GENERAL DECAY RATE OF SOLUTION FOR NONLINEAR LOVE-EQUATION WITH INFINITE MEMORY

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Abstract. In this paper, we consider a nonlinear Love-equation with infinite memory. By certain properties of convex functions, we use an appropriate Lyapunov functional to find a very general rate of decay for energy (2.3).

1. Introduction

Denote \( y = y(x,t), y' = y_t = \frac{\partial y}{\partial t}(x,t), y'' = y_{tt} = \frac{\partial^2 y}{\partial t^2}(x,t), y_x = \frac{\partial y}{\partial x}(x,t), y_{xx} = \frac{\partial^2 y}{\partial x^2}(x,t), x \in \Omega = (0,L), L > 0, t > 0. \) In this article, we consider a nonlinear Love-equation in the form

\[
y'' - \left(y_x + y'_x + y''_x\right)x + \int_{-\infty}^{t} \mu(t-s)y_{xx}(s)ds = F[y] - \left(F[y]\right)_x + f(x,t), \quad x \in \Omega, \quad 0 < t < T, \tag{1.1}
\]

where

\[
F[y] = F\left(x,t,y,y_x,y'_x\right) \in C^1\left([0,1] \times \mathbb{R}^+ \times \mathbb{R}^4\right), \tag{1.2}
\]

The given functions \( \mu, f \) are specified later. With \( F = F(x,t,y_1,\ldots,y_4) \), we put \( D_1F = \frac{\partial F}{\partial y_1}, D_2F = \frac{\partial F}{\partial y_2}, D_{i+2}F = \frac{\partial F}{\partial y_{3i}}, \) with \( i = 1,\ldots,4. \)

Equation (1.1) satisfies the homogeneous Dirichlet boundary conditions:

\[
y(0,t) = y(L,t) = 0, \quad t > 0, \tag{1.3}
\]

and the following initial conditions

\[
y(x,-t) = y_0(x,t), \quad y'(x,0) = y_1(x). \tag{1.4}
\]

To deal with a wave equation with infinite history, we assume that the kernel function \( \mu \) satisfies the following hypothesis:

(Hyp1:) \( \mu : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is a non-increasing \( C^1 \) function such that

\[
1 - \int_0^\infty \mu(s)ds = l > 0, \quad \mu(0) > 0. \tag{1.5}
\]

and there exists an increasing strictly convex function \( H : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) of \( C^1(\mathbb{R}^+) \) satisfying

\[
H(0) = H'(0) = 0, \quad \text{and} \quad \lim_{t \to \infty} H'(t) = \infty. \tag{1.6}
\]
such that
\[ \int_0^\infty \frac{\mu(s)}{H^{-1}(-\mu'(s))} ds + \sup_{s \in \mathbb{R}^+} \frac{\mu(s)}{H^{-1}(-\mu'(s))} ds < \infty \]  
(1.7)

(Hyp2:) \( y_0(0), y_1 \in H^1_0 \cap H^2 \);

We need the following assumptions on source forces:

(Hyp3:) \( f \in H^1((0, 1) \times (0, T)) \);
(Hyp4:) \( F \in C^1([0, 1] \times [0, T] \times \mathbb{R}^4) \), such that \( F(0, t, 0, y_2, 0, y_4) = F(1, t, 0, y_2, 0, y_4) = 0 \) for all \( t \in [0, T] \), \( y_2, y_4 \in \mathbb{R} \).

This kind of systems appears in the models of nonlinear Love waves or Love type waves. It is a generalization of a model introduced by [6], [22] and [23]. The original equation is

\[ u'' - \frac{E}{\rho} u_{xx} - 2\mu^2 w^2 u''_{xx} = 0, \]  
(1.8)

This type of problem describes the vertical oscillations of a rod, was established from Euler’s variational equation of an energy functional associated with (1.8). A classical solution of problem (1.8) with null boundary conditions and asymptotic behavior are obtained by using the Fourier method and method of small parameter. The results are very interesting in the application point of view and, as for as I know, new that is there is no results for equations of Love waves or Love type waves with the presence of finite/infinite memory term (4, 6, 18, 19, 20, 21, 22, 23, 26, 27, ...).

Without infinite memory term, when \( \lambda = 0 \) in (1.1), Triet and his collaborator in [21] considered an IBVP for a nonlinear Kirchhoff-Love equation

\[ u'' - \frac{\partial}{\partial x} \left( u_x + \lambda_1 u_{xt} + u_{xxt} \right) + \lambda u_t = F(x, t, u, u_x, u_t, u_{xt}) \]  
\[ - \frac{\partial}{\partial x} \left[ G(x, t, u, u_x, u_{xt}) \right] + f(x, t), \quad x \in \Omega = (0, 1), \ 0 < t < T, \]  
\[ u(0, t) = u(1, t) = 0, \]  
(1.10)
\[ u(x, 0) = \tilde{u}_0(x), \quad u_t(x, 0) = \tilde{u}_1(x), \]  
(1.11)

where \( \lambda > 0, \lambda_1 > 0 \) are constants and \( \tilde{u}_0, \tilde{u}_1 \in H^1_0 \cap H^2 \); \( f, F \) and \( G \) are given functions. First, under suitable conditions, the existence of a unique local weak solution has been proved and a blow up result for solutions with negative initial energy is also established. A sufficient condition guaranteeing the global existence and exponential decay of weak solutions is given in the last section. This results will be improved in [26], [27] to the Kirchhoff type.

