Abstract

In 1967 Akito Arima spent a year as a visiting professor in the physics department of Rutgers University. In this work we pay tribute to him by discussing topics that we worked on that were directly influenced by his works or were closely related to his interests. These include nuclear Symmetries, magnetic and other moments, analytic expressions in the single \( j \) shell model and schematic interactions.

Contents

1 Introduction 2

2 Symmetries 2

2.1 Isospin 2

2.2 Signature 4

2.3 Seniority 4

3 Nuclear Moments 7

3.1 Magnetic moments 7

3.1.1 Magnetic moments of isotones and isotopes 7

3.1.2 Second order perturbation theory 9

3.1.3 Isoscalar magnetic moments 9

3.2 Quadrupole moments 11

3.2.1 Empirical rule 13

3.3 Isotope shifts 13

4 Redmond Modifications and Counting Pairs 15

5 Closing Remarks 16
1 Introduction

Akito Arima was a visiting professor at Rutgers in 1967. He was not the only Japanese visitor. Also Shiro Yoshida and a bit later a very bright student of Arima-Koichi Yazaki. Yoshida stayed on as a professor for about 5 years and I wrote the following article with him. “Electromagnetic Moments and Transitions Annual Review of Nuclear Science - Vol 22”. I developed lifelong friendships with all these people. I was invited to give one of the banquet speeches at a 1972 conference in Japan honoring Akito. I said diplomatically: “Akito Arima is the most respected nuclear theorist in the world and Shiro Yoshida is the most respected nuclear theorist in Japan”. 

Akito’s work was not only a big influence on the topics that I chose but was also of importance to the experimental group at Rutgers - Noemie Koller, Gerfried Kumbartzki, and from Bonn, Karl Heinz Speidel. This group measured magnetic moments of excited states of even-even nuclei and the work of Arima and Hori, well known at that time, provided a solid theoretical background.

When I was postdoc at Princeton Gerry Brown suggested to me and to a then student Harry Mavromatis that we also work on magnetic moments. The Arima-Horie theory was basically first order perturbation theory so we were to do second order. This is important for a closed major shell plus (or minus) one nucleon because for such cases first order vanishes. Ichimura and Yazaki also had worked on this. By the way soon after Gerry Brown left Princeton for Stony Brook an took Akito with him. I recall at conferences Akito would stand up to ask a question: He started with Akito Arima, Stony Brook and Tokyo and then the fearsome question.

Besides magnetic moments I will here discuss topics which paralleled the interest of Arima - at least I hope they did. These include Nuclear Symmetries, quadrupole moments, single \( j \) shell properties and schematic interactions. In the same time period - after 2000, Akito Arima and Yu-Min Zhao wrote many papers on the single \( j \) shell (e.g. number of states of identical particles ), and I had written a few myself. I recall sending emails to Akito about this but got no replies. I related this to Igal Talmi who laughed and said “Akito does not answer anyone's emails.” At the same Akito and Yu-Min were very generous in mentioning my works on these topics and I reciprocated in turn. I hope the next few sections will help to convey how important Arima’s presence both at Rutgers and on the world scene helped to enlivened my life in physics.

2 Symmetries

Akito’s Arima contributions to the subject of symmetries in nuclei is overwhelming. Here we show some work on this topic which we hope would have met with his approval.

2.1 Isospin

Note in Table 1 that the excitation energies of the even \( J \) states in \( ^{42}\text{Ca} \), \( ^{42}\text{Sc} \) and \( ^{42}\text{Ti} \) are early the same. This is evidence of the charge independence of the nuclear force. The fact that odd \( J \) states appear only in \( ^{42}\text{Sc} \) shows the Pauli principle in action. In \( ^{42}\text{Ca} \) and \( ^{42}\text{Ti} \)
we have 2 identical nucleons so we can only have antisymmetric states. This tells us that in the \( j^2 \) configuration states with even \( J \) are antisymmetric. In \(^{42}\text{Sc}\) we do not have identical nucleons so we can have symmetric states. These are the odd \( J \) states. And the there is the multiplicity rule. Even \( J \) states occur 3 times so \((2T + 1) = 3\) and so \( T = 1 \). The odd \( J \) states occur only once so \((2T + 1) = 1\) and so \( T = 0 \).

Figure 1: Energy levels of \(^{42}\text{Ca}\), \(^{42}\text{Sc}\) and \(^{42}\text{Ti}\) shown in order to display the near charge independence of the nuclear force.
Table 1: The spectra of $^{42}$Ca, $^{42}$Sc, and $^{42}$Ti.

|    | $^{42}$Ca | $^{42}$Sc | $^{42}$Ti |
|----|----------|----------|----------|
| $J = 0$ | 0.0000 | 0.0000 | 0.0000 |
| $J = 1$ | - | 0.6110 | - |
| $J = 2$ | 1.5247 | 1.5803 | 1.5546 |
| $J = 3$ | - | 1.4904 | - |
| $J = 4$ | 2.7524 | 2.8153 | 2.6746 |
| $J = 5$ | - | 1.5100 | - |
| $J = 6$ | 3.1893 | 3.2420 | 3.0430 |
| $J = 7$ | - | 0.6163 | - |

2.2 Signature

In the $f_{7/2}$ paper of McCullen et al. \[1\][2] we also discuss briefly signature selection rules. For say $^{48}$Ti we have a system of 2 protons and 2 neutron holes in the $f_{7/2}$ shell. We find that the wave functions are either even or odd under the interchange of protons and neutron holes. This is different from isospin in that a state of even signature and one of odd signature can have the same isospin.

