Massive 4d particle with torsion and conformal mechanics

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Abstract

The consequences of coupling of the torsion (highest curvature) term to the Lagrangian of a massive spinless particle in four-dimensional space time are studied. It is shown that the modified system remains spinless and possesses extended gauge invariance. Though the torsion term does not generate spin, it provides the system with a nontrivial mass spectrum, described by one-dimensional conformal mechanics. Under an appropriate choice of characteristic constants the system has solutions with a discrete mass spectrum.
1 Introduction

The search of Lagrangian models, describing spinning particles, has a long story. Most popular approach in this direction is the formulation of the Lagrangian on the space-time, extended by the anticommuting variables, which upon quantization provide the system with the nontrivial spin. This approach is closely related with the supersymmetric field theories and is essential for the formulation of superparticle systems.

There is another, pure bosonic, approach in which the Lagrangian is formulated on the direct product of the initial space-time by some orbit of Poincaré group. The actions of this sort are, in fact, the particle counterparts of Born-Infeld type systems. This approach seems to be interesting due its visible relation with orbit method of Kirillov-Konstant-Souriau [1]. Most developed investigation of such systems has been presented in Refs. [2]. Notice that the bosonic approach is the only correct in \((2 + 1)\)-dimensional systems, due to anyonic nature of planar particle.

The bosonic approach admits an aesthetically attractive modification, where the additional bosonic variables are encoded in the dependence of the Lagrangian on higher-order derivatives. These systems seem to be interesting not only for their clear geometrical meaning. The higher-derivative parts in their Lagrangians can be generated by some field-theoretical mechanism, e.g. arising as quantum corrections. Investigation of such sort of particle systems became popular after remarkable work of Polyakov [3], where he show, that evaluation of the effective action of \(CP^1\) model minimally coupled to the Chern-Simons field for the charged solitonic excitation results in the action

\[
S_{eff} = \int (m + \frac{\pi}{2\theta} K^2)|d\mathbf{x}|, \quad (1.1)
\]

where \(K^2\) is world-line’s torsion and \(\theta\) is field coupling strength.

Later it was found, that this system describes anyonic analog of Majorana field equations [4]. Due to further studies it became a part of physical folklore that relativistic systems can get nontrivial spin, if one adds to the Lagrangians the higher-derivatives terms (more precisely, the terms, depending on reparametrization invariants (extrinsic curvatures) of world-line). Some significant observations were done in connection with this subject, particularly, in the description of three- and four-dimensional particle with Majorana spectrum [5], \(4d\) massless particles [6]. The relation of the mentioned massless particle model with \(W^-\) algebras has also been established [7].

However, such systems were not studied completely even in the four-dimensional space-time, where only the first and second extrinsic curvatures were considered. In this note we attempt to fill this gap and consider the simplest four-dimensional analog of (1.1) given by the action [8]

\[
S = \int (c_0 + cK^3)ds, \quad ds = |d\mathbf{x}| \equiv s\tau \neq 0
\]

where \(\tau\) is an arbitrary evolution parameter and \(K^3\) is the torsion (highest curvature) of a worldline in 4d space.

We will show that this system possesses interesting properties which make it drastically different from other four- and three-dimensional massive particle systems depending on extrinsic curvatures:

- it has a zero spin;
- it possesses, in addition to reparametrization invariance, the extra gauge degrees of freedom: its classical trajectories are restricted by the condition

\[
\frac{K^2_2 K^3_4}{K^2_1} = |\alpha|, \quad \alpha = c_0/c
\]

while the mass spectrum is described the conformal mechanics with the energy \(\mathcal{E}\),

\[
dq \wedge dp, \quad \mathcal{H} = \frac{q^2}{2} \pm \frac{\alpha^2}{2q^2}, \quad -2\mathcal{E}/\alpha^2 = \left\{ \begin{array}{ll}
M^2/c_0^2 \pm 1, & \text{if } \alpha < 0, \\
\pm M^2/c_0^2 \pm 1, & \text{if } \alpha > 0
\end{array} \right.,
\]

\(M\) is the effective mass and \(c_0\) is the effective speed of light.
where $q = K_1/K_2$, and $M$ denotes the mass of the system.
When $\alpha < 0$, the upper sign corresponds to the time-like trajectories, while other solutions correspond to the space-like ones.

