Interval-Dismantling for Lattices

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Abstract. Dismantling allows for the removal of elements of a set, or in our case lattice, without disturbing the remaining structure. In this paper we have extended the notion of dismantling by single elements to the dismantling by intervals in a lattice. We utilize theory from Formal Concept Analysis (FCA) to show that lattices dismantled by intervals correspond to closed subrelations in the respective formal context, and that there exists a unique kernel with respect to dismantling by intervals. Furthermore, we show that dismantling intervals can be identified directly in the formal context utilizing a characterization via arrow relations and provide an algorithm to compute all dismantling intervals.

Keywords: Formal Concept Analysis · Concept Lattice · Dismantling Intervals · Arrow Relations · Closed Subrelations

1 Introduction

The dismantling of elements in ordered sets, in particular that of irreducible elements, was examined for example in \cite{1,3,6,8,9}. Notably, in \cite{3} Duffus and Rival examined the dismantling of doubly irreducible elements and proved the uniqueness of the DI-core, i.e., the poset that remains when all doubly irreducible elements have been removed. In \cite{4} Farley gave a simpler proof relying on results for semimodular posets.

In Formal Concept Analysis (FCA) the removal of a doubly irreducible element in a concept lattice corresponds to the removal of a single incidence (cross) from its (clarified) formal context, cf. \cite[Prop. 53]{5}. The dismantling of a doubly irreducible element results in a complete sublattice of the original concept lattice. In particular, this sublattice contains all but one of the original concepts.

In this paper, we extend the notion of dismantling from single elements to intervals in order to remove multiple (not necessarily irreducible) concepts at once while preserving the remaining concept lattice. To this end, we make use of the one-to-one correspondence between closed subrelations of a formal context and the complete sublattices of its concept lattice \cite{10}, and, more generally, of the one-to-one correspondence between closed subcontexts of a context and the sublattices of its concept lattice \cite{7}.

Extending dismantling to intervals, we call an interval $[u,v] =: S$ dismantling for a lattice $L$ if $v$ is infimum prime in the filter of $u$, $u$ is supremum prime in
the ideal of \( v \) and \( u, v \not\in \{ \bot, \top \} \). Because infimum (supremum) prime implies infimum (supremum) irreducible, dismantling intervals that consist of a single element are precisely the doubly irreducible elements. We show that an interval \( S \) is dismantling for \( L \) if and only if the incidences of all concepts not in \( S \) form a closed subrelation. Furthermore, we show that the core obtained by the iterative removal of dismantling intervals for a lattice is unique. Where possible, we use the more general notion of an interval \( S \) being quasi-dismantling for a lattice, which allows for \( u, v \in \{ \bot, \top \} \) and show that an interval \( S \) is quasi-dismantling for \( L \) if and only if the objects, attributes and incidences of all concepts not in \( S \) form a closed subcontext.

Finally, we give a characterization of dismantling intervals via arrow relations on the context side and provide an algorithm to determine whether an interval is dismantling using this characterization. Furthermore, the arrow relations provide a way to compute all dismantling intervals for a given context \( K \) without having to compute the concept lattice \( \mathbf{A}(K) \) itself.

This paper is structured as follows: In Section 2 we recollect all required notions from order theory and FCA. In Section 3 we introduce dismantling intervals, develop their theory and prove the one-to-one correspondence between dismantling intervals and closed subrelations. In Section 4 we characterize dismantling intervals via arrow relations on the context side and give an algorithm to compute all dismantling intervals. In Section 5 we give a brief conclusion.

2 Basics

In the following we recall some notions from order theory, cf. \([2]\), and formal concept analysis, cf. \([5]\), and introduce some notations used in this work.

An ordered set (or short order) is a tuple \( L := (L, \leq) \) consisting of a set \( L \) and a reflexive, transitive and antisymmetric relation \( \leq \subseteq L \times L \). A subset \( S \) of an order \( L \) together with the same order relation restricted to \( S \), i.e., \( S := (S, \leq \cap S^2) \), is called suborder of \( L \). We denote this by \( S \subseteq L \). Specific suborders are generated by a single element \( c \in L \): The ideal of \( c \) is defined as \( \langle c \rangle := \{ x \in L \mid x \leq c \} \). The filter of \( c \) is defined as \( \langle c \rangle := \{ x \in L \mid c \leq x \} \). For two elements \( c, d \in L \) with \( c \leq d \) the interval between \( c \) and \( d \) is given by \( [c,d] := \{ x \in L \mid c \leq x \leq d \} \). Further, we define the order \( \bigcap S := (L \setminus S, (L \setminus S)^2 \cap \leq) \). An element \( c \in L \) of an ordered set is called upper bound for a subset \( T \subseteq L \) if \( c \geq x \) for all \( x \in T \). If \( c \) is the unique smallest upper bound of \( T \), we call \( c \) the supremum of \( T \). Analogous, an element \( c \in L \) of an ordered set is called lower bound for a subset \( T \subseteq L \) if \( c \leq x \) for all \( x \in T \). If \( c \) is the unique greatest lower bound of \( T \), we call \( c \) the infimum of \( T \).

