SHARP EXPONENTIAL INEQUALITIES FOR THE ORNSTEIN-UHLENBECK OPERATOR

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Abstract. The optimal constants in a class of exponential type inequalities for the Ornstein-Uhlenbeck operator in the Gauss space are detected. The existence of extremal functions in the relevant inequalities is also established. Our results disclose analogies and dissimilarities in comparison with Adams’ inequality for the Laplace operator, a companion of our inequalities in the Euclidean space.

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1. Introduction and main results

The Ornstein-Uhlenbeck operator \( \mathcal{L} \) is defined as
\[
\mathcal{L}u = \Delta u - x \cdot \nabla u
\]
for a function \( u: \mathbb{R}^n \to \mathbb{R} \), with \( n \in \mathbb{N} \). Here, \( \Delta \) and \( \nabla \) denote the usual Laplace and gradient operators, and the dot \( \cdot \) stands for scalar product in \( \mathbb{R}^n \). The Ornstein–Uhlenbeck operator is, under various respects, the natural counterpart of the Laplace operator when the Euclidean ambient space \( \mathbb{R}^n \), equipped with the Lebesgue measure, is replaced by the Gauss space \( (\mathbb{R}^n, \gamma_n) \). The latter is still \( \mathbb{R}^n \), but endowed with the Gauss probability measure \( \gamma_n \), whose density obeys
\[
d\gamma_n(x) = (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}} dx \quad \text{for } x \in \mathbb{R}^n.
\]

The operator \( \mathcal{L} \) is the infinitesimal generator of the Ornstein-Uhlenbeck semigroup in the Gauss space, defined via the Mehler kernel, see e.g. [37, Section 12.1] or [51, Chapter 2]. Hence, the operator \( \mathcal{L} \) stands with respect to the Ornstein-Uhlenbeck semigroup in the Gauss space that the Laplace operator stands to the heat kernel in the Euclidean space. Also, recall that the classical Dirichlet integral is the Dirichlet form associated with the Laplace operator in the Euclidean space; likewise, the functional
\[
\int_{\mathbb{R}^n} |\nabla u|^2 d\gamma_n
\]
is the Dirichlet form associated with the Ornstein-Uhlenbeck operator in the Gauss space.

The Ornstein-Uhlenbeck operator plays a role in a number of areas and is the subject of a huge literature. The lecture notes [37], the monograph [51] and the survey papers [46] and [11] are excellent sources for an introduction to these topics, as well as for a rich collection of related references.

Here, we are concerned with a peculiar family of Sobolev type inequalities involving the operator \( \mathcal{L} \) in \( (\mathbb{R}^n, \gamma_n) \). The bases for the study of Sobolev inequalities in the Gauss space were laid by L. Gross [31], who proved a first-order inequality for the \( L^2(\mathbb{R}^n, \gamma_n) \) norm of the gradient. The work of Gross paved the way to extensive researches on Sobolev type inequalities in the Gauss space [4, 5, 9, 10, 12, 14, 22, 25, 29, 41, 43, 44]. Inequalities in terms of the Ornstein-Uhlenbeck operator can be found in [6, 24, 50]. The latter papers deal, in fact, with even more general second-order elliptic operators, but are concerned with the somewhat different situation of functions defined in open subsets of \( \mathbb{R}^n \) and vanishing on \( \partial \Omega \).

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Interestingly, the target spaces in the Gaussian Sobolev inequalities for the Ornstein-Uhlenbeck operator substantially differ from those appearing in the Euclidean inequalities for the Laplace operator. A distinctive feature of the former is that, as in the case of first-order inequalities, the gain in the degree of integrability inherited by a function $u$ from that of $Lu$ in the space $(\mathbb{R}^n, \gamma_n)$ is much weaker than that guaranteed by $\Delta u$ in domains of finite Lebesgue measure in $\mathbb{R}^n$.

In this paper, we focus on inequalities for the operator $\mathcal{L}$ in spaces of exponential type $\exp L^\beta(\mathbb{R}^n, \gamma_n)$, with $\beta > 0$. These are Orlicz spaces built upon Young functions equivalent to $e^{\theta x}$ for $t$ near infinity, and equipped with the Luxemburg norm, denoted by $\| \cdot \|_{\exp L^\beta(\mathbb{R}^n, \gamma_n)}$. In these borderline spaces, the increase in the integrability of a function $u$ ensured by the integrability of $Lu$ deteriorates so that membership of $Lu$ to $\exp L^\beta(\mathbb{R}^n, \gamma_n)$ just ensures that $u$ belongs to the same space. Specifically, the Sobolev type inequality

$$\tag{1.4} \|u\|_{\exp L^\beta(\mathbb{R}^n, \gamma_n)} \leq C\|Lu\|_{\exp L^\beta(\mathbb{R}^n, \gamma_n)}$$

holds for some constant $C = C(\beta)$, and for every function $u \in W^1_{\mathcal{L}} \exp L^\beta(\mathbb{R}^n, \gamma_n)$ such that

$$\tag{1.5} m(u) = 0,$$

where $m(u)$ stands for either the mean value $\text{mv}(u)$ or the median $\text{med}(u)$ of $u$ over $(\mathbb{R}^n, \gamma_n)$. Here, $W^1_{\mathcal{L}} \exp L^\beta(\mathbb{R}^n, \gamma_n)$ denotes the Sobolev type space of those functions $u$ such that $Lu$, defined in a suitable weak sense, belongs to $\exp L^\beta(\mathbb{R}^n, \gamma_n)$—see Section 3 for details. Moreover, the target space in inequality (1.4) is optimal, in the sense that the inequality fails if the norm in the space $\exp L^\beta(\mathbb{R}^n, \gamma_n)$ is replaced by any stronger rearrangement-invariant norm of $u$ on the left-hand side, see [17, Section 6]. Let us point out that, however, the target space in (1.4) is still better than that entering parallel inequalities where $Lu$ is substituted by $\nabla u$, or even by $\nabla^2 u$, the matrix of all second-order derivatives of $u$. The optimal target spaces in the relevant inequalities are still of exponential type, but with an exponent smaller than $\beta$, thus exhibiting a loss of integrability for $u$ with respect to $\nabla u$ [20, Proposition 4.4 (iii)] (see also [2, 8, 33] for special cases) or $\nabla^2 u$ [21, Corollary 7.14 (ii)].

The stronger effect of the operator $\mathcal{L}$ is apparently due to the presence of the term $x \cdot \nabla u$ and to the interaction of the function $x$ with the decay of the density (1.2) of the measure $\gamma_n$ near infinity.

Our specific concern is the identification of the optimal constant $\theta$ in the integral inequality, equivalent to (1.4),

$$\tag{1.6} \sup_u \int_{\mathbb{R}^n} \exp(\theta |u|) \, d\gamma_n < \infty,$$

where the supremum is extended over all functions $u$ in $\mathbb{R}^n$ satisfying the constraint

$$\tag{1.7} \int_{\mathbb{R}^n} \text{Exp}^\beta(|Lu|) \, d\gamma_n \leq M$$

for some constant $M > 1$, and subject to the normalization (1.5). Here, $\exp^\beta$ denotes the function defined by $\exp^\beta(t) = e^{\theta t}$ for $t \geq 0$, and $\text{Exp}^\beta$ its convex envelope, namely the largest convex function not exceeding $\exp^\beta$. Obviously, $\text{Exp}^\beta$ agrees with $\exp^\beta$ near infinity for every $\beta > 0$, and globally if $\beta \geq 1$.

Problem (1.5)–(1.7) can be regarded as a Gaussian analogue of (a special case of) that solved by D. R. Adams for the classical Laplacian in the Euclidean setting [1]—see also the related contributions [3, 27, 28, 32, 35, 38, 45]. Adams’ result is in turn a second-order version of Moser’s inequality [42] in the limiting case of the Sobolev embedding theorem. A first-order companion to problem (1.5)–(1.7), where the Ornstein-Uhlenbeck operator is replaced by the plain gradient in the Gauss space, has recently been addressed in [18].

A trait shared by all the results on exponential inequalities alluded to above, and by their numerous variants and extensions, is the existence of a threshold value in the exponential integrand of $u$, which dictates the validity or not of the inequality in question. This phenomenon also shapes the problem at hand here. The threshold for the constant $\theta$ in (1.6) only depends on $\beta$, and equals

$$\tag{1.8} \theta_\beta = \frac{2}{\beta}.$$
In this connection, recall that Adams’ inequality tells us that, if \( \Omega \) is an open bounded subset of \( \mathbb{R}^n \), with \( n \geq 3 \), then
\[
(1.9) \quad \sup_u \int_\Omega \exp^{\frac{n}{n-2}}(\alpha_n |u|) \, dx < \infty,
\]
where
\[
\alpha_n = \frac{1}{\omega_n} \left( \frac{4\pi^{\frac{n}{2}}}{\Gamma \left( \frac{n}{2} - 1 \right)} \right)^{\frac{n}{n-2}},
\]
\( \omega_n \) is the Lebesgue measure of the unit ball in \( \mathbb{R}^n \), \( \Gamma \) denotes the gamma function, and the supremum is extended over all compactly supported functions \( u \) in \( \Omega \) such that
\[
(1.10) \quad \int_\Omega |\Delta u|^{\frac{n}{2}} \, dx \leq 1.
\]
Both constants \( \alpha_n \) on the left-hand side of inequality (1.9) and 1 on the right-hand side of (1.10) are sharp: inequality (1.9) fails if either of them is increased, while the other one is unchanged.

By contrast, the conclusions about inequality (1.6) are multifaceted, and sensitive to the parameter \( \beta \), but independent of the dimension \( n \). They can be summarized as follows. If \( \beta \in (0, 1] \), then inequality (1.6) holds with \( \theta \leq 2\beta \) for any choice of \( M \), and fails, again for any \( M \), when \( \theta > 2\beta \). On the other hand, if \( \beta > 1 \), then for any \( M \) inequality (1.6) holds with \( \theta < 2\beta \) and fails with \( \theta > 2\beta \); the value of \( M \) has a role only when \( \theta = 2\beta \), in which case inequality (1.6) holds for sufficiently small \( M \) and does not hold if \( M \) is too large. This is the content of our first main result.

**Theorem 1.1 [Integral form].** Let \( n \geq 1 \).

Part 1. Assume that \( \beta \in (0, 1] \).

1. (i) If \( 0 \leq \theta < \frac{2\beta}{\beta} \), then inequality (1.6) holds for every \( M > 1 \).
1. (ii) If \( \theta > \frac{2\beta}{\beta} \), then inequality (1.6) fails for every \( M > 1 \). In particular, there exists a function \( u \) obeying (1.5) and (1.7) that makes the integral in (1.6) diverge.

Part 2. Assume that \( \beta \in (1, \infty) \).

2. (i) If \( 0 < \theta < \frac{2}{\beta} \), then inequality (1.6) holds for every \( M > 1 \).
2. (ii) If \( \theta = \frac{2}{\beta} \), then there exists \( M > 1 \) such that inequality (1.6) holds, and there exists \( M > 1 \) such that (1.6) fails.
2. (iii) If \( \theta > \frac{2}{\beta} \), then inequality (1.6) fails for every \( M > 1 \). In particular, there exists a function \( u \) obeying (1.5) and (1.7) that makes the integral in (1.6) diverge.

A variant of problem (1.5)–(1.7) is the subject of the next theorem, where constraint (1.7) is replaced by its norm-form twin. The resultant inequality reads
\[
(1.11) \quad \sup_u \int_{\mathbb{R}^n} \exp^{\beta}(\theta |u|) \, d\gamma_n < \infty,
\]
where the supremum is extended over all functions \( u \) in \( \mathbb{R}^n \) fulfilling the inequality
\[
(1.12) \quad \|Lu\|_{L^{\frac{n}{\beta}}(\mathbb{R}^n, \gamma_n)} \leq 1,
\]
and condition (1.5). Here, \( B \) is any Young function such that
\[
(1.13) \quad B(\tau) = Ne^{\tau^\beta} \quad \text{for } \tau > \tau_0,
\]
for some constants \( N > 0 \) and \( \tau_0 > 0 \).

The flavour of our result on this problem is similar to that of Theorem 1.1, save that now the job of the parameter \( M \) in (1.7) is performed by the behaviour near zero of the function \( B \) in (1.13).

**Theorem 1.2 [Norm form].** Let \( n \geq 1 \).

Part 1. Assume that \( \beta \in (0, 1] \).

1. (i) If \( 0 < \theta \leq \frac{2}{\beta} \), then inequality (1.11) holds for every \( N > 0 \) and for every Young function \( B \) as in (1.13).
(1.ii) If $\theta > \frac{2}{\beta}$, then inequality (1.11) fails for every $N > 0$ and for every Young function $B$ as in (1.13).
In particular, there exists a function $u$ obeying (1.5) and (1.12) that makes the integral in (1.11) diverge.

Part 2. Assume that $\beta \in (1, \infty)$.

(2.i) If $0 < \theta < \frac{2}{\beta}$, then inequality (1.11) holds for every $N > 0$ and for every Young function $B$ as in (1.13).

(2.ii) If $\theta = \frac{2}{\beta}$, then for every $N > 0$, there exist a Young function $B$ as in (1.13) such that inequality (1.11) holds, and a Young function $B$ such that inequality (1.11) fails.

(2.iii) If $\theta > \frac{2}{\beta}$, then inequality (1.11) fails for every $N > 0$ and for every Young function $B$ as in (1.13).
In particular, there exists a function $u$ obeying (1.5) and (1.12) that makes the integral in (1.11) diverge.

A question that naturally arises is the existence of extremal functions in the inequalities considered so far. Namely, the question of whether the supremum in (1.6) and (1.11) is attained or not. We give an affirmative answer to this question in the critical case when $\theta = \frac{2}{\beta}$. Accordingly, we consider values of the parameter $\beta \in (0, 1]$. In the light of Theorems 1.1 and 1.2, this guarantees that inequality (1.6) or (1.11) holds for any constant $M$ or any Young function $B$ as in (1.13), respectively. As will be clear from the proof, the problem in the subcritical regime when $\theta < \frac{2}{\beta}$ is easier, and can be approached along the same lines.

**Theorem 1.3 [Existence of maximizers].** Let $n \geq 1$ and $\beta \in (0, 1]$.

(i) The supremum in (1.6) is attained for $\theta = \frac{2}{\beta}$ and for every $M > 1$.

(ii) The supremum in (1.11) is attained for $\theta = \frac{2}{\beta}$ and for every Young function $B$ as in (1.13).

With regard to the Euclidean setting, the existence of extremals in Moser’s first-order inequality is well known: the 2-dimensional case goes back to [26], whereas an exhaustive proof for arbitrary dimensions $n$ has only recently been accomplished in [23], where, in particular, arguments from the earlier paper [34] are made rigorous. Instead, a parallel question for Adams’ inequality (1.9) for the Laplace operator seems to be only solved in the special case when $n = 4 - [36]$.

We conclude by tackling the limiting situation when $\beta$ is formally sent to infinity in condition (1.12), namely when $Lu$ is subject to the constraint

$$(1.14) \quad \|Lu\|_{L^\infty(\mathbb{R}^n, \gamma_n)} \leq 1.$$ 

The constant $\theta_\beta$ degenerates to 0 in the limit as $\beta \to \infty$. Such a behaviour hints that some singular phenomenon should be expected. This is in fact the case, since the piece of information contained in (1.14) does not imply that $u \in L^\infty(\mathbb{R}^n, \gamma_n)$. It turns out that, instead,

$$\|u\|_{\text{expexp } L(\mathbb{R}^n, \gamma_n)} \leq C\|Lu\|_{L^\infty(\mathbb{R}^n, \gamma_n)}$$ 

for some constant $C$, and for every function $u$ fulfilling (1.5). Here, $\text{expexp } L(\mathbb{R}^n, \gamma_n)$ denotes the Orlicz space built upon a Young function equivalent to $e^{ct}$ for $t$ near infinity, and is optimal among all rearrangement-invariant target spaces - see [17].

The problem thus emerges of detecting the values of the constant $\eta$ such that

$$(1.15) \quad \sup_u \int_{\mathbb{R}^n} \exp(\eta |u|) \gamma_n < \infty,$$

where now the supremum is extended over all functions $u$ in $\mathbb{R}^n$ satisfying assumption (1.14) and normalized as in (1.5).

A new threshold value appears, which is given by

$$(1.16) \quad \eta_\infty = 2.$$ 

As shown by our last main result, this value is not admissible in inequality (1.15).

**Theorem 1.4 [L\(^\infty\) norm].** Let $n \geq 1$.

(i) If $0 < \eta < 2$, then inequality (1.15) holds.

(ii) If $\eta \geq 2$, then inequality (1.15) fails. In particular, there exists a function $u$ obeying (1.14) and (1.5) that makes the integral in (1.15) diverge.
Extremal functions \( u \) in inequality (1.15) can be shown to exist for every \( \eta \in (0, 2) \). As in the case of inequality (1.11) with \( \theta < \frac{2}{3} \), a proof of this fact is analogous to, and simpler than that of Theorem 1.3, and will be omitted.

2. OUTLINE OF THE APPROACH

Here, we sketch the main ideas and ingredients employed in the proofs of our results, and point out some technical difficulties. Precise definitions of some mathematical objects mentioned below are given in the subsequent sections.

**Gaussian isoperimetric inequality.** The point of departure of our approach is a sharp pointwise estimate, in rearrangement form, for every admissible function \( u : \mathbb{R}^n \to \mathbb{R} \) in terms of \( \mathcal{L}u \). Such an estimate in turn rests upon the isoperimetric inequality in the Gauss space, which tells us that half-spaces are the (only) minimizers for Gauss perimeter among all measurable subsets of \( \mathbb{R}^n \) with prescribed Gauss measure [12, 48]. Recall that the Gauss perimeter \( P_{\gamma_n}(E) \) of a measurable set \( E \subset \mathbb{R}^n \) can be defined by

\[
P_{\gamma_n}(E) = \mathcal{H}_{\gamma_n}^{n-1}(\partial^M E),
\]

where

\[
d\mathcal{H}_{\gamma_n}^{n-1}(x) = (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}} d\mathcal{H}^{n-1}(x),
\]

\( \partial^M E \) denotes the essential boundary of \( E \) and \( \mathcal{H}^{n-1} \) is the \((n-1)\)-dimensional Hausdorff measure. An analytic formulation of the Gaussian isoperimetric inequality makes use of the isoperimetric function \( I \), also called isoperimetric profile, of the Gauss space. The function \( I \) is the largest function that renders the inequality

\[
I(\gamma_n(E)) \leq P_{\gamma_n}(E)
\]

true for every measurable set \( E \subset \mathbb{R}^n \).

**Rearrangement estimate.** The rearrangement estimate in question reads

\[
0 \leq u^c(s) - u^c(\frac{1}{2}) \leq \Theta(s) \int_0^s (\mathcal{L}u)^*_+(r) \, dr + \int_s^{\frac{1}{2}} (\mathcal{L}u)^*_+(r) \Theta(r) \, dr \quad \text{for } s \in (0, \frac{1}{2}]
\]

and

\[
0 \leq u^c(\frac{1}{2}) - u^c(1-s) \leq \Theta(s) \int_0^s (\mathcal{L}u)^*_-(r) \, dr + \int_s^{\frac{1}{2}} (\mathcal{L}u)^*_-(r) \Theta(r) \, dr \quad \text{for } s \in (0, \frac{1}{2}].
\]

Here, \( u^c : (0, 1) \to \mathbb{R} \) denotes the signed decreasing rearrangement of \( u \) with respect to the Gauss measure, and \( (\mathcal{L}u)^*_+ \) and \( (\mathcal{L}u)^*_- \) the decreasing rearrangements of the positive and the negative parts of \( \mathcal{L}u \), respectively, with respect to the same measure. Moreover, \( \Theta : (0, \frac{1}{2}] \to [0, \infty) \) is the function defined as

\[
\Theta(s) = \int_s^{\frac{1}{2}} \frac{dr}{I(r)^2} \quad \text{for } s \in (0, \frac{1}{2}].
\]

Bounds in the spirit of (2.2) and (2.3) are rooted in the work of Maz’ya [39, 40] and Talenti [49], who made use of isoperimetric inequalities to estimate solutions to boundary value problems for classes of elliptic equations—including the Laplace equation—in subsets of the Euclidean space. Results in a similar vein for solutions to Dirichlet problems, with homogeneous boundary conditions, for elliptic equations on subsets of the Gauss space are the subject of [6]. The specific inequalities (2.2)-(2.3) can be proved following ideas of [6] and [15]. A detailed proof can be found in [17, Theorem 3.2]. Let us mention that the same inequalities hold for more general linear partial differential operators enjoying the same ellipticity property, with respect to the Gauss measure, as the Ornstein-Uhlenbeck operator. Hence, results parallel to those stated in Section 1 hold for these operators. We limit ourselves to dealing with the Ornstein-Uhlenbeck operator for ease of presentation.

The special form of the extremal sets in the Gaussian isoperimetric inequality entails that inequalities (2.2)-(2.3) hold as equalities, provided that the function \( u \) only depends on one of the coordinates of \( x \in \mathbb{R}^n \), and \( \mathcal{L}u \) is monotone and odd in this variable. Critical features of this kind of functions \( u \) in connection with this property are the fact that the level sets of both \( u \) and \( \mathcal{L}u \) are half-spaces—the
isoperimetric sets in the Gaussian isoperimetric inequality—and the constancy of their gradient on the boundary of their level sets.

Since both the functionals of $u$, appearing in (1.6), (1.11) and (1.15), and the constraints on $Lu$ prescribed by (1.7), (1.12), (1.14), are rearrangement-invariant, inequalities (2.2)–(2.3) enable us to reduce the original problems to one-dimensional inequalities. Let us emphasize that no loss of information occurs in the involved constants after this reduction, inasmuch as inequalities (2.2)–(2.3) turn into equalities for the class of functions described above.

**Sharp Orlicz-Hölder inequality.** The proof of the resultant one-dimensional inequalities exploits a Hölder type inequality for couples of functions in Orlicz spaces. The precise constant in this inequality is again critical. To this purpose, we have to resort to a special form of the Hölder inequality in Orlicz spaces, where two different kinds of norms—the Luxemburg and the Orlicz norm—are applied to the integrand

$$\int_{\Omega} \Phi(\theta |t|) \, d\gamma \, dt \leq C \int_{\Omega} \Theta(\phi) \, d\gamma \, dt,$$

where $\Theta$ is the function given by (2.4), and $\|\| \|_{L^{\tilde{B}}(s, \frac{1}{2})}$ stands for the Orlicz norm on $(s, \frac{1}{2})$ built upon the Young conjugate $\tilde{B}$ of a Young function $B$ which agrees with $e^{t^\beta}$, or with $e^{t^\gamma}$, near infinity. This is a very subtle task, inasmuch as neither $\Theta$ nor $\tilde{B}$ take an explicit form. In addition, the Orlicz norm of a function is not just defined via an integral. Instead, it requires the solution of a constrained minimization problem for an integral functional. This notwithstanding, we are able to derive the first two terms of an asymptotic expansion of function (2.5) as $s \to 0^+$, which suffices for our purposes. Very precise estimates are required in this derivation. With the asymptotic expansion at our disposal, the proof of inequality (1.11) follows quite easily. On the other hand, problem (1.6)–(1.7) is reduced to (1.11)–(1.12). The proof of inequality (1.15) relies upon an analogous argument.

