Self consistent hydrodynamic description of the plasma wake field excitation induced by a relativistic charged-particle beam in an unmagnetized plasma

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Abstract
A self-consistent nonlinear hydrodynamic theory is presented of the propagation of a long and thin relativistic electron beam, for a typical plasma wake field acceleration configuration in an unmagnetized and overdense plasma. The random component of the trajectories of the beam particles as well as of their velocity spread is modelled by an anisotropic temperature, allowing the beam dynamics to be approximated as a 3D adiabatic expansion/compression. It is shown that even in the absence of the nonlinear plasma wake force, the localisation of the beam in the transverse direction can be achieved owing to the nonlinearity associated with the adiabatic compression/rarefaction and a coherent stationary state is constructed. Numerical calculations reveal the possibility of the beam focussing and defocussing, but the lifetime of the beam can be significantly extended by the appropriate adjustments, so that transverse oscillations are observed, similar to those predicted within the thermal wave and Vlasov kinetic models.

Keywords: plasma wakefield, relativistic beams in plasma, relativistic hydrodynamics

(Some figures may appear in colour only in the online journal)

1. Introduction

The concept of plasma-based particle accelerators, capable of accelerating charged particles to extremely high energies (beyond 10 TeV) has been put forward more than three decades ago and massive theoretical, computational and experimental efforts have been made in this field since then. One of the proposed schemes, often referred to as the plasma wake field (PWF) accelerator consists in the injection of a relativistic electron (or positron) beam into the plasma, which creates a vacancy in the electron population of the background plasma while the plasma ions remain in place due to their large mass. After the passage of the beam, usually referred to as the driving beam or simply the driver, in the wake of the latter, plasma electrons rush in to restore the quasineutrality which produces a very large amplitude electrostatic plasma wave that propagates with the same velocity as the beam that has created it. The longitudinal electric field of such plasma wave can be used to accelerate particles to the energies that are higher that that of the driving beam. When
the latter is properly bunched, and the beam density is of the order of $10^{17}$–$10^{18}$ cm$^{-3}$, an acceleration gradient ranging from 0.01 to 1 GeV cm$^{-1}$ can be achieved, yielding the energy of the accelerated particles to be 3–4 times bigger that that of the driving beam [1]. The dynamics of the driving beam in the direction transverse to its propagation is important not only for the feasibility of such scheme, but also for the quality (i.e., lower emittance and higher brightness) of the relativistic particle bunch generated in it. A high-intensity electron beam of finite extent can be subjected longitudinally to a two-stream instability, while in the transverse direction it can undergo self-focusing or defocusing, filamentation, and self-pinching. As for the quality of the bunch produced this way, transverse PWF can be used for its focussing which contrasts the spreading due to the thermal emittance of the bunch. However, more generally, the concomitant presence of both the transverse PWF and thermal spreading of the bunch can be adequately used for the bunch manipulation. In particular, for sufficiently long bunches the PWF acts on the driving bunch itself, leading to the self-consistent PWF excitation (the feedback of the wake field on the driver). Therefore, in such a physical situation the concomitance of transverse PWF and thermal spreading leads to the beam self modulation. On longer time scales, under suitable conditions, the latter can evolve in an unstable way (self modulation instability).

