RIEFFEL TYPE DISCRETE
DEFORMATION OF FINITE
QUANTUM GROUPS

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Abstract. We introduce a discrete deformation of Rieffel type for
finite (quantum) groups. Using this, we give an example of a finite
quantum group $A$ of order 18 such that neither $A$ nor its dual can
be expressed as a crossed product of the form $C(G_1) \rtimes_{\tau} G_2$ with
$G_1$ and $G_2$ ordinary finite groups. We also give a deformation of
finite groups of Lie type by using their maximal abelian subgroups.

1. Introduction

Since the work of Woronowicz [26, 27], the theory of compact quan-
tum groups, notably the deformation theory of compact Lie groups,
has been intensively studied and is now quite well understood (see e.g.
[28, 19, 12, 13, 21, 20]). However, this is not the case for finite quant-
num groups. Both as objects of great mathematical interest, like finite
groups, and as objects with potential important applications in theo-
retical physics [2, 3], the theory of finite quantum groups calls for more
efforts of study. To start with, the theory needs an interesting supply
of examples, which are still lacking so far, though a few non-trivial
examples have been studied [8, 1, 4, 14].

In this paper, we construct a class of finite quantum groups by in-
roducing a discrete deformation of Rieffel type for finite (nonabelian)
groups. In fact, just as in our earlier paper [24], this deformation can
be applied to finite quantum groups as well, not just finite groups.
This construction is motivated by Rieffel’s deformation of compact Lie
groups [16], which has its origins in the Weyl-von Neumann quan-
tization (also called Moyal product) (cf [15]). As a matter of fact,
our formula for the discretely deformed product (see Definition 2.1) is
an exact analog of the product formula of von Neumann and Rieffel
[13, 13]. In [14] (resp. [24]), actions of finite dimensional vector spaces
are used to deform Lie groups (resp. compact quantum groups) into
new quantum groups. In this paper, we use actions of finite abelian
groups to deform finite groups (and finite quantum groups) into new finite quantum groups. Because of the nature of the objects we deal with, we are spared the analytical complications met in [14, 16, 24] for the actions of continuous abelian groups (viz. vector spaces). Hence the arguments in this paper are of a purely algebraic nature. Though the constructions of this paper are direct analogs of [13, 16, 24] adapted to the actions of finite abelian groups, the proofs of the main results given there do not directly generalize to the new situation. Thus we have to develop different proofs for our main results. The main cause of this is that many facts on Euclidean geometry of \( \mathbb{R}^d \) as used in [13] do not have generalizations to finite abelian groups (e.g. orthogonal complements, polar decomposition of operators, etc), though one can develop to some extent the “Euclidean geometry” on a finite abelian group with “inner product” given by a pairing which identifies itself with its Pontryagin dual.

The construction of deformation in this paper in the dual form is an example of Drinfeld’s twistings [3], just as the constructions in [16, 17, 24] (cf also [11, 12]). This again shows the relationship between the Rieffel type deformations (generalizations of the Weyl-von Neumann quantization) and the Drinfeld’s twistings. For Kac algebras, the most general form of twistings in the sense of Drinfeld [3] is studied by Enock and Vainerman [3, 18], following the work of Landstad and Raeburn [10, 9] on deformations of locally compact groups. Hence in the dual picture our construction constitutes a distinguished class of twistings of Kac algebras in the sense of Enock and Vainerman [3, 18]. Instead of imposing rather complicated cocycle conditions in addition to the existence of an abelian subgroup, such as the approach in [1, 3, 18], our construction of deformation is canonically associated with the abelian subgroup, and it is a natural generalization of the Weyl-von Neumann-Rieffel deformation. As a matter of fact, the twist \( F \) (see formula (3.16)) for the dual of our construction does not satisfy the 2-cocycle condition on the Kac algebra, but the pseudo-2-cocycle condition, which is equivalent to the condition that the associated twisted coproduct is coassociative, a minimal requirement. It is interesting to note that it is not clear how to see that \( F \) is a pseudo-2-cocycle directly in the dual picture! Also, unlike [1, 3], our construction does not give rise to the 8 dimensional quantum group of Kac-Palyutkin [8]. Note also that in [14], more specific examples along the lines of [1, 3, 18] are given; it is also shown there that the \( K_0 \)
ring of Hopf algebra is invariant under twists, which is obvious for our construction.

The plan of this paper is as follows. As preparation for Sect. 3, we construct in Sect. 2 a deformed $C^*$-algebra $A_J$ for every finite dimensional unital $C^*$-algebra $A$ that is endowed with an action $\alpha$ of a finite abelian group $H$, where $J$ is a skew-symmetric automorphism on $H$. See Theorem 2.7. The construction in this section parallels the one in [15]. In Sect. 3, for every finite quantum group $A$ containing a finite abelian subgroup $T$, we construct an action $\alpha$ of $H$ on the $C^*$-algebra $A$ and show that the deformation $A_J$ is also a finite quantum group containing $T$ as a subgroup, where $H = T \oplus T$, $J = S \oplus (-S)$, and $S$ is a skew-symmetric automorphism on $T$. See the main result Theorem 3.2. This theorem parallels the main results in [16, 24], and is announced in Section 2 of [25] without proof. At the end of this section, we discuss the relationships of this construction with Drinfeld’s twistings [4] and twistings of Landstad and Enock-Vainerman [9, 18, 23]. In Sect. 4, we construct a non-trivial finite quantum group $A$ of order 18 such that neither $A$ nor its dual can be expressed as a crossed product of the form $C(G_1) \rtimes \tau G_2$, where $G_1$ and $G_2$ are ordinary finite groups. Finally, in Sect. 5, we deform finite groups of Lie type using their maximal abelian subgroups (tori).

A Convention on Terminology. When $A = C(G)$ is a Woronowicz Hopf $C^*$-algebra, we also call $A$ a compact quantum group, referring to the abstract dual $G$. Hence a representation of the quantum group $A$ is a representation of $G$ in the sense of [27], which is also called a corepresentation of the Woronowicz Hopf $C^*$-algebra $A$ (cf. also [21]); while a representation of the algebra $A$ has an obvious different meaning.

2. Deformation of algebras via actions of finite abelian groups

In this section, we adapt the construction of the monograph of Rieffel [15] to the situation of actions of finite abelian groups on $C^*$-algebras (as opposed to actions of $\mathbb{R}^d$ considered there by Rieffel). Namely, for every quadruple $(A, H, \alpha, J)$ consisting of a finite dimensional unital $C^*$-algebra $A$, an action $\alpha$ of a finite abelian group $H$ on the $C^*$-algebra $A$, and a skew-symmetric automorphism $J$ (with respect to a Pontryagin pairing—see definition below) on $H$, we construct a deformed unital $C^*$-algebra $A_J$. It is not our intention to generalize in detail
everything in [15] to this setting. As a matter of fact, many results in [15] do not generalize to this setting. Our primary task in this section is to give some details of those results that are needed in the next section for the deformation of finite quantum groups, the main one being the construction of the C*-algebra $A_J$ mentioned above. We will also briefly indicate some other results that might be useful elsewhere.

