COLOCALIZATION FUNCTORS IN DERIVED CATEGORIES AND TORSION THEORIES

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Abstract. Let $R$ be a ring and let $\mathcal{A}$ be a hereditary torsion class of $R$-modules. The inclusion of the localizing subcategory generated by $\mathcal{A}$ into the derived category of $R$ has a right adjoint, denoted $\text{Cell}_\mathcal{A}$. In [2], Benson shows how to compute $\text{Cell}_\mathcal{A}R$ when $R$ is a group ring of a finite group over a prime field and $\mathcal{A}$ is the hereditary torsion class generated by a simple module. We generalize Benson’s construction to the case where $\mathcal{A}$ is any hereditary torsion class on $R$. It is shown that for every $R$-module $M$ there exists an injective $R$-module $E$ such that:

$$H^n(\text{Cell}_\mathcal{A}M) \cong \text{Ext}^{n-1}_{\text{End}_R(E)}(\text{Hom}_R(M,E), E) \text{ for } n \geq 2$$

1. Introduction

Let $R$ be a ring and let $\mathcal{D}_R$ be the (unbounded) derived category of chain complexes of left $R$-modules. Fix a class $\mathcal{A}$ of objects of $\mathcal{D}_R$. We recall some definitions of Dwyer and Greenlees from [3]. An object $N$ of $\mathcal{D}_R$ is $\mathcal{A}$-null if $\text{Ext}^*_R(A, N) = 0$ for every $A \in \mathcal{A}$. An object $C$ of $\mathcal{D}_R$ is $\mathcal{A}$-cellular if $\text{Ext}_R^*(C, N) = 0$ for every $\mathcal{A}$-null $N$. An $\mathcal{A}$-cellular object $C$ is an $\mathcal{A}$-cellular approximation of $X \in \mathcal{D}_R$ if there is a map $\mu : C \to X$ such that $\text{Ext}_{\mathcal{D}_R}^*(A, \mu)$ is an isomorphism for all $A \in \mathcal{A}$. Finally, an $\mathcal{A}$-null object $N$ is an $\mathcal{A}$-nullification of $X$ if there is a map $\nu : X \to N$ which is universal among maps in $\mathcal{D}_R$ from $X$ to $\mathcal{A}$-null objects. Denote an $\mathcal{A}$-cellular approximation of $X$ by $\text{Cell}_\mathcal{A}X$ and an $\mathcal{A}$-nullification of $X$ by $\text{Null}_\mathcal{A}X$.

The following properties are easy to check. A map $\mu : C \to X$ is an $\mathcal{A}$-cellular approximation of $X$ if and only if it is universal among all maps from $\mathcal{A}$-cellular objects to $X$. There is an exact triangle $\text{Cell}_\mathcal{A}X \to X \to \text{Null}_\mathcal{A}X$ whenever $\text{Cell}_\mathcal{A}X$ or $\text{Null}_\mathcal{A}X$ exists. An $\mathcal{A}$-cellular approximation of some object $X$ is unique up to isomorphism and the same goes for an $\mathcal{A}$-nullification of $X$.

Now suppose $\mathcal{A}$ is a set, then it turns out that the full subcategory of $\mathcal{A}$-cellular objects is the localizing subcategory generated by $\mathcal{A}$ (see [4] and [7, 5.1.5]). Moreover, when $\mathcal{A}$ is a set the inclusion functor of the full subcategory of $\mathcal{A}$-cellular objects into $\mathcal{D}_R$ has a right adjoint, which is $\text{Cell}_\mathcal{A}X$ for every $X \in \mathcal{D}_R$; see [7] or [6]. Hence $\text{Cell}_\mathcal{A}$ can be constructed as a colocalization functor (the right adjoint of an inclusion functor), and it follows that $\mathcal{A}$-cellular approximation and $\mathcal{A}$-nullification exist for any object of $\mathcal{D}_R$.

Similarly, when $\mathcal{A}$ is a set there exists a left adjoint to the inclusion of the full subcategory of $\mathcal{A}$-null objects, see for example Neeman’s book [11, Section 9]. This functor is in fact $\mathcal{A}$-nullification and it is a localization functor (the left adjoint to an inclusion functor).

One method for calculating $\mathcal{A}$-cellular approximations is the formula given by Dwyer and Greenlees in [3], which holds whenever $\mathcal{A} = \{A\}$ and $A$ is a perfect complex. This was later generalized by Dwyer, Greenlees and Iyengar in [4]. A new method for calculating the $\mathcal{A}$-cellular approximation for $R$-modules has been constructed by Benson in [2], dubbed $k$-squeezed resolutions. This method can be applied whenever $\mathcal{A}$ is a set of simple
modules and $R$ is an Artinian ring. One major benefit of Benson’s construction is that it allows for explicit calculations.

As we will see, it is more natural to use Benson’s method to construct the $A$-nullification of a module, rather than its $A$-cellular approximation. We generalize Benson’s construction so that it applies whenever $A$ is a hereditary torsion class of modules. A hereditary torsion class of modules is a class of modules that is closed under submodules, quotient modules, coproducts and extensions. The main result of this paper is the following.

**Theorem 1.1.** Let $T$ be a hereditary torsion class on left $R$-modules. For every left $R$-module $M$ there exists an injective left $R$-module $E$ such that the complex

$$\text{RHom}_{\text{End}_R(E)}(\text{Hom}_R(M, E), E)$$

is a $T$-nullification of $M$. In particular, the differential graded algebra $\text{REnd}_{\text{End}_R(E)}(E)$ is a $T$-nullification of $R$.

The formula given in the abstract follows immediately from the distinguished triangle $\text{Cell}_T M \rightarrow M \rightarrow \text{Null}_T M$ mentioned above.

The layout of this paper is as follows. The necessary background on hereditary torsion classes and the background on cellular approximations and nullifications is given in Section 2. In Section 3 we describe the construction of nullification with respect to a hereditary torsion class and prove Theorem 1.1. We study the case where $R$ is an Artinian ring in Section 4. This section offers a different proof to a result of Benson ([2, Theorem 5.1]). Finally, Section 5 provides several examples.

1.A. **Notation and Terminology.** By a ring we always mean an associative ring with a unit, not necessarily commutative. Unless otherwise noted all modules considered are left modules. A triangle always means an exact (distinguished) triangle in the unbounded derived category of left $R$-modules, denoted $D_R$. A complex is always a chain-complex of $R$-modules. For complexes we use the standard convention that subscript grading is the negative of the superscript grading, i.e. $\square_{-i} = \square^i$. It is taken for granted that every $R$-module is a complex concentrated in degree 0 and with zero differential. A complex $X$ is bounded-above if for some $n$ and for all $i > n$, $H_i(X) = 0$. For complexes $X$ and $Y$ the notation $\text{Hom}_R(X, Y)$ stands for the usual chain complex of homomorphisms. The notation $\text{RHom}_R(-, -)$ stands for the derived functor of the $\text{Hom}_R(-, -)$ functor. By $\text{End}_R(M)$ we mean the endomorphisms ring of an $R$-module $M$. The symbol $\simeq$ stands for quasi-isomorphism of complexes.

