New result and some open problems on the primitive degree of nonnegative tensors

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Abstract

In this paper, we show that the exponent set of nonnegative primitive tensors with order \( m \geq 3 \) and dimension \( n \) is \( \{1, 2, \ldots, (n-1)^2 + 1\} \), and propose some open problems for further research.

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1 Introduction and a survey

A nonnegative square matrix \( A = (a_{ij}) \) of order \( n \) is nonnegative primitive (or simply, primitive) if \( A^k > 0 \) for some positive integer \( k \). The least such \( k \) is called the primitive exponent (or simply, exponent) of \( A \) and is denoted by \( \gamma(A) \).

Since the work of Qi [9] and Lim [6], the study of tensors which regarded as the generalization of matrices, the spectra of tensors (and hypergraphs) and their various applications has attracted much attention and interest.

As is in [9], an order \( m \) dimension \( n \) tensor \( \mathbb{A} = (a_{i_1i_2...i_m})_{1 \leq i_j \leq n} \) over the complex field \( \mathbb{C} \) is a multidimensional array with all entries \( a_{i_1i_2...i_m} \in \mathbb{C} \) \((1, \ldots, i_m \in [n] = \{1, \ldots, n\})\).

In [1] and [2], Chang et al investigated the properties of the spectra of nonnegative tensors, defined the irreducibility of tensors and the primitivity of nonnegative tensors (as Definition 1.1), and extended many important properties of primitive matrices to primitive tensors.

Definition 1.1. (2, 5) Let \( \mathbb{A} \) be a nonnegative tensor with order \( m \) and dimension \( n \), \( x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n \) a vector and \( x^{[r]} = (x_1^r, x_2^r, \ldots, x_n^r)^T \). Define the map \( T_{\mathbb{A}} \) from \( \mathbb{R}^n \) to \( \mathbb{R}^n \) as: \( T_k(x) = (Ax)^{[\frac{k}{m-1}]} \). If there exists some positive integer \( r \) such that \( T_k^{[r]}(x) > 0 \) for all nonnegative nonzero vectors \( x \in \mathbb{R}^n \), then \( \mathbb{A} \) is called primitive and the smallest such integer \( r \) is called the primitive degree of \( \mathbb{A} \), denoted by \( \gamma(\mathbb{A}) \).

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Recently, Shao [12] defined the general product of two n-dimensional tensors as follows.

**Definition 1.2.** Let $A$ (and $B$) be an order $m \geq 2$ (and $k \geq 1$), dimension $n$ tensor, respectively. Define the general product $AB$ to be the following tensor $D$ of order $(m-1)(k-1)+1$ and dimension $n$:

$$d_{\alpha_1...\alpha_{m-1}} = \sum_{i_2,...,i_m=1}^{n} a_{ii_2...i_m} b_{i_2\alpha_1} ... b_{i_m\alpha_{m-1}} \quad (i \in [n], \alpha_1, \ldots, \alpha_{m-1} \in [n]^{k-1}).$$

The tensor product is a generalization of the usual matrix product, and satisfies a very useful property: the associative law ([12], Theorem 1.1). With the general product, when $k = 1$ and $B = x = (x_1, \ldots, x_n)^T \in \mathbb{C}^n$ is a vector of dimension $n$, then $AB = Ax$ is still a vector of dimension $n$, and for any $i \in [n]$,

$$(AB)_i = (Ax)_i = \sum_{i_2,...,i_m=1}^{n} a_{ii_2...i_m} x_{i_2} \ldots x_{i_m}.$$  

As an application of the general tensor product defined by Shao [12], Shao presented a simple characterization of the primitive tensors. Now we give the definition of “essentially positive” which introduced by Pearson.

**Definition 1.3.** ([7], Definition 3.1) A nonnegative tensor $A$ is called essentially positive, if for any nonnegative nonzero vector $x \in \mathbb{R}^n$, $Ax > 0$ holds.

**Proposition 1.4.** ([12], Proposition 4.1) Let $A$ be an order $m$ and dimension $n$ nonnegative tensor. Then the following three conditions are equivalent:

(i). For any $i, j \in [n], a_{ij...j} > 0$ holds.

(ii). For any $j \in [n], Ae_j > 0$ holds (where $e_j$ is the $j^{th}$ column of the identity matrix $I_n$).

(iii). For any nonnegative nonzero vector $x \in \mathbb{R}^n, Ax > 0$ holds.

By Proposition 1.4, the following Definition 1.5 is equivalent to Definition 1.3.

**Definition 1.5.** ([12], Definition 4.1) A nonnegative tensor $A$ is called essentially positive, if it satisfies one of the three conditions in Proposition 1.4.

Let $Z(A)$ be the tensor obtained by replacing all the nonzero entries of $A$ by one. Then $Z(A)$ is called the zero-nonzero pattern of $A$ (or simply the zero pattern of $A$). In [12], Shao showed the following characterization and defined the primitive degree by using the properties of tensor product and the zero patterns.

**Proposition 1.6.** ([12], Theorem 4.1) A nonnegative tensor $A$ is primitive if and only if there exists some positive integer $r$ such that $A^r$ is essentially positive. Furthermore, the smallest such $r$ is the primitive degree of $A$, $\gamma(A)$.

The concept of the majorization matrix of a tensor introduced by Pearson is very useful.

**Definition 1.7.** ([7], Definition 2.1) The majorization matrix $M(A)$ of the tensor $A$ is defined as $(M(A))_{ij} = a_{ij...j}, i, j \in [n]$. 

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By Definition 1.5, Proposition 1.6 and Definition 1.7 the following characterization of the primitive tensors was easily obtained.

**Proposition 1.8.** ([13], Remark 2.6) Let \( A \) be a nonnegative tensor with order \( m \) and dimension \( n \). Then \( A \) is primitive if and only if there exists some positive integer \( r \) such that \( M(A^r) > 0 \). Furthermore, the smallest such \( r \) is the primitive degree of \( A \), \( \gamma(A) \).

On the primitive degree \( \gamma(A) \), Shao proposed the following conjecture for further research.

**Conjecture 1.9.** ([12], Conjecture 1) When \( m \) is fixed, then there exists some polynomial \( f(n) \) on \( n \) such that \( \gamma(A) \leq f(n) \) for all nonnegative primitive tensors of order \( m \) and dimension \( n \).

In the case of \( m = 2 \) (\( A \) is a matrix), the well-known Wielandt’s upper bound tells us that we can take \( f(n) = (n - 1)^2 + 1 \).

Recently, the authors [13] confirmed Conjecture 1.9 by proving Theorem 1.10.

**Theorem 1.10.** ([13], Theorem 1.2) Let \( A \) be a nonnegative primitive tensor with order \( m \) and dimension \( n \). Then its primitive degree \( \gamma(A) \) \( \leq (n - 1)^2 + 1 \), and the upper bound is tight.

In fact, Theorem 1.10 not only confirmed the existence of \( f(n) \), but also showed that \( f(n) \) is a quadratic function of \( n \), independent of \( m \), furthermore the expression of \( f(n) \) with the tensor case is the same with the matrix case.

Let \( m, n \) be positive integers with \( m \geq 2, n \geq 2 \), and the exponent set of primitive tensors with order \( m \) and dimension \( n \) be 

\[ E(m, n) = \{k \mid \text{there exists a primitive tensor } A \text{ of order } m \text{ dimension } n \text{ such that } k = \gamma(A)\} \]

Now it is natural to consider the question to completely determine \( E(m, n) \).

By Theorem 1.10 we know \( E(m, n) \subseteq [(n - 1)^2 + 1] \). Clearly, \( E(2, n) = E_n \), where \( E_n = \{k \mid \text{there exists a primitive matrix } A \text{ of order } n \text{ such that } \gamma(A) = k\} \).