It has been shown that this signature property leads to several selection rules. For example, for the electric quadrupole operator the $B(E2)$ between states of opposite signature is proportional to $(ep + en)$, and between states of the same signature to $(ep - en)$. The quadrupole moment of the $2^+$ state is proportional to $(ep - en)$. It turns out that the $2^+ + i$ state of Ti in the single $j$ shell calculation has odd signature, but the $2^+ + 2$ state has even signature. Hence $B(E2)$ from the $J = 0$ ground state to $2(1)$ goes as $(ep + en)^2$ whilst to $2(2)$ as $(ep - en)^2$.

Consider next the double Gamow-Teller operator $(\sigma t_+)(\sigma t_-)$ connecting $^{48}$Ca to $^{48}$Ti. Zamick and Moya de Guerra [3] showed that in this single $j$ shell model the transition to the $2(1)$ state (negative signature) vanishes because of the signature selection rule. On the other hand a transition to the $2(2)$ (positive signature) state is allowed. This simple model shows that there might be surprises when calculating double beta decay transitions.

2.3 Seniority

Some of the well-known statements and theorems concerning states of good seniority are

1. The seniority is roughly the number of identical particles not coupled to zero. Hence, for a single nucleon the seniority $v$ is equal to 1. For two nucleons in a $J = 0$ state we have $v = 0$, but for $J = 2, 4, 6$, etc., $v = 2$. For three nucleons there is one state with seniority $v = 1$, which must have $J = j$; all other states have seniority $v = 3$. 
2. The number of seniority-violating interactions is \( [(2j - 3)/6] \), where the square brackets mean the largest integer contained therein. For \( j = 7/2 \) there are no seniority-violating interactions, while for \( j = 9/2 \) there is one.

3. With seniority-conserving interactions, the spectra of states of the same seniority is independent of the particle number.

4. At midshell we cannot have any mixing of states with seniorities \( v \) and \( v + 2 \); one can mix \( v \) and \( v + 4 \) states.

Figure 2: We quote from the paper [4]:

III. SPECIAL BEHAVIORS FOR \( I = 4^+ \) AND
6\(^+\) STATES OF THE \( g_{9/2}^{4} \) CONFIGURATION

For a system of four identical nucleons in the \( g_{9/2} \) shell, the possible seniorities are \( v = 0, 2, \) and \( 4 \), with \( v = 0 \) occurring only for a state of total angular momentum \( I = 0 \). There is also a \( v = 4 \) state with \( I = 0 \).

For \( I = 4 \) and \( 6 \), we can have three states, one with seniority \( v = 2 \) and two with seniority \( v = 4 \). For the two \( v = 4 \) states we have at hand, we can construct different sets of \( v = 4 \) states by taking linear combinations of the original ones. If the original ones are \((4)_1 \) and \((4)_2 \), we can form

\[
(4)_A = a(4)_1 + b(4)_2,
\]

\[
(4)_B = -b(4)_1 + a(4)_2,
\]

with \( a^2 + b^2 = 1 \). The set \((4)_A, (4)_B \) is as valid as the original set.

However, we here note that if we perform a matrix diagonalization with any two-body interaction—seniority conserving or not—one state emerges which does not depend on what the interaction is. The other two states are, in general, mixtures of \( v = 2 \) and \( v = 4 \) which do depend on the interaction. The values of the coefficients of fractional parentage (cfp's) of this unique state of seniority 4 are shown in Table I. The states with \( J_0 \neq 4.5 \) all have seniority \( v = 3 \). Note that in this special \( v = 4 \) state there is no admixture of states with \( J_0 = j = 9/2 \), be they \( v = 1 \) or \( v = 3 \). Again, no matter what
two-body interaction is used, this $I = 4$ state remains a unique state.

