Orbital and strongly orbital spaces

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Abstract

We say that a (countably dimensional) topological vector space $X$ is orbital if there is $T \in L(X)$ and a vector $x \in X$ such that $X$ is the linear span of the orbit $\{T^n x : n = 0, 1, \ldots\}$. We say that $X$ is strongly orbital if, additionally, $x$ can be chosen to be a hypercyclic vector for $T$. Of course, $X$ can be orbital only if the algebraic dimension of $X$ is finite or infinite countable. We characterize orbital and strongly orbital metrizable locally convex spaces. We also show that every countably dimensional metrizable locally convex space $X$ does not have the invariant subset property. That is, there is $T \in L(X)$ such that every non-zero $x \in X$ is a hypercyclic vector for $T$. Finally, assuming the Continuum Hypothesis, we construct a complete strongly orbital locally convex space.

As a byproduct of our constructions, we determine the number of isomorphism classes in the set of dense countably dimensional subspaces of any given separable infinite dimensional Fréchet space $X$. For instance, in $X = \ell_2 \times \omega$, there are exactly 3 pairwise non-isomorphic (as topological vector spaces) dense countably dimensional subspaces.

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1 Introduction

All vector spaces in this article are over the field $\mathbb{K}$ being either the field $\mathbb{C}$ of complex numbers or the field $\mathbb{R}$ of real numbers. As usual, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, $\mathbb{N}$ is the set of positive integers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Throughout the article, all topological spaces are assumed to be Hausdorff. Recall that a Fréchet space is a complete metrizable locally convex space. For a topological vector space $X$, $L(X)$ is the algebra of continuous linear operators on $X$ and $X'$ is the space of continuous linear functionals on $X$. Symbol $GL(X)$ stands for the group of invertible $T \in L(X)$ for which $T^{-1} \in L(X)$. For $T \in L(X)$, the dual operator $T' : X' \to X'$ is defined as usual: $T' f = f \circ T$.

By saying countable, we always mean infinite countable. Recall that the topology $\tau$ of a topological vector space $X$ is called weak if $\tau$ is the weakest topology making each $f \in Y$ continuous for some linear space $Y$ of linear functionals on $X$ separating points of $X$. It is well-known and easy to see that a topology of a metrizable infinite dimensional topological vector space $X$ is weak if and only if $X$ is isomorphic to a dense linear subspace of $\omega = \mathbb{K}^\mathbb{N}$.

Recall that a topological vector space $X$ has the invariant subspace property if every $T \in L(X)$ has a non-trivial (=different from $\{0\}$ and $X$) closed invariant subspace. Similarly, a topological vector space $X$ has the invariant subset property if every $T \in L(X)$ has a non-trivial (=different from $\emptyset$, $\{0\}$ and $X$) closed invariant subset. The problem whether $\ell_2$ has the invariant subspace property is known as the invariance of subspace problem and remains perhaps the most famous open problem in operator theory. The problem whether $\ell_2$ has the invariant subset property is also open. It is worth noting that Read [18] and Enflo [13] (see also [5]) showed independently that there are separable infinite dimensional Banach spaces, which do not have the invariant subspace property. In fact, Read [19, 20] demonstrated that $\ell_1$ does not have the invariant subset property. It is worth noting that Atzmon [2, 3] constructed an infinite dimensional nuclear Fréchet space without the invariant subspace property. All existing constructions of operators on Banach spaces with no invariant subspaces are rather sophisticated. On the other hand, examples of separable non-complete normed spaces with or without the invariant subspace or subset property are relatively easy to construct.
Recall that $x \in X$ is called a hypercyclic vector for $T \in L(X)$ if the orbit

$$O(T, x) = \{ T^n x : n \in \mathbb{Z}_+ \}$$

is dense in $X$. Similarly, $x$ is called cyclic for $T$ if the linear span of $O(T, x)$ is dense in $X$. Clearly, $T$ has no non-trivial invariant subsets (respectively, subspaces) precisely when every non-zero vector is hypercyclic (respectively, cyclic) for $T$. If the space in question is countably dimensional, there is an unusual way to approach the invariant subspace/subset property.

**Definition 1.1.** We say that a topological vector space $X$ is orbital if there exist $T \in L(X)$ and $x \in X$ such that $X = \text{span}(O(T, x))$. We say that $X$ is strongly orbital if there exist $T \in L(X)$ and $x \in X$ such that $x$ is a hypercyclic vector for $T$ and $X = \text{span}(O(T, x))$.

Note that an orbital space always has either finite or countable dimension. Since finite dimensional topological vector spaces support no hypercyclic operators, every strongly orbital space is countably dimensional. We start with the following easy observation.

**Lemma 1.2.** Each strongly orbital topological vector space does not have the invariant subset property.

**Proof.** Let $X$ be a topological vector space and $T \in L(X)$ and $x \in X$ be such that $x$ is a hypercyclic vector for $T$ and $X = \text{span}(O(T, x))$. Due to Wengenroth [20], $p(T)(X)$ is dense in $X$ for every non-zero polynomial $p$. It suffices to show that $T$ has no non-trivial invariant subsets. Take $y \in X \setminus \{0\}$. Since $X = \text{span}(O(T, x))$, there is a non-zero polynomial $p$ such that $y = p(T)x$. Hence

$$O(T, y) = O(T, p(T)y) = p(T)(O(T, x))$$

and therefore $O(T, y)$ is dense in $X$ since each $p(T)(X)$ is dense in $X$ and $O(T, x)$ is dense in $X$. Thus every non-zero vector is hypercyclic for $T$, which means that $T$ has no non-trivial invariant subsets. \qed

The following two propositions are known facts and are basically a compilation of certain results from [8] and [15]. We present their proofs for the sake of convenience.

**Proposition 1.3.** Every normed space $X$ of countable algebraic dimension is strongly orbital and therefore does not have the invariant subset property.

**Proof.** Let $B$ be the completion of $X$. Then $B$ is a separable infinite dimensional Banach space. According to Ansari [11] and Bernal–Gonzáles [8], there is a hypercyclic $T \in L(B)$. That is, there is $x \in B$ such that $\{ T^n x : n \in \mathbb{Z}_+ \}$ is dense in $B$. Let $Z$ be the linear span of $T^n x$ for $n \in \mathbb{Z}_+$. Since both $Z$ and $X$ are dense countably dimensional linear subspaces of $B$, according to Grivaux [13], there is $S \in GL(B)$ such that $S(Z) = X$. Since $Z$ is invariant for $T$, $X$ is invariant for $STS^{-1}$. Hence, the restriction $A = STS^{-1} |_X$ belongs to $L(X)$, $O(A, Sx) = S(O(T, x))$ is dense and $X = \text{span}(O(A, Sx))$. Thus $X$ is strongly orbital. By Lemma 1.2, $X$ does not have the invariant subspace property. \qed

**Proposition 1.4.** In every separable infinite dimensional Fréchet space $X$, there is a dense linear subspace $E$ such that $E$ has the invariant subspace property.

**Proof.** There is a dense linear subspace $E$ of $X$ ($E$ can even be chosen to be a hyperplane [9]) such that every $T \in L(E)$ has the shape $\lambda I + S$ with $\lambda \in \mathbb{K}$ and $\dim S(X) < \infty$. Trivially, such a $T$ has a one-dimensional invariant subspace. \qed

### 1.1 Results

We partially extend Proposition 1.3 from the class of normed spaces to the class of metrizable locally convex (topological vector) spaces.

**Theorem 1.5.** Every metrizable locally convex space $E$ of countable algebraic dimension does not have the invariant subset property.
It turns out that not every metrizable locally convex space \( X \) of countable algebraic dimension is strongly orbital or even orbital. In order to formulate the result neatly, we split the class \( \mathcal{M} \) of infinite dimensional metrizable locally convex spaces into 4 subclasses. Given \( X \in \mathcal{M} \) and an increasing sequence \( \{p_n\}_{n \in \mathbb{N}} \) of seminorms on \( X \) defining the topology of \( X \), we have the following 4 possibilities:

\((A_0)\) \( X/\ker p_1 \) is finite dimensional and \( \ker p_n/\ker p_{n+1} \) is finite dimensional for each \( n \in \mathbb{N} \);

\((A_1)\) there is \( n \in \mathbb{N} \) such that \( \ker p_n = \{0\} \);

\((A_2)\) \( X/\ker p_1 \) is infinite dimensional, \( \ker p_n/\ker p_{n+1} \) is finite dimensional for each \( n \in \mathbb{N} \) and is non-zero for infinitely many \( n \in \mathbb{N} \);

\((A_3)\) \( \ker p_n/\ker p_{n+1} \) is infinite dimensional for infinitely many \( n \in \mathbb{N} \).

It is an easy exercise (left to the reader) to show that exactly one of the conditions \((A_0) – (A_3)\) is satisfied for each \( X \) and \( \{p_n\} \) and that each of the conditions \((A_0) – (A_3)\) does not depend on the choice of an increasing sequence \( \{p_n\}_{n \in \mathbb{N}} \) of seminorms defining the topology of \( X \in \mathcal{M} \). Thus, conditions \((A_0) – (A_3)\) split the class \( \mathcal{M} \) into four disjoint subclasses \( \mathcal{M}_j \) with \( 0 \leq j \leq 3 \), where \( X \in \mathcal{M}_j \) if for some \((=\text{for each})\) increasing sequence \( \{p_n\}_{n \in \mathbb{N}} \) of seminorms on \( X \) defining the topology of \( X \), the condition \((A_j)\) is satisfied.

**Remark 1.6.** It is easy to show that the following alternative way of defining the classes \( \mathcal{M}_j \) holds. Namely, let \( X \in \mathcal{M} \). Then

- \( X \in \mathcal{M}_0 \) if and only if the topology of \( X \) is weak;
- \( X \in \mathcal{M}_2 \) if and only if \( X \) possesses a continuous norm;
- \( X \in \mathcal{M}_3 \) if and only if \( X \) possesses a continuous norm and both \( Y \) and \( X/Y \) are infinite dimensional;
- \( X \in \mathcal{M}_3 \) if and only if the topology of \( X \) can be defined by an increasing sequence \( \{p_n\}_{n \in \mathbb{N}} \) of seminorms such that \( \ker p_n/\ker p_{n+1} \) is infinite dimensional for every \( n \in \mathbb{N} \).

**Theorem 1.7.** Let \( E \) be a metrizable locally convex space of countable algebraic dimension. Then

\[
E \text{ is orbital } \iff E \text{ is strongly orbital } \iff E \notin \mathcal{M}_2.
\]

The above theorem characterizes (strongly) orbital metrizable countably dimensional locally convex topological vector spaces. Proposition \[\text{[14]}\] indicates that completeness is an essential difficulty in constructing operators with no non-trivial invariant subspaces. Countable algebraic dimension is often perceived as almost incompatible with completeness. Basically, there is only one complete topological vector space of countable dimension, most analysts are aware of. Namely, the locally convex direct sum \( \varphi \) (see \[\text{[21]}\]) of countably many copies of the one-dimensional space \( \mathbb{K} \) has countable dimension and is complete. In other words, \( \varphi \) is a vector space of countable dimension endowed with the topology defined by the family of all seminorms. It is easy to see that every linear subspace of \( \varphi \) is closed, which leads to the following observation. It features (many times) in the literature, see, for instance, \[\text{[16]}\]. We present the proof for the sake of completeness.

**Proposition 1.8.** The space \( \varphi \) does not support a cyclic operator with dense range (and therefore has the invariant subspace property).

**Proof.** Assume that \( T \in L(\varphi) \) has dense range and \( x \in \varphi \). We have to show that \( x \) is not a cyclic vector for \( T \). If \( T^n x \) for \( n \in \mathbb{Z}_+ \) are not linearly independent, then \( \text{span}(O(T,x)) \) is finite dimensional and therefore \( x \) is not a cyclic vector for \( T \). If \( T^n x \) for \( n \in \mathbb{Z}_+ \) are linearly independent, then \( E = \text{span}(O(T,x)) \neq T(E) = \text{span}(O(T,Tx)) \). Indeed, in this case \( T(E) \) is a hyperplane in \( E \). Hence \( T(E) \neq \varphi \). Since every linear subspace of \( \varphi \) is closed, \( T(E) \) is not dense in \( \varphi \). Since \( T \) has dense range, \( E \) is not dense in \( \varphi \). Since \( E = \text{span}(O(T,x)) \), \( x \) is not a cyclic vector for \( T \).

**Remark 1.9.** The space \( \varphi \) is orbital. If one is after an analogue of Lemma \[\text{[12]}\] for the invariant subspace property, a modification of orbitality is needed. Indeed, one can show that if \( X \) is a countably dimensional locally convex space \( x \in X \), \( T \in L(X) \), \( X = \text{span}(O(T,x)) \) and the point spectrum of the dual operator \( T' \) (we have to take the complexification of \( T' \) in the case \( \mathbb{K} = \mathbb{R} \)) is empty, then \( X \) does not have the invariant subspace property. More specifically, \( T \) does not have non-trivial invariant subspaces. The proof goes along the same lines as in Lemma \[\text{[12]}\].
In the proof of Proposition 1.3 we have used the fact (due to Grivaux [15]) that isomorphisms on a separable infinite dimensional Banach space act transitively on the set of dense countable linearly independent sets. This property for more general spaces is studied in [23]. In this paper we are interested in the following corollary of this result. Namely, it immediately follows that $GL(X)$ acts transitively on the set $\mathcal{E}(X)$ of dense countably dimensional subspaces of a separable Banach space $X$. Equivalently, countably dimensional normed spaces are isomorphic as topological vector spaces precisely when their completions are isomorphic. We extend this result to Fréchet spaces. The answer turns out to be rather unexpected. Again, we split the class $\mathcal{F}$ of separable infinite dimensional Fréchet spaces into four subclasses

$$\mathcal{F}_j = \mathcal{F} \cap \mathcal{M}_j \text{ for } 0 \leq j \leq 3.$$ 

Using the well-known facts that every infinite dimensional Fréchet space, whose topology is weak, is isomorphic to $\omega$ and that every isomorphic to $\omega$ subspace of a Fréchet space is complemented, we see that Remark 1.9 implies the following fact.

**Remark 1.10.** Let $X \in \mathcal{F}$. Then $X \in \mathcal{F}_0$ if and only if $X$ is isomorphic to $\omega$ and $X \in \mathcal{F}_2$ if and only if $X$ is isomorphic to $Y \times \omega$, where $Y$ is infinite dimensional and possesses a continuous norm.

The following result provides the number of isomorphism classes of dense countably dimensional linear subspaces of a Fréchet space. From now on, for a topological vector space $X$,

$$\mathcal{E}(X)$$

the set of dense countably dimensional linear subspaces of $X$.

**Theorem 1.11.** Let $X \in \mathcal{F}$. Then

1. $X \in \mathcal{F}_0 \iff \mathcal{E}(X) \cap \mathcal{M}_0 \neq \emptyset \iff \mathcal{E}(X) \subseteq \mathcal{M}_0$. Furthermore, if $X \in \mathcal{F}_0$, then every $F, G \in \mathcal{E}(X)$ are isomorphic (as topological vector spaces);
2. $X \notin \mathcal{F}_0 \iff \mathcal{E}(X) \cap \mathcal{M}_1 \neq \emptyset$ and every $E, F \in \mathcal{E}(X) \cap \mathcal{M}_1$ are isomorphic. Furthermore, $X \in \mathcal{F}_1 \iff \mathcal{E}(X) \subseteq \mathcal{M}_1$;
3. $X \in \mathcal{F}_2 \cup \mathcal{F}_3 \iff \mathcal{E}(X) \cap \mathcal{M}_2 \neq \emptyset$. If $X \in \mathcal{F}_3$, then every $E, F \in \mathcal{E}(X) \cap \mathcal{M}_2$ are isomorphic. If $X \in \mathcal{F}_2$, then $\mathcal{E}(X) \cap \mathcal{M}_2$ splits into (exactly) two isomorphism classes;
4. $X \in \mathcal{F}_3 \iff \mathcal{E}(X) \cap \mathcal{M}_3 \neq \emptyset$. If $X \in \mathcal{F}_3$, then the number of isomorphism classes in $\mathcal{E}(X) \cap \mathcal{M}_3$ is uncountable.

Note that isomorphism classes in $\mathcal{E}(X)$ are in the natural one-to-one correspondence with the orbits in $\mathcal{E}(X)$ under the natural action of $GL(X)$. For instance, (1.11.2) implies that $GL(X)$ acts transitively on $\mathcal{E}(X) \cap \mathcal{M}_1$. The above theorem will be proved in subsequent sections. Right now, we derive the following corollary.

**Corollary 1.12.** For $X \in \mathcal{F}$ let $\nu(X)$ be the number of isomorphism classes of spaces in $\mathcal{E}(X)$. Then $\nu(X) = 1$ if $X \in \mathcal{F}_0 \cup \mathcal{F}_1$, $\nu(X) = 3$ if $X \in \mathcal{F}_2$ and $\nu(X)$ is uncountable if $x \in \mathcal{F}_3$.

**Proof.** We use Theorem 1.11 From (1.11.1) it follows that $\nu(X) = 1$ if $X \in \mathcal{F}_0$. According to (1.11.2), $\nu(X) = 1$ if $X \in \mathcal{F}_1$. By (1.11.2) and (1.11.3), $\nu(X) = 3$ if $X \in \mathcal{F}_2$. Indeed, in the notation of Theorem 1.11, $\mathcal{E} = (\mathcal{E} \cap \mathcal{M}_1) \cup (\mathcal{E} \cap \mathcal{M}_2)$ with $\mathcal{E} \cap \mathcal{M}_1$ providing one isomorphism class and $\mathcal{E} \cap \mathcal{M}_2$ consisting of two isomorphism classes. Finally, (1.11.4) implies that $\nu(X)$ is uncountable if $X \in \mathcal{F}_3$. \qed

**Remark 1.13.** Theorem 1.11 completes nicely the main result of [23], which says that for $X \in \mathcal{F}$, $GL(X)$ acts transitively on the set of dense countable linearly independent subsets of $X$ if and only if $X \in \mathcal{F}_1$. Note also that $\nu(X)$ (defined as in Corollary 1.12) does not exceed $2^{\aleph_0}$ for every $X \in \mathcal{F}$. We conjecture that $\nu(X) = 2^{\aleph_0}$ for each $X \in \mathcal{F}_3$.

