Pion Generalized Parton Distributions with covariant and Light-front constituent quark models

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We investigate the model dependence of no-helicity flip generalized parton distribution of the pion upon different approaches for the quark-hadron and quark-photon vertexes, in the spacelike region. In order to obtain information on contributions from both the valence and non-valence regions, we compare results for spacelike momentum transfers obtained from i) an analytic covariant model with a bare quark-photon vertex, ii) a Light-front approach with a quark-photon vertex dressed through a microscopic vector-meson model and iii) a Light-front approach based on the Relativistic Hamiltonian Dynamics. Our comparisons lead to infer the same dynamical mechanism, the one-gluon-exchange dominance at short distances, as a source of both the electromagnetic form factor at large momentum transfer and the parton distribution close to the end-points. The expected collinear behavior of the gener-
alized parton distributions at high momentum transfer, i.e. a maximum for $x \sim 1$, is also illustrated, independently of the different approaches. Finally a comparison with recent Lattice calculations of the gravitational form factors is presented.

PACS numbers: 12.39.Ki, 14.40.Aq, 13.40.-f, 11.10.St

I. INTRODUCTION

In recent years a growing interest in the study of the Deeply Virtual Compton Scattering (DVCS) has motivated an impressive amount of work aimed at the extraction of the so-called Generalized Parton Distributions (GPD’s) from experimental data (see, e.g., Refs. [1, 2, 3, 4, 5, 6] for recent reviews). In principle, GPD’s allow one to achieve an unprecedented level of detail on the knowledge of hadronic states.

Naturally, the pion GPD should represent a test ground of any approach that addresses the issue of obtaining a detailed description of hadron structure, and this explains the wealth of papers devoted to such a task (see, e.g., [4, 7, 8, 9, 10, 11, 12, 13, 14, 15]). In what follows, we focus on the GPD’s that do not depend upon the helicities of the constituents, namely we analyze the pion isoscalar and isovector GPD’s, as defined, e.g. in [11].

Aim of our paper is the investigation of the model dependence of those no-helicity flip (chiral-even) GPD of the pion upon different relativistic approaches in the spacelike region, i.e. for negative values of $t = (p' - p)^2$, where $p$ and $p'$ are the initial and final four-momenta of the pion, respectively. In particular, the study of the GPD’s in the valence and non-valence regions (see the following Section) is emphasized by the choice of three different models that explore different kinematical regions: i) a covariant analytic constituent quark (CQ) model, that covers the whole kinematical domain and allows us to interpolate between the other two models; ii) two phenomenological models, elaborated within a Light-front (LF) framework (see e.g., [16, 17, 18] for a review), which have a smaller kinematical range of applicability, namely one addresses the non-valence region and the other the valence one.

The first model is analytic and covariant, and depends upon the mass of the constituents and a parameter, fixed by the decay constant of the pion. The main ingredients of such an approach are: i) the Bethe-Salpeter amplitude (BSA) of the pion, modeled through an analytic Ansatz in the Minkowski space, ii) the Mandelstam formula [19] (or Impulse
Approximation formula) for the matrix elements of the current operator, and iii) a bare quark-photon vertex. A peculiar feature of our Ansatz for the pion BSA is given by the symmetry under the exchange of the constituent momenta. A first version of such a model was adopted in Ref. [20] to investigate the frame dependence of the description of the electromagnetic (em) pion form factor, putting in evidence the possibility to study the non-valence content of the pion by using a suitable reference frame. In the present work, we consider a natural extension of the model, that features a better end-point behavior of the BSA, as well.

A second model, developed within the LF Dynamics and already applied to the em pion form factor in both the space- and timelike regions [21], is still based on the Mandelstam formula. However, this model retains only the analytic structure given by the poles of the Dirac propagators in the analytic integration over \( k^{-} = k^{0} - k^{3} \), i.e. the minus component of the constituent four-momentum appearing in the loop formula. An important consequence of the \( k^{-} \)-integration can be reached in a frame where the plus component of the virtual-photon four-momentum is different from zero, i.e. \( \Delta^{+} = \Delta^{0} + \Delta^{3} \neq 0 \). Indeed, in this frame the contributions in the valence and non-valence regions can be obtained, allowing an investigation of the Fock components of the hadronic state (see [4, 17, 18, 22, 23] for an overview of the Fock expansion of a hadron state, within the LF framework). Another relevant feature of this model, that has a fundamental impact in the timelike region, is the quark-photon vertex dressed by a microscopic version of the vector meson model (VMD) [21]. Finally, as explained in detail in [21], the model lives in the non-valence region, in the limit of vanishing pion.

A third model is constructed within the LF Relativistic Hamiltonian Dynamics (LFHD), where the Poincaré covariance is fully satisfied (see, e.g. [16] for a detailed review). In particular the rotational covariance is fulfilled through the introduction of the Melosh rotations and the proper definition of the total intrinsic angular momentum. At the present stage, the model explores only the valence region.

The paper is organized as follows: in Sec. [II] a brief resumé of the general properties of the pion isospin-dependent GPD’s is presented; in Sec. [III] the Fock decomposition of the GPD’s is discussed, in view of a frame-dependent analysis; in Sec. [IV] a covariant CQ model, that allows an analytic evaluation of the pion GPD’s is described; in Sec. [V A] a first CQ Light-front model, with a quark-photon vertex dressed by a microscopic version of
the vector meson dominance model, is presented; in Sec. V B, the LFHD model, where the full Poincaré covariance is implemented is described. Finally in Sec. VI and Sec. VII the results are discussed and the conclusions drawn.

II. PION GPD’S: KINEMATICS AND GENERAL FORMALISM

In the spacelike region, let us first illustrate the kinematics of the DVCS process with the symmetric momenta convention shown in Fig. 1 (see [12] for the reduction of the DVCS diagram to the one presented in Fig. 1 and the pioneering paper [24] for the DIS regime). For on-mass-shell pions, i.e. \( p'^2 = p^2 = m^2_\pi \), and adopting standard notations (see, e.g. [4, 6])

\[
\begin{align*}
t &= \Delta^2 = (p' - p)^2, \\
\xi &= -\frac{\Delta \cdot n}{2P \cdot n} = -\frac{\Delta^+}{2P^+} = \frac{p^+ - p'^+}{p^+ + p'^+}, \quad (|\xi| \leq 1), \\
x &= \frac{k \cdot n}{P \cdot n} = \frac{k^+}{P^+}, \quad (1 \geq x \geq -1),
\end{align*}
\]

where \( n \) is a light-like 4-vector, such that \( v^+ = n \cdot v = v^0 + v^3 \) (the scalar product is defined as \( a \cdot b = (a^+ b^- + a^- b^+)/2 - a_\perp \cdot b_\perp \)), \( P = \frac{1}{2}(p' + p) \), and \( k \) is the average momentum of the active quark, i.e. the one that interacts with the photon (see Fig. 1). Notice that \( p^+ \) and \( p'^+ \) are necessarily positive, while \( \Delta^+ \geq 0 \) is taken by choice. From Eq. 1 one can trivially obtain the following useful relations

\[
p'^+ = \frac{\Delta^+}{2}(1 - \frac{1}{\xi}), \quad p^+ = -\frac{\Delta^+}{2}(1 + \frac{1}{\xi}).
\]

As it is well known, the variable \( x \) allows one to single out i) the valence region (where one has only contributions diagonal in the Fock space, cf the following Sec. III) given by the union of two intervals: \( x \in [-1, -|\xi|] \) (corresponding to an active antiquark) and \( x \in [|\xi|, 1] \) (corresponding to an active quark), and ii) the non-valence region, \( x \in [-|\xi|, |\xi|] \). In Fig. 2(a), it is shown a representative of the contribution with an active quark in the kinematical region \( x \in [|\xi|, 1] \), (all the constituents have a plus-component of their-own momentum bounded from above by the corresponding quantity of the parent pion). In Fig. 2(b), it is shown a contribution from a pair-production process, non diagonal in the Fock space. In Appendix A a more detailed kinematical discussion is given. Finally, as a short detour, let us remind that the pion BSA, integrated over the minus component of the quark momentum,
yields the two-body Fock contribution to the pion state, notably non-vanishing only in the valence sector (see [18]).

\[ \Delta \]

\[ k + \frac{\Delta}{2} \quad k - \frac{\Delta}{2} \]

\[ p' = P + \frac{\Delta}{2} \quad k - P \]

\[ k - P \quad p = P - \frac{\Delta}{2} \]

FIG. 1: Diagrammatic representation of the pion GPD, with four-momenta definitions.

Within the QCD-evolution framework, the valence region is called DGLAP [25] region, while the non-valence one is called the ERBL [26, 27] region.

In the interval \([|\xi|, 1]\), the relation between the LF momentum fraction, \(x_q\), of the active constituent in the initial pion (with the support \([0, 1]\)) and the variable \(x\) defined in Eq. (1), is given by

\[ x_q = \frac{k^+ - \Delta^+/2}{p^+} = \frac{k^+ - \Delta^+/2}{P^+ - \Delta^+/2} = \frac{x + \xi}{1 + \xi} = \frac{x - |\xi|}{1 - |\xi|}. \tag{3} \]

The isospin-dependent GPD’s (see, e.g. [4, 7, 11, 13]) are the matrix elements of light-cone bilocal operators separated by a light-like distance, \(z^2 = z^+ z^- - |z_\perp|^2 = 0\), evaluated.
between pion states with different initial and final momenta. In the light-cone gauge, where $A_{\text{gluon}} \cdot n = 0$ and the gauge link becomes unity, one can introduce isoscalar and isovector combinations for the off-forward ($t \neq 0$), non-diagonal ($\xi \neq 0$) GPD’s, as follows

$$H^{I=0}_{\pi \pm}(x, \xi, t) = \int \frac{dz^{-}}{4\pi} e^{ixP^{+}z^{-}} \langle \pi^{\pm}(p') \rangle |\tilde{\psi}_{q}(-\frac{1}{2}z)\gamma \cdot n \psi_{q}(\frac{1}{2}z) |\pi^{\pm}(p)\rangle |_{\tilde{z}=0}$$

$$= \frac{1}{2} \left[ H^{u}_{\pi \pm}(x, \xi, t) + H^{d}_{\pi \pm}(x, \xi, t) \right]$$

(4)

and

$$H^{I=1}_{\pi \pm}(x, \xi, t) = \int \frac{dz^{-}}{4\pi} e^{ixP^{+}z^{-}} \langle \pi^{\pm}(p') \rangle |\bar{\psi}_{q}(-\frac{1}{2}z)\gamma \cdot n \tau_{3}\psi_{q}(\frac{1}{2}z) |\pi^{\pm}(p)\rangle |_{\tilde{z}=0}$$

$$= \frac{1}{2} \left[ H^{u}_{\pi \pm}(x, \xi, t) - H^{d}_{\pi \pm}(x, \xi, t) \right] \pm \frac{1}{2} \left[ H^{u}_{\pi \pm}(x, \xi, t) - H^{d}_{\pi \pm}(x, \xi, t) \right]$$

(5)

where $\tilde{\zeta} \equiv \{ z^+, z_{\perp} \}$, while $\psi_{q}(z)$ and $\tau_{3}\psi_{q}(z)$ are the following doublets of quark field

$$\begin{pmatrix}
U(z) \\
D(z)
\end{pmatrix}, \quad \begin{pmatrix}
U(z) \\
-D(z)
\end{pmatrix},$$

(6)

respectively. In Eqs. (4) and (5), following [7], instead of the Cartesian components, $\pi^0, \pi^1, \pi^2$, (adopted in [4, 11, 13]), the charged pions have been introduced, viz

$$|\pi^{\pm}\rangle = \frac{|\pi^{1}\rangle \pm i|\pi^{2}\rangle}{\sqrt{2}} , \quad |\pi^{0}\rangle = |\pi^{3}\rangle.$$  

(7)

The functions $H^{u}(x, \xi, t)$ and $H^{d}(x, \xi, t)$ are $u$ and $d$ GPD’s, respectively, and contain quark and antiquark contributions (cf. the parton interpretation of $H^{q}$, e.g., in [1, 4] and Fig. 2). It is worth noting that $H^{q}(x, \xi, t)$ has the support $x \in [-1, 1]$. Finally, due to the isospin symmetry one has

$$H^{u}_{\pi^{+}} = H^{d}_{\pi^{-}},$$

(8)

and combining charge and isospin symmetry (G-parity) one gets

$$H^{u}_{\pi^{+}}(x, \xi, t) = -H^{u}_{\pi^{-}}(-x, \xi, t) = -H^{d}_{\pi^{+}}(-x, \xi, t).$$

(9)

In what follows, we deal with a charged pion and the subscript $\pi^{+}$ in the quark GPD’s is dropped out whenever no ambiguity is present.

