Rational bicubic Ball for image interpolation

N A Zulkifli¹, S A A Karim², A Shafie³ and M Sarfraz⁴

¹, ³ Fundamental and Applied Sciences Department, Institute of Autonomous System Universiti Teknologi PETRONAS, Bandar Seri Iskandar, 32610 Seri Iskandar, Perak DR, Malaysia.
² Fundamental and Applied Sciences Department and Centre for Smart Grid Energy Research (CSMER), Universiti Teknologi PETRONAS, Bandar Seri Iskandar, 32610 Seri Iskandar, Perak DR.
⁴ Department of Information Science, College of Computing Sciences & Engineering, Kuwait University, Safat, Kuwait.

2 samsul_ariffin@utp.edu.my

Abstract. In this study, a new rational bicubic Ball with six free parameters is proposed for image interpolation. These free parameters can be used to refine the shape of the surface. This rational bicubic Ball function is constructed by using tensor product approach. The proposed scheme is tested to grayscale image upscale. An efficient algorithm is employed to upscales the original image with factor 2 and 4. The effectiveness of proposed scheme is measured in terms of Peak Signal-to-Noise Ratio (PSNR), Root Mean Square Error (RMSE) and Feature SIMilarity (FSIM) index including comparison with some existing schemes. Numerical and graphical results for image interpolation are presented by using MATLAB. We found out the proposed scheme resulting with higher PSNR and FSIM and smaller in RMSE. Thus, the new rational bicubic Ball with six parameters is better than the existing schemes.

1. Introduction

Image interpolation is one of the important parts in image processing as it is required before the image are subject to further processing [1]. It is a widely used tool in image processing tasks such as zooming, shrinking, rotating and geometric corrections. Majeed et al. [2] has investigated the application of rational cubic Ball (cubic/cubic) with two parameters for craniofacial reconstruction by utilizing the rational cubic Ball (curve) and reconstruct the curve of the images after doing the image detection to the outer part of the given images. History stated that the rational cubic Ball has been introduced by Ball in 1974 [3]. The main usage of the function is to design Boeing fuselage. In addition, the common method for image interpolation is bi-cubic spline interpolation that is well documented in MATLAB as interp2 and imresize built-in functions [1]. This scheme is implemented in application image interpolation for instance image upscaling with factor 2 and 4. For example, if the original image has a size of about 256x256 pixels, then the image output is upgraded by a two-size factor of 512x512 pixels.

This is an active topic in an image processing area with a recent challenge to find the best function for upscaling images to specific needs such as PSNR, RMSE and FSIM. There are many methods that can be used for image interpolation. The standard method is cubic and bicubic spline interpolation [4, 5, 6, 7, 8, 9, 10] as well as rational cubic B-spline [11, 12]. But, there are no free parameter in the spline description, the user cannot alter the surface and resulting image. To overcome this, many researchers have suggested various types of rational spline and wavelet-based approach to tackle image interpolation problem. However, among them used rational quartic spline, rational cubic spline with linear denominator and rational cubic spline with quadratic denominator. In fact, their method suffers from the fact that some of it needs the modification of the first partial derivative, considering only true function
value (where the first derivative is not supplied), and only have at most two parameters. Thus, this scheme constructs new rational bicubic Ball function with six parameters that has the capability to produce smooth interpolation surface with less error. It will be used for image interpolation i.e. image upscaling (grayscale image) through an efficient algorithm. The main objectives of study are stated below:

a) To propose a new rational bicubic Ball with six parameters.
b) To apply the proposed scheme in application of image interpolation, image upscaling i.e. grayscale.

2. Cubic Ball with three parameters

In this section, we will discuss the construction of proposed rational Ball with six parameters wherein can be used to interpolate the surface and image. The new method has been extended from Karim [13] using tensor product approach.

