Anosov diffeomorphisms constructed from $\pi_k(\text{Diff}(S^n))$

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Abstract

We construct Anosov diffeomorphisms on manifolds that are homeomorphic to infranilmanifolds yet have exotic smooth structures. These manifolds are obtained from standard infranilmanifolds by connected summing with certain exotic spheres. Our construction produces Anosov diffeomorphisms of high codimension on infranilmanifolds with irreducible exotic smooth structures.

1. Introduction

Let $M$ be a compact smooth $n$-dimensional Riemannian manifold. Recall that a diffeomorphism $f$ is called Anosov if there exist constants $\lambda \in (0,1)$ and $C > 0$ along with a $df$-invariant splitting $TM = E^s \oplus E^u$ of the tangent bundle of $M$, such that, for all $m \geq 0$,

$$
\|df^m v\| \leq C \lambda^m \|v\|, \quad v \in E^s,
$$

$$
\|df^{-m} v\| \leq C \lambda^m \|v\|, \quad v \in E^u.
$$

If either fiber of $E^s$ or $E^u$ has dimension $k$ with $k \leq \lfloor n/2 \rfloor$, then $f$ is called a codimension $k$ Anosov diffeomorphism. See, for example, [17] for background on Anosov diffeomorphisms.

The classification of Anosov diffeomorphisms is an outstanding open problem. All currently known examples of manifolds that support Anosov diffeomorphisms are homeomorphic to infranilmanifolds. It is an interesting question to study existence of Anosov diffeomorphisms on manifolds that are homeomorphic to infranilmanifolds yet have exotic smooth structure.

Farrell and Jones have constructed [11] codimension 1 Anosov diffeomorphisms on higher dimensional exotic tori, that is, manifolds that are homeomorphic to tori but have non-standard smooth structure. The current paper should be considered a sequel to [11]. We formulate our main result below. (See Section 2 for the definitions of the Gromoll groups and expanding endomorphism.)

**Theorem 1.1.** Let $M$ be an $n$-dimensional ($n \geq 7$) infranilmanifold; in particular, $M$ can be a nilmanifold. Let $L: M \to M$ be a codimension $k$ Anosov automorphism. Assume that $M$ admits an expanding endomorphism $E: M \to M$ that commutes with $L$. Let $\Sigma$ be a homotopy sphere from the Gromoll group $\Gamma_{k+1}^n$, then the connected sum $M \# \Sigma$ admits a codimension $k$ Anosov diffeomorphism.

**Remark 1.2.** Not all exotic smooth structures on infranilmanifolds come from exotic spheres. For example, obstruction theory gives smooth structures on tori that are not even PL-equivalent [15] (note that by the Alexander trick all exotic tori that come from exotic spheres are PL-equivalent). In particular, there are three different smooth structures on $\mathbb{T}^3$ (see [24, p. 227]). It would be very interesting to see if these manifolds support Anosov diffeomorphisms.
Our construction of Anosov diffeomorphisms on exotic infranilmanifolds is different from that of Farrell and Jones and gives Anosov diffeomorphisms of high codimension. Of course, one can multiply the Anosov diffeomorphism of Farrell and Jones on $\mathbb{T}^n \# \Sigma$ by an Anosov automorphism of an infranilmanifold $M$ to obtain a higher codimension Anosov diffeomorphism on $(\mathbb{T}^n \# \Sigma) \times M$. Then one can show using smoothing theory (in conjunction with [10, 20]) that $(\mathbb{T}^n \# \Sigma) \times M$ is not diffeomorphic to any infranilmanifold if $\Sigma$ is an exotic sphere. (The authors would like to thank the referee for bringing this to our attention.) The advantage of our construction is that it gives higher codimension Anosov diffeomorphisms on manifolds with irreducible smooth structure, that is, on manifolds that are not diffeomorphic to a smooth Cartesian product of two lower dimensional closed smooth manifolds. The following is a straightforward consequence of Theorem A.1 that we prove in the Appendix.

**Proposition 1.3.** If $M$ is an $n$-dimensional closed oriented infranilmanifold and $\Sigma$ is a homotopy $n$-sphere such that $M \# \Sigma$ is not diffeomorphic to $M$, then $M \# \Sigma$ is irreducible.

If $M$ is an $n$-dimensional ($n \neq 4$) nilmanifold and $\Sigma$ is not diffeomorphic to the standard sphere $S^n$, then $M \# \Sigma$ is not diffeomorphic to $M$ (see [12, Lemma 4]). For an infranilmanifold the situation is more involved. We have the following proposition.

**Proposition 1.4.** Let $M$ be an $n$-dimensional ($n \neq 4$) orientable infranilmanifold with a $q$-sheeted cover $N$ which is a nilmanifold. Let $\Sigma$ be an exotic homotopy sphere of order $d$ from the Kervaire–Milnor group $\Theta_n$. Then $M \# \Sigma$ is not diffeomorphic to any infranilmanifold if $d$ does not divide $q$. In particular, $M \# \Sigma$ is not diffeomorphic to $M$ if $d$ does not divide $q$.

**Proof.** We proceed via proof by contradiction. Assume that $M$ is diffeomorphic to $M \# \Sigma$. It was shown in [20] that an isomorphism between the fundamental groups of a pair of closed infranilmanifolds is always induced by a diffeomorphism. Hence, by precomposing the assumed diffeomorphism $M \to M \# \Sigma$ with an appropriate self-diffeomorphism of $M$, we obtain that $M$ and $M \# \Sigma$ are diffeomorphic via a diffeomorphism inducing the identity isomorphism of the fundamental group. (The fundamental groups of $M$ and $M \# \Sigma$ are canonically identified.) By lifting this diffeomorphism to the covering space $N \to M$, we see that $N$ and $N \# q\Sigma$ are also diffeomorphic, and hence $q\Sigma$ is diffeomorphic to $S^n$ because of Farrell and Jones [12, Lemma 4]. Therefore, $d$ divides $q$, which is the contradiction proving that $M$ is not diffeomorphic to $M \# \Sigma$. Also $M \# \Sigma$ is not diffeomorphic to any other infranilmanifold by the result from [20].

In the next section, we provide brief background on Anosov automorphisms and the Gromoll filtration of the group of homotopy spheres. Then we proceed with the proof of Theorem 1.1. The last section is devoted to examples to which Theorem 1.1 applies. In particular, we establish the following result.

**Proposition 1.5.** Let $F$ be a finite group. Then, for any sufficiently large number $k$, there exists a flat Riemannian manifold $M$ with holonomy group $F$ and a homeomorphic irreducible smooth manifold $N$ such that

1. $N$ supports a codimension $k$ Anosov diffeomorphism;
2. $N$ is not diffeomorphic to any infranilmanifold; in particular, $N$ is not diffeomorphic to $M$. 

2. Background

2.1. Anosov automorphisms and expanding endomorphisms

Let $G$ be a simply connected nilpotent Lie group equipped with a right invariant Riemannian metric. Let $\tilde{L} : G \to G$ be an automorphism of $G$ such that $D\tilde{L} : g \to g$ is hyperbolic, that is, the absolute values of its eigenvalues are different from 1. Assume that there exists a cocompact lattice $\Gamma \subset G$ preserved by $\tilde{L}$, $\tilde{L}(\Gamma) = \Gamma$. Then $\tilde{L}$ induces an Anosov automorphism $L$ of the nilmanifold $M = G/\Gamma$.

If $\tilde{E} : G \to G$ is an automorphism such that the eigenvalues of $D\tilde{E} : g \to g$ are all greater than 1 and $\tilde{E}(\Gamma) \subset \Gamma$, then $\tilde{E}$ induces a finite-to-one expanding endomorphism of the nilmanifold $M$.

