Differential geometry/Mathematical economics

Some characterizations of the quasi-sum production models with proportional marginal rate of substitution

_Certaines caractérisations des modèles de production quasi-somme avec un taux marginal de substitution proportionnelle_

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1. Introduction

The notion of production function is a key concept in both macroeconomics and microeconomics, being used in the mathematical modeling of the relationship between the output of a firm, an industry, or an entire economy, and the inputs that have been used in obtaining it. Generally, production function is a twice differentiable mapping \( f : \mathbb{R}_{+}^n \rightarrow \mathbb{R}_{+} \), \( f = f(x_1, \ldots, x_n) \), where \( f \) is the quantity of output, \( n \) is the number of the inputs and \( x_1, \ldots, x_n \) are the factor inputs. A production function \( f \) is called quasi-sum \([3,5]\) if there are strict monotone functions \( G, h_1, \ldots, h_n \) with \( G' > 0 \) such that

\[
    f(x) = G(h_1(x_1) + \ldots + h_n(x_n)),
\]

where \( x = (x_1, \ldots, x_n) \in \mathbb{R}_{+}^n \). We note that these functions are of great interest because they appear as solutions to the general bisymmetry equation, being related to the problem of consistent aggregation \([1]\).

Among the family of production functions, the most famous is the so-called Cobb–Douglas production function. A generalized Cobb–Douglas production function depending on \( n \)-inputs is given by
Theorem

In function where and of \[7\].

The function \[7\] is strictly increasing and \(h\) is a homogeneous function of any given degree \(p\), is said to be a homothetic production function [7]. It is easy to see that a production function \(f\) can be identified with the graph of \(f\), i.e. the nonparametric hypersurface of \(\mathbb{R}^{n+1}\) defined by

\[
L(x_1, \ldots, x_n) = (x_1, \ldots, x_n, f(x_1, \ldots, x_n))
\]

and called the production hypersurface of \(f\) (see [9,11]). Motivated by some recent classification results concerning production hypersurfaces [2,5,7,8,12], in the present work we classify quasi-sum production functions with a proportional marginal rate of substitution and investigate the existence of such production models whose production hypersurfaces have null Gauss–Kronecker curvature or null mean curvature. We recall that, if \(f\) is a production function with \(n\) inputs \(x_1, x_2, \ldots, x_n\), \(n \geq 2\), the elasticity of production with respect to a certain factor of production \(x_i\) is defined as

\[
E_{x_i} = \frac{x_i}{f} f_{x_i}
\]

and the marginal rate of technical substitution of input \(x_j\) for input \(x_i\) is given by

\[
\text{MRS}_{ij} = \frac{f_{x_j}}{f_{x_i}},
\]

where the subscripts denote partial derivatives of the function \(f\) with respect to the corresponding variables. A production function satisfies the proportional marginal rate of substitution property if

\[
\text{MRS}_{ij} = \frac{x_i}{x_j}, \text{ for all } 1 \leq i \neq j \leq n.
\]

In the last section of the paper we will prove the following theorem that generalizes the results from [10].

**Theorem 1.1.** Let \(f\) be a quasi-sum production function given by (1). Then:

i. The elasticity of production is a constant \(k_i\) with respect to a certain factor of production \(x_i\) if and only if \(f\) reduces to

\[
f(x_1, \ldots, x_n) = A \cdot x_i^{k_i} \cdot \exp \left( D \sum_{j \neq i} h_j(x_j) \right),
\]

where \(A\) and \(D\) are positive constants.

ii. The elasticity of production is a constant \(k_i\) with respect to all factors of production \(x_i, i = 1, \ldots, n\), if and only if \(f\) reduces to the generalized Cobb–Douglas production function given by (2).

iii. The production function satisfies the proportional marginal rate of substitution property if and only if it reduces to the homothetic generalized Cobb–Douglas production function given by

\[
f(x_1, \ldots, x_n) = F \left( \prod_{i=1}^{n} x_i^k \right),
\]

where \(k\) is a nonzero real number.

iv. If the production function satisfies the proportional marginal rate of substitution property, then:

iv1. The production hypersurface has vanishing Gauss–Kronecker curvature if and only if, up to a suitable translation, \(f\) reduces to the following generalized Cobb–Douglas production function with constant return to scale:

\[
f(x_1, \ldots, x_n) = A \cdot \prod_{i=1}^{n} x_i^{\frac{1}{2}}.
\]

iv2. The production hypersurface cannot be minimal.

iv3. The production hypersurface has vanishing sectional curvature if and only if, up to a suitable translation, \(f\) reduces to the following generalized Cobb–Douglas production function:

\[
f(x_1, \ldots, x_n) = A \cdot \prod_{i=1}^{n} \sqrt{x_i}.
\]
2. Preliminaries on the geometry of hypersurfaces

For general references on the geometry of hypersurfaces, we refer to [4].

