A general framework for applying FGLM techniques to linear codes

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Abstract

We show herein that a pattern based on FGLM techniques can be used for computing Gröbner bases, or related structures, associated to linear codes. This Gröbner bases setting turns out to be strongly related to the combinatorics of the codes.

Introduction

It is well known that the complexity of Gröbner bases computation heavily depends on the term orderings, moreover, elimination orderings often yield a greater complexity. This remark led to the so called FGLM conversion problem, i.e., given a Gröbner basis with respect to a certain term ordering, find a Gröbner basis of the same ideal with respect to another term ordering. One of the efficient approaches for solving this problem, in the zero-dimensional case, is the FGLM algorithm (see [11]).

The key ideas of this algorithm were successfully generalized in [12] with the objective of computing Gröbner bases of zero-dimensional ideals that are determined by functionals. In fact, the pioneer work of FGLM and [12] was the Buchberger-Möller’s paper (cf. [9]). Authors of [1] used the approach of [9] and some ideas of [11] for an efficient algorithm to zero-dimensional schemes in both affine and projective spaces. In [4] similar ideas of using a generalized FGLM algorithm as a pattern algorithm were presented in order to compute Gröbner basis of ideals of free finitely generated algebras. In particular, it is introduced the pattern algorithm for monoid and group algebras. In [3] a more general pattern algorithm which works on modules is introduced, many things behind

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of this idea of using linear algebra are formalized, notions like “Gröbner technology” and “Gröbner representations” are used. There are other approaches which also generalized similar ideas to some settings, behind of all these works is the essential fact of using linear algebra techniques to compute in “Gröbner bases schemes”.

The main goal of this paper is to show the application of techniques to linear codes like the ones in FGLM and subsequent works, which comes from an specification of the pattern algorithm for monoid algebras given in [12], i.e. by taking an algebra associated to a linear code.

1 Preliminaries

The case of the algebra associated to a linear code we are going to introduce is connected with an ideal of a free commutative algebra; therefore, we will restrict ourselves to the formulation of a pattern algorithm for a free commutative algebra. Similar settings can be performed in a free associated algebra or over modules (see [3 43]).

Let $X := \{x_1, \ldots, x_n\}$ be a finite set of variables, $[X]$ the free commutative monoid on $X$, $K$ a field, $I$ an ideal of $K[X]$, $I(F)$ the ideal of $K[X]$ generated by $F \subset K[X]$, $K[X]/I$ the residue class algebra of $K[X]$ module $I$. Let us denote by 1 the empty word in $[X]$, $L(u)$ the length of the word $u \in [X]$, and $Card(C)$ the cardinal of the set $C$. Let now $\prec$ be a semigroup total well ordering on $[X]$ (such an ordering is also called admissible), then for $f \in K[X] \setminus \{0\}$, $T_\prec(f)$ is the maximal term of $f$ with respect to $\prec$, $LC_\prec(f)$ is the leading coefficient of $f$ with respect to $\prec$. Similarly, for $F \subset K[X]$, $T_\prec(F)$ is the set of maximal terms of non-zero polynomials in $F$, $T_\prec(F)$ is the semigroup ideal generated by $T\{F\}$. Moreover, for the sake of simplicity in notation, $U_\prec(F)$ will be used instead of $U(T_\prec(F))$, where $U$ lies in $\{G, N, B, I\}$. Of course, given an ideal $I$ and two different admissible orderings $\prec_1$ and $\prec_2$, in general we have $U(T_\prec_1(I)) \neq U(T_\prec_2(I))$. Notwithstanding this strong dependency on $\prec$, while a single admissible ordering $\prec$ is considered, so that no confusion arise, we will often simply write $U(F)$ for $U_\prec(F)$.

