Explicit invariant measures for infinite dimensional SDE driven by Lévy noise with dissipative nonlinear drift I

Sergio Albeverio∗
Luca Di Persio†
Elisa Mastrogiacomo‡
Boubaker Smii§

Abstract

We study a class of nonlinear stochastic partial differential equations with dissipative nonlinear drift, driven by Lévy noise. Our work is divided in two parts. In the present part I we first define a Hilbert-Banach setting in which we can prove existence and uniqueness of solutions under general assumptions on the drift and the Lévy noise. We then prove a decomposition of the solution process in a stationary component and a component which vanishes asymptotically for large times in the $L^p$–sense, $p \geq 1$. The law of the stationary component is identified with the unique invariant probability measure of the process.

In part II we will exhibit the invariant measure as the limit of explicit invariant measures for finite dimensional approximants.

We shall present a general discussion of such explicit invariant measures involving, in particular, ground state transformation for Lévy driven processes and relations to invariant measures for Ornstein-Uhlenbeck like processes with Lévy noise.

Examples and applications are also provided.

Key words: SPDEs, dissipative drift, invariant measures, dissipative systems, process driven by Lévy noise, stationary components, Gibbs-type measures, explicit invariant measures, ground state transformation.

∗Dept. Appl. Mathematics, University of Bonn, HCM, BiBoS, IZKS. albeverio@uni-bonn.de
†University of Verona, Department of Computer Science, strada Le Grazie, 15, Verona, Italia. luca.dipersio@univr.it
‡Università degli Studi di Milano Bicocca, Dipartimento di Statistica e Metodi Quantitativi, Piazza Ateneo Nuovo, 1 20126 Milano Italia. elisa.mastrogiacomo@unimib.it
§King Fahd University of Petroleum and Minerals, Dept. Math. and Stat., Dhahran 31261, Saudi Arabia. boubaker@kfupm.edu.sa
1 Introduction

Stochastic differential equations for processes with values in infinite dimensional spaces have been studied in the literature under different assumptions on the coefficients and the driving noise term. They are intimately related with stochastic partial differential equations, looking upon the processes as taking values in the infinite dimensional state space expressing their dependence on the space variable. Among the by now numerous books on these topics for Gaussian noise let us mention [46, 47], [56], [99], [60], see also, e.g., [37], [83, 84], [99]. For the case of non Gaussian noises see [93], [78].

In [13] a study was initiated concerning a class of non linear stochastic differential equations with Lévy noise and a drift term consisting of a linear unbounded space-dependent part (typically a Laplacian) and an unbounded non linear part of the dissipative type and of at most polynomial growth at infinity.

This class is of particular interest since it contains the case of Fitz Hugh Nagumo equations with space dependence, on a bounded domain of \( \mathbb{R}^n \) or on bounded networks with 1-dimensional edges. Such equations are of interest in a number of areas including neurobiology and physiology, see, e.g., [5], [109, 110], [112]. It is also related with the stochastic quantization equation in quantum field theory, mostly studied with Gaussian noise, see [67], [87], [24], [44], [45], [19], [12], and references therein. See also [11] for related equations with Lévy noise. For the SPDE equations of the FitzHug-Nagumo type with Lévy noise studied in [13], existence and uniqueness of solutions was proven, as well as existence and uniqueness of invariant measures. Moreover asymptotic small noise expansion for the solutions have been established in the same paper [13].

In the present paper we further study the framework set up in [13] and provide a decomposition of the solution process in the sum of a stationary component and a component which vanishes asymptotically for large times in the \( L^p \)-sense, \( 1 \leq p < \infty \), with respect to the underlying probability measure.

The law of the stationary component is then identified with the unique invariant measure of the solution process.

This is an extension of a result that was proved before in the case of Gaussian noise in [6].

In part II we shall show that the invariant measure is the limit of explicitly given finite dimensional measures and we relate them in certain cases to the invariant measure for an Ornstein-Uhlenbeck process driven by a finite dimensional Lévy process in \( \mathbb{R}^n \).

We shall provide a general discussion of explicit invariant measures for Lévy driven processes in finite dimensions extending in particular the linear drift case of [101, 107] to the case of non linear drift.

For the derivation of these relations we shall use an adaptation to the case of Lévy noise of methods developed before in a diffusion setting (with Gaussian noise) which go under the name of \( h \)-transform, see [51], or “ground state transformation”, see, e.g., [18] and [94].

In the finite dimensional case this has been discussed in [30], where the general case of finite dimensional Lévy noise has been considered.
We relate in particular the work in [30] to work on solutions of martingale problems and relations to weak solutions of Lévy driven SDE. Our explicit invariant measures in finite dimensions considerable extend previously known examples in [27], [33].

The structure of part I of this series of two papers is as follows.

In Section 2 the setting for the infinite dimensional stochastic differential equations with linear drift and Lévy noise is described. Cylindrical Lévy processes are hereby introduced and the invariant measure is discussed.

In Section 3 the results on existence and uniqueness of solutions of such equations with non linear drift and of corresponding invariant probability measure are recalled.

In Section 4 the basic theorem giving the additive decomposition of the solution process in a stationary and an asymptotically small component is formulated and proven. Its proof uses a double indexed finite dimensional approximant, one, $m$, referring to a Yosida approximation of the non linear term (which we allow to be non globally Lipschitz, typically with polynomial growth at infinity!) and the other one, $n$, referring to $n$-dimensional approximations of the SPDE.

In the proof results on stochastic convolution (referring to [93]) and Gronwall’s type estimates are exploited.

Applications to the study of models described by PDE’s of the FitzHugh-Nagumo type with Lévy noise will be presented in part II.

2 The infinite dimensional Ornstein-Uhlenbeck process driven by Lévy noise

There is an increasing interest in the study of stochastic evolution equations driven by Lévy noise. In this section we concentrate on the linear stochastic differential equation

$$
\begin{align*}
\frac{dX(t)}{dt} &= AX(t)dt + dL(t), \quad t \geq 0, \\
X(0) &= x \in \mathcal{H},
\end{align*}
$$

where $\mathcal{H}$ is a real separable Hilbert space, $(L(t))_{t \geq 0}$ is an infinite dimensional cylindrical symmetric Lévy process and $A$ is a self-adjoint operator generating a $C_0$-semigroup. In particular we are going to recall a few results concerning the well-posedness of the above equation (Sec. 2.1) and the existence and uniqueness of explicit invariant measures (Sec. 2.2), in the case where $A$ satisfies Hypothesis 2.1 below, mainly following [95].

2.1 Cylindrical Lévy process

Let us recall that a Lévy process $(L(t))_{t \geq 0}$ with values in a real separable Hilbert space $\mathcal{H}$ is a process on some stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, having stationary independent increments, càdlàg trajectories in $\mathcal{H}$, and such that $L(0) = 0$, $\mathbb{P}$-a.s. One has that

$$
\mathbb{E}[e^{i(L(t),u)}] = \exp(-t\psi(u)), \quad u \in \mathcal{H}, \ t \geq 0,
$$

where $\psi$ is the characteristic exponent of $L$.
where the exponent \( \psi \) can be expressed by the following infinite dimensional Lévy-Khintchine formula:

\[
\psi(u) = \frac{1}{2} \langle Qu, u \rangle - i \langle a, u \rangle - \int_{\mathcal{H}} \left( e^{i(u,y)} - 1 - \frac{i(u,y)}{1 + |y|^2} \right) \nu(dy), \quad u \in \mathcal{H}.
\]

Here \( Q \) is a symmetric non-negative trace class operator on \( \mathcal{H} \), \( a \in \mathcal{H} \) and \( \nu \) is the Lévy measure or jump intensity measure associated to \((L(t))_{t \geq 0}\), i.e. \( \nu \) is a \( \sigma \)-finite Borel measure on \( \mathcal{H} \) such that \( \nu(\{0\}) = 0 \) and

\[
\int_{\mathcal{H}} (|y|^2 \wedge 1) \nu(dy) < \infty.
\]

We shall say that \( L(t) \) is generated by the triplet \((Q, \nu, a)\). Let us remark that the third term on the right hand side of (3) can also be written with the term \(-i \langle u, y \rangle \) replaced by \(-i \langle u, y \rangle \chi_D(y)\) with \( D \) the unit ball in \( \mathcal{H} \) and \( a \) replaced by \( y \).

