ROUND FOLD MAPS ON MANIFOLDS REGARDED AS THE TOTAL SPACES OF LINEAR AND MORE GENERAL BUNDLES

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Abstract. (Stable) fold maps are fundamental tools in studying a generalized theory of the theory of Morse functions on smooth manifolds and its application to geometry of the manifolds. It is important to construct explicit fold maps systematically to study smooth manifolds by the theory of fold maps easy to handle. However, such constructions have been difficult in general.

Round fold maps are defined as stable fold maps such that the sets of all the singular values are concentric spheres and it was first introduced in 2012–2014. The author studied algebraic and differential topological properties of such maps and their manifolds and constructed explicit round fold maps. For example, the author succeeded in constructing such maps on manifolds regarded as the total spaces of bundles over smooth homotopy spheres by noticing that smooth homotopy spheres admit round fold maps whose singular sets are connected and more generally, new such maps on manifolds regarded as the total space of circle bundles over another manifold admitting a round fold map. In this paper, as an advanced work, we construct new explicit round fold maps on manifolds regarded as the total spaces of bundles such that the fibers are closed smooth manifolds and that the structure groups are linear and more general bundles over a manifold admitting a round fold map.

Singularities of differentiable maps; singular sets, fold maps. and Differential topology. 57R45 and 57N15

1. Introduction

Fold maps are fundamental tools in studying a generalization of the theory of Morse functions and its application to geometry of manifolds.

A fold map is defined as a smooth map such that each singular point is of the form

\[(x_1, \cdots, x_m) \mapsto (x_1, \cdots, x_{n-1}, \sum_{k=n}^{m-i} x_k^2 - \sum_{k=m-i+1}^{m} x_k^2)\]

for two positive integers \(m \geq n\) and an integer \(0 \leq i \leq m - n + 1\) and a Morse function is regarded as a fold map, for example. For a fold map from a closed smooth manifold of dimension \(m\) into a smooth manifold of dimension \(n\) without boundary, the following two hold.

1. The set of all the singular points (the singular set) is a closed smooth submanifold of dimension \(n - 1\) of the source manifold.

2. The restriction map to the singular set is an immersion of codimension 1.

We also note that if the restriction map to the singular set is an immersion with normal crossings, then it is stable (stable maps are important in the theory of global singularity; see [6] for example).
Constructions of explicit fold maps will help us to study smooth manifolds by using the theory of fold maps which are easy to handle and it is very difficult to construct explicit fold maps in general, although existence problems for fold maps have been solved under various conditions. However, such fold maps with good properties were constructed as we will introduce the following.

In [1], [5], [16], [17] and [19], special generic maps, which are fold maps whose singular points are of the form

\[(x_1, \cdots, x_m) \mapsto (x_1, \cdots, x_{n-1}, \sum_{k=n}^{m} x_k^2)\]

for two positive integers \(m \geq n\), were studied. Special generic maps are not so difficult to construct. They were constructed by constructing local \(C^\infty\) maps on manifolds with boundaries and gluing them together. For example, by such methods, some special generic maps on homotopy spheres including standard spheres are obtained. Furthermore, manifolds admitting special generic maps were classified under restrictions on the dimensions of source and target manifolds and the fundamental groups of source manifolds.

Later, in [8] and [9] the author introduced round fold maps, which will be mainly studied in this paper. A round fold map is defined as a fold map satisfying the following three.

1. The singular set is a disjoint union of standard spheres.
2. The restriction to the singular set is an embedding.
3. The set of all the singular values is a disjoint union of spheres embedded concentrically.

For example, some special generic maps on homotopy spheres are round fold maps whose singular sets are connected (see Example 1 (1) later and also [16]).

In [9], homology groups and homotopy groups of manifolds admitting round fold maps were studied. Some examples of round fold maps and the diffeomorphism types of their source manifolds were given by the author in [8], [10], [11] and [12]. For example, we have obtained round fold maps on manifolds admitting bundle structures over the \(n\)-dimensional \((n \geq 2)\) standard sphere and manifolds represented as connected sums of manifolds admitting bundle structures over the \(n\)-dimensional \((n \geq 2)\) standard sphere whose fibers are diffeomorphic to the \((m-n)\)-dimensional standard sphere \(S^{m-n}\) \((m > n)\). In [11] and [12], as new answers, we have obtained new round fold maps on closed manifolds admitting bundle structures over (exotic) homotopy spheres or ones over more general manifolds.

In the last two papers, as a useful tool to construct new round fold maps, a \(P\)-operation has been introduced. Especially, in these papers, a lot of round fold maps from manifolds having bundle structures such that the fibers are circles were obtained. In this paper, as a generalized work of [11] and [12], we apply \(P\)-operations to construct more explicit round fold maps on manifolds having bundle structures such that the structure groups are linear and act on the fibers smoothly. This paper is organized as the following.

In section 2, we recall round fold maps and some terminologies on round fold maps such as axes and proper cores. We also recall a \(C^\infty\) trivial round fold map. We introduce results on the diffeomorphism types of manifolds admitting \(C^\infty\) trivial round fold maps shown by the author in [8] and [10].
In section 3, we recall a locally $C^\infty$ trivial round fold map, which is a round fold map satisfying a kind of triviality around the connected components of the set of singular values. We recall $P$-operations defined in \[11\], which are operations used to construct new round fold maps on manifolds having the structures of manifolds admitting bundle structures over manifolds admitting locally $C^\infty$ trivial round fold maps. More precisely, a $P$-operation consists of four steps; we decompose the given round fold map, confirm that the restrictions of the bundle over the given manifold to the obtained pieces of the source manifold of the round fold map are trivial, construct maps on these pieces and glue these maps together. A construction of a round fold map by a $P$-operation requires us that the bundle is not so complex.

In section 4, as main works of the present paper, we apply $P$-operations to construct new round fold maps on manifolds regarded as the total spaces of bundles whose fibers are closed smooth manifolds and whose structure groups are linear and act on the fibers smoothly (linear bundles) and more general bundles over manifolds admitting locally $C^\infty$ trivial round fold maps. These works are regarded as extensions of works \[10\] and \[12\] by the author. In these works, we mainly consider bundles whose fibers are circles and we have obtained a lot of new round fold maps and manifolds. In these works, the theory of the classification of circle bundles, which is the most fundamental part of the theory of characteristic classes of vector bundles discussed in \[14\]. In the present paper, we consider some appropriate situations and obtain new round fold maps and their source manifolds through Theorems 1-8 with Examples 2-8. For such general studies, as an essential tool, we use more general theory of characteristic classes of linear bundles of \[14\].

Throughout this paper, manifolds and maps between manifolds are smooth and of class $C^\infty$ unless otherwise stated. The base spaces and fibers of bundles in this paper are smooth manifolds and the structure groups of the bundles act on the fibers smoothly unless otherwise stated.

Moreover, let $M$ be a closed (smooth) manifold of dimension $m$, let $N$ be a (smooth) manifold of dimension $n$ with no boundary, let $f : M \to N$ be a (smooth) map and let $m \geq n \geq 1$. We denote the singular set of $f$, which is defined as the set consisting of all the singular points of $f$, by $S(f)$. We call the set $f(S(f))$ the singular value set of $f$. We call an inverse image $f^{-1}(p) \in M$ a fiber of $f$ and if the point $p \in N$ is a regular value of $f$, then we call it a regular fiber of $f$.

## 2. Round fold maps

In this section, we review round fold maps. See also \[9\].

### 2.1. Terminologies on round fold maps.

**Definition 1** (round fold maps \((9)\)), $f : M \to \mathbb{R}^n$ $(m \geq n \geq 2)$ is said to be a round fold map if $f$ is $C^\infty$ equivalent to a fold map $f_0 : M_0 \to \mathbb{R}^n$ on a closed $C^\infty$ manifold $M_0$ such that the following three hold.

1. The singular set $S(f_0)$ is a disjoint union of $(n-1)$-dimensional standard spheres and consists of $l \in \mathbb{N}$ connected components.
2. The restriction map $f_0|_{S(f_0)}$ is an embedding.
3. Let $D^n_r := \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid \sum_{k=1}^n x_k^2 \leq r\}$. Then the set $f_0(S(f_0))$ is represented as the disjoint union $\bigcup_{k=1}^l \partial D^n_k$.

We call $f_0$ a normal form of $f$. We call a ray $L$ from $0 \in \mathbb{R}^n$ an axis of $f_0$ and $D^n_L$ the proper core of $f_0$. Suppose that for a round fold map $f$, its normal form
\[ f_0 \text{ and diffeomorphisms } \Phi : M \to M_0 \text{ and } \phi : \mathbb{R}^n \to \mathbb{R}^n, \phi \circ f = f_0 \circ \Phi. \] Then for an axis \( L \) of \( f_0 \), we also call \( \phi^{-1}(L) \) an axis of \( f \) and for the proper core \( D^n_{1/2} \) of \( f_0 \), we also call \( \phi^{-1}(D^n_{1/2}) \) a proper core of \( f \).

Let \( f : M \to \mathbb{R}^n \) be a round fold map, let \( P \) be a proper core of \( f \) and let \( L \) be an axis of the map \( f \). Then, \( f^{-1}(\mathbb{R}^n - \text{Int}P) \) has a bundle structure over \( \partial P \) such that the fiber is diffeomorphic to the manifold \( f^{-1}(L) \) and that \( f_{(f^{-1}(\partial P))} : f^{-1}(\partial P) \to \partial P \) defines a subbundle of the previous bundle \( f^{-1}(\mathbb{R}^n - \text{Int}P) \). In this situation, we can define a \( C^\infty \) trivial round fold map.

**Definition 2.** In this situation, a round fold map \( f \) is said to be \( C^\infty \) trivial if we can take the bundle \( f^{-1}(\mathbb{R}^n - \text{Int}P) \) as a trivial bundle.

We introduce some known examples of round fold maps with their source manifolds.

**Example 1.**

1. Special generic maps from \( m \)-dimensional homotopy spheres into the Euclidean space of dimension \( n \geq 2 (m \geq n) \) such that the singular sets are spheres and that the singular value sets are embedded spheres are round fold maps. They are \( C^\infty \) trivial. The \( m \)-dimensional standard sphere \( S^m \) admits such a round fold map into \( \mathbb{R}^n \). In section 5 of [16], such a fold map from \( m \)-dimensional \( (2 \leq m < 4, m \geq 5) \) homotopy sphere into \( \mathbb{R}^2 \) is constructed.

