The global existence theorem for quasi-linear wave equations with multiple speeds, II

Dedicated to Professor Thomas C. Sideris on the occasion of his 60th birthday

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Abstract

The Cauchy problem is studied for systems of quasi-linear wave equations with multiple speeds in two space dimensions. Using the method of Klainerman and Sideris together with the localized energy estimate, we give an alternative proof of a beautiful result of Hoshiga and Kubo.

Key Words: global existence, quasi-linear wave equations, non-resonance
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1 Introduction

In his previous paper [3] with the same title as above, the present author considered the problem of global existence of small solutions to the Cauchy problem for systems of quasi-linear wave equations with multiple speeds. Relying entirely upon the Klainerman-Sideris method [8], he aimed at giving a unified proof of the beautiful results of Hoshiga and Kubo [6] and Yokoyama [17] whose proofs built upon point-wise (in space and time) hard estimates of the fundamental solutions. This aim of [3] was left unaccomplished. Indeed, in the case of three space dimensions the same result as Yokoyama [17] was obtained without any reliance upon such hard point-wise estimates of the fundamental solution. In the case of two space dimensions, however, by the Klainerman-Sideris method, the author was only able to obtain a global existence result under the very restrictive assumption that initial data were compactly supported. The purpose of revisiting [3] is to remove this assumption and obtain the same result as Hoshiga and Kubo [6]. We rest our proof on the method of Klainerman and Sideris together with two other
ingredients: the space-time $L^2((0,\infty) \times \mathbb{R}^2)$ estimate and the $\dot{H}^{1/2}(\mathbb{R}^2)$ estimate. In this way, we avoid using the Hardy-type inequality of Lindblad (Lemma 1.2 of [9], Lemma 3.3 of [7]), while in [3] we resorted to using it under the condition that the solutions were compactly supported in the space variables for every fixed time.

This paper is organized as follows. We explain the notation in the next section, and then state the main theorem in Section 3. Useful Sobolev-type inequalities and crucial estimates of the null forms are collected in Section 4. Weighted $L^2(\mathbb{R}^2)$-norms of the second or higher-order derivatives are shown to be bounded by generalized energies in Section 5. In Section 6, we carry out higher-order energy estimates, space-time $L^2$-estimates, $\dot{H}^{1/2}(\mathbb{R}^2)$ estimates, and lower-order energy estimates to complete the proof of the main theorem.

2 Notation

As in the previous paper [3], we follow Sideris and Tu [14] to use the notation. Repeated indices are summed if lowed and uppered. Greek indices range from 0 to 2 (space dimensions), and roman indices from 1 to $m$. We shall consider systems of $m$ quasi-linear equations. Points in $(0, \infty) \times \mathbb{R}^2$ are denoted by $(x^0, x^1, x^2) = (t, x)$. In addition to the usual partial differential operators $\partial_\alpha = \partial/\partial x^\alpha$ ($\alpha = 0, 1, 2$) with the abbreviation $\partial = (\partial_0, \partial_1, \partial_2) = (\partial_t, \nabla)$, we use the generator of Euclid rotation $\Omega = x^1 \partial_2 - x^2 \partial_1$ and of space-time scaling $S = x^\alpha \partial_\alpha$. The set of these 5 vector fields is denoted by $\Gamma = \{\Gamma_0, \Gamma_1, \ldots, \Gamma_4\} = \{\partial, \Omega, S\}$. We employ the multi-index notation in Sideris and Tu [14] to mean $\Gamma^a := \Gamma_{a_0} \cdots \Gamma_{a_\kappa}$ for $a = (a_1, \ldots, a_\kappa)$, a sequence of indices $a_i \in \{0, \ldots, 4\}$ of length $|a| = \kappa$. It is convenient to set $\Gamma^a = 1$ if $|a| = 0$. Suppose that $b$ and $c$ are disjoint subsequences of $a$, allowing that $|b| = 0$ or $|c| = 0$. We say $b + c = a$ if $|b| + |c| = |a|$, $b + c < a$ if $|b| + |c| < |a|$

The D’Alembertian, which acts on vector-valued functions $u : (0, \infty) \times \mathbb{R}^2 \to \mathbb{R}^m$, is denoted by

$$\Box = \text{Diag}(\Box_1, \ldots, \Box_m), \quad \Box_k = \frac{\partial^2}{\partial t^2} - c_k^2 \Delta.$$  

In what follows, we suppose that each of the $m$ propagation speeds is different
from the other $m - 1$. Without loss of generality, we assume

$$0 < c_1 < c_2 < \cdots < c_m.$$  

Associated with this operator, the standard and generalized energies are defined as

$$E_1(u(t)) = \frac{1}{2} \sum_{k=1}^{m} \int_{\mathbb{R}^2} \left( |\partial_t u^k(t,x)|^2 + c_k^2 |\nabla u^k(t,x)|^2 \right) dx,$$

$$E_\kappa(u(t)) = \sum_{|a| \leq \kappa - 1} E_1(\Gamma^a u(t)), \quad \kappa = 2, 3, \ldots$$

Allowing a higher-order energy to grow polynomially in time but bounding a lower-order one uniformly in time, we build up a series of estimates of the generalized energies. The auxiliary norm

$$M_\kappa(u(t)) = \sum_{k=1}^{m} \sum_{|a|=2}^{\kappa - 1} \sum_{|b| \leq \kappa - 2} \| \langle c_k t - |x| \rangle \partial^a \Gamma^b u^k(t) \|_{L^2(\mathbb{R}^2)}$$

plays an intermediate role. Here and later on as well we use the notation $\langle A \rangle = \sqrt{1 + |A|^2}$ for a scalar or a vector $A$. We also use

$$N(u(t)) := \left( \sum_{k=1}^{m} \left( \| D_x^{1/2} u^k(t) \|_{L^2(\mathbb{R}^2)} + \| D_x^{-1/2} \partial_t u^k(t) \|_{L^2(\mathbb{R}^2)} \right) \right)^{1/2},$$

$$N_\kappa(u(t)) := \sum_{|a| \leq \kappa} N(\Gamma^a u(t)).$$

3 Result

We consider the Cauchy problem for a system of quasi-linear wave equations

$$\Box u = F(\partial u, \partial^2 u) \quad \text{in} \ (0, \infty) \times \mathbb{R}^2$$

subject to the initial data

$$u(0) = \varphi, \quad \partial_t u(0) = \psi.$$  

We assume that the $k$-th component of the vector function $F$ takes the form

$$F^k(\partial u, \partial^2 u) = G^k(u, u, u) + H^k(u, u, u),$$

where

$$G^k(u, v, w) = G_{ijkl}^{k, \alpha \beta \gamma} \partial_\alpha u^i \partial_\beta v^j \partial_\gamma \partial_\delta w^l,$$

$$H^k(u, v, w) = H_{ijkl}^{k, \alpha \beta \gamma} \partial_\alpha u^i \partial_\beta v^j \partial_\gamma \partial_\delta w^l.$$
for real constants $G_{ijl}^{k,\alpha\beta\gamma\delta}$ and $H_{ijl}^{k,\alpha\beta\gamma\delta}$. We refer to a term as non-resonant if $(i,j,l) \neq (k,k,k)$ in its coefficient. The remaining ones are said to be resonant.

