APPROXIMATION OF ALGEBRAIC RICCATI EQUATIONS WITH GENERATORS OF NONCOMPACT SEMIGROUPS

JAMES CHEUNG

Abstract. In this work, we demonstrate that the Bochner integral representation of the Algebraic Riccati Equations (ARE) are well-posed without any compactness assumptions on the coefficient and semigroup operators. From this result, we then are able to determine that, under some assumptions, the solution to the Galerkin approximations to these equations are convergent to the infinite dimensional solution. Going further, we apply this general result to demonstrate that the finite element approximation to the ARE are optimal for weakly damped wave semigroup processes in the $H^1(\Omega) \times L^2(\Omega)$ norm. Optimal convergence rates of the functional gain for a weakly damped wave optimal control system in both the $H^1(\Omega) \times L^2(\Omega)$ and $L^2(\Omega) \times L^2(\Omega)$ norms are demonstrated in the numerical examples.

1. Introduction

In recent years, the reinforcement learning community has been rediscovering the usefulness of optimal control in the context of determining action policies. In general, reinforcement learning is involved with the direct numerical approximation of the dynamic programming principle. A simplification of the Hamilton-Jacobi-Bellman equations in this framework can be made if (1) the system is linear, and if (2) the cost functional is quadratic. Under these conditions, these equations can be reduced to the well-known Differential Algebraic Riccati Equations (DARE). In the infinite time horizon limit, the DARE becomes the Algebraic Riccati Equations (ARE). The ARE has a long standing history in optimal and control theory. Aside from its direct application in LQR problems, it has been used also in nonlinear control through linearization and gain scheduling strategies. In addition, the ARE is solved at every time step whenever an implicit time-stepping scheme is used to discretize the DARE. Furthermore, these equations are the basis of the widely used continuous time Kalman filter. These broad implications of the ARE make it a very important problem to study in control theory, and by extension, reinforcement learning.

The development of a numerical theory that allows for the derivation of optimal error estimates have proved difficult in the past. Most existing results in the literature are concerned with the general convergence properties of the approximation of the ARE to its infinite dimensional solution (c.f. [2, 4, 6, 3]) rather than the determination of error estimates. Some developments in deriving error estimates for parabolic control problems can be found in [17, 19], where suboptimal error bounds were derived. Optimal error estimates were then derived for this parabolic problem in [5] by leveraging the work done in [7] on the Bochner integrability of the mild representation of the ARE. One of the limitations of the previously cited work is that the abstract theory presented in its discussion does not apply to systems that are represented by noncompact semigroups. This is a severe limitation since many interesting control problems involve these systems - i.e. the control of weakly damped waves. The derivation of error estimates on the Galerkin approximations of these systems is the subject of study in this paper.

The importance of the control and state estimation of systems of weakly damped wave equations cannot be understated. They can be encountered in transmission lines in the form of the Telegrapher’s equation [16], axial vibration systems [27], stabilization of vibrations in piezoelectric materials [21], and in the control of electromagnetic waves [18], among many other applications. The theory presented in this work can be utilized in the design of efficient controllers for these kinds of systems.

Our paper is structured in the following manner. Section 2 will provide a brief introduction to the ARE and LQR control problems in the Hilbert space setting. In Section 3, we demonstrate that the Bochner integral representation of the ARE is well-posed for non-compact semigroups. Using this result, we utilize the Brezzi-Rappaz-Raviart theorem to demonstrate the convergence of Galerkin approximation solution of the ARE to the infinite dimensional solution. Subsequently, we leverage the theory presented in Sections 2-4...
to derive error estimates in Section 5. We then present the computational implementation and numerical convergence rates for the functional gain of a weakly damped wave optimal control problem in Section 6. And finally, we conclude our paper with a discussion of our results in Section 7.

2. Problem Setting

Throughout this paper, we will let $H$ be a separable Hilbert space, from which we will let $V \subset H$ be a dense subspace that is continuously embedded in $H$. Letting $H'$ denote the dual space of $H$ and $V'$ denote the dual space of $V$, we have that $V \subset H \subset H' \subset V'$. On these spaces, we will denote

$$\langle \cdot, \cdot \rangle_{\Omega} : H \times H \to \mathbb{R}_+ \text{ or } V \times V \to \mathbb{R}_+$$

as the inner-product for elements in $H$ or $V$, and

$$\langle \cdot, \cdot \rangle_{\Omega} : H' \times H \to \mathbb{R}_+ \text{ or } V' \times V \to \mathbb{R}_+$$

as the duality pairing for elements belonging to $H'$ and $H$ or $V'$ and $V$.

In this work, we will make frequent use of semigroup theory. To that end, we will define the generator $A$ of $\mathcal{S}$ and its associated semigroup. Let $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset H \to H$ be a bounded linear operator that generates a semigroup $\mathcal{S}(t) \in \mathcal{L}(H)$. The semigroup $\mathcal{S}(t) \in \mathcal{L}(H)$ then can be seen as the solution operator of the following abstract dynamical system

$$\begin{cases}
\frac{d}{dt}z(t) = Az(t) & \text{for } t \in (0, \infty) \\
z(0) = \phi,
\end{cases}$$

for any $\phi \in H$. Thus, the notation

$$z(t) = \mathcal{S}(t)\phi \quad \forall \phi \in H$$

is well-defined. We will refer the reader to [15, 22] for additional results in semigroup theory. We will now move on to describe the problem of interest in the following subsection.

2.1. The Linear Quadratic Regulator. This work is concerned with the numerical approximation of the solution to the Algebraic Riccati Equation that arises from the linear quadratic control of systems governed by partial differential equations. Of particular interest are processes whose semigroups are not compact. To that end, we will begin by defining the quadratic cost functional $J(z, u) : H \times U \to \mathbb{R}_+$.

Let $U$ be a Hilbert space that contains all possible values of the control input at any time $t \in \mathbb{R}_+$. Additionally, let $Q \in \mathcal{L}(H)$ and $R \in \mathcal{L}(U)$ be positive definite weighting operators. Then we can define the cost functional as follows

$$J(z, u) := \lim_{t \to \infty} \int_0^t (z(s), Qz(s))_\Omega + (u(s), Ru(s))_\Omega \, ds,$$

where $u(s) \in U$ is the control input and $z(t) \in H$ is the state of the system at time $t \in \mathbb{R}_+$. The state $z(s) \in H$ is given by the dynamical system

$$\begin{cases}
\frac{d}{dt}z(t) = Az(t) + Bu(t) & \text{for } t \in (0, \infty) \\
z(0) = \phi,
\end{cases}$$

where $B : U \to Z$ is the actuator function that drives the system using the control input. The optimal control problem is then to seek a minimizer $u \in L^2((0, \infty); U)$ such that $J(z, u)$ is minimized, where the state $z \in L^2((0, \infty); H)$ is constrained by (2).

It is known, [11] Chapter 6] that the optimal control at any moment at time can be given by

$$u_{opt}(t) = -Kz(t),$$

where $K : H \to U$ is the functional gain defined by

$$K = R^{-1}B^*\Sigma.$$

$\Sigma \in \mathcal{L}(H)$ is then the solution of the Algebraic Riccati Equation (ARE):

$$\langle \phi, A\phi \rangle_\Omega + \langle A\phi, \phi \rangle_\Omega - \langle R^{-1}B^*\Sigma \phi, B^*\Sigma \phi \rangle_\Omega + \langle Q\phi, \phi \rangle_\Omega = 0 \quad \forall \phi \in \mathcal{D}(A).$$
So, in short, one determines the solution of (4) and then applies the feedback law (3) to synthesize the optimal feedback controller to minimize the infinite horizon cost functional (1).

It can subsequently shown that (4) can be equivalently be represented in the strong Bochner form, i.e.,

\[
\Sigma = \lim_{t \to \infty} \int_0^t S^*(s) (E + \Sigma F \Sigma) S(s) ds,
\]

where \( F := BR^{-1}B^* \) and \( E := Q \) are positive definite operators. The well-posedness of the Bochner integral representation of the ARE (BI-ARE) where \( S(t) \in L(H) \) is a non-compact semigroup is the topic of study in the following section.

3. Well-Posedness of the BI-ARE

In this section, we are concerned with establishing the well-posedness of (5) under the assumptions of exponential stability and the existence of a sequence of compact operators that converges to \( S(t) \in L(H) \). To that end, we begin with the first assumption, stated in the following.

**Assumption 1** (Exponential Stability and Analyticity). The analytic semigroup \( S(t) \in L(X) \) and its adjoint \( S^*(t) \in L(X) \), with \( X = V, H \), satisfies

\[
\|S(t)\|_{L(X)} = \|S^*(t)\|_{L(X)} \leq Me^{-\alpha t}
\]

for some \( \alpha, M \in \mathbb{R}_+ \) and for all \( t \in \mathbb{R}_+ \).

