ON SYMPLECTIC LEAVES AND INTEGRABLE SYSTEMS IN
STANDARD COMPLEX SEMISIMPLE POISSON-LIE GROUPS

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Abstract. We provide an explicit description of symplectic leaves of a simply
connected connected semisimple complex Lie group equipped with the stan-
dard Poisson-Lie structure. This sharpens previously known descriptions of
the symplectic leaves as connected components of certain varieties. Our main
tool is the machinery of twisted generalized minors. They also allow us to
present several quasi-commuting coordinate systems on every symplectic leaf.
As a consequence, we construct new completely integrable systems on some
special symplectic leaves.

1. Introduction

Let $G$ be a simply connected connected semisimple complex Lie group supplied
with the standard Poisson-Lie structure. It is well known (see e.g., [1, 3, 4]) that the
symplectic leaves in $G$ are closely related to double Bruhat cells, the intersections
of double cosets of two opposite Borel subgroups in $G$. Double Bruhat cells were
studied in [2]; one of the main tools developed there was a family of regular functions
on them called twisted (generalized) minors. In the present paper, we provide some
applications of these functions to the study of symplectic leaves in $G$ and integrable
systems on them (some results in this direction were obtained in [4, 6, 8]). One of
our main goals is to bring the machinery of twisted minors to the attention of the
experts in the field. We believe these functions should have further applications to
the study of symplectic leaves and integrable systems.

It was shown in [2] that twisted minors give rise to a family of toric charts
in every double Bruhat cell. These charts were used in [3] for determining the
connected components of real double Bruhat cells. In a similar spirit, we obtain
here an explicit description of symplectic leaves in $G$ (Theorem 2.3); furthermore,
using twisted minors we produce toric charts in every symplectic leaf. This sharpens
the results in [3, 5, 8], where the symplectic leaves were characterized as connected
components of some subvarieties of the double Bruhat cells.

Our second main result (Theorem 2.6) asserts that certain twisted minors quasi-
commute with each other (that is, their Poisson bracket is a scalar multiple of their
product), and gives an explicit expression for their Poisson bracket. (In a different
context, this calculation will also appear in a forthcoming paper by A. Berenstein
and one of the authors (A.Z.) devoted to the study of quantum double cells.)

As an application of Theorem 2.6, we show in Corollary 2.7 that the twisted
minors can be used to construct integrable systems on some special symplectic
leaves. We are unable to match these integrable systems with any known ones; we
call for experts to try to recognize them. It should be possible to extend the method of Corollary 2.7 to produce more examples of integrable systems; we believe that this method deserves further study.

The paper is organized as follows. In Section 2, we provide necessary background and state our main results. Their proofs are given in Sections 3 and 4.

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2. Main results

2.1. Poisson manifolds and symplectic leaves. We start by recalling some basic definitions. Let $M$ be a smooth manifold. Denote by $C^\infty(M)$ the set of smooth complex valued functions on $M$. Recall that $M$ is a Poisson manifold if it is equipped with a bilinear map $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ called Poisson bracket, which makes $C^\infty(M)$ into a Lie algebra and satisfies the Leibniz identity $\{f,gh\} = f\{g,h\} + \{f,h\}g$ for $f,g,h \in C^\infty(M)$. In all the cases we consider $M$ will be a complex algebraic variety; then a Poisson bracket extends uniquely to a Poisson bracket on the space of rational functions on $M$.

A smooth map $\varphi : M \to N$ between two Poisson manifolds is called Poisson if $\varphi^*\{f,g\}_N = \{\varphi^*(f),\varphi^*(g)\}_M$ for every $f,g \in C^\infty(N)$; here $\{\cdot,\cdot\}_M$ and $\{\cdot,\cdot\}_N$ are the Poisson brackets on $M$ and $N$, respectively. The Poisson structure on the product $M \times N$ of two Poisson manifolds is defined by

$$\{f,g\}(x,y) = \{f(x,.),g(x,.)\}_M(x) + \{f(.,g(.,.)\}_N(y)$$

for $f,g \in C^\infty(M \times N)$.

For a function $f$ on a Poisson manifold $M$, the Hamiltonian vector field $X_f$ is the vector field associated to the derivation $\{f,\cdot\}$. For a point $p \in M$, the restrictions of Hamiltonian vector fields to $p$ form a vector subspace $V_p$ inside the tangent vector space $T_pM$. This defines a distribution $V$ on $M$. This distribution is known to be integrable (see e.g. [3]). Thus, every Poisson manifold $M$ is a disjoint union of connected symplectic manifolds $S_{\alpha}$, such that $T_pS_{\alpha} = V_p$ for every $p \in S_{\alpha}$ and the symplectic form $\omega_{\alpha}$ on $S_{\alpha}$ is given by

$$\omega_{\alpha}(X_f,X_g)(p) = \{f,g\}(p) \text{ for } p \in S_{\alpha}, f,g \in C^\infty(M).$$

The manifolds $S_{\alpha}$ are called the symplectic leaves of $M$. Clearly, every inclusion $S_{\alpha} \to M$ is Poisson.

2.2. Standard Poisson-Lie structure on a semisimple Lie group. Recall that a Poisson-Lie group is a Lie group $G$ equipped with a Poisson bracket such that the multiplication map $G \times G \to G$ is Poisson.

Example 2.1. Let $G = SL_2 = \{ \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} : x_{11}x_{22} - x_{12}x_{21} = 1 \}$; then we can define (see e.g. [3]) a family of Poisson structures on $G$ parameterized by $d \in \mathbb{C}$ by

$$\{x_{12},x_{11}\} = dx_{11}x_{12}, \quad \{x_{21},x_{11}\} = dx_{11}x_{21}, \quad \{x_{22},x_{11}\} = 2dx_{12}x_{21}, \quad \{x_{12},x_{21}\} = 0, \quad \{x_{22},x_{12}\} = dx_{12}x_{22}, \quad \{x_{22},x_{21}\} = dx_{21}x_{22}. $$
We will indicate that $SL_2$ is equipped with the above Poisson structure by writing $SL_2^{(d)}$.

