Relative locality and gravity’s weight on worldline fuzziness

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Abstract. We use the example of the much-studied $\kappa$-Minkowski noncommutative spacetime for illustrating a novel approach toward the analysis of the possible implications of spacetime noncommutativity. Our starting point is the proposal that spacetime noncommutativity is most naturally introduced within the manifestly-covariant formulation of quantum mechanics. This allows us to obtain a crisp characterization of the relativity of spacetime locality present in $\kappa$-Minkowski theories. And we also develop a novel description of how $\kappa$-Minkowski noncommutativity affects the fuzziness of worldlines.

1. Spacetime noncommutativity and covariant formulation of quantum mechanics

$\kappa$-Minkowski noncommutative spacetime[1, 2, 3, 4, 5],

$$[x_j, x_0] = i\ell x_j, \quad [x_j, x_k] = 0,$$

has attracted very intense interest over the last twenty years, but the issues we are interested in remain largely unexplored. We would like to characterize a class of physical effects linked with the description of spacetime fuzziness codified by $\kappa$-Minkowski noncommutativity, and it is at this point not even clear which of our current theories could make room for this noncommutativity. Evidently classical mechanics cannot be our choice, since its formalization provides no room for noncommutativity of coordinates. And even giving a formulation of $\kappa$-Minkowski spacetime noncommutativity in a quantum-mechanics setup is not straightforward. This is due to the fact that in $\kappa$-Minkowski the time coordinate should be an operator that does not commute with the spatial-coordinate operators, but in the standard setup of quantum mechanics we are not in the situation of time being described by an operator that commutes with the spatial-coordinate operators: in the standard setup of quantum mechanics time is not an observable at all, it just plays the role of evolution parameter.

We believe that it was indeed because of this mismatch between the nature of time in quantum mechanics and the properties of the $\kappa$-Minkowski time coordinate that progress did not materialize for formulating some of the observable spacetime consequences of $\kappa$-Minkowski that we are here interested in. We here summarize and offer an overall perspective on our results reported in Refs. [6, 7, 8] whose main starting point is recent progress on a manifestly-covariant formulation of quantum mechanics, which matured significantly over the last decade [9, 10, 11, 12]. In this powerful reformulation of quantum mechanics both the spatial
coordinates and the time coordinate play the same type of role. And there is no “evolution”, since dynamics is codified in a constraint, just in the same sense familiar for the covariant formulation of classical mechanics (see, e.g., chapter 4 of Ref. [13]). Spatial and time coordinates are well-defined operators on a “kinematical Hilbert space”, which is just an ordinary Hilbert space of normalizable wave functions [11]. And on the kinematical Hilbert space one has standard momenta \( \pi_0, \pi_1 \) conjugate to the spacetime coordinates \( q_0, q_1 \) [11]

\[
[\pi_0, q_0] = i, \quad [\pi_1, q_1] = -i, \quad [\pi_1, q_0] = [\pi_0, q_1] = 0 , \tag{2}
\]
specializing to the case of a 2D spacetime.

For free massless particles the physical Hilbert space will be characterized by states satisfying the constraint

\[
H\psi \equiv (\pi_0^2 - \pi_1^2)\psi = 0
\]

where \( \psi \) is then a state on the physical Hilbert space.

We focus on 2D \( \kappa \)-Minkowski, so that its noncommutativity is fully encoded in

\[
[x_1, x_0] = i\ell x_1 , \tag{3}
\]

and we observe [6, 7, 8] that this 2D-\( \kappa \)-Minkowski defining commutator (3) is satisfied by posing a relationship between \( \kappa \)-Minkowski coordinates and the phase-space observables of the covariant formulation of quantum mechanics (the ones of Eq. (2), here viewed merely as formal auxiliary operators [6]):

\[
x_1 = e^{\ell \pi_0} q_1 , \quad x_0 = q_0 . \tag{4}
\]