Duality and embedding then allows us to demonstrate the following.

**Proposition 1.** If Assumption 1 is satisfied, we have that

1. \( \|S(t)\|_{L(V')} = \|S^*(t)\|_{L(V')} \leq Me^{-\alpha t} \)
2. \( \|S(t)\|_{L(V;H)} = \|S^*(t)\|_{L(V;H)} \leq Me^{-\alpha t} \)
3. \( \|S(t)\|_{L(H;V')} = \|S^*(t)\|_{L(H;V')} \leq Me^{-\alpha t} \),

where the constants \( \alpha, M \in \mathbb{R}_+ \) are the same as those prescribed in Assumption 1.

**Proof.** Let \( \Pi \in L(V) \) be an arbitrary operator, then the first statement is verified by seeing that

\[
\|\Pi\|_{L(V)} = \sup_{v \in V'} \frac{\langle \Pi v, w \rangle}{\|v\|_{V'}, \|w\|_V} = \sup_{v \in V'} \frac{\langle \Pi v, w \rangle}{\|v\|_{V'}, \|w\|_V} = \|\Pi\|_{L(V')}.
\]

Similarly, the second statement can be verified by seeing that

\[
\|\Pi\|_{L(H;V')} = \sup_{v \in V} \frac{\|\Pi v\|_H}{\|v\|_{V'}} \leq \sup_{v \in V} \frac{\|\Pi v\|_H}{\|v\|_V} = \|\Pi\|_{L(V')}.
\]

Finally, the third statement can be verified by seeing that

\[
\|\Pi\|_{L(H;V')} = \sup_{v \in H} \frac{\|\Pi v\|_V'}{\|v\|_{V'}} \leq \sup_{v \in H} \frac{\|\Pi v\|_{V'}}{\|v\|_{V'}} = \|\Pi\|_{L(V')}.
\]

This concludes the proof. \( \square \)

This proposition allows us to bound the semigroup in the norms of interest in the analysis of the BI-ARE. We now make the following assumption.

**Assumption 2** (Compact-Analytic Approximation). There exists a family of exponentially stable compact analytic semigroups \( S_\epsilon(t) \in L(X) \) that satisfies the convergence property

\[
\lim_{\epsilon \to 0} \|S(t) - S_\epsilon(t)\|_{L(X)} = 0.
\]

Additionally there exists constants \( \alpha_\epsilon, M_\epsilon \) parametrized by \( \epsilon \) such that preservation of exponential stability (POES) is satisfied, i.e.,

\[
\|S_\epsilon(t)\|_{L(X)} \leq M_\epsilon e^{-\alpha_\epsilon t},
\]

and \( \lim_{\epsilon \to 0} \{\alpha_\epsilon, M_\epsilon\} = \{\alpha, M\} \) for \( X = H, V \).
This assumption allows us to apply the theory of [5] to extend the well-posedness of the BI-ARE to the case of non-compact semigroups through a density argument. The statement of the theorem is then given in the following.

**Theorem 1 (Well-Posedness of BI-ARE with Bounded Exponential Semigroups).** Let \( E, F \in \mathcal{L}(V';V) \) be symmetric positive definite operators. If Assumptions [A] and [B] are satisfied, then there exists a unique solution \( \Sigma \in \mathcal{L}(V) \cap \mathcal{L}(V') \cap \mathcal{L}(H) \) to the BI-ARE

\[
\Sigma = \lim_{t \to \infty} \int_0^t S^*(s) (E + \Sigma F \Sigma) S(s) ds.
\]

Furthermore, \( \Sigma \in \mathcal{L}(X) \) satisfies the following bounds

\[
\|\Sigma\|_{\mathcal{L}(X)} \leq \frac{M^4}{2\alpha} \|E\|_{\mathcal{L}(X)},
\]

with \( X = H, V, V' \).

**Proof.** In this proof, we will use an analogous argument to those found in [7, 5] to determine the convergence of the solution of an approximate problem to the exact problem.

Let \( \Sigma_\epsilon \in \mathcal{L}(V) \) be the solution of

\[
\Sigma_\epsilon = \lim_{t \to \infty} \int_0^t S_\epsilon^*(s) (E + \Sigma_\epsilon F \Sigma_\epsilon) S_\epsilon(s) ds.
\]

An inspection then allows to see formally that \( \lim_{\epsilon \to 0} \Sigma_\epsilon \to \Sigma \) in some sense. From [5, Theorem 2], we have that

\[
\|\Sigma_\epsilon\|_{\mathcal{L}(V)} \leq \frac{M^4}{2\alpha} \|E\|_{\mathcal{L}(V)}.
\]

Since by Assumption [B] \( \lim_{\epsilon \to 0} \{\alpha_\epsilon, M_\epsilon\} \) is well-defined and bounded, this implies that \( \lim_{\epsilon \to 0} \Sigma_\epsilon \phi = \Sigma \phi \) is bounded in \( V \) for all \( \phi \in V \). To see this, we set \( y_\epsilon = \Sigma_\epsilon \phi \) for any arbitrary \( \phi \in V \). Then since \( y_\epsilon \in V \), we have that the limit \( \lim_{\epsilon \to 0} y_\epsilon = y \) exists since \( V \) is closed by the definition of a Hilbert space [8, Chapter 4]. This \( y \in V \) is associated with \( y = \Sigma \phi \) for any \( \phi \in V \), hence \( \lim_{\epsilon \to 0} \Sigma_\epsilon \to \Sigma \) pointwise. We will show that this limit is convergent strongly in \( \mathcal{L}(V) \) in the following.

First, let us define

\[
\Sigma_{\epsilon,t} := \int_0^t S^*(s) (E + \Sigma_{\epsilon,t} F \Sigma_{\epsilon,t}) S(s) ds.
\]

From [7, Theorem 2.4], we know that there exists a unique solution \( \Sigma_{\epsilon,t} \in \mathcal{L}(V) \) that satisfies this equation. Let us subsequently define

\[
\Sigma_t := \int_0^t S^*(s) (E + \Sigma F \Sigma) S(s) ds.
\]

We will now demonstrate that for any \( t \in \mathbb{R}_+ \), we can send \( \Sigma_{\epsilon,t} \in \mathcal{L}(V) \) to \( \Sigma_t \).

The Uniform Boundedness Principle [8, Theorem 5-3.2] implies that the following constants are bounded

\[
\alpha_* := \sup_{\epsilon > 0} \{\alpha_\epsilon\}_{\epsilon > 0} \quad M_* := \sup_{\epsilon > 0} \{M_\epsilon\}_{\epsilon > 0} \quad \rho_* := \sup_{\epsilon > 0} \left\{\|\Sigma_\epsilon\|_{\mathcal{L}(V)}\right\}_{\epsilon > 0}.
\]

Defining

\[
\iota(\epsilon) := M_* \int_0^t \|S(s) - S_\epsilon(s)\|_{\mathcal{L}(V)} e^{-\alpha_* s} ds,
\]

and

\[
\eta(\epsilon) := \|\Sigma_{\epsilon,t} - \Sigma_t\|_{\mathcal{L}(V)} e^{-\alpha_* t}.
\]

This implies that

\[
\eta(\epsilon) \leq \iota(\epsilon).
\]

Letting \( \epsilon \to 0 \) we get that

\[
\|\Sigma_{\epsilon,t} - \Sigma_t\|_{\mathcal{L}(V)} \to 0.
\]

Therefore \( \Sigma_{\epsilon,t} \to \Sigma_t \) in \( \mathcal{L}(V) \).
we have that
\[
\|\Sigma_t - \Sigma_{\epsilon, t}\|_{\mathcal{L}(V)} \\
\leq \int_0^t \|S^*(s) (E - \Sigma_t F \Sigma_t) S(s) - S^*_\epsilon (s) (E - \Sigma_{\epsilon, t} F \Sigma_{\epsilon, t}) S_\epsilon(s)\| \, ds \\
\leq \int_0^t \|S^*(s) - S^*_\epsilon(s)\|_{\mathcal{L}(V)} \|E\|_{\mathcal{L}(V)} \|S(s) - S_\epsilon(s)\|_{\mathcal{L}(V)} \, ds \\
+ \int_0^t \|S^*(s)\|_{\mathcal{L}(V)} \|E\|_{\mathcal{L}(V)} \|S(s) - S_\epsilon(s)\|_{\mathcal{L}(V)} \, ds \\
+ \int_0^t \|S^*(s) - S^*_\epsilon(s)\|_{\mathcal{L}(V)} \|S_\epsilon(s)\|_{\mathcal{L}(V)} \, ds \\
+ \int_0^t \|S^*(s)\|_{\mathcal{L}(V)} \|\Sigma_t - \Sigma_{\epsilon, t}\|_{\mathcal{L}(V)} \|F\|_{\mathcal{L}(V)} \|\Sigma_{\epsilon, t}\|_{\mathcal{L}(V)} \|S_\epsilon(s)\|_{\mathcal{L}(V)} \, ds \\
+ \int_0^t \|S^*(s)\|_{\mathcal{L}(V)} \|\Sigma_t\|_{\mathcal{L}(V)} \|F\|_{\mathcal{L}(V)} \|\Sigma_t - \Sigma_{\epsilon, t}\|_{\mathcal{L}(V)} \|S_\epsilon(s)\|_{\mathcal{L}(V)} \, ds \\
+ \int_0^t \|S^*(s)\|_{\mathcal{L}(V)} \|\Sigma_t\|_{\mathcal{L}(V)} \|F\|_{\mathcal{L}(V)} \|S(s) - S_\epsilon(s)\|_{\mathcal{L}(V)} \, ds.
\]

