OPTIMAL INVESTMENT-REINSURANCE STRATEGY IN THE CORRELATED INSURANCE AND FINANCIAL MARKETS

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ABSTRACT. Within the correlated insurance and financial markets, we consider the optimal reinsurance and asset allocation strategies. In this paper, the risk asset is assumed to follow a general continuous diffusion process driven by a Brownian motion, which correlates to the insurer’s surplus process. We propose a novel approach to derive the optimal investment-reinsurance strategy and value function for an exponential utility function. To illustrate this, we show how to derive the explicit closed strategies and value functions when the risk asset is the CEV model, 3/2 model and Merton’s IR model respectively.

1. Introduction. In worldwide financial markets, insurance companies are playing increasingly important roles. They receive insurance premiums and pay claims to the policyholders. In the meantime, they disperse risk and make profits through reinsurance and allocating insurance premium on various financial assets. There have been extensive literature studying the optimal investment and reinsurance problem, where the insurer’s surplus is often assumed to be independent of the financial asset prices. However, they might be affected by some common external factors, such as economic environment, worldwide pandemic and so on. Wells et al.[13] found out, during an economic recession, most economic activities such as production, employment, business earnings and profits, investment spending all fall down. A large number of companies go bankrupt, people’s incomes and financial assets shrink. Meanwhile, fraudulent insurance claims, unemployment and sickness of policyholders dramatically increase, which tends to result in surges in claims costs. Another example, during COVID-19 pandemic, stock markets worldwide reported their largest single-week declines since the 2008 financial crisis on 28 February 2020. By March stock markets declined over 30%, implied volatilities of equities and oil have spiked to crisis levels. On the other hand, the auto insurance premium income and the claims frequency go down due to the decrease of new car sale and lockdown order. According to ABI (Association of British Insurers), travel insurers expect to pay out at least 275 million pounds for trip cancellations, which exceed the payouts for all of last year and hit the highest record. But the new premium

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income is dragged down due to the sudden drop in the number of tourists. In the life and health insurance industry, reduction in premium and increasing pressure for treatment and death claims seriously impacted insurance industry. Hence, in this paper we make a reasonable assumption that the risk asset price and the insurance surplus process are correlated.

Under this assumption, there have been some authors conducting studies on the optimal reinsurance and assets allocation strategies. For example, Jang and Kim [8] studied this problem under regime switching. They found that the optimal reinsurance and asset allocations are affected by future market conditions and the changes of correlation between stock prices and insurance claims. Wang et al. [12] assumed the risk asset to be a CEV model with the constant elasticity parameter \( \beta \), they maximized the expected utility of the insurer’s and the reinsurer’s terminal wealth, where the Legendre transform and dual theory are adopted to solve the Hamilton-Jacobi-Bellman (HJB) equations. Finally, they derived the optimal strategies and value functions in the special cases \( \beta = 0 \) and \( \beta = -1 \).

Then Zhao et al. [16] partially extended the work of Wang et al. [12] by taking the defaultable security into consideration but simultaneously confined the risk asset price to a geometric Brownian motion (GBM). Under the mean-variance criterion, the extended Hamilton-Jacobi-Bellman systems for the post-default case and the pre-default case are established and solved respectively. Bi and Cai [2] considered the optimal investment-reinsurance problem in the presence of state dependent risk aversion and VaR constraints. As for other related research in correlated markets, see Gu et al. [5], Huang et al. [7], Qian and Lin [11] etc.

In most existing literature, the risk asset model is usually described by some classical well-known process such as geometric Brownian motion (GBM), Cox-Ingersoll-Ross (CIR) process, constant elasticity of variance (CEV) process and so on. See, for instance, Gu et al. [6], Xiao et al. [14], Gao [4], Zhao et al. [17], A et al. [1], Zhang [15] as well as the aforementioned literature. This is partially due to their analytical tractability. However, the optimal reinsurance and asset allocation strategy is not easy to find when the stock price and insurance premium are correlated, even for a analytically tractable risk model. As shown by Wang et al. [12], applying stochastic optimal control on the expected exponential utility of terminal wealth leads to a non-linear second order HJB equation. The authors used Legendre transform and dual theory to solve the non-linear problem. However, this method could not apply unless \( \beta \) is equal to 0 or \(-1\).

This present paper extends the existing results in two aspects. First, we assume the risk asset price follows a general continuous diffusion process driven by a Brownian motion, rather than a specified model. This assumption makes model flexible and allows our research to have wider applications in reality. Second, we assume the risk asset price and the insurer’s surplus are linearly correlated. To work out the non-linear second-order PDE deduced from HJB equation, we propose a novel approach to transform it into a linear problem. Then by using Duhamel’s principle, the optimal strategies and value functions have the integral expressions, where the integrands are moments of some given stochastic process. Moreover, the moments are easy to work out for the general risk model mentioned above. To illustrate this, we list three examples, the CVE process, 3/2 model and Merton’s IR model, as the risk asset model. As long as we obtain the moments in the integrands, we can explicitly express the optimal strategies and value functions in analytical form.
The remainder of the paper is organized as follows. In Section 2, we introduce the dynamic models of the surplus process, the riskless asset and the risk asset in a correlated insurance and financial market. In Section 3, under a general risk asset model and an exponential utility function, we derive the HJB equations and give a universal approach to solve the optimization problem. In Section 4, we consider several risk asset examples, such as the CEV model, 3/2 model, and Merton’s IR model. Using the method in Section 3, we can easily express the optimal strategies and value functions in analytic forms. In Section 5, we demonstrate some numerical examples to analyze parameters sensitivity of the optimal strategies.

2. Model setup. Let \((\Omega, \mathcal{F}, P)\) be a complete probability space indexed by a finite horizon \([0, T]\), where \(\mathcal{F} := \{\mathcal{F}_t\}_{t \geq 0}\) is the right-continuous, \(P\)-complete filtration generated by two independent standard Brownian motions \(\{W_0(t)\}\) and \(\{W_t\}\), a Poisson process \(\{N(t)\}\) and a family of random variables \(\{Y_i, i \geq 1\}\). \(E[\cdot]\) and \(Var[\cdot]\) are the expectation and variance with respect to \((w.r.t.)\) \(P\) respectively.