An ordered set \( L \) is called a lattice if for any two elements \( c, d \in L \) there is an infimum \( c \land d \) and a supremum \( c \lor d \) in \( L \). It is called a complete lattice if all subsets \( X \subseteq L \) have an infimum \( \bigwedge X \) and a supremum \( \bigvee X \) in \( L \). If \( L \) is a lattice, we call \( S \) a sublattice of \( L \) if \( \{ a, b \in S \Rightarrow a \lor b \in S \text{ and } a \land b \in S \} \) holds. We call \( S \) a complete sublattice if for every subset \( T \subseteq S \) also \( \bigvee T \in S \) and \( \bigwedge T \in S \). In finite lattices this requirement translates into top \( (\top) \) and bottom \( (\bot) \) element.
of \( L \) being included in \( S \). For an element \( c \in L \) we define \( c^* := \bigvee \{ x \in L \mid x < c \} \) and \( c^+ := \bigwedge \{ x \in L \mid x > c \} \). We call \( c \in L \) supremum-irreducible if \( c \) has exactly one lower neighbor, meaning \( c^* < c \). An element \( c \in L \) is called infimum-irreducible if \( c \) has exactly one upper neighbor, meaning \( c^+ > c \). We call \( c \) doubly-irreducible if it is both, supremum-irreducible and infimum-irreducible. A formal context \( K = (G, M, I) \) is called reduced if all formal concepts of \( K \) are in a one-to-one correspondence to the sublattices of \( P(G) \) with \( X \subseteq G \) and \( X' = m' \). Otherwise \( m \) is called reducible. If all objects and all attributes of a formal context \( K \) are irreducible, \( K \) is called reduced. The concept lattice of a reduced formal context is isomorphic to the concept lattice of the context with additional reducible objects or attributes. In a reduced context \( K = (G, M, I) \) the object concepts \( \gamma g \) are the supremum-irreducible concepts and the attribute-concepts \( \mu m \) are the infimum-irreducible concepts in \( \mathfrak{B}(K) \).

An object \( g \in G \) is called irreducible, if there is no set of objects \( X \subseteq G \) with \( g \not\in X \) and \( X' = g' \). Otherwise, \( g \) is called reducible. Analogous, an attribute \( m \in M \) is called irreducible, if there is no set of attributes \( X \subseteq M \) with \( m \not\in X \) and \( X' = m' \). Otherwise \( m \) is called reducible. If all objects and all attributes of a formal context \( K \) are irreducible, \( K \) is called reduced. The concept lattice of a reduced formal context is isomorphic to the concept lattice of the context with additional reducible objects or attributes. In a reduced context \( K = (G, M, I) \) the object concepts \( \gamma g \) are the supremum-irreducible concepts and the attribute-concepts \( \mu m \) are the infimum-irreducible concepts in \( \mathfrak{B}(K) \).

For a formal context \( K = (G, M, I) \) a subrelation of \( I \) is a subset \( J \subseteq I \). \( J \) is called closed subrelation if all formal concepts of \( (G, M, J) \) are formal concepts of \( K \) as well, i.e., if \( \mathfrak{B}(G, M, J) \subseteq \mathfrak{B}(K) \). A formal context \( S = K_{H,N} := (H, N, J) \) with \( H \subseteq G, N \subseteq M \) and \( J = I \cap (H \times N) \) is called subcontext of \( K = (G, M, I) \) and denoted by \( S \subseteq K \). If instead \( J \subseteq I \cap (H \times N) \) and all formal concepts of \( S \) are also formal concepts of \( K \), \( S \) is called closed subcontext of \( K \). The closed subrelations of a formal context \( K \) are in a one-to-one correspondence to the complete sublattices of \( \mathfrak{B}(K) \) [10]: If \( J \) is a closed relation of \( (G, M, I) \), then \( \mathfrak{B}(G, M, J) \) is a complete sublattice of \( \mathfrak{B}(G, M, I) \) with \( J = \bigcup \{ A \times B \mid (A, B) \in \mathfrak{B}(G, M, J) \} \). Conversely, for every complete sublattice \( U \) of \( \mathfrak{B}(G, M, I) \) the relation \( J := \bigcup \{ A \times B \mid (A, B) \in U \} \) is closed and \( \mathfrak{B}(G, M, J) = U \). Similarly, the closed subcontexts of a finite formal context \( K \) are in a one-to-one correspondence to the sublattices of \( \mathfrak{B}(K) \) [7]: If \( (H, N, J) \) is a closed subcontext of \( (G, M, I) \), then \( \mathfrak{B}(H, N, J) \) is a sublattice of \( \mathfrak{B}(G, M, I) \).
with $H = \bigcup \{A \mid (A, B) \in \mathfrak{B}(H, N, J)\}$, $N = \bigcup \{B \mid (A, B) \in \mathfrak{B}(H, N, J)\}$ and $J = \bigcup \{A \times B \mid (A, B) \in \mathfrak{B}(H, N, J)\}$. Conversely, for every sublattice $U$ of $\mathfrak{B}(G, M, I)$ the context $(H, N, J) = (\bigcup_{(A, B) \in U} A, \bigcup_{(A, B) \in U} B, \bigcup_{(A, B) \in U} A \times B)$ is a closed subcontext of $(G, M, I)$ and $\mathfrak{B}(H, N, J) = U$. In the following all sets are considered finite.

