Meromorphy of solutions for a wide class of ordinary differential equations of Painlevé type

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We prove the meromorphy of solutions for a wide class of ordinary differential equations. These equations are given by invariant manifolds of non-linear partial differential equations integrable by the inverse scattering method. Some higher analogues of the Painlevé equations are considered as examples.

I. INTRODUCTION

The six Painlevé equations are distinguished among the other second-order ordinary differential equations (ODE) \( w''_{zz} = f(z, w, w'_z) \) whose right-hand side is a rational function of \( w \) and \( w'_z \), by the absence of non-polar movable singularities of their solutions (that is, singularities whose position depends on the initial data). In the simplest cases of the first Painlevé equation

\[
 w''_{zz} = 6w^2 + z
\]  

(1)

and the second Painlevé equation

\[
 w''_{zz} = 2w^3 + zw + \alpha,
\]  

(2)

this absence is equivalent to the property of global meromorphy of all their solutions with arbitrary initial conditions

\[
 w(z_0) = a, \quad w'_z(z_0) = b,
\]  

(3)

where \( z_0, a \) and \( b \) are any complex constants.

The Painlevé ODE are currently being applied to a wide variety of problems in mathematics and mathematical physics. But the problem of rigorously proving the absence of non-polar movable singularities of their solutions turned out to be difficult. Original proofs by Painlevé and his followers appeared to be incomplete. Gaps were also found in a number of subsequent attempts to give a satisfying proof. The meromorphy of all solutions of the simplest Painlevé ODE \((1)\) and \((2)\) with initial conditions \((3)\) was rigorously justified for the first time only in 1999 by Hinkkanen and Laine [1]. A glimpse to the rather dramatic story of proving the absence of non-polar movable singularities of solutions of the Painlevé equations can be found in the introduction of the recent paper [2], along with references to the literature.

A principally new proof of the global meromorphy of solutions of \((1)\) and \((2)\) was also given in [2]. It is based on the well-known fact that these ODE describe self-similar solutions of the Korteweg–de Vries (KdV) equation

\[
 u_t + uu'_x + u'''_{xxx} = 0
\]  

(4)

and the modified Korteweg–de Vries (mKdV) equation

\[
 w_t - 6w^2w'_x + w''''_{xxx} = 0
\]  

(5)

respectively. The evolution equations \((4)\) and \((5)\) are integrable by inverse scattering transform (IST) and, by the results in [4], all their local holomorphic solutions can be extended to the whole \( x \)-plane as globally meromorphic functions of the spatial variable. In a similar vein, the global meromorphy of solutions of the initial-value problems was proved in [2] for the infinite hierarchies of higher analogues of the first and second Painlevé ODE and reproved for the fourth Painlevé ODE.

The main result of the present paper formally consists in demonstrating in sections III–V that the global meromorphy of solutions of the initial-value problems can be proved in a uniform and simple way for a very large class of non-linear ODE which determine the so-called invariant manifolds of solutions of KdV \((4)\), mKdV \((5)\), the complexified non-linear Schrödinger (NLS) equation

\[
 p''_t = -ip''_{xx} + 2ip^2q, \quad q'_t = iq'''_{xx} - 2ipq^2
\]  

(6)

and the Sawada–Kotera (SK) equation \((5)\)

\[
 u'_t = u'''''_{xxxx} - 30uu'''_{xxx} - 30u'u''_{xx} + 180u''u^2.
\]  

(7)
As in \[2\], the main step in the proof of global meromorphy consists in using the meromorphic extendibility with respect to \(x\) of all local holomorphic solutions of the evolution equations \([3 - 7]\). However, in contrast with \([2]\), we consider not only ODE for self-similar solutions of IST-integrable evolution equations but also ODE that determine the so-called symmetries (and, more generally, invariant manifolds) of these evolution equations. Note that the ODE for self-similar solutions are very special representatives of this class. They correspond to the so-called classical symmetries.

Our scheme for proving the meromorphy of solutions can be extended to many other simultaneous solutions of ODE and IST-integrable evolution equations. The main and almost the only obstacle is the necessity to establish that all local holomorphic solutions of the integrable partial differential equation admit a global meromorphic extension with respect to one of the independent variables. However, this was already done in \([2, 4 - 9]\) for many equations including \([3 - 7]\).

The main part of the paper begins in the next section, where we give the necessary auxiliary information concerning some properties of systems of evolution equations integrable by inverse scattering transform.

II. SYMMETRIES AND INvariant MANIFOLDS OF EVOLUTION EQUATIONS

Consider a system of evolution equations
\[
(u_i')' = F^i(t,x,u^1,...,u^m) , \quad 1 \leq i \leq m ,
\]
where \(F^i, \ 1 \leq i \leq m\), are locally analytic functions of \(t, x, u^m = \left(\frac{\partial}{\partial x}\right)^m u^m\). (Here and in what follows, subscripts of the components of solutions stand for the orders of their derivatives with respect to the spatial independent variable \(x\).

**Definition 1** A symmetry of the system \([8]\) is a system of evolution equations
\[
(u_i')' = G^i(t,x,u^1,...,u^m) , \quad 1 \leq i \leq m ,
\]
with locally analytic right-hand sides \(G^i\) satisfying the commutation condition
\[
\frac{\partial G^i}{\partial t} + \sum_{k=1}^{m} \sum_{j=1}^{k} \frac{\partial G^j}{\partial u^j} \left(\frac{d}{dx}\right)^l F^j = \sum_{k=1}^{m} \sum_{j=1}^{k} \frac{\partial F^j}{\partial u^j} \left(\frac{d}{dx}\right)^l G^j .
\]

Note that \(\tau\) does not occur in the right-hand side of \([9]\). Furthermore, it follows directly from the definition that all linear combinations (with constant coefficients \(a^i_j\)) of \(\sum_{j=1}^{n} a^i_j G^j, \ 1 \leq i \leq m\), of the right-hand sides \(G^i\) of symmetries \([9]\) of the system \([8]\) are again right-hand sides of symmetries of this system.

Symmetries with functions of the form
\[
G^i = W^i(t,x,u^1,...,u^m,u^1,...,u^m,F^1,...,F^m)
\]
(where \(F^i = F^i(t,x,u^1,...,u^m)\)) are referred to as classical symmetries of the system \([8]\) of evolution equations. All other functions \(G^i\) determine higher symmetries of \([8]\). (The existence of higher symmetries is characteristic for integrable systems of evolution equations.) Systems of ODE given by the stationary parts \(u'_i = 0\) of the symmetries \([9]\) are particular cases of systems of ODE that determine invariant manifolds of the system \([8]\). Note that self-similar solutions of partial differential equations (integrable or not) are determined by the invariant manifolds corresponding to classical symmetries of these equations.

