TOEPLITZ OPERATORS ON THE FOCK SPACE IN A SYMMETRICALLY-NORMED IDEAL

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Abstract: We look at Toeplitz operators $T_\nu$ on the Fock Space (also known as the Segal-Bargmann space) which have a positive Borel measure $\nu$ as a symbol. We characterize when $(T_\nu)^s$ for $0 < s \leq 1$ is in the symmetrically normed ideal associated with any given symmetric norming function.

Keywords: Toeplitz Operator; Fock Space; Symmetrically-Normed Ideal; Symmetric Norming Function.

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1. Introduction and Preliminaries

1.1. Fock Space and Toeplitz Operators. Let $| \cdot |$ be the usual Euclidean norm and $d\mu$ be the measure on $\mathbb{C}^n$ defined by

$$d\mu(z) = \frac{e^{-|z|^2}}{\pi^n}dV(z)$$

where $dV$ is the standard volume measure on $\mathbb{C}^n$. Let $\langle \cdot, \cdot \rangle$ be the inner product on $L^2(\mathbb{C}^n, d\mu)$ defined by

$$\langle f, g \rangle = \int_{\mathbb{C}^n} f(z)\overline{g(z)}d\mu(z) \text{ for all } f, g \in L^2(\mathbb{C}^n, d\mu)$$

and $\| \cdot \|_2$ be the corresponding induced norm. Then the Fock space (also known as the Segal-Bargmann space) $H^2 = H^2(\mathbb{C}^n, d\mu)$ is defined as the set of all functions in $L^2(\mathbb{C}^n, d\mu)$ that are analytic on $\mathbb{C}^n$. It is well-known in the literature that $H^2$ is a closed subspace of $L^2(\mathbb{C}^n, d\mu)$ with respect to $\| \cdot \|_2$. Thus $H^2$ is itself a Hilbert Space with respect to $\langle \cdot, \cdot \rangle$.

Let $\mathcal{L}(H^2)$ be the set of all bounded operators on $H^2$ with operator norm denoted by $\| \cdot \|$ and let $P: L^2(\mathbb{C}^n, d\mu) \rightarrow H^2$ be the orthogonal projection of $L^2(\mathbb{C}^n, d\mu)$ onto $H^2$. Then for any $f \in L^\infty(\mathbb{C}^n, dV)$, the Toeplitz operator on $H^2$ with symbol $f$ is the operator $T_f$ defined by

$$T_f(g) = P(fg) \text{ for all } g \in H^2.$$ 

A straightforward calculation shows $\|T_f\| \leq \|f\|_\infty$. Thus $T_f \in \mathcal{L}(H^2)$. For any $A \in \mathcal{L}(H^2)$, let $A^*$ denote the adjoint of $A$. Then by direct calculation,

$$(T_f)^* = T_{f^*} \text{ and } f \geq 0 \text{ implies } T_f \geq 0.$$
More generally if \( \nu \) is a complex Borel measure on \( \mathbb{C}^n \) such that
\[
\int_{\mathbb{C}^n} |K(z, w)| e^{-|w|^2} d|\nu|(w) < \infty \text{ for all } z \in \mathbb{C}^n
\] (1.1)
then we define the Toeplitz operator \( T_\nu \) on \( H^2 \) by
\[
(T_\nu(g))(z) = \int_{\mathbb{C}^n} K(z, w)g(w)e^{-|w|^2} d\nu(w) \text{ for all } z \in \mathbb{C}^n \text{ and for all } g \in H^2
\]
where \( K(z, w) = e^{z \overline{w_1} + \cdots + z_n \overline{w_n}} \) is the reproducing kernel of \( H^2 \) [4]. Because of (1.1) and the Cauchy-Schwarz inequality for any \( \delta > 0 \) and \( z \in \mathbb{C}^n \), let \( B(z, \delta) = \{ w \in \mathbb{C}^n : |w - z|_\infty < \delta \} \) and \( \overline{B(z, \delta)} = \{ w \in \mathbb{C}^n : |w - z|_\infty \leq \delta \} \). For any \( \alpha > 0 \), let \( \tilde{\nu}_\alpha \) be the function on \( \mathbb{C}^n \) defined by
\[
\tilde{\nu}_\alpha(z) = \int_{\mathbb{C}^n} e^{-\alpha |z-w|^2} d\nu(w) \text{ for all } z \in \mathbb{C}^n.
\]
For any \( r > 0 \) we define the function \( \hat{\nu}_r : \mathbb{C}^n \rightarrow \mathbb{C} \) by
\[
\hat{\nu}_r(z) = \nu(B(z, r)) \text{ for all } z \in \mathbb{C}^n.
\]
For any \( r > 0 \), the \( r \)-lattice of \( \mathbb{C}^n \) is defined as \( \{ r(b_1 + ic_1, \ldots, b_n + ic_n) : \{b_j\}_{j=1}^n, \{c_j\}_{j=1}^n \subseteq \mathbb{Z} \} \). Since the \( r \)-lattice of \( \mathbb{C}^n \) is countable, we can write the \( r \)-lattice of \( \mathbb{C}^n \) as a sequence \( \{a_j\}_{j=1}^\infty \). We also point out the following two useful facts about \( \{a_j\}_{j=1}^\infty \); the first of which is easy to prove and the second is due to [11] Theorem 11.1:
\[
\mathbb{C}^n = \bigcup_{j=1}^\infty B(a_j, \frac{r}{2}) = \bigcup_{j=1}^\infty B(a_j, r) \text{ and } \mu \left( B \left( a_j, \frac{r}{2} \right) \cap B \left( a_k, \frac{r}{2} \right) \right) = 0 \text{ if } k \neq j. \tag{1.3}
\]
We will now state some important properties about Toeplitz operators and the Fock space that we will need.

**Lemma 1.1.** Suppose \( \nu \) is a positive Borel measure on \( \mathbb{C}^n \), \( r > 0 \) and \( \alpha > 0 \). Then
\[
\hat{\nu}_r(z) \leq e^{\alpha 2nr^2} \tilde{\nu}_\alpha(z) \text{ for all } z \in \mathbb{C}^n.
\]
A proof of Lemma 1.1 can be easily made using (1.2) and the proof of [7] Lemma 2.2.

For any \( z, w \in \mathbb{C}^n \), let \( k_{z}(w) = \frac{K(w, z)}{\sqrt{K(z, z)}} = K(w, z)e^{-|z|^2} \). The next result we will need is Theorem 1.2, which is [8] Theorem 8.4.
**Theorem 1.2.** There exists $\delta > 0$ such that if $0 < \rho < \delta$ then for some positive constants $C_1(\rho)$ and $C_2(\rho),$ 

$$C_1(\rho)\|g\|_2^2 \leq \sum_{j=1}^{\infty} |(g, k_{b_j})|^2 \leq C_2(\rho)\|g\|_2^2$$  \hspace{1cm} (1.4)

for all $g \in H^2$ where $\{b_j\}_{j=1}^{\infty}$ is the $\rho$-lattice of $\mathbb{C}^n.$

1.2. **Symmetric norming functions, Symmetrically-normed ideals and s-numbers.**

We will now define symmetric norming functions and symmetrically-normed ideals. For this section we let $\mathcal{H}$ denote any separable complex Hilbert space, $\mathcal{L}(\mathcal{H})$ denote the set of all bounded linear operators on $\mathcal{H}$ with operator norm denoted by $\|\cdot\|_{\mathcal{H}}, \mathcal{L}^e(\mathcal{H})$ be the set of all compact operators on $\mathcal{H}$ and, given any $A \in \mathcal{L}(\mathcal{H}),$ let $\sigma(\mathcal{H})$ denote the spectrum of $A$ as an element of $\mathcal{L}(\mathcal{H}).$ Since we will be working with sequences, we will from now on say a sequence of real numbers $\{c_j\}_{j=1}^{\infty}$ is nonincreasing if $c_{j+1} \leq c_j$ for all $j \geq 1$ and nondecreasing if $c_j \leq c_{j+1}$ for all $j \geq 1.$

1.2.1. **s-numbers.** We will define now the s-numbers of a bounded linear operator on $\mathcal{H}$ and give some important results about the s-numbers. We will first define the numbers $\lambda_j(A)$ ($j = 1, 2, \ldots$) for any $A \in \mathcal{L}(\mathcal{H}).$ A point $\lambda \in \sigma(\mathcal{H}),$ where $A \in \mathcal{L}(\mathcal{H})$ is self-adjoint, is called a point of the condensed spectrum of $A$ if $\lambda$ is either an accumulation point of $\sigma(A)$ or an eigenvalue of $A$ of infinite multiplicity.

