A STABILITY INDEX FOR DETONATION WAVES
IN MAJDA'S MODEL FOR REACTING FLOW

GREGORY LYNG AND KEVIN ZUMBRUN

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Abstract. Using Evans function techniques, we develop a stability index for weak and strong detonation waves analogous to that developed for shock waves in [GZ, BSZ], yielding useful necessary conditions for stability. Here, we carry out the analysis in the context of the Majda model, a simplified model for reacting flow; the method is extended to the full Navier–Stokes equations of reacting flow in [Ly, LyZ]. The resulting stability condition is satisfied for all nondegenerate, i.e., spatially exponentially decaying, weak and strong detonations of the Majda model in agreement with numerical experiments of [CMR] and analytical results of [Sz, LY] for a related model of Majda and Rosales. We discuss also the role in the ZND limit of degenerate, sub-algebraically decaying weak detonation and (for a modified, “bump-type” ignition function) deflagration profiles, as discussed in [GS.1–2] for the full equations.

Section 1. Introduction

In one-dimensional, Lagrangian coordinates, the Navier–Stokes equations of reacting flow for a one-step reaction may be written in the abstract form

\begin{equation}
\begin{aligned}
& u_t + f(u)_x = (B(u)u_x)_x + kq\varphi(u)z, \\
& z_t = (D(u,z)z_x)_x - k\varphi(u)z,
\end{aligned}
\end{equation}

where \( u, f, q \in \mathbb{R}^n, B \in \mathbb{R}^{n \times n}, z, k, D, \varphi \in \mathbb{R}^1 \), and \( k > 0 \) (model (8.81) of [Z.3] with particle velocity set to zero). Here, vector \( u \) comprises the gas-dynamical variables of specific volume, particle velocity, and total energy, and \( z \) measures mass fraction of unburned reactant: more generally, “progress” of a single reaction involving multiple reactants. The first equation thus models kinematic and the second equation reaction effects. The function \( \varphi(u) \) is an “ignition function”, monotone increasing in temperature, and usually assumed for fixed density to be zero below a certain ignition temperature and positive above. The vector \( q \) comprises quantities produced in reaction, in particular heat released. The coefficient \( k \) corresponds to reaction rate, while coefficients \( B \) and \( D \) model transport effects of, respectively, viscosity and heat conduction, and species diffusion. Multi-step reactions may be

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modeled by the same equations with vectorial reaction variable \( z \in \mathbb{R}^m \), and coefficients \( q, D, \varphi, k \) modified accordingly; see Section 9. For further discussion, see, e.g., [CF,FD,GS.1–2,Z.3,Ly].

Under different conditions at \( x = \pm \infty \), there can result a variety of types of waves solving (1.1): nonreactive gas-dynamical shock and rarefaction solutions \( (z \equiv 0, \text{ or } z \equiv \text{constant} \text{ and } \varphi \equiv 0) \), and traveling combustion waves consisting of weak and strong detonations, weak and strong deflagrations, and Chapman–Jouget detonation and deflagration waves, which are limiting cases dividing weak and strong branches. Roughly speaking, detonations are compressive waves analogous to shock waves in nonreactive gas dynamics, while deflagrations are expansive solutions analogous to rarefactions; for a fixed left-hand state, there are weak and strong branches of right-hand sides corresponding to waves of each type. Chapman-Jouget waves occur at the special parameters for which strong and weak branches coalesce. We refer the reader to [CF,FD,G,M.4,GS.1–2] for a detailed discussion of these solutions of the traveling wave ODE for (1.1) and their roles in Riemann solutions/time-asymptotic behavior for the initial value problem under various assumptions on \( B, D \).

Similarly as in the case of “real”, e.g., van der Waals, gas dynamics [BE,MeP], the multitude of possible such elementary waves leads to a multitude of possible time-asymptotic states, and these must be classified according to stability. However, the assessment of stability is a complicated undertaking. Up to now, essentially all analyses have been carried out for one of three simplified models: (i) the Zeldovich–von Neumann-Doering (ZND) model, for which \( B \) and \( D \) are set identically zero in (1.1) [Er.1–6,CF,LS], (ii) the still further simplified Chapman–Jouget (CJ) or “square-wave” model, for which \( B \) and \( D \) are set identically zero and \( k \) is taken to be infinite, so that combustion waves become surfaces of discontinuity [Er.7,F,etc.], or (iii) The Majda model, for which \( u \) is taken to be a scalar, and \( B \) and \( D \) are set to 1 and 0 (or sometimes 1), respectively [M.4,LLT,LYi,RV], or the Majda–Rosales model [CMR,Sz,LY,Li.1–7], a closely-related cousin in which \( -z_x \) is substituted for \( z_t \) everywhere in (1.1). At one time, it seems to have been believed, based on analysis of the ZND case \( B \equiv D \equiv 0 \) (see, e.g., [CF]) that weak detonations and strong deflagrations were unstable, the other types stable at least in moderate parameter ranges. This conjecture on weak detonation is now widely agreed to be false when viscosity and other effects are taken into account, see [CMR,Sz,LY], or more general discussion in [FD]. However, rigorous analysis of stability for the full model (1.1), or comparison with stability for the ZND or CJ approximations, remain important open problems.

The purpose of the present paper is to initiate a larger-scale study of these problems by the introduction, in the simple setting of the Majda model, of new Evans function techniques developed recently in the study of stability of viscous shock profiles (see, e.g., [GZ,ZH,BSZ,ZS,Z.3]). In contrast to the methods of past analyses of the Majda model, these techniques were designed for the study of systems, \( u \in \mathbb{R}^n, n > 1 \), so may be applied also in the case of the full, reacting Navier–Stokes
equations. As a roadmap for discussions to follow, we point out that the ZND limit $B = \varepsilon B_0$, $D = \varepsilon D_0$, $\varepsilon \to 0$ is equivalent by the spatial rescaling $x \to x/\varepsilon$ to the small-$k$ limit $k \to 0$ with $B$, $D$ held fixed. The CJ, or square-wave limit in the viscous setting ($B$, $D$ fixed) is ambiguous, corresponding to intermediate values of $k$, but large spatial scale; see discussion of [Z.3], Appendix A.3. The large $k$ limit $k \to \infty$ (with, necessarily, $u_i$, $u^i \to u_-$ to allow a connection) corresponds roughly to the high activation-energy limit described in, e.g., [B], pp. 22–25, which in the ZND setting leads to a square-wave approximation theory, and is generally associated with instability, and other anomalous behavior; see, e.g., [Er.7,AT,BL,BN,LS].

A tool that has proved useful for the study of stability in the related cases of van der Waals gas dynamics and multiphase flow [GZ,Z.6] is a one-dimensional stability index originally introduced by J. Evans [E.1–4] in the context of nerve axon equations, and generalized in various directions in, e.g., [J.1,AGJ,PW,GZ,BSZ]. A topological index relating evolutionary (PDE) dynamics of a traveling wave to dynamics of the associated traveling wave ODE, the stability index is based on the Evans function [E.1–4,AGJ,PW,GZ,BSZ,etc.] $D(\lambda)$, an analytic function playing the role of a characteristic function for the linearized operator $L$ about the wave. Defined as a Wronskian of solutions of the eigenvalue equation for $L$ decaying at plus and minus spatial infinity, the Evans function vanishes at $\lambda$ if and only if there exists a solution of the eigenvalue equation decaying at both infinities, i.e., $\lambda$ is an eigenvalue. More precisely, zeroes of $D$ agree in both location and multiplicity with eigenvalues of $L$ [GJ.1–2].

For traveling waves, there is always an eigenvalue at $\lambda = 0$, corresponding to translational invariance of the underlying PDE, which in the simplest setting is multiplicity one. In this situation (which will be the case here), $D(0) = 0$, but $D'(0) \neq 0$. Moreover, if the evolution equation is well-posed in the sense that the linearized operator about the wave generates a $C^0$ semigroup, then (by standard resolvent estimates [Pa]) $D$ cannot vanish for $\lambda$ real and sufficiently large. Since $D$, properly constructed respects complex symmetry, $\bar{D}(\lambda) = D(\bar{\lambda})$, as does the eigenvalue equation itself, we have in particular that $D$ is real-valued when restricted to the real axis; thus, $\text{sgn } D(\lambda)$ has a well-defined limit as $\lambda \to +\infty$ along the real axis, which we will denote as $\text{sgn } D(+\infty)$. The stability index is then defined as

\begin{equation}
\Gamma := \text{sgn } D'(0)D(+\infty).
\end{equation}

Evidently, the stability index detects the parity of the number of real roots on the nonnegative real axis, $\Gamma$ positive corresponding to even parity, and $\Gamma$ negative to odd parity. Since complex roots appear in conjugate pairs, this is in fact the parity of the number of all roots (real and complex) in the unstable half-plane $\text{Re } \lambda > 0$, and thus gives partial information on spectral stability, defined as nonexistence of eigenvalues in this region. In particular, $\Gamma > 0$ is seen to be a necessary condition for stability. The value of this index comes from the fact that it can be related to geometric information about the phase portrait of the traveling-wave ODE, which fact ultimately derives from the correspondence at $\lambda = 0$ between the eigenvalue equation and the linearized traveling-wave ODE.
To obtain a concrete result, of course, requires information about the existence problem, and this can in general be obtained only in simple situations, e.g. scalar reaction–diffusion equations, $2 \times 2$ parabolic conservation laws, $3 \times 3$ conservation laws with real viscosity, etc., for which the connection problem is a planar dynamical system, or else [J.1,AGJ,etc.] in some singular limit for which the connection problem can be broken into separately computable “fast” and “slow” problems.

In this paper we show, in the simple context of the Majda model, that the methods introduced in [GZ,BSZ] for the study of stability of viscous shock waves, may, with slight modifications, be applied also in the study of stability of detonations to: (i) construct an analytic Evans function on the set $\Re \lambda \geq 0$ for the linearized operator about the detonation wave, and (ii) in both the strong and weak detonation cases, compute an expression for $\Gamma$ in terms of quantities associated with the traveling-wave ODE. In this simple setting, the connection problem is planar, and we can in fact do more, obtaining a complete evaluation of $\Gamma$; the result, for both weak and strong detonations, is $\Gamma > 0$, consistent with stability.

This is consistent with prior results of [L,LLT,LYi] for small-amplitude strong detonations in the small-$q$ limit, and of [RV] for arbitrary amplitude strong detonations in the small-$k$ (ZND) limit. It is also consistent with results on weak detonations of the Majda–Rosales model obtained in [Sz] for small-amplitude waves with intermediate $k$ and in [LY] for arbitrary amplitude waves in the large-$k$ limit; however, as far as we know, ours is the first analytical result on stability of weak detonations for the Majda model.

These are partial stability results in that they do not rule out instability; on the other hand, they are of general applicability, and are obtained using a relatively small amount of information about the system under study. In particular, they generalize to the full reactive Navier–Stokes case (see discussion below), whereas analyses of the integro-differential Majda–Rosales model clearly do not. Moreover, they do have the interesting implication that transition to instability, if it occurs, must result from a pair of complex conjugate eigenvalues crossing the imaginary axis, typically signaling a Poincaré–Hopf bifurcation to a time-periodic solution, consistent with the experimentally and numerically observed phenomenon of “galloping” detonations [MT,FW,MT,AIT,AT,F.1–2, p. 161,BMR,S,Li.6]. It would be very interesting to search numerically for such instabilities in the high activation-energy limit $k \to \infty$; see, e.g., [Br.1–2,BrZ] for an efficient numerical algorithm.