2.1. Rational Bi-Cubic Ball with Six Parameters

This section discusses the construction of the rational cubic Ball with three parameters [13]. Assume that the data sets \{(x_i, f_i), i = 1, 2, ..., n\} is given where \(x_1 < x_2 < \cdots < x_n\) with the first derivative \(d_i, i = 1, 2, ..., n\). Let, \(h_i = x_{i+1} - x_i, \Delta_i = (f_{i+1} - f_i)/h_i\) and \(\theta = (x - x_i)/h_i, 0 \leq \theta \leq 1\). On each subinterval \(x \in [x_i, x_{i+1}], i = 1, 2, ..., n - 1\). Thus, the rational cubic Ball interpolant with three parameters, \(\alpha_i, \beta_i, \gamma_i, i = 1, 2, ..., n - 1\) is defined as follow

\[
S(x) \equiv S_i(\theta) = \frac{P_i(\theta)}{Q_i(\theta)}
\]

where,

\[
P_i(\theta) = \alpha_i f_i (1 - \theta)^2 + B_i (1 - \theta)^2 \theta + C_i (1 - \theta) \theta^2 + \beta_i f_{i+1} \theta^2
\]

\[
Q_i(\theta) = \alpha_i (1 - \theta)^2 + \gamma_i (1 - \theta) \theta + \beta_i \theta^2
\]

The rational function satisfies \(C^1\) continuity as follows:

\[
S(x_i) = f_i \quad \quad S(x_{i+1}) = f_{i+1}
\]

\[
S'(x_i) = d_i \quad \quad S'(x_{i+1}) = d_{i+1}
\]

Equation (1) can be simply derive using condition in Equation (2) with the unknown variable \(A_i, B_i, i = 1, 2, ..., n - 1\), are written as

\[
\therefore B_i = \gamma_i f_i + \alpha_i h_i d_i
\]

\[
\therefore C_i = \gamma_i f_{i+1} - \beta_i h_{i+1} d_{i+1}
\]

It can be verified when \(x = x_i\), then \(\theta = 0\) and \(x = x_{i+1}\), then \(\theta = 1\). Thus, the rational cubic Ball defined in (1) can be written as:

\[
S(x) = \frac{k(\theta)}{l(\theta)}
\]

where,

\[
k(\theta) = \alpha f_i (1 - \theta)^2 + (\gamma f_i + \alpha h_i d_i)(1 - \theta)^2 \theta + (\gamma f_{i+1} - \beta h_i d_{i+1})(1 - \theta) \theta^2 + \beta f_{i+1} \theta^2
\]

\[
l(\theta) = \alpha_i (1 - \theta)^2 + \gamma_i (1 - \theta) \theta + \beta_i \theta^2
\]

2.2. Rational Bi-Cubic Ball with Six Parameters

The univariate Ball given in (1) is extended to bivariate cases by adopting similar approach in [8]. The rational bicubic function over each rectangular patch \([x_i, x_{i+1}] \times [y_j, y_{j+1}]\), where \(i = 1, 2, ..., n - 1; j = 0, 1, ..., m - 1\) is defined as below.

\[
S_{i,j}(x, y) = A_i(\theta) H_{i,j} A_j(\theta)
\]
\[ H_{i,j} = \begin{bmatrix} F_{i,j} & F_{i,j+1} & F^y_{i,j} & F^y_{i,j+1} \\ F_{i+1,j} & F_{i+1,j+1} & F^y_{i+1,j} & F^y_{i+1,j+1} \\ F^x_{i,j} & F^x_{i,j+1} & F^{xy}_{i,j} & F^{xy}_{i,j+1} \\ F^x_{i+1,j} & F^x_{i+1,j+1} & F^{xy}_{i+1,j} & F^{xy}_{i+1,j+1} \end{bmatrix} \]

where,

\[
a_0(\theta) = \frac{\alpha_{i,j}(1 - \theta)^2 + \gamma_{i,j}(1 - \theta)^2}{q_i(\theta)}
\]

\[
a_1(\theta) = \frac{\beta_{i,j}(1 - \theta)^2}{q_i(\theta)}
\]

\[
a_2(\theta) = \frac{\alpha_{i,j} h_i(1 - \theta)^2}{q_i(\theta)}
\]

