Effects of Nonlinear Dispersion Relations on Non-Gaussianities

Amjad Ashoorioon\textsuperscript{a} Diego Chialva\textsuperscript{b} and Ulf Danielsson\textsuperscript{a}

\textsuperscript{a}Institutionen för fysik och astronomi Uppsala Universitet, Box 803, SE-751 08 Uppsala, Sweden and
\textsuperscript{b}Université de Mons, Service de Mecanique et gravitation, Place du Parc 20, 7000 Mons, Belgium

(Dated: January 19, 2013)

We investigate the effect of non-linear dispersion relations on the bispectrum. In particular, we study the case were the modified relations do not violate the WKB condition at early times, focusing on a particular example which is exactly solvable: the Jacobson-Corley dispersion relation with quartic correction with positive coefficient to the squared linear relation. We find that the corrections to the standard result for the bispectrum are suppressed by a factor \( \frac{H^2}{p_c^2} \) where \( p_c \) is the scale where the modification to the dispersion relation becomes relevant. The modification is mildly configuration-dependent and equilateral configurations are more suppressed with respect to the local ones, by a factor of one percent. There is no configuration leading to enhancements. We then analyze the results in the framework of particle creation using the approximate gluing method of Brandenberger and Martin, which relates more directly to the modeling of the trans-Planckian physics via modifications of the vacuum at a certain cutoff scale. We show that the gluing method overestimates the leading order correction to the spectrum and bispectrum by one and two orders, respectively, in \( \frac{H}{p_c} \). We discuss the various approximation and conclude that for dispersion relations not violating WKB at early times the particle creation is small and does not lead to enhanced contributions to the bispectrum. We also show that in many cases enhancements do not occur when modeling the trans-Planckian physics via modifications of the vacuum at a certain cutoff scale. Most notably they are only of order \( O(1) \) when the Bogolyubov coefficients accounting for particle creation are determined by the Wronskian condition and the minimization of the uncertainty between the field and its conjugate momentum.

PACS numbers: 98.80.Cq
Keywords: 

I. INTRODUCTION

The interest in non-Gaussian features of density perturbations has been mounting in recent years. It is prompted by the upcoming release of the data of unprecedentedly accurate experiments such as Planck and by more advanced research activities. In particular, a notable interest descends from the possible sensitivity of higher-point amplitudes on high-energy physics, namely the existence of new physics at high energy scales. This is due to the fact that higher correlation functions are suggested to be more sensitive to such high energy effects than the power spectrum itself.

Non-Gaussianity is the non-zero value of higher-points (> 2) functions \(^1\) of the comoving curvature perturbation \( \zeta \). In particular, here we will be interested in the bispectrum, which is obtained from the three-point function of the curvature perturbation. The contributions to this correlator arise from different type of diagrams depending on how we define the curvature perturbation (that is, if we use non-linear field redefinitions). In particular we are interested in the connected three-point function and in the effects that modification of the high-energy theory can impart to the standard slow-roll result. The outcomes of this analysis are important in two respects: for the observation of such effects but also for the potential risk for perturbation theory, which might break down (trans-Planckian issue).

The three-point function for the conventional single-field slow-roll models of inflation was originally calculated in \([1]\) and found to be slow-roll suppressed. Any non-Gaussian signature is therefore a smoking-gun for deviation from the orthodox picture. The effect one would generally expect are two-fold:

- a peculiar shape function\(^2\)
- an enhancement factor for some specific configurations.

\(^{1}\) Electronic address: amjad.ashoorioon@fysast.uu.se
\(^{2}\) Electronic address: diego.chialva@umons.ac.be
\(^{3}\) Electronic address: ulf.danielsson@physics.uu.se
\(^{4}\) The two-point one is related to the power spectrum.
\(^{2}\) That is a peculiar dependence of the result on the external momenta, leading to a particular shape for the graph of this function.
Generally these effects are combined, and we have particular enhancement factors when the external momenta of the three point function (in momentum space) have specific configurations.

Various methods have been employed to model high energy modifications to the standard theory. For example, the effects of the trans-Planckian physics have been implemented via a stringy-modified space-time uncertainty relation \[2\], or utilizing the concept of a boundary action, which provides the boundary conditions for the perturbations fields, subject to the renormalization from higher energy scales \[3, 4\]. A particularly interesting method is related to the choice of boundary conditions (the so-called “choice of vacuum”) for the solution of the equation of motion of the Mukhanov variable at the new physics hyper-surface (NPHS), which corresponds to the time the physical momentum of perturbations reaches the scale \(\Lambda\) \[5, 6\]. In this case, the power spectrum exhibits superimposed oscillations whose amplitude is given by \(H/\Lambda\), where \(H\) is the Hubble parameter during the inflation. The bispectrum is instead found to be modified by enhancement factors for the enfolded/flattened configuration of momenta, whose real magnitude depends however on the value of the Bogolyubov parameters calculated within this approach \[6–10\]. In such a configuration, two of the momenta are collinear with the third one in the momenta triangle.

In an alternative approach to model the effects of trans-Planckian physics, we can mainly focus on here, the equation of motion for the perturbations are modified and new dispersion relations, which differ from the linear one for physical momenta larger than a fixed scale of new physics \(p_c\), are considered \[12\]. Such modified dispersion relations represent the violation of Lorentz invariance and were also employed to analyze the possible effect of trans-Planckian physics on black hole radiation \[13\]. Modified dispersion relations of the kind we will investigate here, can also be derived naturally from the recently proposed Horava-Lifshitz gravity \[14, 16\] which is a non-relativistic renormalizable modification of the gravity in the UV region. They also arise in effective theory of single field inflation when the scalar perturbations propagate with a small sound speed \[17\].