Existence of Anosov automorphism is a strong condition on $G$. Still there are plenty of non-toral examples in dimensions 6, 8 and higher (see, for example, [7] and references therein).

Infranilmanifolds are finite quotients of nilmanifolds obtained through the following construction. Consider a finite group $F$ of automorphisms of $G$. Then the semidirect product $G \rtimes F$ acts on $G$ by affine transformations. Consider a torsion-free cocompact lattice $\Gamma$ in $G \rtimes F$. Let $M$ be the orbit space $G/\Gamma$. This space is naturally a manifold since $\Gamma$ is torsion-free. It is known that $\Gamma \cap G$ is a lattice in $G$ and has finite index in $\Gamma$. Hence, $G/\Gamma \cap G$ is a nilmanifold that finitely covers $M$.

A hyperbolic automorphism $\tilde{L} : G \to G$ with $\tilde{L} \circ \Gamma \circ \tilde{L}^{-1} = \Gamma$ induces an infranilmanifold Anosov automorphism $L : M \to M$.

An expanding endomorphism $\tilde{E} : G \to G$ with $\tilde{E} \circ \Gamma \circ \tilde{E}^{-1} \subset \Gamma$ induces an infranilmanifold expanding endomorphism $E : M \to M$.

Remark 2.1. With the definitions above, it is clear that the covering automorphisms $\tilde{L}$ and $\tilde{E}$, of $L$ and $E$ from Theorem 1.1, commute as well. We will use this fact in the proof of Theorem 1.1.

Remark 2.2. More generally, Anosov automorphisms (and expanding endomorphisms) can be constructed starting from affine maps $x \mapsto v \cdot \tilde{L}(x)$ for some fixed $v \in G$. In the case of a single automorphism of a nilmanifold this does not give anything new as one can change the group structure of the universal cover by moving the identity element to the fixed point of the affine map. (In the case of a single automorphism of an infranilmanifold or a higher rank action, affine maps do give new examples as explained in [8] and [16], respectively.)

2.2. Gromoll filtration of Kervaire–Milnor group

Recall that a homotopy $n$-sphere $Σ$ is a smooth manifold that is homeomorphic to the standard $n$-sphere $S^n$. The set of all oriented diffeomorphism classes of homotopy $n$-spheres ($n \geq 5$) is a finite abelian group $Θ_n$ under the operation $\#$ of connected sum. This group was introduced and studied by Kervaire and Milnor [18].

A simple way of constructing a homotopy $n$-sphere $Σ$ is to take two copies of a closed disk $D^n$ and paste their boundaries together by some orientation-preserving diffeomorphism $f : S^{n-1} \to S^{n-1}$.

This produces a homotopy sphere $Σ_f$. It is easy to see that if $f$ is smoothly isotopic to $g$, then $Σ_f$ is diffeomorphic to $Σ_g$. Therefore, the map $f \mapsto Σ_f$ factors through to a map $F : π_0(\text{Diff}(S^{n-1})) \to Θ_n$, which is known, due to [6, 23], to be a group isomorphism for $n \geq 6$. 
View $S^{n-1}$ as the unit sphere in $\mathbb{R}^n$. Consider the group $\text{Diff}_k(S^{n-1})$ of orientation-preserving diffeomorphisms of $S^{n-1}$ that preserve first $k$ coordinates. The image of this group in $\Theta_n$ is a Gromoll subgroup $\Gamma_{k+1}^n$. Gromoll subgroups form a filtration

$$\Theta_n = \Gamma_1^n \supset \Gamma_2^n \supset \ldots \supset \Gamma_n^n = 0.$$ 

Cerf [6] has shown that $\Gamma^n_1 = \Gamma^n_2$ for $n \geq 6$. Antonelli, Burghelea and Kahn [1] have shown that $\Gamma^{2m-2}_{2m-2} \neq 0$ for $m \geq 4$ and that $\Gamma^{2m+1}_{2m+1} \neq 0$ for $m$ not of the form $2^j - 1$, where $v(m)$ denotes the maximal number of linearly independent vector fields on the sphere $S^{2m+1}$. For some more non-vanishing results and explicit lower bounds on the order of $\Gamma^n_k$, see [1].

2.3. Connected summing with $\Sigma \in \Gamma_{k+1}^n$

Consider the group $\text{Diff}(S^{n-1-k}, B)$ of diffeomorphisms of $S^{n-1-k}$ that are identity on an open ball $B$. Note that any element of $\pi_k(\text{Diff}(S^{n-1-k}, B))$ is represented by a diffeomorphism of $D^k \times D^{n-1-k}$, $D^{n-1-k} \equal{} S^{n-1-k} \setminus B$, which preserves the first $k$ coordinates and is the identity near the boundary. The space of such diffeomorphisms will be denoted by $\text{Diff}_k(D^k \times D^{n-1-k}, \partial)$. There are natural inclusions

$$\pi_k(\text{Diff}(S^{n-1-k}, B)) \hookrightarrow \pi_0(\text{Diff}(S^{n-1})) \hookrightarrow \pi_0(\text{Diff}(S^{n-1})).$$

Denote by $i$ the composition of these inclusions.

It is not very hard to show [2, Lemma 1.13] that

$$F(i(\pi_k(\text{Diff}(S^{n-1-k}, B)))) = \Gamma_{k+1}^n.$$ 

Thus, given $\Sigma \in \Gamma_{k+1}^n$ and a manifold $M$, we can realize $M \# \Sigma$ in the following way. Remove a disk $D^k \times D^{n-k}$ from $M$. Consider the diffeomorphism $g \in \text{Diff}_k(D^k \times D^{n-1-k}, \partial)$ which extends by the identity map to $g \in \text{Diff}_k(D^k \times \partial D^{n-k}, \partial)$, which represents $\Sigma$ in $\pi_k(\text{Diff}(S^{n-1-k}, B))$. Form the connected sum $M \# \Sigma$ by gluing $D^k \times D^{n-k}$ back in using the identity map on $\partial(D^k) \times D^{n-k}$ and using $g$ on $D^k \times \partial(D^{n-k})$.

**Remark 2.3.** It is important that $\bar{g}$ is identity not only on the boundary of $D^k \times D^{n-1-k}$ but also in a neighborhood of the boundary. This is needed for the smooth structure to be well defined avoiding the problem at the corners.

3. The proof of Theorem 1.1

3.1. The construction of the smooth structure

Recall that $M = G/\Gamma$ is an infranilmanifold and $\Sigma \in \Gamma_{k+1}^n$. First, we explain our model for $M \# \Sigma$ and provide the basic construction of the diffeomorphism $f: M \# \Sigma \to M \# \Sigma$. Then we explain how to modify $f$ to obtain an Anosov diffeomorphism.

Start with an Anosov automorphism $L: M \to M$ with $k$-dimensional stable distribution $E^s$ (in case the unstable distribution is $k$-dimensional, consider $L^{-1}$ instead). Choose coordinates in a small neighborhood $U$ of a fixed point that comes from the identity element id in $G$ so that $L$ is given by the formula

$$L(x, y) = (L_1(x), L_2(y)), \quad (x, y) \in U \cap L^{-1}U,$$

where $x$ is $k$-dimensional, $L_1$ is contracting and $L_2$ is expanding.

Next we choose a product of two disks $R_0^+ = D_0^+ \times C_0$ in the positive quadrant $\{(x, y): (x, y) > 0\}$ in the proximity of the fixed point $(0, 0)$. Here $D_0^+$ is $k$-dimensional and $C_0$
is \((n-k)\)-dimensional. Consider 

\[ R_0^{- \text{def}} = \{ (-x,y): (x,y) \in R_0^+ \} \overset{\text{def}}{=} D_0 \times C_0, \]

and \( R_0^+ \overset{\text{def}}{=} D_0 \times C_0 \), where \( D_0 \) is the convex hull of \( D_0^+ \) and \( D_0^- \). Also let \( U_0^+ \), \( U_0^- \) and \( U_0 \), \( U_0 \supset U_0^+ \cap U_0^- \) be small neighborhoods of \( R_0^+ \), \( R_0^- \) and \( R_0 \), respectively.

