CLASSIFICATION OF EQUIVARIANT VECTOR BUNDLES OVER TWO-TORUS

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Abstract. We exhaustively classify topological equivariant complex vector bundles over two-torus under a compact Lie group (not necessarily effective) action. It is shown that inequivalent Chern classes and isotropy representations at (at most) six points are sufficient to classify equivariant vector bundles except a few cases. To do it, we calculate homotopy of the set of equivariant clutching maps.

1. Introduction

In the previous paper [K1], we have exhaustively classified topological equivariant complex vector bundles over two-sphere under a compact Lie group (not necessarily effective) action. To do it, we have calculated homotopy of equivariant clutching maps. Two-sphere has relatively small number of effective finite group actions and those actions deliver typical equivariant simplicial complex structures (for example, regular polyhedron). By using this simplicial complex structure, we have reduced the calculation from the whole 1-skeleton to an edge, and this makes the calculation possible. But, readers might have doubted if our method would apply to other spaces.

In this paper, we would do the same thing over two-torus. However, a finite group action on two-torus does not have a typical equivariant simplicial complex structure. Also, the calculation of homotopy of equivariant clutching maps is not possible. But, readers might have doubted if our method would apply to other spaces.

To state our results, let us introduce notations. Let $\Lambda$ be a lattice of $\mathbb{R}^2$. Let $\text{Iso}(\mathbb{R}^2, \Lambda)$ be the group $\{A \in \text{Iso}(\mathbb{R}^2) | A(\Lambda) = \Lambda\}$. Let $\text{Aff}(\mathbb{R}^2)$ and $\text{Aff}(\mathbb{R}^2, \Lambda)$ be sets of affine isomorphisms $\bar{x} \mapsto A\bar{x} + \bar{c}$ for $\bar{x}, \bar{c} \in \mathbb{R}^2$, $A \in \text{Iso}(\mathbb{R}^2)$ and $\bar{c} \in \text{Iso}(\mathbb{R}^2, \Lambda)$, respectively. Let $\text{Aff}_{t}(\mathbb{R}^2, \Lambda)$ be the set of all translations in $\text{Aff}(\mathbb{R}^2, \Lambda)$. Let $\pi : \mathbb{R}^2 \to \mathbb{R}^2/\Lambda$, $\bar{x} \mapsto [\bar{x}]$ be the quotient map, and let $\text{Aff}(\mathbb{R}^2/\Lambda)$ be the group of affine isomorphisms of $\mathbb{R}^2/\Lambda$, i.e. the group of all $[x] \mapsto [A\bar{x} + \bar{c}]$'s for $A \in \text{Iso}(\mathbb{R}^2, \Lambda)$. Let $\text{Aff}(\mathbb{R}^2)$ and $\text{Aff}(\mathbb{R}^2/\Lambda)$ (and their subgroups) act naturally on $\mathbb{R}^2$ and $\mathbb{R}^2/\Lambda$, respectively. Let $\pi_{\text{aff}} : \text{Aff}(\mathbb{R}^2, \Lambda) \to \text{Aff}(\mathbb{R}^2/\Lambda)$, $A\bar{x} + \bar{c} \mapsto [A\bar{x} + \bar{c}]$.

Hereafter, we identify $\text{Aff}_{t}(\mathbb{R}^2, \Lambda)$ with $\mathbb{R}^2$ so that $\ker \pi_{\text{aff}} = \Lambda$. Let $\text{Aff}_{t}(\mathbb{R}^2/\Lambda)$ be the group $\pi_{\text{aff}}(\text{Aff}_{t}(\mathbb{R}^2, \Lambda))$. Let a compact Lie group $G$ act affinely (not necessarily effectively) on two torus $\mathbb{R}^2/\Lambda$ through a homomorphism $\rho : G \to \text{Aff}(\mathbb{R}^2/\Lambda)$. For a bundle $E$ in $\text{Vect}_G(\mathbb{R}^2/\Lambda)$ and a point $x$ in $\mathbb{R}^2/\Lambda$, denote by $E_x$ the isotropy $G_x$-representation on the fiber at $x$. Put $H = \ker \rho$, i.e. the kernel of the $G$-action on $\mathbb{R}^2/\Lambda$. Let $\text{Irr}(H)$ be the set of characters of irreducible $H$-representations which has a $G$-action defined as

$$(g \cdot \chi)(h) = \chi(g^{-1}hg)$$

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for $h \in H$, $g \in G$, and $\chi \in \text{Irr}(H)$. For $\chi \in \text{Irr}(H)$ and a compact Lie group $K$ satisfying $H < K < G$ and $K \cdot \chi = \chi$, a $K$-representation $W$ is called $\chi$-isotypical if $\text{res}_H^K W$ is $\chi$-isotypical, and denote by $\text{Vect}_K(\mathbb{R}^2/\Lambda, \chi)$ the set
\[
\{ E \in \text{Vect}_K(\mathbb{R}^2/\Lambda) \mid E_x \text{ is } \chi\text{-isotypical for each } x \in \mathbb{R}^2/\Lambda \}
\]
for $\chi \in \text{Irr}(H)$ where $\mathbb{R}^2/\Lambda$ has the restricted $K$-action. In the previous paper [K1], classification of $\text{Vect}_C(\mathbb{R}^2/\Lambda)$ has been reduced to classification of $\text{Vect}_{G_2}(\mathbb{R}^2/\Lambda, \chi)$ for each $\chi \in \text{Irr}(H)$.

Now, we would list all possible finite $\rho(\chi)$’s up to conjugacy, and then assign a fundamental domain to each $\rho(\chi)$. Cases of nonzero-dimensional $\rho(\chi)$ are relatively simple and separately dealt with as special cases.

### Table 1.1. Finite subgroups of $\text{Aff}(\mathbb{R}^2/\Lambda)$

| $R$ | $R/R_t$ | $\Lambda_t$ | $\Lambda$ |
|-----|---------|-------------|----------|
| $R_t \times R/R_t$ | $\mathbb{Z}_2, D_2, D_2$, $\mathbb{Z}_4$ | $\Lambda^{sq}$ | a square lattice |
| $R_t \times R/R_t$ | $D_2, D_2, D_3, D_3, D_6$, $D_6, \mathbb{Z}_3, \mathbb{Z}_3, 2$ | $\Lambda^{eq}$ | an equilateral triangle lattice |
| $R_t \times R/R_t$ | ($[a]$) | $\Lambda^a$ | $(m_1, 0, (0, m_2))$ for $m_1, m_2 \in \mathbb{N}$ |

The classification of $\text{Vect}_{G_2}(\mathbb{R}^2/\Lambda, \chi)$ is highly dependent on the $G_2$-action on the base space $\mathbb{R}^2/\Lambda$, i.e. on the image $\rho(G_2)$, and classification is actually given case by case according to $\rho(G_2)$. So, we need to describe $\rho(G_2)$ in a moderate way. For this, we would list all possible finite $\rho(G_2)$’s up to conjugacy, and then assign a fundamental domain to each $\rho(G_2)$. Cases of nonzero-dimensional $\rho(G_2)$ are relatively simple and separately dealt with as special cases.

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| $R_t \times R/R_t$ | ($[a]$) | $\Lambda^a$ | $(m_1, 0, (0, m_2))$ for $m_1, m_2 \in \mathbb{N}$ |

Cases of $R/R_t = D_1, D_{1,4}$ are explained in Section 2. In each row, pick one of the groups in the $R/R_t$ entry, and call it $Q$. And, denote by $L_t$ the $\Lambda_t$ entry in the row. Also, pick any sublattice in $L_t$ which is preserved by $Q$ and satisfies the restriction stated in the $\Lambda$ entry in the row, and call it just $\Lambda$. Then, $Q, L_t, \Lambda$ determine the semidirect product $\text{Aff}(L_t) \rtimes \text{Aff}(Q)$ in $\text{Aff}(\mathbb{R}^2/\Lambda)$. Denote it by $R$. So defined $R$ satisfies $R_t = \text{Aff}(L_t), L_t = R/R_t = Q$. In this way, $R/R_t, \Lambda, \Lambda_t$ entries of each row determine the unique $R$. Since each finite group in $\text{Aff}(\mathbb{R}^2/\Lambda)$ for some $\Lambda$ is conjugate to one of these $R$, it is assumed that $\rho(G_2) = R$ for some $R$ of Table 1.1 in this section. In the below, it is observed that two groups with the same $R/R_t$ and $\Lambda_t$-entries behave similarly even though their $\Lambda$’s are different.
To each finite group $R$ in Table 1.1 we assign a polygonal area $|P_R|$ in Table 1.2 called a two-dimensional fundamental domain according to its $R/R_t$ and $\Lambda_t$. Let $P_4^{sq}$, $P_4^{eq}$, $P_3^{eq}$, $P_6^{eq}$ be gray colored areas defined by regular polygons in Figure 1.1 where dots are points of $\Lambda^{eq}$ or $\Lambda^{sq}$. And, let $P_4^{eq}, P_4^{eq}$ be convex hulls of

$$\{O, \left(0, \frac{\sqrt{3}}{2}\right), \left(\frac{1}{4}, \frac{\sqrt{3}}{2}\right), \left(\frac{1}{4}, 0\right)\}, \quad \{O, \left(-\frac{1}{8}, \frac{3\sqrt{3}}{8}\right), \left(\frac{5}{8}, \frac{3\sqrt{3}}{8}\right), \left(\frac{3}{4}, \frac{\sqrt{3}}{4}\right)\},$$

respectively.

| $R/R_t$ | $\Lambda_t$ | $|P_R|$ |
|---------|-------------|--------|
| $\mathbb{Z}_2$ | $\Lambda^{eq}$ | $\frac{1}{2}P_4^{eq} \cup \left(\frac{1}{4}, 0\right) + \frac{1}{2}P_4^{eq}$ |
| $D_2, D_2, D_4, Z_4, D_4$ | $\Lambda^{eq}$ | $P_4^{eq}$ |
| $D_2, Z_4, D_3, Z_4, D_4$ | $\Lambda^{eq}$ | $P_3^{eq}$ |
| $A_3, D_4$ | $\Lambda^{eq}$ | $P_6^{eq}$ |
| $A_3, D_4$ | $\Lambda^{eq}$ | $\frac{1}{2}c + \left(\frac{1}{4}P_4^{eq} \cup \left(\frac{1}{4}, 0\right) + \frac{1}{2}P_4^{eq}\right)$ |
| $A_3 + \tilde{c} = 0$ | $\Lambda^{eq}$ | $\frac{1}{2}c + P_4^{eq}$ |
| $A_3 + \tilde{c} = 0$ | $\Lambda^{eq}$ | $\frac{1}{2}c + P_4^{eq}$ |
| $D_1, A_3, D_4$ | $\Lambda^{eq}$ | $\frac{1}{2}c + \left(\frac{1}{4}P_4^{eq} \cup (0, -\frac{1}{2}) + \frac{1}{2}P_4^{eq}\right)$ |
| $A_3 + \tilde{c} = I_0$ | $\Lambda^{eq}$ | $\frac{1}{2}c + \left(\frac{1}{4}, -\frac{1}{2}\right) + P_4^{eq}$ |
| $A_3 + \tilde{c} = I_0$ | $\Lambda^{eq}$ | $\frac{1}{2}c + \left(\frac{1}{4}, -\frac{1}{2}\right) + P_4^{eq}$ |

Table 1.2. Two-dimensional fundamental domain

In dealing with equivariant vector bundles over two-torus, we need to consider isotropy representations at a few (at most six) points of $\mathbb{R}^2/\Lambda$. We would specify those points. For this, we endow $|P_R|$ with the natural simplicial complex structure denoted by $P_R$, and we would label its vertices and edges. For simplicity, we allow that a face of a two-dimensional simplicial complex need not be a triangle, and we consider an $n$-gon as the simplicial complex with one face, $n$ edges, $n$ vertices. Define $i_R$ as the integer such that $|P_R|$ is an $i_R$-gon. Put $\tilde{v}^0 = O$ when $R/R_t \neq D_1, D_1, D_4$ where $O$ is the origin of $\mathbb{R}^2$. Put $\tilde{v}^0 = 1/2\tilde{c}$ when $R/R_t = D_1, D_1, D_4$ with $A\tilde{c} + \tilde{c} = 0$, and put $\tilde{v}^0 = 1/2\tilde{c} - 1/2I_0$ when $R/R_t = D_1, D_1, D_4$ with $A\tilde{c} + \tilde{c} = I_0$. Label vertices of $P_R$ as $\tilde{v}$ so that $\tilde{v}^0, \tilde{v}^1, \tilde{v}^2, \ldots, \tilde{v}^{n-1}$ are consecutive vertices arranged in the
clockwise way around \( P_R \). We use \( \mathbb{Z}_{i_R} \) as the index set of vertices. Also, denote by \( \vec{v}^i \) the edge of \( P_R \) connecting \( \vec{v}^- \) and \( \vec{v}^+ \) for \( i \in \mathbb{Z}_{i_R} \). For any two points \( \vec{a}, \vec{b} \) in \( P_R \), denote by \([\vec{a}, \vec{b}]\) the line connecting \( \vec{a} \) and \( \vec{b} \). Also for any points \( \bar{a}^i \)’s for \( i = 0, \ldots, n \) in \( |P_R| \), we denote by \([\bar{a}^0, \ldots, \bar{a}^n]\) the union of \( [\bar{a}^i, \bar{a}^{i+1}] \)’s for \( i = 0, \ldots, n-1 \).

To each finite group \( R \) in Table 1.1, we assign the union \( \bar{D}_R \) of lines in Table 1.3 called a one-dimensional fundamental domain according to its \( R/R_t \) and \( \Lambda_t \).

Here, \( b(\sigma) \) is the barycenter of \( \sigma \) for any simplex \( \sigma \). For each \( \bar{D}_R \) in Table 1.3 express \( \bar{D}_R \) as \([\bar{a}^0, \ldots, \bar{a}^i, \ldots, \bar{a}^{n-1}]\) where we respect the order of \( \bar{a}^i \)’s for each \( \bar{a}^i \) is the number of \( \bar{a}^i \)’s. If \([\bar{a}^0, \bar{a}^1] \subset \bar{v}^0 \) for \( 0 \leq i_0 \leq i_0 + n_R - 1 \), then put \( d^i = \bar{a}^{i-i_0} \) for each \( i_0 \leq i \leq i_0 + n_R - 1 \). In addition to these, put \( d^{-1} = b(\bar{P}_R) \). Denote \( \{i \in \mathbb{Z} | i_0 \leq i \leq i_0 + n_R - 1 \} \) by \( I_{R_t} \) and denote \( \bar{I} = \{i_0 + n_R - 1 \} \), \( I_{R_t} \setminus \{1 \} \) by \( I_{R_t} \), respectively. So, the set of all \( \bar{d}^i \)’s and \( D_R \) are expressed as \( \{\bar{d}^i \} \in I_{R_t}^\pm \) and \( \bigcup_{i \in I_{R_t}} [\bar{d}^i, \bar{d}^{i+1}] \), respectively. For example, in the case of \( R/R_t = D_{2,3}, \bar{d}^1 = b(\bar{v}^1), \bar{d}^2 = \bar{v}^2, \bar{d}^3 = \bar{v}^3, \bar{d}^4 = \bar{v}^0, \bar{d}^5 = \bar{v}^1 \) and \( I_R = \{1, 2, 3, 4, 5 \} \). Also, denote by \( \bar{d}^i \) the image \( \pi(\bar{d}^i) \) for each \( i \in I_{R_t} \). These \( \bar{d}^i \)’s are wanted points. The \( \sim \) of entry of Table 1.3 is explained later.

When \( R = \rho(\mathcal{G}_\chi) \), we simply denote \( P_R, \bar{D}_R, i_R, I_R \) by \( P_\rho, \bar{D}_\rho, i_\rho, I_\rho \), respectively. Next, we define a semigroup with which we would describe isotropy representations at \( (\bar{d}^i)_{i \in I_{R_t}^+} \) of bundles in \( \text{Vect}_{\mathcal{G}_\chi}(\mathbb{R}^2/\Lambda, \chi) \).

**Definition 1.1.** Let \( A_{\mathcal{G}_\chi}(\mathbb{R}^2/\Lambda, \chi) \) be the set of \((|I_{R_t}^+|)\)-tuple \((W_{\bar{d}^i})_{i \in I_{R_t}^+}\)’s such that

1. \( W_{\bar{d}^i} \in \text{Rep}(\mathcal{G}_\chi_{\bar{d}^i}) \) for each \( i \in I_{R_t}^+ \),
2. \( W_{\bar{d}^{-1}} \) is \( \chi \)-isotypical,
3. \( W_{\bar{d}_i} \cong g(W_{\bar{d}_i}) \) for some \( g \in \mathcal{G}_\chi, i, i' \in I_\rho \),
4. \( \text{res}_{\mathcal{G}_\chi(\bar{d}^{-1})} W_{\bar{d}_i} \cong \chi(\bar{d}^{-1}) \text{res}_{\mathcal{G}_\chi(\bar{d}^{-1})} W_{\bar{d}_i} \) for each \( i \in I_\rho \),
5. \( \text{res}_{\mathcal{G}_\chi(\bar{d}^{-1})} W_{\bar{d}_i} \cong \chi(\bar{d}^{-1}) \text{res}_{\mathcal{G}_\chi(\bar{d}^{-1})} W_{\bar{d}_i} \) for each \( i \in I_{R_t}^- \).
Here, \( C(\bar{x}) = \pi([\bar{x}, b(P)]) \) for \( \bar{x} \in \partial|P| \), and \(|e^1|\) is the image \( \pi([e^1]) \). Also, \((G_\chi)_{C(\bar{x})}\) and \((G_\chi)_{|e^1|}\) are subgroups of \( G_\chi \) fixing \( C(\bar{x}) \) and \(|e^1|\), respectively. And, let \( p_{\text{vect}} : \text{Vect}_{G_\chi}(\mathbb{R}^2/\Lambda, \chi) \rightarrow A_{G_\chi}(\mathbb{R}^2/\Lambda, \chi) \) be the map defined as \([E] \mapsto (E_p)_{p \in \mathbb{T}^2}\).

Groups \((G_\chi)_{C(\bar{x})}\) and \((G_\chi)_{|e^1|}\) are calculated in Section 3 and \( p_{\text{vect}} \) is well defined.

Now, we can state our results. Let \( c_1 : \text{Vect}_{G_\chi}(\mathbb{R}^2/\Lambda, \chi) \rightarrow H^2(\mathbb{R}^2/\Lambda) \) be the map defined as \([E] \mapsto c_1(E)\). Denote by \( l_p \) the number \(|\rho(G_\chi)|\).

**Theorem A.** Assume that \( R/R_t = \mathbb{Z}_n \) for \( n = 1, 2, 3, 4, 6 \) when \( \rho(G_\chi) = R \) for some \( R \) of Table 1. Then,

\[
p_{\text{vect}} \times c_1 : \text{Vect}_{G_\chi}(\mathbb{R}^2/\Lambda, \chi) \rightarrow A_{G_\chi}(\mathbb{R}^2/\Lambda, \chi) \times H^2(\mathbb{R}^2/\Lambda)
\]

is injective. For each element in \( A_{G_\chi}(\mathbb{R}^2/\Lambda, \chi) \), the set of the first Chern classes of bundles in its preimage under \( p_{\text{vect}} \) is equal to the set \( \{\chi(id)(l_p k + k_0) \mid k \in \mathbb{Z}\} \) where \( k_0 \) is dependent on it.

**Theorem B.** Assume that \( R/R_t = \mathbb{D}_1 \) with \( A\bar{c} + \bar{c} = \bar{l}_0 \) when \( \rho(G_\chi) = R \) for some \( R \) of Table 1. For each element in \( A_{G_\chi}(\mathbb{R}^2/\Lambda, \chi) \), its preimage under \( p_{\text{vect}} \) has two elements which have the same Chern class.

**Theorem C.** Assume that finite \( \rho(G_\chi) \) is not equal to one of groups appearing in the above two theorems. Then, \( p_{\text{vect}} \) is an isomorphism.

**Theorem D.** Assume that \( \rho(G_\chi) \) is one-dimensional. Then, there exists a circle \( C \) in \( \mathbb{R}^2/\Lambda \) such that the map \( G_\chi \times (G_\chi)_C \) \( C \rightarrow \mathbb{R}^2/\Lambda \), \([g, x]\mapsto gx\) for \( g \in G_\chi \), \( x \in C \) is an equivariant isomorphism where \( (G_\chi)_C \) is the subgroup of \( G_\chi \) preserving \( C \). And,

\[
\text{Vect}_{G_\chi}(\mathbb{R}^2/\Lambda, \chi) \rightarrow \text{Vect}_{(G_\chi)_C}(C, \chi), \quad E \mapsto E|_C
\]

is an isomorphism.

In Theorem 3 two bundles in each preimage are not distinguished by Chern classes and isotropy representations, so it might be regarded as an exceptional case. In [K2], we explain for this case by showing that the two bundles are pull backs of equivariant vector bundles over Klein bottle which are distinguished by Chern classes and isotropy representations. Since equivariant complex vector bundles over circle are classified in [CKMS], Theorem D gives classification of \( \text{Vect}_{G_\chi}(\mathbb{R}^2/\Lambda, \chi) \) when \( \rho(G_\chi) \) is one-dimensional.

This paper is organized as follows. In Section 2 we list all finite subgroups of \( \text{Aff}(\mathbb{R}^2/\Lambda) \) up to conjugacy. In Section 3 we prove Theorem D. In Section 4 we consider \( \mathbb{R}^2/\Lambda \) as a quotient space of the underlying space of an equivariant simplicial complex which will be denoted by \( \mathcal{K}_\rho \). And, we investigate equivariance of \( \mathbb{R}^2/\Lambda \) by calculating isotropy groups at some points which come from vertices or barycenters of \( \mathcal{K}_\rho \). In Section 5 we consider an equivariant vector bundle over \( \mathbb{R}^2/\Lambda \) as an equivariant clutching construction of an equivariant vector bundle over \( \mathcal{K}_\rho \). For this, we define equivariant clutching map. In Section 6 we investigate relations between homotopy of equivariant clutching maps, \( \text{Vect}_{G_\chi}(\mathbb{R}^2/\Lambda, \chi) \), and \( A_{G_\chi}(\mathbb{R}^2/\Lambda, \chi) \). From these relations, it is shown that our classifications in most cases are obtained by calculation of homotopy of equivariant clutching maps. In Section 7 we review the concept and results of equivariant pointwise clutching map from the previous paper [K]. and supplement these with two more cases. In calculation of homotopy of equivariant clutching maps, equivariant pointwise clutching map plays a key role because an equivariant clutching map can be considered as a continuous collection of equivariant pointwise clutching maps. In Section 8 we recall two useful lemmas on fundamental groups from [K]. In Section 9 we prove technical results needed in dealing with equivariant clutching maps through equivariant pointwise clutching maps. In Section 10 and 11 we prove Theorem A, B, C.
2. Finite Subgroups of \( \text{Aff}(\mathbb{R}^2/\Lambda) \)

In this section, we list all finite subgroups of \( \text{Aff}(\mathbb{R}^2/\Lambda) \) up to conjugacy, and explain for two-dimensional fundamental domain.

First, we introduce notations on the group of affine isomorphisms on a torus. Let \( \Lambda \) be a lattice of \( \mathbb{R}^2 \). Let \( \text{Iso}(\mathbb{R}^2, \Lambda) \) be the group \( \{ A \in \text{Iso}(\mathbb{R}^2) \mid A(\Lambda) = \Lambda \} \), and let \( \text{Aff}(\mathbb{R}^2, \Lambda) \) be the set of affine isomorphisms \( \bar{x} \mapsto A\bar{x} + \bar{c} \) for \( \bar{x}, \bar{c} \in \mathbb{R}^2 \) and \( A \in \text{Iso}(\mathbb{R}^2, \Lambda) \). Let \( \text{Aff}_2(\mathbb{R}^2, \Lambda) \) and \( \text{Aff}_0(\mathbb{R}^2, \Lambda) \) be sets of all translations and all orientation preserving affine isomorphisms in \( \text{Aff}(\mathbb{R}^2, \Lambda) \), respectively. Let \( \pi : \mathbb{R}^2 \to \mathbb{R}^2/\Lambda, \bar{x} \mapsto [\bar{x}] \) be the quotient map, and let \( \text{Aff}(\mathbb{R}^2/\Lambda) \) be the group of affine isomorphisms of \( \mathbb{R}^2/\Lambda \), i.e. the group of all \( [\bar{x}] \mapsto [A\bar{x} + \bar{c}] \)'s for \( A \in \text{Iso}(\mathbb{R}^2, \Lambda) \).

Let \( \text{Aff}(\mathbb{R}^2) \) and \( \text{Aff}(\mathbb{R}^2/\Lambda) \) (and their subgroups) act naturally on \( \mathbb{R}^2 \) and \( \mathbb{R}^2/\Lambda \), respectively. And, a compact Lie group \( R \) is said to act naturally on \( \mathbb{R}^2 \) or \( \mathbb{R}^2/\Lambda \) if the action satisfies the following two conditions:

1. \( R \) is a subgroup of \( \text{Aff}(\mathbb{R}^2) \) or \( \text{Aff}(\mathbb{R}^2/\Lambda) \),
2. the \( R \)-action is the action as a subgroup of \( \text{Aff}(\mathbb{R}^2) \) or \( \text{Aff}(\mathbb{R}^2/\Lambda) \), respectively.

Let \( \pi_{\text{aff}} : \text{Aff}(\mathbb{R}^2, \Lambda) \to \text{Aff}(\mathbb{R}^2/\Lambda), \bar{x} + \bar{c} \mapsto [A\bar{x} + \bar{c}] \). We identify \( \text{Aff}_2(\mathbb{R}^2, \Lambda) \) with \( \mathbb{R}^2 \) so that \( \ker \pi_{\text{aff}} = \Lambda \). Let \( \text{Aff}_0(\mathbb{R}^2, \Lambda) \) and \( \text{Aff}_0(\mathbb{R}^2/\Lambda) \) be groups \( \pi_{\text{aff}}(\text{Aff}(\mathbb{R}^2, \Lambda)) \) and \( \pi_{\text{aff}}(\text{Aff}(\mathbb{R}^2/\Lambda)) \), respectively.

Next, we introduce notations on finite subgroups of \( \text{Aff}(\mathbb{R}^2/\Lambda) \). Let \( R \) be a finite subgroup of \( \text{Aff}(\mathbb{R}^2/\Lambda) \), i.e. the finite group \( R \) acts naturally on \( \mathbb{R}^2/\Lambda \). Let \( R_0 \) and \( R_t \) be \( R \cap \text{Aff}_0(\mathbb{R}^2/\Lambda) \) and \( R \cap \text{Aff}_0(\mathbb{R}^2/\Lambda) \), respectively. Put \( \Lambda_t = \pi_{\text{aff}}(\text{Aff}(\mathbb{R}^2/\Lambda)) \). Note that \( R_0 \triangleleft R \) and \( R_t \triangleleft \Lambda_t \) so that \( R_t \) is isomorphic to \( \mathbb{Z}_{m_2} \times \mathbb{Z}_{m_2} \) because \( \Lambda_t \) has two generators \( l_1, l_2 \) such that \( \Lambda = \langle m_1, m_2 \rangle \) for some \( m_1, m_2 \in \mathbb{N} \). For each element \( g = [A\bar{x} + \bar{c}] \) of \( R_t \), it can be shown that \( A(\Lambda_t) = \Lambda_t \) by normality of \( R_t \) in \( R \), i.e. \( R/R_t \) preserves \( \Lambda_t \) in addition to \( \Lambda \). Through the map

\[
R \to \text{Iso}(\mathbb{R}^2, \Lambda), \quad [A\bar{x} + \bar{c}] \mapsto A,
\]

\( R/R_t \) is isomorphic to the subgroup \( \{ A \in \text{Iso}(\mathbb{R}^2, \Lambda) \mid [A\bar{x} + \bar{c}] \in R \} \), and we will use the notation \( R/R_t \) to denote the subgroup. For simplicity, \( R/R_t \) is confused with the subgroup \( \pi_{\text{aff}}(R/R_t) \) of \( \text{Aff}(\mathbb{R}^2/\Lambda) \) according to context.

For a compact subgroup \( R \) of \( \text{Aff}(\mathbb{R}^2/\Lambda) \) and its natural action on \( \mathbb{R}^2/\Lambda \), we define conjugacy. Given an affine isomorphism

\[
\bar{\eta} : \mathbb{R}^2 \to \mathbb{R}^2, \quad \bar{x} \mapsto A'\bar{x} + \bar{c}'
\]

for some \( A' \in \text{Iso}(\mathbb{R}^2), \bar{c}' \in \mathbb{R}^2 \), it induces the affine isomorphism

\[
\eta : \mathbb{R}^2/\Lambda \to \mathbb{R}^2/A'(\Lambda), \quad [\bar{x}] \mapsto [A'\bar{x} + \bar{c}']
\]

and the group isomorphism

\[
\eta^* : \text{Aff}(\mathbb{R}^2/\Lambda) \to \text{Aff}(\mathbb{R}^2/A'(\Lambda)), \quad B \mapsto \eta B \eta^{-1}.
\]

Then, \( \eta^*(R) \) is called conjugate to \( R \), and the natural action of \( \eta^*(R) \) on \( \mathbb{R}^2/A'(\Lambda) \) is called the conjugate action induced by \( \bar{\eta} \). It can be easily checked that

\[
(2.1) \quad \eta^*(R_t) = \eta^*(R_t) \quad \text{and} \quad A'(\Lambda_t) = \Lambda_t
\]

where \( A_t \) is equal to \( \pi_{\text{aff}}^{-1}(\eta^*(R_t)) \) for \( \pi_{\text{aff}} : \text{Aff}(\mathbb{R}^2, A'(\Lambda)) \to \text{Aff}(\mathbb{R}^2/A'(\Lambda)) \). For a subgroup in \( \text{Aff}(\mathbb{R}^2, \Lambda) \) and the isomorphism \( \bar{\eta} \), we can similarly define the conjugate subgroup in \( \text{Aff}(\mathbb{R}^2, A'(\Lambda)) \).