Since our proof is based on the energy integral method, we naturally suppose the symmetry condition

$$G_{ijl}^{k,\alpha\beta\gamma\delta} = G_{ijl}^{k,\alpha\delta\beta\gamma} = G_{ijl}^{l,\alpha\beta\gamma\delta}.$$  \hspace{1cm} (3.4)

We are now in a position to recall the null condition in the setting of multiple speeds which Agemi and Yokoyama [1] proposed: For every $k = 1, \ldots, m$ there holds

$$G_{kkk}^{k,\alpha\beta\gamma\delta} X_\alpha X_\beta X_\gamma X_\delta = H_{kkk}^{k,\alpha\beta\gamma\delta} X_\alpha X_\beta X_\gamma = 0$$  \hspace{1cm} (3.5)

for all $X = (X_0, X_1, X_2) \in \{ X \in \mathbb{R}^{1+2} : X_0^2 - c_k^2(X_1^2 + X_2^2) = 0 \}$. The main theorem of this paper reads as follows.

**Main Theorem.** Assume the different-speed condition (2.1), the symmetry condition (3.4), and the null condition (3.5). Let $\kappa \geq 9$. Then there exist positive constants $\varepsilon$ and $A$ with the following property: If initial data is small so that

$$E_{\kappa}^{1/2}(u(0)) \exp \left( AE_{\kappa}^{1/2}(u(0))(E_{\kappa}^{1/2}(u(0)) + N_{\kappa-2}(u(0))) \right) < \varepsilon$$  \hspace{1cm} (3.6)

may hold, then the problem (3.1)-(3.2) has a unique global in time solution satisfying

$$E_{\kappa}(u(t)) \leq 4E_{\kappa}(u(0))(1 + t)^{C\varepsilon^2}, \quad E_{\kappa-2}(u(t)) < 4\varepsilon^2,$$  \hspace{1cm} (3.7)

$$\sum_{|\alpha| \leq \kappa-3} \| (x)^{-1} \partial^\alpha u \|_{L^2((0,T) \times \mathbb{R}^2)} \leq C\varepsilon \log(2 + T),$$  \hspace{1cm} (3.8)

$$N_{\kappa-2}(u(t)) \leq N_{\kappa-2}(u(0)) + C\varepsilon^2 E_{\kappa}^{1/2}(u(0))(1 + t)^{(1/4)+C\varepsilon^2}$$  \hspace{1cm} (3.9)

for all $t, T > 0$.

**Remark.** The quantities $E_{\kappa}(u(0))$, $E_{\kappa-2}(u(0))$, and $N_{\kappa-2}(u(0))$ depend on the size of the initial data $(\varphi, \psi)$. Indeed, for given data $(\varphi, \psi)$, we can calculate the derivatives of the solution $u$ at $t = 0$ up to the $\kappa$-th order by using the equation (3.1). In this way, we can determine these three quantities explicitly.
4 Preliminaries

In this section, we collect several lemmas concerning commutation relations, some estimates of the null forms, and the Sobolev-type inequalities.

We begin with the commutation relations. Let $[\cdot,\cdot]$ be the commutator. In addition to the well-known facts

$$[\partial_\alpha, \Box] = 0, \quad [\Omega, \Box] = 0, \quad \text{and} \quad [S, \Box] = -2\Box,$$

(4.1)

we need the commutation relations of the vector fields $\Gamma$ with respect to the nonlinear terms. Recall the nonlinear terms $G = (G^1, \ldots, G^m)$ and $H = (H^1, \ldots, H^m)$ defined in (3.3). Part (i) of the following lemma implies that the null structure is preserved upon differentiation, and Part (ii) together with (4.1) inductively shows that, for any $a$, the nonlinear term of the equation (4.4) also possesses the null structure.

**Lemma 4.1** (i) For any $\Gamma^a$, the following equalities hold:

$$\Gamma^a G(u, v, w) = \sum_{b+c+d+e=a} G_e(\Gamma^b u, \Gamma^c v, \Gamma^d w),$$

(4.2)

$$\Gamma^a H(u, v, w) = \sum_{b+c+d+e=a} H_e(\Gamma^b u, \Gamma^c v, \Gamma^d w).$$

(4.3)

Here each $G_e$ (resp. $H_e$) is a cubic nonlinear term of the form which $G$ (resp. $H$) has in (3.3). In particular, $G_e = G$, $H_e = H$ if $b + c + d = a$ in (4.2)-(4.3). Moreover, if the original nonlinearities $G$ and $H$ have the null structure (3.5), then so does each of new nonlinearities $G_e$ and $H_e$.

(ii) Let $u$ be a smooth solution of (3.1)-(3.3). Then, for any $\Gamma^a$, the equalities

$$\Box \Gamma^a u = \sum_{b+c+d+e=a} G_e(\Gamma^b u, \Gamma^c u, \Gamma^d u)$$

$$+ \sum_{b+c+d+e=a} H_e(\Gamma^b u, \Gamma^c u, \Gamma^d u) - [\Gamma^a, \Box] u$$

(4.4)

hold.

**Proof.** See Lemma 4.1 of Sideris and Tu [14].

The next lemma, which crucially comes into play in the estimates of lower-order energies, is the statement of gain of additional decay in nonlinearities with the null structure (3.5).
Lemma 4.2 For any smooth scalar functions \( u, v, w \) and \( z \), the following inequalities hold for \( r \geq c_k t/2 \):

\[
G_{k,k,k}^{k,\alpha,\beta,\gamma,\delta} \frac{\partial}{\partial x} u \frac{\partial}{\partial y} v \frac{\partial}{\partial z} w
\leq C(t)^{-1} \left[ \| \frac{\partial u}{\partial t} \| \| \frac{\partial^2 w}{\partial t^2} \| + \| \frac{\partial u}{\partial t} \| \| \frac{\partial w}{\partial t} \| + \langle c_k t - r \rangle \| \frac{\partial u}{\partial t} \| \| \frac{\partial^2 w}{\partial t^2} \| \right],
\]

\[
G_{k,k,k}^{k,\alpha,\beta,\gamma,\delta} \frac{\partial}{\partial x} u \frac{\partial}{\partial y} v \frac{\partial}{\partial z} w
\leq C(t)^{-1} \left[ \| \frac{\partial u}{\partial t} \| \| \frac{\partial w}{\partial t} \| + \| \frac{\partial^2 u}{\partial t^2} \| \| \frac{\partial w}{\partial t} \| + \langle c_k t - r \rangle \| \frac{\partial^2 u}{\partial t^2} \| \| \frac{\partial w}{\partial t} \| \right],
\]

\[
G_{k,k,k}^{k,\alpha,\beta,\gamma,\delta} \frac{\partial}{\partial x} u \frac{\partial}{\partial y} v \frac{\partial}{\partial z} w
\leq C(t)^{-1} \left[ \| \frac{\partial u}{\partial t} \| \| \frac{\partial w}{\partial t} \| + \| \frac{\partial^2 u}{\partial t^2} \| \| \frac{\partial w}{\partial t} \| + \langle c_k t - r \rangle \| \frac{\partial^2 u}{\partial t^2} \| \| \frac{\partial w}{\partial t} \| \right],
\]

\[
G_{k,k,k}^{k,\alpha,\beta,\gamma,\delta} \frac{\partial}{\partial x} u \frac{\partial}{\partial y} v \frac{\partial}{\partial z} w
\leq C(t)^{-1} \left[ \| \frac{\partial u}{\partial t} \| \| \frac{\partial w}{\partial t} \| + \| \frac{\partial^2 u}{\partial t^2} \| \| \frac{\partial w}{\partial t} \| + \langle c_k t - r \rangle \| \frac{\partial^2 u}{\partial t^2} \| \| \frac{\partial w}{\partial t} \| \right],
\]

\[
H_{k,k,k}^{k,\alpha,\beta,\gamma,\delta} \frac{\partial}{\partial x} u \frac{\partial}{\partial y} v \frac{\partial}{\partial z} w
\leq C(t)^{-1} \left[ \| \frac{\partial u}{\partial t} \| \| \frac{\partial w}{\partial t} \| + \| \frac{\partial^2 u}{\partial t^2} \| \| \frac{\partial w}{\partial t} \| + \langle c_k t - r \rangle \| \frac{\partial^2 u}{\partial t^2} \| \| \frac{\partial w}{\partial t} \| \right].
\]

Proof. We have only to mimic the proof of Lemma 5.1 of Sideris and Tu [14]. □

The following lemma is concerned with Sobolev-type inequalities.