The existence/nonexistence, exponential decay of solutions and blow up results for viscoelastic wave equations with finite history, have been extensively studied and many results have been obtained by many authors (see [2], [3], [8], [9], [12], [14], [15], [31], [33] ...).

Concerning problems with infinite history, we mention the work [8] in which considered the following semi-linear hyperbolic equation, in a bounded domain of \( \mathbb{R}^3 \),

\[ u'' - K(0) \Delta u - \int_0^\infty K'(s) \Delta u(t - s) ds + g(u) = f, \]
with $K(0), K(\infty) > 0, K' \leq 0$ and gave the existence of global attractors for the problem. Next in [28], the authors considered a fourth-order suspension bridge equation with nonlinear damping term and source term. The authors found necessary and sufficient condition for global existence and energy decay results without considering the relation between $m$ and $p$. Moreover, when $p > m$, they gave sufficient condition for finite time blow-up of solutions. The lower bound of the blow-up time is also established.

Recently, in [24], the authors studied a three-dimensional (3D) visco-elastic wave equation with nonlinear damping, supercritical sources and prescribed past history, $t \leq 0$ in

$$u'' - k(0)\Delta u - \int_0^\infty k'(s)\Delta u(t-s)ds + |u'|^{m-1}u' = |u|^{p-1}u,$$

where the relaxation function $k$ is monotone decreasing with $k(\infty) = 1, m \geq 1, 1 \leq p < 6$. When the source is stronger than dissipations, i.e. $p > \max\{m, \sqrt{k(0)}\}$, they obtained some finite time blow-up results with positive initial energy. In particular, they obtained the existence of certain solutions which blow up in finite time for initial data at arbitrary energy level. (see [13], [16], [17], [23], [25], . . .).

In addition to the existence results in Theorem 2.3 and Theorem 2.8 obtained by a new combined methods, our decay rate, which is very general, obtained in the fourth section, Theorem 3.1 extend that obtained in [31] and [19], where they established a general decay rate result for relaxation functions satisfying

$$g'(t) \leq -H(g(t)), t \geq 0, \quad H(0) = 0$$

for a positive function $H \in C^1(\mathbb{R}^+) \text{ and } H$ is linear or strictly increasing and strictly convex $C^2$ function on $(0, r], 1 > r$. This improves the conditions introduced by [2] on the relaxation functions

$$g'(t) \leq -\chi(g(t)), \quad \chi(0) = \chi'(0) = 0$$

where $\chi$ is a non-negative function, strictly increasing and strictly convex on $(0, k_0], k_0 > 0$. They required that

$$\int_0^{k_0} \frac{dx}{\chi(x)} = +\infty, \quad \int_0^{k_0} \frac{x dx}{\chi(x)} < 1, \quad \lim_{s \to 0^+} \inf \frac{\chi(s)/s}{\chi'(s)} > \frac{1}{2}$$

and proved a decay result for the energy in a bounded domain. In addition to these assumptions, if

$$\lim_{s \to 0^+} \sup \frac{\chi(s)/s}{\chi'(s)} < 1,$$

then, in this case, an explicit rate of decay is given.

2. The existence of solution

We define the weak solution to of (1.1)–(1.4) as follows.

**Definition 2.1.** A function $y$ is said to be a weak solution of (1.1)–(1.4) on $[0, T]$ if

$$y, y', y'' \in L^\infty(0, T; H_0^1 \cap H^2),$$
such that \( y \) satisfies the variational equation
\[
\int_{\Omega} y''w \, dx + \int_{\Omega} (y_x + y'_x + y''_x)w_x \, dx \\
- \int_{\Omega} \int_{0}^{\infty} \mu(s)y_x(t-s) \, dw \, dx \\
= \int_{\Omega} fw \, dx + \int_{\Omega} F[y]w \, dx + \int_{\Omega} F[y_x]w_x \, dx,
\]
for all test function \( w \in H^1_0 \), for almost all \( t \in (0, T) \).

The following famous and widely used technical lemma will play an important role in the sequel.

**Lemma 2.2.** ([22]) For any \( v \in C^1 (0, T, H^1_0) \) we have
\[
\int_{\Omega} \int_{0}^{\infty} \mu(s)v_{xx}(t-s)v'(t) \, ds \, dx \\
= \frac{1}{2} \frac{d}{dt} \int_{\Omega} |v(t) - v(t)|^2 \, dx \\
- \frac{1}{2} \int_{\Omega} \mu'(s) |v(t) - v(t)|^2 \, dx.
\]

Now, we state the existence of a local solution for (1.1)–(1.4).

**Theorem 2.3.** ([32], Theorem 2.3) Let \( y_0(0), y_1 \in H^1_0 \cap H^2 \) be given. Assume that (Hyp1)–(Hyp4) hold. Then Problem (1.1)–(1.4) has a unique local solution \( y \) and \( y, y', y'' \in L^\infty (0, T^*; H^1_0 \cap H^2) \),

for some \( T^* > 0 \) small enough.

