Minimal Energy Cost to Initialize a Quantum Bit with Tolerable Error

Yu-Han Ma,1 Jin-Fu Chen,1,2,3 C. P. Sun,1,2,∗ and Hui Dong1, †

1Graduate School of China Academy of Engineering Physics, No. 10 Xibeiwang East Road, Haidian District, Beijing, 100193, China
2Beijing Computational Science Research Center, Beijing 100193, China
3School of Physics, Peking University, Beijing, 100871, China
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Landauer’s principle imposes a fundamental limit on the energy cost to perfectly initialize a classical bit, which is only reached under the ideal operation with infinite-long time. The question on the cost in the practical operation for a quantum bit (qubit) has been posted under the constraint by the finiteness of operation time. We discover a raise-up of energy cost by \( L(\epsilon) \) for a finite-time \( \tau \) initialization with an error probability \( \epsilon \). The thermodynamic length \( L(\epsilon) \) between the states before and after initializing in the parametric space increases monotonously as the error decreases. For example, in the constant dissipation coefficient \( \gamma_0 \) case, the minimal additional cost is 0.997\( k_B T/\gamma_0 \) for \( \epsilon = 1 \% \) and 1.288\( k_B T/\gamma_0 \) for \( \epsilon = 0.1 \% \). Furthermore, the optimal protocol to reach the bound of minimal energy cost is proposed for the qubit initialization realized via a finite-time isothermal process.

\*Corresponding author.
†Also at No. 6 Xibeiwang East Road, Haidian District, Beijing, 100193, China.

Introduction. – Initializing memory is a necessary process in computation [1–6] for further information processing, and inevitably requires an expense of the energy. Landauer derived a fundamental limit of energy cost for such process that a minimal \( k_B T \ln 2 \) (\( k_B \) is the Boltzmann constant) heat will be dissipated to the environment of temperature \( T \) for erasing one bit information [1, 7–11]. Such limit is only reached with two idealities, infinite-long operation time and perfect initialization, which are unfortunately impossible for practical devices.

The quantum bit (qubit) is the basic unit of quantum computation, which is believed to surpass the capability of classical computation [5, 12–14]. The short coherence time makes the first ideality of infinite-time unpractical \([15, 16]\). Fast computation processes require initializing qubits within the time shorter than the coherence time if the qubits are reused in these processes. A finite-time Landauer’s principle has been in turn proposed that the cost in quantum computation process is quantified by the thermodynamic length \( L \), which is firstly tuned from \( \lambda_0 = 0 \) to \( \lambda_f = \lambda_m \) when it is in contact with a thermal bath of temperature \( T \). Then the energy level spacing of the qubit is tuned back to 0 quantum adiabatically. In the ideal erasure process with \( \lambda_m \rightarrow \infty \), the qubit is completely erased to the logical state \( "0" \) with the population in the logical state \( "1" \) of temperature \( T \) is tuned back to 0 quantum adiabatically. In the ideal erasure process with \( \lambda_m \rightarrow \infty \), the qubit is completely erased to the logical state \( "0" \) with the population in the logical state \( "1" \) of temperature \( T \) is tuned back to 0 quantum adiabatically.

The question is to ask what is the minimal energy cost to initialize a qubit with tolerable errors within the coherence time? In this Letter, we tackle this problem by exploiting the geometry framework of quantum thermodynamic \([26, 27]\) to study the finite-time information erasure in a qubit. The erasure here is realized by driving the qubit in a thermal bath. When the driving is not applied infinite-slowly, the additional work required in the erasure process is quantified by the thermodynamic length \( L \) \([26, 28–33]\), which depends on the error probability \( \epsilon \). We discover an analytical trade-off relation among the energy cost, erasure time, and error probability with the asymptotic behavior of \( L \). For applications, we demonstrate the error-probability-dependent optimal erasure protocol to achieve the minimal energy cost for qubit initialization.

