An information aggregation method for hierarchical system reliability analysis with insufficient reliability information based on Bayesian Melding Method

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Abstract. In practical engineering, the reliability analysis of the hierarchical system structure is a hot topic, which has made great progress in recent years. From former experiments, there exists multi-source reliability information for the complex system. In the literature, the Bayesian Melding Method is a useful tool to merge the subsystem and system information based on the deterministic structure. However, when the system reliability information is insufficient, how to integrate the limited information for the system reliability analysis remains a problem to be solved. In this paper, an information aggregation method is developed for the hierarchical system to make full use of the limited reliability information based on the traditional Bayesian Melding Method. Three main steps are involved in the proposed method, i.e. distribution assumption, structure reconstruction, and reliability inference, based on which the existed reliability information can be fully merged and partial missing reliability information can be acquired through the system structure. Finally, the benefit illustration and effectiveness verification of the proposed method are given in the numerical example.

1. Introduction
As the complexity of the system structure increases, the reliability analysis has become more and more important for pursuing the high performance and quality of the complex system [1, 2]. In engineering, the hierarchical system is the important representative of complex systems. Therefore, in order to comprehensively analyze the system reliability, it is of great value to consider the various reliability information of the hierarchical system which is made up of multiple components, subsystems.

Generally, the multi-source information can be divided into two categories. One is the experts’ opinions by experience and the other is the historical data from former experiments. For the first category, Bedford et al. [3] emphasized the important role of experts and pointed out that experts’ opinions can be further extracted as the important information for the reliable system design. Burgman et al. [4] also studied the status and performance of experts. In their work, the feedback of experts
was investigated. Besides, they gave the conclusion that when there was lack of time and money when analyzing the system, the experienced experts can give better advice of the parameter information. Rufo et al. [5] proposed that the experts’ opinions can be merged when there existed much experts’ information. For the second category, the experimental or historical data can also be obtained and used for the reliability analysis. Castet et al. [6] analyzed the in-orbit failure data of numerous satellites and the lifetime distributions can be acquired by the statistical calculation method. Guo et al. [7] also modelled the satellite reliability through the statistical analysis of the failure data and compared the Bayesian method with the nonparametric method. Thus, according to the above two categories, the reliability information can be acquired when there exists large amount of experimental data or experts’ judgment. However, it is still an important issue as for how to make full use of the obtained reliability information.

Research about fully utilizing the information has made certain achievements in recent years. Poole et al. [8] first proposed the Bayesian Melding Method (BMM) to merge the subsystem and system information. The empirical information of the whole system was finally updated through the system structure. Guo et al. [9] developed the BMM to an iterative scheme so that the updated information can be further propagated and merged until the convergence. Besides, different strategies are discussed for the system reliability inference. Yang et al. [10, 11] improved the method proposed by Guo et al. by combining the pre- and post- model information for non-deterministic models. In their work, the traditional BMM is extended by using the sampling importance resampling method and the pooling weight is used as a hyper-parameter obeying the uniform distribution. In addition, Guo et al. [12] used the heterogeneous multi-level information for the system reliability assessment based on the BMM, which further improved the traditional BMM for multi-level systems. Zhao et al. [13] discussed the residual life of components of on-orbit satellites by fusing the multi-source information. Based on the multi-level system structure, Xu et al. [14] proposed a multi-level multi-source lifetime information integration method which considered all the lifetime information of the system and was applied into the reliability-based design optimization of the satellite system. Among the above research, the information of all the components should be known so that the information integration can be conducted. However, in practical engineering, it is time-consuming and cost-expensive to get all the information of the hierarchical system. Thus, how to realize the information integration of the system with limited reliability information remains to be solved.

This paper is dedicated to dealing with the information aggregation for the hierarchical system reliability analysis with the insufficient reliability information. The main contributions of this paper are summarized as follows. Three steps are included in the proposed information aggregation method. First, distributions are assumed for those bottom components with missing information. Second, according to the information existence of middle components, the hierarchical system structure is reconstructed and double-level modules are formed, based on which the traditional BMM can be further applied. Third, the system reliability inference is implemented based on the above two steps by fully propagating the reliability information from lower levels into the system level. To sum up, the proposed method addresses the information integration problem by considering whether the reliability information of different components in different levels is existed.