2. Let \( F \neq \emptyset \) be a closed manifold. Let \( M \) be a closed manifold of dimension \( m \) regarded as the total space of an \( F \)-bundle over \( S^n \) \( (m \geq n \geq 2) \). In [8], in the case where \( F \) is the standard sphere \( S^{m-n} \), a round fold map \( f : M \to \mathbb{R}^n \) such that the following four hold has been constructed and it has been shown that any manifold admitting such a map is regarded as the total space of an \( S^{m-n} \)-bundle over \( S^n \).
   - (a) \( f \) is \( C^\infty \) trivial.
   - (b) The singular set \( S(f) \) has two connected components.
   - (c) For an axis \( L \) of \( f \), \( f^{-1}(L) \) is diffeomorphic to the cylinder \( F \times [-1, 1] = S^{m-n} \times [-1, 1] \).
   - (d) Two connected components of the fiber of a point in a proper core of \( f \) is regarded as fibers of the \( F \)-bundle over \( S^n \).

   In [11], a round fold map \( f : M \to \mathbb{R}^n \) satisfying all the conditions but the second condition just before has been constructed and it has been shown that any manifold admitting such a map is regarded as the total space of an \( F \)-bundle over \( S^n \).

3. A special generic map \( f \) from an \( m \)-dimensional closed manifold \( M \) into \( \mathbb{R}^n \) such that the following two hold appears in [16], for example.
   - (a) The restriction map \( f|_{S(f)} \) is an embedding.
   - (b) \( S(f) \) is a disjoint union of two copies of the \((n-1)\)-dimensional standard sphere.

   For example, the product of \( S^{n-1} \) and an \((m-n+1)\)-dimensional homotopy sphere admits such a map which is also \( C^\infty \) trivial.

3. **Locally \( C^\infty \) Trivial Round Fold Maps and \( P \)-Operations**

We recall locally \( C^\infty \) trivial round fold maps and \( P \)-operations. See also [10] and [12].
Definition 3. Let $f : M \to \mathbb{R}^n$ be a round fold map. Assume that for any connected component $C$ of $f(S(f))$ and a small closed tubular neighborhood $N(C)$ of $C$ such that $\partial N(C)$ is the disjoint union of two connected components $C_1$ and $C_2$, $f^{-1}(N(C))$ has the structure of a trivial bundle over $C_1$ or $C_2$ and $f|_{f^{-1}(C_1)} : f^{-1}(C_1) \to C_1$ and $f|_{f^{-1}(C_2)} : f^{-1}(C_2) \to C_2$ give the structures of subbundles of the bundle $f^{-1}(N(C))$. Then $f$ is said to be locally $C^\infty$ trivial. We call the fiber $F_C$ of the bundle $f^{-1}(N(C))$ the normal fiber of $C$ corresponding to the bundle $f^{-1}(N(C))$. Assume that $C_1$ is in the bounded connected component of $\mathbb{R}^n - C_2$ and we denote the fiber of the subbundle $f^{-1}(C_1)$ by $\partial_0 F_C$.

Maps in Example 1 (1) are locally $C^\infty$ trivial. Ones in Example 1 (2) are constructed as locally $C^\infty$ trivial maps in [8] and [11]. We can construct maps on the product of the $(n-1)$-dimensional standard sphere and a homotopy sphere explained in the last part of Example 1 (3) as locally $C^\infty$ trivial maps.

We can construct a locally $C^\infty$ trivial round fold map in the following method. We use the method in the proof of Proposition 2 and other scenes of the present paper.

Let $l' \in \mathbb{N}$. Let $\{E_j\}_{j=1}^{l'}$ be a family of compact manifold of dimension $m-n+1$ such that the boundary of $E_j$ is a disjoint union of two closed manifolds $F_j$ and $F_{j+1}$, that $F_{l'+1}$ is empty and that except $F_0$ and $F_{l'+1}$, all the manifolds in the family $\{F_j\}$ are non-empty. There exist a positive integer $l$ and a sequence of integers $\{k_j\}_{j=1}^{l'}$ of integers such that $k_1 = 1$ and $k_{l'} = l$ hold and that the inequality $k_j < k_{j+1}$ holds for any integer $1 \leq j \leq l' - 1$. We can construct a Morse function $\tilde{f}_j : E_j \to [k_j - \frac{1}{2}, k_{j+1} - \frac{1}{2}]$ satisfying the following three conditions:

1. For any integer $1 \leq j \leq l'$, On $F_j$, $\tilde{f}_j$ is constant and minimal if $F_j$ is non-empty and on $F_{j+1}$, $\tilde{f}_j$ is constant and maximal if $F_{j+1}$ is non-empty.
2. The minimum of $\tilde{f}_j$ is $k_j - \frac{1}{2}$ if $F_j$ is non-empty. If $F_j$ is empty, then by the assumption, $j = 1$ holds and in this case, the minimum of $\tilde{f}_1$ is 1. The maximum of $\tilde{f}_j$ is $k_{j+1} - \frac{1}{2}$ if $j \neq l'$ holds and an integer $l$ larger than $k_l$ if $j = l'$ holds. Respectively, the image $\tilde{f}_j(\text{Int}E_j)$ of the interior $\text{Int}E_j$ of $E_j$ is the open interval $(k_j - \frac{1}{2}, k_{j+1} - \frac{1}{2})$.
3. Singular points of $\tilde{f}_j$ are always in the interior $\text{Int}E_j$ of $E_j$ and at distinct singular points, the values are always distinct. Furthermore, the set of all the singular values consists of all the integers larger than $k_j - \frac{1}{2}$ and smaller than $k_{j+1} - \frac{1}{2}$ if $j \neq l'$ holds and all the integers larger than $k_j - \frac{1}{2}$ and not larger than $l$ if $j = l'$ holds.

We obtain a family of maps $\{\tilde{f}_j \times \text{id}_{S^{n-1}} : E_j \times S^{n-1} \to [k_j - \frac{1}{2}, k_{j+1} - \frac{1}{2}] \times S^{n-1}\}_{j=1}^{l'}$. If $F_1$ is non-empty, then by gluing the family of maps and the projection $p : D^{n-1}_\frac{1}{2} \times F_1 \to D^{n-1}_\frac{1}{2}$ together properly, we obtain a desired round fold map; for a non-negative real number $t$, we regard $\{t\} \times S^{n-1}$ as $\partial D^{n-1}_t$ by identifying $(t, x) \in \{(t) \times S^{n-1}\} \subset \partial D^{n-1}_t$ with $(\frac{t}{|t|}, x) \in D^{n-1}_t$. If $F_1$ is empty, then by gluing the family $\{\tilde{f}_j \times \text{id}_{S^{n-1}} : E_j \times S^{n-1} \to [k_j - \frac{1}{2}, k_{j+1} - \frac{1}{2}] \times S^{n-1}\}_{j=1}^{l'}$ of maps, we obtain a desired round fold map similarly.

We call such a construction a locally trivial spinning construction.

The following proposition has been shown in [8] and also in [11].
Proposition 2 \((\ref{8})\). Let \(m, n \in \mathbb{N}\), \(n \geq 2\) and \(m \geq 2n\). Any manifold represented as a connected sum of \(l \in \mathbb{N}\) closed manifolds regarded as the total spaces of \(S^{m-n}\)-bundles over \(S^n\) admits a locally \(C^\infty\) trivial round fold map \(f\) into \(\mathbb{R}^n\) such that the following four hold.

1. All the regular fibers of \(f\) are disjoint unions of finite copies of \(S^{m-n}\).
2. The number of connected components of \(S(f)\) and the number of connected components of the fiber of a point in a proper core of \(f\) are \(l + 1\).
3. For any connected component \(C\) of \(f(S(f))\) and a small closed tubular neighborhood \(N(C)\) of \(C\), \(f^{-1}(N(C))\) is regarded as the total space of a trivial bundle over \(C\) as in Definition 3 such that a normal fiber \(F_C\) of \(C\) corresponding to the bundle \(f^{-1}(N(C))\) is diffeomorphic to a disjoint union of a finite number of the following two manifolds.
   
   \((a)\) \(D^{m-n+1}\).
   \((b)\) \(S^{m-n+1}\) with the interior of a union of disjoint three \((m - n + 1)\)-dimensional standard closed discs removed.
4. All the connected components of the fiber of a point in a proper core of \(f\) are regarded as fibers of the \(S^{m-n}\)-bundles over \(S^n\) and a fiber of any \(S^{m-n}\)-bundle over \(S^n\) appeared in the connected sum is regarded as a connected component of the fiber of a point in a proper core of \(f\).

Proof of Proposition 2. We only prove the proposition in the case where \(f\) is locally \(C^\infty\) trivial, since we can similarly prove this in the case where \(f\) is \(C^\infty\) trivial. We assume that \(f(M)\) is diffeomorphic to \(D^n\) and we can prove the theorem similarly if \(f(M)\) is not diffeomorphic to \(D^n\).

We may assume that \(f : M \to \mathbb{R}^n\) is a normal form. Let \(S(f)\) consist of \(l\) connected components. Set \(P_0 := D^n_{\frac{1}{2}}\) and \(P_k := D^n_{k + \frac{1}{2}} - \mathrm{Int}D^n_{k - \frac{1}{2}}\) for an integer \(1 \leq k \leq l\). Then \(f^{-1}(P_k)\) is regarded as the total space of a trivial bundle over \(\partial D^n_{k - \frac{1}{2}}\) or \(\partial D^n_{k + \frac{1}{2}}\) such that the fibers are diffeomorphic to a compact manifold, which we denote by \(E_k\) and that the submersions \(f|_{f^{-1}(\partial D^n_{k - \frac{1}{2}})}\) and \(f|_{f^{-1}(\partial D^n_{k + \frac{1}{2}})}\) make the submanifolds subbundles of the bundle \(f^{-1}(P_k)\); we denote the fibers of these two subbundles by \(E_k^1 \subset E_k\) and \(E_k^2 \subset E_k\), respectively.