2.1. Surplus process. The insurer’s surplus process is given by the classical Cramér-Lundberg risk model perturbed by a diffusion

\[
dR(t) = c_0 dt + \beta_0 dW_0(t) - d \left( \sum_{i=1}^{N(t)} Y_i \right),
\]

where \(c_0 > 0\) is the premium rate and \(\beta_0\) can be regarded as the volatility of the insurer’s surplus. \(\{N(t)\}_{t \geq 0}\) is a homogeneous Poisson process with intensity \(\lambda_N > 0\) and the claim sizes \(\{Y_i, i \geq 1\}\) are independent and identically distributed \((i.i.d.)\) positive random variables with the common distribution \(F(y)\). Denote the first and second moments of \(Y_i\) by \(E[Y_i] = \mu_Y\) and \(E[Y_i^2] = \sigma_Y^2\), both of which are assumed to be finite. Thus \(\sum_{i=1}^{N(t)} Y_i\) is a compound Poisson process representing the cumulative amount of claims during \([0, t]\). We assume that \(\{N(t)\}\) is independent of the claim sizes \(\{Y_i, i \geq 1\}\). We adopt the expected value principle to calculate the insurance premium, so the premium rate is \(c_0 = (1 + \theta)\lambda_N \mu_Y\), where \(\theta > 0\) stands for the safety loading coefficient of the insurer.

The insurer can purchase proportional reinsurance or acquire new business to hedge against the insurance risk. Let \(\pi_q(t) \geq 0\) represent proportional reinsurance/new business level at time \(t\). Suppose that the reinsurance premium is \((1 - \pi_q(t))(1 + \eta)\lambda_N \mu_Y\) due to the expected value principle, where \(\eta\) \((\eta > \theta)\) is the reinsurer’s safety loading. Note that for the insurance company, \(\pi_q(t) \in [0, 1]\) corresponds to a proportional reinsurance cover, and the insurer should transfer part of the premium to the reinsurer at the rate \((1 - \pi_q(t))(1 + \eta)\lambda_N \mu_Y\): \(\pi_q(t) > 1\) would mean that the insurance company take a new insurance business from other companies. The insurer receives the premium at the rate \((\pi_q(t) - 1)(1 + \eta)\lambda_N \mu_Y\) for additional new business.

If taking the reinsurance into account, the surplus process of the insurer, still denoted by \(R(t)\), is

\[
dR(t) = [(\theta - \eta) + (1 + \eta)\pi_q(t)]\lambda_N \mu_Y dt + \beta_0 dW_0(t) - \pi_q(t)d \sum_{i=1}^{N(t)} Y_i.
\]
By stochastic optimal control theory, we define the value function can be stated as
\[ \gamma > \]
with the constant risk aversion parameter described by a general diffusion process
\[ b(s) \]
where the drift coefficient \( b(s) \) and the diffusion coefficient \( \sigma(s) \) are continuous on \([0, \infty)\), and the SDE (1) has a unique strong solution valued on \([0, \infty)\). The correlation coefficient of \( W_t \) and \( W_0(t) \) are denoted by \( \rho \), i.e., \( E[W_0(t)W_t] = \rho t \).

Denote by \( X^\pi(t) \) the whole wealth process of the insurer. The insurer is assumed to invest \( \pi_s(t) \) on the risk asset at time \( t \) and invest \( X^\pi(t) - \pi_s(t) \) on the riskless asset. A trading strategy is a pair process \( \pi(t) = \{\pi_q(t), \pi_s(t)\} \in [0, T] \). The wealth process \( \{X^\pi(t)\}_{t \in [0, T]} \) related to \( \pi(t) \) when starting with an initial wealth satisfies

\[
dX^\pi(t) = dR(t) + \frac{X^\pi(t) - \pi_s(t)}{B(t)} dB(t) + \pi_s(t) dS_t
\]

\[
= \left\{ \left[ (\theta - \eta) + (1 + \eta) \pi_q(t) \right] \lambda_N Y_i + rX^\pi(t) + \pi_s(t) \left( \frac{b(S_t)}{S_t} - r \right) \right\} dt
\]

\[
+ \beta_0 dW_0(t) + \sigma(S_t) \frac{\pi_s(t)}{S_t} dW(t) - \pi_q(t) d \sum_{i=1}^{N(t)} Y_i,
\]

with \( X^\pi(0) = x_0 \).

**Definition 2.1.** For any fixed \( t \in [0, T] \), a trading strategy \( \pi(t) = \{\pi_q(u), \pi_s(u)\}_{u \in [t, T]} \) is said to be admissible if it satisfies

1. \( (\pi_q(u), \pi_s(u)) \) is \( \mathcal{F} \)-predictable;
2. \( \forall u \in [t, T], \pi_q(u) \geq 0 \) and \( E \left[ \int_t^T (\pi_q^2(u) + \pi_s^2(u)) \right] < \infty \);
3. \( (\pi(t), X^\pi(t)) \) is the unique strong solution to the stochastic differential equation (2).

For any initial condition \( (t, x, s) \in [0, T] \times \mathbb{R} \times \mathbb{R} \), the corresponding set of all admissible strategies is denoted by \( \Pi \).

**3. Optimal control problem under the exponential utility function.** In this paper, we assume the insurer aims to maximize the profits under the exponential utility function of terminal wealth. The utility function is

\[ U(x) = -\frac{1}{\gamma} e^{-\gamma x}, \]

with the constant risk aversion parameter \( \gamma > 0 \). The optimal dynamic portfolio can be stated as

\[ \max_{(\pi_q, \pi_s) \in \Pi} E_U(X(T)). \]