Let $\tau \subset [X]$ be a semigroup ideal of $[X]$, i.e., for $u \in [X]$ and $t \in \tau$, $tu \in \tau$. Then, it is well known that $\tau$ has a unique subset $G(\tau)$ of irredundant generators (probably infinite). In the case of $I$ a zero-dimensional ideal, for $\tau = T(I)$, $G(\tau)$ is always finite. We are going to introduce for $\tau$ some notation and terminology, which are similar to those introduced in [12].

\[
\begin{align*}
Pred(w) := \{u \in [X] \mid \exists x \in X (w = ux)\} & \quad \text{(the set of predecessors of } w), \\
N(\tau) := \{s \in [X] \mid s \notin \tau\} & \quad \text{(outside of } \tau), \\
B(\tau) := \{w \in \tau \mid Pred(w) \subset N(\tau)\} & \quad \text{(border of } \tau), \\
I(\tau) := \tau \setminus B(\tau) & \quad \text{(interior of } \tau).
\end{align*}
\]

We remark that $w \in \tau$ lies in $G(\tau)$ if and only if all its proper divisors are in $N(\tau)$ (that is if $Pred(w) \subset N(\tau)$). In the following proposition, some basic results concerning $\tau$ and its regions are summarized. Although they are very easy to prove, their importance is crucial for FGLM techniques.
Proposition 1 (Properties of the semigroup ideal regions).

i. For each \( w \in \tau \) there exist \( u \in X \) and \( v \in B(\tau) \) s.t. \( w = vu \).

ii. For \( x \in X \):
   (a) If \( u \in N(\tau) \), then \( ux \in N(\tau) \cup B(\tau) \).
   (b) If \( u \in B(\tau) \), then \( ux \in B(\tau) \cup I(\tau) \).
   (c) If \( u \in I(\tau) \), then \( ux \in I(\tau) \).

iii. \( N(\tau), N(\tau) \cup G(\tau), N(\tau) \cup B(\tau) \) are order ideals, i.e., if \( u \) belongs to one of these subsets and \( v \) divides \( u \), then \( v \) also belongs to the corresponding sets.

Theorem 1 (The vector space of canonical forms modulo an ideal). Let \( \text{Span}_K(N_\prec(I)) \) be the \( K \)-vector space whose basis is \( N_\prec(I) \). Then the following holds:

i. \( K[X] = I \oplus \text{Span}_K(N_\prec(I)) \) (this sum is considered as a direct sum of vector spaces).

ii. For each \( f \in K[X] \) there is a unique polynomial of \( \text{Span}_K(N_\prec(I)) \), denoted by \( \text{Can}(f,I,\prec) \) such that \( f - \text{Can}(f,I,\prec) \in I \); moreover:
   (a) \( \text{Can}(f,I,\prec) = \text{Can}(g,I,\prec) \) if and only if \( f - g \in I \).
   (b) \( \text{Can}(f,I,\prec) = 0 \) if and only if \( f \in I \).

iii. There is a \( K \)-vector space isomorphism between \( K[X]/I \) and \( \text{Span}_K(N_\prec(I)) \) (the isomorphism associates the class of \( f \) modulo \( I \) with the canonical form \( \text{Can}(f,I,\prec) \) of \( f \) modulo \( I \)).

\( \text{Can}(f,I,\prec) \) is called the canonical form of \( f \) modulo \( I \). We use simply \( \text{Can}(f,I) \) if the ordering used is clear from the context.

We assume the readers to be familiar with definition and properties of Gröbner bases (see [2] for an easy to read introduction to Gröbner bases).

Proposition 2 (Characterization of zero-dimensional ideals). Let \( G \) be a Gröbner basis of \( I \) with respect to \( \prec \). Then, \( I \) is a zero-dimensional ideal (i.e. \( \dim_K K[X]/I < \infty \)) if and only if \( N_\prec(G) \) is finite. Moreover, in such a case, \( \dim_K K[X]/I = \text{Card}(N_\prec(G)) \).