Work cost for initializing a qubit. – The qubit is physically modeled as a two-level system, and the information encoded by the logical state \( "0" \) ("1") is represented by the ground state \( |g\rangle \) (excited state \( |e\rangle \)) of the two-level system. Before the erasure process, we assume that the qubit contains one bit of information and stays at

Figure 1. Schematic of finite-time information erasure for a quantum bit (qubit). The ground state and excited state of the qubit represent the logical states "0" and "1", respectively. The energy level spacing of the qubit \( \lambda \) is firstly tuned from \( \lambda_0 = 0 \) to \( \lambda_f = \lambda_m \) when it is in contact with a thermal bath of temperature \( T \). Then the energy level spacing of the qubit is tuned back to 0 quantum adiabatically. In the ideal erasure process with \( \lambda_m \rightarrow \infty \), the qubit is completely erased to the logical state \( "0" \) with the population in the logical state \( "1" \) of temperature \( T \) is tuned back to 0 quantum adiabatically. In the ideal erasure process with \( \lambda_m \rightarrow \infty \), the qubit is completely erased to the logical state \( "0" \) with the population in the logical state \( "1" \) of temperature \( T \) is tuned back to 0 quantum adiabatically.
the maximum mixed state \( \rho_i = (|e\rangle \langle e| + |g\rangle \langle g|)/2 \). In 
the ideal erasure, the qubit is restored into the logical state \( ^0 \rangle \) perfectly, namely \( \rho_f = |g\rangle \langle g| \). However, 
in a practical process, the qubit is generally erased to \( \rho_f = |e\rangle \langle e| + (1 - \epsilon)|g\rangle \langle g| \) with an error probability \( \epsilon \).

To implement the initialization, the energy level spacing of the two-level system is tuned by a control parameter \( \lambda(t) \) under the Hamiltonian \( H(t) = \lambda(t)\sigma_z/2 \) with the Pauli matrix \( \sigma_z = |e\rangle \langle e| - |g\rangle \langle g| \). The Planck’s constant is 
taken as \( \hbar = 1 \) hereafter for brevity. As illustrated in 
Fig. 1, the whole information erasure process is designed 
with two steps as follows. (i) Population reduction 
by increasing the energy level spacing \( \lambda \) from \( \lambda_0 = 0 \) to \( \lambda_m \).
The qubit is coupled to a thermal bath of inverse 
temperature \( \beta = 1/(k_B T) \). (ii) Energy reset 
by quantum adiabatically decreasing the energy level spacing of the qubit to \( \lambda_0 \) with no thermal bath. The first 
step is designed to reduce the population in the excited state, 
and the second step aims to restore the system’s 
Hamiltonian for future operation.

The total work performed in the erasure process with 
duration \( \tau \) is \( W(\tau) = \int_0^\tau W dt \) with the erasure power \( W \equiv \text{tr}(\rho H) \) \[34, 35\], which is explicitly 

\[
W = \frac{\dot{\lambda}(t)}{2} [2p_e(t) - 1].
\]

The probability for the qubit on the excited state \( p_e(t) \) 
is governed by the master equation \[36\] \( \dot{p}_e = \mathcal{L}(p_e) \). For the qubit in contact with a bosonic heat bath, \( \mathcal{L}(p_e) \) reads \[36\],

\[
\mathcal{L}(p_e) = \gamma \left\{ n(\lambda) - [2n(\lambda) + 1] p_e \right\},
\]

where \( n(\lambda) = 1/(e^{\beta \lambda} - 1) \) is the average particle number 
of the bath mode with energy \( \lambda \), and the dissipation coefficient \( \gamma = \gamma(\lambda) \) is determined by the bath spectral 
\[36\].

To evaluate the finite-time effect, we define the irreversable 
work \( W_{ir}(\tau) \equiv W(\tau) - \Delta F = \int_0^\tau W_{ir}(t) dt \), where 
\( \Delta F = \beta^{-1} [\ln 2 - S(\epsilon)] \) \[37\] is the free energy change 
between the states of the qubit before and after initialization 
\( S(\epsilon) = -\epsilon \ln \epsilon - (1 - \epsilon) \ln(1 - \epsilon) \) is the Shannon entropy 
of the final state). The irreversible power \[32, 33\] is 

\[
\dot{W}_{ir} = \frac{\dot{\lambda}(t)}{2} \{ p_e(t) - p^\text{eq}_e[\lambda(t)] \},
\]

where \( p^\text{eq}_e[\lambda(t)] = e^{-\beta \lambda(t)}/[1 + e^{-\beta \lambda(t)}] \) is the excited-state population in the instantaneous thermal equilibrium distribution. Finding the minimal energy cost for erasing the information stored a qubit is now converted to deriving 
the lower bound of the irreversible work.