### 3 Dismantling Intervals for a Lattice

To identify the intervals that can be removed from a (concept) lattice without disturbing the remaining structure, we introduce the notions of dismantling and quasi-dismantling intervals for a lattice, extending the usual notion of dismantling single elements.

**Definition 1.** Let $L$ be a lattice and $[u, v] = S \leq L$ an interval of $L$. We call $S$ quasi-dismantling for $L$ if $u$ is supremum-prime in $v$ and $v$ is infimum-prime in $u$. If $u \neq \bot$ and $v \neq \top$ we call $S$ dismantling for $L$.

See Figure 1 for a visualization and an example for Definition 1.

On the context side the removal of a set of concepts $S$ corresponds to the removal of all incidences (crosses) that only belong to concepts in the respective interval. We call the remaining context (objects, attributes, incidences) the S-removed context (objects, attributes, incidences).

**Definition 2.** Let $\mathbb{K} = (G, M, I)$ be a formal context and $S \subseteq \mathfrak{B}(\mathbb{K})$ a set of formal concepts. We call

- $I_S := I \setminus \left( \bigcup_{(A, B) \in S} A \times B \setminus \bigcup_{(A, B) \in \mathfrak{B}(\mathbb{K}) \setminus S} A \times B \right)$ S-removed incidences,
- $G_S := G \setminus \left( \bigcup_{(A, B) \in S} A \setminus \bigcup_{(A, B) \in \mathfrak{B}(\mathbb{K}) \setminus S} A \right)$ S-removed objects,
- $M_S := M \setminus \left( \bigcup_{(A, B) \in S} B \setminus \bigcup_{(A, B) \in \mathfrak{B}(\mathbb{K}) \setminus S} B \right)$ S-removed attributes.
In the following we let $\mathcal{K}_S := (G_S, M_S, I_S)$ be the S-removed context for $\mathcal{K}$.

There is a simpler representation for the S-removed objects, attributes and incidences:

**Lemma 1.** Let $\mathcal{K} = (G, M, I)$ be a formal context and $S \subseteq \mathcal{B}(\mathcal{K})$ a set of formal concepts, then $I_S = \bigcup_{(A,B) \in \mathcal{B}(\mathcal{K}) \setminus S} A \times B$, $G_S = \bigcup_{(A,B) \in \mathcal{B}(\mathcal{K}) \setminus S} A$ and $M_S = \bigcup_{(A,B) \in \mathcal{B}(\mathcal{K}) \setminus S} B$.

**Proof.** We show the proof for the S-removed incidences. The proofs for the other two sets are analogous.

$I \setminus \left( \bigcup_{(A,B) \in S} A \times B \setminus \bigcup_{(A,B) \in \mathcal{B}(\mathcal{K}) \setminus S} A \times B \right)$

$= \bigcup_{(A,B) \in \mathcal{B}(\mathcal{K}) \setminus S} A \times B \setminus \left( \bigcup_{(A,B) \in S} A \times B \right)$

$= \left( \bigcup_{(A,B) \in \mathcal{B}(\mathcal{K}) \setminus S} A \times B \right) \setminus \bigcup_{(A,B) \in S} A \times B$

$= \bigcup_{(A,B) \in \mathcal{B}(\mathcal{K}) \setminus S} A \times B$.

If we consider an interval $S$, we see that $S$ being quasi-dismantling corresponds to obtaining a closed subcontext on $S$-removal. More precisely, for an interval $S \subseteq \mathcal{B}(\mathcal{K})$ the S-removed context $\mathcal{K}_S$ for $\mathcal{K}$ is a closed subcontext if and only if $S$ is quasi-dismantling for $\mathcal{B}(\mathcal{K})$.