Amusingly, this state does not appear in the compilation of seniority-classified cfp’s of Bayman and Lande [20] or de Shalit and Talmi [5]. We should emphasize that, although different, the Bayman–Lande cfp’s are perfectly correct (as are the ones of de Shalit and Talmi, whose cfp’s are also different from those of Bayman and Lande [20]). But then, why do they not obtain the unique state that we have shown above? Bayman and Lande use group theoretical techniques to obtain the cfp’s, diagonalizing the following Casimir operator for $Sp(2j + 1)$:

$$G(Sp_{2j+1}) = \frac{1}{2j + 1} \sum_{\text{odd } k=1}^{2j} (-1)^k (2k + 1)^{3/2} [U^k U^{k \dagger}]_0,$$  \hspace{1cm} (3)

where $U^k_q = \sum_{i=1}^N U^k_q(i)$ and $U$ is the Racah unit tensor operator

$$\langle \Psi_{j'}^m | U_q^k | \Psi_{j}^m \rangle = \delta_{jj'} (kjqm|jm').$$  \hspace{1cm} (4)

The two seniority $v = 4$ states are degenerate with such an interaction and, since there is no seniority mixing, we can have arbitrary linear combinations of the $4^+$ states. Only by using an interaction which removes the degeneracy and violates seniority, do we learn about the special state in Table I.

\begin{table}[h]
\centering
\caption{A unique $J = 4, v = 4$ cfp for $j = 9/2$.}
\begin{tabular}{ll}
\hline
$J_0$ & $(j^3 J_0 j) j^4 I = 4, v = 4)$ \\
\hline
1.5 & 0.1222 \\
2.5 & 0.0548 \\
3.5 & 0.6170 \\
4.5 ($v = 1$) & 0.0000 \\
4.5 ($v = 3$) & 0.0000 \\
5.5 & -0.4043 \\
6.5 & -0.6148 \\
7.5 & -0.1597 \\
8.5 & 0.1853 \\
\hline
\end{tabular}
\end{table}
3 Nuclear Moments

3.1 Magnetic moments

With \( n \) nucleons of one kind there are simple formulas for nuclear moments in a single \( j \) shell. For example all \( g \) factors should be the same. From this it follows that for states of the same \( J \) the magnetic moments should be the same. The magnetic moment of a free neutron (in units of nuclear magnetons) is \( \mu_n = -1.913 \) and that of a free proton is \( \mu_p = +2.793 \). In a single \( j \) shell of neutrons with \( j = l + 1/2 \) the magnetic moments are predicted to be the same as those of a free neutron-namely \(-1.913\); for protons it is \((2.793 + l)(\mu_n)\). Here \( L \) is the orbital angular momentum.

The single particle magnetic moments, commonly called the Schmidt moments are given here:

1. for an odd proton:
   - \( \mu = j - 1/2 + \mu_p \) for \( j = l + 1/2 \)
   - \( \mu = j/(j + 1)(j + 3/2 - \mu_p) \) for \( j = l - 1/2 \)

2. for an odd neutron:
   - \( \mu = \mu_n \) for \( j = l + 1/2 \)
   - \( \mu = -j/(j + 1)\mu_n \) for \( j = l - 1/2 \).

We can discuss these in a more physical manner. The magnetic moment of a free neutron (in units of nuclear magnetons) is \( \mu_n = -1.913 \) and that of a free proton is \( \mu_p = +2.793 \). In a single \( j \) shell of \( n \) neutrons with \( j = l + 1/2 \) the magnetic moments are predicted to be the same as those of a free neutron-namely \(-1.913\); for \( n \) protons it is \((2.793 + l)\). Here \( l \) is the orbital angular momentum. For a \( j = l - 1/2 \) neutron we have a quantum effect so that the magnetic moment is only minus that of a free neutron in the large \( j \) limit. In general it is \(-j/(j + 1)\) that of a free neutron.

In the sixties Arima was already famous for the Arima-Horie theory for quenching magnetic moments which is basically first order perturbation theory \[5\][6]. I am showing a figure that I like (Figure 3) because it has both theorists and experimentalists at Rutgers and Bonn testing out the Arima-Horie theory of quenching.

3.1.1 Magnetic moments of isotones and isotopes

In the single shell model, all factors (magnetic moment /angular momentum) are the same. Many people are under the impression that in Arima-Horie that is also true of the quenched \( g \) factors. But that is not the case. Rather they are predicted to lie on a straight line with a negative slope. Experimental confirmation of this is shown beautifully in the figure. The slope for states of even nuclei is different than for the ground states of odd nuclei. This is at it should be. One might think one could also apply this to the Calcium isotopes but there intrude states come in to spoil the picture, especially of the \( g \) factors of the \( 2^+ \) states \[7][8]. For example, for the state of \(^{44}\text{Ca}\) the \( g \) factor in the \( f_{7/2} \) model is about \(-0.5\), whilst that
of a highly deformed intruder state is about +0.5. We explain the measured result of close to zero for this state by assuming a 50% admixture of the shell model and intruder state (see Figure 4) [9].

Figure 3: Triangles: factors of ground states of odd isotopes. Circles: g factors of 2^+ states of even isotopes. Light lines: theory.

