A Nonlinear Small-Gain Theorem for Large-Scale Time Delay Systems

Shanaz Tiwari, Yuan Wang, and Zhong-Ping Jiang

Abstract—This paper extends the nonlinear ISS small-gain theorem to a large-scale time delay system composed of three or more subsystems. En route to proving this small-gain theorem for systems of differential equations with delays, a small-gain theorem for operators is examined. The result developed for operators allows applications to a wide class of systems, including state space systems with delays.

I. INTRODUCTION

One of the most powerful tools in stability analysis and control design of interconnected systems is the small-gain theory. The first small-gain theorem in the context of input-to-state stability (ISS) was developed in [7]. A variant of the ISS small-gain theorem was given in [1] in terms of asymptotic gains. The authors of [5] presented an ISS-type small-gain theorem in terms of operators that recovers the case of state space form, with applications to several variations of the ISS property such as input-to-output stability, incremental stability and detectability for interconnected systems. Initialized by the work [2] and [17], the small-gain theory was extended to systems composed of three or more subsystems. In the recent work [8], a cyclic small-gain theorem was provided to deal with the input-to-output stability (IOS) properties for large-scale interconnected systems.

The goal of the current work is to develop the small-gain results for large-scale systems with time-delays appearing in both the subsystems and the interconnections. Systems with delays arise naturally from many practical applications such as networked control systems. As a consequence, time-delay systems and control have received much attention in recent years, see for instance, [3], [6], and [13]. In a series of recent work [10], [11], and [12], various notions related to ISS were studied for systems with delays. As in the case for systems without delays, small-gain theorems provide natural tools for stability analysis of interconnected systems. In [16], a Razumikhin-type theorem on stability analysis was presented for systems with delays by using the nonlinear small-gain theorem. In [14], a small-gain theorem was developed to solve a stabilization problem of a force-reflecting telerobtics system with time delays. In [9] a small-gain theorem was given for a wide class of systems including systems with time delays.

The previous work on small-gain theorems for time-delay systems focused on systems composed of two subsystems. Our main contribution will be to present a small-gain theorem for interconnected systems composed of three or more subsystems with time delays. Instead of carrying out our proofs for systems of differential equations with delays, we develop a small-gain theorem for input/output operators. This is an approach adopted in [5]. The advantage of doing so is that it allows one to develop small-gain theorems for a wide class of systems including systems of differential equations with delays and possibly certain types of hybrid systems.

Notations. Throughout this work, we use $| \cdot |$ to denote the Euclidean norm of vectors, and $\| \cdot \|_I$ to denote the essential supremum norm of measurable and locally essentially bounded functions defined on the interval $I$. For $\phi = (\phi_1, \cdots, \phi_k)$ defined on an interval $I$, we let $\| \phi \|_I = \max_{1 \leq i \leq k} \{ \| \phi_i \| \}$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $K$ if it is continuous, positive definite, and strictly increasing; and is of class $K_\infty$ if it is also unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $KCL$ if for each fixed $t \geq 0$, $\beta(\cdot, t)$ is of class $K$, and for each fixed $s \geq 0$, $\beta(s, t)$ decreases to 0 as $t \to \infty$.

II. PRELIMINARIES

Let $\theta \geq 0$, and let $C[-\theta, 0]$ denote the Banach space of continuous functions defined on $[-\theta, 0]$, equipped with the norm $\| \cdot \|_{[-\theta, 0]}$. For a continuous function $q : [-\theta, b) \to \mathbb{R}$, where $b > 0$, define $q_t(s) := q(t + s)$. Then, for each $t \in [0, b)$, $q_t \in C[-\theta, 0]$.

Consider a nonlinear system with time delays described by

$$
(S_{\Sigma}): \dot{x}(t) = f(x_t, v_t, u(t)), \quad t \geq 0, \quad (1)
$$

where

- for each $t$, $x(t) \in \mathbb{R}^n$, $v(t) \in \mathbb{R}^p$ and $u(t) \in \mathbb{R}^m$;
- $f : X \times \mathcal{V}_0 \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz and completely continuous (see [4] for definition), where $X = (C[-\theta, 0])^n$ and $\mathcal{V}_0 = (C[-\theta, 0])^p$.

The input of the system is given by $(v(t), u(t))$ where $v : [-\theta, \infty) \to \mathbb{R}^p$ is continuous and $u : [0, \infty) \to \mathbb{R}^m$ is measurable and locally essentially bounded. Trajectories of the systems are absolutely continuous functions defined on some interval that satisfy (1) almost everywhere. We let $\mathcal{U}$ denote the collection of measurable and locally essentially bounded functions, and let $\mathcal{V} = (C[-\theta, \infty])^p$.

