On some combinatorial formulae coming from Hessian Topology

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Abstract

The interaction between combinatorics and algebraic and differential geometry is very strong. While researching a problem of Hessian topology, we came across a series of identities of binomial coefficients, which are useful for proving a topological property of certain spaces whose elements are graphs of a class of hyperbolic polynomials. These identities are proven by different methods in combinatorics.

Keywords. Binomial coefficients, homogeneous polynomials, isotopy

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1 Introduction

Examples of the interaction between combinatorics and algebraic geometry are abundant: the polynomial method that was recently surveyed by Terence Tao [15]; the Combinatorial Nullstellensatz of Noga Alon [1]; the affirmative answer to the conjecture of Read and Rota-Heron-Welsh by June Huh [12]; and, Matthew Baker and Serguei Norine’s graph-theoretic analogue of the classical Riemann-Roch theorem [6].

All of these examples seem to indicate that the interaction is one way. This, of course, is not true. However, the use of combinatorics in the proof of theorems in algebraic geometry is more subtle. Examples of combinatorial arguments in algebraic geometry are: computing maximum possible number of ordinary double points for a surface of degree 5 by Arnaud Beauville [7]; giving an explicit combinatorial formula for the structure constants of the Grothendieck ring of a Grassmann variety with respect to its basis of Schubert structure sheaves by Anders Skovsted Buch [8]; and, the study of Donaldson-Thomas invariants by Benjamin Young [16]. The interaction of
differential topology and geometry with combinatorics is even more obvious due to discrete Morse theory [9] and combinatorial differential geometry by Robin Forman [10]; and, lattice gauge theory by Kenneth G. Wilson [17].

In this article, we prove as a result of some combinatorial identities that the graphs of the class of hyperbolic polynomials studied in [2] have a topological property which guarantees the existence of a number, depending on the degree, of different connected components of hyperbolic homogeneous polynomials. This fact is relevant for the description of the Hessian topology of hyperbolic polynomials.

The paper is organised as follows: In Section 2, we give a detailed account of the problem on quadratic differential forms that start our interest in combinatorial identities; in Section 3, we present some combinatorial formulae involving the binomial coefficients that, to our knowledge, are new; in Section 4, we apply these formulae to prove that some quadratic differential forms are isotopic; and in the last section we consider an alternative approach to prove isotopy.

2 Problem on Hessian Topology

Let us consider $H^n[x, y] \subset \mathbb{R}[x, y]$ the set of real homogeneous polynomials of degree $n \geq 1$ in two variables. The graph of any $f \in H^n[x, y]$ contains the origin of $\mathbb{R}^3$. The polynomial $f$ is called hyperbolic (elliptic) if its graph is a surface with only hyperbolic (elliptic) points off the origin.

The second fundamental form $II_f$ of a hyperbolic homogeneous polynomial $f$ defines two asymptotic lines at each point of $\mathbb{R}^2$. Moreover, it defines two continuous asymptotic fields of lines without singularities on $\mathbb{R}^2$ that extend to the origin with a singularity. Since both of these fields of lines are transverse at each point of $\mathbb{R}^2$, their indexes at the origin coincide. Consequently, this index will be called the index of the field of asymptotic lines at the origin, and it will be denoted by $i_0(II_f)$.

A hyperbolic isotopy between two smooth hyperbolic quadratic differential forms $\omega$ and $\delta$ on $\mathbb{R}^2$ that extend themselves to the origin with a singularity is a smooth map

$$\Psi : \mathbb{R}^2 \times [0, 1] \to \mathcal{Q}, \quad (x, y, t) \mapsto \Psi_t(x, y),$$

where $\mathcal{Q}$ is the space of real quadratic forms on the plane and the following conditions hold: $\Psi_0(x, y) = \omega(x, y), \Psi_1(x, y) = \delta(x, y)$ and $\Psi_t(x, y)$ is a smooth hyperbolic quadratic differential form on $\mathbb{R}^2$, which extends at the
origin with a singularity. In this case, we will say that \( \omega \) and \( \delta \) are hyperbolic isotopic.

If the second fundamental forms of two hyperbolic homogeneous polynomials of degree \( n \) are hyperbolic isotopic, then the indexes of their fields of asymptotic lines at the origin coincide.