Here, we consider problem (1.1) with \( p \geq 2 \) and \( f \equiv 0 \) with the boundary conditions (1.3) and the initial conditions (1.4). We introduce the energy functional \( E(t) \) associated with our problem
\[
E(t) = \frac{1}{2} \int_{\Omega} |y|^2 \, dx + \frac{1}{2} \int_{\Omega} |y_x|^2 \, dx + J(t),
\]
where
\[
J(t) = \int_{\Omega} \left( 1 - \int_{0}^{\infty} \mu(s) \, ds \right) |y_x|^2 \, dx \\
+ \int_{\Omega} \int_{0}^{\infty} \mu(s) |y_x(t) - y_x(t-s)|^2 \, ds \, dx \\
+ \frac{1}{p} \int_{\Omega} |y_x|^p \, dx - \frac{1}{p} \int_{\Omega} |y|^p \, dx
\]

Now, we introduce the stable set as follows (see [29], [30])
\[
W = \{ y \in H^1_0 \cap H^2 : I(t) > 0, J(t) < d \} \cup \{0\}
\]
where
\[
I(t) = \int_{\Omega} \left( 1 - \int_{0}^{\infty} \mu(s) \, ds \right) |y_x|^2 \, dx \\
+ \int_{\Omega} \int_{0}^{\infty} \mu(s) |y_x(t) - y_x(t-s)|^2 \, ds \, dx
\]
\[ + \int_{\Omega} |y_x|^p dx - \int_{0}^{L} |y|^p dx \]  

We notice that the mountain pass level \( d \) given in (2.5) defined by

\[ d = \inf \{ \sup_{y \in H^1_0 \cap H^2 \setminus \{0\}} J(\nu y) \} \]  

Also, by introducing the so called "Nehari manifold"

\[ N = \{ y \in H^1_0 \cap H^2 \setminus \{0\} : I(t) = 0 \} \]

It is readily seen that the potential depth \( d \) is also characterized by

\[ d = \inf_{y \in N} J(t) \]  

This characterization of \( d \) shows that

\[ \text{dist}(0, N) = \min_{y \in N} \|y\|_{H^1_0 \cap H^2} \]  

It is note hard to see this Lemma.

**Lemma 2.4.** Suppose that (Hyp1) holds. Let \( y \) be solution of our equation. Then the energy functional (2.3) is a non-increasing function, i.e., for all \( t \geq 0, \nu > 0 \),

\[ \frac{d}{dt} E(t) \leq -\int_{\Omega} |y_x|^2 dx + \frac{1}{2} \int_{0}^{\infty} \int_{0}^{\infty} \mu'(s) |y_x(t) - y_x(t-s)|^2 dsdx \]

**Proof.** Multiplying (1.1) with \( p \geq 2 \) and \( f \equiv 0 \) by \( y'(x, t) \) and integrating over \( \Omega \), using Lemma 2.2 we obtain th result. \[ \square \]

As in [32], we will prove the invariance of the set \( W \). That is if for some \( t_0 > 0 \) if \( y(t_0) \in W \), then \( y(t) \in W, \forall t \geq t_0 \), beginning by the existence of the potential depth in the next Lemma.

**Lemma 2.5** ([32], Lemma 3.2). \( d \) is positive constant.

**Lemma 2.6** ([32], Lemma 3.3). \( W \) is a bounded neighborhood of 0 in \( H^1_0 \cap H^2 \).

Now, we will show that our local solution \( y \) is global in time, for this purpose it suffices to prove that the norm of the solution is bounded, independently of \( t \), this is equivalent to prove the following theorem.

**Theorem 2.7** ([32], Theorem 3.4). Suppose that (Hyp1) and

\[ C_{p}(1-p) \left( \frac{2p}{p-2} E(0) \right)^{(p-2)} < l. \]  

hold, where \( C \) is the best Poincare’s constant. If \( y_0(0) \in W, y_1 \in H^1_0 \), then the solution \( y \in W, \forall t \geq 0 \).

The next Theorem shows that our local solution is global in time.

**Theorem 2.8.** Suppose that (Hyp1), \( p \geq 2 \) and (2.11) hold. If \( y_0(0) \in W, y_1 \in H^1_0 \). Then the local solution \( y \) is global in time such that \( y \in G_T \) where

\[ G_T = \left\{ y : \begin{array}{l} y \in L^\infty (\mathbb{R}^+; H^1_0 \cap H^2), \vspace{0.1cm} \\ y' \in L^\infty (\mathbb{R}^+; H^1_0) \end{array} \right\}. \]  

(2.12)
Proof. Now, it suffices to show that the following norm

\[ \int_\Omega |y'|^2 dx + \int_\Omega |y_x|^2 dx, \tag{2.13} \]

is bounded independently of \( t \).