With the assumptions on $f : X \times \mathcal{V}_0 \times \mathbb{R}^m \to \mathbb{R}^n$ given above, it can be shown that for each $v(\cdot) \in \mathcal{V}$ and each
\( u(\cdot) \in \mathcal{U} \), the map \( F : \mathcal{X} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n \) given by
\[
F(\xi, t) := f(x, v_t, u(t))
\]
is bounded on any compact set of \( \mathcal{X} \times [0, \infty) \) and is locally Lipschitz in \( \xi \), uniformly in \( t \), for all \( t \) in any compact set (see Lemma 7.11 in Appendix A.) It can also be shown that \( F \) is completely continuous in \( \xi \), uniformly in \( t \), for all \( t \) in any compact set. Consequently, for each \( v \in \mathcal{V} \), \( u \in \mathcal{U} \), and each continuous function \( \xi \) defined on \([-\theta, 0]\), there is a unique trajectory of \( \xi \) corresponding to \( v \) and \( u \) that satisfies the initial condition \( x_0(\cdot) = \xi(\cdot) \) (see [4]). We denote this trajectory by \( x(t, \xi, v, u) \), and its maximum interval by \([-\theta, T_{x,v,u}^{\max}]\).

Our reason for considering \( \dot{x}(t) = f(x, v_1, u(t)) \) instead of the cases \( \dot{x}(t) = f(x_t, u(t)) \) or \( \dot{x}(t) = f(x_t, v_1) \) is that we want to allow an interconnected system to have feedbacks involving time delays (c.f. Section III), and the input signals we want to allow an interconnected system to have feedbacks and a state stable for all \( x \), the initial condition global stability \( F \) and the (AG) conditions.

It can be seen that the ISS property implies both the (GS) property if there exist \( \gamma^u(\cdot) \) and \( \gamma^v(\cdot) \) and a \( \mathcal{KL} \)-function \( \beta(\cdot) \) such that
\[
|x(t, \xi, v, u)| \leq \beta \left( \left[ x(t) \right]_{[-\theta, 0]} \right),
\]
\[
|\sigma^v(\|v\|) + \gamma^u(u_{0, \infty})| \leq \sum_{i=0}^{\infty} \gamma^u(u_{i, \infty}) \leq \sum_{i=0}^{\infty} \gamma^u(u_{i, \infty}) \leq \gamma^u(u_{\infty}) \leq \gamma^u(u_{\infty})
\]
for all \( t \geq 0 \).

Definition 2.2: The system \((\Sigma_u)\) is said to satisfy the global stability (GS) property if there exist \( \sigma^v(\cdot), \sigma^u(\cdot), \sigma^u(\cdot) \) and \( \sigma^u(\cdot) \) such that for all \( t \geq 0 \),
\[
|x(t, \xi, v, u)| \leq \max \left\{ \sigma^v(\|v\|), \sigma^u(u_{0, \infty}) \right\}
\]
for all \( u \in \mathcal{U} \) and \( v \in \mathcal{V} \).

It can be seen that the ISS property implies both the (GS) and the (AG) conditions.

Lemma 2.4: The AG condition as given in (5) is equivalent to
\[
\lim_{t \to \infty} |x(t, \xi, v, u)| \leq \max \left\{ \lim_{t \to \infty} \gamma^u(u(t)), \gamma^u(u_{\infty}) \right\}.
\]

Due to the length restriction, we omit the proof of this lemma.

### III. A SMALL-GAIN THEOREM FOR TIME-DELAY SYSTEMS IN STATE-SPACE FORM

Consider a large-scale interconnected system composed of \( n \) subsystems:
\[
\dot{x}_1(t) = f_1((x_1(t), (v_2)_{t}, (v_3)_{t}, \ldots, (v_k)_{t}, u_1(t)), \dot{x}_2(t) = f_2((x_2(t), (v_1)_{t}, (v_3)_{t}, \ldots, (v_k)_{t}, u_2(t)), \ldots, \dot{x}_k(t) = f_k((x_k(t), (v_1)_{t}, (v_2)_{t}, \ldots, (v_{k-1})_{t}, u_k(t))
\]
subject to the interconnection
\[
v_i = x_i, \quad 1 \leq i \leq k.
\]

For each \( 1 \leq i \leq k \), assume that each \( f_i \) is locally Lipschitz jointly on all of its entries, and for each \( i \) and each \( t, x_i(t) \in \mathbb{R}^{n_i}, v_i(t) \in \mathbb{R}^{n_i}, \) and \( u_i(t) \in \mathbb{R}^{m_i} \).

For any \( K \)-function \( \rho \), we say that \( \rho < id \) if \( \rho(s) < s \) for all \( s > 0 \).

**Theorem 1**: Suppose that for each \( x_i \)-subsystem of (7), there exist \( K \)-functions \( \sigma_i, \gamma_{ij} \) and \( \gamma_i^u \) such that the following properties hold:

- the GS property:
\[
|\gamma_{ij}(\|v_j\|_{\theta, 0})| \leq \gamma_{ij}(\|v_j\|_{\theta, \infty}), \quad \gamma_i^u(\|u_j\|_{\theta, \infty})
\]
for all \( t \geq 0 \), and

- the AG property:
\[
|\gamma_{ij}(\|v_j\|_{\theta, \infty})| \leq \gamma_{ij}(\|v_j\|_{\theta, \infty})
\]