Let $G$ be a simply-connected connected semisimple complex Lie group with the Lie algebra $\mathfrak{g}$. Let $f_i, \alpha_i^+, e_i$ ($i = 1, \ldots, r$) be the Chevalley generators of $\mathfrak{g}$, and $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$ be the corresponding triangular decomposition. Let $\alpha_1, \ldots, \alpha_r \in \mathfrak{h}^*$ be the simple roots of $\mathfrak{g}$. The Cartan matrix $A = (a_{ij})$ is given by $a_{ij} = \langle \alpha_i^+, \alpha_j \rangle$. We fix a diagonal matrix $D$ with positive diagonal entries $d_1, \ldots, d_r$, which symmetrizes $A$, i.e., $d_i a_{ij} = d_j a_{ji}$ for all $i$ and $j$. For $t \in \mathbb{C}$, define

$$x_i(t) = \exp(te_i), \quad x_i(t) = \exp(tf_i).$$

The canonical inclusions $\varphi_i : SL_2 \to G$ are defined by

$$\varphi_i \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = x_i(t), \quad \varphi_i \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = x_i(t).$$

The standard Poisson-Lie structure on $G$ is uniquely determined by the requirement that every map $\varphi_i : SL_2^{(d)} \to G$ is Poisson. (The uniqueness is easy to see; an explicit construction using Manin triples can be found in [3] or [4].)

2.3. Generalized minors and their twists. To state our main results we need to introduce more notation and recall some results of [2]. Let $N_-, H, N$ be the subgroups of $G$, which correspond to $\mathfrak{n}_-, \mathfrak{h}$ and $\mathfrak{n}$. We set $B_- = HN_-$ and $B = HN$ to be the pair of opposite Borel subgroups. The set $G_0 = N_- H N$ is a Zariski open subset of $G$ consisting of the elements $x \in G$, which have (a unique) Gaussian decomposition $x = [x]_-[x]_0[x]_+$, with $[x]_- \in N_-$, $[x]_0 \in H$, and $[x]_+ \in N$.

The weight lattice $P$ consists of elements $\gamma \in \mathfrak{h}^*$ such that $\langle \alpha_i^+, \gamma \rangle \in \mathbb{Z}$ for all $i$. Every weight $\gamma \in P$ defines a multiplicative character $a \mapsto a^\gamma$ of $H$, defined by $\exp(h)\gamma = e^{(h, \gamma)}$ for $h \in \mathfrak{h}$. The basis of fundamental weights $\omega_1, \ldots, \omega_r$ in $P$ is defined by $\langle \alpha_i^+, \omega_j \rangle = \delta_{ij}$.

The Weyl group $W$ is defined by $W = \text{Norm}_G(H)/H$. It acts on $H$ by $a^w = w^{-1}aw$, and the following formula defines the action of $W$ on $P$:

$$a^{w(\gamma)} = (a^{-1}aw)^\gamma, \quad \text{for } a \in H, w \in W, \gamma \in P.$$  

The Weyl group is generated by simple reflections $s_1, \ldots, s_r$ acting on weights by $s_i(\gamma) = \gamma - \langle \alpha_i^+, \gamma \rangle \alpha_i$. If $w = s_{i_1} \ldots s_{i_m}$ is a shortest possible expression of $w$ as a product of simple reflections, then $(i_1, \ldots, i_m)$ is called a reduced word of $w$, and $m$ is called the length of $w$ and denoted by $\ell(w)$.

For every $w \in W$, we define a special representative $\overline{w}$ of $w$ in $\text{Norm}_G(H)$ as follows. Let

$$\overline{s}_i = \varphi_i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$  

Then, if we also require $\overline{w_1} \overline{w_2} = \overline{w_1 w_2}$ as long as $\ell(w_1) + \ell(w_2) = \ell(w_1 w_2)$, it is not difficult to see that $\overline{w}$ is well defined for every $w \in W$ (see [3]).

For $u, v \in W$, define the double Bruhat cell $G^{u,v}$ to be

$$G^{u,v} = BuB \cap B_- v B_-.$$  

By [2] Theorem 1.1, $G^{u,v}$ is a smooth algebraic variety of dimension $\ell(u) + \ell(v) + r$ (recall that $r$ is the rank of $\mathfrak{g}$).

For $x \in G_0$ and a fundamental weight $\omega_i$, define

$$\Delta_i(x) = [x]_0^{\omega_i}.$$
It is shown in [3] that $\Delta_i$ extends to a regular function on $G$. For type $A_r$ (when $G = SL_{r+1}$), this is just the principal $i \times i$ minor of a matrix $x$.

For any pair $u, v \in W$, the corresponding generalized minor is a regular function on $G$ given by

$$\Delta_{u \omega_i, \omega_i}(x) = \Delta_i(\mathbf{F}^{-1} x \mathbf{F}).$$

It is shown in [3] that these functions are well defined, that is they depend only on the weights $u \omega_i$ and $\omega_i$, and do not depend on the particular choice of $u$ and $v$.

Define an involutive automorphism $x \mapsto x^\theta$ on $G$ by

$$a^\theta = a^{-1} \quad (a \in H), \quad x_i(t)^\theta = x_i(1/t), \quad x_i(1/t)^\theta = x_i(t).$$

The twist map for $u, v \in W$ is a birational isomorphism $x \mapsto x'$ between $G^{u,v}$ and $G^{u^{-1}, v^{-1}}$ given by (see [3, Theorem 1.6])

$$x' = \left( [\mathbf{F}^{-1} x]_t \mathbf{F}^{-1} x v [x v^{-1}]_t^{-1} \right)^\theta. \tag{2.1}$$

A double reduced word of $(u, v)$ is a word $i = (i_1, \ldots, i_m)$ of length $m = \ell(u) + \ell(v)$ in the alphabet $\{1, \ldots, r\} \cup \{\bar{1}, \ldots, \bar{r}\}$ such that the subword of $i$ consisting of all letters from $[1, \ldots, r]$ is a reduced word of $u$, and the subword consisting of all letters from $[\bar{1}, \ldots, \bar{r}]$ is a reduced word of $v$. For $i = 1, \ldots, r$, we denote $\varepsilon(i) = +1$ and $\varepsilon(\bar{i}) = -1$, and set $|i| = |\bar{i}| = i$.

