Spaces of flattenings of spheres

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Abstract

The spaces of flattenings of a simplicial sphere played a key role in the study of existence and uniqueness of differentiable structures on a simplicial sphere. In this paper, we will establish that the spaces of flattenings of some simplicial spheres and show that they have the homotopy type of the orthogonal group.

Keywords: Regular cell complex, Oriented matroids, Simplicial spheres, Flattenings

1 Introduction

The space of flattenings $F(L)$ associated to a simplicial $k$-sphere $L$ is the space of all simplicial embeddings of the cone $cL$ of $L$ in $\mathbb{R}^{k+1}$. The space of flattenings of simplicial spheres was first studied by S.S Cairns [1] in the early 1940s. In [1], Cairns proved that the space of all flattenings of a simplicial 2-sphere which have an orientation preserving isomorphism onto a given triangulation is path connected.

The space of flattenings associated to a simplicial 1-sphere is homotopy equivalent to the orthogonal group $O(2)$ as we will prove in Theorem 17 [2]. The only known positive result about the topology of $F(L)$ when $\dim(L) > 2$ is the theorem of N.H Kuiper [3] where he proved that $F(L)$ has the homotopy type of $O(n+1)$ when $L$ is the boundary of the $(n+1)$--simplex. In [4], Milin showed that there exist simplicial sphere $L$ of dimension 3 whose subset of $F(L)$ consisting of flattenings which have an orientation preserving isomorphism onto a given triangulation is not path connected.
Let $\Delta^n$ denote an $n$-simplex. We will establish that the space of flattenings associated to $\partial \Delta^n \ast \partial \Delta^1$ has the homotopy type of an orthogonal group.

Let $L$ be a simplicial $k$-sphere and $CF(L)$ denote the quotient of $F(L)$ by $\text{GL}_{k+1}$ the invertible $(k+1) \times (k+1)$ matrices. To establish the above statement, we will first prove that $CF(L)$ is contractible. In Section 5, we will introduce a poset $P(L)$ called The poset of oriented matroid flattenings of $L$ and a poset stratification map $\pi : CF(L) \to P(L)$. We will prove that $P(L)$ is contractible when $L$ is either a simplicial 1-sphere, $\partial \Delta^{n+1}$ the boundary of an $(n+1)$-simplex or $\partial \Delta^1 \ast \partial \Delta^{n+1}$.

In Section 6, we will prove for the above simplicial spheres that $\{\pi^{-1}(M) : M \in P(L)\}$ is a totally normal cellular decomposition of $CF(L)$. Theorem 10 will then conclude that $\|P(L)\|$ can be embedded in $CF(L)$ as a deformation retract.

2 Oriented Matroids

We will view elements of $\mathbb{R}^n$ as $1 \times n$ row vectors so that $X$ is the rowspace of a $r \times n$ matrix. Suppose $X \in \text{Gr}(r, \mathbb{R}^n)$ so that $X = \text{Rowspace}(v_1 v_2 v_3 \ldots v_n)$. We consider the following function $\chi : [n]^r \to \{+, -, 0\}$ associated to $X$

$$\chi(i_1, i_2, \ldots, i_r) = \text{sign}(\det(v_{i_1} v_{i_2} \cdots v_{i_r}))$$

The collection $\{\pm \chi\}$ is independent of the choice of basis vectors for $X$. The resulting functions ($\pm \chi$) defines a rank $r$ oriented matroid.

In general, an oriented matroid can be obtained from an arrangement of pseudospheres as evident by the following theorem. Figure 1 illustrates an arrangement of pseudospheres.

**Theorem 1 ([5])** The Topological Representation Theorem (Folkman-Lawrence 1978) The rank $r$ oriented matroids are exactly the sets $(E, \mathbb{V}^*)$ arising from essential arrangements of pseudospheres in $S^{r-1}$.

A detailed introduction to the theory of oriented matroids can be found in the in the book [6]. Associated to a rank $r$ oriented matroid on $n$ elements are the functions $\pm \chi : [n]^r \to \{+, -, 0\}$ called the chirotopes. Let $\{+, -, 0\}$ be a poset with the partial order $0 < -$ and $0 < +$.

