TIGHT MAPS, A CLASSIFICATION

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Abstract. Tight maps was introduced along tight homomorphisms by Burger, Iozzi and Wienhard with aims towards maximal representations. In this paper we classify all tight maps into classical Hermitian symmetric spaces and give a partial result for the exceptional spaces.

1. Introduction

Let \((X_i, \omega_i), i = 1, 2,\) be Hermitian symmetric spaces of noncompact type paired with some choice of invariant Kähler forms. A map \(\rho: X_1 \to X_2\) is called totally geodesic if the image of a geodesic in \(X_1\) is a geodesic in \(X_2\). A totally geodesic map \(\rho: X_1 \to X_2\) satisfies

\[
\sup_{\Delta \in X_1} \int_{\Delta} \rho^* \omega_2 \leq \sup_{\Delta \in X_2} \int_{\Delta} \omega_2
\]

where the supremum is taken over triangles with geodesic sides. We say that the map is tight if equality holds in (1.1).

Tight maps was extensively studied in [2] together with the notion of tight homomorphisms between Hermitian Lie groups. The latter is defined in greater generality but the case of Hermitian Lie groups was of special interest. The motivation for this is applications in the theory of maximal representations of surface groups. Let \(\Gamma\) be a surface group, \(G_1, G_2\) Hermitian Lie groups and \(\rho: \Gamma \to G_1\) and \(f: G_1 \to G_2\) homomorphisms. Assume that \(f\) is positive. Then \(f \circ \rho\) is maximal if and only if \(\rho\) is maximal and \(f\) is tight [1].

In [2] it was conjectured that a tight map \(\rho: X_1 \to X_2\) between irreducible spaces must be (anti-) holomorphic if \(X_1\) is of rank greater than one. It was also conjectured that \(\rho\) must be (anti-) holomorphic if \(X_1\) is not of tube type. Together these conjectures are equivalent to saying that \(\rho\) must be (anti-) holomorphic if \(X_1\) is not the Poincaré disc. In this paper we prove this for \(X_2\) classical.

Theorem 1.1. Let \(X_1\) and \(X_2\) be irreducible Hermitian symmetric spaces. Assume that \(X_1\) is not the Poincaré disc and that \(X_2\) is classical. If \(\rho: X_1 \to X_2\) is a tight map, then it is (anti-) holomorphic.

Together with previous results in [7], this yields a complete classification of tight maps from irreducible Hermitian symmetric spaces into...
classical ones. We also get a partial result for the exceptional Hermitian symmetric spaces.

**Theorem 1.2.** Let $\mathcal{X}'$ be the exceptional Hermitian symmetric space associated to the symmetric pair $(\mathfrak{e}_{6(-14)}, \mathfrak{so}(10) + \mathbb{R})$. Further let $\mathcal{X}$ be an irreducible Hermitian symmetric space of rank at least two. If $\rho: \mathcal{X} \to \mathcal{X}'$ is a tight map, then it is (anti-) holomorphic.

2. Preliminaries

2.1. Notation. We start by settling the notation that will be used throughout the paper. We denote by $\mathcal{X}$ Hermitian symmetric spaces, by $G$ the identity component of the isometry group of $\mathcal{X}$, by $K$ the stabilizer of a chosen basepoint $0$ and by $\mathfrak{g}$ the Lie algebra of $G$ with Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. We identify $\mathfrak{t}_0 \mathcal{X} \simeq \mathfrak{p}$ and we use the letters $X, Y$ to denote either tangent vectors or elements of $\mathfrak{g}$. Further we use brackets $\langle \cdot, \cdot \rangle$ to denote the invariant Riemannian metric on $\mathcal{X}$, normalized such that the holomorphic sectional curvature is $-1$, as well as the Killing form of $\mathfrak{g}$, it should be clear from the context which is meant. In the calculations we will make with the Killing form for matrix algebras we will be free to renormalize it, we will do this as $\langle X, Y \rangle = \text{tr}(X^t Y)$ to make calculations simpler. The invariant complex structure on $\mathcal{X}$ is denoted by $J$ and the element in the center of $\mathfrak{k}$ inducing the complex structure on $\mathfrak{p}$ by $Z$. We will denote by $J'$ the set of $G$-invariant complex structures on $\mathcal{X}$, and by $J'$ a minimal subset with the property that $J' \cup -J' = J$. Finally, we denote by $\omega$ the associated Kähler form defined by $\omega(X, Y) = \langle JX, Y \rangle$. By the indexation it should be clear which spaces, groups etc. belong together. We say that a Lie group is Hermitian if it is the identity component of an isometry group of a Hermitian symmetric space of noncompact type or a finite covering group of such.

2.2. Equivalence and equivalent formulations. A totally geodesic map $\rho_\mathcal{X}: \mathcal{X}_1 \to \mathcal{X}_2$ correspond to a Lie algebra homomorphism $\rho_\mathfrak{g}: \mathfrak{g}_1 \to \mathfrak{g}_2$. In turn, a Lie algebra homomorphism lifts to a group homomorphism $\rho_G: \tilde{G}_1 \to G_2$ for some finite cover $\tilde{G}_1$ of $G_1$. This will allow us to treat the problem from three perspectives. We will denote by the same letter the corresponding maps and group and algebra homomorphisms. We will later give conditions for Lie algebra and group homomorphisms to correspond to tight maps. To avoid overuse of the phrase ”corresponding to” we will talk about holomorphic homomorphisms when we mean that the corresponding totally geodesic map is holomorphic. We will also say that a Lie algebra is of tube type when the corresponding symmetric space has the property and say that maps and algebra homomorphisms are positive when the corresponding group homomorphism is.
We say that two maps \( \rho, \rho' : X_1 \to X_2 \) are equivalent if there is a \( g \in G_2 \) such that \( \rho = g \cdot \rho' \). Two homomorphisms \( \rho, \rho' : g_1 \to g_2 \) are equivalent if there is a \( g \in G_2 \) such that \( \rho = Ad(g) \cdot \rho' \). Finally we say that two homomorphisms \( \rho, \rho' : \tilde{G}_1 \to G_2 \) are equivalent if there is a \( g \in G_2 \) such that \( \rho = g \cdot \rho' \).

2.3. Outline of the proof. The main idea in the proof of Theorem 1.1 relies on the following three facts. First, for compositions of homomorphisms \( h \circ f : g_1 \to g_2 \to g_3 \) we have, up to some technicalities, that \( h \circ f \) is tight if and only if both \( f \) and \( h \) are tight. Second, we know that we can tightly and holomorphically embed the Poincaré disc into any Hermitian symmetric space. Third, the tight maps from the Poincaré are fully classified with a quite explicit description in [2].

To prove Theorem 1.1 we consider the following composition \( \rho \circ \iota : su(1,1) \to g_1 \to g_2 \) where we have chosen \( \iota \) tight and holomorphic. We then have that \( \rho \) is tight if and only if \( \rho \circ \iota \) is tight. As \( \rho \circ \iota \) is a map from \( su(1,1) \) to \( g_2 \) we can use the classification of tight maps from \( su(1,1) \) to see for which \( \rho \) it is tight. In practice this is possible to calculate for \( g_1 \) of low rank and \( g_2 \) classical. We do these calculations for \( g_1 = sp(4), sp(4) \oplus su(1,1) \) and \( su(2,1) \) and conclude that if \( \rho \) is non-holomorphic \( \rho \circ \iota \) will not be tight and hence \( \rho \) will not be tight. We choose these because from the classification of tight holomorphic maps in [7] we know that in each Hermitian Lie algebra except \( su(1,1) \) we can tightly and holomorphically embed either \( sp(4), sp(4) \oplus su(1,1) \) or \( su(2,1) \). We will then consider factorizations \( \rho \circ \eta : g_1 \to g_2 \to g_3 \) with \( g_1 \) being either \( sp(4), sp(4) \oplus su(1,1) \) or \( su(2,1) \) and \( \eta \) a tight and holomorphic embedding. If \( \rho \) would be tight and nonholomorphic so would \( \rho \circ \eta \) be. This contradicts the previous calculation. This proves that all tight maps into classical Hermitian symmetric spaces must be (anti-) holomorphic.

In section 3 we use continuous bounded cohomology to investigate when the composition of two maps is tight. In section 4 we recall some facts from representation theory that will be needed. In section 5 we do the calculations mentioned above for \( sp(4), sp(4) \oplus su(1,1) \) and \( su(2,1) \). In section 6 we complete the proof of Theorem 1.1.

3. Continuous bounded cohomology

In this section we recall some of the theory of continuous bounded cohomology. We will use this to answer questions concerning when the composition of two maps is tight. For a thorough review of the theory see [10].

