Measurement schemes for the spin quadratures on an ensemble of atoms

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We consider how to measure collective spin states of an atomic ensemble based on the recent multi-pass approaches for quantum interface between light and atoms. We find that a scheme with two passages of a light pulse through the atomic ensemble is efficient to implement the homodyne tomography of the spin state. Thereby, we propose to utilize optical pulses as a phase-shifter that rotates the quadrature of the spins. This method substantially simplifies the geometry of experimental schemes.

To establish a method for determining the state of a quantum system is an important step for progress in modern physics. For light fields on a few well-defined modes, the quantum-state tomography to determine the density matrix of the system has been developed mainly based on the quadrature measurement implemented by a balanced homodyne detection [1, 2]. The quadrature measurement is a standard tool in the approaches for continuous variable quantum information [3]. However, outside of the optical systems it seems difficult to tell how to implement such a measurement.

It has been known that a continuous variable description of the system can be applied on the collective-spin state of an ensemble of massive particles, and there exist various approaches for implementing continuous variable quantum information processing with ensembles of atoms [4, 5]. In particular, optical accessibility and coherent properties of the atomic ensembles are thought to be useful for implementing quantum memory of light and constructing a quantum interface to transfer the quantum state between the light and atoms [5, 6, 7, 8, 9, 10, 11].

In order to construct a quantum interface, one of the central atom-light interaction is the Faraday-Rotation (FR) interaction [6, 12]. On one hand, this interaction provides a two-mode-coupling gate to essentially implement any quadratic Hamiltonian interaction [12, 13]. While the design of the interaction Hamiltonian and achievable fidelities in the gate operations of interface are widely investigated as well as generation schemes of quantum entanglement [4, 15], how to estimate such a gate operations experimentally is less concerned [11, 16, 17, 18]. One of the reasons for this might be the lack of an established measurement scheme to probe the spin state such as the homodyne measurement in optics. The measurement of the spin state and state reconstruction to visualize its quantum nature in the form of the quasi-probability distribution are interesting objective in its own right [19]. On the other hand, the FR interaction corresponds to a unitary operation used in a classical model for an indirect measurements [20], and recently several models of quantum measurements have been proposed to demonstrate the relation between an indirect measurement and its back action [21, 22, 23].

Until recently the construction of the quantum interface based on the multiple use of the FR interaction [3, 10, 11] has been thought to be impractical due to the requirement of a long delayline (DL) to store the pulse until an interaction is completed. However, an experimental demonstration of the FR interaction using a short pulse light of a few hundred-nano-second width and an ensemble of Laser-cooled atoms has been reported [24], and thus the required length of the DL is thought to be a feasible size. In this report we consider measurement schemes for the tomographic reconstruction of collective spin states and make a link to the measurement theory.

Let us write the Stokes operator of a pulsed light propagating along with z axis

\[
S_x = \frac{\hat{a}_H^\dagger \hat{a}_H - \hat{a}_V^\dagger \hat{a}_V}{2},
\]

\[
S_y = \frac{\hat{a}_D^\dagger \hat{a}_D - \hat{a}_D^\dagger \hat{a}_D}{2},
\]

\[
S_z = \frac{\hat{a}_H^\dagger \hat{a}_H - \hat{a}_V^\dagger \hat{a}_V}{2},
\]

(1)

where \(\hat{a}_{H(V)}\) is the annihilation operator of the pulse with horizontally (vertically) polarization, which is related to the operators for the diagonally polarization \(\hat{a}_D = (\hat{a}_H + \hat{a}_V)/\sqrt{2}\), \(\hat{a}_{D^\dagger} = (\hat{a}_H + \hat{a}_V)/\sqrt{2}\), and the circularly polarization \(\hat{a}_{\pm} = (\hat{a}_H \mp i\hat{a}_V)/\sqrt{2}\). The Stokes operators satisfy the angular momentum commutation relation, \([S_x, S_y] = iS_z\). If the horizontally polarized light is considered to be a strong local oscillator (LO) field with almost fixed average photon number \(n_H = \langle \hat{a}_H^\dagger \hat{a}_H \rangle \gg 1\) and phase \(\phi\), i.e. \(\hat{a}_H \approx \sqrt{n_H}e^{-i\phi}\), we can write the difference of the diagonally polarized photon numbers as

\[
S_y \approx \frac{\sqrt{n_H}}{2}(\hat{a}_V e^{-i\phi} + \hat{a}_V^\dagger e^{i\phi})
\]

\[
= \sqrt{\frac{n_H}{2}}(\hat{x}_V \cos \phi - \hat{p}_V \sin \phi),
\]