Besides allowing a meaningful description of the noncommutativity of the \( \kappa \)-Minkowski time coordinate, this setup leads straightforwardly [6, 8] to a description of translation transformations in \( \kappa \)-Minkowski:

\[
T_{\mu} \triangleright f(x_0, x_1) \longleftrightarrow -ia^\mu \left[ \pi_\mu, f(q_0, q_1 e^{\ell \pi_0}) \right] , \tag{5}
\]

One can then study [6, 8], within this setup, the properties of boost transformations in \( \kappa \)-Minkowski, thereby obtaining two other strongly-characterizing results. The first result [6, 8] concerns the form of the operator that should enforce the Hamiltonian constraint for free particles

\[
H_\ell = \left( \frac{2}{\ell} \right)^2 \sinh^2 \left( \frac{\ell \pi_0}{2} \right) - e^{-\ell \pi_0} \pi_1^2 . \tag{6}
\]

and the second result [6, 8] concerns the measure for integration over momenta, needed for evaluating scalar products when working in the “momentum representation”:

\[
d\pi_0 d\pi_1 \longrightarrow d\pi_0 d\pi_1 e^{-\ell \pi_0} , \tag{7}
\]

2. **Fuzzy points, translation transformations and relative locality**

Evidently within our description a point will be identified with a state in the kinematical Hilbert space that gives rather well determined values to \( x_0 \) and \( x_1 \). A class of states which is well suited for exploring the properties of \( \kappa \)-Minkowski fuzziness is the one of gaussian states on our kinematical Hilbert space. We denote them by \( \Psi_{q_0, q_1}(\pi_\mu; \pi_\mu, \sigma_\mu) \) and they are specified by functions of the variables \( \pi_\mu \) parametrized by \( \pi_\mu, \sigma_\mu \), and \( \pi_\mu \):

\[
\Psi_{q_0, q_1}(\pi_\mu; \pi_\mu, \sigma_\mu) = N e^{-\frac{(\pi_0 - \bar{\pi}_0)^2}{4\sigma_0^2} - \frac{(\pi_1 - \bar{\pi}_1)^2}{4\sigma_1^2}} e^{i\pi_0 q_0 - i\pi_1 q_1} , \tag{8}
\]
where \(N\) is a normalization constant. Of course, our main focus of attention will be on establishing how the \(\kappa\)-Minkowski scale \(\ell\) affects the results, since this is going to be our indicator of the difference between classical Minkowski spacetime and \(\kappa\)-Minkowski.

Following the analysis we reported in Ref. [6] one finds that the \(\kappa\)-Minkowski scale \(\ell\) turns out to play a particularly significant role in the properties of the coordinate \(x_1\), for which we find

\[
\langle x_1 \rangle = \langle \eta_1 \rangle e^{\ell_0} \frac{\sqrt{\pi}}{2}, \quad \delta x_1 = e^{\ell_0} \left[ \frac{1}{4\ell_1^2} + \frac{\eta_1^2}{1 - e^{2\ell_0^2}} \right]^{1/2},
\]

whereas for the \(\kappa\)-Minkowski time coordinate \(x_0\) one has

\[
\langle x_0 \rangle = \eta_0 - \frac{i\ell}{2}, \quad \delta x_0 = \frac{1}{2\ell_0} \tag{10}
\]

(Comments on the imaginary contribution \(-i\ell/2\) to \(\langle x_0 \rangle\) are given in Ref. [6]).

Our next task is to act with a translation transformation on our gaussian state and see what changes as a result. For the expectation values here of interest this is equivalently done by acting with the translation transformation on the operators \(x_1\) and \(x_0\). Following again our work in Ref. [6] one finds that

\[
\langle T_{\alpha\mu} \triangleright x_0 \rangle = \eta_0 - a_0 - \frac{i\ell}{2}, \quad \delta (T_{\alpha\mu} \triangleright x_0) = \frac{1}{2\ell_0}, \tag{11}
\]

and

\[
\langle T_{\alpha\mu} \triangleright x_1 \rangle = (\eta_1 - a_1) e^{\ell_0} \frac{e^{2\ell_0^2}}{\ell_0^2}, \quad \delta (T_{\alpha\mu} \triangleright x_1) = e^{\ell_0} \left[ \frac{1}{4\ell_1^2} + (\eta_1 - a_1)^2 \left( 1 - e^{2\ell_0^2} \right) \right]^{1/2}. \tag{12}
\]