- When $\alpha < -1/2$, the solutions with space-like trajectories possess a discrete spectrum with massive and tachionic sectors, while the solutions with time-like trajectories possess continuous spectrum containing massive, massless, and tachionic solutions.

The paper is arranged as follows:

In Section 2 we construct the Hamiltonian system corresponding to the model under consideration \cite{1}. We show that the system possesses an extended gauge invariance and a zero spin. The geometry of its classical trajectories and quantum spectrum (in the Euclidean space) are defined by the one-dimensional conformal mechanics (with a repulsive potential).

In Section 3 we reformulate the Hamiltonian system constructed in Section 2 in the Minkowski space and consider the properties of its mass spectrum.

\section{Hamiltonian formulation}

In this section we give the Hamiltonian formulation of the model \cite{1}. Recall that the extrinsic curvatures $K_a$ of a (non-null) curve in four-dimensional space can be defined via the Frenet equations for the moving frame $\mathbf{e}_a$ :

\begin{equation}
\dot{x} = s \mathbf{e}_1, \quad \dot{\mathbf{e}}_a = k_a \mathbf{e}_b, \quad \mathbf{e}_a \mathbf{e}_b = \eta_{ab}, \quad a, b, c = 1, 2, 3, 4; \tag{2.1}
\end{equation}

\begin{equation}
k_a \eta_{cb} = \begin{pmatrix}
0 & k_1 & 0 & 0 \\
-k_1 & 0 & k_2 & 0 \\
0 & -k_2 & 0 & k_3 \\
0 & 0 & -k_3 & 0
\end{pmatrix}, \quad k_{a-1} = sK_{a-1}, \tag{2.2}
\end{equation}

where $x$ are coordinates of four-dimensional space, and $\mathbf{e}_a$ are elements of the moving frame.

While $K_1, K_2$ are positive quantities, the sign of the highest curvature $K_3$ (torsion) is not uniquely determined by the Frenet equations \cite{2}. Without loss of generality we assume below that $K_3 > 0$.

In the Euclidean space $\eta_{ab} = \delta_{ab}$, so the Frenet equations read

\begin{equation}
\dot{\mathbf{e}}_a = k_a \mathbf{e}_{a+1} - k_{a-1} \mathbf{e}_{a-1}, \quad \mathbf{e}_0 = \mathbf{e}_5 \equiv 0. \tag{2.3}
\end{equation}

One can transform the Frenet equations in the Euclidean space into those in the Minkowski space, performing the following transition

\begin{equation}
(\mathbf{e}_\underline{2}, k\underline{2}, k\underline{3}, s) \rightarrow (i\mathbf{e}_\underline{2}, ik\underline{2}, ik\underline{3}, (-i)^\delta s) \tag{2.4}
\end{equation}

for some index $\underline{2}$.

Indeed, this transformation preserves the form of the matrix in \cite{2}, while the element $\mathbf{e}_\underline{2}$ becomes time-like: $\mathbf{e}_\underline{2}^2 = -1$. So, we can give our basic derivations for the Euclidean case, reformulating them for the Minkowski space for a final analysis.

It follows from \cite{2} that

\begin{equation}
s = \sqrt{-x^2}, \quad k_a = \dot{\mathbf{e}}_a \mathbf{e}_{a+1} = \sqrt{\mathbf{e}_a^2 - k_{a-1}^2}. \tag{2.5}
\end{equation}

Thus, the Lagrangian appearing in the action \cite{1} in the Euclidean space can be replaced by the following one

\begin{equation}
L = c_0 s + c \sqrt{\mathbf{e}_3^2 - k_2^2} + \mathbf{p}(x - s \mathbf{e}_1) + \sum_i \mathbf{p}_{i-1} (\dot{\mathbf{e}}_{i-1} - k_{i-1} \mathbf{e}_i + k_{i-2} \mathbf{e}_{i-2}) - \sum_{i,j} d_{ij} (\mathbf{e}_i \mathbf{e}_j - \delta_{ij}), \tag{2.6}
\end{equation}

where $x, \mathbf{p}, \mathbf{e}_1, \mathbf{p}_{i-1}, s, k_i, d_{ij}$ are independent variables, $i, j = 1, 2, 3$. 

