Maximal estimates for the Schrödinger equation with orthonormal initial data

Neal Bez 1 · Sanghyuk Lee 2 · Shohei Nakamura 3

Abstract
For the one-dimensional Schrödinger equation, we obtain sharp maximal-in-time and maximal-in-space estimates for systems of orthonormal initial data. The maximal-in-time estimates generalize a classical result of Kenig–Ponce–Vega and allow us to obtain pointwise convergence results associated with systems of infinitely many fermions. The maximal-in-space estimates simultaneously address an endpoint problem raised by Frank–Sabin in their work on Strichartz estimates for orthonormal systems of data, and provide a path toward proving our maximal-in-time estimates.

Keywords Maximal estimate · Strichartz estimate · Orthonormal systems

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1 Introduction and main results

1.1 Pointwise convergence for the Schrödinger equation

In one spatial dimension, consider the free Schrödinger equation
\[
\begin{align*}
\begin{cases}
    i \partial_t u + \partial_x^2 u = 0, & (t, x) \in \mathbb{R}^{1+1}, \\
    u(0, x) = f(x),
\end{cases}
\end{align*}
\]
whose solution we denote by \( u(t, x) = e^{it\partial_x^2} f(x) \). The problem of identifying the smallest exponent \( s > 0 \) for which
\[
\lim_{t \to 0} e^{it\partial_x^2} f(x) = f(x) \quad \text{a.e. } x \in \mathbb{R}
\]
holds for all \( f \in H^s(\mathbb{R}) \) originated in the famous paper by Carleson [11]. Here, \( H^s(\mathbb{R}) = (1 - \partial_x^2)^{-\frac{s}{2}} L^2(\mathbb{R}) \) is the inhomogeneous Sobolev space of order \( s \). It follows from [11] that (1.2) holds true as long as \( s \geq \frac{1}{4} \), and shortly afterwards the problem found a complete solution when Dahlberg and Kenig [17] showed that \( s \geq \frac{1}{4} \) is necessary for (1.2).

The standard way to tackle this pointwise convergence problem is to consider maximal-in-time estimates of the form
\[
\| \sup_t |e^{it\partial_x^2} f| \|_{L_t^q(\mathbb{R})} \leq C \| f \|_{H^s}
\]
or its local variants, since standard arguments allow one to deduce (1.2) for all \( f \in H^s(\mathbb{R}) \). Whilst local space-time bounds of this type suffice for the purpose of deducing (1.2), a particularly strong global form of such an estimate,
\[
\| \sup_{t \in \mathbb{R}} |e^{it\partial_x^2} f| \|_{L_t^4(\mathbb{R})} \leq C \| f \|_{\dot{H}^{\frac{1}{4}}}
\]
was obtained by Kenig–Ponce–Vega [26]. Here, \( \dot{H}^s(\mathbb{R}) = (-\partial_x^2)^{-\frac{s}{2}} L^2(\mathbb{R}) \) is the homogeneous Sobolev space of order \( s \). Since the estimate (1.3) is scaling invariant and \( \| f \|_{\dot{H}^{\frac{1}{4}}} \lesssim \| f \|_{H^{\frac{1}{4}}} \), we note that, by an elementary rescaling argument, (1.3) is equivalent to the corresponding estimate with the inhomogeneous norm \( H^{\frac{1}{4}} \) on the right-hand side.

Whilst the equation (1.1) describes the behavior of a single quantum particle, the present work is motivated by recent investigations of Chen–Hong–Pavlović [12,13] and Lewin–Sabin [28,29] into the dynamics of a system of infinitely many fermions. In the case of a finite number \( N \) of particles (in one spatial dimension), such a system is modelled by \( N \) orthonormal functions \( u_1, \ldots, u_N \) in \( L^2(\mathbb{R}) \) satisfying a system of Hartree equations
\[
i \partial_t u_k = (-\partial_x^2 + w \ast \rho) u_k \quad (k = 1, \ldots, N),
\]

where $\rho = \sum_{j=1}^{N} |u_j|^2$ represents the total density of particles and $w$ is an interaction potential.

In this work, we initiate the study of the pointwise convergence problem for a system of infinitely many fermions. As we shall soon see, our progress in this direction hinges on establishing a generalization of (1.3) for orthonormal systems (possibly infinite) of initial data $(f_j)$. A natural form of such an estimate is

$$
\left\| \sum_j v_j |e^{it\partial_x^2} f_j|^2 \right\|_{L_x^2 L_t^\infty(\mathbb{R}^{1+1})} \leq C \|v\|_{\ell^\beta} (1.5)
$$

with the constant $C$ independent of the orthonormal system $(f_j)$ in $\dot{H}^{1/2}(\mathbb{R})$ and $v = (v_j)$ in $\ell^\beta$, where $\beta \geq 1$. Clearly (1.5) is equivalent to the square function estimate

$$
\left\| \left( \sum_j |e^{it\partial_x^2} f_j|^2 \right)^{1/2} \right\|_{L^4 L_t^\infty(\mathbb{R}^{1+1})} \leq C \left( \sum_j \|f_j\|_{\dot{H}^{1/2}}^{2\beta} \right)^{1/2\beta}
$$

for orthogonal systems $(f_j)$ in $\dot{H}^{1/2}(\mathbb{R})$, and this reduces to (1.3) in the case where the system of initial data consists of a single function. It is also apparent that (1.5) follows from (1.3) with $\beta = 1$ via the triangle inequality, and that such estimates get stronger as we increase $\beta$. Our first main result establishes the optimal value of $\beta$, although we have to pay a small price to obtain such a sharp result in the sense that our estimates are of weak type.

**Theorem 1.1** The estimate

$$
\left\| \sum_j v_j |e^{it\partial_x^2} f_j|^2 \right\|_{L_x^{2,\infty} L_t^\infty(\mathbb{R}^{1+1})} \leq C \|v\|_{\ell^\beta} (1.6)
$$

holds for all systems of orthonormal functions $(f_j)$ in $\dot{H}^{1/2}(\mathbb{R})$ and $v = (v_j)$ in $\ell^\beta$ if and only if $\beta < 2$. Moreover, the restricted weak-type estimate

$$
\left\| \sum_j v_j |e^{it\partial_x^2} f_j|^2 \right\|_{L_x^{2,\infty} L_t^\infty(\mathbb{R}^{1+1})} \leq C \|v\|_{\ell^{2,1}}
$$

also fails.

In the above statement, $L^{2,\infty}$ denotes weak $L^2$ and $\ell^{2,1}$ is a (sequence) Lorentz space; the reader may consult, for example, [42] for further details.

As one would expect, a maximal estimate of the form (1.6) implies pointwise convergence of quantities of the form $\sum_j v_j |e^{it\partial_x^2} f_j|^2$ whenever the system $(f_j)$ is orthonormal in $\dot{H}^{1/2}(\mathbb{R})$ and $(v_j)$ belongs to $\ell^\beta$ with $\beta < 2$. As we now turn to describe, a more natural formulation of such a result is in terms of the density function
\( \rho_{\gamma(t)} \) of the solution \( \gamma(t) \) to the operator-valued Hartree-type equation

\[
\begin{aligned}
&\begin{cases}
  i \partial_t \gamma = [-\partial_x^2, \gamma], & (t, x) \in \mathbb{R}^{1+1}, \\
  \gamma|_{t=0} = \gamma_0.
\end{cases} \\
\end{aligned}
\] (1.7)

Here, \( \gamma_0 \) is a self-adjoint and bounded operator on \( L^2(\mathbb{R}) \), \([\cdot, \cdot]\) denotes the commutator, and the solution \( \gamma \) is given by \( \gamma(t) = e^{-it\partial_x^2} \gamma_0 e^{it\partial_x^2} \). This is the free version of the operator-valued Hartree-type equation

\[
\begin{aligned}
&\begin{cases}
  i \partial_t \gamma = [-\partial_x^2 + w \ast \rho_\gamma, \gamma], & (t, x) \in \mathbb{R}^{1+1}, \\
  \gamma|_{t=0} = \gamma_0.
\end{cases} \\
\end{aligned}
\] (1.8)

associated with (1.4), \( \rho_\gamma \) is the so-called the density function, formally defined by \( \rho_\gamma(x) = \gamma(x, x) \) where (with the typical abuse of notation) \( \gamma(x, y) \) is the integral kernel of \( \gamma \). The operator-theoretic viewpoint allows one to rigorously formulate the problem in the case of infinitely many particles (see, for example, the discussion in [28]).

Here we consider the convergence of the solution \( \gamma(t) \) to the initial data \( \gamma_0 \) in terms of pointwise convergence of the associated density functions

\[
\lim_{t \to 0} \rho_{\gamma(t)}(x) = \rho_{\gamma_0}(x) \quad \text{a.e.} \quad x \in \mathbb{R},
\] (1.9)

and we seek as large a class of initial data \( \gamma_0 \) as possible. The Schatten classes \( \mathcal{C}^\beta(\mathcal{H}) \), associated with a given Hilbert space \( \mathcal{H} \), provide a natural setting in order to quantify progress on this problem thanks to their monotonicity property, \( \mathcal{C}^{\beta_1}(\mathcal{H}) \subseteq \mathcal{C}^{\beta_2}(\mathcal{H}) \) provided \( \beta_1 \leq \beta_2 \) (we refer the reader forward to Sect. 2 for the definition of Schatten classes).

