functions on \(G(A^\infty_{\sigma})\) (resp. \(G(F_p), G(F_{p,j})\)), where the convolution integral is defined by the Haar measure giving the group \(K\) (resp. \(K_p\) resp. \(O_{F_{p,j}}\)) measure \(1\).
- We write \(H^\text{unr}(G(F_p))\) (resp. \(H^\text{unr}(G(F_{p,j}))\)) for the spherical Hecke algebra, i.e. the algebra of \(K_p\) (resp. \(G(O_{F_{p,j}}))\)-bilinear functions in \(H(G(F_p))\) (resp. \(H(G(F_{p,j}))\)).
- Define \(\phi_j \in H^\text{unr}(G(F_{p,j}))\) to be the characteristic function of the double coset \(G(O_{F_{p,j}}) \cdot \mu(p^{-1}) \cdot G(O_{F_{p,j}})\) in \(G(F_{p,j})\).
- \(N_{\infty} := [\Pi^G_{\infty}] \cdot |\pi_0(G(F_\infty)/Z(F_\infty))| = 2^{n-1} \cdot 2 = 2^n\).
- \(\text{ep}(\tau_\infty \otimes \xi) := \sum_{\ell = 0}^{n} (2 \ell - 1) \dim H(\text{Lie} G(F_\infty), K_\infty \otimes \rho)\) for an irreducible admissible representation \(\tau_\infty\) of \(G(F_\infty)\).
- \((\chi, \chi)\) is the central character datum given as \(\chi := (Z(A^\infty_{\sigma}) \cap K) \times Z(F_\infty)\) and \(\chi := \chi_\xi\) (extended from \(Z(F_\infty)\) to \(X\) trivially on \(Z(A^\infty_{\sigma}) \cap K\).

**Remark 7.2.** It is worth pointing out that \(K_\infty\) is a subgroup of index \(|\pi_0(G(F_\infty)/Z(F_\infty))|\) in some \(K'_\infty\), which is a product of \(Z(F_\infty)\) and a maximal compact subgroup of \(G(F_\infty)\) and that \(\text{ep}(\tau_\infty \otimes \xi)\) is defined in terms of \(K_\infty\), not \(K'_\infty\). When \(\tau_\infty\) belongs to the discrete series \(L\)-packet \(\Pi^G_{\infty}\), the representation \(\tau_\infty\) decomposes into a direct sum of irreducible (Lie \(G(F_\infty), K_\infty)\)-modules, the number of which is equal to \(|\pi_0(G(F_\infty)/Z(F_\infty))|\), cf. paragraph below Lemma 3.2 in [Kot92a]. So \(\text{ep}(\tau_\infty \otimes \xi) = (-1)^{(n+1)/2} |\pi_0(G(F_\infty)/Z(F_\infty))| = (-1)^{(n+1)/2} 2\) for \(\tau_\infty \in \Pi^G_{\infty}\).

Let \(E_{\text{dil}}(G)\) denote the set of representatives for isomorphism classes of \((G,K)\)-unramified elliptic endoscopic triples of \(G\). For each \((H,s,\eta) \in E_{\text{dil}}(G)\) we make a fixed choice of an \(L\)-morphism \(\eta: \!^L H \to \!^L G\) extending \(\eta_0\) (which exists since the derived group of \(\text{Sp}_{2g}\) is simply connected). We recall the notation \(\iota(H,G) \in \mathbb{Q}\) and \(h^{\text{Hil}} = h^{H,0,p} h^{H}_{\rho} h^{\text{Hil}}_{\infty} \in \mathbb{H}(H(A_F))\) for the principal endoscopic triple \((H,s,\eta) = (G^*,1,\text{id})\), which is all we need. For other endoscopic triples, the reader is referred to [7.3], second display on p.180 and second display on p.186 of [Kot90]; this is adapted to a little more general setup as ours in [KSZ]). We have \((G,G^*) = 1\) and \(h^{\text{Hil}} = h^{H,0,p} h^{H}_{\rho} h^{\text{Hil}}_{\infty} \in \mathbb{H}(G(A_F), \chi^{-1})\) given as follows.

- \(h^{G,0,p}_F \in \mathbb{H}(G^*(A^\infty_F))\) is an endoscopic transfer of the function \(f^{\infty,p}\) to the inner form \(G^*(h^{G,0,p})\) can be made-under \(Z(A^\infty_F) \cap K^p\) by averaging over it).
- \(h^{G,*}_F \in \mathbb{H}^\text{unr}(G(F_p))\) is the base change transfer of \(\phi_j \in \mathbb{H}^\text{unr}(G(F_{p,j}))\) to \(\mathbb{H}^\text{unr}(G(F_p)) = \mathbb{H}^\text{unr}(G(F_p))\),
- \(h^{G,*}_{\infty} \in \mathbb{H}(G(F_\infty), \chi^{-1}) = [\Pi^G_{\infty}]^{-1} \sum_{\tau_\infty \in \Pi^G_{\infty}} f_{\tau_\infty}\), i.e. the average of pseudo-coefficients for the discrete series \(L\)-packet \(\Pi^G_{\infty}\).
- As shown by [Kot92a, Lem. 3.2], \(h^{G,*}_{\infty} \) is \(N_{\infty}^{-1}\) times the Euler-Poincaré function for \(\xi\) (defined in §6 but using \(K_\infty\) of this section): \(\text{Tr} \pi_\infty(h^{G,*}_{\infty}) = N_{\infty}^{-1} \text{ep}(\pi_\infty \otimes \xi), \pi_{\infty} \in \text{Irr}(G(F_\infty)).\)

The main result of [KSZ] is the following (which works for every Shimura variety of abelian type such that the center of \(G\) is an induced torus and every Shimura variety of Hodge type), where the starting point is Kisin’s proof of the Langlands-Rapoport conjecture for all Shimura varieties of abelian type [Kis13]. When \(F = Q\) it was already shown by [Kot90,Kot92b].

**Theorem 7.3.** Let \(f^{\infty} \in \mathbb{H}(G(A^\infty_F)/K)\). Suppose that \(p\) and \(\rho\) are as above such that

- \((G,K)\) is unramified at \(p\),
- \(f^{\infty} = f^{\infty,p} f_p\) with \(f^{\infty,p} \in \mathbb{H}(G(A^\infty_F)/K^p)\) and \(f_p = 1_{K_p}\).

\(^{11}\)The discussion there is correct in our setting, but it can be false for non-discrete series representations (which are allowed in that paper). For instance when \(\xi\) and \(\pi_\infty\) are the trivial representation in the notation of [Kot92a], it is obvious that \(\tau_\infty\) cannot decompose further even if \(|\pi_\infty(G(F_\infty)/Z(F_\infty))| > 1\).
Then there exists a positive integer $j_0$ (depending on $\xi$, $f^\infty$, $\rho$, $p$) such that for all $j \geq j_0$ we have
\begin{equation}
\tau_i \mathrm{Tr}((f^\infty \mathrm{Frob}_p^\rho, H_i(S_K, L_\xi)) = \sum_{H \in \ell_{\text{ell}}(G)} i(G, H)ST_{\text{ell}, H}^{\infty}(h^H).
\end{equation}

8. Galois representations in the cohomology

Let $\pi$ be a cuspidal $\xi$-cohomological automorphic representation of $\mathrm{GSp}_{2n}(\mathbb{A}_F)$ satisfying condition (St), and fix the place $v_{S_0}$ in that condition. Define an inner form $G$ of $\mathrm{GSp}_{2n}$ as in §2; when $[F : \mathbb{Q}]$ is even, we take $v_{S_0}$ in that definition to be the fixed place $v_{S_0}$. Let $\pi^\circ$ be a transfer of $\pi$ to $G(\mathbb{A}_F)$ via Proposition 6.3 (which applies thanks to Remark 6.4) so that $\pi^\circ|v_{S_0} = \pi^\circ|v_{S_0} = \pi^\circ_{v_{S_0}}$ is an unramified twist of the Steinberg representation, and $\pi^\circ_{v_{S_0}}$ is $\xi$-cohomological. The aim of this section is to compute a certain $\pi^\circ_{v_{S_0}}$-isotopical component of the cohomology of the Shimura variety $S$ attached to $(G, X)$.

Let $A(\pi^\circ)$ be the set of (isomorphism classes of) cuspidal automorphic representations $\tau$ of $G(\mathbb{A}_F)$ such that $\tau|v_{S_0} \cong \pi^\circ_{v_{S_0}} \otimes \delta$ for an unramified character $\delta$ of the group $G(F_{v_{S_0}})$, $\tau|v_{S_0} \cong \pi^\circ_{v_{S_0}}$ and $\tau|_{S_0}$ is $\xi$-cohomological. Let $K \subset G(\mathbb{A}_F)$ be a sufficiently small decomposed compact open subgroup such that $\pi^\circ_{v_{S_0}}$ has non-zero $K$-invariant vector. Let $S_{\text{bad}}$ be the set of prime numbers $p$ for which either $p = 2$, the group $\mathrm{Res}_{F/\mathbb{Q}} G$ is ramified or $K_p = \prod_{v|p} K_v$ is not hyperspecial. Define $\rho^\text{shim}_2 = \rho^\text{shim}_2(\pi^\circ)$ to be the virtual Galois representation
\begin{equation}
(-1)^{n(n+1)/2} \sum_{x \in \mathbb{A}(\pi^\circ)} \sum_{i=0}^{n(n+1)} (-1)^i \text{Hom}(\tau^\infty|v_{S_0}, H_i^\infty(S_K, L_\xi)) \in K_0(\mathbb{Q}_p/\Gamma),
\end{equation}
where $K_0(\mathbb{Q}_p/\Gamma)$ is the Grothendieck group of the category of continuous representations of $\Gamma$ on finite-dimensional $\mathbb{Q}_p$-vector spaces, which are unramified at almost all the places. We compute in this section $\rho^\text{shim}_2$ at almost all $F$-places not dividing a prime in $S_{\text{bad}}$.

Let $Z$ denote the center of $\mathbb{Z}$ determined by $\xi$ as in condition (cent) of the last section. Let $i \in Z_{\geq 0}$. We write $H^1_i(S_K, L_\xi)$ for the $L^2$-cohomology of $S_K$ with coefficient $L_\xi$. By taking a direct limit we obtain an admissible $G(\mathbb{A}_F)$-representation $H^1_i(S_K, L_\xi)$. Similarly we define the compact support cohomology $H^1_i(S_K, L_\xi)$ and the intersection cohomology $I^1(S_K, L_\xi)$ as well as their direct limits $H^1_i(S, L_\xi)$ and $I^1(S, L_\xi)$. The latter two are equipped with commuting actions of $G(\mathbb{A}_F)$ and $\Gamma$.

**Lemma 8.1.** For every $\tau \in A(\pi^\circ)$ the following hold.

1. For every $i \in Z_{\geq 0}$, we have $H^1_i(S_K, L_\xi)[\tau^\infty] \cong H^1_i(S, L_\xi)[\tau^\infty]$ as $G(\mathbb{A}_F)$-representations.

2. For every finite place $x \neq v_{S_0}$ of $F$, if $\tau$ is unramified at $x$ then $\tau_x|\text{sim}^{\text{shim}}_{\text{temp}}$ is tempered and unitary.

**Proof.** (1) This is clear by compactness of $S_K$ if $F \neq \mathbb{Q}$. Suppose $F = \mathbb{Q}$. We claim that the finite part of $\tau$ does not appear in any parabolic induction of an automorphic representation on a proper Levi subgroup of $\mathrm{GSp}_{2n}(\mathbb{A})$. It suffices to check the analogous claim with $\tau^\circ$ (as in Lemma 2.6) and $\mathrm{Sp}_{2n}$ in place of $\tau$ and $\mathrm{GSp}_{2n}$. The latter claim follows from Arthur’s main result [Art13, Thm. 1.5.2], Corollary 2.2, and the strong multiplicity one theorem for general linear groups. As a consequence of the claim, the $\tau^\infty$-isotypic part of the $L^2$-terms of Franke’s spectral sequence consider Franke’s spectral sequence, [Fra98, Thm. 19] or [Wal97, Thm. 4.7], is zero unless $p = 0$, in which case it is identified with $H^1_i(S, L_\xi)[\tau^\infty]$ (note that a discrete automorphic representation with a Steinberg component up to a character is cuspidal, cf. [Lab99, Prop. 4.5.4]). Hence the spectral sequence degenerates and part (1) follows immediately.

(2) Let $\omega : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ denote the common central character of $\tau$, $\pi^\circ$, and $\pi$. The central characters are the same as they are equal at almost all places.) Since $F$ is totally real, $\omega = \omega_0^{|La|}$ for a finite character $\omega_0$ and a suitable $a \in \mathbb{C}$. Since $\pi_{v_{S_0}}$ is $\xi$-cohomological, we must have $a = -\omega$. Then $\tau_x|\text{sim}^{\text{shim}}_{\text{temp}}$ has unitary central character and is essentially tempered by Lemma 2.7. We are done since $\tau_x|\cong \tau_x$.

**Proposition 8.2.** For almost all finite $F$-places $v$ not dividing a prime number in $S_{\text{bad}}$ and all sufficiently large integers $j$, we have $\mathrm{Tr}_{\rho^\text{shim}_2}^j(\mathrm{Frob}_v^\rho) = (\pi^\circ|v)\mathrm{Tr}^{\infty}_{\pi^\circ}(f_j)$. Moreover the virtual representation $\rho^\text{shim}_2$ is a true representation.

For $a \in \mathbb{Z}_{\geq 1}$ and $m \in \mathbb{Z}_{\geq 1}$ let $l_{a,m} : \mathrm{GL}_{am} \to \mathrm{GL}_{an}$ denote the block diagonal embedding.
Corollary 8.3. Let $\rho^\text{shim}_2$ be as above. For almost all finite $F$-places $v$ where $\pi_v^\natural$ is unramified and, $\rho^\text{shim}_2(Frob_v)|_{\GL_{q_v}} \sim q_v^{(n+1)/4} \cdot \iota_{a(v)}(\sin(\nu_{g,v})) \cdot (Frob_v)$ in $\GL_{q_v}$.

Proof of Corollary 8.3. Write $\gamma_2 = \rho^\text{shim}_2(Frob_v)|_{\GL_{q_v}}$ and $\gamma_2 = i_{a(v)}(\sin(\nu_{g,v})) \cdot (Frob_v)$. We have ([Kot94a, (2.1.1)])

$$\Tr \pi_v^\natural(f) = q_v^{(n+1)/4} \cdot \Tr (\sin(\nu_{g,v})) \cdot (Frob_v).$$

By Proposition 8.2 we get $\Tr (\gamma_1^j) = \Tr (\gamma_2^j)$ for $j$ sufficiently large. Consequently $\gamma_1$ and $\gamma_2$ are $\GL_{q_v}$-St.-conjugate.

Proof of Proposition 8.2. We imitate arguments from [Kot92a]. We consider a function $f$ on $G(\mathbb{A}_F)$ of the form $f = f_\infty \otimes f_{p,v} \otimes f_{\infty,v}$, where the components are chosen in the following way:

- We let $f_\infty$ be $N_{\infty}^{-1}$ times an Euler-Poincaré function for $\xi$ (and $K_\infty$) on $G(F_\infty)$ so that

$$\Tr f_\infty(f_\infty) = N_{\infty}^{-1} \cdot \ep(\tau_\infty \otimes \xi) = N_{\infty}^{-1} \cdot \sum_{i=0}^{\infty} (-1)^i \dim H^i(g,K_\infty,\tau_\infty \otimes \xi).$$

- We pick a Hecke operator $f_{\infty,v} \in \mathcal{H}(G(\mathbb{A}_F)|K_{\infty})$ such that for all automorphic representations $\tau$ of $G(\mathbb{A}_F)$ with $\tau_{\infty,K} \neq 0$ and $\Tr f_\infty(f_\infty) \neq 0$ we have

$$\Tr f_{\infty,v}(f_{\infty,v}) = \begin{cases} 1 & \text{if } \tau_{\infty,v} \cong \tau_{\infty,v}^\natural \\ 0 & \text{otherwise} \end{cases}$$

This is possible since there are only finitely many such $\tau$ (one of which is $\tau_\infty$).

- We let $f_{p,v}$ be a Lefschetz function from Equation (A.4) of Appendix B. There exists a finite set of primes $\Sigma$ such that $f_\infty$ decomposes $f_\infty = f_\Sigma \otimes f_{\infty,\Sigma}$ with $f_\Sigma \in \mathcal{H}(G(F_\Sigma)/K_\Sigma)$, $f_{\infty,\Sigma} = 1_{K_\Sigma} \in \mathcal{H}(G(\mathbb{A}_F)|\Sigma)$, and $K_{\Sigma}$ a product of hyperspecial subgroups. In the rest of the proof we consider only finite places $v$ such that $v \notin \Sigma$, $v \not\mid \ell$.

The stabilized Langlands-Kottwitz formula (Theorem 7.3) simplifies as

$$\iota^{-1} \Tr (if_{\infty} \cdot \Frob_v, H_v(S_K,\ell,\xi)) = \ST_{\text{ell},\xi}(h^G).$$

for the same reason as in the proof of Lemma 6.2 due to the special component at the place $v_{\infty}$ of the function $f$. Indeed $f_{p,v}$ is a stabilizing function (Lemma A.7), thus the stable orbital integral of $h^G$ vanishes unless $H = G^\vee$. Note that $h_{\text{ell}}^G$ can be chosen to be a Lefschetz function thanks to Lemma A.4.

To interpret the above formula on the group $G$, we recall from Lemmas 6.1 and 6.2 that

$$\tau_{\text{exp},\lambda}^G = \ST_{\text{ell},\lambda}(f^G), \quad f = \prod_{p} f_p \in \mathcal{H}(G(\mathbb{A}_F),\chi^{-1}),$$

where $f^* \in \mathcal{H}(G(\mathbb{A}_F),\chi^{-1})$ denote a transfer of $f$.

In the following we choose $f$ in such a way that $f^G$ and $h^G$ have the same stable orbital integrals.

Away from the places $\{v|p,\infty\}$, the functions $f_{\Sigma,\infty,p}$ and $h_{\Sigma,\infty,p}$ are both obtained from the endoscopic transfer of $f_{\infty,p}$ on $G(\mathbb{A}_F)|G^\vee(\mathbb{A}_F)$ to $G^\vee(\mathbb{A}_F)$. Recall that the only finite place where $G$ is possibly not quasi-split is $v_{\infty}$, thus we can in fact take $f_{\infty,p,v_{\infty}} = f_{\infty,p,v_{\infty}} = h_{\infty,p,v_{\infty}}$. At the place $v_{\infty}$ the functions $f_{\infty}$ and $h_{\infty}$ can be taken to be a Lefschetz function (Equation (A.4)). Consider the infinite places of $F$, where $f_{p,v}$ and $h_{p,v}$ are $N_{\infty}^{-1}$ times an Euler-Poincaré functions associated to $\xi$ on the inner forms $G(F_{\infty})$ and $G^\vee(F_{\infty})$, respectively. An explicit computation of orbital integrals shows that $f_{p,v}$ and $h_{p,v}$ are associated, cf. [Kot90, p.182] and [Kot92a, Lem. 3.1]. At the primes dividing $p$, the function $h_{p,v} \in \mathcal{H}(G(F_p))$ is obtained from $\phi_{\Sigma}$ via the Frobenius twisted endoscopic transfer from $\phi_{\Sigma}$ down to $G^\vee(F_p)$. Note that $G$ is quasi-split at $p$, and thus $G^\vee(F_p) = G(F_p)$. The twisted endoscopic transfer $\mathcal{H}_{\text{uni}}(G(F_p)) \rightarrow \mathcal{H}_{\text{uni}}(G(F_p))$ is (for the principal endoscopic group) equal to the unramified base change (use the explicit formula in [Kot90, p.179] that defines the twisted transfer). Therefore we may (and do) take $h_{p,v} = \phi_{\Sigma}$.

Consequently, $h_{p,v}$ and $f_{\Sigma}$ have the same orbital integrals. By (8.6) and (8.7), we have

$$\iota^{-1} \Tr (if_{\infty} \cdot \Frob_v, H_v(S_K,\ell,\xi)) = \tau_{\text{exp},\lambda}^G(f) = \sum_{\tau \in \text{Aut}_{\lambda,\wedge}(G)} m(\tau) \Tr \tau f \cdot \Tr \tau v f.$$
unramified twists of each other. In particular \( \tau_v \cong \pi_v^n \) so \( \text{Tr} \tau_v(f_j) = \text{Tr} \pi_v^n(f_j) \). Applying (8.4) and (8.5), we identify the right hand side of (8.8) with

\[
N^{-1} \sum_{\tau \in A(\mathbb{N}^2)} m(\tau) \text{ep}(\tau^{-1} \otimes \xi) \text{Tr} \pi^n(f_j) = (-1)^{n(n+1)/2} a(\pi^n) \text{Tr} \pi^n(f_j).
\]

By the choice of our function \( f \) the left hand side of (8.8) is equal to the trace of \( \text{Frob}_v^j \) on \( (-1)^{n(n+1)/2} \rho^{\text{shim}}_2 \).

To sum up,

\[
\text{Tr} (\text{Frob}_v^j, \rho_2^{\text{shim}}) = a(\pi^n) \text{Tr} \pi^n(f_j).
\]

It remains to show that \( \rho^{\text{shim}}_2 \) is not just a virtual representation but a true representation. To this end we will show that \( \rho^{\text{shim}}_2 \) is concentrated in the middle degree \( n(n+1)/2 \). There are natural maps from \( H^i(S, \mathcal{L}_\xi) \) to each of \( \text{IH}^i(S, \mathcal{L}_\xi) \) and \( H^i_2(S, \mathcal{L}_\xi) \). Compatibility with Hecke correspondences implies that both maps are \( G(A_\mathbb{F}^{\infty}) \)-equivariant; the first map is moreover equivariant for the action of \( \Gamma \) (which commutes with the \( G(A_\mathbb{F}^{\infty}) \)-action). By Zucker’s conjecture, which is proved in the articles \([\text{Loo88}, \text{LR91}, \text{SS90}]\), the \( L^2 \)-cohomology \( H^i_2(S, \mathcal{L}_\xi) \) is naturally isomorphic to the intersection cohomology \( \text{IH}^i(S, \mathcal{L}_\xi) \) so that we have a \( G(A_\mathbb{F}^{\infty}) \)-equivariant commutative diagram:

\[
\begin{array}{ccc}
H^i_1(S, \mathcal{L}_\xi) & \longrightarrow & \text{IH}^i(S, \mathcal{L}_\xi) \\
\downarrow & & \downarrow \\
H^i_2(S, \mathcal{L}_\xi)
\end{array}
\]

The diagram together with Lemma 8.1 yields a \( \Gamma \)-equivariant isomorphism

\[
H^i_1(S, \mathcal{L}_\xi)[\tau^{\infty}] \cong \text{IH}^i(S, \mathcal{L}_\xi)[\tau^{\infty}].
\]

In view of condition (cent), the intersection complex defined by \( \xi \) is pure of weight \( -w \). Hence the action of \( \text{Frob}_v \) on \( \text{IH}^i(S, \mathcal{L}_\xi) \) is pure of weight \( -w + i \), cf. the proof of [Mor10, Rem. 7.2.5]. On the other hand, part (2) of Lemma 8.1 (with \( v = y \)) implies that \( \tau_v|_{\text{sim}}|^{w/2} = n_v^n|_{\text{sim}}|^{w/2} \) is tempered and unitary. Combining with (8.10) and (8.3), we conclude that \( \text{IH}^i(S, \mathcal{L}_\xi)[\tau^{\infty}] = 0 \) unless \( i = n(n+1)/2 \). The proof of the proposition is complete.

Remark 8.4. The last paragraph in the above proof simplifies significantly if \( F \neq \mathbb{Q} \), in which case \( S_F \) is proper (thus the compact support cohomology coincides with the intersection cohomology). We could have taken a slightly different path to prove our main results on Galois representations by first proving everything when \( F = \mathbb{Q} \) and then deal with the case \( F = \mathbb{Q} \) by the patching argument of §4 below (thus avoiding Zucker’s conjecture and arithmetic compactifications of \( S_F \)).

Corollary 8.5. If \( \tau \in A(\mathbb{N}^2) \) then \( \tau^{\infty} \) belongs to the discrete series \( L \)-packet \( \Pi^G_{\tau}(F_\infty) \).

Proof. The preceding proof shows that \( H^i_2(S_K, \mathcal{L}_\xi)[\tau^{\infty}] \) is nonzero only for \( i = n(n+1)/2 \). Hence \( H^i(\text{Lie } G(F_\infty), K'_\infty; \tau^{\infty}_K \otimes \xi) \) does not vanish exactly for \( i = n(n+1)/2 \). Since \( \tau^{\infty}_K \) is \( \xi \)-cohomological and unitary, we see from [SR99, Thm. 1.8] that \( \tau^{\infty}_K \) must appear in the Vogan–Zuckerman classification given in [VZ84], where the authors compute the Lie algebra cohomology explicitly. It follows from this that \( \tau^{\infty}_K \) is a discrete series representation, thus \( \tau \in \Pi^G_{\tau}(F_\infty) \).
9. Removing the unwanted multiplicity

Let \( \pi \) be a cuspidal \( \xi \)-cohomological automorphic representation of \( \text{GSp}_{2n}(\mathbb{A}_F) \) satisfying (St) and (spin-reg). We constructed in Section 7 a certain Shimura datum \((G,X)\) depending on the place \( v_F \). Let \( \pi^k \) be the transfer of \( \pi \) to the inner form \( G(\mathbb{A}_F) \) (Proposition 6.3). Using \( \pi^k \) we constructed in Section 8 a Galois representation \( \rho^\text{shim}_2 \) of dimension \( a(\pi^k) \cdot 2^n \). We want to get rid of the unwanted multiplicity \( a(\pi^k) \). The traces in Proposition 8.2 suggest, but it is a priori not clear, that \( \rho^\text{shim}_2 \) is the \( a(\pi^k) \)-th power of an irreducible representation \( \rho_2 \). The goal of this section is to prove that this is actually the case. The main ingredient to remove the unwanted multiplicity is a lemma originating in a letter to Taylor to Clozel, which is recalled in the proof below and applicable only when \( \xi \) satisfies the spin-regularity condition (cf. Definition 1.3).

**Lemma 9.1.** Let \( \rho^\text{shim}_2 \) be the representation defined in Equation (8.1) and Proposition 8.2.

(i) There exists a semisimple \( \ell \)-adic representation \( \rho_2 : \Gamma \to \text{GL}_{2n}(\mathbb{Q}_\ell) \), unique up to isomorphism, such that \( i_a(\pi^k) \circ \rho_2 \cong (\rho^\text{shim}_2)^{n_a} \).

(ii) \( \rho_2(Frob_v)_{ss} \) is conjugate to \( q_v^{(n+1)/4} \) spin-regular in \( \text{GL}_{2^n}(\mathbb{Q}_\ell) \) for almost all finite \( F \)-places \( v \) where \( \rho_2 \) is unramified.

**Proof.** Write \( a := a(\pi^k) \). Corollary 8.3 implies (ii) as soon as there exists a \( \rho_2 \) as in (i). To verify (i) it suffices to check the following two conditions, cf. [HT01, Lem. 1.2.2]: (a) For almost all finite \( F \)-places \( v \) where \( \rho^\text{shim}_2 \) is unramified, every eigenvalue of \( \rho^\text{shim}_2(Frob_v) \) has multiplicity at least \( a \), (b) the \( \mathbb{C}_\ell \)-span of the Lie algebra of \( \rho^\text{shim}_2(\Gamma) \) contains an element which has \( 2^n \) distinct eigenvalues each with multiplicity \( a \).

Corollary 8.3 again implies condition (a). By [Sen73, Thm. 1], as in the proof of [HT01, Prop. VII.1.8], checking condition (b) boils down to showing that \( \rho^\text{shim}_2 \), with respect to some embedding \( j : F \hookrightarrow \mathbb{Q}_\ell \), has \( 2^n \) distinct Hodge-Tate numbers each with multiplicity \( a \). Recall the \( 2^n \)-dimensional representation \( \rho_1 \) from §3, cf. the diagram below. It is easy to see that \( \rho_1 \) has \( 2^n \) distinct Hodge-Tate numbers with respect to \( j = i \circ v_{\infty} \), where \( v_{\infty} \) is the infinite place where \( \phi_{v_{\infty}} \) is spin-regular. Indeed, in view of Diagram (9.1), it is enough to show that \( \text{spin} \circ \rho_{a|_E} \) has regular Hodge-Tate cocharacter with respect to \( i \circ v_{\infty} \). This follows from the spin-regularity condition via part (iii) of Theorem 2.4.

We claim that there exists a finite extension \( E/F \) such that \( (\rho^\text{shim}_2)_{ss}|_{E} \) and \( (i_a \circ \rho_1)_{ss}|_{E} \) are isomorphic up to a character twist. The claim implies the desired statement on the Hodge-Tate numbers of \( \rho^\text{shim}_2 \), finishing the proof of the lemma.

Let us prove the claim. We have a commutative diagram

\[
\begin{array}{cccc}
\Gamma & \xrightarrow{i_a \circ \rho_1} & \text{GSpin}_{2n+1}(\mathbb{Q}_\ell) & \text{spin} & \text{GL}_{2n}(\mathbb{Q}_\ell) & \text{PGL}_{2^n}(\mathbb{Q}_\ell) \\
\rho_1 & \xrightarrow{i_a \circ \rho_1} & \text{SO}_{2n+1}(\mathbb{Q}_\ell) & \text{spin} & \text{PGL}_{2^n}(\mathbb{Q}_\ell) & \text{PGL}_{2^n}(\mathbb{Q}_\ell) \\
\rho_2 & \xrightarrow{i_a \circ \rho_1} & \text{SO}_{2n+1}(\mathbb{Q}_\ell) & \text{spin} & \text{PGL}_{2^n}(\mathbb{Q}_\ell) & \text{PGL}_{2^n}(\mathbb{Q}_\ell) \\
\end{array}
\]

By Diagram (9.1), the projective representation \( i_a \circ \rho_1 \) is also equal to

\[
i_a \circ \rho_1 \Rightarrow \text{SO}_{2n+1}(\mathbb{Q}_\ell) \xrightarrow{i_a \circ \text{spin}} \text{PGL}_{2^n}(\mathbb{Q}_\ell).
\]

By Theorem 2.4 we have for almost all finite \( F \)-places,

\[
\rho_1(Frob_v)_{ss} \sim i_a(\phi_{\pi^k})_{ss}(Frob_v) \in \text{SO}_{2n+1}(\mathbb{Q}_\ell).
\]

In (9.3), and in Formulas (9.4)–(9.7) below, we mean with ~ conjugacy with respect to the group indicated on the right hand side of the formula. By (9.3)

\[
i_a(\text{spin}(\rho_1(Frob_v))_{ss}) \sim i_a(\text{spin}(i_a(\phi_{\pi^k}))_{ss}(Frob_v)) \in \text{PGL}_{2^n}(\mathbb{Q}_\ell)
\]

as well. By combining Formula (9.4) with Formula (9.2) we get

\[
i_a \circ \rho_1(Frob_v)_{ss} \sim i_a(\phi_{\pi^k}(Frob_v)) \in \text{PGL}_{2^n}(\mathbb{Q}_\ell).
\]

Restricting representations \( \pi^k \subset \pi_\xi \) is known to correspond, in the unramified case, to composition with the natural surjection of dual groups \( \text{GSpin}_{2n+1}(\mathbb{Q}_\ell) \to \text{SO}_{2n+1}(\mathbb{Q}_\ell) \) (see, e.g. Bin Xu [Xua, Lem. 5.2]). Thus

\[
i_a(\text{spin}(i_a(\phi_{\pi^k})))_{ss}(Frob_v) \sim i_a(\text{spin}(i_a(\phi_{\pi^k}))_{ss}(Frob_v)) \in \text{PGL}_{2^n}(\mathbb{Q}_\ell).
\]
The last equation holds true for almost all finite $F$-places $v$, possibly further excluding finitely many places. Write $\rho_2^{\text{shim}}$ for the composition of $\rho_2^{\text{shim}}$ with $\text{GL}_{2n}(\overline{Q}_\ell) \to \text{PGL}_{2n}(\overline{Q}_\ell)$. Combining Formula (9.5), (9.6) and Corollary 8.3, we get

$$\text{Frob}_v \circ \rho_1(\text{Frob}_v)_{ss} \sim \text{Frob}_v(\text{spin}(\nu_{\text{Frob}_v})) = \rho_2^{\text{shim}}(\text{Frob}_v)_{ss} \in \text{PGL}_{2n}(\overline{Q}_\ell),$$

for almost all finite $F$-places $v$. We apply Lemma 4.6 to complete the proof of the claim. \qed

10. Galois representations with values in the $\text{GSpin}$ group

Let $\pi$ be a cuspidal automorphic representation of $\text{GSp}_{2n}(\mathbb{A}_F)$ satisfying the same conditions as at the start of §9. Denote by $\omega_\pi$ its central character. We showed in Lemma 9.1 that there exists a $2^n$-dimensional Galois representation $\rho_2: \Gamma \to \text{GL}_{2n}(\overline{Q}_\ell)$. In this section we combine the results from Sections 2–9 to show that the representation $\rho_2$ factors through the spin representation $\text{GSpin}_{2n+1}(\overline{Q}_\ell) \to \text{GL}_{2n}(\overline{Q}_\ell)$.

Denote by $S_{\text{bad}}$ the finite set of rational primes $p$ such that either $p = 2$ or $p$ is ramified in $F$ or $\pi_v$ is ramified at a place $v$ of $F$ above $p$. (As we commented in introduction, it shouldn’t be necessary to include $p = 2$ in view of [KMP].) In the theorem below the superscript $C$ designates the $C$-normalization in the sense of [BGIL]. Statement (ii) of the theorem will be upgraded in the next section to include all $v$ which are not above $S_{\text{bad}} \cup \{\ell\}$. In the following section (iii.c) of Theorem A will be shown, which will then complete the proof of Theorem A after switching back to the $L$-normalization.

**Theorem 10.1.** Let $\pi$ be a cuspidal $\xi$-cohomological automorphic representation of $\text{GSp}_{2n}(\mathbb{A}_F)$ satisfying the conditions (St) and (spin-reg). Let $\ell$ be a prime number and $i: C \to \overline{Q}_\ell$ a field isomorphism. Then exists a continuous representation $\rho^C_\pi = \rho^C_\pi_{ss}: \Gamma \to \text{GSpin}_{2n+1}(\overline{Q}_\ell)$, unique up to conjugation by $\text{GSpin}_{2n+1}(\overline{Q}_\ell)$, satisfying (ii.b), (ii). (a), (ii) of Theorem A with $\rho^C_\pi$ in place of $\rho_\pi$, as well as the following (which are slightly modified from (i), (ii), (iii.a) of Theorem A due to a different normalization).

(i) For any automorphic $\text{Sp}_{2n}(\mathbb{A}_F)$-subrepresentation $\pi^\flat$ of $\pi$, its associated Galois representation $\rho^\flat_\pi$ is isomorphic to $\rho^C_\pi$, composed with the projection $\text{GSpin}_{2n+1}(\overline{Q}_\ell) \to \text{SO}_{2n+1}(\overline{Q}_\ell)$. The composition $[N \circ \rho^C_\pi]: \Gamma \to \text{GSpin}_{2n+1}(\overline{Q}_\ell) \to \text{GL}_n(\overline{Q}_\ell)$ corresponds to $\omega(\pi)$, $\xi$, $-n^{[n]+1}/2$ by global class field theory.

(ii) There exists a finite set of prime numbers $S$ with $S \supset S_{\text{bad}}$ such that for all finite $F$-places $v$ which are not above $S \cup \{\ell\}$, $\rho^C_{\pi,v}$ is unramified and the element $\rho^C_\pi(\text{Frob}_v)_{ss}$ is conjugate to the Satake parameter of $\pi_{\nu,\ell}$ by a $\sim n^{[n]+1}/4$ in $\text{GSpin}_{2n+1}(\overline{Q}_\ell)$.

(iii.a) $\mu_{\text{HT}}(\rho_{\pi,v}^C, \xi, y) = \frac{1}{2} \mu_{\text{HT}}(\xi, y) + \frac{1}{2} \text{sim}$. (See Definitions 1.6 and 1.9.)

**Remark 10.2.** Property (vi) fails for $\text{GSpin}_{2n+1}$-valued Galois representations in general. Statement (iii.c) from Theorem A will be established in Section 12, and statement (iii) will be strengthened to Thm. A.(ii) in Section 11.

**Proof.** We have the automorphic representation $\pi$ of $\text{GSp}_{2n}(\mathbb{A}_F)$. Consider

- $\pi^\flat \subset \pi$ a cuspidal automorphic $\text{Sp}_{2n}(\mathbb{A}_F)$-subrepresentation from Lemma 2.6;
- $\pi^n$ the transfer of $\pi$ to the group $G(\mathbb{A}_F)$ from Proposition 6.3;
- $\rho_1: \Gamma \to \text{SO}_{2n+1}(\overline{Q}_\ell)$ the Galois representation from Theorem 2.4;
- $\rho_1^{\text{ss}}$, a lift of $\rho_1^{\text{ss}}$ to the group $\text{GSpin}_{2n+1}(\overline{Q}_\ell)$ from Proposition 3.5;
- $\rho_1: \Gamma \to \text{GL}_{2n}(\overline{Q}_\ell)$ the composition of $\rho_1^{\text{ss}}$ with the spin representation.
- $\rho_2^{\text{shim}}: \Gamma \to \text{GSpin}_{2n+1}(\overline{Q}_\ell)$ the representation introduced in (8.1) (see also Proposition 8.2);
- $\rho_2: \Gamma \to \text{GL}_{2n}(\overline{Q}_\ell)$ the Galois representation so that $\mu_\pi(\rho_2) = \rho_2^{\text{shim}}$, from Lemma 9.1; (see also Figure 1 in the introduction). We have

For almost all finite $F$-places $v$ the semisimple parts $\rho_1(\text{Frob}_v)_{ss}$ and $\rho_2(\text{Frob}_v)_{ss}$ are conjugate in the group $\text{PGL}_{2n}(\overline{Q}_\ell)$ (Lemma 9.1 and Proposition 8.2);

- The representation $\rho_1$ is strongly irreducible by Proposition 3.8.