Definition 1 (Border basis). The border basis of \( I \) with respect to \( \prec \) is the subset \( B(I,\prec) \subseteq I \) defined by:
\[
B(I,\prec) := \{ w - \text{Can}(w,I,\prec) \mid w \in B_\prec(I) \} \quad (\text{the } B\text{-basis of } I).
\]

Note that the \( B \)-basis of \( I \) is a Gröbner basis of \( I \) that contains the reduced Gröbner basis.
1.1 Matphi matrices and Gröbner representation

The word Matphi appears for the first time in [11] to denote a procedure that computes a set of matrices (called matphi matrices) s.t. there is one matrix for each variable in \( X \) and they describe the multiplication structure of the quotient algebra \( K[X]/I \), where \( I \) is a zero dimensional ideal. We often refer to this set of matrices as the matphi structure.

**Definition 2 (Gröbner representation, Matphi structure).** Let \( I \) be a zero-dimensional ideal of \( K[X] \), let \( s = \text{dim}(K[X]/I) \). A Gröbner representation of \( I \) is a pair \((N, \phi)\) consisting of

i. \( N = \{N_1, \ldots, N_s\} \) s.t. \( K[X]/I = \text{Span}_K(N) \), and

ii. \( \phi := \{\phi(k) \mid 1 \leq k \leq n\} \), where \( \phi(k) \) are the square matrices \( \phi(k) := (a_{ij}^{k})_{ij} \) s.t. for all \( 1 \leq i \leq s \), \( N_i x_k \equiv \sum_j a_{ij}^{k} N_j \).

\( \phi \) is called the matphi structure and the \( \phi(k) \)'s the matphi matrices.

See [3] for a more general treatment of these concepts. Note that the matphi structure is independent of the particular set \( N \) of representative elements of the quotient \( K[X]/I \). In addition, the matphi matrices allow to obtain the class of any product of the form \( N_i x_k \) as a combination of the representative elements (i.e. as a linear combination of the basis \( N \) for the vector space \( K[X]/I \)).

2 The FGLM pattern algorithm

In this section we present a generalization of the FGLM algorithm for free commutative algebras, which allows to solve many different problems and not only the classic FGLM conversion problem. The procedure we are presenting is based on a sort of black box pattern: in fact, the description of the steps 5 and 6 is only made in terms of their input and output. More precisely, we are assuming that a term ordering \( \preceq_1 \) is fixed on \( [X] \), \( I \) is a zero-dimensional ideal (without this restriction the algorithm does not terminate), and that the \( K \)-vector space \( \text{Span}_K(N_{\preceq_1}(I)) \) is represented by giving

- a \( K \)-vector space \( E \) which is endowed of an effective function

\[
\text{LinearDependency}[v, \{v_1, \ldots, v_r\}]
\]

which, for each finite set \( \{v_1, \ldots, v_r\} \subset E \) of linearly independent vectors and for each vector \( v \in E \), returns the value defined by

\[
\begin{cases} 
\{\lambda_1, \ldots, \lambda_r\} \subset K & \text{if } v = \sum_{i=1}^{r} \lambda_i v_i, \\
\text{False} & \text{if } v \text{ is not a linear combination of } \{v_1, \ldots, v_r\}.
\end{cases}
\]

- an injective morphism \( \xi : \text{Span}_K(N_{\preceq_1}(I)) \rightarrow E \).
This informal approach allows a free choice of a suitable representation of the space $\text{Span}_K(N_{\prec_1}(I))$ regarding an efficient implementation of these techniques and a better complexity. Moreover, as an aside effect, it enables us to present this generalization in such a way that it can be applied on several more particular patterns and helps to make key ideas behind the FGLM algorithm easier to understand. Let us start making some references to some subroutines of the algorithm.

**InsertNexts**\([w, \text{List}, \prec]\) inserts properly the products \(wx\) (for \(x \in X\)) in \(\text{List}\) and sorts it by increasing ordering with respect to the ordering \(\prec\). The reader should remark that **InsertNexts** could count the number of times that an element \(w\) is inserted in \(\text{List}\), so \(w \in N_{\prec_2}(I) \cup B_{\prec_2}(I)\) if and only if this number coincide with the number of variables in the support of \(w\), otherwise, it means that \(w \in T_{\prec_2}(I) \setminus T_{\prec_2}(G)\), see [11], this criteria can be used to know the boolean value of the test condition in Step 4 of the Algorithm 1.

