Study of the splash: A theoretical perspective

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Impacts of drops on liquids are ubiquitous in nature and in a range of applications in healthcare, agriculture and industry. They can lead to splash and generation of secondary droplets important for a range of coating, dispersal or contaminant problems. The physics of splash, despite being more than a century old problem, still has a number of unanswered questions. We study the sequence of events occurring upon drop impact on a deep, inviscid liquid pool. We particularly focus on the air cavity formed below the liquid surface and the liquid sheet of the crown, which forms and rises above the surface. Using combined momentum and energy analyses, we derive the prediction of the temporal evolution of the air cavity and the maximum depth it reaches. We derive and an expression for the sheet velocity profile of the crown. We also derive the expression for the unsteady sheet thickness profile of the crown. Leveraging the knowledge we gained on the unsteady crown sheet velocity and thickness profiles, we derive the analytical expressions for the temporal evolution of the crown diameter and crown height. We also show how the crown and cavity radial expansions are coupled. The cavity and crown analysis employed in the present study provides a new understanding of the traditionally know cavity-crown system started in the seminal studies by Worthington in the early 20th century [1, 2, 3, 4].

1 Background and open questions

Impacts of water drops on liquid layers are ubiquitous in nature, from rain, pesticide drops impacting puddles, to raindrops impacting waste water reservoirs or ocean shores. With sufficient energy, such impacts can lead to a splash, producing secondary droplets dispersing the contaminants or chemicals in the sessile bath [5, 6, 7]. In industrial applications, the control of spray, thermal coating, and more, all require understanding of splash [8, 9, 10].

Upon impact on a pool, a liquid drop of diameter \( d_0 \) and velocity \( u_0 \) can either merge with the bath, splash, or create a bubble, and jets depending on its Weber number, \( \text{We} = \frac{\rho u_0^2 d_0}{\sigma} \), with \( \rho \) the density and \( \sigma \) the surface tension of the liquid. The Weber number quantifies the relative importance of the kinetic energy of the impacting drop to its surface energy. When a splash on a deep pool occurs, myriads of secondary droplets are ejected. First, an air cavity is formed below the liquid surface (figure 1) and a cylindrical liquid sheet resembling a crown rises above the surface (figure 1). As the crown rises, azimuthal instabilities emerge, leading to the formation of ligaments themselves breaking into secondary droplets (figure 1). Upon reaching maximum height, the crown begins to collapse back into the pool. Eventually the cavity collapses also. A high pressure stagnation point can be created in the underlying cavity, forming an upward rising jet. The seemingly simple droplet impact on a liquid pool thus shows a rich behaviour. Since the seminal work of Worthington [1, 2, 3, 4] and Edgerton [11], the rich physics governing liquid-on-liquid impacts and the resulting splash have been subjected to immense interest. Several studies focused on drop impacts on thin liquid layers in which the layer thickness, \( h \), is smaller than the impacting drop diameter, \( d_0 \), i.e. \( h \ll d_0 \) [5, 12, 13, 14, 15, 16]. Relatively less attention has been paid to impacts on deep liquid layers however [17, 18, 19, 20]. We next discuss the remaining open questions involved.
Figure 1: (a) Crown formed when a milk drop impacts a colored thin film [11]. (b) Schematic of the cavity-crown system formed upon drop impact on the liquid pool, which we consider in the present study.

1.1 Cavity

The maximum depth of the cavity, \( r_m \), created when a drop impacts a liquid layer made of the same fluid was examined by [17]. The author provided the first quantification of \( r_m \) for impacts of water droplets on water pools by performing an energy balance analysis on the drop and cavity systems. [21] assumed that all the drop energy is converted to cavity potential energy, and derived an expression for \( r_m \) in terms of the drop diameter \( d_0 \) and its impacting velocity \( u_0 \) as \( r_m = \left( \frac{g}{3} \right)^{1/4} d_0^{3/4} u_0^{1/2} \). [22] modified the expression of \( r_m \) obtained by [21] by introducing a prefactor based on fit with experimental data. [23] and [24] performed numerical simulations to track the cavity depth with time. Despite all this prior work on the maximum cavity depth, a theoretical model predicting the entire cavity growth temporal evolution, not just its maximum size is still lacking.

1.2 Crown

[25] attributed the crown formation to the discontinuity in the velocity distribution at the interface using a quasi-one-dimensional approximation theory and derived an expression for the temporal evolution of the crown diameter formed from drop impacts on thin, inviscid layers. [13] extended the discontinuity theory of [24] to predict the crown motion and shape in two dimensions. [26] also analyzed the crown diameter temporal evolution for impacts on thin, inviscid fluid layers and found their results to be in agreement with those of [25]. These results were verified numerically by [27]. Several studies aimed to understand the crown rim instability [28, 29, 30, 31, 32], with a particular focus on impacts on thin, inviscid layers.

1.3 Coupled cavity-crown system

Despite the attention paid to impacts on thin films discussed above, analytical expressions for the time evolution of the crown diameter and height for impacts on deep pools are still missing. Moreover, a focus on the complete sequence of events from cavity to crown to secondary droplets is also still missing. For example, the crown sheet velocity and thickness profiles, which are critical to the capture of the crown evolution in time and space are still unknown. Moreover, the link between the cavity and crown dynamics also remains unknown. Yet, a deeper understanding of the splash system has numerous applications such as understanding environmental processes such as raindrops impacting oil spills on the surface of the ocean [33, 34, 35] as well as in healthcare, agriculture and waste-water treatment plants where pathogen or contaminant-bearing pools result
in air contamination via the ejection of contaminated droplets and aerosols [36, 37, 38, 39, 40, 41, 42].

In this paper, we aim to elucidate the complete picture of the coupled cavity-crown system, in a theoretical manner. Throughout this study, we focus particularly on elucidating and capturing the cavity and crown dynamics, till the time of their maximum depth and height, respectively. We also restrict our attention to impacts with Weber numbers high enough to enable crown splash ($W_e > 500$), yet low enough to prevent canopy closure and eventual bubble formation. For the highest Weber numbers, canopy closure and bubble formation produce distinct signatures of drops discussed already in prior work [43].

1.4 Outline

In this paper, we aim to theoretically elucidate the full dynamics of splash from impact of a drop on a deep pool, focusing on the regime that produces secondary droplets from crown rim destabilization. To do so:

1. We derive an expression for the cavity growth temporal evolution (Section 2);
2. We establish the nature of the coupling between the cavity and crown radial expansions (Section 3.1) and derived an expression for the crown diameter temporal evolution;
3. We derive the crown sheet velocity and thickness spatio-temporal profiles necessary to elucidate the crown height evolution (Section 3.4 - 3.5) and
4. We derive the temporal evolution of the crown height (Section 3.6 - 3.7).