In the early theoretical works [1], the feedback of the wake field on the driver and the resulting spatial evolution of the beam were not considered. Later, a selfconsistent theory based on the thermal wave model (TWM) [2] was presented and applied to the study of the interaction between the PWF and the driving relativistic electron beam, in an unmagnetized overdense plasma (whose density exceeds that of the beam) and in the case of a long beam ($L_\parallel \gg L_\perp$, where $L_\parallel$ and $L_\perp$ are its longitudinal and transverse characteristic lengths, respectively). Such model successfully described the beam’s self-focusing and self-pinching equilibria, when the spot size of the beam was larger and smaller than the wavelength of the wake field, respectively. More recently, this approach has been extended to the case of self consistent PWF excitation in a cold magnetised plasma for both long nonlaminar warm relativistic driving beam (quantum-like domain of TWM) [3, 4] and relatively cold relativistic quantum driving beam (quantum domain) [5, 6]. Similarly, the self-modulation of a relativistic charged-particle beam as thermal wave envelope and the possibility of its destabilization was studied in [7] using the quantum-like description provided by the TWM in which the beam dynamics is governed by a Zakharov-type system of equations, comprising a Poisson-like equation for the wake potential and a nonlinear Schrödinger equation governing the spatiotemporal evolution of the thermal wave envelope, whose dispersion coefficient is proportional to the beam thermal emittance. In the strongly nonlocal regime, an Ernâkov–Pinney type equation for the evolution of the beam’s cross-section was derived, and the possibilities were discussed of an instability leading to the beam blowup and of a stable self-modulation in the form of sausage-like transverse oscillations. Fully similar results have been obtained by using a kinetic approach, where the study of the self consistent PWF excitation has been provided by the Vlasov–Poisson-type system of coupled equations [8, 9]. In addition, with the same kinetic approach, the self-modulation instability has been successfully described in recent works [10].

In the present paper, we derive a hydrodynamic description for the interaction between a long relativistic electron beam with an unmagnetized, overdense plasma. Similarly to the TWM, the random component of the transverse motion of the beam particles is modelled by the finite temperature of the fluid, which is taken to be anisotropic. We demonstrate that in the case of a small but finite variation in the direction of the propagation, a nonlinearity associated with the adiabatic thermodynamic evolution of the beam produces the localisation of the latter. Such coherent structure is in the state of an unstable equilibrium that, upon a small perturbation, may either collapse or disperse when the number of beam particles is, respectively, bigger or smaller than that in the stationary state. However, a fine tuning of the initial beam profile may sufficiently postpone these instabilities until other processes, not included in our model, enter the picture and determine the actual evolution of the beam. For a thermodynamic process that is predominantly 2D the related nonlinearity vanishes and, on the long timescale, the system is governed by the nonlinear plasma wake force (given by the gradient of the effective potential) that is responsible also for the betatron-like oscillations observed in the earlier works [7–10].

2. Fluid and Vlasov descriptions

The system under study consists of a quiescent plasma pierced by a relativistic beam of electrons, denoted in the rest of the paper by the subscripts $p$ and $b$, respectively. The beam propagates along the $z$ axis with a relativistic speed $u$, $c - u \ll c$ and it is relatively long, $L_{b\parallel} \gg L_{b\perp}$, where $L_{b\parallel}$ and $L_{b\perp}$ are the characteristic lengths of inhomogeneity of the beam in the directions parallel and perpendicular to its direction of propagation, respectively. Expressing the electric and magnetic field in terms of the electrostatic and vector potentials, $E = - \nabla \phi - \partial A / \partial t$ and $B = - \nabla \times A$ and using the Coulomb gauge $\nabla \cdot A = 0$, the Poisson’s equation and the Ampère’s law take the familiar form

$$\nabla^2 \phi = \epsilon / \varepsilon_0 (n_b + n_p - n_i),$$

$$\left( \partial^2 / \partial t^2 - c^2 \nabla^2 \right) A = - \nabla \partial \phi / \partial t - \left( \epsilon / \varepsilon_0 \right) (n_b v_b + n_p v_p),$$

where $-e$ is the electron charge, $n_b$, $n_p$, and $n_i$ are the densities of the beam and plasma electrons, and of the plasma ions, $v_b$, $v_p$, are the hydrodynamic velocities of the beam and plasma electrons, while the temporal variation is considered as sufficiently rapid so that the plasma ions are immobile. We restrict our study to regimes in which the temporal variation in the reference frame comoving with the beam is sufficiently slow and we conveniently introduce the variables

$$\vec{r}' = \vec{r} - \vec{z}_{\parallel} u t, \quad t' = t, \quad \vec{v}_{b}' = \vec{v}_{b} - \vec{z}_{\parallel} u,$$
which implies $\partial / \partial t = \partial / \partial t' - u \partial / \partial z'$. Furthermore, we assume that the electric field is sufficiently weak so that the fluid velocities $\vec{v}_b$ and $\vec{v}_{b}'$ are nonrelativistic, i.e. we adopt the following scaling

$$
\partial / \partial t' \sim v_{b}' \cdot \nabla \sim v_p \cdot \nabla \sim \epsilon u \partial / \partial z, \quad \text{where } \epsilon \ll 1.
$$

