Guarantees for Spontaneous Synchronization on Random Geometric Graphs.

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Abstract

The Kuramoto model is an important mathematical model to study the synchronization phenomena in complex networks. The rich theory behind the rigorous treatment of the synchronization phenomenon has been connected to many branches of science. In particular, the study of the energy function associated with the Kuramoto model is relevant in the context of non-convex optimization for machine learning and data science.

In this work, we use tools from random matrix theory, random graphs, and statistics to derive rigorous guarantees for the random geometric graph on the sphere to synchronize with probability of success tending to one as the number of nodes tends to infinity. To the best of our knowledge, this is the first rigorous result for the Kuramoto model on random geometric graphs.

1 Introduction

Spontaneous synchronization is an ubiquitous phenomenon in nature. The experimental investigation of this phenomenon dates back to the work of Huygens in 1665, who observed that two pendulums held by the same wooden beam tend to swing in anti-phase after a certain amount of time, regardless their initial position. Similar phenomena have been observed in many branches of science [26].

In a seminal work [18], Kuramoto introduced the first mathematical tractable model to study synchronization in a system of all-to-all phase oscillators. Natural generalizations for complex networks were considered in the literature, but their analysis remains not fully understood. We point out that different versions of the Kuramoto model have been studied with applications in electric power networks, wireless networks, neuroscience [19][10][24], to name a few.

In this work, we study the following model: Consider a graph $G$ with adjacency matrix $A$ with entries $a_{ij} \in \{0, 1\}$. Each node $i$ is associated with a phase $\theta_i \in (0, 2\pi]$ that varies with the time $t$ and a natural frequency $\omega_i \in \mathbb{R}$. The system of differential equations below represents the Kuramoto model associated
with the graph $G$

$$\forall i \in [n] : \frac{d\theta_i(t)}{dt} = w_i - \frac{K}{n} \sum_{i,j=1}^{n} a_{ij} \sin(\theta_i - \theta_j),$$

where $K$ is the coupling strength.

In order to give a more intuitive description of the model, we describe an alternative viewpoint due to Dörfler, Chertkov, and Bullo [10]. For simplicity, in this manuscript, we only discuss the homogeneous Kuramoto model, where $w_i$ is zero for all $1 \leq i \leq n$. The coupled oscillators system can be interpreted as a system of particles that tends to minimize the following energy function

$$E_o(\theta) := \sum_{i,j} a_{ij} (1 - \cos(\theta_i - \theta_j)),$$

and the standard Cauchy problem associated with this energy function can be written as

$$\forall i \in [n] : \frac{d\theta_i(t)}{dt} = -\frac{\partial E_o(\theta_1, \ldots, \theta_n)}{\partial \theta_i}, \quad \theta_i = \theta_i(0).$$

From the equation above, the intuitive meaning of Kuramoto model is that the evolution moves according to the gradient flow of the energy function, i.e., the direction of maximum decrease of $E_o$. With this gradient flow formulation, we study the relationship between the graph $G$ and the synchronization phenomenon associated to the topology of $G$. Mathematically, this means that we are interested in determining whether the energy function $E_o(\theta)$ admits a local minimum that is not the global minimum, also called a spurious minimum. It is easy to see that the global minimum corresponds to $\theta_1 = \ldots = \theta_n = 0$ (up to a global shift). The challenge here is that the function is not convex so it is necessary to do a deeper analysis of the optimization landscape associated with the energy function. In fact, there is a significant body of work dedicated to the study of the optimization landscape of non-convex functions because such problems often appear in machine learning and data science (e.g. [13, 14]), some works have been directly connected to the study of energy functions such as the one treated in this work [2][20].

1.1 Kuramoto Model on random graphs

A trivial necessary condition for the oscillators to synchronize is that the underlying graph is connected. However, this is far from being sufficient. There are many examples of networks that do not synchronize even if all nodes are connected over half of the other nodes. There are also known sparse networks that synchronize [20][27]. Moreover, it is known that the addition of a single edge can create spurious local minima [20].