Up to conjugacy, it suffices to consider only isometries on two torus instead of all affine isomorphisms on two torus. Let \( \text{Isom}(\mathbb{R}^2/\Lambda) \) be the isometry group of \( \mathbb{R}^2/\Lambda \) where \( \mathbb{R}^2/\Lambda \) delivers the metric induced by the usual metric on \( \mathbb{R}^2 \). And, put \( \text{Isom}_0(\mathbb{R}^2/\Lambda) = \text{Isom}(\mathbb{R}^2/\Lambda) \cap \text{Aff}_0(\mathbb{R}^2/\Lambda) \) and \( \text{Isom}_0(\mathbb{R}^2/\Lambda) = \text{Aff}_t(\mathbb{R}^2/\Lambda) \).
Now, we would list all finite subgroup of $\text{Aff}(\mathbb{R}^2/\Lambda)$ up to conjugacy. First, we explain for upper two rows of Table 1.1.

**Proposition 2.1.** Let a finite group $R$ act naturally on $\mathbb{R}^2/\Lambda$ for some $\Lambda$. Assume that $R_0/R_t$ is nontrivial. Then, the $R$-action is conjugate to one of the following:

1. the subgroup $R$ in $\text{Isom}(\mathbb{R}^2/\Lambda)$ such that $\Lambda_t = \Lambda^{eq}$ and $R/R_t$ is one of $\mathbb{Z}_2, D_{2,2}, D_2, \mathbb{Z}_4, D_4$

where $\Lambda$ is a square sublattice of $\Lambda^{eq}$ preserved by $R/R_t$.

2. the subgroup $R$ in $\text{Isom}(\mathbb{R}^2/\Lambda)$ such that $\Lambda_t = \Lambda^{eq}$ and $R/R_t$ is one of $\mathbb{Z}_2, D_2, D_{2,3}, D_3, D_{3,2}, \mathbb{Z}_6, D_6$

where $\Lambda$ is an equilateral triangle sublattice of $\Lambda^{eq}$ preserved by $R/R_t$.

In both cases, $R$ is equal to the semidirect product $R_t \rtimes R/R_t$. Two $\mathbb{Z}_2$-actions of (1) and (2) are conjugate.

**Proof.** We may assume that $R$ is a subgroup of $\text{Isom}(\mathbb{R}^2/\Lambda)$ for some lattice $\Lambda$. So, $R/R_t \subset O(2)$. We start with $R_0$ instead of the whole $R$. Since $R_0/R_t$ is a nontrivial subgroup of $O(2)$, it is a nontrivial cyclic group. Pick $g_0$ in $R_0$ such that $g_0$ gives a generator in $R_0/R_t$. Let $g_0(x) = [A_0 x + c_0]$ for some $A_0 \in O(2)$, $c_0 \in \mathbb{R}^2$. Put $c_1 = - (\text{id} - A_0)^{-1}c_0$, and put

$$\bar{\eta} : \mathbb{R}^2 \to \mathbb{R}^2, \quad \bar{x} \mapsto A' \bar{x} + \bar{c}' \quad \text{with} \quad A' = \text{id}, \quad \bar{c}' = \bar{c}_1$$

where $\text{id} - A_0$ is invertible because $A_0$ is a nontrivial rotation. Here, note that $A'(\Lambda) = \Lambda$, $A'(\Lambda_t) = \Lambda_t$, and $\eta^*(R_t) = R_t$. In the remaining proof, we use the conjugate $\eta^*(R)$-action induced by $\bar{\eta}$, and denote $\eta^*(R)$ just by $R$. Since the action by $\eta^*(g_0)$ is equal to $[A_0 \bar{x}]$, $R_0$ contains $R_0/R_t$. Since $R_t \cap (R_0/ R_t)$ is trivial and $R_t < R_0, R_0$ is the semidirect product of $R_0/R_t$ and $R_t$. It is well known that if a lattice in $\mathbb{R}^2$ is preserved by a nontrivial rotation, then it is a square or an equilateral triangle lattice. Since $R_0/R_t$ is nontrivial and $R_0/R_t$ preserves $\Lambda$ and $\Lambda_t$, both lattices are simultaneously square or equilateral triangle lattices. Here, we use the fact that a sublattice of a square lattice can not be an equilateral triangle lattice and vice versa. Take a suitable conformal linear isomorphism $\bar{\eta}^* : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\bar{\eta}^*(\Lambda_t) = \Lambda^{eq}$ or $\Lambda^{eq}$. In the remaining proof, we use the conjugate $\eta^*(R)$-action induced by $\bar{\eta}^*$, and denote $\eta^*(R), \bar{\eta}^*(\Lambda_t)$ just by $R, \Lambda, \Lambda_t$, respectively. Since $R_0/R_t$ preserves $\Lambda_t$, $R_0/R_t$ becomes one of $\mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_2, D_2, \mathbb{Z}_6, D_6$ according to $\Lambda_t = \Lambda^{eq}$ or $\Lambda^{eq}$, respectively.

If $R = R_0$, then proof is done. So, assume that $R_0 \leq R$. Let $g(x) = [A \bar{x} + \bar{c}]$ be an arbitrary element of $R - R_0$. Then, $g^2(x) = [\bar{x} + A \bar{c} + \bar{c}]$ because $R/R_t$ is a dihedral group. Here, note that $A \bar{c} + \bar{c}$ is in $\Lambda_t$, and that $\eta^*(\eta^*(g_0)) = \eta^*(g_0)$ because $\bar{\eta}$ is conformal. Similar to $g^2(x)$, we have $(g \eta^*(g_0))^2(x) = [\bar{x} + A \bar{c} + \bar{c}]$ so that $A A_0 \bar{c} + \bar{c}$ is in $\Lambda_t$. By these two elements $A \bar{c} + \bar{c}$ and $A A_0 \bar{c} + \bar{c}$ of $\Lambda_t$, we have $A (\text{id} - A_0) \bar{c} \in \Lambda_t$. Since $A(\Lambda_t) = \Lambda_t$, this gives us $(\text{id} - A_0) \bar{c} \in \Lambda_t$. So, $\bar{c} \in \Lambda_t$, and $A_0 \bar{c} + \bar{c}$ is in $\Lambda_t$ because $(\text{id} - A_0)^{-1} = \text{id} + A_0 + \cdots + A_0^{n-1}$ and $A_0(\Lambda_t) = \Lambda_t$ when $A_0$ has order $n$. Since $g(x) = [A \bar{x} + \bar{c}]$ with $\bar{c} \in \Lambda_t$, the affine map $[A \bar{x}]$ is in $R$. So, we obtain that $R/R_t$ is in $R$. Since $R_t \cap (R/ R_t)$ is trivial and $R_t < R$, $R$ is the semidirect product of $R/R_t$ and $R_t$. Since $R/R_t$ preserves $\Lambda_t$, possible actions of $R/R_t$ are very restrictive and are conjugate to one of $D_{2,2}, D_2, D_4$, or $D_2, D_{2,3}, D_3, D_{3,2}, D_6$ induced by some suitable isometry on $\mathbb{R}^2$ preserving $\Lambda_t$ according to $\Lambda_t = \Lambda^{eq}$ or $\Lambda^{eq}$, respectively.

Conjugacy of two $\mathbb{Z}_2$-actions of (1) and (2) is easy. □

To each finite group $R$ in the previous proposition, we assign the two-dimensional fundamental domain $|P_R|$ with which we describe the $R$-action.
Corollary 2.2. To each finite group $R$ in the previous proposition, we assign the area $|PR|$ in Table 1.2. Then, the $\pi_{\text{aff}}^{-1}(R)$-orbit of $|PR|$ cover $\mathbb{R}^2$ and there is no interior intersection between different areas in the orbit. Also, the $R$-orbit of $\pi(\mathbb{R}^2)$ cover $\mathbb{R}^2/\Lambda$ and there is no interior intersection between different areas in the orbit.

Proof. For each $R \subset \text{Aff}(\mathbb{R}^2/\Lambda)$ in the previous proposition, $\pi_{\text{aff}}^{-1}(R)$ is equal to $\Lambda_t \times R/R_t$ because $R$ is equal to $R_t \times R/R_t$ and $\Lambda_t = \pi_{\text{aff}}^{-1}(R_t)$. So, $\pi_{\text{aff}}^{-1}(R) = \Lambda_t \cdot R/R_t$ and proof of the first argument is done easily case by case. And, the second argument follows from the first.

Next, we need investigate the case of trivial $R_0/R_t$. Before it, we need an elementary lemma.

Lemma 2.3. Let $\Lambda$ be a lattice of $\mathbb{R}^2$. Let $A$ be an orient reversing element of $\text{Iso}(\mathbb{R}^2, \Lambda)$ such that $A^2 = \text{id}$. Then, the group $\langle A \rangle$ in $\text{Iso}(\mathbb{R}^2, \Lambda)$ is linearly conjugate to one of $D_1$ or $D_{1,4}$ in $\text{Iso}(\mathbb{R}^2, \Lambda^{sq})$.

Proof. We may assume that $\langle A \rangle$ acts naturally on $\mathbb{R}^2$ with $A \in O(2)$. If $\Lambda$ is a square lattice, then it might be assumed that it is $\Lambda^{sq}$ by a linear conformal map. For $u_1 = (1, 0)$ and $u_2 = (0, 1)$, $A$ is determined by $A(u_1)$, and possible $A(u_1)$’s are $\pm u_1, \pm u_2$ because $A$ is a reflection and preserves $\Lambda^{sq}$. By a suitable rotation, the action of $\langle A \rangle$ is conjugate to $D_1$ or $D_{1,4}$ in $\text{Iso}(\mathbb{R}^2, \Lambda^{sq})$.

Next, if $\Lambda$ is not a square lattice but has an orthogonal basis, then it might be assumed that $\{u_1, c \cdot u_2\}$ with $c > 1$ is a basis of $\Lambda$ by a linear conformal map. So, possible $A(u_1)$’s are $\pm u_1$ and the action of $\langle A \rangle$ is conjugate to $D_1$ in $\text{Iso}(\mathbb{R}^2, \Lambda^{sq})$ induced by a suitable linear isomorphism.

Before we go further, we state an elementary fact. If two elements $\bar{w}_1$ and $\bar{w}_2$ in $\Lambda^* = \Lambda - O$ are linearly independent and have the same smallest length, then $\{\bar{w}_1, \bar{w}_2\}$ is a basis of $\Lambda$. We return to our proof. Assume that $\Lambda$ does not have an orthogonal basis. Let $\bar{w}_0$ be an element of $\Lambda^*$ with the smallest length. If $A\bar{w}_0 \neq \pm \bar{w}_0$, then $\bar{w}_0$ and $A\bar{w}_0$ are linearly independent so that they become a basis by the previous elementary fact. By a suitable linear isomorphism, the action is conjugate to $D_{1,4}$ in $\text{Iso}(\mathbb{R}^2, \Lambda^{sq})$. Next, assume that $A\bar{w}_0 = \pm \bar{w}_0$. Let $\bar{w}$ be an element of $\Lambda - \bar{w}_0$ which has the smallest distance to $\mathbb{R} \cdot \bar{w}_0$. Let $L$ be the line parallel to $\mathbb{R} \cdot \bar{w}_0$ containing $\bar{w}$. Pick an element $\bar{w}_1$ of $L \cap \Lambda$ which has the smallest length. Put $\bar{w}_2 = \mp A\bar{w}_1$. Since $\Lambda$ does not have an orthogonal basis, $\bar{w}_1 \neq \bar{w}_2$. Also, $|\bar{w}_1 - \bar{w}_2| = |\bar{w}_0|$ because $\bar{w}_1$ has the smallest length in $L \cap \Lambda$. And, the convex hull of $\{O, \bar{w}_1, \bar{w}_2\}$ contains no element of $\Lambda$ except $O, \bar{w}_1, \bar{w}_2$ so that $\{\bar{w}_1, \bar{w}_2\}$ is a basis for $\Lambda$. If we put $\bar{w}_2 = A\bar{w}_1 = \mp \bar{w}_2'$, then $\{\bar{w}_1, \bar{w}_2\}$ is a basis of $\Lambda$ with the same length such that $A\bar{w}_1 = \bar{w}_2$. By a suitable linear isomorphism, the action of $\langle A \rangle$ is conjugate to the action of the $D_{1,4}$ in $\text{Iso}(\mathbb{R}^2, \Lambda^{sq})$. Therefore, we obtain a proof.

To deal with the case of trivial $R_0/R_t$, we define some notations. When $R/R_t = D_1$ or $D_{1,4}$, let $l_0$ be $(1, 0)$ or $(1, 1)$, and let $l_0^\perp$ be $(0, 1)$ or $(-1/2, 1/2)$, respectively. For those two cases, let $L$ in $\mathbb{R}^2$ be the line fixed by $R/R_t$, and let $L^\perp$ be the line perpendicular to $L$ passing through $O$. Here, $l_0$ is an element of $(\Lambda^{sq})^* \cap L$ with the smallest length, and the length of $l_0^\perp$ is the distance from $L$ to $\Lambda^{sq} - L$. For a line $L'$ and a vector $c'$ (possibly zero) in $\mathbb{R}^2$ with $c' \parallel L'$, the composition of the reflection in $L'$ and the translation by $c'$ is called the glide through $L'$ and $c'$. Now, we can explain for lower three rows of Table 1.2.

Proposition 2.4. Let a finite group $R$ act naturally on $\mathbb{R}^2/\Lambda$ for some $\Lambda$. Assume that $R_0/R_t$ is trivial. Then, the $R$-action is conjugate to one of the following:
Proof. First, if \( R/R_t \) is trivial, then \( R = R_t \) and we easily obtain (1) by a suitable linear isomorphism. For (2), assume that \( R/R_t \) is nontrivial. Then, \( R/R_t \) in \( \text{Aff}(\mathbb{R}^2, \Lambda) \) is order two, and there exists a linear isomorphism \( \bar{\eta} : \mathbb{R}^2 \to \mathbb{R}^2 \) such that \( \bar{\eta}(\Lambda_t) = \Lambda^\text{eq} \) and the conjugate action on \( \mathbb{R}^2 \) induced by \( \bar{\eta} \), and denote \( \bar{\eta}^*(R) \), \( \bar{\eta}(\Lambda) \), \( \bar{\eta}(\Lambda_t) \) just by \( R, \Lambda, \Lambda_t \), respectively. Then, \( \Lambda_t = \Lambda^\text{eq} \) is obtained by \( \bar{\eta} \). Pick an element \( g \) in \( R - R_t \) which is \( [A\bar{c} + \bar{c}] \) with \( \langle A \rangle = D_1 \) or \( D_{1,4} \). By \( g^2 \in R_t \), we obtain \( A\bar{c} + \bar{c} \in \Lambda^\text{eq} \) which is fixed by \( A \), i.e. \( A\bar{c} + \bar{c} \in L \). Since \( [A\bar{c} + (\bar{c} + \lambda)] \in R \) for each \( \lambda \in \Lambda^\text{eq}, \) \( A(\bar{c} + \lambda) + (\bar{c} + \lambda) \) is also in \( \Lambda^\text{eq} \) so that we may assume that \( A\bar{c} + \bar{c} \) is equal to 0 or \( l_0 \). Then, \( \bar{c} \) should be in \( L^+ \) or \( l_0 + L^+ \) because \( A \) is the reflection in \( L \), and (2) is easily obtained. \( \square \)

**Figure 2.1.** Some polygonal areas with glides

**Corollary 2.5.** To a finite group \( R \) in the previous proposition, we assign the two-dimensional fundamental domain \( |P_R| \) in Table 1.3. Then, the \( \pi^{-1}_{\text{aff}}(R) \)-orbit of \( |P_R| \) cover \( \mathbb{R}^2 \) and there is no interior intersection between different areas in the orbit. Also, the \( R \)-orbit of \( \pi(|P_R|) \) cover \( \mathbb{R}^2/\Lambda \) and there is no interior intersection between different areas in the orbit. In Figure 2.7, we illustrate gray colored \( |P_R| \) for some cases.

**Proof.** If \( R = R_t \), then proof is easy. Otherwise, \( \pi^{-1}_{\text{aff}}(R) \) is equal to \( \langle \Lambda_t, A\bar{c} + \bar{c} \rangle \), and proof is done similar to Corollary 2.2 \( \square \)
In summary, any finite subgroup $R$ in $\text{Aff}(\mathbb{R}^2/\Lambda)$ for some $\Lambda$ is conjugate to one of the groups in Table 1.1.

3. ONE-DIMENSIONAL $\rho(G_\chi)$ CASE

In this section, we prove Theorem D. First, we would list all one-dimensional subgroups of $\text{Aff}(\mathbb{R}^2/\Lambda)$ for some $\Lambda$ up to conjugacy. Let $S^1$ be the rotation group $\{[x + (t, 0)] \in \text{Aff}(\mathbb{R}^2/\Lambda^\sq n) \mid t \in [0, 1]\}$. Let $M(a)$ be the matrix $\begin{pmatrix} 1 & a \\ 0 & -1 \end{pmatrix}$ for $a \in \mathbb{Z}$.

Lemma 3.1. Let a compact one-dimensional group $R$ act naturally on $\mathbb{R}^2/\Lambda$ for some $\Lambda$. Then, $R$ is conjugate to one of the following subgroups in $\text{Aff}(\mathbb{R}^2/\Lambda^\sq n)$:

1. $\langle S^1, [x + (0, \frac{1}{n})] \rangle$ for some $n \in \mathbb{N}$,
2. $\langle S^1, [x + (0, \frac{1}{n})], [-x + (0, t_1)] \rangle$ for some $n \in \mathbb{N}$, $t_1 \in [0, 1]$,
3. $\langle K, [M(a)x + (0, t_2)] \rangle$ or $\langle K, [-M(a)x + (0, t_2)] \rangle$ for some $a \in \mathbb{Z}$, $t_2 \in [0, 1]$ where $K$ is equal to one of (1), (2) and $[\pm M(a)x + (0, t_2)]^2 \in K$.

Proof. If $R$ is connected, then it is easy that $R$ is conjugate to $S^1$ in $\text{Aff}(\mathbb{R}^2/\Lambda^\sq n)$. So, if $R$ is not connected, then we may assume that $R$ is a subgroup in $\text{Aff}(\mathbb{R}^2/\Lambda^\sq n)$ whose identity component is $S^1$. For arbitrary $g = [Ax + \overline{c}] \in R$ and $z \in S^1$, $\overline{g} z \overline{g}^{-1} = (g\overline{g}^{-1})g\overline{x}$ where $g\overline{g}^{-1} \in S^1$ by normality of $S^1$ in $R$. Since $S^1$-orbits in $\mathbb{R}^2/\Lambda^\sq n$ are horizontal, the right term of (*) is horizontal when $g\overline{g}^{-1}$ moves in $S^1$. So, $g$ moves the horizontal $S^1$-orbit $z[x]$ for $z \in S^1$ to a horizontal $S^1$-orbit. From this, we can observe that $A$ sends the $x$-axis to the $x$-axis itself, i.e. the $(2, 1)$ entry of $A$ should be zero. Moreover, observe that $A$ has a finite order. From this and $A(\Lambda^\sq n) = \Lambda^\sq m$, we can show that possible $A$’s are $\pm id$, $\pm M(a)$ for $a \in \mathbb{Z}$ by simple calculation. Also, we may assume that the first entry of $\overline{c}$ is 0 because $zg \in R$ for each $z \in S^1$. By these arguments, we would show that possible $R$ is one of (1)~(3).

First, assume that the $R$-action is orient preserving. So, $A$ of each $g = [Ax + \overline{c}]$ in $R$ is equal to $\pm id$. Since the subgroup of elements of the form $[x + (0, t)]$ in $R$ is cyclic, it is generated by an element of the form $[x + (0, 1/n)]$ for some $n \in \mathbb{N}$. So, if $R$ contains no element of the form $[-x + (0, t_1)]$, then $R$ is equal to (1). And, if $R$ contains an element of the form $[-x + (0, t_1)]$, then $R$ is equal to (2).

Next, assume that the $R$-action is not orient preserving. Then, $R$ contains an element $[\pm M(a)x + (0, t_2)]$ for some $a \in \mathbb{Z}$, $t_2 \in [0, 1]$. So, $R$ is equal to $\langle R_0, [\pm M(a)x + (0, t_2)] \rangle$. From this, we obtain this lemma because $R_0$ is equal to (1) or (2). 

For each action of the above lemma, we can find a circle in two-torus by which equivariance of the torus is simply expressed.

Proposition 3.2. Let a compact one-dimensional group $R$ act naturally on $\mathbb{R}^2/\Lambda$ for some $\Lambda$. Then, there exists a circle $C$ in the torus such that the map $R \times_{R_C} C \rightarrow \mathbb{R}^2/\Lambda$, $[g, x] \mapsto gx$ for $g \in R$, $x \in C$ is an equivariant isomorphism where $R_C$ is the subgroup of $R$ preserving $C$.

Proof. It suffices to find $C$ such that $R \cdot C = \mathbb{R}^2/\Lambda$ and $gC \cap C = \emptyset$ for any $g \in R - R_C$. As in the previous lemma, we may assume that $R$ is contained in $\text{Aff}(\mathbb{R}^2/\Lambda^\sq n)$, and that its identity component is $S^1$. When the $R$-action is orient-preserving, we may assume that $R$ is equal to (1) or (2) in Lemma 3.1. Let $C$ be the circle which is the image of the $y$-axis by $\pi$. Then, it is checked that $R$-orbits of $C$ are all translations of $C$, and from this it is easily shown that $C$ becomes a wanted circle. Therefore, we obtain a proof.
When the $R$-action is not orient-preserving, we prove it only for the case of $R = \langle K, [M(a)x + (0, t_2)] \rangle$ when $K$ is equal to (2) of Lemma 3.1. Proof for the other case is similar. Let $C$ be the image of $y = -\frac{2}{\pi} x$ (or $x = 0$ when $a = 0$) by $\pi$. Then, it is checked that $R$-orbits of $C$ are all translations of $C$, and from this it is easily shown that $C$ becomes a wanted circle. Therefore, we obtain a proof. □

By using this proposition, we can prove Theorem 1.

**Proof of Theorem 1** We may assume that $\Lambda = \Lambda^\text{aff}$ and $\rho(G_\chi)$ is one of groups in Lemma 3.4. The torus $\mathbb{R}^2/\Lambda$ is equivariantly isomorphic to $G_\chi \times (G_\chi)_c C$ for the circle $C$ of Proposition 3.2. So, the map

$$\text{Vect}_{(G_\chi)_c}(C, \chi) \to \text{Vect}_{G_\chi}(\mathbb{R}^2/\Lambda, \chi), \quad [F] \mapsto [G_\chi \times (G_\chi)_c F]$$

is the inverse of our map. □

4. $\mathbb{R}^2/\Lambda$ as the quotient of an equivariant simplicial complex for a finite subgroup of $\text{Aff}(\mathbb{R}^2/\Lambda)$

In this section, we consider $\mathbb{R}^2/\Lambda$ as the quotient of the underlying space of an equivariant simplicial complex which will be denoted by $\mathcal{K}_R$. Given a finite subgroup $R$ of $\text{Aff}(\mathbb{R}^2/\Lambda)$, we investigate equivariance of $\mathbb{R}^2/\Lambda$ by calculating isotropy groups at some points of it.

Let $\mathcal{F}_R$ be a set of $g \cdot |P_R|$'s for $g \in \pi^{-1}_\text{aff}(R)$ such that

1. $|P_R| \in \mathcal{F}_R$,
2. $\bigcup_{g \in \pi^{-1}_\text{aff}(R)} \pi(g \cdot |P_R|) = \mathbb{R}^2/\Lambda$,
3. $\pi(g \cdot |P_R|) \cap \pi(h \cdot |P_R|)$ has no interior point for any different $g \cdot |P_R|$ and $h \cdot |P_R|$ in $\mathcal{F}_R$.

Such an $\mathcal{F}_R$ exists by Corollary 2.2 and Corollary 2.5. We pick an $\mathcal{F}_R$ for each $R$. We would consider the disjoint union $\bigcup_{g \in \pi^{-1}_\text{aff}(R)} \pi(g \cdot |P_R|)$ as the underlying space of a two-dimensional simplicial complex. As in Introduction, we allow that a face of a two-dimensional simplicial complex need not be a triangle, and we consider an $n$-gon as the simplicial complex with one face, $n$ edges, $n$ vertices. Then, denote by $\mathcal{K}_R$ the natural simplicial complex structure of $\bigcup_{g \in \pi^{-1}_\text{aff}(R)} \pi(g \cdot |P_R|)$. We denote simply by $\pi$ the quotient map from $|\mathcal{K}_R|$ to $\mathbb{R}^2/\Lambda$ sending each point $\tilde{x}$ in some $g \cdot |P_R|$ $\subset |\mathcal{K}_R|$ to $\pi(\tilde{x})$ in $\mathbb{R}^2/\Lambda$ where $\tilde{x}$ is regarded as a point in $\mathbb{R}^2$. By definition, $\pi|_{\pi^{-1}_\text{aff}(R)}$ is bijective for each $g \cdot |P_R| \in \mathcal{F}_R$ so that the $R$-action on $\mathbb{R}^2/\Lambda$ induces the $R$-action on $|\mathcal{K}_R|$ and $|\mathcal{K}_R|$ such that $\pi$ is equivariant. In general, $\pi$ does not induce the equivariant simplicial complex structure on $\mathbb{R}^2/\Lambda$ from $\mathcal{K}_R$, as the example illustrated in Figure 5.1.

Next, we define some notations on $|\mathcal{K}_R|$. We use notations $\tilde{v}, \tilde{e}, \tilde{f}$ to denote a vertex, an edge, a face of $|\mathcal{K}_R|$, respectively. We use the notation $\tilde{x}$ to denote an arbitrary point of $|\mathcal{K}_R|$. And, we use notations $v, x, [e], b(e), [f], b(f)$ to denote images $\pi(\tilde{v}), \pi(\tilde{x}), \pi([\tilde{e}]), \pi(b(\tilde{e})), \pi([\tilde{f}]), \pi(b(\tilde{f}))$, respectively. Denote by $f^{-1}$ the face of $K_R$ such that $|f^{-1}| = |P_R|$. In Introduction, we have already defined vertices $\tilde{v}^i$’s and edges $\tilde{e}^i$’s of $f^{-1}$.

For simplicity, denote $\pi(\tilde{v}^i), \pi([\tilde{e}^i]), \pi(b(\tilde{e}^i)), \pi(b(f^{-1}))$ by $v^i, [e^i], b(e^i), b(f^{-1})$, respectively. Define the integer $j_R$ as the cardinality of $\pi^{-1}(v^i)$ for $i \in \mathbb{Z}$. Let $B$ be the set of barycenters of faces in $K_R$ on which $R$ acts transitively by definition of $\mathcal{F}_R$, and $B$ is confused with $\pi(B)$.

We would calculate isotropy subgroups at some points of $\mathbb{R}^2/\Lambda$. For this, we define some notations on isotropy subgroups. For each $x \in \mathbb{R}^2/\Lambda$, $R_x$ is considered as a subgroup of $\text{Iso}(T_x \mathbb{R}^2/\Lambda)$. Since the tangent space $T_x \mathbb{R}^2/\Lambda$ at each $x \in \mathbb{R}^2/\Lambda$ inherits the vector space structure and the usual coordinate system from $\mathbb{R}^2$, $\text{Iso}(\mathbb{R}^2)$ might be identified with $\text{Iso}(T_x \mathbb{R}^2/\Lambda)$. With this identification, $R_x/R_x$ is also regarded as a subgroup of $\text{Iso}(T_x \mathbb{R}^2/\Lambda)$ according to context. Here, we can observe
that \( R_x \subset R/R_t \) in \( \text{Iso}(T_x, \mathbb{R}^2/\Lambda) \) for each \( x \in \mathbb{R}^2/\Lambda \). Similarly, we can define subgroups \( \mathbb{Z}_n, D_n, D_{n,1} \) in \( \text{Iso}(T_x, \mathbb{R}^2/\Lambda) \). For the calculation, we give the natural simplicial complex structure on \( \mathbb{R}^2 \) by considering \( \mathbb{R}^2 \) as the union \( \bigcup_{y \in \pi^{-1}_0(R)} \{ y \} \cdot |R_t| \), and we denote it by \( K_{R^2} \).

First, we calculate \( R_{b,(f^{-1})} \). For a lattice \( \Lambda' \) in \( \mathbb{R}^2 \), denote by \( \text{Area}(\Lambda') \) the area of the parallelogram spanned by two basis vectors in \( \Lambda' \). Then, the area of \( \mathbb{R}^2/\Lambda \) is equal to \( \text{Area}(\Lambda_t) \cdot |R_t| \) so that we have

\[
(4.1) \quad |R_t| = \frac{\text{Area}(|P_R|)}{\text{Area}(\Lambda_t)} \cdot |B|
\]

because the area of \( \mathbb{R}^2/\Lambda \) is equal to \( \text{Area}(|P_R|) \cdot |B| \) where \( \text{Area}(|P_R|) \) is the area of \( |P_R| \).

**Proposition 4.1.** For each \( R \) in Table 4.1, \( R_{b,(f^{-1})} \) is listed in Table 4.1.

| \( R/R_t \) | \( \Lambda_t \) | \( \text{Area}(|P_R|)/\text{Area}(\Lambda_t) \) | \( R_{b,(f^{-1})} \) |
|---|---|---|---|
| \( \mathbb{Z}_2 \) | \( \Lambda^{\text{id}} \) | 1/2 | \( \langle \text{id} \rangle \) |
| \( D_{2,2} \) | 1/4 | \( \langle \text{id} \rangle \) |
| \( D_2 \) | 1/4 | \( \langle \text{id} \rangle \) |
| \( Z_4 \) | 1/4 | \( \langle \text{id} \rangle \) |
| \( D_4 \) | 1/4 | \( \langle \text{id} \rangle \) |
| \( \mathbb{Z}_2 \) | \( \Lambda^{\text{id}} \) | 1 | \( R/R_t \) |
| \( \langle \text{id} \rangle \) | \( \Lambda^{\text{id}} \) | 1 | \( \langle \text{id} \rangle \) |
| \( D_1, D_{1,4}, \) | \( \Lambda^{\text{id}} \) | 1/2 | \( \langle \text{id} \rangle \) |
| \( A\bar{e} + \bar{e} = 0 \) | \( \Lambda^{\text{id}} \) | 1/2 | \( \langle \text{id} \rangle \) |

**Table 4.1.** \( R_{b,(f^{-1})} \)

**Proof.** As in proofs of Corollary 2.2, 2.5 we can calculate \( |B/R_t| = |\pi^{-1}(B)/\Lambda_t| \) case by case because \( \pi^{-1}(B) \) is the set of barycenters of faces of \( K_{R^2} \). By using this, we can calculate \( R_{b,(f^{-1})} \) in the below.