The remainder of this paper is organized as follows. In Section 2, the traditional BMM is given in brief. In Section 3, the proposed information aggregation method for the hierarchical system with insufficient information is described at length, involving the three main steps. The numerical example is presented to demonstrate and verify the effectiveness of the proposed method in Section 4. In Section 5, the conclusions and the future prospects are discussed.

2. Bayesian Melding Method
The Bayesian Melding Method proposed by Poole et al. [8] is to merge distributions of the system through the system model. Define the system to be a double-level system structure which is composed of several subsystems and the system. The system model is deterministic as the condition of the BMM, which relates the input and output parameters.
Denote the system model as $\phi = M(\theta)$, where $\theta$ is the subsystem parameter vector, $\phi$ is the system parameter vector, and $M$ is the model function that reflects the relationship between the subsystems and system parameters. For example, if $\theta$ and $\phi$ are the reliability parameter, then $M$ is the logical relationship, e.g., the parallel relationship of subsystems. The model $M$ may be noninvertible if the dimension of $\phi$ is less than that of $\theta$, i.e., many values of the subsystem parameter may correspond to one value of the system parameter. Generally, $\theta$ and $\phi$ can also be regarded as the random variables with probability density functions (PDF) $q_i(\theta)$ and $q_j(\phi)$ respectively. To be more clear, $q_i(\theta)$ is denoted as the empirical subsystem PDF and $q_j(\phi)$ is named as the empirical system PDF. These two distributions usually come from experienced experts or historical data extraction. Based on the system model $\phi = M(\theta)$ and the empirical subsystem PDF $q_i(\theta)$, the new distribution of $\phi$ can be derived as follows.

$$q^*_i(\phi) = q_i[M^{-1}(\phi)]|J(\phi)|$$

(1)

where $J(\phi)$ is the Jacobian matrix and $q^*_i(\phi)$ is named as the induced system PDF. Equation (1) is applied to reversible models. However, in most cases, the system model is irreversible. Thus, some numerical methods (e.g., nonparametric density estimation method) are commonly used for irreversible models.

Through the above description, there are two distributions about one variable $\phi$. The key step of the BMM is to merge the two different distributions of $\phi$. How to combine multiple distributions from different sources has received considerable attention in recent years. There are two main approaches. One is the linear pooling approach and has the form:

$$T(q_1, q_2, \ldots, q_k) \propto \sum_{\alpha=1}^{k} \alpha q_\alpha \sum \alpha_i = 1, \alpha_i > 0$$

(2)

where $q_1, \ldots, q_k$ are the individual distributions, $\alpha_i$ is the pooling weight, and $T$ is the pooling operator. The other is called the logarithmic approach and has the form:

$$T(q_1, q_2, \ldots, q_k) \propto \prod_{i=1}^{k} q_i^{\alpha_i} \sum \alpha_i = 1, \alpha_i > 0$$

(3)

Research shows that the logarithmic pooling approach is the only operator due to its external Bayesianity [8]. Thus, in most cases, the logarithmic pooling approach is used more widely than the linear pooling approach, especially for the system reliability analysis. For the BMM, the empirical system PDF $q_2(\phi)$ and the induced system PDF $q^*_i(\phi)$ are pooled by the logarithmic pooling approach, which has the following form.

$$q^*_i(\phi) = k_\alpha q_i^*(\phi)^\alpha q_2(\phi)^{1-\alpha}$$

(4)

where $k_\alpha$ is the normalization constant, $\alpha$ is the pooling weight that is usually set to be 0.5, and $q^*_i(\phi)$ is named as the pooled system PDF. The normalization constant can be calculated according to the probability theory as:

$$k_\alpha = \frac{1}{\int q^*_i(\phi)^\alpha q_2(\phi)^{1-\alpha} d\phi} \forall \alpha \in [0,1]$$

(5)

Thus, the information of subsystems (i.e. $q^*_i(\phi)$) and the information of the system (i.e. $q_2(\phi)$) are merged. Further, in order to make subsystems contain the information of the system, the system information is propagated into subsystems. So the new obtained subsystem distribution is denoted as the updated subsystem PDF $q_\theta(\theta)$, which has the form as follows.

$$q_\theta(\theta) = k_\alpha q_i(\theta)^{\alpha} q_2[M(\theta)]^{1-\alpha}.$$  

(6)

To sum up, the main process of the BMM is described as above. The empirical distributions of the system and subsystems are all updated through the information integration process. However, the traditional BMM is only applied for double-level systems and the empirical information may be quite
difficult to get in the practical engineering for lack of sufficient knowledge. Therefore, this paper proposes a novel method to address the information integration problem with insufficient information for the system reliability analysis.