In the course of our development, we also discuss existence of deflagrations for a modified “bump-type” ignition function, and the appearance in the ZND limit of degenerate, spatially subalgebraically decaying families of weak and strong detonations, as described for the full equations in [GS.1]. Stability of these waves, and their significance in Riemann solutions, are discussed in Sections 8 and 4, respectively.

**Discussion and open problems.** In [Ly,LyZ], the methods of this paper are extended to the full Navier–Stokes equations of reacting flow, with possibly multi-species reaction. In this case, the connection problem is high-dimensional, and the actual evaluation of the geometric stability conditions is carried out analytically.
only in the small-$k$ (ZND) limit, for which the connection problem has been closely studied in [GS.1–2]. However, the geometric condition itself is valid for arbitrary model parameters, and could be studied numerically at the same time as the connection problem; thus, the results again reduce the question of stability to state of the art existence theory. Extensions to multi-dimensional stability, in the spirit of viscous shock calculations of [ZS], are given in [Z.3,JLy.1–2].

As pointed out in [Z.3], Appendix A.3, it is relatively straightforward using the methods of [ZH,Z.2–4,MZ.1–3] to show that spectral stability of nondegenerate detonation waves implies linearized and nonlinear orbital stability, both for the Majda model\(^1\) and for the full, reacting Navier–Stokes equations; this will be a topic of future work. A very interesting open problem is to carry out a complete spectral stability analysis in the spirit of the matched asymptotic analysis of the Majda model in [RV] for the full reactive Navier–Stokes equations in the ZND limit, both in one- and multi-dimensions. The Majda model has the simplifying feature of scalar kinetics; recent singular perturbation-type techniques developed in the shock wave context in [PZ,FreSz] may be helpful in attacking the full, system case. As discussed in Section 8, it would also be interesting to study further the stability of (generically appearing) degenerate weak deflagrations and of degenerate weak and strong detonations appearing in the ZND limit.

In actual detonations, multidimensional, geometric effects become important; for example, a converging detonation wave is destabilized, while diverging, or expansive waves are stabilized, and this is another important feature to understand; see, e.g. [MR,B,BM,M,Li.7]. There is also the problem of trying to understand the bifurcation to quasi-steady behaviors, such as time-oscillatory “galloping”, or “spinning” detonations in the multidimensional case, in terms of spectral information given by the Evans function. The challenge of this problem, similarly as the original stability analysis, comes from the fact that there is no spectral gap between neutral point spectrum and essential spectrum of the linearized operator about the wave, so that bifurcations are of mathematically nonstandard type; see the appendix of [BMR] for an interesting related discussion in the context of the ZND model. Useful survey of the latter two topics may be found in [FD,LS,S,Li.6]. Finally, we mention the (presumably numerical) problem of cataloguing possible Riemann solutions within the class of viscous profiles in the more general situations discussed in Section 9 of multi-species reactions or reaction-dependent equation of state; this appears to be an interesting and physically important direction for further study.

**Plan of the paper.** In Section 2, we give a brief description of the Majda model, and the associated Chapman–Jouget diagram. In Sections 3–4, we discuss existence of profiles and the implications for solutions of the Riemann problem. In Sections 5–7 we construct an Evans function for the linearized operator about the wave, and carry out the described analysis of the stability index. In Section 8, we discuss the

\(^1\)For the Majda model with $D = 1$, as pointed out in [LLT], this result may be obtained much more simply by Sattinger’s method of weighted norms [Sat].
case of degenerate, spatially subalgebraically decaying profiles and in Finally, in Section 9, we describe extensions to multi-species reactions and reaction-dependent equation of state.

Section 2. The Majda model

Hereafter, we restrict attention to the Majda model

\[
\begin{align*}
\begin{cases}
  u_t + f(u)_x &= u_{xx} + kq\varphi(u)z, \\
  z_t &= -k\varphi(u)z,
\end{cases}
\end{align*}
\]

or, equivalently,

\[
\begin{align*}
\begin{cases}
  (u + qz)_t + f(u)_x &= u_{xx}, \\
  z_t &= -k\varphi(u)z,
\end{cases}
\end{align*}
\]

a simplified, scalar version of (1.1) simulating the dynamics of one-dimensional combustion within a single characteristic family [M.4]. Here, \( u \in \mathbb{R}^1 \) is a lumped gas-dynamical variable combining aspects of specific volume, particle velocity and temperature, and \( z \) corresponds to mass fraction of reactant as before.

Following [M.4], we take

\[
\begin{align*}
  f'(u) > 0, \quad f''(u) > 0,
\end{align*}
\]

and \( q > 0 \), corresponding to an exothermic reaction. However, in place of the “step-type” function of [M.4], we take a modified, “bump-type” ignition function

\[
\varphi \in C^1, \quad \begin{cases}
  \varphi(u) = 0 \quad \text{for} \quad u \leq u_i \text{ or } u \geq u^i, \\
  \varphi(u) > 0 \quad \text{for} \quad u_i < u < u^i.
\end{cases}
\]

This choice is motivated by the physical parametrization of temperature with respect to velocity \( u \) in the traveling-wave phase portrait of the ZND model, and ensures that the traveling-wave equations for (2.1) agree with the reduced system obtained in [GS.1] by asymptotic analysis in the ZND limit; see [GS.1], pp. 979–981. In particular, it allows for existence of weak deflagration profiles, as the step-type ignition function does not; see Section 3, below.

More precisely, interpreting \( u \) as particle velocity we take

\[
\varphi = \psi(T(u)),
\]

where \( T(u) \) denotes temperature, \( \psi \) is a standard, step-type ignition function, and \( \psi \equiv 0 \) below some ignition temperature \( T_i \) and positive above, and \( T(u) \) is quadratic, concave-down, with \( T(u_i) = T(u^i) = T_i \). This agrees qualitatively with the physical dependence of temperature on velocity along the one-dimensional flow of
the traveling wave ODE for the ZND model. For example, for an ideal gas with gas constant $\gamma > 1$, the physical dependence is

\begin{equation}
T(u) = -\gamma M^2 u^2 + (\gamma M^2 + 1)u,
\end{equation}

where $M = |u_+ - s|/|c_+|$, $0 < M < 1$, denotes Mach number of the specific detonation under consideration, $c_+$ the sound speed at the righthand state; see [LyZ], eq. (1.17).

We seek traveling waves $(u, z) = (\bar{u}(x - st), \bar{z}(x - st))$ connecting states $(u_-, z_-)$ and $(u_+, z_+)$, with

\begin{equation}
z_+ := z(+\infty) = 1, \quad z_- := z(-\infty) = 0,
\end{equation}

and

\begin{equation}
u_i < u_- := u(-\infty) < u^i; \quad u_+ := u(+\infty) \leq u_i \text{ or } \geq u^i,
\end{equation}

hence

\begin{equation}
\varphi_- := \varphi(u_-) > 0; \quad \varphi_+ := \varphi(u_+) = 0:
\end{equation}

that is, combustion waves moving from left to right, leaving completely burned gas in their wake.

**The Traveling-wave equation.** The traveling wave ODE for (2.2) is

\begin{equation}
\begin{cases}
u' = f(u) - f(u_-) - sqz - s(u - u_-), \\
z' = (k/s)\varphi(u)z.
\end{cases}
\end{equation}

Thus, necessary conditions for existence of a profile are the modified Rankine–Hugoniot condition

\begin{equation}
[f] = s([u] + q),
\end{equation}

along with

\begin{equation}
\varphi_+ : \varphi(u_+) = 0, \quad \text{i.e. } u_+ \leq u_i \text{ or } \geq u^i,
\end{equation}

together assuring that $(u_+, z_+) = (u_+, 1)$ is a rest state for (2.10).

**Types of Waves.** Examining Figure 1, cases A and B, below, we find that the possible solutions of (2.11) may be described as follows.

**Proposition 2.1.** For fixed $u_+$, $s > s_+(u_+)$, there exist two states $u_- > u_+$ for which (2.11) is satisfied. For $s = s_+$, there exists one solution. For $s < s_+$, there exist no solutions $u_- > u_+$. Here, $s_+ > f'(u_+)$. 

Figure 1A. Rest states \( u_- > u_+ \) (detonation case).

Figure 1B. Rest states \( u_- < u_+ \) (deflagration case).

**Proposition 2.2.** For fixed \( u_+ \), \( s < s^*(u_+) \), there exist two states \( u_- < u_+ \) for which (2.11) is satisfied. For \( s = s^* \), there exists one solution. For \( s > s^* \), there exist no solutions \( u_- < u_+ \). Here, \( s^* < f'(u_+) \).

Figure 1, known as a Chapman–Jouget diagram, is the basis for the standard classification of combustion waves. Denote

\[
(2.13) \quad a_\pm := f'(u_\pm).
\]

**Definition:** A combustion wave \((\bar{u}, \bar{z})\) is called a *detonation* if \( u_- > u_+ \). It is called a *strong detonation* if the Lax characteristic condition holds,

\[
(2.14) \quad a_- > s > a_+.
\]

This corresponds to the larger of the two solutions in Proposition 2.1. It is called a *weak detonation* if

\[
(2.15) \quad s > a_-, a_+,
\]

corresponding to the smaller of the two solutions. This type is undercompressive. The boundary case

\[
(2.16) \quad a_- = s > a_+
\]

is called a Chapman–Jouget detonation, and has special significance in the theory: specifically, in idealized circumstances, it is the wave expected to be time-asymptotically selected in the "ignition problem" of initial data consisting of a large initializing pulse; see [FD, Li.2–4], or discussion at the end of this section.
**Definition:** A combustion wave \((\bar{u}, \bar{z})\) is called a *deflagration* if \(u_- < u_+\). It is called a *weak deflagration* if

\[
a_+ > a_- > s,
\]

corresponding to the larger of the two solutions, and *strong deflagrations* if

\[
a_+ > s > a_-
\]

corresponding to the smaller. The former is again undercompressive, the latter of “reverse-Lax” type. The boundary case

\[
a_+ > s = a_-
\]
is called a Chapman–Jouget deflagration.

The minimum detonation speed \(s_*\) and the maximum deflagration speed \(s^*\) are called Chapman-Jouget (CJ) speeds. Note that they determine a minimum strength for combustion waves connecting to \(u_+\). Indeed, there is a band of speeds \((s_*, s^*)\) about the characteristic speed \(f'(u_+)\) for which neither detonation nor deflagration connections exist, in sharp contrast to the inert-gas, shock wave case. Thus, for fixed \(q > 0\), all combustion waves are “strong” in the sense that they differ appreciably from acoustic signals.

**Section 3. Existence of profiles.**

The Chapman–Jouget classification scheme concerns possible endstates that may be connected by a traveling wave. We next consider the question of existence of a traveling-wave profile, reviewing and extending results of [M.4,Sz,Li.3,LY].