Table 1 lists all symbols used in the present study and their meaning.

2 Cavity

In this section, we aim to elucidate the dependence of the cavity depth on $W_e$, and derive an expression for its temporal evolution. Throughout our analysis, we consider the cavity to be hemispherical of radius $r_c$, with center at the drop impact point, positioned at the original, undisturbed water surface level (figure 2a).

2.1 Cavity depth temporal analysis

When the drop impacts the fluid, a fraction $\alpha$ of the total energy of the drop, $E$, is transferred to the cavity energy, $U$. The drop energy, $E$, consists of the drop kinetic energy, $E_k$, and the drop surface energy, $E_\sigma$, with

\[
E = E_k + E_\sigma = \frac{1}{12} \rho \pi d_0^3 u_0^2 + \pi \sigma d_0^2.
\]  

(1)

The cavity energy consists of four contributions, the kinetic energy, $U_k$, the potential energy, $U_p$, the surface energy, $U_\sigma$, and the viscous dissipation energy, $U_\mu$. Hence, the total energy of the cavity is

\[
U = U_p + U_\sigma + U_\mu + U_k.
\]  

(2)

$U_p, U_\sigma, U_\mu$ and $U_k$ can be obtained in a similar manner as in [17] and [18], where $U_p$ is the work required to lift the liquid displaced by the cavity to the original undisturbed surface of the pool, $U_\sigma$ is the surface energy of the hemispherical cavity, $U_\mu$ is the total energy dissipated for an irrotational flow with a fluid of constant density and $U_k$ is the kinetic energy of the liquid surrounding the cavity. As shown in Appendix A, assuming an axisymmetric hemispherical cavity geometry, with center at the drop impact point at the original, undisturbed water surface level, $U_p, U_\sigma, U_\mu$ and $U_k$, are obtained from Equations 57-73 (Appendix A) and read

\[
U_p = \frac{1}{4} \pi g \rho r_c^4,
\]  

(3)
| Symbol | Definition |
|--------|------------|
| R      | Radial coordinate |
| Z      | Vertical coordinate |
| $u_0$  | Impacting drop velocity |
| $d_0$  | Impacting drop diameter |
| $\Omega_0 = \pi d_0^3/6$ | Impacting drop volume |
| $\rho$ | Impacting drop density |
| $\sigma$ | Impacting drop surface tension |
| $\mu$  | Impacting drop viscosity |
| $\tau_{im} = d_0/u_0$ | Impact timescale |
| $\tau_{cap} = \sqrt{\rho \Omega_0/\pi \sigma}$ | Capillary timescale |
| $\alpha$ | Fraction of the impacting drop energy transferred to the cavity |
| $\alpha_c$ | Fraction of the impacting drop energy transferred to the crown |
| We = $\rho u_0^2 d_0/\sigma$ | Impacting drop Weber number |
| Fr = $u_0^2/\rho g d_0$ | Impacting drop Froude number |
| Re = $\rho u_0 d_0/\mu$ | Impacting drop Reynolds number |
| Bo = $\rho g d_0^2/\sigma$ | Impacting drop Bond number |
| Oh = $\mu/\sqrt{\rho \sigma}$ | Impacting drop Ohnesorge number |
| $E$ | Impacting drop total energy |
| $E_k$ | Impacting drop kinetic energy |
| $E_\sigma$ | Impacting drop surface energy |
| $U_p$ | Cavity potential energy |
| $U_\sigma$ | Cavity surface energy |
| $U_\mu$ | Cavity bulk dissipation energy |
| $U_k$ | Cavity kinetic energy |
| $C_p$ | Crown potential energy |
| $C_\sigma$ | Crown surface energy |
| $R_c = r_c/d_0$ | Non-dimensional cavity depth |
| $R_c^m$ | Maximum non-dimensional cavity depth |
| $T = t/\tau_{cap}$ | Non-dimensional time, using the capillary timescale |
| $T_i = t/\tau_{im}$ | Non-dimensional time, using the impact timescale, $d_0/u_0$ |
| $T_c = t_c/\tau_{cap}$ | Non-dimensional time taken by cavity to reach its maximum depth |
| $T_m = t_m/\tau_{cap}$ | Non-dimensional time taken by the crown to reach its maximum height |
| $L = l/d_0$ | Non-dimensional crown diameter |
| $S = s/d_0$ | Non-dimensional crown height |
| $S_m = s_m/d_0$ | Non-dimensional maximum crown height |
| $H = h/d_0$ | Non-dimensional crown sheet thickness profile, function of $T$ and $Z$ |
| $B = b/d_0$ | Non-dimensional crown rim thickness |
| $V_S = v_s/u_0$ | Non-dimensional crown sheet velocity profile, function of $T$ and $Z$ |

Table 1: Symbols used in the present study.
Re since the fluid is inviscid and energy conservation then reads the vertical coordinate and $r_c$ is the cavity radius.

Thus, $U_c = 2\sigma \pi r_c^2$, 

$$U_\mu = 8\pi \mu \int_0^1 r_c r_c^2 dt,$$  

$U_k = \frac{\pi}{3} \rho r_c^3 r_c^2$.  

Recalling that a fraction $\alpha$ of impact energy is converted into cavity energy with $\alpha E = U$, energy conservation then reads

$$\alpha \left( \frac{1}{12} \rho \pi d_0^3 u_0^2 + \sigma \pi d_0^2 \right) = \frac{1}{4} \pi \rho d_0^4 + 2\pi \sigma d_0^2 + 8\pi \mu \int_0^1 r_c r_c^2 dt + \frac{\pi}{3} \rho r_c^3 r_c^2.$$  

Non-dimensionalizing Equation 7 using the variables introduced in table 1 leads to

$$\alpha \left( \frac{\text{We}}{12} + 1 \right) = \frac{\text{We}}{4\text{Fr}} R_c^4 + 2 R_c^2 + 8 \sqrt{\frac{6\text{We}}{\text{Re}}} \int_0^T R_c \dot{R}_c^2 dT + 2 R_c^3 \dot{R}_c^2.$$  

The ratio of $U_\mu$ and $U_p$ is the ratio of the third and first terms on the right-hand side of Equation 8 and at maximum depth reads

$$\frac{U_\mu}{U_p} = \left[ \frac{32\sqrt{6\text{Fr}}}{\sqrt{\text{WeRe}(R_c^m)^4}} \right] \int_0^{T_m} R_c \dot{R}_c^2 dT.$$  