For vanishing $\xi$ and $t$, one has the following partonic decomposition (cf. [1, 4])

$$H^{u}(x, 0, 0) = \theta(x) \ u(x) - \theta(-x) \ \bar{u}(-x),$$

$$H^{d}(x, 0, 0) = \theta(x) \ d(x) - \theta(-x) \ \bar{d}(-x).$$

(10)
Equations (8) and (9) together with the partonic interpretation lead to the well known relations between the standard parton distribution functions (let us remind that the relations pertain to active quarks), viz

\[ u_{\pi^+}(x) = d_{\pi^-}(x), \quad u_{\pi^+}(x) = d_{\pi^-}(x). \] (11)

The symmetry property of \( H^{I=0,1}(x,\xi,t) \) (see, e.g. [4, 7]) under the transformation \( x \rightarrow -x \), that just reflects i) the charge-conjugation \( (p \rightarrow -p \text{ and } p' \rightarrow -p') \) and ii) the isospin invariance, reads (reminding Eqs. (8) and (9))

\[ H^{I=0}(x,\xi,t) = \frac{1}{2} [H^u(x,\xi,t) - H^u(-x,\xi,t)] = -H^{I=0}(-x,\xi,t), \] (12)
\[ H^{I=1}(x,\xi,t) = \frac{1}{2} [H^u(x,\xi,t) + H^u(-x,\xi,t)] = H^{I=1}(-x,\xi,t). \] (13)

Therefore the two GPD’s are odd or even in \( x \) depending upon the isospin combination. In addition, under the transformation \( \xi \rightarrow -\xi \), that amounts to apply a time-reversal transformation (since we have to exchange the initial and final pion momenta) and to exploit Hermiticity, one has (see, e.g. [4, 7])

\[ H^I(x,\xi,t) = H^I(x,-\xi,t), \] (14)

namely \( H^I(x,\xi,t) \) must be even in \( \xi \).

From Eqs. (11) and (12) one has

\[ H^u(x,\xi,t) = H^{I=0}(x,\xi,t) + H^{I=1}(x,\xi,t), \]
\[ H^d(x,\xi,t) = H^{I=0}(x,\xi,t) - H^{I=1}(x,\xi,t). \] (15)

As well known, the following sum rules hold (note a different overall factor with respect to [11, 13] due to our choice of dealing with a charged pion, cf Eq. (7))

\[ \int_{-1}^{1} dx \ H^{I=1}(x,\xi,t) = \int_{-1}^{1} dx \ H^u(x,\xi,t) = F_\pi(t), \] (16)
\[ \int_{-1}^{1} dx \ x \ H^{I=0}(x,\xi,t) = \int_{-1}^{1} dx \ x \ H^u(x,\xi,t) = \frac{1}{2} \left[ \theta_2(t) - \xi^2 \theta_1(t) \right]. \] (17)

In Eq. (16), \( F_\pi(t) \) is the pion em form factor (see Appendix C); while, according to the Ref. [28], in Eq. (17) \( \theta_1(t) \) and \( \theta_2(t) \) are the gravitational form factors (see also, e.g., [7, 11, 14]), that enter in the parametrization of the matrix elements of the quark part of the energy-momentum tensor (notice that in the chiral limit one has \( \theta_1(0) - \theta_2(0) = \mathcal{O}(m_\pi^2) \)). It should
be pointed out that the sum rule (17) for $t = 0$ and $\xi = 0$ yields the longitudinal-momentum sum rule for the pion, i.e. $< x_q >$, as numerically illustrated in Sec. VI.

For vanishing $\xi$ and $t$, one can exploit i) Eqs. (12) and (13) and ii) the partonic decomposition (cf. Eq. (11)) obtaining

$$H^{I=0}(x,0,0) = \frac{1}{2} [H^u(x,0,0) + H^d(x,0,0)] = \frac{1}{2} [H^u(x,0,0) - H^u(-x,0,0)]$$

$$= \theta(x) \frac{1}{2} [u(x) + \bar{u}(x)] - \theta(-x) \frac{1}{2} [\bar{u}(-x) + u(-x)],$$

and

$$H^{I=1}(x,0,0) = \frac{1}{2} [H^u(x,0,0) - H^d(x,0,0)] = \frac{1}{2} [H^u(x,0,0) + H^u(-x,0,0)]$$

$$= \theta(x) \frac{1}{2} [u(x) - \bar{u}(x)] - \theta(-x) \frac{1}{2} [\bar{u}(-x) - u(-x)].$$

Analogous relations, with singlet, $q(x) + \bar{q}(x)$, and valence, $q(x) - \bar{q}(x)$, combinations, for the $d$-quark can be easily obtained, by using Eq. (11) (see also [7]). It is worth noting that for $\xi = \Delta^+ = 0$ the ERBL region shrinks to zero and the variable $x$ reduces to $x_q$ (Eq. (3)). Finally, from Eq. (16) one has a normalization for the valence combination $u_v(x) = u(x) - \bar{u}(x)$ given by $\int_0^1 \! dx \, u_v(x) = 1$.

It should be pointed out that the parton distributions represent a bridge toward the chiral-even transverse-momentum dependent (TMD) distribution, $f_1(x,|k_\perp|^2)$ (see, e.g. [29, 30, 31] for the nucleon case), as shown by the following relation

$$q(x) = \int \! d|k_\perp|^2 \, f_1^q(x,|k_\perp|^2), \quad (x \geq 0).$$

Furthermore, it is worth noting that an experimental access to $f_1(x,|k_\perp|^2)$ and to other TMD’s is a fundamental step in order to understand the correlations between constituents inside the pion, and eventually the dynamics.

To complete this brief resumé of the general formalism, we have to mention that the sum rules in Eqs. (16) and (17), are the lowest order of the moments of the isovector and isoscalar GPD’s. In particular, $H^{I=1}(x,\xi,t)$ (see Eq. (13)) has only even moments, while $H^{I=0}(x,\xi,t)$ (see Eq. (12)) has only odd moments. Moreover, it turns out (see, e.g., [4]) that the $n$-th Mellin moments of the GPD’s are polynomials of $\xi$ with highest power $n$ for even moments and $n+1$ for odd moments, i.e. only even powers of $\xi$ appear, as expected from Eq. (14). It is worth noting that the so-called polynomiality follows from general properties, like
Hermiticity, covariance, parity and time-reversal invariance \[1, 2\]. The isospin-dependent moments are given by \((j \geq 0)\)

\[
\int_{-1}^{1} dx \ x^{2j} \ H^{I=1}(x, \xi, t) = \sum_{i=0}^{j} A_{2j+1,2i}^{I=1}(t)(2\xi)^{2i}, \tag{21}
\]

\[
\int_{-1}^{1} dx \ x^{2j+1} \ H^{I=0}(x, \xi, t) = \sum_{i=0}^{j+1} A_{2j+2,2i}^{I=0}(t)(2\xi)^{2i}. \tag{22}
\]

In particular, numerical calculations of i) \(F_{\pi}(t) = A_{1,0}^{I=1}(t)\) and ii) \(A_{2,0}^{I=0} = \theta_2(t)/2\) and \(A_{2,2}^{I=0} = -\theta_1(t)/8\), will be presented in Sec. VI.

In conclusion, approaches that satisfy the basic field-theoretic assumptions underlying polynomiality, i.e. extended Poincarè covariance, automatically fulfill the conditions (21) and (22). In general, such a property is an important test of consistency of the model.

III. FOCK DECOMPOSITION

Let us introduce the Fock expansion of the pion state, taking care of the colorless feature of each component and including the amplitudes inside the kets to simplify the notations in this Section, (see, e.g. \[17, 18\]), viz

\[
|\pi\rangle = |q\bar{q}\rangle + |q\bar{q}; g\rangle + |q\bar{q}; q\bar{q}\rangle + \ldots \tag{23}
\]

Then one can decompose the GPD’s in terms of their Fock contents (see also \[4, 22\]), i.e. one can rewrite Eqs. (4) and (5) by using e.g.

\[
H^q(x, \xi, t) = \sum_{n} \langle \pi; n|\Gamma_D^q|\pi; n\rangle \theta(|x| - |\xi|)\theta(1 - |x|) + \theta(|\xi| - |x|) \left[ \sum_{n} \langle \pi; n|\Gamma_{ND}^q|\pi; n + 2\rangle \theta(\xi) + \sum_{n} \langle \pi; n + 2|\Gamma_{ND}^q|\pi; n\rangle \theta(-\xi) \right] + \ldots \tag{24}
\]

where \(n\) indicates the number of quarks and antiquarks, \(\Gamma_D\) and \(\Gamma_{ND}\) are the diagonal and non diagonal, in the Fock space, terms of the current operator, and dots represent all the other transition matrix elements, possibly containing states with gluons. The diagonal terms yield contributions to the valence region (DGLAP region), while the non diagonal ones have to be considered in the non-valence region (ERBL region). In Eq. (24), we have shown only transitions involving fermionic fields, and this explains the selection rule \(\Delta n = 0, 2\).
In a simple picture of a hadron, the valence state has a dominant role at the hadron scale, and this leads naturally to associate the DGLAP region with this Fock component.

The same decomposition can be applied to the em and gravitational form factors, and to all the t-dependent "generalized" form factors appearing in Eqs. (21) and (22). Clearly, this kind of decomposition could allow a deeper understanding of the dynamics related to the components beyond the valence one. As a simple application, let us consider the em form factor. From Eqs. (16) and (24), retaining only the fermionic transitions, one has

\[ F^{(v)}(\xi, t) = 2 \int_{|\xi|}^{1} \ dx \ H^{I=1}(x, \xi, t) = 2 \sum_{n}^{\text{Fock}} \int_{|\xi|}^{1} \ dx \ \langle \pi; \ n|\Gamma_{D}^{I=1}|\pi; \ n \rangle, \]  

\[ F^{(nv)}(\xi, t) = 2 \int_{|\xi|}^{1} \ dx \ H^{I=1}(x, \xi, t) \]
\[ = 2 \sum_{n}^{\text{Fock}} \int_{0}^{|\xi|} \ dx \ \left[ \theta(\xi) \langle \pi; \ n|\Gamma_{ND}^{I=1}|\pi; \ n+2 \rangle + \theta(-\xi) \langle \pi; \ n+2|\Gamma_{ND}^{I=1}|\pi; \ n \rangle \right]. \]  

(25)

The valence term, \( F^{(v)}(\xi, t) \), receives the largest contribution from the valence component of the pion state, but it does not give the full result in the whole kinematical range, as indicated by the residual dependence upon \( \xi \). The non-valence term, \( F^{(nv)}(\xi, t) \), is due to contributions like the pair-production mechanism, see Fig. 2(b). The sum of Eqs. (25) and (26) leads to the full result, viz

\[ F_{\pi}(t) = F^{(v)}(\xi, t) + F^{(nv)}(\xi, t) \]  

(27)

and it is independent of \( \xi \) and function of \( t \) only. One can also express the invariance of the sum under changes of \( \xi \) as:

\[ \frac{\partial^{m}}{\partial \xi^{m}} F^{(v)}(\xi, t) = -\frac{\partial^{m}}{\partial \xi^{m}} F^{(nv)}(\xi, t), \]  

(28)

with \( m \geq 1 \). It is worth noting that all the derivatives of \( F_{\pi}(t) \) are independent upon \( \xi \), and therefore relations like the one in Eq. (28) can be generalized, i.e.

\[ \frac{\partial^{m}}{\partial \xi^{m}} \frac{\partial^{\ell}}{\partial t^{\ell}} F^{(v)}(\xi, t) = -\frac{\partial^{m}}{\partial \xi^{m}} \frac{\partial^{\ell}}{\partial t^{\ell}} F^{(nv)}(\xi, t), \]  

(29)

with \( m \geq 1 \) and \( \ell \geq 0 \). As a consequence, with the help of Eq. (16), one can deduce interesting sum rules for the partial derivatives of \( H^{I=1}(x, \xi, t) \).

Let us remind that calculations of the elastic form factors have been performed in different frames. In particular, it has been chosen i) the Drell-Yan frame, where \( \Delta^{+} = 0 \) and therefore
ξ = 0 (see Ref. [17] for generalities on the Drell-Yan frame), or ii) a Breit frame (i.e. \( \Delta^+ = -\Delta^- \)) where \( \Delta_\perp = 0 \) (see Ref. [32] for an extended discussion of the motivations for adopting such a frame), and then ξ follows a kinematical trajectory in the \((\xi, t)\)-plane given by \(|\xi| = 1 / \sqrt{1 - 4m^2_\pi / t}\) (see below Eq. (33)). In the first case, the em form factor is saturated by the valence contribution, because of \( \xi = 0 \) (cf Eqs. (27) and (25)), while in the second frame both valence and non-valence terms contribute, since \( \xi \) does not vanish, but changes with \( t \). For \( -t^2 >> m^2_\pi \) the value of \( \xi \) approaches 1, and therefore the non-valence term saturates the em form factor (cf Eqs. (27) and (26)). In model calculations this general behavior was indeed observed [20]. It is understood that for an experimental investigation of the whole \((\xi, t)\)-plane, different kinematical conditions are needed, also exploiting the helpful properties of the LF boosts (see, e.g. [16]).

Following the same spirit, one could extend this analysis to the other form factors that appear in Eqs. (21) and (22), i.e. one can consider the partial derivatives of the valence and non-valence contribution to the generalized form factors \( A_I^{2j+0,2i}(t) \) and \( A_I^{2j+1,2i}(t) \), obtaining final relations that have the same structure as the ones in Eqs. (28) and (29).

**IV. COVARIANT MODEL OF THE PION WITH PAULI-VILLARS REGULATORS**

In Ref. [20], an analytic covariant model, symmetric in the exchange of the constituent four-momenta (see Refs. [33, 34] for previous non symmetric versions) was adopted for evaluating the em form factor. In this work, a direct extension of the symmetric covariant model to DVCS is exploited for calculating the no-helicity flip GPD, in the spacelike interval \( 0 \geq t \geq -10 \) (GeV/c)^2.

