The Four-fermi Coupling of the Supersymmetric Non-linear $\sigma$-model on $G/S \otimes \{U(1)\}^k$

Shogo Aoyama*
Department of Physics
Shizuoka University
Ohya 836, Shizuoka
Japan

March 27, 2022

Abstract

The reducible Kähler coset space $G/S \otimes \{U(1)\}^k$ is discussed in a geometrical approach. We derive the formula which expresses the Riemann curvature of the reducible Kähler coset space in terms of its Killing vectors. The formula manifests the group structure of $G$. On the basis of this formula we establish an algebraic method to evaluate the four-fermi coupling of the supersymmetric non-linear $\sigma$-model on $G/S \otimes \{U(1)\}^k$ at the low-energy limit. As an application we consider the supersymmetric non-linear $\sigma$-model on $E_7/SU(5) \otimes \{U(1)\}^3$ which contains the three families of $10 + 5^* + 1$ of $SU(5)$ as the pseudo NG fermions. The four-fermi coupling constants among different families of the fermions are explicitly given at the low-energy limit.

To appear in Nuclear Physics B

*e-mail: spsaoya@ipc.shizuoka.ac.jp
1 Introduction

The non-linear $\sigma$-model on a coset space $G/H$ is a low-energy effective theory of the Nambu-Goldstone(NG) bosons generated when a large group symmetry $G$ at high energy spontaneously breaks into a small one at low energy. If the supersymmetry exists at that high energy and if it survives in the spontaneous breaking, the NG bosons are accompanied by superpartners, called the pseudo NG fermions. They are described by the supersymmetric non-linear $\sigma$-model on a coset space $G/H$ which is kählerian. In the beginning of the 80’s Buchmüller, Love, Peccei and Yanagida proposed to identify the massless pseudo NG fermions with light quarks and leptons and used the supersymmetric non-linear $\sigma$-model as a low-energy effective theory for the grand unification. Recently in ref. this idea was revived to explain the neutrino mass observed in the SuperKamiokande experiment. They proposed the supersymmetric non-linear $\sigma$-model on the Kähler coset space $E_7/SU(5) \otimes \{U(1)\}^3$ as a theory which naturally accomodates the three families of right-handed neutrinos. Namely the model contains three families of $10 + 5^* + 1$ and $5$ of $SU(5)$ as the NG supermultiplet $(\phi^\alpha, \psi^{\alpha\beta})$. Interactions among the pseudo NG fermions take place through the four-fermi coupling

$$R_{\alpha\alpha'}^{\sigma\delta}(\phi^{\alpha}_f, \bar{\phi}^{\alpha}_f)(\bar{\psi}^{\sigma\alpha}_j \psi^{\beta}_j)(\bar{\psi}^{\delta\beta}_j \psi^{\delta\beta}_j)$$

in which $R_{\alpha\alpha'}^{\sigma\delta}$ is the Riemann curvature of the coset space and $f$ is a constant giving a mass scale. It is phenomenologically the most interesting part of the model. The aim of this paper is to establish a practical method to calculate the Riemann curvature of $E_7/SU(5) \otimes \{U(1)\}^3$.

$E_7/SU(5) \otimes \{U(1)\}^3$ is a fairly complicated coset space. The complication comes in twofold. Firstly the coset space is reducible. One can calculate the holomorphic Killing vectors and the Kähler potential of the reducible coset space. But they take more cumbersome forms than those of the irreducible coset space. Secondly the homogeneous group includes plural $U(1)$s as direct products. In such a case the complex structure of the Kähler coset space is not unique and the metric depends on as many free parameters as $U(1)$s. These subjects on the reducible Kähler coset space were extensively studied by the Kyoto group in ref. The general method to construct the Kähler potential of the reducible coset space was given. The Riemann curvature may be calculated by differentiating that Kähler potential by coordinates, in principle. But it is too involved.

In this paper we employ an alternative formalism to do this more directly, which was proposed in ref. It is based on the Killing potentials instead of the Kähler potential, which are also characteristic for the Kähler coset space $G/H$. In ref. the formalism was developed for the irreducible case, and the Riemann curvature was given in terms of the holomorphic Killing vectors with no derivative by coordinates. (See eq. (4.14).) Once given a concrete form of the holomorphic Killing vectors, one can directly calculate the Riemann curvature by that formula. In this paper we extend this formalism to study the reducible Kähler coset space along the same line. We will be particularly
interested in the Riemann curvature at the low-energy limit \( f \to \infty \), i.e., \( R_{\alpha \gamma \beta \delta}(0,0) \). It gives the four-fermi coupling constants at low energy, which depend on as many free parameters as \( U(1) \)s of the homogeneous subgroup. Knowing the dependence explicitly is very interesting from the phenomenological point of view.

In Section 2 we summarize the geometry of the Kähler coset space \( G/H \). In Section 3 we briefly explain the generalized CCWZ formalism\[7\][8]. It enables us to construct the holomorphic Killing vectors, the metric and the Kähler potential for the reducible case. In Section 4 the alternative formalism based on the Killing potentials is presented which allows us to calculate the metric and the Riemann curvature more directly. We first of all review the formalism which was developed for the irreducible case\[9\]. Then it will be extended to the reducible case. We derive the general formula which expresses the Riemann curvature of the reducible Kähler coset space in terms of the holomorphic Killing vectors.(See eq. (4.32).) In Section 5 the Riemann curvatures of \( SU(3)/\{U(1)\}^2 \) and \( E_7/SU(5) \otimes \{U(1)\}^3 \) are evaluated to the leading order of \( \frac{1}{f} \) by this general formula.

2 The geometry of the Kähler manifold

In this section we briefly review on the Kähler manifold, giving our notation. Consider a real \( 2N \)-dimensional Riemann manifold \( \mathcal{M} \) with local coordinates \( \phi^a = (\phi^1, \phi^2, \ldots, \phi^{2N}) \). The line element of the manifold is given by

\[
\text{d}s^2 = g_{ab} \text{d}\phi^a \text{d}\phi^b. \tag{2.1}
\]

\( \mathcal{M} \) is a Kähler manifold if it is endowed with a complex structure which is covariantly constant:

\[
J^a_{\ b;\ c} = 0, \tag{2.2}
\]

and satisfies \( J^a_b J^b_c = -\delta^a_c \). We assume the metric \( g_{ab} \) to be of type \((1,1)\), i.e.,

\[
g_{ab} = g_{cd} J^c_a J^d_b. \tag{2.3}
\]

(A tensor of more general type \((r,s)\) will be discussed soon later.) The symplectic structure \( J_{ab} \) is given by \( J_{ab} = g_{ae} J^e_b \). By (2.2) it is closed:

\[
J_{ab,c} + J_{bc,a} + J_{ca,b} = 0. \tag{2.4}
\]

When the Kähler manifold is a coset space \( G/H \), there is a set of Killing vectors

\[
R^{Aa} = (R^{1a}, R^{2a}, \ldots, R^{Da}), \tag{2.5}
\]

with \( D = \dim G \), which represents the isometry \( G \). They satisfy

\[
\mathcal{L}_{R^{Aa}} R^{Ba} = R^{Ab} R^{Ba}_{\ b} - R^{B}_{\ b} R^{Aa}_{\ b} = f^{ABC} R^{Ca} \tag{2.6}
\]
\[ \mathcal{L}_{R^A} g_{ab} = R^{Ac} g_{ab,c} + R^{Ac}_{\text{c},a} g_{cb} + R^{Ac}_{\text{c},b} g_{ca} = 0 \]  
(2.7)

\[ \text{(Isometry)}, \]

\[ \mathcal{L}_{R^A} J^a_b = R^{Ac} J^a_{b,c} - R^{Aa}_{\text{c},b} J^c_b + R^{Ac}_{\text{c},b} J^a_b = 0, \]  
(2.8)

in which \( \mathcal{L}_{R^A} \) is the Lie-derivative with respect to \( R^A \), and \( f^{ABC} \) are structure constants of the isometry group \( G \).

Any vector \( v^a \) can be projected onto the \((1, 0)\) and \((0, 1)\) types by

\[ \frac{1}{2}(1 - iJ)^a_b v^b, \quad \frac{1}{2}(1 + iJ)^a_b v^b. \]  
(2.9)

A tensor of the \((r, s)\) type is obtained as a multi-product of these projected vectors. We may locally set the complex structure to be

\[ J^a_b = \begin{pmatrix} -i\delta^a_\beta & 0 \\ 0 & i\delta^\alpha_\beta \end{pmatrix}, \]  
(2.10)

with \( \alpha, \bar{\alpha} = 1, 2, \ldots, N \). Then the respective vectors in (2.9) may be written as \( N \)-dimensional complex vectors \( v^a \) and \( v^\bar{\alpha} \). The line element (2.1) is written as

\[ ds^2 = g^a_{\alpha \bar{\beta}} d\phi^a d\phi^\bar{\beta}, \]

by (2.3). The closure of the symplectic structure given by (2.4) reads

\[ g^a_{\alpha \beta, \gamma} = g_{\gamma \beta, \alpha}, \quad g^a_{\alpha \bar{\beta}, \bar{\gamma}} = g_{\alpha \bar{\gamma}, \bar{\beta}}. \]  
(2.11)

Then it follows that there exists a real scalar \( K(\phi, \bar{\phi}) \), called Kähler potential such that

\[ g^a_{\alpha \beta} = K_{\alpha \beta}. \]  
(2.12)

Furthermore (2.8) and (2.10) imply that the Killing vectors are holomorphic:

\[ R^A_{\beta, \alpha} = 0, \quad R^A_{\bar{\beta}, \bar{\alpha}} = 0. \]  
(2.13)

Then (2.6) and (2.7) reduce respectively to

\[ \mathcal{L}_{R^A} R^{B\alpha}_\alpha = R^{A\beta} R^{B\alpha}_{\beta, \alpha} - R^{B \beta} R^{A\alpha}_{\beta, \alpha} = f^{ABC} R^{C\alpha}, \]  
(2.14)

and

\[ \mathcal{L}_{R^A} g^a_{\alpha \bar{\beta}} = R^a_{\alpha, \beta} + R^a_{\bar{\beta}, \alpha} = 0, \]  
(2.15)

with \( R^a_{\alpha} = g^a_{\alpha \bar{\beta}} R^\bar{\beta} \) and \( R^a_{\bar{\alpha}} = g_{\bar{\beta} \alpha} R^\alpha \). From (2.15) we may find real scalars \( M^A(\phi, \bar{\phi}) \), called Killing potentials, such that

\[ R^a_{\alpha} = iM^A_{\alpha}, \quad R^a_{\bar{\alpha}} = -iM^A_{\bar{\alpha}}. \]  
(2.16)
As shown in ref. [10], they transform as the adjoint representation of the group $G$ by the Lie-variation

$$\mathcal{L}_{\mathbf{R}^A} M^B = R^{\alpha A} M^B_{;\alpha} + R^{\alpha \bar{A}} M^B_{;\bar{A}} = f^{ABC} M^C.$$  \hfill (2.17)

A manipulation of (2.17) with (2.16) leads us to write the Killing potentials in terms of the Killing vectors [9] [11]:

$$M^A = -i \frac{1}{N_{adj}} f^{ABC} R^{B\alpha} R^{C\bar{B}} g_{\alpha\bar{B}}.$$  \hfill (2.18)

Here we have used the normalization

$$f^{ABC} f^{ABD} = 2 N_{adj} \delta^{CD}.$$  \hfill (2.18)

These Killing potentials characterize the Kähler manifold no less than the Kähler potential, if it is a coset space $G/H$.