As a consequence of Theorem 1.1, we have the following.

**Corollary 1.2** If \( \gamma_0 \in \mathcal{C}^\beta(\dot{H}^{1/4}(\mathbb{R})) \) is self-adjoint with \( \beta < 2 \), then the density functions \( \rho_{\gamma(t)}(x) \) and \( \rho_{\gamma_0}(x) \) are well defined and satisfy (1.9).

Formal considerations indicate that if \( \gamma_0 \in \mathcal{C}^\beta(\dot{H}^{1/4}(\mathbb{R})) \), then

\[
\rho_{\gamma(t)}(x) = \sum_j v_j |e^{it\partial_x^2} f_j(x)|^2
\]

for an appropriate system \( (f_j)_j \) of orthonormal functions in \( \dot{H}^{1/4}(\mathbb{R}) \) and coefficients \( (v_j)_j \) in \( \ell^\beta \). In the infinite-rank case, some care is required to ensure that this is the case (and in what precise sense); we postpone such discussion to Sect. 4. The role of the maximal estimate in Theorem 1.1 is to allow us to deduce pointwise convergence in the infinite-rank case from the finite-rank case. The statement that (1.9) holds for \( \gamma_0 \in \mathcal{C}^1(\dot{H}^{1/4}(\mathbb{R})) \) is equivalent to the following:

\[
\lim_{t \to 0} |e^{it\partial_x^2} f(x)| = |f(x)| \quad \text{a.e.} \quad x \in \mathbb{R}, \quad \forall f \in \dot{H}^{1/4}(\mathbb{R}).
\] (1.10)
Hence, recalling \( C^1(\dot{H}^{1/4}(\mathbb{R})) \subset C^\beta(\dot{H}^{1/4}(\mathbb{R})) \) for \( \beta > 1 \), the pointwise convergence (1.9) with some \( \beta > 1 \) can be seen as a significant improvement of (1.10).

### 1.2 Strichartz estimates for orthonormal systems of data

Next, we introduce our second main result in this paper concerning Strichartz estimates for the Schrödinger equation for orthonormal systems of initial data. The result is of interest for two reasons; firstly, it addresses a problem left open in recent work of Frank et al. [21] and Frank–Sabin [22,23], and secondly it provides a path to proving our maximal estimates in Theorem 1.1. We now seek to clarify these comments.

The classical Strichartz estimates for the one-dimensional free Schrödinger propagator \( e^{it\partial_x^2} \) state that for all \( 2 \leq q, r \leq \infty \) satisfying \( \frac{2}{q} + \frac{1}{r} = \frac{1}{2} \) (we say \( (q,r) \) is an admissible pair in that case), the estimate

\[
\| e^{it\partial_x^2} f \|_{L^q_t L^r_x(\mathbb{R}^{1+1})} \leq C \| f \|_{L^2(\mathbb{R})} \tag{1.11}
\]

holds true for \( f \in L^2(\mathbb{R}) \); see, for example, [24,25,43]. Recently, this classical setting was significantly generalized to estimates of the form

\[
\left\| \sum_j v_j |e^{it\partial_x^2} f_j|^2 \right\|_{L_t^{q/2} L_x^{r/2}(\mathbb{R}^{1+1})} \leq C \| v \|_{\ell^\beta} \tag{1.12}
\]

for systems of orthonormal functions \( (f_j)_j \) in \( L^2(\mathbb{R}) \) and \( v = (v_j)_j \) in \( \ell^\beta \). For admissible pairs with finite values of \( r \), the optimal value of \( \beta \) has been determined as follows.

**Theorem 1.3** [21,22] Let \( q, r \geq 2 \) satisfy

\[
\frac{2}{q} + \frac{1}{r} = \frac{1}{2}, \quad 2 \leq r < \infty.
\]

Then (1.12) holds if and only if \( \beta \leq \frac{2r}{r+2} \).

One can notice that the endpoint case \( (q,r) = (4,\infty) \) is missing in Theorem 1.3. It was observed in [21] that \( \beta < 2 \) is necessary at the endpoint. On the other hand, as far as the authors are aware, there are no non-trivial positive results at the endpoint which improve upon the trivial case \( \beta = 1 \) (which follows from (1.11) and the triangle inequality). Moreover, it was conjectured by Frank and Sabin [23] that the desired endpoint estimate

\[
\left\| \sum_j v_j |e^{it\partial_x^2} f_j|^2 \right\|_{L_t^2 L_x^\infty(\mathbb{R}^{1+1})} \leq C \| v \|_{\ell^\beta} \tag{1.13}
\]

should hold for some \( \beta \in (1,2) \). In a similar spirit to Theorem 1.1, we obtain the optimal range of \( \beta \) for a weak-type version of (1.13).
Theorem 1.4 The estimate
\[
\left\| \sum_j v_j |e^{it\partial_x^2} f_j|^2 \right\|_{L^2_t L^\infty_x(\mathbb{R}^{1+1})} \leq C \|v\|_{\ell^\beta}
\] (1.14)
holds for all systems of orthonormal functions \((f_j)_j\) in \(L^2(\mathbb{R})\) and \(v = (v_j)_j\) in \(\ell^\beta\) if and only if \(\beta < 2\).

We note that we have been able to obtain the strong-type estimate (1.13) in the range \(\beta \leq \frac{4}{3}\); see the remarks at the end of Sect. 3. We also remark that our proof of Theorem 1.4 is robust enough to permit generalization to, say, fractional Schrödinger equations. We refrain from stating such results here, and refer the reader forward to Sect. 3. Such a generalization is key to our proof of the maximal estimates in Theorem 1.1 since we employ an idea due to Kenig–Ponce–Vega [26] that has the effect of switching the roles of space and time at the cost of replacing the classical Schrödinger propagator with the fractional Schrödinger propagator of order 1/2.

Organisation Section 2 contains some preliminaries and a more detailed overview of our approach to proving our main results. In Sect. 3 we prove Theorem 1.4, and in Sect. 4 we prove Theorem 1.1 and Corollary 1.2. Finally, we collect some additional remarks in Sect. 5.

2 Preliminaries and overview

2.1 Preliminaries

We begin by recalling the definition of the Schatten spaces. For \(\beta \in [1, \infty)\), \(\mathcal{C}^\beta(\mathcal{H})\) is the set of all compact operators \(A\) on the Hilbert space \(\mathcal{H}\) such that \(\|A\|_{\mathcal{C}^\beta} = \|(s_j(A))_j\|_{\ell^\beta} < \infty\), where \((s_j(A))_j\) are the singular values of the operator \(A\). Although the case \(\beta = \infty\) will not arise in the present work, we recall that this is the space of bounded linear operators on \(\mathcal{H}\) with the usual operator norm. In fact, most important for us will be the cases \(\beta = 2\) and \(\beta = 4\), in which case explicit computations will be available. Indeed, for Hilbert–Schmidt integral operators of the form

\[
Af(x) = \int_{\mathbb{R}^d} K(x, y) f(y) \, dy
\]

with \(K \in L^2(\mathbb{R}^d \times \mathbb{R}^d)\), the \(\mathcal{C}^2\) norm is given by \(\|A\|_{\mathcal{C}^2} = \|K\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)}\). For the case \(\beta = 4\), we use the fact that \(\|A\|_{\mathcal{C}^4} = \|A^* A\|_{\mathcal{C}^2}\).

Schatten spaces arise naturally via duality when studying Strichartz estimates for orthonormal systems of initial data. For example, the following is a special case of the duality principle of Frank–Sabin [22, Lemma 3] (strictly speaking, the result in [22] is stated for pure Lebesgue spaces).
Proposition 2.1 Let $S : L^2(\mathbb{R}) \to L^{q,\infty}_t L^r_x(\mathbb{R}^{1+1})$ be a bounded linear operator for some $q > 2$ and $r \geq 2$, and let $\beta \geq 1$. Then

$$\left\| \sum_j v_j |Sf_j|^2 \right\|_{L^{q/2,\infty}_t L^{r/2}_x(\mathbb{R}^{1+1})} \leq C\|v\|_{\ell^\beta}$$

holds for all orthonormal systems $(f_j)_j$ in $L^2(\mathbb{R})$ and $v = (v_j)_j$ in $\ell^\beta$, if and only if

$$\|\mathcal{W}S\mathcal{W}^{*}\|_{C^{\beta'}} \leq C\|W\|_{L^{q_0,\infty}_t L^{r_0}_x}^2$$

holds for all $W \in L^{q_0,2}_t L^{r_0}_x(\mathbb{R}^{1+1})$, where $\frac{1}{q} + \frac{1}{q_0} = \frac{1}{2}$ and $\frac{1}{r} + \frac{1}{r_0} = \frac{1}{2}$. Here we identify $L^{\infty,\infty}_t$ with $L^{\infty}_t$.

