On semilinear Tricomi equations with critical exponents or in two space dimensions

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Abstract

This paper is a complement of our recent works on the semilinear Tricomi equations in [8] and [9]. For the semilinear Tricomi equation
\[ \partial_t^2 u - t \Delta u = |u|^p \]
with initial data
\[ (u(0, \cdot), \partial_t u(0, \cdot)) = (u_0, u_1), \]
where \( t \geq 0, x \in \mathbb{R}^n (n \geq 3), p > 1, \) and \( u_i \in C_0^\infty(\mathbb{R}^n) (i = 0, 1), \) we have shown in [8] and [9] that there exists a critical exponent \( p_{\text{crit}}(n) > 1 \) such that the solution \( u \), in general, blows up in finite time when \( 1 < p < p_{\text{crit}}(n) \), and there is a global small solution for \( p > p_{\text{crit}}(n) \).

In the present paper, firstly, we prove that the solution of \( \partial_t^2 u - t \Delta u = |u|^p \) will generally blow up for the critical exponent \( p = p_{\text{crit}}(n) \) and \( n \geq 2 \), secondly, we establish the global existence of small data solution to \( \partial_t^2 u - t \Delta u = |u|^p \) for \( p > p_{\text{crit}}(n) \) and \( n = 2 \). Thus, we have given a systematic study on the blowup or global existence of small data solution \( u \) to the equation \( \partial_t^2 u - t \Delta u = |u|^p \) for \( n \geq 2 \).

Keywords: Tricomi equation, critical exponent, blowup, global existence, Strichartz estimate.

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1 Introduction

In this paper, we continue to be concerned with the global existence or blowup of solutions \( u \) to the semilinear Tricomi equation
\[
\begin{align*}
\partial_t^2 u - t \Delta u &= |u|^p, \\
u(0, \cdot) &= f(x), \\
\partial_t u(0, \cdot) &= g(x),
\end{align*}
\]  
(1.1)

where \( t \geq 0, x = (x_1, \ldots, x_n) \in \mathbb{R}^n (n \geq 2), p > 1, \) and \( u_i \in C_0^\infty(B(0, M)) (i = 0, 1) \) with \( B(0, M) = \{ x : |x| = \sqrt{x_1^2 + \ldots + x_n^2} < M \} \) and \( M > 1. \) For the local well-posedness and optimal regularities of solution \( u \) to problem (1.1), the readers may consult [18–21, 29] and the

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references therein. In [8]- [9], we have determined a critical exponent $p_{\text{crit}}(n)$ and a conformal exponent $p_{\text{conf}}(n)$ ($> p_{\text{crit}}(n)$) for (1.1) as follows (corresponding to the case of $m = 1$ in the generalized equation Tricomi equation $\partial_t^2 u - t^m \Delta u = |u|^p$). $p_{\text{crit}}(n)$ is the positive root of the algebraic equation

$$(3n - 2)p^2 - 3np - 6 = 0,$$  

(1.2)

and $p_{\text{conf}}(n) = \frac{3n+6}{3n-2}$. It is shown in [8] that for all $n \geq 2$, the solution $u$ of (1.1) generally blows up in finite time when $1 < p < p_{\text{crit}}(n)$, and meanwhile $u$ exists globally when $p \geq p_{\text{conf}}(n)$ for small initial data and $n \geq 2$. In [9], we prove that the small data solution $u$ of (1.1) exists globally when $n \geq 3$ and $p_{\text{crit}}(n) < p < p_{\text{conf}}(n)$. Therefore, collecting the results in [8]- [9], we have given a detailed study on the blowup or global existence of small data solution $u$ to problem (1.1) for $n \geq 3$ except $p = p_{\text{crit}}(n)$, and for $n = 2$ with $p \geq p_{\text{conf}}(n)$ except $p_{\text{crit}}(n) < p < p_{\text{conf}}(n)$. In this paper, firstly, we establish the finite time blowup result for problem (1.1) when $n \geq 2$ and $p = p_{\text{crit}}(n)$, secondly, we prove the global existence of small data solution $u$ to problem (1.1) when $n = 2$ and $p_{\text{crit}}(n) < p < p_{\text{conf}}(n)$.

**Theorem 1.1.** Let $n \geq 2$ and $p = p_{\text{crit}}(n)$. Suppose that the initial data $f, g \in C^\infty_c(\mathbb{R}^n)$ are non-negative and positive somewhere, then problem (1.1) admits no global solution $u$ with

$$u \in C^1([0, \infty), H^1(\mathbb{R}^n)) \cap C([0, \infty), L^2(\mathbb{R}^n)).$$

**Theorem 1.2.** Let $n = 2$. For $p_{\text{crit}}(n) < p \leq p_{\text{conf}}(n)$, suppose that the initial data $(f, g)$ satisfy

$$\sum_{|\alpha| \leq 1} (\|Z^{\alpha} f\|_{\dot{H}^s(\mathbb{R}^2)} + \|Z^{\alpha} g\|_{\dot{H}^{s-\frac{2}{p}}(\mathbb{R}^2)}) < \varepsilon,$$  

(1.3)

where $\varepsilon > 0$ is a sufficiently small constant, $s = 1 - \frac{4}{3(p-1)}$, and $Z = \{\partial_1, \partial_2, x_1 \partial_2 - x_2 \partial_1\}$. Then problem (1.1) admits a global solution $u$ such that

$$u \in L^q_1 L^p_2 L^2(\mathbb{R}^1 \times \mathbb{R}^2), \ \text{i.e.,} \ \left(\int_0^\infty \left( \int_0^\infty \left( \int_0^\infty |u(t, r \cos \theta, r \sin \theta)|^{2p} \, dr \right) \frac{d\theta}{2\pi} \right)^{\frac{1}{p}} \right)^{\frac{1}{q}} < \infty,$$

where $x_1 = r \cos \theta$ and $x_2 = r \sin \theta$ with $r \geq 0$ and $\theta \in [0, 2\pi]$, $q = \frac{p(p-1)}{3p-11} > 2$ for $p_{\text{crit}}(2) < p \leq \frac{7}{3}$; $q = \frac{(3p+1)(p-11)}{11-3p} > 2$ for $\frac{7}{3} < p \leq \frac{4+\sqrt{17}}{6}$; $q = \frac{4-1}{6-3p} > 2$ for $\frac{4+\sqrt{17}}{3} < p \leq p_{\text{conf}}(2) = 3$.

**Remark 1.1.** For the semilinear wave equation $\partial_t^2 u - \Delta u = |u|^p$ ($p > 1$), the critical exponent $p_0(n)$ in Strauss’ conjecture (see [26]) is determined by the algebraic equation $(n-1)p_0^2(n) - (n+1)p_0(n) - 2 = 0$ (so far the global existence of small data solution $u$ for $p > p_0(n)$ or the blowup of solution $u$ for $1 < p < p_0(n)$ have been proved in [4]- [6], [12]- [13], [23] and the references therein). The finite time blowup for the critical wave equations $\partial_t^2 u - \Delta u = |u|^{p_0(n)}$ has been established in [4], [12], [22], and [31]- [32], respectively. Motivated by the techniques in [31] and [8], we prove the blowup result for the critical semilinear Tricomi equation in (1.1).

**Remark 1.2.** For brevity, we only study the semilinear Tricomi equation instead of the generalized semilinear Tricomi equation $\partial_t^2 u - t^m \Delta u = |u|^p$ ($m \in \mathbb{N}$) in problem (1.1). In fact, by the methods in Theorem 1.1 and Theorem 1.2, one can establish the analogous results to Theorem 1.1-Theorem 1.2 for the generalized semilinear Tricomi equation.

**Remark 1.3.** It follows from a direct computation that $p_{\text{crit}}(2) = \frac{3+\sqrt{133}}{4}$ and $p_{\text{conf}}(2) = 3$ in Theorem 1.2.
For \( n = 1 \), the linear equation \( \partial_t^2 u - t \partial_x^2 u = 0 \) is the well-known Tricomi equation which arises in transonic gas dynamics. There are extensive results for both linear and semilinear Tricomi equations in \( n \) space dimensions (\( n \in \mathbb{N} \)). For instance, with respect to the linear Tricomi equation, the authors in [1], [28] and [30] have computed its fundamental solution explicitly; with respect to the semilinear Tricomi equation \( \partial_t^2 u - t \Delta u = f(t, x, u) \), under some certain assumptions on the function \( f(t, x, u) \), the authors in [7] and [14]-[17] have obtained a series of interesting results on the existence and uniqueness of solution \( u \) in bounded domains under Tricomi, Goursat or Dirichlet boundary conditions respectively in the mixed type case, in the degenerate hyperbolic setting or in the degenerate elliptic setting; with respect to the Cauchy problem of semilinear Tricomi equations, the authors in [18–21] established the local existence as well as the singularity structure of low regularity solutions in the degenerate hyperbolic region and the elliptic-hyperbolic mixed region, respectively.

In addition, by establishing some classes of \( L^p-L^q \) estimates for the solution \( v \) of linear equation \( \partial_t^2 v - t \Delta v = F(t, x) \), the author in [29] obtained some results about the global existence or the blowup of solutions to problem (1.1) when the exponent \( p \) belongs to a certain range, however, there was a gap between the global existence interval and the blowup interval. By establishing the Strichartz inequality and the weighted Strichartz inequality for the linear Tricomi equation, respectively, we have shown the global existence of small data solution \( u \) to problem (1.1) for \( p > p_{crit}(n) \) \( (n \geq 3) \) in [8]-[9].

We now comment on the proof of Theorem 1.1 and Theorem 1.2. To prove Theorem 1.1, we define the function \( G(t) = \int_{\mathbb{R}^n} u(t, x) \, dx \). By applying some crucial techniques for the modified Bessel function as in [11, 20], and motivated by [31] and [8], we can derive a Riccati-type ordinary differential inequality for \( G(t) \) through a delicate analysis of (1.1), which is stronger than the ordinary differential inequality in [8] (see (2.1) of [8]). From this and Lemma 2.1 in [31], the blowup result for \( p = p_{crit}(n) \) in Theorem 1.1 is established under the positivity assumptions of \( f \) and \( g \). To prove the global existence result in Theorem 1.2, we require to establish angular Strichartz estimates for the Tricomi operator \( \partial_t^2 - t \Delta \) as in the treatment on the 2-D linear wave operator in [24]. In this process, a series of inequalities are derived by applying an explicit formula for the solutions of linear Tricomi equations and by utilizing some basic properties of related Fourier integral operators and some classical results in harmonic analysis. Based on the resulting Strichartz inequalities and the contractible mapping principle, we eventually complete the proof of Theorem 1.2. Here we point out that compared with the techniques of [24] for deriving the Strichartz inequality with angular mixed-norm of 2-D linear wave equation, due to the influences of degeneracy and variable coefficients in the linear equation, it is more involved and complicated to give the related analysis on the resulting Fourier integral operator from linear Tricomi equation.