By Proposition 4.4 there exists a character $\chi: \Gamma \to \overline{Q}_\ell$ such that $\rho_2$ is conjugate to $\rho_1 \otimes \chi$. By construction the representation $\rho_1$ has image inside $\text{GSpin}_{2n+1}(\overline{Q}_\ell)$, and consequently the representation $\rho_2$ has image in $\text{GSpin}_{2n+1}(\overline{Q}_\ell)$ as well. Thus $\rho_2$ induces a representation $\rho_2^C: \Gamma \to \text{GSpin}_{2n+1}(\overline{Q}_\ell)$ such that $\rho_2 = \text{spin} \circ \rho_2^C$.

**Proposition 10.3** (Steinberg). *In a semisimple algebraic group $H$ over an algebraically closed field of characteristic $0$, two semisimple elements are conjugate if and only if they are conjugate in all linear representations of $H$.*

**Proof.** Theorem 3 of Steinberg’s article [Ste78]. \qed
Remark 10.4. As a further precision to Proposition 10.3, Steinberg remarks at the end of [Ste78] that, for simply connected groups, it suffices to check conjugacy in the fundamental representations. If furthermore the group is of type B or D it suffices to check conjugacy in the spin representation [for simply connected types A and C it suffices to check conjugacy in the standard representation, cf. Proposition B.1].

Corollary 10.5. Two semisimple elements \( x_1, x_2 \in G\text{Spin}_{2n+1}(\overline{Q}_\ell) \) are conjugate if and only if \( \text{spin}(x_1) \) is conjugate to \( \text{spin}(x_2) \) in \( G\text{L}_{2n}(\overline{Q}_\ell) \) and \( N(x_1) = N(x_2) \).

Proof. Since the only if part is obvious, it is enough to verify the if part. Write \( x_i = z_i g_i \) with \( g_i \in \text{Spin}_{2n+1}(\overline{Q}_\ell) \) and \( z_i \in \overline{Q}_\ell^\times \), for \( i = 1, 2 \). Assume \( \text{spin}(x_1) \) and \( \text{spin}(x_2) \) are conjugate in \( G\text{L}_{2n}(\overline{Q}_\ell) \). Write \( H \) for \( G\text{O}_{2n} \) or \( G\text{Sp}_{2n} \) according as \( n \) is 0,3 or 1,2 modulo 4. Write \( \text{spin}: H(\overline{Q}_\ell) \to \overline{Q}_\ell^\times \) for the factor of similitude. Lemma 0.1 tells us that the spin representation has image inside \( H(\overline{Q}_\ell) \) and that \( N(x_1) = N(x_2) \) implies \( \text{spin}(x_1) = \text{spin}(x_2) \). By Proposition B.1 the elements \( \text{spin}(x_1), \text{spin}(x_2) \in H(\overline{Q}_\ell) \) are \( H(\overline{Q}_\ell) \)-conjugate. This expands to \( (z_1)^2 \equiv \text{spin}(x_1) \equiv \text{spin}(x_2) \equiv (z_2)^2 \), and therefore \( z_1 = \pm z_2 \in \overline{Q}_\ell^\times \). Since we can, if necessary, replace \( g_2 \) by \(-g_2 \in H(\overline{Q}_\ell) \), we may assume \( z_1 = z_2 = z \). Then \( \text{spin}(g_1) = z^{-1} \text{spin}(x_1) \) is conjugate to \( \text{spin}(g_2) = z^{-1} \text{spin}(x_2) \). By (the remark below Proposition 10.3, \( g_1 \) and \( g_2 \) are conjugate. It follows that \( x_1 \) is conjugate to \( x_2 \) as well. \( \square \)

We have the semisimple elements \( \rho_n^C(\text{Frob}_v)_{hs} \) and \( i\phi_n(\text{Frob}_v) \) in \( G\text{Spin}_{2n+1}(\overline{Q}_\ell) \). We know that \( \text{spin}(\rho_n^C(\text{Frob}_v)_{hs}) \) and \( \rho_n(\text{Frob}_v)_{hs} \cdot \text{spin}(i\phi_n(\text{Frob}_v)) \) are conjugate in \( G\text{L}_{2n}(\overline{Q}_\ell) \) by Lemma 9.1. We claim that

\[
N(\text{spin}(\rho_n^C(\text{Frob}_v)_{hs})) = N(\rho_n(\text{Frob}_v)_{hs} \cdot \text{spin}(i\phi_n(\text{Frob}_v))).
\]

Once the claim is verified, by Corollary 10.5, \( \rho_n^C(\text{Frob}_v)_{hs} \) is conjugate to \( \rho_n(\text{Frob}_v)_{hs} \cdot \text{spin}(i\phi_n(\text{Frob}_v)) \) in \( G\text{Spin}_{2n+1}(\overline{Q}_\ell) \). This proves statement (ii)'.

We prove the claim. Possibly after conjugation by an element of \( G\text{L}_{2n}(\overline{Q}_\ell) \), the image of \( \text{spin} \rho_n \) lies in \( G\text{O}_{2n} \) (resp. \( G\text{Sp}_{2n} \)) if \( n(n+1)/2 \) is even (resp. odd), for some symmetric (resp. symplectic) pairing on the underlying \( 2n \)-dimensional space. Again by the same lemma, we may assume that \( \text{spin}(i\phi_n(\text{Frob}_v)) \) also belongs to \( G\text{O}_{2n} \) (resp. \( G\text{Sp}_{2n} \)). Hereafter we let the central characters \( \omega_n \) or \( \omega_n^{-1} \) also denote the corresponding Galois characters via class field theory. For almost all \( v \) we have the following isomorphisms.

\[
(\text{spin}(\rho_n^C)_{hs})^\vee \cong |n(n+1)/4| \cdot \text{spin}(\phi_n(\text{Frob}_v))^\vee \cong |n(n+1)/4| \cdot \text{spin}(\phi_n)^\vee \\
\cong |n(n+1)/4| \cdot \text{spin}(\phi_n) \otimes \omega_n^{-1} \cong |n(n+1)/2| \cdot \text{spin}(\phi_n) \otimes \omega_n^{-1},
\]

where \( \phi_n(\text{Frob}_v) := \phi_n(\text{Frob}_v) \otimes \omega_n \) by definition. With this definition, the second isomorphism is directly verified using the equality \( N(\phi_n(\text{Frob}_v)) = \omega_n \) (from functoriality of the Satake isomorphism with respect to the central embedding \( G_m \to G\text{Spin}_{2n} \)) and the isomorphism \( \text{spin}(\phi_n(\text{Frob}_v)) \to \text{spin}(\phi_n)^\vee \otimes \text{spin}(\phi_n)^{-1} \) induced from the pairing \( (\cdot, \cdot) \) that defines \( \text{GO}_{2n} \) (resp. \( \text{GSp}_{2n} \)). The above isomorphisms imply that

\[
\text{spin}(\rho_n^C(\text{Frob}_v)_{hs}) \cong \text{spin}(\rho_n^C)^\vee \otimes |n(n+1)/2| \omega_n.
\]

Since \( \text{spin}(\rho_n^C) \) is strongly irreducible, it follows via Lemma 4.2 that \( \text{spin}(\rho_n^C) \) with image in \( \text{GO}_{2n} \) (resp. \( \text{GSp}_{2n} \)) followed by the similitude character of \( \text{GO}_{2n} \) (resp. \( \text{GSp}_{2n} \)) is equal to \( |n(n+1)/2| \omega_n \). Thus, by Lemma 0.1,

\[
N(\text{spin}(\rho_n^C)) = |n(n+1)/2| \omega_n,
\]

proving the second part of (i'). Evaluating at unramified places, we obtain (10.1), finishing the proof of the claim.

We show statement (ii)'. It only remains to check the first part. By Theorem 2.4 and the preceding proof of (ii)', we have for almost all unramified places \( v \) that

\[
\text{i} \phi_n(\text{Frob}_v) \sim \text{i} \phi_n(\text{Frob}_v) \sim \rho_n(\text{Frob}_v)_{hs},
\]

where we also used that the Satake parameter of the restricted representation \( \pi_v^h \subset \pi_v \) is equal to the composition of the Satake parameter of \( \pi_v \) with the natural surjection \( G\text{Spin}_{2n+1}(\mathbb{C}) \to \text{SO}_{2n+1}(\mathbb{C}) \) (cf. [Xua, Lem. 5.2]). Hence \( \rho_n^C(\text{Frob}_v)_{hs} \sim \rho_n(\text{Frob}_v)_{hs} \) for almost all \( F \)-places \( v \). Consequently std \( \rho_n^C \sim \text{std} \rho_n \sim \rho_n^C \). By Proposition B.1 the representation \( \rho_n^C \) is \( \text{SO}_{2n+1}(\overline{Q}_\ell) \)-conjugate to the representation \( \rho_n^C \). This proves the first part of (ii)'.

We prove statement (v). The first part follows from (i) and Lemma 3.3. The second part is a consequence of Proposition 3.8 since \( \text{spin} \circ \rho_n^C \) is a character twist of \( \rho_1 \).
The Galois representation in the cohomology $H^1(S_K, L)$ is potentially semistable by Kisin [Kis02, Thm. 3.2]. The representation $\rho_{\ell}^{\text{C}}$ appears in this cohomology and is therefore potentially semistable as well. This proves the first assertion of statement (iii).

To verify (iii), let $\mu_{HT}(\rho_{\ell}^{\text{C}}): G_m \rightarrow \text{GSp}_{2n+1, \overline{Q}_\ell}$ be the Hodge-Tate cocharacter of $\rho_{\ell}^{\text{C}}$ (Definition I.6). Similarly, we let $\mu_{HT}(\rho_{\ell}^{\text{H}}): G_m \rightarrow \text{GO}_{2n+1, \overline{Q}_\ell}$ be the Hodge-Tate cocharacter of $\rho_{\ell}^{\text{H}}$. We need to check that $\mu_{HT}(\rho_{\ell}^{\text{C}}, y) = \mu_{\text{Hodge}}(\xi, y)$. In fact it is enough to check the equalities

\begin{align}
q \mu_{HT}(\rho_{\ell}^{\text{C}}, y) &= q \mu_{\text{Hodge}}(\xi, y), \\
N \circ \mu_{HT}(\rho_{\ell}^{\text{C}}, y) &= N \circ \mu_{\text{Hodge}}(\xi, y) + \frac{n(n+1)}{4},
\end{align}

namely after applying the natural surjection $q^*: \text{GSp}_{2n+1} \rightarrow \text{GO}_{2n+1} \times \text{GL}_1$ given by $(q, N)$, since $q^*$ induces an injection on the set of conjugacy classes of cocharacters. (The map $X_r(T_{\text{GSp}}) \rightarrow X_r(T_{\text{GO}})$ induced by $q^*$ is an isogeny since $T_{\text{GSp}} \rightarrow T_{\text{GO}}$ is an isogeny, so the map is still injective after taking quotients using the common Weyl group.) Let $\nu: F \rightarrow \mathbb{C}$ be an embedding, such that $\nu y$ induces the place $v$. It is easy to see that $q \mu_{HT}(\rho_{\ell}^{\text{C}}, y) = \mu_{HT}(\rho_{\ell}^{\text{H}}, y)$ and $q \mu_{\text{Hodge}}(\xi, y) = \mu_{\text{Hodge}}(\xi, y)$ from (1.1) and Lemmas 110 and 117. Thus (10.2) follows from Theorem 2.4 (iii). The proof of (10.3) is similar: we have the following equalities in $X_r(G_m) = \mathbb{Z}$.

\[ N \circ \mu_{HT}(\rho_{\ell}^{\text{C}}, y) = \mu_{HT}(N \circ \rho_{\ell}^{\text{C}}, y). \]

where the Hodge cocharacter of $\omega_{\ell}$ at $y$ is denoted by $\mu_{\text{Hodge}}(\omega_{\ell}, y)$. Indeed, the first, second, and third equalities follow from Lemma 17, part (i) of the current theorem (just proved above), and the fact that $N \circ \omega_{\ell} = \phi_{\omega_{\ell}}$. The last fact comes from the description of the central character of an $L$-packet; see condition (ii) in [Lan89, §3]. The third equality also uses the easy observation that $N \circ \nu = 2$ under the canonical identification $X_r(G_m) = \mathbb{Z}$.

We prove statement (iii.b). So we assume that $v | \ell$ and that $K_\ell$ is either hyperspecial or contains an Iwahori subgroup of $\text{GSp}_{2n}(F \otimes \mathbb{Q}_\ell)$. We use the following proposition of Conrad:

**Proposition 10.6** (Conrad). Consider a representation $\sigma: \Gamma_v \rightarrow \text{GSp}_{2n+1}(\mathbb{Q}_\ell)$ satisfying a basic $p$-adic Hodge theory property $P \in \{ \text{de Rham, crystalline, semistable} \}$. There exists a lift $\sigma': \Gamma_v \rightarrow \text{GSp}_{2n+1}(\mathbb{Q}_\ell)$ that satisfies $P$ if $\sigma$ admits a lift $\sigma'': \Gamma_v \rightarrow \text{GSp}_{2n+1}(\mathbb{Q}_\ell)$ which is Hodge-Tate.

**Proof.** (see also Winterberger [Win95]). Combine Proposition 6.5 with Corollary 6.7 of Conrad's article [Con13]. (Conrad has general statements for central extensions of algebraic groups of the form $[Z \rightarrow H' \rightarrow H]$; we specialized this case to our setting.)

Write $\omega_{\ell}' := \omega_{\ell}|^{-n(n+1)/2}$. Let $\text{rec}(\cdot)$ denote the $\ell$-adic Galois representation corresponding to a Hecke character of $\mathbb{A}_K^\times$ via global class field theory. The product rec$(\omega_{\ell}')\rho_{\ell}^{\text{C}}: \Gamma \rightarrow \text{GO}_{2n+1}(\mathbb{Q}_\ell)$ is the Galois representation corresponding to the automorphic representation $\omega_{\ell}' \otimes \rho_{\ell}^{\text{C}}$ of $\mathbb{A}_E^\times \times \text{Sp}_{2n}(\mathbb{A}_E^\times)$, cf. Theorem 2.4. According to Frobenius elements at the unramified places, and using Chebotarev and Brauer-Nesbitt, the representations std $\circ q' \circ \rho_{\ell}^{\text{C}}$ and std $\circ \text{rec}(\omega_{\ell}')\rho_{\ell}^{\text{C}}$ are isomorphic. Hence $q' \circ \rho_{\ell}^{\text{C}}$ and $\text{rec}(\omega_{\ell}')\rho_{\ell}^{\text{C}}$ are $\text{GO}_{2n+1}(\mathbb{Q}_\ell)$-conjugate by Proposition B.1. We make two observations:

- We saw that $\rho_{\ell}^{\text{C}}$ is potentially semistable. In particular $\text{rec}(\omega_{\ell}')\rho_{\ell}^{\text{C}}$ has a lift which is a Hodge-Tate representation (namely $\rho_{\ell}^{\text{C}}$ is such a lift).
- Under the assumption of (iii.b), $\rho_{\ell}^{\text{C}}$ is known to be crystalline (resp. semistable) by (iv) and (v) of Theorem 2.4. We have $\sim(K_\ell) = (\mathcal{O}_F \otimes \mathbb{Z}_p)^{\times}$ if $K_\ell$ contains an Iwahori subgroup and thus the character $\text{rec}(\omega_{\ell}')$ is crystalline at all $v | \ell$. Therefore $\text{rec}(\omega_{\ell}')\rho_{\ell}^{\text{C}}$ is crystalline (resp. semistable) as well.

By Proposition 10.6, the local representation $\text{rec}(\omega_{\ell}')\rho_{\ell}^{\text{C}}|_{\mathfrak{p}_v}$ has a lift $r_v := \text{rec}(\omega_{\ell}')\rho_{\ell}^{\text{C}}|_{\mathfrak{p}_v}: \Gamma_v \rightarrow \text{GSpin}_{2n+1}(\mathbb{Q}_\ell)$ which is crystalline/semistable. Since $r_v$ and $\rho_{\ell}^{\text{C}}|_{\mathfrak{p}_v}$ are both lifts of $\text{rec}(\omega_{\ell}')\rho_{\ell}^{\text{C}}|_{\mathfrak{p}_v}$, the representations $r_v$ and $\rho_{\ell}^{\text{C}}|_{\mathfrak{p}_v}$ differ by a quadratic character $\chi_v$. Hence statement (iii.b) follows.

We prove statement (vi). Let $\rho': \Gamma \rightarrow \text{GSpin}_{2n+1}(\mathbb{Q}_\ell)$ be such that for almost all $\mathcal{F}$-places $v$ where $\rho'$ and $\rho_{\ell}^{\text{C}}$ are unramified, $\rho'(\text{Frob}_v)_{\mathfrak{p}_v}$ is conjugate to $\rho_{\ell}^{\text{C}}(\text{Frob}_v)_{\mathfrak{p}_v}$. Since $\text{sp} \circ \rho_{\ell}^{\text{C}}$ is conjugate to $\rho_1$, it is strongly irreducible by Proposition 3.8. Proposition 3.4 tells us that $\rho_{\ell}^{\text{C}}$ is $\text{GSpin}_{2n+1}(\mathbb{Q}_\ell)$-conjugate to $\rho'$. Observe that statement (vi) combined with (ii) implies the uniqueness of $\rho_{\ell}^{\text{C}}$ up to conjugacy by $\text{GSpin}_{2n+1}(\mathbb{Q}_\ell)$. Statement (iv) reduces to checking that $\rho_{\ell}^{\text{C}}: \Gamma \rightarrow \text{GO}_{2n+1}(\mathbb{Q}_\ell)$ is totally odd since the covering map $\text{GSpin}_{2n+1} \rightarrow \text{GO}_{2n+1}$ induces an isomorphism on the level of Lie algebras and the adjoint action of $\text{GSpin}_{2n+1}$.
factors through that of $\GO_{2n+1}$. We already know from Theorem 2.4.[vi] that $\rho_{\eta_x}$ is totally odd, so the proof is complete.

II. Compatibility at unramified places

Let $\pi$ be an automorphic representation of $\GSp_{2n}(\A_F)$, satisfying the same conditions as in Section 9. In this section we identify the representation $\rho_{\eta,\nu}$ from Theorem 10.1 at all places $v$ not above $S_{\text{bad}} \cup \{\ell\}$.

**Proposition 11.1.** Let $v$ be a finite $F$-place such that $p := v|_Q$ does not lie in $S_{\text{bad}} \cup \{\ell\}$. Then $\rho_{\eta,\nu}^v$ is unramified at $v$ and $\rho_{\eta,\nu}^v(\text{Frob}_v)$ is conjugate to the Satake parameter of $i\phi_{\eta,\nu}|_{\text{sim}}{\sim}^{-n(n+1)/4}$.

**Proof.** Let $\pi^v$ be a transfer of $\pi$ to the inner form $G(\A_F)$ of $\GSp_{2n}(\A_F)$ (Proposition 6.3). Let $B(\pi^v)$ be the set of cuspidal automorphic representations $\tau$ of $G(\A_F)$ such that

- $\tau_{S_p}$ and $\pi_{S_p}$ are isomorphic up to a twist by an unramified character,
- $\tau_{\nu}$ and $\pi_{\nu}$ are isomorphic,
- $\tau_v$ is unramified,
- $\tau_{\infty}$ is $\xi$-cohomological.

To compare with the definition of $A(\pi^v)$, notice that the condition at $v$ is different. We define an equivalence relation $\approx$ on the set $B(\pi^v)$ by declaring that $\tau_1 \approx \tau_2$ if and only if $\tau_2 \in A(\tau_1)$ (hence, $\tau_1 \approx \tau_2$ if and only if $\tau_1,\nu \cong \tau_2,\nu$). Define a (true) representation of $\pi^v$ by $C_{\eta,\nu}(\text{Frob}_v)$ as in place of $A(\pi^v)$. We know from 10.1 that each $\rho_{\pi^v}(\tau)$ is strongly irreducible. Since $\rho_{\pi^v}(\tau)$ and $\rho_{\pi^v}(\tau)$ have the same Frobenius trace at almost all places for $\tau \in B(\pi^v)$, we deduce from Corollary 8.3 that $\rho_{\pi^v}(\tau) \cong \rho_{\pi^v}(\tau)$. Hence $\rho_{\pi^v} \cong C_{\eta,\nu}(\text{Frob}_v)$.

We adapt the argument of Proposition 8.2 to the slightly different setting here. Consider the function $f$ on $G(\A_F)$ of the form $f = f_{n,\nu} \otimes f_{S_p} \otimes 1_{S_p} \otimes f_{\infty,\nu}$, where $f_{n,\nu}$ and $f_{S_p}$ are as in the proof of that proposition, and $f_{\infty,\nu}$ is such that, for all automorphic representations $\tau$ of $G(\A_F)$ with $\tau_{\nu} \cong 0$ and $\text{Tr}_{\tau_{\nu}}(f_{\nu}) \cong 0$, we have

$$\text{Tr}_{\tau_{\nu}}(f_{\nu}) = \begin{cases} 1 & \text{if } \tau_{\nu} \cong 0 \\ 0 & \text{otherwise}. \end{cases}$$

Arguing as in the paragraph between (8.9) and (8.10), but with $B(\pi^v)$, $B(\pi^v)$, and $C_{\eta,\nu}$ in place of $A(\pi^v)$, $A(\pi^v)$, and $C_{\eta,\nu}$, we obtain

$$\text{Tr}(\text{Frob}_v, \rho_{\pi^v}) = \sum_{\tau \in B(\pi^v)} a(\tau) \text{Tr}(\tau_{\nu})(f') = \sum_{\tau \in B(\pi^v)} a(\tau) q_v^{-n(n+1)/4} \text{Tr}(\text{spin}(\phi_{\tau_{\nu}})(\text{Frob}_v)), $$

where the last equality comes from (8.3). Thus the proof boils down to the next lemma (Lemma II.2). Indeed the lemma and the last equality imply that $\text{Tr}_{\rho_{\pi^v}}(\text{Frob}_v)$ is equal to $q_v^{-n(n+1)/4} \text{Tr}(\text{spin}(\phi_{\tau_{\nu}})(\text{Frob}_v))$. As in the proof of Theorem 10.1, since $\rho_{\pi^v}(\tau) = \text{spin} \circ \phi_{\tau}$ by construction, this shows that the semisimple parts of $\rho_{\pi^v}(\text{Frob}_v)$ and $\phi_{\tau\nu}(\text{Frob}_v)$ are conjugate.

**Lemma 11.2.** With the above notation, if $\tau \in B(\pi^v)$ then $\tau_{\nu} \approx \eta_{\nu}^{\pm 1}$.

**Proof.** Let $\tau'$ and $\pi'$ denote transfers of $\tau$ and $\pi'$ from $G$ to $\GSp_{2n}$ via part (2) of Proposition 6.3; the assumption there is satisfied by Corollary 8.5 (in fact we can just take $\pi'$ to be $\pi$). In particular $\pi_{\nu}' \cong \tau_x$ and $\pi_{\nu}' \cong \pi_{\nu}'$ at all finite places $x$ where $\tau_x$ and $\pi_x$ are unramified. In particular this is true for $x = v$, so it suffices to show that $\tau_{\nu}' \approx \eta_{\nu}^{\pm 1}$.

By [Xua, Thm. 1.8] we see that $\tau'$ and $\pi'$ belong to global $L$-packets $\Pi_1 = \otimes_{\nu} \Pi_{1,\nu}$ and $\Pi_2 = \otimes_{\nu} \Pi_{2,\nu}$ (as constructed in that paper), respectively, such that $\Pi_1 = \Pi_2 \otimes \omega$ for a quadratic Hecke character $\omega : F^\times / A_F^\times \to \{\pm 1\}$ (which is lifted to a character of $\GSp_{2n}(\A_F)$ via the similitude character). Since each local $L$-packet has at most one unramified representation by [Xua, Prop. 4.4.(3)] we see that for almost all finite places $x$, we have $\tau_x' \cong \pi_x' \otimes \omega_x$. Hence $\pi_{\nu}' \cong \pi_{\nu}' \otimes \omega_{\nu}$. (Recall that $\tau_{\nu} \cong \eta_{\nu}^{\pm 1}$ by the initial assumption at almost all $x$.) This implies through (ii) of Theorem 10.1 that $\text{spin}(\rho_{\eta_x}) \cong \text{spin}(\rho_{\eta_x}) \otimes \omega$ (viewing $\omega$ as a Galois character via class.
field theory). Since $\text{spin}(\rho_n)$ is strongly irreducible by the same theorem, it follows that $\omega = 1$. Hence the unramified representations $\tau_r^+$ and $\pi_r^+$ belong to the same local $L$-packet, implying that $\tau_r^+ \simeq \pi_r^+$ as desired. □

12. Proof of crystallineness

In this section we prove that $\rho_n^C$ is crystalline at $\mathfrak{v}|\ell$ when $\ell \not\in S_{\text{bad}}$. (This was shown in Theorem 10.1 only up to an unspecified quadratic character.) The first step of the proof is to reduce the problem to an auxiliary datum $(H_\delta, Y_\delta)$ of Hodge type with $H_\delta^{\text{der}} \cong G_{\text{der}}$. In the second step we adapt some well-known arguments from the PEL case (e.g. [HT01, III.2], [TY07, p.477]) to the Hodge type case. Theorem A will be fully established at the end of this section.

Carayol constructed in [Car86a] the datum $(H_\delta, Y_\delta)$ in case $n = 1$. In our construction of $(H_\delta, Y_\delta)$ we imitate Carayol quite closely; see especially p.163 of [loc. cit.].

Let $D$ be the quaternion algebra over $F$, unique up to isomorphism, which exactly ramifies at all $v \in V_\infty \setminus \{v_\infty\}$ and no finite places (resp. at no finite places other than $v_\infty$) if $[F : \mathbb{Q}]$ is odd (resp. even). Let $\delta$ be a negative element of $\mathbb{Q}$ and fix a square root $\sqrt{\delta} \in \mathbb{C}$. Then $E_\delta := F(\sqrt{\delta})$ is a CM quadratic extension of $F$. Let $z \mapsto z^*$ denote the complex conjugation on $E_\delta$. Write $T_\delta$ for the torus $\text{Res}_{E_\delta/F} \mathbb{G}_m$. Inside $T_\delta$ we have the subtorus $U_\delta$ defined by the equation $z^2 = 1$. The center $Z$ of $G$ is the torus $\mathbb{G}_m.F$. Define $T := \text{Res}_{E/F} \mathbb{G}_m$. We define the amalgated product

$$G_\delta := (\text{Res}_{F/Q} G)^* T_\delta,$$

defined by taking the quotient of $(\text{Res}_{F/Q} G) \times T_\delta$ by the action of $T$ via the formula $z \cdot (g, t) = (g z^* t^{-1} z, t)$, for all $Q$-algebras $R$, where $z \in T(R), g \in \text{Res}_{F/Q} G(R)$, and $t \in T_\delta(R)$. We define the morphism $\text{sim}_3 : G_\delta \to T \times U_\delta$ by $(g, t) \mapsto (\text{sim}_3(g) t^\vee, t/f^\vee)$. Define $H_\delta$ to be the horizontal image under $\text{sim}_3$ of the subtorus $\mathbb{G}_m \times U_\delta \subset T \times U_\delta$. The kernel of $\text{sim}_3$ is identified with the derived subgroup of $\text{Res}_{F/Q} G$. Moreover the map $t \mapsto t/f^\vee$ from $T_\delta$ to $U_\delta$ is onto. Thus we have the following commutative diagram with exact rows.

$$
\begin{array}{ccc}
1 & \to & \text{Res}_{F/Q} G^{\text{der}} \\
& \downarrow & \downarrow \\
1 & \to & \text{Res}_{F/Q} G^{\text{der}}
\end{array}
$$

and $\text{Res}_{F/Q} G^{\text{der}} = (H_{\delta})^{\text{der}} \simeq (G_{\delta})^{\text{der}}$.

We define the hermitian symmetric domain for $H_\delta$. If $v \in V_\infty$ is an embedding $F \to \mathbb{R}$, we extend it to the embedding $v_\delta : E_\delta \to \mathbb{C}$, $a + b \sqrt{\delta} \mapsto v(a) + v(b) \sqrt{\delta}$. From these extensions $\{v_\delta \mid v \in V_\infty\}$, we obtain an isomorphism $T_\delta(\mathbb{R}) = E_\delta^{\times} \cong \prod_{v \in V_\infty} \mathbb{C}^{\times}$. We define the morphism $h_\delta : \mathbb{C}^\times \to T_\delta(\mathbb{R})$ by $h_\delta(z) = (1, z, \ldots, z) \in \prod_{v \in V_\infty} \mathbb{C}^{\times}$ (the first coordinate corresponds to the fixed place $v_\infty \in V_\infty$). Define the morphism $h_\delta : \mathbb{C}^\times \to G_{\delta}(\mathbb{R}) = G(\mathbb{R}) \times_{F_{\delta,R}} E_\delta^{\times}$. Then $z \in \mathbb{C}^{\times}$ is sent to $(h(z), h_{T_\delta}(z))$.

It is straightforward to see that $h_\delta$ has image in $H_{\delta,R}$. The resulting map $\mathbb{C}^\times \to G_{\delta}(\mathbb{R})$ is again denoted by $h_{\delta}$. Write $Y_\delta$ for the $H_{\delta}(\mathbb{R})$-conjugacy class of $h_{\delta}$. It is routine to check that $(H_\delta, Y_\delta)$ is a Shimura datum, which is of abelian type since the reductive group is of type C. This also follows from Lemma 12.2 below.

We would like to check that $(H_\delta, Y_\delta)$ is moreover of Hodge type. Put $D_\delta := D \otimes_F E_\delta$. Let $l \mapsto \mathfrak{t}$ be the involution of the second kind, which is the tensor product of the canonical involution on $D$ with the conjugation $c$ on $E_\delta$. Choose $\varepsilon \in D$ an invertible element with $\mathfrak{t} = \varepsilon$ (will be specified later). We define $l^* = \varepsilon^{-1} l \mathfrak{t}$, for $l \in D_\delta$. Then $l \mapsto l^*$ is an involution of the second kind on $D$. Let $W$ be the left $D_\delta$-module whose underlying space is $D_\delta^\vee$ on which $D_\delta$ acts via left multiplication. We define a pairing

$$(12.1) \quad \psi_{l^*} : W \times W \to \mathbb{Q}, \quad (v, w) \mapsto \text{Tr}_{E/F}(\sqrt{\delta} \cdot \text{Tr}_{D_\delta/E_\delta}(v^{l^*} \varepsilon w)),
$$

where $\text{Tr}_{D_\delta/E_\delta}$ is the reduced trace, and $v^{l^*}$ is the transpose of the vector $v^*$ obtained from $v$ by applying $c^{-1}$ on each coordinate. Then $\psi_{l^*}$ is a non-degenerate alternating form on $W$, and we have for all $l \in D_\delta$ the relation $\psi_{l^*}(l v, w) = \psi_{l^*}(v, l^* w)$.

**Lemma 12.1.** There is an inclusion $\psi : H_\delta \to \text{GSp}_Q(W, \psi_{l^*})$.

**Proof.** To construct the desired map we start by realizing $G$ as a concrete matrix group. We introduce an $F$-linear paring (where $^t$ means transpose),

$$\psi_l : D^{\infty} \times D^{\infty} \to F, \quad (v, w) \mapsto \text{Tr}_{D/F}(v^{l^*} \varepsilon w).$$

Define the group $G(D^{\infty}, \psi_l)$ over $F$ whose set of $R$-points over each $F$-algebra $R$ is given by

$$\{ g \in \text{GL}_D(D^{\infty} \otimes_F R) \mid \forall v, w \in (D \otimes_F R)^\infty, \exists \lambda(g) \in R^\times : \psi_l(g v, g w) = \lambda(g) \psi_l(v, w) \}.$$
The equality $\psi_t(gv,gw) = \lambda(g)\psi_t(v,w)$ is equivalent to $g'\circ g = \lambda(g)e$, or to $\overline{g} = \lambda(g)$. (Since $GL_n(D^\ell \otimes F) \cong GL_n(D^\ell \otimes F)$, the left multiplication of $g$ on $v$ and $w$ becomes the right multiplication if $g$ is viewed as an element of $GL_n(D \otimes F)$.) We claim that, as algebraic groups over $F$,

$$G \cong G(D^n,\psi_t).$$

Suppose that $F'$ is an extension field of $F$ such that $D \otimes F' \cong M_2(F')$. The canonical involution $l \mapsto \overline{l}$ on $D$ goes to the canonical involution $m \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} m \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on $M_2(F')$ up to conjugation. From this it is evident that $G(D^n,\psi_t)$ is isomorphic to $\text{GSp}_{2n}$ over $F'$. Taking $F' = F$ we see that $G(D^n,\psi_t)$ is a form of the split group $\text{GSp}_{2n}$ over $F$, which then should be an inner form (as $\text{GSp}_{2n}$ has no nontrivial outer form). If $v$ is a place of $F$ such that $D$ splits at $v$ then $G(D^n,\psi_t) \cong \text{GSp}_{2n}$ over $F_v$. If $v$ is an infinite place different from $v_\infty$ then $D$ is the Hamitlon quaternioin algebra over $\mathbb{R}$. In this case it is standard that $G(D^n,\psi_t)(F_v)$ is compact modulo center. (The subgroup on which $\lambda = 1$ is the compact from of $\text{Sp}_{2n}([\mathbb{R}])$.) This determines the local invariant classifying $G(D^n,\psi_t)$ as an inner form of $\text{GSp}_{2n}$ at all places but $v_\infty$. This suffices to determine the invariant of $G(D^n,\psi_t)$ at $v_\infty$ from (7.1). Since $\ker \{ F, \text{PSP}_{2n} \} = 1$ we conclude that $G \cong G(D^n,\psi_t)$ over $F$.

Henceforward we identify $G = G(D^n,\psi_t)$. Define

$$\psi^F_{\delta\varepsilon} : D^n \times D^n \rightarrow F, \quad (v,w) \mapsto \text{Tr}_{E/F}((\sqrt{\delta} \cdot \text{Tr}_{D^{\ell}/E}(v^T \cdot \varepsilon w)),

so that we have $\psi_{\delta\varepsilon} = \text{Tr}_{E/F} \circ \psi^F_{\delta\varepsilon}$, and for all $v, w \in D^n \subset D^n = W$,

$$\psi^F_{\delta\varepsilon}(v,w) = \text{Tr}_{E/F}((\sqrt{\delta} \cdot \psi_t(v,w)).$$

Thus we have inclusions of $F$-algebraic groups

$$G = G(D^n,\psi_t) \subset \text{GSp}_F(D^n,\psi_t) \subset \text{GSp}_F(D^n,\psi_{\delta\varepsilon}),$$

which we can extend to obtain a $\mathbb{Q}$-embedding

$$G_\delta = (\text{Res}_{F_\delta/F} G) \times_{\text{Sp}_F} \text{GSp}_F(D^n,\psi_{\delta\varepsilon}) \rightarrow \text{Res}_{E/F} \text{GSp}_E(D^n,\psi^E_{\delta\varepsilon}).$$

Let $\mu : \text{Res}_{F_\delta/F} \text{GSp}_F(D^n,\psi_{\delta\varepsilon}) \rightarrow \text{Res}_{E/F} \text{GSp}_E(D^n,\psi^E_{\delta\varepsilon})$ be the factor of similitudes. We claim that

$$\text{Image}[\mu \circ a] [h_t] \subset \text{GSp}_E(D^n,\psi^E_{\delta\varepsilon}).$$

Let $R$ be a $\mathbb{Q}$-algebra. Let $x \in H_\delta(R)$. Write $x = (g,t) \in G(F \otimes \mathbb{Q}) \times (E_F \otimes \mathbb{Q})^\times$. We compute $

\mu(a(g,t)) = \mu(g) = \mu(g) \mu(t).$

We see from the formula defining $\psi^E_{\delta\varepsilon}$ that $\mu(g) = \text{sim}(g)$ and $\mu(t) = \overline{t}$. The definition of $H_\delta(R)$ means that $\text{sim}(g) \overline{t} \in R^\times$. Hence (12.3) is true. In other words, $x \in H_\delta(R)$ preserves the pairing $\psi^E_{\delta\varepsilon}$, thus also $\psi_{\delta\varepsilon}$, by a scalar in $R^\times$. So we have natural inclusions

$$H_\delta \xrightarrow{a} \mu^{-1} \text{GSp}_E(D^n,\psi_{\delta\varepsilon}).$$

By taking composition we obtained the desired map $\square$. Carayol shows in [Car86a, 2.2.4] that, for a particular choice of $\varepsilon$, the paring $D_\delta \times D_\delta \rightarrow \mathbb{Q}$ defined by the same formula as in (12.3) is a polarization of the Hodge structure

$$C^{n,\varepsilon} \ni z = a + bi \mapsto \left((\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix})z,1,\ldots,1,z^{-1},\ldots,1 \end{smallmatrix}\right) \in \text{GL}_2(F \otimes \mathbb{R}) \times (F \otimes \mathbb{R})^\times.$$

We fix henceforth this choice of $\varepsilon$. Then $(W \otimes \mathbb{R}, \psi_{\delta\varepsilon}, h_\delta)$ is a polarized Hodge structure as well. In particular the map $\psi$ from Lemma 12.1 induces an embedding of $(H_\delta, Y_\delta)$ into the Siegel Shimura datum for $\text{GSp}_E(W, \psi_{\delta\varepsilon}) \cong \text{GSp}_{6n}(F \otimes \mathbb{Q})$. Consequently we have shown:

**Lemma 12.2.** The Shimura datum $(H_\delta, Y_\delta)$ is of Hodge type.

**Proposition 12.3.** Let $\pi$ and $\rho_{\ell}^{\mathcal{H}}$ be as in Theorem 10.1. As at the start of §8, choose a compact open subgroup $K = K' \times_{K'} G \subset G(A_F^{\infty})$ such that $(\pi_{\mathcal{H}} \otimes \mathcal{V})^K \neq 0$ for some fixed choice of $\mathcal{V}$. Suppose that $(G,K)$ is unramified at $\ell$ and that $t \in S_{\text{bad}}$. Then $\rho_{\ell}^{\mathcal{H}}$ is crystalline at the places dividing $\ell$.