Notation In the remainder of the paper, we will use the notation $A \lesssim B$ to mean $A \leq CB$ for an appropriate constant $C$. We reserve the notation $P$ for the frequency projection operator given by $\hat{P}f(\xi) = \chi(\xi)\hat{f}(\xi)$, where $\chi \in C^\infty$ is supported on $[-1, 1]$ and identically 1 on $[-1/2, 1/2]$.

Next, we introduce the propagator $U_\alpha$ for the fractional Schrödinger equation of order $\alpha \in \mathbb{R}_+ \setminus \{1\}$,

$$U_\alpha f(t, x) = e^{it|\partial_x|^{\alpha}} f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(x\xi + t|\xi|^{\alpha})} \hat{f}(\xi) \, d\xi$$

for appropriate functions $f : \mathbb{R} \to \mathbb{C}$. Here,

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-ix\xi} \, d\xi$$

is the Fourier transform of $f$, and we use the notation $|\partial_x| = (-\partial_x^2)^{1/2}$.

Although each of our main results stated in the Introduction concern the classical Schrödinger propagator $U_2$, as we shall see momentarily, the case $\alpha = 1/2$ will also play an important role.

2.2 Overview of our proofs

We shall begin by proving the Strichartz estimates for orthonormal systems contained in Theorem 1.4. Since the desired estimate (1.14) is scaling invariant, it suffices to prove the analogous estimate with $U_2$ replaced by frequency-localized version $U_2P$. Thus, in light of Proposition 2.1, we will consider estimates on $\|WU_2P^2U_2^{*}\mathcal{W}\|_{C^{\beta'}}$ with $\beta' > 2$. In order to capitalize on the time decay of the kernel of $U_2P^2U_2^{*}$, we perform an appropriate dyadic decomposition of the operator. By establishing an appropriate range of $C^2$ and $C^4$ estimates for each operator arising in this decomposition, we shall be able to obtain the desired estimates on $\|WU_2P^2U_2^{*}\mathcal{W}\|_{C^{\beta'}}$ via a bilinear interpolation argument inspired by ideas in [25].
Next, we shall turn to the proof of Theorem 1.1. Here, we employ a trick due to Kenig–Ponce–Vega [26] where they reduced the maximal-in-time estimate (1.3) for $U_2$ to a Strichartz estimate (maximal-in-space estimate) for $U_{1/2}$. More precisely, by an elementary changes of variables, note that

$$2e^{-it\partial_x^2} \frac{\partial_t}{\partial_x} |\partial_x|^{-\frac{1}{2}} f(x) = \frac{1}{2\pi} \int_0^\infty e^{it|\eta|^2} \left( e^{ix|\eta|^2} \hat{f}(|\eta|^{\frac{1}{2}}) + e^{-ix|\eta|^2} \hat{f}(-|\eta|^{\frac{1}{2}}) \right) d\eta$$

$$= e^{ix|\partial_t|^2} \frac{\partial_t}{\partial_x} |\partial_x|^{-\frac{3}{8}} f_+(t) + e^{-ix|\partial_t|^2} |\partial_t|^{-\frac{3}{8}} f_-(t), \quad (2.1)$$

where $f_\pm$ are given by

$$\hat{f}_\pm(\eta) = |\eta|^{-\frac{1}{4}} 1_{(0, \infty)}(\eta) \hat{f}(\pm |\eta|^{\frac{1}{2}}).$$

Fortunately, our proof of Theorem 1.4 is sufficiently robust to allow us to obtain, in a straightforward manner, the desired Strichartz estimate for $U_{1/2}$. In fact, we present a somewhat general result in Proposition 3.1 which allows us to deduce $L_t^4 L_x^\infty$ orthonormal Strichartz estimates for both $U_2$ and $U_{1/2}$. Using the above trick of Kenig–Ponce–Vega we are able to obtain Theorem 1.1; however, an additional step is required to overcome the fact that orthonormal structure is not preserved under the transformation $f \mapsto f_\pm$ (see Lemma 4.2).

To show the sharpness of Theorem 1.1, we will employ a semi-classical limit argument and show the failure of the induced estimate by a geometric argument based on the existence of Nikodym sets with zero Lebesgue measure.

# 3 Proof of Theorem 1.4

First, we observe that in order to prove (1.14), by an elementary rescaling argument, it suffices to prove the frequency localized estimate

$$\left\| \sum_j \nu_j |U_2 P f_j|^2 \right\|_{L_{t,x}^2 L_x^\infty(\mathbb{R}^{1+1})} \lesssim \|\nu\|_{\ell^\beta}. \quad (3.1)$$

Next, we have

$$U_2 \partial_x^2 U_2^* F(t, x) = \int_{\mathbb{R}^{1+1}} K(t - t', x - x') F(t', x') dt' dx', \quad \text{for suitable test functions } F : \mathbb{R}^{1+1} \to \mathbb{C},$$

where the integral kernel $K$ has the decay property $|K(t, x)| \lesssim (1 + |t|)^{-\frac{1}{2}}$ uniformly in $x$. In fact, one can check from a direct computation that

$$K(t, x) = \int_{\mathbb{R}} \chi(\xi)^2 e^{i(x\xi + t|\xi|^2)} d\xi.$$
and such a decay estimate is a consequence of a stationary phase argument. Our argument for proving (3.1) only uses the above two properties of the operator $U_2$. For this reason, and for use in our forthcoming proof of Theorem 1.1, we consider a more general bounded linear operator $S$ from $L^2(\mathbb{R})$ to $L^4,\infty,\infty(\mathbb{R}^{1+1})$ such that $SS^*$ is given by

$$SS^* F(t, x) = \int_{\mathbb{R}^{1+1}} K(t - t', x - x') F(t', x') \, dt' dx' \quad (3.2)$$

on a suitable class of test functions $F : \mathbb{R}^{1+1} \rightarrow \mathbb{C}$, where the kernel satisfies

$$\sup_{x \in \mathbb{R}} |K(t, x)| \lesssim (1 + |t|)^{-\frac{1}{2}} \quad (t \in \mathbb{R}). \quad (3.3)$$

**Proposition 3.1** Suppose $\beta < 2$. Under the assumptions (3.2) and (3.3), the estimate

$$\left\| \sum_j v_j |Sf_j|^2 \right\|_{L^2,\infty,\infty(\mathbb{R}^{1+1})} \lesssim \|v\|_{\ell^\beta}$$

holds for all orthonormal systems $(f_j)_j$ in $L^2(\mathbb{R})$ and $v = (v_j)_j$ in $\ell^\beta$.

Thanks to Proposition 2.1, our goal is equivalent to

$$\|WSS^* W\|_{\ell^{\beta'}(L^2)} \lesssim \|W\|_{L^4,2(L^2)} \quad (3.4)$$

for all $W \in L^4,2(L^2(\mathbb{R}^{1+1}))$. To establish this, we first decompose the operator by using the dyadic partition of unity

$$\chi + \sum_{l=1}^\infty \psi(2^{-l} \cdot) \equiv 1,$$

where $\psi \in C^\infty((-\frac{1}{2}, -\frac{1}{2}) \cup (\frac{1}{2}, 2))$ is a suitable nonnegative even function. In view of the kernel representation (3.2), we break up the operator $SS^*$ as follows

$$SS^* F(t, x) = \int_{\mathbb{R}^{1+1}} K(t - t', x - x') F(t', x') \, dt' dx' := T_0 F(t, x) + \sum_{l \geq 1} T_l F(t, x),$$

where

$$T_0 F(t, x) = \int_{\mathbb{R}^{1+1}} \chi(t - t') K(t - t', x - x') F(t', x') \, dt' dx',$$

$$T_l F(t, x) = \int_{\mathbb{R}^{1+1}} \psi(2^{-l}(t - t')) K(t - t', x - x') F(t', x') \, dt' dx',$$

and thus the integral kernel of $T_l$ is given by

$$K_l(t - t', x - x') = \psi(2^{-l}(t - t')) K(t - t', x - x').$$
In order to estimate each term \( \| W T_l W \|_{\ell^p(L^2)} \), we use the following.