3. Information aggregation method for the hierarchical system reliability analysis
In this section, the proposed information aggregation method based on the tradition BMM is discussed in detail. First the overall description of the method is summarized. Then the three main steps are given at length to describe the specific strategies and algorithms.

3.1. General description about the proposed method
In engineering, most multi-level hierarchical system structures are made up of multiple components, subsystems and so on. In this paper, in order to describe clearly, the word “element” is used for components and subsystems, and each element is supposed to be independent. The hierarchical system structure used in this paper is shown in figure 1.

![Figure 1. Multi-level hierarchical system structure.](image)

Due to the complexity of the system structure, it is difficult to get all the information of elements. Besides, especially for the system, it requires high cost and large amount of time to obtain the reliability information. Thus it is worth utilizing the limited information to acquire the system reliability information and compensate for the missing information. Three main steps are included in the proposed information aggregation method, as shown in figure 2. First, for those root elements without empirical information, the distributions are first assumed for further information integration. Second, according to the middle elements, the multi-level hierarchical system structure is reconstructed in order to simplify the complex system so that the known information can be used fully. Finally, the reliability inference is conducted for the system element based on the reconstructed structure and the traditional BMM.
3.2. Distribution assumption for partial root elements

Take figure 1 as an example to illustrate the proposed method, the system structure is divided into $N$ levels. The first level is named as Level 1, which is composed of bottom components in the system. The level number increases until the system level which is the top level. The $j$th element in the $i$th level is denoted as $\theta_i^j$. For example, $\theta_2^3$ is the third element in level 2. For those elements in the $i$th level ($i \geq 2$) which are made up of several elements in the $(i-1)$th level, they are defined as the child element and elements belonging to them are defined as the parent elements. Elements that have no parent elements are denoted as the root element (e.g. element $\theta_1^j$). In this paper, to further distinguish the elements, three categories of elements are formed, i.e. root element, middle element, and system element. The root element has been defined for those elements with no parent elements. Middle elements refer to the elements that are in the level 2 to level $N-1$. Those elements with no parent elements but in the level 2 to level $N-1$ belong to root elements rather than middle elements, which are not shown in figure 1. As the name implies, the system element refers to the only one element $\theta_N^1$.

Because the number of root elements is quite large in the practical system, it is hard to get all the empirical information for root elements. In this paper, the missing distribution information is assumed at first for root elements so that the information integration can be conducted based on the traditional BMM and the detailed information integration process will be discussed in the following sections. Besides, the reliability information of each element is only considered in this paper. Denote $\omega_i^j$ as the reliability parameter of the element $\theta_i^j$. Generally, the distribution of $\omega_i^j$ is denoted by the Beta distributions with the following form:

$$f(x; a, b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}$$

(7)

where $a$ and $b$ are the parameters of the Beta distribution. For those root elements with no empirical reliability information, the non-informative distributions are used such as the Beta distribution with $a = 2$ and $b = 2$. To sum up, if the root elements do not have the empirical distribution, then the non-informative distribution is applied. For those middle elements and the system element, the specific description will be discussed in the following sections.

3.3. Reconstruction of the hierarchical system

For those complex systems, the information of part middle elements may be missing, thus how to utilize the known the information fully is a problem to be solved. In this paper, the system structure is reconstructed according to the existence of the middle elements. Take a three-level system structure as an example to illustrate the process of structure reconstruction, as shown in figure 3.
Figure 3. A three-level system structure.

Suppose the information of the element $\theta_i^2$ is missing. Element $\theta_i^1$, $\theta_i^3$, and $\theta_i^4$ are the parent elements of $\theta_i^2$. Thus, there exists the logical relationship between the element $\theta_i^2$ and its parent elements (i.e. $\theta_i^1$, $\theta_i^3$, and $\theta_i^4$). Besides, element $\theta_i^2$ and $\theta_i^2$ are the parent elements of the element $\theta_i^3$, thus the relationship between the element $\theta_i^3$ and its parent elements is known.