**NextTerm**\([\text{List}]\) removes the first element from \(\text{List}\) and returns it.

**Algorithm 1** (FGLM pattern algorithm).

1. **Input:** \(\prec_2\), a term ordering on \([X]\); \(\xi : \text{Span}_K(N_{\prec_1}(I)) \rightarrow E\).
2. **Output:** \(\text{rGb}(I, \prec_2)\), the reduced Gröbner basis of \(I\) w.r.t. the ordering \(\prec_2\).
3. 1. \(G := \emptyset\); \(\text{List} := \{1\}\); \(N := \emptyset\); \(r := 0\);
4. 2. While \(\text{List} \neq \emptyset\) do
5. 3. \(w := \text{NextTerm}[\text{List}]\);
6. 4. If \(w \notin T_{\prec_2}(G)\) (if \(w\) is not a multiple of any element in \(G\)) then
7. 5. \(v := \xi(\text{Can}(w, I, \prec_1))\);
8. 6. \(\Lambda := \text{LinearDependency}[v, \{v_1, \ldots, v_r\}]\);
9. 7. If \(\Lambda \neq \text{False}\) then \(G := G \cup \{w - \sum_{i=1}^r \lambda_i w_i\}\) (where \(\Lambda = (\lambda_1, \ldots, \lambda_r)\))
10. 8. else \(r := r + 1\);
11. 9. \(w_r := w\); \(N := N \cup \{w_r\}\);
12. 10. \(\text{List} := \text{InsertNexts}[w_r, \text{List}, \prec_2]\);
13. 11. Return\([G]\).

**Remark 1.**

1. A key idea in algorithms like FGLM is to use the relationship between membership to an ideal \(I\) and linear dependency modulo \(I\), namely \(\forall c_i \in K, s_i \in K[X]:\)
   \[
   \sum_{i=1}^r c_i s_i \in I \setminus \{0\} \iff \{s_1, \ldots, s_r\} \text{ is linearly dependent modulo } I.
   \]
   This connection with linear algebra was used for the first time in Gröbner bases theory since the very beginning (see [5]).
2. Since each element of \(N_{\prec_2}(I) \cup B_{\prec_2}(I)\) belongs to \(\text{List}\) at some moments of the algorithm and \(\text{List} \subset N_{\prec_2}(I) \cup B_{\prec_2}(I)\) at each iteration of the algorithm, it is clear that one can compute \(\mathcal{B}(I, \prec_2)\) or the Gröbner representation \((N_{\prec_2}(I), \phi)\) of \(I\) just by eliminating Step 4 of the algorithm and doing from Step 5 to Step 11 with very little changes in order to built those structures instead of \(\text{rGb}(I, \prec_2)\).
iii. Note that Step 5 and 6 depends on the particular setting. In Step 5 it is necessary to have a way of computing $\text{Can}(w, I, \prec_1)$ and the corresponding element in $E$, while in Step 6 we need an effective method to decide linear dependency.

iv. Complexity analysis of this pattern algorithm can be found in [4] for the more general case of free associative algebras, and for a more general setting in [3]. Of course, having a pattern algorithm as a model, it is expected that for particular applications, one could do modification and specification of the steps in order to improve the speed and decrease the complexity of the algorithm by taking advantage of the particular structures involved.

2.1 The change of orderings: a particular case

Suppose we have an initial ordering $\prec_1$ and the reduced Gröbner basis of $I$ for this ordering, now we want to compute by the FGLM algorithm the new reduced Gröbner basis for a new ordering $\prec_2$. Then the vector space $E$ is $K^s$, where $s = \dim(K[X]/I)$. In Step 5, $\text{Can}(w, I, \prec_1)$ can be computed using the reduced Gröbner basis $\text{rGb}(I, \prec_1)$ and the coefficients of this canonical form build the vector of $E$ corresponding to this element (the image by the morphism $\xi$). Then Step 6 is performed using pure linear algebra.