Assume further that the set of cyclic small-gain conditions hold:
\[
\gamma_{i_1i_2} \circ \gamma_{i_2i_3} \circ \cdots \circ \gamma_{i_ni_1} < id
\]
for all \( 2 \leq r \leq k, \quad 1 \leq i_j \leq k, \quad i_j \neq i_j \neq j \). Then the interconnected system (7)-(8) is forward complete, and it admits the AG and GS properties with \( u = (u_1, u_2, \ldots, u_k) \) as inputs. That is, there exist class \( K \)-functions \( \sigma(\cdot), \gamma^u(\cdot) \) such that:
\[
|x(t)| \leq \max \left\{ \sigma(\|x\|_{\theta, 0}), \gamma^u(\|u\|_{\theta, \infty}) \right\}
\]
for all \( t \geq 0 \), and
\[
\lim_{t \to \infty} |x(t)| \leq \gamma^u(\|u\|_{\theta, \infty})
\]

Note that for any \( \rho_1, \rho_2 \in K, \rho_1 \circ \rho_2 < id \) if and only if \( \rho_2 \circ \rho_1 < id \). Consequently, to verify the set of small-gain conditions (11) for all choices of \( \gamma_{i_1i_2} \circ \gamma_{i_2i_3} \circ \cdots \circ \gamma_{i_ni_1} \), for which \( r \geq 2, \) and \( 1 \leq i_j \leq k, \) \( i_j \neq i_j \neq j \), it is sufficient to verify (11) for all choices of those \( \gamma_{i_1i_2} \circ \gamma_{i_2i_3} \circ \cdots \circ \gamma_{i_ni_1} \), with \( i_k < i \leq i_2, \ldots, i_r \).

When \( k = 2 \), the set of small-gain conditions (11) becomes the usual small-gain condition: \( \gamma_{12} \circ \gamma_{21} < id \).
For the case of \( k = 3 \), the set of small-gain conditions (11) becomes the following:

\[
\gamma_{12} \circ \gamma_{21} < \text{id}, \quad \gamma_{13} \circ \gamma_{31} < \text{id}, \quad \gamma_{23} \circ \gamma_{32} < \text{id};
\]

\[
\gamma_{12} \circ \gamma_{23} \circ \gamma_{31} < \text{id}, \quad \gamma_{13} \circ \gamma_{23} \circ \gamma_{21} < \text{id}.
\]

In the special case when the subsystems in (11) are free of the external signals \( u_i(\cdot) \), the interconnected system becomes

\[
\begin{aligned}
\dot{x}_1(t) &= f_1((x_1)_t, (v_2)_t, (v_3)_t, \ldots, (v_k)_t), \\
\vdots
\end{aligned}
\]

\[
\begin{aligned}
\dot{x}_k(t) &= f_k((x_k)_t, (v_1)_t, (v_2)_t, \ldots, (v_{k-1})_t),
\end{aligned}
\]

subject to the interconnection

\[
v_i = x_i, \quad 1 \leq i \leq k.
\]

The following is then an immediate consequence of Theorem 11.

**Corollary 3.1:** Suppose that for each \( x_i \)-subsystem of (14), there exist \( \mathcal{K} \)-functions \( \sigma_i \) and \( \gamma_{ij} \) (\( j \neq i, 1 \leq j \leq k \)) such that the following properties hold:

- the GS property:
  \[|x_i(t)| \leq \max_{i \neq j} \left\{ \sigma_i \left( \|x_i\|_{[-\theta, 0]} \right), \gamma_{ij} \left( \|v_j\|_{[-\theta, \infty]} \right) \right\};\]
  for all \( t \geq 0 \), and

- the AG property:
  \[
  \lim_{t \to \infty} |x_i(t)| \leq \max_{i \neq j} \left\{ \gamma_{ij} \left( \|v_j\|_{[-\theta, \infty]} \right) \right\}.
  \]

Assume further that the set of cyclic small-gain conditions (11) holds for all \( 2 \leq r \leq k \), \( 1 \leq i_j \leq k \), \( i_j \neq i_j' \) if \( j \neq j' \). Then, the interconnected system (14–15) is globally asymptotically stable in the following sense:

- for some \( \sigma \in \mathcal{K} \), \( |x(t)| \leq \sigma \left( \|x\|_{[-\theta, 0]} \right) \) for all \( t \geq 0 \);
- \( \lim_{t \to \infty} |x(t)| = 0 \). \( \square \)

**A. An Example**

In what follows, we consider an example of a system composed of three subsystems with delays (without \( u \) for simplicity). Let \( \Delta > 0 \) be a constant time-delay.

