A New Single-Source Shortest Path Algorithm for Nonnegative Weight Graph

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The single-source shortest path problem is a classical problem in the research field of graph algorithm. In this paper, a new single-source shortest path algorithm for nonnegative weight graph is proposed. The algorithm can compress multi-round Fibonacci heap operations to one round to save running time relative to Dijkstra’s algorithm using Fibonacci heap. The time complexity of the algorithm is also $O(m+n\log n)$ in the worst case, where $m$ is the number of edges and $n$ is the number of nodes. However, the bound can be linear in some cases, for example, when edge weights of a graph are all the same and the hop count of the longest shortest path is much less than $n$.

We use c to represent the actual execution round number of the algorithm when the algorithm solves the single-source shortest path problem for a class of nonnegative weight graph. Based on the theoretical analyses, we demonstrate that the algorithm can perform better than Dijkstra’s algorithm using Fibonacci heap in average situation only if $n$ is large enough and the expectation of $c$ is no more than $cn$, where $c$ is a positive constant which is less than 1. Because it is so difficult to accurately describe the probability distribution of $c$ for all possible problem instances, it is still an open problem whether the algorithm of this paper is faster than Dijkstra’s algorithm using Fibonacci heap in average situation only if $n$ is large enough.

Categories and Subject Descriptors: F.2.2 [Analysis of Algorithms and Problem Complexity]: Nonnumerical Algorithms and Problems; G.2.2 [Discrete Mathematics]: Graph Theory—graph algorithms

General Terms: Algorithms, Design, Theory, Verification

Additional Key Words and Phrases: Single-source shortest path problem, nonnegative weighted graph, average performance

1. INTRODUCTION

The single-source shortest path problem is a classical problem in the research field of graph algorithm. Different kinds of algorithms have been presented to solve this problem. Among them, Dijkstra’s algorithm [Dijkstra 1959] is probably the best-known classical algorithm for a nonnegative weight graph. The time complexity of Dijkstra’s algorithm using Fibonacci heap which is an optimal implementation of Dijkstra’s algorithm in a comparison model [Fredman and Tarjan 1987] and has the best known bound for an arbitrary nonnegative weight graph is $O(m+n\log n)$ until now, where $m$ is the number of edges and $n$ is the number of nodes. Dijkstra’s algorithm is easy to understand and impressively efficient.

Among the different kinds of algorithms for single-source shortest path problem, there is also a special algorithm (i.e. breadth-first algorithm). It can solve the problem in $O(m+n)$ running time, but it mainly applies to the situation in which the edge weights of directed graph are all the same and positive [Cormen et al. 2009]. Although the application scope of breadth-first algorithm is very limited, it suggests the optimal time complexity of all the algorithms for single-source shortest path problem. Based on this characteristic, this paper tries to extend the application of breadth-first thought to explore more efficient algorithm which can apply to more general situation.

This paper adopts information diffusion mode to simulate the process of breadth-first search. By introducing the idea of uniform speed, this paper proposes a generic single-source shortest path algorithm which extends the idea of breadth-first search algorithm for...
positive weight graph. The basic idea of the algorithm is as follows: firstly, it sets the path distance between source node and itself as zero. After that, the source node transmits information to each of its neighbor nodes along the direction of edges. Every piece of information spreads at the same speed \( u_{\text{step}} \). Meanwhile, each piece of information computes the path distance it has passed through and records the latest retransmission node. Every node receives the first arriving information and retransmits it to each of its neighbor nodes along the direction of edges. At the same time, it records the path distance which the first arriving information has passed through and the latest retransmission node which the information holds. When there is no more information to transmit and receive in the node system of the directed graph, the algorithm terminates. In all the processes, information can only spread from initial point to end point along a directed edge. In the algorithm, because every piece of information spreads at the same speed, the information which spreads along the shortest path must be the first to arrive. Then the path distance and latest retransmission node which the first arriving information holds are respectively the distance of shortest path and the related precursor node.

In view of theoretical analysis, we demonstrate that the generic algorithm can obtain correct results in a sequential system only if \( u_{\text{step}} \) is more than zero and no more than \( W_{\text{min}} \) which is the minimal weight value of the graph. The time complexity of the generic algorithm is \( O(m + \frac{W_{\text{max}}}{u_{\text{step}}} + 1)n \) where \( W_{\text{max}} \) is the maximal weight value of the graph. Then for a class of directed positive weigh graph whose \( \frac{W_{\text{max}}}{u_{\text{step}}} \) is less than a constant, the generic algorithm is a linear algorithm for single-source shortest path problem. The generic algorithm tells us that it is also possible to solve single-source shortest path problem for directed positive weigh graph in linear time, not only the integer weigh graph.

In the execution process, we find the generic algorithm can deal with more than one node in one round loop. Based on the property and the inspiration of Dijkstra’s algorithm, we propose an optimization algorithm using Fibonacci heap which is applicable to nonnegative weight graph and can deal with more than one node in one round loop. In the optimization algorithm, we attempt to compress multi-round Fibonacci heap operations to one round to save running time. The time complexity of the optimization algorithm is also \( O(m+n\log n) \) in the worst case, where \( m \) is the number of edges and \( n \) is the number of nodes. However, the bound can be linear in some case, for example, when the edge weights of a graph are all the same and the hop count of the longest shortest path is much less than \( n \).

We use \( c \) to represent the actual execution round number of the optimization algorithm when the algorithm solves the single-source shortest path problem for a class of nonnegative weight graph. Based on the theoretical analyses, we demonstrate that the algorithm can perform better than Dijkstra’s algorithm using Fibonacci heap in average situation only if \( n \) is large enough and the expectation of \( c \) is no more than \( cn \), where \( c \) is a positive constant which is less than 1. Because it is so difficult to accurately describe the probability distribution of \( c \) for all possible problem instances, it is still an open problem whether the optimization algorithm is faster than Dijkstra’s algorithm using Fibonacci heap in average situation only if \( n \) is large enough.

The remainder of this paper is organized as follows: in Section 2, this paper compares the work with the related work on single-source shortest path problem; in Section 3, we present the problem description; in Section 4, we present the generic algorithm considering
uniform speed diffusion; in Section 5, we demonstrate the correctness of the generic algorithm of this paper; in Section 6, we analyze the time complexity and space complexity of the generic algorithm; in Section 7, we present the optimization algorithm which optimizes the generic algorithm using Fibonacci heap to obtain a better sequential time bound; finally, we conclude this paper in Section 8.