Throughout this section, $A$ will denote a finite dimensional unital C*-algebra on which a finite abelian group $H$ acts by *-automorphisms $\alpha$. The group operation of $H$ is written additively. Let

$$H \times H \rightarrow \mathbb{T}, \quad (s, t) :\mapsto < s, t >$$

be a pairing (with values in the circle group $\mathbb{T}$) that identifies $H$ with its Pontryagin dual $\hat{H}$ (we call such a pairing a Pontryagin pairing). More precisely, identifying $H$ with $\mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_l\mathbb{Z}$, where $n_1, n_2, \cdots, n_l$ are (not necessarily distinct) natural numbers, a pairing is given by

$$< s, t >= < (s_1, \cdots, s_l), (t_1, \cdots, t_l) > = e^{2\pi i(s_1t_1/n_1 + s_2t_2/n_2 + \cdots + s_lt_l/n_l)} \quad (2.1)$$

where $s_k, t_k \in \mathbb{Z}, k = 1, \cdots, l$.

Let $\text{End}(H)$ be the ring of endomorphisms of the group $H$ and $\text{GL}(H)$ the group of automorphisms on $H$ (which is the same as the group of invertible elements in the ring $\text{End}(H)$). Using a Pontryagin pairing $< s, t >$ on $H$ above, we can define the notion of transpose $J^t$ of an endomorphism $J \in \text{End}(H)$. More generally, if $G$ and $H$ are two finite abelian groups endowed with Pontryagin pairings, then every group homomorphism $J$ from $G$ to $H$ admits a transpose $J^t$, which is a homomorphism from $H$ to $G$. Throughout this section, we assume that $H$ admits a nontrivial skew-symmetric automorphism. (Note that some finite abelian groups do not have such automorphisms! But the examples of groups we consider later in this paper do.) We can also define the group of orthogonal automorphisms $O(H)$ in the evident manner. Just as in the case of a vector space, by choosing $l$ cyclic generators of $H$, one for each of the subgroup $H_k \cong \mathbb{Z}/n_k\mathbb{Z}$ of $H$, we can also represent elements of $\text{End}(H)$ in terms of matrices with entries consisting of group homomorphisms from $H_j$ to $H_k$, $j, k = 1, \cdots, l$. With each choice of cyclic generators of $H$, $\text{GL}(H)$ and $O(H)$ can be identified with the sets of invertible and orthogonal matrices, respectively. Note that the skew-symmetry of the matrix $J$ is independent of the choice of the generators of $H$. 
For any finite group $H$, we will use $\int$ to denote the normalized Haar integral on $H$, i.e.

$$
\int_{s \in H} f(s) = \frac{1}{|H|} \sum_{s \in H} f(s),
$$

(2.2)

where $|H|$ is the number of elements in $H$ and $f$ is a function on $H$ taking values in some vector space. We will see that the normalization is convenient for the constructions of our deformed algebra and quantum group. The symbol $\int_{s_1, s_2, \ldots, s_k \in H}$ will denote the corresponding $k$-th fold integral.

For the convenience of the reader, we recall here the orthogonality relations for group characters on $H$, namely the relations

$$
\int_{t \in H} < s, t > = \delta_{s,0},
$$

(2.3)

where $< s, t >$ is a Pontryagin pairing on $H$.

We will also need the Fourier inversion formula for $A$-valued functions $F(s)$ on $H$, which we recall also, as we will be using it a number of times,

$$
\tilde{F} = F, \text{ i.e., } |H| \int_{s,t \in H} F(s) < s, -t > < t, x > = F(x).
$$

(2.4)

(The inversion formula is easily seen to be a consequence of the orthogonality relations (2.3).) In particular, we have,

$$
|H| \int_{s,t \in H} F(s) < s, t > = F(0).
$$

(2.5)

**Definition 2.1.** (cf [13, 15]) Let $J \in \text{End}(H)$ be an endomorphism on $H$. The **deformed product** $\times_J$ (or $\times^J$) on $A$ is defined by

$$
a \times_J b = |H| \int_{s,t \in H} \alpha_s(a)\alpha_t(b) < Js, t >, \quad a, b \in A,
$$

(2.6)

where the products on the right hand side is in $A$. Let $A_J$ (or $A^J$) denote $(A, \times_J)$.

The number $|H|$ in the above formula insures that the deformed algebra $A_J$ is unital (see (2) of the next proposition). The observant reader might have noticed that in the above formula we have chosen $J$ to appear in the dual pairing $\langle \cdot, \cdot \rangle$ instead of in the action $\alpha$, as is done in [13, 16, 24] (cf also von Neumann [13]). We do this because if we replace $J$ with $hJ$, where $h$ is any real number, the above formula...
still make sense. But $\alpha_{H,s}$ does not make sense for finite group $H$ acting on $A$. See also the remarks at the end of this section.

**Proposition 2.2.** (1). For any $J \in \text{End}(H)$, the deformed product $\times_J$ is associative.

(2). If $J \in \text{GL}(H)$, then the unit of $A$ continues to be the unit of $A_J$.

**Proof.** (1). This is the analog of Theorem 2.14 of Rieffel [15]. However the proof given there does not work for finite abelian groups because their subgroups do not have orthogonal complements. We give a much simpler proof of this result. The key is to make the correct change of variables. We compute, by applying change of variables twice (the second change of variable is a little bit tricky),

$$(a \times_J b) \times_J c = |H| \int_{s,t \in H} \alpha_s(a \times_J b) \alpha_t(c) < Js, t >$$

$$= |H|^2 \int_{s,t,u,v \in H} \alpha_{s+u}(a) \alpha_{s+v}(b) \alpha_t(c) < Js, t > < Ju, v >$$

$$= |H|^2 \int_{s,t,u,v \in H} \alpha_s(a) \alpha_{s-u+v}(b) \alpha_t(c) < J(s - u), t > < Ju, v >$$

$$= |H|^2 \int_{s,t,u,v \in H} \alpha_s(a) \alpha_{s+u+v}(b) \alpha_t(c) < Ju, t > < J(s - u), v >,$$

which, after exchanging the roles of $t$ and $v$,

$$= |H|^2 \int_{s,t,u,v \in H} \alpha_s(a) \alpha_{u+v}(b) \alpha_t(c) < Ju, v > < J(s - u), t >$$

$$= |H|^2 \int_{s,t,u,v \in H} \alpha_s(a) \alpha_{u+v}(b) \alpha_{v+u}(c) < Ju, v > < Js, t >.$$

On the other hand, expanding $a \times_J (b \times_J c)$ we see that

$$a \times_J (b \times_J c) = |H|^2 \int_{s,t,u,v \in H} \alpha_s(a) \alpha_{t+u}(b) \alpha_{t+v}(c) < Ju, v > < Js, t >.$$

(2). If $J \in \text{GL}(H)$, then $< Js, t >$ is a Pontryagin pairing for $H$, hence we can use (2.5) (replacing $< s, t >$ in (2.5) by $< Js, t >$),

$$a \times_J 1 = |H| \int_{s,t \in H} \alpha_s(a) < Js, t > = \alpha_0(a) = a.$$

Similarly,

$$1 \times_J b = b.$$
This proves the proposition. Q.E.D.

