The helicity modulus in gauge field theories

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We consider the 4d compact $U(1)$ theory with Wilson action and characterize its phase diagram using the notion of electromagnetic flux, instead of the more usual magnetic monopole. Taking inspiration from the flux picture, we consider the helicity modulus (h.m.) for this theory, and show that it is an order parameter for the confinement deconfinement phase transition. We extend the definition of the h.m. to an Abelian projected Yang-Mills theory, and discuss its behavior in $SU(2)$.

1. Introduction

Consider a theory characterized by the presence of a phase transition as one parameter $\beta$ (the inverse temperature or some coupling of the theory) varies through some particular value $\beta_c$. A usual way to describe the transition is provided by the so-called order parameter, defined as a function of $\beta$ which is zero in one phase and non-zero in the other. The meaning of this definition is two-fold: on one hand, such a function has certainly a point of non-analyticity, which reflects the underlying non-analyticity of the free energy of the system (in the thermodynamic limit); on the other hand, this behavior can sometimes be used to relate the phase transition with the spontaneous breaking of some symmetry (implicitly or explicitly defined) of the model, thus providing an appealing physical picture of the transition itself.

In this paper we introduce an order parameter for the 4d compact $U(1)$ Abelian theory, and study it numerically for the Wilson action

$$ S = -\beta \sum_P \cos \theta_P $$

where $\beta$ is the inverse bare coupling and $\theta_P$ is the plaquette angle. This theory is characterized by a strong coupling confined phase and a weak coupling Coulomb phase; in the confined phase, in analogy with ordinary superconductors, the $U(1)$ gauge symmetry is supposed to be spontaneously broken. Our order parameter will make use exclusively of the notion of electromagnetic flux (instead than the more usual magnetic monopole) which we now introduce.

2. The flux in Abelian theories

Let $L$ be the lattice size of a 4d hypercubic lattice with periodic boundary conditions (p.b.c.). Our definition of the flux $\Phi_{\mu\nu}$ through a given $(\mu, \nu)$ orientation is:

$$ \Phi_{\mu\nu} = \frac{1}{L^2} \sum_{(\mu, \nu) \text{ planes}} \sum_P \theta_P^{\mu\nu} - \pi, \pi $$

where $[\theta_P]_{-\pi, \pi}$ is the plaquette angle reduced to the interval $[-\pi, \pi]$. A double sum is present:

- the internal $\sum_{P_{\mu\nu}}$ is the sum over the plaquettes in a single plane, and is $2\pi k$ ($k \in \mathbb{Z}$) valued because of the p.b.c.
- the external average $\frac{1}{L^2} \sum_{\text{planes}}$ over all parallel planes of the given orientation, is non-trivial because the flux through different planes can change due to the presence of magnetic monopoles (herein lies the connection with the usual picture). The allowed values for $\Phi_{\mu\nu}$ are thus multiples of $2\pi/L^2$.

Monitoring the flux distribution $\nu(\phi)$ in the two phases, one observes that $\nu(\phi)$ is Gaussian (centered at $\phi = 0$) in the confined phase, while it is peaked around multiples of $2\pi$ in the Coulomb phase (defining so-called flux sectors); in the thermodynamic limit, tunnelling between flux sectors becomes completely suppressed. This behavior
already provides a characterization of the phase transition.

3. Response to an external flux

One can now ask what is the response of the system to an external electromagnetic flux, in the two phases. The corresponding flux free energy is a straightforward extension to the Abelian case of \( 't \) Hooft’s non-Abelian twist free energy \([2]\). Like the latter \([3]\), it has a characteristic behavior in each phase. We impose an extra flux \( \phi \in \mathbb{R} \) to the following stack of plaquettes (this is only one possible choice):

\[
\text{stack} = \{ \theta_{P_{\mu \nu}} | \mu = 1, \nu = 2; x = 1, y = 1 \}. \tag{3}
\]

The partition function of the system becomes

\[
Z(\phi) = \int D\theta e^{\beta(\sum_{\text{stack}} \cos(\theta_{P} + \phi) + \sum_{\text{other plaquettes}} \cos \theta_{P})} \tag{4}
\]

where ‘stack’ is the complement of the stack, i.e., consists of all the other unchanged plaquettes. Alternatively (and more conveniently from a numerical point of view), it is possible to perform a change of variables such that the extra flux is spread through all the plaquettes with the given orientation \( (\mu, \nu) \), leading to

\[
Z(\phi) = \int D\theta e^{\beta(\sum_{(\mu, \nu) \text{ planes}} \cos(\theta_{P} + \frac{\pi k}{L}) + \sum_{(\mu, \nu) \text{ planes}} \cos \theta_{P})} \tag{5}
\]

where \( (\mu, \nu) \) indicates all the other orientations, through which no extra flux is imposed.