**Linearized rest points.** Linearizing (2.10) about the critical points \((u_\pm, z_\pm)\), we obtain

\[
\left( \begin{array}{c} u' \\ z' \end{array} \right) = \left( \begin{array}{cc} \alpha_\pm & -sq \\ 0 & (k/s)\varphi(u_\pm) \end{array} \right) \left( \begin{array}{c} u \\ z \end{array} \right),
\]

where

\[
\alpha_\pm := a_\pm - s.
\]

(Note: because of the structure of (2.10), we cannot without loss of generality set \(s = 0\) as in the conservative, shock case.) Here, we have used the fact that \(d\varphi(u_+) = z_- = 0\), so that \(d\varphi(u_+)z_+ = 0\). Thus, the eigenvalues associated with the linearized equations are

\[
\gamma_\pm = \alpha_\pm, \quad (k/s)\varphi(u_\pm).
\]
Moreover, weak and strong detonation profiles decay exponentially to scalar $a$ exist, are unique up to translation, decaying exponentially to difficult to see that flow along the CM is attracting $u < u_i$ as $c/|x|$ both. Neither weak nor strong deflagration profiles exist.

In the generic case, referring to (2.9), (2.12), we find that $(u_+, z_+)$ is a saddle–attractor in the detonation case, and a saddle–repellor in the deflagration case, with center manifold in both cases lying in direction $(qs, \alpha_+)$ tangent to the null-cline $u' = 0$ (recall, $\varphi_ + = 0$), while $(u_-, z_-)$ is a repellor in the strong detonation or weak deflagration case, a saddle in the weak detonation or strong deflagration case. In the exceptional, Chapman–Jouget case $\alpha_- = 0$ that weak and strong values for $u_-$ coalesce, $(u_-, z_-)$ becomes a saddle–repellor, with center manifold lying in the gas-dynamical direction $(1, 0)$; see (3.1). Considering the traveling wave equation as a scalar ODE $u' = f''(u_-) u^2 + O(|u|^3) - sqz$, forced by the solution $z = z_0 e^{(k/s)t}$ of the decoupled exponential growth equation $z' = (k/s)z$, we find that solutions are generically governed by the dominant part $u' = f''(u_-) u^2$, growing algebraically as $c/|x|$; the sole exception is the unique solution lying along the unstable manifold, $-sqz \sim (k/s)u$, for which both $u$ and $z$ exhibit exponential growth with rate $c(k/s)t$.

Finally, note that the center manifold (CM) at $(u_+, z_+)$ is a manifold of rest points, hence there are no orbits connecting to $(u_+, z_+)$ along the CM. That is, for $u < u_i$ or $u > u_i$, ODE (2.10) reduces to

\[
\begin{align*}
\begin{cases}
  u' = f(u) - f(u_-) - s(u - u_-) - sqz, \\
  z' = 0,
\end{cases}
\end{align*}
\]

a scalar ODE with parameter $z$, for which $u_+$ is a nondegenerate rest point by the fact that $f'(u_+) \neq s$, a consequence of Propositions 2.1–2.2: an attractor in the detonation case (with a unique incoming orbit), a repellor in the deflagration case (with no incoming orbit).

Collecting the above discussion, we have the preliminary observation:

**Lemma 3.1.** In the generic situation $u_+ \neq u_i$, $u_+ \neq u^i$, detonation profiles, if they exist, are unique up to translation, decaying exponentially to $(u_+, z_+)$ as $x \to +\infty$. Moreover, weak and strong detonation profiles decay exponentially to $(u_-, z_-)$ as $x \to -\infty$, while Chapman–Jouget detonation profiles generically decay algebraically, as $c/|x|$. For fixed $u_+$, $s$, there may exist a weak or a strong detonation, but not both. Neither weak nor strong deflagration profiles exist.

In the degenerate case that $u_+ = u_i$ or $u_+ = u^i$, the center manifold of rest point $(u_+, z_+)$ consists of equilibria only on the “upper” side $z > z_+$; indeed, it is not difficult to see that flow along the CM is attracting in either the detonation $(u = u_i)$ or deflagration $(u = u^i)$ case, with rate of approach slower than any algebraic order. For, approximating $z \sim 1$ and $u - u_+ \sim (z - z_+)(sq/\alpha_+)$, we may estimate the order of approach by consideration of the scalar ODE

\[
(z - z_+)' = (k/s)\varphi(u_+ + (sq/\alpha_+)(z - z_+)) \geq 0.
\]
In the detonation case $\alpha_+ < 0$, for example, this becomes $\tilde{z}' = (k/s)\varphi(u_i + (sq/|\alpha_+|)|\tilde{z}|)$, where $\tilde{z} := z - z_+$. Noting that $\varphi(u_i + h) = o(h^k)$ for any algebraic order $k$, we obtain the result by comparison with ODE $\tilde{z}' = |\tilde{z}|^k$ for $\tilde{z} \leq 0$. In particular, deflagrations now become possible in principle, while detonations now become possibly nonunique. More precisely, we have:

**Lemma 3.2.** In the degenerate case $u_+ = u_i$, strong detonation orbits, if they exist, occur as a one-parameter family of subalgebraically decaying orbits bounded on the upper side by a unique exponentially decaying strong detonation, and on the lower side by a pair of orbits consisting of a gas-dynamical shock orbit followed by a subalgebraically decaying weak detonation orbit. Decaying exponentially to $(u_+, z_+)$ as $x \to +\infty$. Weak detonation profiles, if they exist, are still unique up to translation, but may be either subalgebraically decaying as $x \to +\infty$, in which case they bound a one-parameter family of strong detonation orbits as described above, or exponentially decaying as $x \to +\infty$, in which case there is no strong detonation profile connecting to $u_+$. In the degenerate case $u_+ = u^i$, there always exists a unique, weak deflagration profile, decaying subalgebraically as $x \to +\infty$; strong deflagration profiles, however, do not exist. The Chapman–Jouget cases are similar, but generically exhibit algebraic decay $\sim c/|x|$ as $x \to -\infty$, as described in Lemma 3.1.

We defer the proof to the following subsection. Exponential decay is what is needed to apply the general Evans function machinery of [GZ]. The “sonic” case for shock waves, analogous to the Chapman–Jouget case has been treated in [H.1–3,HZ] by explicit calculation, and it appears likely that this approach should generalize to combustion; however, we will not treat that issue here, ignoring for the moment the case of Chapman–Jouget detonations in our Evans function analyses. In the case that $u_+ = u_i$ or $u^i$, for which profiles in general approach $(u_+, z^+)$ as $x \to +\infty$ at subalgebraic rate, it is not possible by current techniques to define the stability index, and a different approach must be taken in analyzing stability.

**Phase Plane Analysis.** A more detailed description of existence properties may be obtained by examination of the phase plane, as depicted in Figure 2, below. Following [LY], consider the nullcline

\[(3.4) \quad F(u) := f(u) - f(u_+) - s(u - u_+) - sq(z - z_+) = 0\]

on which $u' \equiv 0$. By convexity of $f$, this intersects the $z = 0$ line at the two points $u_-$ and the $z = 1$ line at the two points $u_+$ described in Propositions 2.1 and 2.2; from here forward, denote these by $u_+ < u_- < u^- < u^+$. By positivity of $sq$, the nullcline $F = 0$ is a graph over $u$, whence on the physical, invariant region $z \geq 0$, the region $F \leq 0$ above the nullcline is attracting (recall, $z' \geq 0$ for $z \geq 0$) so that any flow entering this region for $u > u_i$ remains there and (since $u' = F < 0$) eventually crosses the vertical half-line $u = u_i$, $z > 0$. The value $z = \hat{z}$ at which it strikes at $u = u_i$ will be the terminal $z$-value, since $z' \equiv 0$ for $u \leq u_i$, and the
terminal $u$-value $\hat{u}$ will be the unique point $\hat{u} < u_i$ at which the nullcline intersects $z = \hat{z}$.

From these basic considerations, we may conclude already that strong deflagrations are under no circumstances possible, since these require a passage from $u_-$ to $u^+$, and these points are separated on $z \geq 0$ by the region $F \leq 0$ from which orbits can escape only to the left. Weak deflagrations $u^- \to u^+$, if they exist, must lie entirely within the complement $F > 0$, hence are monotone increasing in both $z$ and $u$. It is easily seen that they in fact exist if and only if $u^+ = u^i$ (recall, we have already shown that they do not exist for $u^+ > u^i$). More generally, there always exists a connection from $(u^-,0)$ to the rest point $(u^i,z^*)$ determined by the intersection of the nullcline $F = 0$ and the vertical line $u = u^i$. For, tracing backward along the center manifold leading to $(u^i,z^*)$, we see that it is trapped in $F \geq 0$, $z \geq 0$, hence must terminate at the corner $(u^-,0)$ of this wedge.

Likewise, weak detonations $u_- \to u_+$, if they exist, are monotone in both $z$ (increasing) and $u$ (decreasing), lying entirely within $F \leq 0$. For, the unique orbit originating from saddle $u_-$ into $z \geq 0$ lies along the unstable manifold, pointing into the region $F \leq 0$, from which it can exit only at a rest point along the lefthand edge of the nullcline $F = 0$. Moreover, denoting by $\hat{z}$ the value of $z$ at which this special orbit strikes $u_i$, we find that: (i) there exists a weak detonation profile if and only if $\hat{z} = 1$, necessarily unique. (ii) there exists a unique strong detonation profile if and only if $\hat{z} < 1$. (iii) there exists a one-parameter family of degenerate strong detonation profiles if and only if $u_+ = u_i$, $\hat{z} = 1$, and the weak detonation profile approaches $(u_+,1)$ along the center manifold; if the weak detonation profile approaches along the stable manifold, then there are no other connections. (Note that $\hat{z} \geq 1$ when $u_+ = u_i$, since the nullcline is a lower barrier for the unstable manifold.) (iv) for $\hat{z} > 1$, there exist no detonation profiles, neither weak nor
strong.

For, in case (iv), this orbit (the unstable manifold from \((u-,0)\)) serves as a barrier separating \((u+,1)\) from all orbits in \(z > 0\) originating from \((u^+,0)\). This is also the case when there exists a weak detonation profile connecting to \((u+,1)\) along the stable manifold. If \(\hat{z} \leq 1\) and the unstable manifold of \((u-,0)\) does not coincide with the stable manifold of \((u+,1)\), then the latter must lie entirely above the former; tracing it backwards, we find that it must either approach the rest point \((u^-,0)\) within \(F \leq 0\), corresponding with a monotone strong detonation profile, or else exit \(F \leq 0\) along its righthand edge, in which case they must approach \((u^-,0)\) from within the wedge \(F \geq 0, z \geq 0\), by the same argument used to prove existence of weak deflagrations. If, along with this nondegenerate strong detonation profile, there exists also a degenerate weak detonation profile entering \((u+,1)\) along a center manifold, i.e., \(u_+ = u_i\) and \(\hat{z} = 1\), then trapped between these two orbits is a one-parameter family of degenerate strong detonation profiles also entering along the center manifold. This verifies claims (ii)–(iv), while claim (i) is evident.