**Lemma 2.** Let $\mathcal{K} = (G, M, I)$ be a formal context and $\mathcal{B}(\mathcal{K})$ its corresponding concept lattice. Let $S = [u, v] \subseteq \mathcal{B}(\mathcal{K})$ be an interval. Then, $S$ is quasi-dismantling for $\mathcal{B}(\mathcal{K})$ if $\mathcal{K}_S = (G_S, M_S, I_S)$ is a closed subcontext of $\mathcal{K}$.

**Proof.** "$\Rightarrow": We show the contraposition: Assume $(G_S, M_S, I_S)$ is no closed subcontext. By definition holds $G_S \subseteq G$, $M_S \subseteq M$ and $I_S \subseteq I$. Then there exists some $c \in \mathcal{B}(\mathcal{K}_S)$ such that $c \notin \mathcal{B}(\mathcal{K})$. Since $\mathcal{B}(\mathcal{K}_S)$ is a lattice generated from $\mathcal{B}(\mathcal{K}) \setminus S$ there exist $x, y \in \mathcal{B}(\mathcal{K}) \setminus S$ such that $x \lor y = c$ or $x \land y = c$ in $\mathcal{B}(\mathcal{K})$. In case $x \lor y = c$: Since $\mathcal{B}(\mathcal{K})$ is a lattice it follows that there exists some $z \in \mathcal{B}(\mathcal{K})$ with $z \notin \mathcal{B}(\mathcal{K}_S)$ and $z = x \lor y$. Thus, $z \in S = [u, v]$ and therefore $z \geq u$ in $\mathcal{B}(\mathcal{K})$. Because $x, y \notin S$ we have $x, y \not\geq u$. Hence, $u$ is not supremum-prime in $[v]$ and $S$ is not quasi-dismantling. The case $x \land y = c$ is analogous.

"$\Leftarrow":$ We show the contraposition: Assume $S$ is not quasi-dismantling. Then $u$ is not supremum-prime in $[v]$ or $v$ is not infimum-prime in $[u]$. In case that $u$ is not supremum-prime in $[v]$: There exist $x, y \in [v]$ such that $z := x \lor y \geq u$, $x \not\geq u$ and $y \not\geq u$. Thus, $x, y \notin S$ and $z \in S$. Therefore, $z \notin \mathcal{B}(\mathcal{K}_S)$. There is some supremum $c = x \lor y$ in $\mathcal{B}(\mathcal{K}_S)$. Because the intent of $c$ is the intent of $z$ by [5, Thm. 3] we have $c \notin \mathcal{B}(\mathcal{K})$. Thus, $(G_S, M_S, I_S)$ is no closed subcontext. The case that $v$ is not infimum-prime in $[u]$ is analogous.

In particular, for a dismantling interval $S$ we have $G_S = G$ and $M_S = M$ (because $\top, \bot \in \mathcal{B}(\mathcal{K}) \setminus S$) and therefore the correspondence in this case is to closed subrelations.

The removal of a quasi-dismantling interval leaves the remaining lattice intact with respect to supremum and infimum:
Lemma 3. Let $\mathcal{L}$ be a lattice and $S \leq \mathcal{L}$ an interval. If $S$ is quasi-dismantling for $\mathcal{L}$, then $\mathcal{L} \setminus S$ is a lattice. In particular, $\mathcal{L} \setminus S$ is a sublattice of $\mathcal{L}$.

Proof. Let $x, y, z \in \mathcal{L}$ with $z = x \lor y$. Because $S$ is quasi-dismantling if $x, y \not\in S$ then $z \not\in S$. Analogously for $z = x \land y$. □

Corollary 1. If $S$ is dismantling for $\mathcal{L}$ then $\mathcal{L} \setminus S$ is a complete sublattice of $\mathcal{L}$.

Combining the previous lemmas, it follows for an interval $S$ which is quasi-dismantling in a lattice $\mathcal{B}(\mathcal{K})$ that the removal of $S$ from $\mathcal{B}(\mathcal{K})$ is isomorphic to the concept lattice of the $S$-removed context $\mathcal{K}_S$.

Theorem 1. Let $\mathcal{K} = (G, M, I)$ be a formal context and $\mathcal{B}(\mathcal{K})$ its corresponding concept lattice. Let $S \leq \mathcal{B}(\mathcal{K})$ be an interval. If $S$ is quasi-dismantling for $\mathcal{L}$, then

$$\mathcal{B}(\mathcal{K}) \setminus S = \mathcal{B}(\mathcal{K}_S).$$

Proof. We know from Lemma 3 that $\mathcal{B}(\mathcal{K}) \setminus S$ is a sublattice of $\mathcal{B}(\mathcal{K})$. Further, from Lemma 2 follows that $\mathcal{B}(\mathcal{K}_S)$ is a sublattice of $\mathcal{B}(\mathcal{K})$. Both contain exactly the concepts of $\mathcal{B}(\mathcal{K})$ that are not included in $S$. □

An example for a lattice with a dismantling interval is given in Figure 2. The concept lattice of $\mathcal{K}_S$ is the concept lattice of $\mathcal{K}$ without the interval $S$.