The subset of \( H^n \mathbb{R}[x, y] \), constituted by hyperbolic polynomials, is a topological subspace of \( \mathbb{R}[x, y] \), denoted by \( Hyp(n) \). The topology of this space has been studied as part of the subject, known as Hessian Topology, introduced in [3, 14, 4], and named by V. I. Arnold [2, 5 problems 2000-1, 2000-2, 2001-1, 2002-1]. In fact, Arnold stated the following conjecture [2, p.1067], [5, p.139]:

“The number of connected components of the space of hyperbolic homogeneous polynomials of degree \( n \) increases as \( n \) increases (at least as a linear function of \( n \)).”

In relation to this, he proved as an application of his characterization of hyperbolic polynomials in polar coordinates [2, p.1031], that the homogeneous polynomials of degree \( n \)

\[
f(x, y) = (x^2 + y^2)^{\frac{m-n}{2}} \text{Re}(x + iy)^m
\]

are hyperbolic if \( m \leq n < m^2 \) and \( n - m \) is even. Moreover, he obtains that \( \iota_0(\mathbb{I}_f) = \frac{2-m}{2} \). This family of polynomials play a fundamental roll in finding the connected components mentioned in the conjecture. In the present article, we prove as an application of our combinatorial formulae that the second fundamental form \( \mathbb{I}_f \) of \( f \) is hyperbolic isopy to the second fundamental form of \( P(x, y) := \text{Re}(x + iy)^m \), Theorem 3.

3 Combinatorial Identities

Let us start by presenting the combinatorial identities that we talk about. These identities occur naturally in the development of a direct proof of Theorem 3 but we decided to isolate them from their original context in order to be more general.

**Theorem 1** Let \( m \geq 2 \) be a natural number. For each integer number \( 0 \leq j \leq \frac{m-1}{2} \) the following expression is fulfilled:

\[
(-1)^j \left[ \binom{m-1}{2j} + \sum_{k=0}^{j-1} \left( \binom{m-1}{2k} \binom{m-1}{2j-2k} - \binom{m-1}{2k+1} \binom{m-1}{2j-2k-1} \right) \right] = \binom{m-1}{j}.
\]
Theorem 2 If \( m \geq 2 \) is an even natural number, then for each integer number \( 0 \leq j \leq \frac{m-2}{2} \) the following expressions are true:

\[
(1) \quad (-1)^{\frac{m}{2}+j-1} \left[ -\binom{m-1}{2j-1} + \sum_{k=0}^{j-1} \left[ \binom{m-1}{2k+2j} \binom{m-1}{2k+1} - \binom{m-1}{2k} \binom{m-1}{2k+2j-1} \right] \right] = \binom{m-1}{j + \frac{m}{2} - 1}.
\]

\[
(2) \quad (-1)^{\frac{m}{2}} \left( 1 - m + \sum_{k=0}^{\frac{m}{2}-2} \binom{m-1}{2k+1} \left[ \binom{m-1}{2k+2} - \binom{m-1}{2k} \right] \right) = \binom{m-1}{\frac{m}{2}}.
\]

Proof of Theorem 1

Using the formula

\[
(m-k) \binom{m}{k} = (k+1) \binom{m}{k+1},
\]

several times, the expression \( 1 \) is equivalent to the expression:

\[
(3) \quad (-1)^j \left[ \binom{m-1}{2j} + \sum_{k=0}^{j-1} \frac{4k-2j+1}{m} \binom{m}{2k+1} \binom{m}{2j-2k} \right] = \binom{m-1}{j}.
\]

By using the formula of the alternating sum of consecutive binomial coefficients,

\[
(4) \quad (-1)^r \binom{m-1}{r} = \sum_{k=0}^{r} (-1)^k \binom{m}{k},
\]

the expression \( 5 \) can be written as:

\[
(5) \quad (-1)^j \sum_{k=0}^{\frac{m}{2}-2} \frac{4k-2j+1}{m} \binom{m}{2k+1} \binom{m}{2j-2k} = \sum_{k=1}^{j} (-1)^{k+1} \binom{m}{j+k}.
\]