In what follows, we fix $u, v \in W$ and a double reduced word $i$ of $(u, v)$. We append $r$ entries $i_{m+1}, \ldots, i_{m+r}$ to $i$ by setting $i_{m+j} = \bar{j}$. For $k = 1, \ldots, m$, we set

$$u_{\geq k} = \prod_{\ell = m, \ldots, k} s_{|i_{\ell}|}, \quad u_{< k} = \prod_{\ell = 1, \ldots, k-1} s_{|i_{\ell}|},$$

where the notation implies that the index $\ell$ in the first (resp. second) product is decreasing (resp. increasing). We also set $u_{\geq k} = e, u_{< k} = v$ for $k = m+1, \ldots, m+r$. For example, if $i = (1, 2, 2, 3, 3, 2)$ then $u_{\geq 4} = s_1 s_3, u_{< 4} = s_1 s_2$.

For every $k = 1, \ldots, m + r$, we set

$$\gamma^k = u_{\geq k} \omega_{|i_k|}, \quad \delta^k = v_{< k} \omega_{|i_k|}$$

and introduce a regular function $M_k$ on $G^{u,v}$ by setting

$$M_k(x) = \Delta_{\gamma^k, \delta^k}(x'), \tag{2.2}$$

where $x'$ is given by (2.3). We refer to the family $M_1, \ldots, M_{m+r}$ as twisted minors associated with a reduced word $i$. Their significance stems from the following result (see [3, Theorems 1.2, 1.9, 1.10 and formula (1.21)]).

**Theorem 2.2.** The map $x_i : H \times \mathbb{C}^m \to G$ given by

$$x_i(a ; t_1, \ldots, t_m) = a x_i(t_1) \cdots x_i(t_m)$$

restricts to a birational isomorphism between a complex torus $H \times (\mathbb{C} - \{0\})^m$ and a Zariski open subset $U_i = \{ x \in G^{u,v} : M_k(x) \neq 0 \text{ for } 1 \leq k \leq m + r \}$ of the double Bruhat cell $G^{u,v}$. Furthermore, for $k = 1, \ldots, m + r$ and $x = x_i(a ; t_1, \ldots, t_m) \in U_i$, we have

$$M_k(x) = a^{-\nu_k} \prod_{1 \leq \ell < k} t_{\ell}^{(\alpha_{i\ell}, u_{\geq \ell} \gamma^\ell) \prod_{1 \leq \ell < k} t_{\ell}^{(\alpha_{i\ell}, v_{< \ell+1} \delta^\ell) \prod_{k \leq \ell \leq m} t_{\ell}^{(\alpha_{i\ell}, u_{\geq \ell} \gamma^\ell) \prod_{k \leq \ell \leq m} t_{\ell}^{(\alpha_{i\ell}, v_{< \ell+1} \delta^\ell)}}. \tag{2.3}$$
2.4. Symplectic leaves of the standard Poisson-Lie structure. We are now ready to describe symplectic leaves of the standard Poisson-Lie structure on \( G \). We retain the above terminology and notation.

Given \( u, v \in W \), let \( H^{u,v} \) denote the subtorus of \( H \) formed by elements \((a^u)^{-1} \cdot a^v\) for \( a \in H \). We also denote by \( I(u, v) \) the set of all indices \( i \) such that \( u \omega_i = v \omega_i = \omega_i \).

**Theorem 2.3.** For every \( u, v \in W \), the set
\[
S^{u,v} = \{ x \in G^{u,v} : [\pi^{-1} x]_0 \cdot ([xv^{-1}]v)_0 \in H^{u,v}, \ [\pi^{-1} x]_0^i = 1 \ \text{for all} \ i \in I(u, v) \}
\]
is a symplectic leaf in \( G \). Every symplectic leaf in \( G \) is of the form \( S^{u,v} \cdot a \) for some \( u, v \in W \) and \( a \in H \).

In the process of the proof of Theorem 2.3, we show that every double reduced word of \((u, v)\) gives rise to a toric chart in \( S^{u,v} \). More precisely, we prove the following.

**Proposition 2.4.** For every double reduced word \( i = (i_1, \ldots, i_m) \) of \((u, v)\), the intersection \( S^{u,v} \cap U_i \) is a dense Zariski open subset of \( S^{u,v} \), and the mapping
\[
x \mapsto (M_1(x), \ldots, M_m(x), [\pi^{-1} x]_0 \cdot ([xv^{-1}]v)_0)
\]
is a biregular isomorphism between \( S^{u,v} \cap U_i \) and the complex torus \((\mathbb{C} - \{0\})^m \times H^{u,v}\). In particular, the symplectic leaf \( S^{u,v} \) has (complex) dimension \( m + \dim H^{u,v} \), where \( m = \ell(u) + \ell(v) \).

Note that the set
\[
(2.4) \quad \tilde{S}^{u,v} = \{ x \in G^{u,v} : [\pi^{-1} x]_0 \cdot ([xv^{-1}]v)_0 \in H^{u,v} \}
\]
has been known to be a union of finite number of symplectic leaves (see [3, 6, 8] and Proposition 2.4 below). The new result in Theorem 2.3 is an explicit description of the connected components of \( \tilde{S}^{u,v} \). As a consequence of Theorem 2.3, we obtain the following corollary.

**Corollary 2.5.** The number of connected components of \( \tilde{S}^{u,v} \) is equal to \( 2^{|I(u, v)|} \).

2.5. Poisson brackets of twisted minors and integrable systems on special symplectic leaves. Let \((\gamma, \gamma')\) denote the \( W \)-invariant scalar product on \( h^* \) such that \((\alpha_i, \gamma) = d_i \langle \alpha_i^\vee, \gamma \rangle\) for all \( i \) and \( \gamma \). Assume we are given \( u, v \in W \) and a double reduced word \( i \) of \((u, v)\), and let \( M_1, \ldots, M_{m+r} \) be twisted minors associated with \( i \). Here is our next main result.

