One-point correlation functions of the two-periodic weighted Aztec diamond in mesoscopic limit

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Abstract

One-point correlation functions were studied for the Aztec diamond with alternating weights in the limit when the size of the diamond tends to infinity in such a way that the correlation radius in the ordered region is of the same order as the linear size of the region.

1 Introduction

This paper is the first one in a series of papers where the behavior of height functions in dimer models is studied in the thermodynamic limit in the vicinity of a “shrinking” ordered region.

For dimers in the Aztec diamond with alternating weights this is the limit when the size of the diamond goes to infinity in such a way that the linear size of the ordered region is of the order of the correlation radius. The study of one-point correlation functions demonstrates that in this limit there is a new process describing correlation functions.

In a followup paper it will be shown that this is part of a more general phenomenon of a scaling limit when ordered regions vanish into a point at the appropriate rate.

A similar phenomenon was observed numerically for the six-vertex model with \( \Delta < -1 \) in [BR22] where it was called the mesoscopic scaling limit.

Now we recall some basic facts about tilings of the Aztec diamond and corresponding dimer models.

The Aztec diamond is part of a square grid shown in Figure 1. The uniform probability measure on tilings on this region was studied in [CEP96]. The two-periodic weighted Aztec diamond [CJ16] is a probability measure on tilings on this region with periodic weights and \( 2 \times 2 \) fundamental domain. It is equivalent to the dimer model on the dual graph with two weights, 1 and \( a \), alternating along the lattice (Figure 2 and 3). It is one of the simplest models which exhibits all three phases: frozen, disordered and ordered [KOS06].

Equations of the limiting frozen-disordered and disordered-ordered boundaries for dimer models on a hexagonal lattice are studied in [KO07]. For the two-periodic weighted Aztec diamond case see [CJ16] and references within.

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In [CJ16], Chhita and Johansson derived an explicit formula (3.1) for the inverse Kasteleyn matrix of the weighted two-periodic Aztec diamond, and computed its leading order asymptotics in the limit when the weights are fixed, and the lattice size of the Aztec diamond tends to infinity. They showed that inside disordered and ordered regions the leading term of the inverse Kastelyen matrix for the Aztec diamond coincides with the inverse Kasteleyn matrix for a plane and derived the local Gibbs measure. They also computed subleading terms in the asymptotics near the frozen-disordered and disordered-ordered boundaries. They showed that at the disordered-ordered boundary the subleading term is of order $n^{-1/3}$ and computed it in terms of the Airy kernel. This gives rise to typical Airy waves in the correlation functions.

1.1 Informal description of our results

We look at the limit when the size of the diamond tends to infinity and the weights tend to the uniform weights in such a way that the correlation length inside the ordered region is of the order of the linear size of the ordered region.

More specifically, as in [CJ16] we assume the weights are 1 and $a < 1$. If $n$ is the linear size of the whole diamond we consider the limit when $1 - a = O(n^{-1/2})$. The region of the Airy asymptotic of correlation functions found in [CJ16] in this case is of the same order as the width of the ordered region. We study the correlation functions in this “mesoscopic” limit.

The ordered region macroscopically shrinks to a point in this limit and there is no longer a ordered region as such. We show that the leading order term of the one-point correlation functions along a diagonal at a distance of order $n^{1/2}$ from the center of the Aztec diamond is $1/4$. This means that at any vertex, the probabilities of having a dimer on the four edges incident to the vertex are equal. The subleading order term is a constant term of order $n^{-1/2} \log n$. The next term has order $n^{-1/2}$ and it depends on the rescaled coordinates. This last term is the focus of this paper.

We show that terms of the order $n^{-1/2}$ in the the one-point correlation function are no longer described by an Airy process; instead they are given by another double integral. We give precise integral formulas for the subleading order terms. We expect
to see similar asymptotic behavior away from the diagonal, except possibly on the horizontal and vertical lines through the center, where for finite weights there are cusp points on the disordered-ordered boundary.

In addition, we simulated a large number of random domino tilings. We compare the theoretical one-point correlation functions with the results of these simulations.

In this paper we primarily present the simulation results and give an overview of our computations. The details of the computations are presented in [Bai].

In the upcoming paper, we will study two-point correlations.

\[ e_2 = (-1, 1) \quad e_1 = (1, 1) \]

**Figure 2:** The two periodic Aztec diamond lattice of size \( n = 4 \) showing the sublattices \( \mathcal{W}_0, \mathcal{W}_1, B_0 \) and \( B_1 \). Edges surrounding a green square have weight \( a \); other edges have weight 1. Two fundamental domains are marked in blue.

### 1.2 Definition of model

In this paper we study the limit shape for random domino tilings of an Aztec diamond with two-periodic weights. Let \( \Gamma \) denote the dimer lattice, and let \( \Gamma^\vee \) denote the dual lattice. Each vertex of \( \Gamma \) marks the center of a face of \( \Gamma^\vee \). The graph with its bipartite structure and the structure of weights is shown in Figure 2. An example of a tiling of \( \Gamma \) and the corresponding dimer configuration is given in Figure 3.

Because each fundamental domain [KOS06] is a \( 2 \times 2 \) square region, the linear size of the diamond should be a multiple of 2. In this paper for simplicity we only consider the case where the linear size is a multiple of 4. Let \( n = 4m \) denote the linear size of \( \Gamma \).\(^1\)

\(^1\)We will mostly follow the same setup as Chhita and Johansson [CJ16].
Figure 3: The two periodic Aztec diamond lattice of size \( n = 4 \) with a dimer matching, and its dual domino tiling in blue. Weight \( a \) dominos are shaded in green.

So on each slice containing either all black or all white vertices we will have either \( 4m \) segments between vertices or \( 4m - 1 \) segments between vertices and two half segments at the sides. We will use Euclidean coordinates \((x_1, x_2)\) with \( x_i \in \mathbb{Z} \) ranging from 0 to \( 2n = 8m \) as in Figure 2, with the vertices being at points satisfying \( x_1 + x_2 \equiv 1 \mod 2 \). Thus the linear Euclidean length of the diamond along the axes \( x_1 \) and \( x_2 \) is \( 2n = 8m \).

We fix a bipartite structure on \( \Gamma \) by coloring the vertices black and white (see Figure 2). We denote the set of white vertices by \( \mathcal{W} \) and the set of black vertices by \( \mathcal{B} \) where we set

\[
\mathcal{W} = \{(x_1, x_2) \in \Gamma : x_1 \equiv 1 \mod 2, x_2 \equiv 0 \mod 2\}
\]
\[
\mathcal{B} = \{(y_1, y_2) \in \Gamma : y_1 \equiv 0 \mod 2, y_2 \equiv 1 \mod 2\}.
\]

Then we split the white vertices \( \mathcal{W} \) into two sublattices \( \mathcal{W}_0 \) and \( \mathcal{W}_1 \), and the black vertices \( \mathcal{B} \) into \( \mathcal{B}_0 \) and \( \mathcal{B}_1 \) according to the values of \( x_1 + x_2 \mod 4 \) and \( y_1 + y_2 \mod 4 \) respectively as follows (and as is shown in Figure 2).

\[
\mathcal{W}_{e_1} = \{(x_1, x_2) \in \mathcal{W} : x_1 + x_2 \equiv 2e_1 + 1 \mod 4\}
\]
\[
\mathcal{B}_{e_2} = \{(y_1, y_2) \in \mathcal{B} : y_1 + y_2 \equiv 2e_2 + 1 \mod 4\}
\]

Let \( e_1 = (1, 1) \) and \( e_2 = (-1, 1) \). There is a \( 2 \times 2 \) fundamental domain which when embedded in the graph consists of a vertex \( w \in \mathcal{W}_0 \) along with \( w + e_1, w + e_2 \) and \( w + e_1 + e_2 \).