The integrand in Equation 9, $R_c \dot{R}_c^2 = 0$ for $T = 0$ and $T = T_m$. $R_c \dot{R}_c^2$ is also positive and bounded above by $R_c \dot{R}_c^2 = M$. $\dot{R}_c$ is positive and monotonically increases from 0 to $R_c^m$. $\dot{R}_c$ is positive and monotonically decreases from $U_i$ to 0, where $U_i$ is the initial cavity radial velocity, immediately after formation of the cavity. Thus, $M < R_c^m U_i^2$. Consequently, Equation 9 leads to

$$\frac{U_\mu}{U_p} < \frac{32\text{Fr} R_c^m U_i^2}{\sqrt{6\text{WeRe}(R_c^m)^4}}.$$  

Since the fluid is inviscid and $\text{Re}$ is high, we expect $U_\mu/U_p$ obtained from Equation 10 to be small throughout the entire duration till the cavity reaches maximum depth. Hence, we neglect $U_\mu$ compared to the cavity potential energy, $U_p$, for the remainder of the section. Consequently, Equation 8 simplifies to

$$\alpha \left( \frac{\text{We}}{12} + 1 \right) = \frac{\text{We}}{4\text{Fr}} R_c^4 + 2 R_c^2 + 2 R_c^3 \dot{R}_c^2.$$  

Thus,

$$\dot{R}_c = \sqrt{\frac{\alpha \left( \frac{\text{We}}{12} + 1 \right) - \frac{\text{We}}{4\text{Fr}} R_c^4 - 2 R_c^2}{2 R_c^3}}.$$  

5
Figure 3: (a) Comparison of the cavity depth temporal evolution obtained from the numerical solution (Equation 12) and the analytical approximation (Equation 14) (b) Comparison of the numerical and analytical approximation, when the cavity depth is scaled by \( \text{We}^{1/5} \).

Equation (12) represents the differential equation governing the cavity depth temporal evolution, which can be numerically solved to obtain the exact solution which governs the cavity evolution. We can obtain an exact analytical expression for the maximum cavity depth \( R_c = 0 \) from Equation (12) as

\[
R_c^m = \sqrt{\frac{2}{\text{Bo}}} \sqrt{4 + \alpha_k \text{Bo} \left( \frac{\text{We}}{12} \right) - 2}
\]  

(13)

where \( \text{Bo} = \text{We}/\text{Fr} \) is the droplet Bond number.

2.1.1 Analytical solution

At early time the impulse of the impact is converted into creation of a shock in the form of a cavity interface, most of the energy at that time is kinetic, leading to an extension of the cavity wall, a motion that is increasingly resisted by the surface and gravitational forces. Here, we do not model these complex forces, but consider the energy contributions associated with these physical constraints: \( \alpha_p = \frac{U_p}{E}, \alpha_s = \frac{U_s}{E} \) and \( \alpha_k = \frac{U_k}{E} \). The cavity kinetic energy \( \alpha_k \) dominates in the initial time following drop impact. However, as the cavity grows, \( \alpha_p, \alpha_s \) increase and become dominant at later time closer to the maximum cavity extension. To simplify Equation (12), we assume that \( \alpha_p + \alpha_s = \alpha_{p,s} E \) where \( \alpha_{p,s} \) is a constant.

Thus, Equation (12) simplifies to

\[
\frac{R_c}{\text{We}^{1/5}} = \left( \frac{5}{4} \sqrt{\frac{\alpha_k}{6}} \right)^{2/5} T^{2/5}.
\]

where \( \alpha_k = \alpha - \alpha_{p,s} \) is the drop energy imparted to the cavity kinetic energy, averaged over the time from the initial impact of the drop when the cavity is formed, till the cavity reaches maximum depth.

In order to make comparisons between the numerical solution Equation (12) and the analytical solution Equation (14), we have to approximate the fraction of drop energy converted to the cavity energy \( \alpha \), and also the fraction converted to the cavity kinetic energy \( \alpha_k \). Unlike [44] who assume that 100% of the drop energy goes into the cavity; we assume that 50% of the drop energy goes into the cavity \( (\alpha \approx 0.5) \) and the remaining 50% is imparted to the crown. Further of the energy imparted to the cavity, we assume that on average, the potential and surface energy account for half of the cavity total energy \( (\alpha_{p,s} \approx 0.25) \) and the cavity kinetic energy accounts for the other
half ($\alpha_k \approx 0.25$). We account for an error of $\approx 3\%$ in all fractions. We further validate this choice of $\alpha_k$ with an experimental study performed by [22] at the end of this section. From the experiments done by [17, 45], we note that the cavity reaches maximum depth at approximately one capillary timescale $T_{mc}^m \approx 1$. To validate our numerical and analytical solutions, we hence show plots till $T \sim O(1)$. Figure 3a shows that the numerical solution (Equation 12) and the analytical approximation (Equation 14) show a good agreement for the entire duration till the cavity reaches maximum depth, for a wide range of impact Weber numbers. Figure 3b shows that when scaled with $We^{1/5}$, the numerical solution for all Weber numbers collapses on to the analytical solution; indicating the validity of the scaling law $R_c \sim We^{1/5}$.

Note that a similar power law to Equation 14 was proposed by [46], but for collapsing spherical cavities instead of the expanding ones considered herein. Later, [22] proposed a similar scaling of the cavity depth evolution with time, but did not pay attention to the cavity surface and potential energy contributions.

When the cavity reaches its maximum depth, Equation 14 leads to

$$R_c^m = \left( \frac{5}{4} \sqrt{\frac{\alpha_k}{6}} \right)^{2/5} (T_{mc}^m)^{2/5} We^{1/5}. \quad (15)$$

where $T_{mc}^m$ is the time taken by the cavity to reach its maximum depth.

To further validate our choice of $\alpha_k \approx 0.25 \pm 0.03$, we can compare the magnitude of the prefactor in Equation 15 with an experimental study done by [22]. Assuming $\alpha_k \approx 0.25, T_{mc}^m \approx 1$ leads to

$$R_c^m / We^{1/5} \approx 0.58. \quad (16)$$

This is very close to what [22] obtained (note that if we consider droplet diameter to be $\approx 4\text{mm}$, $We/Fr \approx Bo \approx 2$ for water-on-water impacts. This conversion can be used to convert $Fr$ to $We$ in the formula mentioned in [22])

$$R_c^m / We^{1/4} \approx 0.63. \quad (17)$$

The reasonable agreement between Equation 16 and Equation 17 further justifies our choice of the energy partition $\alpha_k \approx 0.25 \pm 0.03$.