2. **Background on Hereditary Torsion Theories and Cellular-Approximation, Nullification and Completion**

2.A. **Hereditary Torsion Theories.** Below is a recollection of the definition and main properties of hereditary torsion theories. A thorough review of this material can be found in [11].

**Definition 2.1.** A hereditary torsion class $T$ is a class of $R$-modules that is closed under submodules, quotient modules, coproducts and extensions. Closure under extensions means that if $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ is a short exact sequence with $M_1$ and $M_3$ in $T$, then so is $M_2$. The modules in $T$ will be called $T$-torsion modules (or just torsion modules when the torsion theory is clear from the context). The class of torsion-free modules $F$ is the class of all modules $F$ satisfying $\text{Hom}_R(C, F) = 0$ for every $C \in T$. The pair $(T, F)$ is referred to as a hereditary torsion theory. To every hereditary torsion theory there is an associated radical $t$, where $t(M)$ is the maximal torsion submodule of $M$. Note that $M/t(M)$ is therefore torsion-free.
Every hereditary torsion class $\mathcal{T}$ has an injective cogenerator (see [11, VI.3.7]). This means there exists an injective module $E$ such that a module $M$ is torsion if and only if $\text{Hom}_R(M, E) = 0$. It is also important to note that in any hereditary torsion theory, the class of torsion-free modules is closed under injective hulls (see [11, VI.3.2]). Thus, if $F$ is a torsion-free module then the injective hull of $F$ is also torsion-free.

**Definition 2.2.** Let $(\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory and let $t$ be the associated radical. An $R$-module $M$ is called $\mathcal{F}$-closed if, for every left ideal $a \subset R$ such that $R/a \in \mathcal{T}$, the induced map $M = \text{Hom}_R(R, M) \to \text{Hom}_R(a, M)$ is an isomorphism. The inclusion of the full subcategory of $\mathcal{F}$-closed modules has a left adjoint $M \mapsto M_\mathcal{F}$. The module $M_\mathcal{F}$ is called the module of quotients of $M$ (see [11, IX.1]). The unit of this adjunction has the following properties: the kernel of the map $M \to M_\mathcal{F}$ is $t(M)$, $M_\mathcal{F}$ is torsion-free and the cokernel of this map is a torsion module.

2.B. Cellular-Approximation, Nullification and Completion. The following recalls the basic properties of cellular approximation, as well as the definition of completion given by Dwyer and Greenlees in [3].

**Definition 2.3.** Let $R$ be a ring and let $\mathcal{A}$ be a class of $R$-complexes. We say an $R$-complex $X$ is $\mathcal{A}$-complete if $\text{Ext}^n_R(N, X) = 0$ for any $\mathcal{A}$-null object $N$. An $R$-complex $C$ is an $\mathcal{A}$-completion of $X$ if $C$ is $\mathcal{A}$-complete and there is an $\mathcal{A}$-equivalence $X \to C$. It is easy to see that an $\mathcal{A}$-completion of a complex $X$ is unique up to an isomorphism in $\mathcal{D}_R$. As in [3], we denote an $\mathcal{A}$-completion of $X$ by $X^\mathcal{A}$.

The following criterion for nullification is usually easier to check than the original definition. Its proof is easy and therefore omitted.

**Lemma 2.4.** Let $R$ be a ring and let $\mathcal{A}$ be a class of $R$-complexes. A complex $N$ is an $\mathcal{A}$-nullification of $X$ if there is a triangle $C \to X \to N$ such that $C$ is $\mathcal{A}$-cellular and $N$ is $\mathcal{A}$-null. In this case it also follows that $C$ an $\mathcal{A}$-cellular approximation of $X$.

Recall that when $\mathcal{A}$ is a set, the full subcategory of $\mathcal{A}$-cellular objects of $\mathcal{D}_R$ is the localizing subcategory generated by $\mathcal{A}$, denoted $\langle \mathcal{A} \rangle$, is the smallest full triangulated subcategory of $\mathcal{D}_R$ that is closed under triangles, direct sums and retracts. Closure under triangles means that for every distinguished triangle in $\mathcal{D}_R$, if two of the objects are in the localizing subcategory, so is the third. The proof of the following lemma is clear.

**Lemma 2.5.** Let $\mathcal{A}$ be a class of $R$-complexes, then every object of $\langle \mathcal{A} \rangle$ is $\mathcal{A}$-cellular. If $\mathcal{B}$ is another class of $R$-complexes such that $\langle \mathcal{A} \rangle = \langle \mathcal{B} \rangle$, then $\mathcal{A}$-cellular approximation is the same as $\mathcal{B}$-cellular approximation.

**Remark 2.6.** In [3] $\mathcal{A}$-cellular complexes were called $\mathcal{A}$-torsion while the term $\mathcal{A}$-cellular was reserved for complexes in $\langle \mathcal{A} \rangle$. When $\mathcal{T}$ is a hereditary torsion theory, the two terms agree (by Lemma 2.8 below).

2.C. Cellular-Approximation with respect to a Hereditary Torsion Theory. Let $\mathcal{T}$ be a hereditary torsion class. It is not immediately apparent that $\mathcal{T}$-cellular approximation exists. Below, in Lemma 2.8, we show that $\langle \mathcal{T} \rangle$ is the same as the localizing subcategory generated by a set $\mathcal{A}_\mathcal{T}$. This immediately implies that $\mathcal{T}$-cellular approximation and $\mathcal{T}$-nullification exist for any $R$-complex, see Corollary 2.9.

**Definition 2.7.** Given a hereditary torsion class of $R$-modules $\mathcal{T}$, we denote by $\mathcal{A}_\mathcal{T}$ the set of all cyclic $\mathcal{T}$-torsion modules.
Lemma 2.8. Let \( T \) be a hereditary torsion class, then every \( T \)-torsion module is \( A_T \)-cellular and hence \( \langle T \rangle = \langle A_T \rangle \).

