The critical energy of the Ising double chain - the explanation for melting

Klaus Mika

Institut für Festkörperforschung, Forschungszentrum Jülich, 52425 Jülich, Germany

Abstract

The Ising double chain has been overlooked for more than 80 years, since one was mostly interested in phase transitions. We consider two types of degeneracy, spin occupation degeneracy and "bond" occupation degeneracy. We show that bond occupation on "open nets" is of particular importance for the understanding of melting. The number of open nets on a double chain is determined analytically. If only one bond site of a net is occupied, this already produces a definite number of bonds on sites, bonds that do not belong to the net. All nets on a given chain of length \( n \) then yield a total number \( i_{\text{tot}}^{(n)} \) of bonds, allowing to calculate a characteristic energy \( u_{\text{ch}}^{(n)} \), an approximation of the critical energy \( u_c \) of the infinite double chain. We finally present an explanation for the critical energy \( u_{\text{KG}}^c \) of the kagomé lattice in terms of the critical energies \( u_{\text{TR}}^c \) and \( u_{\text{HC}}^c \) of the triangular and honeycomb lattices.

Keywords: Ising double chain, open nets, critical energy, melting

1. Introduction

J. W. Gibbs made a fundamental statement at the beginning of his work on heterogeneous equilibria around 1875, which he however never applied in his later work [1]: *The comprehension of the laws which govern any material system is greatly facilitated by considering the entropy and energy of the system in the various states of which it is capable.* It is possible that the reason for this statement were the Carnot cycles, which have always been treated in variables leading to four hyperbolas instead of a rectangle. Also, at that time, microcanonical description was not yet known.

In the present work, we will consider the Ising double chain. A double chain of length \( n \) has the form shown in Fig. 1a. Each lattice point is characterized (Fig. 1b) by spin values +1 and -1.

The two-dimensional Ising model was solved by L. Onsager [2] and B. Kaufman [3]. They used a canonical description: From the partition function, they obtained the entropy \( S \) as a function of temperature \( T \) and the critical temperature \( T_c \). The special case of the linear chain, solved by E. Ising [4], did not lead to a critical temperature. The double chain has rarely been considered, perhaps because it has no critical temperature and therefore no phase transition, and as such is believed uninteresting. We will show that it actually possesses some interesting properties.

To describe the Ising double chain, we introduce the concept of "bonds", that connect neighbouring lattice points with opposite spins (Fig. 1c). For \( n = 1 \), we have already 16 different spin arrangements, but 8 different bond arrangements (Fig. 2). The double chain does not allow the free occupation of its bond sites with bonds. This has to be taken into account when placing bonds on the chain; the bond arrangement has to obey the constraint that each cell must have an even number of bonds.

We will not use a canonical description, i.e. we ignore the temperature, nor do we follow a complete microcanonical approach; instead, we limit ourselves to two special types of microcanonical
Fig. 1: a) Ising double chain of length \( n = 7 \) with \( N = 16 \) spins (i.e. lattice points) and \( L = 22 \) bond sites (i.e. cell sides). b) Example of a possible arrangement of spins on the double chain (dots and crosses correspond to spin values of +1 and -1 respectively). c) Bonds on the double chain shown in Fig. 1b.

Fig. 2: The 16 and 8 possible spin and bond arrangements respectively on a double chain of length \( n = 1 \).

description: spin occupation degeneracy, and in particular bond occupation degeneracy. While spin occupation always leads to long-range order (mean-field theory), we shall see that bond occupation always produces disorder. By applying bond occupation to the double chain, we will obtain the entropy \( S \) as a function of the energy \( u \) and will gain access to the critical energy \( u_c \). At \( u = u_c \), the bond entropy \( S \) is 0 and grows to \( \ln 2 \) as \( u \to 0 \).

We will evaluate a special type of bond arrangement in order to obtain directly the characteristic energy \( u_{ch}^{(n)} \) of a double chain of finite length \( n \). With increasing \( n \), \( u_{ch}^{(n)} \) converges rapidly to the critical energy \( u_c \) of the infinite chain.

2. Two different types of occupation

We use the following normalization to denote the internal energy \( u \) of the Ising systems: \( u = 1 \) for the ferromagnetic ground state and \( u = 0 \) for the state of total disorder. Therefore the entropy \( S(u) \) per particle satisfies \( S(1) = 0 \) and has its maximum value \( S(0) = \ln 2 \) at \( u = 0 \). Each of the \( N \) spins
can take the values +1 and -1. We will further differentiate the expressions for the entropies by an upper index, $S^{(0)}$ and $S^{(1)}$, to designate the specific form taken near $u = 0$ and $u = 1$ respectively. The internal energy $u$ is given by the number $i$ of bonds of neighboring spins with opposite sign

$$u = 1 - 2i/L,$$

where $L$ is the total number of bond sites. Hence $u = 0$ for $i = L/2$.