**Lemma 3.2** For \( l \geq 1 \) we have the following estimates with \( C \) independent of \( l \):

\[
\begin{align*}
\| W_1 T_l W_2 \|_{\ell^2(L^2)} & \leq C \| W_1 \|_{L^4_t L^2_x} \| W_2 \|_{L^4_t L^2_x}, \quad (3.5) \\
\| W_1 T_l W_2 \|_{\ell^4(L^2)} & \leq C 2\left(\frac{1}{p_1} - \frac{1}{p_2}\right) \| W_1 \|_{L^{p_1}_t L^2_x} \| W_2 \|_{L^{p_2}_t L^2_x}, \quad (3.6)
\end{align*}
\]

provided that \( p_1 \) and \( p_2 \) satisfy \((\frac{1}{p_1}, \frac{1}{p_2}) \in [0, \frac{1}{2}]^2 \) and

\[
\frac{1}{p_1} + \frac{1}{p_2} \geq \frac{1}{4}, \quad \frac{1}{p_1} - \frac{1}{2 p_2} \leq \frac{1}{4}, \quad \frac{1}{p_2} - \frac{1}{2 p_1} \leq \frac{1}{4}.
\]

**Proof of Lemma 3.2** For (3.5), since

\[
|K_l(t - t', x - x')| \lesssim \psi(2^{-l}(t - t'))|t - t'|^{-1/2}
\]

follows from (3.3), we may use Young’s convolution inequality to obtain

\[
\begin{align*}
\| W_1 T_l W_2 \|_{\ell^2(L^2)}^2 & = \int |W_1(t, x)|^2 |K_l(t - t', x - x')|^2 |W_2(t', x')|^2 \mathrm{d}t' \mathrm{d}x' \\
& \lesssim \| W_1 \|_{L^4_t L^2_x}^2 \| W_2 \|_{L^4_t L^2_x}^2
\end{align*}
\]

uniformly in \( l \).

For (3.6), first note that

\[
\| W_1 T_l W_2 \|_{\ell^4(L^2)}^4 = \| \overline{W_2 T_l^*} \|_{\ell^2(L^2)}^2 \| W_1 \|_{L^4_t L^2_x}^2 \| W_2 \|_{L^4_t L^2_x}^2,
\]

and after a few lines of computation we see that

\[
\overline{W_2 T_l^*} |W_1|^2 T_l W_2[F](t, x) = \int_{\mathbb{R}^{l+1}} \mathcal{K}_l(t, t'', x, x'') F(t'', x'') \mathrm{d}t'' \mathrm{d}x'',
\]

where the integral kernel \( \mathcal{K}_l(t, t'', x, x'') \) is given by

\[
\overline{W_2(t, x)} \int_{\mathbb{R}^{l+1}} K_l(t - t', x - x') |W_1(t', x')|^2 K_l(t' - t'', x' - x'') \mathrm{d}t' \mathrm{d}x' W_2(t'', x'').
\]

Thanks to (3.3) (and relabelling \( t = t_1 \) and \( t'' = t_4 \)) we see

\[
\begin{align*}
\| W_1 T_l W_2 \|_{\ell^4(L^2)}^4 & = \int_{\mathbb{R}^{l+1}} \int_{\mathbb{R}^{l+1}} |\mathcal{K}_l(t_1, t_4, x, x'')|^2 \mathrm{d}t_1 \mathrm{d}t_4 \mathrm{d}x'' \\
& \lesssim \int \int \| W_2(t_1, \cdot) \|_{L^2_x}^2 L_l(t_1, t_4)^2 \| W_2(t_4, \cdot) \|_{L^2_x}^2 \mathrm{d}t_1 \mathrm{d}t_4,
\end{align*}
\]
where

\[ L_l(t_1, t_4) = \int_{\mathbb{R}} 2^{-l/2} \psi(2^{-l}(t_1 - t')) \| W_1(t', \cdot) \|_{L^2_x}^2 2^{-l/2} \psi(2^{-l}(t' - t_4)) \, dt'. \]

By multiplying out the square of \( L_l(t_1, t_4) \), we have

\[ \left\| W_1 T_l W_2 \right\|_{C^4(L^2)}^4 \lesssim \Lambda_l(h_2, h_1, h_1, h_2), \tag{3.7} \]

where \( h_k(t) = \left\| W_k(t, \cdot) \right\|_{L^2_x}^2, k = 1, 2 \), and \( \Lambda_l \) is the 4-linear form given by

\[ \Lambda_l(g_1, g_2, g_3, g_4) := 2^{-2l} \int_{\mathbb{R}^4} \psi(2^{-l}(t_1 - t_2)) \psi(2^{-l}(t_1 - t_3)) \times \psi(2^{-l}(t_2 - t_4)) \psi(2^{-l}(t_3 - t_4)) \prod_{i=1}^4 g_i(t_i) \, dt_i. \]

We shall prove the desired estimates (3.6) at the three points \((p_1, p_2) = (2, 2), (4, \infty), \) and \((\infty, 4)\), and then employ multilinear interpolation to deduce the estimates in the claimed region. The estimate at \((p_1, p_2) = (2, 2)\) follows immediately from (3.7) and the fact that \( \| \psi \|_{L^\infty} \lesssim 1 \). Thus, by symmetry, it suffices to prove (3.6) at \((p_1, p_2) = (4, \infty)\).

From \( \| \psi \|_{L^\infty} \lesssim 1 \) and \( \| \psi(2^{-l}\cdot) \|_{L^1} \sim 2^l \), we see

\[ |\Lambda_l(g_1, g_2, g_3, g_4)| \lesssim 2^{-2l} \| g_1 \|_{L^\infty} \| g_3 \|_{L^\infty} \| g_4 \|_{L^\infty} \]
\[ \times \int_{\mathbb{R}^4} \psi(2^{-l}(t_1 - t_2)) \psi(2^{-l}(t_1 - t_3)) \psi(2^{-l}(t_2 - t_4)) g_2(t_2) \prod_{i=1}^4 dt_i \]
\[ \sim 2^l \| g_1 \|_{L^\infty} \| g_2 \|_{L^1} \| g_3 \|_{L^\infty} \| g_4 \|_{L^\infty}. \]

By symmetry, we also have

\[ |\Lambda_l(g_1, g_2, g_3, g_4)| \lesssim 2^l \| g_1 \|_{L^\infty} \| g_2 \|_{L^\infty} \| g_3 \|_{L^1} \| g_4 \|_{L^\infty}, \]

and so it follows from interpolation between these two bounds that

\[ |\Lambda_l(g_1, g_2, g_3, g_4)| \lesssim 2^l \| g_1 \|_{L^\infty} \| g_2 \|_{L^2} \| g_3 \|_{L^2} \| g_4 \|_{L^\infty}. \]

In particular, from (3.7) this implies

\[ \left\| W_1 T_l W_2 \right\|_{C^4(L^2)}^4 \lesssim 2^l \| h_1 \|_{L^2}^2 \| h_2 \|_{L^\infty}^2 = 2^l \| W_1 \|_{L^4_x L^4_x}^4 \| W_2 \|_{L^\infty_x L^\infty_x}^4 \]

which gives (3.6) at \((p_1, p_2) = (4, \infty)\). \(\Box\)
Proof of Proposition 3.1} By interpolating between (3.5) and (3.6), for each \( 2 < \beta' \leq 4 \), there exists \( \delta(\beta') > 0 \) such that \( \lim_{\beta' \to 2} \delta(\beta') = 0 \) and that the estimate

\[
\| W_{1} T_{1} W_{2} \|_{c^{\beta'}(L^{2})} \lesssim 2^{\left(1 - \frac{1}{p_{1}} - \frac{1}{p_{2}}\right)l_{1}} \| W_{1} \|_{L^{p_{1}}L^{2}} \| W_{2} \|_{L^{p_{2}}L^{2}} \tag{3.8}
\]

holds for all \( p_{1}, p_{2} \) satisfying \( |(\frac{1}{p_{1}}, \frac{1}{p_{2}}) - (\frac{1}{4}, \frac{1}{4})| \leq \delta(\beta') \).

Now fix \( \beta_{*} < 2 \) such that \( \beta_{*} \in (2, 4) \) and define the bilinear operator \( \mathcal{J} \) by

\[
\mathcal{J}(W_{1}, W_{2}) := (W_{1} T_{1} W_{2})_{l \geq 1}.
\]

Then (3.8) shows \( \mathcal{J} : L^{p_{1}}L^{2}_{x} \times L^{p_{2}}L^{2}_{x} \to \ell^{\infty}_{\mu(p_{1}, p_{2})}(c^{\beta_{*}}(L^{2})) \) is bounded for all \((\frac{1}{p_{1}}, \frac{1}{p_{2}}) \) in \( \delta(\beta_{*}) \)-neighborhood of \((\frac{1}{4}, \frac{1}{4}) \). Here, \( \mu(p_{1}, p_{2}) = \frac{1}{p_{1}} + \frac{1}{p_{2}} - \frac{1}{2} \) and, for a general Banach space \( X \) and sequence \((g_{l})_{l} \subset X \), the norm is given by

\[
\| (g_{l})_{l} \|_{\ell^{p}_{\mu}(X)} = \left( \sum_{l} 2^{p \mu l} \| g_{l} \|_{X}^{p} \right)^{1/p}
\]

for \( p < \infty \), and \( \| (g_{l})_{l} \|_{\ell^{p}_{\mu}(X)} = \sup_{l} 2^{\mu l} \| g_{l} \|_{X} \). Therefore, a bilinear real interpolation argument (see [4, Exercise 5, Page 76]) reveals that \( \mathcal{J} : L^{4,2}L^{2}_{x} \times L^{4,2}L^{2}_{x} \to \ell^{1}(c^{\beta_{*}}(L^{2})) \) is bounded, which in particular means

\[
\sum_{l \geq 1} \| W_{T_{l}} W \|_{c^{\beta_{*}}(L^{2})} \lesssim \| W \|_{L^{4,2}L^{2}_{x}}^{2}.
\]