A dimer configuration is a perfect matching on vertices of \( \Gamma \) connected by edges. A domino consists of two adjacent faces of \( \Gamma^\vee \). Dimer configurations of \( \Gamma \) are in bijection with domino tilings of \( \Gamma^\vee \). A dimer configuration of \( \Gamma \) and the corresponding domino tiling of \( \Gamma^\vee \) is shown in Figure 3.
We assign a weight to each edge. The weights of edges connecting vertices in one fundamental domain are \(a\), and the weights of edges connecting vertices in neighboring fundamental domains are 1. This is shown diagrammatically in Figure 2, where the edges of weight \(a\) are the edges surrounding a face labelled with \(a\), and the other edges have weight 1.

Let \(\Omega\) denote the set of all dimer configurations on \(\Gamma\). Define the partition function

\[
Z = \sum_{D \in \Omega} \prod_{e \in D} w(e).
\]

Then the Gibbs measure is defined as

\[
\text{Prob}(D) = \frac{\prod_{e \in D} w(e)}{Z}
\]

for \(D \in \Omega\).

Let \(e_i = (w_i, b_i), 1 \leq i \leq k\) be edges of \(\Gamma\). The \(k\)-point correlation function is

\[
\rho(w_1, b_1; w_2, b_2; \ldots; w_k, b_k) = \sum_{D \in \Omega} \text{Prob}(D) \sigma_{w_1, b_1}(D) \ldots \sigma_{w_k, b_k}(D)
\]

where

\[
\sigma_{w, b}(D) = \begin{cases} 
1 & \text{if } (w, b) \in D \\
0 & \text{otherwise.}
\end{cases}
\]

This is the probability of a random dimer configuration containing the dimers on edges \(\{e_i\}_{1 \leq i \leq k}\).

1.3 Kasteleyn solution

We give a brief overview of the Kasteleyn method. The Kasteleyn matrix for the two periodic Aztec diamond is defined as follows. For each edge \(e\) of \(\Gamma\), let \(w(e)\) be the weight of the edge \(e\), as shown in Figure 2. We define a Kasteleyn-Perkus orientation \(\epsilon\) on the edges of \(\Gamma\) as shown in Figure 4. Then for \(x \in \mathbb{W}\) and \(y \in \mathbb{B}\) we define the Kasteleyn matrix \(K_a\)\(^2\) as having entries

\[
K_a(y, x) = \begin{cases} 
w(x, y)\epsilon(x, y) & \text{if there is an edge between } x \text{ and } y \\
0 & \text{otherwise}
\end{cases}
\]

The values of \(K_a(y, x)\) for edges \((x, y)\) in a fundamental domain are shown in Figure 5. Let \(K_a^{-1}\) denote the inverse matrix.

For edges \(e_i = (w_i, b_i), 1 \leq i \leq k\), the \(k\)-point correlation function has been shown [KOS06] to be

\[
\rho(w_1, b_1; w_2, b_2; \ldots w_k, b_k) = \left( \prod_{i=1}^{k} K_a(b_i, w_i) \right) \det(K_a^{-1}(w_i, b_j))_{1 < i, j < k}
\]

\(^2\)In [CJ16] \(K_{a,1}\) is used instead of \(K_a\).
When we are looking at a single edge, we will denote this edge by \((x, y)\), where \(x \in \mathcal{W}\) and \(y \in \mathcal{B}\). Then the one-point correlation is given by \(\rho(x, y) = K_a(y, x)K_a^{-1}(x, y)\). This is the probability that a random dimer configuration contains the dimer \((x, y)\).

\[\text{Figure 4: The Kasteleyn-Perkus orientation shown on a fundamental domain.}\]

\[\text{Figure 5: The entries } K_a(y, x) \text{ of the Kasteleyn matrix for vertices } x \in \mathcal{W}, y \in \mathcal{B} \text{ connected by an edge, shown for a fundamental domain.}\]

1.4 Height functions

It is known that dimers on a bipartite planar graph can be described in terms of height functions. A height function is a function on faces of the graph (vertices of the dual cell complex). It can be defined relative to a reference matching. The height change between adjacent faces is determined by whether there is a dimer between the faces. Figure 6 shows a height function on the faces of the Aztec diamond.

1.5 Translation invariant weighted graphs

Let \(\Gamma\) be a \(\mathbb{Z}^2\)-periodic bipartite planar graph with fundamental region \(\Gamma_0\). Take \((s, t) \in \mathbb{R}\). In [KOS06], Kenyon, Okounkov and Sheffield constructed an ergodic Gibbs measure on \(\Gamma\) with fixed slope \((s, t)\) (see [KOS06] for definitions and details). This slope is the gradient of the limiting height function.
Figure 6: A height function for an Aztec diamond dimer configuration, determined by the rules shown on the right. Heights have been multiplied by 4 for convenience.

Furthermore, they show that these ergodic Gibbs measures can have distinctly different qualitative properties depending on the slope. Specifically, there are three distinct types of behavior, referred to as the frozen (or solid) phase, the disordered (or liquid or rough) phase and the ordered (or gas or smooth) phase. In the frozen phase there are no fluctuations between height functions, it is “frozen”. In the disordered phase correlations of differences of height functions decay at large distances as a power of the distance. In the ordered phase correlations of the height differences decay exponentially.

The characteristic polynomial of the fundamental region $\Gamma_0$ and its amoeba are defined in [KOS06]. The authors show that the corresponding ergodic Gibbs measure of slope $(s, t) \in \mathbb{R}$ is frozen if the Legendre dual $(H, V)$ lies in the closure of an unbounded complementary component of the amoeba of the characteristic polynomial, disordered if $(H, V)$ lies in the interior of the amoeba and ordered if $(H, V)$ lies in the closure of a bounded complementary component of the amoeba.

For these ergodic Gibbs measures on the infinite $\mathbb{Z}^2$-periodic bipartite planar graph $\Gamma$, the one-point correlation of the edge $(x, y)$ (i.e. the probability of $(x, y)$ being occupied by a dimer) is $K(y, x)K^{-1}(x, y)$, where $K$ is the Kasteleyn matrix and $K^{-1}$ is its inverse. An integral formula for $K^{-1}$ is given in [KOS06].

Let $(H, V)$ be the Legendre dual of the slope $(s, t)$ ($H$ and $V$ are often referred to as magnetic coordinates). Let us describe $K^{-1}$ explicitly for the two-periodically weighted $\mathbb{Z}^2$ lattice with the same weighting as the two-periodic weighted Aztec diamond described above. Let $K_{a,H,V}$ denote the Kasteleyn matrix on $\mathbb{Z}^2$ with two-periodic weights as above with slope $(s, t)$, and corresponding magnetic coordinates $(H, V)$. Let $C_r$ be a positively oriented contour of radius $r$ centered at the origin. Then

$$K^{-1}_{a,H,V}(w, b + u e_1 + v e_2) = \frac{1}{(2\pi i)^2} \int_{C_{eH}} \frac{dz}{z} \int_{C_{eV}} \frac{dw}{w} (K_a(z, w)^{-1})_{e_1 e_2} z^u w^v$$

where $K_a^{-1}(z, w)_{e_1 e_2} = Q_a(z, w)_{e_1 e_2}/P_a(z, w)$ with

$$P_a(z, w) = -2 - 2a^2 - aw^{-1} - aw - az^{-1} - az$$
and
\[ Q_a(z, w) = \begin{pmatrix} i(a + w) & -(a + z) \\ -(a + z^{-1}) & i(a + w^{-1}) \end{pmatrix}. \]

