CSP for binary conservative relational structures

ALEXANDR KAZDA

ABSTRACT. We prove that whenever \( A \) is a 3-conservative relational structure with only binary and unary relations, then the algebra of polymorphisms of \( A \) either has no Taylor operation (i.e., CSP(\( A \)) is \( \text{NP} \)-complete), or it generates an SD(\( \land \)) variety (i.e., CSP(\( A \)) has bounded width).

1. Introduction

In the last decade, the study of the complexity of the constraint satisfaction problem (CSP) has produced several major results due to universal algebraic methods (see, e.g., [6], [13] and [3]).

In this paper, we continue in this direction and look at clones of polymorphisms of finite 3-conservative relational structures with relations of arity at most two (a generalization of conservative digraphs). We show that whenever such a structure admits a Taylor operation, its operational clone actually generates a meet semidistributive (SD(\( \land \))) variety, so the corresponding CSP problem is solvable by local consistency checking (also known as the bounded width algorithm). Since relational structures without Taylor operations yield \( \text{NP} \)-complete CSPs, we obtain a rather simple dichotomy of CSP complexity in this case.

There have been numerous papers published on the behavior of conservative relational structures. We have been mostly building on three previous results: First, Andrei Bulatov proved in [5] the dichotomy of CSP complexity for 3-conservative relational structures, for which Libor Barto recently offered a simpler proof in [2]. Meanwhile, Pavol Hell and Arash Rafiey obtained a combinatorial characterization of all tractable conservative digraphs [9], and have observed that all tractable digraphs must have bounded width.

2. Preliminaries

A relational structure \( A \) is any set \( A \) together with a family of basic relations \( \mathcal{R} = \{ R_i \mid i \in I \} \) where \( R_i \subset A^{n_i} \). We will call the number \( n_i \) the arity of \( R_i \).

Presented by M. Maroti.
Received December 7, 2011; accepted in final form November 11, 2014.
2010 Mathematics Subject Classification: Primary: 08A02; Secondary: 03C05, 68R05.
Key words and phrases: meet-semidistributivity, clone, constraint satisfaction problem, weak near unanimity.

Supported by the Czech-Polish cooperation grant 7AMB13PL013 “General algebra and applications” and the GAČR project 13-01832S.
As usual, we will consider only finite structures (and finitary relations) in this paper.

Let $R$ be an $m$-ary relation and $f: A^n \to A$ an $n$-ary operation. We say that $f$ preserves $R$ if whenever we have elements $a_{ij} \in A$ such that

$$(a_{i1}, a_{i2}, \ldots, a_{im}) \in R, \text{ for } 1 \leq i \leq n,$$

then we also have

$$(f(a_{i1}, \ldots, a_{i1}), \ldots, f(a_{im}, \ldots, a_{im})) \in R.$$ 

If $\mathcal{R}$ is a set of relations, then we denote by $\text{Pol}(\mathcal{R})$ the set of all operations on $A$ that preserve all $R \in \mathcal{R}$. On the other hand, if $\Gamma$ is a set of operations on $A$, then we denote by $\text{Inv}(\Gamma)$ the set of all relations that are preserved by each operation $f \in \Gamma$.

One of the most important notions in CSP is primitive positive definition. If we have relations $R_1, \ldots, R_k$ on $A$, then a relation $S$ on $A$ is primitively positively defined using $R_1, \ldots, R_k$ if there exists a logical formula defining $S$ that uses only conjunction, existential quantification, symbols for variables, predicates $R_1, \ldots, R_k$, and the symbol for equality “$=$”.

Observe that the set $\text{Pol}(\mathcal{R})$ is closed under composition and contains all the projections, that is, it is an operational clone. If $\mathbb{A} = (A, \mathcal{R})$ is a relational structure, then $\text{Inv}(\text{Pol}(\mathcal{R}))$ consists of precisely all the relations that can be primitively positively defined using the relations from $\mathcal{R}$ (see the original works of Bodnarchuk [4] and Geiger [8], or the survey [14] for a proof of this statement, as well as a more detailed discussion of the correspondence between $\text{Pol}$ and $\text{Inv}$). We will call $\text{Inv}(\text{Pol}(\mathcal{R}))$ the relational clone of $\mathbb{A}$.