(4)

The plasma density $n_p$ and fluid velocity $\vec{v}_p$ are readily calculated from the appropriate hydrodynamic equations. We consider a regime in which the density of the beam scales as $n_b \sim n_p - n_i \sim n_i v_p / c$ and, as a consequence, the Lorentz forces acting on an electron in the beam does not exceed the Coulomb force, $|\vec{e}_i u \times \vec{B}_0| \lesssim |\vec{E}|$, provided $u \sim c$. We take also that the plasma electrons are nonrelativistic and that the potential $\phi$ is sufficiently small,

$$
eq \phi / m_0 u^2 \lesssim \epsilon (\partial / \partial z) / \nabla_z,
$$

(5)

$m_0$ being the electron rest mass, which permits us to neglect all convective nonlinearities and also the Lorentz force in the momentum equation for the plasma electrons. We study a regime in which the background plasma is cold, and the continuity and momentum equations for plasma electrons take the simple form

$$
-u \partial n_p / \partial z + n_p \nabla \cdot \vec{v}_p = 0,
$$

(6)

$$
-u \partial \vec{v}_p / \partial z = (e / m_0) (\nabla \phi - u \partial \phi / \partial z),
$$

(7)

$n_p$ being the unperturbed density of plasma electrons. Eliminating $\vec{v}_p$ and using the Poisson’s equation (1), we have

$$
(\omega_{pp}^2 + u^2 \partial^2 / \partial z^2) \nabla^2 \phi = (\epsilon / \epsilon_0) u^2 \partial^2 n_p / \partial z^2,
$$

where $\omega_{pp}^2 = \sqrt{n_p e^2 / m_0 \epsilon_0}$ is the plasma frequency of the background plasma. With the same accuracy, from $z$-components of the momentum equation (7) yielding $v_{p_z} = (e / m_0 u) (\phi - u A_z)$, and of the Ampere’s law (2) we get

$$
(\omega_{pp}^2 + u^2 \partial^2 / \partial z^2 - c^2 \nabla^2) (\phi - u A_z) + c^2 \nabla^2 \phi = (\epsilon / \epsilon_0) u^2 n_p.
$$

(8)

Finally, combining equations (8) and (9) we obtain

$$
(\omega_{pp}^2 + u^2 \partial^2 / \partial z^2) (\omega_{pp}^2 + u^2 \partial^2 / \partial z^2 - c^2 \nabla^2) (\phi - u A_z) = (\epsilon / \epsilon_0) [\omega_{pp}^2 + (u^2 - c^2) \partial^2 / \partial z^2] u^2 n_p,
$$

(10)

which, for a long beam, $\nabla \gg \partial / \partial z$, whose velocity is relativistic, $u \approx c$, simplifies to the well known result

$$
(\omega_{pp}^2 - c^2 \nabla^2) (\phi - u A_z) = (\epsilon / \epsilon_0) u^2 n_p.
$$

(11)

The density of the beam, $n_b$, is calculated from the well-known textbook expression for the relativistic Vlasov equation,

$$
\frac{\partial f_b}{\partial t} + \vec{v} \cdot \frac{\partial f_b}{\partial \vec{r}} - e (\vec{E} + \vec{v} \times \vec{B}) \frac{\partial f_b}{\partial \vec{p}} = 0,
$$

(12)

where $f_b(t, \vec{r}, \vec{p})$ is the distribution function of the beam electrons, while $\vec{v}$ and $\vec{p}$ are the electrons’ velocity and momentum, that are mutually related as

$$
\psi = \frac{\vec{p}}{m_0 \sqrt{1 + p^2 / (m_0 c^2)}}, \quad \vec{p} = \frac{m_0 \vec{v}}{\sqrt{1 - v^2 / c^2}}.
$$

(13)