In this paper we focus on examples where one can obtain the exact solutions to the field equations and therefore has control over subtle effects. As the evolution of the mode function after the modified dispersion relation has entered the linear regime can be mapped to an excited state in the NPHS approach, using the exact solutions we have a precise understanding and quantification of particle creation and can understand the reason that lies behind the presence or absence of the enhancement factors in the bispectrum. The general result that we find is that, in absence of violation of the WKB condition at early times, the enhancement factors are not present in the bispectrum. On the other hand one would expect a more favorable situation for dispersion relations with early time WKB violation. However, as we have found no example of such modified dispersion relations where the field equation was exactly solvable, we discuss this case only briefly and leave the analysis of this point to future research.

We also show how the enhancement factors are at maximum of order \(O(1)\) in some of the emergent approaches to trans-Planckian physics in the NPHS framework. Notably, this occurs in the case where the Bogolyubov coefficients accounting for particle creation are determined by the minimization of the uncertainty relation between the field and its conjugate momentum and by the Wronskian condition \[5\], which yields the largest correction to the spectrum.

We will first study a minimal modification to the linear dispersion relation, known as Jacobson-Corley (JC) \[13\], both solving the field equations exactly, and via approximation methods. The JC dispersion relation has a positive quartic correction to the linear term and there is no WKB violation at early times\(^3\).

We find no modulation in the modified spectrum of perturbations, and the amplitude of perturbations is damped quadratically with the increase of the ratio \(\frac{H}{p_c}\). This is explained in terms of the very small particle creation, due to the absence of WKB violation at early times.

We then turn to the analysis of the modification to the bispectrum: using the exact solution to compute the three-point function, we find no large enhancement factors, and no interference terms as the one studied in \[8–10\]. We then try to understand and interpret these results in the language of particle creation using the gluing approximation of Brandenberger and Martin \[12\] for the solutions. We show that the gluing method overpredicts the magnitude of modification.

The outline of the paper is as follows: we review the formalism, the notation and the techniques to solve the field equations in section \([1]\). We then study the JC modified dispersion relation in section \([III]\) first we solve exactly the field equations in section \([III A]\) then we study the two- and three-point functions in sections \([III B]\) and \([III C]\). We then try to explain and interpret the results in the framework of particle creation, and make contact with the NPHS framework in section \([IV]\). We finally discuss the general results, present the outlooks regarding modified dispersion relations with early time violation of WKB condition, and conclude in section \([V]\).

---

\(^3\) The only WKB violation occurs when the modes exit the horizon at late times, when the dispersion relation has become approximately the standard linear one.
II. FORMALISM AND NOTATION.

The starting point for our analysis is the definition of the curvature perturbation of the comoving hypersurface

\[ \zeta(\eta, x) = \int \frac{d^3k}{(2\pi)^3} \zeta_\mathbf{k}(\eta) e^{i\mathbf{k} \cdot \mathbf{x}}, \]

where \( \zeta_\mathbf{k}(\eta) \) is defined in terms of the mode function \( u_\mathbf{k}(\eta) \) as

\[ \zeta_\mathbf{k}(\eta) = \frac{u_\mathbf{k}(\eta)}{z}, \]

with equation of motion

\[ u_\mathbf{k}'' + \left( \omega(\eta, \mathbf{k})^2 - \frac{z''}{z} \right) u_\mathbf{k} = 0 \]

where \( z = \frac{a}{a} \) and \( \omega(\eta, \mathbf{k}) \) is the comoving frequency as read from the effective action. In the standard formalism \( \omega(\eta, \mathbf{k}) \) is taken to be equal to \( k \equiv |\mathbf{k}| \). Instead, a modified dispersion relation would correspond to considering different dependence of \( \omega \) on \( \mathbf{k}, \eta \). For isotropic backgrounds, which we will confine ourselves to, one expects \( u_\mathbf{k}(\eta) \) and \( \omega(\eta, \mathbf{k}) \) to depend only on \( k \). Hence, we may drop the arrow symbol at some points.

The three-point function is given by the formula

\[ \langle \zeta(x_1) \zeta(x_2) \zeta(x_3) \rangle = -2\text{Re} \left( \int_{\eta_{\text{in}}}^{\eta} d\eta' i\langle \psi_{\text{in}} | \zeta(x_1) \zeta(x_2) \zeta(x_3) H_I(\eta') | \psi_{\text{in}} \rangle \right) \]

where \( H_I = -\int d^3x a^3(\frac{\dot{\phi}}{M})^4 H k^2 \partial^2 \zeta' \) is the interaction Hamiltonian as defined in [1]. \( \eta_{\text{in}} \) and \( |\psi_{\text{in}}\rangle \) are the initial time and state (vacuum). The standard result in slow-roll inflation is obtained with \( \eta_{\text{in}} = -\infty \), and \( |\psi_{\text{in}}\rangle \) is taken to be the Bunch-Davies vacuum.

In momentum phase space the three-point function becomes [1]

\[ \langle \zeta_{\mathbf{k}_1}(\eta) \zeta_{\mathbf{k}_2}(\eta) \zeta_{\mathbf{k}_3}(\eta) \rangle = 2\text{Re} \left( -i(2\pi)^3 \delta(\sum_i \mathbf{k}_i)(\frac{\dot{\phi}}{H})^4 \frac{H}{M^2} \int_{\eta_{\text{in}}}^{\eta} d\eta' \frac{a(\eta')^3}{k_3^3} \prod_{i=1}^{3} \partial_{\eta'} G_{\mathbf{k}_i}(\eta, \eta') + \text{permutations} \right) \]

where \( \eta \) is taken to be a late time when all three functions \( \zeta_\mathbf{k} \) are outside the horizon [5]. \( G_{\mathbf{k}}(\eta, \eta') \) is the Whightman function defined as:

\[ G_{\mathbf{k}}(\eta, \eta') \equiv \frac{H^2}{\dot{a}^2} \frac{u_k(\eta) u_k^*(\eta')}{a(\eta) a(\eta')} \]