Contrary to the common perception, there is an abundance of complete topological vector spaces of countable dimension.

**Theorem 1.14.** There is a complete locally convex topological vector space $X$ of countable algebraic dimension such that $X$ does not have the invariant subspace property.
Thus, $D_p x f$ span of $A$ disk

Recall that a subset $D$ of a locally convex space $X$ is called a disk if $D$ is bounded, convex and balanced (=is stable under multiplication by any $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$). The symbol $X_D$ stands for the space $\text{span}(D)$ endowed with the norm $p_D$ being the Minkowski functional of the set $D$. Boundeness of $D$ implies that the topology of $X_D$ is stronger than the one inherited from $X$. A disk $D$ in $X$ is called a Banach disk if the normed space $X_D$ is complete. It is well-known that a sequentially complete disk is a Banach disk, see, for instance, [9]. In particular, a compact or sequentially compact disk is a Banach disk.

The following result features as Lemma 4.1 in [23].

**Lemma 2.1.** Let $\{x_n\}_{n \in \mathbb{Z}_+}$ be a convergent to 0 sequence in a sequentially complete locally convex space $X$. Then the set $D = \left\{ \sum_{n=0}^{\infty} a_n x_n : a \in \ell_1, \|a\|_1 \leq 1 \right\}$ is a Banach disk. Moreover, $E = \text{span}\{x_n : n \in \mathbb{Z}_+\}$ is a dense linear subspace of the Banach space $X_D$.

We say that a continuous seminorm $p$ on a locally convex space $X$ is non-trivial if

$$\text{ker } p = \{ x \in X : p(x) = 0 \}$$

has infinite codimension in $X$. If $p$ is a continuous seminorm on a locally convex space $X$, we say that $A \subset X$ is $p$-independent if $p(z_1 a_1 + \ldots + z_n a_n) \neq 0$ for any $n \in \mathbb{N}$, any pairwise different $a_1, \ldots, a_n \in A$ and any non-zero $z_1, \ldots, z_n \in \mathbb{K}$. In other words, vectors $x + \text{ker } p$ for $x \in A$ are linearly independent in $X/\text{ker } p$.

**Lemma 2.2.** Let $X$ be a Fréchet space, $\{X_j\}_{j \in \mathbb{N}}$ be a decreasing sequence of closed linear subspaces of $X$ and for each $j \in \mathbb{N}$, let $A_j$ and $B_j$ be dense countable subsets of $X_j$. Then there is a Banach disk $D$ in $X$ such that for every $j \in \mathbb{N}$, both $A_j$ and $B_j$ are dense subsets of the Banach space $(X_D \cap X_j, p_D)$.

**Proof.** Let $d$ be a translation invariant metric defining the topology of $X$. For each $j \in \mathbb{N}$, let $C_j$ be the set of all linear combinations of the elements of $A_j \cup B_j$ with rational coefficients. Obviously, $C_j$ is countable. Pick a map $f_j : \mathbb{N} \to C_j$ such that $f_j^{-1}(x)$ is an infinite subset of $\mathbb{N}$ for every $x \in C$. Since $A_j$ and $B_j$ are dense in $X_j$, we can find maps $\alpha_j : \mathbb{N} \to A_j$ and $\beta_j : \mathbb{N} \to B_j$ such that $d(2^m(f_j(m) - \alpha_j(m)), 0) < 2^{-j-m}$ and $d(2^m(f_j(m) - \beta_j(m)), 0) < 2^{-j-m}$ for every $m \in \mathbb{N}$. Since $A_j$ and $B_j$ are countable, we can write $A_j = \{x_{j,m} : m \in \mathbb{N}\}$ and $B_j = \{y_{j,m} : m \in \mathbb{N}\}$. Using metrizability of $X$, we can find a sequence $\{\gamma_{j,m}\}_{m \in \mathbb{N}}$ of positive numbers such that $d(\gamma_{j,m} x_{j,m}, 0) < 2^{-j-m}$ and $d(\gamma_{j,m} y_{j,m}, 0) < 2^{-j-m}$ for every $m \in \mathbb{N}$. Enumerating the countable set

$$\bigcup_{j,m \in \mathbb{N}} \{2^m(f_j(m) - \alpha_j(m)), 2^m(f_j(m) - \beta_j(m)), \gamma_{j,m} x_{j,m}, \gamma_{j,m} y_{j,m}\}$$

as one (convergent to 0) sequence and applying Lemma 2.1 to this sequence, we find that there is a Banach disk $D$ in $X$ such that $X_D$ contains $A_j$ and $B_j$, the linear span of $A_j \cup B_j$ is $p_D$-dense in $X_D \cap X_j$ and $f_j(m) - \alpha_j(m) \to 0$ and $f_j(m) - \beta_j(m) \to 0$ as $m \to \infty$ in $X_D$ for each $j < n$. The $p_D$-density of the linear span of $A_j \cup B_j$ in $X_D \cap X_j$ implies the $p_D$-density of $C_j$ in $X_D \cap X_j$. Taking into account that $f_j^{-1}(x)$ is infinite for every $x \in C_j$ and that $\alpha_j$ takes values in $A_j$, $p_D$-density of $C_j$ in $X_D \cap X_j$ and the relation $p_D(f_j(m) - \alpha_j(m)) \to 0$ imply that $A_j$ is $p_D$-dense in $X_D \cap X_j$. Similarly, $B_j$ is $p_D$-dense in $X_D \cap X_j$. Thus $D$ satisfies all required conditions. \[\square\]
Applying Lemma 2.2 with \(X_j = X, A_j = A\) and \(B_j = B\) for every \(j \in \mathbb{N}\), we obtain the following result.

**Lemma 2.3.** Let \(A\) and \(B\) be dense countable subsets of a Fréchet space \(X\). Then there is a Banach disk \(D\) in \(X\) such that both \(A\) and \(B\) are dense subsets of the Banach space \(X_D\).

Applying the above lemma with \(A = B\), we obtain the following result.

**Corollary 2.4.** Let \(A\) be a dense countable subset of a Fréchet space \(X\). Then there is a Banach disk \(D\) in \(X\) such that \(A\) is a dense subset of the Banach space \(X_D\).

If \(p\) is a continuous seminorm on a locally convex space \(X\) and \(f \in X'\), we denote
\[
p^*(f) = \sup \{|f(x)| : x \in X, p(x) < 1\}.
\]
Clearly \(X'_p = \{f \in X' : p^*(f) < \infty\}\) is a linear subspace of \(X'\) and \(p^*\) is a norm on \(X'_p\). It is easy to see that \((X'_p, p^*)\) is a Banach space. The following result is Lemma 5.1 in [23].

**Lemma 2.5.** Let \(p\) be a continuous seminorm on a locally convex space \(X\) and \(E\) be a countably dimensional subspace of \(X\) such that \(E \cap \ker p = \{0\}\). Then there exist a Hamel basis \(\{u_n\}_{n \in \mathbb{N}}\) in \(E\) and a sequence \(\{f_n\}_{n \in \mathbb{N}}\) in \(X'_p\) such that \(f_n(u_m) = \delta_{n,m}\) for every \(m, n \in \mathbb{N}\).

The following result features as Lemma 4.3 in [23].

**Lemma 2.6.** Let \(X\) be a separable Fréchet space and \(p\) be a non-trivial continuous seminorm on \(X\). Then for every dense countable set \(A \subseteq X\), there is \(B \subseteq A\) such that \(B\) is \(p\)-independent and dense in \(X\).

**Corollary 2.7.** Let \(X\) be a separable Fréchet space and \(p\) be a non-trivial continuous seminorm on \(X\). Then there is a dense countably dimensional subspace \(E\) of \(X\) such that \(E \cap \ker p = \{0\}\).

**Proof.** Since \(X\) is separable, there is a dense countable subset \(A\) of \(X\). By Lemma 2.6, there is a dense in \(X\) \(p\)-independent subset \(B\) of \(A\). Clearly \(E = \text{span}(B)\) has all desired properties.

**Lemma 2.8.** Let \(X\) be a Fréchet space \(\{x_n\}_{n \in \mathbb{N}}\) and \(\{f_n\}_{n \in \mathbb{N}}\) be sequences in \(X\) and \(X'\) respectively and \(\{p_n\}_{n \in \mathbb{N}}\) be a sequence of seminorms on \(X\) defining the topology of \(X\). If for every \(k \in \mathbb{N}\),
\[
c_k = \sum_{n=1}^{\infty} \hat{p}_k(f_n, x_n) < \infty,
\]
then the formula
\[
Tx = \sum_{n=1}^{\infty} f_n(x) x_n
\]
defines a continuous linear operator on \(X\). Furthermore, if \(c_k < 1\) for every \(k \in \mathbb{N}\), then the operator \(I + T\) is invertible.

**Proof.** It is easy to see that for every \(x \in X\) and \(k, n \in \mathbb{N}\), \(p_k(f_n(x)x_n) \leq p_k(x)\hat{p}_k(f_n, x_n)\). It follows that the series in the last display converges absolutely in \(X\) and \(p_k(Tx) \leq c_k p_k(x)\) for every \(x \in X\) and every \(k \in \mathbb{N}\). Hence \(T\) is a well-defined continuous linear operator on \(X\).

Assume now that \(c_k < 1\) for every \(k \in \mathbb{N}\). Define \(S : X \to X\) by the formula
\[
S = \sum_{n=0}^{\infty} (-T)^n.
\]
Since \(p_k(Tx) \leq c_k p_k(x)\) for every \(x \in X\) and every \(k \in \mathbb{N}\), we have \(p_k(T^n x) \leq c_k^n p_k(x)\) for every \(x \in X\) and every \(k, n \in \mathbb{N}\). Since \(c_k < 1\), the series of operators in the above display converges pointwise. By the uniform boundedness principle, \(S\) is a continuous linear operator. It is a routine exercise to check that \(S(I + T) = (I + T)S = I\). That is, \(I + T\) is invertible.
Lemma 2.9. Let $X$ be a separable Fréchet space, whose topology is given by an increasing sequence $\{p_n\}_{n \in \mathbb{N}}$ of seminorms and let $\{X_n\}_{n \in \mathbb{Z}_+}$ be a sequence of closed linear subspaces of $X$ such that $X_{n+1} \subseteq X_n \cap \ker p_{n+1}$ and $X_n/(X_n \cap \ker p_{n+1})$ is infinite dimensional for each $n \in \mathbb{Z}_+$. Then there exist $\{u_{n,k} : n, k \in \mathbb{Z}_+\} \subset X$ and $\{f_{n,k} : n, k \in \mathbb{Z}_+\} \subset X'$ such that

\[
\text{span}\{u_{m,k} : m \geq n, \ k \in \mathbb{Z}_+\} = X_n \text{ for every } n \in \mathbb{Z}_+; \tag{2.1}
\]

for every $n \in \mathbb{Z}_+$, $u_{n,k}$ for $k \in \mathbb{Z}_+$ are $p_{n+1}$-independent;

\[
p^*_n(f_{n,k}) < \infty \text{ for every } n \in \mathbb{N} \text{ and } k \in \mathbb{Z}_+; \tag{2.2}
\]

\[
f_{n,k}(u_{m,j}) = \delta_{n,m}\delta_{k,j} \text{ for every } n, k, m, j \in \mathbb{Z}_+. \tag{2.3}
\]

Proof. Since $X_n/(X_n \cap \ker p_{n+1})$ is infinite dimensional, $p_{n+1}$ is a non-trivial seminorm on the separable Fréchet space $X_n$. By Lemma 2.6 there is a $p_{n+1}$-independent dense sequence $\{v_{n,k}\}_{k \in \mathbb{Z}_+}$ in $X_n$. Fix a bijection $\alpha = (\alpha_1, \alpha_2) : \mathbb{N} \to \mathbb{Z}_+^2$. We shall construct inductively $w_j \in X$ and $g_j \in X'$ for $j \in \mathbb{N}$ such that for every $k \in \mathbb{N},$

\[
w_k \in v_{\alpha_1(1),\alpha_2(1)}(1) + \text{span}\{v_{\alpha_1(j),\alpha_2(j)} : j < k, \ \alpha_1(j) \geq \alpha_1(k)\}; \tag{2.5}
\]

\[
p^*_n(\alpha_1(j)+1)(g_k) < \infty; \tag{2.6}
\]

\[
g_k(w_k) = 1; \tag{2.7}
\]

\[
g_k(w_j) = 0 \text{ if } j < k; \tag{2.8}
\]

\[
g_j(w_k) = 0 \text{ if } j < k. \tag{2.9}
\]

We start by setting $w_1 = v_{\alpha_1(1),\alpha_2(1)}$ and observing that $w_1 \notin \ker p_{n+1}$, where $n = \alpha_1(1)$. Indeed, this follows from $p_{n+1}$-independence of $\{v_{n,k}\}_{k \in \mathbb{Z}_+}$. By the Hahn–Banach theorem, there is $g_1 \in X'$ such that $p^*_n(g_1) < \infty$ and $g_1(w_1) = 1$. Obviously, (2.5) with $k = 1$ are satisfied. Thus we have our basis of induction.

Assume now that $m \geq 2$ and $w_j \in X$, $g_j \in X'$ for $j < m$ satisfy (2.5) for every $k \leq m$. We have to construct $w_m$ and $g_m$ such that (2.9) hold for $k = m$. First, we define

\[
w_m = v_{\alpha_1(m),\alpha_2(m)} - \sum_{j=1}^{m-1} g_j(v_{\alpha_1(m),\alpha_2(m)})w_j. \tag{2.10}
\]

Let $n = \alpha_1(m)$ and $j < m$. By (2.6) with $k = j$, $p^*_n(\alpha_1(j)+1)(g_j) < \infty$ and therefore $g_j$ vanishes on $\ker p_{\alpha_1(j)+1}(m) \supseteq X_{\alpha_1(j)+1}$. Since $v_{\alpha_1(m),\alpha_2(m)} \in X_n$, we see that $g_j(v_{\alpha_1(m),\alpha_2(m)}) = 0$ if $\alpha_1(j) < n = \alpha_1(m)$. Thus (2.5) for $k < m$ and (2.10) imply (2.5) for $k = m$. The latter together with $p_{j+1}$-independence of $\{v_{j,k}\}_{k \in \mathbb{Z}_+}$ ensures that the vectors $w_k$ for $s \leq m$, $\alpha_1(s) \leq n$ are $p_{n+1}$-independent. By the Hahn–Banach theorem, there exists $g_m \in X'$ such that $p^*_n(g_m) < \infty$, $g_m(w_m) = 1$ and $g_m(w_j) = 1$ whenever $j < m$ and $\alpha_1(j) \leq n$. If $j < m$ and $\alpha_1(j) > n$, then $w_j \in X_{\alpha_1(j)} \subseteq X_{n+1} \subseteq \ker p_{n+1}$ and therefore $g_m(w_j) = 0$ since $g_m$ vanishes on $\ker p_{n+1}$. Thus (2.6), (2.7) and (2.8) with $k = m$ are satisfied. Finally, using (2.10) and (2.7) with $k = m$, one can easily verify that (2.9) for $k = m$ is also satisfied. This completes the inductive construction of the sequences $\{w_n\}$ and $\{g_n\}$ satisfying (2.5) for every $k \leq m$.

Now we set $u_{\alpha_1(j),\alpha_2(j)} = w_j$ and $f_{\alpha_1(j),\alpha_2(j)} = g_j$. Clearly, (2.6) implies (2.3), while (2.7) implies (2.4). Next, (2.2) follows from $p_{j+1}$-independence of $\{v_{j,k}\}_{k \in \mathbb{Z}_+}$ and (2.5). Finally, (2.5) ensures that $\text{span}\{u_{m,k} : m \geq n, \ k \in \mathbb{Z}_+\} = \text{span}\{v_{m,k} : m \geq n, \ k \in \mathbb{Z}_+\}$.

We also need the following lemma, which helps to verify the completeness of locally convex spaces. It is a variation of a pretty standard fact, see the completeness chapter in [22]. We include its proof for the sake of convenience.

Lemma 2.10. Let $(E, \tau)$ be a complete locally convex space, $F$ be a linear subspace of $E$ and $Q$ be a collection of seminorms on $F$ defining a locally convex topology $\theta$ on $F$ stronger than the one inherited from $E$ : $\theta \supseteq \tau|_F$. Assume also that for every $q \in Q$, the ball $B_q = \{x \in F : q(x) \leq 1\}$ is $\tau|_F$-closed in $F$ and that for every $\theta$-Cauchy net $\{x_\alpha\}$ in $F$ its $\tau$-limit belongs to $F$ : $\lim_\tau x_\alpha \in F$. Then $(F, \theta)$ is complete.
Lemma 2.11. Let \( n \) be an element of \( X \) defining the topology of \( X \) for every \( t \). Fix temporarily \( \beta \in D \) such that \( \beta > \alpha_0 \). Then we can write
\[
\{ x_\alpha : \alpha > \alpha_0 \} \subseteq W = \{ x \in F : q(x - x_\beta) \leq \varepsilon \}.
\]
Since \( B_q \) is \( \tau \)-closed in \( F \), \( W \) is also \( \tau \)-closed in \( F \). Since \( \{ x_\alpha \} \) is \( \tau \)-convergent to \( 0 \), from the above display it follows that \( 0 \in W \). Hence \( q(x_\beta) \leq \varepsilon \). Since \( \beta \in D \) is an arbitrary element such that \( \beta > \alpha_0 \), we have \( q(x_\beta) \leq \varepsilon \) for every \( \beta > \alpha_0 \). Hence \( q(x_\alpha) \to 0 \). Since \( q \in Q \) is arbitrary, \( \lim_0 x_\alpha = 0 \), as required. \( \square \)

2.1 An obstacle to embedding of Fréchet spaces

In the proof of Theorem 1.11, we will need to establish that a certain Fréchet space is non-isomorphic to a subspace of another given Fréchet space. Our approach is pretty standard. If \( p \) is a seminorm on a vector space \( X \), \( q \) is a seminorm on a vector space \( Y \), symbol \( L_{p,q}(X,Y) \) stands for the space of linear maps \( T : X \to Y \) for which there is \( c > 0 \) such that \( q(Tx) \leq cp(x) \) for every \( x \in X \). The minimal possible \( c \) with this property will be denoted \( \pi_{p,q}(T) \). For \( n \in \mathbb{Z}_+ \) and \( T \in L_{p,q}(X,Y) \), we denote
\[
\alpha_n(p,q,T) = \inf \{ \pi_{p,q}(T-S) : S \in L_{p,q}(X,Y), \dim S(X) \leq n \}.
\]
The numbers \( \alpha_n(p,q,T) \) are usually called approximation numbers. Clearly, they form a decreasing sequence of non-negative numbers.