However, solving the full Vlasov equation can be very tedious in a general case. In this paper, we do not account for the kinetic effects, most notably the wave breaking of the Langmuir wave associated with the plasma wake and the resonant particle effects (such as the trapping and acceleration of resonant electrons), and we solve instead the appropriate moments of the Vlasov equation (12). Multiplying the Vlasov equation (12) by $1, \vec{p},$ and $v_p p_{j}$, respectively, and integrating for the entire momentum space, we obtain the relativistic fluid equations of continuity and momentum, and the equation for the $(i, j)$ component of the pressure tensor

$$
\partial n_b / \partial t + \nabla \cdot (n_b \vec{v}_b) = 0,
$$

(14)

$$
(\partial / \partial t + \vec{v}_b \cdot \nabla) \vec{p}_b = -e (\vec{E} + \vec{v}_b \times \vec{B}) - (\nabla \cdot \vec{P}) / n_b,
$$

(15)

$$
\partial P_{k,j} / \partial t + \nabla \cdot (P_{k,j} \vec{v}_b) + P_{k,j} \partial \nabla_{\psi b} / \partial \psi_j + P_{j,k} \partial \nabla_{\psi b} / \partial \psi_k = 0,
$$

(16)

where we have used the standard notation from tensor algebra $\nabla \cdot \vec{P} = \epsilon_{ij} \partial P_{i,j} / \partial \psi_j$. In the equations (16) for the stress tensor in a collisionless plasma, we have neglected both the thermal convection and the off-diagonal terms of $\vec{P}$, i.e. we have $\vec{P} = P_{i} (I - \vec{e}_i \vec{e}_i) + P_{\parallel} \vec{e}_i \vec{e}_i$, where $I$ is a unit tensor, viz $L_{\alpha, \beta} = \delta_{\alpha, \beta}$ and $\delta_{\alpha, \beta}$ is the Kronecker delta. We rewrite these equations using the variables defined in equation (3). When the fluid velocity $\vec{v}_b'$ is nonrelativistic, in the momentum equation (15) we may set

$$
\vec{E} + \vec{v}_b \times \vec{B} \approx \vec{E} + u \vec{e}_z \times \vec{B} \approx -\nabla (\phi - u A_z).
$$

(17)

and our fluid equations, rewritten through the variables defined in equation (3), readily obtain the form

$$
\partial n_b / \partial t' + \nabla \cdot (n_b \vec{v}_b') = 0,
$$

(18)

$$
(\partial / \partial t' + \vec{v}_b' \cdot \nabla) \vec{p}_b' = e \nabla (\phi - u A_z) - (\nabla \cdot \vec{P}) / n_b,
$$

(19)

$$
\partial P_{k,j} / \partial t' + \nabla \cdot (P_{k,j} \vec{v}_b') + P_{k,j} \nabla_{\psi b}' = 0,
$$

(20)

$$
\partial P_{i,j} / \partial t' + \nabla \cdot (P_{i,j} \vec{v}_b') + 2P_{i} \nabla_{\psi b}' = 0.
$$

(21)

The pressures $P_{i}$ and $P_{\parallel}$ will be calculated using the Grad’s approximation, expressed through the equilibrium solution of the Vlasov equation and using a limited number of its moments. As the equilibrium solution we use the Jüttner distribution with a particle drift, which is one of the several relativistic versions of the Maxwellian distribution function that can be found in the literature. In the case of an anisotropic electron temperature, it has the form [11]:

$$
f_0(\vec{p}_z, p_{j}) = C \exp[-(\alpha_1 + \beta_{un}) (\gamma - \beta_{un} p_{j}/m c)] - (\alpha_1 - \beta_{un}) \sqrt{1 + \gamma^2 (p_{j}/m_0 c - \beta_{un} \gamma)^2}],
$$

(22)
\[ \alpha_{\parallel} = m_0 c^2 / T_{0,\parallel}, \quad \alpha_{\perp} = m_0 c^2 / T_{0,\perp}, \]
\[ \gamma = \sqrt{1 + (p_{\perp}^2 + p_{\parallel}^2)/m_0^2 c^2}, \]
\[ \gamma_u = 1/\sqrt{1 - \beta_u^2}, \quad \beta_u = u/c, \]
\[ \text{(23)} \]

and \( C \) is determined from the normalisation, \( \int_{-\infty}^{\infty} d^3\vec{p} f_0 = n_0 \).