We define the bispectrum following [18]

\[ \langle \zeta_{\mathbf{k}_1}(\eta) \zeta_{\mathbf{k}_2}(\eta) \zeta_{\mathbf{k}_3}(\eta) \rangle = \delta(\sum_i \mathbf{k}_i)(2\pi)^3 F(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \eta), \]

where the translational invariance has imposed the conservation of momentum. Scale-invariance requires that the function, \( F \), to be a homogeneous function of degree \( -6 \) and the rotational invariance imposes it to be only a function of two variables, say \( x_2 \equiv k_2/k_1 \) and \( x_3 \equiv k_3/k_1 \). To avoid counting the same configuration twice, it is further assumed that \( x_3 \leq x_2 \). The inequality \( 1 - x_2 \leq x_3 \) also comes from the triangle inequality.

The function \( F \) contains a lot of information about the source of non-Gaussianity and could be used to distinguish among different models. The limit in which the configuration is such that \( x_3 \approx 0 \) and \( x_2 \approx 1 \) is recognized in the literature as the local configuration. The one in which \( x_2 \approx x_3 \approx 1 \) is known as equilateral one.

Inflationary models in which non-linearity is developed beyond the horizon, tend to produce a more local type of non-Gaussianity. On the other hand, for the ones that the correlation is among the modes with comparable wavelength, equilateral type of non-Gaussianities tend to be produced [18]. These modes will exit the horizon around the same time.

---

1. \( \phi \) being the inflaton field.
2. In the case of modified dispersion relations, at this time the relation has become effectively the standard linear one.
A. Solving the field equation.

In the case of a modified dispersion relation, equation (3) is often solved by approximation [12]. Two methods have been used: one is the WKB approximation. This approach is convenient, as it is independent of small parameters in the dispersion relation, because it only requires a slow variation of $U = \omega - \frac{a'}{a}$ in time, i.e. [20].

\[
\left| \frac{Q}{U^2} \right| = \left| -\frac{U''}{2U^3} + 3 \frac{U'^2}{4U^4} \right| \ll 1 \quad (8)
\]

The other is the gluing procedure of Brandenberger and Martin, where approximated solutions valid in different regions of behavior of $\omega$ are glued together asking for the continuity of the functions and their first derivatives [12].

In the NPHS approach, enhancement terms appear in the bispectrum due to the interference terms between negative and positive frequency parts of the Wightman function as one starts from a non-Bunch-Davies vacuum. In the case of modified dispersion relations, similar interference term will generally arise solving (3) using the gluing procedure. However, in all cases the actual presence/absence of enhancement depends on the magnitude of the second Bogolyubov parameter $\beta_{\vec{k}}$ [8–10].

With the knowledge of the exact solution in some specific examples, one can gain more precise information about the shape of the bispectrum. We want to explore these features in the case of modified dispersion relations and in particular verify the presence/absence of enhancement factors and their difference with the NPHS case where one employs the cut-off. To do this, in the following we will consider a particular (but very illustrative) example where the exact solution can be found. We then analyze and interpret the results in the framework of particle creation with the approximation methods described above, and discuss the differences.

III. CORLEY-JACOBSON DISPERSION RELATION WITH $b_1 > 0$. NO VIOLATION OF WKB APPROXIMATION.

In this section, we will focus on Jacobson-Corley dispersion relation with positive quartic correction. We assume that at very high physical momenta, $p \gg p_c$, the linear dispersion relation $\omega = p$ gets modified as follows:

\[
\omega_{\text{phys}}^2 = F^2(p) = p^2 + \frac{b_1 p^4}{p_c^4} \quad (9)
\]

The dispersion relation is motivated by studies in black-hole physics [13], and was later used to study the effect of trans-Planckian physics on cosmological perturbations [12, 19]. Here, we will find an exact analytic form of the solution and will derive the effect of this modified dispersion relation on the two and three point functions analytically. Then, we study the same problem making use of approximation techniques and discuss the results.

A. Exact solution of the field equations

In usual quantum field theory, the cosmological perturbations satisfy eq.(3) as the equation of motion. A modified dispersion relation of the kind of Corley-Jacobson entails the replacement

\[
k^2 \Rightarrow a^2(\eta)F^2 \left( \frac{k}{a(\eta)} \right) = k^2 + b_1 \frac{k^4}{p_c^4 a^2(\eta)} \quad (10)
\]

Then in de-Sitter space-time, where $a = -\frac{1}{\eta}$, equation (3) reads:

\[
u''_k(\eta) + (k^2 + \epsilon k^4 \eta^2 - \frac{2}{\eta^2})u_k = 0 \quad (11)
\]

where,

\[
\epsilon \equiv \frac{\sqrt{b_1} H}{p_c} \quad (12)
\]
The above differential equation has two independent closed-form exact solutions that can be given in terms of WhittakerW(a, b, z) and its conjugate WhittakerW*(a, b, z) functions. For brevity, we will show these two functions as WW(a, b, z) and WW*(a, b, z) from now on, respectively. The exact solution is then:

\[
    u_k = \frac{C_1}{\sqrt{-\eta}} WW\left(\frac{i}{4e} \frac{3}{4}, -i\epsilon k^2 \eta^2\right) + \frac{C_2}{\sqrt{-\eta}} WW^*\left(\frac{i}{4e} \frac{3}{4}, i\epsilon k^2 \eta^2\right)
    = \frac{C_1}{\sqrt{-\eta}} WW\left(\frac{i}{4e} \frac{3}{4}, -i\epsilon k^2 \eta^2\right) + \frac{C_2}{\sqrt{-\eta}} WW\left(-\frac{i}{4e} \frac{3}{4}, i\epsilon k^2 \eta^2\right).
\]  

(13)

The above solution is subject to the Wronskian condition

\[
    u(\eta)u'^*(\eta) - u^*(\eta)u'(\eta) = i,
\]

which impose the following constraint on \(C_1\) and \(C_2\):

\[
    2i|C_1|^2 \exp\left(\frac{\pi i}{4e}\right)\epsilon k^2 - 2i|C_2|^2 \exp\left(\frac{\pi i}{4e}\right)\epsilon k^2 = i,
\]

(15)

This condition would not uniquely determine \(C_1\) and \(C_2\). One has to make extra assumptions to determine \(C_1\) and \(C_2\). We will assume that the mode function approaches the positive frequency WKB at early times. As it was shown by \[12\], this choice will minimize the energy density too.

