Research article

On the pulsating \((m, c)\)-Fibonacci sequence

Kittipong Laipaporn, Kiattiyot Phibul, Prathomjit Khachorncharoenkul

School of Science, Walailak University, Nakhon Si Thammarat, 80160, Thailand

**Abstract**

In this paper, we study new ideas in the generalization of additive and multiplicative pulsating Fibonacci sequences. Then, we construct two types of pulsating Fibonacci sequences of the \(m\)th order. Moreover, the closed forms of the two sequences are derived by basic linear algebra.

1. Introduction

K. T. Atanassov et al. [1, 2, 3] proposed the following four 2-Fibonacci sequences in “additive and multiplicative” schemes:

\[
\begin{align*}
a_0 &= a, b_0 = b, a_1 = c, b_1 = d, \\
a_{n+2} &= a_{n+1} + b_n, \\
\beta_{n+2} &= a_{n+1} + \beta_n, \\
\alpha_{n+2} &= a_{n+1} + a_n, \\
\gamma_{n+2} &= a_{n+1} + \gamma_n.
\end{align*}
\]

where \(a_0 = a, b_0 = b, \gamma_0 = c\), \(a_1 = d, b_1 = e, \beta_1 = f\),

\[
\begin{align*}
a_{n+2} &= x_{n+1} + x_n, \\
\beta_{n+2} &= x_{n+1} + y_n, \\
\alpha_{n+2} &= x_{n+1} + x_n, \\
\gamma_{n+2} &= x_{n+1} + y_n.
\end{align*}
\]

In this paper, we study new ideas in the generalization of additive and multiplicative pulsating Fibonacci sequences. Then, we construct two types of pulsating Fibonacci sequences of the \(m\)th order. Moreover, the closed forms of the two sequences are derived by basic linear algebra.

**Keywords:**

- Fibonacci sequence
- Pulsating sequence
- Diagonalization
- Eigenvalue
- Matrix decomposition

\[a_0 = a, b_0 = b, \gamma_0 = c, \quad a_1 = d, b_1 = e, \beta_1 = f, \quad a_{n+2} = x_{n+1} + x_n, \quad \beta_{n+2} = x_{n+1} + y_n, \quad \gamma_{n+2} = x_{n+1} + y_n\]
Theorem 4. \(a_{k+1} = \beta a_k + \gamma a_{k-1}\), \(\beta, \gamma \in \mathbb{R}\) for any \(a, b, c \in \mathbb{R}\) and \(k \in \mathbb{N}\). These sequences are said to be the \((a, b, c)\)-pulsated Fibonacci sequence and the \((a, b, c)\)-pulsated Fibonacci sequence, respectively.

Note that the sequence (6) is the sequence (2) with regard to every even subscript. Moreover, by adding the initial conditions to (6), it becomes the sequence (7). In 2014, K. T. Atanasov [11] provided an extension of the sequence (6), namely, the \((a_1, a_2, \ldots, a_m)\)-pulsated Fibonacci sequence, which is defined as follows:

\[
\begin{align*}
\alpha_{1,0} &= a_1, \alpha_{2,0} = a_2, \ldots, \alpha_{m,0} = a_m, \\
\alpha_{2,1} &= a_1, \alpha_{2,3} = a_2, \ldots, \alpha_{2m+1} = \sum_{i=1}^{m} \alpha_{1,2i}, \\
\alpha_{3,2} &= a_1, \alpha_{3,4} + \alpha_{2m+1} - a_{m-1}, \\
\alpha_{3,2k+2} &= a_1, \alpha_{3,2k+4} + \alpha_{2m+1} - a_{m-2k-1},
\end{align*}
\]

for any non-negative integers \(j, k, m\) such that \(a_1, a_2, \ldots, a_m \in \mathbb{R}\) and \(1 \leq j \leq m\). In 2015, A. Suvanrai [12] transformed the sequence (6) into a multiplicative version, and in 2017, he [13] introduced a new idea, i.e., a generalization of pulsating Fibonacci sequences called multiplicative pulsating \(m\)-Fibonacci sequences. His inspiration may have been the \(m\)-fold multiplication of the term \(a\) in (6). The sequence can be defined as follows:

Let \(a_1, a_2, \ldots, a_m\) be real numbers, with

\[
\begin{align*}
\alpha_{1,0} &= a_1, \alpha_{2,0} = a_2, \ldots, \alpha_{m,0} = a_m, \\
\alpha_{2,1} &= a_1, \alpha_{2,3} = a_2, \ldots, \alpha_{2m+1} = \sum_{i=1}^{m} \alpha_{1,2i}, \\
\alpha_{3,2} &= a_1, \alpha_{3,4} + \alpha_{2m+1} - a_{m-1}, \\
\alpha_{3,2k+2} &= a_1, \alpha_{3,2k+4} + \alpha_{2m+1} - a_{m-2k-1},
\end{align*}
\]

for all \(j \in \{1, 2, \ldots, m\}\) and the non-negative integer \(k\).

All closed forms in [9, 10, 11, 12, 13] were proved by mathematical induction, which is not the only method for obtaining the closed forms. Furthermore, this method can be used when patterns of the closed forms are predicted. Nevertheless, we confront recurrence relation problems that require various methods to solve. Thus, basic linear algebra is used in our work.

In this paper, we apply basic linear algebra, especially the matrix method, eigenvalues and eigenvectors, to find the closed form of two new types of pulsating \(m\)-Fibonacci sequence. We show some results in Section 2 which are essential to acquiring the main result. Section 3 is devoted to the main result. First, we study the sequence (9) in additive schemes in Theorem 4. The sequence in Theorem 4 is a generalization of the \((a, b)\)-pulsated Fibonacci sequence in (6). Next, we take the logarithm function of sequence (9) when all values of \(a_{i,0}\) are positive and apply Theorem 4 to provide the closed form in [13] which is shown in Corollary 6. In the last section, we discuss our work and future research.

2. Preliminaries

In this section, we establish some lemmas to obtain our main result, i.e., Theorem 4. First, we introduce some important notations. Throughout this paper, let \(I_n\) be an \(m\times m\) identity matrix and \(J_n\) be an \(m\times m\) matrix in which every entry is one. Let \(K_n\) be an \(m\times m\) reversal matrix that is a permutation matrix in which \(k_{i,j} = 1\) for \(i = 1, 2, \ldots, m\) and all other entries are zero.

Now, the following lemmas are used to find the closed form of the sequence (10) in Theorem 4.

**Lemma 1.** For any real number \(c\) and integer \(m \geq 2\), the eigenvalues of matrix \(U = (c + 1)J_m - I_m\) are \((c + 1)m - 1\) of multiplicity 1 and \(1\) of multiplicity \(m - 1\) and the corresponding eigenvectors are \([1]_{m=1}\) and \([1']_{m=1}\) for \(j = 1, 2, \ldots, m - 1\), where

\[
u_j' = \begin{cases} 
-1; & j = 1 \\
1; & j = j + 1 \\
0; & \text{otherwise}
\end{cases} \quad \forall j.
\]

**Proof.** Note that \(U[1]_{m=1} = [(c + 1)m - 1]_{m=1}\) is the eigenvalue of the matrix \(U\) with the corresponding eigenvector \([1]_{m=1}\). Next, we will show that \(U[1']_{m=1} = -[1']_{m=1}\) for all \(j \in \{1, 2, \ldots, m - 1\}\).

Let \(U = [u_{ij}]\) and \(u_{ij}'\) be the \(i\)th column of \(U[1]_{m=1}\). Then,

\[
u_j' = \sum_{i=1}^{m} u_{ij} = -u_{11} + u_{(j+1)} \begin{cases} 
-c + u_{1(j+1)}; & j = 1 \\
-(c + 1) + u_{1(j+1)}; & i \neq 1
\end{cases} \quad \forall j.
\]

Hence, we have the desired result.