By stochastic optimal control theory, we define the value function \( V(t, x, s) \) by

\[ V(t, x, s) = \sup_{(\pi_q, \pi_s) \in \Pi} E_U(X(T)) \mid X(t) = x, S_t = s, 0 \leq t \leq T, \]
with $V(T, x, s) = U(x)$. According to Fleming and Soner\cite{3}, if $V(t, x, s) \in C^{1,2,2}([0,T] \times \mathbb{R} \times \mathbb{R})$, then $V(t, x, s)$ satisfies the following HJB equation

$$\sup_{(\pi_t, \pi_s) \in \Pi} \mathcal{A}^{\pi_t, \pi_s} V(t, x, s) = 0, \ 0 \leq t \leq T, \quad (3)$$

with the boundary condition $V(T, x, s) = U(x)$. $\mathcal{A}^{\pi_t, \pi_s}$ is the generator of the wealth process controlled by $\Pi$. Denote

$$\mathcal{A}^{\pi_t, \pi_s} V(t, x, s) = V_t + \left\{ [(\theta - \eta) + (1 + \eta) \pi_q] \lambda_N \mu_Y + r x + \pi_s \left( \frac{b(s)}{s} - r \right) \right\} V_x$$

$$+ b(s) V_s + \frac{1}{2} \left( \beta_0^2 + \sigma^2(s) \frac{\pi_s^2}{s^2} + 2 \rho \beta_0 \sigma(s) \pi_s \right) V_{xx}$$

$$+ \frac{1}{2} \sigma^2(s) V_{ss} + \left( \sigma^2(s) \frac{\pi_s}{s} + \rho \sigma(s) \beta_0 \right) V_{sx}$$

$$+ \lambda_N E [V(t, x - \pi_q Y, s) - V(t, x, s)], \quad (4)$$

where $V_t, V_s, V_x, V_{xx}, V_{ss}$ represent the partial derivatives of the value function. According to (4), (3) can be rewritten as

$$\sup_{\pi_s \in \Pi} \left\{ V_t + \left\{ [(\theta - \eta) + (1 + \eta) \pi_q] \lambda_N \mu_Y + r x + \pi_s \left( \frac{b(s)}{s} - r \right) \right\} V_x$$

$$+ b(s) V_s + \frac{1}{2} \left( \beta_0^2 + \sigma^2(s) \frac{\pi_s^2}{s^2} + 2 \rho \beta_0 \sigma(s) \pi_s \right) V_{xx}$$

$$+ \frac{1}{2} \sigma^2(s) V_{ss} + \left( \sigma^2(s) \frac{\pi_s}{s} + \rho \sigma(s) \beta_0 \right) V_{sx}$$

$$+ \lambda_N E [V(t, x - \pi_q Y, s) - V(t, x, s)] \right\} = 0.$$  

Rearranging the expression above, we have

$$V_t + ((\theta - \eta) \lambda_N \mu_Y + r x) V_x + b(s) V_s + \frac{1}{2} \beta_0^2 V_{xx} + \rho \sigma(s) \beta_0 V_{xx} + \frac{1}{2} \sigma^2(s) V_{ss}$$

$$+ \sup_{\pi_s \in \Pi} \left\{ \pi_s \left( \frac{b(s)}{s} - r \right) V_x + \frac{1}{2} \left( \sigma^2(s) \frac{\pi_s^2}{s^2} + 2 \rho \beta_0 \sigma(s) \pi_s \right) V_{xx} + \sigma^2(s) \frac{\pi_s}{s} V_{sx} \right\}$$

$$+ \sup_{\pi_s \in \Pi} \left\{ (1 + \eta) \pi_q \lambda_N \mu_Y V_x + \lambda_N E [V(t, x - \pi_q Y, s) - V(t, x, s)] \right\} = 0. \quad (5)$$

The first order maximizing condition for the optimal investment strategy is

$$\pi^* = -\frac{s \left( b(s) - r s \right) V_x + \rho s \beta_0 \sigma(s) V_{xx} + s \sigma^2(s) V_{sx}}{s \sigma^2(s) V_{xx}}. \quad (6)$$

Substituting (6) into the HJB equation (5), after simplification, we get

$$V_t + b(s) V_s + r x V_x + \rho \sigma(s) \beta_0 V_{xx} + \frac{1}{2} \sigma^2(s) V_{ss} + \frac{1}{2} \beta_0^2 V_{xx}$$

$$- \frac{\left( \sigma^2(s) V_{xx} + \rho \beta_0 \sigma(s) V_{xx} + (b(s) - r s) V_x \right)^2}{2 \sigma^2(s) V_{xx}} + (\theta - \eta) \lambda_N \mu_Y V_x$$

$$+ \sup_{\pi_s \in \Pi} \left\{ (1 + \eta) \pi_q \lambda_N \mu_Y V_x + \lambda_N E [V(t, x - \pi_q Y, s) - V(t, x, s)] \right\} = 0. \quad (7)$$

We try to conjecture a solution of HJB equation (7) in the following form

$$\begin{cases} V(t, x, s) = -\frac{1}{\gamma} \exp\{-\gamma x e^{(T-t)} + u(t, s)}; \\
V(T, x, s) = -\frac{1}{\gamma} e^{-\gamma x}. \end{cases}$$
A direct calculation yields
\[
\begin{align*}
V_t &= (\gamma_x t e^{r(T-t)} + u_t)V, \\
V_x &= -\gamma e^{r(T-t)}V, \\
V_s &= u_s V, \\
V_{xx} &= \gamma^2 e^{2r(T-t)}V, \\
V_{ss} &= (u_s^2 + u_{ss})V, \\
V_{xx} &= -\gamma e^{r(T-t)}u_s V, \\
E[V(t, x - \pi_q Y, s) - V(t, x, s)] &= \left(\phi(\pi_q e^{r(T-t)}) - 1\right) V,
\end{align*}
\]
where \(\phi(\tau) := E[e^{rT}]\). Plugging the above derivatives into (7) and after simplification, it yields
\[
\begin{align*}
\left\{ u_t + \frac{1}{2} \sigma^2(s) u_s + rs u_s - \frac{(b(s) - rs) - \rho \beta \sigma(s) \gamma e^{r(T-t)})^2}{2 \sigma^2(s)} \right\} V + \sup_{\pi_q \in \Pi} g(t, \pi_q) V = 0, \\
- (\theta - \eta) \lambda N \mu_Y \gamma e^{r(T-t)} + \frac{1}{2} \beta^2 \gamma e^{2r(T-t)} V,
\end{align*}
\]
where \(g(t, \pi_q) = -(1 + \eta) \pi_q \lambda N \gamma e^{r(T-t)} + \lambda N (\phi(\pi_q e^{r(T-t)}) - 1)\). Consider \(h(t, \pi_q) := g(t, \pi_q)V\), it’s easy to check
\[
\begin{align*}
\frac{\partial h(t, \pi_q)}{\partial \pi_q} &= \left\{ -(1 + \eta) \lambda N \mu_Y \gamma e^{r(T-t)} + \lambda N \gamma e^{r(T-t)} E\left[Y e^{\pi_q e^{r(T-t)}Y}\right]\right\} V, \\
\frac{\partial^2 h(t, \pi_q)}{\partial \pi_q^2} &= \left\{ \lambda N \gamma^2 e^{2r(T-t)} E\left[Y^2 e^{\pi_q e^{r(T-t)}Y}\right]\right\} V < 0.
\end{align*}
\]
From (9), \(h(t, \pi_q)\) is concave in \(\pi_q\), there is a maximizer \(\pi_q\) satisfying the equation
\[
-(1 + \eta) \mu_Y + \phi'(n) = 0, \quad (10)
\]
where \(n = \pi_q e^{r(T-t)}\). According to Liang et al.\([10]\), we have the following result.