The motivation to study the synchronization phenomenon in random graphs is to study the behavior of oscillators on a typical graph. In this spirit, the
standard Erdős-Rényi random graphs, in which an edge is placed between every pair of vertices independently with probability $p$, were considered to quantify the typical behaviour of networks in the homogeneous Kuramoto model, i.e., when all $\omega_i$ are equal to zero [17, 20].

In this work, we also focus on the homogeneous case, and analyse the existence of spurious local minima when the graph $G$ is drawn at random with some geometric structure behind, the so called random geometric graphs on the sphere. These are graphs with $n$ vertices sampled by generating $n$ points in the unit sphere, and placing an edge between every pair of vertices with Euclidean distance smaller than a certain threshold. The random geometric graph on the sphere is an interesting model because it captures the property that two nodes are more likely to connect if both are connected with a third node. Similar models have been studied in the literature [8, 12, 19], but we are not aware of any rigorous mathematical guarantees.

In contrast to the Erdős-Rényi random graph model studied in [17, 20], the adjacency matrix of the random geometric graph does not enjoy independence among entries or columns, which requires different tools in their analysis. Our main contribution is to give a rigorous treatment for the random geometric graphs by exploring its connections with random matrix theory and recent work on hypothesis testing among random graph models [21]. Our main result establishes that a random geometric graph with $n$ nodes drawn uniformly at random from the unit sphere $\mathbb{S}^{d-1}$ synchronizes, with high probability, as long as $d$ is sufficiently large compared to $n$. We state here a simplified statement

**Theorem 1. (Simplified Statement of the Main Result Theorem 4)** Let $G$ be a random geometric graph with $n$ nodes on the unit sphere $\mathbb{S}^{d-1}$ such that each edge is connected with probability $p$. Then if $np = \Omega(\log^2(n))$ and $d = \tilde{\Omega}(n^2p^2)$, the graph $G$ synchronizes with high probability. Here $\tilde{\Omega}(\cdot)$ hides polylogarithmic factors.

A more detailed version and its proof is presented in Section 3. We end up this section with a discussion about our main result. It shows that if $p = \tilde{\Theta}\left(\frac{1}{n}\right)$ and $d = \tilde{\Omega}(n^2p^2) = \tilde{\Omega}(1)$, then the random geometric graph on the sphere $\mathbb{S}^{d-1}$ with connectivity probability $p$ synchronizes with high probability. On the other hand, it is natural to expect that if $p$ grows and the other parameters are fixed, then $G$ also synchronizes with high probability. If we could formally prove that this monotonicity holds, then our result would imply that if we fix any $p = \tilde{\Omega}\left(\frac{1}{n}\right)$ and $d = \tilde{\Omega}(1)$, then the graph synchronizes. This is essentially the best we can hope for (up to logarithmic factors) because below this threshold the graph will not be connected with non-vanishing probability.

The challenge is that the proof technique relies in comparisons between the geometric and Erdős-Rényi random graph models which impose upper bounds on $p$. While one expects that, on average, larger $p$ helps synchronization (even if it might make a geometric graph more easily distinguishable from an Erdős-Rényi one), we were not able to establish such a monotonicity result, and we leave it as an open problem. An important difficulty is that adding edges, in a worst case sense, may destroy the synchronization property.
Notation
The standard unit sphere in $\mathbb{R}^d$ is denoted by $\mathbb{S}^{d-1}$ as usual. The matrix $J_n$ represents the $n \times n$ all ones matrix and $I_n$ denotes the $n \times n$ identity matrix. For vectors $u, v$ of same size, the inner product $\langle u, v \rangle$ corresponds to the standard inner product, $\|v\|_2$ denotes the standard Euclidean norm. For a matrix $A$, $\|A\|$ denotes the standard spectral norm. We consider the standard "Big-O" notation (Landau notation) and we write $\tilde{\Omega}(\cdot)$ to hide log factors.

