New variants of the Bell-Kochen-Specker theorem*

Adán Cabello†
José M. Estebananz
Guillermo García-Alcaine

Departamento de Física Teórica,
Universidad Complutense, 28040 Madrid, Spain.

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Abstract

We discuss two new demonstrations of the Bell-Kochen-Specker theorem: a state-independent proof using 14 propositions in $\mathbb{R}^4$, based on a suggestion made by Clifton, and a state-specific proof involving 5 propositions on the singlet state of two spin-$\frac{1}{2}$ particles.

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*Phys. Lett. A 218, 115 (1996).
†Electronic address: fite1z1@sis.ucm.es
We recently found [1] a proof of the Bell-Kochen-Specker (BKS in the following) theorem [2, 3] using 18 vectors in $\mathbb{R}^4$. While the paper was in press, Clifton [4] suggested to us a modification allowing a further reduction of this number. In the first part of this paper we will discuss Clifton’s idea. In the second part we will take advantage of the properties of the singlet state of two spin-$\frac{1}{2}$ particles to find a state-specific BKS proof involving 5 propositions.

The BKS theorem asserts the impossibility of hidden variables theories such that the values $v(\mathbf{u}_i)$ of propositions $P_{\mathbf{u}_i}$ represented by projectors $|\mathbf{u}_i\rangle \langle \mathbf{u}_i|$ obey the following requisites:

(a) *In an individual system each proposition $P_{\mathbf{u}_i}$ has a unique value, 0 (“no”) or 1 (“yes”), that is independent of any other compatible observables being considered jointly (non-contextuality).*

(b) *For each set of rank 1 projectors whose sum is the unit matrix (in the $n$-dimensional Hilbert space of the states of the system), the value of one and only one of the corresponding propositions is 1, and the values of the remaining $n − 1$ propositions are 0.*

Conditions (a) and (b) imply a third one:

(c) *Given two sets of propositions represented by projectors over the same subspace, $\sum_i |\mathbf{u}_i\rangle \langle \mathbf{u}_i| = \sum_i |\mathbf{v}_i\rangle \langle \mathbf{v}_i|$, the sums of values must be equal, $\sum_i v(\mathbf{u}_i) = \sum_i v(\mathbf{v}_i)$.*

The implication is evident: if we have two different sets of projectors summing up to the unit matrix, $\sum_{i=1}^n |\mathbf{u}_i\rangle \langle \mathbf{u}_i| = \sum_{i=1}^n |\mathbf{v}_i\rangle \langle \mathbf{v}_i| = 1$, and both sets share a common element, say $\mathbf{u}_1 = \mathbf{v}_1$, premise (a) implies that the value of the corresponding proposition in both sets must be the same (0 or 1); then premise (b) implies that the sums of the values of both sets of propositions on the complementary subspace must also be equal (1 or 0, respectively), $\sum_{i=2}^n v(\mathbf{u}_i) = \sum_{i=2}^n v(\mathbf{v}_i)$. The same argument can be used again if more vectors are common.

By choosing among eqs. (1–9) of ref. [4] four couples of equations, each couple with a common vector, and equating the sums of values over the
subspaces complementary to these common vectors, we reduce the system to only 5 equations with 14 vectors, for instance,

\[ v(0, 0, 1, 0) + v(1, 1, 0, 0) + v(1, -1, 0, 0) = v(0, 1, 0, 0) + v(1, 0, 1, 0) + v(1, 0, -1, 0) , \]

\[ v(1, -1, -1, 1) + v(1, 1, 0, 0) + v(0, 0, 1, 1) = v(1, 1, 1, 1) + v(1, 0, -1, 0) + v(0, 1, 0, -1) , \]

\[ v(0, 0, 1, 0) + v(0, 1, 0, 0) + v(1, 0, 0, 1) = v(1, -1, -1, 1) + v(1, 1, 1, 1) + v(0, 1, -1, 0) , \]

\[ v(1, 1, -1, 1) + v(1, 0, 1, 0) + v(0, 1, 0, -1) = v(1, 1, 1, -1) + v(1, 0, 0, 1) + v(0, 1, -1, 0) , \]

\[ v(1, 1, -1, 1) + v(1, 1, 1, -1) + v(1, -1, 0, 0) + v(0, 0, 1, 1) = 1 . \] 

We can now formulate the following version of the BKS theorem:

There is no set of values \( v(u_i) \) verifying eqs. \((1-5)\).}

The proof involves a parity argument: if we add these five equations, each value \( v(u_i) \) appears either twice on the same side of the resulting equation, with an even contribution (0 or 2), or once on each side, with a cancellation of both contributions; the extra term 1 on the right-hand side makes it impossible to satisfy the equality.