Define \( R_i, R_i^+ = D_i^+ \times C_i, \ R_i^- = D_i^- \times C_i, \ U_i^+, \ U_i^- \) and \( U_i \) as images of \( R_0, R_0^+, R_0^-, U_0^+, U_0^- \) and \( U_0 \) under \( L^i \), \( i = 1, 2 \), respectively. We make our choices in such a way that \( U_0, U_1 \) and \( U_2 \) are disjoint.

Let \( g \in \text{Diff}_k(D_1^{-} \times \partial C_1, \partial) \) be a diffeomorphism representing \( \Sigma \). Glue \( \Sigma \) in, along the boundary of \( R_1^+ \), using \( g \) as described in the previous section. Glue \( -\Sigma \) in, along the boundary of \( R_1^- \) using \( g^{-1} \) considered as a diffeomorphism in \( \text{Diff}_k(D_1^+ \times \partial C_1, \partial) \). To be more precise, we identify \( R_1^+ \) and \( R_1^- \) via a translation \( t: R_1^+ \rightarrow R_1^- \) (rather than a reflection!) and glue \( -\Sigma \) in using \( t \circ g^{-1} \circ t^{-1} \). Finally, glue \( \Sigma \) in, along the boundary of \( R_2^+ \), using \( L \circ g \circ L^{-1} \).

The resulting manifold \( M \# 2 \Sigma \# - \Sigma \) is diffeomorphic to \( M \# \Sigma \). Note that in the course of the construction of \( M \# 2 \Sigma \# - \Sigma \) the leaves of the unstable foliation undergo some cutting and pasting, which results in a smooth foliation that we call \( W^u \). Locally the leaves of \( W^u \) are given by the same formula \( x = \text{const} \). In contrast, the stable foliation is being torn apart.

3.2. The construction of the diffeomorphism \( f \)

Consider an open set \( V_0 \) that is the union of \( U_1^+ \), \( U_1^- \) and a small tube joining them as shown in Figure 1. Let \( V_1 = L(V_0) \).

We proceed with definition of \( f: M \# 2 \Sigma \# - \Sigma \rightarrow M \# 2 \Sigma \# - \Sigma \). From now on we will be using the same notation for various regions in \( M \# 2 \Sigma \# - \Sigma \) as that for \( M \) with a tilde on top. For example, \( U_1^+ \) stands for \( U_1^+ \) with homotopy sphere \( \Sigma \) glued in along the boundary of \( R_1^+ \).

Let \( f(p) = L(p) \) unless \( p \) belongs to \( \tilde{U}_0, \ U_1^+ \) or \( \tilde{V}_0 \). We need to define \( f \) differently on \( \tilde{U}_0, \ U_1^+ \) and \( \tilde{V}_0 \) as the smooth structure on \( L(U_0), U_1^+ \) and \( V_0 \) has been changed.

The restriction \( f|_{\tilde{U}_1^+}: \tilde{U}_1^+ \rightarrow \tilde{U}_2^+ \) can be naturally induced by \( L: U_1^+ \rightarrow U_2^+ \) because of the way \( \tilde{U}_1^+ \) and \( \tilde{U}_2^+ \) were defined.

To define \( f|_{\tilde{U}_0}: \tilde{U}_0 \rightarrow \tilde{U}_1 \), we interpret \( \tilde{U}_1 \) as \( U_1 \) with \( R_1 \) being removed and then being glued back in via a diffeomorphism \( h \) that is equal to \( g \) along the boundary of \( R_1^+ \), \( g^{-1} \) along the boundary of \( R_1^- \) and identity elsewhere.

The diffeomorphism \( h \) is the concatenation of \( g \) and \( g^{-1} \), and is easily seen to represent identity in \( \pi_k(\text{Diff}(\partial C_1)) \). Indeed one can explicitly construct an isotopy from \( h \) to \( \text{Id} \) in \( \text{Diff}_k(R_1) \) by joining the translation \( t: R_1^+ \rightarrow R_1^- \) (from the definition of \( g^{-1} \in \text{Diff}_k(D_1^+, \partial C_1) \)) to \( \text{Id}: R_1^- \rightarrow R_1^+ \). It follows that \( \tilde{U}_1 \) is diffeomorphic to \( \tilde{U}_1 \) by a diffeomorphism \( f_1 \) that fixes the first \( k \) coordinates and is equal to the identity map near \( \partial \tilde{U}_1 = \partial U_1 \). Define 

\[ f|_{\tilde{U}_0} \overset{\text{def}}{=} f_1 \circ L. \]

Similar to above, \( V_0 \) is diffeomorphic to \( \tilde{V}_0 \) by a diffeomorphism \( f_2 \) that fixes the first \( k \) coordinates and is identity near \( \partial V_0 = \partial \tilde{V}_0 \). Define 

\[ f|_{\tilde{V}_0} \overset{\text{def}}{=} L \circ f_2^{-1}. \]

It is clear that our definitions coincide on the boundary of the regions so that diffeomorphism \( f \) is well defined. Also note that \( f \) preserves foliation \( W^u \).

3.3. Modifying \( f \) to get an Anosov diffeomorphism on \( M \# \Sigma \)

We fix a Riemannian metric on \( M \# 2 \Sigma \# - \Sigma \) in the following way. On \( (M \# 2 \Sigma \# - \Sigma) \setminus (\tilde{U}_1^+ \cup \tilde{U}_1^- \cup \tilde{U}_2^+) \) we use a Riemannian metric induced by a right invariant Riemannian metric on \( G \).
such that $F \subset \text{Iso}(G)$, where $F$ is from Subsection 2.1. We extend it to $\tilde{U}_1^+ \cup \tilde{U}_1^- \cup \tilde{U}_2^+$ in an arbitrary way.

Let $W = \tilde{U}_0 \cup \tilde{U}_1^+ \cup \tilde{V}_0$. The foliation $W^u$ is uniformly expanding everywhere but in $W$. (In fact, with a suitable choice of the Riemannian metric $W^u$ is expanding on $\tilde{U}_1^+$ as well, but we will not use this fact.) The number

$$\alpha(f) \overset{\text{def}}{=} \inf_{v \in T_x W^u, v \neq 0, x \in W} \left( \frac{\|Df^3 v\|}{\|v\|} \right)$$

measures maximal possible contraction along $W^u$ that occurs as a point passes through $W$. Therefore, if one can guarantee that the first return time to $W$

$$N_1(f) \overset{\text{def}}{=} \inf_{x \in W} \{i : i \geq 3, f^i(x) \in W\}$$

is large compared with $\alpha(f)$, then $W^u$ will be expanding for $f$. 

**Figure 1.** The construction of the diffeomorphism $f$. 
Next we will construct a cone field on $M \# 2\Sigma^- - \Sigma$ that would give us the stable bundle.