This paper is organized as follows: In Section 2, we complete the proof of Theorem 1.1. In Section 3, some Strichartz estimates with angular mixed norms for the linear Tricomi equation are established. In Section 4, by applying the results in Section 3, Theorem 1.2 is proved.

## 2 Proof of Theorem 1.1

Before starting the proof of Theorem 1.1, we cite a blowup lemma from [31].

**Lemma 2.1.** Let \( p > 1 \), \( a \geq 1 \), and \( (p - 1)a = q - 2 \). Suppose \( G \in C^2[0, T) \) satisfies that for \( t \geq T_0 > 0 \),

\[
G(t) \geq K_0 (t + M)^a,
\]  

(2.1)
On semilinear Tricomi equations with critical exponents or in two space dimensions

\[ G''(t) \geq K_1 (t + M)^{-q} G(t)^p, \]  

(2.2)

where \( K_0, K_1, T_0 \) and \( M \) are some positive constants. Fixing \( K_1 \), there exists a positive constant \( c_0 \), independent of \( M \) and \( T_0 \) such that if \( K_0 \geq c_0 \), then \( T < \infty \).

With this lemma and \( G(t) = \int_{\mathbb{R}^n} u(t, x) \, dx \), our subsequent tasks are to derive (2.1) and (2.2) for the solution \( u \) of problem (1.1). It follows from Section 2 of [8] that

\[ G''(t) = \int_{\mathbb{R}^n} |u(t, x)|^p \, dx \geq C(M + t)^{-\frac{3}{2}n(p-1)} |G(t)|^p. \]  

(2.3)

This means that (2.2) holds for \( q = \frac{3}{2}n(p-1) \). Next, strongly motivated by the techniques in [31] and [8], we focus on the derivation of (2.1), which is divided into the following three steps:

**Step 1. Some reductions**

Let

\[ \bar{u} = \frac{1}{\omega_n} \int_{S^{n-1}} u(t, r\theta) \, d\theta \]

be the spherical average of \( u \). Then applying the spherical average on both sides of (1.1) yields

\[ \partial_t^2 \bar{u} - t \Delta \bar{u} = |\bar{u}|^p. \]  

(2.4)

By Daboux’s identity, one has \( \Delta \bar{u} = \Delta \bar{u} \). On the other hand, it follows from Hölder’s inequality that

\[ |\bar{u}| = \left| \frac{1}{\omega_n} \int_{S^{n-1}} u(t, r\theta) \, d\theta \right| \leq \left( \frac{1}{\omega_n} \int_{S^{n-1}} |u|^p \, d\theta \right)^{\frac{1}{p}} \left( \int_{S^{n-1}} \frac{d\theta}{\omega_n} \right)^{\frac{1}{p'}} \leq \left( |u|^p \right)^{\frac{1}{p'}}. \]

This, together with (2.4), yields

\[ \partial_t^2 \bar{u} - t \Delta \bar{u} \geq |\bar{u}|^p. \]  

(2.5)

Thus we can assume that \( u \) is radial since the blowup of \( \bar{u} \) obviously yields the blowup of \( u \). Let \( \omega \in \mathbb{R}^n \) be a unit vector. The Radon transform of \( u \) with respect to the variable \( x \) is defined as

\[ R(u)(t, \rho) = \int_{x \cdot \omega = \rho} u(t, x) \, dS_x, \]  

(2.6)

where \( \rho \in \mathbb{R} \), \( dS_x \) is the Lebesgue measure on the hyper-plane \( \{ x : x \cdot \omega = \rho \} \). From (2.6) and the radial assumption of \( u(t, \cdot) \), it is easy to see

\[ R(u)(t, \rho) = \int_{\rho}^{\infty} u(t, \rho \omega + x') \, dS_x \]

\[ = c_n \int_{|\rho|}^{\infty} \int_{|\rho|}^{\infty} u(t, \rho)(r^2 - \rho^2)^{\frac{n-3}{2}} \, r \, dr. \]  

(2.7)

Obviously, \( R(u)(t, \rho) \) is independent of \( \omega \).

**Step 2. The lower bound of \( R(u) \)**

From Page 3 of [10], we have

\[ R(\Delta u)(t, \rho) = \partial_{\rho}^2 R(u)(t, \rho). \]  

(2.8)
Since $u$ is a solution of (1.1), it follows from (2.8) that $R(u)$ solves
\[
\begin{aligned}
\partial_t^2 R(u) - t \partial_ho^2 R(u) &= R(|u|^p), \quad (t, \rho) \in \mathbb{R}_+^{1+1}, \\
R(u)(0, \rho) &= R(f), \quad \partial_t R(u)(0, \rho) = R(g).
\end{aligned}
\]

By Lemma 2.1 in [30] and Theorem 3.1 in [28], we have
\[
R(u)(t, \rho) = C \int_0^1 v_{R(f)}(\phi(t)s, \rho)(1 - s^2)^{-\frac{3}{2}} ds + C t \int_0^1 v_{R(g)}(\phi(t)s, \rho)(1 - s^2)^{-\frac{3}{2}} ds \\
+ C \int_0^t \int_{\rho - \phi(t) + \phi(s)}^{\rho + \phi(t) - \phi(s)} (\phi(t) + \phi(s) + \rho - \rho_1)^{-\gamma} (\phi(t) + \phi(s) - \rho + \rho_1)^{-\frac{1}{6}} \gamma \frac{1}{6} \frac{1}{6} 1, z \\
\times R(|u|^p)(s, \rho_1) d\rho_1 ds,
\]
(2.9)

where $C > 0$ is a constant, $z = \frac{(\rho - \rho_1 + \phi(t) - \phi(s))(\rho - \rho_1 - \phi(t) + \phi(s))}{(\rho - \rho_1 + \phi(t) + \phi(s))(\rho - \rho_1 - \phi(t) - \phi(s))}$ with $\phi(t) = \frac{2}{3} t^2$, $F\left(\frac{1}{6}, \frac{1}{6}, 1, z\right)$ is the hypergeometric function, and the function $v_{\phi}$ solves the 1-D wave equation
\[
\begin{aligned}
\partial_t^2 v - \partial_{\rho}^2 v &= 0, \quad (t, \rho) \in \mathbb{R}_+^{1+1}, \\
v(0, \rho) &= \varphi, \quad \partial_t v(0, \rho) = 0.
\end{aligned}
\]

Note that (2.7) together with the non-negativity of $f$ and $g$ shows $R(f) \geq 0$ and $R(g) \geq 0$. In addition, by D’Alembert’s formula, we obtain $v_{R(f)} \geq 0$ and $v_{R(g)} \geq 0$. Hence,
\[
R(u)(t, \rho) \geq C_\gamma \int_0^t \int_{\rho - \phi(t) + \phi(s)}^{\rho + \phi(t) - \phi(s)} (\phi(t) + \phi(s) + \rho - \rho_1)^{-\frac{1}{6}} (\phi(t) + \phi(s) - \rho + \rho_1)^{-\frac{1}{6}} \\
\times F\left(\frac{1}{6}, \frac{1}{6}, 1, z\right) R(|u|^p)(s, \rho_1) d\rho_1 ds.
\]

Note that
\[
z = \frac{(\phi(t) - \phi(s))^2 - (\rho - \rho_1)^2}{(\phi(t) + \phi(s))^2 - (\rho - \rho_1)^2} \in [0, 1].
\]

Then by page 59 of [3], we arrive at
\[
F\left(\frac{1}{6}, \frac{1}{6}, 1, z\right) = \frac{1}{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right)} \int_0^1 t^{-\frac{5}{6}}(1 - t)^{-\frac{1}{6}}(1 - zt)^{-\frac{5}{6}} dt \\
\geq \frac{1}{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right)} \int_0^1 t^{-\frac{5}{6}}(1 - t)^{-\frac{1}{6}} dt \\
= \frac{1}{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right)} B\left(\frac{1}{6}, \frac{5}{6}\right) \\
= \frac{1}{\Gamma(1)} = 1.
\]

Therefore,
\[
R(u)(t, \rho) \geq C \int_0^t \int_{\rho - \phi(t) + \phi(s)}^{\rho + \phi(t) - \phi(s)} ((\phi(t) + \phi(s))^2 - (\rho - \rho_1)^2)^{-\frac{1}{6}} R(|u|^p)(s, \rho_1) d\rho_1 ds.
\]
(2.10)
Notice that the support of \( u(s, \cdot) \) is contained in the ball \( B(0, M + \phi(s)) =: \{ x \in \mathbb{R}^n : |x| \leq M + \phi(s) \} \). On the other hand, if \(|\rho_1| > M + \phi(s)\), then for any vector \( y \in \mathbb{R}^n \) which is perpendicular to \( \omega \), one has

\[
|\rho_1 \omega + y| = \sqrt{\rho_1^2 + y^2} \geq |\rho_1| > \phi(s) + M.
\]

This yields that for \(|\rho_1| > M + \phi(s)\),

\[
\mathbf{R}(|u|^p)(s, \rho_1) = \int_{\{ y : y \cdot \omega = 0 \}} u(s, \rho_1 \omega + y) dS_y = 0.
\]

Thus, \( \text{supp} \mathbf{R}(|u|^p)(s, \cdot) \subseteq B(0, M + \phi(s)) \) holds. From now on, we can assume \( \rho \geq 0 \). If

\[
0 \leq \phi(s) \leq \phi(s_1) =: \frac{\phi(t) - \rho - M}{2}, \tag{2.11}
\]

then

\[
\rho + \phi(t) - \phi(s) \geq \phi(s) + M, \quad \rho - \phi(t) + \phi(s) \leq - (\phi(s) + M).
\]

From this, we arrive at

\[
\mathbf{R}(u)(t, \rho) \geq C \int_0^{s_1} \int_{-\rho - \phi(t) + \phi(s)}^{\rho + \phi(t) - \phi(s)} \left( (\phi(t) + \phi(s))^2 - (\rho - \rho_1)^2 \right)^{-\frac{1}{\beta}} \mathbf{R}(|u|^p)(s, \rho_1) d\rho_1 ds
\]

\[
= C \int_0^{s_1} \int_{-\infty}^{\infty} \left( (\phi(t) + \phi(s))^2 - (\rho - \rho_1)^2 \right)^{-\frac{1}{\beta}} \mathbf{R}(|u|^p)(s, \rho_1) d\rho_1 ds. \tag{2.12}
\]

