Covariant three-body equations in $\phi^3$ field theory

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Abstract

We derive four-dimensional relativistic three-body equations for the case of a field theory with a three-point interaction vertex. These equations describe the coupled $2 \rightarrow 2$, $2 \rightarrow 3$, and $3 \rightarrow 3$ processes, and provide the means of calculating the kernel of the $2 \rightarrow 2$ Bethe-Salpeter equation. Our equations differ from all previous formulations in two essential ways. Firstly, we have overcome the overcounting problems inherent in earlier works. Secondly, we have retained all possible two-body forces when one particle is a spectator. In this respect, we show how it is necessary to also retain certain three-body forces as these can give rise to (previously overlooked) two-body forces when used in a $2 \rightarrow 3$ process. The revealing of such hidden two-body forces gives rise to a further novel feature of our equations, namely, to the appearance of a number of subtraction terms. In the case of the $\pi NN$ system, for example, the $NN$ potential involves a subtraction term where two pions, exchanged between the nucleons, interact with each other through the $\pi-\pi$ t-matrix. The necessity of an input $\pi-\pi$ interaction is surprising and contrasts markedly with the corresponding three-dimensional description of the $\pi NN$ system where no such interaction explicitly appears. This illustrates the somewhat unexpected result that the four-dimensional equations differ from the three-dimensional ones even at the operator level.

I. INTRODUCTION

Relativistic quantum field theory is the underlying basis of essentially all models in both nuclear and particle physics. Yet our knowledge of how to solve any but the most simple of models using such a field theory is extremely limited. In this respect, it is useful to recall that, for the case of one- and two-particle systems, the way to sum all possible perturbation graphs is known, though only formally. In particular, the coupled problems of single particle dressing, three-point vertex dressing, and $2 \rightarrow 2$ scattering, is described in terms of the Dyson-Schwinger and Bethe-Salpeter equations. It is important to recognize, however, that despite the

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"text-book" nature of these equations, they do not constitute a solution of the one- and two-particle problem. The crucial point is that the essential ingredient in these equations is the kernel of the $2 \rightarrow 2$ Bethe-Salpeter equation, but this kernel is not known. Although one can calculate contributions to this kernel up to some order in the coupling constant, this is not a satisfactory method for strong interactions where it is necessary to have a non-perturbative approach. In this paper we develop integral equations for the coupled $2 \rightarrow 2$, $2 \rightarrow 3$, and $3 \rightarrow 3$ processes, which, in the case of negligible three-body forces, also provide exact closed equations for the unknown Bethe-Salpeter kernel.

More generally, our goal is to formulate the field theoretic three-body problem where one-, two-, and three-particle states are explicitly coupled, and where three-body forces are taken as a given input (a reasonable first approximation is to take them to be zero). Unfortunately, the formulation of the four-dimensional three-body problem is not as straightforward as that for the two-body case. The basic problem is that "irreducibility", the essential concept used to derive both the Dyson-Schwinger and Bethe-Salpeter equations, cannot be applied to the case of three-particles directly - as we shall see, quite a sophisticated additional analysis of diagram structure is needed. In fact, we find that in all the previous works on this subject, such analyses of diagram structure are either incorrect, incomplete, or missing altogether. Thus, despite the fundamental nature of the problem, there does not appear to be any satisfactory formulation of the three-body problem in four-dimensional relativistic field theory. The goal of this paper, therefore, is to provide just such a formulation.

In order to develop equations for the field theoretic three-body system, we also utilise the concept of irreducibility. A Feynman diagram is said to be $r$-particle reducible if one can draw a continuous curve intersecting exactly $r$ lines, at least one of which is an internal line, and each line being crossed just once, so as to separate initial from final states. A diagram that is not $r$-particle reducible, is said to be $r$-particle irreducible. The Bethe-Salpeter equation then follows from the following argument. Firstly, all $2 \rightarrow 2$ Feynman diagrams are classified according to their 2-particle irreducibility. The sum of all 2-particle irreducible diagrams is defined to be the potential for the process. All the rest of the diagrams have at least one 2-particle cut; moreover, it is clear from the topology of such diagrams, that there must be a unique left-most (or right-most) 2-particle cut. To the left of this left-most cut is a 2-particle irreducible potential, and the Bethe-Salpeter equation follows immediately.

In a pioneering work, Taylor [1] applied such a procedure to the three-body system. He recognised that, unlike the case for 2-particle cuts, 3-particle cuts may not be unique. Particular attention was paid to the case illustrated in Fig. 1(a). Here is shown a diagram that is 2- and 3-particle reducible, and where it is possible to cut three internal lines in two different ways, neither of which precedes the other. It was recognised that to eliminate such non-unique internal 3-particle cuts, one
should remove the two-particle cuts first. Formalizing these considerations into his "last cut" lemma, Taylor used the scheme where 3-particle cuts are exposed only in 2-irreducible amplitudes, to derive four-dimensional relativistic equations for the field theoretic three-body problem.

However, no special attention was given to the case illustrated in Fig. 1(b). Shown is a diagram for the $2 \rightarrow 3$ process that is 3-particle reducible, but 2-particle irreducible. Shown also are two 3-particle cuts, neither of which is to the right of the other, i.e. there is no unique right-most 3-particle cut, despite the fact that two-particle cuts have been removed. This lack of uniqueness in the cutting procedure, necessarily involving at least one external particle, does not seem to have been taken into account by Taylor, or in a number of subsequent works that were based on his "last cut" lemma [2-5]; as a result, the three-body equations developed in all the above mentioned works suffer from the problem of overcounting.

The only previous work where this type of non-uniqueness problem was addressed, is the one by Tucciarone [6]. As a result, he developed a description of the four-dimensional three-body problem where there is no overcounting of diagrams. However, his equations were not expressed in terms of subsystem two-body amplitudes, rather, an explicit perturbation expansion for the inhomogeneous terms is needed. This makes his equations unsatisfactory from a practical point of view.

It should be noted, that the non-uniqueness difficulty does not arise for $3 \rightarrow 3$ processes where coupling to two-particle states does not take place, as for example in formulations of the relativistic four-dimensional three-nucleon problem [7]. This problem also does not arise in time-ordered perturbation theory since, for example, the two cuts in Fig. 1(b) would correspond to the two possible time orderings of the right-most vertices, both of which need to be taken into account.

In the present work, we restrict ourselves to the case of a $\phi^3$ field theory. This is the simplest case having wide applicability; however, the strategy we shall adopt is also applicable to other forms of the interaction. In particular, we overcome the non-uniqueness problem by a special procedure where, in cases like Fig. 1(b), one of the two rightmost vertices is "pulled out" further to the right. Our equations therefore do not suffer from the overcounting problem inherent in the above mentioned works.
A second, but equally important goal of this paper, is to take into account as many Feynman diagrams as possible, consistent with pair-like interactions. That this is not an altogether obvious task, is illustrated in Fig. 2. Fig. 2(a) shows a connected three-body force for the $3 \rightarrow 3$ process. Yet if we join any two initial or final legs into a single leg, i.e. if there is absorption through a $\phi^3$ vertex, then the resulting diagram can be represented through a pair-like interaction, as shown in Fig. 2(b). Since the equations we seek couple $3 \rightarrow 3$ to $3 \leftrightarrow 2$ processes, Fig. 2 illustrates how careful one must be in neglecting three-body forces, otherwise one can inadvertently neglect important two-body contributions as well. This subtlety does not appear to have been noticed before, as all the References [1-5] have neglected three-body forces without comment. Note, that in an otherwise correct formulation, the neglect of such three-body forces, leading to the undercounting of pair-like interactions, would result in equations that have the wrong three-dimensional limit; thus, they would have neither the correct non-relativistic limit, nor would they satisfy three-body unitarity.

In this work we present three-body equations that suffer neither from the overcounting nor the undercounting problems discussed above. A detailed derivation is presented for the case of three identical Bosons. The case of identical particles was chosen because it has all the complications of the most general case. It is also the most difficult case that can arise in practice, where, in particular, pair-like interactions are less straightforward to reveal.

We also present the equations for the case of the $\pi NN$ system where only two of the particles are identical. This system has commanded much attention in the literature although mainly in the three-dimensional sector [8]; in this respect, we note that the recently derived convolution equations of Ref. [9] can be viewed as the three-dimensional limit of the four-dimensional equations we present in this paper.

A novel feature of our equations is the presence of explicit subtraction terms necessary to avoid overcounting. In the $\pi NN$ equations, for example, one of the subtraction terms is the two-pion exchange $NN$ potential where the exchanged pions interact with each other. As the corresponding three-dimensional equations have only the $\pi N$ and $NN$ interactions as input, it is remarkable that a consistent four-dimensional description of the $\pi NN$ system also demands, as input, the $\pi-\pi$ interaction.
II. DERIVATION

For the $m \to n$ process the momentum space Green’s function is defined by

\[
\mathcal{G}(p'_{1} \ldots p'_{n}, p_{1} \ldots p_{m}) = \int \prod_{j=1}^{n} d^4x'_j \prod_{i=1}^{m} d^4x_i \exp \left[ i \sum_{j=1}^{n} x'_j p'_j - i \sum_{i=1}^{m} x_i p_i \right] < 0|T[\phi_{1}(x'_1) \ldots \phi_{n'}(x'_n)\phi_{1}(x_{1}) \ldots \phi_{m}(x_{m})S]|0 >.
\]

(1)

where the S-matrix $S$ is given in terms of the interaction Lagrangian $\mathcal{L}_I$ by

\[
S = T \exp \left[ i \int d^4x \mathcal{L}_I(x) \right].
\]

(2)

For clarity of presentation, we shall usually replace momenta by their labels in an obvious way; thus we write $\mathcal{G}(1' \ldots n', 1 \ldots m)$ instead of $\mathcal{G}(p'_{1} \ldots p'_{n}, p_{1} \ldots p_{m})$. For momentum conservation $\delta$ functions, we use a convention where, for example, $\delta(1'2'3', 12)$ denotes the function $\delta(p'_1 + p'_2 + p'_3 - p_1 - p_2)$. For the case $n = m$, it will also be useful to define $\mathcal{G}_0(1' \ldots n', 1 \ldots n)$ as that part of $\mathcal{G}(1' \ldots n', 1 \ldots n)$ corresponding graphically to $n$ completely disconnected lines connecting initial and final states.