We thus have that
\[
\|\Sigma_t - \Sigma_{\epsilon, t}\|_{\mathcal{L}(V)} \leq 2 \left( \|E\|_{\mathcal{L}(V)} + \rho_\epsilon^2 \|F\|_{\mathcal{L}(V)} \right) \ell(\epsilon) + \rho_\epsilon^2 M_\epsilon^2 \|F\|_{\mathcal{L}(V)} \int_0^t e^{-2\alpha_s s} \|\Sigma_t - \Sigma_{\epsilon, t}\|_{\mathcal{L}(V)} \, ds
\]
after applying Assumption 1. Utilizing the Grönwall-Reid Inequality [9] allows us to see that
\[
\|\Sigma_t - \Sigma_{\epsilon, t}\|_{\mathcal{L}(V)} \leq 2 \left( \|E\|_{\mathcal{L}(V)} + \rho_\epsilon^2 \|F\|_{\mathcal{L}(V)} \right) \ell(\epsilon) e^{\rho_\epsilon^2 M_\epsilon^2 \|F\|_{\mathcal{L}(V)} t} e^{\alpha_s t}.
\]

Strong convergence of $\Sigma_\epsilon$ to $\Sigma$ in $\mathcal{L}(V)$ is established after seeing that $\lim_{\epsilon \to 0} \ell(\epsilon) = 0$ for every $t \in \mathbb{R}_+$. This then proves our assertion that there exists a unique solution $\Sigma \in \mathcal{L}(V)$ of [8] in the strong operator sense. The stability bound [9] with $X = V$ is then established by applying Assumption 2 to [10].

The derivation of [8] with $X = V'$ then follows from duality and embedding arguments. This concludes the proof. \qed

**Remark 1.** We remark that if $M_\epsilon \|E\|_V$ is sufficiently small and $\alpha_s$ is sufficiently large, we can bypass the Grönwall-Reid inequality in the above proof and see that
\[
\left( 1 - \frac{\rho_\epsilon^2 + M_\epsilon^2 \|F\|}{2\alpha_s} \right) \|\Sigma_t - \Sigma_{\epsilon, t}\|_{\mathcal{L}(V)} \leq 2 \left( \|E\|_{\mathcal{L}(V)} + \rho_\epsilon^2 \|F\|_{\mathcal{L}(V)} \right) \ell(\epsilon).
\]

However, the left hand side is not always guaranteed to be positive.

The previous theorem showed us that the solution to [5] preserves smoothness. We can go one step further and demonstrate that $\Sigma$ is, in fact, a compact operator that maps $V'$ to $V$. Though, we remark that this result is not necessary in the determination of error estimates. We demonstrate this in the following.

**Corollary 1.** If Assumption 1 and 2 are satisfied. Then, if in addition, $E, F \in \mathcal{L}(V';V)$ is symmetric and positive definite, then $\Sigma \in \mathcal{L}(V';V)$. Furthermore, the solution of [5] is bounded by
\[
\|\Sigma\|_{\mathcal{L}(V';V)} \leq \frac{M^2}{2\alpha} \|E\|_{\mathcal{L}(V';V)} + \frac{M^{10}}{8\alpha^3} \|F\|_{(V';V)} \|E\|_{\mathcal{L}(V)}^2
\]
in the $\mathcal{L}(V';V)$ norm.
Proof. From (5), we have immediately that
\[
\|\Sigma\|_{L(V';V)} \leq \lim_{t \to \infty} \int_0^t \|S^*(s)\|_{L(V;V)} \|E\|_{L(V';V)} \|S(s)\|_{L(V;V')} \, ds
\]
\[+ \lim_{t \to \infty} \int_0^t \|S^*(s)\|_{L(V;V)} \|\Sigma\|_{L(V';V)} \|F\|_{L(V';V)} \|\Sigma\|_{L(V;V)} \|S(s)\|_{L(V;V')} \, ds.
\]
\[\leq M^2 \|E\|_{L(V';V)} \lim_{t \to \infty} \int_0^t e^{-2\alpha s} \, ds + M^2 \lim_{t \to \infty} \int_0^t e^{-2\alpha s} \|\Sigma\|^2_{L(V;V)} \|F\|_{L(V';V)} \, ds.
\]
We then arrive at the result of the corollary by applying (5). \qed

In the next section, we will discuss the abstract approximation of the BI-ARE for non-compact exponentially stable semigroups.

4. Approximation of the BI-ARE

With the well-posedness of the BI-ARE established, we are now ready to establish some abstract results that will allow us to derive error estimates for general exponentially stable systems. We will first define the finite dimensional approximation and then apply the Brezzi-Rappaz-Raviart theorem [14, Chapter IV, Theorem 3.3] to derive an error estimation inequality from which we will be able to derive error bounds.

4.1. Finite Dimensional Approximation to (5). Let us denote \(\{H_\delta\} \subset H\) as a parametrized set of finite dimensional approximation spaces with projection operators \(\pi_\delta \in L(H;H_\delta)\) defined such that
\[\lim_{\delta \to 0} \|v - \pi_\delta v\|_H = 0 \quad \forall v \in H.\]
The \(\delta \in \mathbb{R}_+\) here denotes an abstract discretization parameter. Using the projection, we define the finite dimensional generator \(A_\delta \in L(H)\) as
\[A_\delta = Q[\pi_\delta A \pi_\delta],\]
where \(Q : H_\delta \to H\) is the injection operator. Subsequently, the finite dimensional semigroup \(S_\delta(t) \in L(H)\) can be defined as the following
\[S_\delta(t) = Q[\exp(A_\delta)].\]
The injection operator \(Q \in L(H_\delta;H)\) will be implied in the remainder of this work. With this, we make the following assumption.

Assumption 3 (Assumptions on the Finite Dimensional Semigroup). Let \(S(t) \in L(H)\) satisfy assumptions 7 and 8 and \(S_\delta(t) \in L(H)\) be its finite dimensional Galerkin approximation. We assume that
\[\lim_{\delta \to 0} \|S(t) - S_\delta(t)\|_{L(H)} = 0,\]
for all \(t \in \mathbb{R}_+.\) We further assume that
\[\|S_\delta(t)\|_{L(H)} \leq M_\delta e^{-\alpha_\delta t},\]
with positive constants \(M_\delta, \alpha_\delta \in \mathbb{R}_+.\)

With the finite dimensional semigroup defined, we define the finite dimensional BI-ARE as the following.
\[\Sigma_\delta = \lim_{t \to \infty} \int_0^t S^*_\delta(s) (E + \Sigma_\delta F \Sigma_\delta) S_\delta(s) \, ds.\]
With the finite dimensional problem defined, we now move on to present the Brezzi-Rappaz-Raviart theorem.
4.2. The Brezzi-Rappaz-Raviart Theorem. Consider the fixed point problem
\[ u = T \mathcal{G}(\lambda, u), \]
where \( u \in X, \lambda \in \Lambda \) belongs to a compact interval in \( \mathbb{R}^+ \), \( T \in \mathcal{L}(Y; X) \) is a linear operator, and \( \mathcal{G} : \Lambda \times X \to Y \) is a \( C^2 \) smooth mapping. Let us introduce \( T_\varepsilon \in \mathcal{L}(Y; X) \) as an operator meant to approximate of \( T \). We then have that the fixed point problem can be approximated by the following:
\[ u_\varepsilon = T_\varepsilon \mathcal{G}(\lambda, u_\varepsilon). \]

Now suppose that there exists another space \( Z \to Y \) with continuous imbedding such that
\[ Du \mathcal{G}(\lambda, u) \in \mathcal{L}(X; Z) \quad \forall \lambda \in \Lambda \text{ and } u \in X. \]
Then, additionally under the convergence assumptions
\[ \lim_{\varepsilon \to 0} \| (T - T_\varepsilon) g \|_X = 0 \quad \forall g \in Y \]
and
\[ \lim_{\varepsilon \to 0} \| T - T_\varepsilon \|_{\mathcal{L}(Z; X)} = 0 \]
the following holds.