Given a relational structure $\mathbb{A}$, an instance of the constraint satisfaction problem $\text{CSP}(\mathbb{A})$ consists of a set of variables $V$ and a set of constraints $C$ where each constraint $C = (S, R)$ has a scope $S \subset V$ and a relation $R \subset A^S$ such that $R$ (after a suitable renaming of variables) is either equality or one of the basic relations of $\mathbb{A}$. A solution of this instance is any mapping $f: V \to A$ such that $f|_S \in R$ for each constraint $(S, R) \in C$. In this paper, we will only consider CSP instances where all relations have scopes of size at most two.

We can draw a CSP instance as a microstructure (also known as a potato diagram): For each variable $x$, we have the potato $B_x \subset A$ equal to the intersection of all the unary constraints on $x$. For each constraint $(\{x, y\}, R)$ of arity two, we draw lines from elements of $B_x$ to elements of $B_y$ that correspond to the relation $R$.

To solve the instance now means to choose in each potato $B_x$ a vertex $b_x$ so that whenever $C = (\{x, y\}, R)$ is a constraint, there is a line in $R$ from $b_x$ to $b_y$ (i.e., $(b_x, b_y) \in R$). See Figure 1 for an example.

If we mark some variables in the CSP instance $I$ as free variables and print out values of these variables in all solutions of $I$, we obtain a relation on $A$. It turns out that there is a straightforward correspondence between CSP
Figure 1. An example of a potato diagram with three variables \( x, y, z \) and three binary relations (instance solution in bold).

\[
B_x \overset{R_1}{\longrightarrow} B_y \overset{R_2}{\longrightarrow} B_z
\]

Figure 2. Constraint network with the free variables \( x \) and \( y \) which defines the relation

\[
S = \{(x, y) \mid \exists s, (s) \in R_1 \land (x, s) \in R_2 \land (y, s) \in R_2\}.
\]

instances with free variables and primitive positive definitions. See Figure 2 for an example of such a correspondence.

Let \( A \) be an algebra. We say that the variety generated by \( A \) is congruence meet-semidistributive (SD(\( \land \)) for short) if for any algebra \( B \) in the variety generated by \( A \) and any congruences \( \alpha, \beta, \gamma \) in \( B \), we have

\[
\alpha \land \beta = \alpha \land \gamma \Rightarrow \alpha \land (\beta \lor \gamma) = \alpha \land \beta.
\]
Given a relational structure $A$, if the $(2, 3)$-consistency checking algorithm as defined in [3] always returns a correct answer to any instance of CSP($A$) (i.e., if there are no false positives), we say that $A$ has bounded width. The following result shows a deep connection between bounded width and congruence meet-semidistributivity.

**Theorem 2.1.** Let $A$ be a finite relational structure containing all the one-element unary relations (constants). Then the following are equivalent:

1. $A$ has bounded width;
2. the variety generated by $(A, \text{Pol}(A))$ is SD($\land$);
3. $\text{Pol}(A)$ contains ternary and quaternary weak near unanimity (WNU) operations with the same polymer, i.e., there exist idempotent $u, v \in \text{Pol}(A)$ such that for all $x, y \in A$, we have
   
   $$u(x, x, y) = u(x, y, x) = u(y, x, x) = v(y, x, x) = \cdots = v(x, x, x, y).$$

**Proof.** (1) $\Rightarrow$ (3): See the upcoming survey [12].

(2) $\Rightarrow$ (1): This is the main result of [3].

(3) $\Rightarrow$ (2): It is enough to observe that the equations for idempotent ternary and quaternary WNU operations with the same polymer fail in any nontrivial variety of modules. Therefore, as shown in [10, Theorem 9.10], the third condition implies congruence meet-semidistributivity (this is true even in the case of infinite algebras, as shown in [11]).

We say that an $n$-ary operation $t$ is Taylor if for every $1 \leq k \leq n$, the operation $t$ satisfies some equation of the form

$$t(u_1, \ldots, u_{k-1}, x, u_{k+1}, \ldots, u_n) \approx t(v_1, \ldots, v_{k-1}, y, v_{k+1}, \ldots, v_n),$$

where $u_i, v_i \in \{x, y\}$ for all $i$ and the (different) variables $x$ and $y$ are both on the $k$-th place (this is a weakest set of linear equations that no projection can satisfy). An algebra $A$ (resp. a relational structure $A$) admits a Taylor operation if there is a Taylor operation in the operational clone of $A$ (resp. in $\text{Pol}(A)$). If $A$ is a relational structure with all constants (one-element unary relations) that does not admit any Taylor operation, then CSP($A$) is known to be NP-complete (see [6, Corollary 7.3] together with [10, Lemma 9.4]).

