A Construction of Metabelian Groups

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Abstract. In 1934, Garrett Birkhoff has shown that the number of isomorphism classes of finite metabelian groups of order \(p^{22}\) tends to infinity with \(p\). More precisely, for each prime number \(p\) there is a family \((M_\lambda)_{\lambda=0,...,p-1}\) of indecomposable and pairwise nonisomorphic metabelian \(p\)-groups of the given order. In this manuscript we use recent results on the classification of possible embeddings of a subgroup in a finite abelian \(p\)-group to construct families of indecomposable metabelian groups, indexed by several parameters, which have upper bounds on the exponents of the center and the commutator subgroup.

A group \(G\) is called \textit{metabelian} if its commutator subgroup \(C_1(G)\) is contained in the center \(C(G)\); it follows that the subgroup embedding \(C_1(G) \subseteq C(G)\) is an isomorphism invariant of the group \(G\). In [1, §15], Birkhoff observes that each such invariant can be realized, and he uses his construction of pairwise nonisomorphic subgroup embeddings to obtain a family, indexed by one parameter \(\lambda = 0, \ldots, p-1\), of pairwise nonisomorphic metabelian \(p\)-groups. We recall his result in Theorem 2.

In fact, the classification of all subgroup embeddings is a problem considered infeasible. It is shown in [3] that for \(n > 6\) the category \(S(\mathbb{Z}/p^n)\) of all embeddings of a subgroup in a \(p^n\)-bounded finite abelian group is of (controlled) wild representation type and hence admits families of indecomposable and pairwise nonisomorphic objects which depend on several parameters. We obtain corresponding statements about families of metabelian groups in Theorem 3 and in Corollary 4.

In these examples and in Birkhoff’s, the exponent of the commutator subgroup is \(p^4\). We show in Theorem 5 that this exponent can be reduced to \(p^3\), at the expense of a higher order and a larger exponent of the group.

Birkhoff’s Construction of Metabelian Groups

Let \(B\) be a finite abelian \(p^n\)-bounded group and \(A\) a subgroup of \(B\) where the embedding is denoted by \(\iota: A \to B\). Define the finite group \(M = G(A \subseteq B)\) as the semidirect product

\[
M = (A \oplus B) \times \psi D
\]

where \(D = \mathbb{Z}/p^n\) is the cyclic group of order equal to the exponent of \(A\), and \(\psi: D \to \operatorname{Aut}(A \oplus B)\) is the group map given by \(\psi(d)(a, b) = (a, b + d\iota(a))\). Note that additive notation is used for the (noncommutative) group operation. Thus, \(M\) is the set \(A \oplus B \oplus D\) with the group operation given by

\[
(a, b, d) + (a', b', d') = (a + a', b + d\iota(a') + b', d + d').
\]

Mathematics Subject Classification (2000): Primary 20D10, Secondary 20K27, 16G60.
An object \((A \subseteq B)\) in the category \(S(\mathbb{Z}/p^n)\) consists of a \(\mathbb{Z}/p^n\)-module \(B\) of finite composition length, together with a submodule \(A\) of \(B\). Thus we are dealing with embeddings \((A \subseteq B)\) of a group \(A\) in a finite abelian \(p^n\)-bounded group \(B\). Morphisms from \((A \subseteq B)\) to \((A' \subseteq B')\) are given by the group maps \(f : B \to B'\) which satisfy the condition that \(f(A) \subseteq A'\) holds. The category \(S(\mathbb{Z}/p^n)\) has the Krull-Schmidt property, so every object has a decomposition as a direct sum of indecomposable objects; this decomposition is unique up to isomorphism and reordering. Via the construction \(G\), families of indecomposable and pairwise nonisomorphic objects in \(S(\mathbb{Z}/p^n)\) give rise to corresponding families of metabelian groups:

**Lemma 1.**

1. The group \(M = G(A \subseteq B)\) has center \(C(M) = B\) and commutator subgroup \(C_1(M) = \iota(A)\). Hence it is a metabelian group.
2. For each \(n \in \mathbb{N}\), the map \(S(\mathbb{Z}/p^n) \to \text{Groups}, (A \subseteq B) \mapsto G(A \subseteq B)\), preserves indecomposable objects, and preserves and reflects isomorphisms.

