Quantum Groups, $q$-Dynamics and Raja ji

R. P. Malik

S. N. Bose National Centre for Basic Sciences,
Block-JD, Sector-III, Salt Lake, Calcutta-700 098, India

Abstract: We sketch briefly the essentials of the quantum groups and their application to the dynamics of a $q$-deformed simple harmonic oscillator moving on a quantum line, defined in the $q$-deformed cotangent (momentum phase) space. In this endeavour, the quantum group $GL_{qp}(2)$- and the conventional rotational invariances are respected together. During the course of this discussion, we touch upon Raja ji’s personality as a critical physicist and a bold and adventurous man of mathematical physics.

The basic idea behind the concept of “deformation” in theoretical physics is quite old one. In fact, the two most successful and well-tested theories of 20th century, namely; the quantum mechanics and the special theory of relativity, can be thought of as the “deformed” versions of their “undeformed” counterparts: the classical mechanics and the Galilean relativity. The deformation parameters in these theories are supposed to be the Planck constant ($\hbar$) and the speed of light ($c$) \cite{1,2} respectively (which turn out to be the two fundamental constants of nature). In the limit when $\hbar \to 0$ and $c \to \infty$, we get back the corresponding “undeformed” physical theories. Long time ago, it was proposed that space-time might become noncommutative \cite{3-5} if we probe the deeper structure of matter with energies much higher than the typical scale of energy for quantum mechanics. Nearly a couple of decades ago, this idea got a shot in its arms in the context of inverse scattering method (and Yang-Baxter equations) applied to the integrable systems \cite{6} and it was conjectured that the deformation of groups based on the quasi-triangular Hopf algebras \cite{7} together with the ideas of noncommutative geometry \cite{8} might provide a “fundamental length” ($l_p$) in the context of space-time quantization. This will complete the trio (i.e. $\hbar, c, l_p$) of fundamental constants of nature and will, thereby, enable us to express physical quantities in terms of these natural units. Recently, there has been an upsurge of interest in the noncommutative spaces \cite{8,9} in the context of branes in string theory and matrix model of M-theory. However, we shall discuss here some aspects of noncommutativity associated with the space-time structure in the framework of quantum groups alone and will not touch upon the noncommutativity associated with the string/M-theory.

Let us begin with a $q$-deformation (where $q$ is a dimensionless quantity), introduced as a noncommutativity parameter for the spacetime coordinates in the $D$- dimensional flat
Minkowski (configuration) manifold, as

\[ x_\mu x_\nu = q \ x_\nu x_\mu, \quad (\mu, \nu = 0, 1, 2, \ldots, D - 1). \]  

(1)

It can be checked that (1) is invariant under the Lorentz boost transformations iff \( \mu < \nu \). Moreover, if we reduce (1) to a two dimensional “quantum plane” in space (i.e. \( \mu = 1, \nu = 2 \))

\[ x y = q y x, \]

(2)

we see that the conventional rotational invariance for a two dimensional “undeformed” plane is violated. In some sense, the homogeneity and isotropy of space-time becomes questionable because of the loss of these two conventional invariances. In the limit \( q \to 1 \), the “quantum plane” reduces to an ordinary plane with its rotational symmetry intact.

It was a challenging problem to develop a consistent \( q \)-deformed dynamics where conventional invariances were respected. In this context, the Lagrangian and Hamiltonian formulation of a \( q \)-deformed dynamics was considered in the tangent and cotangent spaces, defined over 2D \( q \)-deformed configuration space (corresponding to the definition (2)) [10]. In this approach, however, the conventional rotational invariance was lost and the status of a one-dimensional physical system was not clear. On the positive side of this approach, a rigorous \( GL_{qp}(2) \) invariant differential calculus was developed and then it was applied to the construction of a consistent \( q \)-dynamics. In another interesting attempt, a \( q \)-deformation was introduced in the Heisenberg algebra [11]. As a result, it was impossible to maintain the Hermiticity property of the phase variables together. This led to the introduction of a new coordinate variable in the algebra. Consequently, a single point particle was forced to move on two trajectories at a given value of the evolution parameter for \( q \neq 1 \) (which was not found to be a physically interesting feature). In an altogether different approach, a \( q \)-deformation was introduced in the cotangent (momentum phase) space defined over a one-dimensional configuration manifold [12]. In this endeavour, a “quantum-line” was defined in the 2D cotangent manifold as

\[ x(t) \pi(t) = q \pi(t) x(t), \]