A relational structure $A$ is conservative if $A$ contains all possible unary relations. We will call a relational structure $A$ 3-conservative if $A$ contains all one, two and three-element unary relations.

### 3. Red, yellow, and blue pairs

Assume that $A$ is a 3-conservative relational structure that admits a Taylor operation. Then for every pair of vertices $a, b \in A$, there must exist a polymorphism of $A$ that, when restricted to $\{a, b\}$, is a semilattice, majority, or minority. If there was a pair without such a polymorphism, then a result by
Schaefer [15] implies that all the operations in Pol(\(A\)) restricted to \{a, b\} are projections, and so Pol(\(A\)) cannot contain a Taylor term.

Following Andrei Bulatov, we color each pair \{a, b\} \(\subset A\) as follows:

1. If there exists \(f \in \text{Pol}(A)\) that acts as a semilattice operation on \{a, b\}, we color \{a, b\} red, else
2. If there exists \(g \in \text{Pol}(A)\) that acts as the majority operation on \{a, b\}, we color \{a, b\} yellow, else
3. If there exists \(h \in \text{Pol}(A)\) that acts as the minority operation on \{a, b\}, we color \{a, b\} blue.

In [5], Andrei Bulatov proves the Three Operations Proposition (the version we present here has notation changed to be compatible with ours and omits the last part of the original proposition, which we will not need):

**Theorem 3.1.** Let \(A\) be a 3-conservative relational structure. There are polymorphisms \(f(x, y), g(x, y, z),\) and \(h(x, y, z)\) of \(A\) such that for every two-element subset \(B \subset A\):

- \(f|_B\) is a semilattice operation if \(B\) is red, and \(f|_B(x, y) = x\) otherwise;
- \(g|_B\) is a majority operation if \(B\) is yellow; \(g|_B(x, y, z) = x\) if \(B\) is blue, and \(g|_B(x, y, z) = f|_B(f|_B(x, y), z)\) if \(B\) is red;
- \(h|_B\) is a minority operation if \(B\) is blue; \(g|_B(x, y, z) = x\) if \(B\) is yellow, and \(g|_B(x, y, z) = f|_B(f|_B(x, y), z)\) if \(B\) is red.

We omit the proof here; note however that the operations \(f, g,\) and \(h\) can be obtained in a straightforward way by patiently composing terms. (Moreover, one actually does not need the full power of 3-conservativity here; 2-conservativity would suffice.)

**Corollary 3.2.** If \(A\) is such that all its pairs are red or yellow, then \(A\) has bounded width since the operations

\[
\begin{align*}
u(x, y, z) &= g(f(f(x, y), z), f(f(y, z), x), f(f(z, x), y)), \\
v(x, y, z, t) &= g(f(f(f(x, y), z), t), f(f(f(y, z), x), t)), f(f(f(z, x), y), t)),
\end{align*}
\]

are a pair of ternary and quaternary WNUs with the same polymer. If \(x\) and \(y\) are red, then \(u(x, x, y) = v(x, x, x, y) = f(x, y)\), and if \(x\) and \(y\) are yellow, then \(u(x, x, y) = v(x, x, x, y) = x\).

By Theorem 2.1, it is enough to show that if CSP(\(A\)) is not NP-complete, then \(A\) does not have any blue pair of vertices.

We could end our paper at this point and refer the reader to the article [9] which, among other things, shows by combinatorial methods that if \(G\) is a conservative digraph and CSP(\(G\)) is not NP-complete, then all pairs of vertices of \(G\) are either yellow or red. However, we would like to present a short algebraic proof of this statement and generalize it beyond digraphs.
4. Main proof

We proceed by contradiction. For the remainder of this section, let us fix a 3-conservative relational structure $A$ (with unary and binary relations only) that admits a Taylor term, yet there exists a blue pair \{a, b\} $\subset A$.