For specific one-, two- and three-particle Green’s functions we shall write respectively

\[
g(1', 1) = \mathcal{G}(1', 1)
\]

(3)

\[
D(1'2', 12) = \mathcal{G}(1'2', 12)
\]

(4)

\[
G(1'2'3', 123) = \mathcal{G}(1'2'3', 123).
\]

(5)

We shall also need corresponding disconnected propagators where the momentum conserving $\delta$ functions have been factored out; in these cases we write

\[
g(1', 1) = d(1')\delta(1', 1)
\]

(6)

\[
D_0(1'2', 12) = d_1(1)d_2(2)\delta(1', 1)\delta(2', 2)
\]

(7)

\[
G_0(1'2'3', 123) = d_1(1)d_2(2)d_3(3)\delta(1', 1)\delta(2', 2)\delta(3', 3).
\]

(8)

For non-identical particles, we have that

\[
D_0(1'2', 12) = \mathcal{G}_0(1'2', 12)
\]

(9)

\[
G_0(1'2'3', 123) = \mathcal{G}_0(1'2'3', 123).
\]

(10)

We shall work also with t-matrix amplitudes defined in terms of the above Green’s functions by the relation

\[
\mathcal{T}(1' \ldots n', 1 \ldots m) = \prod_{j=1}^{n} d_{j'}^{-1}(j')\mathcal{G}'(1' \ldots n', 1 \ldots m) \prod_{i=1}^{m} d_{i}^{-1}(i)
\]

(11)
Thus we denote any quantity $A$ for the dressed $1 \rightarrow 2$ process, $t$ for the $2 \rightarrow 2$ process, $\Gamma$ for the $1 \rightarrow 3$ process, $M$ for the $2 \rightarrow 3$ process, and $T$ for the $3 \rightarrow 3$ process. Thus

$$f(1'2', 1) = T(1'2', 1)$$
$$t(1'2', 12) = T(1'2', 12)$$
$$\Gamma(1'2'3', 1) = T(1'2'3', 1)$$
$$M(1'2'3', 12) = T(1'2'3', 12)$$
$$T(1'2'3', 123) = T(1'2'3', 123).$$

The basic three-point vertex, without a momentum conserving $\delta$ function is defined as $h$, thus

$$f(1'2', 1) = h(1'2', 1)\delta(1'2', 1).$$

For the disconnected $2 \rightarrow 3$ process it will be sufficient to define

$$F_1(1'2'3', 12) = f(1'3', 1)d_2^{-1}(2)\delta(2', 2)$$
$$F_2(1'2'3', 12) = f(2'3', 2)d_1^{-1}(1)\delta(1', 1).$$

To save on notation further, we shall usually suppress particle labels completely. Thus we denote any quantity $A(1' \ldots n', 1 \ldots m)$ simply by $A$. Note that the order of labels in this definition is important. As occasionally a different order of labels will be needed, it will be useful to introduce the left and right "particle exchange" operators $L_{ij}$ and $R_{ij}$; when acting on any quantity $A$, these operators exchange the $i$th and $j$th labels on the left and right set of external legs, respectively. For example, we have that

$$F_1 = F_1(1'2'3', 12),$$
$$R_{12}F_1 = F_1(1'2'3', 21),$$
$$L_{13}F_1 = F_1(3'2'1', 12).$$

For reversed processes we use a bar notation; thus, $\bar{f}$ is the three-particle vertex for absorption, and $\bar{M}$ is the amplitude for the $3 \rightarrow 2$ process.

We shall also write integrals in symbolic form. For example, for any two quantities $B$ and $A$, describing processes $m \rightarrow k$ and $k \rightarrow n$ respectively, we define $AB$ by

$$AB(p'_{1} \ldots p'_{n}, p_{1} \ldots p_{m}) \equiv \int d^{4}p''_{1} \ldots d^{4}p''_{k} A(p'_{1} \ldots p'_{n}, p''_{1} \ldots p''_{k}, p_{1} \ldots p_{m})B(p''_{1} \ldots p''_{k}, p_{1} \ldots p_{m}).$$

Thus an equation of the form

$$C(p'_{1} \ldots p'_{n}, p_{1} \ldots p_{m}) = \int d^{4}p''_{1} \ldots d^{4}p''_{k} A(p'_{1} \ldots p'_{n}, p''_{1} \ldots p''_{k}, p_{1} \ldots p_{m})B(p''_{1} \ldots p''_{k}, p_{1} \ldots p_{m})$$

is the three-particle vertex.
will be written symbolically as just $C = AB$.

**A. Identical particle system**

We first consider the case of identical particles since this contains all the complications of the most general case. To be definite, we assume that the identical particles are Bosons and that the underlying interaction Lagrangian is given by

$$L_I(x) = \frac{\lambda}{3!} \phi^3(x). \tag{25}$$

Here $\lambda$ is the coupling constant and the factor $1/3!$ is chosen to cancel factors arising from the permutation symmetry of the fields in contractions when using Wick’s theorem. Having defined the model in which we shall work, it will now be assumed that all three-point vertices and particle propagators are dressed, so we consider the skeleton Feynman diagrams of a given process.

For identical Bosons, the Green’s function $G$ of Eq. (1) is symmetric under the interchange of either initial or final labels. In this case we still retain all the above definitions, so the full propagators $D$, $G$, and t-matrices $f$, $\Gamma$, $t$, $M$, and $T$ are likewise symmetric in their initial and final labels. Note, however, that the free propagators $D_0$ and $G_0$ do not have a specific symmetry, and the function $F_1$ is symmetric under the interchange of $i$ and $3$ momentum labels, but does not have a specific symmetry under the exchange of labels 1 and 2.

Quantities without a specific symmetry, will sometimes need to be symmetrized explicitly. Thus, for example, we have that

$$\sum_P D_0(1'2', 12) = G_0(1'2', 12) \tag{26}$$
$$\sum_P G_0(1'2'3', 123) = G_0(1'2'3', 123) \tag{27}$$

where the sum is over all permutations of either the initial or final momenta. More generally, we shall indicate sums over permutations of initial momenta by the letter $R$ (right), and sums over permutations of final momenta by the letter $L$ (left); however, we use the letter $P$ whenever it makes no difference which sum, $R$ or $L$, is taken. Quantities symmetrized in one of these ways are indicated by the appropriate superscript. Thus, for example, if $A$ has three initial and final legs, then

$$A^R(1'2'3', 123) = A(1'2'3', 123) + A(1'2'3', 213) + A(1'2'3', 321) + \ldots \tag{28}$$
$$A^L(1'2'3', 123) = A(1'2'3', 123) + A(2'1'3', 123) + A(3'2'1', 123) + \ldots \tag{29}$$
with similar expressions holding for \( A \) having any number of legs. In general, we can write

\[
A^R \equiv \sum_R A \\
A^L \equiv \sum_L A \\
A^P \equiv \sum_P A = A^R = A^L.
\]

In the case of three-particle states, the symbols \( R_c, L_c, \) and \( P_c \) will be used to indicate sums over the corresponding cyclic permutations of particles.

We shall utilise the idea of irreducibility described in the Introduction. The maximal irreducibility of an amplitude will be indicated by a superscript in round brackets; thus by \( T^{(r)} \) we mean the sum of all possible \( 3 \to 3 \) skeleton Feynman graphs that are simultaneously 1-, 2-, \ldots, \( r \)-particle irreducible.

The symmetry property of identical particle amplitudes necessitates the inclusion of various counting factors, as well as sums over permutations of particle labels whenever amplitudes do not already have the required symmetry. In the Appendix, we discuss the origin of such counting factors, and give particular examples of how to symmetrize amplitudes. As a result, we shall utilise such counting factors and symmetry operations in the discussion below, but with only a minimal comment.

1. The \( 3 \to 3 \) process

We begin by examining the structure of the full amplitude \( T \) for the \( 3 \to 3 \) process. We can expose the one-particle cut through the equation

\[
T = T^{(1)} + \Gamma^{(1)} g \bar{\Gamma}^{(1)}
\]

which is illustrated diagrammatically in Fig. 3. In the same way, we can expose the two-particle cut in \( T^{(1)} \)

\[
T^{(1)} = T^{(2)} + M^{(2)} \left[ \frac{1}{2} D_0 + \frac{1}{2} D_0 t^{(1)} \frac{1}{2} D_0 \right] \bar{M}^{(2)}
\]

\[
= T^{(2)} + \frac{1}{4} M^{(2)} \left[ D_0^P + D_0 t^{(1)} D_0 \right] \bar{M}^{(2)}. \quad (31)
\]

\[\begin{array}{c}
\includegraphics[width=0.2\textwidth]{figure3.png}
\end{array}\]

Figure 3: Graphical representation of the equation \( T = T^{(1)} + \Gamma^{(1)} g \bar{\Gamma}^{(1)} \).
Note that the factors of $1/2$ in the first of the above two expressions are associated with the symmetric nature of the amplitudes $M^{(2)}$ and $t^{(1)}$ under the interchange of the exposed two particles; more generally, exposing $n$ particles between appropriately symmetric amplitudes will result in a counting factor of $1/n!$ - see the Appendix for details. By exposing the first vertex in amplitude $\Gamma^{(1)}$, we can write it as

$$\Gamma^{(1)} = \frac{1}{2} M^{(2)} D_0 f$$

which is illustrated in Fig. 4. Note that because all our vertices are dressed from the beginning, the $2 \rightarrow 3$ amplitude in Eq. (32) is $1$ and $2$-particle irreducible. Substituting Eq. (31) into Eq. (30) and using Eq. (32) we obtain

$$T = T^{(2)} + \frac{1}{4} M^{(2)} D \bar{M}^{(2)}$$

where we have used the fact that

$$t = t^{(1)} + fg\bar{f}$$

and

$$D = D_0^P + D_0 t D_0.$$  

In Eq. (35) we can treat $t$ in two ways. In order to compare with some earlier works we would treat $t$ as a given input to the equations. In the second way we would expose three-particle states in $t$ and thereby obtain coupled equations for the $3 \rightarrow 3$ and $2 \rightarrow 2$ processes. Let us assume for the moment that $t$ is known. Then the only unknowns in Eq. (33) are the amplitudes $T^{(2)}$ and $M^{(2)}$.

