Yangians, finite \( \mathcal{W} \)-algebras
and the
Non Linear Schrödinger hierarchy

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Abstract

We show an algebra morphism between Yangians and some finite \( \mathcal{W} \)-algebras. This correspondence is nicely illustrated in the framework of the Non Linear Schrödinger hierarchy. For such a purpose, we give an explicit realization of the Yangian generators in terms of deformed oscillators.
1 Introduction

$\mathcal{W}$ algebras first showed up in the context of two dimensional conformal field theories[1]. Yangians were first considered and defined in connection with some rational solutions of the quantum Yang-Baxter equation[2].

In this note, we will show that the defining relations of a Yangian are satisfied for a family of finite $\mathcal{W}$ algebras (FWA). In other words, such $\mathcal{W}$-algebras provide Yangian realizations. This remarkable connection between two a priori different types of symmetry deserves in our opinion to be considered more thoroughly. We have already shown how this result can help for the classification of the irreducible finite dimensional representations of FWA’s[3]. Here, we show that the Non Linear Schrödinger Hierarchy is a nice framework where the connection is visualised. For such a purpose, we first study this hierarchy and construct the Yangian generators using a deformed oscillator algebra.

This report is a condensed version of [3, 4].

2 Finite $\mathcal{W}(sl(nm), n.sl(m))$ algebras

The usual notation for a $\mathcal{W}$ algebra obtained by the Hamiltonian reduction procedure is $\mathcal{W}(\mathcal{G}, \mathcal{H})$[5, 6]. More precisely, given a simple Lie algebra $\mathcal{G}$, there is a one-to-one correspondence between the finite $\mathcal{W}$ algebras one can construct in $\mathcal{U}(\mathcal{G})$ and the $sl(2)$ subalgebras in $\mathcal{G}$. Since any $sl(2)$ $\mathcal{G}$-subalgebra is principal in a subalgebra $\mathcal{H}$ of $\mathcal{G}$, it is rather usual to denote the corresponding $\mathcal{W}$ algebra as $\mathcal{W}(\mathcal{G}, \mathcal{H})$.

As an example[7], let us consider the $\mathcal{W}(sl(4), 2.sl(2))$ algebra, where $2.sl(2)$ stands for $sl(2) \oplus sl(2)$. It is made of seven generators $J_i, S_i$ ($i = 1, 2, 3$) and a central element $C_2$ such that:

\[
\begin{align*}
[J_i, J_j] &= i\epsilon_{ij}^k J_k \\
[J_i, S_j] &= i\epsilon_{ij}^k S_k \\
[S_i, S_j] &= -i\epsilon_{ij}^k J_k (2J^2 - C_2 - 4) \\
[C_2, J_i] &= [C_2, S_i] = 0
\end{align*}
\]

(1) with $J^2 = J_1^2 + J_2^2 + J_3^2$

We recognize the $sl(2)$ subalgebra generated by the $J_i$’s as well as an adjoint representation (i.e. $S_i$ generators) of this $sl(2)$ algebra. We note that the $S_i$’s close polynomially on the other generators. If one assumes a
“degree” 1 for the J’s and 2 for the S’s, we remark that the commutator of a degree \(k\) generator with a degree \(j\) one is of degree \(j + k - 1\).

The same type of structure can be remarked, at a higher level, for the class of algebras \(W(sl(nm), n.sl(m))\). This algebra is formed by generators \(W \alpha_k^a (a = 1, \ldots, (n^2 - 1)\) and \(k = 1, \ldots, m\) together with central elements \(C_k\). The \(W\) generators gather into a stack of \(m\) adjoint representations of \(sl(n)\) (the first one being an \(sl(n)\) algebra), indexed by the degree \(k\), and which close polynomially, respecting the degree: \([W^a_k, W^b_j] = P_{k+j-1}^{ab}\) where \(P\) is a polynomial in \(W\) and \(C\) generators.

3 Yangians \(Y(\mathcal{G})\)

Yangians are infinite dimensional quantum groups [2, 3] that correspond to quantization of (half of) the loop algebra of some finite-dimensional Lie algebra \(\mathcal{G}\). As such, it is a Hopf algebra, possesses a R-matrix and can be defined through a \(RTT = TTR\) relation. Here, we will choose an alternative approach (known to be equivalent) that enlights the loop algebra deformation and which is best suited to our purpose. In that context, the Yangian \(Y(\mathcal{G})\) is generated by an infinite stack of adjoint representations of \(\mathcal{G}\), indexed by a degree \(n\) going from 0 to infinity, the degree 0 corresponding to \(\mathcal{G}\) itself, the generators of degree 1 being subject to the following constraints:

\[
\begin{align*}
&f^{bc}_{d} [Q^c_1, Q^d_1] + f^{ca}_{d} [Q^c_1, Q^d_1] + f^{ab}_{c} [Q^c_1, Q^d_1] = f^{a}_{pd} f^{b}_{qx} f^{c}_{ry} f^{xy}_{e} \eta^{de}_{s} s_3(Q^p_0, Q^q_0, Q^r_0) \\
&f^{cd}_{e} ([Q^c_1, Q^d_1], Q^e_1) + f^{ab}_{c} ([Q^c_1, Q^d_1], Q^e_1) = \\
&(f^{a}_{pc} f^{b}_{qz} f^{c}_{ry} f^{d}_{g} f^{e}_{zr} f^{xz}_{y} g + f^{c}_{pe} f^{d}_{qz} f^{a}_{v} f^{b}_{y} f^{z}_{xz}_{v} f^{xz}_{g}) \eta^{eg}_{s} s_3(Q^p_0, Q^q_0, Q^r_1)
\end{align*}
\]

where \(f^{ab}_{c}\) are the totally antisymmetric structure constant of \(\mathcal{G}\), \(\eta^{ab}\) is the Killing form, and \(s_3(,\ldots,\ldots)\) is the totally symmetrized product of 3 terms. It can be shown that for \(\mathcal{G} = sl(2)\), the first constraint is trivially satisfied, while for \(\mathcal{G} \neq sl(2)\), the last constraint follows from the previous one.

Note that the constraints imply only the \(Q_0\) and \(Q_1\) generators, \(Y(\mathcal{G})\) being totally defined once we have said that it is a homogeneous quantization of the loop algebra on \(\mathcal{G}\).

In the following, we will focus on the Yangians \(Y(sl(n))\).
4 \( \mathcal{W}(sl(nm), n.sl(m)) \) as a realisation of \( Y(sl(n)) \).

From the previous definitions of \( \mathcal{W}(sl(nm), n.sl(m)) \) algebras and Yangian \( Y(sl(n)) \), it is natural to look for a relation between these objects. Indeed, it can be proven that

Identifying the generators \( W_k^a \) of the finite \( \mathcal{W}(sl(nm), n.sl(m)) \) algebra with the elements \( Q^{k-1}_k \) \((k = 1, \ldots m)\) of the Yangian \( Y(sl(n)) \), one verifies that the defining relations of a Yangian are satisfied for this \( \mathcal{W} \) algebra, which therefore appears as a realisation of the Yangian \( Y(sl(n)) \).

Owing to the above identification, it is possible to link the irreducible finite dimensional representations of the Yangians (known to be products of evaluation representations) to the Miura map developed in the framework of \( \mathcal{W} \) algebras. We will not develop these points, inviting the interested reader to look at [3, 9] for more information. Instead, we will focus on a “physical” framework where the identification can be “visualised”.

5 Non Linear Schrödinger equation and its hierarchy

5.1 State of the art

We start with the well-known Non Linear Schrödinger equation (NLS) in 1+1 dimension, on the real line:

\[
i \frac{\partial \phi(x,t)}{\partial t} + \frac{\partial^2 \phi}{\partial^2 x} = 2g |\phi|^2 \phi
\]  

(2)

where \( \phi \) is a complex field.

In the following we will look at the “vectorial” version of it:

\[
i \frac{\partial \phi(x,t)}{\partial t} + \frac{\partial^2 \phi}{\partial^2 x} = 2g (\phi^\dagger \cdot \phi) \phi \quad \text{with} \quad \phi = \left( \begin{array}{c} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_N \end{array} \right)
\]  

(3)

The solution to this equation has being given (at the classical level) by Rosales [10]:

\[
\phi(x,t) = \sum_{n=0}^{\infty} (-g)^n \phi_n(x,t) \quad \text{where}
\]  

(4)
\[
\phi_n = \int d^{n+1}q \, d^n p \, \frac{\tilde{\lambda}(p_1)\tilde{\lambda}(p_2) \cdots \tilde{\lambda}(p_n)\lambda(q_n) \cdots \lambda(q_1)\lambda(q_0)}{\prod_{j=1}^{n}(p_j - q_j)(p_j - q_{j-1})} \exp(i\Omega_n)
\]

with \(\lambda\) an arbitrary function (such that the series is well-defined). A remarkable fact is that this solution is also valid at the quantum level \([11]\), or more precisely if one looks for a local field \(\phi\) satisfying

\[
[\phi(x, t), \phi^\dagger(y, t)] = \delta(x - y) \quad \text{and} \quad [\phi(x, t), \phi(y, t)] = 0
\]

where:

\[
\phi(x, t) = e^{iHt}\phi(x, 0)e^{-iHt} \quad \text{with} \quad H = \int dx \, \frac{\partial^\dagger}{\partial x} \cdot \frac{\partial}{\partial x} + g(\phi^\dagger \cdot \phi)^2
\]

then, the solution is given by a series expansion of the above type with

\[
\phi_n = \sqrt{2\pi} \int d^{n+1}q \, d^n p \, \frac{a^\dagger(p_1)a^\dagger(p_2) \cdots a^\dagger(p_n)a(q_n) \cdots a(q_1)a(q_0)}{\prod_{j=1}^{n}(p_j - q_j - i\epsilon)(p_j - q_{j-1} - i\epsilon)} \exp(i\Omega_n)
\]

where now \(a(p)\) and \(a^\dagger(p)\) generate a deformed oscillator algebra (or ZF algebra \([12]\)):

\[
a^\alpha(p)a^\beta(p') = R_{\beta\alpha}^{\mu}(p' - p)a^\mu(p')a^\nu(p) \quad (8)
\]

\[
a^\dagger^\alpha(p)a^\dagger^\beta(p') = a^\dagger^\nu(p')a^\dagger^\mu(p)R_{\mu\nu}^{\alpha\beta}(p' - p) \quad (9)
\]

\[
a^\alpha(p)a^\dagger^\beta(p') = a^\dagger^\mu(p')R_{\alpha\mu}^{\nu}(p - p')a^\nu(p) + \delta_{\alpha}^\beta\delta(p - p') \quad (10)
\]

The construction \([11]\) is understood on the Fock space of this ZF algebra \([13]\). Note that the indices \(\alpha, \beta, \ldots\) run from 1 to \(N\), because the vector \(\phi\) is in the fundamental representation of \(sl(N)\). The R-matrix appearing in this algebra is just the one of \(Y(sl(N))\). This is not surprising, since \(Y(sl(N))\) is a symmetry of NLS. In fact, it has been shown in \([14]\) that, for \(N = 2\), the Yangian generators can be expressed in term of \(\phi\) and Pauli matrices \(t^a\):

\[
J^a = \int dx \, \phi^\dagger(x)t^a\phi(x) \quad (11)
\]

\[
S^a = \frac{i}{2} \int dx \, \phi^\dagger(x)t^a \partial_x \phi(x) - \frac{ig}{2} \int dx dy \, \text{sgn}(y - x) \left(\phi^\dagger(x)t^a\phi(y)\right) \phi^\dagger(x) \cdot \phi(y)
\]

where \(J^a\) stands for the generator \(Q^a_0\), and \(S^a\) for \(Q^a_1\). It is easy to see that this formula is also valid in the general case \(Y(sl(N))\), the generators \(t^a\) being in the fundamental representation of \(sl(N)\).
It is natural to look for the form of the Yangian generators in terms of the ZF generators. We are going to see that the answer, although not so trivial, will reveal the natural link between Yangians and \( W \)-algebras, and will also lead to a very nice formulation of the NLS hierarchy.

5.2 NLS hierarchy

It is interesting to rewrite the Hamiltonian (6), the total momentum \( P = -i \int dx \phi^\dagger \cdot \partial_x \phi \) and the particle number \( N = \int dx \phi^\dagger \cdot \phi \) in term of the ZF algebra. The result is surprisingly simple:

\[
N = \int dp \ a^\dagger(p) \cdot a(p) ; \quad P = \int dp \ p \ a^\dagger(p) \cdot a(p) \quad (12)
\]

\[
H = \int dp \ p^2 \ a^\dagger(p) \cdot a(p) \quad (13)
\]

More generally, one can consider the operators

\[
H_k = \int dp \ p^k \ a^\dagger(p) \cdot a(p) \quad (14)
\]

A very nice result is [4]:

i) \( H_k \) is the Hamiltonian of the \( k \)th equation of the NLS hierarchy.

ii) The local field evolving with \( H_k \) takes the form (4), (7), with now

\[
\Omega_{k,n} = \sum_{j=0}^{n}(xq_j - tq_j^k) - \sum_{j=1}^{n}(xp_j - tp_j^k) \quad (15)
\]

