Decay estimates for Schrödinger heat semigroup with inverse square potential in Lorentz spaces II

Kazuhiro Ishige and Yujiro Tateishi
Graduate School of Mathematical Sciences, The University of Tokyo
3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan

e-mail address:
ishige@ms.u-tokyo.ac.jp (K. Ishige), tateishi@ms.u-tokyo.ac.jp (Y. Tateishi)

Abstract
Let $H := -\Delta + V$ be a nonnegative Schrödinger operator on $L^2(\mathbb{R}^N)$, where $N \geq 2$ and $V$ is a radially symmetric inverse square potential. Let $\|\nabla^\alpha e^{-tH}\|_{(L^p,\sigma \rightarrow L^q,\theta)}$ be the operator norm of $\nabla^\alpha e^{-tH}$ from the Lorentz space $L^{p,\sigma}(\mathbb{R}^N)$ to $L^{q,\theta}(\mathbb{R}^N)$, where $\alpha \in \{0, 1, 2, \ldots\}$. We establish both of upper and lower decay estimates of $\|\nabla^\alpha e^{-tH}\|_{(L^p,\sigma \rightarrow L^q,\theta)}$ and study sharp decay estimates of $\|\nabla^\alpha e^{-tH}\|_{(L^p,\sigma \rightarrow L^q,\theta)}$. Furthermore, we characterize the Laplace operator $-\Delta$ from the viewpoint of the decay of $\|\nabla^\alpha e^{-tH}\|_{(L^p,\sigma \rightarrow L^q,\theta)}$. 
1 Introduction

This paper is concerned with the decay of derivatives of Schrödinger heat semigroups $e^{-tH}$, where $H := -\Delta + V$ is a nonnegative Schrödinger operator in $L^2(\mathbb{R}^N)$. Throughout this paper we assume the following condition $(V_m)$, where $m \in \{1, 2, \ldots, \infty\}$:

\begin{align*}
(i) & \quad V = V(|x|) \text{ in } \mathbb{R}^N \setminus \{0\} \text{ and } V \in C^m((0, \infty)); \\
(ii) & \quad V(r) = \lambda_1 r^{-2} + O(r^{-2+\rho_1}) \text{ as } r \to +0, \\
& \quad V(r) = \lambda_2 r^{-2} + O(r^{-2+\rho_2}) \text{ as } r \to \infty, \\
& \quad \text{for some } \lambda_1, \lambda_2 \in [\lambda_*, \infty) \text{ with } \lambda_* := -(N - 2)^2/4 \text{ and } \rho_1, \rho_2 > 0; \\
(iii) & \quad \sup_{r > 0} \left| r^{\ell+2} \frac{d^\ell}{dr^\ell} V(r) \right| < \infty \text{ for } \ell \in \{1, \ldots, m\},
\end{align*}

and investigate decay estimates of $\|\nabla^\alpha e^{-tH}\|_{(L^p, \sigma \to L^q, \theta)}$, where $\alpha \in \{0, 1, \ldots, m+1\}$. Here $\|\nabla^\alpha e^{-tH}\|_{(L^p, \sigma \to L^q, \theta)}$ is the operator norm from the Lorentz space $L^p,\sigma(\mathbb{R}^N)$ to $L^q,\theta(\mathbb{R}^N)$, that is,

$$
\|\nabla^\alpha e^{-tH}\|_{(L^p, \sigma \to L^q, \theta)} := \sup \left\{ \|\nabla^\alpha e^{-tH} \phi\|_{L^q,\theta(\mathbb{R}^N)} : \phi \in C_c(\mathbb{R}^N) \text{ with } \|\phi\|_{L^p,\sigma(\mathbb{R}^N)} = 1 \right\}.
$$

Here

$$
(p, q, \sigma, \theta) \in \Lambda := \left\{ \begin{array}{cl}
\sigma = 1 & \text{if } p = 1, \quad \sigma = \infty & \text{if } p = \infty \\
1 \leq p \leq q \leq \infty & \sigma, \theta \in [1, \infty] : \theta = 1 & \text{if } q = 1, \quad \theta = \infty & \text{if } q = \infty
\end{array} \right\}.
$$

Nonnegative Schrödinger operators $H := -\Delta + V$ on $L^2(\mathbb{R}^N)$ and their heat semigroups $e^{-tH}$ have been studied by many mathematicians since the pioneering work due to Simon [38]. See e.g. [2], [5], [6], [8], [13], [14], [16]–[20], [26]–[31], [33]–[40], and references therein. (See also the monographs of Davies [7], Grigor’yan [10], and Ouhabaz [32].) The Schrödinger operator with an inverse square potential often appears in the field of Schrödinger operators and nonlinear PDEs such as semilinear parabolic equations, and the decay estimates of $e^{-tH} \phi$ and their derivatives are crucial for the study of the behavior of $e^{-tH} \phi$. For related results on the decay of $\|\nabla^\alpha e^{-tH}\|_{(L^p, \sigma \to L^q, \theta)}$, see e.g. [1], [8], [15], [16], [18], [22], [33], and references therein.

This paper is a continuation of our previous paper [22], where the authors of this paper obtained upper decay estimates of $\|\nabla e^{-tH}\|_{(L^p, \sigma \to L^q, \theta)}$, where $(p, q, \sigma, \theta) \in \Lambda$, under condition $(V_1)$. In this paper we develop the arguments in [22] and obtain upper decay estimates of $\|\nabla^\alpha e^{-tH}\|_{(L^p, \sigma \to L^q, \theta)}$ systematically, where $\alpha \in \{0, 1, \ldots, m+1\}$. Furthermore, we also establish lower decay estimates of $\|\nabla^\alpha e^{-tH}\|_{(L^p, \sigma \to L^q, \theta)}$. Combining both of upper and lower decay estimates, we study sharp decay estimates of $\|\nabla^\alpha e^{-tH}\|_{(L^p, \sigma \to L^q, \theta)}$. Furthermore, we give a new characterization of $-\Delta$ from the view point of the decay of $\|\nabla e^{-tH}\|_{(L^p, \sigma \to L^q, \theta)}$ (see Theorem 1.3).

We introduce some notations. Set $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$. Let $B(x, R) := \{y \in \mathbb{R}^N : |y-x| < R\}$ and $B(x, R)^c := \mathbb{R}^N \setminus B(x, R)$ for $x \in \mathbb{R}^N$ and $R > 0$. For any $r \in [1, \infty]$, let $r'$ be the Hölder conjugate number of $r$, that is,

$$
r' = \frac{r}{r-1} \quad \text{if} \quad 1 < r < \infty, \quad r' = 1 \quad \text{if} \quad r = \infty, \quad r' = \infty \quad \text{if} \quad r = 1.
$$
Let $\Delta_{S^{N-1}}$ be the Laplace-Beltrami operator on $S^{N-1}$. Let $\{\omega_k\}_{k=0}^{\infty}$ be the eigenvalues of
\begin{equation}
-\Delta_{S^{N-1}} Q = \omega Q \quad \text{on} \quad S^{N-1}, \quad Q \in L^2(S^{N-1}).
\end{equation}
Then $\omega_k = k(N + k - 2)$ for $k = 0, 1, 2, \ldots$. Let $\{Q_{k,i}\}_{i=1}^{\infty}$ and $d_k$ be the orthonormal system and the dimension of the eigenspace corresponding to $\omega_k$, respectively. Here
\begin{equation}
d_k = \frac{(N + 2k - 2)(N + k - 3)!}{(N - 2)!k!} = O(k^{N-2}) \quad \text{as} \quad k \to \infty.
\end{equation}

Assume condition $(V_m)$ and let $H := -\Delta + V$ be nonnegative, that is,
\begin{equation}
\int_{\mathbb{R}^N} |\nabla \phi|^2 + V(|x|)\phi^2 \, dx \geq 0, \quad \phi \in C^\infty_c(\mathbb{R}^N \setminus \{0\}).
\end{equation}
The operator $H$ is said subcritical if, for any $W \in C_c(\mathbb{R}^N)$, $H - \epsilon W$ is nonnegative for small enough $\epsilon > 0$. If not, the operator $H$ is said critical. For any $k \in \{0, 1, 2, \ldots\}$, set $A_{1,k} := A_{1,k}^+ + \omega_k$ and
\begin{equation}
A_{2,k} := \begin{cases}
A_{\lambda_2}^- & \text{if } k = 0 \text{ and } H \text{ is critical}, \\
A_{\lambda_2 + \omega_k}^+ & \text{otherwise},
\end{cases}
B_k := \begin{cases}
1 & \text{if } k = 0, \lambda_2 = \lambda_s \text{ and } H \text{ is subcritical}, \\
0 & \text{otherwise}.
\end{cases}
\end{equation}
Here
\begin{equation}
A_\lambda^\pm := -\frac{(N - 2) \pm \sqrt{D_\lambda}}{2} \quad \text{for} \quad \lambda \geq \lambda_s, \quad \text{where} \quad D_\lambda := (N - 2)^2 + 4\lambda. 
\end{equation}

By the standard theory for ordinary differential equations we see that, for any $k \in \{0, 1, 2, \ldots\}$, there exists a unique solution $h_k = h_k(r)$ to the problem
\begin{equation}
h_k'' + \frac{N - 1}{r}h_k' - V_k(r)h_k = 0 \quad \text{in} \quad (0, \infty), \\
h_k(r) = r^{A_{1,k}}(1 + o(1)) \quad \text{as} \quad r \to +0,
\end{equation}
where $V_k(r) := V(r + \omega_k r^{-2}$). (See also Section 2.1.) Notice that $h_k \in L^2(B(0, 1))$. Furthermore, it follows from [20, Theorem 1.1] that $h_k(r) > 0$ for $r > 0$ and
\begin{equation}
h_k(r) = c_k v_k(r)(1 + o(1)) \quad \text{as} \quad r \to \infty, \quad \text{where} \quad v_k(r) := r^{A_{2,k}}(\log r)^{B_k},
\end{equation}
for some $c_k > 0$. For $(k, i) \in \mathcal{K} := \{(k, i) : k = 0, 1, 2, \ldots, i = 1, \ldots, d_k\}$, setting
\begin{equation}
J_{k,i}(x) := h_k(|x|)Q_{k,i\frac{x}{|x|}},
\end{equation}
we see that $J_{k,i}$ is a harmonic function for $H$, that is, $HJ_{k,i} = 0$ in $\mathbb{R}^N \setminus \{0\}$. In particular, the function $h_0$ is said a positive harmonic function for the operator $H$. When $H$ is critical, if $h_0 \notin L^2(\mathbb{R}^N)$, then $H$ is said null-critical: if not, $H$ is said positive-critical. The decay of
the fundamental solution \( p = p(x, y, t) \) corresponding to \( e^{-tH} \) depends on whether \( H \) is either subcritical, null-critical or positive-critical. In particular, if \( H \) is positive-critical, then \( e^{-tH} \phi \) does not necessarily decay as \( t \to \infty \). See [33]. (See also [20].)

In this paper, under condition \((V_m)\), we assume either

\[
\begin{align*}
(i) & \quad H \text{ is subcritical or } (ii) \quad H \text{ is critical and } A_{2,0} > -N/2, \tag{N'}
\end{align*}
\]

and obtain both of upper and lower decay estimates of \( \| \nabla^\alpha e^{-tH} \|_{(L^p, \sigma \to L^q, \theta)} \), where \( \alpha \in \{0, 1, \ldots, m+1\} \) and \((p, q, \sigma, \theta) \in \Lambda\). Case (ii) is in the null-critical one (see also [22, Remark 1.1 (iii)]).

We state some results in this paper. These are obtained as applications of our upper and lower decay estimates in Sections 3 and 4. In what follows, for \((p, p, \sigma, \sigma) \in \Lambda\) and \( t > 0 \), set

\[
\Gamma_{p, \sigma}(t) := \frac{\| h_0 \|_{L^p, \sigma(B(0, \sqrt{t}))}}{h_0(\sqrt{t})} \quad \text{if } h_0 \in L^p, \sigma(B(0, 1)), \quad \Gamma_{p, \sigma}(t) := \infty \quad \text{if } h_0 \not\in L^p, \sigma(B(0, 1)).
\]

The first theorem clarifies the relationship between the decay of \( \| \nabla^\alpha e^{-tH} \|_{(L^p, \sigma \to L^q, \theta)} \) and harmonic functions for \( H \), where \( \alpha \in \{0, 1, 2\} \) and \((p, q, \theta, \sigma) \in \Lambda\).

**Theorem 1.1** Assume conditions \((V_1)\) and \((N')\). Let \((p, q, \sigma, \theta) \in \Lambda\) and \( \alpha \in \{0, 1, 2\} \). Then there exists \( C > 0 \) such that

\[
C^{-1} \Phi_\alpha(t) \leq \| \nabla^\alpha e^{-tH} \|_{(L^p, \sigma \to L^q, \theta)} \leq C \Phi_\alpha(t) \quad \text{for } t > 0. \tag{1.8}
\]

Here

\[
\Phi_\alpha(t) := \begin{cases} 
  t^{-\frac{N}{2}} \Gamma_{p', \sigma'}(t) \Gamma_{q, \theta}(t) & \text{if } \alpha = 0, \\
  t^{-\frac{N}{2}} \Gamma_{p', \sigma'}(t) \left[ \| \nabla^2 h_0 \|_{L^q, \sigma(B(0, \sqrt{t}))} + t^{\frac{N}{2}\theta} \right] & \text{if } \alpha = 1, \\
  t^{-\frac{N}{2}} \Gamma_{p', \sigma'}(t) \left[ \| \nabla^2 h_0 \|_{L^q, \sigma(B(0, \sqrt{t}))} + t^{\frac{N}{2}+\theta} \right] & \text{if } \alpha = 2.
\end{cases}
\]

**Remark 1.1** Let \( V \in C^1([0, \infty)) \) and assume conditions \((V_1)\) with \( \lambda_1, \lambda_2 \in (\lambda_*, \infty) \) and \((N')\). Then the sharp large time decay estimate of \( \| e^{-tH} \|_{(L^p, \sigma \to L^q, \theta)} \) has been already obtained in [14]. Our decay estimate \((1.8)\) with \( \alpha = 0 \) gives the same decay estimate as in [14] and it has a simpler expression.

In the second and the third theorems, under conditions \((V_m)\) and \((N')\), we characterize the Schrödinger operator \( H \) satisfying

\[
\| \nabla^\alpha e^{-tH} \|_{(L^p, \sigma \to L^q, \theta)} \leq Ct^{-\frac{N}{2}} \left( \frac{1}{p} \right)^{-\frac{N}{2}}, \quad t > 0,
\]

for some \( C > 0 \), where \( \alpha \in \{0, 1, \ldots, m\} \) and \((p, q, \sigma, \theta) \in \Lambda\).

**Theorem 1.2** Let \( m \in \{1, 2, \ldots\} \) and assume conditions \((V_m)\) and \((N')\).
(a) For any \((p, q, \sigma, \theta) \in \Lambda\) and \(\alpha \in \{0, 1, \ldots, m+1\}\), there exists \(C_1 > 0\) such that
\[
\|\nabla^\alpha e^{-tH}\|_{(L^p, \sigma \to L^q, \theta)} \leq C_1^{-1} t^{-\frac{N}{2} \left(\frac{1}{p} - \frac{1}{q}\right) - \frac{\alpha}{2}}, \quad t > 0.
\]

(b) Let \(V \neq 0\) in \((0, \infty)\) and \(\alpha \in \{0, 1, \ldots, m+1\}\). Assume that there exists \(C_2 > 0\) such that
\[
\|\nabla^\alpha e^{-tH}\|_{(L^p, \sigma \to L^q, \theta)} \leq C_2 t^{-\frac{N}{2p} - \frac{\alpha}{2}}, \quad t > 0,
\]
for some \((p, \infty, \sigma, \infty) \in \Lambda\). Then \(H\) must be subcritical with \(\lambda_1 \in [\omega_\alpha, \infty) \cup \{0\}\) and \(\lambda_2 \in [\omega_\alpha, \infty)\).

**Theorem 1.3** Assume conditions \((V_\infty)\) and \((N')\). Let \((p, q, \sigma, \theta) \in \Lambda\). Assume that, for any \(\alpha \in \{0, 1, 2, \ldots\}\), there exists \(C > 0\) such that
\[
\|\nabla^\alpha e^{-tH}\|_{(L^p, \sigma \to L^q, \theta)} \leq C t^{-\frac{N}{2} \left(\frac{1}{p} - \frac{1}{q}\right) - \frac{\alpha}{2}}, \quad t > 0.
\]
Then \(V\) must be identically zero in \(\mathbb{R}^N\), that is, \(H = -\Delta\).

See also Theorems 7.3 and 7.4.

Upper and lower decay estimates of \(\|\nabla^\alpha e^{-tH}\|_{(L^p, \sigma \to L^q, \theta)}\) are given in Sections 3 and 4, respectively. These are main ingredients of this paper. In order to obtain the upper decay estimates, we follow the arguments in \([16]\) and \([22]\). For any \(\phi \in C_c(\mathbb{R}^N)\), we find radially symmetric functions \(\{\phi_{k,i}\}_{(k,i) \in K} \subset L^2(\mathbb{R}^N)\) such that
\[
\phi(x) = \sum_{k=0}^{\infty} \sum_{i=1}^{d_k} \phi_{k,i}(|x|)Q_{k,i}\left(\frac{x}{|x|}\right) \quad \text{in} \quad L^2(\mathbb{R}^N).
\]

Let \(H_k := -\Delta + V_k(|x|)\) and set
\[
v_{k,i}(|x|, t) := [e^{-tH_k} \phi_{k,i}](|x|), \quad u_{k,i}(|x|, t) := v_{k,i}(|x|, t)Q_{k,i}\left(\frac{x}{|x|}\right).
\]
Then
\[
[e^{-tH} \phi] (x) = \sum_{k=0}^{\infty} \sum_{i=1}^{d_k} u_{k,i}(x, t) = \sum_{k=0}^{\infty} \sum_{i=1}^{d_k} v_{k,i}(|x|, t)Q_{k,i}\left(\frac{x}{|x|}\right) \quad \text{in} \quad C^2(K)
\]
for compact sets \(K \subset \mathbb{R}^N \setminus \{0\}\) and \(t > 0\) (see \([19]\) and \([21]\)). We study the behavior of derivatives of \(v_{k,i}\) by using the radially symmetry of \(v_{k,i}\), and obtain upper decay estimates of \(\|\nabla^\alpha e^{-tH}\|_{(L^p, \sigma \to L^q, \theta)}\). The introduction of functions \(I^n_k[\cdot]\) and \(J^n_k[\cdot]\) enables us to obtain those decay estimates systematically. See Subsection 2.2 and Section 3.