3
Now we can perform the Legendre transformation for this Lagrangian (referring for details to [10]). The variables \( p_{i-1} \) represent the momenta conjugated to \( e_{i-1} \), whereas the momenta conjugated to \((s, k_{i-1}, d_{ij})\) lead to the trivial constraints

\[
p^s \approx 0, \quad p^{i-1} \approx 0, \quad p^{ij} \approx 0. \tag{2.7}
\]

Setting \( k_3 \neq 0 \) we find that the momentum conjugated to \( e_3 \) is of the form

\[
p_3 = c(\dot{e}_3 - k_2^2)^{-1/2} \dot{e}_3. \tag{2.8}
\]

Taking into account (2.5), we get the constraints

\[
p_3 e_3 \approx 0, \quad p_3 e_1 \approx 0, \quad p_3^2 - (p_3 e_2)^2 - c^2 \approx 0. \tag{2.9}
\]

Then the construction of primary Hamiltonian system becomes straightforward.

To simplify the resulting system, one can stabilize trivial primary constraints (2.7) and exclude them from our considerations, which makes the variables \( s, k_{i-1}, d_{ij} \) lagrangian multipliers. Without loss of generality one can also impose the gauge conditions (see for a details [10])

\[
p_3 e_2 \approx 0, \quad p_2 e_2 \approx 0, \quad p_2 e_1 \approx 0.
\]

After these manipulations we get the Hamiltonian system

\[
\omega = dp \wedge dx + \sum_i dp_i \wedge de_i,
\]

\[
\mathcal{H} = s(pe_1 - c_0) + \sum_i k_{i-1} \phi_{i-1,i} + \frac{k_2}{c_0}(p_3^2 - c^2) + \sum_{i,j} d_{ij}(e_i e_j - \delta_{ij}),
\]

with primary constraints

\[
p_i e_j \approx 0, \quad i \geq j \quad \tag{2.11}
\]

\[
e_i e_j - \delta_{ij} \approx 0 \quad \tag{2.12}
\]

\[
p_3 e_1 \approx 0 \quad \tag{2.13}
\]

\[
\phi_{i-1,i} \equiv p_{i-1} e_i - p_i e_{i-1} \approx 0, \quad p_3^2 - c^2 \approx 0, \tag{2.14}
\]

where variables \( s, k_i, d_{ij} \) play the role of Lagrangian multipliers.

Let us stabilize the primary constraints (2.11)-(2.14). Stabilization of (2.11) gives the following fixation of the Lagrangian multipliers \( d_{ij} \):

\[
2d_{i,j} = k_3 c_{0,i}\delta_{i,j} - s c_0 \delta_{1,i}\delta_{1,j},
\]

so that the equations of motion read

\[
\dot{x} = e_1,
\]

\[
\dot{e}_1 = k_1 e_2,
\]

\[
\dot{e}_2 = -k_1 e_1 + k_2 e_3,
\]

\[
\dot{e}_3 = -k_2 e_2 + k_3 p_3 / c
\]

\[
p_3 = -k_3 c e_3 - k_2 p_2,
\]

\[
p_2 = -k_1 p_1 + k_2 p_3,
\]

\[
p_1 = -s p + k_1 p_2 + s c_0 e_1,
\]

\[
p = 0.
\]