**Lemma 3.1.** Eq.\((10)\) admits a unique positive root \(\omega\), which only depends on the safety loading \(\eta\) and the claim sizes distribution \(F(y)\).

From the lemma 3.1, we get \(\gamma \pi_q(t)e^{r(T-t)} = \omega\), i.e., \(\pi_q(t) = \frac{\omega}{\gamma}e^{-r(T-t)}\). Due to the retention level \(\pi_q(t) \in [0,1]\), we discuss the optimal reinsurance strategy in the following three cases:

**Case 1.** If \(\pi_q(t) = \frac{\omega}{\gamma}e^{-r(T-t)} \leq 1\) for any \(t \in [0, T]\), then the optimal reinsurance strategy is \(\pi^*_q(t) = \pi_q(t), 0 \leq t \leq T\).

**Case 2.** Let \(t_0 = T + \frac{1}{r} \ln \frac{\omega}{\gamma}\), if \(\pi_q(t) = \frac{\omega}{\gamma}e^{-r(T-t)} < 1\) for \(t \in [0, t_0]\) and \(\pi_q(t) = \frac{\omega}{\gamma}e^{-r(T-t)} \geq 1\) for \(t \in [t_0, T]\), then the optimal reinsurance strategy is \(\pi^*_q(t) = \left\{ \begin{array}{ll}
\frac{\omega}{\gamma}e^{-r(T-t)}, & 0 \leq t < t_0, \\
1, & t_0 \leq t \leq T.
\end{array} \right\}
\]

**Case 3.** If \(\pi_q(t) = \frac{\omega}{\gamma}e^{-r(T-t)} > 1\) for any \(t \in [0, T]\), i.e., \(\frac{\omega}{\gamma} \geq e^{\gamma T}\), then the optimal reinsurance strategy is \(\pi^*_q(t) = 1, t \in [0, T]\).
Hence, the expression of \( g(t, \pi_q^*) \) should be
\[
g(t, \pi_q^*) = \begin{cases} 
-(1 + \eta)\lambda_N\mu_Y\omega + \lambda_N(\varphi(\omega) - 1), & \pi_q^*(t) = \pi_q(t), \\
-(1 + \eta)\lambda_N\mu_Y m + \lambda_N(\varphi(m) - 1), & \pi_q^*(t) = 1.
\end{cases}
\]
where \( m = \gamma e^{r(T-t)} \). What we do next is to find the optimal investment strategy. Substituting \( \pi_q^*(t) \) into (8), we rewrite (8) as
\[
u_t + \frac{1}{2}\sigma^2(s)u_{ss} + rsu_s - \frac{\left((b(s) - rs) - \rho\beta_0\sigma(s)\gamma e^{r(T-t)}\right)^2}{2\sigma^2(s)}
\]
\[= -(\theta - \eta)\lambda_N\mu_Y\gamma e^{r(T-t)} + \frac{1}{2}\rho^2\gamma^2 e^{2r(T-t)} + g(t, \pi_q^*),
\]
with the initial condition \( u(T, s) = 0 \). By now, the stochastic control problem has been transformed into a linear PDE. Set
\[
\mathcal{A} := \frac{1}{2}\sigma^2(s) \triangle + rs\nabla,
\]
\[
f(t, s) = -\frac{\left((b(s) - rs) - \rho\beta_0\sigma(s)\gamma e^{r(T-t)}\right)^2}{2\sigma^2(s)} - (\theta - \eta)\lambda_N\mu_Y\gamma e^{r(T-t)}
\]
\[+ \frac{1}{2}\rho^2\gamma^2 e^{2r(T-t)} + g(t, \pi_q^*). \tag{11}
\]
Let \( u(t, s) = \tilde{u}(T-t, s) \), then \( \tilde{u}(t, s) \) satisfies
\[
\tilde{u}_t + \mathcal{A}\tilde{u}(t, s) + f(T-t, s) = 0
\]
with \( \tilde{u}(0, s) = 0 \). Denote \( f(T-t, s) = \tilde{f}(t, s) \), by Duhamel’s principle, \( \tilde{u}(t, s) \) could be represented in terms of \( M_f(t, s) \) as follows
\[
\tilde{u}(t, s) = \int_0^t M_{f_s}(t - z, s)dz, \tag{12}
\]
where \( M_{f_s}(t, s) \) satisfies
\[
\begin{cases}
M_t = AM, \\
M(0, s) = \tilde{f}(z, s).
\end{cases} \tag{13}
\]
For the brief proof of Duhamel’s principle, see the Appendix. Thus, we have
\[
u(t, s) = \int_0^{T-t} M_{f_s}(T-t-z, s)dz.
\]
By now, the optimal problem is reduced to solving the PDE (13). In fact, by Feynman-Kac formula, \( M_{f_s}(t, s) \) admits the following stochastic representation
\[
M_{f_s}(t, s) = \mathbb{E}_s[\tilde{f}(z, S_t^A)] = \mathbb{E}_s[f(T-z, S_t^A)], \tag{14}
\]
Recalling the definitions (11), we have
\[
f(t, s) = -\frac{1}{2}\left(\frac{b(s) - rs}{\sigma(s)}\right)^2 + \rho\beta_0\gamma e^{r(T-t)} \left(\frac{b(s) - rs}{\sigma(s)}\right)
\]
\[= -(\theta - \eta)\lambda_N\mu_Y\gamma e^{r(T-t)} + \frac{1}{2}\rho^2\gamma^2(1 - \rho^2)e^{2r(T-t)} + g(t, \pi_q^*).\]
If letting $h(s) = \frac{b(s) - rs}{\sigma(s)}$, then
\[
f(T - z, s) = -\frac{1}{2}h^2(s) + \rho \beta_0 \gamma e^{rT} h(s) - (\theta - \eta) \lambda_N \mu \gamma e^{rT} + \frac{1}{2} \beta_0^2 \gamma^2 (1 - \rho^2) e^{2rT} + g(T - z, \pi_\theta^*(T - z)).
\]
Using (14),
\[
M_{f_z}(t, s) = -\frac{1}{2} E_s[h^2(S^i_t)] + \rho \beta_0 \gamma e^{rT} E_s[h(S^i_t)] - (\theta - \eta) \lambda_N \mu \gamma e^{rT}
+ \frac{1}{2} \beta_0^2 \gamma^2 (1 - \rho^2) e^{2rT} + g(T - z, \pi_\theta^*(T - z)),
\]
which immediately leads to the solution of $u(t, s)$,
\[
u(t, s) = \int_0^{T-t} M_{f_z}(T - t - z, s) dz
= \int_0^{T-t} \left\{ -\frac{1}{2} E_s[h^2(S^i_z)] + \rho \beta_0 \gamma e^{r(T-t-z)} E_s[h(S^i_z)] \right\} dz
\]
\[-\frac{1}{r}(\theta - \eta) \lambda_N \mu \gamma \left( e^{r(T-t)} - 1 \right) \frac{1}{4r} \beta_0^2 (1 - \rho^2) \gamma^2 \left( e^{2r(T-t)} - 1 \right)
+ \int_t^T g(z, \pi_\theta^*(z)) dz.
\]
That means, as long as $E_s[h(S^i_z)]$ and $E_s[h^2(S^i_z)]$ are known, substituting (16) into $V(t, x, s)$ and $\pi_\theta^*$, we will get the optimal value function and asset allocation strategies. In fact, the first and second order moments of $h(S^i_z)$ are available for almost all those risk asset models widely used in applications. In the following section, we discuss several important examples, and give the explicit closed solutions for $V(t, x, s)$ and $\pi_\theta^*$ in each case.