On the other hand, we study lower decay estimates of \(\|\nabla^\alpha e^{-tH}\|_{(L^p, \sigma \to L^q, \theta)}\) by using the lower Gaussian estimate of the fundamental solution to a parabolic equation with \(A_2\)-weight and by applying a tricky switch of weight (see \([14, 61]\)). See Section 4.
The rest of this paper is organized as follows. In Section 2 we recall some properties of Lorentz spaces and $h_k$. Furthermore, we obtain some preliminary results on $I_n[k]$, $J_n[i]$, and $e^{-tH}$. In Sections 3 and 4, we obtain upper and lower decay estimates of $\|\nabla^\alpha e^{-tH}\|_{(L^p,\sigma \to L^q,\theta)}$, respectively. In Section 5, we prove Theorem 1.1. In Section 6 we prove Theorems 1.2 and 1.3. In Section 7, as typical examples of inverse square potentials, we treat the Hardy potentials and bounded potentials, and clarify the relationship between the corresponding harmonic functions and the decay of $\|\nabla^\alpha e^{-tH}\|_{(L^p,\sigma \to L^q,\theta)}$. Furthermore, we show that the decay of $\|\nabla^\alpha e^{-tH}\|_{(L^p,\sigma \to L^q,\theta)}$ is delicate in the case when $\lambda_1 = \lambda_2 = 0$.

2 Preliminaries

In this section we introduce Lorentz spaces. Furthermore, we recall some results on $h_k$ and $e^{-tH}$, and prove some preliminary results.

Throughout this paper we use the same definition of $e^{-tH}$ and notations as in [22]. In particular, for any positive functions $f$ and $g$ on a set $E$, we write $f \asymp g$ for $x \in E$ if there exists $c > 0$ such that $c^{-1} \leq f(x)/g(x) \leq c$ for $x \in E$. By the letters $C$ and $D$ we denote generic positive constants and they may have different values also within the same line.

2.1 Lorentz spaces

For any measurable function $\phi$ in $\mathbb{R}^N$, we denote by $\mu = \mu(\lambda)$ the distribution function of $\phi$, that is,

$$\mu(\lambda) := \{x \in \mathbb{R}^N : |\phi(x)| > \lambda\} \quad \text{for} \quad \lambda > 0.$$  

Here $|E|$ is the $N$-dimensional Lebesgue measure of $E$ for measurable sets $E$ in $\mathbb{R}^N$. We define the non-increasing rearrangement $\phi^*$ of $\phi$ and the spherical rearrangement $\phi^\sharp$ of $\phi$ by

$$\phi^*(s) := \inf\{\lambda > 0 : \mu(\lambda) \leq s\}, \quad \phi^\sharp(x) := \phi^*(\alpha_N|x|^N),$$

for $s > 0$ and $x \in \mathbb{R}^N$, respectively, where $\alpha_N$ is the volume of the unit ball in $\mathbb{R}^N$. For any $(p, p, \sigma, \sigma) \in \Lambda$, we define the Lorentz space $L^{p,\sigma}(\mathbb{R}^N)$ by

$$L^{p,\sigma}(\mathbb{R}^N) := \{\phi : \phi \text{ is measurable in } \mathbb{R}^N, \|\phi\|_{L^{p,\sigma}} < \infty\},$$

where

$$\|\phi\|_{L^{p,\sigma}} := \begin{cases} \left( \int_{\mathbb{R}^N} \left( |x|^N \phi^\sharp(x) \right)^\sigma \frac{dx}{|x|^N} \right)^\frac{1}{\sigma} & \text{if } 1 \leq \sigma < \infty, \\ \sup_{x \in \mathbb{R}^N} |x|^N \phi^\sharp(x) & \text{if } \sigma = \infty. \end{cases}$$

Here $N/p = 0$ if $p = \infty$. The Lorentz spaces have the following properties:

- $L^{p,p}(\mathbb{R}^N) = L^p(\mathbb{R}^N)$ if $1 \leq p \leq \infty$;
- $L^{p,\sigma_1}(\mathbb{R}^N) \subset L^{p,\sigma_2}(\mathbb{R}^N)$ if $1 \leq p < \infty$ and $1 \leq \sigma_1 \leq \sigma_2 < \infty$. 

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Furthermore, there exists $C > 0$ depending only on $N$ such that
\[
\|f + g\|_{L^p,\sigma} \leq C(\|f\|_{L^p,\sigma} + \|g\|_{L^p,\sigma}) \quad \text{if } f, g \in L^{p,\sigma}(\mathbb{R}^N),
\]
\[
\|fg\|_{L^1} \leq C\|f\|_{L^p,\sigma}\|g\|_{L^p,\sigma'} \quad \text{if } f \in L^{p,\sigma}(\mathbb{R}^N), g \in L^{p',\sigma'}(\mathbb{R}^N),
\]
\[
\|f * g\|_{L^q,\theta} \leq C\|f\|_{L^p,\sigma}\|g\|_{L^r,\sigma} \quad \text{if } f \in L^{p,\sigma}(\mathbb{R}^N), g \in L^{r,\sigma}(\mathbb{R}^N).
\]
Here $(q, q, \theta, \theta), (r, r, s, s) \in \Lambda$ and
\[
\frac{1}{r} + \frac{1}{p} = \frac{1}{q} + 1, \quad \frac{1}{\theta} = \frac{1}{s} + \frac{1}{\sigma}.
\]
(See e.g. [3] and [9].) For any measurable function $f$ in a domain $\Omega$, we say that $f \in L^{p,\sigma}(\Omega)$ if and only if $\tilde{f} \in L^{p,\sigma}(\mathbb{R}^N)$, where $\tilde{f}$ is the zero extension of $f$ to $\mathbb{R}^N$. Furthermore, we write $\|f\|_{L^p,\sigma}(\Omega) = \|\tilde{f}\|_{L^p,\sigma}$. Then, for any $A \in \mathbb{R}$, the function $f_A$ defined by $f_A(x) := |x|^A$ satisfies $f_A \in L^{p,\sigma}(B(0, R))$ for $R > 0$ if and only if $pA + N > 0$ for $1 \leq \sigma < \infty$, $pA + N \geq 0$ for $\sigma = \infty$. (2.1)

Furthermore, for any $R > 0$, under condition (2.1), we have
\[
\|f_A\|_{L^p,\sigma}(B(0, \sqrt{t})) \asymp t^{\frac{N}{2p}} \quad \text{for } t \in (0, R^2].
\]
In particular, for any $k \in \{0, 1, 2, \ldots \}$, by (1.6) we see that
\[
\frac{\|h_k\|_{L^p,\sigma}(B(0, \sqrt{t}))}{h_k(\sqrt{t})} \asymp t^{\frac{N}{2p}} \quad \text{for } 0 < t \leq R^2 \quad \text{if } h_k \in L^{p,\sigma}(B(0, 1)).
\]

### 2.2 Preliminary results on $h_k$

Consider the ordinary differential equation
\[
h'' + \frac{N - 1}{r} h' - V_k(r) h = 0 \quad \text{in } (0, \infty). \tag{2.2}
\]
Then ODE (2.2) has two linearly independent solutions $h_k^+$ and $h_k^-$ such that
\[
h_k^+(r) = v_{k, \lambda_1}^+(r)(1 + o(1)), \quad h_k^-(r) = v_{k, \lambda_1}^-(r)(1 + o(1)),
\]
as $r \to +0$ and $h_k^-(1) = 1$. Here
\[
v_{k, \lambda}^+(r) := r^{A_{\lambda}^+ + \omega_k}, \quad v_{k, \lambda}^-(r) := \begin{cases} r^{\frac{N-1}{2} \log \frac{r}{2}} & \text{if } \lambda = \lambda_* \text{ and } k = 0, \\
r^{A_{\lambda}^- + \omega_k} & \text{otherwise}, \end{cases}
\]
for $\lambda \geq \lambda_*$, where $A_{\lambda}^\pm$ is as in (1.4). Then we have the following two propositions for the solution $h_k$ to (1.5).
Proposition 2.1 Let $m \in \{1, 2, \ldots \}$ and assume conditions $(V_m)$ and $(N')$. Then, for any $\ell \in \{0, 1, \ldots, m+1\}$,

\[
\frac{d^\ell}{dr^\ell} h_k(r) = \begin{cases} 
\frac{d^\ell}{dr^\ell} v_{k,\lambda_1}^+(r) + O \left( r^{-\ell + \rho_1 v_{k,\lambda_1}^+(r)} \right) & \text{as } r \to +0, \\
\left( \frac{d^\ell}{dr^\ell} v_k(r) + o \left( r^{-\ell} v_k(r) \right) \right) & \text{as } r \to \infty.
\end{cases}
\]

Here $v_k$ is as in \[1.6\]. Furthermore, there exists $C > 0$ such that

\[
C^{-1} \leq \frac{h_k(r)}{v_k^+(r)} \leq C \quad \text{in } (0, 1], \\
C^{-1} \leq \frac{h_k(r)}{v_k(r)} \leq C \quad \text{in } (1, \infty),
\]

\[
\left| \frac{d^\ell}{dr^\ell} h_k(r) \right| \leq C(k + 1)^{\ell-1} r^{-\ell} h_k(r) \quad \text{in } (0, \infty),
\]

for $k \in \{0, 1, 2, \ldots \}$ and $\ell \in \{0, 1, \ldots, m+1\}$.

**Proof.** Proposition 2.1 with $\ell \in \{0, 1\}$ follows from \[22\] Propositions 2.1 and 2.2. Then, by the use of equation \[2.2\] we obtain the other desired relations. \[\Box\]

**Proposition 2.2** Assume conditions $(V_1)$ and $(N')$.

(a) Let $(p, p, \sigma, \sigma) \in \Lambda$ be such that $h_0 \in L^{p,\sigma}(B(0,1))$. There exists $C_1 > 0$, independent of $(p, \sigma)$, such that

\[
\Gamma_{p,\sigma}(t) \geq C_1 t^{\frac{N}{p'}} \quad \text{for } t > 0.
\]

(b) There exists $C_2 > 0$ such that

\[
\int_0^r s^{N-1} h_k(s)^2 \, ds \leq C_2(k + 1)^{-1} r^N h_k(r)^2
\]

for $r > 0$ and $k \in \{0, 1, 2, \ldots \}$.

(c) Let $\ell \in \{0, 1, 2, \ldots \}$. Then there exist $C_3 > 0$ and $\gamma > 0$ such that

\[
\frac{h_k(\epsilon r)}{h_\ell(\epsilon r)} \leq C e^{\left( \frac{1}{\gamma} - \gamma \right)} \frac{h_k(r)}{h_\ell(r)}
\]

for $r > 0$, $\epsilon \in (0, 1)$ and $k \in \{\ell + 1, \ell + 2, \ldots \}$. Here $a_+ := \max\{a, 0\}$ for $a \in \mathbb{R}$.

**Proof.** Assertions (a) and (b) follow from \[22\] Proposition 2.4. It suffices to prove assertion (c).

We consider the case of $\ell \in \{1, 2, \ldots \}$. Set

\[
\iota_{\ell,k}(r) := \begin{cases} 
 r^{A_{1,k} - A_{1,\ell}} & \text{for } 0 < r < 1, \\
r^{A_{2,k} - A_{2,\ell}} & \text{for } r \geq 1,
\end{cases}
\]

where $k \in \{\ell + 1, \ell + 2, \ldots \}$. Since $A_{i,k} \geq A_{i,\ell}$ for $i \in \{1, 2\}$, the function $\iota_{\ell,k}$ is monotone increasing in $(0, \infty)$ for $k > \ell$. Furthermore, Proposition 2.1 implies that

\[
\frac{h_k(r)}{h_\ell(r)} \geq \iota_{\ell,k}(r) \quad \text{for } r > 0 \text{ and } k \in \{\ell + 1, \ell + 2, \ldots \}.
\]
Since \( A_{i,k} = k(1 + o(1)) \) as \( k \to \infty \), where \( i = 1, 2 \), we find \( \gamma > 0 \) such that

\[
A_{i,k} - A_{i,\ell} \geq \left( \frac{k}{2} - \gamma \right)_{+} \quad \text{for } k \in \{ \ell + 1, \ell + 2, \ldots \} \text{ and } i \in \{1, 2\},
\]

which implies that

\[
\frac{h_k(\varepsilon r)}{h_{\ell}(\varepsilon r)} \geq \nu_{\ell,k}(\varepsilon r) \leq C \varepsilon^{(\frac{k}{2} - \gamma)_{+}} \nu_{\ell,k}(r) \leq \varepsilon^{(\frac{k}{2} - \gamma)_{+}} \frac{h_k(r)}{h_{\ell}(r)}
\]

for \( r > 0, \varepsilon \in (0, 1) \), and \( k \in \{ \ell + 1, \ell + 2, \ldots \} \). Indeed, the inequality in the above relation holds since

\[
\nu_{\ell,k}(\varepsilon r) \leq \varepsilon^{A_{1,k-A_{1,\ell}}} \nu_{\ell,k}(r) \leq \varepsilon^{(\frac{k}{2} - \gamma)_{+}} \nu_{\ell,k}(r) \quad \text{if } r \leq 1,
\]

\[
\nu_{\ell,k}(\varepsilon r) \leq \varepsilon^{A_{1,k-A_{1,\ell}}} \nu_{\ell,k}(r) \leq \varepsilon^{(\frac{k}{2} - \gamma)_{+}} \nu_{\ell,k}(r) \quad \text{if } \varepsilon r \geq 1,
\]

\[
\nu_{\ell,k}(\varepsilon r) \leq \varepsilon^{A_{1,k-A_{1,\ell}}} \nu_{\ell,k}(1) \leq \varepsilon^{(\frac{k}{2} - \gamma)_{+}} \nu_{\ell,k}(r) \quad \text{if } \varepsilon \leq 1 \leq r.
\]

Thus assertion (c) follows for the case of \( \ell \in \{1, 2, \ldots \} \).

Let \( \ell = 0 \). Similarly to the above argument, by (1.5) and (1.6) we see such that

\[
\frac{h_1(\varepsilon r)}{h_0(\varepsilon r)} \leq \nu_{1,0}(\varepsilon r) \quad \text{for } r > 0, \varepsilon \in (0, 1), \text{ and } k \in \{1, 2, \ldots \}.
\]

Then, by assertion (c) with \( \ell \in \{1, 2, \ldots \} \) and (2.3), we find \( \gamma' > 0 \) such that

\[
\frac{h_k(\varepsilon r)}{h_0(\varepsilon r)} \leq C \varepsilon^{(\frac{k}{2} - \gamma')_{+}} \frac{h_k(r)}{h_0(r)}
\]

for \( r > 0, \varepsilon \in (0, 1), \text{ and } k \in \{1, 2, \ldots \} \). This implies assertion (c) for the case of \( \ell = 0 \). Thus Proposition 2 follows. \( \square \)

In the study of the decay of \( \| \nabla^a e^{-tH} \|_{(L^p, \sigma) \to L^q, \theta)} \), it is crucial to obtain the estimates of the spatial derivatives of \( e^{-tH} \phi \) systematically. For this aim, we introduce functions \( I_k^n[\cdot] \) and \( J_{k,n} \).

For any continuous function \( f \) in \((0, \infty)\), we set

\[
I_k[f](r) := \int_0^r s^{-N+1} h_k(s)^{-2} \left( \int_0^s \tau^{N-1} h_k(\tau)^2 f(\tau) \, d\tau \right) \, ds, \quad k \in \{0, 1, 2, \ldots \}.
\]

Then, under a suitable assumption of \( f \) at \( r = 0 \), we have

\[
\frac{d^2}{dr^2} I_k[f](r) + \frac{N - 1}{r} \frac{d}{dr} I_k[f](r) - \left( V(r) + \frac{\omega_k}{r^2} \right) I_k[f](r) = f(r) \quad \text{in } (0, \infty),
\]

\[
I_k[f](0) = \frac{d}{dr} I_k[f](0) = 0.
\]

We define \( I_k^n[f] \), where \( n \in \{0, 1, 2, \ldots \} \), by

\[
I_k^{n+1}[f](r) := I_k[I_k^n[f]](r), \quad I_k^n[f](r) := f(r).
\]
Furthermore, we set

\[ \begin{aligned}
I^n_k(|x|) := I^n_k[1](|x|), \\
J^n_{k,i}(x) := h_k(|x|)I^n_k(|x|)Q_{k,i} \left( \frac{x}{|x|} \right), \\
\end{aligned} \]

for \( x \in \mathbb{R}^N \setminus \{0\} \).

Notice that \( J_{k,i}(x) = J^0_{k,i}(x) \) (see (1.7)). Then we have:

**Lemma 2.1** Assume the same conditions as in Proposition 2.1. Let \( \ell \in \{0, 1, \ldots, m+1\} \) and \( n \in \{0, 1, 2, \ldots\} \).

(a) If \( \ell \leq 2n \), then there exist \( C_1 > 0 \) and \( D_1 > 0 \) such that

\[ \begin{aligned}
&\left| \nabla^\ell I^n_k[f](|x|) \right| \leq C_1(k+1)^D_1|x|^{2n-\ell} \sup_{0<s<|x|}|f(s)|, & x \in \mathbb{R}^N \setminus \{0\},
\end{aligned} \]

for \( f \in C([0, \infty)) \).