For the amplitude $T^{(2)}$ any three-particle cut is unique as there are no two-particle states in this amplitude. This means that we can immediately write the three-particle Bethe-Salpeter equation for this amplitude

$$T^{(2)} = K + \frac{1}{3!} K G_0 T^{(2)}$$

Figure 4: Graphical representation of the equation $\Gamma^{(1)} = \frac{1}{2} M^{(2)} D_0 f$
where $K$ is the sum of all $3 \to 3$ diagrams that are simultaneously 1-, 2- and 3-particle irreducible. We write the disconnected part of $K$, indicated by subscript $d$, in terms of 1- and 2-particle irreducible two-body potentials $v$ ($\equiv t^{(2)}$)

$$K_d(1'2'3', 123) = \sum_{LcRc} v(2'3', 23)d^{-1}(1')\delta(1', 1)$$

(37)

where $Lc$ and $Rc$ indicate that the sum is taken over cyclic permutations of the left labels (1’2’3’) and right labels (123) respectively (note that the sum is restricted to cyclic permutations because the potentials $v$ are already symmetric in their labels). Note that

$$t^{(1)} = v + \frac{1}{2}vD_{0}t^{(1)}.\tag{38}$$

Defining $V_i$ by

$$V_1(1'2'3', 123) = v(1'3', 13)d^{-1}(2)\delta(2', 2)$$
$$V_2(1'2'3', 123) = v(2'3', 23)d^{-1}(1)\delta(1', 1)$$
$$V_3(1'2'3', 123) = v(1'2', 12)d^{-1}(3)\delta(3', 3),$$

(39-41)

we have that

$$K_d = \sum_{Pc} (V_1 + V_2 + V_3)$$

(42)

where it makes no difference over which labels, left or right, the cyclic permutations are taken. The subscript $i$ of $V_i$ can be considered as labelling the interacting pair. In this respect our convention throughout this paper shall be as in the above equations, i.e., 1 labels the pair (13), 2 labels the pair (23), and 3 labels the pair (12). Note that the definitions of $F_i$ given earlier in Eqs. (18) are consistent with this convention. Further, we have that

$$K = \sum_{Pc} (V_1 + V_2 + V_3) + K_c$$

(43)

where $K_c$ is the connected part of $K$, and as such, is a three-body force. Defining

$$V = \frac{1}{2}(V_1 + V_2 + V_3) + \frac{1}{6}K_c$$

(44)

so that

$$K = \sum_P V.$$ 

(45)

Eq. (36) can be written as

$$T^{(2)} = \sum_P V + VG_0T^{(2)}.$$ 

(46)
In obtaining Eq. (46) we made use of the fact that $T^{(2)}$ is symmetric; it is therefore gratifying to see that this derived equation now implies the symmetry of $T^{(2)}$ explicitly.

In this work, we would like to be able to neglect three-body forces, but as discussed in the Introduction, this needs to be done very carefully. In Eq. (13), we can in fact neglect $K_c$ in the expression for $V$; however, as we shall see shortly, this can be done only because there are no two-body states in the amplitude $T^{(2)}$. As it stands, Eq. (13) does not have a compact kernel so a Faddeev rearrangement needs to be performed. To facilitate this, one can recast Eq. (16) into a more familiar form (for the scattering of three non-identical particles) by defining the t-matrix $\tilde{T}$ through the equation

$$\tilde{T} = V + V G_0 \tilde{T}$$

where

$$T^{(2)} = \sum_P \tilde{T}.$$  \hspace{1cm} (48)

Eq. (17) can then be transformed to give equations of standard Faddeev form. Although the introduction of $\tilde{T}$ is the familiar way of handling Eq. (13), we show that in fact one can work with Eq. (16) directly. To do this, it is more straightforward to work with Green’s functions. We thus write

$$G^{(2)} = G_0^P + G_0 T^{(2)} G_0,$$

$$= G_0^P + G_0 V G^{(2)}.$$  \hspace{1cm} (50)

Both these equations for $G^{(2)}$ and Eq. (13) for $T^{(2)}$ differ from the corresponding ones for non-identical particles only in the symmetrization of the inhomogeneous term; moreover, in the equations for $G^{(2)}$, this symmetrization is of the free Green’s function $G_0$ and is thus particularly simple. Without the $K_c$ term, Eq. (50) is given by

$$G^{(2)} = G_0^P + \frac{1}{2} G_0 (V_1 + V_2 + V_3) G^{(2)}.$$  \hspace{1cm} (51)

We define the components $T_i$ by

$$T_i G_0 = \frac{1}{2} V_i G^{(2)}$$

so that

$$G^{(2)} = G_0^P + \sum_{i=1}^{3} G_0 T_i G_0.$$  \hspace{1cm} (53)

and

$$T^{(2)} = T_1 + T_2 + T_3.$$  \hspace{1cm} (54)

Using Eq. (53) in Eq. (52), we obtain

$$T_i G_0 = \frac{1}{2} V_i G_0^P + \frac{1}{2} V_i \sum_{j=1}^{3} G_0 T_j G_0.$$  \hspace{1cm} (55)
\[(1 - \frac{1}{2} V_i G_0) T_i = \frac{1}{2} V_i G_0 P G_0^{-1} + \frac{1}{2} V_i G_0 \sum_{j \neq i} T_j, \quad (56)\]

and thus the Faddeev equations for the components \( T_i \),

\[ T_i = \frac{1}{2} t_i^R + \frac{1}{2} t_i G_0 \sum_{j \neq i} T_j, \quad (57)\]

where \( t_i = V_i + \frac{1}{2} V_i G_0 t_i \). \quad (58)

Note that the inhomogeneous term of Eq. (57) involves a sum over right (but not left!) permutations. Comparing with Eq. (38), we see that the \( t_i \) are explicitly given by

\[ t_1(1'2'3', 123) = t^{(1)}(1'3', 13)d^{-1}(2)d^{-1}(2', 2) \quad (59)\]
\[ t_2(1'2'3', 123) = t^{(1)}(2'3', 23)d^{-1}(1)d^{-1}(1', 1) \quad (60)\]
\[ t_3(1'2'3', 123) = t^{(1)}(1'2', 12)d^{-1}(3)d^{-1}(3', 3). \quad (61)\]

2. The \( 2 \rightarrow 3 \) process

Although Eqs. (57) and (54) determine the 1- and 2-particle irreducible amplitude \( T^{(2)} \), we still need to determine the amplitude \( M^{(2)} \) before the full \( 3 \rightarrow 3 \) amplitude \( T \) can be specified by Eq. (33).

Now to obtain the full connected \( 2 \rightarrow 3 \) amplitude \( M_c \), we write it in terms of one-particle irreducible and reducible parts:

\[ M_c = M_c^{(1)} + \Gamma^{(1)} g \bar{f}. \quad (62)\]

Similarly we can write

\[ M_c^{(1)} = M_c^{(2)} + \frac{1}{2} M^{(2)} D_0 t^{(1)} . \quad (63)\]

Using Eqs. (32) and (34), we then obtain

\[ M_c = M_c^{(2)} + M^{(2)} \frac{1}{2} D_0 t \]
\[ = M_d^{(2)} \frac{1}{2} D_0 t + M_c^{(2)} (1 + \frac{1}{2} D_0 t), \quad (64)\]

where the disconnected part of \( M^{(2)} \) is clearly given by

\[ M_d^{(2)} = \sum_{L_c} (F_1 + F_2). \quad (65)\]
It is seen, therefore, that all we need now is the connected amplitude $M_c^{(2)}$, and then both the coupled $3 \rightarrow 3$ and $2 \rightarrow 3$ processes will be fully determined.

In the case of amplitude $M_c^{(2)}$, the very first possible three-particle cut is not unique since the amplitude contains two-particles in the initial state. However, here this non-uniqueness is simpler to handle than in the case of amplitudes with two-body intermediate states, since, apart from the two initial particles, $M_c^{(2)}$ has no further two-particle states. In particular, we can avoid the overcounting problems associated with the non-uniqueness of the first three-particle cut, by simply "pulling out" a vertex $f$ on one of the two initial particles. Furthermore, the clearest way to see the structure of $M_c^{(2)}$ is to temporarily expose the bare vertex function $f^0$ that is buried inside this chosen vertex $f$ of the skeleton diagram; namely, we write

$$M_c^{(2)} = \frac{1}{2} [T^{(2)} G_0 F_1^0]_c$$

(66)

where $F_1^0$ is defined as in Eq. (13) but with $f^0$ replacing $f$. Eq. (66) is illustrated in Fig. 5. At this stage it is more straightforward to work in terms of Green’s functions, so we use Eq. (19) to write the alternative form of Eq. (66),

$$M_c^{(2)} = \frac{1}{2} [G_0^{-1} G^{(2)} F_1^0]_c.$$  

(67)

We also introduce the Green’s functions $G_i$ defined by the equations

$$G_1 = D^{(1)}(1'3',13)g(2',2)$$

(68)

$$G_2 = D^{(1)}(2'3',23)g(1',1)$$

(69)

$$G_3 = D^{(1)}(1'2',12)g(3',3).$$

(70)

Note that the $G_i$ are symmetric under the interchange of labels corresponding to the interacting pair. It is easy to see that

$$G_i = G_0^P + \frac{1}{2} G_0 V_i G_i$$

(71)

$$= G_0^P + G_0 t_i G_0$$

(72)

Figure 5: Graphical representation of $M_c^{(2)} = \frac{1}{2} [T^{(2)} G_0 F_1^0]_c$. 