To achieve this, we use (2.3), (2.4) and (2.10) to get

\[
E(0) \geq E(t) = J(t) + \frac{1}{2} \int_\Omega |y'|^2 dx + \frac{1}{2} \int_\Omega |y_x|^2 dx + \int_\Omega y_x y' x dx \geq (p - 2) \left[ \frac{1}{2} \int_\Omega |y'|^2 dx + \frac{1}{p} I(t) \right] + \frac{1}{2} \int_\Omega |y_x|^2 dx + \frac{1}{2} \int_\Omega |y'|^2 dx \]

since \( I(t) \) and \( \int_0^\infty \mu(s)|y_x(t) - y_x(t-s)|^2 ds dx \) are positive, hence

\[
(\frac{(p-2)}{2p}) \int_\Omega |y_x|^2 dx + \frac{1}{2} \int_\Omega |y'|^2 dx \leq CE(0),
\]

where \( C \) is a positive constant depending only on \( p \) and \( l \). This completes the proof. \( \square \)

3. General decay rate

Theorem 3.1. Suppose that (Hyp1), \( p \geq 2 \) and \( (2.11) \) hold. If \( y_0(0) \in H_0^1 \cap H^2, y_1 \in H_0^1 \), Then the energy function (2.3) satisfies

\[
E(t) \leq \kappa_1 H_1^{-1}(\kappa t + \kappa_0), \quad \forall t \geq 0, \tag{3.1}
\]

where

\[
H_1(\tau) = \int_1^\tau (sH'(s))^{-1} ds, \quad \kappa_0, \kappa_1, \kappa > 0. \tag{3.2}
\]

In order to prove the main Theorem 3.1, we need to introduce a several Lemmas.

To this end, let us introduce the functionals

\[
\varphi(t) = \int_\Omega yy' dx + \frac{1}{2} \int_\Omega |y_x|^2 dx + \int_\Omega y_x y'_x dx \tag{3.3}
\]

and

\[
\xi(t) = -\int_\Omega y' \int_0^\infty \mu(s)[y(t) - y(t-s)] ds dx - \int_\Omega y'_x \int_0^\infty \mu(s)[y_x(t) - y_x(t-s)] ds dx dt \tag{3.4}
\]

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Lemma 3.2. Assume that (Hyp1), $p \geq 2$ and \( (2.11) \) hold. Then the functional
\[ \varphi(t) \] introduced in (3.5) satisfies, along the solution, the estimate
\[ \varphi'(t) \leq \int_{\Omega} y'^2 dx + \int_{\Omega} y_x'^2 dx \]
\[ - \left[ \frac{1}{2} - \left( \frac{2p}{(p-2)} \right)^{p-2} E(0) \right] \int_{\Omega} |y_x|^2 dx + \frac{(1-l)}{2l} \int_0^\infty \int_{\Omega} \mu(s) |y_x(t-s) - y_x|^2 ds dx \]

Proof.
\[ \varphi'(t) = \int_{\Omega} y'^2 dx + \int_{\Omega} y_x'^2 dx - \int_{\Omega} y^2 dx + \int_{\Omega} |y|^p dx - \int_{\Omega} |y_x|^p dx \]
\[ + \int_{\Omega} y_x \int_0^\infty \mu(s) y_x(t-s) ds dx \]

The next term can be treated as follows
\[ \int_{\Omega} y_x \int_0^\infty \mu(s) y_x(t-s) ds dx \leq \frac{1}{2} \int_{\Omega} y_x'^2 dx + \frac{1}{2} \int_{\Omega} \left( \int_0^\infty \mu(s) y_x(t-s) ds \right)^2 dx \]
\[ \leq \frac{1}{2} \int_{\Omega} y_x'^2 dx + \frac{1}{2} \int_{\Omega} \left( \int_0^\infty \mu(s) |y_x(t-s) - y_x|^2 ds + |y_x|^2 ds \right) dx \]

Thanks to Cauchy-Schwarz inequality, Young’s inequality to obtain, for some $\nu > 0$,
\[ \int_{\Omega} \left( \int_0^\infty \mu(s) |y_x(t-s) - y_x|^2 ds + |y_x|^2 ds \right) dx \]
\[ \leq \int_{\Omega} \left( \int_0^\infty \mu(s)|y_x(t-s) - y_x|^2 ds \right) dx \]
\[ + 2 \int_{\Omega} \left( \int_0^\infty \mu(s)|y_x(t-s) - y_x|^2 ds \right) \left( \int_0^\infty \mu(s)|y_x|^2 ds \right) dx \]
\[ \leq (1 + \frac{1}{\nu}) \int_{\Omega} \left( \int_0^\infty \mu(s)|y_x(t-s) - y_x|^2 ds \right) dx \]
\[ + (1 + \nu) \int_{\Omega} \left( \int_0^\infty \mu(s) ds |y_x|^2 \right) dx \]
\[ \leq \left( 1 + \frac{1}{\nu} \right) (1-l) \int_{\Omega} \int_0^\infty \mu(s)|y_x(t-s) - y_x|^2 ds dx + (1 + \nu)(1-l)^2 \int_{\Omega} |y_x|^2 dx. \]

Then
\[ \varphi'(t) \leq \int_{\Omega} y'^2 dx + \int_{\Omega} y_x'^2 dx \]
\[ + \left[ \frac{1}{2} + \frac{1}{2} (1 + \nu)(1-l)^2 - 1 \right] \int_{\Omega} |y_x|^2 dx \]
\[ + \frac{1}{2} \left( 1 + \frac{1}{\nu} \right) (1-l) \int_{\Omega} \int_0^\infty \mu(s)|y_x(t-s) - y_x|^2 ds dx \]
\[ + \int_{\Omega} |y|^p dx. \] (3.5)