2. RELATED WORK

The single-source shortest path problem is a classical problem in the research field of graph algorithm. In 1959, Dijkstra proposes the classical Dijkstra’s algorithm [Dijkstra 1959] to solve single-source shortest path problem in $O(m+n^2)$ time. After that, all the efficient single-source shortest path algorithms for general directed or undirected graph are proposed based on Dijkstra’s algorithm. The main idea is combining suitable data structures and Dijkstra’s algorithm to realize lower time bound. Applying Williams’ heap [Williams 1964], we get $O(m \log n)$ time [Thorup 1999]. By applying Fibonacci heaps [Fredman and Tarjan 1987], the time bound become $O(m+n \log n)$. They noted that this was an optimal implementation of Dijkstra’s algorithm in a comparison model since Dijkstra’s algorithm visits the vertices in sorted order [Thorup 1999]. In 2010, Orlin et al. give an $O(m+nK)$ bound for a graph with $K$ distinct positive lengths [Orlin et al. 2010].

When the weights of the graph are nonnegative integer, the single-source shortest path problem has more efficient algorithms. By using Fredman and Willard’s fusion trees, Fredman and Willard give an $O(m \log n)$ randomized bound [Fredman and Willard 1993]. Based on their later atomic heaps, they give an $O(m+n \log n)$ bound [Fredman and Willard 1994]. Based on Thorup’s priority queues, we get an $O(m \log \log n)$ bound and an $O(m+n \log n)$ bound in deterministic linear space [Raman 1996] and an $O(m+n \log n^{1+\varepsilon})$ randomized bound [Raman 1997]. Additionally, in 1999, Thorup [1999] present a $O(m+n)$ time algorithm for undirected positive integer weight graph. There is also a substantial development based on the maximal edge weight $W_{\max}$ [Thorup 1999]. Ahuja, Melhorn, Orlin, and Tarjan give an $O(m+n \sqrt{W_{\max}})$ bound based on their priority queue [Ahuja et al. 1990]. Cherkaevski et al. [1997] reduce the time bound to $O((m+n \sqrt{W_{\max}} \log \log W_{\max})^{1/3+\varepsilon})$ expected time and Raman [1997] gives a further improvement to $O(m+n \log W_{\max})^{1/3+\varepsilon}$ time. In summary, the best bounds are $O(m \log \log n)$ [Thorup 1996] and $O(m+n \min((\log n)^{1/3+\varepsilon}, (\log W_{\max})^{1/4+\varepsilon}))$ [Raman 1997] in the situation [Cormen et al. 2009].

In this paper, we propose a new single-source shortest path algorithm for nonnegative weight graph. The algorithm can compress multi-round Fibonacci heap operations to one round to save running time relative to Dijkstra’s algorithm. The time complexity of the algorithm is also $O(m+n \log n)$ in the worst case, where $m$ is the number of edges and $n$ is the number of nodes. However, based on the theoretical analyses, we demonstrate that for a class of nonnegative weight graph, the algorithm can perform better than Dijkstra’s algorithm using Fibonacci heap in average situation only if $n$ is large enough and the expectation of the actual execution round number of the algorithm is no more than $en$, where $\varepsilon$ is a positive constant which is less than 1. It is still an open problem whether the
algorithm of this paper is faster than Dijkstra’s algorithm using Fibonacci heap in average situation only if \( n \) is large enough.

3. **PROBLEM DESCRIPTION**

For a directed graph \( G=(V,E) \), where \( V \) is the set of nodes and \( E \) is the set of edges. The weight of its edge \((i, j) \in E\) is \( W_{ij} (W_{ij} > 0) \) which represents the length or cost of the edge \((i, j) \in E\). For a path \( p=<v_0, v_1, ..., v_k> \), the path distance (path distance can be also called path cost) of \( p \) can be expressed as follow:

\[
L(p) = \sum_{i=1}^{k} W_{i-1,i}
\]

Where \( L(p) \) represents the path distance of \( p \). The shortest path distance from \( v_i \) to \( v_j \) can be defined as follow:

\[
L_{\text{min}}(v_i, v_j) = \begin{cases} 
\min\{L(p) | v_i \rightarrow v_j\} & \text{if there is a path from } v_i \text{ to } v_j \\
\infty & \text{otherwise}
\end{cases}
\]

The shortest path from \( v_i \) to \( v_j \) is defined as the path whose distance is equal to \( L_{\text{min}}(v_i, v_j) \).

The single-source shortest path problem is finding the shortest paths from the source node \( v_s \) and any other node in a directed graph \( G \).

4. **A GENERIC ALGORITHM CONSIDERING UNIFORM SPEED DIFFUSION**

The single-source shortest path problem is a classical problem in the research field of graph algorithm. Different kinds of algorithms have been presented to solve this problem. Dijkstra’s algorithm is probably the best-known classical algorithm for nonnegative weight graph. It is easy to understand and impressively efficient.

Among the different kinds of algorithms for single-source shortest path problem, there is also a special algorithm (i.e. breadth-first algorithm). It can solve the problem in \( O(m+n) \) running time which outperforms the \( O(m+n\log n) \) running time of Dijkstra’s algorithm using Fibonacci heap, but it mainly applies to the situation in which the edge weights of directed graph are all the same and positive [Cormen et al. 2009]. Although the application scope of breadth-first algorithm is very limited, it suggests the optimal time complexity of all the algorithms for single-source shortest path problem. Based on this characteristic, this paper tries to extend the application of breadth-first thought to explore more efficient algorithm which can apply to more general situation.

If we design the algorithm of single-source shortest path problem based on breadth-first thought, the basic train of thought is as follows: (1) the source node transmits information to each of its neighbor nodes along the direction of edges to inform them of the precursor nodes and distances of the currently known shortest paths from source node to themselves. (2) Every node transmits a new piece of information to its neighbor nodes along the direction of edges if it finds a new piece of information which informs it that there is a shorter path from source node to itself. (3) When there is no more information to transmit and receive in the node system of the directed graph, the algorithm terminates and every node has known the precursor node and distance of the shortest path from source node to itself. In all the processes, the information can only spread from initial point to end point along a directed edge. Because the information that spreads along the path which has the minimum number of edges arrives firstly, every node will usually transmit information
more than once if the path which has the minimum number of edges is not the shortest path from source node to itself. As a result, the aforementioned algorithm usually has higher time complexity than that of Dijkstra’s algorithm using Fibonacci heap in the worst case which is easy to appear. This is the main reason why the aforementioned algorithm is not in widespread use.

Based on the aforementioned analysis, the main reason for high time complexity is that some nodes have a few or many operations of transmitting information because the path which has the minimum number of edges is not usually the shortest path from source node to one node. In other words, if we can guarantee that each node only retransmits the information which spreads along the shortest path from source node to itself, the algorithm based on breadth-first thought can have an acceptable time complexity which could be lower than that of Dijkstra’s algorithm using Fibonacci heap in some situations. This is the key of the extension of breadth-first thought for single-source shortest path problem. In this paper, we introduce the uniform speed constraint of information diffusion under which every piece of information spreads at the same speed \( v_{\text{step}} \). Because of the uniform speed constraint of information diffusion, the information which spreads along the shortest path must be the first to arrive and each node only needs to transmit information once when it receives the first arriving information. As a result, we can get a more acceptable time complexity. Based on the aforementioned consideration, we propose the generic algorithm considering uniform speed diffusion.

