CONVERGENCE AND STRUCTURE THEOREMS FOR ORDER-PRESERVING DYNAMICAL SYSTEMS WITH MASS CONSERVATION

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Abstract. We establish a general theory on the existence of fixed points and the convergence of orbits in order-preserving semi-dynamical systems having a certain mass conservation property (or, equivalently, a first integral). The base space is an ordered metric space and we do not assume differentiability of the system nor do we even require linear structure in the base space. Our first main result states that any orbit either converges to a fixed point or escapes to infinity (convergence theorem). This will be shown without assuming the existence of a fixed point. Our second main result states that the existence of one fixed point implies the existence of a continuum of fixed points that are totally ordered (structure theorem). This latter result, when applied to a linear problem for which 0 is always a fixed point, automatically implies the existence of positive fixed points. Our result extends the earlier related works by Arino (1991), Mierczyński (1987) and Banaji-Angeli (2010) considerably with exceedingly simpler proofs. We apply our results to a number of problems including molecular motor models with time-periodic or autonomous coefficients, certain classes of reaction-diffusion systems and delay-differential equations.

1. Introduction. In this paper we study order-preserving dynamical systems having a certain kind of “mass conservation property” and establish a general theory on the convergence of bounded orbits and the existence of fixed points. We will then apply our general theory to a number of problems including reaction-diffusion equations possibly with nonlocal terms and some delay-differential equations. Our theory implies the existence of multiple stationary and/or time-periodic solutions as an immediate corollary.

Our present study was originally motivated by the analysis of mathematical models for transportation in molecular motor systems and those for chemical reversible...
reaction models. While studying these models and examining earlier works in the literature, we realized that the same convergence results and/or the same existence results can be derived more easily in an abstract framework of order-preserving systems with mass conservation. The advantage of this new approach is that the entire argument is rather straightforward and elementary, yet it can produce useful convergence and existence results without the use of a Lyapunov functional nor spectral analysis.

To be more precise, our abstract general theory roughly states the following:

(i) any orbit either converges to a fixed point or escapes to infinity as time tends to infinity;
(ii) if there exists at least one fixed point (which corresponds to a stationary or a time-periodic solution of the model equations), then there exist infinitely many of them, and the set of all the fixed points is totally ordered, connected and unbounded.

In particular, our general results imply that if the model equation possesses a trivial stationary or time-periodic solution (such as 0), then there are automatically infinitely many nontrivial stationary or time-periodic solutions.

There are some earlier works that deal with a general theory similar to ours, namely order-preserving dynamical systems with mass conservation—or systems with a monotone first integral ([2, 3, 19, 20]). (To be more precise, [2] deals with strongly monotone semiflows on a Banach lattice, [3] deals with strongly monotone semiflows on \( \mathbb{R}^n \), and [19, 20] deal mainly with finite-dimensional irreducible cooperative systems; all of these systems are special cases of what is covered by the present paper.) As we will explain at the end of Section 2, our main results in the present paper are much more general than those earlier results, so that they have a considerably wider range of applications. Furthermore, the proofs of our results are exceedingly simple and elementary, as we will see in Section 3.

Let us also mention the related results in [21, 15], which deal with systems of ODE’s of the cooperative type and establish convergence results that are similar to ours. Cooperative ODE systems are classical examples of finite dimensional order-preserving systems, but since [21, 15] do not assume irreducibility of the systems, the strong comparison principle (F2′) defined in Section 2 does not necessarily hold for their systems. Consequently, our general theory does not cover their results: see Remark 3. In the forthcoming paper [18], we will study such systems and establish a general theory that extends the results in [21, 15] to infinite dimensional systems.

This paper is organized as follows. Section 2 is devoted to the abstract theory, where we state our main theorems—the convergence theorem (Theorem 2.1) and the structure theorem (Theorem 2.2)—and some related results including a Liouville type theorem for bounded entire orbits. We also state the semiflow version of the main theorems (Theorems 2.5 and 2.6), which can be derived easily from Theorems 2.1 and 2.2. At the end of Section 2, we compare our results with earlier results in the literature. The proof of our main theorems will be given in Section 3. In Section 4, we apply our abstract results to a number of problems including molecular motor models, reversible chemical reaction models and two-component competition diffusion systems. In Section 5, we apply our results to a certain class of delay-differential equations.

2. Basic concepts and results. We first present our abstract results in the framework of discrete-in-time semi-dynamical systems (map dynamics). Parallel results
on time-continuous systems (semiﬂows) will then follow easily, which we discuss at the end of this section.

Let \((X, d, \preceq)\) be an ordered metric space, that is, a complete metric space with partial order relation \(\preceq\) which is closed under the limiting procedure:

\[ u_n \preceq v_n \ (n = 1, 2, \ldots), \quad u_n \to u_\infty, \quad v_n \to v_\infty \implies u_\infty \preceq v_\infty. \]

For \(u, v \in X\), we write

\[ u \prec v \quad \text{if} \quad u \preceq v \quad \text{and} \quad u \neq v \]

and let \([u, v]\) denote the order interval \(\{w \in X \mid u \preceq w \preceq v\}\).

We assume that the space \(X\) has the following properties:

(X0) Any point in \(X\) is not isolated from above; that is, for any \(u \in X\) and \(\delta > 0\), there exists \(v\) satisfying \(u \prec v\) and \(d(u, v) < \delta\).

(X1) Any pair of points \(u, v \in X\) has the least upper bound

\[ u \lor v := \min\{w \in X \mid u \preceq w, v \preceq w\}. \quad (1) \]

Furthermore, the map \((u, v) \mapsto u \lor v: X \times X \to X\) is continuous.

The condition (X0) is automatically satisfied if \(X\) is an ordered Banach space, but we do not require any linear structure of \(X\) in the present paper.

Now we consider a dynamical system defined by a map \(F: X \to X\). We assume that

(F0) \(F: X \to X\) is compact and order-compact.

In other words, \(F\) is continuous and maps any bounded set into a relatively compact set. Furthermore, the image of any order interval \([u, v]\) by \(F\) is relatively compact. We also assume that

(F1) \(F\) is order-preserving, namely, \(u \preceq v\) implies \(F(u) \preceq F(v)\);

(F2) \(F(u \lor v) \succ F(u) \lor F(v)\) if \(u \npreceq v\) and \(u \npreceq v\).

Note that (F1) implies \(F(u \lor v) \succeq F(u) \lor F(v)\) for any \(u, v \in X\). The condition (F2) asserts that the equality does not hold if \(u\) and \(v\) are unordered.

As we will show later (see Theorem 2.3), the conditions (F0) and (F2) can be relaxed as follows:

(F0′) \(F\) is continuous and there exists \(m \in \mathbb{N}\) such that \(F^m: X \to X\) is compact and order-compact.

(F2′) There exists \(m \in \mathbb{N}\) such that \(F^m(u \lor v) \succ F^m(u) \lor F^m(v)\) if \(u \npreceq v\) and \(u \npreceq v\).

Such a generalization will be useful when we apply our theory to delay-differential equations; see Section 5.

Next let \(M: X \to \mathbb{R}\) be a continuous map satisfying

(M1) \(u \prec v\) implies \(M(u) < M(v)\);

(M2) \(M(F(u)) = M(u)\) for \(u \in X\).

The condition (M2) implies that \(M(u)\) is a conserved quantity along each orbit generated by \(F\). Thus we may call (M2) “mass conservation” or regard \(M\) as a “first integral” of the dynamical system defined by \(F\). The condition (M1) implies that this conserved quantity \(M(u)\) is strictly increasing in \(u\).
Remark 1. If $X$ is a Banach lattice, then every order interval $[u, v]$ is bounded, therefore the compactness of a map $F: X \to X$ in (F0) automatically implies order-compactness; see, e.g., [27]. In the present paper, however, we do not assume the lattice structure, nor do we even require that $X$ be a linear space.

Remark 2. The combination of (F1) and (F2) (or (F2′)) above is slightly stronger than the usual order-preserving property, but it is weaker than the so-called “strongly monotone property” or the “strongly order-preserving property”. Here a continuous map $F: X \to X$ is called strongly monotone, if $u_1 \prec u_2$ implies $v_1 \prec v_2$ for all $v_1, v_2$ that are sufficiently close to $F(u_1), F(u_2)$, respectively (see [14]). Similarly, a map $F: X \to X$ is called strongly order-preserving, if $u_1 \prec u_2$ implies $F(\bar{u}_1) \prec F(\bar{u}_2)$ for all $\bar{u}_1, \bar{u}_2$ that are sufficiently close to $u_1, u_2$, respectively (see [17]). Obviously, if $F$ is strongly monotone, then it is strongly order-preserving. System (18) in Subsection 4.5 is an example that is not strongly order-preserving but satisfies (F1) and (F2) (more precisely (Φ1) and (Φ2)).

Our main abstract results are the following:

Theorem 2.1 (Convergence theorem). For any $u \in X$, the following alternatives holds:
(a) $F^n(u)$ converges to a fixed point of $F$ as $n \to \infty$; or
(b) $F^n(u)$ escapes to infinity, that is, $\lim_{n \to \infty} d(F^n(u), w) = \infty$ for any $w \in X$.

Theorem 2.2 (Structure theorem). Let $E$ denote the set of all the fixed points of $F$. If $E \neq \emptyset$, then $E$ is a totally ordered and connected set. Furthermore, $E$ is unbounded from above, that is, $E$ has no upper bound.

We remark that the existence of a fixed point of $F$ is not assumed in Theorem 2.1. This theorem asserts, among other things, that the existence of a bounded orbit—whatever it is—implies the existence of a fixed point of $F$. Theorem 2.2 asserts that the existence of a fixed point of $F$ implies the existence of infinitely many of them. The following is another immediate consequence of Theorem 2.1.

Corollary 1. For any integer $m \geq 2$, let $E$ and $E_m$ denote the set of all the fixed points of $F$ and $F^m$, respectively. Then $E = E_m$.

The above corollary states that $F$ possesses no periodic points other than fixed points. Such a statement is not necessarily true for general order-preserving maps. It is a remarkable feature of an order-preserving map with mass conservation. Indeed, if we drop the assumption (M2), there are plenty of examples of (strongly) order-preserving maps that have cycles. Furthermore, the papers [23, 24] give an example of strongly order-preserving map that has a stable cycle. (More precisely, the authors of [23, 24] construct a reaction-diffusion equation with time-periodic coefficient that has a stable time-periodic solution of period $mT$ with $m \geq 2$.)

Finally the following theorem asserts that the assumptions (F0) and (F2) can be somewhat relaxed, which allows our abstract theory to be applicable to some classes of delay-differential equations (see subsections 5.1 and 5.2).