Next, we discuss the dynamics of the liquid sheet forming a crown, above the surface of the pool (figure 1).

### 3 Crown

To understand the growth of the crown (figure 1) resulting from drop impact, it is necessary to understand what governs the temporal evolution of the crown sheet height, $s(t)$, and its diameter, $l(t)$. We assume the expanding crown to be a thin, axisymmetric and cylindrical sheet as shown in Figure 1. We denote the radial coordinate $r$ and the vertical coordinate, $z$, both defined with origin at the point of impact of the drop on the undisturbed pool interface The thin film assumption allows us to neglect the variation of the sheet velocity field, $v_s(z, t)$, on film thickness, i.e., it is independent of $r$. The top of the crown sheet, at $z = s$, is bounded by a toroidal rim of thickness $b(t)$ and mass, $\rho \pi b^2 / 4$. As the crown sheet continues to transfer fluid to the rim, with velocity $v_s(z = s, t)$, ligaments grow on the rim periphery, subsequently leading to the ejection of secondary droplets (figure 1).

#### 3.1 Radial expansion of the crown

Careful observation of the crown experiments done before [17, 18, 45] reveals that the crown radial expansion is governed by the cavity radial expansion. We denoted $L$ to be the cavity diameter and $R_c$ to be the cavity radial extent. Thus,

$$L(T) = 2R_c(T) \quad (18)$$
Figure 4: (a) Comparison of the crown diameter temporal evolution obtained from the numerical solution (Equation 19) and the analytical approximation (Equation 20) (b) Comparison of the numerical and analytical approximation, when the crown diameter is scaled by $\frac{W_e^{1/5}}{5}$. 

Equation 18 combined with Equation 12 leads to the following differential equation for $L(T)$

$$
\hat{L} = 2 \sqrt{\frac{\alpha_c (\frac{W_e}{12} + 1) - \frac{W_e}{64Ft} L^4 - L^2}{\frac{1}{4} L^3}}.
$$

(19)

Approximating Equation 19 in a similar manner as done for the cavity leads to the analytical expression for $L(T)$

$$
\frac{L}{W_e^{1/5}} = 2 \left( \frac{5}{4} \right) \frac{\alpha_c}{6} \left( \frac{\alpha_c}{6} \right)^{2/5} T^{2/5}.
$$

(20)

Figure 4a shows that the numerical solution (Equation 19) and the analytical approximation (Equation 20) show a good agreement for the entire duration till the cavity reaches maximum depth. Figure 4b shows that when scaled with $W_e^{1/5}$, the numerical solution for all Weber numbers collapses on to the analytical solution; indicating the validity of the scaling law $L(T) \sim W_e^{1/5}$.

### 3.2 Crown maximum height: static energy analysis

Since the crown is viscous and its potential energy contribution is insignificant compared to its total energy [17, 18], we assume the surface energy to be the most dominant crown energy partition. We let $l_m$ be the maximum crown diameter and $s_m$ be the maximum crown height. Recalling that the drop energy is the expression on the left-hand side of Equation 7, or in non-dimensional form, is the left-hand side of Equation 8, and taking $\alpha_c$ to be the fraction of the impacting drop energy transferred to the crown,

$$
\alpha_c (\frac{W_e}{12} + 1) = 2L_m S_m = 4R_m^m S_m,
$$

(21)

where $R_m^m$ is the maximum cavity depth, $S_m$ is the maximum crown height and $l_m$ is the maximum crown diameter; all in non-dimensional form. Using the exact solution for $R_m^m$ obtained using Equation 13, Equation 21 gives

$$
S_m = \frac{\alpha_c}{4R_m^m} \left( \frac{W_e}{12} + 1 \right)
$$

(22)
where $R_{cm}^n = \sqrt{\frac{2}{\text{Bo}} \left[ \sqrt{4 + \alpha \text{Bo}} \left( \frac{\text{We}}{12} \right)^{2/5} - 2 \right]}$.

Combining Equations 15 and 21 and noting that in the Weber number range we consider (500 - 1500); $\text{We}/12 + 1 \approx \text{We}/12$, we obtain an approximated analytical expression for $S_m$ as

$$S_m = \frac{\alpha_c}{48} \left( \frac{4}{5T_{cm}} \right)^{2/5} \left( \frac{6}{\alpha_k} \right)^{1/5} \text{We}^{4/5} \approx H_m \text{We}^{4/5} \quad (23)$$

where $H_m = \left( \frac{4}{5T_{cm}^n} \right)^{2/5} \left( \frac{\alpha_c}{48} \right)^{1/5}$ for the water-on-water impacts considered herein. Note that $T_{cm}, \alpha_k$ have been defined in Section 2.

### 3.3 Vertical expansion of the crown

In this first order analysis of the crown vertical rise, we choose a control volume around the rim including its ligaments and neglect the mass and momentum flux it loses due to secondary droplet ejection. Owing to the high $Re$ considered in the present study, viscous effects on the crown sheet motion are neglected.

The crown sheet height evolution can be determined by mass conservation and momentum balance on the rim control volume. This control volume obviously moves with non-constant speed. The mass of the toroidal rim increases with the fluid flux entering the rim from the sheet. Mass conservation reads

$$\frac{\pi}{4} \frac{\partial (lb^2)}{\partial t} = (v_s(s,t) - \dot{s})lh(s,t). \quad (24)$$

The rise in the vertical direction is resisted by interfacial and gravitational forces acting on the rim. The crown sheet is assumed to be cylindrical and curvature effects are neglected in the vertical momentum balance. Consequently, momentum balance in the vertical direction reads

$$\frac{\pi}{4} \frac{\partial}{\partial t} (lb^2 \dot{s}) = -\frac{2\sigma l}{\rho} - \frac{\pi}{4} lb^2 g + lh(s,t)(v_s(s,t) - \dot{s})v_s(s,t). \quad (25)$$

where $\rho$ is the density and $\sigma$ is the surface tension of the drop and the pool fluids. Combining Equations 24 and 25 leads to

$$\frac{\pi}{4} lb^2 \ddot{s} = -\frac{2\sigma l}{\rho} - \frac{\pi}{4} lb^2 g + lh(s,t)(v_s(s,t) - \dot{s})^2, \quad (26)$$

Non-dimensionalizing Equations 24 - 26 using $d_0$ as the characteristic lengthscale, $u_0$ as the characteristic speed and the capillary time, $\sqrt{\rho d_0/\pi \sigma}$, as characteristic timescale, we obtain

$$\frac{3\pi}{2} LB^2 \dot{S} = 2L - \frac{\pi}{4} LB^2 \text{Bo} \left( V_S(S,T) - \sqrt{\frac{6}{\text{We}}} \right)^2 - \left( V_S(S,T) - \sqrt{\frac{6}{\text{We}}} S \right)^2 \cdot (27)$$

for mass conservation and momentum balance, respectively. Here, $\text{Bo} = \rho gd_0^2/\sigma$ is the drop Bond number (table 1).