**Definition 2 ([7])** Let $\mathcal{N} = (\pm \chi_1)$ and $\mathcal{M} = (\pm \chi_2)$ be two rank $r$ oriented matroids. We say that $\mathcal{N} \leq \mathcal{M}$ if and only if $\chi_1 \leq \chi_2$ or $\chi_1 \leq -\chi_2$. The oriented matroid $\mathcal{M}$ is said to weak map to $\mathcal{N}$.

**Definition 3 ([7])** MacP($p, n$) denotes the poset of all rank $p$ oriented matroids on elements $\{1, 2, \ldots, n\}$, with weak map as the partial order. The poset is called the MacPhersonian [7].
We have explained how to obtain a rank $r$ oriented matroid on $n$ elements from a rank $r$ subspace of $\mathbb{R}^n$. That is, there is a function $\mu : \text{Gr}(r, \mathbb{R}^n) \to \text{MacP}(r, n) : X \to (\pm \chi_X)$.

The following Proposition and Theorem are from the work of the author in [8], [2].

**Proposition 4** ([8]) Let $M \in \text{MacP}(2, n)$. Then $\partial \mu^{-1}(M) = \bigcup_{N \subset M} \mu^{-1}(N)$

**Theorem 5** ([8]) $\{\mu^{-1}(M) : M \in \text{MacP}(2, n)\}$ is a regular cell decomposition of $\text{Gr}(2, \mathbb{R}^n)$.

### 3 Cellular stratified spaces

**Definition 6** ([9]) A globular $n$-cell is a subset $D$ of $D^n$ containing $H = \text{Int}(D^n)$. We call $D \cap \partial D^n$ the boundary of $D$ and denote it by $\partial D$. The number $n$ is called the globular dimension of $D$.

A globular $n$-cell was introduced by Tamaki [9] as an extension of closure of $n$-cells to non-closed cells.

**Definition 7** ([9]) Let $X$ be a topological space. For a non-negative integer $n$, an $n$-cell structure on a subspace $e \subset X$ is a pair $(D, \varphi)$ of a globular $n$-cell $D$ and a continuous map

$$\varphi : D \to X$$

satsisfying the following conditions:

- $\varphi(D) = \bar{e}$ and $\varphi : D \to \bar{e}$ is a quotient map.
- The restriction $\varphi : H \to e$ is a homeomorphism.
Definition 8 ([9]) Let $X$ be a topological space and $P$ be a poset with the Alexandroff topology. A stratification of $X$ indexed by $P$ is an open continuous map

$$\pi : X \to P$$

satisfying the condition that for each $\lambda \in P$, $e_\lambda = \pi^{-1}(\lambda)$ is connected and locally closed. $X$ is called a cellular stratified space if each $e_\lambda$ is homeomorphic to an open ball.

Definition 9 ([9], [10]) Let $X$ be a cellular stratified space. $X$ is called totally normal if for each globular $n$-cell $(D_\lambda, \varphi)$, and $e_\lambda = \varphi(\text{Int}(D_\lambda))$

1. If $e_\lambda \cap e_\mu \neq \emptyset$, then $e_\lambda \subseteq e_\mu$.
2. There exists a structure of a regular cell complex on $S^{n-1}$ containing $\partial D_\lambda$ as a cellular stratified subspace of $S^{n-1}$.
3. For each cell $e$ in the cellular stratification on $\partial D_\lambda$, there exists a cell $e_\eta$ in $X$ and a map $b : D_\eta \to \partial D_\lambda$ such that $b(\text{Int}(D_\eta)) = e$ and $\varphi_\lambda \circ b = \varphi_\eta$.

Theorem 10 ([9]) For a totally normal cellular stratified space $X$ with stratification $\pi : X \to P$, there is an embedding of $\|P\|$ as a strong deformation retract of $X$.

4 Flattenings

Definition 11 ([11], [4]) Let $L$ be a triangulation of a $k$-sphere, and let $cL$ be a simplicial cone over $L$. A flattening of $L$ is an embedding $\psi : cL \to \mathbb{R}^{k+1}$ that maps the cone vertex to the origin and it is linear on simplices of $cL$.