Let $H \geq 0$ and let $\eta = \sup \sigma(\mathcal{H}).$ If $\eta$ is in the condensed spectrum of $H,$ then we put 

$$\lambda_j(H) = \eta \hspace{1cm} (j = 1, 2, \ldots).$$

If $\eta$ does not belong to the condensed spectrum of $H,$ then it is an eigenvalue of finite multiplicity. In this case we put 

$$\lambda_j(H) = \eta \hspace{1cm} (j = 1, 2, \ldots, p)$$

where $p$ is the multiplicity of the eigenvalue $\eta.$

In the latter case the remaining numbers $\lambda_j(H)$ ($j = p + 1, \ldots$) are defined by 

$$\lambda_{p+j}(H) = \lambda_j(H_1) \hspace{1cm} (j = 1, 2, \ldots)$$

where the operator $H_1$ is given by 

$$H_1 = H - \eta P,$$  \hspace{1cm} (1.5)

and $P$ is the orthoprojector onto the eigenspace of the operator $H$ corresponding to the eigenvalue $\eta.$ For any $A \in \mathcal{L}(\mathcal{H}),$ we define the s-numbers of $A,$ $s_j(A)$ ($j = 1, 2, \ldots$), by 

$$s_j(A) = \lambda_j(H) \hspace{1cm} (j = 1, 2, \ldots)$$

where $H = (A^*A)^{\dagger}.$ The definition given above can also be found in [4, page 59]. We will denote the sequence of s-numbers of $A$ as $\{s_j(A)\}_{j=1}^{\infty}$ enumerated so that $\{s_j(A)\}_{j=1}^{\infty}$ is nonincreasing and so as to include multiplicities.

We will now give some important results about s-numbers. The next lemma is based on the definition of s-numbers.

**Lemma 1.3.** Let $H, C, D \in \mathcal{L}(\mathcal{H}).$ Then 

1. $s_1(D) = \|D\|_{\mathcal{H}}.$
2. $s_j(CD(C^*)^j) \leq \|C\|_{\mathcal{H}}^2 s_j(|D|)$ for all $j \geq 1.$
3. $s_j(D) = s_j(|D|)$ for all $j \geq 1.$
4. If $0 \leq C \leq D,$ then $s_j(C) \leq s_j(D)$ for all $j \geq 1.$
The next result is very useful and can easily be proved using Functional Calculus and the material above.

**Proposition 1.4.** Let $A \in \mathfrak{L}(\mathcal{H})$ such that $A \geq 0$. Then for any nonnegative function $f$ which is bounded on $\mathbb{C}$, continuous at 0 and satisfies $f(0) = 0$, the following statements hold:

1. $f(A) \geq 0$
2. $f(A) \in \mathfrak{L}(\mathcal{H})$
3. $\{f(s_j(A)) : j \geq 1\} = \{s_j(f(A)) : j \geq 1\}$.
4. If $f$ is nondecreasing on $\sigma(H(A))$, then $f(s_j(A)) = s_j(f(A))$ for all $j \geq 1$.

**1.2.2. Symmetric norming functions and symmetrically-normed ideals.** Let $\hat{c}$ be the set of all sequences of real numbers with only a finite number of nonzero terms. A function $\Phi : \hat{c} \rightarrow [0, \infty]$ is called a symmetric norming function if $\Phi$ is a norm that has the following properties:

- $\Phi(1, 0, 0, \ldots) = 1$
- $\Phi(\{\xi_j\}_{j=1}^{\infty}) = \Phi(\{|\xi_{\theta(j)}|\}_{j=1}^{\infty})$ for any bijection $\theta : \mathbb{N} \rightarrow \mathbb{N}$ and any $\{\xi_j\}_{j=1}^{\infty} \in \hat{c}$.

A norm $|\cdot|_c$ defined on a two-sided ideal $C$ in $\mathfrak{L}(\mathcal{H})$ is called a symmetric norm if the following two conditions hold:

- For any $A, B \in \mathfrak{L}(\mathcal{H})$ and $D \in C$, $|ADB|_c \leq \|A\|_\mathcal{H}|D|_c|B|_\mathcal{H}$
- For any rank one operator $T$, $|T|_c = \|T\|_\mathcal{H}$.

If in addition $C \neq \{0\}$ and $C$ is a Banach space with respect to $|\cdot|_c$, then $C$ is called a symmetrically-normed ideal. For any symmetric norming function $\Phi$, let

$$C_\Phi = \{A \in \mathfrak{L}(\mathcal{H}) : \Phi(\{s_j(A)\}_{j=1}^{\infty}) < \infty\}.$$ 

By [4, page 80-82], $C_\Phi$ is a symmetrically-normed ideal with corresponding symmetric norm $|\cdot|_\Phi$ defined by

$$|A|_\Phi = \Phi(\{s_j(A)\}_{j=1}^{\infty}).$$

One of the main properties of symmetric norming functions is given in the following lemma.

**Lemma 1.5.** Let $\Phi$ be a symmetric norming function and let $\{\xi_j\}_{j=1}^{\infty}, \{\eta_j\}_{j=1}^{\infty} \in \hat{c}$. If

$$|\xi_j| \leq |\eta_j| \text{ for all } j \geq 1,$$

then

$$\Phi(\{\xi_j\}_{j=1}^{\infty}) \leq \Phi(\{\eta_j\}_{j=1}^{\infty}).$$

A proof of this property can be found in [4, page 71-72]. It follows from Lemma 1.5 that if $\xi^m := \{\xi_1, \ldots, \xi_m, 0, 0, \ldots\}$ where $\xi = \{\xi_j\}_{j=1}^{\infty}$ is any sequence of real numbers and $m \geq 1$, then $\Phi(\xi^m)_{m=1}^{\infty}$ is a nondecreasing sequence. So we extend the definition of $\Phi$ by defining

$$\Phi(\{\xi_j\}_{j=1}^{\infty}) = \lim_{m \rightarrow \infty} \Phi(\xi^m)$$

for any sequence of real numbers $\{\xi_j\}_{j=1}^{\infty}$. With (1.6), one can easily generalize Lemma 1.5 to the set of all sequences of real numbers.

Let $\hat{c}^* = \{\{\xi_j\}_{j=1}^{\infty} \in \hat{c} : \xi_j \neq 0 \text{ for one or more } j\}$. For any two symmetric norming functions $\Psi$ and $\Phi$, we will as in [4, page 76] say $\Phi$ and $\Psi$ are equivalent if

$$\sup_{\{\xi_j\}_{j=1}^{\infty} \in \hat{c}^*} \frac{\Phi(\{\xi_j\}_{j=1}^{\infty})}{\Psi(\{\xi_j\}_{j=1}^{\infty})} < \infty \text{ and } \sup_{\{\xi_j\}_{j=1}^{\infty} \in \hat{c}^*} \frac{\Psi(\{\xi_j\}_{j=1}^{\infty})}{\Phi(\{\xi_j\}_{j=1}^{\infty})} < \infty.$$  

We will write $\Phi \sim \Psi$ to mean $\Phi$ and $\Psi$ are equivalent.
Here are two examples of symmetric norming functions:

- For any $p > 0$, let $\Phi_p : \mathbb{C} \to [0, \infty)$ be defined by
  \[ \Phi_p(\{\xi_j\}_{j=1}^{\infty}) = \left( \sum_j |\xi_j|^p \right)^{\frac{1}{p}}. \]
  Then if $p \geq 1$, $\Phi_p$ is a symmetric norming function called the Schatten $p$-norm.

- Let $\Phi_\infty : \mathbb{C} \to [0, \infty)$ be defined by
  \[ \Phi_\infty(\{\xi_j\}_{j=1}^{\infty}) = \sup_{j \geq 1} |\xi_j|. \]
  Then $\Phi_\infty$ is a symmetric norming function. In fact by [4, page 77]
  \[ \Phi \sim \Phi_\infty \text{ if and only if } \sup_{n \in \mathbb{N}} \Phi(\{\chi_j^{(n)}\}_{j=1}^{\infty}) < \infty \] (1.8)
  where by definition
  \[ \chi_j^{(n)} = \begin{cases} 1 & \text{if } j \leq n \\ 0 & \text{if } j > n. \end{cases} \]

2. Main Result and Motivation

The purpose of this paper is to prove the following Theorem:

**Theorem 2.1.** Let $\nu$ be a positive Borel measure on $\mathbb{C}^n$ and $T_\nu$ be the corresponding Toeplitz operator. Let $0 < s \leq 1$, $r > 0$ and $\{a_j\}_{j=1}^{\infty}$ be the $r$-lattice of $\mathbb{C}^n$. Then for any symmetric norming function $\Phi$,

\[ (T_\nu)^s \in C_\Phi \text{ if and only if } \Phi(\{\hat{\nu}_r(a_j)^s\}_{j=1}^{\infty}) < \infty. \]

Toeplitz operators on various spaces and domains have been studied by many different people; see for e.g. [7, 5, 15]. Kehe Zhu and Josh Isralowitz together gave necessary and sufficient conditions for when $T_\nu$ as above is in the Schatten $p$-classes $C_{\Phi_p}$ for all $p \geq 1$ [7]. Specifically, [7, Theorem 4.4], the following holds:

**Theorem 2.2.** Let $\nu$ be a positive Borel measure on $\mathbb{C}^n$ and $T_\nu$ be the corresponding Toeplitz operator. Let $p \geq 1$, $r > 0$ and $\{a_j\}_{j=1}^{\infty}$ be the $r$-lattice of $\mathbb{C}^n$. Then

\[ T_\nu \in C_{\Phi_p} \text{ if and only if } \Phi_p(\{\hat{\nu}_r(a_j)\}_{j=1}^{\infty}) < \infty. \]

Since the sets $C_{\Phi_p}$ are symmetrically-normed ideals for all $p \geq 1$, this paper will generalize the above theorem. We should also point out that one can define Schatten $p$-classes $C_{\Phi_p}$ for $0 < p < 1$. However for such $p$, $C_{\Phi_p}$ is not a symmetrically-normed ideal, which is why we are not concerned in this paper with such $p$.