\[
a_3(\theta) = \frac{-\beta_{i,j} h_i(1 - \theta)^2}{q_i(\theta)}
\]

\[
\hat{a}_0(\theta) = \frac{\alpha_{i,j}(1 - \theta)^2}{q_i(\theta)}
\]

\[
\hat{a}_1(\theta) = \frac{\beta_{i,j}(1 - \theta)^2}{q_i(\theta)}
\]

\[
\hat{a}_2(\theta) = \frac{\alpha_{i,j} h_i(1 - \theta)^2}{q_i(\theta)}
\]

\[
\hat{a}_3(\theta) = \frac{-\beta_{i,j} h_i(1 - \theta)^2}{q_i(\theta)}
\]

with,

\[
q_i(\theta) = \alpha_{i,j}(1 - \theta)^2 + \gamma_{i,j}(1 - \theta)\theta + \beta_{i,j}(1 - \theta)^2
\]

\[
q_j(\theta) = \hat{a}_{i,j}(1 - \theta)^2 + \hat{\gamma}_{i,j}(1 - \theta)\theta + \hat{\beta}_{i,j}(1 - \theta)^2
\]

After some derivation, the proposed rational bicubic Ball surface satisfies:

\[ S_{i,j}(0,0) = F_{i,j} \]

\[ S_{i,j}(1,0) = F_{i+1,j} \]

\[ S_{i,j}(0,1) = F_{i,j+1} \]

\[ S_{i,j}(1,1) = F_{i+1,j+1} \]  

(6)

**Table 1.** List of abbreviation of equation (6).

| Symbol      | Definition                                |
|-------------|-------------------------------------------|
| \( F_{i,j} \) | First partial derivative                  |
| \( F^x_{i,j} \) | First partial derivative on x directions |
| \( F^y_{i,j} \) | First partial derivative on y directions |
| \( F^{xy}_{i,j} \) | Mixed partial derivatives at the interior points |
| \( z_{i,j} \) | Input pixel indexes for grayscale intensity |
| \( \hat{z}_{i,j} \) | Interpolating input pixels                |
3. Application in image interpolation
This section will explain on how image interpolation works in its application for instance image upscaling with factor two and four. Algorithm of image is explained in figure 1.

3.1. Steps of image scaling
Figure 2 shows the flowchart for the algorithm to be implemented in image interpolation by using the proposed rational bicubic Ball function.

![Flowchart steps of image upscaling.](image)

**Step 1:** Find the function, $\hat{z}_{i,j} = F(i,j)$, $i = 1, 2, ..., m$, $j = 1, 2, ..., n$ where $(i,j)$ represent input pixels indexes and $z_{i,j}(0-255)$ be their corresponding grayscale intensity which interpolate the given pixels.

**Step 2:** Construct rectangular mesh for given input pixels using proposed function defined in (6).

**Step 3:** Obtaining upscale image by following transformation with scaling factor, $k = 2, 4$

$$
\begin{pmatrix}
x' \\
y'
\end{pmatrix} = 
\begin{pmatrix}
\frac{nk}{n-1} & 0 \\
0 & \frac{mk-1}{m-1}
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix} + 
\begin{pmatrix}
\frac{n(1-k)}{n-1} \\
\frac{m(1-k)}{m-1}
\end{pmatrix}
$$

where $(x,y) =$ input; $(x',y') =$ output

**Step 4:** Identify the rectangles with the missing grayscale intensity values. The original pixels and their intensity values of input image are at vertex of rectangular mesh.