Proof. Clearly, every cyclic \( T \)-torsion module is \( A_T \)-cellular. Therefore every direct sum of cyclic \( T \)-torsion modules is \( A_T \)-cellular. Let \( M \) be a \( T \)-torsion module, then there is a surjection \( C(M) = \bigoplus_{m \in M} R/\text{ann}(m) \to M \). Since every hereditary torsion theory is closed under submodules, \( R/\text{ann}(m) \) is \( T \)-torsion for every \( m \in M \). Clearly \( C(M) \) is \( A_T \)-cellular and \( T \)-torsion. Next we build a resolution \( X \) of \( M \) using \( A_T \)-cellular modules. Let \( X_0 = C(M) \), and let \( d_0 : X_0 \to M \) the map defined above. The kernel of \( d_0 \) is \( T \)-torsion, so there is an epimorphism \( C(\ker(d_0)) \to \ker(d_0) \). Let \( X_1 = C(\ker(d_0)) \) and let \( d_1 \) be the composition \( X_1 \to \ker(d_0) \to X_0 \). In this way \( X \) is built inductively and it is clear that \( X \) is quasi-isomorphic to \( M \). By construction, \( X \) is in the localizing subcategory generated by \( A_T \).

Corollary 2.9. Let \( T \) be a hereditary torsion class, then \( T \)-cellular approximation, \( T \)-nullification and \( T \)-completion exist for every complex. Moreover, a complex \( X \) is \( T \)-cellular if and only if \( X \in \langle T \rangle \).

Proof. Lemma 2.8 implies that \( T \)-cellular approximation is the same as \( A_T \)-cellular approximation. As mentioned in Section 1, \( A_T \)-cellular approximation exists for every complex. The proof of the other claims is similar.

Lemma 2.10. Let \( T \) be a hereditary torsion class.

1. If \( X \) is a \( T \)-cellular complex then the homology groups of \( X \) are \( T \)-torsion \( R \)-modules.

2. If \( X \) is a bounded-above complex such that the homology groups of \( X \) are \( T \)-torsion then \( X \) is \( T \)-cellular.

Proof. Let \( C \) be the full subcategory of \( D_R \) containing all objects whose homology groups are \( T \)-torsion \( R \)-modules. The properties of a hereditary torsion theory show that \( C \) is localizing subcategory. Since \( C \) contains \( T \), then \( C \) also contains \( \langle T \rangle \). This proves the first statement.

Now suppose \( X \) is a bounded-above complex and that \( H_i X \in T \) for all \( i \). Because \( X \) is bounded-above, \( X \) belongs to the localizing subcategory generated by the homology groups of \( X \) (see for example [3, 5.2]). Since the homology groups of \( X \) all belong to \( \langle T \rangle \), so does \( X \).

Remark 2.11. If \( R \) is a commutative Noetherian ring, then a complex \( X \) is \( T \)-cellular if and only if all the homology groups of \( X \) are \( T \)-torsion. This easily follows from a result of Neeman [9, Theorem 2.8]. However, Example 5.2 shows a noncommutative ring \( R \) and a complex \( X \) such that \( H_i(X) \) is \( T \)-torsion for all \( i \) but \( X \) is not \( T \)-cellular.

3. Nullification Construction

In [2], Benson gives a construction called \( k \)-squeezed resolution which yields \( k \)-cellular approximations over the ring \( kG \), where \( k \) is a prime field and \( G \) is a finite group. We generalize Benson’s construction so as to produce \( T \)-cellular approximations over any ring \( R \), where \( T \) is a hereditary torsion class. In fact, we give two isomorphic constructions.

Nullification Construction 3.1. Let \( \langle T, F \rangle \) be a hereditary torsion theory with radical \( t \). For an \( R \)-module \( M \) we construct the \( T \)-nullification of \( M \) as a cochain complex \( I^0 \xrightarrow{d} I^1 \xrightarrow{d} I^2 \xrightarrow{d} \cdots \) inductively.
Let $M^0 = M$, let $F^0 = M^0/t(M^0)$ and let $I^0$ be the injective hull of $F^0$. Note that since $F^0$ is torsion-free, so is $I^0$. We proceed by induction, set

$$M^{n+1} = I^n/F^n, \quad F^{n+1} = M^{n+1}/t(M^{n+1})$$

and let $I^{n+1}$ be the injective hull of $F^{n+1}$. Again $I^{n+1}$ is torsion-free because $F^{n+1}$ is. The differential $d : I^n \to I^{n+1}$ is the composition $I^n \to M^{n+1} \to F^{n+1} \to I^{n+1}$. The image of $d : I^n \to I^{n+1}$ is $F^{n+1}$ and therefore $d \circ d = 0$. Denote the resulting complex by $I$. The natural map $M \to I^0$ extends to a map of complexes $M \to I$.

**Nullification Construction 3.2.** For an $R$-module $M$ we construct a cochain complex $J^0 \xrightarrow{d^0} J^1 \xrightarrow{d^1} J^2 \xrightarrow{d^2} \cdots$ inductively.

Let $Q^0 = M$, let $N^0 = (Q^0)_F$ and let $J^0$ be the injective hull of $N^0$. Denote by $d^{-1}$ the map $M \to J^0$. Now proceed by induction, set

$$Q^{n+1} = J^n/\text{im}(d^n), \quad N^{n+1} = (Q^{n+1})_F$$

and let $J^{n+1}$ be the injective hull of $J^n$. The differential $d^n : J^n \to J^{n+1}$ is the composition $J^n \to Q^{n+1} \to N^{n+1} \to J^{n+1}$. Clearly, $d^{n+1} \circ d^n = 0$. Denote the resulting complex by $J$. The natural map $M \to J^0$ extends to a map of complexes $M \to J$. Note that for every $n$, $J^n$ is torsion-free because $N^n$ is.

**Lemma 3.3.** Let $J$ be the complex constructed from $M$ in 3.2, then $H_0(J) \cong M_F$.

**Proof.** It easily follows from the definition of an $\mathcal{F}$-closed module that any injective torsion-free module is $\mathcal{F}$-closed, therefore $J^0$ is $\mathcal{F}$-closed. For any $\mathcal{F}$-closed module $K$ there is an isomorphism $K \cong K_\mathcal{F}$ (see [11, page 198]), therefore $(J^0)_\mathcal{F} \cong J^0$ and $(M_F)_\mathcal{F} \cong M_F$.

The module of quotients functor is left exact (see [11, page 199]). Hence applying the module of quotients functor to the sequence $M_\mathcal{F} \to J^0 \to J^0/M_F$ yields an exact sequence:

$$0 \to M_F \to J^0 \to (J^0/M_F)_\mathcal{F}$$

We see that $J^0/M_F$ is torsion-free, because it is isomorphic to a submodule of the torsion-free module $(J^0/M_F)_\mathcal{F}$.