By (2.11), one has

\[
\phi(t) + \phi(s) + \rho - \rho_1 \leq \phi(t) + \phi(s) + \rho + \phi(s) + M \leq \phi(t) + \phi(t) - \rho - M + \rho + M \leq 2\phi(t),
\]

\[
\phi(t) + \phi(s) - \rho + \rho_1 \leq \phi(t) + \phi(s) - \rho + \phi(s) + M \leq 2(\phi(t) - \rho).
\]

Together with this and (2.12), we deduce

\[
\mathbf{R}(u)(t, \rho) \geq C \int_0^{s_1} \int_{-\infty}^{\infty} \left( \phi(t) - \rho \right)^{-\frac{1}{\beta}} \phi(t)^{-\frac{1}{\beta}} \mathbf{R}(|u|^p)(s, \rho_1) d\rho_1 ds
\]

\[
= \left( \phi(t) - \rho \right)^{-\frac{1}{\beta}} \phi(t)^{-\frac{1}{\beta}} \int_0^{s_1} \int_{-\infty}^{\infty} \int_{\{ y' : y' \cdot \omega = 0 \}} u(s, \rho_1 \omega + y') dS_{y'} d\rho_1 ds
\]

\[
= \left( \phi(t) - \rho \right)^{-\frac{1}{\beta}} \phi(t)^{-\frac{1}{\beta}} \int_0^{s_1} \int_{\mathbb{R}^n} |u(s, y)|^p dy ds. \tag{2.13}
\]

On the other hand, by (2.17) of [8], one has

\[
\int_{\mathbb{R}^n} |u(s, y)|^p dy \geq Cs^\frac{p}{\beta} (M + \phi(s))^{n - 1 - \frac{n}{2}p}. \tag{2.14}
\]

Substituting (2.14) into (2.13) yields

\[
\mathbf{R}(u)(t, \rho) \geq C \left( \phi(t) - \rho \right)^{-\frac{1}{\beta}} \phi(t)^{-\frac{1}{\beta}} \int_0^{s_1} s^\frac{p}{\beta} (M + \phi(s))^{n - 1 - \frac{n}{2}p} ds
\]

\[
= C \left( \phi(t) - \rho \right)^{-\frac{1}{\beta}} \phi(t)^{-\frac{1}{\beta}} \int_0^{s_1} s^{\frac{p}{\beta} + \frac{n}{2} - \frac{n}{2}p} ds. \tag{2.15}
\]
To guarantee that the integral in (2.15) is convergent, we shall need

\[ \frac{p}{2} + \frac{3}{2} \left( n - 1 - \frac{2p}{n} \right) > -1. \]

This is achieved by \( p = p_{\text{crit}}(n) < p_{\text{conf}}(n) = \frac{3n+6}{3n-2} \) and direct computation. Thus we conclude that for \( n \geq 2 \),

\[ R(u)(t, \rho) \geq C (\phi(t) - \rho)^{-\frac{1}{2}} (\phi(t) - \rho - M)^{n-1 - \frac{2p}{n} + \frac{p+2}{3}}. \]  \hspace{1cm} (2.16)

**Step 3. The lower bound of \( \int_{\mathbb{R}^n} |u(t, x)|^{p} dx \)**

Following (2.16) of [31], one can introduce the transformation

\[ T(f)(\rho) = \frac{1}{|\phi(t) - \rho + M|^{\frac{n}{2}}} \int_{\rho}^{\phi(t)+M} f(r)|r - \rho|^{\frac{n+2}{2}} dr \]

and derive

\[ \|T(f)\|_{L^p} \leq C\|f\|_{L^p}. \]  \hspace{1cm} (2.17)

In fact, if \( n \geq 3 \), then it is easy to see that

\[ |T(f)(\rho)| \leq \frac{2}{2|\phi(t) - \rho + M|} \int_{2\rho - (\phi(t)+M)}^{\phi(t)+M} |f(r)|dr \leq 2M(|f|)(\rho), \]

where \( M(|f|) \) is the maximal function of \( f \). Hence there exists a constant \( C > 0 \) such that (2.17) holds.

For \( n = 2 \), at first we prove that \( T \) maps \( L^\infty \) to \( L^\infty \) and \( L^1 \) to \( L^{1,\infty} \) (weak \( L^1 \) space), respectively. If so, by the Marcinkiewicz interpolation theorem, then (2.17) holds for \( n = 2 \).

In fact, it follows from a direct computation that for \( \rho > 0 \),

\[ |T(f)(\rho)| = \frac{1}{|\phi(t) - \rho + M|^{\frac{n}{2}}} \int_{\rho}^{\phi(t)+M} f(r)|r - \rho|^{\frac{n}{2}} dr \]

\[ \leq \frac{\|f\|_{L^\infty([0,\phi(t)+M])}}{|\phi(t) - \rho + M|^{\frac{n-1}{2}}} \int_{\rho}^{\phi(t)+M} |r - \rho|^{-\frac{n}{2}} dr \]

\[ = 2\|f\|_{L^\infty([0,\phi(t)+M])} \frac{1}{|\phi(t) - \rho + M|^{\frac{n}{2}}} |\phi(t) - \rho + M|^{\frac{1}{2}} \]

\[ = 2\|f\|_{L^\infty([0,\phi(t)+M])}, \]

which yields the \( L^\infty - L^\infty \) estimate of operator \( T \). Next we derive the \( L^1 - L^{1,\infty} \) estimate of \( T \).

Suppose \( f \in L^1([0, \phi(t) + M]) \). Let

\[ g(\rho) = \frac{1}{|\phi(t) - \rho + M|^{\frac{n}{2}}}, \]

\[ h(\rho) = \int_{\rho}^{\phi(t)+M} f(r)|r - \rho|^{-\frac{n}{2}} dr. \]
Denote $d_{\varphi}(\alpha) = \left| \{ 0 \leq \rho \leq \phi(t) + M : \varphi(\rho) > \alpha \} \right|$ as the distribution function of $\varphi$. It is known that for $0 < \alpha < \infty$ and measurable functions $f_1, f_2$

$$d_{f_1f_2}(\alpha) \leq d_{f_1}(\alpha^{\frac{1}{2}}) + d_{f_2}(\alpha^{\frac{1}{2}}).$$

Note that

$$d_{f}(\alpha^{\frac{1}{2}}) = \left| \{ 0 \leq \rho \leq \phi(t) + M : g(\rho) > \alpha \} \right| = \frac{1}{\alpha}.$$

In addition,

$$|h(\rho)| \leq \int_{0}^{\rho(t)+M} |f(r)||r-\rho|^{-\frac{1}{2}} dr = f \ast \frac{1}{|r|^{\frac{1}{2}}}.$$

Since $\frac{1}{|r|^2} \in L^{2,\infty}([0, \phi(t) + M])$ and $f \in L^1([0, \phi(t) + M])$, by Young’s inequality, we have $h \in L^{2,\infty}([0, \phi(t) + M])$. Therefore,

$$\alpha d_{gh}(\alpha) \leq \alpha d_{f}(\alpha^{\frac{1}{2}}) + \alpha d_{h}(\alpha^{\frac{1}{2}}) \leq C,$$

which means $T(f)(\rho) = g(\rho)h(\rho) \in L^{1,\infty}([0, \phi(t) + M])$. Then an application of Marcinkiewicz interpolation theorem yields

$$\|T(f)\|_{L^p([0,\phi(t)+M])} \leq C_0 \|f\|_{L^p([0,\phi(t)+M])}, \quad (2.18)$$

where $C_0 > 0$ is a uniform constant independent of $t$. Due to $\text{supp} u(t, \cdot) \subseteq [0, \phi(t) + M]$, the inequality (2.18) is enough for the application in the proof of Theorem 1.1.

Applying (2.17) or (2.18) to the function

$$f(r) = \begin{cases} |u(t, r)|r^{\frac{n-1}{p}}, & r \geq 0, \\ 0, & r < 0, \end{cases}$$

we have

$$\int_{0}^{\phi(t)+M} \left( \frac{1}{|\phi(t) - \rho + M|^{\frac{n-1}{2}}} \int_{\rho}^{\phi(t)+M} |u(t, r)|r^{\frac{n-1}{p}} |r-\rho|^{\frac{n-1}{2}} dr \right)^p d\rho \quad (2.19)$$

$$\leq C \int_{0}^{\infty} |u(t, r)|^p r^{n-1} dr = C \int_{\mathbb{R}^n} |u(t, x)|^p dx. \quad (2.20)$$

When $r \geq \rho$, we arrive at

$$\frac{n-1}{r^p} = r^{\frac{n-1}{2}} \frac{n-1}{r^p} \frac{n-1}{2} \geq \begin{cases} r^{\frac{n-1}{p}} \frac{n-1}{p} - \frac{n-1}{2}, & 1 < p \leq 2, \\ r^{\frac{n-1}{2}} (\phi(t) + M)^{\frac{n-1}{p}} - \frac{n-1}{2}, & p > 2. \end{cases}$$

Next we only treat the case of $1 < p \leq 2$ since the treatment for $p > 2$ is completely similar. When $1 < p \leq 2$, it follows from (2.19) that
\[ \int_0^{\phi(t)+M} \left( \frac{1}{|\phi(t) - \rho + M|^{\frac{n-1}{2}}} \int_\rho^{\phi(t)+M} |u(t, r)| r^{\frac{n-1}{2}} |r - \rho|^{\frac{n-3}{2}} \, dr \right)^p \rho^{(n-1)(1-p/2)} \, d\rho \leq C \int_{\mathbb{R}^n} |u(t, x)|^p \, dx. \] (2.21)

On the other hand,
\[ R(u)(t, \rho) = c_n \int_{|\rho|}^\infty u(t, r) (r^2 - \rho^2)^{\frac{n-3}{2}} r \, dr \leq c_n \int_{|\rho|}^\infty u(t, r) r^{\frac{n-1}{2}} (r - \rho)^{\frac{n-3}{2}} \, dr. \] (2.22)

Substituting (2.22) into (2.21) yields
\[ \int_{\mathbb{R}^n} |u(t, x)|^p \, dx \geq C \int_0^{\phi(t)+M} \frac{(R(u)(t, \rho))^p}{(\phi(t) - \rho + M)^{\frac{n-1}{2}}} \rho^{(n-1)(1-p/2)} \, d\rho. \] (2.23)

By the bound of \( R(u) \) in (2.16), we deduce
\[ \int_{\mathbb{R}^n} |u(t, x)|^p \, dx \geq C \int_0^{\phi(t)+M} \frac{(\phi(t) - \rho)^{\gamma} \phi(t) - \rho - M)^p \rho^{(n-1)(1-p/2)} \, d\rho. \] (2.24)

Note that for \( p = p_{\text{crit}}(n) \),
\[ \left( \frac{n}{2} - \frac{1}{3} \right)^p - \frac{n}{2} = 1. \]