Notation 12 Let $L$ be a simplicial $k$-sphere. We denote as in [11] by $F(L)$ the space of all flattenings of $L$. Also, the group $GL_{k+1}$ of invertible $(k+1) \times (k+1)$ matrices acts on $F(L)$; the quotient space denoted by $CF(L)$ is the configuration space of $L$.

The space $F(L)$ is an open subset of $\mathbb{R}^{(k+1)|\text{Vert}(L)|}$, and so has a natural smooth manifold structure. The space of flattenings comes up in the problem of existence and uniqueness of differentiable structures on triangulated manifolds (see [12], [3]).

We will show that $CF(L)$ is contractible when $L$ is a simplicial 1-sphere, and so, $F(L)$ has the homotopy type of $O(2)$. Some few other non-trivial results that are known about the topology of $CF(L)$ and $F(L)$ are as follows.

Theorem 13 ([1]) Let $L$ be a triangulated 2-sphere. Then $CF(L)$ is path connected.

For $\text{dim}(L) \geq 3$, Cairns [1] also showed that $CF(L)$ can be empty. When the dimension of $L$ is greater than 2, Milin [4] obtained the following negative result about the topology of $CF(L)$. 
Theorem 14  

There exists a 3 dimensional simplicial sphere $L$ such that $CF(L)$ is disconnected.

So far, for $n > 2$ the only known positive result about the homotopy type of $F(L)$ is the following result of Kuiper.

Theorem 15  

Let $\partial \Delta^{n+1}$ be the boundary of an $(n+1)$-simplex. Then $F(\partial \Delta^{n+1})$ has the homotopy type of $O(n+1)$.

Corollary 16  

Let $\partial \Delta^{n+1}$ be the boundary of an $(n+1)$-simplex. Then any two smoothings of $\partial \Delta^{n+1}$ are diffeomorphic.

As in Theorem 15, we also obtain the following positive result for the simplicial sphere $\partial \Delta^1 \ast \partial \Delta^{n+1}$.

Theorem 17  

Let $\partial \Delta^{n+1}$ be the boundary of an $(n+1)$-simplex. Then $F(\partial \Delta^1 \ast \partial \Delta^{n+1})$ has the homotopy type of $O(n+2)$. Let $L$ be a simplicial 1-sphere. Then $F(L)$ has the homotopy type of $O(2)$.

Corollary 18  

Let $\partial \Delta^{n+1}$ be the boundary of an $(n+1)$-simplex. Then any two smoothings of $\partial \Delta^1 \ast \partial \Delta^{n+1}$ are diffeomorphic.

5 Oriented matroid flattenings

Let $L$ be a triangulated of a $k$-sphere, and $\psi : cL \to \mathbb{R}^{k+1}$ a flattening of $L$. Then the arrangement of vectors $(\psi(v) : v \in \text{Vert}(L))$ determines a rank $k+1$ oriented matroid $M$. Definition 19 gives a combinatorial abstraction for oriented matroids obtained from flattenings of a simplicial sphere.

Definition 19  

Let $L$ be a simplicial sphere of dimension $k$. An oriented matroid flattening of $L$ is a rank $k+1$ oriented matroid $M$ satisfying the following:

1. The elements of $M$ are the vertices of $L$.
2. The set of vertices in a simplex are independent.
3. The set of vertices in a simplex has no other elements in its convex hull.

Notation 20  

The poset of all oriented matroid flattenings of $L$ is denoted by $P(L)$.

Proposition 21  

Let $L$ be a simplicial sphere and $\partial \Delta^n$ the boundary of an $n$-simplex. Then $\|P(L)\|$ is contractible when $L$ is either a simplicial 1-sphere, $\partial \Delta^n$ or $\partial \Delta^1 \ast \partial \Delta^n$. 
Proof The poset $P(\partial \Delta^n)$ consists of a point. Let $P(\partial \Delta^n) = \{ M_n \}$. For the sphere $\partial \Delta^1 \ast \partial \Delta^n$, $P(\partial \Delta^1 \ast \partial \Delta^n)$ has a minimum; given by the join $M_1 \oplus M_n$ of two oriented matroids.

In the case when $L$ is a simplicial 1-sphere, this will follow by induction on the number of vertices in $\text{Vert}(L)$. Let $L_n$ denote a simplicial 1-sphere on $n$ vertices. We know that $P(L_3)$ consists of a point say $M_0 = (\pm \chi_0)$. In the following argument, we will consider chirotopes with positive value on the basis $\{1, 2\}$.