In a Breit frame, one has \( \Delta^0 = 0 \), i.e. \( \Delta^+ = -\Delta^- \), and \( p^\prime = -p = \Delta/2 \). By choosing \( \Delta^+ \geq 0 \), and reminding that

\[
p^\prime^- = \frac{m^2_\pi + |\Delta_\perp|^2/4}{p^+}, \quad p^- = \frac{m^2_\pi + |\Delta_\perp|^2/4}{p^+},
\]

\[
\Delta^- = p^\prime^- - p^- = -\Delta^+ \frac{m^2_\pi + |\Delta_\perp|^2/4}{p^+ p^+} = \Delta^+ \frac{m^2_\pi + |\Delta_\perp|^2/4}{(p^+ + \Delta^+) p^+},
\]

one gets

\[
p^+ = \frac{-\Delta^+ + \sqrt{-\Delta^2 + 4m^2_\pi}}{2} = -\frac{\Delta^+}{2} (1 + \frac{1}{\xi}),
\]
Then the following relation holds (notice that $2 \, P^+ = \sqrt{-\Delta^2 + 4m_\pi^2}$)

$$\Delta^2 = -\Delta^+ - |\Delta_\perp|^2 = -4\xi^2 P^+ - |\Delta_\perp|^2 = -\xi^2 \left( -\Delta^2 + 4m_\pi^2 \right) - |\Delta_\perp|^2, \quad (32)$$

which leads to a constraint on the maximal value for the variable $\xi$. As a matter of fact, in the spacelike region $-\Delta^2 + 4m_\pi^2 \neq 0$ and one has

$$\xi^2 = \frac{-\Delta^2 - |\Delta_\perp|^2}{-\Delta^2 + 4m_\pi^2}. \quad (33)$$

Then, the maximum value of $\xi^2$ is found for $\Delta_\perp = 0$, viz

$$\xi^2 \leq \frac{-\Delta^2}{-\Delta^2 + 4m_\pi^2} \leq 1. \quad (34)$$

For $m_\pi = 0$ (and $\Delta^2 \neq 0$), one has

$$\xi^2 = 1 + \frac{|\Delta_\perp|^2}{\Delta^2}. \quad (35)$$

If one additionally chooses a frame where $\Delta_\perp = 0$ (i.e. only $\Delta_z \neq 0$), then $\xi = -1$ and therefore, in this extreme case, only the non-valence region contributes.

A basic ingredient in the analytic covariant model of Ref. [20] is the pion BS amplitude, that can be quite well approximated by retaining only the pseudo-scalar Dirac structure (see, e.g., [35]), namely

$$\Psi(k - P, p) = -\frac{m}{f_\pi} \, S(k - \Delta/2) \, \gamma^5 \, \Lambda(k - P, p) \, S(k - P), \quad (36)$$

where $\frac{m}{f_\pi}$ is the quark-pion coupling, as suggested by a simple effective Lagrangian (see, e.g. [24]), $f_\pi = 92.4$ MeV the pion decay constant, $m$ and $S(k)$ are the mass and the Dirac propagator of the constituent quark (CQ), respectively. In Eq. (36), $\Lambda(k - P, p)$ is a scalar function that describes the momentum-dependent part of the coupling between the constituents and the spin-0 system and plays the role of the Pauli-Villars regulator of the otherwise divergent integrals that yield GPD’s or the em form factor. In particular in this work we adopt two symmetric (in the exchange of the CQ four-momenta) covariant forms: i) the one considered in Ref. [20], and based on the following sum

$$\Lambda_1(k - P, p) = C_1 \left\{ \frac{1}{[(k - \Delta/2)^2 - m_R^2 + ie]} + \frac{1}{[(P - k)^2 - m_R^2 + ie]} \right\}, \quad (37)$$
and ii) a natural extension based on a product, viz

\[
\Lambda_2(k - P, p) = C_2 \frac{1}{[(k - \Delta/2)^2 - m_R^2 + i\epsilon]} \frac{1}{[(P - k)^2 - m_R^2 + i\epsilon]} .
\]  

(38)

This product-form provides a more realistic transverse-momentum fall-off, as seen from the expected behavior of the BS amplitude obtained by using a simple (one-boson-exchange) kernel (see, e.g., [36]), and this has a sizable impact on both the high-momentum tail of the EM form factor and the end-point behavior of the parton distribution, as shown in the results presented in Sec. VI. We can anticipate that the most favorable comparison with the experimental data of the EM form factor is obtained by using the product-form, as also expected if one follows a pQCD analysis, where a one-gluon exchange represents the leading contribution to the kernel [27, 37].

In both expressions, once the constituent mass \( m \) is chosen, \( m_R \) is determined by fitting the experimental value for \( f_\pi \) (cf [20]), while the constants \( C_1 \) and \( C_2 \) are fixed by exploiting the charge normalization, as discussed below.

As a final comment on the Dirac structure that appears in Eq. (36), we remind that it leads to the standard Melosh rotation for a pair of fermions coupled to a total spin \( S = 0 \) (see [38]), once we consider the valence wave function, defined as follows (see, e.g. [18])

\[
\Phi_{val}(\kappa^+, \kappa_\perp, p) = -\frac{m}{f_\pi} \int \frac{d\kappa^-}{2\pi} S_{on}(\kappa - p) \gamma^5 \Lambda(\kappa, p) S_{on}(\kappa^-) \Lambda(\kappa, p^-) \Lambda(\kappa, p),
\]  

(39)

where \( S_{on}(\kappa) = (\kappa_\perp^2 - m^2 + i\epsilon) \) with \( \kappa_\perp^+ \equiv \{\kappa_\perp^- = (m^2 + |\kappa_\perp|^2)/\kappa^+, \kappa^+, \kappa_\perp\} \).

The no-helicity flip GPD for the pion is calculated in one-loop approximation (triangle diagram cf. Fig. 1) with the BS amplitude of Eq. (36) and the symmetrical forms shown in Eqs. (37) and (38). In particular, the \( u \)-quark GPD is given in Impulse Approximation by

\[
H^u(x, \xi, t) = -i N_c R \times \int \frac{d^4k}{(2\pi)^4} \delta(P^+ - k^+) \ V^+(k, p, p') \Lambda(k - P, p') \ \Lambda(k - P, p) ,
\]  

(40)

where \( N_c = 3 \) is the number of colors, \( R = 2m^2/f_\pi^2 \) and

\[
V^+(k, p, p') = Tr \left\{ S(k - P) \gamma^5 \ S\left(k + \frac{\Delta}{2}\right) \gamma^+ \ S\left(k - \frac{\Delta}{2}\right) \gamma^5 \right\} .
\]  

(41)

The presence of the delta-function in Eq. (40), given the kinematical relations in Eq. (11), imposes the correct support \([-|\xi|, 1]\) for the variable \( x \) as discussed in details in Appendix [33] (note that \( H^d \) has the support \([-1, |\xi|]\) for the variable \( x \)). A relevant feature in the analysis
of the GPD, as well as in the case of the em form factor, is given by the instantaneous term present in $S(k)$. As a matter of fact, the Dirac propagator can be decomposed using the LF kinematics as follows \[17\]

$$S(k) = \frac{k + m}{k^2 - m^2 + i\epsilon} = S_{on}(k) + \frac{\gamma^+}{2k^+} = \frac{k_{on} + m}{k^+(k^+ - k_{on}^+ + \frac{\epsilon}{k^+})} + \frac{\gamma^+}{2k^+}, \quad (42)$$

where the second term, proportional to $\gamma^+$, is an instantaneous one in the LF time. It should be pointed out that the instantaneous contribution to the GPD is produced only by the spectator fermion (in the present example an antifermion), i.e. by $S(k - P)$. Indeed, the instantaneous terms pertaining to the other propagators do not contribute, because of the property $(\gamma^+)^2 = 0$. In our symmetric model, the instantaneous term of Eq. (42) contributes to $H^u(x,\xi,t)$ both in the valence and in the non-valence region (see Eqs. (B10)-(B15)), since we take fully into account the analytic structure of the symmetric vertex function (for a different approach, where such an analytic structure is disregarded see \[12\]).

The pion em form factor is obtained by using the sum rule (16):

$$F_\pi(t) = -\frac{1}{N_c} \frac{R}{(p^+ + p^+)} \int \frac{d^4k}{(2\pi)^4} V^+(k,p,p') \Lambda(k-P,p') \Lambda(k-P,p). \quad (43)$$

The last expression for $F_\pi(t)$ can be extracted directly from the Mandelstam formula for the matrix elements of the em current \[19\] (see, e.g., \[39\], \[40\]), as well. Notice that the model preserves current conservation, as discussed in \[20\].

The normalization of the form factor, Eq. (43), allows us to determine $C_1$ and $C_2$ in Eqs. (37) and (38). Such a charge normalization represents the impulse approximation of the normalization condition in the fully interacting BS theory \[19\], \[41\].

A standard analytic integration on $k^-$ (see Appendix B for details) leads to the following decomposition of $H^u(x,\xi,t)$ in valence and non-valence contributions

$$H^u(x,\xi,t) = H^u_{(v)}(x,\xi,t)\theta(|\xi| - (1 - x) + H^u_{(nv)}(x,\xi,t)\theta(|\xi| - x)\theta(|\xi| + x) \quad (44)$$

Notice that $H^u_{(v)}$ and $H^u_{(nv)}$ are given in Appendix B for the two momentum dependences shown in Eqs. (37) and (38).

The $d$-quark GPD can be obtained reminding Eq. (9).

Within our covariant model the valence component $H^u_{(v)}$ in Eq. (44) is an approximation to the diagonal terms in Eq. (24), while the component $H^u_{(nv)}$ contains the contribution
of the pair-production mechanism from an incoming virtual photon with $\Delta^+ > 0$ and approximates the non diagonal terms.

An interesting approximation of the contribution to GPD in the valence region can be obtained once the analytic structure of the BS amplitude is disregarded and only the poles of the propagators are retained in the integration over $k^-$ (see Appendix B). As a matter of fact, see Eq. (B7), within the mentioned approximation

$$H_{(v)}(x, \xi, t) \sim H_{(v)on}(x, \xi, t) = -\frac{N_c R^4}{(2\pi)^3} \int d\kappa_\perp \int_0^{p^+} d\kappa^+ \frac{\delta[P^+(1-x) - \kappa^+]}{\kappa^+(p^+ - \kappa^+)(p^+ - \kappa^+)}$$

$$\times Tr[O^+(\kappa^-_{on})] \Lambda(\kappa, \gamma^+)(\kappa^-_{on}) \Lambda(\kappa, p')|_{\kappa^-_{on}} \Lambda(\kappa, \gamma^+)(\kappa^-_{on}) \Lambda(\kappa, p')|_{\kappa^-_{on}} (\kappa^2 + |\kappa_\perp|^2)/\kappa^+,$$

where $\kappa = P - k$, $\kappa^-_{on} = (m^2 + |\kappa_\perp|^2)/\kappa^+$ and

$$Tr[O^+(\kappa^-_{on})] = Tr \{[\kappa^-_{on} + m] [(\kappa^-_{on} + m)] \gamma^+ [(\kappa^-_{on} + m)] \} .$$

Moreover, if in Eq. (45) we identify the following ratio

$$\frac{\Lambda(\kappa, p)|_{\kappa^-_{on}}}{[p^- - \kappa^-_{on} - (p - \kappa)_-]}$$

with a model LF wave function, then the final expression coincides with the result obtained within a LFHD approach (see the following Sec V B, since the trace $Tr[O^+(\kappa^-_{on})]$ generates the correct Melosh-rotation factor [38]. We would stress that the identification is meaningful once the analytic structure of the BS amplitude is disregarded.

V. LIGHT-FRONT MODELS OF THE PION

In this Section we present models that at different extent i) fulfill the Poincaré covariance and ii) take into account the Fock components of the pion state beyond the valence contribution. A first important difference between the models is given by the frame we choose. In the approach we call Mandelstam-inspired LF model, a Breit frame, where $\Delta_\perp = 0$, is considered. This choice was followed in Ref. [21] in order to perform a microscopical calculation of the em pion form factor in both the space- and timelike regions. It should be pointed out that such a frame leads to consider contributions from a pair-production mechanism, differently from what happens in a Drell-Yan frame, where $\Delta^+ = 0$. This second frame is the one adopted in the second approach illustrated in this Section, based on a LF Hamiltonian Dynamics description of the pion state (see, e.g. [16] for a general review of LFHD).
A. Mandelstam-inspired LF Model

In Ref. [21] an approach was elaborated to calculate the em form factor of the pion starting from a covariant expression of the matrix elements of the current given by the Mandelstam formula [19] (cf also Eq. (40)). Moreover, a microscopic VMD was used for dressing the quark-photon vertex. The dynamical inputs of such an approach were the wave functions of both the pion and vector mesons, taken as eigenstates of the relativistic CQ square mass operator of Ref. [42], which includes both confinement, through a harmonic oscillator potential, and $\pi - \rho$ splitting through a Dirac-delta interaction in the pseudoscalar channel. In what follows we apply the same approach for evaluating the no-helicity flip GPD’s.