**Proof.** Since spin $\rho_{\ell}^{\mathcal{H}}$ appears as a subspace of $\text{H}(G_{\mathcal{K}, L_\mathcal{C}})$ for $j = n(n+1)/2$, it is enough to show that the $\Gamma$-representation $\text{H}(G_{\mathcal{K}, L_\mathcal{C}})$ is crystalline at the $\ell$-adic places for $j \geq 0$. This is immediate if $L_\mathcal{C}$ is a constant sheaf (i.e. if $\xi$ is trivial) to treat the general case by a more involved argument by reducing to the crystalliness of the universal abelian scheme over the Hodge-type Shimura varieties associated with $(H_\delta, Y_\delta)$.

**Step 0. Preliminaries.** Let $v$ be an $F$-place dividing $\ell$. We start by fixing the choice of the negative element $\delta \in \mathbb{Q}$ such that $Q(\sqrt{\delta}/\mathbb{Q}$ is split above $\ell$. Then $E_v = F(\sqrt{\delta})$ is split above $v$. Write $\Gamma_{E_v} = \text{Gal}(\overline{Q}/E_v)$. We may identify the local Galois groups $\Gamma_{E_v} \times E_v = \Gamma_v$. Since $\ell$ is unramified in $E_v$ ($\ell$ is unramified in $F$ because $\ell \notin S_{\text{bad}}$),
the groups $G_\delta$ and $H_\delta$ are unramified over $Q_{\ell}$. Let $K_{G_\delta,\ell}$ be the image of $K \times (O_{F} \otimes_{Z} Z_{\ell})^\times$ in $G_0(Q_{\ell})$. Then $K_{G_\delta,\ell}$ is a hyperspecial subgroup of $G_0(Q_{\ell})$ and $K_{H_\delta,\ell} := K_{G_\delta,\ell} \cap H_0(Q_{\ell})$ is a hyperspecial subgroup of $H_0(Q_{\ell})$. Note that $K_{H_\delta,\ell} \cap H_\delta^{ad}(Q_{\ell}) = K_\ell \cap H_\delta^{der}(F_{\ell})$ via $H_\delta^{der} = \text{Res}_{F/Q} G^{der}$.

We extend the representation $\xi = \otimes_{i \in I} \xi_i$ of $(\text{Res}_{F/Q} G)_{\mathbb{C}}$ to a representation of the larger group $G_{\mathbb{C}} \supset \text{Res}_{F/Q} G$, with values in $\text{GL}_{N_i,\mathbb{C}}$. Let, for each infinite place $y$ of $F$, $\omega_y : G_{\mathbb{m},\mathbb{C}} \rightarrow G_{\mathbb{m}, \mathbb{C}}$ be the central character of $\xi_y$. Condition (cen) tells us that the central character $\omega$ of $\xi$ is the base change to $\mathbb{C}$ of the character $F^{\times} \rightarrow \mathbb{C}^{\times}$ given by $x \mapsto x^{\omega}$. We extend this character to $F_{\overline{\ell}}$ using the same formula: $\omega^y : F_{\overline{\ell}}^{\times} \rightarrow \mathbb{C}^{\times}$, $x \mapsto x^{\omega}$. Let $g \in \text{Res}_{F/Q} G(C)$ and $t \in T_\ell(C)$, then the formula $\xi_{t g}(g, t) = \xi(g) \omega^y(t)$ gives a well-defined representation $\xi_{t g} : G_{\mathbb{m}, \mathbb{C}} \rightarrow \text{GL}_{N_i, \mathbb{C}}$ extending $\xi$ on $(\text{Res}_{F/Q} G)_{\mathbb{C}}$. The restriction of $\xi_{t g}$ to $H_\delta$ is to be denoted by $\xi_{t g}$. 

Let $S_{K_{H_\delta}}$ denote the Shimura variety attached to the datum $(H_{\delta}, Y_{\delta}, K_{H_\delta})$, defined over the same reflex field $F$. Write $\xi_{G_\delta,\ell}$ for the composition of $\xi_{G_0,\overline{\ell}}$ with $G_\delta(Q_{\overline{\ell}}) \rightarrow \text{GL}_N(Q_{\overline{\ell}})$ with the projection $G_\delta(A_{\overline{\ell}}^\infty) \rightarrow G_\delta(Q_{\overline{\ell}}) \subset G_0(Q_{\overline{\ell}})$. Define $\xi_{\ell}$ (resp. $\xi_{H_\delta,\ell}$) to be the restriction of $\xi_{G_\delta,\ell}$ to $(\text{Res}_{F/Q} G)(A_{\overline{\ell}}^\infty)$ (resp. $H_\delta(A_{\overline{\ell}}^\infty)$). Let us recall that the $\ell$-adic local system $L_\ell$ is defined as follows. The composite map

$$K \hookrightarrow G(A_{\overline{\ell}}^\infty) \twoheadrightarrow G(F_{\ell}) \twoheadrightarrow GL_N(Q_{\ell}),$$

where the second map is the projection onto the $\ell$-component, is trivial on $Z(F) \cap K$ and thus on the closure of $Z(F) \cap K$, if $K'$ is sufficiently small. (Recalling that $Z(F) = F^{\times}$, we know that $Z(F) \cap K$ is a finite index subgroup of $G_{\overline{\ell}}$.) Since $\xi_{\ell}$ is a power of the norm map on $O_{F,\overline{\ell}}^*$, which has image $\pm 1$, it is clear that $\xi_{\ell}$ is trivial on $Z(F) \cap K$ for small enough $K'$.

It is standard, as explained in [Car86b, 2.14], to construct a lisse $\overline{Q}_{\ell}$-sheaf $L_{\ell}$ on $S_{K'}$ from (12.4) using the fact that the Galois group of $S_{K'}$ over $S_K$ has Galois group $K/K' = K/K'(Z(F) \cap K)$ for each finite index normal subgroup $K' \subset K$, where $K := K/Z(F) \cap K$. (For the construction we need to know that $\xi_{\ell}$ is trivial on $Z(F) \cap K$.) Similarly, using the map $K_{H_\delta} \hookrightarrow H_\delta(A_{\overline{\ell}}^\infty) \twoheadrightarrow H_0(F_{\ell}) \twoheadrightarrow GL_N(Q_{\ell})$, we define a lisse $\overline{Q}_{\ell}$-sheaf $L_{S_{K_{H_\delta}}} \ell$ as well as $K_{H_\delta} := K_{H_\delta}/Z_{H_\delta}(Q) \cap K_{H_\delta}$. 

**Step 1. Reduction to the case of Hodge type.** Let $S_{K_{H_\delta}}$ (resp. $S_{K_{H_\delta}^\infty}$) denote the neutral geometrically connected component of $S_K$ (resp. $S_{K_{H_\delta}}$). As $K$ and $K_{H_\delta}$ are unramified at $\ell$, we know that $S_{K_{H_\delta}}$ and $S_{K_{H_\delta}^\infty}$ are defined over $F^\ell$, the maximal extension of $F$ in $\overline{F}$ which is unramified at all $\ell$-adic places, as explained on [Kis10, p.1003]. We need some more notation to relate the difference between $S_{K_{H_\delta}}$ and $S_{K_{H_\delta}^\infty}$ to $S_{K_{H_\delta}}$ and $S_{K_{H_\delta}^\infty}$ for the projective system of $S_K$ (resp. $S_{K_{H_\delta}}$) over all sufficiently small $K$ (resp. $K_{H_\delta}$). Similarly $S^+$ and $S_{K_{H_\delta}}^{+\infty}$ are defined. Write $G(F)$ for the preimage of the neutral component of $G^{ad}(F, \infty)$ under the natural map $G(F) \rightarrow G^{ad}(F, \infty)$. Write $Z(F)^\infty$ (resp. $G(F)^\infty$) for the closure of $Z(F)$ (resp. $G(F)$) in $G^{ad}(F, \infty)$. Define $\mathfrak{U}(\text{Res}_{F/Q} G) := G(A_{\overline{\ell}}^\infty) / Z(F)$ and $\mathfrak{U}(\text{Res}_{F/Q} G)^{\infty} := G(F)^\infty / Z(F)^\infty$. Similarly we define $\mathfrak{U}(H_\delta)$ and $\mathfrak{U}(H_\delta)^\infty$ with $H_\delta$ in place of $\text{Res}_{F/Q} G$ as on page 1001 of [Kis10]. (Our definition coincides with his by Hilbert 90.) We have

$$(12.5) \quad \mathfrak{U}(\text{Res}_{F/Q} G)^{\infty} \simeq \mathfrak{U}(\text{Res}_{F/Q} G)^{\infty} \simeq \mathfrak{U}(H_\delta)^{\infty} \simeq \mathfrak{U}(H_\delta),$$

where the first and third maps are induced by the inclusion of the derived subgroup [Del79, 2.115], cf. [Kis10, 3.3.1]. We have a $G(A_{\overline{\ell}}^\infty)$-equivariant isomorphism of $F^\ell$-schemes

$$(12.6) \quad S = [\mathfrak{U}(\text{Res}_{F/Q} G) \times S^+] / \mathfrak{U}(\text{Res}_{F/Q} G)^{\infty}$$

and similarly for $S_{H_\delta}$ and the exact analogue holds for $S_{H_\delta}$ [Del79, 2.7.11, 2.7.13], cf. [Kis10, p.1004]. Passing to finite levels, we see that if

$$(12.7) \quad \overline{K} \cap \mathfrak{U}(\text{Res}_{F/Q} G)^{\infty} = \overline{K}_{H_\delta} \cap \mathfrak{U}(H_\delta)^{\infty},$$

the equality is tested via (12.5) then $S_{K_{H_\delta}}^+ \simeq S_{K_{H_\delta}^\infty}$ as $F^\ell$-schemes with equivariant $\mathfrak{U}(\text{Res}_{F/Q} G)^{\infty} \simeq \mathfrak{U}(H_\delta)^{\infty}$-actions.

Let $L_{(H^\delta, \ell)}$ and $L_{(H^\delta, \ell)}^+$ denote the restrictions of $L_{(H^\delta, \ell)}$ and $L_{(H^\delta, \ell)}^+$ to $S^+_{K_{H_\delta}}$ and $S^+_{K_{H_\delta}^\infty}$, respectively. Under hypothesis (12.7) we claim that $L_{(H^\delta, \ell)}^+ \simeq L_{(H^\delta, \ell)}^+$ under the isomorphism $S_{K_{H_\delta}}^+ \simeq S_{K_{H_\delta}^\infty}^+$. For verification let us observe that the Galois group of $S^+$ over $S_{K_{H_\delta}}^+$ is

$$(12.8) \quad \overline{K} \cap \mathfrak{U}(\text{Res}_{F/Q} G)^{\infty} = \overline{K}_{der} \cap \mathfrak{U}(\text{Res}_{F/Q} G)^{\infty},$$

where $\overline{K}_{der} := K \cap G^{der}(A_{\overline{\ell}}^\infty) / K \cap Z^{der}(A_{\overline{\ell}}^\infty)$ and $Z^{der} := Z \cap G^{der}$. Again the analogue is true for the Galois group of $S^+_{H_\delta}$ over $S^+_{K_{H_\delta}}$. The lisse sheaf $L_{(H^\delta, \ell)}^+$ (resp. $L_{(H^\delta, \ell)}^+$) corresponds to the representation of $\overline{K}_{der} \cap \mathfrak{U}(\text{Res}_{F/Q} G)^{\infty}$.
small open compact subgroup of $\mathbb{Z}$ subgroup of a Galois representation depends only on the restriction to the inertia subgroup $[\text{Fon94}, \text{Expô III, 5.1.5}]$, under hypothesis (12.7), we deduce that $H_i^\ell(S_K, L^\ell)$ is crystalline at all $\ell$-adic places if and only if $H_i^\ell(S_{K_{H_0}}, L^\ell_{H_0})$ is crystalline at all $\ell$-adic places.

For our purpose, it remains to see that given a sufficiently small $K$, there exists $K_{H_0}$ (possibly after shrinking $K$) such that (12.7) is satisfied. To this end, write $K = K^c K_S$ for a finite set of rational primes $S$ excluding $\ell$ but including all ramified primes of the extension $\mathbb{Q}(\sqrt{5})/\mathbb{Q}$ such that $K^c \subset G(\mathbb{A}_F^{\infty})$ is the product of hyperspecial subgroups $K^c$ away from $S$ while $K_S \subset G(F_S)$ is an open compact subgroup. (Here $F_S$ denotes the product of $F_v$, where $v$ is a place of $F$ above some prime in $S$.) Put $K^c := K \cap G^{\text{der}}(\mathbb{A}_F^{\infty})$, which is an open compact subgroup of $H^\text{der}(\mathbb{A}_F^{\infty})$. It suffices to take $K_{H_0}$ to be the subgroup of $H^\text{der}(\mathbb{A}_F^{\infty})$ generated by $K^c$, a sufficiently small open compact subgroup of $Z_{H_0}(\mathbb{A}_F^{\infty})$, and a hyperspecial subgroup of $H_0(\mathbb{Q}_p)$ containing $K^c \cap G^{\text{der}}(F_v)$ at each finite prime $v$ away from $S$. (This construction is symmetric so that one can start from $K_{H_0}$ and find $K$ such that (12.7) is true.)

**Step 2. Proof of crystallineness for Hodge type.** Henceforth we concentrate on verifying that $H_i^\ell(S_{K_{H_0}}, L^\ell_{H_0})$ is a crystalline representation of $\Gamma_{F_v}$ for each $j \geq 0$ and each $\ell$-adic place $v$. Consider the embedding $\psi : (H_0, L^\ell_{H_0}) \rightarrow \text{(GSp}_{2m}(\mathbb{Z}))$ that was constructed earlier (so $m = 4n[F : \mathbb{Q}]$). Deligne proves in [Del71, Thm. 1.15] that there exists a compact open subgroup $K_{GSP} \subset \text{GSp}_{2m}(\mathbb{Z})$ such that $K_{GSP} \cap H_0(\mathbb{A}_F^{\infty}) \subset K_{H_0}$ and the induced map $i : S_{K_{GSP}} \rightarrow S_{K_{GSp}}$ is a closed immersion over $E_\psi$, where $S_{K_{GSp}}$ is the Shimura variety attached to $(\text{GSp}_{2m}(\mathbb{Z}), K_{GSp})$. Write $\mathcal{H}_{K_{H_0}}$ for the smooth integral model for $S_{K_{H_0}}$ over $\mathbb{O}_{K_{H_0}}$ constructed by [Kis10]. Then $i$ extends to a map $i(p) : \mathcal{H}_{K_{H_0}} \rightarrow \mathcal{H}_{GSp}$ (which may not be a closed immersion), where $\mathcal{H}_{GSp}$ is a closed subscheme of the moduli space of polarized abelian varieties with a fixed degree of polarization and a prime-to-$\ell$ level structure given by $K^{\text{GSP}}$. (This corresponds to the scheme $\mathcal{H}_0^{\text{GSP}}(\mathbb{S}^+)(\mathbb{C})$ on [Kis10, p.993]. See loc. cit. for the construction of $i(p)$.) Write $u : S_{K_{H_0}} \rightarrow \mathcal{H}_{K_{H_0}}$ for the pullback of the universal abelian scheme via $i(p)$.

On the other hand, let $\varepsilon_{\text{std}} = \text{std} \circ \psi$ be the representation $H_0 \rightarrow \text{GSp}_{2m} \subset \text{GL}_{2m}$. Then $\varepsilon_{\text{std}}$ satisfies the condition (cent). Let $L_{\text{std}} = L_{\text{std}, K_{H_0}}$ be the $\ell$-adic local system on $\mathcal{H}_{K_{H_0}}$ attached to $\varepsilon_{\text{std}}$. Then we have $L_{\text{std}} = R^1 u_! Q_{\varepsilon_{\text{std}}} \otimes \mathcal{H}_{K_{H_0}}$, cf. [LS15, Lem. 3.1]. Write $V$ for the underlying $2m$-dimensional vector space of the representation $\varepsilon_{\text{std}}$. Since there exist integers $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ such that the representation $\varepsilon_{\text{std}}$ appears as a direct summand of the representation $V^{\otimes k_1} \otimes (V^{\vee})^{\otimes k_2}$ [see [DMOS82, Prop. 1.3.1]], we obtain an inclusion of local systems as a direct summand:

$$L_{\ell} \subset L_{\text{std}}^{\otimes k_1} \otimes L_{\text{std}}^{\otimes k_2} = (R^1 u_! Q_{\varepsilon_{\text{std}}})^{\otimes k_1} \otimes (R^1 u_! Q_{\varepsilon_{\text{std}}})^{\otimes k_2}.$$  

Let $s : \mathcal{H}_{K_{H_0}} \rightarrow \mathcal{H}_{GSp}$ be the abelian scheme $\mathcal{H}_{K_{H_0}}^{\otimes k_1} \times \mathcal{H}_{K_{H_0}}^{\otimes k_2}$. Note that $R^1 s_! Q_{\ell} = \Lambda^q (Q_{\text{std}}^{\otimes k_1} \otimes Q_{\text{std}}^{\otimes k_2})$. In particular $L_{\ell} \subset R^1 s_! Q_{\ell}$ as a direct summand. Write $\mathcal{B}_{K_{H_0}}$ for the generic fibre of $\mathcal{H}_{K_{H_0}}$.

Now it is clearly enough to show that $H_i^\ell(S_{K_{H_0}}, R^1 s_! Q_{\ell})$ is crystalline for each $j \geq 0$. To this end, we intend to employ the $\Gamma_{E_\delta}$-equivariant Leray spectral sequence

$$E_2^{p,q} : H_i^\ell(S_{K_{H_0}}, R^1 s_! Q_{\ell}) \Rightarrow H_i^{p+q}(\mathcal{B}_{K_{H_0}}, Q_{\ell}).$$

**Step 2.1. When $F \neq \mathbb{Q}$.** In this case $G$, $G_{\delta}$, and $H_{\delta}$ are anisotropic modulo center over $F$. The smooth integral model $\mathcal{H}_{K_{H_0}}$ over $\mathcal{E}_E \otimes \mathcal{Z}_{\ell}$ is proper by work of Dong Uk Lee [Lee12, Thm. 1.1] (see also [MPH, 4.2]). Hence $\mathcal{H}_{K_{H_0}}$, and thus also $\mathcal{B}_{K_{H_0}}$, is smooth and proper over $\mathcal{E}_E \otimes \mathcal{Z}_{\ell}$. By Tsuji [Tsu99] the cohomology $H^p+q(\mathcal{B}_{K_{H_0}}, Q_{\ell})$ is crystalline at the places dividing $\ell$. On the other hand, (12.8) degenerates due to the $\Gamma_{E_\delta}$-equivariance of (12.8) and Deligne's work on the Weil conjectures since the $E_2^{p,q}$-term in (12.8) is pure of weight $p + \pi$ at almost all finite places of $E_\delta$ (away from $\ell$). Therefore $H_i^\ell(S_{K_{H_0}}, R^1 s_! Q_{\ell})$ for $j \geq 0$ is crystalline as desired.

**Step 2.2. When $F = \mathbb{Q}$.** Pick $F'/\mathbb{Q}$ a real quadratic extension in which $\ell$ splits. Let $\pi_{F'}$ be the weak base change of $\pi$ to $\text{GSp}_{2m}(\mathbb{A}_F')$ (Proposition 6.5), and let $\rho^\text{ind}_{F'}$, the corresponding Galois representation. Then
spin \circ \rho^{C}_{\pi}(\Gal_{F/F'}) \simeq \spin \circ \rho^{C}_{\pi_{F'}}$ by Theorem 10.1(iii). Since $\rho^{C}_{\pi_{F'}}$ was shown in Step 2-1 to be crystalline at both $\ell$-adic places, the same is true for $\rho^{C}_{\pi_{\Gal_{\overline{\mathbb{Q}}}/\mathbb{Q}_L}}$.

\begin{flushright}
\qed
\end{flushright}

**Theorem 12.4.** *Theorem A is true.*

**Proof.** Apply Theorem 10.1, Propositions 11.1 and 12.3 to $|\sim^h|^{n(n+1)/4}$, which satisfies (St) and (spin-reg) and is $\xi$-cohomological by condition (L-coh) on $\pi$.

\begin{flushright}
\qed
\end{flushright}

13. The eigenvariety

In this section we make the first step to remove the spin-regularity assumption in Theorem A. We use the eigenvariety of Loeffler [LoeII]. See also [Tail5, Che04, Che09, BC09], from which we copied many arguments. Moreover, Daniel Snaith wrote his PhD thesis [Sna] about overconvergent automorphic forms for $\GSp_{2n}$.

Fix a finite $\ell$-place $v_{St}$ and let $\pi$ be a cohomological cuspidal automorphic representation of $\GSp_{2n}(\mathbb{A}_F)$ satisfying (St) at the place $v_{St}$. The goal of this section is to construct, under technical assumptions (Even) and (Aux1-2), the Galois representation $\rho^{C}_{\pi}: \Gamma \to \GSp_{2n+1}(\mathbb{Q}_L)$ attached to $\pi$ by approximating $\pi$ with points on the eigenvariety satisfying conditions (St), (L-Coh) and which are spin-regular at a certain infinite place $v_{\infty}$.

Our exposition is somewhat complicated by some technical issue coming from the center (which is $GL_1$ over $F$) when trying to ensure that classical points are Zariski dense on the eigenvariety. To get around the issue we basically work over the weight space on which the center acts through a finite quotient.

**Assumptions.** Throughout this section we assume

(Even) The degree $[F : \mathbb{Q}]$ is even.

(Aux1-1) There exists an embedding $\lambda: F \to \overline{\mathbb{Q}}_\ell$, which induces an $\ell$-prime $v_{\lambda}$ above $\ell$ with $v_{\lambda} \neq v_{St}$.

If $v_{St} \nmid \ell$, this condition is void. If $v_{St} | \ell$, the requirement is that there are at least two $F$-places above $\ell$.

(Aux1-2) The representation $\pi_{\lambda}$ has an invariant vector under the upper triangular Iwahori group $I_{\lambda}$ of $\GSp_{2n}(F_{\lambda})$.

**Notation.** We begin with a list of notation.

- $\iota: \mathbb{C} \to \overline{\mathbb{Q}}_\ell$ is an isomorphism, $\mathbb{C}_\ell$ is the completion of $\overline{\mathbb{Q}}_\ell \supset E$.
- $\lambda: F \to \overline{\mathbb{Q}}_\ell$ is fixed as in assumption (Aux1-1).
- $v_{\infty}$ is the infinite $F$-place given by $[r^{-1} \circ \lambda]: F \to \mathbb{C}$.
- $\xi$ is an algebraic representation of $G_{c,F_{\infty}}$ such that $\pi_{v_{\infty}}$ is $\xi$-cohomological.
- $W_{v_{\infty}}$ is an irreducible representation of the group $\prod_{v} F \to \mathbb{C}_{\infty,v_{\infty}} G_{c,c',v_{\infty}}$, chosen such that the representation $\pi_{v_{\infty}} = \otimes_{v\neq v_{\infty},v_{\infty}} \pi_{v}$ is $W_{v_{\infty}}$-cohomological.
- $E/\mathbb{Q}_\ell$ is a finite Galois extension contained in $\overline{\mathbb{Q}}_\ell$, large enough so that $E$ contains all the images of the elements of $\Hom_{\mathbb{Q}}(F,\overline{\mathbb{Q}}_\ell)$ and $\xi \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_\ell$ can be defined over $E$.
- If $\mathcal{X}$ is a rigid variety over $E$, we write $\mathcal{X}(\overline{\mathbb{Q}}_\ell)$ for the set $\bigcup_{E'} \mathcal{X}(E')$ where $E'$ ranges over the finite extensions of $E$ contained in $\overline{\mathbb{Q}}_\ell$. If $x \in \mathcal{X}(E)$ is a point, we write $\kappa(x)$ for the residue field at $x$. We write $|\mathcal{X}|$ for the set of closed points in $\mathcal{X}$.
- If $v$ is a finite $F$-place we write $\kappa(v)$ for the residue field at $v$ and $q_v = \# \kappa(v)$.
- By (Even), there exists an inner form $G/F$ of $\GSp_{2n,F}$ which is anisotropic modulo center at infinity and split at all finite $F$-places (Section 7). We identify $G_{c} \otimes_{F} F_{v} = \GSp_{2n,F_{v}}$ for all finite $F$-places $v$. Via this identification we obtain the model $G_{c,\Omega_{v}}$ of $G_{c}$ that corresponds to $GSp_{2n,F_{v}}$.
- We normalize the Haar measure on $G_{c}(\mathbb{A}_{F})$ such that $\GSp_{2n}(\overline{\mathbb{Q}}_\ell) \subset G_{c}(\mathbb{A}_{\infty})$ has volume 1, and the compact group $G_{c}(F \otimes \mathbb{R})$ has total volume 1.
- Let $c \in Z^{1}(\Gamma,\GSp_{2n})$ be the cocycle that defines the form $G_{c}$ of $\GSp_{2n,F}$.
- We write $T$ for a maximal torus of $G_{c}$ that corresponds to the diagonal torus in $\GSp_{2n,F}$ at the finite place $\lambda$. We define $T^{1} = G_{c}^{1} \cap T$.
- $K = \prod_{v} K_{v} \subset G_{c}(\mathbb{A}^{\infty})$ is a decomposed compact open subgroup such that $(\pi^{\infty})^{K} = 0$.
- $S_{bad}$ is the finite set of prime numbers $p$ different from $\ell$ and such that $K_{p} = \prod_{v|p} K_{v}$ is not hyperspecial.
- We write $S = S_{f,\ell,bad}$ for the set of $F$-places that are either infinite, finite and above a prime number in $S_{bad}$, or the finite place $\lambda$.
- Write $\Gamma_{S} = \Gal(F_{S}/F)$ where $F_{S} \subset \overline{\mathbb{Q}}$ is the maximal extension of $F$ that is unramified away from $S$.
- $P \subset G_{c,\Omega_{S}}$ is the upper triangular Borel subgroup.
- $Z(G_{c}) = G_{c,m}^{1}$ is the center of $G_{c}$.
- $I_{\lambda} := \{ g \in G_{c,\Omega_{G}}(\Omega_{F_{S}}) \mid \mathfrak{p} \in P(\kappa(\lambda)) \} \subset G_{c}(F_{S})$; we assume $K_{\lambda} \supset I_{\lambda}$.
Lemma 13.2

We recall the notions "Zariski dense" and "accumulate". Let \( G_{c,F_1} = \text{GSp}_{2n}(\mathbb{Q}_{\ell}) \).

- \( H^+ = H^+_0 \otimes H(G_c(\mathbb{A}_F^\infty))^\vee \).
- \( e^1 = \text{vol}(K^{-1})^{-1}1_{F_1} \in H^+ = H(G_c(\mathbb{A}_F^\infty)) \), \( e_1 = \text{vol}(I_\ell)^{-1}1_{I_\ell} \in H^+_0 \) and \( e = e_1 \otimes e_1 \in H^+ \).
- \( T = H^+_0 \otimes_E \bigotimes_{v \neq s} e_v H(G_{c,F_v})e_v \).
- \( V_{n,\text{alg}} \text{/} E \) is the one-dimensional \( T(O_{F_1}) \)-representation given by taking the \( O_{F_1} \)-points of the highest weight of the representation \( \xi \). By [Loell, Lem. 3.9.3] there exists a smooth character \( V_{n,\text{sm}} \) such that \( e_1(V_{m,k} \otimes \mathbb{R}) \neq 0 \), for large enough \( k \). Define \( V_{n} = V_{n,\text{alg}} \otimes V_{n,\text{sm}} \).

The integer \( k(V_n) \) is the least integer \( k \) such that the representation \( V_n \) is \( k \)-analytic, i.e., the corresponding character \( T(O_{F_1}) \rightarrow E^\times \) is analytic on the cosets of \( \mathfrak{X} \), i.e., all the vectors in the representation space are \( k \)-analytic in the sense of [Loell, sect. 2.2].

- We write \( \eta = \text{diag}(\omega_1, \omega_2^2, \ldots, \omega_1^n, \omega_2^n, \ldots, \omega_1^1) \in T(F_1) \) where \( \omega_1 \in F_1 \) is a uniformizer.
- We have \( X^*(T) = \mathbb{Z}^{n+1} \) where the \( w \in X^*(T) \) correspond to tuples \( (c, (k_i)_{i=1}^n) \). We call a weight \( w \in X^*(T) \) dominant if the corresponding integers have \( k_1 \geq k_2 \geq \ldots \geq k_n \geq 0 \) (no condition on \( c \)).
- We write \( X^*(T)_\text{dom} = X^*(T) \cap X^*(T)_{\text{dom}} \). We call \( w \) dominant regular if the inequalities are strict: \( k_1 > k_2 > \ldots > k_n > 0 \). We write \( X^*(T)_{\text{dom, reg}} \) for the dominant regular weights.
- If \( \mu \in G(F_1) \) is the monoid generated by the group \( I_\ell \) and the elements \( \mu(\omega_1) \) for \( \mu \in X(T)_\text{\text{dom, reg}} \).
- Let \( \mathcal{P} \) be the set of positive (resp. simple) roots of \( G_{c,F_1} = \text{GSp}_{2n,Q_{\ell}} \) and write \( \rho = \frac{1}{2} \sum_{\alpha \in \mathcal{P}^+, \alpha^{\vee} > 0} \alpha \in X^*(T) \).

The Weight Spaces \( \mathcal{W} \) and \( \mathcal{W}^\text{arith} \). For each affinoid \( A \)-algebra \( A \), define \( \mathcal{W}_T(A) \) to be the set of locally \( F_1 \)-analytic \( \text{Hom}(\mathcal{O}_F, A) \) with values in \( \mathcal{W} \). By [Eme, sect. 6.4], \( \mathcal{W}_T \) is representable by a rigid space over \( E \). This space is the weight space. To simplify notation, we often write \( \mathcal{W} = \mathcal{W}_T \) and \( \mathcal{W}_1 = \mathcal{W}_T \).

Let \( A \) be an affinoid algebra. A weight \( w \in \mathcal{W}(A) \) is

1. \text{(i) algebraic} if it lies in the image of \( X^*(T) \) to \( \mathcal{W} \);
2. \text{(ii) locally algebraic} if there exists an open subgroup \( U \subset T(O_{F_1}) \) such that \( w|_U = w_{\text{alg}}|_U \) for some algebraic weight \( w_{\text{alg}} \in \mathcal{W} \);
3. \text{(iii) arithmetic} if there exists a subgroup \( U \) of \( Z(O_F) \) of finite index such that \( w|_U = 1 \);
4. \text{(iv) finite on the center} if there exists an open subgroup \( U \) of \( Z(O_F) \) such that \( w|_U = 1 \).

In (ii), we call \( w_{\text{alg}} \) the algebraic part of \( w \). We write \( \mathcal{W}^\text{alg} \subset \mathcal{W}^\text{alg} \) for the subsets of \( \mathcal{W} \) corresponding to (i) and (ii). We write \( \mathcal{W}^\text{arith}(A) \) (resp. \( \mathcal{W}^\text{arith}(\mathcal{O}_F) \)) for the set of arithmetical (resp. finite on the center) \( A^\times \)-valued weights of \( T(O_{F_1}) \). Loeﬄer shows [Loell, Prop. 3.6.2] that \( \mathcal{W}^\text{arith} \) (resp. \( \mathcal{W}^\text{arith} \)) is representable by a rigid analytic space over \( E \). If \( \mathcal{W}^\text{arith} \subset \mathcal{W} \) is a subset, then we write \( \mathcal{W}^\text{arith} = \bigcap \mathcal{W}^\text{arith} \).

We will later see that the eigenvariety covers the weight space \( \mathcal{W}^\text{arith} \). If \( F = Q \) the (locally) algebraic points are not dense in \( \mathcal{W} \). In fact \( \mathcal{W} \subset \mathcal{W}^\text{arith} \) is closed, and because any \( U \subset Z(O_F) \) of finite index is Zariski dense, we have \( \mathcal{W}^\text{arith} = \mathcal{W}^\text{arith} \).

We call a subset \( C \subset X^*(T)_1 \otimes \mathbb{R} \) an affine cone if there exist an integer \( d \in \mathbb{Z}_{\geq 0} \), linear functionals \( \phi_1, \ldots, \phi_d \in (X^*(T)_1 \otimes \mathbb{R})^* \), and positive real numbers \( c_1, \ldots, c_d \in \mathbb{R}_{>0} \), such that

\[
C = \{ x \in X^*(T)_1 \otimes \mathbb{R} \mid \text{ for all } 1 \leq i \leq d: \phi_i(x) > c_i \}.
\]

If \( C \subset X^*(T)_1 \otimes \mathbb{R} \) is an affine cone and \( \star \subset \{ \text{algebraic, \text{arith}, \text{affine}} \} \) a subset, we write \( \mathcal{W}^\text{aff} \subset \mathcal{W}^\text{aff} \) for the set of weights \( w \in \mathcal{W}^\text{aff} \) whose algebraic part \( w_{\text{alg}} \in X^*(T)_1 \otimes X^*(Z)_1 \) lies in \( (X^*(T)_1 \otimes C) \times X^*(Z)_1 \).

Density of (Locally) Algebraic Points. We recall the notions "Zariski dense" and "accumulate". Let \( \mathcal{X} \) be a rigid space over \( E \), \( \mathcal{I} \subset \mathcal{X} \) a subset. We call the set \( \mathcal{I} \subset \mathcal{X} \) Zariski dense if for every analytically closed subspace \( \mathcal{I} \subset \mathcal{X} \) with \( \mathcal{I} \subset [\mathcal{X}] \) we have \( \mathcal{I}_\text{red} = \mathcal{I}_\text{red} \). Let \( x \in \mathcal{X} \) be a point. Then \( \mathcal{X} \) accumulates at \( x \) if there is a basis of affinoid neighborhoods \( B_x \) of \( x \) such that for each \( U \subset B_x \) the set \( \mathcal{I} \cap [U] \) is Zariski dense in \( U \).

Lemma 13.1 [Ludl14, lemma 2.7] [Con99, 2.2.9]). Let \( \mathcal{X} \) be a rigid analytic space and \( \mathcal{Y} \) an irreducible component of \( \mathcal{X} \). Assume that \( \mathcal{I} \) is an accumulation subset of \( \mathcal{X} \) and that \( \mathcal{I} \cap [\mathcal{Y}] \) is non-empty. Then \( \mathcal{I} \cap [\mathcal{Y}] \) is Zariski dense in \( \mathcal{Y} \).

Lemma 13.2 ([Che04, lemma 6.2.8]). Let \( X \rightarrow Y \) be a finite morphism between noetherian schemes such that each irreducible component of \( X \) maps surjectively to an irreducible component of \( Y \). Then for every dense subset of \( Y \) its inverse image in \( X \) is dense.
If $\star \subset [\text{alg.}1, \text{alg.}, \text{arith}]$ is a subset, then we write $\mathcal{W}^*E \subset [\mathcal{W}]$ for the set of those weights $w \in [\mathcal{W}]^*$ whose image lies in $E^* \subset \mathcal{Q}_E$.

**Lemma 13.3.** Let $C \subset X(T_1)\mathbb{R}$ be a non-empty affine cone. Let $\mathcal{O}$ be an irreducible component of $\mathcal{W}$. The subset $\mathcal{W}^{\text{alg.}} \cap [\mathcal{O}] \subset [\mathcal{O}]$ is Zariski dense and accumulates at the set $\mathcal{W}^{\text{alg.}} \cap [\mathcal{O}]$.

**Proof.** We reduce the lemma to the weight space of $Z \times T_1$. We have the following sequence on weight spaces

$$\text{Hom}(T(O_F)_1/Z(O_F)_1, T_1(O_F)_1) \twoheadrightarrow \mathcal{W}^{\text{alg.}}_T \rightarrow Z_x T_1 \rightarrow T,$$

where $\psi$ is induced by multiplication $Z \times T_1 \rightarrow T$. The similitude factor $GSp_{2m} \rightarrow \mathbb{G}_m$ induces an isomorphism

$$T(O_F)_1/Z(O_F)_1, T_1(O_F)_1 \twoheadrightarrow O^{	imes}_{F_1}/O^{	imes}_{F_2}.$$

We show that the map $\psi$ surjects onto the space $\mathcal{W}^{\text{alg.}}_Z \subset [\mathcal{O}]/[\mathcal{O}^\vee]$ of arithmetic weights $w: T_1(O_F)_1 \times Z(O_F)_1 \rightarrow \mathcal{Q}_E$ that are trivial on the diagonal $\{z \} \subset T_1(O_F)_1 \times Z(O_F)_1$. Let $x_1, \ldots, x_d \in T(O_F)_1$ be representatives for a basis of the $\mathbb{F}_2$-vector space $O^{	imes}_{F_1}/O^{	imes}_{F_2}$. Given $w: T_1(O_F)_1 \times Z(O_F)_1/\{z \} \rightarrow \mathcal{Q}_E$, we can choose square roots $w(x_1^{1/2}), \ldots, w(x_d^{1/2}) \in \mathcal{Q}_E$, and use these to extend $w$ to a morphism $\bar{w}: T(O_F)_1 \rightarrow \mathcal{Q}_E$. Since $T_1(O_F)_1 \mathcal{Z}(O_F)_1 \subset T(O_F)_1$ is open, $\bar{w}$ is locally analytic as soon as $w$ is locally analytic. Finally, if $w$ is finite on the center, then so is $\bar{w}$.