Let $\Sigma^{n+1}$ denote a subposet of $P(L_{n+1})$ consisting of $M'$ such that $M' \setminus \{n+1\}$ is an element of $P(L_n)$. An oriented matroid in $\Sigma^{n+1}$ is thus an extension of an oriented matroid $M$ in $P(L_n)$ by an element $n+1$, with $n+1$ lying in the convex hull of $\{1, n\}$.

There is a poset map $P(L_{n+1}) \rightarrow \Sigma^{n+1}$ obtained as composition of some poset maps as given below. Let $f_0 : P(L_{n+1}) \rightarrow P(L_{n+1})$ defined as:

$$f_0(\chi)(B) = \begin{cases} 
\chi(B) & \text{if } B \neq (n, 1) \\
0 & \text{if } B = (n, 1) \text{ and } \chi(n, 1) \in \{0, -\} \\
+ & \text{if } B = (n, 1) \text{ and } \chi(n, 1) = +
\end{cases}$$

Let $P_0 = f_0(P(L_{n+1}))$. The poset map $f_0$ is a lowering homotopy, and so $\|P_0\|$ is homotopy equivalent to $\|P(L_{n+1})\|$. We again consider another poset map $f_1 : P_0 \rightarrow P_0$ defined as:
\[ f_1(\chi)(B) = \begin{cases} \chi(B) & \text{if } B \neq (n,1) \\ + & \text{if } B = (n,1) \end{cases} \]

The image of \( f_1 \) is denoted is given by \( f_1(P_0) = \Sigma^{n+1} \). The poset map \( f_1 : P_0 \to P_0 \) is a raising homotopy, and so \( \|P_0\| \) is homotopy equivalent to \( \|\Sigma^{n+1}\| \). The poset map \( \Sigma^{n+1} \to P(L_n) \) induces a homotopy equivalence between \( \|\Sigma^{n+1}\| \) and \( \|P(L_n)\| \).

For a simplicial sphere \( L \), there is a stratification map \( \mu_0 : CF(L) \to P(L) \).

**Conjecture 22** Let \( L \) be a simplicial sphere of dimension at least 2. Then \( \|P(L)\| \) is contractible.

### 6 Topology of space of flattenings of some spheres

Let \( \mu' : Gr(r, \mathbb{R}^{r+2}) \to \text{MacP}(r, r+2) \) and \( \mu : Gr(2, \mathbb{R}^n) \to \text{MacP}(2, n) \). Let \( \mu_0 : CF(L) \to P(L) \) be the restriction of \( \mu' \) to \( CF(L) \) when \( L = \partial \Delta^1 \ast \partial \Delta^{r-1} \) or the restriction of \( \mu \) when \( L \) is a simplicial 1-sphere on \( n \) vertices.

The stratification map \( \mu_0 : CF(L) \to P(L) \) gives a decomposition of \( CF(L) \) into semi-algebraic sets \( \{\mu_0^{-1}(M) : M \in P(L)\} \). When \( L \) is a simplicial 1-sphere or \( L = \partial \Delta^1 \ast \partial \Delta^n \), we will show that the decomposition is a totally normal cellular decomposition.

We have the following commutative diagram

\[
\begin{array}{ccc}
Gr(r, \mathbb{R}^{r+2}) & \xrightarrow{\mu'} & \text{MacP}(r, r+2) \\
V \downarrow V^\perp & & \downarrow M \to M^* \\
Gr(2, \mathbb{R}^{r+2}) & \xrightarrow{\mu} & \text{MacP}(2, r+2) \\
\end{array}
\]

The commutativity of the diagram follows from the fact that \( V = (I_r|A) \in Gr(r, \mathbb{R}^{r+2}) \) if and only if \( V^\perp = \text{Rowspan}(-A^T|I_2) \in Gr(2, \mathbb{R}^{r+2}) \). The oriented matroid \( M^* \) is called the dual of \( M \).

The map \( Gr(r, \mathbb{R}^{r+2}) \to Gr(2, \mathbb{R}^{r+2}) : V \mapsto V^\perp \) is a homeomorphism and the poset map \( \text{MacP}(r, r+2) \to \text{MacP}(2, r+2) : M \mapsto M^* \) is a poset isomorphism.

