No Signalling and
Probabilistic Quantum Cloning

Lucien Hardy\textsuperscript{1} and David D. Song\textsuperscript{2}

Department of Physics, Clarendon Laboratory
Parks Road, University of Oxford
Oxford OX1 3PU, U.K.

\textbf{Abstract}
We show that the condition of no faster-than-light signalling restricts the number of quantum states that can be cloned in a given Hilbert space. This condition leads to the constraints on a probabilistic quantum cloning machine (PQCM) recently found by Duan and Guo.

\textsuperscript{1}E-mail: l.hardy1@physics.ox.ac.uk
\textsuperscript{2}E-mail: d.song1@physics.ox.ac.uk
J.S. Bell showed, in his famous paper [1], the non-local nature of quantum mechanics. Nevertheless, this non-locality cannot be used to communicate faster-than-light due to indistinguishability between two mixtures. On the other hand, if quantum states can be cloned perfectly, special relativity would inevitably be violated. After Wootters and Zurek had shown [2] that nonorthogonal quantum states cannot be cloned, various authors [3, 4, 5] showed that the approximate cloning is possible, i.e. with fidelity less than one. Gisin demonstrated [6] that the imperfect cloning still does not violate special relativity by showing the bounds on the fidelity of QCM for the no-signalling constraint and the fidelity of the approximate QCM are equal. Recently, Duan and Guo (DG) showed [7, 8] that linearly independent quantum states can be cloned perfectly sometimes, a probabilistic quantum cloning. In this paper, we show that the conditions on no faster-than-light signalling lead to the constraints of DG on probabilistic quantum cloning machines (PQCM).

Consider arbitrary, but not necessarily orthogonal, states for Bob, $|B_n\rangle$ where $n = 1$ to $N$, in a Hilbert space of dimension $N$. Bob’s states can be realised by Alice’s measurements on the following entangled states.

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} |n\rangle_A |B_n\rangle$$

where $|n\rangle_A$ are orthogonal states. We will consider two possible measurements by Alice. She can either measure onto the $|n\rangle_A$ basis (call this measurement $A_1$) or onto some other basis (call this measurement $A_2$). When she measures $A_1$ the states $|B_1\rangle, \cdots, |B_N\rangle$ are prepared for Bob and when she measures $A_2$ she prepares a different set of states which we will call $|B_{N+1}\rangle, \cdots, |B_{2N}\rangle$ for Bob. It is clear from (1) that the vectors $|B_{N+1}\rangle, \cdots, |B_{2N}\rangle$ are linearly dependent on the vectors $|B_1\rangle, \cdots, |B_N\rangle$. Furthermore, it is also clear that one of the second set of states, $|B_{N+1}\rangle$ say, can be any linearly dependent state. Let us suppose Bob has a PQCM such that

$$\text{Input } |B_a\rangle \xrightarrow{\text{prob } \neq 0} \text{Output } |B_{\alpha}\rangle^\otimes \mu$$

(2)
where $\mu$ is large and we assume this PQCM works for some subset of $\alpha = 1, \cdots, 2N$. Furthermore, the PQCM is such that when it succeeds in producing $\mu$ copies Bob knows that it has succeeded. He can disregard those cases where it does not succeed. Note that if Bob has a PQCM which only makes a single clone with some non-zero probability then repeated application of this machine will lead to one which makes many copies though with a much smaller probability.

Assume that Alice and Bob share a large number of pairs of entangled pairs $|\psi\rangle_{AB}$. Let us imagine that Alice wants to communicate a 0 or a 1 to Bob. If she wants to communicate a 0 (1) she measures $A_1$ ($A_2$) on all her particles. Bob can now use his PQCM to attempt to infer which measurement Alice made. Since sufficiently many entangled pairs are shared in each group, Bob’s PQCM is guaranteed to produce $\mu$ (which is a large number) copies for at least one of the pairs. Consider one such pair where Bob has been successful in producing $\mu$ copies of particle $B$. Bob can attempt to establish what the state of these copies is by making appropriate measurements. Suppose Bob’s PQCM can in fact only clone $N + 1$ states of $|B_n\rangle$, say $|B_1\rangle, \cdots, |B_{N+1}\rangle$. For inputs $|B_k\rangle$ where $k = N + 2$ to $2N$ (which the PQCM cannot clone) the general output state can be written