In the slow-driving regime with \( \gamma \tau \gg 1 \), the irreversible 
power to the first order of \( 1/(\gamma \tau) \) is \[37\]

\[
\dot{W}_{ir} = \beta \gamma^{-1} \frac{(1 - e^{-\beta \lambda}) e^{-\beta \lambda}}{(1 + e^{-\beta \lambda})^2} \lambda^2.
\]

The typical form of the dissipation coefficient \( \gamma = \gamma_0 \lambda^\alpha \) will be used in the following discussion, where \( \gamma_0 \) is a constant. And \( \alpha \in [0, 1) \), \( \alpha = 1 \), and \( \alpha > 1 \) correspond 
to sub-Ohmic, Ohmic, and super-Ohmic spectral, respectively \[36, 38, 39\].

Energy-time-error trade-off. – The irreversible work \( W_{ir} \) is bounded from below by \( W_{ir} \geq L^2/\tau \) with the thermodynamic length \( L \equiv \int_0^\tau \sqrt{\dot{W}_{ir}} dt \) \[26, 28–33, 40, 41\], which is the geometric distance in the parametric space and independent of the specific erasure protocol of \( \lambda(t) \). The thermodynamic length for the erasure process is obtained explicitly as 
\( \mathcal{L}(\epsilon) = \beta (\alpha - 1) \gamma_0^{-1} f_\alpha(\epsilon) \) with the dimensionless function \[37\]

\[
f_\alpha(\epsilon) \equiv \int_0^{\ln(\epsilon^{-1} - 1)} \sqrt{(1 - e^{-x}) e^{-x} \epsilon^{-x}/(1 + e^{-x})^\alpha} dx.
\]

The upper limit of the integral in Eq. (5) reflects the 
dependence of \( \mathcal{L}(\epsilon) \) on the error probability \( \epsilon \). The precise lower bound for irreversible work is denoted as 
\( W_{ir}^\text{min}(\epsilon) \equiv L^2/\tau \). Particularly, in the constant dissipation coefficient case \( \alpha = 1 \), 
\( W_{ir}^\text{min} = 0.997 k_B T/(\gamma_0 \tau) \) for \( \epsilon = 1% \) and \( W_{ir}^\text{min} = 1.288 k_B T/(\gamma_0 \tau) \) for \( \epsilon = 0.1% \).

Utilizing the asymptotic behavior of the thermodynamic length \( |\mathcal{L}(\epsilon) - \mathcal{L}(0)| \propto \epsilon^{\ln^{-\alpha}(\epsilon^{-1})} \) we obtain the 
main result of this Letter \[37\]

\[
W_{ir} \tau \frac{\mathcal{L}^2(0)}{\mathcal{L}^2(0) + \mu_\alpha \epsilon^{\ln^{-\alpha}(\epsilon^{-1})}} \geq 1.
\]

Here, \( \mathcal{L}(0) \) is the thermodynamic length for the perfect 
erasing (\( \epsilon = 0 \)), and \( \mu_\alpha \equiv 4/f_\alpha(0) \) is a dimensionless constant 
determined solely by the bath spectral. This inequality 
quantitatively reveals a trade-off relation among irreversible work \( W_{ir} \), erasure time \( \tau \), and error probability \( \epsilon \). For the special case of the perfect erasing (\( \epsilon = 0 \)), 
such a trade-off recovers the result \( W_{ir} \geq \mathcal{L}^2(0)/\tau \) obtained in the recent studies on finite-time Landauer’s principle \[7, 18–22, 42, 43\].

We plot the analytical lower bound \( W_{ir}^\text{min} \equiv [1 - \mu_\alpha \epsilon^{\ln^{-\alpha}(\epsilon^{-1})}]/\mathcal{L}^2(0)/\tau \) as the surface in Fig. 2 (a) for the Ohmic spectral \( \alpha = 1 \) case. In the simulation, 
we always use the following parameters \( \gamma_0 = 1 \) and \( \beta = 1 \). The monotonocity of the surface indicates that 
more extra energy cost is required to accomplish the erasure 
with higher accuracy and shorter operation time. The trade-offs in the cases with \( \alpha = 0, 2 \) are illustrated in 
Supplementary Materials (SM) \[37\].
The optimal erasure protocol. – The optimal erasure protocol $\lambda(t)$ applied to initialize the qubit for minimal work satisfies $\sqrt{W_{ir}} = L(\epsilon)/\tau$, namely,

$$\frac{d\lambda}{dt} = L(\epsilon) \left[ \frac{\beta (1 - e^{-\beta \lambda}) e^{-\beta \lambda}}{\gamma_0 \lambda^\alpha (1 + e^{-\beta \lambda})^3} \right]^{\frac{1}{2}},$$