Lemma 4.3 The following inequalities hold for any smooth vector-valued function \( u : (0, \infty) \times \mathbb{R}^2 \to \mathbb{R}^m \), provided that the norms on the right-hand side are finite:

\[
\langle r \rangle^{1/2} |\partial u(t, x)| \leq CE_k^{1/2}(u(t)),
\]

\[
\langle r \rangle^{1/2} |c_j t - r|^{1/2} |\partial^2 u(t, x)| \leq CE_k^{1/2}(u(t)) M_k^{1/2}(u(t)),
\]

\[
\langle r \rangle^{1/2} |c_j t - r| |\partial^2 u(t, x)| \leq CM_k(u(t)).
\]

Moreover, for any \( p \) with \( 2 < p < \infty \), there exists a constant \( C \) depending only on \( p \) such that the inequality

\[
\langle t \rangle^{(1/2)-(1/p)} \| u(t) \|_{L^p(\mathbb{R}^2)} \leq CE_k^{1/4}(u(t)) M_k^{1/2}(u(t))
\]
Proof. The first three inequalities are proved in Lemma 1 of Sideris [12]. For the proof of (4.13), we have only to employ (4.11) and follow the proof of Lemma 2.2 (ii) of Katayama [7].

Remark. The right-hand side of (4.11) takes the “multiplicative” form, which plays an important role in the proof of Lemma 5.3 below.

5 Weighted $L^2$-estimates

It is necessary to bound the weighted $L^2$-norm $M_\kappa(u(t))$ by $E^{1/2}_\kappa(u(t))$ for the completion of the energy integral argument. We carry out this by starting with the next crucial inequality due to Klainerman and Sideris [8], estimating the nonlinear terms carefully, and doing a bootstrap argument.

Lemma 5.1 (Klainerman–Sideris inequality) Let $\kappa \geq 2$. The inequality

\[
M_\kappa(u(t)) \leq C \left( E^{1/2}_\kappa(u(t)) + \sum_{|a| \leq \kappa-2} \|(t+r)\Box^a u(t)\|_{L^2(\mathbb{R}^2)} \right)
\]

holds for any smooth function $u$ with the finite norms on the right-hand side.

Proof. See Lemma 3.1 of Klainerman and Sideris [8] and Lemma 7.1 of Sideris and Tu [14]. Note that their proof is obviously valid for $n = 2$ as well as $n = 3$.

In the following, we denote by $[x]$ the greatest integer not greater than $x$.

Lemma 5.2 Let $u$ be a smooth solution of (3.1)-(3.2). Set $\kappa' = [(\kappa - 1)/2] + 3$. Then for all $|a| \leq \kappa - 2$, it holds that

\[
(t + r)\Box^a u(t)\|_{L^2(\mathbb{R}^2)} \\
\leq C \left( E^{1/4}_\kappa(u(t))M^{1/2}_\kappa(u(t)) \right)^2 E^{1/2}_\kappa(u(t)) \\
+ CE_{\kappa'}(u(t))M_\kappa(u(t)) + CE^{1/2}_{\kappa'}(u(t))E^{1/2}_\kappa(u(t))M_{\kappa'}(u(t)).
\]

Proof. We may focus on the estimate of the $L^2$-norm of $t\Box^a u(t)$ because that of $r\Box^a u(t)$ is treated in a similar (in fact, easier) way. Set $p = [(\kappa - 1)/2], \ldots
so that \( p + 3 = \kappa' \). It immediately follows from (4.4) that

\[
(5.3) \quad t \| \Box^a u(t) \|_{L^2} \leq C \sum_{i,j,l} \sum_{|b| + |c| + |d| \leq \kappa - 2} t \left( \| \partial^b u^j(t) \partial^c u^j(t) \partial^2 \Gamma^d u^l(t) \|_{L^2} + \| \partial^b u^j(t) \partial^c u^j(t) \partial^d u^l(t) \|_{L^2} \right).
\]

For the estimate of the second term on the right-hand side of (5.3), we may suppose \(|b| + |c| \leq p\) without loss of generality. We get

\[
(5.4) \quad \| \partial^b u^j(t) \partial^c u^j(t) \partial^d u^l(t) \|_{L^2} \\
\leq \langle t \rangle^{-1} \langle r \rangle^{1/2} \langle c_j t - r \rangle^{1/2} \partial^b u^i(t) \|_{L^\infty} \times \langle r \rangle^{1/2} \langle c_j t - r \rangle^{1/2} \partial^c u^j(t) \|_{L^\infty} \| \partial^d u^l(t) \|_{L^2} \\
\leq \langle t \rangle^{-1} C \left( E_{\kappa'}^{1/4}(u(t)) M_{\kappa'}^{1/2}(u(t)) \right)^2 E_{\kappa'}^{1/2}(u(t))
\]

by using (4.11). For the first terms on the right-hand side of (5.3), we sort them out into two groups: \(|b| + |c| \leq p\) or \(|d| \leq p - 1\). The first group is estimated as

\[
(5.5) \quad \cdots \leq \langle t \rangle^{-1} \langle r \rangle^{1/2} \partial^b u^i(t) \|_{L^\infty} \| \partial^c u^j(t) \|_{L^\infty} \| \partial^2 \Gamma^d u^l(t) \|_{L^2} \\
\leq C \langle t \rangle^{-1} E_{\kappa'}(u(t)) M_\kappa(u(t))
\]

by (4.10). Otherwise, assuming \(|b| \leq p\) as well as \(|d| \leq p - 1\) without loss of generality, we get

\[
(5.6) \quad \cdots \leq \langle t \rangle^{-1} \langle r \rangle^{1/2} \partial^b u^i(t) \|_{L^\infty} \| \partial^c u^j(t) \|_{L^2} \| \langle r \rangle^{1/2} \langle c_j t - r \rangle \partial^2 \Gamma^d u^l(t) \|_{L^\infty} \\
\leq C \langle t \rangle^{-1} E_{\kappa'}^{1/2}(u(t)) M_\kappa(u(t)) E_{\kappa' - 1}^{1/2}(u(t))
\]

by (4.10), (4.12), which completes the proof of (5.2).

\[\square\]

**Lemma 5.3** Let \( \kappa \geq 9 \), \( \mu = \kappa - 2 \). There exists a small, positive constant \( \varepsilon_0 \) with the following property: Suppose that, for a local smooth solution \( u \) of (3.1)-(3.2), the supremum of \( E_{\mu}^{1/2}(u(t)) \) over an interval \([0, T)\) is sufficiently small so that

\[
(5.7) \quad \sup_{0 \leq t < T} E_{\mu}^{1/2}(u(t)) \leq \varepsilon_0
\]

may hold. Then

\[
(5.8) \quad M_\mu(u(t)) \leq C E_{\mu}^{1/2}(u(t)), \ 0 \leq t < T
\]
and

\[ M_\kappa(u(t)) \leq CE_\kappa^{1/2}(u(t)), \quad 0 \leq t < T \]

hold with a constant \( C \) independent of \( T \).