**Lemma 2.** Let \(U = [u_{ij}]\) be the matrix in Lemma 1. Then, \(U^n = [u_{ij}']\), where

\[
u_{ij}' = \begin{cases} 
\frac{1}{m} \left[ ((c + 1)m - 1)^n - (-1)^n(m - 1) \right] ; & i = j \\
\frac{1}{m} \left[ ((c + 1)m - 1)^n - (-1)^n \right] ; & \text{otherwise}
\end{cases} \quad \forall j.
\]

for any positive integer \(n\). That is, all the main diagonal elements have the same value, and all elements outside the main diagonal have the same value.

**Proof.** By Lemma 1, we know that \(U^n = PD^n P^{-1}\), where

\[
P = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \quad \text{and} \quad D = \text{diag}(\{(c + 1)m - 1\}, \ldots, 1).
\]

On the other hand, the matrices \(P = [p_{ij}]\), \(P^{-1} = [p_{ij}']\) and \(D = [d_{ij}]\) can be written in the form

\[
p_{ij} = \begin{cases} 
1; & 1 \leq i \leq m, j = 1 \\
1; & i = j \in \{2, 3, \ldots, m\} \\
0; & \text{otherwise}
\end{cases} \quad \text{and} \quad p_{ij}' = \begin{cases} 
1/m; & i = 1, 1 \leq j \leq m \\
1/m; & i = j \in \{2, 3, \ldots, m\} \\
0; & \text{otherwise}
\end{cases}
\]

and

\[
d_{ij} = \begin{cases} 
(c + 1)m - 1)^n ; & i = j \\
(-1)^n ; & i = j \in \{2, 3, \ldots, m\} \\
0; & \text{otherwise}
\end{cases}
\]

Thus, for each \(i, j \in \{1, 2, \ldots, m\}\), the \((i, j)\)th entry element of \(U^n\) is

\[
u_{ij}' = \sum_{k=1}^{m} p_{ik} d_{kj} p_{kj}' = \sum_{k=1}^{m} p_{ik} d_{kj} p_{kj}' = \sum_{k=1}^{m} p_{ik} d_{kj} p_{kj}' = \sum_{k=1}^{m} \frac{1}{m} \left[ ((c + 1)m - 1)^n \right] + (-1)^n \sum_{k=1}^{m} p_{ik} d_{kj} p_{kj}'
\]

Since \(PP^{-1} = I_m\) and \(p_{ij} d_{kj} p_{kj}' = (1/m)\), it follows that

\[
\sum_{k=1}^{m} p_{ik} d_{kj} p_{kj}' = \sum_{k=1}^{m} p_{ik} d_{kj} p_{kj}' = \left\{ \begin{array}{ll}
1 - \frac{1}{m}; & i = j \\
1 - \frac{1}{m}; & i \neq j
\end{array} \right.
\]

Hence, we have the desired result.

The structure of \(U^n\) directly affects the following lemma.
Lemma 3. For any real number $c$ and integer $n \geq 3$, let $A = U^{-1} - cU^{-2}J_m$, $B = (I_m - J_m)U^{-n} + U^{n-1}$ and $C = B(J_m - I_m)$ be given. Then,

1. $A = U^{n-1} - cU^{-2}J_m$.
   
   In particular, we have $A = [a_{ij}]$, where
   
   $a_{ij} = \begin{cases} 
   \frac{1}{m}[(c+1)m-1]^{i+j}(-1)^{n-1}(m-1) & : i = j \\
   \frac{1}{m}[(c+1)m-1]^{i}(-1)^{i+j}(m-1) & : i \neq j.
   \end{cases}$

2. $B = c((c+1)m-1)J_m$.
3. $C = c((c+1)m-1)J_m$.