Let \( G \) be a locally compact second countable group and define \( C^k(G, \mathbb{R}) = \{ f : G^{k+1} \to \mathbb{R}, f \text{ is continuous and bounded} \} \). \( C^k(G, \mathbb{R}) \) is naturally equipped with the supremum norm. We define a \( G \)-action on \( C^k(G, \mathbb{R}) \) as follows \( (g \cdot f)(g_0, \ldots, g_k) = f(g^{-1}g_0, \ldots, g^{-1}g_k) \). Denote by \( C^k(G, \mathbb{R})^G \)
the $G$-invariant elements of $C^k(G, \mathbb{R})$. These form a complex

$$0 \to C^0(G, \mathbb{R})^G \to d_0 C^1(G, \mathbb{R})^G \to d_1 C^2(G, \mathbb{R})^G \to d_2 \ldots$$

where $d_{k-1} f(g_0, \ldots, g_k) = \sum_{j=0}^k (-1)^j f(g_0, \ldots, \hat{g_j}, \ldots, g_k)$. We define

$$H^k_{cb}(G, \mathbb{R}) = \text{Ker}(d_k)/\text{Im}(d_{k-1}).$$

The norm on $C^k(G, \mathbb{R})$ induces a seminorm on $H^k_{cb}(G, \mathbb{R})$ by

$$||[f]|| = \inf_{h \in [f]} ||h||.$$

For Hermitian Lie groups this is a norm [3]. A homomorphism $\rho: G \to H$ between groups induces a pullback map between the cohomology groups $\rho^*: H^k_{cb}(H, \mathbb{R}) \to H^k_{cb}(G, \mathbb{R})$, $[f] \mapsto [f \circ \rho]$. From the definition we see that this must always be norm decreasing.

We will from here on restrict our attention to cohomology in degree two and to Hermitian Lie groups. Let $G$ be a Hermitian Lie group and $X$ the associated symmetric space. Then

$$c_\omega(g_0, g_1, g_2) := \int_{\Delta(g_0x_0, g_1x_0, g_2x_0)} \omega$$

defines a cocycle, where $\Delta(g_0x_0, g_1x_0, g_2x_0)$ denotes the triangle with geodesic sides with corners $g_ix_0$ for some point $x_0 \in X$. We will denote the corresponding cohomology class by $\kappa_G$. It is implicit in the definition that $\kappa_G$ depends on the complex structure $J$. We will sometimes use the notation $\kappa_{G,J}$ or write $(G, J)$ for a group with a certain complex structure associated to it when the dependence is crucial. From the definition of this class we see immediatly that $\kappa_{G,-J} = -\kappa_{G,J}$.

The norms $||\kappa_{G,J}^k||$ were computed for the classical case in [5] and equals $r_G \pi$, where $r_G$ is the real rank of the group $G$. Another approach using the Maslov index in [4] covered the exceptional cases.

**Definition 3.1.** Let $\rho: G_1 \to G_2$ be a homomorphism between Hermitian Lie groups. We say that $\rho$ is tight if $||\rho^*\kappa_{G_2}|| = ||\kappa_{G_2}||$.

The following theorem from [2] allows us to translate results concerning tight homomorphisms to tight maps.

**Theorem 3.2.** The homomorphism $\rho: G_1 \to G_2$ is tight if and only if the corresponding totally geodesic map $\rho: X_1 \to X_2$ is tight.

Let $G = G_1 \times \ldots \times G_n$ be a decomposition of $G$ into simple factors and $X = X_1 \times \ldots \times X_n$ be the corresponding decomposition of the symmetric space into irreducible symmetric spaces. The complex structure $J$ on $X$ determines a complex structure $J_i$ on each $X_i$. Recall that for an irreducible Hermitian symmetric space there are two possible choices of complex structure. We have, [2],

$$H^2_{cb}(G) \cong \prod H^2_{cb}(G_i) \cong \prod \mathbb{R}\kappa_{G_i}.$$
and with a slight abuse of notation we write \( \kappa_{G,J} = \sum_i \kappa_{G_i,J_i}^b \). The complex structure \( J \) for \( X \) thus defines an orientation for each \( H^2_{cb}(G_i) \).

**Definition 3.3.** We say that a class \( \alpha \in H^2_{cb}(G) \) is

1. (1) positive if \( \alpha = \sum \mu_i \kappa_{G_i,J_i}^b \) where \( \mu_i \geq 0 \) for all \( i = 1, \ldots, n \), and
2. (2) strictly positive if \( \alpha = \sum \mu_i \kappa_{G_i,J_i}^b \) where \( \mu_i > 0 \) for all \( i = 1, \ldots, n \).

**Definition 3.4.** We say that a homomorphism \( \rho: G_1 \to G_2 \) is (strictly) positive if \( \rho^* \kappa_{G_2} \) is (strictly) positive. In analogous way we also define (strictly) negative classes and homomorphisms.

**Lemma 3.5.** If \( \rho: G_1 \to G_2 \) is holomorphic then it is positive. If it is holomorphic and corresponds to an injective map then it is strictly positive.

**Proof.** We consider the corresponding totally geodesic map \( \rho \), we have
\[
\rho^* \omega_2(X,Y) = g_2(\rho_* X, J_2 \rho_* Y) = g_2(\rho_* X, \rho_* J_1 Y).
\]
Since \( \rho \) is an isometry up to scaling we have that \( \rho^* \omega_2(X,Y) \) is a positive multiple of \( \omega_1(X,Y) \) hence \( \rho \) is positive. If we further assume injectivity we can conclude that this multiple is non-zero hence \( \rho \) is strictly positive. \( \square \)

We will now investigate when compositions of homomorphisms and hence of maps are tight. We start with two lemmas from [2].

**Lemma 3.6.** Let \( f: G_1 \to G_2 \) and \( h: G_2 \to G_3 \) be homomorphisms. Assume \( f \) is tight. If \( h \) is tight and positive or tight and negative then \( h \circ f \) is tight.

**Lemma 3.7.** Let \( \rho: G_1 \to G_2 \) be a tight homomorphism. Then there exist a complex structure for \( X_1 \) such that \( \rho \) is tight and positive.

The following lemma will be very useful.

**Lemma 3.8.** Let \( f: G_1 \to G_2 \) and \( h: G_2 \to G_3 \) be homomorphisms. Assume \( G_2 \) is simple. Then \( h \circ f \) is tight if and only if both \( f \) and \( h \) are tight.

**Proof.** Assume \( h \circ f \) is tight. We have
\[
\| \kappa_{G_3} \| = \| f^* h^* \kappa_{G_3} \| \leq \| h^* \kappa_{G_3} \| \leq \| \kappa_{G_3} \|.
\]
Hence \( \| h^* \kappa_{G_3} \| = \| \kappa_{G_3} \| \) i.e. \( h \) is tight. Since \( G_2 \) is simple we have \( H^2_{cb}(G_2) = \mathbb{R} \kappa_{G_2} \) and hence \( h^* \kappa_{G_3} = \pm \frac{r_{G_3}}{r_{G_2}} \kappa_{G_2} \). We have
\[
r_{G_3} \pi = \| \kappa_{G_3} \| = \| f^* h^* \kappa_{G_3} \| = \| \pm \frac{r_{G_3}}{r_{G_2}} h^* \kappa_{G_2} \| = \frac{r_{G_3}}{r_{G_2}} \| f^* \kappa_{G_2} \|
\]
Hence \( \| f^* \kappa_{G_2} \| = r_{G_2} \pi = \| \kappa_{G_2} \| \), i.e. \( f \) is tight.
Assume that $f$ and $h$ are tight. Since $G_2$ is simple there are only two associated complex structures to it. Hence $h$ is tight and positive or negative. Lemma 3.10 then implies that $h \circ f$ is tight.

What we want is some kind of contrapositive of Lemma 3.6 that $h \circ f$ not tight implies $h$ not tight.

Lemma 3.9. Let $f^J: G_1 \to (G_2, J)$ and $h: (G_2, J) \to G_3$ be homomorphisms. Assume that $f^J$ is tight for all $J \in \mathcal{F}'_2$. If $h \circ f^J$ is nontight for all $J \in \mathcal{F}'_2$ then $h$ is nontight.

Proof. We prove the contrapositive of the statement. If $h$ is tight, there is by Lemma 3.7 a complex structure $J$ on $X_2$ such that $h$ is tight and positive. Either $J$ or $-J$ is in $\mathcal{F}'_2$. In the first case $h$ is tight and positive hence $h \circ f^J$ is tight by Lemma 3.6. In the second case $h$ is tight and negative and again by Lemma 3.6 $h \circ f^{-J}$ is tight.

We have the following lemma from [2].

Lemma 3.10. Let $f: G \to \prod_{i=1}^n G_i$ be a homomorphism, and assume the $G_j$ are simple. Let $p_j: \prod_{i=1}^n G_i \to G_j$ be the projection maps for $j = 1, \ldots, n$. Then $f$ is tight if and only if $p_j \circ f$ is tight and positive for all $j$ or tight and negative for all $j$.

Again we are more interested in when $f$ fails to be tight. For this it suffices that one single $p_j \circ f$ fails to be tight.

Lemma 3.11. Let $f: G \to H$ and $h: H \to L$ be homomorphisms of Hermitian Lie groups. Assume that $h$ is strictly positive. If $f$ is nontight then $h \circ f$ is nontight.

Proof. Let $H = \prod_{i=1}^n H_i$ and $G = \prod_{j=1}^N G_j$ be decompositions of $H$ and $G$ into simple factors. Denote by $\kappa_L, \kappa_{H_i}, \kappa_{G_j}$ the Kähler classes. We have that $h^* \kappa_L = \sum_{i=1}^n \lambda_i \kappa_{H_i}$ where $\sum_{i=1}^n \lambda_i r_{H_i} \leq r_L$ and all $\lambda_i > 0$ since $h$ is strictly positive. We have $f^* h^* \kappa_L = \sum_{i,j} \lambda_i \mu_{ij} \kappa_{G_j}$ where $\sum_j |\mu_{ij}| r_{G_j} \leq r_H$. Now if $f$ is not tight, we have a strict inequality for some $i$. Thus we get $||f^* h^* \kappa_L|| = \sum_{i,j} ||\lambda_i \mu_{ij} \kappa_{G_j}|| = \sum_{i,j} \lambda_i |\mu_{ij}| r_{G_j} \pi \leq \sum_i \lambda_i r_{H_i} \pi \leq r_L \pi = ||\kappa_L||$, i.e. $f \circ h$ is not tight.