By definition of \( |P_R| \) in Table 4.2 \( \text{Area}(|P_R|)/\text{Area}(\Lambda_t) = 1, 1/2, \) or \( 1/4 \). In the case when \( \text{Area}(|P_R|)/\text{Area}(\Lambda_t) = 1, \) it is observed that \( R_t \) acts transitively on \( B \), and the formula (4.1) gives \( |R_t| = |B| \). From these, \( R_t \) acts freely and transitively on \( B \). So, \( R_t \) acts freely and transitively on \( B \). Thus, \( R_{b,(f^{-1})} \cap R = \langle \text{id} \rangle \). That is, \( R_{b,(f^{-1})} \cong R/R_t \) because \( R_t \) is normal in \( R \). Also, since \( R_{b,(f^{-1})} \) is contained in \( R/R_t \) as subgroups of \( \text{Iso}(T_{b,(f^{-1})}, \mathbb{R}^2/\Lambda) \), \( R/R_t \) and \( R_{b,(f^{-1})} \) are equal as subgroups of \( \text{Iso}(T_{b,(f^{-1})}, \mathbb{R}^2/\Lambda) \).

When \( \text{Area}(|P_R|)/\text{Area}(\Lambda_t) = 1/2 \), the formula (4.1) gives \( |R_t| = \frac{1}{2}|B| \) and \( R_t \) acts freely and non-transitively on \( B \). In the below, we will use the notation \( R' \) to denote a subgroup of \( R \) such that \( R' \) acts freely and transitively on \( B \). If such a subgroup \( R' \) exists, then \( R = R_{b,(f^{-1}),R'} \) and \( R_{b,(f^{-1}),R'} \cap R' = \langle \text{id} \rangle \) so that \( |R_{b,(f^{-1}),R'}| = |R|/|R'| \). Moreover, \( R_{b,(f^{-1}),R'} \cong R/R' \) if \( R' \) is normal in \( R \). In the case when \( R/R_t = \mathbb{Z}_2, \mathbb{Z}_6, D_6 \), put \( R' = R_t \times \mathbb{Z}_2 \) which is normal in \( R \). It is easily observed that \( R' \) acts freely and transitively on \( B \). Thus, \( R_{b,(f^{-1}),R'} \cong R/R' \) if \( R' \) is normal in \( R \). In the case when \( R/R_t = \mathbb{Z}_2, \mathbb{Z}_6, D_6 \), respectively. In the case when
When \( R/R_t = D_3 \), put \( R' = \{ b, R_t \} \) so that \( R_{b(f-1)} \) has order 3 and \( R_{b(f-1)} = Z_3 \). In the case when \( R/R_t = D_1, D_{1,4} \), we obtain \( |R| = |B| \) from \( |D_1| = |D_{1,4}| = 2 \) and \( |R| = \frac{1}{2}|B| \) so that \( R \) acts freely and transitively on \( B \), i.e. \( R_{b(f-1)} \) is trivial.

When \( \text{Area}([P_R])/\text{Area}(\Lambda_t) = 1/4 \), the formula (4.1) gives \( |R| = \frac{1}{2}|B| \) and \( R \) acts freely and non-transitively on \( B \). In cases when \( R/R_t = D_2, D_2, Z_4 \) with \( \Lambda_t = \Lambda_{22} \) and \( R/R_t = D_2, D_{2,3} \) with \( \Lambda_t = \Lambda_{21} \), we have \( |R/R_t| = 4 \) so that \( |R| = |B| \). Since \( R \) acts freely and transitively on \( B \), \( R_{b(f-1)} \) is trivial. In the case when \( R/R_t = D_4 \), we similarly have \( |R| = 2|B| \). Since \( R \) acts transitively on \( B \), we have \( |R_{b(f-1)}| = 2 \) and it is checked that \( R_{b(f-1)} = D_{1,2} \).

Now, we calculate \( R_{v^i} \) and \( R_{b(v^i)} \). Denote by \( V \) the set of vertices of \( K_R \), and by \( V \) the set \( \pi(V) \). For two points \( x \) and \( x' \) in \( \mathbb{R}^2/\Lambda \), \( x \sim x' \) means that \( x \) and \( x' \) are in the same \( R \)-orbit.

**Proposition 4.2.** For each \( R \) in Table 4.2, \( R_{v^i} \) is listed in Table 4.2. In \( \sim \) entry, we list all \( v^i \)'s in the same \( R \)-orbit. By \( v^i \sim v^{i'} \), we mean that all \( v^i \)'s are in an orbit.

| \( R/R_t \) | \( \Lambda_t \) | \( [V/R] \) | \( \sim \) | \( R_{v^i} \) |
|----------------|----------------|----------------|----------------|----------------|
| \( Z_2 \) | \( \Lambda_{22} \) | 2 | \( v^0 \sim v^3, v^1 \sim v^2 \) | \( R/R_t \) |
| \( D_2 \) | \( \Lambda_{22} \) | 3 | \( v^1 \sim v^2 \) | \( R/R_t \) |
| \( D_2 \) | \( \Lambda_{22} \) | 3 | \( v^1 \sim v^2 \) | \( R/R_t \) |
| \( Z_4 \) | \( \Lambda_{22} \) | 3 | \( v^1 \sim v^2 \) | \( R/R_t \) |
| \( D_4 \) | \( \Lambda_{22} \) | 1 | \( v^i \sim v^{i'} \) | \( R/R_t \) |
| \( Z_6 \) | \( \Lambda_{22} \) | 1 | \( v^i \sim v^{i'} \) | \( R/R_t \) |
| \( D_6 \) | \( \Lambda_{22} \) | 2 | \( v^i \sim v^{i+2} \) | \( R/R_t \) |
| (id) | \( \Lambda_{22} \) | 1 | \( (id) \) | \( (id) \) |
| \( D_1 \) | \( \Lambda_{22} \) | 2 | \( v^0 \sim v^3, v^1 \sim v^2 \) | \( R/R_t \) |
| \( D_{1,4} \) | \( \Lambda_{22} \) | 2 | \( v^0 \sim v^3, v^1 \sim v^2 \) | \( R/R_t \) |
| \( A\tilde{e} + \tilde{c} = 0 \) | \( \Lambda_{22} \) | 1 | \( v^i \sim v^{i'} \) | \( (id) \) |
| \( A\tilde{e} + \tilde{c} = \bar{b}_0 \) | \( \Lambda_{22} \) | 2 | \( v^0 \sim v^3, v^1 \sim v^2 \) | \( R/R_t \) |

**Table 4.2.** \( R_{v^i} \)

**Proof.** Since \( |V| = i_R|B| \) and \( |\tilde{V}| = j_R|V| \), we obtain

\[
|R_t| = \frac{\text{Area}([P_R])}{\text{Area}(\Lambda_t)} \cdot \frac{j_R}{i_R} \cdot |V|
\]

by (4.1). As in proofs of Corollary 2.2, 2.5, we can calculate

\[
|V/R| = |\pi^{-1}(V)|/\pi^{-1}(\Lambda_t) \quad \text{and} \quad |V/R_t| = |\pi^{-1}(V)|/\Lambda_t|
\]

case by case because \( \pi^{-1}(V) \) is the set of vertices of \( K_{R^2} \). By using this, we can calculate \( R_{v^i} \) in the below.

In the case of \( R/R_t = Z_2 \) with \( \Lambda_t = \Lambda_{22} \), the formula (4.2) gives \( |R_t| = \frac{1}{2}|V| \). From this, we conclude that \( R_t \) acts freely on each \( R_t \)-orbit in \( V \) because \( |V/R_t| = 2 \). Of course, \( R_t \) acts transitively on each \( R_t \)-orbit in \( V \). Since \( |V/R| = 2 \), each \( R_t \)-orbit is also \( R \)-invariant. From this, we have \( R = R_{v^i} \cdot R_t \) and \( R_{v^i} \cap R_t = (\text{id}) \) so that \( R_{v^i} = R/R_t \). Here, \( v^0, v^2 \) are in the same orbit, and \( v^1, v^2 \) are in the same orbit.
In the case of \( R/R_t = D_{2,2} \) with \( \Lambda_t = \Lambda^{eq} \), the formula (1.2) gives \( |\mathcal{E}_t| = \frac{1}{4} |\mathcal{V}| \). From this, we conclude that \( R_t \) acts freely on each \( R_t \)-orbit in \( \mathcal{V} \) because \( |\mathcal{V}/R_t| = 4 \) and \( R_t \) acts transitively on each \( R_t \)-orbit in \( \mathcal{V} \). Since \( |\mathcal{V}/R_t| = 3 \) and \( v^1, v^2 \) are in the same orbit, \( R_t \) acts freely and transitively on both \( R_t \)-orbits of \( v^1, v^2 \) in \( \mathcal{V} \) which are \( R \)-invariant. So, \( R_{e_i} = R/R_t \) for \( i = 0, 3 \). But, each \( R_i \)-orbit of \( v^1 \) for \( i = 1, 2 \) is not \( R \)-invariant. So, we take the index 2 subgroup \( S \) of \( R \) preserving each \( R_t \)-orbit of \( v^1 \) for \( i = 1, 2 \), then \( S = R_{e_i} \cdot R_t \) and \( R_{e_i} \cap R_t = (\text{id}) \) so that \( |\mathcal{E}_{e_i}| \) should be 2 for \( i = 1, 2 \), and it is checked that \( R_{e_i} = D_{1, -4} \) for \( i = 1, 2 \).

In cases when \( R/R_t = D_2, Z_4, D_4 \) with \( \Lambda_t = \Lambda^{eq} \), the formula (1.2) gives \( |\mathcal{E}_t| = \frac{1}{4} |\mathcal{V}| \). From this, \( R_t \) acts freely and transitively on each \( R_t \)-orbit in \( \mathcal{V} \) because \( |\mathcal{V}/R_t| = 4 \). Since \( |\mathcal{V}/R_t| = 4 \) for \( R/R_t = D_2, R_{e_i} = R/R_t \). Since \( |\mathcal{V}/R_t| = 3 \) and \( v^1, v^3 \) are in an \( R \)-orbit for \( R/R_t = Z_4, D_4 \), we have \( R_{e_i} = R/R_t \) for \( i = 0, 2 \). Also, it is checked that \( R_{v^1} = Z_2 \) for \( R/R_t = Z_4, i = 1, 3 \). And, \( R_{v^1} = D_2 \) for \( R/R_t = D_{1, i} \).

In the case of \( R/R_t = D_{2,2} \) with \( \Lambda_t = \Lambda^{eq} \), the formula (1.2) gives \( |\mathcal{E}_t| = |\mathcal{V}| \). From this, we conclude that \( R_t \) acts freely on \( \mathcal{V} \) because \( R_t \) acts transitively on \( \mathcal{V} \). So, \( R_{e_i} = R/R_t \).

In the case of \( R/R_t = D_2, D_{2,2} \) with \( \Lambda_t = \Lambda^{eq} \), the formula (1.2) gives \( |\mathcal{E}_t| = |\mathcal{V}| \). Since \( |\mathcal{V}/R_t| = 2 \), \( R_t \) acts freely and transitively on each \( R_t \)-orbit in \( \mathcal{V} \). Also, since \( |\mathcal{V}/R_t| = 2 \), \( R_{e_i} = R/R_t \).

In the case of \( R/R_t = D_2, D_{2,2} \) with \( A\varepsilon + \bar{\varepsilon} = 0 \), the formula (1.2) gives \( |\mathcal{E}_t| = \frac{1}{2} |\mathcal{V}| \). From this, we conclude that \( R_{e_i} = R/R_t \) because \( |\mathcal{V}/R_t| = |\mathcal{V}/R_t| = 2 \).

Denote by \( \mathcal{E} \) the set of barycenters of edges in \( \mathcal{K}_R \), and by \( \mathcal{E} \) the set \( \pi(\mathcal{E}) \).

**Proposition 4.3.** For each \( R \) in Table 4.3, \( R_{b(e_i)} \) is listed in Table 4.3. And, each \( R_{b(e_i)} \) satisfies \( R_{b(e_i)} \cap R_{b(e_{i-1})} = (\text{id}) \) except \( R/R_t = D_6 \).

**Proof.** As in proofs of Corollary 2.2, 2.5, we can calculate

\[ |\mathcal{E}/R| = |\pi^{-1}(\mathcal{E})/\pi^{-1}(R)| \quad \text{and} \quad |\mathcal{E}/R_t| = |\pi^{-1}(\mathcal{E})/\pi^{-1}(R)| \]

case by case because \( \pi^{-1}(\mathcal{E}) \) is the set of barycenters of edges in \( \mathcal{K}_{R_t} \). By this, we obtain \( |\mathcal{E}/R_t| \)-entry of Table 4.3.

Since \( R_{b(e_i)} \) preserves \( e^1 \), \( R_{b(e_i)} \subset D_{2,2} \) in \( \text{Iso}(T_{b(e_i)} \mathbb{R}^2/\Lambda) \) for some \( l \in \mathbb{Q}^+ \). In cases when \( R/R_t \neq D_1, D_{1,4} \), we can show that a matrix \( A \in D_{2,2} \) is in \( R_{b(e_i)} \subset \text{Iso}(T_{b(e_i)} \mathbb{R}^2/\Lambda) \) if and only if \( Ab(e^1) \in b(e^1) + \Lambda_t \) in \( \mathbb{R}^2 \) because \( R = R_t \times R/R_t \), i.e. each element in \( R \) is of the form \([Ax + \bar{\varepsilon}] \) with \( \bar{\varepsilon} \in \Lambda_t \). By using this, we can calculate \( R_{b(e_i)} \) in Table 4.3. Similarly, we obtain \( R_{b(e_i)} \) in Table 4.3 in cases when \( R/R_t = D_1, D_{1,4} \).

The second statement is checked case by case by Table 4.3. □

Now, we explain for one-dimensional fundamental domain. A closed subset \( \bar{D}_R \) of \( |\mathcal{K}_R(1)| \) is called a one-dimensional fundamental domain if the orbit of \( \pi(\bar{D}_R) \) covers \( \pi([\mathcal{K}_R(1)]) \) and \( \bar{D}_R \) is a minimal set satisfying the property.
Proposition 4.4. Let $R$ be a group in Table 4.3. For each $i \in \mathbb{Z}_{1R}$, $R_{b(e^i)}$ fixes $|e^i|$ if and only if $\mathbb{Z}_2$ is not contained in $R_{b(e^i)}$. And, $\bar{D}_R$ listed in Table 4.3 is a one-dimensional fundamental domain.

Proof. First, we prove the first statement. We can check that it is true for the case of $R/R_t = D_6$. So, we may assume that $R/R_t \neq D_6$. Sufficiency is easy. For necessity, assume that $R_{b(e^i)}$ does not fix $|e^i|$. So, $R_{b(e^i)}$ is nontrivial. Since $R_{b(e^i)}$ preserves $|e^i|$, $R_{b(e^i)} \subset D_{2,1}$ for some $l \in \mathbb{Q}^*$ where we assume that $a_2b$ fixes $|e^i|$. So, $R_{b(e^i)} = \langle a_2b \rangle$ or $\langle -a_2b \rangle$ because $\mathbb{Z}_2$ is not contained in $R_{b(e^i)}$. Since $R_{b(e^i)}$ does not fix $|e^i|$, $R_{b(e^i)} = \langle -a_2b \rangle$. But, $-a_2b$ is orient reversing, and preserves $|e^i|$. So, it fixes $\pi(b(f^{-1}, b(e^i)))$, i.e. $-a_2b \in R_{b(e^i)} \cap R_{b(f^{-1})}$ which is trivial by Proposition 4.3. This is a contradiction.

Since $R$ acts transitively on $B$, each edge of $\mathcal{K}_R$ is in the $R$-orbit of some $e^i$. So, we may assume that any one-dimensional fundamental domain is contained in $[\mathcal{E}/R]$ edges of $|P_R|$, say $e^j$ for $j = 1, \cdots, |\mathcal{E}/R|$. We would select suitable $[\mathcal{E}/R]$ lines in $[\mathcal{K}_R^{(1)}]$ whose union is a candidate for a one-dimensional fundamental domain. For each $j$, if $R_{b(e^j)}$ fixes $|e^j|$, then pick the whole line $|e^j|$, otherwise pick a half of $|e^j|$ because a half of $|e^j|$ should be sent to the other half by $R_{b(e^i)}$. If we define $\bar{D}_R$ as the union of picked lines, then $\bar{D}_R$ satisfies the definition of one-dimensional fundamental domain. In Table 4.3, we have assigned such a $\bar{D}_R$ to each $R$.

In the case of $R/R_t = D_3$, $\bar{D}_R$ is not chosen in this way by a technical reason. But, we can check that $\bar{D}_R$ for the case in Table 4.3 becomes a one-dimensional fundamental domain.

By using the above calculations, we can calculate $R_{e^1}$ and $R_{C(e^2)}$ as promised in Introduction.

Lemma 4.5. For each $\bar{x}$ in $[e^i]$ such that $\bar{x} \neq \bar{e}^i, \bar{e}^{i+1}, b(\bar{e}^i)$, we have $R_{\bar{x}} = \langle \text{id} \rangle$. For $x = \pi(\bar{x})$, $R_x = R_{e^1}$. More precisely,

1. if $R_{b(e^i)} \subset \mathbb{Z}_2$, then $R_x$ is trivial.

Table 4.3. $R_{b(e^i)}$
(2) if $R_{b(e')}$ is $D_{1,4}$ for some $l \in \mathbb{Q}^*$, then $R_x = D_{1,4}$. 
(3) if $R/R_t = D_6$, then $R_x = D_{1,3}, D_{1,3}, D_4$ for $i = 0, 1, 2$, respectively.

**Proof.** First, $R_x$ fixes the whole $|f^{-1}|$ so that $R_x = (\text{id})$.

Next, $R_x$ fixes the whole $|e'|$, especially $R_x \subset R_{b(e')}$. From this, (1) is obtained.

By Proposition 4.4, we obtain (2). (3) is checked by direct calculation. \[ \square \]

Next, we calculate $R_{C(x)}$. If we put $\tilde{C}(\bar{x}) = [\bar{x}, b(P)]$ for $\bar{x} \in \partial P$, then $R_{C(x)}$ is equal to

$$R_{C(x)} = R_{b(f^{-1})} \cap R_x = R_x \subset R_{b(f^{-1})}$$

for the $R$-action on $|\Lambda_R|$. Also, since $R_{b(f^{-1})} = R_{b(f^{-1})}$, we can easily obtain $R_{C(x)}$ by Table 4.1.

We give a remark on injectivity of $\pi|_{D_R}$.

**Remark 4.6.** (1) If $R/R_t \neq (\text{id}), D_1, D_{1,4}$, then it can be checked case by case that any two vertices $\bar{v} \neq \bar{v}'$ in $\tilde{D}_R$ satisfy $v \neq v'$. In this check, it is helpful to note that $v' = v'$ implies $v' \sim v'$. In the remaining cases, $\bar{v} \neq \bar{v}'$ need not imply $v \neq v'$ according to $\Lambda$.

(2) For any two points $\bar{x} \neq \bar{x}'$ in $\tilde{D}_R$, if they are not vertices, then $\pi(\bar{x}) \neq \pi(\bar{x}')$ by the method by which we choose $D_R$.

5. **Equivariant clutching construction**

Let a compact Lie group $G_x$ act affinely and not necessarily effectively on $\mathbb{R}^2/\Lambda$ through a homomorphism $\rho : G_x \to \text{Aff}(\mathbb{R}^2/\Lambda)$. Assume that $\rho(G_x) = R$ for some $R$ in Table 1.1. From now on, we simply denote $K_{\rho(G_x)}, i_{\rho(G_x)}$, $j_{\rho(G_x)}, \cdots$ by $K_{\rho}, P_{\rho}, i_{\rho}, j_{\rho}, \cdots$, respectively. In this section, we would consider an equivariant vector bundle over $\mathbb{R}^2/\Lambda$ as an equivariant clutching construction of an equivariant vector bundle over $|\Lambda|$. For this, we would define equivariant clutching map and its generalization preclutching map. And, we state an equivalent condition under which a preclutching map be an equivariant clutching map. Before these, we need introduce notations on some relevant simplicial complexes. Since we should deal with various cases at the same time, these notations are necessary.

Denote by $L_\rho$ and $\tilde{L}_\rho$ the 1-skeleton $\tilde{K}_{\rho}^{(1)}$ of $\tilde{K}_\rho$ and the disjoint union $\Pi_{\bar{e} \in \bar{e}} \tilde{e}$, respectively. Then, $\tilde{L}_\rho$ is a subcomplex of $\tilde{K}_\rho$, and can be regarded as a quotient of $L_\rho$. These are expressed as two simplicial maps

$$\iota_\rho : L_\rho \to \tilde{K}_\rho, \quad p_\rho : \tilde{L}_\rho \to \tilde{K}_\rho$$

where $\iota_\rho$ is the inclusion, and $p_\rho$ is the quotient map whose preimage of each vertex and edge of $L_\rho$ consists of two vertices and one edge of $\tilde{L}_\rho$, respectively. Two maps on underlying spaces are denoted by

$$\iota_{|\mathcal{L}|} : |L_\rho| \to |\tilde{K}_\rho|, \quad p_{|\mathcal{L}|} : |\tilde{L}_\rho| \to |L_\rho|.$$ 

Here, we give an example of $\iota_{|\mathcal{L}|}$ and $p_{|\mathcal{L}|}$ when $\rho(G_x) = R$ with $R/R_t = (\text{id})$ and $R_t = (\text{id})$, i.e. $\Lambda_t = \Lambda = \Lambda^{(1)}$. In this case, $\tilde{K}_\rho = P_{\rho}$ with $|P_{\rho}| = P_{\rho}^{(1)}$ by Table 1.2. Also, $\tilde{L}_\rho$ is the 1-skeleton of $P_{\rho}^{(4)}$, and $L_\rho$ is disjoint union of four edges of $\tilde{L}_\rho$. These are illustrated in Figure 5.1. In Figure 5.2, one more example of $\tilde{L}_\rho$ and $L_\rho$ is illustrated where its notations are introduced in the below. The $G_x$-action on $\mathbb{R}^2/\Lambda$ naturally induces actions on these relevant simplicial complexes $\tilde{K}_\rho, L_\rho, \tilde{L}_\rho$.

Now, we introduce notations on simplices of relevant simplicial complexes. We use notations $\bar{e}$ and $\bar{e}$ to denote a vertex and an edge of $L_\rho$, respectively. And, we use the notation $\bar{x}$ to denote an arbitrary point in $|L_\rho|$. For simplicity, we often denote by $\bar{v}, \bar{e}, \bar{x}$ images $p_{|\mathcal{L}|}(\bar{v}), p_{|\mathcal{L}|}(\bar{e}), p_{|\mathcal{L}|}(\bar{x})$, respectively. Two edges $\bar{e}, \bar{e}'$ of $L_\rho$ (and their
For notational simplicity, we define $c$.

Notation 5.2.

$\hat{a}$ the shortest line connecting $\hat{b}$ in Figure 5.2. By using these notations, we introduce one-dimensional fundamental domain of $\rho$ whose image by $p$.

Notation 5.1.

$\hat{a}$'s need not be different. So far, we have finished defining $\hat{i}$.

Before it, we introduce a simplicial map. Let $i: \hat{L}_\rho \rightarrow \hat{L}_\rho$ be the simplicial map whose underlying space map $\hat{c}: |\hat{L}_\rho| \rightarrow |\hat{L}_\rho|$ is defined as

for each adjacent $\hat{e}, \hat{e}' \in \hat{L}_\rho$, each point $\hat{x}$ in $|\hat{e}|$ is sent to the point $|\hat{e}'|\hat{x}$ in $|\hat{e}'|$ to satisfy $p(p_{\hat{L}_\rho}(\hat{x})) = p(p_{\hat{L}_\rho}(\hat{c}(\hat{x})))$.

For example, $\hat{e}$ and $\hat{c}(\hat{e})$ are adjacent for any edge $\hat{e} \in \hat{L}_\rho$. Easily, $|\hat{c}|$ is equivariant. For notational simplicity, we define $\hat{c}$ also on edges of $\hat{L}_\rho$ to satisfy $\hat{c}(p_{\hat{L}_\rho}(\hat{e})) = p_{\hat{L}_\rho}(\hat{c}(\hat{e}))$ for each edge $\hat{e}$.

Notation 5.2.

(1) For a vertex $\hat{v}$ in $\hat{L}_\rho$ and its image $v = \pi(\hat{v})$, we label vertices in $\pi^{-1}(v)$ with $\hat{v}_j$ to satisfy

i) $\pi^{-1}(v) = \{\hat{v}_j \mid j \in Z_{j_0}\}$,

ii) $\hat{v}_0 = \hat{v}$,

iii) in each face $\hat{f}_j$ containing $\hat{v}_j$ for $j \in Z_{j_0}$, we can take a small neighborhood $U_j$ of $\hat{v}_j$ so that $\pi(U_j)$'s are arranged in the counterclockwise way around $v$.

(2) For a non-vertex $\hat{x}$ in $|\hat{L}_\rho|$ and its image $x = \pi(\hat{x})$, we label two points in $\pi^{-1}(x)$ with $\{\hat{x}_j | j \in Z_{\hat{x}}\}$ to satisfy $\hat{x}_0 = \hat{x}$.

For simplicity, we denote $(\hat{v}_0^i), (\hat{d}^i)_j$ by $\hat{v}_j^i, \hat{d}_j^i$, respectively.

Now, we introduce superscript $i$ and subscripts $+, -$ for vertices and edges in $\hat{L}_\rho$. Before it, we introduce a simplicial map. Let $\hat{c}: \hat{L}_\rho \rightarrow \hat{L}_\rho$ be the simplicial map whose underlying space map $|\hat{c}|: |\hat{L}_\rho| \rightarrow |\hat{L}_\rho|$ is defined as

for each adjacent $\hat{e}, \hat{e}' \in \hat{L}_\rho$, each point $\hat{x}$ in $|\hat{e}|$ is sent to the point $|\hat{e}'|\hat{x}$ in $|\hat{e}'|$ to satisfy $\pi(p_{\hat{L}_\rho}(\hat{x})) = \pi(p_{\hat{L}_\rho}(\hat{c}(\hat{x})))$.

For example, $\hat{e}$ and $\hat{c}(\hat{e})$ are adjacent for any edge $\hat{e} \in \hat{L}_\rho$. Easily, $|\hat{c}|$ is equivariant. For notational simplicity, we define $\hat{c}$ also on edges of $\hat{L}_\rho$ to satisfy $\hat{c}(p_{\hat{L}_\rho}(\hat{e})) = p_{\hat{L}_\rho}(\hat{c}(\hat{e}))$ for each edge $\hat{e}$.

Notation 5.2.

(1) For a vertex $\hat{v}$ in $\hat{L}_\rho$, we label two vertices in $p_{\hat{L}_\rho}^{-1}(\hat{v})$ with $\hat{v}_\pm$ to satisfy

$p_{\hat{L}_\rho}(\hat{c}(\hat{v}_\pm)) = \hat{v}_1$ and $p_{\hat{L}_\rho}(\hat{c}(\hat{v}_-)) = \hat{v}_0$.

(2) For a non-vertex $\hat{x}$ in $|\hat{L}_\rho|$, we denote the point in $p_{\hat{L}_\rho}^{-1}(\hat{x})$ as $\hat{x}_+$ or $\hat{x}_-$, i.e.

$\hat{x}_+ = \hat{x}_- = \hat{x}$.

For simplicity, denote $\hat{x}_\pm$ for $\hat{x} = \hat{v}_j, \hat{d}_j^i, \hat{d}^i_j$ by $\hat{v}_j^\pm, \hat{d}_j^\pm, \hat{d}^\pm_j$, respectively. So, if $\hat{d}^i_j$ is a barycenter of an edge, then $\hat{d}_j^\pm = \hat{d}_j^\mp$. And, denote by $\hat{d}^i_j$ the edge in $\hat{L}_\rho$, whose image by $p_{\hat{L}_\rho}$ is $\pm^i_j$ for $i \in Z_{\hat{d}}$. Then, we have finished introducing notations in Figure 5.2. By using these notations, we introduce one-dimensional fundamental domain of $|\hat{L}_\rho|$. For any two points $\hat{a}, \hat{b}$ in an edge of $\hat{L}_\rho$, denote by $[\hat{a}, \hat{b}]$ the shortest line connecting $\hat{a}$ and $\hat{b}$ for $M = \bigcup_{\hat{a} \in L_\rho} [\hat{a}, \hat{a}^+ \hat{a}^-] \subset |\hat{K}_\rho|$, we define $M$ in
\[ |\tilde{\mathcal{L}}_\rho| \]

as the disjoint union \( \bigcup_{i \in I^-} [\tilde{d}_i, \tilde{d}_i^{+1}] \) so that \( p_{|\mathcal{C}_L}(\tilde{D}_\rho) = \tilde{D}_\rho \). And, denote by \( \tilde{D}_\rho \) the set \((\pi \circ p_{|\mathcal{C}_L})^{-1}(D_\rho)\) in \( |\tilde{\mathcal{L}}_\rho| \) which is equal to

\[
\left( \bigcup_{i \in I^-} [\tilde{d}_i, \tilde{d}_i^{+1}] \right) \bigcup \left( \bigcup_{i \in I^+} [c([\tilde{d}_i, \tilde{d}_i^{+1}])] \right) \bigcup \left( \bigcup_{v \in D_\rho} (\pi \circ p_{|\mathcal{C}_L})^{-1}(v) \right)
\]

where \( v = \pi(\tilde{v}) \).
For convenience in calculation, we parameterize each edge $\tilde{e}^{i}$ linearly by $s \in [0, 1]$ so that $\tilde{e}_h^{i} = 0$, $b(\tilde{e}^{i}) = 1/2$, $\tilde{e}^{i+1} = 1$, and parameterize each edge $c(\tilde{e}^{i})$ linearly by $s \in [0, 1]$ so that $c((s) = 1 - s$ for $s \in [\tilde{e}^{i}]$. We repeatedly use this parametrization.