Define $E_f^0 \overset{\text{def}}{=} TW^u \subset T(M \# 2\Sigma^- - \Sigma)$. Recall that $E^s \subset TM$ is the stable bundle for $L$. Let $\mathcal{P} \overset{\text{def}}{=} \tilde{V}_0 \cup U_1^+$. For any $x \in (M \# 2\Sigma^- - \Sigma) \setminus \mathcal{P}$, define $E^s(x)$ by identifying $(M \# 2\Sigma^- - \Sigma) \setminus \mathcal{P}$ and $M \setminus (U_1^+ \cup V_0)$. Fix a small $\varepsilon > 0$ and define the cones

$$C(x) \overset{\text{def}}{=} \{ v \in T_x M : \angle(v, E^s) < \varepsilon \}$$

for $x \in (M \# 2\Sigma^- - \Sigma) \setminus \mathcal{P}$. Also define

$$\mathcal{C}(x) \overset{\text{def}}{=} Df^{-1}(C(f(x))), \quad x \in \tilde{V}_0,$$

$$\mathcal{C}(x) \overset{\text{def}}{=} Df^{-2}(C(f^2(x))), \quad x \in U_1^+.$$ 

Note that $Df^{-1}(C(x)) \subset C(f^{-1}(x))$ for $x \notin \mathcal{P}$ and $C(x) \cap E_f^0(x) = \{0\}$ for all $x \in M \# 2\Sigma^- - \Sigma$. Therefore, for any $x \in \mathcal{P}$, the sequence of cones $Df^{-i}(C(x))$, $i > 0$, shrinks exponentially fast toward $E^s$ until the sequence of base points $f^{-i}(x)$ enters $\mathcal{P}$ again. We see that if the first return time

$$N_2(f) \overset{\text{def}}{=} \inf_{x \in \mathcal{P}} \{ i : i \geq 2, f^{-i}(x) \in \mathcal{P} \}$$

is large enough, then there exists $N$ that depends on $\varepsilon$ and various choices we have made in our construction such that

1. $\forall x \in \mathcal{P}, f^{-i}(x) \notin \mathcal{P}$, $i = 2, \ldots, N$;
2. $\forall x \in \mathcal{P}, Df^{-N}(C(x)) \subset C(f^{-N}(x))$;
3. $\exists \lambda > 1 : \forall x \in \mathcal{P}, \forall v \in C(x), \|Df^{-N}v\| \geq \lambda \|v\|$.

Now we modify the cone field in the following way:

$$\mathcal{C}(x) \overset{\text{def}}{=} Df^{-i}(C(f^i(x))), \quad \text{if } x \in f^i(\mathcal{P}) \text{ for } i = 0, \ldots, N - 1,$$

$$\mathcal{C}(x) \overset{\text{def}}{=} C(x), \quad \text{if } x \notin \bigcup_{i=0}^{N-1} f^i(\mathcal{P}).$$

It is clear from our definitions that the new cone field is invariant

$$\forall x, \quad Df^{-1}(\mathcal{C}(f(x))) \subset \mathcal{C}(x)$$

and

$$\exists \mu > 1 : \forall x \text{ and } \forall v \in \mathcal{C}(x), \|Df^{-1}v\| \geq \mu \|v\|.$$ 

These properties imply [17, Section 6.2] that the bundle

$$E_f^1(\cdot) \overset{\text{def}}{=} \bigcap_{i \geq 0} Df^{-i}(\mathcal{C}(f^i(\cdot)))$$

is a $Df$-invariant $k$-dimensional exponentially contracting stable bundle for $f$.

We summarize that $f$ is Anosov, provided that the first return times $N_1(f)$ and $N_2(f)$ are large enough: how large depends only on the choices we have made when constructing $M \# \Sigma$ and $f|\mathcal{P}$.

Recall that $M$ admits an expanding endomorphism $E$ that commutes with $L$. Let $\tilde{E} : G \to G$ be the covering automorphism of $E$. The covering automorphism $\tilde{L}$ of $L$ preserves the lattice of affine transformations $\Gamma_m \overset{\text{def}}{=} \tilde{E}^m \circ \Gamma \circ \tilde{E}^{-m}$, $m \geq 1$, that is, $\tilde{L} \circ \Gamma_m \circ \tilde{L}^{-1} = \Gamma_m$, and hence induces an Anosov automorphism $L_m$ of $M_m \overset{\text{def}}{=} G/\Gamma_m$. Also we have the covering map $p : M_m \to M$ induced by $\text{id} : G \to G$ and the expanding diffeomorphism $H : M \to M_m$ induced by $E^m$.

We repeat our constructions of the exotic smooth structure and the diffeomorphism on $M_m$ using the copy $U_m$ of $U$ in $p^{-1}(U)$ that contains the $\Gamma_m$ orbit of $\text{id}$. Since $U_m$ is isometric to $U$.
and $L_m|_{t_m} = L|_t$, we can repeat the constructions with the same choices as before and obtain a manifold $M_m \# 2\Sigma \# - \Sigma$ together with a diffeomorphism $f_m$. In particular, due to the same choices, $\alpha(f) = \alpha(f_m)$.

Let number $r_m$ be the maximal radius of a ball $B(\text{id}, r_m) \subset G$ that projects injectively into $M_m$. Since $\tilde{E}$ is expanding, $r_m \to \infty$ as $m \to \infty$. It follows that $\min(N_1(f_m), N_2(f_m)) \to \infty$ as $m \to \infty$. Hence, $f_m$ is Anosov for large enough $m$. Since $M_m \# \Sigma$ is diffeomorphic to $M \# \Sigma$, we obtain an Anosov diffeomorphism on $M \# \Sigma$ by conjugating $f_m$ with this diffeomorphism.

4. Examples

Here we collect some examples where the conditions of Theorem 1.1 are satisfied.

4.1. Toral examples

Any Anosov automorphism of the torus $\mathbb{T}^n$ commutes with the expanding endomorphism $s \cdot \text{Id}$, $s > 1$. Thus, we obtain codimension $k$ Anosov diffeomorphisms on $\mathbb{T}^n \# \Sigma$, where $\Sigma \in \Gamma_{k+1}^n$. If $\Sigma$ is not diffeomorphic to $S^n$, then, by the discussion in the paragraph between Propositions 1.3 and 1.4, $\mathbb{T}^n \# \Sigma$ is exotic, and hence, by Proposition 1.3, $\mathbb{T}^n \# \Sigma$ is irreducible. Therefore, we obtain codimension $k$ Anosov diffeomorphisms on exotic irreducible tori whenever $\Gamma_{k+1}^n$ is non-trivial.

4.2. A nilmanifold example

Let $L_1: M_1 \to M_1$ be the Borel–Smale example of Anosov automorphism of a six-dimensional nilmanifold $M_1$, which is a quotient of the product of two copies of Heisenberg group (see [17, Section 4.17], for a detailed construction). It is easy to check that

$$\tilde{E}_1: \begin{pmatrix} 1 & x_1 & z_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & x_2 & z_2 \\ 0 & 1 & y_2 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2x_1 & 4z_1 \\ 0 & 1 & 2y_1 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 2x_2 & 4z_2 \\ 0 & 1 & 2y_2 \\ 0 & 0 & 1 \end{pmatrix}$$

induces an expanding endomorphism $E_1: M_1 \to M_1$ that commutes with $L_1$.

Also let $L_2: \mathbb{T}^{15} \to \mathbb{T}^{15}$ be a codimension 2 Anosov automorphism and let $E_2: \mathbb{T}^{15} \to \mathbb{T}^{15}$ be a conformal expanding endomorphism as in Subsection 4.1.

We get that $L = L_1 \times L_2$ is a codimension 5 Anosov automorphism that commutes with the expanding endomorphism $E = E_1 \times E_2$. It is known that $|\Gamma_6^{21}| \geq 508$. Hence, Theorem 1.1 applies non-trivially to $L$ giving codimension 5 Anosov diffeomorphisms on 21-dimensional exotic nilmanifolds $(M_1 \times \mathbb{T}^{15}) \# \Sigma$, $\Sigma \in \Gamma_6^{21}$. The discussion at the end of Subsection 4.1 gives a reason why these manifolds are exotic and irreducible.

4.3. Infranilmanifold examples

Here the main goal is to show that Theorem 1.1 can be used to produce Anosov diffeomorphisms of high codimension on exotic infratori. We use the term ‘infratorus’ as a synonym to ‘closed flat Riemannian manifold’.