For any integer \(1 \leq k \leq l\) and a diffeomorphism \(\phi_k\) from \(f^{-1}(\partial D^n_{k - \frac{1}{2}}) \subset f^{-1}(P_k)\) onto \(f^{-1}(\partial D^n_{k + \frac{1}{2}}) \subset f^{-1}(P_{k-1})\) regarded as a bundle isomorphism between the two trivial bundles over standard spheres inducing the identification between the base spaces, \(M\) is regarded as \((\cdots((f^{-1}(D^n_{\frac{1}{2}}))\cup_{\phi_1}f^{-1}(P_1))\cup_{\phi_2}f^{-1}(P_2))\cdots)\cup_{\phi_l}f^{-1}(P_l)\) and for any integer \(1 \leq k \leq l\) and a diffeomorphism \(\Phi_k\) from \(f^{-1}(\partial D^n_{k - \frac{1}{2}}) \times F \subset f^{-1}(P_k) \times F\) onto \(f^{-1}(\partial D^n_{k - \frac{1}{2}}) \times F \subset f^{-1}(P_{k-1}) \times F\) regarded as a bundle isomorphism.
between the two trivial \( F \)-bundles inducing \( \phi_k \), \( M' \) is regarded as \( \cdots (f^{-1}(D^n_1) \times F) \cup \Phi \cup (f^{-1}(P_{k+1/2}) \times F) \cup \Phi \cup (f^{-1}(P_k) \times F) \). We construct a map on \( f^{-1}(P_k) \times F \) and \( f^{-1}(P_{k+1/2}) \times F \) such that the fibers are diffeomorphic to \( E_k \times F \). On \( E_k \times F \) there exists a Morse function \( f_k \) such that the following four hold.

1. \( f_k(E_k \times F) \subset [k - \frac{1}{2}, k + \frac{1}{2}] \) and \( f_k(\text{Int}(E_k \times F)) \subset (k - \frac{1}{2}, k + \frac{1}{2}) \) hold.
2. \( f_k(E_k^1 \times F) = \{k - \frac{1}{2}\} \) holds if \( E_k^1 \times F \) is non-empty.
3. \( f_k(E_k^2 \times F) = \{k + \frac{1}{2}\} \) holds if \( E_k^2 \times F \) is non-empty.
4. Singular points of \( f_k \) are in the interior of \( E_k \times F \) and at two distinct singular points, the values are always distinct.

We obtain a map \( \text{id}_{S^{n-1}} \times f_k : S^{n-1} \times E_k \times F \to S^{n-1} \times [k - \frac{1}{2}, k + \frac{1}{2}] \). We can identify \( S^{n-1} \times [k - \frac{1}{2}, k + \frac{1}{2}] \) with \( P_k = D^n_{\frac{1}{2}} \) or \( \text{Int}D^n_{\frac{1}{2}} \) by identifying \((p, t) \in S^{n-1} \times [k - \frac{1}{2}, k + \frac{1}{2}] \) with \( tp \in P_k \) where we regard \( S^{n-1} \) as the unit sphere of dimension \( n - 1 \). By gluing the composition of the projection from \( f^{-1}(D^n_{1/2}) \times F \) onto \( f^{-1}(D^n_{1/2}) \) and \( \text{id}_{S^{n-1}} \times f_k \) together by using the family \( \{\Phi_k\} \) and the family of identifications in the target manifold \( \mathbb{R}^n \), we obtain a new round fold map \( f' : M' \to \mathbb{R}^n \).

In the proof, from a given map \( f : M \to \mathbb{R}^n \), we obtain a new map \( f' : M' \to \mathbb{R}^n \). We call the operation of constructing \( f' \) from \( f \) a \( P \)-operation by \( F \) to the map \( f \). For example, if \( f \) is a map presented in Example 1 (1), then on any manifold having the bundle structure over the source homotopy sphere, we can construct a (locally) \( C^\infty \) trivial round fold map by a \( P \)-operation to the map \( f \).

4. Constructions of round fold maps on manifolds regarded as linear bundles by \( P \)-operations

In this paper, we denote the \( k \)-th orthogonal group by \( O(k) \) and the \( k \)-th special orthogonal group by \( SO(k) \). In this paper, a bundle is said to be linear if the structure group is a subgroup of an orthogonal group. A linear bundle is said to be orientable if the structure group is reduced to a subgroup of a special orthogonal group and we obtain two oriented linear bundles naturally.

We recall known fundamental terms and facts on linear bundles.

For any linear bundle, we can consider its \( k \)-th Stiefel-Whitney class, which is a \( k \)-th cohomology class of the base space whose coefficient ring is \( \mathbb{Z}/2\mathbb{Z} \). For any oriented linear bundle whose structure group is \( SO(k) \), we can consider its Euler class, which is a \( k \)-th cohomology class of the base space whose coefficient ring is \( \mathbb{Z} \). We introduce known facts on classifications of linear bundles without proofs.

**Proposition 3.** Let \( X \) be a topological space regarded as a CW-complex.

1. The 1st Stiefel-Whitney class \( \alpha \in H^1(X; \mathbb{Z}/2\mathbb{Z}) \) of a linear bundle over \( X \) vanishes if and only if the bundle is orientable.
2. For any \( \alpha \in H^2(X; \mathbb{Z}) \), there exists an oriented linear bundle whose structure group is \( SO(2) \) whose Euler class is \( \alpha \) and the 2nd Stiefel-Whitney class \( \bar{\alpha} \in H^2(X; \mathbb{Z}/2\mathbb{Z}) \) of this bundle is the value of the canonical homomorphism from \( \mathbb{Z} \) onto \( \mathbb{Z}/2\mathbb{Z} \), which sends the integer \( k \) to \( k \mod 2 \).
3. Two oriented bundles whose structure groups are \( SO(2) \) over \( X \) are equivalent if and only if the Euler classes are same. Especially, an oriented
Proposition 4. Let \( k = 0, 1, 2, 3 \) and let \( X \) be simple homotopy equivalent to a \( k \)-dimensional CW-complex. Let \( l \) be an integer larger than 2. Then, a linear bundle \( B \) over \( X \) such that the structure group is \( SO(l) \) and that the 2nd Stiefel-Whitney class vanishes is a trivial linear bundle.

In this section, we define a spin bundle as an orientable linear bundle such that the 2nd Stiefel-Whitney class vanishes as in Proposition 3 (4).

A \( k \)-dimensional real (complex) vector bundles are regarded as linear bundles whose structure groups are the groups of all the linear transformations on the fibers. In this paper, we only consider real vector bundles. We also note that the structure groups of \( k \)-dimensional real vector bundles are regarded as the groups of all the orthogonal transformations on the \( k \)-dimensional vector spaces and naturally as \( O(k) \).

Any \( S^k \)-bundle whose structure group consists of linear transformations on the fiber \( S^k \) is regarded as a linear bundle whose structure group is the group of all the transformations given by the restrictions of orthogonal transformations on \( \mathbb{R}^{k+1} \) considering \( S^k \) as the unit sphere and as a result \( O(k+1) \). It is naturally a subbundle of an associated real vector bundle whose fiber is a \((k+1)\)-dimensional real vector space. We call such a linear bundle a standard linear bundle. For \( k = 1, 2, 3 \), any \( S^k \)-bundle is regarded as a standard linear bundle whose structure group is \( O(k+1) \) (see [20] for the case \( k = 2 \) and [7] for the case \( k = 3 \)).

First, we review some topological properties of the total spaces of such bundles. For a \( k \)-dimensional manifold \( X \), let us denote by \( TX \) the total space of the tangent bundle of \( X \), which is an important \( k \)-dimensional real vector bundle over \( X \) and for \( k \geq 2 \), let us denote the total space of the unit tangent bundle of \( X \) by \( UTX \), which is obtained as the subbundle of the bundle \( TX \) whose fiber is the unit sphere \( S^{k-1} \subset \mathbb{R}^k \). A manifold is said to be spin if its tangent bundle is spin.

Also, the following Proposition 4 is useful.

**Proposition 4.**

1. Let \( i \geq 1 \) be an integer. For any topological space \( E \) regarded as the total space of a standard linear \( S^i \)-bundle over a topological space \( X \) regarded as a CW complex. Let \( \pi \) be a surjection giving \( E \) the bundle structure over \( X \). Let \( w_{i+1} \in H^{i+1}(X;\mathbb{Z}/2\mathbb{Z}) \) be the \((i+1)\)-th Stiefel-Whitney class of the bundle \( E \) and \( \bigcup w_{i+1} \) be the operation of taking a cup product with \( w_{i+1} \). Then, we have the following exact sequence (the Gysin sequence of the bundle \( \pi : E \to X \)).

\[
\begin{align*}
\pi^* & : H^j(E;\mathbb{Z}/2\mathbb{Z}) \to H^j(X;\mathbb{Z}/2\mathbb{Z}) \\
\pi^* & : H^{j+1}(E;\mathbb{Z}/2\mathbb{Z}) \to H^{j+1}(X;\mathbb{Z}/2\mathbb{Z}) \\
\end{align*}
\]

We also have the following exact sequence in the case where the bundle \( E \) is oriented. We denote the Euler class by \( e \in H^{i+1}(X;\mathbb{Z}) \) and \( \cup e \) be the operation of taking a cup product with \( e \).

\[
\begin{align*}
\pi^* & : H^j(E;\mathbb{Z}) \to H^j(X;\mathbb{Z}) \\
\pi^* & : H^{j+1}(E;\mathbb{Z}) \to H^{j+1}(X;\mathbb{Z}) \\
\end{align*}
\]
(2) Let $X_1$ and $X_2$ be closed manifolds and set $X := X_1 \times X_2$. Then the tangent bundle $TX$ over $X$ is regarded as the Whitney sum of the pull-back of the tangent bundle $TX_1$ over $X_1$ by the canonical projection of $X_1 \times X_2$ onto $X_1$ and the pull-back of the tangent bundle $TX_2$ over $X_2$ by the canonical projection of $X_1 \times X_2$ onto $X_2$.

(3) Let $X$ be a closed manifold and let $X'$ be a manifold regarded as the total space of a standard linear bundle whose fiber is diffeomorphic to the standard sphere of dimension $k \geq 1$.

(a) Let $X$ be orientable. If the bundle $X'$ is (not) orientable, then the tangent bundle $TX'$ of $X'$ is (resp. not) orientable.

(b) Let $k \geq 2$ and let the manifold $X$ be spin. If the bundle $X'$ is spin (not spin), then the tangent bundle $TX'$ is spin (not spin).

For the theory of linear bundles and their characteristic classes including Stiefel-Whitney classes and Euler classes, see also [14] for example.

In [12], we have constructed a lot of explicit round fold maps on manifolds regarded as the total spaces of $S^1$-bundles over a manifold admitting a locally $C^\infty$ trivial round fold map by using P-operations. In this section, we apply P-operations to locally $C^\infty$ trivial round fold maps to construct new round fold maps on manifolds regarded as the total spaces of linear bundles and more general bundles over the original source manifolds.

4.1. Cases for round fold maps between low-dimensional manifolds. First we show the following theorem, which gives more round fold maps and their source manifolds.