Let us first illustrate the kinematics in the adopted frame, where $\Delta_\perp = 0$ (i.e. $\Delta^- = \Delta^2/\Delta^+$) and $\mathbf{p}_\perp = \mathbf{p}'_\perp = 0$. Then in the spacelike region, for $\Delta^+ \geq 0$, one has for $p^+$ and $p'^+$

$$p^+ = \frac{\Delta^+}{2} \left( -1 + \sqrt{1 - 4 \frac{m_\pi^2}{\Delta^2}} \right) = - \frac{\Delta^+}{2} \left( 1 + \frac{1}{\xi} \right),$$

$$p'^+ = \frac{\Delta^+}{2} \left( 1 + \sqrt{1 - 4 \frac{m_\pi^2}{\Delta^2}} \right) = \frac{\Delta^+}{2} \left( 1 - \frac{1}{\xi} \right),$$

(47)

since

$$p'^- = \frac{m_\pi^2}{p'^+}, \quad p^- = \frac{m_\pi^2}{p^+},$$

$$\Delta^- = p'^- - p^- = -\Delta^+ \frac{m_\pi^2}{(p'^+ + \Delta^+) p^+}.$$  

(48)

The following simple relation between $\xi$ and $\Delta^2$ holds

$$\xi = - \frac{\Delta^+}{2 p^+} = - \frac{\Delta^+}{(p'^+ + p^+)} = - \frac{1}{\sqrt{1 - 4 \frac{m_\pi^2}{\Delta^2}}}. \quad (49)$$

It is easily seen that if $m_\pi = 0$ one has $\xi = -1$ for any $\Delta^2$.

Extending the approach of Ref. [21], one can find for the quark GPD the same formal expression of Eq. (40), but i) a microscopic VMD dressing, $\Gamma^\mu(k, \Delta)$, is considered instead of the bare quark-photon vertex, $\gamma^\mu$, and ii) phenomenological Ansatzes for the BS amplitudes in the valence and non-valence regions are adopted. Another basic difference with respect to the analytic model presented in the previous Section, is that only the simple analytic
structure of the Dirac propagators is retained, i.e. the analytic structure is disregarded in the BS amplitudes of both i) the initial and final pion and ii) the VM dressing of the quark-photon vertex. This approximation turns out to be a very effective one in the calculation of the em form factor just in the $\Delta_\perp = 0$ frame [43].

In Ref. [21], a further simplification in the calculation was achieved by a quite natural assumption, namely a vanishing pion mass. Within such an approximation only diagrams with a $q\bar{q}$ production contribute (cf Fig. 2(b)), and this implies the necessity to introduce the VMD dressing. We have to stress that a bare term is missing, due to the vanishing pion mass (cf the discussion in [21]). Therefore, in the quark-photon vertex for the covariant model, Eq. (41), the Dirac matrix $\gamma^+$ is replaced by the plus component of the following four-vector, that microscopically describes a VM dressing. For $t \leq 0$ one has

$$\Gamma^\mu(k, \Delta) = \sqrt{2} \sum_{n,\lambda} \left[ \epsilon_\lambda \cdot \widehat{V}_n(k, P_n) \right] \Lambda_n(k, P_n) \frac{[\epsilon_\lambda^\dagger] f_{V_n}}{(t - M_n^2)},$$

(50)

where $f_{V_n}$ is the decay constant of the n-th VM into a virtual photon (calculated in the model), $P_n^\mu \equiv \{M_n^2/\Delta^+, \Delta^+, 0_\perp\}$ the four-momentum of an on-mass-shell VM with a square mass given by $P_n^2 = M_n^2$ and $\epsilon_\lambda(P_n)$ its polarization. Moreover, the VM BS amplitude is approximated as follows

$$\Psi_{n\lambda}(k, P_n) = \frac{k + m}{k^2 - m^2 + i\epsilon} \left[ \epsilon_\lambda(P_n) \cdot \widehat{V}_n(k, P_n) \right] \Lambda_n(k, P_n) \frac{k - P_n + m}{(k - P_n)^2 - m^2 + i\epsilon},$$

(51)

where $\widehat{V}_n(k, P_n)$ is the proper Dirac structure, and $\Lambda_n(k, P_n)$ the momentum-dependent part, approximated on the LF hyperplane, as discussed below.

In the valence sector, after performing the $k^-$ integration, both pion and VM’s BS amplitudes reduce to 3D amplitudes with one constituent on its mass shell. In [21], the momentum-dependent part of the on-shell VM BS amplitude (that contains on both sides proper Dirac projectors) is described through a LF VM wave function, i.e.

$$P_n^+ \Lambda_n(k, P_n) |_{k^- = k_{on}} = \psi_n(k^+, k_\perp; P_n^+),$$

(52)

and

$$M_0^2(k^+, k_\perp; P_n^+) = P_n^+ \left[ k_{on}^- + (P_n - k)^- \right].$$

In Eq. (52), $\psi_n(k^+, k_\perp; P_n^+)$ is an eigenfunction of the relativistic CQ square mass operator of Ref. [42], as mentioned at the beginning of this Section. Moreover, it is normalized to the probability of the valence Fock state, according to the model elaborated in [21].
The valence component of the pion was modeled adopting an analogous Ansatz. Moreover, in [21] two different calculations were generated by using i) the pion eigenstate of the model in Ref. [42] and ii) the pQCD asymptotic wave function (see, e.g. [27]).

In the non-valence region, namely the only region contributing to the GPD’s for $m_\pi = 0$ (see Eq. (49)), besides the pion valence component in the initial state one has to deal with a non-valence component of the pion state, since the process depicted in Fig. 2(b) can be interpreted as a transition from a state composed by the valence component of the initial pion and the virtual photon, $|q\bar{q}, \gamma^*\rangle$, to a higher Fock component, $|q\bar{q}, q\bar{q}\rangle$, pertaining to the final pion. At level of the pion BS amplitudes, one has to model an off-shell BS amplitude, that takes into account the absorption of the initial pion by an antiquark (according to the case illustrated in Fig. 2(b)). In [21] a simple Ansatz, namely a constant vertex was assumed, like in Ref. [44]. Notice that such a coupling constant is fixed by the normalization of the pion form factor, since the diagram shown in Fig. 2(a) does not contribute, as a consequence of the simplification $m_\pi = 0$.

Within the approach presented in this subsection, since $|\xi| = 1$, (given the vanishing $m_\pi$) the quark GPD has only contribution from $H_{(nv)}^u$, i.e.

$$H^u(x, |\xi| = 1, t) = H_{(nv)}^u(x, |\xi| = 1, t) \theta(1 - x) \theta(1 + x),$$

where, introducing $\kappa = P - k$,

$$H_{(nv)}^u(x, |\xi| = 1, t) = - \sum_n \frac{f_{Vn}}{t - M_n^2} \frac{N_c}{2} \frac{D_{\pi}}{\sqrt{2}} \int_{p^+}^{p'^+} \frac{d\kappa^+}{\kappa^+} \frac{\delta[P^+(1 - x) - \kappa^+]}{(p'^+ - \kappa^+)(p^+ - \kappa^+)} \int d\kappa⊥ \times$$

$$\left\{ \psi_n((p' - \kappa)^+, -\kappa⊥; P_n^+) \frac{[M_n^2 - M_0^2(\kappa^+, \kappa⊥; P_n^+)]}{[t - M_0^2(\kappa^+, \kappa⊥; P_n^+)] + i\epsilon} I_1 + \psi_n^*((p' - \kappa)^+, -\kappa⊥; p'^+) I_2 \right\}$$

(54)

where $D_{\pi}$ is the constant describing the off-shell quark-pion vertex, while $I_1$ and $I_2$ are given by

$$I_1 = - \frac{1}{2} \frac{m}{f_{\pi}} \Lambda((p' - \kappa), p')|_{\kappa^- = p^-, (p' - \kappa)^+}$$

$$\times Tr \left\{ \gamma^+ [(p' - \kappa)^+; on] + m \left[ \tilde{V}_{nz}(p' - \kappa, P_n) \right]_{on} [(\kappa - \kappa)^+; on] \right\},$$

$$I_2 = \frac{1}{2} Tr \left\{ ((\kappa; on) + m) [(p' - \kappa)^+; on] \left[ \tilde{V}_{nz}(p' - \kappa, P_n) \right]_{on} \gamma^+ \right\}$$

$$\times \Lambda_n(p' - \kappa, P_n)|_{\kappa^- = p^-, (p' - \kappa)^+}.$$
The Dirac structure, \([\hat{V}_n^\mu(p' - \kappa, P_n)]_{on}\), where all the constituents are on their own mass-shell, is chosen in order to generate the proper Melosh rotations for \(^3S_1\) states \([38]\). Furthermore, the traces previously shown contain the instantaneous terms (see Eq. (42)) that survive after assuming \(m_\pi = 0\). In order to model the instantaneous part of the vertex functions directly attached to \(\gamma^+\), we performed the following replacements

\[
\frac{m}{f_\pi} \Lambda((p' - \kappa), p')|_{\kappa^- = \gamma^- -(p' - \kappa)^-} \rightarrow C_\pi \psi_\pi(\kappa^+, \kappa_\perp; p'^+) \frac{[m^2 - M^2_0(\kappa^+, \kappa_\perp; p'^+)]}{p'^+} \tag{56}
\]

for the pion, and

\[
\Lambda_n((p' - \kappa), P_n)|_{\kappa^- = \gamma^- -(p' - \kappa)^-} \rightarrow

C_{V,M} \psi_n((p' - \kappa)^+, -\kappa_\perp; P_n^+) \frac{[M^2_n - M^2_0((p' - \kappa)^+, -\kappa_\perp; P_n^+)]}{P_n^+} \tag{57}
\]

for the VM’s, as in \([21]\). In Eqs. (56) and (57), the constants \(C_\pi\) and \(C_{V,M}\) roughly describes the effects of the short-range interaction. Indeed, a relative weight, \(w_{V,M} = C_{V,M}/C_\pi\), can be used as a free parameter. Let us remind that the on-shell part of the BS amplitudes have on the left and right sides the proper Dirac projectors.

Finally, it is worth noting that the results presented in the following Section \([VI\] have been calculated by using all the parameters adopted in \([21]\), but with a CQ mass \(m = 200\) MeV and \(w_{V,M} = -1\) (see \([21]\) for \(m = 265\) MeV and different values for \(w_{V,M}\)). It should be pointed out that only one adjusted parameter is necessary for describing the em form factor in the spacelike region.

The model remains invariant for kinematical transformation, after the approximation we have applied.

**B. Light-front Hamiltonian Dynamics model**

Within a LFHD approach (see \([16]\) for a review of the three forms of the relativistic HD introduced by Dirac in \([45]\)) the Poincaré covariance of the description of the pion can be fully implemented, once the current operator is chosen in order to fulfill the proper commutation rules with respect to all the generators (i.e. both the kinematical and the dynamical ones).

A widely adopted strategy, within the LFHD approach, is to model the em current by using a one-body operator, but in the Drell-Yan frame, i.e. where \(\Delta^+ = 0\). For instance, in this frame the em form factor can be obtained by using only the matrix elements of the
plus component of the current operator, and this allows to overcome some difficulties that manifestly appear for hadrons with angular momentum $\geq 1$ (see [16] and [32] for a general discussion).

In the Drell-Yan frame, $\Delta_\perp \neq 0$ and one can choose $\mathbf{p}'_\perp = -\mathbf{p}_\perp = \Delta_\perp/2$. It is worth noting that only the spacelike region can be addressed, since $\Delta^2 = -|\Delta_\perp|^2$. Moreover, one has $p^+ = p'^+$ and therefore $\xi = 0$ for any $\Delta^2$.

In this section, the LFHD model with CQ’s, already successfully applied for describing the charge form factor and decay constant of the pion [46, 47], is adopted for investigating the DGLAP contribution to the no-helicity flip GPD. This corresponds to consider in the Fock-space expansion of Eq. (24) the diagonal contribution with $n = 2$ constituents (i.e. the valence component, cf also Eq. (39), introducing the explicit representation in terms of overlap of light-cone wave functions (LCWFs) [17, 22]. The quark contribution to the GPD in the region $0 \leq x \leq 1$ can be written in terms of the LCWF $\Psi_\pi(x, \kappa_\perp; \lambda_q, \lambda\bar{q})$ for the quark-antiquark system as

\begin{equation}
H^u(x, \xi = 0, t) = \sum_{\lambda_q, \lambda_q, \lambda\bar{q}} \int \frac{d\mathbf{k}_\perp}{(2\pi)^3} \Psi_\pi^*(x, \mathbf{k}_\perp'; \lambda'_q, \lambda_{\bar{q}}) \times \frac{\bar{u}(x, \mathbf{k}_\perp + \frac{\Delta_\perp}{2}, \lambda'_q) \gamma^+ u(x, \mathbf{k}_\perp - \frac{\Delta_\perp}{2}, \lambda_{\bar{q}})}{\sqrt{k^+ + \frac{\Delta_\perp^2}{2}}} \Psi_\pi(x, \mathbf{k}_\perp; \lambda_q, \lambda_{\bar{q}}) \Psi_\pi(x, \mathbf{k}_\perp; \lambda_q, \lambda_{\bar{q}}),
\end{equation}

where $u(x, \mathbf{k}_\perp, \lambda)$ is a LF Dirac spinor (see, e.g. [38]), and $\lambda_i$ are the spin projections. The perpendicular component of the active quark momenta, $\mathbf{k}_\perp \pm \Delta_\perp/2$, become in the intrinsic frame

\begin{equation}
\kappa_\perp = \mathbf{k}_\perp - (1 - x) \frac{\Delta_\perp}{2}, \quad \kappa'_\perp = \mathbf{k}_\perp + (1 - x) \frac{\Delta_\perp}{2} = \mathbf{k}_\perp + (1 - x) \Delta_\perp,
\end{equation}

with $x$ given by Eq. (1). Notice that in the Drell-Yan frame $x_q = x$ (cf Eq. (3)), since $\xi = 0$.