1.6 Main result

We look at the one-point correlation functions of edges that are a Euclidean distance of order \( m^{1/2} \) from the center \((4m, 4m)\) of the Aztec diamond, near the diagonal in the third quadrant. We define the asymptotic coordinate \( \alpha < 0 \) as follows. Let \( x = (x_1, x_2) \in W_{\varepsilon_1} \) and \( y = (y_1, y_2) \in B_{\varepsilon_2} \) be vertices of an edge of the graph \( \Gamma \), with
\[
\begin{align*}
x_1 &= [4m + 2m^{1/2}aB] + \overline{x_1} \\
x_2 &= [4m + 2m^{1/2}aB] + \overline{x_2} \\
y_1 &= [4m + 2m^{1/2}aB] + \overline{y_1} \\
y_2 &= [4m + 2m^{1/2}aB] + \overline{y_2},
\end{align*}
\]
where the integral parts \( \overline{x_i}, \overline{y_i} \) are order 1. For an edge \((x, y)\) with \( x \in W_{\varepsilon_1} \) and \( y \in B_{\varepsilon_2} \), define
\[
\zeta(x, y) = \begin{cases} 1 & \text{if } x \text{ and } y \text{ lie in the same fundamental domain} \\ -1 & \text{if } x \text{ and } y \text{ lie in adjacent fundamental domains} \end{cases}
\]
Note that if \( x \) and \( y \) lie in the same fundamental domain then the weight of the edge connecting them is \( a \), and if they are in neighboring fundamental domains then the weight of the edge connecting them is 1. The types of dominos with faces centered at \( x \) and \( y \) where \( \zeta(x, y) = 1 \) are shown in Figure 7a and the dominos where \( \zeta(x, y) = -1 \) are shown in Figure 7b.

In the limit as \( m \to \infty \) with \( a = 1 - Bm^{-1/2} \) we compute the following asymptotic formulas for the one-point correlation function \( \rho(x, y) \). For \(-1/\sqrt{2} \leq \alpha < 0\), if \( x = (x_1, x_2) \in W_{\varepsilon_1} \) and \( y = (y_1, y_2) \in B_{\varepsilon_2} \), we have
\[
\rho(x, y) = \frac{1}{4} - \zeta(x, y)Bm^{-1/2}\left(\frac{1}{4\pi} \left(-2 \log(Bm^{-1/2}) + 4 \log 2\right)
+ \psi(\alpha, \varepsilon_1, \varepsilon_2)\right) + O(m^{-1} \log m)
\]
and for \( \alpha < -1/\sqrt{2} \) we have
\[
\rho(x, y) = \frac{1}{4} - \zeta(x, y)Bm^{-1/2}\left(\frac{1}{4\pi} \left(-2 \log(Bm^{-1/2}) + 4 \log 2 - I_0(\alpha, \varepsilon_1, \varepsilon_2)\right)
+ \psi(\alpha, \varepsilon_1, \varepsilon_2)\right) + O(m^{-1} \log m)
\]
where
\[
\psi(\alpha, \varepsilon_1, \varepsilon_2) = -\frac{1}{32\pi^2}(I_1(\alpha, \varepsilon_1, \varepsilon_2) - I_2(\alpha, \varepsilon_1, \varepsilon_2) - I_3(\alpha, \varepsilon_1, \varepsilon_2) + I_4(\alpha, \varepsilon_1, \varepsilon_2)).
\]
and the integrals $I_0(\alpha, \varepsilon_1, \varepsilon_2)$, $I_1(\alpha, \varepsilon_1, \varepsilon_2)$, $I_2(\alpha, \varepsilon_1, \varepsilon_2)$, $I_3(\alpha, \varepsilon_1, \varepsilon_2)$ and $I_4(\alpha, \varepsilon_1, \varepsilon_2)$ are defined below.

The constant $1/4$ in Equations 1.4–1.5 above is the one-point correlation in the limit $a = 1$ at the center of the Aztec diamond. This means that when the weighting is uniform, every type of domino is equally likely to appear in the center.

The term $\frac{1}{4} - \frac{\zeta(x,y)Bm^{-1/2}}{4\pi} (-2 \log(Bm^{-1/2}) + 4 \log 2)$ that appears in Equation 1.4 is the asymptotic of $K_{\alpha}(y,x)K_{\alpha,0,0}^{-1}(x,y)$ defined in Equation 1.1 as $a \to 1$. Note that Equation 1.1 does not depend on $n$. This is the whole plane inverse Kasteleyn matrix for the two-periodic weighted torus with magnetic coordinates $(0,0)$. In [CJ16], the authors showed that $K_{\alpha}(y,x)K_{\alpha,0,0}^{-1}(x,y)$ is the leading order term of $\rho(x,y)$ for $(x,y)$ in the ordered region as $m \to \infty$ with $a$ fixed.

The term $\frac{1}{4} - \frac{\zeta(x,y)Bm^{-1/2}}{4\pi} (-2 \log(Bm^{-1/2}) + 4 \log 2 - I_0(\alpha, \varepsilon_1, \varepsilon_2))$ that appears in Equation 1.5 is the asymptotic of $K_{\alpha}(y,x)K_{\alpha,0,V_0}^{-1}(x,y)$ defined in Equation 1.1 as $a \to \infty$, for $V_0$ as defined in Appendix A.2. This is the whole plane inverse Kasteleyn matrix for the two-periodic weighted torus with magnetic coordinates $(0,V_0)$. In [CJ16], the authors showed that $K_{\alpha}(y,x)K_{\alpha,0,V_0}^{-1}(x,y)$ is the leading order term of $\rho(x,y)$ for $(x,y)$ in the disordered region as $m \to \infty$ with $a$ fixed. The non-zero magnetic coordinates correspond to a non-flat limiting height slope in this region.

While it may appear from these formulas that there is a clear boundary at $\alpha = -1/\sqrt{2}$ that appears to separate an ordered-like region and a disordered-like region, this is in fact not the case. Once the $\psi(\alpha, \varepsilon_1, \varepsilon_2)$ term is taken account of, we see that the first derivative of the one-point correlation at $\alpha = -1/\sqrt{2}$ is in fact continuous. Figure 8 shows $-I_0(\alpha, \varepsilon_1, \varepsilon_2)/(4\pi) + \psi(\alpha, \varepsilon_1, \varepsilon_2)$ as a function of $\alpha$ for different values of $(\varepsilon_1, \varepsilon_2)$.
where \( I_0(\alpha, \varepsilon_1, \varepsilon_2) \) is taken to be 0 for \(-1/\sqrt{2} < \alpha < 1/\sqrt{2}\).

\[ \frac{I_0(\alpha, \varepsilon_1, \varepsilon_2)}{4\pi} + \psi(\alpha, \varepsilon_1, \varepsilon_2) \text{ as a function of } \alpha \text{ for different values of } (\varepsilon_1, \varepsilon_2), \]

where \( I_0(\alpha, \varepsilon_1, \varepsilon_2) \) is taken to be 0 for \(-1/\sqrt{2} < \alpha < 1/\sqrt{2}\). Up to a sign \(-\zeta(x, y)\), these are the parts of the coefficients of \( Bm^{-1/2} \) in Equations 1.4–1.5 that are not constant as a function of \( \alpha \).

