Internal resonance of an elastic body levitated above high-Tc superconducting bulks

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Abstract. In high-Tc superconducting magnetic levitation systems, levitated bodies can keep stable levitation with no contact and no control and thus their damping is very small. Thanks to these features, their applications to various apparatus are expected. However, on account of their small damping, the nonlinearity of electromagnetic levitation force can give notable effects upon motion of the levitated bodies. Therefore this nonlinearity must be taken into account to accurately analyze the dynamical behavior of the levitated bodies. Structures of such a levitated body can show elastic deformation if the large electromagnetic force acts on it. Therefore, we need to deal with the model as an elastic body. As mentioned above, nonlinear characteristics easily appear in this elastic vibration on account of the small damping. Especially when the ratio of the natural frequencies of the eigenmodes is integer, internal resonance can occur. This nonlinear resonance is derived from nonlinear interactions among the eigenmodes of the elastic levitated body. This kind of internal resonance of an elastic body appearing in high-Tc superconducting levitation systems has not been studied so far. This research especially deals with internal resonance of a beam supported at both its ends by electromagnetic forces acting on permanent magnets. The governing equation with the nonlinear boundary conditions for the dynamics of a levitated beam has been derived. Numerical results show internal resonance of the 1st mode and the 3rd mode. Experimental results are qualitatively in good agreement with numerical ones.

1. Introduction
In magnetic levitation systems consisting of high-Tc superconductors and permanent magnets, levitated bodies can keep stable levitation with no contact and no control. Thus their damping is very small. So these systems have little energy loss and can be applied to support mechanisms in machines, such as flywheel systems for energy storage and magnetically levitated trains. However, the nonlinearity of electromagnetic levitation force can give notable effects upon motion of the levitated bodies[1]-[6]. Therefore this nonlinearity must be taken into account to accurately analyze the dynamical behavior of the levitated bodies. In addition, for application to large-scale mechanism, we need to analyze nonlinear dynamics of a large levitated body supported by electromagnetic force at multiple points. Moreover elastic deformation cannot be ignored for light structures. Here we evaluate nonlinear effects of these forces on dynamics of this system through experiment and numerical analysis.

So far in our research, vibration phenomena in several models have been analyzed, such as a rigid body[2], a rotating body[3] and an elastic body[4]-[5]. As to the model of an elastic body, nonlinear vibrations such as primary resonance and combination resonance were analyzed[4][5]. In this research, we focus on internal resonance in an elastic body. Internal
resonance was analyzed in a relating model[6]. Generally, when the ratio of natural frequencies is integer, internal resonance caused by nonlinearity can happen[7]-[8]. First we derive the governing equation with the nonlinear boundary conditions for the dynamics of a levitated beam. They are solved numerically. Numerical results of nonlinear oscillation are compared with our experimental results.

2. Analytical Model and Governing Equations

Our analytical model also used in [5] is shown in figure 1. A beam has two permanent magnets at its both ends. These magnets are supported by electromagnetic forces from high-Tc superconducting bulks with no contact. These bulks are fixed on a vibrator. The beam is assumed to move in the plane made by the axes $x$ and $z$. $m$ and $M$ denote the magnetization of each magnet and its mass, respectively. The magnetization vector is orthogonal to the major axis of the beam in the $x-z$ plane. $\rho$, $A$, $EI$ and $l$ represent the density of the beam, its cross section, its flexural rigidity and the length of its major axis, respectively. The initial position of the beam before the field-cooling is shown by broken lines in figure 1. We define the $x$ axis along the major axis of the beam at initial position. The $z$ axis is defined as vertical. These axes are fixed on the initial position. On the other hand, the $Z$ axis is fixed on the surface of the bulks. Both initial heights of the magnets are $Z_{fc}$. After the field-cooling, the levitated beam balances with the gravity at the equilibrium position. Furthermore, we define the $s$ axis along the neutral axis of the beam.