Now consider the short exact sequence

$$M_F/\text{im}(M) \to Q^1 \to J^0/M_F$$

The module $M_F/\text{im}(M)$ is a torsion module (see Definition 2.2), while the module $J^0/M_F$ is torsion free. From the definition of the radical $t$ it follows that $M_F/\text{im}(M) \cong t(Q^1)$. Therefore $M_F$ is the kernel of $J^0 \to N^1$ and the proof is complete. \(\square\)

**Lemma 3.4.** Let $M$ be an $R$-module, let $I$ be the complex constructed from $M$ in 3.1, let $C$ be a complex such that $C \to M \to I$ is a distinguished triangle and let $J$ be the complex constructed from $M$ in 3.2. Then $C$ is a $T$-cellular approximation of $M$ and both $I$ and $J$ are $T$-nullifications of $M$. In particular, $H_0(\text{Null}_T M) \cong M_F$ for any $R$-module $M$.

**Proof.** We can choose $C$ to be the complex $M \to I^0 \to I^1 \to \cdots$ with $M$ in degree 0. The homology of $C$ is easy to compute: $H^n(C) = t(M^n)$, with $M^n$ as defined in the Nullification Construction 3.1 above. By Lemma 2.10, the complex $C$ is $T$-cellular. The complex $I$ is $T$-null, simply because $I$ is composed of torsion-free injective modules. Thus, by Lemma 2.4, $I$ is a $T$-nullification of $M$ and $C$ is a $T$-cellular approximation of $M$.

Similar reasoning shows that $J$ is a $T$-nullification of $M$. The complex $J$ is $T$-null, simply because $J$ is composed of torsion-free injective modules. The homology of $J$ is:
\[ H^n(J) = t(Q^n) \text{ for } n > 0 \text{ and } H^0(J) = M_J. \] Let \( C' \) be a complex such that there is a distinguished triangle \( C' \rightarrow M \rightarrow J \). The long exact sequence in homology yields:
\[ H^0(C') = t(M), \quad H^1(C') = M_J/M \quad \text{and} \quad H^n(C') = H^{n-1}(J) \quad \text{for } n > 1. \]
Note that \( M_J/M \) is a \( \mathcal{T} \)-torsion module. By Lemma 2.10 the complex \( C' \) is \( \mathcal{T} \)-cellular and hence \( J \) is a \( \mathcal{T} \)-nullification of \( M \) and \( C' \) is a \( \mathcal{T} \)-cellular approximation of \( M \). In particular, \( H_0(\text{Null}_\mathcal{T} M) = H_0(J) \cong M_J. \]

**Remark 3.5.** It follows that the complexes \( I \) and \( J \) in Lemma 3.4 are isomorphic in the derived category of \( R \). In fact, they are isomorphic as complexes. To construct this isomorphism one needs the following property: for any \( R \)-module \( L \) the injective hull of \( L_J \) and the injective-hull of \( L/t(L) \) are the same; this is because \( (L/t(L))_J = L_J \) and \( L/t(L) \) is an essential submodule of \( (L/t(L))_J \) (see [11, IX.2.4]). Using the aforementioned property it is a simple exercise to construct the isomorphism inductively.

Using the construction above we can give a different description of \( \mathcal{T} \)-nullification, the one shown in Theorem 1.1. Before proving Theorem 1.1 it is necessary to note some properties of the functor \( \text{Hom}_R(\_, E) \).

Let \( E \) be an \( R \)-module and let \( \mathcal{E} \) be the endomorphism ring \( \text{End}_R(E) = \text{Hom}_R(E, E) \). The functor \( \text{Hom}_R(\_, E) \) is a contravariant functor from left \( R \)-complexes to left \( \mathcal{E} \)-complexes. This left \( \mathcal{E} \)-action is simply composition on the left with the morphisms in \( \mathcal{E} \). In other words, the left \( \mathcal{E} \)-action on \( \text{Hom}_R(\_, E) \) is induced by the left \( \mathcal{E} \)-action on \( E \) itself. Moreover, the functor \( \text{Hom}_R(\_, E) \) is a contravariant functor, this time from left \( \mathcal{E} \)-complexes to left \( R \)-complexes. Here the left \( R \)-action on \( \text{Hom}_\mathcal{E}(\_, E) \) comes from the left \( R \)-action on \( E \) (which commutes with the left \( \mathcal{E} \)-action on \( E \)). In particular, there is a derived version of this functor: \( \mathbf{R}\text{Hom}_\mathcal{E}(\_, E) : \mathcal{D}_\mathcal{E} \rightarrow \mathcal{D}_R \).

**Proof of Theorem 1.1.** Given an \( R \)-module \( M \), construct a \( \mathcal{T} \)-nullification of \( M \) in the way prescribed in 3.1. This construction results in a cochain complex \( I \), with \( I^n \) being an injective torsion-free module. Let \( E \) be a torsion-free injective \( R \)-module such that for every \( n \), \( I^n \) is a direct summand of a finite direct sum of copies of \( E \). For example, one can take \( E \) to be the product \( \prod_n I^n \). Denote by \( \mathcal{E} \) the endomorphism ring \( \text{End}_R(E) \).

Now consider the triangle \( C \rightarrow M \rightarrow I \). Since \( I \) is a \( \mathcal{T} \)-nullification of \( M \), \( C \) is a \( \mathcal{T} \)-cellular approximation of \( M \). Applying the functor \( \text{Hom}_R(\_, E) \) to this triangle yields a triangle in \( \mathcal{D}_\mathcal{E} \):

\[
\text{Hom}_R(I, E) \rightarrow \text{Hom}_R(M, E) \rightarrow \text{Hom}_R(C, E)
\]

Since \( E \) is injective, \( H_i(\text{Hom}_R(C, E)) \cong \text{Hom}_R(H_i(C), E) \). Since the homology groups of \( C \) are torsion, \( \text{Hom}_R(H_i(C), E) = 0 \). Therefore the map \( \text{Hom}_R(I, E) \rightarrow \text{Hom}_R(M, E) \) is a quasi-isomorphism of \( \mathcal{E} \)-complexes.

Because \( I^n \) is a direct summand of a finite direct sum of copies of \( E \), the \( \mathcal{E} \)-module \( \text{Hom}_R(I^n, E) \) is projective. Thus the map \( \text{Hom}_R(I, E) \rightarrow \text{Hom}_R(M, E) \) is a projective resolution of \( \text{Hom}_R(M, E) \) in the category of \( \mathcal{E} \)-modules. We conclude that the complex \( \text{Hom}_\mathcal{E}(\text{Hom}_R(I, E), E) \) is the derived functor \( \mathbf{R}\text{Hom}_\mathcal{E}(\text{Hom}_R(M, E), E) \).

Because \( I^n \) is a direct summand of a finite direct sum of copies of \( E \), one readily sees that the \( R \)-module \( \text{Hom}_\mathcal{E}(\text{Hom}_R(I^n, E), E) \) is naturally isomorphic to \( I^n \) and therefore \( \text{Hom}_\mathcal{E}(\text{Hom}_R(I, E), E) \cong I \).