The Schatten $p$-classes have been studied by many different people; see for e.g. [6, 7, 15, 16]. Some results that are stated for certain Schatten $p$-classes have been generalized to symmetrically-normed ideals. For instance Shige Toshi Kuroda in [10] generalized the Weyl von-Neumann theorem to symmetrically-normed ideals $C_\Phi$ such that the finite rank operators are dense in $C_\Phi$ and such that $\Phi \sim \Phi_1$.

Furthermore there are symmetrically normed ideals which are very different from the Schatten $p$-classes. For instance it is shown in [4, page 141] how to construct $\Phi$ such that the finite rank operators are not dense in $C_\Phi$. 

Some other symmetrically normed ideals that appear in the literature are the Orlicz ideals, the Marcinkiewicz ideals and the Lorentz ideals; see [2, 3, 13] for instance.

3. IMPORTANT RESULTS ABOUT SYMMETRIC NORMING FUNCTIONS AND SYMMETRICALLY-NORMED IDEALS

We use the same notation in this Section as in Subsection 1.2 above. The next lemma comes from [4, page 76]:

**Lemma 3.1.** For any symmetric norming function $\Phi$ and $\{\xi_j\}_{j=1}^\infty \in \hat{\mathcal{C}}$,

$$\Phi_\infty(\{\xi_j\}_{j=1}^\infty) \leq \Phi(\{\xi_j\}_{j=1}^\infty) \leq \Phi_1(\{\xi_j\}_{j=1}^\infty)$$

We also have the following useful result.

**Proposition 3.2.** Let $\{\lambda_j\}_{j=1}^\infty \in \hat{\mathcal{C}}$ and $\Phi$ be a symmetric norming function. Suppose $\{p_j\}_{j=1}^\infty \subseteq \mathbb{N}$ such that for each $j$ and $v$ with $j \neq v$, we have $p_j \neq p_v$. Then $\Phi(\{\lambda_{p_j}\}_{j=1}^\infty) \leq \Phi(\{\lambda_j\}_{j=1}^\infty)$.

**Proof.** Let $N \geq 1$ and $\Theta$ be a bijection of $\mathbb{N}$ so that $\Theta(j) = p_j$ for all $j = 1, \ldots, N$. Such a $\Theta$ exist since $\{1, 2, \ldots, N\}$ and $\{p_1, p_2, \ldots, p_N\}$ have the same cardinality. Then by Lemma 1.5 and the definition of $\Theta$,

$$\Phi(\lambda_{p_1}, \lambda_{p_2}, \ldots, \lambda_{p_N}, 0, \ldots) = \Phi(|\lambda_{p_1}|, |\lambda_{p_2}|, \ldots, |\lambda_{p_N}|, 0, \ldots)$$
$$= \Phi(\{|\lambda_{\Theta(1)}|, |\lambda_{\Theta(2)}|, \ldots, |\lambda_{\Theta(N)}|, 0, \ldots\})$$
$$\leq \Phi(\{\lambda_{j}\}_{j=1}^\infty).$$

Therefore $\Phi(\lambda_{p_1}, \lambda_{p_2}, \ldots, \lambda_{p_N}, 0, \ldots) \leq \Phi(\{\lambda_j\}_{j=1}^\infty)$. Since $N \geq 1$ was arbitrary, (1.6) yields $\Phi(\{\lambda_{p_j}\}_{j=1}^\infty) \leq \Phi(\{\lambda_j\}_{j=1}^\infty)$. $\square$

We also have the following result, a proof of which can be easily made using [4, Theorem 2.1] and the definition of s-numbers.

**Proposition 3.3.** Let $\Phi$ and $\Psi$ be two symmetric norming functions. Then

$$C_{\Phi} = C_{\Psi} \text{ if and only if } \Phi \sim \Psi.$$  

**Remark 3.4.** An immediate consequence of Lemma 1.3 is $\| \cdot \|_{\Phi_\infty} = \| \cdot \|_{\mathcal{H}}$. Hence $C_{\Phi_\infty} = \mathcal{L}(\mathcal{H})$ and by Proposition 3.3

$$C_{\Phi} = \mathcal{L}(\mathcal{H}) \text{ if and only if } \Phi \sim \Phi_\infty.$$  

Furthermore since $\mathcal{H}$ is separable, (3.1) yields

$$C_{\Phi} \subseteq \mathcal{L}(\mathcal{H}) \text{ if and only if } \Phi \sim \Phi_\infty$$

From now on, we will use the following notation: for any $g, h \in \mathcal{H}$, let $g \otimes_{\mathcal{H}} h$ be the operator on $\mathcal{H}$ defined by $(g \otimes_{\mathcal{H}} h)(v) = \langle v, h \rangle_{\mathcal{H}} g$ where $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is the inner product on $\mathcal{H} \times \mathcal{H}$.

**Proposition 3.5.** Let $\{\lambda_j\}_{j=1}^\infty$ be a sequences of complex numbers. Let $\{e_j\}_{j=1}^\infty$ and $\{b_j\}_{j=1}^\infty$ be two orthonormal sets in $\mathcal{H}$. If $D = \sum_{j=1}^\infty \lambda_j b_j \otimes_{\mathcal{H}} e_j$ and $D$ is bounded, then for any symmetric norming function $\Phi$,

$$|D|_\Phi = \Phi(\{\lambda_j\}_{j=1}^\infty).$$
Proof. By Lemma \[1.3\] \(|D|_\Phi = |(|D|)|_\Phi\). Also by direct calculation \(|D| = \sum_{j=1}^{\infty} |\lambda_j e_j \otimes_H e_j|\). Because of this, we may assume without loss of generality that \(\lambda_j \geq 0\) and \(b_j = e_j\) for all \(j \geq 1\).

By definition of \(D\), \(\sigma_H(D) = \{\lambda_j : j \geq 1\}\) where \(\{\lambda_j : j \geq 1\}\) is the closure of \(\{\lambda_j : j \geq 1\}\) in \([0, \infty)\). Since by definition of the s-numbers \(\{s_j(D) : j \geq 1\} \subseteq \sigma_H(D)\), we have \(\{s_j(D) : j \geq 1\} \subseteq \{\lambda_j : j \geq 1\}\).

Suppose \(\{s_j(D) : j \geq 1\} \subseteq \{\lambda_j : j \geq 1\}\). Recall from Section \[1.2\] that the sequence \(\{s_j(D)\}_{j=1}^{\infty}\) is enumerated so as to include multiplicities. This implies that for each \(j \geq 1\) there exist \(p_j \geq 1\) so that \(s_j(D) = \lambda_{p_j}(D)\) and so that \(j \neq m\) implies \(p_j \neq p_m\). Thus \(|D|_\Phi = \Phi(\{\lambda_{p_j}\}_{j=1}^{\infty})\). By Proposition \[3.2\] \(\Phi(\{\lambda_{p_j}\}_{j=1}^{\infty}) \leq \Phi(\{\lambda_j\}_{j=1}^{\infty})\) and so we have

\(|D|_\Phi \leq \Phi(\{\lambda_j\}_{j=1}^{\infty})\). \hspace{1cm} (3.2)

Assume instead \(\{s_j(D) : j \geq 1\} \notin \{\lambda_j : j \geq 1\}\). Then by the above statements there exist \(j \geq 1\) so that \(s_j(D)\) is an accumulation point of \(\{\lambda_j : j \geq 1\}\). Let \(\epsilon > 0\). We claim that

for each \(j \geq 1\) there exist \(p_j \geq 1\) such that \(s_j(D) - \frac{\epsilon}{2^j} \leq \lambda_{p_j}\) and so that \(m \neq j\) implies \(p_m \neq p_j\). \hspace{1cm} (3.3)

To prove the claim, let \(j_0\) be the smallest positive integer such that \(s_{j_0}(D)\) is an accumulation point of \(\{\lambda_j : j \geq 1\}\).

If \(j_0 = 1\), then there exist a subsequence \(\{\lambda_{p_j}\}_{j=1}^{\infty}\) of \(\{\lambda_j\}_{j=1}^{\infty}\) satisfying \(s_1(D) - \frac{\epsilon}{2^1} \leq \lambda_{p_j}\) for all \(j \geq 1\). Also by definition of the s-numbers, \(s_1(D) = s_{j_0}(D)\) for all \(j \geq 1\). Thus \(s_j(D) - \frac{\epsilon}{2^j} \leq \lambda_{p_j}\) for all \(j \geq 1\) and (3.3) holds.

So assume \(j_0 > 1\). Then \(\{s_j(D) : 1 \leq j \leq j_0 - 1\} \subseteq \{\lambda_j : j \geq 1\}\). It follows, as above, that for each \(j\) with \(1 \leq j \leq j_0 - 1\) there exist \(p_j \geq 1\) so that \(s_j(D) = \lambda_{p_j}\) and so that \(1 \leq m \leq j_0 - 1\) with \(m \neq j\) implies \(p_j \neq p_m\). Let \(v = \max\{p_1, \ldots, p_{j_0-1}\}\). Since \(s_{j_0}(D)\) is an accumulation point of \(\{\lambda_j : j \geq 1\}\), \(s_{j_0}(D)\) is an accumulation point of \(\{\lambda_j : j > v\}\). This means there exist a subsequence \(\{\lambda_{n_j}\}_{j=1}^{\infty}\) of \(\{\lambda_j\}_{j=1}^{\infty}\) such that \(s_{j_0}(D) - \frac{\epsilon}{2^{j_0+1}} \leq \lambda_{n_j}\) for all \(j \geq 1\). Again by definition of the s-numbers, \(s_{j_0}(D) = s_{j}(D)\) for all \(j \geq j_0\). Thus setting \(p_{j_0-1+d} = n_d\) for all \(d \geq 1\), the above work yields \(s_j(D) - \frac{\epsilon}{2^j} \leq \lambda_{p_j}\) for all \(j \geq 1\) and again (3.3) holds.