Remark. This lemma is actually valid for \( \kappa \geq 8 \). We have assumed \( \kappa \geq 9 \) for the latter use.

Proof. Set \( \mu' = [(\mu - 1)/2] + 3 \). Denoting by \( \delta \) the supremum of \( E_\mu^{1/2}(u(t)) \) over the interval \([0, T)\), we see that Lemma 5.1 and Lemma 5.2 imply for \( 0 \leq t < T \)

\[
M_\mu(u(t)) \leq CE_\mu^{1/2}(u(t)) + C\left( E_\mu^{1/4}(u(t))M_\mu^{1/2}(u(t)) \right)^2 E_\mu^{1/2}(u(t)) \]
\[
+ CE_\mu(u(t))M_\mu(u(t)) + CE_\mu^{1/2}(u(t))E_\mu^{1/2}(u(t))M_{\mu'}(u(t)) \]
\[
\leq CE_\mu^{1/2}(u(t)) + C\left( \delta^{1/2}M_\mu^{1/2}(u(t)) \right)^2 E_\mu^{1/2}(u(t)) \]
\[
+ C\delta^2M_\mu(u(t)) + C\delta^2M_{\mu'}(u(t)) \]
\[
\leq CE_\mu^{1/2}(u(t)) + C\delta^2M_\mu(u(t)),
\]

which immediately yields (5.8) if \( \delta \) is sufficiently small.

As for (5.9), we first note that the inequality \( \kappa' := [(\kappa - 1)/2] + 3 \leq \mu \) holds. Proceeding as in (5.10) and using (5.8), we easily see that

\[
M_\kappa(u(t)) \leq CE_\kappa^{1/2}(u(t)) + C\delta^2M_\kappa(u(t)) + CE_\kappa^{1/2}(u(t)),
\]

which yields (5.9). \( \square \)

6 Energy estimates

Following the strategy in Sideris [13] and Sideris and Tu [14], we accomplish the energy integral argument by deriving a pair of coupled differential inequalities for a higher-order energy \( E_\kappa(u(t)) \), \( \kappa \geq 9 \) and a lower-order energy \( E_\mu(u(t)) \), \( \mu = \kappa - 2 \). Since the equation is quasi-linear, we must actually consider modified energies which are equivalent to the original ones for small solutions.

For initial data \( (\varphi, \psi) \), let us assume \( E_\mu^{1/2}(u(0)) < \varepsilon \) for a sufficiently small \( \varepsilon > 0 \) such that \( 2\varepsilon \leq \varepsilon_0 \) (see (5.7) for \( \varepsilon_0 \)). By the standard local existence theorem, we know that a unique smooth solution exists locally in time. Suppose
that $T_0$ is the supremum of all $T > 0$ for which $E^{1/2}_\mu(u(t)) < 2\varepsilon$ for all $0 \leq t < T$.

It is shown that $E^{1/2}_\mu(u(t)) < 2\varepsilon$ on the closed interval $0 \leq t \leq T_0$, therefore we can continue the local solution to all time.

Suppose $0 \leq t < T_0$ in what follows. Denoting by $\langle \cdot, \cdot \rangle$ the scalar product in $\mathbb{R}^m$, we have for each $n = 1, \ldots, \kappa$ ($\kappa \geq 9$)

\begin{equation}
E'_n(u(t)) = \sum_{|a| \leq n-1} \int_{\mathbb{R}^2} \langle \square \Gamma^a u(t), \partial_t \Gamma^a u(t) \rangle dx
= \sum_{1 \leq k \leq m, |a|=n-1} \int_{\mathbb{R}^2} G^{k,\alpha\beta\gamma\delta}_{ijl} \partial_\alpha u^i \partial_\beta u^j \partial_\gamma \partial_\delta \Gamma^a u^l \partial_t \Gamma^a u^k dx
+ \sum_{b+c+d+e=a, |a| \leq n-1, d \neq a} \int_{\mathbb{R}^2} \langle G^e(\Gamma^b u, \Gamma^c u, \Gamma^d u), \partial_t \Gamma^a u \rangle dx
+ \sum_{b+c+d+e=a, |a| \leq n-1, d \neq a} \int_{\mathbb{R}^2} \langle H^e(\Gamma^b u, \Gamma^c u, \Gamma^d u), \partial_t \Gamma^a u \rangle dx
- \int_{\mathbb{R}^2} \langle [\Gamma^a, \square] u, \partial_t \Gamma^a u \rangle dx.
\end{equation}

The loss of derivatives which has occurred in the first term on the right-hand side
is prevented by the symmetry condition (3.4) as follows:

\[
(6.2) \quad \sum_{k=1}^{m} \int_{\mathbb{R}^2} G^{k,\alpha\beta\gamma\delta}_{ijl} \partial_\alpha u^i \partial_\beta u^j \partial_\gamma \partial_\delta \Gamma^a u^l \partial_t \Gamma^a u^k \, dx
\]

\[
= \sum_{k=1}^{m} \int_{\mathbb{R}^2} G^{k,\alpha\beta\gamma\delta}_{ijl} \partial_\gamma \left( \partial_\alpha u^i \partial_\beta u^j \right) \partial_\delta \Gamma^a u^l \partial_t \Gamma^a u^k \, dx
\]

\[
- \int_{\mathbb{R}^2} G^{k,\alpha\beta\gamma\delta}_{ijl} \left[ \partial_\gamma \left( \partial_\alpha u^i \partial_\beta u^j \right) \partial_\delta \Gamma^a u^l \partial_t \Gamma^a u^k \right. \\
\left. + \partial_\alpha u^i \partial_\beta u^j \partial_\gamma \partial_\delta \Gamma^a u^l \partial_t \Gamma^a u^k \right) \, dx
\]

\[
= \partial_t \sum_{k=1}^{m} \int_{\mathbb{R}^2} G^{k,\alpha\beta\gamma\delta}_{ijl} \partial_\alpha u^i \partial_\beta u^j \partial_\gamma \partial_\delta \Gamma^a u^l \partial_t \Gamma^a u^k \, dx
\]

\[
- \sum_{k=1}^{m} \int_{\mathbb{R}^2} G^{k,\alpha\beta\gamma\delta}_{ijl} \partial_\gamma \left( \partial_\alpha u^i \partial_\beta u^j \right) \partial_\delta \Gamma^a u^l \partial_t \Gamma^a u^k \, dx
\]

\[
- \sum_{k=1}^{m} \int_{\mathbb{R}^2} \frac{1}{2} G^{k,\alpha\beta\gamma\delta}_{ijl} \partial_\alpha u^i \partial_\beta u^j \partial_t \left( \partial_\gamma \partial_\delta \Gamma^a u^l \partial_t \Gamma^a u^k \right) \, dx
\]

\[
= \partial_t \sum_{k=1}^{m} \int_{\mathbb{R}^2} \frac{1}{2} G^{k,\alpha\beta\gamma\delta}_{ijl} \eta^\gamma_\lambda \partial_\alpha u^i \partial_\beta u^j \partial_\delta \partial_\gamma \Gamma^a u^l \partial_t \Gamma^a u^k \, dx
\]

\[
- \sum_{k=1}^{m} \int_{\mathbb{R}^2} G^{k,\alpha\beta\gamma\delta}_{ijl} \partial_\gamma \left( \partial_\alpha u^i \partial_\beta u^j \right) \partial_\delta \Gamma^a u^l \partial_t \Gamma^a u^k \, dx
\]

\[
+ \sum_{k=1}^{m} \int_{\mathbb{R}^2} \frac{1}{2} G^{k,\alpha\beta\gamma\delta}_{ijl} \partial_t \left( \partial_\alpha u^i \partial_\beta u^j \right) \partial_\delta \Gamma^a u^l \partial_t \Gamma^a u^k \, dx.
\]