3 FGLM algorithm for monoid rings

The pattern algorithm is presented in [4] for the free monoid algebra, we will restrict here to the commutative case. Let $M$ be a finite commutative monoid generated by $g_1, \ldots, g_n$; $\xi : [X] \rightarrow M$, the canonical morphism that sends $x_i$ to $g_i$; $\sigma \subset [X] \times [X]$, a presentation of $M$ defined by $\xi(\sigma) = \{(w, v) \mid \xi(w) = \xi(v)\}$. Then, it is known that the monoid ring $K[M]$ is isomorphic to $K[X]/I(\sigma)$, where $I(\sigma)$ is the ideal generated by $P(\sigma) = \{w - v \mid (w, v) \in \sigma\}$; moreover, any Gröbner basis $G$ of $I(\sigma)$ is also formed by binomials of the above form. In addition, it can be proved that $\{(w, v) \mid w - v \in G\}$ is another presentation of $M$.

Note that $M$ is finite if and only if $I = I(\sigma)$ is zero-dimensional. We will show that in order to compute $\text{rGb}(I)$, the border basis or the Gröbner representation of $I$, one only needs to have $M$ given by a concrete representation that allows the user to multiply words on its generators; for instance: $M$ may be given by permutations, matrices over a finite field, or by a more abstract way (a complete or convergent presentation). Accordingly, we are going to do the necessary modifications on Algorithm 1 for this case.

We should remark that in this case $\prec_1 = \prec_2$, then at the begining of the algorithm the set $N_{\prec_1}(I)$ is unknown (which is not the case of the change of orderings). It could be precisely a goal of the algorithm to compute a set of representative elements for the quotient algebra.
Now consider the natural extension of $\xi$ to an algebra morphism $(\xi : K[X] \mapsto K[M])$, note that the restriction of $\xi$ to $\operatorname{Span}_K(N_{<1}(I))$ ( $\xi : \operatorname{Span}_K(N_{<1}(I)) \mapsto K[M]$) is an injective morphism; moreover, $\xi(w) = \xi(Can(w, I, \prec_1))$, for all $w \in [X]$. Therefore, the image of $\operatorname{Can}(w, I, \prec_1)$ can be computed as $\xi(w)$, and the linear dependency checking will find out whether $w$ is a new canonical form (i.e. $w \in N_{<1}(I)$) or not (i.e. $w \in T_{<1}(r\operatorname{Gb}(I, \prec_1))$). Hence, Step 5 will be

$$v := \xi(u)g_i, \text{ where } u \in \operatorname{Pred}(w) \text{ and } ux_1 = w.$$  

Moreover, let $w_1, \ldots, w_r$ be elements of $N_{<1}(I)$ and $v_i = \xi(w_i)$, for $1 \leq i \leq r$. Then \textbf{LinearDependency}\{v, \{v_1, \ldots, v_r\}\} can be computed as

$$\begin{cases} v_j & \text{if } v = v_j, \text{ for some } j \in [1, r], \\ \text{False} & \text{otherwise}. \end{cases}$$

Finally, Step 7 changes into:

If $\Lambda \neq \text{False}$ then $G := G \cup \{w - w_j\}$.

\textbf{Remark 2.}  
\begin{enumerate}
\item This example shows that the capability of the $K$-vector space $E$ w.r.t. \textbf{LinearDependency}, that is demanded in the Algorithm 1, is required only on those sets of vectors $\{v_1, \ldots, v_r, v\}$ that are built in the algorithm, which means in this case that \textbf{LinearDependency} is reduced to the Member checking, i.e., $v$ is linear dependent of $\{v_1, \ldots, v_r\}$ if and only if it belongs to this set.
\item When a word $w$ is analyzed by the algorithm, all the elements in $\operatorname{Pred}(w)$ have been already analyzed ($\xi(u)$ is known for any $u \in \operatorname{Pred}(w)$), this is the case whenever $\prec_1$ is an admissible ordering. Therefore, the computation of $\xi(w)$ is immediate.
\end{enumerate}

We will show the case of linear codes as a concrete setting for an application of the FGLM pattern algorithm for monoid rings, where the monoid is given by a set of generators and a way of multiply them.