In this paper we will consider a cylindrical Lévy process \( L = (L(t))_{t \geq 0} \) defined by the orthogonal expansion

\[
L(t) = \sum_{n=1}^{\infty} \beta_n L^n(t)e_n,
\]

where \( e_n \) is an orthonormal basis in \( \mathcal{H} \), \( L^n = (L^n(t))_{t \geq 0} \), \( L^n(0) = 0 \), are independent real valued, symmetric, identically distributed Lévy processes without a Gaussian part, defined on a fixed stochastic basis. Hence we are assuming, in particular, that \( L(t) \) has \( Q \equiv 0 \) and \( a \equiv 0 \). Moreover, \( \beta_n \) is a given (possibly unbounded) sequence of positive real numbers.

We also assume that \((e_n)_{n \in \mathbb{N}}\) is made of the eigenvectors of the leading operator \( A \) in \( \mathcal{H} \), assumed to have purely discrete spectrum. More precisely we make the following assumptions:

**Hypothesis 2.1.** \( A \) is a self-adjoint, strictly negative operator with domain \( D(A) \), which generates a \( C_0 \)-semigroup \( e^{tA} \), \( t \geq 0 \), on \( \mathcal{H} \) and such that there is a fixed basis \((e_n)_{n \in \mathbb{N}}\) in \( \mathcal{H} \) verifying: \((e_n)_{n \in \mathbb{N}} \subset D(A) \), \( Ae_n = -\lambda_n e_n \), with \( \lambda_n > 0 \), for any \( n \in \mathbb{N}^+ \) and \( \lambda_n \uparrow +\infty \).

**Remark 2.2.** From the above assumptions it follows that:

1. the action of \( e^{tA} \) on any element \( u \in \mathcal{H} \) can be written as

\[
e^{tA}u = \sum_{k=1}^{+\infty} e^{-\lambda_k t} \langle u, e_k \rangle, \quad k \in \mathbb{N}, \ t \geq 0.
\]

2. Since the law of \( L^n \), \( n \in \mathbb{N} \) is assumed to be symmetric, independent of \( n \) we have, for any \( n \in \mathbb{N}^+ \), \( t \geq 0 \),

\[
\mathbb{E}[e^{ihL^n(t)}] = e^{-t\psi_R(h)}, \quad h \in \mathbb{R},
\]
with $\psi_R$ being given by

$$\psi_R(h) = \int_{\mathbb{R}} (1 - \cos(hy))\nu_R(\mathrm{d}y), \quad h \in \mathbb{R},$$  

and the Lévy measure $\nu_R$ associated with $L_n$ is symmetric for any $n \in \mathbb{N}$ (i.e. $\nu_R(A) = \nu_R(-A)$, for any Borel subset $A$ of $\mathbb{R}$).

The definition of cylindrical Lévy process in (4) is only formal, since it has of course to be supplied with suitable assumptions on the $\beta_i$. We need namely to give conditions under which the series on the right hand side of (4) converges in $H$. Anyway, we can prove that $(L(t))_{t \geq 0}$ is always a well-defined Lévy process with values into a suitable Hilbert space $U$. To this end, we recall that any infinite dimensional separable Hilbert space $H$ can be identified with the space $\ell_2$, using the basis $(e_n)_{n \in \mathbb{N}}$. In general, for a given sequence $\rho = (\rho_n)_{n \in \mathbb{N}}$ of real numbers, we set

$$\ell_2^\rho := \left\{ (x_n) \in \mathbb{R}^\mathbb{N} : \sum_{n \geq 1} x_n^2 \rho_n^2 < \infty \right\}.$$

The space $\ell_2^\rho$ becomes a Hilbert space with the inner product $\langle x, y \rangle := \sum_{n \geq 1} x_n y_n \rho_n^2$ for $x = (x_n), y = (y_n) \in \ell_2^\rho$. For $\rho_n = 1$, $\forall n \in \mathbb{N}$, we have $\ell_2^\rho = \ell_2$. We quote from [95, Proposition 2.4] the result which provides conditions on $\beta_n, \nu_R$ such that the cylindrical Lévy process of the form given in (4) is well-defined in some Hilbert space $U$.

**Proposition 2.3.** The following conditions are equivalent:

(i) $\sum_{n=1}^{\infty} (\beta_n L_n(t_0))^2 < +\infty$ for some $t_0 > 0$, a.s.;

(ii) $\sum_{n=1}^{\infty} (\beta_n L_n(t))^2 < +\infty$ for any $t > 0$, a.s.;

(iii) $\sum_{n=1}^{\infty} \left( \beta_n^2 \int_{|y|<1/\beta_n} y^2 \nu_R(\mathrm{d}y) + \int_{|y|\geq 1/\beta_n} \nu_R(\mathrm{d}y) \right) < +\infty$.

**Remark 2.4.** According to Proposition 2.3, our cylindrical Lévy process $L$ is a Lévy process taking values in the Hilbert space $U := \ell_2^\rho$, with a properly chosen weight $\rho$. More precisely, we can choose any sequence $\rho = (\rho_n)$ such that

$$\sum_{n=1}^{\infty} \left( \rho_n^2 \beta_n^2 \int_{|y|<1/\beta_n} y^2 \nu_R(\mathrm{d}y) + \int_{|y|\geq 1/\beta_n} \nu_R(\mathrm{d}y) \right) < +\infty.$$

Now let us come back to the Ornstein-Uhlenbeck process described by equation (1). According to Hypothesis 2.1, we may consider equation (1) as an infinite sequence of independent one dimensional equations, i.e.

$$\begin{cases}
\mathrm{d}X^n(t) = -\lambda_n X^n(t) \mathrm{d}t + \beta_n L^n(t), \\
X^n(0) = x_n, \quad n \in \mathbb{N},
\end{cases}$$  

(6)
with \( x = \sum_{n} x_n e_n \in \mathcal{H} \), and \((x_n)_{n\in\mathbb{N}} \in l^2(\mathbb{R})\). The solution of (6) is the stochastic process 
\[
X = \sum_{n\geq 1} X^n e_n 
\]
with components
\[
X^n(t) = e^{-\lambda_n t} x_n + \int_{0}^{t} e^{-\lambda_n (t-s)} \beta_n dL^n(s), \quad n \in \mathbb{N}, \ t \geq 0.
\]

The processes \( X^n \), for \( n \in \mathbb{N} \), can be assumed to be almost surely right-continuous with left limits. Using point (2) in Remark (2.2) and following [101, pag. 105], we can compute their characteristic functions. In particular we have
\[
\mathbb{E}[e^{ihX^n(t)}] = \exp \left( \int_{0}^{t} \psi_R(e^{-\lambda_n (t-s)} \beta_n h) ds \right), \quad h \in \mathbb{R}, \ t \geq 0, \ n \in \mathbb{N},
\]
where \( \psi_R \) is the function defined in (5). In another way, we write
\[
X(t) = e^{tA} x + L_A(t), \tag{7}
\]
where
\[
L_A(t) = \int_{0}^{t} e^{(t-s)A} dL(s) = \sum_{n=1}^{\infty} \left( \int_{0}^{t} e^{-\lambda_n (t-s)} \beta_n dL^n(s) \right) e_n,
\]
and the process \( X \) is \( \mathcal{F}_t \)-adapted and Markovian. Here the convergence of the series in the last member of the equality above is to be meant in probability. More details on the solution of equation (1) are given in [95, Theorem 2.8]. In the following we provide an application of this result to the construction of an invariant probability measure for Ornstein-Uhlenbeck processes driven by a quite general class of symmetric cylindrical Lévy noises, which we shall call, for simplicity, O-U-Lévy processes.