Here \( \eta^\gamma_\lambda := \text{diag}(1, -1, -1) \). Therefore, introducing the modified energy

\[
(6.3) \quad \tilde{E}_n(u(t)) := E_n(u(t))
\]

\[
- \sum_{|\alpha|=n} \int_{\mathbb{R}^2} \frac{1}{2} G^{k,\alpha\beta\gamma\delta}_{ijl} \eta^\gamma_\lambda \partial_\alpha u^i \partial_\beta u^j \partial_\delta \partial_\gamma \Gamma^a u^l \partial_t \Gamma^a u^k \, dx,
\]

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we finally have

\begin{equation}
\tilde{E}_n'(u(t)) = \sum_{b+c+d+e=a} \int_{\mathbb{R}^2} (G_\epsilon(\Gamma^b u, \Gamma^c u, \Gamma^d u), \partial_t \Gamma^a u) dx
- \sum_{|a|=n-1} \int_{\mathbb{R}^2} G_{ijl}^{k,\alpha\beta\gamma\delta} \partial_i (\partial_{\alpha} u^i \partial_{\beta} u^j) \partial_{\gamma} \Gamma^a u^l \partial_{\delta} \Gamma^a u^k dx
+ \sum_{|a|=n-1} 1/2 \int_{\mathbb{R}^2} G_{ijl}^{k,\alpha\beta\gamma\delta} \partial_i (\partial_{\alpha} u^i \partial_{\beta} u^j) \partial_{\gamma} \Gamma^a u^l \partial_{\delta} \Gamma^a u^k dx
+ \sum_{b+c+d+e=a} \int_{\mathbb{R}^2} (H_\epsilon(\Gamma^b u, \Gamma^c u, \Gamma^d u), \partial_t \Gamma^a u) dx
- \int_{\mathbb{R}^2} \langle \Gamma^a, \Box \rangle u, \partial_t \Gamma^a u \rangle dx.
\end{equation}

We also note that, under the smallness of $E^{1/2}_\mu(u(t))$ ($0 \leq t < T_0$) with $\mu = \kappa - 2$, the inequality

\begin{equation}
\frac{1}{2} E_n(u(t)) \leq \tilde{E}_n(u(t)) \leq 2E_n(u(t)), \quad n = 1, \ldots, \kappa
\end{equation}

holds by the Sobolev embedding.

We plan our energy integral method, allowing the higher-order energy $E_\kappa(u(t))$ ($\kappa \geq 9$) to grow polynomially in time but bounding the lower-order energy $E_\mu(u(t))$ ($\mu = \kappa - 2$) uniformly in time. (See (3.7) above.) Let us start with the estimate of the higher-order energy. Setting $n = \kappa$ in (6.4), we have

\begin{equation}
\tilde{E}_\kappa'(u(t)) \leq \sum_{i,j,l} \sum_{|a| \leq 1} \sum_{|b| + |c| + |d| \leq |a|} \| \partial_t \Gamma^b u^i \partial_t \Gamma^c u^j \partial_t \Gamma^d u^l \| \| \partial_t \Gamma^a u \| L^2
+ \sum_{i,j,l} \sum_{|a| \leq 1} \sum_{|b| + |c| + |d| \leq |a|} \| \partial_t \Gamma^b u^i \partial_t \Gamma^c u^j \partial_t \Gamma^d u^l \| \| \partial_t \Gamma^a u \| L^2.
\end{equation}

Set $q = \lfloor \kappa/2 \rfloor$. Note that $q + 3 \leq \mu$ because of $\kappa \geq 9$. Supposing $|b| + |c| \leq q$ without loss of generality, we bound the second term as

\begin{equation}
\| \partial_t \Gamma^b u^i \partial_t \Gamma^c u^j \partial_t \Gamma^d u^l \| L^2
\leq C(t)^{-1} \| \langle r \rangle^{1/2} \langle c_i t - r \rangle^{1/2} \partial_t \Gamma^b u^i \| L^\infty \| \langle r \rangle^{1/2} \langle c_j t - r \rangle^{1/2} \partial_t \Gamma^c u^j \| L^\infty \| \partial_t \Gamma^d u^l \| L^2
\leq C(t)^{-1} \left( E^{1/4}_\mu(u(t)) M^{1/2}_\mu(u(t)) \right)^2 E^{1/2}_\mu(u(t)) \leq C(t)^{-1} E_\mu(u(t)) E^{1/2}_\mu(u(t)).
\end{equation}
Here we have employed (5.8) at the third inequality. As for the first terms on the right-hand side of (6.6), we sort them out into two groups:

\[ |b| + |c| \leq q \text{ or } |d| \leq q - 1. \]

The first group is estimated as in (5.5) and (6.7):

\[
\| \partial^\Gamma b u_i \partial^\Gamma c u_j \partial^2 \Gamma d u_l \|_{L^2} \leq C(t)^{-1} \left( E_{\mu}^{1/4}(u(t)) M_{\mu}^{1/2}(u(t)) \right)^2 M_{\kappa}(u(t)) \leq C(t)^{-1} E_{\mu}(u(t)) E_{\kappa}^{1/2}(u(t)).
\]

Otherwise, assuming \( |b| \leq q \) in addition to \( |d| \leq q - 1 \) without loss of generality, we get as in (5.6)

\[
\| \partial^\Gamma b u_i \partial^\Gamma c u_j \partial^2 \Gamma d u_l \|_{L^2} \leq C(t)^{-1} E_{\mu}^{1/2}(u(t)) M_{\mu}(u(t)) E_{\kappa}^{1/2}(u(t)) \leq C(t)^{-1} E_{\mu}(u(t)) E_{\kappa}^{1/2}(u(t)).
\]

Taking account of the equivalence between \( E_n \) and \( E'_n \), we get from (6.6)-(6.9)

\[
\tilde{E}_{\mu}(u(t)) \leq C(t)^{-1} E_{\mu}(u(t)) E_{\kappa}^{1/2}(u(t)).
\]

**Lower-order Energy.** The crucial part in the proof of global existence is to bound the lower-order energy \( E_{\mu}(u(t)) \) (\( \mu = \kappa - 2 \)) uniformly in time. For the purpose, we exploit the difference of propagation speeds as well as an improved decay rate of solutions inside the cone to sharpen the decay estimates presented above, when \( |a| \leq \mu \). Moreover, the space-time \( L^2((0, \infty) \times \mathbb{R}^2) \) estimate and the \( \dot{H}^{1/2}(\mathbb{R}^2) \) estimate play an auxiliary role.

Set \( c_0 := \min\{c_i/2 : i = 1, \ldots, m\} \) and \( \mu = \kappa - 2 \) (\( \kappa \geq 9 \)). Setting \( n = \mu \) in (6.4), we estimate the resulting terms on the right-hand side. Divide the integral region \( \mathbb{R}^2 \) into two parts: inside the cone \( \{(t, x) : |x| \leq c_0 t\} \) and away from the spatial origin \( \{(t, x) : |x| \geq c_0 t\} \).