Let \( A = (a_{ij}) \) be a nonnegative primitive matrix of order \( n \). In 1950, H. Wielandt [10] first stated the sharp upper bound for \( \gamma(A) \), that is, \( \gamma(A) \leq w_n = (n - 1)^2 + 1 \) for all primitive matrices of order \( n \) and thus \( E_n \subseteq [1, w_n]^o \), where \( a, b \) are positive integers with \( b \geq a \) and \( [a, b]^o = \{k \mid k \text{ is an integer and } a \leq k \leq b\} \). In 1964, A. L. Dulmage and N. S. Mendelsohn [11] revealed the existence of the so-called gaps in the exponent set of primitive matrices, that is, \( E_n \subseteq [1, w_n]^o \), where “gap” is a set of consecutive integers \( [a, b]^o \subset [1, w_n]^o \), such that no matrix \( A \) of order \( n \) satisfying \( \gamma(A) \in [a, b]^o \). In 1981, M. Lewin and Y. Vitek [30] found all gaps in \( [1, \frac{1}{2} w_n] + 1, w_n]^o \), and conjectured that \( [1, \frac{1}{2} w_n]^o \) has no gaps, where \( \lceil x \rceil \) denotes the greatest integer \( \leq x \). In 1985, Shao [11] proved that this Lewin-Vitek Conjecture is true for all sufficiently large \( n \), and the conjecture has one counterexample when \( n = 11 \) since \( 48 \notin E_{11} \). Finally, in 1987, Zhang [14] continued and completed the work. He showed that the Lewin-Vitek Conjecture holds for all \( n \) except \( n = 11 \). Thus the exponent set \( E_n \) for primitive matrices of order \( n \) is completely determined.

For general tensors (the case of \( m \geq 3 \)), in [3], the authors showed that there are no gaps in tensor case when \( m \geq n \geq 3 \). but the tensor cases \( n > m \geq 3 \) and \( m > n = 2 \) are still open.

In this paper, we show that there are no gaps in tensor case \( m \geq 3 \) in Section 3, that is, \( E(m, n) = [(n - 1)^2 + 1] = \{1, 2, \ldots, (n - 1)^2 + 1\} \) when \( m \geq 3 \), and propose some open problems for further research in Section 4.
2 Preliminaries

For proving Conjecture 1.9, the authors [13] defined $j$-primitive and $j$-primitive degree for a nonnegative tensor and obtained the following result.

**Definition 2.1.** ([13], Definition 2.13) Let $\mathbb{A}$ be a nonnegative tensor with order $m$ and dimension $n$. For a fixed integer $j \in [n]$, if there exists a positive integer $k$ such that

$$(M(\mathbb{A}^k))_{u_j} > 0, \text{ for all } u \in [n],$$

then $\mathbb{A}$ is called $j$-primitive and the smallest such integer $k$ is called the $j$-primitive degree of $\mathbb{A}$, denoted by $\gamma_j(\mathbb{A})$.

**Proposition 2.2.** ([13], Proposition 2.14) Let $\mathbb{A}$ be a nonnegative primitive tensor with order $m$ and dimension $n$. Then $\gamma(\mathbb{A}) = \max_{1 \leq j \leq n} \{\gamma_j(\mathbb{A})\}$.

Let $\mathbb{A}$ be a nonnegative tensor with order $m$ and dimension $n$. For positive integers $k$ and $j \in [n]$, a notation $S_k(\mathbb{A}, j)$ was introduced in [3] as follows:

$$S_k(\mathbb{A}, j) = \{u \in [n] \mid (M(\mathbb{A}^k))_{u_j} > 0\}, \quad k = 1, 2, \ldots. \quad (2.1)$$

and by equation

$$(M(\mathbb{A}^{k+1}))_{u_j} = \sum_{i_2, \ldots, i_m = 1}^n a_{u_1 \ldots i_m} (M(\mathbb{A}^k))_{i_2j} \cdots (M(\mathbb{A}^k))_{i_mj},$$

which investigated in [13], the following recurrence relation (2.2) was obtained as follows:

$$S_{k+1}(\mathbb{A}, j) = \{u \in [n] \mid \text{there exist } i_2, \ldots, i_m \in S_k(\mathbb{A}, j) \text{ and } a_{u_1i_2 \ldots i_m} > 0\}. \quad (2.2)$$

According to this relation (2.2), some good properties were obtained.

**Proposition 2.3.** ([3], Lemma 3.1 and Remark 3.2) Let $\mathbb{A}$ be a nonnegative tensor with order $m$ and dimension $n$.

(i) Let $k, l, i, j$ be positive integers such that $1 \leq i, j \leq n$. Suppose that $S_k(\mathbb{A}, i) = S_l(\mathbb{A}, j)$, then $S_{k+r}(\mathbb{A}, i) = S_{l+r}(\mathbb{A}, j)$ holds for every positive integer $r$.

(ii) For any $j \in [n]$, let $k$ be the least positive integer such that $S_k(\mathbb{A}, j) = [n]$. Then for any integer $l \geq k$, $S_l(\mathbb{A}, j) = [n]$.

(iii) For any $j \in [n]$, $\gamma_j(\mathbb{A})$ is the least positive integer $k$ satisfying $S_k(\mathbb{A}, j) = [n]$.

**Remark 2.4.** For convenience, we replace $S_l(\mathbb{A}, n)$ by the notation $S_l(\mathbb{A}, 0)$, and replace $\gamma_n(\mathbb{A})$ by $\gamma_0(\mathbb{A})$, respectively. Under these notations, we have $\gamma(\mathbb{A}) = \max_{0 \leq j \leq n-1} \{\gamma_j(\mathbb{A})\}$ by Proposition 2.2.

In [13], the authors introduce some theoretical concepts of digraphs and matrices.

Let $D = (V, E)$ denote a digraph on $n$ vertices with vertex set $V(D) = V$ and arc set $E(D) = E$. Loops are permitted, but no multiple arcs. A $u \to v$ walk in $D$ is a sequence of vertices $u, u_1, \ldots, u_k = v$ and a sequence of arcs $e_1 = (u, u_1), e_2 = (u_1, u_2), \ldots, e_k = (u_{k-1}, v)$, where the vertices and the arcs are not necessarily distinct. We use the notation $u \to u_1 \to u_2 \to \cdots \to u_{k-1} \to v$ to refer to this $u \to v$ walk. A closed walk is a $u \to v$ walk where $u = v$. A path is a walk with distinct vertices. A cycle is a closed $u \to v$ walk with distinct vertices except for $u = v$. The length of a walk $W$ is the number of arcs in $W$, denoted by $l(W)$. 

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Definition 2.5. ([13], Definition 2.9) Let $D = (V, E)$ denote a digraph on $n$ vertices. A digraph $D' = (V, E')$ is called the reversed digraph of $D$ where $(j, i) \in E'$ if and only if $(i, j) \in E$ for any $i, j \in V$, denoted by $\overrightarrow{D}$.

Let $A = (a_{ij})$ be a square nonnegative matrix of order $n$. The associated digraph $D(A) = (V, E)$ of $A$ (possibly with loops) is defined to be the digraph with vertex set $V = \{1, 2, \ldots, n\}$ and arc set $E = \{(i, j)|a_{ij} \neq 0\}$. The associated reversed digraph $\overrightarrow{D}(A) = (V, E')$ of $A$ (possibly with loops) is defined to be the digraph with vertex set $V = \{1, 2, \ldots, n\}$ and arc set $E' = \{(j, i)|a_{ij} \neq 0\}$. Clearly, the associated reversed digraph of $A$ is the reversed digraph of the associated digraph of $A$.

Proposition 2.6. ([3], Proposition 4.1) Let $A$ be a nonnegative matrix of order $n$, $j(\in [n]), k$ be positive integers, $S_k(A, j) = \{u \in [n]|(A^k)_{uj} > 0\}$. Then

$$S_k(A, j) = \{u \in [n]| \text{there exists a walk of length } k \text{ from } j \text{ to } u \text{ in the digraph } \overrightarrow{D(A)}\}.$$  

Proposition 2.7. ([3], Lemma 4.2) Let $\mathbb{A}$ be a nonnegative tensor with order $m$ and dimension $n$ such that $a_{i_1 i_2 \ldots i_m} = 0$ if $i_2 \cdots i_m \neq i_2 \cdots i_2$ for any $i \in [n]$. Then for any positive integers $j(\in [n])$ and $k$,

$$S_k(\mathbb{A}, j) = S_k(M(\mathbb{A}), j). \quad (2.3)$$

The relations between a tensor $\mathbb{A}$ and the majorization matrix $M(\mathbb{A})$ are important and useful.