3 The CCWZ formalism

In this section we will explain how to construct the holomorphic Killing vectors $R^{\alpha A}$, the metric $g_{\alpha\bar{B}}$ and the Kähler potential $K$, which essentially characterize the Kähler coset space $G/H$. When the Kähler coset space $G/H$ is irreducible, they can be constructed case by case in heuristic ways [4] [12]. But for the reducible case we need a systematic method. It was given by generalizing the CCWZ formalism [1] by the Kyoto group [7] [8]. We briefly sketch this generalized CCWZ formalism.

3.1 The holomorphic Killing vectors

We assume the isometry group $G$ is compact and semi-simple. If a coset space $G/H$ is kählerian, the unbroken subgroup $H$ contains $U(1)$ groups as $H = S \otimes \{U(1)\}^k, k = 1, 2, \ldots, n$, according to the Borel theorem [13]. The generators $T^A$ of $G$ are decomposed as

$$\{T^A\} = \{X^a, S^I, Q^\mu\}, \quad a = 1, 2, \cdots, 2N(= \dim G - \dim H),$$
$$I = 1, 2, \cdots, \dim S(= \dim H - k),$$
$$\mu = 1, 2, \cdots, k,$$  \hfill (3.1)

in which $S^I$ and $Q^\mu$ are generators of $S$ and $U(1)$s respectively, while $X^a$ broken generators. Let us define a central charge as

$$Y = \sum_{\mu=1}^{k} v^\mu Q^\mu \equiv v \cdot Q,$$  \hfill (3.2)

by choosing real coefficients $v^\mu$ such that all the broken generators $X^a$ have non-vanishing $Y$-charges. Then the broken generators $X^a$ can be splitted into
two parts: the generators $X_i$ with positive $Y$-charge and their hermitian conjugates $X^i$ with negative charge, $i, \bar{i} = 1, 2, \ldots, N$. (3.1) is further decomposed as

$$\{T^A\} = \{X_i, X^i, S^I, Q^\mu\}. \quad (3.3)$$

The splitting of the broken generators determines the complex structure $J_{a\bar{b}}$ of the Kähler coset space $G/H$. But the splitting is not unique depending on the definition of the central charge (3.2). It implies arbitrariness of the complex structure of the coset space.

For the decompostion (3.3) the standard application of the CCWZ formalism does not give the holomorphic Killing vectors $R^A\alpha$ satisfying the Lie-algebra (2.14). Hence we extend the isometry group $G$ to the complex one $G^c$ and consider a coset space $G^c/\hat{H}$ with the complex subgroup $\hat{H}$ generated by $X^i, S^I, Q^\mu$.[4]. As explicitly given later, there is an isomorphism between this complex coset space $G^c/\hat{H}$ and $G/H$:

$$G/H \cong G^c/\hat{H}. \quad (3.4)$$

The holomorphic Killing vectors are obtained by applying the CCWZ formalism to the complex coset space $G^c/\hat{H}$. The coset space $G^c/\hat{H}$ is parametrized by complex coordinates $\phi^\alpha, \alpha = 1, 2, \ldots, N$ corresponding to the broken generators $X^i$. Consider a holomorphic quantity

$$\xi(\phi) = e^{\phi \cdot X} \in G^c/\hat{H} \quad (3.5)$$

with

$$\phi \cdot X^\bar{\tau} = \phi^1 X^\bar{1} + \phi^2 X^\bar{2} + \cdots \phi^N X^\bar{N}. \quad (3.6)$$

For an element $g$ of the isometry group $G$, i.e., $g = e^{iA^T A}$, there exists a compensator $\hat{h}(\phi, \bar{\phi}, g) \in \hat{H}$ such that

$$g\xi(\phi) = \xi(\phi') \hat{h}(\phi, \bar{\phi}, g). \quad (3.6)$$

This defines a holomorphic transformation of the coordinates $\phi^\alpha$ which realizes the isometry group non-linearly. When the real parameters $\epsilon^A$ are infinitesimal, (3.7) yields the holomorphic Killing vectors $R^{A\alpha}(\phi)$ as

$$\delta \phi = \phi'^\alpha(\phi) - \phi^\alpha = \epsilon^A R^{A\alpha}(\phi), \quad (3.7)$$

which satisfy the Lie-algebra (2.14).

### 3.2 The metric

Any two points on the coset space $G/H$ can be related by the isometry transformation (3.6). Therefore the line element (2.1) has the same length at any point of the coset space

$$g_{ab}(\phi', \bar{\phi'}) d\phi'^a d\phi'^b = g_{ab}(\phi, \bar{\phi}) d\phi^a d\phi^b. \quad (3.8)$$

\(^1(3.5)\) should have been written as $\xi(\phi) = \exp\left(\frac{i}{f} \phi \cdot X\right)$ with the mass scale parameter $f$. But it is hereinafter set to be one to avoid unnecessary complication.
On the other hand the line element is invariant under general coordinate transformations:

\[ g_{ab}(\phi, \bar{\phi})d\phi^a d\phi^b = g'_{ab}(\phi', \bar{\phi}')d\phi'^a d\phi'^b. \]  

(3.9)

(3.8) and (3.9) require that

\[ g_{ab}(\phi', \bar{\phi}') = g'_{ab}(\phi', \bar{\phi}') \]  

(3.10)

which gives the Killing condition (2.7) in the infinitesimal form.

To construct the metric \( g_{ab} \) which satisfy the condition (3.10) we have recourse to the CCWZ formalism. Consider a quantity

\[ U(\phi, \bar{\phi}) \in G/H, \]  

(3.11)

with \( U^\dagger U = UU^\dagger = 1 \). But the standard parametrization of \( U \), i.e, \( U(\phi, \bar{\phi}) = e^{\phi \cdot X + \bar{\phi} \cdot X} \) does not give the metric of the type (1,1). Therefore we employ the non-standard one, namely

\[ U(\phi, \bar{\phi}) = \xi(\phi)\zeta(\phi, \bar{\phi}), \]  

(3.12)

in which \( \xi(\phi) \) is the element (3.5), while \( \zeta(\phi, \bar{\phi}) \) an element of \( \hat{H} \). We parametrize the latter as

\[ \zeta(\phi, \bar{\phi}) = e^{a(\phi, \bar{\phi}) \cdot X} e^{b(\phi, \bar{\phi}) \cdot S} e^{c(\phi, \bar{\phi}) \cdot Q}, \]  

(3.13)

with

\[ a \cdot X = \sum_{i=1}^{N} a^i X^i, \quad b \cdot S = \sum_{I=1}^{\dim H - k} b^I S^I. \]

Here the function \( b(\phi, \bar{\phi}) \) and \( c(\phi, \bar{\phi}) \) are chosen to be real since their purely imaginary parts can be absorbed into an element of \( H \). Then the parametrization (3.13) is determined by the unitary condition \( U^\dagger U = 1 \) which reads

\[ \xi^\dagger(\bar{\phi})\xi(\phi) = e^{-a(\phi, \bar{\phi}) \cdot X} e^{-2b(\phi, \bar{\phi}) \cdot S} e^{-2c(\phi, \bar{\phi}) \cdot Y} e^{-a(\phi, \bar{\phi}) \cdot X}. \]

(3.12) is an concrete expression of the isomorphism (3.4) relating the respective elements (3.5) and (3.11) of the coset spaces \( G/H \) and \( G^c/\hat{H} \).

The fundamental object to construct the metric \( g_{ab} \) is the Cartan-Maurer 1-form

\[ \omega = U^{-1}dU \]
\[ = e^{l}X^{l} + e^{\bar{l}} \bar{X}^{\bar{l}} + \omega^{l} S^{l} + \omega^{\mu} Q^{\mu}, \]  

(3.14)

with the 1-forms \( e^{l}(\phi, \bar{\phi}), e^{\bar{l}}(\phi, \bar{\phi}), \omega^{l}(\phi, \bar{\phi}) \) and \( \omega^{\mu}(\phi, \bar{\phi}) \) as coefficients of the expansion. In particular \( e^{l}(\phi, \bar{\phi}) \) takes the form

\[ e^{l} = e^{l}_{\alpha} d\phi^{\alpha}, \]
with no $d\phi\bar{\sigma}$, as can be seen from the parametrization (3.12). $e^\tau(\phi, \bar{\phi})$ is its complex conjugate. The components $e^i_\alpha$ and $e^\tau_\alpha$ are vielbeins of the local frame of the coset space. From this it follows that

$$
g_{\alpha\beta} = g_{\bar{\alpha}\bar{\beta}} = 0, \quad g_{\alpha\bar{\beta}} = g_{\bar{\beta}\alpha} = \sum_{i=1}^{N} y_i(v) e^i_\alpha e^\tau_{\bar{\beta}},
$$

(3.15)

where $y_i(v)$ is the positive $Y$-charge (3.2) of the broken generator $X^i$:

$$
[Y, X^i] = -y_i(v) X^i, \quad [Y, X^\tau] = y_i(v) X^\tau.
$$

(3.16)

By the transformation (3.6) $U$ transforms as

$$
g U(\phi, \bar{\phi}) = U(\phi', \bar{\phi}') h(\phi, \bar{\phi}, g),
$$

(3.17)

with a compensator $h \in H$. Then $e^i$ transform homogeneously as

$$
e^i(\phi', \bar{\phi}') = \rho^{ij}(h(\phi, \bar{\phi}, g), g) e^j(\phi, \bar{\phi}),
$$

(3.18)

in which $\rho^{ij}(h, g)$ is the $N$-dimensional representation of the subgroup $H$. Consequently the metric (3.15) satisfies the transformation property (3.8) under (3.6) or equivalently (3.17). Furthermore (3.16) guarantees the closure property of the metric (2.11). If the Kähler coset space $G/H$ is reducible, the broken generators $X^i$ are decomposed into irreducible sets under the subgroup $H$, each of which may have a different $Y$-charge due to the Schur’s Lemma.

It can be also shown that one can write the metric (3.15) as (2.12) with the Kähler potential

$$
K(\phi, \bar{\phi}) = \sum_{\mu=1}^{k} v^\mu c^\mu(\phi, \bar{\phi}),
$$

(3.19)

where $c^\mu$ are the functions appearing in the parametrization (3.13) and $v^\mu$ are the coefficients defining the $Y$-charge (3.2).