2.2 Invariant measure for the infinite dimensional O-U-Lévy driven processes

As usual we say that a probability measure \( \mu \) on a complete separable metric (i.e. polish) space \( \mathcal{H} \) is invariant with respect to a Markov semigroup \((P_t)_{t\geq 0}\), with transition probability kernel \( P_t(x,dy) \), \( x,y \in \mathcal{H} \) on \( \mathcal{H} \) if for any Borel subset \( \Gamma \subset E \) and any \( t \geq 0 \) we have \( \mu(\Gamma) = \int \mu(dx) P_t(x, \Gamma) \). We say shortly \( \mu \) is an invariant measure for \((P_t)_{t\geq 0}\). See, e.g., [93, chapter 16] for equivalent formulations of this property.

**Proposition 2.5.** Assume Hypothesis 2.1. Moreover, assume that \( \beta_n, n \in \mathbb{N} \), is a bounded sequence and that the symmetric Lévy measure \( \nu_R \) appearing in (5) satisfies
\[
\int_{1}^{+\infty} \log(y) \nu_R(dy) < \infty.
\]
Finally, assume that
\[ \sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty. \]

Then the Lévy driven Ornstein-Uhlenbeck process \( X = (X(t))_{t \geq 0} \) given by \((7)\) admits a unique invariant measure, in the sense that the process \( X(t) \) is invariant under the Markov transition semigroup associated with \( X(t) \).

**Proof.** The proof proceeds basically on the same line of [95, Proposition 2.11]. To show that there exists an invariant measure we first notice that (according to [102, Theorem 17.5]) that each one dimensional Ornstein-Uhlenbeck process \( X^n(t) \) has an invariant measure \( \mu_n \) which is the law of the random variable
\[ \int_0^\infty e^{-\lambda_n u} \beta_n dL_n(u), \]
having characteristic function
\[ \hat{\mu}_n(h) = \exp \left( -\int_0^\infty \psi_R(e^{-\lambda_n u} \beta_n h) du \right), \quad h \in \mathbb{R}. \]

Let us consider the product measure \( \mu = \prod_{n \geq 1} \mu_n \) on \( \mathbb{R}^N \). This is the law of the family \((\xi_n)_{n \in \mathbb{N}}\) of independent random variables, where
\[ \xi_n = \int_0^\infty e^{-\lambda_n u} \beta_n dL_n(u), \quad n \geq 1. \]

We underline that \( \xi_n \) is an infinite divisible real-valued random variable. Now define \( \xi = \sum_{n=1}^{\infty} \xi_n e_n \). Then the random variable \( \xi \) takes values in \( \mathcal{H} \) if and only if the Lévy measures \( \nu_n \) of \( \xi_n \) verify
\[ \sum_{n=1}^{\infty} \int_\mathbb{R} (1 \wedge y^2) \nu_n(dy) < \infty. \] (8)

The latter comes from the definition of \( \xi_n \) and the condition (iii) in Proposition 2.3. Exploiting the result in [95, Proposition 2.11], we deduce that condition (8) is satisfied, hence we have \( \mu(\mathcal{H}) = 1 \). It is possible to prove that \( \mu \) is the unique invariant measure of \( X \) given by \((7)\), by showing that, for any \( x \in \mathcal{H} \),
\[ \lim_{t \to +\infty} X(t) = \xi \] (9)
in probability. To prove this we first assume \( x = 0 \) in \((7)\). In this case
\[ X^n(t) = \int_0^t e^{-\lambda_n(t-s)} \beta_n dL^n(t). \]
Now we recall the following identity:

\[ E \left[ e^{ih \int_s^t g(u) dL^n(u)} \right] = \exp \left( \int_s^t \psi_R(g(u)h) du \right), \quad h \in \mathbb{R}, \ 0 \leq s \leq t, \]

which holds for any real continuous function \( g \) on \([s, t] \), see [102, pag. 105]. Then if we compute the characteristic function of \( \xi_n \) we see that \( X^n(t) \) and \( \xi_n(t) \) have the same law. This fact allows us to estimate the following quantity:

\[ a_t(t) := \mathbb{P}(|X(t) - \xi|^2 > \epsilon), \quad \text{for any } \epsilon > 0, \ t > 0. \]

In fact, since \( X^n \) and \( \xi_n \) have the same law we can write

\[
|X(t) - \xi|^2 = \left| \int_0^t e^{(t-s)A} dL(s) - \sum_{n=1}^{\infty} \xi_n(t)e_n \right|^2 \\
= \left| \sum_{n=1}^{\infty} \int_0^t e^{-\lambda_n(t-s)} \beta_n dL^n(s) - \sum_{n=1}^{\infty} \int_0^\infty e^{-\lambda_n s} \beta_n dL^n(s) \right|^2 \\
= \left| \sum_{n=1}^{\infty} \beta_n^2 \int_0^t e^{-\lambda_n s} \beta_n dL^n(s) \right|^2.
\]

Moreover, (see [95, pag. 11]) it is possible to prove that the law of the random variable in the last term in the previous expression coincides with the one of

\[ \sum_{n=1}^{\infty} e^{-2\lambda_n t} \xi_n^2. \]

We then have, for any \( t > 0, \)

\[ a_t(t) := \mathbb{P} \left( \sum_{n=1}^{\infty} e^{-2\lambda_n t} \xi_n^2 > \epsilon \right) \leq \mathbb{P} \left( e^{-\lambda_1 t} \sum_{n=1}^{\infty} \xi_n^2 > \epsilon \right) = \mathbb{P}(|\xi|^2 > e^{-2\lambda_1 t} \epsilon). \]

Letting \( t \to +\infty \) we find \( \lim_{t \to +\infty} a(t) = 0, \) for any \( \epsilon > 0. \) This proves the claim (9) for the case \( x = 0. \) The general case, \( x \neq 0, \) can easily be obtained by translation. \( \square \)

**Remark 2.6.** For further use (see Section 3.2), we emphasize that from the proof of the claim (9) we can also see that, for any \( t > 0, \) the random variables

\[ \int_0^t e^{(t-s)A} dL(s) \quad \text{and} \quad \xi_t := \sum_{n=1}^{\infty} \int_0^t e^{-\lambda_n s} \beta_n dL^n(s) \]

have the same law. Hence we obtain that the random variable

\[ L_A(+) := \lim_{t \to +\infty} \int_0^t e^{(t-s)A} dL(s) \]

is well-defined and its law coincides with the one of the random variable \( \xi. \)
3 The stochastic semilinear differential equation

3.1 The state equation: existence and uniqueness of solutions

In this section we concentrate on a semilinear SDE driven by Lévy noise, thus extending the setting of Section 2 to include a non linear drift. Following basically the setting of [13] and [93], we consider a stochastic differential equation of the form

\[
\begin{aligned}
\left\{ \begin{array}{l}
dX(t) = AX(t) \, dt + F(X(t)) \, dt + B dL(t), \quad t \geq 0 \\
X(0) = x \in D(F)
\end{array} \right.
\end{aligned}
\]

where the stochastic process \(X = (X(t))_{t \geq 0}\) takes values in a real separable Hilbert space \(H\), \(A\) is a linear operator from a dense domain \(D(A)\) in \(H\) into \(H\) which generates a \(C_0\)-semigroup of strict negative type. \(B\) is a linear bounded operator from a suitable Hilbert space \(U\) (which is to be precised, see Remark 3.1 below) into \(H\). \(F\) is a mapping from \(D(F) \subset H\) into \(H\), continuous, nonlinear, Fréchet differentiable and such that

\[
\langle F(u) - F(v) - \eta(u - v), u - v \rangle < 0, \quad \text{for some } \eta > 0
\]

and all \(u, v \in D(F)\), where \(\langle , \rangle\) is the scalar product in \(H\).

