AMPLE LINE BUNDLES ON BLOWN UP SURFACES

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Abstract. Given a smooth complex projective surface $S$ and an ample divisor $H$ on $S$, consider the blow up of $S$ along $k$ points in general position. Let $H'$ be the pullback of $H$ and $E_1, \ldots, E_k$ be the exceptional divisors. We show that $L = nH' - E_1 - \ldots - E_k$ is ample if and only if $L^2$ is positive provided the integer $n$ is at least 3.

Introduction.

In this note we give an answer to the following question: Given a smooth projective surface $S$ over $C$ and an ample divisor $H$ on $S$, consider the blow up $f : S' \rightarrow S$ of $S$ along $k$ points in general position. Let $H' = f^*H$ and $E_1, \ldots, E_k$ be the exceptional divisors. When is the divisor

$$L = nH' - \sum_{i=1}^{k} E_i$$

ample?

We show that the condition $L^2 > 0$, which clearly is necessary, is also sufficient provided the integer $n$ is at least 3. Note that the answer to this question has been unknown even in the case of $S = \mathbb{P}^2$. The basic idea is to study the situation on the surface $S$ with variational methods.

Shortly after this work has been completed the author learned that Geng Xu obtained a similar result in the case of $S = \mathbb{P}^2$ independently.

It’s a pleasure to thank Rob Lazarsfeld, who introduced me to this circle of ideas.

Proofs.

The main technical tool is an estimate on the self-intersection of moving singular curves established by Ein, Lazarsfeld and Xu in the context of Seshadri constants of ample line bundles on smooth surfaces (cf. [EL], 1.2, and [Laz], 5.16). The precise statement is:

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**Proposition.** Let \( \{C_t\}_{t \in \Delta} \) be a 1–parameter family of reduced irreducible curves on a smooth projective surface \( X \), and \( y, y_1, \ldots, y_r \in X \) be distinct points such that \( \text{mult}_y C_t \geq m_i \) for all \( t \in \Delta \) and \( i = 1, \ldots, r \). Suppose there exist \( t, t' \) with \( \text{mult}_y C_t = m > 0 \) and \( y \not\in C_{t'} \). Then

\[
(C_t)^2 \geq m(m-1) + \sum_{i=1}^{r} m_i^2.
\]

Using this Proposition we can prove:

**Theorem.** Let \( S' \) be as above and \( a > 2 \) be a rational number. Consider the \( \mathbb{Q} \)-divisor

\[
M = aH' - \sum_{i=1}^{k} E_i.
\]

Then the following hold:

1. If \( M^2 = a^2H^2 - k \geq 2 \), then \( M \) is ample on \( S' \).

2. If \( M^2 = a^2H^2 - k \geq 1 \), then \( M \) is positive on all curves \( C' \subset S' \) for which \( j \) exists with \( C'.E_j \geq 2 \).

**Proof.** Suppose the theorem is not true, and choose a curve \( C' \subset S' \) such that \( M.C' \leq 0 \). Consider \( C = f(C') \). Defining \( m_i = \text{mult}_{p_i}(C) \), we may suppose that \( m_1 \geq \cdots \geq m_k \). Since \( M.C' \leq 0 \), we have

\[
\sum_{i=1}^{k} m_i \geq a(H.C). \tag{*}
\]

Now we may assume that

- \( C \) passes through all the points \( p_i \), i.e. \( m_i \geq 1 \)
- \( C \) is irreducible and reduced
- \( C \) moves, since the \( p_i \) are in general position

Here \( C \) moves even in the strong sense, that is, fixing \( p_1, \ldots, p_{k-1} \), the curve \( C \) still moves in a family of curves satisfying (\#). To see this simply observe that any curve on \( S \) lies in one of countably many families, but no neighbourhood of \( p_k \) is covered by countably many curves.

Finally we claim that a general member of this family has sufficiently big multiplicity at \( p_1, \ldots, p_{k-1} \). But any member satisfies (*), so this follows from semicontinuity.