Consider the system described by the equations,

\[
\begin{aligned}
\dot{x}_1(t) &= -3x_1(t) + \frac{v_2^2(t - \Delta)}{1 + v_2^2(t - \Delta)}, \\
\dot{x}_2(t) &= -\frac{3}{2} x_2(t) + v_3^3(t - \Delta), \\
\dot{x}_3(t) &= -2x_3(t) + v_4^2(t - \Delta),
\end{aligned}
\]

with the interconnection \( v_i = x_i \) for \( i = 1, 2, 3 \). Each of the \( x_i \)-subsystems satisfies the AG and GS conditions. The gain functions can be chosen as:

\[
\sigma_1(s) = 7s, \quad \sigma_2(s) = 4s, \quad \sigma_3 = 3s, \quad \text{and}
\]

\[
\gamma_{12} = \frac{s^2}{2(1 + s^2)}, \quad \gamma_{23} = s^3, \quad \gamma_{31} = s^2.
\]

We show the computations for the calculation of \( \sigma_1(s) \) and \( \gamma_{12}(s) \). The other gain functions can be calculated in a similar manner. Let \( w_1(t) = \frac{v_2^2(t - \Delta)}{1 + v_2^2(t - \Delta)} \). We can now rewrite the first equation of the system as

\[
\dot{x}_1(t) = -3x_1(t) + w_1(t).
\]

The solution of this linear system satisfies

\[
|x_1(t)| \leq |x_{10}| e^{-3t} + \frac{1}{3} \|w_1\|_{[-\theta, \infty)].
\]

Using the fact that \( a + b < \max \{(1 + \varepsilon)^{-1}a, (1 + \varepsilon)\} \) for every \( \varepsilon > 0 \) we get that

\[
|x_1(t)| \leq \max \left\{ 7 |x_{10}| e^{-3t}, \frac{1}{2} \|w_1\|_{[-\theta, \infty]} \right\}
\]

by letting \( \varepsilon = \frac{1}{3} \). To verify the small-gain condition, it is enough to show that \( \gamma_{12} \circ \gamma_{23} \circ \gamma_{31} < \text{id} \). By calculation,

\[
\gamma_{12} \circ \gamma_{23} \circ \gamma_{31}(s) = \frac{s^{12}}{2(1 + s^{12})}.
\]

The desired small-gain condition \( \frac{s^{12}}{2(1 + s^{12})} < s \) is equivalent to \( \frac{s^{12}}{2(1 + s^{12})} < s + s^{13} \). The inequality can be verified by considering the two cases of \( s \leq 1 \) and \( s > 1 \). By the small-gain theorem, the interconnected system given in (16) is globally asymptotically stable.

**IV. INPUT/OUTPUT OPERATORS**

To prove the small-gain theorem for systems of differential equations with delays as stated in the previous section, we first consider the more general case of the small-gain theorem for input/output operators, an approach used in the work [5]. Our results established for operators allow small-gain theorems to be developed for several situations; systems of differential equations with delays being just one particular application.

**A. Small-Gain Theorem for Operators**

We say that a triple \((\tau, y, u)\) is a trajectory if \( \tau \in [0, \infty) \), \( u = (u_1, \ldots, u_q) : [0, \tau) \to \mathbb{R}^q \) is measurable and locally essentially bounded, and \( y = (y_1, \ldots, y_p) : [0, \tau) \to \mathbb{R}^p \) is continuous.

Note that the trajectories are defined in an abstract way, and no underlying relation is presupposed between the functions \( u \) and \( y \).

For an initialized system as in (11) with \( x_0(s) = \xi(s) \) on \([-\theta, 0], \), let \( \tilde{u} = (\tilde{v}, \tilde{u}) \), \( y(t) = x(t, \xi, \tilde{v}, \tilde{u}) \), then for any \( 0 < \tau < T_{\xi, \tilde{v}, \tilde{u}}^{\max} \), the triple \((\tau, y, \tilde{u})\) can be identified as a trajectory defined in this section.

**Proposition 4.1:** Consider a trajectory \((\tau, y, u)\) for which \( \tau = \infty \). Assume the following conditions hold:

- there exist some class \( \mathcal{K} \)-functions \( \gamma_{ij}(\cdot), \gamma_{ij}(\cdot) \), and a constant \( c \geq 0 \), such that
  \[
  |y_i(t)| \leq \max_{j \neq i} \left\{ c, \gamma_{ij} \left( \|y_j\|_{[0, t]} \right), \gamma_{ij} \left( \|u\|_{[0, t]} \right) \right\}
  \]
  for all \( t \geq 0 \); and

it holds that
\[
\lim_{t \to \infty} |y_i(t)| \leq \max_{i \neq j} \left\{ \gamma_{ij} \left( \lim_{t \to \infty} |y_j(t)| \right), \gamma_i^u \left( \|u\|_{[0,\infty)} \right) \right\}.
\] (18)

Assume further, the small-gain condition:
\[
\gamma_{i_1 i_2} \circ \gamma_{i_2 i_3} \circ \cdots \circ \gamma_{i_p i_1} < \text{id},
\] (19)
for all \(2 \leq \tau \leq p, i_j \neq i_{j'}\) whenever \(j \neq j'\). Then there exist \(K\)-functions \(\tilde{\sigma}_i(e), \tilde{\gamma}_i^u()\) and \(\tilde{\gamma}_i^u()\) such that:
\[
|y_i(t)| \leq \max \left\{ \tilde{\sigma}_i(c), \tilde{\gamma}_i^u \left( \|u\|_{[0,\infty)} \right) \right\}
\] (20)
and
\[
\lim_{t \to \infty} |y_i(t)| \leq \tilde{\gamma}_i^u \left( \|u\|_{[0,\infty)} \right).
\] (21)

To prove the proposition we first consider the following lemma. Note that this lemma holds for any \(\tau \in [0,\infty]\) though the proposition only requires the lemma for the special case when \(\tau = \infty\). We include in our consideration the case when \(\tau < \infty\) to develop a result applicable to interconnected systems as in Section III when forward completeness is not known a priori.