The connection between \(A\) and \(F\) consists in requiring that \(\omega > \eta\) (so that \(A + F\) is maximal dissipative or \(m\)-dissipative in the sense of [46, pag. 73], i.e. the range of \(\lambda - (A + F)\) is \(H\), for some - and consequently all - \(\lambda > 0\)).

We assume that \(L\) is a cylindrical Lévy process according to the description given in Section 2.1 i.e.

\[
L(t) = \sum_{n=1}^{\infty} \beta_n L^n(t) e_n,
\]

with \((\beta_n)_{n \in \mathbb{N}}\) a given (possibly unbounded) sequence of positive real numbers, \((L^n)_{n \in \mathbb{N}}\) a sequence of real valued, symmetric, i.i.d. Lévy processes without Gaussian part and \((e_n)_{n \in \mathbb{N}}\) an orthonormal basis of \(H\). Moreover, we will work under Hypothesis 2.1 hence assuming that \((e_n)_{n \in \mathbb{N}}\) is made of eigenvectors of \(A\).

Remark 3.1. According to Remark 2.4 identifying \(H\) with the space \(\ell^2\), the cylindrical Lévy process \(L(t)\) is a well-defined Lévy process taking values in the Hilbert space \(U := \ell^2_\rho\) for a suitable sequence \(\rho = (\rho_n)_{n \in \mathbb{N}}\). This means that

\[
\sum_{k=1}^{\infty} (\beta_k^2 L^n(t)^2 \rho_k^2) < \infty.
\]

We assume that \(B\) is a linear bounded operator acting on \(U\) with values in \(H\). Through the identification of \(H\) with \(\ell^2\) we can think that \(B\) can be written in the following form:

\[
BL(t) = \sum_{k=1}^{\infty} b_k \beta_k L^n(t) e_k,
\]

with \((b_k)_{k \in \mathbb{N}}\) such that \(\sup_{n \in \mathbb{N}} \frac{b_k}{\rho_k} < \infty\).
Let us provide an example for the setting \( \mathcal{B} \) and such that the corresponding Lévy measure \( \nu \) require Remark 3.2. Let \( A \) be a process on \( \mathcal{B} \) and that \( F \) maps bounded subsets of \( \mathcal{B} \) into bounded subsets of \( \mathcal{H} \). We also assume that the stochastic convolution \( L_A(t) \) of \( B dL(t) \) with \( S(t) = e^{tA} \), i.e.

\[
L_A(t) = \int_0^t S(t-s)B \, dL(s), \quad t \geq 0,
\]

has a càdlàg version in \( D(-A_B)^\alpha \) for some \( \alpha \in [0,1) \). Finally, we impose the following condition: for all \( T > 0 \)

\[
\int_0^T |F(L_A(t))|_{BdL} dt < \infty, \quad \mathbb{P} - a.s.
\]

(cf. [93, p. 183] for more details). It follows from our assumptions that \( L_A \) is square-integrable, \( \mathcal{F}_t \)-adapted (where \( \mathcal{F}_t \) is the natural \( \sigma \)-algebra associated with \( L \)) (see [93, p. 163]).

**Remark 3.2.** We give two cases where the property \( (13) \) of \( (L_A(t))_{t \geq 0} \) holds:

1. \( \mathcal{B} \) is a Hilbert space and \( S(t) \) is a contraction semigroup in this space \( \mathcal{B} \) and \( B L(t) \) takes values in \( \mathcal{B} \); see, e.g. [93, p.158, Theorem 9.20].

2. \( S(t) \) is analytic in \( \mathcal{B} \) and \( B L \) has càdlàg trajectories in \( D((-A_B)^\alpha) \), for some \( \alpha \in [0,1) \), where \( A_B \) is the restriction of \( A \) to \( \mathcal{B} \), see, e.g. ([93, p. 163, Prop. 9.28]).

**Example 3.3.** Let us provide an example for the setting \( \mathcal{H}, \mathcal{U}, \mathcal{B}, L, A, Q, F \) where both conditions \( (13) \) and \( (14) \) hold. Let \( \Lambda \subset \mathbb{R}^n \), bounded and open, \( n \in \mathbb{N} \), and let \( \mathcal{H} = \mathcal{U} := L^2(\Lambda) \). Let \( F \) be of the form of a multinomial of odd degree \( 2n+1 \), \( n \in \mathbb{N} \), i.e. \( F \) is a mapping of the form \( F(u) = g_{2n+1}(u) \), where \( g_{2n+1} : \mathbb{R} \to \mathbb{R} \), is a polynomial of degree \( 2n+1 \) with first derivative bounded from above, see [13]. It follows that \( D(F) = L^{2(2n+1)}(\Lambda) \subset L^2(\Lambda) \). We take \( \mathcal{B} := L^{2p}(\Lambda) \) with \( p \geq 2n+1 \). Let \( A = \Lambda \) be the Laplacian in \( L^2(\Lambda) \) with Neumann boundary conditions on the boundary \( \partial \Lambda \). Let \( Q = 1 \) as an operator in \( \mathcal{H} \) and let \( L \) be a process on \( \mathcal{H} = L^2(\Lambda) \) of the type described in Remark 3.2 and such that the corresponding Lévy measure \( \nu \) satisfies

\[
\int_{L^2(\Lambda)} |x|_{W^{\beta,2p(2n+1)}} \nu(dx) < +\infty,
\]

where \( W^{\beta,2p(2n+1)} \) is a fractional Sobolev space with given index \( \beta > 0 \); moreover we require \( \int_{|y| \leq 1} |y|^2 \nu(dy) + \nu(y) |y| \geq 1 \) \( \infty \). From [13] (see [93, Prop. 6.9]) it follows that \( L(t) \in D((-A_{2p(2n+1)})^\gamma) \subset L^{2p(2n+1)}(\Lambda) \) for some \( \gamma > 0 \) and has càdlàg trajectories in \( D((-A_{2p(2n+1)})^\gamma) \). Here \( A_{2p(2n+1)} \) denotes the generator of the heat semigroup with Neumann boundary conditions operating in \( L^{2p(2n+1)} \). Then by [93, Prop. 9.28, p.163 and Theorem 10.15, pag.187], \( L_A \) is well-defined and satisfies \( (13) \) and \( (14) \).
Now we are ready to state the main result of this section, which concerns with existence and uniqueness of solutions for equation (10). We refer to [13, Theorem 4.9] for the proof.

**Theorem 3.4.** Assume \( A, Q, F, L \) satisfy all previous assumptions and let \( L_A \) satisfy conditions (13) and (14) above. Then there exists a unique càdlàg mild solution of (10) in the sense of being adapted, càdlàg in \( B \), for any \( x \in B \) and satisfying almost surely

\[
X(t) = S(t) x + \int_0^t S(t - s) F(X(s)) \, ds + L_A(t), \quad t \geq 0, \ x \in D(F),
\]

with \( X(t) \in D(F) \) for all \( t \geq 0 \).

For each \( x \in H \) there exists a generalized solution of (10), i.e. there exists \( \{X_n\}_{n \in \mathbb{N}} \), \( X_n \in B \), \( X_n \) unique mild adapted solutions of (10) with \( X_n(0) = x \) such that \( |X_n(t) - X(t)|_H \to 0 \) on each bounded interval, as \( n \to \infty \).

Moreover \( X(t) \) defines Feller families on \( B \) and on \( H \), in the sense that the Markov semigroup \( P_t \) associated with \( X(t) \) maps for any \( t \geq 0 \), \( C_b(H) \) into \( C_b(H) \) and \( C_b(B) \) into \( C_b(B) \).