**Optimality.** As suggested by the considerations above, the proof of the optimality of our results exploits suitable trial functions $u : \mathbb{R}^n \to \mathbb{R}$ depending on one variable only. Heuristically, these functions should be chosen in such a way that the equality holds in the Orlicz-Hölder inequality mentioned above, for each $s \in (0, \frac{1}{2})$. This requirement, however, prevents $Lu$ from belonging to the prescribed Orlicz space. In order to restore membership in the latter space, a truncation argument is introduced, which results in trial functions $u$ attaining the equality in the Hölder inequality at least asymptotically. Since $L$ is a second-order differential operator, plain truncation of functions cannot be employed, and has to be replaced by a smooth truncation operation. In order not to violate the integral or the norm constraint on $Lu$, an ad hoc smooth truncation operator has to be introduced.

**Maximizers.** The question of the existence of extremals, for $\beta \in (0, 1)$, in the inequalities discussed in Theorems 1.1 and 1.2 displays some peculiar features. Their proofs rest upon a compactness argument, which, besides standard lower semicontinuity theorems, applies thanks to a uniform integrability property of maximizing sequences of functions. The uniform integrability is guaranteed by a slightly enhanced integrability result, of independent interest, for functions fulfilling condition (1.7) or (1.12). Indeed, we show that the supremum in (1.6) or (1.11), respectively, for the threshold value $\theta = \frac{2}{\beta}$, is still finite even if the integrand $e^{\frac{t}{\beta} |u|^\beta}$ is multiplied by a function $\varphi(|u|)$, where $\varphi(t)$ diverges to infinity at a sufficiently mild rate when $t \to \infty$. Such a result holds in sharp contrast with inequality (1.9), for which any improvement in this direction is known to fail. This is possible thanks to the special nature of the exponential type norms and measure entering the game. Let us stress that this augmentation neither contradicts the optimality of the target space $\exp L^{\beta}(\mathbb{R}^n, \gamma_n)$ in inequality (1.4), nor the sharpness of the constant $\frac{2}{\beta}$ in (1.6) or (1.11). Actually, under the condition to be imposed on $\varphi$, the function $e^{\frac{t}{\beta} |u|^\beta}$ is still equivalent to $e^{\frac{t}{\beta} |u|^\beta}$, in the sense of Young functions. Moreover, under the same condition, the function $e^{\beta t^\beta}$ grows faster than $e^{\beta t^\beta}$ near infinity for every $\theta > \frac{2}{\beta}$.

**Article structure.** We begin by recalling the necessary background on functions and function spaces in the next section. Section 4 has a technical content: it is devoted to the asymptotic expansion of the function in (2.5) and of related functions. The reduction to one-dimensional problems is performed in Section 5. Section 6 deals with the smooth truncation operator, which is applied in the construction of trial functions showing the optimality of our results. The proofs of Theorems 1.1–1.4 are accomplished in
Section 7. In the final Section 8, we establish the improved integrability property, which is the key step in the proof of Theorem 1.3.

3. Function spaces

Rearrangements. Let \((\mathcal{R}, \nu)\) be a probability space, namely a measure space \(\mathcal{R}\) endowed with a probability measure \(\nu\). Assume that \((\mathcal{R}, \nu)\) is non-atomic. In fact, we shall just be concerned with the case when \(\mathcal{R}\) is either \(\mathbb{R}^n\) endowed with the Gaussian measure \(\gamma_n\), or \((0, 1)\) endowed with the Lebesgue measure. In the latter case, the measure will be omitted in the notation. More generally, we shall simply write \(\mathcal{R}\) instead of \((\mathcal{R}, \nu)\) when no ambiguity can arise. The notation \(\mathcal{M}(\mathcal{R}, \nu)\) is employed for the space of real-valued, \(\nu\)-measurable functions on \(\mathcal{R}\). Also, \(\mathcal{M}_+(\mathcal{R}, \nu)\) stands for the subset of its nonnegative functions.

Let \(\phi \in \mathcal{M}(\mathcal{R}, \nu)\). The decreasing rearrangement \(\phi^* : [0, 1] \to [0, \infty]\) of \(\phi\) is given by

\[
\phi^*(s) = \inf \{t \geq 0 : \nu(\{x \in \mathcal{R} : |\phi(x)| > t\}) \leq s\} \quad \text{for } s \in [0, 1].
\]

Similarly, the signed decreasing rearrangement \(\phi^\circ : [0, 1] \to [-\infty, \infty]\) of \(\phi\) is defined as

\[
\phi^\circ(s) = \inf \{t \in \mathbb{R} : \nu(\{x \in \mathcal{R} : \phi(x) > t\}) \leq s\} \quad \text{for } s \in [0, 1].
\]

Note that, in particular,

\[
\text{med}(\phi) = \phi^\circ\left(\frac{1}{2}\right)
\]

for every function \(\phi \in L^1(\mathbb{R}, \nu)\). The space \(L^A(\mathcal{R}, \nu)\) is defined as

\[
L^A(\mathcal{R}, \nu) = \left\{ \phi \in \mathcal{M}(\mathcal{R}, \nu) : \int_{\mathcal{R}} A\left(\frac{|\phi|}{\lambda}\right) d\nu < \infty \quad \text{for some } \lambda > 0 \right\}.
\]

The space \(L^A(\mathcal{R}, \nu)\) is a Banach space with respect to the Luxemburg norm defined by

\[
\|\phi\|_{L^A(\mathcal{R}, \nu)} = \inf \left\{ \lambda > 0 : \int_{\mathcal{R}} A\left(\frac{|\phi|}{\lambda}\right) d\nu \leq 1 \right\}
\]

for a function \(u \in L^A(\mathcal{R}, \nu)\). The Luxemburg norm is equivalent to the Orlicz norm given by

\[
\|\phi\|_{L^A(\mathcal{R}, \nu)} = \sup\left\{ \int_{\mathcal{R}} \phi \psi d\nu : \int_{\mathcal{R}} \tilde{A}(|\psi|) d\nu \leq 1 \right\}
\]

for a function \(\phi \in L^A(\mathcal{R}, \nu)\). Here, \(\tilde{A} : [0, \infty) \to [0, \infty]\) denotes the Young conjugate of \(A\), defined as

\[
\tilde{A}(t) = \sup\{\tau : \tau - A(t) : \tau \geq 0\} \quad \text{for } t \geq 0,
\]

which is also a Young function. Notice that, if \(a : [0, \infty) \to [0, \infty]\) is the non-decreasing left-continuous function such that

\[
A(t) = \int_0^t a(\tau) d\tau \quad \text{for } t \geq 0,
\]

then \(\tilde{A}\) admits the representation formula

\[
\tilde{A}(t) = \int_0^t a^{-1}(\tau) d\tau \quad \text{for } t \geq 0,
\]

where \(a^{-1}\) denotes the (generalized) left-continuous inverse of \(a\). Young’s inequality tells us that

\[
t\tau \leq A(t) + \tilde{A}(\tau) \quad \text{for } t, \tau \geq 0,
\]

and follows from the very definition of Young conjugate. Equality holds in (3.3) if and only if either \(t = a^{-1}(\tau)\) or \(\tau = a(t)\).

Both the Luxemburg norm and the Orlicz norm are rearrangement invariant. Hence,

\[
\|\phi\|_{L^A(\mathcal{R}, \nu)} = \|\phi^*\|_{L^A(0, 1)} = \|\phi^\circ\|_{L^A(0, 1)} \quad \text{and} \quad \|\phi\|_{L^A(\mathcal{R}, \nu)} = \|\phi^*\|_{L^A(0, 1)} = \|\phi^\circ\|_{L^A(0, 1)}
\]
for every function $\phi \in L^A(\mathcal{R}, \nu)$.

A sharp form of the Hölder inequality in Orlicz spaces reads

\begin{equation}
\int_{\mathcal{R}} \phi \psi \, d\nu \leq \|\phi\|_{L^A(\mathcal{R}, \nu)} \|\psi\|_{L^\tilde{A}(\mathcal{R}, \nu)}
\end{equation}

for $\phi \in L^A(\mathcal{R}, \nu)$ and $\psi \in L^{\tilde{A}}(\mathcal{R}, \nu)$.

If $\phi \in L^A(\mathcal{R}, \nu)$ and $E \subset \mathcal{R}$ is a measurable set, we use the abridged notations

$\|\phi\|_{L^A(E)} = \|\phi \chi_E\|_{L^A(\mathcal{R}, \nu)}$ and $\|\phi\|_{L^A(E)} = \|\phi \chi_E\|_{L^A(\mathcal{R}, \nu)}$.

In particular,

\begin{equation}
\|1\|_{L^A(E)} = \nu(E) \tilde{A}^{-1}(1/\nu(E)),
\end{equation}

where $\tilde{A}^{-1}$ denotes the (generalized) right-continuous inverse of $\tilde{A}$.

**The isoperimetric function in Gauss space.** The Gaussian isoperimetric function $I$, also called the Gaussian isoperimetric profile, of the space $(\mathbb{R}^n, \gamma_n)$ plays a pivotal role in our approach. Its name stems from the fact that it governs the isoperimetric inequality in Gauss space - see [12, 48]. The function $I: [0, 1] \rightarrow [0, \infty)$ obeys

\begin{equation}
I(s) = \frac{1}{\sqrt{2\pi}} e^{-s^2/2} \quad \text{for } s \in (0, 1),
\end{equation}

and $I(0) = I(1) = 0$, where $\Phi: \mathbb{R} \rightarrow (0, 1)$ is the function defined as

\begin{equation}
\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-\tau^2/2} \, d\tau \quad \text{for } t \in \mathbb{R}.
\end{equation}

Note that

\begin{equation}
I(\Phi(t)) = -\Phi'(t) \quad \text{for } t \in \mathbb{R}.
\end{equation}

**Sobolev type spaces.** The Sobolev space $W^{1,2}(\mathbb{R}^n, \gamma_n)$ is defined as

$W^{1,2}(\mathbb{R}^n, \gamma_n) = \{ u \in L^2(\mathbb{R}^n, \gamma_n) : u \text{ is weakly differentiable and } |\nabla u| \in L^2(\mathbb{R}^n, \gamma_n) \}.$

Similarly,

$W^{2,2}(\mathbb{R}^n, \gamma_n) = \{ u \in L^2(\mathbb{R}^n, \gamma_n) : u \text{ is twice weakly differentiable and } |\nabla u|, |\nabla^2 u| \in L^2(\mathbb{R}^n, \gamma_n) \}.$

The operator $\mathcal{L}$ is defined on functions $u \in W^{2,2}(\mathbb{R}^n, \gamma_n)$ via equation (1.1). Moreover, one has that $\mathcal{L}: W^{2,2}(\mathbb{R}^n, \gamma_n) \rightarrow L^2(\mathbb{R}^n, \gamma_n)$, and

\begin{equation}
\int_{\mathbb{R}^n} \nabla u \cdot \nabla v \, d\gamma_n = - \int_{\mathbb{R}^n} v \, \mathcal{L} u \, d\gamma_n
\end{equation}

for every $\phi \in W^{1,2}(\mathbb{R}^n, \gamma_n)$, see e.g. [37, Theorem 13.1.3].

As customary, equation (3.9) enables one to extend the operator $\mathcal{L}$ outside its natural domain, and to define it on all functions $u \in W^{1,2}(\mathbb{R}^n, \gamma_n)$ such that there exists a function $f \in L^2(\mathbb{R}^n, \gamma_n)$ fulfilling

\begin{equation}
\int_{\mathbb{R}^n} \nabla u \cdot \nabla v \, d\gamma_n = - \int_{\mathbb{R}^n} v \, f \, d\gamma_n
\end{equation}

for every $v \in W^{1,2}(\mathbb{R}^n, \gamma_n)$. This function space will be denoted by $W^\mathcal{L} L^2(\mathbb{R}^n, \gamma_n)$. On setting

$\mathcal{L} u = f$

for $u \in W^\mathcal{L} L^2(\mathbb{R}^n, \gamma_n)$, one thus has that $\mathcal{L}: W^\mathcal{L} L^2(\mathbb{R}^n, \gamma_n) \rightarrow L^2(\mathbb{R}^n, \gamma_n)$.

More generally, given an Orlicz space $L^A(\mathbb{R}^n, \gamma_n)$ contained in $L^2(\mathbb{R}^n, \gamma_n)$, we define

$W^\mathcal{L} L^A(\mathbb{R}^n, \gamma_n) = \{ u \in W^\mathcal{L} L^2(\mathbb{R}^n, \gamma_n) : \mathcal{L} u \in L^A(\mathbb{R}^n, \gamma_n) \}$.
4. Asymptotic expansions

A substantial problem to be faced in attacking inequalities (1.6) and (1.11) is the computation of certain Orlicz norms of functions depending on the function \( \Phi \) and on the exponent \( \beta \). The norms in question do not admit an expression in a closed form. However, we can to describe their asymptotic behaviour in an accurate form that enables us to circumvent this problem. The asymptotic expansions which come into play in this connection are collected in this section.

Given a function \( F \) defined in some neighbourhood of a point \( t_0 \in [-\infty, \infty] \), and \( k \in \mathbb{N} \), we write

\[
F(t) = E_1(t) + \cdots + E_k(t) + \cdots
\]

as \( t \to t_0 \) to denote that

\[
\lim_{t \to t_0} F(t) = 1 \quad \text{if } k = 1,
\]

and

\[
\lim_{t \to t_0} \frac{F(t) - [E_1(t) + \cdots + E_j(t) + \cdots]}{E_{j+1}(t)} = 1 \quad \text{for } 1 \leq j \leq k - 1, \text{ otherwise.}
\]

If \( F(t) = E_1(t) + E_2(t) + E_3(t) + \cdots \) as \( t \to t_0 \) and \( E_1(t) \) is positive on a neighbourhood of \( t_0 \), then for every \( \sigma \in \mathbb{R} \), one has

\[
F(t)^\sigma = E_1(t)^\sigma + \sigma E_1(t)^{\sigma-1} E_2(t) + \cdots \quad \text{as } t \to t_0.
\]

Furthermore

\[
\log(F(t)) = \log(E_1(t)) + \frac{E_2(t)}{E_1(t)} + \cdots \quad \text{as } t \to t_0.
\]

The behaviour of the function \( \Phi \) defined by (3.7) was analyzed in [18, Lemma 5.1]. It tells us that

\[
\log(\frac{1}{\Phi(t)}) = \frac{t^2}{2} + \log t + \cdots \quad \text{as } t \to \infty
\]

and

\[
-\Phi'(t) = t\Phi(t) + \frac{\Phi(t)}{t} + \cdots \quad \text{as } t \to \infty.
\]

The following asymptotic expansion for the isoperimetric function \( I \) can be derived through equations (3.8) and (4.4).

**Lemma 4.1.** Let \( I \) be the function defined by (3.6). Then

\[
I(s) = s \sqrt{2 \log \frac{1}{s} - s \log \log \frac{1}{s} + \cdots} \quad \text{as } s \to 0^+.
\]

The content of the lemma below is an estimate for an expression involving the isoperimetric function \( I \), which will be exploited on various occasions.

**Lemma 4.2.** Let \( I \) be the function defined by (3.6). Then

\[
\frac{s}{I(s)^2} \geq \frac{1}{2s \log \frac{1}{s}} \quad \text{for } s \in (0, \frac{1}{2}],
\]

Inequality (4.7) appears in [7, Lemma 4.2]. We provide a proof for completeness.

**Proof.** Since the function \( \Phi: [0, \infty) \to (0, \frac{1}{2}] \) is bijective, equation (3.8) ensures that inequality (4.7) is equivalent to

\[
2\Phi(t)^2 \log \frac{1}{\Phi(t)} - \Phi'(t)^2 \geq 0 \quad \text{for } t \geq 0.
\]

Owing to expansions (4.4) and (4.5), it is easily verified that the left-hand side of (4.8) tends to zero as \( t \to \infty \). Therefore, inequality (4.8) will follow if we show that the function on its left-hand side is decreasing. The derivative of this function equals

\[
2\Phi'(t) \left[ 2\Phi(t) \log \frac{1}{\Phi(t)} - \Phi(t) + t\Phi'(t) \right] \quad \text{for } t > 0.
\]
Notice that here we have made use of the equality $\Phi''(t) = -t\Phi'(t)$ for $t \in \mathbb{R}$. Denote the function in the square bracket in (4.9) by $F(t)$. Inasmuch as $\Phi'(t) < 0$ for $t > 0$, it suffices to show that $F(t) > 0$ for $t > 0$. We have that

$$
F'(t) = \Phi'(t) \left[ 2 \log \frac{1}{\Phi(t)} - 2 - t^2 \right] \quad \text{for } t > 0.
$$

Let us analyze the sign of $F'$. Denote the function in the square bracket in (4.10) by $G(t)$. We claim that $G$ is increasing on $(0, \infty)$. Indeed,

$$
G'(t) = -2 \frac{\Phi'(t)}{\Phi(t)} - 2t \quad \text{for } t > 0.
$$

Hence, $G'(t) > 0$ for $t > 0$, thanks to the inequality $-\Phi'(t) > t\Phi(t)$ for $t > 0$, see [16, Lemma 3.4]. Furthermore,

$$
\lim_{t \to 0^+} G(t) = 2(\log 2 - 1) < 0.
$$

On the other hand, owing to equation (4.4),

$$
\lim_{t \to \infty} G(t) = \lim_{t \to \infty} \left[ \left( t^2 + 2 \log t + \cdots \right) - 2 - t^2 \right] = \infty.
$$

Thereby, there exists a unique $t_0 > 0$ satisfying $G(t_0) = 0$. Consequently, $F$ is increasing on $(0, t_0)$ and decreasing on $(t_0, \infty)$. Since

$$
\lim_{t \to 0^+} F(t) = \log 2 - \frac{1}{2} > 0 \quad \text{and} \quad \lim_{t \to \infty} F(t) = 0,
$$

we can thus conclude that $F(t) > 0$ for $t > 0$. \hfill \Box

An asymptotic expansion near zero for the function $\Theta$, defined by (2.4), is stated in the next lemma, and can be deduced via Lemma 4.1, equation (4.2) and L'Hôpital's rule.

**Lemma 4.3.** Let $\Theta$ be the function defined by (2.4). Then

$$
\Theta(s) = \frac{1}{2s \log \frac{1}{s}} - \frac{\log \log \frac{1}{s}}{4s \left( \log \frac{1}{s} \right)^2} + \cdots \quad \text{as } s \to 0^+.
$$

The following lemma is a consequence of Lemma 4.3, and concerns the asymptotic behaviour near zero of the function $\Lambda$: $(0, \frac{1}{2}] \to [0, \infty)$ given by

$$
\Lambda(s) = \int_s^{\frac{1}{2}} \Theta(r) \, dr + s\Theta(s) \quad \text{for } s \in (0, \frac{1}{2}].
$$

**Lemma 4.4.** Let $\Lambda$ be the function defined by (4.12). Then

$$
\Lambda(s) = \frac{1}{2} \log \log \frac{1}{s} + \cdots \quad \text{as } s \to 0^+.
$$

The next result is contained in [19, Lemma 3.4].

**Lemma 4.5.** Let $\beta > 0$, $N > 0$ and let $B$ be a Young function obeying $(1.13)$. Let $b: [0, \infty) \to [0, \infty)$ be the left-continuous function such that

$$
b(t) = \int_0^t b(\tau) \, d\tau \quad \text{for } t \geq 0.
$$

Then

$$
b^{-1}(t) = (\log t)^{\frac{1}{\beta}} + \frac{1 - \beta}{\beta^2} (\log t)^{\frac{1}{\beta} - 1} \log \log t - \frac{\log N\beta}{\beta} (\log t)^{\frac{1}{\beta} - 1} + \cdots \quad \text{as } t \to \infty.
$$

From equation (4.2) and Lemma 4.3, one obtains the following lemma.

**Lemma 4.6.** Let $\beta > 0$, $N > 0$ and let $B$ be a Young function obeying $(1.13)$. Then

$$
\Theta(s)sB^{-1}\left( \frac{1}{s} \right) = \frac{1}{2} (\log \frac{1}{s})^{\frac{1}{\beta} - 1} - \frac{1}{4} (\log \frac{1}{s})^{\frac{1}{\beta} - 2} \log \log \frac{1}{s} + \cdots \quad \text{as } s \to 0^+.
$$

Lemmas 4.1 and 4.3 enable one to derive the result below.
**Lemma 4.7.** Let $I$ and $\Theta$ be the functions defined by (3.6) and (2.4). Then
\begin{equation}
I \left( \Theta^{-1}(t) \right)^2 = \frac{1}{2t^2 \log t} - \frac{5 \log \log t}{4t^2 (\log t)^2} + \cdots \quad \text{as } t \to \infty.
\end{equation}

Combining Lemmas 4.5 and 4.7 yields the expansion which is the subject of the next result.

**Lemma 4.8.** Let $\beta > 0$, $N > 0$ and let $B$ be a Young function obeying (1.13). Let $b$ be the function appearing in (4.13). Let $I$ and $\Theta$ be the functions defined by (3.6) and (2.4). Assume that $\lambda > 0$. Then
\begin{equation}
b^{-1}(\lambda t) I(\Theta^{-1}(t))^2 = \frac{1}{2t} (\log t)^{\frac{1}{\beta} - 1} + \left( \frac{1 - \beta}{2\beta^2} - \frac{5}{4} \right) \frac{1}{t} (\log t)^{\frac{1}{\beta} - 2} \log \log t + \cdots \quad \text{as } t \to \infty.
\end{equation}

Given $\sigma > 0$, let $\Psi_\sigma : (1, \infty) \to (0, \infty)$ be the function defined by
\begin{equation}
\Psi_\sigma(t) = \int_1^t \frac{(\tau - 1)^\sigma}{\tau} \, d\tau \quad \text{for } t > 1.
\end{equation}

Elementary considerations yield the following asymptotic expansion for $\Psi_\sigma$ as $t \to \infty$.

**Lemma 4.9.** Let $\sigma > 0$ and let $\Psi_\sigma$ be the function defined by (4.17). Then
\begin{equation}
\Psi_\sigma(t) = \frac{1}{\sigma} t^\sigma - \begin{cases} \frac{c}{\sigma - 1} t^{\sigma - 1} + \cdots & \text{if } \sigma \in (1, \infty) \\ \log t + \cdots & \text{if } \sigma = 1 \\ c + \cdots & \text{if } \sigma \in (0, 1) \end{cases} \quad \text{as } t \to \infty,
\end{equation}
for some constant $c$ depending on $\sigma$.

The next result provides us with a formula for Orlicz norms, which generalizes [19, Lemma 3.5].