Now, the sum on the right-hand side is divided into the sums of the even and odd terms. The sum on the left-hand side is divided into the sum of the first \( \frac{j}{2} \) terms, if \( j \) is even \( (\frac{j+1}{2} \) if \( j \) is odd), and the remaining terms, but the terms are listed in reverse order.

\[
\sum_{r=1}^{\frac{j}{2}} \frac{1-4r}{m} \left( \binom{m}{j-2r+1} \binom{m}{j+2r} \right) + \sum_{r=0}^{\frac{j-1}{2}} \frac{4r+1}{m} \left( \binom{m}{j+2r+1} \binom{m}{j-2r} \right) = \sum_{r=1}^{\frac{j}{2}} (-1)^r \binom{m}{j+2r} + \sum_{r=0}^{\frac{j-1}{2}} \binom{m}{j+2r+1}.
\]
Then, by associating corresponding terms of both sides, we obtain the equation

\[ \sum_{k=1}^{j} \left[ (-1)^k \binom{m}{j + k} \left( 1 + \frac{1 - 2k}{m} \binom{m}{j - k + 1} \right) \right] = 0. \] (7)

We present now a proof of (7) by means of recurrence relations techniques. It was given by C. Merino-López [13]. By using the alternating sum (6) we have

\[
m \sum_{k=0}^{j} (-1)^k \binom{m}{j + k} = \begin{cases} 
(m - 2j) \binom{m}{2j} - (m - j) \binom{m}{j} & \text{if } j \text{ is even}, \\
-(m - 2j) \binom{m}{2j} - (m - j) \binom{m}{j} & \text{if } j \text{ is odd}. 
\end{cases}
\]

By formula (4), the last equality becomes

\[
m \sum_{k=0}^{j} (-1)^k \binom{m}{j + k} = \begin{cases} 
(2j + 1) \binom{m}{2j + 1} - (j + 1) \binom{m}{j + 1} & \text{if } j \text{ is even}, \\
-(m - 2j) \binom{m}{2j} - (m - j) \binom{m}{j} & \text{if } j \text{ is odd}. 
\end{cases}
\]

Replacing the last equality in (7) we obtain

\[
\sum_{k=1}^{j+1} (-1)^{k+1} (2k - 1) \binom{m}{k + j} \binom{m}{j - k + 1} = (j + 1) \binom{m}{j + 1}.
\]

Denote by \( T(m, j) \) the left-hand side of the last expression. Using Stifel’s identity, \( \binom{m}{j} = \binom{m - 1}{j} + \binom{m - 1}{j - 1} \), we verify that \( T(m, j) \) satisfies the recurrence relation

\[
T(m, j) = T(m - 1, j) + T(m - 1, j - 1) + \binom{m - 1}{j}^2 + 2 \sum_{k=1}^{j} (-1)^k \binom{m - 1}{j - k} \binom{m - 1}{j + k},
\] (8)

Now, applying again Stifel’s formula to the function

\[
F(m - 1, j) = \binom{m - 1}{j}^2 + 2 \sum_{k=1}^{j} (-1)^k \binom{m - 1}{j - k} \binom{m - 1}{j + k},
\]
it can be verified that $F$ satisfies the recurrence relation

$$F(m, j) = F(m - 1, j) + F(m - 1, j - 1).$$

(9)

Stifel’s identity establishes that the binomial coefficients also satisfy (9). Because $\binom{m}{j}$ and $F(m, j)$ satisfy the same initial conditions, we conclude that $F(m, j) = \binom{m}{j}$. So, the recurrence relation (8) becomes

$$T(m, j) - T(m - 1, j) - T(m - 1, j - 1) = \binom{m - 1}{j}.$$  

But, this relation is also fulfilled by $(j + 1)\binom{m}{j+1}$ with the same initial conditions, thus $T(m, j) = (j + 1)\binom{m}{j+1}$ and (7) is proved. □

**Proof of Theorem 2**

Note that using formula (4) on the left side of (2), it results in

$$(-1)^{\frac{m}{2} + j - 1} \left[ \sum_{k=0}^{n} \frac{m - 4k - 2j - 1}{m} \binom{m}{2k + 1} \binom{m}{2k + 2j} \right] = \binom{m - 1}{j + \frac{m}{2} - 1}. (10)$$