### 3.2 Existence and uniqueness of an invariant measure

In the following we deal with the asymptotic behavior of the Markov semigroup corresponding to equation (10). We will see that in our case, in addition to existence and uniqueness of the invariant measure \( \mu \) (whose notion has been recalled in Section 2), we can prove that it is also exponentially mixing. We quote from [93, chapter 16] the definition of exponentially mixing invariant measure. Let \( Lip(H) \) be the space of all real valued Lipschitz continuous functions \( \psi : H \to \mathbb{R} \) endowed with norm \( \|\psi\|_\infty + \|\psi\|_{Lip} \), with \( \|\psi\|_\infty \) the sup-norm and

\[
\|\psi\|_{Lip} := \sup_{x \neq y} \frac{|\psi(x) - \psi(y)|}{|x - y|_H};
\]

\( \|\psi\|_{Lip} \) is said to be the smallest Lipschitz constant for \( \psi \), see [93, p.16]. We say that an invariant measure \( \mu \) is exponentially mixing with exponent \( \omega > 0 \) and bound function \( c : H \to (0, +\infty) \) with respect to a Markov semigroup \( \{P_t\}_{t \geq 0} \) if

\[
\left| P_t \psi(x) - \int_H \psi(y) \, d\mu(y) \right| \leq c(x) e^{-\omega t} \|\psi\|_{Lip}, \quad \forall x \in H, \ \forall t > 0, \ \psi \in Lip(E).
\]

If \( \mu \) is exponentially mixing, then \( P_t(x, \Gamma) \to \mu(\Gamma) \) as \( t \uparrow +\infty \) for any Borel subset \( \Gamma \) of \( E \) (cfr. [10] and [93, p. 288]).

We have the following from [93, Theorem 16.6 p.293]:

**Theorem 3.5.** Let us consider the SDE (10) under the assumptions on \( A \) and \( F \) given at the beginning of the Subsection 3.1 and under the assumptions of Theorem 3.4. Assume, in addition:

\[
\sup_{t \geq 0} \mathbb{E} (|L_A(t)|_H + |F(L_A(t))|_H) < +\infty.
\]
Then there exists a unique invariant measure $\mu$ for the Markov semigroup $(P_t)_{t \geq 0}$ on $\mathcal{H}$ associated with the mild solution $X$ of (10) ($(P_t)_{t \geq 0}$ gives the transition probabilities for $X$).

$\mu$ is exponentially mixing with exponent $\omega + \eta$ and a bound function $c$ of linear growth in the sense that

$$|c(x)| \leq C(|x| + 1),$$

for some constant $C > 0$ and for all $x \in \mathcal{H}$.

4 Decomposition of the solution process in a stationary and an asymptotically small component

Let us consider the cylindrical Lévy process $L = (L(t))_{t \geq 0}$, as in Sections 2 and 3. Following [93, p. 295], let $L(t), t \in \mathbb{R}$ be the corresponding double-sided process such that $L(t) = L(t), t \geq 0$ and $L(-t), t \geq 0$ is a process independent of $L(t), t \geq 0$ and such that all finite dimensional distributions of $L(t)$ coincide with those of $L(t), t \geq 0$.

Our aim now is to split the solution $X$ of the equation (10) into the sum of a stationary process $r$ and an asymptotically (for $t \to +\infty$) vanishing process $v$. To this end we will split the solution $X^{(n)}(0, m) \in \mathbb{N}$, of the approximating equation into the sum of a stationary process $r^{(n)}$ and an asymptotically vanishing process $v^{(n)}$ satisfying some suitable properties.

Let $\{B_n\}_{n \in \mathbb{N}}$ be a sequence of finite dimensional subspaces of the Banach space $B$ introduced in Sect.3 and $\{\Pi_n\}$ a sequence of self-adjoint operators from $\mathcal{H}$ onto $B_n$ such that $\Pi_n x \to x$ in $B$, for arbitrary $x \in B$. The existence of such spaces can be proved as in [117], Prop.3.

Moreover let $F, m \in \mathbb{N}$ be the $m$-th Yosida approximation of $F$ (i.e. $F_m := m(mI - F)^{-1}$). We know that $F_m$ is Lipschitz continuous and it satisfies the following estimates:

$$|F_m(x) - F(x)|_H \to 0, \quad m \to \infty, x \in D(F)$$

and

$$|F_m(x)|_H \leq |F(x)|_H, \quad x \in D(F), \forall m \in \mathbb{N}$$

and

$$\langle F_m(x) - F_m(y), x - y \rangle \leq m|x - y|^2, \quad \forall m \in \mathbb{N}.$$ 

For any $n, m \in \mathbb{N}$ we consider the following families of equations

$$\begin{cases}
    dX^{(n)}_m(t) = AX^{(n)}_m(t)dt + \Pi_n F_m(\Pi_n X^{(n)}_m(t))dt + BdL(t), \\
    X^{(n)}_m(0) = \Pi_n x \in \mathcal{H},
\end{cases}$$

and

$$\begin{cases}
    dX^{(n)}(t) = AX^{(n)}(t)dt + \Pi_n F(\Pi_n X^{(n)}(t))dt + BdL(t), \\
    X^{(n)}(0) = \Pi_n x \in \mathcal{H},
\end{cases}$$

(21)
which can be seen as approximating problems relative to (10).

There exists a well-established theory on stochastic evolution equations in Hilbert spaces, see, for example, Da Prato and Zabczyk [7], that we shall apply in order to show that, for any \( n, m \in \mathbb{N} \), equation (20), admits a unique solution \( X_m^{(n)} \). The precise statement concerning the well-posedness of problems (20) and (21) can be found in [7, Propositions 5.4 and 5.5]. Moreover, it is possible to prove an existence and uniqueness result for equation (10) in Section 3.1 through an approximating procedure on finite dimensional spaces and with a Lipschitz continuous nonlinearity. For more details concerning these results compare [7, Section 5], where the case of Gaussian noise is treated. In the following we investigate the asymptotic properties of the mild solution of (10). In particular, we are going to prove that its unique solution admits a characterization in terms of a stationary process \( r \) and a process \( v \) which vanishes at \( t \to \infty \). To this end, we proceed by splitting the solution \( X_m^{(n)} \) of the approximating problems into the sum of a stationary process \( r_m^{(n)} \) and a vanishing process \( v_m^{(n)} \) satisfying suitable properties. Let us define the two sequences of processes \( r_m^{(n)} \) and \( v_m^{(n)} \) respectively, in the sense of the definition (11) below for \( r_m^{(n)} \) and \( v_m^{(n)} = X_m^{(n)}(t) - r_m^{(n)}(t) \), as solutions of the equations:

\[
\begin{align*}
    r_m^{(n)}(t) &= \int_{-\infty}^{t} S(t-s) \Pi_n F_m \Pi_n (r_m^{(n)}(s)) ds + \bar{L}_A^{\infty,n}(t) \\
    v_m^{(n)}(t) &= S(t)x - \int_{-\infty}^{0} e^{(t-s)A} \Pi_n F_m \Pi_n (r_m^{(n)}(s)) ds - \int_{-\infty}^{0} e^{(t-s)A} \Pi_n B \bar{L}(s) \\
    &\quad + \int_{0}^{t} e^{(t-s)A} [\Pi_n F_m (\Pi_n X_m^{(n)}(s)) ds - \Pi_n F_m (\Pi_n r_m^{(n)}(s))] ds,
\end{align*}
\]

where \( S(t) \) is as in Section 2. \( F_m, \Pi_n \) are as above and \( \bar{L}_A^{\infty,n}(t), t \in \mathbb{R} \), is defined by

\[
L_A^{a,n}(t) := \int_{-a}^{t} S(t-s) \Pi_n B d\bar{L}(s), \quad t \geq -a, \quad a \geq 0
\]