5.3 Yangian and deformed oscillators

In view of the formulae (7) and (11), it is clear that trying to reconstruct the Yangian generators (in term of oscillators) by direct calculation is a difficult task. Fortunately, the commutation relations of the oscillators with these generators have being given in [14], and it is simpler to seek for operators that fulfill these relations (see [4] for details). To simplify the presentation, we drop the indices \( \alpha, \beta, \ldots \) and the momenta, and use indices refering to spaces instead. For instance, (10) reads \( a_1 a_2^\dagger = a_2^\dagger R_{21} a_1 + \delta_{12} \). For \( sl(2) \), a careful calculation leads to (the general case can be found in [4]):

\[
J^a = \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n!} J^n_a \quad \text{and} \quad S^a = \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n!} S^n_a \quad (16)
\]
\[ J_n^a = a_{1\ldots n}^\dagger T_{1\ldots n}^a a_{n-1} \quad \text{with} \quad T_{1\ldots n}^a = \sum_{j=1}^n (-1)^{j-1} \binom{n-1}{j-1} t_j^a = \sum_{j=1}^n a_j^a t_j^a \]

\[ S_n^a = a_{1\ldots n}^\dagger \bar{T}_{1\ldots n}^a a_{n-1} \quad \text{with} \quad \bar{T}_{1\ldots n}^a = \sum_{j=1}^n \alpha_j^a \left( p_j t_j^a - i g f_{abc}^a \sum_{i=1}^j t_i^b t_i^c \right) \]

where \( a_{1\ldots n}^\dagger = a_1^\dagger (p_1) a_2^\dagger (p_2) \cdots a_n^\dagger (p_n) \), \( a_{n-1} = a_n (p_n) a_{n-1} (p_{n-1}) \cdots a_1 (p_1) \) and the integration on \( p_1, p_2, \ldots, p_n \) is implied in \( J^a \) and \( S^a \).

The formulae (16) provide a construction for the Yangian generators in terms of the ZF algebra. In this formulation, it is easy to see that the Yangian generators commute with \( H_k \), so that the Yangian symmetry of the whole NLS hierarchy is manifest.

### 5.4 \( \mathcal{W} \)-algebra in NLS hierarchy

Let us now focus on the Fock space \( \mathcal{F} \) associated to the ZF algebra. It can be decomposed into a direct sum of subspaces with fixed particle number: \( \mathcal{F} = \bigoplus_{m=0}^\infty \mathcal{F}_m \). Since the Yangian generators commute with the particle number, we can consider their restriction to any \( \mathcal{F}_m \). On that subspace, the infinite series (16) truncate, because the products of more than \( m \) \( a \)'s identically vanish on \( \mathcal{F}_m \). We are thus considering polynomials in \( a \)'s and \( a^\dagger \)'s of order less than \( m+1 \). This implies that there is only a finite number of independent Yangian generators (considered as operators on \( \mathcal{F}_m \)), the other ones being in their enveloping algebra. In other words, we get a polynomial algebra which satisfies the defining relation of the Yangian \( Y(sl(N)) \): it is related to the \( \mathcal{W}(sl(mN), Nsl(m)) \) algebra. More precisely, the Hamiltonians \( H_k \) (\( k \leq m \)) have to be added and generate a \( m \)-dimensional center of this polynomial algebra, while the \( \mathcal{W}(sl(mN), Nsl(m)) \) algebra possesses only \( m-1 \) central generators. One thus needs a constraint to connect these two algebras. It appears to be \( H_1 = 0 \), i.e. the vanishing of the total momentum. On the corresponding Fock space \( \mathcal{F}_m^{(\text{red})} \), the Yangian will be represented by a \( \mathcal{W} \)-algebra. Note that this constraint is just the one introduced in [3, 4] when looking at Yangian evaluation representations as \( \mathcal{W} \) representations (see [4] for details).

In the special case of \( m = 1 \), since \( S_1^a = 0 \) (on \( \mathcal{F}_1^{(\text{red})} \)), we recover a \( sl(N) \) algebra.
6 Conclusion

There is a natural correspondence between $\mathcal{W}(sl(mn), n.sl(m))$ algebras and Yangians $Y(sl(n))$. It is well illustrated in the framework of NLS hierarchy, where the use of a deformed oscillator algebra makes explicit the Yangian symmetry of this hierarchy. On the Fock space of the ZF algebra, the action of the Yangian is represented (for fixed particle number $m$ and vanishing total momentum) by a $\mathcal{W}(sl(mn), n.sl(m))$ algebra. This is visualised in the figure.

Figure 1: Action of the Yangian on the reduced Fock space. On each subspace $\mathcal{F}^{(red)}_m$ (fixed particle number $m$ and vanishing total momentum), it is realized by a finite $\mathcal{W}$ algebra which acts on $\mathcal{F}^{(red)}_m$.

This construction can be generalised to the case of NLS on a half-line. In that case, Yangians are replaced by twisted Yangians [15], the ZF algebra by an enlarged deformed oscillator algebra which takes into account the boundary properties [16], and the $\mathcal{W}$ algebra by a folded $\mathcal{W}$ algebra [17]. This work is under investigation [18].
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