For nonrelativistic temperatures \( \alpha_{\parallel,\perp} \gg 1 \), the Jüttner distribution (22) reduces to a shifted Maxwellian, with appropriate relativistic corrections to the electron mass and temperature
\[ f_0(\vec{p}_\perp, p_z) = \frac{n_0}{(2\pi m_0)^2 T_{0,\perp} T_{0,\parallel}} \exp \left[ -\frac{\vec{p}_\perp^2}{2m_0 T_{0,\perp}} - \frac{(p_z - m_0 u)^2}{2m_0 T_{0,\parallel} \gamma_u^2} \right]. \]
\[ \text{(24)} \]

A reliable approximation for the distribution function in the presence of finite potentials \( \phi \) and \( A_z \), referred to as the Grad’s moment approximation, is obtained when in the equilibrium distribution (24) we make the substitutions \( n_0 \rightarrow n_b, \quad \vec{p} \rightarrow \vec{p} - \vec{p}_b, \) and \( T_{0,\perp,\parallel} \rightarrow T_{\perp,\parallel} \). Perpendicular and parallel pressures \( P_{\perp} \) and \( P_{\parallel} \) are then readily calculated as the appropriate integrals of such distribution function, viz
\[ P_{\perp} = n_b T_{\perp} / \gamma_u \quad \text{and} \quad P_{\parallel} = n_b T_{\parallel} \gamma_u. \]
\[ \text{(25)} \]

From these expressions for the perpendicular and parallel pressures we note that \( P_{\perp} / P_{\parallel} \sim \gamma_u^2 T_{\parallel} / T_{\perp} \). This may be a very big number and permits us to estimate the relevant scalings of the beam dynamics in the following way:

In the regime \( n_b e(\phi - u A_z) \sim P_{\perp} \ll P_{\parallel} \), the parallel convection is negligible, viz \( v_{\parallel} p_z \partial / \partial z \ll v_{\parallel} \partial / \partial z \), when
\[ \partial / \partial z \ll (T_{\perp} / T_{\parallel}) \gamma_u^{-2} \nabla. \]
\[ \text{(26)} \]

Equations (20) and (21) for the perpendicular pressures \( P_{\perp} \) and \( P_{\parallel} \) remain coupled even for a weak \( z \)-dependence, equation (26), provided \( P_{\parallel} \partial v_{\parallel} / \partial z \gtrsim P_{\perp} \nabla \cdot v_{\parallel} \), which corresponds to
\[ \partial / \partial z \gtrsim (T_{\perp} / T_{\parallel}) \gamma_u^{-2} \nabla. \]
\[ \text{(27)} \]

Conversely, equation (21) for \( P_{\parallel} \) is decoupled and can be discarded when \( P_{\parallel} \partial v_{\parallel} / \partial z \ll P_{\perp} \nabla \cdot v_{\parallel} \), which corresponds to
\[ \partial / \partial z \ll (T_{\perp} / T_{\parallel}) \gamma_u^{-2} \nabla. \]
\[ \text{(28)} \]

Finally, using \( P_{\parallel} = \gamma_u^2 P_{\perp} (T_{\perp} / T_{\parallel}) \) with \( T_{\parallel}/T_{\parallel} \approx \) constant, and making an appropriate combination of equations (20), (21), and (18), we obtain
\[ (\partial / \partial t + v_{\parallel} \cdot \nabla)(n_b \gamma_u^2 P_{\parallel} \gamma_u) = 0, \]
\[ \text{(29)} \]

which is, essentially, the equation of state for an adiabatic thermodynamic process. Here, from the scalings (27) and (28), we note that we have \( \kappa = 5/3 \) for a not too long beam (27), and \( \kappa = 2 \) for an extremely long beam (28).

