DISCRETE FRACTIONAL INTEGRATION OPERATORS ALONG THE PRIMES

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Abstract. We prove that the discrete fractional integration operators along the primes
\[ T^\lambda_p f(x) := \sum_p f(x - p) \cdot \frac{p^\lambda \cdot \log p}{p^\lambda} \]
are bounded \( \ell^p \to \ell^{p'} \) whenever \( \frac{1}{p'} < \frac{1}{p} - (1 - \lambda), \ p > 1 \). Here, the sum runs only over prime \( p \).

1. Introduction

The topic of this paper is discrete harmonic analysis, with a focus on discrete analogues of fractional integral operators. While elementary arguments link the discrete operators,
\[ J_{d,\lambda} f(x) := \sum_{0 \neq m \in \mathbb{Z}^d} \frac{f(x - m)}{|m|^{d\lambda}} \]
to their continuous analogues – which leads to the expected range of norm estimates: \( J_{d,\lambda} \) maps \( \ell^p(\mathbb{Z}^d) \to \ell^{p'}(\mathbb{Z}^d) \) when \( 1/p' \leq 1/p - (1 - \lambda) \) – the problem becomes much more subtle upon the introduction of radon behavior, where the primary objects of consideration (in the one-dimensional setting) are of the form
\[ I^s_{\lambda} f(x) := \sum_{m \geq 1} \frac{f(x - m^s)}{m^\lambda}. \]

While it is expected that \( I^s_{\lambda} \) maps \( \ell^p \to \ell^q \) when
\begin{itemize}
  \item \( 1/p > 1 - \lambda \) and
  \item \( \frac{1}{p'} \leq \frac{1}{p} - \frac{1 - \lambda}{s}, \)
\end{itemize}
significant number-theoretic complications aside from the \( s = 2 \) case, treated in [2, 6, 9, 10], have made it difficult to obtain these full range of exponents; see [7] for a discussion of these number-theoretic complications. Indeed, Pierce’s work on

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fractional radon transforms [8] along curves of the form (say)
\[ \sum_{0 \neq m \in \mathbb{Z}^d} \frac{f(x - m, y - Q(m))}{|m|^{d + \lambda}}, \]
Q a quadratic form, further established the link between the quadratic nature of the the curves in question and the ability to obtain wide range of $\ell^p$ estimates for fractional radon transforms.

In this short note, we explore the case of fractional radon transforms along the primes,
\[ (1.1) \quad T_\lambda^P f(x) := \sum_p \frac{f(x - p)}{p^\lambda} \cdot \log p; \]
these fractional radon transforms do not have a quadratic nature to the operator (the sum runs over prime $p$, and the presence of the logarithm is a normalizing factor, appearing from density considerations).

Nevertheless, drawing upon the techniques of [3], we prove the following theorem.

**Theorem 1.2.** Suppose that $p > 1$ and $1/p' < 1/p - (1 - \lambda)$. Then (1.1) maps $\ell^p \to \ell^{p'}$.

**Remark 1.3.** The hard restriction $\frac{1}{p'} < 1/p - (1 - \lambda)$ is an artifact of the real interpolation method we use; a soft inequality is expected, and would follow from complex methods. We do not pursue this issue here.

1.1. **Notation.** Here and throughout, $e(t) := e^{2\pi it}$. We let $\mu$ and $\phi$ denote the Möbius and totient functions, respectively. A key estimate is the lower bound
\[ (1.4) \quad \phi(q) \gtrsim q^{1-\epsilon} \]
valid for any $\epsilon > 0$.

For $k \geq 1$ we let
\[ K_k(x) := \frac{1}{2^k} \sum_{p \leq 2^k} \delta_p(x) \cdot \log p, \]
where the sum runs over only prime $p$.

We will make use of the modified Vinogradov notation. We use $X \lesssim Y$, or $Y \gtrsim X$, to denote the estimate $X \leq CY$ for an absolute constant $C$. We use $X \approx Y$ as shorthand for $Y \lesssim X \lesssim Y$. We also make use of big-O notation: we let $O(Y)$ denote a quantity that is $\lesssim Y$. If we need $C$ to depend on a parameter, we shall indicate this by subscripts, thus for instance $X \lesssim_p Y$ denotes the estimate $X \leq C_p Y$ for some $C_p$ depending on $p$. We analogously define $O_p(Y)$. 
2. The Argument

By an appeal to the triangle inequality, Theorem 1.2 will follow from the following proposition.