4 The Riemann curvature

The Riemann curvature of the Kähler manifold is given by

$$
R_{\alpha\sigma,\beta\tau} = g_{\sigma\tau} \Gamma_{\alpha\beta,\beta}^{\eta} - g_{\alpha\tau} \Gamma_{\sigma\beta,\beta}^{\eta} - g_{\alpha\beta} \Gamma_{\sigma\tau,\beta}^{\eta} + g_{\alpha\beta} \Gamma_{\sigma\tau,\tau}^{\eta},
$$

(4.1)

To obtain it explicitly we have to compute the metric $g_{\alpha\beta}$ in the first place. It may be done with (3.15) by calculating the vielbeins $e^i_\alpha$ or with (2.12) by calculating the Kähler potential (3.19). Either calculation is already complicated. It is further complicated to take the derivative $g_{\alpha\sigma,\beta\tau}$ to obtain the Riemann curvature. Hence in this section we will study a method which enables us to calculate the Riemann curvature in a more direct way.
4.1 The irreducible case

When the Kähler manifold $G/H$ is irreducible, all the broken generators $X^i$ have the same $Y$-charge $y(v) (> 0)$. Then the metric (3.15) becomes simple:

$$g_{\alpha \overline{\beta}} = y(v) \sum_{i=1}^{N} e^i_{\alpha} e^i_{\overline{\beta}},$$

(4.2)

the value of which at the origin of the manifold is

$$g_{\alpha \overline{\beta}} \mid_{\phi = \overline{\phi} = 0} = y(v) \delta_{\alpha \overline{\beta}}.$$  

(4.3)

It was the Killing condition (2.14) that allows us to write the metric in the form of (4.2). The Killing condition can be satisfied also by giving the metric in terms of the Killing vectors (2.5): $g^{\alpha \overline{\beta}} \propto R^{\alpha A} R^{A \overline{\beta}}$. Fixing the free parameter by the initial condition (4.3) we then have

$$g^{\alpha \overline{\beta}} = \frac{1}{y(v)} R^{\alpha A} R^{A \overline{\beta}},$$

(4.4)

which should be equivalent to the metric given by (4.2). Here we have used

$$R^{\alpha A} \mid_{\phi = 0} = i \delta^{\alpha A}, \quad R^{A \overline{\sigma}} \mid_{\overline{\phi} = 0} = -i \delta^{A \overline{\sigma}},$$

(4.5)

which are obvious by the construction in Subsection 3.1. For other components of the metric we have

$$g^{\alpha \overline{\beta}} = R^{\alpha A} R^{A \overline{\beta}} = 0, \quad g^{\alpha \overline{\sigma}} = R^{A \overline{\sigma}} R^{A \alpha} = 0.$$  

(4.6)

With (4.4) the Affine connection becomes

$$\Gamma^{\eta}_{\alpha \overline{\beta}} = g^{\overline{\sigma} \eta} g^{\alpha \overline{\sigma} \beta} = -g^{\overline{\sigma} \eta \beta} g_{\alpha \overline{\sigma}} = -\frac{1}{y(v)} R^A_{\beta} R^{A \eta \alpha} = -\frac{1}{y(v)} R^A_{\alpha} R^{A \eta \beta}.$$  

(4.7)

by using the property (2.11). Putting this into (4.1) we have

$$R_{\alpha \overline{\sigma} \beta \overline{\eta}} = g_{\overline{\sigma} \eta} (-\frac{1}{y(v)} R^A_{\beta \overline{\sigma} \alpha}) = g_{\overline{\sigma} \eta} (-\frac{1}{y(v)} R^A_{\overline{\beta} \sigma \alpha}) = -\frac{1}{y(v)} R^A_{\beta \overline{\sigma} \alpha}.$$  

(4.8)

The second equality is due to

$$R^A_{\beta} R^{A \eta} = y(v) \delta^\eta_{\beta},$$

(4.9)
following from (4.4). Multiplying the Lie-algebra (2.14) by $R^A_{\gamma}$ or $R^{A\gamma}$ yields

$$R^{B\beta}_{\quad \gamma} = \frac{1}{y(v)} f^{ABC} R^{C\beta}_{\quad \gamma} R^A_{\gamma}$$  \hfill (4.10)

owing to (4.7), or

$$-R^{B\alpha}_{\quad A\gamma} R^{A\beta}_{\quad \alpha} = f^{ABC} R^{C\beta}_{\quad \gamma} R^A_{\gamma}.$$  \hfill (4.11)

The former is rewritten as

$$R^{B}_{\quad \beta,\gamma} = \frac{1}{y(v)} f^{ABC} R^{C\beta}_{\quad \gamma} R^A_{\gamma},$$  \hfill (4.12)

while the latter becomes

$$f^{ABC} R^{C\beta}_{\quad \gamma} R^A_{\gamma} = 0,$$  \hfill (4.13)

because we have

$$R^{A\gamma}_{\quad R^{A\beta}_{\quad \alpha}} = R^{A\gamma}_{\quad R^{A\beta}_{\quad \alpha}} = g^{\beta\alpha} R^{A\gamma}_{\quad R^A_{\gamma}} = -g^{\beta\alpha} R^{A\gamma}_{\quad R^A_{\gamma}} = 0,$$

due to (4.6), (2.15) and (4.9). With (4.12) the Riemann curvature (4.8) takes the form

$$R_{\alpha\beta\gamma\delta} = \frac{1}{y(v)^3} f^{ABE} R^A_{\alpha} R^B_{\beta} \cdot f^{CDE} R^C_{\beta} R^D_{\gamma}$$  \hfill (4.14)

$$= R^A_{\beta\alpha\gamma\delta}.$$

The last equality follows from the symmetry of the Affine connection (4.7), or directly from the Jacobi identity of the structure constants

$$-f^{ADC} f^{BCE} + f^{BDC} f^{ACE} = f^{ABC} f^{CDE},$$  \hfill (4.15)

and (4.13). Contrary to (4.1) this manifests the isometry $G$ and includes no derivative with respect to the coordinates. By using it the Riemann curvature can be calculated algebraically, once given a concrete form of the Killing vectors $R^{A\alpha}$ which are proper to the Kähler manifold $G/H$. Thus (4.14) gives a more practical formula than (4.1) for physical applications.

### 4.2 The reducible case

When the Kähler manifold $G/H$ is reducible, the broken generators $X^i$ are decomposed into irreducible sets under the subgroup $H$. Each irreducible set has a different $Y$-charge. The metric (3.15) satisfies the initial condition
\[ g_{\alpha\overline{\beta}}\Big|_{\phi=\overline{\phi}=0} = g_{\overline{\beta}\alpha}\Big|_{\phi=\overline{\phi}=0} = \begin{pmatrix} y_1(v) & 0 \\ y_2(v) & \ddots \\ 0 & \ddots & y_N(v) \end{pmatrix}. \tag{4.16} \]

Therefore the formula (4.4) is no longer correct in this case. We have to generalize the whole arguments in the previous subsection.

First of all, with \( U \) given by (3.12) and a real symmetric matrix \( P \) we define the quantity

\[ \Delta = UPU^{-1} \]

in the adjoint representation of the isometry group \( G \). By (3.17) it transforms as

\[ \Delta(\phi', \overline{\phi}') = g\Delta(\phi, \overline{\phi})g^{-1}, \]

or equivalently

\[ \mathcal{L}_{R^A}\Delta = i[T^A, \Delta], \tag{4.17} \]

if \( P \) satisfies

\[ hPh^{-1} = P. \tag{4.18} \]

With this \( \Delta \) the metric \( g^{ab} \) is found as a solution to the Killing condition (2.7)

\[ g^{ab} = g^{\alpha a} = R^A a \Delta^{AB} R^{Bb} \equiv R^a \Delta R^b. \tag{4.19} \]

In the complex basis it reads

\[ \begin{align*}
  g^{\alpha\overline{\beta}} &= g^{\overline{\beta}\alpha} = R^\alpha \Delta \overline{R^\beta}, \\
  g^{\alpha\beta} &= R^\alpha \Delta R^\beta, \\
  g^{\overline{\alpha}\overline{\beta}} &= \overline{R^\alpha \Delta R^\beta}. \tag{4.20} 
\end{align*} \]

We now assume the real symmetric matrix \( P \) to have non-vanishing elements only in the diagonal blocks corresponding to the broken generators \( X^a = (X^i, X^\overline{i}) \) such that

\[ P^\overline{\gamma} = P^{\overline{\beta}} = \begin{pmatrix} y_1(v)^{-1} & 0 \\ y_2(v)^{-1} & \ddots \\ 0 & \ddots & y_N(v)^{-1} \end{pmatrix}. \tag{4.21} \]
Then $P$ satisfies (4.18) because the diagonal elements are decomposed into irreducible sets under the subgroup $H$ by the $Y$-charge. Evaluate these metrics in (4.20) at the origin of the coset space by (4.5) and (4.21). We find that they all satisfy the same initial conditions as the metrics given in (3.15). Thus both metrics are equivalent\footnote{The equivalence is alternatively stated as $e^a = (R_a U P)^a d\phi^a + (R_a \bar{U} P)^a d\phi^\alpha$, with $e^a = (e^\alpha, e^\beta)$ defined by the Cartan-Maurer 1-form (3.14). Namely both sides have the same Lie-derivatives with respect to $R^A a$ and the same values at the origin of the coset space. The author is indebted to T. Kugo for the discussion on this comment.}, and we have

$$R^\alpha \Delta R^\beta = 0, \quad R^\alpha \Delta R^\beta = 0. \quad (4.22)$$

This generalization of the metric requires to modify the formula (4.7)\sim (4.14) in the previous subsection. Rewrite (4.20) as

$$R^\alpha \Delta R^\beta = \delta^\alpha_\beta, \quad R^\alpha \Delta R^\beta = \delta^\alpha_\beta, \quad R^\alpha \Delta R^\beta = 0, \quad R^\alpha \Delta R^\beta = 0, \quad (4.23)$$

using (4.22). Differentiate them by the coordinates to find

$$R^\alpha \Delta (R^\beta),_\gamma = 0, \quad R^\alpha \Delta (R^\beta),_\gamma = 0, \quad (4.24)$$

$$R^\alpha \Delta (R^\beta),_\gamma = 0, \quad R^\alpha \Delta (R^\beta),_\gamma = 0. \quad (4.25)$$

With the metric (4.20) the Affine connection (4.7) changes the form as

$$\Gamma_{\alpha\beta}^\gamma = R_{\alpha,\beta} \Delta R^\gamma = -R_{\alpha} (\Delta R^\gamma),_\beta,$$

owing to (4.23) and (4.24). Then the Riemann curvature becomes

$$R_{\alpha\beta}^\gamma = \Gamma_{\alpha\beta}^\gamma$$

$$= \frac{-[R_{\alpha,\bar{\beta}}(\Delta R^\gamma),_\beta + R_{\alpha}(\Delta R^\gamma),_\beta,\gamma]}{}$$

$$= \frac{-[(R_{\alpha} \Delta)_{\beta} R^\gamma,_{\bar{\beta}} + (R_{\alpha} \Delta)_{\bar{\beta}} R^\gamma,_{\beta}]}{}.$$ \hspace{1cm} (4.26)

By (4.24) and (2.15) the first piece changes the form as

$$(R_{\alpha} \Delta)_{\beta} R^\gamma,_{\bar{\beta}} = (R_{\alpha} \Delta)_{\beta} R^\gamma,_{\bar{\beta}}$$