This observation together with (2.23) yields
\[ \int_{\mathbb{R}^n} |u(t, x)|^p \, dx \geq C \int_0^{\phi(t)+M} \frac{\rho^{(n-1)(1-p/2)} \phi(t)^{-\epsilon}}{(\phi(t) - \rho - M)^{p/2}} \, d\rho. \] (2.25)

Note that the term \( \ln(\phi(t) - M + 1) \) can be sufficiently large when \( t \) is large, and if the power of \( t \) in the right hand side of (2.25) satisfies
\[ \sigma = \frac{3}{2} \left( n - 1 - \frac{np}{2} \right) > -1, \] (2.26)
On semilinear Tricomi equations with critical exponents or in two space dimensions

then there is a large constant $K_0 > 0$ such that for large $t > 0$ and $p = p_{\text{crit}}(n)$,

$$G''(t) = \int_{\mathbb{R}^n} |u(t,x)|^p \, dx \geq K_0 t^{\frac{n}{2} + \frac{3}{2} \left( n - 1 - \frac{n}{2} \right)} \geq CK_0(t + M)^{\frac{n}{2} + \frac{3}{2} \left( n - 1 - \frac{n}{2} \right)},$$

and

$$G(t) \geq CK_0(t + M)^{\frac{n}{2} + \frac{3}{2} \left( n - 1 - \frac{n}{2} \right)}.$$  \hspace{1cm} (2.27)

Next we turn to verify (2.26). By the condition

$$p = p_{\text{crit}}(n) < p_{\text{conf}}(n) = \frac{3n + 6}{3n - 2}.$$  

direct computation yields

$$\sigma = \frac{3}{2}(n - 1) - \frac{1}{2} \left( \frac{3}{2} n - 1 \right) p$$

$$> \frac{3}{2}(n - 1) - \frac{1}{2} \left( \frac{3}{2} n - 1 \right) p_{\text{conf}}(n)$$

$$= \frac{3n}{4} - 3.$$  

If $n \geq 3$, then

$$\sigma > \frac{3}{4} > -1.$$  

If $n = 2$, then

$$\sigma = \frac{3}{2}(1 - \frac{2}{3} p_{\text{crit}}(2)) = \frac{3 - \sqrt{33}}{4} > -1.$$  

Hence (2.26) is valid for all $n \geq 2$. By (2.27) and (2.3), choosing $a = \frac{n}{2} + \frac{3}{2} \left( n - 1 - \frac{n}{2} \right) + 2$ and $q = \frac{3n}{2}(p - 1)$ with $p = p_{\text{crit}}(n)$ in Lemma 2.1, then all the assumptions of Lemma 2.1 hold. Therefore, Theorem 1.1 is shown by Lemma 2.1.

3 Strichartz estimates in angular mixed norm spaces

Before establishing Strichartz estimates for the linear Tricomi operator, we recall two important results. The first one is a minor variant of [13, Lemma 3.8], and the second one comes from [2, Theorem 1.2].

Lemma 3.1. Let $\beta \in C_0^\infty ((1/2, 2))$ and $\sum_{j = -\infty}^{\infty} \beta (2^{-j} \tau) \equiv 1$ for $\tau > 0$. Define the Littlewood-Paley operators as

$$G_j(t, x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \beta \left( 2^{-j} |\xi| \right) \hat{G}(t, \xi) \, d\xi, \quad j \in \mathbb{Z}.$$  

Then

$$\|G\|_{L_t^2 L_x^q} \leq C \left( \sum_{j = -\infty}^{\infty} \|G_j\|_{L_t^2 L_x^q}^2 \right)^{1/2}, \quad 2 \leq q < \infty, \quad 1 \leq s \leq \infty,$$
D.-Y. He, I. Witt, and H.-C. Yin

and

$$\left( \sum_{j=\infty}^{\infty} \|G_j\|_{L_p^r L_p^r}^2 \right)^{1/2} \leq C \|G\|_{L_p^r L_p^r}, \quad 1 < p \leq 2, \quad 1 \leq r \leq \infty.$$  

**Lemma 3.2.** Suppose that $1 \leq p < q \leq \infty$. Let $T : L^p(\mathbb{R}) \to L^q(\mathbb{R})$ be a bounded linear operator which is defined by

$$T f(x) = \int_{\mathbb{R}} K(x, y) f(y) dy,$$

where $K(x, y)$ is locally integrable. Define

$$\tilde{T} f(x) = \int_{-\infty}^{\infty} K(x, y) f(y) dy.$$

Then

$$\|\tilde{T} f\|_{L^q} \leq C_{p,q} \|T\|_{L^p \to L^q} \|f\|_{L^p}.$$

To prove Theorem 1.2, we shall require to get certain Strichartz estimates in $\mathbb{R}^{1+2}_+$ for 2-D linear Tricomi operator. For this purpose, we study the following linear Cauchy problem

$$\begin{cases}
\partial_t^2 u - t \triangle u = F(t, x), & (t, x) \in \mathbb{R}^{1+2}_+,
 u(0, \cdot) = f(x), \quad \partial_t u(0, \cdot) = g(x),
\end{cases}$$  

(3.1)

Note that the solution $u$ of (3.1) can be written as

$$u(t, x) = v(t, x) + w(t, x),$$

where $v$ solves the homogeneous problem

$$\begin{cases}
\partial_t^2 v - t \triangle v = 0, & (t, x) \in \mathbb{R}^{1+2}_+,
 v(0, \cdot) = f(x), \quad \partial_t v(0, \cdot) = g(x),
\end{cases}$$  

(3.2)

and $w$ solves the inhomogeneous problem with zero initial data

$$\begin{cases}
\partial_t^2 w - t \triangle w = F(t, x), & (t, x) \in \mathbb{R}^{1+2}_+,
 w(0, \cdot) = 0, \quad \partial_t w(0, \cdot) = 0.
\end{cases}$$  

(3.3)

Let $\dot{H}^s(\mathbb{R}^2)$ denote the homogeneous Sobolev space with norm

$$\|f\|_{\dot{H}^s(\mathbb{R}^2)} = \| |D_x|^s f \|_{L^2(\mathbb{R}^2)},$$

where

$$|D_x| = \sqrt{-\Delta}.$$

It follows from [29] that the solution $v$ of (3.2) can be expressed as

$$v(t, x) = V_1(t, D_x) f(x) + V_2(t, D_x) g(x),$$

where $V_1(t, D_x)$ and $V_2(t, D_x)$ are certain operators.
where the symbols $V_j(t, \xi)$ ($j = 1, 2$) of the Fourier integral operators $V_j(t, D_x)$ are

$$V_1(t, |\xi|) = \frac{\Gamma(\frac{5}{6})}{\Gamma(\frac{1}{6})} \left[ e^{\frac{2}{3}t} H_+ \left( \frac{5}{6}, \frac{5}{3}; z \right) + e^{-\frac{2}{3}t} H_- \left( \frac{5}{6}, \frac{5}{3}; z \right) \right] \tag{3.4}$$

and

$$V_2(t, |\xi|) = \frac{\Gamma(\frac{5}{6})}{\Gamma(\frac{1}{6})} \left[ e^{\frac{2}{3}t} H_+ \left( \frac{5}{6}, \frac{5}{3}; z \right) + e^{-\frac{2}{3}t} H_- \left( \frac{5}{6}, \frac{5}{3}; z \right) \right], \tag{3.5}$$

here $z = 2i\phi(t)|\xi|$, $i = \sqrt{-1}$, and $H_\pm$ are smooth functions of the variable $z$. By [27], one knows that for $\beta \in \mathbb{N}_0^n$,

$$|\partial_\xi^\beta H_+ (\alpha, \gamma; z)| \leq C(\phi(t)|\xi|)^{\alpha - \gamma} (1 + |\xi|^2)^{-\frac{|\beta|}{2}} \quad \text{if } \phi(t)|\xi| \geq 1, \tag{3.6}$$

$$|\partial_\xi^\beta H_- (\alpha, \gamma; z)| \leq C(\phi(t)|\xi|)^{-\alpha} (1 + |\xi|^2)^{-\frac{|\beta|}{2}} \quad \text{if } \phi(t)|\xi| \geq 1. \tag{3.7}$$

We only estimate $V_1(t, D_x)f(x)$ since the estimation on $V_2(t, D_x)g(x)$ is similar. Indeed, up to a factor of $t \phi(t)^{-\frac{2}{3}} = C\phi(t)^{-\frac{2}{3}}$, the powers of $t$ appearing in $V_1(t, D_x)f(x)$ or $V_2(t, D_x)g(x)$ are the same.

Choose a cut-off function $\chi(s) \in C^\infty(\mathbb{R})$ with $\chi(s) = \begin{cases} 1, & s \geq 2 \\ 0, & s \leq 1. \end{cases}$ Then

$$V_1(t, |\xi|) \hat{f}(\xi) = \chi(\phi(t)|\xi|) V_1(t, |\xi|) \hat{f}(\xi) + (1 - \chi(\phi(t)|\xi|)) V_1(t, |\xi|) \hat{f}(\xi) =: \hat{v}_1(t, \xi) + \hat{v}_2(t, \xi). \tag{3.8}$$

By (3.4), (3.6) and (3.7), we derive that

$$v_1(t, x) = C \left( \int_{\mathbb{R}^n} e^{i(x-x_0+\phi(t)|\xi|)} a_{11}(t, \xi) \hat{f}(\xi) d\xi + \int_{\mathbb{R}^n} e^{i(x-x_0-\phi(t)|\xi|)} a_{12}(t, \xi) \hat{f}(\xi) d\xi \right), \tag{3.9}$$

where $C > 0$ is a generic constant, and for $\beta \in \mathbb{N}_0^n$,

$$|\partial_\xi^\beta a_{1l}(t, \xi)| \leq C_{l|\beta|} |\xi|^{-|\beta|} (1 + \phi(t)|\xi|)^{-\frac{1}{2}}, \quad l = 1, 2. \tag{3.10}$$

Next we analyze $v_2(t, x)$. It follows from [3] or [29] that

$$V_1(t, |\xi|) = e^{-\frac{2}{3}t} \Phi \left( \frac{1}{6}, \frac{1}{3}; z \right),$$

where $\Phi$ is the confluent hypergeometric function which is analytic with respect to the variable $z = 2i\phi(t)|\xi|$. Then

$$\left| \partial_\xi \left\{ (1 - \chi(\phi(t)|\xi|)) V_1(t, |\xi|) \right\} \right| \leq C(1 + \phi(t)|\xi|)^{-\frac{1}{2}} |\xi|^{-\frac{1}{2}}.$$