According to the definition in section 3.2 for the reliability parameters, define the relationship between $\theta_i^1$, $\theta_i^2$, $\theta_i^3$, and $\theta_i^4$ as

$$\omega_i^2 = M_i'(\omega_i^1, \omega_i^3, \omega_i^4)$$ (8)

Besides, the relationship between $\theta_i^2$, $\theta_i^3$, and $\theta_i^4$ is defined as

$$\omega_i^3 = M_i'(\omega_i^1, \omega_i^2)$$ (9)

Replace $\omega_i^2$ in equation (9) with equation (8), there has

$$\omega_i^3 = M_i'(M_i'(\omega_i^1, \omega_i^3, \omega_i^4), \omega_i^2)$$ (10)

From equation (10), it can be seen that $\omega_i^3$ is related to the parameter $\omega_i^1$, $\omega_i^2$, and $\omega_i^3$ directly. Therefore, the element $\theta_i^2$ can be ignored, i.e. the element $\theta_i^1$, $\theta_i^3$, and $\theta_i^3$ point to the element $\theta_i^4$ directly, and the structure is reconstructed as shown in figure 4.

Figure 4. Reconstruction of the three level system structure.

Thus, if the information of part middle elements is unknown, then those elements are ignored when reconstructing the system structure. It should be noted that for those middle elements with no information, non-informative distribution is not used, which is different from the root elements. The main rules of the system reconstruction is as follows.

1. Give the relationship between parent elements and the child elements.
2. Perform the following rules for middle elements from level 2 to level $N-1$.
3. If the reliability information of the middle element $\theta_i^j$ is unknown, then the parent elements of $\theta_i^j$ point to its child element directly.
4. According to the new formed relationship between elements, reconstruct the hierarchical system.
3.4. Reliability inference for the system element

After reconstructing the hierarchical system structure, the newly formed structure is further decomposed into multiple double-level blocks. Take figure 1 as an example and suppose the information of the element $\theta_2$ is missing. The decomposed structure of the system is shown in figure 5.

![Decomposition of the hierarchical system.](image)

In order to integrate all the known information fully, the traditional BMM is adopted for the double-level blocks. The information integration is from the bottom blocks to the top block according to the level number. Because the middle elements are included in the two blocks at the same time, they will be updated twice. Once the block is updated, then the empirical distribution should be replaced by the updated distribution so that the information from lower levels can be propagated into the higher levels. Through the information propagation process, elements in level $N - 1$ will be updated and contain more information from other levels.

In most situations, the reliability information of the system element is hard to obtain due to the limitation of time and cost. Thus, this paper proposes to utilize the reliability information of lower levels (i.e. level 2 to level $N - 1$) to conduct the system reliability inference. Based on the reconstructed and decomposed structure, the system element $\theta_N$ and its parent elements in level $N - 1$ are made up of a double-level block. Because the reliability information of $\theta_N$ is missing, the BMM cannot be used for the block. According to the updated information of elements in level $N - 1$ and the double-level block structure, the induced system distribution can be obtained, which can then be used as the reliability information of the system element.

In conclusion, through the information integration process, the missing reliability information can be obtained by utilizing the known information. The main steps of the proposed information aggregation method are summarized as follows and the flowchart is shown in figure 6.

**Step 1:** Input all the known reliability information.

**Step 2:** Assume the non-informative distributions for partial root elements with no reliability information.

**Step 3:** According to the middle elements, reconstruct the hierarchical system structure.

**Step 4:** Initialize the level number as $i = 1$.

**Step 5:** If $i < N - 1$, then perform **Step 6** to **Step 8**. Otherwise, go to **Step 9**.

**Step 6:** Apply the BMM into double-level blocks between level $i$ and level $i + 1$.

**Step 7:** Replace the empirical distribution with the updated distribution of elements in level $i$ and level $i + 1$.

**Step 8:** $i = i + 1$ and go back to **Step 5**.
Step 9: System reliability inference. Obtain the induced system distribution based on the updated elements in level $N - 1$.

Step 10: Output the system reliability information and end the process.

Figure 6. The flowchart of the information aggregation.

4. Case study
In this section, a numerical example is used to demonstrate the proposed method. The problem description and results analysis are discussed in the following sections.

4.1. Problem description
The multi-level system structure used for the case study is shown in figure 7. The logical relationship between different elements is described as follows.

1. $\theta^1_1$ and $\theta^1_2$ are in series;
2. $\theta^1_1$ and $\theta^1_2$ are in parallel first, and then in series with $\theta^1_3$ and $\theta^1_6$;
3. $\theta^2_1$ and $\theta^2_2$ are in parallel first, and then in series with $\theta^2_3$;
4. $\theta^2_1$ and $\theta^2_2$ are in parallel;
5. $\theta^3_1$ and $\theta^3_2$ are in series first, and then in parallel with $\theta^3_3$. 