(2)

where we defined the quadrature operators of the vertical mode, \(\hat{x}_V \equiv (\hat{a}_V + \hat{a}_V^\dagger)/\sqrt{2}\) and \(\hat{p}_V \equiv (\hat{a}_V - \hat{a}_V^\dagger)/\sqrt{2}\). Similarly, the difference of the circularly polarized photon numbers becomes

\[
S_z = -i(\hat{a}_V \hat{a}_H - \hat{a}_H^\dagger \hat{a}_V)/2
\]

\[
\approx -\sqrt{\frac{n_H}{2}}(\hat{x}_V \sin \phi + \hat{p}_V \cos \phi).
\]

(3)
From Eqs. (2) and (3), one can see that the normalized Stokes operators $s_y = S_z/\sqrt{n_H/2}$, $s_z = S_y/\sqrt{n_H/2}$ play the role of the canonical continuous variables, $[s_y, s_z] = i$. The measurement of the polarization difference gives the statistics of the variable, and it acts as the quadrature measurement of quantum optical mode [see FIG. 1 (a)].

In the optical homodyne tomography, the phase difference between the signal and LO fields is given by the amount of $\phi$ modulated by the noise of $\delta \in [0, \pi]$. However, it might be difficult to separate the two polarization components and/or apply the phase shift on each of the polarization modes with sufficient resolution individually when the intensities of the two polarizations are highly different so that $n = (\hat{a}_v^\dagger \hat{a}_v) \ll n_H$. For such a collinear propagation of the light, the double homodyne measurement might be effective. In this measurement the $S_z$ measurement and $S_y$ measurement are performed on the pulses equally split by the non-polarized beamsplitter as in FIG. 1 (a). It is known that the double homodyne measurement gives the Husimi-Q function and is tomographic complete [2]. In order to see this, let us write the density operator of the vertical light with the P representation, $\hat{\rho} = \int d^2 \alpha P(\alpha) |\alpha\rangle \langle \alpha|$ where $|\alpha\rangle$ represents the coherent state. The $x$-quadrature and $p$-quadrature distributions of $|\alpha\rangle$ are given by $\langle x |\alpha\rangle^2 = \exp\{-x - \sqrt{2}\Re[\alpha]^2\}/\sqrt{\pi}$ and $\langle p |\alpha\rangle^2 = \exp\{-p - \sqrt{2}\Im[\alpha]^2\}/\sqrt{\pi}$, respectively. Due to the beamsplitter transformation $|\alpha\rangle |0\rangle_{\text{vac}} \rightarrow |\alpha/\sqrt{2}\rangle |\alpha/\sqrt{2}\rangle_{\text{vac}}$, the double homodyne measurement that measures $x$ quadrature and $p$ quadrature on each of the modes yields the probability distribution $Q(x, p) \equiv \int d^2 \alpha P(\alpha) \langle x |\alpha/\sqrt{2}\rangle^2 |p |\alpha/\sqrt{2}\rangle_{\text{vac}}^2 = \int d^2 \alpha P(\alpha) \exp \{-|x + ip - \alpha|^2\}/\pi$. This convolution of the P function corresponds to the Husimi-Q function since $\langle \alpha |\rho |\alpha\rangle/\pi = \langle \alpha | \int \alpha' P(\alpha') |\alpha'/\sqrt{2}\rangle^2 |\alpha/\sqrt{2}\rangle^2_{\text{vac}} |\alpha' |\alpha\rangle/\pi = Q(\Re[\alpha], \Im[\alpha])$. Therefore, one can directly reconstruct the quantum state in the form of the Husimi-Q function.