2.1. Spin occupation

If $k$ spin sites are occupied by spins (-1), the degeneracy is $g(k) = \binom{N}{k}$, and the spin entropy

$$S_s(k) = \frac{1}{N} \ln g(k).$$

(2.2)

For large $N$, we have

$$S_s(k) = - \frac{k}{N} \ln \frac{k}{N} - \left(1 - \frac{k}{N}\right) \ln \left(1 - \frac{k}{N}\right).$$

(2.3)

Let $p = k/N$ be the probability to find a spin (-1). Then the probability to find a bond site occupied is $w = 2pq$, where $q = 1 - p$. The total number of bonds is $i = Lw$, and with $u = 1 - 2i/L = 1 - 4pq$ we obtain $p = (1 - \sqrt{u})/2$. The entropy $S_s(u)$ therefore is

$$S_s(u) = -\frac{1}{2}(1 - \sqrt{u}) \ln \frac{1}{2}(1 - \sqrt{u}) - \frac{1}{2}(1 + \sqrt{u}) \ln \frac{1}{2}(1 + \sqrt{u}),$$

(2.4)

i.e. the mean-field entropy, valid for all Ising systems (Fig. 3).

![Fig. 3: Lower curve: entropy $S_s(u)$ of the mean-field theory (Eq. 2.4). At $u = 0$ we have a none-zero slope of $S_s(u)$ leading to a cut-off tangent. Upper curve: entropy of the linear chain (Eq. 2.7). Central curve: entropy of the double chain (Eq. 3.3).](image)

Near $u = 0$ we have

$$S_s^{(0)}(u) = \ln 2 - \frac{1}{2}u,$$

(2.5)

i.e. a finite slope at $u = 0$, the mean-field $T_c$, which is a cut-off tangent and has no physical meaning, except for the infinite-dimensional Ising system, where $S_s(u)$ is the exact entropy. At $u = 1 + \varepsilon$ ($\varepsilon \ll 1$), we have

$$S_s^{(1)}(u) = -\frac{\varepsilon}{4} \ln \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + O(\varepsilon^2 \ln \varepsilon).$$

(2.6)

This result holds for all Ising systems with two and higher dimensions.
2.2. Bond occupation

The linear chain allows the free occupation of bond sites with bonds, i.e. there are no constraints \(^4\). Let \(i = k\) bonds be occupied and \(u = 1 - 2k/L\), then \(S_i(u) = S(u)\) and

\[
S(u) = -\frac{1}{2} (1 - u) \ln \frac{1}{2} (1 - u) - \frac{1}{2} (1 + u) \ln \frac{1}{2} (1 + u).
\]  

(2.7)

Near \(u = 0\) we have

\[
S^{(0)}(u) = \ln 2 - \frac{1}{2} u^2.
\]  

(2.8)

As bond sites on the linear chain can be occupied freely, no additional energy can occur and the entropy is the maximum possible entropy of all Ising systems. The entropy of spin occupation is the minimum entropy, because it uses the maximum energy per occupation with one spin. (Here we consider only the ferromagnetic branch.)

3. The Ising double chain

3.1. Thermodynamic results

The double chain of length \(n\) has \(N = 2n + 2\) spin sites and \(L = 3n + 1\) bond sites (see Fig. 1a for \(n = 7\)). The internal energy \(u\) and the entropy \(S_{dc}(u)\) can, for large \(n\), be written in terms of the parameter \(x\), where \(x\) is related to the temperature \(T\) by

\[
\lambda(x) = \frac{1}{2} \left[ 1 + x^2 + \sqrt{(1 - x^2)^2 + 4x^4} \right],
\]  

(3.1)

\[
u(x) = x \left\{ 1 + \frac{1}{3\lambda(x)} (1 - x^2) \left[ 1 + \frac{5x^2 - 1}{\sqrt{(1 - x^2)^2 + 4x^4}} \right] \right\},
\]  

(3.2)

\[
S_{dc}(x) = -\frac{3}{4} u(x) \ln \frac{1 + x}{1 - x} + \ln 2 - \frac{3}{4} \ln (1 - x^2) + \frac{1}{2} \ln \lambda(x),
\]  

(3.3)

The mean-field (Eq. 2.4), the linear chain (Eq. 2.7) and the double chain (Eq. 3.3) entropies are displayed for comparison in Fig. 3.