Proposition 2.1. For any $p > 1$,

$$\|K_k * f\|_{\ell^p'} \lesssim_\epsilon 2^{-k(1/p - 1/p' - \epsilon)} \|f\|_{\ell^p}.$$  

Since our range of exponents is open, it suffices to prove a restricted weak-type estimate:

$$\langle K_k * 1_F, 1_G \rangle \lesssim_\epsilon |Q| \cdot \left( \frac{|F|}{|Q|} \right)^{1-\epsilon} \cdot \left( \frac{|G|}{|Q|} \right)^{1-\epsilon}$$  

whenever $F, G \subset Q := Q_k := \{|x| \lesssim 2^k\}$.

It is (2.2) to which we turn.

We next recall the following multi-frequency multiplier theorems, which in turn grew out of [2].

2.1. A Multi-Frequency Multiplier Theorem for Ionescu-Wainger Type Multipliers. The results of this section appear as the special one-dimensional case of [4, Theorem 5.1], a refinement of [2, Theorem 1.5] (which would also be adequate for our purposes).

Theorem 2.3 (Special Case). Suppose that $m(\xi)$ is an $L^p(\mathbb{R})$ multiplier with norm $A$:

$$\| (m(\xi)f(\xi))^\vee \|_{L^p(\mathbb{R})} \leq A \| f \|_{L^p(\mathbb{R})}.$$  

Let $\rho > 0$ be arbitrary (for applications, we will take $0 < \rho \ll_p 1$). Then, for every $N$, there exists an absolute constant $C_\rho > 0$ so that one may find a set of rational frequencies

$$\left\{ \frac{a}{q} \text{ reduced} : q \leq N \right\} \subset \mathcal{U}_N \subset \left\{ \frac{a}{q} \text{ reduced} : q \leq C_\rho e^{N_\rho} \right\},$$

so that

$$\left( \sum_{\theta \in \mathcal{U}_N} m(\alpha - \theta)\eta_N(\alpha - \theta)\hat{f}(\beta) \right)^\vee$$

has $\ell^p$ norm $\lesssim_{\rho, p} A \cdot \log N$. Here, $\eta_N$ is a smooth bump function supported in a ball centered at the origin of radius $\lesssim e^{-N^{2p}}$.

Remark 2.4. This result should be contrasted with the strongest analogous multiplier theorem for general frequencies, which accrues a norm loss of

$$(\text{Number of Frequencies})^{1/2 - 1/p},$$

even in the special case when $m \in \mathcal{V}^2(\mathbb{R})$ has finite 2-variation, see [1, Lemma 2.1].
2.2. Decompositions. Set \( \mathcal{U} := \mathcal{U}_0 C_0 \) for some \( C_0 \gg_p 1 \), where \( \mathcal{U} \) is as in Theorem 2.3, and choose \( 0 < \rho = \rho_p \ll 1 \). With these choices in mind, decompose \( f = 1_F \) via the Fourier transform as

\[
\hat{f} = \hat{f}_1 + \hat{f}_2,
\]

where

\[
\hat{f}_1(\alpha) := \sum_{\theta \in \mathcal{U}} \eta_k(\alpha - \theta) \cdot \hat{f}(\alpha),
\]

for \( \eta \) a compactly supported bump function that is one in a neighborhood of the origin, and \( \eta_k(t) := \eta(2^{k \rho'} t) \) for some \( 1 \gg \rho' \gg \rho \).

Note that for any \( p > 1 \) so that (2.5)

\[
\|f_1\|_{\ell^p} \lesssim \log k \cdot \|f\|_{\ell^p}.
\]

We will also decompose the Fourier transform of \( K_k \)

\[
\widehat{K}_k(\alpha) := \frac{1}{2^k} \sum_{p \leq 2^k} e(-p\alpha) \cdot \log p.
\]

To do so, we recall the following approximation result of [5].