For the generic algorithm considering uniform speed diffusion, the main train of thought can be described as follow: At the beginning of algorithm, we set the path distance from source node to itself as zero and set the path distances from source node to other nodes as \( +\infty \). We adopt global clock \( t \) to calculate the globally identical propagation distance, \( d_{su} \), of all the spreading information. The propagation distance, \( d_{su} \), is equal to \( t \times v_{\text{step}} \). In the algorithm, the global clock \( t \) is corresponding to the round number of the loop. The \( d_{su} \) will continuously increase with the execution of the algorithm. Because all the information spreads at the same speed \( v_{\text{step}} \), all the information spread the identical distance, \( d_{su} \), at the same time. Then if the shortest path distance of a node is \( d_{\min} \), the piece of information which spread along the shortest path from source node has arrived at the node when \( d_{\min} \) is no more than \( d_{su} \).

When a node finds its holding distance is no more than \( d_{su} \), it means the first piece of information has arrived at itself. Then it transmits information to its neighbor nodes. In the process, every piece of information directly calculates the total path distance, \( d_{st} \), which it will pass through if it arrives at the destination node at the moment that it is retransmitted. For example, if we use \( d_{si} \) to represent the holding distance of node \( i \) when the node transmits information to its neighbor nodes. For node \( j \) which is a neighbor of node \( i \), the distance of the transmission information, \( d_{st} \), is equal to the sum of \( d_{si} \) and \( W_{ij} \) which is the weight of edge \((i, j) \in E \). In addition, every piece of information will compare the \( d_{st} \) with the current path distance which the destination node holds. If \( d_{st} \) is shorter, it will set the currently holding distance of destination node as \( d_{st} \) and set the currently holding precursor node as the information transmission node. In contrast, if the currently holding distance is shorter, it does nothing. In the process, every node only records the shortest distance which it has learnt. This is because only the shortest distance can be the real shortest path distance. The design will greatly reduce the time complexity of the algorithm. Based on this
design, we can find an information transmission is equivalent to a relaxation operation in Dijkstra’s algorithm.

By theoretical analysis, we demonstrate that only if \( 0 < v_{\text{step}} \leq W_{\text{min}} \), the currently holding distance of a node is equal to the shortest path distance from source node to itself when the holding distance is no longer than \( d_{su} \) and is longer than \( d_{su} - v_{\text{step}} \) even if the generic algorithm runs in a sequential system (please see the demonstration of Theorem 2). Then every node only needs to transmit information to its neighbor nodes once when its holding distance is no longer than \( d_{su} \) and is longer than \( d_{su} - v_{\text{step}} \) and the algorithm will obtain the correct computing result in the end only if \( 0 < v_{\text{step}} \leq W_{\text{min}} \).

Based on the aforementioned analyses, the generic algorithm should make \( v_{\text{step}} \) satisfy the condition of Theorem 2, namely, \( 0 < v_{\text{step}} \leq W_{\text{min}} \). And in every round loop, the generic algorithm should check every node which is possible to transmit information in the current round loop to search the nodes which really satisfy the condition of transmitting relaxation information, namely, the holding distance is no longer than \( d_{su} \) and is longer than \( d_{su} - v_{\text{step}} \), and make the qualified nodes transmit relaxation information. Then at the end of every round loop, add \( d_{su} \) by \( v_{\text{step}} \), which means that all the spreading information spreads another \( v_{\text{step}} \) distance, to go into the next round loop. Repeat the aforementioned processes until the termination of the algorithm. A simple example is given below:

In the example, S is the source node and \( v_{\text{step}} \) is equal to 1. For a piece of relax information, if it finds its total path distance is shorter than the currently holding distance of the destination node, it will reset the currently holding distance and precursor of the destination node. In contrast, if the currently holding distance of the destination node is shorter, it does nothing. Then for one node, such as node \( h \) in the example, if there have been some neighbor nodes transmitting information to it after one round loop, the currently holding distance will be the shortest total path distance of all the transmission information

\[
\begin{align*}
\text{Node } S \text{ transmits relax information to its neighbor nodes} & \quad t=0, d_{su}=0 \\
\text{Node } i \text{ transmits relax information to its neighbor nodes} & \quad t=1, d_{su}=1 \\
\text{Node } j \text{ transmits relax information to its neighbor nodes} & \quad t=2, d_{su}=2 \\
\text{Node } k \text{ transmits relax information to its neighbor nodes} & \quad t=3, d_{su}=3 \\
\text{Node } h \text{ transmits relax information to its neighbor nodes} & \quad t=4, d_{su}=4 \\
\text{The algorithm terminates} & \quad t=5, d_{su}=5
\end{align*}
\]
which is transmitted to it. The more detailed description of the generic algorithm is given as follows:

For the directed graph $G=(V,E)$, we use an array Adj which includes $|V|$ lists to represent the adjacency list of $G$ which corresponds to the structure. Each list of Adj corresponds to a node of $G$. For every node $u \in V$, the list Adj[$u$] includes every edge which satisfies the condition that $(u,v) \in E$. Each element of a list in Adj has three attributes {id, W, edge_pointer_next}, where the id represents the label of the endpoint node of a direct edge, the W represents the weight of the edge and the edge_pointer_next is used to point to the subsequence element in the list. If there is no subsequence element in the list, the edge_pointer_next will be set as NULL.

We establish an array Node which including $|V|$ elements that corresponds to the nodes of $G$. Each Node element has five attributes {id, distance, precursor, pointer_next, pointer_precursor}, where the id is the label of a node, the distance records the distance of the shortest path from the source node to itself, the precursor records the label of the precursor node in the shortest path from the source node to itself, pointer_next and pointer_precursor are two node pointers which are used to conveniently establish a temporary doubly-linked list in the algorithm. The pointer_next is used to point to its subsequence node in the temporary doubly-linked list and the pointer_precursor is used to point to its precursor node in the temporary doubly-linked list. If there is no precursor node in the temporary doubly-linked list, the pointer_precursor will be set as NULL, and if there is no subsequence node in the temporary doubly-linked list, the pointer_next will be set as NULL. In the array Node, we set the id of Node[$u$] as $u$.

**ALGORITHM 1. Single-source Shortest Path Computation**

**Input:** The adjacency list of $G$, Adj, the source node id, S and the propagation speed, $v_{step}$.

**Output:** For every node $u \in V$, output the distance and precursor node id of the shortest path from the source node to node $u$.