Note that this statement does not hold for intervals that are not dismantling for the lattice. See Figure 3 for a counterexample.

Theorem 1 is a generalization of the following proposition concerning the dismantling of doubly irreducible lattice elements.

Proposition 1 (Prop. 53 [5]). If $a = (g'', g') = (m', m'')$ is a doubly irreducible concept of a clarified context $(G, M, I)$, then

$$\mathcal{B}((G, M, I)) \setminus \{a\} = \mathcal{B}(G, M, I \setminus \{(g, m)\}).$$

Propositions 2 and 3 clarify how Proposition 1 and Theorem 1 are connected.
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Proposition 2. Let $\mathbb{K} = (G, M, I)$ be a formal context and $\mathcal{B}(\mathbb{K})$ its corresponding concept lattice. Let $S \leq \mathcal{B}(\mathbb{K})$ be an interval with $|S| = 1$. $S$ is dismantling for $\mathcal{B}(\mathbb{K})$ if and only if $S$ is doubly irreducible.

Proposition 3. Let $\mathbb{K} = (G, M, I)$ be a formal context and $\mathcal{B}(\mathbb{K})$ its corresponding concept lattice. Let $S \leq \mathcal{B}(\mathbb{K})$ be an interval with $|S| = 1$. $S$ is quasi-dismantling if and only if

- $S$ is doubly irreducible or
- $S = \top$ and $S$ is supremum-irreducible or
- $S = \bot$ and $S$ is infimum-irreducible or
- $S = \bot = \top$.

If we consider multiple intervals at once, one direction of Lemma 2 still holds:

Lemma 4. Let $\mathbb{K} = (G, M, I)$ be a formal context, $S_1, \ldots, S_k$ be intervals in $\mathcal{B}(\mathbb{K})$. If $S_1, \ldots, S_k$ are quasi-dismantling, then $(G_{S_1 \cup \ldots \cup S_k}, M_{S_1 \cup \ldots \cup S_k}, I_{S_1 \cup \ldots \cup S_k})$ is a closed subcontext of $\mathbb{K}$.

Proof. We show the contraposition. Assume $(G_{S_1 \cup \ldots \cup S_k}, M_{S_1 \cup \ldots \cup S_k}, I_{S_1 \cup \ldots \cup S_k})$ is no closed subcontext. By definition, we have $G_{S_1 \cup \ldots \cup S_k} \subseteq G$, $M_{S_1 \cup \ldots \cup S_k} \subseteq M$ and $I_{S_1 \cup \ldots \cup S_k} \subseteq I$. Then there exists $c \in \mathcal{B}(\mathbb{K}_{S_1 \cup \ldots \cup S_k})$ with $c \notin \mathcal{B}(\mathbb{K})$. Hence, there exist $x, y \in \mathcal{B}(\mathbb{K})$ such that $c = x \lor y$ or $c = x \land y$ in $\mathcal{B}(\mathbb{K}_{S_1 \cup \ldots \cup S_k})$. In case $c = x \lor y$ in $\mathcal{B}(\mathbb{K}_{S_1 \cup \ldots \cup S_k})$ there exists $z \in \mathcal{B}(\mathbb{K})$ such that $z = x \lor y$. Since $z \in S_i \cup \ldots \cup S_k$ we have $z \in S_i$ for some $i$. The rest follows analogous to the proof of Lemma 2. 

However, not all closed subcontexts (and therefore sublattices of the corresponding concept lattice) can be obtained via a quasi-dismantling interval or a set of quasi-dismantling intervals for the corresponding lattice, see e.g. Figure 4.
Fig. 4. A closed subrelation (middle) that can not be obtained as via dismantling intervals. The interval \([\{3\}, \{a,b\}], \{(2,3,4),\{b\}\}]\) is not dismantling, and neither \((\{3\}, \{a,b\}\]) nor \((\{2,3,4\}, \{b\}\]) are doubly irreducible.

Fig. 5. Smallest (non-trivial) lattice that has no dismantling interval.

Further note, that not every lattice contains a quasi-dismantling interval besides the trivial one (the complete lattice) or any dismantling interval at all, see for example Figure 5.

However, there is always a unique smallest lattice that can be obtained by iteratively removing all dismantling intervals, as shown in Theorem 2. To this end, we make use of the following lemma concerning the dismantlability of intervals upon removing one of them.

**Lemma 5.** Let \(L\) be a lattice and \(S_1, S_2\) dismantling intervals for \(L\) such that \(S_2 \not\subseteq S_1\). Then, \(S_2 \setminus S_1\) is a dismantling interval for \(L \setminus S_1\).