By a theorem of Auslander and Kuranishi [3] there exists an infratorus with holonomy group $F$ for any finite group $F$. There are only finitely many groups of a given order $q$. Hence, there exists a positive integer $k(q)$ such that, for any finite group $F$ of order $q$, there is an infratorus with holonomy group $F$ of dimension $k(q)$.
Proposition 4.1. Given a finite group $F$ of order $q$ and an integer $k > k(q)$, there exists an (orientable) infratorus $M$ with holonomy group $F$ and a homeomorphic irreducible smooth manifold $N$ such that

1. $N$ supports a codimension $k$ Anosov diffeomorphism;
2. $N$ is not diffeomorphic to any infranilmanifold; in particular, $N$ is not diffeomorphic to $M$.

Clearly, Proposition 4.1 implies Proposition 1.5.

4.3.1. Existence of a commuting expanding endomorphism. We start our proof of Proposition 4.1 by showing that the assumption from Theorem 1.1 about the existence of a commuting expanding endomorphism is satisfied for some positive power of a given Anosov automorphism.

Proposition 4.2. Let $M$ be an infratorus of dimension $n$ and $L: M \to M$ be an Anosov automorphism. Then there exists an expanding affine transformation (cf. Remarks 2.2 and 4.7) that commutes with some positive power of $L$ and has a fixed point in common with this power of $L$.

Proof. Let $\Gamma$ be the fundamental group of $M$. Then $\Gamma$ can be identified with the group of deck transformations of $\mathbb{R}^n$. By classical Bieberbach theorems, we know that the intersection of $\Gamma$ and the group of translations of $\mathbb{R}^n$ is a normal free abelian group of finite index in $\Gamma$. We pick our basis for $\mathbb{R}^n$ so that this group is $\mathbb{Z}^n \subset \mathbb{R}^n$. The finite quotient group $F = \Gamma / \mathbb{Z}^n$ is the holonomy group of $M$. The holonomy representation of $F$ into $\text{GL}_n(\mathbb{R})$ is faithful and contained in $\text{GL}_n(\mathbb{Z})$. Thus, we can identify $F$ with a subgroup of $\text{GL}_n(\mathbb{Z})$.

We have the corresponding exact sequence

$$0 \to \mathbb{Z}^n \to \Gamma \to F \to 1.$$ 

Every element $\gamma \in \Gamma$ has the form $x \mapsto g_\gamma x + u_\gamma$, where $g_\gamma \in F$, $u_\gamma \in \mathbb{R}^n$.

Claim 4.3. Let $q$ be the order of $F$. There exists $u_0 \in \mathbb{R}^n$ such that if we denote by $t$ the translation $x \mapsto x + u_0$, then

$$t \circ \gamma \circ t^{-1} \subseteq \left(\frac{1}{q}\mathbb{Z}\right)^n \rtimes \text{GL}_n(\mathbb{Z}).$$

Proof of Claim 4.3. Since $(u_\gamma \mod \mathbb{Z}^n) \in \mathbb{T}^n$ depends only on $g_\gamma$ the function $\gamma \mapsto (u_\gamma \mod \mathbb{Z}^n)$ factors through to a function

$$\bar{u}: F \to \mathbb{T}^n.$$ 

This function can be easily seen to be a crossed homomorphism, that is, $\bar{u}(gh) = g\bar{u}(h) + \bar{u}(g)$ for all $g, h \in F$.

Let $\pi: \mathbb{T}^n \to (1/q)\mathbb{T}^n \overset{\text{def}}{=} \mathbb{R}^n/(1/q)\mathbb{Z}^n$ be the natural projection. The class of $\bar{u}$ in $H^1(F, \mathbb{T}^n)$ vanishes after projecting to $H^1(F, (1/q)\mathbb{T}^n)$. This can be seen as follows. Define

$$u_0 \overset{\text{def}}{=} -\frac{1}{q} \sum_{g \in F} u_g,$$
where \( \hat{g} \) is any lift of \( g \) to \( \Gamma \). Then a direct computation shows that
\[
\left( gu_0 - u_0 \mod \frac{1}{q} \mathbb{Z}^n \right) = \pi(\bar{u}(g))
\]
for any \( g \in F \). This means that \( \pi \circ \bar{u} \) is a principal crossed homomorphism. Now take any \( \gamma \in \Gamma \),
\[
t \circ \Gamma \circ t^{-1}(x) = g_\gamma x - g_\gamma u_0 + u_\gamma + u_0.
\]
But
\[
\left( u_\gamma - g_\gamma u_0 + u_0 \mod \frac{1}{q} \mathbb{Z}^n \right) = \pi(\bar{u}(g_\gamma)) - \left( g_\gamma u_0 - u_0 \mod \frac{1}{q} \mathbb{Z}^n \right) = 0,
\]
and the claim follows.

Therefore, by changing the origin of \( \mathbb{R}^n \), we may assume from now on that \( \Gamma \) is a subgroup of \( ((1/q)\mathbb{Z})^n \rtimes \text{GL}_n(\mathbb{Z}) \).

Fix an integer \( s \equiv 1 \mod q \), \( s > 1 \). Using group cohomology, Epstein and Shub [9] showed that there exists a monomorphism \( \phi: \Gamma \rightarrow \Gamma \) that fits into the commutative diagram
\[
\begin{array}{cccccc}
0 & \rightarrow & \mathbb{Z}^n & \rightarrow & \Gamma & \rightarrow & F & \rightarrow & 1 \\
& & \downarrow{s \cdot \text{Id}} & & \downarrow{\varphi} & & \downarrow{\text{Id}_F} & & \\
0 & \rightarrow & \mathbb{Z}^n & \rightarrow & \Gamma & \rightarrow & F & \rightarrow & 1.
\end{array}
\]
This implies that there exists an expanding map of \( M \) that is covered by an affine map of the form \( x \mapsto s \cdot x + e_0 \). Our goal is to find an expanding map of \( M \) that lifts to an origin-preserving expanding conformal map.

Define a function \( \theta: F \rightarrow (1/q)\mathbb{Z}^n \) by
\[
\theta(g) \overset{\text{def}}{=} s \cdot u_\hat{g} - u_{\varphi(\hat{g})},
\]
where \( \hat{g} \) is any lift of \( g \) to \( \Gamma \).

It is straightforward to check that \( \theta \) is well defined and, actually, a crossed homomorphism. It is also easy to see that \( \theta \), in fact, takes values in \( \mathbb{Z}^n \). Indeed, recall that \( s \equiv 1 \mod q \). Therefore,
\[
\theta(g) \equiv u_\hat{g} - u_{\varphi(\hat{g})} \equiv u_{\hat{g} \circ \varphi(\hat{g})^{-1}} \mod \mathbb{Z}^n.
\]
The latter expression is zero because \( \hat{g} \circ \varphi(\hat{g})^{-1} \) is a pure translation.

Since the inclusion \( \mathbb{Z}^n \hookrightarrow (1/q)\mathbb{Z}^n \) induces multiplication by \( q \) on cohomology and the abelian group \( H^1(F; (1/q)\mathbb{Z}^n) \) has exponent \( q \) (see [4, p. 85, 10.2]), the composite
\[
\theta: F \rightarrow \mathbb{Z}^n \hookrightarrow \frac{1}{q} \mathbb{Z}^n
\]
is a principal crossed homomorphism; that is, there exists a vector \( v \in (1/q)\mathbb{Z}^n \) such that
\[
\theta(g) = gv - v.
\]
In fact, by a direct computation one can check that
\[
v = -\frac{1}{q} \sum_{g \in F} \theta(g).
\]
Now consider the conformal expanding affine transformation \( E \) of \( \mathbb{R}^n \) defined by
\[
E(x) \overset{\text{def}}{=} s \cdot x + v.
\]
CLAIM 4.4. Conjugation by $E$ restricted to $\Gamma$ is $\varphi$; that is,

$$\varphi(\gamma) = E \circ \gamma \circ E^{-1}$$

for each $\gamma \in \Gamma$.