**Lemma 4.2**: Consider a trajectory \((\tau, y, u)\), where \(\tau \in [0,\infty]\). Suppose that for this trajectory, condition (17) holds for all \(t \in [0,\tau]\). Assume the gain functions \(\gamma_{ij}\) satisfy the small-gain condition (19). Then there exist \(\tilde{\sigma}_i, \tilde{\gamma}_i^u \in K\) such that
\[
|y_i(t)| \leq \max \left\{ \tilde{\sigma}_i(c), \tilde{\gamma}_i^u \left( \|u\|_{[0,\infty)} \right) \right\}
\] (22)
for all \(t \in [0,\tau]\).

The lemma can be proved by induction on \(p\). The case of \(p = 2\) is in fact part of the known small-gain theorem for systems with two subsystems (though not stated in the form for operators). Due to the length restriction, we omit the proof of this lemma.

**B. Proof of Proposition 4.3**

The first part of the proposition follows immediately from Lemma 4.2 with \(\tau = \infty\). The proof of the limit property of the proposition is also inductive. For the sake of saving space, we skip the proof for the case of \(p = 2\).

Instead of treating the general inductive step to pass from \(p\) to \(p + 1\), we just go through the case of \(p = 3\).

For each \(i, j = 1, 2, 3\), let \(\gamma_{ij}\) and \(\gamma_i^u\) be \(K\)-functions such that the small-gain condition holds for \(\{\gamma_{ij}\}\). Let \((\infty, \gamma, u)\) be a trajectory satisfying all assumptions of the proposition with the given gain functions \(\{\gamma_{ij}\}\) and \(\{\gamma_i^u\}\). Furthermore, assume that \(\|u\|_{[0,\infty)} < \infty\). Let \(b_i \) denote \(\lim_{t \to \infty} |y_i(t)|\). Property (20) implies that \(b_i < \infty\) for \(i = 1, 2, 3\). By the assumption in (18), we have:
\[
b_1 \leq \max \left\{ \gamma_{12}(b_2), \gamma_{13}(b_3), \gamma_1^u \left( \|u\|_{[0,\infty)} \right) \right\},
\]
\[
b_2 \leq \max \left\{ \gamma_{21}(b_1), \gamma_{23}(b_3), \gamma_2^u \left( \|u\|_{[0,\infty)} \right) \right\}
\]
and
\[
b_3 \leq \max \left\{ \gamma_{31}(b_1), \gamma_{32}(b_2), \gamma_3^u \left( \|u\|_{[0,\infty)} \right) \right\}.
\]

We eliminate \(b_3\) from the first two inequalities as in the following:
\[
b_1 \leq \max \left\{ \gamma_{12}(b_2), \gamma_{13} \circ \gamma_{31}(b_1), \gamma_{13} \circ \gamma_{32}(b_2), \gamma_{13} \circ \gamma_3^u \left( \|u\|_{[0,\infty)} \right), \tilde{\gamma}_1^u \left( \|u\|_{[0,\infty)} \right) \right\},
\]
\[
b_2 \leq \max \left\{ \gamma_{21}(b_1), \gamma_{23} \circ \gamma_{31}(b_1), \gamma_{23} \circ \gamma_{32}(b_2), \gamma_{23} \circ \gamma_3^u \left( \|u\|_{[0,\infty)} \right), \tilde{\gamma}_2^u \left( \|u\|_{[0,\infty)} \right) \right\}.
\]

By the small-gain condition, \(\gamma_{ij} \circ \gamma_{j1}(s) < s\) for all \(s > 0\), we get:
\[
b_1 \leq \max \left\{ \gamma_{12}(b_2), \gamma_{13} \circ \gamma_{32}(b_2), \gamma_{13} \circ \gamma_3^u \left( \|u\|_{[0,\infty)} \right), \tilde{\gamma}_1^u \left( \|u\|_{[0,\infty)} \right) \right\},
\]
\[
b_2 \leq \max \left\{ \gamma_{21}(b_1), \gamma_{23} \circ \gamma_{31}(b_1), \gamma_{23} \circ \gamma_3^u \left( \|u\|_{[0,\infty)} \right), \tilde{\gamma}_2^u \left( \|u\|_{[0,\infty)} \right) \right\}.
\]

Now define
\[
\tilde{\gamma}_{ij}(s) := \max \{\gamma_{ij}(s), \gamma_{ij} \circ \gamma_{j1}(s)\},
\]
\[
\tilde{\gamma}_i^u(s) := \max \{\gamma_i(s), \gamma_i \circ \gamma_i^u(s)\}.
\]