4. Representation theory

In this section we will recall some facts from representation theory that will be needed in section 5. As there are quite a lot of different notions we start with a subsection going through the basics while settling the notation.

4.1. Notation. Let $\mathfrak{g}$ be a complex semisimple Lie algebra of rank $n$ with root space decomposition $\mathfrak{h} + \sum_{\alpha \in A} \mathfrak{g}_\alpha$. Here $\mathfrak{h}$ is a maximal abelian subalgebra, $A \subset \mathfrak{h}^\ast$ the set of roots and $\mathfrak{g}_\alpha$ subspaces of $\mathfrak{g}$ such that $[H, X] = \alpha(H)X$ for all $H \in \mathfrak{h}$, $X \in \mathfrak{g}_\alpha$. There is a subset
{α₁, ..., αₙ} ⊂ A called simple roots defined by the property that any α ∈ A can be written as α = ∑ᵢ₌₁ⁿ aᵢαᵢ with either all aᵢ nonpositive or all aᵢ nonnegative integers. We get a partial ordering on the set of roots by saying that α = ∑ᵢ₌₁ⁿ aᵢαᵢ ≥ 0 if the aᵢ ≥ 0, and that α ≥ β if α − β ≥ 0. Using the Killing form we define H° α by ⟨H° α, H⟩ = α(H) for all H ∈ ℱ. The coroots are the elements H₁ = 2H° H₁/H₁H₁.

A complex representation of g is a homomorphism ρ: g → gl(V) where V is a complex vector space. We will often refer to V as the representation. We say that two representations ρ: g → gl(V) and ρ′: g → gl(V) are equivalent if ρ(·) = gρ′(·)g⁻¹ for some g ∈ GL(V).

A representation is said to be irreducible if the only ρ(g)-invariant subspaces of V are {0} and V itself. An arbitrary representation V decomposes into a sum of irreducible ones, though the decomposition is not necessarily unique. The equivalence classes of irreducible representations appearing however, are unique. A weight ω is an element of ℱ⁺ paired with a subspace V₀ ⊂ V such that ρ(H)v = ω(H)v for all v ∈ V₀ and H ∈ ℱ. The vector v is called a weight vector. The partial ordering of the roots gives us a partial ordering of the weights of a representation. An irreducible representation is determined up to equivalence by its highest weight. There is a set {ω₁, ..., ωₙ} ⊂ ℱ⁺ called fundamental weights defined as the dual base of the simple coroots, i.e. ωᵢ(Hαⱼ) = δᵢⱼ. Each weight can be written as a sum of fundamental weights with rational coefficients. We denote such weights as ω₁,...,ωₙ = ∑ᵢ₌₁ⁿ mᵢωᵢ and a corresponding weight vector by v₁,...,vₙ. We denote a representation with highest weight ω₁,...,ωₙ by ρₗ⁽₁,...,ₙ⁾. We will sometimes refer to it as a representation with highest weight (m₁,...,mₙ). The Weyl group W of g acts on ℱ⁺ and is generated by the reflections β → β − 2⟨β,α⟩/⟨α,α⟩α, α ∈ A. To see what weights appear in a representation V with highest weight ω one starts by considering the set W · ω. These points are the corners of a convex set C ⊂ ℱ⁺. The weights appearing are those ω + ∑ᵢ₌₁ⁿ aᵢαᵢ, aᵢ ∈ ℤ, that lie in C. Let ρ₁: g₁ → gl(V₁) and ρ₂: g₂ → gl(U₁) be representations. We will denote by ρ₁ ⊗ ρ₂: g₁ ⊗ g₂ → gl(V ⊗ U₁) the representation defined by ρ₁ ⊗ ρ₂(X,Y)v ⊗ u := ρ₁(X)v ⊗ u + v ⊗ ρ₂(Y)u. If ρ₁ and ρ₂ are irreducible, so is ρ₁ ⊗ ρ₂.

The following lemma allows us to use the powerful machinery of representation theory of complex semisimple Lie algebras.

**Lemma 4.1.** There is a one to one correspondence between complex representations of a Hermitian Lie algebra g and complex representations of its complexification g°C = g ⊗₁ℝ C.

**Proof.** Let ρ: g → gl(V) be an irreducible complex representation. We get the corresponding representation ρ°C: g°C → gl(V), X ⊗ z → zρ(X). If V has no ρ(g)-invariant subspaces the action of a larger algebra will not have any invariant subspaces. If we instead start with an
irreducible representation $\rho: \mathfrak{g}^C \to gl(V)$ we get a representation of $\mathfrak{g}$ by restriction. Assume $\rho$ is not irreducible, i.e. there is a nontrivial subspace $W \subset V$ invariant under $\rho|_{(\mathfrak{g})}$. Consider the action of $\rho(X \otimes z)$ on a vector $w \in W$. We have $\rho(X \otimes z)w = z\rho(X)w$. We have $\rho(X)w \in W$ by assumption and hence $z\rho(X)w \in W$ since $W$ is a complex vector space. Thus $W$ is $\rho(\mathfrak{g}^C)$-invariant contradicting the irreducibility of $\rho$. □

We denote by $\rho_{m_1 \ldots m_n}$ the irreducible complex representation of a Hermitian Lie algebra $\mathfrak{g}$ that corresponds the representation $\rho^C_{m_1 \ldots m_n}$ of $\mathfrak{g}^C$.

4.2. On equivalence. It should be remarked that the notion of tightness is not welldefined for equivalence classes of representations. We illustrate with the following example. Let $f, d: su(1, 1) \to su(1, 1) \oplus su(1, 1)$ be the homomorphisms $f(X) = (X, -X^t)$, $d(X) = (X, X)$ inducing the maps $z \mapsto (z, \bar{z})$, $z \mapsto (z, z)$ from $\mathbb{D}$ to $\mathbb{D} \times \mathbb{D}$. The first is easily seen to be nontight and the second tight. However, both representations are in the same equivalence class of representations.

The equivalence class of an irreducible representation $\rho: \mathfrak{g} \to su(p, q)$ contains at most two equivalence classes of homomorphisms, the (homomorphism) equivalence class of $\rho$ and, in the case $p = q$ the (homomorphism) equivalence class of $\rho$ composed with $\theta$. Here $\theta$ is the antiholomorphic isomorphism of $su(p, p)$ induced by conjugation with the element $\begin{pmatrix} 0 & 1_p \\ -1_p & 0 \end{pmatrix} \in GL(2p, \mathbb{C})$. The homomorphism $\theta$ correspond to a tight map. Thus tightness is welldefined for equivalence classes of irreducible representations into $su(p, q)$ by Lemma 3.8.

For an equivalence class of a sum of representations we may not be able to say that it is tight. We can however say that it contains no tight equivalence classes of homomorphisms in the following setting.

An equivalence class of a sum of irreducible representations $\sum \rho^i: \mathfrak{g} \to \oplus su(r_i, s_i)$ contains no tight homomorphisms if one $\rho^i$ is nontight. This follows by Lemma 3.10. This is all we will need.

4.3. Regular subalgebras. A regular subalgebra of a complex semisimple Lie algebra $\mathfrak{g}$ is a subalgebra that agrees with the root space decomposition. More precisely, we choose a subset $B \subset A$ such that

1. if $\alpha, \beta \in B$ then $\alpha - \beta \notin B$,
2. $B$ is linearly independent in $\mathfrak{h}^*$. 

Let $C = \sum_{\alpha \in B} \mathbb{Z}\alpha \cap A$ and define the subalgebra 

$$ \mathfrak{g}(B) := \sum_{\alpha \in B} \mathbb{C}H_\alpha + \sum_{\alpha \in C} g_\alpha $$

We say that a subalgebra of $\mathfrak{g}$ is a regular subalgebra if it can be constructed in this way. For a real semisimple Lie algebra $\mathfrak{g}$ we say that a
A subalgebra \( g' \) is **regular subalgebra** if it can be written as \( g' = g^C(B) \cap g \) where \( g^C(B) \) is a (complex) regular subalgebra of the complexification \( g^C \).

For Hermitian Lie algebras we would like regular subalgebras to be of Hermitian type also. We can ensure this by adding the condition that

(3) each connected component in the Dynkin diagram of \( B \) contains exactly one noncompact root.

### 4.4. Decomposing representations.

**Lemma 4.2.** Let \( \rho: g \to gl(V) \) be a complex representation of a Hermitian Lie algebra \( g \) containing only \( su(p,q) \) and \( sp(2m) \) as simple factors. Then there exists an indefinite Hermitian form on \( V \) that is invariant under \( \rho(g) \).

For \( g \) simple this can be seen by realising an irreducible representation as a subspace \( V \) of the standard representation tensored to some power. The standard representation carries a Hermitian form which has a natural extension to the tensor product. Then one has to check that the highest weight vector is not a null vector. From this it is not hard to deduce that the whole space \( V \) must be nondegenerate. The general case follows from that irreducible representations of semisimple algebras are tensor products of the the representations of the simple factors. The following theorem together with Lemma 3.10 allows the reduction to homomorphisms which are irreducible when considered as representations.