\section{FGLM algorithm for linear codes}

For the sake of simplicity we will stay in the case of binary linear codes, where more powerfull structures for applications are obtained as an output of the corresponding FGLM algorithm (for a general setting see \cite{4, 5}). From now on we will refer to linear codes simply as codes.

Let $\mathbb{F}_2$ be the finite field with 2 elements. Let $C$ be a binary code of dimension $k$ and length $n$ ($k \leq n$), so that the $n \times (n-k)$ matrix $H$ is a parity check matrix ($c \cdot H = 0$ if and only if $c \in C$). Let $d$ be the minimum distance of the code, and $t$ the error-correcting capability of the code ($t = \lceil \frac{d-1}{2} \rceil$, where $[x]$ denotes the greater integer less than $x$). Let $B(C, t) = \{y \in \mathbb{F}_2^n \mid \exists c \in C \ (d(c, y) \leq t)\}$, it is well known that the equation $eH = yH$ has a unique solution $e$ with weight($e$) $\leq t$, for $y \in B(C, t)$.

Let us consider the free commutative monoid $[X]$ generated by the $n$ variables $X := \{x_1, \ldots, x_n\}$. We have the following map from $X$ to $\mathbb{F}_2^n$: $\psi : \eta \mapsto$
$X \to \mathbb{F}_2^n$, where $x_i \mapsto e_i$ (the $i$-th coordinate vector). The map $\psi$ can be extended in a natural way to a morphism from $[X]$ onto $\mathbb{F}_2^n$, where $\psi(\prod_{i=1}^n x_i^{\beta_i}) = (\beta_1 \mod 2, \ldots, \beta_n \mod 2)$.

A binary code $C$ defines an equivalence relation $R_C$ in $\mathbb{F}_2^n$ given by $(x, y) \in R_C$ if and only if $x - y \in C$. If we define $\xi(u) := \psi(u)H$, where $u \in [X]$, the above congruence can be translated to $[X]$ by the morphism $\psi$ as $u \equiv w$ if and only if $(\psi(u), \psi(w)) \in R_C$, that is, if $\xi(u) = \xi(w)$. The morphism $\xi$ represents the transition of the syndromes from $\mathbb{F}_2^n$ to $[X]$; therefore, $\xi(w)$ is the “syndrome” of $w$, which is equal to the syndrome of $\psi(w)$.

**Definition 3 (The ideal associated with a binary code).** Let $C$ be a binary code. The ideal $I(C)$ associated with $C$ is

$$I(C) := \langle \{w - u \mid \xi(w) = \xi(u)\} \rangle \subset K[X].$$

5 The algorithm for binary codes

The monoid $M$ is set to be $\mathbb{F}_2^{n-k}$ (where the syndromes belong to). Doing $g_i := \xi(x_i)$, note that $M = \mathbb{F}_2^{n-k} = \langle g_1, \ldots, g_n \rangle$. Moreover, $\sigma := R_C$, hence $I(\sigma) = I(C)$. Let $\prec$ be an admissible ordering. Then the FGLM algorithm for linear codes can be used to compute the reduced Gröbner basis, the border basis, or the Gröbner representation for $\prec$.

**Algorithm 2 (FGLM for binary codes).**

**Input:** $n, H$ the parameters for a given binary code, $\prec$ an admissible ordering.

**Output:** $\text{rGb}(I(C), \prec)$.

1. **List** := \{1\}, $N := \emptyset$, $r := 0$, $G := \{\}$;
2. **While** List $\neq \emptyset$ do
3. \quad $w := \text{NextTerm}[\text{List}]$;
4. \quad If $w \notin T(G)$;
5. \quad \quad $v := \xi(w)$;
6. \quad \quad $\Lambda := \text{Member}[v, \{v_1, \ldots, v_r\}]$;
7. \quad \quad If $\Lambda \neq \text{False}$ then $G := G \cup \{w - w_3\}$;
8. \quad \quad else $r := r + 1$;
9. \quad \quad \quad $v_r := v'$;
10. \quad $w_r := w$, $N := N \cup \{w_r\}$;
11. \quad $\text{List} := \text{InsertNext}[w_r, \text{List}]$;
12. **Return**$|G|$. 