Because $m$ is even, we make the substitutions $m = 2r$ and $j = r - n$ to obtain

$$(-1)^{2r - n - 1} \left[ \sum_{k=0}^{n} \frac{(2n - 4k - 1)}{2r} \binom{2r}{2k + 1} \binom{2r}{2k + 2r - 2n} \right] = \binom{2r - 1}{2r - n - 1}. \quad (11)$$

By using the **symmetry identity** $\binom{a}{b} = \binom{a}{a - b}$ (see [11]) in the last expression, and changing back $2r = m$, the result is

$$(-1)^{n} \left[ \sum_{k=0}^{n} \frac{(4k - 2n + 1)}{m} \binom{m}{2k + 1} \binom{m}{2n - 2k} \right] = \binom{m - 1}{n}. \quad (11)$$

Note that the term $k = n$ on the left-hand side sum equals $\binom{m - 1}{2n}$. Then, expression (11) is (5), and thus, equation (2) is proved.
We finish with a proof of equation (3). The following expression is derived from formula (4):

\[
\left( \frac{m-1}{2k+2} \right) - \left( \frac{m-1}{2k} \right) = \frac{m-4k-3}{m} \left( \frac{m}{2k+1} \right) \left( \frac{m}{2k+2} \right).
\]

Then, equation (3) is equivalent to

\[
(-1)^{\frac{m}{2}} \left( \sum_{k=0}^{m-1} \frac{(m-4k-3)}{m} \left( \frac{m}{2k+1} \right) \left( \frac{m}{2k+2} \right) \right) = \left( \frac{m-1}{2} \right). \quad (12)
\]

But, when we replace \( j = 1 \) in (10), we retrieve (12). \( \square \)

4 An application: isotopic quadratic forms

In the following analysis, we consider a polynomial \( f \in \mathbb{R}[x,y] \) as a Hamiltonian function with Hamiltonian vector field \( \nabla f = (f_y, -f_x) \) on \( \mathbb{R}^2 \). The field of Hessian matrices \( \text{Hess} f = \left( \begin{array}{cc} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{array} \right) \) determines at each point \( p \in \mathbb{R}^2 \) a bilinear form. That is,

\[
\text{Hess}_p : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}
\]

\[
\text{Hess}_p(X,Y) = X(\text{Hess}(p))Y^t,
\]

where \( X,Y \in \mathbb{R}^2 \) and the index \( t \), means the transpose of the vector \( Y \).

Thus, for any homogeneous polynomial \( P \in H^m[x,y] \) we define the following application:

\[
\nabla PHessP : H^u[x,y] \to H^{2m+u-4}[x,y],
\]

\[
Q \mapsto \nabla PHessP \nabla Q^t.
\]

A straightforward computation implies that

\[
\nabla PHessP \nabla Q^t = P_{xx}P_yQ_y + P_{yy}P_xQ_x - P_{xy}(P_xQ_y + P_yQ_x).
\]

The following inequality plays an important role in the proof of Theorem 3

\[
\nabla PHessP \nabla Q^t(p) \leq 0, \quad p \in \mathbb{R}^2.
\]

We inform the reader that there are well-known techniques for dealing with the aforementioned inequality, and we present this alternative approach in the last section. But the purpose of this current paper is to highlight the combinatorial identities that we found, and their possible use in algebraic and differential geometry or elsewhere.
Theorem 3 Let \( P(x, y) = \Re(x + iy)^m \) and \( Q(x, y) = (x^2 + y^2)^k \) be homogeneous polynomials of degree \( m \geq 2 \) with \( 2k \geq 2 \) and \( m \leq n < m^2 \). Then, the quadratic form \( II_f \) is hyperbolic isotopic to the quadratic form \( II_P \), where \( f = QP \).

Proof. A direct computation implies that the second fundamental form of the product \( f = QP \) has the expression:

\[
II_{PQ} = PII_Q + 2dPdQ + QII_P,
\]

where \( dP = P_x dx + P_y dy \), \( dQ = Q_x dx + Q_y dy \) and \( dPdQ \) is the quadratic differential form defined by the product of \( dP \) and \( dQ \).

