Universal free energy distribution in the critical point of a random Ising ferromagnet

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We discuss the non-self-averaging phenomena in the critical point of weakly disordered Ising ferromagnet. In terms of the renormalized replica Ginzburg-Landau Hamiltonian in dimensions $D < 4$ we derive an explicit expression for the probability distribution function (PDF) of the critical free energy fluctuations. In particular, using known fixed-point values for the renormalized coupling parameters we obtain the universal curve for such PDF in the dimension $D = 3$. It is demonstrated that this function is strongly asymmetric: its left tail is much more slow than the right one.

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I. INTRODUCTION

It is well known that the presence of weak quenched disorder in a ferromagnetic system can essentially modify its critical properties in the vicinity of the phase transition point such that new universal critical exponents may set in \cite{1–5}. On the other hand in recent years it is argued that due to the presence of disorder the statistical properties of some thermodynamical quantities at the critical point can become non-self-averaging \cite{6-9}. The aim of the present study is to demonstrate that due to the presence of weak disorder the statistics of the free energy fluctuations in the critical point of the Ising ferromagnet is described by a nontrivial universal distribution function.

Away from the critical point at scales much bigger than the correlation length $\xi_c$ the situation is sufficiently simple: here the system could be considered as a set of essentially independent regions with the size $\xi_c$, and for that reason one could naively expect that the free energy distribution function must be Gaussian. In fact, besides the central Gaussian part (the “body”) this distribution has asymmetric and essentially non-Gaussian tails \cite{9}. Approaching the critical point one finds that the range of validity of the Gaussian body shrinks while the tails are getting of the same order as the body. Finally, when the correlation length becomes of the order of the system size (in the critical point) the free energy distribution function turns into a universal curve.

Present investigation of the critical free energy fluctuations is performed in terms of the renormalized replica Ginzburg-Landau Hamiltonian in dimensions $D < 4$ which allows to derive the explicit expression for their probability distribution function (PDF). In particular, using known fixed-point values for the renormalized coupling parameters we obtain the universal curve for such PDF in the dimension $D = 3$ (eq.\textsuperscript{21}, Figure\textsuperscript{1}).

II. RENORMALIZATION GROUP REPLICA APPROACH

We consider the continuous version of the Ising ferromagnet in terms of the random temperature $D$-dimensional Ginzburg-Landau (GL) Hamiltonian:

$$H[\phi, \xi] = \int d^{D}x \left[ \frac{1}{2} \nabla \phi(x)^{2} + \frac{1}{2} (\tau - \xi(x))\phi^{2}(x) + \frac{1}{4} g \phi^{4}(x) \right]$$

where $\phi(x)$ are scalar fields, $\tau = (T - T_c)/T$ is the dimensionless temperature parameter and $g$ is the usual GL coupling parameter. The independent random quenched parameters $\xi(x)$ are described by the Gaussian distribution with $\xi(x) = 0$ and $\xi^{2}(x) = 2u$ where the parameter $u$ describes the strength of the disorder.

For a given realization of the disorder the partition function of the considered system is

$$Z[\xi] = \int D\phi \exp(-H[\phi, \xi]) = \exp(-F[\xi]) \quad (2)$$

where $\int D\phi$ denotes the integration over all configurations of the fields $\phi(x)$ and $F[\xi]$ is the (disorder realization dependent) free energy of the system.

The distribution function of the random free energy $F[\xi]$ can be analyzed by studying the moments of the partition function. Taking the integer $n$-th power of the expression in eq.\textsuperscript{2} and performing the Gaussian averaging over the disorder parameters $\xi(x)$ we get the replica partition function

$$Z[n][\xi] = Z(n) = \int D\phi_1 ... \int D\phi_n \exp(-H_n[\phi]) \quad (3)$$

where the replica Hamiltonian

$$H_n[\phi] = \int d^{D}x \left[ \frac{1}{2} \sum_{a=1}^{n} \nabla \phi_a(x)^{2} + \frac{1}{2} \gamma \sum_{a=1}^{n} \phi_a^{2}(x) + \frac{1}{4} g \sum_{a=1}^{n} \phi_a^{4}(x) - \frac{1}{4} u \sum_{a,b=1}^{n} \phi_a^{2}(x)\phi_b^{2}(x) \right]$$

where

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depends on \( n \) interacting fields \( \phi \equiv \{ \phi_1, ... \phi_n \} \).