8
In many cases of FGLM applications a good choice of the ordering $\prec$ is a crucial point in order to solve a particular problem. In the following theorem it is shown the importance of using a total degree compatible ordering (for example the Degree Reverse Lexicographic). Let us denote by $<_T$ a total degree compatible ordering.

**Theorem 2 (Canonical forms of the vectors in $B(C,t)$).** Let $C$ be a code and let $G_T$ be the reduced Gröbner basis with respect to $<_T$. If $w \in [X]$ satisfies 
\[
\text{weight}(\psi(\text{Can}(w,G_T))) \leq t \text{ then } \psi(\text{Can}(w,G_T)) \text{ is the error vector corresponding to } \psi(w).
\]
On the other hand, if \(\text{weight}(\psi(\text{Can}(w,G_T))) > t\) then $\psi(w)$ contains more than $t$ errors.

**Proof.** If we assume that weight(\(\psi(\text{Can}(w,G_T))\)) \(\leq t\) then, we can infer at once that $\psi(w) \in B(C,t)$ and $\psi(\text{Can}(w,G_T))$ is its error vector (notice that $\xi(w) = \xi(\text{Can}(w,G_T))$ and the unicity of the error vector).

Now, if weight(\(\psi(\text{Can}(w,G_T))\)) \(> t\), we have to prove that $\psi(w) \notin B(C,t)$. It is equivalent to show that weight(\(\psi(\text{Can}(w,G_T))\)) \(\leq t\) if $\psi(w) \in B(C,t)$. Let $\psi(w)$ be an element of $B(C,t)$ and let $e$ be its error vector then, weight($e$) \(\leq t\). Let $w_e$ be the squarefree representation of $e$. Note that weight($e$) coincides with the total degree of $w_e$; accordingly, $L(w_e) \leq t$. On the other hand, $\text{Can}(w,G_T) <_T w_e$, which implies that $L(\text{Can}(w,G_T)) \leq L(w_e)$ (because $<_T$ is degree compatible). Hence, weight(\(\psi(\text{Can}(w,G_T))\)) \(\leq L(\text{Can}(w,G_T)) \leq t\).

\(\square\)

The computation of the error-correcting capability of the code $t$ can be done in the computing process of Algorithm 2 (see the example in Section 5.1 and 7). The previous theorem allows us to use the computed reduced Gröbner basis for solving the decoding problem in general binary codes, but also with such a powerful tool available, it is expected to be able to study the structure of the codes, like some combinatorics properties. Some possible examples are the permutation-equivalence of codes (see 5), and some problems related with binary codes associated with the set of cycles in a graph (finding the set of minimal cycles and a minimal cycle basis of the cycles of a graph), see 6.

To generalize Theorem 2 for non binary linear codes have some conflicts with the needed ordering; however, the FGLM algorithm can be still used to compute the border basis or a Gröbner representation for the ideal $I(C)$ and it will be possible to solve the problems that one can solve with the reduced Gröbner basis in the case of binary codes. Those problems are explained in 7. In addition, 5 contains some results and examples about the application of this setting to general linear codes and, in binary codes, for studying the problems of decoding and the permutation-equivalence.

### 5.1 An example

Let $C$ be the linear code over $\mathbb{F}_2$ determined by the parity check matrix $H$ given below. The set $C$ of codewords is given in the right hand side. The minimum distance is $d = 3$, so, $t = 1$, the numbers of variables is 6, $<_T$ is set to be the
Degree Reverse Lexicographic ordering with $x_{i+1} >_T x_i$. Only essential parts of the computation will be described.

\[
H = \begin{bmatrix}
1 & 1 & 1 \\
1 & 0 & 1 \\
0 & 1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

\[C = \{(0, 0, 0, 0, 0, 0), (1, 0, 1, 1, 0, 0), (1, 1, 1, 0, 0, 1), (1, 0, 1, 0, 1, 0), (0, 0, 1, 0, 1, 1), (0, 1, 0, 1, 1, 0), (0, 0, 1, 0, 1, 0), (1, 0, 0, 1, 1, 1)\}.
\]