This 14-vector set (or any of the many others that we can obtain similarly) leads to a proof of the BKS theorem based on the explicit use of (c), according to Clifton’s suggestion. Condition (c) is a direct consequence of (a) and (b), and does not impose any new requirement on hidden variables. Its use is not an artifice to leave out some propositions when counting the number of them involved in the proof, because no concrete value for the propositions eliminated is assumed; the proof stands, whatever the values of the omitted propositions.

Let us now consider a system of two spin-\( \frac{1}{2} \) particles and choose the basis formed by the eigenvectors of \( \sigma_x^{(1)} \otimes \sigma_x^{(2)} \) (i.e., the vectors \( \text{up} \otimes \text{up}, \text{up} \otimes \text{down}, \text{down} \otimes \text{up}, \text{down} \otimes \text{down} \)). In any individual system prepared in the singlet state, \( (0, 1, -1, 0) \) (we omit, as before, the normalization constant), we have \( v(0, 1, -1, 0) = 1 \), and the values for propositions over any orthogonal direction must be zero \( v(1, 1, 1, -1) = 0, v(-1, 1, 1, 1) = 0 \), for instance).
Replacing these values into eqs. (7,8) of ref. [1] (or into eqs. (4,5) in this paper, respectively, but the intermediate step to obtain (4) from (8,9) in [1] is actually unnecessary) we obtain

\[ v(1,1,-1,1) + v(1,-1,0,0) + v(0,0,1,1) = 1, \tag{6} \]

\[ v(1,1,-1,1) + v(1,0,1,0) + v(0,1,0,-1) = 1. \tag{7} \]

The hidden variables values for the four non-repeated propositions in (6,7) satisfy the following relation in the singlet state:

\[ v(1,-1,0,0) + v(0,0,1,1) + v(1,0,1,0) + v(0,1,0,-1) = 1. \tag{8} \]

The proof of (8) is straightforward: first, the value of a factorizable proposition (like the ones appearing in (8)) is the product of the values of its factors,

\[ v((a,b)^{(1)} \otimes (c,d)^{(2)}) = v((a,b)^{(1)}) v((c,d)^{(2)}) ] \]

secondly, if in the singlet state we measure the spin component of each particle along the same (arbitrary) direction, the results are perfectly correlated (always opposite), and the same relations must exist between the values of propositions in any deterministic hidden variables theory, \( v((1,0)^{(1)}) = v((0,1)^{(2)}) \), \( v((1,1)^{(1)}) = v((1,-1)^{(2)}) \), etc... Therefore the left-hand side of (8) can be written as follows:

\[
v((1,0)^{(1)}) v((1,-1)^{(2)}) + v((0,1)^{(1)}) v((1,1)^{(2)}) \\
+ v((1,1)^{(1)}) v((1,0)^{(2)}) + v((1,-1)^{(1)}) v((0,1)^{(2)}) \\
= [v((1,0)^{(1)}) + v((0,1)^{(1)})] \times [v((1,1)^{(1)}) + v((1,-1)^{(1)})] = 1. \tag{9} \]

The last equality in (9) is a consequence of condition (b) in the 2-dimensional space of the spin states of the first particle, q.e.d.

\footnote{Asking if the two-particle system is in the state \((a,b)^{(1)} \otimes (c,d)^{(2)}\) is the same as asking if particle 1 is in the state \((a,b)\) \textit{and} particle 2 is in the state \((c,d)\): the answer is 1 (yes) only if the values of both factor propositions are 1, and is 0 otherwise. The same conclusion can be reached as a consequence of the Kochen-Specker product rule for compatible observables \( v(AB) = v(A) v(B) \); in particular \( v(|u\rangle \langle u|^{(1)} \otimes |w\rangle \langle w|^{(2)}) = v(|u\rangle \langle u|^{(1)} \otimes 1^{(2)}) v(1^{(1)} \otimes |w\rangle \langle w|^{(2)}) = v(|u\rangle \langle u|^{(1)} ) v(|w\rangle \langle w|^{(2)}) \), where the last equality reflects the fact that the proposition represented by the projector \(|u\rangle \langle u|^{(1)} \otimes 1^{(2)}\) corresponds to the first particle being in the state \(|u\rangle\), whatever the state of the second particle.}
The first two vectors in eq. (8) are not orthogonal to the last two, and the corresponding projectors do not commute; moreover, there is no common eigenstate to the four projectors. Then, we can neither prepare a system in a state free of dispersion for the four propositions, nor check experimentally the relation by a simultaneous measurement of them; in this sense this relation is different from those that implement condition (b) \( (\sum_{i=1}^{4} |u_i\rangle \langle u_i| = 1 \Rightarrow \sum_{i=1}^{4} v(u_i) = 1) \), for which there are states simultaneously free of dispersion for each tetrad of compatible projectors, and which can (in principle) be experimentally verified in any state. But in deterministic hidden variables theories non-compatible observables can have well definite values in the same individual system, and therefore eq. (8), although not consequence of a resolution of the identity, is a legitimate relation between hidden variables values, based on the properties of the singlet state.