As in [15], let $A_u$ be the spectral subspace of $u \in H$:

$$A_u = \{ a \in A \mid \alpha_s(a) = < u, s > a, s \in H \}.$$  

(2.7)

**Proposition 2.3.** Let $J \in GL(H)$ be a skew-symmetric automorphism: $J^t = -J$. Let $a \in A_u, b \in A_v$ (the spectral subspace of $A$ corresponding to $u, v \in H$). Then

$$a \times_J b = < J^{-1}u, v > ab.$$  

(2.8)

**Proof.** This is the analog of Proposition 2.22 in [15]. Instead of the Poisson summation formula, as is used in the proof of 2.22 in [15], we apply the Fourier inversion formula to the last line of the following computation:

$$a \times_J b = |H| \int_{s,t \in H} \alpha_s(a) \alpha_t(b) < Js, t > = |H| \int_{s,t \in H} < s, u > a < t, v > b < Js, t >$$

$$= |H| \int_{s,t \in H} < J^{-1}s, u > < t, -v > < s, -t > ab = < J^{-1}(-v), u > ab.$$  

That is

$$a \times_J b = < J^{-1}u, v > ab$$

by skew-symmetry of $J$. Q.E.D.

**Remark.** Note that if $A$ is commutative, then $a \times_J b = < 2J^{-1}u, v > b \times_J a$, where $a, b$ are as in the above proposition. Hence, we see that if $2J^{-1} \neq 0$ and if the action $\alpha$ is non-trivial, then the algebra $A_J$ is noncommutative, even if $A$ is a commutative algebra. The condition $2J^{-1} = 0$ is related with the characteristic 2 phenomenon (see the last two sections for examples concerning this).

**Proposition 2.4.** Let $J \in End(H)$ be a skew-symmetric homomorphism: $J^t = -J$. Then under the involution $*$ of the algebra $A$, we have

$$(a \times_J b)^* = b^* \times_J a^*$$

for $a, b \in A_J$. Hence $A_J$ is a $*$-algebra.

**Proof.** Use $< Jy, x > = < y, J^t x > = < y, -Jx >$. Q.E.D.

Consider the Hilbert $A$-module $E = C(H) \otimes A$ under the $A$-valued inner product

$$< f, g >_A = \int_{x \in H} f^*(x)g(x), \quad f, g \in C(H) \otimes A,$$  

(2.9)
where \( C(H) \) is the algebra of complex valued functions on \( H \). Note that as a tensor product of two \( C^* \)-algebras, \( E \) is also a \( C^* \)-algebra and \( H \) acts on it by translation:

\[
\tau_s(f)(x) = f(x - s). \tag{2.10}
\]

If \( J \) is a skew-symmetric automorphism, then from the propositions above, \( E_J = (E, \times J) \) is a unital \( * \)-algebra. Let \( L \) denote the left regular multiplication on \( E_J \):

\[
L_f g = f \times_J g. \tag{2.11}
\]

Under these assumptions, we have (cf 4.2, 4.3, 4.6 of [15])

**Proposition 2.5.** The left regular multiplication \( L \) is a faithful unital \( * \)-representation of the \( * \)-algebra \( E_J \) by bounded operators on the Hilbert \( A \)-module \( E \). More precisely, we have \( L_f = 0 \) if and only if \( f = 0 \), and

\[
\langle f \times_J^* g, h \rangle_A = \langle g, f \times_J h \rangle, \quad f \in E_J, \ g, h \in E, \tag{2.12}
\]

\[
||L_f|| \leq \int_{s \in H} ||f(s)|| = ||f||_1, \quad f \in E_J. \tag{2.13}
\]

**Proof.** The identity (2.13) is a straightforward checking without going into the complications such as involved in the proof of 4.2 in Rieffel [15]. We leave this to the reader.

We show that \( L \) is faithful. Let \( L_f = 0 \). Hence

\[
\langle f \times_J^* g, f \times_J g \rangle_A = 0, \quad g \in E.
\]

Let \( g \) be the unit element of the \( C^* \)-algebra \( E \), \( g(s) = 1 \), \( s \in H \). Then by Proposition 2.2, \( g \) is the unit of \( E_J \). Hence

\[
\langle f \times_J^* g, f \times_J g \rangle_A = \langle f, f \rangle_A = 0.
\]

Note that \( g \) plays two roles in here: as the unit of the algebra \( E_J \) and as an vector in the Hilbert \( A \)-module \( E \). Hence \( f = 0 \).

The proof of the inequality (2.13) is the same as (and easier than) the proof of 4.3 in [15] (see also 4.6 of [15]). For the convenience of the reader, we sketch the proof here. A short computation shows that

\[
L_f(g) = f \times_J^* g = \int_{s \in H} f(s) U_s(g), \quad \text{i.e.,} \quad L_f = \int_{s \in H} f(s) U_s,
\]

where \( U_s \) is the unitary operator on the Hilbert module \( E \) defined by

\[
U_s(g)(x) = \langle J(x - s), x > \tilde{g}(J(x - s)),
\]

\( \tilde{g} \) being the inverse Fourier transform of \( g \) (Plancherel’s theorem). Hence (2.13) is immediate. Q.E.D.
Let us come back to our algebra $A_J$. For $a \in A_J = A$, the element $\tilde{a}$ of $E_J$ defined by $\tilde{a}(s) = \alpha_s(a)$ is zero if and only if $a = 0$. Using the above result, we can define a $C^*$-norm on $A_J$ as follows.

**Definition 2.6.** Let $J$ be a skew-symmetric automorphism on $(H, \langle , \rangle)$. The deformed $C^*$-norm $|| \cdot ||_J$ on $A_J$ is defined by

$$||a||_J = ||L_{\tilde{a}}||, \ a \in A_J,$$

where $||L_{\tilde{a}}||$ is the operator norm of $L_{\tilde{a}}$ on the Hilbert $A$-module $E$.