**Theorem 2** (Brezzi–Rappaz–Raviart). Assume that conditions (13), (14), and (15) are satisfied. In addition, assume that \( \mathcal{G} : \Lambda \times X \to Y \) is a \( C^2 \) operator, with \( D^2 \mathcal{G} \) bounded on all subsets of \( \Lambda \times X \). Then, there exists a neighborhood \( \mathcal{O} \) of the origin in \( X \) and for \( \varepsilon \leq \varepsilon_0 \) small enough a unique \( C^2 \)-function \( \lambda \in \Lambda \to u_\varepsilon(\lambda) \in X \) such that
\[ (\lambda, u_\varepsilon(\lambda); \lambda \in \Lambda) \text{ is a branch of nonsingular solutions of (12)} \]
and
\[ u(\lambda) - u_\varepsilon(\lambda) \in \mathcal{O} \quad \forall \lambda \in \Lambda. \]
Furthermore, there exists a constant \( K > 0 \) independent of \( \varepsilon \) and \( \lambda \) with:
\[ \| u(\lambda) - u_\varepsilon(\lambda) \|_X \leq K \| (T - T_\varepsilon) \mathcal{G}(\lambda, u(\lambda)) \|_X \quad \forall \lambda \in \Lambda. \]

Using this theorem, we are able to derive an abstract error estimation inequality that we will use to determine error estimates of Galerkin approximations to the BI-ARE.

4.3. Error Estimation Inequality. Let us define the Lyapunov operator \( L(\cdot) \in \mathcal{L} \left( \mathcal{L}(H) \right) \) as the following
\[ L(\Pi) := \lim_{t \to \infty} \int_0^t S^*(s)\Pi S(s) ds \quad \forall \Pi \in \mathcal{L}(H), \]
where the Lyapunov operator can be seen as the special \( F = 0 \) case of (5). We can now demonstrate the following.

**Proposition 2.** The Lyapunov operator \( L \in \mathcal{L}(H) \) is a continuous operator.

**Proof.** Utilizing Assumption 1 we see that
\[ \| L(\Pi) \| \leq \lim_{t \to \infty} \| S^*(s) \|_{\mathcal{L}(H)} \| \Pi \|_{\mathcal{L}(H)} \| S(s) \|_{\mathcal{L}(H)} ds \leq M^2 \| \Pi \|_{\mathcal{L}(H)} \lim_{t \to \infty} \int_0^t e^{-2\alpha s} ds \leq \frac{M^2}{4\alpha} \| \Pi \|_{\mathcal{L}(H)}. \]
This concludes the proof. \( \square \)

Let us now define
\[ G(\Pi) := E + \Pi F \Pi \quad \forall \Pi \in \mathcal{L}(H). \]
It then becomes clear that \( G : \mathcal{L}(H) \to \mathcal{L}(H) \) is infinitely Fréchet differentiable with respect to \( \Pi \in \mathcal{L}(H) \), and furthermore,
\[ D\Pi G(\Pi) \in \mathcal{L}(H) \quad \forall \Pi \in \mathcal{L}(H). \]
Since the domain of $D_\Pi G(\Pi)$ is continuously embedded in the domain of $L(\cdot) \in \mathcal{L}(\mathcal{L}(H))$.

Let us now define the finite dimensional Lyapunov operator $L_\delta(\cdot) \in \mathcal{L}(\mathcal{L}(H))$ as

$$L_\delta(\Pi) := \lim_{t \to \infty} \int_0^t S_\delta^*(s) \Pi S_\delta(s) \, ds \quad \forall \Pi \in \mathcal{L}(H).$$

Using Assumption 3 and integrating the exponential decay term, we have that

$$\|(L - L_\delta)\Pi\|_{\mathcal{L}(H)} \leq \lim_{t \to \infty} \int_0^t \left( \|S^*(s) - S_\delta^*(s)\|_{\mathcal{L}(H)} \|\Pi\|_{\mathcal{L}(H)} \left( \|S(s)\|_{\mathcal{L}(H)} + \|S_\delta(s)\|_{\mathcal{L}(H)} \right) \right) \, ds$$

Hence

$$\|(L - L_\delta)\Pi\|_{\mathcal{L}(H)} \leq \max \{M, M_\delta\} \max_{t \in \mathbb{R}_+} \|S(t) - S_\delta(t)\|_{\mathcal{L}(H)} \|\Pi\|_{\mathcal{L}(H)}.$$

An application of Theorem 2 with $X, Y, Z = \mathcal{L}(H), \Lambda = 1$, (17), and (18) gives us the following.

**Theorem 3.** If Assumptions 1, 2, and 3 are satisfied, then there exists a solution $\Sigma_\delta \in \mathcal{L}(H)$ to (11). Furthermore, there exists a positive constant $K \in \mathbb{R}_+$ for a sufficiently small $\delta \in \mathbb{R}_+$ such that

$$\|\Sigma - \Sigma_\delta\|_{\mathcal{L}(H)} \leq K \|L - L_\delta\|_{\mathcal{L}(L(H); \mathcal{L}(H))} \|G(\Sigma)\|_{\mathcal{L}(H)}.$$  

**Proof.** From Theorem 2 we have immediately that there exists a solution $\Sigma_\delta \in \mathcal{L}(H)$ that satisfies (11). Furthermore, it follows from the same theorem that

$$\|\Sigma - \Sigma_\delta\|_{\mathcal{L}(H)} \leq C \|(L - L_\delta) G(\Sigma)\|_{\mathcal{L}(H)},$$

for some positive constant $C \in \mathbb{R}_+$. The result follows from the definition of the semigroup and the assumption that $S(t) \in \mathcal{L}(V; H)$. We illustrate this by demonstrating that

$$\|L\Pi\|_{\mathcal{L}(H)} \leq \lim_{t \to \infty} \int_0^t \|S^*(s)\Pi S(s)\|_{\mathcal{L}(H)} \, ds$$

$$\leq \lim_{t \to \infty} \int_0^t \|S^*(s)\|_{\mathcal{L}(V'; H)} \|\Pi\|_{\mathcal{L}(V'; H')} \|S(s)\|_{\mathcal{L}(H; V')} \, ds$$

$$\leq \lim_{t \to \infty} \int_0^t 2 \|S^*(s)\|_{\mathcal{L}(V'; H)} \|\Pi\|_{\mathcal{L}(V'; H')} \, ds$$

$$\leq \lim_{t \to \infty} \frac{2M}{\alpha} \|\Pi\|_{\mathcal{L}(V'; V)}$$

for all $\Pi \in \mathcal{L}(V'; H)$ after applying Proposition 1 and seeing that

$$\|S(s)\|_{\mathcal{L}(H; V')} = \|S^*(s)\|_{\mathcal{L}(V'; V')} \leq \|S^*(s)\|_{\mathcal{L}(V'; H)}.$$  

Applying the same logic to $\|L_\delta\Pi\Pi_{\mathcal{L}(H)}$ for all $\Pi \in \mathcal{L}(V'; V)$ allows us to see that the bound presented in this theorem is well-defined.

This theorem will be the fundamental result that will allow us to determine the optimal convergence rates that we seek in the remainder of this work.

5. **Finite Element Approximation of the Weakly Damped Wave BI-ARE**

In this section we will leverage the theory discussed in the previous section to derive optimal convergence rates for the functional gain associated with the LQR control of weakly damped waves. To this end, we will begin our discussion by defining the topological vector spaces of interest.
5.1. **Notation.** Throughout this section, we will define \( \Omega \subset \mathbb{R}^n \) to be a domain with an infinitely differentiable boundary which we will denote \( \Gamma := \partial \Omega \). We will then define \( L^2(\Omega) \) as the standard space of square integrable functions, i.e.

\[
L^2(\Omega) := \left\{ v(x) : \int_{\Omega} v^2 \, dx < \infty \right\}.
\]

The \( L^2(\Omega) \) norm will subsequently be defined as

\[
\|v\|^2_{L^2(\Omega)} := \int_{\Omega} v^2 \, dx.
\]

Additionally, we will define \( H^k(\Omega) \) as the Sobolev space of \( k \)-times weakly differentiable functions that have bounded square integrals, i.e.,

\[
H^k(\Omega) := \left\{ v \in L^2(\Omega) : \sum_{|\beta|=0}^{k} \int_{\Omega} (D^\beta v)^2 \, dx < \infty \right\},
\]

where we have used the multinomial notation \( D^\beta (\cdot) \) to denote the partial derivatives of \( k \)-th order. The \( H^k(\Omega) \) norm and seminorm are then defined as

\[
\|v\|^2_{H^k(\Omega)} = \sum_{|\beta|=0}^{k} \|D^\beta v\|^2_{L^2(\Omega)}, \quad \text{and} \quad |v|^2_{H^k(\Omega)} := \sum_{|\beta|=k} \|D^\beta v\|^2_{L^2(\Omega)}
\]

respectively. Of special interest is the space \( H^1_0(\Omega) \), which we define as

\[
H^1_0(\Omega) := \left\{ v \in H^1(\Omega) : D^\beta v|_\Gamma = 0 \quad \forall |\beta| \leq k \right\}.
\]