**Remark 3.6.** As noted in Theorem 1.1 nullification \( \mathbf{R}\text{Hom}_\mathcal{E}(E) \) and therefore
\[
R_\mathcal{T} \cong H^0(\text{Null}_\mathcal{T} R) \cong H^0(\mathbf{R}\text{End}_\mathcal{E}(E)) = \text{End}_\mathcal{E}(E)
\]
This isomorphism recovers \([11] \text{IX.3.3}\), where it is stated that there is an injective \(R\)-module \(E\) such that \(R Two \equiv \text{End}_{\text{End}_R(E)}(E)\). Also note that \(\text{Null}_T R\) is quasi-isomorphic to a differential graded algebra. This also follows from a result of Dwyer \([5] \text{Proposition 2.5}\)], where it is shown that for any set of complexes \(\mathcal{A}\), the complex \(\text{Null}_A R\) is quasi-isomorphic to a differential graded algebra.

**Remark 3.7.** Let \(X = \cdots \to X_n \to X_{n-1} \to \cdots\) be a complex such that there exists some \(m\) for which \(X_n = 0\) for all \(n > m\). Then it is possible to generalize the Nullification Construction \([3,1]\) to give the \(T\)-nullification of \(X\). Moreover, this generalized construction of \(\text{Null}_T X\) can be done in such a way that for \(n > m\) \((\text{Null}_T X)_n = 0\), while for \(n \leq m\) \((\text{Null}_T X)_n\) is a finite direct sum of torsion-free injective modules. Therefore \((\text{Null}_T X)_n\) is itself a torsion-free injective for \(n \leq m\). Now it is easy to see that the proof of Theorem \([11]\) works for \(\text{Null}_T X\) as well and yields the same result. Namely, there exists an injective \(R\)-module \(E\) such that

\[
\text{Null}_T X \cong R \text{Hom}_{\text{End}_R(E)}(\text{Hom}_R(X, E), E)
\]

Clearly, this result carries over to any bounded-above complex \(X\).

Say an injective module \(E\) is *sufficient to compute the \(T\)-nullification of \(M\)* if

\[
\text{Null}_T M \cong R \text{Hom}_E(\text{Hom}_R(M, E), E)
\]

where \(E = \text{End}_R(M)\). Given an \(R\)-module \(M\) one can use the proof of Theorem \([11]\) to construct an injective module \(E\) which is sufficient to compute the \(T\)-nullification of \(M\). However there are other injective modules sufficient to compute the \(T\)-nullification of \(M\), as shown by the following proposition.

**Proposition 3.8.** Let \(M\) be an \(R\)-module and let \(E\) be an injective cogenerator of \(T\). Denote by \(E\) the ring \(\text{End}_R(E)\).

1. If the \(E\)-module \(\text{Hom}_R(M, E)\) has a resolution composed of finitely generated projective modules in each degree, then \(E\) is sufficient to compute the \(T\)-nullification of \(M\).

2. There exists an ordinal \(\alpha\) such that the module \(E' = \prod_{\alpha \leq \alpha} E\) is sufficient to compute the \(T\)-nullification of \(M\).

**Proof.** Let \(P\) be a finitely generated projective \(E\)-module, then it is easy to see that \(\text{Hom}_R(\text{Hom}_E(P, E), E)\) is naturally isomorphic to \(P\). Now let \(F\) be a projective resolution of \(\text{Hom}_R(M, E)\) over \(E\) and assume \(F\) is composed of finitely generated projective modules in each degree. Then \(\text{Hom}_R(\text{Hom}_E(F, E), E)\) is naturally isomorphic to \(F\).

The quasi-isomorphism \(\eta : F \to \text{Hom}_R(M, E)\) induces a map \(\text{Hom}_E(\text{Hom}_R(M, E), E) \to \text{Hom}_E(F, E)\). Composing with the natural map \(M \to \text{Hom}_E(\text{Hom}_R(M, E), E)\) yields a map \(\mu : M \to \text{Hom}_E(F, E)\). It is easy to see that \(\text{Hom}_R(\mu, E)\) is the quasi-isomorphism \(\eta\).

Consider the triangle \(C \to M \xrightarrow{\mu} \text{Hom}_E(F, E)\). Clearly, \(\text{Hom}_E(F, E)\) is \(T\)-null. Since \(\text{Hom}_R(\mu, E)\) is a quasi-isomorphism, \(\text{Hom}_R(C, E)\) is quasi-isomorphic to zero. This implies \(\text{Hom}_R(H_i(C), E) = 0\) for all \(i\). Since \(E\) is an injective cogenerator for \(T\), \(H_i(C)\) is torsion for all \(i\). Clearly \(C\) is bounded-above and so, by Lemma \([2,10]\) \(C\) is \(T\)-cellular. We conclude that \(\text{Hom}_E(F, E)\) is a \(T\)-nullification of \(M\) and \(E\) is sufficient to compute the \(T\)-nullification of \(M\).

We now turn our attention to the second item in the proposition. By \([11] \text{VI.3.9}\), every torsion-free module has a monomorphism to some direct product of copies of \(E\). In particular, every torsion-free injective is a isomorphic to a direct summand of some direct product of copies of \(E\).
Let $I$ be the complex described in the Nullification Construction\[3.1\]. Let $E'$ be a direct product of copies of $E$ such that for every $n$, $I^n$ is isomorphic to a direct summand of $E'$. Clearly the $\text{End}_R(E')$-complex $\text{Hom}_R(I, E')$ is a projective resolution of $\text{Hom}_R(M, E')$ which is composed of finitely generated projective modules in every degree. Hence $E'$ is sufficient to compute the $\mathcal{T}$-nullification of $M$. \hfill $\square$

4. TORSION THEORIES AND CELLULAR APPROXIMATION IN ARTINIAN RINGS

Throughout this section $R$ is an Artinian ring and $S$ is a set of non-isomorphic simple modules of $R$. Define a class $\mathcal{F}$ of $R$-modules by $\mathcal{F} = \{ F \mid \text{Hom}_R(S, F) = 0 \text{ for all } S \in S \}$, and define a class $\mathcal{T}$ by setting $\mathcal{T} = \{ M \mid \text{Hom}_R(M, F) = 0 \text{ for all } F \in \mathcal{F} \}$. By \[11\ VIII.3\], the pair $(\mathcal{T}, \mathcal{F})$ forms a hereditary torsion theory (alternatively, one can easily deduce this from Lemma 4.3 below). Because $R$ is Artinian, every hereditary torsion theory of $R$-modules is generated by a set of simple modules (see \[11\ VIII\]), so this context covers all hereditary torsion theories over $R$. In this section we give several results regarding $\mathcal{T}$-nullification. We also give a different proof for a result of Benson \[2\] Theorem 5.1 in Corollary 4.5.