Summarizing the above, we have the following main result of the section:

**Theorem 2.7.** Let $A$ be a finite dimensional unital $C^*$-algebra. Let $H$ be a finite abelian group acting on $A$ by automorphisms. Let $J \in GL(H)$ be a skew-symmetric automorphism: $J^t = -J$. Then $A_J$ is a unital $C^*$-algebra under the norm $|| \cdot ||_J$.

**Remarks.** (1). Note that on any finite dimensional $*$-algebra, there can be at most one $C^*$-norm. Hence the $C^*$-norm defined above is the unique one on $A_J$.

(2). Note that we need $J$ to be a skew-symmetric automorphism in order to define the $C^*$-norm on $A_J$, while in Rieffel [15], $J$ can be any skew-symmetric endomorphism on a vector space. Also note that we do not have the analog of Theorems 2.15 and 6.5 of Rieffel [15]. Namely,

$$(A_J)_K = A_{J+K}$$

is not true in general. However, using the orthogonality relations for group characters (see (2.3)), one can easily prove the following proposition.

**Proposition 2.8.** Under the assumption of the theorem above, $(A_J)_{-J} = A$.

Now we can state the analogs of Theorems 2.10, 5.7, 5.8, 5.12 and 7.7 of Rieffel [15].

**Proposition 2.9.** Let $J \in GL(H)$ be a skew-symmetric automorphism. Let $\alpha$ and $\beta$ be actions of $H$ on $A$ and $B$ respectively. Let $\theta : A \rightarrow B$ be an equivariant homomorphism.

(1). $\theta$ is still an equivariant homomorphism from $A_J$ to $B_J$ (denote this homomorphism by $\theta_J$, called the deformation of $\theta$);

(2). $\theta$ is injective (resp. surjective) if and only if $\theta_J$ is.
(3). Let $I$ be an ideal of $A$ that is invariant under the action $\alpha$. Let $Q = A/I$, and let $\alpha$ also denote the action of $H$ on $Q$, so we have an equivariant exact sequence

$$0 \rightarrow I \rightarrow A \rightarrow Q \rightarrow 0.$$ 

Then the corresponding sequence (see (1) above)

$$0 \rightarrow I_J \rightarrow A_J \rightarrow Q_J \rightarrow 0$$

is also exact.

The proofs of these analogs follows directly from our definitions and the key assumption that $A$ is a finite dimensional $C^*$-algebra. We leave the checking to the reader. The reader is advised not to look up the proofs in [15] for clues (for otherwise the reader would be mislead to complicate the proofs of these analogs), but to simply think about our definitions.

Remarks. (1). If we replace $J$ with $\hbar J$ in the construction above, a number of things in this section are still true, where $\hbar$ is real, and

$$< h J s, t > = e^{2\pi i h(s_1 t_1/n_1 + s_2 t_2/n_2 + \cdots + s_l t_l/n_l)},$$

using the above identification of $H$ with the concrete abelian group as a direct sum of cyclic groups. For any skew-symmetric $J$, $A_{hJ}$ is an associative $*$-algebra, but it may not have a unit or a $C^*$-norm even if $J$ is an automorphism. So it not clear how one constructs strict deformation quantization.

(2). For practical purposes of next section, we have restricted $A$ to be a finite dimensional $C^*$-algebra. If we remove this restriction, then the proofs of all the above results, except Proposition 2.9, are still valid (Of course, in Theorem 2.7, we need a completion to obtain a $C^*$-algebra). We believe that Proposition 2.9 is still true in this case.

3. Deformation of finite quantum groups via finite abelian subgroups

In the theory of finite groups, the finite groups of Lie type are one of the most important classes of finite groups. In view of the fact that classical Lie groups have $q$-deformation, a natural question in this connection is

Problem 3.1. Do finite groups of Lie type have an analog of $q$-deformation into finite quantum groups?
This problem seems to be out of reach at the moment. In this section, we construct a deformation of Rieffel type for finite groups (as well as for finite quantum groups) that contain an abelian subgroup. This deformation is not the analog of the $q$-deformation, it is dual to Drinfeld twistings of the quantized universal enveloping algebras. We will see this at the end of this section.

We start with a finite quantum group $G = (A, \Phi)$ (in the sense that $A$ is the “function space” $C(G)$, where $\Phi$ is the coproduct on $A$ [27]). Assume that its maximal subgroup $X(A) = \{ *$-homomorphisms from $A$ into $\mathbb{C}\}$ contains an abelian subgroup $T$ with a nontrivial skew-symmetric automorphisms $S$. So there is a surjective morphism of Hopf $C^*$-algebras $\pi : A \longrightarrow C(T)$. Let

$$H := T \oplus T,$$

and let

$$J := S \oplus (-S)$$

be the skew-symmetric automorphism on $H$. Define an action $\alpha$ of $H$ on the $C^*$-algebra $A$ as follows:

$$\alpha_{(s,u)} = \lambda_s \rho_u,$$

where

$$\lambda_s = (E_{-s} \pi \otimes id) \Phi, \quad \rho_u = (id \otimes E_u \pi) \Phi,$$

$id$ being the identity map on $A$ and $E_u$ the evaluation functional on $C(T)$ corresponding to $u$. Using results of the previous section, we obtain a deformed $C^*$-algebra $A_J$ with new product $\times_J$ defined by (see formula (2.1))

$$a \times_J b = |T|^2 \int_{s,t,u,v \in T} \alpha_{(s,u)}(a) \alpha_{(t,v)}(b) < St, t > < -Su, v >,$$

where $a, b \in A$ and $< s, t >$ is a Pontryagin pairing on $T$. The main result of this section is (cf [16, 24])

**Theorem 3.2.** Under the same coproduct $\Phi$ of $A$, the deformation $(A, \times_J)$ is still a finite quantum group containing $T$ as a subgroup.

**Remarks on the proof.** We will show that $A_J$ satisfies the axioms of a finite dimensional Hopf $C^*$-algebra as given in Kac and Palyutkin [8], instead of the ones given in Appendix 2 of Woronowicz [27], though they are equivalent to each other. The proof is a modification of the proof of Theorem 3.9 of [24]. Unlike that theorem, because $A$ is of finite dimension here, we do not need to consider the analogs of Propositions
3.2 and 3.8 in [24], which are essential steps for the proof of that theorem. The subtlety in our situation here is that the method used there in the treatment of the deformed coproduct does not work anymore, because the existence of orthogonal complements is used in an essential way there (but, as pointed out before, a subgroup of a finite abelian group needs not have an orthogonal complement). To deal with the deformed coproduct, we will show that the heuristic computation on page 471 of [16] can be made rigorous in our setting (replacing the compact Lie group \( G \) there by our finite quantum group).