**Lemma 4.10.** Let $B$ be a finite-valued Young function of the form (4.13), with $b$ strictly increasing and such that $b(0) = 0$. Assume that the function $h$ belongs to $\mathcal{M}_+(0, \frac{1}{2})$ and does not vanish identically. Then,
\begin{equation}
\|h\|_{L^\tilde{B}(s, \frac{1}{2})} = \int_s^{\frac{1}{2}} b^{-1}(\xi_s h(r)) h(r) \, dr \quad \text{for } s \in (0, \frac{1}{2}),
\end{equation}
where $\xi_s > 0$ is uniquely defined by
\begin{equation}
\int_s^{\frac{1}{2}} B(b^{-1}(\xi_s h(r))) \, dr = 1 \quad \text{for } s \in (0, \frac{1}{2}).
\end{equation}

**Proof.** Let $s \in (0, \frac{1}{2})$. By the definition of the Orlicz norm,
\begin{equation}
\|h\|_{L^\tilde{B}(s, \frac{1}{2})} = \sup \left\{ \int_s^{\frac{1}{2}} f(r) h(r) \, dr : \int_s^{\frac{1}{2}} B(|f(r)|) \, dr \leq 1 \right\}.
\end{equation}

Let $\xi_s > 0$ and $f \in \mathcal{M}_+(0, \frac{1}{2})$. By Young’s inequality (3.3),
\begin{equation}
\int_s^{\frac{1}{2}} h(r) f(r) \, dr \leq \int_s^{\frac{1}{2}} B \left( \frac{f(r)}{\xi_s} \right) \, dr + \int_s^{\frac{1}{2}} \tilde{B}(\xi_s h(r)) \, dr.
\end{equation}

Let us define
\begin{equation}
f_s(r) = \xi_s b^{-1}(\xi_s h(r)) \quad \text{for } r \in (s, \frac{1}{2}).
\end{equation}

By the equality cases in Young’s inequality (3.3),
\begin{equation}
f_s h = \xi_s h b^{-1}(\xi_s h) = B(b^{-1}(\xi_s h)) + \tilde{B}(\xi_s h),
\end{equation}
whence
\begin{equation}
f_s h = B \left( \frac{f_s}{\xi_s} \right) + \tilde{B}(\xi_s h).
\end{equation}

Now, assume that $\xi_s$ obeys (4.19), namely
\begin{equation}
\int_s^{\frac{1}{2}} B \left( \frac{f_s(r)}{\xi_s} \right) \, dr = 1.
\end{equation}
Observe that, since $B(b^{-1})$ is strictly increasing, there exists a unique $\xi_s > 0$ fulfilling condition (4.19). Integrating both sides of equation (4.21) over $(s, \frac{1}{2})$ and recalling equation (4.22) yield
\[
\int_s^{\frac{1}{2}} f_s(r)h(r) \, dr = 1 + \int_s^{\frac{1}{2}} \widetilde{B}(\xi_s h(r)) \, dr,
\]
whence
\[
\sup \left\{ \int_s^{\frac{1}{2}} f(r)h(r) \, dr : \int_s^{\frac{1}{2}} B\left(\frac{f}{\xi_s}\right) \leq 1 \right\} = \int_s^{\frac{1}{2}} f_s(r)h(r) \, dr.
\]
Therefore, owing to (4.20),
\[
\|h\|_{L^\infty([s, \frac{1}{2}])} = \sup \left\{ \int_s^{\frac{1}{2}} f(r)h(r) \, dr : \int_s^{\frac{1}{2}} B\left(\frac{f}{\xi_s}\right) \leq 1 \right\} = \frac{1}{\xi_s} \int_s^{\frac{1}{2}} f_s(r)h(r) \, dr = \int_s^{\frac{1}{2}} b^{-1}(\xi_s h(r))h(r) \, dr.
\]
Equation (4.18) hence follows.

In the special case when $h = \Theta$, the conclusions of Lemma 4.10 can be rephrased as in the next lemma. In what follows, given $t > 0$, we denote by $\lambda_t$ the unique positive number such that
\[
(4.23) \quad \int_0^t B(b^{-1}(\lambda_t \tau)) I(\Theta^{-1}(\tau))^2 \, d\tau = 1.
\]

**Lemma 4.11.** Let $B$ be as in Lemma 4.10, let $\Theta$ be given by (2.4) and $\lambda_{\Theta(s)}$ by (4.23), with $t = \Theta(s)$. Then
\[
(4.24) \quad \|\Theta\|_{L^\infty([s, \frac{1}{2}])} = \int_0^{\Theta(s)} b^{-1}(\lambda_{\Theta(s)} \rho) \rho I(\Theta^{-1}(\rho))^2 \, d\rho \quad \text{for } s \in (0, \frac{1}{2}).
\]

**Proof.** For each $s \in (0, \frac{1}{2})$, the function $\Theta$ is strictly decreasing in $(s, \frac{1}{2})$, and maps this interval onto $(0, \Theta(s))$. The conclusion thus follows from Lemma 4.10, on choosing $h = \Theta$, and making the change of variable $\rho = \Theta(r)$ in the integrals in (4.18) and (4.19). The equality $dr = -I(\Theta^{-1}(\rho))^2 \, d\rho$ has to be exploited here.

An asymptotic expansion for the function $\lambda_t$, defined via equation (4.23), is provided by the following lemma in the case when the Young function $B$ fulfills condition (1.13).

**Lemma 4.12.** Let $B$ be a Young function of the form (1.13) and let $\lambda_t > 0$ be defined by (4.23) for $t > 0$. Then the function $t \mapsto \lambda_t$ is decreasing on $(0, \infty)$, and
\[
(4.25) \quad \lambda_t = \begin{cases} (2 - 2\beta)(\log t)^{1 - \frac{1}{\beta}} + \cdots & \text{if } \beta \in (0, 1), \\ \frac{2}{\log \log t} + \cdots & \text{if } \beta = 1, \\ \frac{\lambda + \cdots}{t} & \text{if } \beta \in (1, \infty) \end{cases} \quad \text{as } t \to \infty.
\]

for some $\lambda > 0$, depending on $\beta$.

**Proof.** The monotonicity of the function $t \mapsto \lambda_t$ follows from equation (4.23) and the fact that the function $B \circ b^{-1}$ is increasing.

In order to prove expansion (4.25), we begin by observing that
\[
B(b^{-1}(t)) = \frac{t}{\beta}[b^{-1}(t)]^{1 - \beta} \quad \text{for sufficiently large } t.
\]

Consequently, if $\beta \neq 1$, then by Lemma 4.5 and equation (4.2) with $\sigma = 1/\beta$,
\[
(4.26) \quad B(b^{-1}(t)) = \frac{1}{\beta} t(\log t)^{\frac{1}{\beta} - 1} + \frac{(1 - \beta)^2}{\beta^3} t(\log t)^{\frac{1}{\beta} - 2} \log \log t + \cdots \quad \text{as } t \to \infty.
\]

On the other hand, if $\beta = 1$, then $b = B$ near infinity, whence $B(b^{-1}(t)) = t$ for large $t$. 

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Owing to expansion (4.26) and Lemma 4.7, for every \( \varepsilon \in (0, 1) \) there exists \( t_0 > 0 \) such that
\[
B(b^{-1}(t)) \geq \frac{t}{\beta} (\log t)^{\frac{1}{\beta} - 1} \quad \text{and} \quad I(\Theta^{-1}(t))^2 \geq (1 - \varepsilon) \frac{1}{2t^2 \log t}
\]
and
\[
B(b^{-1}(t)) \leq \frac{1 + \varepsilon}{\beta} t (\log t)^{\frac{1}{\beta} - 1} \quad \text{and} \quad I(\Theta^{-1}(t))^2 \leq \frac{1}{2t^2 \log t}
\]
for \( t > t_0 \).

Assume first that \( \beta \in (1, \infty) \). We claim that \( \lim_{t \to \infty} \lambda_t > 0 \). Assume, by contradiction, that \( \lim_{t \to \infty} \lambda_t = 0 \). Inequalities (4.28) ensure that
\[
\int_0^\infty B(b^{-1}(\tau)) I(\Theta^{-1}(\tau))^2 \, d\tau < \infty.
\]
Therefore, by the dominated convergence theorem and the fact that \( b^{-1}(0) = 0 \),
\[
1 = \lim_{t \to \infty} \int_0^t B(b^{-1}(\lambda_t \tau)) I(\Theta^{-1}(\tau))^2 \, d\tau \leq \int_0^\infty \lim_{t \to \infty} B(b^{-1}(\lambda_t \tau)) I(\Theta^{-1}(\tau))^2 \, d\tau = 0.
\]
This contradiction proves our claim, and hence also equation (4.25).

Assume next that \( \beta \in (0, 1] \). We claim that \( \lim_{t \to \infty} \lambda_t = 0 \). Suppose, by contradiction, that \( \lim_{t \to \infty} \lambda_t = \lambda > 0 \). Fix \( \varepsilon \in (0, 1) \), and let \( t_0 \) be as above. Set
\[
t'_0 = \max\{\lambda, 1\} t_0.
\]
By (4.23), Fatou’s lemma and inequalities (4.27),
\[
1 \geq \int_0^\infty B(b^{-1}(\lambda t)) I(\Theta^{-1}(t))^2 \, dt \geq \int_0^{t'_0} B(b^{-1}(\lambda t)) I(\Theta^{-1}(t))^2 \, dt \geq \frac{1 - \varepsilon}{2\beta} \lambda \int_{t'_0}^\infty (\log \lambda t)^{\frac{1}{\beta} - 1} \frac{1}{t \log t} \, dt.
\]
Inasmuch as \( 1/\beta - 1 \geq 0 \), the latter integral diverges, and we obtain a contradiction. Our claim is thus established.

Now, let \( \beta \in (0, 1) \). Equation (4.25) will follow if we show that
\[
\lim_{t \to \infty} \frac{\lambda_t}{(2 - 2\beta)(\log t)^{1 - \frac{1}{\beta}}} = 1.
\]
Assume that equation (4.29) does not hold. Then there exist \( \delta > 0 \) and an increasing sequence \( \{t_k\} \) such that \( t_k \to \infty \) and either
\[
\lambda_{t_k} \geq (1 + \delta)(2 - 2\beta)(\log t_k)^{1 - \frac{1}{\beta}}
\]
or
\[
\lambda_{t_k} \leq (1 - \delta)(2 - 2\beta)(\log t_k)^{1 - \frac{1}{\beta}}
\]
for \( k \in \mathbb{N} \). Assume first that (4.30) is satisfied. Then
\[
\lim_{k \to \infty} \lambda_{t_k} t_k = \infty
\]
and
\[
\lim_{k \to \infty} \frac{\log t_k}{\log \lambda_{t_k}} = \infty.
\]
Fix \( \varepsilon \in (0, 1) \). By combining equation (4.26) and Lemma 4.7 with the piece of information that \( \lambda_t \to 0 \), we conclude that there exists \( t_0 > 0 \) such that (4.27) and (4.28) hold for \( t > t_0 \), and
\[
\lambda_t < 1 \quad \text{for} \ t > t_0.
\]
By (4.32), there exists $k_0 \in \mathbb{N}$ such that, if $k \geq k_0$, then $t_k \geq t_0$ and $t_k \lambda_{t_k} > t_0$. Thereby,

$$1 = \int_0^{t_k} B(b^{-1}(\lambda_{t_k} \tau)) I(\Theta^{-1}(\tau))^2 \, d\tau \geq \int_0^{t_k} B(b^{-1}(\lambda_{t_k} \tau)) I(\Theta^{-1}(\tau))^2 \, d\tau$$

(4.35)

$$\geq \frac{1 - \varepsilon}{2\beta} \lambda_{t_k} \int_{t_0/\lambda_{t_k}}^{t_k} \left( \frac{\log \lambda_{t_k} \tau}{\tau \log \tau} \right)^{\frac{1}{\beta} - 1} \, d\tau \quad \text{for } k \geq k_0.$$  

The change of variables $\xi = \log \tau / \log \frac{1}{\lambda_{t_k}}$ yields

$$\int_{t_0/\lambda_{t_k}}^{t_k} \left( \frac{\log \lambda_{t_k} \tau}{\tau \log \tau} \right)^{\frac{1}{\beta} - 1} \, d\tau = \left( \frac{\log 1}{\lambda_{t_k}} \right)^{\frac{1}{\beta} - 1} \int_{\log \frac{t_0}{\lambda_{t_k}}}^{\log \frac{t_k}{\lambda_{t_k}}} \frac{(\xi - 1)^{\frac{1}{\beta} - 1}}{\xi} \, d\xi$$

(4.36)

$$= \left( \frac{\log 1}{\lambda_{t_k}} \right)^{\frac{1}{\beta} - 1} \left[ \frac{1}{\beta - 1} \left( \log \frac{t_k}{\lambda_{t_k}} \right) - \frac{1}{\beta - 1} \left( \frac{\log t_0}{\lambda_{t_k}} \right) \right]$$

for $k \geq k_0$, where $\Psi_1$ is the function defined as in (4.17). Observe that

$$\lim_{k \to \infty} \frac{\log \frac{t_0}{\lambda_{t_k}}}{\log \frac{1}{\lambda_{t_k}}} = 1,$$

whence, by (4.17),

$$\lim_{k \to \infty} \Psi_1 \left( \frac{\log \frac{t_k}{\lambda_{t_k}}}{\log \frac{1}{\lambda_{t_k}}} \right) = 0.$$  

On the other hand, it follows from equation (4.33) and Lemma 4.9 that

$$\Psi_1 \left( \frac{\log \frac{t_k}{\lambda_{t_k}}}{\log \frac{1}{\lambda_{t_k}}} \right) = \frac{\beta}{1 - \beta} \left( \frac{\log t_k}{\log \lambda_{t_k}} \right)^{\frac{1}{\beta} - 1} + \cdots \quad \text{as } k \to \infty.$$  

Coupling (4.35) with (4.30) and (4.36), and making use of (4.37) and (4.38) enable one to deduce that

$$1 \geq (1 + \delta)(1 - \varepsilon) \frac{1 - \beta}{\beta} \lim_{k \to \infty} \left( \frac{\log t_k}{\log \lambda_{t_k}} \right)^{1 - \frac{1}{\beta}} \left[ \frac{1}{\beta - 1} \left( \log \frac{t_k}{\lambda_{t_k}} \right) - \frac{1}{\beta - 1} \left( \frac{\log t_0}{\lambda_{t_k}} \right) \right] = (1 + \delta)(1 - \varepsilon).$$

A contradiction follows from this chain, provided that $\varepsilon$ is chosen so small that $(1 + \delta)(1 - \varepsilon) > 1$.

Assume next that equation (4.31) is in force. Therefore, fixing $\varepsilon \in (0, 1)$, there exists $t_0 \geq \varepsilon$ such that equation (4.28) holds for $t > t_0$ and (4.34) is fulfilled. We claim that there exists $k_0 \in \mathbb{N}$ such that

$$t_k \leq \frac{t_0}{\lambda_{t_k}} \quad \text{for } k \geq k_0.$$  

Suppose, by contradiction, that there exists an increasing subsequence of $\{t_k\}$, denoted again by $\{t_k\}$, satisfying $t_k \lambda_{t_k} > t_0$ for $k \in \mathbb{N}$. Thus,

$$1 = \int_0^{t_0/\lambda_{t_k}} B(b^{-1}(\lambda_{t_k} \tau)) I(\Theta^{-1}(\tau))^2 \, d\tau + \int_{t_0/\lambda_{t_k}}^{t_k} B(b^{-1}(\lambda_{t_k} \tau)) I(\Theta^{-1}(\tau))^2 \, d\tau$$

(4.40)

$$= J_1(t_k) + J_2(t_k) \quad \text{for } k \in \mathbb{N}.$$  

Since $B$ is a Young function, we have that $B(t) \leq tb(t)$ for $t > 0$. Consequently,

$$J_1(t_k) = \int_0^{t_0/\lambda_{t_k}} B(b^{-1}(\lambda_{t_k} \tau)) I(\Theta^{-1}(\tau))^2 \, d\tau \leq \lambda_{t_k} \int_0^{t_0/\lambda_{t_k}} b^{-1}(\lambda_{t_k} \tau) \tau I(\Theta^{-1}(\tau))^2 \, d\tau$$

(4.41)

$$\leq \lambda_{t_k} b^{-1}(t_0) \int_0^{t_0} \tau I(\Theta^{-1}(\tau))^2 \, d\tau + \lambda_{t_k} b^{-1}(t_0) \int_{t_0}^{t_0/\lambda_{t_k}} \tau I(\Theta^{-1}(\tau))^2 \, d\tau \quad \text{for } k \in \mathbb{N}. $$
The first integral on the rightmost side of (4.41) is trivially convergent. Moreover, since $t_0$ was chosen in such a way that $t_0 \geq e$ and (4.28) holds for every $\tau > t_0$, the second integral on the rightmost side of (4.41) can be estimated as

$$
\int_{t_0}^{\tau I(\Theta^{-1}(\tau))} \frac{d\tau}{2\tau \log \tau} \leq \frac{1}{2} \log \log \left( \frac{t_0}{\lambda_{t_k}} \right) \text{ for } k \in \mathbb{N}.
$$

Consequently, since $\lambda_{t_k} \to 0$ as $k \to \infty$,

(4.42) \quad $$
\lim_{k \to \infty} J_1(t_k) = 0.
$$

On making use of equations (4.28) and (4.36), we infer that

(4.43) \quad $$
J_2(t_k) \leq \frac{1 + \varepsilon}{2\beta} \lambda_{t_k} \left( \log \frac{1}{\lambda_{t_k}} \right)^{\frac{\beta}{\sigma} - 1} \Psi_{\frac{1}{\beta} - 1} \left( \frac{\log t_k}{\log \lambda_{t_k}} \right) \text{ for } k \in \mathbb{N}.
$$

Observe, that for every $\sigma > 0$,

$$
\Psi_{\sigma}(t) \leq \int_{1}^{t} \tau^{\sigma - 1} d\tau = \frac{1}{\sigma} t - \frac{1}{\sigma} \quad \text{for } t \in [1, \infty).
$$

Hence,

(4.44) \quad $$
\sigma t^{-\sigma} \Psi_{\sigma}(t) \leq 1 - t^{-\sigma} \leq 1 \quad \text{for } t \in [1, \infty).
$$

Form equations (4.40), (4.42),(4.43) and (4.31), we deduce that

$$
1 = \lim_{k \to \infty} [J_1(t_k) + J_2(t_k)] \leq (1 + \varepsilon)(1 - \delta) \lim_{k \to \infty} \frac{1 - \beta}{\beta} \left( \frac{\log t_k}{\log \lambda_{t_k}} \right)^{1 - \frac{\beta}{\sigma}} \Psi_{\frac{1}{\beta} - 1} \left( \frac{\log t_k}{\log \lambda_{t_k}} \right).
$$

This chain, coupled with estimate (4.44), yields $1 \leq (1 + \varepsilon)(1 - \delta)$, and hence a contradiction, provided that $\varepsilon$ is chosen small enough. Thus, inequality (4.39) is established. As a consequence,

(4.45) \quad $$
1 = \int_{0}^{t_k} B(b^{-1}(\lambda_{t_k} \tau)) I(\Theta^{-1}(\tau)) \frac{d\tau}{2\tau \log \tau} \leq \int_{0}^{t_k} B(b^{-1}(\lambda_{t_k} \tau)) I(\Theta^{-1}(\tau)) \frac{d\tau}{2\tau \log \tau} = J_1(t_k)
$$

for $k \geq k_0$. Equation (4.45), coupled with (4.42), leads to a contradiction again.

The case when $\beta = 1$ is analogous and even simpler. The details are omitted, for brevity. \hfill \Box

With Lemma 4.12 at our disposal, we are able to derive an asymptotic expansion for the norm $\|\Theta\|_{L^B(s, \frac{1}{2})}$ as $s \to 0^+$.

**Lemma 4.13.** Let $B$ be a Young function obeying (1.13) for some $\beta > 0$ and $N > 0$, and let $\Theta$ be the function defined by (2.4). Then

(4.46) \quad $$
\|\Theta\|_{L^B(s, \frac{1}{2})} = \frac{\beta}{2} \left( \log \frac{1}{s} \right)^{\frac{1}{\beta}} + \begin{cases} 
\frac{-2+3\beta}{4-\beta} \left( \log \frac{1}{s} \right)^{\frac{1}{\beta} - 1} \log \log \frac{1}{s} + \cdots & \text{if } \beta \in (0, 1), \\
\frac{-2}{\beta} \left( \log \log \frac{1}{s} \right)^{2} + \cdots & \text{if } \beta = 1 \quad \text{as } s \to 0^+, \\
c_{\beta,N} + \cdots & \text{if } \beta \in (1, \infty)
\end{cases}
$$

where $c_{\beta,N}$ is a constant depending on $\beta$ and $N$.

**Proof.** Given $s \in (0, \frac{1}{2})$, set $t = \Theta(s)$. By Lemma 4.11,

$$
\|\Theta\|_{L^B(s, \frac{1}{2})} = \int_{0}^{t} b^{-1}(\lambda_{t} \tau) I(\Theta^{-1}(\tau)) \frac{d\tau}{2\tau \log \tau} \quad \text{for } s \in (0, \frac{1}{2}),
$$

where $\lambda_{t} > 0$ is uniquely defined by (4.23).

Let $\beta \in (0, 1]$. Owing to Lemma 4.12, the function $t \mapsto \lambda_{t}$ is decreasing and $\lambda_{t} \to 0$ as $t \to \infty$. Thus, there exists $t_0 > e$ such that $\lambda_{t} < 1$ for $t > t_0$. Notice that, owing to equation (4.25), one has that $t > t_0/\lambda_{t}$ if $t$ is sufficiently large. Consequently,

(4.47) \quad $$
\|\Theta\|_{L^B(s, \frac{1}{2})} = \int_{0}^{t_0/\lambda_{t}} b^{-1}(\lambda_{t} \tau) I(\Theta^{-1}(\tau)) \frac{d\tau}{2\tau \log \tau} + \int_{t_0/\lambda_{t}}^{t} b^{-1}(\lambda_{t} \tau) I(\Theta^{-1}(\tau)) \frac{d\tau}{2\tau \log \tau} = J_1(t) + J_2(t)
$$
for large \( t \), depending on \( t_0 \). Let us focus on \( J_1 \) first. By Lemma 4.7, we may assume that \( t_0 \) is so large that
\[
(4.48) \quad tI(\Theta^{-1}(t))^2 \leq \frac{1}{2t \log t} \quad \text{for } t > t_0.
\]
One has that
\[
0 \leq J_1(t) \leq b^{-1}(t_0) \left( \int_0^{t_0} \tau I(\Theta^{-1}(\tau))^2 \, d\tau + \int_{t_0/\lambda_1}^{t_0} \tau I(\Theta^{-1}(\tau))^2 \, d\tau \right) \quad \text{for } t > t_0.
\]
Note that the first integral on the rightmost side of the last equation is finite and independent of \( t \) and, simultaneously,
\[
(4.47) \quad c(t_0) = \int_0^{t_0} \tau I(\Theta^{-1}(\tau))^2 \, d\tau.
\]
Next, by inequality (4.48),
\[
\int_{t_0/\lambda_1}^{t_0} \tau I(\Theta^{-1}(\tau))^2 \, d\tau \leq \frac{1}{2} \int_{t_0/\lambda_1}^{t_0} \frac{d\tau}{\tau \log \tau} \leq \frac{1}{2} \log \log \frac{t_0}{\lambda_1} \quad \text{for } t > t_0.
\]
Altogether,
\[
(4.49) \quad 0 \leq J_1(t) \leq \frac{b^{-1}(t_0)}{2} \log \log \frac{1}{\lambda_1} + \cdots \quad \text{as } t \to \infty.
\]
Let us now consider \( J_2 \). First, assume that \( \beta \in (0, 1) \). Fix \( \varepsilon > 0 \). By Lemmas 4.5 and 4.7, we may assume that \( t_0 \) is so large that
\[
b^{-1}(t) \leq (\log t)^{\frac{1}{2}} + (1 + \varepsilon) \frac{1 - \beta}{\beta^2} (\log t)^{\frac{1}{2} - 1} \log \log t
\]
and
\[
b^{-1}(t) \geq (\log t)^{\frac{1}{2}} + (1 - \varepsilon) \frac{1 - \beta}{\beta^2} (\log t)^{\frac{1}{2} - 1} \log \log t
\]
and, simultaneously,
\[
(4.50) \quad tI(\Theta^{-1}(t))^2 \leq \frac{1}{2t \log t} - (1 - \varepsilon) \frac{5 \log \log t}{4t(\log t)^2}
\]
and
\[
(4.51) \quad tI(\Theta^{-1}(t))^2 \geq \frac{1}{2t \log t} - (1 + \varepsilon) \frac{5 \log \log t}{4t(\log t)^2}
\]
for \( t > t_0 \). Thus, on setting
\[
J_{21}(t) = \int_{t_0/\lambda_1}^{t} \frac{(\log \lambda_1 \tau)^{\frac{1}{2}}}{\tau \log \tau} \, d\tau, \quad J_{22}(t) = \int_{t_0/\lambda_1}^{t} \frac{(\log \lambda_1 \tau)^{\frac{1}{2} - 1} \log(\lambda_1 \tau)}{\tau \log \tau} \, d\tau,
\]
\[
J_{23}(t) = \int_{t_0/\lambda_1}^{t} \frac{(\log \lambda_1 \tau)^{\frac{1}{2}} \log \log \tau}{\tau (\log \tau)^2} \, d\tau, \quad J_{24}(t) = \int_{t_0/\lambda_1}^{t} \frac{(\log \lambda_1 \tau)^{\frac{1}{2} - 1} \log(\lambda_1 \tau) \log \log \tau}{\tau (\log \tau)^2} \, d\tau
\]
for large \( t \), we deduce that
\[
(4.52) \quad J_2(t) \leq \frac{1}{2} J_{21}(t) + (1 + \varepsilon) \frac{1 - \beta}{2\beta^2} J_{22}(t) + (\varepsilon - 1) \frac{5}{4} J_{23}(t) + (\varepsilon^2 - 1) \frac{5(1 - \beta)}{4\beta^2} J_{24}(t)
\]
and
\[
(4.53) \quad J_2(t) \geq \frac{1}{2} J_{21}(t) + (1 - \varepsilon) \frac{1 - \beta}{2\beta^2} J_{22}(t) - (1 + \varepsilon) \frac{5}{4} J_{23}(t) + (\varepsilon^2 - 1) \frac{5(1 - \beta)}{4\beta^2} J_{24}(t)
\]
for large \( t \).