**Theorem 2.6.** On every symplectic leaf in a double Bruhat cell \( G^{u,v} \), the standard Poisson bracket between twisted minors is given by
\[
\{ M_k, M_{k'} \} = ((\gamma^k, \gamma'^{k'}) - (\delta^k, \delta'^{k'})) M_k \cdot M_{k'}
\]
for \( 1 \leq k \leq k' \leq m + r \).

Let \( u \) be an arbitrary element of the Weyl group \( W \). Using Theorem 2.6, we shall construct a family of completely integrable systems on the symplectic leaf \( S^{u,u} \), one for each reduced word \( j = (j_1, \ldots, j_{\ell(u)}) \) of \( u \). To do this, we first associate with \( j \) a double reduced word \( i = (i_1, \ldots, i_{\ell(u)}) \) of \((u, u)\) (where \( m = 2\ell(u) \)) by \( i_{2k-1} = j_k \) and \( i_{2k} = j_k \) for \( 1 \leq k \leq \ell(u) \). Let \( M_1, \ldots, M_{m+r} \) be twisted minors associated with \( i \).
Corollary 2.7. The twisted minors $M_{2k-1}$ for $k = 1, \ldots, \ell(u)$ form a completely integrable system on the symplectic leaf $S^{u,u}$, i.e. they are independent on $S^{u,u}$, Poisson commute with each other, and the cardinality $\ell(u)$ of this family is equal to $\frac{1}{2} \dim S^{u,u}$.

Remark 2.8. M. Gekhtman and M. Shapiro informed us that they have proved (in an ongoing joint work with A. Vainshtein) that the twist isomorphism (2.1) between double cells $G^{u,u}$ and $G^{u^{-1},v^{-1}}$ is an anti-isomorphism of Poisson manifolds (this means that it becomes Poisson if we change the sign of the Poisson bracket on $G^{u^{-1},v^{-1}}$). Thus, Corollary 2.7 also gives rise to a family of integrable systems on symplectic leaves $S^{u,u}$ formed by (non-twisted) generalized minors.

Example 2.9. Let $u = v = w_0$ be the longest element in $W$. The corresponding double Bruhat cell $G^{w_0,w_0}$ is an open set in $G$ given by

$$G^{w_0,w_0} = \{ x \in G : \Delta_{w_0 w_0} (x) \neq 0, \Delta_{w_0 w_0} (x) \neq 0 \text{ for all } i \}.$$ 

Let $i \mapsto i^*$ be an involution on the index set $\{1, \ldots, r\}$ induced by the action of $-w_0$ on fundamental weights: that is, we have $w_0 (\omega_i) = -\omega_{i^*}$. As an easy consequence of Theorem 2.3, the symplectic leaf $S^{w_0,w_0}$ is given by

$$S^{w_0,w_0} = \{ x \in G^{w_0,w_0} : \Delta_{w_0 w_0} (x) = \Delta_{w_0 w_0} (x) \text{ for all } i \}.$$ 

In particular, for $G = SL_n$, we have

$$G^{w_0,w_0} = \{ x \in G : \Delta_{[n+1-i,n],[1,i]} (x) \neq 0, \Delta_{[1,i],[n+1-i,n]} (x) \neq 0 \text{ for all } i \}$$

and

$$S^{w_0,w_0} = \{ x \in G : \Delta_{[n+1-i,n],[1,i]} (x) = \Delta_{[1,i],[n+1-i,n]} (x) \neq 0 \text{ for all } i \}$$

where $\Delta_{I,J}(x)$ is the minor of a matrix $x$ with the row set $I$ and the column set $J$, and $[a,b]$ stands for the set $\{a, a+1, \ldots, b\}$. The dimension of $S^{w_0,w_0}$ is equal to $2\ell(w_0)$; for $SL_n$, this amounts to $n(n-1)$.

To illustrate Corollary 2.7 consider the symplectic leaf $S^{w_0,w_0}$ in $G = SL_3$. Its dimension is 6, and it is given in $G$ by the conditions

$$x_{13} = \Delta_{[1,2],[2,3]} (x) \neq 0, \quad x_{13} = \Delta_{[2,3],[1,2]} (x) \neq 0,$$

where $x_{ij}$ denotes the $(i,j)$-entry of the matrix $x$. Choose the reduced word $j = (1,2,1)$ of $w_0$. Using an explicit description of the twist map $x \mapsto x'$ given in [2, Example 4.6], the restrictions to $S^{w_0,w_0}$ of the corresponding twisted minors $M_1, M_3$ and $M_5$ can be calculated as follows:

$$M_1(x) = \Delta_{3,1} (x') = \frac{1}{x_{13}},$$

$$M_3(x) = \Delta_{[2,3],[1,2]} (x') = \frac{1}{x_{31}},$$

$$M_5(x) = \Delta_{2,2} (x') = \frac{\Delta_{23}(x',[1,3],[1,2]) - x_{13} \Delta_{[2,3],[1,2]} (x)}{x_{13} x_{31}}.$$ 

By Corollary 2.7, these functions form a completely integrable system on $S^{w_0,w_0}$. In view of Remark 2.8 the matrix entries $x_{31}, x_{13},$ and $x_{22}$ also form a completely integrable system on $S^{w_0,w_0}$.
3. PROOFS OF THE RESULTS ON SYMPLECTIC LEAVES

We retain the terminology and notation of Section 2. In particular, $G$ is a simply-connected connected semisimple complex Lie group with the standard Poisson-Lie structure. In this section, we prove Theorem 2.3, Proposition 2.4 and Corollary 2.5. Our starting point is the following description of symplectic leaves in $G$.

**Proposition 3.1.** The symplectic leaves in $G$ are the connected components of the sets $\overline{S^u} \cdot a$ for some $u, v \in W$ and $a \in H$, where $\overline{S^u}$ is given by (3.3).

Proposition 3.1 appeared in \[3, 6, 8\]. We still would like to outline a proof in order to make the presentation more self-contained, and also since this gives us a convenient occasion to introduce some notation needed later.

**Proof.** We deduce Proposition 3.1 from the following description of symplectic leaves which is essentially due to M. Semenov-Tyan-Shanskiı\[7\]. Let us identify $G^*$ with the double cosets of $G \times G$.