(b) There exist positive constants \( C_2, C_3, \) and \( D_2 \) such that

\[ \begin{aligned}
t^{-n} \frac{|\nabla^\ell J^n_{k,i}(x)|}{h_k(\sqrt{t})} &\leq C_2(k+1)^D_2 t^{-n}|x|^{2n-\ell} \frac{h_k(|x|)}{h_k(\sqrt{t})} \\
&\leq C_3(k+1)^D_2 |x|^{-\ell} \frac{h_0(|x|)}{h_0(\sqrt{t})}
\end{aligned} \]

(2.4)

for \( x \in B(0, \sqrt{t}), t > 0, \) and \( (k,i) \in K \).

**Proof.** Applying elliptic regularity theorems to (1.1), for any \( \beta \in \{0, 1, \ldots, \ell\} \), we have

\[ \begin{aligned}
&\left| \nabla^\beta Q_{k,i} \left( \frac{x}{|x|} \right) \right| \leq C(k+1)^D|x|^{-\beta}, & x \in \mathbb{R}^N \setminus \{0\}, \ (k,i) \in K.
\end{aligned} \]

(2.5)

Furthermore, it follows from Proposition 2.2 (c) that

\[ \frac{h_k(|x|)}{h_k(\sqrt{t})} \leq C \frac{h_0(|x|)}{h_0(\sqrt{t})}, \ x \in \mathbb{R}^N \setminus \{0\}. \]

Then Proposition 2.1 together with Proposition 2.2 (b) implies the desired inequalities. \( \square \)

### 2.3 Estimates of \( J^n_{k,i} \)

We collect estimates of \( J^n_{k,i} \), which are used in the rest of this paper. Due to (1.5), for \( \alpha \in \{0, 1, \ldots, m+1\} \), we divide the behavior of \( h_0 \) near 0 into the following two cases, which depends on whether \( \partial^\alpha h_0 \) is degenerate at \( r = 0 \) or not:

(A) \( h_0(r) \asymp r^A \) as \( r \to 0 \) for \( A \in \{0, 1, \ldots\} \) with \( A \leq \alpha - 1 \);

(B) \( h_0(r) \asymp r^A \) as \( r \to 0 \), where \( A \in \{0, 1, \ldots\} \) with \( A \leq \alpha - 1 \).
Let \((p, q, \sigma, \theta) \in \Lambda, (k, i) \in \mathcal{K},\) and \(n \in \{0, 1, 2, \ldots\} \). Set \(h_k^{(\gamma)}(r) := r^{-\gamma}h_k(r)\) for \(\gamma \in \mathbb{R}\) and
\[
\Gamma_{p,\sigma}^k(t) := \frac{\|h_k\|_{L^p(B(0, \sqrt{t}))}}{h_k(\sqrt{t})} \quad \text{if} \quad h_k \in L^p(\mathbb{R}), \quad \Gamma_{p,\sigma}^k(t) := \infty \quad \text{if} \quad h_k \notin L^p(\mathbb{R}).
\]
Then, by Proposition 2.1, Proposition 2.2, and Lemma 2.1 we have:

**Case \((A_0)\):** Consider case \((A_0)\). By Proposition 2.1 we find \(R_1 > 0\) such that
\[
|\nabla^\ell h_0| \leq C h_0^{(\ell)} \times \partial^\ell h_0 \leq |\nabla^\ell h_0|, \quad x \in B(0, R_1), \quad \ell \in \{0, \ldots, \alpha\}.
\]
This together with (2.4) implies that
\[
t^{-n} \frac{\|\nabla^\ell \mathcal{F}_{n,\alpha} \|_{L^q(E)}}{h_k(\sqrt{t})} \leq C \frac{\|h_0^{(\ell)}\|_{L^q(E)}}{h_0(\sqrt{t})} \leq C \frac{\|\nabla^\ell h_0\|_{L^q(E)}}{h_0(\sqrt{t})} \quad (2.7)
\]
for measurable sets \(E \subset B(0, \sqrt{t}) \cap B(0, R_1)\) and \(\ell \in \{0, \ldots, \alpha\}\).

**Case \((B_0)\):** Consider case \((B_0)\). Let \(R_2 > 0\). Then
\[
\Gamma_{p,\sigma}^k(t) \times \Gamma_{q,\theta}^k(t) \times \frac{N}{2p}, \quad \Gamma_{q,\theta}^k(t) \times \Gamma_{q,\theta}^k(t) \times \frac{N}{2q}, \quad (2.8)
\]
for \(0 < t \leq \sqrt{R_2}\). Furthermore, by (2.4) we have
\[
t^{-n} \frac{\|\nabla^\ell \mathcal{F}_{n,\alpha} \|_{L^q(E)}}{h_k(\sqrt{t})} \leq C t^{-\frac{\ell}{2}|E|^\frac{N}{2q}} \leq C t^{\frac{N}{2q}} \quad (2.9)
\]
for measurable sets \(E \subset B(0, \sqrt{t}) \cap B(0, R_2)\) and \(\ell \in \{0, \ldots, \alpha\}\).

Similarly, we divide the behavior of \(h_0\) at the space infinity into the following two cases:

**Case \((A_0')\):** Consider case \((A_0')\). By Proposition 2.1 we find \(R_3 > 0\) such that
\[
|\nabla^\ell h_0| \leq C h_0^{(\ell)} \times \partial^\ell h_0 \leq |\nabla^\ell h_0|, \quad x \in B(0, R_3)^c, \quad \ell \in \{0, \ldots, \alpha\}.
\]
Then, similarly to (2.7), we have
\[
t^{-n} \frac{\|\nabla^\ell \mathcal{F}_{n,\alpha} \|_{L^q(E)}}{h_k(\sqrt{t})} \leq C \frac{\|h_0^{(\ell)}\|_{L^q(E)}}{h_0(\sqrt{t})} \leq C \frac{\|\nabla^\ell h_0\|_{L^q(E)}}{h_0(\sqrt{t})} \quad (2.11)
\]
for measurable sets \(E \subset B(0, \sqrt{t}) \setminus B(0, R_3)\) and \(\ell \in \{0, \ldots, \alpha\}\).

**Case \((B_0')\):** Consider case \((B_0')\). Let \(R_4 > 0\). Then
\[
\Gamma_{p,\sigma}^k(t) \times \Gamma_{q,\theta}^k(t) \times \frac{N}{2p}, \quad \Gamma_{q,\theta}^k(t) \times \Gamma_{q,\theta}^k(t) \times \frac{N}{2q}, \quad h_k(\sqrt{t}) \geq Ct^{\frac{k}{2}}, \quad (2.12)
\]
for \(t \geq \sqrt{R_4}\). Furthermore, by (2.4) we have
\[
t^{-n} \frac{\|\nabla^\ell \mathcal{F}_{n,\alpha} \|_{L^q(E)}}{h_k(\sqrt{t})} \leq C t^{-\frac{\ell}{2}|E|^\frac{N}{2q}} \leq C t^{\frac{N}{2q}} \quad (2.13)
\]
for measurable sets \(E \subset B(0, \sqrt{t}) \setminus B(0, R_4)\) and \(\ell \in \{0, \ldots, \alpha\}\).
2.4 Estimates of solutions

We obtain some results on the behavior of $e^{-tH}\phi$, where $\phi \in C_c(\mathbb{R}^N)$. The following two propositions follow from the same arguments as in the proof of Propositions 3.1 and 4.2, respectively.

**Proposition 2.3** Let $m \in \{1, 2, \ldots\}$ and assume conditions $(V_m)$ and $(N')$.

(a) Let $(p, q, \sigma, \theta) \in \Lambda$, $\alpha \in \{0, 1, \ldots, m + 1\}$, $\beta \in \{0, 1, 2, \ldots\}$, and $\delta \in (0, 1]$. Then there exist $C_1 > 0$ and $C_2 > 0$ such that

$$t^\beta \left\| \partial_t^\beta \nabla^\alpha e^{-tH}\phi \right\|_{L^{p,\theta}(B(0, \delta\sqrt{t}))} \leq C_1 t^{-\frac{N}{p} \left(1 - \frac{1}{q}\right) - \frac{\alpha}{2}} \left[ \frac{\|h_0\phi\|_{L^1(B(0, \sqrt{t}))}}{h_0(\sqrt{t})} + t^{\frac{\alpha}{p}} \left\|\phi\right\|_{L^{p,\sigma}(B(0, \delta\sqrt{t}))} \right]$$

$$\leq C_2 t^{-\frac{N}{p} \left(1 - \frac{1}{q}\right) - \frac{\alpha}{2}} \Gamma_{p',\sigma'}(t) \left\|\phi\right\|_{L^{p,\sigma}}$$

for $\phi \in C_c(\mathbb{R}^N)$ and $t > 0$.

(b) Let $(p, q, \sigma, \theta) \in \Lambda$. Then there exist $C_3 > 0$ and $C_4 > 0$ such that

$$\left\| e^{-tH}\phi \right\|_{L^{p,\theta}} \leq C_3 t^{-\frac{N}{p} \left(1 - \frac{1}{q}\right) - \frac{\alpha}{2}} \left[ \frac{\|h_0\phi\|_{L^1(B(0, \sqrt{t}))}}{h_0(\sqrt{t})} + t^{\frac{\alpha}{p}} \left\|\phi\right\|_{L^{p,\sigma}(B(0, \delta\sqrt{t}))} \right]$$

$$\leq C_4 t^{-\frac{N}{p} \left(1 - \frac{1}{q}\right) - \frac{\alpha}{2}} \Gamma_{p',\sigma'}(t) \left\|\phi\right\|_{L^{p,\sigma}}$$

for $\phi \in C_c(\mathbb{R}^N)$ and $t > 0$.

**Proposition 2.4** Assume the same conditions as in Proposition 2.3. Furthermore, assume that $h_0 \in L^{p,\sigma}(B(0, 1)) \cap L^{p',\sigma'}(B(0, 1))$ for some $(p, p, \sigma, \sigma) \in \Lambda$. Let $\|\phi\|_{p,\sigma} \leq 1$ and $v_{k,i}$ be as in (1.12) and set

$$w_{k,i}(x, t) := \frac{v_{k,i}(x, t)}{h_k(x)} \quad \text{for} \quad (x, t) \in \mathbb{R}^N \times (0, \infty).$$

Then, for any $\beta \in \{0, 1, 2, \ldots\}$, there exist $C > 0$ and $\delta \in (0, 1]$ such that

$$t^\beta |\partial_t^\beta w_{k,i}(x, t)| \leq CM_{k,i} t^{-\frac{N}{p} \left(1 - \frac{1}{q}\right) - \frac{\alpha}{2}} \frac{\Gamma_{p',\sigma'}(t)}{h_k(\delta\sqrt{t})}$$

(2.14)

for $x \in B(0, \delta\sqrt{t})$, $t > 0$, and $(k, i) \in \mathcal{K}$. Here $M_{k,i} := \|Q_{k,i}\|_{L^{\infty}(\mathbb{S}^{N-1})}$.

Furthermore, we have:

**Proposition 2.5** Assume the same conditions as in Proposition 2.4. Let $\|\phi\|_{p,\sigma} \leq 1$ and $u_{k,i}$ be as in (1.12). Then, for any $\alpha \in \{0, 1, \ldots, m + 1\}$, there exist $C > 0$, $D > 0$, and $\delta \in (0, 1)$ such that

$$|\nabla^\alpha u_{k,i}(x, t)| \leq C(k + 1)^D t^{-\frac{N}{p} \left(1 - \frac{1}{q}\right) - \frac{\alpha}{2}} \frac{\Gamma_{p',\sigma'}(t)|x|^{-\alpha}}{h_k(\delta\sqrt{t})}$$

(2.15)

for $x \in B(0, \delta\sqrt{t})$, $t > 0$, and $(k, i) \in \mathcal{K}$.
Proof. It follows from \cite[Proposition 4.2]{22} that
\[
\partial_t^\beta w_{k,i}(|x|, t) = \partial_t^\beta w_{k,i}(0, t) + I_k[\partial_t^{\beta+1} w_{k,i}(\cdot, t)](|x|)
\]
for \((x, t) \in \mathbb{R}^N \times (0, \infty), \beta \in \{0, 1, 2, \ldots\},\) and \((k, i) \in K\). Repeating this relation, we have
\[
\begin{align*}
\partial_t^\beta w_{k,i}(|x|, t) &= w_{k,i}(0, t) + \partial_t w_{k,i}(0, t)I_k(|x|) + I_k^2[\partial_t^2 w_{k,i}(\cdot, t)](|x|) \\
&= \sum_{\ell=0}^{n-1} \partial_t^\ell w_{k,i}(0, t)I_k^\ell(|x|) + I_k^n[\partial_t^n w_{k,i}(\cdot, t)](|x|)
\end{align*}
\]
(2.16)
for \((x, t) \in \mathbb{R}^N \times (0, \infty), n \in \{1, 2, \ldots\},\) and \((k, i) \in K\). This implies that
\[
\begin{align*}
u_{k,i}(x, t) &= h_k(|x|)w_{k,i}(|x|, t)Q_{k,i}\left(\frac{x}{|x|}\right) = \sum_{\ell=0}^{n-1} \nu_{k,i}^\ell(x, t) + R_{k,i}^n(x, t)
\end{align*}
\]
(2.17)
for \((x, t) \in \mathbb{R}^N \times (0, \infty), n \in \{1, 2, \ldots\},\) and \((k, i) \in K\), where
\[
\begin{align*}
u_{k,i}^\ell(x, t) &= \partial_t^\ell w_{k,i}(0, t)J_k^\ell(|x|), \\
R_{k,i}^n(x, t) &= h_k(|x|)I_k^n[\partial_t^n w_{k,i}(\cdot, t)](|x|)Q_{k,i}\left(\frac{x}{|x|}\right).
\end{align*}
\]
Then, for any \(n \in \{1, 2, \ldots\}\) and \(\ell \in \{0, 1, \ldots, n-1\}\), by Proposition \(2.1, \ \text{Lemma 2.1, \text{and} (2.18)}\), we have
\[
|\nabla^\alpha \nu_{k,i}^\ell(x, t)| \leq C(k+1)Dt^{-\frac{N}{2}-\frac{\alpha}{2}t} \Gamma_{\nu', \sigma'}(t) \frac{h_k(|x|)}{h_k(\delta \sqrt{t})}|\nabla^\alpha J_{k,i}^\ell(x)|
\]
\[
\leq C(k+1)Dt^{-\frac{N}{2}-\frac{\alpha}{2}t} \Gamma_{\nu', \sigma'}(t) |x|^{2\ell-\alpha} \frac{h_k(|x|)}{h_k(\delta \sqrt{t})}
\]
\[
\leq C\delta^{2\ell}(k+1)Dt^{-\frac{N}{2}-\frac{\alpha}{2}t} \Gamma_{\nu', \sigma'}(t) |x|^n \frac{h_k(|x|)}{h_k(\delta \sqrt{t})},
\]
(2.18)
\[
|\nabla^\alpha R_{k,i}^n(x, t)| \leq C(k+1)Dt^{-\frac{N}{2}-\frac{\alpha}{2}t} \Gamma_{\nu', \sigma'}(t) |x|^{n-\alpha} \frac{h_k(|x|)}{h_k(\delta \sqrt{t})}
\]
\[
\leq C\delta^{2n-\frac{\alpha}{2}}(k+1)Dt^{-\frac{N}{2}-\frac{\alpha}{2}t} \Gamma_{\nu', \sigma'}(t) \frac{h_k(|x|)}{h_k(\delta \sqrt{t})} \quad \text{if} \quad \alpha \leq 2n,
\]
for \(x \in B(0, \delta \sqrt{t}), \ t > 0,\) and \((k, i) \in K\). Here the constant \(\delta\) is as in Proposition \(2.1\) with \(\beta = n\). Taking large enough \(n \in \{0, 1, 2, \ldots\}\) so that \(2n \geq \alpha\), by \(1.3\) and \(2.17\) we obtain inequality \(2.13\) for \(x \in B(0, \delta \sqrt{t})\) and \(t > 0\). Thus Proposition \(2.5\) follows. \(\square\)

3 Upper decay estimates

In this section we study upper decay estimates of \(\|\nabla^\alpha u_{\nu, \sigma}\|_{(L^p, \sigma) \rightarrow L^q, \theta})\) and prove the following theorem, which is one of the main ingredients of this paper.
Theorem 3.1 Let \( m \in \{1, 2, \ldots \} \) and assume conditions \((V_m)\) and \((N')\). Let \((\ell, p, q, \sigma, \theta) \in \Lambda\) and \(\alpha \in \{0, 1, \ldots, m + 1\}\).