Near \(x = 0\) (the state of total disorder) we have \(\lambda(x) = 1 + x^4, u(x) = x\) and \(S_{dc}^{(0)}(x) = \ln 2 - 3/4 x^2\), or

\[
S_{dc}^{(0)}(u) = \ln 2 - \frac{1}{2} \frac{L}{N} u^2.
\]  

(3.4)

This result holds for all Ising systems \(^5\).

At \(x = 1 - \varepsilon\) (the ferromagnetic ground state) we have \(\lambda(x) = 2 - 3\varepsilon + 2\varepsilon^2, u(x) = 1 - 1/3 \varepsilon^2\) and \(S_{dc}^{(1)}(x) = 1/4 \varepsilon^2 (\ln 2 + 1/2) - 1/4 \varepsilon^2 \ln \varepsilon\), or

\[
S_{dc}^{(1)}(u) = -\frac{3}{8} (1 - u) \ln (1 - u) + O(1 - u).
\]  

(3.5)

These results have been obtained previously in \(^5\), but we have taken the opportunity to correct some errors in the expressions for \(\lambda, u\) and \(S_{dc}(x)\) (for \(x = 1 - \varepsilon\)). The expression given for \(S_{dc}^{(1)}(u)\) (here Eq. 3.5) was however error free.

Remarkably, starting from \(u = 1\), Eq. (2.6) gives a lower entropy increase than does Eq. (3.5). The reason is subtle and the explanation goes as follows. When leaving the ground state, each Ising system
evolves towards states that maximize the entropy. In other words, the new states are such that the energy increase is minimized. In the ground state of the double chain, there are no (energy-)bonds on the bond sites. Considering bond occupation, the smallest energy quantity, that can be added to the chain is achieved by occupying one bond site, which - in accordance with the constraint introduced in Section 1 - requires a second suitable bond site to be occupied as well (see Fig. 4).

Fig. 4: Bond occupation leaving the ground state requires at least two bond sites occupied

Fig. 5: Spin occupation leaving the ground state requires at least three bond sites occupied

In contrast, if we consider spin occupation (see Fig. 5), leaving the ground state would yield a minimum energy increase corresponding to the addition of three bonds. Bond occupation is therefore the correct approach to the physics of the double chain near the ground state, because it is the only way to minimize the energy increase and to minimize the entropy increase. This is the profound reason why we give so much importance to the bond occupation approach, rather than the spin occupation scenario, for the treatment of the behavior of the double chain in the vicinity of the ground state \( u = 1 \). Notice however that this argumentation holds only for the double chain and not for the full two-dimensional lattice as treated by Onsager.

Consequently, let us now treat the spin distribution on the double chain, applying bond occupation near the ferromagnetic ground state \( u = 1 \). For this purpose, we must consider all the different possibilities to distribute the double bond over the double chain. For instance, in Fig. 4 where the double chain is of length \( n = 5 \), there are five configurations. They each contain an equal number of \(+1\) and \(-1\) spins: the spin system is disordered near \( u = 1 \). This argument holds for any value \( n \) of the chain length and one can conclude that even for the smallest excitation above the ground state in the double chain, the spin system is disordered.

3.2. Open nets

An open net on a double chain connects all lattice points and contains no closed loops (see Fig. 6). It has length \( 2n + 1 \), the number of bond sites on the net. It should be stressed that the open net set is a set of bond sites and not of bonds, in other words, the open net can carry different numbers of bonds, varying from zero to a maximum number. Fig. 6 shows four examples of open nets for a double chain with \( n = 7 \) with 15 bond sites each.

Fig. 6: Four examples of possible open nets on a double chain of length \( n = 7 \).

We shall have to consider all possible open nets on a given double chain of length \( n \). Their number \( l(n) \) can be calculated analytically. Let \( a_n \) and \( f_n \) be the number of open nets with and without vertical net-bond site at length \( n \). The sequential construction is schematically represented in Fig. 7.