In this section, the symbol \( \Omega \) stands for the set of decreasing sequences \( r = \{ r_n \}_{n \in \mathbb{Z}_+} \) of positive numbers such that \( \{nr_n\} \) is bounded. For \( r \in \Omega \), the symbol \( \mathcal{F}_r \) stands for the class of \( X \in \mathcal{F} \) such that for every continuous seminorm \( p \) on \( X \) and \( j \in \mathbb{N} \), there is a continuous seminorm \( q \) on \( X \) satisfying \( q \geq p \) and
\[
\lim_{n \to \infty} \frac{n^2 \alpha_n(q,p,\text{Id}_X)}{r_n} = 0.
\]
It is an easy exercise to see that \( \mathcal{F}_r \) is a subclass of the class \( \mathcal{N} \) of nuclear Fréchet spaces and \( \mathcal{F}_r = \mathcal{N} \) if \( n^{-k} = O(r_n) \) for some \( k \in \mathbb{N} \). It is easy (see [17]) to verify that if \( Z \) is a subspace of \( Y \) and \( T \in L_{p,q}(X,Y) \) satisfies \( T(X) \subseteq Z \), then \( \alpha_n(p,q,T) \leq \alpha_n(p,q,T) \leq (n+1)\alpha_n(p,q,T) \), where \( \overline{T} \) denotes \( T \) considered as an element of \( L_{p,q}(X,Z) \). It follows that each of the classes \( \mathcal{F}_r \) is closed under passing to a closed linear subspace. We will not need this, but it is worth noting that each \( \mathcal{F}_r \) is also closed under quotients.

Lemma 2.11. Let \( \{ X_n \}_{n \in \mathbb{N}} \) be a sequence in \( \mathcal{F} \setminus \mathcal{F}_0 \). Then there is \( r \in \Omega \) such that \( X_n \notin \mathcal{F}_r \) for every \( n \in \mathbb{N} \).

Proof. Since \( X_n \) is non-isomorphic to \( \omega \), there is an increasing sequence \( \{ p_{n,m} \}_{m \in \mathbb{N}} \) of seminorms on \( X_n \) defining the topology of \( X_n \) such that \( X_n/\ker p_{n,1} \) is infinite dimensional. Then \( \alpha_t(p_{n,k},p_{n,m},\text{Id}_{X_n}) > 0 \) for every \( t \in \mathbb{Z}_+ \) and \( k,m,n \in \mathbb{N} \) satisfying \( k < m \). Now we can pick \( r \in \Omega \) such that \( nr_n \to 0 \) and
\[
\lim_{t \to \infty} \frac{r_t}{\alpha_t(p_{n,m},p_{n,k},\text{Id}_{X_n})} = 0 \quad \text{whenever } k,m,n \in \mathbb{N} \text{ and } k < m.
\]
Since \( \{ p_{n,m} \}_{m \in \mathbb{N}} \) defines the topology of \( X_n \), we have \( X_n \notin \mathcal{F}_r \). \( \square \)

Lemma 2.12. Let \( r \in \Omega \), \( X \in \mathcal{F} \) and \( \{ p_n \}_{n \in \mathbb{N}} \) be an increasing sequence of seminorms defining the topology of \( X \) such that \( \ker p_n/\ker p_{n+1} \) is infinite dimensional for each \( n \in \mathbb{N} \). Then there exists a closed linear subspace \( Y \) of \( \ker p_1 \) such that \( Y \in \mathcal{F}_r \) and \( Y \cap \ker p_n \) is infinite dimensional for every \( n \in \mathbb{N} \).
Proof. Denote $X_n = \ker p_n$ for $n \in \mathbb{N}$ and $X_0 = X$. By Lemma 2.3 there exist \{\(u_{n,k} : n, k \in \mathbb{Z}_+\) \}\subset X and \{\(f_{n,k} : n, k \in \mathbb{Z}_+\) \}\subset X’ such that (2.1,2.2) are satisfied. By Corollary 2.4 there exists a Banach disk \(D\) in \(X\) such that \(u_{n,k} \in X_D\) for every \(n, k \in \mathbb{Z}_+\). For each \(n \in \mathbb{N}\), let \(q_n\) be the Minkowski functional of the set \(D + \{x \in X : p_n(x) \leq 1\}\). It is easy to see that each \(q_n\) is equivalent to \(p_n\) and therefore \{\(q_n\)\} is an increasing sequence of seminorms defining the topology of \(X\). It also has an extra convenient property: \(q_n(x) \leq p_D(x)\) for every \(x \in X\). Let \(\beta : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}\) be a bijection. Now for \(A \subseteq \mathbb{N} \times \mathbb{N}\), we can define a linear map \(S_A : X \rightarrow X\) by the formula

\[
S_Ax = \sum_{(n,k) \in A} \frac{r_{\beta(n,k)} f_{n+1,2k}(x)}{2^{\beta(n,k)} q_{n+2}^*(f_{n+1,2k}) p_D(u_{n,k})} u_{n,k}.
\]

We also set \(S = S_{\mathbb{N} \times \mathbb{N}}\). First, we shall verify that each \(S_A\) is a well-defined continuous linear operator. To this end, consider the linear map \(S_{A,n} : X \rightarrow X\) given by

\[
S_{A,n}x = \sum_{k:(n,k) \in A} \frac{r_{\beta(n,k)} f_{n+1,2k}(x)}{2^{\beta(n,k)} q_{n+2}^*(f_{n+1,2k}) p_D(u_{n,k})} u_{n,k}.
\]

Since for each \(x \in X\), \(|f_{n+1,2k}(x)| \leq q_{n+2}(x) q_{n+2}^*(f_{n+1,2k})\). Plugging this estimate into the above display, we immediately see that the above series converges absolutely in the Banach space \(X_D\) and therefore in \(X\). It follows that \(S_{A,n}\) is well-defined. Furthermore

\[
p_D(S_{A,n}x) \leq q_{n+2}(x) \sum_{k:(n,k) \in A} \frac{r_{\beta(n,k)} f_{n+1,2k}(x)}{2^{\beta(n,k)} q_{n+2}^*(f_{n+1,2k}) p_D(u_{n,k})} \leq 2\cdot 2^{-j(A,n)} r_{j(A,n)} q_{n+2}(x),
\]

where \(j(A,n) = \min\{s \in \mathbb{N} : \beta^{-1}(s) \in A \cap \{n\} \times \mathbb{Z}_+\}\). It immediately follows that \(S_{A,n}\) are continuous. Furthermore, \(S_{A,n}(X) \subseteq X_n\). Since \(q_n\) vanishes on \(X_n\) it follows that the series \(S_Ax = \sum_{n=1}^{\infty} S_{A,n}x\) converges absolutely in the Fréchet space \(X\). Thus \(S_A\) is well defined. The continuity of each \(S_{A,n}\) and the uniform boundedness principle imply the continuity of \(S_A\). Moreover, for \(x \in X\) and \(m \in \mathbb{N}\), we can use the above display to see that

\[
q_m(S_Ax) = q_m\left(\sum_{n=0}^{\infty} S_{A,n}x\right) = q_m\left(\sum_{n=0}^{m-1} S_{A,n}x\right) \leq \sum_{n=0}^{m-1} q_m(S_{A,n}x) \leq \sum_{n=0}^{m-1} p_D(S_{A,n}x)
\]

\[
\leq 2\sum_{n=0}^{m-1} 2^{-j(A,n)} r_{j(A,n)} q_{n+2}(x) \leq 2q_{m+1}(x) \sum_{n=0}^{m-1} 2^{-j(A,n)} r_{j(A,n)} \leq 4q_{m+1}(x) 2^{-t(A)} r_{t(A)},
\]

where \(t(A) = \min\{s \in \mathbb{N} : \beta^{-1}(s) \in A\}\).

Observe that according to (2.1,2.2), \(S(X_{n+1}) \subseteq X_n\) for \(n > 1\) and \(S\) vanishes on \(X_1\). Now we define \(Y\) as the closure in \(X\) of

\[
G = \left\{\sum_{n=1}^{\infty} x_n : x_n \in X_n \text{ and } Sx_{n+1} = x_n \text{ for } n \in \mathbb{N}\right\}.
\]

Note that each series in the definition of \(G\) converges absolutely since \(x_n \in X_n\) and each \(q_k\) vanishes on \(X_n\) for \(s \geq k\). First, we shall show that \(G \cap X_n\) is infinite dimensional for every \(n \in \mathbb{N}\). Let \(n \in \mathbb{N}\) and \(m\) be an odd positive integer. Using (2.1,2.3) and the definition of \(S\) we see that \(Su_{n,m} = 0\) and for each \(j \in \mathbb{Z}_+\) there is \(c_j > 0\) such that \(c_0 = 1\) and \(S(c_{j+1} u_{n+j,2j+1/m}) = c_j u_{n+j,2j/m}\). Since \(u_{n,k} \in X_n\), it follows that \(w_{n,m} = \sum_{j=0}^{\infty} c_j u_{n+j,2j/m} \in G\). Using (2.1,2.3) once again, we see that \(f_{n,m}(w_{n,m'}) = \delta_{m,m'}\) whenever \(m\) and \(m'\) are odd positive integers. Thus \{\(w_{n,m} : m + 1 \in 2\mathbb{N}\)\} is an infinite linearly independent subset of \(G \cap X_n\). Since \(G \subseteq Y\), \(Y \cap X_n\) is infinite dimensional.
Now let $m \geq 2$ and $x = \sum_{n=1}^{\infty} x_n \in G$, where $x_n \in X_n$ and $Sx_{n+1} = x_n$ for $n \in \mathbb{N}$. Let $u = \sum_{n=1}^{\infty} x_n$. Using the equalities $Sx_{n+1} = x_n$ and $Sx_1 = 0$ as well as the inclusions $x_k \in X_k$, we obtain

$$x - u \in \ker p_{m+1} \quad \text{and} \quad x - Su \in \ker p_m.$$ 

It follows that up to moving along the kernels of the corresponding seminorms, $\Id_G$ coincides with $S|_G$ as elements of $L_{p_m+1,p_m}(G,G) = L_{q_m+1,q_m}(G,G)$. Since $G$ is dense in $Y$, we have

$$\alpha_k(q_{m+1},q_m,\Id_Y) = \alpha_k(q_{m+1},q_m,S|_Y) \quad \text{for each } k \in \mathbb{Z}_+.$$ 

Now for each $k \in \mathbb{Z}_+$, we can consider $A_k = \mathbb{N} \times \mathbb{N} \setminus \{(n,j) \in \mathbb{N} \times \mathbb{N} : \beta(n,j) \leq k\}$. Obviously $|\mathbb{N} \times \mathbb{N} \setminus A_k| = k$ and $t(A_k) = k + 1$. By \[ \text{q} \text{m}(S_{A_k}x) \leq 2^{1-k}r_{k+1}x_{k+1}(x) \text{ for every } x \in X \text{ (and therefore for every } x \in Y). \]

Since the linear operator $S - S_{A_k}$ has rank $k$, the above display yields

$$\alpha_k(q_{m+1},q_m,\Id_Y) = \alpha_k(q_{m+1},q_m,S|_Y) \leq 2^{1-k}r_{k+1}.$$ 

Since the sequence $\{q_m\}$ defines the topology of $Y$, $Y \in \mathcal{F}_{[r]}$. \qed

\section{A specific countable topological space}

We call a sequence trivial if it is eventually stabilizing.

\begin{lemma}
There exists a regular topology $\tau$ on $\mathbb{Z}$ such that the topological space $\mathbb{Z}_\tau = (\mathbb{Z}, \tau)$ has the following properties:

(a) $f : \mathbb{Z} \to \mathbb{Z}$, $f(n) = n + 1$ is a homeomorphism of $\mathbb{Z}_\tau$ onto itself;

(b) $\mathbb{Z}_+^* \text{ is dense in } \mathbb{Z}_\tau$;

(c) for every $z \in \mathbb{C} \setminus \{0,1\}$, $n \mapsto z^n$ is non-continuous as a map from $\mathbb{Z}_\tau$ to $\mathbb{C}$;

(d) $\mathbb{Z}_\tau$ has no non-trivial convergent sequences.
\end{lemma}

\begin{proof}
Consider the Hilbert space $\ell_2(\mathbb{Z})$ and the bilateral weighted shift $T \in \mathcal{L}(\ell_2(\mathbb{Z}))$ given by $Te_n = e_{n-1}$ if $n \leq 0$ and $Te_n = 2e_{n-1}$ if $n > 0$, where $\{e_n\}_{n \in \mathbb{Z}}$ is the canonical orthonormal basis in $\ell_2(\mathbb{Z})$. Symbol $H_\sigma$ stands for $\ell_2(\mathbb{Z})$ equipped with its weak topology $\sigma$. Clearly, $T \in GL(\ell_2(\mathbb{Z}))$ and therefore $T \in GL(H_\sigma)$. According to Chan and Sanders [12], there is $x \in \ell_2(\mathbb{Z})$ such that the set $O = \{T^n x : n \in \mathbb{Z}_+\}$ is dense in $H_\sigma$. Then $Y = \{T^n x : n \in \mathbb{Z}\}$ is also dense in $H_\sigma$. We equip $Y$ with the topology inherited from $H_\sigma$ and transfer it to $\mathbb{Z}$ by declaring the bijection $n \mapsto T^n x$ from $\mathbb{Z}$ to $Y$ a homeomorphism.

Since $\sigma$ is a completely regular topology, so is the just defined topology $\tau$ on $\mathbb{Z}$. Since $T \in GL(H_\sigma)$ and $Y$ is a subset of $\ell_2(\mathbb{Z})$ invariant for both $T$ and $T^{-1}$, $T$ is a homeomorphism from $Y$ onto itself. Since $T(T^n x) = T^{n+1} x$, it follows that $f$ is a homeomorphism of $\mathbb{Z}_\tau$ onto itself. Since $O$ is dense in $H_\sigma$ implies the density of $\mathbb{Z}_+^*$ in $\mathbb{Z}_\tau$.

Observe that the sequence $\{|T^n x|\}_{n \in \mathbb{Z}}$ is strictly increasing and $|T^n x| \to \infty$ as $n \to +\infty$. Indeed, the inequality $|Tu| \geq |u|$ for $u \in \ell_2(\mathbb{Z})$ follows from the definition of $T$. Hence $\{|T^n x|\}_{n \in \mathbb{Z}}$ is increasing. Assume that $|T^{n+1} x| = |T^n x|$ for some $n \in \mathbb{Z}$. Then, by definition of $T$, $T^n x$ belongs to the closed linear span $L$ of $e_n$ with $n < 0$. The latter is invariant for $T$ and therefore $T^n x \in L$ for $m \geq n$, which is incompatible with the $\sigma$-density of $O$. Next, if $|T^n x|$ does not tend to $\infty$ as $n \to +\infty$, the sequence $\{|T^n x|\}_{n \in \mathbb{Z}_+}$ is bounded. Since every bounded subset of $\ell_2(\mathbb{Z})$ is $\sigma$-nowhere dense, we have again obtained a contradiction with the $\sigma$-density of $O$.

In order to show that $X$ has no non-trivial convergent sequences, it suffices to show that $Y$ has no non-trivial convergent sequences. Assume that $\{T^n x\}_{k \in \mathbb{Z}_+}$ is a non-trivial convergent sequence in $Y$. Without loss of generality, we can assume that the sequence $\{n_k\}$ of integers is either strictly increasing or strictly decreasing. If $\{n_k\}$ is strictly increasing the above observation ensures that $|T^{n_k} x| \to \infty$ as $k \to \infty$. Since every $\sigma$-convergent sequence is bounded, we have arrived to a contradiction. If $\{n_k\}$ is strictly decreasing, then by the above observation, the sequence $\{|T^{n_k} x|\}$ of positive numbers is also strictly decreasing and therefore converges to $c \geq 0$. Then $|T^l x| > c$ for every $l \in \mathbb{Z}$. Since $\{T^{n_k} x\}$ $\sigma$-converges to $T^m x \in Y$, the
upper semicontinuity of the norm function with respect to \( \sigma \) implies that \( \|T^n x\| \leq c \) and we have arrived to a contradiction.