13
where the symbol $P_i$ indicates a sum over permutations of (left or right) momenta of the corresponding interacting pair:

\[
G_{0}^{P_i} = (1 + P_{13})G_0 = D_0^P(1'3', 13)g(2', 2) \quad (73)
\]
\[
G_{0}^{P_2} = (1 + P_{23})G_0 = D_0^P(2'3', 23)g(1', 1) \quad (74)
\]
\[
G_{0}^{P_3} = (1 + P_{12})G_0 = D_0^P(1'2', 12)g(3', 3). \quad (75)
\]

It is then evident that

\[
G_0F_1 = \frac{1}{2}G_1F_1^0, \quad (76)
\]

and using Eq. (71) to write \((1 - \frac{1}{2}G_0V)G_1 = G_0^{P_1}\), we obtain

\[
(1 - \frac{1}{2}G_0V)G_0F_1 = \frac{1}{2}G_0^{P_1}F_0^0 = G_0F_0^0 \quad (77)
\]

from which follows that

\[
F_0^0 = (1 - \frac{1}{2}V_1G_0)F_1. \quad (78)
\]

Using this result in Eq. (74),

\[
M_{c}^{(2)} = \frac{1}{2}[G_0^{-1}G^{(2)}(1 - \frac{1}{2}V_1G_0)F_1]_c. \quad (79)
\]

Writing the alternative form of Eq. (50),

\[
G^{(2)} = G_0^P + G^{(2)}V G_0, \quad (80)
\]

the previous expression becomes

\[
M_{c}^{(2)} = \frac{1}{2}G_0^{-1}G^{(2)}(V_2/2 + V_3/2 + K_c/6)G_0F_1. \quad (81)
\]

where the term $G_0^{-1}G_0^P F_1$ has been dropped as it has no connected part. To obtain a scattering equation for $M_{c}^{(2)}$ we multiply Eq. (81) on the left by $1 - VG_0$ and note that Eq. (50) implies

\[
(G_0^{-1} - V)G^{(2)} = G_0^{-1}G_0^P = \sum_P \delta(1', 1)\delta(2', 2)\delta(3', 3). \quad (82)
\]

Thus

\[
M_{c}^{(2)} = V G_0 M_{c}^{(2)} + \frac{1}{2} \sum_P (V_2/2 + V_3/2 + K_c/6)G_0F_1. \quad (83)
\]

With the derivation of this equation, we have, at least formally, solved the double counting problem. Eq. (83) is an integral equation whose solution provides $M_{c}^{(2)}$, and consequently, the full Green’s functions for both the $2 \to 3$ and $3 \to 3$ process; of course, this equation has kernels $V_i$ which are non-compact, but this is an easily
overcome formality. In a three-dimensional time-ordered approach, we would get, instead of Eq. (83), the equation

\[ M_c^{(2)} = V G_0 M_c^{(2)} + \frac{1}{2} \sum_L (V_2/2 + V_3/2 + K_c/6) G_0 F_1 + \frac{1}{2} \sum_L (V_1/2 + V_3/2 + K_c/6) G_0 F_2, \tag{84} \]

which, in contrast to Eq. (83), is symmetric with respect to the particle labels 1 and 2. The seeming asymmetry of Eq. (83) implies that there must be two-body forces, hidden within the three-body force \( K_c \), that restore the symmetry. Revealing these hidden two-body forces is therefore of great conceptual importance, quite apart from questions of their importance in practical calculations.

The amplitude \( K_c \) is one-, two- and three-particle irreducible, and it is connected. In this sense, it is a genuine three-body force and it would be tempting to follow the common procedure of neglecting this term. This, however, would be a major error: as illustrated in Fig. 6, the combination \( K_c G_0 F_1 \), entering Eq. (83), in fact contains pair-like interactions when viewed as having the very first vertex as \( F_2 \).

Let us specify more precisely how this dual nature of the one diagram in Fig. 6 manifests itself in our theory. To do this, we need to relate our four-dimensional three-body problem to the corresponding formulation in three-dimensional time-ordered perturbation theory. This can be done as follows. Let \( \mathcal{F}(p_1 \ldots p_n, q_1 \ldots q_m) \) be some Feynman diagram contributing to the Green’s function \( G \) of Eq. (1), and define \( \tilde{\mathcal{F}}(E, p_1 \ldots p_n, q_1 \ldots q_m) \) by the expression

\[ \tilde{\mathcal{F}}(E, p_1 \ldots p_n, q_1 \ldots q_m) = \int \delta(E - \sum_{i=1}^n p_i^0) dp_1^0 \ldots dp_n^0 \]

\[ \mathcal{F}(p_1 \ldots p_n, q_1 \ldots q_m) \delta(E - \sum_{i=1}^m q_i^0) dq_1^0 \ldots dq_m^0. \tag{85} \]

Note that the integrations in Eq. (85) are in fact with respect to the relative energies of the Feynman diagram. It can then be shown, that \( \tilde{\mathcal{F}} \) is the sum of all possible orderings of a time-ordered perturbation graph topologically identical to \( \mathcal{F} \), together with additional terms due to the advanced part of the two-time
Green’s function \([10]\). Thus if we take the set of Feynman diagrams specified by \(K_cG_0F_1\), then

\[
\tilde{K}_cG_0F_1 = \int \delta(E - p_1^0 - p_2^0 - p_3^0)dp_1^0dp_2^0dp_3^0
\]

\[
K_cG_0F_1(p_1p_2p_3, q_1q_2)\delta(E - q_1^0 - q_2^0)dq_1^0dq_2^0
\]

includes the sum of all possible orderings of corresponding time-ordered perturbation theory graphs; amongst these are diagrams whose graphical representation is given by Figs. 6(a) and (b). Now in time-ordered perturbation theory, the diagram of Fig. 6(b), is unambiguously a two-body rescattering in the presence of a spectator, and is therefore included in any three-dimensional approach; indeed, neglecting this diagram while keeping other two-body contributions would lead to violation of three-body unitarity. For these reasons it is of utmost importance that what we have called pair-like contributions in \(K_cG_0F_1\) be retained.

To examine more closely which pair-like diagrams of the form \(V_iG_0F_2\) also belong to the three-body force term \(K_cG_0F_1\), we expand \(V_i\) by order of the interaction:

\[
V_i = V_i^{[2]} + V_i^{[4]} + \ldots
\]

(87)

where \(V_i^{[n]}\) is the \(n\)-th order potential term. The example of Fig. 6 shows already that the fourth-order term, \(V_i^{[4]}\), gives a diagram that is a member of \(K_cG_0F_1\). Clearly, therefore, all higher order terms of \(V_i\) will likewise give diagrams belonging to \(K_cG_0F_1\). This leaves only the lowest order term, \(V_i^{[2]}\), as the single exception as it does not belong to \(K_cG_0F_1\). The corresponding terms \(V_1^{[2]}G_0F_2\) and \(V_3^{[2]}G_0F_2\), illustrated in Fig. 7, are just appropriately symmetrized versions of the basic graph

\[
B(1'2'3', 12) = \delta(1'2'3', 12)d(1'n)d(2'n)h(1'1'', 1)h(-1'' - 2'', -2')h(3'2'', 2),
\]

(88)

shown as the bottom left diagram of Fig. 7.

\[
V_1^{[2]}G_0F_2 = \begin{array}{c}
2' \\
3' \\
1' \\
\hline
\end{array} + \begin{array}{c}
2' \\
1' \\
\hline
\end{array}
\]

\[
V_3^{[2]}G_0F_2 = \begin{array}{c}
3' \\
2' \\
1' \\
\hline
\end{array} + \begin{array}{c}
3' \\
2' \\
\hline
\end{array}
\]

Figure 7: Terms of the form \(V_iG_0F_2\) that do not belong to \(K_cG_0F_1\). The bottom left diagram is denoted by \(B\) in the text.
Taking the above discussion into account, we can write

\[ K_c G_0 F_1 = \sum_{L_c} \left[ (V_1 - V_1^{[2]}) + (V_3 - V_3^{[2]}) \right] G_0 F_2 + K_{1c} G_0 F_1 \]  

(89)

where \( K_{1c} \) is made up of graphs which do not give rise to a two-body interaction when multiplied from the right by \( G_0 F_1 \) with a subsequent "pulling out" of vertex \( F_2 \). Note that the combination \( K_{1c} G_0 F_1 \), despite its notation, is completely symmetric; moreover, diagrams belonging to \( K_{1c} G_0 F_1 \) do not give rise to pair-like interactions when either \( F_1 \) or \( F_2 \) is pulled out to the right. This suggests a possible classification of 2 \( \rightarrow \) 3 diagrams: (i) those that give pair-like interactions when either of the two vertices is pulled out - the only possible case is shown in Fig. 4, (ii) those that give pair-like interactions when just one of the two vertices is pulled out; this case is given by

\[ \left[ (V_1 - V_1^{[2]}) + (V_3 - V_3^{[2]}) \right] G_0 F_2 + \left[ (V_2 - V_2^{[2]}) + (V_3 - V_3^{[2]}) \right] G_0 F_1, \]

and (iii) those that do not give rise to pair-like interactions - \( K_{1c} G_0 F_1 \). Defining

\[ B_0 = \frac{1}{2} (V_1^{[2]} + V_3^{[2]}) G_0 F_2, \]

(90)

and using Eq. (89), we can write Eq. (83) as

\[ (1 - VG_0) M_c^{(2)} = \frac{1}{2} \sum_{L} \left[ \frac{1}{2} (V_2 + V_3) G_0 F_1 + \frac{1}{2} (V_1 + V_3) G_0 F_2 - B_0 \right] + \frac{1}{2} K_{1c} G_0 F_1 \]

(91)

where, because of the left permutation sum, \( B_0 \) can also be replaced by \( 2B \). By symbolic form, this result differs from the corresponding result in time-ordered perturbation theory, just in the subtraction of the term \( B_0 \). This can be understood because, in our covariant case, the right most vertices are not time-ordered relative to each other, so that \( B_0 \) is contained in both the first and second terms on the rhs of Eq. (91). Thus one subtraction of \( B_0 \) is necessary to avoid overcounting. The potential \( V \) on the lhs of these equations still contains the three-body force \( K_c \) as given by Eq. (44). Now, however, we can safely neglect this term as it is multiplied on the right by a connected amplitude, and thus cannot give rise to pair-like interactions. With the understanding that all genuine three-body forces are to be neglected, Eq. (91) provides a linear integral equation for the amplitude \( M_c^{(2)} \) where all potentials are pair-like. We now need to perform a Faddeev rearrangement in order to obtain a compact kernel. This we cannot do directly to Eq. (91) because of the sum over left permutations.