By the continuous embedding for $p \geq 2$, we have
\[ \int_{\Omega} |y|^p dx \leq c^p \left( \int_{\Omega} |y_x|^2 dx \right)^{p/2} \]
\[ \leq c^p \left( \int_{\Omega} |y_x|^2 dx \right)^{(p-2)/2} \int_{\Omega} |y_x|^2 dx \]
where

\[ \nu > \frac{b}{4} \]

and for all \( \nu > 0 \)

\[ b = \frac{1 - l}{4 \nu} + (2 \nu + \frac{1}{4 \nu})(1 - l) + 2 \nu (1 - l)^2 \left( \frac{8}{1 - l} E(0) + 2 m_0^2 \right)^{(p-2)/2} > 0. \]

**Proof.** We have

\[
\xi'(t) = \int_0^\infty \int_0^t \mu(s) [y_x(s) - y_x(t-s)] ds dx
- \int_0^\infty \mu(s) y_x(t-s) ds \int_0^t \mu(s) [y_x(t) - y_x(t-s)] ds dx
- \int_\Omega |y|^{p-2} y \int_0^\infty \mu(s) [u(t) - u(t-s)] ds dx
+ \int_\Omega |y_x|^{p-2} y \int_0^\infty \mu(s) [y_x(t) - y_x(t-s)] ds dx
- \int_\Omega y' \int_0^\infty \mu'(s) [y(t) - y(t-s)] ds dx - (1 - l) \int_\Omega y_x^2 dx.
\]

For \( \nu > 0 \), we have

\[
\int_\Omega y_x \int_0^\infty \mu(s) [y_x(t) - y_x(t-s)] ds dx
\leq \nu \int_\Omega |y_x|^2 dx + \frac{1 - l}{4 \nu} \int_\Omega \int_0^\infty \mu(s) [y_x(t) - y_x(t-s)]^2 ds dx,
\]
and
\[
\int_0^\infty \mu(s) y_x(t-s) ds \int_0^\infty \mu(s) |y_x(t) - y_x(t-s)| ds dx
\]

\[
\leq 2(1-l)\nu^2 \int_\Omega |y_x|^2 dx + (2\nu + \frac{1}{4\nu})(1-l) \int_\Omega \int_0^\infty \mu(s) |y_x(t) - y_x(t-s)|^2 ds dx,
\]

we have the estimate
\[
\int_\Omega |y(t) - y(t-s)|^2 dx
\]

\[
\leq 2 \int_\Omega |y_x(t)|^2 dx + 2 \int_\Omega |y_x(t-s)|^2 dx
\]

\[
\leq 4 \sup_{s>0} \int_\Omega |y_x(t)|^2 dx + 2 \sup_{\tau < 0} \int_\Omega |y_x(\tau)|^2 dx
\]

\[
\leq \sup_{s>0} \int_\Omega |y_x(t)|^2 dx + 2 \sup_{\tau < 0} \int_\Omega |y_0(t)|^2 dx
\]

\[
\leq \frac{8}{1-t} E(0) + 2m_0^2,
\]

(3.7)

where \(\int_\Omega |y_0(t)|^2 dx \leq m_0, m_0 \geq 0, \quad \forall \tau > 0. \) Since \(p \geq 2,\) we have by using (3.6) and the previous estimate

\[
\int_\Omega |y|^{p-2} y \int_0^\infty \mu(s) |y(t) - y(t-s)| ds dx
\]

\[
\leq \nu \int_\Omega \left( \int_0^\infty \mu(s) |y(t) - y(t-s)| ds \right)^p dx + c(\nu) \int_\Omega |y|^p dx
\]

\[
\leq (1-l)^{p-1} \int_0^\infty \mu(s) |y(t) - y(t-s)|^p ds dx + c(\nu) \int_\Omega |y|^p dx
\]

\[
\leq (1-l)^{p-1} c \left( \int_0^\infty \mu(s) \left( \int_\Omega |y_x(t) - y_x(t-s)|^2 dx \right)^{p/2} ds \right) + c(\nu) \int_\Omega |y|^p dx
\]

\[
\leq (1-l)^{p-1} c \left( \int_0^\infty \mu(s) \left( \int_\Omega |y_x(t) - y_x(t-s)|^2 dx \right)^{p/2} ds \right) + c(\nu) \left( \frac{2p}{p-2} E(0) \right)^{(p-2)/2} \int_\Omega |y_x|^2 dx.
\]

Similarly to estimate

\[
\int_\Omega |y_x|^{p-2} y_x \int_0^\infty \mu(s) |y_x(t) - y_x(t-s)| ds dx
\]

\[
\leq \nu \int_\Omega \left( \int_0^\infty \mu(s) |y_x(t) - y_x(t-s)| ds \right)^p dx + c(\nu) \int_\Omega |y_x|^p dx
\]

\[
\leq (1-l)^{p-1} \int_0^\infty \mu(s) |y_x(t) - y_x(t-s)|^p ds dx + c(\nu) \int_\Omega |y_x|^p dx
\]

\[
\leq (1-l)^{p-1} c \left( \int_0^\infty \mu(s) \left( \int_\Omega |y_x(t) - y_x(t-s)|^2 dx \right)^{p/2} ds \right) + c(\nu) \int_\Omega |y_x|^p dx
\]
A combination of all estimates gives
\[
\nu(1-l)^{p-1}c\left(\frac{8}{1-l}E(0) + 2m_0^2\right)^{(p-2)/2} \int_0^\infty \mu(s) \int_0^\infty |y_x(t) - y_x(t-s)|^2 \, dx
\]
\[+ c(\nu) \int_\Omega |y_x|^2 \, dx.\]