Proposition 2.8. ([12], Corollary 4.1) Let $\mathbb{A}$ be a nonnegative tensor with order $m$ and dimension $n$. If $M(\mathbb{A})$ is primitive, then $\mathbb{A}$ is also primitive.

Proposition 2.9. ([13], Corollary 3.4) Let $\mathbb{A}$ be a nonnegative primitive tensor with order $m$ and dimension $n$ such that $a_{i_1 i_2 \ldots i_m} = 0$ if $i_2 \cdots i_m \neq i_2 \cdots i_2$ for any $i \in [n]$. If $M(\mathbb{A})$ is primitive, then $\gamma(\mathbb{A}) = \gamma(M(\mathbb{A}))$.

Now we define the nonnegative tensor $\mathbb{A}_0 = (a_{i_1 i_2 \ldots i_m})_{1 \leq i_j \leq n}$ ($j=1, \ldots, m$) with order $m$ and dimension $n$ such that

(i). The majorization matrix $M(\mathbb{A}_0)$ is given by

$$M_1 = \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 & 1 \\
1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0 \\
0 & 0 & \cdots & 0 & 1 & 0
\end{pmatrix}.$$  

(ii). $a_{i_1 i_2 \ldots i_m} = 0$, if $i_2 \cdots i_m \neq i_2 \cdots i_2$ for any $i \in [n]$.

It is well-known that $M_1$ is primitive and the primitive exponent $\gamma(M_1) = (n - 1)^2 + 1$. Then by Propositions 2.8 $\sim$ 2.9 it can easily be seen that the tensor $\mathbb{A}_0$ is primitive and its primitive degree $\gamma(\mathbb{A}_0) = (n - 1)^2 + 1$. Furthermore, there are more good properties on tensor $\mathbb{A}_0$ as follows.

Proposition 2.10. ([3], Proposition 4.5) Let $\mathbb{A}_0$ be the nonnegative primitive tensor with order $m$ and dimension $n$ defined as above. Then $\gamma_{n-1}(\mathbb{A}_0) = n^2 - 3n + 3$. 

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Proposition 2.11. ([3], Remark 4.6 and Proposition 4.9) Let $k$ be a positive integer with $1 \leq k \leq n^2 - 3n + 2$. Then $S_1(\mathbb{A}_0, n-1), S_2(\mathbb{A}_0, n-1), \ldots, S_k(\mathbb{A}_0, n-1)$ are pairwise distinct proper subsets of $[n]$, and $S_{n^2-3n+2}(\mathbb{A}_0, n-1) = [n-1]$.

For convenience, we let $\lfloor a \rfloor_n$ denote the least positive integer $t$ with $t \equiv a \pmod{n}$, and let a set $S = \{a_1, \ldots, a_s\}$ (mod $n$) denote the set $S = \{\lfloor a_1 \rfloor_n, \ldots, \lfloor a_s \rfloor_n\}$.

Proposition 2.12. Let $k$ be a positive integer and $1 \leq k \leq n^2 - 3n + 2$.

(i) If $k = (n-1)q + r$ with $q \geq 0$ and $1 \leq r \leq n-1$, then

$$S_k(\mathbb{A}_0, n-1) = \{r - q - 1, \ldots, r - 1, r\} \pmod{n} \quad (2.4)$$

and $|S_k(\mathbb{A}_0, n-1)| = q + 2$.

(ii) Let $t(\geq 1), j(\geq 0)$ be integers. Then for any positive integer $t$ and any integer $j \in \{0, 1, \ldots, n-2\}$, we have $S_{t+(n-1-j)}(\mathbb{A}_0, j) = S_t(\mathbb{A}_0, n-1)$.

Proof. (i). By the definition of $\mathbb{A}_0$ and Propositions 2.6–2.7, we have

$$S_k(\mathbb{A}_0, n-1) = S_k(M(\mathbb{A}_0), n-1) \equiv \{u \in [n] \mid \text{there exists a walk of length } k \text{ from } n-1 \text{ to } u \text{ in the digraph } \overrightarrow{D(M(\mathbb{A}_0))}\}.$$

Now we can obtain the conclusion (i) from the digraph $\overrightarrow{D(M(\mathbb{A}_0))} = \overrightarrow{D(M_1)}$ (See Figure 1) by simple calculation.

![Figure 1. digraph $\overrightarrow{D(M(\mathbb{A}_0))}$](image)

(ii). By the definition of $\mathbb{A}_0$, an easy computation shows that

$$\{n, 1\} = S_1(\mathbb{A}_0, n-1) = S_2(\mathbb{A}_0, n-2) = \cdots = S_{n-1}(\mathbb{A}_0, 1) = S_n(\mathbb{A}_0, 0).$$

It implies that

$$S_1(\mathbb{A}_0, n-1) = S_{n-j}(\mathbb{A}_0, j) \text{ for any } j \in \{0, 1, \ldots, n-2\}. \quad (2.5)$$

Then for any integer $r \geq 1$, by (i) of Proposition 2.3, we have

$$S_{1+r}(\mathbb{A}_0, n-1) = S_{n-j+r}(\mathbb{A}_0, j) \text{ for any } j \in \{0, 1, \ldots, n-2\}. \quad (2.6)$$

Combining (2.5) and (2.6), (ii) holds. \qed

Remark 2.13. For convenience, we regard a set $\{n, 1\}$ as a consecutive positive integers subset of $\{1, 2, \ldots, n\}$. Then it implies that $\{n-1, n, 1\}$, $\{n, 1, 2\}$, $\{n, 1, 2, 3\}$ and so on, and thus $S_k(\mathbb{A}_0, n-1)$ are consecutive positive integers subsets of $\{1, 2, \ldots, n\}$ for all $k \in [n^2 - 3n + 2]$ by the result (i) of Proposition 2.12.
3 The exponent set of nonnegative primitive tensors

In this section, we will show that there are no gaps in tensor case \( m \geq 3 \), that is, \( E(m,n) = [(n-1)^2 + 1] \) when \( m \geq 3 \). It implies that the results of the case \( m \geq 3 \) is totally different from the case \( m = 2 \).

Let \( s_1, s_2, \ldots, s_l \) be \( l \) elements which are not necessarily to be distinct. For convenience, we let \( \{s_1, s_2, \ldots, s_l\} \) denote the set of all different elements from \( s_1, s_2, \ldots, s_l \).

Let \( k (\geq 1), n (\geq 2), q (\geq 0), r (\geq 1) \) be integers, \( A_0 = (a_{i_1i_2\ldots i_m})_{1 \leq i_j \leq n} \) be a nonnegative tensor with order \( m \) and dimension \( n \) defined in Section 2, \( k = (n-1)q + r \) with \( 1 \leq r \leq n - 1 \), then \( 0 \leq q \leq n - 3 \). We define the nonnegative tensor \( A_k = (a^{(k)}_{i_1i_2\ldots i_m})_{1 \leq i_j \leq n} \) with order \( m \) and dimension \( n \) such that

(i). \( M(A_k) = M(A_0) = M_1 \);
(ii). \( a^{(k)}_{i\alpha} = 1 \) if \( i \in [n] \{ r - q, r - q + 1, \ldots, r, r + 1 \} \) (mod \( n \)) and \( \alpha = i_2 \cdots i_m \in [n]^{m-1} \) with \( \{i_2, \ldots, i_m\} = \{r - q, r\} \) (mod \( n \));
(iii). \( a^{(k)}_{i_1i_2\ldots i_m} = 0 \), except for (i) and (ii).