Proposition 4.1. The relational clone of $A$ contains the relation

$$R = \{(b, b, b), (a, a, b), (a, b, a), (b, a, a)\}.$$ 

Proof. Consider the ternary relation

$$R = \{(t(a, a, b), t(a, b, a), t(b, a, a)) | t \in \text{Pol}_3(A)\},$$ 

where $\text{Pol}_3(A)$ denotes all the ternary polymorphisms of $A$. It is easy to see that the relation $R$ lies in $\text{Inv}(\text{Pol}(A))$. Since the projections $\pi_1, \pi_2, \pi_3$, and the local minority $h$ belong to $\text{Pol}_3(A)$, substituting these polymorphisms for $t$ yields that $(a, a, b), (a, b, a), (b, a, a)$, and $(b, b, b)$ lie in $R$.

Since the set $\{a, b\}$ is invariant under all polymorphisms of $A$, we have $R \subset \{a, b\}^3$. Assume now that $R$ contains more than the four elements given above. If $(a, a, a) \in R$, then there exists $t \in \text{Pol}_3(A)$ that acts as a majority on $\{a, b\}$, and $a, b$ should have been yellow. If $(b, b, a) \in R$, then there exists some $t$ such that

$$t(a, a, b) = t(a, b, a) = b \quad \text{and} \quad t(b, a, a) = a.$$ 

Since $\{a, b\}$ is not red, $t(b, b, a) = t(b, a, b) = a$ and $t(a, b, b) = b$ (otherwise one of $t(x, x, y), t(x, y, x)$, or $t(y, x, x)$ would be a semilattice operation). But then $f(x, y, z) = h(t(x, y, z), t(y, z, x), t(z, x, y))$ is a majority operation on $\{a, b\}$, a contradiction.

We can handle the cases $(b, a, b), (a, b, b) \in R$ in a similar fashion. \hfill $\square$

If $(x, y, z)$ is a triple of free variables of the CSP instance $I$ and $s$ is a solution of $I$, then we say that $s$ realizes the triple $(c, d, e) \in A^3$ if $s(x) = c$, $s(y) = d$, and $s(z) = e$. We say that $I$ realizes some triple of elements if there exists a solution $s$ of $I$ that realizes this triple.

Since $R$ lies in the relational clone of $A$, it follows that $R$ lies in the relational clone of the structure $\overline{A}$ obtained from $A$ by adding all the unary and binary relations in the relational clone of $A$ as basic relations. (We will need to use these relations to make our induction work.) Hence, there is a CSP($\overline{A}$) instance $I$ and three variables $x, y, z$ such that $I$ with free variables $x, y, z$ realizes precisely all the triples in $R$. Let $\{B_j | j \in J\}$ be the potatoes in the constraint network of $I$. Choose $I$ so that the sum of the sizes of its potatoes is minimal among all possible CSP($\overline{A}$) instances realizing $R$.

Observe that if $s_1, s_2, s_3$ are solutions of $I$ and $p$ is a ternary operation preserving all the relations used in $I$, then $p(s_1, s_2, s_3)$ is also a solution of $I$.

Observation 4.2. For every $j \in J$, we have $|B_j|$ equal to 2 or 3.
Proof. It is easy to see that the potatoes for $x, y,$ and $z$ are all equal to \{a, b\}.

Assume that there is a potato $B_j$ with at least four distinct elements. Let $s_{aab}, s_{aba},$ and $s_{baa}$ be solutions of $I$ realizing the triples $(a, a, b)$, $(a, b, a)$, and $(b, a, a)$.

Now let $B_j' = \{s_{aab}(j), s_{aba}(j), s_{baa}(j)\}$. If we replace $B_j$ by $B_j'$ (using a unary constraint), we get a smaller instance $I'$ which keeps the solutions $s_{aab}, s_{aba},$ and $s_{baa}$. We know that $\overline{A}$ has a polymorphism $h$ which is the minority on \{a, b\}. Therefore, $h(s_{aab}, s_{aba}, s_{baa})$ is a solution of $I'$ that realizes the triple $(b, b, b)$. The instance $I'$ then realizes precisely all the elements of $R$, a contradiction with the minimality of $I$.