The next term estimated as
\[- \int_\Omega y' \int_0^\infty \mu'(s)|y(t) - y(t-s)| \, ds \, dx \leq \nu \int_\Omega |y'|^2 \, dx + \frac{c(\mu(0)}{4\nu} \int_\Omega \int_0^\infty (-\mu'(s))|y(t) - y(t-s)|^2 \, ds \, dx.
\]
A combination of all estimates gives
\[
\xi'(t) \leq \nu \int_\Omega |y_x|^2 \, dx + \frac{1-l}{4\nu} \int_\Omega \int_0^\infty \mu(s)|y_x(t) - y_x(t-s)|^2 \, ds \, dx
\]
\[+ 2\nu(1-l)^2 \int_\Omega |y_x|^2 \, dx + \frac{c(\mu(0)}{4\nu} \int_\Omega \int_0^\infty (-\mu'(s))|y(t) - y(t-s)|^2 \, ds \, dx.
\]
Then, for \(\nu < (1-l)\)
\[
\xi'(t) \leq -a \int_\Omega |y_x|^2 \, dx - (1-l - \nu) \int_\Omega |y'|^2 \, dx
\]
\[+ b \int_\Omega \int_0^\infty \mu(s)|y_x(t) - y_x(t-s)|^2 \, ds \, dx
\]
\[+ \frac{c(\mu(0)}{4\nu} \int_\Omega \int_0^\infty (-\mu'(s))|y(t) - y(t-s)|^2 \, ds \, dx.
\]
where by (2.11), we have
\[
a = c(\nu)\left(1 + 2(1-l)^2 - \left(\frac{2p}{(p-2)}E(0)\right)^{(p-2)/2}\right) > 0
\]
and for all \(\nu > 0\), we have
\[
b = \frac{1-l}{4\nu} + \frac{2(1-l)}{4\nu} + \frac{2\nu(1-l)^{p-1}c\left(\frac{8}{1-l}E(0) + 2m_0^2\right)^{(p-2)/2}}{4\nu} > 0.
\]
Let define the functional
\[
L(t) = \varepsilon_1 E(t) + \varphi(t) + \varepsilon_2 \xi(t), \quad \varepsilon_1, \varepsilon_2 > 0
\]
We need the next lemma, which means that there is equivalent between the Lyapunov and energy functions

\[\square\]
Lemma 3.4. For $\varepsilon_1, \varepsilon_2 > 1$, we have

$$L \sim E,$$

(3.9)

Proof. By (3.8) we have

$$|L(t) - \xi_1 E(t)| \leq |\varphi(t)| + \varepsilon_2 |\xi_1(t)|$$

$$\leq \int_{\Omega} |yy'| dx + \frac{1}{2} \left( \int_{\Omega} |y|^2 dx \right)^{1/2} \left( \int_{\Omega} |y'|^2 dx \right)^{1/2}$$

$$\leq \frac{1}{2} \left( \int_{\Omega} |y|^2 dx \right) + \frac{1}{2} \left( \int_{\Omega} |y'|^2 dx \right)$$

$$\leq \frac{c}{2} \left( \int_{\Omega} |y_x|^2 dx \right) + \frac{1}{2} \left( \int_{\Omega} |y'|^2 dx \right)$$

Similarly

$$\int_{\Omega} |y_x y_x'| dx \leq \frac{1}{2} \left( \int_{\Omega} |y_x|^2 dx \right) + \frac{1}{2} \left( \int_{\Omega} |y_x'|^2 dx \right)$$

and

$$\int_{\Omega} |y'_x \int_0^\infty \mu(s)[y(t) - y(t - s)] ds| dx$$

$$\leq \frac{1}{2} \left( \int_{\Omega} |y'|^2 dx \right) + \frac{c}{2} \int_{\Omega} \int_0^\infty \mu(s)|y_x(t) - y_x(t - s)|^2 ds dx$$

and

$$\int_0^t \int_{\Omega} |y'_x \int_0^\infty \mu(s)[y_x(t) - y_x(t - s)] ds| dx$$

$$\leq \frac{1}{2} \int_0^t \left( \int_{\Omega} |y_x'|^2 dx \right) d\tau + \frac{1}{2} \int_0^t \int_{\Omega} \int_0^\infty \mu(s)|y_x(t) - y_x(t - s)|^2 ds dx d\tau$$

and

$$\int_0^t \int_{\Omega} |y''_x \int_0^\infty \mu(s)[y_x(t) - y_x(t - s)] ds| dx$$

$$\leq \frac{1}{2} \int_0^t \left( \int_{\Omega} |y'_x|^2 dx \right) d\tau + \frac{1}{2} \int_0^t \int_{\Omega} \int_0^\infty \mu(s)|y_x(t) - y_x(t - s)|^2 ds dx d\tau$$

Then, for some $c > 0$, we have

$$|L(t) - \varepsilon_1 E(t)| \leq c(E(t))$$
Therefore, we can choose \( \varepsilon_1 \) so that
\[
L(t) \sim E(t). 
\] (3.10)