We have \( a_0 = 1, f_0 = 0 \) and \( a_1 = 3, f_1 = 1 \) (Fig. 8), chain and net for \( n = 0 \) being represented by a single vertical bond site.
Eq. 7: Extension of open nets \( n \) (dashed lines represent right (last) cell of these nets) to open nets \( n + 1 \):

upper left: Cells at right end of nets \( n \) with vertical bond site at \( n \).
upper right: Four possible extensions (to \( n + 1 \)) of nets \( n \) with vertical bond site at \( n \). Note: The fourth extension is obtained shifting the vertical bond in the third extension to the right end.
lower left: Cell on right end of nets \( n \) without vertical bond site at \( n \).
lower right: Three possible extensions (to \( n + 1 \)) of nets \( n \) without vertical bond site at \( n \). Note: These extensions are identical to the first to third extension of nets \( n \) with vertical bond site at \( n \).

Fig. 8: The four open nets on the double chain of length \( n = 1 \).

It is apparent from Fig. 7 that for \( n > 1 \) one has an extension rule encapsulated by

\[
\begin{align*}
o_{n+1} &= 3o_n + 2f_n, \\
f_{n+1} &= o_n + f_n.
\end{align*}
\]  
(3.6)  
(3.7)

The eigenvalues of this system are \( \lambda_\pm = 2 \pm \sqrt{3} \), and \( l(n) = A\lambda^n_+ + B\lambda^n_- \) with \( A, B \) fixed by \( l(0) = 1 \) and \( l(1) = 4 \). It follows that

\[
l(n) = \frac{1}{2\sqrt{3}}[(2 + \sqrt{3})^{n+1} - (2 - \sqrt{3})^{n+1}].
\]  
(3.8)

Furthermore, the \( l(n) \) obey the recursion relation: \( l(n+1) = 4l(n) - l(n-1) \), their exact values are listed in Table 1 for \( n = 0, \ldots, 10 \). We note, that the ratio \( l(n+1)/l(n) \) converges rapidly to \( \lambda_+ = 2 + \sqrt{3} \approx 3.732050808 \ldots \), meaning that \( l(n) \) grows as \( \lambda^n_+ \). For \( n \geq 7 \), the ratio agrees with its asymptotic value to seven decimal places or better.

It was shown in [5] that, near \( u = 0 \) and for large \( n \), each open net on the Ising systems contributes the same amount to \( S_b(u) \):

\[
S_b^{(0)}(u) = \ln 2 - \frac{1}{2} \left( \frac{L}{N} \right)^2 u^2,
\]  
(3.9)

"thermodynamic" entropy of Ising systems given in Eq. (3.4).

We have dealt with this discrepancy in [5] and for the sake of completeness we briefly describe the procedure. One calculates the mean entropy \( S_b(u) \) from the entropy of all open-net configurations (all possible nets with all possible combinations of bond arrangements on the nets’ bond sites), introducing a set of weighting parameters for each of the net entropies. These parameters are determined in turn by demanding that \( S_b(u) \) be maximized near \( u = 0 \). This yields the entropy \( S_b(u) \) with the correct behaviour (Eq. 3.4) near \( u = 0 \), and the critical energy \( u_c = \sqrt{2/3} \) for the double chain from \( S_b(u_c) = 0 \). Parenthetically, in [5] we have analogously treated the periodic square chains with 4,
6, 8, and 10 rows, obtaining for each of these systems the same entropy function $S_b(u)$, and from $S_b(u_c) = 0$ - the same critical energy $u_c = \sqrt{1/2}$, i.e. the Onsager-Kaufman solution for the infinite two-dimensional Ising square system.

For the problem at hand, we again consider all possible open nets and all of their bond sites, but we limit ourselves to occupation of only one net-bond site at the time, leaving $2n$ net-bond sites empty. This means that, for large $n$, $S_b(u)$ is approximately 0, thus avoiding the necessity of calculating explicitly $S_b(u)$. In compliance with the constraint that each cell of the chain has to have an even number of bonds, we can obtain directly the characteristic energy $u_{\text{ch}}^{(n)}$ of a double chain of length $n$ without calculating the entropy. For large $n$, the characteristic energies $\{u_{\text{ch}}^{(n)}\}$ converge to the critical energy $u_c = \sqrt{2/3}$ of the infinite double chain, the value found in [5].

We note, that for the linear chain and for large $n$ the ratio $L/N = 1$ in Eqs. (3.4) and (3.9), and that therefore the discrepancy discussed above does not exist.

3.3. Occupation of open nets by one bond

If there is no bond on a net, the not-net-bond sites are also empty. If, however, there is one bond on the net, there has also to be at least one more bond on not-net-bond sites in order to comply with the constraint discussed in Section 1 (for $n = 1$ see Fig. 9).