Theorem 2.3. The conclusions of Theorems 2.1 and 2.2 remain true if the assumption (F0) and (F2) are relaxed by (F0′) and (F2′), respectively.

Remark 3. As mentioned in Section 1, the papers [21, 15] prove convergence results similar to our Theorem 2.1 for finite-dimensional cooperation systems (that is, ODE systems of the cooperative type), possibly with time-periodic coefficients. As they
do not assume that their cooperation system to be irreducible, their systems do not necessarily satisfy the strong comparison principle (F2'). Consequently, our Theorem 2.1 does not fully cover their results. Let us also note that the papers [28, 31] extends the results of [21] to cooperation systems with almost periodic in time coefficients. The arguments in [21, 15, 28, 31] rely heavily on the particular structure of finite-dimensional cooperation systems, which form a special class of order-preserving systems. In the forthcoming paper [18], we will establish a general theory that extends the results in [21, 15] to infinite dimensional systems.

Let us also present some related results on ancient orbits and entire orbits. A sequence of points \( \{ \ldots, u_{-2}, u_{-1}, u_0 \} \) is called an ancient orbit emanating from \( u_0 \) if
\[
  u_{n+1} = F(u_n) \quad (n = -1, -2, -3, \ldots).
\]
A two-sided sequence \( \{ \ldots, u_{-2}, u_{-1}, u_0, u_1, u_2, \ldots \} \) is called an entire orbit if
\[
  u_{n+1} = F(u_n) \quad (n \in \mathbb{Z}).
\]

**Theorem 2.4.** Let \( \{ \ldots, u_{-2}, u_{-1}, u_0 \} \) be an ancient orbit. Then either of the following holds:

(a) There exists a fixed point \( v \) of \( F \) such that \( u_n = v \ (n = 0, -1, -2, \ldots) \);

(b) \( u_n \) escapes to infinity as \( n \to -\infty \), that is, \( \lim_{n \to -\infty} d(u_n, w) = \infty \) for any \( w \in X \).

An immediate consequence of the above theorem is the following:

**Corollary 2** (Liouville type result). Let \( \{ u_n \}_{n \in \mathbb{Z}} \) be a bounded entire orbit. Then \( u_n = v \ (n \in \mathbb{Z}) \) for some fixed point of \( F \).

**Results on time-continuous systems**

Next we present results for time-continuous systems (semiflows) that are parallel to Theorems 2.1 and 2.2 above. Let \( X \) be an ordered metric space as before, and let \( \Phi = \{ \Phi_t \}_{t \geq 0} \) be a semiflow on \( X \), namely \((t, u) \mapsto \Phi_t(u)\) is a continuous map from \([0, \infty) \times X\) to \( X \) with the following property:
\[
  \Phi_0(u) = u, \quad \Phi_t(\Phi_s(u)) = \Phi_{t+s}(u) \quad \text{for all } u \in X, \ t, s \geq 0.
\]

Now we assume \( \Phi \) has the following properties that correspond to (F0)–(F2) and (M2) :

\[
  (\Phi 0) \quad \Phi_t: X \to X \text{ is compact and order-compact for each } t > 0;  
  (\Phi 1) \quad \Phi_t \text{ is order-preserving, namely, } u \preceq v \text{ implies } \Phi_t(u) \preceq \Phi_t(v) \text{ for each } t > 0;  
  (\Phi 2) \quad \Phi_t(u \lor v) > \Phi_t(u) \lor \Phi_t(v) \text{ if } u \not\preceq v \text{ and } u \not\succeq v \text{ for each } t > 0;  
  (M2) \quad M(\Phi_t(u)) = M(u) \text{ for } u \in X, \ t > 0.
\]

Our results are stated as follows:

**Theorem 2.5** (Convergence theorem). For any \( u \in X \), the following alternatives holds:

(a) \( \Phi_t(u) \) converges to an equilibrium point as \( t \to \infty \); or

(b) \( \Phi_t(u) \) escapes to infinity, that is, \( \lim_{n \to \infty} d(F^n(u), w) = \infty \) for any \( w \in X \).

\[
  \lim_{n \to \infty} d(F^n(u), w) = \infty \quad \text{for any } w \in X.
\]
Theorem 2.6 (Structure theorem). Let $E$ denote the set of all the equilibrium points of $\Phi$. If $E \neq \emptyset$, then $E$ is a totally ordered and connected set. Furthermore, $E$ is unbounded from above, that is, $E$ has no upper bound.

Remark 4. If $\{\Phi_t\}_{t \geq 0}$ is a strongly order-preserving (or strongly monotone) semiflow having a certain compactness property, then the “almost everywhere convergence theorem” of Hirsch [14] asserts that the $\omega$-limit set of “almost every” bounded orbit is contained in the set of equilibria. Under some extra hypotheses on $\Phi_t$ such as analyticity and stability of equilibrium points, Takač [30] and Smith and Thieme [29] show that almost every bounded orbit converges to an equilibrium point. There is also an earlier work by Polacik [22], who obtained such results for abstract semilinear parabolic evolution systems. However, these results do not exclude the possibility that some “thin set” of orbits do not stabilize at all. An easy counterexample is an advection-diffusion equation $u_t = u_{xx} + cu_x + f(u)$ on a circle $\mathbb{R}/\mathbb{Z}$, which can possess rotating waves if $c \neq 0$. Our Theorem 2.5 above shows that convergence occurs for all bounded orbits if we assume mass conservation.

Let us also note that the following are immediate consequence of Theorem 2.4 and Corollary 2. Here we omit the definition of ancient orbits and entire orbits for the semiflow.

Theorem 2.7. Let $\{p(t); t \leq 0\}$ be an ancient orbit of $\Phi$. Then either of the following holds:

(a) There exists an equilibrium point $v$ of $\Phi$ such that $p(t) = v$ ($t \leq 0$);
(b) \[ \lim_{t \to -\infty} d(p(t), w) = \infty \] for any $w \in X$.

Corollary 3 (Liouville type result). The only bounded entire orbits of $\Phi$ are equilibrium points.

Before closing this section, let us mention related earlier works on order-preserving systems with mass conservation and explain the difference between those results and ours.

Arino [2] considers an order-preserving semiflow $\{\Phi_t\}_{t \geq 0}$ in a $B$-lattice—which is a special case of an ordered metric space with $(X0)$, $(X1)$ in the present paper—satisfying conditions similar to (but slightly stronger than) $(\Phi0)$-$(\Phi2)$ and $(M1)$, $(M2)$. Under the additional assumption that 0 is an equilibrium point (i.e. $\Phi_t(0) \equiv 0$), the author shows that for any $u \in X$ having the same mass as 0 (i.e. $M(u) = M(0)$), the orbit $\{\Phi_t(u)\}_{t \geq 0}$ converges to 0 as $t \to \infty$. In the setting of the present paper, this result can be restated as follows: assume that the orbit $\{\Phi_t(u_0)\}_{t \geq 0}$ is precompact in $X$ and that there exists an equilibrium point $v$ such that $M(u_0) = M(v)$. Then $\Phi_t(u_0) \to v$ as $t \to \infty$. Our convergence theorem (Theorem 2.5) is clearly stronger than the above result as we do not need to assume the existence of an equilibrium point $v$ to start with. The existence of such an equilibrium point is part of our conclusion, not the assumption.

Mierczyński [19, 20] studies finite-dimensional cooperative ODE systems with a “first integral” (or “mass conservation” in our terminology). Assuming that the system is $C^1$ and irreducible (which implies strong monotonicity of the system), the author establishes a convergence results similar to our Theorem 2.5. The paper [20] also studies ancient orbits and proves results similar to our Theorem 2.7. Under some additional geometrical assumptions on the gradient of the first integral (the map $M$ in our terminology) [20] derives a kind of dichotomy theorem that states
that either all the orbits escape to infinity or all the orbits converge to some equilibria. This dichotomy theorem, in particular, implies the existence of infinitely many equilibria, which is thus closely related to our structure theorem (Theorem 2.6). Banaji-Angeli [3] extends some of the results in [20] to more general strongly monotone finite-dimensional systems and establishes results that are similar to our convergence theorem (Theorem 2.5). They show that if 0 is an equilibrium, then any positive orbit that is relatively close to 0 also converges to some positive equilibrium point, which implies the existence of infinitely many equilibrium points. Thus result is also related to our structure theorem (Theorem 2.6).

Note that the arguments in [19, 20] and [3] are limited to finite-dimensional systems, and the $C^1$ regularity of the first integral (the functional $M$ in our terminology) plays an important role. Our simpler proof in the present paper does not need such assumptions. We do not even require the base space $X$ to have linear structure. Furthermore, our main results hold not only for semiflows but also for time discrete systems (Theorems 2.1 and 2.2), thus allowing the theory to be applicable to systems with time-periodic coefficients.

3. Proof of the theorems.

3.1. Time-discrete systems. We first prove Theorem 2.1 (convergence theorem), then Theorem 2.2 (structure theorem). The proof of the latter is slightly more involved. We start with the following simple lemmas.

Lemma 3.1. Let $\bar{u}_1$, $\bar{u}_2$ be fixed points of $F$ satisfying $\bar{u}_1 \neq \bar{u}_2$. Then either $\bar{u}_1 \prec \bar{u}_2$ or $\bar{u}_1 \succ \bar{u}_2$.

Proof. Assume the contrary. Then from (F2) it follows that

$$F(\bar{u}_1 \lor \bar{u}_2) \succ F(\bar{u}_1) \lor F(\bar{u}_2) = \bar{u}_1 \lor \bar{u}_2.$$ 

Hence by (M1) we have $M(F(\bar{u}_1 \lor \bar{u}_2)) > M(\bar{u}_1 \lor \bar{u}_2)$, which contradicts (M2).

Lemma 3.2. Let $v$ be an $\omega$-limit point of $u \in X$. Then $M(v) = M(u)$.

Proof. Since $v$ is an $\omega$-limit point of $u$, there exist $n_1 < n_2 < n_3 \ldots \rightarrow \infty$ such that $F^{n_j}(u) \rightarrow v$ as $j \rightarrow \infty$. Hence by (M2) and the continuity of $M$, we have $M(v) = M\left(\lim_{j \rightarrow \infty} F^{n_j}(u)\right) = \lim_{j \rightarrow \infty} M(F^{n_j}(u)) = M(u)$.

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Assuming that $F^n(u)$ ($n = 1, 2, \ldots$) does not escape to infinity, we will show that $F^n(u)$ converges to a fixed point of $F$ as $n \rightarrow \infty$. Suppose that there exists a bounded subsequence $\{F^{n_j}(u)\}_{j=1, 2, \ldots}$ of $F^n(u)$ converging to a point in $X$, which we denote by $v$. Let us show that $v$ is a fixed point of $F$.