### 3. Numerical results

Now, under the scalings (26) and (27), or (28), we can rewrite our basic equations (10) and (18)–(21) as
\[ [(c^2 / \omega_{pp}) n_{\perp} - 1] U_w = -m_0 u^2 / n_0 n_b, \]
\[ \text{(30)} \]

\[ \partial n_b / \partial t + \nabla \cdot (n_b v_b) = 0, \]
\[ \text{(31)} \]

\[ m_0 \gamma_u n_b \partial / \partial t + v_{\parallel} \cdot \nabla v_{\parallel} = \nabla \cdot (U_w - \tau n_b^{-1}), \]
\[ \text{(32)} \]

where \( U_w = e(\phi - u A_z), \tau = (T_{\parallel} / n_b^{-1})[\kappa / (\kappa - 1)], \) and \( \kappa \) takes the values \( 5/3 \) and 2 under the scalings (27) and (28), respectively. Using the scaled variables \( t'' = t'/T, \quad v_{\parallel}'' = v_{\parallel}/V, \quad r_{\parallel}'' = r_{\parallel}/R, \quad U_w = U_w/W, \) and \( n_b'' = n_b/N, \) where
\[ N = n_b \left( n_{bc} n_{bc} T_{bc} / m_0 u^2 \right)^{1 / (\kappa - 1)}, \]
\[ W = m_0 u^2 / n_{bc}, \quad V = u \sqrt{N / \gamma_u n_{\perp}}, \]
\[ T = c / \omega_{pp} \sqrt{N / N}, \quad R = c / \omega_{pp}, \]
\[ \text{(33)} \]

where \( n_{bc} \) and \( T_{bc} \) are the beam density and temperature in the centre of the structure, respectively, our equations (30)–(32) are rewritten in a dimensionless form and with no physical parameters involved, viz
\[ (\nabla^2 + 1) U_w = -n_b, \]
\[ \text{(34)} \]

\[ \partial n_b / \partial t + \nabla \cdot (n_b v_b) = 0, \]
\[ \text{(35)} \]

\[ (\partial / \partial t + v_{\parallel} \cdot \nabla) v_{\parallel} = \nabla \cdot (U_w - n_b^{-1}), \]
\[ \text{(36)} \]

where for simplicity, the notations \( '' \) are omitted hereafter. It is worth noting that in the derivation of the Poisson’s-like equation (11) or (34), which accounts for the plasma response to the beam density \( n_b \), we have neglected small terms of the order \( \partial / \partial t \) and \( v_{\parallel} \cdot \nabla \) (the latter are assumed to be of the same order as the convective terms \( v_{\parallel} \cdot \nabla \) in the hydrodynamic equations for the beam). Thus, equations (34)–(36) describe essentially an adiabatic evolution of the potential \( U_w \) in which the plasma instantaneously responds to the slow changes of the beam density \( n_b \). A stationary solution (denoted with the subscript ’0’), with \( \partial / \partial t = 0 \) is readily obtained, substituting \( n_{bc} = U_w^{1/\kappa} \) into the Poisson’s-like equation (34), yielding an equation from which all physical parameters have been scaled out, viz
\[ (\nabla^2 + 1) U_{w0} + U_w^{1/\kappa} = 0. \]
\[ \text{(37)} \]

When the beam is extremely long, i.e. in the strictly 2D case (28) with \( \kappa = 2 \), the above equation (37) is linear and can not possess a localised solution that is well-behaved at \( r = 0 \). An appropriate localisation in such strictly 2D regime is possible only in the presence of a finite fluid velocity, viz \( v_{\parallel} \neq 0 \), when the convective nonlinearity associated with the curlfree component of \( v_{\parallel} \) produces a nonlinear plasma wake force on the slow time scale. The latter drives the modulational instability which, eventually, saturates into a coherent soliton state described in [7–10].
Conversely, in the presence of a finite dependence along $z$, (27), our equation (37) contains a nonlinear term $U_{w_0}^{2/2}$ which produces the localisation of the solution. Physically, the localisation comes from the 3D adiabatic process described by the appropriate equation of state (29). The cylindrically symmetric solution of equation (37) is displayed in figure 1.