Figure 4: Source: Phys. Rev C 68, 061302(R) (2003).
3.1.2 Second order perturbation theory

My first foray into this subject of magnetic moments was actually not on the work above – i.e. first order perturbation theory. Rather with Gerry Brown and a Princeton student Harry Mavromatis we dealt with cases where first order perturbation theory was zero and we had to go to second order – much more complicated [10][11][12]. Work on second order was also done in Japan by Ichimura and Yazaki [13]. We deal with a closed major shell plus a nucleon e.g. $^{17}\text{O}$, $^{17}\text{F}$, $^{41}\text{Ca}$, $^{41}\text{Sc}$. These calculations involved a lot of complicated Feynman diagrams. The results were in the right direction to remove the discrepancy from theory and experiment.

The calculations were done first with the Kallio-Koltveit (KK) interaction [14] which does not contain a central interaction and then with the more realistic Hamada Johnson (HJ) interaction [15]. Note that with KK the corrections for mirror pairs are equal and opposite i.e. there is no isoscalar correction. However with HJ, which contains a tensor interaction we do get an isoscalar correction in second order perturbation theory. This result was proved by the authors. The above interactions were soon superseded by the Kuo-Brown matrix elements [16].

Table 2: The calculated sums for the two mirror pairs and the results for the second order corrections to the magnetic moments of the eight nuclei [17].

| Nucleus | Ground state | Spin | Experimental moment in nucleon magnetons | Schmidt moment in nuclear magnetons | Exp.–Sch. (column 4–column 5) | 2nd order correction calculated with H.–J. | 2nd order correction calculated with K.–K. | Calculated (with H.J.) moments (columns 8, 9) |
|---------|--------------|------|-----------------------------------------|-----------------------------------|-------------------------------|---------------------------------|---------------------------------|---------------------------------|
| 1 $^{15}_{7}\text{N}$ | $\frac{1}{2}$ proton | $\frac{1}{2}$ | -0.28 | -0.36 | -0.02 | -0.10 | -0.10 | -0.36 |
| 2 $^{15}_{8}\text{O}$ | $\frac{1}{2}$ neutron | $\frac{1}{2}$ | 0.72 | 0.64 | +0.08 | +0.14 | +0.10 | +0.78 |
| Total | rows 1 and 2 | | 0.44 | 0.38 | +0.06 | +0.04 | 0.00 | +0.42 |
| 3 $^{17}_{9}\text{O}$ | $\frac{1}{2}$ proton | $\frac{1}{2}$ | 4.72 | 4.79 | -0.07 | -0.30 | -0.16 | 4.49 |
| 4 $^{17}_{8}\text{O}$ | $\frac{1}{2}$ neutron | $\frac{1}{2}$ | -1.89 | -1.91 | +0.02 | +0.28 | +0.16 | -1.63 |
| Total | rows 3 and 4 | | 2.83 | 2.88 | -0.05 | -0.02 | 0.00 | 2.86 |
| 5 $^{39}_{20}\text{Ca}$ | $\frac{1}{2}$ proton | $\frac{1}{2}$ | 0.39 | 0.12 | +0.27 | - | -0.26 | - |
| 6 $^{39}_{20}\text{Ca}$ | $\frac{1}{2}$ neutron | $\frac{1}{2}$ | - | 1.15 | - | - | +0.26 | - |
| 7 $^{41}_{21}\text{Sc}$ | $\frac{1}{2}$ proton | $\frac{1}{2}$ | - | 5.79 | - | - | -0.28 | - |
| 8 $^{41}_{20}\text{Ca}$ | $\frac{1}{2}$ neutron | $\frac{1}{2}$ | -1.59 | -1.91 | 0.32 | - | 0.28 | - |

3.1.3 Isoscalar magnetic moments

Isoscalar magnetic moments have been extensively discussed by S. S. Yeager, L. Zamick, Y.Y. Sharon and S.J.Q. Robinson [18] Isoscalar magnetic moments are much closer to the Schmidt
values than the isovector ones. Nevertheless, there are small but systematic deviations. It was noted by Talmi \[19\] that “The experimental values of $\langle S \rangle$ seems to follow a simple rule. They are always smaller in absolute value than the values calculated in $jj$ coupling.” Arima, however, noted \[20\] that the smallness of the isoscalar deviation is due to the small isoscalar spin coupling (0.44) relative to that of the isovector coupling (2.353). If one divides the deviation by the lowest order result one can get a rather large ratio even in the isoscalar case, even up to 50%.