Finally, we consider the operator \( W_{1} T_{0} W_{2} \). Note that, in a similar manner to the proof of Lemma 3.2(1), we have

\[
\| W_{T_{0}} W \|_{c^{\beta_{*}}(L^{2})}^{2} = \int |W(t, x)|^{2} \chi^{2}(t - t') |K(t - t', x - x')|^{2} |W(t', x')|^{2} \, dtd'dx'dx'
\]

\[
\lesssim \int \| W(t, \cdot) \|_{L^{4}_{x}}^{2} \chi^{2}(t - t') \| W(t', \cdot) \|_{L^{4}_{x}}^{2} \, dtd'
\]

\[
\lesssim \| W \|_{L^{4}_{x}L^{4}_{x}}^{4}
\]

which certainly implies

\[
\| W_{T_{0}} W \|_{c^{\beta_{*}}(L^{2})} \lesssim \| W \|_{L^{4,2}L^{2}_{x}}^{2}
\]

via embeddings. Hence, by the triangle inequality, we have

\[
\| W_{S} S^{*} W \|_{c^{\beta_{*}}(L^{2})} \leq \| W_{T_{0}} W \|_{c^{\beta_{*}}(L^{2})} + \sum_{l \geq 1} \| W_{T_{l}} W \|_{c^{\beta_{*}}(L^{2})} \lesssim \| W \|_{L^{4,2}L^{2}_{x}}^{2} \tag{3.9}
\]

which yields our goal (3.4) for all \( \beta_{*} \in (2, 4) \). This suffices to prove Proposition 3.1. \( \square \)
Proof of Theorem 1.4  As indicated at the start of Sect. 3, for the sufficiency part of Theorem 1.4, we obtain (3.1) as an immediate consequence of Proposition 3.1 with $S = U_2 P$, and a standard rescaling argument to remove the frequency cut-off $P$ yields (1.14).

To see the necessity of $\beta < 2$, it is enough to show the failure of (1.14) with $\beta = 2$ thanks to the inclusion relation between Schatten spaces. Suppose (1.14) holds true with $\beta = 2$, then from Proposition 2.1, we also have

$$
\|W_1 U_2 U_2^* W_2\|_{C^2(L^2)} \lesssim \|W_1\|_{L_t^{4,2} L_x^2} \|W_2\|_{L_t^{4,2} L_x^2}.
$$

On the other hand, the kernel of $UU^*$ is given by $C|t - t'|^{-\frac{1}{2}} e^{-\frac{|x - x'|^2}{4(t - t')}}$ and therefore

$$
\|W_1 U_2 U_2^* W_2\|^2_{C^2(L^2)} = C \int_{\mathbb{R}^{1+1}} \int_{\mathbb{R}^{1+1}} |W_1(t, x)|^2 |t - t'|^{-\frac{1}{2}} |W_2(t', x')|^2 \, dt \, dx \, dt' \, dx'.
$$

Choosing $W_1 = W_2 = 1_{[-1,1]}$, we get a contradiction because $\|W_1 U_2 U_2^* W_2\|_{C^2(L^2)} = \infty$. This establishes the necessity of the condition $\beta < 2$ for (1.14). \(\square\)

Remark (1)  Ideas in the proof of Proposition 3.1 yield the bound

$$
\left\| \sum_j v_j e^{i\xi \partial_x^2} f_j \right\|_{L_t^{\frac{4}{3}} L_x^{\frac{4}{3}}(\mathbb{R}^{1+1})} \lesssim \|v\|_{\ell^\frac{4}{3}} \tag{3.10}
$$

for orthonormal systems $(f_j)_j$ in $L^2(\mathbb{R})$ and $v = (v_j)_j$ in $\ell^\frac{4}{3}$, which, in particular, is a Lorentz-space improvement of the strong-type estimate

$$
\left\| \sum_j v_j e^{i\xi \partial_x^2} f_j \right\|_{L_t^2 L_x^\infty(\mathbb{R}^{1+1})} \lesssim \|v\|_{\ell^\frac{4}{3}}.
$$

Indeed, if one computes $\|WU_2 U_2^* W\|_{C^4(L^2)}$ directly as we did in the above, then we obtain

$$
\|WU_2 U_2^* W\|^4_{C^4} \lesssim \int_{\mathbb{R}^4} |t_1 - t_2|^{-\frac{1}{2}} |t_1 - t_3|^{-\frac{1}{2}} |t_2 - t_4|^{-\frac{1}{2}} |t_3 - t_4|^{-\frac{1}{2}} \prod_{i=1}^4 h(t_i) \, dt_i, \tag{3.11}
$$

where $h(t) = \|W(t, \cdot)\|^2_{L^2_x}$. We regard the right-hand side of (3.11) as an 8-linear rank-one Brascamp–Lieb form and we may use Barthe’s characterisation in [2] of the associated Brascamp–Lieb polytope in the rank-one case and Christ’s observations in [35] on extending classical Brascamp–Lieb estimates to Lorentz spaces to conclude

$$
\|WU_2 U_2^* W\|_{C^4(L^2)} \lesssim \|W\|^2_{L_t^{4,8} L_x^2}, \tag{3.12}
$$

or equivalently, (3.10). We refer the reader to [3,6,9,10] for further details regarding the Brascamp–Lieb inequality and its Lorentz space refinement.
(II) If one can appropriately exploit the orthogonality of the $T_l$, rather than the application of the triangle inequality in (3.9), it seems possible to upgrade (1.14) to a strong-type estimate. For instance, it seems reasonable to expect that for all $\beta \in [1, 2]$ we have

$$
\| WU P^2 U^* W \|_{C^\beta(L^2)} \leq \| W T_0 W \|_{C^\beta(L^2)} + \left( \sum_{l \geq 1} \| W T_l W \|_{C^\beta(L^2)}^\beta \right)^{1/\beta}.
$$

(3.13)

Indeed, if $\beta = 2$ it is easy to see that (3.13) holds in this case. Also, (3.13) for $\beta = 1$ is an easy consequence of the triangle inequality. However, it is not clear to us how to interpolate these two estimates.

Under the assumption that (3.13) holds for all $\beta \in [1, 2]$, it follows from our argument to show (1.14) that

$$
\| W U U^* W \|_{C^\beta(L^2)} \lesssim \| W \|_{L^4, 2}^2, 
$$

(3.14)

for $\beta < 2$ arbitrary close to 2. On the other hand, one has a Lorentz improvement if $\beta' = 4$ as in (3.12). Interpolating (3.14) and (3.12), one would obtain the desired strong-type estimate for any $\beta < 2$.

(III) Assuming the more general decay hypothesis

$$
\sup_{x \in \mathbb{R}} |K(t, x)| \lesssim (1 + |t|)^{-\sigma} \quad (t \in \mathbb{R})
$$

for some $\sigma > 0$, one may easily generalize our argument in the proof of Proposition 3.1 to obtain

$$
\left\| \sum_j v_j|Sf_j|^2 \right\|_{L^{q/2, \infty}_t L^{\infty}_x(\mathbb{R}^{1+1})} \lesssim \| v \|_{\ell^\beta}
$$

for orthonormal systems $(f_j)_j$ in $L^2(\mathbb{R})$ and $v = (v_j)_j$ in $\ell^\beta$, where $q = \max\{\frac{2}{\sigma}, 4\}$ and $\beta < 2$. It is also clear from an inspection of the proof that the domain of the spatial variable may be generalized.

4 Proofs of Theorem 1.1 and Corollary 1.2

Recalling the identity (2.1), our first step in establishing Theorem 1.1 is to observe the following analogue of Theorem 1.4.

**Theorem 4.1** Suppose $\beta < 2$. Then the estimate

$$
\left\| \sum_j v_j U_\frac{1}{2} |\partial_x|^{-\frac{3}{8}} f_j \right\|_{L^2_t, \infty L^\infty_x(\mathbb{R}^{1+1})} \lesssim \| v \|_{\ell^\beta}
$$

holds for all systems of orthonormal functions $(f_j)_j$ in $L^2(\mathbb{R})$ and $v = (v_j)_j$ in $\ell^\beta$. 
Proof We invoke Proposition 3.1 with \( S = U \frac{1}{2} |\partial_x|^{-\frac{3}{8}} P \). It is clear that (3.2) holds with
\[
K(x, t) = \int_{\mathbb{R}} \chi(\xi)^2 |\xi|^{-3/4} e^{i(x\xi + t|\xi|^{1/2})} \, d\xi,
\]
and the desired decay estimate (3.3) holds thanks to work of Kenig–Ponce–Vega [26, Lemma 2.7].