**Theorem 1.** Let $M$ be a closed manifold of dimension $2 \leq m \leq 4$, $f : M \to \mathbb{R}^n$ ($m \geq n \geq 2$) be a locally $C^\infty$ trivial round fold map. Let $M'$ be a manifold regarded as the total space of a linear bundle whose fiber is a closed manifold $F \neq \emptyset$. Then we have the following.

1. Let $n = 4$. Then by a P-operation by $F$ to $f$, we can obtain a locally $C^\infty$ trivial round fold map from $M'$ into $\mathbb{R}^n = \mathbb{R}^4$.

2. Let $(m, n) = (4, 3)$ and let $M$ be connected. We assume that for a connected component $C_0$ of the singular value set $f(S(f))$ and a small closed tubular neighborhood $N(C_0)$ as in Definition 3, the 2nd Stiefel-Whitney class of the restriction of the bundle $M'$ above to the image of a section of the trivial $\partial \tau_F C_0$-bundle vanishes and that for any connected component $C$ of the singular value set $f(S(f))$ and a small closed tubular neighborhood $N(C)$ as in Definition 3, the restriction of the bundle $M'$ above to the normal fiber $F_C$ of $C$ corresponding to the bundle $f^{-1}(N(C))$ is orientable. Then by a P-operation by $F$ to the map $f$, we can obtain a locally $C^\infty$ trivial round fold map from $M'$ into $\mathbb{R}^n$.

3. Let $(m, n) = (4, 2)$. We assume that for any connected component $C$ of the singular value set $f(S(f))$ and a small closed tubular neighborhood $N(C)$, the restriction of the bundle $M'$ to $f^{-1}(N(C))$ is spin. Then by a P-operation by $F$ to the map $f$, we can obtain a locally $C^\infty$ trivial round fold map from $M'$ into $\mathbb{R}^n$.

4. Let $(m, n) = (3, 3)$ and let $M$ be connected. We assume that for a connected component $C$ of the singular value set $f(S(f))$ and a small closed tubular neighborhood $N(C)$ as in Definition 3, the 2nd Stiefel-Whitney class of the
restriction of the bundle $M'$ to the image of a section of the trivial $\partial_0F_{C_0}$-bundle vanishes. Then by a $P$-operation by $F$ to the map $f$, we can obtain a locally $C^\infty$ trivial round fold map from $M'$ into $\mathbb{R}^n$.

(5) Let $(m, n) = (3, 2)$. We assume that for any connected component $C$ of the singular value set $f(S(f))$ and a small closed tubular neighborhood $N(C)$, the restriction of the bundle $M'$ to $f^{-1}(N(C))$ is spin. Then by a $P$-operation by $F$ to the map $f$, we can obtain a locally $C^\infty$ trivial round fold map from $M'$ into $\mathbb{R}^n$.

(6) Let $(m, n) = (2, 2)$ and let $M$ be connected. We assume that for a connected component $C_0$ of the singular value set $f(S(f))$ and a small closed tubular neighborhood $N(C_0)$ as in Definition 3, the restriction of the bundle $M'$ to the image of a section of the trivial $\partial_0F_{C_0}$-bundle is orientable. Then by a $P$-operation by $F$ to the map $f$, we can obtain a locally $C^\infty$ trivial round fold map from $M'$ into $\mathbb{R}^n$.

Proof. By virtue of Proposition 2, to prove the statements, it suffices to show that for any connected component $C$ of the singular value set $f(S(f))$ and a small closed tubular neighborhood $N(C)$ as in Definition 3, the restriction of the bundle $M'$ over $M$ to $f^{-1}(N(C))$ is trivial. Note that $f^{-1}(N(C))$ is regarded as a trivial bundle whose fiber is $F_C$ as mentioned in Definition 3. Note also that $F_C$ is regarded as an $(m-n+1)$-dimensional CW-complex simple homotopy equivalent to an $(m-n)$-dimensional CW-complex and that $f^{-1}(N(C))$ is regarded as an $m$-dimensional CW-complex simple homotopy equivalent to the product of $S^{n-1}$ and the $(m-n)$-dimensional CW-complex before.

We prove the first case. $n = 4$ or $(m, n) = (4, 4)$ is assumed. In this case, $f^{-1}(N(C))$ is a compact 4-dimensional manifold diffeomorphic to the product of $S^3$ and the closed interval and regarded as a CW-complex simple homotopy equivalent to $S^3$. We have $H^1(f^{-1}(N(C)); \mathbb{Z}/2\mathbb{Z}) \cong H^2(f^{-1}(N(C)); \mathbb{Z}) \cong \{0\}$ and from Proposition 3 (4), the restriction of the bundle $M'$ over $M$ to $f^{-1}(N(C))$ is trivial.

We prove the second case. $(m, n) = (4, 3)$ is assumed. Thus, $f^{-1}(N(C))$ is a compact 4-dimensional manifold and regarded as a CW-complex simple homotopy equivalent to a 3-dimensional CW-complex and as the product of the 2-dimensional sphere $S^2$ and the compact surface $F_C$ with non-empty boundary. It is assumed that for a connected component $C_0$ of the singular value set $f(S(f))$ and a small closed tubular neighborhood $N(C_0)$, the 2nd Stiefel-Whitney class of the restriction of the bundle above to $f^{-1}(N(C_0))$ vanishes and $M$ is connected. Moreover, it is assumed that for any connected component $C$ of the singular value set $f(S(f))$ and a small closed tubular neighborhood $N(C)$ as in Definition 3, the restriction of the bundle $M'$ above to the normal fiber $F_C$ of $C$ corresponding to the bundle $f^{-1}(N(C))$ is orientable. We have $H^1(f^{-1}(N(C)); \mathbb{Z}/2\mathbb{Z}) \cong H^1(F_C; \mathbb{Z}/2\mathbb{Z})$ and $H^2(f^{-1}(N(C)); \mathbb{Z}/2\mathbb{Z}) \cong H^2(C; \mathbb{Z}/2\mathbb{Z}) \oplus H^2(F_C; \mathbb{Z}/2\mathbb{Z}) \cong H^2(C; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ for any connected component $C$ of the singular value set $f(S(f))$. By these facts, for any connected component $C$ of the singular value set $f(S(f))$ and a small closed tubular neighborhood $N(C)$, the 2nd Stiefel-Whitney class of the restriction of the bundle $M'$ to $f^{-1}(N(C))$ vanishes. From Proposition 3 (4), the restriction of the bundle $M'$ over $M$ to $f^{-1}(N(C))$ is spin and trivial.

We prove the third and fifth cases. In each case, the result follows from the assumption that for any connected component $C$ of the singular value set $f(S(f))$ and a small closed tubular neighborhood $N(C)$ as in Definition 3, the restriction of
the bundle $M'$ to $f^{-1}(N(C))$, which is regarded as a CW-complex simple homotopy equivalent to a CW-complex of dimension 2 or 3, is a spin bundle together with Proposition 3 (4).

We can prove the fourth and sixth cases by applying methods similar to that of the second case. So we omit the proof. □

By considering specific cases of some cases of Theorem 1, as a corollary, we have the following.

**Corollary 1.** Let $M$ be a closed manifold of dimension $2 \leq m \leq 4$, $f : M \to \mathbb{R}^n$ $(m \geq n \geq 2)$ be a locally $C^\infty$ trivial round fold map. Let $M'$ be a manifold regarded as the total space of a linear bundle whose fiber is a closed manifold $F \neq \emptyset$. Then we have the following.

1. Let $(m, n) = (3, 2), (4, 2), (4, 3)$. If the bundle above is a spin bundle, then by a P-operation by $F$ to the map $f$, we have a locally $C^\infty$ trivial round fold map from $M'$ into $\mathbb{R}^n$.
2. Let $(m, n) = (3, 3)$. If the 2nd Stiefel-Whitney class of the bundle above vanishes, then by a P-operation by $F$ to the map $f$, we have a locally $C^\infty$ trivial round fold map from $M'$ into $\mathbb{R}^n$.
3. Let $(m, n) = (2, 2)$. If the bundle above is an orientable bundle, then by a P-operation by $F$ to the map $f$, we have a locally $C^\infty$ trivial round fold map from $M'$ into $\mathbb{R}^n$.

**Example 2.** (1) In the situation of the former part of Example 1 (2), let $(m, n) = (4, 3)$ and the given map $f$ be $C^\infty$ trivial and locally $C^\infty$ trivial or in the situation of Proposition 1, let $(m, n) = (4, 2)$ and $l = 1$.

The source manifold $M$ in the example is $S^2 \times S^2$ or a manifold regarded as the total space of a non-trivial $S^2$-bundle over $S^2$, which is not spin. For both cases, we have $H^2(M; \mathbb{Z}) \cong \mathbb{Z}^2$. Let $M = S^2 \times S^2$, which is naturally regarded as the total space of a trivial $S^2$-bundle over $S^2$ and let $\alpha, \beta \in H^2(M; \mathbb{Z}) \cong \mathbb{Z}^2$ be generators represented by the base space and the fiber of the trivial bundle, respectively. Let $a, b \in \mathbb{Z}$ and consider the Whitney sum of two real (oriented) vector bundles of dimension 2 whose Euler classes are $a\alpha$ and $2b\beta$, respectively. We immediately have the subbundle whose fiber is the unit sphere $S^3$ (we denote the total space by $M'$) and the Euler class of the bundle is 2ab times a generator of the cohomology group $H^4(M; \mathbb{Z}) \cong \mathbb{Z}$ and by Proposition 4 (1), we have $H^4(M'; \mathbb{Z}) \cong \mathbb{Z}/|2ab|\mathbb{Z}$.