The \( H^1_0(\Omega) \) coincides with the \( H^1(\Omega) \) norm. Throughout the remainder of this work, we will define

\[
H := L^2(\Omega) \times H^2(\Omega) \cap H^1_0(\Omega) \quad \text{and} \quad V^k := H^{k+1}(\Omega) \times H^k_0(\Omega)
\]

as the product spaces of interest. Their respective norms are defined as

\[
\|v\|^2_H := \|v_1\|^2_{L^2(\Omega)} + \|v_2\|^2_{L^2(\Omega)} \quad \text{and} \quad \|v\|^2_{V^k} := \|v_1\|^2_{H^{k+1}(\Omega)} + \|v_2\|^2_{H^k(\Omega)}.
\]

respectively for \( v = (v_1, v_2) \) in \( H \) or \( V^k \). Additionally, on \( V^k \), we define the operator \( D^\beta_{k,l} \) as

\[
D^\beta_{k,l} g := \begin{pmatrix} D^\beta u \\ D^\gamma v \end{pmatrix} \quad \forall g := \begin{pmatrix} u \\ v \end{pmatrix} \in V^k
\]

where \( |\beta| \leq k \) and \( |\gamma| \leq l \).

Finally, we denote \( W^k_h \) as a \( C^0(\Omega) \) finite element space such that the best approximation assumption is satisfied, i.e.,

\[
\min_{\phi_h \in W^k_h} \|v - \phi_h\|_{H^m(\Omega)} \leq C h^{k-m} |v|_k \quad \forall v \in H^k(\Omega),
\]

for some positive constant \( C \in \mathbb{R}_+ \). We will define the best approximation as

\[
\pi^k_h v := \arg \min_{\phi_h \in W^k_h} \|v - \phi_h\|_{H^m(\Omega)} \quad \forall v \in H^k(\Omega).
\]

We then define the product space

\[
V^k_h := W^k_h \times W^k_h.
\]

for any \( k \in \mathbb{N} \). To simplify the exposition of this paper, we will not consider geometric nonformity.
5.2. The Algebraic Riccati Equation. We will now define the generators and semigroups of interest for the purpose of determining the error estimates for the finite element approximation of the ARE associated with the control problem defined in the previous paragraph. Let \( A(\cdot) : H^2(\Omega) \cap H_0^1(\Omega) \rightarrow L^2(\Omega) \) be defined as \( c^2 \Delta(\cdot) \) with \( c \in \mathbb{R}_+ \) and the Laplacian \( \Delta(\cdot) \), then the generator for the weakly damped wave process can be defined as the following operator

\[
\Lambda \left( \begin{array}{c} v \\ w \end{array} \right) := \left[ \begin{array}{cc} 0 & v \\ Aw & -\gamma v \end{array} \right], \quad v = 0 \text{ on } \Gamma
\]

with \( \gamma \in \mathbb{R}_+ \). \( \Lambda : V_0^2 \rightarrow [L^2(\Omega)]^2 \) can be seen as an unbounded operator on \( H \). As such, we can employ an eigenvector decomposition to determine that \( \Lambda \) is a generator of an analytic semigroup \( S(t) \in \mathcal{L}(H) \) \[20\] Lemma 2.2]. We demonstrate this in the following.

**Lemma 1.** Let \( \Lambda \in V_0^2 \rightarrow [L^2(\Omega)]^2 \) be defined as in \( \Theta \), then \( \Lambda \) is a generator of an analytic \( C_0 \) semigroup \( S(t) \in \mathcal{L}(H) \) and there exists a positive constant \( \alpha \) such that

\[
\| S(t) \|_{\mathcal{L}(X)} = \| S^*(t) \|_{\mathcal{L}(X)} \leq Me^{-\alpha t},
\]

where \( X = H, V_k^{k+1} \) for all \( t \in \mathbb{R}_+ \), with \( k \in \mathbb{N} \).

**Proof.** It is known in the literature, e.g. in \[10, 23\] that there exists constants \( M, \alpha \in \mathbb{R}_+ \) such that

\[
\| S(t) \|_{\mathcal{L}(H)} \leq Me^{-\alpha t}.
\]

We arrive at the desired bounds by seeing that

\[
D^2_{k+1, k} S(t)v = S(t) D^2_{k+1, k} v \quad \forall v \in V_k^{k+1},
\]

and hence

\[
\left\| D^2_{k+1, k} S(t)v \right\| \leq Me^{-\alpha t} \| v \|_{V_k^{k+1}}.
\]

We now analyze the eigenvalues of \( \Lambda \). Let \( \mu_n \in \mathbb{R}_+ \) be defined as the negative of the eigenvalues of the suboperator \( A \). Additionally, let us define

\[
\Lambda(\mu_n) := \left[ \begin{array}{cc} 0 & 1 \\ -\mu_n & -\gamma \end{array} \right].
\]

Then we have for all \( g \in H \) that

\[
(sI - \Lambda)^{-1} g := \sum_{n=1}^{N} (sI_2 - \Lambda(\mu_n))^{-1} g_n,
\]

where \( I_2 \in \mathbb{R}^{2 \times 2} \) is the identity operator in \( \mathbb{R}^2 \) and \( g_n := \langle g, \phi_n \rangle \psi_n \) is the spectral decomposition of \( g \in H \) with respect to the basis functions \( \phi_n, \psi_n \in H \) generated from \( \Lambda^* \) and \( \Lambda \) respectively. Hence, it suffices to study the eigenvalues of \( \Lambda(\mu_n) \) for all \( n \in \mathbb{N} \). To that end, we see that

\[
(sI_2 - \Lambda(\mu_n))^{-1} = \frac{1}{s^2 + \gamma s + \mu_n} \left[ \begin{array}{cc} s + \gamma & -\mu_n \\ 1 & s \end{array} \right].
\]

The roots of \( p(s) := s^2 + \gamma s + \mu_n \), e.g. the eigenvalues of \( \Lambda(\mu_n) \) are given by the quadratic formula to be

\[
\lambda_+(\mu_n) := -\frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - \mu_n} \quad \text{and} \quad \lambda_-(\mu_n) := -\frac{\gamma}{2} - \sqrt{\frac{\gamma^2}{4} - \mu_n}.
\]

Since \( \gamma \) is strictly positive, and \( \mu_n \) is strictly positive \[13\] Theorem 3, pp. 361], we have that \( \Re(\lambda_+(\mu_n)) < 0 \) and \( \Re(\lambda_-(\mu_n)) < 0 \) for any \( \mu_n \). This implies that \( \sigma(\Lambda) \in \mathbb{C}_- \) and that there exists a sector

\[
T_\theta := \{ s \in \mathbb{C} : s \neq 0, \left| \arg s \right| < \theta \},
\]

where \( \theta \in \left( \frac{\pi}{2}, \pi \right] \), where \( (sI - \Lambda)^{-1} \) remains bounded. This concludes the proof.

We now demonstrate that Assumption 2 is satisfied for the weakly damped wave process in the following.
Lemma 2. There exists a semigroup \( S_\epsilon(t) \in \mathcal{L}(H) \) parametrized by \( \epsilon \in \mathbb{R}_+ \) such that
\[
\lim_{\epsilon \to 0} \|S(t) - S_\epsilon(t)\|_{\mathcal{L}(V_k^{k+1})} = 0.
\]
Furthermore, there exist positive constants \( M_\epsilon, \alpha_\epsilon \in \mathbb{R}_+ \) such that
\[
\|S_\epsilon(t)\|_{\mathcal{L}(X)} \leq M_\epsilon e^{-\alpha_\epsilon t},
\]
with \( X = H, V_k^{k+1} \).

Proof. Let us define \( N_\epsilon : H \to C^\infty(\Omega) \) as by the convolution product
\[
N_\epsilon g := \eta_\epsilon * g(x) \quad \forall g \in V_k^{k+1},
\]
where \( \eta_\epsilon \) is a mollifier, c.f. [II §2.28]. We will then define \( S_\epsilon(t) \in \mathcal{L}(V_k^{k+1}) \) by
\[
S_\epsilon(t)g := S(t)N_\epsilon g \quad \forall g \in V_k^{k+1}.
\]
From inspection, we have that \( N_\epsilon(\cdot) \) is a compact operator by the Rellich-Kondrachov Theorem [II Theorem 6.3] since it maps \( V_k^{k+1} \to C^\infty(\Omega) \). It then follows that \( S_\epsilon(t) \) is compact since it is a continuous operator on \( V_k^{k+1} \), hence it maps compact sets into compact sets. These observations allows us to conclude that \( S_\epsilon(t) \in \mathcal{L}(V_k^{k+1}) \) is a compact operator.