Let us denote by \( \omega \) the quadratic form \( 2dPdQ(x, y) + Q(x, y)II_P(x, y) \). We recall that the discriminant of a quadratic form, \( \det adx^2 + 2bdxdy + cdy^2 \), is \( b^2 - ac \). Firstly, we shall prove that \( \omega \) is a hyperbolic quadratic form on \( \mathbb{R}^2 \), that is, that its discriminant \( \Delta_\omega \) is positive on \( \mathbb{R}^2 \). After some computations, we have that

\[
\Delta_\omega = -Q^2 \det(Hess P) + \frac{1}{4}(P_x Q_y - P_y Q_x)^2 - 2Q(\nabla PHess P \nabla Q^\top).
\]

On one hand, the term \( -Q^2 \det(Hess P) \) is negative off the origin since \( P \) is a hyperbolic polynomial and \( Q \) is positive off the origin. On the other hand, the term \( \Delta_{dPdQ} = \frac{1}{4}(P_x Q_y - P_y Q_x)^2 \) is nonnegative on the plane. Finally, the proof follows from Lemma 4.

Now, we shall prove that \( \omega \) and \( \omega + \delta = II_f \) are hyperbolic isotopic, where \( \delta \) is the quadratic form \( P(x, y)II_Q(x, y) \). Consider the isotopy

\[
\Phi_t(x, y) = \omega(x, y) + t\delta(x, y).
\]

Because \( \omega(x, y) = \omega_1 dx^2 + 2\omega_2 dxdy + \omega_3 dy^2 \) and \( \delta(x, y) = \delta_1 dx^2 + 2\delta_2 dxdy + \delta_3 dy^2 \) we have that the discriminant of \( \Phi_t(x, y) \) is

\[
\Delta_{\Phi_t} = \omega_3^2 - \omega_1 \omega_3 + t(2\omega_2 \delta_2 - \omega_1 \delta_3 \omega_3 \delta_1) + t^2(\delta_2^2 - \delta_1 \delta_3).
\]

Note that \( \omega_3^2 - \omega_1 \omega_3 + (2\omega_2 \delta_2 - \omega_1 \delta_3 \omega_3 \delta_1) = (\delta_2^2 - \delta_1 \delta_3) > 0 \) and \( \omega_3^2 - \omega_1 \omega_3 > 0 \) on \( \mathbb{R}^2 \) since \( \omega \) and \( \omega + \delta \) are hyperbolic on \( \mathbb{R}^2 \). So, for \( t \in [0, 1] \) we have that the discriminant \( \Delta_{\Phi_t} \) is positive on \( \mathbb{R}^2 \).

The next step is to prove that the quadratic differential forms \( QII_P + 2dPdQ \) and \( QII_P \) are hyperbolic isotopic. To do that, consider the isotopy \( \Psi_t(x, y) = QII_P + 2t dPdQ(x, y) \), where \( t \in [0, 1] \). We can see that the discriminant of the quadratic differential form \( QII_P + 2dPdQ(x, y) \) is

\[
\Delta_{\Psi_t} = -Q^2 \det(Hess P) + t^2 \Delta_{dPdQ} - 2tQ(\nabla PHess P \nabla Q^\top),
\]

which is positive on \( \mathbb{R}^2 \) by the next Lemma.
Lemma 4 Let \( m, k \in \mathbb{Z} \) such that \( m \geq 2 \) and \( k \geq 1 \). Then, the homogeneous polynomials \( P, Q \) of degree \( m \) and \( 2k \) as previously defined satisfy inequality (13).

Proof of Lemma 4. We only present the case when \( m \) is even. The odd case is analogous. In order to prove that inequality (13) holds for the polynomials \( P \) and \( Q \) we consider the polynomial expression \( \nabla P \, Hess P(\nabla Q)^t \) and prove that

\[
P_x(Q_yP_{xy} - Q_xP_{yy}) + P_y(Q_xP_{xy} - Q_yP_{xx}) = 2k m^2(m - 1)(x^2 + y^2)^{k+m-2}.
\]

Since \( m \) is even, a straightforward computation shows that

\[
Q_yP_{xy} - Q_xP_{yy} = 2k(x^2 + y^2)^{k-1} \left[ \sum_{j=0}^{m-1} (-1)^j \frac{(m-1)m!}{(2j)!(m - 2j - 1)!} x^{m-2j-1} y^{2j} \right].
\]