The subspace $\mathcal{W}_Z^{\text{alg.}} \subset [\mathcal{O}]$ is closed and also open because it is the complement of a closed subset:

$$\mathcal{W}^{\text{alg.}}_Z - \{(z, w) \in [\mathcal{O}] \times T_1 \mid (z - 1) = w(-1) \} \subset [\mathcal{O}] \times T_1.$$

Thus the image of $\psi$ is a union of connected components.

We show that $\psi$ is locally an open immersion. Let $\varepsilon > 0$ be a positive number. Let $w \in [\mathcal{O}]$ be a point, consider the open $\mathcal{U} = \{u \in [\mathcal{O}] \mid |u(x_i) - w(x_i)| < \varepsilon \} \subset \mathcal{T}(\mathcal{O}_E)$ (where the $x_i$ are as above). Assume $u, u' \in \mathcal{U}$ have $u(x) = u'(x)$, i.e. $u|_{T_1(O_F)_1}Z(O_F)_1 = u'|_{T_1(O_F)_1}Z(O_F)_1$. Let $x$ be one of the representatives $x_1, \ldots, x_d$. Let $t = u(x) - w(x)$, let $t' = u'(x) - w(x)$, so $|t| < \varepsilon$ and $|t'| < \varepsilon$. We compute

$$\frac{|u(t) - u'(t)|}{|u(x)|} = \frac{|u(x) + t - u(x) - t'|}{|u(x)|} < \frac{\varepsilon}{|u(x)|}.$$  

If moreover $\varepsilon < |w(x)|$, then $|t'| < |w(x)|$ thus $|u(x)| = |u'(x)|$ and hence $|u(x)/u'(x) - 1| < \varepsilon/|w(x)|$. Thus, by taking $\varepsilon$ sufficiently small, $u(x)/u'(x)$ lies in an arbitrarily small neighborhood of 1, and we can make sure that this neighborhood does not contain $\varepsilon$ for any $x \in \{x_1, x_2, \ldots, x_d\}$. Consequently, the mapping $u^{-1}u': T(O_F)_1 \times Z(O_F)_1 \rightarrow \mathcal{Q}_E$, which we know takes values in $\{z \} \subset \mathcal{O}_E$, must be trivial. Thus the map $\psi|_{[\mathcal{O}]}$ is injective and $\psi$ is locally an open immersion.

We have $\mathcal{W}^{\text{alg.}}_Z = [\mathcal{O}]^{\varepsilon \times \ell}$. By the argument for [Che04, Lem. 2.7] (cf. [Ta11, Prop. 3.1.3]) the subset $\mathcal{W}^{\text{alg.}}_T \subset [\mathcal{O}]$ is Zariski dense and accumulates at the algebraic weights. We have

$$\psi \left( \mathcal{W}_T^{\text{alg.}} \cap [\mathcal{O}] \right) \supset \left( \mathcal{W}_T^{\varepsilon \times \ell} \times \mathcal{W}^{\text{alg.}}_T \right) \cap \psi([\mathcal{O}]).$$

Hence the set $\psi(\mathcal{W}_T^{\text{alg.}} \cap [\mathcal{O}])$ is Zariski dense in $\psi([\mathcal{O}]) \subset \mathcal{W}^{\text{alg.}}_Z$ and accumulates at $(\mathcal{W}_T^{\varepsilon \times \ell} \times \mathcal{W}^{\text{alg.}}_T) \cap \psi([\mathcal{O}])$. Because $\psi$ is finite, surjective from $\mathcal{X}$ onto $\psi([\mathcal{O}])$, and $\psi^{-1}(\mathcal{W}_T^{\text{alg.}}) = [\mathcal{O}]^{\varepsilon \times \ell}$, Lemma 13.2 (cf. discussion at [Che09, top. 6]) implies that the set $\mathcal{W}_T^{\text{alg.}} \cap [\mathcal{O}]$ is Zariski dense in $[\mathcal{O}] \cap [\mathcal{O}]$. Because $\psi$ is locally an open immersion, the set $\mathcal{W}_T^{\text{alg.}} \cap [\mathcal{O}]$ accumulates at the set $\psi^{-1}(\mathcal{W}_T^{\varepsilon \times \ell} \times \mathcal{W}^{\text{alg.}}_T) \cap [\mathcal{O}]$. Let $w \in \mathcal{W}_T^{\text{alg.}} \cap [\mathcal{O}]$, $U \subset T(O_F)_1$ an open subgroup, and $w^{\text{alg.}} \in \mathcal{W}^{\text{alg.}}_T$ such that $w|_{U} = w^{\text{alg.}}|_{U}$. Then $w = (w^{\text{alg.}})^{-1}$ is a character of $T(O_F)_1$ of finite order with values in $E^\times$. Since $\mathcal{W}_T^{\text{alg.}}$ accumulates at $w^{\text{alg.}}$ there exist arbitrarily small affinoid neighborhoods $\mathcal{U} \subset \mathcal{W}^{\text{arith.}}$ of $w^{\text{alg.}}$ such that $\mathcal{W}_T^{\text{alg.}} \cap [\mathcal{O}]$ is dense in $\mathcal{U}$. Then $w \in \mathcal{W}_T^{\text{alg.}} \cap [\mathcal{O}] \subset \mathcal{W}_T^{\text{alg.}} \cap \mathcal{X}$ is dense, and $\mathcal{X}$ is an arbitrarily small affinoid neighborhood of $w \in \mathcal{W}_T^{\text{alg.}}$. This completes the proof.

The identity element in $\mathcal{W}^{\text{arith.}}(\mathcal{W}^{\text{arith.}})$ gives us the universal locally analytic character $\Delta_{\mathcal{W}^{\text{arith.}}}(T(O_F)_1)$ with values in $\mathcal{O}^{\text{arith.}}(\mathcal{W}^{\text{arith.}})^\times$. More generally, if $\mathcal{U} \subset \mathcal{W}^{\text{arith.}}$ is an affinoid open, we write $\Delta_{\mathcal{U}}$ for the corresponding locally analytic character.
Spaces of \(\ell\)-adic automorphic forms. Let \(R\) be an \(E\)-algebra and \(W\) be an \(R\)-module with left \(\mathbb{I}\)-action. We define

\[
\mathcal{L}(W \otimes W_\infty) = \left\{ \phi: G_c(F)\backslash G_c(A_F^\infty) \to W \otimes W_\infty \right\}
\]

(cf. [Loeffl, Def. 3.2.2], definition of \(F(E)\) in [Chen09, Sect. 2.15, p.13]). The space \(\mathcal{L}(W \otimes W_\infty)\) is an \(R\)-module and has a smooth action of the monoid \(G_c(A_F^\infty) \times \mathbb{I}\) defined by \((\gamma \circ \phi)(g) = \gamma \circ \phi(g \gamma)\), for all \(\gamma \in G_c(A_F^\infty) \times \mathbb{I} \subset G_c(A_F^\infty)\) and all functions \(\phi \in \mathcal{L}(W \otimes W_\infty)\).

For the readers who are comparing our definition with Loeffler’s: In our case the group \(G_c(F_\infty)\) is connected, so there is no need to consider the identity component group \(G_c(F_\infty)^\ast\). Let us check that indeed the group \(G_c(F_\infty)\) is connected if \(F = \mathbb{Q}\), the general case being similar: The mapping \(\text{GSp}_{2n,e}(\mathbb{R}) \to \text{PSp}_{2n,c}(\mathbb{R})\) is onto by Hilbert 90. The kernel is a central subgroup \(\mathbb{R}^\times\), which is contained in \(\text{GSp}_{2n,e}(\mathbb{R})^\ast\) since the similitude map has positive values on \(\mathbb{R}^\times\). On the other hand, \(\text{GSp}_{2n,e}(\mathbb{R})\) is connected as the \(\mathbb{R}\)-points of a connected anisotropic reductive group over \(\mathbb{R}\). Thus \(\text{GSp}_{2n,e}(\mathbb{R})\) is connected.

Let \(V'\subset \mathcal{W}^\text{arith}\) be an affinoid subset. We write \(\Delta^\ast\colon (\mathcal{O}_{F_1}) \to \mathcal{O}^{\text{rig}}(\mathcal{W})^\ast\) for the universal character, and \(k(\mathcal{O}^{\text{rig}}(\mathcal{W}))\) for the least integer such that \(\Delta^\ast\) is \(k\)-analytical [Loeffl, sect. 2.3]. Let \(k\) be an integer that is larger than both \(k(V'_{\infty})\) and \(k(\mathcal{W})\). We define [cf. Loeffl, Def. 3.7.1] the space of \(k\)-overconvergent automorphic forms of type \(e\) and weight \(\mathcal{U}', V_{\mathcal{U}'}\) to be the space \(M(e, \mathcal{U}', V_{\mathcal{U}'}, W_{\infty}, k) = e \mathcal{L}(C(\mathcal{U}', V_{\mathcal{U}'}, k) \times W_{\infty}^\infty)\), where \(C(\mathcal{U}', V_{\mathcal{U}'}, k)\) is the locally analytical principal series, as defined in [Loeffl, Def. 3.2.2].

Let \(M\) be a Banach module over a non-archimedean Banach algebra \(A\). Then \(M\) is said to satisfy \((Pr)\) if there exists an \(A\)-Banach module such that \(M \otimes \mathbb{N}\) is isomorphic (but not necessarily isometrically isomorphic) to an orthornormalizable \(A\)-Banach module.

**Theorem 13.4** (cf. [Loeffl, Thm. 3.7.2]). The spaces \(M(e, \mathcal{U}', V_{\mathcal{U}'}, W_{\infty}, k)\) have the following properties.

(i) Each space \(M(e, \mathcal{U}', V_{\mathcal{U}'}, W_{\infty}, k)\) is a Banach module over \(\mathcal{O}^{\text{rig}}(\mathcal{W})\) with property \((Pr)\) and is endowed with an action of \(e \mathcal{H}^+(G_c)e\) by continuous \(\mathcal{O}^{\text{rig}}(\mathcal{W})\)-linear operations.

(ii) The maps \(C(\mathcal{U}', V_{\mathcal{U}'}, k) \to C(\mathcal{U}', V_{\mathcal{U}'}, k + 1)\) induce maps \(i_k\) from the spaces \(M(e, \mathcal{U}', V_{\mathcal{U}'}, W_{\infty}, k)\) to the spaces \(M(e, \mathcal{U}', V_{\mathcal{U}'}, W_{\infty}, k + 1)\) which are \(\mathcal{O}^{\text{rig}}(\mathcal{W})\)-linear, \(e \mathcal{H}^+(G_c)e\)-equivariant, injective and have dense image.

(iii) The construction commutes with base change of reduced affinoids, in the sense that if \(\mathcal{U}' \subset \mathcal{U}\) and \(k \geq \max(k(\mathcal{U}_0), k(V'_{\infty}))\), then

\[
M(e, \mathcal{U}', V_{\mathcal{U}'}, W_{\infty}, k) \otimes_{\mathcal{O}^{\text{rig}}(\mathcal{W})} \mathcal{O}^{\text{rig}}(\mathcal{W}') = M(e, \mathcal{U}', V_{\mathcal{U}'}, W_{\infty}, k)
\]

as \(\mathcal{O}^{\text{rig}}(\mathcal{W}')\)-Banach module, and this isomorphism is compatible with \(e \mathcal{H}^+(G_c)e\)-action.

(iv) If \(T \in e \mathcal{H}^+(G_c)\) is supported in \(\mathcal{U} \setminus \{x \in \mathcal{U} \mid d_\mathcal{H}(\phi(x), 1) \leq 1\}\), and \(k \geq \max(k(\mathcal{U}_0), k(\mathcal{W}'))\), then the action of \(T\) on \(M(e, \mathcal{U}', V_{\mathcal{U}'}, W_{\infty}, k + 1)\) in fact defines a continuous map \(T_k\) to \(M(e, \mathcal{U}', V_{\mathcal{U}'}, W_{\infty}, k + 1)\) and if \(k \geq \max(k(\mathcal{U}_0), k(\mathcal{W}')) + 1\), we have a commutative diagram

\[
\begin{array}{ccc}
M(e, \mathcal{U}', V_{\mathcal{U}'}, W_{\infty}, k) & \xrightarrow{T} & M(e, \mathcal{U}', V_{\mathcal{U}'}, W_{\infty}, k - 1) \\
M(e, \mathcal{U}', V_{\mathcal{U}'}, W_{\infty}, k + 1) & \xrightarrow{i_k} & M(e, \mathcal{U}', V_{\mathcal{U}'}, W_{\infty}, k) \\
\end{array}
\]

where \(i_k\) is as above. In particular, the endomorphism of \(M(e, \mathcal{U}', V_{\mathcal{U}'}, W_{\infty}, k)\) defined by \(T\) is compact.

(v) Let \(T\) be as in (iv) and let \(\lambda \in \mathbb{E}^\ast\) with \(\lambda \neq 0\). The natural mapping \(M(e, \mathcal{U}', V_{\mathcal{U}'}, W_{\infty}, k) \to M(e, \mathcal{U}', V_{\mathcal{U}'}, W_{\infty}, k + 1)\) restricts to an endomorphism on the generalized \(\lambda\)-eigenspaces for \(T\).

**Proof.** Properties (i) to (iv) are proved in Theorem 3.7.2 of [Loeffl] and Property (v) is Corollary 3.7.3 in Loeffler. We should mention that Loeffler works with a slightly different space of \(\ell\)-adic automorphic forms (the space where \(W_\infty\) is the trivial representation). However, his proofs are valid also in our context. \(\square\)

**The Eigenvariety.** Using the spaces of \(\ell\)-adic automorphic forms from the previous subsection we construct an adjustment of the eigenvariety by Loeffler from [Loeffl] for the group \(G_c\) over the field \(F\) with respect to the finite prime \(\mathfrak{p} := \lambda\), the parabolic subgroup equal to \(P\) defined above (which is just the upper triangular Borel subgroup of \(G_c, F_1 = \text{GSp}_{2n, Q_\mathbb{A}}\)) and the locally algebraic representation \(V_{\mathcal{U}'}\) as selected above.

Let us explain why the adjustment is needed. In Loeffler’s setting for the group \(G_c\), classical points on the eigenvariety correspond to automorphic representations that are cohomological for the algebraic representation \((\mathcal{O}_{F,1}^{\infty} \otimes \mathcal{O}_{\mathcal{W}'}^{\infty})\otimes \mathcal{E}_{\mathcal{W}'}\) of the group \(G_c(C) \cong \prod_{\mathcal{W}'} \text{GSp}_{2n}(C)\) [see Loeffl, Def. 3.8.2]. Consequently, if \(\tau\) is a cohomological automorphic representation corresponding to a classical point on Loeffler’s eigenvariety, the
representation \( \tau_v (v = v_\infty) \) is in the lowest weight discrete \( L \)-packet. Thus \( \pi \) need not define a point on the eigenvariety, and can’t always be interpolated. It is possible to use Loeffler’s eigenvariety for the group \( \text{Res}_{\mathbb{F}/\mathbb{Q}} G_c \) over \( \mathbb{Q} \) (instead of \( G_c \) over \( \mathbb{F} \)); then \( \pi \) will define a point. However, in this case the Hodge-Tate weights vary at all infinite places, hence the required Hodge-Tate results in families are not available [Berger-Colmez [BC08] only consider families in which the Hodge-Tate numbers lie in a bounded interval]. Fortunately, Chevaller has dealt with a similar issue in [Che09] for eigenvarieties for unitary groups. We follow his approach by introducing \( \ell \)-adic automorphic forms that are cohomological for the fixed representation \( W_\infty \) at the infinite places \( v \neq v_\infty \). In this context we are able to both prove Lemma 13.i6 and find a classical point on the eigenvariety corresponding to \( \pi \).

We are currently in the following situation. For each affinoid \( U \subset \mathcal{W}^{\text{ar-th}} \) we have the Banach module \( M = M(e, U, V_\lambda, W_\infty, k) \) over \( \mathcal{O}^{\text{rig}}(U) \) satisfying (P\text{r}), the commutative algebra \( \mathcal{T} \) with an action on \( M \), the compact operator \( I_{U, V_\lambda, k} \) acting as a compact endomorphism on \( M \). Finally, if \( \mathring{U} \subset U \subset \mathcal{W}^{\text{ar-th}} \) are affinoid subdomains there exists a \( T \)-equivariant \( \mathcal{O}^{\text{rig}}(U) \)-morphisms \( \alpha_{\mathring{U}, U} : M(U) \to M(\mathring{U}) \otimes_{\mathcal{O}^{\text{rig}}(U)} \mathcal{O}^{\text{rig}}(\mathring{U}) \), which is a link [see LoeII, Def. 3.11.5]. These links satisfy the compatibility condition \( \alpha_{\mathring{U}_2, \mathring{U}_1} \circ \alpha_{\mathring{U}_3, \mathring{U}_2} = \alpha_{\mathring{U}_3, \mathring{U}_1} \) whenever \( \mathring{U}_1 \supset \mathring{U}_2 \supset \mathring{U}_3 \) are affinoid subdomains of \( \mathcal{W}^{\text{ar-th}} \) [see LoeII, Lem. 3.12.2].

Let \( E'/E \) be a discretely valued extension, and let \( \mathcal{W} \subset \mathcal{W}^{\text{ar-th}} \) be an open affinoid or a point. By definition an \( E \)-algebra morphism \( \beta : T \to E' \) is an \( E \)-value of \( \text{eigenvalues} \) for \( M(e, U, V_\lambda, W_\infty, k) \) if there is a point \( P \in \mathcal{H}(E') \) (giving a morphism \( \mathcal{O}^{\text{rig}}(\mathcal{H}(E')) \to E' \)) and a nonzero vector \( m \in M(e, U, V_\lambda, W_\infty, k) \otimes_E E' \) such that \( \alpha m = \beta(\alpha)m \) for all \( \alpha \in T \) [LoeII, Def. 3.11]. We call \( \beta \) of \( \text{finite slope} \) if \( \beta(U_{1, V_\lambda}) \neq 0 \) (cf. LoeII, Def. 3.4.2 and Def. 3.11). The spaces \( M(e, U, V_\lambda, W_\infty, k) \) can now be used to construct the eigenvariety \( \mathcal{E} \) as in [LoeII, Thm. 13.2.3], endowed with an \( E \)-algebra morphism \( \Psi : T \to \mathcal{O}^{\text{rig}}(\mathcal{E}) \) and the weight morphism \( \zeta : \mathcal{E} \to \mathcal{W}^{\text{ar-th}} \).

**Classical Points on \( \mathcal{E} \).** We have \( F \subset E \subset \overline{Q}_{\mathbb{E}} \cong \mathbb{C} \) and the embedding \( F \subset \mathbb{C} \) singles out the infinite place \( v_\infty \) of \( F \). If \( W_{\text{alg}} \) is an irreducible algebraic representation of \( G_c \) over \( E \), we write \( (W_{\text{alg}}, W_\infty) \) for the complex representation \( W_{\text{alg}} \otimes_E U \otimes W_\infty \). Let \( \tau_\infty \) be an irreducible continuous representation of \( G_c(F_\infty) \). Then \( \tau_\infty \) is \( W_\infty \)-allowable if \( (W_{\text{alg}}, W_\infty) \otimes_{\mathcal{O}^{\text{rig}}(E)} G_c(F_\infty) \neq 0 \) for some \( W_{\text{alg}} \) (cf. [LoeII, Def. 3.8.2] Eme06, Def. 3.1.3). An automorphic representation \( \tau \) of \( G_c(\mathbb{A}_E) \) is \( W_\infty \)-allowable if \( \tau_\infty \) is.

In [LoeII, sect. 3.9] Loeffler investigates the classical subspaces of the spaces of overconvergent automorphic forms corresponding to the fibers of \( \mathcal{E} \to \mathcal{W}^{\text{ar-th}} \) above points of \( \mathcal{W} \subset \mathcal{W}^{\text{ar-th}} \) that are locally algebraic. If \( w \in \mathcal{W}_\text{alg} = \mathcal{W} \cap \mathcal{W}_\text{dom} \) and \( w \) is defined over \( E' \), then have the space

\[
M = M(e, w, V_\lambda, W_\infty, k) = c \mathcal{L}(C(w, V_\lambda, k) \otimes W_\infty, k)
\]

of overconvergent forms of weight \( w \) (cf. Definition 13.1). Observe that \( M(e, w, V_\lambda, W_\infty, k) \cong M(e, V_\lambda, w, W_\infty, k) \) by the properties of the eigenvariety machine [LoeII, Thm. 3.11.2,[ii]], the finite slope \( E' \)-valued eigenvalues of \( M \) are in bijection with the fiber above \( w \) of the mapping \( \mathcal{E} \to \mathcal{W}^{\text{ar-th}} \).

We define the classical subspace

\[
M(e, w, V_\lambda, W_\infty, k) = c \mathcal{L}(\text{Ind}(V_\lambda \otimes w)^{\text{cl}}_E \subset c \mathcal{L}(\text{Ind}(V_\lambda \otimes w)_E) = M(e, w, V_\lambda, W_\infty, k).
\]

For the definition of the space of \( k \)-analytic vectors in the induced representation \( \text{Ind}(V_\lambda \otimes w) \) we refer to [LoeI, §22, p.198]. Its classical subspace \( \text{Ind}(V_\lambda \otimes w)^{\text{cl}}_k \) is defined in [LoeII, Def. 2.5.1.3] to be \( U_{\text{sm}, k} \otimes U_{\text{alg}} \), where \( U_{\text{sm}, k} \) and \( U_{\text{alg}} \) are smooth and algebraically induced representations of \( I_k \). Loeffler shows in [LoeII, Prop. 2.5.3] that \( \text{Ind}(V_\lambda \otimes w)^{\text{cl}}_k \) is the space of \( U(\text{Lie}G) \)-finite vectors in \( \text{Ind}(V_\lambda \otimes w)_k \).

By going through Loeffler’s proof for [LoeII, Thm. 3.9.2], with the minor adjustment that we have the module \( W_\infty \), the space of classical forms \( M(e, V_\lambda, w, W_\infty, k)^{\text{cl}} \) of \( M \) is seen to be isomorphic to

\[
\begin{pmatrix}
\mathbb{C} \\
\tau_v \\
\pi_v \quad \text{finite } \tau \text{-places } v \neq \lambda
\end{pmatrix} \otimes (U_{\text{sm}} \otimes \pi_\lambda) \otimes (I(U_{\text{alg}}, W_\infty) \otimes \pi_\infty(\text{Frob}))^{m(\pi)}
\]

where the direct sum ranges over all automorphic representations \( \pi \) of \( G_c(F_\infty) \) and \( m(\pi) \) is the multiplicity of \( \pi \) in the discrete spectrum of \( L^2 \)-automorphic forms on \( G_c(\mathbb{A}_E) \) (with fixed central character). In particular, only the \( W_\infty \)-allowable \( \pi \) contribute.

**Theorem 13.5** [LoeII, Cor. 3.9.4]. Let \( \tau \) be an \( W_\infty \)-allowable automorphic representation of \( G_c(\mathbb{A}_E) \) such that \( \tau_\infty \) is isomorphic to a subquotient of a representation that is parabolically induced from \( P \). Assume \( e^{k+1} \neq 0 \). Then there exists a locally algebraic character \( V_\lambda \) of \( T(\mathcal{O}_F) \) and, for all \( k \geq k(\chi) \), a non-zero \( \mathcal{H}^\lambda \)-invariant finite slope subspace of \( M(e, I, V_\lambda, W_\infty, k)^{\text{cl}} \), which is isomorphic as an \( e^k \mathcal{H}^\lambda e^k \)-module to a direct sum of copies of \( e^k \mathcal{H}^\lambda e^k \).
Let $\tau$ be an $W_{\infty}$-allowable automorphic representation of $G_c(\mathbb{A}_F)$. Then we say that $\tau$ corresponds to the point $c \in \mathcal{E}(E)$ if the $E$-valued system of eigenvalues $\beta_c : T \to E$ attached to $c$ satisfies $\beta_c(T) = iTrT \in \mathcal{O}_E$ for all $T \in T$. Note that for given $\tau$ there could exist several corresponding $c$, and vice-versa. We call a point $c \in \mathcal{E}(E)$ classical if it corresponds to an $W_{\infty}$-allowable automorphic representation. We write $\mathcal{E}^{cl} \subset \mathcal{E}$ for the subset of all classical points. We have $\zeta(\mathcal{E}^{cl}) \subset \mathcal{W}^{lalg,E}$.

We now seek to classify the classical points that arise in $\mathcal{E}^{cl}$. Let $\tau$ be an automorphic representation of $G_c(\mathbb{A}_F)$ such that $\tau|_{K^1} \neq 0$, $\tau_{\infty}$ is $W_{\infty}$-allowable with respect to an algebraic representation $W_{alg}$ of $G_c(F_{\text{non}})$. Since $\tau_{\lambda_1} \neq 0$, the representation $\tau_{\lambda_1}$ is isomorphic to a subquotient of a parabolically induced representation from $P$. Following Loeffler’s proof for Theorem 13.5, we get the following construction for $V_{\tau}$. We have $(W_{alg} \otimes \tau_{\infty})^{\mathcal{O}(F_{\text{non}})} \neq 0$. By highest weight theory, $W_{alg}$ arises from parabolic induction from some algebraic character $\tau_{ralg}$ of $T(O_{F_1})$. By [Loeffl, Lem. 3.9.3] there exists a smooth character $V_{\tau,sm}$ such that $\zeta_\lambda(U_{\text{non}},k \otimes \tau_1) \neq 0$, for large enough $k$. (Here $U_{\text{non}}$ is a representation induced from $V_{\tau,sm}$. See loc. cit. for the precise definition.) Then we take $V_{\tau} = V_{ralg} \otimes V_{\tau,sm}$. Then Loeffler shows that, as $e^1\mathcal{H}^{lalg,E}$-module, $\tau$ appears in $M(e,l, V_i, W_{\text{non}}, k)^{cl}$. We obtain the following criterion:

**Proposition 13.6.** If $V_{\tau} = V_{\tau_0} \otimes w$ for some $w \in \mathcal{W}^{l}$, then there exists a classical point $c \in \mathcal{E}^{cl}$ corresponding to $\tau$.

By Proposition 13.6 the set of classical points $c \in \mathcal{E}^{cl}$ on the eigenvariety corresponding to the automorphic representation $\tau$ that was fixed at the beginning of the section (p. 36) is non-empty (take $w = 1$). We fix once and for all a choice of such a point $c_\tau \in \mathcal{E}^{cl}$.

**Theorem 13.7 (Loeffler).** Let $w \in \mathcal{W}^{lalg,E}_{\text{dom}}$, and let $w_{alg} \in \mathcal{W}^{lalg,E}_{\text{dom}}$ be the algebraic part of $w$. Let $\beta \in \mathcal{O}_E^*$ be such that $\nu_{l}(\beta) = \inf_{a > 0} (-1 + a^\nu_{l}(1)) \nu_{l}(\alpha(a))$. Then the generalized $\beta$-eigenspace of $\phi = 1_{f_1, l_{alg}}$ acting on $M(e,l, V_{\tau_0} \otimes w, W_{\text{non}}, k)$ is contained in $M(e,l, V_{\tau_0} \otimes w, W_{\text{non}}, k)^{cl}$.\hfill $\square$

**Proof.** In fact, the results in [Loeffl, Thm. 3.9.6] Loeffler shows that if $\beta$ is a ‘small slope’, then the generalized $\beta$-eigenspace is contained in the classical subspace as stated in the theorem. In [Loeffl, Prop. 2.6.4] Loeffler shows that the above bound on $\nu_{l}(\beta)$ implies that $\beta$ is a ‘small slope’, and thus the stated theorem follows from Loeffler’s results.$^{13}$

Let $M \in \mathcal{Z}_{\geq 0}$ be an integer. We define the cone $C_M \subset X^*(T_1) \otimes \mathbb{R}$ by

\[(1.3.2) \quad \{ x \in X^*(T_1) \otimes \mathbb{R} \mid \forall \alpha \in \Delta : -(1 + \alpha^\vee(x)) \cdot \nu_{l}(\alpha(\eta)) > M \}.
\]

Let $C \subset X^*(T_1) \otimes \mathbb{R}$ be an affine cone. We call the cone $C$ admissible if $C \cap C_M \neq \emptyset$ for all sufficiently large integers $M$. We write $\mathcal{E}^{cl}_C$ for the set of all $c \in \mathcal{E}^{cl}$ whose weight lies in $\mathcal{W}^{lalg,E}_C$. We put $\mathcal{E}^{cl} = \mathcal{E}^{cl}_C \cap \mathcal{E}^{cl}$ and $\mathcal{E}^{cl}_C = \mathcal{E}^{cl} \cap \mathcal{E}^{cl}_C$.

**Theorem 13.8 (Loeffl, Cor. 3.13.3) [Che04, sect. 6.4.5].** Assume that $C \subset X^*(T_1) \otimes \mathbb{R}$ is an admissible affine cone. Let $\mathcal{F}$ be a component of $\mathcal{W}^{l}$. The subset $\mathcal{E}^{cl}_C \cap \mathcal{Z}^{-1}(\mathcal{F}) \subset \mathcal{Z}^{-1}(\mathcal{F})$ is Zariski dense and accumulates at the set of classical points $\mathcal{E}^{cl}_C \cap \mathcal{Z}^{-1}(\mathcal{F})$.

**Proof.** We prove the theorem by copying Loeffler’s argument, taking into account the adjustment that we work with $\mathcal{E}^{cl}$ instead of $\mathcal{E}$ (otherwise our Theorem 13.8 would be false as stated). Moreover, Loeffler does not state the accumulation property.

Let $\mathcal{Z}$ be the spectral variety attached to the choices $\eta$, $e^1$ and $V_{\tau'}$. Let $\mathcal{F}^{lalg,E}$ (resp. $\mathcal{F}^{lalg}$) be the preimages in $\mathcal{E}^{cl}$ (resp. $\mathcal{F}^{lalg} = \mathcal{Z} \times \mathcal{F}$ with $\mathcal{F}^{l}$) of the irreducible component $\mathcal{F}$. We have a finite map $\mu : \mathcal{Z}^{lalg} : \mathcal{Z} \times \mathcal{F}^{lalg} \to \mathcal{F}$ and the usual projections onto weight space $\zeta = \zeta^{lalg} : \mathcal{Z}^{lalg} \to \mathcal{Z}$ and $\delta = \delta^{lalg} : \mathcal{F}^{lalg} \to \mathcal{F}$. These maps satisfy the compatibility $\zeta = \delta \circ \mu$.

Assume $\mathcal{F}^{lalg}$ is a component of $\mathcal{Z}^{lalg}$. Then $\delta(\mathcal{F}^{lalg} \cap \mathcal{Z}^{lalg}) \subset \mathcal{Z}$ is Zariski open. By [Buz07, Thm. 4.6] the spectral variety admits an admissible covering by its open affinoids $\mathcal{Y}$ such that $\delta|_{\mathcal{Y}}$ is finite and surjects onto an open affinoid $\mathcal{U}$ of $\mathcal{F}$, and $\mathcal{Y}$ is a connected component of the pullback of $\mathcal{U}$. The morphism $\delta|_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{U}$ is then flat, hence open [Bos09, Cor. 7.2]. Consequently $\delta(\mathcal{F}^{lalg} \cap \mathcal{Z}^{lalg})$ is open.

By Lemma 13.3 there exists a point $w \in \mathcal{W}^{lalg,E}_C \cap \delta(\mathcal{F}^{lalg})$. Let $z$ be a point in $\mathcal{Z}^{lalg} \cap \mathcal{F}^{lalg}$ lying above $w$, so that $\delta(z) = w$. By the construction of the spectral variety, $\mathcal{Z}$ has a cover by affinoids $\mathcal{W}$ such that $\mathcal{W}' = \delta|_{\mathcal{Z}}(\mathcal{W})$ is affine in $\mathcal{Z}$, $\delta : \mathcal{W} \to \mathcal{W}'$ is finite and flat, and $\mathcal{W}$ is a connected component of $\delta^{-1}(\mathcal{W}')$. Fix a $\mu$ in

---

13We should mention that Tabi [Tab92, §3.2] points out a small error in the proof of Loeffler’s theorem 3.9.6 in case the group is non-split. Since $G_c$ is split at $\lambda$, Loeffler’s theorem is correct in our setting. Tabi corrects Loeffler’s result for groups that are only quasi-split at $\ell$; see [Tab92, Lem. 3.2.1] and the discussion there.
this cover that contains \( z \). Since \( \mathcal{U} \) is quasi-compact and \( \Psi(\mathfrak{I}_U \eta_U) \) does not vanish on \( \mu^{-1}(\mathcal{U}) \), the function \( \text{ord}_d \Psi(\eta) \) has a supremum \( M \). Let \( C_M \) be the cone defined in (13.2). By definition of \( C_M \), if \( w \in \mathfrak{S}_{C_M}^{\text{alg}} \) then \( -(1 + \alpha \gamma(w^{\text{alg}})) \cdot \psi(\alpha(\eta)) > M \) for all roots \( \alpha \in \Delta \). By Lemma 13.3 there exists an affinoid \( \mathcal{U}' \subset \mathcal{U} \) such that \( \mathfrak{S}_{C_M \cap C}^{\mathfrak{S}_{C_M}} \cap \mathcal{U}' \) is dense in \( \mathcal{U}' \). We define \( \mathcal{Y} = \mathfrak{T}_M(\mathcal{U}) \cap \mu^{-1}(\mathcal{U}) \subset \mathfrak{S}_{\mathfrak{T}_M}^{\mathfrak{S}_{C_M}} \), which is an affinoid neighborhood of \( \mu^{-1}(\mathcal{U}) \) in \( \mathfrak{S}_{\mathfrak{T}_M} \). The set \( [\mathfrak{Y}] \cap \mu^{-1}(\mathfrak{S}_{C_M}) \) is Zariski dense in \( \mathfrak{Y} \). By the classicality criterion, Theorem 13.7, the intersection \( \mathfrak{Y} \cap \mu^{-1}(\mathfrak{S}_{C_M}) \) is contained in \( \mathfrak{Y} \cap \mathfrak{S}_{\mathfrak{T}_M}^{\mathfrak{S}_{C_M}} \). In particular \( \mathfrak{Y} \cap \mathfrak{S}_{C_M}^{\mathfrak{S}_{C_M}} \) is Zariski dense in \( \mathfrak{Y} \cap \mathfrak{S}_{\mathfrak{T}_M}^{\mathfrak{S}_{C_M}} \). Thus \( \mathfrak{S}_{C_M} \) is Zariski dense in \( \mathfrak{Y} \). Because the irreducible component \( \mathfrak{S}_{\mathfrak{T}_M} \) is arbitrary, the first part of the theorem follows.

The argument for the accumulation property is similar: Assume \( c \in \mathfrak{S}_{\mathfrak{T}_M} \); let \( w \) be the weight of \( c \), and \( z \) be the image of \( c \) in the spectral variety. Let \( \mathcal{U}', \mathcal{U} \) be as in the above proof. Consider the \( \mathcal{U}' \subset \mathcal{U} \) such that \( \mathfrak{S}_{C_M \cap C}^{\mathfrak{S}_{C_M}} \cap \mathcal{U}' \subset \mathcal{U}' \) is dense. Since \( \mathfrak{T}_M \) is a finite morphism, a lemma of Taïbi [Taïbi, Lemma 3.1.2] shows that the connected components of \( f^{-1}(\mathcal{U}') \) form a basis for the canonical topology on \( \mathfrak{S}_{\mathfrak{T}_M} \). By Lemma 13.2 the set of classical points is dense in these components. □

**Regularity assumptions on weights.** The spin representation induces a morphism from the weight space of \( T \) to the weight space \( \mathfrak{S}_{\mathfrak{T}_M}^{\mathfrak{S}_{C_M}} \) of the diagonal torus \( G_m \) of \( GL_{2n} \), as follows. By duality, spin induces a \( \mathfrak{T}_M \)-morphism \( \mathfrak{T}_M^{-1} : (\mathfrak{T}_M^{-1}(\mathfrak{T}_M)) \rightarrow (\mathfrak{T}_M^{-1}(\mathfrak{T}_M)) \). The map \( \mathfrak{T}_M^{-1} \) descends to a mapping \( (\mathfrak{T}_M^{-1}(\mathfrak{T}_M)) \rightarrow (\mathfrak{T}_M^{-1}(\mathfrak{T}_M)) \). If \( A/E \) is affinoid, \( w \in \mathfrak{S}_{\mathfrak{T}_M}(A), w: T(O_{\mathfrak{T}_M}) \rightarrow A^\ast \), we define \( \mathfrak{T}_M(w) := w \circ \mathfrak{T}_M \in \mathfrak{S}_{\mathfrak{T}_M}(A) \). The construction is functorial in \( A \) so we have a morphism of rigid spaces: \( \mathfrak{S}_{\mathfrak{T}_M} \rightarrow \mathfrak{S}_{\mathfrak{T}_M} \).

We call a weight \( w \in \mathfrak{S}_{\mathfrak{T}_M} \) *spin-regular* if its image in \( \mathfrak{S}_{\mathfrak{T}_M} \) is regular (i.e. for all indices \( 1 \leq i, j \leq 2n, i \neq j \), the components at \( i \) and \( j \) are different). Similarly, we call a weight \( w \in \mathfrak{S}_{\mathfrak{T}_M} \) *local algebraic regular* if its image under the standard representation gives a regular weight for \( GL_{2n+1} \). We write \( \mathfrak{S}_{\mathfrak{T}_M}^\ast \) (resp. \( \mathfrak{S}_{\mathfrak{T}_M}^{\mathfrak{T}_M} \)) for the set of all algebraic arithmetic weights which are regular for \( \mathfrak{T}_M \) (resp. dominant and regular for \( \mathfrak{T}_M \)) where \( \mathfrak{T}_M \subset \{ \text{spin, std} \} \) and \( c \subset \{ \text{arith, alg, alg, alg, alg, alg, alg} \} \).

We write \( \mathfrak{S}_{\mathfrak{T}_M}^{\mathfrak{T}_M} \) for the set of \( x \in \mathfrak{S}_{\mathfrak{T}_M} \) such that \( \mathfrak{T}_M(x) \in \mathfrak{S}_{\mathfrak{T}_M}^{\mathfrak{T}_M} \) where \( x \subset \{ \text{spin, std} \} \).