with the dimensionless time $\tilde{t} \equiv t/\tau$. The exact optimal protocol is numerically obtained by solving the above differential equation with the boundary conditions $\lambda(\tilde{t} = 0) = 0$ and $\lambda(\tilde{t} = 1) = \lambda_m$. In Fig. 2 (b), we illustrate the optimal protocols for cases with $\alpha = 1$ for $\epsilon = 10^{-1}$ (black solid curve), $\epsilon = 10^{-2}$ (blue dashed curve), and $\epsilon = 10^{-4}$ (red dash-dotted curve), respectively. The optimal protocols in the cases with $\alpha = 0, 2$ are demonstrated in SM [37]. In the initial stage $(t/\tau \ll 1)$ of the erasure process, we find the optimal protocols satisfy a universal scaling $\lambda(\tilde{t} \ll 1) \sim \tilde{t}^{2/(3-\alpha)}$ by solving Eq. (7) approximately [37]. The inset figure shows that the optimal protocol scales as $\lambda \sim \tilde{t}$ in the present case ($\alpha = 1$). Additionally, we stress that the solution of Eq. (7) with respect to $\tilde{t}$ is independent of the erasure time. Therefore, for given erasure processes (fixed $\alpha, \epsilon$) with different duration $\tau$, the required optimal protocol $\lambda(t)$ can be directly obtained by performing variable substitution $\tilde{t} \rightarrow t/\tau$ on the fixed optimal protocol (with respect to $\tilde{t}$).

Numerical validations of the trade-off. – We solve the exact minimal irreversible work numerically from Eqs. (2) and (3) to validate the analytical trade-off in Eq. (6). The irreversible work is illustrated in Fig. 3(a) as a function of the erasure time with $\epsilon = 10^{-4}$. The red squares represent the numerical results corresponding to the optimal erasure protocol, and the analytical lower bound $\tilde{W}^{\min}_{ir}$ is plotted with red dash-dotted curve. The analytical trade-off, exhibiting the typical $1/\tau$-scaling of irreversibility [7, 11, 28, 44–47], is in good agreement with the numerical results in the long-time regime of $\gamma_0 \tau \gg 1$. In the short-time regime (beyond the slow-driving regime), the higher order terms of $1/(\gamma_0 \tau)$ in the expansion of $\tilde{W}_{ir}$ can not be ignored anymore [37], and thus the minimal irreversible work deviates from the $1/\tau$-scaling [46–48]. To demonstrate the dependence of irreversible work on the erasure protocol, the exact numerical irreversible work related to two erasure protocols, $\lambda(t) = \lambda_m (t/\tau)^2$ and $\lambda(t) = \lambda_m t/\tau$ ($\lambda_m = \beta^{-1} \ln(\epsilon^{-1} - 1)$), are illustrated with the blue dot curve and green triangle curve, respectively. The irreversible work corresponding to these two protocols are larger in comparison with the minimal irreversible work achieved with the optimal protocol (red squares). Since the irreversible work can not be less than the analytical lower bound $\tilde{W}^{\min}_{ir}$ with any erasure protocol, we denote the light gray area below the red dash-dotted curve as the inaccessible regime of the irreversible work. In this sense, the light red area above the red dash-dotted curve is accessible.

The exact minimal irreversible work as a function of the error probability is marked with the red squares in Fig. 3 (b) for fixed duration $\tau = 200$. The irreversible work increases for lower the error probability. Similar to Fig. 3 (a), the two areas separated by the red dashed curve (analytical lower bound of Eq. (6)) represent the accessible and inaccessible regions. The black dash-dotted curve represents the precise bound of the irreversible work $\tilde{W}^{\min}_{ir}$ characterized by the exact thermodynamic length, which agrees well with the exact numerical results in the whole plotted range of $\epsilon$. The fact that the red dashed curve ($\tilde{W}^{\min}_{ir}$) approaches to the black dash-dotted curve ($\tilde{W}^{\min}_{ir}$) in the low error region.
probability regime ($\epsilon \ll 1$) is consistent with the approximation condition used to obtain Eq. (6). In addition, we introduce the following normalized quantity $\Delta W_{\text{ir}}(\epsilon) \equiv \frac{W_{\text{ir}}^{\min}(\epsilon) - W_{\text{ir}}^{\min}(10\epsilon)}{W_{\text{ir}}^{\min}(0)}$ to evaluate the additional work required to reduce the error probability $\epsilon$ by an order of magnitude. As demonstrated in the inset figure of Fig. 3 (b), $\Delta W_{\text{ir}}$ decreases rapidly with the error probability. We remark that it typically requires less additional work to reduce the error probability in the low-$\epsilon$ regime, noticing the plateau at $\epsilon \leq 10^{-4}$ region.