(3)

where \( t \) is a commuting real evolution parameter and \( x(t) \) and \( \pi(t) \) are the phase space variables. In relation (3), the conventional rotational invariance is maintained because a rotation does not mix a coordinate with its momentum. However, a rigorous differential calculus was not developed in this approach and dynamics was discussed by exploiting the on-shell conditions alone. It was also required that the solutions to equations of motion should be such that the quantum-line (3) is satisfied for all values of the evolution parameter. This way of deformation was generalized to the multi-dimensional systems [13-15]

\[ x_\mu x_\nu = x_\nu x_\mu, \quad \pi_\mu \pi_\nu = q \pi_\nu \pi_\mu, \quad x_\mu \pi_\nu = q \pi_\nu x_\mu, \]

(4)

and the dynamics of (non)relativistic systems was discussed by exploiting the on-shell conditions only. Prof. G. Rajasekaran (popularly known as “Rajaji” in the physics community
of India) is blessed with a very critical mind. Not only he is critical about others’ work, he is self-critical too. In fact, he was very much critical about these relations in Eq. (4) and argued that there must be some quantum group symmetry behind this choice of relations. His criticism spurred our interest in this problem a great deal. As a result, we were able to find that, under the following transformations for the pair(s) of phase variable(s):

$$(x_0, \pi_0), (x_1, \pi_1) \ldots \ldots (x_{D-1}, \pi_{D-1})$$

$$x_\mu \rightarrow A x_\mu + B \pi_\mu,$$
$$\pi_\mu \rightarrow C x_\mu + D \pi_\mu; \quad (5)$$

where $A, B, C, D$ are the elements of a $2 \times 2$ matrix belonging to the quantum group $GL_{qp}(2)$ and obeying the braiding relations in rows and columns (with $q, p \in C/\{0\}$) as:

$$AB = p BA, \quad AC = q CA, \quad BC = (q/p) CB, \quad BD = q DB,$$
$$CD = p DC, \quad AD - DA = (p - q^{-1}) BC = (q - p^{-1}) CB, \quad (6a)$$

the relations (4) remain invariant for any arbitrary ordering of $\mu$ and $\nu$ if the parameters of the group are restricted to obey $pq = 1$. In fact, relations (4) respect the conventional Lorentz invariance as well as the quantum group $GL_{q,q^{-1}}(2)$

$$AB = q^{-1} BA, \quad AC = q CA, \quad BC = q^2 CB, \quad BD = q DB,$$
$$CD = q^{-1} DC, \quad AD = DA, \quad (6b)$$

invariance together if we assume the commutativity of elements $A, B, C, D$ of the above quantum group with the phase variables $x_\mu$ and $\pi_\mu$. It will be noticed that the relationship (6b) has been derived from (6a) for $pq = 1$ (and $GL_{q,q^{-1}}(2) \neq GL_q(2)$). In fact, relations (4) and symmetry transformations (6a,6b) were exploited for the discussion of a consistent $q$-dynamics for some physical systems in the multi-dimensional phase space [16].

To develop a consistent $q$-dynamics in the 2D phase space for a one dimensional simple harmonic oscillator (1D-SHO), we shall exploit the definition of a quantum-line (3) in the phase space. This relationship remains invariant under the conventional rotations as well as the following quantum group symmetry transformations

$$\begin{pmatrix} x \\ \pi \end{pmatrix} \rightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ \pi \end{pmatrix}, \quad (7)$$

where $A, B, C, D$ are the elements of the quantum group $GL_{qp}(2)$ that obey relations (6a). It will be noticed that the quantum-line (3) is also invariant under transformations corresponding to the quantum group $GL_q(2)$ which is endowed with elements $(A, B, C, D)$ that obey relations (6a) for $q = p$ [13]. In fact, both these quantum groups possess identity, inverse, closure property and associativity under the binary operation $(\cdot)$ as the matrix multiplication. However, these quantum groups form what are known as pseudo-groups. To elaborate this point, let us examine the group properties of the simpler group $GL_q(2)$. The identity element $(I)$ of this group can be defined from its typical general element $(T)$ as:

$$T_{ij} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL_q(2), \quad \text{and} \quad T_{ij} \rightarrow \delta_{ij} \equiv I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (8)$$
for \( A = D = 1 \) and \( B = C = 0 \). The determinant of \( T \) (\( \det T \)) turns out to be the central for this group in the sense that it commutes with all the elements \((A, B, C, D)\)

\[
\det T = AD - qBC = DA - q^{-1}CB,
\]

\((\det T) (A, B, C, D) = (A, B, C, D) (\det T)\),

as can be seen from the \( q \)-commutation relations of elements \( A, B, C, D \) belonging to the quantum group \( GL_q(2) \). Now the inverse of \( T \) can be defined as

\[
T^{-1} = \frac{1}{AD - qBC} \begin{pmatrix} D & -q^{-1}B \\ -qC & A \end{pmatrix} \equiv \frac{1}{DA - q^{-1}CB} \begin{pmatrix} D & -q^{-1}B \\ -qC & A \end{pmatrix},
\]

because \( T \cdot T^{-1} = T^{-1} \cdot T = I \). The closure property \((T \cdot T' = T'')\), under matrix multiplication, can be seen by taking two matrices \( T \) and \( T' \in GL_q(2) \) and demonstrating

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = \begin{pmatrix} A'' & B'' \\ C'' & D'' \end{pmatrix} \in GL_q(2),
\]

where it is assumed that elements of the above two matrices, belonging to the quantum group \( GL_q(2) \), commute among themselves (i.e., \([T_{ij}, T_{kl}'] = 0\)). Exploiting the closure property, it can be checked that the associativity property is also satisfied, i.e.,

\[
T \cdot (T' \cdot T'') = (T \cdot T') \cdot T''.
\]

The requirement that \([T_{ij}, T_{kl}'] = 0\), entails upon quantum group (e.g. \( GL_q(2) \)) to be a pseudo-group. This is because of the fact that: (i) the elements of the product \( T^2 = T \cdot T \) do not belong to the quantum group (e.g. \( GL_q(2) \)), and (ii) the condition \([T_{ij}, T_{kl}'] = 0\) is not taken into account while defining the inverse matrix \((T^{-1})\) where the elements of \( T^{-1} \) and \( T \) are taken to be non-commuting in the proof: \( T \cdot T^{-1} = T^{-1} \cdot T = I \).

In our work [16], a consistent \( q \)-dynamics was developed where conventional rotational (and/or Lorentz) symmetry invariance, together with a quantum group symmetry invariance, was maintained. A \( GL_q(2) \) invariant differential calculus, consistent with the Yang-Baxter equations, was developed in 2D (momentum) phase space and then it was applied for the discussion of dynamics of some physical systems. As an example here, we begin with the following Lagrangian for the 1D-SHO [16]

\[
L = P \dot{x}^2 - Q x^2,
\]

where \( \dot{x}, x \) are the velocity and position variables (with \( x \dot{x} = pq \ddot{x}x \)) and \( P \) and \( Q \) are the parameters which are, in general, non-commutative. A general discussion for the least action principle leads to the definition and derivation of the canonical momentum (\( \pi \)) and the Euler-Lagrange equation of motion \( (EOM) \) as

\[
\pi = \frac{1}{p} \frac{\partial L}{\partial \dot{x}}, \quad \ddot{\pi} = \frac{1}{p} \frac{\partial L}{\partial x}.
\]
The differential calculus (with \(dx = \dot{x}dt, d\pi = \dot{\pi}dt\) where \(t\) is a real commuting evolution parameter) leads to the derivation of the following basic relations [16]

\[
x \pi = q \pi x, \quad \dot{x} \dot{\pi} = q \dot{\pi} \dot{x}, \quad \pi \dot{x} = p \dot{x} \pi,
\]

\[
\pi \dot{\pi} = pq \dot{\pi} \pi, \quad x \dot{\pi} = q \pi x + (pq - 1) \dot{x} \pi,
\]