**Definition 2** We say that a system of ODE
\[
u^i_N(t,x,u^1,...,u^m,...,v^1_{N-1},...v^m_{N-1}) = (i = 1, ... m)
\]
where the functions \(S^i\) are analytic near the points
\[
(t,x,u^1,...,u^m,...,v^1_{N-1}) = (t_0,x_0,a^1,...,a^m,...,a^1_{N-1})
\]
\((t_0,x_0,a^1,...,a^m,...,a^1_{N-1}) \) are arbitrary complex constants), determines an invariant manifold of the system \([8]\) if the following relations hold on all solutions \(u(t,x)\) of \([10]\) \((1 \leq i \leq m)\):
\[
\frac{\partial S^i}{\partial t} + \sum_{j=1}^{m} \left[ \frac{\partial S^i}{\partial u^j} F^j + \sum_{i=1}^{N-1} \frac{\partial S^i}{\partial u^i} \left(\frac{d}{dx}\right)^l F^j \right] = \left(\frac{d}{dx}\right)^N S^i ,
\]
\((12)\)

The relations \((12)\) are clearly necessary for the existence of a simultaneous local analytic solution of the systems \((8)\) and \((10)\).

We now assume that \(S^i\) are local analytic functions near the points \((11)\) and the relations \((12)\) hold. Consider the set of all analytic solutions of the system of ODE \((10)\) in a neighborhood of the point \((t_0,x_0) \in \mathbb{C}^2\) satisfying the conditions
\[
u^i_{N_1}(t_0,x_0) = a^i_{N_1} \quad 1 \leq i \leq m , \quad 0 \leq n_1 \leq N_1 - 1 .
\]

We claim that this set contains a unique solution of \((8)\). Indeed, differentiating each component of the system of ODE \((10)\) with respect to \(t\) and using \((12)\), we see that the functions
\[
w^i(x,t) = (u^i)' - F^i(t,x,u^1,...,u^m) , \quad 1 \leq i \leq m ,
\]
satisfy the following system of linear ODE in a neighborhood of the point \((t_0,x_0) \in \mathbb{C}^2\) \((1 \leq i \leq m , \quad 0 \leq n_1 \leq N_1 - 1)\):
\[
\left(\frac{d}{dx}\right)^{N_1} w^i = \sum_{j=1}^{m} \left[ \frac{\partial S^i}{\partial u^j} w^j + \sum_{i=1}^{N-1} \frac{\partial S^i}{\partial u^i} \left(\frac{d}{dx}\right)^l w^j \right] .
\]

For every fixed \(t\) in a neighborhood of \(t_0\), these ODE (in the independent variable \(x\)) are resolved with respect to the highest derivatives and their coefficients are analytic functions. By the uniqueness theorem, our task reduces to choosing the initial data
\[
u^i_{N_1}(t_0,x_0) = v^i_{N_1}(t) \quad (1 \leq i \leq m , \quad 0 \leq n_1 \leq N_1 - 1)
\]
of the considered solution of \((10)\) in such a way that the functions \(w^i(x,t)\) and all their derivatives in \(x\) of orders up to \(N_1 - 1\) vanish identically in a neighborhood of \(t_0\) for \(x = x_0\). But this
can be done in a unique way, namely, letting \( v^{(m)}(t) \) be the unique solution with initial conditions \((13)\) of the system of first-order ODE
\[
\left( v^{(m)} \right)_t = C^{(m)}(t,x_0,v^{10}, \ldots, v^{mn}), \quad v^{(N)(N-1)}, \ldots, v^{(m)(m-1)}),
\]
where the right-hand side is defined as \( \left( \frac{d}{dt} \right)^n F(t,x,u^1, \ldots, u^m) \bigg|_{x=x_0} \) with all the occurring derivatives of \( u^i \) of order greater than \( N_j - 1 \) (if any) expressed in terms of the functions \( v^{(m)}(t) \) by means of the system \((10)\) and its differential consequences. Thus we arrive at the following result.

**Theorem 1** For any complex numbers \(a^j_{n_i}, (1 \leq i \leq m, 0 \leq n_i \leq N_j - 1)\) and all sufficiently small \( \delta_1, \delta_2 > 0 \) there is a unique simultaneous analytic solution \( u(t,x) \) in the bidisk
\[
\{ (x,t) \in \mathbb{C}^2 | |t-t_0| < \delta_1, |x-x_0| < \delta_2 \} \tag{15}
\]
of the systems \((8)\) and \((10)\) with initial conditions \((13)\).

**Remark 1** This theorem was essentially proved in section 3.1 of \([10]\), although in the infinitely differentiable case instead of our analytic version. For the reader’s convenience, we gave a self-contained proof of Theorem 1 which is simpler than the proof in \([10]\).

### III. MEROMORPHY OF SOLUTIONS OF THE STATIONARY PARTS OF SYMMETRIES FOR KDV AND MKDV

We first explain our scheme of proving the meromorphy of solutions of ODE (arising as invariant manifolds of evolution equations) on the example of the hierarchy of the first Painlevé equation \((1)\). This hierarchy can be written in the form
\[
x - t u + L_n(u,u_1, \ldots, u_n) + \sum_{j=1}^{n-1} \mu_j L_j(u,u_1, \ldots, u_j) = 0, \tag{16}
\]
where \( \mu_j \) are arbitrary complex numbers and \( L_j \) are the so-called Lenard polynomials, which are defined recursively as
\[
L_1 = u_2 + \frac{u^2}{2}, \quad L_2 = u_4 + \frac{5}{3} u_2^2 + \frac{5}{6} u_3, \quad L_3 = \frac{5}{18} u_3^2,
\]
and are identically equal to zero for \( u = 0 \).

This hierarchy of ODE is known \([11], [12]\) as the hierarchy of massive \((2n+1,2)\) string equations. It is also referred to as the \( P^2 \)-hierarchy of the first Painlevé equation (see, for example, \([12], [13]\)).

Differentiating both parts of the ODE \((16)\) with respect to \( x \), we obtain the stationary parts of symmetries of the KdV equation \((4)\):
\[
1 - tu_1 + \frac{d}{dx} \left( L_n(u,u_1, \ldots, u_n) + \sum_{j=1}^{n-1} \mu_j L_j(u,u_1, \ldots, u_j) \right) = 0.
\]

These are linear combinations of the stationary part of the classical Galilean symmetry \( u^{\prime}_t = 1 - tu_1 \) of the KdV equation and the stationary parts of representatives of the well-known hierarchy of commuting higher symmetries
\[
u^{\prime}_1 = u_3 + uu_1 = K_3[u], \quad u^{\prime}_2 = \frac{d}{dx}(L_2) = K_5[u],
\]
where \( u \) is any solution of ODE (arising as invariant manifolds of evolution equations) on the example of the hierarchy of the first Painlevé equation \((1)\), the only rigorous and detailed proof of the meromorphy of solutions of \((16)\) prior to \([2]\) was given for \( P^2 \) in \([15]\). We note that Shimomura \([16]\) announced a proof of meromorphy of solutions of \((16)\) for all positive integers \( n \) in 2004, but the subsequent authors \([17] - [19]\) still regard the question of rigorous proof for \( n > 2 \) as open.
Remark 3 Simultaneous solutions of the ODE (16) for \( n = 2 \) and the KdV equation (4) are equivalent [15] to solutions of a pair of isomonodromic Hamiltonian systems \( H^{9/2} \) with two degrees of freedom belonging to a hierarchy (written out by Kimura [20]) of degenerations of the classical two-dimensional Garnier system.