The numerical results confirm that the analytical trade-off (red dashed curve) approaches the precise lower bound (black dash-dotted curve) with the error probability $\epsilon \ll 1$. Beyond the regime $\epsilon \ll 1$ or $\gamma \tau \gg 1$, one can observe from Fig. 2 (a) and (b) that all the exact minimal irreversible work (red squares) located above the red dashed curve. This implies that our analytical trade-off (6) may have a wider applicable scope beyond the slow-driving regime.

Conclusions and discussions. – In this Letter, we studied the finite-time information erasure in a qubit with tolerable errors. A universal trade-off among irreversible work, erasure time, and error probability is obtained for non-equilibrium erasure processes characterized by the thermodynamic length. This trade-off relation reveals that reducing the erasure time and error probability require additional energy cost. For practical purposes, we found the optimal erasure protocol associated with the minimal work cost to initialize a qubit. This study paves the way for analyzing the finite-time information processing with the thermodynamic geometry approach, and shall bring new insights to the practical optimization of gate control in quantum computation. As possible extensions, the influences of different bath spectral, quantum coherence of the qubit [22, 42, 49], and fast-driving of the erasure process [33, 43, 50, 51] on the trade-off obtained in the current work can be taken into future consideration.

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Note added: After completion of this work, we became aware of a recent related work by Zhen et al. [52]

Figure 3. (a) Irreversible work as a function of the erasure time $\tau$ in the Ohmic spectral case ($\alpha = 1$), where $\epsilon = 10^{-4}$ is fixed. The red squares represent the exact minimal irreversible work associated with the optimal erasure protocol. The blue dot curve (green triangle curve) is obtained numerically with the erasure protocol chosen as $\lambda(t) = \lambda_m(t/\tau)^2$ ($\lambda(t) = \lambda_m t/\tau$), where $\lambda_m = \beta^{-1} \ln(\epsilon^{-1} - 1)$. The red dashed curve is plotted with the analytical lower bound $W_{\text{ir}}^{\min}$. (b) Minimal irreversible work as a function of error probability $\epsilon$ in the case with $\alpha = 1$, where $\tau = 200$ is fixed. The red squares are obtained numerically with the optimal erasure protocol, while the black dash-dotted curve and red dashed curve represent the precise lower bound $W_{\text{ir}}^{\min} = \mathcal{L}(\epsilon)/\tau$ and the analytical lower bound $W_{\text{ir}}^{\min}$, respectively. The additional work $\Delta W_{\text{ir}}(\epsilon)$ required to reduce the error probability $\epsilon$ by an order of magnitude is plotted in the inset figure. In this figure, we use $\beta = 1$ and $\gamma_0 = 1$.
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Please see Supplementary Materials for the detailed calculations of the free energy change $\Delta F$ and the irreversible power $\dot{W}_\text{ir}$ in Sec. I; In Sec. II, the asymptotic expression of the thermodynamic length is discussed, and the trade-offs for $\alpha = 0, 2$ are plotted; The optimal era- surse protocols in the cases of $\alpha = 0, 2$ are demonstrated in Sec. III.

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Supplementary Materials for "Minimal Energy Cost to Initialize a Quantum Bit with Tolerable Error?"

Yu-Han Ma,1 Jin-Fu Chen,1,2,3 C. P. Sun,1,2,∗ and Hui Dong1,†
1Graduate School of China Academy of Engineering Physics, No. 10 Xibeiwang East Road, Haidian District, Beijing, 100193, China
2Beijing Computational Science Research Center, Beijing 100193, China
3School of Physics, Peking University, Beijing, 100871, China
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In Sec. I of the Supplementary Materials, we show the detailed calculations of the free energy change $\Delta F$ and the irreversible power $\dot{W}_{ir}$ in the erasure process. The asymptotic expression of the thermodynamic length is derived in Sec. II, and the trade-offs in the cases with $\alpha = 0, 2$ are demonstrated. Finally, in Sec. III, we show the optimal erasure protocols in the cases with $\alpha = 0, 2$ and present the analytical discussion on the optimal protocol in the initial stage.