Proof. By Lemma 2, we have

$U^{n-1}J_m = \frac{1}{m}[((c+1)m-1)^{i+j}(-1)^{n-1}(m-1)]J_m$

$+ \frac{m-1}{m}[(c+1)m-1]^{i+j}(-1)^{i+j}(m-1)]J_m$

$= ((c+1)m-1)^{i+j}(\frac{1}{m} + \frac{m-1}{m})J_m$

$= ((c+1)m-1)^{i+j}J_m.

Since $U = (c+1)J_m - I_m$, we determine that $J_m - I_m = U - cJ_m$. Then, we obtain

$A = U^{n-2}(J_m - I_m) = U^{n-2}(U - cJ_m) = U^{n-1} - cU^{n-2}J_m$.

Thus, by Lemma 2, we can write $A = [a_{ij}m]$, where

$a_{ij} = \begin{cases} 
\frac{1}{m}[(c+1)m-1]^{i+j}(-1)^{n-1}(m-1) & : i = j \\
\frac{1}{m}[(c+1)m-1]^{i}(-1)^{i+j}(m-1) & : i \neq j.
\end{cases}$

2. Since $U^{n-2}$ is a symmetric matrix, it follows that $J_mU^{n-2} = U^{n-2}J_m$. By the feature of $U^{n-2}J_m$, we have

$B = (I_m - J_m)U^{n-1} + U^{n-1} = U^{n-2} - U^{n-2}J_m + U^{n-1}$

$= -A + U^{n-1} = c((c+1)m-1)^{i+j}J_m$

3. By the structure of $J_m$, we can see that $J_m^2 = mj_m$. We obtain

$C = B(J_m - I_m) = c((c+1)m-1)^{i+j}(J_m - I_m)$

$= c((c+1)m-1)^{i+j}(J_m^2 - I_m) = c(m-1)((c+1)m-1)^{i+j}J_m$.

3. Main results

In this section, we will prove the main result, i.e., Theorem 4. This theorem plays an important role in Corollary 5 and Corollary 6.

Let $a_1, a_2, \ldots, a_m$ and $b$ be real numbers, with

$a_{1,1} = a_1, a_{1,2} = a_2, \ldots, a_{m,0} = a_m,$

$a_{1,2k+1} = a_{2,2k+1} = \cdots = a_{m,2k+1} = c \sum_{i=1}^{m} a_{i,2k},$  \hspace{1cm} (10)

$a_{j,2k+2} = a_{j,2k+1} + \sum_{i=1}^{m} a_{i,j}$

for any non-negative integers $j, k, m$ such that $1 \leq j \leq m$ and $c \neq 0$. This sequence is called an additive pulsating $(m,c)$-Fibonacci sequence.

Theorem 4. For all positive integers $j, k, m$ such that $j \leq m, k \geq 2$ and $c \in \mathbb{R}\setminus\{0\}$, the closed form of the additive pulsating $(m,c)$-Fibonacci sequence is

$a_{1,2k-1} = a_{2,2k-1} = \cdots = a_{m,2k-1} = c((c+1)m-1)^{k-1}\sum_{i=1}^{m} a_i,$

$a_{j,2k} = \frac{c}{m} \left[\sum_{i=1}^{m} a_i \right] + \frac{c}{m} ((c+1)m-1)^{k-1}\sum_{i=1}^{m} a_i.$

Proof. Define a linear transformation $T: \mathbb{R}^{2m} \rightarrow \mathbb{R}^{2m}$ by

$T(p_1, p_2, \ldots, p_m, q_1, q_2, \ldots, q_m) = (x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m), \ldots,$

where $x_j = q_j + \sum_{i=1}^{m} p_i$ and $y_j = (c(m-1)) \sum_{i=1}^{m} q_i + \sum_{i=1}^{m} q_i$ for all $j = 1, 2, \ldots, m$.