**Step 5:** Estimate the derivative at each vertex of rectangular (input pixels) by method discussed.

**Step 6:** Estimate the missing grayscale intensity value in by using proposed function defined in (5).

**Step 7:** Display the result of interpolating image and compare the performance of proposed method.

**Output:** PSNR, RMSE and FSIM.

Figure 1. Flowchart steps of image upscaling.
The proposed rational bicubic Ball with six parameters is tested for image upscaling with factor 2 and 4 using five different tested images such as cameraman, baboon, couple, gold hill and thumbprint as shown in figure 2. The original image is given with size 256×256 pixels. Later on, the image will be upcaled up to factor 2 and 4. The comparison of propose scheme has done with some existing schemes such as with three standard method in MATLAB: nearest neighbor, bilinear spline, bicubic spline and one from existing scheme: Karim and Saaban [6].

![Figure 2](image)

*Figure 2. Original tested images; (a)cameraman (b)baboon (c)couple (d)gold hill (e)thumbprint.*

3.2. Image Quality Assessment

There are three ways of measurement that can be used to evaluate results of image upscaling in terms of PSNR, RMSE and FSIM. The definition of each terms is defined as below:

a) Peak Signal-to-Noise Ratio (PSNR)

\[
PSNR = 10 \log_{10} \frac{255^2}{MSE}
\]

where,

\[
MSE = \frac{1}{mn} \sum_{i=0}^{n} \sum_{j=0}^{m} |z_{ij} - \bar{z}_{ij}|^2
\]

b) Root Mean Square Error (RMSE)

\[
RMSE = \sqrt{\frac{1}{mn} \sum_{i=0}^{n} \sum_{j=0}^{m} |z_{ij} - \bar{z}_{ij}|^2}
\]

c) Feature SIMilarity (FSIM) index

\[
FSIM = \frac{\sum_{x \in \Omega} S_{PC}(x)S_{m}(x)}{\sum_{x \in \Omega} P_{PC}(x)}
\]

where,

\[
S_{PC}(x) = [S_{PC}(x)]_1, \quad S_{PC}(x) = \frac{2PC_1(x)PC_2(x)+T_1}{PC_1^2(x)+PC_2^2(x)+T_1}, \quad S_{m}(x) = \frac{2G_1(x)G_2(x)+T_2}{G_1^2(x)+G_2^2(x)+T_2}
\]

| Symbol | Definition |
|--------|------------|
| $PC_1$ | Phase congruency of the original image. |
| $PC_2$ | Phase congruency of the interpolation image. |
| $G_1$  | Gradient magnitude of the original image. |
| $G_2$  | Gradient magnitude of the interpolation image. |
| $T_1$  | Constant to increase the stability of $S_{PC}$. |
| $T_2$  | Positive constant depending on the dynamic range of gradient magnitude values. |
4. Result and discussion

In this section, results from implementation of the proposed scheme is discussed with validation stated in figure 2.

4.1. Parameter selection

The proposed bivariate rational Ball have six free parameters that can be altered to produce quality image with higher in PSNR and FSIM value and lower in RMSE value. The simulation undertaken by PSNR as a stopping criterion as shown in the flowchart (figure 3). This step is done repeatedly and performed on each different image to get the suitable parameter value for each six parameters.

![Figure 3. Flowchart steps of parameter selection.](image)

Table 3 shows all the parameter values that have been set for every tested image and the explanation about selection of parameter values done in table 4. The list of parameter values with symbol: P1, ..., P24, that used for simulation is shown in table 4, but only for tested image cameraman as an overview of the simulation step. For set parameter P1-P15 are setup with value 0.1 (starting value) and 0.2 (increment value) at the selected parameter by pattern as shown in the table. This has resulting parameter P13 given the highest PSNR value which then conducting to modification of parameter value with increment pattern at $\alpha_{i,j}$, $\beta_{i,j}$, $\gamma_{i,j}$, $\hat{\alpha}_{i,j}$, $\hat{\beta}_{i,j}$ and decrement pattern at $\gamma_{i,j}$, $\hat{\gamma}_{i,j}$. Therefore, the parameter values for image ‘cameraman’ are set as follows, $\alpha_{i,j} = 0.7$, $\beta_{i,j} = 0.7$, $\gamma_{i,j} = 0.09$, $\hat{\alpha}_{i,j} = 0.7$, $\hat{\beta}_{i,j} = 0.7$ and $\hat{\gamma}_{i,j} = 0.09$. Overall results of image quality assessments; PSNR, RMSE and FSIM are shown in Table 5 with comparison with some existing schemes. Based on Table 5, image upscaling with factor 2 and 4 shows that the proposed scheme gives highest PSNR and lowest in RMSE for all images. Meanwhile, for FSIM assessment of image upscaling with factor 2 and 4 resulting highest PSNR for all images except image ‘Thumbprint’ and except image ‘Cameraman’ and ‘Gold Hill’ respectively.