**Theorem 4.3.** Let \( U \) be a complex vector space equipped with an indefinite Hermitian form \( F \). Further let \( \rho: g \to su(U,F) \) be a homomorphism, with \( g \) containing only \( su(p,q) \) and \( sp(2m) \) as simple factors. If \( \rho \) is not irreducible viewed as a representation on \( U \) there is a decomposition \( U = \bigoplus_{i=1}^{n} U_i \) such that \( F|_{U_i} \) is nondegenerate for all \( i \). This means that \( \rho = (\rho^1, ..., \rho^n): g \to \bigoplus_i su(r_i, s_i) \subset su(U,F) \) where the inclusion is holomorphic and all \( \rho^i \) are irreducible representations.

**Proof.** Start with an arbitrary decomposition into irreducible invariant subspaces \( U = \bigoplus_{i \in I} U_i \). There are two possibilities, either \( F|_{U_i} \) is degenerate for all \( i \) or there exists some \( i \) such that it is nondegenerate. In the second case we claim that \( U_i^\perp \), the orthogonal complement of \( U_i \), is an invariant subspace. Taking \( v \in U_i^\perp \), we have

(4.1) \[
0 = F(\rho(X)u, v) + F(u, \rho(X)v) = F(u, \rho(X)v)
\]

for all \( u \in U_i \), since \( U_i \) is invariant. This proves the claim. We can now take a decomposition of \( U_i^\perp \) into irreducible subspaces to get a new decomposition of \( U = U_i \oplus \bigoplus_{j \in I'} V_j \). Here \( \oplus^\perp \) denotes a direct sum that is orthogonal with respect to \( F \). Repeating this process we get,
with some reindexing, $U = U_1 \oplus^+ \ldots \oplus^+ U_n \oplus^+ \bigoplus_{j \in V} V_j$ where all the $U_i$ are nondegenerate and all the $V_j$ are degenerate irreducible subspaces.

Our next claim is that for an irreducible degenerate space $V_j$ we have $F|_{V_j} \equiv 0$. Since $F|_{V_j}$ is degenerate we have $v \in V_j$ such that $v \perp V_j$. If $\dim(V_j) = 1$ we are done, if not let $X \in \mathfrak{g}$ be such that $\rho(X)v \not\in \mathbb{C}v$.

We have that $0 = F(\rho(X)v, u) + F(v, \rho(X)u) = F(\rho(X)v, u)$ for all $u \in V_j$, which means that $\rho(X)v \perp V_j$. Using the fact that $V_j$ is irreducible and repeating this process we can conclude that $V_j \perp V_j$.

Since $F$ is a nondegenerate form there must be some $k \in I'$ such that $F|_{V_j \oplus V_k} \neq 0$. We claim that $F|_{V_j \oplus V_k}$ is nondegenerate. Assume otherwise, then there are vectors $u_j \in V_j$ and $u_k \in V_k$ such that $u_j + u_k \perp V_j \oplus V_k$. We have $0 = F(u_j + u_k, u) = F(u_j, u)$ for all $u \in V_k$ which implies $u_j \perp V_k$. We have further

$$0 = F(\rho(X)u_j, u) + F(u_j, \rho(X)u) = F(\rho(X)u_j, u)$$

for all $u \in V_k$ which means that $\rho(X)u_j \perp V_k$. Repeating this we get $V_j \perp V_k$ which means $F|_{V_j \oplus V_k} \equiv 0$ contradicting our assumption.

So the situation is that we have two irreducible invariant degenerate spaces $V, W$ such that $V \oplus W$ is nondegenerate. We claim that $\dim V = \dim W =: n$ and that the signature of $F|_{V \oplus W}$ is $(n, n)$. This follows from that the maximal dimension of a null subspace of a space of signature $(p, q)$ is $\min(p, q)$.

From Lemma 4.2 we know that there exists an indefinite Hermitian form $F_V$ of say signature $(p, q)$ on $V$ invariant under $\rho(\mathfrak{g})$. Take a basis $\{v_i\}_{i=1}^n$ for $V$ orthonormal with respect to $F_V$ with the first $p$ vectors positive, and let $\{w_j\}_{j=1}^n$ be some choice of basis for $W$. Representing $F$ as a matrix with respect to the basis $\{v_1, \ldots, v_n, w_1, \ldots, w_n\}$ we have $F(u_1, u_2) = u_1^T \begin{pmatrix} 0 & C \\ C^* & 0 \end{pmatrix} u_2$ for some invertible matrix $C$. Choosing a new basis $\{w_j = C^{-1}w_j\}_{j=1}^n$ for $W$ we can represent $F$ with the matrix $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. An arbitrary element $\rho(X)$ leaves the subspaces $V$ and $W$ invariant. This means it is of the form $\rho(X) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. It preserves the form $F$ which implies that $B = -A^*$. We have further that $A$ preserves the form $F_V$ which means that $A$ is of the form $A = \begin{pmatrix} K & Z \\ Z^* & L \end{pmatrix}$ with $K \in M_{p,p}(\mathbb{C}), L \in M_{q,q}(\mathbb{C})$ skewsymmetric and $\text{tr}(K + L) = 0$.

Now consider the following basis, let $\{e_1, \ldots, e_{2n}\} = \{v_1 + w_1, \ldots, v_p + w_p, v_{p+1} - w_{p+1}, \ldots, v_n - w_n, v_1 - w_1, \ldots, v_p - w_p, v_{p+1} + w_{p+1}, \ldots, v_n + w_n\}$.

With respect to this basis we have that $\rho(X) = \begin{pmatrix} K & Z & 0 & 0 \\ Z^* & L & 0 & 0 \\ 0 & 0 & K & Z \\ 0 & 0 & Z^* & L \end{pmatrix}$.
We see that the subspaces spanned by $\{e_i\}_{i=1}^n$ and $\{e_j\}_{j=n+1}^{2n}$ are invariant and a quick calculation shows that they are nondegenerate. We have now established an inductive procedure of picking out orthogonal nondegenerate subspaces out of an arbitrary decomposition. This proves the theorem. □

4.5. Criteria for tightness. In each Hermitian symmetric space $\mathcal{X}$ we can holomorphically and isometrically embed $D^{r_{\mathcal{X}}}$, the product of $r_{\mathcal{X}}$ Poincaré discs $\mathbb{D}$, where $r_{\mathcal{X}}$ is the rank of $\mathcal{X}$. Composing this embedding with the diagonal embedding $d: \mathbb{D} \to D^{r_{\mathcal{X}}}$ we get a tight and holomorphic map from $\mathbb{D}$ into $\mathcal{X}$ known as a diagonal disc. These maps are tight and play an important role in the following theorem from [2].

**Theorem 4.4.** Let $\rho: g_1 \to g_2$ be a homomorphism between Hermitian Lie algebras. Further let $d_1: su(1, 1) \to g_1$ be homomorphisms corresponding to diagonal discs. Then $\rho$ corresponds to a tight and positive map if and only if

$$\langle \rho d_1 Z_{su(1,1)} , Z_2 \rangle = \langle d_2 Z_{su(1,1)} , Z_2 \rangle .$$

Combining Theorem 4.4 with Lemma 3.7 we have the following corollary.

**Corollary 4.5.** Let $\rho: (g_1, J) \to g_2$ be a homomorphism between Hermitian Lie algebras. Further let $d_1: su(1, 1) \to g_1$ and $d_2: su(1, 1) \to g_2$ be homomorphisms corresponding to diagonal discs. Then $\rho$ corresponds to a tight map if and only if there is a $J \in J'_1$ such that

$$|\langle \rho d_1' Z_{su(1,1)} , Z_2 \rangle | = |\langle d_2 Z_{su(1,1)} , Z_2 \rangle | .$$

**Theorem 4.6.** Let $\rho: g \to su(r, s)$ be a homomorphism and an irreducible representation, where $g$ is a Hermitian Lie algebra with simple factors of type $su(p, q)$ and $sp(2p)$. Further let $\iota: g_0 \to g$ be a tight, injective and holomorphic homomorphism, where $g_0 = su(1, 1)$ or $su(1, 1) \oplus su(1, 1)$. Let $\rho = \sum \rho^j$ be a decomposition into irreducible representations. Each $\rho^j$ defines a homomorphism $\rho^j: g_0 \to su(r_i, s_i)$ for some $(r_i, s_i)$. If there is one nontight $\rho^j$ then $\rho$ is nontight.

**Proof.** By Theorem 1.3 we can factor $\rho \circ \iota$ as

$$\begin{array}{ccc}
g & \xrightarrow{\rho} & su(r, s) \\
\iota \downarrow & & \downarrow \iota' \\
g_0 & \xrightarrow{\sum \rho^j} & \oplus su(r_i, s_i) \\
\end{array}$$

with $\iota'$ injective and holomorphic. Lemma 3.5 thus implies that $\iota'$ is strictly positive. If there is one $\rho^j$ nontight Lemma 3.10 implies that $\sum \rho^j$ is nontight. Lemma 3.11 then implies that $\iota' \circ \sum \rho^j$ is nontight. Since the diagram commutes this implies that $\rho \circ \iota$ is nontight. Next
we want to apply Lemma 3.9 to conclude that \( \rho \) is nontight. This requires that we can construct tight homomorphisms \( \iota^J : \mathfrak{g}_0 \to (\mathfrak{g}, J) \) for all \( J \in \mathcal{J}' \) such that \( \rho \circ \iota^J \) is nontight. Let \( \mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_n \) be a decomposition into simple factors and \( J = (J_1, \ldots, J_n) \) the original complex structure for which \( \iota = (\iota_1, \ldots, \iota_n) \) is tight and such that \( \rho \circ \iota \) is equivalent to \( \rho \circ \iota^J \) as a representation. By the previous argument this means that \( \rho \circ \iota \) is nontight. We can then construct \( \iota^J \) for any \( J \in \mathcal{J}' \) by changing the complex structure in one simple factor at a time.