**Proof.** Let \( F, G \in A_J \otimes A_J \), and let \( \times_J \) also denote the product on \( A_J \otimes A_J \). Using formula (2.1) we can find the formula for the product in the \( C^* \)-algebra \( A_J \otimes A_J \) in terms of product in the \( C^* \)-algebra \( A \otimes A \), with the summation (integration) over repeated indices:

\[
F \times_J G = |T|^4 \int \gamma(s, u, s', u') (F) \gamma(t, v, t', v') (G) < L(s, u, s', u'), (t, v, t', v') >.
\]

where \( \gamma = \alpha \otimes \alpha \) is the tensor product action of \( H \oplus H \) on \( A \otimes A \), \( L = J \oplus J \) is the corresponding skew-symmetric automorphism on \( H \oplus H \), and

\[
< (s, u, s', u'), (t, v, t', v') > = < s, t > < u, v > < s', t' > < u', v' > .
\]

Note that this identity is easily verified on tensors of the form

\[
F = a_1 \otimes a_2, \quad G = b_1 \otimes b_2.
\]

Since \( A \) is finite dimensional, this gives an isomorphism of \( * \)-algebras

\[
(A \otimes A)_J = A_J^\alpha \otimes A_J^\alpha.
\]

It is easy to see this is actually an isomorphism of \( C^* \)-algebras (see Remark (1) after Theorem 2.7). This isomorphism is the analog of Corollary 2.2 of [16], where a more complicated proof is needed.

For \( a, b \in A_J \), we have by (3.6) and (3.3) (cf [16])

\[
\Phi(a) \times_J \Phi(b) = \Phi(a) \times_L \Phi(b)
\]

\[
= |T|^4 \int_{s, u, v, s', u', t, t', v'} (\lambda_s \rho_u \otimes \lambda_s' \rho_u') (\Phi(a)) (\lambda_t \rho_v \otimes \lambda_t' \rho_v') (\Phi(b))
\]

\[
< Ss, t > < -Su, v > < Ss', t' > < -Su', v' > ,
\]

which, by 2.7 of [24]

\[
= |T|^4 \int_{s, u, v, s', u', t, t', v'} (\lambda_s \rho_{-s'} \otimes \rho_v) (\Phi(a)) (\lambda_t \otimes \lambda_{-v'} \rho_{v'}) (\Phi(b))
\]

\[
< Ss, t > < -Su, v > < Ss', t' > < -Su', v' > .
\]
Making change of variables \( u - s' \mapsto u, t' - v \mapsto t' \), and using (2.4) twice (note that both \(<-Su, v>\) and \(<Ss', t'>\) are Pontryagin pairings on \( T \! \)), the last expression

\[
= |T|^4 \int_{s,u,t,v,s',u',t',v' \in T} (\lambda_s \rho_u \otimes \rho_{u'})(\Phi(a))(\lambda_t \otimes \lambda_{t'} \rho_{v'})(\Phi(b)) < Ss, t > < -Su, v > < Ss', t' > < -Su', v' >
\]

\[
= |T|^2 \int_{s,t,u,v \in T} (\lambda_s \rho_u)(\Phi(a))(\lambda_t \rho_{v})(\Phi(b)) < Ss, t > < -Su, v >
\]

which, by 2.7 of [24]

\[
= |T|^2 \int_{s,t,u,v \in T} \Phi(\lambda_s \rho_u(a))\Phi(\lambda_t \rho_{v}(b)) < Ss, t > < -Su, v >
\]

\[
= \Phi(a \times_J b).
\]

That is

\[
\Phi(a) \times_J \Phi(b) = \Phi(a \times_J b).
\]

As in [16], the action \( \alpha \) restrict to an action on \( C(T) \) and \( \pi \) is equivariant. From Proposition 2.3 of the last section, this gives a surjective homomorphism \( \pi_J \) from \( A_J \) onto \( C(T)_J \). It is also clear that

\[
(\pi_J \otimes \pi_J)\Phi_J = \Phi_T \pi_J,
\]

(3.8)

where \( \Phi_T \) is the coproduct on \( C(T) \). However the method used in [16] for the proof of

\[
C(T)_J = C(T)
\]

(3.9)

does not work here, because it uses a result of [15] which is not true for finite abelian groups. We can prove this directly as follows. For \( f \in C(T) \), we have

\[
\alpha_{(s,u)}(f) = \lambda_s \rho_u(f) = \lambda_{s-u}(f).
\]

Hence

\[
f \times_J g = |T|^2 \int_{s,t,u,v \in T} \lambda_{s-u}(f)\lambda_{t-v}(g) < Ss, t > < -Su, v >
\]

\[
= |T|^2 \int_{s,t,u,v \in T} \lambda_s(f)\lambda_t(g) < Ss, t > < Su, v > < Ss, t >
\]

\[
= fg,
\]

where we have used the orthogonality relations of the characters of a finite abelian group. This shows that \( T \) will still be a subgroup of \( A_J \) once \( A_J \) is shown to be a quantum group.
The counit of $A_J$ is defined by
\[ \varepsilon_J = \varepsilon_T \pi_J, \tag{3.10} \]
where $\varepsilon_T$ is the counit of $C(T)$. So as a linear map, $\varepsilon_J$ is the same as $\varepsilon$. The coinverse $\kappa_J$ on $A_J$ is defined to be the same as $\kappa$. The identity
\[ \kappa_J(a \times_J b) = \kappa_J(b) \times_J \kappa_J(a) \tag{3.11} \]
is a direct consequence of the fact that $\kappa \alpha(s,u) = \alpha(u,s) \kappa$ (cf 2.8 of [24]) and the skew-symmetry of $S$.