In the limit of \(\eta \to -\infty\), the exact equation of motion is reduced to the following form

\[
    u''_k(\eta) + \epsilon^2 k^4 \eta^2 u(\eta) = 0,
\]

(16)

and its approximate positive frequency WKB solution is:

\[
    u_k(\eta) \approx \frac{1}{\sqrt{2}\omega(\eta)} \exp\left(-i \int \omega(\eta) d\eta\right) = \frac{1}{k\sqrt{-2\epsilon\eta}} \exp\left(\frac{i\epsilon k^2 \eta^2}{2}\right).
\]

(17)

The limit of the exact solution (13) for large time can be found writing the Whittaker functions in terms of KummerU functions and using the asymptotic form of the latter for large argument \[21\]. We obtain

\[
    u_k(\eta) = \frac{C_1}{\sqrt{-\eta}} \exp\left(\frac{i\epsilon k^2 \eta^2}{2}\right)(-i\epsilon k^2 \eta^2)^{5/4} U\left(\frac{i}{4e} \frac{3}{4}, -i\epsilon k^2 \eta^2\right)
    + \frac{C_2}{\sqrt{-\eta}} \exp\left(-\frac{i\epsilon k^2 \eta^2}{2}\right)(i\epsilon k^2 \eta^2)^{5/4} U\left(-\frac{i}{4e} \frac{3}{4}, i\epsilon k^2 \eta^2\right)
    \approx \frac{C_1}{\sqrt{-\eta}} (-i\epsilon k^2 \eta^2)^{\frac{5}{4}} \exp\left(\frac{i\epsilon k^2 \eta^2}{2}\right) + \frac{C_2}{\sqrt{-\eta}} (i\epsilon k^2 \eta^2)^{\frac{5}{4}} \exp\left(-\frac{i\epsilon k^2 \eta^2}{2}\right),
\]

(18)

where we see that the positive frequency WKB solution corresponds to the choice \(C_2 = 0\). Using the Wronskian condition, (15), \(C_1\) is fixed up to a phase:

\[
    C_1 = \frac{\exp\left(\frac{\pi i}{8e}\right)}{\sqrt{2\epsilon k}},
\]

(19)

so that our exact complete solution to the mode equation is, finally,

\[
    u_k(\eta) = \frac{\exp\left(\frac{\pi i}{8e}\right)}{\sqrt{-2\epsilon\eta k}} WW\left(\frac{i}{4e} \frac{3}{4}, -i\epsilon k^2 \eta^2\right).
\]

(20)

---

\[6\] The extra factor of \((-i\epsilon k^2 \eta^2)^{\frac{5}{4}}\) is the subleading correction to the WKB approximation in the limit of \(\eta \to -\infty\).
FIG. 1: The figure shows the dependence of the power spectrum in de-Sitter space to the parameter \( \epsilon \), which quantifies the ratio of Hubble parameter over the momentum scale \( p_c \) around which the behavior of the dispersion relation changes.

### B. Two-point function and power spectrum

With the exact mode function in equation (18) one calculates the power-spectrum

\[
P_R(k, \epsilon) = \frac{k^3 H^2}{2\pi^2 \dot{\phi}^2} \frac{\left| u_k(\eta) \right|^2}{a(\eta)} \bigg|_{k \to 0}.
\]

(21)

The spectrum is found to be still scale-invariant (no dependence on \( k \)), but the standard result now shows corrections depending on \( \epsilon \):

\[
P_R(\epsilon) = \frac{H^4}{\dot{\phi}^2} \frac{\exp\left(- \frac{\pi}{4} \epsilon \right)}{16 \pi \epsilon^{3/2} \Gamma\left(\frac{5}{4} - \frac{i}{4} \epsilon\right) \Gamma\left(\frac{5}{4} + \frac{i}{4} \epsilon\right)}.
\]

(22)

The plot of the power spectrum in terms of \( \epsilon \) is shown in figure (1). The scale-invariance of the power spectrum is again due to the scale-invariance of the de-Sitter background. The corrections to the standard slow-roll are

\[
P_R(\epsilon) \approx \left( \frac{H^2}{2\pi \dot{\phi}} \right)^2 \left( 1 - \frac{5}{4} \epsilon^2 \right).
\]

(23)

These corrections are appearing at second order in \( \epsilon \).