Fig. 9: Occupation of the four open nets $n = 1$ by one bond each. left: the four nets. right: the three possibilities to place one bond on each net (thick lines). The dashed lines represent the chain-bond sites not lying on the net, which have to be occupied as well, in order to comply with the constraint of an even number of bonds for each cell.
To obtain $i^{(n)}_{\text{tot}}$, we shall need, for fixed $n$, all nets, and we have to take into account occupation of each net-bond site of each of these nets with one bond at the same time (see Fig. 9 for $n = 1$). Thus, we get a total number of bonds $i^{(n)}_{\text{tot}}$ on net- and on not-net-bond sites. For large $n$, bonds on net-bond sites become more and more negligible, and the remaining bonds on the not-net-bond sites will finally determine the critical energy.

For chains of arbitrary length $n$, one analyzes the geometry of the open nets and considers the position of the bond site occupied by the one bond on the net. For fixed $n$, one finds the number of net types and the number of bonds for each type; this makes up the contribution of each net type to $i^{(n)}_{\text{tot}}$. For the sake of brevity, we only present the details of the calculation for the first few ($n = 0, 1, 2, 3$) values of $i^{(n)}_{\text{tot}}$. The procedure is straightforward, but becomes increasingly tedious as the length $n$ of the double chain continues to grow.

For $n = 0$, we have $i^{(0)}_{\text{tot}} = 1$. For $n = 1$, we have (see Fig. 9) $i^{(1)}_{\text{tot}} = 24$. For $n = 2$, the 15 nets can be divided into two types: 6 nets of the first type and 9 nets of the second type (Fig. 10). Each net of the first type contributes 13 bonds and each net of the second type contributes 11 bonds to $i^{(2)}_{\text{tot}}$ (Fig. 11). The result is $i^{(2)}_{\text{tot}} = 6 \times 13 + 9 \times 11 = 177$ bonds.

Similarly, for $n = 3$, we find four net types (see Fig. 12) and $i^{(3)}_{\text{tot}} = 1000$. The numbers $i^{(n)}_{\text{tot}}$ obtained for $n = 4, \ldots, 10$ are given in Table 1.

![Fig. 10: Double chain of length $n = 2$: 6 nets of the first type and 9 nets of the second type.](image)

![Fig. 11: Contribution of one single net ($n = 2$) to $i^{(2)}_{\text{tot}}$ (two examples - see Fig. 10). upper line: contribution of one of the nets of the first type, counting from left to right, $2 + 3 + 3 + 3 + 2 = 13$ bonds. lower line: contribution of one of the nets of the second type, counting from left to right, $2 + 3 + 2 + 2 + 2 = 11$ bonds.](image)

![Fig. 12: The four types of nets for a double chain of length $n = 3$. From left to right: 6 nets of the first type contribute 22 bonds each to $i^{(3)}_{\text{tot}}$, 2 nets of the second type contribute 20 bonds, 30 nets of the third type contribute 18 bonds, and 18 nets of the fourth type contribute 16 bonds. Therefore, $i^{(3)}_{\text{tot}} = 6 \times 22 + 2 \times 20 + 30 \times 18 + 18 \times 16 = 1000$ bonds.](image)
3.4. The critical energy

As a first step, we introduce the mean number $i(n)$ of bonds (bonds on not-net-bond sites plus those occupying the one net-bond site) per net-bond site. With the number $l(n)$ of open nets and with the length $2n + 1$ of each net we have

$$i(n) = \frac{l(n)}{2n + 1}, \quad (3.10)$$

These numbers are listed in Table 1 for the first 10 values of $n$.

With the number $L = 3n + 1$ of bond sites on the chain and $u_{ch}^{(n)} = 1 - 2i(n)/L$ (Eq. 2.1), we get characteristic energies $u_{ch}^{(n)}$ shown in Table 1 and in Fig. 13. For sufficiently large $n$, $u_{ch}^{(n)}$ converges nicely to the critical energy $u_c = \sqrt{2/3}$ of the infinite double chain.

Fig. 13: The characteristic energies $u_{ch}^{(n)}$ for $n = 0, 1, \ldots, 10$ converging to the critical energy $u_c = \sqrt{2/3} \approx 0.816496 \ldots$ of the infinite double chain.