$d_{su}=0$; Node[S]->distance=0; Node[S]-> precursor=S;

for all $u \neq S$ do Node[$u$]->distance=$+\infty$;

Create Doublylinkedlist(send_list);

Insert Doublylinkedlist(send_list, Node[S]); /*Insert the node in the list tail*/

While (!EmptyDoublylinkedlist (send_list)) do

Current_pointer=head_pointer(send_list);

List_tail_pointer=tail_pointer(send_list);

/*Search the nodes which satisfy the condition of transmitting information*/

While (Current_pointer!=0) do

if (Current_pointer->distance<=$d_{su}$) then

/*The information which spreads along the shortest path has arrived*/

$u=$Current_pointer->id;

for every neighbor node $v$ of Node[$u$] do

if (Node[$v$]->distance==$+\infty$) Insert Node[$v$] in the tail of send_list without the change of List_tail_pointer;

end

Make Node[$u$] transmit relaxation information to Node[$v$] along the direction of the edge; if the total path distance of the relaxation information is shorter than the currently holding distance of Node[$v$], reset the currently holding distance and precursor of Node[$v$];

end

if (Current_pointer==List_tail_pointer) then

Current_pointer=0;

else

end

end

end

end
At the beginning of the generic algorithm, $t=0$, $d_{su}=0$ and the currently holding distance of the source node, $d_{su}=0$. Because $d_{su}$ is added by $v_{step}$ in the end of every round loop, $d_{su}=t^*v_{step}$ is always valid. In every round loop, the generic algorithm will compare $d_{su}$ with the currently distance of every node which is in the node linked list, but not newly inserted into the list in the round loop to find the nodes that satisfy the condition of transmitting information. This is because the newly inserting nodes cannot satisfy the condition of transmitting information in current round loop owing to $0 < v_{step} \leq W_{min}$. Until the next round loop, it is possible that the newly inserting nodes satisfy the condition of transmitting information. If the generic algorithm finds a node whose currently holding distance is no longer than $d_{su}$, it will make the node transmit relaxation information to its neighbor nodes. If the holding distance of a neighbor node is reset as a constant from $+\infty$, the node will be inserted into the tail of the node linked list. After the information transmission, the node of information transmission will be removed from the node linked list. Based on these disposing principle, the node linked list only stores the nodes which have not transmitted relaxation information and whose currently holding distances are less than $+\infty$. Because the currently holding distances of the nodes in the node linked list only compare with $d_{su}$, not compare with each other, it is possible that the generic algorithm deals with more than one node in one round loop. The generic algorithm terminates when it finds there are no nodes in the node linked list. It means that the generic algorithm has computed all the shortest path distances of the nodes which connect with the source node in the graph.

In the process of computing the shortest path, for an arbitrary node, its relaxation operation, namely, transmitting relaxation information to all its neighbor nodes is significant only when its holding distance is equal to the distance of the shortest path from the source node to itself. Then the core of a single-source shortest path algorithm is how to judge the holding distance of a node is equal to the distance of the shortest path from the source node to itself, namely, how to judge when a node should transmit relaxation information to all its neighbor nodes. Dijkstra’s algorithm can judge the distance of the node whose holding distance is the shortest one among the nodes which have not transmitted relaxation information is equal to its shortest distance in every round loop. Then in every round loop, Dijkstra’s algorithm can only make one node transmit relaxation information. Instead, based on the idea of uniform speed diffusion, this paper proposes a new determinant law, Theorem 2 for judging whether the holding distance of a node is equal to the distance of the shortest path from the source node to itself. Based on the new
determinant law, the generic algorithm only needs to compare the holding distances of each node which are possible to transmit relaxation information in current round loop with the \(d_{su}\) to find the qualified nodes of transmitting information in every round loop. Then it is possible that there is more than one node to be deal with in one round loop. As a result, the generic algorithm can have less time complexity than Dijkstra’s algorithm in some situations. For the generic algorithm, the correctness is determined by the validity of the new determinant law, Theorem 2. The detailed proof of Theorem 2 is given in Section 5.

5. THEORETICAL PROOF OF THE CORRECTNESS

For arbitrary node \(i\), we use \(d_{imin}\) to represent the actual distance of shortest path from source node to itself. And we use \(d_{ic}\) to represent the currently holding distance of node \(i\) in the algorithm. With the execution of the generic algorithm, \(d_{ic}\) will be smaller and smaller until it is equal to \(d_{imin}\).

**Theorem 1.** For a directed graph \(G=(V,E)\), where \(V\) is the set of nodes and \(E\) is the set of edges. The weight of its edge \((i,j)\in E\) is \(W_{ij}\) \((W_{ij} > 0)\) which represents the length or cost of the edge \((i,j)\in E\). Suppose \(p=<v_1, v_2, …,v_k>\) is the shortest path from \(v_1\) to \(v_k\). For arbitrary \(i, j\) \((1 \leq i \leq j \leq k)\), suppose \(p_{ij}=<v_i, v_{i+1}, …,v_j>\) is the sub-path of \(p\) from \(v_1\) to \(v_j\). Then \(p_{ij}\) is the shortest path from \(v_1\) to \(v_j\) [Cormen et al. 2009].

**Proof** if we divide path \(p\) as \(v_1 \rightarrow v_j \rightarrow v_k\), \(L(p) = L(p_{ij}) + L(p_{jk})\). Suppose that there is a path from \(v_1\) to \(v_j\), \(p_{ij}\) and \(L(p_{ij}) < L(p_{jk})\). Then \(v_j \rightarrow v_j \rightarrow v_j\) is a path from \(v_1\) to \(v_k\), and the path distance \(L(p_{ij}) + L(p_{jk}) + L(p_{jk})\) is less than \(L(p)\). This is in contradiction with the assumption that \(p\) is the shortest path from \(v_1\) to \(v_k\). Then we can conclude that Theorem 1 is valid.

**Theorem 2.** For arbitrary node \(i\) in graph \(G\), \(d_{ic} = d_{imin}\) when \(d_{su} - v_{step} < d_{ic} \leq d_{su}\) on condition of \(0 < v_{step} \leq W_{min}\) even if the generic algorithm runs in a sequential system.

**Proof** Suppose that for node \(i\) which is different from source node, \(d_{ic} \neq d_{imin}\) (i.e. \(d_{ic} > d_{imin}\) or \(d_{ic} < d_{imin}\)) when \(d_{su} - v_{step} < d_{ic} \leq d_{su}\) on condition of \(0 < v_{step} \leq W_{min}\).

(i) Suppose that for arbitrary node \(i\) which is different from source node, \(d_{ic} > d_{imin}\) when \(d_{su} - v_{step} < d_{ic} \leq d_{su}\) \((d_{su} = t_0 * v_{step}, t_0 \geq 1)\) on condition of \(0 < v_{step} \leq W_{min}\). We assume that the shortest path from source node to node \(i\) is \(p_{min}=(p_{min}=<v_s, v_0, …,v_i, v_i>)\). Given Theorem 1 and Equation (1), we know that \(d_{imin} = d_{imin} + W_{ji}\) where \(W_{ji}\) is the weight of edge \((j,i)\in E\).