Stabilizing the remaining primary constraints, we get the following first-stage secondary constraints

\[
p e_2 \approx 0, \quad p_1 e_3 \approx 0, \quad p_3 p_2 \approx 0. \tag{2.17}
\]
From the orthogonality of \((e_i, p_3/c)\) to \(p_2\), which follows from (2.14), (2.11), (2.17) we conclude that
\[
p_2 \approx 0. \tag{2.18}
\]
Stabilizing the first and second constraints from (2.17) and the constraint (2.18), we get
\[
k_2 = q k_1, \quad k_3 = s c_0 / c q^2, \quad p_1 = q p_3, \tag{2.19}
\]
where
\[
1/q \equiv p e_3 / c_0 \neq 0. \tag{2.20}
\]
Then we get
\[
p = c_0 e_1 + c_0 e_3 / q + p p_3, \tag{2.22}
\]
where
\[
p \equiv p p_3 / c^2. \tag{2.23}
\]
Consistency of the equations of motion (2.16) with the Frenet equations implies that
\[
e_a = \{e_i, e_4 \equiv p_3 / c\} : \quad e_a e_b = \delta_{ab}. \tag{2.24}
\]
Therefore, the initial Hamiltonian system can be formulated purely in terms of the moving frame \(e_a\), coordinates \(x\), and momentum \(p\), while the relations (2.24), (2.22) play the role of constraints.

Taking into account the constraint (2.22), we get the expression for the rotation generators:
\[
J = p \vee x + \sum_i p_i \vee e_i = p \vee (x - q p_3 / c_0) \tag{2.25}
\]
and the Casimirs
\[
p^2 = (c p)^2 + c_0^2 (1 + 1/ q^2), \quad W^2 = (p \vee J)^2 = 0. \tag{2.26}
\]
Thus, the system possesses a zero spin despite the dependence of the initial Lagrangian on higher derivatives.

The expression for rotation generators (2.25) hints us to introduce the “effective” coordinate
\[
X \equiv x - q p_3 / c_0 : \quad \dot{X} = s p / c_0, \quad p = \text{const} \tag{2.27}
\]
whose evolution equations look similar to the convenient relation between velocity and momentum of a massive particle (note, that in spite of these relations the mass of the system is not equal to \(c_0\)).

The reduction of the initial Hamiltonian system (2.10) by the constraints (2.24), (2.22), (2.20) leads to the following unconstrained system
\[
\omega^{\text{red}} = dp \wedge dX + c_0^2 dp \wedge dq, \quad \mathcal{H}^{\text{red}} = \frac{c_0^2}{c_0} \left( \frac{c_0^2}{2 c q^2} - \frac{p^2 - c_0^2}{2 c^2} \right). \tag{2.28}
\]
The external curvatures of the system are related with the modular “coordinate” \(q\) as follows
\[
\frac{K_2}{K_1} = q, \quad K_3 = \frac{c_0}{q c^0}, \quad \Rightarrow \quad \frac{K_2^2 K_3}{K_1^2} = \frac{c_0}{c}. \tag{2.29}
\]
So, the system under consideration possesses extended gauge invariance since the trajectories with the same ratio \(K_1 / K_2\) are gauge equivalent.

Reducing the system (2.28) by \(p\), to exclude the trivial part of dynamics (2.27), we get that the evolution of the parameter \(q\) is described by the textbook mechanical system
\[
dp \wedge dq, \quad \mathcal{H}_0 = p^2 / 2 + \frac{c_0^2}{2 c q^2}, \quad \mathcal{E} = \frac{p^2 - c_0^2}{2 c^2}. \tag{2.30}
\]
This system can be immediately integrated at the classical level

\[ q^2 = \frac{(p^2 - c_0^2)s^2}{2c^2} + \frac{2}{p^2 - c_0^2}, \quad p^2 > c_0^2, \]

as well as at the quantum one.

Thus, adding of the torsion term increases the absolute value of momenta of the system with respect to the torsionless system, \( p^2 \geq c_0^2 \). The system (2.30) is known in literature as a conformal mechanics, due to the symmetry of its action under conformal transformations generated by

\[ \mathcal{H}_0 = \frac{p^2}{2} + \frac{c^2}{2c_0^2 q^2}, \quad \mathcal{D} = pq, \quad \mathcal{K} = q^2 / 2. \] (2.31)

The “energy” spectrum is continuous and has the lowest bound \( E = 0 \) which is not normalizable. In [12] the conformal symmetry of this system has been used for solving the problem of the ground state.

Recently this mechanism has been found to be adequate to the problem of motion of charged particle in the field of a charged black hole [13]. Due to this observation the interest to the conformal mechanics and its supergeneralizations [14], is renewed in the context of study of probe D0-brane dynamics in the external D-brane field (see e.g. [15] and refs therein). Conformal mechanics arises in our model a different context: it defines local gauge symmetry of the particle system.