Remark 4 By [21], the set of these simultaneous solutions contains the familiar Gurevich–Pitaevskii special solution [introduced in (22), (23)] of the KdV equation. This special solution describes the generation of the so-called dispersive shock waves for a wide class of problems with small dispersion in the leading order. The same special solution of (16) was considered in the early nineties of the last century (see [21], [22], [24]) in connection with the problems of quantum gravitation theory (11), (23), (26) and the description of blow-up regimes in the random matrix theory (27), (28). Many publications in the last quarter century were devoted to studying various properties of simultaneous solutions of ODE and KdV (first of all, the Gurevich–Pitaevskii special solution); see [29], [39] and references therein in addition to those given above. Other representatives of the hierarchy of simultaneous solutions of (16) and (4) are also related to the description of formation of dispersive shock waves in degenerate cases [13], [29].

Our scheme can be used in an equally simple way to prove the meromorphy in \( x \) of all solutions of initial-value problems for the following hierarchy of stationary parts of symmetries of KdV (4):

\[
2u + xu_1 - 3t(u_3 + uu_1) + \frac{d}{dx} \left( \sum_{j=1}^{n} \mu_j L_j(u, u_1, \ldots, u_2) \right) = 0,
\]

(23)

where \( \mu_j \) are arbitrary complex constants. Here the stationary part of the classical scaling symmetry \( u'_c = 2u + xu_1 - 3t(u_3 + uu_1) \) [14, §5.2] is added to a linear combination of the stationary parts of the symmetries [19]. Being the stationary part of a symmetry of KdV, the ODE (23) determines an invariant manifold of KdV. Assuming that \( \mu_n \neq 0 \), we see from Theorem 1 that every initial condition \( \tau_{0} \) determines a unique simultaneous holomorphic solution of the ODE (23) and the KdV equation (4) in a neighbourhood \( \tau_{0} \) of an arbitrary point \((t_0, x_0)\). By [4], any such solution of (4) extends meromorphically to the strip \( \tau_{0} \). Hence all solutions of (23) are meromorphic with respect to \( \tau \) in this strip and, in particular, meromorphic on the whole \( \tau \)-plane for every fixed \( \tau \). This proves the following theorem.

Theorem 3 Let \( \mu_1, \ldots, \mu_n, x_0, t_0 \) and \( b_1, \ldots, b_n \) be any complex constants. Suppose that \( n > 1 \) and \( \mu_n \neq 0 \). Then the unique solution of the ODE (23) with \( \tau = t_0 \) and initial conditions (20) is meromorphic with respect to \( x \) on the whole complex plane \( \mathbb{C} \).

The least representative of the hierarchy of ODE (23) corresponds to the case when \( \mu_1 = \cdots = \mu_n = 0 \). It coincides with the stationary part of the classical scaling symmetry of the KdV equation (4).

\[
2u + xu_1 - 3t(u_3 + uu_1) = 0.
\]

(24)

The proof of Theorem 5 shows that if \( t \neq 0 \), then any solution of this equation (that is, the solution with initial conditions \( u_0 \mid_{x=x_0} = c_k \), \( 0 \leq k \leq 2 \), for arbitrary complex constants \( x_0, c_0, c_1, c_2 \)) is also meromorphic on the whole plane. The change of variables

\[
z = \frac{x}{(-3t)^{1/3}}, \quad u = (-3t)^{2/3} v(z)
\]

reduces this ODE as well as the KdV equation (4) to the \( \tau \)-independent ODE

\[
v'''_zzz + (v + z)v'_z + 2v = 0.
\]

The familiar Miura transform \( v = 6(f'_u - f^2) \) sends the solutions of this ODE to solutions of the second Painlevé equation (2), which has many applications to the problems of nonlinear mathematical physics.

Remark 5 Simultaneous solutions of the KdV equation (4) and the representative of the hierarchy (23) with \( n = 2 \) and \( \mu_2 \neq 0 \) determine simultaneous solutions (see [40]) of a pair of isomonodromic Hamiltonian systems \( H^{9/2} \) with two degrees of freedom belonging to Kawamuko’s list [41]. This list of Hamiltonian systems also consists of degenerations of the classical isomonodromic Garnier pair, but they are different from the hierarchy of its degenerations written out in [20].

Remark 6 Special simultaneous solutions of (4) and the representative of (23) with \( n = 2 \) and \( \mu_2 \neq 0 \) were studied in [42]–[44]. They give a universal description [42] of the corrective influence of small dispersion on the process of transforming weak discontinuities of solutions of ideal hydrodynamic equations into strong discontinuities. It is also clear that solutions of other higher representatives of the hierarchy of ODE (23) also describe this influence in certain degenerate cases, for example, in the case considered in [45, §3.2].

Note that the proof in [2] of the global meromorphy property for all solutions of the higher analogues of the second Painlevé equation (2) can also be simplified. Indeed, differentiating these analogues with respect to \( x \), we have the following hierarchy of ODE (\( n = 2, 3, \ldots \)):

\[
\frac{d}{dx} \left( t(2w^3 + w_2) + xw + \sum_{j=2}^{n} v_j P_j(w, w_1, \ldots, w_2) \right) = 0,
\]

(25)

where \( v_j \) are arbitrary complex numbers, \( v_n \neq 0 \). These ODE are obtained by adding the stationary part of the classical scaling symmetry \( w'_c = 3t(w_3 - 6w^2) + xw_2 + w \) of the mKdV equation (5) to a linear combination of the stationary parts of higher symmetries of this equation (see, for example, [46]),

\[
w'_c = \frac{d}{dx} P_n(w, w_1, \ldots, w_2) = \frac{d}{dx} \left( \frac{d}{dx} + w \right) L_n[w - w^2],
\]

where
where \( L_\mu[u] = L_\mu(u, u_1, \ldots, u_{2n}) \) are the Lenard polynomials \([17]\). Using Theorem 4 and the result of [4] about the global meromorphic extensibility with respect to \( x \) of all local holomorphic solutions of the mKdV equation, we see that for arbitrary complex parameters \( t = t_0, x_0 \) and \( a_i (i = 0, 1, \ldots, 2n) \) the solution of (23) with initial conditions \( u_i(x_0) = a_i (i = 0, 1, \ldots, 2n) \) is meromorphic on the whole complex \( x \)-plane. Hence, for arbitrary complex values of \( a_0, t = t_0, x_0 \) and \( a_i (i = 0, 1, \ldots, 2n-1) \), the solution of the \( n \)th equation of the second Painlevé hierarchy

\[
t(w_2 - 2w^3) + xw + \sum_{j=1}^n v_j P_j(w, w_1, \ldots, w_j) = a_n \tag{26}
\]

with initial conditions \( u_i(x_0) = a_i (i = 0, 1, \ldots, 2n-1) \) is meromorphic on the whole complex \( x \)-plane. Moreover, any local holomorphic solution of (26) is meromorphic with respect to \( x, t \) in some strip \([21]\).