Finally, let \( z \in \mathbb{C} \setminus \{0, 1\} \) and \( f : \mathbb{Z} \to \mathbb{C} \), \( f(n) = z^n \). It remains to show that \( f \) is not continuous as a function on \( \mathbb{Z}_r \). Equivalently, it is enough to show that the function \( g : Y \to \mathbb{C} \), \( g(T^n x) = z^n \) is non-continuous. Assume the contrary. First, consider the case \( |z| \neq 1 \). In this case the topology on the set \( M = \{z^n : n \in \mathbb{Z}\} \) inherited from \( \mathbb{C} \) is the discrete topology. Continuity of the bijection \( g : Y \to M \) implies then that \( Y \) is also discrete, which is not the case: the density of \( Y \) in \( H_\sigma \) ensures that \( Y \) has no isolated points. It remains to consider the case \(|z| = 1, z \neq 1\). In this case the closure \( G \) of \( \{z^n : n \in \mathbb{Z}\} \) is a closed subgroup of the compact abelian topological group \( \mathbb{T} \). Since \( x \) is a hypercyclic vector for \( T \) acting on \( H_\sigma \) and \( z \) generates the compact abelian topological group \( G \), [24 Corollary 4.1] implies that \( \{(T^n x, z^n) : n \in \mathbb{Z}_+\} \) is dense in \( H_\sigma \times G \). Hence the graph of \( g \) is dense in \( Y \times G \). Since \( G \) is not a singleton, the latter is incompatible with the continuity of \( g \). The proof is complete. \( \Box \)

**Remark 2.14.** It is easy to see that the topology \( \tau \) on \( \mathbb{Z} \) constructed in the proof of Lemma 2.13 does not agree with the group structure. That is \( \mathbb{Z}_r \) is not a topological group. Indeed, it is easy to see that the group operation \( + \) is only separately continuous on \( \mathbb{Z}_r \), but not jointly continuous. As a matter of curiosity, it would be interesting to find out whether there exists a topology \( \tau \) on \( \mathbb{Z} \) satisfying all conditions of Lemma 2.13 and turning \( \mathbb{Z} \) into a topological group.

### 3. Isomorphisms of countably dimensional spaces in \( \mathcal{M} \)

The following theorem is Theorem 3.1 in [23].

**Theorem 3.1.** The group \( GL(\omega) \) acts transitively on the set \( \mathcal{E}(\omega) \) of dense countably dimensional subspaces of \( \omega \).

**Corollary 3.2.** Every two countably dimensional spaces in \( \mathcal{M}_0 \) are isomorphic.

**Proof.** Let \( X \) and \( Y \) be two countably dimensional spaces in \( \mathcal{M}_0 \). Since \( X \) and \( Y \) are metrizable and strong (or uniform) weak topologies, the completions of both \( X \) and \( Y \) are isomorphic to \( \omega \). Thus, without loss of generality, both \( X \) and \( Y \) are dense countably dimensional linear subspaces of \( \omega \). It remains to apply Theorem 3.1. \( \Box \)

**Lemma 3.3.** Let \( X \) be a Fréchet space isomorphic to \( \omega \), \( E \) and \( F \) be countably dimensional subspaces of \( X \) such that both \( X/E \) and \( X/F \) are infinite dimensional. Then there is \( T \in GL(X) \) such that \( T(E) = F \).

**Proof.** Since, up to an isomorphism, there is only one infinite dimensional Fréchet space whose topology is weak (it is \( \omega \)), each of the spaces \( E, F, \omega/E \) and \( \omega/F \) is isomorphic to \( \omega \). Since every closed linear subspace of \( \omega \) is complemented \([9]\), \( X = E \oplus G = F \oplus H \), where \( G \) and \( H \) are isomorphic to \( \omega \). Thus we can, without loss of generality, assume that \( X = \omega \times \omega \) and \( E \) and \( F \) are dense (countably dimensional) subspaces of \( \omega \times \{0\} \). By Theorem 3.1 there is \( R \in GL(\omega) \) such that \((R \oplus I)(E) = F\). Hence \( R \oplus I \) is a required isomorphism. \( \Box \)

#### 3.1 Isomorphisms of spaces in \( \mathcal{M}_1 \)

The following theorem, generalizing the main result of [15], is Theorem 1.5 in [23].

**Theorem 3.4.** Let \( X \) be a Fréchet space, \( p \) be a non-trivial continuous seminorm on \( X \), and \( A \) and \( B \) be two countable dense subsets of \( X \) such that both \( A \) and \( B \) are \( p \)-independent. Then there exists \( R \in GL(X) \) such that \( R(A) = B \) and \( Rx = x \) for every \( x \in ker p \).

**Corollary 3.5.** Let \( E, F \in \mathcal{M}_1 \) be countably dimensional. Then \( E \) and \( F \) are isomorphic if and only if their completions \( \overline{E} \) and \( \overline{F} \) are isomorphic.
Proof. If \( T : E \to F \) is an isomorphism, then the extension of \( T \) by continuity provides an isomorphism of \( \overline{E} \) and \( \overline{F} \). Conversely, assume that \( \overline{E} \) and \( \overline{F} \) are isomorphic. Then, without loss of generality, we can assume that both \( E \) and \( F \) are dense countably dimensional subspaces of the same Fréchet space \( X \). Since \( E, F \in \mathcal{M}_1 \), there exist continuous seminorms \( p_1 \) and \( p_2 \) on \( X \) such that \( E \cap \ker p_1 = F \cap \ker p_2 = \{0\} \). Hence \( E \cap \ker p = F \cap \ker p = \{0\} \), where \( p = p_1 + p_2 \). By Lemma 2.2, there are \( A_1 \subseteq E \) and \( B_1 \subseteq F \) such that both \( A_1 \) and \( B_1 \) are \( p \)-independent and dense in \( X \). Pick a dense Hamel basis \( A \) in \( E \) and a dense Hamel basis \( B \) in \( F \) such that \( A_1 \subseteq A \) and \( B_1 \subseteq B \). Since \( E \cap \ker p = F \cap \ker p = \{0\} \), \( A \) and \( B \) are \( p \)-independent. Moreover, \( A \) and \( B \) are dense in \( X \) and countable. By Theorem 3.4, there is \( R \in GL(X) \) such that \( R(A) = B \). Obviously, \( R(E) = F \). Hence \( R|_E : E \to F \) is an isomorphism. \( \Box \)

### 3.2 Isomorphisms of spaces in \( \mathcal{M}_3 \)

The following result features as Lemma 2.2 in [23].

**Lemma 3.6.** Let \( \varepsilon > 0 \), \( X \) be a locally convex space, \( D \) be a Banach disk in \( X \), \( Y \) a closed linear subspace of \( X \), \( \Lambda \subseteq Y \cap X_D \) be a dense subset of \( Y \) such that \( \Lambda \) is \( p_D \)-dense in \( Y \cap X_D \), \( p \) be a continuous seminorm on \( X \), \( L \) be a finite dimensional subspace of \( X \) and \( T \in L(X) \) be a finite rank operator such that \( T(Y) \subseteq Y \cap X_D \). Then

1. For every \( u \in Y \cap X_D \) such that \( (u + L) \cap \ker p = \emptyset \), there are \( f \in X' \) and \( v \in Y \cap X_D \) such that \( p^*(f) = 1 \), \( f|_L = 0 \), \( p_D(v) < \varepsilon \), \( (I + R)u \in \Lambda \) and \( \ker (I + R) = \{0\} \), where \( Rx = Tx + f(x)v \).

2. For every \( u \in Y \cap X_D \) such that \( (u + (I + T)(L)) \cap \ker p = \emptyset \), there are \( f \in X' \), \( a \in \Lambda \) and \( v \in Y \cap X_D \) such that \( p^*(f) = 1 \), \( f|_L = 0 \), \( p_D(v) < \varepsilon \), \( (I + R)a = u \) and \( \ker (I + R) = \{0\} \), where \( Rx = Tx + f(x)v \).

**Theorem 3.7.** Let \( X \) be a Fréchet space, whose topology is defined by an increasing sequence \( \{p_n\}_{n \in \mathbb{N}} \) of seminorms. Assume also that \( \{X_n\}_{n \in \mathbb{Z}_+} \) is a sequence of closed infinite dimensional linear subspaces of \( X \) such that \( X_0 = X \) and \( X_n \subseteq X_{n-1} \cap \ker p_n \) for every \( n \in \mathbb{N} \). For each \( n \in \mathbb{Z}_+ \) let \( A_n \) and \( B_n \) be \( p_{n+1} \)-independent countable dense subsets of \( X_n \). Then there exists \( R \in GL(X) \) such that \( R(A_n) = B_n \) for every \( n \in \mathbb{Z}_+ \).

**Proof.** By Lemma 2.2 we can find a Banach disk \( D \) in \( X \) such that for every \( j \in \mathbb{Z}_+ \), both \( A_j \) and \( B_j \) are dense subsets of the Banach space \( (X_D \cap X_j, p_D) \). Let \( q_n \) be the Minkowski functional of \( D \cup \{x \in X : p_n(x) \leq 1\} \). It is easy to see that each \( q_n \) is a seminorm equivalent to \( p_n \). Hence \( \{q_n\} \) is another increasing sequence of seminorms on \( X \) defining the topology of \( X \) and \( \ker p_n = \ker q_n \) for \( n \in \mathbb{N} \). In particular, \( X_n \subseteq X_{n-1} \cap \ker q_n \) for every \( n \in \mathbb{N} \) and \( A_n \) and \( B_n \) are \( q_{n+1} \)-independent countable dense subsets of \( X_n \) for \( n \in \mathbb{Z}_+ \). The point of introducing the seminorms \( q_n \) is their extra property:

\[
q_n(x) \leq p_D(x) \quad \text{for every } n \in \mathbb{N} \text{ and } x \in X_D.
\]

First, observe that the sets \( A_j \) are pairwise disjoint. Indeed, let \( j < l \), \( x \in A_j \) and \( y \in A_l \). Since \( A_l \subseteq X_l \subseteq \ker p_l \), \( p_l(y) = 0 \). Since \( A_j \) is \( p_{j+1} \)-independent \( p_l(x) \geq p_{j+1}(x) > 0 \). Thus \( x \neq y \) and therefore \( A_j \) are pairwise disjoint. Similarly, \( B_j \) are pairwise disjoint. Hence for \( A = \bigcup_{j=0}^\infty A_j \) and \( B = \bigcup_{j=0}^\infty B_j \), the maps

\[
\tau : A \to \mathbb{Z}_+, \quad \tau(a) = j \text{ if } a \in A_j \text{ and } \sigma : B \to \mathbb{Z}_+, \quad \sigma(b) = j \text{ if } b \in B_j
\]

are well-defined. Fix bijections \( \alpha : \mathbb{N} \to A \) and \( \beta : \mathbb{N} \to B \) and a sequence \( \{\varepsilon_k\}_{k \in \mathbb{N}} \) of positive numbers satisfying \( \sum_{k=1}^\infty \varepsilon_k < 1 \).

We shall construct inductively sequences \( \{s_j\}_{j \in \mathbb{N}} \) and \( \{m_j\}_{j \in \mathbb{N}} \) of natural numbers, \( \{v_j\}_{j \in \mathbb{N}} \) in \( X \),
\{f_j\}_{j \in \mathbb{N}} \text{ in } X' \text{ and } \{T_k\}_{k \in \mathbb{Z}_+} \text{ in } L(X) \text{ such that } T_0 = 0 \text{ and for every } k \in \mathbb{N},
\begin{align}
T_kx &= \sum_{j=1}^k f_j(x)v_j; \\
m_j &\neq m_t \text{ and } s_j \neq s_t \text{ for } 1 \leq j < l \leq k; \\
\tau(\alpha(m_j)) &= \sigma(\beta(s_j)) \text{ for } 1 \leq j \leq k; \\
\{1, \ldots, j\} &\subseteq \{m_1, \ldots, m_2j\} \cap \{s_1, \ldots, s_2j\} \text{ whenever } 2j \leq k; \\
(I + T_k)\alpha(m_j) &= \beta(s_j) \text{ for } 1 \leq j \leq k; \\
v_j \in X_{r_j} \cap X_D, \quad p_D(v_j) < \varepsilon_j \text{ and } q_{r_j+1}(f_j) = 1 \text{ for } 1 \leq j \leq k, \text{ where } r_j = \tau(\alpha(m_j)).
\end{align}

$T_0 = 0$ serves as the basis of induction. Assume now that $n \in \mathbb{Z}_+$ and that $s_j, m_j, v_j, f_j$ and $T_j$ for $j \leq 2n$ satisfying (3.1)-(3.6) with $k \leq 2n$ are already constructed. We shall construct $s_j, m_j, v_j, f_j$ and $T_j$ with $j \in \{2n + 1, 2n + 2\}$ such that (3.1)-(3.6) hold for every $k \leq 2n + 2$.

Let $j \leq 2n$ and $r \in \mathbb{N}$. By (3.6), $v_j \in X_{r_j}$ with $r_j = \tau(\alpha(m_j))$. Since $q_r$ vanishes on $X_r$, $q_r(v_j) = 0$ and therefore $\hat{q}_r(f_j, v_j) = 0$ if $r \leq r_j$. On the other hand, if $r > r_j$, then $q_r^*(f_j) \leq q_{r_j+1}^*(f_j) = 1$ and $q_r(v_j) \leq p_D(v_j) < \varepsilon_j$. Hence $\hat{q}_r(f_j, v_j) < \varepsilon_j$ if $r \leq r_j$. Thus in any case
\begin{align}
\hat{q}_r(f_j, v_j) < \varepsilon_j \quad \text{for every } r \in \mathbb{N}.
\end{align}

The estimate (3.7) and Lemma 2.8 together with the inequality $\sum_{k=1}^\infty \varepsilon_k < 1$ and formula (3.1) ensure that $I + T_{2n}$ is invertible. Furthermore, if $r \leq r_j$, then $v_j \in X_r \subseteq \ker q_r$, while if $r > r_j$, then $q_r^*(f_j)$ is finite and therefore $f_j$ vanishes on $\ker q_r \supseteq X_r$. It follows that
\begin{align}
X_r \text{ and } \ker q_r \text{ are invariant for } T_j \text{ for every } r \in \mathbb{N} \text{ and each } j.
\end{align}

Since each $v_j$ belongs to $X_D$, each $T_j$ takes values in $X_D$. Now we define
\begin{align}
m_{2n+1} &= \min(\mathbb{N} \setminus \{m_j : j \leq 2n\}) \quad \text{and } \quad r = \tau(\alpha(m_{2n+1})).
\end{align}

It is a routine exercise to see that all conditions of the first part of Lemma 3.6 with $u = \alpha(m_{2n+1})$, $p = q_{r+1}$, $Y = X_r$, $L = \operatorname{span} \{\alpha(m_j) : j \leq 2n\}$, $T = T_{2n}$, $\varepsilon = \varepsilon_{2n+1}$ and $\Lambda = B_r \setminus \{\beta(s_j) : j \leq 2n\}$ are satisfied. Thus Lemma 3.6 provides $f_{2n+1} \in X'$ and $v_{2n+1} \in X_D \cap X_r$ such that $q_{r+1}^*(f_{2n+1}) = 1$, $f_{2n+1}|_L = 0$, $p_D(v_{2n+1}) < \varepsilon_{2n+1}$ and $(I + T_{2n+1})u \in \Lambda$, where $T_{2n+1}x = T_{2n}x + f_{2n+1}(x)v_{2n+1}$. The inclusion $(I + T_{2n+1})u \in \Lambda$ means that $(I + T_{2n+1})u = \beta(s_{2n+1})$ with $s_{2n+1} \in \mathbb{N} \setminus \{s_j : j \leq 2n\}$. Since $u = \alpha(m_{2n+1})$ and $f_{2n+1}|_L = 0$, (3.5) with $k = 2n$ implies that (3.5) with $k = 2n + 1$ is also satisfied. The equality $\tau(\alpha(m_{2n+1})) = \sigma(\beta(s_{2n+1})) = r$ gives (3.3) for $k = 2n + 1$. The properties (3.1), (3.2) and (3.6) for $k = 2n + 1$ are satisfied by construction. We postpone the discussion of (3.4) for later.

Define
\begin{align}
s_{2n+2} &= \min(\mathbb{N} \setminus \{s_j : j \leq 2n + 1\}) \quad \text{and } \quad r' = \sigma(\beta(s_{2n+2})).
\end{align}

It is a routine exercise to see that all conditions of the second part of Lemma 3.6 with $u = \beta(s_{2n+2})$, $p = q_{r'+1}$, $Y = X_r$, $L = \operatorname{span} \{\alpha(m_j) : j \leq 2n+1\}$, $T = T_{2n+1}$, $\varepsilon = \varepsilon_{2n+2}$ and $\Lambda = A_r \setminus \{\alpha(m_j) : j \leq 2n+1\}$ are satisfied. Thus Lemma 3.6 provides $f_{2n+2} \in X'$, $a \in \Lambda$ and $v_{2n+2} \in X_D \cap X_{r'}$ such that $q_{r'+1}^*(f_{2n+2}) = 1$, $f_{2n+2}|_L = 0$, $p_D(v_{2n+2}) < \varepsilon_{2n+2}$ and $(I + T_{2n+2})a = u$, where $T_{2n+2}x = T_{2n+1}x + f_{2n+2}(x)v_{2n+2}$. The inclusion $a \in \Lambda$ means that $a = \alpha(m_{2n+2})$ with $m_{2n+2} \in \mathbb{N} \setminus \{m_j : j \leq 2n + 1\}$. Since $u = \beta(s_{2n+2})$ and $f_{2n+2}|_L = 0$, (3.5) with $k = 2n + 1$ implies that (3.5) with $k = 2n + 2$ is also satisfied. The equality $\tau(\alpha(m_{2n+2})) = \sigma(\beta(s_{2n+2})) = r'$ gives (3.3) for $k = 2n + 2$. The properties (3.1), (3.2) and (3.6) for $k = 2n + 2$ are satisfied by construction. It remains to notice that (3.4) for $k = 2n + 2$ follows from (3.4) for $k = 2n$ and from the equalities (3.9) and (3.10). This concludes the inductive construction.