Now, we describe an equivariant vector bundle over $\mathbb{R}^2/\Lambda$ as an equivariant clutching construction of an equivariant vector bundle over $|\mathcal{K}_\rho|$. Let $V_B$ be a $G_X$ vector bundle over $B$ such that $(\text{res}^G_B V_B)|_{b(\tilde{f})}$ is $\chi$-isotypical at each $b(\tilde{f})$ in $B$.

Especially, $(\text{res}^G_B V_B)|_{b(\tilde{f})}$'s are all isomorphic. Also, if we denote by $V_f$ the isotropy representation of $V_B$ at each $b(\tilde{f})$ in $B$, then $V_B \cong G_X \times_{(G_X)\times_{(f)}} V_f$ because $G_X$ acts transitively on $B$. And, we define $\text{Vect}_{G_X}(\mathbb{R}^2/\Lambda, \chi)_{V_B}$ as the set

$$\{ E \in \text{Vect}_{G_X}(\mathbb{R}^2/\Lambda, \chi) \mid E|_B \cong V_B \}.$$

Similarly, $\text{Vect}_{G_X}(|\mathcal{K}_\rho|, \chi)_{V_B}$ is observed. Observe that $\text{Vect}_{G_X}(|\mathcal{K}_\rho|, \chi)_{V_B}$ has the unique element $[F_{V_B}]$, for the bundle $F_{V_B} = G_X \times_{(G_X)\times_{(f-1)}} (|\tilde{f}^{-1}| \times V_{f-1})$ because $|\mathcal{K}_\rho| \cong G_X \times_{(G_X)\times_{(f-1)}} |\tilde{f}^{-1}|$ is equivariant homotopically equivalent to $B$. Henceforward, we use trivializations

$$|\tilde{f}| \times V_f \quad \text{for} \quad (\text{res}^G_B G_X|_{\tilde{f}} F_{V_B})|_{|\tilde{f}|},$$
$$|\tilde{e}| \times V_f \quad \text{for} \quad (\text{res}^G_B G_X|_{\tilde{e}} F_{V_B})|_{|\tilde{e}|}$$

for each face $\tilde{f}$ and edge $\tilde{e} \in \tilde{f}$. Then, each $E \in \text{Vect}_{G_X}(\mathbb{R}^2/\Lambda, \chi)_{V_B}$ can be constructed by gluing $F_{V_B} \cong \pi^*E$ along edges through

$$|\tilde{e}| \times V_f \to |c(\tilde{e})| \times V_f, \quad (p|_{|\tilde{e}|}((\tilde{e}), u) \mapsto (p|_{|\tilde{e}|}(|c(\tilde{e})), \varphi|_{|\tilde{e}|}(\tilde{e})) u)$$

via some continuous maps

$$\varphi|_{|\tilde{e}|} : |\tilde{e}| \to \text{Iso}(V_f, V_f')$$

for each edge $\tilde{e}, \tilde{e} \in |\tilde{e}|$, $u \in V_f$ where $\tilde{e} = p|_{|\tilde{e}|}(\tilde{e})$ and $\tilde{e} \in \tilde{f}$, $c(\tilde{e}) \in \tilde{f}'$. The union $\Phi = \bigcup \varphi|_{|\tilde{e}|}$ is called an equivariant clutching map of $E$ with respect to $V_B$. When we use the phrase ‘with respect to $V_B$', it is assumed that we use the bundle $F_{V_B}$ and its trivialization (5.1) in gluing. This construction of the bundle is called an equivariant clutching construction, and $E$ is denoted by $E^\Phi$ to stress that it is constructed through $\Phi$. Here, note that $\Phi$ is defined on $p|_{|\tilde{e}|}^* F_{V_B}$ over $|\tilde{L}_p|$. That is why we define $\tilde{L}_p$. Sometimes, we regard $\Phi$ as a map

$$p|_{|\tilde{e}|}^* F_{V_B} \to p|_{|\tilde{e}|}^* F_{V_B}, \quad (\tilde{x}, u) \mapsto (|c(\tilde{x})), \Phi(\tilde{x}) u)$$

by using trivialization (5.1) for each $(\tilde{x}, u) \in |\tilde{e}| \times V_f$ where $\tilde{e} \in \tilde{f}$. Also, note that $\Phi$ should be equivariant, i.e.

$$g \Phi(\tilde{x})^g^{-1} = \Phi(\tilde{x})$$

for all $g \in G_X$, $\tilde{x} \in |\tilde{L}_p|$ because $\Phi$ gives an equivariant vector bundle. An equivariant clutching map of some bundle in $\text{Vect}_{G_X}(\mathbb{R}^2/\Lambda, \chi)_{V_B}$ with respect to $V_B$ is called simply an equivariant clutching map with respect to $V_B$, and let $\Omega_{V_B}$ be the set of all equivariant clutching maps with respect to $V_B$. We also need to restrict an equivariant clutching map in $\Omega_{V_B}$ to $\tilde{L}_p$. We explain for this. Let $\Omega_{\tilde{L}_p, V_B}$ be the set

$$\{ \Phi|_{\tilde{L}_p} \mid \Phi \in \Omega_{V_B} \}.$$

If two equivariant clutching maps coincide on $\tilde{L}_p$, then they are identical by equivariance and definition of one-dimensional fundamental domain. So, the restriction map $\Omega_{V_B} \to \Omega_{\tilde{L}_p, V_B}$, $\Phi \mapsto \Phi|_{\tilde{L}_p}$ is bijective, and we obtain isomorphism
Proposition 6.1. Our classification of the paper is based on these relations. Before it, we state basic homotopic, then $E$-bundle though it need not be an equivariant vector bundle. And, if $\Phi$ also satisfies an equivariant clutching map. We will answer the same question for a preclutching map in $E$. We explain for this more precisely. As we have seen in [Ki], if $\Phi$ satisfies Condition $G$ if and only if there is a $\Phi$-isomorphism with respect to $C$. A function $\Phi$ in $C^0(\hat{\mathcal{V}}(\hat{\mathcal{V}}), \mathcal{V})$ or a function $\Phi_{\hat{\mathcal{V}}}$ in $C^0(\hat{\mathcal{V}}, \mathcal{V})$ is an extension of $\Phi$ if and only if

\begin{align*}
\Phi(\hat{v}_0, -) \cdots \Phi(\hat{v}_j, +) \cdots \Phi(\hat{v}_0, +) = \text{id}
\end{align*}

for $j \in \mathbb{Z}$. We explain for this more precisely. As we have seen in [Ki], if $\Phi$ satisfies Condition N1, N2., then the bundle $E^\Phi$ is well-defined and becomes an inequivariant vector bundle though it need not be an equivariant vector bundle. And, if $\Phi$ also satisfies Condition E1., then $E^\Phi$ becomes an equivariant vector bundle so that $\Phi$ is an equivariant clutching map. We will answer the same question for a preclutching map in $C^0(\hat{\mathcal{V}}), \mathcal{V})$ and only if

\begin{enumerate}
\item $\Phi(\hat{\mathcal{V}}(\hat{\mathcal{V}}), \mathcal{V}) = \Phi(\hat{\mathcal{V}})^{-1}$ for each $\hat{\mathcal{V}} \in \hat{\mathcal{V}}$, $\mathcal{V} \in \mathcal{V}$.
\item For each vertex $\hat{v} \in \hat{K}$, $\mathbb{Z}$, $\Phi(\hat{v}_0, -) \cdots \Phi(\hat{v}_j, +) \cdots \Phi(\hat{v}_0, +) = \text{id}$
\end{enumerate}

We call a map $\Phi$ in $\hat{\mathcal{V}}(\hat{\mathcal{V}}), \mathcal{V})$ or a function $\Phi_{\hat{\mathcal{V}}}$ in $C^0(\hat{\mathcal{V}}, \mathcal{V})$ an extension of $\Phi$ if and only if $\Phi$ is the extension of $\Phi_{\hat{\mathcal{V}}}$. Now, we investigate relations between $\pi_0(\Omega_{\mathcal{V}}), \mathcal{V}, G_{\chi}(\mathcal{V}, \mathcal{V})$, $A_{\chi}(\mathcal{V}, \mathcal{V})$.

Our classification of the paper is based on these relations. Before it, we state basic facts on equivariant vector bundles from the previous paper.

**Proposition 6.1.** For two maps $\Phi$ and $\Phi'$ in $\Omega_{\mathcal{V}}$, if $\Phi$ and $\Phi'$ are equivariantly homotopic, then $E^\Phi$ and $E^{\Phi'}$ are isomorphic equivariant vector bundles.

When we consider a map in $\Omega_{\mathcal{V}}$ as a map defined on $p^*_{\mathcal{V}} \mathcal{V}_{\mathcal{V}}$, we have the following result:

**Proposition 6.2.** For two maps $\Phi$ and $\Phi'$ in $\Omega_{\mathcal{V}}$, $[E^\Phi] = [E^{\Phi'}]$ in $\mathcal{V}, G_{\chi}(\mathcal{V}, \mathcal{V})$ if and only if there is a $G_{\chi}$-isomorphism $\Theta : \mathcal{V} \to \mathcal{V}$ such that $(p^*_{\mathcal{V}} \Theta) \Phi = \Phi'(p^*_{\mathcal{V}} \Theta)$ where $p^*_{\mathcal{V}} \Theta : p^*_{\mathcal{V}} \mathcal{V}_{\mathcal{V}} \to p^*_{\mathcal{V}} \mathcal{V}_{\mathcal{V}}$ is the pull-back of $\Theta$.

\begin{align*}
\begin{array}{ccc}
p^*_{\mathcal{V}} F_{\mathcal{V}} & \xrightarrow{p^*_{\mathcal{V}} \Theta} & p^*_{\mathcal{V}} F_{\mathcal{V}} \\
\downarrow & & \downarrow \\
p^*_{\mathcal{V}} F_{\mathcal{V}} & \xrightarrow{p^*_{\mathcal{V}} \Theta} & p^*_{\mathcal{V}} F_{\mathcal{V}}
\end{array}
\end{align*}
Consider the map \( \iota \colon \pi_0(\Omega_{V_B}) \to \text{Vect}_{G_\chi}(\mathbb{R}^2/\Lambda, \chi) \) mapping \([\Phi]\) to \([E^\Phi]\). This is well defined by Proposition 6.1. And, the map \( p_\Omega \colon \pi_0(\Omega_{V_B}) \to \text{A}_{G_\chi}(\mathbb{R}^2/\Lambda, \chi) \) defined as \( p_\Omega = p_{\text{vect}} \circ \iota \) satisfies the following diagram:

\[
\begin{array}{ccc}
\pi_0(\Omega_{V_B}) & \xrightarrow{\iota} & \text{Vect}_{G_\chi}(\mathbb{R}^2/\Lambda, \chi) \\
\xrightarrow{p_\Omega} & & \xrightarrow{p_{\text{vect}}} \\
A_{G_\chi}(\mathbb{R}^2/\Lambda, \chi) & & \\
\end{array}
\]

We might consider the map \( p_\Omega \) as defined also on \( \pi_0(\Omega_{\bar{D}_p, V_B}) \) because \( \pi_0(\Omega_{V_B}) \cong \pi_0(\Omega_{\bar{D}_p, V_B}) \). By using \( p_\Omega \), we would decompose \( \Omega_{V_B} \). Let \( \pi_\nu : \Omega_{\bar{D}_p, V_B} \to \pi_0(\Omega_{\bar{D}_p, V_B}) \) be the natural quotient map. For different elements in \( \text{A}_{G_\chi}(\mathbb{R}^2/\Lambda, \chi) \), their preimages through \((p_\Omega \circ p_\pi^{-1})\) do not intersect so that we obtain a decomposition of \( \Omega_{D_p, V_B} \). We describe this decomposition more precisely. For each \((W_d)_{i \in I^+_p} \in \text{A}_{G_\chi}(\mathbb{R}^2/\Lambda, \chi)\), put

\[
V_B = G_\chi \times_{(G_\chi)_{d-1}} W_{d-1}, \quad F_{V_B} = G_\chi \times_{(G_\chi)_{d-1}} ([f^{-1}] \times W_{d-1}),
\]

and by using these define \( \Omega_{D_p, (W_d)_{i \in I^+_p}} \) as \((p_\Omega \circ p_\pi^{-1})((W_d)_{i \in I^+_p})\). Henceforward, we will use these \( V_B \) and \( F_{V_B} \) when we deal with \( \Omega_{D_p, (W_d)_{i \in I^+_p}} \). Then, given a bundle \( V_B \), the set \( \Omega_{D_p, V_B} \) is equal to the disjoint union

\[
\bigcup_{(W_d)_{i \in I^+_p} \in \text{A}_{G_\chi}(\mathbb{R}^2/\Lambda, \chi) \text{ with } W_{d-1} = f_{d-1}} \Omega_{D_p, (W_d)_{i \in I^+_p}}.
\]

From this decomposition, it suffices to focus on \( \Omega_{D_p, (W_d)_{i \in I^+_p}} \) to understand \( \Omega_{D_p, V_B} \).

In general, calculation of \( \pi_0(\Omega_{D_p, V_B}) \) do not give classification of equivariant vector bundles even for \( S^2 \). However, in many cases it gives classification of equivariant vector bundles as we have seen in [K]. Let \( c_1 : \pi_0(\Omega_{V_B}) \to H^2(\mathbb{R}^2/\Lambda), \Phi \mapsto c_1(E^\Phi) \) be the Chern class.

**Lemma 6.3.**

1. Assume that \( \pi_0(\Omega_{D_p, (W_d)_{i \in I^+_p}}) \) is nonempty for each \((W_d)_{i \in I^+_p} \in \text{A}_{G_\chi}(\mathbb{R}^2/\Lambda, \chi)\), and that \( c_1 : \pi_0(\Omega_{D_p, (W_d)_{i \in I^+_p}}) \to H^2(\mathbb{R}^2/\Lambda) \) is injective for each \((W_d)_{i \in I^+_p} \in \text{A}_{G_\chi}(\mathbb{R}^2/\Lambda)\). Then, \( p_{\text{vect}} \) is surjective, and

\[
p_{\text{vect}} \times c_1 : \text{Vect}_{G_\chi}(\mathbb{R}^2/\Lambda, \chi) \to \text{A}_{G_\chi}(\mathbb{R}^2/\Lambda, \chi) \times H^2(\mathbb{R}^2/\Lambda)
\]

is injective.

2. Assume that \( \pi_0(\Omega_{D_p, (W_d)_{i \in I^+_p}}) \) consists of exactly one element for each \((W_d)_{i \in I^+_p} \in \text{A}_{G_\chi}(\mathbb{R}^2/\Lambda, \chi)\). Then

\[
p_{\text{vect}} : \text{Vect}_{G_\chi}(\mathbb{R}^2/\Lambda, \chi) \to \text{A}_{G_\chi}(\mathbb{R}^2/\Lambda, \chi)
\]

is isomorphic.

By this lemma, we only have to calculate \( \pi_0(\Omega_{D_p, (W_d)_{i \in I^+_p}}) \) to classify equivariant vector bundles in many cases. When we can not apply this lemma, we should apply Proposition 6.2 directly. In fact, we can apply this lemma except the case when \( \rho(G_\chi) \) is equal to \( D_1 \) with \( A\bar{c} + \bar{c} = l_0 \) as we shall see in the below.
7. EQUIVARIANT POINTWISE CLUTCHING MAP

Let $\Phi$ be an equivariant clutching map of an equivariant vector bundle $E$ in $\text{Vect}_{G_{\chi}}(\mathbb{R}^2/L)$, i.e., the map $\Phi$ glues $F_{V_0}$ along $|L_\rho|$ to give $E$. Let us investigate this gluing process pointwise. For each $x \in |L_\rho|$ and $x = \pi(x)$, let $x = \pi^{-1}(x) = \{x_j | j \in \mathbb{Z}_m\}$ for some $m$. Then, the map $\Phi$ glues the $(G_{\chi})_x$-bundle $(\text{res}_{(G_{\chi})_x}) E_x \big|_x$ to give the $(G_{\chi})_x$-representation $E_x$, and call this process equivariant pointwise gluing. Here, we can observe that $(G_{\chi})_x < (G_{\chi})_z$ for each $j \in \mathbb{Z}_m$ and

$$\text{res}_{(G_{\chi})_x} E_x \cong (F_{V_0})_{x_j} \quad (7.1)$$

by equivariance of $\Phi$. In dealing with equivariant clutching maps, technical difficulties occur in equivariant pointwise gluings because $\Phi$ is just a continuous collection of equivariant pointwise gluings at points in $|L_\rho|$. In this section, we review the concept and results of equivariant pointwise clutching map from the previous paper [K1], and supplement these with two more cases. To deal with equivariant pointwise gluing, we need the concept of representation extension. For compact Lie groups $N_1 < N_2$, let $V$ be a $N_2$-representation and $W$ be an $N_1$-representation. Then, $V$ is called an representation extension or $N_2$-extension of $W$. For example, $E_x$ is a $(G_{\chi})_x$-extension of $(F_{V_0})_{x_j}$ for each $j \in \mathbb{Z}_m$ by (7.1). And, let $\text{ext}^N_{N_1} W$ be the set

$$\{V \in \text{Rep}(N_2) \mid \text{res}_{N_1}^N V \cong W\}.$$ 

Let a compact Lie group $N_2$ act on a finite set $x = \{x_j | j \in \mathbb{Z}_m\}$ for $m \geq 2$, and let $N_0$ and $N_1$ be the kernel of the action and the isotropy subgroup $(N_2)_x$ of $N_0$, respectively. Let $F$ be an $N_2$-vector bundle over $x$. Assume that $(\text{res}_{N_0}^N F)|_{x_j}$'s are all isomorphic (not necessarily $N_0$-isotypical) for $j \in \mathbb{Z}_m$. Consider an arbitrary map

$$\psi : x \to \Pi_{j \in \mathbb{Z}_m} \text{Iso}(F_{x_j}, F_{x_{j+1}})$$

such that $\psi(x_j) \in \text{Iso}(F_{x_j}, F_{x_{j+1}})$. Call such a map pointwise preclutching map with respect to $F$. By using $\psi$, we glue $F_{x_j}$'s, i.e., a vector $u$ in $F_{x_j}$ is identified with $\psi(x_j)u$ in $F_{x_{j+1}}$. Let $F/\psi$ be the quotient of $F$ through this identification, and let $p_\psi : F \to F/\psi$ be the quotient map. Let $\iota_\psi : F_{x_0} \to F/\psi$ be the composition of the natural injection $\iota_{x_0} : F_{x_0} \to F$ and the quotient map $p_\psi$.

$$\text{(7.2)}$$

We would find conditions on $\psi$ under which the quotient $F/\psi$ inherits an $N_2$-representation structure from $F$ and the map $\iota_\psi$ becomes an $N_1$-isomorphism from $F_{x_0}$ to $\text{res}_{N_1}^N(F/\psi)$. For notational simplicity, denote

$$\psi(x_j) \cdots \psi(x_{j+1}) \psi(x_j)u$$

by $\psi^{j'-j+1}u$ for $u \in F_{x_j}$ and $j \leq j'$ in $\mathbb{Z}$.

**Lemma 7.1.** For a pointwise preclutching map $\psi$ with respect to $F$, the quotient $F/\psi$ carries an $N_2$-representation structure so that $p_\psi$ is $N_2$-equivariant and the map $\iota_\psi$ is an $N_1$-isomorphism if and only if the following conditions hold:

1. $\psi^m = \text{id}$ in $\text{Iso}(F_{x_j})$ for each $j \in \mathbb{Z}_m$. So, $\psi$ is well defined for all $j \in \mathbb{Z}_m$.
2. $\psi^{j_2-j_1} = g \psi(x_{j_1})g^{-1}$ in $F_{x_{j_1}}$ for each $j_1 \in \mathbb{Z}_m$, $g \in N_2$ when $g^{-1}x_{j_1} = x_{j_2}$ and $g x_{j_2+1} = x_{j_3}$ for some $j_2, j_3 \in \mathbb{Z}_m$. 


A pointwise preclutching map satisfying conditions (1), (2) of Lemma 7.5 is called an equivariant pointwise clutching map with respect to \( F \). Let \( A \) be the set of all equivariant pointwise clutching maps with respect to \( F \), and an \( N_2 \)-representation \( W \) is called determined by \( \psi \in A \) with respect to \( F \) if \( W \cong F/\psi \).

**Remark 7.2.** Here, we give an example of equivariant pointwise clutching map by using the bundle \( F_{V_{\rho}} \) over \( |L_p| \) of Section 5. For each \( \bar{x} \in |L_p| \) and \( x = \pi(\bar{x}) \), let \( \bar{\mathfrak{K}} = \pi^{-1}(\bar{x}) = \{ \bar{x}_j | j \in \mathbb{Z}_m \} \) for some \( m \). Put \( F = (\text{res}^G_{H_2} F_{V_{\rho}})_{|\bar{\mathfrak{K}}} \). Since \( (\text{res}^G_{H_2} F_{V_{\rho}})_{|\bar{\mathfrak{K}}} \)'s are all isomorphic, \( (\text{res}^N_{H_2} F)_{|\bar{\mathfrak{K}}} \)'s are all isomorphic. Given a \( \Phi \) in \( \Omega_{V_{\rho}} \), we can define a map \( \psi \in A \) by using \( \Phi \) as follows:

1. If \( \bar{x} \) is not a vertex and \( \bar{x} = p|L_p(\bar{x}) \), then \( \psi(\bar{x}_0) = \Phi(\bar{x}) \) and \( \psi(\bar{x}_1) = \Phi(\pi(\bar{x})) \).

2. If \( \bar{x} \) is a vertex, then
   \[
   \psi_{\bar{x}}(\bar{x}_j) = \Phi(\bar{x}_{j,+}) \quad \text{and} \quad \psi_{\bar{x}}^{-1}(\bar{x}_j) = \Phi(\bar{x}_{j,-})
   \]
   for each \( j \in \mathbb{Z}_{j_{\rho}} \).

Then, \( (\text{ext}^2)^p \) is determined by \( \psi \).

To calculate \( \pi_0(\Omega_{V_{\rho}}) \) later, we need to understand topology of \( A \) because an equivariant clutching map is a continuous collection of equivariant pointwise gluings. First, the zeroth homotopy of \( \Omega \) is related to \( \text{ext}^N_{N_1} F_{Z_2} \) as follows:

**Proposition 7.3.** Assume that \( N_2 \) acts transitively \( \mathfrak{K} \). Then, the map
\[
\pi_0(\Omega) \to \text{ext}^N_{N_1} F_{Z_2}, \quad [\psi] \mapsto F/\psi
\]
is bijective.

**Remark 7.4.** In Lemma 7.1 and Proposition 7.5, 7.6, we show that Proposition 7.3 also holds for the following cases:

1. \( N_2 = N_1 = N_0 \) with \( m = 2, 4 \),
2. \( N_2/N_0 \cong \mathbb{Z}_2 \) and \( N_2/N_0 \) acts freely on \( \mathfrak{K} \) with \( m = 4 \).

Let \( A^j \) be the set
\[
\{ \psi(\bar{x}_j) \mid \psi \in A \}
\]
and let \( A^{j,j'} \) be the set
\[
\{ (\psi(\bar{x}_j), \psi(\bar{x}_{j'})) \mid \psi \in A \}.
\]
In the below, it will be witnessed that \( A \) is homeomorphic to \( A^j \) or \( A^{j,j'} \) in many cases in the below. For example, the evaluation map
\[
\mathcal{A} \to A^j \to A^{j,j'}, \quad \psi \mapsto \psi(\bar{x}_j)
\]
is homeomorphic by Lemma 7.1(1) when \( m = 2 \). Meanwhile, we also need to restrict our arguments on \( \mathcal{A} \) to \( \{ \bar{x}_j, \bar{x}_{j'} \} \) as follows: let \( A^{j,j'} \), with \( j \neq j' \) be the set of equivariant pointwise clutching maps with respect to the \( (N_2)_{|\{\bar{x}_j, \bar{x}_{j'}\}} \)-bundle \( (\text{res}^N_{H_2} F_{\{\bar{x}_j, \bar{x}_{j'}\}})_{|\{\bar{x}_j, \bar{x}_{j'}\}} \) where \( (N_2)_{|\{\bar{x}_j, \bar{x}_{j'}\}} \) is the subgroup preserving \( \{\bar{x}_j, \bar{x}_{j'}\} \).

And, let \( A^{j,j'}_k \) for \( k = j, j' \) be the set
\[
\{ \psi(\bar{x}_k) \mid \psi \in A^{j,j'} \}.
\]
Then, we obtain a useful lemma.

**Lemma 7.5.** The map \( \text{res}_{j,j'} : \mathcal{A} \to A^{j,j'} \) with \( \psi \mapsto \text{res}_{j,j'}(\psi) \) is
\[
\text{res}_{j,j'}(\psi)(\bar{x}_j) = \psi^{-j}(\bar{x}_j), \quad \text{res}_{j,j'}(\psi)(\bar{x}_{j'}) = \psi^{j-j'}(\bar{x}_{j'})
\]
is well defined. And,
\[ \text{res}_{N_{2}}(x_{j},x_{j'})^{F/\psi} \cong \text{res}_{N_{2}}(x_{j},x_{j'})^{F}(x_{j},x_{j'}) / \text{res}_{j,j'}(\psi) \]
for each \( \psi \in A \).

**Proof.** The injection from \( \text{res}_{N_{2}}(x_{j},x_{j'})^{F}(x_{j},x_{j'}) \) to \( F \) induces the isomorphism through \( p_{\text{res}_{j,j'}(\psi)} \) and \( p_{\psi} \).

For \( \psi \in A \), denote by \( A_{\psi} \) and \( A_{\psi}^{0} \) the path component of \( A \) containing \( \psi \) and the set \( \{ \psi'(\bar{x}_{0})|\psi' \in A_{\psi} \} \), respectively. Then, each path component is expressed by isomorphism groups.

**Lemma 7.6.** For each \( \psi \in A \), \( A_{\psi} \) is homeomorphic to
\[ \text{ISO}_{N_{2}}(F/\psi) / \text{ISO}_{N_{2}}(F/\psi). \]

In some special cases, we can understand \( A \) and \( A_{\psi} \) more precisely.

**Lemma 7.7.** If \( m = 2 \), then \( A \) is homeomorphic to \( A_{\psi}^{0} \). Moreover, if \( N_{2} = N_{1} = N_{0} \), then \( A \) is homeomorphic to \( A_{\psi}^{0} = \text{ISO}_{N_{2}}(F_{\bar{x}_{0}},F_{\bar{x}_{1}}) \).

**Proposition 7.8.** Assume that \( N_{2} = \langle N_{0},a_{0} \rangle \) with some \( a_{0} \in N_{2} \) such that \( a_{0} \bar{x}_{j} = \bar{x}_{j+1} \) for each \( j \in \mathbb{Z}_{m} \) so that \( N_{2}/N_{0} \cong \mathbb{Z}_{m} \) and \( N_{1} = N_{0} \). Then,
1. A pointwise preclutching map \( \psi \) with respect to \( F \) is in \( A \) if and only if \( \psi^{m} = \text{id}, \psi(\bar{x}_{0}) \in \text{ISO}_{N_{0}}(F_{\bar{x}_{0}},F_{\bar{x}_{1}}), \psi(\bar{x}_{j}) = a_{0}^{j}\psi(\bar{x}_{0})a_{0}^{-j} \) for each \( j \in \mathbb{Z}_{m} \).
2. \( A, A_{\psi} \) are homeomorphic to \( A_{\psi}^{0}, A_{\psi}^{0} \), respectively.
3. If \( F_{\bar{x}_{0}} \) is \( N_{0} \)-isotypical, then \( A_{\psi} \) is simply connected for each \( \psi \) in \( A \).

**Proposition 7.9.** Assume that \( N_{2} = \langle N_{0},a_{0},b_{0} \rangle \) with some \( a_{0},b_{0} \in N_{2} \) such that \( a_{0} \bar{x}_{j} = \bar{x}_{j+1} \) and \( b_{0} \bar{x}_{j} = \bar{x}_{j-1} \) for each \( j \in \mathbb{Z}_{m} \) so that \( N_{2}/N_{0} \cong \mathbb{Z}_{2} \times \mathbb{Z}_{2} \) and \( N_{1} = N_{0} \). Then, \( A \) is homeomorphic to \( A_{\psi}^{0.3} \).

For two points \( \bar{x}_{j}, \bar{x}_{j'} \) in \( x \), \( \bar{x}_{j} \sim \bar{x}_{j'} \) means that \( \bar{x}_{j} \) and \( \bar{x}_{j'} \) are in an \( N_{2} \)-orbit.

**Proposition 7.10.** Assume that \( N_{2} = \langle N_{0},\alpha_{1},\alpha_{2},\alpha_{3} \rangle \) with some \( \alpha_{j} \)'s in \( N_{2} \) such that \( \alpha_{1} \bar{x}_{j} = \bar{x}_{j+1}, \alpha_{2} \bar{x}_{j} = \bar{x}_{j+2}, \alpha_{3} \bar{x}_{j} = \bar{x}_{j+3} \) for each \( j \in \mathbb{Z}_{m} \) with \( m = 4 \) so that \( N_{2}/N_{0} \cong \mathbb{Z}_{2} \times \mathbb{Z}_{2} \) and \( N_{1} = N_{0} \). Then, \( A \) is homeomorphic to \( A_{\psi}^{0.3} \).

**Proposition 7.11.** Suppose that \( N_{2}/N_{0} \cong \mathbb{Z}_{2} \) and \( N_{2}/N_{0} \) acts freely on \( x \) with \( m = 4 \).