If \( M \) is a hypersurface of the Euclidean space \( \mathbb{R}^{n+1} \), then it is known that the Gauss map \( v : M \to S^n \) maps \( M \) to the unit hypersphere \( S^n \) of \( \mathbb{R}^{n+1} \). With the help of the differential \( dv \) of \( v \) it can be defined a linear operator on the tangent space \( T_p M \), denoted by \( S_p \) and known as the shape operator, by \( g(S_p v, w) = g(dv(v), w) \), for \( v, w \in T_p M \), where \( g \) is the metric tensor on \( M \) induced from the Euclidean metric on \( \mathbb{R}^{n+1} \). The eigenvalues of the shape operator are called principal curvatures. The determinant of the shape operator \( S_p \), denoted by \( K(p) \), is called the Gauss–Kronecker curvature. When \( n = 2 \), the Gauss–Kronecker curvature is simply called the Gauss curvature, which is intrinsic due to famous Gauss’s Theorem Egregium. The trace of the shape operator \( S_p \) is called the mean curvature of the hypersurfaces. In contrast to the Gauss–Kronecker curvature, the mean curvature is extrinsic, which depends on the immersion of the hypersurface. A hypersurface is said to be \textit{minimal} if its mean curvature vanishes identically. We recall now the following lemma which will be used in the proof of Theorem 1.1.

**Lemma 2.1.** (See [4].) For the production hypersurfaces defined by (3) and \( w = \sqrt{1 + \sum_{i=1}^{n} f_{i}^{2}} \), we have:

i. The Gauss–Kronecker curvature \( K \) is given by

\[
K = \frac{\text{det}(f_{i,k,l})}{w^{n+2}}. 
\]

(11)

ii. The mean curvature \( H \) is given by

\[
H = \frac{1}{n} \sum_{i=1}^{n} \frac{\partial}{\partial x_{i}} \left( \frac{f_{i}}{w} \right). 
\]

(12)

iii. The sectional curvature \( K_{ij} \) of the plane section spanned by \( \frac{\partial}{\partial x_{i}}, \frac{\partial}{\partial x_{j}} \) is

\[
K_{ij} = \frac{f_{i,k,l} f_{j,k,l} - f_{i,k,l}^{2}}{w^{2} \left( 1 + f_{i}^{2} + f_{j}^{2} \right)}. 
\]

(13)

3. Proof of Theorem 1.1

Let \( f \) be a quasi-sum production function given by (1). Then we have

\[
f_{i}(x) = G'(u)h_i'(x_i)
\]

(14)

with \( u = h_1(x_1) + \ldots + h_n(x_n) \) and from (14) we derive

\[
f_{i,k,l} = G''(h_i)h_i'^{2} + G'h_i'', \quad i = 1, \ldots, n, 
\]

(15)

\[
f_{i,k,l} = G''h_i'\delta_{ij}, \quad i \neq j.
\]

(16)

i. We first prove the left-to-right implication. If the elasticity of production is a constant \( k_i \) with respect to a certain factor of production \( x_i \), then from (4) we obtain

\[
f_{i} = k_i \frac{f}{x_i}.
\]

(17)

Using now (1) and (14) in (17) we get

\[
\frac{G'}{G} = k_i \frac{1}{x_i h_i'}.
\]

(18)

By taking the partial derivative of (18) with respect to \( x_j, \ j \neq i \), we obtain

\[
h_j \frac{G''G - (G')^{2}}{G^{2}} = 0.
\]

Now, taking into account that \( h_j \) is a strict monotone function, we find

\[
G(u) = C \cdot e^{Du},
\]

(19)
for some positive constants $C$ and $D$. Hence from (18) and (19) we obtain
\[
h_i(x_i) = \frac{k_i}{D} \ln x_i + A_i, \tag{20}
\]
where $A_i$ is a real constant. Finally, combining (1), (19) and (20) we get a function of the form (7), where $A = Ce^{D\cdot A_i}$. The converse can be verified easily by direct computation.

ii. The assertion is an immediate consequence of i.