In order to close the system of Equations 20, 27, and 28 we however still need to estimate the crown sheet velocity, $V_S(Z,T)$, and thickness, $H(Z,T)$, profiles. We determine these two fields next.
3.4 Crown sheet velocity profile

We assume the crown sheet to be thin and axisymmetric. Owing to the thin film approximation, the crown sheet velocity field, \( v_s(z,t) \), is assumed to be uniform across the thickness of the film, thus, independent of \( r \). Hence, the velocity profile in non-dimensional form satisfies the Navier-Stokes equation as

\[
\frac{\partial v_s(Z,T_i)}{\partial T_i} + v_s(Z,T_i) \frac{\partial v_s(Z,T_i)}{\partial Z} = - \frac{\partial P(Z,T_i)}{\partial Z} + \frac{1}{\Re} \frac{\partial^2 v_s(Z,T_i)}{\partial Z^2},
\]

(29)

where \( v_s = v_s(z,t)/u_0 \) is the non-dimensional sheet velocity profile, \( T_i = tu_0/d_0 \) is the time non-dimensionalized by the impact timescale and \( P = \rho u_0^2 \) is the pressure non-dimensionalized by the characteristic dynamic pressure. Owing to the high \( \Re \) considered, viscous effects are neglected. Moreover, since we do not focus on the extremely early stage of the splash prior to crown formation, we also neglect compressibility effects. We assume the pressure in the sheet to be the ambient characteristic dynamic pressure. Thus, \( \Re \approx \sqrt{\We} \).

Having determined the velocity profile in the sheet, to close Equations (20), (27-28), we still require \( \epsilon \), which is the offset time required for the onset of sheet formation post impact. A similar timescale was discussed for drop impacts on solid surfaces [47].

3.5 Crown thickness profile

3.5.1 Sheet thickness formulation

Mass conservation applied on any section of the cylindrical crown sheet leads to

\[
\frac{\partial (h(z,t)l(t))}{\partial t} + \frac{\partial (v_s(z,t)h(z,t)l(t))}{\partial z} = 0
\]

(32)

Since we are looking at times \( T_i \gg \epsilon \), from (31), the sheet velocity can be expressed as \( v_s(z,t) = z/t \). Using this sheet velocity profile along with the crown diameter temporal variation obtained in Equation (20) and non-dimensionalizing the lengthscale using \( d_0 \), Equation (32) leads

\[
\frac{7H}{5} + T_i \frac{\partial H}{\partial T_i} + Z \frac{\partial H}{\partial Z} = 0
\]

(33)
This can be expressed as

\[
\left( T_i, Z, -\frac{7H}{5} \right) \left( \frac{\partial H}{\partial T_i}, \frac{\partial H}{\partial Z}, -1 \right) = 0. \tag{34}
\]

and can be solved by the method of characteristics to yield a functional ansatz for the sheet thickness as

\[
H(Z, T_i) = \frac{1}{T_i^{3/5}} F \left( \frac{Z}{T_i} \right) \tag{35}
\]

Considering a control volume encompassing any section of the cylindrical crown sheet, the momentum balance equation reads

\[
\frac{1}{\rho} \sum F = \frac{d}{dt} \left( \int_{CV} \mathbf{v}_s dV \right) + \int_{CS} \mathbf{v}_s (\mathbf{v}_s \cdot \mathbf{n}) dA \tag{36}
\]

where \( \mathbf{v}_s(z, t) = v_s(z, t) \) is the crown sheet velocity. Term I in Equation 36 is zero, since the surface tension force is symmetric on both sides of the control volume. The gravitational force is negligibly small due to the negligible mass of the differential control volume chosen. We expect Term II to be zero due to the insignificant mass of the crown (this is the same reason why the crown potential energy is seen to be negligible as in \[18\]). Consequently, Term III should be zero, i.e. the momentum entering and leaving any cross section of the crown, per unit of time, are equal, with

\[
\rho v_s(z, t)^2 A(z) = \rho v_s(z + dz, t)^2 A(z + dz) \tag{37}
\]

Using \( A(z) = \pi l(t) h(z, t) \), \( v_s(z, t) = z/t \) and non-dimensionalizing the length scale by \( d_0 \),

\[
Z^2 H(Z) = (Z + dZ)^2 H(Z + dZ) \tag{38}
\]

Using the Taylor Series expansion of \( H(Z + dZ) \) and neglecting \( dZ^2 \), upon expanding (38)

\[
Z^2 \frac{\partial H}{\partial Z} + 2ZH = 0 \tag{39}
\]

Using the functional ansatz for \( H \) obtained in Equation 35 in (39) and denoting \( \phi = Z/T \)

\[
\phi \frac{dF}{d\phi} + 2F = 0 \tag{40}
\]

Integrating Equation 40 we obtain a solution for \( f(\phi) \) as

\[
F(\phi) = \frac{c_1}{\phi^2} = c_1 \left( \frac{Z}{T} \right)^{-2} \tag{41}
\]

where \( c_1 \) is a constant. Thus, from Equations 35, 41, the sheet thickness profile is

\[
H(Z, T_i) = c_1 \frac{T_i^{3/5}}{Z^2} \left[ \frac{T_i}{\sqrt{\text{We}} / 6} \right]^{3/5} \tag{42}
\]

with \( c_1 \approx 0.043 \), also derived in Appendix B using mass conservation at initial times.

Finally, using Equation 42 we can now further confirm that the Term II in Equation 36 is indeed negligible. Using \( dV = \pi l(t) h(z, t) dz \), the integrand of Term II is

\[
v_s dV = (\pi z h(t)) dz, \tag{43}
\]
Non-dimensionalizing Equation\[43\] using $d_0$ as the lengthscale and $T = tu_0/d_0$; and using $L(T_i)H(Z,T_i) = 2c_1\left(\frac{5}{4}\sqrt{\alpha_k}\right)^{2/5} \frac{T_i}{Z}$ from Equations 20, 42, 43 becomes

\[V_SdV = \frac{2\pi c_1}{Z} \left(\frac{5}{4}\sqrt{\alpha_k}\right)^{2/5}.\]  \hspace{1cm} (44)

which is independent of time. Thus, Term II in Equation \[36\] is zero and can be neglected. This ends the justification of Equation \[37\] and the derived sheet thickness profile (Equation 42).