One of the properties of mollification is that
\[
\lim_{\epsilon \to 0} N_\epsilon g = g \quad \forall g \in V_k^{k+1}.
\]
Using this, we have that
\[
\|S(t)g - S_\epsilon(t)g\|_{V_k^{k+1}} \leq \|S(t)\|_{\mathcal{L}(V_k^{k+1})} \|g - N_\epsilon g\|_{V_k^{k+1}} \quad \forall V_k^{k+1}.
\]
Sending \( \epsilon \to 0 \) proves the first assertion of this lemma.

The second assertion is proved by seeing that
\[
\|S_\epsilon(t)v\|_{V_k^{k+1}} = \|S(t)N_\epsilon v\|_{V_k^{k+1}} \leq Me^{-\alpha t} \|N_\epsilon v\|_{V_k^{k+1}} \leq Me^{-\alpha t} \|v\|_{V_k^{k+1}}
\]
after applying Lemma 1. This concludes the proof. \( \square \)

With Lemmas 1 and 2 we can now state the following.

Theorem 4. Let \( A : V_0^2 \to [L^2(\Omega)]^2 \), be defined as
\[
A \left( \begin{array}{c} v \\ w \end{array} \right) := \left[ \begin{array}{cc} 0 & v \\ Aw & -\gamma v \end{array} \right], \quad v = 0 \text{ on } \Gamma,
\]
for all \( (v, w) \in V_0^2 \) be the generator of the weakly damped wave semigroup \( S(t) \in \mathcal{L}(X) \), with \( X = H, V_k^{k+1}, (V_k^{k+1})' \). Additionally, let \( E, F \in \mathcal{L}(V_k^{k+1})' \cap \mathcal{L}(H) \) be positive definite operators on \( H \). Then there exists a unique solution \( \Sigma \in \mathcal{L}(X) \) that satisfies the following BI-ARE
\[
\Sigma = \lim_{t \to \infty} \int_0^t S^*(s) (E + F \Sigma) S(s) ds,
\]
where \( X = H, V_k^{k+1}, (V_k^{k+1})' \). Furthermore, the solution satisfies the following stability bound
\[
\|\Sigma\|_{\mathcal{L}(X)} \leq \frac{M^4}{2\alpha} \|E\|_{\mathcal{L}(X)},
\]
where \( M, \alpha \in \mathbb{R}_+ \) are the constants in the exponential stability bound of \( S(t) \in \mathcal{L}(H) \).

With the well-posedness of the problem satisfied, we are ready to derive an error estimate for the Galerkin approximation of the Algebraic Riccati Equation [5] with \( S(t) \in \mathcal{L}(H) \) being the weakly damped wave semigroup.
5.3. Error Estimates of the Finite Element Approximation. Let us define \( A_h : \mathcal{L}(H) \) as the generator of the finite element approximation of the weakly damped wave semigroup \( S_h(t) \in \mathcal{L}(H) \). The operator \( A_h \) is a Galerkin projection defined by
\[
A_h := \langle v, \pi_h^0 A^0 \pi_h^0 v \rangle \quad \forall v \in H.
\]
The semigroup \( S_h(t) \in \mathcal{L}(H) \) is known to be exponentially stable, i.e., there exists positive constants \( M_h, \alpha_h \in \mathbb{R}_+ \) such that
\[
\| S_h(t) \|_{\mathcal{L}(H)} \leq M_h e^{-\alpha_h t}.
\]
From [24, Corollary 8.4-1], we can state the following.

**Lemma 3.** Let \( S(t) \in \mathcal{L}(H) \) be defined the weakly damped wave semigroup generated by \( A \in V_0^2 \rightarrow [L^2(\Omega)]^2 \) as defined in [24]. Additionally, let \( S_h(t) \in \mathcal{L}(H) \) be the semigroup generated by \( A_h \in \mathcal{L}(H) \), as defined in (20). Then we have that
\[
\lim_{h \to 0} \| S(t) - S_h(t) \|_H = 0.
\]

Lemma 3 shows us that our Galerkin approximation satisfies Assumption 3 on the finite element approximation \( S_h(t) \). Now that all three of the assumptions made in the previous section are satisfied, we can now present an error estimate on \( H \).

**Theorem 5.** Let \( A_h \in \mathcal{L}(H) \) be defined as in (20), then there exists a solution \( \Sigma_h \in \mathcal{L}(H) \) to the finite dimensional BI-ARE
\[
\Sigma_h := \lim_{t \to \infty} \int_0^t S_h^*(t) (E + \Sigma F \Sigma) S_h(t).
\]
Furthermore, we have that
\[
\| \Sigma - \Sigma_h \|_{\mathcal{L}(H)} \leq C h^{k-1} \| E \|_{\mathcal{L}(V_k^{k+1})} \| F \|_{\mathcal{L}(V_k^{k+1})},
\]
where \( k - 1 \) is the polynomial order of the basis functions in the approximation space.

**Proof.** The well-posedness of (21) comes directly from Theorem 3. From the same theorem, we have that
\[
\| \Sigma - \Sigma_h \|_{\mathcal{L}(H)} \leq K \| (L - L_h) \|_{\mathcal{L}(\{(V_{k+1})', V_k^{k+1} \}, \mathcal{L}(H))} \| G(\Sigma) \|_{\mathcal{L}(\{(V_{k+1})', V_k^{k+1} \})},
\]
where we have defined \( L_h : \mathcal{L}(\{(V_{k+1})', V_k^{k+1} \}) \to H \) as the following finite-dimensional Lyapunov operator
\[
L_h \Pi := \lim_{t \to \infty} \int_0^t S_h^*(t) \Pi S_h(t) \quad \forall \Pi \in \mathcal{L}(\{(V_{k+1})', V_k^{k+1} \}).
\]
From [12], we know that
\[
\| S^*(t) - S_h^*(t) \|_{\mathcal{L}(V_{k+1}^k, H)} \leq C h^{k-1}
\]
and hence also
\[
\| S(t) - S(t) \|_{\mathcal{L}(H, (V_{k+1}^k)' \})} \leq C h^{k-1}
\]
for some constant \( C \in \mathbb{R}_+ \) and any time \( t \in \mathbb{R}_+ \). From this, we have that
\[
\| (L - L_h) \Pi \|_{\mathcal{L}(H)} \leq \lim_{t \to 0} \int_0^t \| S^*(s) \Pi S(s) - S_h^*(s) \Pi S_h(s) \|_{\mathcal{L}(H)} ds
\]
\[\leq \lim_{t \to 0} \int_0^t \| S^*(s) - S_h^*(s) \|_{\mathcal{L}(V_{k+1}^k, H)} \| \Pi \|_{\mathcal{L}(\{(V_{k+1})', V_k^{k+1} \})} \| S(s) \|_{H, (V_{k+1}^k)' \}) ds
\]
\[+ \lim_{t \to 0} \int_0^t \| S^*(s) \|_{\mathcal{L}(V_{k+1}^k, H)} \| \Pi \|_{\mathcal{L}(\{(V_{k+1})', V_k^{k+1} \})} \| S(s) - S_h(s) \|_{H, (V_{k+1}^k)' \}) ds
\]
\[\leq C h^{k-1} \| \Pi \|_{\mathcal{L}(\{(V_{k+1})', V_k^{k+1} \})} \lim_{t \to \infty} \int_0^t M e^{\alpha s} + M e^{\alpha h s} ds
\]
\[\leq C h^{k-1} \| \Pi \|_{\mathcal{L}(\{(V_{k+1})', V_k^{k+1} \}).
\]
for all $\Pi \in \mathcal{L}(V_{k+1}^h):V_{k+1}^h$). This then implies that
\[
\|\Sigma - \Sigma_h\|_{\mathcal{L}(H)} \leq K\|(L - L_h)G(\Sigma)\|_{\mathcal{L}(H)} \\
\leq C h^{k-1}\|\Sigma\|_{\mathcal{L}(V_{k+1}^h)}\|E\|_{\mathcal{L}(V_{k+1}^h)}\|\Sigma\|_{\mathcal{L}(V_{k+1}^h)} \\
\leq C \frac{M h^{k-1}}{4\alpha^2}\|E\|_{\mathcal{L}(V_{k+1}^h)}\|\Sigma\|_{\mathcal{L}(V_{k+1}^h)}.
\]
This concludes the proof. \hfill \Box

With the error bound derived, we move on to present numerical results in the following section.