Now we check the antipodal property
\[ m_J(id_J \otimes \kappa_J) \Phi_J = I_J \varepsilon_J = m_J(\kappa_J \otimes id_J) \Phi_J, \tag{3.12} \]
By 2.8 and 2.6 of [24], we have for coefficients $a_{ij} \in A$ of a unitary representation $(a_{ij})$ of the quantum group $A$ that
\[ m_J(id_J \otimes \kappa_J) \Phi_J(a_{ij}) = m_J(id_J \otimes \kappa_J) \Phi(a_{ij}) \]
\[ = |T|^2 \int_{s,u,t,v \in T} m(\alpha(s,u) \otimes \alpha(t,v)) \kappa) \Phi(a_{ij}) < J(s,u), (t,v) > \]
\[ = |T|^2 \int_{s,u,t,v \in T} m(id \otimes \kappa)(\lambda_s \rho_u \otimes \lambda_v \rho_t) \Phi(a_{ij}) < Ss, t > < -Su, v > \]
\[ = |T|^2 \int_{s,u,t,v \in T} m(id \otimes \kappa)(\lambda_s \rho_{u-v} \otimes \rho_t) \Phi(a_{ij}) < Ss, t > < -Su, v > \]
\[ = |T|^2 \int_{s,u,t,v \in T} m(id \otimes \kappa)(\lambda_s \rho_u \otimes \rho_t) \Phi(a_{ij}) < Ss, t > < -Su, v >, \]
which, using (2.5), noting that $< -Su, v >$ is a Pontryagin pairing on $T$,
\[ = |T| \int_{s,t \in T} m(id \otimes \kappa)(\lambda_s \otimes \rho_t) \Phi(a_{ij}) < Ss, t > \]
\[ = |T| \int_{s,t \in T} m(\lambda_s \otimes \kappa \rho_t)(\sum_k a_{ik} \otimes a_{kj}) < Ss, t > \]
\[ = |T| \int_{s,t \in T} \sum_{k,l,r} E_{-s}(\pi(a_{il}))a_{lk}a_{k,j}^* E_t(\pi(a_{rj})) < Ss, t > \]
\[ = |T| \int_{s,t \in T} (E_{-s} \otimes E_t) \Phi_T(\pi(a_{ij})) < Ss, t > \]
\[ = |T| \int_{s,t \in T} E_{-s+t}(\pi(a_{ij})) < Ss, t > = |T| \int_{s,t \in T} E_t(\pi(a_{ij})) < Ss, t >, \]
which, using once again (2.3),

\[ E_0(\pi(a_{ij})) = \epsilon_T(\pi(a_{ij})) = \epsilon(a_{ij}) = \epsilon_f(a_{ij}). \]

That is, on \( A_J \) (note that \( A_J = A \) as a vector space), \( m_J(id_J \otimes \kappa_J)\Phi_J = I_J \epsilon_J \).

Similarly \( m_J(\kappa_J \otimes id_J)\Phi_J = I_J \epsilon_J \).

Q.E.D.

We will denote the coproduct of \( A_J \) by \( \Phi_J \). Note that because of Proposition 2.8, \( A_J \) can be deformed back to the original quantum group \( (A_J)_{-J} = A \).

Remarks. (1). In the above, we assumed that \( A \) is a finite quantum group instead of a more general compact quantum group. In the latter case, we do not know how to show directly that \( A_J \) is still a compact quantum group. The main difficulty is to rigorously define the coproduct. Note that we can not define an analog of the map \( \varrho \) as given near 3.3 of [24] (see also [16]). One possible approach is to use the Krein duality as given in our paper [23].

(2). Just as in [16, 24], the Haar measure of \( A \) is still the Haar measure of \( A_J \). One can easily see this from the uniqueness of the (left and right invariant) Haar measure and the fact that the coproduct of \( A_J \) is that of \( A \). This proof is also valid for the infinite dimensional cases of [16, 24] if we work with the dense Hopf \( * \)-algebra, and it is much easier than the proof in [16].

(3). From the above remark and the orthogonality relations for characters of irreducible representations, we see that the irreducible representations of the quantum group \( A \) are still irreducible representations of the quantum group \( A_J \). From these, we see that the representation ring of the quantum group \( A \) is invariant under deformation, just as in [16, 24] and [14].

Now we describe the construction above in the dual picture. Let \( B \) be a finite dimensional Hopf \( C^* \)-algebra with coproduct \( \Phi \). Let \( T \) be an abelian subgroup of the group of group-like elements of \( B \). Endow \( T \) with a Pontryagin pairing. For notational convenience, we now assume that the group operation on \( T \) is multiplicative instead of additive, and for a skew symmetric automorphism \( S \) on \( T \), the matrix \( -S \) will denote the automorphism \( (-S)x = (Sx)^{-1} \). As before, let \( J = (S, -S) \) be the skew-symmetric automorphism on \( H = T \times T \). Then \( B \) is a Hopf \( C^* \)-algebra under the original \( C^* \)-algebra structure, original counit and
antipode, and the deformed coproduct given by
\[ \Phi_J(b) = |T|^2 \int_{s,u,t,v \in T} (\beta(s,u) \otimes \beta(t,v)) \Phi(b) < Ss,t > < (Su)^{-1},v > \] (3.13)
where
\[ \beta(s,u)(b) = sbu^{-1}. \] (3.14)

Using orthogonality relations for group characters, we have that
\[ (|T| \int_{s,t \in T} (s \otimes t) < Ss,t >)^{-1} = |T| \int_{u,v \in T} (u^{-1} \otimes v) < Su,v > . \] (3.15)

We can also check that
\[ F = |T| \int_{s,t \in T} (s \otimes t) < Ss,t > \] (3.16)
is a unitary element in \( B \otimes B \). We can now rewrite \( \Phi_J(b) \) as
\[ \Phi_J(b) = F \Phi(b) F^{-1}. \] (3.17)

This shows that our deformation of finite quantum groups in this dual picture is an analog of the twistings of quantized universal enveloping algebra of Drinfeld [4, 11, 12], just as the ones in [16, 24]. The element \( F \) is not a 2-cocycle, though both \((\epsilon \otimes 1)(F) = 1\) and \((1 \otimes \epsilon)(F) = 1\) are satisfied. But since the twist \( \Phi_J \) is coassociative because of Theorem 3.2, \( F \) is a pseudo-2-cocycle in the sense of [5, 18]. However, it is not clear how to verify directly that \( F \) satisfies the pseudo-2-cocycle condition for a general noncommutative and noncocommutative Hopf \( C^* \)-algebra \( B \) without passing to the dual \( A \) of \( B \) (see Proposition 2.2 and Theorem 3.2)! The above twisting of \( B \) can be viewed as the canonical one among the ones considered in [4, 18] that are associated with a finite abelian subgroup \( T \). We will call the twist \( F \) the Weyl-von Neumann-Rieffel twist associated with \((T,S)\), and denote the twist of \( B \) by \( B^S \) (to distinguish it from \( A_J \)). To gain a better understanding of this construction, it is desirable to solve the following (cf remark (3) above and [14]):

**Problem 3.3.** Characterize the finite quantum groups that can be obtained as deformations of finite groups in the manner above. Find their isomorphic invariants.
4. A finite quantum group of order 18

Let \( T = (\mathbb{Z}/n\mathbb{Z})^{2k} \) with canonical generators \( \epsilon_j, j = 1, \ldots, 2k \) (\( \epsilon_j \) is the element with \( j \)-th component 1 and the other components 0). Let \( \beta \) be the action of \( \mathbb{Z}/2\mathbb{Z} \) on \( T \) which exchanges \( \epsilon_j \) and \( \epsilon_{j+k}, j = 1, \ldots, k \). Let \( G \) be the corresponding semi-direct product of \( T \) by \( \mathbb{Z}/2\mathbb{Z} \) under this action: \( G = T \rtimes_\beta \mathbb{Z}/2\mathbb{Z} \). Let \( A = C(G) \). Consider the skew-symmetric automorphism \( S \) of \( T \) defined by

\[
S(\epsilon_j) = -\epsilon_{k+j}, \quad S(\epsilon_{k+j}) = \epsilon_j, \quad j = 1, \ldots, k.
\]

(4.1)

Alternatively, \( S \) has matrix representation

\[
\left( \begin{array}{cc} 0 & I_k \\ -I_k & 0 \end{array} \right)
\]

with respect to the generators (or “basis”) \( \epsilon_j, j = 1, \ldots, 2k \), where \( I_k \) represents the identity transformation on the group \( (\mathbb{Z}/n\mathbb{Z})^k \). Then we are in the position to apply Theorem 3.2 to obtain a noncommutative deformation \( A_J \), where \( J = S \oplus (-S) \) (see the remark after Proposition 2.3).