Table 3: In Table 2 of the work of Yeager et al \[18\] we show a table of empirical magnetic moments as well as shell model calculations and Schmidt estimates. The close agreement of the latter with experiment should be noted.

| Nuclei | $J$  | Measured $g$-factors | Large-scale shell model | Single-$j$ model |
|--------|------|----------------------|-------------------------|-----------------|
| $^2$H  | $1^+$| 0.8574382828(9)      | 0.88<sup>a</sup>         | 0.88 ($s_{1/2}$) |
| $^6$Li | $1^+$| 0.8220473(6)         | 0.87<sup>b</sup>         | 0.63 ($p_{3/2}$) |
| $^{10}$B | $3^+$| 0.60021493(2) | 0.61<sup>b</sup>         | 0.63 ($p_{3/2}$) |
|        | $1^+$| 0.63(12)             | 0.77<sup>b</sup>         | 0.63 ($p_{3/2}$) |
| $^{14}$N | $1^+$| 0.40376100(6) | 0.32<sup>b</sup>         | 0.37 ($p_{1/2}$) |
| $^{18}$F | $3^+$| 0.59(4)              | 0.62<sup>c</sup>         | 0.58 ($d_{5/2}$) |
|        | $5^+$| 0.572(6)             | 0.58<sup>c</sup>         | 0.58 ($d_{5/2}$) |
| $^{22}$Na | $3^+$| 0.582(6)             | 0.59<sup>c</sup>         | 0.58 ($d_{5/2}$) |
|        | $1^+$| 0.523(11)            | 0.52<sup>c</sup>         | 0.58 ($d_{5/2}$) |
| $^{26}$Al | $5^+$| 0.561(8)             | 0.57<sup>c</sup>         | 0.58 ($d_{5/2}$) |
| $^{38}$K | $3^+$| 0.457(2)             | 0.41<sup>c</sup>         | 0.42 ($d_{3/2}$) |
| $^{46}$V | $3^+$| 0.55(1)              | 0.58<sup>d</sup>         | 0.55 ($f_{7/2}$) |
| $^{58}$Cu | $1^+$| 0.52(8)              | 0.63<sup>e</sup>         | 0.63 ($p_{3/2}$) |

<sup>a</sup> See table 1.

<sup>b</sup> With PJI interaction \[16\]; full $p$ shell.

<sup>c</sup> With USDA interaction \[17\]; full $sd$ shell.

<sup>d</sup> With GXPF1 interaction \[18\]; full $fp$ shell.

<sup>e</sup> With GXPF1 interaction, up to 4 particles excited from $f_{7/2}$ orbit.
3.2 Quadrupole moments

For quadrupole moments there is also an $n$ dependent simple formula for ground states of odd nuclei in a single $j$ shell

$$Q(n) = [(2j + 1 - 2n)/(2j - 1)]Q(sp)$$

Note that for a single hole $n = 2j$. The formula becomes $Q = -Q(sp)$. I.e. the quadruple moment of a hole is minus that of a particle. We can understand this another way. A nuclear moment is the expectation value of a moment operator in a state with $M = J$,

$$Q^2 = \langle \Psi_J | Q_J^2 | \Psi_J \rangle.$$ 

To create a hole nucleus in a state with $M = J$ we have to remove a nucleon from a closed shell with $M = -J$. The value of $Q$ for a closed shell is zero-this is the the sum of $Q$ for the hole nucleus and the nucleon removed. The value of $Q^2$ in a state with $M = J$ is the same as it is for $M = -J$ – namely $Q(sp)$. So we have $Q(\text{hole}) + Q(sp) = 0$ or $Q(\text{hole}) = -Q(sp)$. For magnets moments we have the opposite the value for $-J$ is minus that for $+J$. Thus we have 2 minus signs and $\mu(\text{hole}) = \mu(sp)$.

As a first example of the sturdiness of the shell model we look at the work of Ruiz et al. [21] on measurements and theoretical analysis quadrupole moments of odd $A$ nuclei in the “f-p” region. They measured the quadruple moments of the $J = 7/2^-$ ground states of Calcium isotopes with $A = 43, 45, 47$ which have ground state spins $J = 7/2^-$. They did not do $A = 41$ but this case could be obtained from another source. They also obtained results for $A = 49, 51$ with $J = 3/2^-$ spins.

A starting point for $A = 41$ to 47 would be the $f_{7/2}$ shell while for $A = 49, 51$ it would be the p$_{3/2}$ shell. The theoretical calculations were performs with many interactions and different model spaces. The latter include complete pf space, (pf + g$_{9/2}$) and breaking the $^{40}$Ca core by allowing 2p-2h admixtures. They use effective charges of 1.5 for the protons and 0.5 for the neutrons. In general the calculations are in excellent agreement with the measurements. We will not go into further details about the calculations except to say that they involve an enormous number of shell model configurations.

Rather in Fig 5 we show the quadrupole moments vs. $A$ and show the the remarkable result that the measured moments from $A = 41$ to $A = 47$ lie, to an excellent approximation on a straight line. As noted in the introduction this is exactly what a single $j$ calculation predicts. To repeat $Q = (2j - 1 - 2n)/(2j - 1) \times Q(s.p.)$ This simple result seems to survive the large shell attack. For $A = 51$ the measured quadrupole moment $Q = +0.04$ b. It is nearly equal and opposite of that for $A = 49 Q = -0.04$ b. This is the prediction of the simplest shell model in which $A = 49$ consist of a single p$_{3/2}$ neuron and $A = 51$ of a p$_{3/2}$ hole.
Before leaving this section we should mention that a purist might say that the real prediction of single $j$ is that all the charge quadrupole moments are zero because the neutrons have no charge. We have to assign an effective charge to the neutrons, popular choice being $e_{\text{eff}} = 0.5$. But note that even the large space calculations including those of \cite{21} require effective charges in order to get agreement with experiment. In first order perturbation theory the effective charge comes from $\Delta N = 2$ excitations. For example for $^{41}$Ca excitations from 0p to 1p; from 0d to 0g, 1d, and 2s. As large as model spaces are in \cite{21} and in nearly all other calculations these configurations are not present and one needs to insert effective charges.