Table 3. Parameter value for image upscaling.

| Image     | $\alpha_{i,j}$ | $\beta_{i,j}$ | $\gamma_{i,j}$ | $\hat{\alpha}_{i,j}$ | $\hat{\beta}_{i,j}$ | $\hat{\gamma}_{i,j}$ |
|-----------|----------------|---------------|----------------|-----------------------|-----------------------|----------------------|
| Cameraman | 0.7            | 0.7           | 0.09           | 0.7                   | 0.7                   | 0.09                 |
| Baboon    | 1.0            | 1.0           | 0.01           | 1.0                   | 1.0                   | 0.01                 |
| Couple    | 0.5            | 0.5           | 0.03           | 0.5                   | 0.5                   | 0.03                 |
| Gold Hill | 0.5            | 0.5           | 0.02           | 0.5                   | 0.5                   | 0.02                 |
| Thumbprint| 0.8            | 0.9           | 0.01           | 0.8                   | 0.9                   | 0.01                 |

Table 4. Impact Parameter towards PSNR.
| Set  | $\alpha_{i,j}$ | $\beta_{i,j}$ | $\gamma_{i,j}$ | $\bar{\alpha}_{i,j}$ | $\bar{\beta}_{i,j}$ | $\bar{\gamma}_{i,j}$ | PSNR  |
|------|----------------|----------------|----------------|----------------------|-------------------|-------------------|-------|
| P1   | 0.1            | 0.1            | 0.1            | 0.1                  | 0.1               | 0.1               | 40.15 |
| P2   | 0.1            | 0.1            | 0.2            | 0.1                  | 0.2               | 0.1               | 40.16 |
| P3   | 0.1            | 0.2            | 0.1            | 0.1                  | 0.2               | 0.1               | 40.15 |
| P4   | 0.2            | 0.1            | 0.1            | 0.2                  | 0.1               | 0.1               | 40.10 |
| P5   | 0.2            | 0.1            | 0.1            | 0.1                  | 0.1               | 0.1               | 40.14 |
| P6   | 0.1            | 0.2            | 0.1            | 0.1                  | 0.1               | 0.1               | 40.15 |
| P7   | 0.1            | 0.1            | 0.2            | 0.1                  | 0.1               | 0.1               | 40.12 |
| P8   | 0.1            | 0.1            | 0.2            | 0.1                  | 0.1               | 0.1               | 40.16 |
| P9   | 0.1            | 0.1            | 0.1            | 0.1                  | 0.2               | 0.1               | 40.14 |
| P10  | 0.1            | 0.1            | 0.1            | 0.1                  | 0.2               | 0.1               | 40.11 |
| P11  | 0.1            | 0.1            | 0.2            | 0.1                  | 0.2               | 0.1               | 40.15 |
| P12  | 0.2            | 0.2            | 0.2            | 0.1                  | 0.1               | 0.1               | 40.15 |
| P13  | 0.2            | 0.2            | 0.1            | 0.2                  | 0.2               | 0.1               | 40.18 |
| P14  | 0.2            | 0.1            | 0.2            | 0.2                  | 0.1               | 0.2               | 40.12 |
| P15  | 0.1            | 0.2            | 0.2            | 0.1                  | 0.2               | 0.2               | 40.10 |
| P16  | 0.2            | 0.2            | 0.1            | 0.2                  | 0.2               | 0.1               | 40.18 |
| P17  | 0.3            | 0.3            | 0.1            | 0.3                  | 0.3               | 0.1               | 40.19 |
| P18  | 0.4            | 0.4            | 0.1            | 0.4                  | 0.4               | 0.1               | 40.19 |
| P19  | 0.5            | 0.5            | 0.1            | 0.5                  | 0.5               | 0.1               | 40.20 |
| P20  | 0.6            | 0.6            | 0.1            | 0.6                  | 0.6               | 0.1               | 40.20 |
| P21  | 0.7            | 0.7            | 0.1            | 0.7                  | 0.7               | 0.1               | 40.20 |
| P22  | 0.7            | 0.7            | 0.09           | 0.7                  | 0.7               | 0.09              | 40.21 |
| P23  | 0.7            | 0.7            | 0.08           | 0.7                  | 0.7               | 0.08              | 40.21 |
| P24  | 0.7            | 0.7            | 0.07           | 0.7                  | 0.7               | 0.07              | 40.21 |