Applying the renormalization group (RG) method to analyze the Hamiltonian \( \mathcal{H} \), in dimensions \( D = 4 - \epsilon \) one does not encounter in the one loop approximation the fixed point (FP) with both non-zero coordinates \( u^* \neq 0 \), \( g^* \neq 0 \): this is because the system of equations for the fixed points is degenerate on the one-loop level \([1, 3, 4]\).

This fixed point appears in the next, two–loop approximation. However, the degeneracy of the one–loop equations leads to the \( \sqrt{\epsilon} \)-expansion \([2, 4]\). Being qualitatively correct, this expansion appears to be of no use if the accurate quantitative results at \( D = 3 \) are needed \([10-12]\).

Alternatively, RG equations for the Hamiltonian \( \mathcal{H} \) have been analyzed directly at \( D = 3 \) using the minimal subtraction \([13]\) and massive \([14]\) RG schemes. To evaluate the divergent perturbation series in the renormalized couplings, appropriate resummation technique has been used. Results of the five loop calculations based on the minimal subtraction scheme at \( D = 3 \) are given in Ref. \([12]\). In the massive RG scheme, the most accurate results are obtained within accuracy of six loops in Ref. \([15]\). In particular, using two different resummation schemes: based on (i) conformal mapping and (ii) Padé approximants, the following estimates for the FP values were obtained, respectively:

\[
\begin{align*}
(i): & \quad u_\ast \simeq 2.14, \quad g_\ast \simeq 6.28, \\
(ii): & \quad u_\ast \simeq 1.98, \quad g_\ast \simeq 6.12,
\end{align*}
\]

\( \text{cf. Eqs. } (3.12) \text{ and } (3.15) \text{ of Ref. } [15] \) (here, instead of the notations for the coupling constants \( u^* \) and \( g^* \) of Ref. \([15]\) the FP values \([7, 6]\) are given for the appropriately rescaled renormalized couplings: \( \pi u^* \to u_\ast \) and \( \frac{\pi}{\nu} g^* \to g_\ast \).

Note, that these results for the 3D random Ising model stable FP coordinates are far less accurate than those for the \( O(m) \) symmetrical FP of the \( m \)-vector model. Further discussion and comparison of contributions of different orders of perturbations theory and interplay of different resummation schemes may be found in Refs. \([12, 10]\).

For further calculations of the critical free energy distribution function we will take just the average of the two FP values \([6, 6]\):

\[
\begin{align*}
u \ast \simeq 2.06, \quad \phi \ast \simeq 6.20.
\end{align*}
\]

### III. CRITICAL FREE ENERGY DISTRIBUTION FUNCTION

The idea of the further (somewhat heuristic) calculations of the critical free energy distribution function is in the following. According to the general approach of the RG theory of critical phenomena in the vicinity of the phase transition point the total free energy \( F \) of the system can be decomposed into two essentially different contributions:

\[
F = V f_0 + V |\tau|^{2-\alpha},
\]

where \( V = L^D \) is the volume of the system (\( L \) is its linear size), \( f_0 \) is the regular (background) free energy density (which remains finite and non-singular at \( T = T_c \)) and \( \alpha \) is the specific heat critical exponent. The second term \( \mathcal{F} = L^D |\tau|^{2-\alpha} \) represents the fluctuating part of the free energy which is singular at the critical point \( \tau = 0 \) and it is this part which is calculated in terms of the RG theory. Taking into account the standard relation among the critical exponents, \( D\nu = 2 - \alpha \) (where \( \nu \) is the critical exponent of the correlation length) one notes that at the critical point, when the correlation length becomes of the order of the system size, \( R_c \sim |\tau|^{-\nu} \sim L \) the fluctuating part of the free energy \( \mathcal{F} \sim L^{(D\nu+\alpha-2)/\nu} \sim O(\ln L) \) is getting non extensive with the volume of the system. It is the distribution function of the random quantity \( \mathcal{F} \) in the critical point which we are going to derive in this paper.