We next consider an “empirical” limit for expectation of the isoscalar spin operator $\langle \sigma \rangle$. In the single particle model (i.e. Schmidt) the value for $j = L - 1/2$ is $-j/(j + 1)$. While the value for $j = L + 1/2$ is one. For the most part the measured values lie between these 2 limits and this has been called an empirical rule. Occasionally some one comes up with an exception. In the work of Kramer et al. \cite{22}, the magnetic moment of $^{21}$Mg is measured, which when combined with the moment of $^{21}$F yields an isoscalar magnetic moment and an expectation value of the spin operator.

They find a value of $\langle \sigma \rangle = 1.15(2)$. They call this an anomalous result. We pointed out however that the “empirical rule” is not a theoretical rule. In LS coupling the value of the spin operator is

$$\langle \sigma \rangle = \frac{S(S + 1) + J(J + 1) - L(L + 1)}{(J + 1)}.$$  \hspace{1cm} (3.1)

The smallest value is $-2SJ/(J + 1)$. And the largest value is $2S$. So we can get in principle get values of $\langle \sigma \rangle$ that are greater than one. The empirical rule is not a theoretical rule \cite{32}.
3.2.1 Empirical rule

As shown in Fig 6 from the work of P.W. Zhao et al. [23] we have the strange case where without pairing we get a complicated behavior of $Q$ vs $N$ but when pairing is included one gets a linear behavior as in the single $j$ shell. However the linear curve has more entries than are present for an $h_{11/2}$ shell.

![Graph showing Q vs N](image)

Figure 6: $Q$ vs. $N$.

3.3 Isotope shifts

In Fig 7 we show measured values of isotope shifts in the Argon Isotopes by Blau et al. [24] (open circles). Also shown are spherical Hartree-Fock calculations in closed triangles, as well as a formula by Zamick [25] and by Talmi [26] [27] which will soon be discussed.
Figure 7: Isotope shifts in the odd $^N_3$Ar Isotopes. The filled circles correspond to the experiment of Blaum et al. [24] and the dashed line the Zamick formula [25]. The triangles correspond to spherical Hartee-Fock calculation [26][27].

Note that the data shows a lot of even-odd staggering but the HF calculations do not. The Zamick-Talmi calculations yield excellent fits to the data and have the even-odd scattering features well under control. In order to get the even-odd staggering Zamick [25] introduced a 2 body effective radius operator in addition to the one body term.

We simply make the assumption that the effective charge radius operator has a two-body part as well as one body part

$$\delta r_{\text{eff}}^2 = \sum_i O(i) + \sum_{i<j} V(i,j)$$  \hspace{1cm} (3.2)

where the symbol $V$ for the two-body part has been written to suggest the similarity with the two-body potential, since both are scalars.

The problem of evaluating this operator for $n$ particles in the $j = f_{7/2}$ shell is exactly the same problem as calculating the binding energies of nuclei whose configuration consists of several nucleons in a single $j$ shell. This problem has been solved and used with great success by the “Israeli group” including de-Shalit, Racah, Talmi, Thieberger, and Unna [28]. In analogy with their binding energy formula we get for the change in charge radius

$$\delta r^2(40 + n) = nC + \frac{n(n-1)}{2} \alpha + \left[ \frac{n}{2} \right] \beta,$$  \hspace{1cm} (3.3)

where

$$\left[ \frac{n}{2} \right] = \begin{cases} \frac{n}{2} & \text{for even } n \\ \frac{n-1}{2} & \text{for odd } n. \end{cases}$$  \hspace{1cm} (3.4)
The parameter $C$ comes from the one-body part and is equal to $\delta r^2(41)$, the difference in charge radius of $^{41}\text{Ca}$ and $^{40}\text{Ca}$. The quantities $\alpha$ and $\beta$ come from the two-body part

\begin{align*}
\alpha &= -\frac{2(j + 1)\bar{E}_2 - E_0}{2j + 1}, \\
\beta &= \frac{2(j + 1)(\bar{E}_2 - E_0)}{2j + 1},
\end{align*}

where

\begin{align*}
E_0 &= \langle j^2J = 0|V|j^2J = 0 \rangle \\
\bar{E}_2 &= \frac{\sum_{J \neq 0}(2J + 1)\langle j^2J|V|j^2J \rangle}{\sum_{J \neq 0}(2J + 1)}.
\end{align*}

4 Redmond Modifications and Counting Pairs

Akito Arima and Yu-Min Zhao have written many papers concerning relations of states in the single $j$ shell. One example is “Number of States for Nucleons in a single $j$ shell” [29]. I was involved in this kind of business as well and was delighted by the generous references to my works by Arima and Zhao.