We have now verified the claims of Lemma 3.2; more, we have characterized the existence of weak and strong detonation profiles in terms of the height \(\hat{z}\) at which the unstable manifold of \((u-,0)\) strikes \(u_i\) (and in degenerate cases also the direction): equivalently, in terms of the Melnikov separation function

\[
d(u_+, s, k, q) := \hat{z} - 1.
\]

The following monotonicity properties will be helpful in completing our description of existence.

**Proposition 3.3.** The function \(d(\cdot)\), more generally, the height of the unstable manifold of \((u-,0)\) at any value \(u_i \leq u < u_-\), is monotone nondecreasing in \(u_+, k\), and, for \(\hat{z} \leq 1\), in \(q\), and is nonincreasing in \(s\). Moreover, this monotonicity is strict except in the special case that \((u_i, \hat{z})\) is a rest point, i.e., lies on the nullcline \(F = 0\).

**Proof.** To establish monotonicity, we show in each case that the vector field \((F, G), G := (k/s)\varphi(u)z\), determining the flow of the traveling wave ODE, within the region \(F \leq 0, z \geq 0, u_i < u < u_-\) rotates clockwise strictly monotonically with respect to the parameter \(\beta\) under consideration, or, equivalently,

\[
d_\beta := \det \begin{pmatrix} F & \partial F / \partial \beta \\ G & \partial G / \partial \beta \end{pmatrix} \leq 0,
\]

with \(< 0\) corresponding to monotone increase, and \(> 0\) to monotone decrease. For example, \(d_k\) is simply \(FG/k < 0\), \(d_q\) is \(Gs(z-z_+) < 0\), and \(d_{u_+} = G\alpha_+ < 0\), by the fact that \(\alpha_+ < 0\) (weak detonation). Finally,

\[
d_s := \det \begin{pmatrix} F & -q(z-z_+) - (u-u_+) \\ G & -(k/s^2)\varphi(u)z \end{pmatrix} = \det \begin{pmatrix} F & (1/s)(F - (f(u) - f(u_+))) \\ G & -(1/s)G \end{pmatrix} = (-G/s)(2F - (f(u) - f(u_+))) > 0.
\]
This gives strict comparison principles on $u_i < u < u_-$, not only for the unstable manifolds, but for any orbits lying in $F \leq 0, z \geq 0$ ($1 \geq z \geq 0$ in the case of $q$), $u_i < u < u^i$. Noting that $\partial u_-/\partial s < 0$, $\partial u_-/\partial q$, $\partial u_-/\partial u_+ > 0$, and $\partial u_-/\partial k = 0$, we thus obtain the claimed strict monotonicity on $u_i < u < u_-$. It follows that there is an orbit lying strictly between two stable manifolds under consideration for $u_i < u < u^i$. Noting that $\partial u^- - /\partial s < 0$, $\partial u^- /\partial q$, $\partial u^- /\partial u_+ > 0$, and $\partial u^- /\partial k = 0$, we thus obtain the claimed strict monotonicity on $u_i < u < u^i$. It follows that there is an orbit lying strictly between two stable manifolds under consideration for $u_i < u < u^i$, and this implies strict monotonicity at $u = u_i$ unless this third orbit collides with the stable manifold at $u_i$, which can only happen if $(u_i, z_*)$ is a rest point.

(Note: the above argument applies also in the Chapman–Jouget case.)

**Remark 3.4.** In the nondegenerate case $u_+ \neq u_i$, we may express the derivatives of $d$ at a profile $d = 0$ in the standard way (see, e.g., [GH,HK]) as Melnikov integrals

$$
\frac{\partial d}{\partial \beta} = -\int_{-\infty}^{\infty} e^{-\int_0^x \operatorname{Tr}(\partial(F,G)/\partial(u,z))(\bar{u},\bar{z})(y))dy} \det \begin{pmatrix} F & \partial F/\partial \beta \\ G & \partial G/\partial \beta \end{pmatrix}(\bar{u}, \bar{z})(x)dx,
$$

and

$$
= \int_{-\infty}^{\infty} e^{-\int_0^x (\alpha + k\varphi)(\bar{u}(y))dy} \det \begin{pmatrix} \bar{u}_x & \bar{u}_x - u_+ \\ \bar{z}_x & \bar{z}_x - z_+ \end{pmatrix}(x)dx.
$$

**Remark 3.5.** Alternatively, following [M.4], we may fix the lefthand state state $u_-$ and consider $d$ as a function of $(k, q, u_-, s)$. It is a straightforward exercise to verify that $d$ then becomes monotone nondecreasing in $u_-$ and $k$, but nonincreasing in $q$ (now for all values of $\hat{z}$). Monotonicity in $s$ is lost.

Our main conclusions regarding existence are summarized in the following two propositions.

**Proposition 3.6.** For each $u_+$, there holds one of two possibilities: either (i) for each speed $s$ greater than or equal to the minimal, Chapman–Jouget detonation speed $s^*$ for which detonations can occur, there exists a single exponentially decaying strong detonation profile, or (ii) there exists a threshold speed $\hat{s} \geq s^*$ at which there exists a (necessarily) unique exponentially decaying weak detonation profile, above which there exists a single exponentially decaying strong detonation profile, and below which there exists neither weak nor strong profile. Where it is defined, $s^*$ is monotone increasing in $u_+$. If $u_+ \neq u_-$, then strong detonation profiles, when they exist, are the only detonation profiles that occur. If $u_+ = u_-$, on the other hand, then whenever there exists an exponentially decaying strong detonation, it bounds from above a degenerate family of strong detonation profiles, bounded below by a
degenerate weak detonation profile and a gas-dynamical shock profile, as described in Lemma 3.2. Weak detonations are always monotone in \( u \) (decreasing) and \( z \) (increasing).

Weak deflagrations exist if and only if \( u^+ = u^i \), and are monotone increasing in both \( u \) and \( z \); moreover, they are always degenerate, decaying sub-algebraically as \( x \to +\infty \). Strong deflagrations do not occur.

Proposition 3.7. For fixed model parameters \( f, \varphi, q \), there exists \( k_0 > 0 \) such that, in Proposition 3.6, only case (i) occurs for \( 0 < k < k_0 \), while, for \( k \geq k_0 \), case (ii) always occurs for some choice of \( (u_+, s) \).

Proof of Proposition 3.6. It is straightforward to see, in the infinite-speed limit \( u^- \to u_+ + q \), that \( \hat{z} < 1 \) if \( u_+ \neq u^i \), and \( \hat{z} = 1 \) if \( u_+ = u^i \), with the weak detonation profile entering \( u_+ = u^i \) along the center manifold. If \( u_+ + q < u^i \) is out of range, then we consider instead the maximum speed limit \( u^- \to u^i \), observing that \( \hat{z} \to 0 \) is again less than 1. As speed \( s \) is decreased, \( \hat{z} \) increases monotonically, so that either \( \hat{z} \leq 1 \) for all \( s \), with the unstable manifold of \( u^- \) lying always below the stable manifold of \( u_+ \), or else there is a unique transition point \( s^* \) as described in the theorem at which the two manifolds coincide, below which \( \hat{z} > 1 \) and there are no detonation connections. That \( s^* \) is monotone in \( u_+ \) follows from the fact that \( \hat{z} \) decreases with \( s \) but increases with \( u_+ \). The remaining conclusions about detonations, and the conclusions about weak and strong deflagrations, follow from the discussion above Proposition 3.3.

Proof of Proposition 3.7. As \( k \to 0 \), the unstable manifold of \((u_-,0)\) approaches the nullcline \( F = 0 \), as can be seen either by direct calculation, or using singular perturbation techniques as in [GS.1]. Likewise, the stable manifold of rest point \((u^i, z^*)\) (recall: the intersection of the nullcline and \( u = u^i \)) approaches a horizontal line. Thus, the unstable manifold of \( u_- \) is trapped between the nullcline and the stable manifold of \((u^i, z^*)\), hence must approach \((u^i, z^*)\) along the center manifold tangent to the nullcline, decaying subalgebraically as \( x \to +\infty \); see the discussion of case \( u_+ = u^i \) for details. (In particular, it connects to \((u_+, 1)\) if and only if \( u_+ = u^i \).) From these observations, and \( \hat{z} = z^* \leq 1 \), we find that we are in case (i). Moreover, this argument is uniform in model parameters, yielding a global result for \( k \) less than some threshold \( k_0 \) as described. That case (ii) occurs for all \( k \geq k_0 \) follows by monotonicity of \( \hat{z} \) with respect to \( k \).

Section 4. Riemann solutions and the CJ shift.

We conclude our discussion of existence by cataloging solutions of Riemann problems involving waves with viscous profiles. In a Riemann problem, we prescribe data \( U_L = (u_L, z_L) \) for \( x \leq 0 \) and \( U_R = (u_R, z_R) \) for \( x > 0 \), and seek a sequence of waves \((U_L, U_1), (U_1, U_2), \ldots, (U_m, U_R)\) with increasing speeds, progressing from
left state $U_L$ to right state $U_R$, consisting of gas-dynamical shock or rarefaction waves, weak or strong detonations, or weak deflagrations, each possessing a viscous profile, for which the endstates $U_j = (u_j, z_j)$, $j = 1, \ldots, m$, $R$ are individually valid time-asymptotic states, i.e., $z_j = 0$ if $u_i < u_j < u^i$. As discussed, e.g., in [AMPZ], such solutions represent possible time-asymptotic states for solutions of (2.1), (2.2), as obtained by formal, matched asymptotic expansion with the usual, hyperbolic scaling $x, t \to \varepsilon x, \varepsilon t$.

By the one-sided nature of the Majda model, it is easily seen that the only interesting situation is the one $z_L = 0, z_R = 1$ described in the introduction. For, the reverse situation $z_L = 1, z_R = 0$ clearly admits no solution, there being no waves along which $z$ increases, nor, for the same reason, does the situation $z_L = z_R = 1$ with $u_L$ and $u_R$ lying in different components of $(-\infty, u_i] \cup [u^i, +\infty)$. In these cases, the asymptotic behavior has a different, diffusive scaling not captured by the Riemann solution. For $z_L = z_R = 0$ on the other hand, or $z_L = z_R = 1$ and $u_L$ and $u_R$ lying in the same component of $(-\infty, u_i] \cup [u^i, +\infty)$, the solution is just the gas-dynamical solution for the conservation law $u_t + f(u)_x = u_{xx}$ with the value of $z$ held fixed. We therefore fix $z_L = 1, z_R = 0$ for the remainder of our discussion.

**Case I. (deflagration: $u_L < u^i \leq u_r$).** Denote by $u_{CJ}$ the Chapman–Jouget deflagration speed associated with $u^+ = u_R$. Then, there are two subcases. *Ia.* For $u_{CJ} < u_L < u^i$, the solution consists of a weak deflagration from $u_L$ to $u^i$, followed by a (possibly zero strength) fluid-dynamical rarefaction with $z \equiv 1$ from $u^i$ to $u_R$. *Ib.* For $u_L < u_{CJ}$, the solution consists of a (possibly zero strength) fluid-dynamical rarefaction from $u_L$ to $u_{CJ}$, followed by a Chapman–Jouget deflagration from $u_{CJ}$ to $u^i$, followed by a fluid-dynamical rarefaction with $z \equiv 1$ from $u^i$ to $u_R$. Note that weak deflagrations may be followed by gas-dynamical waves, whereas the CJ deflagration may be both followed by and preceded by another wave, so long as the preceding wave is a rarefaction.