### 3.6 Temporal evolution of the crown height: Numerical solution

Based on the experiments of the crown done by [18, 45], we approximate the crown initial height and diameter to be $S_0 \approx 1/2, L_0 \approx 3/2$. The initial vertical crown velocity at this time is about $\dot{S} \approx S_0/T_0$. Since the crown rim thickness is negligible, the ratio of the gravitational force term to the interfacial tension force in Equation 25, $B^2 Bo \ll 1$. Hence, we can neglect the gravitational force term for obtaining the analytical expression for the crown height temporal evolution.

Taking the sheet velocity profile given in Equation 31 in terms of the capillary timescale to be $V(Z,T) = \frac{Z}{\sqrt{6 \ We}}$ and consequently $V(S,T) = \frac{S}{\sqrt{T/We}}$, the sheet thickness profile obtained in Equation 42 and the crown diameter temporal evolution obtained in Equation 19 or Equations 20, Equations 27 and 28 can be solved numerically to obtain the solution for the crown height temporal evolution $S(T)$.

### 3.7 Temporal evolution of the crown height: Analytical solution

To obtain an analytical solution for the crown height temporal evolution, we use the analytical expression for the crown diameter temporal evolution derived in Equation 20, in Equation 27 leading to

\[\pi \frac{\partial (T^{2/5} B^2)}{\partial T} = c_1 \left(\frac{We}{6}\right)^{3/10} \left(\frac{S}{T} - \dot{S}\right) \frac{T}{S^2}.\]  \hspace{1cm} (45)

Integrating Equation \[45\] leads to

\[\frac{\pi}{4} \left(T^{2/5} B^2\right) = c_1 \left(\frac{We}{6}\right)^{3/10} \left(\frac{T}{S} + a_1\right),\]  \hspace{1cm} (46)

where $a_1$ is a constant.

Subsequently, using Equation \[46\] in Equation 28 and neglecting the gravitational term leads to

\[\left(c_1 \left(\frac{We}{6}\right)^{3/10} \frac{T}{S} + a_1\right) \dot{S} = -\frac{1}{3} T^{2/5} + c_1 \left(\frac{We}{6}\right)^{3/10} \left(\frac{T}{S^2}\right) \left(\frac{S}{T} - \dot{S}\right)^2.\]  \hspace{1cm} (47)

Denoting $c_2 = c_1 \left(\frac{We}{6}\right)^{3/10}$, Equation \[47\] can be integrated (details in Appendix C) to obtain

\[\frac{a_1}{c_2} \left(\frac{S}{T}\right) \exp\left(\frac{a_1}{c_2} \left(\frac{S}{T}\right)\right) = \frac{a_1}{c_2} \exp\left(\frac{(-25/6) T^{12/5} + 42a_2 + 42a_3 T}{42c_2 T}\right).\]  \hspace{1cm} (48)

where $a_2, a_3$ are constants to be determined from initial conditions.
3.7.1 Initial conditions

Using the same initial conditions for $S(T_0), \dot{S}(T_0)$ and $B(T_0)$ as mentioned in Section 3.6, we obtain

\[ a_1 = -c_1 \left( \frac{\text{We}}{6} \right)^{3/10} \frac{T_0}{S_0} = -2c_1 \left( \frac{6}{\text{We}} \right)^{1/5} \]  
\[ (49) \]

Thus, using the initial conditions for $S, \dot{S}$ in Equation 48, we have for the constants $a_2, a_3$:

\[ a_2 = -\left( \frac{5}{6^{4/5}} \right) \frac{1}{\text{We}^{6/5}} \]  
\[ (50) \]

\[ a_3 = c_1 \left( \frac{\text{We}}{6} \right)^{3/10} \left( \frac{\ln \left( \frac{1}{2\text{e}} \frac{\sqrt{\text{We}}}{6} \right)}{10} \right) + \left( \frac{1}{7} \right) \left( \frac{1}{6^{3/10}} \right) \left( \frac{10}{\text{We}^{6/10}} \right) \]  
\[ (51) \]

3.7.2 Solution for $S(T)$

Thus, using Equations 48 - 51, we obtain the full solution for $S(T)$, which can be expressed as

\[ S(T) = -\frac{T}{2} \sqrt{\frac{\text{We}}{6}} \psi(K(T)) \]  
\[ (52) \]

where $\psi(X)$ gives the principal solution for $W$ in $X = W \exp(W)$ and is also called Lambertw function. More details regarding the Lambertw function are provided in Appendix C. In Equation 52, $K(T)$ is

\[ K(T) = -2\sqrt{\frac{6}{\text{We}}} \exp \left[ \left( -\frac{1}{6^{7/10}} \right) \frac{25}{42c_1 \text{We}^{3/10}} T^{7/5} + \frac{a_2}{c_1} \left( \frac{6}{\text{We}} \right)^{3/10} \frac{1}{T} + \frac{a_3}{c_1} \left( \frac{6}{\text{We}} \right)^{3/10} \right] \]  
\[ (53) \]

For the range of $\text{We}$ considered in the present study, as validated in figure 5, $K(T)$ can be approximated to a simpler form given by $K_{\text{approx}}(T)$, thus simplifying Equation 52 to

\[ S(T) \approx -\frac{T}{2} \sqrt{\frac{\text{We}}{6}} \psi(K_{\text{approx}}(T)) \]  
\[ (54) \]

where

\[ K_{\text{approx}}(T) = -\exp \left[ \left( -\frac{1}{6^{7/10}} \right) \frac{25}{42c_1 \text{We}^{3/10}} T^{7/5} - 1 \right] \]  
\[ (55) \]
Figure 6: (a) Comparison of the crown height temporal evolution obtained from its numerical solution (as mentioned in Section 3.6), and the approximated analytical solutions (Equations 52, 54). (b) Comparison of the crown height scaled solution $S(T)/\sqrt{We}$ obtained numerically (as mentioned in Section 3.6) and analytically (Equation 56). The inset of this figure shows that $\psi(K_{\text{approx}}(T)) \sim We^{3/10}$ which leads to $S(T) \sim We^{4/5}$.

Figure 6a shows the comparison between the crown height temporal evolution obtained from its numerical solution (as mentioned in Section 3.6), the analytical solution (Equation 52) and its approximated form (Equation 54). We can see that a good agreement is seen for all the Weber numbers.