By Proposition 2.8, it’s easy to see the tensor \( A_k \) is primitive from the fact that \( M(A_k) = M(A_0) = M_1 \) is primitive. Now we investigate the relations between the tensors \( A_k \) and \( A_0 \).

**Proposition 3.1.** Let \( k (\geq 1), n (\geq 2), q (\geq 0), r \) be integers, and \( k = (n-1)q + r \in [n^2 - 3n + 2] \) with \( 1 \leq r \leq n - 1 \), \( A_0 \) and \( A_k \) defined as above. Then for any integer \( t \in [k] \), we have

\[
S_t(A_k, n - 1) = S_t(A_0, n - 1). 
\] (3.1)

**Proof.** Now we prove (3.1) holds for any \( t \in [k] \) by induction on \( t \).

It’s obvious that \( S_1(A_k, n - 1) = S_1(A_0, n - 1) = \{1, n\} \) from (2.1) and \( M(A_k) = M(A_0) \). Assume that \( S_t(A_k, n - 1) = S_t(A_0, n - 1) \) holds for \( t < k \). Now we only need show that \( S_{t+1}(A_k, n - 1) = S_{t+1}(A_0, n - 1) \) holds.

**Case 1:** \( \{r - q - 1, r\} \) (mod \( n \) \( \not\subseteq S_t(A_0, n - 1) \).

We note that the recurrence relation (2.2) and the definition of \( A_k \), then

\[
S_{t+1}(A_k, n - 1) = \{u \in [n] \mid \ exists \ i_2, \ldots, i_m \in S_t(A_k, n - 1) and \ a_{u_i\ldots i_m}^{(k)} > 0\}
\]

\[
= \{u \in [n] \mid \ exists \ i_2, \ldots, i_m \in S_t(A_0, n - 1) and \ a_{ui\ldots i_m}^{(k)} > 0\}
\]

\[
= \{u \in [n] \mid \ exists \ v \in S_t(A_0, n - 1) and \ a_{uv\ldots v} > 0\}
\]

\[
\cup \{u \in [n] \mid \ exists \ i_2, \ldots, i_m \in S_t(A_0, n - 1) such \ that \ i_2 \cdots i_m \neq i_2 \cdots i_2 and \ a_{ui\ldots i_m}^{(k)} > 0\}
\]

\[
= S_{t+1}(A_0, n - 1) \cup \emptyset
\]

\[
= S_{t+1}(A_0, n - 1).
\]

**Case 2:** \( \{r - q - 1, r\} \) (mod \( n \) \( \subseteq S_t(A_0, n - 1) \).

From the Remark 2.13 we know that \( S_t(A_0, n - 1) \) is a consecutive positive integers subset of \( \{1, 2, \ldots, n\} \). Thus

\[
\{r - q - 1, r - q, \ldots, r\} \pmod{n} \subseteq S_t(A_0, n - 1)
\]

or

\[
\{r, r + 1, \ldots, r - q - 1\} \pmod{n} \subseteq S_t(A_0, n - 1)
\]

holds.
If \( \{r-q-1, r-q, \ldots, r\} \pmod{n} \subseteq S_t(A_0, n-1) \), then \( S_k(A_0, n-1) \subseteq S_t(A_0, n-1) \) and 
\[ |S_t(A_0, n-1)| = \left\lfloor \frac{k}{n} \right\rfloor + 2 \leq |S_k(A_0, n-1)| = q + 2 \] by the result (i) of Proposition 2.12 and 
\( t < k \). Then \( S_t(A_0, n-1) = S_k(A_0, n-1) \), it implies a contradiction since \( S_t(A_0, n-1) \neq S_k(A_0, n-1) \) by \( t < k \) and Proposition 2.11. It follows that 
\[ \{r, r+1, \cdots, r-q-1\} \pmod{n} \subseteq S_t(A_0, n-1). \]
Thus by the definition of \( A_0 \) and Propositions \ref{prop:2.6} \( \sim \) \ref{prop:2.7} we have 
\[ \{r+1, r+2, \cdots, r-q\} \pmod{n} \subseteq S_{t+1}(A_0, n-1). \] (3.2)

Then by the definition of \( A_k \), the assumption and (3.2), we can see that 
\[
S_{t+1}(A_k, n-1) = \{u \in [n] | \text{there exist } i_2, \ldots, i_m \in S_t(A_k, n-1) \text{ and } a^{(k)}_{u_{i_2} \cdots i_m} > 0\}
\[= \{u \in [n] | \text{there exist } i_2, \ldots, i_m \in S_t(A_0, n-1) \text{ and } a^{(k)}_{u_{i_2} \cdots i_m} > 0\}
\[= \{u \in [n] | \text{there exist } v \in S_t(A_0, n-1) \text{ and } a_{uv \cdots v} > 0\}
\[\cup \{u \in [n] | \text{there exist } i_2, \ldots, i_m \in S_t(A_0, n-1) \text{ such that } i_2 \cdots i_m \neq i_2 \cdots i_2 \text{ and } a^{(k)}_{u_{i_2} \cdots i_m} > 0\}
\[= S_{t+1}(A_0, n-1) \cup [n] \setminus \{r-r-q, r-q+1, \ldots, r+1\} \pmod{n}\]
\[\subseteq S_{t+1}(A_0, n-1).\]

Combining the above two cases, the equation (3.1) is proved for all \( t \in [k] \). \( \Box \)

**Proposition 3.2.** Let \( k \geq 1 \), \( n \geq 2 \), \( q \geq 0 \), \( r \) be integers, and \( k = (n-1)q + r \in [n^2 - 3n + 2] \)
with \( 1 \leq r \leq n - 1 \), \( A_0 \) and \( A_k \) defined as above. Then for any integer \( t \in [k] \), we have 
\[ S_t(A_k, j) = S_t(A_0, j), \text{where } j \in \{0,1,\ldots,n-2\} \] (3.3)

**Proof.** Now we show (3.3) holds for any \( t \in [k] \) by the following two cases.

**Case 1:** \( 1 \leq t \leq n - 1 \).

Now we prove (3.3) holds for any \( t \) (\( 1 \leq t \leq n - 1 \)) by induction on \( t \).

Clearly, \( S_1(A_k, j) = \{j + 1\} = S_1(A_0, j) \) for any \( j \in \{0,1,2,\ldots,n-2\} \). Now we assume that \( S_t(A_k, j) = S_t(A_0, j) \) holds for any \( j \in \{0,1,2,\ldots,n-2\} \) and \( t < n - 1 \). Then we show that \( S_{t+1}(A_k, j) = S_{t+1}(A_0, j) \) holds for any \( j \in \{0,1,2,\ldots,n-2\} \).

By observing and the result (ii) of Proposition 2.12 we know for any \( t \) with \( 2 \leq t \leq n - 1 \),
\[
S_t(A_0, j) = \left\{ \begin{array}{ll}
(j + t), & j \in \{0,1,\ldots,n-t-1\}; \\
S_{t-(n-1-j)}(A_0, n-1) = \{t + j - n, t + j - n + 1\} \pmod{n}, & j \in \{n-t,\ldots,n-2\}.
\end{array} \right.
\]

We note that \( (t + j - n + 1) - (t + j - n) \equiv 1 \pmod{n} \), but \( (r - q - 1) - r \not\equiv 1 \pmod{n} \)
by \( 0 \leq q \leq n - 3 \). Thus \( \{r - q - 1, r\} \pmod{n} \subseteq S_t(A_0, j) \) if and only if \( q = 0 \) and 
\( r = k \), then \( \{r - q - 1, r\} \pmod{n} = \{k - 1, k\} \pmod{n} \). It is a contradiction by the fact 
\( 1 \leq t + j - n + 1 < t \leq k \) for \( j \in \{n-t,\ldots,n-2\} \).

The above arguments imply that \( \{r-q-1, r\} \pmod{n} \not\subseteq S_t(A_0, j) \) for \( j = 0,1,\ldots,n-2 \).