Application of Algorithm 2: List := \{\}; \(N := \{\}; r := 0; w := 1; \xi(1) = (0, 0, 0); N := N \cup \{1\} = \{1\}; \xi(N) := \{(0, 0, 0)\}; List := \{x_1, x_2, x_3, x_4, x_5, x_6\}; w := x_1; \xi(x_1) = (1, 1, 1); N := \{x_1\}; \xi(N) := \{(0, 0, 0), (1, 1, 1)\};

After analyzing \(x_6\) we are at the following stage:

\[
N := \{1, x_1, x_2, x_3, x_4, x_5, x_6\}, \text{ and } \text{List} = \{x_1^2, x_1 x_2, x_1 x_3, x_1 x_4, x_1 x_5, x_1 x_6, x_2^2, x_2 x_3, x_2 x_4, x_2 x_5, x_2 x_6, x_3^2, x_3 x_4, x_3 x_5, x_3 x_6, x_4^2, x_4 x_5, x_4 x_6, x_5^2, x_5 x_6, x_6^2\}.
\]

There is still one element left in \(N\) because there are 7 elements in \(N\) of a total of 8 \((2^6-3)\). Taking the elements of List from \(x_1^2\) to \(x_1 x_5\) they are a linear combination of elements already in \(N\) (their syndromes are in the list of syndromes computed \(\xi(N)\)). Therefore, \(G := \{x_1^2 - 1, x_1 x_2 - x_5, x_1 x_3 - x_4, x_1 x_4 - x_3, x_1 x_5 - x_2\}\), for example \(x_1 x_2 - x_5\) is obtained, because when \(w = x_1 x_2\), first note that \(\text{Pred}(w) \subset N\), which means that it is either a new irreducible element or a head of a binomial of the reduced basis. Then \(\xi(x_1 x_2)\) is computed and we got that \(\xi(x_1 x_2) = \xi(x_5)\). This means that \(x_1 x_2 - x_5\) belongs to \(G\). Also \(x_1 x_2\) is the first minimal representation which is not in \(N\), this implies that \(t = \text{weight} (\psi(x_1 x_2)) - 1\) (see [7]). The next element in List, \(w = x_1 x_6\), is the last element that will be included in \(N\) and the corresponding multiples will be included in List.

From this point, the algorithm will just take elements from List and it analyzes in each case whether it is in \(T(r\text{Gb}(I(C),<_T))\) (like \(x_2 x_3\)) or in \(T(r\text{Gb}(I(C),<_T) \setminus T(r\text{Gb}(I(C),<_T))\) (like \(x_1 x_2 x_6\)), this process is executed until the List is empty when the last element \(x_1 x_6^2\) of the list is analyzed. Finally, the reduced Gröbner basis for \(_T\) is

\[G := \{x_1^2 - 1, x_1 x_2 - x_5, x_1 x_3 - x_4, x_1 x_4 - x_3, x_1 x_5 - x_2, x_2^2 - 1, x_2 x_3 - x_1 x_6, x_2 x_4 - x_6, x_2 x_5 - x_1, x_2 x_6 - x_4, x_3^2 - 1, x_3 x_4 - x_1, x_3 x_5 - x_6, x_3 x_6 - x_5, x_4^2 - 1, x_4 x_5 - x_1 x_6, x_6 x_4 - x_2, x_5^2 - 1, x_5 x_6 - x_3, x_6^2 - 1\}.
\]

Now let us assume that a vector \(y = (1, 1, 1, 0, 1, 0)\) is received, the corresponding word is \(w = x_1 x_2 x_3 x_5\). Then we compute \(w_e = \text{Can}(w, G) = x_3\). As \(\text{weight}(\psi(w_e)) = 1\) \((t = 1\) is the error-correcting capability\); therefore, the error vector is \(e = (0, 0, 1, 0, 0, 0)\), and the codeword is \(c = (1, 1, 0, 0, 1, 0)\).

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