**Lemma 2.6.** For any \( A \gg 1 \), there exists a \( C = C(A) \) so that one may decompose

\[
\widehat{K}_k(\alpha) = L'_k(\alpha) + E_k(\alpha),
\]

where

\[
L'_k(\alpha) := \sum_t L'_{k,t}(\alpha)
\]

with

\[
L'_{k,t}(\alpha) := \sum_{q \approx 2^t} \frac{\mu(q)}{\phi(q)} \sum_{(a,q) = 1} V_k(\alpha - a/q) \chi_t(\alpha - a/q),
\]

and \( |E_k(\alpha)| \lesssim k^{-A} \) pointwise. Here, \( \chi_t(\alpha) := \chi(2^{Ct} \alpha) \) is a compactly supported bump function, and \( V_k(\alpha) := \int_0^1 e(-2^k t \cdot \alpha) \, dt \).

For our purposes, we will replace \( L'_k \) with \( L_k := \sum_t L_{k,t} \), where

\[
L_{k,t}(\alpha) := \sum_{q \approx 2^t} \frac{\mu(q)}{\phi(q)} \sum_{(a,q) = 1} V_k(\alpha - a/q) \varphi_k(\alpha - a/q),
\]

where \( \varphi \) is a compactly supported bump function, and \( \varphi_k(\alpha) := \varphi(2^k(1-\epsilon) \alpha) \).

Specifically, we have the following lemma.

**Lemma 2.7.** The following estimate holds:

\[
\sup_{\alpha} |L_k(\alpha) - L'_k(\alpha)| \lesssim 2^{-ek}.
\]
Proof. It suffices to show that
\[ |L_{k,t}(\alpha) - L'_{k,t}(\alpha)| \lesssim 2^{-(\epsilon-1)t} \cdot 2^{-ck}. \]
The key point is that if \( \varphi_k(\alpha) - \chi_t(\alpha) \) does not vanish, then \( 2^{(\epsilon-1)k} \lesssim |\alpha| \), so that
\[ |V_k(\alpha)| \lesssim (2^k|\alpha|)^{-1} \lesssim 2^{-ck}. \]
The result then follows from the fact that \( \{\chi_t(\cdot - a/q) : (a, q) = 1, q \approx 2^t\} \) are disjointly supported, taking into account the decay of the totient function, (1.4). □

With these decompositions in hand we turn to the proof.

Proof. To prove (2.2) it suffices to bound
\[ |K_k * f| \leq M_1 f + M_2 f, \quad f = 1_F \]
where for any \( p > 1 \)
\[ \|M_1 f\|_{p'} \lesssim \frac{k^2}{2^{k(2/p-1)}} \cdot \|f\|_p \]
and
\[ \|M_2 f\|_2 \lesssim k^{-C} \cdot \|f\|_2, \]
where \( C \) may be adjusted to be as large as we wish.

Our decomposition is as follows:
\[ K_k * f = K_k * f_1 + \left( L_k \hat{f}_2 \right)^\vee + \left( E_k \hat{f}_2 \right)^\vee. \]
We set \( M_1 f := K_k * f_1 \); by interpolating between the \( \ell^1 \to \ell^\infty \) bound of \( \frac{k}{2^k} \), and the trivial \( \ell^2 \to \ell^2 \) bound, we see that
\[ \|K_k * f\|_{\ell^p'} \lesssim \frac{k}{2^{k(2/p-1)}} \cdot \|f\|_{\ell^p}, \quad p > 1 \]
which leads to the estimate (2.8), taking into account (2.5).

The contribution of the term involving \( E_k \) is absorbed into \( M_2 f \), but contributes a negligible bound, as we are free to choose \( A \) in Lemma 2.6 as large as we wish; it suffices to show that \( L_k \hat{f}_2 \) satisfies the \( \ell^2 \) estimate, (2.9).

In particular, we need to estimate
\[ \sum_t \|L_{k,t} \hat{f}_2\|_2 = \sum_{t: 2^t \geq k^{C_0}} \|L_{k,t} \hat{f}_2\|_2; \]
using the decay of the totient function, (1.4), a bound of \( k^{\epsilon-C_0} \) is obtained, which yields the result. □
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