We then have
\[
b_1 \leq \max \left\{ \tilde{\gamma}_{12}(b_2), \tilde{\gamma}_1^u \left( \|u\|_{[0,\infty)} \right) \right\},
\]
\[
b_2 \leq \max \left\{ \tilde{\gamma}_{21}(b_1), \tilde{\gamma}_2^u \left( \|u\|_{[0,\infty)} \right) \right\},
\]
and consequently,
\[
b_1 \leq \max \left\{ \gamma_{12} \circ \tilde{\gamma}_{21}(b_1), \gamma_{12} \circ \tilde{\gamma}_2^u \left( \|u\|_{[0,\infty)} \right), \tilde{\gamma}_1^u \left( \|u\|_{[0,\infty)} \right) \right\}.
\]

It can be shown that \(\tilde{\gamma}_{12}\) and \(\tilde{\gamma}_{21}\) satisfy the small-gain condition \(\gamma_{12} \circ \gamma_{21} < \text{id}\), and thus,
\[
b_1 \leq \tilde{\gamma}_{12}^u \left( \|u\|_{[0,\infty)} \right),
\]
where \(\tilde{\gamma}_{12}^u(s) = \max \{\gamma_{12} \circ \tilde{\gamma}_{21}(s), \tilde{\gamma}_1^u(s)\}\). Similarly, one sees that
\[
b_2 \leq \tilde{\gamma}_{22}^u \left( \|u\|_{[0,\infty)} \right),
\]
where \(\tilde{\gamma}_{22}^u(s) = \max \{\tilde{\gamma}_{21} \circ \gamma_{21}(s), \tilde{\gamma}_2^u(s)\}\).

With the obtained estimates on \(b_1\) and \(b_2\), one has
\[
b_3 \leq \max \left\{ \gamma_{31} \tilde{\gamma}_1^u \left( \|u\|_{[0,\infty)} \right), \gamma_{32} \tilde{\gamma}_2^u \left( \|u\|_{[0,\infty)} \right), \gamma_3^u \left( \|u\|_{[0,\infty)} \right) \right\}.
\]

From this it follows that
\[
b_3 \leq \tilde{\gamma}_3^u \left( \|u\|_{[0,\infty)} \right),
\]
where \(\tilde{\gamma}_3^u(s) = \max \{\gamma_{31} \tilde{\gamma}_1^u(s), \gamma_{32} \tilde{\gamma}_2^u(s), \gamma_3^u(s)\}\). This concludes the proof for the case of \(p = 3\).
C. Proof of Theorem 7

Let \( u \) and \( \xi = (\xi_1, \ldots, \xi_k) \) be given. Consider the corresponding trajectory \( x(t) = (x_1(t), \ldots, x_k(t)) \) of the interconnected system defined on the maximum interval \([0, T_{\xi,u}^\infty)\). Let \( T = T_{\xi,u}^\infty \). Then one has the following for each \( i \):

\[
|x_i(t)| \leq \max_{i \neq j} \left\{ \sigma_i \left( \|\xi_i\|_{-\theta, 0} \right), \gamma_{ij} \left( \|x_j\|_{-\theta, 0} \right), \gamma_i^u \left( \|u\|_{0, T} \right) \right\}.
\]

Observe that

\[
\|x_j\|_{-\theta, 0} \leq \max \left\{ \|x_j\|_{-\theta, 0}, \|x_j\|_{0, 0} \right\}.
\]

Hence,

\[
|x_i(t)| \leq \max_{i \neq j} \left\{ \sigma_i \left( \|\xi_i\|_{-\theta, 0} \right), \gamma_{ij} \left( \|x_j\|_{-\theta, 0} \right), \gamma_i^u \left( \|u\|_{0, T} \right) \right\}.
\]

Let \( c = \max_{i \neq j} \left\{ \sigma_i \left( \|\xi_i\|_{-\theta, 0} \right), \gamma_{ij} \left( \|x_j\|_{-\theta, 0} \right) \right\} \). Then we have

\[
|x_i(t)| \leq \max_{i \neq j} \left\{ c, \gamma_{ij} \left( \|x_j\|_{0, i} \right), \gamma_i^u \left( \|u\|_{0, T} \right) \right\}.
\]

And thus we can apply Lemma 4.2 to \((T, x, u)\) to get

\[
|x_i(t)| \leq \max \left\{ \sigma_i(c), \gamma_i^u \left( \|u\|_{0, T} \right) \right\} \quad \text{for all } i \in \mathbb{N} \text{ and all } t \in [0, T).
\]

V. A Remark on ISS for Time-Delay Systems

Notice that our main result Theorem 7 was presented in terms of gain functions in the context of the GS and AG properties instead of the gain functions appearing in an ISS estimate of the type (3). In the delay-free case, it is well-known that the ISS property defined by (3) is equivalent to the combination of the GS and AG properties defined by (4) and (5) (see [15]). With this equivalence relation, one sees that a small-gain theorem in terms of the gain functions in the GS and AG properties also leads to a small-gain theorem in terms of ISS gain functions as in (3). However, it is not clear at this stage if the combination of the GS and the AG conditions is equivalent to the ISS property for time-delay systems.