Then by the definition of \( A_k \), and the assumption, we can see that
\[
S_{t+1}(A_k, j) = \{u \in [n] | \text{there exist } i_2, \ldots, i_m \in S_t(A_k, j) \text{ and } a^{(k)}_{u_{i_2} \cdots i_m} > 0\}
\[= \{u \in [n] | \text{there exist } i_2, \ldots, i_m \in S_t(A_0, j) \text{ and } a^{(k)}_{u_{i_2} \cdots i_m} > 0\}
\[\cup \{u \in [n] | \text{there exist } v \in S_t(A_0, j) \text{ and } a_{uv \cdots v} > 0\}
\[\cup \{u \in [n] | \text{there exist } i_2, \ldots, i_m \in S_t(A_0, j) \text{ such that } i_2 \cdots i_m \neq i_2 \cdots i_2 \text{ and } a^{(k)}_{u_{i_2} \cdots i_m} > 0\}
\]
\[ = S_{t+1}(\mathcal{A}_0, j) \cup \emptyset \]
\[ = S_{t+1}(\mathcal{A}_0, j). \]

**Case 2:** \( t \geq n. \)

Now we prove (3.3) holds for any \( n \leq t \leq k \) by induction on \( t. \)

By (ii) of Proposition 2.12, \( S_n(\mathcal{A}_0, j) = S_{j+1}(\mathcal{A}_0, n-1) = \{j, j+1\} \pmod{n} \) for any \( j \in \{0, 1, \ldots, n-2\}. \) On the other hand, by Proposition 2.7, the result of Case 1 and (2.2), we have
\[ S_{n-1}(\mathcal{A}_0, j) = \begin{cases} \{n-1\}, & j = 0; \\ \{j-1, j\} \pmod{n}, & 1 \leq j \leq n-2. \end{cases} \]
and
\[ S_n(\mathcal{A}_k, j) = \{u \in [n] \mid \text{there exist } i_2, \ldots, i_m \in S_{n-1}(\mathcal{A}_0, j) \text{ and } a^{(k)}_{ui_2 \cdots i_m} > 0\} \]
\[ = \{u \in [n] \mid \text{there exist } i_2, \ldots, i_m \in S_{n-1}(\mathcal{A}_0, j) \text{ and } a^{(k)}_{ui_2 \cdots i_m} > 0\} \]
\[ = \begin{cases} \{1, n\}, & j = 0; \\ \{j, j+1\} \pmod{n}, & 1 \leq j \leq n-2. \end{cases} \]
Thus \( S_n(\mathcal{A}_k, j) = S_n(\mathcal{A}_0, j). \)

Now we assume that \( S_t(\mathcal{A}_k, j) = S_t(\mathcal{A}_0, j) \) holds for \( n \leq t < k \) for any \( j \in \{0, 1, \ldots, n-2\}. \) Then we only need show that \( S_{t+1}(\mathcal{A}_k, j) = S_{t+1}(\mathcal{A}_0, j) \) holds for any \( j \in \{0, 1, \ldots, n-2\}. \) By the assumption, the fact that \( 1 \leq t - (n-1-j) < k, \) (ii) of Proposition 2.12 and Proposition 3.1, we have
\[ S_t(\mathcal{A}_k, j) = S_t(\mathcal{A}_0, j) = S_{t-(n-1-j)}(\mathcal{A}_0, n-1) = S_{t-(n-1-j)}(\mathcal{A}_k, n-1). \]
Thus by (i) of Proposition 2.3, Proposition 3.1 and (ii) of Proposition 2.12 we have
\[ S_{t+1}(\mathcal{A}_k, j) = S_{t+1-(n-1-j)}(\mathcal{A}_k, n-1) \]
\[ = S_{t+1-(n-1-j)}(\mathcal{A}_0, n-1) \]
\[ = S_{t+1}(\mathcal{A}_0, j). \]

Combining the above two cases, we complete the proof. \( \square \)

**Theorem 3.3.** Let \( k \) be a positive integer with \( 1 \leq k \leq n^2 - 3n + 2. \) Then
(i). \( \gamma_j(\mathcal{A}_k) = k + n - j \) where \( j = 0, 1, \ldots, n-1. \)
(ii). \( \gamma(\mathcal{A}_k) = k + n. \)

**Proof.** Firstly, we show \( \gamma_{n-1}(\mathcal{A}_k) = k + 1. \) By Proposition 3.1 and (i) of Proposition 2.12, we have
\[ S_k(\mathcal{A}_k, n-1) = S_k(\mathcal{A}_0, n-1) = \{r - q - 1, r - q, \ldots, r\} \pmod{n}. \]
And by Proposition 2.7 and (i) of Proposition 2.12 we have
\[ \{r - q, r - q + 1, \ldots, r + 1\} \pmod{n} \subseteq S_{k+1}(M(\mathcal{A}_0), n-1) = S_{k+1}(\mathcal{A}_0, n-1). \]
Thus by Proposition 3.1, we have
\[ S_{k+1}(\mathcal{A}_k, n-1) = \{u \in [n] \mid \text{there exist } i_2, \ldots, i_m \in S_k(\mathcal{A}_k, n-1) \text{ and } a^{(k)}_{ui_2 \cdots i_m} > 0\} \]
\[ = \{u \in [n] \mid \text{there exist } i_2, \ldots, i_m \in S_k(\mathcal{A}_0, n-1) \text{ and } a^{(k)}_{ui_2 \cdots i_m} > 0\} \]
\[ = \{u \in [n] \mid \text{there exist } v \in S_k(\mathcal{A}_0, n-1) \text{ and } a_{uv \cdots v} > 0\} \]
\[ \cup \{u \in [n] \mid \text{there exist } i_2, \ldots, i_m \in S_k(\mathcal{A}_0, n-1) \text{ such that} \]
\[ \cdots \]
Let by constructing primitive tensor $\gamma$ we have
\[
S_{k+1}(A_0, n - 1) \cup ([n] \setminus \{r - q, r - q + 1, \ldots, r + 1\}) \pmod{n}
\]

On the other hand, we have $S_t(A_k, n - 1) = S_t(A_0, n - 1) \notin [n]$ holds for all $1 \leq t \leq k$ by Proposition 3.1 and Proposition 2.11. Thus $\gamma_{n-1}(A_k) = k + 1$ by (iii) of Proposition 2.3.

Now we show $\gamma_j(A_k) = k + n - j$ for any $j \in \{0, 1, \ldots, n - 2\}$.

By Propositions 3.1–3.2, we have
\[
S_1(A_k, n - 1) = S_2(A_k, n - 2) = S_3(A_k, n - 3) = \cdots = S_{n-1}(A_k, 1) = S_n(A_k, n) = \{1, n\}.
\]

Thus by (i) of Proposition 2.3 and the definition of $j$-primitive degree, we have
\[
S_1(A_k, n - 1) = S_2(A_k, n - 2) \quad \Rightarrow \quad \left\{ \begin{array}{ll}
([n] \neq )S_{1+r}(A_k, n - 1) = S_{2+r}(A_k, n - 2), & \text{if } 1 \leq r \leq k - 1; \\
([n] = )S_{1+k}(A_k, n - 1) = S_{2+k}(A_k, n - 2), & \text{if } r = k.
\end{array} \right.
\]

By Proposition 2.2 or Remark 2.4 we have
\[
\gamma_j(A_k) = \max_{0 \leq j \leq n-1} \{\gamma_j(A_k)\} = \gamma_0(A_k) = k + n,
\]
and we complete the proof of (ii) immediately. \hfill \Box

**Theorem 3.4.** Let $m$, $n$ be positive integers with $m \geq 3$, $n \geq 2$. Then $E(m, n) = [(n-1)^2+1]$.

**Proof.** Let $t$ be any positive integer with $1 \leq t \leq (n - 1)^2 + 1$, we will complete the proof by constructing a primitive tensor $B_t$ with order $m$ and dimension $n$ such that $\gamma(B_t) = t$. We consider the following two cases.

**Case 1:** $1 \leq t \leq n$.

It is well known that there exists a primitive matrix $A_t$ of order $n$ such that $\gamma(A_t) = t$.