Let us identify $G^*$ with the double cosets of $G \times G$ such that $x \in H/H_0$.

For every $x \in H_0$, let $x \mapsto [x^{-1} 1]_0$ be the subgroup of $G_0$ such that $[x^{-1} 1]_0 = [x^{-1} 0]_0 = [x^{-1}]_0$.

We deduce Proposition 3.1 from the following description of symplectic leaves, which is essentially due to M. Semenov-Tyan-Shanskiı\[7\]. Let us identify $G^*$ with the double cosets of $G \times G$. It remains to show that the intersections of $G^*$ and the double cosets of $G^*$ in $G \times G$.

By the definition, two elements $x$ and $y$ of $G$ belong to the same double coset in $G^*/(G \times G)/G^*$ if and only if

$$y = bxb' = x' x'$$

for some $b, b' \in B$ and $b'_-, b'_+ \in B_-$ such that $[b]_0 = [b_-]_0$ and $[b']_0 = [b_+]_0$. In particular, $x$ and $y$ must belong to the same double Bruhat cell $G^u u'$.

To shorten the notation, let us define for $x \in G^u v'$

$$h(x) = [v^{-1} x]_0, \quad h'(x) = ([v^{-1}]_0)^v.$$ 

One easily checks that

$$h(xa) = h(x)a, \quad h'(xa) = h'(x)a, \quad h(ax) = a^x h(x), \quad h'(ax) = a^x h'(x)$$

for every $a \in H$.

Now suppose that $x, y \in G^u v'$ satisfy (3.1). Using (3.3), we obtain

$$h(y) = h([b]_0 \cdot x \cdot [b']_0) = [b]_0 \cdot h(x) \cdot [b']_0$$

and

$$h'(y) = h'([b]_0 \cdot x \cdot [b']_0) = [b]_0 \cdot h'(x) \cdot [b']_0.$$ 

Therefore, $h(y) h'(y)$ and $h(x) h'(x)$ belong to the same coset in $H/H^u v'$.

Conversely, suppose $h(y) h'(y)$ and $h(x) h'(x)$ belong to the same coset in $H/H^u v'$, i.e. we have $a^y h(y) h'(y) = a^x h(x) h'(x)$ for some $a \in H$. Setting

$$a' = a^y h'(y) h'(x)^{-1} = a^x h(x) h(y)^{-1},$$

we obtain

$$(x, x) \in G^* (\overline{\pi h(x)}, \overline{v^{-1}}^{-1} h'(x)) G^* = G^* (\overline{a\pi h(x)}, \overline{v^{-1}}^{-1} h'(x) a') G^*$$

$$= G^* (\overline{\pi h(y)}, \overline{v^{-1}}^{-1} h'(y)) G^* = G^* (y, y) G^*.$$
Thus \(x, y \in G^{u,v}\) satisfy (3.1) if and only if
\[ h(x)h'(x) \text{ and } h(y)h'(y) \text{ belong to the same coset in } H/H^{u,v}. \]
At the same time, (2.3), (3.2) and (3.3) imply that \(x, y \in G^{u,v}\) belong to the same set \(\tilde{S}^{u,v}.a\) if and only if (3.4) holds. This completes the proof of Proposition 3.1. \(\square\)

Recall that \(I(u, v)\) is the set of indices \(i\) for which \(u\omega_i = v\omega_i = \omega_i\). Let \(\tilde{H}^{u,v}\) denote the subtorus of \(H\) given by
\[ \tilde{H}^{u,v} = \{ h \in H : h^{\omega_i} = 1 \text{ for } i \in I(u, v) \}. \]
Clearly, \(H^{u,v}\) is a subtorus of \(\tilde{H}^{u,v}\). In view of (3.2), we have
\[ \tilde{S}^{u,v} = \{ x \in G^{u,v} : h(x)h'(x) \in H^{u,v} \}, \]
(3.5)
\[ S^{u,v} = \{ x \in \tilde{S}^{u,v} : h(x) \in \tilde{H}^{u,v} \}. \]
(3.6)

**Lemma 3.2.** For every \(x \in G^{u,v}\), we have \(h(x)h'(x)^{-1} \in \tilde{H}^{u,v}\).

**Proof.** Let \(i \in I(u, v)\). Then we have
\[ h(x)^{\omega_i} = ([\pi^{-1}x]_0)^{\omega_i} = \Delta_{u\omega_i, \omega_i}(x) = \Delta_{\omega_i, \omega_i}(x), \]
and
\[ h'(x)^{\omega_i} = ([\pi^{-1}x]_0)^{\omega_i} = \Delta_{\omega_i, \omega_i}(x) = \Delta_{\omega_i, \omega_i}(x) = h(x)^{\omega_i}, \]
implying our statement. \(\square\)

Let us denote
\[ \Sigma = \{ a \in H : a^2 = e \} = \{ a \in H : a^{\omega_i} = \pm 1 \text{ for all } i \}, \]
and \(\Sigma^{u,v} = \Sigma \cap \tilde{H}^{u,v}\). Thus, \(\Sigma\) is a finite Abelian group isomorphic to \((\mathbb{Z}/2\mathbb{Z})^r\), and \(\Sigma^{u,v}\) is a subgroup of index \(2|I(u,v)|\) in \(\Sigma\). By (3.3) and (3.5), \(\Sigma\) acts on \(\tilde{S}^{u,v}\) by right translations. Using (3.4), we see that \(\Sigma^{u,v}\) takes \(S^{u,v}\) into itself; furthermore, if \(a\) and \(a'\) belong to different cosets in \(\Sigma/\Sigma^{u,v}\) then \(S^{u,v} \cdot a \cap S^{u,v} \cdot a' = \emptyset\). On the other hand, Lemma 3.2 and (3.3) imply that \(h(x)^2 \in \tilde{H}^{u,v}\) for every \(x \in \tilde{S}^{u,v}\). It follows that every \(\Sigma\)-orbit in \(\tilde{S}^{u,v}\) has non-empty intersection with \(S^{u,v}\). We conclude that \(\tilde{S}^{u,v}\) is the disjoint union of \(2|I(u,v)|\) right translates of \(S^{u,v}\) by a set of representatives of \(\Sigma/\Sigma^{u,v}\). Since \(S^{u,v}\) is closed in \(\tilde{S}^{u,v}\), it is also open, hence is a union of connected components of \(\tilde{S}^{u,v}\).