We claim that the random variable \( L_A^{a,n}(t) \) is well-defined, for any \( t \in \mathbb{R} \). In fact, \( L_A^{a,n}(t), t \geq -a \), can be split into the following sum:

\[
L_A^{\infty,n}(t) = \lim_{a \to -\infty} \int_{-a}^{0} S(t-s) \Pi_n B d\bar{L}(s) + \int_{0}^{t} S(t-s) \Pi_n B d\bar{L}(s)
\]

and the second term is easily seen to be well-defined for any \( t \geq -a \). Concerning the first term we notice that, for any \( a \geq 0, t \geq -a \)

\[
\int_{-a}^{0} S(t-s) \Pi_n B d\bar{L}(s) = \sum_{k=1}^{n} \left( \int_{-a}^{0} e^{-\lambda_k(t-s)} b_k \beta_k d\bar{L}^k(s) \right) e_k = \\
\sum_{k=1}^{n} e^{-\lambda_k t} \left( \int_{-a}^{0} e^{\lambda_k s} b_k \beta_k d\bar{L}^k(s) \right) e_k = - \sum_{k=1}^{n} e^{-\lambda_k t} \left( \int_{0}^{a} e^{\lambda_k s} b_k \beta_k d\bar{L}^k(s) \right) e_k.
\]
Now, taking into account Remark 4.6 in Subsection 2.2 we see that
\[
\lim_{a \to +\infty} \sum_{k=1}^{\infty} \left( \int_{0}^{a} e^{-\lambda_k s} b_k \beta_k dL^k(s) \right) e_k
\]
is a well-defined $\mathcal{H}$-valued random variable; hence
\[
\bar{L}_{A}^{\infty,n}(t) = \int_{-\infty}^{0} e^{(t-s)A} \Pi_n B d\bar{L}(s), \quad t \in \mathbb{R}
\]
and, taking $n \to +\infty$:
\[
\bar{L}_{A}^{\infty}(t) := \int_{-\infty}^{t} e^{(t-s)A} B d\bar{L}(s)
\]
are well-defined too, for any $t \in \mathbb{R}$.

We give the following notion of solution for equation (22).

**Definition 4.1.** An $\mathcal{F}_t$-adapted process $r_{m}^{(n)}$ is said to be a mild solution to equation (22) if it satisfies the integral equation (22) for any $t \in \mathbb{R}$.

**Theorem 4.2.** For any $n, m \in \mathbb{N}$, there exists a unique mild solution $r_{m}^{(n)}$ to the equation (22), such that
\[
\sup_{t \in \mathbb{R}} \mathbb{E}|r_{m}^{(n)}(t)|_{\mathcal{H}}^{p} \leq C_p, \quad (24)
\]
for every $p \geq 2$ and for some positive constant $C_p$ (independent on $n$ and $m$). Further, $r_{m}^{(n)}$ is a stationary process, that is, for every $h \in \mathbb{R}^+$, $k \in \mathbb{N}$, any $-\infty < t_1 \ldots \leq t_k < +\infty$ and any $A_1, \ldots, A_k \in \mathcal{B}(\mathcal{H})$ we have
\[
\mathbb{P}(r_{m}^{(n)}(t_1+h) \in A_1, \ldots, r_{m}^{(n)}(t_k+h) \in A_k) = \mathbb{P}(r_{m}^{(n)}(t_1) \in A_1, \ldots, r_{m}^{(n)}(t_n) \in A_k).
\]

**Proof.** Let us first prove the uniqueness: Assume that $(x(t))_{t \in \mathbb{R}}$ and $(y(t))_{t \in \mathbb{R}}$ are solutions of equation (22). Dissipativity of $A + \Pi_n F_m(\Pi_n)$, which follows from the assumptions, implies
\[
d|y(t) - x(t)|^2 = \langle A(x(t) - y(t)) + \Pi_n F_m(\Pi_n x(t)) - \Pi_n F_m(\Pi_n y(t)), x(t) - y(t) \rangle \; dt
\]
and, by using Gronwall’s lemma, we deduce that for any $\xi > 0$ and $t \geq -\xi$ the following inequality holds
\[
|y(t) - x(t)|^2 \leq |x(-\xi) - y(-\xi)|^2 e^{-2(\omega - \eta)(t+\xi)}.
\]
Letting $\xi \to +\infty$ we conclude that $x(t) = y(t)$ for any $t \in \mathbb{R}$.

For the existence of a solution $r_{m}^{(n)}(t)$ to (22) let $r_{m}^{(n)}(t, -\xi), \xi > 0$ be the unique solution of the equation:
\[
\begin{align*}
\frac{dr_{m}^{(n)}(t, -\xi)}{dt} &= Ar_{m}^{(n)}(t) dt + \Pi_n F_m(\Pi_n r_{m}^{(n)}(t, -\xi)) dt + \Pi_n B dL(t), \quad t \geq -\xi, \\
r_{m}^{(n)}(-\xi, -\xi) &= e^{-\xi A} x
\end{align*}
\]
Let us note that equation (25) has a unique solution being a Cauchy problem with Lipschitz coefficients, see, e.g., [33] for details. From the smoothness properties of the stochastic convolution, one can assume that \( r_m^{(n)}(\cdot; -\xi) \) admits a \( \mathcal{B} \)-càdlàg version, which we still denote by \( r_m^{(n)}(\cdot; -\xi) \). Consequently, for any \( t \in \mathbb{R} \), we can define \( r_m^{(n)}(t) \) as the limit of \( r_m^{(n)}(t; -\xi) \) for \( \xi \to +\infty \) and this turns out to be the solution of equation (22). Now we will prove that, for any \( t \geq -\xi, \xi > 0 \) and \( p \geq 2 \) the following estimate holds
\[
\sup_{-\xi \leq t \leq 0} \mathbb{E}||r_m^{(n)}(t; -\xi)||_\mathcal{H}^p < C_p,
\]
where \( C_p \) is a positive constant independent on \( n, m \) and \( \xi \), but possibly depending on \( p \). For simplicity, and without loss of generality, we consider the case \( p = 2a, a \in \mathbb{N} \). We want to apply Itô’s formula to the processes \( |r_m^{(n)}(t, -\xi)|^2 \). To this end, we recall the expressions for the first and second derivatives of the function \( F(x) := |x|^{2a} \).

We have
\[
\nabla F(x) = 2a|x|^{2(a-1)}x,
\]
\[
\frac{1}{2}Tr(Q\nabla F^2(x)) = aTr(Q)|x|^{2(a-1)} + (a - 1)a|x|^{2(a-2)}|Bx|^2.
\]

Hence
\[
d|r_m^{(n)}(t, -\xi)|^{2a} = \langle 2a|r_m^{(n)}(t, -\xi)|^{2(a-1)}r_m^{(n)}(t, -\xi), dr_m^{(n)}(t, -\xi) \rangle
\]
\[
+ aTr(Q)|r_m^{(n)}(t, -\xi)|^{2(a-1)}dt + (a - 1)a|r_m^{(n)}(t, -\xi)|^{2(a-2)}|B r_m^{(n)}(t, -\xi)|^2 dt.
\]