The following result thus follows from Theorem 5 and the commutativity of the diagram described above.

**Theorem 23** Let \( M \in \text{MacP}(r, r+2) \) be a rank \( r \) oriented matroid on \( r+2 \) elements, and \( \mu' : Gr(r, \mathbb{R}^{r+2}) \to \text{MacP}(r, r+2) \). Then \( \{\mu'^{-1}(M) : M \in \text{MacP}(r, r+2)\} \) is a regular cell decomposition of \( Gr(r, \mathbb{R}^{r+2}) \).

**Proposition 24** Let \( L \) be a simplicial sphere and \( \mu_0 : CF(L) \to P(L) \) a stratification map. If \( L \) is a simplicial 1-sphere or \( L = \partial \Delta^1 \ast \partial \Delta^n \), then the decomposition \( \{\mu_0^{-1}(M) : M \in P(L)\} \) is a totally normal cellular decomposition of \( CF(L) \).
Proof 1. Suppose $L$ is as given above. It was proven in Proposition 4 that if $N, M \in P(L)$ such that $N < M$, then $\mu_0^{-1}(N) \subseteq \mu_0^{-1}(M)$. So, the decomposition $\{\mu_0^{-1}(M) : M \in P(L)\}$ is normal.

2. In Theorem 5, it was proven that $\{\mu^{-1}(M) : M \in \text{MacP}(2, |\text{Vert}(L)|)\}$ is a regular cell decomposition of $Gr(2, \mathbb{R}^{\text{Vert}(L)})$. Similarly, we have in Theorem 23 that $\{\mu^{-1}(M) : M \in \text{MacP}(r, r+2)\}$ is a regular cell decomposition of $Gr(r, \mathbb{R}^{r+2})$.

If $L$ is a simplicial 1-sphere, and $M \in P(L)$, then $\partial\mu^{-1}(M)$ is a regular cellular cell complex homeomorphic to a sphere. Let $\mu_0^{-1}(M)$ denote the closure of $\mu_0^{-1}(M)$ in $CF(L)$. Then $\partial\mu^{-1}(M)$ contains $\partial\mu_0^{-1}(M)$ as a cellular stratified subspace. Similarly when $L = \partial\Delta^1 \ast \partial\Delta^n$ and $M \in P(L)$, $\partial(\mu')^{-1}(M)$ contains $\partial\mu_0^{-1}(M)$ as a cellular stratified subspace.

3. $D_M = \mu_0^{-1}(M)$, and let $\varphi_M$ be the restriction to $D_M$ of the characteristic map of the cell $\mu^{-1}(M)$ if $L$ is a simplicial 1-sphere or restriction of the characteristic map of $(\mu')^{-1}(M)$ if $L = \partial\Delta^1 \ast \partial\Delta^n$.

For a cell $e$ in the boundary of $D_M$, there exists an oriented matroid $N$ in $P(L)$ such that $N < M$ and $\mu_0^{-1}(N) = e$. The map $b : D_N \to \partial D_M$ is given by $b = (\varphi_M)^{-1} \circ \varphi_N$.

Proof of Theorem 17 Suppose $L$ is a simplicial 1-sphere or $L = \partial\Delta^1 \ast \partial\Delta^n$. The decomposition $\{\mu_0^{-1}(M) : M \in P(L)\}$ is a totally normal cellular decomposition of $CF(L)$ by Proposition 24. It thus follows from Theorem 10 that $\|P(L)\|$ is a deformation retract of $CF(L)$. We know from Proposition 21 that $\|P(L)\|$ is contractible. Hence, $CF(L)$ is contractible.

Suppose $L$ is a simplicial 1-sphere. We know that $F(L)|_H \cong \text{GL}_2(\mathbb{R}) \times CF(L)$. Hence $F(L)$ has the homotopy type of $O(2)$. Similarly, if $L = \partial\Delta^1 \ast \partial\Delta^n$, then $F(L) \cong \text{GL}_{n+1}(\mathbb{R}) \times CF(L)$. Hence $F(L)$ has the homotopy type of $O(n+1)$.

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