4. Examples of risk asset models.

4.1. CEV model. Assume the risk asset price follows a CEV process with drift $b(s) = \kappa s$ and diffusion coefficient $\sigma(s) = \delta s^{\beta+1}$, i.e.,
\[
dS_t = S_t(\delta dt + \delta S_t^\beta dW_t),
\]
where $\kappa, \delta, \beta$ are constants. The CEV model is capable of reproducing the volatility smile observed in the empirical data and hence extensively used in finance field. Moreover, CEV specification nests some important models like the Ornstein-Uhlenbeck process ($\beta = 1$), the CIR process ($\beta = \frac{1}{2}$) and the geometric Brownian motion ($\beta = 0$). In this subsection, we exclude the simple case $\beta = 0$, and assume $\beta < 0$.

In this case, the infinitesimal generators of $S^A_t$ is
\[
\mathcal{A} = \frac{1}{2} \delta^2 s^{2\beta+2} \triangle + rs \nabla,
\]
and the according function $h(s) = \frac{\delta - \beta}{\delta} s^{-\beta}$. So we need to compute $E_s[(S^A_t)^{-2\beta}]$ and $E_s[(S^A_t)^{-\beta}]$. Setting $Y_t = \frac{1}{\sqrt{\pi r}} (S^A_t)^{-2\beta}$, by Itô formula, we see $Y_t$ is a CIR process as follows
\[
dY_t = k(\theta - Y_t) dt + \delta \sqrt{Y_t} dW_t,
\]
where \( k = 2r\beta, \theta = \frac{(2\beta+1)\delta^2}{4r\beta} \), the initial value \( y_0 := \frac{1}{4r\beta} s^{-2\beta} \). For a CIR process, the first order moment is obvious,

\[
E_{y_0}[Y_t] = y_0 e^{-kt} + \theta(1 - e^{-kt}),
\]

then

\[
E_s[(S_t^{A})^{-2\beta}] = 4\beta^2 E[Y_t|Y_0] = \frac{1}{4\beta^2} s^{-2\beta}
\]

\[
= s^{-2\beta}e^{-2r\beta t} + \frac{(2\beta + 1)\delta^2}{2r} (1 - e^{-2r\beta t}). \tag{17}
\]

As for \( E_s[(S_t^{A})^{-\beta}] \), it suffices to derive the \( \frac{1}{2} \) order moment of \( Y_t \) since \( E_s[(S_t^{A})^{-\beta}] = 2|\beta|E_{y_0}[\sqrt{Y_t}] \). On the one hand, we have

\[
E_{y_0}[\sqrt{Y_t}] = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1 - E_{y_0}[e^{-\xi Y_t}]}{\xi^2} d\xi. \tag{18}
\]

See [9], Section 1.1.12. On the other hand, \( E[e^{-\xi Y_t}|Y_0] = \frac{1}{4\beta^2} s^{-2\beta} \) equals to

\[
\left( \frac{4r\beta}{\xi\delta^2 (1 - e^{-2r\beta t}) + 4r\beta} \right)^{\frac{3}{2}} \exp \left( -\frac{r s^{-2\beta} \xi e^{-2r\beta t}}{\beta\xi\delta^2 (1 - e^{-2r\beta t}) + 4r\beta^2} \right). \tag{19}
\]