By (3.3), the triangle inequality for \(\Phi\), Lemma \[1.3\], Lemma \[3.1\] and Proposition \[3.2\]

\[|D|_\Phi - \epsilon \sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^{j} \leq |D|_\Phi - \epsilon \Phi \left(\left\{ \left(\frac{1}{2}\right)^{j} \right\}_{j=1}^{\infty} \right) \leq \Phi \left(\left\{ s_j(D) - \frac{\epsilon}{2^j} \right\}_{j=1}^{\infty} \right) \leq \Phi \left(\{\lambda_{p_j}\}_{j=1}^{\infty} \right) \leq \Phi \left(\{\lambda_j\}_{j=1}^{\infty} \right).
\]

Since \(1 = \sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^{j}\) and \(\epsilon > 0\) was arbitrary, we again have (3.2).

Let \(N \geq 1\) and \(P_N = \sum_{j=1}^{N} \lambda_j e_j \otimes_H e_j\). Then \(0 \leq P_N \leq D\). It follows from Lemma \[1.3\] and Lemma \[1.3\] that \(|P_N|_\Phi \leq |D|_\Phi\). Moreover, \(|P_N|_\Phi = \Phi(\lambda_1, \lambda_2, \ldots, \lambda_N, 0, \ldots)\) by definition of the s-numbers. This implies \(\Phi(\lambda_1, \lambda_2, \ldots, \lambda_N, 0, \ldots) \leq |D|_\Phi\). Since \(N \geq 1\) was arbitrary, (1.6) implies

\(\Phi \left(\{\lambda_j\}_{j=1}^{\infty} \right) \leq |D|_\Phi\).

Therefore by the above and (3.2), \(|D|_\Phi = \Phi \left(\{\lambda_j\}_{j=1}^{\infty} \right)\).
The next result, which is [4, Theorem 5.1] will be used to prove Proposition 3.7 below.

Lemma 3.6. Let \( \Phi \) be an arbitrary symmetric norming function such that \( \Phi \sim \Phi_\infty \). If an operator \( A \in \mathcal{L}(\mathcal{H}) \) is the weak limit of a sequence \( \{A_n\}_{n=1}^\infty \) from \( C_\Phi \) and if \( \sup_n |A_n|_\Phi < \infty \), then \( A \in C_\Phi \) and \( |A|_\Phi \leq \sup_n |A_n|_\Phi \).

Proposition 3.7. Let \( 0 < s \leq 1 \) and \( \Phi \) be a symmetric norming function. If \( \{H_j\}_{j=1}^\infty \subseteq \mathcal{L}(\mathcal{H}) \) then there exists a \( C > 0 \) so that
\[
\left\| \sum_{j=1}^\infty H_j \right\|_\Phi^s \leq C 2^{1-s} \sum_{j=1}^\infty |H_j|^s_\Phi.
\]

Proof. Clearly we may assume \( \sum_{j=1}^\infty |H_j|^s_\Phi < \infty \). Thus by Lemma 3.1 and the fact that \( 0 < s \leq 1 \),
\[
\left\| \sum_{j=1}^\infty H_j \right\|_\mathcal{H}^s \leq \left( \sum_{j=1}^\infty \|H_j\|_\mathcal{H} \right)^s \leq \sum_{j=1}^\infty \|H_j\|_\mathcal{H}^s = \sum_{j=1}^\infty \|H_j\|_\Phi^s \leq \sum_{j=1}^\infty \|H_j\|^s_\Phi \leq 2^{1-s} \sum_{j=1}^\infty |H_j|^s_\Phi.
\]

Thus if \( \Phi \sim \Phi_\infty \),
\[
\left\| \sum_{j=1}^\infty H_j \right\|_\Phi^s \leq C_1 \sum_{j=1}^\infty \|H_j\|_\mathcal{H}^s \leq C_1 2^{1-s} \sum_{j=1}^\infty |H_j|^s_\Phi
\]

for some \( C_1 > 0 \).

Now assume \( \Phi \sim \Phi_\infty \). Then by Remark 3.4, \( \sum_{j=1}^\infty \|H_j\|^s_\Phi < \infty \) implies \( |H_j|^s \in \mathcal{L}(\mathcal{H}) \) for all \( j \geq 1 \). So by Proposition 1.4, \( H_j \in \mathcal{L}(\mathcal{H}) \) which itself gives us \( H_j \in \mathcal{L}(\mathcal{H}) \) for every \( j \geq 1 \). Hence for each \( j \geq 1 \), there exist two orthonormal sets \( \{\eta_q^{(j)}\}_{q=1}^\infty \) and \( \{\sigma_p^{(j)}\}_{p=1}^\infty \) in \( \mathcal{H} \) such that \( H_j = \sum_{p=1}^\infty s_p(H_j) \eta_p^{(j)} \otimes \mathcal{H} \sigma_p^{(j)} \) and \( \lim_{p \to \infty} s_p(H_j) = 0 \).

Let \( M \geq 1 \) and \( N \geq 1 \). Let \( F_N^{(j)} = \sum_{p=1}^N s_p(H_j) \eta_p^{(j)} \otimes \mathcal{H} \sigma_p^{(j)} \). Then by [12, Lemma 3.1],
\[
\left\| \sum_{j=1}^M F_N^{(j)} \right\|_\Phi^s \leq 2^{1-s} \sum_{j=1}^M \left\| F_N^{(j)} \right\|_\Phi^s.
\]

It is also true by definition of the \( s \)-numbers and Proposition 1.4 that
\[
\left| F_N^{(j)} \right|_\Phi = \Phi(|s_1(H_j)|^s, |s_2(H_j)|^s, \ldots, |s_N(H_j)|^s, 0, \ldots)
\]
for all \( j \geq 1 \). It follows from Lemma 1.5 that
\[
\left\| F_N^{(j)} \right\|_\Phi^s \leq \|H_j\|^s_\Phi
\]
for all \( j \geq 1 \). Then combining (3.5) and (3.6) gives us
\[
\left\| \sum_{j=1}^M F_N^{(j)} \right\|_\Phi^s \leq 2^{1-s} \sum_{j=1}^M \|H_j\|^s_\Phi \leq 2^{1-s} \sum_{j=1}^\infty \|H_j\|^s_\Phi \text{ for all } N \geq 1.
\]

Furthermore since \( \lim_{p \to \infty} s_p(H_j) = 0 \) for all \( j \geq 1 \), \( \{F_N^{(j)}\}_{N=1}^\infty \) converges to \( H_j \) with respect to the norm topology. Then the Stone-Weierstrass Theorem and the continuous functional
Proof. By Lemma 1.1 and Lemma 1.5, calculus \([14, \text{page 62}]\) yields \(\left\{ \sum_{j=1}^{M} F_N^{(j)} \right\}_{N=1}^{\infty} \) converges to \(\left| \sum_{j=1}^{M} H_j \right|^s\) with respect to the norm topology. Then by (3.7) and Lemma 3.6,

\[
\left| \left( \sum_{j=1}^{M} H_j \right)^s \right|_\Phi \leq 2^{1-s} \sum_{j=1}^{\infty} \|H_j|^s\|_\Phi \text{ for any } M \geq 1.
\]

Moreover since \(\sum_{j=1}^{\infty} \|H_j|^s\|_\Phi < \infty\), (3.4) implies \(\sum_{j=1}^{\infty} \|H_j\|_H < \infty\). It follows that \(\left\{ \sum_{j=1}^{M} H_j \right\}_{M=1}^{\infty}\) converges to \(\sum_{j=1}^{\infty} H_j\) in the norm topology. Thus another application of the continuous functional calculus \([14, \text{page 62}]\) and the Stone-Weierstrass theorem shows \(\left\{ \left| \sum_{j=1}^{M} H_j \right|^s \right\}_{M=1}^{\infty}\) converges to \(\left| \sum_{j=1}^{\infty} H_j \right|^s\) in the norm topology. Then by another application Lemma 3.6,

\[
\left| \sum_{j=1}^{\infty} H_j \right|^s \leq 2^{1-s} \sum_{j=1}^{\infty} \|H_j|^s\|_\Phi.
\]

Therefore with \(C = \max\{C_1, 1\}\), we have

\[
\left| \sum_{j=1}^{\infty} H_j \right|^s \leq C 2^{1-s} \sum_{j=1}^{\infty} \|H_j|^s\|_\Phi.
\]

\[\square\]

4. PROOF OF MAIN RESULT

From now on \(\otimes = \otimes_H^2\) where \(\otimes_H\) is defined in Section 3 for any separable complex Hilbert space \(H\).