Clearly, the matrix representation of $T$ with respect to the standard basis is

$Q = \left[ I_m - J_m \right] \left[ c(m-1)J_m \right].$

Let $k \in \mathbb{N}$ and $X_k = \begin{bmatrix} a_{1,2k-2} & \cdots & a_{m,2k-2} & a_{1,2k-1} & \cdots & a_{m,2k-1} \end{bmatrix}$ and $QX_k = X_k+1$.

Hence, the main result will be obtained, that is, $X_k = Q^{k-1}X_1$ where $k$ is greater than 1.

To obtain the features of $X_k$, we will find the structure of $Q^{k-1}$. For the matrix $U$ in Lemma 1, we know that

$I_m - J_m + U = cJ_m$ and $(I_m - J_m + U)(J_m - I_m) = c(m-1)J_m$.

In case $k = 2$, it is easy to see that $X_2 = QX_1$ and we have

$a_{j,3} = c((c+1)m-1) \sum_{i=1}^{m} a_i$ and $a_{j,2} = c \sum_{i=1}^{m} a_i$ where $1 \leq j \leq m$.

From now on, we consider $k \geq 3$. In facts, the matrix $Q$ can be rewritten in the following form:

$Q = \begin{bmatrix} I_m & 0_n \end{bmatrix} \begin{bmatrix} I_m - J_m & I_m \end{bmatrix} \begin{bmatrix} 0_m & I_m \end{bmatrix} \begin{bmatrix} I_m - I_m & 0_m \end{bmatrix}.$

Since the inverse of a matrix $\begin{bmatrix} I_m & 0_n \end{bmatrix} \begin{bmatrix} I_m - J_m & I_m \end{bmatrix} \begin{bmatrix} 0_m & I_m \end{bmatrix} \begin{bmatrix} I_m - I_m & 0_m \end{bmatrix}$, it implies that

$Q^{k-1} = \begin{bmatrix} I_m & 0_n \end{bmatrix} \begin{bmatrix} I_m - J_m & I_m \end{bmatrix} \begin{bmatrix} 0_m & I_m \end{bmatrix} \begin{bmatrix} I_m - I_m & 0_m \end{bmatrix} \begin{bmatrix} I_m & 0_n \end{bmatrix} \begin{bmatrix} I_m - J_m & I_m \end{bmatrix} \begin{bmatrix} 0_m & I_m \end{bmatrix} \begin{bmatrix} I_m - I_m & 0_m \end{bmatrix}.$

where $A$, $B$ and $C$ are defined as in Lemma 3.

In order to make it easier to prove, we set $X_1 = \begin{bmatrix} Y_1 & Y_2 \end{bmatrix}$ where

$Y_1 = [a_1, a_2, \ldots, a_m]^T$ and $Y_2 = \left[ \sum_{i=1}^{m} a_i \right]$. Then,

$X_k = Q^{k-1}X_1 = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} I_m - J_m & I_m \end{bmatrix} \begin{bmatrix} 0_m & I_m \end{bmatrix} \begin{bmatrix} I_m - I_m & 0_m \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} I_m - J_m & I_m \end{bmatrix} \begin{bmatrix} 0_m & I_m \end{bmatrix} \begin{bmatrix} I_m - I_m & 0_m \end{bmatrix}.$

where $k$ is greater than 1. After that, we will calculate the element $s_j$ by Lemma 2 and Lemma 3. By the features of $A$, we see that all its main diagonal entries are equal and the entries outside the main diagonal are equal. To write clear and concise content, we utilize $a_{ij}$ for the main diagonal entry and $a_{ij}$ for the other entry. Thus, we have
\[ AY_i = A \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \sum_{i=1}^{m} a_i & a_1 a_2 + a_2 \sum_{i=1}^{m} a_i & \cdots & a_1 a_m + a_2 \sum_{i=1}^{m} a_i \end{bmatrix}^T. \]