$$= g_{\gamma,\bar{\beta}} (R_{\alpha} \Delta)_{\beta} R^\gamma,_{\bar{\beta}}$$

$$= \frac{-g_{\gamma,\bar{\beta}} (R_{\alpha} \Delta)_{\beta} R^\gamma,_{\bar{\beta}}}{\gamma,\bar{\beta}}.$$\hspace{1cm} (4.27)
On the other hand the second piece of (4.26) is calculated as
\[(R_\alpha \Delta, \beta) \pi \bar{\pi} R^\gamma = g^\pi \{ f^{ABC}(R_\alpha \Delta)^A \bar{\pi} R^B B(R_\sigma \Delta)^C \\
+ f^{ABC}(R_\alpha \Delta)^A R^B B(R_\sigma \Delta)^C \\
+ f^{ABC}(R_\alpha \Delta)^A R^B B(R_\sigma \Delta)^C \}, \tag{4.28} \]
by means of (A.10). Putting (4.27) and (4.28) together into (4.26) we have
\[R_{\alpha \pi \beta \bar{\pi}} = -(R_\alpha \Delta) \pi R^\pi \cdot R_\pi \Delta \pi R_\beta - f^{ABC}(R_\alpha \Delta)^A R^B B(R_\pi \Delta)^C - f^{ABC}(R_\alpha \Delta)^A R^B B(R_\pi \Delta)^C. \tag{4.29} \]
Calculate the first piece further as
\[(R_\alpha \Delta) \pi R^\pi \cdot R_\pi \Delta \pi R_\beta = (R_\alpha \Delta) \pi R^\pi \cdot g^{\pi \bar{\pi}} \cdot R_\pi \Delta \pi R_\beta \\
= R_\alpha \Delta R_\pi \rho \cdot g^{\pi \bar{\pi}} \cdot R_\pi \Delta \pi R_\beta \\
= -R_\alpha \Delta \pi R_\pi \cdot g^{\pi \bar{\pi}} \cdot R_\pi \Delta \pi R_\beta, \tag{4.30} \]
in which the second equality is obtained by (4.22) and (2.15), while the third
one by (4.24). Rewrite the last piece of (4.29) as
\[f^{ABC}(R_\alpha \Delta)^A R^B B(R_\pi \Delta)^C = R_\alpha \Delta R_\pi \cdot R_\pi \Delta \pi \bar{\pi} R^\pi, \tag{4.31} \]
by (A.8). By means of (A.7), (A.9) and (A.10) the Riemann curvature (4.29)
turns out to be
\[R_{\alpha \pi \beta \bar{\pi}} = f^{ABC}(R_\alpha \Delta)^A (R_\pi \Delta)^B R^{C \pi} \cdot f^{DEF}(R_\pi \Delta)^D (R_\beta \Delta)^E R^{F \pi} \]
\[+ f^{ABC}(R_\alpha \Delta)^A (R_\pi \Delta)^B R^{C \pi} \cdot f^{DEF}(R_\pi \Delta)^D (R_\beta \Delta)^E R^{F \pi} \]
\[+ f^{ABC}(R_\alpha \Delta)^A (R_\pi \Delta)^B \cdot f^{CDE}(R_\beta \Delta)^D \cdot f^{DEF}(R_\pi \Delta)^D R^{E \pi} \]
\[- f^{ABC}(R_\alpha \Delta)^A (R_\pi \Delta)^B \cdot f^{DEF}(R_\pi \Delta)^D R^{E \pi}(R_\pi \Delta)^F \]. \tag{4.32} \]
This is the generalization of (4.14). If one replaces \( \Delta^{AB} \) by \( \frac{1}{g^{(\alpha \nu)}} \delta^{AB} \), then
(4.32) reduces to (4.14) due to the formula (4.13) which is only valid for the
irreducible case. At this final stage it is worth showing the symmetry property
\[R_{\alpha \pi \beta \bar{\pi}} = R_{\beta \pi \alpha \bar{\pi}} = R_{\alpha \beta \pi \bar{\pi}} \tag{4.33} \]
as a consistency check of our calculations. The demonstration will be given in
Appendix B. There we also show that the formula (4.32) takes an alternative
form such that
\[R_{\alpha \pi \beta \bar{\pi}} = f^{ABC}(R_\alpha \Delta)^A (R_\pi \Delta)^B R^{C \pi} \cdot f^{DEF}(R_\pi \Delta)^D (R_\beta \Delta)^E R^{F \pi} \]
\[+ f^{ABC}(R_\alpha \Delta)^A (R_\pi \Delta)^B R^{C \pi} \cdot f^{DEF}(R_\pi \Delta)^D (R_\beta \Delta)^E R^{F \pi} \]
\[+ f^{ABC}(R_\alpha \Delta)^A (R_\pi \Delta)^B \cdot f^{CDE}(R_\beta \Delta)^D \cdot f^{DEF}(R_\pi \Delta)^D R^{E \pi} \]
\[- f^{ABC}(R_\alpha \Delta)^A (R_\pi \Delta)^B \cdot f^{DEF}(R_\pi \Delta)^D R^{E \pi}(R_\pi \Delta)^F \].
5 Applications

In the $N = 1$ supersymmetric non-linear $\sigma$-model on the Kähler coset space $G/H$, the four-fermi coupling is the most interesting part. When the coset space is reducible, the Riemann curvature depends on the $Y$-charge of the broken generators through the metric (3.15). It takes the form (4.32) which is rather complicated than that of the irreducible coset space. On top of this complication we have another one, if the homogeneous subgroup $H$ contains plural $U(1)$s as $H = H' \otimes \{U(1)\}^k$. Namely, the splitting of the broken generators $X^i$ and $\chi^i$ depends on the constants $\psi^\mu$ of the $Y$-charge (3.2), so that we may have different sets of the NG bosons. Of course a phenomenologically interesting set should be chosen. Then the four-fermi coupling depends on the $Y$-charges of the broken generators $X^i$ in a peculiar way to the choice. It is very interesting from the phenomenological point of view.

As has been explained in the introduction, the most important part of the four-fermi coupling is

$$R_{\alpha\sigma\beta\delta} \big|_{\phi=\bar{\phi}=0} = (\bar{\psi}^{\sigma} \psi^\alpha)(\bar{\psi}^{\delta} \psi^\beta)$$

in the non-linear $\sigma$-model as a low-energy effective theory. We shall present a systematic method to evaluate the four-fermi coupling constants $R_{\alpha\sigma\beta\delta} \big|_{\phi=\bar{\phi}=0}$ by means of (4.32). The method will enable us to fully control the $Y$-charge dependence of $R_{\alpha\sigma\beta\delta} \big|_{\phi=\bar{\phi}=0}$.

By (4.5) and (4.21) we note at first that

$$R^A_{\alpha} \big|_{\phi=\bar{\phi}=0} = -iy_{\alpha}(v)\delta^A_{\alpha}, \quad R^{\alpha}_{\bar{\phi}} \big|_{\phi=\bar{\phi}=0} = iy_{\alpha}(v)\delta^A_{\alpha},$$

and

$$\left(R_{\alpha} \Delta\right)^A \big|_{\phi=\bar{\phi}=0} = -i\delta^A_{\alpha}, \quad \left(R^{\alpha} \Delta\right)^A = -\frac{i}{y_{\alpha}(v)}\delta^A_{\alpha},$$

$$\left(R_{\alpha} \Delta\right)^A \big|_{\phi=\bar{\phi}=0} = i\delta^A_{\alpha}, \quad \left(R^{\alpha} \Delta\right)^A = \frac{i}{y_{\alpha}(v)}\delta^A_{\alpha}.$$
with the normalization $\text{tr}(T^A T^B) = 2\mathcal{N}\delta^{AB}$. The Riemann curvature appears with the indices $\alpha, \sigma, \beta, \delta$ of the three types:

$$
\begin{align*}
\eta([T^\alpha, T^\sigma]) & > 0, & \eta([T^\beta, T^\delta]) & < 0, \\
\eta([T^\alpha, T^\sigma]) & < 0, & \eta([T^\beta, T^\delta]) & > 0, \\
\eta([T^\alpha, T^\sigma]) & = 0, & \eta([T^\beta, T^\delta]) & = 0,
\end{align*}
$$

(5.4)

(5.5)

(5.6)

in which $\eta([ , ])$ is the $Y$-charge of the commutator. (4.32) reads

$$
R_{\alpha\sigma\beta\delta} \mid_{\phi = \bar{\phi} = 0} = \sum_{\eta=1}^{N} f^{\bar{\alpha}\eta} \cdot f^{\bar{\beta}\delta\eta} y_\beta(v) \\
- \sum_{\eta=1}^{N} [f^{\bar{\alpha}\eta} \cdot f^{\bar{\beta}\sigma\eta} y_\eta(v) + f^{\bar{\alpha}\sigma\eta} \cdot f^{\bar{\beta}\delta\eta} \frac{y_\beta(v)y_\delta(v)}{y_\eta(v)}] 
$$

(5.7)

for the case (5.4),

$$
R_{\alpha\sigma\beta\delta} \mid_{\phi = \bar{\phi} = 0} = \sum_{\eta=1}^{N} f^{\bar{\alpha}\sigma\eta} \cdot f^{\bar{\beta}\delta\eta} (y_\beta(v) - y_\eta(v)) \\
- \sum_{\eta=1}^{N} [f^{\bar{\alpha}\eta} \cdot f^{\bar{\beta}\sigma\eta} y_\eta(v) + f^{\bar{\alpha}\sigma\eta} \cdot f^{\bar{\beta}\delta\eta} \frac{y_\beta(v)y_\delta(v)}{y_\eta(v)}] 
$$

(5.8)

for the case (5.5), and

$$
R_{\alpha\sigma\beta\delta} \mid_{\phi = \bar{\phi} = 0} = \sum_{C=1,\mu}^{\text{dim}H} f^{\bar{\alpha}\sigma C} \cdot f^{\bar{\beta}\delta C} y_\beta(v) \\
- \sum_{\eta=1}^{N} [f^{\bar{\alpha}\delta\eta} \cdot f^{\bar{\beta}\sigma\eta} y_\eta(v) + f^{\bar{\alpha}\sigma\eta} \cdot f^{\bar{\beta}\delta\eta} \frac{y_\beta(v)y_\delta(v)}{y_\eta(v)}] 
$$

(5.9)

for the case (5.6). (5.7) can be alternatively obtained by applying the symmetry property (4.33) to (5.8).

### 5.1 $SU(3)/\{U(1)\}^2$

We start with the most simplest reducible coset space $SU(3)/\{U(1)\}^2$ to illustrate our basic strategy. The generators of $SU(3)$ are

$$
\{T^A\} = \{T^1_2, T^1_3, T^2_3, T^1_1, T^1_2, T^3_3, Q, Q'\},
$$

with

$$
Q = \frac{1}{\sqrt{2}}(T^1_1 - T^2_2), \quad Q' = -\sqrt{2}T^3_3,
$$

(5.10)

and the hermitian condition $(T^i_j)^\dagger = T^i_j$. They satisfy the Lie-algebra

$$
[T^i_j, T^l_k] = \delta^i_k T^l_j - \delta^i_j T^l_k.
$$
The quadratic Casimir takes the form
\[
\{T^1_2, T^2_1\} + \{T^3_3, T^3_1\} + \{T^3_2, T^3_2\} + Q^2 + Q'^2,
\]
from which we read the Killing metric \(\delta^{AB}\) in (2.18). The \(U(1)\)-charges \(Q\) and \(Q'\) of the broken generators \(T^j_i\) \((i \neq j)\) as well as their \(Y\)-charges
\[
Y = vQ + v'Q'
\]
are given in Table 1. By means of them the broken generators are splitted in two parts: the generators \(X^i\) with positive \(Y\)-charge and their hermitian conjugates \(X^i\) with negative charge. For illustration we plot the broken generators in the \((Q, Q')\)-charge plane in Figure 1. There are three possibilities to draw the line: \(Y = 0\),

| \(X^i\) | \(Q\)   | \(Q'\) | \(y(X^i)\)          |
|--------|--------|--------|---------------------|
| \(T^1_2\) | \(-\sqrt{2}\) | 0      | \(-\sqrt{2}v\)      |
| \(T^3_1\) | \(-\frac{1}{\sqrt{2}}\) | \(-\frac{3}{2}\) | \(-\frac{1}{\sqrt{2}}v - \frac{\sqrt{3}}{2}v'\) |
| \(T^2_3\) | \(\frac{1}{\sqrt{2}}\)  | \(-\frac{3}{2}\) | \(\frac{1}{\sqrt{2}}v - \frac{\sqrt{3}}{2}v'\) |

Table 1: \(U(1)\)-charges of \(X^7\) in \(SU(3)\).