Suppose that $v \neq F(v)$. Since $M(v) = M(F(v))$ by (M2), the property (M1) implies $v \neq F(v)$ and $v \neq F(v)$. It follows from (F2) that $F(v \lor F(v)) \succ F(v) \lor F^2(v)$, hence

$$M(v \lor F(v)) = M(F(v \lor F(v))) > M(F(v) \lor F^2(v)).$$

On the other hand, (F1) implies $F(F^n(u) \lor F^{n+1}(u)) \succeq F^{n+1}(u) \lor F^{n+2}(u)$ for all $n \in \mathbb{N}$, which, together with (M1) and (M2), shows that $\{M(F^n(u) \lor F^{n+1}(u))\}_{n=1, 2, \ldots}$
is a monotone non-increasing sequence. Hence \( \lim_{n \to \infty} M(F^n(u) \vee F^{n+1}(u)) \) exists. In view of this, and recalling that \( v = \lim_{j \to \infty} F^{n_j+1}(u) \), we see that

\[
M(v \vee F(v)) = M(F(v) \vee F^2(v)) = \lim_{n \to \infty} M(F^n(u) \vee F^{n+1}(u))
\]

This contradicts (2). Therefore \( v \) is a fixed point of \( F \).

Next we show that \( \{F^n(u)\}_{n=1,2,...} \) converges to \( v \). Assume the contrary. Then there exist a constant \( r > 0 \) and a subsequence \( \{F^{n_j}(u)\}_{j=1,2,...} \) such that

\[
d(F^{n_j}(u), v) \leq r, \quad d(F^{n_j+1}(u), v) > r.
\]

Using the same argument as before, we see that \( \{F^{n_j+1}(u)\}_{j=1,2,...} \) converges to a fixed point of \( F \), which we denote by \( w \). By the above inequality, \( d(w, v) \geq r \), therefore \( v \neq w \). Hence by Lemma 3.1 we have \( v \succ w \) or \( v \prec w \). But this is impossible since Lemma 3.2 implies \( M(u) = M(v) = M(w) \). This contradiction shows that the orbit converges to \( v \).

**Proof of Corollary 1.** We only prove that \( E_m \subset E \) since the opposite inclusion is obvious. Let \( \pi \in E_m \). Then we have \( F^m(\pi) = \pi \), which means that the orbit

\[
\{F^n(\pi)\}_{n=0,1,2,...} = \{\pi, F(\pi), \ldots, F^{m-1}(\pi)\}
\]

is a cycle and hence it is bounded. Therefore, by Theorem 2.1, \( F^n(\pi) \) converges to some fixed point \( \nu \) of \( F \) as \( n \to \infty \). Thus we have

\[
\pi = F(\pi) = \cdots = F^{m-1}(\pi) = \nu \in E.
\]

The proof is complete. \( \square \)

Next we prove Theorem 2.2 (structure theorem). Before doing so, we define the notion of stability from above and present some preliminary lemmas.

A fixed point \( \pi \) of \( F \) is said to be **stable from above** if for any \( \varepsilon > 0 \) there exists some \( \delta > 0 \) such that

\[
d(u, \pi) < \delta, \quad u \succ \pi \implies d(F^n(u), \pi) < \varepsilon \quad (n = 1, 2, \ldots).
\]

**Lemma 3.3.** Any fixed point \( \pi \) of \( F \) is **stable from above**.

**Remark 5.** By Lemma 3.4 below, no fixed point of \( F \) is asymptotically stable.

**Proof of Lemma 3.3.** Suppose that \( \pi \) is not stable from above. Then there exists some \( \varepsilon > 0 \) and some converging sequence \( u_k \to \pi \) satisfying \( u_k \succ \pi \) and

\[
d(F^m(u_k), \pi) < \varepsilon \quad (m = 0, \ldots, n_k), \quad d(F^{n_k+1}(u_k), \pi) \geq \varepsilon
\]

for some \( n_k \in \mathbb{N} \). Since (3) implies that \( \{F^{n_k}(u_k)\} \) is a bounded sequence and since \( F \) is a compact map, without loss of generality we may assume that \( F^{n_k+1}(u_k) \) converges to some point \( \pi_0 \). Clearly \( \pi_0 \geq \pi, d(\pi_0, \pi) \geq \varepsilon \), hence \( \pi_0 \succ \pi \). Since \( M(u_k) = M(F^{n_k+1}(u_k)) \), and since \( u_k \to \pi \) as \( k \to \infty \), we see that

\[
M(\pi) = \lim_{k \to \infty} M(u_k) = \lim_{k \to \infty} M(F^{n_k+1}(u_k)) = M(\pi_0).
\]

This, however, contradicts (M1), since \( \pi_0 \succ \pi \). The lemma is proved. \( \square \)

**Lemma 3.4.** For any \( \pi \in E \) and any \( \delta > 0 \) there exists some \( \nu \in E \) satisfying \( \pi \prec \nu \) and \( d(\pi, \nu) < \delta \).
Proof. Suppose that there exist some \( \overline{u} \in E \) and some \( \delta_0 > 0 \) satisfying
\[
\{ u \in E \mid u \succ \overline{u}, \ d(u, \overline{u}) \leq \delta_0 \} = \emptyset. \tag{5}
\]
By (X0), there exists a sequence of points \( v_1, v_2, v_3, \ldots \in X \) satisfying \( v_k \succ \overline{u} \) and \( v_k \to \overline{u} \ (k \to \infty) \). Since \( \overline{u} \) is stable from above by virtue of Lemma 3.3, we have
\[
d(F^n(v_k), \overline{u}) < \delta_0 \quad (n = 1, 2, 3, \ldots) \tag{6}
\]
if \( k \) is chosen sufficiently large. Fix such \( k \). Then by Theorem 2.1, \( F^n(v_k) \) converges to some point \( u_0 \in E \) as \( n \to \infty \). By (6) we have \( d(u_0, \overline{u}) \leq \delta_0 \), hence, by (5), \( u_0 = \overline{u} \).

This contradiction proves the lemma. \( \Box \)

We also need the following lemma. Though this is a well-known property of an ordered set, we give an outline of the proof for the convenience of the reader.

Lemma 3.5. Let \( A \) be a totally ordered and compact subset of \( X \). Then \( A \) has the maximal element.

Proof. Since \( A \) is compact, for each \( n \in \mathbb{N} \) we can choose \( u_{n,1}, \ldots, u_{n,k_n} \in A \) satisfying
\[
A \subset U_{\frac{1}{n}}(u_{n,1}) \cup \cdots \cup U_{\frac{1}{n}}(u_{n,k_n}), \tag{7}
\]
where \( U_{\frac{1}{n}}(u) \) denotes the \( \frac{1}{n} \)-neighborhood of \( u \). Since \( u_{n,1}, \ldots, u_{n,k_n} \) are totally ordered, without loss of generality we may assume that \( u_{n,1} \) is the maximal element. By the compactness of \( A \), the sequence \( \{u_{n,1}\}_{n=1,2,\ldots} \) possesses a convergent subsequence, and let \( u_\infty \) denote its limit. Then it is easily seen that \( u_\infty = \max A \).

Details are omitted.

Now we are ready to prove the structure theorem.

Proof of Theorem 2.2. By Lemma 3.1, \( E \) is totally ordered. We first show that \( E \) is unbounded from above. Assume the contrary. Then \( E \) has an upper bound, which we denote by \( u^+ \). Fix \( u_0 \in E \) arbitrarily and set
\[
A = [u_0, u^+] \cap E.
\]
Since \( F(A) = A \) and since \( F \) is order-compact, \( A \) is a compact subset of \( E \). Therefore, by Lemma 3.5, \( A \) possesses the maximal element, which is also the maximal element of \( E \). This contradicts Lemma 3.4. Thus \( E \) is unbounded from above.

Next we show that \( E \) is connected. Suppose that \( E \) is not connected. Then there exist subsets \( O_1, O_2 \subset E \) that are open in the relative topology of \( E \) such that
\[
O_1, O_2 \neq \emptyset, \quad O_1 \cap O_2 = \emptyset, \quad O_1 \cup O_2 = E.
\]
Choose \( u_1 \in O_1 \) and \( u_2 \in O_2 \) arbitrarily. Since \( E \) is totally ordered, without loss of generality we may assume that \( u_1 \prec u_2 \). Put
\[
B = [u_1, u_2] \cap O_1 \quad (\subset E).
\]
Clearly \( u_1 \in B, u_2 \notin B \) and \( B \) is a totally ordered set in \( X \). Furthermore, since \( B = ([u_1, u_2] \cap E) \setminus O_2 \) and since \( F([u_1, u_2] \cap E) = [u_1, u_2] \cap E, B \) is compact. Hence, by Lemma 3.5, the maximal element of \( B \), denoted by \( \max B \), exists. Clearly \( u_1 \leq \max B \prec u_2 \) since \( u_1 \in B \) and \( u_2 \notin B \). By Lemma 3.4, there exists some convergent sequence \( \overline{v}_k \in O_1 \to \max B \) satisfying \( \max B \prec \overline{v}_k \). The inequality \( \max B \prec \overline{v}_k \) implies \( \overline{v}_k \notin B \). Therefore, \( \overline{v}_k \) cannot belong to \([u_1, u_2]\). Since \( E \) is
totally ordered, this implies \( u_2 < \bar{u}_k \). Letting \( k \to \infty \) yields \( u_2 \leq \max B \). This contradicts the fact that \( \max B \prec u_2 \). Thus \( E \) is connected. This completes the proof of the theorem.

\[ \Box \]

**Proof of Theorem 2.3.** First we show that the conclusion of Theorem 2.1 holds true. Suppose that the orbit \( \{F^n(u)\}_{n=0,1,2,...} \) does not escape to infinity. Then there exists a sequence of integers \( n_1 < n_2 < n_3 < \cdots \to \infty \) such that \( \{F^{n_j}(u)\}_{j=1,2,3,...} \) remains bounded. Replacing \( \{n_j\}_{j=1,2,3,...} \) by its subsequence if necessary, we may assume without loss of generality that \( n_j \equiv m_0 \mod m \) \( (j = 1, 2, 3, \ldots) \) for some \( m_0 \in \{0, \ldots, m-1\} \). Setting \( n_j = k_j m + m_0 \), we see that the sequence \( \{F^{k_j m}(F^{m_0}(u))\}_{j=1,2,3,...} \) remains bounded. Applying Theorem 2.1 to the map \( F^m \), we see that \( F^{km}(F^{m_0}(u)) \) converges to some point \( v \) as \( k \to \infty \), which is a fixed point of \( F^m \). Let us show that \( v \) is also a fixed point of \( F \). Assume the contrary. Then \( v \neq F(v) \). Since \( M(v) = M(F(v)) \), we have \( v \neq F(v) \) and \( v \neq F(v) \). Hence by (F2),

\[
F^m(v \vee F(v)) \succ F^m(v) \vee F^m(F(v)) = v \vee F(v).
\]

This and (M1) imply \( M(F^m(v \vee F(v))) > M(v \vee F(v)) \), which contradicts (M2). This contradiction shows that \( v = F(v) \). Consequently

\[
\lim_{k \to \infty} F^{km+j}(u) = \lim_{k \to \infty} F^{(k-1)m}(F^{m_0}(u)) = v
\]

for each \( j \in \{0, 1, \ldots, m-1\} \). This implies that the orbit \( \{F^n(u)\}_{n=0,1,2,...} \) converges to \( v \), which is a fixed point of \( F \).