**Inside the cone.** Here we exploit an improved decay rate of solutions. The space-time \( L^2((0, \infty) \times \mathbb{R}^2) \) estimate also comes into play. The contribution from the quasi-linear terms is bounded by

\[
\sum_{i,j,l} \sum_{|a| \leq \mu - 1} \sum_{|b|+|c|+|d| \leq |a| \atop d \neq a} \| \partial^\Gamma b u_i \partial^\Gamma c u_j \partial^2 \Gamma d u_l \|_{L^2(t < c_0 t)} \| \partial^\Gamma a u \|_{L^2}.
\]
We may suppose $|b| \leq [\mu/2]$ without loss of generality. It then follows from (4.11) and (5.8) that

\begin{align}
(6.12) \quad & \| \partial^b u^i \partial^c u^j \partial^d u^l \|_{L^2} \\
&\leq \langle t \rangle^{-3/2} \| \langle c; t - r \rangle^{1/2} \partial^b u^i \|_{L^\infty(r < c; t)} \| \partial^c u^j \|_{L^\infty} \langle r \rangle^{-1/2} \| \partial^d u^l \|_{L^2(r < c; t)} \\
&\leq C \langle t \rangle^{-3/2} \left( E_{|b| + 2}^1 (u(t)) M_{|b| + 3}^{1/2} (u(t)) \right) E_{|c| + 3}^{1/2} (u(t)) M_{\mu} (u(t)) \\
&\leq C \langle t \rangle^{-3/2} E_{\kappa}^{1/2} (u(t)) E_{\mu} (u(t)),
\end{align}

where we have used $|b| + 3 \leq [\mu/2] + 3 \leq \mu$, $|c| + 3 \leq \kappa$. Concerning the contribution from the semi-linear parts, we see from (6.4) that it is bounded by

\begin{align}
(6.13) \quad & \sum_{i,j,l} \sum_{|a| \leq \mu - 1} \sum_{|b| + |c| + |d| \leq |a|} \| \partial^b u^i \partial^c u^j \partial^d u^l \|_{L^2(r < c; t)} \| \partial^a u \|_{L^2}.
\end{align}

Assume $|b| + |c| \leq [\mu/2]$ without loss of generality. Proceeding quite differently from how we did in (6.23) of [3], we get

\begin{align}
(6.14) \quad & \| \partial^b u^i (t) \partial^c u^j (t) \partial^d u^l (t) \|_{L^2(r < c; t)} \\
&\leq C \langle t \rangle^{-1} \| \langle c; t - r \rangle^{1/2} \partial^b u^i (t) \|_{L^\infty(r < c; t)} \\
&\quad \times \| \langle c; t - r \rangle^{1/2} \partial^c u^j (t) \|_{L^\infty(r < c; t)} \| \langle r \rangle^{-1} \partial^d u^l (t) \|_{L^2(r < c; t)} \\
&\leq C \langle t \rangle^{-1} \left( E_{|b| + 2}^1 (u(t)) M_{|b| + 3}^{1/2} (u(t)) \right) \left( E_{|c| + 2}^{1/2} (u(t)) M_{|c| + 3}^{1/2} (u(t)) \right) \\
&\quad \times \| \langle r \rangle^{-1} \partial^d u^l (t) \|_{L^2(\mathbb{R}^2)} \\
&\leq C \langle t \rangle^{-1} E_{\mu} (u(t)) \sum_{|d| \leq \mu - 1} \| \langle r \rangle^{-1} \partial^d u^l (t) \|_{L^2(\mathbb{R}^2)}.
\end{align}

The estimate inside the cone has been finished.

*Away from the spatial origin.* Here the difference of propagation speeds comes into play. Moreover, we employ the null condition (3.5) for the estimates of resonance terms.

*Non-resonance.* Let us start with non-resonance terms. Our task is to estimate the contribution from quasi-linear terms

\begin{align}
(6.15) \quad & \sum_{(i,j,l) \neq (k,k,k)} \sum_{|a| \leq \mu - 1} \sum_{|b| + |c| + |d| \leq |a|} \| \partial^b u^i \partial^c u^j \partial^d u^l \partial^a u^k \|_{L^2(r > c; t)}.
\end{align}

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and the contribution from semi-linear terms

\begin{equation}
(6.16) \quad \sum_{(i,j,l) \neq (k,k,k)} \sum_{|a| \leq \mu - 1} \sum_{|b| + |c| + |d| \leq |a|} \| \partial^{b} u^i \partial^{c} u^i \partial^{d} u^j \partial^{a} u^k \|_{L^1(r > c_0 t)}.
\end{equation}

In estimating the $L^1$-norm in (6.15) we separate two cases: $i = j = l$ or otherwise.
In the former case, noting $i \neq k$, we have

\begin{equation}
(6.17) \quad \| \partial^{b} u^i \partial^{c} u^i \partial^{d} u^j \partial^{a} u^k \|_{L^1(r > c_0 t)}
\leq \langle t \rangle^{-3/2} \langle r \rangle^{1/2} \| \partial^{b} u^i \|_{L^\infty} \| \partial^{c} u^i \|_{L^2} 
\times \| \langle c_i t - r \rangle \partial^{2} u^i \|_{L^2} \| \langle r \rangle^{1/2} \langle c_i t - r \rangle^{1/2} \partial^{a} u^k \|_{L^\infty}
\leq C \langle t \rangle^{-3/2} E_{[b]+3}^{1/2} (u(t)) E_{[c]+1}^{1/2} (u(t)) E_{|a|+2}^{1/2} (u(t)) \left( E_{[a]+3}^{1/4} (u(t)) M_{|a|+3}^{1/2} (u(t)) \right)
\leq C \langle t \rangle^{-3/2} E_{[b]+2}^{1/2} (u(t)) E_{[a]+3}^{1/2} (u(t)) M_{|u|+1}^{1/2} (u(t)) E_{[a]+3}^{1/2} (u(t))
\leq C \langle t \rangle^{-3/2} E_{\mu} (u(t)) E_{\mu} (u(t)).
\end{equation}

Otherwise, it is easy to get

\begin{equation}
(6.18) \quad \| \partial^{b} u^i \partial^{c} u^i \partial^{d} u^j \partial^{a} u^k \|_{L^1(r > c_0 t)}
\leq \langle t \rangle^{-3/2} \langle r \rangle^{1/2} \| \partial^{c} u^i \|_{L^\infty} \| \langle c_i t - r \rangle^{1/2} \partial^{2} u^j \|_{L^2} \| \partial^{a} u^k \|_{L^2}
\leq C \langle t \rangle^{-3/2} E_{\mu} (u(t)) E_{\mu} (u(t)).
\end{equation}

As for (6.16), we may suppose $i \neq k$ without loss of generality. We obtain

\begin{equation}
(6.19) \quad \| \partial^{b} u^i \partial^{c} u^j \partial^{d} u^j \partial^{a} u^k \|_{L^1(r > c_0 t)}
\leq \langle t \rangle^{-3/2} \langle r \rangle^{1/2} \langle c_i t - r \rangle^{1/2} \| \partial^{b} u^i \|_{L^\infty(r > c_0 t)}
\| \partial^{c} u^j \partial^{d} u^j \|_{L^1} \| \langle r \rangle^{1/2} \langle c_i t - r \rangle^{1/2} \partial^{a} u^k \|_{L^\infty(r > c_0 t)}
\leq C \langle t \rangle^{-3/2} \left( E_{[b]+2}^{1/4} (u(t)) M_{[b]+3}^{1/2} (u(t)) \right) E_{\mu} (u(t)) \left( E_{[a]+3}^{1/4} (u(t)) M_{[a]+3}^{1/2} (u(t)) \right)
\leq C \langle t \rangle^{-3/2} E_{\mu} (u(t)) E_{\mu} (u(t)).
\end{equation}

Therefore the estimates of non-resonance terms away from the spatial origin have been completed.