### C. Three-point Function

Given the exact mode-function (20), the Wightman function for the positive frequency WKB vacuum is

\[
G_k(\eta, \eta') = \frac{\exp\left(- \frac{\pi}{4} \epsilon \right) \sqrt{\eta \eta'} H^4}{2\epsilon k^2 \dot{\phi}^2} \text{WW} \left( \frac{i}{4\epsilon}, \frac{3}{4} - i \epsilon k^2 \eta^2 \right) \text{WW} \left( -\frac{i}{4\epsilon}, \frac{3}{4}, i \epsilon k^2 \eta'^2 \right).
\]

(24)

To compute the three-point function from equation (19), the first argument of the Wightman function has to be set at a time when the mode \( k \) is outside the horizon, generically taken to be time zero. Then, the Whightman functions must be differentiated with respect to the second argument. We obtain

\[
a(\eta) \partial_\eta G_k^\omega(0, \eta) = \frac{(-1)^{\frac{3}{2}} H^3}{8\epsilon^2 k^2 \dot{\phi}^2 (\eta - \eta') \dot{\phi}} \text{WW} \left( \frac{-i}{4\epsilon}, \frac{3}{4} - i \epsilon k^2 \eta^2 \right) \text{WW} \left( \frac{-i}{4\epsilon}, \frac{3}{4}, i \epsilon k^2 \eta'^2 \right).
\]

(25)
For computing the three-point function, it is convenient to write the WhittakerW functions in terms of KummerM’s based on the following relation which is valid in the principal branch, $-\pi < \arg(z) \leq \pi$.

$$\text{WW}(a, b, z) = \exp\left(-\frac{z}{2}\right) \left(\frac{\Gamma(2b)M\left(\frac{1}{2} - a - b, 1 - 2b, z\right)z^{\frac{1}{2}-b}}{\Gamma\left(\frac{1}{2} - a + b\right)} + \frac{\Gamma(-2b)M\left(\frac{1}{2} - a + b, 1 + 2b, z\right)z^{\frac{1}{2}+b}}{\Gamma\left(\frac{1}{2} - a - b\right)}\right)$$  \hspace{1cm} (26)

We can then exploit the following expansion of the KummerM functions in terms of Bessel-J functions [21]:

$$M(a, b, z) = \Gamma(b)\exp(hz)\sum_{n=0}^{\infty} C_n z^n (-az)^{\frac{1}{2}(1-b-n)}J_{b-1+n}(2\sqrt{-az})$$  \hspace{1cm} (27)

$C_{n+1}$ is given in terms $C_n$, $C_{n-1}$ and $C_{n-2}$ from the following recursive relation

$$C_{n+1} = \frac{1}{n+1} [((1-2h)z - bh)C_n + ((1-2h)z - a h(h+1)(b+n) - 1)C_{n-1} - h(h+1)aC_{n-2}]$$  \hspace{1cm} (28)

$$C_0 = 1, \quad C_1 = -bh, \quad C_2 = -\frac{1}{2}(2h-1)a + \frac{b}{2}(b+1)h^2$$  \hspace{1cm} (29)

Although strictly valid for small $z$, the above expansion becomes the correct expansion of the bispectrum for small $\epsilon$ when the integration over time in the formula (5) for the three-point function is performed. We will also prove this point in section IV 1.

As it appears from equations (25) and (26), the function $a(\eta)\partial_\eta G_\omega^\omega(0, \eta)$ is proportional to $\exp\left(-\frac{-ik^2\eta^2}{2}\right)$. Factoring out this term, one can expand the coefficients in powers of $\epsilon$ after the KummerM functions are substituted with their approximate Bessel series expansion. To second order in $\epsilon$, we have:

$$a(\eta)\partial_\eta G_\omega^\omega(0, \eta) \simeq \frac{H^3}{\phi^2} \exp\left(-\frac{-i k^2 \eta^2}{2} + i \eta k \right) \left[-\frac{1}{2k} - i \eta k \eta^2 \left\{\frac{5}{8} - i \frac{5}{8} \eta^2 - i \frac{12}{10} \eta^4 + \frac{1}{16} \eta^6\right\}\right].$$  \hspace{1cm} (30)

It is now straightforward to obtain the three-point function. We will perform the relevant integration over $\eta$ and expand the final results in series of $\epsilon$ in order to obtain the corrections to the standard ($\epsilon = 0$) spectrum. We show this procedure in detail for the zeroth order term in the square bracket of (30), we then present the final results regarding the other terms in the square brackets.

The zeroth order term in the square bracket of (30) leads to the contribution

$$\langle \zeta_{\eta_1} \zeta_{\eta_2} \zeta_{\eta_3} \rangle_0 = -i(2\pi)^3 \delta^3 \left(\sum \hat{k}_i\right) \left(\frac{\phi}{H}\right)^4 M^{-2} \int_{-\infty}^{0} dt_1 \frac{1}{k_1^3} \frac{-H^3}{8 \delta^6 k_1 k_2 k_3} \exp\left[-i \eta (k_1 + k_2 + k_3) + \frac{i \epsilon \eta}{2}(k_1^2 + k_2^2 + k_3^2)\right] + \text{c.c.} + \text{permutations},$$  \hspace{1cm} (31)

with solution

$$\langle \zeta_{\eta_1} \zeta_{\eta_2} \zeta_{\eta_3} \rangle_0 = 2\Re \left[ \frac{H^6}{\phi^2 M^2 P} \frac{(-1)^{\frac{3}{2}} \pi^3}{\sqrt{2\kappa^2}} \exp\left(-\frac{-i k^2}{2\kappa^2}\right) \text{Erfi} \left[\frac{(1-i)(-k_1+k_2+k_3)}{2\sqrt{\kappa^2}}\right] \right]_{\eta=0} + \text{permutations,}$$  \hspace{1cm} (32)

where $\Re$ indicates the real part and

$$k_1 \equiv k_1 + k_2 + k_3, \quad k_2 \equiv k_1^2 + k_2^2 + k_3^2.$$  \hspace{1cm} (33)

In contrast to the Lorentzian dispersion relation, $\omega = p$, the integrand remains finite at $\eta = -\infty$ even without the change of variable from $\eta \rightarrow \eta + i\epsilon |\eta|$. In this limit the Erfi function tends to $i$. This is interesting, as the adhoc prescription to make sense of the contribution at $-\infty$ is completely taken care of by the modified dispersion relation, which is expected from quantum gravity.