4.1. An Important Theorem. The next theorem will be very useful in proving Theorem 2.1

**Theorem 4.1.** Let \(\rho > 0\), \(\gamma > 0\), \(\alpha > 0\), \(0 < s \leq 1\), and \(\{b_j\}_{j=1}^{\infty}\) be the \(\rho\)-lattice of \(\mathbb{C}^n\). Let \(\nu\) be a positive Borel measure on \(\mathbb{C}^n\) and \(\Phi\) be a symmetric norming function. Then

\[
e^{-s\gamma^2\alpha(2n)} \Phi \left( \{(\tilde{\nu}_\gamma(b_j))^s\}_{j=1}^{\infty} \right) \leq \Phi \left( \{(\tilde{\nu}_\alpha(b_j))^s\}_{j=1}^{\infty} \right) \leq C \Phi \left( \{(\tilde{\nu}_\rho(b_j))^s\}_{j=1}^{\infty} \right)
\]

where \(C = e^{s\alpha n^2 \rho^2} \sum_{m=0}^{\infty} (2m+1)^{2n} e^{-s\gamma^2(2m-1)^2}\).

**Proof.** By Lemma 1.1 and Lemma 1.5

\[
e^{-s\gamma^2\alpha(2n)} \Phi \left( \{(\tilde{\nu}_\gamma(b_j))^s\}_{j=1}^{\infty} \right) \leq \Phi \left( \{(\tilde{\nu}_\alpha(b_j))^s\}_{j=1}^{\infty} \right)
\]

For any \(z \in \mathbb{C}^n\) and any \(A \subseteq \mathbb{C}^n\), let \(A + \{z\} = \{a + z : a \in A\}\). Then by the definition of a \(\rho\)-lattice as in Section 1.1

\(\{b_j\}_{q=1}^{\infty} = \{b_q\}_{q=1}^{\infty} + \{b_j\}\) for all \(j \geq 1\). (4.1)
From the above and (1.3) it follows that $\mathbb{C}^n = \bigcup_{q=1}^{\infty} B(b_j + b_q, \rho)$ for every $j \geq 1$. Hence

$$(\nu_\alpha(b_j))^s = \left( \int_{\mathbb{C}^n} e^{-|z-b_j|^2} d\nu(z) \right)^s \leq \left( \sum_{q=1}^{\infty} \int_{B(b_q+b_j,\rho)} e^{-|z-b_j|^2} d\nu(z) \right)^s \leq \sum_{q=1}^{\infty} \left( \int_{B(b_q+b_j,\rho)} e^{-|z-b_j|^2} d\nu(z) \right)^s \quad (4.2)$$

For any $z \in B(b_q + b_j, \rho)$, we have by (1.2)

$$|z - (b_q + b_j)| \leq \sqrt{2n}|z - (b_q + b_j)|_\infty < \rho\sqrt{2n}.$$ 

Then from the triangle inequality

$$|z - b_j|^2 \geq (|z - (b_j + b_q)| - |b_q|)^2 \geq |b_q|^2 - 2|z - (b_j + b_q)||b_q| \geq |b_q|^2 - 2\rho \sqrt{2n}|b_q|.$$ 

This implies

$$\sum_{q=1}^{\infty} \left( \int_{B(b_q+b_j,\rho)} e^{-|z-b_j|^2} d\nu(z) \right)^s \leq \sum_{q=1}^{\infty} \left( \int_{B(b_q+b_j,\rho)} e^{-|b_q|^2 + 2\alpha \rho \sqrt{2n}|b_q|} d\nu(z) \right)^s \quad (4.3)$$

So by Lemma 1.5, (4.2) and (4.3),

$$\Phi \left( \{ (\nu_\alpha(b_j))^s \}_{j=1}^\infty \right) \leq \Phi \left( \left\{ \sum_{q=1}^{\infty} e^{-\alpha|b_q|^2 + 2\alpha \rho \sqrt{2n}|b_q|} (\nu_\rho(b_q + b_j))^s \right\}_{j=1}^\infty \right).$$

Then by the triangle inequality for $\Phi$ and (4.1),

$$\Phi \left( \left\{ \sum_{q=1}^{\infty} e^{-\alpha|b_q|^2 + 2\alpha \rho \sqrt{2n}|b_q|} (\nu_\rho(b_q + b_j))^s \right\}_{j=1}^\infty \right) \leq \sum_{q=1}^{\infty} \Phi \left( e^{-\alpha|b_q|^2 + 2\alpha \rho \sqrt{2n}|b_q|} \{ (\nu_\rho(b_q + b_j))^s \}_{j=1}^\infty \right)$$

$$= \left[ \sum_{q=1}^{\infty} e^{-\alpha|b_q|^2 + 2\alpha \rho \sqrt{2n}|b_q|} \Phi \left( \{ (\nu_\rho(b_q + b_j))^s \}_{j=1}^\infty \right) \right].$$
By \([1.2]\),
\[
\sum_{q=1}^{\infty} e^{-s\alpha |b_q|^2 + 2s\alpha \rho \sqrt{2\nu}|b_q|} \leq \sum_{q=1}^{\infty} e^{-s\alpha |b_q|^2 + 2s\alpha \rho \rho |b_q|_{\infty}} = \sum_{q=1}^{\infty} e^{-s\alpha |b_q|^2 + 2s\alpha \rho \rho |b_q|_{\infty}} = \sum_{q=1}^{\infty} e^{-s\alpha |b_q|_{\infty} - 2\nu s^2 \rho^2 + 2s\alpha \rho^2}
\]
\[
= \sum_{q=1}^{\infty} e^{s\alpha \rho^2} \sum_{q=1}^{\infty} e^{-s\alpha |b_q|_{\infty} - 2\nu s^2 \rho^2}.
\]

By definition of a \(\rho\)-lattice, we have \(b_q = \rho(j_1(q) + 2l_1(q), \ldots, j_n(q) + 2l_n(q))\) for some integers \(j_1(q), l_1(q), \ldots, j_n(q), l_n(q)\). Hence \(|b_q|_{\infty} = \rho \max \{\left| \frac{j_d(q)}{\rho} \right|, \left| \frac{l_d(q)}{\rho} \right| \}_{d=1}^{n}\). Thus for any \(m \geq 0\), \(|b_q|_{\infty} \leq \rho m\) if and only if \(|j_d(q)| \leq m\) and \(|l_d(q)| \leq m\) for each \(1 \leq d \leq n\). Moreover the number of integers \(j\) satisfying \(|j| \leq m\) is \(2m + 1\). This means the cardinality of the set \(\{q \in \mathbb{N} : |b_q|_{\infty} = \rho m\}\) is not larger than \((2m + 1)^2\). So with \(B_m = \{q \in \mathbb{N} : |b_q|_{\infty} = \rho m\}\) the above statements tells us
\[
e^{-s\alpha \gamma^2(2n)} \Phi \left( \left\{ (\hat{\nu}_{\gamma}(b_j))^s \right\}_{j=1}^{\infty} \right) \leq \Phi \left( \left\{ (\nu_{\gamma}(b_j))^s \right\}_{j=1}^{\infty} \right) \leq C \Phi \left( \left\{ (\nu_{\gamma}(b_j))^s \right\}_{j=1}^{\infty} \right).
\]

Therefore
\[
e^{-s\alpha \gamma^2(2n)} \Phi \left( \left\{ (\hat{\nu}_{\gamma}(b_j))^s \right\}_{j=1}^{\infty} \right) \leq \Phi \left( \left\{ (\nu_{\gamma}(b_j))^s \right\}_{j=1}^{\infty} \right) \leq C \Phi \left( \left\{ (\nu_{\gamma}(b_j))^s \right\}_{j=1}^{\infty} \right).
\]

\(\square\)

4.2. **Proof of Sufficiency.** We will show sufficiency of Theorem 2.1 here by proving
\[
| (T_{a})^s |_{\Phi} \leq C \Phi \left( \left\{ (\nu_{\gamma}(a_j))^s \right\}_{j=1}^{\infty} \right)
\]
for some constant \(C = C(r, s) > 0\).

If \(\Phi \left( \left\{ (\nu_{\gamma}(a_j))^s \right\}_{j=1}^{\infty} \right) = \infty\), then the above clearly holds. So assume \(\Phi \left( \left\{ (\nu_{\gamma}(a_j))^s \right\}_{j=1}^{\infty} \right) < \infty\). It follows from Lemma 3.1 that \(\Phi \left( \left\{ (\nu_{\gamma}(a_j))^s \right\}_{j=1}^{\infty} \right) < \infty\), which further implies \(\|T_{a}\| < \infty\) [7]. Let \(\delta\) be as in Theorem 1.2 and choose \(M \in \mathbb{N}\) with \(M > 2\) and such that \(\frac{1}{M} < \delta\). For ease of notation, let \(\rho = \frac{M}{\delta}\). Let \(\{b_j\}_{j=1}^{\infty} \) be the \(\rho\)-lattice of \(\mathbb{C}^n\), \(\{e_{b_j}\}_{j=1}^{\infty}\) be an orthonormal set in \(H^2\) and let \(B^{(\rho)}\) be the operator on \(H^2\) defined by
\[
B^{(\rho)} = \sum_{j=1}^{\infty} e_{b_j} \otimes k_{b_j}.
\]
Then for any \( f, g \in H^2 \) we have by Bessel’s inequality, the Cauchy-Schwarz inequality and [7, Lemma 2.1],
\[
|\langle B(\rho), f \rangle| \leq \sum_{j=1}^{\infty} |\langle f, e_{b_j} \rangle| |\langle b_j, g \rangle|
\]
\[
= \sum_{j=1}^{\infty} |\langle f, e_{b_j} \rangle| |g(b_j)| e^{-\frac{|b_j|^2}{2}}
\]
\[
\leq \left( \sum_{j=1}^{\infty} |\langle f, e_{b_j} \rangle|^2 \right)^{\frac{1}{2}} \left( \sum_{j=1}^{\infty} |g(b_j)|^2 e^{-\frac{|b_j|^2}{2}} \right)^{\frac{1}{2}}
\]
\[
\leq \|f\|_2 J \left( \sum_{j=1}^{\infty} \int_{B(b_j, \rho)} |g(z)|^2 e^{-\frac{|z|^2}{2}} dV(z) \right)^{\frac{1}{2}}
\]
\[
\leq Jq\|f\|_2 \|g\|_2
\]
where \( J = J(\rho, n) > 0 \) and where \( q \in \mathbb{N} \) is such that every \( z \in \mathbb{C}^n \) lies in at most \( q \) of the sets \( B(b_j, \rho) \). Such an \( q \) clearly exists. Since \( f, g \in H^2 \) were arbitrary, (4.4) implies \( \|B(\rho)\| \leq qJ \) and \( B(\rho) \in \mathcal{L}(H^2) \).