Similarly, under the features of \( U^{k+2} \) in Lemma 2, we utilize \( k^{2} \) \( \sum_{i=1}^{m} a_i \) for the main diagonal entry and \( k^{2} \) \( \sum_{i=1}^{m} a_i \) for the other entry. Then,

\[
U^{k+2}Y_2 = (c \sum_{i=1}^{m} a_i)U^{k+2}[1]_{\text{loc1}} = (c \sum_{i=1}^{m} a_i)\begin{bmatrix} E(u^{k+2}a_1) + (m - 1) E(u^{k+2}a_2) \end{bmatrix}_{\text{loc1}} = \begin{bmatrix} E(u^{k+2}a_1) + (m - 1) E(u^{k+2}a_2) \end{bmatrix}_{\text{loc1}}.
\]

Thus, we let \( j \in \{1, 2, \ldots, m\} \) and \( w_j \) be the \( j \)th row of the matrix \( AY_i + U^{k+2}Y_2 \). Then, by Lemma 3 (1), we have

\[
w_j = a_1 a_2 + a_2 \sum_{i=1}^{m} a_i + c(c + 1)m - 1 (k^{2} \sum_{i=1}^{m} a_i)
\]

\[
= \frac{m - 1}{m} \sum_{i=1}^{m} a_i + c(c + 1)m - 1 (k^{2} \sum_{i=1}^{m} a_i)
\]

\[
= \frac{m - 1}{m} \sum_{i=1}^{m} a_i + c(c + 1)m - 1 (k^{2} \sum_{i=1}^{m} a_i)
\]

\[
= \frac{m - 1}{m} \sum_{i=1}^{m} a_i + c(c + 1)m - 1 (k^{2} \sum_{i=1}^{m} a_i)
\]

\[
= \frac{m - 1}{m} \sum_{i=1}^{m} a_i + c(c + 1)m - 1 (k^{2} \sum_{i=1}^{m} a_i)
\]

\[
= \frac{m - 1}{m} \sum_{i=1}^{m} a_i + c(c + 1)m - 1 (k^{2} \sum_{i=1}^{m} a_i)
\]

\[
= \frac{m - 1}{m} \sum_{i=1}^{m} a_i + c(c + 1)m - 1 (k^{2} \sum_{i=1}^{m} a_i)
\]

Last but not least, since \( AY_i Y_2 = Y_2, J_a Y_2 = mY_2 \), and Lemma 3, we obtain

\[
CY_1 + BY_2 = c(m - 1)(c + 1)m - 1 (k^{2} \sum_{i=1}^{m} a_i) + c((c + 1)m - 1) k^{2} \sum_{i=1}^{m} a_i = c((c + 1)m - 1) k^{2} \sum_{i=1}^{m} a_i.
\]

Finally, we conclude that for any integer \( k \geq 2 \), the matrix \( X_k \) is \([s_j]_{\text{loc1}}\) where

\[
s_j = \begin{cases} \frac{(m - 1)}{m} \sum_{i=1}^{m} a_i + c(c + 1)m - 1 (k^{2} \sum_{i=1}^{m} a_i) : 1 \leq j \leq m \\ \frac{(m - 1)}{m} \sum_{i=1}^{m} a_i + c(c + 1)m - 1 (k^{2} \sum_{i=1}^{m} a_i) : m + 1 \leq j \leq 2m. \end{cases}
\]

This completes the proof.

To illustrate Theorem 4, we provide the following example.

**Example 1.** Let \( a, b, \) and \( c \) be real numbers with

\[
a_0 = a, b_0 = b, \gamma_0 = c.
\]

\[
a_{2n+1} = \beta_{2n+1} = \gamma_{2n+1} = a_{n+1} + b_n + \gamma_n,
\]

\[
a_{2n+2} = a_{2n+1} + b_{2n+1} + \gamma_{2n+1}, \]

\[
\beta_{2n+2} = a_{2n+1} + b_{2n+1} + \gamma_{2n+1}, \]

\[
\gamma_{2n+2} = a_{2n+1} + b_{2n+1} + \gamma_{2n+1}
\]

for every non-negative integer \( n \).