Now, we multiply both sides of the last expression by \( P_x \). The product \( P_x (Q_yP_{xy} - Q_xP_{yy}) \) equals

\[
2k(x^2 + y^2)^{k-1} m^2(m - 1) \left[ \sum_{j=0}^{m-1} (-1)^j \left( \frac{m-1}{2j} \right) x^{m-2j-1} y^{2j} \right]^2.
\]

Developing the squared factor of the last expression we have

\[
P_x (Q_yP_{xy} - Q_xP_{yy}) = 2k (x^2 + y^2)^{k-1} m^2(m - 1)
\]

\[
\left[ x^{2m-2} + \sum_{j=1}^{m} \left( \sum_{k=0}^{j} (-1)^j \left( \frac{m-1}{2k} \right) \left( \frac{m-1}{2j-2k} \right) \right) x^{2m-2j-2} y^{2j} + \right.
\]

\[
\left. \sum_{j=1}^{m-1} \left( \sum_{k=0}^{j-1} (-1)^{j+1} \left( \frac{m-1}{2k+1} \right) \left( \frac{m-1}{2j+2k-1} \right) \right) x^{m-2j-2} y^{m+2j-2} \right]. \tag{15}
\]

By doing a similar computation for \( P_y(Q_xP_{xy} - Q_yP_{xx}) \) we obtain that

\[
P_y(Q_xP_{xy} - Q_yP_{xx}) = 2k (x^2 + y^2)^{k-1} m^2(m - 1)
\]

\[
\left[ y^{2m-2} + \sum_{j=1}^{m} \left( \sum_{k=0}^{j-1} (-1)^{j+1} \left( \frac{m-1}{2k+1} \right) \left( \frac{m-1}{2j+2k-1} \right) \right) x^{2m-2j-2} y^{2j} + \right.
\]

\[
\left. \sum_{j=1}^{m-1} \left( \sum_{k=0}^{j-1} (-1)^{j+1} \left( \frac{m-1}{2k+1} \right) \left( \frac{m-1}{2j+2k-1} \right) \right) x^{2m-2j-2} y^{m+2j-2} \right].
\]
\[
\sum_{j=2}^{m-1} \left( \sum_{k=0}^{m-j} (-1)^{m-j} \left( \begin{array}{c} m-1 \\ 2k+2j-1 \end{array} \right) \left( \begin{array}{c} m-1 \\ m-2k-1 \end{array} \right) x^{m-2j} y^{m+2j-2} \right), \tag{16}
\]

By adding the expressions (15) and (16) we obtain
\[
P_x (Q_y P_{xy} - Q_x P_{yy}) + P_y (Q_x P_{xy} - Q_y P_{xx}) =
\]
\[
= 2k (x^2 + y^2)^{m-1} m^2 (m-1) \left[ x^{2m-2} + \left( \sum_{j=1}^{m-1} A(j) x^{2m-2j-2} y^{2j} \right) + B x^{m-2} y^m + \left( \sum_{j=2}^{m-1} C(j) x^{m-2j} y^{m+2j-2} \right) + y^{2m-2} \right]. \tag{17}
\]

By replacing (1), (3) and (2) in (17) we conclude that
\[
P_x (Q_y P_{xy} - Q_x P_{yy}) + P_y (Q_x P_{xy} - Q_y P_{xx}) =
\]
\[
= 2k (x^2 + y^2)^{k-1} m^2 (m-1) \left[ x^{2m-2} + \left( \sum_{j=0}^{m-1} \left( \begin{array}{c} m-1 \\ j \end{array} \right) x^{2m-2j-2} y^{2j} \right) + \left( \sum_{r=1}^{m-2} \left( \begin{array}{c} m-1 \\ r \end{array} \right) x^{2m-2r-2} y^{2r} + y^{2m-2} \right) \right].
\]

By collecting all the terms inside the square brackets we have
\[
P_x (Q_y P_{xy} - Q_x P_{yy}) + P_y (Q_x P_{xy} - Q_y P_{xx}) =
\]
\[
= 2k (x^2 + y^2)^{k-1} m^2 (m-1) \left[ \sum_{j=0}^{m-1} \left( \begin{array}{c} m-1 \\ j \end{array} \right) x^{2m-2j-2} y^{2j} \right].
\]