Taking the real part and expanding to second order in $\epsilon$, one obtains

$$\langle \zeta_{\eta_1} \zeta_{\eta_2} \zeta_{\eta_3} \rangle_0 \simeq \delta^3 \left(\sum \hat{k}_i\right) \left[\frac{2H^6 \pi^3}{\phi^2 k_1 k_2 k_3 M^2 P} - \epsilon^2 \frac{6H^6 k^4 \pi^3}{\phi^2 k_1 k_2 k_3 M^2 P} \left(\frac{1}{k_1^2} + \frac{1}{k_2^2} + \frac{1}{k_3^2}\right)\right]$$  \hspace{1cm} (34)

The first term in the square brackets is the regular quantum field theory result in absence of any nonlinearity in the dispersion relation. The first correction is proportional to $\epsilon^2$.

We then apply the same procedure of integration and expansion for small $\epsilon$ for the higher order terms in the square brackets of (30). We obtain
and the relative change in the shape function \( F \) is

\[
\frac{\Delta F}{F} = \frac{\Delta F(x_2, x_3)}{F(x_2, x_3)}
\]

- for the term of order \( \epsilon \) in the square brackets

\[
\langle \zeta_1^\alpha \zeta_2^\beta \zeta_3^\gamma \rangle_1 \approx \delta^3 \left( \sum_k \kappa_k \right) \epsilon^2 \frac{12H^6k_1^4\pi^3}{\phi^2 k_1 k_2 k_3 M^2_0} \left( \frac{1}{k_1^2} + \frac{1}{k_2^2} + \frac{1}{k_3^2} \right),
\]

- for the term of order \( \epsilon^2 \) in the square brackets

\[
\langle \zeta_1^\alpha \zeta_2^\beta \zeta_3^\gamma \rangle_2 \approx \delta^3 \left( \sum_k \kappa_k \right) \epsilon^2 \frac{\pi^6(6k_1^4 - 2k_2^3k_1 + k_2^2 + 10k_1^4)}{k_1 k_2 k_3 M^2_0 \phi^2} \left( \frac{1}{k_1^2} + \frac{1}{k_2^2} + \frac{1}{k_3^2} \right),
\]

where

\[
k_c^3 = k_1^3 + k_2^3 + k_3^3.
\]

Having all the contributions, we can compute the total three-point function:

\[
\langle \zeta_1^\alpha \zeta_2^\beta \zeta_3^\gamma \rangle_{\text{tot}} = \langle \zeta_1^\alpha \zeta_2^\beta \zeta_3^\gamma \rangle_0 + \langle \zeta_1^\alpha \zeta_2^\beta \zeta_3^\gamma \rangle_1 + \langle \zeta_1^\alpha \zeta_2^\beta \zeta_3^\gamma \rangle_2,
\]

and the relative change in the shape function \( F \), \( \frac{\Delta F(k_1, k_2, k_3)}{F(k_1, k_2, k_3)} \), to second order in \( \epsilon \), due to Jacobson-Corey dispersion relation, is

\[
\frac{\Delta F(k_1, k_2, k_3)}{F(k_1, k_2, k_3)} = (-5 + \frac{k_c^3}{k_1^2} + \frac{k_c^3}{2k_1^2})\epsilon^2.
\]

We have plotted \( \frac{\Delta F(k_1, k_2, k_3)}{F(k_1, k_2, k_3)} \) in figure 2 setting \( \epsilon = 0.1 \). As one can see from the plot and the formula (39), the general effect up to quartic order in the expansion is very mild and proportional to \( \epsilon^2 \). The modification is slightly configuration-dependent and the equilateral configurations are more suppressed with respect to the local ones by a factor of \( \sim \) one percent. One should also notice that the enfolded configurations are not enhanced with respect to the other ones.

### IV. EXPLANATION OF THE ABSENCE OF ENHANCEMENT FACTORS IN THE FRAMEWORK OF PARTICLE CREATION

We have seen in the previous section that no interference term seems to appear in the bispectrum. This seems to contradict the expectation from the fact that a modified dispersion relation can be understood in terms of a “vacuum” state for the perturbations given by an excited state.

Comprehending this point is crucial in showing that there will not be any enhancement for the enfolded configurations. In particular, the analysis arriving at this result is based on an expansion of the Whittacker function in \( \epsilon \) valid for small \( z = i\epsilon k^2 \eta^2 \): as the integral in the three-point function \( (5) \) goes up to \( \eta \to -\infty \), we would like to understand in other terms the physical reason behind the absence of the interference terms.

To make sure that it does not depend on the particular expansions used, we have performed another analysis without using any series expansion. Finally, we have also considered the approximated solution found with i) the gluing method, ii) the WKB approximation and compared the different approaches.
1. Analysis with complete solution in the region $\omega_{\text{phys}} > H$

In this section we are going to solve the equation exactly in the region $\omega_{\text{phys}} > H$. The solutions are more manageable than before and need not be expanded. We make a change of variables: $\chi = -\sqrt{2k}\eta$, so that (11) becomes

$$\partial_\chi^2 u_k + \left(\frac{\chi^2}{4} + \frac{1}{2\epsilon} - \frac{2}{\chi^2}\right) u_k = 0. \quad (40)$$

We see that for $\chi^2 > 4\epsilon \leftrightarrow k|\eta| > 2$, we can approximate the equation as

$$\partial_\chi^2 u_k + \left(\frac{\chi^2}{4} + \frac{1}{2\epsilon}\right) u_k = 0. \quad (41)$$

The general solution to this differential equation is (19)

$$u_k(\chi) = c_1 E(a, \chi) + c_2 E^*(a, \chi), \quad a \equiv -\frac{1}{2\epsilon}, \quad (42)$$

where $E(\nu, y)$ are complex parabolic cylinder functions (21).