Based on the execution process of the generic algorithm, we can judge \(d_{ic} > d_{imin}\).

If \(d_{jc} \leq d_{imin}\), \(d_{jc} + W_{ji} \leq d_{imin} + W_{ji} = d_{imin} < d_{ic}\).

\[\therefore d_{ic} \leq d_{su} = t_0 * v_{step} \text{ and } W_{ji} \geq W_{min} \geq v_{step}\]

\[d_{jc} \leq d_{imin} < d_{ic} - W_{ji} \leq t_0 * v_{step} - W_{ji} \leq (t_0 - 1) * v_{step}\]

This means that node \(j\) retransmitted information to node \(i\) at \(t_0-1\) round loop and compare \(d_{jc} + W_{ji}\) with \(d_{ic}\). The directed result is \(d_{ic} \leq d_{jc} + W_{ji}\) at \(t_0\) round loop, which is determined by the mechanism of generic algorithm. Then \(d_{ic} \leq d_{imin}\), it is in contradiction
with $d_{ic} > d_{i\text{min}}$. Consequently, we can judge $d_{jc} > d_{j\text{min}}$ if $d_{ic} > d_{i\text{min}}$ when $d_{su} - v_{\text{step}} < d_{ic} \leq d_{su}$ on condition of $0 < v_{\text{step}} \leq W_{\text{min}}$. Based on the same argument, we can judge $d_{oc} > d_{o\text{min}}$ and $d_{sc} > d_{s\text{min}}$.

However, we set the path distance of source node as zero which is equal to the shortest path distance from source node to itself at the beginning of the generic algorithm, namely, $d_{sc} = d_{s\text{min}}$. This means that $d_{sc} > d_{s\text{min}}$ is not possible to appear in the algorithm because it is in contradiction with the initial setting. It also means that $d_{ic} > d_{i\text{min}}$ is not possible to appear in the algorithm when $d_{su} - v_{\text{step}} < d_{ic} \leq d_{su}$ on condition of $0 < v_{\text{step}} \leq W_{\text{min}}$. As a result, we can conclude that $d_{ic} \leq d_{i\text{min}}$ when $d_{su} - v_{\text{step}} < d_{ic} \leq d_{su}$ on condition of $0 < v_{\text{step}} \leq W_{\text{min}}$.

(II) Suppose that for arbitrary node i which is different from source node, $d_{ic} < d_{i\text{min}}$ when $d_{su} - v_{\text{step}} < d_{ic} \leq d_{su}$ (for all $i$, $0 < t_{0} \leq 1$) on condition of $0 < v_{\text{step}} \leq W_{\text{min}}$. We assume that $d_{i}$ is the distance of node $i$ which is different from source node. Then $d_{i} = d_{ic} + W_{ki}$ and $d_{j\text{min}} \leq d_{i\text{min}} + W_{ki}$, where $W_{ki}$ is the weight of edge $(k,i) \in E$. Based on the execution process of the generic algorithm, we can judge $d_{ic} < d_{i\text{min}}$.

If $d_{kc} \geq d_{k\text{min}}$, $d_{ic} = d_{ic} + W_{ki} \geq d_{ic} + W_{ki} \geq d_{ic} + W_{ki} \geq d_{i\text{min}}$ (for all $(i,j) \in E$). This is in contradiction with $d_{ic} < d_{i\text{min}}$. Consequently, we can judge $d_{ic} < d_{i\text{min}}$ if $d_{ic} < d_{i\text{min}}$ when $d_{su} - v_{\text{step}} < d_{ic} \leq d_{su}$ on condition of $0 < v_{\text{step}} \leq W_{\text{min}}$. Based on the same argument, we can judge $d_{ic} < d_{i\text{min}}$ and $d_{sc} < d_{s\text{min}}$.

However, we set $d_{sc} = d_{s\text{min}}$ at the beginning of the generic algorithm. This means that $d_{sc} < d_{s\text{min}}$ is not possible to appear in the algorithm. It also means that $d_{ic} < d_{i\text{min}}$ is not possible to appear in the algorithm when $d_{su} - v_{\text{step}} < d_{ic} \leq d_{su}$ on condition of $0 < v_{\text{step}} \leq W_{\text{min}}$. As a result, we can conclude that $d_{ic} \geq d_{i\text{min}}$ when $d_{su} - v_{\text{step}} < d_{ic} \leq d_{su}$ on condition of $0 < v_{\text{step}} \leq W_{\text{min}}$.

Combining all the aforementioned analyses, we can conclude that for arbitrary node i which is different from source node, $d_{ic} = d_{i\text{min}}$ when $d_{su} - v_{\text{step}} < d_{ic} \leq d_{su}$ on condition of $0 < v_{\text{step}} \leq W_{\text{min}}$. Additionally, we set $d_{sc} = d_{s\text{min}}$ at the beginning of the generic algorithm. Then we can judge that Theorem 2 is valid.

According to Theorem 2, we can know that arbitrary node i only needs to transmit information once when $d_{su} - v_{\text{step}} < d_{ic} \leq d_{su}$ on condition of $0 < v_{\text{step}} \leq W_{\text{min}}$. This is because the information transmission of node i is really valuable only when $d_{ic} = d_{i\text{min}}$. Therefore, we set that every node only transmits information once when $d_{su} - v_{\text{step}} < d_{ic} \leq d_{su}$ and $0 < v_{\text{step}} \leq W_{\text{min}}$ in the generic algorithm.
6. **ANALYSIS OF THE COMPLEXITY**

In the generic algorithm, every node is initialized once and transmits information to all its neighbor nodes along the direction of the edges only once. And every node is inserted into the double-link list once and is removed from the double-link list once. In the double-link list, every node will compare its holding distance with \( d_{su} \) in every round loop. Then the time complexity of the algorithm is determined by the sum of operation frequency of all the five parts. The initialization of the algorithm aggregately needs \( O(n) \) operations. In the algorithm, every node only transmits information to its neighbor nodes along the direction of edges once. For each piece of information, the algorithm needs to compute the total path distance when it arrives at the destination node, compare the total path distance with the holding distance of destination node and reset the holding distance of destination node if necessary. In the process, the total frequency of basic operation is \( O(m) \).

