The Low-lying Dirac Eigenmodes from Domain Wall Fermions
Guofeng Liu*[RBC collaboration]

aDepartment of Physics, Columbia University, New York, NY, 10027, USA

We calculate the low-lying eigenvalues and eigenvectors of the hermitian domain wall Dirac operator on various
gauge backgrounds by Ritz minimization. The mass dependence of these eigenvalues is studied to extract the
physical 4 dimensional $\lambda$, whose spectral density is related to $\langle \tilde{\psi}\psi \rangle$ through the Banks-Casher relation, and $\delta m$
which represents the effects of the residual chiral symmetry breaking in domain wall formalism on a per eigenmode
basis. The topological structure of the underlying gauge field is examined by measuring the $\Gamma_5$ matrix elements
between the low-lying eigenmodes.

1. DIAGONALIZATION METHOD

We use the conjugate-gradient method proposed by Kaukreuter and Simma [1] to calculate the lowest 19 eigenvalues and
eigenvectors of the hermitian domain wall Dirac operator $D_H = \gamma_5 R D$, where $D$ is the domain wall Dirac
operator and $R$ is the reflection operator in $s$ direction (look at [2] for a detailed description of
the conventions).

Table 1
The bare quark masses used for the Wilson ensembles are 0.0, 0.0025, 0.005, 0.0075, 0.01. For the Iwasaki ensemble, only the lowest mass,
0.0005, is different.

| $V$ | Action | $\beta$ | $a^{-1}$/Gev | $L_s$ | #conf. |
|-----|--------|--------|-------------|------|--------|
| 16$^4$ | Wilson | 6.0 | 2.0 | 16 | 32 |
| 16$^4$ | Wilson | 6.0 | 2.0 | 8 | 10 |
| 16$^4$ | Iwasaki | 2.6 | 2.0 | 16 | 55 |

For each of the configurations listed in Table 1, we calculate the eigenvalue spectrum at five different masses. There are two main reasons for
this practice. First, by studying the mass dependence as described in section 2, we can remove from the eigenvalue spectrum a portion of
the residual chiral symmetry breaking effects. For the Iwasaki action at $\beta = 2.6$, the residual mass is very small, which makes the convergence of the
Ritz minimization prohibitively slow at zero bare quark mass. Thus, we had to use nonzero bare quark masses ($5 \times 10^{-4}$ and up) to speed up the
convergence, and then extrapolate to the region of interest.

2. SPECTRAL DENSITY AND CHIRAL SYMMETRY BREAKING

In the continuum limit, the spectral density of Dirac operator is related to the chiral condensate $\langle \tilde{\psi}\psi \rangle$ by the Banks-Casher relation

$$- \langle \tilde{\psi}\psi \rangle = \frac{1}{12V} \frac{\langle |\nu| \rangle}{m} + \frac{1}{12V} \left( \sum_{\lambda \neq 0} \frac{m}{\lambda^2 + m^2} \right)$$

and

$$\lim_{m \to 0} \lim_{V \to \infty} - \langle \tilde{\psi}\psi \rangle = \frac{\pi}{12} \rho(0).$$

Since $D_H$ is a continuous function of $m_f$, we can expand and parameterize its eigenvalue $\Lambda_{H,i}$ as

$$\Lambda_{H,i}^2 = n_{5,i}^2 (\lambda_i^2 + (m_f + \delta m_i)^2) + O(m_f^3).$$

From

$$\frac{dD_H}{dm_f} = -\gamma_5 Q^{(w)},$$

where $-m_f R Q^{(w)}$ is the mass term in the fermion matrix, we can derive

$$- \langle \Lambda_{H,i} | e^{-i \gamma_5 Q^{(w)}} | \Lambda_{H,i} \rangle = \frac{d\Lambda_{H,i}}{dm_f}$$

and

$$\frac{n_{5,i}^2 (m_f + \delta m_i)}{\Lambda_{H,i}}.$$
The quantity \( \langle \bar{\psi} \psi \rangle \) is defined on the boundary for domain wall fermions, and given by

\[
- \langle \bar{q} q \rangle = -\frac{1}{12V} \left\langle \sum_{\Lambda H} \frac{\langle \Lambda H | \gamma_5 Q^{(w)} | \Lambda H \rangle}{\Lambda H} \right\rangle.
\]