Now we consider the operator $T$ given by the formula
\begin{align}
T x = \lim_{k \to \infty} T_k x = \sum_{j=1}^\infty f_j(x)v_j.
\end{align}
Formula (3.7) and Lemma 2.8 show that $T$ is a well-defined continuous linear operator on $X$ such that $I + T$ is invertible. Passing to the limit as $k \to \infty$ in (3.5), we see that $(I + T)\alpha(m_j) = \beta(s_j)$ for every $j \in \mathbb{N}$. By (3.9) and (3.1), $j \mapsto m_j$ and $j \mapsto s_j$ are bijections of $\mathbb{N}$ onto itself. Taking into account that $\alpha : \mathbb{N} \to A$ and $\beta : \mathbb{N} \to B$ are also bijections, we see that $(I + T)(A) = B$. Now according (3.3) we have additionally that $(I + T)(A_j) = B_j$ for every $j \in \mathbb{Z}_+$. Thus $R = I + T$ is an isomorphism we were after.

**Corollary 3.8.** Let $X$ be a Fréchet space, whose topology is defined by an increasing sequence $\{p_n\}_{n \in \mathbb{N}}$ of seminorms. Let also $E$ and $F$ be two dense countably dimensional subspaces of $X$ such that $E \cap \ker p_n = F \cap \ker p_n$ for every $n \in \mathbb{N}$. Then there exists $R \in GL(X)$ such that $R(E) = F$.

**Proof.** For $n \in \mathbb{N}$, let $X_n = E \cap \ker p_n = F \cap \ker p_n$. If there exists $n \in \mathbb{N}$ such that $X_n$ is finite dimensional, then there is a continuous seminorm $q$ on $X$ such that $E \cap \ker q = F \cap \ker q = \{0\}$. Then $E, F \in M_1$. By Corollary 3.5 there is an isomorphism $S : E \to F$. The unique continuous extension of $S$ to $R \in L(X)$ is an isomorphism such that $R(E) = F$.

It remains to consider the case when each $X_n$ is infinite dimensional. An easy application of Lemma 2.6 allows us for every $n \in \mathbb{N}$ to choose dense countable $p_{n+1}$-independent subsets $A_n$ and $B_n$ of $X_n$ such that $E \cap X_n = \text{span}(A_n) \oplus (E \cap X_{n+1})$ and $F \cap X_n = \text{span}(B_n) \oplus (F \cap X_{n+1})$. By Theorem 3.7 there is $R \in GL(X)$ such that $R(A_n) = B_n$ for every $n \in \mathbb{N}$. Since $E$ is the linear span of the union of $A_n$ and $F$ is the linear span of the union of $B_n$, we have $R(E) = F$.

### 3.3 Isomorphisms of spaces in $M_2$

**Lemma 3.9.** Let $X \in F_1$, $E, F$ be dense countably dimensional subspaces of $X \times \omega$ and $E_0 = \{u \in \omega : (0, u) \in E\}$ and $F_0 = \{u \in \omega : (0, u) \in F\}$. Assume also that at least one of the following conditions is satisfied:

(a) $F_0$ and $E_0$ are both finite dimensional;
(b) $\omega/F_0$ and $\omega/E_0$ are both finite dimensional;
(c) each of the 4 spaces $F_0$, $E_0$, $\omega/F_0$ and $\omega/E_0$ is infinite dimensional.

Then there is $T \in GL(X \times \omega)$ such that $T(E) = F$.

**Proof.** Since $X \in F_1$, there is a continuous seminorm $p$ on $X \times \omega$ such that $\ker p = \{0\} \times \omega$. If $F_0$ and $E_0$ are both finite dimensional, we can find a continuous norm $q$ on $X \times \omega$ such that $\ker q \subseteq \{0\} \times \omega$ and $\ker q \cap \{0\} \times (E_0 + F_0) = \{0\}$. It follows that $\ker q \cap E = \{0\}$ and $\ker q \cap F = \{0\}$. Thus $E, F \in M_1$.

By Corollary 3.5 there is an isomorphism $S : E \to F$. Then the unique continuous extension of $S$ to $T \in L(X \times \omega)$ is an isomorphism and $T(E) = F$.

If (b) is satisfied then $Y = F_0 \cap E_0$ is a closed linear subspace of $\omega$ of finite codimension. Then $\omega = Y \oplus L$, where $L$ is a finite dimensional subspace of $\omega$. Thus $X \times \omega$ can be naturally identified with $X_1 \times Y$, where $X_1 = X \times L$. Since $X \in F_1$ and $L$ is finite dimensional, we have $X_1 \in F_1$. Furthermore, $Y$ is isomorphic to $\omega$ and $\{u \in Y : (0, u) \in E\}$ and $\{u \in Y : (0, u) \in F\}$ are both dense in $Y$. Thus, without loss of generality, we may assume that both $E_0$ and $F_0$ are dense in $\omega$. Then by Theorem 3.1 there is $S \in GL(\omega)$ such that $S(E_0) = F_0$. If (c) is satisfied, then by Lemma 3.3 there is $S \in GL(\omega)$ such that $S(E_0) = F_0$.

By Lemma 2.6 applied to a Hamel basis in $F$, there is a dense in $X \times \omega$ $p$-independent subset of $F$. By Zorn’s lemma, there is a dense in $X \times \omega$ maximal $p$-independent subset $A$ of $F$. Then $F$ is an algebraic direct sum of $\text{span}(A)$ and $\{0\} \times F_0$:

$$F = \text{span}(A) \oplus \{0\} \times F_0.$$  

Since $I_X \oplus S$ is an isomorphism of $X \times \omega$ onto itself, $G = (I_X \oplus S)(E)$ is a dense countably dimensional subspace of $X$. Furthermore, $G_0 = \{u \in \omega : (0, u) \in G\}$ coincides with $S(E_0) = F_0$.

By Lemma 2.6 there is a dense in $X \times \omega$ $p$-independent subset of $G$. By Zorn’s lemma, there is a dense in $X \times \omega$ maximal $p$-independent subset $B$ of $G$. Then $G$ is an algebraic direct sum of $\text{span}(B)$ and $\{0\} \times G_0 = \{0\} \times F_0$:

$$G = \text{span}(B) \oplus \{0\} \times F_0.$$
Since both $A$ and $B$ are dense in $X \times \omega$ and $p$-independent, Theorem 3.3 furnishes us with $R \in GL(X \times \omega)$ such that $R(B) = A$ and $Rx = x$ for every $x \in \ker p = \{0\} \times \omega$. Thus using the last two displays, we conclude that $R(G) = F$. Hence $T(E) = F$, where $T \in GL(X \times \omega)$ is given by $T = R \circ (I_X \oplus S)$. 

\subsection*{3.4 Action of $GL(X)$ on subspaces $E \in \mathcal{M}_0$ for $X \in \mathcal{F}_3$}

\textbf{Lemma 3.10.} Let $X \in \mathcal{F}_2 \cup \mathcal{F}_3$. Then $X$ has subspaces isomorphic to $\omega$.

\textit{Proof.} By definition of the classes $\mathcal{F}_2$ and $\mathcal{F}_3$, there is an increasing sequence $\{p_n\}_{n \in \mathbb{N}}$ of seminorms defining the topology of $X$ such that $\ker p_n/\ker p_{n+1} \neq \{0\}$ for every $n \in \mathbb{N}$. Hence, we can pick $x_n \in \ker p_n \setminus \ker p_{n+1}$ for each $n \in \mathbb{N}$. Let $V$ be the closed linear span of $x_n$ for $n \in \mathbb{N}$. It is easy to verify that the series $\sum c_n x_n$ converges for every sequence $c = \{c_n\}$ of complex numbers and that the map $c \mapsto \sum c_n x_n$ provides an isomorphism between $\omega$ and $V$. Thus $X$ has subspaces isomorphic to $\omega$.

\textbf{Lemma 3.11.} Let $X \in \mathcal{F}_3$. Then for every two subspaces $Y, Z$ of $X$ isomorphic to $\omega$, there exists a subspace $W$ of $X$ isomorphic to $\omega$ such that $Y \subset W$, $Z \subset W$ and both $W/Y$ and $W/Z$ are infinite dimensional.

\textit{Proof.} Let now $Y, Z$ be two subspaces of $X$ isomorphic to $\omega$. Then $V = Y + Z$ carries weak topology since its dense subspace $Y + Z$ carries weak topology. Hence $V$ is isomorphic to $\omega$. Since every isomorphic to $\omega$ subspace of a Fréchet space is complemented, $X = V \oplus M$, where $M$ is a closed linear subspace of $X$. It is straightforward to see that $M \in \mathcal{F}_3$ and therefore by Lemma 3.10 there is a closed linear subspace $N$ of $M$ isomorphic to $\omega$. Now $W = V \oplus N$ has all desired properties.

\textbf{Lemma 3.12.} Let $X \in \mathcal{F}_3$. Then $GL(X)$ acts transitively on subspaces of $X$ isomorphic to $\omega$. Furthermore $GL(X)$ acts transitively on countably dimensional subspaces of $E$ of $X$ such that $E \in \mathcal{M}_0$.

\textit{Proof.} Let $Y$ and $Z$ be two subspaces of $X$ isomorphic to $\omega$ and $E$ and $F$ be dense countably dimensional subspaces of $Y$ and $Z$ respectively. In order to prove the lemma, it suffices to find $T \in GL(X)$ such that $T(E) = F$ (the equality $T(Y) = Z$ follows).

By Lemma 3.11, there is a subspace $W$ of $X$ isomorphic to $\omega$ such that $Y \subset W$, $Z \subset W$ and both $W/Y$ and $W/Z$ are infinite dimensional. Since every isomorphic to $\omega$ subspace of a Fréchet space is complemented, $X = W \oplus V$, where $V$ is a closed linear subspace of $X$. By Lemma 3.10 there is $S \in GL(W)$ such that $S(E) = F$. Clearly, $T = S \oplus Id_V \in GL(X)$ and $T(E) = F$.

\subsection*{3.5 Proof of Theorem 1.11}

Obviously subspaces of a topological space with weak topology also carry weak topology. Furthermore the topology of $X$ is weak if $X$ possesses a dense subspace with weak topology. These observation together with Remark 1.6 and Theorem 3.1 immediately imply (1.11.1). For the sake of brevity we denote $E = E(X)$.

By definition, $X \notin \mathcal{F}_0$ if and only if $X$ possesses a non-trivial continuous seminorm. Clearly, $E \subseteq \mathcal{M}_1$ if and only if $X$ possesses a continuous norm, that is $X \in \mathcal{F}_1$. Finally, by Corollary 3.5 every $E, F \in E \cap \mathcal{M}_1$ are isomorphic, which proves (1.11.2).

By Lemma 3.10 every $X \in \mathcal{F}_2 \cup \mathcal{F}_3$ has a closed linear subspace $Y$ isomorphic to $\omega$. Since every isomorphic to $\omega$ subspace of a Fréchet space is complemented, $X = Y \oplus Z$, where $Z$ is a closed linear subspace of $X$. Then $Z$ possesses a non-trivial continuous seminorm $p$ (otherwise $X \in \mathcal{F}_0$). Then the seminorm $q(y + z) = p(z)$ for $y \in Y$ and $z \in Z$ is a non-trivial continuous seminorm on $X$. By Corollary 2.7 there is a dense countably dimensional subspace $E_1$ of $X$ such that $E_1 \cap \ker q = \{0\}$. Let also $E_2$ be a dense countably dimensional subspace of $Y$ and $E = E_1 + E_2$. Obviously, $E \in E$. Since $E_2 \subset Y$, $E_1 \cap \ker q = \{0\}$ and $Y \subseteq \ker q$, $E = E_1 \oplus E_2$ (in algebraic sense) and $E_2 = E \cap Y$. In particular, $E_2$ is closed in $E$ and carries weak topology. Furthermore, $q$ provides (in a natural way) a continuous norm on $E/E_2$. By Remark 1.6 $E \in \mathcal{M}_2$. Thus $E \cap \mathcal{M}_2 \neq \emptyset$. On the other hand, from (1.11.1) and (1.11.2) it follows that $E \cap \mathcal{M}_2 = \emptyset$ if $X \in \mathcal{F}_0 \cup \mathcal{F}_1$. Thus $E \cap \mathcal{M}_2 \neq \emptyset$ if and only if $X \in \mathcal{F}_2 \cup \mathcal{F}_3$.

Now let $X \in \mathcal{F}_2$. Then $X = Y \oplus Z$, where $Z \in \mathcal{F}_1$ and $Y$ is isomorphic to $\omega$. For $E \in \mathcal{E}$, we denote $E_0 = E \cap Y$ and $E_1 = E_0$. Clearly $E \cap \mathcal{M}_2$ is the union of two disjoint subsets $E_1$ and $E_2$, where $E_1 = \{E \in E \cap \mathcal{M}_2 : \dim Y/E_1 < \infty\}$ and $E_2 = \{E \in E \cap \mathcal{M}_2 : \dim Y/E_1 = \infty\}$. By Lemma 3.3 every two
spaces in $\mathcal{E}_1$ are isomorphic and every two spaces in $\mathcal{E}_2$ are isomorphic. Let now $E \in \mathcal{E}_1$ and $F \in \mathcal{E}_2$. In order to show that $\mathcal{E} \cap \mathcal{M}_2$ consists of exactly two isomorphism classes, it remains to verify that $E$ and $F$ are non-isomorphic. Assume the contrary. That is, there is an isomorphism $T \in L(X)$ such that $T(E) = F$.

Since $Z$ possesses a continuous norm and $Y/E_1$ is finite dimensional, $X/E_1$ also possesses a continuous norm. Since $T$ is an isomorphism, $X/T(E_1)$ possesses a continuous norm. Since $E_0$ is a subspace of $E$ carrying weak topology, $T(E_0)$ is a subspace of $F$ carrying weak topology. It is straightforward to verify that every subspace of $F$ carrying weak topology is contained in $F_0 + L$, where $L$ is a finite dimensional subspace of $F$. Thus $T(E_0) \subseteq F_0 + L$. Then $T(E_1) = \overline{T(E_0)} \subseteq \overline{F_0} + L = F_1 + L$. Since $X/T(E_1)$ possesses a continuous norm, $X/F_1 + L$ also possesses a continuous norm. Since $L$ is finite dimensional, $X/F_1$ possesses a continuous norm. The latter is impossible since $Y/F_1$ is an infinite dimensional space carrying weak topology. This contradiction proves that $\mathcal{E} \cap \mathcal{M}_2$ contains exactly two isomorphism classes if $X \in \mathcal{F}_2$.

Let $X \in \mathcal{F}_3$. We have to show that every $E, F \in \mathcal{E} \cap \mathcal{M}_2$ are isomorphic. Indeed, let $E, F \in \mathcal{E} \cap \mathcal{M}_2$. By definition of $\mathcal{M}_2$, there are closed linear subspaces $E_1$ and $F_1$ of $E$ and $F$ respectively such that both $E_1$ and $F_1$ carry weak topology and both $E/E_1$ and $F/F_1$ are infinite dimensional and possess a continuous norm. By Lemma 3.12 there is an isomorphism $T \in L(X)$ such that $T(E_1) = F_1$. Then $F_1$ is a common closed subspace of $T(E)$ and of both $E/F_1$ and $F/F_1$ possess a continuous norm. Hence there is a continuous seminorm $p$ on $X$ such that $F_1 \subseteq \ker p$ and $T(E) \cap \ker p = F \cap \ker p = F$. Using Lemma 2.6, we can find dense in $X$ $p$-independent sets $A$ and $B$ such that $\text{span}(B) \oplus F_1 = T(E)$ and $\text{span}(A) \oplus F_1 = F$ (in algebraic sense). By Theorem 3.3 there is an isomorphism $S \in L(X)$ such that $S(B) = A$ and $Sx = x$ for $x \in \ker p$. Since $F_1 \subseteq \ker p$, it follows that $S(F_1) = F_1$. Hence $R(E_1) = F_1$, where $R = ST$. Next, the relation $\text{span}(B) \oplus F_1 = T(E)$ implies $\text{span}(T^{-1}(B)) \oplus E_1 = E$. Furthermore, $R(T^{-1}(B)) = S(B) = A$. It immediately follows that $R(E) = F$. Thus $E$ and $F$ are isomorphic, which completes the proof of (1.11.3).

It remains to verify (1.11.4). The fact that $X \in \mathcal{F}_3$ if and only if $E \cap \mathcal{M}_3 \neq \emptyset$ is obvious. We have to show that there are uncountably many pairwise non-isomorphic spaces in $\mathcal{E} \cap \mathcal{M}_3$. Let $E_n \in \mathcal{E} \cap \mathcal{M}_3$ for $n \in \mathbb{N}$. It suffices to find a $E \in \mathcal{E} \cap \mathcal{M}_3$ non-isomorphic to each $E_n$. Since $X \in \mathcal{F}_3$, we can choose an increasing sequence $\{p_n\}_{n \in \mathbb{N}}$ of continuous seminorms on $X$ defining the topology of $X$ such that $X_n/X_{n+1}$ is infinite dimensional for every $n \in \mathbb{Z}_+$, where $X_0 = X$ and $X_n = \ker p_n$ for $n \in \mathbb{N}$. Since $E_n \in \mathcal{M}_3$, $E_n \cap X_n \in \mathcal{F}_3$ for every $n \in \mathbb{N}$ and $k \in \mathbb{N}$. By Lemma 2.11 there is a sequence $r \in \Omega$ (see Section 2.1) such that $E_k \cap X_n \notin \mathcal{F}_r$ for every $n \in \mathbb{Z}_+$ and $k \in \mathbb{N}$. By Lemma 2.12 there is a closed subspace $Y$ of $X_1$ such that $Y \times X_n$ is infinite dimensional for every $n \in \mathbb{N}$ and $Y \in \mathcal{F}_r$. Now for each $n \in \mathbb{N}$ we can choose a dense countably dimensional subspace $F_m$ of $Y \times X_m$. By Corollary 2.7 there is a dense in $X$ countably dimensional subspace $F_0$ such that $F_0 \times X_1 = \{0\}$. Let $E$ be the linear span of the union of $F_m$ for $m \in \mathbb{Z}_+$. Clearly $E \in \mathcal{E}$ and $E \cap X_n$ is infinite dimensional for every $n \in \mathbb{N}$. It follows that $E \in \mathcal{M}_3$. It remains to show that $E$ is non-isomorphic to each of $E_n$. Assume the contrary. That is, there are $n \in \mathbb{N}$ and an isomorphism $T \in L(X)$ such that $T(E_n) = E$. Continuity of $T$ implies that there are $c > 0$ and $k \in \mathbb{N}$ such that $p_k(Tx) \leq cp_k(x)$ for every $x \in X$. In particular, $T(X_k) \subseteq X_1$. Hence $T(X_k \cap E_n) \subseteq X_1 \cap E$ and therefore $T(X_k \cap E_{n+1}) \subseteq X_1 \cap E$. That is, $X_k \cap E_n$ is isomorphic to a subspace of $Y$. Since $Y \in \mathcal{F}_r$, it follows that $X_k \cap E_n \in \mathcal{F}_r$. This contradiction completes the proof of (1.11.4) and that of Theorem 1.11.