As a next step, we evaluate the asymptotic behaviour of the terms \( J_{21}(t) - J_{24}(t) \). Let us begin with \( J_{21}(t) \). Via the change of variables \( \log \tau = \xi \log 1 / \lambda_1 \), one obtains that
\[
(4.54) \quad J_{21}(t) = \left( \log \frac{t}{\lambda_1} \right)^{\frac{1}{2}} \int_{\log \frac{t_0}{\lambda_1}}^{\log \frac{t}{\lambda_1}} \left( \xi - 1 \right)^{\frac{1}{2}} \xi^{-\frac{1}{2}} d\xi = \left( \log \frac{t}{\lambda_1} \right)^{\frac{1}{2}} \left[ \Psi_{\frac{1}{2}} \left( \log \frac{t}{\lambda_1} \right) - \Psi_{\frac{1}{2}} \left( \log \frac{t_0}{\lambda_1} \right) \right]
\]
where \( \Psi_{\frac{1}{2}} \) denotes the upper incomplete gamma function.
for large \( t \), where \( \Psi_{\frac{1}{\beta}} \) is defined as in (4.17). Lemma 4.12 implies that

\[
\lambda_t = (2 - 2\beta)(\log t)^{1 - \frac{1}{\beta}} + \cdots \quad \text{as } t \to \infty,
\]

whence

\[
\log \frac{1}{\lambda_t} = \left(\frac{1}{\beta} - 1\right) \log \log t + \cdots \quad \text{as } t \to \infty.
\]

Consequently,

\[
\lim_{t \to \infty} \log \frac{t}{\lambda_t} = \infty \quad \text{and} \quad \lim_{t \to \infty} \log \frac{1}{\lambda_t} = 1.
\]

Lemma 4.9 tells us that

\[
\Psi_{\frac{1}{\beta}}(t) = \beta(t^\frac{1}{\beta} - \frac{1}{1 - \beta} t^{\frac{1}{\beta} - 1} + \cdots \quad \text{as } t \to \infty \quad \text{and} \quad \Psi_{\frac{1}{\beta}}(t) \to 0 \quad \text{as } t \to 1^+.
\]

Therefore, equation (4.54) reads

\[
J_{21}(t) = \beta(\log t)^\frac{1}{\beta} - \frac{1}{1 - \beta} \log \frac{1}{\lambda_t}(\log t)^\frac{1}{\beta} - 1 + \cdots
\]

As for the term \( J_{22} \), the same change of variables as above yields

\[
J_{22}(t) = \left(\log \frac{1}{\lambda_t}\right)^\frac{1}{\beta} - 1 \int_{\log \frac{2}{1 - \beta}}^{\log \frac{2}{\lambda_t}} \frac{1}{\xi} \log \left(\frac{\xi - 1}{\lambda_t} \log \frac{1}{\lambda_t}\right) d\xi
\]

for large \( t \). Hence

\[
J_{22}(t) = \left(\log \frac{1}{\lambda_t}\right)^\frac{1}{\beta} - 1 \int_{\log \frac{2}{1 - \beta}}^{\log \frac{2}{\lambda_t}} \frac{1}{\xi} \log \left(\frac{\xi - 1}{\lambda_t} \log \frac{1}{\lambda_t}\right) d\xi
\]

where \( \Upsilon_{\sigma} : (1, \infty) \to \mathbb{R} \) is the function defined for \( \sigma > 0 \) by

\[
\Upsilon_{\sigma}(t) = \int_{1}^{t} \frac{(\tau - 1)^{\sigma}}{\tau} \log(\tau - 1) d\tau \quad \text{for } t > 1.
\]

Observe that

\[
\Upsilon_{\sigma}(t) = \frac{\beta}{1 - \beta} t^{\frac{1}{\beta} - 1} + \cdots \quad \text{as } t \to \infty.
\]

Furthermore, by Lemma 4.9,

\[
\Psi_{\frac{1}{\beta} - 1}(t) = \frac{\beta}{1 - \beta} t^{\frac{1}{\beta} - 1} + \cdots \quad \text{as } t \to \infty.
\]

Altogether, since \( \Upsilon_{\frac{1}{\beta} - 1}(t) \to 0 \) and \( \Psi_{\frac{1}{\beta} - 1}(t) \to 0 \) as \( t \to 1^+ \), we conclude, via equation (4.57), that

\[
J_{22}(t) = \frac{\beta}{1 - \beta}(\log t)^{\frac{1}{\beta} - 1} \log \left(\frac{\log t}{\log \lambda_t}\right) + \frac{\beta}{1 - \beta}(\log t)^{\frac{1}{\beta} - 1} \log \left(\frac{1}{\lambda_t}\right) + \cdots \quad \text{as } t \to \infty.
\]
Hence,

\[
J_{22}(t) = \frac{\beta}{1 - \beta} (\log t)^{\frac{1}{\beta} - 1} \log \log t + \cdots \quad \text{as } t \to \infty.
\]

The term \(J_{23}(t)\) can be estimated similarly. First, changing variables as above, we get

\[
J_{23}(t) = \left( \log \frac{1}{\lambda_t} \right)^{\frac{1}{\beta} - 1} \int_{\log \frac{t}{\lambda_t}}^{\log t} \frac{(x - 1)^{\frac{1}{\beta}}}{x^2} [\log x + \log \log \frac{x}{\lambda_t}] \, dx
\]

\[
= \left( \log \frac{1}{\lambda_t} \right)^{\frac{1}{\beta} - 1} \left[ \frac{t}{\lambda_t} \log \frac{t}{\lambda_t} - \frac{t}{\lambda_t} \log \frac{1}{\lambda_t} \right] + \left( \log \frac{1}{\lambda_t} \right)^{\frac{1}{\beta} - 1} \log \frac{1}{\lambda_t} \left[ \frac{t}{\lambda_t} \log \frac{t}{\lambda_t} - \frac{t}{\lambda_t} \log \frac{1}{\lambda_t} \right]
\]

for large \(t\), where the functions \(\Upsilon_\sigma: (1, \infty) \to (0, \infty)\) and \(\Psi_\sigma: (1, \infty) \to (0, \infty)\) are defined, for \(\sigma > 0\), by

\[
\Upsilon_\sigma(t) = \int_1^t \frac{(\tau - 1)^{\sigma + 1}}{\tau^2} \log \tau \, d\tau \quad \text{and} \quad \Psi_\sigma(t) = \int_1^t \frac{(\tau - 1)^{\sigma + 1}}{\tau^2} \, d\tau \quad \text{for } t > 1.
\]

It is easily verified that

\[
\Upsilon_1(t) = \frac{1}{1 - \beta} t^{\frac{1}{\beta} - 1} \log t + \cdots \quad \text{and} \quad \Psi_1(t) = \frac{1}{1 - \beta} t^{\frac{1}{\beta} - 1} + \cdots \quad \text{as } t \to \infty.
\]

Also \(\Upsilon_1^{-1}(t) \to 0\) and \(\Psi_1^{-1}(t) \to 0\) as \(t \to 1^+\). Therefore, owing to (4.57), we conclude that

\[
J_{23}(t) = \frac{\beta}{1 - \beta} (\log t)^{\frac{1}{\beta} - 1} \log \log t + \cdots \quad \text{as } t \to \infty.
\]

Finally, we claim that \(J_{24}(t)\) is of lower order than \(J_{21}(t), J_{22}\) and \(J_{23}(t)\) as \(t \to \infty\). Indeed, since \(\lambda_t < 1\), one has that

\[
0 \leq J_{24}(t) \leq \int_e^t \left( \frac{\log \tau}{\tau} \right)^{\frac{1}{\beta} - 1} \left( \log \log \frac{\tau}{e} \right)^2 \, d\tau = Y_{\beta - 2}(\log t)
\]

for large \(t\), where \(Y_\sigma: (1, \infty) \to (0, \infty)\) is defined by

\[
Y_\sigma(t) = \int_1^t \tau^{\sigma - 1} (\log \tau)^2 \, d\tau \quad \text{for } t > 1.
\]

Observe that

\[
Y_\sigma(t) = \begin{cases} \frac{1}{\sigma} t^\sigma (\log t)^2 + \cdots & \text{if } \sigma > 0 \\ \frac{1}{3} (\log t)^3 + \cdots & \text{if } \sigma = 0 \quad \text{as } t \to \infty \\ c + \cdots & \text{if } \sigma < 0 \end{cases}
\]

for a suitable constant \(c \in \mathbb{R}\) depending on \(\sigma\). Therefore,

\[
0 \leq J_{24}(t) \leq \begin{cases} \frac{1}{\sqrt{2\beta}} (\log t)^{\frac{1}{\beta} - 2} (\log \log t)^2 + \cdots & \text{if } \beta \in (0, \frac{1}{2}) \\ \frac{1}{3} (\log t)^3 + \cdots & \text{if } \beta = \frac{1}{2} \quad \text{as } t \to \infty \\ c + \cdots & \text{if } \beta \in \left(\frac{1}{2}, 1\right) \end{cases}
\]

This proves our claim.

On making use of equations (4.58)–(4.61), we infer from (4.52) that

\[
J_2(t) \leq \frac{\beta}{2} (\log t)^{\frac{1}{\beta} + \left( -\frac{1}{2\beta} + (1 + \varepsilon) \frac{1}{2\beta} + (\varepsilon - 1) \frac{5\beta}{4 - 4\beta} \right)} (\log t)^{\frac{1}{\beta} - 1} \log \log t + \cdots \quad \text{as } t \to \infty,
\]

and from (4.53) that

\[
J_2(t) \geq \frac{\beta}{2} (\log t)^{\frac{1}{\beta} + \left( -\frac{1}{2\beta} + (1 - \varepsilon) \frac{1}{2\beta} - (\varepsilon + 1) \frac{5\beta}{4 - 4\beta} \right)} (\log t)^{\frac{1}{\beta} - 1} \log \log t + \cdots \quad \text{as } t \to \infty.
\]
Hence, by the arbitrariness of $\varepsilon$,
\[ J_2(t) = \frac{\beta}{2} \left( \log t \right)^{\frac{1}{\beta}} - \frac{5\beta}{4 - 4\beta} \left( \log t \right)^{\frac{1}{\beta} - 1} \log \log t + \cdots \quad \text{as } t \to \infty. \]

By (4.49), $J_1(t)$ is of lower order than $J_2(t)$ as $t \to \infty$. Hence, we conclude via (4.47) that
\[ \|\Theta\|_{L^\infty(s, \frac{1}{2})} = \frac{\beta}{2} \left( \log \frac{s}{2} \right)^{\frac{1}{\beta}} - \frac{5\beta}{4 - 4\beta} \left( \log \frac{s}{2} \right)^{\frac{1}{\beta} - 1} \log \log s + \cdots \quad \text{as } s \to 0^+, \]
where $t = \Theta(s)$. Thanks to Lemma 4.3 and (4.3), (4.62)
\[ \log t = \log \Theta(s) = \log \frac{1}{s} - \log \log \frac{1}{s} + \cdots \quad \text{as } s \to 0^+. \]

Hence, owing to (4.2), we obtain that
\[ \|\Theta\|_{L^\infty(s, \frac{1}{2})} = \frac{\beta}{2} \left( \log \frac{s}{2} \right)^{\frac{1}{\beta}} - \frac{5\beta}{4 - 4\beta} \left( \log \frac{s}{2} \right)^{\frac{1}{\beta} - 1} \log \log s + \cdots \quad \text{as } s \to 0^+ \]
and (4.46) follows.

Assume next that $\beta = 1$. Fix $\varepsilon > 0$. There exists $t_0 > e$ such that, if $t > t_0$, then $\lambda_t < 1$, equations (4.50) and (4.51) hold, and moreover
\[ b^{-1}(t) = \log t - \log N. \]

If we now define
\[ \mathcal{J}_{21}(t) = \int_{t_0/\lambda_t}^t \frac{\log \lambda_t \tau}{\tau \log \tau} \, d\tau, \quad \mathcal{J}_{22}(t) = \int_{t_0/\lambda_t}^t \frac{d\tau}{\tau \log \tau}, \]
\[ \mathcal{J}_{23}(t) = \int_{t_0/\lambda_t}^t \frac{\log(\lambda_t \tau) \log \tau}{\tau (\log \tau)^2} \, d\tau, \quad \mathcal{J}_{24}(t) = \int_{t_0/\lambda_t}^t \frac{\log \log \tau}{\tau (\log \tau)^2} \, d\tau \]
for large $t$, then
\[ J_2(t) \leq \frac{1}{2} \mathcal{J}_{21}(t) - \frac{1}{2} \log N \mathcal{J}_{22}(t) + (\varepsilon - 1) \frac{5}{4} \mathcal{J}_{23}(t) + (1 - \varepsilon) \frac{5}{4} \log N \mathcal{J}_{24}(t) \]
and
\[ J_2(t) \geq \frac{1}{2} \mathcal{J}_{21}(t) - \frac{1}{2} \log N \mathcal{J}_{22}(t) - (\varepsilon - 1) \frac{5}{4} \mathcal{J}_{23}(t) + (1 + \varepsilon) \frac{5}{4} \log N \mathcal{J}_{24}(t) \]
for large $t$. Let us evaluate the asymptotic behaviour of the terms $\mathcal{J}_{21}(t) - \mathcal{J}_{24}(t)$ as $t \to \infty$. First observe that
\[ \mathcal{J}_{21}(t) = \int_{t_0/\lambda_t}^t \frac{d\tau}{\tau} - \log \frac{1}{\lambda_t} \int_{t_0/\lambda_t}^t \frac{d\tau}{\tau \log \tau} = \log t - \log \frac{t_0}{\lambda_t} - \log \frac{1}{\lambda_t} \left[ \log \log t - \log \log \frac{t_0}{\lambda_t} \right] \]
for large $t$. From Lemma 4.12 we deduce that
\[ \log \frac{1}{\lambda_t} = \log \log \log t + \cdots \quad \text{as } t \to \infty. \]

Consequently,
\[ \mathcal{J}_{21}(t) = \log t - \log \log t \log \log t + \cdots \quad \text{as } t \to \infty. \]

Next,
\[ \mathcal{J}_{22}(t) = \log t + \cdots \quad \text{as } t \to \infty. \]

On the other hand, $\mathcal{J}_{24}(t) \to 0$ as $t \to \infty$ since
\[ 0 \leq \mathcal{J}_{24}(t) \leq \int_{t_0/\lambda_t}^\infty \frac{\log \log \tau}{\tau (\log \tau)^2} \, d\tau \]
for large $t$, and $\lambda_t \to 0$ as $t \to \infty$. It remains to deal with the term $\mathcal{J}_{23}(t)$. We have that
\[ \mathcal{J}_{23}(t) = \int_{t_0/\lambda_t}^t \frac{\log \log \tau}{\tau \log \tau} \, d\tau - \log \frac{1}{\lambda_t} \int_{t_0/\lambda_t}^t \frac{\log \log \tau}{\tau (\log \tau)^2} \, d\tau \]
\[ = \frac{1}{2} \left( \log \log t \right)^2 - \frac{1}{2} \left( \log \log \frac{t_0}{\lambda_t} \right)^2 - \log \frac{1}{\lambda_t} \mathcal{J}_{24}(t) = \frac{1}{2} \left( \log \log t \right)^2 + \cdots \quad \text{as } t \to \infty. \]
Altogether, by equation (4.64),
\[ J_2(t) \leq \frac{1}{2} \log t + (\varepsilon - 1) \frac{5}{8} (\log \log t)^2 + \cdots \quad \text{as } t \to \infty \]
and, by (4.65),
\[ J_2(t) \geq \frac{1}{2} \log t - (1 + \varepsilon) \frac{5}{8} (\log \log t)^2 + \cdots \quad \text{as } t \to \infty. \]
Therefore, by the arbitrariness of \( \varepsilon \),
\[ J_2(t) = \frac{1}{2} \log t - \frac{5}{8} (\log \log t)^2 + \cdots \quad \text{as } t \to \infty. \]
The term \( J_1(t) \) is of lower order than \( J_2(t) \) as \( t \to \infty \) also in this case. Therefore, one infers from (4.47) that
\[ \| \Theta \|_{L^\beta(s, \frac{1}{2})} = \frac{1}{2} \log t - \frac{5}{8} (\log \log t)^2 + \cdots \quad \text{as } s \to 0^+, \]
where \( t = \Theta(s) \). Equation (4.46) hence follows via (4.63).

Finally, let \( \beta \in (1, \infty) \). By Lemma 4.12, the function \( t \mapsto \lambda_t \) is decreasing and \( \lambda_t \to \lambda \) for some \( \lambda > 0 \). Thus, given \( \varepsilon > 0 \), there exists \( t_0 > 0 \) such that
\[ \lambda \leq \lambda_t < \lambda + \varepsilon \quad \text{for } t > t_0. \]
As a consequence,
\[ \int_0^t b^{-1}(\lambda \tau) \tau I(\Theta^{-1}(\tau))^2 \, d\tau \leq \| \Theta \|_{L^\beta(s, \frac{1}{2})} \leq \int_0^t b^{-1}((\lambda + \varepsilon) \tau) \tau I(\Theta^{-1}(\tau))^2 \, d\tau \]
for \( t > t_0 \). Observe that
\[
c_\varepsilon = \lim_{t \to \infty} \left[ \int_0^1 b^{-1}((\lambda + \varepsilon) \tau) \tau I(\Theta^{-1}(\tau))^2 \, d\tau - \frac{\beta}{2} (\log t)^\frac{1}{\beta} \right]
\]
\[= \int_0^1 b^{-1}((\lambda + \varepsilon) \tau) \tau I(\Theta^{-1}(\tau))^2 \, d\tau + \lim_{t \to \infty} \left[ \int_1^t b^{-1}((\lambda + \varepsilon) \tau) \tau I(\Theta^{-1}(\tau))^2 \, d\tau - \int_1^t \frac{1}{2\tau} (\log \tau)^\frac{1}{\beta} - 1 \, d\tau \right]
\]
\[= \int_0^1 b^{-1}((\lambda + \varepsilon) \tau) \tau I(\Theta^{-1}(\tau))^2 \, d\tau + \int_1^\infty \left[ b^{-1}((\lambda + \varepsilon) \tau) \tau I(\Theta^{-1}(\tau))^2 - \frac{1}{2\tau} (\log \tau)^\frac{1}{\beta} - 1 \right] \, d\tau,
\]
where the last integral converges thanks to Lemma 4.8. Hence, by the dominated convergence theorem, one can deduce that
\[
(4.67) \quad \| \Theta \|_{L^\beta(s, \frac{1}{2})} = \frac{\beta}{2} (\log t)^\frac{1}{\beta} + c + \cdots \quad \text{as } t \to \infty,
\]
where \( t = \Theta(s) \) and
\[ c = \lim_{\varepsilon \to 0^+} c_\varepsilon = \int_0^1 b^{-1}(\lambda \tau) \tau I(\Theta^{-1}(\tau))^2 \, d\tau + \int_1^\infty \left[ b^{-1}(\lambda \tau) \tau I(\Theta^{-1}(\tau))^2 - \frac{1}{2\tau} (\log \tau)^\frac{1}{\beta} - 1 \right] \, d\tau.
\]
Equation (4.46) follows from (4.67) and the expansion (4.63). \( \square \)

Given a Young function \( B \) and a constant \( C \in \mathbb{R} \), we define the function \( \mathcal{G}_{B,C} : (0, \frac{1}{2}] \to \mathbb{R} \) by
\[
(4.68) \quad \mathcal{G}_{B,C}(s) = \| \Theta \|_{L^\beta(s, \frac{1}{2})} + \Theta(s) s B^{-1} \left( \frac{1}{2} \right) + C \quad \text{for } s \in (0, \frac{1}{2}].
\]
Combining Lemmas 4.6 and 4.13 with equation (4.2) yields the following asymptotic expansion for the function \( \theta_\beta \mathcal{G}_{B,C} \).

**Corollary 4.14.** Let \( \beta > 0 \) and \( C \in \mathbb{R} \). Assume that \( B \) is a Young function obeying (1.13), and let \( \mathcal{G}_{B,C} \) be the function given by (4.68). Then
\[
(4.69) \quad \left[ \theta_\beta \mathcal{G}_{B,C}(s) \right]^\beta = \log \frac{1}{s} + \begin{cases} 
\frac{2+3\beta}{2-2\beta} \log \log \frac{1}{s} + \cdots & \text{if } \beta \in (0, 1) \\
-\frac{5}{8} (\log \log \frac{1}{s})^2 + \cdots & \text{if } \beta = 1 \quad \text{as } s \to 0^+. \\
2(C + c_{\beta,N})(\log \frac{1}{s})^{1-\frac{1}{\beta}} + \cdots & \text{if } \beta \in (1, \infty)
\end{cases}
\]
Here, \( c_{\beta,N} \) denotes the constant appearing in equation (4.46).
5. Basic estimates

Inequalities (2.2)–(2.3) are crucial for the bounds exhibited in the sequel of this section.

Proposition 5.1. Let $n \in \mathbb{N}$ and $u \in W_{c}L^{2}(\mathbb{R}^{n}, \gamma_{n})$. Then

\begin{equation}
|\text{med}(u) - \text{mv}(u)| \leq \int_{0}^{\frac{1}{2}} (L) u^{*}(r) \Lambda(r) \, dr,
\end{equation}

where $\Lambda$ is defined in (4.12).

