A short comparison between $m_{T2}$ and $m_{CT}$

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**ABSTRACT:** We compare $m_{T2}$ with $m_{CT}$; both are kinematic variables designed to find relationships between masses of pair-produced new states with symmetric decay chains. We find that for massless visible particles $m_{CT}$ equals $m_{T2}$ in a particular limit. We identify advantages and disadvantages to the use of each variable. Tovey’s paper on $m_{CT}$ also introduced a powerful concept of extracting mass information from an analysis at intermediate stages of a symmetric decay chain. We suggest that $m_{T2}$ is a better tool for performing this analysis than $m_{CT}$ due to $m_{T2}$’s better properties under initial state radiation.

**KEYWORDS:** Hadron-Hadron Scattering.
Dark matter’s likely signature in a hadron collider will be missing transverse momentum. The stability of dark matter suggests a charge or conservation law that requires dark matter particles be produced in pairs at colliders. The kinematic variables $m_{T2}$ introduced by Lester and Summers [1] and $m_{CT}$ introduced by Tovey [2] aid in the task of determining the mass of new states that decay to dark matter particles at hadron colliders. Although $m_{T2}$ has been used extensively (see [3, 4, 5, 6, 7] for a few examples), the variable $m_{CT}$ is new but shares many similarities and differences with $m_{T2}$. This note briefly defines $m_{T2}$ and $m_{CT}$, explains when they give identical results, when they differ, and comments on benefits of each in their intended applications.

Both variables assume a pair-produced new-particle state followed by each branch decaying symmetrically to visible states and dark-matter candidates which escape detection and appear as missing transverse momentum. Fig 1 is the simplest example on which we can meaningfully compare the two kinematic quantities. The figure shows two partons colliding and producing some observed initial state radiation (ISR) with four momenta $g$ and an on-shell, pair-produced new state $Y$. On each branch, $Y$ decays to on-shell states $X$ and $v_1$ with masses $m_X$ and $m_{v_1}$, and $X$ then decays to on-shell states $N$ and $v_2$ with masses $m_N$ and $m_{v_2}$. The four-momenta of $v_1$, $v_2$, and $N$ are respectively $\alpha_1$, $\alpha_2$ and $p$, $\beta_1$, $\beta_2$ and $q$ in the other branch. The missing transverse momenta $P_T$ is given by the transverse part of $p + q$.

First we describe $m_{T2}$. The variable $m_{T2}$ accepts three inputs: $\chi$ (an assumed mass of the two particles carrying away missing transverse momenta), $\alpha$ and $\beta$ (the visible momenta of each branch), and $P_T = (p + q)_T$ (the missing transverse momenta). The variable $m_{T2}$ is the minimum mass of the pair of parent particles compatible with the observed particles’ four momenta and an assumed mass for particles carrying away the missing momenta. We can define $m_{T2}$ in terms of the transverse mass of each branch where we minimize the maximum of the two transverse masses over the unknown split between $p$ and $q$ of the

![Figure 1](image.png)

**Figure 1:** A simplest topology with which we compare $m_{T2}$ and $m_{CT}$.  

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overall missing transverse momenta:

$$m_{T2}^2(\chi_N, \alpha, \beta, \not{P}_T) = \min_{p_T+q_T=\not{P}_T} \left[ \max \left\{ m_1^2(\alpha, p), m_2^2(\beta, q) \right\} \right].$$

(1)

In this expression $\chi_N$ is the assumed mass of $N$, $\alpha$ and $\beta$ are the four momenta of the visible particles in the two branches, the transverse mass is given by $m_{T2}^2(\alpha, p) = m_{\alpha}^2 + \chi_N^2 + 2(E_T(p)E_T(\alpha) - p_T \cdot \alpha_T)$ and the transverse energy $E_T(p) = \sqrt{p_T^2 + \chi_N^2}$ is determined from the transverse momentum of $p$ and the assumed mass of the particle associated with momentum $p$. An analytic formula for the case with no transverse ISR can be found in the appendix of [4]. For each event, the quantity $m_{T2}(\chi_N = m_N, \alpha_1 + \alpha_2, \beta_1 + \beta_2, \not{P}_T)$ gives the smallest mass for the parent particle compatible with the event’s kinematics. Under ideal assumptions, the mass of the parent particle $Y$ is given by the end-point of the distribution of this $m_{T2}$ parameter over a large number of events like fig 1. Because a priori we do not know $m_N$, we need some other mechanism to determine $m_N$. 1. We use $\chi$ to distinguish assumed values of the masses ($\chi_Y, \chi_X, \chi_N$) from the true values for the masses ($m_Y, m_X, m_N$). Because of this dependence on the unknown mass, we should think of $max m_{T2}$ as providing a relationship or constraint between the mass of $Y$ and the mass of $N$. This forms a surface in the $(\chi_Y, \chi_X, \chi_N)$ space on which the true mass will lie. We express this relationship as $\chi_Y(\chi_N)$.