Let

\[ f^\pm(w) = \sqrt{1/2 - 2iw} \pm \sqrt{1/2 + 2iw}. \]

We define the following double integrals, where \( x = (x_1, x_2) \in W_{\varepsilon_1} \) and \( y = (y_1, y_2) \in B_{\varepsilon_2} \) and \( \alpha \) is as in Equation 1.2.

\[
I_1(\alpha, \varepsilon_1, \varepsilon_2) = \int_{C_0} dw \int_{C_0'} dz \frac{A_{0,0}^{\varepsilon_1, \varepsilon_2}(w, z)}{i(z - w)} \exp(B^2(-2i(w - z) + \alpha(f^-(w) - f^-(z)))),
\]

\[
I_2(\alpha, \varepsilon_1, \varepsilon_2) = \int_{C_0} dw \int_{C_0'} dz \frac{A_{1,0}^{\varepsilon_1, \varepsilon_2}(w, z)}{i(z - w)} \exp(B^2(-2i(w - z) + \alpha(f^-(w) + f^+(z)))),
\]

\[
I_3(\alpha, \varepsilon_1, \varepsilon_2) = \int_{C_1} dw \int_{C_1'} dz \frac{A_{0,1}^{\varepsilon_1, \varepsilon_2}(w, z)}{i(z - w)} \exp(B^2(-2i(w - z) + \alpha(f^+(w) - f^-(z)))),
\]

\[
I_4(\alpha, \varepsilon_1, \varepsilon_2) = \int_{C_1} dw \int_{C_1'} dz \frac{A_{1,1}^{\varepsilon_1, \varepsilon_2}(w, z)}{i(z - w)} \exp(B^2(-2i(w - z) + \alpha(f^+(w) + f^+(z)))),
\]

where the functions \( A_{ij}^{\varepsilon_1, \varepsilon_2}(w, z) \) are defined in Equation A.1 and the contours are defined below. We also define the single contour

\[ I_0(\alpha, \varepsilon_1, \varepsilon_2) = \int_{-\eta(\alpha)}^{\eta(\alpha)} \frac{1 + (-1)^{\varepsilon_2} \sqrt{1/2 - 2iw} + (-1)^{\varepsilon_1} \sqrt{1/2 + 2iw}}{\sqrt{1/2 - 2iw} \sqrt{1/2 + 2iw}} dw \]

where \( \eta(\alpha) \) is defined in Equation A.2. All of these integrals are real.

We describe the contours \( C_0, C'_0, C_1 \) and \( C'_1 \), shown in Figure 9 for some different values of \( \alpha \). See Appendix A.1.1 for precise definitions. All contours are infinite, and
symmetric under reflection in the imaginary axis. The contours $C_0$ and $C_1$ go to infinity in the lower half plane, while the contours $C'_0$ and $C'_1$ go to infinity in the upper half plane. $C'_0$ is a reflection of $C_0$ in the real axis, and $C'_1$ is a reflection of $C_1$ in the real axis. The contours $C_0$ and $C'_0$ look considerably different depending on the value of $\alpha$. Specifically, if $\alpha < -\frac{1}{\sqrt{2}}$, there is a point on the real axis where the contours intersect, and then they go to the branch cut and travel around the branch cut. If $\alpha = -\frac{1}{\sqrt{2}}$ the contours meet at a point, and if $-\frac{1}{\sqrt{2}} < \alpha < 0$ they do not touch. Contours $C_1$ and $C'_1$ are the same general shape regardless of the value of $\alpha$.

![Contours](image)

**Figure 9:** Contours $C_0$ (blue), $C'_0$ (orange), $C_1$ (blue, dashed) and $C'_1$ (orange, dashed) for different values of $\alpha$. Different scales are used to show the notable features of each.

1.7 Acknowledgments

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We used the source code developed by Keating and Sridhar [KS18a] as the basis for our simulation code.

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2 Simulations

![Figure 10: Domino tiling with \( n = 64 \) and \( a = 0.8 \)](image)

We used Markov chain sampling to produce a large number of sample tilings from the correct probability distribution. One such tiling is shown in Figure 10. We used the source code developed by Keating and Sridhar [KS18a] described in [KS18b], with some modifications. We ran our code on a GTX 1080 Ti GPU.

We look at the experimental one-point correlation of dominos along the diagonal shown in Figure 11 that are of order \( n^{1/2} \) from the center of the Aztec diamond. This is the relative frequency with which each type of domino appears in our samples. Our formulas hold for the part of this diagonal that lies in the third quadrant, but it is straightforward to deduce the formulas for the entire diagonal. There are 8 types of dominos in total (see Figure 7), corresponding to the 8 distinct edges in a fundamental domain. In Equations 1.4 and 1.5, we see that we have 8 different formulas for the one-point correlation, coming from the 4 possibilities for \((\varepsilon_1, \varepsilon_2)\) and two possibilities for \(\zeta(x,y)\), which is defined in Equation 1.3. Figure 12 shows the relative frequencies of each type of domino over 6000000 random domino tilings of size 512. Will we look
more closely at what is happening in the two areas shaded in pink and purple near the center of the figure.

Figure 13 shows the experimental one-point correlations of the eight different types of domino in a region of order $n^{1/2}$ from the center of the Aztec diamond. We have computed one-point correlations for the diagonal from the lower-left to upper-right corner of these plots. Each plot is only over the subset of edges where that domino can exist. The dimers corresponding to the eight types of domino are $(w_0, b_0)$, $(w_1, b_1)$, $(w_0, b_1)$ and $(w_0, b_1)$ of weight $a$, and $(w_0, b_0 - 2e_2)$, $(w_1, b_1 + 2e_2)$, $(w_1, b_0 + 2e_1)$ and $(w_0, b_1 - 2e_1)$ of weight 1, where $w_i \in W_i$ and $b_j \in B_j$ are vertices in the same fundamental domain. The corresponding dominos are shown in Figure 7.

The one-point correlation function $\rho(x, y) = K_a(y, x)K_a^{-1}(x, y)$ is the probability of the domino $(x, y)$ appearing in a random tiling with weights $a$ and 1. In [CJ16] the authors show that the leading order term in the inverse Kasteleyn matrix for an Aztec diamond tiling of fixed weights as its size goes to infinity is the whole plane inverse Kasteleyn matrix corresponding to the ergodic Gibbs measure of fixed slope (see Section 1.5). Let $\nu_a(x, y)$ denote the one-point correlation corresponding to this ergodic Gibbs measure for fixed weight $a$ (see Appendix A.2 for definition).

Recall $\psi(\alpha, \varepsilon_1, \varepsilon_2)$ from Equation 1.6. From our work and [CJ16], we can show that

$$\rho(x, y) - \nu_a(x, y) = Bm^{-1/2}\zeta(x, y)\psi(\alpha, \varepsilon_1, \varepsilon_2) + O(m^{-1}\log m)$$

In Figure 15 we plot $\nu_a(x, y)$, the one-point correlation corresponding to the ergodic Gibbs measure, for the different types of dominos. We also plot $\nu_a(x, y) +$
Figure 12: Experimental one-point correlations of the 8 distinct types of dominos averaged over 6,000,000 random Aztec domino tilings of size $n = 512$. Note that the experimental one-point correlations for dominos of the same weight with $(\varepsilon_1, \varepsilon_2) = (0, 1)$ and $(\varepsilon_1, \varepsilon_2) = (1, 0)$ are very close, which reflects the fact that the theoretical one-point correlations are equal. In the region shaded in pink, the one-point correlations roughly look like the functions plotted in Figure 8 plus a constant term, while in the region shaded in purple, the one-point correlations roughly look like a reflection of the functions plotted in Figure 8 plus a constant term.

Figure 13: Empirical one-point correlations in a region order $m^{-1/2}$ from the center of the Aztec diamond, averaged over multiple samples, for each type of domino. Lighter colors correspond to higher frequency. Faint waves can be seen.
Figure 14: A plot of $\nu_a(x,y)$, the one-point correlation under the ergodic Gibbs measure, for $n = 512$ and $a \approx 0.912$. The blue lines correspond to $(\varepsilon_1, \varepsilon_2) = (0, 1)$ or $(1, 0)$; the orange lines correspond to $(\varepsilon_1, \varepsilon_2) = (0, 0)$ and the green lines correspond to $(\varepsilon_1, \varepsilon_2) = (1, 1)$. Here $(x, y)$ is an edge near the diagonal of the Aztec diamond as shown in Figure 11, and the $x_1$-coordinate of the center of the domino is shown on the horizontal axis.