**Proof:** The first assertion is an immediate consequence of the group operation. If \(f : B \to B'\) gives rise to an isomorphism \((A \subseteq B) \to (A' \subseteq B')\), then \(A\) and \(A'\) have the same exponent, say \(p^m\), and the diagonal map \((f|_{A,A'}, f, 1_{\mathbb{Z}/p^n})\) yields an isomorphism \(G(A \subseteq B) \to G(A' \subseteq B')\). Moreover, the construction \(G\) has a left inverse given by assigning to a metabelian group \(M\) the embedding of the commutator subgroup in the center. Thus if \((A \subseteq B)\) is an object in \(S(\mathbb{Z}/p^n)\) and \(M = G(A \subseteq B)\) then \((A \subseteq B)\) and \((C_1(M) \subseteq C(M))\) are isomorphic objects in \(S(\mathbb{Z}/p^n)\). Hence the second assertion follows. \(\square\)

**Theorem 2** (Birkhoff). *The number of isomorphism classes of metabelian groups of order \(p^{2^2}\) tends to infinity with \(p\).*

**Proof:** Let \(B\) be the finite abelian \(p\)-group generated by elements \(x, y,\) and \(z\) of order \(p^6, p^4,\) and \(p^2\), respectively. For each value of the parameter \(\lambda \in \{0, \ldots, p - 1\}\), let \(A_\lambda\) be the subgroup of \(B\) generated by \(u = p^2x + py + z\) and \(v_\lambda = p^2y + p\lambda z\), as indicated in the diagram.

\[
(A_\lambda \subseteq B)\:\begin{array}{c}
\bullet
\end{array}
\]

It is shown in [1; Corollary 15.1] that the objects \((A_\lambda \subseteq B)\) in \(S(\mathbb{Z}/p^6)\) are indecomposable and pairwise nonisomorphic. By Lemma 1, the family \(G(A_\lambda \subseteq B)\), where \(0 \leq \lambda < p\), consists of \(p\) indecomposable and pairwise nonisomorphic metabelian groups; each has order \(p^{2^2}\). \(\square\)

We observe for later use that the metabelian groups constructed this way all have exponent \(p^6\), which is also the exponent of the center, while the exponent of the commutator subgroup is \(p^4\). It has been shown in [2] that for \(n < 6\), there are only finitely many indecomposable embeddings \((A \subseteq B)\) in \(S(\mathbb{Z}/p^n)\), so under the above construction, the exponent \(p^6\) of the center is minimal. Conversely, for \(n = 6\), there are actually infinitely many indecomposable embeddings in \(S(\mathbb{Z}/p^6)\), up to isomorphism, and hence for each prime number \(p\) there are infinitely many indecomposable finite metabelian \(p\)-groups, up to isomorphism.
WILD REPRESENTATION TYPE

In order to construct metabelian groups indexed by several parameters, we first recall the corresponding situation for subgroup embeddings. Consider the classical problem of finding a normal form for pairs of square matrices of the same size, indexed by this parameter set; here we can even assume that each matrix pair is indecomposable in the sense that it is not equivalent to a pair of proper block diagonal matrices of the same block type. For example, a family indexed by two parameters arises already in the case of $1 \times 1$-matrices: The matrix pairs $([\lambda],[\mu])$, where $\lambda,\mu \in k$, are all pairwise nonequivalent under simultaneous conjugation.

Clearly, a pair $(X,Y)$ of $m \times m$-matrices defines the structure of a $k\langle X,Y \rangle$-module on the vector space $V = k^m$ where $k\langle X,Y \rangle$ is the free algebra in two generators; here the action of the indeterminates $X$ and $Y$ on $V$ is given by multiplication by the matrices with the same names. We denote this module by $(V;X,Y)$. It is easy to see that two pairs of $m \times m$-matrices are equivalent if and only if the two corresponding $k\langle X,Y \rangle$-modules are isomorphic, so the problem of classifying all finite dimensional $k\langle X,Y \rangle$-modules, up to isomorphism, is considered infeasible, too, and hence this classification problem gives rise to the notion of “wild type”.