### 4 Orbitality for countably dimensional $E \in \mathcal{M}$

#### 4.1 Strong orbitality for $E \in \mathcal{M}_0 \cup \mathcal{M}_1$

**Proposition 4.1.** Every countably dimensional $E \in \mathcal{M}_0$ is strongly orbital. In particular, $E$ does not have the invariant subset property.

**Proof.** It is well-known (see, for instance, [10]) that $\omega$ supports a hypercyclic operator. Thus, we can take $S \in L(\omega)$ and $x \in \omega$ such that $x$ is a hypercyclic vector for $S$. Now let $F = \text{span} \{ S^n x : n \in \mathbb{Z}_+ \}$. Then $F$ is a dense countably dimensional subspace of $\omega$. Obviously, $F \in \mathcal{M}_0$ and $F$ is strongly orbital. By Corollary 3.2, $E$ is isomorphic to $F$ and therefore $E$ is strongly orbital as well. \hfill $\square$

The following result is an immediate corollary of Lemma 5.2 in [23].
Lemma 4.2. Let $p$ be a non-trivial continuous seminorm on a separable Fréchet space $X$. Then there exists a hypercyclic $T \in L(X)$ such that $Tx = x$ for every $x \in \ker p$.

Proposition 4.3. Every countably dimensional $E \in \mathcal{M}_1$ is strongly orbital. In particular, $E$ does not have the invariant subset property.

Proof. Pick a dense Hamel basis $A$ in $E$. Let $X$ be the completion of $E$. Then $X \in \mathcal{F}$. Since $E \in \mathcal{M}_1$, there is a continuous seminorm $p$ on $X$ such that $E \cap \ker p = \{0\}$. It follows that $A$ is $p$-independent. By Lemma 4.2, there is a hypercyclic operator $R \in L(X)$ such that $Rx = x$ for each $x \in \ker p$. Let $u \in X$ be a hypercyclic vector for $R$. Let $F = \text{span}(O)$, where $O = O(T, u) = \{T^n u : n \in \mathbb{Z}_+\}$. Observe that $O$ is $p$-independent. Indeed, otherwise $R^n u \in L + \ker p$, where $L = \text{span}\{u, Ru, \ldots, R^{n-1}u\}$ for some $n \in \mathbb{N}$. Since $Rx = x$ for $x \in \ker p$, it follows that $O \subset L + \ker p$. Since $L$ is finite dimensional and $\ker p$ is a closed linear subspace of $X$ of infinite codimension, $L + \ker p$ is a proper closed subspace of $X$. Then the inclusion $O \subset L + \ker p$ contradicts density of $O$ in $X$. Thus $O$ is $p$-independent and therefore $F \cap \ker p = \{0\}$. Hence $F \in \mathcal{M}_1$. Obviously, $F$ is strongly orbital. By Corollary 3.5, $E$ and $F$ are isomorphic. Hence $E$ is strongly orbital. \hfill \bbox

4.2 Strong orbitality for countably dimensional $E \in \mathcal{M}_3$

Lemma 4.4. Let $X$ be a separable Fréchet space, whose topology is defined by an increasing sequence $\{p_n\}_{n \in \mathbb{N}}$ of seminorms. Let also $\{X_n\}_{n \in \mathbb{Z}_+}$ be a sequence of closed linear subspaces of $X$ such that $X_0 = X$, $X_{n+1} \subset X_n \cap \ker p_{n+1}$ and $X_n/(X_n \cap \ker p_{n+1})$ is infinite dimensional for every $n \in \mathbb{Z}_+$. Then there is $T \in L(X)$ and $x \in X$ such that $O(T, x) \cap \ker p_n$ is a dense subset of $X_n$ for every $n \in \mathbb{Z}_+$.

Proof. By Lemma 2.3, there exist $\{u_{n,k} : n, k \in \mathbb{Z}_+\} \subset X$ and $\{f_{n,k} : n, k \in \mathbb{Z}_+\} \subset X'$ such that (2.1), (2.4) are satisfied. By Corollary 2.4, there is a Banach disk $D$ in $X$ such that $u_{n,k} \in D$ for every $n, k \in \mathbb{Z}_+$. For each $n \in \mathbb{N}$, consider $T_n \in L(X)$ defined by the formula

$$T_n x = \sum_{k=0}^{\infty} \frac{-2^{-k} f_{n,k}(x)}{p_{n+1}(f_{n,k}) p_D(u_{n-1,k})} u_{n-1,k}. $$

Since $|f_{n,k}(x)| \leq p_{n+1}(x) p_{n+1}'(f_{n,k}(x))$, the series in the above display converges absolutely in the Banach space $X_D$ and $p_D(T_n x) \leq 2 p_{n+1}(x)$ for every $x \in X$. Hence $T_n : X \to X_D$ is a continuous linear operator and therefore $T_n \in L(X)$. Furthermore, (2.1) ensures that

$$T_n(X) \subset X_{n-1} \text{ for every } n \in \mathbb{N}. \quad (4.1)$$

Since $X_n \subset \ker p_n$ for $n \in \mathbb{N}$, (4.1) guarantees that the series $\sum T_n$ converges pointwise. By the uniform boundedness principle, the formula

$$Tx = \sum_{n=1}^{\infty} T_n x = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{-2^{-k} f_{n,k}(x)}{p_{n+1}(f_{n,k}) p_D(u_{n-1,k})} u_{n-1,k}$$

defines a continuous linear operator $T$ on $X$. For $n \in \mathbb{Z}_+$, let $E_n = \text{span}\{u_{n,k} : k \in \mathbb{Z}_+\}$ and $F_n = \text{span}\{u_{m,k} : m \geq n, k \in \mathbb{Z}_+\}$. Clearly $F_n$ is the sum of $E_m$ for $m \geq n$. By (2.1), $F_n$ is a dense countably dimensional subspace of $X_n$. According to (2.3),

$$T \text{ vanishes on } E_0 \text{ and } S_n = T|_{E_n} : E_n \to E_{n-1} \text{ is an invertible linear operator for each } n \in \mathbb{N}. \quad (4.2)$$

Now pick a countable set $A \subset F_0$ such that $A \cap F_k$ is dense in $F_k$ for every $k \in \mathbb{Z}_+$. By (2.1), each $F_k$ is dense in $X_k$ and therefore $A \cap F_k$ is dense in $X_k$. For each $a \in A$, let $k(a) = \max\{m \in \mathbb{Z}_+ : a \in F_m\}$. Then $a \in F_{k(a)}$ and $A_m = \{a : k(a) = m\}$ is dense in $X_m$ for every $m \in \mathbb{Z}_+$. By definition of $F_m$, for every $a \in A$, there is $r(a) \geq k(a)$ such that $a \in E_{k(a)} + E_{k(a)+1} + \ldots + E_{r(a)}$. In other words,

$$a = \sum_{j=k(a)}^{r(a)} w(j, a), \text{ where } w(j, a) \in E_j.$$
Now by (4.2),

\[ \text{for } u \in E_j, \ T^n u \in E_{j-n} \text{ if } n \leq j \text{ and } T^n u = 0 \text{ if } n > j. \] (4.3)

For \( u \in E_j \) and \( n \in \mathbb{Z}_+ \), we denote \( u^{[n]} = u \) if \( n = 0 \) and \( u^{[n]} = S_{j+n}^{-1} \ldots S_{j+1}^{-1} u \) if \( n > 0 \), where \( S_i \) are defined in (4.2). According to (4.2),

\[ u^{[n]} \in E_{j+n} \text{ and } T^m u^{[n]} = u^{[n-m]} \text{ if } m \leq n. \] (4.4)

For \( a \in A \) and \( n \in \mathbb{Z}_+ \), we denote

\[ a^{[n]} = \sum_{j=k(a)}^{r(a)} u(j,a)^{[n]}. \]

Now we enumerate \( A: A = \{ a_l : l \in \mathbb{N} \} \). Define the sequence \( \{ n_j \}_{j \in \mathbb{N}} \) of non-negative integers by the formula \( n_1 = 1 \) and \( n_{j+1} = n_j + r(a_{j+1}) + j \) for \( j \in \mathbb{N} \). Consider the series

\[ x = \sum_{l=1}^{\infty} a_l^{[n_l]} = \sum_{l=1}^{\infty} \sum_{j=k(a_l)}^{r(a_l)} u(j,a_l)^{[n_l]} \]

First, we show that this series converges in \( X \) and therefore \( x \) is well-defined. Indeed, let \( s \in \mathbb{N} \). By (4.4), \( u(j,a_l)^{[n_l]} \in E_{j+n_l} \subset F_{j+n_l} \subset F_{n_l} \subset F_l \subset X_l \). Hence \( p_s(a_l^{[n_l]}) = 0 \) for \( l \geq s \) and therefore the series in the above display converges absolutely. Now let \( m \in \mathbb{N}, l < m \) and \( k(a_l) \leq j \leq r(a_l) \). Since \( n_m - n_l > r(l) \), (4.3) implies that \( T^{n_m}(u(j,a_l)^{[n_l]}) = 0 \). Hence \( T^{n_m}(a_l^{[n_l]}) = 0 \). Then, using (4.3), we have \( T^{n_m}(a_l^{[n_l]}) = a_l^{[n_l-n_m]} \) for \( l > m \). Hence

\[ T^{n_m} x = a_m + \sum_{l=m+1}^{\infty} a_l^{[n_l-n_m]} = a_m + \sum_{l=m+1}^{\infty} \sum_{j=k(a_l)}^{r(a_l)} u(j,a_l)^{[n_l-n_m]} \]

Since for \( l > m, n_l - n_m > r(a_l) + l > r(a_l) + m, (4.3) \) and the above display imply that

\[ a_m - T^{n_m} x \in F_m \subseteq X_m. \]

Hence \( p_s(a_m - T^{n_m} x) = 0 \) for \( m \geq s \) and therefore \( a_m - T^{n_m} x \to 0 \) in \( X \). Moreover, by (2.1) and (2.2), for every \( j \in \mathbb{Z}_+ \), there is \( l \in \mathbb{Z}_+ \) such that \( T^j x \in X_l \setminus \ker p_{l+1} \). It follows that \( O(T,x) \cap \ker p_n \subseteq X_n \) for every \( n \in \mathbb{N} \). Furthermore, \( T^{n_m} x \in X_k(a_m) \setminus \ker p_{k(a_m)+1} \) for each \( m \in \mathbb{N} \). Since \( A_s = \{ a \in A : k(a) = s \} \) is dense in \( X_s \) for every \( s \in \mathbb{Z}_+ \), the relation \( a_m - T^{n_m} x \to 0 \) ensures that \( \{ T^{n_m} x : k(a_m) = s \} \) is a dense subset of \( X_s \) for each \( s \in \mathbb{Z}_+ \). That is, \( O(T,x) \cap \ker p_n \) is a dense subset of \( X_n \) for every \( n \in \mathbb{Z}_+ \).

**Proposition 4.5.** Let \( E \in \mathcal{M}_3 \) be countably dimensional. Then \( E \) is strongly orbital and therefore does not have the invariant subset property.

**Proof.** Let \( X \) be the completion of \( E \). Since \( E \in \mathcal{M}_3 \), we can pick an increasing sequence \( \{ p_n \}_{n \in \mathbb{N}} \) of continuous seminorms on \( X \) defining the topology of \( X \) such that \( E \cap \ker p_1 \) has infinite codimension in \( E \) and \( E \cap \ker p_{n+1} \) has finite codimension in \( E \cap \ker p_n \) for each \( n \in \mathbb{N} \). Let \( X_0 = X \) and \( X_n = E \cap \ker p_n \) for \( n \in \mathbb{N} \). It is easy to verify that \( X_{n+1} \subseteq X_n \cap \ker p_{n+1} \) and \( X_n/(X_{n} \cap \ker p_{n+1}) \) is infinite dimensional for every \( n \in \mathbb{Z}_+ \). By Lemma 4.4 there are \( T \in L(X) \) and \( x \in X \) such that \( O(T,x) \cap \ker p_n \) is a dense subset of \( X_n \) for every \( n \in \mathbb{Z}_+ \). Clearly \( F = \text{span}(O(T,x)) \) is strongly orbital. On the other hand, \( X_n = E \cap \ker p_n = F \cap \ker p_n \) for each \( n \in \mathbb{N} \). By Corollary 4.8 \( E \) and \( F \) are isomorphic. Hence \( E \) is also strongly orbital.

**4.3 Invariant subset property for countably dimensional \( E \in \mathcal{M}_2 \)**

**Lemma 4.6.** Let \( X \in \mathcal{F} \) be the direct sum \( X = Y \oplus Z \) of its closed linear subspaces such that \( Y \) is isomorphic to \( \omega \) and \( Z \) is non-isomorphic to \( \omega \). Then there is a dense countably dimensional subspace \( E \) of \( X \) such that \( E \) does not have the invariant subset property and \( E \cap Y \) is dense in \( Y \).
Proof. Since $Z$ is non-isomorphic to $\omega$, there is a non-trivial continuous seminorm $p$ on $X$ such that $Y \subseteq \ker p$. Since $Y$ is isomorphic to $\mathbb{K}^\mathbb{N}$, we can pick $\{x_n : n \in \mathbb{N}\} \subset Y$ and $\{f_n : n \in \mathbb{N}\} \subset X'$ such that $f_n|_Z = 0$ for each $n \in \mathbb{N}$, $f_n(x_m) = \delta_{n,m}$ for every $m, n \in \mathbb{N}$ and the map $c = \{c_n\} \mapsto \sum_{k=1}^{\infty} c_n x_n$ is an isomorphism of $\mathbb{K}^\mathbb{N}$ and $Y$.

Pick any dense countably dimensional subspace $F$ of $Z$. By Corollary 2.4, there exists a Banach disk $D$ in $Z$ such that $F \subset Z_D = X_D$. By Lemma 2.5, there exists a Hamel basis $\{x_n\}_{n \in \mathbb{N}_+}$ in $F$ and a sequence $\{f_n\}_{n \in \mathbb{N}}$ in $X'_p$ such that $f_n(x_m) = \delta_{n,m}$ for every $m, n \in \mathbb{Z}_+$. Since $f_n \in X'_p$, $f_n$ vanishes on $Y$ for each $n \in \mathbb{Z}_+$ and therefore $f_n(x_k) = 0$ if $k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$. Since $f_n$ vanishes on $Z$ for $n \in \mathbb{N}$, $f_k(x_n) = 0$ if $k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$. Thus $f_n(x_m) = \delta_{n,m}$ for $m, n \in \mathbb{Z}$. Consider $T \in L(X)$ defined by the formula

$$Tx = \sum_{n=-\infty}^{\infty} c_n f_{n+1}(x) x_n,$$

where $c_n = 1$ for $n \geq 1$ and $c_n = 2^n p(\sum_{j=1}^{n} p(x_j))$ for $n < 0$. Obviously $T|_Y$ is continuous and well defined. From the definition of $c_n$ it follows that for $x \in Z$, the series in the above display converges absolutely in the Banach space $X_D$ and therefore in $X$. Thus $T$ is a well-defined continuous linear operator. Let $G = \text{span} \{x_n : n \in \mathbb{Z}\}$. It is easy to see that $G$ is dense in $X$ and that $T|_G : G \to G$ is a bijective linear map. Thus we can consider its inverse $S = (T|_G)^{-1} : G \to G$. It is straightforward to verify that $S|_{Y \cap G}$ is the forward shift:

$$Sx = \sum_{n=1}^{\infty} f_n(x) x_{n+1} \text{ for } x \in Y \cap G.$$

The operator $S$ on the space $G$ is, of course, a bilateral weighted (forward) shift with respect to the basis $\{x_n\}_{n \in \mathbb{Z}}$.

First, we shall show that there is $w \in Y$ such that $f_1(w) \neq 0$, and $w$ is a hypercyclic vector for $T$. Indeed, let $w_0 = x_1$ and choose a dense in $X$ countable subset $\{w_n : n \in \mathbb{N}\}$ of $G$. We start with a few easy observations. First, from the definition of $T$ it follows that $T^n w \in Z$ for all sufficiently large $n$ and $p_D(T^n w) \to 0$ for every $w \in G$. Second, for each $w \in G$ and $n \in \mathbb{N}$, $S^n w \in Y$ and $f_1(S^n w) = \ldots = f_k(S^n w) = 0$ all for sufficiently large $n$. Using this observations and the fact that $T^n S^m w = T^{n-m} w$ and $T^m S^n w = S^{n-m} w$ for $w \in G$ and $n, m \in \mathbb{Z}_+$, $n > m$, it is a standard exercise to see that if the sequence $\{k_n\}_{n \in \mathbb{Z}}$ of non-negative integers with $k_0 = 0$ grows fast enough, then

(a) the sets $\{k \in \mathbb{N} : f_k(S^{k_n} w_n) \neq 0\}$ for $n \in \mathbb{Z}_+$ are pairwise disjoint and therefore the series $\sum_{n=0}^{\infty} S^{k_n} w_n$ converges in $Y \subset X$ to $w \in Y$;

(b) $T^{k_n} w - w_n \to 0$ in $X$.