**Proof.** We first show that \(S_2 \setminus S_1\) is an interval: Assume \(S_2 \setminus S_1\) is no interval in \(L \setminus S_1\). Then, without loss of generality, there exist \(x, y \in S_2 \setminus S_1\) such that \(z := x \lor y \not\in S_2 \setminus S_1\) in \(L \setminus S_1\). Either \(z = x \lor y\) in \(L\), thus \(z \in S_2\) and therefore \(z \in S_2 \setminus S_1\) (f). or \(z \neq x \lor y\) in \(L\), hence \(w = x \lor y\) in \(L\) with \(w \in S_1\) and since \(S_1\) is dismantling, \(x \in S_1\) or \(y \in S_1\) (f). Thus, \(S_2 \setminus S_1\) is an interval in \(L \setminus S_1\).

It remains to show that \(S_2 \setminus S_1\) is dismantling for \(L \setminus S_1\). Let \([u, v] = S_2 \setminus S_1\) and assume \(S_2 \setminus S_1\) is not dismantling. Then, \(u\) is not supremum-prime in \([v]\) or \(v\) is not infimum-prime in \([u]\). Without loss of generality, assume \(u\) is not supremum-prime in \([v]\). Then, there exist \(x, y \in L \setminus S_1\) such that \(x, y \not\in [u, v]\) and \(x \lor y \in [u, v]\). Hence, \(x \lor y \in S_2\) in \(L\). Because \(S_2\) is dismantling for \(L\)
it follows that $x \in S_2$ or $y \in S_2$ and therefore $u$ is supremum-prime($\ell$). Thus, $S_2 \setminus S_1$ is dismantling in $L \setminus S_1$.

Let $DI(L)$ be the family of all subsets of $L$ that can be obtained by iterated dismantling from $L$, i.e., by iteratively removing dismantling intervals starting from $L$. A smallest element of $DI(L)$ is called a DI-core of $L$.

**Theorem 2.** Let $L$ be a lattice. There exists a unique DI-core.

**Proof.** Let $U, V \in DI(L)$ be two minimal elements in $DI(L)$. Then, there is a minimal upper bound $T \in DI(L)$ of $U$ and $V$, i.e., both $U$ and $V$ are obtained by removing dismantling intervals from $T$. Hence, there are two sequences of intervals $S_1, \ldots, S_k, R_1, \ldots, R_l$ such that $U = T \setminus (S_1 \cup \ldots \cup S_k)$ and $V = T \setminus (R_1 \cup \ldots \cup R_l)$. By iterative application of Lemma 5 we have that $T \setminus (S_1 \cup \ldots \cup S_k \cup R_1 \cup \ldots \cup R_l) \in DI(L)$. \hfill \Box

## 4 Dismantling in the Formal Context

In this section we show that dismantling intervals can be identified directly in the formal context. Based on this, we propose an algorithm to find all dismantling intervals for a given formal context. Thus, we can omit the (expensive) computation of the concept lattice. To this end, we make use of the arrow relations from FCA:

**Definition 3 (Def. 25 [5]).** Let $\mathbb{K} = (G, M, I)$ be a formal context. For an object $g \in G$ and an attribute $m \in M$ we write

\[
\begin{align*}
g \prec m & : \iff (g, m) \notin I \quad \text{and} \\
& \quad \text{if } \exists h \in G \text{ with } g' \subseteq h' \quad \text{and} \quad g' \neq h', \text{ then } (h, m) \in I, \\
g \not\prec m & : \iff (g, m) \notin I \quad \text{and} \\
& \quad \text{if } \exists n \in M \text{ with } m' \subseteq n' \quad \text{and} \quad m' \neq n', \text{ then } (g, n) \in I, \\
g \not\succ m & : \iff g \prec m \quad \text{and} \quad g \not\succ m.
\end{align*}
\]

We now adapt the definitions of $(\cdot)^*, (\cdot)^{\ast}$, to object and attribute concepts,

\[
\begin{align*}
(\gamma g)^{\ast} & := \bigvee \{ c \in \mathcal{B}(\mathbb{K}) \mid c \leq \gamma g \}, \quad \text{and} \\
(\mu m)^{\ast} & := \bigwedge \{ c \in \mathcal{B}(\mathbb{K}) \mid \mu m \leq c \},
\end{align*}
\]

in order to characterize the arrow relations in a formal context as follows [5]:

\[
\begin{align*}
g \not\prec m & \iff \gamma g \wedge \mu m = (\gamma g)^{\ast} \neq \gamma g \\
g \not\succ m & \iff \gamma g \vee \mu m = (\mu m)^{\ast} \neq \mu m
\end{align*}
\]