If some $B_j$ were a singleton, we could simply remove the variable $j$ from $I$. The potatoes that were connected with $B_j$ by binary constraints might need to become smaller, but that is easy to achieve using unary constraints. We would obtain in this way a smaller instance that still realizes $R$. □

Observation 4.3. The pair \{c, d\} is blue for every distinct $c, d \in B_j$ where $j \in J$.

Proof. Assume first that \{c, d\} is red. Without loss of generality, let $f(c, d) = d$, where $f$ is the local semilattice polymorphism from Theorem 3.1. We know that there exists a solution $s$ of $I$ such that $s(j) = d$ (otherwise, we could just delete $d$ from $B_j$). If now $r$ is a solution such that $r(j) = c$, then $f(r, s)$ is also a solution of $I$, and $f(r, s)(j) = d$. What is more, since $f = \pi_1$ on \{a, b\}, the solution $f(r, s)$ realizes the same triple as $r$. Therefore, we can remove $c$ from $B_j$ without losing anything in $R$.

The situation for \{c, d\} yellow is similar. Again, let $s$ and $r$ be solutions such that $s(j) = d$ and $r(j) = c$. Then $g(r, s, s)$ is a solution that realizes the same triple as $r$ and satisfies $g(r, s, s)(j) = d$. We can thus eliminate $c$ from $B_j$. □

We note that one of the main ingredients in the above proof is the fact that the algebra \{d\} absorbs \{c, d\} in the sense of [3].

Observation 4.4. For every $j \in J$, we have $|B_j| = 2$.

Proof. Assume that we have a $j$ such that $B_j = \{c, d, e\}$. As in the proof of Observation 4.2, let $s_{aab}, s_{aba}, s_{baa}$ be some solutions of $I$ realizing $(a, a, b)$, $(a, b, a)$, $(b, a, a)$.

If $\{s_{aab}(j), s_{aba}(j), s_{baa}(j)\} \neq B_j$, then we can make $B_j$ (and therefore $I$) smaller as in the proof of Observation 4.2. Without loss of generality, assume that

$$s_{baa}(j) = c, \quad s_{aba}(j) = d, \quad s_{aab}(j) = e, \quad s_{bbb}(j) = c.$$

Now consider the solution $r = h(s_{baa}, s_{aba}, s_{bbb})$ where $h$ again comes from Theorem 3.1. Since all the pairs in $B_j$ are blue, $r$ is a realization of $(a, a, b)$ such that $r(j) = h(c, d, e) = d$. This means that we can safely delete $e$ from $B_j$ and again get a smaller instance that realizes $R$. □
Let us put together what we know about $I$: We have two-element potatoes everywhere, connected by binary constraints. Now observe that every binary relation on a two element set is invariant under the majority map $m$. Therefore, $I$ must realize the triple
\[ m((b, a, a), (a, b, a), (a, a, b)) = (a, a, a), \]
a contradiction.

We state our result as a theorem:

**Theorem 4.5.** If $A$ is a finite 3-conservative relational structure that admits a Taylor term and all of its basic relations are binary or unary, then the variety generated by $(A, \text{Pol}(A))$ is $SD(\wedge)$.

Translating our result to the CSP complexity setting, we obtain:

**Corollary 4.6** (Dichotomy for 3-conservative CSPs with binary relations). If $A = (A, R_1, \ldots, R_k)$ is a finite 3-conservative relational structure that admits a Taylor term and all of its basic relations are binary or unary, then CSP$(A)$ has bounded width. If $A$ does not admit a Taylor polymorphism, then CSP$(A)$ is NP-complete.

Note, however, that our result cannot be generalized to relational structures with ternary basic relations. Consider the structure $A$ on $\{0, 1\}$ equipped with all possible unary relations of size one and two, plus the three equivalence relations $\alpha$, $\beta$, and $\gamma$. These three equivalences correspond to the following partitions of the...
4-element set \( \{0, 1\}^2 \):

\[
\alpha \iff \{\{(0, 0), (1, 1)\}, \{(0, 1), (1, 0)\}\}, \\
\beta \iff \{\{(0, 0), (0, 1)\}, \{(1, 0), (1, 1)\}\}, \\
\gamma \iff \{\{(0, 0), (1, 0)\}, \{(0, 1), (1, 1)\}\}.
\]

Observe that \( \text{Pol}(A) \) contains the idempotent Taylor term \( p(x, y, z) = x + y + z \), where addition is taken componentwise and modulo 2 (i.e., as in \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)). However, the variety generated by \( A = (\{0, 1\}^2, \text{Pol}(A)) \) is definitely not SD(\( \wedge \)) since \( \alpha, \beta, \gamma \) are congruences of \( A \) such that \( \alpha \wedge \beta = \alpha \wedge \gamma = 0_A \), while \( \alpha \wedge (\beta \lor \gamma) = \alpha \).