More precisely, we determine this cohomology group by using the following Gysin sequence where $\bigcup e$ is the operation of taking a cup product with the Euler class $e \in H^4(M'; \mathbb{Z})$ of the standard linear bundle $M'$ and we often use similar methods in this paper.

$$
\begin{array}{ccc}
\longrightarrow & H^3(M'; \mathbb{Z}) & \longrightarrow & H^0(M; \mathbb{Z}) \cong \mathbb{Z} & \bigcup e & H^4(M; \mathbb{Z}) \cong \mathbb{Z} \\
\longrightarrow & H^4(M'; \mathbb{Z}) & \longrightarrow & H^1(M; \mathbb{Z}) \cong \{0\} & \bigcup e & \\
\end{array}
$$

If $a$ is even, then it is a spin bundle and if $a$ is odd, then it is not a spin bundle. If the bundle is spin, then on the total space $M'$ of the $S^3$-bundle over $M = S^2 \times S^2$, we can construct a round fold map by applying Theorem 1 or Corollary 1 (1) and the resulting source manifold is spin (we can apply
both Theorem 1 and Corollary 1 (1)). Even if the bundle is not spin, then on the total space $M'$ of the $S^2$-bundle over $M = S^2 \times S^2$, we can construct a round fold map by applying Theorem 1 (we cannot apply Corollary 1 (1)) and the resulting source manifold is not spin. More generally, let $a_1, a_2, b_1, b_2 \in \mathbb{Z}$ and consider the Whitney sum of two (oriented) real vector bundles of dimension 2 whose Euler classes are $a_1\alpha + 2b_1\beta$ and $a_2\alpha + 2b_2\beta$, respectively. We immediately have the subbundle whose fiber is the unit sphere $S^3$ (we denote the total space by $M'$), the Euler class of the bundle is $2a_1b_2 + 2a_2b_1$ times a generator of the cohomology group $H^4(M; \mathbb{Z}) \cong \mathbb{Z}$ and we have $H^4(M'; \mathbb{Z}) \cong \mathbb{Z}/(2a_1b_2 + 2a_2b_1\mathbb{Z}$ by applying Proposition 4 (1). Moreover, we can construct a round fold map from $M'$ into $\mathbb{R}^n$ by applying Theorem 1 or Corollary 1 (1).

Furthermore, let $M$ be the total space of a non-trivial $S^2$-bundle over $S^2$ in this situation, or more generally, in the situation of Proposition 1, let $(m, n) = (4, 2)$. By applying Theorem 1, we can obtain explicit locally $C^\infty$ trivial round fold maps similarly. For the latter case, see also Example 6 (1) later.

(2) In the situation of Example 1 (3), let $(m, n) = (4, 3)$. The source manifold $M := S^2 \times S^2$ of dimension $m = 4$ admits a $C^\infty$ trivial and locally $C^\infty$ trivial round fold map into $\mathbb{R}^n = \mathbb{R}^3$. We define $\alpha, \beta \in H^2(M; \mathbb{Z}) \cong \mathbb{Z}^2$ as in the example just before. Let $a_1, a_2, b_1, b_2 \in \mathbb{Z}$ and let the sum $a_1 + a_2$ be even. Consider the Whitney sum of two (oriented) real vector bundles of dimension 2 whose Euler classes are $a_1\alpha + b_1\beta$ and $a_2\alpha + b_2\beta$, respectively. Similarly, we immediately have the subbundle whose fiber is the unit sphere $S^3$ (we denote the total space by $M'$), the Euler class of the bundle is $a_1b_2 + a_2b_1$ times a generator of the cohomology group $H^4(M; \mathbb{Z}) \cong \mathbb{Z}$ and we have $H^4(M'; \mathbb{Z}) \cong \mathbb{Z}/(a_1b_2 + a_2b_1\mathbb{Z}$ by applying Proposition 4 (1). Moreover, we can construct a round fold map from $M'$ into $\mathbb{R}^n$ by applying Theorem 1 or Corollary 1 (1).

4.2. Cases for round fold maps such that regular fibers are disjoint unions of spheres. In the previous subsection, we obtained new round fold maps by applying P-operations to some locally $C^\infty$ trivial round fold maps from $m$-dimensional round fold maps into $\mathbb{R}^n$ under the constraint that $4 \geq m \geq n \geq 2$ holds. As specific cases, we applied P-operations to locally $C^\infty$ trivial round fold maps in the former part of Example 1 (2) and (3) and Proposition 1. Here, for general pairs $(m, n)$ of dimensions, we apply P-operations to round fold maps satisfying these conditions.

**Theorem 2.** Let $m, n \in \mathbb{N}$. Let $m > n \geq 2$ hold. Let an $m$-dimensional connected closed manifold $M$ admit a locally $C^\infty$ trivial round fold map into $\mathbb{R}^n$ satisfying the following conditions as mentioned in Proposition 1.

1. All the regular fibers of $f$ are disjoint unions of finite copies of $S^{m-n}$.
2. The number of connected components of $S(f)$ and the number of connected components of the fiber of a point in a proper core of $f$ coincide.
3. For any connected component $C$ of $f(S(f))$ and a small closed tubular neighborhood $N(C)$ of $C$, $f^{-1}(N(C))$ is regarded as the total space of a trivial bundle over $C$ as in Definition 3 such that a normal fiber $F_C$ of $C$ corresponding to the bundle $f^{-1}(N(C))$ is homeomorphic to a disjoint union of a finite number of the following manifolds.
   (a) $D^{m-n+1}$.
connected component $C$:

Example 3. Let $m$ and $n$ be integers satisfying the relation $m > n \geq 2$. Let $M$ be a manifold regarded as the total space of a trivial $S^{m-n}$-bundle over $S^n$ and let $f : M \to \mathbb{R}^n$ be a locally $C^\infty$ trivial round fold map presented in Example 1 (2).

1. We consider the Whitney sum of the pull-back of a trivial $k_1$-dimensional real vector bundle over $S^{m-n}$ by the canonical projection of the product $M = S^{m-n} \times S^n$ onto $S^{m-n}$ and the pull-back of a $k_2$-dimensional real vector bundle over $S^n$ by the canonical projection of the product $M = S^{m-n} \times S^n$ onto $S^n$. Let $M'$ be the total space of the subbundle of the real vector bundle whose fiber is the $(k_1 + k_2 - 1)$-dimensional unit sphere. Then, the bundle $M'$ is trivial over the fiber of a point in a proper core of $f$. We can apply Theorem 2 to obtain a round fold map from $M'$ into $\mathbb{R}^n$.

In this situation, let $n = 2$. In addition, let $k_2 = 2$. For any integer $k$, we can take the mentioned 2-dimensional (oriented) real

Proof. We consider a normal form $f_0 : M \to \mathbb{R}^n$ of the map $f$. If we restrict the bundle $M'$ over $M$ to the inverse image $f^{-1}(D^n_{\frac{1}{2}})$ of the proper core $D^n_{\frac{1}{2}}$, then it is trivial by the assumption on $M'$.

For any connected component $C$ of the singular value set $f(S(f))$, we denote a small closed tubular neighborhood as in Definition 3 by $N(C)$ and the normal fiber corresponding to the trivial bundle $f^{-1}(N(C))$ explained in Definition 3 by $F_C$. In the definition, the subbundle of the bundle $f^{-1}(N(C))$ whose fiber is $\partial_{\delta} F_C \subset F_C$ is defined and we assume that the restriction of the bundle $M'$ over $M$ to the total space of this subbundle is trivial. Then, by the assumed conditions, the restriction of the bundle to $f^{-1}(N(C))$ is also trivial. More precisely, we have this fact as the following.

By the third condition, the fiber of any connected component of the bundle $f^{-1}(N(C))$ is $D^{m-n+1}$ or $S^{m-n+1}$ with the interior of a union of disjoint three $(m-n+1)$-dimensional standard closed discs removed. By considering the intersection of the fiber of each connected component of the bundle $f^{-1}(N(C))$ and the fiber $F_C$ of the bundle $f^{-1}(N(C))$, we obtain a subbundle of each connected component of the bundle $f^{-1}(N(C))$. By the second condition, the fiber of the resulting bundle is homeomorphic to the sphere $S^{m-n}$ if the fiber of the connected component of $f^{-1}(N(C))$ is homeomorphic to $D^{m-n+1}$ and the fiber of the resulting bundle is homeomorphic to the disjoint union of two copies of the sphere $S^{m-n}$ if the fiber of the connected component of $f^{-1}(N(C))$ is homeomorphic to $S^{m-n+1}$ with the interior of a union of disjoint three $(m-n+1)$-dimensional standard closed discs removed. By considering the homotopy types of the fibers of the connected components of the trivial bundle $f^{-1}(N(C))$, we have the desired fact.

By the induction, if we restrict the bundle $M'$ over $M$ to $f^{-1}(N(C))$ for any connected component $C$ of the singular value set $f(S(f))$, then it is trivial. Thus we have the statement.

$\square$

Example 3. Let $m$ and $n$ be integers satisfying the relation $m > n \geq 2$. Let $M$ be a manifold regarded as the total space of a trivial $S^{m-n}$-bundle over $S^n$ and let $f : M \to \mathbb{R}^n$ be a locally $C^\infty$ trivial round fold map presented in Example 1 (2).

(b) $S^{m-n+1}$ with the interior of a union of disjoint three $(m-n+1)$-dimensional standard closed discs removed.

Thus, on any manifold $M'$ regarded as the total space of a bundle over $M$ such that the restriction to any connected component of the fiber of a point in a proper core of $f$ is trivial, by a P-operation, we have a round fold map into $\mathbb{R}^n$. 
vector bundle over $S^n = S^2$ as a bundle whose Euler class is $k$ times a generator of the group $H^2(S^2; \mathbb{Z}) \cong \mathbb{Z}$. Furthermore, if $k$ is even (odd), then the resulting bundle $M'$ is spin (resp. not spin) and the manifold $M'$ is spin (resp. not spin). Let $k_2 > 2$ hold. We consider the mentioned $k_2$-dimensional real vector bundle over $S^n = S^2$. Thus, if the mentioned $k_2$-dimensional real vector bundle is spin (not spin), then the resulting bundle $M'$ is spin (resp. not spin) and the manifold $M'$ is spin (resp. not spin).

(2) We consider the Whitney sum of the pull-back of the tangent bundle over $S^{m-n}$ by the canonical projection of the product $M = S^{m-n} \times S^n$ onto $S^{m-n}$ and the pull-back of a $k$-dimensional real vector bundle over $S^n$ by the canonical projection of the product $M = S^{m-n} \times S^n$ onto $S^n$. Let $M'$ be the total space of the subbundle of the real vector bundle whose fiber is the $(m-n+k-1)$-dimensional unit sphere. Then, the bundle $M'$ is trivial over the fiber of a point in a proper core of $f$. We can apply Theorem 2 to obtain a round fold map from $M'$ into $\mathbb{R}^n$.

In this situation, let $m$ and $n$ be even and let $k = n$. Then, for a generator $a$ of the cohomology group $H^m(M; \mathbb{Z}) \cong \mathbb{Z}$, we can construct the (oriented) bundles $M'$ over $M$ whose Euler classes are 0 and $4a$. In fact, we can take the mentioned (oriented) $n$-dimensional real vector bundle over $S^n$ as a trivial bundle and also as a tangent bundle over $S^n$. Especially, if $n = 2$ holds, then for any integer $l$, we can construct the bundles $M'$ over $M$ whose Euler class is $2l\alpha$. In fact, we can take the mentioned oriented $2$-dimensional real vector bundle over $S^n = S^2$ as a bundle whose Euler class is $l$ times a generator of the group $H^2(S^2; \mathbb{Z}) \cong \mathbb{Z}$.