Proof. Integrating both sides of inequality (2.2) over the interval $(0, \frac{1}{2})$ and making use of Fubini’s theorem enable us to infer that

\begin{equation}
0 \leq \int_{0}^{\frac{1}{2}} [u^{\circ}(s) - u^{\circ}(\frac{1}{2})] \, ds \leq \int_{0}^{\frac{1}{2}} \Theta(s) \int_{0}^{s} (L) u_{+}^{*}(r) \, dr \, ds + \int_{0}^{\frac{1}{2}} \int_{s}^{\frac{1}{2}} (L) u_{+}^{*}(r) \Theta(r) \, dr \, ds
\end{equation}

\begin{equation}
= \int_{0}^{\frac{1}{2}} (L) u_{+}^{*}(r) \int_{r}^{\frac{1}{2}} \Theta(s) \, ds \, dr + \int_{0}^{\frac{1}{2}} (L) u_{+}^{*}(r) r \Theta(r) \, dr = \int_{0}^{\frac{1}{2}} (L) u_{+}^{*}(r) \Lambda(r) \, dr.
\end{equation}

Analogously, owing to (2.3),

\begin{equation}
0 \leq \int_{0}^{\frac{1}{2}} [u^{\circ}(\frac{1}{2}) - u^{\circ}(1 - s)] \, ds \leq \int_{0}^{\frac{1}{2}} (L) u_{+}^{*}(r) \Lambda(r) \, dr.
\end{equation}

By equations (3.1) and (3.2),

\begin{equation}
\text{mv}(u) - \text{med}(u) = \int_{\mathbb{R}^{n}} u \, d\gamma_{n} - u^{\circ}(\frac{1}{2}) = \int_{0}^{\frac{1}{2}} [u^{\circ}(s) - u^{\circ}(\frac{1}{2})] \, ds
\end{equation}

\begin{equation}
= \int_{0}^{\frac{1}{2}} [u^{\circ}(s) - u^{\circ}(\frac{1}{2})] \, ds - \int_{0}^{\frac{1}{2}} [u^{\circ}(\frac{1}{2}) - u^{\circ}(1 - s)] \, ds.
\end{equation}

Therefore, coupling inequalities (5.2) and (5.3) yields

\begin{equation}
- \int_{0}^{\frac{1}{2}} (L) u_{+}^{*}(r) \Lambda(r) \, dr \leq \text{mv}(u) - \text{med}(u) \leq \int_{0}^{\frac{1}{2}} (L) u_{+}^{*}(r) \Lambda(r) \, dr.
\end{equation}

Hence

\begin{equation}
|\text{mv}(u) - \text{med}(u)| \leq \max \left\{ \int_{0}^{\frac{1}{2}} (L) u_{+}^{*}(r) \Lambda(r) \, dr, \int_{0}^{\frac{1}{2}} (L) u_{+}^{*}(r) \Lambda(r) \, dr \right\},
\end{equation}

and (5.1) follows.

\[\square\]

Lemma 5.2. Let $n \in \mathbb{N}$, let $\Lambda$ be the function defined by (4.12), and let $B$ be a Young function such that $\|\Lambda\|_{L^{B}(0, \frac{1}{2})} < \infty$. Assume that $u \in W_{c}L^{B}(\mathbb{R}^{n}, \gamma_{n})$ and fulfills condition (1.5). Let $\mathcal{G}_{B,C}$ be the function defined by (4.68), where

\begin{equation}
C = 0 \quad \text{if} \quad \text{med}(u) = 0 \quad \text{and} \quad C = \|\Lambda\|_{L^{B}(0, \frac{1}{2})} \quad \text{if} \quad \text{mv}(u) = 0.
\end{equation}

Then

\begin{equation}
|u^{\circ}(s)| \leq \mathcal{G}_{B,C}(s) \|L u\|_{L^{B}(\mathbb{R}^{n}, \gamma_{n})} \quad \text{for} \quad s \in (0, \frac{1}{2}]
\end{equation}

and

\begin{equation}
|u^{\circ}(1 - s)| \leq \mathcal{G}_{B,C}(s) \|L u\|_{L^{B}(\mathbb{R}^{n}, \gamma_{n})} \quad \text{for} \quad s \in (0, \frac{1}{2}].
\end{equation}

Proof. Assume first that $\text{mv}(u) = 0$. By inequality (2.2), we have

\begin{equation}
0 \leq u^{\circ}(s) - u^{\circ}(\frac{1}{2}) \leq \Theta(s) \int_{0}^{s} (L) u^{*}(r) \, dr + \int_{s}^{\frac{1}{2}} (L) u^{*}(r) \Theta(r) \, dr \quad \text{for} \quad s \in (0, \frac{1}{2}].
\end{equation}
Hence, via Hölder’s inequality in the form (3.4) and equation (3.5), we deduce that

$$0 \leq u^\circ(s) - u^\circ(\frac{1}{2}) \leq \Theta(s)\|\mathcal{L}u\|_{L^B(0,1)}\|\chi_{(0,s)}\|_{L^B(0,1)} + \|\mathcal{L}u\|_{L^B(0,1)} \|\Theta\|_{L^B(0,1)}$$

(5.7)

$$= \left(\Theta(s)\varphi^{-1}(\frac{1}{2}) + \|\Theta\|_{L^B(0,1)}\right) \|\mathcal{L}u\|_{L^B(0,1)} \text{ for } s \in (0, \frac{1}{2}].$$

Proposition 5.1 and inequality (3.4) again tell us that

$$|u^\circ(\frac{1}{2})| = |\text{med}(u)| \leq \int_0^{\frac{1}{2}} (\mathcal{L}u)^\ast(r)\Lambda(r)\,dr \leq \|\mathcal{L}u\|_{L^B(0,1)}\|\Lambda\|_{L^B(0,1)}.$$  

(5.8)

Under the current assumption that mv(u) = 0, inequality (5.5) follows from (5.7) and (5.8). On the other hand, under the assumption that med(u) = 0, inequality (5.5) is a straightforward consequence of (5.7).

The proof of (5.6) is analogous: it just rests upon (2.3) instead of (2.2).

**Corollary 5.3.** Let $n \in \mathbb{N}$ and $\beta > 0$, and let $B$ be a Young function as in (1.13). Assume that $\varphi : [0, \infty) \to [0, \infty)$ is a non-decreasing function and that $\theta > 0$. Then

$$\sup_u \int_{\mathbb{R}^n} \exp^\beta(\theta|u|) \varphi(|u|)\,d\gamma_n \leq 2 \int_0^{\frac{1}{2}} \exp^\beta(\theta \mathcal{G}_{B,C}(s)) \varphi(\mathcal{G}_{B,C}(s))\,ds,$$

(5.9)

where the supremum is extended over all functions $u$ satisfying (1.12) and (1.5). Here, $\mathcal{G}_{B,C}$ denotes the function defined by (4.68), with $C$ given according to the alternative (5.4), depending on whether $m = \text{med}$ or $m = \text{mv}$ in (1.5).

**Proof.** Since the function $\tilde{B}(t)$ is equivalent to $t(\log t)^\frac{1}{2}$ near infinity, one has that $\|\Lambda\|_{L^B(0,\frac{1}{2})} < \infty$, and hence the constant $C$ in (5.4) is well defined when $m = \text{mv}$. Given any function $u$ as in the statement, we have that

$$\int_{\mathbb{R}^n} \exp^\beta(\theta|u|) \varphi(|u|)\,d\gamma_n = \int_0^{\frac{1}{2}} \exp^\beta(\theta|u^\circ(s)|) \varphi(|u^\circ(s)|)\,ds + \int_0^{\frac{1}{2}} \exp^\beta(\theta|u^\circ(1-s)|) \varphi(|u^\circ(1-s)|)\,ds.$$  

The conclusion thus follows by Lemma 5.2. \qed

**Corollary 5.4.** Let $n \in \mathbb{N}$, let $\Lambda$ be the function defined by (4.12) and let $\eta > 0$. Then

$$\sup_u \int_{\mathbb{R}^n} \exp \exp(\eta|u|)\,d\gamma_n \leq 2 \int_0^{\frac{1}{2}} \exp(\eta \Lambda(s) + \eta C)\,ds,$$

(5.10)

where the supremum is extended over all functions $u$ obeying (1.14) and (1.5), and $\mathcal{G}_{B,C}$ is as in Corollary 5.3.

**Proof.** Let $B$ be the Young function given by $B(t) = 0$ if $t \in [0, 1]$ and $B(t) = \infty$ if $t \in (1, \infty)$. Thus, $L^B(\mathcal{R}, \nu) = L^\infty(\mathcal{R}, \nu)$ for any probability space $(\mathcal{R}, \nu)$. Moreover, $B^{-1} = 1$ on $(0, \infty)$ and hence

$$\|\Theta\|_{L^B(0,\frac{1}{2})} = \|\Theta\|_{L^1(0,\frac{1}{2})} = \int_s^{\frac{1}{2}} \Theta(r)\,dr \text{ for } s \in (0, \frac{1}{2}].$$

Thus,

$$\Theta(s)\varphi^{-1}(\frac{1}{2}) + \|\Theta\|_{L^B(0,\frac{1}{2})} = \Lambda(s) \text{ for } s \in (0, \frac{1}{2}].$$

By Lemma 4.4, the function $\Lambda$ is integrable on $(0, \frac{1}{2})$. Hence, the constant $C$ appearing in (5.4) is well defined. The rest of the proof is analogous to that of Corollary 5.3, and will be omitted. \qed
6. Sharpness

Here we establish some technical results that are critical in showing the optimality of the conclusions of Theorems 1.1 and 1.2. The nature of the Ornstein-Uhlenbeck operator and of the measure in the Gauss space call for the use of ad hoc smooth truncations of suitable twice weakly differentiable trial functions. Also, the estimates to be derived for the truncated functions are quite subtle. By contrast, a more standard smooth truncation argument suffices in proving the sharpness of the constant $\alpha_n$ in the companion Euclidean inequality (1.9).

We begin with two lemmas, whose proofs rest upon standard properties of weakly differentiable functions. Notice that the proof of formulas (6.4) and (6.5) of the latter also requires the use of the identities $\left(I(\Phi(t))\right)' = -\Phi''(t) = t\Phi'(t)$ for $t > 0$.

**Lemma 6.1.** Let $u \in W^{2,2}(\mathbb{R}^n, \gamma_n)$ and let $\psi \in C^2(\mathbb{R})$ be such that $\psi', \psi'' \in L^\infty(\mathbb{R})$. Then $\psi \circ u \in W_2L^2(\mathbb{R}^n, \gamma_n)$ and

$$L(\psi \circ u) = \psi'(u)Lu + \psi''(u)\nabla u^2.$$  

**Lemma 6.2.** Let $f_0: (0, 1) \to [0, \infty)$ be a function in $L^2(0, 1)$ and let $s_0 \in (0, 1)$. Define the function $f_1: (0, 1) \to \mathbb{R}$ by

$$f_1(s) = \int_{ss_0}^{s_0} \frac{1}{I(\rho)^2} \int_0^\rho f_0 \left( \frac{r}{\rho} \right) \, dr \, d\rho \quad \text{for } s \in (0, \frac{1}{s_0}),$$

where the function $f_0$ is extended by 0 in $(1, \frac{1}{s_0})$, and the integral $\int_{ss_0}^{s_0} \cdots d\rho$ has to be interpreted as $-\int_{ss_0}^{s_0} \cdots d\rho$ if $ss_0 > s_0$. Moreover, let $u: \mathbb{R}^n \to \mathbb{R}$ be the function defined as

$$u(x) = f_1 \left( \frac{\Phi(x_1)}{s_0} \right) \quad \text{for } x \in \mathbb{R}.$$ 

Then $u \in W^{2,2}(\mathbb{R}^n, \gamma_n)$,

$$\nabla u(x) = (h_0(\Phi(x_1)), 0, \ldots, 0)$$

and

$$\Delta u(x) - x \cdot \nabla u(x) = -f_0 \left( \frac{\Phi(x_1)}{s_0} \right)$$

for a.e. $x \in \mathbb{R}^n$. Here $h_0: (0, 1) \to [0, \infty)$ is the function given by

$$h_0(s) = \frac{1}{I(s)} \int_0^s f_0 \left( \frac{r}{s_0} \right) \, dr \quad \text{for } s \in (0, 1).$$

The asymptotic behaviour of the function $f_1$ for functions $f_0$ with a particular behaviour near zero is the subject of the next result. Its proof is elementary, and is omitted. This is not the case for the subsequent Lemma 6.4, which provides us with sharp bounds for a function implicitly defined by prescribing the value of integral on the right-hand side of equation (6.2).

**Lemma 6.3.** Let $\beta > 0$. Assume that the function $f_0: (0, 1) \to [0, \infty)$ satisfies

$$f_0(s) = \left( \log \frac{1}{s} \right)^{\frac{3}{2}} + \cdots \quad \text{as } s \to 0^+.$$ 

Let $f_1$ be the function associated with $f_0$ as in (6.2) for some $s_0 \in (0, 1)$. Then

$$f_1(s) = \frac{\beta}{2} \left( \log \frac{1}{s} \right)^{\frac{3}{2}} + \cdots \quad \text{as } s \to 0^+.$$ 

**Lemma 6.4.** Let $\beta > 0$ and $\varepsilon > 0$. Assume that $f_0: (0, 1) \to [0, \infty)$ is a decreasing function satisfying (6.7). Then the equation

$$\varepsilon = \int_{s_1}^{s_0} \frac{1}{I(\rho)^2} \int_0^\rho f_0 \left( \frac{r}{\rho} \right) \, dr \, d\rho$$
implicitly defines a function $s_1 = s_1(s_0)$, with $s_1: (0, \frac{1}{2}) \to (0, \frac{1}{2})$, such that $s_1(s_0) \in (0, s_0)$,

\begin{equation}
\frac{s_1}{s_0} \to 0 \quad \text{as } s \to 0^+
\end{equation}

and, for any $\varepsilon' > \varepsilon$,

\begin{equation}
\log \frac{s_0}{s_1} \leq \left[2\varepsilon' \left(\frac{1}{\beta} + 1\right) \log \frac{1}{s_0}\right]^{\frac{1}{\alpha}} \quad \text{for } s_0 \text{ near zero.}
\end{equation}

**Proof.** Equation (6.9) actually defines a function $s_1(s_0)$ since, for each $s_0 \in (0, \frac{1}{2})$, the function

$$s \mapsto \int_s^{s_0} \frac{1}{I(\rho)^2} \int_0^\rho f_0 \left(\frac{r}{s_0}\right) \, dr \, d\rho$$

for $s \in (0, s_0)$ is strictly decreasing and, owing to (6.8), it maps $(0, s_0)$ into $(0, \infty)$. Let $\Theta$ be the function given by (2.4).

From equation (6.9) one can easily infer that

$$\varepsilon \leq \left[\frac{s_0}{s_1}\right] = \Theta(s_1) \int_0^{s_0} f_0 \left(\frac{r}{s_0}\right) \, dr = s_0 \Theta(s_1) \int_0^1 f_0(r) \, dr.$$

Notice that the last integral is convergent. By Lemma 4.3, $s_1 \to 0^+$ and $s_1 \Theta(s_1) \to 0$ as $s_0 \to 0^+$. Hence, equation (6.10) follows, inasmuch as

$$\frac{s_1}{s_0} \leq s_1 \Theta(s_1) \int_0^1 f_0(r) \, dr.$$

Next, an application of Fubini’s theorem to the integral on the right-hand side of equation (6.9) tells us that

\begin{equation}
\varepsilon = \int_{s_1}^{s_0} \frac{1}{I(\rho)^2} \int_0^{s_1} f_0 \left(\frac{r}{s_0}\right) \, dr \, d\rho + \int_{s_1}^{s_0} \frac{1}{I(\rho)^2} \int_s^{s_1} f_0 \left(\frac{r}{s_0}\right) \, dr \, d\rho
\end{equation}

\begin{align}
= \left[\Theta(s_1) - \Theta(s_0)\right] \int_0^{s_1} f_0 \left(\frac{r}{s_0}\right) \, dr + \int_{s_1}^{s_0} f_0 \left(\frac{r}{s_0}\right) \left[\Theta(r) - \Theta(s_0)\right] \, dr
\end{align}

\begin{align}
= \Theta(s_1) \int_0^{s_1} f_0 \left(\frac{r}{s_0}\right) \, dr + \int_{s_1}^{s_0} f_0 \left(\frac{r}{s_0}\right) \Theta(r) \, dr - \Theta(s_0) \int_0^{s_0} f_0 \left(\frac{r}{s_0}\right) \, dr.
\end{align}

The third addend on the rightmost side of equation (6.12) tends to 0 as $s_0 \to 0^+$ thanks to Lemma 4.3, since

\begin{equation}
\Theta(s_0) \int_0^{s_0} f_0 \left(\frac{r}{s_0}\right) \, dr = s_0 \Theta(s_0) \int_0^1 f_0(r) \, dr.
\end{equation}

Let us now focus on the second addend. Fixing $\varepsilon' > \varepsilon$, choose $\delta > 0$ so small that $\varepsilon'(1 - \delta)^2 > \varepsilon$. By equations (4.11) and (6.7), there exists $r_0 \in (0, \frac{1}{2})$ such that

$$\Theta(r) > \frac{1 - \delta}{2} \cdot \frac{1}{r \log \frac{1}{r}} \quad \text{and} \quad f_0(r) > (1 - \delta)(\log \frac{1}{r})^{\frac{1}{\beta}} \quad \text{for } r \in (0, r_0).$$

If $s_0$ is sufficiently small, then, by (6.10), $s_0 r_0 > s_1$. Furthermore,

\begin{equation}
\int_{s_1}^{s_0} f_0 \left(\frac{r}{s_0}\right) \Theta(r) \, dr \geq \int_{s_1}^{s_0} f_0 \left(\frac{r}{s_0}\right) \, dr \geq \frac{(1 - \delta)^2}{2} \int_{s_1}^{s_0} \frac{(\log \frac{1}{s_0})^{\frac{1}{\beta}}}{r \log \frac{1}{r}} \, dr.
\end{equation}

The change of variables $\log \frac{1}{r} = t \log \frac{1}{s_0}$ yields

\begin{equation}
\int_{s_1}^{s_0} \frac{(\log \frac{1}{s_0})^{\frac{1}{\beta}}}{r \log \frac{1}{r}} \, dr = \left(\log \frac{1}{s_0}\right)^{\frac{1}{\beta}} \Psi_{\frac{1}{\beta}} \left(\frac{1}{s_1} \log \frac{1}{s_0}\right) - \Psi_{\frac{1}{\beta}} \left(\frac{1}{s_0 r_0} \log \frac{1}{s_0}\right),
\end{equation}

where $\Psi_{\frac{1}{\beta}}$ is the function defined as in (4.17). Clearly

\begin{equation}
\lim_{s_0 \to 0^+} \frac{\log \frac{1}{s_0 r_0}}{\log \frac{1}{s_0}} = 1.
\end{equation}
We claim that

\[(6.17) \quad \lim_{s_0 \to 0^+} \frac{\log \frac{1}{s_1}}{\log \frac{1}{s_0}} = 1.\]

Trivially,

\[
\liminf_{s_0 \to 0^+} \frac{\log \frac{1}{s_1}}{\log \frac{1}{s_0}} \geq 1.
\]

Thus, on setting

\[L = \limsup_{s_0 \to 0^+} \frac{\log \frac{1}{s_1}}{\log \frac{1}{s_0}},\]

equation (6.17) will follow if we show that \(L = 1\). Suppose, by contradiction, that \(L > 1\). Therefore, fixing \(L' \in (1, L)\), there exists a decreasing sequence \(\{s^k_0\}\) such that \(s^k_0 \to 0^+\) and that, on defining \(s^1_1 = s_1(s^k_0)\),

\[
\frac{\log \frac{1}{s_1}}{\log \frac{1}{s_0}} > L' \quad \text{for} \quad k \in \mathbb{N}.
\]

By equations (6.14), (6.15) and (6.16),

\[(6.18) \quad \int_{s^k_1}^{s^0_0} f_0 \left( \frac{r}{s^0_0} \right) \Theta(r) \, dr \geq \frac{(1-\delta)^2}{2} \left( \log \frac{1}{s^0_0} \right)^{\frac{1}{\beta}} \Psi_{\frac{1}{\beta}}(L') \quad \text{for large} \quad k \in \mathbb{N}.
\]

Owing to equation (6.12), the integral on the left-hand side of (6.18) does not exceed

\[
\varepsilon + \Theta(s^k_0) \int_0^{s^0_0} f_0 \left( \frac{r}{s^0_0} \right) \, dr.
\]

By equation (6.13), this quantity tends to \(\varepsilon\) as \(k \to \infty\). On the other hand, the right hand side of (6.18) tends to infinity as \(k \to \infty\). This contradiction establishes (6.17).

Next, as is easily verified,

\[
\Psi_{\frac{1}{\beta}}(s) = \frac{\beta}{1+\beta} (s-1)^{\frac{1}{\beta} + 1} + \ldots \quad \text{as} \quad s \to 1^+.
\]

Hence, owing to (6.16) and (6.17), equation (6.15) tells us that

\[(6.19) \quad \int_{s_1}^{s_1 s_0} \frac{(\log \frac{1 + \beta}{r \log_\frac{r}{s_1}})}{\log \frac{1 + \beta}{r \log_\frac{1}{s_1}}} \, dr = \frac{\beta}{1+\beta} \left( \log \frac{1}{s_0} \right)^{\frac{1}{\beta}} \left[ \left( \log \frac{1 + \beta}{\log \frac{1 + \beta}{s_0} - 1} \right)^{\frac{1}{\beta} + 1} + \ldots \right.
\]

for \(s_0 \to 0^+\). Assume, by contradiction, that (6.11) does not hold. Thus, there exists a sequence \(\{s^k_0\}\) such that \(s^k_0 \to 0^+\) and that, on setting \(s^1_1 = s_1(s^k_0)\),

\[
\log \frac{s^k_0}{s^k_1} > \left[ 2\varepsilon' \left( \frac{1}{\beta} + 1 \right) \log \frac{1}{s^k_0} \right]^{\frac{1}{1 + \beta}} \quad \text{for} \quad k \in \mathbb{N}.
\]

This inequality is equivalent to

\[(6.20) \quad \left( \log \frac{1}{s^k_0} \right)^{\frac{1}{1 + \beta}} > 2\varepsilon' \left( \frac{1}{\beta} + 1 \right) \left( \log \frac{1}{s^k_0} \right)^{-\frac{1}{\beta}} \quad \text{for} \quad k \in \mathbb{N}.
\]

Combining equations (6.12), (6.14), (6.19) and (6.20) yields

\[
\varepsilon \geq \lim_{k \to \infty} \frac{(1-\delta)^2}{2} \int_{s_1}^{s_1 s_0} \frac{(\log \frac{1 + \beta}{r \log_\frac{r}{s_1}})}{\log \frac{1 + \beta}{r \log_\frac{1}{s_1}}} \, dr
\]

\[
\geq \frac{(1-\delta)^2}{2} \frac{\beta}{1+\beta} \left[ 2\varepsilon' \left( \frac{1}{\beta} + 1 \right) - \lim_{k \to \infty} \left( \left( \log \frac{1}{s_0} \right)^{\frac{1}{1 + \beta} + 1} \right) \right] = (1-\delta)^2 \varepsilon',
\]
a contradiction, because of our choice of $\delta$. □

The failure of inequalities (1.6) and (1.11) for exponents $\theta > \frac{2}{3}$ is a consequence of the following proposition.