1. If \( \bar{x}_{j} \sim \bar{x}_{j'} \) and \( \bar{x}_{j'} \sim \bar{x}_{j''} \) for some \( j \neq j' \neq j'' \), then
\[ \text{res}_{j,j'} \times \text{res}_{j',j''} : A \rightarrow A_{j,j'} \times A_{j',j''} \]
is homeomorphic.
2. \( A_{j,j'} \times A_{j',j''} \) is \( N_{0} \)-isotypical, then path components of \( A_{j,j'} \) are simply connected and \( \pi(A_{\psi}) \cong \mathbb{Z} \) for each \( \psi \in A \).
3. \( F/\psi \cong (\text{res}_{N_{2}}^{2}(x_{j},x_{j'}),F)(x_{j},x_{j'}) / \text{res}_{j,j'}(\psi) \) for any \( \psi \in A \).

**Proof.** For simplicity, assume that \( N_{2} = \langle N_{0},a_{0} \rangle \) with some \( a_{0} \in N_{2} \) such that \( a_{0} \bar{x}_{j} = \bar{x}_{j+2} \) for each \( j \in \mathbb{Z}_{4} \). And, we prove this proposition only for \( j = 3, j' = 1, j'' = 0 \). Then, it suffices to show that
\[ p : A \rightarrow A_{3,1}^{3} \times A_{1,0}^{0}, \psi \mapsto \left( \psi(\bar{x}_{0})\psi(\bar{x}_{3}),\psi(\bar{x}_{0}) \right) \]
is homeomorphic to show (1) by Lemma 7.1. First, we show injectivity. Given 
\( \psi(\bar{x}_0)\psi(\bar{x}_1) \) and \( \psi(\bar{x}_3) \), i.e. \( \psi(\bar{x}_0) \) and \( \psi(\bar{x}_3) \), equivariance of \( \psi \) gives us
\[
\psi(\bar{x}_1) = \psi(a_0\bar{x}_3) = a_0\psi(\bar{x}_3)a_0^{-1}.
\]
By \( \psi^4 = id \), \( \psi(\bar{x}_2) \) is also obtained. So, we obtain injectivity. Next, we show surjectivity. Given two arbitrary elements \( A \in \mathcal{A}_0 \) and \( A' \in \mathcal{A}_1 \), \( A^{-1}A' \) is \( N_0 \)-equivariant. We would construct \( \psi \) by Lemma 7.7. By Schur’s Lemma, \( A \equiv \psi \). We would construct \( \psi \) \( A \) s.t. \( \psi(\bar{x}_0) = A \) and \( \psi(\bar{x}_3) = A^{-1}A' \). Define pointwise preclutching map \( \psi \) as
\[
\psi(\bar{x}_0) = A, \\
\psi(\bar{x}_1) = a_0\psi(\bar{x}_3)a_0^{-1}, \\
\psi(\bar{x}_2) = a_0\psi(\bar{x}_0)a_0^{-1}, \\
\psi(\bar{x}_3) = A^{-1}A'.
\]
To show \( \psi \in \mathcal{A} \), we only have to show that
\[
\psi(\bar{x}_0)\psi(\bar{x}_3)\psi(\bar{x}_2)\psi(\bar{x}_1)\psi(\bar{x}_0) = id.
\]
by Lemma 7.1. This is equivalent to
\[
(\ast)
\]
\( \psi(\bar{x}_0)\psi(\bar{x}_3)\psi(\bar{x}_1)\psi(\bar{x}_0) = id. \)
Here, \( \psi(\bar{x}_0)\psi(\bar{x}_3) = A' \) and
\[
\psi(\bar{x}_2)\psi(\bar{x}_1) = a_0\psi(\bar{x}_0)a_0^{-1}, \quad a_0\psi(\bar{x}_3)a_0^{-1} = a_0A'a_0^{-1}.
\]
Since \( A' \in \mathcal{A}_1 \), \( a_0A'a_0^{-1} = A'^{-1} \) by Proposition 7.3(2) so that we obtain \( \ast \). Therefore, we obtain a proof of (1).

Next, we prove (2). Since \( (N_2)_{\{x_1, x_0\}} = N_0 \), we obtain \( \mathcal{A}_0 = \text{Iso}_{N_2}(F_{x_0}, F_{x_1}) \) by Lemma 7.7. By Schur’s Lemma, \( \mathcal{A}_1 \) is path connected and \( \pi_1(\mathcal{A}_1) \cong \mathbb{Z} \). And, since \( (N_2)_{\{x_3, x_1\}} = N_2 \), simply connectedness of each path component of \( \mathcal{A}_1 \) is obtained by Proposition 7.1. Therefore, we obtain a proof of \( \pi(\mathcal{A}_0) \cong \mathbb{Z} \) by (1).

Last, (3) follows from Lemma 7.3.

\[\square\]

**Proposition 7.12.** Assume that \( g\bar{x}_j = \bar{x}_j \) for each \( g \in N_2, j \in \mathbb{Z}_4 \), i.e. \( N_2 = N_0 \). Then,

1. \( \mathcal{A} \) is equal to
   \[
   \{ \psi \mid \psi(\bar{x}_j) \in \text{Iso}_{N_0}(F_{x_j}, F_{x_{j+1}}) \text{ for each } j \in \mathbb{Z}_4, \text{ and } \psi^4 = id \}.
   \]
   And, \( p : \mathcal{A} \to \mathcal{A}_{j,j+1} \times \mathcal{A}_{j',j'+1} \times \mathcal{A}_{j'',j''+1}, \psi \mapsto (\psi(\bar{x}_j), \psi(\bar{x}_j'), \psi(\bar{x}_j'')) \) is bijective for \( j \neq j' \neq j'' \). Here, \( \mathcal{A}_{j,j+1} = \text{Iso}_{N_0}(F_{x_j}, F_{x_{j+1}}) \).

2. If \( F \) is \( N_0 \)-isotypical, then \( \pi_1(\mathcal{A}) \cong \mathbb{Z}^3 \).

**Proof.** Since \( N_2 \) fixes all \( \bar{x}_j \)'s, we have \( N_2 = N_0 \) so that Lemma 7.1 gives \( \mathcal{A}_{j,j+1} = \text{Iso}_{N_0}(F_{x_j}, F_{x_{j+1}}) \). This and Lemma 7.3 proves (1). That is, \( \psi(\bar{x}_j) \) in \( \mathcal{A}_{j,j+1} \) is \( \text{Iso}_{N_0}(F_{x_j}, F_{x_{j+1}}) \) for three \( j \)'s are free and the remaining one is restricted by \( \psi^4 = id \).

Schur’s Lemma and (1) give a proof of (2).

\[\square\]

Since any \( \psi \) in \( \mathcal{A} \) glues all fibers of \( F \) to obtain a single vector space \( F/\psi \), \( \psi \) might be considered to glue each pair of fibers of \( F \). That is, \( \psi \) determines the function \( \bar{\psi} \) defined on \( x \times x - \Delta \) sending a pair \( (x, x') \) to the element \( \bar{\psi}(x, x') \) in \( \text{Iso}(F_x, F_{x'}) \) such that each \( u \) in \( F_x \) is identified with \( \bar{\psi}(x, x')u \) in \( F_{x'} \) by \( \psi \), i.e. \( \psi(x, x') \) satisfies \( p_0(u) = p_0(\bar{\psi}(x, x')u) \) where \( \Delta \) is the diagonal. Call \( \bar{\psi} \) the *saturation* of \( \psi \). Since the index \( j \) is not used in defining \( \bar{\psi} \), it is often convenient to use \( \bar{\psi} \) instead of
ψ. Denote by \( A \) the set \( \{ \psi \mid \psi \in A \} \), and call it the saturation of \( A \). And, denote \( F/\psi, \psi_0 \) also by \( F/\overline{\psi}, \overline{\psi}_0 \), respectively.

8. A Lemma on Fundamental Groups

In this section, we recall two lemmas needed to calculate homotopy of equivariant clutching maps from \( [K] \). One is just a rewriting of Schur’s Lemma. Another is on relative homotopy.

**Lemma 8.1.** For \( \chi \in \text{Irr}(H) \), let \( W \) be a \( \chi \)-isotypical \( H \)-representation. For the natural inclusion \( \iota : \text{Iso}_H(W) \to \text{Iso}(W) \), the map

\[
i_* : \pi_1(\text{Iso}_H(W)) \to \pi_1(\text{Iso}(W))
\]

is equal to the multiplication by \( \chi(id) \) up to sign between two \( Z \)'s.

Here, we recall a notation.

**Notation 8.2.** Let \( X \) be a topological space. For two points \( y_0 \) and \( y_1 \) in \( X \) and a path \( \gamma_0 : [0, 1] \to X \) such that \( \gamma_0(0) = y_0 \) and \( \gamma_0(1) = y_1 \), denote by \( \overline{\gamma_0} \) the function defined as

\[
\overline{\gamma}_0 : \pi_1(X, y_0) \to \pi_1(X, y_1), \quad \sigma \mapsto \gamma_0^{-1} \cdot \sigma \cdot \gamma_0
\]

for \( \sigma \in \pi_1(X, y_0) \).

**Lemma 8.3.** Let \( X \) be a path connected topological space with an abelian \( \pi_1(X) \). Let \( A \) and \( B \) be path connected subspaces of \( X \). Also, let \( t_0 \) and \( t_1 \) denote inclusions from \( A \) and \( B \) to \( X \), respectively. Pick two points \( y_0 \in A \) and \( y_1 \in B \), and a path \( \gamma_0 : [0, 1] \to X \) such that \( \gamma_0(0) = y_0 \) and \( \gamma_0(1) = y_1 \). Then, we have an isomorphism

\[
\Pi : \pi_1(X, y_1)/\{ \overline{\gamma}_0(t_0 \ast \overline{\pi}_1(A, y_0)) + t_1 \ast \overline{\pi}_1(B, y_1) \} \to [[0, 1], 0, 1; X, A, B],
\]

\[
[s] \mapsto [\gamma_0 \cdot s].
\]

9. Equivariant Clutching Maps on One-Dimensional Fundamental Domain

In this section, we find conditions on a preclutching map \( \Phi_{\hat{D}_\rho} \) in \( C^0(\hat{D}_\rho, V_B) \) to guarantee that \( \Phi_{\hat{D}_\rho} \) be the restriction of an equivariant clutching map. By using this, we show that \( \Omega_{\hat{D}_\rho, (W_{\hat{d}_1})_{\in I\hat{I}^*}} \) is nonempty for each \( (W_{\hat{d}_1})_{\in I\hat{I}^*} \in A_{G_\chi}(\mathbb{R}^2/\Lambda, \chi) \).

For these, we define notations on equivariant pointwise clutching maps with respect to the \( (G_\chi)_x \)-bundle \( (\text{res}^G_{G_\chi}(x), F_{V_B})\{x \}^{-1}(z) \) for each \( x \in |\hat{L}_\rho| \) and \( x = \pi(\hat{x}) \), and prove some lemmas on them. Then, we can apply results of Section 7 in dealing with \( \Omega_{\hat{D}_\rho,(W_{\hat{d}_1})_{\in I\hat{I}^*}} \). Concrete calculation of homotopy of equivariant clutching maps for each case is done from the next section.

Henceforth, assume that \( R = \rho(G_\chi) \) for some \( R \) in Table 7.1. First, we define the set \( A_x \) of equivariant pointwise clutching maps for each \( \hat{x} \) in \( |\hat{L}_\rho| \). For each \( \hat{x} \) in \( \hat{L}_\rho \) and \( x = \pi(\hat{x}) \), put \( \mathbf{x} = \pi^{-1}(x) = \{ \hat{x}_j | j \in \mathbb{Z}_m \} \) for \( m = j_\rho \) or 2 according to whether \( \hat{x} \) is a vertex or not. Then, let \( A_x \) be the set of equivariant pointwise clutching maps with respect to the \( (G_\chi)_x \)-bundle \( (\text{res}^G_{G_\chi}(x), F_{V_B})\{x \}^{-1}(z) \). Here, we need to explain for codomain of maps in \( A_x \). For each \( \hat{x} \in |\hat{L}_\rho| \) and \( \psi_{\hat{x}} \in A_{\hat{x}}, \psi_{\hat{x}}(\hat{x}_j) \) is in

\[
\text{Iso} \left( (F_{V_B})_{\hat{x}_j}, (F_{V_B})_{\hat{x}_{j+1}} \right)
\]

for \( j \in \mathbb{Z}_{j_\rho} \) or \( \mathbb{Z}_2 \). If \( \hat{x}_j \in [f] \) and \( \hat{x}_{j+1} \in [f'] \) for some \( f \) and \( f' \), then \( \psi_{\hat{x}}(\hat{x}_j) \) is henceforth regarded as in \( \text{Iso}(V_f, V_{f'}) \). This is justified because we have fixed the trivialization \( (\text{res}^G_{G_\chi}(f), F_{V_B})\{f \}^{-1}(z) \) for each face \( \hat{f} \in \hat{K}_\rho \). We define one
more set $\mathcal{A}_e$ of equivariant pointwise clutching maps for each edge $\tilde{e}$ and its images $\tilde{e} = p_\mathcal{E}(\tilde{e})$, $|e| = \pi(|\tilde{e}|)$. For each $\tilde{x}$ in $|e|$ and $\tilde{x} = p_\mathcal{E}(\tilde{x})$, put

$$x'_0 = \tilde{x}, \quad x'_1 = p_\mathcal{L}(|e|)(\tilde{x}), \quad \text{and} \quad \mathcal{x}' = \{x'_j | j \in \mathbb{Z}_2\}.$$ 

Consider the set $\mathcal{A}_e$ of equivariant pointwise clutching maps with respect to the $(G_\chi)|e|$-bundle $(\text{res}_{G_\chi}|e|) F_{V\chi}|\mathcal{x}'$ where $(G_\chi)|e|$ is the subgroup of $G_\chi$ fixing $|e|$. As for $\mathcal{A}_x$, each map $\psi_x$ in $\mathcal{A}_x$ is considered to satisfy

$$\psi_x(x'_0) \in \text{Iso}(V_f, V_f) \quad \text{and} \quad \psi_x(x'_1) \in \text{Iso}(V_f, V_f)$$

when $\tilde{x}_0' \in |f|$ and $\tilde{x}_1' \in |f'|$ for some $f$ and $f'$. Here, observe that $\mathcal{A}_x$ is in one-to-one correspondence with $\mathcal{A}_g$ for any two $\tilde{x}, \tilde{y}$ in $|e|$, i.e. an element $\psi_{\tilde{x}}$ in $\mathcal{A}_x$ and an element $\psi_{\tilde{y}}$ in $\mathcal{A}_g$ are corresponded to each other when $\psi_{\tilde{x}}(\tilde{x}) = \psi_{\tilde{y}}(\tilde{y})$ for $j \in \mathbb{Z}_2$.

This is because the $(G_\chi)|e|$-bundle $(\text{res}_{G_\chi}|e|) F_{V\chi}|\mathcal{x}'$ is all isomorphic regardless of $\tilde{x} \in |e|$. It is very useful to identify all $\mathcal{A}_x$'s for $\tilde{x} \in |e|$ in this way, so we denote the identified set by $\mathcal{A}_e$. That is, each element $\psi_x$ in $\mathcal{A}_e$ is considered as contained to $\mathcal{A}_x$ for any $\tilde{x} \in \hat{e}$ according to the context.

Next, we would define a $G_\chi$-action on saturations. First, we define notations on saturations. For each $x = \pi(\tilde{x})$ and $\psi_x \in \mathcal{A}_x$, denote saturations of $\mathcal{A}_x$ and $\psi_x$ by $\mathcal{A}_x$ and $\mathcal{A}_x$, respectively. Since index set is irrelevant in defining $\mathcal{A}_x$, the saturation depends not on $\tilde{x}$ but on $x$. This is why we use the subscript $x$ instead of $\tilde{x}$. For any $g \in G_\chi$, the function $g \cdot \psi_x$ is contained in $\mathcal{A}_x$ where $g \cdot \psi_x$ is defined as

$$(g \cdot \psi_x)(\tilde{y}, \tilde{y}') = g\psi_x(g^{-1}\tilde{y}, g^{-1}\tilde{y}')g^{-1}$$

for any $\tilde{y} \neq \tilde{y}'$ in $\pi^{-1}(gx)$. That is, we obtain $g\mathcal{A}_x = \mathcal{A}_{gx}$. Especially, it is easily shown that $g \cdot \psi_x = \psi_x$ for each $g \in (G_\chi)|e|$ by equivariance of $\psi_x$ so that $\mathcal{A}_{gx} = \mathcal{A}_x$. From this, it is noted that if $g'x = gx$ for some $g', g \in G_\chi$, then $g' \cdot \psi_x = g \cdot \psi_x$ because $g' = g(g^{-1}g')$ with $g^{-1}g' \in (G_\chi)|e|$, i.e. $g \cdot \psi_x$ is determined by not $g$ but $gx$. We have defined a $G_\chi$-action on saturations. Since $\mathcal{A}_x$ and $\mathcal{A}_x$ are in one-to-one correspondence, $\mathcal{A}_x$'s also deliver a $G_\chi$-action induced from $G_\chi$-actions on $\mathcal{A}_x$'s, i.e. $g \cdot \psi_x \in \mathcal{A}_{gx}$ for each $\psi_x \in \mathcal{A}_x$ is defined and $\mathcal{A}_{gx} = g\mathcal{A}_x$. Here, we prove a useful lemma on this action. Before it, we state an elementary fact.

**Lemma 9.1.** Let $G$ be a compact Lie group acting on a topological space $X$. And, let $E$ be an equivariant vector bundle over $X$. Then, $gE_x \cong E_{gx}$ for each $g \in G$ and $x \in X$. Also,

$$\text{res}_{G_\chi \cap G_x} G_x E_x \cong \text{res}_{G_\chi \cap G_x} G_x E_x$$

for any two points $x, x'$ in $X$ which are in the same component of the fixed set $X^{G_x \cap G_x'}$.

Here, we explain for the subscript $g$.

**Definition 9.2.** Let $K$ be a closed subgroup of a compact Lie group $G$. For a given element $g \in G$ and $W \in \text{Rep}(K)$, the $gKg^{-1}$-representation $gW$ is defined to be $V$ with the new $gKg^{-1}$-action

$$gKg^{-1} \times W \rightarrow W, \quad (k, u) \mapsto g^{-1}ku$$

for $k \in gKg^{-1}$, $u \in W$.

**Lemma 9.3.** For each $\tilde{x} \in |\mathcal{L}|$, $x = \pi(\tilde{x})$, $g \in G_\chi$, $\psi_x \in \mathcal{A}_x$, we have an $(G_\chi)|e|$-isomorphism

$$\left(\text{res}_{(G_\chi)|e} G_\chi F_{V\chi}\right)_{\psi_x} ^{g} / g \cdot \psi_x \cong g\left(\left(\text{res}_{(G_\chi)|e} G_\chi F_{V\chi}\right)_{\psi_x} ^{g}\right).$$
Proof. Put \( L = \langle g, (G_x)_x \rangle \), and let \( k_0 \in \mathbb{N} \) be the smallest natural number satisfying \( g^{k_0} \in (G_x)_x \) where such a number exists because \( R = \rho(G_x) \) is finite. To prove this lemma by Lemma 9.1, we would construct an \( L \)-bundle \( \mathcal{F} \) over the orbit \( Lx = \{ x, gx, \ldots, g^{k_0-1}x \} \). Put \( \mathcal{F} = \langle \text{res}_{Lx}^\mathcal{F} \nabla \rangle \) over \( Lx \), and consider the equivariant \( \mathcal{F} \) over the orbit \( Lx \) whose fiber \( \mathcal{F}_{g^kx} \) at \( g^kx \) is equal to \( \mathcal{F}_{g^kx}/(g^k \cdot \psi_x) \) for \( k = 0, \ldots, k_0 - 1 \). And, let \( P : \mathcal{F} \to \mathcal{F} \) be the map such that the restriction of \( P \) to \( \mathcal{F}_{g^kx} \) is equal to \( p_{g^k \cdot \psi_x} : \mathcal{F}_{g^kx} \to \mathcal{F}_{g^kx}/(g^k \cdot \psi_x) \) is the quotient map of \( g^k \cdot \psi_x \). Then, we would define an \( L \)-action on \( \mathcal{F} \) as \( lu = P(lu) \) for each \( l \in L \), \( u \in \mathcal{F} \), and any \( u' \in P^{-1}(u) \) so that \( P \) becomes \( L \)-equivariant. As long as this is well defined, \( P \) is easily shown that it becomes an action because it is defined by the \( L \)-action on \( \mathcal{F} \) through \( P \). So, we would prove well definedness of the action. For this, it suffices to show \( P(lu) = P(lu') \) for each \( l \in L \) and each \( u, u' \) in \( \mathcal{F} \) satisfying \( P(u) = P(u') \). If \( u \in (F_V)_{g^kx} \), and \( u' \in (F_V)_{g^kx} \) for some \( j, j' \), then \( P(u) = P(u') \) is written as

\[
(g^k \cdot \psi_x)(g^k \cdot \psi_x)u = u'.
\]

Note that \( lu \in (F_V)_{g^kx} \), and \( lu' \in (F_V)_{g^kx} \). And, put \( l = g^{k'}l' \) with \( l' \in L_{g^k} \) and some integer \( k' \) so that \( g^{k'}x = g^{k+k'}x \) because \( l' \) fixes \( g^{k'}x \). Remembering that the restriction of \( P \) to \( \mathcal{F}_{g^kx} \) is equal to \( p_{g^{k+k'}x} \), \( P(lu) = P(lu') \) is shown as

\[
(g^{k+k'} \cdot \psi_x)(g^{k} \cdot \psi_x)lu = g^{k+k'}(g^{k} \cdot \psi_x)(g^{k} \cdot \psi_x)g^{k} \cdot l' \cdot l' \cdot g^{k} \cdot \psi_x = g^{k+k'}(g^{k} \cdot \psi_x)(g^{k} \cdot \psi_x)g^{k+k'} \cdot l' \cdot l' \cdot g^{k} \cdot \psi_x = l(g^{k} \cdot \psi_x)(g^{k} \cdot \psi_x)l' \cdot l' \cdot g^{k} \cdot \psi_x = l = lu'.
\]

where we use (*) in the last line. So, we obtain well definedness of \( L \)-action on \( \mathcal{F} \).

By definition, isotropy representations \( \mathcal{F}_x \) and \( \mathcal{F}_{g^kx} \) are equal to representations

\[
(\text{res}_{(G_x)_x}^\mathcal{F} V_n)|_{g^kx} \quad \text{and} \quad (\text{res}_{(G_x)_x}^\mathcal{F} V_n)|_{g^kx}/g \cdot \psi_x,
\]

respectively. Then, the lemma follows from Lemma 9.1. \( \square \)

To investigate \( \Omega_{P_{c^{\psi_k}}}^{\psi_k} \), we need to understand \( A_x \) precisely for each \( \bar{x} \in |L_{c^{\psi_k}}| \). For this, we need prove a basic lemma on relations between \( A_{c^{\psi_k}} \) and \( A_x \) with \( \bar{x} = \bar{v}^i \) or \( \bar{v}^{i+1} \). Also, we prove lemmas on evaluation of equivariant clutching maps.

**Lemma 9.4.** Put \( \psi^k = \pi^{-1}(v^k) = \{ \bar{x}_j : \bar{x}_j = \bar{v}^k \text{ for } j \in \mathbb{Z}_{j_0} \} \) for \( k = i, i+1 \). And, put \( \psi^i = \{ \bar{v}_0, \bar{v}_1 \} \) and \( \psi^{i+1} = \{ \bar{v}_0, \bar{v}_1 \} \).

1. \( A_x \subset A_c \) for each interior \( \bar{x} \in |c^i| \).
2. \( A_{c^{\psi_k}} = A_x \) for each interior \( \bar{x} \neq b(c^{\psi_k}) \) in \( |c^i| \). Moreover, \( A_{c^{\psi_k}} = A_{b(c^{\psi_k})} \) if \( (G_x)_{b(c^{\psi_k})} = (G_x)_x \) for \( x = \pi(\bar{x}) \).
3. \( A^{0}_{c^{\psi_k}} \subset A^{0}_x \) and \( A^{0}_{c^{\psi_k}} \subset A^{0}_x \).
4. For each \( \psi_{c^{\psi_k}} \) in \( A_{c^{\psi_k}} \), we have \( \psi_{c^{\psi_k}}(\bar{v}_0) = \psi_{c^{\psi_k}} \) for the unique \( \psi_{c^{\psi_k}} \) in \( A_{c^{\psi_k}} \), and

\[
(\text{res}_{(G_x)_{\psi^i}}^\mathcal{F} V_n)|_{\psi^i/\psi^i} \equiv (\text{res}_{(G_x)_{\psi^{i+1}}}^\mathcal{F} V_n)|_{\psi^i/\psi^{i+1}}.
\]
For each \( \psi_{v^{i+1}} \) in \( A_{v^{i+1}} \), we have \( \psi_{v^{i+1}}(\vec{v}^{i+1}) = \psi_{v^{i+1}}(\vec{v}^{i+1}) \) for the unique \( \psi_{v^{i+1}} \in A_{v^{i+1}} \), and
\[
\text{res}(G_x)^{v^{i+1}}\{(\text{res}(G_x)^{v^{i+1}}, F_{v^{i+1}})|_{\psi_{v^{i+1}}/\psi_{v^{i+1}}} \} \equiv \{ \text{res}(G_x)^{v^{i+1}}\{(\text{res}(G_x)^{v^{i+1}}, F_{v^{i+1}})|_{\psi_{v^{i+1}}/\psi_{v^{i+1}}} \}
\]

**Proof.** (1) follows from \((G_x)^{v^{i+1}} \subset (G_x), x = \pi(x)\). (2) follows from \((G_x)^{v^{i+1}} = (G_x), x \subset \pi(x)\). Similarly, (3) holds by Lemma 7.4 and \((G_x)^{v^{i+1}} \subset (G_x)^{v^{i+1}}, (G_x)^{v^{i+1}} \subset (G_x)^{v^{i+1}}\) where \((G_x)^{v^{i+1}}\) and \((G_x)^{v^{i+1}}\) are subgroups of \(G_x\) preserving \(v^{i+1}\) and \(v^{i+1}\), respectively. The first statement of (4) follows by (3) and Lemma 7.4. By Lemma 7.5, we have the second statement of (4). Similarly, (5) is also obtained. □

**Lemma 7.5.** For a vertex \( \vec{v} \) in \( \vec{K}_p \) and \( v = \pi(\vec{v}) \), if \( R_e \cong \mathbb{Z}_m \) or \( D_m \) with \( m = |\pi^{-1}(v)| \), then \( \Phi_e \to \Phi_e \) is injective.

**Proof.** Proof is done by Proposition 7.8(2) and 7.9(2). □

**Lemma 7.6.** If \( R/R_t \neq (\pi, D_1) \) with \( A \vec{c} + \vec{c} = \vec{l}_0 \), then the map
\[
A_{v^{i+1}} \to A_{v^{i+1}}^0 \times A_{v^{i+1}}^{-1}, \psi_{v^{i+1}} \mapsto (\psi_{v^{i+1}}(\vec{v}^i_0), \psi_{v^{i+1}}(\vec{v}^{i-1}))
\]
is injective.

**Proof.** In these cases, \( R_{v^{i+1}} \) is not trivial by Table 7.2 and it is checked case by case that \((\psi_{v^{i+1}}(\vec{v}^i_0), \psi_{v^{i+1}}(\vec{v}^{i-1})) \) determines \( \psi_{v^{i+1}} \) by using equivariance of equivariant pointwise clutching maps. In this check, Lemma 7.5 is helpful. □

Now, we state conditions on a preclutching map \( \Phi_{\hat{p}_v} \) in \( C^0(\hat{D}_p, V_B) \) to guarantee that \( \Phi_{\hat{p}_v} \) is restriction of an equivariant clutching map. When \( \Lambda_t = \Lambda^{\hat{p}_v} \), pick two elements \( g_0, g_1 \in G_x \) such that
\[
\rho(g_0)[x] = [x + (1, 0)], \rho(g_1)[x] = [x + (0, 1)].
\]
When \( R/R_t = D_1 \) with \( A \vec{c} + \vec{c} = \vec{l}_0 \), pick an element \( g_2 \in G_x \) such that \( \rho(g_2) = [A \vec{c} + \vec{c}] \) where \( A \vec{c} + \vec{c} \) is the glide through \( \frac{1}{2} \vec{c} + L \) and \( \frac{1}{2}(A \vec{c} + \vec{c}) \). Also, we define a terminology.

**Definition 9.7.** For \( x \in \{c \mid c \in V_B \}, \Phi = \Phi^{c} \in C^0(\hat{D}_p, V_B), \) the map \( \psi_x \) or its saturation \( \hat{\psi}_x \) is called determined by \( \Phi \) if \( \Phi \) satisfies the following condition:
\[
\hat{\psi}_x(p_{c}([c](\hat{x})), p_{c}([c](\hat{x}))) = \Phi(\hat{x})
\]
for each \( \hat{x} \in (\pi \circ p_{c})^{-1}(x) \). The condition is concretely written as

1. if \( \hat{x} \) is not a vertex, then
   \[
   \psi_x(\hat{x}_0) = \Phi(\hat{x}_0), \quad \psi_x(\hat{x}_1) = \Phi(\hat{x}_1),
   \]
2. if \( \hat{x} \) is a vertex, then
   \[
   \psi_x(\hat{x}_j) = \Phi(\hat{x}_j), \quad \psi_x^{-1}(\hat{x}_j) = \Phi(\hat{x}_j).
   \]
for each \( j \in \mathbb{Z}_m \).

**Theorem 9.8.** When \( R = \rho(G_x) \) for some \( R \) in Table 7.1, a preclutching map \( \Phi_{\hat{p}_v} \) in \( C^0(\hat{D}_p, V_B) \) is in \( \Omega_{\hat{D}_p, V_B} \) if and only if there exists the unique \( \psi_x \in A_x \) for each \( \hat{x} \in \hat{D}_p \) and \( x = \pi(\hat{x}) \) satisfying

E2. \( \hat{\psi}_x(p_{c}([c](\hat{x})), p_{c}([c](\hat{x}))) = \Phi(\hat{x}) \) for each \( \hat{x} \in \hat{D}_p \).