We define the tensor $B_t$ to be the nonnegative primitive tensor with order $m$ and dimension $n$ such that $(B_t)_{i_2 \cdots i_m} = 0$ if $i_2 \cdots i_m \neq i_2 \cdots i_2$ for any $i \in [n]$, and $M(B_t) = A_t$. Then $\gamma(B_t) = \gamma(A_t) = t$ by Proposition 2.9.

**Case 2:** $n + 1 \leq t \leq (n - 1)^2 + 1$.

We choose $B_t = A_{t-n}$. Then $\gamma(B_t) = (t - n) + n = t$ by Theorem 3.3. \hfill \Box

## 4 Some open problems for further research

In this section, we will propose some interesting open problems for further research.

Let $A$ be a nonnegative primitive tensor with order $m$ and dimension $n$. By Theorem 1.10, we have $\gamma(A) \leq (n-1)^2 + 1$. It is well-known that when $m = 2$, $\gamma(A) = (n-1)^2 + 1$ if and only if $A \cong M_1$. But the case of $m \geq 3$ is totally different. In fact, we know $A_{n^2-3n+2} \cong A_0$, but $\gamma(A_{n^2-3n+2}) = \gamma(A_0) = (n-1)^2 + 1$ by Proposition 2.9 and Theorem 3.3.

**Question 4.1.** Let $A$ be a nonnegative primitive tensor with order $m$ and dimension $n$. If $\gamma(A) = (n-1)^2 + 1$, can we give a characterization of such $A$?
Let $k$ be a positive integer with $1 \leq k \leq n^2 - 3n + 2$, $A_k$ defined as Section 3. Then by Theorem 3.3, we have

$$\{\gamma_1(A_k), \gamma_2(A_k), \ldots, \gamma_{n-1}(A_k), \gamma_n(A_k)\} = \{k + 1, k + 2, \ldots, k + n\} = [k + 1, k + n]^o. \tag{4.1}$$

Based on (4.1), we propose the following question.

**Question 4.2.** Let $\mathbb{A}$ be a nonnegative primitive tensor with order $m$ and dimension $n$. Does there exist two positive integers $a, b$ with $a < b$ such that $\{\gamma_1(\mathbb{A}), \gamma_2(\mathbb{A}), \ldots, \gamma_{n-1}(\mathbb{A}), \gamma_n(\mathbb{A})\} = [a, b]^o$?

Now we recall the definition of reducibility.

**Definition 4.3.** ([1], Definition 2.1) A tensor $\mathbb{C} = (c_{i_1 \ldots i_m})$ of order $m$ dimension $n$ is called reducible, if there exists a nonempty proper index subset $I \subset \{1, \ldots, n\}$ such that

$$c_{i_1 \ldots i_m} = 0 \quad \forall i_1 \in I, \forall i_2, \ldots, i_m \notin I.$$  

If $\mathbb{C}$ is not reducible, then we call $\mathbb{C}$ irreducible.

When $m = 2$, Definition 4.3 give the definition of reducible matrices and irreducible matrices. It is well-known from the basic relations between matrices and digraphs that a square matrix $A$ is irreducible if and only if its associated digraph $D(A)$ is strongly connected, a matrix $A$ is primitive if and only if $A$ is irreducible and the period of $A$ is 1. It is also well-known that $A$ is primitive if and only if $D(A)$ is primitive, and a digraph $D$ is primitive if and only if $D$ is strongly connected and the greatest common divisor of the lengths of all the cycles of $D$ is 1. Then by the definitions of primitive matrices, irreducible matrices and $j$-primitive, we obtain the following characterization.

**Proposition 4.4.** Let $A$ be a nonnegative matrix of order $n$. Then $A$ is primitive if and only if $A$ is irreducible and there exists some $j \in [n]$ such that $A$ is $j$-primitive.

**Proof.** Necessity is obvious. So we omit it. Now we only show sufficiency. Let $\gamma_j(A) = k$, it implies that $(A^k)_{u,v} > 0$ for any $u \in [n]$, thus there exists a walk of length $k$ from $u$ to $j$ in the associated digraph $D(A)$ by the relation between matrices and digraphs. Now we show there exists a positive integer $l$ such that $(A^l)_{u,v} > 0$ for any $u, v \in [n]$, that is, we need show there exists a walk of length $l$ from $u$ to $v$ in the associated digraph $D(A)$ for any $u, v \in V(D(A))$.

Since $A$ is irreducible, $D(A)$ is strongly connected. Then for any $v$, there exists a path $P$ with length $l_1 (0 \leq l_1 \leq n - 1)$ from $j$ to $v$. Similarly, there exists $w$ such that there exists a walk $Q$ with length $n - 1 - l_1$ from $u$ to $w$. Since $\gamma_j(A) = k$, then there exists a walk $R$ with length $k$ from $w$ to $j$. Thus the walk $Q + R + P$ is a walk of length $(n - 1 - l_1) + k + l_1 = k + n - 1$ from $u$ to $v$. Let $l = k + n - 1$, we complete the proof. \qed

Based on Proposition 4.4, we propose the following conjecture.

**Conjecture 4.5.** Let $\mathbb{A}$ be a nonnegative tensor with order $m$ and dimension $n$. Then $\mathbb{A}$ is primitive if and only if $\mathbb{A}$ is irreducible and there exists some $j \in [n]$ such that $\mathbb{A}$ is $j$-primitive.
Let \( m(\geq 2), n(\geq 2), j \in [n], k(\geq 1) \) be positive integers, \( A \) be a nonnegative tensor with order \( m \) and dimension \( n \). Recall the definitions of \( \gamma_j(A) \), \( S_k(A, j) \) and Proposition 2.3 we know \( \gamma_j(A) \) is the least positive integer satisfying \( S_k(A, j) = [n] \). If \( A \) is primitive, then \( 1 \leq \gamma_j(A) \leq (n - 1)^2 + 1 \) and the upper and lower bounds are tight by Theorem 1.10 and Proposition 2.2. But if \( A \) is not primitive, and there exists some \( j \in [n] \) and positive integer \( k \) such that \( S_k(A, j) = [n] \), that is, there exists some \( j \in [n] \) such that \( A \) is \( j \)-primitive, does \( \gamma_j(A) \) have a tight lower bound and a tight upper bound? It is obvious that \( \gamma_j(A) \geq 1 \). Now we only consider the upper bound of \( \gamma_j(A) \).

Let \( m, n, j(\in [n]) \) be positive integers with \( m \geq 2, n \geq 2 \), and notations \( r(m, n) = \max\{\gamma_j(A) \mid A \) is a nonnegative but not primitive tensor of order \( m \) dimension \( n \) and there exists some \( j \in [n] \) such that \( A \) is \( j \)-primitive \}. \( R_j(m, n) = \{k \mid \) there exists a nonnegative but not primitive tensor \( A \) of order \( m \) dimension \( n \) such that \( A \) is \( j \)-primitive and \( \gamma_j(A) = k \} \).

Now we investigate the properties of \( r(m, n) \) and \( R_j(m, n) \).

**Proposition 4.6.** \( r(2, n) = n^2 - 4n + 6 \).

**Proof.** Let \( A \) be a nonnegative matrix of order \( n \), but \( A \) is not primitive and there exists some \( j \in [n] \) such that \( A \) is \( j \)-primitive. Let \( \gamma_j(A) = k \), it implies that \( (A^k)_{uj} > 0 \) for any \( u \in [n] \), thus there exists a walk of length \( k \) from \( u \) to \( j \) in the associated digraph \( D = D(A) \) for any \( u \in [n] \) by the relation between matrices and digraphs.

Take \( u = j \), we know there exists a closed walk of length \( k \) from \( j \) to \( j \), it implies that \( j \) belongs in some strong connected subdigraph of \( D \). Without loss of generality, we assume a set \( V_1 \) is the maximal subset of \( V(D) \) such that \( j \in V_1 \subseteq V(D) \) and the induced subdigraph \( D_1 = D[V_1] \) is strong connected. Thus for any \( v \not\in V_1 \), there does not exist a walk from \( j \) to \( v \) in \( D \) since there exists a walk of length \( k \) from \( v \) to \( j \) by \( A \) is \( j \)-primitive.