Let
\[
\Gamma(\rho) = (B(\rho))^{*} B(\rho).
\]
This means \( \langle \Gamma(\rho), g, g \rangle = \langle B(\rho) g, B(\rho) g \rangle \) for all \( g \in H^2 \). Hence
\[
\langle \Gamma(\rho), g, g \rangle = \sum_{j=1}^{\infty} |\langle g, b_j \rangle|^2.
\]
So \( \Gamma(\rho) \geq 0 \) and by Theorem 1.2, there exist a constant \( W_1(\rho) > 0 \) so that
\[
W_1(\rho) \|g\|_2^2 \leq \langle \Gamma(\rho), g, g \rangle \text{ for all } g \in H^2.
\]
Thus by [11, Corollary 4.9], \( \Gamma(\rho) \) is invertible on \( H^2 \). So (4.5) implies
\[
T_\nu = \Gamma(\rho)^{-1} \Gamma(\rho) T_\nu \Gamma(\rho)^{-1} = \Gamma(\rho)^{-1} (B(\rho))^{*} B(\rho) T_\nu (B(\rho))^{*} B(\rho) \Gamma(\rho)^{-1}.
\]
We claim that, for some constant \( C_1 = C_1(\rho, n, s) > 0 \),
\[
|\langle (T_\nu)^{s} \rangle_\Phi| \leq C_1 \left| \left( B(\rho) T_\nu (B(\rho))^{*} \right)^{s} \right|_\Phi.
\]
To prove this claim, first note if \( |\langle (B(\rho) T_\nu (B(\rho))^{*})^{s} \rangle_\Phi| = \infty \), then the above clearly holds. So assume instead \( |\langle (B(\rho) T_\nu (B(\rho))^{*})^{s} \rangle_\Phi| < \infty \). If \( \Phi \sim \Phi_\infty \), then by Remark 3.4, Lemma 1.3 and Proposition 1.4 yields
\[
\left| \left( \Gamma(\rho)^{-1} (B(\rho))^{*} B(\rho) T_\nu (B(\rho))^{*} B(\rho) \Gamma(\rho)^{-1} \right)^{*} \right|_\Phi
\]
\[
= \Phi \left( \left\{ s_j \left( \Gamma(\rho)^{-1} (B(\rho))^{*} B(\rho) T_\nu (B(\rho))^{*} B(\rho) \Gamma(\rho)^{-1} \right)^{*} \right\}_j^{\infty} \right)
\]
\[
\leq \|\Gamma(\rho)^{-1} (B(\rho))^{*}\|^{2s} \Phi \left( \left\{ s_j (B(\rho) T_\nu (B(\rho))^{*})^{\infty} \right\}_j^{1} \right)
\]
\[
= \|\Gamma(\rho)^{-1} (B(\rho))^{*}\|^{2s} \left| \left( B(\rho) T_\nu (B(\rho))^{*} \right)^{\infty} \right|_\Phi.
\]
If instead $\Phi \sim \Phi_\infty$ then for some $V > 0$ that only depends on $\Phi$,

$$
\left| \left( \Gamma^{(\rho)^{-1}} (B^{(\rho)})^* B^{(\rho)} T_\nu (B^{(\rho)})^* B^{(\rho)} \Gamma^{(\rho)^{-1}} \right) \right|^\Phi
\leq V \left| \left( \Gamma^{(\rho)^{-1}} (B^{(\rho)})^* B^{(\rho)} T_\nu (B^{(\rho)})^* B^{(\rho)} \Gamma^{(\rho)^{-1}} \right) \right|^\Phi
= V \left| \left( \Gamma^{(\rho)^{-1}} (B^{(\rho)})^* B^{(\rho)} T_\nu (B^{(\rho)})^* B^{(\rho)} \Gamma^{(\rho)^{-1}} \right) \right|^\Phi
\leq V \left| \left( \Gamma^{(\rho)^{-1}} (B^{(\rho)})^* B^{(\rho)} T_\nu (B^{(\rho)})^* B^{(\rho)} \Gamma^{(\rho)^{-1}} \right) \right|^\Phi
\leq V \left| \left( \Gamma^{(\rho)^{-1}} (B^{(\rho)})^* B^{(\rho)} T_\nu (B^{(\rho)})^* B^{(\rho)} \Gamma^{(\rho)^{-1}} \right) \right|^\Phi
$$

where the latter inequality follows by Lemma 3.1. Thus (4.7) holds.

By direct calculation and [9] page 188-191

$$
B^{(\rho)} T_\nu (B^{(\rho)})^* = \sum_{j=1}^\infty \sum_{i=1}^\infty \langle T_\nu k_{b_i}, k_{b_i+b_j} \rangle e_{b_i+b_j} \otimes e_{b_i}.
$$

(4.8)

So with $H_j$ defined by

$$
H_j = \sum_{i=1}^\infty \langle T_\nu k_{b_i}, k_{b_i+b_j} \rangle e_{b_i+b_j} \otimes e_{b_i},
$$

(4.9)

we have

$$
B^{(\rho)} T_\nu (B^{(\rho)})^* = \sum_{j=1}^\infty H_j
$$

(4.10)

and by Proposition 3.7

$$
\left| \left( \sum_{j=1}^\infty H_j \right) \right|^\Phi \leq 2^{1-s} D_1 \sum_{j=1}^\infty ||H_j||^\Phi
$$

(4.11)

for some constant $D_1 > 0$. Combining (4.7), (4.10) and (4.11) together gives us

$$
| (T_\nu)^s \|^\Phi \leq 2^{1-s} C_2 \sum_{j=1}^\infty ||H_j||^\Phi
$$

(4.12)

where $C_2 = C_1 D_1$. By the definition of $H_j$ and Proposition 3.5,

$$
||H_j||^\Phi = \Phi \left( \{ |\langle T_\nu k_{b_i}, k_{b_i+b_j} \rangle|^s \}_{i=1}^\infty \right).
$$

Let $i \geq 1$ and $j \geq 1$. Since $\|T_\nu\| < \infty$ [5] Lemma 4.1] implies

$$
|\langle T_\nu k_{b_i}, k_{b_i+b_j} \rangle| = \left| \int_{\mathbb{C}^n} k_{b_i}(z) k_{b_i+b_j}(z) e^{-|z|^2} d\nu(z) \right| \leq \int_{\mathbb{C}^n} \left| k_{b_i}(z) k_{b_i+b_j}(z) e^{-|z|^2} \right| d\nu(z)
$$

$$
= \int_{\mathbb{C}^n} e^{-|z-b_j|^2} e^{-|z-(b_i+b_j)|^2} d\nu(z)
$$

(4.13)

where the latter equality comes from direct calculation.

Now by the triangle inequality,

$$
\frac{|b_j|}{2} \leq |z - b_i| \text{ or } \frac{|b_j|}{2} \leq |z - (b_i + b_j)| \text{ for all } z \in \mathbb{C}^n.
$$
If \( \frac{|b_j|}{2} \leq |z-(b_i+b_j)| \), then \( \frac{|b_j|^2}{8} + \frac{|z-b_i|^2}{2} \leq \frac{|z-(b_i+b_j)|^2}{2} + \frac{|z-b_i|^2}{2} \). Thus \( \frac{|b_j|^2}{8} + \frac{|z-b_i|^2}{2} + \frac{|z-(b_i+b_j)|^2}{2} \leq |z-(b_i+b_j)|^2 + |z-b_i|^2 \). Likewise if \( \frac{|b_j|}{2} \leq |z-b_i| \) then \( \frac{|b_j|^2}{8} + \frac{|z-b_i|^2}{2} + \frac{|z-(b_i+b_j)|^2}{2} \leq |z-(b_i+b_j)|^2 + |z-b_i|^2 \). In either case,

\[
e^{-\frac{|z-b_i|^2}{2}} e^{-\frac{|z-(b_i+b_j)|^2}{2}} \leq e^{-\frac{|b_j|^2}{4}} e^{-\frac{|z-b_i|^2}{4}} e^{-\frac{|z-(b_i+b_j)|^2}{4}}. \tag{4.14}
\]