Similarly, one has

$$\left| \partial_\xi^\beta \left\{ (1 - \chi(\phi(t)|\xi|)) V_1(t, |\xi|) \right\} \right| \leq C(1 + \phi(t)|\xi|)^{-\frac{1}{2}} |\xi|^{-|\beta|}.$$
Thus we arrive at
\[
v_2(t, x) = C \left( \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \phi(t)\langle \xi \rangle)} a_{21}(t, \xi) \hat{f}(\xi) d\xi + \int_{\mathbb{R}^n} e^{i(x \cdot \xi - \phi(t)\langle \xi \rangle)} a_{22}(t, \xi) \hat{f}(\xi) d\xi \right), \tag{3.10}
\]
where, for $\beta \in \mathbb{N}_0^n$,
\[
|\partial_\xi^\beta a_{2l}(t, \xi)| \leq C_{l\beta} (1 + \phi(t)\langle |\xi| \rangle)^{-\frac{1}{2}} |\xi|^{-|\beta|}, \quad l = 1, 2.
\]
Substituting (3.9) and (3.10) into (3.8) yields
\[
V_1(t, D_x) f(x) = C \left( \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \phi(t)\langle \xi \rangle)} a_1(t, \xi) \hat{f}(\xi) d\xi + \int_{\mathbb{R}^n} e^{i(x \cdot \xi - \phi(t)\langle \xi \rangle)} a_2(t, \xi) \hat{f}(\xi) d\xi \right),
\]
where $a_l$ $(l = 1, 2)$ satisfies
\[
|\partial_\xi^\beta a_l(t, \xi)| \leq C_{l\beta} (1 + \phi(t)\langle |\xi| \rangle)^{-\frac{1}{2}} |\xi|^{-|\beta|}. \tag{3.11}
\]
Next we only treat the integral $\int_{\mathbb{R}^n} e^{i(x \cdot \xi - \phi(t)\langle \xi \rangle)} a_2(t, \xi) \hat{f}(\xi) d\xi$ since the treatment of the integral $\int_{\mathbb{R}^n} e^{i(x \cdot \xi + \phi(t)\langle \xi \rangle)} a_1(t, \xi) \hat{f}(\xi) d\xi$ is similar. Denote
\[
(Af)(t, x) = \int_{\mathbb{R}^n} e^{i(x \cdot \xi - \phi(t)\langle \xi \rangle)} a_2(t, \xi) \hat{f}(\xi) d\xi. \tag{3.12}
\]
We will show that
\[
\|(Af)(t, x)\|_{L^q L^r_1 L^2_2(\mathbb{R}^{1+2})} \leq C \|f\|_{H^s(\mathbb{R}^2)}, \tag{3.13}
\]
where $q \geq 2$ and $r \geq 2$ are some suitable constants related to $s$. One obtains by a scaling argument that those indices in (3.13) should satisfy
\[
\frac{1}{q} + \frac{3}{r} = \frac{3}{2} (1 - s). \tag{3.14}
\]
On the other hand, by another scaling argument similar to Knapp’s counter example, we get the second restriction on the indices in (3.13)
\[
\frac{1}{q} \leq 1 - \frac{3}{2} \cdot \frac{1}{r}. \tag{3.15}
\]
In fact, for small $\delta > 0$, set $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}$ and denote
\[
D = D_\delta = \{ \xi \in \mathbb{R}^2 : |\xi_1 - 1| < 1/2, |\xi_2| < \delta \}.
\]
Let $\hat{f}(\xi) = \chi_D(\xi)$ be the characteristic function of domain $D$. Note that on domain $D$ it holds
\[
|\xi| - \xi_1 = \frac{|\xi|^2 - \xi_1^2}{|\xi| + \xi_1} = \frac{|\xi_2|^2}{|\xi| + \xi_1} \sim \delta^2.
\]
By (3.12), one has
\[
(Af)(t, x) = \int_{\mathbb{R}^2} e^{i(x \cdot \xi - \phi(t)\langle \xi \rangle)} a_2(t, \xi) \hat{f}(\xi) d\xi
\]
\[
= e^{i(x_1 - \phi(t))} \int_{D} e^{i(-\phi(t)\langle |\xi| - \xi_1 \rangle + (x_1 - \phi(t))(\xi_1 - 1) + x_2 \xi_2)} a_2(t, \xi) d\xi. \tag{3.16}
\]
On semilinear Tricomi equations with critical exponents or in two space dimensions

Choose a domain $R$ in $\mathbb{R}_+ \times \mathbb{R}^2$ as

$$R = \{ (t, x) : \phi(t) \leq \delta^{-1}, |x_1 - \phi(t)| \lesssim 1, |x_2| \lesssim \delta^{-1} \}. $$

For $(t, x) \in R$ and $\xi \in D$, then the phase function in (3.16) is essentially equivalent to a constant and we have

$$|(Af)(t, x)| \geq |D|(1 + \delta^{-1})^{-\frac{2}{q}} \sim |D|\delta^\frac{2}{q}.$$

Therefore if we take $s = 0$ in (3.13), then a direct computation yields

$$\frac{\| (Af)(t, x) \|_{L^q_t L^r_x L^2_\xi(\mathbb{R}_+ \times \mathbb{R}^2)}}{\| f \|_{L^2(\mathbb{R}^2)}} \geq \frac{|D|\delta^\frac{2}{q} \| \chi_R \|_{L^q_t L^r_x L^2_\xi(\mathbb{R}_+ \times \mathbb{R}^2)}}{|D|\delta^\frac{2}{q}} \sim \delta^\frac{2}{3} \cdot \frac{2}{3q} \cdot \frac{1}{r}.$$

Since $\delta > 0$ is small, in order to get (3.13), we shall need

$$\frac{2}{3} - \frac{2}{3q} - \frac{1}{r} \geq 0 \iff \frac{1}{q} \leq 1 - \frac{3}{2} \cdot \frac{1}{r},$$

which gives restriction (3.15). Now our task is to prove

**Lemma 3.3.** Let operator $A$ be defined by (3.12). Assume that $(q, r) \neq (\infty, \infty)$,

$$q, r \geq 2 \quad \text{and} \quad \frac{1}{q} \leq 1 - \frac{3}{2} \cdot \frac{1}{r}.$$ 

Then

$$\| (Af)(t, x) \|_{L^q_t L^r_x L^2_\xi(\mathbb{R}_+^{1+2})} \leq C \| f \|_{\dot{H}^s(\mathbb{R}^2)},$$

where $s = 2(\frac{1}{2} - \frac{1}{r}) - \frac{2}{3} \cdot \frac{1}{q}$.

**Proof.** The main step in the proof of (3.17) is to show that

$$\| (Af)(t, x) \|_{L^q_t L^r_x L^2_\xi(\mathbb{R}_+^{1+2})} \leq C \| f \|_{L^2(\mathbb{R}^2)}$$

if $2 \leq q < \infty, 2 \leq r \leq \infty$ and $\hat{f}(\xi) = 0$ if $|\xi| \notin [\frac{1}{2}, 1]$. (3.18)

Indeed, once (3.18) is proved, then by the support condition of $f$, we know that

$$\| (Af)(t, x) \|_{L^q_t L^r_x L^2_\xi(\mathbb{R}_+^{1+2})} \leq C \| f \|_{\dot{H}^s(\mathbb{R}^2)}, \quad s = 2(\frac{1}{2} - \frac{1}{r}) - \frac{2}{3} \cdot \frac{1}{q}. \quad (3.19)$$

This together with Lemma 3.1 yields (3.17).

To prove (3.18), we follow some ideas of [24] and use the interpolation method. The first case is

$q = \infty$ and $s = 1 - \frac{2}{r}$. Since Hardy-Littlewood-Sobolev estimate gives $\dot{H}^{1 - \frac{2}{r}}(\mathbb{R}^2) \subseteq L^q(\mathbb{R}^2)$ for

$2 \leq r < \infty$, we clearly have

$$\| Af \|_{L^q_t L^r_x L^2_\xi(\mathbb{R}_+^{1+2})} \leq C \| Af \|_{L^q_t L^r_x L^2_\xi(\mathbb{R}_+^{1+2})} \leq C \| Af \|_{L^q_t \dot{H}^{1 - \frac{2}{r}}(\mathbb{R}_+^{1+2})}.$$ 

It follows from (3.11) and (3.12) that if $\hat{f}(\xi) = 0$ for $|\xi| \notin [\frac{1}{2}, 1]$,

$$\| Af \|_{L^q_t \dot{H}^{1 - \frac{2}{r}}(\mathbb{R}_+ \times \mathbb{R}^2)}$$
Let $\alpha(t) = \rho \beta(t) a_2(t, \rho)$. Then by (3.24) and the support condition of $c_k$, we have

$$(Af)(t, r(\cos \omega, \sin \omega)) = \left( 2\pi \right)^{-1} \sum_k \left( i^k \int_0^\infty J_k(\tau \rho) c_k(\rho) \rho \, d\rho \right) e^{i k \omega},$$

where $k \in \mathbb{Z}$, and $J_k$ is the $k$-th Bessel function defined by

$$J_k(y) = \frac{(-i)^k}{2\pi} \int_0^{2\pi} e^{iy \cos \theta - i k \theta} \, d\theta.$$
Lemma 3.4. Let \( \hat{\alpha}(t, \xi) \) be defined as above and a number \( N \in \mathbb{N} \) be fixed. Then there is a uniform constant \( C > 0 \), which is independent of the variables \( b \in \mathbb{R} \) and \( r \geq 0 \), so that the following inequalities hold:

\[
\int_{0}^{2\pi} | \hat{\alpha}(b - r \cos \theta) | \, d\theta \leq C(b)^{-N} (1 + \phi(t))^{-\frac{1}{2}} \quad \text{if} \quad 0 \leq r \leq 1 \quad \text{or} \quad |b| \geq 2r; \tag{3.26}
\]

\[
\int_{0}^{2\pi} | \hat{\alpha}(b - r \cos \theta) | \, d\theta \leq C(r^{-1} + r^{-\frac{1}{2}} (r - |b|)^{-\frac{1}{2}}) (1 + \phi(t))^{-\frac{1}{2}} \quad \text{if} \quad r > 1 \; \text{and} \; |b| \leq 2r. \tag{3.27}
\]

Proof of Lemma 3.4. By the definition of function \( \hat{\alpha}_t \), we only need to study the integral

\[
I = \int_{0}^{2\pi} \left| \int_{0}^{\infty} e^{-i(b-r \cos \theta)} \rho \beta(\rho) a_2(t, \rho) \, d\rho \right| \, d\theta.
\]

Case I. \(|b| \geq 2r\)

In this case, we have

\[
I = \int_{0}^{2\pi} \left| \int_{0}^{\infty} (b - r \cos \theta)^N e^{-i(b-r \cos \theta)} \rho \beta(\rho) a_2(t, \rho) \, d\rho \right| (b - r \cos \theta)^{-N} \, d\theta \\
= \int_{0}^{2\pi} \left| \int_{0}^{\infty} (-D_\rho)^N (e^{-i(b-r \cos \theta)}) \rho \beta(\rho) a_2(t, \rho) \, d\rho \right| (b - r \cos \theta)^{-N} \, d\theta \\
= \int_{0}^{2\pi} \left| \int_{0}^{\infty} e^{-i(b-r \cos \theta)} (D_\rho)^N (\rho \beta(\rho) a_2(t, \rho)) \, d\rho \right| (b - r \cos \theta)^{-N} \, d\theta.
\]
which just corresponds to (3.26).