Consider systems of the following type:

\[
\dot{x}(t) = f(x_t, d(t)),
\]

where \( f : \mathcal{X} \times \mathbb{R}^m \to \mathbb{R}^n \) is locally Lipschitz. Assume further that for any bounded sets \( K_x \subset \mathcal{X} \) and \( K_u \subset \mathbb{R}^m \), \( \{f(\xi, u) : \xi \in K_x, u \in K_u\} \) is bounded. (Note that this is not a trivial property since \( K_x \) does not need to be compact.) Although it is still unclear as to whether the combination of the GS and the AG conditions is equivalent to the ISS property for time-delay systems, we nevertheless obtained the following preparatory results (the proofs of which will be given in a more detailed version of this work).

Consider now a system of the following type:

\[
(S_d): \quad \dot{x}(t) = F(x_t, d(t)),
\]

where the disturbance function \( d : \mathbb{R}_{\geq 0} \to [0, 1]^m \) is measurable. We assume that \( f \) is a locally Lipschitz map. Let \( \Omega \) be the set of measurable functions \( d \) with \( |d(t)| \leq 1 \) for all \( t \geq 0 \). For each \( \xi \in \mathcal{X} \) and \( d \in \Omega \), we let \( x(t, \xi, d) \) denote the trajectory of the system corresponding to the initial state \( \xi \) and the disturbance function \( d \).

Definition 5.1: The system \((S_d)\) is said to be globally asymptotically stable (GAS) if

(a) there exists a \( K_\infty \)-function \( \sigma \) such that

\[
|\ddot{x}(t, \xi, d)| \leq \sigma \left( \|\ddot{x}(t, \xi, d)\|_{\theta, 0} \right)
\]

for all \( t \geq 0 \); and

(b) for each trajectory, it holds that \( \lim_{t \to \infty} |\ddot{x}(t, \xi, d)| = 0 \).

Definition 5.2: A system as in (25) is uniformly globally asymptotically stable (UGAS) if it satisfies property (a) in Definition 5.1 and the following holds:

\[
\forall \varepsilon > 0 \exists \kappa > 0 \forall T = T(e, \kappa) \geq 0 \text{ s.t. :}
\]

\[
|\ddot{x}(t, \xi, d)| \leq \varepsilon \text{ for } t \geq T.
\]

Clearly, a system \((S_d)\) is globally asymptotically stable if it is uniformly globally asymptotically stable.

Let \( \varphi : C[-\theta, 0] \to \mathbb{R}_{\geq 0} \) be any locally Lipschitz functional such that \( \varphi(0) = 0 \) where \( 0 \) denotes the zero function. Consider the auxiliary system associated with the system \((S_d)\):

\[
(S_{\varphi}): \quad \dot{x}(t) = f(x_t, \varphi(x_t)d(t)),
\]
where \( d \in \Omega \).

Let \( x_\varphi(t, \xi, d) \) denote the trajectory of \((\Sigma_\varphi)\) with initial state \( \xi \) and input \( d \).

**Proposition 5.3:** Suppose that a system \((\Sigma_u)\) as in (I) satisfies both the (AG) and the (GS) properties. Then there exists a class \( K_{\infty} \) function \( \rho \) which is locally Lipschitz such that with \( \varphi(\xi) = \rho(\|\xi\|_{-\theta, 0}) \), the corresponding auxiliary system \((\Sigma_\varphi)\) as in (24) is globally asymptotically stable. \( \square \)

**Proposition 5.4:** Consider a system \((\Sigma_u)\) as in (I). Suppose:

- the system satisfies the (GS) property; and
- there is a class \( K_{\infty} \) function \( \rho \) which is locally Lipschitz such that with \( \varphi(\xi) = \rho(\|\xi\|_{-\theta, 0}) \), the corresponding auxiliary system \((\Sigma_\varphi)\) as in (26) is uniformly globally asymptotically stable.

Then the system \((\Sigma_u)\) satisfies the ISS property. \( \square \)

In order to show that the combination of the (GS) and the (AG) conditions is equivalent to the ISS property, a crucial step is to determine for the system \((\Sigma_\varphi)\) if the global asymptotic stability property implies uniform global asymptotic stability. This remains a topic for further study.

**VI. CONCLUSION**

The nonlinear ISS small-gain theorem has been generalized to large-scale systems with time-delays. Both state-space form and input-output operators are considered for large-scale system modeling. Under the set of cyclic small-gain conditions, it is shown that the large-scale system enjoys the same type of stability properties as each individual subsystem. Our future work will be directed at applications of this tool to the control of time-delay nonlinear systems.