In the concrete implement of the generic algorithm, this paper establishes a queue to deposit the nodes whose holding distances are less than \(+\infty\) and which have not transmitted information to the neighbor nodes. Every node will be pushed into the queue when the holding distance is reset firstly and will be removed from the queue after transmitting information. This paper adopts doubly-linked list to implement the node queue. Then for every node, the operation frequency of inserting and removing is \( O(1) \). As a result, the total operation frequency of inserting and removing of the generic algorithm is \( O(n) \). Every node in the queue will be checked to compare the holding distance with \( d_{su} \) and judge if the node should transmit information to its neighbor nodes in every round loop. The holding distance of a node is reset firstly when the neighbor node which is the first to satisfy the condition of information transmission transmits information to itself. For example, the holding distance of node \( j \) is set as \(+\infty\) at the beginning. In the \( t_{0} \) \((t_{0} \geq 0)\) round loop, node \( i \), one of neighbor nodes of node \( j \), is the first neighbor node which satisfies the condition of information transmission. Then node \( i \) transmits information to node \( j \) and resets the holding distance as \( d_{j} \) \((d_{j} = d_{i} + W_{ij})\) where \( d_{i} \) \((t_{0} - 1)^{\ast}v_{step} < d_{i} \leq t_{0}^{\ast}v_{step}\) is the currently holding distance of node \( i \) which is equal to \( d_{imin} \) and \( W_{ij} \) is the weight of edge \((i, j) \in E\). Then node \( j \) is pushed into the node queue. Based on Theorem 2, we know that \( d_{j} = d_{imin} \) when node \( j \) transmits information to its neighbor nodes and is removed from the node queue, namely, when \( d_{su} - v_{step} < d_{j} \leq d_{su} \) on condition of \( 0 < v_{step} \leq W_{min} \). We use \( t_{l} \) to represent the round number of the loop when node \( j \) is removed from node queue. Then \( t_{l} \) satisfies the condition of \( (t_{l} - 1)^{\ast}v_{step} < d_{j} \leq d_{j} \leq t_{l}^{\ast}v_{step} \) and the total round number of node \( j \) in node queue, \( t_{j} - t_{0} \) is less than \( d_{j}^{\ast}v_{step} \). Based on the enactment of the generic algorithm, we know that \( d_{j} \leq d_{i} + W_{ij} = d_{imin} + W_{ij} \) \(d_{imin} = \min_{(i, j) \in E}(d_{imin} + W_{ij})\). Then it can be concluded that \( d_{j} \leq d_{imin} + W_{j} \), where \( W_{max} \) is the maximal value of edge weights in graph \( G \). For reducing the time complexity, we set \( v_{step} \) as \( W_{min} \) which is the minimum value of edge weights in graph \( G \). As a result, \( t_{j} - t_{0} \) is less than \( \frac{W_{min}}{v_{step}} \). Then in node queue, the comparing frequency of an arbitrary node including the source node whose comparing
frequency is one is less than $\frac{v_{\text{step}}}{c_{\text{min}}}$... Consequently, the total comparing frequency in node queue for all the node is $O(c_{\text{max}})$.

Based on the aforementioned analysis, we can conclude that the time complexity of the generic algorithm is $O(m + c_{\text{max}} + 1)n)$. Then for a class of directed positive weight graph whose $c_{\text{max}}$ is less than a constant, the generic algorithm is a linear algorithm for single-source shortest path problem and can outperform the Dijkstra’s algorithm using Fibonacci heap. Comparing with the partly linear algorithm for single-source shortest path problem which only apply to undirected positive integer weight graph [Thorup 1999], the generic algorithm has wider application scope because the generic algorithm can apply to arbitrary directed positive weight graph. The generic algorithm tells us that it is also possible to solve single-source shortest path problem for directed positive weigh graph in linear time, not only the integer weigh graph.

In the generic algorithm, we need O(m) storage space to store the adjacency list of G, Adj which corresponds to the structure and O(n) storage space to store the array Node which including $|V|$ elements that corresponds to the nodes of G. Then the space complexity of the algorithm is O(m+n). It is almost the same as Dijkstra’s algorithm.

7. THE OPTIMIZATION ALGORITHM USING FIBONACCI HEAP

In the generic algorithm, we assume that all the information spreads at the same speed $v_{\text{step}} (0 < v_{\text{step}} \leq W_{\text{min}})$. Then in the execution process of the generic algorithm, if the minimal holding distance of all the nodes which have not transmitted relaxation information is $d_{\text{min}}$ after one round information transmission, the next round loop $t_0$ when there is some node transmitting information must satisfy the condition of $(t_0 - 1)v_{\text{step}} < d_{\text{min}} \leq t_0v_{\text{step}} (t_0 > 0)$. Based on Theorem 2, we know that if we assume that all the information spreads at the same speed $v_{\text{step}} (0 < v_{\text{step}} \leq W_{\text{min}})$, the nodes whose holding distances are in the same interval $(t_0 - 1)v_{\text{step}} < d_{\text{min}} \leq t_0v_{\text{step}} (t_0 > 0)$ can transmit relaxation information at the same round loop and it does not affect the correctness of the final result. Based on the property and the inspiration of Dijkstra’s algorithm, we propose an optimization algorithm which optimizes the generic algorithm using Fibonacci heap to obtain a better sequential time bound. In the computing process, the optimization algorithm confirms the $d_{\text{min}}$ using Fibonacci heap and computes the minimal $d_{\text{s}}$ when there is some node transmitting information. Besides, the optimization algorithm attempts to compress multi-round Fibonacci heap operations to one round to save running time. For facilitating the understanding, we explain the optimization algorithm in comparison with the Dijkstra’s algorithm using Fibonacci heap.

In the Dijkstra’s algorithm using Fibonacci heap, the algorithm will find the node whose holding distance is the shortest one among all the nodes which have not transmitted relaxation information from the Fibonacci heap and make the node transmit relaxation information in every round loop. We use $H$ to represent the Fibonacci heap, use $\text{min}[H]$ to represent the pointer of the heap and use $\text{min-node}$ to represent the node which $\text{min}[H]$ points to in $H$. At the beginning of the algorithm, the $\text{min}[H]$ points to the source node. The process of the Dijkstra’s algorithm using Fibonacci heap in every round loop can be described as follows:
Fib-Extract-Min-and-Relax (H)

(i) If \( \text{min}[H] \) is equal to NULL, exit the loop. Otherwise, transmit relaxation information according to \( \text{min-node} \), the algorithm needs either some \text{insert} operations or some \text{decrease key} operations which are both the basic operations of Fibonacci heap.

(ii) If \text{child}[\text{min-node}] is not equal to NULL, remove all the children of \text{min-node} from the child list and add them to the root list of H, then remove \text{min-node} from the root list.

(iii) If H is not empty, merge the root list of H to reduce the tree number and make \( \text{min}[H] \) point to the node whose key value is the minimum. This process also belongs to the basic operation process of Fibonacci heap.