Resonance. The resonance terms remain to be estimated away from the spatial origin. It is just the place where the null condition comes into play. Without the
null condition, the solution may become singular in finite time (see, e.g., Zhou and Han [18]). In view of Lemma 4.2 and (6.4), the estimate is reduced to bounding

\[
\sum_{1 \leq k \leq m} \sum_{|a| \leq \mu - 1} \sum_{|b| + |c| + |d| \leq |a|} \langle t \rangle^{-1} \left( \| \Gamma^{d+1} u_k^\alpha \partial \Gamma^c u_k^\beta \partial^2 \Gamma^d u_k^\kappa \|_{L^2(r_{>0},t)} \right) \\
+ \| \partial \Gamma^b u_k^\alpha \partial \Gamma^c u_k^\beta \partial \Gamma^d u_k^\kappa \|_{L^2(r_{>0},t)} \\
+ \| \langle c_k t - r \rangle \partial \Gamma^b u_k^\alpha \partial \Gamma^c u_k^\beta \partial^2 \Gamma^d u_k^\kappa \|_{L^2(r_{>0},t)} \| \partial \Gamma^a u \|_{L^2} \\
+ \sum_{1 \leq k \leq m} \sum_{|a| \leq \mu - 1} \sum_{|b| + |c| + |d| \leq |a|} \langle t \rangle^{-1} \left( \| \Gamma^{d+1} u_k^\alpha \partial \Gamma^c u_k^\beta \partial \Gamma^d u_k^\kappa \|_{L^2(r_{>0},t)} \right) \| \partial \Gamma^a u \|_{L^2}.
\]

Here, by \( b + 1 \), we mean any sequence of length \( |b| + 1 \).

We proceed differently from how we did in (6.40) of [3]. Using (4.10) and the Hardy inequality of order 1/2, we estimate the first norm on the right-hand side of (6.20) as

\[
\langle t \rangle^{-1/2} \| r^{-1/2} \Gamma^{b+1} u_k^\alpha r^{1/2} \partial \Gamma^c u_k^\beta \langle r \rangle^{1/2} \partial^2 \Gamma^d u_k^\kappa \|_{L^2(r_{>0},t)} \\
\leq C \langle t \rangle^{-1/2} \| r^{-1/2} \Gamma^{b+1} u_k^\alpha \|_{L^2} \| r^{1/2} \partial \Gamma^c u_k^\beta \|_{L^\infty} \| \langle r \rangle^{1/2} \partial^2 \Gamma^d u_k^\kappa \|_{L^\infty} \\
\leq C \langle t \rangle^{-1/2} \| D_x^{1/2} \Gamma^{b+1} u_k^\alpha \|_{L^2} \| E^{1/2}_{|d|+3} (u(t)) \|_{L^\infty} \\
\leq C \langle t \rangle^{-1/2} \| D_x^{1/2} \Gamma^{b+1} u_k^\alpha \|_{L^2} \| E^{1/2}_{|d|} (u(t)) \|_{L^\infty} \\
\leq C \langle t \rangle^{-1/2} \left( \sum_{|a| \leq \mu} \| \Gamma^a u(t) \|_{H^{1/2}} \right) E^{1/2}_{\mu} (u(t)) E^{1/2}_{\kappa} (u(t)).
\]

Assuming \( |b| \leq |c| \) without loss of generality, we estimate the second and third norms on the right-hand side of (6.20) as

\[
\langle t \rangle^{-1/2} \| \langle r \rangle^{1/2} \partial \Gamma^b u_k^\alpha \|_{L^\infty(r_{>0},t)} \| \partial \Gamma^c u_k^\beta \|_{L^\infty} \| \partial \Gamma^d+1 u_k^\kappa \|_{L^2} \\
+ \langle t \rangle^{-1/2} \| \langle r \rangle^{1/2} \partial \Gamma^b u_k^\alpha \|_{L^\infty(r_{>0},t)} \| \partial \Gamma^c u_k^\beta \|_{L^\infty} \| \langle c_k t - r \rangle \partial^2 \Gamma^d u_k^\kappa \|_{L^2} \\
\leq C \langle t \rangle^{-1/2} E^{1/2}_{\mu} (u(t)) E^{1/2}_{\mu} (u(t)).
\]

The remaining terms in (6.20) are estimated as

\[
\langle t \rangle^{-1/2} \| r^{-1/2} \Gamma^{b+1} u_k^\alpha r^{1/2} \partial \Gamma^c u_k^\beta \langle r \rangle^{1/2} \partial^2 \Gamma^d u_k^\kappa \|_{L^2(r_{>0},t)} \\
+ \langle t \rangle^{-1/2} \| \langle r \rangle^{1/2} \partial \Gamma^b u_k^\alpha \|_{L^\infty(r_{>0},t)} \| \partial \Gamma^c u_k^\beta \|_{L^\infty} \| \langle c_k t - r \rangle \partial^2 \Gamma^d u_k^\kappa \|_{L^2(r_{>0},t)} \\
\leq C \langle t \rangle^{-1/2} \left( \sum_{|a| \leq \mu} \| \Gamma^a u(t) \|_{H^{1/2}} \right) E^{1/2}_{\mu} (u(t)) E^{1/2}_{\mu} (u(t)) \\
+ C \langle t \rangle^{-1} E^{1/2}_{\mu} (u(t)) E^{1/2}_{\mu} (u(t)).
\]
thanks to (4.10), (4.11), and the Hardy inequality of order $1/2$. The estimate of (6.20) has been finished.

Collecting the estimates of $\tilde{E}'_\mu(u(t))$ and taking (6.5) into account, we have finally obtained

\begin{equation}
\tilde{E}'_\mu(u(t)) \leq C(t)^{-3/2} \tilde{E}^{1/2}_\kappa(u(t)) \tilde{E}^{3/2}_\mu(u(t)) \\
+ C(t)^{-1} \tilde{E}^{3/2}_\mu(u(t)) \sum_{|a| \leq \mu - 1} \| \langle r \rangle^{-1} \partial \Gamma^a u(t) \|_{L^2(\mathbb{R}^2)} \\
+ C(t)^{-3/2} \tilde{E}_\kappa(u(t)) \tilde{E}_\mu(u(t)) \\
+ C(t)^{-3/2} \tilde{E}^{1/2}_\kappa(u(t)) \tilde{E}_\mu(u(t)) \left( \tilde{E}^{1/2}_\mu(u(t)) \right) + \sum_{|a| \leq \mu} \| \Gamma^a u(t) \|_{\dot{H}^{1/2}}.
\end{equation}