By applying the principle of Hamilton to the analytical model, the governing equation and the boundary conditions are obtained. By assuming bending of the beam is small and the displacement in the horizontal direction $x$ is negligible, these equations are expressed by the relative coordinate $Z$ as below. $\cdot$ and $\cdot\cdot$ show partial differential operators by coordinates $s$ and $t$, respectively. In the following expression, the upper sign is applied at the left end of the beam (at $s = 0$), and the lower sign is applied at the right end (at $s = l$).

$$\rho A \ddot{Z} + EI Z''' = -\rho Ag + \rho AaN^2 \sin Nt$$  \hspace{1cm} (1)

B.C. \hspace{1cm} \begin{align*}
\begin{cases}
M \ddot{Z} \pm EI Z''' + Mg &= MaN^2 \sin Nt + F_Z \\
Z'' &= 0 
\end{cases} \quad \text{at } s=0,l
\end{align*} \hspace{1cm} (2)

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Analytical model of a beam supported by high-Tc superconducting magnetic bearings.}
\end{figure}
where $F_z$ denotes the electromagnetic forces acting on the levitated magnets, which can be evaluated analytically as nonlinear functions of the displacement $Z$ by the flux frozen method [6]. Figure 2 shows the nonlinear relation between the relative coordinate $Z$ and the electromagnetic force $F_z$. It can be noted from this nonlinear relation that the restoring characteristics of the electromagnetic force changes between the softening effect and the hardening effect, according to $Z$.

Now, we rewrite the governing equations by some variable transformation with nondimensionalization as follows.

\[ \ddot{\zeta} + \zeta''' = \alpha \nu^2 \sin \nu t \]  
\( \text{B.C.} \left\{ \begin{array}{l} \frac{\dot{\zeta}}{\beta} + \zeta'' = \alpha \nu^2 \sin \nu t + F_z \zeta' \\ \zeta'' = 0 \text{ at } s=0,1 \end{array} \right. \]

where $\zeta$ denotes the relative coordinate from the equilibrium point. The solution of equation (3) can be expanded by the eigenfunctions as below.

\[ \zeta(s, t) = \sum_{i=1}^{\infty} \eta_i(t) \Phi_i(s) \]  

We obtained the natural frequencies of eigenmodes up to the 3rd, considering nonlinearity of the boundary conditions (4). The mode shapes of vibration are shown in figure 3. By using equation (5) we can finally derive the nonlinear ordinary differential equations of the modes up to the 3rd as below.

\[ \ddot{\eta}_1 + 2\gamma_1 \dot{\eta}_1 + \omega_1^2 \eta_1 + NLT_1(\eta_1, \eta_2, \eta_3) = \alpha_1 \sin \nu t \]
\[ \ddot{\eta}_2 + 2\gamma_2 \dot{\eta}_2 + \omega_2^2 \eta_2 + NLT_2(\eta_1, \eta_2, \eta_3) = 0 \]
\[ \ddot{\eta}_3 + 2\gamma_3 \dot{\eta}_3 + \omega_3^2 \eta_3 + NLT_3(\eta_1, \eta_2, \eta_3) = \alpha_3 \sin \nu t \]

where $NLT$ represents terms caused by nonlinearity in electromagnetic forces. Equations (6) and (8) have nonhomogeneous excitation terms. Therefore, if the excitation frequency $\nu$ is in
the neighborhood of their natural frequencies, the 1st and 3rd modes can be largely excited. On the other hand, equation (7) does not have a nonhomogeneous excitation term. Thus, the 2nd mode cannot get excited even if the excitation frequency \( \nu \) is in the neighborhood of its natural frequency. We focus on internal resonance of the 1st and 3rd modes, which can occur by nonlinear coupling between these linearly uncoupled modes through the \( NLT \) terms in equations (6) and (8).

3. Numerical Simulation

In this section, to investigate effects of nonlinearity of the electromagnetic force on dynamics of the levitated beam, we show results of numerical simulation.

We performed numerical integration of equations (6)–(8) using the Runge-Kutta method. The nondimensional parameters used in our analysis are based on our experimental data. As a result, nondimensional natural frequencies of up to the 3rd mode become as below:

\[
\begin{align*}
\omega_1 &= \lambda_1^2 = 0.767 \\
\omega_2 &= \lambda_2^2 = 1.401 \\
\omega_3 &= \lambda_3^2 = 1.519
\end{align*}
\]

It can be noted that the ratio of the natural frequency of the 1st mode to that of the 3rd mode is approximately one to two.