**Case II. (standard detonation: $u_L < u^i, u_r < u_i$).** In this case, the solution structure depends on whether we are in case (i) or case (ii) as described in Proposition 3.6. Denote by $u_{CJ}$ the Chapman–Jouget detonation speed associated with $u^+ = u_R$. If we are in case (i), then there are two subcases. *IIa.* For $u_{CJ} < u_L < u^i$, the solution consists simply of a strong detonation from $u_L$ to $u_R$. *IIb.* For $u_L \leq u_{CJ}$, the solution consists of a (possibly zero strength) gas-dynamical rarefaction from $u_L$ to $u_{CJ}$, followed by a Chapman–Jouget detonation from $u_{CJ}$ to $u_R$.

If we are in case (ii), there are again two subcases, but with the special role of the CJ detonation, which no longer has a profile, now played by the unique weak detonation possessing a (nondegenerate) profile. Denote by $u_\ast \leq u_{CJ}$ the special left state $u_-$ that is connected to $u_R$ by a nondegenerate weak detonation profile with speed $s_\ast$, and $u^\ast \geq u_{CJ}$ the corresponding value of $u^-$: i.e., the unique state that is connected to $u_\ast$ by a gas-dynamical shock with the same speed $s_\ast$. *IIa*. For $u^\ast < u_L < u^i$, the solution consists simply of a strong detonation from $u_L$ to $u_R$. *IIb*. For $u_L \leq u^\ast$, the solution consists of a (possibly zero strength) gas-dynamical
shock or rarefaction, according as $u_L$ is $\geq$ or $\leq u_*$, from $u_L$ to $u^*$, followed by a weak detonation from $u_*$ to $u_R$.

**Case III. (limiting detonation: $u_L < u^i$, $u_r = u^i$).** In this degenerate case, there exist, along with each nondegenerate strong detonation profile from $u^-$ to $u_i$, a degenerate weak detonation profile from $u_-$ to $u_i$. This leads to nonuniqueness of Riemann solutions for $u_L$ on the interval between $u_{CJ}$ or $u^*$ (in case (i) or (ii), respectively) and $u^i$. Namely, along with the solutions described in case II, we also have solutions obtained by substituting for any strong detonation with speed $s > s^*$ from $u^-$ to $u_+$ a gas-dynamical shock with speed $\leq s$ from $u^-$ to $\hat{u} \leq u_-$, followed by a degenerate weak detonation with speed $\hat{s} \geq s > s^*$ from $\hat{u}$ to $u_+$. (recall: both strong detonation and degenerate weak detonation profiles exist for any speed $> s^*$). There is a milder nonuniqueness at the level of profiles, since any nondegenerate strong detonation profile may be replaced by a degenerate one with the same endstates.

**Case IV. ($u_L \geq u^i$, $u_R > u_i$).** It is easily deduced that this problem has no solution. For, in order that $z$ increase from 0 to 1, there must be an intermediate state within $(u_i, u^i)$, and this can only be reached from $u_L$ by a single gas-dynamical shock. The only wave that can follow a shock is a weak detonation, and no wave can follow a detonation. But, $u_R$, as the right endstate of a weak detonation must then satisfy $u_R \leq u_i$, a contradiction.

**Case V. ($u_L \geq u^i$, $u_R < u_i$).** By the discussion of the previous case, the only possible solution is a gas-dynamical shock from $u_L$ to $u_1$, followed by a weak detonation from $u_- = u_1$ to $u_+ = u_R$ having the special property $u_+ \leq u_i < u_- < u^i \leq u_L \leq u^-$. This never occurs in detonation case (i), but may occur in case (ii) within certain parameter range.

Note that the solvable cases I–III contain the basic propagation problem described in the introduction, of a combustion wave moving from a burned, superignition state at $-\infty$ into an unburned, subignition state at $+\infty$. Likewise, cases II-III and IV contain the ignition problem $U_L = (u_L, 0)$, $U_R = (u_R, 1)$, where $u_L, u_R \leq u_i$ or $u_L, u_R \geq u^i$: a one-sided version obtained by left-right symmetry of the ignition problem $U_L = (u_L, 1) = U_R$ for the full, reacting Navier–Stokes equations, corresponding to a large, pulse-type excitation, initiating combustion, of a quiescent background state. We do not have a useful interpretation of case V. In cases I–II, the Riemann solution is uniquely determined within the class of viscous profiles, while in case III it is not. If we restrict to the class of nondegenerate (exponentially decaying) detonation profiles, then the Riemann solution is uniquely determined also in case III. A special role in the solution structure is played by the detonation profile with minimum speed: the CJ detonation, in case (i), the weak detonation profile with speed $s^*$, in case (ii).

Unique solvability in the class of exponentially decaying profiles suggests that nondegenerate detonation profiles, both weak and strong, at least in usual circumstances are stable. Change in stability would presumably signal bifurcation to more complicated time-asymptotic behavior, perhaps involving time-oscillatory “gallop-
Remark 4.1. (CJ shift) In both propagation and ignition Riemann problems considered above, appearance of a weak detonation profile, case (ii), is associated with a shift in the solution structure from CJ to weak detonation, in contrast with the behaviour predicted by the ZND model. Such a shift is indeed observed in experiments, for appropriate parameter regimes [FD]. Proposition 3.7 states that no such shift occurs for $k < k_0$: that is, it validates the ZND picture for $k$ merely small, and not only in the $k \to 0$ limit. This important observation was made first in [GS.1], in the larger context of the full reacting Navier–Stokes equations, and answers in the negative a conjecture of Majda [M.4].

Remark 4.2. As should be apparent from the above discussion, it is monotonicity of $d$ with respect to $s$ that leads to unique solvability of the Riemann problem in the presence of weak detonation profiles (detonation case (ii)). Likewise, a local analysis in the spirit of [ScSh,ZPM,Fre.1,GZ,ZS] shows that $\partial d/\partial s \neq 0$ is equivalent to linearized well-posedness of the Riemann problem about weak detonation data.

Section 5. Construction of the Evans function.

We now turn to the question of stability, beginning by a careful construction of the Evans function. Consider a nondegenerate, i.e., spatially exponentially decaying traveling wave profile $(\bar{u}(x-st))$ of any type, satisfying the nonsonicity assumption $a_+ \neq s$. The linearized equations of (2.1), (2.2) about $\bar{u}(x-st)$, in moving coordinates $\tilde{x} = x-st$, are

\begin{align}
\{ \begin{array}{l}
u_t - q(k\varphi'(\bar{u})u\bar{z} - k\varphi(\bar{u})z) + (\alpha u)_x = u_{xx} \\
z_t - sz_x = - k\varphi'(\bar{u})u\bar{z} - k\varphi(\bar{u})z,
\end{array} \tag{5.1} \end{align}

\begin{align}
\{ \begin{array}{l}
u_t + qz_t + (\alpha u)_x - sqz_x = u_{xx} \\
z_t - sz_x = - k\varphi'(\bar{u})u\bar{z} - k\varphi(\bar{u})z,
\end{array} \tag{5.2} \end{align}

where

\begin{align}
\alpha := a(\bar{u}) - s, \quad a(u) = df(u). \tag{5.3} \end{align}

The associated eigenvalue equation is thus

\begin{align}
\{ \begin{array}{l}
u'' = (\alpha u)' + \lambda u - q(k\varphi'(\bar{u})u\bar{z} + k\varphi(\bar{u})z), \\
z' = (1/s)(k\varphi'(\bar{u})u\bar{z} + k\varphi(\bar{u})z + \lambda z),
\end{array} \tag{5.4} \end{align}

---

\footnote{In the Remark below Theorem 1 of the reference, the conjecture that $q_{0}^{CR} > \hat{q}$ for any $k_0 > 0$: in our notation, that $\hat{z} > 1$ for $u_+ = u_i$, for any $k > 0$, which would imply the existence of nondegenerate weak detonation profiles for $u_+$ less than but sufficiently close to $u_i$.}
or, alternatively,
\begin{align}
(5.5) \quad \begin{cases} u'' = (\alpha u)' + \lambda u + q(\lambda z - s z'), \\ z' = (1/s)(k\varphi'(\bar{u})u\bar{z} + k\varphi(\bar{u})z + \lambda z), \end{cases}
\end{align}
and the limiting systems at \( x = \pm \infty \) are
\begin{align}
(5.6)_{\pm} \quad \begin{cases} u'' = \alpha_{\pm} u' + \lambda u - qk\varphi_{\pm} z, \\ z' = (1/s)(k\varphi_{\pm} z + \lambda z). \end{cases}
\end{align}
(Recall: \( \varphi_{-} > 0, z_{-} = d\varphi_{+} = \varphi_{+} = 0 \).) Here, \( \alpha_{\pm} \neq 0 \), by the nonsonicity assumption \( a_{\pm} \neq s \).

Seeking solutions \( u = e^{\mu x} U, z = e^{\mu x} Z \) of (5.6)\(_{\pm}\), we obtain the characteristic equation
\begin{align}
(5.7)_{\pm} \quad \begin{pmatrix} \mu^2 - \mu\alpha_{\pm} - \lambda & -qk\varphi_{\pm} \\ 0 & \mu - (k/s)\varphi_{\pm} - \lambda/s \end{pmatrix} \begin{pmatrix} U \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\end{align}
This is readily solved using its block triangular form, to yield growth rates (eigenvalues)
\begin{align}
(5.8)_{\pm} \quad \mu = \frac{\alpha_{\pm} \pm \sqrt{\alpha_{\pm}^2 + 4\lambda}}{2}, \quad \underbrace{(k/s)\varphi_{\pm} + \lambda/s}_{\geq 0},
\end{align}
associated, for all except finitely many points \( \lambda \) where eigenvalues may coincide, with respective normal modes of form
\begin{align}
(5.9)_{\pm} \quad \begin{pmatrix} u \\ u' \\ z \end{pmatrix} = e^{\mu x} \begin{pmatrix} U \\ \mu U \\ Z \end{pmatrix} = e^{\mu_{\pm} x} \begin{pmatrix} 1 \\ \mu_{\pm} \\ 0 \end{pmatrix}, \quad e^{\mu x} \begin{pmatrix} * \\ * \\ 1 \end{pmatrix}.
\end{align}

Inspection of formulae (5.8)\(_{\pm}\)-(5.9)\(_{\pm}\) yields

Lemma 5.1. On \( \{ \Re \lambda > 0 \} \), there are two unstable and one stable modes of (5.7)\(_{\pm}\) at either of \( \pm \infty \),
\( \mu_{1\pm} < 0 < \mu_{2\pm}, \mu_{3\pm} \);
moreover, there exist analytic choices of solutions of form
\begin{align}
\begin{pmatrix} u \\ u' \\ z \end{pmatrix} (\lambda) = \begin{pmatrix} u \\ \mu_{\pm} u \\ 0 \end{pmatrix}, \quad \begin{pmatrix} * \\ * \\ z \end{pmatrix}
\end{align}
spanning the associated stable/unstable manifolds. Moreover, each of these analytic functions has the complex symmetry \( \bar{f}(z) = f(\bar{z}) \).