which restrict any arbitrary general Lagrangian \((L(x, \dot{x}))\) to satisfy

\[
x \frac{\partial L}{\partial \dot{x}} = q \frac{\partial L}{\partial x} \dot{x}, \quad \dot{x} \frac{\partial L}{\partial x} = q \frac{\partial L}{\partial \dot{x}} \dot{x}, \quad \pi \frac{\partial L}{\partial \dot{x}} = p \frac{\partial L}{\partial \dot{x}} \pi, \quad x \frac{\partial L}{\partial \dot{x}} = q \frac{\partial L}{\partial \dot{x}} x + (pq - 1) \dot{x} \frac{\partial L}{\partial \dot{x}}.
\]

(15)

We demand that the Lagrangian (13) should satisfy all the basic conditions listed in (16). As it turns out, there are two interesting sectors of dynamics for 1D-SHO, described by (13). These are: (i) when the parameters of deformations are restricted to satisfy \(pq = 1\), and (ii) when \(pq \neq 1\) for the discussion of solutions to EOM. For the former case, consistent with the differential calculus [16], we obtain the following \(q\)-commutation relations

\[
x \dot{x} = \dot{x} x, \quad P Q = Q P, \quad \xi P = q P \xi, \quad \xi Q = q Q \xi,
\]

(17)

where \(\xi\) stands for \(x, \dot{x}\) (i.e. \(\xi = x, \dot{x}\)). Exploiting these relations, we obtain the following EOM for the system under consideration

\[
\ddot{x} = -P^{-1}Q x \equiv -\omega^2 x, \quad (\omega^2 = P^{-1}Q),
\]

(18)

which has its solution, at any arbitrary value of the evolution parameter \(t\), as

\[
x(t) = e^{i\omega t} A + e^{-i\omega t} B,
\]

(19)

where \(A\) and \(B\) are the non-commuting constants which can be fixed in terms of the initial conditions of the dynamics, as given below

\[
A = \frac{1}{2} \left[ x(0) + \omega^{-1}\dot{x}(0) \right], \quad B = \frac{1}{2} \left[ x(0) - \omega^{-1}\dot{x}(0) \right].
\]

(20)

The consistency requirements of \(GL_{qp}(2)\) invariant differential calculus [16] vis-a-vis relations (17) and (20), lead to

\[
AB = BA, \quad B\omega = \omega B, \quad A\omega = \omega A, \quad x\omega = \omega x, \quad \dot{x}\omega = \omega \dot{x},
\]

(21)

Furthermore, it can be checked that all the \(q\)-commutation relations [16] among \(x, \pi, \dot{x}, \dot{\pi}\) are satisfied at any arbitrary value of \(t\). Thus, we have a completely consistent dynamics in a noncommutative phase space for \(pq = 1\). We have paid a price, however. As it has turned out, all the variables (i.e. \(A, B, \omega, x(0), \dot{x}(0)\)), present in the solution \(x(t)\), are commutative in nature like we have in conventional classical mechanics. Thus, in some sense, the nature of this dynamics is trivial. In other words, as far as the evolution of the system is concerned, we do not see the effect of \(q\)-deformation (or noncommutativity) on the dynamics.
This is the point where boldness and adventure of “Rajaji” (as a mathematical physicist) came to the fore. He argued that we must find out a non-trivial solution to the equation of motion where parameters of the deformation are not restricted to unity (i.e, \( pq \neq 1 \)). Thus, let us consider this non-trivial sector of dynamics for 1D-SHO. The Lagrangian (13) satisfies all but one relations in (16) with the following \( q \)-commutation relations

\[
PQ = QP, \quad \xi Q = pq^2 Q \xi, \quad P \xi = p \xi P, \quad (\xi = x, \dot{x}).
\]  

(22)

The problematic relation of (16) (the last one!) forces us to require:

\[
(pq - 1)(Qx^2 - pq \dot{P}x^2) = 0,
\]