Thus $w$ constructed in this way is a hypercyclic vector for $T$. Since $f_1(w_0) = f_1(S^{k_0} w_0) = 1$, (a) implies that $f_1(S^{k_n} w_n) = 0$ for $n \in \mathbb{N}$ and therefore $f_1(w) = 1$.

Now we set $u_n = T^n w$ for $n \in \mathbb{Z}_+$ and $u_n = \sum_{j=1}^{\infty} f_j(w) x_{j-n}$ for $n \in \mathbb{Z}$, $n < 0$. From the definition of $T$ it follows that

$$T^m u_n = u_{n+m} \text{ for every } n \in \mathbb{Z} \text{ and } m \in \mathbb{Z}_+. \tag{4.5}$$

Now let $E = \text{span} \{u_n : n \in \mathbb{Z}\}$. Since $w$ is a hypercyclic vector for $T$, $E$ is dense in $X$. Since $u_n \in Y$ for $n \leq 0$, $f_j(u_{j-1}) = 1$ for $j \in \mathbb{N}$ and $f_j(u_k) = 0$ for $1 \leq j \leq k$, we see that $\text{span} \{u_n : n \leq 0\}$ is a dense subspace of $Y$. Hence $E \cap Y$ is dense in $Y$. According to (4.5), $T(E) \subseteq E$. It remains to show that $E$ does not have the invariant subset property. In order to show that, it suffices to demonstrate that every $v \in E \setminus \{0\}$ is hypercyclic vector for $T$. Let $v \in E \setminus \{0\}$. Using (4.5) and the equality $u_0 = w$, we see that there is a non-zero polynomials $q$ and $n \in \mathbb{N}$ such that $T^n v = q(T) w$. According to Bourdon [11], $q(T) w$ is a hypercyclic vector for $T$. Hence $T^n v$ is hypercyclic for $T$ and therefore $v$ is hypercyclic for $T$. \hfill \Box

Proposition 4.7. Let $E \in M_2$ be countably dimensional. Then $E$ does not have the invariant subset property.
Assume the contrary. That is, $K\in M$ does not have the invariant subset property according to Proposition 4.7. Finally, by Proposition 4.8, \ref{lem:4.3} and \ref{lem:4.5}. By Lemma 1.2, \ref{lem:4.9}. Proof. Assume that \ref{lem:4.8} does not have the invariant subset property. By Theorem 3.4, there is an existence of a continuous seminorm \( p \) on \( E \) such that \( \ker p \cap E \) is infinite dimensional and \( \ker p \cap E \) carries weak topology. We have to show that \( \ker p \cap E \) is finite dimensional. Since \( (J + zJ)/J \) is finite dimensional, there exist \( u_1, \ldots, u_k \in \mathbb{K}[z] \) such that \( zJ \subseteq J + L \), where \( L = \text{span}\{u_1, \ldots, u_k\} \). Since \( J \) is finite dimensional, there is \( p \in J \) such that \( n = \deg p > \deg u_j \) for every \( j \in \{1, \ldots, k\} \). If \( f \in \mathbb{N} \), \( m \geq n \) and there is \( q \in J \) satisfying \( \deg q = m \), then the inclusion \( zJ \subseteq J + L \) implies that there is \( r \in L \) for which \( zq - r \in J \). Since \( \deg r \leq \deg q \), \( \deg(zq - r) = m + 1 \). Thus \( J \) contains a polynomial of degree \( m + 1 \) whenever \( m \geq n \) and \( J \) contains a polynomial of degree \( m \). Since \( J \) contains a polynomial of degree \( n \), it follows that \( J \) contains a polynomial of degree \( m \) for every \( m \geq n \). Hence \( \dim(\mathbb{K}[z]/J) \leq n \). Proof of Proposition \ref{lem:4.8}. Since \( E \in M_2 \), there exists a non-trivial continuous seminorm \( p \) on \( E \) such that \( F = \ker p \) carries weak topology. Let \( T \in \text{L}(E) \) and \( x \in E \). We have to show that \( \text{span}(O(T,x)) \neq E \). Assume the contrary. That is, \( \text{span}(O(T,x)) = E \). Since \( E \) is infinite dimensional, the vectors \( T^n x \) are linearly independent. Hence the map \( \Phi : \mathbb{K}[z] \to E, \Phi(p) = p(T)x \) is a linear bijection. Consider the map \( S : F \to E/F, Sx = Tx + F \). Then \( S \) is a continuous linear operator from the space \( F \) carrying weak topology to the space \( E/F \) possessing a continuous norm. Since the continuous linear image of a space with weak topology carries weak topology and only finite dimensional spaces carry weak topology and a continuous norm simultaneously, \( S(F) \) is finite dimensional. That is, there is a finite dimensional subspace \( L \) of \( E \) such that \( T(F) \subseteq F + L \). This means that \( (J + zJ)/J \) is finite dimensional, where \( J = \Phi^{-1}(F) = \{p \in \mathbb{K}[z] : p(T)x \in F\} \). Since \( F \) is infinite dimensional and \( \Phi : \mathbb{K}[z] \to E \) is a linear bijection, \( J \) is infinite dimensional. By Lemma \ref{lem:4.9}, \( J \) has finite codimension in \( \mathbb{K}[z] \). Since \( \Phi(J) = F \) and \( \Phi : \mathbb{K}[z] \to E \) is a linear bijection, \( F = \ker p \) has finite codimension in \( E \), which contradicts the non-triviality of \( p \). 4.4 Non-orbitality for \( X \in M_2 \) Proposition 4.8. Let \( E \in M_2 \) be countably dimensional. Then \( E \) is non-orbital. In order to prove the above proposition, we need the following curious elementary lemma. Lemma 4.9. Let \( J \) be a linear subspace of the space \( \mathbb{K}[z] \) of polynomials such that \( (J + zJ)/J \) is finite dimensional. Then either \( J \) is infinite dimensional or \( J \) has finite codimension in \( \mathbb{K}[z] \). Proof. Assume that \( J \) is infinite dimensional. We have to show that the codimension of \( J \) in \( \mathbb{K}[z] \) is finite. Since \( (J + zJ)/J \) is finite dimensional, there exist \( u_1, \ldots, u_k \in \mathbb{K}[z] \) such that \( zJ \subseteq J + L \), where \( L = \text{span}\{u_1, \ldots, u_k\} \). Since \( J \) is infinite dimensional, there is \( p \in J \) such that \( n = \deg p > \deg u_j \) for every \( j \in \{1, \ldots, k\} \). Next, if \( m \in \mathbb{N} \), \( m \geq n \) and there is \( q \in J \) satisfying \( \deg q = m \), then the inclusion \( zJ \subseteq J + L \) implies that there is \( r \in L \) for which \( zq - r \in J \). Since \( \deg r \leq \deg q \), \( \deg(zq - r) = m + 1 \). Thus \( J \) contains a polynomial of degree \( m + 1 \) whenever \( m \geq n \) and \( J \) contains a polynomial of degree \( m \). Since \( J \) contains a polynomial of degree \( n \), it follows that \( J \) contains a polynomial of degree \( m \) for every \( m \geq n \). Hence \( \dim(\mathbb{K}[z]/J) \leq n \). Proof of Proposition\ref{lem:4.8}. Since \( E \in M_2 \), there exists a non-trivial continuous seminorm \( p \) on \( E \) such that \( F = \ker p \) carries weak topology. Let \( T \in \text{L}(E) \) and \( x \in E \). We have to show that \( \text{span}(O(T,x)) \neq E \). Assume the contrary. That is, \( \text{span}(O(T,x)) = E \). Since \( E \) is infinite dimensional, the vectors \( T^n x \) are linearly independent. Hence the map \( \Phi : \mathbb{K}[z] \to E, \Phi(p) = p(T)x \) is a linear bijection. Consider the map \( S : F \to E/F, Sx = Tx + F \). Then \( S \) is a continuous linear operator from the space \( F \) carrying weak topology to the space \( E/F \) possessing a continuous norm. Since the continuous linear image of a space with weak topology carries weak topology and only finite dimensional spaces carry weak topology and a continuous norm simultaneously, \( S(F) \) is finite dimensional. That is, there is a finite dimensional subspace \( L \) of \( E \) such that \( T(F) \subseteq F + L \). This means that \( (J + zJ)/J \) is finite dimensional, where \( J = \Phi^{-1}(F) = \{p \in \mathbb{K}[z] : p(T)x \in F\} \). Since \( F \) is infinite dimensional and \( \Phi : \mathbb{K}[z] \to E \) is a linear bijection, \( J \) is infinite dimensional. By Lemma \ref{lem:4.9}, \( J \) has finite codimension in \( \mathbb{K}[z] \). Since \( \Phi(J) = F \) and \( \Phi : \mathbb{K}[z] \to E \) is a linear bijection, \( F = \ker p \) has finite codimension in \( E \), which contradicts the non-triviality of \( p \). 4.5 Proof of Theorems\ref{thm:1.5} and \ref{thm:1.7} Let \( E \in M \) be countably dimensional. If \( E \notin M_2 \), then \( E \) is strongly orbital according to Propositions\ref{lem:4.1}, \ref{lem:4.3} and \ref{lem:4.5}. By Lemma \ref{lem:1.2}, \( E \) does not have the invariant subset property if \( E \notin M_2 \). If \( E \in M_2 \), then \( E \) does not have the invariant subset property according to Proposition \ref{lem:4.7}. Finally, by Proposition \ref{lem:4.8}, \( E \) is non-orbital if \( E \in M_2 \). This proves Theorems\ref{thm:1.5} and \ref{thm:1.7}
5 Proof of Theorem 1.14

We shall construct an operator $T$ with no non-trivial invariant subspaces by lifting a non-linear map on a topological space to a linear map on an appropriate topological vector space.

5.1 A class of complete countably dimensional spaces

Recall that a topological space $X$ is called completely regular (or Tychonoff) if for every $x \in X$ and a closed subset $F \subset X$ satisfying $x \notin F$, there is a continuous $f : X \to \mathbb{R}$ such that $f(x) = 1$ and $f|_F = 0$. Equivalently, a topological space is completely regular, if its topology can be defined by a family of pseudometrics. Note that any subspace of a completely regular space is completely regular and that every topological group is completely regular.

Our construction is based upon the concept of the free locally convex space \cite{25}. Let $X$ be a completely regular topological space. We say that a topological vector space $L_X$ is a free locally convex space of $X$ if $L_X$ is locally convex, contains $X$ as a subset with the topology induced from $L_X$ to $X$ being the original topology of $X$ and for every continuous map $f$ from $X$ to a locally convex space $Y$ there is a unique continuous linear operator $T : L_X \to Y$ such that $T|_X = f$. It turns out that for every completely regular topological space $X$, there is a free locally convex space $L_X$ unique up to an isomorphism leaving points of $X$ invariant. Thus we can speak of the free locally convex space $L_X$ of $X$. Note that $X$ is always a Hamel basis in $L_X$. That is, as a vector space, $L_X$ consists of formal finite linear combinations of elements of $X$. Identifying $x \in X$ with the point mass measure $\delta_x$ on $X$ ($\delta_x(A) = 1$ if $x \in A$ and $\delta_x(A) = 0$ if $x \notin A$), we can also think of elements of $L_X$ as measures with finite support on the $\sigma$-algebra of all subsets of $X$. Under this interpretation

$$L^0_X = \{ \mu \in L_X : \mu(X) = 0 \}$$

is a closed hyperplane in the locally convex space $L_X$. If $f : X \to X$ is a continuous map, from the definition of the free locally convex space it follows that $f$ extends uniquely to a continuous linear operator $T_f \in L(L_X)$. It is also clear that $L^0_X$ is invariant for $T_f$. Thus the restriction $S_f$ of $T_f$ to $L^0_X$ belongs to $L(L^0_X)$.

According to Uspenskii \cite{25}, $L_X$ is complete if and only if $X$ is Dieudonne complete \cite{14} and every compact subset of $X$ is finite. Since Dieudonné completeness follows from paracompactness \cite{14}, every regular countable topological space is Dieudonné complete. Since every countable compact topological space is metrizable \cite{14}, for a countable $X$, finiteness of compact subsets is equivalent to the absence of non-trivial convergent sequences (a convergent sequence is trivial if it is eventually stabilizing). Note also that a regular countable topological space is automatically completely regular \cite{14} and therefore we can safely replace the term completely regular by regular in the context of countable spaces. Thus we can formulate the following corollary of the Uspenskii theorem.

**Proposition 5.1.** Let $X$ be a regular countable topological space. Then the countably dimensional locally convex topological vector spaces $L_X$ and $L^0_X$ are complete if and only if there are no non-trivial convergent sequences in $X$.

The above proposition provides plenty of complete locally convex spaces of countable algebraic dimension. We also need the shape of the dual space of $L_X$. As shown in \cite{25}, $L_X^\prime$ can be identified with the space $C(X)$ of continuous scalar valued functions on $X$ in the following way. Every $f \in C(X)$ produces a continuous linear functional on $L_X$ in the usual way:

$$\langle f, \mu \rangle = \int f \, d\mu = \sum c_j f(x_j), \quad \text{where} \quad \mu = \sum c_j \delta_{x_j}$$

and there are no other continuous linear functionals on $L_X$. Note also that $L_X^0$ is the kernel of the functional $\langle 1, \cdot \rangle$, where $1$ is the constant $1$ function.
5.2 Operators $S_f$ with no invariant subspaces

The following lemma is the main tool in the proof of Theorem 1.14.

**Lemma 5.2.** Let $\tau$ be a regular topology on $\mathbb{Z}$ such that $f : \mathbb{Z} \to \mathbb{Z}$, $f(n) = n + 1$ is a homeomorphism of $\mathbb{Z}_r = (\mathbb{Z}, \tau)$ onto itself, $\mathbb{Z}_+$ is dense in $\mathbb{Z}_r$ and for every $z \in \mathbb{C} \setminus \{0, 1\}$, $n \mapsto z^n$ is non-continuous as a map from $\mathbb{Z}_r$ to $\mathbb{C}$. Then the operators $T_f$ and $S_f$ are invertible continuous linear operators on $L_{\mathbb{Z}_r}$ and $L^0_{\mathbb{Z}_r}$ respectively and $S_f$ has no non-trivial closed invariant subspaces.

**Proof.** We shall prove the lemma in the case $\mathbb{K} = \mathbb{C}$. The case $\mathbb{K} = \mathbb{R}$ follows easily by passing to complexifications.

We already know that $T_f$ and $S_f$ are continuous linear operators. It is easy to see that $T_f^{-1} = T_{f^{-1}}$ and $S_f^{-1} = S_{f^{-1}}$. Since $f^{-1}$ is also continuous, $T_f$ and $S_f$ have continuous inverses.

Now let $\mu \in L^0_{\mathbb{Z}_r} \setminus \{0\}$. It remains to show that $\mu$ is a cyclic vector for $S_f$. Assume the contrary. Then there is a non-constant $g \in C(\mathbb{Z}_r)$ such that $(S_f^n \mu, g) = (T_f^n \mu, g) = 0$ for every $n \in \mathbb{Z}_+$. Decomposing $\mu$ as a linear combination of point mass measures, we have $\mu = \sum_{k=-l}^l c_k \delta_k$ with $c_k \in \mathbb{C}$. Then $\mu = \sum_{j=0}^{2l} c_{\overline{l}} T_f^j \delta_{-l} = p(T_f)^j \delta_{-l}$, where $p$ is a non-zero polynomial. Then $0 = (T_f^n \mu, g) = (T_f^n \delta_{-l}, p(T_f)^j g)$ for $n \in \mathbb{Z}_+$. Thus the functional $p(T_f)^j g$ vanishes on the linear span of $T_f^n \delta_{-l}$ with $n \in \mathbb{Z}_+$, which contains the linear span of $\mathbb{Z}_+$ in $L_{\mathbb{Z}_r}$. Since $\mathbb{Z}_+$ is dense in $\mathbb{Z}_r$, $p(T_f)^j g$ vanishes on a dense linear subspace and therefore $p(T_f)^j g = 0$. It immediately follows that $T_f^j$ has an eigenvector, which is given by a non-constant function $h \in C(\mathbb{X})$: $T_f^j h = z h$ for some $z \in \mathbb{C}$. Since $T_f$ is invertible, so is $T_f^j$ and therefore $z \neq 0$. It is easy to see that $T_f^j h(n) = h(n + 1)$ for each $n \in \mathbb{Z}$. Thus the equality $T_f^j h = z h$ implies that (up to a multiplication by a non-zero constant) $h(n) = z^n$ for each $n \in \mathbb{Z}$. Since $h$ is non-constant, $z \neq 1$. Thus the map $n \mapsto z^n$ is continuous on $\mathbb{Z}_r$ for some $z \in \mathbb{C} \setminus \{0, 1\}$. We have arrived to a contradiction.

**Proof of Theorem 1.14** Let $\tau$ be the topology on $\mathbb{Z}$ provided by Lemma 2.13. By Proposition 5.1, $E = L^0_{\mathbb{Z}_r}$ is a complete locally convex space of countable algebraic dimension. Let $f : \mathbb{Z} \to \mathbb{Z}$, $f(n) = n + 1$ and $S = S_f \in L(E)$. By Lemmas 5.2 and 2.13, $S$ is invertible and has no non-trivial invariant subspaces. The proof is complete (with an added bonus of invertibility of $S$).

6 Proof of Theorem 1.15

We fix the vector space and the operator from the start. The proof will amount to constructing an appropriate topology on the space.