See [9], Corollary 6.3.4.4. Inserting (19) into (18), we get

\[
E[\sqrt{Y_t}|Y_0] = \frac{1}{4\beta^2} s^{-2\beta} = \frac{1}{2\sqrt{\pi}} \int_0^\infty w(\xi, s, t) d\xi \tag{20}
\]

with

\[
w(\xi, s, t) = \frac{1}{\xi^2} \left( \frac{4r\beta}{\xi\delta^2 (1 - e^{-2r\beta t}) + 4r\beta} \right)^{\frac{3}{2}} \exp \left( -\frac{r s^{-2\beta} \xi e^{-2r\beta t}}{\beta\xi\delta^2 (1 - e^{-2r\beta t}) + 4r\beta^2} \right).
\]

Apply (17) and (20) to (15) and (16), we obtain that

\[
M_{f_\beta}(t, s) = -\frac{(\kappa - r)^2}{2\delta^2} \left[ s^{-2\beta}e^{-2r\beta t} + \frac{(2\beta + 1)\delta^2}{2r} (1 - e^{-2r\beta t}) \right]
\]

\[
-\rho^2 s^{-2\beta} \xi e^{-2r\beta t} \int_0^\infty w(\xi, s, t)d\xi - (\theta - \eta)\lambda_{N\mu_Y} \gamma e^{r z}
\]

\[
+ \frac{1}{2} \rho^2 (1 - \rho^2) \gamma^2 e^{2r z} + g(T - z, \pi^*_q(T - z)),
\]

and

\[
u(t, s) = -\frac{(\kappa - r)^2}{4r\beta \delta^2} \left[ s^{-2\beta} \left( 1 - e^{-2r\beta(T-t)} \right) + \frac{(2\beta + 1)\delta^2}{2r} \left( 2r\beta(T-t) \right) \right]
\]

\[
- \left( 1 - e^{-2r\beta(T-t)} \right) \right] - \rho \frac{\beta_0(\kappa - r)\gamma}{\delta^2} \int_0^{T-t} \int_0^{e^{r(T-t-z)}} e^{r(T-t-z)} w(\xi, s, t)d\xi dz
\]

\[
- \frac{1}{r} (\theta - \eta)\lambda_{N\mu_Y} \gamma (e^{r(T-t) - 1}) + \frac{1}{4r^2} \beta_0(1 - \rho^2) \gamma^2 (e^{2r(T-t) - 1})
\]

\[
+ \int_t^T g(z, \pi^*_q(z)) dz. \tag{21}
\]

Therefore, the optimal value function and strategies in the correlated insurance and financial markets under a CEV risk asset model could be summarized as follows.
Theorem 4.1. Assume that the insurer’s surplus and the risk asset price are linear correlated. In the CEV risk asset model case, the optimal value function under the exponential utility function is

\[ V(t,x,s) = -\frac{1}{\gamma} \exp\{-\gamma xe^{r(T-t)} + u(t,s)\}. \]

The optimal investment strategy is given by

\[
\pi^*_s(t) = \frac{\kappa - r}{\delta^2 s^2 \gamma e^{r(T-t)}} + \frac{su(t,s)}{\gamma e^{r(T-t)}} \frac{\rho \beta_0}{\delta s^2} \\
= \frac{\kappa - r}{\delta^2 s^2 \gamma e^{r(T-t)}} + \frac{(\kappa - r)^2}{2r \delta^2 s^2 \gamma e^{r(T-t)}} \left(1 - e^{-2r \beta(T-t)}\right) \\
+ \frac{2r \rho \beta_0 \beta^2 (\kappa - r)}{\delta \sqrt{\pi s^2}} \int_0^{T-t} \int_0^\infty w_s(\xi, s, t) d\xi dz - \frac{\rho \beta_0}{\delta s^2},
\]  

(22)

where \( u(t,s) \) is given by (21) and \( u_s(t,s), w_s(\xi,s,t) \) are the partial derivatives of \( u(t,s) \) and \( w(\xi,s,t) \) in \( s \). The optimal reinsurance strategies of the insurer are as follows:

(1) If \( \omega \leq \gamma \), the optimal reinsurance strategy is

\[
\pi^*_q(t) = \pi_q(t), 0 \leq t \leq T.
\]

(2) If \( \gamma < \omega < \gamma e^{rT} \), let \( t_0 = T + \frac{1}{r} \ln \frac{s}{\omega} \), the optimal reinsurance strategy is

\[
\pi^*_q(t) = \begin{cases} \frac{\omega}{\gamma} e^{-r(T-t)}, & 0 \leq t < t_0, \\ 1, & t_0 \leq t \leq T. \end{cases}
\]  

(23)

(3) If \( \omega \geq \gamma e^{rT} \), the optimal reinsurance strategy is

\[
\pi^*_q(t) = 1, t \in [0,T].
\]

Remark 1. when \( \beta = -1 \), the risk asset price follows

\[ dS_t = \kappa S_t dt + \delta S_t^3 dW_t, \]

which is a special case of CEV model. In this situation, \( A = \frac{1}{2} \delta^2 \triangle + r s \nabla \) and

\[
\begin{align*}
E[S^4_t] &= e^{rt} s, \\
E[(S^4_t)^2] &= e^{2rt} s^2 + \frac{\delta^2}{2r} e^{2rt} + \frac{s^2}{2r}.
\end{align*}
\]

We have a simple expression for the optimal investment strategy, i.e.,

\[
\pi^*_s(t) = \frac{(\kappa - r)s^2 e^{r(T-t)}}{\gamma \delta^2} \left[1 + \frac{\kappa - r}{2r} (1 - e^{2r(T-t)})\right] + \frac{s \beta_0 \rho (\kappa - r)(T-t)}{\delta} - \frac{s \beta_0 \rho}{\delta}.
\]

4.2. The 3/2 model. The name of 3/2 model mainly comes from the diffusion term \( \sigma(s) = \delta s^2 \), its drift has more flexibility than CEV process, \( b(s) = \kappa (l - s) s \). The dynamic equation of 3/2 model is

\[ dS_t = \kappa (l - S_t) S_t dt + \delta S_t^3 dW_t. \]
By the definition of $\mathcal{A}$ and $h(s)$, we have

$$
\mathcal{A} = \frac{1}{2} s^2 \triangle + rs \nabla,
$$

$$
h(s) = (\frac{\kappa l - r}{\delta}) - \kappa s^2 = \frac{\kappa l - r}{\delta} s^{-\frac{1}{2}} - \frac{\kappa}{\delta} s^{\frac{3}{2}},
$$

$$
h^2(s) = \left(\frac{\kappa l - r}{\delta}\right)^2 s^{-1} + \frac{\kappa^2}{\delta^2} s - \frac{2\kappa(\kappa l - r)}{\delta^2}.
$$

In order to compute the expectations $E_s[h(S^A)]$ and $E_s[h^2(S^A)]$, we need to know $E_s[S^A]^{-\frac{1}{2}}, E_s[S^A]\frac{1}{2}, E_s[S^A]^{-1}$ and $E_s[S^A]$.