Proof. The first claim is immediate from the formulae; the second then follows as described in [GZ,Z.3] by a standard lemma of Kato [K], which asserts that analytic subspaces on a simply connected domain possess an analytic choice of bases. The third statement follows exactly as in [GZ,Z.3] from the observation that the construction of Kato preserves complex symmetry.

Further, we have
Lemma 5.2. The functions $\mu_j^\pm, u_j^\pm$ and $z_j^\pm$ can be analytically extended onto a region
\begin{equation}
\Omega := \{ \Re \lambda \geq -\theta_1 - \theta_2 |\Im \lambda|^2 \}, \quad \theta_1, \theta_2 > 0.
\end{equation}
Moreover, $\mu_j$ are analytic on a neighborhood of $\lambda = 0$, with associated (distinct) analytic eigenvectors.

Proof. A closer look reveals that Lemma 5.1 holds true on $\Omega \setminus B(0,r)$ for arbitrary $r > 0$ and $\theta_j(r) > 0$ sufficiently small. Thus, it is sufficient to show that there exists an analytic extension on $B(0,r)$: in particular, to verify the second claim. At $x = -\infty$, $\varphi_- > 0$, and so, at $\lambda = 0$, the reactive root is $\mu = (k/s) + \lambda/s > 0$, while the kinematic roots are 0 and $\alpha_+ \neq 0$; thus, there is a spectral gap between the single zero mode and strong stable or unstable modes, and so each of these may be analytically continued through the origin. At $x = +\infty$ on the other hand, $\varphi_+ = 0$, and (5.7) becomes block diagonal, decoupling into kinematic and reactive blocks. Within blocks, the roots $\mu$ are distinct at $\lambda = 0$, and the result follows as before.

Remark. For the purposes of this paper, it is sufficient to establish Lemma 5.2 on the smaller set $\Omega = \{ \lambda : \Re \lambda \geq 0 \}$.

Applying the Gap lemma of [GZ,KS], we obtain, finally,

Corollary 5.3. There exist solutions $u_j^\pm(\lambda, x), z_j^\pm(\lambda, x)$ of eigenvalue equation (5.5), analytic on $\Omega, (5.10)$. Moreover, on $\{ \Re \lambda > 0 \}$, $(u_1^+, z_1^+)$ spans the stable manifold at $+\infty$, and $\{(u_2^-, z_2^-), (u_3^-, z_3^-)\}$ the unstable manifold at $-\infty$.

Definition: Following the development of [GZ] we define the Evans function as
\begin{equation}
D(\lambda) := \det \begin{pmatrix} u_1^+ & u_2^- & u_3^- \\ u_1' & u_2' & u_3' \\ z_1^+ & z_2^- & z_3^- \end{pmatrix} \bigg|_{x=0}.
\end{equation}

Proposition 5.4. The Evans function $D$ is analytic on $\Omega$; on the subdomain $\Re \lambda > 0$, its zeroes correspond precisely to eigenvalues of the linearized operator $L$. Moreover, it possesses complex symmetry $D(\bar{z}) = \bar{D}(z)$; in particular, $D(\lambda)$ is real for real $\lambda$.

Proof. Analyticity and complex symmetry are inherited from the properties of the columns of the matrix from which $D$ is evaluated. Recalling that, on $\Re \lambda > 0$, the columns of that matrix span the decaying manifolds of the eigenvalue equation at $\pm\infty$, we find, further, that vanishing of $D$ is equivalent to nontrivial intersection of these manifolds, i.e., existence of a solution decaying at both spatial infinities. Thus, zeroes of $D$ correspond in location with eigenvalues of $L$. That they correspond also in multiplicity follows from a more detailed calculation of Gardner and Jones [GJ.1–2]; see also Section 6 of [ZH].

---

3Here, we make use of the assumed spatial decay of the wave, a hypothesis of the Gap lemma.
Corollary 5.5. A necessary condition for linearized stability is that the number of zeroes of $D$ in the unstable half-plane $\{\lambda: \text{Re } \lambda > 0\}$ be zero; in particular, that it be even.

Section 6. Stability analysis for strong detonations.

Specializing now to the strong detonation case, $\alpha_- > 0 > \alpha_+$, we complete our stability analysis by a computation of the stability index $\Gamma := \text{sgn } D'(0)D(+\infty)$.

Analysis at $\lambda = 0$. At $\lambda = 0$, (5.5) reduces to the linearized traveling wave ODE

\[
\begin{align*}
(6.1) \quad &
\begin{cases}
u'' = (\alpha u)' - sqz', \\
z' = (1/s)(k\varphi'\bar{z}u + k\varphi(\bar{u})z).
\end{cases}
\end{align*}
\]

Take without loss of generality

\[
(6.2) \quad (u_1^+, z_1^+) = (u_3^-, z_3^-) = (\bar{u}_x, \bar{z}_x),
\]

both exponentially decaying modes. The solution $(u_2^-, z_2^-)$ is likewise asymptotically decaying at $-\infty$,

\[
(6.3) \quad u_2^-(\infty) = z_2^-(\infty) = 0,
\]

since, for $\alpha_- > 0$, the unstable modes at $-\infty$ remain strictly unstable as $\lambda \to 0$.

Integrating the first equation of (6.1) from $\pm \infty$ to $x$, we thus obtain the homogeneous first-order equation

\[
(6.4)_\pm \quad u' - \alpha u + sqz = 0,
\]

satisfied by each of $(u_1^+, z_1^+), (u_2^-, z_2^-), (u_3^-, z_3^-)$.

Proposition 6.1. $D(0) = 0$, while

\[
D'(0) = \gamma \Delta
\]

where

\[
(6.5) \quad \gamma = \lim_{M \to +\infty} \det \begin{pmatrix} u_2^- & \bar{u}_x \\ z_2^- & \bar{z}_x \end{pmatrix} (-M) e^{-\int_{-\infty}^{0} (k/s)\varphi(u)(x))}.
\]

and

\[
(6.6) \quad \Delta = [u] + q.
\]
Proof. $D(0) = 0$ is immediate from (6.2). Using the Leibnitz rule, we obtain as usual

\begin{equation}
D'(0) = \det \begin{pmatrix}
\bar{u}_x & u_2^- & y \\
\bar{u}_{xx} & u_2^- & y' \\
\bar{z}_x & z_2^- & \sigma
\end{pmatrix},
\end{equation}

where $y := y^- - y^+$, $\sigma := \sigma^- - \sigma^+$, and $(y^+, \sigma^+) := (\partial/\partial \lambda)(u_1^+, z_1^+)$, $(y^-, \sigma^-) := (\partial/\partial \lambda)(u_3^-, z_3^-)$, satisfy the variational equations (obtained by differentiating (5.5))

\begin{equation}
\begin{cases}
y^{\pm''} = (\alpha y^{\pm})' - sq\sigma^{\pm'} + \bar{u}_x + q\bar{z}_x, \\
\sigma^{\pm'} = (1/s)(k\varphi'()y^{\pm}\bar{z} + k\varphi(\bar{u})\sigma^{\pm} + \bar{z}_x),
\end{cases}
\end{equation}

with

\begin{equation}
(y^{\pm}, \sigma^{\pm})(\pm\infty) = 0.
\end{equation}

Thus

\begin{equation}
y^{\pm'} - \alpha y^{\pm} + sq\sigma^{\pm} = \bar{u} - u_\pm + q(\bar{z} - z_\pm),
\end{equation}

and

\begin{equation}
y - \alpha y + sq\sigma = [u] + q,
\end{equation}

where $[u] := (u_+ - u_-)$. Using (6.4)±, (6.10) to partially eliminate the 2nd row in (6.7), we obtain

\begin{equation}
D'(0) = -\det \begin{pmatrix}
\bar{u}_x & u_2^- \\
\bar{z}_x & z_2^-
\end{pmatrix} ([u] + q)
\end{equation}

where $\gamma := \det \left( \begin{array}{cc}
u_2^- & \bar{u}_x \\
z_2^- & \bar{z}_x
\end{array} \right) |_{x=0}$ is a Wronskian for the linearization about $(\bar{u}, \bar{z})$ of the traveling wave equation (2.10). The evaluation (6.5) then follows by Abel’s formula.$\blacksquare$

Analysis as $\lambda \to +\infty$. Similarly as in [GZ], we next establish
Proposition 6.2. As $\lambda \to \infty$ along the real axis,

\begin{equation}
\text{sgn } D(\lambda) \to \text{sgn } u_1^+(+\infty) \det \begin{pmatrix} u_2^- & u_3^- \\ z_2^- & z_3^- \end{pmatrix} (-\infty)
\end{equation}

Proof. Following [GZ], consider the frozen-coefficient version

\begin{equation}
\begin{cases}
u'' = \alpha(\bar{u})u' + \alpha(\bar{u})_x u + \lambda u - qk\varphi z, \\
z' = (1/s)(k\varphi'(\bar{u})u\bar{z} + k\varphi(\bar{u})z + \lambda z),
\end{cases}
\end{equation}

of (5.5) obtained by evaluating $\bar{u}$ at a fixed point $x_0$, and as usual seek solutions $u = e^{\mu x}U$, $z = e^{\mu x}Z$: that is, solutions of characteristic equation

\begin{equation}
\begin{pmatrix} \mu^2 - \mu\alpha - \alpha_x - \lambda & -qk\varphi \\
-(k/s)\varphi'\bar{z} & \mu - (k/s)\varphi - \lambda/s \end{pmatrix}
\begin{pmatrix} U \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\end{equation}

It is a straightforward exercise to show that, as $\lambda \to +\infty$ along the real axis, (6.15) has two kinematic solutions

\begin{equation}
\mu \sim \pm \lambda^{1/2}, \quad \begin{pmatrix} U \\ Z \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix},
\end{equation}

or $(u, u', z) \sim e^{\mu x}(1, \pm \lambda^{1/2}, 0)$ in phase plane variables, and one reactive solution

\begin{equation}
\mu \sim \lambda/s, \quad \begin{pmatrix} U \\ Z \end{pmatrix} \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\end{equation}

or $(u, u', z) \sim e^{\mu x}(0, 0, 1)$ in phase variables. Thus, the stable and unstable subspaces of the frozen-coefficient equations have a uniform spectral gap of order $\lambda^{1/2}$, and, carefully applying the tracking lemma of [GZ,ZH,Z.3], we may conclude that the stable and unstable manifolds of the variable-coefficient equations (5.5) track the corresponding subspaces of the frozen-coefficient equations to angle $\lambda^{-1/2}$. See [Z.3] and especially [MZ.3] for similar calculations.