By analogy with Eq. (17), one way to proceed would be to define the amplitude \( \tilde{M} \) satisfying the equation

\[ (1 - VG_0) \tilde{M} = \frac{1}{2} \left[ \frac{1}{2} (V_2 + V_3) G_0 F_1 + \frac{1}{2} (V_1 + V_3) G_0 F_2 - 2B \right] \]

(92)
where
\[ M_c^{(2)} = \sum_L \tilde{M}, \] (93)

so that Eq. (92) is the same as Eq. (91) but without the sum over permutations. We may then express \( \tilde{M} \) in terms of the set of equations
\[ \tilde{M} = \sum_i \tilde{M}_i - B, \] (94)
\[ \tilde{M}_i = V_i G_0 \left[ \frac{1}{2} \tilde{M} + \beta_i \right], \] (95)
where
\[ \beta_1 = \frac{1}{4} F_2; \quad \beta_2 = \frac{1}{4} F_1; \quad \beta_3 = \frac{1}{4} (F_1 + F_2). \] (96)

A Faddeev rearrangement then gives the equations
\[ \tilde{M}_i = t_i G_0 (\beta_i - \frac{1}{2} B) + \frac{1}{2} t_i G_0 \sum_{j \neq i} \tilde{M}_j, \] (97)

This method corresponds to a reduction to the different particle case; however, as we now proceed to show, this reduction is not necessary as the sum over permutations in Eq. (91) can also be explicitly evaluated.

The evaluation of \( \sum_L V_2 G_0 F_1 \) and \( \sum_L V_3 G_0 F_1 \) in Eq. (91) is most easily seen diagrammatically, as illustrated in Fig. 8. It is evident that
\[ \sum_L V_2 G_0 F_1 = \sum_L V_3 G_0 F_1. \] (98)

In a similar way, we can consider the graphical representation of the terms \( \sum_L V_1 G_0 F_2 \) and \( \sum_L V_3 G_0 F_2 \). In this way, we deduce that
\[ \sum_L [(V_2 + V_3) G_0 F_1 + (V_1 + V_3) G_0 F_2] \]
\[ = 4V_1 G_0 F_2^R + 4V_2 G_0 F_1^R + 4V_3 G_0 F_1^R \] (99)
where, on the rhs, the potentials \( V_i \) appear as in the different particle case, without left-hand side permutation sums.

Substituting the last equation into Eq. (111), we obtain

\[
(1 - \frac{1}{2} \sum_i V_i G_0 - \frac{1}{6} K_c G_0) M_c^{(2)} = V_i G_0 F_2^R + V_2 G_0 F_1^R + V_3 G_0 F_1^R - B^L + \frac{1}{2} K_{1c} G_0 F_1. \tag{100}
\]

For the description of the \( 2 \to 3 \) process, we can drop the terms involving \( K_c \) and \( K_{1c} \) in this equation. It must be emphasized, however, that once these three-body force terms are neglected, the resulting amplitude \( M_c^{(2)} \) should not be used to calculate the \( 2 \to 2 \) amplitude directly, i.e. through a multiplication on the left by \( \bar{F}_1 G_0 \); such a procedure would certainly miss important pair-like interactions coming from the term \( \bar{F}_1 G_0 K_c \) and, as we shall see shortly, also from the term \( \bar{F}_1 G_0 K_{1c} G_0 F_1 \). At this stage we consider only the \( 2 \to 3 \) process so the neglect of \( K_c \) and \( K_{1c} \) is justified. To lowest order, Eq. (100) gives the expected result \( M_c^{(2)} = B^L \), thus providing a simple check of our result so far, and illustrating the necessity of retaining the subtraction term \( B^L \).

We can now perform the standard Faddeev rearrangement of Eq. (100) by defining

\[
M_c^{(2)} = \sum_i M_i - B^L, \tag{101}
\]

\[
M_i = V_i G_0 \left[ \frac{1}{2} M_c^{(2)} + \alpha \right], \tag{102}
\]

where

\[
\alpha_1 = F_2^R; \quad \alpha_2 = F_1^R; \quad \alpha_3 = F_1^R \text{ or } F_2^R. \tag{103}
\]

With the help of Eq. (58), Eq. (102) can be written in terms of two-body t-matrices as

\[
M_i = t_i G_0 (\alpha_i - B^L) + \frac{1}{2} \sum_{j \neq i} t_j G_0 M_j. \tag{104}
\]

3. The \( 2 \to 2 \) process

In the above discussion, the \( 2 \to 2 \) amplitude \( t \) was assumed to be a known input. In the following discussion, \( t \) will not be assumed to be known, rather, we use the given field theoretic model to write an integral equation for \( t \).

The first steps, separating out the one- and two-particle cuts in \( t \), have already been performed in Eqs. (34) and (38). We therefore begin by examining the structure of the \( 2 \to 2 \) potential \( v \). Since \( v \) is connected and one-particle irreducible, both of the final state particles in \( v \) have separate left-most vertices. In order to
expose three-particle states, we shall "pull out" the left-most vertex on the first particle. This is just the same procedure as was used in Eq. (66) for exposing three-body states in \( M^{(2)} \). We recall, that in the case of \( M^{(2)} \), we pulled out the bare vertex \( f_0 \). However, we soon showed that this resulted in an expression, Eq. (79), where it is the dressed vertex \( f \) that has effectively been pulled out. This result has its origin in the general property of quantum field theory where a careful summation of Feynman diagrams leads to an effective perturbation theory where only skeleton diagrams need be considered.

To underscore this equivalence of working with bare and dressed vertices, we shall this time pull out the left-most dressed vertex on the first particle, thus we write

\[
v = \frac{1}{2} \bar{F}_1 G_0 M',
\]

which is illustrated in Fig. [9]. Here \( M' \) consists of all the diagrams of \( M^{(2)} \) except those that lead to dressing of vertices and particle propagators in the above expression for \( v \) (these restrictions are also sufficient to guarantee the connectedness of \( v \)). Examples of such diagrams belonging to \( M^{(2)} \) but excluded from \( M' \) are given in Fig. [10].

Writing \( M' \) in terms of its connected and disconnected parts,

\[
v = \frac{1}{2} \bar{F}_1 G_0 M'_c + \frac{1}{2} \bar{F}_1 G_0 M'_d
\]

one can simply use Eqs. (100) and (65) for \( M^{(2)}_c \) and \( M^{(2)}_d \) respectively, keeping only those terms that do not give rise to propagator or vertex dressing; in this sense, we write Eq. (106) as

\[
v = \frac{1}{2} \left\{ \bar{F}_1 G_0 M^{(2)}_c \right\}_{ND} + \frac{1}{2} \left\{ \bar{F}_1 G_0 M^{(2)}_d \right\}_{ND}
\]

where subscript \( ND \) stands for "no dressing". The part of \( v \) resulting from \( M^{(2)}_d \) is thus given by

\[
v^{OPE} = \frac{1}{2} \left\{ \bar{F}_1 G_0 M^{(2)}_d \right\}_{ND} = \bar{F}_1 G_0 F^R_2
\]

Figure 9: Graphical representation of \( v = \frac{1}{2} \bar{F}_1 G_0 M' \).
which is just the properly symmetrized one-particle exchange (OPE) potential. The full potential $v$ is thus

$$v = v^{OPE} + \left\{ \frac{1}{2} \tilde{F}_1 G_0 (V_2 G_0 F^R_1 + V_3 G_0 F^R_1 - \sum_L B + \frac{1}{2} K_{1c} G_0 F_1) \\
+ \frac{1}{2} \tilde{F}_1 G_0 \left( \frac{1}{2} V_2 + \frac{1}{2} V_3 + \frac{1}{6} K_c G_0 M^{(2)}_{1c} \right) \right\}_{ND} .$$

(109)

To display the pair-interactions resulting from $K_c$, we use the analogue of Eq. (89),

$$\tilde{F}_1 G_0 K_c = \sum_{R_c} \tilde{F}_2 G_0 \left[ (V_1 - V^{[2]}_1) + (V_3 - V^{[2]}_3) \right] + \tilde{F}_1 G_0 K'_1 .$$

(110)

Note that $K'_{1c} \neq K_{1c}$. Moreover, $\tilde{F}_1 G_0 K'_1 G_0 M^{(2)}_c$ does not have pair-like rescattering contributions (with only one particle as spectator) as can be easily seen by using the lowest order contribution for $M^{(2)}_c$. Similarly, we define the analogues of Eqs. (88) and (90) for the $3 \to 2$ process by

$$\tilde{B}(1'2', 123) = \delta(1'2', 123) d(1'') d(2'') h(1'1'', 1) h(-1'' - 2'', 2) h(2'2'', 3) .$$

(111)

and

$$\tilde{B}_0 = \frac{1}{2} \tilde{F}_2 G_0 (V^{[2]}_1 + V^{[2]}_3) .$$

(112)

respectively. Using Eqs. (111) and (112) in Eq. (109), we obtain

$$v = v^{OPE} + \left\{ \frac{1}{2} \tilde{F}_1 G_0 (V_2 G_0 F^R_1 + V_3 G_0 F^R_1 - \sum_L B + \frac{1}{2} K_{1c} G_0 F_1) \\
+ \frac{1}{2} \tilde{F}_1 G_0 (V_2 + V_3) + \frac{1}{2} \tilde{F}_2 G_0 (V_1 + V_3) - \tilde{B}_0 + \frac{1}{6} \tilde{F}_1 G_0 K'_{1c} G_0 M^{(2)}_c \right\}_{ND} .$$

(113)

At this stage, $M^{(2)}_c$ in this expression is still defined by the exact expression of Eq. (100). However, in contrast to the case of Eq. (107), the factors multiplying $M^{(2)}_c$...
in Eq. (113) ensure that the three-body force contributions to $M_c^{(2)}$ do not give rise to hidden two-body contributions in Eq. (113). We may therefore safely neglect such three-body contributions and thus have $M_c^{(2)}$ described by Eq. (101) rather than Eq. (100).