**Proposition 6.5.** Let $\beta > 0$, $M > 1$ and $\theta > \frac{2}{3}$. Suppose that $B$ is a Young function of the form (1.13) for some $N > 0$. Then, there exists a function $u \in W_L^1(\mathbb{R}, \gamma_n)$ such that $\text{med}(u) = \text{mv}(u) = 0$,

\begin{equation}
\|Lu\|_{L^\beta(\mathbb{R}, \gamma_n)} \leq 1, \quad \int_{\mathbb{R}} \text{Exp}^\beta(|Lu|) \, d\gamma_n \leq M
\end{equation}

and

\begin{equation}
\int_{\mathbb{R}} \text{exp}^\beta(\theta|u|) \, d\gamma_n = \infty.
\end{equation}

**Proof.** Let $t_0$ be a positive number to be chosen later and let $\lambda \in (0, 1)$. Assume that $\psi \in C^2(\mathbb{R})$ is a function such that $\psi = 0$ on $(-\infty, 0]$ and $\psi(t) = t$ for $t \in [1, \infty)$, $\psi'(0) = 0$. Hence, in particular, $\psi'(1) = 1$, $|\psi'| \leq C$ and $|\psi''| \leq C$ for a suitable constant $C > 0$. Define the function $f_0: (0, 1) \to [0, \infty)$ as

\begin{equation}
f_0(s) = (\lambda \log \frac{1}{s})^{\frac{1}{3}} \quad \text{for } s \in (0, 1).
\end{equation}

Let $f_1: (0, \frac{1}{s_0}) \to [0, \infty)$ be the function defined as in (6.2), where $s_0 = \Phi(t_0)$. Define the function $u: \mathbb{R} \to \mathbb{R}$ by

\begin{equation}
u(x) = \text{sgn}(x_1) \psi \left( f_1 \left( \frac{\Phi(|x_1|)}{\Phi(t_0)} \right) \right) \quad \text{for } x \in \mathbb{R}.
\end{equation}

Since $u$ is an odd function of the sole variable $x_1$, it obeys $\text{med}(u) = 0$ and $\text{mv}(u) = 0$. Of course, the verification of the latter assertion requires that $u$ be integrable on $(\mathbb{R}, \gamma_n)$, a property that will follow from the embedding $W_L^1(\mathbb{R}, \gamma_n) \to L^1(\mathbb{R}, \gamma_n)$, once we have shown that $u \in W_L^1(\mathbb{R}, \gamma_n)$. This membership is in its turn a consequence of Lemmas 6.1 and 6.2. Note that the factor $\text{sgn}(x_1)$ and the presence of the absolute value of $x_1$ do not affect this conclusion, since the function $f_1 \left( \frac{\Phi(|x_1|)}{\Phi(t_0)} \right)$ is non-positive for $x_1$ in a neighborhood of zero, and $\psi$ vanishes identically in $(-\infty, 0]$.

Denote by $t_1 > t_0$ the value, depending on $t_0$, which is implicitly defined by the equation

\begin{equation}
1 = f_1 \left( \frac{\Phi(t_1)}{\Phi(t_0)} \right).
\end{equation}

Thereby, equation (6.24) can be rewritten as

\begin{equation}
u(x) = \text{sgn}(x_1) \psi \left( f_1 \left( \frac{\Phi(|x_1|)}{\Phi(t_0)} \right) \right) \quad \text{for } |x_1| > t_1
\end{equation}

\begin{equation}
\psi \left( f_1 \left( \frac{\Phi(|x_1|)}{\Phi(t_0)} \right) \right) \quad \text{for } |x_1| < t_1
\end{equation}

\begin{equation}
0 \quad \text{for } |x_1| \leq t_0.
\end{equation}

Lemmas 6.1 and 6.2 then tell us that

\begin{equation}
Lu(x) = \text{sgn}(x_1) \times \begin{cases} 
- f_0 \left( \frac{\Phi(|x_1|)}{\Phi(t_0)} \right) & \text{for } |x_1| > t_1 \\
- \psi' \left( f_1 \left( \frac{\Phi(|x_1|)}{\Phi(t_0)} \right) \right) f_0 \left( \frac{\Phi(|x_1|)}{\Phi(t_0)} \right) & \text{for } |x_1| = t_1 \\
\psi'' \left( f_1 \left( \frac{\Phi(|x_1|)}{\Phi(t_0)} \right) \right) h_0 \left( \frac{\Phi(|x_1|)}{\Phi(t_0)} \right) & \text{for } |x_1| < t_1
\end{cases}
\end{equation}

\begin{equation}
0 & \text{for } |x_1| \leq t_0,
\end{equation}

where the function $h_0$ is defined as in (6.6). Owing to our bounds on the derivatives of $\psi$,

\begin{equation}
|Lu(x)| \leq \begin{cases} 
f_0 \left( \frac{\Phi(|x_1|)}{\Phi(t_0)} \right) & \text{for } |x_1| > t_1 \\
C f_0 \left( \frac{\Phi(|x_1|)}{\Phi(t_0)} \right) + Ch_0 \left( \frac{\Phi(|x_1|)}{\Phi(t_0)} \right) & \text{for } |x_1| < t_1 \\
0 & \text{for } |x_1| \leq t_0.
\end{cases}
\end{equation}
We begin by observing that $\lambda$ can be chosen in such a way that equation (6.22) holds. Plainly,
\begin{equation}
\int R^n \exp^{\beta}(\theta|u|) \, d\gamma_n \geq \int_0^\infty \exp^{\beta} \left( \theta f_1 \left( \frac{\Phi(x_1)}{\Phi(t_0)} \right) \right) \, d\gamma_n(x_1) = \int_0^{\Phi(t_1)} \exp^{\beta} \left( \theta f_1 \left( \frac{s}{\Phi(t_0)} \right) \right) \, ds.
\end{equation}

Lemma 6.3 tells us that
\begin{equation}
f_1 \left( \frac{s}{\Phi(t_0)} \right) = \frac{1}{\theta_\beta} \left( \lambda \log \frac{1}{s} \right)^\frac{\beta}{\theta_\beta} + \cdots \quad \text{as } s \to 0^+.
\end{equation}

Hence, the integral on the rightmost side of equation (6.29) diverges provided that
\begin{equation}
\lambda \left( \frac{\theta}{\theta_\beta} \right)^\beta > 1.
\end{equation}

The remaining part of the proof is devoted to showing that $t_0$ can be chosen so large that the inequalities in (6.21) are fulfilled as well. Let $B$ be a Young function such that $B(\tau) = N \exp^{\beta}(\tau)$ for $\tau \geq \tau_0$. Then
\begin{equation}
\int R^n B(|Lu|) \, d\gamma_n \leq \int_{0 < |Lu| < \tau_0} B(|Lu|) \, d\gamma_n + N \int_{|Lu| \geq \tau_0} \exp^{\beta}(|Lu|) \, d\gamma_n
\leq B(\tau_0) \int_{|Lu| > 0} \, d\gamma_n + N \int_{|x| > t_0} \exp^{\beta}(|Lu|) \, d\gamma_n.
\end{equation}

Note that
\begin{equation}
\int_{|Lu| > 0} \, d\gamma_n \leq 2\Phi(t_0),
\end{equation}

since the support of $Lu$ is contained in the union of the halfspaces $\{x_1 > t_0\}$ and $\{x_1 < -t_0\}$, both having Gauss measure equal to $\Phi(t_0)$. From equation (6.28) and the change of variables $s = \Phi(x_1)$, we obtain that
\begin{equation}
\int_{|x_1| > t_0} \exp^{\beta}(|Lu|) \, d\gamma_n \leq 2 \int_0^{\Phi(t_1)} \exp^{\beta} \left( f_0 \left( \frac{s}{\Phi(t_0)} \right) \right) \, ds
+ 2 \int_{\Phi(t_1)}^{\Phi(t_0)} \exp^{\beta} \left( Cf_0 \left( \frac{s}{\Phi(t_0)} \right) + Ch_0(s)^2 \right) \, ds.
\end{equation}

The first integral on the right-hand side of inequality (6.33) can be estimated as
\begin{equation}
\int_0^{\Phi(t_1)} \exp^{\beta} \left( f_0 \left( \frac{s}{\Phi(t_0)} \right) \right) \, ds \leq \int_0^{\Phi(t_0)} \exp \left( \lambda \log \frac{\Phi(t_0)}{s} \right) \, ds = \Phi(t_0) \int_0^1 \frac{ds}{s^\lambda} = \frac{\Phi(t_0)}{1 - \lambda}.
\end{equation}

Let us next focus on the second integral on the right-hand side of (6.33). Since $f_0$ is nonnegative and decreasing, its integral average is also decreasing. On the other hand, $s/I(s)$ is increasing. Therefore
\begin{equation}
h_0(s) = \frac{s}{I(s)} \int_0^s f_0 \left( \frac{r}{\Phi(t_0)} \right) \, dr \leq \frac{\Phi(t_0)}{I(\Phi(t_0))} \int_0^{\Phi(t_1)} f_0 \left( \frac{r}{\Phi(t_0)} \right) \, dr
\end{equation}
for $s \in (\Phi(t_1), \Phi(t_0))$. Note that, by (6.25),
\begin{equation}
\int_{\Phi(t_1)}^{\Phi(t_0)} \frac{1}{I(s)^2} \int_0^s f_0 \left( \frac{r}{\Phi(t_0)} \right) \, dr \, ds = 1.
\end{equation}

Hence Lemma 6.4 with $\varepsilon = 1$, $s_0 = \Phi(t_0)$ and $s_1 = \Phi(t_1)$ implies that
\begin{equation}
\frac{\Phi(t_1)}{\Phi(t_0)} \to 0 \quad \text{as } t_0 \to \infty.
\end{equation}

Also, by equations (6.11) and (4.4), there exists a constant $C_{\beta,\lambda}$, depending on $\beta$ and $\lambda$, such that
\begin{equation}
\log \frac{\Phi(t_0)}{\Phi(t_1)} \leq C_{\beta,\lambda} t_0^{2\beta} \quad \text{for large } t_0.
\end{equation}

Next, since
\begin{equation}
\int_0^s \left( \log \frac{1}{s} \right)^\beta \, dr = s \left( \log \frac{1}{s} \right)^\beta + \cdots \quad \text{as } s \to 0^+,
\end{equation}
Finally, we infer from equations (6.40)–(6.42) that
\[
\left(\frac{r}{\Phi(t_0)}\right) \frac{1}{\Phi(t_1)} \int_0^{\Phi(t_1)} f_0 \left(\frac{r}{\Phi(t_0)}\right) dr = \Phi(t_0) \int_0^{\Phi(t_1)} \lambda \left(\log \frac{1}{\Phi(t_1)}\right) \frac{1}{\Phi(t_1)} dr = \left(\lambda \log \frac{\Phi(t_0)}{\Phi(t_1)}\right)^{\frac{1}{2}} + \cdots
\]
as \(t_0 \to \infty\). By equations (3.8) and (4.5), one has that \(I(\Phi(t)) = -\Phi'(t) = t\Phi(t) + \cdots \) as \(t \to \infty\). Hence,
\[
\frac{\Phi(t_0)}{I(\Phi(t_0))} = \frac{1}{t_0} + \cdots \quad \text{as } t_0 \to \infty.
\]
Thanks to equations (6.37)–(6.39), inequality (6.35) implies that
\[
h_0(s)^{2\beta} \leq C_{p,\lambda}^2 t_0 (\frac{2\beta}{\beta+\lambda} - 1) = C_{p,\lambda}^2 (1 - \frac{\beta}{\beta+\lambda}) t_0^{\frac{2\beta}{\beta+\lambda}}
\]
for large \(t_0\) and for \(s \in (\Phi(t_1), \Phi(t_0))\), where \(C_{p,\lambda}\) is a suitable constant depending on \(\beta\) and \(\lambda\). The use of inequality (6.37) yields
\[
f_0 \left(\frac{\Phi(t_1)}{\Phi(t_0)}\right)^{\beta} \leq \lambda C_{p,\lambda} t_0^{\frac{2\beta}{\beta+\lambda}} \quad \text{for large } t_0.
\]
Also, by equation (4.5),
\[
\Phi(t_0) = -\frac{\Phi'(t_0)}{\Phi(t_0)} + \cdots = \frac{1}{t_0 \sqrt{2\pi}} \exp \left(-\frac{t_0^2}{2}\right) + \cdots \quad \text{as } t_0 \to \infty.
\]
Finally, we infer from equations (6.40)–(6.42) that
\[
\int_{\Phi(t_1)}^{\Phi(t_0)} \exp^\beta \left(C f_0 \left(\frac{s}{\Phi(t_0)}\right) + C h_0(s)^2\right) ds \leq \Phi(t_0) \exp \left(C_{p,\lambda} f_0 \left(\frac{\Phi(t_1)}{\Phi(t_0)}\right)^{\beta} + C_{p,\lambda} \sup_{s \in (\Phi(t_1), \Phi(t_0))} h_0(s)^{2\beta}\right)
\]
\[
\leq \frac{1}{t_0 \sqrt{2\pi}} \exp \left(-\frac{t_0^2}{2}\right) + \lambda C_{p,\lambda} C_{p,\lambda} t_0^{\frac{2\beta}{\beta+\lambda}} + C_{p,\lambda} C_{p,\lambda} (1 - \frac{\beta}{\beta+\lambda}) t_0^{\frac{2\beta}{\beta+\lambda}} + \cdots \quad \text{as } t_0 \to \infty,
\]
for some constant \(C_{p,\lambda}\) depending on \(C\) and \(\beta\). Equation (6.43) tells us that its leftmost side tends to 0 as \(t_0 \to \infty\) for any fixed \(\beta\) and \(\lambda\). Thus, combining equations (6.31)–(6.34) and (6.43) enables us to deduce that
\[
\int_{\mathbb{R}^n} B(\|L u\|) d\gamma_n \leq 2B(\tau_0) \Phi(t_0) + \frac{2N}{1 - \lambda} \Phi(t_0) + R(\beta, \lambda, t_0),
\]
where the expression \(R(\beta, \lambda, t_0)\) has the property that \(R(\beta, \lambda, t_0) \to 0\) as \(t_0 \to \infty\).

Moreover, since there exists \(\tau_0 \geq 0\) such that
\[
\exp^\beta(\tau) \leq \begin{cases} 
\frac{-1}{\tau_0} (\exp^\beta(\tau_0) - 1) + 1 & \text{for } \tau \in [0, \tau_0) \\
\exp^\beta(\tau) & \text{for } \tau \in [\tau_0, \infty),
\end{cases}
\]
we have that
\[
\int_{\mathbb{R}^n} \exp^\beta(\|L u\|) d\gamma_n \leq \int_{\{L u = 0\}} d\gamma_n + \int_{\{0 < \|L u\| < \tau_0\}} \exp^\beta(\tau_0) d\gamma_n + \int_{\{|L u| \geq \tau_0\}} \exp^\beta(\|L u\|) d\gamma_n
\]
\[
\leq 1 + \exp^\beta(\tau_0) \int_{\{|L u| > \tau_0\}} d\gamma_n + \exp^\beta(\|L u\|) d\gamma_n
\]
\[
\leq 1 + 2 \exp^\beta(\tau_0) \Phi(t_0) + \frac{1}{1 - \lambda} \Phi(t_0) + R(\beta, \lambda, t_0).
\]
Note that here we have also made use of equations (6.32)–(6.34) and (6.43). Inequalities (6.44) and (6.45) ensure that, given \(\beta > 0\), \(\lambda \in (0, 1)\) and either a function \(B\) as in (1.13) or a number \(M > 1\), the number \(t_0 > 0\) can be chosen large enough for both inequalities in (6.21) to hold. \(\square\)
Propositions 6.6 and 6.7 deal with the case when $\beta > 1$. They imply that there actually exist Young functions $B$ obeying (1.13) for which the threshold value $\frac{2}{\beta}$ of the constant $\theta$ in inequality (1.11) is not attained, and Young functions $B$ for which it is attained.

**Proposition 6.6.** Let $\beta \in (1, \infty)$. For every $N > 0$, there exists $\tau_0 > 0$, a Young function $B$ as in (1.13), and a sequence of functions $\{u_k\} \subset \mathcal{W}^2 \exp \mathcal{L}^B(\mathbb{R}^n, \gamma_n)$, such that $\text{med}(u_k) = \text{mv}(u_k) = 0$,

\[
\|\mathcal{L}u_k\|_{L^B(\mathbb{R}^n, \gamma_n)} \leq 1 \quad \text{for } k \in \mathbb{N}
\]

and

\[
\lim_{k \to \infty} \int_{\mathbb{R}^n} \exp^\beta \left( \frac{2}{\beta} \|u_k\| \right) \, \mathrm{d}\gamma_n = \infty.
\]

**Proof.** Let us set, for brevity of notation, $A = \text{med}(u_k) \exp \mathcal{L}^B(\mathbb{R}^n, \gamma_n)$, such that $\text{med}(u_k) = \text{mv}(u_k) = 0$.

Define the function

\[
A(t) = \left\{
\begin{array}{ll}
0 & \text{for } t \in [0, \tau_0] \\
A(\tau_0) + a(\tau_0)(\tau - \tau_0) & \text{for } t \in (\tau_0, \tau_0'] \\
A(\tau) & \text{for } t \in (\tau_0, \infty),
\end{array}
\right.
\]

where $\tau_0' \in (0, \tau_0)$ is given by

\[
\tau_0' = \tau_0 - \frac{A(\tau_0)}{a(\tau_0)}.
\]

Observe that $A$ is a Young function. Moreover,

\[
a(\tau) = \left\{
\begin{array}{ll}
0 & \text{for } \tau \in (0, \tau_0'] \\
a(\tau_0') & \text{for } \tau \in (\tau_0', \tau_0] \\
a(\tau) & \text{for } \tau \in (\tau_0, \infty),
\end{array}
\right.
\]

and $\frac{a^{-1}(t)}{a^{-1}(t)}$ for $t \in (0, a(\tau_0)]$

\[
\tau_0' = \tau_0 - \frac{A(\tau_0)}{a(\tau_0)}.
\]

Observe that $A$ is a Young function. Moreover,

\[
a(\tau) = \left\{
\begin{array}{ll}
0 & \text{for } \tau \in (0, \tau_0'] \\
a(\tau_0') & \text{for } \tau \in (\tau_0', \tau_0] \\
a(\tau) & \text{for } \tau \in (\tau_0, \infty),
\end{array}
\right.
\]

and $\frac{a^{-1}(t)}{a^{-1}(t)}$ for $t \in (0, a(\tau_0)]$

\[
\tau_0' = \tau_0 - \frac{A(\tau_0)}{a(\tau_0)}.
\]

Observe that, by Lemmas 4.3 and 4.5,

\[
\frac{a^{-1}(t)}{a^{-1}(t)} = \left( \log \frac{1}{2} \right)^\beta + \cdots \quad \text{as } r \to 0^+.
\]

Let $\psi \in C^2(\mathbb{R})$ be a function such that $\psi(t) = t$ for $t \in (-\infty, 1]$, $\psi(t) = 1$ for $t \in [2, \infty)$, $\psi(t) \geq 1$ for $t \in (1, 2]$ and $|\psi'| \leq 1$. Hence, $\psi'(1) = 1$ and there exists a positive constant $C$ such that $|\psi''| \leq C$.

Define, for each $k \in \mathbb{N}$, the function $v_k: \mathbb{R}^n \to \mathbb{R}$ by

\[
v_k(x) = f_1(2\Phi(k)) \psi \left( \frac{f_1(2\Phi(x_1))}{f_1(2\Phi(k))} \right) \quad \text{for } x \in \mathbb{R}^n
\]

and the function $w_k: \mathbb{R}^n \to \mathbb{R}$ as

\[
w_k(x) = -v_k(-x) \quad \text{for } x \in \mathbb{R}^n.
\]

From Lemmas 6.1 and 6.2 one deduces that $v_k, w_k \in W^{2,2}(\mathbb{R}^n, \gamma_n)$ for every $k \in \mathbb{N}$. One can also verify that the function $u_k: \mathbb{R}^n \to \mathbb{R}$, given by

\[
u_k(x) = \left\{
\begin{array}{ll}
v_k(x) & \text{for } x_1 \geq 0 \\
w_k(x) & \text{for } x_1 < 0,
\end{array}
\right.
\]
obeys \( u_k \in W^{2,2}(\mathbb{R}^n, \gamma_n) \), and
\[
\mathcal{L}u_k(x) = \begin{cases} 
 f_1(2\Phi(k)) & \text{for } x_1 > 0 \\
 f_1(2\Phi(k)) \psi \left( \frac{f_1(2\Phi(|x_1|))}{f_1(2\Phi(k))} \right) & \text{for } k \leq |x_1| < t_k \\
 f_1(2\Phi(|x_1|)) & \text{for } |x_1| < k.
\end{cases}
\]
Each function \( u_k \) is odd in variable \( x_1 \), whence \( \text{med}(u_k) = 0 \). The fact that \( \text{mv}(u_k) = 0 \) is a consequence of the same property of \( u_k \). Of course, the existence of \( \text{mv}(u_k) \) requires that \( u_k \) be integrable, and this is a consequence of equation (6.46), and of the embedding \( W_L \exp L^{2}(\mathbb{R}^n, \gamma_n) \to L^{1}(\mathbb{R}^n, \gamma_n) \).

In order to prove property (6.46), denote by \( t_k \) the number implicitly defined by
\[
(6.66) 
 f_1(2\Phi(t_k)) = 2f_1(2\Phi(k)) \quad \text{for } k \in \mathbb{N}.
\]
Notice that \( t_k > k \) for \( k \in \mathbb{N} \). Then, \( u_k \) takes the form
\[
(6.7) 
 u_k(x) = \text{sgn}(x_1) \times \begin{cases} 
 0 & \text{for } |x_1| \geq t_k \\
 -\psi' \left( \frac{f_1(2\Phi(|x_1|))}{f_1(2\Phi(k))} \right) f_0(2\Phi(|x_1|)) & \text{for } k \leq |x_1| < t_k \\
 -f_0(2\Phi(|x_1|)) & \text{for } |x_1| < k,
\end{cases}
\]
where \( h_0 \) is given by (6.6). Since \( |\psi'| \leq 1 \) and \( |\psi''| \leq C \),
\[
(6.8) 
 |\mathcal{L}u_k(x)| \leq \begin{cases} 
 0 & \text{for } |x_1| \geq t_k \\
 f_0(2\Phi(|x_1|)) + C \frac{h_0(\Phi(|x_1|))}{f_1(2\Phi(k))} & \text{for } k \leq |x_1| < t_k \\
 f_0(2\Phi(|x_1|)) & \text{for } |x_1| < k.
\end{cases}
\]
Equation (6.46) will be shown to hold with \( B = N A \). Thanks to equation (6.59) and the change of variables \( \Phi(x_1) = s \),
\[
\int_{\mathbb{R}^n} B(\mathcal{L}u_k) \, d\gamma_n \leq \int_{\{|x_1| < k\}} B \left( f_0(2\Phi(|x_1|)) \right) \, d\gamma_1(x_1)
\]
\[
(6.9) 
 + \int_{\{k \leq |x_1| < t_k\}} B \left( f_0(2\Phi(|x_1|)) + C \frac{h_0(\Phi(|x_1|))}{f_1(2\Phi(k))} \right) \, d\gamma_1(x_1)
\]
\[
= 2N \int_{\Phi(k)}^{\frac{1}{2}} A(f_0(2s)) \, ds + 2N \int_{\Phi(k)}^{\Phi(t_k)} A \left( f_0(2s) + C \frac{h_0(s)^2}{f_1(2\Phi(k))} \right) \, ds.
\]
Consider the first integral on the rightmost side of inequality (6.60). Assume that \( k \) is so large that
\[
(6.10) 
 \Phi(k) < t_0.
\]
Then
\[
\int_{\Phi(k)}^{\frac{1}{2}} A(f_0(2s)) \, ds = \int_{\Phi(k)}^{t_0} A(a^{-1}(\Theta(s))) \, ds \leq \int_{0}^{t_0} A(a^{-1}(\Theta(s))) \, ds = M(t_0).
\]
We claim that \( M(t_0) \to 0 \) as \( t_0 \to 0^+ \). To verify this fact, it suffices to prove that
\[
(6.11) 
 \int_{0}^{t_0} A(a^{-1}(\Theta(s))) \, ds < \infty.
\]
By equation (4.26),
\[
A(a^{-1}(t)) = \frac{1}{\beta} t^{\frac{1}{\beta} - 1} \quad \text{as } t \to \infty,
\]
whence, owing to Lemma 4.3, one deduces that

\[ A \left( a^{-1}(\Theta(s)) \right) = \frac{1}{2\beta s} \left( \log \frac{1}{s} \right)^{3/2 - 2} + \cdots \quad \text{as} \quad s \to 0^+. \]

Hence, (6.63) follows, since we are assuming that \( \beta > 1 \).