Next we show that the conclusions of Theorem 2.2 hold. Applying Theorem 2.2 to \( F^m \), we see that the set of all fixed points of \( F^m \), which we denote by \( E \), is totally ordered, connected and unbounded from above. As we have seen above, any fixed point of \( F^m \) is a fixed point of \( F \), which implies \( E = E \). This completes the proof of the theorem.

\[ \Box \]

Now we prove the theorem on ancient orbits.

**Proof of Theorem 2.4.** Suppose that the alternative (b) does not hold. Then there exists a sequence of negative integers \( n_j \to -\infty \) as \( j \to \infty \) such that \( \{u_{n_j}\}_{j=1,2,3,...} \) is bounded. By the compactness of the map \( F \), we can choose a subsequence, again denoted by \( \{u_{n_j}\}_{j=1,2,3,...} \) such that \( u_{n_j+1} = F(u_{n_j}) \) converges to some point \( v \in X \). Arguing precisely the same way as in the proof of Theorem 2.1, we see that \( M(u_{n_j} \vee u_{n+1}) \) is monotonically non-increasing as \( n \) increases, and that this implies that \( v \) is a fixed point of \( F \). Set

\[
a := M(u_0), \quad b_n := M(u_n \vee u_{n+1}) \quad \text{for} \quad n = -1, -2, -3, \ldots.
\]

Then

\[
M(u_n) = M(v) = a, \quad b_n \geq a \quad \text{for} \quad n = -1, -2, -3, \ldots,
\]

and

\[
b_{-1} \leq b_{-2} \leq b_{-3} \leq \cdots.
\]  

(8)

Since \( u_{n_j+1} \to v \) as \( j \to \infty \) and since \( v \) is a fixed point of \( F \), we have

\[
b_{n_j+1} = M(u_{n_j+1} \vee F(u_{n_j+1})) \to M(v \vee F(v)) = M(v) = a \quad \text{as} \quad j \to \infty.
\]

Combining this, (8) and the inequality \( b_n \geq a \), we obtain \( b_n = a \) \( (n = -1, -2, \ldots) \), that is,

\[
M(u_n \vee u_{n+1}) = M(u_n) = M(u_{n+1}) \quad (n = -1, -2, -3, \ldots).
\]
This implies that \( u_n = u_{n+1} := F(u_n) \) \((n = -1, -2, \ldots)\), hence the alternative (a) holds. The theorem is proved.

3.2. Time-continuous systems. First we prepare the following lemma:

**Lemma 3.6.** For any \( \tau > 0 \), let \( E \) and \( E_\tau \) denote the set of all the equilibrium points and \( \tau \)-periodic points of \( \Phi \), respectively. Then \( E = E_\tau \).

**Proof.** Clearly \( E \subseteq E_\tau \). We show \( E_\tau \subseteq E \). Let \( \pi \in E_\tau \). Then, for any \( m \in \mathbb{N} \), since \( \pi \) is a fixed point of \((\Phi_{\tau/m})^m(= \Phi_\tau)\), by virtue of Corollary 1, it is also a fixed point of \( \Phi_{\tau/m} \). Hence it is a fixed point of \( \Phi_{l\tau/m} \) for all \( l \in \mathbb{N} \). Thus \( \pi \) is a fixed point of \( \Phi_{q\tau} \) for any rational number \( q > 0 \). By continuity, \( \Phi_{\tau}(\pi) = \pi \) for all \( t \geq 0 \), namely, \( \pi \in E \). Hence \( E_\tau \subseteq E \).

Theorem 2.6 immediately follows from Theorem 2.1 and Lemma 3.6.

**Proof of Theorem 2.6.** Fixed \( \tau > 0 \) arbitrarily and let \( E_\tau \) be the set of all the \( \tau \)-periodic points of \( \Phi \). By Lemma 3.6, \( E = E_\tau \). Now put \( F = \Phi_{\tau} \). Since \( E_\tau \) is the set of all the fixed points of \( F \), applying Theorem 2.2 we obtain the conclusion of Theorem 2.6.

**Proof of Theorem 2.5.** We show that, if the orbit \( \{\Phi_t(u)\}_{t \geq 0} \) does not escape to infinity, it converges to an equilibrium point. Suppose that the orbit \( \{\Phi_t(u)\}_{t \geq 0} \) does not escape to infinity. Then there exists a sequence \( t_j \to \infty \) as \( j \to \infty \) such that \( \{\Phi_{t_j}(u)\}_{j=1, 2, \ldots} \) is bounded. Since the map \( \Phi_0 : X \to X \) is compact, by taking a subsequence if necessary, we may assume that \( \Phi_{t_j+1}(u) = \Phi_1(\Phi_{t_j}(u)) \) converges to some point \( z \in X \):

\[
\Phi_{t_j+1}(u) \to z \quad \text{as} \quad j \to \infty.
\]

For each \( j = 1, 2, \ldots \), let \( k_j \) denote the smallest integer satisfying \( k_j \geq t_j + 1 \) and set \( r_j := k_j - (t_j + 1) \). Since \( 0 \leq r_j < 1 \) \((j = 1, 2, \ldots)\), we may assume that \( r_j \) converges to some real number \( r_\infty \in [0, 1] \). Set \( F := \Phi_1 \). Then

\[
F^{k_j}(u) = \Phi_{k_j}(u) = \Phi_{t_j+1+r_j}(u) = \Phi_{r_j}(\Phi_{t_j+1}(u)) \quad (j = 1, 2, \ldots).
\]

Thus, by the continuity of the map \((x, t) \to \Phi_1(x)\), \( F^{k_j}(u) \) converges to \( \Phi_{r_\infty}(z) \). Consequently, by Theorem 2.1. \( \Phi_{n_j}(u) = F^n(u) \) converges to a fixed point \( v \) of \( F := \Phi_1 \) as \( n \to \infty \). By Lemma 3.6, \( v \) is an equilibrium point of \( \Phi \).

It remains to show that \( \Phi_t(u) \to v \) as \( t \to \infty \). Choose any sequence of positive numbers \( t_j \to \infty \) and set \( t_j = n_j + r_j \), \( j = 1, 2, \ldots \), where \( n_j \) is an integer and \( 0 \leq r_j < 1 \). Then

\[
\Phi_{t_j}(u) = \Phi_{r_j}(\Phi_{n_j}(u))
\]

Considering that \( \Phi_{n_j}(u) \to v \) and that \( v \) is an equilibrium point of \( \Phi \), we easily find that \( \Phi_{t_j}(u) \to v \) as \( j \to \infty \). The theorem is proved.

Finally, we prove Theorem 2.7 on ancient orbits.

**Proof of Theorem 2.7.** Choose a real number \( \tau > 0 \) arbitrarily and set \( F := \Phi_{\tau} \). Then the sequence \( \{\ldots, p(-3\tau), p(-2\tau), p(-\tau), p(0)\} \) forms an ancient orbit for the map dynamical system generated by \( F \). Suppose that the alternative (b) does not hold. Then there exists a sequence of real numbers \( t_j \to -\infty \) \((j \to \infty)\) such that \( p(t_j) \) remains bounded as \( j \to \infty \). Arguing as in the proof of Theorem 2.5, we can easily show that there exists a sequence of negative integers \( n_j \to -\infty \) such that \( p(n_j \tau) \) remains bounded as \( j \to \infty \). Then, by Theorem 2.4 we see that \( p(n\tau) = v \) for \( n = 0, -1, -2, \ldots \) for some fixed point of \( F \), which is an equilibrium point of
\[ \Phi \text{ by Lemma 3.6. Since } \tau > 0 \text{ is arbitrary, it follows that } p(t) \text{ is identically an equilibrium point of } \Phi. \text{ The theorem is proved.} \]

4. Applications to diffusion equations. In this section we apply our abstract theory to various types of nonlinear diffusion equations and establish results on the long-time behavior of solutions as well as on the existence of steady-states and time-periodic solutions.

4.1. General strategy. In this subsection, we consider partial differential equations in a rather general setting to explain how our theory is applied. Let \( X \) be an ordered metric space satisfying the conditions in Section 2. First we consider an initial value problem for an abstract evolution equation on \( X \) of the form:

\[ \begin{align*}
\frac{du}{dt} &= A(u), & t > 0, \\
u(0) &= u_0,
\end{align*} \tag{9} \]

where \( A \) is a map from some subset of \( X \) to \( X \).

We assume that (9) is well-posed on \( X \) and defines a compact and order compact semiflow \( \Phi = \{ \Phi_t \}_{t \geq 0} \) on \( X \), namely \( \Phi \) is defined by

\[ \Phi_t(u_0) = u(t; u_0) \quad \text{for } u_0 \in X, t \geq 0, \]

where \( u(t; u_0) \) denotes the solution of (9) with initial data \( u(0) = u_0 \), and the map \( \Phi_t : X \rightarrow X \) is compact and order compact if \( t > 0 \). We also assume that there exists a continuous map \( M : X \rightarrow \mathbb{R} \) satisfying condition (M1) in Section 2. We further assume that \( \Phi \) and \( M \) satisfy conditions (Φ0)-(Φ2) and (M1), (M2). For example, if (9) is a second order semilinear parabolic equation and if \( X \) denotes the space of continuous functions on the domain where the equation is defined, the property (Φ0) follows from parabolic estimates, while the property (Φ1) follows from the maximum principle (see, for example, [25]). The property (Φ2) can easily be shown by the strong maximum principle.

Applying Theorems 2.5 and 2.6 to (9), we obtain the following:

**Theorem 4.1** (autonomous case). (i) Any solution of (9) either converges to some stationary solution of (9) or escapes to infinity as \( t \rightarrow \infty \).