**Lemma 3.5.** Assume (Hyp1) hold. Then there exist strictly positive constants \( c \) such that
\[
L'(t) \leq -\varepsilon_1 E(t) + c \int_{-1}^{1} \int_{-1}^{1} \mu(s)|y_x(t) - y_x(t-s)|^2 dsdx 
\] (3.11)

**Proof.** By Lemma 2.4, Lemma 3.2 and Lemma 3.3 we have
\[
L'(t) = \varepsilon_1 E'(t) + \varphi'(t) + \varepsilon_2 \xi'(t) 
\]

\[
\leq -|2|(1-l) \nu - 1 \int_{0}^{1} |y'|^2 dx - (1 - \frac{\nu}{2}) \varepsilon_1 - 1 \int_{0}^{1} |y_x|^2 dx 
\]

\[
- \left( \frac{l}{2} - \left( \frac{2p}{(p-2)l} E(0) \right)^{(p-2)/2} - \varepsilon_2 a \right) \int_{0}^{1} |y_x|^2 dx 
\]

\[
+ \left( \frac{(1-l)}{2l} + \frac{bE(0)}{4\nu} \right) \int_{0}^{1} \int_{0}^{1} \mu(s)|y_x(t-s) - y_x|^2 dsdx 
\]

where, by (2.11), we have
\[
a = c(\nu) \left( 1 + 2(1-l)^2 - \left( \frac{2p}{(p-2)l} E(0) \right)^{(p-2)/2} \right) > 0 
\]

and for all \( \nu > 0 \)
\[
b = \frac{1-l}{4\nu} + (2\nu + \frac{1}{4\nu}) (1-l) + 2\nu(1-l) c \left( \frac{8}{1-l} E(0) + 2m_0^2 \right)^{(p-2)/2} > 0. 
\]

Now, we choose \( \nu \), and whence this constant is fixed. we can choice \( \varepsilon_1, \varepsilon_2 \) small enough such that for \( p \geq 2 \) there exist \( c > 0 \) and (3.11) yields. \( \square \)

**Lemma 3.6.** Assume that (Hyp1) hold. Then, there exist \( \gamma, \gamma_0 > 0 \) such that for all \( t > 0 \)
\[
\int_{0}^{1} \int_{0}^{1} \mu(s)|y_x(t) - y_x(t-s)|^2 dsdx \leq \frac{-\gamma E'(t)}{H'(\gamma_0 E(t))} + \gamma_0 E(t) 
\] (3.12)

**Proof.** Let \( H^* \) be the convex conjugate of \( g \) in the sense of Young (see [1], pages 61-64), then
\[
H^*(s) = s(H'^{-1}(s) - H[(H')^{-1}(s)] 
\]

\[
\leq s(H'^{-1}(s), \ s \in (0, H'(r)) 
\] (3.13)

and satisfies the following Young's inequality
\[
AB \leq H^*(A) + H(B), \ A \in (0, H'(r)), B \in (0, r]. 
\] (3.14)

for
\[
A = H^{-1} \left( -r_2 \mu'(s) \int_{0}^{1} |y_x(t) - y_x(t-s)|^2 dx \right) 
\]

\[
B = \frac{r_1 H'(\gamma_0 E(t)) \mu(s) \int_{0}^{1} |y_x(t) - y_x(t-s)|^2 dx}{H^{-1} \left( -r_2 \mu'(s) \int_{0}^{1} |y_x(t) - y_x(t-s)|^2 dx \right)}. 
\]
Then, for $r_1, r_2 > 0$, we have (see [17])

\[
\int_0^\infty \mu(s) |y_x(t) - y_x(t - s)|^2 ds dx
\]

\[
= \frac{1}{r_1 H'(\gamma_0 E(t))} \int_0^\infty \left\{ H^{-1} \left( - r_2 \mu'(s) \int_\Omega |y_x(t) - y_x(t - s)|^2 dx \right) \times \right.
\]

\[
\left. \frac{r_1 H'(\gamma_0 E(t)) \mu(s) \int_\Omega |y_x(t) - y_x(t - s)|^2 dx}{H^{-1} \left( - r_2 \mu'(s) \int_\Omega |y_x(t) - y_x(t - s)|^2 dx \right)} \right\} ds
\]

\[
\leq - \frac{r_2}{r_1 H'(\gamma_0 E(t))} \int_0^\infty \mu'(s) |y_x(t) - y_x(t - s)|^2 ds dx
\]

\[
+ \frac{1}{r_1 H'(\gamma_0 E(t))} \int_0^\infty \frac{H'(r_1 H'(\gamma_0 E(t)) \mu(s) \int_\Omega |y_x(t) - y_x(t - s)|^2 dx}{H^{-1} \left( - r_2 \mu'(s) \int_\Omega |y_x(t) - y_x(t - s)|^2 dx \right)} ds
\]