Let us next focus on the second integral on the rightmost side of inequality (6.60). Let \( h_0 \) be the function defined by (6.6). Assume that \( k \) is large enough for inequality (6.61) to hold. Via equation (6.56) and the monotonicity of \( f_1 \) we obtain that

\[
\int_{\Phi(t_k)}^{\Phi(k)} A \left( f_0(2s) + C \frac{h_0(s)^2}{f_1(2\Phi(k))} \right) ds = \int_{\Phi(t_k)}^{\Phi(k)} A \left( f_0(2s) + 2C \frac{h_0(s)^2}{f_1(2\Phi(k))} \right) ds \\
\leq \int_{\Phi(t_k)}^{\Phi(k)} A \left( f_0(2s) + 2C \frac{h_0(s)^2}{f_1(2\Phi(k))} \right) ds \leq \int_0^{t_0} \exp(\beta) \left( f_0(2s) + 2C \frac{h_0(s)^2}{f_1(2\Phi(k))} \right) ds.
\]

Our next task is to show that the integral on the right-hand side of (6.64) tends to 0 as \( t_0 \to 0^+ \). From equation (6.54), Lemma 4.1 and L'Hôpital's rule, one can infer that

\[
h_0(s) = \frac{1}{\sqrt{2}} \left( \log \frac{1}{s} \right)^{3/2 - 2} + \cdots \quad \text{as} \quad s \to 0^+.
\]

Hence, thanks to Lemma 6.3 and inequality (6.54),

\[
f_1(s) = \frac{\beta}{2} \left( \log \frac{1}{s} \right)^{3} + \cdots \quad \text{as} \quad s \to 0^+.
\]

Coupling expansions (6.65) and (6.66) yields

\[
\frac{h_0(s)^2}{f_1(2\Phi(k))} = \frac{1}{\beta} \left( \log \frac{1}{s} \right)^{3/2 - 1} + \cdots \quad \text{as} \quad s \to 0^+.
\]

The asymptotic expansion (6.67) and (6.54) ensure that \( f_0(2s) \) is the leading term in the sum \( f_0(2s) + 2C \frac{h_0(s)^2}{f_1(2\Phi(k))} \). Therefore, by equation (4.2) with \( \sigma = \beta \),

\[
\left( f_0(2s) + 2C \frac{h_0(s)^2}{f_1(2\Phi(k))} \right)^\beta = f_0(2s)^\beta + \beta 2C f_0(2s)^{\beta - 1} \frac{h_0(s)^2}{f_1(2\Phi(k))} + \cdots
\]

\[
= f_0(2s)^\beta + 2C + \cdots \quad \text{as} \quad s \to 0^+.
\]

Notice that the second equality relies upon equations (6.54) and (6.67). Now, equations (6.68) and (6.65) tell us that

\[
\exp(\beta) \left( f_0(2s) + 2C \frac{h_0(s)^2}{f_1(2\Phi(k))} \right) \leq e^{4C} \exp(\beta) \left( f_0(2s) \right) = e^{4C} A(a^{-1}(\Theta(s))) \quad \text{for} \quad s \in (0, t_0),
\]

provided that \( t_0 \) is sufficiently small. Therefore, from equations (6.64) and (6.69) we conclude that

\[
\int_{\Phi(t_k)}^{\Phi(k)} A \left( f_0(2s) + C \frac{h_0(s)^2}{f_1(2\Phi(k))} \right) ds \leq e^{4C} M(t_0),
\]

if \( k \) is so large that inequality (6.61) holds. Here, \( M(t_0) \) is the function defined by (6.62). From equations (6.60), (6.62) and (6.70) one obtains that

\[
\int_{\mathbb{R}^n} B(|L u_k|) \, d\gamma_n \leq 2N \left( 1 + e^{4C} \right) M(t_0),
\]

provided that \( k \) is large enough for inequality (6.61) to be fulfilled. By the definition of the Luxemburg norm, if \( t_0 \) is chosen so small that \( 2N \left( 1 + e^{4C} \right) M(t_0) \leq 1 \), then inequality (6.46) holds for sufficiently large \( k \).

It remains to establish property (6.47). Since, by our assumptions on \( \psi \),

\[ |u_k(x)| \geq f_1(2\Phi(k)) \quad \text{for} \quad x_1 \geq k, \]

we have that

\[
\int_{\mathbb{R}^n} \exp(\theta_k|u_k|) \, d\gamma_n \geq \int_{\{x_1 > k\}} \exp(\theta_k f_1(2\Phi(k))) \, d\gamma_n = \Phi(k) \exp(\theta_k f_1(2\Phi(k))).
\]
An application of Fubini’s theorem tells us that
\[ f_1(2s) = \int_s^t \frac{1}{t(\rho)^2} \int_0^\rho f_0(2r) \, dr \, d\rho \geq \int_s^t \frac{1}{t(\rho)^2} \int_0^\rho f_0(2r) \, dr \, d\rho = \int_s^t f_0(2r) \, dr \int_r^t \frac{1}{t(\rho)^2} \, d\rho \, dr \]
(6.72)
\[ = \int_s^t f_0(2r) \Theta(r) \, dr = \int_s^{\tau_0} f_0(2r) \Theta(r) \, dr + \int_{\tau_0}^t f_0(2r) \Theta(r) \, dr \quad \text{for } s \in [0, t_0). \]

By equations (6.52) and (6.53), the change of variables \( t = \Theta(r) \) in the first integral on the rightmost side of equation (6.72) results in
\[ \int_s^{t_0} f_0(2r) \Theta(r) \, dr = \int_{\Theta(s)}^{\Theta(t)} a^{-1}(t) tI(\Theta^{-1}(t))^2 \, dt \quad \text{for } s \in [0, t_0). \]

From Lemma 4.8, we infer that, if \( \tau_0 \) is sufficiently large, then
\[ a^{-1}(t) tI(\Theta^{-1}(t))^2 \geq \frac{1}{2t} (\log t)^{\frac{1}{2} - 1} - c_\beta \frac{1}{t} (\log t)^{\frac{1}{2} - 2} \log \log t, \]
provided that the constant \( c_\beta > 5/4 + (\beta - 1)/2\beta^2 \), and \( t > a(\tau_0) \). Therefore,
\[ \int_s^{t_0} f_0(2r) \Theta(r) \, dr \geq \frac{1}{2} \int_{\Theta(s)}^{\Theta(t)} \frac{1}{t} (\log t)^{\frac{1}{2} - 1} \, dt - c_\beta \int_{\Theta(s)}^{\Theta(t)} \frac{1}{t} (\log t)^{\frac{1}{2} - 2} \log \log t \, dt \]
\[ \geq \frac{\beta}{2} \left[ \log \Theta(s) \right]^{\frac{1}{2}} - \frac{\beta}{2} \left[ \log a(\tau_0) \right]^{\frac{1}{2}} - c_\beta \int_{\Theta(s)}^{\Theta(t)} \frac{1}{t} (\log t)^{\frac{1}{2} - 2} \log \log t \, dt \]
for \( s \in [0, t_0) \). Altogether,
\[ f_1(2s) \geq \frac{\beta}{2} \left[ \log \Theta(s) \right]^{\frac{1}{2}} + \lambda(\tau_0) \quad \text{for } s \in (0, t_0), \]
where we have set
\[ \lambda(\tau_0) = \int_{\Theta(s)}^{\Theta(t)} f_0(2r) \Theta(r) \, dr - \frac{\beta}{2} \left[ \log a(\tau_0) \right]^{\frac{1}{2}} - c_\beta \int_{\Theta(s)}^{\Theta(t)} \frac{1}{t} (\log t)^{\frac{1}{2} - 2} \log \log t \, dt. \]

Let us analyze the asymptotic behaviour of \( \lambda(\tau_0) \) as \( \tau_0 \to \infty \). Owing to equations (6.49)–(6.51),
\[ \int_{\Theta(s)}^{\Theta(t)} f_0(2r) \Theta(r) \, dr = \left( \tau_0 - \frac{A(\tau_0)}{a(\tau_0)} \right) \int_{\Theta(s)}^{\Theta(t)} \Theta(r) \, dr. \]

Therefore, by the change of variables \( t = \Theta(r) \), L’Hôpital’s rule and Lemma 4.7,
\[ \int_{\Theta(s)}^{\Theta(t)} \Theta(r) \, dr = \int_0^{a(\tau_0)} tI(\Theta^{-1}(t))^2 \, dt = \frac{1}{2} \log \log a(\tau_0) + \cdots = \frac{\beta}{2} \log \tau_0 + \cdots \quad \text{as } \tau_0 \to \infty. \]

Also \( A(\tau_0)/a(\tau_0) \to 0 \) as \( \tau_0 \to \infty \). Thus, equation (6.75) can be rewritten as
\[ \int_{\Theta(s)}^{\Theta(t)} f_0(2r) \Theta(r) \, dr = \frac{\beta}{2} \tau_0 \log \tau_0 + \cdots \quad \text{as } \tau_0 \to \infty. \]

Next,
\[ \left[ \log a(\tau_0) \right]^{\frac{1}{2}} = \tau_0 + \cdots \quad \text{as } \tau_0 \to \infty \]
and, since the last integral on the right-hand side of equation (6.74) tends to 0 as \( \tau_0 \to \infty \), we conclude that
\[ \lambda(\tau_0) = \frac{\beta}{2} \tau_0 \log \tau_0 + \cdots \quad \text{as } \tau_0 \to \infty. \]

In particular, \( \lambda(\tau_0) \to \infty \) as \( \tau_0 \to \infty \). Therefore, given any \( \lambda > 0 \), one can chose \( \tau_0 \) so large that
\[ (6.76) \]
\[ \lambda(\tau_0) > \lambda. \]

Inequalities (6.73), (6.76) and an elementary inequality yield
\[ \left[ \theta \beta f_1(2s) \right]^{\beta} \geq \log \Theta(s) + \beta \theta \beta \lambda \left[ \log \Theta(s) \right]^{1 - \beta} \quad \text{for } s \in (0, t_0). \]
Therefore, we deduce from (6.71) that

\[ \int_{\mathbb{R}^n} \exp^\beta(\theta|u|) \, d\gamma_n \geq \Phi(k)\Theta(\Phi(k)) \exp\left(2\lambda [\log \Theta(\Phi(k))]^{1-\frac{1}{\beta}}\right) \]

for every \( k \) obeying (6.61). Thanks to Lemma 4.3 and equation (4.4),

\[ \Phi(k)\Theta(\Phi(k)) = \frac{1}{k^2} + \cdots \quad \text{as} \quad k \to \infty. \]

and

\[ \log \Theta(\Phi(k)) = \frac{k^2}{2} + \cdots \quad \text{as} \quad k \to \infty. \]

These asymptotic expansions ensure that the right-hand side of inequality (6.77) tends to infinity as \( k \to \infty \). Property (6.47) hence follows. \( \square \)

**Proposition 6.7.** Let \( \beta \in (1, \infty) \). For any \( N > 0 \), there exists \( \tau_0 > 0 \) and a Young function \( B \) of the form (1.13) such that

\[ \sup_u \int_{\mathbb{R}^n} \exp^\beta(\theta|u|) \, d\gamma_n < \infty, \]

where the supremum is extended over all functions \( u \in W_L \exp L^\beta(\mathbb{R}^n, \gamma_n) \) fulfilling conditions (1.5) and (1.12).

**Proof.** Thanks to Corollary 5.3,

\[ \sup_u \int_{\mathbb{R}^n} \exp^\beta(\theta|u|) \, d\gamma_n \leq 2 \int_0^1 \exp^\beta(\theta \mathcal{G}_{B,C}(s)) \, ds, \]

for any Young function \( B \), where \( \mathcal{G}_{B,C} \) is the function defined by (4.68) and \( C \) the constant given by (5.4). Let \( \lambda > 0 \). We claim that for any \( N > 0 \), there exists \( \tau_0 > 0 \) and a Young function \( B \) of the form (1.13) such that

\[ \mathcal{G}_{B,C}(s) \leq \frac{\beta}{2} \left( \log \frac{1}{s} \right) ^\frac{2}{3} - \lambda \quad \text{for} \quad s \text{ near zero}. \]

Set \( A(\tau) = \exp^\beta(\tau) \) for \( \tau \geq 0 \), and define the function \( \mathcal{A} : [0, \infty) \to [0, \infty) \) by

\[ \mathcal{A}(\tau) = \begin{cases} \tau A(\tau) & \text{for } \tau \in (0, \tau_0) \\ A(\tau_0) & \text{for } \tau \in [\tau_0, \infty), \end{cases} \]

where the number \( \tau_0 > 0 \) will be specified later. Observe that the function \( B \), defined as \( B = NA \), is a Young function provided that \( \tau_0 \) is large enough. The left-continuous derivative of \( B \), denoted by \( b \), obeys

\[ b(\tau) = \begin{cases} \frac{N A(\tau_0)}{\tau_0} & \text{for } \tau \in (0, \tau_0) \\ \frac{Na(\tau)}{Na(\tau_0)} & \text{for } \tau \in (\tau_0, \infty) \end{cases} \quad \text{and} \quad b^{-1}(t) = \begin{cases} 0 & \text{for } t \in (0, N \frac{A(\tau_0)}{\tau_0}) \\ \tau_0 & \text{for } t \in (N \frac{A(\tau_0)}{\tau_0}, Na(\tau_0)) \\ a^{-1}\left(\frac{t}{N}\right) & \text{for } t \in (Na(\tau_0), \infty), \end{cases} \]

where \( a \) stands for the derivative of \( A \) and \( b^{-1} \) for the generalized left-continuous inverse of \( b \). Lemmas 4.11 and 4.12 tell us that

\[ \|\Theta\|_{L^\beta(s, 2)} = \int_0^{\Theta(s)} b^{-1}(\lambda \Theta(\tau)) \sqrt{\tau I(\Theta^{-1}(\tau))^2} \, d\tau \quad \text{for} \quad s \in (0, \frac{1}{2}), \]

where the function \( \lambda \Theta(s) \to \lambda \) as \( s \to 0^+ \), for some constant \( \lambda > 0 \). From Lemma 4.8 we infer that there exists \( t_0 > 0 \) such that

\[ a^{-1}\left(\frac{2t}{N}\right) t I(\Theta^{-1}(t))^2 \leq \frac{1}{2t} (\log t)^{\frac{3}{2}} \quad \text{for} \quad t > t_0. \]
Now, choose \( \tau_0 \) so large that \( N \alpha(\tau_0) > 2 \lambda \tau_0 \), and \( s_0 = s_0(\tau_0) \) in such a way that \( \lambda \alpha(s) \in (\lambda, 2\lambda) \) and \( 2\lambda \Theta(s) > N \alpha(\tau_0) \) for \( s \in (0, s_0) \). Then,

\[
\| \Theta \|_{L^B(s, 1)} \leq \int_0^{\Theta(s)} b^{-1}(2\lambda \tau) \tau I(\Theta^{-1}(\tau))^2 \, d\tau
\]

\[
= \tau_0 \int_{2\lambda \tau_0}^{N \lambda \tau_0} \tau I(\Theta^{-1}(\tau))^2 \, d\tau + \int_{\lambda \tau_0}^{\Theta(s)} a^{-1} \left( \frac{2\lambda}{N} \right) \tau I(\Theta^{-1}(\tau))^2 \, d\tau
\]

\leq \tau_0 \int_{\lambda \tau_0}^{N \lambda \tau_0} \tau I(\Theta^{-1}(\tau))^2 \, d\tau + \frac{1}{2} \int_{\lambda \tau_0}^{\Theta(s)} \frac{1}{2} \tau \left( \log \tau \right)^{1/2} \, d\tau
\]

\[
= \frac{\beta}{2} \left[ \log \Theta(s) \right]^{1/2} + \nu(\tau_0) \quad \text{for } s \in (0, s_0),
\]

where we have set

\[
(6.82) \quad \nu(\tau_0) = - \frac{\beta}{2} \left[ \log \left( \frac{N}{2\lambda} a(\tau_0) \right) \right]^{1/2} + \tau_0 \int_{\lambda \tau_0}^{N \lambda \tau_0} \tau I(\Theta^{-1}(\tau))^2 \, d\tau.
\]

Observe that, thanks to Lemma 4.3, we may assume that \( s_0 \) is so small that \( \log \Theta(s) \leq \log \frac{1}{s} \) for \( s \in (0, s_0) \). Therefore, it suffices to show that \( \nu(\tau_0) \to -\infty \) as \( \tau_0 \to \infty \). To this purpose, first notice that, since \( a(\tau) = \beta \tau^{-1} \exp(\tau) \),

\[
(6.83) \quad \left[ \log \left( \frac{N}{2\lambda} a(\tau_0) \right) \right]^{1/2} = \tau_0 + \cdots \quad \text{as } \tau_0 \to \infty.
\]

Next, let us show that the second addend on the right-hand side of (6.82) tends to 0 as \( \tau_0 \to \infty \). From L’Hôpital’s rule and Lemma 4.7, we deduce that

\[
\int_0^t \tau I(\Theta^{-1}(\tau))^2 \, d\tau = \frac{1}{2} \log t + C + \cdots \quad \text{as } t \to \infty,
\]

for some constant \( C \). By formula (4.3),

\[
\frac{1}{2} \log \left( \frac{N}{2\lambda} a(\tau_0) \right) = \frac{1}{2} \log \left( \tau_0^{\beta} + (\beta - 1) \log \tau_0 + \cdots \right) = \frac{\beta}{2} \log \tau_0 + \frac{\beta - 1}{2} \tau_0^{-\beta} \log \tau_0 + \cdots \quad \text{as } \tau_0 \to \infty
\]

and

\[
\frac{1}{2} \log \left( \frac{N}{2\lambda} A(\tau_0) \right) = \frac{1}{2} \log \left( \tau_0^{\beta} - \log \tau_0 + \cdots \right) = \frac{\beta}{2} \log \tau_0 - \frac{1}{2} \tau_0^{-\beta} \log \tau_0 + \cdots \quad \text{as } \tau_0 \to \infty.
\]

Therefore

\[
\tau_0 \int_{\lambda \tau_0}^{N \lambda \tau_0} \tau I(\Theta^{-1}(\tau))^2 \, d\tau = \frac{\beta}{2} \tau_0^{1-\beta} \log \tau_0 + \cdots \quad \text{as } \tau_0 \to \infty.
\]

This asymptotic expansion ensures that the expression on the left-hand side tends to 0 as \( \tau_0 \to \infty \). This piece of information, combined with equations (6.82) and (6.83), ensures that \( \nu(\tau_0) \to -\infty \) as \( \tau_0 \to \infty \). Thus, given \( \lambda > 0 \), there exists \( \tau_0 > 0 \) such that \( B \) is a Young function and, simultaneously, \( \nu(\tau_0) < -\lambda - C \).

Estimate (6.81) therefore yields

\[
(6.84) \quad \| \Theta \|_{L^B(s, 1)} \leq \frac{\beta}{2} \left( \log \frac{1}{s} \right)^{1/2} - \lambda - C \quad \text{for } s \in (0, s_0)
\]

for some \( s_0 = s_0(\tau_0) \). Next, owing to Lemma 4.6, we have that \( \Theta(s) s B^{-1}(\frac{1}{s}) \to 0 \) as \( s \to 0^+ \). This fact, coupled with inequality (6.84), implies that

\[
G_{B,C}(s) = \| \Theta \|_{L^B(s, 1)} + \Theta(s) s B^{-1}(\frac{1}{s}) + C \leq \frac{\beta}{2} \left( \log \frac{1}{s} \right)^{1/2} - \lambda,
\]

provided that \( s \) is sufficiently small. Inequality (6.80) is thus established. From inequality (6.80) we infer that

\[
\left[ \theta_\beta G_{B,C}(s) \right]^{1/2} \leq \log \frac{1}{s} - 2\lambda \left( \log \frac{1}{s} \right)^{1/2} + \cdots \quad \text{as } s \to 0^+.
\]
7. Proofs of Theorems 1.1, 1.2 and 1.4

The following lemmas are variants of [18, Lemmas 6.1-6.3]. There, the function $\exp^\beta$ appears in the place of its convex envelope $\text{Exp}^\beta$, which enters Lemmas 7.1-7.3 below. The proofs of the latter only require minor modifications, and will be omitted.

**Lemma 7.1.** Let $\beta > 0$. Assume that $B$ is a Young function satisfying condition (1.13) for some $N \in (0, 1)$. Then there exists a constant $M > 1$ such that

$$\|\phi\|_{L^B(\mathbb{R}^n, \gamma_n)} \leq 1$$

for every function $\phi \in \mathcal{M}(\mathbb{R}^n)$ fulfilling

$$\int_{\mathbb{R}^n} \text{Exp}^\beta(|\phi|) \, d\gamma_n \leq M.$$

**Lemma 7.2.** Let $\beta > 0$. Assume that $B$ is a Young function satisfying condition (1.13) for some $N > 0$. Then there exists a constant $M > 1$ such that inequality (7.2) holds for every function $\phi \in \mathcal{M}(\mathbb{R}^n)$ fulfilling condition (7.1).

**Lemma 7.3.** Let $\beta > 0$ and $M > 1$. There exists a Young function $B$ of the form (1.13) such that inequality (7.1) holds for every function $\phi \in \mathcal{M}(\mathbb{R}^n)$ satisfying (7.2).

We are now in a position to accomplish the proofs of our inequalities for the Ornstein-Uhlenbeck operator.

**Proof of Theorem 1.2.** Let $N > 0$ and let $B$ be a Young function of the form (1.13). The choice $\varphi(t) = 1$ in Corollary 5.3 tells us that

$$\sup_u \int_{\mathbb{R}^n} \exp^\beta(|u|) \, d\gamma_n \leq 2 \int_0^1 \exp^\beta(\theta \mathcal{G}_{B,C}(s)) \, ds,$$

where the supremum is extended over all functions $u \in W_L \exp^L(\mathbb{R}^n, \gamma_n)$ satisfying (1.12) and (1.5), and $C$ is defined by (5.4). Let $\theta_{B}$ be the constant given by (1.8).

If $\beta \in (0, 1)$ and $\theta = \theta_{B}$, then Corollary 4.14 ensures that the integral on the right-hand side of (7.3) converges, since

$$\int_0^1 \left( \log \frac{1}{s} - 2 + 3\beta \log \log \frac{1}{s} \right) \, ds = \int_0^1 \left( \log \frac{1}{s} \right)^{\frac{2+3\beta}{2-2\beta}} \frac{ds}{s} < \infty.$$

If $\beta = 1$ and $\theta = \theta_{1}$, then Corollary 4.14 implies that the integral on the right-hand side of (7.3) converges, inasmuch as

$$\int_0^1 \left( \log \frac{1}{s} - \frac{5}{4} \left( \log \log \frac{1}{s} \right)^2 \right) \, ds = \int_0^1 \exp \left( - \frac{5}{4} \left( \log \log \frac{1}{s} \right)^2 \right) \frac{ds}{s} < \infty.$$

Part (1.i) is thus established.