\( Z(\phi) \) is \( 2\pi \) periodic in \( \phi \). Fig. 1 shows the behavior of the free energy \( F(\phi) = -\log Z(\phi) \) of the system in the two phases. In the confined phase (where the flux can freely change) the system is insensitive to the presence of an external flux \( (F(\phi) = \text{const.}) \). In the Coulomb phase the free energy is described perfectly by the ansatz

\[
F(\phi) = -\log \sum_k e^{-\frac{2\pi}{\beta} (\phi - 2\pi k)^2} \tag{6}
\]

which says that around each flux-sector the dependence on the external flux is quadratic, as expected classically, and that we must consider the contribution of all the sectors at the same time. \( \beta_R \) is the only parameter in this equation. It replaces \( \beta \) in the classical expression \([4]\), and therefore plays the role of a renormalized coupling \([5]\).

4. The helicity modulus

The last step in the construction of the order parameter is to note that the physical information contained in Fig. 1 can be rephrased in a more concise way: if instead of the whole curve (as a function of \( \phi \)) we consider only the \emph{curvature} of \( F(\phi) \) at the origin (or at any other point), we get a function of \( \beta \) which is always zero in the confined phase \( (F(\phi) = \text{const.}) \) and is different from zero in the Coulomb phase; that is, it is an order parameter. This construction was already known in the context of the 2d XY model \([6]\), where the name ‘helicity modulus’ was first introduced. In our context we define the helicity modulus

\[
h(\beta) = \left. \frac{\partial^2 F(\phi)}{\partial \phi^2} \right|_{\phi=0} \tag{7}
\]

which can be related to \( \beta_R \) via Eq. 6. Computing explicitly the double derivative (with \( F(\phi) \) defined as per Eq. 6), one gets

\[
h(\beta) = \frac{1}{L} \left( \sum_{(\mu, \nu) \text{ planes}} \beta \cos \theta_{P} \right) - \frac{1}{L} \left( \sum_{(\mu, \nu) \text{ planes}} \beta \sin \theta_{P} \right)^2 \tag{8}
\]

In Fig. 2 we plot this observable for different volumes: its behavior is qualitatively consistent with an order parameter. Moreover, it is shown in \([6]\) that the drop to zero in the confined phase is exponential in \( (\beta_c - \beta) \), providing convincing evidence that the transition is \( 1^{\text{st}} \) order.

5. Extension to non-Abelian theories

We now turn our attention to Yang-Mills theories, which display a finite temperature transition
If we assume that (1), following [5], the long range properties of the theory can be well described by an effective Abelian theory, then we expect that the Abelian gauge ensemble obtained after a suitable gauge fixing and Abelian projection must change from confining to deconfined across $T_c$. Therefore, the helicity modulus measured in the projected ensemble should present a discontinuity at $T_c$. Moreover, under the further assumption (2) that the effective action which describes the projected ensemble is Wilson-like

$$S_{\text{eff}} = \beta_{\text{eff}} \sum_P \cos \theta_{P_{\text{proj}}},$$

where $\beta_{\text{eff}}$ is an effective coupling whose value could be determined by Inverse Monte Carlo, the helicity modulus is the same as in Eq.(8), which we rewrite for convenience as

$$h(\beta) = \beta_{\text{eff}} H_1(\beta) - \beta_{\text{eff}}^2 H_2(\beta)$$

where $H_1(\beta)$ and $H_2(\beta)$ are given by

$$H_1(\beta) = \frac{1}{V} \langle \sum_{(\mu, \nu) \text{ planes}} \cos \theta_P \rangle$$

$$H_2(\beta) = \frac{1}{V} \langle (\sum_{(\mu, \nu) \text{ planes}} \sin \theta_P)^2 \rangle.$$  \hspace{1cm} (11)

To perform the Abelian projection, we use the Maximal Abelian Gauge [6], defined by

$$\sum_{x, \mu} Tr(U_\mu(x)\sigma_3 U^\dagger_\mu(x)\sigma_3) \text{ maximum}$$

where the maximum must be found over the set of all gauge transformations. Not knowing the value of $\beta_{\text{eff}}$, we measure separately $H_1(\beta)$ and $H_2(\beta)$. In fact, we can try to infer $\beta_{\text{eff}}$ from the requirement that Eq.(10) represents an order parameter. The value $\beta_{\text{eff}} = 1.0$ around the transition region works well, as shown Fig.3.

The similarity with Fig.2 is remarkable, but finite size effects here are more suggestive of a second order phase transition, as they should be. This behavior can be considered as a strong support of both assumptions (1) and (2). Our measurements of the same observable Eq.(11) after fixing to another gauge do not show a similar behavior. This presumably singles out the Maximal Abelian Gauge as yielding an effective action particularly local and close to the Wilson action. Fixing to another gauge, the effective action contains sizeable additional terms, and the expression for the helicity modulus Eq.(7) differs appreciably from Eq.(10).

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