6. Numerical Implementation and Observed Convergence Rates

In this section, we present the model problem on which we test the validity of the results that we have presented in the earlier section. We begin by defining the model control system, its numerical implementation, and finally the observed convergence rates of the functional gain in the $H^1(\Omega) \times L^2(\Omega)$ norm. Additionally, we will also show that the functional gain also converges on the optimal order of $\mathcal{O}(h^k)$ in the $L^2(\Omega) \times L^2(\Omega)$ norm despite not having proved this result.

6.1. Definition of the Control System. Let us define the domain as $\Omega := (-1, 1)$, and the boundary as the endpoints $\Gamma = \{-1\} \cup \{1\}$. The model problem we approximate in this section is then defined as
\[
\min_{w(t) \in L^2((0,\infty);R)} \lim_{t \to \infty} \int_0^t \left[ \left( \int_{-1}^1 q(x)w(s)dx \right)^2 + Ru^2(s) \right] ds
\]
subject to
\[
\begin{align*}
\partial_t w(x,t) &= v(x,t) & \text{for } (x,t) &\in (-1,1) \times (0,\infty) \\
\partial_t v(x,t) &= c^2 \partial_{xx} w(x,t) - \gamma v(x,t) + b(x)u(t) & \text{for } (x,t) &\in (-1,1) \times (0,\infty) \\
w(x,0) &= w_0(x) & \text{for } x &\in (-1,1) \\
v(x,0) &= v_0(x) & \text{for } x &\in (-1,1) \\
w(-1) &= w(1) &= 0 & \text{for } t \in (0,\infty),
\end{align*}
\]
for some arbitrary initial conditions $w_0 \in H_0^1(-1,1)$, and $v_0 \in L^2(-1,1)$ and homogeneous Dirichlet boundary conditions. We map this control problem into our theory by setting
\[
A \begin{bmatrix} w \\ v \end{bmatrix} := \begin{bmatrix} c^2 \partial_{xx} w - \gamma v \\ v \end{bmatrix}, \quad w = 0 \text{ on } \{-1\} \cap \{1\} \\
B := \begin{bmatrix} 0 \\ b(x) \end{bmatrix}, \quad Q := \begin{bmatrix} q(x) \int_{-1}^1 q(x) (\cdot) dx & 0 \\ 0 & 0 \end{bmatrix}, \quad R := \frac{1}{10}.
\]
In the numerical examples, we will vary the values of $c, \gamma$ as well as the definitions of the functions $b(x), q(x)$ to evaluate the performance of the finite element convergence.

6.2. Numerical Implementation. Let us define $\{\phi_n\}_{n=1}^N$ to be the basis of $W_h^{k-1}$, where $N := \# \{W_h^{k-1}\}$ is the dimensionality of the finite element approximation space. We will now define the finite dimensional approximation to our operators in the control system. First, we will define
\[
[M_h]_{ij} = \langle \phi_i, \phi_j \rangle_{(-1,1)} \quad i,j = 1, \ldots, N
\]
as the mass matrix. Then, we define
\[
[A_h]_{ij} = \langle \phi_i, c^2 \partial_{xx} \phi_j \rangle_{(-1,1)} \quad i,j = 1, \ldots, N
\]
as the stiffness matrix. Following this, the finite element approximation of $A$ can then be defined as
\[
[A_h] = \begin{bmatrix} 0 \\ [M_h]^{-1}[A_h] \\ -\gamma I_h \end{bmatrix}.
\]
where \([I_h] \) is the \(N \times N\) identity matrix. We then define

\[
[p_h]_{ij} := \left\langle \phi_i(x), q(x) \int_{-1}^{1} q(x) \phi_j(x) dx \right\rangle_{(−1,1)} \quad i, j = 1, \ldots, N,
\]

\[
[b_h] := \left[ \int_{-1}^{1} b(x) \phi_1(x) dx, \ldots, \int_{-1}^{1} b(x) \phi_N(x) dx \right] [M_h]^{-1},
\]

\[
[Q_h] := \left[ \begin{array}{c} [p_h] \\ 0 \\ 0 \end{array} \right] \quad \text{and} \quad [B_h] := \left[ \begin{array}{c} 0 \\ [b_h] \end{array} \right].
\]

With these operator approximations defined, we can now pose the finite dimensional ARE as to seek a \([\Sigma_h] \in \mathbb{R}^{N \times N}\) such that

\[
[A_h]^T[\Sigma_h] + [\Sigma_h][A_h] - [\Sigma_h][B_h]R^{-1}[B_h]^T[\Sigma_h] + [Q_h] = 0.
\]

The solution of (22) coincides with its BI-ARE solution due to finite dimensionality. The functional gain is then computed using the formula

\[
[\kappa_h] := -R^{-1}[M_h]^{-1}[B_h]^T[\Sigma_h],
\]

and the functional approximation of \(\kappa_h(x)\) can be recovered by using the formula

\[
\kappa_h(x) := \sum_{i=1}^{2N} [\kappa_h]\Phi_i(x),
\]

where \(\Phi_i(x) \in V_h^{k-1}\) is the i-th basis function of the product approximation space \(V_h^{k-1} := W_h^{k-1} \times W_h^{k-1}\). This formula is what is used to determine the approximation in the numerical examples presented in the remainder of this paper. The functional gain is a precise indication of the accuracy of both the finite element \(L^2(-1,1)\) projection operator, as well as the accuracy of the approximation to the solution of (5). We now state the following Lemma and refer the reader to [7, Lemma 4.1] for the proof.

**Lemma 4.** Assume that the assumptions in Theorem 3 are satisfied. Additionally assume that \(B \in \mathcal{L}(\mathbb{R}^m; V_{k+1}^{k+1})\), then there exists a positive constant \(C \in \mathbb{R}_+\) such that

\[
\|K - K_h\|_{\mathcal{L}(\mathbb{R}^m; \mathbb{R}^m)} \leq C h^{k-1} \|B\|_{\mathcal{L}(\mathbb{R}^m; V_{k+1}^{k+1})}.
\]

In the examples presented below, the author’s personal code was utilized to generate the operator matrices defined above. Additionally, the \texttt{linalg.solve_continuous_are()} function in SciPy was utilized to solve (22).

### 6.3. Numerical Convergence Rates.

We present two numerical examples in this section.

#### 6.3.1. Example 1. Low Gradient, High Dissipation.

In this example, we set \(c = \gamma = 1\), and \(b(x) = q(x) := \cos(\frac{\pi}{2} x)\). This example demonstrates that optimal convergence is achieved when we have low gradients in the weighting and actuator functions and high dissipation in the system. The numerical results are presented in the following tables. We see agreement between the numerically computed convergence rates and our theoretical results, albeit with numerical precision error past \(10^{-8}\). Optimal convergence in \(L^2(\Omega) \times L^2(\Omega)\) was also observed.

| Linear Elements | \(\|K - K_h\|_{L^2(\Omega) \times L^2(\Omega)}\) | \(\|K - K_h\|_{H^1(\Omega) \times L^2(\Omega)}\) | \(L^2(\Omega) \times L^2(\Omega)\) Rate | \(H^1(\Omega) \times L^2(\Omega)\) Rate |
|-----------------|--------------------------|--------------------------|----------------|----------------|
| \(h\)           |                          |                          |                |                |
| 0.5             | 0.0347184                | 0.364411                 | -              | -              |
| 0.25            | 0.00848587               | 0.181998                 | 2.0326         | 1.0016         |
| 0.125           | 0.00210928               | 0.0909734                | 2.0083         | 1.0004         |
| 0.0625          | 0.000526556              | 0.0454835                | 2.0021         | 1.0001         |
| 0.03125         | 0.000131591              | 0.0227413                | 2.0005         | 1.0000         |
### 6.3.2. Experiment 2: Sharp Gradients and Low Dissipation

In this example, we set \( c = 1, \gamma = 10^{-4} \) and \( b(x) = q(x) := \exp(-\frac{1}{x-2}) \). This example demonstrates that optimal convergence can also be achieved in the more difficult case where we have sharp gradients in the weighting and actuator functions as well as low dissipation in the system. The numerical results are presented in the following tables. Again, we see agreement between the numerically computed convergence rates and our theoretical results. Optimal convergence in \( L^2(\Omega) \times L^2(\Omega) \) was also observed.