Figure 1: The splitting of the broken generators

each of which gives a different splitting:

I \quad \{X^i\} = \{T^2_1, T^3_1, T^3_2\};

II \quad \{X^i\} = \{T^3_3, T^3_1, T^3_2\};

III \quad \{X^i\} = \{T^3_2, T^3_1, T^3_1\};
Taking the case I we proceed with the argument. With the identification $X^1 = T_1^2$, $X^2 = T_1^3$, $X^3 = T_2^3$, $Q = X^q$, $Q' = X^{q'}$, the non-trivial part of the Lie-algebra reads

\[
\begin{align*}
[X^2, X^3] &= X^1, & [X^3, X^2] &= X^1, \\
[X^1, X^3] &= X^2, & [X^7, X^1] &= X^7, \\
[X^1, X^2] &= X^3, & [X^7, X^1] &= X^7, \\
[X^1, X^7] &= \sqrt{2}Q, & [X^7, X^1] &= \sqrt{2}Q, \\
\end{align*}
\]

and their hermitian conjugates. The holomorphic Killing vectors $R^{A\alpha}$ are easily obtained by studying (3.6) in the fundamental representation:

\[
\begin{align*}
R^{T_1} &= i, & R^{11} &= -i(\phi^1)^2, & R^{q_1} &= -\sqrt{2}i\phi^1, \\
R^{T_2} &= 0, & R^{21} &= -i\phi^1(\phi^2 + \frac{1}{3}\phi^1\phi^3), & R^{q_2} &= 0, \\
R^{T_3} &= 0, & R^{31} &= i(\phi^2 + \frac{1}{3}\phi^1\phi^3), & R^{q_3} &= -\frac{2}{3}i\phi^2, \\
R^{T_4} &= -i\phi^3, & R^{12} &= -i\phi^1(\phi^2 + \frac{1}{3}\phi^1\phi^3), & R^{q_4} &= -\frac{2}{3}i\phi^3, \\
R^{V_2} &= i, & R^{22} &= -i[(\phi^2)^2 + \frac{1}{3}(\phi^1\phi^3)^2], & R^{q_5} &= -\frac{2}{3}i\phi^2, \\
R^{V_3} &= \frac{i}{\sqrt{2}}\phi^1, & R^{32} &= -\frac{i}{\sqrt{2}}\phi^1(\phi^2 - \frac{1}{3}\phi^1\phi^3), & R^{q_6} &= -\frac{2}{3}i\phi^3, \\
R^{T_0} &= 0, & R^{13} &= -i(\phi^2 - \frac{1}{3}\phi^1\phi^3), & R^{q_7} &= -\frac{2}{3}i\phi^2, \\
R^{V_0} &= 0, & R^{23} &= -i\phi^1(\phi^2 - \frac{1}{3}\phi^1\phi^3), & R^{q_8} &= -\frac{2}{3}i\phi^3, \\
R^{V_3} &= i, & R^{33} &= -i(\phi^3)^2, & R^{q_9} &= -\frac{2}{3}i\phi^3.
\end{align*}
\]

The Riemann curvature $R_{\alpha\beta\gamma\delta}\Big|_{\phi=\bar{\phi}=0} (\equiv G_{\alpha\beta\gamma\delta})$, given by (5.2), is calculated by using (5.3) with the commutators (5.11). The Riemann curvature of this type appears in the coset space $E_7/SU(5) \otimes \{U(1)^3\}$ which we will study in the next subsection. The result is given in Tables 8 and 9 there.

### 5.2 $E_7/SU(5) \otimes \{U(1)^3\}$

The generators of $E_7$ are decomposed as

\[
\{T^A\} = \{E_i^{ab}, T^i_a, E^a, E_{ab}, T^a_i, E_a, T^i_a, T^j_i, T\}
\]

in the basis of the subgroup $SU(5) \otimes SU(3) \otimes U(1)$. Here $a, b, \cdots$ and $i, j, \cdots$ are indices of $SU(5)$ and $SU(3)$ running over $1 \sim 5$ and $1 \sim 3$ respectively. They have $SU(5) \otimes SU(3)$ quantum numbers

\[
(10, 3^*), (5^*, 3), (5, 1), (10^*, 3), (5, 3^*), (5^*, 1), (24, 1), (1, 8), (1, 1),
\]

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in the order of (5.12). The non-trivial part of the $E_7$ algebra takes the form \[ [E_{i}^{ab}, E_{j}^{cd}] = \varepsilon_{ijk}\varepsilon^{abde}T_{e}^{k}, \quad [E_{a}, E_{j}^{bc}] = 0, \]

\[ [T_{a}^{i}, E_{j}^{bc}] = \delta_{j}^{i}(\delta_{a}^{c}E_{b} - \delta_{a}^{b}E_{c}), \quad [T_{a}^{i}, E_{b}^{c}] = 0, \]

\[ [E_{a}, E_{b}] = 0, \]

\[ [E_{ab}^{i}, E_{j}^{cd}] = \delta_{j}^{i}(\delta_{a}^{c}T_{b}^{d} + \delta_{b}^{d}T_{a}^{c} - \delta_{a}^{d}T_{b}^{c} - \delta_{a}^{b}T_{d}^{c}) - \delta_{ab}(T_{j}^{i} + \sqrt{\frac{2}{15}}\delta_{j}^{i}T), \]

\[ [E_{a}, E_{b}] = \sqrt{\frac{6}{5}}\delta_{a}^{b}T - T_{a}^{a}, \]

\[ [E_{a}, E_{bc}] = \delta_{c}^{a}T_{b}^{i} - \delta_{b}^{a}T_{c}^{i}, \]

\[ [T_{a}^{i}, T_{b}^{j}] = -\delta_{a}^{b}T_{j}^{i} + \delta_{b}^{i}T_{a}^{j} + 2\sqrt{\frac{2}{15}}\delta_{j}^{i}\delta_{a}^{b}T, \]

\[ [T_{a}^{i}, E_{bc}^{j}] = -\frac{1}{2}\varepsilon_{ijk}\varepsilon_{abcd}E_{k}^{d}, \]

\[ [T_{a}^{i}, E_{b}] = -E_{ab}^{i}, \quad \text{h.c.}, \]

with \[ \varepsilon_{ijk}^{*} = \varepsilon_{ijk}, \quad \varepsilon_{abde}^{*} = \varepsilon^{abde}, \quad \delta_{ab}^{cd} = \delta_{a}^{c}\delta_{b}^{d} - \delta_{a}^{d}\delta_{b}^{c}. \]

In the basis of the smaller subgroup $SU(5) \otimes \{U(1)\}^{3}$ the generators are further decomposed as

\[ \{T^{A}\} = \{X^{7}, X^{i}, S^{\ell}, Q^{\mu}\}, \]

with

\[ \{X^{7}\} = \{E_{i}^{ab}, E_{a}, T_{a}^{i}, T_{a}^{j} (i > j)\}, \]

\[ \{X^{i}\} = \{E_{ab}^{i}, E_{a}, T_{a}^{i}, T_{a}^{j} (i < j)\}, \]

\[ \{S^{\ell}\} = \{T_{a}^{b} (\sum_{a}T_{a}^{a} = 0)\}, \]

\[ \{Q^{\mu}\} = \{T, Q, Q'\}. \]

Here $Q$ and $Q'$ are the $U(1)$s contained in $SU(3)$ which was given by (5.10).

The quadratic Casimir takes the form

\[ C = \frac{1}{2}\{E_{i}^{ab}, E_{a}^{i}\} + \{T_{a}^{i}, T_{a}^{i}\} + \{E_{a}^{i}, E_{a}\} \]

\[ + \{T_{1}^{2}, T_{1}^{2}\} + \{T_{3}^{1}, T_{1}^{3}\} + \{T_{3}^{2}, T_{2}^{3}\} \]

\[ + \{T_{a}^{b}, T_{a}^{a}\} + T^{2} + Q^{2} + Q'^{2}. \]

\{X^{7}\} and \{X^{i}\} are broken generators of $E_{7}/SU(5) \otimes \{U(1)\}^{3}$. In the supersymmetric model on $E_{7}/SU(5) \otimes \{U(1)\}^{3}$ there are pseud NG fermions corresponding to them. Among of them the pseud NG fermions having the same $SU(5) \otimes SU(3)$ quantum numbers as $E_{i}^{ab}, T_{a}^{i}$ and $T_{i}^{j}(i > j)$ are identified with the three families of quarks and leptons $\psi^{ab}_{i}, \psi^{i}_{a}$ and $\psi^{i}_{b}$. The $Y$-charge is made of the $U(1)$ charges in \{Q^{\mu}\}:

\[ Y = \alpha T + \beta Q + \gamma Q'. \]
$$X^i$$ | $T$ | $Q$ | $Q'$ | $y(X^i)$
---|---|---|---|---
$E_{ab}^1$ | $1$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{\sqrt{6}}$ | $\alpha + \frac{1}{\sqrt{2}} \beta + \frac{1}{\sqrt{6}} \gamma$
$E_{ab}^2$ | $1$ | $-\frac{1}{\sqrt{2}}$ | $\frac{1}{\sqrt{6}}$ | $\alpha - \frac{1}{\sqrt{2}} \beta + \frac{1}{\sqrt{6}} \gamma$
$E_{ab}^3$ | $1$ | $0$ | $\frac{\sqrt{2}}{3}$ | $\alpha - \frac{\sqrt{2}}{3} \gamma$
$T_{a}^1$ | $2$ | $-\frac{1}{\sqrt{2}}$ | $\frac{1}{\sqrt{6}}$ | $2\alpha - \frac{1}{\sqrt{2}} \beta - \frac{1}{\sqrt{6}} \gamma$
$T_{a}^2$ | $2$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{\sqrt{6}}$ | $2\alpha + \frac{1}{\sqrt{2}} \beta - \frac{1}{\sqrt{6}} \gamma$
$T_{a}^3$ | $2$ | $0$ | $\frac{\sqrt{3}}{3}$ | $2\alpha + \frac{\sqrt{3}}{3} \gamma$
$E_{a}^1$ | $3$ | $0$ | $0$ | $3\alpha$
$T_{a}^1$ | $0$ | $-\sqrt{2}$ | $0$ | $-\sqrt{2} \beta$
$T_{a}^2$ | $0$ | $-\sqrt{3}$ | $\frac{3}{2}$ | $-\frac{\sqrt{3}}{2} \beta - \frac{\sqrt{3}}{2} \gamma$
$T_{a}^2$ | $0$ | $\frac{1}{\sqrt{2}}$ | $\frac{3}{2}$ | $\frac{1}{\sqrt{2}} \beta - \frac{3}{2} \gamma$

Table 2: $U(1)$-charges of $X^7$ in $E^7$.

These $U(1)$-charges are given in Table 3.

The splitting of the broken generators $\{X^7\}$ and $\{X^i\}$ changes according to the orientation of the plane $Y = 0$ in the $(T, Q, Q')$-charge space. The splitting (5.15) is valid only when the vector coefficients $(\alpha, \beta, \gamma)$ are chosen such that

$$y_i(v) \equiv Y(X^7) > 0, \quad \text{for all } i,$$

for instance,

$$\alpha = 1, \quad \beta < 0, \quad \gamma < 0, \quad |\beta| = |\gamma| << 1.$$

We proceed with the argument in this special splitting, since the pseudo NG fermions are then neatly identified with the three families of quarks and leptons.