Figure 4 shows numerical results of frequency response under the circumstance that nondimensional excitation frequency \( \nu \) is close to \( \omega_3 \). The left and right graphs in this figure show the amplitude of the 1st mode and that of the 3rd mode, respectively. If the system were linear, resonance would only occur in the 3rd mode (the right graph) whose resonant frequency is close to the excitation one. However, figure 4 shows resonance not only in the 3rd mode but also in the 1st mode (the left one) whose resonant frequency is far from the excitation one. This derives from nonlinear coupling between two modes, or nonlinearity of electromagnetic force. This can derive from nonlinear coupling between the two modes, or nonlinearity of electromagnetic force.

Figures 5 and 6 show time histories and their spectra of the 1st mode and 3rd mode. Excitation frequency \( \nu \) is 1.532 and 1.512 in figure 5 and figure 6, respectively. In figure 5, excitation of the two modes can be clearly found. It should be further noted that the amplitude of the 1st mode is much larger than that of the 3rd mode, though the excitation frequency is close to the resonant frequency of the 3rd mode. This can be caused by energy transfer from the excitation source to the 1st mode by way of the linearly resonated 3rd mode through nonlinear coupling between the two modes. Although figure 5 shows steady-state oscillations with their amplitudes constant, in figure 6 the amplitudes of the two modes increase and decrease alternatively with a constant period. This is called almost periodic motion. This motion means that there is a continual exchange of energy between the two modes. Both figures show internal resonance between the 1st mode and the 3rd mode. Especially under certain conditions, almost periodic motions appear as in figure 5.
Figure 4. Numerical results of frequency responses.

Figure 5. Numerical results of time histories and their FFT spectra of the modes, showing internal resonance ($\nu = 1.532$).

Figure 6. Numerical results of time histories and their FFT spectra of the modes, showing almost periodic motion ($\nu = 1.512$).
4. Experiment

![Experimental setup diagram]

Figure 7. Experimental setup.

Figure 7 shows our experimental setup. Two high-$T_c$ superconducting bulks used in our experiments were ceramic YBCO samples of melt-quench type. Permanent magnets consist of Nd, Fe, and B. An elastic bar with one magnet on each end was levitated stably after the field-cooling by liquid nitrogen. We gave the superconducting bulks vertical excitation. The displacements of center of the beam $Z_1$ and the base excitation $Z_0$ were measured by laser displacement sensors. Their results were analyzed by a FFT analyzer.

By experiment of free vibration, we found the natural frequencies of this system up to 3rd:

$$f_1 = 12.50 \text{ Hz} \quad (12)$$
$$f_2 = 21.25 \text{ Hz} \quad (13)$$
$$f_3 = 24.75 \text{ Hz} \quad (14)$$

It can be noted that the ratio of the natural frequency of the 1st mode to that of the 3rd mode is approximately one to two.

Figure 8 shows the frequency response under the circumstance that the excitation frequency is close to the natural frequency of the 3rd mode. The left and right graphs show the amplitude of the 1st mode and that of the 3rd mode, respectively. Occurrence of internal resonance can be found just as in numerical results shown in figure 4.

Figure 9 and figure 10 show time histories and their spectra at the excitation frequencies $f = 24.875\text{Hz}$ and $25\text{Hz}$, respectively. Figure 10 shows excitation of the two modes with their amplitudes constant. The amplitude of the 1st mode is much larger than that of the 3rd mode. Figure 10 also shows resonance of the two modes, but their amplitudes change with some constant period. These results are qualitatively in good agreement with numerical ones shown in figures 5 and 6.
Figure 8. Experimental results of frequency responses.

Figure 9. Experimental results of time histories and their FFT spectra, showing internal resonance ($f = 25\text{Hz}$).

Figure 10. Experimental results of time history and their FFT spectrum, showing almost periodic motion ($f = 24.875\text{Hz}$).
5. Conclusion
Nonlinear dynamics of a beam with magnets at its ends levitated above high-Tc superconducting bulks have been investigated. Numerical results considering nonlinearity of the electromagnetic force show internal resonance of the 1st mode and the 3rd mode if the ratio of the natural frequency of the 1st mode to that of the 3rd mode is approximately one to two. The 1st mode can be largely resonated even if the excitation frequency is far from its resonant frequency because of nonlinear coupling of the two modes. Under specific conditions unsteady oscillations of the two modes, that is almost periodic motion, can also occur. Experimental results are qualitatively in good agreement with numerical ones. Thus, in mechanical design of HTSC levitation systems, it should be noted that slower large oscillation can appear even if the excitation frequency is far from the resonant frequencies of lower modes.

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