The stability of the adiabatic evolution of such solution with respect to small perturbations can be assessed if we linearise equations (34)–(36) around the stationary solution, setting $n_0 = n_{w_0} + \epsilon n_0$, $\tilde{v}_0 = \epsilon \tilde{v}_0$, and $U_w = U_{w_0} + \epsilon U_w$, when to the first order in the (small) bookkeeping parameter $\epsilon$ we obtain

$$n_{b1} = - (\nabla_\perp^2 - 1) U_{w_1}, \quad \nabla_\perp^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r}$$

$$- \omega^2 n_{b1} + \frac{1}{r} \frac{\partial}{\partial r} \left( n_{b0} r \frac{\partial}{\partial r} [U_{w_1} - (\kappa - 1)n_{b0} - n_{b1}] \right) = 0.$$  

(38)

(39)

Here we have applied the Fourier transformation in time and assumed a cylindrically symmetric perturbation. Multiplying equation (39) by $r |U_{w_1}^\kappa - (\kappa - 1)n_{b0}^{\kappa-2} n_{b1}|^2$, where $k$ denotes the complex conjugate, doing some simple manipulations and taking that all quantities vanish at $r \to \infty$, we arrive at

$$\omega^2 = \int_0^\infty r \mathrm{d}r \int_0^\infty r \mathrm{d}r |U_{w_0}^\kappa - (\kappa - 1)n^{\kappa-2} n_{b1}|^2$$

$$\int_0^\infty r \mathrm{d}r |U_{w_0}^\kappa|^2 + |\partial U_{w_1}/\partial r|^2 - (\kappa - 1)n_{b0}^{\kappa-2} |n_{b1}|^2.$$  

(40)

Obviously, the right-hand side of equation (40) is a real quantity and its numerator is positive definite, which implies that small perturbations of the stationary solution are either purely growing/damped ($\omega^2 < 0$) or purely oscillating functions of time ($\omega^2 > 0$), depending on the sign of the denominator. Using the following simple estimate, we can indicate that both oscillating and growing/damped solutions may exist. If we estimate the stationary beam profile by a rectangular shape, $n_{b0}(r) \approx n_{b0} h(r_m - r)$ with $n_{b0} \approx 4$ and $r_m \approx 2.5$, we can rewrite equation (39) simply as

$$(\nabla_\perp^2 + k_0^2)(\nabla_\perp^2 + k_1^2)U_{w_1} = 0,$$  

when $r < r_m$

$$(\nabla_\perp^2 - 1)U_{w_1} = 0,$$  

when $r > r_m$.  

(41)

where

$$k_\pm^2 = \frac{1}{2(\kappa - 1)n_{b0}^\kappa} \left[ \omega^2 + n_{b0} - (\kappa - 1)n_{b0}^{\kappa-1} \pm \sqrt{[\omega^2 + n_{b0} - (\kappa - 1)n_{b0}^{\kappa-1}]^2 + 4 \omega^2(\kappa - 1)n_{b0}^{\kappa-1}}. $$

(42)

A small perturbation is now readily written as $U_{w_1} = a_1 J_0(k_\pm r) + a_- J_0(k_\pm r)$ for $r < r_m$ and $U_{w_1} = b K_0(r)$ for $r > r_m$, where $a_\pm$ and $b$ are constants of integrations. From the requirements that $U_{w_1}, \partial U_{w_1}/\partial r$, and $\nabla_\perp^2 U_{w_1}$ are continuous functions at the edge of the beam, $r = r_m$ (i.e. that the Coulomb and pressure forces on the electron fluid are finite and that there are no surface charges at the edge of the beam) and eliminating the constants of integration $a_\pm$ and $b$ we readily obtain the dispersion relation for the frequency of the linear mode $D(\omega^2) = 0$, where

$$D(\omega^2) = k_\pm(1 + k_\pm^2) J_0(k_\pm r_m) J_0(k_\pm r_m)$$

$$- k_\pm^2 K_0(r_m).$$  

(43)