Let us write the collective spin operator composed of an ensemble of spin-one-half particles defined by the population difference between the spin-up state $|\uparrow\rangle$ and spin-down state $|\downarrow\rangle$ quantized in $x$ direction as

$$
\begin{align*}
J_x &= \frac{1}{2} (|\uparrow\rangle \langle \uparrow| - |\downarrow\rangle \langle \downarrow|)/2,
J_y &= \frac{1}{2} (|\downarrow\rangle \langle \uparrow| + |\uparrow\rangle \langle \downarrow|)/2,
J_z &= i(|\downarrow\rangle \langle \downarrow| - |\uparrow\rangle \langle \uparrow|)/2.
\end{align*}
$$

We assume the number of the particles $J = 2 \langle J_y^2 + J_z^2 \rangle \gg 1$ and the spins are almost polarized in $x$ direction, e.g. $(J_z) \sim \frac{1}{2}$. Noting that the angular-momentum commutation relation holds $[J_y, J_z] = iJ_x$ the normalized spin operators, $j_y = J_y/\sqrt{J}J$ and $j_z = J_z/\sqrt{J}$, act as the canonical continuous variable, $[j_y, j_z] = i$, similar to the case of the Stokes operator. We refer to $j_y$ and $j_z$ as the spin quadratures.

The action of the FR interaction due to the passage of the light through the ensemble is described by the unitary operator

$$
U(\alpha t) = \exp(-i\alpha t S_z J_z),
$$

where $\alpha$ is a coupling constant and $t$ is an interaction time. This FR interaction rotates both of the light and spin quadratures about the light propagating axis ($z$ axis). For the quantum interface, the parameters are selected to attain $\kappa \equiv \alpha t \sqrt{SJ} = 1$. With the normalized operators we concerns the elementary gate interaction

$$
U_z = \exp(-i\kappa J_z).
$$

In the recent proposal [11] (see also [13, 14]), a swapping gate that exchanges $(s_y, s_z)$ and $(j_y, j_z)$ can be implemented by the three passages of the light through the
atomic-spin ensemble as in FIG. 1 (a). Obviously, if
the swapping is performed, the measurement of the light
quadrature implies the measurement of the quadrature
of the initial spin state. Hence, after the swapping op-
eration, one can apply the quantum optical measurement
on the light in order to determine the density operator of
the initial spin state. Then an application of the double
homodyne measurement as in FIG. (a) enables to recon-
struct the spin state in the form of the Husimi–Q function
directly.

This measurement might be preferable to show the
better-than-classical performance of the transfer/storage
gate with the input of the coherent states [11, 17, 18].
The measured probability distribution \( p(\alpha|\rho_m) |\alpha\rangle \)
enables us to calculate the average fidelity \( F(E, \eta, \lambda) \equiv \int d^2 \alpha p_\lambda(\alpha) |\sqrt{\eta} |\langle \alpha|\rho_m|\alpha\rangle |\sqrt{\eta}\rangle \), where \( E \) represents
the gate action and \( p_\lambda(\alpha) = \frac{1}{\lambda} \exp(-\frac{1}{\lambda} |\alpha|^2) \) is a Gaussian
probability distribution. If one can find a pair of posi-
tive numbers \( (\lambda, \eta) \) s.t., \( F(E, \eta, \lambda) > F_c(\eta, \lambda) = \frac{\eta + \lambda}{1 + \eta + \lambda} \)
then the gate is capable of transmitting quantum correla-
tion since the gate cannot be simulated by any measure-
ment-and-prepare scheme. While \( \lambda \) is introduced to test the
better-than-classical performance within a finite input
state distribution, \( \eta \) is introduced to take into account the
non-unit–gain (non-unitary) effect of the experiments.
It is shown [18] that the independent measurements of the quadratures, \( S_y \) and \( S_z \), are also useful to check the
criterion since a lower bound of the fidelity to a
coherent state can be estimated from the expectation
values and variances of the two quadratures. In this
case, we may calculate the average quadrature mean-
square deviation \( \delta(E, \eta, \lambda) = \int p_\lambda(\alpha) \text{Tr} \{ E |\alpha\rangle \langle \alpha| \} |(s_y - \sqrt{\eta} \text{Re}[\alpha e^{i\phi}]^2 + (s_z - \sqrt{\eta} \text{Im}[\alpha e^{-i\phi}]^2 - 1)\} d^2 \alpha \). The
inequality \( \delta(E, \eta, \lambda) < \frac{1}{1 + \eta + \lambda} \) is a sufficient condition for
the non-classical performance of the gate.