**Case II.** $0 \leq r \leq 1$

For $|b| > 2$, it is reduced to Case I. For $|b| \leq 2$, by a direct computation, we have

$$|I| \leq \int_0^{2\pi} \left| \int_{1/4}^{1/2} \rho \beta(\rho) (1 + \phi(t)\rho)^{-\frac{1}{B}} d\rho \right| d\theta \leq C (1 + \phi(t))^{-\frac{1}{B}} \leq C (1 + \phi(t))^{-\frac{1}{B}} (b)^{-N}.$$

**Case III.** $r > 1$ and $|b| \leq 2r$

In this case, we intend to prove that

$$\int_0^{\pi/4} |\hat{a}_t(b - r \cos \theta)| d\theta + \int_{3\pi/4}^{\pi} |\hat{a}_t(b - r \cos \theta)| d\theta \leq Cr^{-\frac{1}{B}} (r - |b|)^{-\frac{1}{B}} (1 + \phi(t))^{-\frac{1}{B}}, \quad (3.28)$$

and

$$\int_{\pi/4}^{3\pi/4} |\hat{a}_t(b - r \cos \theta)| d\theta \leq r^{-1} (1 + \phi(t))^{-\frac{1}{B}}. \quad (3.29)$$

To show (3.28), it only suffices to estimate the first integral in (3.28). Let $u = 1 - \cos \theta$, we then have

$$\int_0^{\pi/4} |\hat{a}_t(b - r \cos \theta)| d\theta = \int_0^{1 - \frac{r^2}{2}} |\hat{a}_t(b - r + ru)| \frac{du}{\sqrt{2u - u^2}}$$

$$\leq C \int_0^{1 - \frac{r^2}{2}} |\hat{a}_t(b - r + ru)| \frac{du}{\sqrt{u}}. \quad (3.30)$$

We further set $\bar{u} = ru$, then the last integral in (3.30) can be controlled by

$$r^{-\frac{1}{B}} \int_0^{\infty} |\hat{a}_t(b - r + \bar{u})| \frac{d\bar{u}}{\sqrt{\bar{u}}}$$

$$= r^{-\frac{1}{B}} \int_0^{\infty} \left| \int_{-\infty}^{\infty} e^{-i(b-r+\bar{u})} \rho \beta(\rho) a_2(t, \rho) d\rho \right| \frac{d\bar{u}}{\sqrt{\bar{u}}} =: r^{-\frac{1}{B}} II. \quad (3.31)$$

If $|r - |b|| \geq 2$, then

$$II = \int_0^{r - |b|/2} \left| \int_{-\infty}^{\infty} e^{-i(b-r+\bar{u})} \rho \beta(\rho) a_2(t, \rho) d\rho \right| \frac{d\bar{u}}{\sqrt{\bar{u}}}$$

$$+ \int_{r - |b|/2}^{\infty} \left| \int_{-\infty}^{\infty} e^{-i(b-r+\bar{u})} \rho \beta(\rho) a_2(t, \rho) d\rho \right| \frac{d\bar{u}}{\sqrt{\bar{u}}} =: II_1 + II_2. \quad (3.32)$$
For $II_1$, we can repeat the analysis in Case I and integrate by parts to get

$$II_1 \leq C \int_0^{r-|b|/2} |b - r + \bar{u}|^{-N} (1 + \phi(t))^{-\frac{1}{6}} \frac{d\bar{u}}{\sqrt{u}}$$

$$\leq C |b - r|^{-N} (1 + \phi(t))^{-\frac{1}{6}} \int_0^{r-|b|/2} \frac{d\bar{u}}{\sqrt{u}}$$

$$\leq C |b - r|^{-N} (1 + \phi(t))^{-\frac{3}{6}}. \quad (3.33)$$

For $II_2$, integrating by parts yields

$$II_2 \leq C (1 + \phi(t))^{-\frac{1}{6}} \int_0^\infty \langle b - r + \bar{u} \rangle^{-N} \frac{d\bar{u}}{\sqrt{u}}$$

$$\leq C (1 + \phi(t))^{-\frac{1}{6}} |r - |b||^{-\frac{1}{2}} \int_0^\infty \langle b - r + \bar{u} \rangle^{-N} d\bar{u}$$

$$\leq C (1 + \phi(t))^{-\frac{1}{6}} |b - r|^{-\frac{1}{2}}. \quad (3.34)$$

If $|r - |b|| \leq 2$, then by similar computation,

$$\int_0^\infty \frac{|\hat{\alpha}_t(b - r + \bar{u})|}{\sqrt{u}} d\bar{u} \leq C \int_0^\infty \langle b - r + \bar{u} \rangle^{-N} \frac{d\bar{u}}{\sqrt{u}}$$

$$\leq C (1 + \phi(t))^{-\frac{1}{6}} (r - |b|)^{-\frac{1}{2}}. \quad (3.35)$$

Thus it follows from (3.30)-(3.35) that the proof of (3.28) is finished.

To show (3.29), we set $u = r \cos \theta$. Then

$$\int_{\pi/4}^{3\pi/4} |\hat{\alpha}_t(b - r \cos \theta)| d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{-\infty}^\infty e^{-i\rho(b-u)} \alpha(t, \rho) d\rho \frac{du}{r \sin \theta}$$

$$\leq C r^{-1} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1 + \phi(t))^{-\frac{1}{6}} (b - u)^{-N} d\rho$$

$$\leq C r^{-1} \int_{-\infty}^\infty (1 + \phi(t))^{-\frac{1}{6}} (u)^{-N} d\rho$$

$$\leq C (1 + \phi(t))^{-\frac{1}{6}} r^{-1}. $$

Collecting all the analysis above in Case I-Case III yields the proof of Lemma 3.4. \qed

By Lemma 3.4, we have

**Claim.** For $\delta > 0$, there is a constant $C_\delta > 0$, which is independent of $t \in \mathbb{R}_+$ and $r \geq 0$, such that

$$\int_{-\infty}^\infty \left( \int_0^{2\pi} (1 + \phi(t))^{\frac{1}{2}} \langle \phi(t) - s \rangle^{\frac{3}{2} - \delta} |\hat{\alpha}_t(\phi(t) - s - r \cos \theta)| d\theta \right)^2 ds \leq C_\delta. \quad (3.36)$$

**Proof of claim.** If $0 \leq r \leq 1$, or $|\phi(t) - s| \geq 2r$, then for $\delta > 0$, it is easy to see that (3.26) yields the expected estimate (3.36).
If $|\phi(t) - s| \leq 2r$ and $r > 1$, then by (3.27), one has

$$
\int_{\phi(t)-2r}^{\phi(t)+r} \left( \int_0^{2\pi} (1 + \phi(t)) \frac{1}{r} (\phi(t) - s) \left| \frac{\partial}{\partial \theta} \phi(t) - s \right| \cos \theta \right)^2 d\theta \right) ds
$$

$$
\leq \int_{\phi(t)-2r}^{\phi(t)+r} \left( (1 + \phi(t)) \frac{1}{r} (\phi(t) - s) \left| (1 + \phi(t)) \frac{1}{r} (r^{-1} + r^{-\frac{1}{2}} (r - |\phi(t) - s|)^{-\frac{1}{2}}) \right)^2 \right) ds
$$

$$
\leq 2 \int_{\phi(t)-2r}^{\phi(t)+r} (r^{-1-2\delta} + \langle \phi(t) - s \rangle^{1-2\delta} r^{-1} \langle r - |\phi(t) - s| \rangle^{-1}) ds. \quad (3.37)
$$

Next we treat the integral in (3.37). By $r > 1$, we have

$$
\int_{\phi(t)-2r}^{\phi(t)+2r} r^{-1-2\delta} ds = \frac{4}{r^{2\delta}} \leq C_\delta.
$$

For the second part of the integral in (3.37), if $|\phi(t) - s| \leq 1$ and $r > 1$, then

$$
\int_{\phi(t)-2r}^{\phi(t)+2r} \langle \phi(t) - s \rangle^{1-2\delta} r^{-1} \langle r - |\phi(t) - s| \rangle^{-1} ds \leq C \int_{\phi(t)-1}^{\phi(t)+1} r^{-1} ds \leq \frac{C}{r} \leq C_\delta.
$$

If $|\phi(t) - s| > 1$ and $r > 1$, we then set $\eta = |\phi(t) - s|$ and derive that for $0 < \delta < 1/2$, the integral in (3.37) can be controlled by

$$
\left| \int_{\phi(t)-2r}^{\phi(t)+r} \langle \phi(t) - s \rangle^{1-2\delta} r^{-1} \langle r - |\phi(t) - s| \rangle^{-1} ds \right|
$$

$$
\leq C \int_1^{2r} r^{-1} \eta^{1-2\delta} \langle r - \eta \rangle^{-1} d\eta
$$

$$
\leq C \int_1^{2r} \frac{\eta^{1-2\delta}}{1 + |r - \eta|} d\eta
$$

$$
\leq C \left( \int_1^{r} \frac{\eta^{1-2\delta}}{1 + r - \eta} d\eta + \int_r^{2r} \frac{\eta^{1-2\delta}}{1 + \eta - r} d\eta \right). \quad (3.38)
$$

Note that

$$
\int_1^{r} \frac{\eta^{1-2\delta}}{1 + r - \eta} d\eta = \ln(2 + r) + (1 - 2\delta) \int_1^{r} \ln(1 + r - \eta) \eta^{-2\delta} d\eta
$$

$$
\leq \ln(2 + r) + (1 - 2\delta) \ln r \int_1^{r} \eta^{-2\delta} d\eta \quad (3.39)
$$

$$
\leq \ln(2 + r) + r^{1-2\delta} \ln r
$$

and

$$
\int_r^{2r} \frac{\eta^{1-2\delta}}{1 + \eta - r} d\eta = \eta^{1-2\delta} \ln(1 + \eta - r) \bigg|_r^{2r} - \int_r^{2r} \ln(1 + \eta - r) \eta^{1-2\delta} d\eta
$$

$$
\leq (2r)^{1-2\delta} \ln(1 + r). \quad (3.40)
$$

Then collecting (3.38), (3.39) and (3.40) yields

$$
\int_1^{2r} r^{-1} \eta^{1-2\delta} \langle r - \eta \rangle^{-1} d\eta \leq \frac{\ln(1 + r)}{r^{2\delta}} \leq C_\delta.
$$