(a) There exists \( C_1 > 0 \) such that
\[
\|\nabla^\alpha e^{-tH} \|_{(L^{p, \sigma} \to L^{q, \theta}(B(0, \sqrt{t})^c))} \leq C_1 t^{-\frac{\alpha}{2}} \left(1 - \frac{1}{2}\right)^{\frac{\alpha}{2}} (\Gamma_{p', \sigma'})^t, \quad t > 0.
\]

(b) There exists \( C_2 > 0 \) such that
\[
\|\nabla^\alpha e^{-tH} \|_{(L^{p, \sigma} \to L^{q, \theta}(B(0, \sqrt{t})^c))} \leq C_2 t^{-\frac{\alpha}{2}} \Gamma_{p', \sigma'}(t) \left[J_\alpha(t) + t^{\frac{\alpha}{2} - \frac{\alpha}{2}}\right], \quad t > 0,
\]
where
\[
J_\alpha(t) := \sum_{0 \leq k + 2n \leq \alpha} \sum_{i=1}^{d_k} \left|t^{-n} \left\|\nabla^\alpha J_{k,i}^n \right\|_{L^q(B(0, \sqrt{t}))} / h_k(\sqrt{t})\right|.
\]

Proof. Assertion (a) follows from Proposition 2.3 (a). Then, for the proof of assertion (b), it suffices to prove
\[
\|\nabla^\alpha e^{-tH} \|_{(L^{p, \sigma} \to L^{q, \theta}(B(0, \sqrt{t})^c))} \leq \frac{C_t^{-\frac{\alpha}{2}} \Gamma_{p', \sigma'}(t) \left[J_\alpha(t) + t^{\frac{\alpha}{2} - \frac{\alpha}{2}}\right]}{t > 0}. \tag{3.1}
\]
Let \( \phi \in C_c(\mathbb{R}^N) \) be such that \( \|\phi\|_{p, \sigma} \leq 1 \) and set \( u := e^{-tH} \phi \). We use the same notations as in (1.11), (1.12), and (1.13). For any \( \ell \in \{0, 1, \ldots\} \), we set
\[
[R_\ell u](x, t) := \sum_{k=\ell}^{\infty} \sum_{i=1}^{d_k} u_{k,i}(x, t) = u(x, t) - \sum_{k=0}^{\ell-1} \sum_{i=1}^{d_k} u_{k,i}(x, t).
\]
Let \( \alpha \in \{0, 1, \ldots, m + 1\} \) and \( \delta \in (0, 1) \) be small enough. We observe from (1.2) and (2.15) that
\[
|\nabla^\alpha [R_\ell u](x, t)| \leq \sum_{k=\ell}^{\infty} \sum_{i=1}^{d_k} |\nabla^\alpha u_{k,i}(x, t)| \leq C t^{-\frac{\alpha}{2}} \Gamma_{p', \sigma'}(t)|x|^{-\alpha} \sum_{k=\ell}^{\infty} (k + 1)^D \frac{h_k(|x|)}{h_k(\delta^{\sqrt{t}})} \tag{3.2}
\]
for \( x \in B(0, \delta^{\sqrt{t}}) \setminus \{0\} \) and \( t > 0 \). On the other hand, by Proposition 2.2 (iii), we find \( \gamma > 0 \) such that
\[
\frac{h_k(|x|)}{h_k(\delta^{\sqrt{t}})} = \frac{t^{\frac{\alpha}{2}}}{h_k(\delta^{\sqrt{t}}) h_k(|x|)} / h_k(\delta^{\sqrt{t}}) \leq C e^{(\frac{\alpha}{2} - \gamma)} \frac{h_k(|x|)}{h_k(\delta^{\sqrt{t}})} \tag{3.3}
\]
for \( x \in B(0, e\delta^{\sqrt{t}}) \setminus \{0\}, \ t > 0, \ e \in (0, 1), \) and \( k \in \{\ell + 1, \ell + 2, \ldots\} \). Due to the relation that \( h_\ell(r/2) \approx h_\ell(r) \approx h_\ell(2r) \) for \( r > 0 \) (see (1.3) and (1.6)), taking small enough \( \epsilon > 0 \) if necessary, we see that
\[
\sum_{k=\ell}^{\infty} (k + 1)^D \frac{h_k(|x|)}{h_k(\delta^{\sqrt{t}})} \leq C h_\ell(|x|) [\ell + 1]^D + \sum_{k=\ell+1}^{\infty} \epsilon^{(\frac{\alpha}{2} - \gamma)} (k + 1)^D \tag{3.3}
\]
for $x \in B(0, \epsilon \delta \sqrt{t}) \setminus \{0\}$ and $t > 0$. By \((3.2)\) and \((3.3)\) we obtain
\[
|\nabla^\alpha [R_\ell u](x, t)| \leq C t^{-\frac{N}{2}} \Gamma_{p', \sigma'}(t) \frac{h_2^\alpha(|x|)}{h_\ell(\sqrt{t})} \tag{3.4}
\]
for $x \in B(0, \epsilon \delta \sqrt{t}) \setminus \{0\}$ and $t > 0$. Here $h_2^\alpha$ is as in Subsection 2.3. On the other hand, by Proposition 2.3 (a) with $q = \theta = \infty$ we have
\[
\|\nabla^\alpha u(\cdot, t)\|_{L^{q, \theta}(B(0, \epsilon \delta \sqrt{t}), B(0, \epsilon \delta \sqrt{t}))} \leq C \|\nabla^\alpha u(\cdot, t)\|_{L^\infty(B(0, \epsilon \delta \sqrt{t}}) t^{N/2} \leq C t^{-\frac{N}{2} + \frac{N}{2q} - \frac{\alpha}{2}} \Gamma_{p', \sigma'}(t) \tag{3.5}
\]
for $t > 0$.

We study the behavior of $\nabla^\alpha u$ in $B(0, \epsilon \delta \sqrt{t})$ by using the arguments in Subsection 2.3.

Step 1: Consider case \((A_\alpha)\), that is, $h_0(r) \neq r^A$ as $r \to 0$ for $A \in \{0, 1, \ldots\}$ with $A \leq \alpha - 1$. Let $R_1$ be as in Subsection 2.3. It follows from \((2.6)\) that
\[
|\nabla^\alpha h_0(|x|)| \geq h_0^\alpha(|x|) > 0, \quad x \in B(0, R_1) \setminus \{0\}. \tag{3.6}
\]
Then, by \((3.4)\) with $\ell = 0$ we have
\[
|\nabla^\alpha u(x, t)| = |\nabla^\alpha [R_0 u](x, t)| \leq C t^{-\frac{N}{2}} \Gamma_{p', \sigma'}(t) \frac{h_0^\alpha(|x|)}{h_0(\sqrt{t})} \leq C t^{-\frac{N}{2}} \Gamma_{p', \sigma'}(t) \frac{|\nabla^\alpha h_0(|x|)|}{h_0(\sqrt{t})}
\]
for $x \in [B(0, \epsilon \delta \sqrt{t}) \cap B(0, R_1)] \setminus \{0\}$ and $t > 0$. This implies that
\[
\|\nabla^\alpha u(\cdot, t)\|_{L^{q, \theta}(B(0, R))} \leq C t^{-\frac{N}{2}} \Gamma_{p', \sigma'}(t) \frac{\|\nabla^\alpha h_0\|_{L^{q, \theta}(B(0, R))}}{h_0(\sqrt{t})} \tag{3.7}
\]
for $R \in (0, R_1]$ and $t > 0$ if $R \leq \epsilon \delta \sqrt{t}$.

Let $R_2 > 1$ be such that $R_2 \in (R_1, \infty)$. We observe from \((3.4)\) with $\ell = 0$ that
\[
|\nabla^\alpha u(x, t)| = |\nabla^\alpha [R_0 u](x, t)| \leq C t^{-\frac{N}{2}} \Gamma_{p', \sigma'}(t)|x|^{-\alpha} h_0(|x|) \leq C t^{-\frac{N}{2}} \Gamma_{p', \sigma'}(t) \frac{h_0^\alpha(|x|)}{h_0(\sqrt{t})}
\]
for $x \in [B(0, \epsilon \delta \sqrt{t}) \cap B(0, R_2)] \setminus B(0, R_1)$ and $t > 0$. This together with \((3.6)\) implies that
\[
\|\nabla^\alpha u(\cdot, t)\|_{L^{q, \theta}(B(0, R) \setminus B(0, R_1))} \leq C t^{-\frac{N}{2}} \Gamma_{p', \sigma'}(t) \frac{\|\nabla^\alpha h_0\|_{L^{q, \theta}(B(0, R_1))}}{h_0(\sqrt{t})} \leq C t^{-\frac{N}{2}} \Gamma_{p', \sigma'}(t) \frac{\|\nabla^\alpha h_0\|_{L^{q, \theta}(B(0, R_1))}}{h_0(\sqrt{t})} \tag{3.8}
\]
for $R \in [R_1, R_2)$ and $t > 0$ if $R \leq \epsilon \delta \sqrt{t}$. Combining \((3.7)\) and \((3.8)\), we see that
\[
\|\nabla^\alpha u(\cdot, t)\|_{L^{q, \theta}(B(0, R))} \leq C t^{-\frac{N}{2}} \Gamma_{p', \sigma'}(t) \|\nabla^\alpha h_0\|_{L^{q, \theta}(B(0, \sqrt{t}))} \leq C t^{-\frac{N}{2}} \Gamma_{p', \sigma'}(t) J_\alpha(t) \tag{3.9}
\]
for $R \in (0, R_3)$ and $t > 0$ if $R \leq \epsilon \sqrt{t}$.
Step 2: Consider case \((B_\alpha)\), that is, \(h_0(r) \asymp r^A\) as \(r \to 0\), where \(A \in \{0,1,\ldots\}\) with \(A \leq \alpha - 1\). Then \(\alpha \geq 1\) and \(\lambda_1 = \omega_A \geq 0\) (see (1.3)). Let \(R_2\) as in the above. Since \(A_{1,\alpha} \geq A_{\omega_\alpha} = \alpha\), it follows that
\[
h^{(\alpha)}_\alpha(|x|) \leq C|x|^{A_{1,\alpha} - \alpha} \leq C \min\{R, \sqrt{T}\}^{A_{1,\alpha} - \alpha}, \quad x \in B(0, R) \cap B(0, \sqrt{T}),
\]
for \(R \in (0, R_2)\). Let \(T > 1\). By (1.5) and (1.6) we have
\[
h^{(\alpha)}_\alpha(\sqrt{t}) \geq \begin{cases} 
C t^{\frac{\alpha}{2}} & \text{for } 0 < t < T, \\
C h_0(\sqrt{t}) & \text{for } t > T \text{ in case } (A'_\alpha), \\
C t^{\frac{\alpha}{2}} & \text{for } t > T \text{ in case } (B'_\alpha).
\end{cases}
\]
These together with (2.8), (2.12), and (3.4) imply that
\[
\|\nabla^{(\alpha)} [R_{1,\alpha}(\cdot, t)] \|_{L^{3,\theta}(B(0, R))} \leq C t^{-\frac{N}{2}} \Gamma_{\theta'}(t) \|h^{(\alpha)}_\alpha\|_{L^3,\theta(0, R)} \leq C t^{-\frac{N}{2}} \Gamma_{\theta'}(t) \|\Psi_T(t)\|
\]
for \(R \in (0, R_2)\) and \(t > 0\) if \(R < \epsilon \delta \sqrt{t}\). Here
\[
\Psi_T(t) := \begin{cases} 
\frac{N}{2} n - \frac{\alpha}{2} & \text{if } 0 < t \leq T, \\
h_0(\sqrt{t})^{-1} & \text{if } t > T \text{ in case } (A'_\alpha), \\
\frac{N}{2} n - \frac{\alpha}{2} & \text{if } t > T \text{ in case } (B'_\alpha).
\end{cases}
\]
Let \(k \in \{0,1,\ldots,\alpha - 1\}\) and \(i \in \{1,\ldots,d_k\}\). Let \(n_k \in \{0,1,2,\ldots\}\) be such that
\[
\frac{\alpha - k}{2} \leq n_k < 1 + \frac{\alpha - k}{2}. 
\]
By (2.17) and (2.18) we have
\[
|\nabla^{\alpha} u_{k,i}(x, t)| \leq \sum_{t=0}^{n_k - 1} |\nabla^{\alpha} u_{k,i}^t(x, t)| + |\nabla^{\alpha} R_{k,i}^{n_k}(x, t)|
\leq C \sum_{t=0}^{n_k - 1} |\nabla^{\alpha} u_{k,i}^t(x, t)| + C t^{-\frac{N}{2} - n_k} \frac{\Gamma_{\theta'}(t)}{h_k(\delta \sqrt{t})} |x|^{2n_k - \alpha} h_k(|x|)
\]
for \(x \in B(0, \delta \sqrt{t})\) and \(t > 0\). On the other hand, it follows from (3.13) that \(2n_k - \alpha + A_{1,k} \geq 2n_k - \alpha + k \geq 0\) and
\[
|x|^{2n_k - \alpha} h_k(|x|) \leq C|x|^{2n_k - \alpha + A_{1,k}} \leq C \min\{R_2, \sqrt{T}\}^{2n_k - \alpha + A_{1,k}} 
\]
for \( x \in B(0, R_2) \cap B(0, \sqrt{t}) \). Furthermore, similarly to (3.10), by (3.13) we have
\[
C t^{\frac{1}{2} + n_k} h_k(\sqrt{t}) \geq \begin{cases} 
C t^{\frac{1}{2} + n_k}h_k(\sqrt{t}) & \text{if } 0 < t < T, \\
C t^{\frac{1}{2}} h_0(\sqrt{t}) & \text{if } t > T \text{ in case } (A_0'), \\
C t^{\frac{1}{2}} h_k(\sqrt{t}) & \text{if } t > T \text{ in case } (B_0'), \\
\end{cases} 
\] (3.16)
for \( t > 0 \). By (3.12), (3.14), (3.15), and (3.16) we obtain
\[
\| \nabla^\alpha u_{k,i}(\cdot,t) \|_{L^q,\theta(B(0,R))} \leq C t^{-\frac{N}{2}} \Gamma_{p',\sigma'}(t) \left[ \sum_{k=0}^{n_k-1} t^{-\ell} \frac{\| \nabla^\alpha J_{k,i}^\ell \|_{L^q,\theta(B(0,\sqrt{t}))}}{h_k(\sqrt{t})} + C \Psi_T(t) \right] 
\] (3.17)
for \( R \in (0, R_2) \) and \( t > 0 \) if \( 0 < R < \delta \sqrt{t} \). Therefore, thanks to (3.11), (3.13), and (3.17), we observe that
\[
\| \nabla^\alpha u(\cdot,t) \|_{L^q,\theta(B(0,R))} \leq \sum_{k=0}^{\alpha-1} \sum_{i=1}^{d_k} \| \nabla^\alpha u_{k,i}(\cdot,t) \|_{L^q,\theta(B(0,R))} + \| \nabla^\alpha [R_\alpha u](\cdot,t) \|_{L^q,\theta(B(0,R))} 
\leq C t^{-\frac{N}{2}} \Gamma_{p',\sigma'}(t) \left[ \sum_{0 \leq k+2 \ell \leq \alpha} \sum_{k=1}^{d_k} t^{-\ell} \frac{\| \nabla^\alpha J_{k,i}^\ell \|_{L^q,\theta(B(0,\sqrt{t}))}}{h_k(\sqrt{t})} + \Psi_T(t) \right] 
\leq C t^{-\frac{N}{2}} \Gamma_{p',\sigma'}(t) \left[ J_{\alpha}(t) + \Psi_T(t) \right] 
\] (3.18)
for \( R \in (0, R_2) \) and \( t > 0 \) if \( R < \delta \sqrt{t} \).

Combining (3.5), (3.9), and (3.18), in both of cases \((A_\alpha)\) and \((B_\alpha)\), we see that
\[
\| \nabla^\alpha u(\cdot,t) \|_{L^q,\theta(B(0,R))} \leq C t^{-\frac{N}{2}} \Gamma_{p',\sigma'}(t) \left[ J_{\alpha}(t) + \Psi_T(t) \right] 
\] (3.19)
for \( R \in (0, R_2) \) and \( t > 0 \) if \( R < \delta \sqrt{t} \). This implies that inequality (3.11) holds for \( t \in (0, T) \). It remains to prove inequality (3.1) for \( t \in (T, \infty) \).

Step 3: Consider case \((A_\alpha')\), that is, \( h_0(r) \propto r^A \) as \( r \to \infty \) for \( A \in \{0,1,\ldots\} \) with \( A \leq \alpha - 1 \). Taking large enough \( R_2 > 0 \) if necessary, we see that \( R_2 \geq R_3 + 1 \), where \( R_3 \) is as in Subsection 2.3. We can assume, without loss of generality, that \( \epsilon \delta \sqrt{T} > R_2 \). By (2.10) and (3.31) with \( \ell = 0 \), applying the same argument as in Step 1 (see also (3.7)), we have
\[
\| \nabla^\alpha u(\cdot,t) \|_{L^q,\theta(B(0,\epsilon \delta \sqrt{T}) \setminus B(0,R_2))} \leq C t^{-\frac{N}{2}} \Gamma_{p',\sigma'}(t) \frac{\| \nabla^\alpha h_0 \|_{L^q,\theta(B(0,\sqrt{T}))}}{h_0(\sqrt{T})} 
\leq C t^{-\frac{N}{2}} \Gamma_{p',\sigma'}(t) J_{\alpha}(t), \quad t > T. 
\] (3.20)
Since $|\nabla^\alpha h_0(x)| \asymp |x|^{-\alpha}h_0(|x|) > 0$ in $B(0, R_3)^c$, by (3.19) we have
\[
\|\nabla^\alpha u(\cdot, t)\|_{L^q,\theta(B(0,R_2),B(0,\sqrt{t}))} \leq Ct^{\frac{\alpha}{2}}\Gamma_p,\sigma'(t)\left[J_\alpha(t) + \frac{\|\nabla^\alpha h_0\|_{L^q,\theta(B(0,R_2),B(0,\sqrt{t}))}}{h_0(\sqrt{t})}\right] + t^{\frac{\alpha}{2}} - \frac{\alpha}{2}, \quad t > T.
\]
By (3.20), (3.21), and (3.22) we see that
\[
\|\nabla^\alpha u(\cdot, t)\|_{L^q,\theta(B(0,\sqrt{t}))} \leq Ct^{\frac{\alpha}{2}}\Gamma_p,\sigma'(t)\left[J_\alpha(t) + t^{\frac{\alpha}{2}} - \frac{\alpha}{2}\right], \quad t > T.
\]
This implies that inequality (3.11) holds for $t > T$ in case $(A'_\alpha)$.