**Proof of Claim 4.4.** We have the following sequence of equalities:

$$E \circ \gamma \circ E^{-1} = (s \cdot x + v) \circ (g_\gamma x + u_\gamma) \circ \left( \frac{1}{s} \cdot x - \frac{v}{s} \right) = g_\gamma x - g_\gamma v + v + s \cdot u_\gamma$$

$$= g_\gamma x - \theta(g_\gamma) + s \cdot u_\gamma = g_\gamma x + s \cdot u_\gamma - (s \cdot u_\gamma - u_{\varphi(\gamma)}) = \varphi(\gamma).$$

Consider the $q$-fold composition

$$E^q(x) = s^q \cdot x + \tilde{v}, \quad \text{where } \tilde{v} \equiv (s^{q-1} + s^{q-2} + \ldots + 1)v.$$

Denote by $\mathcal{E}$ the map of $M$ induced by $E^q$. Since $s^{q-1} + s^{q-2} + \ldots + 1 \equiv 0 \mod q$, we have that $\tilde{v} \in \mathbb{Z}^n$. Hence $\mathcal{E}(\mathbf{o}) = \mathbf{o}$, where $\mathbf{o}$ denotes the image of the new origin of $\mathbb{R}^n$ (after applying Claim 4.3) under the covering projection $\mathbb{R}^n \to M$.

Now let $\tilde{L} : \mathbb{R}^n \to \mathbb{R}^n$ be a lift of $L$. It has the form $x \mapsto Ax + x_0$ with respect to the new origin for $\mathbb{R}^n$ (after applying Claim 4.3), where $x_0 \in \mathbb{R}^n$ and $A \in \text{GL}_n(\mathbb{Z})$ is a hyperbolic matrix. Conjugation by $\tilde{L}$ induces an automorphism $\psi : \Gamma \to \Gamma$ yielding the following commutative diagram:

$$
\begin{array}{cccc}
0 & \longrightarrow & \mathbb{Z}^n & \longrightarrow & \Gamma & \longrightarrow & F & \longrightarrow & 1 \\
\downarrow A & & \downarrow \psi & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}^n & \longrightarrow & \Gamma & \longrightarrow & F & \longrightarrow & 1.
\end{array}
$$

Since $F$ is finite, by replacing $L$ and $\tilde{L}$ by some finite powers, we can assume that the induced automorphism of $F$ is identity. This implies that

$$\forall \gamma \in \Gamma, \quad A \circ g_\gamma = g_\gamma \circ A. \quad \text{(\ast)}$$

Proceeding analogously to how we defined $\theta$ using $\varphi$, we define $\omega : F \to (1/q)\mathbb{Z}^n$ by

$$\omega(g) \overset{\text{def}}{=} Au_\gamma - u_{\psi(g)}.$$

Again, it is straightforward to check that $\omega$ is well defined and, due to (\ast), a crossed homomorphism.

Let $\tilde{A}$ be the image of $A$ in the finite group $\text{GL}_n(\mathbb{Z}_q)$. By replacing $L$ (and $\tilde{L}$ accordingly) by a further positive power, we may also assume that $\tilde{A} = \text{Id}$. We can check that the image of $\omega$ is contained in $\mathbb{Z}^n$. Indeed, write

$$Au_\gamma - u_{\psi(\gamma)} = (Au_\gamma - u_\gamma) + (u_\gamma - u_{\psi(\gamma)}).$$

The first summand is in $\mathbb{Z}^n$ because $u_\gamma \in (1/q)\mathbb{Z}^n$ and $\tilde{A} = \text{Id}$. The second summand equals $u_{\gamma \circ \psi(\gamma)^{-1}}$ and belongs to $\mathbb{Z}^n$ since $\gamma \circ \psi(\gamma)^{-1}$ is a pure translation.

Hence, just as for $\theta$, we get that

$$\forall g \in F, \quad \omega(g) = gw - w,$$

where $w \in (1/q)\mathbb{Z}^n$.

Now consider the affine Anosov diffeomorphism $\mathbb{A}$ of $\mathbb{R}^n$ defined by

$$\mathbb{A}(x) \overset{\text{def}}{=} Ax + w.$$
CLAIM 4.5. Conjugation by $A$ restricted to $\Gamma$ is $\psi$, that is,

$$\psi(\gamma) = A \circ \gamma \circ A^{-1}$$

for each $\gamma \in \Gamma$.

Proof of Claim 4.5. The claim follows from the computation below that uses the commutativity property (*).

$$A \circ \gamma \circ A^{-1} = (Ax + w) \circ (g_\gamma x + u_\gamma) \circ (A^{-1}x - A^{-1}w)$$

$$= A \circ g_\gamma \circ A^{-1}x - A \circ g_\gamma \circ A^{-1}w + Au_\gamma + w = g_\gamma x - g_\gamma w + w + Au_\gamma$$

$$= g_\gamma x - \omega(g) + Au_\gamma = g_{\psi(\gamma)}x - (Au_\gamma - u_{\psi(\gamma)}) + Au_\gamma = \psi(\gamma).$$

Consider the $q$-fold composition

$$A^q(x) = A^q x + \hat{w}, \text{ where } \hat{w} \overset{\text{def}}{=} A^{q-1}w + A^{q-2}w + \ldots + w.$$ 

Denote by $A$ the Anosov diffeomorphism of $M$ induced by $A^q$. Since $\bar{A} = \text{Id}$ we have that $Au \equiv u \mod \mathbb{Z}^n$ for any $u \in (1/q)\mathbb{Z}^n$. Thus, $\hat{w} \equiv 0 \mod \mathbb{Z}^n$; that is, $\hat{w} \in \mathbb{Z}^n$ and hence $A(o) = o$. (Recall that $o \in M$ is the image of $0 \in \mathbb{R}^n$.)

Therefore, the conformal expanding endomorphism $\mathcal{E}: M \to M$ and the Anosov diffeomorphism $A: M \to M$ have a common fixed point $o$, and hence commute.

CLAIM 4.6. The diffeomorphism $A$ is affinely conjugate to a positive power of $L$.

Proof of Claim 4.6. It is clearly sufficient to find a vector $a \in \mathbb{R}^n$ such that the translation $T: x \mapsto x + a$ has the following properties:

1. $T \circ A^q \circ T^{-1} = \tilde{L}^q$;
2. $T \circ \gamma = \gamma \circ T$ for every $\gamma \in \Gamma$.

Define vector $\hat{u}$ by the equation

$$\tilde{L}^q x = A^q x + \hat{u}.$$ 

Set $a = (\text{Id} - A^q)^{-1}(\hat{u} - \hat{w})$. (Recall that $A$ has no eigenvalues of length 1.) A straightforward calculation verifies the first property. Another straightforward calculation shows that the second property is equivalent to

$$\forall \gamma \in \Gamma, \ g_\gamma a = a.$$ 

Using (*), this is equivalent to

$$\forall \gamma \in \Gamma, \ g_\gamma(\hat{u} - \hat{w}) = \hat{u} - \hat{w}.$$ 

To verify this, we use that

$$\tilde{L}^q \circ \gamma = \psi^q(\gamma) \circ \tilde{L}^q \text{ and } A^q \circ \gamma = \psi^q(\gamma) \circ A^q$$

for all $\gamma \in \Gamma$. Expanding these equations yields

$$\hat{u} + A^q u_\gamma = g_{\psi^q(\gamma)} \hat{u} + u_{\psi^q(\gamma)} \text{ and } \hat{w} + A^q u_\gamma = g_{\psi^q(\gamma)} \hat{w} + u_{\psi^q(\gamma)}$$

for all $\gamma \in \Gamma$. And by subtracting the second equation from the first yields

$$\hat{u} - \hat{w} = g_{\psi^q(\gamma)}(\hat{u} - \hat{w}) = g_\gamma(\hat{u} - \hat{w}).$$


Denote by $T$ the conjugacy between $A$ and $L^q$ induced by $T$. To complete the proof, we expand $\mathcal{E} \circ A = A \circ \mathcal{E}$ as

$$
\mathcal{E} \circ T^{-1} \circ L^q \circ T = T^{-1} \circ L^q \circ T \circ \mathcal{E},
$$

which means that $L^q$ commutes with the expanding endomorphism $T \circ \mathcal{E} \circ T^{-1}$.