iii. Assume first that $f$ satisfies the proportional marginal rate of substitution property. Then from (5), (6) and (14) we derive $x_i h'_i = x_j h'_j, \forall i \neq j$. Hence we conclude that there exists a nonzero real number $k$ such that: $x_i h'_i = k, i = 1, \ldots, n$, and therefore we obtain
\[
h_i(x_i) = k \ln x_i + C_i, \quad i = 1, \ldots, n, \tag{21}
\]
for some real constants $C_1, \ldots, C_n$. Now, from (1) and (21) we derive
\[
f(x) = G \left( k \sum_{i=1}^{n} \ln x_i + \vec{A} \right),
\]
where $\vec{A} = \sum_{i=1}^{n} C_i$ and hence we find
\[
f(x) = (G \circ \ln) \left( A \cdot \prod_{i=1}^{n} x_i^j \right), \tag{22}
\]
where $A = e^\vec{A}$. Therefore we get a production function of the form (8), where $F(u) = (G \circ \ln)(A \cdot u)$.

The converse is easy to verify.

iv1. We first prove the left-to-right implication. If the production hypersurface has null Gauss–Kronecker curvature, then from (11) we get
\[
\det(f_{x_i x_i}) = 0. \tag{23}
\]
On the other hand, the determinant of the Hessian matrix of $f$ is given by [6]
\[
\det(f_{x_i x_i}) = (G')^n \prod_{i=1}^{n} h_i'' + (G')^{n-1} G'' \sum_{i=1}^{n} h_i'' + \cdots + h_i'' h_i'' \cdots h_n''. \tag{24}
\]
By using (21), (23) and (24), we obtain
\[
(-1)^n (G')^{n-1} k^n (G' - k n G'') = 0.
\]
But $G' > 0$ and $k \neq 0$ and hence we derive
\[
\frac{G''}{G'} = \frac{1}{kn} \tag{25}
\]
After solving (25) we find
\[
G(u) = C \cdot n k e^{\frac{u}{kn}} + D \tag{26}
\]
for some constants $C, D$ with $C > 0$. Combining (22) and (26), after a suitable translation, we conclude that the function $f$ reduces to the form (9). The converse follows easily by direct computation.

iv2. Let us assume that the production hypersurface is minimal. Then we have $H = 0$ and from (12) we derive
\[
\sum_{i=1}^{n} f_{x_i x_i} \left( 1 + \sum_{i=1}^{n} f_{x_i}^2 \right) - \sum_{i,j=1}^{n} f_{x_i x_j} f_{x_i x_j} = 0
\]
which reduces to
\[
\sum_{i=1}^{n} f_{x_i x_i} + \sum_{i \neq j} \left( f_{x_i x_j}^2 - f_{x_i x_j} f_{x_i x_j} \right) = 0. \tag{27}
\]
By introducing (14), (15) and (16) in (27), we get
\[
G'' \sum_{i=1}^{n} (h_i)^2 + G' \sum_{i=1}^{n} h_i'' + (G')^3 \sum_{i \neq j} (h_i)^2 h_j'' = 0. \tag{28}
\]

By using now (21) in (28) and taking into account that \( k \neq 0 \), we obtain
\[
(kG'' - G') \sum_{i=1}^{n} \frac{1}{x_i^2} - k^2 (G')^3 \sum_{i \neq j} \frac{1}{x_i^2 x_j^2} = 0. \tag{29}
\]

But the only solution to the equation (29) is \( G(u) = \text{constant} \), which is a contradiction because \( G' > 0 \). Hence the production hypersurface cannot be minimal.

iv). Assume first that the production hypersurface has \( K_{ij} = 0 \). Then from (13) we get
\[
f_{x_k x_l} f_{x_l x_j} - f_{x_k x_j}^2 = 0. \tag{30}
\]

By introducing (14), (15) and (16) into (30), since \( G' \neq 0 \), we obtain
\[
[(h_i)^2 h_j'' + (h_j)^2 h_i'']G'' + h_i'h_j'G' = 0. \tag{31}
\]

By using now (21) in (31) and taking into account that \( k \neq 0 \), we obtain
\[
\frac{G''}{G'} = \frac{1}{2k}. \tag{32}
\]

After solving (32) we get
\[
G(u) = 2 k C e^{\frac{n}{2}} + D \tag{33}
\]
for some constants \( C, D \) with \( C > 0 \). Finally, combining (22) and (33), after a suitable translation, we conclude that the function \( f \) reduces to the Cobb–Douglas production function given by (10). The converse is easy to verify by direct computation.

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