According to the general ideas of the RG theory of critical phenomena, in the vicinity of the critical point at small but non-zero value of the temperature parameter \( \tau \) the considered system can be regarded as a set of \( N \sim V/R^D \) essentially independent “cells” with the size of the order of the correlation length \( R_c \). The internal degrees of freedom of the cells are integrated out providing the renormalized FP values for the coupling parameters \( g \to g_\ast \) and \( u \to u_\ast \) eq. \([7]\), as well as for the temperature parameter \( \tau \to \tau_R = \tau R_c^{\alpha/\nu} \). Thus, the effective (renormalized) Hamiltonian of the cell of the size \( R_c \) is

\[
(9)
\]

\begin{align}
H_R &= R_c^D \left[ \frac{1}{2} \sum_{a=1}^{n} \phi_a^2 + \frac{1}{4} g_\ast \sum_{a,b=1}^{n} \phi_a^4 - \frac{1}{4} u_\ast \sum_{a,b=1}^{n} \phi_a^2 \phi_b^2 \right].
\end{align}

Correspondingly, in the critical point at \( \tau = 0 \) and \( R_c = L \) the partition function of the considered system can be estimated as:

\[
Z(n) \simeq Z_0^n \prod_{a=1}^{n} \left[ \int_{-\infty}^{\infty} d\phi_a \right] \exp \left\{ L^D \left[ -\frac{1}{4} g_\ast \sum_{a=1}^{n} \phi_a^4 + \frac{1}{4} u_\ast \sum_{a,b=1}^{n} \phi_a^2 \phi_b^2 \right] \right\}.
\]

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defines the distribution function \( P(f) \) of the finite (\( L \)-independent) fluctuating part of the free energy \( f \) in the critical point:

\[
\tilde{Z}(n) = \int_{-\infty}^{+\infty} df \, P(f) \exp(-nf) \, .
\]

By definition, the fluctuating part of the free energy \( f \) is the mere difference between the total free energy \( F \) and its self-averaging part \( F_0 \), cf. eqs. (8), (12):

\[
f = F - F_0 \, .
\]

By simple transformations the replica partition function, eq.(13), can be represented as follows:

\[
\tilde{Z}(n) = \int_{-\infty}^{+\infty} \frac{d\eta}{\sqrt{8\pi u_*}} \exp\left\{-\frac{1}{4u_*} \eta^2 \right\} G^n(\eta)
\]

where

\[
G(\eta) = \int_{-\infty}^{+\infty} d\phi \exp\left\{\frac{1}{2} \eta \phi^2 - \frac{1}{4} g_* \phi^4 \right\} .
\]

Performing analytic continuation from the integer \( n \) to arbitrary complex values, according to eqs.(14) and (16), the distribution function \( P(f) \) can be obtained by the inverse Laplace transform:

\[
P(f) = \int_{-i\infty}^{+i\infty} \frac{ds}{2\pi i} \int_{-\infty}^{+\infty} \frac{d\eta}{\sqrt{8\pi u_*}} \exp\left\{-\frac{1}{4u_*} \eta^2 + s f \right\} G^n(\eta)
\]

or

\[
P(f) = \int_{-i\infty}^{+i\infty} \frac{dt}{\sqrt{8\pi u_*}} \exp\left\{t - \frac{1}{4u_*} \eta^2(t) \right\} \frac{\delta(\ln[G(\eta)] + f)}{G[\eta(t)]} .
\]