An additive category $\mathcal{A}$ is controlled $k$-wild provided there are full subcategories $\mathcal{C} \subseteq \mathcal{B} \subseteq \mathcal{A}$ such that the subquotient $\mathcal{B}/\langle \mathcal{C} \rangle$ of $\mathcal{A}$ is equivalent to the category $k\langle X,Y \rangle$-mod. Here $\langle \mathcal{C} \rangle$ is the categorical ideal in $\mathcal{B}$ of all morphisms which factor through a (finite) direct sum of objects in $\mathcal{C}$. It is shown in [3, Theorem 2] that for $n > 6$ the category $\mathcal{S}(\mathbb{Z}/p^n)$ is controlled $k$-wild where $k$ is the field $\mathbb{Z}/p$. In this case, the category $\mathcal{B}$ consists of objects $M$ which are in between the objects $I$ and $J$ in the sense that $I^\ell \subseteq M \subseteq J^\ell$ holds for some $\ell \in \mathbb{N}$, and $\mathcal{C}$ consists of the single object $I$. The two objects $I$ and $J$ are as follows.

\[
I = \begin{array}{cccc}
& & & \\
& \bullet & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& \bullet & & \\
& & & \\
& & & \\
& & & \\
\end{array} \quad \quad J = \begin{array}{cccc}
& & & \\
& & & \\
& \bullet & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{array}
\]

Thus, if $I = (I_1 \subseteq I_0)$ and $J = (J_1 \subseteq J_0)$ then the abelian group $J_0$ is generated by three elements $x, y, z$ of order $p^7, p^4, p^2$, respectively, and $I_0$ is the subgroup generated by $px, y, z$. The subgroup $J_1$ is generated by $p^3x - py$ and $py - z$, and $I_1 = pJ_1$. The equivalence $\mathcal{B}/\langle I \rangle \to k\langle X,Y \rangle$-mod is established by providing a full and dense functor $F: \mathcal{B} \to k\langle X,Y \rangle$-mod and a construction which assigns to the $k\langle X,Y \rangle$-module $(V;X,Y)$ the object $S_{(V;X,Y)}$. If $V = k^m$ then $S_{(V;X,Y)}$ is also denoted by $S_{X,Y}$ and satisfies $I^{2m} \subseteq S_{X,Y} \subseteq J^{2m}$. The functor is left inverse to the construction in the sense that the $k\langle X,Y \rangle$-modules $F(S_{(V;X,Y)})$ and $(V;X,Y)$ are isomorphic. For more details on the construction (which is obtained by taking fibre products in a homomorphism category), we refer the reader to the discussion following [3, Theorem 2] and the presentation of the construction $\Phi$ after [3, Proposition 1]. As a consequence of the above, the functor $F$ preserves indecomposable objects and reflects isomorphisms (and clearly, as an additive functor, $F$ reflects indecomposable objects and preserves isomorphisms).
METABELIAN GROUPS INDEXED BY SEVERAL PARAMETERS

We have seen in Lemma 1 that the construction $G$ preserves indecomposability and preserves and reflects isomorphisms. As a consequence we obtain

Theorem 3. Let $m$ be a natural number and let $(X, Y)$ and $(X', Y')$ be two pairs of $m \times m$-matrices with coefficients in the field $\mathbb{Z}/p$.

1. The metabelian group $G(S_{X, Y})$ is indecomposable if the matrix pair $(X, Y)$ is not equivalent under simultaneous conjugation to a pair of proper block diagonal matrices of the same block type.