Since the crown maximum height depends on Weber number as $S_m \sim We^{4/5}$, we also expect the crown height to scale as $We^{4/5}$ at all times. From the inset of figure 6b which shows that $\psi(K_{\text{approx}}(T)) \sim We^{3/10}$. Thus, we have

$$\frac{S(T)}{We^{4/5}} \sim T \frac{1}{2\sqrt{6} We^{3/10}} \psi(K_{\text{approx}}(T)) = \text{independent of We}$$

(56)

This scaling law $S(T) \sim We^{4/5}$ has been verified numerically in figure 6b which shows that the numerical solutions for all Weber numbers collapses on the analytical solution (Equation 56) when scaled with $We^{4/5}$.

4 Conclusions

In conclusion, we address the sequence of events accompanying the impact of a liquid droplet on a deep, inviscid liquid pool, namely: cavity, crown and wave swell formation. We obtain expressions for the maximum cavity depth, $R_{mc}$ in terms of the impacting droplet properties, shown in Equation 13. At $We > 500$, we show that $R_{mc}^m$ scales as $We^{1/5}$ (Equation 15). We show that the cavity depth $R_c$ scales with the non-dimensionalized capillary time as $R_c \sim We^{1/5} T^{2/5}$ (Equation 14). Previous studies [46, 22] obtained expressions relating the maximum cavity depth to the time it takes to reach such depth, but did not propose an expression for the entire cavity temporal evolution.

We show that the crown diameter expansion is identical to the cavity diameter expansion, thus linking the cavity and crown dynamics. We propose a similarity solution for the crown sheet velocity profile (Equation 31). We show that the velocity profile is similar to the one obtained for impact on solid surfaces [47], but requires the introduction of a time offset $\epsilon$. Subsequently, we proposed a unified functional form for the spatial and temporal variation of the crown sheet thickness profile (Equation 42) and showed that the functional ansatz for the sheet thickness profile

$$\psi(K_{\text{approx}}(T)) \sim We^{3/10}$$

(56)
shows the scaling $HT_{1}^{7/5} = F(Z/T)$. We showed that such a similarity scaling is robust for a wide range of Weber numbers.

Using the similarity solutions obtained for the crown sheet thickness and velocity, we obtain an analytical solution for the crown height temporal evolution. We show that the crown height $S(T)$ scales with the impacting Weber number as $S(T) \sim We^{4/5}$. The crown height analytical solution shows good agreement with the numerically obtained data for the entire duration of the splash, i.e. the rise and subsequent fall of the crown and is robust for a wide range of Weber numbers, further confirming the validity of the derived crown diameter, sheet velocity and sheet thickness profiles.

Thus, the cavity and crown analysis employed in the present study paves way for a new understanding of the traditionally know cavity-crown system formalized in the seminal studies by Worthington [1, 2, 3, 4].

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6 Appendix A

The schematic of the spherical cavity of radius $r = r_c$ formed in the pool is shown in figure 2a, where $r$ is the radial coordinate and $z$ is the vertical coordinate defined therein.

6.1 Cavity potential energy

The gravitational potential energy of the cavity at its maximum depth, $r_c = r_c^m$, is the work required to bring the liquid content of the cavity to the original undisturbed surface of the pool. This work is given by

$$U_p = \pi g \rho \int_{0}^{r_m} ((r_m^c)^2 - z^2) zdz = \frac{1}{4} \pi g \rho (r_m^c)^4.$$ (57)

6.2 Cavity surface energy

Assuming the expanding cavity to be axisymmetric and hemispherical with its center at the point of impact of the drop on the pool, the cavity surface energy at its maximum depth is obtained from its area as

$$U_\sigma = 2\pi \sigma (r_m^c)^2.$$ (58)

6.3 Cavity viscous energy

The boundary conditions on the flow velocity, $v(r, \theta) = v_r(r, \theta)\hat{r} + v_\theta(r, \theta)\hat{\theta}$ around the expanding cavity are

$$\|v_{r \to \infty}\| = 0 \quad \text{and} \quad \|v_{r=r_c}\| = r_c.$$ (59)

As $Re \gg 1$, and the flow of interest is far from solid boundaries, we can express the velocity field as a gradient of potential $v(r, \theta) = \nabla \phi$. Combined with the divergence free condition imposed by continuity, $\nabla \cdot v = 0$, leads to a Laplace equation on the potential $\phi$:

$$\nabla^2 \phi = 0.$$ (60)

Unlike [22] where the velocity potential form is assumed to be independent of $\theta$, we assume the potential which satisfies Equation (60) to have the following form as assumed in [18]

$$\phi = A \cos \theta \frac{r}{r_c}.$$ (61)
Hence,

\[ v_r = -\frac{\partial \phi}{\partial r} = A \frac{\cos \theta}{r^2}, \]

(62)

and

\[ v_\theta = -\frac{1}{r} \frac{\partial \phi}{\partial \theta} = A \frac{\sin \theta}{r^2} \]

(63)

The resultant speed \(|v| = \sqrt{v_r^2 + v_\theta^2}\) is then \(A/r^2\). Close to the cavity wall, \(r = r_c\), \(|v| = \dot{r}_c\). Thus

\[ A \frac{r_c}{\dot{r}_c} = \dot{r}_c. \]

(64)

From Equations [61] and [64] we obtain the velocity potential, as in [18],

\[ \phi = \dot{r}_c r_c^2 \frac{\cos \theta}{r}, \]

(65)

with

\[ v_r = \frac{\dot{r}_c^2}{r^2} r_c \cos \theta, \]

(66)

\[ v_\theta = \frac{\dot{r}_c^2}{r^2} r_c \sin \theta, \]

(67)

\[ ||v|| = \frac{\dot{r}_c^2}{r^2} r_c. \]

(68)

The total rate of viscous dissipation, \(W\), in this fluid is given by [48] and is calculated in [18] as

\[ W = -\mu \int_{0}^{r_c} \left( \frac{\partial}{\partial r} (\phi \nabla \phi) \right)_d dS = \frac{\pi}{2} \mu \int_{0}^{r_c} \dot{r}_c^2 (2\pi r_c \sin \theta) r_c d\theta = 8\pi \mu r_c \dot{r}_c^2, \]

(69)

where \(dS\) is the elemental surface area on the hemispherical cavity and is shown in figure 2.

Hence the dissipation energy, \(U_\mu\), during the time \(t_m\), needed for the cavity to reach maximum depth, \(r_m\), is

\[ U_\mu = 8\pi \mu \int_{0}^{t_m} r_c \dot{r}_c^2 dt, \]

(70)

### 6.4 Cavity kinetic energy

The kinetic energy in the liquid pool surrounding the cavity is

\[ U_k = \frac{\rho}{2} \int \int \int v^2 dV. \]

(71)

Using \(v = \nabla \phi\) and the identity \(\nabla \cdot (\phi \nabla \phi) = v^2\) in (71),

\[ U_k = \frac{\rho}{2} \int \int \phi \nabla \phi \cdot n dS = -\frac{\rho}{2} \int \int \left( \frac{\partial \phi}{\partial r} \right)_d dS, \]

(72)

where \(n\) is the outward normal vector to the elemental area \(dS\), shown in figure 2.