Step 4: Consider case $(B'_\alpha)$, that is, $h_0(r) \asymp r^A$ as $r \to \infty$, where $A \in \{0, 1, \ldots\}$ with $A \leq \alpha - 1$. Since $A_{2,\alpha} \geq A_{\alpha}^\pm = \alpha$, similarly to (3.11), by (3.4) we have
\[
\|\nabla^\alpha [R_\alpha u(\cdot, t)]\|_{L^q,\theta(B(0,\sqrt{t}),B(0,\sqrt{t}))} \leq Ct^{\frac{\alpha}{2}}\Gamma_p,\sigma'(t)\left[J_\alpha(t) + t^{\frac{\alpha}{2}} - \frac{\alpha}{2}\right], \quad t > T. \tag{3.22}
\]
Let $k \in \{0, 1, 2, \ldots, \alpha - 1\}$ and $i \in \{1, \ldots, d_k\}$. Let $n_k \in \{0, 1, 2, \ldots\}$ be as in (3.13). It follows from (1.10) that $2n_k - \alpha = A_{2,k} \geq 2n_k - \alpha + k \geq 0$ and
\[
|x|^{2n_k - \alpha}h_k(|x|) \leq C|x|^{2n_k - \alpha + A_{2,k}}(\log |x|)^{B_k} \leq Ct^{n_k - \frac{\alpha}{2}}h_k(\sqrt{t})
\]
for $x \in B(0, \sqrt{t}) \cap B(0, R_2)^c$. Then, similarly to (3.17), we obtain
\[
\|\nabla^\alpha u_{k,i}(\cdot, t)\|_{L^q,\theta(B(0,\sqrt{t}) \setminus B(0, R_2))} \leq Ct^{\frac{\alpha}{2}}\Gamma_p,\sigma'(t)\left[\sum_{\ell=0}^{n_k-1} t^{-\ell}\frac{\|\nabla^\alpha J_{k,i}\|_{L^q,\theta(B(0,\sqrt{t}))}}{h_k(\sqrt{t})} + t^{\frac{\alpha}{2}} - \frac{\alpha}{2}\right], \quad t > T. \tag{3.23}
\]
Similarly to (3.18), by (3.13), (3.22), and (3.23) we obtain
\[
\|\nabla^\alpha u(\cdot, t)\|_{L^q,\theta(B(0,\sqrt{t}) \setminus B(0, R_2))} \leq \sum_{\alpha-1}^{d_k} \sum_{k=0}^{t} \sum_{i=1}^{d_k} \|\nabla^\alpha u_{k,i}(\cdot, t)\|_{L^q,\theta(B(0,\sqrt{t}) \setminus B(0, R_2))} + \|\nabla^\alpha [R_\alpha u(\cdot, t)]\|_{L^q,\theta(B(0,\sqrt{t}) \setminus B(0, R_2))}
\leq Ct^{\frac{\alpha}{2}}\Gamma_p,\sigma'(t)\left[\sum_{0 \leq k + 2\ell \leq \alpha} \sum_{i=1}^{d_k} \sum_{0 \leq \ell = \alpha}^{\min(\alpha, \ell)} t^{-\ell}\frac{\|\nabla^\alpha J_{k,i}\|_{L^q,\theta(B(0,\sqrt{t}))}}{h_k(\sqrt{t})} + t^{\frac{\alpha}{2}} - \frac{\alpha}{2}\right]
\leq Ct^{\frac{\alpha}{2}}\Gamma_p,\sigma'(t)\left[J_\alpha(t) + t^{\frac{\alpha}{2}} - \frac{\alpha}{2}\right], \quad t > T. \tag{3.24}
\]
Combining (3.5), (3.19), and (3.24), we obtain
\[
\|\nabla^\alpha u(\cdot, t)\|_{L^q,\theta(B(0,\sqrt{t}))} \leq Ct^{\frac{\alpha}{2}}\Gamma_p,\sigma'(t)\left[J_\alpha(t) + t^{\frac{\alpha}{2}} - \frac{\alpha}{2}\right], \quad t > T.
\]
This implies that inequality (3.11) holds for $t > T$ in case $(B'_\alpha)$. Then assertion (b) follows, and the proof of Theorem 3.1 is complete. \(\square\)
Remark 3.1 Let $H = -\Delta$. Then, for any $k, n \in \{0, 1, 2, \ldots\}$, we find $C_{k,n} > 0$ such that

$$I^p_k(|x|) = C_{k,n}|x|^{2n}, \quad x \in \mathbb{R}^N.$$  

Furthermore, $h_k(|x|) = |x|^k$ and $h_k(|x|)Q_{k,i}(x/|x|)$ is a homogeneous polynomial of degree $k$. These mean that

$$|\nabla^\alpha J^p_{k,i}(x)| \leq C|x|^{k+2n-\alpha} \quad \text{if} \quad \alpha \leq k + 2n, \quad |\nabla^\alpha J^p_{k,i}(x)| = 0 \quad \text{if} \quad \alpha > k + 2n,$$  

for $x \in \mathbb{R}^N$. Then, by Theorem 3.1, for any $(p, q, \sigma, \theta) \in \Lambda$ and $\alpha \in \{0, 1, 2, \ldots\}$, we obtain the well-known decay estimate for $\nabla^\alpha e^tH$,

$$\|\nabla^\alpha e^t\|_{(L^p, \sigma \rightarrow L^q, \theta)} \leq Ct^{-\frac{\alpha}{2}}(\frac{1}{\alpha-\frac{\sigma}{2}})^{\frac{\alpha}{2}}, \quad t > 0.$$

4 Lower decay estimates

In this section we study lower decay estimates of $\|\nabla^\alpha e^{-tH}\|_{(L^p, \sigma \rightarrow L^q, \theta)}$. Let

$$L^p_\sigma := \{f \in L^p : f \text{ is radially symmetric in } \mathbb{R}^N\},$$

$$L^p_{k,i} := \{fQ_{k,i} \in L^p : f \text{ is radially symmetric in } \mathbb{R}^N\},$$

where $(k, i) \in K$ and $[fQ_{k,i}](x) = f(x)Q_{k,i}(x/|x|)$ for $x \in \mathbb{R}^N \setminus \{0\}$.

Theorem 4.1 Let $m \in \{1, 2, \ldots\}$ and assume conditions $(\mathcal{V}_m)$ and $(\mathcal{N})$. Let $(p, q, \sigma, \theta) \in \Lambda$, $\alpha \in \{0, 1, \ldots, m+1\}$, and $k \in \{0, 1, 2, \ldots\}$. Then there exist $C > 0$ and $\delta \in (0, 1)$ such that

$$\|\partial^\alpha \|_{(L^p, \sigma \rightarrow L^q, \theta)} \geq c \frac{\Gamma \theta^\alpha \sigma(t)}{h_k(\sqrt{t})} \left[ C^{-1}\|\partial^\alpha h_k\|_{L^q, \theta}(E) - Ct^{-1}\|h_k^{(\alpha-2)}\|_{L^q, \theta}(E) \right]_+, \quad (4.1)$$

$$\|\nabla^\alpha e^{-tH}\|_{(L^p_{k,i} \rightarrow L^q_{k,i})} \geq c \frac{\Gamma \theta^\alpha \sigma(t)}{h_k(\sqrt{t})} \left[ C^{-1}\|\nabla^\alpha h_k\|_{L^q, \theta}(E) - Ct^{-1}\|h_k^{(\alpha-2)}\|_{L^q, \theta}(E) \right]_+, \quad (4.2)$$

for $t > 0$ and measurable sets $E \subset B(0, \delta \sqrt{t})$.

Proof. Let $t > 0$ and fix it. Let $k \in \{0, 1, 2, \ldots\}$. Assume that $h_k \in L^p, \sigma'(B(0, 1))$. Then we find $c > 0$ and a radially symmetric nonnegative function $\phi \in C_c(\mathbb{R}^N)$ such that $1/2 \leq \|\phi\|_{L^p, \sigma} \leq 1$ and

$$\int_{B(0, \sqrt{t})} h_k(y)\phi(y) dy \geq c\|h_k\|_{L^p', \sigma'(B(0, \sqrt{t}))}. \quad (4.3)$$

Here the constant $c$ is independent of $t$. Let

$$v(|x|, \tau) := [e^{-\tau H_k}\phi](|x|), \quad w(|x|, \tau) := \frac{[e^{-\tau H_k}\phi](|x|)}{h_k(|x|)}. \quad (4.4)$$

Then it follows that

$$[e^{-\tau H_k}\phi](x) = v(|x|, \tau)Q_{k,i} \left(\frac{x}{|x|}\right), \quad \text{where} \quad \phi_{k,i}(x) := \phi(|x|)Q_{k,i} \left(\frac{x}{|x|}\right). \quad (4.5)$$
For any $n \in \{0, 1, 2, \ldots \}$, set

$$
\nu_{k,n}(|x|) := |x|^{-n} \nu_k(|x|) = |x|^{-n} h_k(|x|)^2,
$$

for $(x, \tau) \in \mathbb{R}^{N+n} \times (0, \infty)$. Then $w$ satisfies

$$
\partial_\tau w = \partial_\tau w = \frac{1}{r} \partial_\tau (r^{N-1} \nu_k(r) \partial_\tau w) = \frac{1}{rN-n} \partial_\tau (r^{N+n-1} \nu_k(r) \partial_\tau w)
$$

(4.6)

in $\mathbb{R}^{N+n} \times (0, \infty)$, where $r = |x|$.

On the other hand, it follows from condition (N') and (1.4) that $2A_{1,k} \geq -N + 2$ and $2A_{2,k} \geq 2A_{2,0} > -N$. Taking large enough $n$ if necessary, we see that

$$
-N - n < 2A_{1,k} - n < N + n,
-N - n < 2A_{2,k} - n < N + n.
$$

These imply that $\nu_{k,n}$ is an $A_2$-weight in $\mathbb{R}^{N+n}$. Then we apply \cite[Theorem 1.2]{20} (see also \cite{6}) to obtain

$$
w(|x|, t) \geq C \int_{\mathbb{R}^{N+n}} \frac{w(|y|, 0)}{\sqrt{\omega_{n,k}(B(x, \sqrt{t}))} \sqrt{\omega_{n,k}(B(y, \sqrt{t}))}} \exp\left(-\frac{|x - y|^2}{Ct}\right) \nu_{k,n}(|y|) dy
$$

for $(x, t) \in \mathbb{R}^{N+n} \times (0, \infty)$. Here

$$
B(x, r) := \{y \in \mathbb{R}^{N+n} : |x - y|_{N+n} < r\}, \quad \omega_{n,k}(B(x, r)) := \int_{B(x,r)} \nu_{k,n}(y) dy.
$$

It follows from Proposition \cite[2.2]{} (b) that

$$
\omega_{n,k}(B(0, r)) \leq C \int_0^r \nu_k(r) r^{N-1} dr \leq C \int_{B(0,r)} \nu_k(|x|) dx \leq Cr^N h_k(r), \quad r > 0.
$$

Then, recalling $w(x, 0) = \phi(r)/h_k(r)$ with $r = |x|$, we have

$$
w(x, t) = w(x, t) \geq C \int_{B(0, \sqrt{t})} \frac{w(|y|, 0)}{\sqrt{\omega_{n,k}(B(x, \sqrt{t}))} \sqrt{\omega_{n,k}(B(y, \sqrt{t}))}} \nu_{k,n}(y) dy
$$

$$
\geq C \int_{B(0, \sqrt{t})} \frac{w(y, 0)\nu_{k,n}(y)}{\sqrt{\omega_{n,k}(B(0, 2\sqrt{t}))} \sqrt{\omega_{n,k}(B(0, 2\sqrt{t}))}} dy
$$

$$
\geq Ct^{-\frac{N}{2}} \nu_k(\sqrt{t})^{-1} \int_0^{\sqrt{t}} \phi(r) h_k(r) r^{-n} \nu_k(r) r^{N+n-1} dr
$$

$$
\geq Ct^{-\frac{N}{2}} \nu_k(\sqrt{t})^{-1} \int_0^{\sqrt{t}} \phi(r) h_k(r) r^{N-1} dr
$$

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for \( x \in B(0, \sqrt{t}) \). This together with (4.3) implies that

\[
  w(x, t) \geq C^{-1} t^{-\frac{N}{2}} \frac{\Gamma_{p', \sigma'}(t)}{h_k(\sqrt{t})} \int_{B(0, \sqrt{t})} h_k(y) \phi(y) \, dy \geq C^{-1} t^{-\frac{N}{2}} \frac{\Gamma_{p', \sigma'}(t)}{h_k(\sqrt{t})}
\]

for \( x \in B(0, \sqrt{t}) \).

On the other hand, similarly to (2.16), taking small enough \( \delta \in (0, 1) \), we obtain

\[
  w(|x|, t) = \sum_{\ell=0}^{n-1} \partial_{\ell}^\alpha w(0, t) I_k^\alpha(|x|) + I_k^\alpha [\partial_{\ell}^\alpha w(\cdot, t)](|x|), \quad x \in \mathbb{R}^N,
\]

where \( n = 1, 2, \ldots \). Applying Proposition 2.4 with \( H \) replaced by \( H_k \), for any \( \beta \in \{0, 1, 2, \ldots \} \), we have

\[
  t^\beta |\partial_{\ell}^\beta w(x, t)| \leq Ct^{-\frac{N}{2}} \frac{\Gamma_{p', \sigma'}(t)}{h_k(\sqrt{t})} \leq Ct^{-\frac{N}{2}} \frac{\Gamma_{p', \sigma'}(t)}{h_k(\sqrt{t})}
\]

for \( x \in B(0, \delta \sqrt{t}) \). Then, taking small enough \( \epsilon \in (0, 1) \) and applying Lemma 2.1 by (4.7), (4.8) with \( n = 1 \) and (4.9) we see that

\[
  w(0, t) \geq w(|x|, t) - I_k [\partial_{\ell} w(\cdot, t)](|x|)
\]

\[
  \geq C^{-1} t^{-\frac{N}{2}} \frac{\Gamma_{p', \sigma'}(t)}{h_k(\sqrt{t})} - Ct^{-\frac{N}{2}} \frac{\Gamma_{p', \sigma'}(t)}{h_k(\sqrt{t})} |x|^2 \geq C^{-1} t^{-\frac{N}{2}} \frac{\Gamma_{p', \sigma'}(t)}{h_k(\sqrt{t})}
\]

for \( x \in B(0, \epsilon \delta \sqrt{t}) \). By (4.8), (4.9), and (4.10), taking large enough \( n \) if necessary, we apply Proposition 2.1 and Lemma 2.1 to obtain

\[
  |\partial_{\ell}^\alpha v(|x|, t)| = |\partial_{\ell}^\alpha [h_k(|x|)w(|x|, t)]|
\]

\[
  \geq |w(0, t)||\partial_{\ell}^\alpha h_k(|x|)| - C \sum_{\ell=1}^{n} t^{-\frac{N}{2} - \ell} \frac{\Gamma_{p', \sigma'}(t)}{h_k(\sqrt{t})} |x|^{2\ell - \alpha} h_k(|x|)
\]

\[
  \geq t^{-\frac{N}{2}} \frac{\Gamma_{p', \sigma'}(t)}{h_k(\sqrt{t})} \left[ C^{-1} |\partial_{\ell}^\alpha h_k(|x|)| - Ct^{-1} |x|^{2\ell - \alpha} h_k(|x|) \right]
\]

for \( x \in B(0, \delta \sqrt{t}) \setminus \{0\} \). This implies that

\[
  \|\partial_{\ell}^\alpha e^{-tH_k}\|_{(L^p, \sigma) \rightarrow L^q, \theta(E)} \geq \frac{\|\partial_{\ell}^\alpha v(\cdot, t)\|_{L^q, \theta(E)}}{\|\phi\|_{L^p, \sigma}}
\]

\[
  \geq t^{-\frac{N}{2}} \frac{\Gamma_{p', \sigma'}(t)}{h_k(\sqrt{t})} \left[ C^{-1} \|\partial_{\ell}^\alpha h_k\|_{L^q, \theta(E)} - Ct^{-1} |h_k^{(\alpha - 2)}\|_{L^q, \theta(E)} \right]_+
\]

for measurable sets \( E \subset B(0, \delta \sqrt{t}) \). Thus inequality (4.11) holds when \( h_k \in L^{p', \sigma'}(B(0, 1)) \).

On the other hand, it follows from (4.1), (4.5), and (4.8) that

\[
  [e^{-tH} \phi_{k,i}](x) = \sum_{\ell=0}^{n-1} \partial_{\ell}^\ell w(0, t) J_{k,i}^\ell(|x|) + Q_{k,i}(x/|x|)h_k(|x|)I_k^\alpha [\partial_{\ell}^\ell w(\cdot, t)](|x|), \quad x \in \mathbb{R}^N \setminus \{0\}.
\]
Similarly to (4.11), we have
\[
|\nabla^\alpha e^{-tH}\phi_{k,i}(x)| \geq |w(0, t)||\nabla^\alpha f^0_{k,i}(x)| - C \sum_{l=1}^n t^{-\frac{n}{2}} - t \frac{\Gamma^k_{p',\sigma'}(t)}{h_k(\sqrt{t})} |x|^{2-\alpha} h_k(|x|)
\]
\[
\geq t^{-\frac{n}{2}} \frac{\Gamma^k_{p',\sigma'}(t)}{h_k(\sqrt{t})} \left[ C^{-1} |\nabla^\alpha f^0_{k,i}(x)| - C t^{-1} |x|^{2-\alpha} h_k(|x|) \right]
\]
for \( x \in B(0, \delta \sqrt{t}) \setminus \{0\} \). Then, similarly to (4.12), we obtain inequality (4.2) when \( h_k \in L^{p',\sigma'}(B(0, 1)) \).

If \( h_k \not\in L^{p',\sigma'}(B(0, 1)) \), then we approximate the potential \( V \) by bounded radially symmetric potentials and apply the above arguments. Then we deduce that inequalities (4.1) and (4.2) hold with \( \Gamma^k_{p',\sigma'}(t) = \infty \) when \( h_k \not\in L^{p',\sigma'}(B(0, 1)) \). Thus Theorem 4.1 follows. \( \square \)

We apply Theorem 4.1 to obtain the following theorem.

**Theorem 4.2** Let \( m \in \{1, 2, \ldots\} \) and assume conditions (V_m) and (N').

(a) Let \( \alpha \in \{0, 1, \ldots, m + 1\} \) and \( k \in \{0, 1, 2, \ldots\} \). Assume that
\[
C^{-1} r^{-\alpha} h_k(r) \leq |\partial^\alpha_r h_k(r)| \leq C r^{-\alpha} h_k(r) \quad \text{for} \quad R_1 < r < R_2,
\]
for some \( 0 \leq R_1 < R_2 \leq \infty \). Then there exist \( C_1 > 0 \) and \( \delta_1 \in (0, 1) \) such that
\[
\|\partial^\alpha_r e^{-tH}\|_{(L^p,\sigma \rightarrow L^q,\sigma(E))} \geq C_1 \left[ \frac{1}{\sqrt{t}} \frac{\Gamma^k_{p',\sigma'}(t)}{\|\partial^\alpha_r h_k\|_{L^q,\sigma(E)}} \right]^{\frac{1}{\alpha}}
\]
for measurable sets \( E \subset \{ x \in B(0, \delta_1 \sqrt{t}) : R_1 < |x| < R_2 \} \) and \( t > 0 \).