Hence the claim is proved. \qed
It follows from Claim, (3.25) and Hölder’s inequality that
\[ \|Af\|_{L^q_w L^2_v} \leq C \delta \sum_k \int_{-\infty}^{\infty} \left| (1 + \phi(t))^{-\frac{1}{q}} \langle \phi(t) - s \rangle^{-\frac{1}{2} + \delta} c_k(s) \right|^2 ds. \]

For any \( q \geq 2 \), we can choose a constant \( \delta > 0 \) such that
\[ \sigma := q \left( 1 - \frac{3}{2} \delta \right) > 1. \]

Then by Minkowski’s inequality, we have that for \( q \geq 2 \),
\[ \|Af\|_{L^q_w L^2_v} \leq \left( \sum_k \int_{-\infty}^{\infty} \left( \int_0^\infty \left| (1 + \phi(t))^{-\frac{1}{q}} \langle \phi(t) - s \rangle^{-\frac{1}{2} + \delta} |q| dt \right)^{\frac{1}{q}} ds \right)^{\frac{1}{2}}. \]

To handle (3.42), we require to compute
\[ \int_0^\infty \left| (1 + \phi(t))^{-\frac{1}{q}} \langle \phi(t) - s \rangle^{-\frac{1}{2} + \delta} |q| dt. \]

If \( s \leq 0 \), then by (3.41), we arrive at
\[ \int_0^\infty \left| (1 + \phi(t))^{-\frac{1}{q}} \langle \phi(t) - s \rangle^{-\frac{1}{2} + \delta} \right|^{\frac{1}{q}} dt \leq C \int_0^\infty (1 + t)^{-\frac{3q}{2} \left( \frac{q}{2} - \delta \right)} dt \leq C. \]

If \( s > 0 \), we then write \( s = \phi(\bar{s}) \) and conclude
\[
\int_0^\infty \left| (1 + \phi(t))^{-\frac{1}{q}} \langle \phi(t) - \phi(\bar{s}) \rangle^{-\frac{1}{2} + \delta} \right|^{\frac{1}{q}} dt
\]
\[= C \int_0^\infty \left| (1 + t)^{-\frac{1}{q}} \langle t - \bar{s} \rangle^{-\frac{1}{2} + \delta} \right|^{\frac{1}{q}} dt
\]
\[\leq C \int_0^\infty (1 + |t - \bar{s}|)^{-\frac{\alpha}{2} \left( \frac{q}{2} - \frac{1}{2} + \delta \right)} \left( 1 + t \right)^{-\frac{\alpha}{2}} dt. \]

By (3.41), in order to estimate (3.44), we only need to compute the following integral for \( \bar{s} > 0 \),
\[ \int_0^\infty (1 + |t - \bar{s}|)^{-\alpha} (1 + t)^{-\beta} dt, \quad \alpha + \beta > 1. \]

A direct computation yields
\[
\int_0^\infty (1 + |t - \bar{s}|)^{-\alpha} (1 + t)^{-\beta} dt = \int_0^{\bar{s}} (1 + |t - \bar{s}|)^{-\alpha} (1 + t)^{-\beta} dt + \int_{\bar{s}}^{\infty} (1 + |t - \bar{s}|)^{-\alpha} (1 + t)^{-\beta} dt
\]
\[\leq C \left( \int_0^{\bar{s}} (1 + t)^{-\alpha - \beta} dt + \int_{\bar{s}}^{\infty} (1 + |t - \bar{s}|)^{-\alpha - \beta} dt \right)
\]
\[\leq C \left( 2 - 2 \left( 1 + \frac{\bar{s}}{2} \right)^{-\alpha - \beta} + 1 \right) \leq C. \]
where the amplitude function $a$ satisfies
\[
\left| \partial_\xi a(t, s, \xi) \right| \leq C \left( 1 + \phi(t) |\xi| \right)^{\frac{1}{q'}} \left( 1 + \phi(s) |\xi| \right)^{-\frac{1}{q}} |\xi|^{-\frac{2}{q} - \frac{2}{r}}. \tag{3.46}
\]

By a dual argument similar to the proof of Lemma 3.4 in [8], we can prove that if $\hat{F}(\tau, \xi) = 0$ when $|\xi| \notin \left[ \frac{2}{r}, 1 \right]$, then
\[
\left\| \int_\mathbb{R} V_2(t, D_x) V_1(\tau, D_x) F(\tau, x) d\tau \right\|_{L^q_t L^r_x L^2_\phi(\mathbb{R}^+ \times \mathbb{R}^2)} \leq C \| F \|_{L^2_t L^q_x L^r_x L^2_\phi(\mathbb{R}^+ \times \mathbb{R}^2)}, \tag{3.47}
\]
where
\[
\frac{1}{q} + \frac{3}{r} = \frac{1}{q} + \frac{3}{r} \tag{3.47}
\]
and
\[
\frac{1}{q} \leq 1 - \frac{3}{2} \cdot \frac{1}{r}, \quad \frac{1}{q} \leq 1 - \frac{3}{2} \cdot \frac{1}{r}. \tag{3.48}
\]

Then an application of Lemma 3.2 yields that for $\hat{F}(\tau, \xi) = 0$ when $|\xi| \notin \left[ \frac{2}{r}, 1 \right]$, \[
\left\| w \right\|_{L^q_t L^r_x L^2_\phi(\mathbb{R}^+ \times \mathbb{R}^2)} \leq C \| F \|_{L^2_t L^q_x L^r_x L^2_\phi(\mathbb{R}^+ \times \mathbb{R}^2)}. \tag{3.49}
\]

Utilizing Lemma 3.1 to remove the restriction on the support of $\hat{F}$ in (3.49), we then get the following estimate for problem (3.3).

**Lemma 3.5.** Let $w$ be the solution of (3.3). If $q, r, \tilde{q}, \tilde{r} \geq 2$ and satisfy (3.47)-(3.48), then
\[
\left\| w \right\|_{L^q_t L^r_x L^2_\phi(\mathbb{R}^+ \times \mathbb{R}^2)} \leq C \| F \|_{L^2_t L^q_x L^r_x L^2_\phi(\mathbb{R}^+ \times \mathbb{R}^2)}. \tag{3.49}
\]
4 Proof of Theorem 1.2

First we consider the linear problem (3.1). Recall the definition of vector fields \( \{ Z \} \) in Theorem 1.2, then by Lemma 3.3 and energy estimates, we have

\[
\sum_{|\alpha| \leq 1} (\| Z^{\alpha} u \|_{L^q(L^r_{|x|}H^s(\mathbb{R}^+ \times \mathbb{R}^2))} + \| Z^{\alpha} u \|_{L^\infty(I^\theta(L^\tilde{q}(\mathbb{R}^2)\cdots + L^\tilde{r}(\mathbb{R}^2)\cdots))}) \leq C \sum_{|\alpha| \leq 1} (\| Z^{\alpha} f \|_{H^s(\mathbb{R}^2)} + \| Z^{\alpha} g \|_{H^{s-\frac{2}{p}}(\mathbb{R}^2)} + \| Z^{\alpha} F \|_{L^q_{|x|}L^r(\mathbb{R}^+ \times \mathbb{R}^2))}),
\]

where \( q, r, \tilde{q}, \tilde{r} \geq 2 \) satisfy (3.48) and

\[
\frac{1}{q} + \frac{3}{r} = \frac{1}{\tilde{q}} + \frac{3}{\tilde{r}} - 2 = \frac{3}{2}(1 - s), \quad \frac{1}{q} + \frac{3}{2r} \leq 1, \quad \frac{1}{\tilde{q}} + \frac{3}{2\tilde{r}} \leq 1.
\]

Note that the nonlinear term in (1.1) is \(|u|^p\), we then have \( q = \tilde{q}' \) and \( r = \tilde{r}' \). This together with condition (4.2) yields

\[
s = 1 - \frac{4}{3(p - 1)}.
\]

Meanwhile the conditions on \( r \) and \( q \) become

\[
\frac{1}{q} + \frac{3}{r} = \frac{2}{p - 1}, \quad \frac{1}{q} + \frac{3}{2r} \leq 1.
\]

**Case I. Choosing \( r = p \) in (4.3) and (4.4)**

In this case, it follows from (4.3) that \( q = \frac{p(p - 1)}{3 - p} \). Then

\[
\frac{1}{q} + \frac{3}{2r} \leq 1 \iff 2p^2 - 3p - 3 \geq 0 \iff p \geq p_{\text{crit}}(2).
\]

From Lemma 3.1 we require \( q \geq 2 \), which leads to

\[
\frac{p(p - 1)}{3 - p} \geq 2 \implies p \geq 2.
\]

This condition is fulfilled by \( p > p_{\text{crit}}(2) \) and Remark 1.3. Furthermore, we have

\[
q' = \frac{q}{p} = \frac{p - 1}{3 - p}.
\]

On the other hand, by Lemma 3.1 we also require \( 1 \leq \tilde{q}' \leq 2 \) and \( 1 \leq \tilde{r}' \leq 2 \), which is equivalent to \( 2 \leq p \leq 7/3 \). In addition, one needs

\[
\frac{1}{q'} + \frac{3}{2r} = \frac{1}{q} \leq 1,
\]

which holds by \( q' \geq 1 \). Collecting all these observations above, we intend to prove the global existence of \( u \) to problem (1.1) by an iteration argument in the range

\[
p_{\text{crit}}(2) < p \leq \frac{7}{3},
\]
provided the initial data are small. More specifically, let \( u_0 \) solve the Cauchy problem (3.2), we then define \( u_k \) (\( k \geq 1 \)) by solving

\[
\begin{align*}
&\frac{\partial^2}{\partial t^2} u_k - t \Delta u_k = |u_{k-1}|^p, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2 \\
&u_k(0, \cdot) = f(x), \quad \partial_t u_k(0, \cdot) = g(x).
\end{align*}
\]

(4.5)

The first step is to show that if

\[
\sum_{|\alpha| \leq 1} (\| Z^\alpha f \|_{\dot{H}^s(\mathbb{R}^2)} + \| Z^\alpha g \|_{\dot{H}^{s-\frac{3}{4}}(\mathbb{R}^2)}) < \varepsilon, \quad s = 1 - \frac{4}{3(p-1)},
\]

(4.6)

and \( \varepsilon > 0 \) is small enough, then

\[
M_k = \sum_{|\alpha| \leq 1} (\| Z^\alpha u_k \|_{L_t^q L_x^p(\mathbb{R}_+ \times \mathbb{R}^2)} + \| Z^\alpha u_k \|_{L_t^\infty \dot{H}^s(\mathbb{R}_+ \times \mathbb{R}^2)})
\]

is uniformly small, where \( q = \frac{p(p-1)}{3-p} \) and \( s = 1 - \frac{4}{3(p-1)} \).