(ii) If there exists at least one stationary solution for (9), then (9) possesses infinitely many stationary solutions and the set of stationary solutions is a totally ordered, unbounded and connected subset of \( X \).

By statement (ii) of Theorem 4.1, if (9) possesses some trivial stationary solution, such as 0, then there exist nontrivial stationary solutions for (9). Particularly, in the case where \( X \) is a linear space and \( A(u) \) is linear, since 0 is a trivial stationary solution, we obtain the existence of nontrivial stationary solutions for (9).

From statement (i) of Theorem 4.1, we see that (9) does not possess a time-periodic solution that is not stationary.

Next we apply our results to the time-periodic problem. Let us consider the problem of the form:

\[ \begin{align*}
\frac{du}{dt} &= A(u, t), & t > 0, \\
u(0) &= u_0,
\end{align*} \tag{10} \]

where \( A \) is \( T \)-periodic in \( t \) for some \( T > 0 \).
We assume that (10) is well-posed on \( X \) and let \( F \) be the time \( T \)-map associated with (10), namely,
\[
F(u_0) = u(T; u_0) \quad \text{for} \quad u_0 \in X,
\]
where \( u(t; u_0) \) denotes the solution of (10) with initial data \( u_0 \in X \). We assume that all the assumptions in Section 2 are fulfilled. Then, Theorems 2.1 and 2.2 imply the following:

**Theorem 4.2** (time-periodic case). (i) Any solution of (10) either converges to some \( T \)-periodic solution of (10) or escapes to infinity as \( t \to \infty \).

(ii) If there exists at least one \( T \)-periodic solution for (10), then (10) possesses infinitely many \( T \)-periodic solutions and the set of \( T \)-periodic solutions is a totally ordered, unbounded and connected subset of \( X \).

By statement (ii) of Theorem 4.2, the existence of at least one trivial \( T \)-periodic solution implies the existence of nontrivial \( T \)-periodic solutions. Especially, in the case where \( X \) is a linear space and \( A(u, t) \) is linear in \( u \), since 0 is a trivial \( T \)-periodic solution for (10), we obtain the existence of nontrivial \( T \)-periodic solutions for (10).

From statement (i) of Theorem 4.2, we see that (10) possesses no subharmonic solution: in other words, there exists no periodic solution whose minimal period is \( mT \) with \( m \in \mathbb{N}, m \geq 2 \).

### 4.2. Molecular motor model.

First, let us consider the following cooperative system, which comes from a model for intracellular transportation by molecular motors:
\[
\begin{align*}
\frac{\partial u_1}{\partial t} & = \frac{\partial}{\partial x} \left( \sigma_1 \frac{\partial u_1}{\partial x} + \psi_1(x) u_1 \right) - a_1(x) u_1 + a_2(x) u_2, \quad x \in (0, 1), \ t > 0, \\
\frac{\partial u_2}{\partial t} & = \frac{\partial}{\partial x} \left( \sigma_2 \frac{\partial u_2}{\partial x} + \psi_2(x) u_2 \right) + a_1(x) u_1 - a_2(x) u_2, \quad x \in (0, 1), \ t > 0, \\
\sigma_1 \frac{\partial u_i}{\partial x} + \psi_i'(x) u_i & = 0, \quad x = 0, 1, \ t > 0, \ i = 1, 2,
\end{align*}
\]
where \( \sigma_i > 0 \) is a constant and \( a_i(x) \geq 0, \neq 0 \), \( \psi_i(x) \) are smooth functions. It is assumed that the molecular motor is two-headed and its state switches between state 1 and state 2. For each \( t \geq 0 \), \( u_1(x, t) \) and \( u_2(x, t) \) denote the probability density at position \( x \). Thus one has \( u_1(x, t), u_2(x, t) \geq 0 \) and
\[
\int_0^1 (u_1(x, t) + u_2(x, t)) \, dx = 1, \quad t \geq 0.
\]

Derivation of system (12) from a mass transport viewpoint is given in the paper [9] by Chipot, Kinderlehrer and Kowalczyk. For a mathematical analysis and for further references we refer to [7, 12, 13] and the references therein. In what follows, for convenience, we consider all nonnegative solutions of (12) without setting (13).

Set \( X = (C([0, 1]))_+^2 \), where \( C([0, 1]))_+ \) denotes the set of nonnegative continuous functions on \([0, 1]\). Then \( X \) is an ordered metric space endowed with metric induced by the uniform convergence topology and order relation defined by
\[
u \preceq v \quad \text{if} \quad u_i(x) \leq v_i(x), \ x \in [0, 1], \ i = 1, 2
\]
for \( u = (u_1, u_2), v = (v_1, v_2) \in X \). Note that the symbol \( u \prec v \) then means that \( u \preceq v \) and that either \( u_1(x_0) < v_1(x_0) \) or \( u_2(x_0) < v_2(x_0) \) holds at some point \( x_0 \in [0, 1] \). The least upper bound \( u \vee v \) of \( u, v \) is defined by
\[
u \vee v (x) = \{ \max\{u_1(x), v_1(x)\}, \max\{u_2(x), v_2(x)\}\}, \ x \in [0, 1].
\]
By the standard a priori estimate it is known that (12) defines a compact semiflow on $X$, which we denote by $\{\Phi_t\}_{t \geq 0}$. Furthermore, it follows from the comparison principle and the strong maximum principle that, if $$u_i(x, 0) \leq v_i(x, 0), \quad x \in [0, 1], \; i = 1, 2, \quad u(x, 0) \neq v(x, 0)$$ hold for solutions $u(x, t) = (u_1(x, t), u_2(x, t))$, $v(x, t) = (v_1(x, t), v_2(x, t))$ of (12), then $$u_i(x, t) < v_i(x, t), \quad x \in [0, 1], \; t > 0, \; i = 1, 2.$$ We define a continuous map $M : X \to \mathbb{R}$ by $$M(u) = \int_0^1 (u_1(x) + u_2(x)) \, dx \quad \text{for } u = (u_1, u_2) \in X.$$ Since (12) is a linear problem, $0 = (0, 0)$ is a stationary solution of (12) and therefore, by statement (ii) of Theorem 4.1, there exists some non-zero stationary solution $\pi = (\pi_1(x), \pi_2(x)) > 0$ of (12). Clearly $\lambda \pi$ is also a stationary solution of (12) for any $\lambda > 0$ and from the strong maximum principle it follows that $$\pi_i(x) > 0, \quad x \in [0, 1], \; i = 1, 2.$$ Let $u(x, t) = (u_1(x, t), u_2(x, t))$ be an arbitrary solution of (12) and put $$\mu = \max \{u_i(x, 0)/\pi_i(x) \mid x \in [0, 1], \; i = 1, 2\}.$$ Then $$u(x, 0) \leq \mu \pi(x), \quad x \in [0, \infty)$$ and hence, by virtue of the comparison principle, $$u(x, t) \leq \mu \pi(x), \quad x \in [0, \infty), \; t > 0,$$ which shows that $u(\cdot, t)$ is bounded from above by $\mu \pi$ for all $t \geq 0$.

Thus Theorem 4.1 implies the following:

**Proposition 1 (autonomous model).** (i) (12) possesses a unique (up to multiplication by positive constant) positive stationary solution $\pi(x) = (\pi_1(x), \pi_2(x))$.

(ii) Any solution $u(x, t)$ of (12) converges to some stationary solution $\lambda \pi(x)$ in $(C([0, 1], \mathbb{R}))^2$ as $t \to \infty$, where a constant $\lambda$ is determined by the initial data $u(\cdot, 0)$ as $\lambda = M(u(\cdot, 0))/M(\pi)$.

Next we apply our result to a time-periodic system. More specifically, we consider a flashing ratchet model, proposed by [16, 10], which is represented as a Fokker-Plank equation with a time-periodic potential $\psi(x, t)$:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \frac{\partial}{\partial x} \left( \sigma \frac{\partial u}{\partial x} + \psi_x(x, t) u \right), \quad x \in (0, 1), \; t > 0, \\
\sigma \frac{\partial u}{\partial x} + \psi_x(x, t) u &= 0, \quad x = 0, 1, \; t > 0,
\end{align*}
\]  

(15)

where $\sigma > 0$ is a constant and $\psi(x, t)$ is a smooth function which is $T$-periodic in $t$ for some $T > 0$. Here the molecular motors are represented by the probability density $u(x, t)$ and thus $u(x, t) \geq 0$ and $$\int_0^1 u(x, t) \, dx = 1, \quad t \geq 0.$$ In what follows, as for (12), we consider all nonnegative solutions of (15).
Now we set \( X = C([0, 1])_+ \) endowed with metric induced by the uniform convergence topology and order relation defined by
\[
u \preceq \mu \quad \text{if} \quad \nu(x) \leq \mu(x), \quad x \in [0, 1]
\]
for \( \nu, \mu \in X \) and put
\[
M(\nu) = \int_0^1 \nu(x) \, dx \quad \text{for} \quad \nu \in X.
\]
Then, by the strong maximum principle, we easily see that the assumptions of Theorem 4.2 are all satisfied. Furthermore, by a comparison argument similar to what we have done for (12), we see that all nonnegative solutions remain bounded as \( t \to \infty \). Thus, we obtain the following:

**Proposition 2** (time-periodic model).
(i) (15) possesses a unique (up to multiplication by positive constant) positive time \( T \)-periodic solution \( \pi(x, t) \).
(ii) Any solution \( u(x, t) \) of (15) converges to some time \( T \)-periodic solution \( \lambda \pi(x, t) \) in \( C([0, 1])_+ \) as \( t \to \infty \), where a constant \( \lambda \) is determined by the initial data as
\[
\lambda = M(u(\cdot, 0))/M(\pi(\cdot, 0)).
\]

The above proposition implies, in particular, that (15) possesses no subharmonic solution, that is, there is no periodic solution whose minimal period is \( mT \) with \( m \in \mathbb{N}, m \geq 2 \).

We remark that we can relax the smoothness assumption on the coefficients of equations (12) and (15), by setting, for example, \( X = L^2([0, 1])_+ \) or \( X = (L^2([0, 1])_+)^2 \) instead of \( C([0, 1])_+ \) or \( (C([0, 1])_+)^2 \), where \( L^2([0, 1])_+ \) denotes the set of square-integrable nonnegative functions on \([0, 1] \).