Assume next that $\beta \in (1, \infty)$ and $\theta < \theta_{B}$. Then Part (2.i) follows via Corollary 4.14, which tells us that the integral on the right-hand side of (7.3) converges, since

$$\int_0^1 \exp \left( \frac{\theta'}{\theta_{B}} \log \frac{1}{s} \right) \, ds = \int_0^1 \frac{s^{\theta'}}{s} \, ds < \infty$$

for every $\theta' \in (\theta, \theta_{B})$.

Parts (1.ii) and (2.iii) follow from Proposition 6.5. Finally, Part (2.ii) is a consequence of Propositions 6.7 and 6.6.
**Proof of Theorem 1.1.** Lemma 7.3 tells us that there exists \( N > 0 \) and a Young function \( B \) of the form (1.13) such that every function \( u \in W_2^2 \) satisfies (1.7) also obeys (1.12). Parts (1.i) and (2.i) thus follow from the respective parts of Theorem 1.2.

Parts (1.ii) and (2.iii) are the subject of Proposition 6.5.

Consider now Part (2.ii). Let \( N \in (0, 1) \). By Theorem 1.2, Part (2.ii), there exists a Young function \( B \) of the form (1.13) which renders inequality (1.11) true. On the other hand, Lemma 7.1 tells us that (2.ii) thus follow from the respective parts of Theorem 1.2.

Notice that the last integral converges, since, by Lemma 4.1 and equation (4.2),

\[
\int_0^1 \exp \left( \frac{\eta}{2} \log \log \frac{1}{s} \right) \, ds < \infty,
\]

if \( \eta \in (\eta, 2) \). This Proves part (i).

In order to establish Part (ii), choose \( f_0 = 1 \) and \( s_0 = 1/2 \) in definition (6.2) of the function \( f_1 \). Let \( v: \mathbb{R}^n \to \mathbb{R} \) be the function given by

\[
v(x) = f_1(2\Phi(x_1)) \quad \text{for} \quad x \in \mathbb{R}^n,
\]

and let \( w: \mathbb{R}^n \to \mathbb{R} \) be the function defined as \( w(x) = -v(-x) \) for \( x \in \mathbb{R}^n \). By Lemma 6.2, we have that \( v, w \in W^{2,2}(\mathbb{R}^n, \gamma_n) \). One can also verify that the function \( u: \mathbb{R}^n \to \mathbb{R} \), defined as

\[
u(x) = \begin{cases} v(x) & \text{for } x_1 \geq 0 \\ w(x) & \text{for } x_1 < 0, \end{cases}
\]

also belongs to \( W^{2,2}(\mathbb{R}^n, \gamma_n) \), and

\[
Lu(x) = -\text{sgn}(x_1) \quad \text{for a.e. } x \in \mathbb{R}^n.
\]

Therefore, the function \( u \) satisfies condition (1.14). Moreover, since \( u \) is odd, \( \text{med}(u) = \text{mv}(u) = 0 \). One has that

\[
\int_{\mathbb{R}^n} \exp(2|u|) \, d\gamma_n = 2 \int_0^1 \exp(2f_1(2s)) \, ds.
\]

We claim that

\[
f_1(2s) = \frac{1}{2} \log \log \frac{1}{s} + C + \cdots \quad \text{as } s \to 0^+,
\]

for some constant \( C > 0 \). Indeed, from the definition of \( f_1 \) one obtains that

\[
\lim_{s \to 0^+} \left( f_1(2s) - \frac{1}{2} \log \log \frac{1}{s} \right) = \lim_{s \to 0^+} \left( \int_s^1 \frac{r}{I(r)^2} \, dr - \int_s^{1/2} \frac{1}{2r \log \frac{1}{r}} \, dr - \frac{1}{2} \log \log 2 \right)
\]

\[
= -\frac{1}{2} \log \log 2 + \int_0^{1/2} \left( \frac{r}{I(r)^2} - \frac{1}{2r \log \frac{1}{r}} \right) \, dr.
\]

Notice that the last integral converges, since, by Lemma 4.1 and equation (4.2),

\[
\frac{r}{I(r)^2} = \frac{1}{2r \log \frac{1}{r}} + \frac{\log \log \frac{1}{r}}{4r \log^2 \frac{1}{r}} + \cdots \quad \text{as } r \to 0^+.
\]
Furthermore, Lemma 4.2 implies that the integral in question is positive. Equation (7.7) hence follows, since \( \frac{1}{2} \log \log 2 < 0 \). Equations (7.5) and (7.6) imply that
\[
\int_{\mathbb{R}^n} \exp \exp(2|u|) \, d\gamma_n = \infty.
\]
This concludes the proof of Part (ii). \( \square \)

### 8. Improved inequalities and existence of maximizers

The key step in our proof of Theorem 1.3 on the existence of extremals in inequalities (1.6) and (1.11), for \( \beta \in (0,1] \) and \( \theta = \frac{2}{\beta} \), is an improved version of these inequalities. The improvement amounts to allowing integrands of the function \( u \) in (1.6) and (1.11) which grow slightly faster than \( \exp^\beta(\frac{2}{\beta}|u|) \) as \( |u| \) tends to \( \infty \). Namely, we show that if \( \varphi : [0, \infty) \to [0, \infty) \) is an increasing function that diverges to \( \infty \) as \( t \to \infty \) with a sufficiently mild growth, then
\[
\sup_{u} \int_{\mathbb{R}^n} \exp^\beta(\frac{2}{\beta}|u|) \varphi(|u|) \, d\gamma_n < \infty,
\]
where the supremum is extended over all functions \( u \in \mathcal{W}_C \exp L^\beta(\mathbb{R}^n, \gamma_n) \) satisfying condition (1.5), and either (1.7) or (1.12). This is the content of the next result, of independent interest, where an explicit condition on the function \( \varphi \) for inequality (8.1) to hold is offered. Let us emphasize that, by contrast, Adams’ inequality (1.9) in Euclidean domains does not admit any enhancement of this kind.

**Theorem 8.1 [Improved integrability].** Let \( \varphi : [0, \infty) \to [0, \infty) \) be a non-decreasing function. Assume that either \( \beta \in (0,1) \) and
\[
\int_{0}^{\infty} t^{-1 - \frac{5\beta^2}{2 - 2\beta}} \varphi(t) \, dt < \infty,
\]
or \( \beta = 1 \) and
\[
\int_{0}^{\infty} \exp \left( -\frac{\beta}{3} + \epsilon (\log t)^2 \right) \varphi(t) \, dt < \infty
\]
for some \( \epsilon > 0 \). Then, inequality (8.1) holds for any constant \( M > 1 \), or for any Young function \( B \) obeying (1.13), according to whether \( u \) is subject to constraint (1.7) or (1.12).

**Proof.** By Lemma 7.3, there exists a Young function \( B \) obeying (1.13) such that condition (1.7) implies (1.12) for any \( u \in \mathcal{W}_C \exp L^\beta(\mathbb{R}^n, \gamma_n) \). Therefore, it suffices to prove inequality (8.1) under the assumption that the supremum is extended over all functions \( u \) subject to constraint (1.12).

Corollary 5.3 yields
\[
\sup_{u} \int_{\mathbb{R}^n} \exp^\beta(\theta_\beta|u|) \varphi(|u|) \, d\gamma_n \leq 2 \int_{0}^{1} \exp^\beta(\theta_\beta \mathcal{G}_{B,C}(s)) \varphi(\mathcal{G}_{B,C}(s)) \, ds,
\]
where the supremum is extended over all functions \( u \in \mathcal{W}_C \exp L^\beta(\mathbb{R}^n, \gamma_n) \) satisfying conditions (1.5) and (1.12). Here, \( \mathcal{G}_{B,C} \) denotes the function defined by (4.68) and \( C \) is the constant given by (5.4). From Lemmas 4.6 and 4.13 we infer that
\[
\mathcal{G}_{B,C}(s) \leq \frac{\beta}{2} \left( \log \frac{1}{s} \right)^{\frac{1}{\beta}} \text{ for } s \text{ near zero}.
\]
Furthermore, Corollary 4.14 implies that, for any \( \delta > 0 \),
\[
\left[ \theta_\beta \mathcal{G}_{B,C}(s) \right]^\beta \leq \log \frac{1}{s} + \begin{cases} \left( \delta - \frac{2 + 3\beta}{2 - 2\beta} \right) \log \log \frac{1}{s} & \text{if } \beta \in (0,1) \\ \left( \delta - \frac{5\beta}{2} \right) \left( \log \log \frac{1}{s} \right)^2 & \text{if } \beta = 1 \end{cases} \text{ for } s \text{ near zero}.
\]
Assume first that \( \beta \in (0,1) \). By equations (8.5) and (8.6), the integral on the right-hand side of (8.4) converges if
\[
\int_{0}^{1} \exp \left( \log \frac{1}{s} + \left( \delta - \frac{2 + 3\beta}{2 - 2\beta} \right) \log \log \frac{1}{s} \right) \varphi \left( \frac{\beta}{2} \left( \log \log \frac{1}{s} \right)^{\frac{1}{\beta}} \right) \, ds < \infty.
\]
As shown by a change of variables, this condition is equivalent to
\[ \int_0^\infty t^{-1-\frac{\alpha^2}{2\alpha \delta} + \beta \delta} \varphi(t) \, dt < \infty. \]

Thanks to assumption (8.2), the latter condition holds with \( \delta = \varepsilon / \beta \).

Assume now that \( \beta = 1 \). Owing to equations (8.5) and (8.6) again, the integral on the right-hand side of (8.4) converges provided that
\[ \int_0^\infty \exp \left( \frac{1}{\delta} \left( \log 2 - \frac{\delta}{4} \right) (\log \log 2)^2 \right) \varphi \left( \frac{1}{\delta} \log 2 \right) \, ds < \infty. \]

After changing variables, this condition turns out to be equivalent to
\[ \int_0^\infty \exp \left( (\log t + \log 2)^2 \right) \varphi(t) \, dt < \infty, \]
which, by assumption (8.3), holds for any \( \delta < \varepsilon \).

Besides Theorem 8.1, a few (special cases) of classical results of functional analysis are needed in the proof of Theorem 1.3. They are recalled below, for the reader’s convenience.

**Theorem A [Equi-integrability (de la Vallée-Poussin)].** Let \((\mathcal{R}, \nu)\) be a probability space. Then a sequence \(\{u_k\} \subseteq L^1(\mathcal{R}, \nu)\) is equi-integrable if and only if there exists a continuous function \(\psi: [0, \infty) \rightarrow [0, \infty)\), satisfying \(\lim_{t \rightarrow \infty} \psi(t)/t = \infty\), such that
\[ \sup_k \int_{\mathcal{R}} \psi(|u_k|) \, d\nu < \infty. \]

**Theorem B [Convergence in \(L^1\) (Vitali)].** Let \((\mathcal{R}, \nu)\) be a probability space, let \(u \in L^1(\mathcal{R}, \nu)\) and let \(\{u_k\}\) be a sequence in \(L^1(\mathcal{R}, \nu)\) such that \(u_k \rightarrow u\) in measure. If the sequence \(\{u_k\}\) is equi-integrable, then \(u_k \rightarrow u\) in \(L^1(\mathcal{R}, \nu)\).

The semicontinuity theorem stated below is a consequence of [30, Theorem 8, Section 1.1].

**Theorem C [Semicontinuity].** Let \(g: \mathbb{R}^n \rightarrow [0, \infty)\) be a convex function. Then the functional defined as
\[ \int_{\mathbb{R}^n} g(|u|) \, d\gamma_n \]
is sequentially lower semicontinuous with respect to the weak*-convergence in \(L^1(\mathbb{R}^n, \gamma_n)\).

The next result is a special case of the Banach-Alaoglu theorem. With this regard, note that the space \(\exp L^\beta(\mathbb{R}^n, \gamma_n)\) is the dual of the separable space \(L(\log L)^{1/2}(\mathbb{R}^n, \gamma_n)\).

**Theorem D [Weak*-compactness in \(\exp L^\beta(\mathbb{R}^n, \gamma_n)\) (Banach-Alaoglu)].** Assume that \(\{u_k\}\) is a bounded sequence in \(\exp L^\beta(\mathbb{R}^n, \gamma_n)\). Then there exist a function \(u \in \exp L^\beta(\mathbb{R}^n, \gamma_n)\) and a subsequence of \(\{u_k\}\), still denoted by \(\{u_k\}\), such that \(u_k \rightarrow u\) in the weak*-topology of \(\exp L^\beta(\mathbb{R}^n, \gamma_n)\).

We conclude our preliminaries by stating a result proved in [19, Lemma 4.1] on the convergence of medians and mean values.

**Lemma 8.2.** Let \(u \in W^{1,1}(\mathbb{R}^n, \gamma_n)\) and let \(\{u_k\}\) be a sequence in \(W^{1,1}(\mathbb{R}^n, \gamma_n)\) such that \(u_k \rightarrow u\) in \(L^1(\mathbb{R}^n, \gamma_n)\). Then, there exists a subsequence of \(\{u_k\}\), still denoted by \(\{u_k\}\), such that
\[ \lim_{k \rightarrow \infty} \text{m}(u_k) = \text{m}(u). \]
Here, \(\text{m}(\cdot)\) stands for either mean value or median.

**Proof of Theorem 1.3.** We treat Parts (i) and (ii) simultaneously, since their proofs only require minor variants. Let \(M > 1\), or let \(B\) be a Young function satisfying equation (1.13). Denote by \(\{u_k\}\) a maximizing sequence for (1.6), or for (1.11), and by \(S\) the supremum in (1.6) or (1.11). Set \(f_k = \mathcal{L}u_k\) for \(k \in \mathbb{N}\). Thus, \(\text{m}(u_k) = 0\),
\[ \int_{\mathbb{R}^n} \exp^\beta(|f_k|) \, d\gamma_n \leq M, \]
(8.7)
or

$$\|f_k\|_{L^B(\mathbb{R}^n, \gamma_n)} \leq 1$$

for \(k \in \mathbb{N}\), and

$$\lim_{k \to \infty} \int_{\mathbb{R}^n} \exp^\beta (\theta_\beta |u_k|) \, d\gamma_n = S,$$

where \(\theta_\beta\) is defined by (1.8). Throughout this proof, indices will be preserved when passing to a subsequence, for simplicity of notation. Owing to Lemma 7.3, condition (8.7) implies (8.8) for some (possibly different) Young function \(B\) fulfilling (1.13). We claim that

$$\int_{\mathbb{R}^n} |\nabla u_k|^2 \, d\gamma_n \leq 2 \int_0^{\frac{1}{2}} \left( \frac{1}{I(s)} \int_0^s (f_k)_+^*(r) \, dr \right)^2 \, ds \quad \text{for } k \in \mathbb{N}. \tag{8.10}$$

Here, \(f_k^*\) denotes the decreasing rearrangement of \(f_k\). Since the operators \(\nabla\) and \(L\) are invariant under additive constants, it suffices to prove our claim under the assumption that \(\text{med}(u_k) = 0\). Set \(\mu_k(t) = \gamma_n(\{x \in \mathbb{R}^n : u_k(x) > t\})\) for \(k \in \mathbb{N}\). Thanks to [17, Proof of Theorem 3.1],

$$- \frac{d}{dt} \int_{\{u_k > t\}} |\nabla u_k|^2 \, d\gamma_n \leq - \frac{\mu_k(t)}{I(\mu_k(t))^2} \left( \int_0^{\mu_k(t)} (f_k)_+^*(r) \, dr \right)^2 \quad \text{for } t > 0, \tag{8.11}$$

for \(k \in \mathbb{N}\). Integrating both sides of inequality (8.11) over \((0, \infty)\) tells us that

$$\int_{\{u_k > 0\}} |\nabla u_k|^2 \, d\gamma_n \leq 2 \int_0^{\frac{1}{2}} \left( \frac{1}{I(s)} \int_0^s (f_k)_+^*(r) \, dr \right)^2 \, ds \quad \text{for } k \in \mathbb{N}. \tag{8.12}$$

Note that the derivation of inequality (8.12) also rests upon the change of variables \(s = \mu_k(t)\) and on the inequality \(\mu_k(0) \leq \frac{1}{2}\). Combining inequality (8.12) with an analogous estimate on the set \(\{u_k < 0\}\), with \((f_k)_- \) replaced by \((f_k)_+ \), yields

$$\int_{\mathbb{R}^n} |\nabla u_k|^2 \, d\gamma_n \leq 2 \int_0^{\frac{1}{2}} \left( \frac{1}{I(s)} \int_0^s (f_k)_+^*(r) \, dr \right)^2 \, ds + 2 \int_0^{\frac{1}{2}} \left( \frac{1}{I(s)} \int_0^s (f_k)_-^*(r) \, dr \right)^2 \, ds \quad \text{for } k \in \mathbb{N}. \tag{8.13}$$

Inequality (8.10) follows from (8.13).

By Hölder’s inequality in the form (3.4) and equation (3.5),

$$\int_0^s f_k^*(r) \, dr \leq \|f_k\|_{L^B(0,1)} \|\chi(0,s)\|_{L^{\tilde{B}}(0,1)} = \|f_k\|_{L^B(\mathbb{R}^n, \gamma_n)} s B^{-1}(\frac{1}{2}) \quad \text{for } s \in (0, 1). \tag{8.14}$$

Coupling inequalities (8.10) and (8.14) yields

$$\|\nabla u_k\|_{L^2(\mathbb{R}^n, \gamma_n)} \leq \|f_k\|_{L^B(\mathbb{R}^n, \gamma_n)} \left( 2 \int_0^{\frac{1}{2}} \left( \frac{s B^{-1}(\frac{1}{2})}{I(s)} \right)^2 \, ds \right)^{\frac{1}{2}} \quad \text{for } k \in \mathbb{N}. \tag{8.15}$$

Owing to equations (1.13) and (4.6), the integral on the right-hand side of (8.15) converges. Hence, there exists a constant \(C\) such that

$$\|\nabla u_k\|_{L^2(\mathbb{R}^n, \gamma_n)} \leq C \quad \text{for } k \in \mathbb{N}. \tag{8.16}$$

Since the embedding \(W^{1,2}(\mathbb{R}^n, \gamma_n) \to L^1(\mathbb{R}^n, \gamma_n)\) is compact - see e.g. [47, Theorem 7.3] - there exist a function \(u \in L^1(\mathbb{R}^n, \gamma_n)\) and a subsequence of \(\{u_k\}\) such that \(u_k \to u\) in \(L^1(\mathbb{R}^n, \gamma_n)\) and

$$u_k(x) \to u(x) \quad \text{for a.e. } x \in \mathbb{R}^n. \tag{8.17}$$

Lemma 8.2 ensures that, on taking a subsequence, if necessary, \(m(u_k) \to m(u)\), whence \(m(u) = 0\). Next, by inequality (8.16) and the reflexivity of the space \(L^2(\mathbb{R}^n, \gamma_n)\), there exist a subsequence of \(\{u_k\}\) and a function \(V : \mathbb{R}^n \to \mathbb{R}^n\) such that \(V \in L^2(\mathbb{R}^n, \gamma_n)\) and \(\nabla u_k \to V\) weakly in \(L^2(\mathbb{R}^n, \gamma_n)\). Hence, \(u \in W^{1,2}(\mathbb{R}^n, \gamma_n)\) and \(\nabla u = V\). Also, by inequality (8.8) and Theorem D, there exist a function \(f \in \exp L^\beta(\mathbb{R}^n, \gamma_n)\) and a subsequence of \(\{f_k\}\) such that \(f_k \to f\) in the weak*-topology of \(\exp L^\beta(\mathbb{R}^n, \gamma_n)\). In
particular, $f_k \rightharpoonup f$ weakly in $L^2(\mathbb{R}^n, \gamma_n)$. By equation (3.10), given any function $v \in W^{1,2}(\mathbb{R}^n, \gamma_n)$, one has that
\begin{equation}
\int_{\mathbb{R}^n} \nabla u_k \cdot \nabla v \, d\gamma_n = - \int_{\mathbb{R}^n} u_k v \, d\gamma_n \quad \text{for } k \in \mathbb{N}.
\end{equation}

Passing to the limit as $k \to \infty$ in equation (8.18) tells us that $\mathcal{L} u = f$. Furthermore, inasmuch as $f_k \rightharpoonup f$ weakly* in $\exp L^\beta(\mathbb{R}^n, \gamma_n)$, we have that $f_k \rightharpoonup \mathcal{L} u$ in the weak*-topology of $L^1(\mathbb{R}^n, \gamma_n)$. Therefore, if constraint (8.7) is in force, then by Theorem C
\begin{equation}
\int_{\mathbb{R}^n} \exp^\beta(|\mathcal{L} u|) \, d\gamma_n \leq \liminf_{k \to \infty} \int_{\mathbb{R}^n} \exp^\beta(|f_k|) \, d\gamma_n \leq M \quad \text{for } k \in \mathbb{N}.
\end{equation}

On the other hand, under constraint (8.8), by the weak* lower semicontinuity of the norm in $L^B(\mathbb{R}^n, \gamma_n)$, (8.8),
\begin{equation}
\|\mathcal{L} u\|_{L^B(\mathbb{R}^n, \gamma_n)} \leq \liminf_{k \to \infty} \|f_k\|_{L^B(\mathbb{R}^n, \gamma_n)} \leq 1.
\end{equation}

We conclude by showing that
\begin{equation}
\int_{\mathbb{R}^n} \exp^\beta(\theta_\beta |u|) \, d\gamma_n = S.
\end{equation}

Thanks to Theorem 8.1, there exists a continuous increasing function $\varphi: [0, \infty) \to [0, \infty)$ such that $\lim_{t \to \infty} \varphi(t) = \infty$ and
\begin{equation}
\int_{\mathbb{R}^n} \exp^\beta(\theta_\beta |u_k|) \varphi(|u_k|) \, d\gamma_n \leq C \quad \text{for } k \in \mathbb{N},
\end{equation}

for some constant $C$. Consider the function $\psi: [0, \infty) \to [0, \infty)$ defined as
\begin{equation}
\psi(t) = t \varphi \left( \frac{1}{\theta_\beta} (\log t)^{\frac{1}{2}} \right) \quad \text{for } t \geq 1
\end{equation}

and $\psi(t) = 0$ for $t \in [0, 1)$. Then, $\lim_{t \to \infty} \psi(t)/t = \infty$. Moreover, by inequality (8.20),
\begin{equation}
\sup_{k \in \mathbb{N}} \int_{\mathbb{R}^n} \psi \left( \exp^\beta(\theta_\beta |u_k|) \right) \, d\gamma_n = \sup_{k \in \mathbb{N}} \int_{\mathbb{R}^n} \exp^\beta(\theta_\beta |u_k|) \varphi(|u_k|) \, d\gamma_n \leq C.
\end{equation}

As a consequence of Theorem A, the sequence of functions $\{\exp^\beta(\theta_\beta |u_k|)\}$ is equi-integrable in $L^1(\mathbb{R}^n, \gamma_n)$. Also, by property (8.17), this sequence converges a.e. in $\mathbb{R}^n$, and hence in measure, to the function $\exp^\beta(\theta_\beta |u|)$. Thus, from equation (8.9) and of Theorem B, one infers that
\begin{equation}
S = \lim_{k \to \infty} \int_{\mathbb{R}^n} \exp^\beta(\theta_\beta |u_k|) \, d\gamma_n = \int_{\mathbb{R}^n} \exp^\beta(\theta_\beta |u|) \, d\gamma_n,
\end{equation}

whence equation (8.19) follows. Altogether, we have shown that $u$ is a maximizer for (1.6) or (1.11). $\square$

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