(3) Let $m \geq 4$ and let $n = m-2 \geq 2$ in this situation. We consider the Whitney sum of the pull-back of a $k_1$-dimensional vector bundle over $S^2$ which is spin by the canonical projection of the product $M = S^{m-n} \times S^n = S^{m-2} \times S^2$ onto $S^{m-n} = S^{m-2}$ and the pull-back of a $k_2$-dimensional real vector bundle over $S^{m-2}$ by the canonical projection of the product $M = S^2 \times S^{m-2}$ onto $S^{m-2}$. Let $M'$ be the total space of the subbundle of the real vector bundle whose fiber is the $(k_1 + k_2 - 1)$-dimensional unit sphere. Then, the bundle $M'$ is trivial over the fiber of a point in a proper core of $f$. We can apply Theorem 2 to obtain a round fold map from $M'$ into $\mathbb{R}^n$.

As a specific case, let $n = m - 2 = 4$ or $m = 6$ and let $k_1 = 2$ and $k_2 = m - 2 = 6 - 2 = 4$. For any integer $k$, we can take a $k_1$-dimensional (oriented) real vector bundle whose Euler class is $k$ times a generator of the cohomology group $H^2(S^2; \mathbb{Z}) \cong \mathbb{Z}$ of the base space $S^2$. For any integer $k$, we can take a $k_2$-dimensional (oriented) real vector bundle whose Euler class is $k$ times a generator of the cohomology group $H^{m-2}(S^{m-2}; \mathbb{Z}) \cong H^4(S^4; \mathbb{Z}) \cong \mathbb{Z}$ of the base space $S^{m-2} = S^4$. We consider the Whitney sum of the pull-backs defined before. The resulting real vector bundle is of dimension $k_1 + k_2 = 2 + 4 = 6$ and for any integer $k$, we can obtain this vector bundle so that the Euler class is $2k$ times a generator of the cohomology group $H^m(M; \mathbb{Z}) \cong H^m(S^2 \times S^{m-2}; \mathbb{Z}) \cong \mathbb{Z}$. It follows that for any integer $k$, we can obtain the total space $M'$ of the subbundle of the real vector bundle whose fiber is the 5-dimensional unit sphere satisfying $H^6(M'; \mathbb{Z}) \cong \mathbb{Z}/|2k|\mathbb{Z}$. 


Theorem 3. Let $m, n \in \mathbb{N}$. Let $m > n \geq 2$ hold. Let an $m$-dimensional connected closed manifold $M$ admit a locally $C^\infty$ trivial round fold map $f$ into $\mathbb{R}^n$ which is special generic and whose image is diffeomorphic to the cylinder $S^{n-1} \times [-1, 1]$ as presented in Example 1 (3). Thus, on any manifold $M'$ regarded as the total space of a bundle over $M$ such that for any $(n-1)$-dimensional standard sphere $C'$ embedded in the interior of the image $f(M)$, the restriction to the image of a section of the trivial bundle given by $f|_{f^{-1}(C')} : f^{-1}(C') \to C'$ is trivial, by a $P$-operation to the original map $f$, we have a round fold map into $\mathbb{R}^n$.

Proof. Let $C$ be a connected component of the singular value set $f(S(f))$ and we take a small closed tubular neighborhood $N(C)$ as presented in Definition 3. The obtained bundle $f^{-1}(N(C))$ is a trivial bundle and the normal fiber $F_C$ corresponding to the bundle $f^{-1}(N(C))$ is the $(m-n+1)$-dimensional standard closed disc $D^{m-n+1}$. By the extra assumption on the bundle $M'$ over $M$ that for any $(n-1)$-dimensional standard sphere $C'$ embedded in the interior of the image $f(M)$, the restriction to the image of a section of the trivial bundle given by $f|_{f^{-1}(C')} : f^{-1}(C') \to C'$ is trivial, the fact that the previous trivial bundle is a subbundle of the bundle $f^{-1}(N(C))$ and the fact that the fiber of the trivial bundle $f^{-1}(N(C))$ is diffeomorphic to the disc $D^{m-n+1}$ and contractible, if we restrict the bundle $M'$ to $f^{-1}(N(C))$, then it is trivial. This completes the proof. □

Example 4. Let $m$ and $n$ be integers satisfying the relation $m > n \geq 2$ as assumed in Example 3. Let $\Sigma$ be an $(m-n+1)$-dimensional homotopy sphere and let $M = \Sigma \times S^{n-1}$. We consider a round fold map $f : M \to \mathbb{R}^n$ presented in Example 1 (3) or in the assumption of Theorem 3.

1. We consider the Whitney sum of the pull-back of a $k_1$-dimensional real vector bundle over $\Sigma$ by the canonical projection of the product $M = \Sigma \times S^{n-1}$ onto $\Sigma$ and the pull-back of a trivial $k_2$-dimensional real vector bundle over $S^{n-1}$ by the canonical projection of the product $M = \Sigma \times S^{n-1}$ onto $S^{n-1}$. Let $M'$ be the total space of the subbundle of the real vector bundle whose fiber is the $(k_1 + k_2 - 1)$-dimensional unit sphere. Then, the bundle $M'$ is trivial over the image of the section of the bundle $f^{-1}(C')$ in Theorem 3 and we can apply Theorem 3.

2. We consider the Whitney sum of the pull-back of the tangent bundle of $S^{n-1}$ by the canonical projection of the product $M = \Sigma \times S^{n-1}$ onto $S^{n-1}$ and the pull-back of a $k$-dimensional real vector bundle over $\Sigma$ by the canonical projection of the product $M = \Sigma \times S^{n-1}$ onto $\Sigma$. Let $M'$ be the total space of the subbundle of the real vector bundle whose fiber is the $(n + k - 1)$-dimensional unit sphere. Then, the bundle $M'$ is trivial over the image of the section of the bundle $f^{-1}(C')$ in Theorem 3 and we can apply Theorem 3.

In this situation, let $m$ and $n - 1$ be even. Then, for a generator $\alpha$ of the homology group $H^m(M; \mathbb{Z}) \cong \mathbb{Z}$, we can construct the bundles $M'$ over $M$ whose Euler classes are 0 and $4\alpha$. In fact, we can take the mentioned $k$-dimensional (oriented) real vector bundle over $\Sigma$ as a trivial bundle of dimension $k = m - n + 1$ and also as a tangent bundle of $\Sigma$, which is of dimension $k = m - n + 1$. Especially, if $m - (n - 1) = 2$ or $m - n = 1$ holds, then for any integer $l$, we can construct the bundle $M'$ over $M$ whose Euler class is $2l\alpha$. In fact, we can take the mentioned $k$-dimensional (oriented)
holds. Thus, we have a family of manifolds \( \{ \mathcal{C} \} \) be a locally connected component neighborhood \( \mathcal{N} \) regarded as the total spaces of linear bundles over \( C \). Let \( f \) be a round fold map into the plane. In this paper, we construct such maps on manifolds \( \mathcal{C} \). We have the following.

4.3. Cases for round fold maps into the plane. In [12], we have obtained a lot of round fold maps by P-operations by the circle \( S^1 \) to a locally \( C^\infty \) trivial round fold map into the plane. In this paper, we construct such maps on manifolds regarded as the total spaces of (more general) linear bundles by P-operations.

We introduce a class of round fold maps first introduced in [9].

**Definition 4.** Let \( f : M \to \mathbb{R}^n \) be a round fold map, and let \( R \) be a commutative group.

Let \( P \) be a proper core of \( f \). Then, \( f^{-1}(\mathbb{R}^n - \text{Int} P) \) has a bundle structure mentioned just before Definition 2. \( f \) is said to be homologically \( R \)-trivial if for a bundle \( f^{-1}(\mathbb{R}^n - \text{Int} P) \), the following diagram commutes for the canonical projection \( p : \partial P \times f^{-1}(L) \to \partial P \), the projection of the bundle \( \pi : f^{-1}(\mathbb{R}^n - \text{Int} P) \to \partial P \) and two isomorphisms of homology groups \( \Phi \) and \( \phi \) for any integer \( j \).

\[
\begin{align*}
H_j(E'; R) & \xrightarrow{\Phi} H_j(\partial P \times f^{-1}(L); R) \\
\downarrow \pi_* & \downarrow p_* \\
H_j(\partial P; R) & \xrightarrow{\phi} H_j(\partial P; R)
\end{align*}
\]

We have the following.

**Theorem 4.** Let \( M \) be a closed manifold of dimension \( m \geq 2 \) and let \( f : M \to \mathbb{R}^2 \) be a locally \( C^\infty \) trivial and homologically \( \mathbb{Z} \)-trivial round fold map such that for any connected component \( C \) of the singular value set \( f(S(f)) \), a small closed tubular neighborhood \( N(C) \) and \( F_C \) as in Definition 3, \( H^1(F_C; \mathbb{Z}) \cong H^2(F_C; \mathbb{Z}) \cong \{0\} \) holds. Thus, we have a family of manifolds \( \{ M_K \} \subset H_1(f^{-1}(L); \mathbb{Z}/2\mathbb{Z}) \oplus H_2(f^{-1}(L); \mathbb{Z}/2\mathbb{Z}) \) regarded as the total spaces of linear bundles over \( M \) and a family of round fold maps \( \{ f_K : M_K \to \mathbb{R}^n \} \subset H_1(f^{-1}(L); \mathbb{Z}/2\mathbb{Z}) \oplus H_2(f^{-1}(L); \mathbb{Z}/2\mathbb{Z}) \).

Furthermore, we have the following three statements.

1. We have the linear bundles \( M_K \) as standard linear bundles whose fibers are the standard sphere \( S^k \) of dimension \( k > 1 \) and we can construct the maps \( f_K \) so that they are homologically \( \mathbb{Z} \)-trivial.

2. Let the manifold \( f^{-1}(L) \) be spin and for a proper core \( P \) of \( f \), the manifold \( f^{-1}(\mathbb{R}^n - \text{Int} P) \) be not spin. In this case, \( f \) is not \( C^\infty \) trivial. If the linear bundles \( M_K \) are standard linear bundles whose fibers are diffeomorphic to the standard sphere \( S^k \) of dimension \( k > 1 \), then for an element \( K \in H_1(f^{-1}(L); \mathbb{Z}/2\mathbb{Z}) \oplus H_2(f^{-1}(L); \mathbb{Z}/2\mathbb{Z}) \), the map \( f_K \) is not \( C^\infty \) trivial.
(3) Let the manifolds $f^{-1}(L)$ and $f^{-1}(\mathbb{R}^n - \text{Int}P)$ be spin. In this case, we can construct the map $f_K$ so that it is not $C^\infty$ trivial for any element $K = (c, 0) \in H_1(f^{-1}(L); \mathbb{Z}/2\mathbb{Z}) \oplus H_2(f^{-1}(L); \mathbb{Z}/2\mathbb{Z})$ where $c \in H_1(f^{-1}(L); \mathbb{Z}/2\mathbb{Z})$ is not zero.