Now using the dissipativity of \( A + F \), for sufficient small \( \epsilon > 0 \), we get
\[
d|r_m^{(n)}(t, -\xi)|^{2a} = 2a|r_m^{(n)}(t, -\xi)|^{2(a-1)}\langle Ar_m^{(n)}(t, -\xi) + \Pi_n F_m \Pi_n (r_m^{(n)}(t, -\xi)), r_m^{(n)}(t, -\xi) \rangle dt
\]
\[
+ 2a|r_m^{(n)}(t, -\xi)|^{2(a-1)}\langle \Pi_n BdL(t), r_m^{(n)}(t, -\xi) \rangle dt
\]
\[
+ aTr(Q)|r_m^{(n)}(t, -\xi)|^{2(a-1)}dt + (a - 1)a|r_m^{(n)}(t, -\xi)|^{2(a-2)}|B r_m^{(n)}(t, -\xi)|^2 dt
\]
\[
\leq -2(\omega - \eta)|r_m^{(n)}(t, -\xi)|^{2a}_\mathcal{H} + 2a|r_m^{(n)}(t, -\xi)|^{2(a-1)}\langle F(0), r_m^{(n)}(t, -\xi) \rangle dt
\]
\[
+ C_{a,Q}|r_m^{(n)}(t, -\xi)|^{2(a-1)} + 2a|r_m^{(n)}(t, -\xi)|^{2(a-1)}\langle \Pi_n BdL(t), r_m^{(n)}(t, -\xi) \rangle dt
\]
\[
\leq -2(\omega - \eta)|r_m^{(n)}(t, -\xi)|^{2a}_\mathcal{H} + 2a|r_m^{(n)}(t, -\xi)|^{2(a-1)}\langle \Pi_n BdL(t), r_m^{(n)}(t, -\xi) \rangle dt
\]
\[
\leq -2(\omega - \eta - \epsilon)|r_m^{(n)}(t, -\xi)|^{2a}_\mathcal{H} + \frac{1}{\epsilon}C_{a,Q,F(0)}dt + 2a|r_m^{(n)}(t, -\xi)|^{2(a-1)}\langle \Pi_n BdL(t), r_m^{(n)}(t, -\xi) \rangle dt.
\]

and, integrating over \([-\xi, t], t \geq -\xi \) we obtain:
\[
|r_m^{(n)}(t, -\xi)|^{2a} \leq e^{-2a\xi\omega}|x|^{2a} - 2(\omega - \eta - \epsilon) \int_{-\xi}^{t} |r_m^{(n)}(s, -\xi)|^{2a}_\mathcal{H} ds
\]
\[
+ \int_{-\xi}^{t} |r_m^{(n)}(s, -\xi)|^{2(a-1)}\langle \Pi_n BdL(s), r_m^{(n)}(s, -\xi) \rangle + \frac{1}{\epsilon}C_{a,Q,F(0)}(t + \xi).
\]
Notice that the term
\[ \int_{-\xi}^{t} |y^{(n)}_{m}(s; \omega)|^2(\omega-t)BdL(s), r^{(n)}_{m}(s; \omega)| \]

is a square integrable martingale with mean 0, so that taking the expectation of both members in the previous inequality we obtain
\[ \mathbb{E}|r^{(n)}_{m}(s; \omega)|^{2a} \leq e^{-2(\omega-t)|x|^{2a}} + \frac{1}{\omega}C_{a,Q,F(0)}(\xi + t) - 2(\omega - \eta + \epsilon) \int_{-\xi}^{t} \mathbb{E}|r^{(n)}_{m}(r; \omega)|^{2a}ds . \]

Then applying Gronwall’s lemma we get
\[ \mathbb{E}|r^{(n)}_{m}(t; \omega)|^{2a} \leq [e^{-2a(\omega-t)|x|^{2a}} + \frac{1}{\omega}C_{a,Q,F(0)}(\xi + t)]e^{-2a(\omega-\eta)} \xi \]
where \( C \) is a suitable constant independent on \( m, n \) and \( \xi \).

Moreover, in a similar way we can prove that, for any fixed \( t \in \mathbb{R} \), the sequence \( \{r^{(n)}_{m}(t; \omega)\} \) is a Cauchy sequence, uniformly on \( t \).

Now let \( 0 \leq \gamma \leq \xi \). We need to estimate the norm:
\[ \sup_{t \geq -\gamma} \mathbb{E}|r^{(n)}_{m}(t; \omega) - r^{(n)}_{m}(t; -\gamma)|^{2a} . \]

To this end we notice that the process \( y^{(n)}_{\xi,\gamma}(t) := r^{(n)}_{m}(t; \omega) - r^{(n)}_{m}(t; -\gamma) \) can be written as:
\[ y^{(n)}_{\xi,\gamma}(t) = (e^{-\xi A} - e^{-\gamma A})x + \int_{-\xi}^{\gamma} e^{(\gamma-s)A}\Pi_{n}F_{m}(\Pi_{n}r^{(n)}_{m}(s; \omega))ds \]
\[ + \int_{-\xi}^{\gamma} e^{(\gamma-s)A}BdL(s) \]
\[ + \int_{-\gamma}^{t} e^{(t-s)A}[\Pi_{n}F_{m}(\Pi_{n}r^{(n)}_{m}(s; \omega)) - \Pi_{n}F_{m}(\Pi_{n}r^{(n)}_{m}(s; -\gamma))]ds. \]

Moreover, the stochastic differential of \( y^{(n)}_{\xi,\gamma}(t) \) is given by
\[ dy^{(n)}_{\xi,\gamma}(t) = Ay^{(n)}_{\xi,\gamma}dt + [F_{m}(r^{(n)}_{m}(t; \omega)) - F_{m}(r^{(n)}_{m}(t; -\gamma))]dt, \]

since the three terms in (27) do not depend on \( t \). By applying Itô’s formula to \( |y^{(n)}_{\xi,\gamma}|^{2a} \) and reasoning as above we get
\[ \mathbb{E}|y^{(n)}_{\xi,\gamma}(t)|^{2a} \leq |e^{-\xi x}|^{2a} + \mathbb{E}|y^{(n)}_{\xi,\gamma}(\gamma)|^{2a} - 2(\omega - \eta) \int_{-\xi}^{t} \mathbb{E}|y^{(n)}_{\xi,\gamma}(r)|^{2a}ds . \]
Gronwall’s lemma then implies
\[
\sup_{t \geq -\gamma} \mathbb{E}|y_{\xi, \gamma}(t)|^{2a} \leq (\mathbb{E}|y_{\xi, \gamma}(-\gamma)|^{2a} + |e^{-\xi A}x|^{2a})e^{2(\omega-\eta)(\xi+t)} \leq C,
\]
where \(C\) is a constant independent on \(\xi\). We then conclude that for any \(t \in \mathbb{R}\) the limit \(r_m^{(n)}(t) := \lim_{\xi \to \infty} r_m^{(n)}(t; -\xi)\) exists in \(L^2(\Omega, \mathbb{P}; \mathcal{B})\) and moreover,
\[
\sup_{t \geq -\xi} \mathbb{E}|r_m^{(n)}(t)|^{2a} \leq C.
\]
In addition, by the condition at \(t = -\xi\) in (25), we deduce that \(\lim_{t \to -\infty} r_m^{(n)}(t) = 0\).

Finally, for any \(n, m \in \mathbb{N}\) we are going to show that the process \(r_m^{(n)}\) is stationary. In order to prove this statement, we adapt to our case the argument given in [79]. In particular, we introduce the following Picard iteration:
\[
\begin{cases}
r_m^{(n,0)}(t) = x \\
r_m^{(n,k+1)}(t) = \int_{-\infty}^{t} e^{(t-s)A} \Pi_n F_m(\Pi_n r_m^{(n,k)}(s)) ds + L_{A,-\infty}(t).
\end{cases}
\]

We notice that the limit \(\lim_{k \to \infty} r_m^{(n,k)}(t) = \tilde{r}_m^{(n)}(t)\) exists (see, e.g., [46]) and it is a stationary process. The crucial point is that \(\tilde{r}_m^{(n)}\) and \(r_m^{(n)}\) coincide. In fact, if we pass to the limit in (28), we see that \(r_m^{(n)}\) solves equation (22) so that, by uniqueness, \(\tilde{r}_m^{(n)} \equiv r_m^{(n)}\). Consequently, \(r_m^{(n)}\) is stationary. This completes the proof of theorem (4.2). \(\square\)

Taking into account our assumptions on the double sided convolution process \(\tilde{L}_A(t)\) we will discuss existence and uniqueness of a mild solution for the equation for the process \(v_m^{(n)}\) and we will show that it vanishes, in a suitable sense, as \(t \to +\infty\).