Dispersion function $D(\omega^2)$ is displayed in figure 2. We can readily see that there exist two unstable modes featuring $\omega^2 < 0$, with the more unstable one corresponding to the branch point of the characteristic wavenumbers $k_\pm$, see equation (42). Besides these, owing to the periodicity of the Bessel functions $J_0$ and $J_1$, there is also a sequence of oscillating solutions with purely real frequencies, i.e. with $\omega^2 > 0$. A numerical solution of the dispersion relation $D(\omega^2) = 0$, using equation (42) with $n_{b0} = 4$ and $r_m = 2.5$ yields $\omega^2 = -9, -0.633, 1.820, 8.415, 18.49, 31.82, 48.36, ...$

Using the normalisations defined in equation (33) and $\kappa = 3/2$, we find that the effective width of the stationary

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Cylindrically symmetric stationary structure, found as the solution of equation (37) for $U_{w_0}$ (red) and the corresponding $n_{b0}$ (black), in a three-dimensional case with $\kappa = 5/3$.}
\end{figure}
state of the beam is \( r_m \sim 2.5 e / \omega_{pp} \), while the characteristic frequencies and growthrates of its linear perturbations scale (in physical units) as \( \omega \sim \omega_{pp} (n_{e0} / \gamma_e n_{b0})^{2} (T_{b0} / m_0 c^2) \sim \omega_{pp} / \gamma_e^2 \).

We have also solved numerically the full system of equations (34)–(36), see figures 3 and 4. The solution appears to be spreading when the beam contains less electrons than the stationary state shown in figure 1, and collapsing when it contains more electrons than the stationary state. Conversely, when we adopted an initial condition that contained the same number of electrons as the stationary state but with a different spatial profile, the solution remained stable for a considerably longer time, and even performed a limited number of oscillations (one or two) before the final collapse or spreading took place. Such behaviour comes from the fact that, with our somewhat arbitrary choice of the initial beam profile, besides the stable oscillating linear modes we have inevitably excited also the linearly unstable modes that eventually prevailed in our numerical example. However, the growthrate scales as \( \sim \omega_{pp} / \gamma_e^2 \) and in an experiment we can finely tune the initial beam profile so that such blowup or dispersion of the beam (i.e. the selffocussing or defocussing) is sufficiently delayed so that other physical effects enter the picture, including the two-stream instability, longitudinal (paraxial) variation, and the nonlinear plasma wake force that can be responsible for the creation of different coherent nonlinear states such as 2D solitons.

It is important to note that the maximum value (in physical units) of the stationary solution (37), displayed in figure 1, is given by \( n_{e0} \approx 4N \). Using the normalisations equation (33), this readily yields the condition for the stationary state in the form \( T_{b0} / m_0 c^2 = C n_{e0} / n_{b0} \) with \( C \approx 0.63 \). This expression coincides with that found earlier in the Vlasov description, using a macroscopic theory and the virial theorem, see equation (35) in [8], where the same functional dependence was obtained with the value \( C \approx 1/2 \). Such small discrepancy probably comes from the deviation of our solution (37) from the exact Gaussian, that was used in [8] as the initial condition.

4. Conclusions

In this paper we presented a self-consistent nonlinear hydrodynamic theory of the propagation of a long and thin relativistic electron beam through an unmagnetized and overdense plasma, in a configuration that is typical for the plasma...
wakefield acceleration scheme. The random component of the trajectories of the beam particles and of their velocities was modelled by an effective anisotropic temperature. It was demonstrated that in the presence of a finite velocity spread in the parallel direction and for not too long a beam, the beam dynamics could be approximated as a fully 3D adiabatic expansion/compression. The resulting nonlinearity provided the localisation of the beam in the transverse direction and produced a coherent stationary state, even when the effects of the nonlinear plasma wake force are small or absent. The linear analysis revealed that a small perturbation of such coherent state could be unstable, resulting either in the defocussing or focussing of the beam when the number of the beam particles was initially larger or smaller than that of the stationary state. Numerical calculations demonstrated that the lifetime of the beam could be significantly extended by the appropriate fine tuning, so that the transverse oscillations predicted earlier within the TWM, could be observed. Conversely, in a system that is predominantly two dimensional, this kind of thermodynamic nonlinearity vanishes and on the long timescale the system is governed by the plasma wake force responsible also for the betatron-like oscillations observed in the earlier works [7–10].

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