In Eq. (113) terms of the form $V_i^G_0 M_c^{(2)}$ may be eliminated using Eqs. (102) and (103), and the symmetry of $M_c^{(2)}$ may be used to reduce $\bar{B}_0$ to $2\bar{B}$. Thus, neglecting ”safe” three-body forces, we obtain

$$v = v^{OPE} + \frac{1}{2} \left\{ -\bar{F}_1 G_0 \sum L B + \frac{1}{2} \bar{F}_1 G_0 K_{1c} G_0 F_1 + \bar{F}_1 G_0 M_2 + \bar{F}_2 G_0 M_1 \\
+ (\bar{F}_1 + \bar{F}_2) G_0 M_3 - 2\bar{B} G_0 M_c^{(2)} - \bar{F}_2 G_0 V_i^G_0 F_2^R - \bar{F}_2 G_0 V_3 G_0 F_1^R \right\}_{ND}. \quad (114)$$

We still need to identify pair-like interactions that might be present in the term involving the three-body force $K_{1c}$; that is, we seek hidden expressions of the form $\bar{F}_i G_0 V_j^G_0 F_j$, including cases where the initial and final momenta are permuted. In this respect, we note that all such pair-like interactions can be expressed as one of the four terms $\bar{F}_i G_0 V_j^G_0 F_j$ $(i,j = 1, 2)$; for example,

$$\bar{F}_1 G_0 V_2 G_0 F_1 = \bar{F}_1 G_0 V_3 G_0 F_1, \quad \bar{F}_2 G_0 V_1 G_0 F_2 = \bar{F}_2 G_0 V_3 G_0 F_2, \quad L_{12} \bar{F}_1 G_0 V_3 G_0 F_1 = \bar{F}_2 G_0 V_3 G_0 F_1,$$

etc.

(of course cases like $\bar{F}_2 G_0 V_2 G_0 F_1$ do not arise as our vertices are dressed from the beginning). We note, therefore, that the last two terms of Eq. (114) may be simplified:

$$\frac{1}{2} (\bar{F}_2 G_0 V_1 G_0 F_2^R + \bar{F}_2 G_0 V_3 G_0 F_1^R) = \bar{F}_2 G_0 V_3 G_0 (F_2 + F_1) \quad (115)$$

$$= \bar{F}_2 G_0 V_3 G_0 F_2^R \quad (116)$$

the result being illustrated in Fig. 11. Note that this term is not symmetric under the interchange of its left labels. This implies that there must be a compensating term in Eq. (114) that restores the symmetry. Indeed we will see that this compensating term can be found in $\bar{F}_1 G_0 K_{1c} G_0 F_1$.

Central to the term $\bar{F}_1 G_0 K_{1c} G_0 F_1$ is the three-body force $K_{1c}$. Although $K_{1c}$ has been defined [by Eq. (89)] so that no graph of $K_{1c} G_0 F_1$ is expressible as vertex $F_2$ followed by a two-body potential, the full term $\bar{F}_1 G_0 K_{1c} G_0 F_1$, nevertheless, involves just such a two-body potential. In Fig. 12 we give an example of a graph, belonging to $K_{1c}$, which, on being sandwiched between $\bar{F}_1 G_0$ and $G_0 F_1$, is of the form $\bar{F}_2 G_0 V_3 G_0 F_2$. It is important that such pair-like interactions are not neglected in the theory.
To identify all such pair-like interactions in $\bar{F}_1 G_0 K_1 c G_0 F_1$, it is sufficient to examine the four terms $\bar{F}_1 G_0 V_3 G_0 F_j$ ($i, j = 1, 2$). In this respect, we note that terms of the form $V_2 G_0 F_1$ or $V_3 G_0 F_1$ cannot belong to $K_1 c G_0 F_1$ since $K_1 c$ is a three-body force. Likewise, terms of the form $V_1 G_0 F_2$ or $V_3 G_0 F_2$ cannot belong to $K_1 c G_0 F_1$ just by the property of $K_1 c$ given below Eq. (89). Consequently, out of the above mentioned four terms, two of them, $\bar{F}_1 G_0 V_3 G_0 F_1$ and $\bar{F}_1 G_0 V_3 G_0 F_2$, cannot be of the type $\bar{F}_1 G_0 K_1 c G_0 F_1$. This leaves only two candidates that are consistent with the definition of $K_1 c$, namely $\bar{F}_2 G_0 V_3 G_0 F_2$ and $\bar{F}_2 G_0 V_3 G_0 F_1$. However, even these have contributions which do not belong to $F_1 G_0 K_1 c G_0 F_1$. In particular, we need to exclude from $\bar{F}_2 G_0 V_3 G_0 F_2$ and $\bar{F}_2 G_0 V_3 G_0 F_1$ terms of the form $\bar{F}_1 G_0 V_3 G_0 F_1$ and $F_1 G_0 V_3 G_0 F_2$, as well as terms corresponding to vertex dressing (that these are the only terms to be excluded may be checked explicitly by using Eq. (114) to specify $V_3$). The only such terms in our theory are given by the amplitudes $W$, $X$ and $Y$ defined by Fig. (13), together with the same amplitudes but with the labels on the two external right legs interchanged. The diagram of Fig. (13)(a), defining amplitude $W$, corresponds to two-particle exchange where the exchanged particles

Figure 12: (a) Example of a graph belonging to the three-body force $K_1 c$. (b) The graph of (a) sandwiched between operators $\bar{F}_1 G_0$ and $G_0 F_1$; the resulting graph belongs to $\bar{F}_1 G_0 K_1 c G_0 F_1$, but is nevertheless of the two-body type $\bar{F}_2 G_0 V_3 G_0 F_2$. 

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J. Phys. 13 (1980) 1453 \title{Graphs that need to be subtracted from the two-body rescattering term $F_2G_0V_3G_0F_2$ as they do not belong to $F_1G_0K_1cG_0F_1$. (a) Interacting two-particle exchange, $W$; in the s-channel, the circle represents the amplitude $t' = t^{(1)} - v^{OPE}$. (b) Crossed two-particle exchange, $X$. (c) Vertex dressing graph, $Y$.}

Interact through the interaction $t'$, where

$$t' = t^{(1)} - v^{OPE}. \tag{117}$$

Here the subtraction of $v^{OPE}$ is needed to eliminate contributions to vertex dressing. The diagram of Fig. 13(b), defining amplitude $X$, corresponds to a "crossed two-particle exchange" potential; here the exchanged particles do not interact. The diagram of Fig. 13(c), defining amplitude $Y$, corresponds to vertex dressing and must be excluded from our final expressions. In summary, we can write that

$$\frac{1}{4} \tilde{F}_1G_0K_1cG_0F_1 = \tilde{F}_2G_0V_3G_0F_2^R - W^R - 2X - Y^R + \frac{1}{4} \tilde{F}_1G_0K_2cG_0F_1. \tag{118}$$

where $K_{2c}$ is a three-body force which when sandwiched between $\tilde{F}_1G_0$ and $G_0F_1$, does not result in a pair-like interaction. Note that the amplitude $X$ is symmetric and is thus equal to its exchange term; this and other symmetry aspects of this expression are discussed in detail in the Appendix.

On substituting the results of Eqs. (116) and (118) into Eq. (114), the unsymmetrical terms of Fig. 11 are eliminated, and we obtain

$$v = v^{OPE} + \left\{ \frac{1}{2} [\tilde{F}_1G_0M_2 + \tilde{F}_2G_0M_1 + (\tilde{F}_1 + \tilde{F}_2)G_0M_3] \right. $$

$$- \frac{1}{2} \tilde{F}_1G_0 \sum_L B - BG_0M_c^{(2)} - W^R - 2X - Y^R \bigg\} \tag{119}$$

where we have neglected the term involving the three-body force $K_{2c}$. In this expression, we can further evaluate

$$\frac{1}{2} \tilde{F}_1G_0 \sum_L B = X + (1 + L_{12}R_{12})Y. \tag{120}$$

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As written, Eq. (119) specifies that dressing terms inside the curly brackets need to be removed. This we can now do explicitly by identifying all possible dressing terms, and then including subtraction terms in order to remove their contribution. Some vertex dressing contributions have already been revealed in the $Y$ amplitudes of Eqs.(119) and (120). The only other vertex dressings arise from the first iteration of the terms $\bar{F}_i G_0 M_j$ in Eq. (119). Thus, with the $M_j$ evaluated to first order using Eqs. (104),

$$\frac{1}{2} [\bar{F}_1 G_0 M_2 + \bar{F}_2 G_0 M_1 + (\bar{F}_1 + \bar{F}_2) G_0 M_3] \quad \text{(lowest order)}$$

$$= \frac{1}{2} [\bar{F}_1 G_0 t_2 G_0 F_1^R + \bar{F}_2 G_0 t_1 G_0 F_2^R + (\bar{F}_1 + \bar{F}_2) G_0 t_3 G_0 F_1^R]$$

$$= (\bar{F}_1 + \bar{F}_2) G_0 t_3 G_0 (F_1 + F_2) = \sum_{LR} \bar{F}_2 G_0 t_3 G_0 F_2. \quad (121)$$