**Remark 7** The hierarchy of the second Painlevé equation and its relation to the hierarchy of ODE (23) was earlier described in [47] - [49].

Our scheme can also be used to prove the global meromorphy of solutions of ODE given by the stationary parts of non-local symmetries of the KdV equation [4]. (The right-hand side of a non-local symmetry can depend not only on \( t, x, u, u_1, \ldots, u_m \) but also on the integrals of polynomials in the function \( u \) and its derivatives with respect to \( x \).) For example, consider the simplest non-local symmetry of [4], the so-called master symmetry

\[
\begin{align*}
\alpha = -3t & \left[ u_5 + \frac{5u_3u_2}{3} + \frac{10u_4u_1}{3} + \frac{5u_5u_1}{6} \right] \\
& + x(u_3 + uu_1) + 4u_2 + \frac{4u_2^2}{3} + \frac{u_1u_3}{3} = G^1(t, x, u, \ldots, u_5, v),
\end{align*}
\]

where

\[
\begin{align*}
v_x &= u, \tag{28} \\
v_i &= -u_2 - \frac{u_2^2}{2}. \tag{29}
\end{align*}
\]

The right-hand side \( G^1(t, x, u, \ldots, u_5, v) \) of this symmetry, which was introduced in the end of section 2 of [50], satisfies the following identity

\[
\frac{\partial G^1}{\partial t} - (u_3 + uu_1) \frac{\partial G^1}{\partial u} - \left( u_2 + \frac{u_2^2}{2} \right) \frac{\partial G^1}{\partial v} - \\
- \sum_{j=1}^5 \frac{\partial G^1}{\partial u_j} \left( \frac{d}{dx} \right)^j (u_3 + uu_1) = -u_1 G^1 - u \frac{dG^1}{dx} - \frac{d^2G^1}{dx^2}. \tag{30}
\]

The stationary part \( G^1(t, x, u, \ldots, u_5, v) = 0 \) of the master symmetry (27), after dividing both parts by \((-3t)\), takes the form

\[
g^1(t, x, u, \ldots, u_5, v) = 0, \tag{31}
\]

where the function

\[
g^1(t, x, u, \ldots, u_5, v) = u_5 + \frac{5u_3u_2}{3} + \frac{10u_4u_1}{3} + \frac{5u_5u_1}{6} - \\
\frac{1}{3t} \left\{ x(u_3 + uu_1) + 4u_2 + \frac{4u_2^2}{3} + \frac{u_1u_3}{3} \right\}
\]

is not the right-hand side of a symmetry of KdV. Nevertheless, by [30], the following equality holds on solutions of the ODE (31):

\[
\frac{\partial g^1}{\partial t} - (u_3 + uu_1) \frac{\partial g^1}{\partial u} - \left( u_2 + \frac{u_2^2}{2} \right) \frac{\partial g^1}{\partial v} - \\
- \sum_{j=1}^5 \frac{\partial g^1}{\partial u_j} \left( \frac{d}{dx} \right)^j (u_3 + uu_1) = - \left( u_1 g^1 + u \frac{d g^1}{dx} + \frac{d^2 g^1}{dx^2} \right).
\]

This equality means that the function

\[
S^2(t, x, u, u_1, \ldots, u_4, v) = - \frac{5u_3u_2}{3} - \frac{10u_4u_1}{3} - \frac{5u_5u_1}{6} + \\
+ \frac{1}{3t} \left\{ x(u_3 + uu_1) + 4u_2 + \frac{4u_2^2}{3} + \frac{u_1u_3}{3} \right\}
\]

satisfies the following condition:

\[
\frac{\partial S^2}{\partial t} = \frac{\partial S^2}{\partial u} (u_3 + uu_1) - \frac{\partial S^2}{\partial v} (u_2 + u_2^2/2) + \\
+ \sum_{i=1}^4 \frac{\partial S^2}{\partial u_i} \left( \frac{d}{dx} \right)^i (u_3 + uu_1) = - \left( \frac{d}{dx} \right)^5 (u_3 + uu_1).
\]

Combining this with obvious identity

\[
\frac{\partial S^1}{\partial t} = \frac{\partial S^1}{\partial u} (u_3 + uu_1) = - \frac{d}{dx} (u_2 + u_2^2/2)
\]

for the function \( S^1 = u \) and using Definition 2, we see that the system of ODE consisting of (28) and the equation

\[
u_5 = - \frac{5u_3u_2}{3} - \frac{10u_4u_1}{3} - \frac{5u_5u_1}{6} + \\
+ \frac{1}{3t} \left\{ x(u_3 + uu_1) + 4u_2 + \frac{4u_2^2}{3} + \frac{u_1u_3}{3} \right\} \tag{32}
\]

determines an invariant manifold of the system of evolution equations (29) and (4). Hence, by Theorem 1 for any six complex constants \( a \) and \( b_i (i = 0, \ldots, 4) \) there is a unique simultaneous analytic solution of the system of evolution equations (29, 4) and the system of ODE (33) in a neighborhood of any fixed point \((t_0, x_0)\) with initial conditions

\[
v(t_0, x_0) = a, \quad u_i(t_0, x_0) = b_i (i = 0, \ldots, 4). \tag{33}
\]

Using the uniqueness of this solution, the result of [4] on the meromorphic extension of its component \( u \) (a solution of the KdV equation (4)) to the strip \([21]\), and the shape of the ODE (32), we obtain the following theorem.

**Theorem 4** For any values of the complex constants \( x_0, t_0 \neq 0, a \) and \( b_i (i = 0, \ldots, 4) \), the solution of the system of ODE (28, 32) with \( t = t_0 \) and initial conditions (33) is meromorphic with respect to \( x \) on the whole complex plane \( C \).
An analogous theorem can be proved by the same scheme for the system of ODE consisting of (28) and the stationary part
\[ G^2(t,x,u,\ldots,u_5,v) = 0 \] (34)
of the following symmetry of the KdV equation (4):
\[ u_{\text{station}} = G^2(t,x,u,\ldots,u_5,v) \]
This symmetry generalizes the master symmetry (27) to the case of arbitrary complex constants \( k_1 \) and \( k_2 \).

Theorem 5 Let \( t_0 \neq 0 \) and \( k_1, k_2 \) be arbitrary complex parameters. Then, for any values of the complex constants \( a, b_i \) (if \( i = 0, \ldots, 4 \)), the system of ODE (28), (35) with \( t = t_0 \) and initial conditions (33) is meromorphic with respect to \( x \) on the whole complex plane \( \mathbb{C} \).

Remark 8 Recently Adler [51] described a simultaneous solution of the system of evolution equations (29), (4) and the system of ODE (28), (34). This solution provides an example of an exact solution of the so-called first Gurevich–Pitaevskii problem [52, §8] concerning solutions of the KdV equation (4) with step-like initial data possessing different constant limits as \( x \to \pm \infty \).