In the supersymmetric $\sigma$-model on $E_7/SU(5) \otimes \{U(1)^3$ the Riemann curvature $R_{\alpha\sigma\beta\delta}$ is a $SU(5)$-covariant tensor. We shall be interested in the four-fermi coupling of the three families of quarks and leptons alone. Then the relevant part of the Riemann curvature appears with the $SU(5)$-content of the following types:

$$R_{\alpha\sigma\beta\delta} \bigg|_{\phi = \phi^0 = 0} \equiv G^{\alpha\sigma\beta\delta} \sim \left\{ \begin{array}{c}
(5^*, 5, 5^*, 5), \\
(10, 10^*, 10, 10^*), \\
(1, 1, 1, 1), \\
(5^*, 5, 10, 10^*), \\
(10, 10^*, 5^*, 5), \\
(5, 10, 5^*, 10^*), \\
(5^*, 10^*, 5, 10), \\
(1, 1, 5^*, 5), \\
(5^*, 5, 1, 1), \\
(1, 5, 5^*, 1), \\
(5^*, 1, 5, 1), \\
(1, 1, 10, 10^*), \\
(10, 10^*, 1, 1), \\
(1, 10^*, 10), \\
(10, 1, 1, 10^*).
\end{array} \right.$$
We evaluate $G^{\alpha \beta \delta}$ in components by means of (5.2) with (5.3) and (5.13). 
(See Appendix C.) The results are summarized in Tables 3~16 for six types of the Riemann curvature

$(5^*, \ 5, \ 5^*, \ 5), \ (10,10^*, 10,10^*), \ (1, \ 1, \ 1, \ 1),
(5^*, \ 5,10,10^*), \ (1, \ 1, \ 5^*, \ 5), \ (1, \ 1, 10,10^*)$.

Other types of the Riemann curvature are obtained by applying the symmetry property (4.33) to these results.

|        | $(1)_{(2)}$ | $(1)_{(3)}$ | $(2)_{(3)}$ | $(2)_{(1)}$ | $(3)_{(1)}$ | $(3)_{(2)}$ |
|--------|-------------|-------------|-------------|-------------|-------------|-------------|
| $(1)_{(2)}$ | 0          | 0           | 0           | $\delta_a \delta_y(T^2_c)$ | 0           | 0           |
| $(1)_{(3)}$ | 0          | 0           | 0           | 0           | $\delta_a \delta_y(T^2_c)$ | 0           |
| $(2)_{(3)}$ | 0          | 0           | 0           | 0           | 0           | $\delta_a \delta_y(T^2_c)$ |
| $(2)_{(1)}$ | $\delta_a \delta_y(T^2_c)$ | 0           | 0           | 0           | 0           | 0           |
| $(3)_{(1)}$ | 0          | $\delta_a \delta_y(T^2_c)$ | 0           | 0           | 0           | 0           |
| $(3)_{(2)}$ | 0          | 0           | $\delta_a \delta_y(T^2_c)$ | 0           | 0           | 0           |

Table 3: The Riemann curvature $G^{(i)}(j)^{(b)}(c)^{(d)}$ of the type $(5^*, 5, 5^*, 5)$ with $i \neq j$. 

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$$\left( \begin{array}{c} \text{(1)} \left( \begin{array}{c} c \\ 1 \end{array} \right) \left( \begin{array}{c} d \\ 1 \end{array} \right) \\ \text{(2)} \left( \begin{array}{c} c \\ 2 \end{array} \right) \left( \begin{array}{c} d \\ 2 \end{array} \right) \\ \text{(3)} \left( \begin{array}{c} c \\ 3 \end{array} \right) \left( \begin{array}{c} d \\ 3 \end{array} \right) \end{array} \right) \right)$$

$$\left( \begin{array}{c} \text{(1)} \left( \begin{array}{c} a \\ 1 \end{array} \right) \left( \begin{array}{c} b \\ 1 \end{array} \right) \\ \text{(2)} \left( \begin{array}{c} a \\ 2 \end{array} \right) \left( \begin{array}{c} b \\ 2 \end{array} \right) \\ \text{(3)} \left( \begin{array}{c} a \\ 3 \end{array} \right) \left( \begin{array}{c} b \\ 3 \end{array} \right) \end{array} \right) \right)$$

$$\frac{\delta_{ab} \delta_{ef} y(E_{1}^{ef})}{y(T_{1}^{2})} - \frac{\delta_{abf} y(E_{1}^{ef}) y(E_{2}^{gh})}{y(T_{2}^{2})}$$

$$\frac{\delta_{ab} \delta_{gh} y(E_{1}^{ef})}{y(T_{1}^{2})} - \frac{\delta_{abf} y(E_{1}^{ef}) y(E_{3}^{gh})}{y(T_{3}^{2})}$$

$$\frac{\delta_{ab} \delta_{gh} y(E_{2}^{ef})}{y(T_{2}^{2})} - \frac{\delta_{abf} y(E_{2}^{ef}) y(E_{3}^{gh})}{y(T_{3}^{2})}$$

Table 4: The Riemann curvature $G^{(i)}(a)(b)(j)$ of the type $(5^*, 5, 5^*, 5)$

Table 5: The Riemann curvature $G^{(i)}(a)(b)(j)(ef)$ of the type $(10, 10^*, 10, 10^*)$ with $i < j$. 

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Table 6: The Riemann curvature $G^{(ab)}_{ij}(c)_{gh}^{(ef)}$ of the type $(10, 10^*, 10, 10^*)$ with $i > j$. 

|   | $(e_f)_{1}^{(1)gh}$ | $(e_f)_{2}^{(1)gh}$ | $(e_f)_{3}^{(1)gh}$ |
|---|---------------------|---------------------|---------------------|
| $(a_b)_{(2)}^{(1)cd}$ | $d_{a_b}^{(e_f)_{1}^{(1)gh} y(E_{1ef})} - d_{a_b}^{(e_f)_{1}^{(1)gh} y(E_{1eh})}$ | 0                   | 0                   |
| $(a_b)_{(3)}^{(1)cd}$ | $d_{a_b}^{(e_f)_{1}^{(1)gh} y(E_{1ef})} - d_{a_b}^{(e_f)_{1}^{(1)gh} y(E_{1eh})}$ | 0                   | 0                   |

Table 7: The Riemann curvature $G^{(ab)}_{ij}(c)_{gh}^{(ef)}\ y^{(k)}$ of the type $(10, 10^*, 10, 10^*)$. 

|   | $(e_f)_{1}^{(1)gh}$ | $(e_f)_{2}^{(1)gh}$ | $(e_f)_{3}^{(1)gh}$ |
|---|---------------------|---------------------|---------------------|
| $(a_b)_{(1)}^{(1)cd}$ | $d_{a_b}^{(e_f)_{1}^{(1)gh} y(E_{1ef})} - d_{a_b}^{(e_f)_{1}^{(1)gh} y(E_{1eh})}$ | $d_{a_b}^{(e_f)_{1}^{(1)gh} y(E_{1ef})} - d_{a_b}^{(e_f)_{1}^{(1)gh} y(E_{1eh})}$ | $d_{a_b}^{(e_f)_{1}^{(1)gh} y(E_{1ef})} - d_{a_b}^{(e_f)_{1}^{(1)gh} y(E_{1eh})}$ |
| $(a_b)_{(2)}^{(2)cd}$ | $d_{a_b}^{(e_f)_{1}^{(1)gh} y(E_{1ef})} - d_{a_b}^{(e_f)_{1}^{(1)gh} y(E_{1eh})}$ | $d_{a_b}^{(e_f)_{1}^{(1)gh} y(E_{1ef})} - d_{a_b}^{(e_f)_{1}^{(1)gh} y(E_{1eh})}$ | $d_{a_b}^{(e_f)_{1}^{(1)gh} y(E_{1ef})} - d_{a_b}^{(e_f)_{1}^{(1)gh} y(E_{1eh})}$ |
| $(a_b)_{(3)}^{(3)cd}$ | $d_{a_b}^{(e_f)_{1}^{(1)gh} y(E_{1ef})} - d_{a_b}^{(e_f)_{1}^{(1)gh} y(E_{1eh})}$ | $d_{a_b}^{(e_f)_{1}^{(1)gh} y(E_{1ef})} - d_{a_b}^{(e_f)_{1}^{(1)gh} y(E_{1eh})}$ | $d_{a_b}^{(e_f)_{1}^{(1)gh} y(E_{1ef})} - d_{a_b}^{(e_f)_{1}^{(1)gh} y(E_{1eh})}$ |
\[ G_{ij}^{(1)}(1)_{il}^{(1)}(m)_{jl}^{(2)} = \begin{array}{cccc}
(\frac{1}{2})(\frac{3}{1}) & (\frac{1}{3})(\frac{3}{1}) & (\frac{1}{3})(\frac{2}{1}) & (\frac{2}{3})(\frac{2}{1}) \\
(\frac{1}{2})(\frac{3}{1}) & 0 & y(T^2_1) & 0 & 0 & 0 & 0 \\
(\frac{1}{3})(\frac{3}{2}) & y(T^2_1) & 0 & 0 & 0 & 0 & 0 \\
(\frac{1}{3})(\frac{2}{1}) & 0 & 0 & 0 & y(T^2_1) & 0 & 0 \\
(\frac{2}{3})(\frac{3}{2}) & 0 & 0 & y(T^2_1) & 0 & 0 & 0 \\
(\frac{2}{3})(\frac{2}{1}) & 0 & 0 & 0 & 0 & 0 & -\frac{y(T^2_1)y(T^1_1)}{y(T^2_1)} \\
(\frac{2}{3})(\frac{3}{2}) & 0 & 0 & 0 & -\frac{y(T^2_1)y(T^1_1)}{y(T^2_1)} & y(T^2_1) & 0 \\
\end{array} \]

Table 8: The Riemann curvature \( G^{(1)}_{ij}^{(1)}(1)_{il}^{(1)}(m)_{jl}^{(2)} \) of the type \((1, 1, 1, 1)\) with \( y([T^i_1, T^k_1]) \neq 0 \).

\[ G_{ij}^{(1)}(1)_{il}^{(1)}(m)_{jl}^{(2)} = \begin{array}{ccc}
(\frac{1}{2})(\frac{2}{1}) & (\frac{1}{3})(\frac{3}{2}) & (\frac{2}{3})(\frac{3}{2}) \\
(\frac{1}{2})(\frac{2}{1}) & 2y(T^2_1) & y(T^2_1) & -\frac{y(T^2_1)y(T^1_1)}{y(T^2_1)} \\
(\frac{1}{3})(\frac{3}{2}) & y(T^2_1) & 2y(T^2_1) & y(T^2_1) \\
(\frac{2}{3})(\frac{3}{2}) & -\frac{y(T^2_1)y(T^1_1)}{y(T^2_1)} & y(T^2_1) & 2y(T^2_1) \\
\end{array} \]

Table 9: The Riemann curvature \( G^{(1)}_{ij}^{(1)}(1)_{il}^{(1)}(m)_{jl}^{(2)} \) of the type \((1, 1, 1, 1)\).
|        | \( (cd)^1_1(2)_{ef} \) | \( (cd)^1_1(3)_{ef} \) | \( (cd)^2_2(3)_{ef} \) |
|--------|------------------------|------------------------|------------------------|
| \( (1)_a(2)^b \) | \(-\delta^b_a\delta_{ef}y(E_{1}^{cd})\) | 0                      | 0                      |
|         | \(-\delta^b_a\delta_{ef}y(E_{1}^{cd})\) | 0                      | 0                      |
| \( (1)_a(3)^b \) | 0                      | \(-\delta^b_a\delta_{ef}y(E_{2}^{cd})\) | 0                      |
|         | \(-\delta^b_a\delta_{ef}y(E_{2}^{cd})\) | \(-\delta^b_a\delta_{ef}y(E_{2}^{cd})\) | 0                      |
| \( (2)_a(3)^b \) | 0                      | 0                      | \(-\delta^b_a\delta_{ef}y(E_{2}^{cd})\) |
|         | \(-\delta^b_a\delta_{ef}y(E_{2}^{cd})\) | \(-\delta^b_a\delta_{ef}y(E_{2}^{cd})\) | \(-\delta^b_a\delta_{ef}y(E_{2}^{cd})\) |