The solution has two asymptotic regions, A and B, corresponding to $\frac{\chi^2}{4} > |a|$ and $\frac{\chi^2}{4} < |a|$. Using the property of parabolic cylinder functions (21), one finds in A

$$E(a, \chi) \sim \frac{2^\frac{1}{4}}{\chi^2|k|^{\frac{3}{2}}|\eta|^{\frac{1}{2}}} (\sqrt{2k}\eta)^\frac{1}{4} e^{i\phi_2} k^2 + i\eta + i\phi_2, \quad (43)$$

where $\phi_2 \equiv \arg(\Gamma(\frac{1}{2} - i\frac{\phi_2}{2}))$. We reproduce the correct WKB asymptotic (see (19)) choosing

$$c_1 = \frac{e^{-i\phi_2 - i\phi_2}}{2^{\frac{1}{4}} e^{\frac{1}{4}k^2}}, \quad c_2 = 0. \quad (44)$$

In region B we use the asymptotic of $E(a, \chi)$ for $\chi \ll a$, obtaining

$$u_k \sim c_1 \frac{1}{2^{\frac{1}{4}}} \frac{|\Gamma(\frac{1}{4} - i\frac{\phi_2}{2})|^{\frac{1}{4}}}{|\Gamma(\frac{1}{4} - i\frac{\phi_2}{2})|} e^{i\frac{\pi}{4}} e^{-ik\eta + O(\epsilon^2\eta^2}). \quad (45)$$

We therefore find that no term proportional to $e^{ik\eta}$ appears in the mode function, $u_k$. Since the Wightman function is proportional to $u_k^*$, we are assured that no interference term will appear in the Wightman function. We will now make a comparison with the result obtained using the gluing and the WKB approximations.

2. Analysis using gluing and WKB methods

In this section we approximate the solution using the gluing method. We also discuss the WKB approach. As for the gluing, to start with we notice that there are three different asymptotic regions for equation (11) and its solution:

$$(k|\eta|)^2 < 2 \quad \text{region I}$$

$$2 < (k|\eta|)^2 < \epsilon^{-2} \quad \text{region II}$$

$$(k|\eta|)^2 > \epsilon^{-2} \quad \text{region III} \quad (46)$$

One approximates the solution in the various regions in (19) and asks for the continuity of the mode functions and their first derivatives across the boundaries of the regions. We start from region III, where, being $k|\eta| > \epsilon^{-1}$, equation (11) can be approximated by (16). The asymptotic solution in region III is given by (17), where we have taken only the positive frequency mode.

In region II we have instead a solution of the form

$$u_k = \frac{\alpha_k}{\sqrt{2k}} e^{-ik\eta} + \frac{\beta_k}{\sqrt{2k}} e^{ik\eta}. \quad (47)$$
The coefficients $\alpha_k, \beta_k$ are fixed imposing the continuity conditions at the time $\eta_c$ given by

$$\eta_c = -\frac{1}{\epsilon k},$$

which separates regions II and III. We find

$$\begin{cases}
\alpha_k = e^{-\frac{\eta_c}{\epsilon k}} \left(1 + i \frac{\eta_c}{\epsilon k} \right) \\
\beta_k = -ie^{i\frac{\eta_c}{\epsilon k}} \left(\frac{\eta_c}{\epsilon k} \right)
\end{cases},$$

(49)

where we have used $k|\eta_c| = \epsilon^{-1}$. We can now investigate the appearance of enhancements, expected from the interference terms that would appear in (5) due to the form of the solution (47). In fact, the integrand of the bispectrum given by (5) is proportional to the product of three Wightman functions. The integral domain over $\eta$ can be divided into regions I, II and III where we can use the approximate solutions in each region. In particular, because of the form of the solution in region II, see (47), in parallel with the results of [9] one finds enhancement factors multiplying the rest of the bispectrum from the interference terms. However, from (49) these factors are of the order of

$$|\beta_k k \eta_c| \sim \epsilon |k \eta_c| = 1.$$  

(50)

That is, there is no large enhancement factor, even before taking into account the suppression that comes from the 2D projection on the CMB surface.

Furthermore, this method also overestimates the corrections to the power spectrum. This can be seen by comparing to the WKB-method, where there is no distinction between regions II and III. By matching the WKB solution, valid there, with the growing and decaying modes solving the equation of motion in region I, one readily finds that there are no interference terms, and that the spectrum is corrected only at the order $\epsilon^2$, in agreement with the result obtained with the exact solution. Failure of gluing method in obtaining the correct amount of modification to the power spectrum was encountered in [22] too.

**a. Contact with the NPS method and importance of the Wronskian condition**

The NPHS procedure for modifying the vacuum in presence of trans-Planckian physics makes direct contact with the result we have found in this section via the gluing method, if we identify the cutoff with the scale $p_c$. In that case, the evolution in region III is accounted for by an excited state implementing boundary conditions for the perturbations that lead to the conditions (49) at the $k$–dependent junction time (48).

Also in the NPHS scenario where the Bogolyubov parameter is determined by the minimization of the uncertainty relation between the field and its conjugate momentum and by the Wronskian condition, there is actually no large enhancement factor. This descends from the fact that in that case [5]

$$|\beta_k|_{\text{NPHS}} = (2|k \eta_c|)^{-1} \sim \frac{H}{\Lambda} = \epsilon$$

(51)

where we have used the definition (48). Furthermore, in other prescriptions for the choice of vacuum in line with the NPHS approach [19], the coefficient $\beta_k$ is even further suppressed, as powers of $\frac{H}{\Lambda}$. Therefore, it seems that in most of the emergent cases in the cutoff procedure the second Bogolyubov factor is very constrained and will not lead to any large enhancement factor.