Repeating the proofs of Theorems 2 and 5, we arrive at the following result concerning the whole hierarchy of higher analogues of the system of ODE (28) and
\[ G^2(t,x,u,\ldots,u_5,v) + \frac{d}{dx} \left( L_m(u,u_1,\ldots,u_{2n}) + \sum_{j=1}^{n-1} \mu_j L_j(u,u_1,\ldots,u_{2j}) \right) = 0, \]
where \( L_j(u,u_1,\ldots,u_{2j}) \) are the Lenard polynomials (17).

Theorem 6 Let \( t_0, k_1, k_2 \) and \( \mu_j \) be arbitrary complex parameters. Then, for any values of the complex constants \( a, b_i \) (if \( i = 0, \ldots, 4 \)) and \( t = t_0 \) with \( n > 2 \) satisfying the initial conditions
\[ v(t_0, x_0) = a, \quad u_i(t_0, x_0) = b_i \quad (i = 0, \ldots, 2n) \]
is meromorphic with respect to \( x \) on the whole complex plane \( \mathbb{C} \).

IV. MEROMORPHY OF SOLUTIONS OF THE STATIONARY PARTS OF SYMMETRIES FOR THE COMPLEXIFIED NON-LINEAR SCHröDINGER EQUATION

Along with the hierarchy (26) of the second Painlevé equation (2) (this hierarchy is related to symmetries of the mKdV equation (5)), there is another hierarchy, which seems to be first described in [12]. The \( n \)th element of this hierarchy is a system of two ODE of the form
\[ \sum_{j=2}^{n} \mu_j g_j[p,q] - 2t_{p1} - ixp = 0, \]
\[ \sum_{j=2}^{n} \mu_j \psi_j[p,q] - 2t_{q1} + ixq = 0, \]
where \( \mu_j \) are arbitrary complex constants, \( \mu_n \neq 0, n > 1 \). It is obtained by adding the stationary part of the classical Galilean symmetry
\[ p_{\tau 0} = -2t_{p1} - ip, \quad q_{\tau 0} = -2t_{q1} + iq \]
of the complexified NLS equation (6) to a linear combination of the stationary parts of autonomous symmetries of (6),
\[ p_{\tau 0} = g_j[p,q], \quad q_{\tau 0} = \psi_j[p,q], \]
whose right-hand sides
\[ g_0[p,q] = ip, \quad \psi_0[p,q] = -iq, \]
\[ g_1[p,q] = p_1, \quad \psi_1[p,q] = q_1, \]
\[ g_2[p,q] = -ip + 2ip^2 q, \quad \psi_2[p,q] = iq - 2ip q^2, \]
\[ g_3[p,q] = -p_3 + 6p q q_1, \quad \psi_2[p,q] = -q_3 + 6pq q_1, \ldots \]
are related [33] by the recursive formula
\[ g_{j+1} = -\frac{d}{dx} g_j + 2ip \left( \frac{d}{dx} \right)^{-1} (p \psi_j + q g_j), \]
\[ \psi_{j+1} = i \frac{d}{dx} \psi_j - 2iq \left( \frac{d}{dx} \right)^{-1} (p \psi_j + q g_j). \]
Here the integrals \( \left( \frac{d}{dx} \right)^{-1} (p \psi_j + q g_j) \) are uniquely determined by requiring that they are equal to zero for \( p = q = 0 \). It is easily provable by induction that the functions \( g_j[p,q] \) and \( \psi_j[p,q] \) are polynomials of the form
\[ g_j[p,q] = A_j p_j + F_j(p_{j-2}, q_{j-2}, \ldots, p, q), \]
\[ \psi_j[p,q] = B_j q_j + G_j(p_{j-2}, q_{j-2}, \ldots, p, q) \]
for some non-zero constants \( A_j, B_j \) such that the expressions
\[ p \psi_j + q g_j \]
and
\[ p \psi_j + q g_j = \frac{d}{dx} \left( P_j(p_{j-1}, q_{j-1}, \ldots, p, q) \right), \]
\[ \frac{d}{dx} \psi_j - q \frac{d}{dx} g_j = \frac{d}{dx} \left( Q_j(p_j, q_j, \ldots, p, q) \right) \]
of some polynomials \( P_j(p_{j-1}, q_{j-1}, \ldots, p, q) \) and \( Q_j(p_j, q_j, \ldots, p, q) \).

By shifting the independent variables \( x, t \), the lowest representative of this hierarchy of systems of ODE can be written without loss of generality in the form
\[ -i \mu_2 [p_2 - 2p^2 q] - 2t_{p1} - ip = 0, \]
\[ i \mu_2 [q_2 - 2p q^2] - 2t_{q1} + iq = 0. \]


The familiar change of variables

\[ k = pq, \quad l = \frac{q}{q} \]  

(41)

(see, for example, the formula (1.5) in [2]) transforms the complexified NLS equation (45) to the system of long water waves with purely imaginary time,

\[ k_t' = i(2k'_l + 2k'_l - k_2), \quad l_t' = i(l_2 + 2l'_l - 2k'_l). \]  

(42)

(The change of variables (41) actually sends the solutions of the last system to solutions of a somewhat more general system

\[ p_t' = -ip_2 + 2i(p^2 q + b(t)p), \quad q_t' = iq_2 - 2i p^2 - b(t)q. \]

But the simple transformation

\[ p(t) = \exp \left( \int_{t_0}^{t} b(\tau) d\tau \right) P, \quad q(t) = \exp \left( - \int_{t_0}^{t} b(\tau) d\tau \right) Q \]

again reduces them to solutions \((P, Q)\) of the complexified NLS equation (46).

The change of variables (41) transforms the system of ODE (40) to the following system of two first-order ODEs depending on a complex parameter \(\gamma\):

\[ k_1 = 2l + \frac{2it}{\mu_2} - \gamma, \quad l_1 = -l - \frac{2it}{\mu_2} + 2k + i \frac{2it}{\mu_2}. \]  

(43)

Indeed, it follows from (40) that

\[ (p_t'q - q_t'p)_t = \frac{2itk_1}{\mu_2} \]

or, after integration,

\[ p_t'q - q_t'p = \frac{2itk_1}{\mu_2} - \gamma. \]

Here \(\gamma\) is independent of \(t\) since \((p, q)\) is a solution of the complexified NLS equation (46). Hence the functions (41) constructed from any solution of (40) satisfy the first equation in (43). A direct verification shows that they also satisfy the second equation in (43). In its turn, the system (43) is related by the formulas

\[ k = -\nu, \quad l = -u - \frac{it}{\mu_2} \]

with the equivalent system of ODE

\[ u_t = u^2 + 2v - \frac{ix}{\mu_2} + \frac{r^2}{\mu_2^2}, \quad v_t' = -2uv + \gamma, \]

which coincides with the system of ODE (69), (70) in [54, §5.1]. The component \(u\) of a solution of this system satisfies the ODE

\[ u_2 = 2u^3 + 2u \left( \frac{r^2}{\mu_2^2} - \frac{ix}{\mu_2} \right) - \frac{i}{\mu_2} + 2\gamma, \]

which is related by the trivial transformations

\[ w = \left( \frac{i \mu_2}{2} \right)^{1/3} u, \quad z = \left( \frac{i \mu_2}{2} \right)^{-1/3} \left( x + \frac{it^2}{\mu_2} \right) \]

to the second Painlevé equation (2). Hence the hierarchy of systems of ODE (36) may indeed be regarded as a hierarchy of higher analogues of (2). Its Solutions are possibly related by the change of variables (41) to solutions of another second Painlevé equation introduced in [54].