Let \( \theta : \mathfrak{g}_0 \to \mathfrak{g}_0 \) be the antiholomorphic isomorphism. We construct our new tight and holomorphic embedding \( \tilde{\iota} := (\iota_1 \circ \theta, \iota_2, \ldots, \iota_n) : \mathfrak{g}_0 \to (\mathfrak{g}, \tilde{J}) \). Define \( f : \mathfrak{g}_0 \oplus \mathfrak{g}_0 \to \mathfrak{g}, \quad (X, Y) \mapsto (\iota_1(X), \iota_2(Y), \ldots, \iota_n(Y)) \). We can factor our embeddings \( \iota = f \circ (\mathrm{Id}, \mathrm{Id}), \tilde{\iota} = f \circ (\theta, \mathrm{Id}) \). We need to show that the representations \( \rho \circ \iota = \rho \circ f \circ (\mathrm{Id}, \mathrm{Id}) \) and \( \rho \circ \tilde{\iota} = \rho \circ f \circ (\theta, \mathrm{Id}) \) are equivalent. With the factorisations we see that it is sufficient to show that \( \eta \circ (\mathrm{Id}, \mathrm{Id}) \) is equivalent to \( \eta \circ (\theta, \mathrm{Id}) \) for a representation \( \eta : \mathfrak{g}_0 \oplus \mathfrak{g}_0 \to \mathfrak{gl}(V) \). Let \( \mathfrak{g}_0 = \mathfrak{su}(1, 1) \oplus \mathfrak{su}(1, 1) \). An irreducible representation \( (\rho_k, V^k) \) of \( \mathfrak{su}(1, 1) \) is the restriction of an irreducible representation of \( \mathfrak{sl}(2, \mathbb{C}) \). \( V^k \) decomposes as \( V^k = \oplus_{j=0}^k V_{k-2j} \) with \( \rho_k(H)v_l = ilv_l \) for \( v_l \in V_l \) and \( H = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{su}(1, 1) \). An irreducible representation of \( \mathfrak{g}_0^{\otimes 2} = \mathfrak{su}(1, 1)^{\otimes 4} \) is the tensor product of four irreducible representations of \( \mathfrak{su}(1, 1) \), i.e. \( \eta = \rho_{k_1} \otimes \rho_{k_2} \otimes \rho_{k_3} \otimes \rho_{k_4} : \mathfrak{su}(1, 1)^{\otimes 4} \to \mathfrak{gl}(V^{k_1} \otimes V^{k_2} \otimes V^{k_3} \otimes V^{k_4}) \). We show that any weight space for \( \eta \circ (\mathrm{Id}, \mathrm{Id}) \) appears for \( \eta \circ (\theta, \mathrm{Id}) \) too. For each weight space \( V_a \otimes V_b \otimes V_c \otimes V_d \) of \( \eta \) there is also \( V_{-a} \otimes V_{-b} \otimes V_c \otimes V_d \). We have

\[
\eta \circ (\mathrm{Id}, \mathrm{Id})(H, H)v_{a,b,c,d} = \eta(H, H, H, H)v_{a,b,c,d} \\
= \rho_{k_1} \otimes \rho_{k_2} \otimes \rho_{k_3} \otimes \rho_{k_4}(H, H, H, H)(v_a \otimes v_b \otimes v_c \otimes v_d) \\
= \rho_{k_1}(H)v_a \otimes v_b \otimes v_c \otimes v_d + \ldots + v_a \otimes v_b \otimes v_c \otimes \rho_{k_4}(H)v_d \\
= i(a + b + c + d)v_{a,b,c,d}
\]

and

\[
\eta \circ (\theta, \mathrm{Id})(H, H)v_{-a,-b,c,d} = \eta(-H, -H, H, H) \\
= \rho_{k_1} \otimes \rho_{k_2} \otimes \rho_{k_3} \otimes \rho_{k_4}(-H, -H, H, H)(v_{-a} \otimes v_{-b} \otimes v_c \otimes v_d) \\
= \rho_{k_1}(-H)v_{-a} \otimes v_{-b} \otimes v_c \otimes v_d + \ldots + v_{-a} \otimes v_{-b} \otimes v_c \otimes \rho_{k_4}(H)v_d \\
= i(a + b + c + d)v_{-a,-b,c,d}
\]

Hence as representations of \( \mathfrak{su}(1, 1) \oplus \mathfrak{su}(1, 1), \eta \circ (\mathrm{Id}, \mathrm{Id}), V^{k_1} \otimes \ldots \otimes V^{k_4} \) and \( \eta \circ (\theta, \mathrm{Id}), V^{k_1} \otimes \ldots \otimes V^{k_4} \) decomposes into the same weight spaces. This implies that they are equivalent representations. The case \( \mathfrak{g}_0 = \mathfrak{su}(1, 1) \) is done in the same way. \[\square\]
We now have the tools we need to determine if a representation corresponds to a tight map. We will also need the following elementary fact about holomorphic maps between Hermitian symmetric spaces. It says roughly that if a map between two irreducible Hermitian symmetric spaces is holomorphic in one direction, it is holomorphic in all directions.

**Theorem 4.7.** Let \( \rho : \mathfrak{g}_1 \to \mathfrak{g}_2 \) be a homomorphism between simple Hermitian Lie algebras. If there is one nonzero vector \( X \in \mathfrak{p}_1 \) such that \( \rho([Z_1, X]) = [Z_2, \rho(X)] \) then this is true for all \( X \in \mathfrak{p}_1 \).

**Proof.** Consider \( X' = [Y, X] \) for some \( Y \in \mathfrak{k}_1 \). Using that \( Z_i \) is in the center of \( \mathfrak{k}_i \) and the Jacobi identity we have

\[
\rho([Z_1, X']) = \rho([Z_1, [Y, X]]) = -\rho([X, [Z_1, Y]] + [Y, [X, Z_1]]) = -\rho([Y, [Z_1, X]]) = [\rho(Y), \rho([Z_1, X])] = \rho([Y, Z_2]) - [Z_2, \rho(X)] = [Z_2, \rho([Y, X])].
\]

We thus have that \( \rho \) is holomorphic in the \( X' \)-direction. The fact that \( \mathfrak{p}_1 \) is an irreducible representation of \( \mathfrak{k}_1 \) now proves the theorem. \( \square \)

5. Some key low rank cases

5.1. **Representations of \( su(1, 1) \).** The root system of \( sl(2, \mathbb{C}) \) is \( A = \{ \alpha, -\alpha \} \) and the fundamental weight is \( \omega = \frac{\alpha}{2} \). The irreducible complex representation of highest weight \( k \) can be realised as the symmetric tensor product \( V^{\otimes k} \) of the standard representation \( V \cong \mathbb{C}^2 \). Restricting these representations to the real form \( su(1, 1) \) we get the irreducible complex representations of \( su(1, 1) \). Let \( \mathfrak{S}_k \) be the symmetric group. It acts on \( V^{\otimes k} \) by permuting the factors. For \( v, w \in V \) we define

\[
v^l w^{k-l} := \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \sigma(v \otimes \ldots \otimes v \otimes w \otimes \ldots \otimes w) \in V^{\otimes k}
\]

where there are \( l \) copies of \( v \) and \( k - l \) copies of \( w \) in the product. Let \( \{e_1, e_2\} \) be an orthonormal basis for \( V \) with respect to \( F \), the Hermitian form invariant under \( su(1, 1) \), \( e_1 \) being positive and \( e_2 \) negative. We can extend \( F \) to a Hermitian form on \( V^{\otimes k} \) in the natural way, \( F(v_1 \otimes \ldots \otimes v_k, w_1 \otimes \ldots \otimes w_k) := \prod_i F(v_i, w_i) \). \( su(1, 1) \) preserves this form. If \( k = 2l \) is even we have an orthogonal basis \( \{f_0, f_1, \ldots, f_k\} = \{e_1^k, e_1^{k-2}e_2, \ldots, e_1^{k-l}e_2, \ldots, e_1e_2^{k-l}\} \) for \( V^{\otimes k} \) with \( \{f_0, f_1, \ldots, f_l\} \) positive and \( \{f_{l+1}, f_{l+2}, \ldots, f_k\} \) negative. Hence \( F \) is of signature \( (l + 1, l) \) and the representation defines a homomorphism into \( su(l + 1, l) \). If \( k = 2l - 1 \) is odd we have an orthogonal basis \( \{h_1, \ldots, h_{k+1}\} = \{e_1^{2l-1}, e_1^{2l-3}e_2, \ldots, e_1e_2^{2l-2}, e_1^{2l-4}e_2, \ldots, e_2^{2l-1}\} \) for \( V^{\otimes k} \) with \( \{h_1, h_2, \ldots, h_l\} \) positive and \( \{h_{l+1}, h_{l+2}, \ldots, h_k\} \) negative.
Hence $F$ is of signature $(l, l)$ and the representation defines a homomorphism into $su(l, l)$.

**Theorem 5.1.** Let $\rho_k : su(1, 1) \to su(p, q)$ be a homomorphism and an irreducible complex representation of highest weight $k$. Then $\rho_k$ is tight if and only if $k$ is odd.