(b) Let \( \alpha \in \{0, 1, \ldots, m + 1\} \). Then there exist \( C_2 > 0 \) and \( \delta_2, \delta_3 \in (0, 1) \) with \( \delta_2 < \delta_3 \) such that
\[
\|\partial^\alpha_r e^{-tH}\|_{(L^p,\sigma \rightarrow L^q,\sigma(E))} \geq C_2 \left[ \frac{1}{\sqrt{t}} \frac{\Gamma^k_{p',\sigma'}(t)}{\|\partial^\alpha_r h_k\|_{L^q,\sigma(E)}} \right]^{\frac{1}{\alpha}}
\]
for measurable sets \( E \subset B(0, \delta_3 \sqrt{t}) \cap B(0, \delta_2 \sqrt{t})^c \) and \( t > 0 \). In particular,
\[
\|\nabla^\alpha e^{-tH}\|_{(L^p,\sigma \rightarrow L^q,\sigma)} \geq \|\partial^\alpha_r e^{-tH}\|_{(L^p,\sigma \rightarrow L^q,\sigma(E))} \geq C_3 \left[ \frac{1}{\sqrt{t}} \frac{\Gamma^k_{p',\sigma'}(t)}{\|\partial^\alpha_r h_k\|_{L^q,\sigma(E)}} \right]^{\frac{1}{\alpha}}, \quad t > 0.
\]

**Proof.** We prove assertion (a). Assume (4.13). Let \( \delta \in (0, 1) \) be as in Theorem 4.1 and \( 0 < \epsilon \leq 1 \). It follows that
\[
t^{-1} h_k^{\alpha-2}(|x|) = t^{-1} r^{-\alpha+2} h_k(|x|) \leq (\epsilon \delta)^2 |x|^{-\alpha} h_k(|x|) \leq C(\epsilon \delta)^2 |\partial^\alpha_r h_k(|x|)|
\]
for measurable sets \( E \subset \{ x \in B(0, \epsilon \delta \sqrt{t}) : R_1 < |x| < R_2 \} \). Taking small enough \( \epsilon \in (0, 1) \) if necessary, by Theorem 4.1 and (4.16) we have
\[
\|\partial^\alpha_r e^{-tH}\|_{(L^p,\sigma \rightarrow L^q,\sigma(E))} \geq t^{-\frac{n}{2}} \frac{\Gamma^k_{p',\sigma'}(t)}{h_k(\sqrt{t})} \left[ C^{-1} \|\partial^\alpha_r h_k\|_{L^q,\sigma(E)} - C \epsilon^2 \delta^2 \|\partial^\alpha_r h_k\|_{L^q,\sigma(E)} \right]^{\frac{1}{\alpha}}
\]
\[
\geq C^{-1} t^{-\frac{n}{2}} \frac{\Gamma^k_{p',\sigma'}(t)}{h_k(\sqrt{t})} \|\partial^\alpha_r h_k\|_{L^q,\sigma(E)}
\]
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for measurable sets $E \subset \{ x \in B(0, \epsilon \delta \sqrt{t}) : R_1 < |x| < R_2 \}$ and $t > 0$. Thus assertion (a) follows.

We prove assertion (b). Let $\delta' \in (0, \delta)$. Taking large enough $k \in \{0, 1, 2, \ldots \}$, by Propositions 2.1 and 2.2 we see that

$$C^{-1}r^{-\alpha} h_k(r) \leq \left| \frac{d^\alpha}{dr^\alpha} h_k(r) \right| \leq C r^{-\alpha} h_k(r) \quad \text{for} \quad r > 0,$$

$$\Gamma^k_{p', \sigma'}(t) \geq C^{-1} t^{-\frac{k}{2p'}}, \quad \frac{\|\partial^\alpha h_k\|_{L^q, \theta(E)}}{h_k(\sqrt{t})} \geq C^{-1} t^{-\frac{q}{2}}|E|^\frac{1}{q} \quad \text{for} \quad t > 0,$$

where $E$ is a measurable set in $B(0, \epsilon \delta \sqrt{t}) \cap B(0, \epsilon \delta' \sqrt{t})^c$. Then assertion (a) implies that

$$\|\partial^\alpha e^{-tH}\|_{(L^p, \sigma \rightarrow L^q, \theta(E))} \geq \|\partial^\alpha e^{-tH}\|_{(L^p, \sigma \rightarrow L^q, \theta(E))} \geq C^{-1} \|\partial^\alpha e^{-tH_k}\|_{(L^p, \sigma \rightarrow L^q, \theta(E))} \geq C^{-1} t^{-\frac{k}{2p'}} - \frac{\alpha}{2} |E|^\frac{1}{q}$$

for measurable sets $E \subset B(0, \epsilon \delta \sqrt{t}) \cap B(0, \epsilon \delta' \sqrt{t})^c$ and $t > 0$, where $i \in \{1, \ldots, d_k\}$. Then we have inequality (4.14). Inequality (4.15) easily follows from (4.14). Thus we obtain assertion (b), and the proof of Theorem 4.2 is complete. $\square$

5 Proof of Theorem 1.1

In this section, combining the results in the previous sections, we prove Theorem 1.1.

Proof of Theorem 1.1 Let us consider the case $\alpha = 0$. It follows from Theorem 3.1 and Proposition 2.2 (a) that

$$\|e^{-tH}\|_{(L^p, \sigma \rightarrow L^q, \theta)} \leq C t^{-\frac{k}{2}} \sum \left[ \|h_0\|_{L^q, \theta(B(0, \sqrt{t}))} + \frac{t^{\frac{k}{2}}}{h_0(\sqrt{t})} \right] \leq C t^{-\frac{k}{2}} \sum \|h_0\|_{L^q, \theta(B(0, \sqrt{t}))}, \quad t > 0.$$ 

Let $\delta \in (0, 1)$ be small enough. By Theorem 4.2 we see that

$$\|e^{-tH}\|_{(L^p, \sigma \rightarrow L^q, \theta)} \geq \|e^{-tH}\|_{(L^p, \sigma \rightarrow L^q, \theta(B(0, \delta \sqrt{t})))} \geq C t^{-\frac{k}{2}} \sum \|h_0\|_{L^q, \theta(B(0, \delta \sqrt{t}))} \geq C t^{-\frac{k}{2}} \sum \|h_0\|_{L^q, \theta(B(0, \delta \sqrt{t}))} \geq C t^{-\frac{k}{2}} \sum \|h_0\|_{L^q, \theta(B(0, \delta \sqrt{t}))}, \quad t > 0.$$ 

Then Theorem 1.1 follows in the case $\alpha = 0$. So it suffices to prove Theorem 1.1 in the case $\alpha \in \{1, 2\}$.

Step 1: We prove that

$$\|\nabla^\alpha e^{-tH}\|_{(L^p, \sigma \rightarrow L^q, \theta)} \leq C \Phi_\alpha(t), \quad t > 0,$$

where $\alpha \in \{1, 2\}$. We can assume, without loss of generality, that $\Phi_\alpha(t) < \infty$ for some $t > 0$. Then

$$h_0 \in L_{p', \sigma'}(B(0, \sqrt{t})), \quad \nabla^\alpha h_0 \in L^{q, \theta}(B(0, \sqrt{t})),$$

These imply that

$$h_0 \in L_{p', \sigma'}(B(0, R)), \quad \nabla^\alpha h_0 \in L^{q, \theta}(B(0, R)), \quad \nabla^\alpha J_{1,i} \in L^{q, \theta}(B(0, R)),$$

$$\Phi_\alpha(t) \leq J_\alpha(t) < \infty.$$
for $R > 0$ and $t > 0$, where $i = 1, \ldots, N$. It follows from Theorem 3.1 that

\[
\begin{align*}
\|\nabla e^{-tH}\|_{(L^p, \sigma \rightarrow L^q, \theta)} \\
&\leq Ct^{-\frac{N}{2}} \Gamma_{p', \sigma'}(t) \left[ \Phi_1(t) + \sum_{i=1}^{N} \frac{\|\nabla J_{1,i}\|_{L^q, \theta(B(0, \sqrt{t}))}}{h_1(\sqrt{t})} \right], \\
\|\nabla^2 e^{-tH}\|_{(L^p, \sigma \rightarrow L^q, \theta)} \\
&\leq Ct^{-\frac{N}{2}} \Gamma_{p', \sigma'}(t) \left[ \Phi_2(t) + t^{-1} \frac{\|\nabla^2 J_{0,1}\|_{L^q, \theta(B(0, \sqrt{t}))}}{h_0(\sqrt{t})} + \frac{\|\nabla^2 h_2\|_{L^q, \theta(B(0, \sqrt{t}))}}{h_2(\sqrt{t})} \right], \quad t > 0.
\end{align*}
\] (5.4)

Following the arguments in Subsection 2.2, we divide the behavior of $h_0$ near 0 into cases $(A_2)$ and $(B_2)$. In case $(A_2)$, by (2.7) and (5.4) we find $R_1 > 0$ such that (5.11) holds for $t \in (0, \sqrt{R_1})$. In case $(B_2)$, for any $R_2 > 0$, by (2.9) and (5.4) we have (5.1) for $t \in (0, \sqrt{R_2})$. These imply that (5.11) holds in $(0, T_1)$ for some $T_1 > 0$.

Let $T_2 \in (T_1, \infty)$ be large enough. By (5.3) we see that

\[
J_\alpha(t) + t^\frac{N}{2} - \frac{p}{2} \leq C, \quad \Phi_\alpha(t) \geq t^\frac{N}{2} - \frac{p}{2} \geq C, \quad \text{for} \quad t \in (T_1, T_2).
\]

Then Theorem 3.1 implies that (5.11) holds for $T_1 < t \leq T_2$.

Next we divide the behavior of $h_0$ at the space infinity into cases $(A_2')$ and $(B_2')$. Consider case $(A_2')$. Let $R_3 > 0$ be as in (2.10). Taking large enough $T_2 > 0$ if necessary, by (2.11) and (5.3) we have

\[
\begin{align*}
\sum_{i=1}^{N} \frac{\|\nabla J_{1,i}\|_{L^q, \theta(B(0, \sqrt{t}))}}{h_1(\sqrt{t})} &\leq \sum_{i=1}^{N} \frac{\|\nabla J_{1,i}\|_{L^q, \theta(B(0, \sqrt{t}) \setminus B(0, R_3))} + C}{h_1(\sqrt{t})} \\
&\leq C \frac{\|\nabla h_0\|_{L^q, \theta(B(0, \sqrt{t}) \setminus B(0, R_3))} + 1}{h_0(\sqrt{t})} \leq C \frac{\|\nabla h_0\|_{L^q, \theta(B(0, \sqrt{t}))}}{h_0(\sqrt{t})}, \\
t^{-1} \frac{\|\nabla^2 J_{0,1}\|_{L^q, \theta(B(0, \sqrt{t}))}}{h_0(\sqrt{t})} &\leq t^{-1} \frac{\|\nabla^2 J_{0,1}\|_{L^q, \theta(B(0, \sqrt{t}) \setminus B(0, R_3))} + C}{h_0(\sqrt{t})} \\
&\leq C \frac{\|\nabla^2 h_0\|_{L^q, \theta(B(0, \sqrt{t}) \setminus B(0, R_3))} + 1}{h_0(\sqrt{t})} \leq C \frac{\|\nabla^2 h_0\|_{L^q, \theta(B(0, \sqrt{t}))}}{h_0(\sqrt{t})}, \\
\frac{\|\nabla^2 h_2\|_{L^q, \theta(B(0, \sqrt{t}))}}{h_2(\sqrt{t})} &\leq \frac{\|\nabla^2 h_2\|_{L^q, \theta(B(0, \sqrt{t}) \setminus B(0, R_3))} + C}{h_2(\sqrt{t})} \\
&\leq C \frac{\|\nabla^2 h_0\|_{L^q, \theta(B(0, \sqrt{t}) \setminus B(0, R_3))} + 1}{h_0(\sqrt{t})} \leq C \frac{\|\nabla^2 h_0\|_{L^q, \theta(B(0, \sqrt{t}))}}{h_0(\sqrt{t})},
\end{align*}
\]

for large enough $t \geq T_2$. These together with (5.4) imply that inequality (5.1) holds for $t \geq T_2$ in case $(A_2')$.

Consider case $(B_2')$. Let $R_4 > 0$ be large enough. By (2.12) and (2.13), taking large enough
\( T_2 \) if necessary, we have

\[
\sum_{i=1}^{N} \frac{\|\nabla J_{1;i}\|_{L^q(B(0,\sqrt{t}))}}{h_1(\sqrt{t})} \leq \sum_{i=1}^{N} \frac{\|\nabla J_{1;i}\|_{L^q(B(0,\sqrt{t}) \setminus B(0,R_4))} + C}{h_1(\sqrt{t})} \leq Ct^{\frac{N}{2t} - \frac{1}{2}},
\]

\[
t^{-\frac{1}{2}} \frac{\|\nabla^2 J_{0,1}\|_{L^q(B(0,\sqrt{t}))}}{h_0(\sqrt{t})} \leq t^{-\frac{1}{2}} \frac{\|\nabla^2 J_{0,1}\|_{L^q(B(0,\sqrt{t}) \setminus B(0,R_4))} + C}{h_0(\sqrt{t})} \leq Ct^{\frac{N}{2t} - 1},
\] (5.5)

\[
\frac{\|\nabla^2 h_2\|_{L^q(B(0,\sqrt{t}))}}{h_2(\sqrt{t})} \leq \frac{\|\nabla^2 h_2\|_{L^q(B(0,\sqrt{t}) \setminus B(0,R_4))} + C}{h_2(\sqrt{t})} \leq Ct^{\frac{N}{2t} - 1} + Ct^{-1},
\]

for \( t \geq T_2 \). By (5.3) and (5.5) we see that inequality (5.1) holds for \( t \geq T_2 \) in case \((B'_2)\). Therefore we obtain (5.1) for \( t \geq T_2 \), and the proof of (5.1) is complete.

**Step 2:** We prove

\[
\|\nabla^\alpha e^{-tH}\|_{(L^p,\sigma \rightarrow L^q,\theta)} \geq C^{-1} t^{-\frac{1}{2}} \Phi_\alpha(t), \quad t > 0,
\] (5.6)

where \( \alpha \in \{1,2\} \). We can assume, without loss of generality, that

\[
\|\nabla^\alpha e^{-t_* H}\|_{(L^p,\sigma \rightarrow L^q,\theta)} < \infty \quad \text{for some } t_* > 0.
\] (5.7)

We also prove that (5.3) holds for \( R > 0 \) and \( t > 0 \) under assumption (5.7). Similarly to Step 1, we divide the behavior of \( h_0 \) near 0 into cases \((A_2)\) and \((B_2)\). Let \( R_i \) \((i = 1, 2, 3, 4)\) be as in the above.

Consider case \((A_2)\). By Theorem 2.2 (a), 2.6, and 2.7 we see that

\[
\|\nabla^\alpha e^{-tH}\|_{(L^p,\sigma \rightarrow L^q,\theta)} \geq \frac{\|\partial^\alpha e^{-tH}\|_{(L^p,\sigma \rightarrow L^q,\theta)(B(0,R))}}{h_0(\sqrt{t})} \geq \frac{C^{-1} t^{-\frac{N}{2}} \Gamma_{\alpha}(t) \|\nabla^\alpha h_0\|_{L^q,B(0,R)}}{h_0(\sqrt{t})} \geq C^{-1} t^{-\frac{N}{2}} \Gamma_{\alpha}(t) \|\nabla^\alpha J_{1,i}\|_{L^q,B(0,R)} \]

for \( 0 < R < \delta \sqrt{t} \) with \( R < R_1 \) and \( t > 0 \). This together with (5.7) implies (5.2) and (5.3). On the other hand, it follows from Proposition 2.2 (a) and (2.6) that

\[
t^{\frac{N}{2t} - \frac{1}{p}} \leq Ct^{-\frac{1}{2}} \frac{\|h_0\|_{L^p,B(0,\sqrt{t})}}{h_0(\sqrt{t})} \leq C \frac{\|h_0^{(\alpha)}\|_{L^q,B(0,\sqrt{t})}}{h_0(\sqrt{t})} \leq C \frac{\|\nabla^\alpha h_0\|_{L^q,B(0,\sqrt{t})}}{h_0(\sqrt{t})}
\] (5.9)

for \( 0 < t \leq R_1^2 \). By (5.8) and (5.9) we see that (5.6) holds for \( t \in (0,R_1^2) \).

Consider case \((B_2)\). By Theorem 4.1 and 2.3 we have

\[
\|\nabla^\alpha e^{-tH}\|_{(L^p,\sigma \rightarrow L^q,\theta)} \geq \frac{\|\nabla^\alpha e^{-tH}\|_{(L^p,\sigma \rightarrow L^q,\theta)(B(0,R))}}{h_0(\sqrt{t})} \geq \frac{C^{-1} t^{-\frac{N}{2}} \Gamma_{\alpha}(t) \|\nabla^\alpha J_{k,i}\|_{L^q,B(0,R)}}{h_0(\sqrt{t})} \geq C^{-1} t^{-\frac{N}{2}} \Gamma_{\alpha}(t) \|\nabla^\alpha h_0\|_{L^q,B(0,\sqrt{t})} \]

(5.10)
for $0 < R < \delta \sqrt{t}$ with $R < R_2$ and $t > 0$, where $k \in \{0, 1\}$ and $i \in \{1, \ldots, d_k\}$. This together with (5.7) implies (5.2) and (5.3).