**Proposition 4.3.** Under the assumptions given in Theorem 3.4 and in Theorem 3.5 we have that for any \(n, m \in \mathbb{N}\) there exists a unique mild solution \((v_m^{(n)}(t))_{t \geq 0}\) of equation (23). Moreover, for any \(p \geq 1\), we have the following bound:
\[
\sup_{t \geq 0} \mathbb{E}|v_m^{(n)}(t)|^p \leq C_p,
\]
where \(C_p\) is a positive constant independent of \(n\) and \(m\). In addition, we have the following limit
\[
\lim_{t \to +\infty} \mathbb{E}|v_m^{(n)}(t)|^p = 0, \quad \text{for any } p \geq 1.
\]

**Proof.** The existence and uniqueness of a mild solution for the process \((v_m^{(n)})\) follow straightforward by results in finite dimensions, see e.g., [31].
Without loss of generality we can assume that \( p = 2a \) for \( a \in \mathbb{N} \). By the dissipativity of the mapping \( \Pi_n F_m \Pi_n \) and the fact that \( A \leq -\omega \), we get

\[
d|v^{(n)}_m(t)|^{2a} = d|X^{(n)}_m(t) - r^{(n)}_m(t)|^{2a} = 2a \langle A v^{(n)}_m(t) + \Pi_n F_m (\Pi_n X^{(n)}_m(t)) - \Pi_n F_m (\Pi_n r^{(n)}_m(t)), v^{(n)}_m(t) \rangle |v^{(n)}_m(t)|^{2a-2} dt - 2a\omega |v^{(n)}_m(t)|^{2a} + 2a\eta |v^{(n)}_m(t)|^{2a},
\]

so that integrating on \([0, t]\) and applying Gronwall’s lemma, we obtain

\[
\sup_{t \in [0, T]} \mathbb{E}|v^{(n)}_m(t)|^{2a} \leq e^{-(\omega - \eta)T} \mathbb{E}|v^{(n)}_m(0)|^{2a} \leq C_a e^{-(\omega - \eta)T} \left[ x_H^{2a} + \mathbb{E} \int_{-\infty}^{0} |e^{-sA} \Pi_n F_m (\Pi_n r^{(n)}_m(s))|^{2a} ds + \right.
\]
\[
\left. + \int_{-\infty}^{0} e^{-sA} \Pi_n B dL(s) \right]^{2a} \leq C_a e^{-(\omega - \eta)T} \left[ K_a + \sup_{t \geq 0} \mathbb{E} (|L_A(t)|_H + |F(L_A(t))|_H) \right],
\]

where \( C_a, K_a \) are positive constants depending only on \( a \). Now from this inequality and the assumptions in Theorem 3.5, we deduce that

\[
\sup_{t \in [0, T]} \mathbb{E}|v^{(n)}_m(t)|^{2a} \leq C_a e^{-\omega T} (K_a + C), \tag{29}
\]

with \( \omega - \eta > 0 \), the result now follow by letting \( T \to +\infty \).

The next result states that \( r^{(n)}_m \) and \( v^{(n)}_m \) converge respectively to stochastic processes \( r \) and \( v \) in \( L^p(\Omega, C([0, T]; \mathcal{B})) \), \( p \geq 1, T > 0 \), where \( \mathcal{B} \) is as in Sect.3, moreover it also shows additional properties of \( r, v \).

**Proposition 4.4.** There exist a stationary process \( r \) and a process \( v \) in \( L^p(\Omega; C([0, T]; \mathcal{B})) \) such that

\[
\lim_{n, m \to \infty} r^{(n)}_m(t) = r(t)
\]
\[
\lim_{n, m \to \infty} v^{(n)}_m(t) = v(t).
\]

Further, for any \( p \geq 1 \), \( \lim_{t \to +\infty} \mathbb{E}|v(t)|^p = 0 \).

**Proof.** Again without loss of generality, we assume that \( p = 2a, a \in \mathbb{N} \). For the convergence of the sequence \( r^{(n)}_m \) and \( v^{(n)}_m \) the proof is by contradiction. Assume that there exists \( \varepsilon > 0 \) such that, for all \( m, n \in \mathbb{N} \):

\[
\sup_{k, k' > n, j, j' > m} \mathbb{E}|r^{(k)}_j(t) - r^{(k')}_{j'}(t)|^{2a}_H > 2\varepsilon.
\]
Since the difference of two stationary processes is stationary, the expression on the left hand side is independent of time \(t\). By choosing \(t\) large enough, thus making \(E|v_j^{(k)}(t)|^{2a}\) and \(E|v_{j'}^{(k')}(t)|^{2a}\) sufficiently small, it is easy to show that

\[
\sup_{t \geq 0} \sup_{k, k' > n, j, j' > m} E|X_j^{(k)}(t) - X_{j'}^{(k')}(t)|^{2a} > \varepsilon.
\]

But this contradicts the fact that \(\lim_{n,m \to \infty} X_m^{(n)}(t)\) exists. As a consequence of the convergence of the sequence \(r_m^{(n)}\) and \(X_m^{(n)}\) we obtain the convergence of \(v_m^{(n)}\).

Now let us show that:

\[
\lim_{t \to +\infty} E|v(t)|^{2a} = 0,
\]

where \(v(t) := X(t) - r(t), t \geq 0\). For this we have that:

\[
E|v(t)|^{2a} = E|X(t) - r(t)|^{2a} \leq c_a E|X(t) - X_m^{(n)}(t)|^{2a} + c_a E|X_m^{(n)}(t) - r_m^{(n)}(t)|^{2a} + c_a E|r_m^{(n)}(t) - r(t)|^{2a},
\]

for some strictly positive constant \(c_a\) depending only on \(a\). If \(n, m\) are large enough, then the first and third terms are less than \(\varepsilon/c_a\), uniformly in \(t \geq 1\). The second term is less than \(\varepsilon/c_a\), for \(t > T(\varepsilon)\), for some \(T(\varepsilon)\) independent of \(n, m\) for all \(n, m\) large. Combining the previous three terms we have shown that \(E|v(t)|^{2a} < \varepsilon\) for sufficient large positive \(t\).

From Theorem 3.5 we had already the existence of the invariant measure for the process \(X\). We shall now prove that is given by the law of the stationary process \(r\):

**Theorem 4.5.** The invariant measure for the process \((X(t))_{t \geq 0}\), is given by the law \(\mathcal{L}(r(t))\) of the stationary process \(r\).

**Proof.** It suffices to prove that the law \(\mathcal{L}(r(t))\) of \(r(t)\) is an invariant measure for \(X\), which is implied by the stationarity of \(r(t)\). Moreover, exploiting the uniqueness of invariant measures for \(X\), see Theorem 3.5, we have that \(\mathcal{L}(r(t))\) is the unique invariant measure for \(X\).

**Acknowledgments**

This work was supported by King Fahd University of Petroleum and Minerals under the project \#IN121060. The authors gratefully acknowledges this support.

We thank Stefano Bonaccorsi and Luciano Tubaro at the University of Trento for many stimulating discussions.

The authors would also like to gratefully acknowledge the great hospitality of various institutions. In particular for the first author CIRM and the Mathematics Department of the University of Trento; for him and the fourth author King Fahd University of Petroleum and Minerals at Dhahran; for the second, third and fourth authors IAM and HCM at the University of Bonn, Germany.
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_S. Albeverio_
Dept. Appl. Mathematics, University of Bonn,
HCM; KFUPM, BiBoS, IZKS

_L. Di Persio_
University of Verona, Department of Computer Science,
Italia

_E. Mastrogiacomo_