Let $\Omega$ be the set of isomorphism classes of simple modules of $R$ and let $S'$ be the complement of $S$ in $\Omega$. We denote by $E$ the product of the injective hulls of the simple modules in $S'$ and denote by $P$ the direct sum of the projective covers of those simple modules. We show that $E$ is an injective cogenerator of $\mathcal{T}$ and that being $\mathcal{T}$-cellular is the same as being $S$-cellular.

**Lemma 4.1.** Let $C$ be a cyclic $R$-module such that $\text{Hom}_R(C, E) = 0$, then $C$ is $S$-cellular.

**Proof.** Since $R$ is Artinian, $C$ admits a composition series

$$0 = C_0 \subset C_1 \subset \cdots \subset C_m = C,$$

where all the quotients $C_i/C_{i-1}$ are simple modules. We next show that $C_i/C_{i-1} \in S$ for all $i$. Suppose that for some $i$, $C_i/C_{i-1} \cong S'$ for some $S' \in S'$. Let $x \in C_i \setminus C_{i-1}$, then the cyclic module generated by $x$ has $S'$ as a quotient. This implies the submodule $Rx$ of $C$ has a non-zero map to $E(S')$ - the injective hull of $S'$. Clearly such a map can be lifted to a non-zero map $C \to E$, in contradiction. Therefore $C_i/C_{i-1} \cong S$ for some $S \in S$. Now a simple inductive argument on $i$ shows that $C_i \in (S)$ for every $i$, and hence $C$ is $S$-cellular. \hfill $\square$

**Corollary 4.2.** A complex $X$ is $\mathcal{T}$-cellular if and only if $X$ is $S$-cellular.

**Proof.** We need to show that $\langle T \rangle = \langle S \rangle$. Since $S \subset T$, we only need to show that $T \subset \langle S \rangle$. By Lemma 2.8 it is enough to show that every cyclic $R$-module is $S$-cellular, but that is immediate from Lemma 4.1. \hfill $\square$

**Lemma 4.3.** The module $E$ is an injective cogenerator for $\mathcal{T}$.

**Proof.** Let $\mathcal{U}$ be the class of modules $M$ such that $\text{Hom}_R(M, E) = 0$. Then $\mathcal{U}$ is a hereditary torsion theory. Because $\text{Hom}_R(S, S') = 0$ for every $S \in S$ and $S' \in S'$, we see that $\text{Hom}_R(S, E(S')) = 0$, where $E(S')$ is the injective envelope of $S'$. Hence $E \in \mathcal{F}$ and therefore $\mathcal{U}$ contains $\mathcal{T}$.

Next, let $M$ be in $\mathcal{U}$. To show that $M$ is a $\mathcal{T}$-torsion module it is enough to show that every cyclic submodule of $M$ is a torsion module, because $M$ is a quotient of the direct sum of its cyclic submodules. So let $C$ be a cyclic submodule of $M$. Since $E$ is injective, it follows that $\text{Hom}_R(C, E) = 0$. By Lemma 4.1 $C$ is $S$-cellular. Therefore $C$ is $\mathcal{T}$-cellular and by Lemma 2.10 $C$ is $\mathcal{T}$-torsion. \hfill $\square$
Lemma 4.4. For any complex $X$, $\text{Ext}^*_R(P, X) = 0$ if and only if $\text{Ext}^*_R(X, E) = 0$.

Proof. This is known when $X$ is a finitely generated $R$-module, see Benson’s book [1, 1.7.6 &1.7.7]. Now suppose $X$ is any $R$-module. Since $P$ is a finitely generated projective module, $\text{Hom}_R(P, X) = 0$ if and only if $\text{Hom}_R(P, X') = 0$ for every finitely generated submodule $X'$ of $X$. Similarly, because $E$ is injective, $\text{Hom}_R(X, E) = 0$ if and only if $\text{Hom}_R(X', E) = 0$ for every finitely generated submodule $X'$ of $X$. Hence the lemma holds for any $R$-module. Finally, let $X$ be any complex, then $\text{Ext}^*_R(P, X) = \text{Hom}_R(P, H_*(X))$. Similarly $\text{Ext}^*_R(X, E) = \text{Hom}_R(H^*(X), E)$. □

Corollary 4.5. For any $R$-module $M$, a $\mathcal{T}$-nullification of $M$ is also a $P$-completion of $M$ and is therefore given by

$$\text{Null}_\mathcal{T}M \simeq R\text{Hom}_{\text{End}_R(P)}(\text{Hom}_R(P, R), \text{Hom}_R(P, M))$$

Proof. Consider the triangle $\text{Cell}_\mathcal{T}M \to M \xrightarrow{\nu} \text{Null}_\mathcal{T}M$. Lemma 4.3 implies that $E$ is $\mathcal{T}$-null and therefore $\text{Ext}^*_R(\text{Cell}_\mathcal{T}M, E) = 0$. By Lemma 4.4 $\text{Cell}_\mathcal{T}M$ is $P$-null and $\nu$ is a $P$-equivalence.

It remains to show that $\text{Null}_\mathcal{T}M$ is $P$-complete. Let $I$ be the $\mathcal{T}$-nullification of $M$ described in 3.1. The full subcategory of $P$-complete objects in $\mathcal{D}_R$ is closed under isomorphisms, completion of triangles, products and retracts. Denote this subcategory by $\mathcal{C}$. From Lemma 4.4 we see that $E \in \mathcal{C}$ and therefore every product of $E$ is also in $\mathcal{C}$. Lemma 4.3 and [11, VI.3.9] imply that every torsion-free module is a submodule of a product of copies of $E$. Since $I^n$ is injective, it is a direct summand of some product of copies of $E$, hence $I^n$ is also an object of $\mathcal{C}$.

Let $I(n)$ denote the cochain complex $I^0 \to I^1 \to \cdots \to I^n$. An inductive argument shows that $I(n) \in \mathcal{C}$. There is a triangle $I \to \prod_n I(n) \xrightarrow{\phi^{-1}} \prod_n I(n)$, where the map $\phi$ is induced by the maps $I(n+1) \to I(n)$. Hence $I$ is $P$-complete.

By Dwyer and Greenlees [3, Theorem 2.1], the $P$-completion of an $R$-module $M$ is given by

$$M_P \simeq R\text{Hom}_{\text{End}_R(P)}(\text{Hom}_R(P, R), \text{Hom}_R(P, M))$$

□

Corollary 4.5 above implies Benson’s formula for $\mathcal{T}$-cellular approximation given in [2, Theorem 5.1]. This corollary also explains the connection between Benson’s formula and Dwyer and Greenlees formula for $P$-completion from [3, Theorem 2.1].