Tovey [2] recently defined a new variable $m_{CT}$ which has many similarities to $m_{T2}$. The variable is defined as

$$m_{CT}^2(\alpha_1, \beta_1) = (E_T(\alpha_1) + E_T(\beta_1))^2 - (\alpha_{1T} - \beta_{1T})^2.$$  

(2)

Tovey’s goal is to identify another constraint between masses in the decay chain. He observes that in the rest frame of $Y$ the momentum of the back-to-back decay products $X$ and $v_1$ is given by

$$(k_*(m_Y, m_X, m_{v_1}))^2 = \frac{(m_Y^2 - (m_{v_1} + m_X)^2)(m_Y^2 - (m_{v_1} - m_X)^2)}{4m_Y^2}$$

(3)

where $k_*$ is the two-body excess momentum parameter (2BEMP) 3. In the absence of transverse ISR ($g_T = 0$) and if the visible particles are effectively massless ($m_{v_1} = 0$), Tovey observes that $max m_{CT}(\alpha_1, \beta_1)$ is given by $2k_*$; this provides an equation of constraint between $m_Y$ and $m_X$. Tovey observes that if we could do this analysis at various stages along the symmetric decay chain all the masses could be determined.

The big advantage of $m_{CT}$ is in its computational simplicity. Also, $m_{CT}$ is intended to only be calculated once per event instead of at a variety of choices of $\chi$. In contrast, $m_{T2}$

1The true $m_Y$ and $m_N$ cannot be found in the case where $Y$ undergoes a three-body ($X$ is off-shell) through kinks in $m_{T2}$ [5, 6, 8] or when combined with endpoints from other distribution (like $max m_{\alpha}$) [7].

2In principle this surface would be considered a function of $\chi_Y(\chi_X, \chi_N)$, but $m_{T2}$ makes no reference to the mass of $X$ and the resulting constraints are therefore independent of any assumed value of the mass of $X$.

3Tovey refers to this as the 2-body mass parameter $M_2$. We feel calling this a mass is a bit misleading so we are suggesting 2BEMP.
is a more computationally intensive parameter to compute; but this is aided by the use of a shared repository of community tested C++ libraries found at [9].

How are these two variables similar? Both $m_{CT}$ and $m_{T2}$, in the absences of ISR, are invariant under back-to-back boosts of the parent particles momenta [6]. The variable $m_{CT}$ equals $m_{T2}(\chi = 0)$ in the special case where $\chi = 0$ and when the visible particles are massless ($\alpha^2_1 = \beta^2_1 = 0$) and there is no transverse ISR ($g_T = 0$)

$$m_{CT}(\alpha_1, \beta_1) = m_{T2}(\chi = 0, \alpha_1, \beta_1, \mathbf{P}_T = (p + q + \alpha_2 + \beta_2)T) \quad \text{if} \quad \alpha^2_1 = \beta^2_1 = 0. \quad (4)$$

$$= 2(\alpha_1T \cdot \beta_1T + |\alpha_1T||\beta_1T|). \quad (5)$$

The $m_{CT}$ side of the equation is straight forward. The $m_{T2}$ side of the expression can be derived analytically using the formula for $m_{T2}$ given in [4]; we also outline a short proof in the appendix. Eq(4) uses a $m_{T2}$ in an unconventional way; we group the observed momenta of the second decay products into the missing transverse momenta. In this limit, both share an endpoint of $2k_* = (m^2_Y - m^2_X)/m_Y$. To the best of our knowledge, this endpoint was first pointed out by Cho et.al. [5] 4. We find it surprising that a physical relationship between the masses follows from $m_{T2}$ evaluated at a non physical $\chi$. In the presence of ISR, eq(4) is no longer an equality. Furthermore in the presence of the ISR, the end point of the distribution given by either side of eq(4) exceeds $2k_*$. In both cases, we will need to solve a combinatoric problem of matching visible particles to their decay order and branch of the event which is beyond the scope of this paper.