E3. for each \( x, x' \in \hat{D}_p \) and their images \( \pi(x) = \pi(x') \), if \( x' = g x \) for some \( g \in G_x \), then \( \psi_{x'} = g \cdot \psi_x \).

A set \( (\psi_x)_{x \in \hat{D}_p} \) with \( \psi_x \in A_x \) is called determined by \( \Phi_{\hat{p}_v} \in C^0(\hat{D}_p, V_B) \) if \( \Phi_{\hat{p}_v} \) is in \( \Omega_{\hat{D}_p, V_B} \) and each \( \psi_x \) is the unique element of this theorem.
Proof. For necessity, it suffices to construct a map \( \Phi \) in \( \Omega_{V_B} \) such that \( \Phi|_{\tilde{D}_\rho} = \Phi|_{\tilde{D}_\rho} \). For this, we would define \( \Phi \) on \( \tilde{D}_\rho \), and then extend the domain of definition of \( \Phi \) to the whole \( |\tilde{L}_\rho| \). First, we define \( \Phi \) on \( \tilde{D}_\rho \) so that each \( \psi_x \) is determined by \( \Phi \). Then, Condition E2. says that \( \Phi|_{\tilde{D}_\rho} = \Phi|_{\tilde{D}_\rho} \). Next, we define \( \Phi(\hat{x}) = g^{-1}\Phi(g\hat{x})g \) for each \( \hat{x} \in |\tilde{L}_\rho| \) and some \( g \in G_\chi \) such that \( g\hat{x} \) is in \( \tilde{D}_\rho \). We need prove well definedness of this. Assume that \( \hat{y} = g\hat{x} \) and \( \hat{y}' = g'y \) are in \( \tilde{D}_\rho \) for two elements \( g, g' \) in \( G_\chi \) so that \( \hat{y}' = g'g^{-1}\hat{y} \). And, let \( y \) and \( y' \) be images of \( \hat{y} \) and \( \hat{y}' \) through \( \pi \circ\rho|_{\tilde{L}_\rho} \), respectively. Then, \( y' = g'g^{-1}y \). These give \( \psi_{y'} = (g'g^{-1}) \cdot \psi_y \) by Condition E3. From this, we obtain

\[
g^{-1}\Phi(\hat{y})g' = g^{-1}\psi_{y'}(p|_{\tilde{L}_\rho}(\hat{y}'), p|_{\tilde{L}_\rho}(|c|(\hat{y}')))g' = g^{-1}\psi_y(p|_{\tilde{L}_\rho}(\hat{y}), p|_{\tilde{L}_\rho}(|c|(\hat{y})))g = g^{-1}\Phi(\hat{y})g
\]

where we use equivariance of \( |c| \). So, well definedness is proved. It is easily checked that \( \Phi \) satisfies Condition N1., N2., E1. Therefore, \( \Phi \) is the wanted equivariant clutching map.

For sufficiency, assume that \( \Phi|_{\tilde{D}_\rho} = \Phi|_{\tilde{D}_\rho} \) for some \( \Phi \in \Omega_{V_B} \). Then, we should choose the unique \( \psi_x \in A_x \) for each \( \hat{x} \in \tilde{D}_\rho \) satisfying Condition E2. and E3. When we show that \( \psi_x \)'s satisfy Condition E3., it suffices to consider only vertices \( \hat{x} \)'s by definition of \( \tilde{D}_\rho \). In \( \sim \) entry of Table \[13\] vertices in the same orbit are listed.

Proof for sufficiency is different according to \( R \), especially choices of \( \psi_x \)'s.

First, assume that \( R/R_\ell \) is not equal to one of following groups:

\[
D_2 \text{ with } A_1 = \Lambda_\emptyset, D_{2,3} \text{ with } A_1 = \Lambda_\emptyset, \text{ and } D_1, D_{1,4}.
\]

At each \( \hat{x} \in \tilde{D}_\rho \), the unique \( \psi_x \) in \( A_x \) is determined by \( \Phi \) because \( \Phi \) satisfies Condition N1., N2., E1. Moreover, it can be checked that \( \psi_x \) is the unique element satisfying Condition E2. for each \( \hat{x} \) by Lemma \[9.6\]. So, it suffices to show that \( \psi_x \)'s satisfy Condition E3. However, since \( \Phi \) is equivariant, \( \psi_x \)'s satisfy Condition E3.

Second, assume that \( R/R_\ell = D_2 \) or \( D_{2,3} \) with \( A_1 = \Lambda_\emptyset \). Observe that there exists the unique \( \psi_x \) satisfying Condition E2. for each \( \hat{x} \in \tilde{D}_\rho - \{\hat{v}^2\} \) or \( \tilde{D}_\rho - \{\hat{v}^3\} \) according to \( R/R_\ell = D_2 \) or \( D_{2,3} \) by Lemma \[9.6\] respectively. For the remaining one point, pick the unique element as

\[
\psi_{v^2} = g \cdot \psi_{v^2}
\]

in \( \tilde{A}_{v^2} \) by Condition E3. when \( R/R_\ell = D_2 \) and \( gv^3 = v_2 \) for some \( g \in G_\chi \), or

\[
\psi_{v^3} = g' \cdot \psi_{v^3}
\]

in \( \tilde{A}_{v^3} \) by Condition E3. when \( R/R_\ell = D_{2,3} \) and \( gv^3 = v_1 \) for some \( g' \in G_\chi \). By definition, these two are determined by \( \Phi \) because \( \Phi \) is equivariant. So, these two satisfy Condition E2. From this, all \( \psi_x \)'s satisfy Condition E2. By definition, these two also satisfy Condition E3., and this implies that all \( \psi_x \)'s satisfy Condition E3. by \( \sim \) entry of Table \[13\]. Uniqueness of \( \psi_{v^2} \) and \( \psi_{v^3} \) is guaranteed by definition because \( \psi_{v^2} \) and \( \psi_{v^3} \) should satisfy Condition E3.

Third, assume that \( R/R_\ell = D_1 \) with \( A\hat{c} + \hat{c} = 0 \) or \( D_{1,4} \). Observe that there exists the unique \( \psi_x \) satisfying Condition E2. for each \( \hat{x} \in \tilde{D}_\rho - \{\hat{v}^0, \hat{v}^1\} \) by Lemma \[9.6\]. For the remaining two points, pick unique elements

\[
\psi_{v^0} = g_0^{-1} \cdot \psi_{v^0}, \quad \psi_{v^1} = g_0^{-1} \cdot \psi_{v^2} \text{ in } \tilde{A}_{v^0}, \tilde{A}_{v^2};
\]

when \( R/R_\ell = D_1 \) with \( A\hat{c} + \hat{c} = 0 \), and

\[
\psi_{v^0} = g_1^{-1} \cdot \psi_{v^0}, \quad \psi_{v^1} = g_0^{-1} \cdot \psi_{v^3} \text{ in } \tilde{A}_{v^0}, \tilde{A}_{v^3};
\]
when $R/R_1 = D_{1,4}$, respectively. Then, the remaining proof is the same with the second case.

Last, assume that $R/R_1 = \langle \text{id} \rangle$ or $D_1$ with $A\overline{c} + \overline{c} = \overline{t}_0$. Observe that there exists the unique $\psi_x$ satisfying Condition E2. for each $\overline{x} \in D_\rho = \{ \overline{v}^2, \overline{v}^3, \overline{v}^0 \}$ by Lemma 9.9. Since $D_\rho = \{ \overline{v}^2, \overline{v}^3, \overline{v}^0 \}$ has no vertex, these $\psi_x$’s satisfy Condition E3. Then, we should show unique existence of $\psi_x$ for $\overline{x} \in \{ \overline{v}^2, \overline{v}^3, \overline{v}^0 \}$ to satisfy Condition E2., E3. To show uniqueness, first assume existence so that all $\psi_x$’s for $\overline{x} \in \{ \overline{v}^2, \overline{v}^3, \overline{v}^0 \}$ satisfy Condition E2., E3. Then, Condition E2. says that

\[ (*) \quad \psi_{\bar{x}}(\overline{v}^2) = \Phi_{D_\rho}(\overline{v}^2), \quad \psi_{\bar{x}}(\overline{v}^3) = \Phi_{D_\rho}(\overline{v}^3), \quad \psi_{\bar{x}}(\overline{v}^0) = \Phi_{D_\rho}(\overline{v}^0). \]

And, Condition E3. says that that

\[ (**) \quad \tilde{\psi}_{\bar{x}} = g_1 \cdot \tilde{\psi}_{\bar{x}}, \quad \tilde{\psi}_{\bar{x}} = \left\{ \begin{array}{ll} g_0^{-1} \tilde{\psi}_{\bar{x}} & \text{when } R/R_1 = \langle \text{id} \rangle, \\ (g_2g_1)^{-1} \tilde{\psi}_{\bar{x}} & \text{when } R/R_1 = D_1 \text{ with } A\overline{c} + \overline{c} = \overline{t}_0. \end{array} \right. \]

By $\psi_{\bar{x}}(\overline{v}^2)$ of ($*$) and $\tilde{\psi}_{\bar{x}}$ of ($**$), we have

\[ (***) \quad \psi_{\bar{x}}(\overline{v}^2) = g_1^{-1} \Phi_{D_\rho}(\overline{v}^2)g_1 \]

because $g_1 \overline{v}^2 = \overline{v}^2$. Similarly, by $\psi_{\bar{x}}(\overline{v}^0)$ of ($*$) and $\tilde{\psi}_{\bar{x}}$ of ($**$), $\psi_{\bar{x}}(\overline{v}^0)$ is equal to

\[ (****) \quad \left\{ \begin{array}{ll} g_0 \Phi_{D_\rho}(\overline{v}^0)g_0^{-1} & \text{when } R/R_1 = \langle \text{id} \rangle, \\ g_2g_1 \Phi_{D_\rho}(\overline{v}^0)(g_2g_1)^{-1} & \text{when } R/R_1 = D_1, A\overline{c} + \overline{c} = \overline{t}_0. \end{array} \right. \]

By ($*$), ($**$), ($****$), $\psi_{\bar{x}}$ is uniquely determined by values of $\Phi_{D_\rho}$ so that $\tilde{\psi}_{\bar{x}}$, $\psi_{\bar{x}}$ are also uniquely determined by Condition E3. So, we obtain uniqueness. Proof of existence is easy. Define $\psi_{\bar{x}}$ as in the proof of uniqueness. By equivariance of $\Phi$, we have $\psi_{\bar{x}} \in A_{\bar{x}}$. And, define $\psi_{\bar{x}}$ and $\psi_{\bar{x}}$ as in ($**$) so that $\psi_{\bar{x}}$’s satisfy Condition E3. Since $\Phi$ is equivariant, $\psi_{\bar{x}}$ is determined by $\Phi$ by definition of $\psi_{\bar{x}}$. So, $\psi_{\bar{x}}$ satisfies Condition E2. Again by equivariance of $\Phi$, $\psi_{\bar{x}}$ and $\psi_{\bar{x}}$ satisfy Condition E2. Therefore, we obtain a proof. \( \square \)

**Remark 9.9.** This theorem holds even though we might omit the word ‘unique’ in the statement of the theorem because uniqueness is not used in the proof of necessity.

By using Theorem 9.8, we would describe $\Omega_{D_\rho, V_B}$ through $A_{\bar{x}}$’s. Define the following set of equivariant pointwise clutching maps on $d^\rho$’s.

**Definition 9.10.** Denote by $A_{\bar{x}}(\mathbb{R}^2/\Lambda, V_B)$ the set

\[
\{ (\psi_{\bar{d}})_{\bar{d}} \in T_\rho \mid \psi_{\bar{d}} \in A_{\bar{d}} \text{ and } \psi_{\bar{d}} = g \cdot \tilde{\psi}_{\bar{d}} \text{ if } d^\rho = gd^\rho \text{ for some } g \in G_\chi \}. 
\]

For $(\psi_{\bar{d}})_{\bar{d}} \in A_{\bar{x}}(\mathbb{R}^2/\Lambda, V_B)$ and $\Phi_{D_\rho} \in \Omega_{D_\rho, V_B}$, the element $(\psi_{\bar{d}})_{\bar{d}} \in T_\rho$ is determined by $\Phi_{D_\rho}$ if $\psi_{\bar{d}}$’s are unique elements determined by $\Phi_{D_\rho}$ in Theorem 9.8.

Also, for $(W_{\bar{d}})_{\bar{d}} \in A_{\bar{d}}(\mathbb{R}^2/\Lambda, \chi)$ and $(\psi_{\bar{d}})_{\bar{d}} \in A_{\bar{x}}(\mathbb{R}^2/\Lambda, V_B)$, the element $(W_{\bar{d}})_{\bar{d}} \in T_\rho$ is determined by $(\psi_{\bar{d}})_{\bar{d}} \in T_\rho$ if $W_{\bar{d}}$ is determined by $\psi_{\bar{d}}$ for each $i \in I_\rho$.

**Corollary 9.11.** The set $\Omega_{D_\rho, V_B}$ is equal to the set

\[
\left\{ \Phi_{D_\rho} = \bigcup_{i \in I_\rho} \varphi^i \in C^0(\hat{D}_\rho, V_B) \left| \begin{array}{l} \varphi^i(\hat{d}^i_+) = \psi_{\bar{d}^i} (d^i), \quad \varphi^i(\hat{d}^{i+1}) = \psi_{\bar{d}^{i+1}} (d^{i+1}), \quad \varphi^i(\hat{d}) \in A_{\bar{d}}^0, \\
\text{for each } i \in I_\rho, \quad \hat{d} \in [\hat{d}^i, \hat{d}^{i+1}], \quad \text{and some } (\psi_{\bar{d}})_{\bar{d}} \in A_{\bar{x}}(\mathbb{R}^2/\Lambda, V_B) \right. \right\}. 
\]
Proof. To prove this corollary, we would rewrite Theorem 9.8 by using \( A_e \) and \( \bar{A}_G (\mathbb{R}^2 / \Lambda, V_B) \). By Theorem 9.8, a preclutching map \( \Phi \) is in \( \Omega_{D_\rho, V_B} \) if and only if a set \( \Psi = \langle \psi \rangle_{\rho \in D_\rho} \) is determined by \( \Phi \). As we have seen in the proof of the theorem, \( gx = x' \) with \( x \neq x' \) in Condition E3. is possible only when \( x \) and \( x' \) are \( d_i \)'s (of course, more precisely when they are images of vertices). So, \( \Psi \) is determined by \( \Phi \) if and only if \( \langle \psi \rangle_{\rho \in D_\rho} \) satisfies Condition E2. and E3. and \( \Psi - \langle \psi \rangle_{\rho \in D_\rho} \) satisfies Condition E2. Here, \( \langle \psi \rangle_{\rho \in D_\rho} \) satisfies Condition E2. if and only if

\[
(*) \quad \phi^i(\bar{d}_i^+) = \psi_{\rho_i}(\bar{d}_i) \quad \text{and} \quad \phi^i(\bar{d}_i^{-1}) = \psi_{\rho_i}^{-1}(\bar{d}_i^{-1})
\]

for each \( i \in I_\rho^- \). Lemma 9.3 says that \((*)\) implies \( \phi^i(\bar{x}) \in A_{\rho\rho_i}(\mathbb{R}^2 / \Lambda) \) for \( \bar{x} = \bar{d}_i^+, \bar{d}_i^- \). So, \((*)\) could be redundantly rewritten as

\[
(**) \quad \phi^i(\bar{d}_i^+) = \psi_{\rho_i}(\bar{d}_i), \quad \phi^i(\bar{d}_i^{-1}) = \psi_{\rho_i}^{-1}(\bar{d}_i^{-1}), \quad \text{and} \quad \phi^i(\bar{x}) \in A_{\rho\rho_i}(\mathbb{R}^2 / \Lambda)
\]

for \( \bar{x} = \bar{d}_i, \bar{d}_i^- \). And, \( \langle \psi \rangle_{\rho \in D_\rho} \) satisfies Condition E3. if and only if

\[
(***) \quad \langle \psi \rangle_{\rho \in D_\rho} \in \bar{A}_G (\mathbb{R}^2 / \Lambda, V_B).
\]

Next, we deal with \( \langle \psi \rangle_{\rho \in D_\rho} \) s in \( \Psi - \langle \psi \rangle_{\rho \in D_\rho} \), i.e. \( \bar{x} \)'s in \( \bar{d}_i, \bar{d}_i^- \) for some \( i \in I_\rho^- \). They satisfy Condition E2. if and only if \( \psi_{\rho_i}(\bar{x}) = \phi^i(\bar{x}) \) for \( \bar{x} \) and \( i \) such that \( \bar{x} \in (\bar{d}_i^+, \bar{d}_i^-) \). And, this is satisfied if and only if \( \phi^i(\bar{x}) \in A_{\rho\rho_i}(\mathbb{R}^2 / \Lambda) \) and we have chosen \( \psi_{\rho_i} \)'s such that \( \psi_{\rho_i}(\bar{x}) = \phi^i(\bar{x}) \) for each \( i, \bar{x} \). In summary, three conditions of this, \((**), (***)\) are equivalent conditions for \( \Psi \) to be determined by \( \Phi \). Therefore, we obtain a proof.

By using this corollary, we would show nonemptiness of \( \Omega_{D_\rho, (W_{d_i})_{i \in I_\rho^+}} \). For this, we need a lemma.

**Lemma 9.12.** For each \( (W_{d_i})_{i \in I_\rho^+} \in \bar{A}_G (\mathbb{R}^2 / \Lambda, \chi) \), if we put

\[
V_B = G_\chi \times (G_\chi)^{-1} \times W_{d_i}, \quad F_{\Psi} = G_\chi \times (G_\chi)^{-1} \times W_{d_i},
\]

then we can pick an element \( \langle \psi \rangle_{\rho \in D_\rho} \) in \( \bar{A}_G (\mathbb{R}^2 / \Lambda, V_B) \) which determines \( (W_{d_i})_{i \in I_\rho^+} \).

**Proof.** For each \( i \in I_\rho^+ \), put \( x = (x_1, \ldots, x_{d_i}) \) \( x_j = \bar{d}_j \) for \( j \in \mathbb{Z} \), \( m = j_0 \) or 2. Put \( F_1 = (\text{res}_{\rho \in (G_\chi)^{-1}})^{-1} \times W_{d_i} \), and \( F_2 = (\chi \times \chi)^{-1} \times W_{d_i} \). Since \( W_{d_i} \) is \( \chi \)-isotopic and \( G_\chi \) fixes \( \chi \), \( (\text{res}_{\rho \in (G_\chi)^{-1}})^{-1} \times W_{d_i} \)'s are all isomorphic regardless of \( j \). So, the \( (G_\chi)^{-1} \)-bundle \( F_1 \) satisfies the assumption on \( F \) in Section 7. Note that

\[
(F_1)_{x_0} \cong \text{res}_{\rho \in (G_\chi)^{-1} \times W_{d_i}} \quad \text{because} \quad (G_\chi)_{\rho \in D_\rho} = (G_\chi)_{x_0}.
\]

This implies

\[
(F_1)_{x_0} \cong \text{res}_{\rho \in (G_\chi)^{-1} \times W_{d_i}} \quad \text{by Definition 11.1(4), i.e.} \quad W_{d_i} \text{ is a (}G_\chi)^{-1} \text{-extension of (}F_1)_{x_0} \text{). So, Proposition 7.3 says that there exists an element } \psi_{d_i} \text{ in } \bar{A}_G, \text{ which determines } W_{d_i}. \text{ If } d_i' = g \cdot d_i \text{ for some } g \in G_\chi, \text{ then the element } g \cdot \psi_{d_i} \text{ in } \bar{A}_G, \text{ satisfies}
\]

\[
F_{d_i'} / g \cdot \psi_{d_i} \cong W_{d_i'} \quad \text{by Lemma 9.3 and Definition 11.1(3). So, we may assume that } \psi_{d_i} = g \cdot \psi_{d_i}. \text{ Then,}
\]

\[
(\psi_{d_i})_{i \in I_\rho^+} \in \bar{A}_G (\mathbb{R}^2 / \Lambda, V_B) \text{ which determines } (W_{d_i})_{i \in I_\rho^+}. \text{ Therefore, we obtain a proof.}
\]

**Proposition 9.13.** For each \( (W_{d_i})_{i \in I_\rho^+} \in \bar{A}_G (\mathbb{R}^2 / \Lambda, \chi), \Omega_{D_\rho, (W_{d_i})_{i \in I_\rho^+}} \) is nonempty.
Proof. By Lemma 9.12, we can pick an element $(\psi_{\bar{d}})_{\bar{d}} \in \tilde{A}_{G_\chi}(\mathbb{R}^2/\Lambda, V_B)$ which determines $(W_{\bar{d}}')_{\bar{d}} \subset D_\rho$. For $[\bar{d}^i, \bar{d}^i+1] \subset D_\rho$, Lemma 9.14 says that both $\psi_{\bar{d}^i}(\bar{d}^i)$ and $\psi_{\bar{d}^i+1}(\bar{d}^i+1)$ are in $(A_{\chi})^0$ which determine $\text{res}_{(G_\chi)_{\bar{d}^i}} W_{\bar{d}^i}$ and $\text{res}_{(G_\chi)_{\bar{d}^i+1}} W_{\bar{d}^i+1}$, respectively. Since these two are isomorphic by definition of $A_{G_\chi}(\mathbb{R}^2/\Lambda, \chi)$, Proposition 9.3 says that $\psi_{\bar{d}^i}(\bar{d}^i)$ and $\psi_{\bar{d}^i+1}(\bar{d}^i+1)$ are in the same component of $(A_{\chi})^0$. So, we can pick a function $\varphi^i : [\bar{d}^i, \bar{d}^i+1] \rightarrow (A_{\chi})^0 \subset \text{Iso}(V_{\bar{d}^i-1}, V_{\bar{d}^i})$ such that $\varphi^i(\bar{d}^i) = \psi_{\bar{d}^i}(\bar{d}^i)$ and $\varphi^i(\bar{d}^i+1) = \psi_{\bar{d}^i+1}(\bar{d}^i+1)$ for each $i \in I_\rho^-$. From this, $\Phi_{D_\rho} = \bigcup_{i \in I_\rho^-} \varphi^i$ is in $\Omega_{D_\rho, V_B}$ by Corollary 9.11. Therefore, we obtain a proof. \qed

10. The case of $\mathcal{R}/R_t \neq (id), D_1$ with $A\varepsilon + \varepsilon = \bar{t}_0$

Now, we are ready to calculate homotopy of equivariant clutching maps.

Theorem 10.1. If $\rho(G_\chi)$ is one in Table 10.1 then $\pi_0(\Omega_{\tilde{D}_{\rho}, (W_{\bar{d}}')_{\bar{d}} \in I_\rho^+})$ is one point set for each element $(W_{\bar{d}}')_{\bar{d}} \in A_{G_\chi}(\mathbb{R}^2/\Lambda, \chi)$.

| $\mathcal{R}/R_t$ | $\Lambda_\varepsilon$ |
|-------------------|---------------------|
| $D_{2,2}, D_2, D_4$ | $\Lambda^\infty$ |
| $D_2, D_2, D_3, D_6, D_{3,2}$ | $\Lambda^{eq}$ |
| $D_1$ with $A\varepsilon + \varepsilon = 0$, $D_{1,4}$ | $\Lambda^\infty$ |

Table 10.1. $\rho(G_\chi)$ such that $\pi_0(\Omega_{\tilde{D}_{\rho}, (W_{\bar{d}}')_{\bar{d}} \in I_\rho^+})$ consists of one element

Proof. We divide these groups in the table into two categories. The first are the following six kinds of groups: $D_{2,2}, D_2, D_4$ when $\Lambda_\varepsilon = \Lambda^\infty$, and $D_2, D_6, D_{3,2}$ when $\Lambda_\varepsilon = \Lambda^{eq}$. The second are the remaining. We deal with two categories one after another. When $\mathcal{R} = \rho(G_\chi)$ is given, put

$$V_B = G_{\chi} \times_{(G_\chi)_{d^{-1}}} W_{d^{-1}}, \quad F_{V_B} = G_{\chi} \times_{(G_\chi)_{d^{-1}}} ([s^{-1}] \times W_{d^{-1}})$$

for each $(W_{\bar{d}}')_{\bar{d}} \in I_\rho^+ \subset A_{G_\chi}(\mathbb{R}^2/\Lambda, \chi)$.

Assume that $\mathcal{R} = \rho(G_\chi)$ is in the first category. By Table 10.3 $\bar{d}^i$’s are all vertices. For two arbitrary $\Phi_{D_\rho} = \bigcup_{i \in I_\rho^-} \varphi^i$ and $\Phi'_{D_\rho} = \bigcup_{i \in I_\rho^-} \varphi'^i$ in $\Omega_{\tilde{D}_{\rho}, (W_{\bar{d}}')_{\bar{d}} \in I_\rho^+}$, let $(\psi_{\bar{d}^i})_{\bar{d}^i} \in I_\rho^-$ and $(\psi'_{\bar{d}^i})_{\bar{d}^i} \in I_\rho^+$ in $A_{G_\chi}(\mathbb{R}^2/\Lambda, V_B)$ be two elements determined by $\Phi_{D_\rho}$ and $\Phi'_{D_\rho}$, respectively. We would construct a homotopy connecting $\Phi_{D_\rho}$ and $\Phi'_{D_\rho}$ in $\Omega_{\tilde{D}_{\rho}, (W_{\bar{d}}')_{\bar{d}} \in I_\rho^+}$. First, we show that we may assume that $(\psi'_{\bar{d}^i})_{\bar{d}^i} = (\psi_{\bar{d}^i})_{\bar{d}^i}$. Since $\psi_{\bar{d}^i}$ and $\psi'_{\bar{d}^i}$ for $i \in I_\rho$ determine the same representation $W_{\bar{d}^i}$, these two are in the same path component of $A_{\bar{d}^i}$ by Proposition 9.3. Take paths $\gamma^i : [0, 1] \rightarrow A_{\bar{d}^i}$ for $i \in I_\rho$ such that $\gamma^i(0) = \psi_{\bar{d}^i}$ and $\gamma^i(1) = \psi'_{\bar{d}^i}$. If $d^i = g \cdot d^i$ for some $g \in G_{\chi}$, then $g \cdot \gamma^i$ satisfies $(g \cdot \gamma^i)(0) = \psi'_{\bar{d}^i}$ and $(g \cdot \gamma^i)(1) = \psi'_{\bar{d}^i}$ by Definition 9.10. So, we may assume that $\gamma^i = g \cdot \gamma^i$ if $d^i = g \cdot d^i$. Here, these $\gamma^i$’s are well defined because elements of $A_{\bar{d}^i}$ are fixed by $(G_\chi)_{d^i}$. Recall that the parametrization on $e^i = [e^i, e^{i+1}]$ by $s \in [0, 1]$ satisfies $e^i_s = 0, b(e^i) = 1/2, e^{i+1} = 1$ for $i \in I_\rho^-$. Then, we construct a homotopy $L^i(s, t) : [e^i_s, e^{i+1}] \times [0, 1] \rightarrow A_{\bar{d}^i}$, for each $i \in I_\rho^-$ as

$$L^i(s, t) = \gamma^i((1-3s)t)(d^i) \quad \text{for } s \in [0, \frac{1}{3}],$$

$$L^i(s, t) = \varphi'^i(3s-1) \quad \text{for } s \in \left[\frac{1}{3}, 1\right],$$

$$L^i(s, t) = \gamma^{i+1}((3s-2)t-1)(d^{i+1}) \quad \text{for } s \in \left[\frac{2}{3}, 1\right].$$


If we put $L = \bigcup_{t \in I^e} L^t$, then $L_t$ for each $t \in [0,1]$ is in $\Omega_{D_{\rho}, (W_0^e), I^e \setminus I^e_0}$ by Corollary 9.11 and $L$ connects $\Phi^*_{D_{\rho}}$ with $L_1$ in $\Omega_{D_{\rho}, (W_0^e), I^e \setminus I^e_0}$ which determines $(\psi_{\rho^e})_{I^e \setminus I^e_0}$. So, we may put $\Phi^*_{D_{\rho}} = L_1$, and we may assume that $\Phi_{D_{\rho}}$ and $\Phi', D_{\rho}$ determine the same element $(\psi_{\rho^e})_{I^e \setminus I^e_0}$ in $A_{G_0}(\mathbb{R}^2/\Lambda, V_B)$.

As in the proof of Proposition 9.13, $\psi_{\rho^e}(\bar{d}^3)$ and $\psi_{\rho^e}^{-1}(\bar{d}^3)$ are in the same component $(A_{\rho})^0_{\bar{v}_i^e}$ for some element $\bar{v}_i^e$ in $A_{\rho^e}$. Since $\varphi^e$ and $\varphi^e$ have values in $A_{\rho^e}$ by Corollary 9.11 and they connect $\psi_{\rho^e}(\bar{d}^3)$ with $\psi_{\rho^e}^{-1}(\bar{d}^3)$, $\varphi^e$ and $\varphi^e$ have values in $(A_{\rho^e})^0_{\bar{v}_i^e}$. Here, note that $(A_{\rho^e})_{\bar{v}_i^e}$ is simply connected by Lemma 1.5 and Proposition 7.8(3). By simply connectedness, we can obtain a homotopy $L^e(s, t) : [\bar{d}^e_{i^e}, \bar{d}^e_{i^e} + 1] \times [0,1] \rightarrow A_{\rho^e}^0$ for $i \in I^e_0$ as

$$L^e(0, s) = \psi_{\rho^e}(\bar{d}^3), \quad L^e(1, s) = \psi_{\rho^e}(\bar{d}^3), \quad L^e(s, 1) = \varphi^e(s),$$

for $s, t \in [0,1]$. Then, $L' = \bigcup_{i \in I^e_0} L^e$ connects $\Phi_{D_{\rho}}$ and $\Phi'_{D_{\rho}}$ in $\Omega_{D_{\rho}, (W_0^e), I^e \setminus I^e_0}$ by Corollary 9.11. Therefore, we obtain a proof for the first category. Here, we remark that simply connectedness is critical in obtaining $L'$.