5. Examples

Example 5.1. Let $I$ be a two-sided ideal of $R$ such that $I$ is finitely generated as a left $R$-module. An $R$-module $M$ will be called $I$-torsion if for every $m \in M$ there exists some $n$ such that $I^n m = 0$. It is not difficult to show that the class of $I$-torsion modules forms a hereditary torsion class $\mathcal{T}$ (see [11, VI.6.10]). Using Lemma 2.8 it is easy to conclude that $\langle \mathcal{T} \rangle = \langle R/I \rangle$. Hence $\mathcal{T}$-cellular approximation is the same as $R/I$-cellular approximation and the same goes for nullification. Note that in this case the radical $t$ associated with $\mathcal{T}$ has a simple description: for any $R$-module $M$

$$t(M) = \text{colim}_{n \to \infty} \text{Hom}_R(R/I^n, M)$$

Now suppose $R$ is a commutative Noetherian ring. Dwyer and Greenlees have shown in [3] that $R/I$-cellular approximation computes $I$-local cohomology, namely that there is a natural isomorphism $H^*_I(M) \cong H^*(\text{Cell}_{R/I} M)$. Recall there is an isomorphism:

$$H^*_I(M) \cong \text{colim}_{n \to \infty} \text{Ext}^*_R(R/I^n, M)$$
These facts show that $\mathcal{T}$-cellular approximation is the derived functor of the radical $t$. Moreover, in this case an object $X \in \mathcal{D}_R$ is $\mathcal{T}$-cellular if and only $H_n(X)$ is $\mathcal{T}$-torsion for all $n$, see [3, 6.12].

**Example 5.2.** Here is an example of a case where $\mathcal{T}$-cellular approximation is not the derived functor of the associated radical. Let $G$ be the symmetric group on 3 elements, let $k$ be the field $\mathbb{Z}/3\mathbb{Z}$ and let $R$ be the group ring $k[G]$. There is an augmentation map $R \to k$, where $k$ has the trivial $G$-action. Let $I$ be the augmentation ideal. As before, denote the class of $I$-torsion modules by $\mathcal{T}$ and the associated radical by $t$. Since $R$ is an Artinian ring, the sequence $I \supseteq I^2 \supseteq I^3 \supseteq \cdots$ stabilizes. So there is a fixed index $m$ such that $t(M) = \text{Hom}_R(R/I^m, M)$ for every $R$-module $M$. Therefore, the derived functors of the torsion radical $t$ are the functors $\text{Ext}^*_R(R/I^m, -)$. In particular, $\text{Ext}^*_R(R/I^m, R) = 0$ for all $i > 0$, because $R$ is injective. On the other hand, a calculation using Benson’s methods from [2] shows that $H^n(\text{Cell}_t R)$ is non-zero for infinitely many values of $n$; thereby showing that $\text{Cell}_t$ is not the derived functor of $t$. We describe this calculation next.

From the surjection $G \to \mathbb{Z}/2$ one sees that $R$ has two simple modules, the trivial module $k$ and a one dimensional simple module $\omega$. As a left module, $R \cong E_k \oplus E_\omega$ where $E_k$ and $E_\omega$ are the injective hulls of $k$ and $\omega$ respectively. The module $E_k$ has a composition series

$$k \subset B \subset E_k,$$

where $B/k \cong \omega$ and $E_k/B \cong k$.

The composition series for $E_\omega$ is

$$\omega \subset B' \subset E_\omega,$$

where $B'/\omega \cong k$ and $E_\omega/B' \cong \omega$.

In addition $E_\omega/\omega \cong B$ and $E_k/k \cong B'$. Since $E_\omega$ is $k$-null (see Lemma [4,3], then

$$\text{Null}_t R \cong \text{Null}_t E_\omega \oplus \text{Null}_t E_k \cong E_\omega \oplus \text{Null}_t E_k$$

So we need only compute $\text{Null}_t E_k$. Applying Construction [3,1] to the module $E_k$ we get the complex $I$ which is $E_\omega \xrightarrow{d} E_\omega \xrightarrow{d} E_\omega \xrightarrow{d} \cdots$ where $d$ is the composition $E_\omega \to \omega \to E_\omega$. Hence $H^n(\text{Null}_t E_k) = k$ for $n > 1$ and therefore $H^n(\text{Cell}_t R)$ is non-zero for infinitely many values of $n$. In fact

$$H^n(\text{Cell}_t R) = \begin{cases} k, & n = 0; \\ 0, & n = 1; \\ k, & n > 1. \end{cases}$$

It is important to note that in this case, a complex $X$ such that $H_n(X)$ is $\mathcal{T}$-torsion for all $n$ need not be $\mathcal{T}$-cellular. Consider, for example, the complex $R^\wedge_t$. As we explain below, the homology groups $H_n(R^\wedge_t)$ are $\mathcal{T}$-torsion for all $n$. On the other hand, the $\mathcal{T}$-equivalences $\text{Cell}_t R \to R$ and $R \to R^\wedge_t$ show that $\text{Cell}_t R$ is $\mathcal{T}$-equivalent to $R^\wedge_t$. If $R^\wedge_t$ was $\mathcal{T}$-cellular, then $R^\wedge_t$ would have been quasi-isomorphic to $\text{Cell}_t R$, because a $\mathcal{T}$-equivalence between $\mathcal{T}$-cellular complexes is a quasi-isomorphism. As we show next, the complex $R^\wedge_t$ has no homology in negative degrees and so cannot be quasi-isomorphic to $\text{Cell}_t R$.

It remains to explain the properties of $R^\wedge_t$ used above. From Corollary [1,2] we learn that $R^\wedge_t \cong R^\wedge_k$ and $\text{Cell}_t R \cong \text{Cell}_k R$. Without going into details, combining [3, 5.9] with [3, 4.3] shows that

$$R^\wedge_k \cong \text{RHom}_R(\text{Cell}_k R, R)$$

This immediately implies that $R^\wedge_t$ has no homology in negative degrees. We next show that $H_n(R^\wedge_t)$ is $\mathcal{T}$-torsion for all $n$. Since $E_\omega$ is a $\mathcal{T}$-null module, $\text{Ext}^*_R(E_\omega, R^\wedge_t) = 0.$
Recall that $R$ is a group-algebra and therefore $E_\omega$ is also the projective cover of $\omega$. Because $E_\omega$ is projective we have
\[
\text{Ext}_R^{-n}(E_\omega, R^\wedge_T) \cong \text{Hom}_R(E_\omega, H_n(R^\wedge_T))
\]
Hence, by Lemma 4.3 $\text{Ext}_R^{0}(H_n(R^\wedge_T), E_\omega) = 0$. Lemma 4.3 shows $E_\omega$ is an injective cogenerator for $T$, therefore $H_n(R^\wedge_T)$ is $T$-torsion.