Table 10: The Riemann curvature \( G^{(i)_a}(b)_j(\gamma)^{cd}(\zeta)_e(f) \) of the type \( (5^*, 5, 10, 10^*) \) with \( i < j \).

|        | \( (cd)^1_1(1)_{ef} \) | \( (cd)^1_1(3)_{ef} \) | \( (cd)^2_2(2)_{ef} \) |
|--------|------------------------|------------------------|------------------------|
| \( (1)_a(1)^b \) | \(-\delta^b_a\delta_{ef}y(E_{1}^{cd})\) | 0                      | 0                      |
|         | \(-\delta^b_a\delta_{ef}y(E_{1}^{cd})\) | 0                      | 0                      |
| \( (1)_a(2)^b \) | 0                      | \(-\delta^b_a\delta_{ef}y(E_{1}^{cd})\) | 0                      |
|         | \(-\delta^b_a\delta_{ef}y(E_{1}^{cd})\) | \(-\delta^b_a\delta_{ef}y(E_{1}^{cd})\) | 0                      |
| \( (2)_a(2)^b \) | 0                      | 0                      | \(-\delta^b_a\delta_{ef}y(E_{1}^{cd})\) |
|         | \(-\delta^b_a\delta_{ef}y(E_{1}^{cd})\) | \(-\delta^b_a\delta_{ef}y(E_{1}^{cd})\) | \(-\delta^b_a\delta_{ef}y(E_{1}^{cd})\) |

Table 11: The Riemann curvature \( G^{(i)_a}(b)_j(\gamma)^{cd}(\zeta)_e(f) \) of the type \( (5^*, 5, 10, 10^*) \) with \( i > j \).
Table 12: The Riemann curvature $G^{(i)}_{a(j)}^{(k)}_{(l)}$ of the type $(5^*, 5, 10, 10^*)$.

|        | $(1)_{a(1)}^{(b)}$ | $(1)_{a(2)}^{(b)}$ | $(1)_{a(3)}^{(b)}$ | $(2)_{a(1)}^{(b)}$ | $(2)_{a(2)}^{(b)}$ | $(2)_{a(3)}^{(b)}$ | $(3)_{a(1)}^{(b)}$ | $(3)_{a(2)}^{(b)}$ | $(3)_{a(3)}^{(b)}$ |
|--------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| $(1)_{a(1)}^{(b)}$ | 0                 | 0                 | 0                 | $-\delta_a f(T_3^2)y(T_3^2)$ | 0                 | 0                 | 0                 | 0                 | 0                 |
| $(1)_{a(2)}^{(b)}$ | 0                 | 0                 | 0                 | 0                 | 0                 | 0                 | $-\delta_a f(T_3^2)y(T_3^2)$ | 0                 | 0                 |
| $(1)_{a(3)}^{(b)}$ | 0                 | 0                 | 0                 | 0                 | 0                 | 0                 | 0                 | 0                 | 0                 |
| $(2)_{a(1)}^{(b)}$ | $\delta_a y(T_3^2)$ | 0                 | 0                 | 0                 | 0                 | 0                 | 0                 | 0                 | 0                 |
| $(2)_{a(2)}^{(b)}$ | 0                 | 0                 | 0                 | 0                 | 0                 | 0                 | 0                 | 0                 | 0                 |
| $(2)_{a(3)}^{(b)}$ | 0                 | 0                 | 0                 | 0                 | 0                 | 0                 | 0                 | 0                 | 0                 |
| $(3)_{a(1)}^{(b)}$ | 0                 | 0                 | 0                 | 0                 | 0                 | 0                 | 0                 | 0                 | 0                 |
| $(3)_{a(2)}^{(b)}$ | 0                 | 0                 | 0                 | 0                 | 0                 | 0                 | 0                 | 0                 | 0                 |

Table 13: The Riemann curvature $G^{(i)}_{a(j)}^{(k)}_{(l)}$ of the type $(1, 1, 5^*, 5)$ with $y([T_j, T_l]) \neq 0$. 

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Table 14: The Riemann curvature $G^{(i)}_{(j)(k)(l)}$ of the type $(1, 1, 5^*, 5)$.

|                | $(^1_a)(^1_1)$ | $(^2_a)(^2_2)$ | $(^3_a)(^3_3)$ |
|----------------|----------------|----------------|----------------|
| $(^1_2)(^2_1)$ | $\delta^b_a y(T^2_2)$ | $-\delta^b_a y(T^2_2) y(T^2_3) / y(T^2_4)$ | 0              |
| $(^1_3)(^3_1)$ | $\delta^b_a y(T^3_3)$ | 0              | $-\delta^b_a y(T^3_3) y(T^3_4) / y(T^3_5)$ |
| $(^2_3)(^3_2)$ | 0              | $\delta^b_a y(T^3_5)$ | $-\delta^b_a y(T^3_3) y(T^3_4) / y(T^3_5)$ |

Table 15: The Riemann curvature $G^{(i)}_{(j)(k)(l)}$ of the type $(1, 1, 10, 10^*)$ with $y([T^i_j, T^k_l]) \neq 0$.

|                | $(^1_a)(^1_2)$ | $(^2_a)(^2_3)$ | $(^3_a)(^3_4)$ | $(^1_a)(^1_3)$ | $(^2_a)(^2_2)$ | $(^3_a)(^3_4)$ |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| $(^1_2)(^3_1)$ | 0              | 0              | $\delta^b_a y(T^2_2)$ | 0              | 0              | 0              |
| $(^1_3)(^2_1)$ | 0              | 0              | 0              | 0              | 0              | $\delta^b_a y(T^2_2)$ |
| $(^1_3)(^3_2)$ | 0              | 0              | 0              | $-\delta^b_a y(E^2_{abc}) y(T^2_2) / y(E^2_{abc})$ | 0              | 0              |
| $(^2_3)(^3_1)$ | $-\delta^b_a y(E^2_{abc}) y(T^2_2) / y(E^2_{abc})$ | 0              | 0              | 0              | 0              | 0              |
| $(^1_2)(^3_2)$ | 0              | 0              | 0              | 0              | 0              | 0              |
| $(^2_3)(^3_1)$ | 0              | 0              | 0              | 0              | 0              | 0              |

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| \((\frac{1}{2})_1\) | \((\frac{1}{2})_2\) | \((\frac{1}{2})_3\) |
|---|---|---|
| \(-\delta_{ab} \frac{y(T_{1}^{a} y(E_{b})}{y(E_{a})}\) | \(\delta_{ab} y(T_{1}^{a})\) | 0 |
| \((\frac{1}{3})_1\) | \((\frac{1}{3})_2\) | \((\frac{1}{3})_3\) |
| \(-\delta_{ab} \frac{y(T_{1}^{a} y(E_{b})}{y(E_{a})}\) | 0 | \(\delta_{ab} y(T_{1}^{a})\) |
| \((\frac{2}{3})_1\) | \((\frac{2}{3})_2\) | \((\frac{2}{3})_3\) |
| 0 | \(-\delta_{ab} \frac{y(T_{1}^{a} y(E_{b})}{y(E_{a})}\) | \(\delta_{ab} y(T_{1}^{a})\) |

Table 16: The Riemann curvature \(G^{(i) (j) (ab) (cd)}\) of the type \((1,10, 10^*)\).

6 Conclusions

In this paper we have discussed the reducible Kähler coset space \(G/S \otimes \{U(1)\}^k\) in the geometrical approach generalizing the arguments in ref. [9]. We have expressed the Riemann curvature of the coset space in terms of the Killing vectors as \((4.32)\). It is the most important formula in this paper. We have been then interested in the four-fermi coupling of the supersymmetric non-linear \(\sigma\)-model on \(G/S \otimes \{U(1)\}^k\), to the leading order of \(\frac{1}{f}\). It is given by evaluating the Riemann curvature at the origin of the coset space. We have established the group theoretical method to do this by using the formula \((4.32)\). Otherwise the calculation would be too complicated. Concrete calculations have been done for \(SU(3)/\{U(1)\}^2\) and \(E_7/SU(5) \otimes \{U(1)\}^3\). The results of the last Kähler coset space is phenomenologically interesting, since they give four-fermi coupling constants among the three families of \(10 + 5^* + 1\) of \(SU(5)\) in the supersymmetric non-linear \(\sigma\)-model on \(E_7/SU(5) \otimes \{U(1)\}^3\). Among them those involving the three families of right-handed neutrinos are particularly interesting and have been given in Tables 8,9,13~16. The dependence of the three \(U(1)\)-charges of the NG pseudo fermions are explicit in these results. Of course, we may take another set of \(U(1)\)s, say \(Q_1, Q_2, Q_3\), than \(T, Q, Q'\), for instance, those which remain unbroken in the breaking process

\[ E_7 \xrightarrow{U(1)} E_6 \xrightarrow{U(1)} SO(10) \xrightarrow{U(1)} SU(5) \]

as in ref. [5]. The results given by Tables 3~16 are still valid if one defines the the Y-charge as

\[ Y = \alpha Q^1 + \beta Q^2 + \gamma Q^3. \]
and replaces Table 2 for \( y(X^i) \) by a new table with \( Q^1, Q^2, Q^3 \). It is desired to carry out a phenomenological study by tuning the three arbitrary parameters \( \alpha, \beta, \gamma \).

**Acknowledgements**

The author would like to thank T. Yanagida for reviving the interest in the subject. He is grateful to T. Kugo for the valuable discussions on the reducible Kähler coset space and reading the manuscript.
Appendix A

We derive the useful formulae for the calculation in Subsection 4.2. We start with covariantization of the Lie-algebra (2.14):

\[ R^{A\alpha} R^{B\beta}_{\alpha} - R^{B\alpha} R^{A\beta}_{\alpha} = f^{ABC} R^{C\beta}. \]

Multiplying both sides by \((R_\gamma \Delta)^B\) or \((R_{\bar{\gamma}} \Delta)^B\) and lower the index \(\beta\) to get

\[ R^{A\alpha} \cdot R_\gamma \Delta R^{\beta}_{\bar{\gamma},\alpha} = f^{ABC} (R_\gamma \Delta)^B R^{C}_{\bar{\gamma}}, \quad (A.1) \]

or

\[ R^{A\alpha} \cdot R_{\bar{\gamma}} \Delta R^{\beta}_{\gamma,\alpha} = f^{ABC} (R_{\bar{\gamma}} \Delta)^B R^{C}_{\beta}, \quad (A.2) \]

by using (4.23). Noting that

\[ R^{\bar{\gamma}} \Delta R^{\beta}_{\gamma,\alpha} = - R^{\bar{\gamma}} \Delta R^{\beta}_{\alpha,\gamma} = R^{\gamma} \Delta R^{\beta}_{\gamma,\alpha}, \quad (A.3) \]

and

\[ R^{\gamma} \Delta R^{\beta}_{\gamma,\alpha} = - R^{\gamma} \Delta R^{\beta}_{\gamma,\alpha} = R^{\gamma} \Delta R^{\beta}_{\gamma,\alpha}, \quad (A.4) \]

by (4.24) and (2.15), we write (A.1) and (A.2) respectively as

\[ R^{A\alpha} \cdot R_\gamma \Delta \bar{R}^{\beta}_{\gamma,\alpha} = - f^{ABC} (R_\gamma \Delta)^B R^{C}_{\bar{\gamma}}, \quad (A.5) \]

and

\[ R^{A\alpha} \cdot R_{\bar{\gamma}} \Delta \bar{R}^{\beta}_{\gamma,\alpha} = f^{ABC} (R_{\bar{\gamma}} \Delta)^B R^{C}_{\beta}. \quad (A.6) \]