Proof. From the assumption that $H^1(F_C; \mathbb{Z}) \cong H^2(F_C; \mathbb{Z}) \cong \{0\}$ holds, we have $H^2(f^{-1}(N(C)) \times F_C; \mathbb{Z}) \cong \{0\}$. Here, we consider the situation of Definition 4 and abuse notation there. Let $k_0 \in H_2(\partial P; \mathbb{R}) \cong \mathbb{Z}$ be a generator of the group. For $K \in H_1(f^{-1}(L); \mathbb{Z}/2\mathbb{Z})$, let us regard the tensor product $k_0 \otimes K$ as an element of $H_2(\partial P \times f^{-1}(L); \mathbb{R})$ by considering the natural identification and we denote the 2nd homology class $\Phi^{-1}(k_0 \otimes K) \in H_2(f^{-1}(\mathbb{R}^n - \text{Int}P); \mathbb{Z}/2\mathbb{Z})$ by $K'$.

By applying Proposition 3 (2), we can obtain a manifold $M_K$ regarded as the total space of a linear bundle whose structure group is $SO(2)$ such that the 2nd Stiefel-Whitney class is the dual of $H_2(f^{-1}(\mathbb{R}^n - \text{Int}P); \mathbb{Z}/2\mathbb{Z})$ and construct the desired round fold map $f_K : M_K \to \mathbb{R}^n$. By constructing the manifold $M_K$ as the total space of a standard linear bundle whose fiber is diffeomorphic to $S^k$ satisfying $k > 1$, we easily have the first statement of the latter three statements too. The former part of the second statement of the latter three statements is clear. In the situation of this statement, there exist a class $K$ and the corresponding manifold $M_K$ regarded as the total space of a linear bundle whose structure group is $SO(2)$ such that the 2nd Stiefel-Whitney class of the bundle is the dual of $K' \in H_2(f^{-1}(\mathbb{R}^n - \text{Int}P); \mathbb{Z}/2\mathbb{Z})$ and that this 2nd Stiefel-Whitney class and that of the tangent bundle of the manifold $f^{-1}(\mathbb{R}^n - \text{Int}P)$ do not coincide (set $K = 0$ for example). Thus, we have a round fold map $f_K : M_K \to \mathbb{R}^n$ which is not $C^\infty$ trivial since the total space of the bundle obtained by the restriction of the bundle $M_K$ to $f^{-1}(\mathbb{R}^n - \text{Int}P)$ is not spin by Proposition 4 (3). Last, in the situation of the last statement of the three statements, the resulting manifold $M_K$ is not spin, for an proper core $P_K$ of the resulting round fold map $f_K$, the inverse image $f^{-1}(\mathbb{R}^n - \text{Int}P_K)$ is not spin, and for an axis $L_K$ of the resulting map $f_K : M_K \to \mathbb{R}^n$, the inverse image $f_K^{-1}(L_K)$ is spin. Thus, we obtain the last statement of the latter three.

Example 5. (1) In the situation of the explanation of a locally trivial spinning construction introduced after Definition 3, let $n = 2$ and $l' = 3$. Moreover, let $E_1$ and $E_2$ be a manifold homeomorphic to the standard sphere of dimension $k \geq 4$ with the interior of disjoint three smoothly embedded $k$-dimensional standard closed discs removed; moreover, let $F_2$ be the disjoint union of two copies of a $(k-1)$-dimensional homotopy sphere and let $F_1$ and $F_3$ be a $(k-1)$-dimensional homotopy sphere and let $E_3$ be a $k$-dimensional standard closed disc. By performing the construction, we have a round fold map satisfying the assumption of Theorem 4.

(2) In the situation of Example 1 or Theorem 3, let $m = n = 2$. Then $M$ is the torus $S^1 \times S^1$. We may apply Theorem 3 or 4 to obtain a round fold map from a manifold $M'$ regarded as the total space of a standard linear bundle over the torus $M$ whose fiber is diffeomorphic to the standard sphere of dimension $k \geq 3$; especially, we can take the linear bundle and the manifold $M'$ that are not spin. Furthermore, we can obtain a resulting round fold map satisfying the assumption of the previous example by a P-operation. This resulting map satisfies the assumption of Theorem 4 (2).
4.4. Other cases. First, by applying Proposition 4 (2), we easily have the following proposition.

**Proposition 5.** Let $X_1$ and $X_2$ be topological spaces and let $\pi_i : X_1 \times X_2 \to X_i$ be the canonical projection ($i = 1, 2$). Let $B_i$ be regarded as the total space of a real vector bundle over $X_i$. Assume also that the following two hold.

1. The vector bundle $B_1$ over $X_1$ is trivial.
2. The Whitney sum of the vector bundle $B_2$ over $X_2$ and a trivial real vector bundle over $X_2$ of dimension not larger than that of the vector bundle $B_1$ is a trivial vector bundle over $X_2$.

Then, the Whitney sum of the vector bundle over $X_1 \times X_2$ defined as the pull-back of the bundle $B_1$ by the projection $\pi_1$ and the bundle defined as the pull-back of the bundle $B_2$ by the projection $\pi_2$ is a trivial vector bundle over $X_1 \times X_2$.

By virtue of Propositions 2 and 5, we immediately have the following.

**Proposition 6.** Let $M$ be a closed manifold of dimension $m \geq 2$, let $f : M \to \mathbb{R}^n$ ($m \geq n \geq 2$) be a locally $C^\infty$ trivial round fold map. Let $M'$ be a manifold regarded as the total space of a standard linear bundle over $M$ such that the following two hold.

1. For any connected component $C$ of the singular value set $f(S(f))$ and a small closed tubular neighborhood $N(C)$ as in Definition 3, the restriction of the bundle $M'$ to $f^{-1}(N(C))$, which is regarded as the total space of a trivial bundle over $C$, is equivalent to the Whitney sum of the following two real vector bundles $E_1$ and $E_2$, where $F_C$ is the normal fiber corresponding to the trivial bundle $f^{-1}(N(C))$.
   a. The pull-back $E_1$ of a real vector bundle over $C$ by the projection of the trivial bundle $f^{-1}(N(C))$ over $C$.
   b. The pull-back $E_2$ of a real vector bundle over $F_C$ by the canonical projection of $f^{-1}(N(C))$, regarded as $C \times F_C$, onto $F_C$.
2. One of the previous two bundles $E_1$ ($E_2$) is trivial and the Whitney sum of the other bundle $E_2$ (resp. $E_1$) and a trivial real vector bundle of dimension not larger than that of the trivial real vector bundle $E_1$ (resp. $E_2$) is trivial.

Then, by a $P$-operation to the map $f$, we can construct a locally $C^\infty$ trivial round fold map $f' : M' \to \mathbb{R}^n$.

We have the following theorem.

**Theorem 5.** Let $m, n \in \mathbb{N}$ and $m \geq n \geq 2$. Let $M$ be a closed manifold of dimension $m$ and let $f : M \to \mathbb{R}^n$ be a locally $C^\infty$ trivial round fold map. Let $M'$ be a manifold regarded as the total space of a standard linear bundle over $M$ such that for any connected component $C$ of the singular value set $f(S(f))$ and a small closed tubular neighborhood $N(C)$, the restriction of the bundle $M'$ over $M$ to $f^{-1}(N(C))$ as in Definition 3 is equivalent to the unit tangent bundle $UTf^{-1}(N(C))$ of $f^{-1}(N(C))$. Assume that either of the following two holds.

1. $n = 2, 4, 8$ and the Whitney sum of the tangent bundle of the normal fiber $F_C$ corresponding to the bundle $f^{-1}(N(C))$ and a trivial real vector bundle of dimension $n - 1$ is trivial.
2. The tangent bundle of the normal fiber $F_C$ corresponding to the bundle $f^{-1}(N(C))$ is trivial. In other word, $F_C$ is parallelizable.
Then, by a P-operation by $S^{m-1}$ to the map $f$, we can obtain a locally $C^\infty$ trivial round fold map $f' : M' \to \mathbb{R}^n$.

**Proof.** For any positive integer $k$, the tangent bundle $TS^k$ of the sphere $S^k$ is stably parallelizable, or the Whitney sum of the bundle $TS^k$ and a trivial real vector bundle of dimension 1 over $S^k$ is trivial. Moreover, for $k = 1, 3, 7$, the tangent bundle is trivial. By virtue of Proposition 6, in the situation of this theorem, the tangent bundle $Tf^{-1}(N(C))$ and the unit tangent bundle $UTf^{-1}(N(C))$ is trivial. From Proposition 2, we have a round fold map $f' : M' \to \mathbb{R}^n$ by applying a P-operation. \qed

**Example 6.** (1) We can apply Theorem 5 to the map $f : M \to \mathbb{R}^n$ in Proposition 1 to construct a round fold map on the total space $UTM$ of the unit tangent bundle of $M$. In the situation of Proposition 1, let the integers $m$ and $n$ be even. In this situation, the Euler class of the tangent bundle $TM$ and the unit tangent bundle $UTM$ is $4l$ times a generator of the cohomology group $H^m(M; \mathbb{Z}) \cong \mathbb{Z}$ and we have $H^m(UTM; \mathbb{Z}) \cong \mathbb{Z}/4l\mathbb{Z}$. In the case where $(m, n) = (4, 2)$ is assumed, we can obtain such maps also by applying Theorem 1. See also Example 2 (1).

(2) Let $m, n \in \mathbb{N}$ and let $n \geq 2$. Let $M$ be a closed manifold of dimension $m$ admitting a locally $C^\infty$ trivial round fold map $f : M \to \mathbb{R}^n$. Assume that one of the following three holds.