2 Preliminaries on the Kuramoto Model

We will establish spontaneous synchronization of the graphs drawn from our random graph model by showing that they satisfy, with high probability, explicit properties that imply synchronization. In what follows we describe two such sets of properties. One approach, developed to analyse the landscape of certain optimization problems arising from the Burer-Monteiro relaxation to semidefinite programming (see [5, 6]), yields a weaker estimate than the one on this paper, and is available in an earlier version of our work [15].

A second approach, based on very recent remarkable progress by Kassabov, Strogatz and Townsend [17] on the problem of synchronization of Erdős-Rényi graphs, provides better sharper guarantees. They established a sufficient condition for the graph to synchronize based on the so called complex order parameters $\rho_k := \frac{1}{n} \sum_{j=1}^{n} e^{ik\theta_j}$, for $k \in \mathbb{N}$. These parameters encode information about the landscape of $E(\theta)$, for example, $\rho_1 = 1$ corresponds to the state of pure synchrony. Their main result in this direction is the following theorem

**Theorem 2.** [17, Theorem 6] Let $G$ be a graph with $n$ vertices and fix $p_o$ such that $0 < p_o < 1$. Let $A$ and $L$ be the associated adjacency, Laplacian matrix respectively, and define $\Delta_A = A - p_o J_n$, $\Delta_L = L + p_o J_n - np_o I_n$. If the three conditions below are satisfied

1. $\frac{\|\Delta_A\|}{np_o} < \frac{1}{12}$,
2. $\frac{\|\Delta_L\|}{np_o} < \frac{1}{4}$,
3. $\frac{\pi/4}{\sin^{-1}\left(\frac{\|\Delta_A\|}{np_o}\right)} > \frac{\log(n/6)}{\log^2(\frac{n/6}{np_o})} + 1$,

then the homogeneous Kuramoto model on the graph $G$ globally synchronizes.

In the next section, we prove that these conditions are satisfied for random geometric graphs on the sphere in the regime stated in Theorem 4.

3 The Random Geometric Graph on the Sphere

We start by formally defining the random geometric graph on the sphere. We also include the definition of the Erdős-Rényi random graph for the sake of completeness.
Definition 1 (Erdős-Rényi Random Graph [11]). Let $n \in \mathbb{N}$ and consider $p$ a fixed probability. The random graph $G(n, p)$ is drawn as follows: For each pair $\{i, j\} \in [n] \times [n]$ with $i \neq j$, an edge is placed with probability $p$ independently.

Definition 2 (Random Geometric Graph on the Sphere [22,23]). Let $n \in \mathbb{N}$ and $X_1, \ldots, X_n \in \mathbb{S}^{d-1}$ be points drawn independently and uniformly at random on the unit sphere $\mathbb{S}^{d-1}$. We define the random geometric graph $G(n, p, d)$ as the graph on the vertex set $[n]$ such that for $i \neq j$, the pair $\{i, j\} \in [n] \times [n]$ is an edge of the graph if and only if $\langle X_i, X_j \rangle \geq t_{p,d}$, where $t_{p,d} \in [-1,1]$ is related to $p$ by

$$p = \mathbb{P}(\langle X_i, X_j \rangle \geq t_{p,d}).$$

Equivalently, $\{i, j\}$ is an edge of the graph if and only if the Euclidean distance between $X_i$ and $X_j$ satisfies $\|X_i - X_j\|_2 \leq \sqrt{2 - 2t_{p,d}}$.

A significant body of work in statistics [3,4,7,21] is dedicated to the following question: If we receive one sample from a random graph that is either sampled from $G(n, p, d)$ or $G(n, p)$, is it possible to determine, with high probability, from which graph it was sampled? In other words, can we do hypothesis testing with the probability of error going to zero as $n$ goes to infinity?