From these considerations, we find that

\begin{equation}
\begin{pmatrix} u_1^+ \\ u_1'^+ \\ z_1^+ \end{pmatrix} \sim \begin{pmatrix} 1 \\ \sqrt{\lambda} \\ 0 \end{pmatrix} \eta_+(x),
\end{equation}

\begin{equation}
\begin{pmatrix} u_2^- \\ u_3^- \\ z_2^- \\ z_3^- \end{pmatrix} \sim \begin{pmatrix} 0 & 1 \\ 0 & -\sqrt{\lambda} \\ 1 & 0 \end{pmatrix} \eta_-(x),
\end{equation}
where $\eta_+(x)$ is a real nonvanishing scalar function, and $\eta_-(x)$ is a real nonsingular $2 \times 2$ matrix function. Thus,

$$
D \sim \det \begin{pmatrix}
1 & 0 & 1 \\
\sqrt{\lambda} & 0 & -\sqrt{\lambda} \\
0 & \lambda & 0
\end{pmatrix} \eta_+ \det \eta_-
$$

$$
= \eta_+ \det \eta_-
$$

$$
= u^+_1 \det \begin{pmatrix}
u_2^- & u_3^- \\
z_2^- & z_3^-
\end{pmatrix},
$$

from which we may deduce the claim by reality of $D$ and nonvanishing of $\eta_+, \det \eta_-$. Alternatively, following [BSZ], we may first deduce by by standard Gärding-type energy estimates that there exist no zeroes of $D$/eigenvalues of $L$ for sufficiently large real $\lambda$, uniformly, independent of (bounded) model parameters. Performing the homotopy $f, k \to 0$ reduces the eigenvalue equations to the simple case

$$
\begin{cases}
u'' = \lambda \nu, \\
z' = (\lambda/s)z,
\end{cases}
$$

for which the claim follows by an elementary direct calculation. ■

**The stability index.** Combining our calculations at 0 and $+\infty$, we have

**Corollary 6.3.** For nondegenerate strong detonations, the stability index $\Gamma := \sgn D'(0)D(+\infty)$ is well-defined, and satisfies

$$
(6.16) \quad \Gamma = \sgn ([\nu] + q)\gamma^2 \bar{u}_x(+\infty) > 0.
$$

Thus, the number of zeroes of $D$ on the unstable half-plane is even, consistent with stability.

**Proof.** $D(\lambda)$ is real for real $\lambda$, and, by the previous lemma, nonvanishing for real $\lambda$ sufficiently large. Thus, $\sgn D(\lambda)$ is independent of $\lambda$ for real $\lambda$ sufficiently large, and $\sgn D(+\infty)$ is well-defined. Moreover, since $D$ is analytic, the number of positive real roots of $D$ is even or odd according as $\Gamma$ is positive or negative (indeterminate parity if $\Gamma = 0$, corresponding to $D'(0) = 0$). In fact, since nonreal roots occur in conjugate pairs, by complex symmetry of $D$, the sign of $\Gamma$ determines the parity of the number of zeroes of $D$ on the entire unstable half-plane $\{\lambda : \text{Re} \lambda > 0\}$. Thus, it is sufficient to verify (6.16) in order to establish the claim.

To establish the formula $\Gamma = \sgn ([\nu] + q)\gamma^2 \bar{u}_x(+\infty)$, it is sufficient to show that expression (6.13) is independent of $\lambda$ for real $\lambda \geq 0$; evaluating at $\lambda = 0$ and combining with the formula of Proposition 6.1 for $D'(0)$, we then obtain the result. Here, we are using also the fact that

$$
\sgn \gamma = \sgn \det \begin{pmatrix}
u_2^- & \nu_3^- \\
z_2^- & z_3^-
\end{pmatrix}(0) = \sgn \det \begin{pmatrix}
u_2^- & \nu_3^- \\
z_2^- & z_3^-
\end{pmatrix}(-\infty) \neq 0
$$
as a nontrivial (since its columns are by construction independent as \( x \to -\infty \)) Wronskian for the linearized traveling-wave ODE.

To establish independence of (6.13) with respect to \( \lambda \), we may prove a projection lemma as in [GZ], namely that projection onto \( u \) coordinate is full rank on the unstable subspace, and projection onto \((u, z)\) coordinates is full rank on the stable subspace of the coefficient matrix of the limiting eigenvalue equation. This is straightforward in the present case, using the limiting structure \((1, \mu, 0)^t\) of unstable eigenvectors, and

\[
\begin{pmatrix}
1 & * \\
\mu & * \\
0 & 1
\end{pmatrix}
\]

of stable eigenvectors at \( x \to \pm \infty \).

Finally, the conclusion \( \text{sgn } ([u] + q) \gamma^2 \bar{u}_x(+\infty) > 0 \) follows from the facts that:
(i) \( \bar{u}_x(+\infty) < 0 \) for all strong detonation profiles, by our earlier phase plane analysis,
(ii) \([u] + q = (1/s)[f(u)] < 0\) by \([u] < 0, s > 0\), and the fact that \( f \) is strictly monotone increasing, and (iii) \( \gamma \neq 0 \). \qed

**Remark 6.4.** A similar result has been shown in the ZND limit \((k \text{ sufficiently small})\) in [Ly,LyZ], for the full, Navier–Stokes equations of reacting flow with an ideal gas equation of state. Multi-dimensional analogues are given in [Z.3, JLy.1–2].

**Remark 6.5.** A determinant analogous to the Evans function was used by Erpenbeck [Er.2, Er.4] to study stability of discontinuous strong detonation fronts within the framework of the ZND model; moreover, he obtained an interesting stability criterion by examining the high frequency limit, opposite to what we do (and outside the low-frequency regime in which the ZND model is expected to describe model (1.1)) [Er.3]; for further discussion, see [Z.3], Appendix A.3, or [JLy.2]. The stability criteria of Erpenbeck–Majda for shock waves [Er.1, M.1–3], of Erpenbeck for detonations [Er.1], and the Evans function criterion of Evans–Alexander–Gardner–Jones for reaction–diffusion equations [E.1–4, J.1, AGJ] are thus seen to be facets of the same general principle/stability criterion, applied to equations of varying type. For further discussion, see [ZS].

**Section 7. Stability analysis for weak detonations.**

We next consider the interesting weak detonation case \( \alpha_- < 0, \alpha_+ < 0 \), for which the speed of the detonation everywhere exceeds the rate of propagation of gas dynamical signals. These correspond to undercompressive, saddle–saddle connections and connect \( u_+ \) only to special states \( u_- \), as discussed in Section 3. As usual, we assume nondegeneracy, i.e., spatially exponential decay.

Repeating the stability analysis of the previous section, we now find that, at \( \lambda = 0 \), the mode \((u_2^-, z_2^-)\) is not asymptotically decaying, but asymptotically constant, \( (7.1) \)

\[
\begin{pmatrix}
u_2^- \\ u_2^- \\ z_2^-
\end{pmatrix} \longrightarrow \begin{pmatrix}1 \\ 0 \\ 0
\end{pmatrix}
\]
as \( x \to -\infty \) (see (5.7)). Thus for \( j = 2 \), we obtain in place of (6.4) the equation (recall, \( \alpha_- < 0 \))

\[
(7.2) \quad u_2' - \alpha u_2^- + sqz_2^- = -\alpha_- = |\alpha_-|.
\]

**Proposition 7.1.** For nondegenerate weak detonations, \( D(0) = 0 \), while

\[
(7.3) \quad D'(0) = (\partial d/\partial s)(u_+, s)|\alpha_-| < 0,
\]

where \( d(u_+, s) \) is the Melnikov separation function defined in (3.5), \( \partial d/\partial s \) is as given in (3.9), and \( \alpha \) as usual denotes \( a - s = df(\bar{u}) - s \).

**Proof.** Following the steps of the previous case, we obtain in place of (6.12):

\[
(7.4) \quad D'(0) = \det \begin{pmatrix}
\bar{u}_x & u_2^- & y \\
0 & |\alpha_-| & [u] \\
\bar{z}_x & z_3^- & \sigma \\
\end{pmatrix}_{|x=0},
\]

\( y, \sigma \) as defined previously. (Note: the only difference is in the new 2,2 entry \( |\alpha_-| \) coming from the righthand side of (7.2).) This in turn gives

\[
(7.5) \quad D'(0) = |\alpha_-| \det \begin{pmatrix}
\bar{u}_x & \bar{y} \\
\bar{z}_x & \bar{\sigma} \\
\end{pmatrix}_{|x=0},
\]

where \( \bar{y} := y - \left( \frac{|u|}{|\alpha_-|} \right) u_2^- \), \( \bar{\sigma} := \sigma - \left( \frac{|u|}{|\alpha_-|} \right) z_2^- \). But, \( I = \det \begin{pmatrix}
\bar{u}_x & \bar{y} \\
\bar{z}_x & \bar{\sigma} \\
\end{pmatrix}_{|x=0}, \)

just as in the standard, undercompressive shock case treated in [GZ], can be recognized as the Melnikov integral (3.9), by expressing \( I = I^- - I^+ \), where

\[
I^\pm := \begin{pmatrix}
\bar{u}_x & \bar{y}^\pm \\
\bar{z}_x & \bar{\sigma}^\pm \\
\end{pmatrix}_{|x=0},
\]

with \((\bar{y}^+, \bar{\sigma}^+) = (y^+, \sigma^+)\) and \((\bar{y}^-, \bar{\sigma}^-) = (y^-, \sigma^-) - ([u]/|\alpha_-|)(u_2^-, z_2^-)\), and observing that \( I^\pm \) (since do \( \bar{y}^\pm, \bar{\sigma}^\pm \)) satisfy the same ODE

\[
I' - (\alpha + k\varphi)I = \det \begin{pmatrix}
\bar{u}_x & (\bar{u} - u_+) + q(\bar{z} - z_+) \\
\bar{z}_x & (\bar{z}_x/s) \\
\end{pmatrix}_{|x=0},
\]

and using Duhamel’s principle to express \( I^\pm \) as the integral from \( \pm \infty \) to 0 of the integrand in the first line of (3.9); see, e.g., [GZ] for details. The evaluation in the second line follows from the identity \(-s(\bar{u} - u_+) - sq(\bar{z} - z_+) = \bar{u}' - (f(\bar{u}) - f(u_+))\), a rearrangement of the traveling-wave ODE, and elementary calculation. The sign of the integral then follows from \( \bar{z}_x > 0, \bar{u}_x < 0, \) and \( f' > 0 \).
Corollary 7.2. For nondegenerate weak profiles, there holds

\[ \Gamma = \text{sgn} \left| \alpha_- \left( \frac{\partial d}{\partial (u_+, s)} \right) \frac{\bar{u}_x(\pm\infty)}{\bar{z}_x(\pm\infty)} > 0, \right. \]

consistent with stability.

**Proof.** The result of Proposition 6.2 is independent of the case, and so the evaluation (6.13) of \( \text{sgn} \left. D(\pm\infty) \right| \) remains valid, as does the homotopy to \( \lambda = 0 \). However, \( \text{det} \begin{pmatrix} \bar{u}_x^- & \bar{u}_x^+ \\ \bar{z}_x^- & \bar{z}_x^+ \end{pmatrix} \) now becomes asymptotic to the explicitly prescribed endstates

\[
\text{det} \begin{pmatrix} 1 & \bar{u}_x^- \\ 0 & \bar{z}_x^- \end{pmatrix} (-\infty) = \bar{z}_x(-\infty),
\]

giving a combined result of \( \text{sgn} \left. D(\pm\infty) \right| = \text{sgn} \left. \bar{u}_x(\pm\infty) \bar{z}_x(-\infty) \right| < 0 \). Combining with our previous calculation of \( \text{sgn} \left. D'(0) \right| < 0 \), we obtain the result. □

**Remark 7.3.** As in [GZ], notice the relation between the key Melnikov integral \( \partial d/\partial s \) and well-posedness of the Riemann problem, Remark 4.2.