I. FREE ENERGY CHANGE AND IRREVERSIBLE POWER

A. Free energy change

The free energy change $\Delta F$ between the states of the qubit before and after initializing is equal to the work done in the quasi-static erasure process. In the quasi-static population reduction process, the qubit always satisfies the thermal equilibrium distribution $p_{eq}(\lambda) = e^{-\beta \lambda}/(1 + e^{-\beta \lambda})$. When the energy level spacing of the qubit is tuned from $\lambda_0 \rightarrow \lambda_m$, the corresponding quasi-static work $W_{qs}^{(1)} \equiv \int_{\lambda_0}^{\lambda_m} \left[p_{eq}(\lambda) - 1/2\right] d\lambda$ is obtained as

$$W_{qs}^{(1)} = \beta^{-1} \ln \left(\frac{1 + e^{-\beta \lambda_0}}{1 + e^{-\beta \lambda_m}}\right) - \frac{1}{2} (\lambda_m - \lambda_0).$$  (1)

Then the energy level spacing of the qubit is tuned back to $\lambda_0$ adiabatically in the energy reset process, and the work done $W_{qs}^{(2)} \equiv \int_{\lambda_m}^{\lambda_0} \left[p_{eq}(\lambda_m) - 1/2\right] d\lambda$ is

$$W_{qs}^{(2)} = \frac{e^{-\beta \lambda_m}}{1 + e^{-\beta \lambda_m}} (\lambda_0 - \lambda_m) + \frac{1}{2} (\lambda_m - \lambda_0).$$  (2)

The total quasi-static work done in these two processes $W_{qs} \equiv W_{qs}^{(1)} + W_{qs}^{(2)}$ is obtained as

$$W_{qs} = \Delta F = \beta^{-1} \left[ \ln 2 - S(\epsilon) \right],$$  (3)

Here, we have chosen $\lambda_0 = 0$, and defined $\epsilon \equiv e^{-\beta \lambda_m}/(1 + e^{-\beta \lambda_m})$ as the error probability. $S(\epsilon) = -\epsilon \ln \epsilon - (1 - \epsilon) \ln (1 - \epsilon)$ is the Shannon entropy of the final state, and $\beta^{-1} \ln 2$ is the work cost in the ideal erasure process ($\epsilon = 0$) as stated by Landauer’s principle [1]. Since the entropy $S(\epsilon)$ is an increasing function for $0 \leq \epsilon \leq 1/2$, more work is required to achieve lower error probability (smaller $\epsilon$).

B. Irreversible power

When the qubit is in contact with the bosonic heat bath, the evolution of the excited-state population is governed by the master equation [2]

$$\dot{p}_e = \gamma(\lambda) \left\{ n(\lambda) - [2n(\lambda) + 1] p_e \right\},$$  (4)

∗ suncp@gscsep.ac.cn
† hdong@gscsep.ac.cn
where \( n(\lambda) = 1/(e^{\beta \lambda} - 1) \) is the average particle number of the bath mode with energy \( \lambda \), and the dissipation coefficient \( \gamma(\lambda) \) is determined by the bath spectral \([3]\). With the equilibrium distribution \( p_e^\alpha(\lambda) = e^{-\beta \lambda}/(1 + e^{-\beta \lambda}) \), Eq. (4) is rewritten as

\[
\left[ 1 + \frac{1 - 2p_e^\alpha(\lambda)}{\gamma(\lambda)\tau} \frac{d}{dt} \right] p_e = p_e^\alpha(\lambda), \tag{5}
\]

with the dimensionless time \( \bar{t} \equiv t/\tau \). The series expansion solution of this equation with respect to \( \gamma(\lambda)\tau \) follows as

\[
p_e = \left[ 1 + \frac{1 - 2p_e^\alpha(\lambda)}{\gamma(\lambda)\tau} \frac{d}{dt} \right]^{-1} p_e^\alpha(\lambda) = \sum_{n=0}^{\infty} \left[ -\frac{1 - 2p_e^\alpha(\lambda)}{\gamma(\lambda)\tau} \frac{d}{dt} \right]^n p_e^\alpha(\lambda). \tag{6,7}
\]

In the slow-driving regime with \( \gamma(\lambda)\tau \gg 1 \), we obtain the excited population \( p_e(t) \) to the first order of \( 1/\gamma(\lambda)\tau \) as

\[
p_e \approx p_e^\alpha(\lambda) - \frac{1 - 2p_e^\alpha(\lambda)}{\gamma(\lambda)} \frac{\partial p_e^\alpha}{\partial \lambda} \dot{\lambda}. \tag{8}
\]