We will find the closed form of this sequence using the process in Theorem 4. The above sequence leads us to define a linear transformation \( T \) on \( \mathbb{R}^3 \) by

\[
T(p_1, p_2, p_3, q_1, q_2, q_3) = (q_1 + p_1 + p_2 + q_1 + p_3, p_1 + p_3, p_1 + p_3, y, y, y),
\]

where \( y = 2(p_1 + p_2 + p_3) + (q_1 + q_2 + q_3) \). Then, the matrix \( Q \) in Theorem 4 is

\[
\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 \end{bmatrix}
\]

and we can see that \( U = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \). In addition, by Lemma 2, it follows that

\[
U^n = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5^n & 0 & 0 \\ 0 & (-1)^n & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \beta_3 & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 5^n & 0 & 0 \\ 0 & (-1)^n & 0 \end{bmatrix}
\]

\[
= \begin{bmatrix} \frac{1}{3} s_0^n + 2(-1)^n s_1^n & 1 & \frac{1}{3} s_0^n + 2(-1)^n s_1^n \\ \frac{1}{3} s_0^n + 2(-1)^n s_1^n & 1 & \frac{1}{3} s_0^n + 2(-1)^n s_1^n \end{bmatrix}
\]

and by Lemma 3, if we let \( a_1 = \frac{1}{3} s_0^n + 2(-1)^n s_1^n, a_2 = \frac{1}{3} s_0^n - (-1)^n s_1^n, b_1 = 2 (\frac{1}{3} s_0^n + 2(-1)^n s_1^n) \) and \( b_2 = \frac{1}{3} \), then we have

\[
\begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix} = B = \begin{bmatrix} b_1 & b_2 & b_2 \\ b_2 & b_2 & b_2 \end{bmatrix} \begin{bmatrix} c \end{bmatrix} = \begin{bmatrix} s_0 & d_1 \end{bmatrix}
\]

From the initial conditions in (11),

\[
X_1 = \begin{bmatrix} a & b & c & a + b + c & a + b + c \end{bmatrix}
\]

which implies that

\[
X_n = Q^{n-1} X_1 = \begin{bmatrix} (a + b + c) \end{bmatrix}
\]

Hence, the closed form of this sequence is
\[ a_{2m-1} = \beta_{2m-1} = \gamma_{2m-1} = 5^{m-1}(a + b + c), \]
\[ a_{2m+2} = \frac{1}{3}(-1)^m(-2a + b + c) + \frac{1}{3}5^{m-1}(a + b + c), \]
\[ \beta_{2m-2} = \frac{1}{3}(-1)^m(a + b + c) + \frac{1}{3}5^{m-1}(a + b + c), \]
\[ \gamma_{2m-2} = \frac{1}{3}(-1)^m(a + b - 2c) + \frac{1}{3}5^{m-1}(a + b + c) \]

where \( n \) is any positive integer.

Next, by replacing \( c \) with 1 in the sequence (10), we obtain the following sequence: Let \( a_1, a_2, \ldots, a_n \) be any real numbers, with
\[
a_{1n} = a_1, a_{2n} = a_2, \ldots, a_{mn} = a_m,
\]
\[
a_{1,2k+1} = a_{2,2k+1} = \cdots = a_{m,2k+1} = \sum_{i=1}^{m} a_{i,2k}, \quad (12)
\]
\[
a_{j,2k+1} = a_{j,2k+1} + \sum_{i \neq j}^{m} a_{i,2k}
\]

for any non-negative integers \( j, k, m \) such that \( 1 \leq j \leq m \). Thus, the corresponding closed form of this sequence is provided by the following corollary.