Note that the expression inside the square brackets equals the binomial \((x^2 + y^2)^{m-1}\). So, we conclude
\[
P_x (Q_y P_{xy} - Q_x P_{yy}) + P_y (Q_x P_{xy} - Q_y P_{xx}) = 2k m^2 (m-1) (x^2 + y^2)^{k+1} \]
\[\square\]
5 Alternative Approach

Let us present a much faster way of obtaining inequality (13). Consider Euler’s Lemma:

Let $f \in H^n[x, y]$ be a homogeneous polynomial of degree $n$. Then,

$$nf(x, y) = xf_x(x, y) + yf_y(x, y).$$

By doing a recursive use of this Lemma we obtain that

$$P_x(Q_yP_{xy} - Q_xP_{yy}) + P_y(Q_xP_{xy} - Q_yP_{xx}) = \frac{n}{m-1}Q(HessP).$$

So, $\nabla P Hess P(\nabla Q)^t \leq 0$ since $P$ is a hyperbolic polynomial and $Q$ is positive on $\mathbb{R}^2$.

However, as mentioned before, we wanted to used the combinatorial identities of Theorem 1 and 2 as a way of putting combinatorics to the service of algebraic and differential geometry and with the hope that they can be used elsewhere.

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References

[1] N. Alon, Combinatorial Nullstellensatz, Combin. Probab. Comput. 8 no. 1-2 (1999) 7-29.

[2] V.I. Arnold, Astroidal geometry of hypocycloids and the Hessian topology of hyperbolic polynomials, Russian Math. Surveys 56 no. 6 (2001) 1019-1083.

[3] V. I. Arnold, Remarks on Parabolic Curves on Surfaces and the Higher-Dimensional Möbius-Sturm Theory, (Russian) Funktsional. Anal. i Prilozhen. 31 (1997) 3-18, 95; Translation in Funct. Anal. Appl. 310 no. 4 (1997) 227-239.

[4] V. I. Arnold, On the problem of realization of a given Gaussian curvature function, Topol. Methods Nonlinear Anal. 11 no. 2 (1998) 199-206.

[5] V. I. Arnold, Arnold’s Problems, Springer, Berlin 2004.
[6] M. Baker and S. Norine, Riemann-Roch and Abel-Jacobi Theory on a Finite Graph, Adv. Math. 215 (2007) 766-788.

[7] A. Beauville, Sur le nombre maximum de points doubles d’une surface dans $\mathbb{P}^3 (\mu(5) = 31)$, Journées de Géométrie Algébrique d’Angers, Juillet 1979/Algebraic Geometry, Angers (1979) 207-215, Alphen aan den Rijn–Germantown, Md.: Sijthoff & Noordhoff (1981).

[8] A. S. Buch, A Littlewood-Richardson rule for the $K$-theory of Grassmannians, Acta Math. 189 (2002) 37-78.

[9] R. Forman, Morse theory for cell complexes, Adv. Math. 134 (1998) 90-145.

[10] R. Forman, Combinatorial differential topology and geometry, New Perspectives in Algebraic Combinatorics, Math. Sci. Res. Inst. Publ. 38 (1999) 177-206.

[11] R. L. Graham, D. E. Knuth and O. Patashnik, Concrete Mathematics: A Foundation for Computer Science, Addison-Wesley, 1994.

[12] J. Huh, Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs, J. Amer. Math. Soc. 25 (2012) 907-927.

[13] C. Merino-López, Personal communication.

[14] D. A. Panov, Parabolic Curves and Gradient Mappings, Trudy Mat. Inst. Steklova 221 (1998) 271-288; English transl., Proc. Steklov Inst. Math 221 (1998) 261-278.

[15] T. Tao, Algebraic combinatorial geometry: the polynomial method in arithmetic combinatorics, incidence combinatorics, and number theory, EMS Surv. Math. Sci. 1 no. 1 (2014) 1-46.

[16] B. Young, Generating functions for colored 3D Young diagrams and the Donaldson-Thomas invariants of orbifolds (with an appendix by Jim Bryan), Duke Math. J. 152 (2010) 115-153.

[17] K. G. Wilson, Confinement of quarks, Phys. Rev. D 10 (1974) 2445-2459.

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