To complete the proofs of Theorem 2.3 and Corollary 2.5, it remains to show that \(S^{u,v}\) is connected. This in turn is a consequence of Proposition 2.3 since the connectedness property of a complex algebraic variety is preserved by passing to a dense Zariski open subset. So it remains to prove Proposition 2.3.

In what follows, we fix \(u, v \in W\) and a double reduced word \(\bar{1} = (i_1, \ldots, i_m)\) of \((u, v)\). Recall that we append \(r\) entries \(i_{m+1}, \ldots, i_{m+r}\) to \(\bar{1}\) by setting \(i_{m+j} = j\). For any \(j\), let \(k(j)\) denote the smallest index \(k\) with \(|i_k| = j\); thus, \(1 \leq k(j) \leq m\) for \(j \notin I(u, v)\), and \(k(j) = m + j\) for \(j \in I(u, v)\).

**Lemma 3.3.** The variety \(S^{u,v}\) can be identified with the subvariety of pairs \((x, a) \in G^{u,v} \times H^{u,v}\) satisfying
\[ M_{m+j}(x) = \begin{cases} 1 & \text{if } j \in I(u, v); \\ a^{-\omega_j} \prod_{i \notin I(u,v)} M_{k(i)}(x)^{-\langle \alpha_i^\vee, \omega_j \rangle} & \text{if } j \notin I(u, v). \end{cases} \]
(3.7)
Proof. By (3.3) and (3.6), \( S^{u,v} \) can be identified with the subvariety of pairs \((x, a) \in G^{u,v} \times H^{u,v}\) such that

\[
h(x) \in \tilde{H}^{u,v}, \quad h(x)h'(x) = a,
\]
or, equivalently,

\[
h(x)^{\omega_j} = 1 \quad (j \in I(u,v)), \quad h(x)^{\omega_j}h'(x)^{\omega_j} = a^{\omega_j} \quad (j = 1, \ldots, r).
\]

It remains to show the equivalence of (3.8) and (3.7).

We claim that

\[
h(x)^{\omega_j} = M_{m+j}(x)^{-1}, \quad h'(x)^{v^{-1}\omega_j} = M_{k(j)}(x)^{-1}
\]

for any \( x \in G^{u,v} \) and \( j = 1, \ldots, r \). Indeed, the definition (2.1) of the twist isomorphism \( x \mapsto x' \) between \( G^{u,v} \) and \( G^{u,v-1} \) implies at once that

\[
[x'\varpi]_0 = (w^{-1}x|0)^{-1}, \quad [w^{-1}^{-1}x'|0] = (x\varpi^{-1}|0)^{-1}.
\]

The equality (3.9) is then a direct consequence of definitions (3.2) and (2.2).

Since every fundamental weight \( \omega_j \) can be written as \( \sum_i (\alpha_i^+, v\omega_j) \cdot v^{-1}\omega_i \), the equality (3.3) implies that

\[
h'(x)^{\omega_j} = \prod_i M_{k(i)}(x)^{-(\alpha_i^+, v\omega_j)}.
\]

Substituting this and the first equality in (3.4) into (3.8), we conclude (after a slight simplification) that it is indeed equivalent to (3.7).

Lemma 3.3 together with Theorem 2.3 immediately prove that the map \( x \mapsto (M_1(x), \ldots, M_m(x), h(x)h'(x)) \) provides an isomorphism between \( S^{u,v} \cap U_1 \) and \((\mathbb{C} - \{0\})^m \times H^{u,v} \). To prove Proposition 2.4, it remains to show that \( S^{u,v} \cap U_1 \) is dense in \( S^{u,v} \). Since \( S^{u,v} \) is smooth, it suffices to show that the complement of this set is of (complex) codimension at least one. In other words, we claim that \( S^{u,v} \cap \{M_k = 0\} \) is of codimension at least one in \( S^{u,v} \) for any \( k = 1, \ldots, m+r \). But this is clear from the defining equations (3.7): the twisted minors \( M_{m+j} \) and \( M_{k(i)} \) vanish nowhere on \( S^{u,v} \), while the remaining \( M_k \) just do not appear in (3.7). Proposition 2.4, Theorem 2.3 and Corollary 2.3 are proved.

4. PROOFS OF THEOREM 2.0 AND COROLLARY 2.7

In this section, we again fix two elements \( u, v \in W \) and a double reduced word \( i = (i_1, \ldots, i_m) \) of \((u, v)\). By Theorem 2.3, a generic element \( x \in G^{u,v} \) has a unique factorization \( x = x_1(a_1; t_1, \ldots, t_m) \), so each \( t_k \) as well as \( a^\gamma \) for any weight \( \gamma \), can be viewed as a rational function on \( G^{u,v} \). The Poisson brackets between these functions are given as follows.

**Lemma 4.1.** We have

\[
\{t_k, t_{k'}\} = \varepsilon(i_k)(\alpha_{i_k}, \alpha_{i_{k'}})t_k t_{k'}, \quad \text{for } k < k',
\]

\[
\{t_k, a^{\gamma}\} = (\alpha_{i_k}, \gamma) t_k a^{\gamma}, \quad \text{for any weight } \gamma
\]

\[
\{a^{\gamma}, a^{\gamma'}\} = 0, \quad \text{for any weights } \gamma, \gamma'.
\]

A slight modification of this lemma can be found in [1]. To make the exposition more self-contained, we outline the proof.
Proof. Consider the Poisson-Lie group $SL_2^{(d)}$ (see Example 2.1), and two symplectic leaves in it given by

$$S_+ = \{ \left( \begin{array}{cc} p & q \\ 0 & p^{-1} \end{array} \right) : p, q \neq 0 \}, \quad S_- = \{ \left( \begin{array}{cc} p & 0 \\ q & p^{-1} \end{array} \right) : p, q \neq 0 \};$$

in the notation of Theorem 2.5, we have $S_+ = S^{v,w_0}$ and $S_- = S^{w_0,v}$. In both these leaves, the Poisson structure is given by $\{ q, p \} = dpq$; to indicate the dependence on $d$, we will write $S_{d+} = S_d^{(d)}$.