(a) $m = n + 1$.
(b) $m = n + 2$ and for any connected component $C$ and a small closed tubular neighborhood $N(C)$ as in Definition 3, the tangent bundle of the normal fiber $F_C$ corresponding to the bundle $f^{-1}(N(C))$ is orientable.
(c) $m = n + 3$ and for any connected component $C$ and a small closed tubular neighborhood $N(C)$ as in Definition 3, the tangent bundle of the normal fiber $F_C$ corresponding to the bundle $f^{-1}(N(C))$ is spin. In this case, for any connected component $C$ and a small closed tubular neighborhood $N(C)$ as in Definition 3, the tangent bundle of the normal fiber $F_C$ corresponding to the bundle $f^{-1}(N(C))$ is always trivial. We can apply Theorem 5 to the map $f : M \to \mathbb{R}^n$ to construct a round fold map on the total space $UTM$ of the unit tangent bundle of $M$.

We also have the following.

**Theorem 6.** Let $M$ be a closed manifold of dimension $2 \leq m \leq 4$, let $f : M \to \mathbb{R}^n$ ($m \geq n \geq 2$) be a locally $C^\infty$ trivial round fold map. Let $M'$ be a manifold regarded as the total space of the subbundle of a normal bundle obtained by considering an immersion of the manifold $M$ into an Euclidean space of codimension $k > 2$ whose fiber is the unit sphere of dimension $k - 1$. Assume that for any connected component $C$ and a small closed tubular neighborhood $N(C)$ as in Definition 3, the normal fiber $F_C$ corresponding to the bundle $f^{-1}(N(C))$ is orientable. Then, on the manifold $M'$, by a P-operation to the original map $f$, we can obtain a round fold map into $\mathbb{R}^n$.

**Proof.** As discussed in Example 6 (2), the tangent bundle of the normal fiber $F_C$ corresponding to the bundle $f^{-1}(N(C))$ is always trivial and the tangent bundle $Tf^{-1}(N(C))$ of the manifold $f^{-1}(N(C))$ is also trivial by virtue of Proposition 6.
From this fact, the restriction of the bundle $M'$ over $M$ to $f^{-1}(N(C))$ is orientable and spin. By Proposition 3 (4), the obtained bundle over $f^{-1}(N(C))$ is trivial. We can apply Proposition 2 and this completes the proof. □

Last, we prove two theorems. Before that, we define a \textit{trivial} embedding of a standard sphere into a manifold as an embedding of the sphere into the interior of the latter manifold which is smoothly isotopic to an unknot in the interior of a standard closed disc embedded in the interior of the manifold.

**Theorem 7.** Let $m, n \in \mathbb{N}$ satisfying $m \geq n \geq 2$. Let $M$ be a closed connected manifold of dimension $m$ and let $f : M \to \mathbb{R}^n$ be a locally $C^\infty$ trivial round fold map. Let $P$ be a proper core of $f$ and we define a compact manifold $\bar{M}$ of dimension $m$ as the union of $f^{-1}(\mathbb{R}^n - \text{Int}P)$ and some connected components of the manifold $f^{-1}(P)$. Assume also that $f(M)$ is diffeomorphic to $\mathbb{D}^n$; in other words, $f(M)$ must be diffeomorphic to $\mathbb{D}^n$ and $\bar{M}$ must include at least one connected component of $f^{-1}(P)$.

Let $M'$ be a manifold regarded as the total space of a bundle over $\bar{M}$ whose fiber is a closed connected manifold $F \neq \emptyset$ such that the restriction of the bundle to the previous manifold $\bar{M}$ is a trivial bundle. Assume also that (the embedding of) the inverse image $f^{-1}(\partial f(M))$ of the boundary $\partial f(M)$ is a trivial embedding into $M$. Then by a P-operation by $F$ to the map $f$, we can obtain a round fold map $f' : M' \to \mathbb{R}^n$ such that (the embedding of) the inverse image $f'^{-1}(\partial f'(M'))$ of the boundary $\partial f'(M')$ is a trivial embedding into $M'$.

**Proof.** From Proposition 2 and the assumption that the restriction of the bundle $M$ to $M \supset f^{-1}(\mathbb{R}^n - \text{Int}P)$ is trivial, we can construct a locally $C^\infty$ trivial round fold map $f' : M' \to \mathbb{R}^n$ by a P-operation by $F$ to the map $f$. To show that we can construct such a map satisfying the additional property, we study the structure of obtained map by noticing the definition of a P-operation or the construction demonstrated in the proof of Proposition 2. We abuse notation and terminologies in the proof of Proposition 2.

We may regard the given map $f$ as a normal form. By the definition of a P-operation and the mentioned bundle structure of the restriction of the bundle $M'$ over $M$, in the proof of Proposition 2, we can choose the bundle isomorphism $\Phi_k (k \neq 1)$ as the product of the identification map $\phi_k$ from $f^{-1}(\partial D^n_{k-\frac{1}{2}}) \subset f^{-1}(P_k)$ onto $f^{-1}(\partial D^n_{k-\frac{1}{2}}) \subset f^{-1}(P_{k-1})$ and the identity map $\text{id}_F$. We can choose the bundle isomorphism $\Phi_1$ so that its restriction to the restriction of the bundle $M'$ over $M$ to $(\partial f^{-1}(\mathbb{D}^n_1)) \cap \text{Int}M \subset \partial f^{-1}(\mathbb{D}^n_1) \subset f^{-1}(P_1)$ is the product of the identification map between the resulting base spaces and the identity map $\text{id}_F$. Moreover, (the embedding of) the inverse image $f^{-1}(\partial f(M))$ of the boundary $\partial f(M)$ into $\bar{M}$ is assumed to be trivial. By virtue of these facts, we can construct the map $f'$ so that (the embedding of) the inverse image $f'^{-1}(\partial f'(M'))$ of the boundary $\partial f'(M')$ into $\bar{M} \times F$ is smoothly isotopic to the restriction of the section of the trivial bundle $\bar{M} \times F$ over $\bar{M}$ to $f^{-1}(\partial f(M)) \subset \bar{M}$. The image of this restriction map is regarded as $f^{-1}(\partial f(M)) \times \{p\} \subset \bar{M} \times F$ where $p$ is a point in $F$. Hence, (the embedding of) the inverse image $f'^{-1}(\partial f'(M'))$ of the boundary $\partial f'(M')$ into $\bar{M} \times F$ is a trivial embedding into the total space $\bar{M} \times F$ of a resulting trivial bundle and $M'$. This completes the proof. □
Example 7. Maps in Proposition 1 satisfy the assumption of Theorem 7. In [10, EXAMPLE 2], by P-operations by $S^1$ to a map $f$ in Proposition 1 the author obtained a lot of round fold maps and source manifolds under the assumption that $n = 2$ and $m - n \geq 3$ hold (see also [10, THEOREM 4]). By virtue of Theorem 7, by such a P-operation by $S^1$ to the map $f$, we can obtain a round fold map $f' : M' \to \mathbb{R}^n$ such that (the embedding of) the inverse image $f'^{-1}(\partial f'(M'))$ of the boundary $\partial f'(M')$ is a trivial embedding into $M'$. More generally, if $M'$ is regarded as the total space of a bundle whose structure group is $SO(2)$, we can perform the construction similarly.

On the other hand, we also have the following theorem.

Theorem 8. Let $m, n \in \mathbb{N}$ satisfying $m \geq n \geq 2$. Let $M$ be a closed connected manifold of dimension $m$ and let $f : M \to \mathbb{R}^n$ be a locally $C^\infty$ trivial round fold map. Let $C_0$ be the connected component of the boundary $\partial f(M)$ of the image $f(M)$ bounding the unbounded connected component of the set $\mathbb{R}^n - \text{Int} f(M)$ and assume also that (the embedding of) the inverse image $f^{-1}(C_0)$ of the component $C_0$ into $M$ is not null-homotopic.

Furthermore, let $F \neq \emptyset$ be a connected manifold such that the group $\pi_{n-1}(F)$ is zero and let $M'$ be a manifold regarded as the total space of an $F$-bundle such that for any connected component $C$ of the singular value set $f(S(f))$ and a small closed tubular neighborhood $N(C)$ as in Definition 3, the restriction of the bundle to the space $f^{-1}(N(C))$ is trivial and that the homomorphism from $\pi_{n-2}(F)$ into $\pi_{n-2}(M')$ induced by the natural inclusion $i$ is injective.

Then by a P-operation by $F$ to the map $f$, we can obtain a round fold map $f' : M' \to \mathbb{R}^n$. Furthermore, for the connected component $C_0'$ of the boundary $\partial f'(M')$ of the image $f(M')$ bounding the unbounded connected component of the set $\mathbb{R}^n - \text{Int} f'(M')$ such that (the embedding of) the inverse image $f'^{-1}(C_0')$ of the component $C_0'$ into $M'$ is also null-homotopic.

Proof. We have the following homotopy exact sequence

$$
\begin{array}{c}
\to & \pi_{n-1}(F) \cong \{0\} & \to & \pi_{n-1}(M') & \to & \pi_{n-1}(M) \\
\to & \pi_{n-2}(F) & \xrightarrow{i_*} & \pi_{n-2}(M') & \to & \pi_{n-2}(M')
\end{array}
$$

Since the last homomorphism is assumed to be injective, The homomorphism $\pi_{n-1}(M'; \mathbb{Z})$ into $\pi_{n-1}(M)$ is an isomorphism.

From this, we immediately have the result.

Example 8. We review the construction of the map presented in the former part of Example 1 (2) done in [8] and [11] in the case where the source manifold $M$ is a manifold regarded as the total space of an $S^1$-bundle over $S^2$ which is not homeomorphic to $S^3$. In these proofs, essentially, P-operations are used.

More precisely, in the proof of Proposition 2, set $l = 1$ or consider a P-operation to a map presented in Example 1 (1) from $S^2$ into the plane. In the last step, we need to take the diffeomorphism $\Phi_1$ used in the proof appropriately. $\Phi_1$ is regarded as a bundle isomorphism between two trivial $S^1 \sqcup S^1$-bundles over $S^1$. By considering the structure of the $S^1$-bundle over $S^2$ and well-known facts on the diffeomorphism group of $S^1 \times S^1$, we can construct a round fold map $f : M \to \mathbb{R}^2$ so that the inverse image $f^{-1}(\partial f(M))$ of the boundary $\partial f(M)$ of the image $f(M)$
is smoothly isotopic to the fiber of a point in a proper core of $f$ and that the fiber of the point is also regarded as a fiber of the $S^1$-bundle $M$ over $S^2$.

As a result, we have a round fold map $f : M \to \mathbb{R}^2$ satisfying the assumption of Theorem 8. Furthermore, for example, set $F = S^k$ for $k \geq 2$, we can construct a desired round fold map into the plane on any manifold $M'$ regarded as the total space of an linear $F$-bundle over $M$ which is orientable.

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