**Table 5.** Image quality assessment for image upscaling.

| PSNR | Factor 2 | Factor 4 |
|------|----------|----------|
|      | NN | BL | BC | KS | BB | NN | BL | BC | KS | BB |
| Baboon | 37.96 | 37.75 | 38.26 | 38.20 | 38.38 | 36.54 | 36.53 | 36.70 | 36.66 | 36.70 |
| Thumbprint | 35.88 | 35.56 | 35.94 | 35.96 | 36.18 | 35.17 | 35.11 | 35.14 | 35.22 | 35.24 |
| Couple | 38.80 | 38.49 | 38.98 | 39.01 | 39.16 | 37.08 | 37.02 | 37.24 | 37.30 | 37.38 |
| Cameraman | 39.99 | 39.42 | 39.87 | 40.08 | 40.22 | 38.59 | 38.26 | 38.48 | 38.60 | 38.65 |
| Gold Hill | 39.71 | 39.49 | 40.21 | 40.13 | 40.35 | 37.54 | 37.43 | 37.79 | 37.82 | 37.94 |
| RMSE | Factor 2 | Factor 4 |
|      | NN | BL | BC | KS | BB | NN | BL | BC | KS | BB |
| Baboon | 3.22 | 3.30 | 3.11 | 3.14 | 3.07 | 3.80 | 3.80 | 3.73 | 3.75 | 3.73 |
| Thumbprint | 4.10 | 4.25 | 4.07 | 4.06 | 3.96 | 4.45 | 4.48 | 4.48 | 4.42 | 4.41 |
| Couple | 2.93 | 3.03 | 2.87 | 2.86 | 2.81 | 3.57 | 3.60 | 3.50 | 3.48 | 3.45 |
| Cameraman | 2.55 | 2.73 | 2.59 | 2.53 | 2.49 | 3.00 | 3.12 | 3.04 | 2.99 | 2.98 |
| Gold Hill | 2.64 | 2.70 | 2.49 | 2.51 | 2.45 | 3.39 | 3.43 | 3.29 | 3.28 | 3.23 |
| FSIM | Factor 2 | Factor 4 |
|      | NN | BL | BC | KS | BB | NN | BL | BC | KS | BB |
| Baboon | 0.8546 | 0.8063 | 0.8503 | 0.8483 | 0.8622 | 0.6269 | 0.6245 | 0.6607 | 0.6575 | 0.6763 |
| Thumbprint | 0.8347 | 0.7650 | **0.8484** | 0.8206 | 0.8433 | 0.4548 | 0.4130 | 0.4527 | 0.4471 | 0.4658 |
| Couple | 0.8660 | 0.8367 | 0.8653 | 0.8610 | **0.8697** | 0.6528 | 0.7034 | 0.7257 | 0.7167 | 0.7274 |
| Cameraman | 0.8642 | 0.8343 | 0.8588 | 0.8580 | **0.8650** | 0.6919 | 0.7245 | 0.7369 | 0.7304 | 0.7330 |
| Gold Hill | 0.8981 | 0.8771 | 0.9004 | 0.8941 | **0.9013** | 0.7131 | 0.7549 | 0.7782 | 0.7660 | 0.7759 |
Figure 4. Image cameraman interpolates with factor 2; (a) nearest neighbor, (b) bilinear, (c) bicubic, (d) Karim and Saaban [9], (e) proposed
Figure 5. Image cameraman interpolation for factor 4; (a) nearest neighbor, (b) bilinear, (c) bicubic, (d) Karim and Saaban [9], (e) proposed
5. Conclusion
An efficient algorithm with a newly constructed rational bi-cubic Ball with six parameters is developed for image upscaling for grayscale image. These six free parameters are manipulated to refine shape of surface and to produce the best image upscaling. The numerical comparison of rational bi-cubic Ball with existing scheme has been done on details and the results show the proposed scheme gives comparable result with some advantages. Based on overall result, it can be seen clearly the proposed scheme capable to improve image quality after rescaling to factor two and four compared to Karim and Saaban [6] and conventional method in MATLAB with less error. Moreover, this paper justified with some critical comparison in term of PSNR, RMSE and FSIM for image interpolation. For further research we propose new techniques determine parameter value automatically by using genetic algorithm (GA). Furthermore, the proposed scheme can be implemented in different application of image interpolation such as image zooming and image rotation.