For a precise analysis of the convergence as a function of $n$, it could be useful to include values $i_{tot}^{(n)}$ for $n > 10$. However, given the rapid convergence displayed in Table 1 (at $n = 10$, the characteristic energy is within a fraction of a per cent to its asymptotic value), the cumbersome task of calculating exactly $i_{tot}^{(n)}$ may not be worth the effort. A few scaling rules are easily obtained from our analysis: $l(n) \propto \lambda_n^1$, $i_{tot}^{(n)} \propto n^2 \lambda_n^1$, and $i(n) \propto n$.

4. Conclusions

The basic idea of this work is to use energy and entropy to describe the Ising systems, applying a general suggestion of Gibbs dating back to 1875. For the Ising double chain of length $n$, we obtain
analytically the value of the characteristic energy \( \{ i_{ch}^{(n)} \} \) which serves as a good approximation to the critical energy \( u_c \) with increasing \( n \). The special status of the critical energies is exemplified in Appendix A. Our approach uses a bond occupation framework which not only describes the degeneracy of the linear chain, but also of all other Ising systems. The origin of the physical mechanism of melting arises from the presence of a critical energy \( u_c \) away from the ground state energy \( u = 1 \). The essential point is, that already one bond on the bond sites of an open net produces bonds on not-net-bond sites, leading to a deeper understanding of melting: The critical energy is the minimum averaged amount of energy, that has to be added to the chain, if only one of its – randomly chosen – bond sites is to be occupied.

On the basis of our special type of microcanonical description, further research concerning phase transition and critical energy should be possible, for instance the investigation of other (e.g. 3-dimensional) Ising arrangements like the periodic square chain with four rows ("double ladder"), and possibly the description of the influence of the geometric parameters of these arrangements on their physical properties, such as the critical energy.

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| \( n \) | \( l(n) \) | \( i_{tot}^{(n)} \) | \( i(n) \) | \( u_{ch}^{(n)} \) |
|---|---|---|---|---|
| 0 | 1 | 1 | -1 | 1 |
| 1 | 4 | 24 | 2 | 0 |
| 2 | 15 | 177 | 2.36 | 0.3257 |
| 3 | 56 | 1 000 | 2.5510 | 0.4898 |
| 4 | 209 | 5 013 | 2.6651 | 0.5900 |
| 5 | 780 | 23 504 | 2.7304 | 0.6577 |
| 6 | 2 911 | 105 629 | 2.7912 | 0.7062 |
| 7 | 10 864 | 461 072 | 2.8294 | 0.7428 |
| 8 | 40 545 | 1 970 281 | 2.8585 | 0.7713 |
| 9 | 151 316 | 8 281 600 | 2.8806 | 0.7942 |
| 10 | 564 719 | 34 399 495 | 2.9007 | 0.8129 |

Table 1: The exact results, \( l(n) \) (Eq 3.8) and \( i_{tot}^{(n)} \); the derived results, \( i(n) \) (Eq 3.10) and \( u_{ch}^{(n)} \) (Eq 2.1) for \( n = 0, 1, \ldots, 10 \).
Appendix A. The critical energy of the kagomé net

The particular role of the critical energy can be appreciated with a look at the known two-dimensional Ising lattices. The four classical Ising systems (Onsager (ON) [2, 3], triangular (TR) [6, 7, 8, 9, 10], honeycomb (HC) [10, 11], kagomé (KG) [12]) have the following critical energies [13], normalized in [5]:

\[
\begin{align*}
    u_{\text{ON}}^c &= \frac{1}{\sqrt{2}}, \\
    u_{\text{TR}}^c &= \frac{2}{3}, \\
    u_{\text{HC}}^c &= \frac{4}{3\sqrt{3}}, \\
    u_{\text{KG}}^c &= \frac{1}{6} + \frac{1}{\sqrt{3}}.
\end{align*}
\]  
(A.1)

We will consider only the kagomé lattice.

Fig. 14: The unit cell of the kagomé lattice. It can be divided into eight triangles, two belonging to the triangular lattice and six (not shown) to the honeycomb lattice.

The unit cell of the kagomé net has six bond sites (Fig. 14). All six bond sites cannot be occupied and, since \( i \) must be even, only four bond sites per unit cell can at most be occupied, leading to an antiferromagnetic ground state at \( u = -\frac{1}{3} \). The unit cell can be divided into eight triangles, two belonging to the triangular and six (not shown) to the hexagonal net. The critical energy therefore satisfies:

\[
    u_{\text{KG}}^c = \frac{2}{8} u_{\text{TR}}^c + \frac{6}{8} u_{\text{HC}}^c.
\]  
(A.2)

Corresponding results for \( u_{\text{TR}}^c \) and \( u_{\text{HC}}^c \) are still lacking.
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