Clearly, if we cannot distinguish between the $G(n, p)$ and $G(n, p, d)$, then the random graph $G(n, p, d)$ must synchronize with high probability since we know that the $G(n, p)$ does. Recall that the definition of total variation distance between two random graphs $G$ and $G'$ with set of vertices $V$ is defined by

$$\text{TV}(G, G') = \max \{\mathbb{P}(G \in G') - \mathbb{P}(G' \in G)\},$$

where $G$ is any set of graphs over $V$. In our context, the relevant result is the following

Theorem 3. [21, Theorem 1.2] For any fixed constant $\alpha > 0$, if $\frac{\alpha}{n} < p < \frac{1}{2}$ and $d = \Omega(n^3p^2)$, then

$$\lim_{n \to \infty} \text{TV}(G(n, p), G(n, p, d)) = 0.$$
Lemma 1. Let $A$ be the adjacency matrix of a graph sampled from the $G(n, p)$ with i.i.d Bernoulli $p$ random variables above the diagonal. Suppose that $np > \log(n)$. Then there exist absolute constants $K, c > 0$ such that, with probability at least $1 - e^{-t^2/c}$,

$$\|A - E_A\| \leq K\sqrt{np} + t.$$ 

We provide a short sketch of the proof.

**Proof.** By Theorem 2.1 in [25], we know that $E\|A - E_A\|$ is at most a constant times the expectation of the maximum between the Euclidean norms of the columns. Based on this fact, an easy computation gives that $E\|A - E_A\| \leq C\sqrt{np}$. To get the tail bound, we apply Talagrand convex Lipschitz concentration inequality [28, Theorem 5.2.16].

The second preliminary proposition that we need is the following.

**Proposition 1.** [21, Proposition 1.3] For any constant $\alpha > 0$, there exist constants $C_{\alpha}, C_{\epsilon}, C_d > 0$ such that if $\alpha n \leq p \leq \frac{1}{4}$ and $d \geq C_d(n^2p^2 + \log^4 n)\log^4 n$, then, for any $\epsilon \geq C_{\epsilon}\sqrt{\frac{1}{d}(np + \log n)}\log^4 n$, there exists a probability space such that $G_- \sim G(n, p(1 - \epsilon))$, $G \sim G(n, p, d)$ and $G_+ \sim G(n, p(1 + \epsilon))$ satisfy, with probability at least $1 - n^{-\Omega(\log n)}$, $G_- \subseteq G \subseteq G_+$. For simplicity, we assume that both random graphs have self-loops, i.e, every node may be connected with itself. We remark that this assumption does not change the energy function. We isolate the main technical proof in the following lemma, it verifies the first condition of Theorem 2.

**Lemma 2.** Let $A$ be the adjacency matrix of a graph sampled from $G(n, p, d)$. There exists absolute constants $K, C_{\epsilon}, C_d > 0$ such that if $np > \log(n)^2$ and $d \geq C_d(n^2p^2 + \log^4(n))\log^4(n)$, then, with probability at least $1 - n^{-\Omega(1)}$,

$$\frac{\|\Delta A\|}{np} < \sqrt{\frac{2}{np}} \left( K + \max \left( 4\frac{C_{\epsilon}}{\sqrt{C_d}}, 24 \right) \right).$$

**Proof.** We choose $C_{\epsilon}, C_d > 0$ to be the constants in Proposition 1 and $K > 0$ as in Lemma 1. We fix

$$\epsilon := \max \left( \frac{C_{\epsilon}}{\sqrt{C_d}}, 6 \right) \sqrt{\frac{2}{np}} \geq C_{\epsilon}\sqrt{\frac{1}{d}(np + \log n)}\log^4 n. \quad (1)$$