2. The two groups $G(S_{X, Y})$ and $G(S_{X', Y'})$ are isomorphic if and only if the two pairs of matrices are equivalent under simultaneous conjugation. □

We demonstrate this correspondence in the case $m = 1$ where we obtain a two parameter family of indecomposable and pairwise nonisomorphic metabelian groups. Given a pair of $1 \times 1$-matrices $([\lambda], [\mu])$, where $0 \leq \lambda, \mu < p$, the corresponding subgroup embedding $S_{[\lambda], [\mu]} = (A_{\lambda, \mu} \subseteq B)$ in $S(\mathbb{Z}/p^2)$ is as follows. The abelian group $B$ is generated by elements $x_1, x_2, y_1, y_2, z_1, \text{ and } z_2$ of order $p^7, p^6, p^4, p^2, \text{ and } p^2$, respectively, and the subgroup $A_{\lambda, \mu}$ is generated by $u_1 = p^3 x_1 + p y_1 + p y_2 + z_2,\ u_2(\lambda, \mu) = p^2 x_2 + p \lambda y_1 + p \mu y_2 + \lambda z_1 + \mu z_2,\ v_1 = p^2 y_1 + p z_1, \text{ and } v_2 = p^2 y_2 + p z_2$; these elements have order $p^4, p^4, p^2, \text{ and } p^2$, respectively, and $\mu_* = \mu + 1$. The embedding $(A_{\lambda, \mu} \subseteq B)$ can be pictured as follows.

\[
(A_{\lambda, \mu} \subseteq B): \quad \begin{array}{c}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\lambda & \mu & \lambda \mu
\end{array}
\]

By applying the construction $G$, we obtain the metabelian group $G(A_{\lambda, \mu} \subseteq B)$. It is generated by 11 elements

\[ u_1, u_2(\lambda, \mu), v_1, v_2, \quad x_1, x_2, y_1, y_2, z_1, z_2, \quad d \]

of order $p^4, p^4, p^2, p^2, p^7, p^6, p^4, p^4, p^2, p^2, \text{ and } p^4$, respectively, and hence $G(A_{\lambda, \mu} \subseteq B)$ has order $p^{41}$. The generators commute with each other with the exception of the following four pairs for which we list the commutators (still using additive notation).

\[
[u_1, d] = (dp^3) x_1 + (dp) y_1 + (dp) y_2 + (d) z_2 \\
[u_2(\lambda, \mu), d] = (dp^2) x_2 + (d) y_1 + (dp(\mu + 1)) y_2 + (d) z_1 + (dp) z_2 \\
v_1, d = (dp^2) y_1 + (dp) z_1 \\
v_2, d = (dp^2) y_2 + (dp) z_2
\]

We have shown the following result:

Corollary 4. For each prime number $p$, there are at least $p^2$ indecomposable and pairwise nonisomorphic metabelian groups of order $p^{41}$. □
Commutator Subgroups with Small Exponents

In the above example, the exponent of the groups constructed is $p^7$, while the exponent of the commutator subgroup is $p^3$. In the remainder of this note we specify metabelian groups for which the commutator subgroups all have even smaller exponent $p^3$; this is at the expense of the order and of the exponent of the group.

Theorem 5.

1. For each prime $p$, there are at least $p$ indecomposable and pairwise nonisomorphic metabelian groups of order $p^{31}$ and exponent $p^7$ such that the exponent of the commutator subgroup is $p^3$.

2. For each prime $p$, there are at least $(p^2)$ indecomposable and pairwise nonisomorphic metabelian groups of order $p^{60}$ and exponent $p^8$ such that the exponent of the commutator subgroup is $p^3$.

Proof: According to [4], the two families of embeddings $(A_\lambda \subseteq B)_{0 \leq \lambda < p}$ and $(C_{\lambda, \mu} \subseteq D)_{0 \leq \lambda < \mu < p}$ pictured below consist of indecomposable and pairwise nonisomorphic objects in $S(\mathbb{Z}/p^7)$ and in $S(\mathbb{Z}/p^8)$, respectively; clearly the subgroups are bounded by $p^3$. Now apply Lemma 1. □

\[
\begin{align*}
(A_\lambda \subseteq B): & \quad \bullet \bullet \bullet \bullet \bullet \\
& \quad \lambda \\
(C_{\lambda, \mu} \subseteq D): & \quad \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \\
& \quad \lambda \mu
\end{align*}
\]

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