The irreversible power \( \dot{W}_ir = \dot{\lambda}(t)(p_e - p_e^\alpha) \) is thus explicitly obtained as

\[
\dot{W}_ir = -\frac{1 - 2p_e^\alpha(\lambda)}{\gamma(\lambda)} \frac{\partial p_e^\alpha}{\partial \lambda} \dot{\lambda}^2 \tag{9}
\]

\[
= \frac{\beta}{\gamma(\lambda)} \frac{1 - e^{-\beta \lambda}}{(1 + e^{-\beta \lambda})^3} \dot{\lambda}^2. \tag{10}
\]

II. ASYMPTOTIC EXPRESSION OF THE THERMODYNAMIC LENGTH

In this section, we discuss the asymptotic expression of the thermodynamic length \( \mathcal{L}(\epsilon) \) with respect to \( \epsilon \), and obtain approximately an analytical trade-off among irreversible work, erasure time, and error probability.

In the slow-driving regime, by using Eq. (9) and the typical dissipation coefficient \( \gamma(\lambda) = \gamma_0 \lambda^\alpha \) \([2]\), the thermodynamic length \( \mathcal{L} \equiv \int_0^\tau \sqrt{\dot{W}_ir} \, dt \) \([4]\) of the population reduction process is obtained in terms of \( \epsilon \) as

\[
\mathcal{L}(\epsilon) = \sqrt{\beta^{(\alpha - 1)} \gamma_0^{-1}} \int_0^{\ln(\epsilon^{-1} - 1)} \sqrt{\frac{(1 - e^{-x}) e^{-x} x^\alpha}{x^\alpha (1 + e^{-x})^3}} \, dx \tag{11}
\]

\[
= \sqrt{\beta^{(\alpha - 1)} \gamma_0^{-1}} f_\alpha(\epsilon). \tag{12}
\]

For the perfect erasure \( \epsilon = 0 \), one has \( \mathcal{L}(0) = \sqrt{\beta^{(\alpha - 1)} \gamma_0^{-1}} f_\alpha(0) \) with

\[
f_\alpha(0) = \int_0^\infty \sqrt{\frac{(1 - e^{-x}) e^{-x} x^\alpha}{x^\alpha (1 + e^{-x})^3}} \, dx = \begin{cases} 1.1981 & \alpha = 0 \\ 0.9433 & \alpha = 1 \\ 1.0914 & \alpha = 2 \end{cases}. \tag{13}
\]

For general erasure processes, \( f_\alpha(\epsilon) \) can be re-expressed in term of \( f_\alpha(0) \) as \( f_\alpha(\epsilon) = f_\alpha(0) - I_\alpha(\epsilon) \) with

\[
I_\alpha(\epsilon) \equiv \int_{\ln(\epsilon^{-1} - 1)}^\infty \sqrt{\frac{(1 - e^{-x}) e^{-x} x^\alpha}{x^\alpha (1 + e^{-x})^3}} \, dx. \tag{14}
\]
Figure 1: $L(0) - L(\epsilon)$ as functions of $\epsilon$ with $\alpha = 0, 1, 2$. In this plot, we choose $\beta = 1, \gamma_0 = 1$.

We further use a new integral variable $y \equiv e^{-x}$ to rewrite Eq. (14) as

$$I_\alpha(\epsilon) = \int_0^\epsilon (-\ln y)^{-\alpha/2} \sqrt{\frac{1 - y}{y(1 + y)^3}} dy,$$

which is expanded into series form with respect to $y$ as

$$I_\alpha(\epsilon) = \int_0^\epsilon (-\ln y)^{-\alpha/2} \left[ \frac{1}{\sqrt{y}} - 2\sqrt{y} \mathcal{O}(y^{3/2}) \right] dy.$$

Noticing $\sqrt{(1 - y)/(y(1 + y)^3)} \leq \sqrt{1/y}$, we find $I_\alpha(\epsilon)$ is bounded from below by the first term of its series expansion as

$$I_\alpha(\epsilon) \leq \int_0^\epsilon (-\ln y)^{-\alpha/2} \frac{1}{\sqrt{y}} dy \leq 2\sqrt{\epsilon} \left[ \ln (\epsilon^{-1}) \right]^{-\frac{\alpha}{2}} \leq 2\sqrt{\epsilon} \left[ \ln (\epsilon^{-1}) \right]^{-\frac{\alpha}{2}}.$$