Taking the complex conjugation of them gives

\[ R^{A\gamma} \Delta \bar{R}^{\beta}_{\gamma,\alpha} = - R^{A\gamma} \cdot R_{\bar{\gamma}} \Delta \bar{R}^{\beta}_{\gamma,\alpha} = - f^{ABC} (R_{\bar{\gamma}} \Delta)^B R^{C}_{\beta}, \quad (A.7) \]

and

\[ R^{A\gamma} \cdot R_{\bar{\gamma}} \Delta \bar{R}^{\beta}_{\gamma,\alpha} = f^{ABC} (R_{\bar{\gamma}} \Delta)^B R^{C}_{\beta}. \quad (A.8) \]

That (A.5) and (A.7) satisfy the Killing condition (2.15) can be easily checked by (4.17), i.e.,

\[ R^{A\alpha} \Delta_{\alpha} + R^{A\bar{\alpha}} \Delta_{\bar{\alpha}} = i[T^A, \Delta], \]

with \((T^A)^{BC} = -if^{ABC}\). Multiplying both sides of (A.6) and (A.8) respectively by \((R_\eta \Delta)^A\) and \((R_{\bar{\eta}} \Delta)^A\) we get

\[ R_{\bar{\eta}} \Delta \bar{R}^{\beta}_{\eta,\alpha} = f^{ABC} (R_{\bar{\eta}} \Delta)^A R^{B}_{\bar{\eta}} (R_\eta \Delta)^C \quad (A.9) \]

and

\[ R_{\eta} \Delta \bar{R}^{\beta}_{\eta,\alpha} = f^{ABC} (R_{\eta} \Delta)^A R^{B}_{\eta} (R_{\bar{\eta}} \Delta)^C \quad (A.10) \]

owing to (4.23).
Appendix B

We will check the symmetry property $R_{\alpha \beta \gamma \delta} = R_{\alpha \delta \beta \gamma}$ of the r.h.s. of (4.32). Put it in the form

$$R_{\alpha \beta \gamma \delta} = [f^{ABC}(R_\alpha \Delta)^A(R_\gamma \Delta)^B R^{C\gamma} \cdot f^{DEF}(R_\sigma \Delta)^D(R_\delta \Delta)^E R^{F\sigma}] + f^{ABC}(R_\alpha \Delta)^A(R_\eta \Delta)^B R^{C\eta} \cdot f^{DEF}(R_\sigma \Delta)^D(R_\delta \Delta)^E R^{F\sigma} + [f^{ABC}(R_\alpha \Delta)^A(R_\sigma \Delta)^B \cdot f^{CDE} R^D_\beta (R_\sigma \Delta)^E - R_\alpha \Delta \beta R_\sigma \cdot R_\sigma \Delta \beta R^{F\gamma}], \quad \text{(B.1)}$$

remembering (4.31). The first bracket is already symmetric under the interchange of $\sigma$ and $\delta$. Therefore we are left with the second bracket to examine. The anti-symmetric sum of its first piece by interchanging $\sigma$ and $\delta$ becomes

$$f^{ABC}(R_\alpha \Delta)^A(R_\sigma \Delta)^B \cdot f^{CDE} R^D_\beta (R_\sigma \Delta)^E - f^{ABC}(R_\alpha \Delta)^A(R_\sigma \Delta)^B \cdot f^{CDE} R^D_\beta (R_\delta \Delta)^E = f^{ABC}(R_\alpha \Delta)^A R^B_\beta \cdot f^{CDE} (R_\sigma \Delta)^D (R_\delta \Delta)^E, \quad \text{(B.2)}$$

by using the Jacobi identity of the structure constants (4.15). On the other hand that of the second piece is given by

$$(R_\sigma \Delta \beta R_\eta - R_\delta \Delta \sigma R_\eta) \cdot R_\alpha \Delta \beta R^{\eta \gamma} = f^{ABC}(R_\sigma \Delta)^A(R_\delta \Delta)^B \cdot f^{CDE} (R_\alpha \Delta)^D R^{E \gamma}, \quad \text{(B.3)}$$

This is easily checked as follows. Note at first that

$$R_\sigma \Delta \beta R_\eta + R_\sigma \Delta R_{\eta \delta} = 0,$$

from (4.24). Then plug (A.7) in this to find

$$R_\sigma \Delta \beta R_\eta - R_\delta \Delta \sigma R_\eta = f^{ABC}(R_\sigma \Delta)^A R^{B \gamma}_\delta (R_\sigma \Delta)^C. \quad \text{(B.4)}$$

Multiplying both sides by $R_\alpha \Delta \beta R^{\eta \gamma}$ and using (A.8) yields (B.3). From (B.2) and (B.3) the second bracket of (B.1) is also symmetric under the interchange of $\sigma$ and $\delta$. Thus we have $R_{\alpha \beta \gamma \delta} = R_{\alpha \delta \beta \gamma}.$

Next we examine the symmetry property $R_{\alpha \beta \gamma \delta} = R_{\beta \delta \alpha \gamma}$. We rewrite the Riemann curvature (4.29) as

$$R_{\alpha \beta \gamma \delta} = -(R_\alpha \Delta)^A \gamma R^\gamma \cdot (R_\eta \Delta) \eta R^\eta \gamma + f^{ABC}(R_\alpha \Delta)^A R^{B \gamma}_{\delta \eta} (R_\sigma \Delta)^C \quad \text{(B.4)}$$

by using the Killing condition (2.15). Then with (A.5),(A.9),(A.10),(4.30) and (4.31) it becomes

$$R_{\alpha \beta \gamma \delta} = [f^{ABC}(R_\alpha \Delta)^A(R_\gamma \Delta)^B R^{C\gamma} \cdot f^{DEF}(R_\sigma \Delta)^D(R_\delta \Delta)^E R^{F\sigma}] + f^{ABC}(R_\beta \Delta)^A(R_\gamma \Delta)^B R^{C\gamma} \cdot f^{DEF}(R_\sigma \Delta)^D(R_\delta \Delta)^E R^{F\sigma} + [f^{ABC}(R_\alpha \Delta)^A(R_\delta \Delta)^B \cdot f^{CDE} (R_\beta \Delta)^D R_{\gamma \delta}^{E \gamma} - f^{ABC}(R_\alpha \Delta)^A R^{B \beta}_\delta (R_\sigma \Delta)^C \cdot f^{DEF} (R_\sigma \Delta)^D R^{E \gamma}_\delta (R_\delta \Delta)^F]. \quad \text{(B.5)}$$
The first bracket is symmetric under the interchange of $\alpha$ and $\beta$. The symmetry of the second bracket can be shown similarly to the previous demonstration of $R_{\alpha\beta\gamma\delta} = R_{\alpha\delta\beta\gamma}$.

**Appendix C**

We show how to evaluate the Riemann curvature $G^{(ab)(ie)(ej)(gh)}$ of the type $(10, 10^*, 10, 10^*)$ by (5.3) and (5.13). For $i > j$ we have $y([E^{i}_{ab}, E^{i}_{cd}]) < 0$. By (5.8) non-trivial components of the Riemann curvature are

$$G^{(ab)(ie)(ej)(gh)} = tr([E^{ab}_{ij}, E^{i}_{cd}][E^{ef}_{ij}, E^{f}_{gh}])y(E^{ef}_{ij}) - y(T_{ij})$$

$$- tr([E^{ab}_{ij}, E^{ef}_{ij}][E^{i}_{cd}, E^{f}_{gh}])y(E^{ef}_{ij})y(E^{gh}_{ij})/y([E^{ab}_{ij}, E^{i}_{ij}])$$

$$= \delta_{cd}^{ab} \delta_{gh}^{ef} y(E^{ef}_{ij}) - \delta_{cdgh}^{abef} y(E^{ef}_{ij}) y(E^{gh}_{ij})/y([E^{ab}_{ij}, E^{i}_{ij}]).$$

For $i < j$ we have $y([E^{ab}_{ij}, E^{i}_{cd}]) > 0$. By (5.7) non-trivial components of the Riemann curvature are

$$G^{(ab)(ie)(ej)(gh)} = tr([E^{ab}_{ij}, E^{i}_{cd}][E^{i}_{ef}, E^{f}_{gh}])y(E^{ef}_{ij})$$

$$- tr([E^{ab}_{ij}, E^{i}_{ef}][E^{i}_{cd}, E^{f}_{gh}])y(E^{ef}_{ij})y(E^{gh}_{ij})/y([E^{ab}_{ij}, E^{i}_{ij}])$$

$$= \delta_{cd}^{ab} \delta_{gh}^{ef} y(E^{ef}_{ij}) - \delta_{cdgh}^{abef} y(E^{ef}_{ij}) y(E^{gh}_{ij})/y([E^{ab}_{ij}, E^{i}_{ij}]).$$

The same result is also obtained by interchanging the indices as

$$G^{(ab)(ie)(ej)(gh)} = G^{(ef)(gh)(ij)(cd)}$$

and calculating it by (5.8). For $i = j$ we have $y([E^{ab}_{ij}, E^{i}_{cd}]) = 0$. Non-trivial components of the Riemann curvature $G^{(ab)(ie)(ej)(gh)}$ are evaluated by (5.9). For $k < i$

$$G^{(ab)(ie)(ej)(gh)} = tr([E^{ab}_{ij}, E^{i}_{cd}][E^{i}_{ef}, E^{k}_{gh}])y(E^{ef}_{ij})$$

$$- tr([E^{ab}_{ij}, E^{i}_{ef}][E^{i}_{cd}, E^{k}_{gh}])y(E^{ef}_{ij})y(E^{gh}_{ij})/y([E^{ab}_{ij}, E^{i}_{ij}]).$$

By the formula

$$tr([E^{ab}_{ij}, E^{i}_{cd}][E^{ef}_{ij}, E^{k}_{gh}]) = -\delta_{cdgh}^{abef} + \delta_{gh}^{cd} \delta_{ef}^{ij},$$

it becomes

$$G^{(ab)(ie)(ej)(gh)} = \delta_{gh}^{cd} \delta_{ef}^{ij} y(E^{ef}_{ij}) - \delta_{cdgh}^{abef} y(E^{ef}_{ij}) y(E^{gh}_{ij})/y([E^{ab}_{ij}, E^{i}_{ij}]).$$
For \( k > i \)

\[
G^{(ab)(i)(ef)(gh)}_{(cd)}(k) = \delta^{ab}_{gh} \delta^{ef}_{cd} y(E^{i}_{i}) - \delta^{abe}_{cdgh} y(E^{i}_{i}) y(E^{ef}_{k})/y([E^{ab}_{i}, E^{ef}_{k}]),
\]

by applying the symmetry property

\[
G^{(ab)(i)(ef)(gh)}_{(cd)}(k) = G^{(ef)(gh)(ab)(cd)}_{(cd)}(k)
\]
to the above result. Of course this can be obtained by a direct calculation. Finally for \( k = i \)

\[
G^{(ab)(i)(ef)(gh)}_{(cd)}(i) = tr([E^{ab}_{i}, E^{i}_{cd}][E^{ef}_{i}, E^{i}_{gh}]) y(E^{ef}_{i})
\]

\[
= (\delta^{ab}_{ch} \delta^{ef}_{gd} + \delta^{ab}_{cg} \delta^{ef}_{dh} + \delta^{ab}_{dh} \delta^{ef}_{cg} + \delta^{ab}_{gd} \delta^{ef}_{ch}) y(E^{ef}_{i}).
\]

Other types of the Riemann curvature are obtained similarly.
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