Figure 15: The dashed lines show $\nu_a(x,y)$ as plotted in Figure 14. The solid lines show $\nu_a(x,y) + B m^{-1/2} \zeta(x,y) \psi(\alpha, \varepsilon_1, \varepsilon_2)$. The blue lines correspond to $(\varepsilon_1, \varepsilon_2) = (0, 1)$ or $(1, 0)$; the orange lines correspond to $(\varepsilon_1, \varepsilon_2) = (0, 0)$ and the green lines correspond to $(\varepsilon_1, \varepsilon_2) = (1, 1)$. As $n \to \infty$, the solid lines on the second plot will tend to the lines in Figure 8 up to scaling and translation, and the solid lines on the first plot will tend to reflections of these lines up to scaling and translation. Here $n = 512$, and $\alpha$ ranges from $-5$ to $5$. 
For dominoes with \((\varepsilon_1, \varepsilon_2) = (1, 0)\), we can show that both \(\nu_a(x, y)\) and \(\psi(\alpha, \varepsilon_1, \varepsilon_2)\) are the same as for the neighboring domino with \((\varepsilon_1, \varepsilon_2) = (0, 1)\). The one point correlations in the first quadrant can be deduced from our formulas by symmetry, and are also plotted.

Let \(\overline{\rho}(x, y)\) denote the experimental one-point correlations of the domino with faces centered at \(x\) and \(y\) for our simulated tilings (recall that \(\rho(x, y)\) denotes the corresponding theoretical one-point correlation). This is the relative frequency with which this domino appears in our randomly sampled tilings. We will compare \(\overline{\rho}(x, y)\) to \(\nu_a(x, y) + Bm^{-1/2}\zeta(x, y)\psi(\alpha, \varepsilon_1, \varepsilon_2)\). The reason for this is simply that the magnitude of the coefficients of the lower order terms of \(\nu_a(x, y)\) are large, and we would need to use a much larger \(n\) to reduce the error to be negligible compared to the terms we are interested in. However due to computational limitations we have been unable to perform simulations with such a large \(n\).

There are some strange kinks in the graph around \(\alpha = 1/\sqrt{2}\), due to the fact that the whole plane inverse Kasteleyn matrix has a discontinuity in the derivative at \(\alpha = -1/2B^{-1}n^{1/2}\sqrt{1-2c}\), while \(\psi(\alpha, \varepsilon_1, \varepsilon_2)\) has a discontinuity in the derivative at \(\alpha = -1/\sqrt{2}\). As \(n \to \infty\), \(-1/2B^{-1}n^{1/2}\sqrt{1-2c} \to -1/\sqrt{2}\).
Figure 16 shows the observed one-point correlations (i.e. relative frequencies) of some of the different types of domino on the diagonal in the third and first quadrants. We also show \( \nu_a(x, y) + Bm^{-1/2}\zeta(x, y)\psi(\alpha, \varepsilon_1, \varepsilon_2) \) and \( \nu_a(x, y) \) for comparison. In Figure 17 we plot the same quantities with \( \nu_a(x, y) \) subtracted from each point, in order to better see the agreement between theoretical and experimental results. In both plots, the \( x_1 \)-coordinate of the center of the domino is marked on the \( x \)-axis. We see that for small \( |\alpha| \), we have good agreement between \( \overline{\rho}(x, y) - \nu_a(x, y) \) and \( Bm^{-1/2}\zeta(x, y)\psi(\alpha, \varepsilon_1, \varepsilon_2) \).

![Figure 16: Observed one-point correlations](image1)

![Figure 17: Combined plots](image2)

### 3 Outline of computations

Here we give an outline of our computations. Full details can be found in [Bai]. We start with the following formula from Chhita and Johansson [CJ16].

**Theorem 3.1** (Chhita and Johansson [CJ16]). For \( n = 4m \), \( 0 < a < 1 \), \( x = (x_1, x_2) \in W_{\varepsilon_1}, y = (y_1, y_2) \in B_{\varepsilon_2} \) with \( \varepsilon_1, \varepsilon_2 \in \{0, 1\} \), \( 0 < x_1, x_2, y_1, y_2 < n \), the entries of \( K_a^{-1}((x_1, x_2), (y_1, y_2)) \) are given by

\[
K_a^{-1}((x_1, x_2), (y_1, y_2)) = K_{a,0,0}^{-1}((x_1, x_2), (y_1, y_2)) - \left( B_{\varepsilon_1,\varepsilon_2}(a, x_1, x_2, y_1, y_2) \right.
\]

\[
- \frac{i}{a} (-1)^{\varepsilon_1+\varepsilon_2} (B_{1-\varepsilon_1,\varepsilon_2}(1/a, 2n-x_1, x_2, 2n-y_1, y_2) + B_{\varepsilon_1,1-\varepsilon_2}(1/a, x_1, 2n-x_2, y_1, 2n-y_2) + B_{1-\varepsilon_1,1-\varepsilon_2}(a, 2n-x_1, 2n-x_2, 2n-y_1, 2n-y_2))
\]

\]
where $K_{a,0,0}^{-1}((x_1, x_2), (y_1, y_2))$ is defined in Equation 1.1 and the remaining terms are defined in Equations A.7–A.10.

### 3.1 Asymptotics of $K_{a,0,0}^{-1}((x_1, x_2), (y_1, y_2))$

$K_{a,0,0}^{-1}$ is the whole plane inverse Kasteleyn matrix for the ordered phase, where the limiting height function is flat. It is defined in Equation 1.1. Note that it depends only on $a$ and not directly on $m$. First we show the following relations between $K_{a,0,0}^{-1}((x_1, x_2), (y_1, y_2))$ for the 8 types of domino (which are shown in Figure 7).

\[(3.2)\quad K_{a,0,0}^{-1}(w_0, b_1) = K_{a,0,0}^{-1}(w_1, b_0) = i K_{a,0,0}^{-1}(w_1, b_1) = i K_{a,0,0}^{-1}(w_0, b_0),\]

\[(3.3)\quad K_{a,0,0}^{-1}(w_0, b_1 - 2e_1) = K_{a,0,0}^{-1}(w_1, b_0 + 2e_1) = i K_{a,0,0}^{-1}(w_1, b_1 + 2e_2) = i K_{a,0,0}^{-1}(w_0, b_0 - 2e_2)\]

and

\[(3.4)\quad a K_{a,0,0}^{-1}(w_0, b_1) + K_{a,0,0}^{-1}(w_0, b_1 - 2e_1) = 1/2\]

This means that for the ergodic Gibbs measure, the probability of a dimer only depends on the weight, and not the type of dimer, and that the probabilities of dimers around any particular vertex sum to 1 (as expected). So we only need to find the asymptotics of one of these integrals. We use the residue theorem to write the double contour integral as a single integral, and then deform the contour to obtain a real integral. We find that

\[(3.5)\quad a K_{a,0,0}^{-1}(w_0, b_1) = \frac{1}{\pi} \int_{z_1}^{z_2} \frac{a + z}{\sqrt{4z^2 - (2(a + a^{-1})z + z^2 + 1)^2}} dz\]

where

\[
z_1 = -(a + a^{-1}) + \sqrt{(a + a^{-1})^2 - 1}
\]

\[
z_2 = -(a + a^{-1}) + \sqrt{(a + a^{-1})^2 - 1}.\]

The endpoints $z_1$ and $z_2$ are asymptotes of the integrand. After some computation, we obtain

\[(3.6)\quad a K_{a,0,0}^{-1}(w_0, b_1) = \frac{1}{4} + \frac{a - a^{-1}}{2\pi(a + a^{-1})} \int_0^1 \frac{1}{\sqrt{(1 - 4c^2y^2)(1 - y^2)}} dy\]

where $c = 1/(a + a^{-1})$. This is an elliptic integral, so we can compute

\[a K_{a,0,0}^{-1}(w_0, b_1) = \frac{1}{4} - \frac{h}{2\pi}(-\log h + 2 \log 2) + O(h^2 \log h)\]

where $a = 1 - h$, and hence from Equations 3.2–3.4, for an edge $(x, y)$ with $x \in \mathbb{W}, y \in \mathbb{B}$ we have

\[K_a^{-1}(y, x) K_{a,0,0}^{-1}(x, y) = \frac{1}{4} - \frac{\zeta(x, y)}{2\pi} h(-\log h + 2 \log 2) + O(h^2 \log h).\]

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We see that the subleading order term is $O(h \log h)$. This means that for the ergodic Gibbs measure, as $a \to \infty$ the difference between the probability of a weight $a$ domino and a weight 1 domino appearing in the ordered region is $O(h \log h)$.