**Section 8. Stability of degenerate waves.**

Our methods, in the basic form presented here, require spatially exponential decay of the profile, so do not apply to the remaining class of degenerate, subalgebraically decaying profiles.\(^4\) Nonetheless, we can make one or two comments of a general nature regarding their stability.

*Degenerate detonation profiles.* Degenerate detonation profiles occur only in the special case \( u_+ = u_i \), and generically do not appear in Riemann solutions; in this sense, they may be considered a technical curiosity arising from the physically artificial cutoff at \( u_i \) of the ignition function (for further discussion, see Section 9). When they do appear, they occur as a one-parameter family of degenerate strong detonation profiles, bounded above by a nondegenerate, and presumably stable, strong detonation profile, and below by a nondegenerate gas-dynamical shock profile followed by a degenerate weak detonation profile.

This is somewhat reminiscent of the one-parameter families of overcompressive shocks studied for nonstrictly hyperbolic conservation laws in [Fre.2,L,FreL]. As in that situation, degenerate strong detonations cannot be singly orbitally stable, but only as a family; moreover, in the small viscosity limit (i.e., the true ZND limit, without rescaling), there is the same situation that the \( L^1 \) difference between profiles corresponding to fixed orbits goes to zero, so that stability if it exists at all cannot be expected to be uniform with respect to \( L^1 \) in the small viscosity limit. (Note that, since subalgebraically decaying, they are infinitely far in \( L^1 \) from the bounding, exponentially decaying strong detonation.) On the other hand, conservation of mass

\(^4\)See however [H.1–2,HZ,SS] for extensions to more general situations.
(in this case, of \( \int (u + qz) \, dx \)) no longer determines uniquely the time-asymptotic profile, and it is not clear that any degenerate profile should be expected to be stable under perturbation even for fixed viscosity.

In any case, at the level of Riemann solutions, degenerate strong detonations cannot be distinguished from the bounding strong detonation, so give no contradiction with the conclusions of [CF] based on the ZND model. On the other hand, as we saw in Section 4, existence of degenerate weak detonation profiles does lead to nonuniqueness in solutions of the Riemann problem for the special case \( u_+ = u_i \).

One might hope, therefore, that they could be discarded on grounds of instability, thereby validating ZND conclusions for \( k \) sufficiently small.

Degenerate deflagration profiles. Contrary to the situation of degenerate detonations, weak deflagrations are always of degenerate type, and play an important role in the solution of Riemann problems. In particular, stability of (degenerate) weak deflagration profiles is necessary for any \( k > 0 \) in order that there exist a general Riemann solution composed of stable viscous profiles, hence expected from the ZND point of view. Thus, stability vs. instability of these waves has important philosophical consequences, and is deserving of further study. We point out only that, discarding cross terms in the linearized equations, all remaining terms for weak deflagrations are favorable for a basic \( L^2 \) energy estimate, in contrast to the detonation case: the term \( (\alpha u)_x \), because \( \alpha_x \geq 0 \) (expansivity) due to monotonicity of the wave, and the term \( qkd\varphi\tilde{z}u \) because \( d\varphi \leq 0 \) in the deflagration regime.

The stability of degenerate waves of either type (detonation or deflagration) is likely to be sensitive due to the subexponential rate of decay of the background profile, as for example in the analogous case of KPP waves; see, e.g., [He].

Section 9. Extensions.

Finally, we discuss various elaborations that can be accommodated in the basic model without affecting the analysis:

Multi-species and multiple reactions. Combustion involving more than one reactant or reaction can be modeled abstractly, following [FD], by the use of progress variables

\[
\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m),
\]

each \( 0 \leq \lambda_j \leq 1 \) denoting the progress toward completion of a single idealized reaction. As described in [FD], the same framework can be used to describe arbitrarily complex reactions in a compact form, with the \( \lambda_j \) now representing linear combinations of progress variables for several simpler reactions. The progress variables satisfy a rate equation

\[
\dot{\Lambda} = R(\Lambda, u),
\]

modeling the composite chemical reactions, and enter the kinematic equations via

\[
(u - Q\Lambda)_t + (f(u)) = u_{xx},
\]
where \( Q = (q_1, q_2, \ldots, q_m) \) denote heats of reaction, \( q_j > 0 \) for \textit{exothermic reactions}, for \( q_j < 0 \) for \textit{endothermic reactions}.

For comparison, consider first a single reaction \( A \to B \). This is represented in the present notation by a single progress variable \( \lambda_1 \), with rate equation

\[
\dot{\lambda}_1 = k(\lambda_1, u)(1 - \lambda_1),
\]

a typical physical rate \( k \) being the Arrhenius rate

\[
k(\lambda_1, u) = e^{-\frac{E_0}{RT}},
\]

where \( T \) is temperature, \( R \) is the gas constant, and \( E_0 \) is \textit{activation energy}: the threshold determining ignition temperature. In the simple model considered previously, \( z \) corresponds to \( (1 - \lambda_1) \), and \( k \phi \) to the rate function \( k(\cdot, \cdot) \). Modification of the Arrhenius rate by a low-temperature cutoff \( T_i \) is a standard device in the combustion literature, circumventing the “cold boundary” problem that the Arrhenius kinetics do not permit a stationary unburned state and therefore preclude the existence of traveling wave profiles.

A more realistic model of combustion consisting of two, consecutive reactions, \( A \to B, B \to C \), is modeled by the rate equations

\[
\begin{align*}
\dot{\lambda}_1 &= k_1(\lambda, u)(1 - \lambda_1), \\
\dot{\lambda}_2 &= k_2(\lambda, u)(\lambda_1 - \lambda_2),
\end{align*}
\]

with either (i) \( q_1, q_2 > 0 \) (exothermic/exothermic), or (ii) \( q_1 > 0, q_2 < 0 \) (exothermic/endothermic). As above, a standard choice of rate function would be \( k_j = \varphi_j(u)k_j \), with \( k_j \) constant, and \( \varphi_j \) ignition functions of the usual form. As pointed out in [FD], the traveling wave structure in case (i) is very similar to that for a single reaction; on the other hand, (ii) introduces important new phenomena at the ZND level, explaining e.g. an observed experimental shift from CJ to weak detonation in ignition problem. See [FD], p. 168–173, for further discussion. Three or more reactions can easily be included, but do not appear to result in significant new phenomena [FD].

It is straightforward to show that the main conclusions of the previous sections remain valid for equations (9.6), or indeed any system of reactions having such a triangular structure: more generally, for any reaction system in which \( \Lambda = (1, \cdots, 1) \) is stable above the ignition temperature(s) as a rest point of rate equation (9.6). In particular, we recover the result for strong detonations that the stability index is always positive, consistent with stability. Likewise, we recover the same expression for \( \Gamma \) in the case of weak detonations, but now the Melnikov integral for \( (\partial d/\partial s)_{(u+),s} \) is more complicated and its sign (apparently) no longer explicitly evaluable. It would be very interesting to investigate structure and stability of traveling waves for the 2–reaction model above, especially in light of anomalous behavior (e.g., a shift in
the ignition problem from CJ to weak detonation) predicted in [FD,B] in the the ZND setting, in the high–activation energy limit.

**Reaction-dependent equation of state.** As pointed out in [CHT], a more realistic assumption is that the gas-dynamical equation of state (EOS) \( f \) depend not only on \( u \), but also on \( z \), through the chemical makeup of the gas. With this change, the linearized equations (5.1), (5.2) become

\[
\begin{align*}
    u_t - q(k\varphi'(\bar{u})u\bar{z} - k\varphi(\bar{u})z) + (\alpha u)_x + (\beta z)_x &= u_{xx} \\
    z_t - sz_x &= - k\varphi'(\bar{u})u\bar{z} - k\varphi(\bar{u})z,
\end{align*}
\]

(9.7)

\[
\begin{align*}
    u_t + qz_t + (\alpha u)_x + (\beta z)_x - sqz_x &= u_{xx} \\
    z_t - sz_x &= - k\varphi'(\bar{u})u\bar{z} - k\varphi(\bar{u})z,
\end{align*}
\]

(9.8)

where \( \alpha := f_u(\bar{u}, \bar{z})(x) - s \) and \( \beta := f_v(\bar{u}, \bar{z})(x) \). It is straightforward to check that all of our stability analysis goes through as before in this more general setting, to yield exactly the same geometric stability conditions. In particular, \( \Gamma > 0 \) holds always for a strong detonation profile if it exists. Likewise, the expression derived for \( \Gamma \) in the weak detonation case, including our calculation of the Melnikov integral \( \partial d/\partial s \), remains valid, though the sign of the integral may change depending on the details of the equation of state.

Regarding the existence problem, and the related question of stability of weak detonation profiles, we are led naturally to a simple condition,

\[
f_z \leq 0,
\]

(9.9)

under which all of the conclusions of the paper generalize to the case of a reaction-dependent EOS. For, this implies that \( F_z = f_z - qs < 0 \), where \( F = 0 \) as in (3.4) defines the nullcline \( u' = 0 \) for the traveling-wave equations, and so the nullcline is again a graph over \( z \). This was essentially the only property used in the qualitative analysis of the phase portrait, and so we recover all related conclusions, including the important property of monotonicity in \( u \) and \( z \) of weak detonation profiles. Likewise, (9.9) implies that \( f(\bar{u}, \bar{z}) - f(u_+, z_+) > 0 \) along weak detonation profiles, since \( f_u > 0 \), \( f_z \leq 0 \) and \( \bar{u} \) and \( \bar{z} \) are, respectively, monotone decreasing and increasing. Thus, we obtain the key monotonicity in \( s \) asserted in Proposition 3.3, now in the restricted case \( \bar{z} \leq 1 \), in particular, the conclusion that \( \partial d/\partial s < 0 \) at a profile \( d = 0 \), along with the other stated monotonicity results (the arguments for which are unaffected by dependence of \( f \) on \( z \)), and thereby the remaining conclusions of the paper.

On the other hand, when the monotonicity condition (9.9) is violated, we see no reason why anomalies in both existence and stability theory might not occur, and this seems an important direction for further investigation. In the full, reacting Navier–Stokes equations, the flux functions, taking gas composition into account,
are just the convex averages, weighted by $z$, of corresponding fluxes for pure unburned and pure burned gas, and similarly for the coefficient $c_v$ relating temperature to internal energy. We conjecture that the corresponding monotonicity condition in this physical setting reduces to the condition that the gas constant $\gamma(1)$ of the unburned gas be less than the gas constant $\gamma(0)$ of the burned gas, assuming that each separately obeys the EOS of an ideal, polytropic gas: i.e., from the kinetic theory of gases point of view, the average number of internal degrees of freedom per molecule $n$ decrease upon reaction (recall: $\gamma = (n + 2)/n$; see, e.g., [Ba], pp. 37–45). The derivation of an analog of condition (9.9) by analysis of the ZND limit we regard as another very interesting open problem.

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Department of Mathematics, University of Michigan, Ann Arbor, MI 48109-1109
E-mail address: glyng@umich.edu

Department of Mathematics, Indiana University, Bloomington, IN 47405-4301
E-mail address: kzumbrun@indiana.edu