As we have mentioned, the optical homodyne tomog-
raphy for collinearly propagating light may not be feasible
due to the difficulty of the phase rotation. For the tomog-
raphy of the spin state this is not matter if one can apply
the phase rotation on the spin state before the swapping
operation. For the implementation of the spin rotation,
we propose to apply the FR interaction of Eq. (5) with
a circularly polarized pulse in the strongly polarized di-
rection \( (x \text{ direction}) \). We refers to this phase rotation
as the fictitious-magnetic-field phase-shifter (FMPS)
because its action is the same as the Larmor precession by
a static magnetic field [26, 27]. Due to the optical means,
the FMPS has a good local addressability and its action
can be quicker than the magnetic-resonance rotation in-
duced by the RF field. Note that the rotation angle can
be a sizable amount whereas the rotation angle due to
the interaction of Eq. (6) is considered to be small.

This FMPS enables us to substantially change the ge-
ometry of the experimental setup and will be helpful to
manage a limited optical access of the atomic-spin en-
sembles possibly in a vacuum chamber. Figure 1(b) illus-
trates a setup of swapping gate with the FMPS. In
this scheme, the second passage of the light in \( y \text{ direction}
in FIG. 1(a) can be replaced with the counter propagation
along with \( z \text{ axis} \) provided that the spins are \( \pi/2 \)
rotated about \( x \text{ axis} \) after the first passage. Then addi-
tional \( -\pi/2 \) rotation and further counter propagation
complete the swapping gate. The counter propagation
scheme is favorable to earn the mode coupling as well as
to manage the limited optical access. The single quan-
trature measurement of any spin-quadrature angle can be
executed by initially applying the phase rotation \( \theta \) onto
the spins via the FMPS as schematically shown in FIG.
1(c). This provides a possible scheme for the homodyne
tomography of the spin state.

![FIG. 2: (a) The two passages of the light pulse through the atomic ensemble exchange a single quadrature of the light with a single quadrature of spin. The swapped single quadrature is measured by a homodyne detector. This measurement with various phase rotation \( \theta \) enables to perform the homodyne tomography of the spin state. (b) Folding setup for the homodyne tomography of the spin state with the help of \( \pi/2 \) rotation.](image-url)
spectively. The partial swapping transforms this function as \( W_J(j_y, j_z) W_S(s_y, s_z) \rightarrow W'' = W_J(j_{y''} + s_{y''}, -s_{y''}) W_S(s_{y''} + j_{z''}, -j_{z''}). \) Then the measurement statistics of \( z \)-component of the light quadrature is given by \( \int W'' d s_{y''} d j_{y''} d j_{z''} = \int W_J(s_{y''}, -s_{y''}) d s_{y''}. \) This corresponds to the quadrature distribution of \( j_{y''}. \) Hence, the partial swapping together with the phase shifter enables us to perform the homodyne tomography to reconstruct the spin state. The two-pass interaction is referred to as the noiseless quadrature transducer \( \hat{W}_J \) associated with the contractive state measurement \( [14, 21, 22]. \) If we fold this two-pass scheme exploiting the FMPS then we can prepare a well-squeezed light pulse, a better measurement is known that a better state transfer can be performed if the spin state with one DL and two interactions with a phase shifter enables us to perform the homodyne tomography and presented possible modifications for the measurement schemes. We are hoping that those measurement schemes naturally form a part of the multi-pass approach for quantum interface and provide an established tool for the homodyne tomography of the spin state as well as for the test of the quantum measurement theory.

In conclusion, we have considered how to measure the quantum state of atomic collective spins based on the multi-pass approach for quantum interface and quantum optical homodyne measurements. Thereby, we have proposed to utilize the FR interaction as a phase shifter for the homodyne tomography, and presented possible modifications for the measurement schemes. We are hoping that those measurement schemes naturally form a part of the multi-pass approach for quantum interface and provide an established tool for the homodyne tomography of the spin state as well as for the test of the quantum measurement theory.

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