By the definition of the standard Poisson structure on $G$, a double reduced word $i$ of $(u, v)$ gives rise to a Poisson map

$$\varphi_i : S_{\text{sign}(c(i))}^{(d)} \times \cdots \times S_{\text{sign}(c(i_m))}^{(d)} \to G^{u,v}$$

given by

$$\varphi_i(g_1, \ldots, g_m) = \varphi_{|i_1|}(g_1) \cdots \varphi_{|i_m|}(g_m).$$

We use the standard coordinates $(p_k, q_k)$ in each factor $S_{\text{sign}(c(i_k))}^{(d)}$. An easy calculation using commutation relations [2, (2.5)] shows that

$$\varphi_i^*(\alpha^\gamma) = \prod_{\ell=1}^m \rho_{\ell}^{(\alpha_{\gamma_i}^\ell, \gamma)}, \quad \varphi_i^*(t_k) = q_k p_k^{-\epsilon(i_k)} \prod_{\ell=k+1}^m p_{\ell}^{-\epsilon(i_{\gamma_i}^\ell, i_{\gamma_i}^{\ell+1})}.$$ 

On the other hand, the only nonzero Poisson brackets between the coordinates $p_1, q_1, \ldots, p_m, q_m$ are $\{ q_k, p_k \} = d_{i_k} p_k q_k$. Recalling that $(\alpha_i, \gamma) = d_i (\alpha_i^\gamma, \gamma)$, we obtain our statement.

We will say that two functions $f, g$ on a Poisson manifold quasi-commute if $\{ f, g \} = cfg$ for some constant $c$. In this situation, we denote $c = \{ f, g \}$.

Lemma 4.4 asserts in particular that all functions $t_k$ and $a^\gamma$ on $G^{u,v}$ quasi-commute with each other. Clearly, any two monomials in quasi-commuting variables quasi-commute, and the pairing $\{ f, g \}$ is skew-symmetric and bilinear in the following sense: $\langle f_1 f_2, g \rangle = \langle f_1, g \rangle + \langle f_2, g \rangle$. Taking into account (2.3), we conclude that all twisted minors $M_k$ quasi-commute with each other. To prove Theorem 2.6, we only need to show that

$$\langle M_k, M_{k'} \rangle = (\gamma^k, \gamma^{k'}) - (\delta^k, \delta^{k'}) \text{ for } k < k'. $$

We set

$$\gamma^k_\ell = u_{\ell-1}^{-1} \gamma^k_\ell, \quad \delta^k_\ell = v_{\ell+1}^{-1} \delta^k_\ell;$$

in particular, we have

$$\gamma^k_1 = u^{\gamma^k}, \quad \gamma^k_{|i_k|}, \quad \delta^k_{|i_k|} = \omega_{|i_k|}, \quad \delta^k_m = v^{-1} \delta^k. $$

An easy inspection shows that we can rewrite (2.3) as

$$M_k = a^{u^{\gamma^k}} \prod_{\ell=1}^{k-1} c^k_\ell \prod_{\ell=k}^m d^k_\ell, $$

where the exponents $c^k_\ell$ and $d^k_\ell$ are determined from

$$c^k_\ell \alpha_{|i_\ell|} = \gamma^k_\ell - \gamma^k_{\ell+1}, \quad d^k_\ell \alpha_{|i_\ell|} = \delta^k_\ell - \delta^k_{\ell+1}.$$

To simplify calculations, we perform the following monomial change of variables: replace each $t_k$ by

$$y_k = \begin{cases} t_k & \text{if } \varepsilon(i_k) = +1; \\ a^{-\alpha(i_k)}t_k & \text{if } \varepsilon(i_k) = -1. \end{cases}$$

(4.7)

As an immediate consequence of Lemma 4.1, we get

(4.8) $\langle y_k, y_{k'} \rangle = \varepsilon(i_k)(\alpha_{|i_k|}, \alpha_{|i_{k'}|})$ for $k < k'$ and $\varepsilon(i_k) = \varepsilon(i_{k'})$,

(4.9) $\langle y_k, y_{k'} \rangle = 0$ for $\varepsilon(i_k) \neq \varepsilon(i_{k'})$,

(4.10) $\langle y_k, a\gamma \rangle = (\alpha_{|i_k|}, \gamma)$ for any weight $\gamma$,

(4.11) $\langle a\gamma, a\gamma' \rangle = 0$ for any weights $\gamma, \gamma'$.

Rewriting $M_k$ in these new variables, we get

$$M_k = a^{-\omega_{|i_k|}} \prod_{\ell=1}^{k-1} y_{\ell}^{c_{k\ell}} \prod_{\ell=k}^{m} y_{\ell}^{d_{k\ell}};$$

the only thing to check here is that the weight $(-w\gamma^k)$ in (4.3) transforms into

$$-w\gamma^k + \sum_{\ell=1}^{k-1} c_{k\ell}\alpha_{|i_{\ell}|} = -w\gamma^k + \sum_{\ell=1}^{k-1} (\gamma^k_{\ell} - \gamma^k_{\ell+1}) = -w\gamma^k + \gamma_1^k - \gamma_k^k = -\omega_{|i_k|}.$$

Now everything is ready for the proof of (4.4). We have

$$\langle M_k, M_{k'} \rangle = \langle a^{-\omega_{|i_k|}}, M_{k'} \rangle + \langle M_k, a^{-\omega_{|i_{k'}|}} \rangle + \sum_{\ell=1}^{k-1} y_{\ell}^{c_{k\ell}} \prod_{\ell'=1}^{k'-1} y_{\ell'}^{d_{k'\ell'}} + \sum_{\ell=1}^{k'-1} y_{\ell}^{c_{k'\ell}} \prod_{\ell'=1}^{k-1} y_{\ell'}^{d_{k\ell'}} + \sum_{\ell=1}^{m} y_{\ell}^{c_{k\ell}} \prod_{\ell'=k}^{m} y_{\ell'}^{d_{k'\ell'}} + \sum_{\ell=1}^{m} y_{\ell}^{c_{k'\ell}} \prod_{\ell'=k'}^{m} y_{\ell'}^{d_{k\ell'}}$$