Writing the above equation in terms of $m$ for our limit $a = 1 - Bm^{-1/2}$ as $m \to \infty$, we have

\begin{equation}
K_a^{-1}(y, x)K^{-1}_{a,0,0}(x, y) = \frac{1}{4} - \frac{\zeta(x, y)}{2\pi} Bm^{-1/2}( - \log(Bm^{-1/2}) + 2 \log 2) + O(m^{-1} \log m).
\end{equation}

3.2 Asymptotics of remaining terms

Let $\Gamma_r$ denote a positively oriented contour of radius $r$ centered at the origin. The terms $B_{\xi_1, \xi_2}(a, x_1, x_2, y_1, y_2)$, $B_{1-\xi_1, \xi_2}(1/a, 2n-x_1, x_2, 2n-y_1, y_2)$, $B_{\xi_1, 1-\xi_2}(1/a, x_1, 2n-x_2, y_1, 2n-y_2)$ and $B_{1-\xi_1, 1-\xi_2}(a, 2n-x_1, 2n-x_2, 2n-y_1, 2n-y_2)$ are double contour integrals over contours $\Gamma_r$ and $\Gamma_{1/r}$ where $\sqrt{2c} < r < 1$ with $c = 1/(a + a^{-1})$.

First we look at these four integrals. We use saddle point analysis to find the asymptotic behaviour.

Recalling that $a$ depends on $m$, we first show that the saddle points of the exponential parts of the integrands are at a distance of order $m^{-1}$ from $\pm i$. We expand the exponent in the integrands locally near $\pm i$ and find the saddle points and contours of steepest descent of the leading order term. We call these local saddle points and local contours of steepest descent respectively. Let $\eta$ be as defined in Equation A.2, and let $\eta'$ be as defined in Equation A.3. The local saddle points are as follows.\(^4\)

\begin{itemize}
  \item $B_{\xi_1, \xi_2}(a, x_1, x_2, y_1, y_2)$
  \hspace{1cm} In the $\omega_1$-plane, there are four local saddle points at $\omega_1 = \pm i \pm B^2 m^{-1} \eta$, and in the $\omega_2$-plane, there are four local saddle points at $\omega_2 = \pm i \pm B^2 m^{-1} \eta$.

  \item $B_{1-\xi_1, \xi_2}(1/a, 2n-x_1, x_2, 2n-y_1, y_2)$
  \hspace{1cm} In the $\omega_1$-plane, there are four local saddle points at $\omega_1 = \pm i \pm B^2 m^{-1} \eta$, and in the $\omega_2$-plane, there are two local saddle points at $\omega_2 = \pm (i \pm B^2 m^{-1} \eta')$.

  \item $B_{\xi_1, 1-\xi_2}(1/a, x_1, 2n-x_2, y_1, 2n-y_2)$
  \hspace{1cm} In the $\omega_1$-plane, there are two local saddle points at $\omega_1 = \pm (i - B^2 m^{-1} \eta')$, and in the $\omega_2$-plane, there are four local saddle points at $\omega_2 = \pm i \pm B^2 m^{-1} \eta$.

  \item $B_{1-\xi_1, 1-\xi_2}(a, 2n-x_1, 2n-x_2, 2n-y_1, 2n-y_2)$
  \hspace{1cm} In the $\omega_1$-plane, there are two local saddle points at $\omega_1 = \pm (i - B^2 m^{-1} \eta')$, and in the $\omega_2$-plane, there are two local saddle points at $\omega_2 = \pm (i + B^2 m^{-1} \eta')$.
\end{itemize}

We then move the contours so that they pass through these local saddle points, and in a neighborhood of the local saddle points they agree with the local contours of $\pm i$.

\(^4\)Note that for $\alpha = -1/\sqrt{2}$ we have $\eta = 0$, so the four saddle points at $\pm i \pm B^2 m^{-1} \eta$ become two degenerate saddle points at $\pm i$. 

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steepest descent. When the asymptotic coordinate $\alpha$ defined in Equation 1.2 satisfies $\alpha < -1/\sqrt{2}$, we must cross the contours in $B_{\varepsilon_1,\varepsilon_2}(a, x_1, x_2, y_1, y_2)$ over each other as shown in Figure 18, picking up a single integral contribution.

We show that the only significant contributions to the double integrals over the new contours come from a neighborhood of $(i, i) \in \mathbb{C}^2$ and a neighborhood of $(-i, -i) \in \mathbb{C}^2$, and moreover that these contributions are the same. Similarly, the only significant contributions to the new single integral come from neighborhoods of $\pm i \in \mathbb{C}$, and these contributions are equal.

So we can write the integrals as integrals over contours in a neighborhood of $i$ in each variable, plus lower order terms. We take an asymptotic expansion of the whole of each integrand around $i$ in each variable, do a change of variables, and throw away terms of order $m^{-1}$. We then extend the local steepest descent contours in the new variables to infinity, which has a negligible effect on the value of the integral. These contours are given in Appendix A.1.1, and are shown graphically in Figure 9. We find that

$$K_a(y, x)B_{\varepsilon_1,\varepsilon_2}(a, x_1, x_2, y_1, y_2)$$

$$= \left\{ \begin{array}{ll}
- \frac{\zeta(x, y)}{32\pi^2} Bm^{-1/2}I_1(\alpha, \varepsilon_1, \varepsilon_2) + O(m^{-1}) & \text{if } -1/\sqrt{2} < \alpha < 0 \\
- \frac{\zeta(x, y)}{32\pi^2} Bm^{-1/2}(I_1(\alpha, \varepsilon_1, \varepsilon_2) + 8\pi I_0(\alpha, \varepsilon_1, \varepsilon_2)) + O(m^{-1}) & \text{if } \alpha < -1/\sqrt{2},
\end{array} \right.$$ 

$$\frac{i}{a}(-1)^{\varepsilon_1+\varepsilon_2}K_a(y, x)B_{1-\varepsilon_1,\varepsilon_2}(1/a, 2n - x_1, x_2, 2n - y_1, y_2)$$

$$= - \frac{\zeta(x, y)}{32\pi^2} Bm^{-1/2}I_2(\alpha, \varepsilon_1, \varepsilon_2) + O(m^{-1}),$$

$$\frac{i}{a}(-1)^{\varepsilon_1+\varepsilon_2}K_a(y, x)B_{\varepsilon_1,1-\varepsilon_2}(1/a, x_1, 2n - x_2, y_1, 2n - y_2)$$

$$= - \frac{\zeta(x, y)}{32\pi^2} Bm^{-1/2}I_3(\alpha, \varepsilon_1, \varepsilon_2) + O(m^{-1}),$$

and

$$K_a(y, x)B_{1-\varepsilon_1,1-\varepsilon_2}(a, 2n - x_1, 2n - x_2, 2n - y_1, 2n - y_2)$$

$$= - \frac{\zeta(x, y)}{32\pi^2} Bm^{-1/2}I_4(\alpha, \varepsilon_1, \varepsilon_2) + O(m^{-1}).$$

Below we will give some formulas that are part of our results but are messy, as well as precise definitions of the contours of integration for the integrals $I_j(\alpha, \varepsilon_1, \varepsilon_2)$.
Figure 18: The deformed contours for the integral $B_{\varepsilon_1, \varepsilon_2}(a, x_1, x_2, y_1, y_2)$ with $B = 1$, $\alpha = -3$, $m = 64$.