Being the stationary part of a symmetry of (6), the system of ODE (46) determines an invariant manifold of (6). By Theorem 1 for any complex constants \(a_0, \ldots, a_{2n-1}, 0\) and \(b_0, \ldots, b_{2n-1}, 0\) there is a unique simultaneous holomorphic solution of the system of ODE (36) and the evolutionary system (6) in a neighborhood \((t_0, x_0) \in \mathbb{C}^2\) with initial conditions

\[ p_j(t_0, x_0) = a_j, \quad q_j(t_0, x_0) = b_j \quad (j = 0, \ldots, n - 1). \]

(44)

By [4], every holomorphic solution of (6) in the bidisk extends meromorphically in \(x, t\) to the strip \(|t - t_0| < \delta_1\). Hence the following theorem holds.

**Theorem 7** For any complex constants \(t_0, x_0, a_0, \ldots, a_{2n-1}, 0\) and \(b_0, \ldots, b_{2n-1}, 0\), the solution \((p, q)\) of the system of ODE (36) with \(t = t_0\) and initial conditions (44) is meromorphic with respect to \(x\) on the whole complex plane \(\mathbb{C}\).

**Remark** After the reduction \(p = \delta q\) with real \(\delta\), some solutions of the ODE \(\beta(q_3 - 6\delta |q|^2 q_1) - 2t q_1 - i x q = 0\) in this hierarchy coincide with the Haberman–Sun special solution of the NLS equation \(-i q_3 = q_2 - 2\delta |q|^2 q\). This solution tends to the Pearcey integral \(\text{const} \int_{\mathbb{R}} \exp[-i(\beta \lambda^4 + 4t \lambda^2 + 2x \lambda)] d\lambda\) as \(\delta \to 0\) and describes the influence of small non-linearities on the high-frequency asymptotics near the cusps of caustics. Other solutions of this ODE with \(\delta = -1\) give a principal-order description of the influence of small dispersion on the processes of falling self-steepening of pulses, which are typical for non-linear geometric optic approximations. By [59], special solutions of other representatives of the hierarchy of ODE (36) give a hierarchy of solutions of the NLS equation \(-i q_3 = q_2 - 2\delta |q|^2 q\) which describe the influence of small non-linearities on the high-frequency asymptotics under surgeries of caustics.

Our scheme gives an equally simple way to prove the global meromorphy in \(x\) of the solutions of initial-value problems for the hierarchy of systems of ODE

\[ \sum_{j=0}^{n} \mu_j g_j[p, q] + 2it(p_2 - 2p^2 q) - xp_1 - p = 0, \]

(45)

\[ \sum_{j=2}^{n} \mu_j \psi_j[p, q] - 2it[q_2 - 2pq^2] - xq_1 - q = 0, \]

where \(\mu_j\) are any complex constants and \(\mu_n \neq 0\). These systems of ODE are obtained by adding the stationary part of the classical scaling symmetry

\[ p_{x \tau} = 2it(p_2 - 2p^2 q) - xp_1 - p, \quad q_{x \tau} = -2it(q_2 - 2pq^2) - xq_1 - q \]

(46)
of the complexified NLS equation \(6\) to a linear combination of the stationary parts of the autonomous symmetries \(37\).

**Remark 10** As already mentioned in \(60\), it seems that the set of solutions of \((45)\) with \(n = 3\) contains the special solution of the NLS equation \(-iq_t = q_2 - 2 \delta [q^2] q\) described in \(61\). (When \(\delta = 0\), this solution reduces to \(\alpha \exp[j \beta \lambda^3 - t \lambda^2 + \lambda \delta \lambda] d\lambda\), where \(\alpha\) and \(\beta\) are real constants.)

In view of the possibility of shifts in \(x\), there is no loss of generality in writing the lowest representative of the hierarchy of ODE \((45)\) in the form of the system

\[
2i[pt - 2p^2 q] - xp_1 - p(1 - x \mu_0) = 0, \\
-2i[q_t - 2pq^2] - xq_1 - q(1 + x \mu_0) = 0.
\]

The change of variables \((41)\) transforms simultaneous solutions of the complexified NLS equation \((6)\) and this system to simultaneous solutions of the evolutionary system \((42)\) and the system of ODE

\[
2i[2k_1 l + 2k_1^2 - k_2] = -2k - xk_1^2, \\
2i[l_2 + 2l_1^2 - k_1^2] = -l - xl_1.
\]

For \(t \neq 0\), this system can be reduced to the fourth Painlevé equation; see the end of section 1 in \(2\). (Integrating the last equation in \((47)\) with respect to \(x\), we have

\[
k = \frac{l_1 + l_2}{2} + \frac{x l - a}{4 t},
\]

where the integration constant \(a\) is independent of \(t\) since the functions \(k(t,x)\) and \(l(t,x)\) satisfy the system \((42)\). Substituting the expression \((48)\) for \(k\) into the first equation in \((47)\) and making the changes of variables

\[
z = \frac{(i)^{1/2} x}{2 t^{1/2}}, \\
w = \frac{(i)^{1/2}}{2 t^{1/2}} [l - ix \frac{2 t}{2 t}],
\]

we arrive at the ODE

\[
w_{zz} = 6w^2 w_z + 12zw w'_z + 4w^2 + 4zw + 4(c^2 + a)w_z,
\]

which can also be obtained by differentiation with respect to \(z\) of the canonical fourth Painlevé ODE

\[
w_{zz} = \frac{(w')^2}{2w} + \frac{3w^3}{2} + 4zw^2 + 2(c^2 + a)w + \frac{b}{2w}
\]

depending on the two parameters \(a\) and \(b\). Therefore the systems of ODE \((45)\) may naturally be regarded as higher analogues of the fourth Painlevé equation \((50)\). This hierarchy differs from the hierarchy of higher analogues of \((50)\) considered earlier in \(54\), \(52\), but can possibly be related to it by the formulas \((41)\).

We omit the proof of the following theorem since it contains nothing principally new compared to the proof of Theorem \(3\).

**Theorem 8** For every \(n > 2\) and any complex constants \(t_0, \varphi_0, a_{00}, \ldots, a_{2(n-1)0}\) and \(b_{00}, \ldots, b_{2(n-1)0}\), the solution \((p,q)\) of the system of ODE \((45)\) with \(t = t_0\) and initial conditions \((44)\) is meromorphic with respect to \(x\) on the whole complex plane \(\mathbb{C}\).

As in the case of non-local symmetries of the KdV equation \((3)\), our general scheme can be used to justify the meromorphy of all solutions of initial-value problems for systems of ODE given by the stationary parts of non-local symmetries of the NLS system \((6)\). We illustrate this by an example, which can obviously be generalized in the same sense as Theorems \(5\) and \(6\) generalize Theorem \(4\).