$$\sum_{l=1}^{N+1} c^k_l |B_l\rangle^{\otimes \mu} + d^k \langle \phi^k |$$

where $l = 1$ to $N + 1$ and where the state $|\phi^k\rangle$ has zero overlap with $|B_l\rangle^{\otimes \mu}$ for $l = 1$ to $N + 1$. To attempt to establish the state of the clones Bob divides them up into $N + 1$ groups and makes projective measurements on each clone onto the state $|B_l\rangle$ for the $l$th group. If all the clones in the $l$th group successfully project onto $|B_l\rangle$ then Bob can conclude that output state of the $\mu$ clones behaved as state $|B_l\rangle^{\otimes \mu}$. In such a case he will place a $\sqrt\cdot$ in the $|B_l\rangle^{\otimes \mu}$ column. If none of the groups have this property then he can place a $\sqrt\cdot$ in the $|\phi\rangle$ column. It is very unlikely that more than one group will have this property (but if this does happen we can put a $\sqrt\cdot$ in the $|\phi\rangle$ column instead). The idea that the state may behave as $|B_l\rangle^{\otimes \mu}$ is motivated by the fact that we can write
any general output state down as in (3). However, we should be a little careful with this equation. It is possible that terms other than $|B_l\rangle^{\otimes \mu}$ could lead to a $\sqrt{\cdot}$ in column $l$. Equation (3) should only be regarded as motivating the idea that the state may behave as one of $|B_l\rangle^{\otimes \mu}$.

Let us consider the most general case as shown in the table 1.

| Input | Output |
|-------|--------|
| $|B_1\rangle$ | $\sqrt{\cdot}$ | $|B_1\rangle^{\otimes \mu}$ | $|B_{N-1}\rangle^{\otimes \mu}$ | $|B_N\rangle^{\otimes \mu}$ | $|B_{N+1}\rangle^{\otimes \mu}$ | $|\phi\rangle$ |
| $\vdots$ | $\cdot \cdot \cdot$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $|B_{N-1}\rangle$ | $\sqrt{\cdot}$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | |
| $|B_N\rangle$ | | $\sqrt{\cdot}$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $|B_{N+1}\rangle$ | $\sqrt{\cdot}$ | $\cdot \cdot \cdot$ | $\sqrt{\cdot}$ | $\sqrt{\cdot}$ | $\sqrt{\cdot}$ | $\sqrt{\cdot}$ |
| $|B_{N+2}\rangle$ | $\sqrt{\cdot}$ | $\cdot \cdot \cdot$ | $\sqrt{\cdot}$ | $\sqrt{\cdot}$ | $\sqrt{\cdot}$ | $\sqrt{\cdot}$ |
| $\vdots$ | $\vdots$ | $\cdot \cdot \cdot$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $|B_{2N}\rangle$ | $\sqrt{\cdot}$ | $\cdot \cdot \cdot$ | $\sqrt{\cdot}$ | $\sqrt{\cdot}$ | $\sqrt{\cdot}$ | $\sqrt{\cdot}$ |

Table 1

This table shows all the possible places $\sqrt{\cdot}$’s could be placed. For the inputs which can be cloned there can only be one $\sqrt{\cdot}$ in the corresponding row. For the other cases there could be a non-zero probability for a $\sqrt{\cdot}$ in any of the columns including the $|\phi\rangle$ column.

Now let us consider how Alice may try to communicate faster-than-light to Bob. Take a case in which Bob’s cloning machine tells him that he has successfully cloned $\mu$ copies of particle $B$ (of course this will only actually be true if the input was one of $|B_1\rangle$ to $|B_{N+1}\rangle$).

Define $P(n|A_i)$ as the probability that Bob places a $\sqrt{\cdot}$ in the $|B_l\rangle^{\otimes \mu}$ column given that Alice measured $A_i$. Bob can proceed in the following way. If he places a $\sqrt{\cdot}$ in a column corresponding to $l = 1$ to $N$ he guesses that Alice sent a 0 and if he places a $\sqrt{\cdot}$ in column $N + 1$ he guesses that Alice sent a 1. The conditional probabilities for his inferring a 0 or a 1 can be read off from the table. They are

$$P_0(A_1) \equiv \sum_{n=1}^{N} P(n|A_1) = 1, \quad P_1(A_1) \equiv P(N + 1|A_1) = 0 \quad (4)$$
\[ P_0(A_2) \equiv \sum_{n=1}^{N} P(n|A_2) \leq 1 - P(N+1|A_2), \quad P_1(A_2) \equiv P(N+1|A_2) \neq 0 \]  

(5)

Although Bob will sometimes guess wrongly, if this process is repeated many times Bob can be sure of guessing correctly.

We know that faster-than-light signalling is not possible and hence it cannot be possible to clone more than \( N \) states. Note further, that we can also see from this argument that the states which can be cloned must all be linearly independent. To see this imagine that the first \( N - 1 \) of the states \( |B_n\rangle \) are linearly independent and the \( N \)th is linearly dependent on these. Then we could run the whole argument again replacing \( N \) by \( N' = N - 1 \) throughout and again arrive at a contradiction. Duan and Guo’s PQCM [7, 8] has the constraint that only linearly independent quantum states can be cloned perfectly with non-zero probability. Therefore the condition for no signalling yields the constraint on a PQCM.

This proof that the no faster-than-light signalling constraint leads to the constraint on probabilistic cloning is stronger in some respects than Gisin’s proof relating to deterministic cloning since we do not make any assumptions about the nature of the output state of the cloning machine (other than that it successfully clones in those cases we require it to successfully clone).

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