**Proof.** To see if these representations are tight we need to calculate the image of the complex structure. With respect to the basis $\{e_1, e_2\}$ we can write the complex structure as the matrix $Z = \frac{i}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The action on $V^{\otimes k}$ is $\rho_k(Z)(e_1^{m}e_2^{n}) = \frac{1}{2}(k - 2m)(e_1^{m}e_2^{k-m})$. With respect to the basis $\{f_i\}$ we get the matrix $\rho_k(Z) = \frac{i}{2}\text{diag}(k, k - 4, ..., -k, k - 2, k - 6, ..., 2 - k) = \frac{i}{2} \begin{pmatrix} A_k & 0 \\ 0 & B_k \end{pmatrix}$ for $k$ even. The block form is with respect to the positive respectively the negative base vectors and will be used later. For $k$ odd we get $\rho_k(Z) = \frac{i}{2}\text{diag}(k, k - 4, ..., 2 - k, k - 2, k - 6, ..., -k) = \frac{i}{2} \begin{pmatrix} C_k & 0 \\ 0 & D_k \end{pmatrix}$ for $k$ odd. For $su(n, n)$ the complex structure is $Z_{n,n} = \frac{i}{2} \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$ and for $su(n+1, n)$ it is $Z_{n,n+1} = \frac{i}{2n+1} \begin{pmatrix} nI_{n+1} & 0 \\ 0 & -(n + 1)I_n \end{pmatrix}$. A quick calculation shows that $|\langle \rho_{2l-1}d_1Z, Z_{l,l}\rangle| = |\langle d_2Z, Z_{l,l}\rangle|$ and $|\langle \rho_{2l}d_1Z, Z_{l+1,l}\rangle| \neq |\langle d_2Z, Z_{l+1,l}\rangle|$. Hence by Corollary 1.5 $\rho_k$ is tight if and only if $k$ is odd. □

5.2. **Representations of $su(1, 1) \oplus su(1, 1)$**. An irreducible representation $(\rho, W)$ of $su(1, 1) \oplus su(1, 1)$ is the tensor product of two irreducible representations $(\rho_1, V)$ and $(\rho_2, U)$ of $su(1, 1)$. If $V$ is equipped with a $su(1, 1)$-invariant Hermitian form of signature $(a, b)$ and $U$ one of signature $(c, d)$ then $V \otimes W$ has a canonical $su(1, 1) \oplus su(1, 1)$-invariant Hermitian form of signature $(ac + bd, ad + bc)$, so $\rho$ defines a homomorphism into $su(ac + bd, ad + bc)$.

**Theorem 5.2.** Let $\rho = \rho_k \otimes \rho_l : su(1, 1) \oplus su(1, 1) \to su(r, s)$ be a homomorphism and an irreducible complex representation. Then $\rho$ is tight if and only if $k$ is odd and $l = 0$ or vice versa.

**Proof.** Let $\iota_1$ denote the diagonal embedding of $su(1, 1)$ into $su(1, 1) \oplus su(1, 1)$, $X \mapsto (X, X)$, and $\iota_2$ the map $X \mapsto (X, -X^t)$. Then $\iota_1$ is tight for $su(1, 1) \oplus su(1, 1)$ equipped with the complex structure $(Z, Z)$, and $\iota_2$ is tight for $su(1, 1) \oplus su(1, 1)$ equipped with the complex structure $(Z, -Z)$.

We first consider the case $k, l$ both odd or both even. Composing $\rho \circ \iota_1$, we get a representation of $su(1, 1)$ that decomposes into irreducible representations of highest weights $k+l, k+l-2, ..., |k-l|$, see [6] p. 332.
All of these are of even highest weight hence $\rho$ is nontight Theorem 5.1 and 4.6.

Now consider the case of $\ell = 2p - 1$ odd and $k = 2q$ even. We have that $(r, s) = (p(q+1)+pq, pq+p(q+1)) = (2pq+p, 2pq+p)$. Depending on the choice of complex structure on $su(1,1) \oplus su(1,1)$ we have two different diagonal discs $\iota_1$ and $\iota_2$. We will now choose a natural basis for $V$ in terms of the bases $\{f_j\}, \{h_i\}$ defined in the previous subsection. We do this so that the first $2pq$ vectors are positive and such that we can easily see how $su(1,1)$ acts by considering various tensor products of $A_k, B_k, C_l, D_l$, the matrices defined in the proof of Theorem 5.1 with appropriately sized identity matrices. Let

$$\begin{align*}
\{e_{i+pj}\} &= \{f_j \otimes h_i | i = 1, \ldots, p, j = 0, \ldots, q\}, \\
\{e_{i+pj-p}\} &= \{f_j \otimes h_i | i = p+1, \ldots, 2p, j = q+1, \ldots, 2q\}, \\
\{e_{i+pj+2pq}\} &= \{f_j \otimes h_i | i = p+1, \ldots, p, j = 0, \ldots, q\}, \\
\{e_{i+pj+p+pq}\} &= \{f_j \otimes h_i | i = 1, \ldots, p, j = q+1, \ldots, 2q\}.
\end{align*}$$

With this choice of basis we get the following matrix description

$$\rho \circ \iota_1(Z) = \frac{i}{2} \text{diag}(A_k \otimes I_p + I_{q+1} \otimes C_l, B_k \otimes I_p + I_q \otimes D_l, A_k \otimes I_p + I_{q+1} \otimes D_l, B_k \otimes I_p + I_q \otimes C_l).$$

Therefore

$$\langle \rho \circ \iota_1(Z), Z_{l(2k+1),l(2k+1)} \rangle = \frac{1}{4}(tr(A_k)p + tr(C_l)(q+1) + tr(B_k)p + tr(D_l)q - tr(A_k)p - tr(D_l)(q+1) - tr(B_k)p - tr(C_l)q)$$

$$= \frac{1}{4}(tr(C_l) - tr(D_l)) = \frac{1}{4}(p - (-p)) = \frac{p}{2}.$$

For the other choice of complex structure we have the matrix description

$$\rho \circ \iota_2(Z) = \frac{i}{2} \text{diag}(A_k \otimes I_p + I_{q+1} \otimes -C_l, B_k \otimes I_p + I_q \otimes -D_l, A_k \otimes I_p + I_{q+1} \otimes -D_l, B_k \otimes I_p + I_q \otimes -C_l).$$

Thus

$$\langle \rho \circ \iota_2(Z), Z_{l(2k+1),l(2k+1)} \rangle = \frac{1}{4}(tr(A_k)p - tr(C_l)(q+1) + tr(B_k)p - tr(D_l)q$$

$$- tr(A_k)p + tr(D_l)(q+1) - tr(B_k)p + tr(C_l)q)$$

$$= -\frac{1}{4}(tr(C_l) + tr(D_l)) = -\frac{p}{2}.$$

We have

$$\langle d(Z), Z_{p(2q+1),p(2q+1)} \rangle = \langle Z_{p(2q+1),p(2q+1)}; Z_{p(2q+1),p(2q+1)} \rangle = \frac{p}{2}(2q+1).$$

From Corollary 4.5 we have that $\rho$ is tight if and only if $\frac{p}{2} = \frac{p}{2}(2q+1)$ or equivalently $k = 2q = 0$. 

5.3. **Representations of\( sp(4, \mathbb{R}) \).** For \( sp(4, \mathbb{C}) \) we have the root system \( \Lambda = \{ \pm \alpha_1, \pm \alpha_2, \pm (\alpha_1 + \alpha_2), \pm (2\alpha_1 + \alpha_2) \} \). We differ from the notation in [7] and [9] and let \( \alpha_2 \) denote the longer noncompact simple root. We have the fundamental weights \( \omega_1 = \frac{2\alpha_1 + \alpha_2}{2} \) and \( \omega_2 = \alpha_1 + \alpha_2 \). From [11], [7] we know that the only irreducible representation corresponding to a holomorphic map is \( \rho_{10} \) and that this map is tight.

**Theorem 5.3.** Let \( \rho: sp(4, \mathbb{R}) \to su(p, q) \) be a homomorphism and an irreducible complex representation. If it is tight then it is (anti-) holomorphic.

**Proof.** From [7] we know that there are two tight regular subalgebras of \( sp(4, \mathbb{R}) \). The first is \( \mathfrak{g}(\alpha_1 + \alpha_2) \cong su(1, 1) \) and the second is \( \mathfrak{g}(\{\alpha_2, 2\alpha_1 + \alpha_2\}) = \mathfrak{g}(\alpha_2) \oplus \mathfrak{g}(2\alpha_1 + \alpha_2) \cong su(1, 1) \oplus su(1, 1) \).

We divide the representations into two types. The first type are the representations with highest weight \((0, l)\). The second type is the remaining representations, with highest weight \((k, l)\) where \( k \neq 0 \). We exclude the case \((k, l) = (1, 0)\) which is tight and holomorphic.

To see that a representation of the first type is nontight we restrict it to the tightly embedded regular subalgebra \( \mathfrak{g}(\alpha_1 + \alpha_2) \). We then move over to the complexification of the algebras and the representation. We have \( \rho^C(H_{\alpha_1+\alpha_2}) v_{0,l} = l(\alpha_1 + \alpha_2)(H_{\alpha_1+\alpha_2}) v_{0,l} = 2l v_{0,l} \). So we know that \( \rho^C \) branches into at least one representation of \( \mathfrak{g}(\alpha_1 + \alpha_2)^C \) with even highest weight. By Theorem 5.1 and 4.6 this means that \( \rho \) is nontight.