**V. DISCUSSION, OUTLOOKS AND CONCLUSION**

Quantum gravity effects are expected to modify the standard Lorentzian dispersion relation at very high energies, close to the Planck scale. One notable example is Hořava-Lifshitz gravity [14, 15], where higher order derivative corrections yield a modified dispersion relation. Such high energy modifications could leave detectable signatures in the temperature fluctuations of the CMBR, allowing an experimental detection of this aspect of high energy theories.

Using the illustrative example of the Corley-Jacobson (JC) modified dispersion relation with positive coefficient, we were able to investigate the modification to the bispectrum in the case where there is no violation of the WKB condition at early times, and to compare these with the expectations deriving from the NPHS approach. The field equations for the JC dispersion relation could be solved exactly, so that the complete formula for the bispectrum, in series of the ratio $\epsilon = \frac{H}{p_c}$, can be obtained. In this ratio, $H$ is the Hubble parameter during inflation, while $p_c$ is the scale at which the modification to the linear dispersion relation become important.
It has been found that the leading correction to the standard bispectrum is suppressed by a factor $\epsilon^2$ and no configuration of momenta leads to compensating factors that enhance the correction. The modification is slightly more pronounced for the equilateral configurations by a factor of one percent with respect to the local ones. The analysis of the bispectrum in two different approximations and in the framework of particle creation leads to the conclusion that if the modified dispersion relation does not violate the WKB condition at early times, the particle production is too small to generate large modifications to the bispectrum. In particular, using the WKB approximation gives results in qualitative and quantitative agreement with those obtained with the exact solution, while the gluing method proposed by Brandenberger and Martin, which relates more directly with the NPNS approach to trans-Planckian physics, is in partial disagreement.

One would expect more favorable possibilities when the modified dispersion relations do violate the WKB condition at early times. However, this point needs to be verified, as the real presence of enhancements could depend strictly on the form of the dispersion relations. We leave this to future investigation.

We have also shown that in most cases within the New Physics Hyper-surface approach to modeling trans-Planckian physics, where at the scale $p_e$ of new physics the theory is cutoffed and a boundary condition is imposed on the fields, there is no occurrence of large enhancement factors in the bispectrum, not even for the enfolded configurations studied in [9]. Most notably, the enhancement is only of order $O(1)$ when the Bogolyubov coefficients accounting for particle creation are determined by minimizing the uncertainty relation and by the Wronskian condition, which yields the largest correction to the power spectrum.

Acknowledgments

A.A. is supported by the Göran Gustafsson Foundation. U.D. is supported by the the Göran Gustafsson Foundation and the Swedish Research Council (VR). Diego Chialva is supported by a Postdoctoral F.R.S.-F.N.R.S. research fellowship via the Ulysses Incentive Grant for the Mobility in Science (promoter at the Université de Mons: Per Sundell). A.A. acknowledges useful discussions with M. M. Sheikh-Jabbari and G. Shiu.

[1] J. M. Maldacena, JHEP 0305, 013 (2003) [arXiv:astro-ph/0210603].
[2] R. Easther, B. R. Greene, W. H. Kinney and G. Shiu, Phys. Rev. D 64, 103502 (2001) [arXiv:hep-th/0104102].
[3] A. Kempf, Phys. Rev. D 63, 083514 (2001). astro-ph/0009209.
[4] A. Ashoorioon, A. Kempf and R. B. Mann, Phys. Rev. D 71, 03503 (2005) [arXiv:astro-ph/0410139].
[5] A. Ashoorioon and R. B. Mann, Nucl. Phys. B 716, 261 (2005) [arXiv:gr-qc/0411056].
[6] A. Ashoorioon, J. L. Hovebo, R. B. Mann, Nucl. Phys. B 727, 63-76 (2005). gr-qc/0504135.
[7] S. F. Hassan, M. S. Sloth, Nucl. Phys. B 674, 434-458 (2003). hep-th/0204110.
[8] K. Schalm, G. Shiu, J. P. van der Schaar, JHEP 0404, 076 (2004). hep-th/0401164.
[9] B. R. Greene, K. Schalm, G. Shiu, J. P. van der Schaar, JCAP 0502, 001 (2005). hep-th/0411217.
[10] U. H. Danielsson, Phys. Rev. D 66, 023511 (2002) [arXiv:hep-th/0203198].
[11] U. H. Danielsson, JHEP 0207, 040 (2002) [arXiv:hep-th/0205227].
[12] V. Bozza, M. Giovannini, G. Veneziano, JCAP 0305, 001 (2003). hep-th/0302184.
[13] R. Easther, B. R. Greene, W. H. Kinney and G. Shiu, Phys. Rev. D 66, 023518 (2002) [arXiv:hep-th/0204129].
[14] X. Chen, M. x. Huang, S. Kachru and G. Shiu, JCAP 0701, 002 (2007) [arXiv:hep-th/0605015].
[15] R. Holman and A. J. Tolley, JCAP 0805, 001 (2008) [arXiv:0710.1302 [hep-th]].
[16] P. D. Meerburg, J. P. van der Schaar and M. G. Jackson, JCAP 1002, 001 (2010) [arXiv:0910.4986 [hep-th]].
[17] A. Ashoorioon, G. Shiu, JCAP 1103, 025 (2011). [arXiv:1012.3392 [astro-ph.CO]].
[18] A. Ashoorioon, J. L. Hovebo, R. B. Mann, Nucl. Phys. B 716, 261 (2005) [arXiv:gr-qc/0411056].
[19] J. Martin and R. H. Brandenberger, Phys. Rev. D 63, 123501 (2001) [arXiv:hep-th/0005209].
[20] S. Corley and T. Jacobson, Phys. Rev. D 54, 1568 (1996) [arXiv:hep-th/9601073].
[21] M. Abramowitz and I. Stegun, HandBook of Mathematical Functions, Dover Publication, INC., New York, 1970.