Let $\epsilon > 0$ be small enough. By Theorem 4.2 (b), (2.8), and (5.10) we obtain

\[
\|\nabla e^{-tH}\|_{(L_{p,\sigma} \to L_{q,\theta})} \geq (1 - \epsilon)\|\nabla e^{-tH}\|_{(L_{p,\sigma} \to L_{q,\theta}(B(0,\delta\sqrt{t})))} + \epsilon\|\nabla e^{-tH}\|_{(L_{p,\sigma} \to L_{q,\theta}(B(0,\sqrt{t})))}
\]

\[
\geq C^{-1}t^{-\frac{\alpha}{2}\left(\frac{1}{p} - \frac{1}{q}\right) - \frac{1}{2}} + C^{-1}t^{-\frac{\alpha}{2p}}\frac{\|\nabla J_{0,1}\|_{L_{q,\theta}(B(0,\sqrt{t}))}}{h_0(\sqrt{t})} - C\epsilon t^{-\frac{\alpha}{2}\left(\frac{1}{p} - \frac{1}{q}\right) - \frac{1}{2}}
\]

\[
\geq C^{-1}t^{-\frac{\alpha}{2}\left(\frac{1}{p} - \frac{1}{q}\right) - \frac{1}{2}} + C^{-1}t^{-\frac{\alpha}{2p}}\frac{\|\nabla h_0\|_{L_{q,\theta}(B(0,\sqrt{t}))}}{h_0(\sqrt{t})}
\]

\[
\geq C^{-1}t^{-\frac{\alpha}{2p}}\left[\frac{\|\nabla h_0\|_{L_{q,\theta}(B(0,\sqrt{t}))}}{h_0(\sqrt{t})} + t^{-\frac{\alpha}{2p}}\right] \geq C^{-1}\epsilon \Phi_1(t), \quad t \in (0, R_2^2).
\]

Similarly, we have

\[
\|\nabla^2 e^{-tH}\|_{(L_{p,\sigma} \to L_{q,\theta})}
\]

\[
\geq (1 - (N + 1)\epsilon)\|\nabla^2 e^{-tH}\|_{(L_{p,\sigma} \to L_{q,\theta})} + \epsilon\|\nabla^2 e^{-tH}\|_{(L_{p,\sigma} \to L_{q,\theta})} + \epsilon \sum_{i=1}^{N}\|\nabla^2 e^{-tH}\|_{(L_{p,\sigma} \to L_{q,\theta})}
\]

\[
\geq C^{-1}t^{-\frac{\alpha}{2}\left(\frac{1}{p} - \frac{1}{q}\right) - 1} + C^{-1}t^{-\frac{\alpha}{2p}}\frac{\|\nabla^2 h_0\|_{L_{q,\theta}(B(0,\sqrt{t}))}}{h_0(\sqrt{t})} - C(N + 1)\epsilon t^{-\frac{\alpha}{2}\left(\frac{1}{p} - \frac{1}{q}\right) - 1}
\]

\[
\geq C^{-1}t^{-\frac{\alpha}{2}\left(\frac{1}{p} - \frac{1}{q}\right) - 1} + C^{-1}t^{-\frac{\alpha}{2p}}\left[\frac{\|\nabla h_0\|_{L_{q,\theta}(B(0,\sqrt{t}))}}{h_0(\sqrt{t})} + \sum_{i=1}^{N}\frac{\|\nabla^2 J_{1,1}\|_{L_{q,\theta}(B(0,\sqrt{t}))}}{h_0(\sqrt{t})}\right] + t^{-\frac{\alpha}{2p}}\geq C^{-1}\epsilon \Phi_2(t)
\]

for $t \in (0, R_2^2)$. These imply that (5.6) holds for $t \in (0, R_2^2)$.

Combining the arguments in cases (A2) and (B2), we find $T_1 > 0$ such that (5.6) holds for $t \in (0, T_1)$. Furthermore, we see that (5.2) and (5.3) hold. Then, for any $T_2 > T_1$, we have

\[
\|\nabla^\alpha e^{-tH}\|_{(L_{p,\sigma} \to L_{q,\theta})} \geq C t^{-\frac{\alpha}{2}\left(\frac{1}{p} - \frac{1}{q}\right) - \frac{\alpha}{2}} \geq C^{-1} \geq C^{-1} \Phi_\alpha(t), \quad t \in [T_1, T_2],
\]

which implies that (5.6) holds for $t \in [T_1, T_2]$.

It remains to prove (5.6) for $t > T_2$. We divide the proof in cases (A2') and (B2'). Consider case (A2'). Similarly to (5.8), taking large enough $T_2$ if necessary, by Theorem 4.2 (a) and (2.10) we have

\[
\|\nabla^\alpha e^{-tH}\|_{(L_{p,\sigma} \to L_{q,\theta})} \geq \|\partial^\alpha V^\alpha(t)\|_{L_{q,\theta}(B(0,\sqrt{t}) \setminus B(0, R_3))}
\]

\[
\geq C^{-1}t^{-\frac{\alpha}{2p}}\frac{\|\nabla^\alpha h_0\|_{L_{q,\theta}(B(0,\sqrt{t}) \setminus B(0, R_3))}}{h_0(\sqrt{t})}
\]

\[
\geq C^{-1}t^{-\frac{\alpha}{2p}}\frac{\|\nabla^\alpha h_0\|_{L_{q,\theta}(B(0,\sqrt{t}) \setminus B(0, R_3))}}{h_0(\sqrt{t})} \geq C^{-1}t^{-\frac{\alpha}{2p}}\frac{\|\nabla^\alpha h_0\|_{L_{q,\theta}(B(0,\sqrt{t}) \setminus B(0, R_3))}}{h_0(\sqrt{t})}
\]

\[
\geq C^{-1}t^{-\frac{\alpha}{2p}}\frac{\|\nabla^\alpha h_0\|_{L_{q,\theta}(B(0,\sqrt{t}) \setminus B(0, R_3))}}{h_0(\sqrt{t})} \geq C^{-1}t^{-\frac{\alpha}{2p}}\frac{\|\nabla^\alpha h_0\|_{L_{q,\theta}(B(0,\sqrt{t}) \setminus B(0, R_3))}}{h_0(\sqrt{t})}
\]
for $t \geq T_4$. Furthermore, by Proposition 2.2 (a), (2.10), and (2.11) we have

$$\sum_{i=1}^{N} \frac{\|\nabla^{\alpha} J_{1,k}\|_{L^{2,\theta}(B(0,\sqrt{T}))}}{h_1(\sqrt{T})} \leq \sum_{i=1}^{N} \frac{\|\nabla^{\alpha} J_{1,k}\|_{L^{2,\theta}(B(0,\sqrt{T}) \setminus B(0,R_3))} + C}{h_1(\sqrt{T})} \leq C \frac{\|\nabla^{\alpha} h_0\|_{L^{2,\theta}(B(0,\sqrt{T}))} + 1}{h_0(\sqrt{T})} \leq C \frac{\|\nabla^{\alpha} h_0\|_{L^{2,\theta}(B(0,\sqrt{T}) \setminus B(0,R_3))}}{h_0(\sqrt{T})} \geq C t^{-\frac{\alpha}{2}} \frac{\|h_0\|_{L^{2,\theta}(B(0,\sqrt{T}) \setminus B(0,R_3))}}{h_0(\sqrt{T})} \geq C t^{-\frac{\alpha}{2}} \Gamma_{\theta}(t) \geq C t^{\frac{N}{2} - \frac{\alpha}{2}},$$

for $t \geq T_4$. These imply that (5.6) holds for $t \geq T_4$.

Consider case (B_2). Similarly to (5.10), by Theorem 4.2 and (2.8) we have

$$\|\nabla^{\alpha} e^{-tH}\|_{(L^p,\sigma) \rightarrow L^{q,\theta}} \geq \|\nabla^{\alpha} e^{-tH}\|_{(L^p,\sigma) \rightarrow L^{q,\theta}(B(0,\sqrt{T}))},$$

$$\geq t^{-\frac{\alpha}{2}} \langle \frac{\gamma_{p,\sigma}(t)}{h_k(\sqrt{T})} \rangle \left[ C^{-1} \|\nabla^{\alpha} J_{k,i}\|_{L^{2,\theta}(B(0,\sqrt{T}))} - C t^{-1} \|h_k^{(\alpha-2)}\|_{L^{q,\theta}(B(0,\sqrt{T}))} \right] + C^{-1} t^{-\frac{\alpha}{2}} \|\nabla^{\alpha} J_{k,i}\|_{L^{2,\theta}(B(0,\sqrt{T}))} - C^{-1} t^{-\frac{\alpha}{2}} \left( \frac{1}{p-1} \right) < \frac{\alpha}{2},$$

for $t \geq T_4$, where $k \in \{0,1\}$ and $i \in \{1, \ldots, d_k\}$. Applying the same argument as in case (B_2) we see that (5.6) holds for $t > T_4$. Therefore we deduce that (5.6) holds for $t > 0$. Thus Theorem 1.1 follows. □

6 Proofs of Theorems 1.2 and 1.3

Proof of Theorem 1.2. Assertion (a) follows from Theorem 1.2 (b). It suffices to prove assertion (b). Assume (1.9) and let $\alpha \in \{0,1,\ldots,m+1\}$. Consider the case of $A_{1,0} < \alpha$. If $A_{1,0} \notin \{0,1,\ldots,\alpha-1\}$, then, by Proposition 2.1 we find $R_1 > 0$ such that

$$\|\nabla^{\alpha} h_0(|x|) \times |\partial_{\xi}^{\alpha} h_0(|x|)| \times |x|^{A_{1,0} - \alpha}, \quad x \in B(0, R_1) \setminus \{0\}.$$

This implies that $\partial_{\xi}^{\alpha} h_0 \notin L^{\infty}(B(0,R))$ for $R > 0$. Then, by Theorem 1.2 (a) we see that

$$\|\nabla^{\alpha} e^{-tH}\|_{(L^p,\sigma) \rightarrow L^{q,\theta}} \geq \|\nabla^{\alpha} e^{-tH}\|_{(L^p,\sigma) \rightarrow L^{q,\theta}(B(0,\sqrt{T}))} \geq C t^{-1} \|\partial_{\xi}^{\alpha} e^{-tH}\|_{(L^p,\sigma) \rightarrow L^{q,\theta}(B(0,\sqrt{T}))} = \infty,$$

which contradicts (1.9). This implies that $A_{1,0} \geq \alpha$ if $A_{1,0} \notin \{0,1,\ldots,\alpha-1\}$.

If $A_{1,0} \in \{1,2,\ldots,\alpha-1\}$, then $\alpha \geq 1$, $\lambda_1 \geq \omega_1 > 0$, and $0 < A_{1,1} - A_{1,0} < 1$ (see e.g. [17, Lemma 4.2]). Applying the above argument again, we see that $\partial_{\xi}^{\alpha} h_1 \notin L^{\infty}(B(0,R))$ for $R > 0$. Then, by Theorem 1.2 (a) we see that

$$\|\nabla^{\alpha} e^{-tH}\|_{(L^p,\sigma) \rightarrow L^{q,\theta}} \geq \|\nabla^{\alpha} e^{-tH}\|_{(L^p,\sigma) \rightarrow L^{q,\theta}(B(0,\sqrt{T}))} \geq C t^{-1} \|\partial_{\xi}^{\alpha} e^{-tH}\|_{(L^p,\sigma) \rightarrow L^{q,\theta}(B(0,\sqrt{T}))} = \infty,$$
which contradicts (1.9). This implies that $A_{1,0} \not\in \{1, 2, \ldots, \alpha - 1\}$. We deduce that $A_{1,0} \in \{\alpha, \infty\} \cup \{0\}$, that is, $\lambda_1 \in \{\omega_\alpha, \infty\} \cup \{0\}$.

Consider the case of $A_{2,0} < \alpha$. If $A_{2,0} \not\in \{0, 1, 2, \ldots, \alpha - 1\}$, then, by Proposition 2.1, we find $R_2 > 0$ such that

$$|\nabla^\alpha h_0(x)| \approx |\partial_x^\alpha h_0(x)| \approx h_0^\alpha(x), \quad x \in B(0, R_2)^c.$$

By Theorem 4.1, we see that $\lambda \geq 2$ for large enough $R$. This together with (1.9) implies that $\alpha \leq A_{2,1} < A_{2,0} + 1 \leq \alpha$. This is a contradiction. So we see that $A_{2,0} \in \{\alpha, \infty\} \cup \{0\}$.

We prove that $A_{2,0} \in [\alpha, \infty)$ if $\alpha \geq 1$. Let $A_{2,0} = 0$ and $R > 0$. By Proposition 2.2 (a) and Theorem 4.1, we see that

$$\|\nabla^\alpha e^{-tH}\|_{(L^p, \sigma) \to L^\infty} \geq \|\partial_x^\alpha e^{-tH}\|_{(L^p, \sigma) \to L^\infty(B(0, \delta_0 \sqrt{t}) \cap B(0, R_2)^c)}$$

$$\geq Ct \frac{\|h_0^\alpha\|_{L^\infty(B(0, \delta_0 \sqrt{t}) \cap B(0, R_2)^c)}}{h_0^{\alpha/2}} \geq Ct^{-\frac{\alpha}{2p}} - \frac{L_2^{\alpha/2}}{2}$$

for large enough $t > 0$. This together with (1.9) implies that $\alpha \leq A_{2,1} < A_{2,0} + 1 \leq \alpha$. This is a contradiction. So we see that $A_{2,0} \in \{\alpha, \infty\} \cup \{0\}$.

We prove that $A_{2,0} \in [\alpha, \infty)$ if $\alpha \geq 1$. Let $A_{2,0} = 0$ and $R > 0$. By Proposition 2.2 (a) and Theorem 4.1, we see that

$$\|\nabla e^{-tH}\|_{(L^p, \sigma) \to L^\infty} \geq \|\nabla e^{-tH}\|_{(L^p, \sigma) \to L^\infty(B(0, R_1 + 1) \setminus B(0, R))}$$

$$\geq t^{-\frac{\alpha}{2}} \frac{\|\nabla h_0\|_{L^\infty(B(0, R_1 + 1) \setminus B(0, R))}}{h_0^{\alpha/2}} \geq C^{-1} t^{-\frac{\alpha}{2}} \left[ C^{-1} \|\nabla h_0\|_{L^\infty(B(0, R_1 + 1) \setminus B(0, R))} - C^{-1} \|h_0\|_{L^\infty(B(0, R_1 + 1) \setminus B(0, R))} \right]^{1/2}$$

for large enough $t > 0$. This together with (1.9) implies that $\|\nabla h_0\|_{L^\infty(B(0, R_1 + 1) \setminus B(0, R))} = 0$. Since $R$ is arbitrary, we observe that $h_0$ is a constant function in $\mathbb{R}^N$, which contradicts that $V \neq 0$ in $\mathbb{R}^N$ (see (1.5)). Thus $A_{2,0} \neq 0$. Therefore we see that $A_{2,0} \in [\alpha, \infty)$, that is, $H$ is subcritical and $\lambda_2 \geq \omega_\alpha$ (see (1.3)). Thus Theorem 1.2 follows. □

**Proof of Theorem 1.3** Since $\alpha$ is arbitrary, we apply a similar argument as in that of the proof of Theorem 1.2 to see that $\lambda_1 = \lambda_2 = 0$. Furthermore, for $R > 0$, we have

$$\|\nabla e^{-tH}\|_{(L^p, \sigma) \to L^{q, \sigma}} \geq C^{-1} t^{-\frac{\alpha}{2}} \left[ C^{-1} \|\nabla h_0\|_{L^{q, \sigma}(B(0, R_1 + 1) \setminus B(0, R))} - C^{-1} \right]^{1/2}$$

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for large enough $t > 0$, instead of (6.2). Then, similarly to the proof of Theorem 1.2, we observe from (1.10) that $h_0$ is a constant function in $\mathbb{R}^N$. This means that $V \equiv 0$ in $\mathbb{R}^N$. Thus Theorem 1.3 follows. □

7 Applications

We study the decay of $\|\nabla^\alpha e^{-tH}\|_{(L^p,\sigma \to L^q,\theta)}$ for some typical potentials.

7.1 Hardy potentials

In this subsection we consider the case when

$$V(r) = \lambda r^{-2}, \quad r > 0,$$

with $\lambda \geq \lambda_* = -(N-2)^2/4$ and $\lambda \neq 0$. (See Remark 3.1 for the case of $\lambda = 0$.) Set

$$A_k := \frac{-(N-2) + \sqrt{(N-2)^2 + 4(\lambda + \omega_k)}}{2}, \quad k \in \{0, 1, 2, \ldots \}.$$

We remark that, under (7.1), $H = -\Delta + V$ is subcritical if $\lambda > \lambda_*$ and it is critical if $\lambda = \lambda_*$. Then

$$A_{1,k} = A_{2,k} = A_k, \quad h_k(|x|) = |x|^{A_k},$$

for $k \in \{0, 1, 2, \ldots \}$ and $\lambda \geq \lambda_*$. Furthermore, $A_0 \neq 0$ by $\lambda \neq 0$.

**Theorem 7.1** Assume (7.1) with $\lambda \geq \lambda_* = -(N-2)^2/4$ and $\lambda \neq 0$. Let $\alpha \in \{0, 1, \ldots \}$ and $(p, q, \sigma, \theta) \in \Lambda$. For $A \in \mathbb{R}$, set $f_A(x) := |x|^A$ for $x \in \mathbb{R}^N \setminus \{0\}$.