For the model calculation, we use a phenomenological LCWF which satisfies Poincaré covariance and is eigenstate of the total angular momentum operator in the Light-front dynamics. As outlined in Ref. [46], these properties can be fulfilled by constructing the wave function as the product of a momentum wave function $\psi(x, \mathbf{k}_\perp)$, which is spherically symmetric and invariant under permutations, and a spin wave function, which is uniquely
determined by symmetry requirements. Therefore, within LFHD one has

\[ \Psi_{\pi}(x, \kappa; \lambda_q, \lambda_{\bar{q}}) = \psi_{\pi}(x, \kappa) \sum_{\mu_q, \mu_{\bar{q}}} \left( \frac{1}{2} \mu_q \frac{1}{2} \mu_{\bar{q}} \right) [00] D_{\mu_q \lambda_q}^{1/2} \left[ R_{M}(\kappa) \right] D_{\mu_{\bar{q}} \lambda_{\bar{q}}}^{1/2} \left[ R_{M}(-\kappa) \right], \]  

(60)

where \( \kappa \equiv \{ \kappa_{\perp}, \kappa_z \} \) with

\[ \kappa_z = M_0(x, \kappa_{\perp}) \left( x - \frac{1}{2} \right), \]  

(61)

and the free mass defined by

\[ M_0^2(x, \kappa_{\perp}) = \frac{m^2 + |\kappa_{\perp}|^2}{x(1-x)}. \]  

(62)

The spin-dependent part contains the Melosh rotations \( R_{M}(\kappa) \) which convert the instantform spins of both quark and antiquark into LF spins and ensure the rotational invariance of the pion wave function. The representation of the Melosh rotation is explicitly given by

\[ D_{1/2}{^\lambda \mu} [R_{M}(\kappa)] = \langle \lambda \left| R_{M}(\kappa) \right| \mu \rangle = \langle \lambda \left| m + xM_0(x, \kappa_{\perp}) - i\sigma \cdot (\hat{z} \times \kappa_{\perp}) \right| \mu \rangle. \]  

(63)

For the momentum-dependent part of the pion wave function we adopt the following exponential form used in Refs. [46, 47]

\[ \psi_{\pi}(x, \kappa_{\perp}) = \left[ 2(2\pi)^3 \right]^{1/2} \left( \frac{M_0(x, \kappa_{\perp})}{4x(1-x)} \right)^{1/2} \frac{1}{\pi^{3/4} \beta^{3/2}} \exp \left( -\kappa_{\perp}^2/(2\beta^2) \right). \]  

(64)

The wave function in Eq. (64) is normalized as

\[ \int_0^1 dx \int \frac{d\kappa_{\perp}}{2(2\pi)^3} |\psi_{\pi}(x, \kappa_{\perp})|^2 = 1 \]

(reminding that \( d\kappa_z = dx \ M_0(x, \kappa_{\perp})/[4x(1-x)] \)), and depends on the free parameter \( \beta \) and the quark mass \( m \), which have been fitted to the pion charge radius and decay constant.

Inserting the model wave function of Eq. (60) in the LCWF overlap representation of GPD in Eq. (58), one obtains

\[ H^u(x, \xi = 0, t) = \int \frac{d\kappa_{\perp}}{2(2\pi)^3} \psi_{\pi}(x, \kappa_{\perp}) \psi_{\pi}(x, \kappa_{\perp}) \frac{m^2 + \kappa_{\perp}^2}{x(1-x)} M_0(x, \kappa_{\perp}) M_0(x, \kappa_{\perp}). \]  

(65)

In the forward limit \( \Delta^u \to 0 \), the Melosh rotation matrices combine to the identity matrix and one obtains the ordinary parton distribution as momentum density distribution given by the square of the momentum-dependent part of the wave function [17], i.e. for \( x \geq 0 \) one gets

\[ u(x) = \int \frac{d\kappa_{\perp}}{2(2\pi)^3} |\psi(x, \kappa_{\perp})|^2. \]  

(66)
VI. RESULTS AND DISCUSSION

In this Section the results obtained from the different models described in the previous Sections are presented and discussed. Let us first illustrate the actual values of the parameters entering the three models.

For the covariant model (Sec. IV) the CQ mass and the pion mass have values $m = 220$ MeV and $m_\pi = 140$ MeV, respectively. It should be pointed out that, for some runs, the value $m_\pi = 0$ has been used in order to match the vanishing pion mass adopted for the Mandelstam-inspired model (see Sec. VA). This change will be adequately emphasized whenever applied (in this case the CQ mass is a little bit lowered, i.e. $m = 210$ MeV). The parameter $m_R$ present in the pion Bethe-Salpeter amplitudes is fixed through the pion decay constant, obtaining $m_R = 600$ MeV for the sum-form (Eq. (37)) and $m_R = 1200$ MeV for the product-form (Eq. (38)).

In the Mandelstam-inspired model, as already mentioned, all the parameters are the same ones used in [21], except for i) $w_{VM} = -1$ that yields the relative weight of the instantaneous contributions and ii) the CQ mass, $m = 200$ MeV, i.e. the one adopted in [48] within the same approach for the very detailed description of the nucleon em form factors in both the spacelike and timelike region. As already mentioned, given the complexity of the calculation a simplifying assumption of a vanishing pion mass has been also added. Finally, in the VM dressing of the quark-photon vertex (cf Eq. (50)) up to 20 isovector mesons have been considered in order to have a good convergence even for $t = -10$ (GeV/c)$^2$.

In the LFHD model (see Sec. V B), a CQ mass $m = 250$ MeV and a wave-function parameter $\beta = 319.4$ MeV have been used in order to reproduce the pion charge radius $(r_{ch} = 0.670 \pm 0.02$ fm) and the pion decay constant [47].

First of all, the theoretical models have been compared with available experimental data, in particular the pion em form factor in the spacelike region.

In Fig. 3, it is shown the ratio between the spacelike form factors, calculated by using our models, and the monopole form factor $F_{mon} = 1 /(1 + |t|/m_\rho^2)$ ($m_\rho = .770$ GeV). The relevance of such a presentation of the form factor is twofold: i) dividing by $F_{mon}$ allows one to avoid the log plot that hinders a detailed analysis, ii) more important, one can immediately discriminate between models that produce a divergent charge density at short distances and models that do not (cf, e.g., [51]), since their fall-off is more rapid than $F_{mon}$.
FIG. 3: Pion form factor vs $-t$. Thin dashed line: covariant symmetric model of Ref. [20], with the momentum dependence of the pion Bethe-Salpeter amplitude given by the sum-form of Eq. (37), and $m_\pi = 140$ MeV. Double-dot-dashed line: calculation performed within the LF Mandelstam-inspired model (cf Sec. V A), by using an asymptotic pion wave function [27] with $m_\pi = 0$, and adopting a CQ mass of $m = 200$ MeV (notice that in [21] $m = 265$ MeV). Thick solid line: monopole fit to Lattice data as obtained in Ref. [52], arbitrarily extended in this figure from $-4$ (GeV/c)$^2$ to $-10$ (GeV/c)$^2$ (see text). Thick dot-dashed line: faster-than-monopole fit to Lattice data as obtained in Ref. [52], arbitrarily extended in this figure from $-4$ (GeV/c)$^2$ to $-10$ (GeV/c)$^2$ (see text). Dot-dashed line: the same as the double-dot-dashed line, but with a non perturbative pion wave function, eigenstate of the squared LF mass operator of Ref. [42]. Dotted line: the same as the thin dashed line, but with the product-form of Eq. (38) for the pion Bethe-Salpeter amplitude. Thick dashed line: LFHD model (cf Sec. V B) with a Gaussian pion wave function and the proper Melosh rotations. Experimental data: full dots from the collection of Ref. [49]; open squares, TJLAB data from Ref. [50].

For the sake of completeness, we have also displayed two different fits (thick solid and dot-dashed lines in Fig. 3) to the Lattice data as obtained in Ref. [52]. In that paper, Lattice data have been extrapolated to the experimental pion mass, and they were described up to $t = -4$ (GeV/c)$^2$ both in terms of i) a monopole function $F^{\text{lat}}_\pi(t) = 1/[1 - t/M^2(m_\pi^{\text{phys}})]$ with $M(m_\pi^{\text{phys}}) = 0.727$ GeV and ii) a function with a fall-off faster than the monopole one, i.e. $F^{\text{lat}}_\pi(t) = 1/[1 - t/(p M^2(m_\pi^{\text{phys}}))]^p$ with $p = 1.173 \pm 0.058$ and $M(m_\pi^{\text{phys}}) = 0.757 \pm 0.018$ GeV. In Fig. 3 the Lattice results have been arbitrarily extended by using the previous
functions from $t = -4 \text{ (GeV/c)}^2$ to $t = -10 \text{ (GeV/c)}^2$, with a quite reasonable outcome.

To show the sensitivity of the covariant model of Sec. IV upon the change of the pion mass, a comparison between calculations performed with a vanishing pion mass and with $m_\pi = 140$ MeV is presented in Fig. 4. These calculations are helpful in view of the following comparisons with the Mandelstam-inspired model, where the value $m_\pi = 0$ has been adopted. It is interesting to notice from Figs. 3 and 4 that the sum-form for the BS amplitude is unable to accurately describe the experimental form factor at high values of $|t|$.

In order to illustrate the frame dependence of the Fock decomposition of the form factor, in Fig. 5 the valence and non-valence contributions to the pion form factor (Eq. (27)) within the covariant model based on the product-form and $m_\pi = 0$ are presented. Such a choice for $m_\pi$ is suggested (cf Eq. (35)) by the need to explore the whole range $0 \leq |\xi| \leq 1$. The sum of the two contributions becomes $\xi$ independent, and the result is shown in Fig. 4 by the dot-dashed line. Figure 5 allows us to disentangle the valence and non-valence contributions. Indeed, different values of $\xi$ correspond to different choices of the frame (let us remind that $\xi = 0$ corresponds to the Drell-Yan frame and $\xi = -1$ to the frame where $\Delta_\perp = 0$). Moreover, it is worth noting that the operator “number of constituents” does not...
FIG. 5: Left Panel: valence contribution, $F^{v}_\pi(|\xi|, t)$, to the pion form factor (see Eq. \[25\]), evaluated within the covariant symmetric model of Sec. IV by using the product-form for the momentum-dependent part of the Bethe-Salpeter amplitude (cf Eq. \[38\]) and choosing $m_\pi = 0$ for covering the whole range $0 \leq |\xi| \leq 1$, according to Eq. \[35\]. Right Panel: the same as in the Left Panel, but for the non-valence contribution, $F^{nv}_\pi(|\xi|, t)$, (see Eq. \[26\]). Note the different orientations of the axes in the two Panels, for a straightforward selection of the relevant regions.

commute with the whole set of the Poincaré generators, and therefore a change of frame alters the non-valence content. The knowledge of valence and non-valence contributions in the plane $(\xi, t)$ could impose new constraints to models that aim to go beyond the standard CQM.

After completing the analysis of the pion form factor within our models, in Fig. 6 the isovector GPD for positive $x$, namely $H^{I=1}(x, 0, 0) = u_v(x)/2$ (see Eq. \[19\]), is shown as a function of $x \equiv x_q$ (since for $\xi = 0$ one recovers the longitudinal momentum fraction, Eq. \[3\]). It should be pointed out that, at this stage of our analysis, no evolution has been
applied. The effects of the evolution for the parton distribution will be considered elsewhere, together with a study of the evolution for the whole GPD. Calculations for the covariant model of Sec. IV and the LFHD model of Sec. V B are shown in Fig. 6. Notice that the Mandelstam-inspired LF model presently allows predictions only for $|\xi| = 1$. In order to extend to $\xi = 0$ this approach, a non-vanishing value of $m_{\pi}$ and a bare term, besides the VMD one, should be considered. Thus one can take into account the contribution depicted in Fig. 2(a), that produces the valence term in $H^{I=1}(x, \xi, t)$, but new, non-trivial parameters have to be added (cf the nucleon case in $[48]$).