Therefore we can apply the Proposition and obtain the estimate

\[
C \cdot C \geq m_1^2 + \cdots + m_{k-1}^2 + m_k(m_k - 1),
\]
and hence combined with the Hodge-Index-Theorem
\[
\left( \sum_{i=1}^{k} m_i \right)^2 \geq a^2(H.C)^2 \geq a^2 H^2 \cdot C^2 \geq a^2 H^2 \left( \sum_{i=1}^{k} m_i^2 - m_k \right).
\]

By (\(*\)), (\(\ast\ast\)) and the assumption \(a > 2\) we may assume \(k \geq 2\) in the following.

Suppose for the time being that \(C\) is not smooth at one of the \(p_j\), which is the case if and only if \(C'.E_j \geq 2\). Then \(m_1 \geq 2\), and (\(\ast\ast\)) contradicts the following Lemma:

**Lemma.** Let \(k \geq 2\) and \(x_1, \ldots, x_k \in \mathbb{Z}\) be integers with \(x_1 \geq \cdots \geq x_k \geq 1\) and \(x_1 \geq 2\). Then we have
\[
(k + 1) \sum_{i=1}^{k} x_i^2 > \left( \sum_{i=1}^{k} x_i \right)^2 + x_k(k + 1).
\]

**Proof of the Lemma.** We argue by induction on \(k \geq 2\).

For \(k = 2\) one proves
\[
3(x_1^2 + x_2^2) - (x_1 + x_2)^2 - 3x_2 > 0
\]
by minimizing this expression with respect to \(x_2\). From the inductive hypothesis, we then obtain
\[
(k + 1) \sum_{i=1}^{k} x_i^2 > kx_k^2 + \sum_{i=1}^{k} x_i^2 + \left( \sum_{i=1}^{k-1} x_i \right)^2 + kx_k
\]
\[
= \left( \sum_{i=1}^{k} x_i \right)^2 + x_k(k + 1) - x_k^2 - 2 \cdot \sum_{i=1}^{k-1} x_i x_k - x_k + kx_k^2 + \sum_{i=1}^{k} x_i^2
\]
\[
= \left( \sum_{i=1}^{k} x_i \right)^2 + x_k(k + 1) + \sum_{i=1}^{k-1} (x_i - x_k)^2 + x_k^2 - x_k.
\]

So what we need to show is
\[
\sum_{i=1}^{k-1} (x_i - x_k)^2 + x_k^2 \geq x_k,
\]
but this is obvious.

\(\square\)

This proves the second part of the Theorem. To prove the first part it remains to exclude the case \(m_1 = \cdots = m_k = 1\). But then (\(\ast\ast\)) reads
\[
k^2 \geq H^2 \cdot a^2(k - 1),
\]
contradicting the assumptions on \(a\).

\(\square\)
**Corollary.** Let $L$ be as in the introduction. Then $L$ is ample if and only if $L^2 > 0$.

**Proof.** It clearly suffices to prove the if–part. So suppose $L^2 > 0$ and that $L$ is not ample. Then by the Theorem we know that $L^2 = 1$, i.e. $n^2 H^2 = k + 1$, and that there exists an irreducible reduced curve $C \subset S$ which is smooth at all the $p_i$ satisfying $k \geq n(H.C)$.

We claim that $k = n(H.C)$ holds. Otherwise we have $L.C' < 0$. Consider the surface $\hat{S}$ obtained from $S'$ by contracting the exceptional divisor $E_j$, where $j$ is an index such that $C$ passes smoothly through $p_j$. The image $\hat{L}$ of $L$ then satisfies $\hat{L}^2 = L^2 + 1 = 2$, hence it is ample by the Theorem. But this contradicts $L.C' + 1 = \hat{L}.\hat{C} \leq 0$ for the image $\hat{C}$ of $C'$.

Therefore we conclude $k + 1 = n^2 H^2 = n(H.C) + 1$, but this is impossible since besides $n \neq 1$ also $H^2$ and $(H.C)$ are integers.

\[ \square \]

**Remark.** The example of a line in $\mathbb{P}^2$ through any two points shows that we cannot drop the assumption $n \geq 3$ in general. On the other hand an analysis of the proof shows that the Corollary still holds in the case $n \geq 2$ if two general points on $S$ can not be joined by a curve $C$ with $(H.C) = 1$, which is true e.g. whenever $H^2 \geq 2$.

**References**

[EL] L. Ein and R. Lazarsfeld, *Seshadri constants on smooth surfaces*, Journées de Géometrie Algébrique d’Orsay, Astérisque 218 (1993), 177-186.

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