4.1 Proof of Theorem 1.1 (Sufficiency part)

A minor snag which arises when using (2.1) is that the transformation \( f \mapsto f_\pm \) does not preserve the orthogonal structure and the orthonormality of \((f_j)_j\) does not always ensure the orthonormality of \((f_j,+)_j\) and \((f_j,-)_j\). To recover this, we introduce the reflection operator \( R \) given by
\[
R \varphi(t, x) := \varphi(-t, -x)
\]
and establish the following.

Lemma 4.2 (I) For each \( f \in L^2 \),
\[
\sqrt{2}(1 \pm R)U_2 |\partial_x|^{-\frac{3}{8}} f(t, x) = e^{ix|\partial_x|^{\frac{1}{2}} |\partial_t|^{-\frac{3}{8}} f_\pm(t)} + e^{-ix|\partial_x|^{\frac{1}{2}} |\partial_t|^{-\frac{3}{8}} f_\pm^*(t)},
\]
where
\[
\hat{f}_\pm(\eta) = \frac{1}{\sqrt{2}} |\eta|^{-\frac{1}{4}} (I_{(0, \infty)}(\eta) \hat{f}(|\eta|^{\frac{1}{2}}) \pm I_{(-\infty, 0)}(\eta) \hat{f}(-|\eta|^{\frac{1}{2}}))
\]
\[
\hat{f}_{\pm}^*(\eta) = \frac{1}{\sqrt{2}} |\eta|^{-\frac{1}{4}} (I_{(0, \infty)}(\eta) \hat{f}(-|\eta|^{\frac{1}{2}}) \pm I_{(-\infty, 0)}(\eta) \hat{f}(|\eta|^{\frac{1}{2}}))
\]
(II) Suppose \((f_j)_j\) is a orthonormal system in \( L^2 \). Then each of the families \((f_j,+)_j\), \((f_j,-)_j\), \((f_j^*)_j\) and \((f_j^*)_j\) is orthonormal in \( L^2 \).

Proof First we show (4.1). Following the idea in (2.1), we have
\[
2RU_2 |\partial_x|^{-\frac{3}{4}} f(t, x) = 2e^{it\partial_x^2} |\partial_x|^{-\frac{3}{4}} [f(-\cdot)](x)
\]
\[
= \frac{1}{2\pi} \int_0^\infty e^{ix|\eta|^{\frac{1}{2}}} e^{-i\eta t} |\eta|^{-\frac{5}{8}} \hat{f}(-|\eta|^{\frac{1}{2}}) \, d\eta + \frac{1}{2\pi} \int_0^\infty e^{-ix|\eta|^{\frac{1}{2}}} e^{-i\eta t} |\eta|^{-\frac{5}{8}} \hat{f}(|\eta|^{\frac{1}{2}}) \, d\eta
\]
\[
= \frac{1}{2\pi} \int_{-\infty}^0 e^{ix|\eta|^{\frac{1}{2}}} e^{i\eta t} |\eta|^{-\frac{5}{8}} \hat{f}(-|\eta|^{\frac{1}{2}}) \, d\eta + \frac{1}{2\pi} \int_{-\infty}^0 e^{-ix|\eta|^{\frac{1}{2}}} e^{i\eta t} |\eta|^{-\frac{5}{8}} \hat{f}(|\eta|^{\frac{1}{2}}) \, d\eta,
\]
from which we obtain (4.1).
Next, let us see the orthonormality. Using Parseval’s identity,

\[ 4\pi \langle f_{j,+}, f_{k,+} \rangle_{L^2} = \int_\mathbb{R} |\eta|^{-\frac{1}{2}} \left( \mathbf{1}_{(0,\infty)}(\eta) \hat{f}_j(\eta) \frac{1}{\sqrt{2}} \hat{f}_k(\eta) + \mathbf{1}_{(-\infty,0)}(\eta) \hat{f}_j(-\eta) \frac{1}{\sqrt{2}} \hat{f}_k(-\eta) \right) d\eta \]

\[ = \int_0^\infty |\eta|^{-\frac{1}{2}} \left( \hat{f}_j(\eta) \frac{1}{\sqrt{2}} \hat{f}_k(\eta) + \hat{f}_j(-\eta) \frac{1}{\sqrt{2}} \hat{f}_k(-\eta) \right) d\eta, \]

where we performed a simple change of the variable for the second term in the last equality. Then the change of variable \( \xi = \eta \) yields

\[ 4\pi \langle f_{j,+}, f_{k,+} \rangle_{L^2} = 2 \int_0^\infty \left( \hat{f}_j(\xi) \frac{1}{\sqrt{2}} \hat{f}_k(\xi) + \hat{f}_j(-\xi) \frac{1}{\sqrt{2}} \hat{f}_k(-\xi) \right) d\xi = 4\pi \langle f_j, f_k \rangle_{L^2}, \]

and hence \( \langle f_{j,+}, f_{k,+} \rangle_{L^2} = \langle f_j, f_k \rangle_{L^2} \). By a very similar calculation we also have \( \langle f_{j,-}, f_{k,-} \rangle_{L^2} = \langle f_j, f_k \rangle_{L^2} \) Finally, since \( f^*_\pm = f(\cdot)\pm \), we may deduce the corresponding identities for \( (f^*_j,+) \) and \( (f^*_j,-) \).

**Proof of the sufficiency part of Theorem 1.1** By writing \( 2U_2 = (1+R)U_2 + (1-R)U_2 \), and applying Lemma 4.2 and the triangle inequality, we have

\[ \left\| \sum_j \nu_j |U_2| \partial_x |\sqrt{2} f_j| \right\|_{L^2_t L^\infty_x} \lesssim N_+ + N_+^* + N_- + N_-^* \]

where

\[ N_\pm := \left\| \sum_j \nu_j e^{ix|\partial_t|^{3/8}} |\partial_t|^{-\frac{3}{8}} f_{j,\pm}| \right\|_{L^2_t L^\infty_x}, \]

\[ N_\pm^* := \left\| \sum_j \nu_j e^{-ix|\partial_t|^{3/8}} |\partial_t|^{-\frac{3}{8}} f_{j,\pm}^*| \right\|_{L^2_t L^\infty_x}. \]

Hence, applying Theorem 4.1, for \( \beta < 2 \) we get

\[ \left\| \sum_j \nu_j |U_2| \partial_x |\sqrt{2} f_j| \right\|_{L^2_t L^\infty_x} \lesssim \|\nu\|_{\ell^\beta}. \]
4.2 Proof of Theorem 1.1 (Sharpness)

Our goal is to show that the estimate
\[
\left\| \sum_j v_j |e^{i\alpha^2 t} |\partial_x|^\frac{1}{2} f_j |^2 \right\|_{L^2_x L^\infty_t(\mathbb{R}^{1+1})} \lesssim \|v\|_{L^{2,1}}
\]
for systems of orthonormal functions \((f_j)_j\) in \(L^2(\mathbb{R})\) and \(v = (v_j)_j\) in \(L^{2,1}\) is false. If this estimate were true, by a semi-classical limiting argument, we may induce the following maximal estimate for the (weighted) velocity average of the kinetic transport equation
\[
\left\| \int_{\mathbb{R}} f(x - tv, v) \frac{dv}{|v|^{1/2}} \right\|_{L^2_x L^\infty_t(\mathbb{R}^{1+1})} \lesssim \|f\|_{L^{2,1}_{x,v}} \quad (4.2)
\]
for any \(f \in L^{2,1}_{x,v}(\mathbb{R}^{1+1})\). We refer the reader to [5,36] for further details of such a limiting procedure.

Proof that (4.2) fails Suppose \(N \subset [-10,10]^2\) has Lebesgue measure zero and contains a unit line segment whose angle from the vertical line is at most \(\pi/4\) through every point of \(\{(x,0) : x \in [-1,1]\}\). Sets with the latter geometric property are often referred to as Nikodym sets and the existence of such sets with Lebesgue measure zero goes back to [34] (see also [44] for further discussion and an explicit construction).