The interpretation here of course is such that the \(x_0, x_1\) are operators characterizing the distance of a given (fuzzy) point from the frame origin of some observer Alice, and then \(T_{\alpha\mu} \triangleright x_0, \ T_{\alpha\mu} \triangleright x_1\) are the operators that characterize the distance of that point from the frame origin of an observer Bob, purely translated with respect to Alice. And accordingly one can deduce the relation between the mean values and uncertainties in positions among two distant observers in relative rest by comparing (10) to (11) and comparing (9) to (12). This we did in detail in Ref. [6]. We here summarize the main message of that analysis, focusing on the case of two fuzzy points in \(\kappa\)-Minkowski as described by two distant observers: one of the points is near observer Alice, while the other one is near observer Bob, purely translated with respect to Alice. By comparing (10) to (11) and comparing (9) to (12) one then finds two main features [6]: (i) the same point appears to be more fuzzy to a distant observer than to a nearby observer, (ii) the point at Alice is not described as being at Alice in the coordinatization of spacetime of observer Bob, and vice versa the point at Bob is not described as being at Bob in the coordinatization of spacetime of observer Alice. The second feature, (ii), is essentially already known from previous studies of relative locality in the classical limit [14, 15]: one can have consistently relativistic theories where pairs of points found to be coincident by a nearby observer (or, as in the case here considered, a point found to coincide with the origin of that observer) are instead described as noncoincident if one uses the inferences about those points by a distant observer. Feature (i) was established for the first time in our Ref. [6], and is a feature of relative locality for the fuzziness of points in a quantum spacetime. The emerging picture is fully relativistic (though in the sense of deformed relativistic symmetries [16, 17, 18, 19]): all observers in \(\kappa\)-Minkowski perceive their origin as the point of lowest fuzziness and attribute to distant points fuzziness proportional to the distance.
3. Fuzziness of $\kappa$-Minkowski worldlines

In the previous section we worked on the kinematical Hilbert space. For our next (and here last) task we must progress to the level of the physical Hilbert space. We want to now use $\kappa$-Minkowski noncommutativity as a way to model the influence of quantum-gravity effects on the fuzziness of worldlines. This should be viewed in the context of attempts to describe the dynamics of matter particles as effectively occurring in an “environment” of short-distance quantum-gravitational degrees of freedom: for propagating particles with wavelength much larger than the Planck length, when it may be appropriate to integrate out these quantum-gravitational degrees of freedom, the main residual effect of short-distance gravity could indeed be an additional contribution to the fuzziness of worldlines, which one could model with spacetime noncommutativity.

For working on the physical Hilbert space we follow (as we did in Ref. [8]) the prescription adopted in Ref. [11]: we obtain the needed feature of invariance of physical observables under the action of the constraint $H_\ell$ by introducing a new scalar product that projects all the orbit of the gauge transformation generated by $H_\ell$ on the same state. For massless particles, on which we here focus, this amounts to inserting the Hamiltonian constraint as follows

\[ \langle \psi | \phi \rangle_{H_\ell} = \langle \psi | \delta (H_\ell) \Theta(\pi_0) | \phi \rangle \]  

(13)

where $\langle \rangle$ is the scalar product on the kinematical Hilbert space and $\Theta(\pi_0)$ specifies a restriction [11] to positive-energy solutions of the constraint.

The next hurdle we must face concerns the identification of an observable suitable for the characterization of the fuzziness of the worldline. The apparently obvious choices, $x_1$ and $x_0$, are actually not suitable for this task, since they are not self-adjoint operators on our physical Hilbert space (in particular they do not commute with $H_\ell$). We propose to remedy this by focusing on the following “intercept operator” $\mathcal{A}$:

\[ \mathcal{A} = e^{\ell\pi_0} \left( q_1 - \mathcal{V} q_0 - \frac{1}{2} [q_0, \mathcal{V}] \right) \]  

(14)

where $\mathcal{V}$ is short-hand for $\mathcal{V} = (\partial H_\ell / \partial \pi^0)^{-1} \partial H_\ell / \partial \pi^1$.