(a) There exists $C_1 > 0$ such that

$$\|\nabla^\alpha e^{-tH}\|_{(L^p,\sigma \to L^q,\theta)} \geq C_1 t^{-\frac{N}{2}} \left(\frac{1}{p} - \frac{1}{q}\right)^{-\frac{\alpha}{2}}, \quad t > 0.$$

(b) Assume that $A_0 \notin \{2, 4, \ldots \}$. Then $\|\nabla^\alpha e^{-tH}\|_{(L^p,\sigma \to L^q,\theta)} < \infty$ for some $t > 0$ if and only if

$$f_{A_0} \in L^{p',\sigma'}(B(0,1)), \quad f_{A_0}^{(\alpha)} \in L^{q,\theta}(B(0,1)).$$

Furthermore, under assumption (7.2), there exists $C_2 > 0$ that

$$\|\nabla^\alpha e^{-tH}\|_{(L^p,\sigma \to L^q,\theta)} \leq C_2 t^{-\frac{N}{2}} \left(\frac{1}{p} - \frac{1}{q}\right)^{-\frac{\alpha}{2}}, \quad t > 0.$$  \hspace{1cm} (7.3)

(c) Assume that $A_0 = 2\gamma$ for some $\gamma \in \{1, 2, \ldots \}$. Then $\|\nabla^\alpha e^{-tH}\|_{(L^p,\sigma \to L^q,\theta)} < \infty$ for some $t > 0$ if and only if

$$\text{either } \alpha \leq 2\gamma \text{ or } f_{A_1}^{(\alpha)} \in L^{q,\theta}(B(0,1)).$$

Furthermore, under assumption (7.4), inequality (7.3) holds for some $C_2 > 0$.
Proof. Assertion (a) follows from Theorem 4.2 (b). We prove assertions (b) and (c). Let $(k, i) ∈ K$. By Theorem 4.1 we have
\[
\|\nabla^α e^{-tH}\|_{(L^p,σ ⇒ L^q,θ)} ≥ \|\nabla^α e^{-tH}\|_{(L_{k,i}^p,σ ⇒ L_{k,i}^q,θ)} ≥ t^{-\frac{γ}{2}} \frac{Γ_{p',σ'}(t)}{h_k(\sqrt{t})} \left[ C^{-1} \|\nabla^α J_{k,i}(E)\|_{L^q,θ(E)} - Ct^{-1}∥f_{k}^{(α)}(t)∥_{L^q,θ(E)} \right] + (7.5)
\]
for measurable sets $E ⊂ B(0, δ\sqrt{t})$ and $t > 0$, where $δ ∈ (0, 1)$ is as in Theorem 4.1. Furthermore, for any $(k, i) ∈ K$ and $α, n ∈ \{0, 1, 2, \ldots\}$, by Lemma 2.4 (b) we have
\[
t^{-n} \frac{\|\nabla^α J_{k,i}(x)\|}{h_k(\sqrt{t})} ≤ Ct^{-n} |x|^{2n-α} \frac{h_k(|x|)}{h_k(\sqrt{t})} ≤ Ct^{-\frac{2n-α}{2}} |x|^{2n-α}, \quad x ∈ R^N \setminus \{0\}, \quad t > 0. (7.6)
\]
Assume that $A_0 ∉ \{2, 4, \ldots\}$. Since $|\nabla^α x|^A_0| ≠ 0$ in $R^N \setminus \{0\}$, taking small enough $ε ∈ (0, δ)$, by (7.5) we have
\[
\|\nabla^α e^{-tH}\|_{(L^p,σ ⇒ L^q,θ)} ≥ t^{-\frac{γ}{2}} \frac{Γ_{p',σ'}(t)}{h_0(\sqrt{t})} \left[ C^{-1} \|\nabla^α f_{A_0}(x)\|_{L^q,θ(D_R(t))} - C(εδ)^2 \|f_{A_0}^{(α)}(t)\|_{L^q,θ(D_R(t))} \right] + (7.7)
\]
for small enough $R > 0$, where $D_R(t) := B(0, εδ\sqrt{t}) \cap B(0, R)^c$. Since $R$ is arbitrary, we see that (7.2) holds if $\|\nabla^α e^{-tH}\|_{(L^p,σ ⇒ L^q,θ)} < ∞$ for some $t > 0$. Furthermore, since $A_k + 2n - α ≥ A_0 - α$, if $f_{A_0}^{(α)} ∈ L^q,θ(B(0, 1))$, by (2.1) we see that
\[
\sum_{0≤k+2n≤α} \sum_{i=1}^{d_k} t^{-n} \frac{\|\nabla^α J_{k,i}\|_{L^q,θ(B(0,\sqrt{t}))}}{h_k(\sqrt{t})} ≤ Ct^{-\frac{A_0}{2}} \|f_{A_0}^{(α)}\|_{L^q,θ(B(0,\sqrt{t}))} ≤ Ct^{-2γ}, \quad t > 0.
\]
This together with Theorem 3.1 (b) implies assertion (b). Similarly, we see that assertion (c) holds in the case of $α ≤ 2γ$.

Consider the case when $A_0 = 2γ$ with $γ ∈ \{1, 2, \ldots\}$ and $α > 2γ ≥ 2$. Assume that $f_{A_1}^{(α)} ∈ L^q,θ(B(0, 1))$. It follows from $λ > 0$ that $A_1 < A_0 + 1$ (see e.g. [17] Lemma 4.2). Since $\nabla^α |x|^A_1 ≠ 0$ in $R^N \setminus \{0\}$, similarly to (7.7), taking small enough $ε ∈ (0, δ)$ if necessary, we have
\[
\|\nabla^α e^{-tH}\|_{(L^p,σ ⇒ L^q,θ)} ≥ t^{-\frac{γ}{2}} \frac{Γ_{p',σ'}(t)}{h_1(\sqrt{t})} \left[ C^{-1} \|\nabla^α f_{A_1}(x)\|_{L^q,θ(D_R(t))} - C(εδ)^2 \|f_{A_1}^{(α)}(t)\|_{L^q,θ(D_R(t))} \right] +
\]
for small enough $R > 0$. This implies that $f_{A_1}^{(α)} ∈ L^q,θ(B(0, 1))$ if $\|\nabla^α e^{-tH}\|_{(L^p,σ ⇒ L^q,θ)} < ∞$ for some $t > 0$. Furthermore, since $A_1 < A_0 + 1$, if $f_{A_1}^{(α)} ∈ L^q,θ(B(0, 1))$, then $f_{A_0}^{(α-2)} ∈ L^q,θ(B(0, 1))$. 30
On the other hand, since \( h_{A_0} \) is a homogeneous polynomial of degree \( 2\gamma \), we see that \( |\nabla^\alpha f_{A_0}(x)| = C|\nabla^\alpha f_{A_0}(|x|)| \equiv 0 \) in \( \mathbb{R}^N \) if \( \alpha > 2\gamma \). These together with (7.6) imply that

\[
\sum_{0 \leq k+2n \leq m \leq \alpha} \sum_{i=1}^{d_k} t^{-n} \frac{||\nabla^\alpha J_{n,i}||}{h_k(\sqrt{t})} = \sum_{0 < 2n \leq \alpha} t^{-n} \frac{||\nabla^\alpha J_{0,i}||}{h_0(\sqrt{t})} + \sum_{0 \leq k+2n \leq \alpha} \sum_{i=1}^{d_k} t^{-n} \frac{||\nabla^\alpha J_{k,i}||}{h_k(\sqrt{t})} \leq C t^{-\frac{\alpha_0}{2}} \frac{A_0}{||f_{A_0}(\alpha)||_L^\theta(B(0,\sqrt{t}))} + C t^{-\frac{\alpha_1}{2}} \frac{A_1}{||f_{A_1}(\alpha)||_L^\theta(B(0,\sqrt{t}))} \leq C t^{-\frac{\alpha_0}{2}}, \quad t > 0.
\]

Then, by Theorem 3.1 (b) we have assertion (c). Thus Theorem 7.1 follows. \( \square \)

### 7.2 Bounded potentials

We consider the case when \( V \in C^m([0, \infty)) \). Then \( \lambda_1 = 0 \) and \( h_k \) has no singularity at \( x = 0 \). In Theorem 7.2 we treat the following two cases:

(A) either \( A_{2,0} \notin \{0, 2, 4, \ldots \} \) or \( A_{2,0} \geq \alpha \),

(B) \( A_{2,0} \in \{2, 4, \ldots \} \) and \( A_{2,0} < \alpha \).

The case when \( A_{2,0} = 0 \) and \( \alpha > 1 \) is discussed later.

**Theorem 7.2** Let \( V \in C^m([0, \infty)) \) for some \( m \in \{0, 1, 2, \ldots \} \). Assume conditions \( (V_m) \) and \( (N') \). Let \( \alpha \in \{0, 1, \ldots, m+1\} \), \( (p, q, \sigma, \theta) \in \Lambda \), and \( T > 0 \). Then

\[
|\nabla^\alpha e^{-tH}||_{L^p(H \rightarrow L^q)} \leq t^{-\frac{N}{2}} \frac{1}{p - \frac{\alpha}{2}}, \quad 0 < t \leq T. \tag{7.8}
\]

Furthermore,

\[
|\nabla^\alpha e^{-tH}||_{L^p(H \rightarrow L^q)} \times \begin{cases} t^{-\frac{N}{2}} \frac{\|h_0^{(\alpha)}\|}{h_0(\sqrt{t})}, & \text{in case (A)}, \\ t^{-\frac{N}{2}} \frac{\|h_0^{(\alpha)}\|}{h_0(\sqrt{t})} + t^{-\frac{N}{2}} \frac{\|h_1^{(\alpha)}\|}{h_1(\sqrt{t})}, & \text{in case (B)} \end{cases} \tag{7.9}
\]

for \( t > T \). Here \( h_k^{(\alpha)}(x) = (1 + |x|)^{-\alpha} h_k(|x|) \) for \( k = 0, 1 \).

**Proof.** It follows from \( V \in C^m([0, \infty)) \) that, for any \( R > 0 \) and \( k \in \{0, 1, 2, \ldots \} \), \( |\nabla^\alpha h_k| \) is bounded in \( B(0, R) \) for \( \alpha \in \{0, 1, \ldots, m+1\} \). Then, by Theorems 3.1 and 4.2 (b) we easily obtain relation (7.8). In case (A) we find \( R_1 > 0 \) such that

\[
|\nabla^\alpha h_0(|x|)| \leq h_0^{(\alpha)}(|x|), \quad x \in B(0, R_1).
\]

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Furthermore, Proposition 2.2 (a) implies that
\[
\frac{\|\tilde{h}_0^{(2)}\|_{L^{p,\theta}(B(0,\sqrt{T}))}}{h_0(\sqrt{T})} \geq (1 + \sqrt{T})^{-\alpha} \frac{\|h_0\|_{L^{p,\theta}(B(0,\sqrt{T}))}}{h_0(\sqrt{T})} \geq C t^{\frac{N}{2\gamma} - \frac{\alpha}{2}}, \quad t \geq T.
\]

Since \(\nabla^\alpha h_k\) is bounded in \(B(0, R_1)\), by Lemma 2.1 (b) we apply Theorem 3.1 Theorem 4.2 and Proposition 2.2 (c) to obtain
\[
\|\nabla^\alpha e^{-tH}\|_{(L^{p,\theta} \rightarrow L^{q,\theta})} \approx t^{-\frac{N}{2p}} \Gamma^{p',\sigma'}(t) \left[\frac{\|\tilde{h}_0^{(2)}\|_{L^{p,\theta}(B(0,\sqrt{T}))}}{h_0(\sqrt{T})} + t^{\frac{N}{2p} - \frac{\alpha}{2}}\right]
\]
\[
\approx t^{-\frac{N}{2p}} \Gamma^{p',\sigma'}(t) \frac{\|\tilde{h}_0^{(2)}\|_{L^{p,\theta}(B(0,\sqrt{T}))}}{h_0(\sqrt{T})}, \quad t \geq T.
\]

(7.10)

On the other hand, in case (B) we see that \(A_{2,0} \in \{2, 4, \ldots\}\) and \(0 < A_{2,1} - A_{2,0} < 1\). Then we find \(R_2 > 0\) such that
\[
|\nabla^\alpha h_1(x)| \asymp h_1^{(\alpha)}(|x|), \quad x \in B(0, R_2)^c.
\]

Then, by Lemma 2.1 and Proposition 2.2 (c) we have
\[
t^{-n} \frac{\|\nabla^\alpha J_{k,1}(x)\|}{h_k(\sqrt{T})} \leq C \frac{\|h_1^{(\alpha)}(|x|)\|}{h_1(\sqrt{T})} \quad \text{if} \quad k \geq 1,
\]
\[
t^{-n} \frac{\|\nabla^\alpha J_{0,1}(x)\|}{h_0(\sqrt{T})} \leq C \frac{|x|^2 h_0^{(\alpha)}(|x|)}{t h_0(\sqrt{T})} \leq C \frac{h_1^{(\alpha)}(|x|)}{h_1(\sqrt{T})} \quad \text{if} \quad n \geq 1,
\]

for \(x \in B(0, \sqrt{T}) \cap B(0, R_2)^c\) and \(0 \leq k + 2n \leq \alpha\). Since
\[
\frac{\|\tilde{h}_1^{(\alpha)}\|_{L^{p,\theta}(B(0,\sqrt{T}))}}{h_1(\sqrt{T})} \geq C t^{\frac{N}{2p} - \frac{\alpha}{2}}, \quad t \geq T,
\]
similarly to (7.10), we apply Theorem 3.1 Theorem 4.2 and Proposition 2.2 (c) to obtain inequality (7.9) in case (B). Thus Theorem 7.2 follows. \(\square\)

Under the assumptions of Theorem 7.2, the exact large time decay rate of \(\|\nabla^\alpha e^{-tH}\|_{(L^{p,\theta} \rightarrow L^{q,\theta})}\) for bounded potentials is obtained by the combination of (7.8), (7.9), and
\[
\Gamma^{p',\sigma'}(t) \asymp \begin{cases} 
-\frac{A_{2,0}}{2} (\log t)^{-B_0} & \text{if} \quad p < p_s, \\
-\frac{A_{2,0}}{2} (\log t)^{\frac{1}{\sigma'}} & \text{if} \quad p = p_s, \\
n^{\frac{N}{2p}(1 - \frac{1}{\sigma'})} & \text{if} \quad p > p_s,
\end{cases}
\]

for large enough \(t > 0\). Here \(p_s := N/(N + A_{2,0}), q_\alpha := N/(\alpha - A_{2,0})\) and \(B_0\) is as in (1.3).

Finally, we discuss the large time decay of \(\|\nabla^\alpha e^{-tH}\|_{(L^{p,\theta} \rightarrow L^{q,\theta})}\) in the case when \(\lambda_2 = 0\) and \(\alpha \in \{1, 2, \ldots, m + 1\}\) under some additional conditions.
Theorem 7.3 Let \( m \in \{0, 1, 2, \ldots \} \) and \( V \in C^m([0, \infty)) \) satisfy
\[
\frac{d^\ell}{dr^\ell} V(r) = a(1 + o(1)) \frac{d^\ell}{dr^\ell} r^{-\kappa} \quad \text{as} \quad r \to \infty
\]
for \( \ell \in \{0, 1, \ldots, m \} \), where \( a \neq 0 \) and \( \kappa > 2 \). Assume conditions \((V_m)\) and \((N)\). Let \((p, q, \sigma, \theta) \in \Lambda \) and \( \alpha \in \{1, \ldots, m + 1\} \). Set
\[
\eta_\alpha(|x|) := \begin{cases}
(1 + |x|)^{-\kappa + 2 - \alpha} & \text{if } 2 < \kappa < N, \\
(1 + |x|)^{-N + 2 - \alpha} \log(|x| + 2) & \text{if } \kappa = N, \\
(1 + |x|)^{-N + 2 - \alpha} & \text{if } \kappa > N.
\end{cases}
\]
Then
\[
\|\nabla^\alpha e^{-tH}\|_{(L^{p,\sigma} \to L^{q,\theta})} \asymp t^{-\frac{N}{2q}} \left[ \|\eta_\alpha\|_{L^{q,\theta}(B(0,\sqrt{t}))} + t^{\frac{N}{2q} - \frac{N - 2}{2}} \right]
\]
for large enough \( t > 0 \).

**Proof.** For any \( k \in \{0, 1, 2, \ldots \} \), it follows from \( A_{2,k} = k \) that \( h_k(r) = F_k(r) \) for \( r > 0 \), where
\[
F_k(r) := r^k \left[ 1 + \int_0^r s^{-2k - N + 1} \left( \int_0^s t^{k + N - 1} V(t) h_k(t) \, dt \right) ds \right].
\]
Indeed, \( F_k \) satisfies (1.5). Since \( h_k \in C^\alpha(\mathbb{R}^N) \) and \( |x|^k Q_{k,i}(x/|x|) \) is a homogeneous polynomial of degree \( k \), by (7.11) we see that
\[
\left| \|\nabla^\alpha h_0\|_{L^{q,\theta}(B(0,\sqrt{t}))} \right| \asymp \|\eta_\alpha\|_{L^{q,\theta}(B(0,\sqrt{t}))},
\]
\[
t^{-n} \|\nabla^\alpha J_{k,i}^\alpha\|_{L^{q,\theta}(B(0,\sqrt{t}))} \leq C \|\eta_\alpha\|_{L^{q,\theta}(B(0,\sqrt{t}))} + Ct^{\frac{N}{2q}},
\]
for large enough \( t > 0 \), where \((k,i) \in \mathcal{K}\) and \( n \in \{0, 1, \ldots \} \) with \( 0 \leq k + 2n \leq \alpha \). These together with Theorems 3.1 and 4.2 imply the desired inequality. Thus Theorem 7.3 follows. \( \square \)

Similarly we have:

**Theorem 7.4** Let \( V \in C^m([0, \infty)) \) for some \( m \in \{0, 1, 2, \ldots \} \). Assume conditions \((V_m)\) and \( A_{2,0} = 0 \). Furthermore, assume that
\[
\tau^{-N} V(\tau) \in L^1((0, \infty)), \quad \int_0^\infty \tau^{-N} V(\tau) h_0(\tau) \, d\tau \neq 0.
\]
Let \((p, q, \sigma, \theta) \in \Lambda \) and \( \alpha \in \{1, 2, \ldots \} \). Then
\[
\|\nabla^\alpha e^{-tH}\|_{(L^{p,\sigma} \to L^{q,\theta})} \asymp t^{-\frac{N}{2q}} \left[ \|\eta_\alpha\|_{L^{q,\theta}(B(0,\sqrt{t}))} + t^{\frac{N}{2q} - \frac{N - 2}{2}} \right]
\]
for large enough \( t > 0 \), where \( \bar{\eta}_\alpha(|x|) := (1 + |x|)^{-N + 2 - \alpha} \).

By Theorems 7.3 and 7.4 we see that the large time decay rate of \( \|\nabla^\alpha e^{t\Delta}\|_{(L^{p,\sigma} \to L^{q,\theta})} \) varies discontinuously with respect to perturbations of the potential \( V \) if \( \alpha \neq 0 \). See also Theorem 1.3

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