Hence, the kinetic energy of the liquid in the immediate vicinity of the cavity is

\[ U_k = -\frac{\rho}{2} \int \int \left( \frac{\partial \phi}{\partial r} \right)_d dS = \frac{\rho}{2} \int \int_{r=r_c} \dot{r}_c r_c^2 (\cos^2 \theta) 2\pi r_c \sin \theta r_c d\theta = \frac{\pi}{3} \rho r_c^3 \dot{r}_c^2. \]

(73)
Figure 7: (a) Schematic of the drop at one impact timescale, $\tau_{im} = d_0/u_0 = 1$, after it contacts the pool surface.

### 7 Appendix B

Figure 7a shows the schematic of the impacting drop, one impact time, $\tau_{im} = d_0/u_0 = 1$, after it contacts the pool. In this analysis, all length scales are non-dimensionalized by the droplet diameter. At this time, the cavity has not yet reached its hemispherical shape, and is cylindrical in shape. A similar schematic was assumed by [49] for drop impacts on deep pools, without considering the cylindrical shape of the cavity at the very early times. The volume occupied by the cavity is denoted by $V_1$, with a corresponding depth $R_0$ at one impact time. As the cavity forms, it displaces fluid upwards to create a crown of volume $V_3$, height $S_0$ and diameter $L_0$ at one impact time. We assume that the sum of volumes $V_1 + V_2$ is approximately equal to the volume of a spherical cap of height $S_0 + R_0$ and cap radius $L_0/2$. Using the above approximations,

$$V_2 = V_3,$$

$$V_3 = \frac{\pi L_0^2}{4} S_0 - V_1.$$  \hspace{1cm} (74)

$$V_1 + V_2 = \frac{\pi}{6} (S_0 + R_0) \left( \frac{3L_0^2}{4} + (S_0 + R_0)^2 \right).$$  \hspace{1cm} (75)

Using Equations 74-76, we get a cubic solution for the cavity depth at one impact timescale, $R_0$ as

$$(S_0 + R_0) \left( \frac{3L_0^2}{4} + (S_0 + R_0)^2 \right) = \frac{3}{2} L_0^2 S_0$$  \hspace{1cm} (77)

As mentioned in Section 3.6, we assume the initial conditions of the crown height and diameter as

$$S_0 = 1/2; \ L_0 = 3/2$$

Using $S_0, L_0$ obtained from Equation 78 in 77, the only real solution of $R_0$ is found to be

$$R_0 = 0.25.$$  \hspace{1cm} (79)

This value is very close to the data at initial times reported by [13] and also to the experimentally observed value of $R_0 \approx 0.23 \pm 0.01$ in the present study, as shown in figure 7a. For the rest of the analysis, we use the experimentally observed value of $R_0 \approx 0.23$, by noting that this value can be well captured by our theory (Equation 79).

We further denote the thickness of the crown at one impact timescale by $H_0 = H(Z,1)$. From
\[ H_0 = H(Z,1) = \frac{c_1}{Z^2} \]  

(80)

At one impact time, we assume the crown to be cylindrical with uniform thickness \( H_0 = H(S_0,1) \). Thus, using Equation (80) along with this approximation and equating the volume of the cavity and crown

\[ \frac{c_1}{S_0^2} L_0 S_0 \approx \frac{L_0^3}{4} R_0. \]  

(81)

Thus,

\[ c_1 \approx \frac{S_0 L_0 R_0}{4}. \]  

(82)

Using (78), (79) in (82),

\[ c_1 \approx 0.043. \]  

(83)

This value of \( c_1 \) is used for the sheet thickness profile obtained in Equation (42).

8 Appendix C

8.1 Crown height \( S(T) \) analytical solution

\[ \left( c_2 \frac{T}{S} + a_1 \right) \ddot{S} = -\frac{1}{3} T^{2/5} + c_2 T \left( \frac{S}{T} - \dot{S} \right)^2 \]  

(84)

Equation (84) can be expressed as

\[ \frac{-c_2 T^2 \dot{S}^2 + 2c_2 S \ddot{S} T + c_2 T^2 S \dot{S}}{S^2} + a_1 T \ddot{S} + \frac{1}{3} T^{7/5} - c_2 = 0 \]  

(85)

Terms I, II, III in Equation (85) can be integrated separately to obtain

\[ \frac{c_2 T^2 \dot{S}}{S} + a_1 T \ddot{S} - a_1 \dot{S} + \frac{5}{36} T^{12/5} - c_2 T + a_2 = 0 \]  

(86)

Denoting \( \phi = S/T \), Equation (86) can be expressed as

\[ \frac{c_2 \ddot{\phi}}{\phi} + a_1 \ddot{\phi} + \frac{5}{36} T^{2/5} + \frac{a_2}{T^2} = 0 \]  

(87)

Integrating Equation (87) we obtain

\[ c_2 \ln \phi + a_1 \phi + \frac{25}{252} T^{-7/5} - \frac{a_2}{T} = a_3 \]  

(88)

Equation (88) leads to

\[ \frac{a_1}{c_2} \phi \exp \left( \frac{a_1 \phi}{c_2} \right) = \frac{a_1}{c_2} \exp \left( -\frac{(25/6) T^{12/5} + 42 a_2 + 42 a_3 T}{42 c_2 T} \right) \]  

(89)

Replacing \( \phi = S/T \), we obtain Equation (48).
8.2 Lambertw function: \( \psi(X) \)

The Lambertw function, \( \psi(X) \), gives the principal solution for \( W(X) \) in \( X = W(X)\exp[W(X)] \) and takes real values in the domain \( X \geq -1/e \). Figure 8a shows the plot of the Lambertw function for the domain range valid for the present study. Some important properties of the Lambertw function are

- The \( n \)th derivatives \( \psi^{(n)}(X) \) can be expressed as \( \psi^{(n)}(X) = \frac{\psi^{(n-1)}(X)}{X^n [1 + \psi(X)]^{2n-1}} \).
- The integral can be expressed as \( \int \psi(X) dX = X \left( \psi(X) - 1 + \frac{1}{\psi(X)} \right) \).
- The Lambertw function has the real valued series expansion \( \psi(X) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}n^{n-2}}{(n-1)!}X^n \).

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