The inequality above follows easily from the choice of $d$ and $np$. The need for the constant 6 will become clear later. Notice that the hypothesis of Proposition 1 is satisfied, so if $G^- := G(n, p(1 - \epsilon))$ and $G^+ := G(n, p(1 + \epsilon))$ are Erdős-Rényi random graphs, then there exists a probability space such that $G^- \subseteq G \subseteq G^+$ with the required probability. Now we define $L := D - A$, the Laplacian matrix associated to $G$, and $L^- := D^- - A^-$, $L^+ := D^+ - A^+$ the Laplacian matrices of $G^-$ and $G^+$ respectively. Recall that the quadratic form of a Laplacian
matrix can always be written as $\frac{1}{2} \sum_{i,j} \in E(G) (v_i - v_j)^2$. Hence the property $G^- \subseteq G \subseteq G^+$ implies in the following monotone relation: For all vectors $v$,
\[ v^T L^- v \leq v^T L v \leq v^T L^+ v. \] (2)

Therefore, for an arbitrary unit vector $v \in \mathbb{R}^n$, we have
\[
v^T (A - \mathbb{E}A)v = v^T Av - pv^T J_n v = v^T Dv - v^T L v - pv^T J_n v \\
\leq v^T D^+ v - v^T L^- v - pv^T J_n v + v^T \mathbb{E}[A^-] v - v^T \mathbb{E}[A^+] v \\
= v^T (D^+ - D^-) v + v^T (A^- - \mathbb{E}[A^-]) v - pv^T J_n v \\
\leq v^T (D^+ - D^-) v + v^T (A^- - \mathbb{E}[A^-]) v,
\]
we used here the self-loop simplification, $\mathbb{E}A = pJ_n$ and $\mathbb{E}[A^-] = p(1 - \varepsilon)J_n$. In the second line we use property (2) and the fact that $G \subseteq G^+$ implies $D_{ii} \leq D^+_{ii}$ for every $i \in [n]$. The last line then follows because $v^T J_n v \geq 0$ for every $v \in \mathbb{R}^n$.

In an analogous way, we obtain
\[
v^T (A - \mathbb{E}A)v \geq v^T (D^- - D^+) v + v^T (A^+ - \mathbb{E}[A^+]) v.
\]

Putting everything together, we obtain
\[
\|A - \mathbb{E}A\| = \sup_{\|v\|_2 = 1} |v^T (A - \mathbb{E}A)v| \\
\leq \|D^+ - D^-\| + \max \left\{ \|A^- - \mathbb{E}[A^-]\|, \|A^+ - \mathbb{E}[A^+]\| \right\}.
\] (3)

By Lemma 1 we obtain that
\[
\|A^- - \mathbb{E}[A^-]\| \leq K \sqrt{np(1 - \varepsilon)}, \\
\|A^+ - \mathbb{E}[A^+]\| \leq K \sqrt{np(1 + \varepsilon)},
\]
with the desired probability and $K > 0$ is the constant in Lemma 1. For the first term in the right hand side of (3), we claim that $\|D^+ - D^-\| \leq 4\varepsilon n$ with the desired probability. Indeed, each entry in the diagonal matrix $D^+ - D^-$ is the sum of $n$ independent Bernoulli random variables:
\[
(D^+ - D^-)_{ii} = \sum_{j=1}^{n} X_{ij} \quad \text{for} \quad X_{ij} = \begin{cases} 1 & \text{if } A^+_{ij} = 1 \text{ and } A^-_{ij} = 0, \\ 0 & \text{else,} \end{cases}
\]
and
\[
\mathbb{P}(X_{ij} = 1) = 1 - (\mathbb{P}(A^+_{ij} = 0) + \mathbb{P}(A^-_{ij} = 1)) = 1 - (1 - p(1 + \varepsilon) + p(1 - \varepsilon)) = 2\varepsilon.
\]