The optimization algorithm is proposed based on the inspiration of Dijkstra’s algorithm. In the optimization algorithm of this paper, we use \( k_{\text{min}} \) to represent the key value of \text{min-node} which \( \text{min}[H] \) points to and use \text{child}[x] to represent the pointer which points to the child node list of node x. We add a Boolean quantity, flag as a new attribute of H. At the beginning of the optimization algorithm, we make \( \text{min}[H] \) point to the source node and set flag as false. The process of the optimization algorithm in every round loop can be described as follows:

Fib-Extract-Min-and-Relax-Optimization (H, \( v_{\text{step}} \))

(i) If \( \text{min}[H] \) is equal to NULL, exit the loop. Otherwise, if flag is false, go to (iii), else if \( v_{\text{step}} \) is equal to zero, \( y=k_{\text{min}} \), else if \( v_{\text{step}} \) is larger than zero, \( y=\left\lfloor k_{\text{min}} / v_{\text{step}} \right\rfloor \times v_{\text{step}} \), if \( y<k_{\text{min}} \), \( y=y+v_{\text{step}} \). Then go to (iii);

(ii) If the key value of the node, \text{min-node} which \( \text{min}[H] \) points to is no more than y, go to (iii), else go to (viii);

(iii) Transmit relaxation information according to \text{min-node} and the algorithm needs either some \text{insert} operations or some \text{decrease key} operations which are both the basic operations of Fibonacci heap. In the process, every node which is added to the root list of H will be added to the front of \text{min-node}, namely, the node will be the left brother of \text{min-node}.

(iv) If flag is false, go to (vi). Otherwise, if \text{child}[\text{min-node}] is not equal to NULL and the key value of the node, z which \text{child}[\text{min-node}] points to is no more than y, go to (v), else go to (vi);

(v) Remove node z from the child list of \text{min-node} and add it to the back of \text{min-node}, namely, node z will be the right brother of \text{min-node}. At the same, if node z has a right brother, make the \text{child}[\text{min-node}] point to the right brother of node z. Then go to (iv). If node z has no right brother, set \text{child}[\text{min-node}] as NULL and go to (vii);

(vi) If \text{child}[\text{min-node}] is not equal to NULL, directly remove all the children of \text{min-node} from the child list and add them to the front of \text{min-node}, namely, the last node of the child list will be the left brother of \text{min-node} if we regard the node which \text{child}[\text{min-node}] points to as the first node of the child list. In the process, every child node does not compare its key value with y;

(vii) Remove \text{min-node} from the root list of H. If there is a right brother of \text{min-node}, make \( \text{min}[H] \) point to the right brother. If there is not a right brother, set \( \text{min}[H] \) as NULL. If \( \text{min}[H] \) is not equal to NULL and flag is true, go to (ii), else go to (viii);
(viii) If H is not empty, consolidate the root list of H to reduce the tree number and make min[H] point to the node whose key value is the minimum. This process also belongs to the basic operation process of Fibonacci heap.

(ix) If min[H] is not equal to NULL and either the key value of child[min-node] or the key value of the right brother of min-node is equal to k_min, set flag as true, else set flag as false.

In every round loop of the optimization algorithm, the algorithm computes the minimal d_su when there is some node transmitting relaxation information based on d_min (i.e. k_min) and makes some qualified nodes transmit relaxation information. In the process, the optimization algorithm attempt to make more qualified nodes transmit information by some added checking operations. This is in order to compress multi-round Fibonacci heap operations to one round to save running time. After one round information transmission, the optimization algorithm will confirm the new d_min using Fibonacci heap. The optimization algorithm terminates when there are no nodes in the node linked list.

In computing process, the optimization algorithm will judge whether checking the child nodes and the right brothers of min-node in the next round loop. If the algorithm finds a new node which satisfies the condition of transmitting information in the next round loop, then check the child nodes and right brothers in the next round loop. Otherwise, execute the next round loop as the Dijkstra's algorithm using Fibonacci heap. In the process of checking the child nodes and the right brothers of min-node, the optimization algorithm will check the child nodes of min-node to judge whether the child nodes satisfy the condition of transmitting relaxation information until it finds a child node which does not satisfy the condition. Besides, the optimization algorithm will also traverse the root list to find the root nodes which satisfy the condition of transmitting relaxation information until it finds a root node which does not satisfy the condition. The reason for the design as above is to make the optimization algorithm possible to deal with more than one node in one round loop and the checking operations will not take too much time in the worst case. If the optimization algorithm can deal with more than one node in one round loop, the round number of the loop of the optimization algorithm will decrease and the scale of the Fibonacci heap could be smaller in the subsequent loops. The direct result is the time complexity can be lower than that of the Dijkstra's algorithm using Fibonacci heap.

In the worst case, the optimization algorithm also deals with only one node in one round loop. The optimization algorithm will pay O(2n) more time than the Dijkstra's algorithm using Fibonacci heap in n round loops. However, the time complexity of the optimization algorithm is still O(m+nlogn) which is better than the generic algorithm when n is very large. And when n is large enough, the total checking cost can be ignored because O(nlogn) is much larger than O(2n). Then we can think that the time complexity of the optimization algorithm in the worst case is almost the same as that of the Dijkstra's algorithm using Fibonacci heap when n is large enough. Although the optimization algorithm is a little slower than the Dijkstra's algorithm using Fibonacci heap in the worst case, it can be much faster than the Dijkstra's algorithm using Fibonacci heap in some other cases. For instance, if graph G is a graph whose edge weighs are all the same, the optimization algorithm can solve the single-source problem in O(m+clogn+2n) time where c is the hop count of the longest shortest path in the graph. If c is much less than n, the O(m+clogn+2n) will be
closed to linear time which is much faster than that (i.e. \( O(m+n \log n) \)) of the Dijkstra’s algorithm using Fibonacci heap.

Comparing with Dijkstra’s algorithm using Fibonacci heap, the basic accelerating idea of the optimization algorithm is to compress multi-round Fibonacci heap operations to one round to save running time. In the compressing process, there are no other differences between the optimization algorithm and Dijkstra’s algorithm using Fibonacci heap except that the optimization algorithm needs to pay the checking cost. We can regard it as a kind of time deal. Then only if the saving time cost is larger than the checking time cost, the optimization algorithm will perform better than Dijkstra’s algorithm using Fibonacci heap.

We use \( c \) to represent the actual round number of the execution loop of the optimization algorithm in computing process. Because the optimization algorithm deal with at least one node in one round loop and the optimization algorithm executes at least two round loop, the \( c \) is be located in the interval \([2, n]\). In the computing process of shortest path, the Fibonacci heap dose three kinds of operations in the consolidating process of the root list: (i) merge the root list to reduce the tree number; (ii) confirm the new min-node by comparison; (iii) reconstruct the root list. Among the operations, the latter two kinds of operations totally cost \( O(2 \log n) \) [Fredman and Tarjan 1987]. If the optimization algorithm deposes more than one node in one round loop, it will reduce total cost for confirming the new min-node and reconstructing the root in the algorithm. Actually, although the size of Fibonacci heap in Dijkstra’s algorithm using Fibonacci heap is dynamic, the average size of \( n \) round loop should also be \( \Theta(n) \), namely, the average size is increasing with \( n \), not a constant. This determines the consolidating operations of Fibonacci heap in Dijkstra’s algorithm using Fibonacci heap averagely cost \( \Theta(\log n) \) time in one round loop. Based on the aforementioned reason, we assume that in the consolidating process of the root list, the optimization algorithm can averagely save \( \Theta(\log n) \) time relative to Dijkstra’s algorithm using Fibonacci heap if it disposes one more node in one round loop. Then if the optimization algorithm executes \( c \) round loop in computing process, it can save \( \Theta((n-c) \log n) \) time in the consolidating process of the root list relative to the Dijkstra’s algorithm using Fibonacci heap.