Consider the system of ODE related to the master symmetry

\[
p'_{5}\mu = G(t, x, p, q, v, p_1, p_2, p_3), \\
v'_{5}\mu = \Psi(t, x, p, q, v, q_1, q_2, q_3)
\]

of the complexified NLS equation \((6)\), where

\[
G(t, x, p, q, v, p_1, p_2, p_3) = 2t(p_3 - 6pq p_1) + ip(xq - 2p^2 q) + 2ip_1 - 2ipv,
\]

\[
\Psi(t, x, p, q, v, q_1, q_2, q_3) = 2t(q_3 - 6pq q_1) - iq(xq - 2q^2 p) - 2i q_1 + 2iqv,
\]

\[
v' = pq,
\]

\[
v' = i(q_1 p - p_1 q).
\]

(The right-hand sides of this symmetry are obtained by formally applying the right-hand sides of the recursion relations \((39)\) to the right-hand sides of the equations of the classical scaling symmetry \((46)\). The basic fact of compatibility of the system \((51)\) with the system of evolution equations \((6)\) means that the following identities hold for the functions \((52)\) and \((53)\):

\[
\frac{\partial G}{\partial t} + i(p_2 - 2p^2 q) \frac{\partial G}{\partial p} - i(p_2 - 2p^2 q) \frac{\partial G}{\partial q} + \frac{\partial \Psi}{\partial t} + i(p_2 - 2p^2 q) \frac{\partial \Psi}{\partial p} + i(p_2 - 2p^2 q) \frac{\partial \Psi}{\partial q} + i(q_1 p - p_1 q) \frac{\partial \Psi}{\partial q} + i(q_1 p - p_1 q) \frac{\partial \Psi}{\partial q} = i \left( \frac{\partial G}{\partial x} \right)^2 - 4ipq G + 2ip^2 \Psi,
\]

\[
\frac{\partial \Psi}{\partial t} + i(p_2 - 2p^2 q) \frac{\partial \Psi}{\partial p} + i(p_2 - 2p^2 q) \frac{\partial \Psi}{\partial q} + i(q_1 p - p_1 q) \frac{\partial \Psi}{\partial q} + i(q_1 p - p_1 q) \frac{\partial \Psi}{\partial q} = i \left( \frac{\partial \Psi}{\partial x} \right)^2 - 4ip\Psi q + 2i \Psi^2.
\]
where

\[ G^1(t, x, p, q, p_1, p_2, p_3, v) = \]
\[ = p_3 - 6pqp_1 + \frac{ix(p_2 - 2p^2q) + 2ip_1 - 2ipv}{2t} \]

\[ \Psi^1(t, x, p, q, q_1, q_2, q_3, v) = \]
\[ = q_3 - 6pqq_1 + \frac{-i(xq_2 - 2q^2p) - 2iq_1 + 2iqv}{2t} \]

It follows from the identities \((56), (57)\) that on all solutions of the closed system of ODE \((54), (58)\) we have

\[
\frac{dG^1}{dt} = \frac{-dG^1}{dx^2} + 4ipqG^1 + 2ip^2\Psi^1, \\
\frac{d\Psi^1}{dt} = \frac{-d\Psi^1}{dx^2} - 4ipq\Psi^1 - 2iq^2G^1. 
\]

(59)

In their turn, the equalities \((59)\) mean that the expressions

\[ S^1(t, x, p, q, v, p_1, p_2) = G^1 - p_3 = \]
\[ = -6pqp_1 + \frac{ix(p_2 - 2p^2q) + 2ip_1 - 2ipv}{2t} \]

and

\[ S^2(t, x, p, q, v, q_1, q_2) = \Psi^1 - q_3 = \]
\[ = -6pqq_1 + \frac{-i(xq_2 - 2q^2p) - 2iq_1 + 2iqv}{2t} \]

satisfy the following identities on all solutions of the system \((54), (58)\):

\[
\frac{\partial S^1}{\partial t} - i(p_2 - 2p^2q)\frac{\partial S^1}{\partial p} + \\
+ i(q_2 - 2q^2p)\frac{\partial S^1}{\partial q} + i(p_1 - p_1q)\frac{\partial S^1}{\partial v} - \\
- i\sum_{j=1}^{3} \left[ \frac{\partial S^1}{\partial p_j} \frac{d}{dx} \right] j (p_2 - 2p^2q) - \frac{\partial S^1}{\partial q_j} \left( \frac{d}{dx} \right) j (q_2 - 2q^2p) = \\
= i \left( \frac{d}{dx} \right)^3 (p_2 - 2p^2q), \\
\frac{\partial S^2}{\partial t} - i(p_2 - 2p^2q)\frac{\partial S^2}{\partial p} + \\
+ i(p_2 - 2p^2q)\frac{\partial S^2}{\partial q} + i(p_1 - p_1q)\frac{\partial S^2}{\partial v} - \\
- i\sum_{j=1}^{3} \left[ \frac{\partial S^2}{\partial p_j} \frac{d}{dx} \right] j (p_2 - 2p^2q) - \frac{\partial S^2}{\partial q_j} \left( \frac{d}{dx} \right) j (q_2 - 2q^2p) = \\
= -i \left( \frac{d}{dx} \right)^3 (q_2 - 2q^2p). 
\]

Moreover, the function \(S^3(p, q) = pq\) satisfies the obvious equality

\[
\frac{\partial S^3}{\partial t} - i(p_2 - 2p^2q)\frac{\partial S^3}{\partial p} + i(q_2 - 2q^2p)\frac{\partial S^3}{\partial q} = i\frac{d}{dx} (q_1p - p_1q). 
\]

By Definition\(2\) these three equalities mean that the system of three non-linear ODE \((54), (55)\) determines an invariant manifold of the system of evolution equations \((6), (55)\). Combining this with Theorem\(1\) and the result of \([4]\) about the global meromorphy of all local holomorphic solutions of the complexified NLS equation \((6)\), we deduce the following theorem by our standard scheme.

**Theorem 9** For any values of the complex constants \(t_0 \neq 0, x_0, a_i\) and \(b_i\) \((i = 0, 1, 2)\), the solution of the system of ODE \((54), (58)\) with \(t = t_0\) and initial conditions

\[ p_i(t_0, x_0) = a_i, \quad q_i(t_0, x_0) = b_i \quad (i = 0, 1, 2) \]

is meromorphic with respect to \(x\) on the whole complex plane \(\mathbb{C}\).  

**Remark 11** The set of solutions of the stationary part of the symmetry \((51)\) with the reduction \(p = -q^2\) contains the even \((in x)\) simultaneous solutions of the equation

\[ 2t(q_3 - 6pqq_1) - i(xq_2 - 2q^2p) - 2iq_1 - 2iq \int_0^x |q^2(t, \zeta)| d\zeta \]

(60)

and the focusing NLS equation \(q'_1 = i(q_{1xx} + 2|q_1|^2)\) which universally describe the corrective influence of small dispersion on the self-focusing of non-linear geometric optic approximations \([63]\) in spatially one-dimensional problems with sufficiently small non-linearities. These simultaneous solutions of \((60)\) and the focusing NLS equation also arise in the description of rogue waves of infinite order \([64]\). As indicated in \([64]\), they satisfy an ODE which is a particular case of the second term of the third Painlevé hierarchy written out in \([65]\). See also \([66]\–\[71]\) for the properties of these solutions and their generalizations.