If $(\gamma g)^{\ast} \neq \gamma g$ then the object $g$ is irreducible in the formal context and $\gamma g$ is supremum-irreducible in the corresponding concept lattice. Analogously, if
Let \( K = (G, M, I) \) with \( g \in G \) and \( m \in M \). Then:

1. \( \gamma g \) supremum irreducible in \((\mu m)\) if and only if
   \[ \exists n \in M : g \not\succ n \text{ in } K_{|m',M} \text{ (attribute-clarified)} \text{ and} \]
   \[ \exists h \in G : h \not\succ m \text{ in } K_{|G,g'} \text{ (object-clarified)}. \]
2. \( \mu m \) supremum irreducible in \((\gamma g)\) if and only if
   \[ \exists h \in G : h \not\succ m \text{ in } K_{|G,g'} \text{ (object-clarified)} \text{ and} \]
   \[ \exists n \in M : g \not\succ n \text{ in } K_{|m',M} \text{ (attribute-clarified)}. \]

Proof. This follows directly from [5, Lemma 13] with \( \mathfrak{A}(K_{|m',M}) = (\mu m) \) and \( \mathfrak{A}(K_{|G,g'}) = (\gamma g) \).

Based on this equivalence, we propose a characterization for supremum-prime and infimum-prime concepts in the formal context.

**Proposition 4.** Let \( K = (G, M, I) \) be a formal context, \( g \in G, m \in M \), then

1. \( \gamma g \) is supremum prime in \((\mu m)\) if and only if
   \[ i) \exists n \in M : g \not\succ n \text{ in } K_{|m',M} \text{ (attribute-clarified)} \text{ and} \]
   \[ ii) \exists n \in M : g \not\succ n \text{ in } K_{|m',M} \text{ (attribute-clarified)}. \]
2. \( \mu m \) is infimum prime in \((\gamma g)\) if and only if
   \[ i) \exists h \in G : h \not\succ m \text{ in } K_{|G,g'} \text{ (object-clarified)} \text{ and} \]
   \[ ii) \exists h \in G : h \not\succ m \text{ in } K_{|G,g'} \text{ (object-clarified)}. \]

Proof. We show the first part of the statement:

"\( \Leftarrow \)" We show this by contraposition. Assume \( \gamma g \) is not supremum prime in \((\mu m)\) but is supremum irreducible (otherwise use Lemma 6). Hence, \( \exists c_1, c_2 \in (\mu m), c_1 \neq c_2, \gamma g \not\subseteq c_1, c_2, \gamma g \subseteq c_1 \lor c_2 \). Let \( c_i = \mu k_i \) with \( k_i \in M \) such that there is no \( l \in M \) with \( \mu l > \mu k_i \) and \( \gamma g \not\subseteq \mu l \), i.e., choose maximal attribute concepts not larger than \( \gamma g \).

We show that \( g \not\succ k_i \) using the characterization \( \gamma g \lor \mu k_i = (\mu k_i)^* \neq \mu k_i \).

The second part, \((\mu k_i)^* \neq \mu k_i \), is fulfilled by choice of \( \mu k_i \). If we assume that \( \gamma g \lor \mu k_i \neq (\mu k_i)^* \), then \((\mu k_i)^* \neq \gamma g \lor \mu k_i \) and thus there exists some \( l \in M \) with \( l \in \text{int}(\mu k_i)^*, l \not\in \text{int}(\gamma g), l \in \text{int}(\mu k_i)(f) \). Hence, \( g \not\succ k_i \) in \( K_{|m',M} \).

"\( \Rightarrow \)" We show this by contraposition. Assume \( k, n \in M, k \neq n, g \not\succ n, g \not\succ k \) in \( K_{|m',M} \) (clarified). From \( g \not\succ n \) we have \( \gamma g \lor \mu n = (\mu n)^* \neq \mu n \) and thus \( \mu n \not\subseteq \gamma g \). Analogously, \( g \not\succ k \) implies \( \gamma g \lor \mu k = (\mu k)^* \neq \mu k \) and thus \( \mu k \not\subseteq \gamma g \). Since \( \gamma g \lor \mu k = (\mu k)^* \) and \( \gamma g \lor \mu n = (\mu n)^* \), we have \( \mu k \not\subseteq \mu n \) and \( \mu n \not\subseteq \mu k \). Thus, we have \( \mu k \lor \mu n \supseteq \gamma g \).

The second part of the statement can be shown analogously.

Now, two questions arise. First, given we have a formal context \( K \) and an interval \([\gamma g, \mu m]\) between an object concept and an attribute concept, is this interval dismantling in \( \mathfrak{B}(K) \)? And second, given a formal context \( K \), which are the dismantling intervals in the corresponding concept lattice \( \mathfrak{B}(K) \)?