Let us denote the \(\delta\)-neighbourhood of \(N\) by \(N_\delta\) for each \(\delta > 0\); we shall test \((4.2)\) on the characteristic function \(f = 1_{N_\delta}\). Thanks to the geometric property of the Nikodym set \(N\), for any \(x \in [-1,1]\) there exist \(t(x) \in \mathbb{R}\) and a unit interval \(I(x)\) such that
\[
\{(x - v(-t(x)), 1) : v \in I(x)\}
\]
is contained in \(N\). Here, we use the notation \(\omega' := |\omega|^{-1}\omega\). Note that, \(t(x) \in [-1,1]\), thanks to the restriction of the angle to the vertical, and furthermore we have \(I(x) \subset [-20,20]\) since \(N \subset [-10,10]^2\). Thus, for any \(x \in \mathbb{R}\), the above yields
\[
\left\| \int_{\mathbb{R}} 1_{N_\delta}(x - tv, v) \frac{dv}{|v|^{1/2}} \right\|_{L^\infty_t} \geq 1_{[-1,1]}(x) \int_{\mathbb{R}} 1_{N_\delta}(x - t(x)v, v) \frac{dv}{|v|^{1/2}} \gtrsim 1,
\]
and hence
\[
\left\| \int_{\mathbb{R}} 1_{N_\delta}(x - tv, v) \frac{dv}{|v|^{1/2}} \right\|_{L^2_x L^\infty_t} \gtrsim 1,
\]
with implicit constants uniform in \(\delta > 0\). On the other hand, since \(N\) has zero Lebesgue measure, we have \(\|1_{N_\delta}\|_{L^{2,1}_{x,v}} \sim |N_\delta|^{1/2} \to 0\) as \(\delta \to 0\). This establishes that \((4.2)\) is false. \(\Box\)

Whilst it was rather easier to establish the failure of the case \(\beta = 2\) in Theorem 1.4 by using duality and explicit computations using the Hilbert–Schmidt norm, it seems
unclear how to proceed along similar lines for the necessity part of Theorem 1.1. More precisely, in order to see that (1.6) fails with \( \beta = 2 \), by duality it suffices to show the failure of the estimate

\[
\| W_1 |\partial_x|^{-\frac{1}{4}} U_2 U_2^* |\partial_x|^{-\frac{1}{4}} W_2 \|_{C^2(L^2)} \lesssim \| W_1 \|_{L^4_2 L_t^2} \| W_2 \|_{L^4_2 L_t^2}.
\]

Thanks to the presence of the derivatives \( |\partial_x|^{-\frac{1}{4}} \), however, as far as we aware, it does not seem easy to have a convenient formula for the integral kernel of \( |\partial_x|^{-\frac{1}{4}} U_2 U_2^* |\partial_x|^{-\frac{1}{4}} \).

The alternative approach we took in the above using a semi-classical limiting argument circumvents this issue and moreover allows us to show the failure of the restricted weak-type estimate.

### 4.3 Proof of Corollary 1.2

For a given Hilbert space \( \mathcal{H} \) and a unit vector \( g \in \mathcal{H} \), we define \( \Pi_g : \mathcal{H} \to \mathcal{H} \) to be the orthogonal projection onto the span of \( g \) given by\( \Pi_g \phi := \langle \phi, g \rangle g \). Note that for any compact operator \( \gamma_0 \) on \( \mathcal{H} \), in particular \( \gamma_0 \in C^\beta(\mathcal{H}), \beta < 2 \), one can find \( (\nu_j) \) and orthonormal system \( (g_j) \) in \( \mathcal{H} \) such that \( \gamma_0 = \sum_j \nu_j \Pi_{g_j} \) thanks to the singular value decomposition.

For \( \gamma_0 \in C(\dot{H}^{1/4}(\mathbb{R})) \) and its evolution \( \gamma(t) = e^{-it\partial_x^2} \gamma_0 e^{it\partial_x^2} \) under (1.7), first we clarify the meaning of the density functions \( \rho_{\gamma_0} \) and \( \rho_{\gamma(t)} \). In the finite-rank case \( \gamma_0 = \sum_{j=1}^N v_j \Pi_{g_j} \), the integral kernel is given by

\[
(x, y) \mapsto \sum_{j=1}^N v_j g_j(x) g_j(y)
\]

and thus we have

\[
\rho_{\gamma_0}(x) = \sum_{j=1}^N v_j |g_j(x)|^2.
\]

In the infinite-rank case, some care is required and we proceed via Lieb’s generalization of the Sobolev inequality

\[
\left\| \sum_j v_j |\partial_x|^{-\frac{1}{4}} f_j |^2 \right\|_{L^2(\mathbb{R})} \lesssim \| v \|_{\ell^1} \| v \|_{\ell^\infty} \tag{4.3}
\]

for orthonormal systems \( (f_j) \) in \( L^2(\mathbb{R}) \) and coefficients \( v = (v_j) \) in \( \ell^1 \cap \ell^\infty \) (see [31]). We may replace the right-hand side of (4.3) by \( \| v \|_{\ell^2} \) (using, for example, [42, Ch. 5, Theorem 3.13]) and, in view of the inclusion \( \ell^\beta \subseteq \ell^{2,1} \) for any \( \beta < 2 \), we

\[\text{If } \beta < 2 \text{ and } (v_j^*) \text{ is the sequence } (|v_j|) \text{ permuted in a decreasing order, we have } \| v \|_{\ell^\beta} \lesssim (\sum_{j \geq 1} (v_j^*)^\beta^\prime \cdot j^{-\beta^\prime/2} \cdot j^{-\beta^\prime/2})^{1/\beta^\prime} \lesssim \sup_{j \geq 1} j^{1/2} v_j^* = \| v \|_{\ell^2,\infty} \text{ and therefore, by duality, } \ell^\beta \subseteq \ell^{2,1}.\]
have
\[ \left\| \sum_j v_j |\partial_x|^{-\frac{1}{2}} f_j \right\|_{L^2(\mathbb{R})} \lesssim \|v\|_{\ell^\beta} \tag{4.4} \]
for orthonormal systems \((f_j)_j\) in \(L^2(\mathbb{R})\), \(v = (v_j)_j\) in \(\ell^\beta\), and \(\beta < 2\).

We now fix \(\beta < 2\) and approximate \(\gamma_0 = \sum_{j=1}^{\infty} v_j \Pi_\gamma g_j \in \mathcal{C}^{\beta}(\dot{H}^{\frac{1}{2}}(\mathbb{R}))\), for \(v \in \ell^\beta\) and orthonormal vectors \(g_j \in \dot{H}^{\frac{1}{2}}(\mathbb{R})\), by the sequence of finite-rank operators \((\gamma_0^N)_N\) given by \(\gamma_0^N = \sum_{j=1}^{N} v_j \Pi_\gamma g_j\). For \(M > N\), we obtain
\[ \|\rho_{\gamma_0^M} - \rho_{\gamma_0^N}\|_2 = \left\| \sum_{j=N+1}^{M} v_j |g_j|^2 \right\|_2 \lesssim \left( \sum_{j=N+1}^{M} |v_j|^\beta \right)^{\frac{1}{\beta}} \]
from (4.4), and therefore \((\rho_{\gamma_0^N})\) is a Cauchy sequence in \(L^2(\mathbb{R})\). Thus, we define \(\rho_{\gamma_0} = \sum_{j=1}^{\infty} v_j |g_j|^2 \in L^2(\mathbb{R})\) as the limit of \((\rho_{\gamma_0^N})\) in \(L^2(\mathbb{R})\). Since orthonormality of \((f_j)_j\) is preserved under the action of \(e^{it\partial_x^2}\) for each \(t \in \mathbb{R}\), we may repeat the above to define the density function \(\rho_{\gamma(t)} = \sum_{j=1}^{\infty} v_j |e^{it\partial_x^2} g_j|^2 \in L^2_x(\mathbb{R}).\)

**Proof of Corollary 1.2** Fix \(\beta < 2\) and \(\gamma_0 \in \mathcal{C}^{\beta}(\dot{H}^{\frac{1}{2}}(\mathbb{R}))\), and let \(\gamma(t) = e^{-it\partial_x^2} \gamma_0 e^{it\partial_x^2}\).

Clearly, it suffices to prove
\[ \lim_{t \to 0} \sup \|\rho_{\gamma(t)} - \rho_{\gamma_0}\|_{L^2_{t,x}} = 0. \tag{4.5} \]
As in the discussion preceding this proof, we approximate \(\gamma_0 = \sum_{j=1}^{\infty} v_j \Pi_\gamma g_j\) by the finite-rank operator \(\gamma_0^N = \sum_{j=1}^{N} v_j \Pi_\gamma g_j\), and define \(\gamma^N(t) = e^{-it\partial_x^2} \gamma_0^N e^{it\partial_x^2}\). Then we claim that
\[ \lim_{N \to \infty} \|\rho_{\gamma(t)} - \rho_{\gamma^N(t)}\|_{L^2_{t,x} L^\infty} = 0. \tag{4.6} \]
To see this we make use of Theorem 1.1 as follows. By (1.6), we have
\[
\|\rho_{\gamma(t)} - \rho_{\gamma^N(t)}\|_{L^2_{t,x} L^\infty} = \left\| \sum_{j=N+1}^{\infty} v_j |e^{it\partial_x^2} g_j|^2 \right\|_{L^2_{t,x} L^\infty} \lesssim \left( \sum_{j=N+1}^{\infty} |v_j|^\beta \right)^{\frac{1}{\beta}}.
\]
Since \(\gamma_0 \in \mathcal{C}^{\beta}(\dot{H}^{\frac{1}{2}}(\mathbb{R}))\) we have \(v \in \ell^\beta\) and hence (4.6) follows.