**Corollary 13.9.** Let \( \ast \) be a subset of \{spin, std\}. Then

(1) The set \( \mathfrak{S}_{\mathfrak{T}_M}^{\mathfrak{T}_M} \subset \mathfrak{S}_{\mathfrak{T}_M} \) is Zariski dense and accumulates at the classical points \( \mathfrak{S}_{\mathfrak{T}_M}^{\mathfrak{T}_M} \).

(2) The set \( \mathfrak{S}_{\mathfrak{T}_M}^{\mathfrak{T}_M} \subset \mathfrak{S}_{\mathfrak{T}_M} \) is Zariski dense and accumulates at the locally algebraic points \( \mathfrak{S}_{\mathfrak{T}_M}^{\mathfrak{T}_M} \).

**Proof.** Choose an admissible affine \( C \subset X^\ast(T_1) \otimes \mathbb{R} \) such that \( \mathfrak{S}_{C}^{\mathfrak{S}_{C_M}} \subset \mathfrak{S}_{\mathfrak{T}_M}^{\mathfrak{T}_M} \). Then \( \mathfrak{S}_{\mathfrak{T}_M}^{\mathfrak{T}_M} \) is contained in \( \mathfrak{S}_{\mathfrak{T}_M}^{\mathfrak{T}_M} \) and the corollary follows from Theorem 13.8 and Lemma 13.3.

**Galois representations at classical points.** Let \( c : G_m \rightarrow GSp_{2n} \) be a dominant cocharacter (with respect to the upper triangular Borel subgroup). Let \( \eta_c \) be the highest weight representation of \( GSp_{2n+1} \) attached to \( c \). Consider the subalgebra \( \mathfrak{T}_c \subset \mathfrak{T}_T \) of functions which take values in the ring of integers \( \mathcal{O}_E \) (rather than \( E \)). For each finite \( F \)-place \( v \in S \), define \( \chi_v = i(q_v^{\ast}(c) \cdot \mathfrak{T}_c) \) in the ring \( \mathcal{O}_L[T]^v \) of \( W \)-invariant regular functions on \( \mathcal{O}_L \), where \( W \) is the Weyl group of \( T \) and \( \mathcal{O}_L \) is the dual torus of \( T \). Let \( v \) be a finite \( F \)-place which is not in \( S \). Write \( S \) for the Tate transformation \( H(G(F_v)/G(O_{\mathfrak{T}_M}, \mathcal{O}_L)) \rightarrow \mathcal{O}_L[T]^v \). By [Grothendieck, (3.12)] (Gross normalizes \( \chi_v \) differently) we have

\[
S^{-1}(\chi_v) = \sum_{u \in X^\ast(T)^v} \tilde{d}_c(u) \cdot \mathbf{1}_{K_v, p(\varnothing), K_v}.
\]

where \( \varnothing \in F_v \) is a uniformizer at \( v \) and the coefficients \( \tilde{d}_c(u) \) are integers. We define the Hecke operator

\[
(13.3) \quad T_v(v) = \sum_{u \in X^\ast(T)^v \setminus \text{finite } F \text{-places } w \neq S \setminus \{v\}} \tilde{d}_c(u) \cdot \mathbf{1}_{K_v, p(\varnothing), K_v} \otimes \mathbf{1}_{K_v} \in \mathfrak{T}_{\mathcal{O}_L}^v.
\]

Let \( c \in \mathfrak{S}_{\mathfrak{T}_M} \) be a point and \( \beta \) its corresponding system of eigenvalues. Assume that \( \eta_c \) is highest weight representation of \( GSp_{2n+1} \) with \( c \in X^\ast(T)^v \). Write \( N = \dim(\eta_c) \). We say that \( c \) has a *Galois representation at \( c \) if there exists a representation \( \rho_c^\ast : \Gamma_S \rightarrow GL_N(\mathcal{O}_L) \) which is unramified away from \( S \) and such that \( \text{Tr} \rho_c^\ast(\text{Frob}_v) = \beta(\mathbf{T}_v(v)) \) for all \( v \in S \).
Interpolating Galois representations. Let $\zeta : G_{m} \to \text{GSp}_{2n}$ be a dominant cocharacter (with respect to the upper triangular Borel subgroup). Let $\eta_{\zeta}$ be the highest weight representation of $\text{GSp}_{2n+1}$ attached to $\zeta$. Let $f : \mathcal{X} \to \mathbb{E}^{f}$ be a morphism of rigid spaces with $\mathcal{X}$ reduced. Let $\mathcal{C} \subset f^{-1}(\mathbb{E}^{\text{std}})$ be a subset of points such that each point $c \in \mathcal{C}$ has a Galois representation for $c$ and $\mathcal{C} \subset \mathcal{X}$ is Zariski dense.

**Proposition 13.10 (Chenevier).** There exists a continuous pseudocharacter $T_{\eta}$ of $\Gamma_{S}$ with values in $O^\text{rig}(\mathcal{X})$ such that at every $c \in \mathcal{C}$ the pseudocharacter $T_{\eta}$ specializes to the trace character of $\rho_{c}^{\text{rig}}$.

**Remark 13.11.** On the ring $O^\text{rig}(\mathcal{X})$ we put the coarsest locally convex topology such that all the restriction maps $O^\text{rig}(\mathcal{X}) \to O^\text{rig}(\mathcal{Y})$, $\mathcal{Y} \subset \mathcal{X}$ affinoid subdomain, are continuous, where $O^\text{rig}(\mathcal{Y})$ is equipped with its usual Banach algebra topology (cf. [BC09, Def. 4.2.2]).

**Proof.** Using [Loeffl, Prop. 3.3.3, Prop. 3.5.2] we see that

$$\Psi(T_{C_{Q}}) \subset O^\text{rig}(\mathcal{X})^{\leq 1} := \{f \in O^\text{rig}(\mathcal{X}) \mid |f| \leq 1\},$$

and moreover $\Psi : T \to O^\text{rig}(\mathcal{X})^{\leq 1}$ is continuous. Write $\Psi_{x}$ for $\Psi : T \to O^\text{rig}(\mathcal{X})^{\leq 1}$.

We check that the ring $O^\text{rig}(\mathcal{X})^{\leq 1}$ is compact. Bellaïche–Chenevier define [BC09, Def. 7.2.10] that a rigid analytic space $\mathcal{Y}$ over $\mathcal{Q}_{p}$ is nested if it has an admissible covering by open affinoids $\mathcal{Y}_{U}$ such that $\mathcal{Y}_{U} \subset \mathcal{Y}_{U+1}$ and that the natural map $O^\text{rig}(\mathcal{Y}_{U+1}) \to O^\text{rig}(\mathcal{Y})$ is compact. They mention that any point of nested spaces is again nested, and $A_{1}$ and $G_{m}$ are nested spaces. Moreover, any space that is finite over a nested space is again nested [BC09, 7.2.11.(i)]. Loeffl shows in [Loeffl, Prop. 3.6.2] that $\mathcal{W}^\text{an}$ is finite étale over a rigid ball. Consequently $\mathcal{W}^\text{an}$ is nested. Since, by point (i), the eigenvariety $\mathcal{E}$ is finite over $\mathcal{W}^\text{an} \times \mathbb{C}_{m}$, it is nested. Bellaïche–Chenevier show [BC09, 7.2.11.(iii)] that for any reduced and nested rigid analytic space $\mathcal{Y}$, the ring of bounded functions $O^\text{rig}(\mathcal{Y})^{\leq 1}$ is compact. Thus $O^\text{rig}(\mathcal{X})^{\leq 1}$ is compact.

The rigid space $\mathcal{X}$ need not be nested, so there is no a priori reason that $O^\text{rig}(\mathcal{X})^{\leq 1}$ is compact. However, $O^\text{rig}(\mathcal{X})$ is Hausdorff [BC09, below Lem. 7.2.11], and thus $f(O^\text{rig}(\mathcal{X})^{\leq 1}) \subset O^\text{rig}(\mathcal{X})$ is a compact and closed subring containing the algebra $\Psi_{x}(T_{C_{Q}}) \subset O^\text{rig}(\mathcal{X})$. At each classical point $c \in \mathcal{C}$ and each finite $F$-place $v \not\in S$, since $\text{dim}(\tau_{c,v}^{\text{rig}})$ is $1$ at $w \not\in S_{F,\text{bad}} \cup \{v\}$, we have

$$\text{Tr} \tau_{c}(\zeta_{c}(v)) = q_{v}^{m} \cdot \text{Tr}(\eta_{\zeta} \circ \phi_{\eta_{\zeta}})(\text{Frob}_{v}) = i^{-1} \text{Tr} \rho_{c}^{\text{rig}}(\text{Frob}_{v}) \in \mathbb{C},$$

which implies $\Psi(T_{C_{Q}}(v))(c) = \text{Tr} \rho_{c}^{\text{rig}}(\text{Frob}_{v})$. Write $H$ for the compact ring $f(O^\text{rig}(\mathcal{X})^{\leq 1}) \subset O^\text{rig}(\mathcal{X})$. Defining $a_{c} : = \Psi_{x}(T_{C_{Q}}(v)) \in H$, we have for all points $c \in \mathcal{C}$, $\text{Tr} \rho_{c}^{\text{rig}}(\text{Frob}_{v}) = a_{c}(c)$, which is Chenevier’s condition (H) in [Che04, sect. 7]. Now if we have the proof of [Che04, Prop. 7.1.1] and adapt it to the setting where the coefficient field $E$ is not algebraically closed. By compactness of $H$, the mapping $\psi : H \to \prod_{E \in \mathcal{E}} \cdots$ is a homeomorphism onto its image. We define $\Psi_{x} : T \to \prod_{E \in \mathcal{E}} \cdots$, $\sigma \to (\text{Tr} \rho_{c}^{\text{rig}}(\psi \circ \eta_{\zeta}))(c)$. Choose embeddings $i_{c} : \kappa(c) \to \overline{Q}_{E}$ of $E$-algebras for each $c \in \mathcal{C}$. Write $i$ for the product of the $i_{c}$. For each Frobenius element $\text{Frob}_{v}$, $v \in S$, $(\text{Tr} \rho_{c}^{\text{rig}}(\text{Frob}_{v}) \circ i_{E}) \in \prod_{E \in \mathcal{E}} \overline{Q}_{E}$ lies in the image of $i$. Since $i \circ \psi : H \to \prod_{E \in \mathcal{E}} \overline{Q}_{E}$ is injective, has closed image, and the Frobenius elements are dense in $\Gamma_{S}$, $i \circ \psi$ induces a map $T_{\eta}$ from $\Gamma_{S}$ to $O^\text{rig}(\mathcal{X})$. The map $T_{\eta}$ is easily checked to be a pseudocharacter (check it on the dense subset $\mathcal{C} \subset \mathcal{X}$).

**Step 1.** Construct the pseudocharacter for the standard representation and the spinor norm, and construct also corresponding big Galois representations on (covers of parts of) the eigenvariety. Let $c$ be a classical point in $\mathcal{E}$ such that one of the corresponding automorphic representations $\tau_{c}$ of $G_{c}(\mathbb{A}_{F})$ is cohomological for an algebraic representation $\zeta_{c}$ of $(\text{Res}_{E/F}(\mathbb{G}_{m})_{\mathbb{Q}_{c}})$ whose highest weight maps to a regular weight of $\Gamma_{S_{2n+1}}(\mathbb{Q}_{E})$ under the standard representation $\text{GSpin}_{2n+1}(\mathbb{Q}_{E}) \to \text{GL}_{2n+1}(\mathbb{Q}_{E})$ (which factors through $\text{SO}_{2n+1}(\mathbb{Q}_{E})$). Choose $\tau_{c}^{\text{rig}} \subset \tau_{c}$, an automorphic representation of $\Gamma_{S_{2n+1}}(\mathbb{A}_{F})$ contained in $\tau_{c}^{\text{rig}}$. By Tait’s recent result [Tait5] we can attach a (formal) Arthur parameter $\psi_{c}$ to $\tau_{c}^{\text{rig}}$. As in the quasi-split case, cf. Lemma 2.1 and Corollary 2.2, the parameter $\psi_{c}$ is simple and generic, and thus given by a self-dual unitary cuspidal automorphic representation $\tau_{c}^{\text{rig}} \subset \text{GL}_{2n+1}(\mathbb{A}_{F})$ such that $\tau_{c,\text{un}}$ is the Steinberg representation. Using Theorem 2.4 we attach to $\tau_{c,\text{un}}$ a Galois representation $\rho_{c,\text{std}}^{\text{rig}} : \Gamma_{S} \to \text{GL}_{2n+1}(\mathbb{Q}_{E})$ such that for all $v \in S$, $\rho_{c,\text{std}}^{\text{rig}}(\text{Frob}_{v}) = \text{GL}_{2n+1}(\mathbb{Q}_{E})$. Theorem 13.4 shows that each classical point $c \in \mathcal{E}$ has a Galois representation for ‘std’ (cf. below (13.3)). By Proposition 13.10 there exist unique

We are now in place to interpolate the standard representation. By Corollary 13.9 the subset $\mathcal{E}_{\text{std}}^{\text{Ld}} \subset \mathcal{E}^{\text{rig}}$ is Zariski dense. Take $\mathcal{X} = \mathcal{E}^{\text{rig}}$, $f = \text{id} : \mathcal{E}^{\text{rig}} \to \mathcal{E}^{\text{rig}}$, and $\mathcal{C} = \mathcal{E}_{\text{std}}^{\text{Ld}}$. Then Equation (13.4) shows that each classical point $c \in \mathcal{E}_{\text{std}}^{\text{Ld}}$ has a Galois representation for ‘std’ (cf. below (13.3)).
continuous pseudocharacter $T_{\text{std}}: \Gamma_S \to \mathcal{O}^{\text{rig}}(E^f)$ such that for all $c \in \mathfrak{g}_{\text{std}}^{\text{Ld}}$ we have $T_{\text{std}, c} = \text{Tr}(\rho_c^{\text{std}})$ at the residue field $k(c)$ of $c \in \mathfrak{g}_{\text{std}}^{\text{Ld}}$.

Even though we have not yet constructed the Galois representation $\rho_c^{\text{G}}: \Gamma_S \to \text{GSpin}_{2n+1}(\mathbb{Q}_c)$ for a dense set of classical points, its spinor norm $N \circ \rho_c^{\text{G}}$ is easily constructed using class field theory and the central character of $\zeta_c$ for $c \in \mathfrak{g}_{\text{std}}^{\text{Ld}}$ (cf. Theorem A(1)). By interpolating we obtain the character $T_{\text{cG}}: \Gamma_S \to \mathcal{O}^{\text{rig}}(E^f)^{x}$. Let $x \in E^f$ be a point, $\kappa(x)$ the residue field at $x$ and let $\kappa(x)$ be an algebraic closure of $\kappa(x)$. We may (and often do) identify $\kappa(x) = \mathbb{Q}_c$ as $E$-algebras. By Taylor’s theorem [Tay91, Thm. 1], the pseudocharacter $T_{\text{std}} \otimes \kappa(x)$ arises from the trace of a Galois representation $r_{\text{std}, x}: \Gamma_S \to \text{GL}_{2n+1}(\kappa(x))$. We write $\mathfrak{g}_{\text{std}}^{\text{cG}} \subset \mathcal{O}^{\text{rig}}$ for the locus of those $x \in \mathcal{O}^{\text{rig}}$ such that $r_{\text{std}, x}$ is (absolutely) irreducible.

**Lemma 13.12** (Chenevier, cf. [Che04, p. 205]). The subset $\mathfrak{g}_{\text{std}}^{\text{cG}} \subset \mathcal{O}^{\text{rig}}$ is Zariski open.

**Proof.** For an arbitrary $x \in \mathcal{O}^{\text{rig}}$, Chenevier shows [Che04, Lemma 7.2.2] that $r_{\text{std}, x}$ is irreducible if and only if there exist elements $a_1, a_2, \ldots, a_{2n+1} \in \Gamma_S$ (with $m = 2n + 1$) such that the $m^2 \times m^2$ matrix $(Tr r_{\text{std}, x}(a_i \sigma_j))_{i,j=1}^{m^2} \in \mathcal{M}_{m^2}(\kappa(x))$ is invertible. Define $Z$ to be the intersection of the sets $V(\det(T_{\text{std}}(a_i \sigma_j))_{i,j=1}^{m^2})$ inside $\mathcal{O}^{\text{rig}}$, where $(a_i)$ ranges over the set of all $m^2$-tuples of elements in $\Gamma_S$, and $V(\det(T_{\text{std}}(a_i \sigma_j))_{i,j=1}^{m^2})$ denotes the locus in $\mathcal{O}^{\text{rig}}$ where the determinant vanishes. By Chenevier’s lemma, $\mathfrak{g}_{\text{std}}^{\text{cG}}$ is the complement of $Z$ in $\mathcal{O}^{\text{rig}}$, and hence is open. □

By Lemma 13.12 we have an open immersion $\mathfrak{g}_{\text{std}}^{\text{cG}} \subset \mathcal{O}^{\text{rig}}$. The representation $r_{\text{std}, c_{\text{G}}} \subset \mathcal{O}^{\text{rig}}$ is irreducible by Taylor-Yoshida (Theorem 2.4.(vii)), and hence $\mathfrak{g}_{\text{std}}^{\text{cG}}$ is a Zariski open neighborhood of the point $c_{\text{G}}$ in $\mathfrak{g}_{\text{std}}^{\text{Ld}}$. By the accumulation property, there exists an affinoid neighborhood $\mathcal{U} \hookrightarrow \mathfrak{g}_{\text{std}}^{\text{Ld}}$ of $c_{\text{G}}$ in which the set $\mathfrak{g}_{\text{spin, std}}^{\text{Ld}} \cap \mathcal{U}$ is dense. Let $\mathcal{U} \subset \mathcal{U}'$ be an irreducible component of $\mathcal{U}'$ with $c_{\text{G}} \in \mathcal{U}$. By Lemma 13.1, $\mathfrak{g}_{\text{spin, std}}^{\text{Ld}} \cap \mathcal{U}$ is dense in $\mathcal{U}$. Write $R$ for the Tate algebra such that $\mathcal{U} = \text{Sp}(R)$. Let $I \subset \mathcal{O} \Gamma_S$ be the kernel of the pseudocharacter $T_{\text{std}}|_{\mathcal{U}}$, i.e.

$$I = \{ x \in \mathcal{O} \Gamma_S \mid \forall y \in \mathcal{O} \Gamma_S \ T_{\text{std}}(xy) = 0 \}.$$

By Rouquier [Rou96, Thm. 5.1] the ring $A_{\text{std}} := \mathcal{O} \Gamma_S / I$ is an Azumaya algebra, and there is a unique (up to conjugation) representation $r_{\text{std}}(\mathcal{U}): \mathcal{O} \Gamma_S \to A_{\text{std}}$ whose reduced trace is equal to $T_{\text{std}}|_{\mathcal{U}}$. The Azumaya algebra $A_{\text{std}}$ has a unique topology making it into a Banach module over $R$ of finite type. This topology is induced from choosing a surjection of $R^N$ onto $A_{\text{std}}$ (cf. [BGR84, Prop. 3.7.3.3]).

**Lemma 13.13** (Chenevier, cf. proof of [Che04, Lem. 7.2.4]). The representation $r_{\text{std}}(\mathcal{U}): \Gamma_S \to A_{\text{std}}^\times$ is continuous.

**Proof.** Let $a_1, a_2, \ldots, a_{2n+1} \in \Gamma_S$ such that the elements $r_{\text{std}}(\mathcal{U})(a_i)$ generate the algebra $A_{\text{std}}$ as $R$-module. Then the map $\varphi': A_{\text{std}} \to R^n, v \mapsto \text{Trd}(r_{\text{std}}(\mathcal{U})(a_i) v)_{i=1}^{2n+1}$ is a linear $R$-injection because the reduced trace, denoted by $\text{Trd}$, is non-degenerate on an Azumaya algebra. Then $\varphi'$ is a homeomorphism onto its image, which is a closed submodule of $R^n$ [BGR84, 3.7.3.3]. Thus, to check that $r_{\text{std}}(\mathcal{U})$ is continuous, it suffices to check the continuity of $\varphi' \circ r_{\text{std}}(\mathcal{U})$. Since the projection of $\varphi' \circ r_{\text{std}}(\mathcal{U})$ onto the $i$-th copy of $R$ is given by $a \mapsto \text{Trd}(r_{\text{std}}(\mathcal{U})(a_i) \sigma)$, the desired continuity follows from the continuity of $T_{\text{std}}$. The topology on $A_{\text{std}}^\times$ is induced from the embedding $A_{\text{std}}^\times \to A_{\text{std}}^\times, x \mapsto (x, x^{-1})$. Hence the mapping $r_{\text{std}}(\mathcal{U})(\sigma) \mapsto r_{\text{std}}(\mathcal{U})(\sigma^{-1})$ are continuous. The second mapping is the first map composed with $\sigma \mapsto \sigma^{-1}$ and hence continuous. □

The Azumaya algebra $A_{\text{std}}$ is locally trivial for the étale topology on $\text{Spec}(R)$ [Mil80, Prop. IV.2.1]. Thus [Mil80, remark 2.2(b)] there exists a Zariski open neighborhood $\text{Spec}(R_r)$ of the closed point $c_r$ (with $r \in R$ and $R_r = R[1/r]$) and a finite étale morphism $R_r \to R_r'$ such that $A_{\text{std}} \otimes_R R_r'$ splits. Define the Tate algebra $S_r := R(x)/\langle x - 1 \rangle$ over $R_r$. Define $S_r' := R_r' \otimes_{R_r} S_r$, then $S_r'$ is finite étale over $S_r$ and hence a Tate algebra as well. Write $\mathcal{U}' = \text{Sp}(S_r')$. Let $B$ be the normalization of $S_r'$. Then $B$ is a Tate algebra over $E$, a normal domain and we have a representation $r_{\text{std}}(B)$ induced from $A_{\text{std}} \otimes_R B$. We write $\mathcal{F} = \text{Sp}(B)$.

Summarizing, we have constructed the following maps

$$\mathcal{F} \twoheadrightarrow \mathcal{U} \quad \text{étale map that splits the Azumaya algebra } A_{\text{std}} \quad \text{irred. cmp. of affinoid neighborhood of } c_r \quad \mathfrak{g}_{\text{std}}^{\text{cG}} \quad \text{irreducibility locus of } T_{\text{std}} \quad \mathcal{O}^{\text{rig}}$$

We write $\mathcal{F}$ for the composition $\mathcal{F} \twoheadrightarrow \mathfrak{g}_{\text{std}}^{\text{cG}}$. We choose $x_r \in \mathcal{O}^{\text{rig}}$ a point such that $\delta(x_r) = c_r$. 

\[
\begin{align*}
\mathcal{F} &\twoheadrightarrow \mathcal{U} \\
\text{étale map that splits } A_{\text{std}} &\quad \text{irred. cmp. of affinoid neighborhood of } c_r \\
\mathfrak{g}_{\text{std}}^{\text{cG}} &\quad \text{irreducibility locus of } T_{\text{std}} \\
\mathcal{O}^{\text{rig}} &
\end{align*}
\]
Step 2. Construction of the pseudocharacter attached to the spin representation. We define the following sets of classical points:

- $\mathcal{E}_{\text{St},c} \subset \mathcal{E}$ is the set of $c \in \mathcal{E}$ such that the corresponding automorphic representations satisfy (St).
- Let $\star \in \{\text{spin}, \text{std}\}$. We write $\mathcal{E}_{\text{St},\star} = \mathcal{E}_{\text{St}} \cap \mathcal{E}_{\star}.$
- Let $\star \in \{\text{St}, \text{spin}, \text{std}\}$. We write $\mathcal{E}_{\star} \subset \mathcal{E}$ for the set of $c \in \mathcal{E}$ such that $\delta(c) \in \mathcal{E}_{\star}$.

Note, for the first bullet, that if $\tau_0$ is the Steinberg representation. The representations $\tau^c_0$ are tame inertia group. Fix a tame inertia character $\ell$ and $\tau_0$ is quasi-split). This implies that $\tau_0$ satisfies (St), then $\tau_0$ is irreducible. Then $\delta^{-1}(\mathcal{S}_x) = \delta^{-1}(\mathcal{S}_x)$ is Zariski dense by Lemma 13.14.

Lemma 13.14. Let $\star \subset \{\text{spin}, \text{std}\}$. The subset $\mathcal{E}_{\text{St},\star} \subset \mathcal{E}$ is Zariski dense and accumulates at all the classical points $\mathcal{E}_{\star}$.

Proof. Recall that $\mathcal{U} \subset \mathcal{E}_{\star}$ was constructed in such a way that $\mathcal{U}_{\text{St},\star} \subset \mathcal{U}$ is dense in $\mathcal{U}$ (see above 13.5)). Write $\mathcal{U} = \mathcal{E}_{\text{St},\text{std}} \cap \mathcal{U}$. By definition $\mathcal{E}_{\text{St},\text{std}} = f^{-1}(\mathcal{U})$. The mapping $\delta: \mathcal{E}_{\text{St}} \to \mathcal{U}$ is finite and $\mathcal{E}_{\text{St}}, \mathcal{U}$ are integral. By Lemma 13.2, $\delta^{-1}(\mathcal{U}) \subset \mathcal{E}$ is Zariski dense. Similarly, $\delta^{-1}(\mathcal{U})$ accumulates at $\mathcal{E}_{\text{std}}$: If $x \in \mathcal{E}_{\text{std}}$ is a point, then there is an affine neighborhood $\mathcal{U}_x \subset \mathcal{U}$ such that $\mathcal{U}_x \cap [\mathcal{U}]_x$ is dense in $\mathcal{U}_x$. By Lemma 13.1 we may assume that $\mathcal{U}_x$ is irreducible. Then $\delta^{-1}(\mathcal{U}_x) = \delta^{-1}(\mathcal{U}_x)$ is Zariski dense by Lemma 13.2. Since $\delta$ is finite, $\delta^{-1}(\mathcal{U}_x)$ is affinoid.

To show that the sets $\mathcal{E}_{\text{St},\star} \subset \mathcal{E}$ are Zariski dense and accumulate at $\mathcal{E}_{\star}$, we first construct a nilpotent operator $N$ on $\mathcal{E}$. We distinguish between two cases.

First Case $v_{\text{St}} \mid \ell$. Write $L := F_{v_{\text{St}}}$. Denote by $W_L$ the Weil group of $L$, $I_L$ its inertia subgroup, and $I_L^{\text{tame}}$ the tame inertia group. Fix a tame inertia character $\ell: I_L^{\text{tame}} \to \mathbb{Z}_l$. By Grothendieck’s monodromy theorem in families in the “$\ell \neq p$ case” [BC09, 7.8.14], there exists a unique nilpotent element $N \in M_{2n+1}(B)$ such that $r_{\text{std}}(B)(\sigma) = \exp(t_{\sigma}(\alpha)N)$ for all $\sigma$ in some fixed open subgroup of the inertia subgroup $I_L \subset W_L$.

Lemma 13.15. (v) Let $c \in \mathcal{E}_{\text{std}}$. If $(\mathcal{N}_c)^{2n} = 0 \in A_{\text{std}} \otimes \kappa(c)$ then $\mathcal{N}_c$ is isomorphic to a twist by a character of the Steinberg representation.

Proof. We have the Galois representation $r_{\text{std},\star}: \Gamma_S \to \text{GL}_{2n+1}(\mathbb{Q}_L)$ attached to $T_{\text{std}} \otimes \kappa(c)$. Then $\mathcal{N}_c$ is the nilpotent operator in the Weil-Deligne representation attached to $r_{\text{std},\star}|_{I_L}$. If $(\mathcal{N}_c)^{2n} = 0$, then this nilpotent operator is of maximal order. Consequently, the Weil-Deligne representation of $r_{\text{std},\star}|_{I_L}$ is indecomposable. This means that the GL$_{2n+1}(\mathcal{A}_L)$ automorphic representation $r_{\star}$ attached by Tabi, has a $\mathcal{N}_c$-component which is isomorphic to a twist of the Steinberg representation. Then $\mathcal{N}_c$ is also the Steinberg representation by Lemma 2.1 (since the local packet in [Tail5] is the same as Arthur’s at places, in particular at $v_{\text{St}}$, where $G_{\ell}$ is quasi-split). This implies that $\mathcal{N}_c$ is a twist of the Steinberg representation.

Second Case $v_{\text{St}} \nmid \ell$. We need $\ell$-adic Hodge theory results in families. Write $L$ for the local field $\mathcal{F}_{v_{\text{St}}}$, and write $W = B^{2n+1}$ for the space of the representation $r_{\text{std}}(B)|_{I_L}$. Assuming certain technical conditions on $\mathcal{V}$, Theorem C of Berger–Colmez [BC08] states that the $L \otimes \mathcal{O}_B$, $B$-module $\mathcal{V}(\mathcal{A}_L)$ is locally free of rank $2n+1$.

We now explain the technical conditions that Berger–Colmez impose on $\mathcal{V}$, and recall a result of Chenevier which shows that they are satisfied in our setting.

First we check that $\mathcal{V}$ is indeed “une famille de représentations $\ell$-adiques” in the sense of Berger–Colmez [BC08, sect. 2.3]. Since $\mathcal{E}$ is reduced, quasi-compact and quasi-separated, Chenevier shows [Che09, Cor. 3.17] that there exists a pair $(\Lambda, \mathcal{B})$ where $\mathcal{B} \subset B$ is an $\mathcal{O}_L$-subalgebra, topologically of finite type and such that $\mathcal{B}[[1/\ell]] = B$, and $\Lambda \subset \mathcal{V}$ is an $\mathcal{B}$-submodule, free of finite rank, equipped with a continuous $\mathcal{I}$-action, such that $\Lambda[[1/\ell]] = \mathcal{B}$ as $B[[1/\ell]]$-module. In particular $\mathcal{V}$ is a family in the sense of Berger–Colmez.

Second, by the assumptions made at the beginning of this section, the infinite place $v_{\infty}$ corresponds via the isomorphism $i$ to a place $\lambda$ above $\ell$ which is different from $v_{\text{St}}$. Since we restricted the Galois representation to the decomposition group at $v_{\text{St}}$ (as opposed to the place $\lambda|\ell$), the Hodge-Tate weights are constant in the family of representations $\mathcal{V}$ of $\Gamma_{v_{\text{St}}}$. By the construction of the endomorphism $\mathcal{E}^{\mathcal{I}}$, the classical points correspond to automorphic representations which have Iwahori fixed vectors at all the $F$-places above $\ell$. In particular, the family $\mathcal{V}$ is semi-stable at a dense set of classical points (Theorem 2.4.(v)). Berger–Colmez require in their
Theorem C that the representation is semi-stable with Hodge-Tate weights in some fixed integral interval $I \subset \mathbb{Z}$. As our weights are constant, this condition is verified.

At this point we have checked all the conditions in Theorem C of Berger-Colmez for our $V$, hence their theorem applies, so the module $D_{\text{std}}(V)$ is locally free of rank $2n+1$. Let $x \in \mathcal{X}$ be a point. By [BC08, Thm. C.4] we have $\kappa(x) \otimes_B D_{\text{std}}(V) \cong D_{\text{std}}(V_x)$. Moreover, the ring $B_\mathbb{A}$ comes with a nilpotent operator $N$ and a Frobenius $q$. Thus, on $D_x = D_{\text{std}}(V_x)$ we have a nilpotent $L \otimes q_1 B$-endomorphism $N_q : D_x \to D_x$, and an action of elements $\sigma \in W_L$, given by $q^{-\tau_{\sigma}}$ where $\tau_{\sigma} : W_L \to \mathbb{Z}$ is the map so that $\sigma \in W_L$ acts on the residue field of $\mathcal{X}$ via $x \mapsto q^{\tau_{\sigma}(x)}$ with $q$ the cardinality of the residue field of $L$. Similarly, on the module $D_{\text{std}}(V)$ the element $N \in B_\mathbb{A}$ defines a family of operators $N_\mathcal{X}$. Since $\kappa(x) \otimes_B D_{\text{std}}(V) \cong D_{\text{std}}(V_x)$ at the points $x \in \mathcal{X}$, the operator $N$ interpolates the $N_\mathcal{X}$ that we considered above.

**Lemma 13.16.** $\rho_{\mathcal{S}t}[\ell]$ Let $c \in \mathcal{H}^{\text{cl}}_{\text{spin}}$. If $(N_\mathcal{X})^2 \nu = 0 \in A_{\text{std}} \otimes \kappa(c)$ then $\tau_{\mathcal{S}t,\mathcal{S}t}$ is isomorphic to a twist by a character of the Steinberg representation.

**Proof.** This can be proved in the same way as in Lemma 13.15 using local-global compatibility at the places dividing $\ell$ (Theorem 2.4.(iv)).

We drop the assumption that either $v_{\mathcal{S}t} | \ell$ or $v_{\mathcal{S}t}| \ell$ and return to the general setting.

**Proposition 13.17.** Let $\star \in \text{[spin, std]}$. The subset $\mathcal{H}^{\text{cl}}_{\mathcal{S}t,\star} \subset \mathcal{H}$ is Zariski dense and accumulates at all the classical points $\mathcal{H}^{\text{cl}}$.

**Proof.** Since $\mathcal{N}$ is a nilpotent $(2n+1) \times (2n+1)$-matrix, $\mathcal{N}^{2n+1}$ vanishes on $\mathcal{X}$. Consider the locus where $\mathcal{N}$ has maximal order, $\mathcal{N}_{\text{max}} := \{ x \in \mathcal{X} | N_\mathcal{X}^{2n+1} \neq 0 \} \subset \mathcal{X}$. Then $\mathcal{N}_{\text{max}} \subset \mathcal{X}$ is open and nonempty (since $x_\mathcal{X} \in \mathcal{N}_{\text{max}}$) thus dense since $\mathcal{X}$ is irreducible. By Lemmas 13.15 and 13.16 we have the inclusion $\mathcal{N}_{\text{max}} \cap \mathcal{H}^{\text{cl}}_{\mathcal{S}t,\star} \subset \mathcal{H}^{\text{cl}}_{\mathcal{S}t,*}$. Since $\mathcal{N}_{\text{max}} \subset \mathcal{X}$ is open, the lemma follows.

Let $c \in \mathcal{H}^{\text{cl}}_{\mathcal{S}t,\text{spin, std}}$. The automorphic representations $\tau_{\mathcal{G}}$ of $G_\kappa(A_\mathbb{F})$ corresponding to $c$ satisfies (Coh), (\text{spin-reg}) and (\text{St}). Thus there exists a weak transfer of elements $\psi_{\mathcal{G}}$ by Proposition 6.3. By Theorem A, we can attach to $\psi_{\mathcal{G}}$ a Galois representation $\rho_{\mathcal{G}} := \rho_{\mathcal{G}}^\nu : \Gamma_\mathcal{G} \to \text{GSpin}_{2n+1}(\mathbb{Q}_L)$, such that

\begin{equation}
\rho_{\mathcal{G}} \big| (\text{Frob}_v)_{\nu} \sim q_v^{-n(n+1)/4} (\text{spin o } \psi_{\mathcal{G}}) (\text{Frob}_v) \in \text{GSpin}_{2n+1}(\mathbb{Q}_L).
\end{equation}

**Proposition 13.18.** There exists a unique pseudocharacter $T_{\text{spin}} : \Gamma_\mathcal{G} \to \mathcal{O}^{\text{inv}}(\mathcal{X})$ such that $T_{\text{spin}} \otimes \kappa(c) \equiv \text{Tr(\text{spin o } } \rho_{\mathcal{G}}^\nu)$ for all $c \in \mathcal{H}^{\text{cl}}_{\mathcal{S}t,\text{spin, std}}$.

**Proof.** By Proposition 13.17 the subset $\mathcal{H}^{\text{cl}}_{\mathcal{S}t,\text{spin, std}} \subset \mathcal{H}$ is Zariski dense. Thus the proposition follows from Proposition 13.10 and Equation (13.7).

**Step 3.** Construction of the GSpin-valued representation $\rho_{\mathcal{G}}^\nu$. Write $L$ for the function field of $\mathcal{X}$ (so $L = \text{Frac}(B)$). Let $\mathcal{L}$ be an algebraic closure of $L$. Since $B$ is a Tate algebra, it is complete for the Gauss norm. The fields $L$ and $\mathcal{L}$ inherit a Hausdorff topology from (extension of) this Gauss norm. From the pseudocharacter $T_{\text{spin}}$ we obtain a continuous semisimple representation $r_{\text{spin}}(\mathcal{L}) : \Gamma_\mathcal{G} \to \text{GL}_{2n}(\mathcal{L})$ by [Ty91, Thm. 1].

Let $x \in \mathcal{X}$ be a point. The local ring $B_x$ is henselian by [FvdP04, Prop. 7.1.8]; also see the last two lines on p. 192 there. We write $B_x^{\text{sh}}$ for the strict henselization of the local ring $B_x$ at $x$. Since $B_x$ is a local normal domain, its strict henselization $B_x^{\text{sh}}$ is also a local normal domain [St16, Tag 06D1].