5. Closing remarks

It is challenging to obtain a digraph that would have a tractable CSP, yet would not have bounded width (though such beasts do exist; see the original argument in [1], or the construction in [7]). We give a partial explanation for this phenomenon: such digraphs needs to admit some nonconservative binary or ternary operation, while avoiding ternary and quaternary WNU.

As we have seen, our result about 3-conservative binary structures is quite tight. However, some generalizations might still be possible. At the moment, we do not know if our result holds for 2-conservative digraphs and if Theorem 4.5 holds when we drop the finiteness condition. We suspect that the answer to both questions will be negative, but the counterexamples might turn out to be illuminating.

Acknowledgments. The author thanks Libor Barto for motivating him, and thanks the reviewer for pointing out how the final part of the proof could be simplified.

References

[1] Atserias, A.: On digraph coloring problems and treewidth duality. European J. Combin. 29, 796–820 (2008)
[2] Barto, L.: The dichotomy for conservative constraint satisfaction problems revisited. In: 26th Annual IEEE Symposium on Logic in Computer Science (LICS 2011), pp. 301–310. IEEE, Washington (2011)
[3] Barto, L., Kozik, M.: Constraint satisfaction problems of bounded width. In: 50th Annual IEEE Symposium on Foundations of Computer Science (FOCS 2009), pp. 595–603. IEEE, Atlanta (2009)
[4] Bodnarchuk, V., Kaluzhin, L., Kotov, V., Romov, B.: Galois theory for Post algebras. I. Cybernetics 5, 243–252 (1969)
[5] Bulatov, A.A.: Complexity of conservative constraint satisfaction problems. ACM Trans. Comput. Log. 12, 24:1–24:66 (2011)
[6] Bulatov, A., Jeavons, P., Krokhin, A.: Classifying the complexity of constraints using finite algebras. SIAM J. Comput. 34, 720–742 (2005)
[7] Bulín, J., Delić, D., Jackson, M., Niven, T.: On the reduction of the CSP dichotomy conjecture to digraphs. In: C. Schulte (ed.) Principles and Practice of Constraint Programming. Lecture Notes in Computer Science, vol. 8124, pp. 184–199. Springer, Berlin (2013)
[8] Geiger, D.: Closed systems of functions and predicates. Pacific J. Math. 27, 95–100 (1968)
[9] Hell, P., Rafiey, A.: The dichotomy of list homomorphisms for digraphs. In: Proceedings of the Twenty-Second Annual ACM-SIAM Symposium on Discrete Algorithms, pp. 1703–1713. SIAM (2011)
[10] Hobby, D., McKenzie, R.: The Structure of Finite Algebras. American Mathematical Society (1988)
[11] Kearnes, K.A., Kiss, E.W.: The shape of congruence lattices. Mem. Amer. Math. Soc. 222 (2013)
[12] Kozik, M., Krokhin, A., Valeriote, M., Willard, R.: Characterizations of several Maltsev conditions. Algebra Universalis 73, 205–224 (2015)
[13] Larose, B., Tesson, P.: Universal algebra and hardness results for constraint satisfaction problems. Theoret. Comput. Sci. 410, 1629–1647 (2009)
[14] Pöschel, R.: A general Galois theory for operations and relations and concrete characterization of related algebraic structures. Report, vol. R-01/80. Akademie der Wissenschaften der DDR, Institut für Mathematik, Berlin (1980). www.math.tu-dresden.de/~poeschel/poePUBLICATIONSpdf/poeREPORT80.pdf
[15] Schaefer, T.J.: The complexity of satisfiability problems. In: Proceedings of the Tenth Annual ACM Symposium on Theory of Computing (STOC 1978), pp. 216–226. ACM, New York (1978)

ALEXANDR KAZDA
IST Austria, Am Campus 1, 3400 Klosterneuburg, Austria
e-mail: alex.kazda@gmail.com