We also remark that there are earlier related results concerning (12) and (15). The paper [7] deals with (12) and proves results that are basically the same as our Proposition 1 above. Their proof relies on the spectrum theory of compact linear operators. The paper [10] deals with (15) and proves results that are basically the same as our Proposition 2 above. Their proof relies on the entropy analysis. On the other hand, Propositions 1 and 2 follow immediately from a more general result without relying on further information such as spectrum, entropy and Floquet exponents. Therefore it is easy to extend the results in Propositions 1 and 2 to more general equations including nonlinear equations.

### 4.3. Reversible chemical reaction model

In this subsection we consider the following reaction-diffusion system which models a reversible chemical reaction between two mobile reactants \( A \) and \( B \). See [4, 11] and references therein for details.

\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= d_1 \Delta u_1 - \alpha (r_A(u_1) - r_B(u_2)), \quad x \in \Omega, \ t > 0, \\
\frac{\partial u_2}{\partial t} &= d_2 \Delta u_2 + \beta (r_A(u_1) - r_B(u_2)), \quad x \in \Omega, \ t > 0, \\
\frac{\partial u_i}{\partial \nu} &= 0, \quad x \in \partial \Omega, \ t > 0, \ i = 1, 2,
\end{align*}
\]
(16)

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with smooth boundary \( \partial \Omega \), \( \nu \) is the outward normal at each point of \( \partial \Omega \), \( d_1, d_2, \alpha, \beta > 0 \) are constants and \( r_A(u), r_B(u) \) are strictly increasing continuous functions satisfying \( r_A(0), r_B(0) = 0 \). Here, for each \( t \geq 0, \ u_1(x, t), u_2(x, t) \geq 0 \) represent the concentrations of \( A, B \) at \( x \in \Omega \), respectively.
Now we set \( X = (C(\overline{\Omega}))^2 \) and define the metric induced by the uniform convergence topology and the order relation (14) replacing \([0, 1]\) by \( \overline{\Omega}\). We further put
\[
M(u) = \int_{\Omega} \left( u_1(x)/\alpha + u_2(x)/\beta \right) dx \quad \text{for } u = (u_1, u_2) \in X.
\]

Take \( v_0 = (a_0, b_0) \in [0, \infty)^2 \) arbitrarily and let \( v(t) = (v_1(t), v_2(t)) \) denote a solution of (16) satisfying \( v(0) = (a_0, b_0) \). Then, since condition (M2) in Section 2 holds, we have
\[
v(t) \in \{(a, b) \in [0, \infty)^2 | a/\alpha + b/\beta = a_0/\alpha + b_0/\beta\}, \quad t > 0,
\]
which shows that a solution whose initial value is a constant function is bounded. Furthermore, for any \( u_0 \in X \), if we choose \( v_0 \in [0, \infty)^2 \) satisfying
\[
u_0 \preceq v_0,
\]
then the comparison principle implies
\[
u(\cdot, t) \preceq v(t) \quad t > 0,
\]
where \( u(x, t), v(t) \) is a solution of (16) with initial data \( u(\cdot, 0) = u_0 \), \( v(0) = v_0 \), respectively. This shows that any solution of (16) is bounded.

Denote by \( E \) the set of stationary solutions of (16). Clearly
\[
\{(a, b) \in [0, \infty)^2 | r_A(a) = r_B(b)\} \subset E.
\]

Applying Theorem 4.1, we obtain the following:

**Proposition 3.** Let \( E \) denote the set of all the stationary solutions of (16). Then,
(i) \( E = \{(a, b) \in [0, \infty)^2 | r_A(a) = r_B(b)\} \);
(ii) any solution \( u(x, t, u_1(x, t), u_2(x, t)) \) of (16) converges to some stationary solution \((\pi, \delta) \in E \in (C(\overline{\Omega}))^2 \) as \( t \to \infty \), which is determined by the initial data as \( M((u_1(\cdot, 0), u_2(\cdot, 0)) = \Omega(\pi/\alpha + \delta/\beta) \).

In [4], Bothe and Hilhorst studied (16) and proved the convergence of solutions as reaction rates \( \alpha, \beta \) tend to infinity. The limiting problem is given by a single diffusion equation with nonlinear diffusion. They also described the asymptotic behavior of solutions as \( t \to \infty \) by using the existence of a Lyapunov function (entropy).

On the other hand, our method is applicable to more general problems. For example, we can consider a problem of the form
\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= d_1 \Delta u_1 - \alpha (r_A(x,t,u_1) - r_B(x,t,u_2)), \quad x \in \Omega, \ t > 0, \\
\frac{\partial u_2}{\partial t} &= d_2 \Delta u_2 + \beta (r_A(x,t,u_1) - r_B(x,t,u_2)), \quad x \in \Omega, \ t > 0, \\
\frac{\partial u_i}{\partial \nu} &= 0, \quad x \in \partial \Omega, \ t > 0, \ i = 1, 2,
\end{align*}
\]
where the functions \( r_A, r_B \) are nondecreasing in \( u_1 \) and \( u_2 \), respectively, and are \( T \)-periodic in \( t \) and satisfy \( r_A(x,t,0) = r_B(x,t,0) = 0 \). For such a system, our Theorem 4.2 immediately implies the existence of time \( T \)-periodic solutions and convergence of all solutions to time \( T \)-periodic solutions.
4.4. Cooperative reaction-diffusion system. The next example is a cooperative reaction-diffusion system found in [8], whose special cases include (12) and (16):

\[
\begin{align*}
\frac{\partial u_i}{\partial t} & = \text{div}(\sigma_i \nabla u_i + u_i \nabla \psi_i) + \alpha_i \sum_{j=1}^{m} \lambda_{ij} r_j(u_j, x), & x \in \Omega, \ t > 0, \ i = 1, \ldots, m, \\
\sigma_i \frac{\partial u_i}{\partial \nu} + u_i \frac{\partial \psi_i}{\partial \nu} & = 0, & x \in \partial \Omega, \ t > 0, \ i = 1, \ldots, m,
\end{align*}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with smooth boundary \( \partial \Omega \), \( \nu \) is the outward normal at each point of \( \partial \Omega \), \( \sigma_i > 0 \) and \( \alpha_i > 0 \) are constants and \( \lambda_{ij} \) are constants such that

\[
\lambda_{ii} \leq 0, \quad \lambda_{ij} \geq 0 \text{ if } i \neq j, \quad \sum_{i=1}^{m} \lambda_{ij} = 0
\]

and such that the matrix \( (\lambda_{ij}) \) is irreducible. We assume that functions \( \psi_i(x), r_i(u, x) \) are smooth and \( r_i(u, x) \) is nondecreasing in \( u \) and satisfies \( r_i(0, x) = 0 \). As is the case of previous examples, we consider nonnegative solutions for (17).

Now we set \( X = (C(\Omega)_+)^m \) associated with the order relation

\[
u \preceq u \quad \text{if} \quad u_i(x) \leq v_i(x), \ x \in \overline{\Omega}, \ i = 1, \ldots, m
\]

for \( u = (u_1, \ldots, u_m), \ v = (v_1, \ldots, v_m) \in X \). Put

\[
M(u) = \int_{\Omega} \sum_{i=1}^{m} u_i(x)/\alpha_i \ dx \quad \text{for } u = (u_1, \ldots, u_m) \in X.
\]

Note that \( 0 = (0, \ldots, 0) \) is a stationary solution of (17). Therefore Theorem 4.1 implies the following:

**Proposition 4.**

(i) The set of stationary solutions of (17) is a nonempty, totally ordered, unbounded connected subset of \( (C(\Omega)_+)^m \).

(ii) Any solution of (17) either converges to some stationary solution or escapes to infinity in \( (C(\Omega)_+)^m \) as \( t \to \infty \).

In [8], Chipot, Hilhorst, Kinderlehrer and Olech proved \( L^1 \)-contraction property for solutions of (17). They then proved the existence of stationary solutions for the case where (17) is linear, especially the case where \( r_i(u, x) \equiv u \). Our theorems are also applicable to nonlinear problems.

Finally let us mention that, similarly to (16), applying Theorem 4.2 we can consider the case where problem (17) is time-periodic, more precisely, the case where constants \( \sigma_i, \lambda_{ij} \) and functions \( \psi_i, r_i \) depend on \( t \) and are \( T \)-periodic in \( t \). In this case, our Theorem 4.2 immediately yields the existence of time \( T \)-periodic solutions and convergence to time \( T \)-periodic solutions.

4.5. Competition diffusion system with two species. Now we consider the following competition diffusion system of two species \( A \) and \( B \).

\[
\begin{align*}
\frac{\partial u_1}{\partial t} & = d_1 \Delta u_1 - r(x, t) u_1 u_2, & x \in \Omega, \ t > 0, \\
\frac{\partial u_2}{\partial t} & = d_2 \Delta u_2 - kr(x, t) u_1 u_2, & x \in \Omega, \ t > 0, \\
\frac{\partial u_i}{\partial \nu} & = 0, & x \in \partial \Omega, \ t > 0, \ i = 1, 2,
\end{align*}
\]
where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial \Omega$, $\nu$ is the outward normal at each point of $\partial \Omega$. $d_1, d_2, k > 0$ are constants and $r(x, t) > 0$ are continuous functions that are $T$-periodic in $t$. We refer to Bothe [5] and to Bothe and Pierre [6] for the study of related systems with $r(x, t) = \text{constant}$. Here, for each $t \geq 0$, $u_1(x, t), u_2(x, t) \geq 0$ represent the concentrations of $A, B$ at $x \in \overline{\Omega}$, respectively. Therefore we consider nonnegative solutions to (18).

We set $X = (C(\overline{\Omega}), \lambda)^2$ with the metric induced by the uniform convergence topology. Here we define the order relation in $X$ by

$$u \preceq v \quad \text{if} \quad u(x) \leq v(x), \quad x \in \overline{\Omega}$$

for $u = (u_1, u_2), v = (v_1, v_2) \in X$. We further put

$$M(u) = \int_{\Omega} (k u_1(x) - u_2(x)) \, dx \quad \text{for} \quad u = (u_1, u_2) \in X.$$

Let us mention that (18) is well-posed on $X$ and the map $F$ on $X$ defined by (11) satisfies (F1), (F2). However $F$ is not strongly order-preserving. Here $F$ is called strongly order-preserving if

$$u \prec v \quad \text{implies} \quad F(\tilde{u}) \prec F(\tilde{v}) \quad \text{for all} \quad \tilde{u} \in U, \tilde{v} \in V,$$

where $U, V$ is some neighborhood of $u, v$, respectively. Indeed, for a solution $u(x, t) = (u_1(x, t), u_2(x, t))$ of (18),

$$\text{if} \quad u_2(\cdot, 0) = 0, \quad \text{then} \quad u_2(\cdot, t) = 0 \quad \text{for all} \quad t > 0.$$

On the other hand, the strong maximum principle implies that

$$\text{if} \quad u_2(\cdot, 0) \geq 0, \neq 0, \quad \text{then} \quad u_2(x, t) > 0 \quad \text{for all} \quad x \in \overline{\Omega}, \; t > 0.$$

Therefore, if we choose $u = (u_{01}, 0), v = (v_{01}, 0) \in X$ satisfying $u_{01}(x) \leq v_{01}(x)$ and $u_{01}(x) \neq v_{01}(x)$, then $u \prec v$ holds but $F(\tilde{u}) \prec F(\tilde{v})$ cannot hold for $\tilde{u} = (u_{01}, 0)$, $\tilde{v} = (v_{01}, v_{02})$ with $v_{02}(x) \geq 0, \neq 0$. Thus the existing theory of strongly order-preserving dynamical systems as [1] cannot be applied to (18).