The expectation of $S^A$ equipped with the infinitesimal generator $\mathcal{A}$ is easy to work out, meanwhile, $S^A$ is the reciprocal of the CIR process $Y_t$ with the infinitesimal generator $\frac{1}{2} \delta^2 s^{-1} \triangle - (r s^{-1} + \delta^2) \nabla$. Hence, we have

$$
E_s[S^A] = e^{rz},
$$

$$
E_s[(S^A)^{-\frac{1}{2}}] = E_s[Y_t] = \frac{\delta^2}{r} + \left(\frac{1}{s} - \frac{\delta^2}{r}\right) e^{-rz}.
$$

For the $\frac{1}{2}$ and $-\frac{1}{2}$ order moment of $S^A$, we refer to [9], Corollary 6.3.4.4 and Section 1.1.12, then we have

$$
E_s[(S^A)^{\frac{1}{2}}] = E_s[Y_t]^{-\frac{1}{2}}
$$

$$
= \frac{1}{\sqrt{\pi}} \int_0^\infty \left\{ \left( \frac{\kappa l - r}{\delta^2} + 2r \right) T \exp \left( -\frac{s(2r\xi e^{-rz} - 2r\xi)\kappa l - r}{\delta^2} + 2r \right) \right\} d\xi,
$$

$$
E_s[(S^A)^{-\frac{1}{2}}] = E_s[Y_t]^{\frac{1}{2}}
$$

$$
= \frac{1}{2\sqrt{\pi}} \int_0^\infty \left\{ \left( \frac{\kappa l - r}{\delta^2} + 2r \right) T \exp \left( -\frac{s(2r\xi e^{-rz} - 2r\xi)\kappa l - r}{\delta^2} + 2r \right) \right\} d\xi.
$$

Inserting all the expressions of $E_s[S^A]^{-\frac{1}{2}}, E_s[S^A]^{\frac{1}{2}}, E_s[S^A]^{-1}$ and $E_s[S^A]$ into (16), we get

$$
u(t, s) =
$$

$$
\frac{\rho \beta_0 \gamma \kappa l - r}{\delta \sqrt{\pi}} \int_0^T \int_0^\infty \left\{ \left( \frac{\kappa l - r}{\delta^2} + 2r \right) T \exp \left( -\frac{s(2r\xi e^{-rz} - 2r\xi)\kappa l - r}{\delta^2} + 2r \right) \right\} d\xi dz
$$

$$
- \frac{\rho \beta_0 \gamma \kappa l - r}{2\delta \sqrt{\pi}} \int_0^T \int_0^\infty \left\{ \left( \frac{\kappa l - r}{\delta^2} + 2r \right) T \exp \left( -\frac{s(2r\xi e^{-rz} - 2r\xi)\kappa l - r}{\delta^2} + 2r \right) \right\} d\xi dz
$$

$$
+ \frac{2r\kappa l - r}{2r^2} \left( \frac{\kappa l - r}{\delta^2} \right)^2 (T - t) + \frac{\kappa l - r}{2r^2\delta^2 s} \left( e^{-r(T - t) - 1} \right)
$$

$$
- \frac{\kappa^2}{2r\delta^2 \left( e^{(T - t) - 1} \right) - \frac{1}{r} (\theta - \eta) \lambda_N \eta \gamma (e^{(T - t) - 1})}
$$

$$
+ \frac{1}{4r} \beta_0^2 (1 - \rho^2) \gamma^2 (e^{2r(T - t) - 1}) + \int_t^T g(z, \pi_q(z)) dz.
$$

(24)
Theorem 4.2. Assume that the insurer’s surplus and the risk asset price are linear correlated. In the 3/2 risk asset model case, the optimal value function \( V(t, x, s) \) and the optimal investment strategy \( \pi^*_s(t) \) are similar to Theorem 4.1 except for replacing \( u(t, s) \) and \( u_s(t, s) \) therein by (24) and its derivative.

4.3. Merton’s IR model. Merton’s IR model is another non-affine stochastic model with \( b(s) = \kappa(l - s)s \) and \( \sigma(s) = \delta s \). Assume the risk asset model follows Merton’s IR model, according the definition of the infinitesimal generator \( A = \frac{1}{2} \delta^2 s^2 \Delta + rs \nabla \). The corresponding process \( S_t^A \) has an exponential form

\[
S_t^A = s \exp \left\{ (r - \frac{\delta^2}{2}) t + \delta W_t \right\}.
\]

By the definition of \( h(s) \),

\[
E_s[h(S_t^A)] = \frac{\kappa l - r}{\delta} - \frac{\kappa}{\delta} E_s[S_t^A] = \frac{\kappa l - r}{\delta} - \frac{\kappa s}{\delta} e^{rz},
\]

\[
E_s[h^2(S_t^A)] = \left( \frac{\kappa l - r}{\delta} \right)^2 + \frac{\kappa^2 s^2}{\delta^2} E_s[(S_t^A)^2] - \frac{2k(\kappa l - r)}{\delta^2} E_s[S_t^A]
\]

\[
= \left( \frac{\kappa l - r}{\delta} \right)^2 + \frac{\kappa^2 s^2}{\delta^2} e^{(2r + \delta^2)z} - \frac{2k(\kappa l - r)}{\delta^2} se^{rz}.
\]