The comparison in Fig. 6 shows the difficulty of the sum-form, Eq. (37), for the pion Bethe-Salpeter amplitude to give a realistic parton distribution, i.e. to have a vanishing value at the end-points. Reminding that, for $\xi = 0$, the presence of the delta-function in Eq. (40) and the kinematical relations in Eq. (1) impose the correct support $[0, 1]$ for the variable $x$ (remind that for $\xi = 0$, one has $x = x_q$), the sum-form produces a discontinuity at the end-points, i.e. an infinite derivative. It is instructive to correlate such a drawback to the one already seen in Fig. 3, where the sum-form is not able to reproduce the em form factor at high values of $(-t)$. Indeed, in both cases, the high momentum part of the valence component of the pion state is involved. As a matter of fact, for $x = 0$ and $x = 1$, the intrinsic three-momentum becomes infinite (cf Eqs. (61) and (62)), and therefore small distances are involved, just as in the case of the tail of the em form factor, where the influence upon the small-$r$ part of the pion wave function is felt. The more realistic behavior of the product-form (38), can be ascribed to a $|k_\perp|$ fall-off like the one dictated by a BS kernel dominated by a one-gluon-exchange, as already pointed out in Sec. IV. An important, final remark is the clear shift towards small $x$ of the curves evaluated within the covariant model, while the prediction obtained within the LFHD model is symmetric with respect to $x = 1/2$. Such an interesting difference could be explained by the fact that the full covariance of the model of Sec. IV together with its dynamical content, related to the adjusted parameter $m_R$, could take into account some effects beyond the pure $q\bar{q}$ component of the pion state. First, one should note that the valence component, Eq. (39), generates a quark distribution symmetric with respect to $x = 1/2$ and a probability definitely less than 1: for the sum-form $P_{val} = 0.78$ and for the product-form $P_{val} = 0.84$ (see also $[20]$). Then, by using the Fock decomposition of the pion state (see, e.g., $[4]$ for a general discussion), one immediately recognizes contributions from both the $q\bar{q}$ component, (i.e. the valence component) and
Thin dashed line: covariant model of Sec. IV, calculated by using the sum-form, Eq. (37), for the pion Bethe-Salpeter amplitude and $m_\pi = 140$ MeV. Dotted line: the same as the thin dashed line, but for the product form, Eq. (38). Thick dashed line: LFHD model of Sec. V B with a Gaussian pion wave function and the proper Melosh rotations. The variable $x$, given in Eq. (1), coincides with the usual LF longitudinal fraction $x_q$, since $\xi = 0$ (see text below Eq. (10)).

from other components with more constituents (see e.g. the instantaneous contributions in Eqs. (B10), (B11) and (B12) and the analysis in [55]). Thus, the active quark shares the longitudinal momentum of the pion with more than one spectator parton, belonging to the Fock space configuration beyond the valence one. Therefore, the shift toward values of $x$ less than 1/2 is expected, since our covariant model contains more physical effects than the basic one. In particular, for a non vanishing pion mass the average longitudinal momentum fraction for the sum-form is $<x_q> \sim 0.483$ and for the product-form is $<x_q> \sim 0.471$, i.e. quite similar, but a little bit different from 1/2. As a simple cross-check we have reobtained those values also from $A_{1,0}^L(0) = <x_q>$ (cf Eq. (22) with $j = 0$ and Eq. (18)).

A more detailed analysis of the parton distribution can be achieved by using the chiral-even TMD distribution, $f_1(x, |k_\perp|)$, see Eq. (20). In Fig. 7, the TMD distributions calculated within the covariant model by using the different BS amplitudes of Eqs. (37) and (38) are shown. In order to avoid log plot, $f_1(x, |k_\perp|)$ has been divided by $G(|k_\perp|) = 1/(1 + |k_\perp|^2/m_\rho^2)^4$, (with $m_\rho = 770$ MeV)). Clearly, the product-form has a $|k_\perp|$ fall-off faster than the sum-form does, i.e. low transverse-momentum partons are favored in the first case.
FIG. 7: Transverse-momentum dependent function, \( f_1(x,|k_{\perp}|^2)/G(|k_{\perp}|) \), with \( G(|k_{\perp}|) = 1/(1 + |k_{\perp}|^2/m^2)^4 \). Left Panel: sum-form of the Bethe-Salpeter amplitude (see Eq. (37)). Right Panel: the same as in the Left Panel, but for the product-form of the Bethe-Salpeter amplitude (see Eq. (38)). The normalization is given by \( \int_0^1 dx \int d|k_{\perp}| \, f_1(x,|k_{\perp}|^2) = 1 \), and \( k_{\perp} \) means \( |k_{\perp}| \).

The analysis of both the generalized form factors involved in the second moment of the isoscalar pion GPD, i.e. \( A_{I=0}^{2,0}(t) \) and \( A_{I=0}^{2,2}(t) \) (cf Eq. (22) with \( j = 0 \)), has to be performed necessarily within the covariant analytic model of Sec. IV. This is obvious if we look at Eq. (17), where the polynomiality imposes a square dependence upon \( \xi \), and therefore one needs a model that covers an extended range for the variable \( \xi \). Indeed, for each value of \( t \), we have first numerically checked the parabolic behavior against \( \xi \), and then we have extracted the coefficients of the parabolic fit getting the values of \( A_{I=0}^{2,0}(t) \) and \( A_{I=0}^{2,2}(t) \). Figure 8 shows a comparison between i) recent results from Lattice QCD, extrapolated to the physical pion mass \( 53, 54 \), ii) our covariant calculations evaluated with both \( m_\pi = 0 \) and \( m_\pi = m_{phys} \) by using the sum- and the product-form for the BS amplitude (Eqs. (37) and (38)) and iii) the
LFHD result (see Sec. V B) for $A_{2,0}^{I=0}(t)$ only, since this approach at the present stage allows one to perform calculations exclusively for $\xi = 0$. Indeed, the ratios $A_{2,0}^{I=0}(t)/A_{2,0}^{I=0}(0)$ and $A_{2,2}^{I=0}(t)/A_{2,2}^{I=0}(0)$ are presented in order to get rid of the evolution (see Ref. [13] for a detailed discussion of this issue). The Lattice calculations are described through a monopole form, $1/(1-t/M_{2,t}^2)$, as obtained in [53] from the analysis of their Lattice data, without evolution and with evolution in the $\overline{MS}$ scheme at the scale $\mu = 2$ GeV. In particular, we have used the following values: $M_{2,0} = 1.329 \pm 0.058$ GeV and $M_{2,2} = 0.89 \pm 0.25$ GeV, corresponding to an analysis of the Lattice data that satisfies the low-energy theorem, i.e. $A_{2,0}^{I=0}(0) = -4A_{2,2}^{I=0}(0)$.

The uncertainties on the previous masses generate the shaded areas in the Left and Right panels in Fig. 8.

Unfortunately, i) the available range of $(-t)$ (we refrained to enlarge the interval as we did in the case of the em form factor, since we do not have experimental data yielding confidence in an arbitrary extension of the monopole fit) and ii) the large uncertainties in the Lattice calculations of $A_{2,2}^{I=0}$ do not allow us to elaborate too much on the comparison between our phenomenological models and the Lattice results. On the other hand, for large values of $|t|$ the calculations obtained by using the covariant model with the product-form and $m_\pi = 140$ MeV could give some insight on the expected behavior of the Lattice calculations, since one could argue that the covariant model with the product-form phenomenologically contains at some extent dynamical features typical of QCD, like the one-gluon-exchange dominance at small distances. In order to complete the information, in Table I the values of $A_{2,0}^{I=0}(0)$ and $A_{2,2}^{I=0}(0)$ are shown. It is worth noting that while the Lattice calculations largely fulfill the low-energy theorem, as already mentioned, our calculations do not. Furthermore, it should be pointed out that for small $t$ the disagreement between Lattice data and the calculation with the covariant approach at some extent is an expected one, since the mechanism responsible for the confinement is not present in our model, and therefore we have a free propagation of the $q\bar{q}$ pair. A possible solution could be elaborated following the suggestion in Ref. [56], where a covariant model without the disturbing free propagation of the $q\bar{q}$ pair was proposed and applied to the em decays of the vector mesons.

The previous figures have illustrated ”integral” properties of the pion GPD’s, like em form factor and the generalized ones, or the parton distribution, i.e. $H^{I=1}(x, 0, 0)$. In the following figures, the isoscalar and isovector GPD’s are shown in the plane $(x, t)$ with $-1 \leq x \leq 1$ and $-10 \text{ (GeV/c)}^2 \leq t \leq 0$, but with fixed values for $\xi$, as dictated by the two phenomenological
TABLE I: Gravitational form factors at $t = 0$, (cf Eq. (22) with $j = 0$), obtained i) within the covariant model of Sec. IV and both the sum- and the product-form for the BS amplitude, Eqs. (37) and (38); and ii) from the Lattice data of Ref. [53].

|         | Sum $m_\pi = 0$, Sum $m_\pi = m_{phys}$ | Product $m_\pi = m_{phys}$ | Latt. no evol. $m_\pi = m_{phys}$ | Latt. with $\overline{MS}$ evol. $m_\pi = m_{phys}$ |
|---------|--------------------------------------|-----------------------------|-----------------------------------|-------------------------------|
| $A_{2,0}^{I=0}(0)$ | 0.4828 | 0.4833 | 0.4707 | 0.4710 | 0.365 | 0.261 |
| $A_{2,2}^{I=0}(0)$ | -0.0307 | -0.0272 | -0.0357 | -0.0327 | -0.092 | -0.066 |

FIG. 8: Left Panel: the ratio $A_{2,0}^{I=0}(t)/A_{2,0}^{I=0}(0)$, involving the generalized form factor $A_{2,0}^{I=0}(t)$ that appears in the second moment of the isovector GPD $H^{I=0}$ (cf Eqs. (17) and (22)) as a function of $t$. Solid line: sum-form for the pion Bethe-Salpeter amplitude, Eq. (37), and $m_\pi = 0$. Dashed line: the same as the solid line, but with $m_\pi = 140$ MeV. Dot-dashed line: product-form for the pion Bethe-Salpeter amplitude, Eq. (38), and $m_\pi = 0$. Dotted line: the same as the dash-dotted line, but with $m_\pi = 140$ MeV. Thick long-dashed line: LFHD model (cf Sec. V B) with a Gaussian pion wave function and the proper Melosh rotations. Shaded area: results from Lattice QCD [53] (see text). Right Panel: the same as the Left Panel, but for $A_{2,2}^{I=0}(t)/A_{2,2}^{I=0}(0)$.

models, namely $|\xi| = 1$ for the Mandelstam-inspired model (Sec. V) and $\xi = 0$ for the LFHD model (Sec. V B), respectively. The covariant model (Sec. IV), in its two versions for the momentum dependence (Eqs. (37) and (38)), will be compared to the results for the two phenomenological models, that, in some sense, represent two extrema, in the Fock language: the first model is basically related to the non-valence (ERBL) region, the second
one is related to the valence (DGLAP) domain. In order to cover the whole range of \( \xi \) for the given interval of \( t \) (i.e. \(-10 \text{ (GeV/c)}^2 \leq t \leq 0\)) the covariant model has been evaluated by assuming \( m_\pi = 0 \), as already pointed out. Finally, let us stress that the GPD’s are divided by \( F_{\text{mon}} \), as in the case of the em form factor, for avoiding log plot and for emphasizing as many details as possible. In Fig. 9 the results of the covariant symmetric model are shown for \( |\xi| = 1 \), in order to be compared with the calculations performed by using the Mandelstam-inspired model, presented in Fig. 10. We remind that the phenomenological model has a photon-quark vertex dressed by a microscopical VMD, as discussed in Sec. V.

In Fig. 11 the no-helicity flip GPD’s of the covariant symmetric model are shown for \( \xi = 0 \), allowing a comparison with the calculations performed by using the LFHD model, presented in Fig. 12. For \( \xi = 0 \), where only the valence component is acting and \( |x| = x_q \), a nice feature, stemming from the figures of both isoscalar and isovector GPD’s, is shared by all the presented models: in the limit of large \(|t|\) the collinearity clearly emerges, as shown by the migration of the maximum (minimum) value from \(|x| \approx 1/2\) for \( t = 0 \) toward \(|x| \sim 1\) for \(|t| \to \infty\). Such a behavior can be easily understood in the LFHD model, since non vanishing contributions to GPD (cf Eq. 65) can be obtained if \( \kappa'_{\perp} \) in Eq. 59 does not depend too much from \( \Delta_{\perp} \), namely \( x \sim 1 \) (notice that for \( x \) exactly 1, the free mass blows up and the wave functions become vanishing, as well as GPD’s). Correspondingly, for \(|\xi| = 1\), where only the non-valence component is acting, the relevance of the \( x \) region around \( \pm 1 \) can be explained by the pair-production mechanism. For simplicity, let us consider large values of \(|t|\), that amount to large values of \( \Delta^+ = \Delta_z \) (remind that in the Breit frame \( \Delta^0 = 0 \)). Then, using \( \Delta^+ = k^+_q + k^+_\bar{q} = k_{zq} + k_{\bar{z}q} \approx 2k_{zq} \geq 0 \) (given our choice for the sign of \( \Delta^+ \)), and the fact that each quark in the pair is almost on its mass-shell, we can approximate \( 2k^+ = k^+_q - k^+_\bar{q} \approx 2E_q = 2\sqrt{m^2 + |\vec{k}_q|^2} \). Thus, one can see that, when \( \xi = -1 \), \( x \) becomes close to 1 for \( \Delta_z \gg m \), since \( x = k^+ / P^+ = 2k^+ / \Delta^+ \sim E_q / k_{zq} \to 1 \). The case \( x = -1 \) can be obtained for \( \Delta^+ \leq 0 \).