Let me give one example “New Relations for coefficients of fractional parentage”, where we simplify a recursion relation due to Redmond [30].

Figure 8: We quote from the paper [30]:

A recursion formula for cfp’s due to Redmond [1] is presented in the books of de Shalit and Talmi [2] on p. 528, and Talmi [3] on p. 274. It can be written as follows

\begin{align*}
(n + 1)[f^n(\alpha J_0)jJ]|f^{n+1}[\alpha J_0]J|f^n(\alpha_2J_1)J]j^{n+1}[\alpha_2J_0]J \\
= \delta_{\alpha_1\alpha_0}\delta_{J_1J_0} + n(-1)^{j_0+j_1}(2J_0 + 1)(2J_1 + 1)\sum_{J_2J_1}{J_2jJ_1 \choose JjJ_0} \\
&\times [f^{-1}(\alpha_2J_2)jJ_0]|f^J[\alpha J_0]|f^{J-1}(\alpha_2J_2)jJ_1].
\end{align*}

In the above, square bracket designates the principal parent used to calculate the cfp. Actually, the principal parent sometimes loses its significance because in some cases more than one principal parent can yield the same cfp. In tables of cfp’s, the principal parent is usually not listed. The quantities in parentheses $(\alpha J_0)$ are listed. The cfp with $(\alpha J_0)$ is the probability amplitude that a system of $(n + 1)$ identical particles can be separated into a system of $n$ particles with quantum numbers $(\alpha_0J_0)$ and a single nucleon.
We here present the equivalent of the Redmond recursion relation, but for cfp's classified by the seniority quantum number $v$ and for which there are no redundancies. Here is our formula

$$(n + 1) \sum_v [f^r(v_0J_0)jI_r]|J_{r+1}v_1J_1][f^r(v_1J_1)jI_1]|J_{r+1}v_1J_1]$$

$$= \delta_{J_r,1}\delta_{v_0,v_1} + n(-1)^{v_0+j_1}\sqrt{(2J_0 + 1)(2J_1 + 1)} \sum_{J_0,J_1,J_2} \left\{ \begin{array}{ccc} J_2 & j & J_1 \\ I_r & j & J_0 \end{array} \right\}$$

$$\times [f^{r-1}(v_2J_2)jJ_0][f^rJ_0]f^{r-1}(v_2J_2)jJ_1]|J_{r+1}v_1J_1].$$

This differs from the Redmond formula inasmuch as there is now a sum on the left-hand side of the equation over $v_r$. Note that $I_r$ is fixed. Basically, then, the sum is over all states that are present which have angular momentum $I_1$ for the $(n + 1)$-particle system.

5 Closing Remarks

Already in 1967, when Akito Arima was at Rutgers he was a big name on the world scene. Indeed when Gerry Brown went from Princeton to Stony Brook he took Arima with him as well as Tom Kuo. But as they say, the best was yet to come. I am sure the summary of all his accomplishments will appear somewhere in this compendium so I won’t mention them. Well maybe a couple - pseudo spin and the interacting boson model. And in service-president of the University of Tokyo and head of Ricken. Rather I would like to dwell on the fun time it was to be a nuclear physicist in New Jersey around 1967. Princeton and Rutgers had a joint seminar called Nuclear News run by Rubby Sherr. Amongst the faculty, post-docs, senior grad students and visitors at that time were the following:

- Princeton: Gerry Brown, Tom Kuo, Tony Green, Chun WA Wong, George Bertsch, Felix Wong, Alex Lande Yitzhak Sharon, Julian Noble, Harry Mavromatis;

- Rutgers: Joe Ginocchio, Aldo Covello, Giovanni Sartoris, George Ripka and oh yes me.

And for icing on the cake. Shiro Yoshida, Akito Arima and Koichi Yazaki. Arima’s discussions were appreciated not only by the theorists but also the experimentalists - Noemie Koller, George Temmer, Rubby Sherr and others. Rutgers had a major program of measuring magnetic moments of excited states. Those were great times and Akito was a major contributor to the fun we all had.
References

[1] B. F. Bayman, J. D. McCullen, and Larry Zamick Phys. Rev. Lett. 11, 215 (1963) - Published 1 September 1963.

[2] J. D. McCullen, B. F. Bayman, and Larry Zamick Phys. Rev. 134, B515 (1964) - Published 11 May 1964.

[3] L. Zamick and E. Moya de Guerra Phys. Rev. C 34, 290 (1986) - Published 1 July 1986.

[4] A. Escuderos and L. Zamick. Seniority conservation and seniority violation in the g9/2 shell. Phys Rev C 73, 044302 (2006).