The vertex dressing terms arise from the one-particle exchange potential part of $t_3$, that is,

$$\sum_{LR} \bar{F}_2 G_0 v_{3\text{OP}}^E G_0 F_2 = \sum_{LR} Y, \quad (122)$$

and these must be subtracted from Eq. (119). Putting this together, the potential for the $2 \rightarrow 2$ process is

$$v = v_{\text{OP}}^E + \frac{1}{2} [\bar{F}_1 G_0 M_2 + \bar{F}_2 G_0 M_1 + (\bar{F}_1 + \bar{F}_2) G_0 M_3] - \bar{B} G_0 M_{e}^{(2)} - W^R - 3X - \sum_{LR} Y. \quad (123)$$

B. The $\pi NN$ system

Since the equations we have derived for identical particles contain all the complications of the most general case, it is now essentially a formality to write down the corresponding equations for other systems involving three-point vertices. Here we would like to apply our formalism to the case of the $\pi NN$ system which, because of its fundamental nature, has been under intensive investigation for many years [8]. It is thus assumed that we are dealing with a field theory of nucleons and pions where the interaction is given by a $\pi NN$ vertex.

The discussion of the previous section does not need to be modified in any essential way. However, one does need to take into account that now one has only two identical particles, and moreover, that these two are Fermions. In this respect, we need to modify our definitions of the various symmetry operations introduced in the previous section. Although the symbols $L$ and $R$ still have the meaning of sums over permutations, now the sums extend only over the two nucleon labels, and with the additional condition that the ”exchanged” term enters with a minus
Choosing the convention that label 3 always refers to the pion, Eqs. (28) and (29) are now modified to

\[
A^R(1'2'3', 123) = A(1'2'3', 123) - A(1'2'3', 213) \quad (124)
\]

\[
A^L(1'2'3', 123) = A(1'2'3', 123) - A(2'1'3', 123) \quad (125)
\]

with similar relations holding for quantities having only nucleons in initial or final states.

1. \(\pi NN \rightarrow \pi NN\)

The amplitude \(T\) for the \(\pi NN \rightarrow \pi NN\) process is given, as for the identical particle case, by Eq. (33). The 2-particle irreducible amplitude \(T^{(2)}\) is again the sum of Faddeev components \(T_k\) as in Eq. (54); however, the equations for these components are now given by

\[
T_i = t_i^R + t_i G_0 \sum_{k \neq i} T_k, \quad (126)
\]

\[
T_3 = t_3 + \frac{1}{2} t_3 G_0 \sum_{k \neq 3} T_k, \quad (127)
\]

with

\[
t_i = V_i + V_i G_0 t_i, \quad (128)
\]

\[
t_3 = V_3 + \frac{1}{2} V_3 G_0 t_3, \quad (129)
\]

where, in the above, \(i = 1, 2\), and \(k = 1, 2, 3\). Note that \(t_i\) is the \(\pi N\) amplitude where the pion interacts with the \(i\)th nucleon, the other nucleon being a spectator. Similarly \(t_3\) is the \(NN\) amplitude in the presence of a spectator pion.

2. \(NN \rightarrow \pi NN\)

With three-body forces neglected, the integral equation for the \(NN \rightarrow \pi NN\) 2-particle irreducible amplitude \(M^{(2)}_c\) is given by a slightly modified form of Eq. (100), namely

\[
M^{(2)}_c - (V_1 + V_2 + \frac{1}{2} V_3) G_0 M^{(2)}_c = V_1 G_0 F^R_2 + V_2 G_0 F^R_1 + V_3 G_0 F^R_1 - B^{LR} \quad (130)
\]

where no 1/2 factors appear in front of \(V_1\) and \(V_2\) since they correspond to potentials between distinguishable particles (\(\pi N\)). The amplitude \(B\) now takes on the form shown in Fig. 14, and sums over both left and right nucleon exchanges are now also needed.
To effect the Faddeev rearrangement of Eq. (130) we define the amplitude components by appropriately modified Eqs. (104)

\[ M_i = V_i G_0 [M^{(2)}_c + F^R_j], \quad (j \neq i) \]  
\[ M_3 = V_3 G_0 [\frac{1}{2} M^{(2)}_c + F^R_1], \]  
\[ M^{(2)}_c = \sum_k M_k - B^{LR}. \]  

In the above equations, as in the two equations below, we use the convention that \( i, j = 1, 2 \) while \( k = 1, 2, 3 \). The rearranged equations are then given by

\[ M_3 = t_3 G_0 (F^R_1 - \frac{1}{2} B^{LR}) + \frac{1}{2} \sum_{k \neq 3} t_3 G_0 M_k, \]  
\[ M_i = t_i G_0 (F^R_j - B^{LR}) + \sum_{k \neq i} t_i G_0 M_k \quad (j \neq i). \]  

3. \( NN \rightarrow NN \)

For \( NN \) scattering, there can be no contributions that are 1-particle reducible in the specified model where all interactions are generated by the \( \pi NN \) vertex. The full \( NN \) t-matrix, \( t_{NN} \), is thus given by the Bethe-Salpeter equation of Eq. (38), which in the present case takes the form

\[ t_{NN} = v_{NN} + \frac{1}{2} v_{NN} D_0 t_{NN}. \]  

Here \( D_0 \) is the dressed two-nucleon propagator, and the \( NN \) potential, \( v_{NN} \), is specified in a similar way to Eq. (123), namely

\[ v_{NN} = v_{NN}^{OPE} + \hat{F}_1 G_0 M_2 + \hat{F}_2 G_0 M_1 + (\hat{F}_1 + \hat{F}_2) G_0 M_3 \]  
\[ -B^{LR} G_0 M^{(2)}_c - W^{R}_{\pi\pi} - W^{LR}_{\pi N} - W^{R}_{NN} - X^R - Y^{LR}. \]  

where \( v_{NN}^{OPE} \) is now the \( NN \) one-pion exchange potential and the terms \( W_{\pi\pi}, W_{\pi N}, W_{NN}, X^R \) and \( Y \) are illustrated in Fig. 14. Each of the three different \( W \) amplitudes
Figure 15: Graphs that need to be subtracted from the NN potential in the relativistic formulation of the πNN system. (a) $W_{\pi\pi}$, (b) $W_{\pi N}$, (c) $W_{NN}$, (d) $X$, and (e) $Y$. The small circles represent the t-matrix amplitudes: (a) $t_{\pi\pi}$, (b) $t_{\pi N}^{(1)}$, and (c) $t_{NN} - v_{NN}^{OPE}$, each being the appropriate version of Eq. (117).

Involves the exchange of two particles that interact with each other through the appropriate version of the t-matrix $t'$ of Eq. (117). Thus the pions of amplitude $W_{\pi\pi}$ interact through the full π-π t-matrix $t_{\pi\pi}$, the amplitude $W_{\pi N}$ involves π-N scattering through the so-called background term, $t_{\pi N}^{(1)}$, and $W_{NN}$ involves NN scattering through $t_{NN} - v_{NN}^{OPE}$. Note that $W_{NN}$ does contain u-channel one-pion exchange, although t-channel one-pion exchange is forbidden.

It is important to note that the amplitude $t_{NN}$ enters our equations both as an input and as an output. In this respect, we note that the t-matrix $t_3$, defined formally by Eq. (61), in our case is just the NN t-matrix $t_{NN}$ in the presence of a spectator pion:

$$t_3(1'2'3', 123) = t_{NN}(1'2', 12)d_{\pi}^{-1}(3)\delta(3', 3).$$

Similarly, Eq. (41) becomes in our case

$$V_3(1'2'3', 123) = v_{NN}(1'2', 12)d_{\pi}^{-1}(3)\delta(3', 3),$$

and Eq. (129) can thus be identified with Eq. (136). Since $t_3$ enters as an input to Eqs. (127) and (134), so does $t_{NN}$. Similarly, $t_{NN}$ is needed as an input to calculate $W_{NN}$. At the same time, $t_{NN}$ is the output amplitude of the Bethe-Salpeter equation Eq. (136). Ideally, the input and output $t_{NN}$ need to be self-consistent. This might be achievable through an iterative process where the output of one iteration becomes the input of the next iteration. Alternatively, a great numerical simplification is afforded by sacrificing the self-consistency and using an externally constructed $t_{NN}$ for the input. This may be reasonable if the observables of interest are not very sensitive to the details of the NV channel.
With the possible exception of $t_{NN}$, the two-body scattering t-matrices entering the calculation of the $W$-amplitudes form part of the externally constructed input to our equations. It seems remarkable that the $\pi-\pi$ t-matrix is a necessary input to our equations. No such input is needed for the corresponding three-dimensional theory of the $\pi NN$ system, nor has it appeared in previous formulations of four-dimensional $\pi NN$ equations. In the present work, the appearance of this input arises through the careful retention of all $\pi N$ interactions where one nucleon is a spectator.

**SUMMARY**

We have derived four-dimensional relativistic three-body equations based on $\phi^3$ field theory. These equations couple the amplitudes for $2 \rightarrow 2$, $2 \rightarrow 3$, and $3 \rightarrow 3$ processes, and can be considered as the natural extension of the Dyson-Schwinger and Bethe-Salpeter equations to the three-particle sector. With the assumption that three-body forces can be either neglected or given explicitly, these three-body equations form a closed set of equations for the above amplitudes, and provide the exact non-perturbative solution of the field theoretical problem. In particular, these equations provide an answer to the long standing question of how to determining the kernel of the $2 \rightarrow 2$ Bethe-Salpeter equation in a non-perturbative way. If three-body forces are not given, then in the spirit of our approach, one could expose four-particle states in the three-body forces, and thus express them in terms of four-particle equations where now four-body forces are either neglected or explicitly given. Although one can, in this way, envisage a whole hierarchy of equations, involving ever higher multi-particle forces, there is a large amount of evidence that three-body forces are small for most processes; thus, the three-body equations developed in this paper, should form a solid theoretical starting point for the investigation of relativistic few-body processes.