**Lemma 13.19.** The representation $r_{\text{spin}}(\mathcal{L})$ is strongly irreducible.

**Proof.** Let $x \in \mathcal{X}^{\text{cl}}_{\mathcal{S}t,\text{spin, std}}$ be a point where $\mathcal{X}$ is smooth. There is a corresponding $\ell$-adic representation $r_{\ell} : \Gamma \to \text{GL}_{2n}(\kappa(\mathcal{X}))$. By extending the base $\ell$-adic field $E$ if necessary, we may assume that $r_{\ell}$ has coefficient field $\kappa(\mathcal{X})$. The pseudocharacter $T_{\text{spin}}$ localized at $x$ gives rise to a true representation $r : \Gamma \to \text{GL}_{2n}(B_x^{\text{sh}})$ [Rouquier and Nyssen’s theorem [Rou96, Nys96]]. Let $\Gamma' \subset \Gamma$ be an open subgroup. Taking the group algebra, we obtain a morphism of $B_x^{\text{sh}}$-algebras $r_{\Gamma'} : B_x^{\text{sh}}[\Gamma'] \to M_{2n}(B_x^{\text{sh}})$. The residual representation $r_{\Gamma'} \otimes_{B_x^{\text{sh}}} B_x^{\text{sh}}/m_x$ coincides with $\text{spin o } \rho_{\mathcal{G}}^\nu |_{\Gamma'}$ (their trace characters agree). Hence $r_{\Gamma'} \otimes B_x^{\text{sh}}/m_x$ is absolutely irreducible. By Wedderburn’s theorem, the map $r_{\Gamma'} \otimes B_x^{\text{sh}}/m_x$ has image equal to $M_{2n}(B_x^{\text{sh}}/m_x)$. By Nakayama’s lemma, the map $r_{\Gamma'}$ is surjective, so $r_{\Gamma'} \otimes \mathcal{L}$ is surjective as well. Hence $r_{\text{spin}}(\mathcal{L})_{\Gamma'}$ is irreducible for all $\Gamma' / \Gamma$.

By taking the trace of the dual representation $r_{\text{std}}(B)^{\vee}$, we obtain the pseudocharacter $T_{\text{std}}^{\vee}$ with values in $B$. In fact, $T_{\text{std}}$ is equal to $T_{\text{std}}^{\vee}$ because $T_{\text{std}} \otimes \kappa(c) = T_{\text{std}}^{\vee} \otimes \kappa(c)$ for all $c \in \mathcal{H}^{\text{cl}}_{\text{reg}}$ and $\mathcal{H}^{\text{cl}}_{\text{std}}$ is Zariski dense in $\mathcal{X}$.
\( (\text{Lemma } 13.14) \) Write \( r_{\text{std}}(L) = r_{\text{std}}(B) \otimes_B L \). We have \( \text{Tr } r_{\text{std}}(L) = \text{Tr } r_{\text{std}}(L) \), and therefore \( r_{\text{std}}(L) \) induces a representation \( r_{SO}: \Gamma \to SO_{2n+1}(L) \).

Propositions 3.5 and 4.4 are true (without changing the proofs) more generally for any coefficient field \( C \), which is of characteristic 0, algebraically closed and has a Hausdorff topology. In particular we can lift \( r_{SO}(L) \) to a continuous representation \( \tilde{r}_{SO}(\Gamma): \Gamma \to \text{GSpin}_{2n+1}(\bar{L}) \).

**Lemma 13.20.** Let \( \varphi: \text{GL}_{N_1}(B) \to \text{GL}_{N_2}(B) \) be a \( B \)-morphism, and assume that \( M \in \text{GL}_{N_1}(\bar{T}) \) is some matrix with \( \text{Tr}(M^j) \in B \) for all integers \( j \geq 1 \). Then \( \text{Tr}(\varphi(M)^j) \in B \) for all \( j \geq 1 \).

**Proof.** Since the map \( \varphi \) sends \( M_{\text{std}} \) to \( \varphi(M)_{\text{std}} \), we may and do assume \( M \) is semisimple. Inside the group \( \text{GL}_{N_1}(\bar{T}) \) the matrix \( M \) has the same characteristic polynomial as its companion matrix

\[
\text{Comp}_M = \begin{pmatrix}
0 & 0 & \cdots & 0 & -s_0 \\
0 & 0 & \cdots & 0 & -s_1 \\
0 & 0 & \cdots & 0 & -s_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & -s_{n-1}
\end{pmatrix} \in \text{GL}_{N_1}(\bar{T}),
\]

where the \( s_i \) are (up to signs) the symmetric functions in the eigenvalues of \( M \). The traces \( \text{Tr}(M^j) \) are the power sums in the eigenvalues of \( M \). The Newton identities express the symmetric functions in terms of these power sums, and vice versa. The coefficients of these expressions only involve rational numbers, so the \( s_i \) lie in \( B \supseteq \mathbb{Q} \). Hence \( \text{Comp}_M \) lies in \( \text{GL}_{N_1}(B) \). The matrix \( \varphi(\text{Comp}_M) \) in \( \text{GL}_{N_1}(B) \) has the same characteristic polynomial as the matrix \( \varphi(M) \in \text{GL}_{N_1}(\bar{T}) \). Therefore \( \text{charpol}(\varphi(M)) = \text{charpol}(\varphi(\text{Comp}_M)) \) has coefficients in \( B \). The lemma now follows by applying the Newton identities again.

**Proposition 13.21.** There exists a continuous character \( \chi^\prime: \Gamma_\mathcal{S} \to \mathbb{C}^n \) such that the \( 2^n \)-dimensional representations \( \chi \otimes \text{spin } \circ \tilde{r}_{SO}(\Gamma) \) and \( r_{\text{spin}}(\bar{T}) \) are conjugate.

**Proof.** Write \( T_{SO} \) (resp. \( T_{GL} \)) for the maximal torus of \( SO_{2n+1} \) (resp. \( GL_{2n+1} \)), and \( W_{SO} \) (resp. \( W_{GL} \)) for its Weyl group. Write \( X^*(T_{GL}) \otimes \text{dom} \) for the dominant weights with respect to the upper triangular Borel subgroup, and if \( \xi \in X^*(T_{GL}) \otimes \text{dom} \), \( \chi_{\xi} \) for the corresponding highest weight representation. By Steinberg’s theorem \( [\text{Ste78}] \) (cf. Proposition 10.3 and the remark following it) the finite mapping \( T_{SO}/W_{SO} \to T_{GL}/W_{GL} \) is injective on \( \mathbb{C} \)-points and therefore a closed immersion. In particular the mapping on global sections

\[
\Omega[\text{Tr } \chi_{\xi} \mid \xi \in X^*(T_{GL}) \otimes \text{dom}] = \mathcal{O}(T_{GL}/W_{GL}) \to \mathcal{O}(T_{SO}/W_{SO})
\]

is surjective. Let \( s: \text{PGL}_{2n} \to \text{GL}_N \) be a linear representation. Consider the representation \( \eta = [s \circ \text{spin}] : SO_{2n+1,\mathbb{Q}} \to \text{GL}_{N,\mathbb{Q}} \). Since \( \text{Tr } \eta \in \mathcal{O}(T_{SO}/W_{SO}) \), we get from (13.8) a finite sum \( \text{Tr } \eta = \sum_{i=1}^{k} a_i \cdot \text{Tr } (\eta_i \circ \text{std}) \) where the \( \eta_i \) are certain irreducible representations of \( GL_{2n+1,\mathbb{Q}} \) and the \( a_i \) are rational numbers. We define the pseudocharacter \( T_{\eta}: \Gamma \to B \) by \( T_{\eta}(v) = \sum_{i=1}^{k} a_i \cdot \text{Tr } (\eta_i \circ r_{\text{std}}(B))(v) \). Let \( v \in S \). We have for all \( c \in \mathcal{Z}_\text{std} \) and all integers \( j \geq 1 \),

\[
\text{Tr } r_{\text{std}}(B)(Frob_v^j) \otimes_B \chi(c) \sim \text{Tr } \eta_{\circ \text{spin}}(\phi_{\xi_{\ell,v}}^j)(Frob_v^j) \in \mathcal{O}(T_{SO}/W_{SO})
\]

(we fixed an embedding \( \chi(c) \in \mathcal{O}_\mathcal{S} \)). Hence also, for \( i = 1, \ldots, k \),

\[
(\eta_i \circ r_{\text{std}}(B)(Frob_v^j)) \otimes_B \chi(c) \sim \text{Tr } i(\eta_i \circ \text{std}) \otimes_B \phi_{\xi_{\ell,v}}^j(Frob_v^j) \in \mathcal{O}(T_{SO}/W_{SO})
\]

Taking traces and the linear combination \( \sum_{i=1}^{k} a_i \cdot (\ast) \) on both sides, we get for all \( j \geq 1 \)

\[
T_{\eta}(\text{Frob}_v^j) \otimes_B \chi(c) = \text{Tr } (s \circ \text{spin } \circ \phi_{\xi_{\ell,v}}^j)(\text{Frob}_v^j) \in \mathcal{O}_\mathcal{S}.
\]

On the other hand we have the representation \( r_{\text{spin}}(\bar{T}) \), which we can compose with \([s \circ \text{spin}]: \text{GL}_{2n}(\mathbb{Q}) \to \text{GL}_N(\mathbb{Q})\) (where \( P \) is the surjection of \( GL_{2n} \) onto \( \text{PGL}_{2n} \)). Since \( \text{Tr } r_{\text{spin}}(\bar{T})(\text{Frob}_v^j) \) lies in \( B \) (it is equal to \( T_{\text{spin}}(\text{Frob}_v^j) \in B \)), we get from Lemma 13.20 the integrality

\[
\text{Tr } (s \circ P \circ r_{\text{spin}}(\bar{T}))(\text{Frob}_v^j) \in B,
\]

for all \( j \geq 1 \). At the classical points \( c \in \mathcal{Z}_\text{std,spin} \) we have

\[
\text{Tr } (s \circ P \circ r_{\text{spin}}(\bar{T}))(\text{Frob}_v^j) \otimes_B \chi(c) \in \mathcal{O}(T_{SO}/W_{SO})
\]

(13.9)
The right hand sides of (I3.10) and (I3.9) agree, thus
\[ (\forall c \in \mathcal{X}^{\text{cl}, \text{spin}}_{\text{std}, \text{spin}} : \ T_j (\text{Frob}_j^L) \otimes_B \kappa(c) = \text{Tr} (s \circ P \circ r_{\text{spin}}(\text{Frob}_j)) (\text{Frob}_j^L) \otimes_B \kappa(c), \]
\[ j \leq 1 \]
\[ T_j (\text{Frob}_j^L) = \text{Tr} (s \circ P \circ r_{\text{spin}}(\text{Frob}_j)) (\text{Frob}_j^L) \in B. \]

Expand $T_j (\text{Frob}_j^L)$:
\[ T_j (\text{Frob}_j^L) = \sum_{i=1}^k a_i \cdot \text{Tr} (\eta_i \circ r_{\text{rad}}(B)) (\text{Frob}_j^L) = \sum_{i=1}^k a_i \cdot \text{Tr} (\eta_i \circ \text{std} \circ r_{\text{SO}}(L)) (\text{Frob}_j^L) \]
\[ = \text{Tr} (\eta \circ r_{\text{SO}}(L)) (\text{Frob}_j^L) = \text{Tr} (s \circ \text{spin} \circ r_{\text{SO}}(L)) (\text{Frob}_j^L). \]

The conclusion is that for all $j \geq 1$
\[ \text{Tr} (s \circ P \circ r_{\text{spin}}(\text{Frob}_j^L)) (\text{Frob}_j^L) = \text{Tr} (s \circ \text{spin} \circ r_{\text{SO}}(L)) (\text{Frob}_j^L), \]
and hence
\[ \left( s \circ P \circ r_{\text{spin}}(\text{Frob}_j) \right)_{\text{ss}} \sim \left( s \circ \text{spin} \circ r_{\text{SO}}(L) \right)_{\text{ss}} \in \text{GL}_N(\ell). \]

Steinberg [Ste78, Thm. 2] proved that two semisimple elements of $\text{PGL}_2(\ell)$ are conjugate if and only if they are conjugate in all linear representations $s$ of $\text{PGL}_2$. Thus for all $v \in S$, (I3.11)
\[ \left( P \circ r_{\text{spin}}(\ell) \right)_{\text{ss}} \sim \left( s \circ r_{\text{SO}}(\ell) \right)_{\text{ss}} \in \text{PGL}_2(\ell). \]

By Lemma 13.19 the representation $r_{\text{spin}}(\ell)$ is strongly irreducible and thus by Proposition 4.4 and (I3.11) there exists some character $\chi : \Gamma \rightarrow \ell^\ast$ such that $\text{spin} \circ r_{\text{SO}}(\ell) = \chi r_{\text{spin}}(\ell). \quad \square$

Let $\chi$ be as in Proposition 13.21 and define $r(\ell) := \chi \cdot r_{\text{SO}}(\ell) : \Gamma \rightarrow \text{GSpin}_{2n+1}(\ell)$. We embed $B_{x_n}^b$ in $\ell$. We would like to show that $r(\ell)$ has image in $\text{GSpin}_{2n+1}(B_{x_n}^b)$ so that we can specialize $r(\ell)$ at the point $x_n \in \mathcal{X}^{\text{cl}}$. Write $r_{\text{std}}(B_{x_n}^b)$ for the representation of the standard $B_{x_n}^b$.

**Lemma 13.22.** We have $r_{\text{std}}(B_{x_n}^b) \simeq r_{\text{std}}(B_{x_n}^b)^\text{v}$. 

**Proof.** In [Car94, Thm. 11] Carayol proves the Brauer-Nesbitt Theorem over local rings, under the condition that the residual representation is absolutely irreducible. Carayol's condition is satisfied for $r_{\text{std}}(B_{x_n}^b)$: We have
\[ r_{\text{std}}(B_{x_n}^b) \otimes B_{x_n}^b/m_{x_n} \cong \rho_{\text{sc}}, \]
which is indeed irreducible by Theorem 2.4.(vii). Thus the equality $\text{Tr} r_{\text{std}}(B_{x_n}^b) = \text{Tr} r_{\text{std}}(B_{x_n}^b)^\text{v}$ implies the lemma. \quad \square

By Lemma 13.22, $r_{\text{std}}(B_{x_n}^b)$ is valued in $\text{SO}(B_{x_n}^b,(\cdot,\cdot))$ for some perfect paring $(\cdot,\cdot)$ on $(B_{x_n}^b)^{2n+1}$. We write $r_{\text{SO}}(B_{x_n}^b) : \Gamma \rightarrow \text{SO}_{2n+1}(B_{x_n}^b)$ for the induced representation. In the rest of this section, whenever we write $\text{SO}_{2n+1}(B_{x_n}^b)$ or $\text{GSpin}_{2n+1}(B_{x_n}^b)$ we refer to the special orthogonal group or general spinor group constructed with respect to this paring.

**Proposition 13.23.** There exists an element $g \in \text{GSpin}_{2n+1}(\ell)$ such that the conjugated representation $g r(\ell) g^{-1}$ has image in $\text{GSpin}_{2n+1}(B_{x_n}^b)$ and $g \circ r(\ell) g^{-1} = r_{\text{SO}}(B_{x_n}^b)$.

**Proof.** Note that the map $\text{GSpin}_{2n+1} \rightarrow \text{SO}_{2n+1}$ is surjective in the étale topology in characteristic prime to two. (The double covering $\text{Spin}_{2n+1} \rightarrow \text{SO}_{2n+1}$ is already surjective.) Since $B_{x_n}^b$ is strictly henselian, the map $\text{GSpin}_{2n+1}(B_{x_n}^b) \rightarrow \text{SO}_{2n+1}(B_{x_n}^b)$ is surjective as well. In particular the square in the diagram below is Cartesian.

\[ \xymatrix{ \Gamma \ar@{->}[r] & \text{GSpin}_{2n+1}(B_{x_n}^b) \ar@{->}[d]^{q} \ar@{->}[r] & \text{GSpin}_{2n+1}(\ell) \ar@{->}[d]^{q} \\
\text{SO}_{2n+1}(B_{x_n}^b) \ar@{->}[r] & \text{SO}_{2n+1}(\ell) } \]

Observe that the $\text{SO}_{2n+1}(\ell)$-valued representations $g \circ r(\ell)$ and $[\text{std} \circ r_{\text{SO}}(B_{x_n}^b)] \otimes_{B_{x_n}^b} \ell$ become isomorphic as $\text{GL}_{2n+1}(\ell)$-valued representations. In the proof of Proposition 13.17 we saw that $\mathcal{F}_{\text{max}}$ is a non-empty open in $\mathcal{X}$. Consequently, $\text{std} \circ q \circ r(\ell)$ is irreducible, and hence semi-simple. Thus, by Proposition B.1 there exist $g \in \text{SO}_{2n+1}(\ell)$ such that $g (q \circ r(\ell)) g^{-1} = r_{\text{SO}}(B_{x_n}^b)$. Choose a lift $\tilde{g}$ of $g$ via the surjection $\text{GSpin}_{2n+1}(\ell) \rightarrow \text{SO}_{2n+1}(\ell)$. Replacing $r(\ell)$ by $\tilde{g} r(\ell) \tilde{g}^{-1}$ we may assume that $q \circ r(\ell)$ and $r_{\text{SO}}(B_{x_n}^b)$ are (not just isomorphic but)
equal. In particular the image of \( r(\overline{L}) \) under \( q \) lands in \( SO_{2n+1}(B_{x_i}^\text{sh}) \). We conclude that the image of \( r(\overline{L}) \) is contained in \( GSpin_{2n+1}(B_{x_i}^\text{sh}) \).

Replace \( r(\overline{L}) \) by its conjugate \( gr(\overline{L})g^{-1} \), as constructed in Proposition 13.23. The residue field \( \kappa(B_{x_i}^\text{sh}) \) of \( B_{x_i}^\text{sh} \) coincides with \( \overline{\mathcal{O}}_q \). We define the representation

\[
\rho_i^\text{C} := r(\overline{L}) \otimes_{B_{x_i}^\text{sh}} \kappa(B_{x_i}^\text{sh}); \quad \Gamma_S \to GSpin_{2n+1}(\overline{\mathcal{O}}_q).
\]

Unwinding the above constructions and definitions, we have

**Proposition 13.24.** We have

\[
\begin{align*}
(i) & \quad \text{Tr}(\text{spin} \circ \rho_i^\text{C}) = T_{\text{spin}} \otimes_B \kappa(x_i) \\
(ii) & \quad \text{Tr}(\text{std} \circ \rho_i^\text{C}) = T_{\text{std}} \otimes_{GSpin(\overline{\mathcal{O}}_q)} \kappa(c_i) \\
(iii) & \quad \text{Tr}(\mathcal{N} \circ \rho_i^\text{C}) = T_{\mathcal{N}} \otimes_{GSpin(\overline{\mathcal{O}}_q)} \kappa(c_i),
\end{align*}
\]

as pseudocharacters \( \Gamma \to \overline{\mathcal{O}}_q \).

**Proof.** We compute

\[
\begin{align*}
\text{Tr}(\text{spin} \circ \rho_i^\text{C}) &= \text{Tr}(r(\overline{L}) \otimes_{B_{x_i}^\text{sh}} \kappa(B_{x_i}^\text{sh})) \\
&= \text{Tr}(\text{spin} \circ r(\overline{L})) \otimes_{B_{x_i}^\text{sh}} \kappa(B_{x_i}^\text{sh}) \quad \text{(Equation (13.12))} \\
&= \text{Tr}(\text{spin}(\overline{L}) \otimes_{B_{x_i}^\text{sh}} \kappa(B_{x_i}^\text{sh}) \quad \text{(Equation (13.2))}) \\
&= T_{\text{spin}} \otimes_B \kappa(x_i) \quad \text{(Equation (13.6)).}
\end{align*}
\]

This proves (i). Using the above computation, we have also

\[
\begin{align*}
\text{Tr}(\text{std} \circ \rho_i^\text{C}) &= \text{Tr}(\text{std} \circ r(\overline{L})) \otimes_{B_{x_i}^\text{sh}} \kappa(B_{x_i}^\text{sh}) \\
&= \text{Tr}(\text{std}(\overline{L}) \otimes_{B_{x_i}^\text{sh}} \kappa(B_{x_i}^\text{sh}) \quad \text{(Equation (13.23))} \\
&= T_{\text{std}} \otimes_{GSpin(\overline{\mathcal{O}}_q)} \kappa(c_i) \quad \text{(Equation (13.6)).}
\end{align*}
\]

This proves (ii). The computation for (iii) is similar.

Let \( v \) be an \( F \)-place that is not in \( S \). Let \( \psi_{\text{spin}}, \psi_{\text{std}}, \psi_{\mathcal{N}} \in X^*(\overline{\mathcal{T}})\text{dom} \) be the highest weights of the representations \( \text{spin}, \text{std} \) and \( \mathcal{N} \) of \( GSpin_{2n+1} \). We write \( \mathcal{T}_{\text{spin}}(v,j) = \mathcal{T}_{\text{std}}(v,j) = \mathcal{T}_{\psi_{\text{spin}}}(v) \) and \( \mathcal{T}_{\mathcal{N}}(v,j) = \mathcal{T}_{\psi_{\text{std}}}(v) \) and \( \mathcal{T}_{\psi_{\mathcal{N}}}(v,j) = \mathcal{T}_{\psi_{\mathcal{N}}}(v) \in \overline{\mathcal{T}} \) (Recall, \( \psi_{\mathcal{N}}(v) \in \overline{\mathcal{T}} \) was defined in (13.3)).

**Corollary 13.25.** For all \( v \in S \) and \( j \in \mathbb{Z}_{\geq 1} \) we have

\[
\begin{align*}
(i) & \quad \text{Tr}(\text{spin} \circ \rho_i^\text{C}(\text{Frob}^j_v)) = (\delta^* \circ \psi_{\text{spin}})(\mathcal{T}_{\text{spin}}(v,j)) \otimes_B \kappa(x_i) \\
(ii) & \quad \text{Tr}(\text{std} \circ \rho_i^\text{C}(\text{Frob}^j_v)) = \psi_{\text{std}}(\mathcal{T}_{\text{std}}(v,j)) \otimes_{GSpin(\overline{\mathcal{O}}_q)} \kappa(c_i) \\
(iii) & \quad \text{Tr}(\mathcal{N} \circ \rho_i^\text{C}(\text{Frob}^j_v)) = \psi_{\mathcal{N}}(\mathcal{T}_{\mathcal{N}}(v,j)) \otimes_{GSpin(\overline{\mathcal{O}}_q)} \kappa(c_i).
\end{align*}
\]

**Proof.** For all \( c \in \mathcal{X}_{\text{St,spin, std}} \) and all \( j \in \mathbb{Z}_{\geq 1} \) we have

\[
\begin{align*}
T_{\text{spin}}(\text{Frob}^j_v) \otimes_B \kappa(c) &= (\delta^* \circ \psi_{\text{spin}})(\mathcal{T}_{\text{spin}}(v,j)) \otimes_B \kappa(c) \quad \text{by Equation (13.7).}
\end{align*}
\]

We established in Proposition 13.17 that \( \mathcal{X}_{\text{St, spin, std}} \subset \mathcal{X} \) is dense. Hence (13.13) holds for all points in \( \mathcal{X} \), so also for \( x_i \in \mathcal{X} \). Part (i) follows from Proposition 13.24 and Equation (13.13). The proofs for (ii) and (iii) are similar, the required ingredients for (ii) are: Equation (13.4) (the required equality at the points \( c \in \mathcal{X}_{\text{std}} \), above Equation (13.5) + Corollary 13.9 (for the density of those points).

**Theorem 13.26.** Let \( \pi \) be as in Theorem B, and assume conditions (Aux\( _{\mathcal{E}}^{-i} \)), \( i = 1, 2, \) and (Even). Then Theorem B is true for \( \pi \).
Proof. Let $\beta : T \to \overline{Q}_l$ be the system of eigenvalues attached to the point $c_{\pi} \in E^T$. Then, for all $T \in T$, $iTr_{\pi}(T) = \beta(T) = \Psi(T) \otimes G_{\ell} \times \kappa(c_{\pi})$. Taking $T = T_{\text{spin}}(v,j)$, we obtain from Corollary 13.25(i),

$$i\eta_{c_{\pi}}^{n(n+1)/4} Tr (\text{spin} \circ \phi_{n_{\pi},p})(\text{Frob}_v) = \Psi(T_{\text{spin}}(v,j)) \otimes G_{\ell} \otimes \kappa(c_{\pi})$$

$$= (\delta^* \circ \Psi_{|G_{\ell}^C})(T_{\text{spin}}(v,j)) \otimes \kappa(c_{\pi})$$

$$= \text{Tr} (\text{spin} \circ \rho_{n_{\pi}}^C)(\text{Frob}_v).$$

for all $v \not\in S$ and all $j \geq 1$. Thus for all $v \not\in S$, we have (a) in

$$i\eta_{c_{\pi}}^{n(n+1)/4} (\text{spin} \circ \phi_{n_{\pi},p})(\text{Frob}_v) \sim (\text{spin} \circ \rho_{n_{\pi}}^C)(\text{Frob}_v) \in \text{GL}_{2n}(\overline{Q}_l)$$

(b) $i(\text{std} \circ \phi_{n_{\pi},p})(\text{Frob}_v) \sim (\text{std} \circ \rho_{n_{\pi}}^C)(\text{Frob}_v) \in \text{GL}_{2n+1}(\overline{Q}_l)$,

(c) $i(\mathcal{N} \circ \phi_{n_{\pi},p})(\text{Frob}_v) = (\mathcal{N} \circ \rho_{n_{\pi}}^C)(\text{Frob}_v) \in \mathcal{O}_{\overline{Q}_l}$. (I3.14)

Statements (b) and (c) are deduced in the same way.

We check that the representation $\rho_{n_{\pi}}^C$ from (I3.12) satisfies properties (i)-(v) of Theorem A. Part (i) follows from (I3.14),(b,c) and Proposition B.1.

By Corollary 10.5, two semisimple elements in the group $\text{GSp}_{2n+1}(\overline{Q}_l)$ are conjugate if and only if they have the same spinor norm and are conjugate in the spin representation. Thus (I3.14),(a,c) implies Part (ii).

We check (iii). The trace character of $\text{spin} \circ \rho_{n_{\pi}}^C$, multiplied by a nonzero integer, agrees with the trace of $\rho_{n_{\pi}}^{\text{Shim}}$ in Proposition 8.2. Thus $\text{spin} \circ \rho_{n_{\pi}}^C$ appears in $H^2(S_F, L_\ell)$, which is potentially semi-stable by Kisin [Kis02, Thm. 3.2]. Statements (iii,a) and (iii,b) can be proved using the same argument we carried out in the proof of part (iii) of Theorem 10.1. Statement (iii,c) follows from Proposition 12.3.

Statement (iv) can be deduced as in the proof of (iv) in Theorem 1.5 via (i) of the theorem.

Finally statements (v) and (vi) follow under the assumption of (HT$_1$) and (HT$_2$) from Lemma 3.3 and Proposition 5.4 respectively. □

14. Patching

In Section 13 we constructed $\rho_{n_{\pi}}$ under conditions (Aux$_{\ell}$-1) and (Aux$_{\ell}$-2) on the prime number $\ell$ and condition (Even) that the degree $[F : Q]$ is even. We use a patching/base change argument to remove these conditions and prove Theorem B.

Let $\pi$ be a cuspidal automorphic representation of $\text{GSp}_{2n}(A_F)$ satisfying (St) and (L-coh). Consider an irreducible sub $\text{Sp}_{2n}(A_F)$-representation $\pi^b$ of $\pi$. We write $\mathcal{I}_1(\pi)$ for the set of finite extensions $F'/F$ such that

- $F'/F$ is soluble and totally real with $[F' : Q]$ even,
- $F'$ has at least two places above $\ell$,
- $F'$ is large enough such that for every $F$-place $\lambda|\ell$ and for every $F'$-place $\lambda'|\lambda$, the $L$-parameter $\phi_{\pi_{\lambda'}}$ and the central character $\omega_{\pi_{\lambda}}$ are unramified when restricted to $W_{F'_{\ell}}$.

For each $F' \in \mathcal{I}_1(\pi)$, Proposition 6.5 provides us with a weak base change $\pi_{F'}$ of $\pi$, which still satisfies (St) and (L-coh). By Lemma 2.7 and its proof, all local components of $\pi^b$ (resp. $\pi$, $\pi_{F'}$) are tempered (resp. essentially tempered). The $L$-parameter $\phi_{\pi_{F'}_{\lambda'}}$ above is uniquely determined by the condition that its $L$-packet contains $\pi^b_{\lambda'}$, cf. [Art03, Thm. 1.5.1].

Lemma 14.1. For each $F' \in \mathcal{I}_1(\pi)$, statements (i)-(v) of Theorem A are true for $\pi_{F'}$ (in place of $\pi$) except that (iii,c) holds under the extra condition that $\pi$ is unramified at all $\ell$-adic places.

Proof. Obviously $F'$ satisfies (Even) and (Aux$_{\ell}$-1). A large part of the proof would be to show that $\pi_{F'}$ satisfies condition (Aux$_{\ell}$-2). For a technical reason we prove only a weaker form, which still suffices for our purpose.

Choose $\pi^b_{F'} \subset \pi_{F'}$ to be any irreducible sub $\text{Sp}_{2n}(A_F)$-representation. We claim that $\phi_{\pi^b_{F'}_{\lambda'}}|W_{F'_{\ell}}$ is unramified for each $F'$-place $\lambda'$ above $\ell$. We obtain a cuspidal automorphic representation $\pi^b_{F'}$ of $\text{GL}_{2n+1}(A_F)$ from $\pi^b$ as in $\S 2$. Write $\pi^b_{F'}$ for the Arthur–Clozel base change of $\pi^b$ to $F'$. It is easy to check that $\pi^b_{F'}$ is a weak base change of $\pi^b$ to $\text{Sp}_{2n}(A_F)$ and has a Steinberg component at a finite place. Since $\pi^b_{F'}$ is a weak transfer of $\pi^b$, by a direct computation of Satake parameters, it follows from Arthur's classification [Art03, Thm. 1.5.2] that the $L$-parameter $\phi_{\pi^b_{F'}_{\lambda'}}$ transfers to $\phi_{\pi^b_{F'}_{\lambda'}}$ at each place $v$ of $F'$ via the standard embedding $\text{SO}_{2n+1}(C) \hookrightarrow \text{GL}_{2n+1}(C)$. In particular the parameter $\phi_{\pi^b_{F'}_{\lambda'}}$ is unramified on $W_{F'_{\ell}}$ for each $\lambda'|\ell$ by the third condition on $F'$ above (and the fact that the Arthur–Clozel base change corresponds to restriction on the Galois side via the local Langlands correspondence).
The claim implies by [Taf12, Lem. 5.1.1] (attributed to Moeglin– Waldspurger) that the (tempered) L-packet for \( \phi_{\pi_F} \) contains a member with Iwahori-fixed vectors, which we denote by \( \sigma_{\pi_F}^b \) at each \( F \)-place \( \lambda \). Define an irreducible admissible representation \( \sigma^b \) of \( \Sp_{2n}(\mathbb{A}_F) \) from \( \pi^b_F \) by changing the \( \lambda' \)-component from \( \pi^b_F \) to \( \sigma_{\pi_F}^b \). Then \( \sigma^b \) is cuspidal automorphic by [Art13, Thm 1.5.2]. (The switching of a local component does not affect automorphy since the global parameter for \( \sigma \) is simple generic, cf. Corollary 2.2. The cuspidality follows from the existence of a Steinberg component.)

By [Xub, Lem. 6.2, Cor. 6.3] we can choose an irreducible smooth representation \( \sigma' \) of \( \GSp_{2n}(F_{\lambda'}) \) such that its restriction to \( \Sp_{2n}(F_{\lambda'}) \) contains \( \sigma_{\pi_f}^b \) as a direct summand. Denote by \( \sigma \) the irreducible admissible representation obtained from \( \pi_F \) by changing the \( \lambda' \)-component to \( \sigma' \) at each \( \lambda' \). We apply [Xua, Cor. 5.6] (taking his \( \tilde{\pi} \) and \( \pi \) to be our \( \sigma \) and \( \sigma^b \)) to see that we may replace \( \sigma \) by a suitable character twist such that \( \sigma \) appears in the \( L^2 \)-discrete spectrum and still contains \( \sigma_{\pi_F}^b \) in its restriction to \( \Sp_{2n}(F_{\lambda'}) \) at every \( \lambda' \). We rename \( \sigma \) to be such a twist.

Since (weak) base change is compatible with base change of central characters, we know in particular that \( \omega_{\pi_f} \) is unramified at all \( F' \)-places above \( \ell \) by the third condition on \( F' \) above. Choose an Iwahori subgroup \( Iw_{\lambda'} \) of \( \GSp_{2n}(F_{\lambda'}) \) and put \( Iw_{\lambda'} := Iw_{\lambda'} \cap \Sp_{2n}(F_{\lambda'}) \). Then the inclusion of \( Iw_{\lambda'} \) in \( \GSp_{2n}(F_{\lambda'}) \) and the similitude character induce isomorphisms

\[
\frac{Iw_{\lambda'}}{Iw_{\lambda'} \cap Z(F_{\lambda'})} \cong \frac{\GSp_{2n}(O_{F_{\lambda'}})}{\Sp_{2n}(O_{F_{\lambda'}}) \cap Z(F_{\lambda'})} \cong O_{F_{\lambda'}}^\times / (O_{F_{\lambda'}}^\times)^2.
\]

Since \( \sigma \) contains a nonzero subspace invariant under \( Iw_{\lambda'} \cap Z(F_{\lambda'}) \), it contains a character \( \chi_{\lambda'} \) of \( Iw_{\lambda'} \). If \( \chi_{\lambda'} \) is nontrivial then we can extend it to a character of \( F'/F^{\text{sep}} \) and then to a character of \( \GSp_{2n}(O_{F_{\lambda'}}) \) via the similitude character. Calling the extension still \( \chi_{\lambda'} \), we see that \( \sigma_{\chi_{\lambda'}} \) has nonzero \( Iw_{\lambda'} \)-invariants.

Globalizing \( \{ \chi_{\lambda'} \}_{\lambda'} \) to a (quadratic) Hecke character \( \chi \) of \( \mathbb{A}_F \), we see that \( \sigma \otimes \chi \) satisfies condition (Aux'-2).

To sum up, the hypotheses of Theorem 13.26 are satisfied for \( F' \) and \( \sigma \otimes \chi \) in place of \( F \) and \( \pi \). Hence there exists a Galois representation \( \rho_{\sigma \otimes \chi} \). We obtain \( \rho_{\sigma} \) by untwisting and then simply put \( \rho_{\rho_{\sigma \otimes \chi}} := \rho_{\sigma} \). It remains to check that (i)-(iv) of Theorem A hold true for \( \rho_{\sigma_{\rho_{\sigma \otimes \chi}}} \) and \( \rho_{\rho_{\sigma \otimes \chi}} \). We have the set of bad places \( \mathcal{S}_{\text{bad}}(\pi_{\rho_{\sigma \otimes \chi}}) \) (resp. \( \mathcal{S}_{\text{bad}}(\sigma \otimes \chi) \)) for \( \pi_{\rho_{\sigma \otimes \chi}} \) (resp. \( \sigma \otimes \chi \)) in the same way as \( \mathcal{S}_{\text{bad}}(\pi) \) for \( \pi \). A priori we have (ii) of the theorem for finite places outside \( \mathcal{S}_{\text{bad}}(\sigma \otimes \chi) \), which may not cover all finite places away from \( \mathcal{S}_{\text{bad}}(\pi_{\rho_{\sigma \otimes \chi}}) \). However we can freely choose the auxiliary character \( \chi \) to be unramified at any finite place not above \( \ell \), so we fully verify part (ii) of Theorem A. (Recall that \( \sigma \) is isomorphic to \( \pi_{\rho_{\sigma \otimes \chi}} \) away from \( \ell \)-adic places by construction.) We easily see (i) and (iv) of the theorem for \( \pi_{\rho_{\sigma \otimes \chi}} \) from those for \( \sigma \otimes \chi \). As before (iii.a) and (iii.b) are deduced as in the proof of Theorem 10.1. Finally we check (iii.c) when \( \pi \) is unramified at all \( \ell \)-adic places. Then Proposition 6.5 ensures that \( \pi_{\rho_{\sigma \otimes \chi}} \) is also unramified at all \( \ell \)-adic places, so condition (Aux'-2) already holds for \( \pi_{\rho_{\sigma \otimes \chi}} \). Applying Theorem 13.26 to \( \pi_{\rho_{\sigma \otimes \chi}} \), we see in particular that \( \pi_{\rho_{\sigma \otimes \chi}} \) is crystalline at \( \ell \)-adic places.

We recall Sorensen’s patching result [Sor08, Thm. 1]. Let \( \mathcal{I} \) be a nonempty collection of finite solvable extensions \( F' \) over \( F \). We say that \( \mathcal{I} \) has uniformly bounded height if the length of the \( \mathbb{Z} \)-module \( \mathbb{Z}[F'/F] \mathbb{Z} \) is uniformly bounded by an integer \( H_F \). For a finite set \( S \) of \( F \)-places, we say that \( \mathcal{I} \) is \( S \)-general if the following property holds: For each field extension \( L \) of \( F' \) (in \( \overline{F} \)) which is a subfield of a member of \( \mathcal{I} \), and for each \( L \)-place \( v \) not lying above any place of \( S \), either (a) \( L \cap F \cap v \) (or \( b \)) there are infinitely many cyclic extensions \( K \) of \( L \) of prime degree such that each \( K \) is a subfield of a member of \( \mathcal{I} \) and \( v \) splits in \( K \).

**Proposition 14.2** (Sorensen). Let \( \mathcal{I} \) be an \( S \)-general collection of finite solvable extensions \( F' \) over \( F \) with uniformly bounded heights in the above sense. Suppose that a collection of isomorphism classes of \( m \)-dimensional semisimple Galois representations \( \{ r_F : \Gamma \rightarrow \text{GL}_m(\overline{Q}_L) \mid F' \in \mathcal{I} \} \) satisfies the following conditions:

- For all \( \sigma \in \text{Gal}(F'/F) \): \( r_{F'} \cong r_{F''} \).
- For all \( F', F'' \in \mathcal{I} \), \( r_{F'}|_{\text{Gal}(F'/F'')} \cong r_{F''} |_{\text{Gal}(F'/F'')} \).

Then there exists a continuous semisimple representation \( r : \Gamma \rightarrow \text{GL}_m(\overline{Q}_L) \) such that for all \( F' \in \mathcal{I} \), we have \( r_{|_{\text{Gal}(F'/F)}} \cong r \).

**Theorem 14.3.** Theorem B is true.

**Proof.** We present a proof under the temporary assumption that \( F \) has at least two \( \ell \)-adic places (i.e. the second condition on \( \mathcal{I}_1(\pi) \)). This assumption will be removed at the very end.