In the case where the visible particle $v_1$ is massive, the two parameters give different end-points

$$\max m_{CT}(\alpha_1, \beta_1) = \frac{m^2_Y - m^2_X}{m_Y} + \frac{m^2_{v_1}}{m_Y}. \quad (6)$$

$$\max m_{T2}(\chi = 0, \alpha_1, \beta_1, \mathbf{P}_T = (p + q + \alpha_2 + \beta_2)T) = \sqrt{m^2_{v_1} + 2(k_*^2 + k_*\sqrt{k_*^2 + m^2_{v_1}})} \quad (7)$$

where $k_*$ is given by eq(3). Unfortunately, there is no new information about the masses in these two end-points. If we solve eq(6) for $m_X$ and substitute this into eq(7) and (3), all dependence on $m_Y$ is eliminated.

Tovey’s idea of analyzing the different steps in a symmetric decay chain to extract the masses is powerful. Up until now, we have been analyzing both variables in terms of the first decay products of $Y$. This restriction is because $m_{CT}$ requires no transverse ISR to give a meaningful endpoint. If we were to try and use $\alpha_2$ and $\beta_2$ to find a relationship between $m_X$ and $m_N$, then we would need to consider the transverse ISR to be $(g + \alpha_1 + \beta_1)T$ which is unlikely to be zero.

We suggest $m_{T2}$ is a better variable with which to implement Tovey’s idea of analyzing the different steps in a symmetric decay chain because of its ISR properties. With and without ISR, $m_{T2}$’s endpoint gives the correct mass of the parent particle when we assume

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4The endpoint given by Cho et.al. is violated for non-zero ISR at $\chi_N < m_N$ and $\chi_N > m_N$. 

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Figure 2: Shows constraints from \( m_{T2} \) used with different combinations as described in eqs(8,9,10) and the \( m_{12} \) described in eq(12). Intersection is at the true mass (97 GeV, 144 GeV, 181 GeV) shown by sphere. Events include ISR but otherwise ideal conditions: no background, resolution, or combinatoric error.

the correct value of the missing-energy-particle’s mass \(^5\). For this reason, \( m_{T2} \) gives a meaningful relationship between masses \((m_Y, m_X, m_N)\) for all three symmetric pairings of the visible particles across the two branches. A relationship between \( m_Y \) and \( m_X \) is given by

\[
\chi_Y(\chi_X) = \max m_{T2}(\chi_X, \alpha_1, \beta_1, \not{P}_T = (p + q + \alpha_2 + \beta_2)_T).
\]  

(8)

A relationship between \( m_X \) and \( m_N \) can be found by computing

\[
\chi_X(\chi_N) = \max m_{T2}(\chi_N, \alpha_2, \beta_2, \not{P}_T = (p + q)_T)
\]

(9)

where we have grouped \( \alpha_1 + \beta_1 \) with the \( g \) as a part of the ISR. A relationship between \( m_Y \) and \( m_N \) can be found by using \( m_{T2} \) in the traditional manner giving

\[
\chi_Y(\chi_N) = \max m_{T2}(\chi_N, \alpha_1 + \alpha_2, \beta_1 + \beta_2, \not{P}_T = (p + q)_T).
\]  

(10)

Lastly, we can form a distribution from the invariant mass of the visible particles on each branch \( m_{12}^2 = (\alpha_1 + \alpha_2)^2 \) or \( m_{12}^2 = (\beta_1 + \beta_2)^2 \). The endpoint of this distribution gives a relationship between \( m_Y, m_X, \) and \( m_N \) given by

\[
\max m_{12}^2 = \frac{(m_Y^2 - m_X^2)(m_X^2 - m_N^2)}{m_X^2}.
\]

\( ^5 \)In principle we could plot the \( m_{T2}(\chi_X, \alpha_1, \beta_1, \not{P}_T = (\alpha_2 + \beta_2 + p + q)_T) \) vs \( \chi_X \) as a function of transverse ISR and the value of \( \chi_X \) at which the end point is constant would give the correct value of \( m_X \); at which point the distributions end point would give the correct \( m_Y \). In practice we probably will not have enough statistics of ISR events.
Solving this expression for $m_Y$ gives the relationship

$$\chi_Y^2(\chi_N, \chi_X) = \frac{\chi_X^2((\max m_{12}^2) + \chi_X^2 - \chi_N^2)}{\chi_X^2 - \chi_N^2}. \quad (12)$$

Fig 2 shows the constraints from eqs$(8,9,10,12)$ in an ideal simulation using $(m_Y = 181 \text{ GeV}, m_X = 144 \text{ GeV}, m_N = 97 \text{ GeV}$), 1000 events, and massless visible particles, and ISR added with an exponential distribution with a mean of 50 GeV. These four surfaces in principle intersect at a single point $(m_Y, m_X, m_N)$ given by the sphere in the figure 2. Unfortunately, all these surfaces intersect the correct masses at a shallow angles so we have a sizable uncertainty along the direction of the sum of the masses and a tight constraints in the perpendicular directions. In other words, the mass differences are well-determined but not the mass scale. From here one could use a shape fitting technique like that described in [7] to find a constraint on the sum of the masses. Tovey’s suggestion for extracting information from these intermediate stages of a symmetric cascade chain clearly provides more constraints to isolate the true mass than one would find from only using the one constraint of eq$(10)$ as described in [5]. However, Tovey’s suggestion is more feasible using the $m_{T2}$ rather than $m_{CT}$ because the constraint surfaces derived from $m_{T2}$ intersect the true masses even with ISR.