The expectation of each entry of $D^+ - D^-$ is $\mathbb{E}[(D^+ - D^-)_{ii}] = 2\varepsilon n$. We now apply Chernoff’s deviation inequality [16][28] to get
\[
\mathbb{P}((D^+ - D^-)_{ii} \geq 4\varepsilon n) \leq \exp \left( -\frac{p}{3} \varepsilon n \right).
\]
By union bound, definition of \( \varepsilon \) in (1) and the fact that \( np > \log^2(n) \), we obtain that the claim holds with probability at least

\[
1 - \exp\left(-\frac{p\varepsilon n}{3} + \log n\right) \geq 1 - \exp\left(-2\sqrt{2}\log n + \log n\right),
\]

the probability converges to one even if \( n = 2 \). Now, for the range of \( d \) in the hypothesis of this lemma, the following holds with the desired probability,

\[
\left\| \Delta_A \right\| \leq K \max(\sqrt{1 - \varepsilon}, \sqrt{1 + \varepsilon}) \frac{4p\varepsilon n}{np} \leq K \sqrt{\frac{2}{np}} + 4\varepsilon.
\]

Recall the choice of \( \varepsilon \) in the beginning to conclude the proof.

We now present the proof of the main theorem.

**Proof of Theorem 4.** As explained above, we verify that the three conditions listed in Theorem 2 are satisfied. Thanks to Lemma 2, we know that \( \left\| \Delta_A \right\| \leq \frac{1}{12} \) with the required probability. To verify the second condition, we apply triangular inequality to obtain

\[
\left\| \Delta_L \right\| = \left\| D - A - ED + EA \right\| \leq \left\| D - ED \right\| + \left\| A - EA \right\| < \left\| D - ED \right\| + \frac{1}{12}.
\]

To deal with the first term in the right hand side, observe that the entries of \( D - ED \) are only nonzero on the diagonal and \( (D - ED)_{ii} = \sum_{j=1}^{n} 1_{\{X_i = X_j\}} \geq t_{p,d} - p \). Conditioned on \( X_i \), the distribution of \( (D - ED)_{ii} \) is a centered binomial random variable and we apply Chernoff’s deviation again to obtain, for every \( \delta > 0 \),

\[
P(\left| (D - ED)_{ii} \right| \geq \delta np) \leq 2 \exp\left(-\frac{\delta^2 np}{3}\right).
\]

We choose \( \delta = \frac{\log n}{\sqrt{np}} \) and union bound over all \( i \in [n] \) to obtain that, with the required probability,

\[
\left\| \Delta_L \right\| \leq \frac{\log n}{\sqrt{np}} + \frac{1}{12} \leq \frac{1}{\sqrt{C_p}} + \frac{1}{12} < \frac{1}{8},
\]

for a sufficiently large \( C_p > 0 \). The reason why \( \frac{1}{8} \) instead of \( \frac{1}{4} \) will become clear later. We proceed to verify the last condition. It is easy to see that for a sufficiently large \( C_p > 0 \), we have

\[
\frac{12 \left\| \Delta_A \right\|}{np} \leq \frac{1}{4 \log n}.
\]

Since the function \( \sin^{-1}(x) \) is non-increasing and satisfies \( \sin^{-1}(x) < 2x \) for \( x \geq 0 \), we obtain that

\[
\sin^{-1}\left(\frac{12 \left\| \Delta_A \right\|}{np}\right) \leq \frac{1}{2 \log n}.
\]
On the other hand, we know that \( \frac{np}{2 \| \Delta L \|} > 4 \) and then

\[
\log \left( \frac{np}{2 \| \Delta L \|} - 1 \right) > \log(3).
\]  

(5)

Combining (4) with (5), we get

\[
\frac{\pi/4}{\sin^{-1} \left( \frac{\| \Delta A \|}{np} \right)} \geq \frac{\pi}{2} \log n > \log \left( \frac{n}{6} - 3 \right) + 1 > \frac{\log(n/6)}{\log(\frac{np}{2 \| \Delta L \|} - 1)} + 1.
\]

We conclude the proof by applying Theorem 2.

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