Next, we deal with $R = \rho(G_0^e)$’s in the second category. Assume that $R/R^e = D_2$ with $A_0$. The calculation for this case also applies for other groups in the second category. In this case, we have

$$D_{\rho} = [b(\bar{e}_2), \bar{v}_2, \bar{v}_1, \bar{v}_2] \text{ with } v^2 \sim v^3, \quad I_0 = \{2, 3, 4, 5, 6\},$$

and

$$\begin{align*}
(G_0^e)_{b(\bar{e}_2)}/H &= D_{1,2}, \\
(G_0^e)_{b(\bar{e}_2)}/H &= D_{1,2}, \\
(G_0^e)_{b(\bar{e}_2)}/H &= D_{1,2}, \\
(G_0^e)_{b(\bar{e}_2)}/H &= D_{1,2}, \\
(G_0^e)_{b(\bar{e}_2)}/H &= D_{1,2},
\end{align*}$$

by Table 3 and tables in Section 3. From these, we especially obtain that path components of $A_{\rho^e}$ for $i = 0, 1, 3$ are all simply connected by Proposition 7.8(3) so that we would focus on $[b(\bar{e}_2), \bar{v}_3]$. To calculate $\Pi_{D_{\rho}, (W_0^e), I^e \setminus I^e_0}$, as in the proof for the first category, we may assume that two arbitrary $\Phi_{D_{\rho}} = \bigcup_{i \in I^e_0} \varphi^e$ and $\Phi'_{D_{\rho}} = \bigcup_{i \in I^e_0} \varphi^e$ in $\Omega_{\rho, (W_0^e), I^e \setminus I^e_0}$ determine the same element $(\psi_{\rho^e})_{I^e \setminus I^e_0}$ in $A_{G_0^e}(\mathbb{R}^2/\Lambda, V_B)$.

We would construct a homotopy $L^e(s, t) : [\bar{d}^e_{i^e}, \bar{d}^e_{i^e} + 1] \times [0,1] \rightarrow A_{\rho^e}^0$ for $i \in I^e_0$ such that $L' = \cup_{i \in I^e_0} L^e$ satisfies

$$\begin{align*}
(1) \quad &L^e_i \in \Omega_{D_{\rho}, (W_0^e), I^e \setminus I^e_0}, \\
(2) \quad &L^e_0 = \Phi_{D_{\rho}} \text{ and } L^e_1 = \Phi'_{D_{\rho}}.
\end{align*}$$

First, we define $L^2(s, t) : \partial([\bar{d}^2_0, \bar{d}^2_1] \times [0,1]) \rightarrow A_{\rho^e}^0$ as

$$\begin{align*}
L^2(1/2, t) &= \psi_{\rho^e}(\bar{d}^3) \quad \text{for } t \in [0,1], \\
L^2(s, 0) &= \varphi^e(s) \quad \text{for } s \in [1/2,1], \\
L^2(s, 1) &= \varphi^e(s) \quad \text{for } s \in [1/2,1], \\
L^2(1, t) &= \psi_{\rho^e}(1-t) \quad \text{for } t \in [0,1], \\
L^2(1, t) &= \varphi^e(t) \quad \text{for } t \in [1/2,1].
\end{align*}$$

Then, we can extend $L^2$ to $[\bar{d}^2_0, \bar{d}^2_1] \times [0,1]$ because $L^2$ on the boundary is trivial in $\pi_1(A_{\rho^e}^0)$. Here, note that $L^2(1, t)^{-1}$ is in $A_{\rho^e}^2 = \text{Isom}(V_{F^c}, V_{F^c}^{-1})$. If we apply Proposition 7.11(1) with $F = F_{V_{\rho^e}}, (-1)\omega$ and $j = 1, j' = 0, j'' = 3$, then $L^2(1, t)$ and $\text{res}_{0,1} \psi_{\rho^e}$ give the unique element in $A_{\rho^e}$ for each $t \in [0,1]$, say $\psi_{\rho^e}^t$. If we put $\psi_{\rho^e}^t = g \cdot \psi_{\rho^e}$ for $g \in G_{\rho^e}$, then five maps $\psi_{\rho^e}, \psi_{\rho^e}^t, \psi_{\rho^e}^t, \psi_{\rho^e}^t, \psi_{\rho^e}^t,$
ψ_{i_p}'s consist of an element of \( A_G(\mathbb{R}^2/\Lambda, V_B) \), call it \( \Psi_t \), which determines \( (W_{d'})_{i \in I_t^p} \) regardless of \( t \). With these, define other \( L^i \)'s on \( \partial([\tilde{d}^2_+, \tilde{d}^3_+] \times [0, 1]) \) as
\[
L^i(s, 0) = \varphi^i(s), \\
L^i(s, 1) = \varphi'^i(s)
\]
for \( i = 3, 4, 5 \) and
\[
\begin{align*}
L^3(0, t) &= \psi_{i_p}'(d^2), \\
L^3(1, t) &= \psi_{i_p}^{-1}(d^2), \\
L^4(0, t) &= \psi_{i_p}'(d^3), \\
L^4(1, t) &= \psi_{i_p}^{-1}(d^3), \\
L^5(0, t) &= \psi_{i_p}(d^2), \\
L^5(1, t) &= \psi_{i_p}^{-1}(d^3)
\end{align*}
\]
for \( s, t \in [0, 1] \). We can extend these \( L^i \)'s to \([\tilde{d}^2_+, \tilde{d}^3_+] \times [0, 1]\) because path components of \( A_v \) for \( i = 0, 1, 3 \) are all simply connected. Then, it can be checked that \( L_i \) determines \( \Psi^t \) and \( L_i \) is in \( \Omega_{p_e,(W_{d'})_{i \in I_t^p}} \) by Corollary 9.11 So, \( L' \) is a wanted homotopy and we obtain a proof.

This theorem gives a proof of Theorem \( \Box \)

**Proof of Theorem \( \Box \) By** Theorem 10.1 and Lemma 6.3 (2), we obtain a proof. \( \Box \)

**Theorem 10.2.** If \( \rho(G_\chi) \) is one in Table 10.3 then
\[
p_{\text{vect}} \times c_1 : \text{Vect}_{G_\chi}(\mathbb{R}^2/\Lambda, \chi) \rightarrow A_{G_\chi}(\mathbb{R}^2/\Lambda, \chi) \times H^2(\mathbb{R}^2/\Lambda, \chi)
\]
is injective. For each element \( (W_{d'})_{i \in I_t^p} \) in \( A_{G_\chi}(\mathbb{R}^2/\Lambda, \chi) \), the set of the first Chern classes of bundles in the preimage of the element under \( p_{\text{vect}} \) is equal to the set \( \{ \chi(\varphi)(l_3k + k_0) \mid k \in \mathbb{Z} \} \) where \( k_0 \) is dependent on \( (W_{d'})_{i \in I_t^p} \).

| \( R/R_t \) | \( A_t \) |
|----------------|----------------|
| \( Z_2 \), \( Z_4 \) | \( A^3 \) |

**Table 10.2.** \( R \)'s with nontrivial cyclic \( R/R_t \)

**Proof.** When \( R = \rho(G_\chi) \) is given, put
\[
V_B = G_\chi \times (G_\chi)_{2-1}, \quad F_{V_B} = G_\chi \times (G_\chi)_{2-1} (|f^{-1}| \times W_{d-1})
\]
for each \( (W_{d'})_{i \in I_t^p} \in A_{G_\chi}(\mathbb{R}^2/\Lambda, \chi) \). We only deal with the case of \( R/R_t = \mathbb{Z}_2 \) and other cases are calculated in a similar way. In this case, we have
\[
\begin{align*}
\tilde{D}_p & = [b(\tilde{e}^1), \tilde{e}^2, \tilde{e}^3, \tilde{b}(\tilde{e}^3)], \\
(G_\chi)_{b(\tilde{e}^1)} / H & = \mathbb{Z}_2, \\
(G_\chi)_{b(\tilde{e}^2)} / H & = \mathbb{Z}_2
\end{align*}
\]
by Table 1.3 and tables in Section 4. And, no two vertices \( v' \) and \( v'' \) are in the same orbit by Table 1.3. By using these, path components of
\[
\tilde{A}_{b(\tilde{e}^1)}, \quad \tilde{A}_{b(\tilde{e}^2)}, \quad (\tilde{A}_{\tilde{e}^3})_{j, j+2}, \quad (\tilde{A}_{\tilde{e}^3})_{j, j+2}
\]
Lemma 9.12. Here, we may assume that $A_j$ for

Pick an element $(\psi_{z'})_{i \in I_\nu}$ in $\mathcal{A}_{\mathcal{C}_\nu}(\mathbb{R}^2/\Lambda, V_B)$ which determines $(W_{z'})_{i \in I_\nu}$ by Lemma 9.11. Here, we may assume that $\psi_{z'}^{-1}(v^2) = \psi_{b(z')}(b(e^1))$ and $\psi_{\nu}(v^3) = \psi_{b(\nu)}(b(e^3))$. We explain for this. By Proposition 9.11 the map

$$\mathcal{A}_{z'} \to (\mathcal{A}_{z'})_{0,3} \times (\mathcal{A}_{z'})_{1,3}, \quad \psi \mapsto \left(\psi^{-1}(v^2), (\psi(v^2)\psi(v^3))^{-1}\right)$$

is homeomorphic, and $(\mathcal{A}_{z'})_{0,3} = \text{Iso}_H(V_{f-1}, V_f)$ is path-connected so that elements of each component of $\mathcal{A}_{z'}$ can take any values of $(\mathcal{A}_{z'})_{0,3}$. Also, since $\mathcal{A}_{b(\nu)} \subset \mathcal{A}_{\nu}$ by Lemma 7.11(1), we may take an element $\psi_{\nu}$ in the component $(\mathcal{A}_{z'})_{\nu}$ such that $\psi_{\nu}^{-1}(v^2) = \psi_{b(\nu)}(b(e^1))$. So, we may assume that

\[(\star) \quad \psi_{\nu}^{-1}(v^2) = \psi_{b(\nu)}(b(e^1)).\]

In the same reason, we may assume that

\[(\star\star) \quad \psi_{\nu}(v^3) = \psi_{b(\nu)}(b(e^3)).\]

As in the proof of Theorem 10.1 each $\Phi_{D_\nu} = \cup_{i \in I\nu} \varphi^i$ in $\Omega_{D_\nu}(W_{z'})_{i \in I_\nu}$ is homotopic to an element which determines $\varphi_{i \in I_\nu}$. So, we may assume that each $\Phi_{D_\nu}$ determines $\varphi_{i \in I_\nu}$ so that it satisfies

\[(\star\star\star) \quad \varphi^1(1/2) = \varphi^1(1) = \psi_{b(\nu)}(b(e^1)), \quad \varphi^3(0) = \varphi^3(1/2) = \psi_{b(\nu)}(b(e^3))\]

by (\star) and (\star\star).

Let $t_2$ be the map from $\Omega_1$ to $\Omega_{D_\nu}(W_{z'})_{i \in I_\nu}$ which sends an arbitrary $\varphi^2$ to the map $\Phi_{D_\nu} = \cup_{i \in I\nu} \varphi^i$ such that

$$\varphi^1 \equiv \psi_{b(\nu)}(b(e^1)), \quad \varphi^2 = \varphi^2, \quad \varphi^3 \equiv \psi_{b(\nu)}(b(e^3))$$

where $\Omega_2$ is defined as

$$\{ \varphi^2 \in C^0([d^1_\nu, d^2_\nu], \text{Iso}_H(V_{f-1}, V_f)) \mid \varphi^2(d^2_\nu) = \psi_{\nu}(v^2_0) \text{ and } \varphi^2(d^1_\nu) = \psi_{\nu}^{-1}(v^2_0) \}.$$

To calculate $\pi_0(\Omega_{D_\nu}(W_{z'})_{i \in I_\nu})$, we would show that

$$\pi_0(t_2) : \pi_0(\Omega_2) \to \pi_0(\Omega_{D_\nu}(W_{z'})_{i \in I_\nu})$$

is bijective because it is is easy to calculate $\pi_0(\Omega_2)$.

First, we would show that each $\Phi_{D_\nu} = \cup_{i \in I\nu} \varphi^i$ satisfying (\star\star\star) is homotopic to an element $\Phi_{D_\nu}' = \cup_{i \in I\nu} \varphi^i$ in $\Omega_{D_\nu}(W_{z'})_{i \in I_\nu}$ such that $\varphi^1 \equiv \psi_{b(\nu)}(b(e^1))$, $\varphi^3 \equiv \psi_{b(\nu)}(b(e^3))$, and $\Phi_{D_\nu}'$ determines $(\varphi_{i \in I_\nu})_{i \in I_\nu}$, i.e. surjectiveness of $\pi_0(t_2)$. For this, we construct a homotopy $L^i(s, t) : [d^i_\nu, d^i_{\nu+1}] \times [0, 1] \to \mathcal{A}_{z'}^0$ for $i \in I_\nu$ such that the homotopy $L = \cup_{i \in I_\nu} L^i$ satisfies

1. $L_t \in \Omega_{D_\nu}(W_{z'})_{i \in I_\nu}$ for each $t \in [0, 1]$,
2. $L_0 = \Phi_{D_\nu}$, and $L_1$ is constant except $[d^i_\nu, d^i_{\nu+1}]$,
3. $L_1$ determines $(\varphi_{i \in I_\nu})_{i \in I_\nu}$.
Define $L^i(s,t) : \partial([\hat{d}^1_i, \hat{d}^{i+1}_i] \times [0,1]) \to \mathcal{A}^0_{\pi^i}$ for $i = 1, 3$ as

$$
L^1(\frac{1}{2}, t) = L^1(s, 1) = \varphi^1(\frac{1}{2}) \quad \text{for } s \in [\frac{1}{2}, 1],
$$

$$
L^1(s, 0) = \varphi^1(s) \quad \text{for } s \in [\frac{1}{2}, 1],
$$

$$
L^3(1, t) = \varphi^3(1 - \frac{1}{t}),
$$

$$
L^3(\frac{1}{2}, t) = L^3(s, 1) = \varphi^3(\frac{1}{2}) \quad \text{for } s \in [0, \frac{1}{2}],
$$

$$
L^3(s, 0) = \varphi^3(s) \quad \text{for } s \in [0, \frac{1}{2}],
$$

$$
L^3(0, t) = \varphi^3(\frac{1}{t})
$$

for $t \in [0, 1]$, and define a homotopy $L^i(s,t) : [\hat{d}^1_i, \hat{d}^{i+1}_i] \times [0,1] \to \mathcal{A}^0_{\pi^i}$ for $i = 2$ as

$$
L^2(s,t) = \gamma_2((1 - 3s)t) \quad \text{for } s \in [0, \frac{1}{3}],
$$

$$
L^2(s,t) = \varphi^2(3s - 1) \quad \text{for } s \in [\frac{1}{3}, \frac{2}{3}],
$$

$$
L^2(s,t) = \gamma_3((3s - 2)t) \quad \text{for } s \in [\frac{2}{3}, 1]
$$

where

$$
\gamma_2 : [0, 1] \to \text{Iso}_H(V_{f_1}, V_{f_2}), \quad t \mapsto \left(\psi_{\varphi^2}(\bar{v}_2)^3 \psi_{\varphi^2}(\bar{v}_2)^3\right)^{-1}\varphi^1(1 - \frac{1}{t}),
$$

$$
\gamma_3 : [0, 1] \to \text{Iso}_H(V_{f_1}, V_{f_2}), \quad t \mapsto \psi_{\varphi^3}(\bar{v}_3)^3 \psi_{\varphi^3}(\bar{v}_3)^3 \varphi^3(\frac{1}{t}).
$$

By (**), $L^i$’s are well defined, and we can easily extend $L^i$’s to the whole $[\hat{d}^1_i, \hat{d}^{i+1}_i] \times [0, 1]$. If we define pointwise pre clutching maps $\psi_{\varphi^1}^i, \psi_{\varphi^3}^i$ for $t \in [0, 1]$ as

$$
\psi_{\varphi^1}^i(\bar{v}_2^i) = \gamma_2(t), \quad \psi_{\varphi^1}^i(\bar{v}_2^i) = \psi_{\varphi^1}(\bar{v}_2),
$$

$$
\psi_{\varphi^3}^i(\bar{v}_3^i) = \psi_{\varphi^3}(\bar{v}_3), \quad \psi_{\varphi^3}^i(\bar{v}_3^i) = \psi_{\varphi^3}(\bar{v}_3),\quad \psi_{\varphi^3}^i(\bar{v}_3^i) = \gamma_3(t),
$$

then they are contained in $A_{\pi^1}, A_{\pi^3}$ by Proposition 7.11 respectively. And, $L_i$ determines $\{\psi_{b(\varphi^1)}, \psi_{b(\varphi^1)}^t, \psi_{b(\varphi^3)}^t, \psi_{b(\varphi^3)}\}$ and $L_i$ is contained in $\Omega_{\hat{d}^1_i, \hat{d}^{i+1}_i}$ by Corollary 9.11.

So, it is easily checked that $L$ is a wanted homotopy.

Next, for two $\Phi_{\bar{D}_\rho} = \bigcup_{i=1}^{\rho-1} \Phi^i_{\bar{D}_\rho}, \Psi_{\bar{D}_\rho} = \bigcup_{i=1}^{\rho-1} \Phi^i_{\bar{D}_\rho}$ such that

1. $\Phi_{\bar{D}_\rho}$ and $\Phi^i_{\bar{D}_\rho}$ determine $\psi_{\varphi^i}$ in $\Omega_{\hat{d}^1_i, \hat{d}^{i+1}_i}$ for $i = 1, 3$,

2. $\varphi^i = \varphi^i \equiv \psi_{b(\varphi^i)}(b(\bar{e}^i))$ for $i = 1, 3$,

assume that there exists a homotopy $L^i(s,t) : [\hat{d}^1_i, \hat{d}^{i+1}_i] \times [0, 1] \to \mathcal{A}^0_{\pi^i}$ for $i \in I^\rho$ such that $L^i = \bigcup_{i \in I^\rho} L^i$ satisfies

1. $L^i_t \in \Omega_{\hat{d}^1_i, \hat{d}^{i+1}_i}$ for each $t \in [0, 1]$,

2. $L^i_0 = \Phi_{\bar{D}_\rho}$ and $L^i_1 = \Phi^i_{\bar{D}_\rho}$,

Then, we would show that there exists a homotopy $L^{i_0}(s,t) : [\hat{d}^1_i, \hat{d}^{i+1}_i] \times [0, 1] \to \mathcal{A}^0_{\pi^i}$ for $i \in I^\rho$ such that $L^{i_0} = \bigcup_{i \in I^\rho} L^{i_0}$ satisfies

1. $L^{i_0}_t \in \Omega_{\hat{d}^1_i, \hat{d}^{i+1}_i}$ for each $t \in [0, 1]$,

2. $L^{i_0}_0 = \Phi_{\bar{D}_\rho}$ and $L^{i_0}_1 = \Phi^i_{\bar{D}_\rho}$,

3. $L^{i_0} \equiv \psi_{b(\varphi^i)}(b(\bar{e}^i))$ for each $t \in [0, 1]$ and $i = 1, 3$,

4. $L^{i_0}$ determines $\psi_{\varphi^i}$ in $\Omega_{\hat{d}^1_i, \hat{d}^{i+1}_i}$ for each $t \in [0, 1]$,

i.e. injectiveness of $\psi_0(\bar{e}_2)$. To begin with, we would show that $[L^2(0, t)]$ and $[L^2(1, t)]$ are trivial in $\pi_1(\text{Iso}_H(V_{f_1}, V_{f_2}))$. Note that $[L^4(\frac{1}{2}, t)]$ is trivial in $\pi_1(\text{Iso}_H(V_{f_1}, V_{f_2}))$ because $[L^4(\frac{1}{2}, t)]$ is in the trivial $\pi_1((A_{\pi^1(\bar{e}^1)})^0(\psi_{b(\varphi^1)}))$ where we regard $[L^4(\frac{1}{2}, t)]$ as a loop. From this, we obtain that $[L^4(1, t)]$ is trivial in $\pi_1(\text{Iso}_H(V_{f_1}, V_{f_2}))$ because $L^4(s,t)$ for $t = 0, 1$ is constant and $L^4$ is defined over $[\hat{d}^1_i, \hat{d}^{i+1}_i] \times [0, 1]$. Let $L^i_t$
determine \( (\psi_t^i)_{i \in I_\rho} \) in \( \tilde{A}_{G_x}(\mathbb{R}^2/\Lambda, V_\rho) \) for \( t \in [0, 1] \) so that

\[
\begin{align*}
L^2(0, t) &= (\psi_{\tilde{t}^2}^2(\psi_{\tilde{t}^2}^1)^{-1})^t \times 1 \times L^2(1, t), \\
L^2(1, t) &= (\psi_{\tilde{t}^3}^2(\psi_{\tilde{t}^3}^1)^{-1})^t \times 0 \times (\psi_{\tilde{t}^3}^2(\psi_{\tilde{t}^3}^1)^{-1})^t \times 0.
\end{align*}
\]

Here, \( [(\psi_{\tilde{t}^2}^2(\psi_{\tilde{t}^2}^1)^{-1})^t \times 1 \times L^2(1, t)] \) is trivial in \( \pi_1(\text{Iso}_H(V_{f^1}, V_{f^2})) \) because it is contained in \( \pi_1((\mathbb{A}_{A^2})_{1,3}) \) and each component of \( \mathbb{A}_{A^2} \) is simply connected by Proposition 7.11. From this and (****), triviality of \( [L^2(1, t)] \) gives triviality of \( [L^2(0, t)] \) in \( \pi_1(\text{Iso}_H(V_{f^1}, V_{f^2})) \). In the same reason, \( [L^2(1, t)] \) is trivial in \( \pi_1(\text{Iso}_H(V_{f^1}, V_{f^2})) \).

By triviality, there exist two homotopies \( F^i(s, t) : [0, 1] \times [0, 1] \to \text{Iso}_H(V_{f^1}, V_{f^2}) \) for \( i = 2, 3 \) such that

\[
\begin{align*}
F^2(s, 0) &= F^2(s, 1) = F^2(0, t) = \psi_{\tilde{t}^2}(\tilde{t}^2_0), \\
F^2(1, t) &= L^2(0, t), \\
F^3(s, 0) &= F^3(s, 1) = F^3(1, t) = \psi_{\tilde{t}^3}(\tilde{t}^3_0), \\
F^3(0, t) &= L^2(1, t),
\end{align*}
\]

Then, we construct a wanted homotopy \( L'' \) as

\[
\begin{align*}
L''(s, t) &= F^2(3s, t) \quad \text{for } s \in [0, \frac{1}{3}], \\
L''(s, t) &= L^2(3s - 1, t) \quad \text{for } s \in [\frac{1}{3}, \frac{2}{3}], \\
L''(s, t) &= F^3(3s - 2, t) \quad \text{for } s \in [\frac{2}{3}, 1], \\
L''(s, t) &= \psi_{\tilde{t}^3}(b(e^3_1)), \\
L''(s, t) &= \psi_{\tilde{t}^3}(b(e^3_2)).
\end{align*}
\]

Since \( \pi_0(\mathbb{A}_2) \) is bijective and \( \pi_0(\mathbb{A}_2) \) is in one-to-one correspondence with \( \pi_1(\text{Iso}_H(V_{f^1}, V_{f^2})) \) by Lemma 8.3, we have \( \pi_0(\Omega_{D_\rho}(w^\rho)_{i \in I_\rho}) \cong \pi_1(\text{Iso}_H(V_{f^1}, V_{f^2})) \) which is isomorphic to \( Z \) by Lemma 5.1. As in the proof of [K3 Theorem A.], it can be shown that Chern classes of equivariant vector bundles determined by different classes in \( \pi_0(\Omega_{D_\rho}(w^\rho)_{i \in I_\rho}) \) are all different, and the statement on the set of possible Chern classes is obtained. Here, we use the fact the \( \rho(G_X) \)-action on \( \mathbb{R}^2/\Lambda \) is orientation-preserving and the cardinality of the \( G_X \)-orbit of \( b(e^2) \) is equal to \( l_\rho \). Then, we obtain a proof by Lemma 6.3.

11. THE CASE OF \( R/R_t = (\text{id}), D_1 \) WITH \( \mathcal{A} + \bar{c} = \bar{l}_0 \)

In this section, we calculate homotopy of equivariant clutching maps in cases of \( R/R_t = (\text{id}), D_1 \) with \( \mathcal{A} + \bar{c} = \bar{l}_0 \). First, we show that we only have to consider equivariant clutching maps with a simple form. Then, we give a discrimination on whether two equivariant clutching maps with the simple form are equivariantly homotopic or not. With these, we can calculate homotopy of equivariant clutching maps in the case of \( R/R_t = (\text{id}) \). But, in the case of \( R/R_t = D_1 \) with \( \mathcal{A} + \bar{c} = \bar{l}_0 \), Lemma 6.3 is not applied and we should also consider \( G_X \)-isomorphisms in addition to equivariant clutching maps as in cases of \( \rho(G_X) \) is \( \mathbb{Z}_n \times Z \) with odd \( n \) or \( \langle -a_n \rangle \) with even \( n/2 \) of [K].

In these two cases, \( D_\rho = [\bar{v}^2, \bar{v}^3, \bar{v}^0] \) with \( v^i \sim v^{i'} \) and \( (G_X)_x = H \) for each \( x \in \mathbb{R}^2/\Lambda \) by Table 13 and Table 14. For each \( (W_{\rho})_{i \in I_\rho} \) in \( A_{G_X}(\mathbb{R}^2/\Lambda, \chi) \), put

\[
V_B = G_X \times_{(G_X)_{d-1}} W_{d-1}, \quad F_{V_0} = G_X \times_{(G_X)_{d-1}} (\mathbb{R}^2/\Lambda) \times W_{d-1}.
\]

Since \( V_f \cong W_{d-1} \) for each \( f \in \bar{K}_\rho \), in these cases, we may assume that all \( V_f \)'s are the same as \( H \)-representation and \( \text{Iso}_H(V_{f^1}, V_{f^2}) = \text{Iso}_H(V_{f^1}) \). By this, \( \mathcal{A}_{e^2} = \text{Iso}_H(V_{f^1}) \), and Corollary 6.11 is rewritten as follows:
Lemma 11.1. A preclutching map $\Phi_{\rho} = \varphi^2 \cup \varphi^3$ in $C^0(\hat{D}_{\rho}, V_B)$ is contained in $\Omega_{\hat{D}_{\rho}(W_d)}$ if and only if $\varphi^i$’s are $\text{Iso}_H(V_{f-i})$-valued and

\[
\begin{align*}
\varphi^2(1)^{-1} \cdot g_0 \varphi^3(1)^{-1} \cdot g_0^{-1} \cdot \varphi^2(0) g_1 \cdot \varphi^3(0) = \text{id} \quad \text{or} \\
\varphi^2(1)^{-1} \cdot (g_2 g_1) \varphi^3(1) (g_2 g_1)^{-1} \cdot g_1^{-1} \varphi^2(0) g_1 \cdot \varphi^3(0) = \text{id}
\end{align*}
\]

according to $R/R = \{\text{id}\}$ or $D_1$ with $A \hat{c} + \hat{c} = I_0$, respectively.

Proof. We prove this lemma only for the case of $R/R = \{\text{id}\}$ because proof for another case is similar. We would prove that $\Phi_{\rho} = \varphi^2 \cup \varphi^3$ is in the set of Corollary 9.11 if and only if $\varphi^i$’s are $\text{Iso}_H(V_{f-i})$-valued and they satisfy (*). Since $A^0_{\rho} = \text{Iso}_H(V_{f-1})$, the condition $\varphi^i(\hat{x}) \in A^0_{\rho}$ of Corollary 9.11 holds if and only if $\varphi^i$’s are $\text{Iso}_H(V_{f-i})$-valued. So, we do not need pay attention to this condition. First, we prove sufficiency. Corollary 9.11 says that

\[
\begin{align*}
\varphi^2(0) = \psi_{\varphi^2}(\hat{v}^2), \\
\varphi^3(0) = \psi_{\varphi^3}(\hat{v}^3), \\
\varphi^3(1) = \psi_{\varphi^3}(\hat{v}^0)
\end{align*}
\]

for some $(\psi_{\varphi^i})_{\in I_\rho}$ in $A_{G_x}(\mathbb{R}^2/\Lambda, V_B)$. Since $\psi_{\varphi^2} = g_1 \cdot \psi_{\varphi^3}$ and $\psi_{\varphi^0} = g_0^{-1} \cdot \psi_{\varphi^3}$, $\psi_{\varphi^i}^{(3)} = \text{id}$ is expressed as (*). For necessity, define a pointwise preclutching map $\psi_{\varphi^3}$ as

\[
\begin{align*}
\psi_{\varphi^3}(\hat{v}^3) = \varphi^3(0), \\
\psi_{\varphi^3}(\hat{v}^1) = g_1^{-1} \varphi^2(0) g_1, \\
\psi_{\varphi^3}(\hat{v}^0) = g_0 \varphi^3(1)^{-1} g_0^{-1}, \\
\psi_{\varphi^3}(\hat{v}^2) = \varphi^2(1)^{-1}.
\end{align*}
\]

Then, (*) says that $\psi_{\varphi^3}$ is in $A_{\varphi^3}$ by Proposition 7.12. If we put $\psi_{\varphi^2} = g_1 \cdot \psi_{\varphi^3}$ and $\psi_{\varphi^0} = g_0^{-1} \cdot \psi_{\varphi^3}$, then it is easily checked that $\varphi^i$’s are in the set of Corollary 9.11. Therefore, we obtain a proof. $
$}

Given $(W_{d'})_{\in I_{\rho}^+}$, we would pick a simple $(\psi_{d'})_{\in I_\rho}$ in $A_{G_x}(\mathbb{R}^2/\Lambda, V_B)$ which determines $(W_{d'})_{\in I_{\rho}}$ as follows:

Lemma 11.2. For each $(W_{d'})_{\in I_{\rho}}$ in $A_{G_x}(\mathbb{R}^2/\Lambda, \chi)$, there exists the unique $(\psi_{d'})_{\in I_{\rho}}$ in $A_{G_x}(\mathbb{R}^2/\Lambda, V_B)$ which determines $(W_{d'})_{\in I_{\rho}}$ and satisfies

\[
\psi_{\varphi^2}(\hat{v}^2) = \psi_{\varphi^3}^{-1}(\hat{v}^3) = \psi_{\varphi^0}(\hat{v}^0) = \text{id}.
\]

Proof. Since $(G_x)x = H$ for each $x \in \mathbb{R}^2/\Lambda$, we note that $W_{d'}$’s are all isomorphic $H$-representations and any element of $A_{\varphi^3}$ for any $\hat{x}$ determines the same $W_{d_{-1}}$.