Denote by $E$ the set of $T$-periodic solutions of (18). Clearly

$$\{(a, 0) \mid a \geq 0\} \cup \{(0, b) \mid b \geq 0\} \subset E.$$

For $u_0 = (u_{01}, u_{02}) \in X$, if we put

$$v_{01} = \max\{u_{01}(x) \mid x \in \overline{\Omega}\}, \quad v_{02} = \max\{u_{02}(x) \mid x \in \overline{\Omega}\},$$

then

$$(v_{01}, 0), (0, v_{02}) \in E, \quad (0, v_{02}) \preceq u_0 \preceq (v_{01}, 0)$$

and hence the comparison principle implies

$$0, v_{02}) \preceq u(\cdot, t) \preceq (v_{01}, 0), \quad t > 0,$$

where $u(x, t)$ is a solution of (18) with initial data $u(\cdot, 0) = u_0$. This shows that any solution of (18) is bounded.

Applying Theorem 4.2, we obtain the following:

**Proposition 5.** Let $E$ denote the set of all $T$-periodic solutions of (18). Then,

(i) $E = \{(a, 0) \mid a \geq 0\} \cup \{(0, b) \mid b \geq 0\}$;
(ii) any solution \( u(x,t) = (u_1(x,t), u_2(x,t)) \) of (18) converges to some \( T \)-periodic solution in \( (C(\Omega), +) \) as \( t \to \infty \), which is determined by the initial data as

\[
\lim_{t \to \infty} u(\cdot, t) = \begin{cases} 
\left( \frac{1}{|\Omega|} M(u_0), 0 \right) & \text{if } M(u_0) > 0, \\
(0, 0) & \text{if } M(u_0) = 0, \\
(0, -\frac{1}{|\Omega|} M(u_0)) & \text{if } M(u_0) < 0
\end{cases}
\]

with \( |\Omega| = \int_{\Omega} dx \).

Proposition 5 (ii) shows that at least one species \( A \) or \( B \) ceases to exist for (18) and the extinct species is determined by the initial concentrations. Note that all \( T \)-periodic solutions of (18) are constant in time.

4.6. Nonlocal evolution equation. Finally we consider a diffusion equation with a nonlocal term, which is given by

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta u + f(u, x, t) - \frac{1}{|\Omega|} \int_{\Omega} f(u, y, t) dy, & x \in \Omega, \ t > 0, \\
\frac{\partial u}{\partial \nu} &= 0, & x \in \partial \Omega, \ t > 0,
\end{aligned}
\]

(19)

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with smooth boundary \( \partial \Omega \), \( \nu \) is the outward normal at each point of \( \partial \Omega \), and \( f(u, x, t) \) is a smooth function that is nonincreasing in \( u \) and \( T \)-periodic in \( t \).

A similar model was discussed by Rubinstein and Sternberg [26] in the case where the function \( f \) is of cubic type and does not depend on \( x \) and \( t \). If \( f \) is a cubic type nonlinearity, the solution may develop transition layers whose behavior can be highly complex in some cases. Since we assume that \( f \) is monotone decreasing in \( u \), the solution does not develop transition layers, and the dynamics are simpler. However, since we allow \( f \) to depend on \( x \) and \( t \), the problem is not so trivial despite the simple dependence of \( f \) on \( u \).

Let \( X = C(\Omega) \) and denote by \( F \) the time \( T \)-map on \( X \) associated with (19) and put

\[
M(u) = \int_{\Omega} u(x) dx \quad \text{for } u \in X.
\]

We remark that, since \( f \) is nonincreasing in \( u \), conditions (F1) and (F2) hold. Indeed, let \( u(x,t), v(x,t) \) be solutions of (19) satisfying

\[
u(x,0) \leq v(x,0), \quad x \in \Omega.
\]

Clearly \( w(x,t) = v(x,t) - u(x,t) \) satisfies

\[
\frac{\partial w}{\partial t} = \Delta w + c(x,t) w - \frac{1}{|\Omega|} \int_{\Omega} c(y,t)w(y,t) dy, \quad x \in \Omega, \ t > 0,
\]

(20)

where

\[
c(x,t) = \int_0^1 f_u(sv(x,t) + (1-s)u(x,t), x,t) ds.
\]

First we will show that

\[
w(x,T) \geq 0, \quad x \in \Omega.
\]

Put \( \tilde{w}(x,t) = e^{-\lambda t} w(x,t) \), where \( \lambda \) is a constant satisfying

\[
\lambda > -\min\{c(x,t) \mid x \in \Omega, t \in [0,T]\}
\]

(21)
and denote by \( (x_0, t_0) \) the minimum point of \( \hat{w}(x, t) \) in the region \( \overline{\Omega} \times [0, T] \), namely,
\[
\hat{w}(x_0, t_0) = \min \{ \hat{w}(x, t) : x \in \Omega, t \in [0, T] \}.
\]
Assume \( \hat{w}(x_0, t_0) < 0 \). Then, since \( \hat{w}(x, 0) = w(x, 0) \geq 0 \) for all \( x \in \overline{\Omega} \), we have \( 0 < t_0 \leq T \) and
\[
\frac{\partial \hat{w}}{\partial t}(x_0, t_0) \leq 0, \quad \Delta \hat{w}(x_0, t_0) \geq 0.
\]
However, the last inequality implies
\[
\frac{\partial \hat{w}}{\partial t}(x_0, t_0) = \Delta \hat{w}(x_0, t_0) + \{c(x_0, t_0) - \lambda\} \hat{w}(x_0, t_0) - \frac{1}{|\Omega|} \int_{\Omega} c(y, t) \hat{w}(y, t) \, dy \\
> \{c(x_0, t_0) - \lambda\} \hat{w}(x_0, t_0) + \frac{1}{|\Omega|} \int_{\Omega} \lambda \hat{w}(x_0, t_0) \, dy \\
= c(x_0, t_0) \hat{w}(x_0, t_0) \geq 0,
\]
which is a contradiction. Thus we have \( \hat{w}(x, t) \geq 0 \), \( x \in \overline{\Omega}, t \in [0, T] \)
and hence (21) holds. Furthermore, by (20)
\[
\frac{\partial w}{\partial t} \geq \Delta w + c(x, t) w, \quad x \in \Omega, \ 0 < t \leq T.
\]
Therefore, if \( u(\cdot, 0) \leq v(\cdot, 0) \) and \( u(\cdot, 0) \neq v(\cdot, 0) \), then it follows from the strong maximum principle for (22) that
\[
w(x, t) > 0, \quad x \in \overline{\Omega}, 0 < t \leq T,
\]
and hence
\[
u(x, T) < v(x, T), \quad x \in \overline{\Omega}.
\]
This shows that the map \( F \) has strong order-preserving property and therefore (F1) and (F2) hold. Thus, by Theorem 4.2, we obtain the following:

**Proposition 6.** Either of the followings holds for a solution \( u(x, t) \) of (19):

(i) \( u(\cdot, t) \) converges to some \( T \)-periodic solution in \( C(\overline{\Omega}) \) as \( t \to \infty \);

(ii) \( \lim_{t \to \infty} \| u(x, t) \|_{C(\overline{\Omega})} = \infty \).

**Proposition 7.** If (19) possesses a \( T \)-periodic solution, it possesses infinity may of them. More precisely, the set of \( T \)-periodic solutions for (19) is a totally ordered, unbounded and connected subset of \( C(\overline{\Omega}) \).

In the special case where \( f \) is independent of \( x \) and depends only on \( u \) and \( t \), then one can easily show that all solutions of (19) are uniformly bounded. To see this, it suffices to show that any constant function of \( \overline{\Omega} \) is a stationary solution of (19). Indeed, bounded, if \( v \) is a constant function, then
\[
\Delta v = 0 \quad \text{in } \Omega, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial \Omega = 0
\]
and
\[
v - \frac{1}{|\Omega|} \int_{\Omega} v(x) \, dx = 0.
\]
Thus we have the following corollary:

**Corollary 4.** In the case \( f = f(u, t) \), any solution \( u(x, t) \) of (19) converges to a constant in \( C(\overline{\Omega}) \) as \( t \to \infty \). More precisely, \( \lim_{t \to \infty} \left\| u(\cdot, t) - \frac{1}{|\Omega|} \int_{\Omega} u_0(x) \, dx \right\|_{C(\overline{\Omega})} = 0 \).
5. **Applications to delay-differential equation.** Our results also apply to delay-differential equation without diffusion and those with diffusion.

5.1. **Delay-differential equation without diffusion.** First we consider a delay-differential equation of the form

\[
\frac{du}{dt} = f(t-1, u(t-1)) - f(t, u(t)), \quad t > 0,
\]

where \( f(t, u) \) is a smooth function that is \( T \)-periodic in \( t \) and strictly increasing in \( u \). This equation was introduced by Arino [2] as an example of a delay equation with a first integral.

We consider the initial value problem for the above equation. The initial condition is given in the form

\[
 u(t) = u_0(t), \quad t \in [-1, 0],
\]

where \( u_0 \) is a continuous function on \([-1, 0]\).

To study the above initial value problem, we set \( X = C([-1, 0]) \) and define the map \( F \) on \( X \) by

\[
 F(u_0)(s) = u(T + s; u_0) \quad \text{for} \quad u_0 \in X, \quad s \in [-1, 0],
\]

where \( u(t; u_0) \) denotes a solution of (23) with initial data \( u_0 \in X \).