### 3 Transition to Minkowski space

In the previous Section we constructed the constrained Hamiltonian system corresponding to the action (1.2) in the Euclidean space. We found that the evolution and the geometry of this system are described in terms of the non-relativistic conformal mechanics with repulsive potential. The purpose of this Section is to consider the relativistic aspects of this system, i.e. to investigate the model (1.2) in the Minkowski space. As we have mentioned in the beginning of Section 2, we can use the results concerning the Euclidean case, since the Frenet formulae in the Euclidean space can be transformed into the ones in Minkowski space with the use of transition (2.4), where \( a \) denotes an element of moving frame which becomes time-like, i.e. \( e_a^2 = -1 \). Correspondingly, for the transition to time-like trajectories we have to choose \( a = 1 \), while for the transition to space-like trajectories we have to choose \( a \neq 1 \). We do not consider here systems with light-like trajectories.

For convenience we will use the notation

\[ p^2 \equiv -M^2, \quad \alpha = \frac{c_0}{c}, \] (3.1)

so that \( M^2 >, =, < 0 \) corresponds to the massive, massless, and tachionic sectors of the system, respectively.

To reformulate the system under consideration in the Minkowski space we have to supply the transition (2.4) with appropriate transformations of characteristic constants \( c_0, c \) and momenta \( p_i \) which preserve the form of the initial action (1.2) and the initial Hamiltonian system, namely,

\[ \begin{align*}
    a = 1 : & \quad (e_1, p_1, k_1, s, c_0) \rightarrow (ie_1, -ip_1, ik_1, -is, ic_0); \\
    a = 2 : & \quad (e_2, p_2, k_2, k_1) \rightarrow (ie_2, -ip_2, ik_2, ik_1), \\
    a = 3 : & \quad (e_3, p_3, k_3, k_2, c) \rightarrow (ie_3, -ip_3, ik_3, ik_2, -ic), \\
    a = 4 : & \quad (c, k_3) \rightarrow (-ic, ik_3). 
\end{align*} \] (3.2)

Indeed, it is easy to see that it is only the orthonormality condition \( e_a e_b = (-1)^a \delta_{ab} \) that is changes in (2.10) under this transformation. Consequently, the reduced system (2.31) with appropriately changed parameters describes the effective Hamiltonian system corresponding to the action (1.2) in the Minkowski space.
The transition (3.2) induces the following transformation of the parameters and coordinates of the reduced system (2.30)

\[
(p, q, \alpha, c_0, s) \rightarrow \begin{cases} 
(p, iq, i\alpha, ic_0, -is) & \text{if } \underline{a} = 1 \\
(p, q, \alpha, c_0, s) & \text{if } \underline{a} = 2 \\
(ip, -iq, i\alpha, c_0, s) & \text{if } \underline{a} = 3 \\
(-p, q, i\alpha, c_0, s) & \text{if } \underline{a} = 4.
\end{cases}
\] (3.3)

Thus, the reduced system reads

\[
dp \wedge dq, \quad \epsilon_a s \left( \frac{p^2}{2} + \frac{\epsilon_a \alpha^2}{2q^2} - \mathcal{E}_a \right) \approx 0,
\] (3.4)

where

\[
\mathcal{E}_a = -(-1)^{3a} \frac{\alpha^2}{2} (M^2/c_0^2 + (-1)^a \alpha), \quad \epsilon_a = \begin{cases} 
1, & \text{if } \underline{a} = 1, 2 \\
-1, & \text{if } \underline{a} = 3, 4.
\end{cases}
\]

The expressions for curvatures (2.29) read

\[
\frac{K_2}{K_1} = (-1)^a q, \quad K_3 = (-1)^2 \frac{\alpha}{q^2}, \quad \Rightarrow \quad \frac{K_2^2 K_3}{K_1^2} = (-1)^a \alpha.
\] (3.5)

Due to positivity of curvatures $K_i$, we conclude that

- $a = 1, 3$, $q < 0$, if $\alpha < 0$,
- $a = 2, 4$, $0 < q$, if $\alpha > 0$.