In summary, we have compared and contrasted $m_{CT}$ with $m_{T2}$. The variable $m_{CT}$ is a special case of $m_{T2}$ given by eq$(4)$ when ISR can be neglected and when the visible particles are massless. In this case, the end-point of this distribution gives $2k_*$, twice the two-body excess momentum parameter $(2\text{BEMP})$. If $m_{v1} \neq 0$, the two distributions have different endpoints but no new information about the masses. In the presence of ISR the two functions are not equal; both have endpoints that exceed $2k_*$. Because of it’s better properties in the presence of ISR, $m_{T2}$ is a better variable for the task of extracting information from each step in the decay chain. Extracting this information requires solving combinatoric problems which are beyond the scope of this paper.

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Appendix: Verifying $m_{T2}$ in eq$(5)$

We derived the $m_{T2}$ side of eq$(5)$ by following the analytic solution given by Barr and Lester in [4]. In this appendix, we outline how to verify that $m_{T2}$ is is indeed given by

$$m_{T2}(\chi = 0, \alpha, \beta, \vec{p}_T = -\alpha_T - \beta_T) = 2(\alpha_T \cdot \beta_T + |\alpha_T||\beta_T|) \quad (13)$$
when $\alpha^2 = \beta^2 = 0$ and $p^2 = q^2 = \chi^2 = 0$ and $q_T = 0$. To do this we note that $m_{T2}$ can also be defined as the minimum value of $(\alpha + p)^2$ minimized over $p$ and $q$ subject to the conditions $p^2 = q^2 = \chi^2$ (on-shell missing energy state), and $(\alpha + p)^2 = (\beta + q)^2$ (equal on-shell parent-particle state), and $(\alpha + \beta + p + q + g)_T = 0$ (conservation of transverse momentum) [7].

The solution which gives eq(13) has $p_T = -\beta_T$ and $q_T = -\alpha_T$ with the rapidity of $p(q)$ equal to the rapidity of $\alpha (\beta)$. We now verify that this solution satisfies all the constraints listed above. Transverse momentum conservation is satisfied trivially: $(\alpha + \beta + p + q)_T = (\alpha + \beta - \alpha - \beta)_T = 0$. The constraint to have the parent particles on-shell can be verified with $2|\alpha_T||p_T| - 2\beta_T \cdot \delta_T = 2|\beta_T||q_T| - 2\tilde{q}_T \cdot \tilde{\delta}_T = 2|\beta_T||\beta_T| + 2\tilde{\alpha}_T \cdot \tilde{\beta}_T$.

Now all that remains is to show that the parent particle’s mass is a minimum with respect to ways in which one splits up the missing transverse momentum between $p_T$ and $q_T$ while satisfying the above constraints. We take $p$ and $q$ to be a small deviation from the stated solution $p_T = -\beta_T + \delta_T$ and $q_T = -\alpha_T - \delta_T$ where $\delta_T$ is the small deviation in the transverse plane. We keep $p$ and $q$ on shell at $\chi = 0$. The terms $p_o$, $p_z$, $q_o$, $q_z$ are maintained at their minimum by keeping the rapidity of $p$ and $q$ equal to $\alpha$ and $\beta$. The condition that the parent particles are on-shell and equal is satisfied for a curve of values for $\delta_T$. The deviation tangent to this curve near $|\delta_T| = 0$ is given by $\delta_T(\lambda) = \lambda \hat{z} \times (\alpha_T|\beta_T| + \beta_T|\alpha_T|)$ where $\times$ is a cross product, $\hat{z}$ denotes the beam direction, and we parameterized the magnitude by the scalar $\lambda$. Finally, we can substitute $p$ and $q$ with the deviation $\delta_T(\lambda)$ back into the expression for the parent particle’s mass $(\alpha + p)^2$ and verify that $2(\alpha_T \cdot \beta_T + |\alpha_T||\beta_T|)$ at $\lambda = 0$ is indeed the minimum with respect to changes in $\lambda$.

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