First, we prove existence. Since $v^i \sim v^{i'}$, it suffices to construct a suitable element in $A_{\varphi^3}$. Define $\psi_{\varphi^3}$ as

\[
\begin{align*}
\psi_{\varphi^3}(\hat{v}^3) &= \text{id}, \\
\psi_{\varphi^3}(\hat{v}^1) &= g_1^{-1} \text{id} g_1, \\
\psi_{\varphi^3}(\hat{v}^0) &= \text{id}, \\
\psi_{\varphi^3}(\hat{v}^2) &= \psi_{\varphi^3}(\hat{v}^1)^{-1} \psi_{\varphi^3}(\hat{v}^3)^{-1} \psi_{\varphi^3}(\hat{v}^0)^{-1}.
\end{align*}
\]

Here, the expression $g_1^{-1} \text{id} g_1$ might seem awkward, but two $g_1$’s act on different spaces because $\text{id}$ is an element of $\text{Iso}(V_{f-1}, V_B)$. So, $g_1^{-1} \text{id} g_1$ needs not be equal to $\text{id}$. By Proposition 7.12 $\psi_{\varphi^3}$ is contained in $A_{\varphi^3}$. Other maps $\psi_{\varphi^2}, \psi_{\varphi^0}$ are obtained by group action from $\psi_{\varphi^3}$ so that $(\psi_{d'})_{\in I_{\rho}}$ is contained in $A_{G_x}(\mathbb{R}^2/\Lambda, V_B)$. Since $\psi_{\varphi^3}(\hat{v}^1) = g_1^{-1} \text{id} g_1$ and $\psi_{\varphi^2} = g_1 \cdot \psi_{\varphi^3}$, we obtain $\psi_{\varphi^2}(\hat{v}^2) = \text{id}$ so that $(\psi_{d'})_{\in I_{\rho}}$ satisfies the lemma.

Uniqueness is easy. Since $\psi_{\varphi^2} = g_1 \cdot \psi_{\varphi^3}$, the condition $\psi_{\varphi^2}(\hat{v}^2) = \text{id}$ gives $\psi_{\varphi^3}(\hat{v}^1) = g_1^{-1} \text{id} g_1$. And, $\psi_{\varphi^4}^{(3)} = \text{id}$ gives (*). So, $\psi_{\varphi^3}$ in the above is the unique map to satisfy the lemma. Other maps $\psi_{\varphi^2}, \psi_{\varphi^0}$ are uniquely obtained by group action from $\psi_{\varphi^3}$. $
$
For the unique element \((\psi_d)_t \in I_p\) of Lemma 11.2, put \(c_0 = \psi^{-1}_\varepsilon(\varepsilon_d)\). So, if \(\Phi_{D_p} = \varphi^2 \cup \varphi^3\) in \(\Omega_{D_p,(W_{d})_{t} \in I_p^+}\) determines \((\psi_d)_t \in I_p\), then we have

\[
\varphi^2(0) = \varphi^2(1) = \varphi^3(0) = id, \quad \varphi^3(1) = c_0.
\]

Pick a loop \(\sigma : [0, 1] \to \text{Iso}_\mathcal{H}(V_{f-1})\) with \(\sigma(0) = \sigma(1) = id\) such that \([\sigma]\) is a generator of \(\pi_1(\text{Iso}_\mathcal{H}(V_{f-1}))\) where \(\pi_1(\text{Iso}_\mathcal{H}(V_{f-1})) \cong \mathbb{Z}\) by Lemma 11.1 and pick a path \(\gamma_0 : [0, 1] \to \text{Iso}_\mathcal{H}(V_{f-1})\) with \(\gamma_0(0) = id\) and \(\gamma_0(1) = c_0\). To calculate \(\pi_0(\Omega_{D_p,(W_{d})_{t} \in I_p^+})\), we might consider only \(\Phi_{D_p}'\)’s with the following simple form:

**Lemma 11.3.** In the case of \(R/R_t = \langle id \rangle\) or \(D_1\) with \(A\bar{c} + \bar{c} = \bar{l}_0\), any \(\Phi_{D_p} = \varphi^2 \cup \varphi^3\) in \(\Omega_{D_p,(W_{d})_{t} \in I_p^+}\) is homotopic to some \(\Phi_{D_p}' = \varphi^2 \cup \varphi^3\) which satisfies \(\varphi^2 \equiv id, \varphi^3 \equiv \sigma^k \gamma_0\) for some \(k \in \mathbb{Z}\).

**Proof.** We only prove the case of \(R/R_t = D_1\) with \(A\bar{c} + \bar{c} = \bar{l}_0\) because proofs are similar. First, we would construct a homotopy \(L'(s, t) : [d^2_s, d^2_{s+1}] \times [0, 1] \to \text{Iso}(V_{f-1})\) for each \(i \in I_{p'} = \{2, 3\}\) such that the homotopy \(L = L^2 \cup L^3\) satisfies

1. \(L_t \in \Omega_{D_p,(W_{d})_{t} \in I_p^+}\) for each \(t \in [0, 1]\),
2. \(L_0 = \Phi_{D_p}\) and \(L_1 \equiv id\),
3. \(L^3(0, 1) = id\) and \(L^3(1, 1) = c_0\).

Let \(\gamma^i : [0, 1] \to \text{Iso}_\mathcal{H}(V_{f-1})\) for \(i = 2, 3\) be paths satisfying \(\gamma^2(0) = \varphi^2(0), \gamma^3(0) = \varphi^3(0), \gamma^3(1) = id\). Define \(L^2\) on \(\partial([d^2_s, d^2_{s+1}] \times [0, 1])\) as

\[
\begin{align*}
L^2(s, 0) &= \varphi^2(s) \quad \text{for } s \in [0, 1], \\
L^2(0, t) &= \gamma^2(t) \quad \text{for } t \in [0, 1], \\
L^2(s, 1) &= id \quad \text{for } s \in [0, 1], \\
L^2(1, t) &= \varphi^2(1 - 2t) \quad \text{for } t \in [0, \frac{1}{2}], \\
L^2(1, 1) &= \gamma^2(2t - 1) \quad \text{for } t \in [\frac{1}{2}, 1],
\end{align*}
\]

and define \(L^3\) on \(\{[d^3_s, d^3_{s+1}] \times 0\} \cup (\partial[d^3_s, d^3_{s+1}] \times [0, 1])\) as

\[
\begin{align*}
L^3(s, 0) &= \varphi^3(s), \\
L^3(0, t) &= \gamma^3(t), \\
L^3(1, t) &= \varphi^2(1 - 2t) \cdot g_1^{-1} L^2(0, t)^{-1} \cdot g_1 \\
(1.1) & \quad (g_2 g_1)L^3(1, t) g_2 g_1^{-1} = L^2(1, t) \cdot L^3(0, t)^{-1} \cdot g_1^{-1} L^2(0, t)^{-1} g_1
\end{align*}
\]

for each \(s, t\). Then, we can extend these to the whole \(D_p \times [0, 1]\) to obtain a wanted homotopy \(L\) by Lemma 11.1. So, any \(\Phi_{D_p} = \varphi^2 \cup \varphi^3\) in \(\Omega_{D_p,(W_{d})_{t} \in I_p^+}\) is homotopic to some \(\Phi_{D_p}' = \varphi^2 \cup \varphi^3\) which satisfies \(\varphi^2 \equiv id, \varphi^3(0) = id, \varphi^3(1) = c_0\).

Next, we would construct a homotopy \(L^h(s, t) : [d^3_s, d^3_{s+1}] \times [0, 1] \to \text{Iso}(V_{f-1})\) for each \(i \in I_{p'}\) so that the homotopy \(L' = L^2 \cup L^3\) satisfies

1. \(L'_t \in \Omega_{D_p,(W_{d})_{t} \in I_p^+}\) for each \(t \in [0, 1]\),
2. \(L'_0 = \Phi_{D_p}'\) and \(L'_1 \equiv id\),
3. \(L^3(0, t) = id, L^3(1, t) = c_0, L^3 = \sigma^k \gamma_0\) for some \(k \in \mathbb{Z}\).

Since \([0, 1], 0, 1; \text{Iso}_\mathcal{H}(V_{f-1}), \text{id}, c_0] \cong \pi_1(\text{Iso}_\mathcal{H}(V_{f-1}))\) by Lemma 11.3, and each class is represented by \(\sigma^k \gamma_0\) for some \(k \in \mathbb{Z}\), a wanted \(L'\) is easily obtained. Therefore, we obtain a proof. \(\square\)

We can determine whether two arbitrary \(\Phi_{D_p}\) and \(\Phi_{D_p}'\) with the simple form are equivariantly homotopic or not.
Proof. First, we prove that \( \psi \) is injective. For each element \( \xi \in \mathcal{D}_e \), let \( \Phi_{\mathcal{D}_e} = \psi^2 \cup \psi^3 \) and \( \Phi'_{\mathcal{D}_e} = \psi^2 \cup \psi^3 \) in \( \Omega_{\mathcal{D}_e, (w')_t} \). Lemma \[11.1\] gives

\[ (*) \quad g^{-1}_1 L^2(0, t) g_1 \cdot L^3(0, t) \cdot L^2(1, t)^{-1} \cdot (g_2 g_1) L^3(1, t) (g_2 g_1)^{-1} = \text{id} \]

for each \( t \) because proofs are similar.

Next, we prove necessity. Let \( L = L^2 \cup L^3 \) be a homotopy between \( \Phi_{\mathcal{D}_e} \) and \( \Phi'_{\mathcal{D}_e} \) in \( \Omega_{\mathcal{D}_e, (w')_t} \). Define \( L^3(s, t) : \partial([d^3_+, d^3_-] \times [0, 1]) \rightarrow \mathcal{A}_{\mathcal{D}_e} \) as

\[ L^3(s, 0) = \phi^3(s), \]
\[ L^3(s, 1) = \phi^3(s), \]
\[ L^3(0, t) = \sigma^3(t), \]

which satisfies \( (*) \). Since \( L^3 \equiv \text{id} \), \( (**) \) gives \( [L^3(1, t)] = [L^3(0, t)^{-1}] \) in \( \pi_1(\text{Iso}_H(V_{f - 1})) \). Then, it is checked that the homotopy \( L^3 \) on \( \partial([d^3_+, d^3_-] \times [0, 1]) \) is trivial in \( \pi_1(\text{Iso}_H(V_{f - 1})) \). So, we can extend \( L^3 \) to the whole \( [d^3_+, d^3_-] \times [0, 1] \). Therefore, we obtain a proof.

By using these lemmas, we can calculate \( \pi_0(\Omega_{\mathcal{D}_e, (w')_t} \mathcal{D}_e) \).

Theorem 11.5. In the case of \( R/R_t = \langle \text{id} \rangle \) or \( D_1 \) with \( \mathcal{A} \), let \( \Phi_{\mathcal{D}_e} = \psi^2 \cup \psi^3 \) and \( \Phi'_{\mathcal{D}_e} = \psi^2 \cup \psi^3 \) in \( \Omega_{\mathcal{D}_e, (w')_t} \). Lemma \[11.3\] gives

\[ \phi^2 = \phi^2 \equiv \text{id}, \quad \phi^3 = \sigma^3 \cdot \gamma_0, \quad \phi'^3 = \sigma'^3 \cdot \gamma_0 \]
for some $k, k' \in \mathbb{Z}$. By Lemma 11.4, $\Phi_{D_\rho}$ is homotopic to $\Phi'_{D_\rho}$ if and only if $k = k'$. So, we obtain

$$
\pi_0(\Omega_{D_\rho,(W_\mu)}|_{t_0}) \cong \mathbb{Z}.
$$

As in Theorem 10.2, the statement on Chern classes is obtained. Therefore, we obtain a proof by Lemma 9.3.

Now, we can prove Theorem A.

**Proof.** Theorem 10.2 and Theorem 11.5 give a prove Theorem A.

Now, we deal with the case of $R/R_t = D_1$ with $A\vec{e} + \vec{e} = \bar{l}_0$.

**Proposition 11.6.** In the case of $R/R_t = D_1$ with $A\vec{e} + \vec{e} = \bar{l}_0$, the homotopy

$$
\pi_0(\Omega_{D_\rho,(W_\mu)}|_{t_0})
$$

for each element $(W_\mu)_{|t_0}$ in $A\mathcal{C}_2(\mathbb{R}^2/\Lambda, \chi)$ has two elements which give equivariant vector bundles with the same Chern class.

**Proof.** First, we obtain that $\pi_0(\Omega_{D_\rho,(W_\mu)}|_{t_0})$ has two elements for each $(W_\mu)_{|t_0} \in A\mathcal{C}_2(\mathbb{R}^2/\Lambda, \chi)$ similarly to the proof of Theorem 11.5.

To prove the second statement of the proposition, let $\Phi_{D_\rho} = \varphi^2 \cup \varphi^3$ in $\Omega_{D_\rho,(W_\mu)}|_{t_0}$ be the element to satisfy

$$
\varphi^2 \equiv \text{id} \quad \text{and} \quad \varphi^3 = \sigma^k \gamma_0
$$

for some $k \in \mathbb{Z}$. We would show that the Chern class of the bundle $E^{\Phi_{D_\rho}}$ is independent of $k$ as in the proof of [KIl Theorem C.]. To resemble the proof, we need merge trivializations over a pair of faces as explained in the below. Denote by $|f^{-1}| \cup_c |f^2|$ the quotient of the disjoint union $|f^{-1}| \amalg |f^2|$ by the identification of two points $p_{\partial[1]}(\hat{x})$ in $|f^{-1}|$ and $p_{\partial[2]}(|c|(\hat{x}))$ in $|f^2|$ for each $\hat{x}$ in $|c^3|$. Since $\varphi^2 \equiv \text{id}$, trivializations of $F_{V_\mu}$ over $|f^{-1}|$ and $|f^2|$ merge into a trivialization over $|f^{-1}| \cup_c |f^2|$. Pick $g_k$’s in $G_\chi$ such that

$$
\mathbb{R}^2/\Lambda = \pi \left( g_1 |f^{-1}| \cup_c g_1 |f^2| \right) \cup \cdots \cup \pi \left( g_n |f^{-1}| \cup_c g_n |f^2| \right)
$$

for $k = 1, \cdots, l_0$ where $l_0 = |\Lambda_t/\Lambda|$ and $g_1 = \text{id}$. Similar to $|f^{-1}| \cup_c |f^2|$, trivializations of $F_{V_\mu}$ over $g_k |f^{-1}|$ and $g_k |f^2|$ are merged through $g_k \varphi^2 g_k^{-1}$. By using these trivializations, we calculate Chern class.

Let $\Phi$ in $\Omega_{V_\mu}$ be the extension of $\Phi_{D_\rho}$. Since $|\hat{c^3}| \cup g_1^{-1} g_2 |c(|\hat{c^3}|)$ give an edge of $|f^{-1}| \cup_c |f^2|$, we investigate $\Phi$ on it to calculate Chern class. Recall that we parameterize $|\hat{c^3}|$ by $s \in [0, 1]$ so that $\hat{c^3}_{0,+} = 0$ and $\hat{c^3}_{0,-} = 1$. By using this, parameterize $g_1^{-1} g_2 |c(|\hat{c^3}|)$ with $[-1, 0]$ to satisfy $g_1^{-1} g_2 |c(\hat{x})| = s - 1$ for each $\hat{x} = s$ in $|\hat{c^3}|$ with $s \in [0, 1]$. Then, $\Phi(s - 1)$ for $s \in [0, 1]$ is equal to

$$
\Phi(g_1^{-1} g_2 |c|(s)) = g_1^{-1} g_2 \Phi(|c|(s)) g_2^{-1} g_1
$$

$$
= g_1^{-1} g_2 \Phi(s)^{-1} g_2^{-1} g_1
$$

$$
= g_1^{-1} g_2 (\sigma^k \gamma_0)(s)^{-1} g_2^{-1} g_1.
$$

Here, note that $\Phi(-1) = g_1^{-1} g_2 \text{id} g_2^{-1} g_1$. Since $\Phi(s) = \sigma^{k} \gamma_0(s)$ on $|\hat{c^3}|$ and $\Phi(s - 1) = g_1^{-1} g_2 (\sigma^k \gamma_0)(s)^{-1} g_2^{-1} g_1$ on $g_1^{-1} g_2 |c(|\hat{c^3}|)$ merge to cancel each other regardless of $k$ in the level of

$$
[-1, 1], -1, 1; \text{Iso}_{H}(V_{f^{-1}}), g_1^{-1} g_2 \text{id} g_2^{-1} g_1, c_0,
$$

Chern class of $E^{\Phi_{D_\rho}}$ is independent of $k \in \mathbb{Z}$. This gives a proof.
Since elements in $\pi_0(\Omega_{D^p, (W_{d^i})}^{1})$ are not discriminated by their Chern classes, we need to consider $G_\chi$-isomorphisms by Proposition 6.2. In the case of $R/R_i = D_1$ with $A\hat{c} + \tilde{\epsilon} = \hat{\ell}_0$, $(\hat{G}_\chi)_s = H$ for each $x \in \mathbb{R}^2/\Lambda$ and $G_\chi$ acts transitively on $B$ so that any $H$-isomorphism of $(\text{res}_{H}^\chi F_{V_{\nu}})|_{f^{-1}}$, which is just a function from $|f^{-1}|$ to $\text{Iso}_H(V_{f^{-1}})$, can be equivariantly extended to $G_\chi$-isomorphism of $F_{V_{\nu}}$ over the whole $|\hat{C}_\rho|$. In this extension, we need restrict our arguments to $|\partial f^{-1}|$ or $\hat{L}_\rho$ because $G_\chi$-isomorphisms are related with equivariant clutching maps which are defined on $\hat{L}_\rho$. A map $\hat{\theta} : |\partial f^{-1}| \to \text{Iso}_H(V_{f^{-1}})$ can be extended to an $H$-isomorphism $\hat{\theta}$ of $(\text{res}_{H}^\chi F_{V_{\nu}})|_{f^{-1}}$ if $[\hat{\theta}]$ is trivial in $\pi_1(\text{Iso}_H(V_{f^{-1}}))$. For such a $\hat{\theta}$, define $\theta' : |\hat{c}^i| \to \text{Iso}_H(V_{f^{-1}})$ as the function $\theta'(s) = \hat{\theta}|_{|c^i|}(p_1(s))$ for each $i \in \mathbb{Z}_{n^p}$ and $s \in |\hat{c}^i|$. For $\theta$ and an extension $\hat{\theta}$ such that $\hat{\theta} = \hat{\theta}|_{|\partial f^{-1}|}$, there exists the $G_\chi$-isomorphism of $F_{V_{\nu}}$ which extends $\hat{\theta}$ to the whole $|\hat{C}_\rho|$. An $G_\chi$-isomorphism $\hat{\Theta}$ of $F_{V_{\nu}}$ satisfying the condition $(p_1^*\hat{\Theta})\Phi(p_1^*\Theta)^{-1} = \Phi'$ of Proposition 6.2 has a simple form at $|\partial f^{-1}|$ as follows:

**Lemma 11.7.** Assume that $R/R_i = D_1$ with $A\hat{c} + \tilde{\epsilon} = \hat{\ell}_0$. Let $\Phi_{D^p} = \varphi^2 \cup \varphi^3$ and $\Phi'_{D^p} = \varphi^2 \cup \varphi^3$ be elements in $\Omega_{D^p, (W_{d^i})}^{1}$ such that

\begin{equation}
(11.2) \quad \varphi^2 = \varphi^2 \equiv \text{id}, \quad \varphi^3(0) = \varphi^3(0) = \text{id}, \quad \varphi^3(1) = \varphi^3(1) = c_0,
\end{equation}

and let $\Phi$ and $\Phi'$ be the extensions of $\Phi_{D^p}$ and $\Phi'_{D^p}$, respectively. Let $\Theta$ be a $G_\chi$-isomorphism of $F_{V_{\nu}}$ satisfying $(p_1^*\hat{\Theta})\Phi(p_1^*\Theta)^{-1} = \Phi'$. Then, there exists a $G_\chi$-isomorphism $\hat{\Theta}'$ of $F_{V_{\nu}}$ which still satisfies $(p_1^*\hat{\Theta}')\Phi(p_1^*\Theta)^{-1} = \Phi'$ and also satisfies $\Theta'(\hat{c}^i) = \text{id}$ for each $i$.

**Proof.** First, we show that $\Theta(\hat{c}^0)$ determines $\Theta(\hat{c}^i)$ for $i = 1, 2, 3$. Put $\theta = \Theta|_{|\partial f^{-1}|}$. By equivariance of $\Theta$, $p_1^*\Theta$ on $|c(\hat{c}^2)| \cup |c(\hat{c}^3)|$ is calculated as follows:

\begin{equation}
(11.3) \quad p_1^*\Theta(s) = p_1^*\Theta(\hat{x}) = g_2 p_1^*\Theta(g_2^{-1}\hat{x})g_2^{-1} \quad \text{for } s = \hat{x} \in |c(\hat{c}^2)| = g_2 \theta^0(1 - s)g_2^{-1},
\end{equation}

\begin{equation}
\quad p_1^*\Theta(s) = p_1^*\Theta(\hat{x}) = g_1 p_1^*\Theta(g_1\hat{x})g_1 \quad \text{for } s = \hat{x} \in |c(\hat{c}^3)| = g_1^{-1} \theta^1(s)g_1.
\end{equation}

where we recall that we parameterize each edge $c(\hat{c}^i)$ linearly by $s \in [0, 1]$ so that $|c|(s) = 1 - s$ for $s \in |\hat{c}^i|$. On $D_{\rho}$, the condition $(p_1^*\Theta)\Phi(p_1^*\Theta)^{-1} = \Phi'$ is written as

\begin{equation}
(11.4) \quad g_2 \theta^0(s)g_2^{-1} \varphi^2(s) \theta^2(s)^{-1} = \varphi^2(s),
\end{equation}

\begin{equation}
\quad g_1^{-1} \theta^1(s)g_1 \varphi^3(s) \theta^3(s)^{-1} = \varphi^3(s)
\end{equation}

by using (11.3). By applying (11.2) and substituting $s = 0, 1$ to these, we obtain

\begin{equation}
\quad g_2 \theta^0(0)g_2^{-1} = \theta^2(0), \quad g_2 \theta^0(1)g_2^{-1} = \theta^2(1),
\end{equation}

\begin{equation}
\quad g_1^{-1} \theta^1(0)g_1 = \theta^3(0), \quad g_1^{-1} \theta^1(1)g_1 = c_0 \theta^3(1)c_0^{-1}.
\end{equation}

That is,

\begin{equation}
(11.5) \quad g_2 \theta^0(\hat{c}^0)g_2^{-1} = \theta(\hat{c}^2), \quad g_2 \theta^0(\hat{c}^0)g_2^{-1} = \theta(\hat{c}^3),
\end{equation}

\begin{equation}
\quad g_1^{-1} \theta^1(\hat{c}^3)g_1 = \theta(\hat{c}^2), \quad g_1^{-1} \theta^1(\hat{c}^3)g_1 = c_0 \theta(\hat{c}^0)c_0^{-1}.
\end{equation}

These show that $\Theta(\hat{c}^0)$ determines $\theta(\hat{c}^i)$ for $i = 1, 2, 3$. Let us express these relations by three maps. Define $\delta^i : \text{Iso}_H(V_{f^{-1}}) \to \text{Iso}_H(V_{f^{-1}})$ for $i = 1, 2, 3$ as

\begin{equation}
A \mapsto g_1c_0Ae_0^{-1}g_1^{-1}, \quad g_2Ag_2^{-1}, \quad g_1^{-1}g_2Ag_2^{-1}g_1,
\end{equation}
respectively. Note that \( \delta^i(\theta^i) = \theta^i \) for \( i = 1, 2, 3 \) by (11.3). Let \( \gamma : [0,1] \to \text{Iso}_H(V_f^{-1}) \) be a path such that \( \gamma(0) = \text{id} \) and \( \gamma(1) = \theta^0 \). By using these, we would define a new map \( \theta' : |\partial f^{-1}| \to \text{Iso}_H(V_f^{-1}) \) from which an extension \( \Theta' \) will be obtained. If we define \( \Theta^i : |\tilde{c}| \to \text{Iso}_H(V_{f^{-1}}) \) for \( i = 0, 3 \) as

\[
\begin{align*}
\theta^0(s) &= \gamma(3s) \quad \text{for } s \in [0, 1/3], \\
\theta^0(s) &= \theta^0(3s - 1) \quad \text{for } s \in [1/3, 2/3], \\
\theta^0(s) &= \delta^3(\gamma(3 - 3s)) \quad \text{for } s \in [2/3, 1], \\
\theta^0(s) &= \delta^3(\gamma(s)) \quad \text{for } s \in [0, 1/3], \\
\theta^0(s) &= \theta^0(3s - 1) \quad \text{for } s \in [1/3, 2/3], \\
\theta^0(s) &= \gamma(3 - 3s) \quad \text{for } s \in [2/3, 1],
\end{align*}
\]

then it is easily checked that both are well-defined and

\[
\theta^0(s) = \text{id}, \quad \theta^3(s) = \text{id}
\]

for \( s = 0, 1 \). If we define \( \theta^1 \) and \( \theta^2 \) by substituting four \( \theta^0 \)'s for \( \theta^i \)'s in (11.3), then \( \theta^0 \)'s give the map \( \theta' : |\partial f^{-1}| \to \text{Iso}_H(V_f^{-1}) \) satisfying \( \theta^0(s) = \theta'|_{|\tilde{c}|}(\tilde{p}_L|_{|\tilde{c}|}(s)) \) for each \( i \) and \( s \). Similarly, if we define \( \tilde{\theta}_i' : |\partial f^{-1}| \to \text{Iso}_H(V_f^{-1}) \) by using \( \gamma_i(s) = \gamma(1 - t(1 - s)) \) instead of \( \gamma \) in definition of \( \theta' \), then it gives a homotopy between \( \theta \) and \( \theta' \). So, \( \Theta = [\theta'] \) in \( \pi_1(\text{Iso}_H(V_f^{-1})) \) is trivial, and \( \theta' \) can be extended to the whole \( |\kappa| \). If \( \Theta' \) is an extension of \( \theta' \), then \( \Theta'(\tilde{c}) = \text{id} \) for each \( i \) by definition of \( \theta' \) and also \( (p_L|_{|\tilde{c}|}\Theta)(p_L|_{|\tilde{c}|}\Theta')^{-1} = \Phi' \) because \( \theta' \) satisfies (11.4) Therefore, \( \Theta' \) is a wanted \( H \)-isomorphism.

Now, we can finish classification.

**Theorem 11.8.** In the case of \( R/R_c = D_1 \) with \( A\tilde{c} + \tilde{c} = \tilde{c}_0 \), the preimage of each \( (W_{d'}')_{d' \in d'} \) in \( A_{d'}(\mathbb{R}^2, \Lambda, \chi) \) under \( p_{\text{vec}} \) has two elements.

**Proof.** By Proposition 11.6 \( \pi_0(\Omega_{D_{d'}}(W_{d'}'))_{d' \in d'} \) has two elements \( [\Phi_{D_{d'}}] \neq [\Phi'_{D_{d'}}] \). We would show that they determine non-isomorphic equivariant bundles. We may assume that \( \Phi_{D_{d'}} = \varphi^2 \cup \varphi^3 \), \( \Phi'_{D_{d'}} = \varphi^2 \cup \varphi^3 \) with \( \varphi^2 = \varphi^3 \equiv \text{id} \), \( \varphi^3(s) = \sigma_{k', \gamma_{00}} \), \( \varphi^3(s) = \sigma_{k', \gamma_{00}} \) for some \( k', \gamma_{00} \in \mathbb{Z} \) by Lemma 11.3. By Lemma 11.4 \( k \) is not congruent to \( k' \) modulo 2. Let \( \Phi \) and \( \Phi' \) be the extensions of \( \Phi_{D_{d'}} \) and \( \Phi'_{D_{d'}} \), respectively. Assume that two bundles \( \varphi^2 \) and \( \varphi^3 \) are \( G \)-isomorphic, i.e.

\[
(p_L|_{|\tilde{c}|}\Theta)(p_L|_{|\tilde{c}|}\Theta')^{-1} = \Phi' \quad \text{for some } \Theta \text{ by Proposition 6.2}.
\]

By Lemma 11.7 we may assume that \( \Theta(\tilde{c}) = \text{id} \) for each \( i \). Put \( \theta = \Theta|_{|\partial f^{-1}|} \). Formula (11.4) says that \( [\theta^0(s)] = [\theta^2(s)] \) in \( \pi_1(\text{Iso}_H(V_f^{-1})) \) because \( \varphi^2 = \varphi^3 \equiv \text{id} \). This gives \( [\theta^1(s)] \equiv [\theta^3(s)] \mod 2 \) in \( \pi_1(\text{Iso}_H(V_f^{-1})) \) because \( [\theta] \) is trivial in \( \pi_1(\text{Iso}_H(V_f^{-1})) \). From this, \( k \equiv k' \mod 2 \) because \( g_{1}^{-1} \theta^1(1 - s)g_{1} \varphi^3(s) \theta^3(s)^{-1} \equiv \varphi^3(s) \) by (11.4). This is a contradiction. Therefore, \( \varphi^2 \) and \( \varphi^3 \) are not \( G \)-isomorphic and we obtain a proof.

\[\square\]

In summary, we obtain a proof of Theorem 13

**Proof.** Proposition 11.6 and Theorem 11.8 gives a proof of Theorem 13

**References**

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