Thus, in both cases the system possesses the solutions described by conformal mechanics with a repulsive ($a = 1, 2$) and an attractive ($a = 3, 4$) potentials. The classical solutions of these systems are of the form

\[
q^2 = \begin{cases} 
\mathcal{E}_a \tilde{s}^2 + (-1)^{|a|} \alpha^2/\mathcal{E}_a & \text{if } \mathcal{E} \neq 0, \\
2\alpha \tilde{s}, & \text{if } \mathcal{E} = 0, \; a = 3, 4
\end{cases}
\]

When the potential is attractive ($a = 3, 4$), the parameter $\tilde{s}$ is defined on the domain

\[
\tilde{s} \in \begin{cases} 
[\alpha/|\mathcal{E}|, \alpha/|\mathcal{E}|], & \text{if } \mathcal{E} < 0, \\
|\alpha/|\mathcal{E}|, \infty[, & \text{if } \mathcal{E} > 0.
\end{cases}
\]

While the quantum spectrum of the mechanics with a repulsive potential is continuous, the spectrum of conformal mechanics with an attractive potential can be both continuous and discrete. The discrete spectrum corresponds to the strongly attractive potential ($|\alpha| > 1/2$) and has an infinite number of energy levels [16].

We first consider the systems with repulsive potential

\[
dp \wedge dq, \quad \mathcal{H}_0 = \frac{p^2}{2} + \frac{\alpha^2}{2q^2}, \quad \frac{2\mathcal{E}}{\alpha^2} = \begin{cases} 
(1 - M^2/c_0^2), & \text{if } \alpha < 0, \; \underline{a} = 0 \\
-(1 + M^2/c_0^2), & \text{if } \alpha > 0, \; \underline{a} = 1
\end{cases}
\] (3.6)

Notice that while in the first case trajectories are time-like, those are space-like in the second case.

Since the energy $\mathcal{E}$ is positive both in classical and quantum cases, we get the restrictions on the admissible value of the mass

- $M^2 < c_0^2$, for $\alpha < 0$ and time-like trajectories ($\underline{a} = 1$),
- $M^2 < -c_0^2$, for $\alpha > 0$, $\underline{a} = 2$. 

Thus, in the first case there are massive, massless, and tachionic states, while in the second case the system is tachionic.

Now we consider the systems with an attractive potential. In this case all trajectories are space-like and are defined by the mechanics
\[ dp \wedge dq, \quad H_0 = \frac{p^2}{2} - \frac{\alpha^2}{2q^2}, \quad \frac{2\mathcal{E}}{\alpha^2} = \begin{cases} -\left(1 + M^2/c_0^2\right) & \text{if } \alpha < 0, \quad q = 3 \\ \left(1 + M^2/c_0^2\right) & \text{if } \alpha > 0, \quad q = 4 \end{cases} \tag{3.7} \]
The spectrum of conformal mechanics with an attractive potential is continuous for \( \mathcal{E} > 0 \) and for \( \mathcal{E} \leq 0, \alpha^2 < 1/4 \) (see [11]).

If the potential is "strongly attractive" \( \alpha^2 > 1/4 \), and \( \mathcal{E} < 0 \), the system has a discrete spectrum with an infinite number of bound states given by the expression [16]
\[ \mathcal{E}_n = -\hbar^2 B^2 \exp\left(-\frac{2\pi n}{\sqrt{\alpha^2 - 1/4}}\right), \quad n = 0, \pm 1, \pm 2, \ldots, \]
where \( B \) is an undefined "phase" factor.
So,
\[ M_n^2/c_0^2 = -1 + (-1)^{sgn\alpha} \left(\frac{\hbar B}{\alpha}\right)^2 \exp\left(-\frac{2\pi n}{\sqrt{\alpha^2 - 1/4}}\right), \quad |\alpha| > 1/2. \tag{3.8} \]
Therefore, for \( \alpha > 0 \) the discrete branch corresponds to pure tachionic states, while for \( \alpha < 0 \) it contains massive, massless, and tachionic sectors.

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