Put them into (16), we have

\[
u(t, s) = - \left[ \frac{1}{2} \left( \frac{\kappa l - r}{\delta} \right)^2 + \frac{\kappa s \rho \beta_0 \gamma e^{(T - t)}}{r^2} \right] (T - t) + \frac{k^2 s^2}{2\delta^2 (2r + \delta^2)} \left( 1 - e^{(2r + \delta^2)(T - t)} \right)
\]

\[
+ \frac{\rho \beta_0 \gamma (\kappa l - r) + \kappa s (\kappa l - r)}{r^2 \delta^2} e^{r(T - t)} - 1 - \frac{1}{r} (\theta - \eta) \lambda N \mu Y \gamma \left( e^{r(T - t)} - 1 \right)
\]

\[
+ \frac{1}{4r} \beta_0^2 (1 - \rho^2) \gamma^2 \left( e^{2r(T - t)} - 1 \right) + \int_t^T g(z; \pi^*_q(z)) \text{dz}.
\]

Theorem 4.3. Assume that the insurer’s surplus and the risk asset price are linear correlated. In the Merton’s IR risk asset model case, the optimal value function \( V(t, x, s) \) and the optimal investment strategy \( \pi^*_s(t) \) are similar to Theorem 4.1 except for replacing \( u(t, s) \) and \( u_s(t, s) \) therein by (25) and its derivative.

5. Numerical analysis. In this section, we provide some numerical examples to analyze the parameters sensitivity of the strategies obtained from the CEV risk model. Here we assume that the claims \( \{ Y_t \} \) are independent and exponentially distributed with the parameter \( 1/\mu_Y \). Throughout the numerical analysis, unless otherwise stated, the parameters are chosen as follows:

\[
\mu_Y = 1, \beta_0 = 0.4, T = 10, r = 0.3, \kappa = 0.4,
\]

\[
\beta = -0.5, s = 5, \gamma = 0.2, \delta = 1, \rho = \pm 1.
\]

According to (23), we see \( \frac{\partial \pi^*_s(t)}{\partial \gamma} \leq 0 \), i.e., the optimal reinsurance strategy \( \pi^*_s(t) \) is decreasing with \( \gamma \). That means, the more risk aversion, the less aggressive the insurer behaves. In this situation, the insurer prefers more reinsurance and less new business to manage risk exposure.

From (22), we study the effect of the correlation \( \rho \) on the optimal investment strategies. If \( \rho > 0 \), the surplus and risk asset’s price are positively correlated, \( \pi^*_s(t) \)
The optimal investment strategy 

\[ \pi_s(t) \]

increases with \( \rho \) when 

\[ \frac{2r}{s} \frac{\beta_0 \beta^2 (\kappa - r)}{\sqrt{s^2}} > \frac{\beta_0}{s} \]

and decreases with \( \rho \) when 

\[ \frac{2r}{s} \frac{\beta_0 \beta^2 (\kappa - r)}{\sqrt{s^2}} < \frac{\beta_0}{s} \].

If \( \rho < 0 \), the surplus and risk asset’s price are negatively correlated, \( \rho \) acts an exactly opposite effect on \( \pi_s(t) \) when 

\[ \frac{2r}{s} \frac{\beta_0 \beta^2 (\kappa - r)}{\sqrt{s^2}} > \frac{\beta_0}{s} \]

and 

\[ \frac{2r}{s} \frac{\beta_0 \beta^2 (\kappa - r)}{\sqrt{s^2}} < \frac{\beta_0}{s} \].

In Figure 1, we first set \( \rho = 1 \), vary \( \beta \) from \(-0.4\), \(-0.5\) to \(-0.6\), then plot the optimal investment strategy \( \pi_s^*(t) \) against \( t \). Figure 1 shows \( \pi_s^*(t) \) is increasing with \( t \), however, a large \( \beta \) does not necessarily mean the increasing or decreasing of investment.

Figure 2 and Figure 3 respectively demonstrate the sensitivity of \( \pi_s^*(t) \) to the parameter \( \beta_0 \) in the condition \( \rho = 1 \) and \( \rho = -1 \). In both cases, \( \beta_0 \) varies from 0, 0.2 to 0.4. We see from Figure 2 that \( \pi_s^*(t) \) decreases with \( \beta_0 \), whereas in Figure 3, \( \pi_s^*(t) \) increases with \( \beta_0 \). This phenomenon could be explained as follows. When the correlation parameter \( \rho \) is positive, a large value of \( \beta_0 \) nests more financial risk from the risk asset, the insurer is willing to invest less on the risk asset. However, when \( \rho \) is negative, the situation just reversed, a large value of \( \beta_0 \) means more profits from investment, hence the insurer prefers to invest more in the risk asset.

With \( \beta = -0.5 \), \( \rho = 1 \), \( s = 5 \), Figure 4 plots \( \pi_s^*(t) \) against \( t \) for \( \kappa = 0.35, 0.45, 0.55 \). \( \pi_s^*(t) \) is an increasing function of \( t \) in each case. We also find at first the optimal investment portion \( \pi_s^*(t) \) becomes less with the increases of \( \kappa \). but as the time elapses,
the situation gradually changes to the opposite. That is to say, a large expected instantaneous rate of return \( \kappa \) does not necessarily imply more or less investment.

![Graph showing the optimal investment strategy](image)

**Figure 5.** Effect of \( \delta \) on the optimal strategy \( \pi^*_s(t) \)

With the same parameter \( \beta = -0.5 \), \( \rho = 1 \), \( s = 5 \), Figure 5 shows \( \pi^*_s(t) \) against \( t \) for \( \delta = 0.8, 0.9, 1 \). Similar to Figure 4, we see that the increasing of \( \delta \) does not necessarily lead to the rise or fall of the optimal investment portion.

**Appendix.**

**Proof of (12).** According to (13), we derive

\[
\tilde{u}_t(t, s) = \frac{\partial}{\partial t} \int_0^t M \dot{f}_z(t - z, s)dz
\]

\[
= M \dot{f}_z(0, s) + \int_0^t \frac{\partial}{\partial t} M \dot{f}_z(t - z, s)dz
\]

\[
= \ddot{f}(z, s) + \int_0^t A M \dot{f}_z(t - z, s)dz
\]

\[
= f(T - t, s) + A \int_0^t M \dot{f}_z(t - z, s)dz
\]

\[
= f(T - t, s) + A \tilde{u}(t, s).
\]

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