To see that a representation of the second type is nontight we consider the restriction to the regular subalgebra \( \mathfrak{g}(2\alpha_1 + \alpha_2) \). Again we move over to the complexifications and we have \( \rho^C(H_{2\alpha_1+\alpha_2}) v_{k,l} = (k \frac{2\alpha_1 + \alpha_2}{2} + l(\alpha_1 + \alpha_2))(H_{2\alpha_1+\alpha_2}) v_{k,l} = (k + l) v_{k,l} \). We also have a weight \( \omega_{k,l-1} \) which when restricted to the subalgebra becomes \( k+l-1 \). One of these numbers is even. If it is \( k+l \) we know that it is even and nonzero, if it \( k+l-1 \) we know that it is nonzero since \( (k, l) = (0, 1) \) is of the first type and \((1, 0) \) excluded. This means that when we branch \( \rho \) to the subalgebra \( \mathfrak{g}(\alpha_1) \oplus \mathfrak{g}(\alpha_2) \) there will appear even \( j \)'s in the decomposition \( \rho|_{\mathfrak{g}(\alpha_1)} \oplus \mathfrak{g}(\alpha_2) = \sum_{(i,j)\in I} \rho_i \otimes \rho_j : su(1, 1) \oplus su(1, 1) \to su(p, q) \). By Theorem 5.2 and 4.6 this means that \( \rho \) is nontight. \( \square \)

5.4. **Representations of \( sp(4, \mathbb{R}) \oplus su(1, 1) \).**

**Theorem 5.4.** Let \( \rho = \rho_{ij} \otimes \rho_k : sp(4, \mathbb{R}) \oplus su(1, 1) \to su(p, q) \) be a homomorphism and an irreducible complex representation. Then \( \rho \) is tight if and only if

\[
(i, j, k) = \begin{cases} 
(1, 0, 0) & \text{or} \\
(0, 0, k) & \text{with } k \text{ odd.}
\end{cases}
\]

In particular, this means that there are no injective, non-holomorphic tight \( \rho \).
Proof. By our previous analysis of representations of $sp(4, \mathbb{R})$ we know that if $(i, j)$ is not equal to $(1, 0)$ or $(0, 0)$ then either

1. there exists a diagonal disc $d$: $su(1, 1) \rightarrow sp(4, \mathbb{R})$ such that $\rho_{ij} \circ d = \sum \rho_l$ with some $l$ even and nonzero or
2. there exists a tight and holomorphic embedding $f$: $su(1, 1) \oplus su(1, 1) \rightarrow sp(4, \mathbb{R})$ such that $\rho_{ij} \circ f = \sum \rho_l \otimes \rho_m$ with some $l$ even and nonzero.

In the first case we consider the composition $(\rho_{ij} \otimes \rho_k) \circ (d \oplus Id): su(1, 1) \oplus su(1, 1) \rightarrow sp(4, \mathbb{R}) \oplus su(1, 1) \rightarrow su(p, q)$.

Since $(\rho_{ij} \otimes \rho_k) \circ (d \oplus Id) = \sum \rho_l \otimes \rho_k$ with some $l$ even and nonzero it is nontight by Theorem 5.2. This implies that $\rho$ is nontight by Theorem 4.6.

In the second case we begin by defining $h$: $su(1, 1) \oplus su(1, 1) \rightarrow sp(4, \mathbb{R}) \oplus su(1, 1)$, $(X, Y) \mapsto (f(X, Y), Y)$.

This a tight and holomorphic embedding. We have

$$\rho \circ h(X, Y) = \rho_{ij} \otimes \rho_k(f(X, Y), Y) = \sum \rho_l \otimes \rho_m \otimes \rho_k(X, Y, Y).$$

However, $\rho_m \otimes \rho_k(Y, Y)$ is not irreducible and can thus be written as $\sum \rho_l(Y)$. Continuing we have

$$\sum \rho_l \otimes \rho_m \otimes \rho_k(X, Y, Y) = \sum \rho_l \otimes \rho_l(X, Y).$$

Since some $l$ is even and nonzero we know by Theorem 5.2 and 4.6 that $\rho$ is nontight. That the map is tight for the remaining cases $(i, j, k) = (1, 0, 0)$ and $(0, 0, odd)$ follows from the classification in [7] and [2].

5.5. Representations of $su(2, 1)$. Let $su(2, 1)^C = sl(3, \mathbb{C}) = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$ be a root space decomposition of $sl(3, \mathbb{C})$. Here $\Delta = \{ \pm \alpha_1, \pm \alpha_2, \pm (\alpha_1 + \alpha_2) \}$ where $\alpha_1$ is chosen as the noncompact simple root. The fundamental weights are $\omega_1 = \frac{2\alpha_1 + \alpha_2}{3}$ and $\omega_2 = \frac{\alpha_1 + 2\alpha_2}{3}$. The Weyl group of $sl(3, \mathbb{C})$ consists of six elements. These send a weight $\omega_{k,l}$ to $\omega_{k,l}$, $\omega_{k-l, l}$, $\omega_{k+1, l}$, $\omega_{k, l-k}$, $\omega_{k, l-k+1}$, and $\omega_{k-l, k}$ respectively.

We recall from [11] that the only irreducible representations of $su(2, 1)$ that are (anti-) holomorphic have highest weight $(1, 0)$ and $(0, 1)$ and that these are tight [7]. From [7] we know that there is a tight regular subalgebra $\mathfrak{g}(\alpha_1) \cong su(1, 1)$.

Theorem 5.5. Let $\rho: su(2, 1) \rightarrow su(p, q)$ be a homomorphism and an irreducible complex representation. If $\rho$ is tight then it is (anti-) holomorphic.

Proof. Let $\rho_{kl}$ be an irreducible representation of $su(2, 1)$ with $k + l \geq 2$. We restrict this representation to $\mathfrak{g}(\alpha_1)$. We now look at the complexifications $\rho_{kl}^C$ and $\rho_{kl}^C|_{\mathfrak{g}(\alpha_1)}$ to see which weights appear. We divide our analysis of the branching into two cases.
In the first we assume that either \( k = 0 \) or \( l = 0 \). Between the weights \( \omega_{k,l} \) and \( \omega_{-l,-k} \) are the weights \( \omega_{k,l} - n(\alpha_1 + \alpha_2) = \omega_{k,l} - n(\omega_1 + \omega_2) = \omega_{k,n,l-n} \) for \( n = 1, \ldots, k+l \). Since \( k+l \geq 2 \) we have in particular that the weights \( \omega_{k-1,l-1} \) and \( \omega_{k-2,l-2} \) appear in the representation. Assume \( l = 0 \), we have that \( \omega_{k,l}(H_{\alpha}) = k \) and \( \omega_{k-1,l-1}(H_{\alpha}) = k-1 \). One of these integers must be even and nonzero since \( k > 1 \). If instead \( k = 0 \) we have \( \omega_{k-2,l-2}(H_{\alpha}) = -2 \). This means that \( \rho|_{\mathfrak{g}(\alpha)} \) contains irreducible representations of even highest weight. Hence \( \rho \) is nontight by Theorem 5.1 and 4.6.

We now consider the case \( k \neq 0, l \neq 0 \). Between the weights \( \omega_{k,l} \) and \( \omega_{k+1,-l-1} \) are the weights \( \omega_{k,l} - n\alpha_2 = \omega_{k,l} - n(2\omega_2 - \omega_1) = \omega_{k+n,l-2n} \) for \( n = 1, \ldots, l \). Since \( l \geq 1 \) we have in particular that the weight \( \omega_{k+1,l-2} \) appear in the representation. We have that \( \omega_{k,l}(H_{\alpha}) = k \) and \( \omega_{k+1,l-2}(H_{\alpha}) = k+1 \) and that one of these integers must be even and nonzero since \( k \geq 1 \). Again this implies that \( \rho \) is nontight by Theorem 5.1 and 4.6.

\[ \Box \]

6. Proof of Main Theorem

Before proving Theorem 1.1 we recall the following two theorems that are consequences of the classification in [7].

**Theorem 6.1.** We can in each simple Hermitian Lie algebra \( \mathfrak{g} \neq \mathfrak{su}(1,1) \) embed tightly and holomorphically either \( \mathfrak{sp}(4), \mathfrak{sp}(4) \oplus \mathfrak{su}(1,1) \) or \( \mathfrak{su}(2,1) \).

**Theorem 6.2.** Each classical simple Hermitian Lie algebra of tube type can be tightly and holomorphically embedded in \( \mathfrak{su}(n,n) \) for some \( n \).

We will also need the following theorem from [2].

**Theorem 6.3.** Each Hermitian Lie algebra \( \mathfrak{g} \) contains a tightly and holomorphically embedded subalgebra \( \mathfrak{g}' \) of tube type with \( \operatorname{rank}(\mathfrak{g}') = \operatorname{rank}(\mathfrak{g}) \). This subalgebra is unique up to inner automorphisms of \( \mathfrak{g} \). Further, if \( \mathfrak{g}' \) is of tube type and \( \rho: \mathfrak{g}' \to \mathfrak{g} \) is a tight homomorphism then \( \rho(\mathfrak{g}') \subset \mathfrak{g}' \).

Having covered a few key low rank algebras in the previous section we will use these prove our main theorem. We will do this through a series of lemmas.

**Lemma 6.4.** Let \( \rho: \mathfrak{g} \to \mathfrak{su}(p,q) \) be a homomorphism between simple Hermitian Lie algebras, where \( \mathfrak{g} \neq \mathfrak{su}(1,1) \). If \( \rho \) is tight then it is (anti-) holomorphic.