Take \( k = 1 \) and \( n = 3 \). Then \( A_J = C(G)_J \) is a finite quantum group of order 18, where by definition, the order of a finite group is the dimension of its function algebra. From the remark after Proposition 2.3, we see that this quantum group is noncommutative and noncocommutative.

We now show that the quantum group \( A_J \) (and its dual \( B_S \)) is not a crossed product of the form \( C(G_1) \rtimes_{\tau} G_2 \), where \( G_1 \) and \( G_2 \) are ordinary finite groups and \( \tau \) is an action of \( G_2 \) on \( C(G_1) \) by automorphisms of quantum groups (the triple \( (C(G_1), G_2, \tau) \) is also called a Woronowicz Hopf \( C^* \)-dynamical system, see \([8, 22]\)).

An easy application of the Mackey Machine shows that the group \( G = (\mathbb{Z}/3\mathbb{Z})^2 \rtimes_\beta \mathbb{Z}/2\mathbb{Z} \) has 6 irreducible representations: 2 one-dimensional representations and 4 two-dimensional representations. Hence by remark (3) after the proof of Theorem 3.2, the quantum group \( A_J \) also has 6 irreducible representations. If \( A_J \) is a crossed product of the form \( C(G_1) \rtimes_{\tau} G_2 \), then we only need to consider four cases: (i) \( |G_1| = 2, |G_2| = 9 \), (ii) \( |G_1| = 9, |G_2| = 2 \), (iii) \( |G_1| = 3, |G_2| = 6 \), (iv) \( |G_1| = 6, |G_2| = 3 \). We claim that in each of the cases (i), (ii) and (iii), the quantum group \( A_J = C(G_1) \rtimes_{\tau} G_2 \) is a group \( C^* \)-algebra of an ordinary group and hence it has 18 irreducible representations instead of 6 (cf. [27], the irreducible representations of a compact quantum group of the form \( C^*(\Gamma) \) are exactly the elements of of the discrete group \( \Gamma \)).
This claim implies that cases (i), (ii) and (iii) cannot happen. To prove the claim we note first that $G_1$ is abelian in each of these three cases. Since $\tau$ is assumed to preserve the Hopf $C^*$-algebra structure of $C(G_1)$, which is isomorphic to $C^*(\hat{G}_1)$ (as a Hopf $C^*$-algebra), by transport of structure, $\tau$ is an action of $G_2$ by automorphisms on the group $\hat{G}_1$. Hence $C(G_1) \rtimes_\tau G_2 \cong C^*(\hat{G}_1 \rtimes_\tau G_2)$ as claimed. Now consider case (iv). If $G_1$ is abelian, then the same reasoning as above leads to a contradiction. If $G_1$ is non-abelian, then $G_1$ is the group $S_3$, hence it has 3 irreducible representations. From the classification of irreducible representations of quantum groups associated with crossed products (see Theorem 3.7 of [22]), $C(G_1) \rtimes_\tau G_2$ has $3 \times 3 = 9$ irreducible representations. This again contradicts the fact that the quantum group $A_J$ has 6 irreducible representations. This shows that case (iv) cannot happen.

Similarly, we show that the dual $B^S$ of the above $A_J$ cannot be expressed in the form $C(G_1) \rtimes_\tau G_2$ either. As above we only need to look at case (iv) above with $G_1 = S_3$ and $\tau$ non-trivial. By remark (3) after the proof of Theorem 3.2 and the analysis in the previous paragraph, the algebra $B^S$ has 6 irreducible representations (which are exactly the irreducible representations of the quantum group $A_J$). By transport of structure, $\tau$ gives an automorphism of order three of the group $S_3$. An examination of the structure of $S_3$ then shows that the action $\tau$ is conjugation by a 3 cycle in $S_3$. From this, using Mackey Machine we see that the algebra $C(S_3) \rtimes_\tau \mathbb{Z}/3\mathbb{Z}$ has 10 irreducible representations instead of 6: 9 one-dimensional representations and 1 three-dimensional representation. This completes the proof.

**Remarks.** (1). The group $D_4 = (\mathbb{Z}/2\mathbb{Z})^2 \rtimes_{\beta} \mathbb{Z}/2\mathbb{Z}$ is the only non-abelian group of order 8 that has a maximal abelian subgroup $T$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$, and $S$ defined above is the only non-trivial skew-symmetric automorphism on $T$. It is easy to see that $C(D_4)_J = C(D_4)$ (see the remark after Proposition 2.3). Since the 8 dimensional quantum group of Kac-Palyutkin [3] clearly contains $T$ as a subgroup, we conclude from Theorem 3.3 that this quantum group is not of the form $C(G)_J$ for any finite group $G$.

(2). By the same method as above, we note that the duals of the two examples of 12 dimensional Kac algebras in [4, 14] (as twists of $D_6$ and $Q_4$ respectively) are not isomorphic to a Kac algebra of the form $C(S_3) \rtimes_\tau \mathbb{Z}/2\mathbb{Z}$ (like case (iv) above, this is the only non-trivial case), where $\mathbb{Z}/2\mathbb{Z}$ acts on $S_3$ non-trivially. The second tensor power of
one of the 2 two-dimensional irreducible representations of the algebra
$L(G)$ of [3] decomposes into a direct sum of one-dimensional subrep-
resentations, but one can verify using [22] that this does not happen
for either of the 2 two-dimensional irreducible representations of the
quantum group $C(S_3) \rtimes \tau \mathbb{Z}/2\mathbb{Z}$. Similarly, we see that the group
of one-dimensional representations of the algebra $L(G)$ of [14] (see Exam-
ple 2.9.(iii) therein) is $\mathbb{Z}/4\mathbb{Z}$, while using [22] we see that the group
of one dimensional representations of the quantum group $C(S_3) \rtimes \tau \mathbb{Z}/2\mathbb{Z}$
is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

5. Deformation of finite groups of Lie type

Instead of constructing $q$-deformations of finite groups of Lie type
(see Problem 3.1), we now construct a Rieffel type deformation for
such finite groups.