(23)

which emerges from the basic condition: \( x\dot{\pi} = q \dot{\pi} x + (pq - 1) \dot{x}\pi \) of Eq. (15). The EOM for the system in this sector (where \( pq \neq 1 \))

\[
\ddot{x} = -\frac{1}{pq^2} P^{-1}Q x \equiv -\omega^2 x, \quad (\omega^2 = \frac{1}{pq^2} P^{-1}Q),
\]

(24)

has the following general solution in terms of constants \( A, B \) and \( \omega \)

\[
x(t) = e^{i\omega t} A + e^{-i\omega t} B.
\]

(25)

However, the restriction (23) is satisfied if and only if: either \( A = 0 \) or \( B = 0 \). With the expression for \( A \) and \( B \), given in (20), this leads to the restriction: \( \dot{x}(0) = \pm \omega x(0) \). Now the general form of the exponential evolution for the 1D-SHO is either

\[
x(t) = e^{i\omega t} A, \quad \text{with} \quad A\omega = pq \omega A,
\]

(26)

or,

\[
x(t) = e^{-i\omega t} B, \quad \text{with} \quad B\omega = pq \omega B.
\]

(27)

As a consequence, the dynamics for \( pq \neq 1 \) does not evolve in the 2D (velocity) phase space but degenerates into a restricted 1D region. In this region, both the restrictions: \( \dot{x}(t) = \pm \omega x(t) \) and \( Q x^2 = pq \dot{P} \dot{x}^2 \) are satisfied for all values of the evolution parameter \( t \). In fact, it can be checked that the evolution of the system, described by Eq. (26) and/or Eq.(27), is such that these conditions are very precisely satisfied.

It will be noticed that, unlike the dynamical sector for \( pq = 1 \), here the constants \( \omega \) and \( A \) (or \( B \)) do not commute with each-other and still there exists a consistent “time” evolution for the system, albeit a restricted one. In fact, as a result of the \( q \)-deformation, the evolution of the 1D-SHO is strictly on a 1D “quantum-line” even-though the whole (velocity) phase space is allowed for its evolution. This new feature is completely different from the discussion of a 1D-SHO in the framework of classical mechanics. It will be a nice idea to discuss the supersymmetric version of this system in the framework of \( q \)-deformed dynamics. It will be very interesting to explore the possibility of \( h \)-deformation over \( q \)-deformation and look for the new aspects of dynamics when \( q \) and \( h \) both are present.

\[\text{‡ The discussion of dynamics in the cotangent (momentum phase) space has been carried out in the Hamiltonian formulation as well [16].}\]
References

[1] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, *Ann. Phys.* **111** (1978) 61, 111.

[2] U. Carow-Watamura, M. Schliecker, M. Scholl and S. Watamura, *Z. Phys. C* **48** (1990) 159, *Int. J. Mod. Phys.* **A6** (1991) 3081.

[3] H. S. Snyder, *Phys. Rev.* **71** (1947) 38.

[4] C. N. Yang, *Phys. Rev.* **72** (1947) 874.

[5] H. Yukawa, *Phys. Rev.* **91** (1953) 415.

[6] See, e.g., for review, L. D. Faddeev, N. Reshetikhin and L. A. Takhtadjan, *Alg. Anal.* **1** (1988) 129.

[7] See, e.g., for review, S. Majid, *Int. J. Mod. Phys.* **A5** (1999) 1.

[8] See, e.g., for review, A. Connes, M. R. Douglas and A. Schwarz: [hep-th/9711162].

[9] See, e.g., for review, N. Seiberg and E. Witten: [hep-th/9908142].

[10] M. Lukin, A. Stern and I. Yakushin, *J. Phys. A: Math Gen* **26** (1993) 5115.

[11] J. Schwenk and J. Wess, *Phys. Lett.* **B291** (1992) 273.

[12] I. Ya. Aref’eva and I. V. Volovich, *Phys. Lett.* **B 264** (1991) 62.

[13] R. P. Malik, *Phys. Lett.* **B316** (1993) 257.

[14] R. P. Malik, *Phys. Lett.* **B345** (1995) 131.

[15] R. P. Malik, *Mod. Phys. Lett.* **A11** (1996) 2871.

[16] R. P. Malik, A. K. Mishra and G. Rajasekaran, *Int. J. Mod. Phys.* **A13** (1998) 4759.