**Proof.** By Theorem 6.1 we can by some \( \iota \) tightly and holomorphically embed \( \mathfrak{g}' = \mathfrak{su}(2,1), \mathfrak{sp}(4,\mathbb{R}) \) or \( \mathfrak{sp}(4,\mathbb{R}) \oplus \mathfrak{su}(1,1) \) depending on \( \mathfrak{g} \). Let us first treat the cases \( \mathfrak{g}' = \mathfrak{su}(2,1) \) or \( \mathfrak{sp}(4,\mathbb{R}) \). If \( \rho \) is tight and nonholomorphic, we have by Lemma 3.8 that \( \rho \circ \iota \) is tight and by Lemma 4.7 nonholomorphic. This contradicts either Theorem 5.5 or
depending on \( g' \). Therefore, if \( \rho \) is tight it must be holomorphic. Now consider the case \( g' = sp(4, \mathbb{R}) \oplus su(1, 1) \). Assume \( \rho \) tight and nonholomorphic. Lemma \ref{tight-implies-holomorphic} implies that both

\[
\rho \circ \iota : sp(4, \mathbb{R}) \to su(p, q) \quad \text{and} \quad \rho \circ \iota : su(1, 1) \to su(p, q)
\]

are nonholomorphic. However, by the classification of tight representations of \( sp(4, \mathbb{R}) \oplus su(1, 1) \) in Theorem \ref{classification-tight-reps} we know that these are sums of representation of the form \( \rho_0 \otimes \rho_{\text{odd}} \) and \( \rho_1 \otimes \rho_0 \). Neither \( \rho_0 \otimes \rho_{\text{odd}} \) nor \( \rho_1 \otimes \rho_0 \) is nonholomorphic in the \( sp(4, \mathbb{R}) \)-factor contradicting our assumptions of \( \rho \) being nonholomorphic.

\begin{lemma}
Let \( \rho : g_1 \to g_2 \) be a homomorphism between simple Lie algebras. Assume that \( g_2 \) is classical and that \( g_1 \neq su(1, 1) \) and of tube type. If \( \rho \) is tight then it is (anti-) holomorphic.
\end{lemma}

\begin{proof}
From Theorem \ref{classification-tight-reps} we know that if \( \rho \) is tight then \( g_2 \) contains a subalgebra \( g_2^T \) of tube type containing the image \( g_1 \). By Theorem \ref{tube-type-tight-embedding} we know that such \( g_2^T \) can be tightly and holomorphically embedded by some \( \iota \) into \( su(n, n) \) for some \( n \). If \( \rho \) is tight and nonholomorphic, we get by Lemma \ref{nonholomorphic-tight-implies-nonholomorphic} that \( \iota \circ \rho \) is tight and nonholomorphic. This contradicts Lemma \ref{nonholomorphic-tight-implies-nonholomorphic}.
\end{proof}

\begin{lemma}
There are no tight homomorphisms \( \rho : su(n, 1) \to so^*(2p) \) for \( n \geq 2, p \geq 4 \).
\end{lemma}

\begin{proof}
Consider the composition \( \rho \circ \iota : su(2, 1) \to su(n, 1) \to so^*(2p) \) where \( \iota \) is the canonical tight inclusion. Then \( \rho \) is tight if and only if \( \rho \circ \iota \) is tight. This reduces our search of tight homomorphisms to the case \( n = 2 \).

If \( p \) is even we consider the composition \( \iota \circ \rho : su(2, 1) \to so^*(2p) \to so^*(2(p + 1)) \), where \( \iota \) is the canonical tight inclusion. Then \( \rho \) is tight if and only if \( \iota \circ \rho \) is tight. This reduces our search of tight homomorphisms to the case \( p \geq 5 \) and odd.

Consider the restriction of \( \rho \) to a tightly embedded regular subalgebra isomorphic to \( su(1, 1) \), as in the proof of Theorem \ref{classification-tight-reps}. Assume that \( \rho \) is tight. Then the image of \( su(1, 1) \) is contained in a maximal subalgebra of \( so^*(2p) \) that is of tube type by Theorem \ref{classification-tight-reps}. From \cite{7} we know that this algebra is \( so^*(2(p - 1)) \). This algebra can be tightly and holomorphically embedded in \( su(p - 1, p - 1) \). We can extend this to a homomorphism \( \iota : so^*(2p) \to su(p, p) \), though this extension is not tight. We get the following commutative diagram of homomorphisms.

\[
\begin{array}{ccc}
su(2, 1) & \xrightarrow{\rho} & so^*(2p) & \xrightarrow{i} & su(p, p) \\
\downarrow \iota_1 & & \downarrow \iota_2 & & \downarrow \iota_3 \\
su(1, 1) & \xrightarrow{\rho_1} & so^*(2(p - 1)) & \xrightarrow{i_1} & su(p - 1, p - 1)
\end{array}
\]
We have by Lemma 3.8 and commutativity of the diagram the following equivalences:

\[ \rho \text{ tight } \iff \rho \circ \iota_1 \text{ tight } \iff \iota_2 \circ \rho| \text{ tight } \iff \rho| \circ \iota_1 \text{ tight } \]

As a representation, \( i \) is just the standard representation. So, as a representation \( i \circ \rho \) decomposes into the same irreducible parts as \( \rho \). Since it is tight, we have by Theorem 5.1 the decomposition into irreducible representations

\[ i \circ \rho = \sum_{k \text{ odd}} n_k \rho_k + n_0 \rho_0. \]

Here the \( n_k \) denote the multiplicities of the representations. Composing with \( \iota_3 \) adds two trivial representations

\[ \iota_3 \circ i \circ \rho = \sum_{k \text{ odd}} n_k \rho_k + (n_0 + 2) \rho_0. \]

From the proof of Theorem 5.5 we know that the only representations of \( su(2, 1) \) that branch into odd- and zero-highest weight representations when restricted to \( su(1, 1) \) are \( \rho_{10}, \rho_{01} \) and \( \rho_{00} \). The first two both branch into \( \rho_1 + \rho_0 \) and the last one to \( \rho_0 \) when restricted to \( su(1, 1) \). Thus \( \rho \) is of the form \( \rho = k_3 \rho_{10} + (n - k) \rho_{01} + l \rho_{00} \). We get \( i \circ \rho \circ \iota_3 = n \rho_1 + (n + l) \rho_0 \). Since the diagram commutes we get

\[ n \rho_1 + (n + l) \rho_0 = \sum_{k \text{ odd}} n_k \rho_k + (n_0 + 2) \rho_0. \]

Thus \( n_1 = n, n_0 + 2 = n + l \) and \( n_k = 0 \) for \( k \neq 1, 0 \). Since \( i \circ \rho \) is tight we have that \( n_1 = n = p - 1 \), for dimensional reasons we have \( 3n + l = 2p \). These together give us \( p - 3 + l = 0 \). But \( p \geq 5 \) and \( l \) is a nonnegative integer. We thus get a contradiction. Hence \( \rho \) can not be tight.

\[ \square \]

**Proof of Theorem 1.1.** The case remaining is homomorphisms \( \rho: g_1 \to g_2 \) where \( g_1 \) is not of tube type. Consider first the case \( \text{rank}(g_1) \geq 2 \).

From Theorem 6.3 we know that \( g_1 \) contains a tightly and holomorphically embedded subalgebra \( g_1^T \) of tube type with \( \text{rank}(g_1^T) = \text{rank}(g_1) \). Lemma 6.5 together with Lemma 3.8 then proves that if \( \rho \) is tight it must be holomorphic.

If \( \text{rank}(g_1) = 1 \) it must be isomorphic to \( su(n, 1) \) for some \( n \geq 2 \). We can then tightly and holomorphically embed \( su(2, 1) \) in \( su(n, 1) \). Lemma 3.8 then allows us to reduce to the case \( \rho: su(2, 1) \to g_2 \). We first consider the case when \( g_2 \) is of tube type. Then we can embed it tightly and holomorphically in some \( su(n, n) \). Lemma 3.8 and Theorem 5.5 then implies that if \( \rho \) is tight it must be (anti-) holomorphic. If \( g_2 \) is not of tube type there are two possibilities. Either \( g_2 \) is \( su(p, q) \) with \( p > q \) or it is \( so^*(2p) \) with \( p \) odd. The first case is covered by Theorem...
Since $so^{*}(6) \simeq su(3,1)$ we can assume $p \geq 5$ in the second case. It is thus covered by Lemma 6.6.

**Proof of Theorem 1.2.** Assume that $\rho: \mathfrak{g} \to e_{6(-14)}$ is tight and non-holomorphic. Let $\mathfrak{g}^T$ be a maximal tube type subalgebra of $\mathfrak{g}$, tightly and holomorphically embedded by Theorem 6.3. Since $\rho$ is tight, $\rho|: \mathfrak{g}^T \to e_{6(-14)}$ is tight. By Theorem 6.3 $\rho(\mathfrak{g}^T) \subset e^T_{6(-14)}$, where $e^T_{6(-14)}$ is a maximal tube type subalgebra of $e_{6(-14)}$. Since $e_{6(-14)}$ is not of tube type $e^T_{6(-14)}$ is a proper subalgebra and hence must be classical. Then $\rho|: \mathfrak{g}^T \to e^T_{6(-14)}$ is tight, nonholomorphic with $\mathfrak{g}^T$ of tube type and $e^T_{6(-14)}$ classical. This contradicts Lemma 6.5.

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