By (2.10), (3.13), we have

\[
\int_0^\infty \mu(s) |y_x(t) - y_x(t - s)|^2 ds dx \leq \frac{2r_2}{r_1 H'(\gamma_0 E(t))} E'(t)
\]

\[
+ \int_0^\infty \left\{ \frac{\mu(s) \int_\Omega |y_x(t) - y_x(t - s)|^2 dx}{H^{-1} \left( - r_2 \mu'(s) \int_\Omega |y_x(t) - y_x(t - s)|^2 dx \right)} \times \right.
\]

\[
\left. \frac{r_1 H'(\gamma_0 E(t)) \mu(s) \int_\Omega |y_x(t) - y_x(t - s)|^2 dx}{H^{-1} \left( - r_2 \mu'(s) \int_\Omega |y_x(t) - y_x(t - s)|^2 dx \right)} \right\} ds
\]

By the fact that $H^{-1}$ is concave and $H^{-1}(0) = 0$, the function $h(s) = \frac{s}{H^{-1}(s)}$, such that for $0 \leq s_1 < s_2$, we have

\[
h(s_1) \leq h(s - 2).
\]

Therefore, using (3.7) to get

\[
H^{-1} \left( \frac{r_1 H'(\gamma_0 E(t)) \mu(s) \int_\Omega |y_x(t) - y_x(t - s)|^2 dx}{H^{-1} \left( - r_2 \mu'(s) \int_\Omega |y_x(t) - y_x(t - s)|^2 dx \right)} \right)
\]

\[
= H^{-1} \left[ \frac{r_1 H'(\gamma_0 E(t)) \mu(s)}{-r_2 \mu'(s)} h \left( - r_2 \mu'(s) \int_\Omega |y_x(t) - y_x(t - s)|^2 dx \right) \right]
\]

\[
\leq H^{-1} \left[ \frac{r_1 H'(\gamma_0 E(t)) \mu(s)}{-r_2 \mu'(s)} h \left( - r_2 \mu'(s) \left( \frac{8}{1 - l} E(0) + 2m_0^2 \right) \right) \right]
\]

\[
\leq H^{-1} \left[ \frac{r_1 \left( \frac{8}{1 - l} E(0) + 2m_0^2 \right) H'(\gamma_0 E(t)) \mu(s)}{H^{-1} \left( - r_2 \mu'(s) \left( \frac{8}{1 - l} E(0) + 2m_0^2 \right) \right)} \right].
\]

Then,

\[
\int_0^\infty \mu(s) |y_x(t) - y_x(t - s)|^2 ds dx
\]

\[
\leq - \frac{2r_2}{r_1 H'(\gamma_0 E(t))} E'(t)
\]

\[
+ \left( \frac{8}{1 - l} E(0) + 2m_0^2 \right) \int_0^\infty \left\{ \frac{\mu(s)}{H^{-1} \left( - r_2 \mu'(s) \left( \frac{8}{1 - l} E(0) + 2m_0^2 \right) \right)} \right\} ds dx
\]
By (Hyp1), we have

\[
\sup_{s \in \mathbb{R}^+} \frac{\mu(s)}{H^{-1}(\gamma_0 E(t))} = \kappa_1 < \infty
\]

\[
\int_{0}^{\infty} \frac{\mu(s)}{H^{-1}(\gamma_0 E(t))} = \kappa_2 < \infty.
\]

Since \(H'^{-1}\) is nondecreasing, we choose \(r_1, r_2\) such that

\[
\int_{\Omega} \int_{0}^{\infty} \mu(s)|y_x(t) - y_x(t - s)|^2 ds dx 
\leq -\frac{2\kappa_2}{H'(\gamma_0 E(t))} E'(t) + \left(\frac{8}{1-l} E(0) + 2m_0^2\right) \int_{0}^{\infty} \frac{\mu(s)}{H^{-1}(\gamma_0 E(t))} ds dx
\]

\[
\leq -\frac{2\kappa_2}{H'(\gamma_0 E(t))} E'(t) + \left(\frac{8}{1-l} E(0) + 2m_0^2\right) \gamma_0 E(t).
\]

This completes the proof. \(\square\)

**Proof.** (of Theorem 3.1) Multiplying (3.11) by \(H'(\gamma_0 E(t))\) and using results in (3.12)

\[
H'(\gamma_0 E(t))L'(t) \leq -\varepsilon_1 H'(\gamma_0 E(t)) E(t) + c H'(\gamma_0 E(t)) \int_{\Omega} \int_{0}^{\infty} \mu(s)|y_x(t) - y_x(t - s)|^2 ds dx 
\]

\[
\leq -[\varepsilon_1 - c\gamma_0] H'(\gamma_0 E(t)) E(t) - c\gamma E'(t).
\]

(3.15)

We choose \(\gamma_0\) small enough so that \(\varepsilon_1 - c\gamma_0 > 0\).

Put

\[
g(t) = H'(\gamma_0 E(t)) L(t) + c\gamma E(t) \sim E(t),
\]

then,

\[
g'(t) \leq -\kappa g(t) H'(\gamma_0 g(t)),
\]

which implies that \((H_1(g))' \geq \kappa\), where

\[
H_1(\tau) = \int_{\tau}^{1} \frac{1}{s H'(\gamma_0 s)} ds, \quad 0 < \tau < 1.
\]

Integrating \((H_1(g))' \geq \kappa\) over \([0, t]\) we get

\[
g(t) \leq H_1^{-1}(\kappa t + \kappa_0)
\]

the equivalence between \(E(t)\) and \(g(t)\) gives the result. \(\square\)
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