From (4.13) and (4.14) we obtain

\[
|\langle T_\nu k_{b_i}, k_{b_i+b_j} \rangle|^s \leq e^{-\frac{|b_j|^2}{16}} \left( \tilde{\nu}_{\frac{1}{4}}(b_i) \right)^s \text{ for every } i \geq 1 \text{ and } j \geq 1. \tag{4.15}
\]

Hence by Lemma 1.5,

\[
||H_j||_\Phi \leq e^{-\frac{|b_j|^2}{16}} \Phi \left( \left\{ \left( \tilde{\nu}_{\frac{1}{4}}(b_i) \right)^s \right\}_{i=1}^{\infty} \right) \text{ for all } j \geq 1. \tag{4.16}
\]

Thus by (4.12),

\[
| (T_\nu) \Phi | \leq 2^{1-s} C_2 \Phi \left( \left\{ \left( \tilde{\nu}_{1/4}(b_i) \right)^s \right\}_{i=1}^{\infty} \right) \sum_{j=1}^{\infty} e^{-\frac{|b_j|^2}{16}}.
\]

By (1.2) and an argument similar to one used in the proof of Theorem 4.1 shows

\[
\sum_{j=1}^{\infty} e^{-\frac{|b_j|^2}{16}} \leq \sum_{j=1}^{\infty} e^{-\frac{|b_j|^2}{16}} \leq \sum_{j=1}^{\infty} (2j+1)^{2n} e^{-\frac{s(\rho^2)}{16}}.
\]

So in fact

\[
| (T_\nu) \Phi | \leq 2^{1-s} C_3 \Phi \left( \left\{ \left( \tilde{\nu}_{\frac{1}{4}}(b_i) \right)^s \right\}_{i=1}^{\infty} \right)
\]

where \( C_3 = C_2 \sum_{j=1}^{\infty} e^{-\frac{|b_j|^2}{16}} \). Then by Theorem 4.1

\[
| (T_\nu) \Phi | \leq C_4 \Phi \left( \left\{ \tilde{\nu}_{\rho}(b_j) \right\}_{j=1}^{\infty} \right)
\]

for some positive constant \( C_4 \).

For any \( j \geq 1 \), (1.3) and the definition of \( \rho \) implies there exist a unique \( p_j \geq 1 \) so that \( a_{p_j} \in B \left( b_j, \frac{\rho}{2} \right) \). Assume \( |b_j - a_{p_j}| = \frac{\rho}{2} \). It follows from the definition of an \( r \)-lattice and the definition of \( \rho \) that \( |Mw - d| = \frac{1}{2} \) for some \( w, d \in \mathbb{Z} \). Thus \( \frac{1}{2} \in \mathbb{Z} \), which is a contradiction. So we must have \( a_{p_j} \in B \left( b_j, \frac{\rho}{2} \right) \). Now if \( p_j = p_q \) for some \( q \neq j \), then by the triangle inequality, \( |b_j - b_q| \leq |a_{p_j} - b_j| + |a_{p_j} - b_q| < \rho \). This is also a contradiction since \( |b_q - b_j| \geq \rho \). Thus \( j \neq q \) implies \( p_j \neq p_q \). We also have \( B(b_j, \rho) \subseteq B(a_{p_j}, \frac{3\rho}{2}) \). It follows that \( \tilde{\nu}_{\rho}(b_j) \leq \tilde{\nu}_{\rho}(a_{p_j}) \). Hence by Lemma 1.5

\[
\Phi \left( \left\{ \tilde{\nu}_{\rho}(b_j) \right\}_{j=1}^{\infty} \right) \leq \Phi \left( \left\{ \tilde{\nu}_{\rho}(a_{p_j}) \right\}_{j=1}^{\infty} \right).
\]

It is also true by Proposition 3.2 and the above statements that

\[
\Phi \left( \left\{ \tilde{\nu}_{\rho}(a_{p_j}) \right\}_{j=1}^{\infty} \right) \leq \Phi \left( \left\{ \tilde{\nu}_{\rho}(a_j) \right\}_{j=1}^{\infty} \right).
\]

Therefore with \( C = C_4 \),

\[
| (T_\nu) \Phi | \leq C \Phi \left( \left\{ \tilde{\nu}_{\rho}(a_j) \right\}_{j=1}^{\infty} \right).
\]
4.3. Proof of Necessity. We will show necessity of Theorem 2.1 here by proving
\[
\Phi \left( \{(\tilde{v}_r(a_j))^s\}_{j=1}^{\infty} \right) \leq V_1 |(T_\nu)^s|_\Phi
\]
for some constant $V_1 = V_1(r, s) > 0$.

If $| (T_\nu)^s |_\Phi = \infty$ then the above clearly holds. So assume $| (T_\nu)^s |_\Phi < \infty$. Then by Remark 3.4 $(T_\nu)^s \in \mathfrak{L}(H^2)$. This implies $((T_\nu)^s)^{\frac{1}{2}} \in \mathfrak{L}(H^2)$. Moreover by definition, $((T_\nu)^s)^{\frac{1}{2}} = T_\nu$.

Thus $T_\nu \in \mathfrak{L}(H^2)$.

Fix $m \in \mathbb{N}$ such that $m \geq 2$ and let $R = mr$. Let $A_m = \{(j_1 + iv_1, \ldots, j_n + iv_n) : (j_q)_{q=1}^n \subseteq \{0, 1, 2, \ldots, m-1\}\}$. For convenience, we write $A_m = \{y_j\}_{j=1}^{m^n}$. For each $y_j \in A_m$, let $\Gamma_{y_j}$ be the subset of $\{a_j\}_{j=1}^{\infty}$ defined by $\Gamma_{y_j} = \{R y + r y_j : y = (u_1 + iv_1, \ldots, u_n + iv_n) \text{ for some } \{u_j\}_{j=1}^{n}, \{v_j\}_{j=1}^{n} \subseteq \mathbb{Z}\}$. It follows that, for any $j \geq 1$, if $\omega, \gamma \in \Gamma_{y_j}$ with $\omega \neq \gamma$ then $|\omega - \gamma|_\infty \geq R$. It is also easy to show that
\[
\{a_j\}_{j=1}^{\infty} = \bigcup_{p=1}^{m^{2n}} \Gamma_{y_p} \quad \text{and} \quad \Gamma_{y_1}, \Gamma_{y_2}, \ldots, \Gamma_{y_{m^{2n}}} \text{ are pairwise disjoint.} \quad (4.17)
\]

Now fix $p$ and write $\Gamma_{y_p} = \{\rho_j\}_{j=1}^{\infty}$ for convenience. Let $\{e_{\rho_j}\}_{j=1}^{\infty}$ be an orthonormal set in $H^2$ and $A$ be the operator defined on $H^2$ by $A = \sum_{j=1}^{\infty} k_{\rho_j} \otimes e_{\rho_j}$. Then by a calculation almost identical to (4.14), $||A|| \leq J$ for some constant $J = J(r, n) > 0$ and $A \in \mathfrak{L}(H^2)$. Let $\{v_q\}_{q=1}^{\infty} = \Gamma(0, \ldots, 0)$ and $\{w_q\}_{q=1}^{\infty} = \Gamma(0, \ldots, 0) \setminus \{(0, \ldots, 0)\}$ where $(0, \ldots, 0)$ is the origin of $\mathbb{C}^n$.

By definition of $\{\rho_j\}_{j=1}^{\infty}$, we have
\[
\{\rho_d + v_q\}_{q=1}^{\infty} = \{\rho_j\}_{j=1}^{\infty} \quad \text{for any } d \geq 1.
\]

Using the above, [9] page 188-191, and direct calculation, we can write $A^* T_\nu A$ as $A^* T_\nu A = D + E$ where
\[
D = \sum_j \langle T_\nu k_{\rho_j}, k_{\rho_j} \rangle e_{\rho_j} \otimes e_{\rho_j} \quad \text{and} \quad E = \sum_q \sum_{j=1}^{\infty} \langle T_\nu k_{\rho_j}, k_{\rho_j + w_q} \rangle e_{\rho_j + w_q} \otimes e_{\rho_j}.
\]

Then from Proposition 3.7
\[
||D||_\Phi \leq W 2^{1-s} (||A^* T_\nu A||_\Phi + ||E||_\Phi)
\]
for some $W > 0$ that is independent of $m$ and $\{\rho_j\}_{j=1}^{\infty}$. A calculation similar to the one used to verify (4.7) shows $||A^* T_\nu A||_\Phi \leq C_1 ||T_\nu||_\Phi$ where $C_1 > 0$ also does not depend on $m$ or $\{\rho_j\}_{j=1}^{\infty}$. This implies
\[
||D||_\Phi \leq 2^{1-s} W C_1 ||T_\nu||_\Phi + W 2^{1-s} ||E||_\Phi. \quad (4.18)
\]

By Proposition 3.5 $||D||_\Phi = \Phi \left( \{(\langle T_\nu k_{\rho_j}, k_{\rho_j} \rangle)^s\}_{j=1}^{\infty} \right)$. Since $T_\nu$ is bounded, $\langle T_\nu (k_z), k_z \rangle = \tilde{\nu}_1(z)$ for all $z \in \mathbb{C}^n$ [7]. Thus $||D||_\Phi = \Phi \left( \{(\tilde{\nu}_1(\rho_j))^s\}_{j=1}^{\infty} \right)$ and by Theorem 4.1.