Output the system reliability information
As shown in Figure 7, the reliability information of elements \( \theta_2^3 \), \( \theta_2^4 \), \( \theta_1^3 \), and \( \theta_1^4 \) is unknown. Element \( \theta_2^4 \) and \( \theta_1^4 \) are the root elements, of which the non-informative distributions are assumed as the Beta distribution with \( a = 2 \) and \( b = 2 \). For the middle element \( \theta_1^3 \), it will be ignored during the reconstruction process. The reliability will be inferred for the element \( \theta_1^3 \). According to the definition in Section 3, the reliability parameter of each element \( \theta_i^j \) is \( \omega_i^j \). The parameters of the Beta distribution for known reliability information are shown in Table 1.

### Table 1. Empirical distributions for known elements.

| Reliability parameters | Parameters | Reliability parameters | Parameters |
|------------------------|------------|------------------------|------------|
| \( \omega_1^1 \)        | 10         | \( \omega_2^3 \)       | 8          |
| \( \omega_2^1 \)        | 7          | \( \omega_2^4 \)       | 4          |
| \( \omega_1^3 \)        | 8          | \( \omega_1^4 \)       | 2          |
| \( \omega_2^2 \)        | 5          | \( \omega_2^6 \)       | 3          |
| \( \omega_1^6 \)        | 12         | \( \omega_2^3 \)       | 9          |
| \( \omega_2^5 \)        | 6          | \( \omega_1^7 \)       | 3          |

4.2. Results analysis

According to Section 3.3, the system structure is reconstructed due to the unknown information of \( \theta_2^3 \). Double-level blocks are formed to integrate all the known information based on Section 3.4. The reconstructed and decomposed system structure are shown in Figure 8.

Figure 8. Four-level system structure reconstruction and decomposition in case study.
Based on the decomposed structure, the BMM can be employed in the double-level blocks from the bottom level to the top level. After the information integration of double-level blocks, the known empirical information will be updated and the updated information will be propagated into the next blocks. Through the information aggregation, the element $\theta^1_2$ and $\theta^2_5$ can get the updated distribution to replace the initial non-informative distribution because the updated distribution contains the information from other known elements. The change of the reliability information for element $\theta^1_5$, $\theta^2_4$, and $\theta^3_1$ is shown in figure 9 (a), (b), and (c), respectively. The induced system distribution of the system element $\theta^4_1$ is shown in figure 9 (d), which is obtained by the kernel density estimation method and compared with the histogram figure.

![Figure 9. Distributions of element $\theta^1_5$, $\theta^2_4$, $\theta^3_1$, and $\theta^4_1$.](image)

From figure 9 (a) and figure 9 (b), it can be seen that the updated PDF is quite different from the non-informative distribution, and the updated PDF of $\theta^1_5$ and $\theta^2_4$ are also different from each other. Thus, it can be concluded that the known information from the system structure can compensate for root elements with no information through the information propagation process. Besides, the relationship between elements leads to the difference of distributions. From figure 9 (c), it is noted that the empirical PDF of the element $\theta^3_1$ is updated finally because the information from other elements is integrated. As shown in figure 9 (d), the kernel density estimation is consistent with the histogram. The induced system distribution is obtained by merging the updated information of elements in level 3 shown in figure 8. Although there exists no information of the system element, through the proposed information aggregation method, the system element can obtain the induced PDF which can be a reference for the further system reliability analysis.

In conclusion, through the proposed method, the non-informative distribution of partial root elements can be updated and the system element will obtain the reliability information through the system structure. The proposed method solves the problem of how to integrate the limited reliability information efficiently.
5. Conclusions
As for the information aggregation of multi-level hierarchical system with limited information, this paper proposes to merge the known reliability information fully for the system reliability analysis. Three main parts are included in the proposed method. First, the non-informative distributions are assumed for partial root elements. Second, the system structure is reconstructed based on the middle elements. Third, the induced system distribution is obtained through the reconstructed and decomposed structure by aggregating the known reliability information from different levels. The case study shows that the proposed method can solve the information aggregation problem and has great significance for the further system analysis.

Acknowledgments
This work was supported in part by National Natural Science Foundation of China under Grant No.51675525 and 11725211.

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