One may notice that $\mathcal{A}$ is describable as an $\ell$-deformed Newton-Wigner operator [20]. And it is well known that within special-relativistic quantum mechanics there is no better estimator of localization than the Newton-Wigner operator (it can only be questioned for localization comparable to the Compton wavelength of the particle [20], but this merely conceptual limit of ideal localization is not relevant for our purposes here [8]).

Let us focus, for conceptual clarity, on the analysis of the properties of $\mathcal{A}$ for the case of $\Psi_{0,0}$, i.e. for $\bar{q}_0 = 0, \bar{q}_1 = 0$. One then easily finds that

\[ \langle \Psi_{0,0} | \mathcal{A} | \Psi_{0,0} \rangle_{H_\ell} = 0 \]

so this is a case where the particle intercepts the observer Alice in her origin.

The fuzziness of this intercept, which reflects the fuzziness of the worldline [7, 8] described by $\Psi_{0,0}$, is characterized by [8]

\[ \delta A^2_{[\ell]} \approx \langle \Psi_{0,0} | \mathcal{A}^2 | \Psi_{0,0} \rangle_{H_\ell} \approx \ell (\pi_0) \sigma^{-2} / 2 \]

(15)

where for simplicity we focused on the leading $\ell$-dependent contributions (index $[\ell]$) and we assumed that $\sigma_1$ is small enough, in comparison to $\sigma_0, \pi_1$, to allow a saddle point approximation in the $\pi_1$ integration; then $\sigma$ (without indices) is the effective gaussian width after the saddle point approximation in $\pi_1$: $\sigma^{-2} \equiv \sigma_1^{-2} + \langle \mathcal{V} \rangle \geq 2 \sigma_0^{-2}$.
In the interpretation we proposed in Ref. [8] we describe Eq. (15) as the fuzziness of the worldline “at Alice” (at the point of crossing the origin of Alice’s reference frame). It is then interesting to establish whether observers reached by the particle at cosmological distances from Alice will observe bigger fuzziness. We characterize such observers as those who are connected to Alice by a pure translation, so that for them the state of the particle is \( \Psi_{a_0,a_1} \), and are, like Alice, such that \( \langle A \rangle = 0 \), i.e. \( \langle \Psi_{a_0,a_1} | A | \Psi_{a_0,a_1} \rangle H_t = 0 \). Unsurprisingly this requires \( a^1 = \langle V \rangle a^0 \). And for these observers one finds

\[
\delta A^2_{|t]} = \left( \langle \Psi_{a^0,\langle V \rangle a^0} | A^2 | \Psi_{a^0,\langle V \rangle a^0} \rangle H_t \right)_{|t]} \approx \left( \frac{\ell(\pi_0)}{2\sigma^2} + \ell^2 a_0^2 \delta a^2 \right) \tag{16}
\]

From this we see that the approach we developed in Refs. [6, 7, 8] leads to the first example of a quantum-spacetime picture (and of an interpretation of relativistic quantum mechanics in such a spacetime) providing the main ingredient of scenarios [21, 22, 23, 24, 25, 26] motivating the idea that quantum-gravity effects could affect worldline fuzziness in ways that inevitably lead to an increase of this fuzziness as the particle propagates. One can in fact interpret our observer Alice, the observer on the worldline for whom the fuzziness of the intercept takes the minimum value, as the observer at the source (where the particle is produced), and then the intercept of the particle worldline with the origin of the reference frames of observers distant from Alice (where the particle could be detected) has bigger uncertainty.

As we stressed in Refs. [7, 8] this formalization of gravity’s contribution to worldline fuzziness could be relevant for an ongoing phenomenological effort aimed at finding experimental evidence of this sort of spacetime-fuzziness effects [21, 22, 23, 24, 25, 26].

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