Now we show the adjacent matrix \( A(D_1) \) is \( j \)-primitive. That is, for any \( u \in V_1 \), let \( W \) be a walk of length \( k \) from \( u \) to \( j \) in \( D \), we only need show \( V(W) \subseteq V_1 \). Otherwise, if there exists vertex \( v \in (V(D) \backslash V_1) \cap V(W) \), then there exists a walk from \( j \) to \( v \) since \( u, j \in V_1 \) and \( D_1 = D[V_1] \) is strong connected, and thus \( v \in V_1 \), it is a contradiction.

Therefore \( A(D_1) \) is primitive by Proposition 4.4 and thus there exists a positive integer \( l \leq (|V_1| - 1)^2 + 1 \) such that there exists a walk of length \( l \) from \( u \) to \( j \) for any \( u \in V_1 \) by the well-known Wieland’s upper bound.

Then \( k \leq n - |V_1| + l \leq |V_1|^2 - 3|V_1| + n + 2 \leq n^2 - 4n + 6 \) by \( |V_1| \leq n - 1 \), and thus \( r(2, n) \leq n^2 - 4n + 6 \) by the definition of \( r(2, n) \).

On the other hand, let \( M_2 \) be a nonnegative matrix of order \( n \) and \( D(M_2) \) be the associated digraph as follows. Clearly, \( M_2 \) and \( D(M_2) \) are not primitive.

\[
M_2 = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 & 0 \\
1 & 0 & 0 & \cdots & 0 & 1 & 0 \\
1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & 0
\end{pmatrix},
\]
exists an integer $i$ such that $\gamma_{n-1}(M_2) \geq n^2 - 4n + 6$.

Combining the above two inequalities, we obtain $r(2, n) = n^2 - 4n + 6$. \hfill \Box

The following necessary condition for a tensor to be primitive is useful.

**Proposition 4.7.** ([13], (i) of Proposition 2.7) Let $A$ be a nonnegative primitive tensor with order $m$ and dimension $n$, $M(A)$ the majorization matrix of $A$. Then for each $j \in [n]$, there exists an integer $i \in [n] \backslash \{j\}$ such that $(M(A))_{ij} > 0$.

**Proposition 4.8.** If $m \geq \lfloor \frac{n-1}{2} \rfloor + 1$, then

$$\left( \frac{n-1}{\lfloor \frac{n-1}{2} \rfloor} \right) + 1 \leq r(m, n) \leq 2^n - 1.$$

**Proof.** Let $S_1, S_2, \ldots, S_k$ be all the subset of $[n] \backslash \{1\}$ with $|S_1| = |S_2| = \ldots = |S_k| = \lfloor \frac{n-1}{2} \rfloor$. It is clear that $k = \left( \frac{n-1}{\lfloor \frac{n-1}{2} \rfloor} \right)$. Let $A = (a_{i_1 \ldots i_m})$ be a nonnegative tensor with order $m$ and dimension $n$, where

$$a_{i_1 i_2 \ldots i_m} = \begin{cases} 
1, & \text{if } i_1 \in S_1, \text{ and } i_2 = \ldots = i_m = 1; \\
1, & \text{if } i_1 \in S_{j+1}, \text{ and } \{i_2, \ldots, i_m\} = S_j \text{ for } j = 1, 2, \ldots, k-1; \\
1, & \text{if } i_1 \in [n], \text{ and } \{i_2, \ldots, i_m\} = S_k; \\
0, & \text{otherwise.} 
\end{cases}$$

By the above definition, we know for any $j \in [n] \backslash \{1\}$, $(M(A))_{ij} = 0$ for any $i \in [n]$. Then $A$ is not primitive by Proposition 4.7.

Now we show $\gamma_1(A) = k + 1$. By (2.1) and (2.2), we have

$$S_1(A, 1) = \{ u \in [n] \mid M(A)_{u1} = a_{u1 \ldots 1} > 0 \} = S_1,$$

$$S_2(A, 1) = \{ u \in [n] \mid \text{there exist } i_2, \ldots, i_m \in S_1(A, 1) \text{ and } a_{ui_2 \ldots im} > 0 \} = S_2,$$

$$\vdots$$

$$S_l(A, 1) = \{ u \in [n] \mid \text{there exist } i_2, \ldots, i_m \in S_{l-1}(A, 1) \text{ and } a_{ui_2 \ldots im} > 0 \} = S_l,$$

$$\vdots$$

$$S_k(A, 1) = \{ u \in [n] \mid \text{there exist } i_2, \ldots, i_m \in S_{k-1}(A, 1) \text{ and } a_{ui_2 \ldots im} > 0 \} = S_k,$$

$$S_{k+1}(A, 1) = \{ u \in [n] \mid \text{there exist } i_2, \ldots, i_m \in S_k(A, 1) \text{ and } a_{ui_2 \ldots im} > 0 \} = [n].$$

Then $\gamma_1(A) = k + 1$ by Proposition 2.3 and thus $r(m, n) \geq \left( \frac{n-1}{\lfloor \frac{n-1}{2} \rfloor} \right) + 1$. 

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**Figure 2.** digraph $D(M_2)$
Next, we show \( r(m, n) \leq 2^n - 1 \). Let \( \mathbb{B} = (b_{i_1...i_m}) \) be a nonnegative tensor with order \( m \) and dimension \( n \), and \( \mathbb{B} \) be not primitive. If \( \mathbb{B} \) is \( j \)-primitive with some \( j \in [n] \), then we denote \( K = \gamma_j(\mathbb{A}) \). Then \( S_1(\mathbb{B}, j), S_2(\mathbb{B}, j), \ldots, S_{K-1}(\mathbb{B}, j) \) are proper subsets of \([n]\) and \( S_K(\mathbb{B}, j) = [n] \).

Now we show \( S_1(\mathbb{B}, j), S_2(\mathbb{B}, j), \ldots, S_K(\mathbb{B}, j) \) are pairwise distinct. Otherwise, there exist \( u, v \) such that \( 1 \leq u < v \leq K \) and \( S_1(\mathbb{B}, j), S_2(\mathbb{B}, j), \ldots, S_u(\mathbb{B}, j), \ldots, S_{v-1}(\mathbb{B}, j) \) are pairwise distinct, but \( S_u(\mathbb{B}, j) = S_v(\mathbb{B}, j) \). Then by (2.2), we have \( S_{u+t+s(u-v)}(\mathbb{B}, j) = S_{u+t}(\mathbb{B}, j) \) for any \( t \) with \( 0 \leq t < v-u \) and nonnegative integer \( s \), and thus there exist nonnegative integers \( s_1, t_1 \) with \( 0 \leq t_1 < v-u \) such that \( K = u+t_1+s_1(v-u) \) and \( S_{K}(\mathbb{B}, j) = S_{u+t_1}(\mathbb{B}, j) \neq [n] \), it is a contradiction. We denote \( S_1(\mathbb{B}, j), S_2(\mathbb{B}, j), \ldots, S_K(\mathbb{B}, j) \) subset of \([n]\) and they are not empty set, so \( K \leq 2^n - 1 \). Thus \( r(m, n) \leq 2^n - 1 \).

\[ \square \]

**Proposition 4.9.** Let \( m \geq 2 \), and \( \mathbb{A} = (a_{i_1...i_m}) \) be a nonnegative tensor with order \( m \) and dimension \( n \), \( \mathbb{B} = (b_{i_1...i_{m+1}}) \) be a nonnegative tensor with order \( m+1 \) and dimension \( n \) where

\[
b_{i_1...i_{m+1}} = \begin{cases} 
a_{i_1...i_m}, & \text{if } i_{m+1} = i_m; \\
0, & \text{otherwise.}
\end{cases}
\]

Then we have

(i). There exists some \( j \in [n] \) satisfying \( \mathbb{A} \) is \( j \)-primitive if and only if there exists some \( j \in [n] \) satisfying \( \mathbb{B} \) is \( j \)-primitive, and \( \gamma_j(\mathbb{B}) = \gamma_j(\mathbb{A}) \).