By comparing Eq. 2 and Eq. 3, we can recognize the parameter \( \lambda_i \) as an eigenvalue of the four dimensional Dirac operator. Here, \( \delta m_i \) represents the contribution to the eigenvalue from the chiral symmetry breaking effects of coupling of the domain walls.

\[
\frac{1}{12V} \left\langle \sum_{\Lambda H} \frac{\langle \Lambda H | \gamma_5 Q^{(w)} | \Lambda H \rangle}{\Lambda H} \right\rangle.
\]

By comparing Eq. 3 and Eq. 4, we can recognize the parameter \( \lambda_i \) as an eigenvalue of the four dimensional Dirac operator. Here, \( \delta m_i \) represents the contribution to the eigenvalue from the chiral symmetry breaking effects of coupling of the domain walls.

![Figure 1. \( \lambda \) and \( \delta m \) distribution for the Wilson ensemble at \( \beta = 6.0, L_s = 16 \).](image1.png)

![Figure 2. \( \lambda \) and \( \delta m \) distribution for the Wilson ensemble at \( \beta = 6.0, L_s = 8 \).](image2.png)

![Figure 3. \( \lambda \) and \( \delta m \) distribution for the Iwasaki ensemble at \( \beta = 2.6, L_s = 16 \).](image3.png)

Fig. 1, 2 and 3 are the distribution of \( \lambda \) and \( \delta m \) for the ensembles listed in Table 1. A rough agreement between the peak of the \( \delta m \) distribution and \( m_{\text{res}} \) from the mid-point term is observed. A nice agreement between the spectral density and \( \langle \bar{\psi} \psi \rangle \) is confirmed except for the peak at \( \lambda = 0 \), which is explained by the abundance of zero modes at finite volume, and is expected to disappear in the infinite volume limit. The comparison between Fig. 1 and 2 shows the expected \( L_s \) dependence of \( \delta m \). Fig. 1 and 3 have the same \( L_s \), volume and strength of coupling but different types of action, which makes the \( \delta m \) an order of magnitude smaller for the latter.

### 3. GAUGE FIELD TOPOLOGICAL STRUCTURE

In the continuum limit, the Dirac operator has \( \gamma_5 \) symmetry, which ensures that the zero modes have chirality of \( \pm 1 \), and the non-zero modes, with chirality 0, are paired by the application of \( \gamma_5 \). The number of zero modes should correspond
to the winding number of the gauge field.

In the domain wall formalism, we display the matrix elements $|\langle \Lambda H_i \mid \Gamma_5 \mid \Lambda H_j \rangle|$ as a 3 dimensional lego plot. This provides a way of visualizing the chiral properties of the domain wall Dirac operator and the topological structure of the underlying gauge field. Fig 4 shows a set of such lego plots for the Wilson ensemble at $\beta = 6.0$, $L_s = 16$. In the first lego plot, the lowest 4 of the eigenvectors are also $\Gamma_5$ eigenvectors, while the rest of them are paired. Fig 4 also shows that during the heat bath evolution, when a configuration transforms from the topological sector with topology 4 to that with topology 2, some complex configurations result in between which can not be unambiguously categorized as belonging to any regular topological sector. Table 2 lists the percentage of this kind of complex configuration for each ensemble in Table 1. The readers are referred to C. Dawson’s poster for a more quantitative study of the complex configurations.

### Table 2

| Action   | $\beta$ | $L_s$ | $m_{res}$         | complex% |
|----------|---------|-------|-------------------|----------|
| Wilson   | 6.0     | 16    | 0.00124(5)        | 50%      |
| Wilson   | 6.0     | 8     | $\sim 0.007$     | 50%      |
| Iwasaki  | 2.6     | 16    | 0.00014(3)        | 10%      |

## 4. CONCLUSIONS

Our prescription of extracting $\lambda$ and $\delta m$ distribution from mass dependence of the eigenvalues of $D_H$ is consistent with the Banks-Casher relation and the usual ways of determining $\langle \bar{\psi} \psi \rangle$ and $m_{res}$. The good chiral properties of domain wall fermions makes them a very good tool for studying the topological structure of the background gauge field. We find that this structure is closely related to the type of action used in generating these configurations.

## 5. ACKNOWLEDGMENT

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## REFERENCES

1. T. Kalkreuter and H. Simma, [hep-lat/9507023](http://arxiv.org/abs/hep-lat/9507023)
2. T. Blum et al., [hep-lat/0007038](http://arxiv.org/abs/hep-lat/0007038)