In the optimization algorithm, every node will lead to at most two unprofitable checking operations. Then the total number of unprofitable checking operations is at most \( 2n \). If a checking operation finds a new node which satisfies the transmission condition in one round loop, the newly discoverable node need another comparison operation before transmitting relaxation information. Then a node will experience at most one discoverable checking operation and at most one comparison operation before transmitting relax information which is described in step (ii) of the optimization algorithm. The total cost is at most \( O(2n) \). At the end of every round loop, the optimization algorithm will execute two checking operations to set flag. This totally needs \( 2c \) operations for \( c \) round loops. Based on the aforementioned analyses, the optimization algorithm will pay at most \( O(6n) \) more time in the checking process relative to the Dijkstra’s algorithm using Fibonacci heap.

If we set \( v_{\text{step}} \) as zero directly and the optimization algorithm performs better than the Dijkstra’s algorithm using Fibonacci heap in average situation, the following inequality must be valid:

\[
\sum_{i=2}^{n} \alpha P(c = i) \frac{(n-i) \log n}{n} > 1
\]  

(3)
Where $\alpha (\alpha > 0)$ is a positive constant which represents the coefficient ratio. Then we have,

$$\sum_{i=2}^{n} P(c = i) * i < n(1 - \frac{1}{\alpha \log n})$$  \hspace{1cm} (4)

If $\sum_{i=2}^{n} P(c = i) * i \leq \varepsilon n < n$ where $\varepsilon$ is a positive constant. Then the Inequality (4) is valid, only if,

$$n > \phi^{1/(1-\varepsilon)}$$  \hspace{1cm} (5)

Where $\phi = (1 + \sqrt{5})/2$ [Fredman and Tarjan 1987]. Based on inequalities (3), (4) and (5), we can conclude that for a class of nonnegative weight graph, the optimization algorithm can perform better than the Dijkstra’s algorithm using Fibonacci heap in average situation only if $n$ is large enough and the expectation of the actual execution round number, $c$ is no more than $\varepsilon n$, where $n$ is the number of the nodes and $\varepsilon$ is a positive constant which is less than 1. Because it is so difficult to accurately describe the probability distribution of $c$ for all possible problem instances, it is still an open problem whether the optimization algorithm is faster than Dijkstra’s algorithm using Fibonacci heap in average situation only if $n$ is large enough.

For a nonnegative weight graph, the optimization algorithm can compute the correct result only if $0 \leq v_{\text{step}} \leq W_{\text{min}}$. In reality, the criterion that is used to judge whether checking the child nodes and the right brothers of min-node in the next round loop of the optimization algorithm is very conservative. That is in order to minimize the cost of computing the value of $y$ in the worst case. Similarly, the criterion that is used to terminate the checking operations in one round loop is also very conservative. That is in order to reduce the cost of profitless checking operations in the worst case. The result is that the optimization algorithm has lower time complexity in the worst case, but has less possibility to perform better than the Dijkstra’s algorithm using Fibonacci heap. The design selection is a kind of trade-off. However, based on the analysis of the complexity of the generic algorithm, we can deduce that when $W_{\text{min}}$ is larger than zero, the less the value of $W_{\text{max}} / W_{\text{min}}$ is, the more nodes could transmit relaxation information in one round loop.

Based on the property, we can revise the optimization algorithm. If the $W_{\text{min}}$ is larger than zero, the less the value of $W_{\text{max}} / W_{\text{min}}$ is, we make the optimization algorithm have more possibility to check the child nodes and the right brothers of min-node in the next round loop and make the optimization algorithm check more nodes which do not satisfy the condition of transmitting relaxation information before terminating the checking operations. The design revision can make the optimization algorithm more possible to perform better than the Dijkstra’s algorithm using Fibonacci heap and have fewer risks to perform very badly. The result is that the revised optimization algorithm can have better average performance than before.

We use $b$ to represent the number of the nodes which the optimization algorithm check but do not satisfy the condition of transmitting relaxation information before terminating the checking operations. Because the root list and the child list of min-node both have at most $\log n$ nodes [Fredman and Tarjan 1987] at the beginning of every round loop, $b$ should
be located in the interval \([1, \log n]\). If we make the optimization algorithm check the root list and the child list of \(\text{min-node}\) in every round loop, the algorithm will pay at most \(O(2n)\) more time which is no more than \(O(2\log n)\) in the worst case than the Dijkstra’s algorithm using Fibonacci heap. The time complexity of the optimization algorithm is still \(O(m+n\log n)\) which is also acceptable. Consequently, if we know the value of \(W_{\text{max}}/W_{\text{min}}\), we can daringly adopt suitable \(b\) to pursue better performance.

8. CONCLUSION

This paper proposes a new single-source shortest path algorithm in average situation for nonnegative weight graph. The algorithm adopts the uniform-speed diffusion mechanism and Fibonacci heap to guarantee the high efficiency and wide suitability. The time complexity of the algorithm is also \(O(m+n\log n)\) in the worst case, where \(m\) is the number of edges and \(n\) is the number of nodes. However, the bound can be linear in some case, for example, when the edge weights of a graph are all the same and the hop count of the longest shortest path is much less than \(n\).

We use \(c\) to represent the actual execution round number of the algorithm when the algorithm solves the single-source shortest path problem for a class of nonnegative weight graph. Based on the theoretical analyses, we demonstrate that the algorithm can perform better than Dijkstra’s algorithm using Fibonacci heap in average situation only if \(n\) is large enough and the expectation of \(c\) is no more than \(\varepsilon n\), where \(\varepsilon\) is a positive constant which is less than 1. Because it is so difficult to accurately describe the probability distribution of \(c\) for all possible problem instances, it is still an open problem whether the algorithm of this paper is faster than Dijkstra’s algorithm using Fibonacci heap in average situation only if \(n\) is large enough.

ACKNOWLEDGMENTS

This work was supported by the National Natural Science Foundation of China (No.61170164, and No. 61472079), the Funds for Distinguished Young Scholars of the Natural Science Foundation of Jiangsu Province (No.BK2012020), and the Program for Distinguished Talents of Six Domains in Jiangsu Province (No.2011-DZ023).

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