Let $q = p^d$ be the power of a prime $p$, and let $F_q$ be the field with $q$
elements. Let $G$ be a finite groups of Lie type over $F_q$, e.g., $GL(n, F_q)$,
$SL(n, F_q)$, $PSL(n, F_q)$ or $Sp(n, F_q)$. Let $T$ be a maximal abelian sub-
group of $G$ ($T$ can be viewed as the points over $F_q$ of a maximal torus of
the algebraic group corresponding to $G$). Apart from trivial cases like
$GL(2, F_2) = SL(2, F_2) = PSL(2, F_2) = S_3$, $T$ has non-trivial skew-
symmetric automorphisms $S$. Letting $J = S \oplus (-S)$, we can use
Theorem 3.2 to obtain a deformation $C(G)_J$. In general, the quan-
tum groups $C(G)_J$ should be non-trivial in the sense of the previous
sections.

Take for example the simplest case $GL(2, F_q)$. Then $T$ consists ma-
trices of the form

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

with $a, b \in F_q^*$. Hence $T$ is isomorphic to the group $(\mathbb{Z}/(q - 1)\mathbb{Z})^2$. As
in the previous section, we have on $T$ the canonical skew-symmetric
automorphism

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$  

Using Theorem 3.2, we can form the deformation $C(GL(2, F_q))_J$. If $q \neq
2, 3$, then the remark after Proposition 2.3 shows that $C(GL(2, F_q))_J$
is noncommutative and noncocommutative.

Apart from trivial cases such as those mentioned in the beginning of
this section, it is not hard to see that $C(G)_J$ is noncommutative and
noncocommutative for an arbitrary finite group of Lie type $G$. However,
it is not so clear how to show that $C(G)_J$ cannot be expressed as crossed
product of the form $C(G_1) \rtimes G_2$ for some finite groups $G_1$ and $G_2$. We plan to study this further in the future.

Since there are many skew-symmetric automorphisms on the maximal abelian subgroup of a finite group of Lie type $G$, $C(G)_J$ is not the analog of $q$-deformation of Drinfeld and Jimbo of infinite Lie groups.

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**References**

[1] Baaj, S. and Skandalis, G.: Unitaires multiplicatifs et dualité pour les produits croisés de $C^*$-algébres, *Ann. Sci. Ec. Norm. Sup.* **26** (1993), 425-488.

[2] Connes, A.: *Noncommutative Geometry*, Academic Press, 1994.

[3] Connes, A.: Noncommutative geometry and reality, *J. Math. Phys.* **36**:11 (1995), 6194-6231

[4] Drinfeld, V. G. : Quasi-Hopf algebras, *Leningrad Math. J.* **1**(6) (1990), 1419-1457

[5] Enock, M. and Vainerman, L.: Deformation of a Kac algebra by an abelian subgroup, *Commun. Math. Phys.* **178** (1996), 571-596.

[6] Hewitt, E. and Ross, K.: *Abstract Harmonic Analysis II*, Springer–Verlag, 1970.

[7] Kac, G.: Certain arithmetic properties of ring groups, *Funct. Anal. Appl.* **6** (1972), 158-160.

[8] Kac, G. and Palyutkin, V.: Finite ring groups, *Trans. Moscow Math. Soc.* **15** (1966), 251-294.

[9] Landstad, M.B.: Quantizations arising from abelian subgroups, *Internat. J. Math.* **5** (1994), 897-936.

[10] Lanstad, M. B. and Raeburn, I.: Twisted dual-group algebras: Equivariant deformations of $C_0(G)$, *J. Funct. Anal.* **132**, 43-85 (1995).

[11] Levendorskii, S.: Twisted algebra of functions on compact quantum group and their representations, *St. Petersburg Math. J.* **3**:2 (1992), 405-423.
[12] Levendorskiı, S. and Soibelman, Y.: Algebra of functions on compact quantum groups, Schubert cells, and quantum tori, Commun. Math. Phys. 139, (1991), 141-170.
[13] von Neumann, J.: Die Eindeutigkeit der Schrödingschen Operatoren, Math. Ann. 104 (1931), 570-578.
[14] Nikshych, Dmitri: $K_0$ rings and twisting of finite dimensional semisimple Hopf algebras, Preprint, National Technical University of Ukraine “Kiev Polytechnic Institute”, 1997.
[15] Rieffel, M.: Deformation quantization for actions of $\mathbb{R}^d$, Memoirs A.M.S. no. 506, 1993.
[16] Rieffel, M.: Compact quantum groups associated with toral subgroups, Contemp. Math. 145 (1993), 465-491.
[17] Rieffel, M.: Non-compact quantum groups associated with abelian subgroups, Commun. Math. Phys. 171 (1995), 181-201.
[18] Vainerman, L. I.: 2-cocycles and twisting of Kac algebras, Commun. Math. Phys. 191:3 (1998), 697-721.
[19] Vaksman, L. and Soibelman, Y.: The algebra of functions on quantum $SU(2)$, Funct. Anal. ego Pril. 223 (1988), 1-14.
[20] Van Daele, A. and Wang, S. Z.: Universal quantum groups, International J. Math 7:2 (1996), 255-264.
[21] Wang, S. Z.: Free products of compact quantum groups, Commun. Math. Phys. 167:3 (1995), 671-692.
[22] Wang, S. Z.: Tensor products and crossed products of compact quantum groups, Proc. London Math. Soc. 71:3 (1995), 695-720.
[23] Wang, S. Z.: Krein duality for compact quantum groups, J. Math. Phys. 38:1 (1997), 524-534
[24] Wang, S. Z.: Deformations of compact quantum groups via Rieffel’s quantization, Commun. Math. Phys. 178:3 (1996), 747-764.
[25] Wang, S. Z.: Problems in the theory of quantum groups, in Quantum Groups and Quantum Spaces, Banach Center Publication 40 (1997), Inst. of Math., Polish Acad. Sci., Editors: R. Budzynski, W. Pusz, and S. Zakrzewski. pp67-78
[26] Woronowicz, S. L.: Twisted $SU(2)$ group. An example of noncommutative differential calculus, Publ. RIMS, Kyoto Univ. 23 (1987), 117-181.
[27] Woronowicz, S. L.: Compact matrix pseudogroups, Commun. Math. Phys. 111 (1987), 613-665.
[28] Woronowicz, S. L.: Tannaka-Krein duality for compact matrix pseudogroups. Twisted $SU(N)$ groups, Invent. Math. 93 (1988), 35-76.

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