From the definition of \(\rho_{\gamma_0}\) and (4.6), for any \(\varepsilon > 0\), we can find \(N_\varepsilon\) such that
\[
\|\rho_{\gamma_0} - \rho_{\gamma_0^{N_\varepsilon}}\|_{L^2_{t,x}}, \|\rho_{\gamma(t)} - \rho_{\gamma(t)^{N_\varepsilon}}\|_{L^2_{t,x} L^\infty} < \varepsilon.
\]
For such \( N_\varepsilon \), we have

\[
\| \limsup_{t \to 0} |\rho_y(t) - \rho_{y_0}| \|_2^2 \infty \leq \| \limsup_{t \to 0} |\rho_y(t) - \rho_{y_N} - \rho_{y_{N_0}}| \|_2^2 \infty + \| \limsup_{t \to 0} |\rho_y - \rho_{y_{N_0}} - \rho_{y_0}| \|_2^2 \infty \leq 2\varepsilon + \| \limsup_{t \to 0} |\rho_y - \rho_{y_{N_0}} - \rho_{y_0}| \|_2^2 \infty.
\]

Since \( g_j \in \dot{H}^{\frac{1}{4}}(\mathbb{R}) \), it follows from Carleson’s result in [11] that

\[
\limsup_{t \to 0} \rho_y\partial_x^2(x) = \sum_{j=1}^{N_\varepsilon} v_j \limsup_{t \to 0} |e^{it\partial_x^2} g_j(x)|^2 = \sum_{j=1}^{N_\varepsilon} v_j |g_j(x)|^2 = \rho_y\partial_x^2(x)
\]

holds almost everywhere. Hence \( \| \limsup_{t \to 0} |\rho_y - \rho_{y_{N_0}} - \rho_{y_0}| \|_2^2 \infty = 0 \) and we obtain (4.5).

## 5 Additional remarks

### 5.1 Carleson’s problem with data in Besov spaces

Even though the Sobolev regularity \( 1/4 \) in the classical version of Carleson’s problem (1.2) is the optimal one, it still seems plausible to obtain a further refinement of the estimate (1.3), in particular, with data in the Besov spaces \( \dot{B}^{1/4}_{2,2}\beta \). For \( \beta > 1 \), we have \( \dot{H}^{1/4} \subset \dot{B}^{1/4}_{2,\beta} \) and thus we would see an improvement in the classical results on Carleson’s pointwise convergence problem in the one-dimensional case. Although we are not able to answer this question here, we may quickly obtain the following related result as an additional application of Theorem 1.1.

**Proposition 5.1** Suppose \( \beta < 2 \). Then the estimate

\[
\| e^{it\partial_x^2} f \|_{L_x^2, BMO_t(\mathbb{R}^{1+1})} \leq C \| f \|_{\dot{B}^{1/4}_{2,2}\beta}
\]

(5.1)

holds for all \( f \in \dot{B}^{1/2}_{2,2}\beta \).

Unfortunately, BMO is strictly larger than \( L^\infty \). (For the definition of BMO and the homogeneous Besov spaces \( \dot{B}^{\frac{1}{2},q}_{p,q} \), we refer the reader to [41].)

**Proof of Proposition 5.1** For each \( j \in \mathbb{Z} \), let \( P_j \) denote the frequency projection operator (with respect to the spatial variable) given by

\[
\hat{P_j f}(\xi) = \varphi(2^{-j} \xi) \hat{f}(\xi).
\]
where $\varphi \in C^\infty_c([-4, 4]\setminus [-\frac{1}{4}, \frac{1}{4}])$ is chosen such that $\varphi \equiv 1$ on $[-2, 2]\setminus [-\frac{1}{2}, \frac{1}{2}]$ and $\sum_{j \in \mathbb{Z}} \varphi(2^{-j} \xi) = 1$ for $\xi \neq 0$. Also, we let $Q_j$ denote the frequency projection operator (with respect to the temporal variable) given by

$$
\widehat{Q_j f}(\tau) = \theta(2^{-j} \tau) \hat{f}(\tau),
$$

where $\theta$ is a similarly chosen bump function which satisfies $\theta \equiv 1$ on $[-16, 16]\setminus [-\frac{1}{16}, \frac{1}{16}]$.

For each fixed $x \in \mathbb{R}$, the support of the (temporal) Fourier transform of $t \mapsto e^{it\partial_x^2} P_j f(x)$ is contained in $\{\tau \in \mathbb{R} : |\tau| \in [2^{2j-4}, 2^{2j+4}]\}$ and therefore the function $t \mapsto e^{it\partial_x^2} P_j f(x)$ is invariant under the action of $Q_{2j}$. It follows from this and the Littlewood–Paley inequality in the temporal variable that

$$
\|e^{it\partial_x^2} f\|_{L^4_x \cap \dot{H}^1} = \left\| \sum_{j \in \mathbb{Z}} Q_{2j} (e^{it\partial_x^2} P_j f) \right\|_{L^4_x \cap \dot{H}^1} \lesssim \left( \sum_{j \in \mathbb{Z}} |Q_{2j} (e^{it\partial_x^2} P_j f)|^2 \right)^{\frac{1}{2}} \lesssim 3 \left( \sum_{j \in \mathbb{Z} + k} |e^{it\partial_x^2} P_j f|^2 \right)^{\frac{1}{2}}.
$$

For each fixed $k = 0, 1, 2, 3$, it is readily checked that $(P_j f/\|P_j f\|_{\dot{H}^\frac{1}{4}})_{j \in \mathbb{Z} + k}$ forms an orthonormal system, and thus Theorem 1.1 implies

$$
\left( \sum_{j \in \mathbb{Z} + k} |e^{it\partial_x^2} P_j f|^2 \right)^{\frac{1}{2}} \lesssim \left( \sum_{j \in \mathbb{Z} + k} \|P_j f\|_{\dot{H}^\frac{1}{4}}^{2\beta} \right)^{\frac{1}{2\beta}},
$$

from which we obtain (5.1).

\section*{5.2 Other dispersion relations}

For simplicity of the exposition, we have stated our main results in the Introduction in terms of the classical Schrödinger operator $U_2$. However, an inspection of our proofs of Theorems 1.1 and 1.4 reveal that generalization to a wider class of dispersive equations is possible with straightforward modifications. As a concrete example, Theorem 4.1 may be generalized to the statement that, for $a > 1$ and $\beta < 2$, we have

$$
\left\| \sum_j \nu_j |U_a^1 \partial_x|^{-\frac{2a-1}{4a}} f_j \right\|_{L^2_x \cap \dot{H}^\frac{1}{4}} \lesssim \|\nu\|_{\ell^\beta}
$$

for all systems of orthonormal functions $(f_j)_j$ in $L^2(\mathbb{R})$ and $\nu = (\nu_j)_j$ in $\ell^\beta$. It is also clear that the identity (2.1) may be appropriately modified to relate $U_a$ with $U_{1/a}$.
with the roles of space and time reversed, and consequently we may deduce that the estimates in Theorems 1.1 and 1.4 hold with $U_2$ replaced by $U_a$ for $a > 1$.

### 5.3 Further discussion

Finally we make additional comments regarding possible development in different directions.

(I) (Higher dimensions) Concerning the classical form of Carleson’s problem (1.2) in higher dimensions, Bourgain [8] showed the necessary regularity condition $s ≥ \frac{1}{2} - \frac{1}{2(d+1)}$, and Du–Guth–Li [19] ($d = 2$) and Du–Zhang [20] ($d ≥ 3$) recently proved that the condition $s > \frac{1}{2} - \frac{1}{2(d+1)}$ suffices, thus leaving open only the endpoint case. Their essentially definitive results built on a number of significant prior work including, for example, [7,27,32,33]. There are also numerous results on variants of Carleson’s problem; see [1,14,15,18,30,37–40]. We believe it is an interesting problem to extend Corollary 1.2 to higher dimensions. However, the arguments in [19,20] are very far from the ones we have used in this article, and obtaining a sharp version of (1.9) for higher dimensions looks very challenging.

(II) (Nonlinear equations) We considered the pointwise convergence (1.9) for the free solution $γ(t) = e^{-it\partial^2_x} γ_0 e^{it\partial^2_x}$ as a first step toward generalization of the classical maximal estimates to orthonormal systems of initial data. However, from the perspective of the quantum mechanics, it is more natural to consider (1.9) with $γ(t)$ which is a solution of the nonlinear equation (1.8). Related to this problem, we note that Compaan–Lucá–Staffilani [16] recently investigated the behavior of the solution to the nonlinear Schrödinger equation as $t → 0$.

(III) (Improving summability with higher regularity) Our pointwise convergence result in Corollary 1.2 is given under the optimal regularity assumption $s = \frac{1}{4}$. However, there is no reason to restrict ourselves to the specific regularity exponent when dealing with orthonormal systems of initial data. In fact, it seems to be natural to expect a gain of summability in the exponent $β$ by imposing higher regularity. Such kind of tradeoff between regularity and summability has been already observed in [5]. The problem of characterizing $β = β(s)$ for which the pointwise convergence (1.9) holds for $γ_0 ∈ C^β(H^s(\mathbb{R}))$ remains open. Corollary 1.2 only ensures that $β < 2$ is sufficient for all $s ≥ \frac{1}{4}$.

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