**VII. APPENDIX A**

In this section we prove the following:

**Lemma 7.1:** Suppose \( f : X \times V_0 \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is locally Lipschitz, then for any \( v \in V, u \in U \), the map

\[
F : X \times [0, \infty) \rightarrow \mathbb{R}^n, \quad F(\xi, t) := f(\xi, v, u(t))
\]

is locally Lipschitz in \( \xi \), uniformly in \( t \) for all \( t \) in any compact set. \( \square \)

**Proof.** Let \( v \in V, u \in U \) be given. Consider a compact subset \( K \) of \( X \) and a compact interval \([0, T]\). Since \( u \) is locally essentially bounded, there exists some \( L > 0 \) such that \( \|u\|_{[0, T]} \leq L \). Since \( v \) is continuous, \( v \) is uniformly continuous on \([-\theta, T]\). Thus, for any \( \varepsilon > 0 \) given, there exists some \( \delta > 0 \) such that for all \( t \in [0, T] \)

\[
|v_s(x_1) - v_s(x_2)| = |v(t + s_1) - v(t + s_2)| < \varepsilon
\]

for all \( s_1, s_2 \in [-\theta, 0] \). This shows that the family of functions \( \{v_s\}_{0 \leq t \leq T} \) is equicontinuous. It is clear that the family \( \{v_s\}_{0 \leq t \leq T} \) is uniformly bounded. It can also be shown that the family is closed. Thus by the Arzelà-Ascoli theorem, the set \( V = \{v_s\}_{0 \leq t \leq T} \) is a compact subset of \( V_0 \).

Since \( f \) is Lipschitz on \( K \times W \times [-L, L]^m \), there exists some \( M \geq 0 \) such that

\[
|f(\xi_1, w_1, \mu_1) - f(\xi_2, w_2, \mu_2)| \leq M(\|\xi_1 - \xi_2\|_{[-\theta, 0]} + \|w_1 - w_2\|_{[-\theta, 0]} + |\mu_1 - \mu_2|)
\]

for all \( \xi_1, \xi_2 \in K, w_1, w_2 \in W, \) and all \( \mu_1, \mu_2 \in [-L, L]^m \). In particular, for all \( \xi_1, \xi_2 \in K, \) almost all \( t \in [0, T] \),

\[
|f(\xi_1, v, (t) - f(\xi_2, v, u(t))| \leq M \|\xi_1 - \xi_2\|_{[-\theta, 0]},
\]

which means \( |F(\xi_1, t) - F(\xi_2, t)| \leq M \|\xi_1 - \xi_2\|_{[-\theta, 0]} \) for almost all \( t \in [0, T] \). \( \square \)

**REFERENCES**

[1] J.-M. Coron, L. Praly, and A. Teel, Feedback stabilization of nonlinear systems: sufficient conditions and Lyapunov and input-output techniques, in Trends in Control, A. Isidori, ed., Springer-Verlag, 1995.

[2] S. Dashkovski, B. S. Ruffer, and F. R. Wirth, An ISS small gain theorem for general networks, Mathematics of Control, Signals, and Systems, 19 (2007), pp. 93–122.

[3] X. Fan and M. Arcak, Delay robustness of a class of nonlinear systems and applications to communication networks, IEEE Transactions on Automatic Control, 51 (2006), pp. 139–144.

[4] J. Hale and S. Lunel, Introduction to Functional Differential Equations, Springer-Verlag, 1993.

[5] B. Ingalls and E. Sontag, A small-gain theorem with applications to input/output systems, incremental stability, detectability, and interconnections, Journal of the Franklin Institute, 339 (2002), pp. 211–229.

[6] M. Jankovic, Control Lyapunov-Razumikhin functions and robust stabilization of time delay systems, IEEE Transactions on Automatic Control, 46 (2001), pp. 1048–1060.

[7] Z. Jiang, A. Teel, and L. Praly, Small-gain theorem for ISS systems and applications, Mathematics of Control, Signals, and Systems, 7 (1994), pp. 95–120.

[8] Z. P. Jiang and Y. Wang, A generalization of the nonlinear small-gain theorem for large-scale complex systems, in Proc. 7th World Congress on Intelligent Control and Automation (WCICA), 2008.

[9] I. Karafyllis and Z. Jiang, A small-gain theorem for a wide class of feedback systems with control applications, SIAM Journal on Control and Optimization, 46 (2007), pp. 1483–1517.

[10] I. Karafyllis, P. Pepe, and Z.-P. Jiang, Global output stability for systems described by delayed functional differential equations: Lyapunov characterizations, European Journal of Control, 14 (2008), pp. 516–536.

[11] I. Karafyllis, P. Pepe, and Z.-P. Jiang, Input-to-output stability for systems described by retarded functional differential equations, European Journal of Control, 14 (2008), pp. 539–555.

[12] I. Karafyllis, P. Pepe, and Z.-P. Jiang, Stability results for systems described by coupled retarded functional differential equations and functional difference equations, Nonlinear Analysis, Theory, Methods and Applications, (to appear).

[13] F. Mazenc, M. Malisoff, and Z. Lin, Further results on input-state stability for nonlinear systems with delayed feedbacks, Automatica, 44 (2008), pp. 2451–2421.

[14] I. Polushin, A. Tayebi, and H. Marquez, Control schemes for stable teleoperation with communication delay based on ISS small gain theorem, Automatica, 42 (2006), pp. 905–915.

[15] E. Sontag and Y. Wang, New characterizations of the input to state stability property, IEEE Transactions on Automatic Control, 41 (1996), pp. 1283–1294.

[16] A. Teel, Connections between Razumikhin-type theorems and the ISS nonlinear small gain theorem, IEEE Transactions on Automatic Control, 43 (1998), pp. 960–964.

[17] A. R. Teel, Input-to-state stability and the nonlinear small-gain theorem, Preprint (2003).