A Expressions used in the text

A.1 Expressions from Section 1

For $i, j \in \{0, 1\}$ and $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$, define

\[
A_{\varepsilon_1, \varepsilon_2}^{i, j}(w, z) = \frac{-(-1)^{\varepsilon_1 + \varepsilon_2}}{\sqrt{1/2 - 2iw} \sqrt{1/2 + 2iw}} \left( \sqrt{1/2 - 2iz} \sqrt{1/2 + 2iz} \right) \\
+ (-1)^{\varepsilon_1 + \varepsilon_2} \left( \sqrt{1/2 - 2iw} + (-1)^i \sqrt{1/2 - 2iz} \right) \left( (-1)^j \sqrt{1/2 + 2iw} + \sqrt{1/2 + 2iz} \right) \\
+ \left( (-1)^{\varepsilon_1} \sqrt{1/2 - 2iw} + (-1)^{\varepsilon_2 + j} \sqrt{1/2 + 2iz} \right) \\
+ \left( (-1)^{\varepsilon_2} \sqrt{1/2 + 2iz} + (-1)^{\varepsilon_1 + i} \sqrt{1/2 - 2iz} \right) \\
\times \left( \sqrt{1/2 - 2iw} \sqrt{1/2 + 2iz} + (-1)^{i + j} \sqrt{1/2 + 2iw} \sqrt{1/2 - 2iz} \right).
\]

Let $\eta = \eta(\alpha)$ be defined to be the unique complex number with non-negative real part and non-negative imaginary part that satisfies

\[
(A.2) \quad \frac{1}{\sqrt{1/2 - 2i\eta}} + \frac{1}{\sqrt{1/2 + 2i\eta}} = -2/\alpha.
\]
When \(-1/\sqrt{2} < \alpha < 0\), \(\eta \in i(0,1/4)\); when \(\alpha < -1/\sqrt{2}\), \(\eta \in (0,\infty)\); and when \(\alpha = -1/\sqrt{2}\), \(\eta = 0\).

Let \(\eta'\) be the unique complex number that satisfies

\[
\frac{1}{\sqrt{1/2 - 2i\eta'}} - \frac{1}{\sqrt{1/2 + 2i\eta'}} = \frac{2}{\alpha}.
\]

For all \(\alpha < 0\) we have \(\eta' \in i(0,1/4)\).

### A.1.1 Contours

The contours \(C_0, C'_0, C_1\) and \(C'_1\) are defined as follows. Let \(\eta\) be as defined in Equation A.2 and let \(\eta'\) be as defined in Equation A.3. Recall \(f^\pm(w)\) defined in Equation 1.7.

Recall that along a steepest descent contour of a function, its imaginary part is constant. A function has a saddle point when its second derivative is 0. The function 

\[-2iw + \alpha_x f^-(w)\]

has saddle points at \(w = \pm \eta\) and the function 

\[-2iw + \alpha_x f^+(w)\]

has a saddle point at \(w = -\eta'\).

All contours are oriented in the direction of decreasing real part.

For \(-1/\sqrt{2} < \alpha < 0\), let \(C_0\) be the steepest descent contour for 

\[-2iw + \alpha_x f^-(w)\]

that is contained in the negative half plane and passes through the saddle point \(w = -\eta\).

For \(\alpha = -1/\sqrt{2}\) let \(C_0\) be the steepest descent contour for 

\[-2iw + \alpha_x f^-(w)\]

that passes through the saddle point \(w = 0\) and enters the negative half plane at angles of 

\(-\pi/6\) and \(-5\pi/6\).

For \(\alpha < -1/\sqrt{2}\), let \(C_0\) consist of the steepest descent contour for 

\[-2iw + \alpha_x f^-(w)\]

that starts from the branch cut \(i(1/4,\infty)\), passes through the saddle point \(w = -\eta\) and goes to infinity in the third quadrant; the reflection in the imaginary axis of this contour; and a contour that goes around the branch cut \(i(1/4,\infty)\). This contour is shown in detail in Figure 19.

Let \(C'_0\) be the reflection of \(C_0\) in the real axis.

Let \(C_1\) be the steepest descent contour for 

\[-2iw + \alpha_x f^+(w)\]

This passes through 

\(w = -\eta'\) and goes to infinity in the negative half plane.

Let \(C'_1\) be the reflection of \(C_1\) in the real axis.

### A.2 Expressions from Section 2

Let \(x \in \mathcal{W}\) and \(y \in \mathcal{B}\) be vertices near the diagonal in the third quadrant that are connected by an edge. Let \(\xi = ((x_1 + y_1)/2 - n)/n\). We have \(-1 < \xi < 0\). Let \(\omega_c\) be the solution to

\[1 + \xi \left(\frac{\omega}{\sqrt{\omega^2 + 2c}} + \frac{\omega^{-1}}{\sqrt{\omega^{-2} + 2c}}\right) = 0\]

that lies in the first quadrant (including axes). Let \(V_0 = -2\log|G(\omega_c)|\). Recall \(K_{a,H,V}^{-1}(x,y)\) defined in Equation 1.1. Let \(W_a(x,y)\) be defined as

\[
W_a(x,y) = \begin{cases} 
K_{a,0,0}^{-1}(x,y) & \text{if } -1/2\sqrt[3]{1-2c} \leq \xi < 0 \\
K_{a,0,V_0}^{-1}(x,y) & \text{if } \xi < -1/2\sqrt[3]{1-2c}
\end{cases}
\]
where $c = 1/(a + a^{-1})$. This is the whole plane inverse Kasteleyn matrix corresponding to the ergodic Gibbs measure of fixed slope where the weight $a$ is fixed.

Now define $\nu_a(x, y) = K_a(y, x)W_a(x, y)$. This is the one-point correlation corresponding to the ergodic Gibbs measure of fixed slope where the weight $a$ is fixed.

A.3 Expressions from Section 3

Let

$$c = \frac{1}{a + a^{-1}}.$$  

Since we are assuming $a \in (0, 1)$, we have $c \in (0, 1/2)$. For $\omega \in \mathbb{C} \setminus i[-\sqrt{2c}, \sqrt{2c}]$ we define

$$(A.4) \quad \sqrt{\omega^2 + 2c} = i\sqrt{-i(\omega + i\sqrt{2c})}\sqrt{-i(\omega - i\sqrt{2c})}$$

where the square roots on the right hand side are the principal branch of the square root.