Let $E \subset \omega = \mathbb{K}^\mathbb{N}$ be the subspace of all sequences with finite supports. That is, $x \in E$ precisely when $\text{supp}(x) = \{n \in \mathbb{N} : x_n \neq 0\}$ is finite. In other words, $E = \text{span} \{e_n : n \in \mathbb{N}\}$, where $\{e_n\}_{n \in \mathbb{N}}$ is the standard basis in $\omega$. Clearly $E$ is a dense countably dimensional subspace of $\omega$. By Theorem 1.7, $E$ is strongly orbital. That is, there is $T \in L(\omega)$ and a hypercyclic vector $u \in E$ for $T$ such that $E = \text{span}(O(T, u))$. Since $T \in L(\omega)$, the matrix of $T$ (with respect to the standard basis) is row finite. Since $T(E) \subseteq E$, the matrix of $T$ is column finite. Thus the matrix of $T$ is both row and column finite. It follows that there is $T^* \in L(\omega)$, whose matrix is the transpose of the matrix of $T$ and that $T^*(E) \subseteq E$. Of course, the matrix of $T^*$ is also row and column finite. We shall use the $\langle \cdot, \cdot \rangle$ notation for the canonical bilinear form on $\omega \times E$:

$$\langle x, y \rangle = \sum_{n=1}^{\infty} x_n y_n.$$  

Clearly, the restriction of this form to $E \times E$ is symmetric and $\langle Tx, y \rangle = \langle x, T^* y \rangle$ for every $x, y \in E$. We say that a sequence $\{x_n\}_{n \in \mathbb{N}}$ in $E$ has **point finite supports** if every $k \in \mathbb{N}$ can belong only to finitely many sets $\text{supp}(x_n)$.

For every sequence $v = \{v_n\}_{n \in \mathbb{N}}$ in $E$ with point finite supports, we can define the seminorm $p_v$ on $E$:

$$p_v(x) = \sum_{n=1}^{\infty} |\langle x, v_n \rangle|.$$  

---
The point finiteness of the supports of \( v_n \) together with finiteness of the support of \( x \), guarantee that the terms in the above series are zero for all sufficiently large \( n \). By \( P \), we shall denote the set of all seminorms on \( E \) obtained in this way. For \( p \in P \), we denote \( p^{[k]}(x) = p(T^k x) \) for \( k \in \mathbb{Z}_+ \) and \( x \in E \). Clearly, each \( p^{[k]} \) is a seminorm on \( E \). Moreover, \( p^{[k]} \in P \) for every \( p \in P \) and \( k \in \mathbb{Z}_+ \). Indeed, the column finiteness of the matrix of \( T^* \) implies that if a sequence \( v = \{v_n\}_{n \in \mathbb{N}} \) in \( E \) has point finite supports, then the sequence \( \{T^* v_n\}_{n \in \mathbb{N}} \) also has point finite supports. Thus the seminorm \( p_v(T \cdot) \) also belongs to \( P \):

\[
p_v(Tx) = \sum_{n=1}^{\infty} |\langle Tx, v_n \rangle| = \sum_{n=1}^{\infty} |\langle x, T^* v_n \rangle|.
\]

We say that a subset \( Q \) of \( P \) is \( T \)-invariant if \( p^{[1]} = p(T \cdot) \) belongs to \( Q \) for every \( p \in Q \). Obviously, if \( Q \) is a \( T \)-invariant subset of \( P \), then \( T : E \rightarrow E \) is continuous when \( E \) carries the topology \( \tau_Q \) defined by the collection \( Q \) of seminorms. From now on let

\[
\tau_0 \text{ be the topology on } E \text{ inherited from } \omega.
\]

Obviously, the seminorms \( \pi_k(x) = |x_k| \) belong to \( P \) and the collection \( \{\pi_k : k \in \mathbb{N}\} \) defines (on \( E \)) the topology \( \tau_0 \). Furthermore, \( \pi_k^{[s]} \) is also \( \tau_0 \)-continuous for every \( k \in \mathbb{N} \) and \( s \in \mathbb{Z}_+ \). Let

\[
P_0 = \{\pi_k^{[s]} : k \in \mathbb{N}, s \in \mathbb{Z}_+\}.
\]

Then \( P_0 \) is a countable \( T \)-invariant subset of \( P \), which defines the same topology \( \tau_0 \).

We say that \( y \in \omega \) is \( p_v \text{-singular} \) if

\[
\sum_{n=1}^{\infty} |\langle y, v_n \rangle| = \infty.
\]

Lemma 6.1. Let \( Q \) be a subset of \( P \) such that \( \pi_k \in Q \) for each \( k \in \mathbb{N} \) and for every \( y \in \omega \setminus E \), there is \( p \in Q \) for which \( y \) is \( p \)-singular. Then the topological vector space \((E, \tau)\) is complete, where \( \tau = \tau_Q \) is the topology defined by the family \( Q \) of seminorms.

Proof. Clearly \( \omega \) is complete and induces on \( E \) the topology weaker than \( \tau \). Next, it is easy to see that \( \{x \in E : p(x) \leq 1\} \) is \( \tau_0 \)-closed in \( E \) for every \( p \in P \). By Lemma 2.10, the completeness of \((E, \tau)\) will be verified if we show that for every \( \tau \)-Cauchy net \( \{x_\alpha\} \) in \( E \), its \( \tau_0 \)-limit \( \lim_{\tau_0} x_\alpha \) belongs to \( E \).

Let \( \{x_\alpha\} \) be a \( \tau \)-Cauchy net in \( E \), and \( y_\alpha \) be the \( \tau_0 \)-limit of \( \{x_\alpha\} \) in \( \omega \). We have to show that \( x \in E \). Assume the contrary. Then by our assumptions there is \( p_\alpha \in Q \) such that \( x \) is \( p_\alpha \)-singular. Since \( p_\alpha \in Q \), \( \{p_\alpha(x_\alpha)\}_{\alpha \in D} \) is a Cauchy net in \([0, \infty)\). Hence there is a non-negative real number \( c \) such that \( p_\alpha(x_\alpha) \rightarrow c \).

Next, since \( \{x_\alpha\} \) converges to \( x \) in \( \omega \)

\[
c = \lim_{\alpha} p_\alpha(x_\alpha) \geq \lim_{\alpha} \sum_{n=1}^{m} |\langle x_\alpha, v_n \rangle| = \sum_{n=1}^{m} |\langle x, v_n \rangle| \quad \text{for every } m \in \mathbb{N}.
\]

We have arrived to a contradiction with the \( p_\alpha \)-singularity of \( x \), which proves that \( x \in E \). \( \square \)

Let \( G \) be the set of \( x \in E \) such that each \( x_k \) is rational \((x_k \in \mathbb{Q} + i\mathbb{Q} \text{ in the case } K = \mathbb{C})\). Clearly \( G \) is a countable subset of \( E \). Furthermore, it is easy to see that \( G \) is \( \tau \)-dense in \( E \) for every topology \( \tau \) such that \((E, \tau)\) is a topological vector space. \( \quad (6.2)\)

The following lemma is our main instrument.

Lemma 6.2. Let \( x \in \omega \setminus E \) and \( S \) be a countable subset of \( P \) such that \( O(T, u) \) is \( \tau_S \)-dense in \( E \) and \( \tau_0 \subseteq \tau_S \), where \( \tau_S \) is the topology generated by \( S \). Then there exists a sequence \( v = \{v_n\}_{n \in \mathbb{N}} \) in \( E \) with point finite supports such that \( x \) is \( p_v \)-singular and \( O(T, u) \) is \( \tau_{S'} \)-dense in \( E \), where \( S' = S \cup \{p^{[k]}_v : k \in \mathbb{Z}_+\} \).
According to the last two displays, \( n_k > n_{k-1} \) and \( m_k > m_{k-1} \) if \( k \geq 2 \);
\[ x_{m_k} \neq 0 \text{ and } v_k = \frac{1}{|x_{m_k}|}e_{m_k}; \]
\[ r_k(g_k - T^{n_k}u) < \frac{1}{k}; \]
\[ \langle T^l(g_j - T^{n_j}u), v_m \rangle = 0 \text{ for } 0 \leq l < j \leq m \leq k; \]
\[ |\langle T^l(g_j - T^{n_j}u), v_m \rangle| < 2^{-j} \text{ for } 0 \leq l < j < k \text{ and } 1 \leq m < j \leq k. \]

**Basis of induction.** Since \( O(T, u) \) is \( \tau_S \)-dense in \( E \), there is \( n_1 \in \mathbb{N} \) such that \( r_1(T^{n_1}u - g_1) < 1 \). Since \( x \notin E \), we can find \( m_1 \in \mathbb{N} \) such that \( x_{m_1} \neq 0 \) and \( m_1 \notin \text{supp} (g_1 - T^{n_1}u) \). Denote \( v_1 = \frac{e_{m_1}}{|x_{m_1}|} \). Since \( m_1 \notin \text{supp} (g_1 - T^{n_1}u) \), we have \( \langle g_1 - T^{n_1}u, v_1 \rangle = 0 \). Clearly, (6.3) \( \text{and } (6.7) \) with \( k = 1 \) are satisfied.

**Induction step.** Assume now that \( q \geq 2 \) and \( n_j, m_j, v_j \) satisfying (6.3) \( \text{and } (6.7) \) for every \( k < q \) are already constructed. We shall construct \( n_q, m_q \) and \( v_q \) such that (6.3) \( \text{and } (6.7) \) are satisfied with \( k = q \). Since \( O(T, u) \) is \( \tau_S \)-dense in \( E \) and \( \tau_0 \subseteq \tau_S \), we can choose \( n_q \in \mathbb{N} \) such that
\[ n_q > n_{q-1}, r_q(g_q - T^{n_q}u) < \frac{1}{q} \text{ and } |\langle T^l(g_q - T^{n_q}u), v_m \rangle| < 2^{-q} \text{ for } 0 \leq l < q \text{ and } 1 \leq m < q. \]

Since \( x \notin E \), there is \( m_q \in \mathbb{N} \) such that
\[ m_q > m_{q-1}, \quad x_{m_q} \neq 0 \quad \text{and} \quad m_q \notin \bigcup_{0 \leq l < q} \text{supp} (T^l(g_j - T^{n_j}u)). \]

Since \( x_{m_q} \neq 0 \), we can define \( v_q = \frac{e_{m_q}}{|x_{m_q}|} \). Clearly, (6.3) \( \text{and } (6.4) \) with \( k = q \) are satisfied. The condition (6.6) with \( k = q \) follows from (6.9) \( \text{and } (6.4) \) with \( k < q \). Similarly, (6.7) with \( k = q \) follows from (6.8) \( \text{and } (6.7) \) with \( k < q \). This concludes the inductive construction of the sequences \( \{n_k\}_{k \in \mathbb{N}} \), \( \{m_k\}_{k \in \mathbb{N}} \) and \( v = \{v_k\}_{k \in \mathbb{N}} \) satisfying (6.3) \( \text{and } (6.7) \).

By (6.3) \( \text{and } (6.4) \) the supports of \( v_k \) are pairwise disjoint and therefore are point finite. Hence \( p_v \in \mathcal{P} \).

It remains to show that \( O(T, u) \) is \( \tau_{S'} \)-dense in \( E \), where \( S' = S \cup \{p_v^k : k \in \mathbb{Z}_+\} \). By (6.5),
\[ r_k(g_k - T^{n_k}u) \rightarrow 0. \]

Now let \( l \in \mathbb{Z}_+ \). Using (6.6) \( \text{and } (6.7) \), we see that for \( j > l \),
\[ p^{(l)}(g_j - T^{n_j}u) = \sum_{m=1}^{\infty} |\langle T^l(g_j - T^{n_j}u, v_m) \rangle| = \sum_{m=1}^{j} |\langle T^l(g_j - T^{n_j}u, v_m) \rangle| + \left\{ \begin{array}{ll}
\sum_{m=j+1}^{\infty} |\langle T^l(g_j - T^{n_j}u, v_m) \rangle| & \text{if } j < \infty \\
\sum_{m=1}^{\infty} |\langle T^l(g_j - T^{n_j}u, v_m) \rangle| & \text{if } j = \infty
\end{array} \right. \]

According to the last two displays, \( g_j - T^{n_j}u \rightarrow 0 \) in \( \tau_{Q'} \). By (6.2), \( \{g_j : j \in \mathbb{N}\} \) is \( \tau_{Q'} \)-dense in \( E \). Since \( g_j - T^{n_j}u \rightarrow 0 \) in \( \tau_{Q'} \), it follows that \( \{T^{n_j}u : j \in \mathbb{N}\} \) is \( \tau_{Q'} \)-dense in \( E \). Hence \( O(T, u) \) is \( \tau_{Q'} \)-dense in \( E \). \( \square \)

Now we are ready to prove Theorem 1.15. Since \( |\omega \setminus E| = 2^{\aleph_0} \), the Continuum Hypothesis implies that \( |\omega \setminus E| = \aleph_1 \). We shall also denote the first ordinal of cardinality \( \aleph_1 \) by the same symbol \( \aleph_1 \). The equality \( |\omega \setminus E| = \aleph_1 \) allows us to enumerate \( \omega \setminus E \) by the ordinals \( \alpha \) satisfying \( 1 \leq \alpha < \aleph_1; \omega \setminus E = \{x_\alpha \}_{1 \leq \alpha < \aleph_1} \).

We shall construct inductively a chain \( \{P_\alpha\}_{\alpha < \aleph_1} \) of subsets of \( \mathcal{P} \) satisfying the following conditions
\[ P_0 \text{ is the set defined in (6.1);} \]
\[ P_\beta \subseteq P_\alpha \text{ if } \beta < \alpha; \]
\[ P_\alpha \text{ is countable and } T\text{-invariant;} \]
\[ \text{there is } p \in P_\alpha \text{ such that } x_\alpha \text{ is } p\text{-singular if } \alpha \geq 1; \]
\[ O(T, u) \text{ is } \tau_\alpha \text{-dense in } E, \text{ where } \tau_\alpha \text{ is the topology defined by } P_\alpha. \]
We will use the following elementary observation.

If \( \{\tau_s\}_{s \in S} \) is a totally ordered by inclusion collection of topologies on the same set \( X \), then \( \tau = \bigcup \tau_s \) is a topology on \( X \). Furthermore, a set \( A \subseteq X \) is \( \tau \)-dense in \( X \) if and only if \( A \) is \( \tau_s \)-dense in \( X \) for each \( s \in S \). (6.15)

The set \( P_0 \) serves as the basis of induction. Assume now that \( \gamma < \aleph_1 \) and \( P_{\beta} \) for \( \beta < \gamma \) satisfying (6.11–6.14) for each \( \alpha < \gamma \) are already constructed. First, set \( Q_\gamma = \bigcup_{\beta < \gamma} P_\beta \). Then \( Q_\gamma \) is \( T \)-invariant (as a union of \( T \)-invariant sets) and countable as a countable union of countable sets. Furthermore, the topology \( \theta \) generated by \( Q_\gamma \) is the union of \( \tau_\beta \) for \( \beta < \gamma \). By (6.15) and (6.14) for \( \alpha < \gamma \), \( O(T, u) \) is \( \tau_\gamma \)-dense in \( E \). Since \( x_\gamma \in \omega \setminus E \), all conditions of Lemma 6.2 with \( x = x_\gamma \) and \( S = Q_\gamma \) are satisfied. Lemma 6.2 provides a sequence \( v = \{v_n\}_{n \in \mathbb{N}} \) in \( E \) with point finite supports such that \( x_\gamma \) is \( p_v \)-singular and \( O(T, u) \) is \( \tau_\gamma \)-dense in \( E \), where \( \tau_\gamma \) is generated by \( P_\gamma = Q_\gamma \cup \{p^{[k]}_v : k \in \mathbb{Z}_+\} \). Clearly, the just constructed set \( P_\gamma \) satisfies (6.11–6.14) with \( \alpha = \gamma \). This concludes the construction of the chain \( \{P_\alpha\}_{\alpha < \aleph_1} \) of subsets of \( P \) satisfying (6.10–6.14). Now we consider

\[
P^* = \bigcup_{\alpha < \aleph_1} P_\alpha.
\]

Clearly \( P^* \) is \( T \)-invariant as a union of \( T \)-invariant sets. Hence \( T \) is a continuous linear operator on \((E, \tau^*)\), where the topology \( \tau^* \) defined by \( P^* \). Furthermore, since \( P_\alpha \) are totally ordered by inclusion, \( \tau^* \) coincides with \( \bigcup \tau_\alpha \) for \( \alpha < \aleph_1 \). According to (6.11) and (6.15), \( O(T, u) \) is \( \tau^* \)-dense in \( E \). Hence \( u \) is a hypercyclic vector for \( T \in L(E_{\tau^*}) \). Finally, by (6.11) and (6.13), for every \( x \in \omega \setminus E \), there is \( p \in P^* \) such that \( x \) is \( p \)-singular. Now Lemma 6.1 guarantees the completeness of \((E, \tau^*)\). Thus \( E_{\tau^*} = (E, \tau^*) \) is a complete locally convex space, \( T \in L(E_{\tau^*}) \), \( u \) is a hypercyclic vector for \( T \) and \( E = \text{span}(O(T, u)) \). Thus \( E_{\tau^*} \) is complete and strongly orbital.

### 7 Open problems

We have characterized orbital and strongly orbital metrizable locally convex spaces. The following more general problems remains open.

**Question 7.1.** Characterize orbital countably dimensional locally convex spaces. Characterize orbital countably dimensional complete locally convex spaces.

**Question 7.2.** Characterize strongly orbital countably dimensional locally convex spaces. Characterize strongly orbital countably dimensional complete locally convex spaces.

**Question 7.3.** Characterize countably dimensional locally convex spaces having the invariant subspace/subset property. Characterize complete countably dimensional locally convex topological vector spaces having the invariant subspace/subset property.

**Question 7.4.** Can one get rid of the Continuum Hypothesis in Theorem 1.15?
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