**Remark 4.7.** In the course of the proof of Proposition 4.2, we have passed to a positive finite power of $L$ twice (without changing the notation for $L$) to guarantee that the induced homomorphism of $F$ is identity and $\hat{A} \in \text{GL}_n(\mathbb{Z}_q)$ is the identity matrix. Therefore, the actual power of the initial Anosov diffeomorphism that commutes with the constructed expanding endomorphism may be larger than $q$. Note also that if we change for a second time the origin of $\mathbb{R}^n$ to a point lying over $T(o)$, then $T \circ \mathcal{E} \circ T^{-1}$ and $L^q$ are an expanding endomorphism and an Anosov automorphism, respectively.

**4.3.2. The construction of an Anosov automorphism of an infratorus with holonomy group $F$.** Here we present a construction of an Anosov automorphism of $T^s$. See [11, Lemma 1.1] where such a matrix $A_1$ is constructed. Then the $m$-fold product

$$
A_m \defeq \text{Id} \otimes A_1 : \mathbb{Z}^m \otimes \mathbb{Z}^s \rightarrow \mathbb{Z}^m \otimes \mathbb{Z}^s
$$

is an Anosov matrix $A_m \in \text{GL}_{ms}(\mathbb{Z})$ representing a codimension $m$ Anosov automorphism of $T^{ms}$.

The group $F$ acts faithfully on $\mathbb{Z}^m \otimes \mathbb{Z}^s$ by $g \otimes \text{Id}$ for each $g \in F$ and $A_m$ obviously commutes with this action.

The matrix $A_m$ induces an automorphism

$$(A_m)_* : H^2(F, \mathbb{Z}^m \otimes \mathbb{Z}^s) \rightarrow H^2(F, \mathbb{Z}^m \otimes \mathbb{Z}^s),$$

which has a finite order since $H^2(F, \mathbb{Z}^m \otimes \mathbb{Z}^s)$ is a finite group. Therefore, some positive power of $(A_m)_*$ is identity. Hence, after replacing $A_1$ by some positive power, we may assume that $(A_m)_* = \text{Id}$.

Now consider an $s$-fold product $\Gamma^s = \Gamma \times \Gamma \times \ldots \times \Gamma$. It fits into an exact sequence

$$
0 \rightarrow (\mathbb{Z}^m)^s \rightarrow \Gamma^s \rightarrow F^s \rightarrow 1.
$$

Let $\Gamma^{(s)} = p^{-1}(F)$, where $F$ is identified with the diagonal subgroup of $F^s$.

Then we have a short exact sequence

$$
0 \rightarrow \mathbb{Z}^{ms} \rightarrow \Gamma^{(s)} \rightarrow F \rightarrow 1,
$$

where $\Gamma^{(s)}$ is torsion-free and the diagonal action of $F$ on $(\mathbb{Z}^m)^s$ is faithful. Also note that $\mathbb{Z}^{ms}$ can be identified with $\mathbb{Z}^m \otimes \mathbb{Z}^s$ in such a way that the diagonal action of $F$ becomes the action described above on $\mathbb{Z}^m \otimes \mathbb{Z}^s$. Recall that the extension $\Gamma^{(s)}$ determines an element $\theta \in H^2(F, \mathbb{Z}^s)$. And, since $(A_m)_*(\theta) = \theta$, there is an automorphism $A : \Gamma^{(s)} \rightarrow \Gamma^{(s)}$ such that
the following diagram commutes:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathbb{Z}^m \otimes \mathbb{Z}^s & \longrightarrow & \Gamma^{(s)} & \longrightarrow & F & \longrightarrow & 1 \\
\downarrow A_m & & \downarrow A & & \downarrow \text{Id}_F & & \\
0 & \longrightarrow & \mathbb{Z}^m \otimes \mathbb{Z}^s & \longrightarrow & \Gamma^{(s)} & \longrightarrow & F & \longrightarrow & 1
\end{array}
\]

(see [4, p. 94]).

By the Bieberbach theorems $\Gamma^{(s)} = \pi_1(M_F)$, where $M_F$ is an $ms$-dimensional infratorus with holonomy group $F$. Furthermore, $A$ is induced by a codimension $m$ affine Anosov diffeomorphism of $M_F$, which we denote by $A$ as well. After changing, if needed, the origin $0$ of the affine space $\mathbb{R}^n$, $A(0) = 0$. Hence, $A$ induces an Anosov automorphism of $M_F$.

Note that if $s$ is even, then $\Gamma^{(s)}$ acts by orientation-preserving transformations. Therefore, $M_F$ is orientable when $s$ is even.

4.3.3. The proof of Proposition 4.1. It follows from the proof of Farrell and Ontaneda [13, Theorem 3”] that, given $k, u \in \mathbb{Z}^+$, there exists $d(k, u)$ such that, for all integers $d > d(k, u)$, there is an element in the Gromoll group $\Gamma_{k+1}^{4d+3}$ of an odd order larger than $u$.

Take $u = q$ and apply the construction of the previous subsection with $sm = s(k - 1) > 4d(k, u)$ and $s$ even. This gives a codimension $k - 1$ Anosov automorphism $A: M_F \rightarrow M_F$. Let $L_1: \mathbb{T}^\sigma \rightarrow \mathbb{T}^\sigma$ be a codimension 1 Anosov automorphism. Choose $\sigma \in \{2, 3, 4, 5\}$ so that $s(k - 1) + \sigma \equiv 3 \mod 4$.

Let $M = M_F \times \mathbb{T}^\sigma$ and $L = A \times L_1$. Then $L$ is a codimension $k$ Anosov automorphism of the orientable infranilmanifold $M$ with holonomy group $F$. By the construction, $\Gamma_{k+1}^{\dim M}$ has an element $\Sigma$ of order larger than $q$.

Let $N = M \# \Sigma$; then $N$ is not diffeomorphic to $M$ as well as to any other infranilmanifold by Proposition 1.4. By Theorem 1.1 and Proposition 4.2, $N$ supports a codimension $k$ Anosov diffeomorphism; cf. Remark 4.7. Finally, Proposition 1.3 yields that $N$ is irreducible.

Appendix: Irreducible smooth structures

This additional section is devoted to proving the following result.

THEOREM A.1. Let $M$ be an $n$-dimensional closed oriented infranilmanifold, $n \geq 7$, and $\Sigma$ be an exotic $n$-sphere. If $M \# \Sigma = N_1 \times N_2$, that is, is a smooth Cartesian product, where $\dim N_i \geq 1$, $i = 1, 2$, then $M \# \Sigma$ is diffeomorphic to $M$.

Before proving this result, we need some pertinent facts from smoothing theory which can be found in [19].

Let $Y$ be a smooth $n$-dimensional manifold, where $n \geq 5$. A smooth structure on $Y$ is a homeomorphism $\varphi: X \rightarrow Y$, where $X$ is a smooth manifold. Two such structures

$\varphi_i: X_i \rightarrow Y, \quad i = 1, 2$

are concordant, if there exist a smooth manifold $W$ and a homeomorphism

$\Phi: W \rightarrow Y \times [0, 1],$

such that $\partial W = X_1 \sqcup X_2$ and $\Phi|_{X_i} = \varphi_i$.