(ii). \( \mathbb{A} \) is primitive if and only if \( \mathbb{B} \) is primitive, and thus \( \gamma(\mathbb{B}) = \gamma(\mathbb{A}) \).

**Proof.** Firstly, we show \( S_k(\mathbb{B}, j) = S_k(\mathbb{A}, j) \) for \( k \geq 1 \) by induction on \( k \). Clearly,

\[
S_1(\mathbb{B}, j) = \{ u \in [n] | b_{u_{j...j}} > 0 \} = \{ u \in [n] | a_{u_{j...j}} > 0 \} = S_1(\mathbb{A}, j).
\]

Now we assume that \( S_k(\mathbb{B}, j) = S_k(\mathbb{A}, j) \) for \( k \geq 1 \). Then by (2.2), we have

\[
S_{k+1}(\mathbb{B}, j) = \{ u \in [n] | \text{there exist } i_2, \ldots, i_m, i_{m+1} \in S_k(\mathbb{B}, j) \text{ and } b_{u i_2...i_{m+1}} > 0 \}
\]

\[
= \{ u \in [n] | \text{there exist } i_2, \ldots, i_m, i_{m+1} \in S_k(\mathbb{A}, j) \text{ and } b_{u i_2...i_{m+1}} > 0 \}
\]

\[
= \{ u \in [n] | \text{there exist } i_2, \ldots, i_m \in S_k(\mathbb{A}, j) \text{ and } a_{u i_2...i_{m}} > 0 \}
\]

\[
= S_{k+1}(\mathbb{A}, j).
\]

Then \( \mathbb{B} \) is \( j \)-primitive if and only if \( \mathbb{A} \) is \( j \)-primitive, and \( \gamma_j(\mathbb{B}) = \gamma_j(\mathbb{A}) \).

It is obvious that \( \mathbb{A} \) is primitive if and only if for all \( j \in [n] \), \( \mathbb{A} \) is \( j \)-primitive. Thus (ii) holds by (i).

By Proposition 4.9 we obtain the following results and propose some questions immediately.

\[ \square \]

**Proposition 4.10.** Let \( l \) be a large positive integer. Then

(i). \( r(2, n) \leq r(3, n) \leq r(4, n) \leq \ldots \leq r(l, n) \leq r(l+1, n) \).

(ii). \( R_j(2, n) \subseteq R_j(3, n) \subseteq R_j(4, n) \subseteq \ldots \subseteq R_j(l, n) \subseteq R_j(l+1, n) \).

**Question 4.11.** Let \( m(\geq 3), n(\geq 2) \) be positive integers. Then \( r(m, n) = ? \) Can you characterize the extremal tensor?

**Question 4.12.** Does there exist gaps in \( R_j(m, n) \)?

In [12], Shao proposed the concept of strongly primitive.
Definition 4.13. ([12], Definition 4.3) Let $\mathbb{A}$ be a nonnegative tensor with order $m$ and dimension $n$. If there exists some positive integer $k$ such that $\mathbb{A}^k > 0$ is a positive tensor, then $\mathbb{A}$ is called strongly primitive, and the smallest such $k$ is called the strongly primitive degree of $\mathbb{A}$.

Let $\mathbb{A} = (a_{i_1i_2\ldots i_m})$ be a nonnegative tensor with order $m$ and dimension $n$. It is clear that if $\mathbb{A}$ is strongly primitive, then $\mathbb{A}$ is primitive. In fact, it is obvious that in the matrix case ($m = 2$) a nonnegative matrix $A$ is primitive if and only if $A$ is strongly primitive, but in the case $m \geq 3$, Shao gave an example to show these two concepts are not equivalent ([12]). Till now, there are no further research on strongly primitive. For convenience, let $\eta(\mathbb{A})$ be the strongly primitive degree of $\mathbb{A}$.

Proposition 4.14. Let $\mathbb{A} = (a_{i_1i_2\ldots i_m})$ be a nonnegative strongly primitive tensor with order $m$ and dimension $n$. Then for any $\alpha \in [n]^{m-1}$, there exists some $i \in [n]$ such that $a_{i\alpha t} > 0$.

Proof. Without loss of generality, we assume the strongly primitive degree of $\mathbb{A}$, denoted by $\eta(\mathbb{A})$, is $k$. Then $\mathbb{A}^k$ be a positive tensor with order $(m - 1)^k + 1$ and dimension $n$. Therefore for any $i_1 \in [n]$ and any $\alpha_2, \ldots, \alpha_t \in [n]^{m-1}$ where $t = (m - 1)^k - 1$, we can complete the proof by the following equation.

$$(A^k)_{i_1\alpha_2\ldots\alpha_t} = \sum_{i_2, \ldots, i_t = 1}^n (A^{k-1})_{i_1i_2\ldots i_t}a_{i_2\alpha_2}\ldots a_{i_t\alpha_t} > 0. \quad (4.2)$$

Example 4.15. Let $m = n = 3$, $\mathbb{A} = (a_{i_1i_2i_3})$ be a nonnegative tensor with order $m$ and dimension $n$, where $a_{111} = a_{222} = a_{333} = a_{233} = a_{311} = 0$ and other $a_{i_1i_2i_3} = 1$. Then $\eta(\mathbb{A}) = 4$.

Proof. Firstly, we show that $\mathbb{A}^2 = (a_{i_1i_2i_3i_4i_5}^{(2)})$ is a nonnegative tensor with order 5 and dimension 3, and $a_{i_1j_2j_3j_4j_5} > 0$ except for $a_{13333} = a_{21111} = a_{33333} = 0$. We complete the proof by the following four cases and $a_{i_1j_2j_3j_4j_5}^{(2)} = \sum_{i_2, i_3 = 1}^3 a_{i_1i_2i_3}a_{i_2j_2j_3}a_{i_3j_4j_5}$.

Case 1: $j_2 \neq j_3$ and $j_4 \neq j_5$.

Then $a_{i_2j_2j_3} = a_{i_3j_3j_5} = 1$ for any $i_2, i_3 \in \{1, 2, 3\}$. Thus $a_{i_1j_2j_3j_4j_5}^{(2)} = \sum_{i_2, i_3 = 1}^3 a_{i_1i_2i_3} > 0$.

Case 2: $j_2 = j_3$ and $j_4 \neq j_5$.

Then $a_{i_3j_4j_5} = 1$ for any $i_3 \in \{1, 2, 3\}$. Thus

$$a_{i_1j_2j_3j_4j_5}^{(2)} = \sum_{i_2, i_3 = 1}^3 a_{i_1i_2i_3}a_{i_2j_2j_3} \geq \begin{cases} a_{i_123}a_{211}, & \text{if } j_2j_3 = 11; \\ a_{i_113}a_{122} + a_{i_131}a_{322}, & \text{if } j_2j_3 = 22; \\ a_{i_113}a_{133}, & \text{if } j_2j_3 = 33. \end{cases} > 0.$$  

Case 3: $j_2 \neq j_3$ and $j_4 = j_5$.

The proof is similar to the proof of Case 2.

Case 4: $j_2 = j_3$ and $j_4 = j_5$.

In this case, we know only $a_{13333} = a_{21111} = a_{33333} = 0$ by direct computation. Similarly, we can show that $\mathbb{A}^3 = (a_{i_1i_2\ldots i_9}^{(3)})$ is a almost positive tensor with order 9 and dimension 3 except for $a_{233333333}^{(3)} = 0$ by ([12]), but $\mathbb{A}^4, \mathbb{A}^5, \ldots$ are positive tensors. Thus combining the above cases, we know $\eta(\mathbb{A}) = 4$. 

\[\square\]
Question 4.16. Can we define and study the strongly primitive degree, the strongly exponent set, the $j$-strongly primitive of strongly primitive tensors and so on?

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