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References
[1] Charbit, M., & Blanchet, G. 2006. Digital signal and image processing using MATLAB (ISTE).
[2] Majeed, A., Piah, A. R. M., Yahya, Z. R., Abdullah, J. Y., & Rafique, M. (2017). Construction of occipital bone fracture using B-spline curves. Computational and Applied Mathematics, 1-20.
[3] Ball, A. A. 1974. CONSURF. Part one: introduction of the conic lofting tile. Computer-Aided Design, 6(4), 243-249.
[4] Gao, S., Zhang, C., & Zhang, Y. 2009, May. A New Algorithm for Image Resizing Based on Bivariate Rational Interpolation. In International Conference on Computational Science (pp. 770-779). Springer, Berlin, Heidelberg.
[5] Gao, S., Zhang, C., Zhang, Y., & Zhou, Y. 2008, December. Medical image zooming algorithm based on bivariate rational interpolation. In International Symposium on Visual Computing (pp. 672-681). Springer, Berlin, Heidelberg.
[6] Karim, S. A. A., & Saaban, A. (2017). Shape Preserving Interpolation Using Rational Cubic Ball Function and Its Application in Image Interpolation. Mathematical Problems in Engineering, 2017.
[7] Wang, Q., & Tan, J. 2007, October. Multi-focus image fusion algorithm based on rational spline. In 2007 10th IEEE International Conference on Computer-Aided Design and Computer Graphics (pp. 225-229). IEEE.
[8] Yao, X., Zhang, Y., Bao, F., & Zhang, C. 2016. Rational Spline Image Upscaling with Constraint Parameters. Mathematical and Computational Applications, 21(4), 48.
[9] Zhang, C. Q., n.feng Zhang, Y., & Zhang, C. M. 2012. Surface Constraint of a Rational Interpolation and the Application in Medical Image Processing. Research Journal of Applied Sciences, Engineering and Technology, 4(19), 3697-3703.
[10] Zhang, Y., Gao, S., Zhang, C., & Chi, J. 2009. Application of a bivariate rational interpolation in image zooming. Int. J. Innov. Comput. Inform. Cont, 5(11).
[11] Abbas, S., Hussain, M. Z., & Irshad, M. 2016. GA based rational cubic B-spline representation for still image interpolation. Pakistan Journal of Statistics and Operation Research, 12(4), 753-763.
[12] Abbas, S., Hussain, M. Z., & Irshad, M. (2018). Image interpolation by rational ball cubic B-spline representation and genetic algorithm. Alexandria Engineering Journal, 57(2), 931-937.
[13] Karim, S. A. A. (2015). Shape preserving by using rational cubic ball interpolant. *Far East Journal of Mathematical Sciences, 96*(2), 211.