For \( k = 0 \), it follows from Lemma 3.3 and the energy estimate that

\[ M_0 \leq \varepsilon_0. \]

For \( k \geq 1 \), (3.1) yields

\[
M_k \leq \varepsilon + C \varepsilon \sum_{|\alpha| \leq 1} \| Z^\alpha (|u_{k-1}|^p) \|_{L_t^\varphi L_{x}^1 L_x^q(\mathbb{R}_+ \times \mathbb{R}^2)}, \quad \varphi = \frac{p-1}{3-p},
\]

(4.7)

To control the right hand side of (4.7), we note that for a function \( v(x) = v(|x|, \theta) \) \( (x \in \mathbb{R}^2) \) with \( \sum_{|\alpha| \leq 1} |Z^\alpha v| \in L^2_\theta \),

\[
\sum_{|\alpha| \leq 1} |Z^\alpha v|^p \leq C |v|^{p-1} \sum_{|\alpha| \leq 1} |Z^\alpha v|,
\]

Since \( \partial_\theta = x_1 \partial_2 - x_2 \partial_1 \in \{ Z \} \), we have

\[
\| v(|x|, \cdot) \|_{L^\infty_\theta} \leq C \sum_{|\alpha| \leq 1} \| Z^\alpha v(|x|, \cdot) \|_{L^2_\theta},
\]

which derives

\[
\begin{align*}
\| |v|^{p-1} \sum_{|\alpha| \leq 1} |Z^\alpha v| \|_{L^2_\theta} \|_{L_t^\varphi L_x^1} &\leq C \| |v|^{p-1} \|_{L^\infty_\theta} \sum_{|\alpha| \leq 1} \| Z^\alpha v \|_{L^2_\theta} \|_{L_t^\varphi L_x^1} \\
&\leq C \| |v|^{p-1} \|_{L^\infty_\theta} \sum_{|\alpha| \leq 1} \| Z^\alpha v \|_{L^2_\theta} \|_{L_t^\varphi L_x^1} \\
&\leq C \left( \sum_{|\alpha| \leq 1} \| Z^\alpha v \|_{L^2_\theta} \right)^{p-1} \sum_{|\alpha| \leq 1} \| Z^\alpha v \|_{L^2_\theta} \|_{L_t^\varphi L_x^1} \\
&\leq C \left( \sum_{|\alpha| \leq 1} \| Z^\alpha v \|_{L^2_\theta} \right)^p \| L_t^\varphi L_x^1 \|
\end{align*}
\]
On semilinear Tricomi equations with critical exponents or in two space dimensions

\[
\leq C\left(\sum_{|\alpha|\leq 1} \|Z^{\alpha}v\|_{L_t^p L_{|x|}^{p'} L_{\tilde{\theta}}^q}^p\right)^p,
\]
where \( q = p' \).

(4.8)

Thus we have

\[
M_k \leq C_0 \varepsilon + C_1 M_{k-1}^p.
\]

If \( M_{k-1} \leq 2C_0 \varepsilon \), then for small \( \varepsilon > 0 \),

\[
M_k \leq C_0 \varepsilon + C_1 M_{k-1}^{p-1} \leq C_0 \varepsilon + \frac{1}{2} \times 2C_0 \varepsilon \leq 2C_0 \varepsilon.
\]

(4.9)

Define

\[
A_k = \|u_k - u_{k-1}\|_{L_t^q L_{|x|}^{r'} L_{\tilde{\theta}}^2}.
\]

Then by (4.9) and direct computation similar to (4.8), we get that for small \( \varepsilon > 0 \),

\[
A_k \leq CA_k^{p-1}(M_k - 1 + M_k - 2)^{p-1} \leq CA_k^{-1}(2C_0 \varepsilon)^{p-1} \leq 1 + 2C_0 \varepsilon/2.
\]

(4.10)

This means that there exists a function \( u \in L_t^q L_{|x|}^{r'} L_{\tilde{\theta}}^2 \) such that \( u_k \to u \) in \( L_t^q L_{|x|}^{r'} L_{\tilde{\theta}}^2 \). In addition,

\[
\|u\|_{L_t^q L_{|x|}^{r'} L_{\tilde{\theta}}^2} \to 0,
\]

which means \( |u_k|^p \to |u|^p \) in \( L_t^1 L_{|x|}^r L_{\tilde{\theta}}^2 \) and hence in the sense of distribution. Therefore \( u \) is a global weak solution of (1.1) and the proof of Theorem 1.2 is completed for \( p_{\text{crit}}(2) \leq p \leq \frac{7}{3} \).

**Case II. Choosing** \( r = p + \frac{1}{3} \) in (4.3) and (4.4)

In this case, it follows from (4.3) that \( q = \frac{(3p+1)(p-11)}{11-3p} \). Then \( q \geq 2 \) is equivalent to

\[
3p^2 + 4p - 23 \geq 0,
\]

which leads to \( p \geq \frac{\sqrt{109} - 2}{3} \). Note that

\[
\frac{\sqrt{109} - 2}{3} \approx 2.162 < 2.186 \approx \frac{3 + \sqrt{33}}{4} = p_{\text{crit}}(2).
\]

In addition,

\[
\frac{3p+1}{p(11-3p)} \in [1, 2] \iff \frac{13 + \sqrt{193}}{12} \leq p \leq \frac{4 + \sqrt{17}}{3},
\]

and

\[
\frac{3p+1}{3p} \in (1, 2] \iff p \geq \frac{1}{3}.
\]

On the other hand,

\[
\frac{1}{q} + \frac{3}{2r} \leq 1 = \frac{2}{p-1} - \frac{9}{3p+1} + \frac{9}{2(3p+1)} \leq 1 \iff 6p^2 - 7p - 15 \geq 0,
\]

which derives

\[
p \geq \frac{7 + \sqrt{409}}{12} \approx 2.269.
\]
Note that the following condition in Lemma 3.1 is also required

\[ \frac{1}{q'} + \frac{3}{2r'} \leq 1. \]  

(4.10)

Substituting

\[ \frac{1}{q} = 1 - \frac{1}{q'} = 1 - \frac{p(11 - 3p)}{(3p + 1)(p - 1)} \]

and

\[ \frac{1}{r} = 1 - \frac{1}{r'} = \frac{1}{3p + 1} \]

into (4.10) yields

\[ p \leq \frac{19 + \sqrt{433}}{12} \approx 3.317. \]

Since \[ \frac{13 + \sqrt{193}}{12} \approx 2.141 \] and \[ \frac{4 + \sqrt{177}}{3} \approx 2.708 \], we obtain that the admissible range for \( p \) in Case II is

\[ \frac{7 + \sqrt{409}}{12} \leq p \leq \frac{4 + \sqrt{177}}{3}. \]

Hence, we can use (4.1) and repeat the computation from (4.8) to (4.9) to get a global weak solution

\( u \in L^q_t L^{p+1/3}_r L^2_p(\mathbb{R}^{1+2}_+) \) of problem (1.1), where \( q = \frac{(3p+1)(p-11)}{2(3p+1)} \) and \[ \frac{7 + \sqrt{409}}{12} \leq p \leq \frac{4 + \sqrt{177}}{3}. \]

**Case III.** Choosing \( r = p + 1 \) and \( q = \frac{p^2-1}{5-p} \) in (4.3) and (4.4)

In this case, by (4.3) we get \( q = \frac{p^2-1}{5-p} \). Then \( q \geq 2 \) is equivalent to

\[ p^2 + 2p - 11 \geq 0, \]

which derives

\[ p \geq 2\sqrt{3} - 1. \]

In addition,

\[ \frac{p^2 - 1}{p(5-p)} \in [1, 2] \iff \frac{5 + \sqrt{33}}{4} \leq p \leq \frac{5 + 2\sqrt{7}}{3}, \]

and

\[ \frac{p + 1}{p} \in (1, 2] \iff p \geq 1. \]

On the other hand,

\[ \frac{1}{q} + \frac{3}{2r} = 1 - \frac{p(5-p)}{p^2 - 1} + \frac{3}{2(p + 1)} \leq 1 \iff 2p^2 - p - 9 \geq 0, \]

which leads to

\[ p \geq \frac{1 + \sqrt{13}}{4} \approx 2.386. \]

Note that the following condition in Lemma 3.1 is also required

\[ \frac{1}{q'} + \frac{3}{2r'} = 1 - \frac{p(5-p)}{p^2 - 1} + \frac{3}{2(p + 1)} \leq 1. \]
On semilinear Tricomi equations with critical exponents or in two space dimensions

This means
\[ 0 \leq p \leq \frac{7 + \sqrt{73}}{4} \approx 3.886. \]

Since \( 2\sqrt{3} - 1 \approx 2.464 \), \( \frac{5 + \sqrt{33}}{3} \approx 2.686 \) and \( \frac{5 + 2\sqrt{7}}{3} \approx 3.43 > 3 \), the admissible range for \( p \) in Case III is
\[ \frac{5 + \sqrt{33}}{4} \leq p \leq \frac{5 + 2\sqrt{7}}{3}. \]

Then we can use (4.1) and repeat the computation from (4.5) to (4.9) to get a global weak solution \( u \in L^q_t L^p L^2_\theta (\mathbb{R}^{1+2}_+) \), where \( q = \frac{p+1}{6-p} > 2 \) and \( \frac{5 + \sqrt{33}}{4} \leq p \leq 3 \).

Note that \( \frac{7 + \sqrt{409}}{12} < \frac{7}{3} < \frac{4 + \sqrt{17}}{3} < \frac{5 + \sqrt{33}}{4} \) and \( \frac{4 + \sqrt{17}}{3} < 3 \). Then
\[ \left( p_{\text{crit}}(2), \frac{7}{3} \right) \bigcup \left[ \frac{7 + \sqrt{409}}{12}, \frac{4 + \sqrt{17}}{3} \right] \bigcup \left[ \frac{5 + \sqrt{33}}{4}, 3 \right] = (p_{\text{crit}}(2), p_{\text{conf}}(2)). \]

Therefore collecting the proofs in Case I-Case III, we obtain Theorem 1.2.

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On semilinear Tricomi equations with critical exponents or in two space dimensions

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