\[
e^{-sr^2(2n)\Phi} \left( \{(\tilde{\nu}_1(\rho_j))^s\}_{j=1}^{\infty} \right) \leq ||D||_\Phi. \quad (4.19)
\]

Note that for each $q \geq 1$, $|w_q| \geq R$ by (1.2). So using calculations similar to those used to derive (1.14) shows
\[
e^{-\frac{|z-\rho_j|^2}{2}} e^{-\frac{-|z-(\rho_j + w_q)|^2}{2}} \leq e^{-\frac{R^2}{2}} e^{-\frac{|w_q|^2}{2}} e^{-\frac{|z-\rho_j|^2}{2}} e^{-\frac{|z-(\rho_j + w_q)|^2}{2}} \text{ for all } j \geq 1 \text{ and } q \geq 1. \quad (4.20)
\]
For each \(q \geq 1\), let
\[
E_q = \sum_{j=1}^{\infty} \langle T_\nu k_{\rho_j}, k_{\rho_j+w_q} \rangle e_{\rho_j+w_q} \otimes e_{\rho_j}. 
\]
Then \(E = \sum_{q=1}^{\infty} E_q\) and by Proposition \(3.7\) \(|E|^s \Phi \leq 2^{1-s} V_2 \sum_{q=1}^{\infty} ||E_q||^s \Phi\) for some \(V_2 > 0\) that only depends on \(\Phi\). So using \(4.20\) and an argument almost identical to that used to prove \(4.16\) shows
\[
||E||^s \Phi \leq 2^{1-s} V_2 e^{-s \frac{R^2}{16}} \Phi \left( \left\{ \left( \tilde{\nu}_s^{(\nu)}(\rho_j) \right)^s \right\}_{j=1}^{\infty} \right) \left( \sum_{q=1}^{\infty} e^{-s |w_q|^2} \right). 
\]
Since \(\{w_q\}_{q=1}^{\infty} \subseteq \{a_j\}_{j=1}^{\infty}, \sum_{q=1}^{\infty} e^{-s |w_q|^2} \leq \sum_{j=1}^{\infty} e^{-s |a_j|^2} \). Then by an argument similar to one given in the proof of Theorem \(4.1\) \(\sum_{j=1}^{\infty} e^{-s |a_j|^2} \leq \sum_{k=0}^{\infty} (2k+1)^{2n} e^{-s \frac{R^2}{32}}\). Hence
\[
||E||^s \Phi \leq C_2 e^{-s \frac{R^2}{16}} \Phi \left( \left\{ \left( \tilde{\nu}_s^{(\nu)}(\rho_j) \right)^s \right\}_{j=1}^{\infty} \right) 
\]
where \(C_2 = 2^{1-s} V_2 \sum_{k=0}^{\infty} (2k+1)^{2n} e^{-s \frac{R^2}{32}}\).

So by \(4.18\), \(4.19\) and \(4.21\)
\[
e^{-sr^2(2n)} \Phi \left( \left\{ \left( \tilde{\nu}_s^{(\nu)}(\rho_j) \right)^s \right\}_{j=1}^{\infty} \right) \leq WC_1 |(T_\nu)^s| \Phi + 2^{1-s} WC_2 e^{-s \frac{R^2}{16}} \Phi \left( \left\{ \left( \tilde{\nu}_s^{(\nu)}(\rho_j) \right)^s \right\}_{j=1}^{\infty} \right). 
\]
It is also true by Proposition \(3.2\) and Theorem \(4.1\) that
\[
\Phi \left( \left\{ \left( \tilde{\nu}_s^{(\nu)}(\rho_j) \right)^s \right\}_{j=1}^{\infty} \right) \leq \Phi \left( \left\{ \left( \tilde{\nu}_s^{(\nu)}(a_j) \right)^s \right\}_{j=1}^{\infty} \right) \leq C_3 \Phi \left( \left\{ \left( \tilde{\nu}_s^{(\nu)}(a_j) \right)^s \right\}_{j=1}^{\infty} \right) 
\]
where \(C_3 = \sum_{j=0}^{\infty} (2j+1)^{2n} e^{-sr^2(2n)}\). It follows that
\[
e^{-sr^2(2n)} \Phi \left( \left\{ \left( \tilde{\nu}_s^{(\nu)}(\rho_j) \right)^s \right\}_{j=1}^{\infty} \right) \leq WC_1 |(T_\nu)^s| \Phi + C_4 e^{-s \frac{R^2}{16}} \Phi \left( \left\{ \left( \tilde{\nu}_s^{(\nu)}(a_j) \right)^s \right\}_{j=1}^{\infty} \right) 
\]
where \(C_4 = 2^{1-s} C_2 C_3 W\). Since \(W\), \(C_1\) and \(C_4\) are each independent of \(m\) and \(\{\rho_j\}_{j=1}^{\infty}\) we have
\[
e^{-sr^2(2n)} \Phi \left( \left\{ \left( \tilde{\nu}_s^{(\nu)}(a_j) \right)^s \right\}_{w \in \Gamma_{y_p}} \right) \leq WC_1 |(T_\nu)^s| \Phi + C_4 e^{-s \frac{R^2}{16}} \Phi \left( \left\{ \left( \tilde{\nu}_s^{(\nu)}(a_j) \right)^s \right\}_{j=1}^{\infty} \right) 
\]
for each \(y_p \in A_m\).

By \(4.17\) and basic properties of symmetric norming functions,
\[
\Phi \left( \left\{ \left( \tilde{\nu}_s^{(\nu)}(a_j) \right)^s \right\}_{j=1}^{\infty} \right) \leq \sum_{q=1}^{m^{2n}} \Phi \left( \left\{ \left( \tilde{\nu}_s^{(\nu)}(w) \right)^s \right\}_{w \in \Gamma_{y_q}} \right). 
\]
Then \(4.22\) give us
\[
e^{-sr^2(2n)} \Phi \left( \left\{ \left( \tilde{\nu}_s^{(\nu)}(a_j) \right)^s \right\}_{j=1}^{\infty} \right) \leq e^{-sr^2(2n)} \sum_{q=1}^{m^{2n}} \Phi \left( \left\{ \left( \tilde{\nu}_s^{(\nu)}(w) \right)^s \right\}_{w \in \Gamma_{y_q}} \right) 
\]
\[
\leq m^{2n} WC_1 |(T_\nu)^s| \Phi + m^{2n} C_4 e^{-s \frac{R^2}{16}} \Phi \left( \left\{ \left( \tilde{\nu}_s^{(\nu)}(a_j) \right)^s \right\}_{j=1}^{\infty} \right). 
\]
Recall that $R = mr$ where $m \geq 2$. Then choosing $m$ large enough so that $C_Am^{2n}e^{-\frac{sr^2(2n)}{2}} \leq \frac{e^{-sr^2(2n)}}{2}$, we get from the above

\[ e^{-sr^2(2n)}\Phi \left( \{ (\tilde{\nu}_r(a_j))^s \}_{j=1}^{\infty} \right) \leq m^{2n}W_C1 |(T_\nu)^s|_\Phi + \frac{e^{-sr^2(2n)}}{2} \Phi \left( \{ (\tilde{\nu}_r(a_j))^s \}_{j=1}^{\infty} \right) . \]  

(4.23)

Thus for any positive Borel measure $\nu$ such that $\Phi \left( \{ (\tilde{\nu}_r(a_j))^s \}_{j=1}^{\infty} \right) < \infty$, (4.23) gives us

\[ \Phi \left( \{ (\tilde{\nu}_r(a_j))^s \}_{j=1}^{\infty} \right) \leq m^{2n}2e^{-sr^2(2n)}W_C1 |(T_\nu)^s|_\Phi. \]  

(4.24)

However a straightforward approximation argument shows that (4.24) holds for any positive Borel measure $\nu$ satisfying $| (T_\nu)^s |_\Phi < \infty$. Thus with $V_1 = m^{2n}2e^{-sr^2(2n)}W_C1$,

\[ \Phi \left( \{ (\tilde{\nu}_r(a_j))^s \}_{j=1}^{\infty} \right) \leq V_1 | (T_\nu)^s |_\Phi. \]

5. Important Corollaries

By Theorem 4.1, Subsection 4.2 and Subsection 4.3, we actually have the following corollary:

**Corollary 5.1.** Let $0 < s \leq 1$, $\alpha > 0$, $r > 0$ and $\{ a_j \}_{j=1}^{\infty}$ be the $r$-lattice of $\mathbb{C}^n$. Let $\nu$ be a positive Borel measure on $\mathbb{C}^n$ and $T_\nu$ be the corresponding Toeplitz operator. Then there exists positive constants $B_1 = B_1(r,s,n)$, $B_2 = B_2(r,\alpha,s,n)$, $B_3 = B_3(r,\alpha,s,n)$ and $B_4 = B_4(r,s,n)$ so that for any symmetric norming function $\Phi$,

\[ | (T_\nu)^s |_\Phi \leq B_1 \Phi \left( \{ (\tilde{\nu}_r(a_j))^s \}_{j=1}^{\infty} \right) \leq B_1B_2 \Phi \left( \{ (\tilde{\nu}_r(a_j))^s \}_{j=1}^{\infty} \right) \leq B_1B_2B_3 \Phi \left( \{ (\tilde{\nu}_r(a_j))^s \}_{j=1}^{\infty} \right) \leq B_1B_2B_3B_4 | (T_\nu)^s |_\Phi. \]

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