Let $[\varphi]$ denote the concordance class of $\varphi$ and $S(Y)$ be the set of all such classes. Then $S(Y)$ is in natural bijective correspondence with $[Y, \text{Top}/O]$, where $\text{Top}/O$ is an infinite loop space and $[Y, \text{Top}/O]$ denotes the set of all homotopy classes of continuous maps from $Y$ to
Top/O. Note that \([Y, \text{Top/O}]\) is an abelian group. In this way, \(\mathcal{S}(Y)\) acquires an abelian group structure. Given a smooth structure \(\phi: X \to Y\), we let \(\hat{\phi}: Y \to \text{Top/O}\) denote a representative of the corresponding homotopy class. Here are some properties of this correspondence.

1. The map \(\hat{\text{Id}}_Y\) is homotopic to a constant map and \([\text{Id}_Y] = 0\).
2. Let \(\sigma: \Sigma \to S^n\) be an exotic sphere and \(\sigma_Y: Y \# \Sigma \to Y\) be the usual homeomorphism; then \([\hat{\sigma}] = [\hat{\sigma} \circ f_Y]\), where \(f_Y: Y \to S^n\) is a degree 1 map.
3. Let \(\alpha: U \to Y\) denote the inclusion of an open subset \(U \subset Y\); then \([\hat{\phi} \circ \alpha] = [\hat{\phi}_U]\), where \(\varphi_U: \varphi^{-1}(U) \to U\) is the restriction of \(\varphi\) to \(\varphi^{-1}(U)\).
4. Product Structure Theorem. The homeomorphism \(\varphi \times \text{Id}_{\mathbb{R}^m}: X \times \mathbb{R}^m \to Y \times \mathbb{R}^m\) is a smooth structure on \(Y \times \mathbb{R}^m\), and the map \([\varphi]\) in \([\varphi \times \text{Id}_{\mathbb{R}^m}]\) is a bijection of smooth structure sets \(\mathcal{S}(Y) \to \mathcal{S}(Y \times \mathbb{R}^m)\). Furthermore, \([\varphi \times \text{Id}_{\mathbb{R}^m}] = p^*[\hat{\phi}]\), where \(p: Y \times \mathbb{R}^m \to Y\) denotes projection onto the first factor.

We now start the proof of Theorem A.1. Since \(\pi_1(M) = \pi_1(N_1) \times \pi_1(N_2)\) and \(\pi_1(M)\) is a torsion-free, finitely generated and virtually nilpotent group, it follows that \(\pi_1(N_1)\) and \(\pi_1(N_2)\) are also torsion-free, finitely generated and virtually nilpotent groups. By Mal’cev’s work [21] (cf. [24, p. 231]), any such group is the fundamental group of a closed infranilmanifold. Hence, there exist closed infranilmanifolds \(M_1\) and \(M_2\) with \(\pi_1(M_i) = \pi_1(N_i), i = 1, 2\). Note that \(N_i, i = 1, 2\), are aspherical. Therefore, using [10, Theorem 6.3], we obtain homeomorphisms \(f_i: N_i \to M_i\). ([10, Theorem 5.1] (of which Theorem 6.3 is a corollary) was extended to dimension 4 by Freedman and Quinn [14, Section 11.5] and follows from results of Perelman (see, for example, [5]) in dimension 3.) Note that \(M_1\) and \(M_1 \times M_2\) are both closed infranilmanifolds with a specified isomorphism between their fundamental groups. Now the smooth rigidity result of Lee and Raymond [20] yields a diffeomorphism \(M_1 \times M_2 \to M\), which makes the following diagram commute up to homotopy:

\[
\begin{array}{ccc}
M \# \Sigma & \overset{\sigma_M}{\longrightarrow} & M \\
\downarrow & & \downarrow \\
N_1 \times N_2 & \overset{f_1 \times f_2}{\longrightarrow} & M_1 \times M_2.
\end{array}
\]

We also orient \(N_i\) and \(M_i, i = 1, 2\), so that all pertinent maps are orientation-preserving and identify \(M_1 \times M_2\) with \(M\) by the vertical diffeomorphism in the diagram. Using [10] again, we see that \(\sigma_M\) is concordant to \(f_1 \times f_2\); that is,

\([-\sigma_M] = [f_1 \times f_2] \quad \text{in} \quad \mathcal{S}(M)\).

We now complete the proof under the additional assumption that \(\dim N_i \geq 5\) for \(i = 1\) and \(i = 2\). After doing this, we will indicate the modifications needed to prove Theorem A.1 when this assumption is dropped.

Identify \(\mathbb{R}^s\) with an open ball in \(M_2\), where \(s = \dim M_2\). We intend to apply property (3) to the inclusion

\(\alpha: U \overset{\text{def}}{=} M_1 \times \mathbb{R}^s \longrightarrow M_1 \times M_2 = M\),

and the smooth structure

\(\varphi \overset{\text{def}}{=} f_1 \times f_2: N_1 \times N_2 \longrightarrow M_1 \times M_2\).

Note, in this situation, that

\(\varphi_U: \varphi^{-1}(U) \longrightarrow U\)

is the same as

\(f_1 \times (f_2|_V): N_1 \times V \longrightarrow M_1 \times \mathbb{R}^s\),
where $V = f_2^{-1}(\mathbb{R}^n)$. Moreover, the Product Structure Theorem (property (4)) yields that

$$[f_1] = 0 \quad \text{if and only if} \quad [f_1 \times f_2|_V] = 0.$$  

To see that $[f_1 \times f_2|_V] = 0$, we recall that $f_1 \times f_2$ is concordant to $\sigma_M$. This fact together with property (2) yields that

$$[f_1 \times f_2] = [\hat{\sigma}_M] = [\hat{\sigma} \circ f_M] \quad \text{in } [M, \text{Top/O}].$$

Therefore, property (3) yields that

$$[f_1 \times f_2] = [f_1 \times f_2 \circ \alpha] = [\hat{\sigma} \circ f_M \circ \alpha] \quad \text{in } [M_1 \times V, \text{Top/O}].$$

But $f_M \circ \alpha : M_1 \times V \to S^n$ is homotopic to a constant map since $V$ is homotopic to a point and dim $M_1 < n$. Therefore, $[f_1 \times f_2] = 0$ and consequently $[f_1] = 0$. Property (1) implies that $f_1$ is homotopic to a diffeomorphism $f_1 : N_1 \to M_1$. And a completely analogous argument shows that $f_2$ is also homotopic to a diffeomorphism $f_2 : N_2 \to M_2$. Consequently, $N_1 \times N_2 = M_1 \# \Sigma$ is diffeomorphic to $M_1 \times M_2 = M$ which is the posited result.

We finish by briefly indicating how to modify the above argument to complete the proof in general, that is, after dropping the assumption that dim $N_i \geq 5$, $i = 1, 2$.

For this purpose, consider the smooth structure

$$(M \# \Sigma) \times \mathbb{R}^{10} \xrightarrow{\sigma_M \times \text{Id}_{\mathbb{R}^{10}}} M \times \mathbb{R}^{10}.$$  

Because of the Product Structure Theorem, it suffices to show that $[\sigma_M \times \text{Id}_{\mathbb{R}^{10}}] = 0$ in $S(M \times \mathbb{R}^{10})$. This is accomplished by showing that

$$[f_i \times \text{Id}_{\mathbb{R}^{5}}] = 0 \quad \text{in } S(M_i \times \mathbb{R}^{5}), \quad i = 1, 2,$$

which is proved by an argument similar to the one given above, which verified that $[f_i] = 0$ in $S(M_i)$ when dim $M_i \geq 5$, $i = 1, 2$.

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