Note that these have branch cuts for $\omega = it$ with $t < -\sqrt{2c}$ and $t < \sqrt{2c}$ respectively, but for $t < -\sqrt{2c}$ the branch cuts cancel. Define

$$(A.5) \quad G(\omega) = \frac{1}{\sqrt{2c}}(\omega - \sqrt{\omega^2 + 2c}).$$

For even $x_1, x_2$ with $0 < x_1, x_2 < 2n$ define

$$(A.6) \quad H_{x_1, x_2}(\omega) = \frac{\omega^{2m}G(\omega)^{2m-\frac{x_1}{2}}}{G(\omega^{-1})^{2m-\frac{x_2}{2}}}.$$  

Let $\Gamma_r$ denote a positively oriented contour of radius $r$ centered at the origin. For $n = 4m$, $0 < a < 1$, $x = (x_1, x_2) \in W_{\varepsilon_1}$, $y = (y_1, y_2) \in B_{\varepsilon_2}$ with $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$,
\[ 0 < x_1, x_2, y_1, y_2 < n, \text{ and } \sqrt{2} c < r < 1, \text{ define} \]

\[ (A.7) \quad B_{\epsilon_1, \epsilon_2}(a, x_1, x_2, y_1, y_2) = \frac{i^{(x_2-x_1+y_1-y_2)/2}}{(2\pi i)^2} \]
\[ \times \int_{\Gamma_r} \frac{dw_1}{w_1} \int_{\Gamma_1/r} \frac{dw_2}{w_2} \frac{\omega_2}{\omega_2^2 - \omega_1^2} \frac{H_{x_1+1,x_2}(\omega_1)}{H_{y_1,y_2+1}(\omega_2)} \sum_{\gamma_1, \gamma_2=0}^{1} Q_{\gamma_1, \gamma_2}^{\epsilon_1, \epsilon_2}(\omega_1, \omega_2), \]

\[ (A.8) \quad a^{-1} B_{1-\epsilon_1, \epsilon_2}(a^{-1}, 2n - x_1, x_2, 2n - y_1, y_2) = -\frac{i^{(x_1-x_2-y_1-y_2)/2}}{(2\pi i)^2} \]
\[ \times \int_{\Gamma_r} \frac{dw_1}{w_1} \int_{\Gamma_1/r} \frac{dw_2}{w_2} \frac{\omega_2}{\omega_2^2 - \omega_1^2} \frac{H_{x_1+1,x_2}(\omega_1)}{H_{y_1,y_2+1}(\omega_2)} \sum_{\gamma_1, \gamma_2=0}^{1} (-1)^{\epsilon_2+\gamma_2} Q_{\gamma_1, \gamma_2}^{\epsilon_1, \epsilon_2}(\omega_1, \omega_2) \]

\[ (A.9) \quad a^{-1} B_{\epsilon_1, 1-\epsilon_2}(a^{-1}, x_1, 2n - x_2, y_1, 2n - y_2) = -\frac{i^{(y_2-x_1-x_2)/2}}{(2\pi i)^2} \]
\[ \times \int_{\Gamma_r} \frac{dw_1}{w_1} \int_{\Gamma_1/r} \frac{dw_2}{w_2} \frac{\omega_2}{\omega_2^2 - \omega_1^2} \frac{H_{x_1+1,x_2}(\omega_1)}{H_{y_1,y_2+1}(\omega_2)} \sum_{\gamma_1, \gamma_2=0}^{1} (-1)^{\epsilon_1+\gamma_1} Q_{\gamma_1, \gamma_2}^{\epsilon_1, \epsilon_2}(\omega_1, \omega_2) \]

\[ (A.10) \quad B_{1-\epsilon_1, 1-\epsilon_2}(a, 2n - x_1, 2n - x_2, 2n - y_1, 2n - y_2) = -\frac{i^{(y_2+y_1+x_2+x_1)/2}}{(2\pi i)^2} \]
\[ \times \int_{\Gamma_r} \frac{dw_1}{w_1} \int_{\Gamma_1/r} \frac{dw_2}{w_2} \frac{\omega_2}{\omega_2^2 - \omega_1^2} \frac{H_{x_1+1,x_2}(\omega_1)}{H_{y_1,y_2+1}(\omega_2)} \sum_{\gamma_1, \gamma_2=0}^{1} (-1)^{\epsilon_1+\gamma_1+\epsilon_2+\gamma_2} Q_{\gamma_1, \gamma_2}^{\epsilon_1, \epsilon_2}(\omega_1, \omega_2) \]

where \( Q_{\gamma_1, \gamma_2}^{\epsilon_1, \epsilon_2}(\omega_1, \omega_2) \) is defined as follows. We temporarily consider weights \( a \) and \( b \) where \( b \) is not necessarily 1. Let

\[ f_{a,b}(u, v) = (2a^2 uv + 2b^2 uv - ab(-1 + u^2)(-1 + v^2)) \times (2a^2 uv + 2b^2 uv + ab(-1 + u^2)(-1 + v^2)) \]

and define

\[ y^{0,0}_{0,0}(a, b, u, v) = \frac{1}{4(a^2 + b^2)^2 f_{a,b}(u, v)}(2a^7 v^2 - a^5 b^2(1 + u^4 + u^2 v^2 - v^4) - a^3 b^4(1 + 3u^2 + 3v^2 + 2u^2 v^2 + u^4 v^2 + u^2 v^4 - v^4)) \]
\[ - ab^6(1 + v^2 + u^2 + 3u^2 v^2)) \]

\[ y^{0,0}_{0,1}(a, b, u, v) = \frac{1}{4(a^2 + b^2)^2 f_{a,b}(u, v)} a(b^2 + a^2 u^2)(2a^2 v^2 + b^2(1 + v^2 - u^2 + u^2 v^2)) \]

\[ y^{0,0}_{1,0}(a, b, u, v) = \frac{1}{4(a^2 + b^2)^2 f_{a,b}(u, v)} a(b^2 + a^2 v^2)(2a^2 u^2 + b^2(1 - u^2 + v^2 + u^2 v^2)) \]

\[ y^{0,0}_{1,1}(a, b, u, v) = \frac{1}{4 f_{a,b}(u, v)} a(2a^2 u^2 v^2 + b^2(-1 + v^2 + u^2 + u^2 v^2)). \]
Then for $\gamma_1, \gamma_2 \in \{0, 1\}$ define
\[
y_{\gamma_1, \gamma_2}^{0,1}(a, b, u, v) = y_{\gamma_1, \gamma_2}^{0,0}(b, a, u, v^{-1}) \frac{1}{v^2},
\]
\[
y_{\gamma_1, \gamma_2}^{1,0}(u, v) = y_{\gamma_1, \gamma_2}^{0,0}(b, a, u^{-1}, v) \frac{1}{u^2},
\]
\[
y_{\gamma_1, \gamma_2}^{1,1}(u, v) = y_{\gamma_1, \gamma_2}^{0,0}(a, b, u^{-1}, v^{-1}) \frac{1}{u^2 v^2}.
\]
When $b = 1$, we write $y_{\gamma_1, \gamma_2}^{\varepsilon_1, \varepsilon_2}(a, b, u, v) = y_{\gamma_1, \gamma_2}^{\varepsilon_1, \varepsilon_2}(u, v)$. Then define
\[
x_{\gamma_1, \gamma_2}^{\varepsilon_1, \varepsilon_2}(\omega_1, \omega_2) = \frac{G(\omega_1)G(\omega_2)}{\prod_{j=1}^2 \sqrt{\omega_j^2 + 2c \omega_j^{-2} + 2c}} y_{\gamma_1, \gamma_2}^{\varepsilon_1, \varepsilon_2}(G(\omega_1), G(\omega_2))(1 - \omega_1^2 \omega_2^2).
\]
Also let
\[
t(\omega) = \omega \sqrt{\omega^2 + 2c}.
\]
Finally we define
\[
(A.11) \quad Q_{\gamma_1, \gamma_2}^{\varepsilon_1, \varepsilon_2}(\omega_1, \omega_2) = (-1)^{\varepsilon_1 + \varepsilon_2 + \varepsilon_1 \varepsilon_2 + \gamma_1 (1 + \varepsilon_2) + \gamma_2 (1 + \varepsilon_1)} \\
\times t(\omega_1)^{\gamma_1} t(\omega_2)^{\gamma_2} G(\omega_1)^{\varepsilon_1} G(\omega_2)^{\varepsilon_2} x_{\gamma_1, \gamma_2}^{\varepsilon_1, \varepsilon_2}(\omega_1, \omega_2^{-1}).
\]

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