Finally in Fig. 13 the isovector GPD, evaluated within the covariant model adopting the product-form of Eq. 38 and \( m_\pi = 0 \), is shown for the case \( x = |\xi| \) and \( 0 \geq t \geq -10 \text{ (GeV/c)}^2 \). This kinematical region, where the transition from DGLAP to ERBL regimes occurs, should be relevant for the experimental studies of the single spin asymmetry (see, e.g. the discussion in 4). Let us notice that in our covariant analytic model the GPD is continuous at \( x = |\xi| \).
FIG. 9: Upper Left Panel: Isoscalar no-helicity flip GPD from the covariant symmetric model of Sec. [IV] with the sum-form for the Bethe-Salpeter amplitude (Eq. (37)) at $|\xi| = 1$. The value of $\xi$ is fixed by using $m_\pi = 0$ (cf Eq. (35)), for the sake of comparison with the microscopic model of Sec. [V] whose results are shown in Fig. [10]. On the z-axis the ratio with respect to $F_{\text{mon}} = 1/(1 + |t|/m_\rho^2)$ is presented. Upper Right Panel: the same as in the Upper Left Panel, but for the isovector GPD. Lower Panels: the same as in Upper Panels, but for the product-form for the Bethe-Salpeter amplitude (see Eq. (38)).
FIG. 10: Left Panel: Isoscalar no-helicity flip GPD from the Mandelstam-inspired model of Sec. 5 at $|\xi| = 1$ (see text). Right Panel: the same as in the Left Panel, but for the isovector GPD.

From the 3D plots, one can see that the covariant model, in the version with the product-form for the momentum-dependent part of the BS amplitude, is able to reproduce quite satisfactorily the GPD’s evaluated within the two phenomenological models, the Mandelstam-inspired and the LFHD ones, and therefore one could argue that it contains the main ingredients for a realistic descriptions of the constituents inside the pion. In view of this, it appears challenging to test the covariant model (or its refinements [57] based on the Nakanishi representation, see, e.g. [36], for a recent applications to a bosonic system) of the BS amplitude, in comparisons with experimental data, whose analysis requires the knowledge of the pion GPD’s.
FIG. 11: Upper Left Panel: Isoscalar no-helicity flip GPD from the covariant symmetric model of Sec. [IV] with the sum-form for the Bethe-Salpeter amplitude (Eq. (37)) at \( \xi = 0 \), and \( m_\pi = 0 \).
Upper Right Panel: the same as in the Left Panel, but for the isovector GPD. Lower Panels: the same as in the Upper Panels, but for the product-form for the Bethe-Salpeter amplitude (Eq. (38)).
FIG. 12: Left Panel: Isoscalar no-helicity flip GPD from the LFHD model of Sec. V B at $\xi = 0$ (see text). Right Panel: the same as in the Left Panel, but for the isovector GPD.

VII. CONCLUSION

In this paper, we have investigated the no-helicity flip Generalized Parton Distributions of the pion by using three models, based on a description of the pion where constituent quarks with masses between 200 MeV and 250 MeV are considered. In particular, we have evaluated the isoscalar and isovector GPD’s adopting a covariant, analytic model and two Light-front phenomenological models. It is important to notice that the first model, based on 4D Ansatzes for the Bethe-Salpeter amplitudes, allows us to explore the whole kinematical domain of the three variables $x$, $\xi$ and $t$ upon which the GPD’s depend, while the others two are, presently, constrained to a given value of $\xi$. The second model, the Mandelstam-inspired model of Sec. V is a natural extension of the approach proposed in Ref. [21] for a successful investigation of the em form factor of the pion in both the space- and timelike regions. Main features of the model are: i) a microscopical Vector Meson Model dressing
FIG. 13: Isovector no-helicity flip GPD from the covariant symmetric model of Sec. IV, with the product-form for the Bethe-Salpeter amplitude (Eq. (38)) at $|\xi| = x$, and $m_\pi = 0$.

for the quark-photon vertex and ii) proper Ansatzes for the 3D LF projection of the BS amplitudes of both pion and vector mesons, taken as the eigenfunctions of a LF square mass operator [42]. As in [21], the assumption $m_\pi = 0$ is added, and this simplification allows calculations of the GPD’s only for the value $|\xi| = 1$ (cf Eq. (35)), namely the non-valence region covers the whole range $1 \geq x \geq -1$.

On the contrary, the LFHD model of Sec. V B is based on a Poincaré covariant description of the pion, with a proper treatment of the spin wave functions, due to the presence of the Melosh rotations. The momentum part of the pion wave function is given by a Gaussian function, that contains the dynamical input of the model through two adjusted parameters. A bare quark-photon vertex is assumed. It is worth noting that the model yields a description of the GPD’s for $\xi = 0$, i.e. the valence region can be investigated.

The covariant symmetric model of Sec. IV, based on a Mandelstam formula for matrix elements of the operators yielding the isoscalar and isovector GPD’s, allows us to have close
expressions for the physical quantities, since analytic forms for the momentum-dependent part of the Bethe-Salpeter amplitude are adopted and a bare quark-photon vertex is assumed as well. Such a covariant model can be applied for any value of $x$, $\xi$ and $t$, and therefore can be used for interpolating between the two phenomenological models. A peculiar feature is given by the presence of instantaneous terms, both in the valence and non-valence regions, since we fully take into account the analytic structure of the BS amplitude.

The comparison with the em form factor (Fig. 3) suggests that the covariant model with a sum-form of the BS amplitude has a non-realistic increasing behaviour with respect to $F_{\text{mon}}$, for large $|t|$, which leads to a divergent density at short distances, while the version with a product-form together with the LFHD model decrease more rapidly than $F_{\text{mon}}$. Finally, the Mandelstam-inspired model and the Lattice results (red curve in Fig. 3), arbitrarily extended from $t = -4 \ (\text{GeV}/c)^2$ to $t = -10 \ (\text{GeV}/c)^2$, given the analytic form proposed in [52] for extrapolating the Lattice data to the physical $m_\pi$, show a moderate decreasing with respect to $F_{\text{mon}}$, for large $|t|$. Such a comparison for the em form factor and the analysis of the parton distribution in Fig. 6 point to the relevance of the behavior of the pion valence function (or better the momentum part of the BS amplitude) for large transverse momentum.

In particular the product-form, that has a behavior at large transverse momentum $|\mathbf{k}_T|$ compatible with the one suggested by the one-gluon-exchange dominance (see, e.g. [36]), seems to give a consistent description of both the tail of the em form factor and the end-point fall-off of the parton distribution. With respect to this finding, more details can be gained from the investigation of the chiral-even transverse-momentum dependent distribution, as shown in Fig. 7.

Another important step in the characterization of the covariant model is given by the comparison of the generalized form factors with the Lattice results. For the present, the comparison is restricted to the gravitational form factors, $A_{2,0}^{I=0}(t)$ and $A_{2,2}^{I=0}(t)$, that appear in the second moment of the isovector GPD, $H_{I=0}$, (cf Eqs. (17) and (22)). Indeed, for $A_{2,0}^{I=0}(t)$ we have presented results from both our covariant model and the LFHD approach, while for $A_{2,2}^{I=0}(t)$ only the covariant calculations are available (let us remind that calculations with $\xi \neq 0$ are necessary for disentangling both form factors). Unfortunately, since Lattice data have been obtained in a $t$-interval not too wide and are affected by large uncertainties, one cannot yet draw stringent conclusions from the comparison shown in Fig. 8. However, the encouraging agreement between model calculations and Lattice data for both ratios,
\[ A_{0,2}^{f=0}(t)/A_{0,2}^{f=0}(0) \] and \[ A_{2,2}^{f=0}(t)/A_{2,2}^{f=0}(0) \], suggests to extend our analysis also to the spin-flip GPD’s, since Lattice results are available for the lowest moments, in order to explore the onset of the dominance of a one-gluon-exchange mechanism for a light hadron.

To complete our analysis, we have studied the GPD’s in the \((x, t)\) plane for fixed values of \(\xi\), i.e. \(|\xi| = 0, 1, x\). These values are representative of different, interesting cases. The first one, \(\xi = 0\), involves contributions to GPD’s only in the valence region, while the second one involves contributions only from the non-valence one. Finally, the case \(|\xi| = x\) illustrates the transition from the DGLAP region to the ERBL one. The covariant model can explore the whole 3D space of the variables \((x, \xi, t)\) and it is compared with LFHD model for \(\xi = 0\), and with the Mandelstam-inspired model for \(|\xi| = 1\), while for \(|\xi| = x\) shows a smooth transition from DGLAP region to the ERBL one, given the continuity of the model. It should be pointed out that the covariant model with the product-form for the Bethe-Salpeter amplitude exhibits an overall agreement with the Mandelstam-inspired model, for \(|\xi| = 1\), and with the LFHD model, for \(\xi = 0\). Therefore, from these findings one could conjecture that the general shape, illustrated by the previous covariant model and the phenomenological ones, is a typical feature of the pion GPD’s, dictated from both kinematical arguments (cf the discussion at the end of Sec. VI) and the dynamical input reflected by the proper fall-off of the momentum distribution (cf the one-gluon exchange dominance at short distances).

Further analyses, to make more and more realistic the models presented in this paper, are in progress.

**Acknowledgments**

This work was partially supported by the Brazilian agencies CNPq and FAPESP and by Ministero della Ricerca Scientifica e Tecnologica. It is also part of the Research Infrastructure Integrating Activity “Study of Strongly Interacting Matter” (acronym HadronPhysics2, Grant Agreement n. 227431) under the Seventh Framework Programme of the European Community. T. F. acknowledges the hospitality of the Dipartimento di Fisica, Università di Roma ”Tor Vergata” and of Istituto Nazionale di Fisica Nucleare, Sezione Tor Vergata and Sezione di Roma.
APPENDIX A: KINEMATICS

Following the notations of Fig. 1, where $\Delta^+ \geq 0$, one obtains from Eq. (11) that $0 \geq \xi$.

In the valence region, for a quark, one has: i) in the initial state, $p^+ \geq k^+ - \Delta^+/2 \geq 0$, i.e. $P^+ \geq k^+ \geq \Delta^+/2$, (notice that necessarily the spectator constituent is an antiquark, since $0 \geq k^+ - P^+$) and then $1 \geq x \geq -\xi$; ii) in the final state, $p'^+ \geq k^+ + \Delta^+/2 \geq 0$, i.e. $P^+ \geq k^+ \geq -\Delta^+/2$, and then $1 \geq x \geq \xi$. Therefore in the valence region, one gets the interval $1 \geq x \geq -\xi$, and given our choice for $\xi$ one has $1 \geq x \geq |\xi|$.

For an antiquark in the initial pion, the four-momentum is $k + \Delta/2$, while the spectator quark has four-momentum $k + P$. In the final pion, the antiquark four-momentum is $k - \Delta/2$. The antiquark plus components are negative both in the initial and in the final pion. Therefore i) $p^+ \geq -(k^+ + \Delta^+/2) \geq 0$ that leads to $\xi \geq x \geq -1$ and ii) $p'^+ \geq -(k^+ - \Delta^+/2) \geq 0$, i.e. $-\xi \geq x \geq -1$. Summarizing, for an antiquark in the valence region one finds $-|\xi| \geq x \geq -1$.

In the non-valence region, one has to deal with a $q\bar{q}$ production, i.e. $0 > k^+ - \Delta^+/2$ and $k^+ + \Delta^+/2 > 0$ (see Fig. 2), and those constraints translate into $\xi < x < -\xi$. The $q\bar{q}$ annihilation is prevented by the choice of a positive $\Delta^+$. In order to have general extrema, holding for both positive and negative $\Delta^+$, one can write $|\xi| > x > -|\xi|$.

APPENDIX B: INTEGRATION ON $k^-$

In this Appendix, the no-helicity flip GPD for the symmetric covariant models (see Sec. 1V) calculated using Eq. (40) and the momentum dependent part of the BS amplitude, given by Eqs. (37) or (38).

The evaluation of the trace in Eq. (41) can be simplified according to the decomposition of the Dirac propagator shown in Eq. (42) and reminding that $[\gamma^+]^2 = 0$. By introducing the variable $\kappa = P - k$, one has

$$\text{Tr}[O^+(\kappa^-)] = \text{Tr} \left\{ (\not\kappa + m) (\not{p}' - \not{\kappa} + m) \gamma^+ (\not{p} - \not{\kappa} + m) \right\} =$$

$$= \text{Tr}[O^+(\kappa^-_{on})] + \frac{(\kappa^- - \kappa^-_{on})}{2} \text{Tr} \left\{ \gamma^+ [(\not{p}' - \not{\kappa})_{on} + m] \gamma^+ [(\not{p} - \not{\kappa})_{on} + m] \right\} =$$

$$= -4 \left\{ \kappa^+ \left[(p' - \kappa)_{on} \cdot (p - \kappa)_{on} - m^2\right] - (p'^+ - \kappa^+) \left[\kappa_{on} \cdot (p - \kappa)_{on} - m^2\right] +
\right.$$ 

$$\left. - (p^+ - \kappa^+) \left[(p' - \kappa)_{on} \cdot \kappa_{on} - m^2\right]\right\} + 4 \left((\kappa^- - \kappa^-_{on}) \cdot (p^+ - \kappa^+) \cdot (p^+ - \kappa^+) \right)$$

(B1)
where

\[
Tr[\mathcal{O}^+(\kappa^-_\text{on})] = Tr \left\{ (\hat{c}_{\text{on}} + m) (\hat{\phi}' - \hat{\phi})_{\text{on}} + m \right\} \gamma^+ \left[ (\hat{p} - \hat{\phi})_{\text{on}} + m \right] \]  

(B2)

After performing the scalar products, one gets

\[
Tr[\mathcal{O}^+(\kappa^-)] = 4 \quad p'^+ p^+ \quad \kappa^-_\text{on} - \kappa^+ \quad |\Delta_-|^2 + 2 \Delta^+ \quad \kappa_\perp \cdot \Delta_- + 4 \quad (\kappa^- - \kappa^-_\text{on}) \quad (p'^+ - \kappa^+) \quad (p^+ - \kappa^+) 
\]

(B3)

Given the simple expression adopted for the momentum dependence of the BS amplitude (see Eqs. (37) and (38)), the analytic integration on \( k^- \) can be easily performed in Eq. (40).