For arbitrarily fixed \( t_1 > t_2 \geq 0 \), integrating both sides of (23) from \( t_2 \) to \( t_1 \) yields

\[
 u(t_1) + \int_{t_1-1}^{t_1} f(s, u(s)) \, ds = u(t_2) + \int_{t_2-1}^{t_2} f(s, u(s)) \, ds,
\]

which shows

\[
 u(t) + \int_{t-1}^{t} f(s, u(s)) \, ds = u(0) + \int_{-1}^{0} f(s, u(s)) \, ds \quad \text{for} \quad t \geq 0.
\]

Therefore, the map \( M \) on \( X \) defined by

\[
 M(u) = u(0) + \int_{-1}^{0} f(s, u(s)) \, ds \quad \text{for} \quad u \in X
\]

satisfies (M2). Since \( f \) is strictly increasing in \( u \), condition (M1) also holds.

We show that condition (F1) holds. Let \( u_1(t), u_2(t) \) be solutions of (23) satisfying

\[
 u_1(t) \leq u_2(t), \quad t \in [-1, 0].
\]

The monotonicity of \( f \) in \( u \) implies

\[
 \frac{d}{dt} (u_2 - u_1) = f(t-1, u_2(t-1)) - f(t-1, u_1(t-1)) - \{ f(t, u_2(t)) - f(t, u_1(t)) \}
 \geq -\{ f(t, u_2(t)) - f(t, u_1(t)) \}
 = -c(t) \{ u_2(t) - u_1(t) \}, \quad t \in (0, 1],
\]

where

\[
 c(t) = \int_{0}^{1} f_u(t, su_2(t) + (1-s)u_1(t)) \, ds.
\]

Therefore \( u_2(t) - u_1(t) \) is a supersolution of

\[
 \frac{du}{dt} = -c(t) u.
\]
Since 0 is a solution of the above equation and \(0 \leq u_2(0) - u_1(0)\) holds, we have \(0 \leq u_2(t) - u_1(t)\) for \(t \in (0, 1]\), which means
\[
u_1(t) \leq u_2(t), \quad t \in [0, 1].
\]

Repeating this argument we see that
\[
u_1(t) \leq u_2(t), \quad t \in [n, n + 1]
\]
holds for any \(n \in \mathbb{N}\) and hence
\[
u_1(t) \leq u_2(t), \quad t \in [-1, \infty).
\]

Thus \(u_1(t) \leq u_2(t)\) for \(t \in [T - 1, T]\), namely (F1) is fulfilled.

We note that, although (F0) or (F2) does not necessarily hold, (F0') and (F2') hold for some \(m \in \mathbb{N}\). Indeed, first suppose that \(u_1(t), u_2(t)\) are solutions of (23) satisfying \(u_1(t) \leq u_2(t)\) and \(u_1(t) \neq u_2(t)\) in \([-1, 0]\). Then \(u_1(t_0) < u_2(t_0)\) for some \(t_0 \in (-1, 0)\). Since \(f\) is strictly increasing in \(u\), now we have
\[
\frac{d}{dt}(u_2 - u_1)(t_0 + 1) > -c(t_0 + 1)\{u_2(t_0 + 1) - u_1(t_0 + 1)\},
\]
where \(c(t)\) is defined by (26). This shows that
\[
u_2(t) - u_1(t) > 0, \quad t > t_0 + 1
\]
and hence
\[
u_1(t) < u_2(t), \quad t \in [mT - 1, mT]
\]
for \(m \in \mathbb{N}\) satisfying \(mT - 1 > t_0 + 1\). Thus, if we choose \(m \in \mathbb{N}\) satisfying \(mT > 2\), then, for \(u, v \in X\) satisfying \(u \not\leq v\) and \(u \not\geq v\), \(u < u \lor v\) and \(v < u \lor v\) imply
\[
F^m(u)(t) < F^m(u \lor v)(t) \quad \text{and} \quad F^m(v)(t) < F^m(u \lor v)(t), \quad t \in [-1, 0],
\]
which shows that
\[
F^m(u) \lor F^m(v) < F^m(u \lor v).
\]

Thus (F2') holds.

By Arino’s argument in [2], we see that the existence of at least one \(T\)-periodic solution of (23) implies the boundedness of the orbit \(F^n(u_0)\) for all \(u_0 \in X\). In fact, let \(\pi(t)\) be a \(T\)-periodic solution and denote by \(\pi_0(t)\) the restriction of \(\pi(t)\) on the interval \([-1, 0]\). Clearly \(\pi_0 \in X\) is a fixed point of \(F\). First we consider the case where \(\pi_0 \leq u_0\) holds. In this case, by the comparison theorem we have shown above, \(F^n(\pi_0)(t) = \pi(t) \leq F^n(u_0)(t)\) for all \(t > -1\) and \(n \in \mathbb{N}\). Furthermore, since (25) holds and since \(f(t, u)\) is increasing in \(u\), for \(n \in \mathbb{N}\) satisfying \(nT - 1 > 0\) we have
\[
M(u_0) = u_0(0) + \int_{-1}^{0} f(s, u_0(s)) \, ds = F^n(u_0)(t) + \int_{t-1}^{t} f(s, F^n(u_0)(s)) \, ds
\]
\[
\geq F^n(u_0)(t) + \int_{t-1}^{t} f(s, \pi_0(s)) \, ds
\]
\[
\geq F^n(u_0)(t) + \overline{m}, \quad t \in [-1, 0]
\]
with \(\overline{m} = \min\{f(s, \pi(s)) \mid s \in [0, T]\}\) and hence \(\pi_0 \leq F^n(u_0) \leq M(u_0) - \overline{m}\). This means the boundedness of the orbit \(\{F^n(u_0)\}_{n=0,1,2,...}\) in \(X\). In the same way, we obtain the boundedness of the orbit \(\{F^n(u_0)\}_{n=0,1,2,...}\) in \(X\) for the case where \(u_0 \leq \pi_0\) holds. Finally we consider the case where \(u_0\) is an arbitrary element of \(X\). Set
\[
u_0(s) = \min\{u_0(s), \pi(s)\}, \quad \nu^+_0(s) = \max\{u_0(s), \pi(s)\}, \quad s \in [-1, 0],
\]
Then, \( u_0^- \leq u_0 \leq u_0^+ (s) \) implies that \( F^n(u_0^-) \leq F^n(u_0) \leq F^n(u_0^+) \) for \( n = 1, 2, \ldots \).

In the other hand, since \( u_0^- \leq u_0 \leq u_0^+ \) holds, the orbits \( \{F^n(u_0^-)\}_{n=0,1,2,\ldots}, \{F^n(u_0^+)\}_{n=0,1,2,\ldots} \) are both bounded. Therefore, the orbit \( \{F^n(u_0)\}_{n=0,1,2,\ldots} \) is bounded in \( X \).

Applying Theorem 2.3 we obtain the following:

**Proposition 8.** Any solution of the initial value problem for (23) converges to a \( T \)-periodic solution or any solution escapes to infinity as \( t \to \infty \). More precisely, let \( u(t) \) be a solution of (23) for an arbitrary initial data in \( C([-1,0]) \), then the function \( \tau \mapsto u(kT + \tau) \), with \( k \) being a positive integer, converges uniformly in \( t \in [-1,0] \) as \( k \to \infty \) or \( \lim_{k \to \infty} \|u(kT + \tau)\|_\infty = \infty \) for all \( \tau \in [-1,0] \).

**Proposition 9.** If there exists at least one \( T \)-periodic solution for (23), then (23) possesses infinitely many \( T \)-periodic solutions and the set of \( T \)-periodic solutions is a totally ordered, unbounded and connected subset of \( C([-1,\infty)) \) and any solution in \( C([-1,\infty)) \) of (23) converge to some \( T \)-periodic solution.

5.2. Delay-differential equation with diffusion. The final example is a boundary value problem for a delay-differential equation with diffusion of the form

\[
\begin{align*}
\frac{\partial u}{\partial t} & = \text{div}(\sigma \nabla u + u \nabla \psi) + f(t-1,u(x,t-1)) - f(t,u(x,t)), \quad x \in \Omega, \ t > 0, \\
\sigma \frac{\partial u}{\partial \nu} + u \frac{\partial \psi}{\partial \nu} & = 0, \quad x \in \partial \Omega, \ t > 0,
\end{align*}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with smooth boundary \( \partial \Omega \), \( \sigma > 0 \) is a constant. We assume that functions \( \psi(x) \), \( f(t,u) \) are smooth and \( f(t,u) \) is \( T \)-periodic in \( t \) and strictly increasing in \( u \).

As is the case of previous example, we set \( X = C(\bar{\Omega} \times [-1,0]) \) and define the map \( F \) on \( X \) by

\[
F(u_0)(x,s) = u(x,T+s;u_0) \quad \text{for } u_0 \in X, \ x \in \bar{\Omega}, \ s \in [-1,0],
\]

where \( u(x,t;u_0) \) is a solution of (27) satisfying \( u(x,t;u_0) = u_0(x,t) \) for \( x \in \bar{\Omega} \) and \( t \in [-1,0] \).

For arbitrarily fixed \( s_1 > s_2 > 0 \), integrating both sides of (27) on \( \Omega \times (s_2,s_1) \) yields

\[
\int_{\Omega} u(x,s_1) \, dx + \int_{\Omega \times (s_1-1,s_1)} f(t,u(x,t)) \, dxdt = \int_{\Omega} u(x,s_2) \, dx + \int_{\Omega \times (s_2-1,s_2)} f(t,u(x,t)) \, dxdt,
\]

which shows

\[
\int_{\Omega} u(x,s) \, dx + \int_{\Omega \times (s-1,s)} f(t,u(x,t)) \, dxdt = \int_{\Omega} u(x,0) \, dx + \int_{\Omega \times (-1,0)} f(t,u(x,t)) \, dxdt
\]

holds for any \( s \geq 0 \). Therefore, the map \( M \) on \( X \) defined by

\[
M(u) = \int_{\Omega} u(x,0) \, dx + \int_{\Omega \times (-1,0)} f(t,u(x,t)) \, dxdt
\]

satisfies (M2). By the monotonicity of \( f \) in \( u \), condition (M1) is also fulfilled. In the same way as in the previous subsection 5.1, we can show that (F1) holds and, although (F0) or (F2) does not necessarily hold, (F0') and (F2') hold for \( m \in \mathbb{N} \) satisfying \( mT \geq 2 \).

Applying Theorem 2.3 we obtain the following:
Proposition 10.  

(i) Any solution in \( C(\Omega \times [-1, \infty)) \) of (27) either converges to some \( T \)-periodic solution of (27) or escapes to infinity in locally uniform convergence topology as \( t \to \infty \).

(ii) If there exists at least one \( T \)-periodic solution for (27), then (27) possesses infinitely many \( T \)-periodic solutions and the set of \( T \)-periodic solutions is a totally ordered, unbounded and connected subset of \( C(\Omega \times [-1, \infty)) \).

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