#### Linear Elements

| \( h \)      | \( \| \kappa - \kappa_h \|_{L^2(\Omega) \times L^2(\Omega)} \) | \( \| \kappa - \kappa_h \|_{H^1(\Omega) \times L^2(\Omega)} \) | \( L^2(\Omega) \times L^2(\Omega) \) Rate | \( H^1(\Omega) \times L^2(\Omega) \) Rate |
|--------------|-------------------------------------------------|-------------------------------------------------|-------------------------------------|-------------------------------------|
| 0.05         | 0.0157108                                       | 1.36526                                         |                                    |                                    |
| 0.04         | 0.0101083                                       | 1.09756                                         | 1.97627                            | 0.978094                           |
| 0.0333333    | 0.00703711                                      | 0.917138                                       | 1.98637                            | 0.984995                           |
| 0.0285714    | 0.00517615                                      | 0.787298                                       | 1.99244                            | 0.990272                           |
| 0.025        | 0.00396607                                      | 0.689556                                       | 1.9942                             | 0.992725                           |

#### Quadratic Elements

| \( h \)      | \( \| \kappa - \kappa_h \|_{L^2(\Omega) \times L^2(\Omega)} \) | \( \| \kappa - \kappa_h \|_{H^1(\Omega) \times L^2(\Omega)} \) | \( L^2(\Omega) \times L^2(\Omega) \) Rate | \( H^1(\Omega) \times L^2(\Omega) \) Rate |
|--------------|-------------------------------------------------|-------------------------------------------------|-------------------------------------|-------------------------------------|
| 0.05         | 0.0013325                                       | 0.175042                                       |                                    |                                    |
| 0.04         | 0.000687903                                     | 0.112362                                       | 2.96296                            | 1.98661                            |
| 0.0333333    | 0.00039958                                      | 0.0781055                                      | 2.97525                            | 1.99456                            |
| 0.0285714    | 0.000253003                                     | 0.0576058                                      | 2.96471                            | 1.97502                            |
| 0.025        | 0.0001701                                       | 0.0442195                                      | 2.97322                            | 1.9805                             |

#### Cubic Elements

| \( h \)      | \( \| \kappa - \kappa_h \|_{L^2(\Omega) \times L^2(\Omega)} \) | \( \| \kappa - \kappa_h \|_{H^1(\Omega) \times L^2(\Omega)} \) | \( L^2(\Omega) \times L^2(\Omega) \) Rate | \( H^1(\Omega) \times L^2(\Omega) \) Rate |
|--------------|-------------------------------------------------|-------------------------------------------------|-------------------------------------|-------------------------------------|
| 0.05         | 0.000102169                                     | 0.0195019                                       |                                    |                                    |
| 0.04         | 4.44989e-05                                     | 0.010588                                       | 3.72479                            | 2.73721                            |
| 0.0333333    | 2.24081e-05                                     | 0.00638881                                     | 3.76279                            | 2.77077                            |
| 0.0285714    | 1.21041e-05                                     | 0.00402017                                     | 3.9953                             | 3.00501                            |
| 0.025        | 7.10324e-06                                     | 0.00268938                                     | 3.99155                            | 3.01065                            |
In this paper, we have demonstrated that the Bochner integral representation of the Algebraic Riccati Equation is well-posed even when the process semigroup is not compact. Using this new result, we presented an abstract error estimation inequality that can be used to derive error estimates. This new theory was then applied to the case where the process semigroup represents a weakly damped wave process. The new results indicate that the BI-ARE is well-posed for these processes and that the finite element approximation of this BI-ARE is optimally convergent in the natural topology of weakly damped processes. The numerical examples presented corroborate the theory presented in this work.

While this work addresses the problem of deriving error bounds in the natural topology of the weakly damped wave system, there are some open questions that still need to be addressed. Optimality in the $L^2(\Omega) \times L^2(\Omega)$ norm was not proven, despite convergence to be shown to be optimal in this norm in the numerical results. Additionally, this work does not provide any results for unbounded control/estimation problems. These problems are important and we plan to discuss the numerical approximation of the ARE for these problems in future work.

7. Conclusion

| $h$     | $\|\kappa - \kappa h\|_{L^2(\Omega) \times L^2(\Omega)}$ | $\|\kappa - \kappa h\|_{H^1(\Omega) \times L^2(\Omega)}$ | $L^2(\Omega) \times L^2(\Omega)$ Rate | $H^1(\Omega) \times L^2(\Omega)$ Rate |
|---------|-------------------------------------------------|-------------------------------------------------|-------------------------------------|-------------------------------------|
| 0.05    | 1.14795e-05                                     | 0.00284567                                      | -                                   | -                                   |
| 0.04    | 3.69379e-06                                     | 0.00112649                                      | 5.08153                            | 4.1529                              |
| 0.033333 | 1.51158e-06                                     | 0.000481155                                     | 4.90064                            | 4.66576                              |
| 0.0285714 | 9.103e-07                                       | 0.000270276                                     | 3.28988                            | 3.74144                              |
| 0.025    | 1.44916e-06                                     | 0.000164564                                     | -3.48209                            | 3.71559                              |

References

[1] Robert A Adams and John JF Fournier. *Sobolev spaces*. Elsevier, 2003.
[2] HT Banks and Karl Kunisch. The linear regulator problem for parabolic systems. *SIAM Journal on Control and Optimization*, 22(5):684–698, 1984.
[3] Maîtine Bergounioux, Kazufumi Ito, and Karl Kunisch. Primal-dual strategy for constrained optimal control problems. *SIAM Journal on Control and Optimization*, 37(4):1176–1194, 1999.
[4] Jeff Borggaard, John A Burns, Eric Vugrin, and Lizette Zietsman. On strong convergence of feedback operators for non-normal distributed parameter systems. In *2004 43rd IEEE Conference on Decision and Control (CDC)* (IEEE Cat. No. 04CH37601), volume 2, pages 1526–1531. IEEE, 2004.
[5] John A Burns and James Cheung. Optimal convergence rates for galerkin approximation of operator riccati equations. *Numerical Methods for Partial Differential Equations*, 2022.
[6] John A Burns and Kevin P Hul ing. Numerical methods for approximating functional gains in lqr boundary control problems. *Mathematical and Computer Modelling*, 33(1-3):89–100, 2001.
[7] John A Burns and Carlos N Rautenberg. Solutions and approximations to the riccati integral equation with values in a space of compact operators. *SIAM Journal on Control and Optimization*, 53(5):2846–2877, 2015.
[8] Philippe G Ciarlet. *Linear and nonlinear functional analysis with applications*, volume 130. Siam, 2013.
[9] Dean S Clark. Generalization of an inequality of gronwall-reid. *Journal of approximation theory*, 58(1):12–14, 1989.
[10] Monica Conti, Lorenzo Liverani, and Vittorino Pata. On the optimal decay rate of the weakly damped wave equation. *Communications on Pure and Applied Analysis*, 0:--, 2022.
[11] Ruth F Curtain and Hans Zwart. *An introduction to infinite-dimensional linear systems theory*, volume 21. Springer Science & Business Media, 2012.
[12] Todd Dupont. $L^2$-estimates for galerkin methods for second order hyperbolic equations. *SIAM journal on numerical analysis*, 10(5):880–889, 1973.
[13] Lawrence C Evans. *Partial differential equations*, volume 19. American Mathematical Soc., 2010.
[14] Vivette Girault and Pierre-Arnaud Raviart. *Finite element methods for Navier-Stokes equations: theory and algorithms*, volume 5. Springer Science & Business Media, 2012.
[15] Jerome A Goldstein. *Semigroups of linear operators and applications*. Courier Dover Publications, 2017.
[16] Simone Göttlich and Peter Schillen. Numerical discretization of boundary control problems for systems of balance laws: Feedback stabilization. *European Journal of Control*, 35:11–18, 2017.
[17] Kazufumi Ito. Strong convergence and convergence rates of approximating solutions for algebraic riccati equations in hilbert spaces. In *Distributed Parameter Systems*, pages 153–166. Springer, 1987.
[18] Vilmos Komornik. Boundary stabilization, observation and control of maxwell’s equations. Technical report, SCAN-9410325, 1994.
[19] Michael Kroller and Karl Kunisch. Convergence rates for the feedback operators arising in the linear quadratic regulator problem governed by parabolic equations. *SIAM journal on numerical analysis*, 28(5):1350–1385, 1991.
[20] STIG LARSSON, VIDAR THOMÉE, and LARS B. WAHLBIN. Finite-Element Methods for a Strongly Damped Wave Equation. *IMA Journal of Numerical Analysis*, 11(1):115–142, 01 1991.

[21] K Morris and A Özkan Özer. Strong stabilization of piezoelectric beams with magnetic effects. In *52nd IEEE Conference on Decision and Control*, pages 3014–3019. IEEE, 2013.

[22] Amnon Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44. Springer Science & Business Media, 2012.

[23] JEFFREY RAUCH, MICHAEL TAYLOR, and Ralph Phillips. Exponential decay of solutions to hyperbolic equations in bounded domains. *Indiana University Mathematics Journal*, 24(1):79–86, 1974.

[24] Pierre-Arnaud Raviart. *Introduction à l’analyse numérique des équations aux dérivées partielles*. 1983.

[25] Zhijia Zhao, Xiuyu He, Zhigang Ren, and Guilin Wen. Output feedback stabilization for an axially moving system. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 49(12):2374–2383, 2018.