---

14 Only condition (b) is required in [Sor08, Def. 3]. We believe that the possibility for (a) should be allowed; otherwise an \( S \)-general set in Section 1 of that paper may not be \( S \)-general in the sense of loc. cit. In line 5 of the proof of [Sor08, Thm. 1], it should read “Clearly either \( K \in \mathcal{I} \) or \( L_K \) is an \( (S,K) \)-general set of...”
Let \( \pi, \pi^b \), and \( I_1(\pi) \) be as at the start of this section. For each \( F \)-place \( \lambda \) above \( \ell \), let \( d_\lambda \in \mathbb{Z}_{\geq 1} \) denote the order of the image of the inertia subgroup under \( \phi_{\pi^b, \lambda} \). Take \( d \) to be the least common multiple of \( d_\lambda \)'s. Define a subcollection \( \mathcal{I}(\pi) \subset I_1(\pi) \) consisting of \( F' \in \mathcal{I}(\pi) \) such that \( |F' : F| \) divides \( 2d \), so that \( \mathcal{I}(\pi) \) has uniformly bounded height. Choose \( S \) to be the set of all \( \ell \)-adic places of \( F \). We claim that \( \mathcal{I}(\pi) \) is \( S \)-general. Once the claim is proven, we apply Sorensen’s result to \( r_F := \text{spin} \circ \rho_{n_F} \) for \( F' \in \mathcal{I}(\pi) \) to construct
\[
r : \Gamma_F \rightarrow \text{GL}_{2n}(\overline{\mathbb{Q}}_\ell).
\]

Let us verify the claim. Let \( L \) be an extension of \( F \) contained in \( L' \in \mathcal{I}(\pi) \). Without loss of generality we may assume \( L \in \mathcal{I}(\pi) \) so that \( d' := [L' : L] > 1 \). For each \( L \)-place \( \delta \) above an \( \ell \)-adic place \( \lambda \) of \( F \), let \( H_\delta \) denote the image of the inertia subgroup of \( W_{L_\delta} \) under the \( L \)-parameter \( \phi_{\pi^b, \delta} \). The second condition for \( I_1(\pi) \) implies that \( |H_\delta| \) divides \( d' \). An application of Galois theory and weak approximation\(^{15}\) produces a family of extensions \( L(v, w) \) of \( L \), where \( v \) and \( w \) run over places of \( L \) not above \( \ell \), such that

- \( [L(v, w) : L] = d' \),
- \( v \) and \( w \) split completely in \( L(v, w) \),
- the second characterizing property of \( I_1(\pi) \) holds for \( L(v, w) \) (in place of \( F' \)).

By the first condition \( [L(v, w) : \mathbb{Q}] \) is equal to \( [L' : \mathbb{Q}] \). Hence \( L(v, w) \in \mathcal{I}(\pi) \). For each \( v \) and \( w \), fix an intermediate extension \( K(v, w) \) between \( L(v, w) \) and \( L \) such that \( [K(v, w) : L] = 1 \). Then for each fixed \( v \mid \ell \), the collection \( K(v, w) \) for varying \( w \) yields infinitely many non-identical extensions of \( L \) in which \( v \) splits. Therefore \( \mathcal{I}(\pi) \) is \( S \)-general as claimed.

We show that \( r \) factors through \( \text{GSpin}_{2n+1}(\overline{\mathbb{Q}}_\ell) \subset \text{GL}_{2n}(\overline{\mathbb{Q}}_\ell) \) (possibly after conjugation). Then we have a morphism of groups \( \Gamma \rightarrow \pi_0(\text{r}(\Gamma)) \), where \( \pi_0 \) denotes the group of connected components. Let \( \Gamma_F \) be the kernel of the map \( \Gamma \rightarrow \pi_0(\text{r}(\Gamma)) \). Pick \( F'' \in \mathcal{I}(\pi) \) such that \( F'' \) is linearly disjoint to \( F' \). After conjugating \( r \), we may assume that \( r(\Gamma_{F''}) \) is contained in \( \text{GSpin}_{2n+1}(\overline{\mathbb{Q}}_\ell) \). Since the neutral component of the Zariski closure of the image is insensitive to shrinking \( \Gamma \) to an open subgroup, we have \( r(\Gamma_{F''}) = r(\Gamma_F) = r(\Gamma_{F'}) = r(\Gamma_{F''}) \). Consequently,
\[
(\text{14.1}) \quad r(\Gamma_F) = r(\Gamma_{F'}) = r(\Gamma_{F''}) \subset r(\Gamma_{F''}) \subset \text{GSpin}_{2n+1}(\overline{\mathbb{Q}}_\ell).
\]

The subgroups \( \Gamma_F \) and \( \Gamma_{F''} \) generate \( \Gamma_F \) (since \( F' \) and \( F'' \) are linearly disjoint). Hence (14.1) implies \( r(\Gamma_F) \subset \text{GSpin}_{2n+1}(\overline{\mathbb{Q}}_\ell) \). We thus obtain a Galois representation \( \rho_n : \Gamma \rightarrow \text{GSpin}_{2n+1}(\overline{\mathbb{Q}}_\ell) \) such that \( \text{spin} \circ \rho_n = r \).

We now check that properties (i) through (iv) from Theorem A hold for \( \rho_n \). Statements (iii.a) and (iii.b) follow from the arguments for (iii.a) and (iii.b) in the proof of Theorem 10.1 as before. The remaining (i), (ii), (iii,c), and (iv) are either local statements or immediately reduced to local statements, which can be checked by passing to \( F' \in \mathcal{I}(\pi) \) in which the place in question splits completely and applying Lemma 14.1.

Parts (v) and (vi) of Theorem A follow from hypotheses (HT\(_1\)) and (HT\(_2\)) in the same way as in the proof of Theorem 13.26.

Finally we get rid of the temporary assumption that \( F \) has at least two \( \ell \)-adic places. This is done by reducing to the previous case via Proposition 14.2 applied to quadratic extensions of \( F \) in which the unique \( \ell \)-adic place of \( F \) splits.

\[\square\]

15. Galois representations for the exceptional group \( G_2 \)

As an application of our main theorems we realize some instances of the global Langlands correspondence for \( G_2 \) in the cohomology of Siegel modular varieties of genus 3 via theta correspondence, following the strategy of Gross–Savin [GS98]. In particular the constructed Galois representations will be motivic and come in compatible families as such. We work over \( F = \mathbb{Q} \) (as opposed to a general totally real field) mainly because this is the case in [GS98].

More precisely we are writing \( G_2 \) for the split simple group of type \( G_2 \) defined over \( \mathbb{Z} \). Denote by \( G_2^\vee \) the inner form of \( G_2 \) over \( \mathbb{Q} \) which is split at all finite places such that \( G_2^\vee(\mathbb{R}) \) is compact. The dual group of \( G_2 \) is \( G_2(\mathbb{C}) \) and fits in the diagram

\[
\begin{array}{ccccccc}
PGL_2(\mathbb{C}) & \rightarrow & G_2(\mathbb{C}) & \rightarrow & SO_7(\mathbb{C}) & \rightarrow & GL_7(\mathbb{C}) \\
\gamma & & & & & & \\
\downarrow & & & & & & \\
\text{Spin}_7(\mathbb{C}) & \rightarrow & SO_8(\mathbb{C}) & \rightarrow & GL_8(\mathbb{C}) \\
\phi & & & & & & \\
\end{array}
\]

\(^{15}\)The basic idea appears in the proof of [Art13, Lem. 6.2.1], for instance, though the prescribed local properties are somewhat different.
such that $G_2(\mathbb{C}) = SO_7(\mathbb{C}) \cap \text{Spin}_7(\mathbb{C})$. The subgroup $\text{PGL}_2(\mathbb{C})$ is given by a choice of a regular unipotent element of $SO_8(\mathbb{C})$. See [GS98, pp.169-170] for details. Note that the spin representation of $\text{Spin}_7$ is orthogonal and thus factors through $SO_7$. The 8-dimensional representation $G_2(\mathbb{C}) \hookrightarrow \text{GL}_8(\mathbb{C})$ decomposes into 1-dimensional and 7-dimensional irreducible pieces. The former is the trivial representation. The latter factors through $SO_7(\mathbb{C})$. Evidently all this is true with $\overline{Q}_\ell$ in place of $\mathbb{C}$.

The (exceptional) theta lift from each of $G_2$ and $G_2^\vee$ to $\text{PGSp}_6$ using the fact that $(G_2, \text{PGSp}_6)$ and $(G_2^\vee, \text{PGSp}_6)$ are dual reductive pairs in groups of type $E_7$ [GRS97, GS98]. In this section we concentrate on the case of $G_2^\vee$, only commenting on the case of $G_2$ at the end. Every irreducible admissible representation of $G_2(\mathbb{R})$ is finite-dimensional, and both $(\pi, C)$-cohomological and $L$-cohomological since the half sum of all positive roots of $G_2$ is integral. Note that an automorphic representation $\pi$ of $\text{PGSp}_6(\mathbb{A})$ is the same as an automorphic representation of $\text{Sp}_{2n}(\mathbb{A})$ with trivial central character, so we will use them interchangeably. For such a $\pi$ the subgroup $\rho_\pi(\Gamma)$ of $\text{Sp}_{2n}(\overline{Q}_\ell)$ is contained in $\text{Spin}_{2n}(\overline{Q}_\ell)$ by (i) of Theorem A.

**Theorem 15.1.** Let $\sigma$ be an automorphic representation of $G_2^\vee(\mathbb{A})$. Assume that

- $\sigma$ admits a theta lift to an automorphic representation $\pi$ on $\text{PGSp}_6(\mathbb{A})$.
- $\sigma|_{\mathbb{Q}_6}$ is the Steinberg representation at a finite place $\nu|_{\mathbb{Q}_6}$.

Then for each prime $\ell$ and $i: \mathbb{C} \cong \overline{Q}_\ell$, there exists a continuous representation $\rho_{\sigma}: \Gamma \rightarrow G_2(\overline{Q}_\ell)$ such that

1. For every finite place $v \neq \ell$ where $\sigma$ is unramified, $\rho_{\sigma}$ is unramified at $v$. Moreover $(\rho_{\sigma}|_{\mathbb{Q}_v})_{\mathbb{A}} \simeq \phi_{\sigma, v}$ as unramified $L$-parameters for $G_2$.
2. $\rho_{\sigma}$ is de Rham with $\mu_{\mathbb{A}}(\rho_{\sigma}) \cong \mu_{\mathbb{A}}(\phi_{\sigma, v})$. 
3. If $\sigma|_v$ is unramified then $\rho_{\sigma|_v}$ is crystalline.
4. $\zeta \circ \rho_{\sigma} \simeq \rho_\pi$.

Before starting the proof, we recall the basic properties of the theta lift $\pi$. We see from [GS98, §4 Prop. 3.1, §4 Prop. 3.19, §5 Cor. 4.9] that $\pi$ is cuspidal, that $\pi_{\mathbb{Q}_6}$ is the Steinberg representation, and that $\pi|_v$ is unramified whenever $\sigma|_v$ is unramified at a finite place $v$ and the unramified $L$-parameters are related via

$$(\ref{15.1}) \quad \zeta \phi_{\pi|_v} \simeq \phi_{\sigma|_v}.$$  

Furthermore $\pi|_v$ is an $L$-algebraic discrete series representation whose parameter can be explicitly described in terms of $\sigma$ [§3 Cor. 3.9 of loc. cit.]

**Proof.** We apply Theorem B to the above $\pi$ to obtain a continuous representation $\rho_{\pi|_v}: \Gamma \rightarrow \text{Spin}_{2n}(\overline{Q}_\ell)$. Note that spin $\circ \rho_{\pi|_v}$ is a semisimple representation by construction. Since the image of $\rho_{\phi|_v}$ contains a regular unipotent element of $SO_7(\overline{Q}_\ell)$, it follows that $\rho_{\pi|_v}(\Gamma)$ contains a regular unipotent of $\text{Spin}_{2n}(\overline{Q}_\ell)$ and also that of $SO_8(\overline{Q}_\ell)$. By (15.1) and the Chebotarev density theorem, the image of $\rho_{\pi|_v}$ is locally contained in $G_2$ in the terminology of Gross-Savin, so [GS98, §2 Cor. 2.4] implies that $\rho_{\pi|_v}(\Gamma)$ is contained in $G_2(\overline{Q}_\ell)$ (given as $SO_7(\overline{Q}_\ell) \cap \text{Spin}_{2n}(\overline{Q}_\ell)$ for a suitable choice of the embedding $SO_7 \hookrightarrow SO_8$; here spin $: \text{Spin}_{2n} \hookrightarrow SO_8$ is fixed). Hence we have $\rho_{\sigma|_v}$ satisfying (4), namely that $\rho_{\pi|_v} \simeq \zeta \circ \rho_{\sigma|_v}$.

Assertions (1)-(3) of the theorem follow from Theorem B and the fact that the set of Weyl group orbits on the maximal torus (resp. on the cocharacter group of a maximal torus) for $G_2$ maps injectively onto that for Spin$_7$. (The latter can be checked explicitly.)

**Example 2.** There is a unique automorphic representation $\sigma$ of $G_2^\vee(\mathbb{A})$ unramified outside 5 such that $\sigma_5$ is the Steinberg representation and $\sigma|_{\mathbb{Q}_6}$ is the trivial representation [GS98, §1 Prop. 7.12]. Proposition 5.8 in §5 of loc. cit. (via a computer calculation due to Lanksy and Pollack) tells us that $\sigma$ admits a nontrivial theta lift to $\text{PGSp}_6(\mathbb{A})$. Proposition 5.5 in the same section gives another example of nontrivial theta lift but we will not consider it here.

We confirm the prediction of Gross-Savin that a rank 7 motive whose motivic Galois group is $G_2$ is realized in the middle degree cohomology of a Siegel modular variety of genus 3.

**Corollary 15.2.** Let $\sigma$ be as in Example 2. Write $\pi$ for its theta lift. Then $\rho_{\sigma|_v}$ has Zariski dense image in $G_2(\overline{Q}_\ell)$. Moreover spin $\circ \rho_{\sigma|_v}$ is isomorphic to the direct sum of $\eta \circ \rho_{\sigma}$ and the trivial representation. In particular $\eta \circ \rho_{\sigma}$ and the trivial representation appear in the $\pi(\sigma)$-isotypic part in $H^6_\text{c}(S, \overline{Q}_\ell)(3)$, where $S$ is the tower of Siegel modular varieties for $\text{GSp}_6$ and (3) denotes the Tate twist (i.e. the cube power of the cyclotomic character).

\[\footnote{Note that $\sigma\circ_\ell$ is $\xi$-cohomological for $\xi = \sigma|_{\mathbb{Q}_6}$.} \]
Proof. If the image is not dense in $G_2(\mathbb{Q}_F)$ then the proof of [GS98, §2 Prop. 2.3] shows that the Zariski closure of $\rho_\sigma(\Gamma)$ is $\text{PGL}_2$. However we see from the explicit computation of Hecke operators at 2 and 3 on $\sigma$ carried out by Lansky–Pollack [LP02, p.45, Table V] that the Satake parameters at 2 and 3 do not come from $\text{PGL}_2$.\footnote{The authors check [LP02, §4.3] that the Satake parameters do not come from $\text{SL}_2$ via the map $\text{SL}_2(\mathbb{C}) \to G_2(\mathbb{C})$ induced by a regular unipotent element. But the latter map factors through the projection $\text{SL}_2 \to \text{PGL}_2$.} We conclude that $\rho_\sigma(\Gamma)$ is dense in $G_2(\mathbb{Q}_F)$. The second assertion of the corollary is clear from (4) of Theorem 15.1 and the construction of $\rho_{\Sigma}$. \hfill $\square$

Remark 15.3. The Tate conjecture predicts the existence of an algebraic cycle on $S$ which should give rise to the trivial representation in the corollary. Gross and Savin suggest that it should come from a Hilbert modular subvariety of $S$ for a totally real cubic extension of $\mathbb{Q}$. See [GS98, §6] for details.

Remark 15.4. Ginzburg–Rallis–Soudry [GRS97, Thm. B] showed that every globally generic automorphic representation $\sigma$ of $G_2(\mathbb{A})$ admits a theta lift to $\text{PGSp}_6(\mathbb{A})$. (The result is valid over every number field.) Using this, Khare–Larsen–Savin [KLS10] established instances of global Langlands correspondence for $G_2(\mathbb{A})$, the analogue of Theorem 15.1 with $G_2$ in place of $G_2^\vee$, under a suitable local hypothesis. (See Section 6 of their paper for the hypothesis. They prescribe a special kind of supercuspidal representation instead of the Steinberg representation.) In our notation, their $\rho_\sigma$ is constructed inside the $SO_7$-valued representation $\rho_{\Sigma}$, where the point is to show that the image is contained and Zariski dense in $G_2([KLS10, Cor. 9.5])$.\footnote{Thereby they give an affirmative answer to Serre's question on the motivic Galois group of type $G_2$, since it is well known that $\rho_{\Sigma}$ appears in the cohomology of a unitary PEL-type Shimura variety after a quadratic base change, along the way to proving a result on the inverse Galois problem. Sometimes this contribution of [KLS10] is overlooked in the literature.}

16. Automorphic multiplicity

In this section we prove multiplicity one results for automorphic representations of $\text{GSp}_{2n}(\mathbb{A}_F)$ and those of the inner form $G(\mathbb{A}_F)$. For $\text{GSp}_{2n}$ we deduce this from Bin Xu’s multiplicity formula using the strong irreducibility of the associated Galois representations. The result is then transferred to $G(\mathbb{A}_F)$ via the trace formula. This is standard except when the highest weight of $\xi$ is not regular: in that case we need the input from Shimura varieties that the automorphic representations of $G(\mathbb{A}_F)$ of interest are concentrated in the middle degree.

Theorem 16.1. Let $n \geq 2$. Let $\pi$ be a cuspidal automorphic representation of $\text{GSp}_{2n}(\mathbb{A}_F)$ such that conditions (St) and (L-Coh) hold. If $n > 2$ we also assume that $\rho_\omega^b$ satisfies (HT$_1$) and (HT$_2$) for some infinite $F$-place $v_\infty$ and for some cuspidal automorphic sub $\text{Sp}_{2n}(\mathbb{A}_F)$-representation $\pi^b$ of $\pi$, cf. Lemma 2.6. Then $\text{spin} \circ \rho_\pi$ is strongly irreducible. Moreover, the automorphic multiplicity $m(\pi)$ of $\pi$ is equal to 1, i.e. Theorem C is true.

Remark 16.2. Conditions (HT$_1$) and (HT$_2$) may be translated to conditions on the $L$-parameter of $\pi^b_{v_\infty}$ (or on that of $\pi_{v_\infty}$) via Theorem 2.4 (iii).

Proof. Proposition 3.8 tells us that $\text{spin} \circ \rho_\pi$ is strongly irreducible. By Bin Xu [Xua, Prop. 1.7] we have the formula
\[
m(\pi) = m(\pi^b)|Y(\pi)/\alpha(S_\phi)|,
\]
where $m(\pi^b)$ is 1 in our case by Arthur [Art13, Thm. 1.5.2]. The group $Y(\pi)$ is equal to the set of characters $\omega : \text{GSp}_{2n}(\mathbb{A}_F) \to \mathbb{C}^*$ which are trivial on $\text{GSp}_{2n}(F)\mathbb{A}_F^\vee \subset \text{GSp}_{2n}(\mathbb{A}_F)$ and are such that $\pi \simeq \pi \otimes \omega$. The definition of the subgroup $\alpha(S_\phi)$ of $Y(\pi)$ is not important for us: We claim that $Y(\pi) = 1$. Let $\omega \in Y(\pi)$ and let $\chi : \Gamma \to \mathbb{Q}_F^\times$ be the corresponding character via class field theory. Because the representations have the same local components at unramified places, we get from Proposition 5.4 that $\chi \rho_\pi \simeq \rho_\pi$. Thus also $\chi(\text{spin} \circ \rho_\pi) \simeq \text{spin} \circ \rho_\pi$. Since $\text{spin} \circ \rho_\pi$ is strongly irreducible it follows that $\chi = 1$ and hence $\omega$ is trivial as well. \hfill $\square$

We now prove an analogue of Theorem 16.1 for inner forms of $\text{GSp}_{2n,F}$. Since Arthur’s and Bin Xu’s results have not been written up yet for non-trivial inner forms, we only obtain a partial result. Let $G$ be an inner form of $\text{GSp}_{2n,F}$ in the construction of Shimura varieties, cf. §7.

Theorem 16.3. Let $\pi$ be a cuspidal automorphic representation of $G(\mathbb{A}_F)$, such that the conditions (L-coh), (HT$_1$) and (HT$_2$) hold, and for some place $v \in \Sigma^\infty$ we assume that $\pi_{v_\infty}$ is isomorphic to a twist of the Steinberg representation. Then the automorphic multiplicity of $\pi$ is equal to 1.
Proof. Let \( f^G_{∞} := |Π^G_{ξ}|^{-1}(-1)^{||G_{∞}||} f^G_{Lef,∞} \) and \( f^G_v := |Π^G_{ξ}|^{-1}(-1)^{||G_{∞}||} f^G_{Lef,∞} \), where \( f^G_{Lef,∞} \) and \( f^G_{Lef,∞} \) denote the Lefschetz functions on \( G(F_{∞}) \) and \( G(F_v) \) as in Equation (A.5), respectively. At the place \( v_{St} \), we consider the truncated Lefschetz functions \( f^G_{Lef,v} \) and \( f^G_{Lef,v} \) as introduced in (A.4), and put \( f^G_{v} := (-1)^{||G_{∞}||} f^{G}_{Lef,v} \) and \( f^G_v := (-1)^{||G_{∞}||} f^{G}_{Lef,v} \). Let \( f^G_v \in \mathcal{H}(G(F_v^{'},F_v)) \) be arbitrary. Hence \( n \) (resp. \( n' \)) be a discrete automorphic representation of \( G \) (resp. \( G' \)). Note that \( f^G_v \) and \( f^G_v \) are associated at \( v \in \{ v_{St}, \infty \} \) by Lemmas A.4 and A.10 (with the choice of Haar measures and transfer factors explained there). Hence \( f^G_{∞,v} f^G_{v} f^G_{∞,v} \) and \( f^G_{∞,v} f^G_{v} f^G_{∞,v} \). Implicitly we are choosing the local transfer factor at each place \( v \) to be the sign \( e(G_v) \) so that the global transfer factor always equals one (whenever nonzero).

Comparing the trace formulas for \( G \) and \( G' \) and arguing as in the proof of Proposition 6.3, we obtain
\[
\sum_{τ} m(τ) Tr(τ(f^G_{∞,v} f^G_{v} f^G_{∞,v} τ) = \sum_{τ'} m(τ') Tr(τ'(f^G_{∞,v} f^G_{v} f^G_{∞,v} τ'),
\]
where \( τ \) (resp. \( τ' \)) runs over discrete automorphic representations of \( G(\mathbb{A}_F) \) (resp. \( G'(\mathbb{A}_F') \)), and \( m(\cdot) \) denotes the automorphic multiplicity as usual. By Lemma A.12 the functions \( f^G_{∞,v} \) (resp. \( f^G_{∞,v} \)) have non-zero trace against \( \pi_{∞}(\cdot) \) (resp. \( \pi_{∞}(\cdot) \)) only if \( \pi_{∞}(\cdot) \) is an unramified twist of the Steinberg representation. Therefore
\[
|Π^G_{ξ}|^{-1} \sum_{τ} m(τ)(-1)^{||G_{∞}||} \text{ep}(τ_{∞} \otimes ξ) = |Π^G_{ξ}|^{-1} \sum_{τ'} m(τ')(-1)^{||G_{∞}||} \text{ep}(τ_{∞} \otimes ξ),
\]
where \( Π^G_{ξ} \) and \( Π^G_{ξ} \) are discrete \( L \)-packets. In Equation (16.1), each sum ranges over the set of \( ξ \)-cohomological discrete automorphic representations which are Steinberg (up to unramified twist) at \( v_{St} \) and isomorphic to \( \pi_{∞} \) away from \( v_{St} \) and \( \infty \).

For the quasi-split group \( G' \), Corollary 2.8 shows that any \( τ' \) in Equation (16.1) is tempered at infinity, thus \( τ_{∞} \) is a discrete series (and belongs to \( Π^G_{ξ} \)). However Corollary 2.8 does not imply the analogue for \( τ \). For \( τ_{∞} \), Corollary 8.5 implies that \( τ_{∞} \) must be a discrete series representation. (If \( ξ \) has regular highest weight then this does not require the corollary as it follows from a standard fact on Lie algebra cohomology.) Hence \( \text{ep}(τ_{∞} \otimes ξ) = (-1)^{||G_{∞}||} \) and \( \text{ep}(τ_{∞} \otimes ξ) = (-1)^{||G_{∞}||} \) above, allowing us to simplify (16.1) to
\[
|Π^G_{ξ}|^{-1} \sum_{τ} m(τ) = |Π^G_{ξ}|^{-1} \sum_{τ'} m(τ'),
\]
where each sum runs over discrete automorphic representations which are Steinberg (up to unramified twist at \( v_{St} \)) and belong to the specified discrete \( L \)-packet at infinity, and isomorphic to \( \pi_{∞} \) away from \( v_{St} \) and \( \infty \).

At this point it is useful to show the following lemma by the trace formula argument.

**Lemma 16.4.** Suppose that \( F \neq \mathbb{Q} \). We have \( m(τ) = m(τ_{∞} τ'_{∞}) \) and \( m(τ') = m(τ'_{∞} τ_{∞} \prime) \), where \( τ_{∞} \), \( τ'_{∞} \) are arbitrary members of \( Π^G_{ξ} \) or \( Π^G_{ξ} \).

**Proof.** The trace formula for compact quotients applies to \( G \) since \( G \) is anisotropic modulo center over \( F \) (since it is compact modulo center at some infinite place as \( F \neq \mathbb{Q} \)). Consider \( f = f^G_{∞,v} \) and \( f' = f^G_{∞,v} \) in \( \mathcal{H}(G(\mathbb{A}_F)) \) and suppose that \( f_{∞} \) and \( f'_{∞} \) have the same stable orbital integrals. Identifying the geometric sides via stabilization, we have \( \sum_{σ} m(σ) Tr(σf_{∞}) = \sum_{σ} m(σ) Tr(σf'_{∞}) \), where both sums run over the set of (cuspidal) automorphic representations of \( G(\mathbb{A}_F) \). Since \( f^G_{∞} \) is free to choose, we deduce that
\[
\sum_{σ_{∞}} m(σ_{∞} σ_{∞}) Tr(f_{∞} - f'_{∞} | σ_{∞}) = 0,
\]
where the sum runs over the set of irreducible admissible representations of \( G(\mathbb{Q}) \). Since \( f_{∞} - f'_{∞} \) can be any test function with zero stable orbital integrals, it follows that the coefficients \( m(σ_{∞} σ_{∞}) \) must be equal for every \( σ_{∞} \in Π^G_{ξ} \). (Here we use the fact that a stable character containing a member of \( Π^G_{ξ} \) with fewest terms is precisely the sum of the characters of \( σ_{∞} \) in \( Π^G_{ξ} \), each with multiplicity one.) So the lemma is proved for \( G \).

Now we consider the case of \( G' \). Take \( f = f^G_{∞,v} f^G_{v} f^G_{∞} \in \mathcal{H}(G'(\mathbb{A}_F')) \) and \( f' = f^G_{∞,v} f^G_{v} f^G_{∞} \in \mathcal{H}(G'(\mathbb{A}_F')) \), where \( f_{∞,v} \) is arbitrary, \( f_{v} \) is the Lefschetz function at \( v_{St} \), and \( f'_{∞,v} \) (resp. \( f_{∞,v}' \)) is a pseudo-coefficient for \( τ_{∞} \) (resp. \( τ'_{∞} \)). By [Kot92a, Lem. 3.1], \( τ_{∞} \) and \( τ'_{∞} \) have the same stable orbital integrals. They are moreover cuspidal by Lemma A.9. The simple trace formula and its stabilization (Lemmas 6.1 and 6.2) imply that, by arguing similarly as in the proof of Proposition 6.3,
\[
\sum_{σ_{∞} \in Π^G_{ξ}(G')} m(σ) Tr(f_{∞,v}' | σ_{∞}) = \sum_{σ_{∞} \in Π^G_{ξ}(G')} m(σ) Tr(f_{∞,v}' | σ_{∞}).
\]
As in that proof, we also know that \( σ_{∞} \) is essentially tempered if it contributes to the sum on either side. Hence the above equality boils down to \( m(τ_{∞} τ'_{∞}) = m(τ_{∞} τ'_{∞}) \), as we wanted to show. \( \square \)
We have $\pi$ as in the statement of Proposition 16.2 appearing in the left sum of (16.2). So both sides are positive in that equation. In particular there exists $\pi^*$ contributing to the right hand side such that $m(\pi^*) > 0$. Any other $\pi^*$ in the sum is isomorphic to $\pi^*$ at all finite places away from $v_{\infty}$. We also know that $\tau_{v_{\infty}}^*$ and $\tau_{v_{\infty}}^{\pi^*}$ differ by an unramified character and that $\tau_{\infty}$ and $\tau_{\infty}^*$ belong to the same $L$-packet. Now we claim that $\tau_{v_{\infty}}^* = \tau_{v_{\infty}}$. To see this, we apply [Xua, Thm. 1.8] to deduce that $\tau^*$ and $\tau^{\pi^*}$ belong to the same global $L$-packet as in that paper, by the same argument as in the proof of Lemma I.2. Since the local $L$-packet for $GSp_{2n}(F_{v_{\infty}})$ of (any character twist of) the Steinberg representation is a singleton by [Xua, Prop. 4.4, Thm. 4.6], the claim follows.\footnote{Lemma 2.1 implies that the $L$-parameter $W_{F_{v_{\infty}}} \times SU(2) \to SO_{2n+1}(C)$ for the Steinberg representation of $Sp_{2n}(F_{v_{\infty}})$ restricts to the principal representation of $SU(2)$ in $SO_{2n+1}(C)$. A lift of this to a $GSpin_{2n+1}(C)$-valued parameter is attached to the Steinberg representation of $GSp_{2n}(F_{v_{\infty}})$ in Xu’s construction. This is enough to imply that the group $S_\phi$ in [Xua, Prop. 4.4] is trivial.}

We have shown that the right hand side of (16.2) may be summed over $\pi^*$ which is isomorphic to $\pi^*$ away from infinity and belongs to $\Pi^\wedge_{\xi}$ at infinity. Each $\pi^*$ has automorphic multiplicity one by Theorem 16.1 and Lemma 16.4. (Without the lemma, some $m(\pi^*)$ could be zero.) So (16.2) comes down to

$$\sum_{\pi} m(\pi) = |\Pi^\wedge_{\xi}|.$$ 

Recall that the sum runs over $\pi$ such that $\tau_{\infty}^\wedge \cdot \tau_{v_{\infty}}^\wedge \cdot \tau_{v_{\infty}}^\pi \cdot \tau_{v_{\infty}}^{\pi^*} = \Pi^\wedge_{\xi}$, and $\tau_{v_{\infty}}^\pi \cdot \tau_{v_{\infty}}^{\pi^*}$ for an unramified character $\epsilon$. By Lemma 16.4, the contribution from $\tau$ with $\tau_{v_{\infty}}^\pi \cdot \tau_{v_{\infty}}^{\pi^*}$ is already $|\Pi^\wedge_{\xi}| m(\pi)$, and the other $\tau$ (if any) contributes nonnegatively. We conclude that $m(\pi) = 1$. (Moreover, all $\tau$ in the sum with $m(\tau) > 0$ should be isomorphic to $\pi$ at $v_{\infty}$; this gets used in the next corollary.)

Consequently we deduce that the Galois representation $\rho_{\pi^*}$ appears in the cohomology with multiplicity one, cf. §9.

**Corollary 16.5.** In the setting of Theorem 16.3, the integer $a(\pi^*)$ defined in (8.2) is equal to one. (Note that $\pi$ here is $\pi^*$ there.)

**Proof.** The last paragraph in the proof of the preceding theorem shows that for every $\tau \in A(\pi^*)$, we have an isomorphism $\tau_{v_{\infty}} \simeq \tau_{v_{\infty}}$. Since $\tau_{\infty} \in \Pi^\wedge_{\xi}$ for every $\tau \in A(\pi^*)$ by Corollary 8.5, it follows from Theorem 16.3, Lemma 16.4, and Remark 7.2 that

$$a(\pi) = (-1)^{n+1/2} N_{\ Pixel}^{-1} \sum_{\tau \in A(\pi^*)} m(\tau) \cdot \exp(\tau_{\infty} \otimes \xi) = |\Pi^\wedge_{\xi}|^{-1} \sum_{\tau_{v_{\infty}}^\pi \cdot \tau_{v_{\infty}}^{\pi^*} = \Pi^\wedge_{\xi}} m(\tau) = 1.$$ 

\[\Box\]

**Remark 16.6.** The same argument proves Theorem 16.3 for all inner forms $G$ of $GSp_{2n}$ as long as the highest weight of $\xi$ is regular. Note that we used the fact that $G$ is the particular inner form twice when invoking Corollary 8.5 and the trace formula for compact quotients in the proof of Lemma 16.4; let us explain how to get around them for general $G$. The corollary is unnecessary since any $\xi$-cohomological representation belongs to discrete series if the highest weight is regular. Then we employ the simple trace formula with pseudocoeficients for two discrete series representations as in the case of $G^*$ in the proof of Lemma 16.4. Instead of appealing to the essential temperedness, we use the fact that if $Tr(\phi_{v_{\infty}}) \neq 0$ then $\phi_{v_{\infty}}$ is cohomological (this is true even if $\xi$ has non-regular highest weight). Then $\phi_{v_{\infty}}$ has to be in discrete series again by the regularity of highest weight, and the rest of the argument goes through unchanged.

### 17. Meromorphic continuation of the Spin $L$-function

Let $\pi$ be a cuspidal automorphic representation of $GSp_{2n}(A_F)$ unramified away from a finite set of places $S$. The partial spin $L$-function for $\pi$ away from $S$ is by definition

$$L^S(s, \pi, \text{spin}) := \prod_{\tau \in S} 1/\det(1 - q^{s/2} \text{spin}(\phi_{v_{\infty}})).$$ 

Various analytic properties of this function would be accessible if the Langlands functoriality conjecture for the $L$-morphism spin were known. However we are far from it when $n \geq 3$. In particular no results have been known about the meromorphic (or analytic) continuation of $L^S(s, \pi, \text{spin})$ when $n \geq 6$ (see introduction for some results when $2 \leq n \leq 5$).

The aim of this final section is to establish Theorem D on meromorphic continuation for $L$-algebraic $\pi$ under hypotheses (St), (L-coh), and (spin-REG) by applying a potential automorphy theorem, namely Theorem
A of [BLGGT14]. Let $S_{\text{bad}}$ be as in §10. A continuous character $\mu : \Gamma \to GL_1(\overline{M}_{\pi,\lambda})$ is considered totally of sign $\epsilon \in \{\pm 1\}$ if $\mu(c_v) = \epsilon$ for all infinite places $v$ of $F$. More commonly a character totally of sign +1 or -1 is said to be totally odd or even, respectively. For each $\mathbb{Q}$-embedding $w : F \hookrightarrow \mathbb{C}$ define the cocharacter $\mu_{\text{Hodge}}(\phi_{\pi,\lambda}) : GL_1(\mathbb{C}) \to \text{GSpin}_{2n+1}(\mathbb{C})$ by restricting the L-parameter $\phi_{\pi,\mu} : W_F \to \text{LSp}_{2n} \to W_C = GL_1(\mathbb{C})$ (via $w$). Condition (L-coh) ensures that $\mu_{\text{Hodge}}(\phi_{\pi,\lambda})$ is an algebraic cocharacter.

**Proposition 17.1.** Suppose that $\pi$ satisfies (St), (L-coh), and (spin-REG). There exist a number field $M_{\pi}$ and a continuous representation

$$R_{\pi,\lambda} : \Gamma \to GL_{2n}(\overline{M}_{\pi,\lambda})$$

for each finite place $\lambda$ of $M_{\pi}$ such that the following hold. (Write $\ell$ for the rational prime below $\lambda$.)

1. At each place $v$ of $F$ not above $S_{\text{bad}} \cup \{\ell\}$, we have
   $$\text{char}(R_{\pi,\lambda}(\text{Frob}_v)) = \text{char}(\text{spin}(\phi_{\pi,\lambda}(\text{Frob}_v))) \in M_{\pi}[X].$$

2. $R_{\pi,\lambda}|_{\Gamma_v}$ is de Rham for every $v|\ell$. Moreover it is crystalline if $\pi_v$ is unramified and $v \notin S_{\text{bad}}$.

3. For each $v|\ell$ and each $w : F \hookrightarrow \mathbb{C}$ such that $w$ induces $v$, we have $\mu_{\text{HT}}(R_{\pi,\lambda}|_{\Gamma_v}, i w) = i(\text{spin} \circ \mu_{\text{Hodge}}(\phi_{\pi,\lambda})).$

4. $R_{\pi,\lambda}$ is irreducible.

5. $R_{\pi,\lambda}$ maps into $\text{GSp}_{2n}(\overline{M}_{\pi,\lambda})$ (resp. $\text{GO}_{2n}(\overline{M}_{\pi,\lambda})$) for a suitable nondegenerate alternating (resp. symmetric) pairing on the underlying $2n$ dimensional space over $\overline{M}_{\pi,\lambda}$ if $(-1)^{\nu(n+1)/2}$ is odd (resp. even). The multiplier character $\mu_{\lambda} : \Gamma \to GL_1(\overline{M}_{\lambda})$ (so that $R_{\pi,\lambda} \simeq R_{\pi,\lambda} \otimes \mu_{\lambda}$) is totally of sign $(-1)^{\nu(n+1)/2}$.