[5] A. Arima and H. Horie. Configuration mixing and magnetic moments of nuclei. Progress of Theoretical Physics 11,509 (1954); A. Arima and H. Horie. Configuration mixing and magnetic moments of odd nuclei. Progress of Theoretical Physics. 12,623 (1954).

[6] H. Noya, A. Arima and H. Horie. Nuclear moments and configuration mixing, Progress of Theoretical Physics Supplement 8, 33 (1958).

[7] Evidence for 40 Ca core excitations from g factor and B(E2) measurements on the 2+ states of 42,44 Ca, S. Schielke, D. Hohn, K. H. Speidel, O. Kenn, J. Leske, N. Gemein, M. Offer, J. Gerber, P. Maier-Komor, O. Zell, F. Nowacki, Y. Y. Sharon, L. Zamick, Phys. Lett. B. 571, Oct. 2003, 29-35.

[8] Competing Core and Single Particle excitations in the 2+ State in 44Ca, M. J. Taylor, N. Benczer-Koller, G. Kumbartzki, T. J. Mertzimekis, S.J. Q. Robinson, Y. Y. Sharon, L. Zamick, A. E. Stuchbery, C. Hutter, C. W. Beausang, J. J. Ressler and M. A. Caprio, Phys. Lett. B559, (2003) 187.

[9] Core polarization in the light of new experimental g factors of fp shell, N=28, isotones, K. H. Speidel, R. Ernst, O.Kenn, J. Gerber, P. Maier-Komor, N. Benczer-Koller, G. Kumbartzki, L. Zamick, M. S. Fayache, and Y. Y. Sharon, Physical Review C62, September 2000, 031301.

[10] First and Second Order Corrections to the Magnetic Moments of Nuclei Using Realistic Interactions, L. Zamick, H. A. Mavromatis, and G. E. Brown, Nuclear Phys. 80, 545 (1966).

[11] Magnetic Moments of Nuclei with Closed J-J Shells Plus One or Minus One Nucleon, L. Zamick and H. A. Mavromatis, Nucl. Phys. A104, 17 (1967).

[12] H.A. Mavromatis and Larry Zamick, Physics. Letters 20,2, 171 (1966).

[13] M. Ichumira, K. Yazaki Nuclear Phys., 63 (1965), p. 401.

[14] A. Kallio and K. Kolltveit, Nuclear Phys. 53 (1964) 87.

[15] B. T. Hamada and I.D. Johnston, Nuclear Phys. 34 (1962) 382.
[16] T. T. S. Kuo and G. E. Brown, Nucl. Phys. 85, 40 (1966).

[17] H.A. Mavromatis, L. Zamick. Magnetic moment corrections to second order in perturbation theory. Phys. Rev. Volume 20, Issue 2 (1966).

[18] S. Yeager, L. Zamick, Y. Y. Sharon, S. J. Q. Robinson, Europhys. Lett. 88: 52001, 2009.

[19] I. Talmi. Hyperfine Excited States of Nuclei. Gordon and Breach, New York, 1971.

[20] A. Arima. Hyperfine Interactions, volume 4. 1978.

[21] R. F. Garcia Ruiz, et al. Phys. Rev. C 91, 041304(R) (2015).

[22] J. Kramer et al., Phys. Lett. B 678, 465 (2009).

[23] P.W. Zhao, S.Q. Zhang and J. Meng, Physical Review C 89, 011310 (R) 2014.

[24] K. Blaum, et al. Nuclear moments and charge radii of argon isotopes between the neutron-shell closures N = 20 and N = 28. Nuclear Physics A 799 30-45 (2008).

[25] L. Zamick. Annals of Physics, 66 issue 2 784-789 (1971).

[26] I. Talmi, Nucl. Phys. A423 (1984) 189.

[27] A. Klein, B.A. Brown, U. Georg, M. Keim, P. Lievens, R. Neugart, M. Neuroth, R.E. Silverans, L. Vermeeren, ISOLDE Collaboration, Nucl. Phys. A 607 (1996) 1.

[28] I. Talmi and R. Thieberger, Phys. Rev. 103 (1956), 718; S. Goldstein and I. Talmi, Phys. Rev. 105 (1957), 995; R. Thieberger and A. de-Shalit, Phys. Rev. 108 (1957), 378; I. Talmi and I. Unna, Ann. Rev. Nucl. Sci. 10 (1960), 353; I. Talmi, Rev. Mod. Phys. 34 (1962), 104.

[29] Y.M. Zhao and A. Arima, Phys. Rev C 72, 064333 (2005).

[30] The Redmond Recursion Relation with Seniority. L. Zamick and A. Escuderos, Annals of Physics 321 (2006) 987-998.

[31] P.J. Redmond, Proc. R. Soc. London A222 (1954) 84.

[32] L. Zamick, Phys. Rev. C84, 017302 (2011).