By using the LF variables (i.e. \( d^4 \kappa \rightarrow d\kappa^+ d\kappa^- d\kappa_\perp / 2 \)) one obtains

\[
H^u(x, \xi, t) = -iN_c \mathcal{R} \int \frac{d\kappa^+ d\kappa^- d\kappa_\perp}{4(2\pi)^4} \delta \left[ P^+(1 - x) - \kappa^- \right] \frac{Tr[\mathcal{O}^+(\kappa^-)]}{\kappa^+ (p^+ - \kappa^+) (p'^+ - \kappa^+)} \times \\
\left( \kappa^- - \kappa^-_\text{on} + i \frac{\epsilon}{\kappa^+} \right) \left[ p^- - \kappa^- - (p - \kappa^-_\text{on}) + i \frac{\epsilon}{(p^+ - \kappa^+)} \right] \left[ p'^- - \kappa^- - (p' - \kappa^-_\text{on}) + i \frac{\epsilon}{(p'^+ - \kappa^+)} \right] \times \\
\Lambda(\kappa, p') \Lambda(\kappa, p) 
\]

(B4)

where

\[
\kappa^-_\text{on} = \frac{m^2 + |\kappa_\perp|^2}{\kappa^+} \quad (p - \kappa^-)_\text{on} = \frac{m^2 + |p_\perp - \kappa_\perp|^2}{(p^+ - \kappa^+)} \quad (p' - \kappa^-)_\text{on} = \frac{m^2 + |p'_\perp - \kappa_\perp|^2}{(p'^+ - \kappa^+)} 
\]

(B5)

In the integration over the minus component \( \kappa^- \) one faces with the following six poles (coming from the BS amplitudes and the Dirac propagators)

\[
\kappa^-_{1(2)} = \kappa^-_\text{on(R)} - i \frac{\epsilon}{\kappa^+} \\
\kappa^-_{3(4)} = p^- - (p - \kappa^-)_\text{on(R)} + i \frac{\epsilon}{(p^+ - \kappa^+)} \\
\kappa^-_{5(6)} = p'^- - (p' - \kappa^-)_\text{on(R)} + i \frac{\epsilon}{(p'^+ - \kappa^+)} 
\]

(B6)

where \( \kappa^-_R, (p - \kappa^-)_R \) and \( (p' - \kappa^-)_R \) can be obtained from the corresponding quantities in Eq. (B5) by substituting \( m \rightarrow m_R \). Notice that \( \kappa^-_R \) can appear both as a single and as a double pole.

It is easily seen that the analytic integral (B4) is not vanishing only if \( p'^+ \geq \kappa^+ \geq 0 \). Furthermore we can recognize two subinterval i) \( p^+ \geq \kappa^+ \geq 0 \), or valence region, and ii)
\( p^+ \geq \kappa^+ \geq p^+ \), the non-valence region. Let us stress that Eq. (B.4) is vanishing for \( x < -|\xi| \), since in this case \( \kappa^+ = p^+ (1 - x) > p^+ (1 + |\xi|) = p^+ \).

In the valence region, only the poles \( \kappa_1^- \) and \( \kappa_2^- \) belong to the lower semiplane. In the non-valence region, only \( \kappa_5^- \) and \( \kappa_6^- \) belong to the upper semiplane.

To obtain the no-helicity flip GPD in the valence region, let us integrate over \( \kappa^- \) closing the contour in the lower semiplane. The contribution from \( \kappa_1^- \) reads as follows

\[
H^u(x, \xi, t) = -\frac{N_c}{4(2\pi)^3} \int d\kappa_{\perp} \int_0^{p^+} d\kappa^+ \frac{\delta [P^+(1 - x) - \kappa^+]}{\kappa^+ (p^+ - \kappa^+) (p^+ - \kappa^+)} Tr[\mathcal{O}^+(\kappa^-)]
\times \frac{\Lambda(\kappa, p)|_{\kappa_{\text{on}}^-}}{[p^+ - \kappa_{\text{on}}^- - (p - \kappa)^-_{\text{on}}]} \frac{\Lambda(\kappa, p')|_{\kappa_{\text{on}}^-}}{[p^+ - \kappa_{\text{on}}^- - (p - \kappa)^-_{\text{on}}]}
\]

(B7)

where for the sum-form, Eq. (37), one has

\[
\Lambda(\kappa, p)|_{\kappa_{\text{on}}^-} = C_1 \left\{ \frac{1}{(p^+ - \kappa^+)} \left[ \frac{1}{\kappa^+ (\kappa^-_{\text{on}})} + \frac{1}{\kappa^+(\kappa^-_{\text{on}} - \kappa^- R)} \right] \right\}
\]

(B8)

and for the product-form, Eq. (38), one has

\[
\Lambda(\kappa, p)|_{\kappa_{\text{on}}^-} = C_2 \left\{ \frac{1}{(p^+ - \kappa^+)} \left[ \frac{1}{\kappa^+ (\kappa^-_{\text{on}})} + \frac{1}{\kappa^+(\kappa^-_{\text{on}} - \kappa^- R)} \right] \right\}
\]

(B9)

For the sum-form, the pole \( \kappa^- R \) generates a contribution as a single pole and a contribution as a double pole.

The single-pole contribution is given by

\[
H^u_{(v)1}(x, \xi, t) = -\frac{N_c}{4(2\pi)^3} C_1^2 \int d\kappa_{\perp} \int_0^{p^+} d\kappa^+ \frac{\delta [P^+(1 - x) - \kappa^+]}{(\kappa^+)^2 (p^+ - \kappa^+) (p^+ - \kappa^+)} Tr[\mathcal{O}^+(\kappa^-)]
\times \frac{1}{(\kappa^- R - \kappa^-_{\text{on}})} \left[ \frac{1}{[p^+ - \kappa^- R - (p - \kappa)^-_{\text{on}}]} \right]
\times \left\{ \frac{1}{(p^+ - \kappa^+)} \left[ \frac{1}{\kappa^+ (\kappa^-_{\text{on}})} + \frac{1}{\kappa^+(\kappa^-_{\text{on}} - \kappa^- R)} \right] \right\}
\]

(B10)

and the double-pole contribution is given by

\[
H^u_{(v)2}(x, \xi, t) = -\frac{N_c}{4(2\pi)^3} C_1^2 \int d\kappa_{\perp} \int_0^{p^+} d\kappa^+ \frac{\delta [P^+(1 - x) - \kappa^+]}{(\kappa^+)^3 (p^+ - \kappa^+) (p^+ - \kappa^+)} \times \frac{d}{d\kappa^-} \left\{ Tr[\mathcal{O}^+(\kappa^-)] \right\}_{\kappa^- R}
\]

(B11)

For the product-form, the pole \( \kappa^- R \) generates only a double-pole contribution, given by

\[
H^u_{(v)2}(x, \xi, t) = -\frac{N_c}{4(2\pi)^3} C_2^2 \int d\kappa_{\perp} \int_0^{p^+} d\kappa^+ \frac{\delta [P^+(1 - x) - \kappa^+]}{(\kappa^+)^3 (p^+ - \kappa^+) (p^+ - \kappa^+)^2}
\times \]
\[
\frac{d}{d\kappa^-} \left\{ \text{Tr}[O^+ (\kappa^-)] \right\} \frac{1}{(\kappa^- - \kappa^-_{\text{on}}) \left[ p^- - \kappa^- - (p - \kappa)_{\text{on}} \right] \left[ p^- - \kappa^- - (p' - \kappa)_{\text{on}} \right]} \times \frac{1}{\left[ p^- - \kappa^- - (p - \kappa)_{\text{on}} \right] \left[ p^- - \kappa^- - (p' - \kappa)_{\text{on}} \right]} \right\} \bigg|_{\kappa^-_{\text{R}}}
\]

(B12)

The contribution in the non-valence region can be evaluated by considering the poles $\kappa^-_{\text{R}}$ and $\kappa^-_{\text{R}}$. In particular the contribution from $\kappa^-_{\text{R}}$ has the same form for both choices of the BS amplitudes, i.e.

\[
H^u_{(nv)5}(x, \xi, t) = -\frac{N_e \mathcal{R}}{4(2\pi)^3} \int d\kappa_\perp \int_{p^+}^{p'^+} d\kappa^+ \frac{\delta [P^+(1 - x) - \kappa^+]}{\kappa^+(p^+ - \kappa^+) (p'^+ - \kappa^+)^2} \times \text{Tr}[O^+ (p^+ - (p - \kappa)_{\text{on}})] \frac{\Lambda(\kappa, p') \left[ \kappa^+_{\text{on}} - (p' - \kappa)_{\text{on}} \right]}{\left[ (p^+ - (p - \kappa)_{\text{on}}) \left[ (p'^+ - (p' - \kappa)_{\text{on}}) - (p - \kappa)_{\text{on}} \right] \right]} \times \frac{1}{\left[ p^+ - p'^+ + (p' - \kappa)_{\text{R}} - (p - \kappa)_{\text{on}} \right]}
\]

(B13)

while the contributions from $\kappa^-_{\text{R}}$, reads differently for the sum-form, viz

\[
H^u_{(nv)6}(x, \xi, t) = -\frac{N_e \mathcal{R}}{4(2\pi)^3} C_1 \int d\kappa_\perp \int_{p^+}^{p'^+} d\kappa^+ \frac{\delta [P^+(1 - x) - \kappa^+]}{\kappa^+(p^+ - \kappa^+) (p'^+ - \kappa^+)^2} \times \text{Tr}[O^+ (p^+ - (p - \kappa)_{\text{on}})] \frac{\Lambda(\kappa, p') \left[ \kappa^+_{\text{on}} - (p' - \kappa)_{\text{on}} \right]}{\left[ (p^+ - (p - \kappa)_{\text{on}}) \left[ (p'^+ - (p' - \kappa)_{\text{on}}) - (p - \kappa)_{\text{on}} \right] \right]} \times \frac{1}{\left[ p^+ - p'^+ + (p' - \kappa)_{\text{R}} - (p - \kappa)_{\text{on}} \right]}
\]

(B14)

and for the product-form, viz

\[
H^u_{(nv)6'}(x, \xi, t) = -\frac{N_e \mathcal{R}}{4(2\pi)^3} C_2 \int d\kappa_\perp \int_{p^+}^{p'^+} d\kappa^+ \frac{\delta [P^+(1 - x) - \kappa^+]}{(\kappa^+)^2 (p^+ - \kappa^+) (p'^+ - \kappa^+)^2} \times \text{Tr}[O^+ (p^+ - (p - \kappa)_{\text{on}})] \frac{\Lambda(\kappa, p') \left[ \kappa^+_{\text{on}} - (p' - \kappa)_{\text{on}} \right]}{\left[ (p^+ - (p - \kappa)_{\text{on}}) \left[ (p'^+ - (p' - \kappa)_{\text{on}}) - (p - \kappa)_{\text{on}} \right] \right]} \times \frac{1}{\left[ p^+ - p'^+ + (p' - \kappa)_{\text{R}} - (p - \kappa)_{\text{on}} \right]}
\]

(B15)

Summarizing, for the sum-form one has

\[
H^u(x, \xi, t) = \theta(x - |\xi|) \theta(1 - x) \left[ H^u_{(nv)5}(x, \xi, t) + H^u_{(nv)6}(x, \xi, t) + H^u_{(nv)2}(x, \xi, t) \right] + \theta(|\xi| - x) \theta(|\xi| + x) \left[ H^u_{(nv)5}(x, \xi, t) + H^u_{(nv)6}(x, \xi, t) \right]
\]

(B16)

with $H^u_{(nv)6}$ given by Eq. (B14), while for the product-form one gets

\[
H^u_{(nv)6'}(x, \xi, t) = \theta(x - |\xi|) \theta(1 - x) \left[ H^u_{(nv)5}(x, \xi, t) + H^u_{(nv)2}(x, \xi, t) \right] + \theta(|\xi| - x) \theta(|\xi| + x) \left[ H^u_{(nv)5}(x, \xi, t) + H^u_{(nv)6'}(x, \xi, t) \right]
\]

(B17)

with $H^u_{(nv)6'}$ given by Eq. (B15).
The pion electromagnetic form factor is defined by

\[ F_\pi(t) = \frac{1}{2} P^+ \langle \pi^+(p')|J(0)\cdot n|\pi^+(p)\rangle = \int_{-1}^{1} dx H^{I=1}(x, \xi, t) = \]

\[ = \frac{1}{2} \int_{-\infty}^{\infty} dx \int \frac{dz^-}{2\pi} e^{ixP^+z^-} \times \langle \pi^+(p')|\bar{\psi}_q(-\frac{1}{2}z)\gamma \cdot n \tau_3 \psi_q(\frac{1}{2}z)|\pi^+(p)\rangle \bigg|_{\bar{z}=0} \]  

(C1)

where the range of \( x \) has been extended from \([-1, 1]\) to \([-\infty, \infty]\), since \( H^{I=1}(x, \xi, t) \) is vanishing outside the support \([-1, 1]\), given the presence of the delta-function in Eq. (40) and the kinematical relations in Eq. (1) (see, e.g. [1, 4]).

\[ ]

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