To answer the first question Proposition 4 tells us that it suffices to check the arrow relations of \( g \) in \( K_{|m',M} \) and of \( m \) in \( K_{|G,g'} \): If \( g \) only has a single \( \not\succ \) in
Algorithm 1: Compute All Dismantling Intervals

Input: A formal context $\mathcal{K} = (G, M, I)$
Result: The set of all dismantling intervals for $\mathcal{K}$.

1. $U = \emptyset$, $O = \emptyset$
2. for $g \in G$
   3. compute $\mathcal{K}|_{G,g'}$ and clarify objects
   4. compute $\nearrow(\mathcal{K}|_{G,g'}) = \{(h, m) \mid h \nearrow m \text{ in } \mathcal{K}|_{G,g'}\}$
   5. for $m \in g'$
      6. $H_m = (G \times \{m\}) \cap \nearrow(\mathcal{K}|_{G,g'})$
      7. if $H_m = \{(h, m)\}$ and $h \searrow m$ in $\mathcal{K}|_{G,g'}$ then
         8. $U = U \cup \{(g, m)\}$
   9. for $m \in M$ do
      10. compute $\searrow(\mathcal{K}|_{m',M}) = \{(g, n) \mid g \searrow n \text{ in } \mathcal{K}|_{m',M}\}$
      11. for $g \in m'$ do
         12. $N_g = ((g) \times M) \cap \searrow(\mathcal{K}|_{m',M})$
         13. if $N_g = \{(g, n)\}$ and $g \searrow n$ in $\mathcal{K}|_{m',M}$ then
            14. $O = O \cup \{(g, m)\}$
   15. return $\{[\gamma g, \mu m] \mid (g, m) \in O \cap U\}$

If $\mathcal{K}|_{m',M}$ and no additional $\nearrow$, then $\gamma g$ is supremum-prime in $(\mu m)$. Analogously, if $m$ only has a single $\nearrow$ in $\mathcal{K}|_{G,g'}$ and no additional $\searrow$, then $\mu m$ is infimum-prime in $[\gamma g]$. If both conditions hold, then the interval $[\gamma g, \mu m]$ is dismantling in $\mathcal{P}(\mathcal{K})$. Note that, if $\gamma g \not\leq \mu m$ then $g \not\in \mathcal{K}|_{m',M}$ and $m \not\in \mathcal{K}|_{G,g'}$.

In order to compute all dismantling intervals for a given formal context the naive approach is to check all intervals $[\gamma g, \mu m]$ between object concepts and attribute concepts. However, this (essentially iterative) approach results in the repeated computation of the same subcontexts. To prevent this, we instead compute each subcontext only once and for each object concept $\gamma g$ we check which attribute concepts are infimum-prime in $[\gamma g]$, i.e., we check the arrow relations in $\mathcal{K}|_{G,g'}$, and vice versa. More precisely, for each object $g$ we take the attributes $m$ where $\mu m$ is infimum-prime in $[\gamma g)$ and collect them in the set

$$U = \{(g, m) \mid g \in G, \mu m \text{ infimum-prime in } [\gamma g]\}.$$  

Similarly, for each attribute $m$ we take the objects $g$ where $\gamma g$ is supremum-prime in $(\mu m)$ and collect them in the set

$$O = \{(g, m) \mid m \in M, \gamma g \text{ suprermum-prime in } (\mu m)\}.$$  

If a pair $(g, m)$ is in both $U$ and $O$, then the respective interval $[\gamma g, \mu m]$ is dismantling for the lattice $\mathcal{P}(\mathcal{K})$. Note that it suffices to consider the reduced formal context. In Algorithm 1 we present an implementation in pseudo-code.

If we are interested in the dismantling intervals of a lattice $L$, we can simply compute them for its standard context, i.e., for the context $(J(L), M(L), \leq)$ where $J(L)$ are the supremum-irreducible, and $M(L)$ the infimum-irreducible elements of $L$.  

Interval-Dismantling for Lattices
5 Conclusion

In this paper, we introduce the notion of dismantling intervals for a lattice in order to transfer the notion of dismantling doubly irreducible elements to a set of elements. In particular, we show the connection between closed subrelations on the context side and dismantling intervals on the lattice side, and more generally the connection between closed subcontexts and quasi-dismantling intervals. While a lattice can always be shrunk to the trivial empty lattice by removing a quasi-dismantling interval, iteratively removing only dismantling intervals for a lattice results in a unique (not necessarily trivial) smallest sublattice. The dismantling intervals can be found directly in the formal context $K$ with help of the arrow relations. We show how to decide in $K$ if a given interval is dismantling for $B(K)$. Additionally, given $K$, we propose an algorithm to compute all dismantling intervals of $B(K)$ without first computing the concept lattice itself.

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