**Remark 17.2.** Parts (1)-(3) of the lemma imply that the family $\{R_{\pi,\lambda}\}$ is a weakly compatible system of $\lambda$-adic Galois representations, cf. [BLGGT14, §5.1].

**Remark 17.3.** Although we do not need it, we can choose $M_{\pi}$ such that $R_{\pi,\lambda}$ is valued in $GL_{2n}(M_{\pi,\lambda})$ for every $\lambda$. Concretely we may take $M_{\pi}$ to be the field of definition for the $\pi^{\infty}$-isotypic part in the compact support Betti cohomology with $\mathbb{Q}$-coefficients (with respect to the local system arising from $\xi$). When the coefficients are extended to $M_{\pi,\lambda}$ the $\pi^{\infty}$-isotypic part becomes a $\lambda$-adic representation of $\Gamma$ via étale cohomology, which is isomorphic to a single copy of $R_{\pi,\lambda}$ if the coefficients are further extended to $\overline{M}_{\pi,\lambda}$ by Corollary 16.5.

**Proof.** The first four assertions are immediate from Theorem 10.1. The first part of (5) is clear from Lemma 0.1. Let us check the second part of (5). Consider the diagram

$$\begin{array}{ccc}
\Gamma & \xrightarrow{\mu_{\lambda}} & \text{GSpin}_{2n+1}(\overline{M}_{\pi,\lambda}) \\
\downarrow^{N} & \sim & \downarrow^{\text{sim}} \\
GL_1(\overline{M}_{\pi,\lambda}) & \xrightarrow{\text{spin}} & \text{GSp}_{2n}(\overline{M}_{\pi,\lambda}) \text{ or } \text{GO}_{2n}(\overline{M}_{\pi,\lambda}).
\end{array}$$

The outer triangle commutes by the definition of $\mu_{\lambda}$. The right triangle also commutes by Lemma 0.1. Hence $\mu_{\lambda} = N \circ \mu_{\pi}$. By (L-coh) we know that $\omega_{\pi,\lambda}|_{||w||=1/4}$ is the inverse of the central character of some irreducible algebraic representation $\xi_{\lambda}$ at each infinite place $v$. Hence $\omega_{\pi,\lambda}|_{||w||=1/4} : \mathbb{R}^n \to \mathbb{C}^n$ is the $w$-th power map, where $w \in \mathbb{Z}$ is as in (cent) of §7 (in particular $w$ is independent of $v$). Hence $\omega_{\pi,\lambda}|_{||w||=1/4} = \omega_{\pi,n}^{|v(n+1)/2|}$ corresponds via class field theory and $i$ to an even Galois character of $\Gamma$ (regardless of the parity of $w$). On the other hand, $\mu_{\lambda}$ corresponds to $\omega_{\pi}$ via class field theory and $i$ in the $L$-normalization, cf. Theorem A. We conclude that $\mu_{\lambda,\pi}(c_v) = (-1)^{v(n+1)/2}$ for each $v|\infty$.}

Now that we proved the lemma, Theorem A of [BLGGT14] implies

**Corollary 17.4.** Theorem D is true.

**Proof.** The conditions of the theorem in loc. cit. are verified by the above lemma with the following additional observation: the characters $\mu_{\lambda}$ forms a weakly compatible system since they are associated to the central character of $\pi$ (which is an algebraic Hecke character).

**Remark 17.5.** For the lack of precise local Langlands correspondence for general symplectic groups (however note that Bin Xu established a slightly weaker version in [Xua]), we cannot extend the partial spin L-function for $\pi$ to a complete L-function by filling in the bad places. However the method of proof for the corollary yields a finite alternating product of completed L-functions made out of $\Pi$ as in Theorem D, by an argument based on Brauer induction theorem as in the proof of [HSBT10, Thm. 4.2]; this alternating product should be equal to the completed spin L-function as this is indeed true away from $S$. 


Appendix A. Lefschetz functions

We collect some results on Lefschetz functions that are used in the text. In this appendix, $F$ is a non-archimedean local field of residue characteristic $p$, except in Lemma A.12 at the end, where $F$ is global. We collect the required results from the literature and prove some lemmas to deal with small technical difficulties (non-compact center, twisted group).

To help the readers we clarify some terminology. There are three names for the function whose trace computes the Euler-Poincaré characteristic or the Lefschetz number of the group cohomology (resp. Lie algebra cohomology) for a given reductive $p$-adic (resp. real) group: they are called Euler-Poincaré functions, Kottwitz functions, or Lefschetz functions. In the real case one can consider twisted coefficients by local systems. There are small differences between the three functions. Euler-Poincaré and Lefschetz functions are considered on either $p$-adic or real groups, and can be described in terms of pseudo-coefficients for certain discrete series representations. The functions may not be unique but their orbital integrals are well defined. A Kottwitz function mainly refers to a particular function on a $p$-adic group and gives pseudo-coefficients for the Steinberg representations. It is not just characterized by their traces but can be given by an explicit formula, cf. [Kot88, §2]. A generalization of Kottwitz functions on a $p$-adic group is given in [S897] but we will not need it in this paper.

We recall Kottwitz’s result. Let $G$ be a connected reductive group and write $q(G)$ for the $F$-rank of the derived subgroup of $G$.

Proposition A.1 (Kottwitz). Suppose that the center of $G$ is anisotropic over $F$. Then there exists a locally constant compactly supported function $f_{\text{Lef}} = f_{\text{Lef}}^G$ on $G(F)$ such that

(i) For all irreducible admissible unitary representations $\tau$ of $G(F)$ the trace $\text{Tr}(f_{\text{Lef}})$ vanishes unless $\tau$ is either the trivial representation or the Steinberg representation, in which case the trace is equal to $1$ or $(-1)^{q(G)}$ respectively;

(ii) Let $\gamma \in G(F)$ be a semisimple element, and $I$ the neutral component of its centralizer; we give $I(F) \backslash G(F)$ the Euler-Poincaré measure. Then the orbital integral

$$O_\gamma(f_{\text{Lef}}) = \int_{I(F) \backslash G(F)} f_{\text{Lef}}(g^{-1} \gamma g)dg$$

is non-zero if and only if $I(F)$ has a compact center, in which case $O_\gamma(f_{\text{Lef}}) \equiv 1$.

Proof. Kottwitz constructed in [Kot88, Sect. 2] Lefschetz functions $f_{\text{Lef}}$ on $G(F)$. These are functions such that

$$\text{Tr } f_{\text{Lef}}(\tau) = \sum_{i=0}^{\infty} (-1)^i \dim H^i_{\text{cts}}(G(F), \tau),$$

for all irreducible representations $\tau$ of $G(F)$. The cohomology in Equation (A.1) is the continuous cohomology, i.e. the derived functors of $\tau \mapsto \tau^{G(F)}$ on the category of smooth representations of finite length. By the computation of the cohomology spaces $H^i_{\text{cts}}(G(F), \tau)$ in [BW00, Chap. VIII], for unitary $\tau$, the spaces $H^i_{\text{cts}}(G(F), \tau)$ vanish unless $i \geq 0$ and $\tau$ is the trivial representation, or $i = n$ and $\tau$ is the Steinberg representation, in which case $\dim H^i_{\text{cts}}(G(F), \tau) = 1$. \hfill $\square$

We now consider the group $GL_{2n+1}$ equipped with the involution $\theta$ defined by $g \mapsto j^* g^{-1} j$ with $j$ the $(2n+1) \times (2n+1)$ matrix with all entries 0 except those on the antidiagonal, where we put 1. Define $GL^+_{2n+1} := GL_{2n+1} \ltimes \{\theta\}$. In [BLS96] Borel, Labesse, and Schwermer introduced twisted Lefschetz functions. Below we will cite mostly from the more recent article of Chenevier–Clozel [CC09, Sect. 3]. The results we need from them are proven in a general twisted setup but specialize to our case as follows.

Proposition A.2. There exists a function $f_{\text{Lef}}^{GL_{2n+1}, \theta}$ with compact support in the subset $GL_{2n+1}(F)\theta$ of $GL^+_{2n+1}(F)$, such that

(i) For all unitary admissible representations $\tau$ of $GL_{2n+1}(F)$ the trace $\text{Tr}(f_{\text{Lef}}^{GL_{2n+1}, \theta})$ vanishes unless $\tau$ is either the trivial representation or the Steinberg representation, in which case trace is equal to $1$ or $2(-1)^n$ respectively;

(ii) Let $\gamma \in G(F)\theta \in GL^+_{2n+1}(F)$ be a semisimple element, and $I_{\gamma, \theta}$ the neutral component of the centralizer of $\gamma \theta$; we give $I_{\gamma, \theta}(F) \backslash G(F)$ the Euler-Poincaré measure [CC09, p.30] (cf. [Kot88, Sect. I]). Then the twisted orbital integral

$$O_{\gamma \theta}(f_{\text{Lef}}^{GL_{2n+1}, \theta}) = \int_{I_{\gamma, \theta}(F) \backslash G(F)} f_{\text{Lef}}^{GL_{2n+1}, \theta}(g^{-1} \gamma \theta(g))dg,$$
is non-zero if and only if $I_{\gamma\theta}(F)$ has a compact center. Moreover in case $I_{\gamma\theta}(F)$ has compact center, the integral $O_{\gamma\theta}(f_{GL_{2n+1},\theta})$ is equal to 1.

**Proof.** We deduce this from Proposition 3.8 and Corollary 3.10 of [CC09]. In fact, [CC09] states that $\text{TrSt}(f_{GL_{2n+1}}(F)\theta) = (-1)^q(P(1)/\varepsilon(\theta))$. In our case one computes $q(G) = 2n$, $\varepsilon(\theta) = (-1)^n$ and $P(1) = 2$, hence the statement of our proposition. \hfill $\Box$

It is well known (cf. [Art13, I.2] or [Wal10, 1.8]) that $\text{Sp}_{2n,F}$ is a $\theta$-twisted endoscopic group of $GL_{2n+1,F}$.

Let $f_{\text{Sp}_{2n},I}^{\theta}$ be the Lefschetz function on $\text{Sp}_{2n,F}$ from Proposition A.1. We introduce the notion of associated functions. Let $G_0$ be a reductive group over $F$. Suppose that $G_0$ is an endoscopic group for $G_0$ (i.e. $G_0$ is part of an endoscopic datum for $G_0$). We say that the functions $f_0 \in H(G_0'(F))$ and $f_0' \in H(H_0'(F))$ are associated if they have matching orbital integrals in the sense of [KS99, (5.5.1)]. The same definition carries over to the archimedean and adelic setup.

**Lemma A.3.** Define $C = [(F^x/F^{x^2})^{-1}].$ Then the functions $(f_{\text{Sp}_{2n},I}^{\theta}, C \cdot f_{\text{GL}_{2n+1},\theta})$ are associated.

**Proof of Lemma A.3.** We show that for each $\gamma \in \text{Sp}_{2n,F}$ (regularly, semisimple and elliptic), we have

\[ \text{SO}_2(f_{\text{Sp}_{2n},I}^{(\gamma)}) = \sum_{\delta} \Delta(\gamma, \delta)\text{SO}_2(C \cdot f_{\text{GL}_{2n+1},\theta}) \]

where $\delta$ ranges over a set of representatives of the twisted conjugacy classes in $GL_{2n+1}(F)$ which are associated to $\gamma$. Note that the stable orbital integral $\text{SO}_2(f_{\text{Sp}_{2n},I}^{(\gamma)})$ is equal to the number of conjugacy classes in the stable conjugacy class of $\gamma$. Our first claim (1) is that $\Delta(\gamma, \delta) = 1$ for all elliptic $\delta$ associated to $\gamma$. Assuming that this claim is true, the right hand side of Equation (A.2) is the stable twisted orbital integral $\text{SO}_2(C \cdot f_{\text{GL}_{2n+1},\theta})$, which equals, up to multiplication by $C$, the number of twisted conjugacy classes in the stable twisted conjugacy class of $\delta$. Our second claim is that $\text{SO}_2(f_{\text{Sp}_{2n},I}^{(\gamma)}) = \text{SO}_2(C \cdot f_{\text{GL}_{2n+1},\theta})$. Clearly the lemma follows from claims (1) and (2).

Let us now check the two claims. We begin with claim (1). For this we use the formulas in the article of Waldspurger [Wal10, Sect. 10]. The factors $\Delta(\gamma, \delta)$ are complicated and involve many notation to introduce properly, for which we do not have the space to introduce. Thus we use, without introducing, the notation from [Wal10]. By [Wal10, Prop. 1.10]

\[ \Delta_{\text{Sp}_{2n},I}^{\theta}(\gamma, \bar{x}) = \chi(\eta x_D P_I(1) P_{\gamma}(1)) \prod_{\tau \in I^{-}} \text{sgn}_{F/F_{\tau}}(C_{\tau}) \]

Since our conjugacy class is elliptic we have $H = \text{Sp}_{2n}$, the group $H^{-}$ is trivial and the sets $I^{-}, I^{+}$ are empty. A priori, $\chi$ is any quadratic character of $F^{x}$, and each choice defines a different isomorphism class of twisted endoscopic data. The choice affects the L-morphism:

\[ \eta_{\chi}^{-1}H = \text{SO}_{2n+1}(C) \times W_{F} \rightarrow \text{GL}_{2n+1}(C) \times W_{F}, \quad (g, w) \mapsto (\chi(w)g, w). \]

Since the Steinberg $L$-parameter for $GL_{2n+1}(C)$, $W_{F} \times SU_{2}(R) \rightarrow GL_{2n+1}(C)$, is trivial on $W_{F}$ (and $\text{Sym}^{2n}$ on $SU_{2}(R)$), it factors through $\eta_{\chi}$ only for $\chi = 1$ (and not for non-trivial $\chi$). Consequently the transfer factor in Equation (A.3) is equal to 1, and claim (1) is true.

We check claim (2). In Sections 1.3 and 1.4 of [Wal10], Waldspurger describes the (strongly) regular, semisimple (stable) conjugacy classes of $\text{Sp}_{2n}(F)$ in terms of $F$-algebras. More precisely, a regular semisimple conjugacy class in $\text{Sp}_{2n}(F)$ is given by data $(F_{i}, F_{i+1}, c_{i}, x_{i}, I_{i})$, where $I$ is a finite index set; for each $i \in I$, $F_{i}$ is a finite extension of $F$; for each $i \in I$, $F_{i}$ is a commutative $F_{i}$-algebra of dimension $2$; we have $\sum_{i \in I}[F_{i} : F] = 2n$; $\tau_{i}$ is the non-trivial automorphism of $F_{i}/F_{i+1}$; for each $i \in I$, we have an element $c_{i} \in F_{i}^{x}$ such that $\tau_{i}(c_{i}) = -c_{i}$; $x_{i} \in F^{x}$ such that $x_{i} \tau_{i}(x_{i}) = 1$; to the data $(F_{i}, F_{i+1}, c_{i}, x_{i}, I_{i})$, Waldspurger attaches a conjugacy class [Wal10, Eq. (l)] and this conjugacy class is required to be regular.

The data $(F_{i}, F_{i+1}, c_{i}, x_{i}, I_{i})$ should be taken up to the following equivalence relation: The index set $I$ is up to isomorphism; the triples $(F_{i}, F_{i+1}, x_{i})$ are up to isomorphism; the element $c_{i} \in F_{i}^{x}$ is given up to multiplication by the norm group $N_{F/F_{i}}(F_{i}^{x})$. The stable conjugacy class is obtained from $(F_{i}, F_{i+1}, c_{i}, x_{i}, I_{i})$ by forgetting the elements $c_{i}$ [Wal10, Sect. 1.4], and keeping only $(F_{i}, F_{i+1}, c_{i}, x_{i}, I_{i})$.

According to [Wal10, Sect. 1.3] a strongly regular, semisimple twisted conjugacy class in $GL_{2n+1}$ is given by data $(L_{i}, L_{i+1}, y_{i}, y_{D}, I)$ where$^{20}$ a finite index set $J$; for each $j \in J$, $L_{i}$ is a finite extension of $F$; for each $i \in J$, $L_{i}$ is a commutative $L_{i}$-algebra of dimension $2$ over $L_{i+1}$; for each $i \in J$, an element $y_{i} \in L_{i}^{x}$; $2n + 1 = \sum_{i \in J}[L_{i} : F]$.

$^{20}$ To avoid a collision of notation in our exposition, we write $L_{i}, L_{i+1}, y_{i}, y_{D}$ where Waldspurger writes $F_{i}, F_{i+1}, x_{i}, I, x_{D}$. 


As an element of \(F^x\), to the data \((L_i, L_{z_i}, y_i, y_D, I)\) Waldspurger attaches a twisted conjugacy class [Wal10, p.45] and this class is required to be strongly regular.

The data \((L_i, L_{z_i}, y_i, I)\) should be taken up to the following equivalence relation: \((L_i, L_{z_i}, y_i, I)\) are under the same equivalence relation as before for the symplectic group; the elements \(y_i\) are determined up to multiplication by \(N_{L_i/L_{z_i}}(L_i^x)\); \(y_D\) is determined up to squares \(F^x\). The stable conjugacy class of \((L_i, L_{z_i}, y_i, y_D)\) is obtained by taking \(y_i\) up to \(L_{z_i}^x\) and forgetting the element \(y_D\).

By [Wal10, Sect. 1.9] the stable (twisted) conjugacy classes \((L_i, L_{z_i}, x_i, J)\) and \((F_i, F_{z_i}, y_i, I)\) correspond if and only if \((L_i, L_{z_i}, x_i, J) = (F_i, F_{z_i}, y_i, I)\).

The conjugacy class \(\gamma\) (resp. \(\delta\)) is elliptic if the algebra \(F_i\) (resp \(L_i\)) is a field. Let’s assume that this is the case (otherwise there is nothing to prove, because the equation \(\text{SO}_x(\mathcal{L}_{z_i}, F_i) = \text{STO}_x(G(F_{z_i})^1, F_i)\) reduces to \(0 = 0\). By the description above, to refine the stable conjugacy class \((F_i, F_{z_i}, x_i, J)\) to a conjugacy class, is to give elements \(C_i \in F_i^x/N_{F_i/F} F_i^x\) such that \(\tau_i(C_i) = -\tau_i\). For each \(i\) there are 2 choices for this, so we get \(2^g\) conjugacy classes inside the stable conjugacy class.

To refine the stable twisted conjugacy class \((L_{z_i}, L_i, y_i, I)\) to a twisted conjugacy class, is to lift \(y_i\) under the mapping \(L_i^x/N_{L_i/L_{z_i}}(L_i^x) \to L_i^x/L_i^x\). There are \(2^g\) choices for such a lift for each \(i\), we get \(2^g\) choices to refine the collection \((y_i)_{i \in I}\). Finally we also have to choose an element \(y_D \in F^x/F^x\). In total we have \([F^x/F^x]2^g\) twisted conjugacy classes inside the stable conjugacy class. Thus if we take \(C = [F^x/F^x]^{-1}\) the lemma follows.

We also need Lefschetz functions on the group \(GSp_{2m}(F)\) and its nontrivial inner form \(G\). Unfortunately Proposition A.1 does not apply since the center of \(GSp_{2m}(F)\) is not compact. We follow Labesse to construct Kottwitz functions generally for an arbitrary reductive group \(G\) over \(F\). Let \(A\) denote the maximal split torus in the center of \(G\). Define \(\nu: G(F) \to X'(A) \otimes_{\mathbb{R}} \mathbb{R}\) as in [Lab99, 3.8] and put \(G^\nu := \ker \nu\). Consider the exact sequence
\[
1 \to A \to G \to G^\nu := G/A \to 1
\]
and a Lefschetz function \(f^G_{\text{Lef}} \in \mathcal{H}(G\nu)\) as in Proposition A.1. The pullback \(f^G_{\text{Lef}} \circ \nu\) is a function on \(G(F)\). Multiplying with the characteristic function \(1_{G(F)}\), we obtain
\[
(A.4) \quad f^G_{\text{Lef}} := 1_{G(F)} \cdot f^G_{\text{Lef}} \circ \nu \in \mathcal{H}(G(F)).
\]
Let \(G\) denote a quasi-split inner form of \(G\). Suppose that the Haar measures on \(G'(F)\) and \((G\nu)'(F)\) are chosen compatibly in the sense of [Kot88, p.631]. Given a Haar measure on \(A(F)\) this determines Haar measures on \(G(F)\) and \(G^\nu(F)\). Let us normalize the transfer factor between \(G\) and \(G^\nu\) to be \((\nu)(G\nu)\) whenever nonzero the Kottwitz sign \(\epsilon(G)\), which is equal to \((-1)^{\nu(G)}\nu(G)\) by [Kot83, pp.296-297].

**Lemma A.4.** With the above choices, \((-1)^{\nu(G)} f^G_{\text{Lef}}\) and \((-1)^{\nu(G)} f^G_{\text{Lef}}\) are associated.

**Remark A.5.** Our lemma specifies the constant in [Lab99, Prop. 3.9.2] when \(\Theta = 1\) and \(H = G^\nu\).

**Proof.** The proof is quickly reduced to the case when the center is anisotropic, the point being that \(G(F)^1\) and \(G^\nu(F)^1\) are invariant under conjugation. From Proposition A.1 we deduce that \(SO_y^G(f^G_{\text{Lef}})\) is zero if \(y\) is non-elliptic and equals the number of \((G(F))-\text{conjugacy classes in the stable conjugacy class of } y\) if \(y\) is elliptic. Since the same is true for \(G\) it is enough to show that the number of conjugacy classes in a stable conjugacy class is the same for \(y\) and \(\gamma y\) when they are strongly regular and have matching stable conjugacy classes. This follows from the \(p\)-adic case in the proof of [Kot88, Thm. 1].

**Definition A.6** [Lab99, Def. 3.8.1, 3.8.2]. Let \(\phi \in \mathcal{H}(G(F))\). We say that \(\phi\) is \textit{cuspidal} if the orbital integrals of \(\phi\) vanish on all regular non-elliptic semisimple elements, and \textit{strongly cuspidal} if the orbital integrals of \(\phi\) vanish on all non-elliptic elements and if the trace of \(\phi\) is zero on all induced representations from unitary representations on proper parabolic subgroups. The function \(\phi\) is said to be stabilizing if \(\phi\) is cuspidal and if the \(\kappa\)-orbital integrals of \(\phi\) vanish on all semisimple elements for all nontrivial \(\kappa\).

**Lemma A.7.** The function \(f^G_{\text{Lef}}\) is strongly cuspidal and stabilizing. If \(\text{Tr} \pi(f^G_{\text{Lef}}) \neq 0\) for an irreducible unitary representation \(\pi\) of \(G(F)\) then \(\pi\) is an unramified character twist of either the trivial or the Steinberg representation.

**Proof.** The first assertion is [Lab99, Prop. 3.9.1]. By construction \(f^G_{\text{Lef}}\) is constant on \(Z(F) \cap G(F)^1\), which is compact. Since \(\text{Tr} \pi(f^G_{\text{Lef}})\) is the sum of \(\text{Tr} \pi^1(f^G_{\text{Lef}})\) over irreducible constituents \(\pi^1\) of \(\pi(G(F)^1)\), we may choose \(\pi^1\) such that \(\text{Tr} \pi^1(f^G_{\text{Lef}}) \neq 0\). The nonvanishing implies that \(\pi^1\) has trivial central character on \(Z(F) \cap G(F)^1\). Thus \(\pi^1\) descends to a representation \(\pi'\) of \(G'(F)\). Computing \(\text{Tr} \pi^1(f^G_{\text{Lef}})\) by first integrating over \(A(F)\) we know that the result is a nonzero multiple of \(\text{Tr} \pi'(f^G_{\text{Lef}})\). So \(\text{Tr} \pi'(f^G_{\text{Lef}}) \neq 0\). When \(\pi\) is unitary, \(\pi^1\) and thus also \(\pi'\) are unitary. By Proposition A.1, \(\pi'\) is either trivial or Steinberg so the same is true for \(\pi^1\).

Writing \(\pi'\)
CL11 carries Kot92a A.6 CL11 makes sense verbatim when

Assume from now on that $G$ is a non-split inner form of a quasi-split group $G^*$ over $F$. Consider a finite cyclic extension $E/F$ with $\theta$ generating the Galois group. Put $G^* := \text{Res}_{E/F} G^*$ equipped with the evident $\theta$-action. (The case $G^* = \text{GSp}_{2n}$ is used in the main text.)

**Lemma A.8.** The function $f^G_\text{Let}$ is strongly cuspidal and stabilizing on the twisted group $G^* \theta$. There exist constants $c \in \mathbb{C}^\times$ such that the functions $c f^G_\text{Let}$ and $f^G_\text{Let}$ are associated (via base change).

*Proof.* [Lab99, Prop. 3.9.2]. (Take $\theta = 1$, $H = G^*$ in *loc. cit.* for the first assertion.)

Now we turn our attention to Lefschetz functions (also called Euler-Poincaré functions) on connected reductive groups $G$ over $F = \mathbb{R}$. Let $\xi$ be an irreducible algebraic representation of $G \times_{\mathbb{R}} \mathbb{C}$. Denote by $\chi_\xi : Z(\mathbb{R}) \to \mathbb{C}^\times$ the restriction of $\xi^\vee$ to the center $Z(\mathbb{R})$. Write $f_\xi = f^G_\xi \in \mathcal{H}(G(\mathbb{R}), \chi_\xi^{-1})$ for an Euler-Poincaré function associated to $\xi$, characterized by the identity

$$\text{Tr} \pi(f_\xi) = \text{ep}_{\mathbb{G}_a}(\pi \otimes \xi) \defeq \sum_{i=0}^{\infty} (-1)^i \dim \mathcal{H}(\text{Lie } G(\mathbb{R}), \mathbb{K}_\infty; \pi \otimes \xi)$$

for every irreducible admissible representation $\pi$ of $G(\mathbb{R})$ with central character $\chi_\xi$, where $\mathbb{K}_\infty$ is a finite index subgroup in the group generated by the center and a maximal compact subgroup of $G(\mathbb{R})$. (A main example for us is $\tilde{K}_H$ in Definition 1.8.) The formula does not determine $f_\xi$ uniquely but the orbital integrals of $f_\xi$ are well defined. The function $f_\xi$ exists by Clozel and Delorme and can be constructed as the average of pseudo-coefficients as follows. Let $\Pi^G_\xi$ denote the $L$-packet consisting of (the isomorphism classes of) irreducible discrete series representations $\pi$ of $G(\mathbb{R})$ whose central character and infinitesimal character are the same as $\xi^\vee$. Write $f_\pi \in \mathcal{H}(G(\mathbb{R}), \chi_\xi^{-1})$ for a pseudo-coefficient of $\pi$, and $q(G) \in \mathbb{Z}$ for the real dimension of $G(\mathbb{R})/\mathbb{K}_\infty$.

Then $f_\xi$ can be defined by

$$f^G_\xi := (-1)^q(G) \sum_{\pi \in \Pi^G_\xi} f_\pi.$$ 

Let $G^*$ denote a quasi-split inner form of $G$ over $\mathbb{R}$. Then the same $\xi$ gives rise to the functions $f^G_\xi$. Note that Definition A.6 makes sense verbatim when $F$ is archimedean and the function has central character.

**Lemma A.9.** Any pseudo-coefficient $f_\pi$ for a discrete series representation $\pi$ is cuspidal. In particular $f^G_\xi$ and $f^G_\xi$ are cuspidal.

*Proof.* It suffices to check that $f_\pi$ is cuspidal, or equivalently that the trace of every induced representation from a proper Levi subgroup vanishes against $f_\pi$, cf. [Art88a, p.538], but this is true by the construction of pseudo-coefficients. (The cuspidality of $f_\xi$ also follows from the proof of [Kot92a, Lem. 3.1].)

Let $A$ denote the maximal split torus in the center of $G$ (hence also in $G^*$). Equip $G(\mathbb{R})/A(\mathbb{R})$ and $G^*(\mathbb{R})/A(\mathbb{R})$ with Euler-Poincaré measures and $A(\mathbb{R})$ with the Lebesgue measure so that the Haar measures on $G(\mathbb{R})$ and $G^*(\mathbb{R})$ are determined. Define $q(G)$ (resp. $q(G^*)$) to be the real dimension of the symmetric space associated to the derived subgroup of $G(\mathbb{R})$ (resp. $G^*(\mathbb{R})$). Normalize the transfer factor between $G(\mathbb{R})$ and $G^*(\mathbb{R})$ to be $e(G) = (-1)^{q(G)-q(G^*)}$. Write $\Pi^G_\xi$ (resp. $\Pi^G_\xi$) for the discrete series $L$-packet for $G(\mathbb{R})$ (resp. $G^*(\mathbb{R})$) associated to $\xi$, cf. Example below Definition 1.8.

**Lemma A.10.** The functions $(-1)^q(G)\Pi^G_\xi \sum f^G_\xi$ and $(-1)^q(G^*)\Pi^G_\xi \sum f^G_\xi$ are associated.

*Proof.* This follows from the computation of stable orbital integrals in [Kot92a, Lem. 3.1]; see also [CLII, Prop. 3.3] when $A = 1$. 

A similar construction works in the base change context, cf. [CLII]. We are only concerned with a particular case that $G^* = G^* \times \cdots \times G^*$ (the number of copies is $d \in \mathbb{Z}_{\geq 1}$) and $\theta$ is the automorphism $(g_1, \ldots, g_d) \mapsto (g_{d+1}, \ldots, g_1)$. Write $\xi := \xi \otimes \cdots \otimes \xi$. A function $f_\xi \in \mathcal{H}(G^*(\mathbb{R}), \chi_\xi^{-1})$ is said to be a (twisted) Lefschetz function for $\xi$ if

$$\text{Tr} \pi(f_\xi) = \sum_{i=0}^{\infty} (-1)^i \text{Tr} \left( \theta | H^i \left( \text{Lie } G(\mathbb{R})^d, K_{\mathbb{K}_\infty}^d; \pi \otimes \xi \right) \right)$$

for every irreducible admissible representation $\pi$ of $G(\mathbb{R})$ with central character $\chi_\xi$. Definition A.6 carries over to this base change setup as in [Lab99, Def. 3.8.1, 3.8.2].
Lemma A.11. The function $f_E$ is cuspidal.\footnote{When $H^1(R,G_{	ext{ad}}) = 1$ (assuming $A = 1$), Clozel and Labesse prove that $f_E$ is also stabilizing in [Lab99, Thm. A1.1]. However we do not use it in the main text but instead appeal to the fact that the (twisted) Lefschetz function at a finite place is stabilizing.} Moreover $f_E$ and $\tilde{e}f_E^G$ are associated for some $\tilde{e} \in \mathbb{C}^\times$.

Proof. The first assertion is [CLII, Prop. 3.3]. The second assertion follows from [CLII, Prop. 3.3, Thm. 4.1]. (The reference assumes that $A = 1$ but the arguments can be adapted to the case of nontrivial $A$ as in the untwisted setup above). \hfill $\square$

We end this appendix with a global result. We change notation. Let $F$ be a totally real number field and $v_{St}$ a finite $F$-place. Let $G$ be an inner form of either the group $\text{GSp}_{2n,F}$ or the group $\text{Sp}_{2n,F}$.

Lemma A.12. Assume that $n > 1$. Let $\pi$ be a non-abelian discrete automorphic representation of $G(A_F)$, and assume that $\text{Tr} \pi_{\mathcal{V}_G} \left( f_E^G \right) \neq 0$. Then $\pi_{\mathcal{V}_G}$ is an unramified twist of the Steinberg representation.

Remark A.13. The lemma is false for $n = 1$.

Proof. We give the argument only in case $G$ is an inner form of $\text{GSp}_{2n,F}$; the argument for inner forms of $\text{Sp}_{2n,F}$ is similar. By the assumption $\text{Tr} \pi_{\mathcal{V}_G} \left( f_E^G \right) \neq 0$, $\pi_{\mathcal{V}_G}$ is either a twist of the Steinberg representation or a twist of the trivial representation (Lemma A.7). We assume that we are in the second case; after twisting we may assume that $\pi_{\mathcal{V}_G}$ is the trivial representation. Let $G_1 \subset G$ be the kernel of the factor of similitudes. By strong approximation the subset $G_1(F)G_1(F_{v_{St}}) \subset G_1(A_F)$ is dense if $G_1,F_{v_{St}}$ is non-anisotropic. Let us assume this for now. Let $f \in \pi$. Since $\pi_{\mathcal{V}_G}$ is the trivial representation, $f$ is invariant under $G(F_{v_{St}})$. Thus $f$ is $G_1(F)G_1(F_{v_{St}})$-invariant, and hence $G_1(A_F)$ invariant. This implies that $\pi$ is abelian. It remains to check that $G_1,F_{v_{St}}$ is non-anisotropic. In the split case, we have $G_1,F_{v_{St}} \cong \text{Sp}_{2n,F_{v_{St}}}$. In the non-split case, the group $G_1,F_{v_{St}}$ is of the following form. Let $D/F_{v_{St}}$ be the quaternion algebra, and consider the involution on $D$ defined by $\overline{x} = \text{Tr}(x) - x$ where $\text{Tr}$ is the reduced trace. Then $G(1,F_{v_{St}}) \cong \text{Sp}_{4}(D)$ is the group of $g \in \text{GL}_n(D)$ such that $gA_n = c(g)A_n$ for some $c(g) \in F_{v_{St}}$, where $A_n$ is matrix with 0’s everywhere and 1’s on its anti-diagonal [PR94, item (3), p.92]. For $n > 1$ the group $\text{Sp}_{2n}(D)$ has a strict parabolic subgroup and thus $G_1,F_{v_{St}}$ is not anisotropic. \hfill $\square$

Appendix B. Conjugacy in the standard representation

Let $C$ be an algebraically closed Hausdorff topological field of characteristic different from 2. Let $V$ be a finite dimensional $C$-vector space and $\langle \cdot, \cdot \rangle$ a non-degenerate, symmetric or skew-symmetric bilinear form on $V$. Let $H \subset \text{GL}_n(C)$ be the subgroup of elements that preserve $\langle \cdot, \cdot \rangle$ (resp. preserve it up to scalar). Thus, $H(C)$ is either an orthogonal group or a symplectic group (resp. of similitude). Write $\sim$: $H(C) \rightarrow C^\times$ for the factor of similitude. Note sim is non-trivial only for the groups $G(V,\langle \cdot, \cdot \rangle)$ and $\text{GSp}(V,\langle \cdot, \cdot \rangle)$.

Proposition B.1 [Larsen, cf. proof of [Lar94b, Prop. 2.3, Prop. 2.4]]. Let $G$ be a topological group. Consider two continuous morphisms $\phi_1, \phi_2 : G \rightarrow H(C)$ such that $\phi_1$ is semisimple. Then $\phi_1, \phi_2$ are $H(C)$-conjugate if and only if $\text{std} \circ \phi_1 = \text{std} \circ \phi_2$ and $\text{std} \circ \phi_1, \text{std} \circ \phi_2$ are $\text{GL}_n(C)$-conjugate.

Remark B.2. The conclusion of Proposition B.1 fails in general for the special orthogonal group in even dimension. In odd dimension the group $O_{2n+1}(C)$ is $\{ \pm 1 \} \times SO_{2n+1}(C)$ and so the proposition is true for $SO_{2n+1}(C)$ as well.

Proof of Proposition B.1. We consider the group $H = \text{GO}(V,\langle \cdot, \cdot \rangle)$ with $\langle \cdot, \cdot \rangle$ symmetric and non-degenerate, the other groups are treated in a similar fashion. Fix a morphism $\phi : G \rightarrow \text{GO}(V,\langle \cdot, \cdot \rangle)$ with $\text{std} \circ \phi$ semisimple. Write $\chi = \text{sim} \circ \phi : G \rightarrow C^\times$. Consider the set $X_\phi(\chi)$ of $\text{GO}(V,\langle \cdot, \cdot \rangle)$-conjugacy classes of morphisms $\phi' : G \rightarrow \text{GO}(V,\langle \cdot, \cdot \rangle)$ such that $\text{std} \circ \phi' = \text{std} \circ \phi$ and $\chi = \text{sim} \circ \phi'$. We view the space $V$ as a $G$-representation via $\phi$. We have the injection

\[ X_\phi(\chi) \hookrightarrow \text{Isom}_G(V,V^* \otimes \chi)/\text{Aut}_V(V), \quad [\phi', \rho] : \text{std} \circ \phi' \sim \text{std} \circ \phi \mapsto \rho_* \chi, \]

whose image is the set of non-degenerate symmetric pairings taken modulo $\text{Aut}_V(V)$, where the automorphisms $\sigma \in \text{Aut}_V(V)$ act on $\text{Isom}_G(V,V^* \otimes \chi)$ via $\rho \mapsto (\sigma^{-1} \circ (\sigma \circ \rho))$. Using the pairing $\langle \cdot, \cdot \rangle$ on $V$ we can further identify

\[ \text{Isom}_G(V,V^* \otimes \chi)/\text{Aut}_V(V) \sim \text{Aut}_V(V)/\text{Aut}_V(V)-\text{congruence}, \]

where by congruence we mean the action $\tau \mapsto \sigma^{\tau \circ \sigma}$ for $\sigma \in \text{Aut}_V(V)$; here the transpose is defined using $\langle \cdot, \cdot \rangle$. Since $(V,\phi)$ is semisimple we can consider the isotypical decomposition $V = \bigoplus_{i=1}^t V_i^d_i$, and so by Schur's
lemma $\text{End}_G(V) = \prod_{i=1}^t M_{d_i}(C)$. We obtain an embedding

$$X_\chi(\phi) \mapsto \prod_{j=1}^t \text{GL}_{d_j}(C)/\text{GL}_{d_j}(C)\text{-congruence},$$

where two matrices $X, Y \in \text{GL}_{d_j}(C)$ are $\text{GL}_{d_j}(C)$-congruent if there exists a third matrix $g \in \text{GL}_{d_j}(C)$ such that $Y = gXg$. The image of $X_\chi(\phi)$ in the set on the right hand side decomposes along the product and is in each $GL_{d_j}(C)$-factor equal to the set of congruence classes of invertible symmetric matrices. Since $C$ is algebraically closed of characteristic $\neq 2$, these classes have exactly one element. \qed

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