Introducing a new integration variable \( t = \ln G(\eta) \) one gets

\[
P(f) = \int_{-i\infty}^{+i\infty} \frac{dt}{\sqrt{8\pi u_*}} \frac{\exp\left\{t - \frac{1}{4u_*} \eta^2(-f) \right\}}{G'[\eta(-f)]} \delta(t + f) \, .
\]

Thus, the final result for the free energy distribution function is

\[
P(f) = \frac{1}{\sqrt{8\pi u_*}} \exp\left\{-f - \frac{1}{4u_*} \eta^2(-f) \right\} \, \left\{ -\frac{g_*}{u_*} \right\} \frac{d \eta}{\sqrt{8\pi u_*}} \, .
\]

where the function \( G(\eta) \) is defined in eq.(17), its derivative

\[
G'(\eta) = \frac{1}{2} \int_{-\infty}^{+\infty} d\phi \, \phi^2 \exp\left\{\frac{1}{2} \eta \phi^2 - \frac{1}{4} g_* \phi^4 \right\}
\]

and the function \( \eta(-f) \) is defined by the equation

\[
\int_{-\infty}^{+\infty} d\phi \exp\left\{\frac{1}{2} \eta \phi^2 - \frac{1}{4} g_* \phi^4 \right\} = \exp(-f) \, .
\]

Here, the values of the FP couplings \( g_* \) and \( u_* \) are given in eq.(7).

The universal curve for the probability distribution function \( P(f) \), eq.(21), is represented in Figure 1. We see that, like in all the other systems where the free energy PDFs have been calculated \[9, 17, 18\] this function is essentially non-symmetric: the left tail is much more slow than the right one.

A. Asymptotics

Using eqs.(21)–(23) both the left and the right tails of the probability distribution function \( P(f) \) can be derived explicitly.

In the limit \( f \rightarrow -\infty \) the approximate solution of eq.(23) is

\[
\eta(-f) \simeq \sqrt{4g_*}|f| .
\]

Substituting this into eqs.(22) and (21), and neglecting pre-exponential factors one easily gets

\[
P(f \rightarrow -\infty) \sim \exp\left\{-\frac{g_*}{u_*} |f| \right\} \simeq \exp\left\{-3.01 |f| \right\} .
\]

In the opposite limit, \( f \rightarrow +\infty \) the approximate solution of eq.(23) is

\[
\eta(-f) \sim -2\pi \exp\{2f\} .
\]

Substituting this solution into eqs.(22) and (21), with exponential accuracy one gets

\[
P(f \rightarrow +\infty) \sim \exp\left\{2f - \frac{\pi^2}{u_*} \exp\{4f\} \right\} \simeq \exp\left\{2f - 4.79 \exp\{4f\} \right\} .
\]

Thus, according to eqs.(25) and (27), we see that the left tail of the probability distribution function \( P(f) \) is indeed much more slow than the right one.
IV. CONCLUSIONS

In this paper we have derived explicit expression for the probability distribution function of the free energy fluctuations of weakly disordered three-dimensional Ising ferromagnet in the critical point. First of all, it should be stressed that the mere existence of such distribution function in the thermodynamic limit means that the critical free energy fluctuations in the considered system are non-self-averaging. This, of course is not surprising as the values of these critical fluctuations are not extensive with volume of the system. In this respect our analysis differs from that of Refs. [6–8], where behaviour of extensive thermodynamic quantities at $T_c$ was considered.

The other maybe more important result of the present research is that obtained distribution function, eqs. (21)-(23) for the fluctuating part of the free energy $f$, eq. (15), Figure 1, is universal, which means that hopefully it could be verified by e.g. numerical simulations. Of course it must be not so easy to do, as the fluctuations under consideration must be “extracted” at the background of the leading extensive with the volume self-averaging part of the free energy, but nevertheless we hope that nothing is impossible for nowadays numerics...

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