Therefore, the thermodynamic length $\mathcal{L}(0) = \sqrt{\beta^{(\alpha-1)}\gamma_0^{-1}} f_\alpha(\epsilon)$ has a lower bound as

$$\mathcal{L}(\epsilon) \geq \sqrt{\beta^{(\alpha-1)}\gamma_0^{-1}} f_\alpha(0) - 2\sqrt{\epsilon} (-\ln \epsilon)^{-\alpha/2}.$$

We stress that this inequality for $\mathcal{L}(\epsilon)$ is quite tight in the low error probability regime ($\epsilon \ll 1$). As shown in Fig. 1, the asymptotic expression

$$\mathcal{L}(\epsilon) \approx \mathcal{L}(0) - 2\sqrt{\beta^{(\alpha-1)}\gamma_0^{-1}} \epsilon \ln^{-\alpha} (\epsilon^{-1}).$$

of the thermodynamic length (lines) agrees well with the exact numerical results (dots) obtained from Eq. (11).
Substituting Eq. (22) into the bound on the irreversible work $W_{ir} \geq L^2(\epsilon)/\tau$, one obtains

$$W_{ir} \tau \geq L^2(0) - 4L(0) \sqrt{\beta(\beta \lambda) (1 - \beta \lambda)} \gamma_0^{-1} \epsilon \ln^{-\alpha} (\epsilon^{-1}),$$

where the $O(\epsilon)$ term has been ignored. Then, divide $L^2(0)$ on both sides of the above inequality, we find

$$\frac{W_{ir} \tau}{L^2(0)} + \mu_{\alpha} \sqrt{\epsilon \ln^{-\alpha} (\epsilon^{-1})} \geq 1$$

with $\mu_{\alpha} \equiv 4/f_{\alpha}(0)$. This trade-off relation among irreversible work, erasure time, and error probability is presented as $[Eq. (6)]$ in the main text. The equality gives the analytical lower bound $\tilde{W}_{\text{ir}}^\text{min}(\epsilon) \equiv (1 - \mu_{\alpha} \sqrt{\epsilon \ln^{-\alpha} (\epsilon^{-1})})L^2(0)/\tau$ for irreversible work. In Fig. 2, the analytical lower bound ($\tilde{W}_{\text{ir}}^\text{min}$) of the trade-off in $[Eq. (6)]$ of the main text is plotted as the surface for (a) $\alpha = 0$ and (b) $\alpha = 2$.

III. OPTIMAL ERASURE PROTOCOL TO ACHIEVE MINIMUM IRREVERSIBLE WORK

By numerically solving $[Eq. (7)]$ of the main text, the exact optimal protocols for $\alpha = 0, 2$ are plotted in Fig. 3. In the initial stage ($\tilde{t} = t/\tau \ll 1$) of the population reduction process, noticing $\lambda(\tilde{t}) \ll 1$, $[Eq. (7)]$ of the main text is approximated as

$$\frac{d\lambda}{d\tilde{t}} \approx \mathcal{L}(\epsilon) \left[ \frac{\beta(\beta \lambda)(1 - \beta \lambda)}{\gamma_0 \lambda^\alpha (2 - \beta \lambda)^2} \right]^{1/2} \approx \mathcal{L}(\epsilon) \beta^{-1} \sqrt{8\gamma_0} \lambda^{a_{\alpha} - 1}. $$

By straightforward calculation, one obtains the optimal protocol

$$\lambda(\tilde{t} \ll 1) = \left[ 2(3 - \alpha)^2 \gamma_0 \beta^{-2} \mathcal{L}^2(\epsilon) \right]^{1/2} \tilde{t}^{\frac{3-\alpha}{2}} \tilde{t}^{\frac{a_{\alpha} - 1}{2}} ,$$

where the power exponent of $\tilde{t}$ is only determined by the bath spectral parameter $\alpha$. In the initial stage, the scaling of the optimal protocol, $\lambda \sim \tilde{t}^{2/(3-\alpha)}$, is shown in the inset figure of Fig. 3.
Figure 3: Optimal protocol of $\lambda(\tilde{t})$ in the cases with (a) $\alpha = 0$ and (b) $\alpha = 2$ for different error probability $\epsilon$. The black solid curve, blue dashed curve, and red dash-dotted curve represent the optimal protocols for $\epsilon = 10^{-1}$, $\epsilon = 10^{-2}$, and $\epsilon = 10^{-4}$, respectively. The scaling of the optimal protocol in the initial stage $\lambda(\tilde{t} \ll 1) \sim \tilde{t}^{2/(3-\alpha)}$ is illustrated in the inset figure. In this plot, $\gamma_0 = 1$ and $\beta = 1$ are used.

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