We now state our second no-go theorem:

There is no set of values \( v(u_i) \) verifying eqs. (6–8).

The proof rests once more on a parity argument: each proposition appears twice in (6–8), but the sum of the right-hand sides is 3.

We will finish with a reflexion on the number of propositions involved in the proof; we have counted them explicitly using another consequence of requisites (a) and (b):

(d) Given an individual system prepared in a state \( w \) and a set of vectors \( \{u_i\} \) spanning a subspace \( U \) that contains \( w \), then \( \sum_{i} v(u_i) = 1 \).

A reason for this was given in [6]: any vector \( v \) in the subspace complementary to \( U \) is orthogonal to \( w \); \( v(w) = 1 \) and (b) imply \( v(v) = 0 \); therefore the sum of values of any complete set of compatible propositions on the subspace \( U \) must be 1. This can be justified too, without explicitly quoting the values \( v(v) = 0 \), if we keep in mind that the probability of finding the system in a state in the subspace \( U \) is 1. Condition (d) is a consequence of (a) and (b), but in contradistinction to what happened in (c), now the propositions omitted have definite zero values, and therefore it could be argued that using (d) is essentially a way to count the number of vectors appearing in state-specific
proofs \[3\], leaving out the vectors orthogonal to the initial state\[4\].

In our example, both triads of vectors on the left-hand sides of (6,7) span subspaces that contain the singlet \((0, 1, -1, 0)\), and therefore both equations are a direct application of rule (d). If we use this condition, we only need to count the 5 vectors explicitly appearing in (6,7); a count of the number of propositions with definite values involved in the theorem, based strictly on (a) and (b), should also include the omitted vectors \((1, 1, 1, -1), (-1, 1, 1, 1)\) (and perhaps the initial state too).

In the second part of this paper we have proved the impossibility of assigning non-contextual values to a set of propositions in the singlet state. The contradiction arises in only 3 equations involving 5 propositions in \(\mathbb{R}^4\): in terms of *number of propositions*, this state-specific BKS proof is the most economic no-go theorem that we know of.

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\[2\] Starting from a state-specific BKS proof in terms of \(n\)-dimensional vectors we can trivially find a state-specific proof in any dimension \(n + m\): it suffices to append \(m\) zeros to all vectors, including the initial state. If the counting of the number of vectors is based on the explicit use of condition (d), this number is independent of \(m\) (for instance, the 10-vector set of the state-specific proof in dimension \(n = 4\) in ref. \[4\] gives also a 10-vector state-specific proof in \(n = 8\)). The ratio between the number \(f\) of vectors used in a demonstration of the BKS theorem and the dimensionality \(n\) of the space \[4\] is a reasonable measure of the merit of a state-independent proof (where obviously \(f > n\)), but its meaning in state-specific proofs is questionable if the initial state and the vectors orthogonal to it are not counted, as it is done when using condition (d).
References

[1] A. Cabello, J.M. Esteban and G. García-Alcaine, Phys. Lett. A 212 (1996) 183.

[2] J.S. Bell, Rev. Mod. Phys. 38 (1966) 447.

[3] S. Kochen and E.P. Specker, J. Math. Mech. 17 (1967) 59.

[4] R.K. Clifton (private communication).

[5] A. Einstein, B. Podolsky and N. Rosen, Phys. Rev. 47 (1935) 777.

[6] M. Kernaghan and A. Peres, Phys. Lett. A 198 (1995) 1.

[7] J.S. Bell, Phys. 1 (1964)195.

[8] N.D. Mermin, Phys. Rev. Lett. 65 (1990) 3373; Rev. Mod. Phys. 65 (1993) 803.

[9] L. Hardy, Phys. Rev. Lett. 68 (1992) 2981; 71 (1993) 1665; R.K. Clifton and P. Niemann, Phys. Lett. A 166 (1992) 177; S. Goldstein, Phys. Rev. Lett. 72 (1994) 1951.

[10] A. Peres, Phys. Lett. A 151 (1990) 107.