**Space-time $L^2$ estimate.** Two more ingredients are needed to accomplish the energy integration argument. One is the space-time $L^2$ estimate, and the other is the $\dot{H}^{1/2}(\mathbb{R}^2)$ estimate. See the right-hand side of (6.24). For the former, we utilize the localized energy estimate of Smith and Sogge [15]: for $n \geq 1$ and $0 \leq \gamma \leq (n - 1)/2$, there holds

\begin{equation}
\| \beta(\exp(it|D_x|)|g) \|_{L^2(\mathbb{R}; H^\gamma(\mathbb{R}^n))} \leq C \| D_x |^\gamma g \|_{L^2(\mathbb{R}^n)},
\end{equation}

here $\beta \in C_0^\infty(\mathbb{R}^n)$, $C = C(n, \beta, \gamma) > 0$. It is well known (see, e.g, [4], [5], [10]) that this estimate with $\gamma = 0$, together with the Duhamel principle, yields

**Lemma 6.1** Let $n \geq 1$, $\delta > 0$. Suppose that $v$ solves the Cauchy problem $\square v = G$ in $(0, T) \times \mathbb{R}^n$, with data $v(0) = f$, $\partial_t v(0) = g$. Then the estimate

\begin{equation}
\| \langle r \rangle^{-(1/2) - \delta} \partial v \|_{L^2((0,T) \times \mathbb{R}^n)} \leq C \left( \| \nabla f \|_{L^2} + \| g \|_{L^2} + \| G \|_{L^1((0,T); L^2(\mathbb{R}^n))} \right)
\end{equation}

holds.

For $n = 1$ or $n \geq 3$, the estimate (6.26) with $G \equiv 0$ is proved by the multiplier method, as mentioned on page 7 of [4]. On the other hand, the proof of the Smith-Sogge estimate (6.25) builds on Fourier analysis, and it is valid for $n = 2$ as well. This is why we start with (6.25) for the proof of (6.26).

Using (4.4) with $|a| = \mu - 1$ and (6.26) with $\delta = 1/2$, and then proceeding as
in (5.3)-(5.6), we obtain for \( T < T_0 \)

\[
\sum_{|a| \leq \mu - 1} \| \langle r \rangle^{-1} \partial \Gamma^a u \|_{L^2((0, T) \times \mathbb{R}^2)} \\
\leq C E_{\mu}^{1/2}(u(0)) + C \sum_{|b| + |c| + |d| \leq \mu - 1} \int_0^T \left( \| \partial \Gamma^b u^i(t) \partial \Gamma^c u^j(t) \partial^2 \Gamma^d u^l(t) \|_{L^2} \\
+ \| \partial \Gamma^b u^i(t) \partial \Gamma^c u^j(t) \partial \Gamma^d u^l(t) \|_{L^2} \right) dt
\]

\[
\leq C \left( \varepsilon + \varepsilon^3 \int_0^T (1 + t)^{-1} dt \right) \leq C \varepsilon \log(2 + T),
\]

which is the estimate we have seeked for.

\( \dot{H}^{1/2} \) estimate. We use the following basic estimate.

**Lemma 6.2** Let \( v \) be the solution to the Cauchy problem \( \Box v = G \) in \((0, T) \times \mathbb{R}^2\) with data \((v(0), \partial_t v(0)) = (f, g)\). Then there holds that

\[
\| D_x \|^{1/2} v(t) \|_{L^2(\mathbb{R}^2)} + \| D_x \|^{-1/2} \partial_t v(t) \|_{L^2(\mathbb{R}^2)} \\
\leq C \left( \| D_x \|^{1/2} f \|_{L^2} + \| D_x \|^{-1/2} g \|_{L^2} + \| G \|_{L^1((0, T); L^{4/3}(\mathbb{R}^2))} \right).
\]

The proof is elementary, and we may omit it.

Recall the definition (2.2) of \( N_{\mu}(u(t)) \). Using (4.4) with \( |a| \leq \mu \), proceeding as in (5.3)-(5.6), and applying the Hölder inequality and (4.13) with \( p = 8 \), we get for \( t < T_0 \)

\[
N_{\mu}(u(t)) \\
\leq N_{\mu}(u(0)) + C \int_0^t (\tau)^{-3/8} \left( M_{\mu}^{1/2}(u(\tau)) E_{\mu}^{1/4}(u(\tau)) \right)^2 E_{\kappa}^{1/2}(u(\tau)) d\tau \\
\leq N_{\mu}(u(0)) + C \varepsilon^2 \int_0^t (\tau)^{-3/4} E_{\kappa}^{1/2}(u(\tau)) d\tau.
\]

Now we are ready to complete the proof of our main theorem. Since we know \( E_{\mu}^{1/2}(u(t)) < 2 \varepsilon \) (0 \( \leq t < T_0 \)) for a sufficiently small \( \varepsilon \) such that \( 2 \varepsilon \leq \varepsilon_0 \) for \( \varepsilon_0 \) in (5.7), we get from (6.10)

\[
\tilde{E}_{\kappa}(u(t)) \leq \tilde{E}_{\kappa}(u(0)) \langle t \rangle^{C \varepsilon^2},
\]

which, combined with (6.29), yields

\[
N_{\mu}(u(t)) \leq N_{\mu}(u(0)) + C \varepsilon^2 \tilde{E}_{\kappa}^{1/2}(u(0)) \langle t \rangle^{(1/4) + C \varepsilon^2}.
\]
Inserting (6.30) and (6.31) into (6.24) and using the obvious inequality $\tilde{E}_\mu(u(t)) \leq \tilde{E}_\kappa(u(t))$ as well, we have
\begin{equation}
(6.32) \quad \tilde{E}_\mu(u(t)) \leq \tilde{E}_\mu(u(0)) + C \int_0^t \left( (\langle \tau \rangle)^{(3/2) + C\varepsilon^2} \tilde{E}_\kappa(u(\tau)) \right. \\
+ \langle \tau \rangle^{-1 + C\varepsilon^2} \tilde{E}_\kappa^1/2(u(0)) \sum_{|d| \leq \mu - 1} \| \langle \tau \rangle^{-1} \partial^d u(\tau) \|_{L^2(\mathbb{R}^2)} \\
+ \langle \tau \rangle^{-3/2 + C\varepsilon^2} \tilde{E}_\kappa^1/2(u(0)) N_\mu(u(0)) \\
+ \langle \tau \rangle^{-(3/2) + C\varepsilon^2} \tilde{E}_\kappa^1/2(u(0)) \tilde{E}_\mu(u(\tau)) d\tau \right)
\end{equation}
for $0 \leq t < T_0$. By the Gronwall inequality, we obtain
\begin{equation}
(6.33) \quad \tilde{E}_\mu(u(t)) \leq \tilde{E}_\mu(u(0)) \exp \left( C \tilde{E}_\kappa(u(0)) + C \tilde{E}_\kappa^1/2(u(0)) N_\mu(u(0)) \right)
\end{equation}
for $0 \leq t < T_0$. Note that here we have used (6.27) together with the useful technique of dyadic decomposition of the interval $(0, T_0)$, as in page 363 of [16], page 408 of [11], page 13 of [4], and page 11 of [5].

Recalling (6.5) and taking the size condition (3.6) into account, we see that there exists a constant $A$ such that
\begin{equation}
(6.34) \quad E_\mu^1/2(u(t)) \leq 2 E_\mu^1/2(u(0)) \exp \left( A \tilde{E}_\kappa^1/2(u(0)) \left( E_\kappa^1/2(u(0)) + N_\mu(u(0)) \right) \right) < 2\varepsilon
\end{equation}
for $0 \leq t < T_0$. The last inequality proves that the norm $E_\mu^1/2(u(t))$ is strictly smaller than $2\varepsilon$ on the closed interval $[0, T_0]$. The proof of the main theorem has been completed. □

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