Two features distinguish our approach from previous attempts to formulate the relativistic four-dimensional equations. Firstly, we have overcome the overcounting problem, present in all previous works that use the so-called ”last-cut lemma”; to do this, we used a procedure of explicitly pulling out one initial vertex in the $2 \rightarrow 3$ amplitude. Secondly, we have retained all possible two-body interactions taking place in the presence of a third spectator particle. In previous works, it was apparently not realised that three-body forces in the $3 \rightarrow 3$ amplitude can give rise to such two-body forces in the $2 \rightarrow 3$ amplitude. By retaining appropriate three-body forces, we avoid the serious problem of undercounting pair-like interactions which would arise if all three-body forces in the $3 \rightarrow 3$ amplitude were to be neglected.

Another interesting aspect of our four-dimensional equations is that they differ at the ”operator level” from the corresponding three-dimensional three-body equations of time-ordered perturbation theory. This is in contrast to the case
for the two-particle system where the four-dimensional Bethe-Salpeter and threedimensional Lippmann-Schwinger equations differ essentially in the dimension of integrals involved, but otherwise have the same operator form. For the three-body problem, this difference is due to the presence of a number of subtractions terms in our four-dimensional equations which are not present in the three-dimensional equations. These extra terms are necessary to avoid overcounting and arise as a direct consequence of including all the above-mentioned possible two-body interactions. We note that all the previous works that suffer from overcounting [1-5] have four-dimensional equations that are of the same operator form as the corresponding three-dimensional equations.

It should be emphasised that our four-dimensional three-body equations are not only of theoretical interest. With the ever increasing power of current computers, these equations should also provide a timely practical tool for calculating few-body hadronic processes in nuclear and high-energy physics.

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APPENDIX

Here we would like to explain the origin of the counting factors appearing in expressions of Section II dealing with identical particles.

The most common such factor is the inverse factorial 1/n! associated with n-particle intermediate states connecting two amplitudes which are symmetric with respect to label interchanges of these n particles. Although this factor can be derived by applying Wick’s theorem to the Green’s function of Eq. (II), its appearance is not a feature of a four-dimensional approach, rather, it is due to the basic symmetry property of identical particle amplitudes. Indeed, just the same factor appears in three-dimensional approaches where, in fact, the origin of this factor can be more easily seen. In a three-dimensional approach, the Green’s function corresponding to the four-dimensional one of Eq. (II) is given by

\[
\tilde{G} = <0|a(p'_n)\ldots a(p'_1)\frac{1}{E_+ - H}a^\dagger(p_1)\ldots a^\dagger(p_m)|0>
\]

(A1)

where \( H \) is the Hamiltonian and the creation and annihilation operators obey the commutation relations (for Bosons)

\[
[a(p), a^\dagger(p')] = \delta(p - p'); \quad [a^\dagger(p), a^\dagger(p')] = [a(p), a(p')] = 0.
\]

(A2)
Just the same creation and annihilation operators appear in the expansion of the fields $\phi(x)$ in Eq. (11) (here there is no subscript on $\phi$ since all particles are identical):

$$\phi(x) = \int \frac{d\mathbf{p}}{\sqrt{(2\pi)^3 2\omega_0}} [a(\mathbf{p})e^{-ip\cdot x} + a^\dagger(\mathbf{p})e^{ip\cdot x}]. \quad (A3)$$

For three-dimensional Green’s functions of Eq. (A1), the states

$$|p_1 \ldots p_n > = a^\dagger(p_1) \ldots a^\dagger(p_n)|0 > \quad (A4)$$

form a basis of the full Fock space of identical particle states. As is well known [11], the closure relation in this Fock space is given by

$$|0><0| + \sum_{n=1}^{\infty} \int dp_1 \ldots dp_n \frac{1}{n!}|p_1 \ldots p_n><p_1 \ldots p_n| = 1, \quad (A5)$$

and here one can explicitly see the origin of the $1/n!$ factor for $n$-particle intermediate states.

We note that together with the condition $<0|0> = 1$, Eqs. (A2) or Eq. (A3) define the normalization of the states $|p_1 \ldots p_n >$. This normalization is particularly convenient for the case of disconnected graphs as we now demonstrate. Let $G_1(\alpha_i, \beta_i)$ and $G_2(\alpha'_j, \beta'_j)$ denote two four-dimensional Green’s functions that make up the two disconnected parts of the disconnected Green’s function $G_d(\alpha_i, \beta_i; \alpha'_j, \beta'_j)$. Here we have used a short-hand notation for the momentum labels: e.g. $\alpha_i \equiv \{\alpha_1, \alpha_2, \ldots\}$ denote the left-hand momenta of $G_1$ and $\alpha_i \alpha'_j \equiv \{\alpha_1, \alpha_2, \ldots, \alpha'_1, \alpha'_2, \ldots\}$ denote the left-hand momenta of $G_d$. Note that $G_1$, $G_2$ and $G_d$ are symmetric functions of their labels. Now it is straightforward to show that

$$G_d(\alpha_i, \alpha'_j; \beta_i, \beta'_j) = \sum_{\alpha_i \leftrightarrow \alpha'_j; \beta_i \leftrightarrow \beta'_j} G_1(\alpha_i, \beta_i)G_2(\alpha'_j, \beta'_j) \quad (A6)$$

where the sum is over distinct terms resulting from all possible exchanges of momentum labels, as indicated. The above mentioned normalization ensures that this expression appears with no extra counting factors. The result of Eq. (A6) can be obviously generalized to disconnected Green’s functions made up of any number of disconnected pieces, there again being just a sum over exchange of labels with no extra counting factors. In Section II, Eqs. (26), (27), (37) and (65) can be considered as special cases of this result. It must be emphasized, that although Eq. (A6) is presented for four-dimensional Green’s functions, the structure of the result, in terms of a simple sum over label exchanges, is due to the normalization and symmetry properties of the underlying Fock states of Eq. (A4). Thus the same equation holds for three-dimensional Green’s functions, although in that case, the simple product of $G_1$ and $G_2$ in Eq. (A6) needs to be replaced by a convolution integral [12].

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Figure 16: Illustration of how to determine the properly symmetrized amplitudes for (a) non-interacting two-particle exchange, and (b) interacting two-particle exchange.

We now would like to discuss the various counting factors entering the expression of Eq. (118). This will at the same time demonstrate how counting factors can be determined for most other expressions.

The factor of $1/4$ entering the left side of Eq. (118) is associated with the vertex functions $\bar{F}_1$ and $F_1$ as they expose two identical particles each.

Turning to the r.h.s. of Eq. (118), let us show that the amplitude $X$ of Fig. 13(b), constructed simply from four symmetric vertices and without any extra counting factors, is just the properly symmetrized non-interacting two-particle exchange amplitude in our model. In Fig. 16(a) we demonstrate how such a symmetrized two-particle exchange amplitude is constructed. The factor of $1/16$ in Fig. 16(a) results from factors of $1/2$ for each of the four vertices bracketing the symmetrized four-particle propagator. The factor of $1/16$ could also have been obtained by first symmetrizing the products $\bar{F}_1 \bar{F}_2$ and $F_1 F_2$ using Eq. (A6); in each case there would be 6 terms, each giving an equal contribution. One would then need to multiply twice by $1/4!$, since a symmetrized product of two vertices exposes 4 particles in intermediate state. Thus $(6/4!)^2 = 1/16$, as before. Now consider the sum over permutations $P$ of the 4-particle propagator in Fig. 16(a): there will be 24 terms in this sum, however, not all of them will lead to a two-particle exchange amplitude. In fact it is easily seen that 8 of these permutations will result in disconnected dressing graphs, leaving only $24 - 8 = 16$ that are proper two-particle exchanges. Now all of these 16 contributions are equal to each other, so taking into account the previous factor of $1/16$, we obtain that the properly symmetrized non-interacting two-particle exchange amplitude is just given by $X$, the amplitude defined by the single diagram of Fig. 13(b). In a similar way, let’s find the symmetrization factor necessary in front of the interacting two-particle exchange amplitude $W$ defined in Fig. 13(a). The properly symmetrized interacting two-particle exchange is constructed as in Fig. 16(b). We again have the factor of $1/16$ due to the four vertices. This time, however, two of the four particles in the center of the diagram interact. In Fig. 16(b), the symmetrization sum over $L'$
and $R'$ is, as in Eq. (A3), over exchanges of left and right labels leading to distinct contributions. Now out of the $4! \times 4!$ possible left and right permutations, the switching of labels $2 \leftrightarrow 3$, $2' \leftrightarrow 3'$ and $1' \leftrightarrow 4'$, leads to a repetition of already included terms, so they must be excluded. The sum over $L'$ and $R'$ thus extends over only $4!4!/8$ terms. However, from all these terms, there will be cases where the (interacting) legs 2 and 3 will both be connected to the same vertex, thus leading to a vertex dressing graph. Such cases must be excluded. Thus, effectively, there are only 4 possibilities for legs 2 and 3: one leg connected to either of the 2 legs of the top left vertex, the other being connected to either of the 2 legs of the bottom left vertex. Similarly there are 4 possibilities for the $2'$ and $3'$ legs of the middle diagram to connect to the right-hand vertices. In total there can therefore be only $4 \times 4 = 16$ interacting two-particle exchanges, and taking into account the initial $1/16$ factor, we see that the properly symmetrized interacting two-particle exchange is given by the amplitude $W$ of Fig. 13(a) with no counting factors.

This kind of reasoning can be used for other types of diagrams involving identical particles.

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