The last two terms vanish by (4.9). Let us calculate the remaining terms. First of all, we have

$$\langle a^{-\omega_{|i_k|}}, M_{k'} \rangle = \langle a^{-\omega_{|i_k|}} \prod_{\ell=1}^{k'-1} y_{\ell}^{c_{\ell'}} \prod_{\ell=k'}^{m} y_{\ell}^{d_{\ell'}} \rangle = (\omega_{|i_k|}, \sum_{\ell=1}^{k'-1} c_{\ell'}^{\alpha_{|i_{\ell}|}} + \sum_{\ell=k'}^{m} d_{\ell'}^{\alpha_{|i_{\ell}|}})$$

$$= (\omega_{|i_k|}, \sum_{\ell=1}^{k'-1} (\gamma^k_{\ell} - \gamma^k_{\ell+1}) + \sum_{\ell=k'}^{m} (\delta^k_{\ell} - \delta^k_{\ell-1}))$$

$$= (\omega_{|i_k|}, \gamma_1^k - \gamma_k^k + \delta^k_m - \delta^k_{k'-1}) = (\omega_{|i_k|}, u\gamma^k + v^{-1}\delta^k - 2\omega_{|i_{k'}|}).$$

Similarly,

$$\langle M_k, a^{-\omega_{|i_{k'}|}} \rangle = -(u\gamma^k + v^{-1}\delta^k - 2\omega_{|i_{k'}|}, \omega_{|i_{k'}|}).$$
The third term can be calculated as follows:

\[
\left\langle \prod_{\ell=1}^{k-1} y_{\ell}^k, \prod_{\ell'=1}^{k'-1} y_{\ell'}^{k'} \right\rangle = \sum_{\ell=1}^{k-1} \left( \gamma_{\ell}^k - \gamma_{\ell+1}^k, \sum_{\ell'=1}^{k'-1} \left( \gamma_{\ell'}^{k'} - \gamma_{\ell'+1}^{k'} \right) \right) - \sum_{\ell' = \ell+1}^{k'-1} \gamma_{\ell'}^{k'}
\]

\[
= \sum_{\ell=1}^{k-1} \left( \gamma_{\ell}^k - \gamma_{\ell+1}^k, \gamma_{\ell}^{k'} - \gamma_{\ell+1}^{k'} \right)
\]

\[
= \sum_{\ell=1}^{k-1} \left( \gamma_{\ell}^k - \gamma_{\ell+1}^k, \gamma_{\ell}^{k'} - \gamma_{\ell+1}^{k'} \right)
\]

\[
= \sum_{\ell=1}^{k-1} \left( \gamma_{\ell}^k - \gamma_{\ell+1}^k, \gamma_{\ell}^{k'} - \gamma_{\ell+1}^{k'} \right)
\]

\[
= \sum_{\ell=1}^{k-1} \left( \gamma_{\ell}^k - \gamma_{\ell+1}^k, u \gamma^{k'} + \omega_{i_{\ell k'}} - \gamma_{\ell+1}^{k'} \right)
\]

\[
= \left( \gamma_{\ell}^k - \gamma_{\ell+1}^k, u \gamma^{k'} + \omega_{i_{\ell k'}} \right)
\]

\[
= \left( \gamma_{\ell}^k - \gamma_{\ell+1}^k, u \gamma^{k'} + \omega_{i_{\ell k'}} \right)
\]

Here we used the fact that, for \( \ell < k \) and \( \varepsilon(i_{\ell}) = -1 \), we have \( \left( \gamma_{\ell}^k, \gamma_{\ell+1}^{k'} \right) = (s_{i_{\ell k}}) \gamma_{\ell}^k \gamma_{\ell+1}^{k'} \).

The last remaining term is calculated in the same way. We leave the details to the reader and only give the answer:

\[
\left\langle M_k, M_{k'} \right\rangle = (\omega_{i_{\ell k'}}, \gamma_{\ell}^k) - (\delta_{k-1}^k, \omega_{i_{\ell k'}}) .
\]

Since \( (\omega_{i_{\ell k'}}, \gamma_{\ell}^k) = (\gamma_{\ell}^k, \gamma_{\ell}^k) \) and \( (\delta_{k-1}^k, \omega_{i_{\ell k'}}) = (\delta_{k}^k, \delta_{k}^k) \), this completes the proof of Theorem 2.6. \( \square \)

Proof of Corollary 2.7: Since the torus \( H^{u,u} \) is trivial, Proposition 2.4 implies that the twisted minors \( M_1, \ldots, M_m \) form a system of local coordinates on \( S^{u,u} \); in particular, the functions \( M_{2k-1} \) are algebraically independent on \( S^{u,u} \). Moreover, there are exactly \( \frac{1}{2} \dim S^{u,u} \) of them.

It remains to show that the functions \( M_{2k-1} \) pairwise Poisson commute on \( S^{u,u} \). This follows at once from Theorem 2.6 and the following computation for \( k < k' \):

\[
(\gamma^{2k-1}, \gamma^{2k'-1}) = \left( s_{j_{k-1}}, \cdots, s_{j_{k}}, \omega_{j_{k}}, s_{j_{k-1}}, \cdots, s_{j_{k'}} \right) \left( \omega_{j_{k'}}, \omega_{j_{k'}} \right)
\]

\[
= \left( s_{j_{k-1}}, \cdots, s_{j_{k}}, \omega_{j_{k}}, \omega_{j_{k}} \right). \omega_{j_{k'}}\right)
\]

\[
= \left( s_{j_{k}} \cdots s_{j_{k-1}}, \omega_{j_{k}} \right. \omega_{j_{k}} \left. \omega_{j_{k-1}} \right)
\]

\[
= \left( \delta_{k-1}^{2k-1}, \delta_{k}^{2k'-1} \right).
\]

Corollary 2.7 is proved. \( \square \)

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