V. SYSTEMS OF EQUATIONS OF PAINELEVÉ TYPE THAT DETERMINE INVARIANT MANIFOLDS OF THE SAWADA–KOTERA EQUATION, AND MEROMORPHIC OF THEIR SOLUTIONS

It is known that the stationary parts of symmetries determine invariant manifolds of systems of evolution equations. But the converse is not true in general. A striking example of a hierarchy of such invariant manifolds was given by Bagnenerina \([72]\) in the case of the Sawada–Kotera equation \((7)\). The relation of these invariant manifolds to symmetries of \((7)\) is highly non-trivial. The hierarchy is given by the ODE

\[ U_i(u, \ldots, u_{2i}) = 0, \]

(61)

where \(U_i(u, \ldots, u_{2i})\) is a sequence of polynomials

\[ U_0 = u, \quad U_1 = u_2 - 6u^2, \]
\[ U_3 = u_6 - 6(6uu_4 + 10u_1u_3 + 5(u_2)^2) + 36(u^2u_2 + u(u_1)^2 - u^4), \]

\[ \ldots \]

The formula

\[ U_{i+1} = \left( \frac{d^2}{dx^2} \right) J_i - u J_i, \]

(62)
expresses them in terms of the polynomials \( J_i(u, \ldots, u_2) \) with

\[
J_{-1} = -1/6, \quad J_0 = u, \quad J_2 = u_4 - 30u_2 + 6u^3, \\
J_3 = u_6 - 12(uu_4 + u_1u_3 + (u_2)^2) + 252(2u^2u_2 + u(u_1)^2 - 2u^4),
\]

which determine a familiar [73] infinite series of symmetries of the SK equation (7):

\[
u'_a = f_i = \left( \frac{d}{dx} \right) J_i, \quad (63)
\]

Note that \( U_{3i} = J_{3i+1} = 0 \) for every \( i > 0 \). These symmetries satisfy the recursion relation \( f_{i+3} = Rf_i \) with

\[
R = \left[ \frac{d^2}{dx^2} - 24u - 12u_1 \left( \frac{d}{dx} \right)^{-1} \right] \left[ \frac{d^2}{dx^2} - 6u \left( \frac{d}{dx} \right)^{-1} \right]
\]

This relation was found in [74].

We generalize Bagderina’s construction to the case of a system of non-autonomous ODE. Consider the simplest non-autonomous symmetry of the SK equation (7), the classical scaling symmetry

\[
u_x = 5ut_x + xu_x + 2u = 5t[u_5 - 30uu_3 - 30u_1u_2 + 180u^2u_1] + xu_x + 2u.
\]

(The stationary part)

\[
5t[u_5 - 30uu_3 - 30u_1u_2 + 180u^2u_1] + xu_x + 2u = 0 \quad (64)
\]

of this symmetry determines self-similar solutions of the Sawada–Kotera equation by the change of variables

\[
z = \frac{x}{t^{1/5}}, \quad u = \frac{w(z)}{t^{1/5}}.
\]

This change of variables reduces the SK equation (7) and the ODE (63) to the \( t \)-independent non-linear ODE

\[
5(w_3 - 30ww_1 - 30w_1w_2 + 180w^2w_1) + zv_1 + 2w = 0
\]

on the function \( w(z) \), where \( w_f = \frac{d^2w}{dx^2}, \) the derivative in \( x \) of the function

\[
5t[u_4 - 30uu_2 + 60u^3] + xu + v, \quad (65)
\]

where

\[
v_x = u, \quad (66)
\]

cocincides with the right-hand side of the scaling symmetry. Formally acting on (65) by the differential operator in the right-hand side of (63) and dividing the result by 5t, we obtain a function

\[
U_r = u_6 - 6(6uu_4 + 10u_1u_3 + 5u_2^2) + 360(2u^2u_2 + uu_1^2 - u^4) - 6x(u_2 - 6u^2) + 3u_1 - 6uv.
\]

A direct verification shows that the system of ODE consisting of the equations (66) and the non-autonomous ODE

\[
U_r = 0 \quad (67)
\]

determines an invariant manifold (in the sense of Definition [3]) of the evolutionary system consisting of the equation

\[
v_t = u_4 - 30uu_2 + 60u^3 \quad (68)
\]

and the SK equation (7).

The following theorem is proved along the same lines as in Theorems [5, 9].

**Theorem 10** For any complex numbers \( t_0 \neq 0, \ x_0, \ a \) and \( b_i \)

\( i = 0, \ldots, 4, \) the solution of the system of ODE (66), (67) with \( t = t_0 \) and initial conditions (65) is meromorphic with respect to \( x \) on the whole complex plane \( C. \)

**Proof.** In [75], the SK equation (7) was represented in the Lax form \( L = [A, L] \) with differential operators

\[
L = D^3 + uD_x, \ A = -9D_x^4 + 80uD_x^3 + 80u_1D_x^2 - (180u^2 - 60u_2)D_x
\]

doing coprime orders, where \( D = \frac{d}{dx} \). By the main theorem in [5] (with an irrelevant inaccuracy corrected in [6]), all the coefficients of \( L \) (in our case, \( u(t, x) \)) for any local holomorphic solution extend to globally meromorphic functions of the spatial variable \( x \). Since the system of ODE (66), (67) determines an invariant manifold of the system of evolution equations (68), (7), Theorem 11 yields a simultaneous solution of these two systems with initial conditions (63) in a neighborhood of the point \( t = t_0, x = x_0 \). The rest of the proof is clear. \( \blacksquare \)

VI. CONCLUSION

We have demonstrated on a number of examples that the problem of rigorously proving the global meromorphy of solutions of many non-linear ODE or their systems can be solved easily and uniformly by using the general scheme described in the proofs of Theorems [3, 10]. Here are the key assumptions under which this scheme works.

1) The system of non-linear ODE under study is equivalent to a system of ODE of the form (10) such that its right-hand sides are globally analytic in the independent variable \( x \) and it determines an invariant manifold of some evolutionary system of the form (8). Then Theorem 11 guarantees the existence of a simultaneous holomorphic solution in a neighborhood of any given point in \( C^2 \).

2) The solutions of (8) are expressible in terms of solutions of an evolutionary system (usually integrable by inverse scattering transform) with the following meromorphic extension property. Any local holomorphic solution can be extended globally in \( x \) (and locally in other independent variables) to a strip in \( C^2. \)

The proof of this property is the main non-trivial point of our scheme. It is currently unavailable for some integrable
which is related to solutions of (69) [76]–[78], as well as for some matrix analogues of the second Painlevé equation described in [79] and their hierarchies.

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