**Example 5.3.** This example relates $T$-nullification with Cohn localization. We begin by recalling the definition of Cohn localization. Let $S = \{ f_\alpha : P_\alpha \to Q_\alpha \}$ be a set of maps between finitely generated projective $R$-modules. Say a ring map $R \to R'$ is $S$-inverting if $\text{Hom}_R(f, R')$ is an isomorphism for every $f \in S$. A Cohn localization of $R$ with respect to $S$ is a ring map $R \to S^{-1}R$ which is initial among all $S$-inverting ring maps. Note that the definition given here is not the standard definition (see e.g. [5]), but it is equivalent to the standard one.

Let $C_S$ be the set of cones of the maps $f_\alpha$. In [5], Dwyer considers $C_S$-nullification and shows that $H_0(\text{Null}_{C_S} R) = S^{-1}R$ (see [5 3.2]). Combining Dwyer’s results with Theorem 1.1 yields the following proposition.

**Proposition 5.4.** Let $T$ be a hereditary torsion-class of $R$-modules. If $\langle T \rangle = \langle C_S \rangle$ for some set of maps $S$ between finitely generated projective $R$-modules, then

1. $\text{Null}_T(-) \cong (-)\wedge_T$,  
2. the module of quotients functor $(-)\wedge_T$ is exact and  
3. there is an isomorphism $S^{-1}R \otimes_R M \cong M_T$ for every module $M$.

**Proof.** By Theorem 1.1 for every $R$-module $M$ the complex $\text{Null}_T M$ has no homology in positive degrees. By [5 Proposition 3.1], $\text{Null}_T R$ has no homology in negative degrees. Moreover, a result of Miller [8] (see also [5 Proposition 2.10]) shows that for every $R$-module $M$, $\text{Null}_T M \cong \text{Null}_T R \otimes_R M$. This implies that $\text{Null}_T M$ has no homology in negative degrees.

We conclude that for every $R$-module $M$, $\text{Null}_T M$ has homology only in degree zero and therefore, by Lemma 3.4 $\text{Null}_T M$ is quasi-isomorphic to $M_T$. Since the functor $\text{Null}_T$ is exact, so is $(-)\wedge_T$. Since $\text{Null}_T R \cong \text{Null}_{C_S} R$, Dwyer’s result [5 3.2] shows that $R_T \cong S^{-1} R$. Finally, the quasi-isomorphism $\text{Null}_T M \cong \text{Null}_T R \otimes_R M$ implies $S^{-1} R \otimes_R M \cong M_T$. □

**Example 5.5.** This example is of a topological nature. Let $M$ be a discrete monoid and let $k = \mathbb{Z}/p\mathbb{Z}$ for some prime $p$. The ring $R$ we consider is the monoid ring $R = k[M]$, it has a natural augmentation $R \to k$ with augmentation ideal $I$. We also make the following assumptions:

1. The classifying space $BM$ of $M$ has a finite fundamental group.
2. The augmentation ideal $I$ is finitely generated as a left $R$-module.
3. There is a projective resolution $P = \cdots P_2 \to P_1 \to P_0$ of $k$ over $R$ such that every $P_n$ is finitely generated as an $R$-module.

Let $T$ be the hereditary torsion class of $I$-torsion $R$-modules, then $\langle T \rangle = \langle k \rangle$ and hence $\text{Cell}_T \cong \text{Cell}_k$. Denote by $R^\vee$ the left $R$-module $\text{Hom}_k(R, k)$. From the results of Dwyer, Greenlees and Iyengar [4 6.15 and 7.5] it is easy to conclude that $\text{Cell}_k R^\vee$ is quasi-isomorphic to the cochain complex (with coefficients in $k$) of a certain space we describe next. Let $(BM)^{\wedge}_p$ be the Bousfield-Kan $p$-completion of the classifying space of $M$. The space $\Omega(BM)^{\wedge}_p$ is the loop-space of $(BM)^{\wedge}_p$. So, $\text{Cell}_k R^\vee$ is quasi-isomorphic to $C^*(\Omega(BM)^{\wedge}_p; k)$ - the singular cochain complex of $\Omega(BM)^{\wedge}_p$ with coefficients in $k$. By
Theorem 1.1, there exists an injective $R$-module $E$ such that
$$H^n(\Omega(BM)_{p};k) \cong \operatorname{Ext}_{\operatorname{End}_R(E)}^{n-1}(\operatorname{Hom}_R(R^\ell,E),E)$$
for $n \geq 2$.

REFERENCES

1. D. J. Benson, *Representations and cohomology. I*, Cambridge Studies in Advanced Mathematics, vol. 30, Cambridge University Press, Cambridge, 1991, Basic representation theory of finite groups and associative algebras. MR MR1110581 (92m:20005)

2. Dave Benson, *An algebraic model for chains on $\Omega BG_{p}$*, Trans. Amer. Math. Soc. 361 (2009), no. 4, 2225–2242. MR MR2465835 (2009k:55014)

3. W. G. Dwyer and J. P. C. Greenlees, *Complete modules and torsion modules*, Amer. J. Math. 124 (2002), no. 1, 199–220. MR MR1879003 (2003g:55014)

4. W. G. Dwyer, J. P. C. Greenlees, and S. Iyengar, *Duality in algebra and topology*, Adv. Math. 200 (2006), no. 2, 357–402. MR MR2200850 (2006k:55017)

5. William G. Dwyer, *Noncommutative localization in homotopy theory*, Non-commutative localization in algebra and topology, London Math. Soc. Lecture Note Ser., vol. 330, Cambridge Univ. Press, Cambridge, 2006, pp. 24–39. MR MR2222480 (2007g:55010)

6. Emmanuel Dror Farjoun, *Cellular spaces, null spaces and homotopy localization*, Lecture Notes in Mathematics, vol. 1622, Springer-Verlag, Berlin, 1996. MR 1392221 (98f:55010)

7. Philip S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, vol. 99, American Mathematical Society, Providence, RI, 2003. MR MR1944041 (2003j:55018)

8. Haynes Miller, *Finite localizations*, Bol. Soc. Mat. Mexicana (2) 37 (1992), no. 1-2, 383–389, Papers in honor of José Adem (Spanish). MR 1317588 (96k:55009)

9. Amnon Neeman, *The chromatic tower for $D(R)$*, Topology 31 (1992), no. 3, 519–532, With an appendix by Marcel Bökstedt. MR MR1174255 (93h:55018)

10. , *Triangulated categories*, Annals of Mathematics Studies, vol. 148, Princeton University Press, Princeton, NJ, 2001. MR MR1812507 (2001k:55010)

11. Bo Stenström, *Rings of quotients*, Springer-Verlag, New York, 1975, Die Grundlehren der Mathematischen Wissenschaften, Band 217, An introduction to methods of ring theory. MR 0389953 (52 #10782)

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