**Corollary 5.** For all positive integers \( j, k, m \) and \( j \leq m \), the closed form of the pulsating \( m \)-Fibonacci sequence (12) is
\[
a_{1,2k+1} = a_{2,2k+1} = \cdots = a_{m,2k+1} = (2m - 1)^{k-1} \sum_{i=1}^{m} a_i,
\]
\[
a_{j,2k+1} = \frac{(-1)^k}{m} \left( -(m - 1) a_j + \sum_{i \neq j}^{m} a_i \right) + \frac{1}{m} (2m - 1)^{k-1} \sum_{i=1}^{m} a_i.
\]

**Proof.** This follows directly by substituting \( c = 1 \) in the result of Theorem 4.

Moreover, by taking the logarithm of (9), we obtain
\[
\beta_{1,0} = \ln a_1, \beta_{2,0} = \ln a_2, \ldots, \beta_{m,0} = \ln a_m,
\]
\[
\beta_{1,2k+1} = \beta_{2,2k+1} = \cdots = \beta_{m,2k+1} = \ln a_{m,2k+1} = \ln \left( \prod_{i=1}^{m} a_{i,2k} \right), \quad (13)
\]
\[
\beta_{j,2k+2} = \ln a_{j,2k+2} = \ln (a_{j,2k+1} + \sum_{i \neq j}^{m} a_{i,2k}) = \beta_{j,2k+1} + \sum_{i \neq j}^{m} \beta_{i,2k}
\]

for positive real numbers \( a_1, a_2, \ldots, a_m \) and non-negative integers \( j, k, m \) such that \( 1 \leq j \leq m \). If we consider the variable \( \beta \) in terms of \( a \) in (12), then the closed form of (9) is the following corollary.

**Corollary 6.** For all positive integers \( j, k, m \) and \( 1 \leq j \leq m \), the closed form of the multiplicative pulsating \( m \)-Fibonacci sequence (9) is
\[
a_{1,2k+1} = a_{2,2k+1} = \cdots = a_{m,2k+1} = \left( \prod_{i=1}^{m} a_i \right)^{(2m-1)^{k-1}},
\]
\[
a_{j,2k+2} = (a_j) \left( \prod_{i \neq j}^{m} a_i \right)^{(2m-1)^{k-1} - (m-1)^{k-1}},
\]

which is the same as Suvarnmani’s result.

**Proof.** Let each \( \beta_i \) in the sequence (13) be the \( a_{ij} \) in the sequence (12). Applying Corollary 5, we get
\[
\beta_{1,2k+1} = \beta_{2,2k+1} = \cdots = \beta_{m,2k+1} = (2m - 1)^{k-1} \sum_{i=1}^{m} \ln a_i,
\]
\[
\beta_{j,2k+2} = \frac{(-1)^k}{m} \left( -(m - 1) \ln a_j + \sum_{i \neq j}^{m} \ln a_i \right) + \frac{1}{m} (2m - 1)^{k-1} \sum_{i=1}^{m} \ln a_i.
\]

And using the properties of the logarithmic function as the desired result.

**4. Conclusion and discussion**

In the last ten years, many researchers have investigated abundant types of pulsating Fibonacci sequences, which are generalizations of the Fibonacci sequence. They use mathematical induction to verify the closed form of these sequences. This method is beautiful; however, it can only be used when patterns of the closed forms are predicted.

In this paper, we use the matrix method to bridge the gap to find the closed forms. Moreover, we study new ideas in the generalization of additive and multiplicative pulsating Fibonacci sequences. Furthermore, we construct two types of pulsating Fibonacci sequences of the \( m \)th order. These types generalize other past efforts in pulsating Fibonacci sequences.

One future research direction in this area is the extension and characterization of additive and multiplicative pulsating Fibonacci sequences by using permutation of some substrates.

**Declarations**

**Author contribution statement**

K. Laipaporn, K. Phibul, P. Khachorncharoenkul: Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

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The authors declare no conflict of interest.

**Additional information**

No additional information is available for this paper.

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