GENERALIZED BREGMAN AND JENSEN DIVERGENCES
WHICH INCLUDE SOME F-DIVERGENCES

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Abstract. In this paper, we introduce new classes of divergences by extending the definitions of the Bregman divergence and the skew Jensen divergence. These new divergence classes (g-Bregman divergence and skew g-Jensen divergence) satisfy some properties similar to the Bregman or skew Jensen divergence. We show these g-divergences include divergences which belong to a class of f-divergence (the Hellinger distance, the chi-square divergence and the alpha-divergence in addition to the Kullback-Leibler divergence). Moreover, we derive an inequality between the g-Bregman divergence and the skew g-Jensen divergence and show this inequality is a generalization of Lin’s inequality.

Keywords: Bregman divergence, f-divergence, Jensen divergence, Kullback-Leibler divergence, Jeffreys divergence, Jensen-Shannon divergence, Hellinger distance, Chi-square divergence, Alpha divergence, Centroid, Parallelogram, Lin’s inequality.

1. Introduction

Divergences are functions measure the discrepancy between two points and play a key role in the field of machine learning, signal processing and so on. Given a set $S$ and $p, q \in S$, a divergence is defined as a function $D : S \times S \to \mathbb{R}$ which satisfies the following properties.

1. $D(p, q) \geq 0$ for all $p, q \in S$
2. $D(p, q) = 0 \iff p = q$

The Bregman divergence [7] $B_F(p, q) \overset{\text{def}}{=} F(p) - F(q) - \langle p - q, \nabla F(q) \rangle$ and f-divergence [1, 10] $D_f(p\|q) \overset{\text{def}}{=} \sum_i q_i f(q_i / p_i)$ are representative divergences defined by using a strictly convex function, where $F, f : S \to \mathbb{R}$ are strictly convex functions and $f(1) = 0$. Besides that Nielsen have introduced the skew Jensen divergence $J_{F,\alpha}(p, q) \overset{\text{def}}{=} (1 - \alpha)F(p) + \alpha F(q) - F((1 - \alpha)p + \alpha q)$ for a parameter $\alpha \in (0, 1)$ and have shown the relation between the Bregman divergence and the skew Jensen divergence [8, 16].

The Bregman divergence and the f-divergence include well-known divergences. For example, the Kullback-Leibler divergence (KL-divergence) [12] is a type of the Bregman divergence and the f-divergence. On the other hand, the Hellinger distance, the chi-square divergence and the alpha-divergence [5, 9] are a type of the f-divergence. The Jensen-Shannon divergence (JS-divergence) [13] is a type of the skew Jensen divergence.

For probability distributions $p$ and $q$, each divergence is defined as follows.

KL-divergence

$$KL(p\|q) \overset{\text{def}}{=} \sum_i p_i \ln \left( \frac{p_i}{q_i} \right)$$

(1)
Hellinger distance

\[ H^2(p, q) \overset{\text{def}}{=} \sum_i (\sqrt{p_i} - \sqrt{q_i})^2 \]  

(2)

Pearson \( \chi \)-square divergence

\[ D_{\chi^2,P}(p\|q) \overset{\text{def}}{=} \sum_i \frac{(p_i - q_i)^2}{q_i} \]  

(3)

Neyman \( \chi \)-square divergence

\[ D_{\chi^2,N}(p\|q) \overset{\text{def}}{=} \sum_i \frac{(p_i - q_i)^2}{p_i} \]  

(4)

\( \alpha \)-divergence

\[ D_\alpha(p\|q) \overset{\text{def}}{=} \frac{1}{\alpha(\alpha - 1)} \sum_i (p_i^\alpha q_i^{1-\alpha} - 1) \]  

(5)

JS-divergence

\[ JS(p, q) \overset{\text{def}}{=} \frac{1}{2} KL(p\|p + q/2) + \frac{1}{2} KL(q\|p + q/2) \]  

(6)

The main goal of this paper is to introduce new classes of divergences ("g-divergence") and study properties of the g-divergences.

First, we introduce the g-Bregman divergence, the symmetric g-Bregman divergence and the skew g-Jensen divergence by extending the definitions of the Bregman divergence and the skew Jensen divergence respectively. This idea is based on Zhang’s papers [21, 22]. For the g-Bregman divergence, we show some geometrical properties (for triangle, parallelogram, centroids) and derive an inequality between the symmetric g-Bregman divergence and the skew g-Jensen divergence.

Then, we show that the g-Bregman divergence includes the Hellinger distance, the Pearson and Neyman \( \chi \)-square divergence and the \( \alpha \)-divergence. Furthermore, we show that the skew g-Jensen divergence include the Hellinger distance and the \( \alpha \)-divergence. These divergences have not only the properties of \( f \)-divergence but also some properties similar to the Bregman or the skew Jensen divergence.

Finally we derive many inequalities by using an inequality between the g-Bregman divergence and skew g-Jensen divergence.

2. The g-Bregman divergence and the skew g-Jensen divergence

2.1. Definition of the g-Bregman divergence and the skew g-Jensen divergence. By using a injective function \( g \), we define the g-Bregman divergence and the skew g-Jensen divergence.

**Definition 1.** Let \( \Omega \subset \mathbb{R}^d \) be a convex set and let \( p, q \) be points in \( \Omega \). Let \( F(x) \) be a strictly convex function \( F : \Omega \rightarrow \mathbb{R} \).

The Bregman divergence and the symmetric Bregman divergence are defined as

\[ B_F(p, q) \overset{\text{def}}{=} F(p) - F(q) - \langle p - q, \nabla F(q) \rangle \]  

(7)

\[ B_{F,sym}(p, q) \overset{\text{def}}{=} B_F(p, q) + B_F(q, p) = \langle p - q, \nabla F(p) - \nabla F(q) \rangle. \]  

(8)

The symbol \( \langle \cdot, \cdot \rangle \) indicates an inner product.

For the parameter \( \alpha \in \mathbb{R} \setminus \{0, 1\} \), the scaled skew Jensen divergence [13,14] is defined as

\[ sJ_{F,\alpha}(p, q) \overset{\text{def}}{=} \frac{1}{\alpha(1-\alpha)} \left( (1-\alpha)F(p) + \alpha F(q) - F((1-\alpha)p + \alpha q) \right). \]  

(9)
 Nielsen has shown the relation between the Bregman divergence and the Jensen divergence as follows \cite{16}.

\begin{align}
B_F(q,p) &= \lim_{\alpha \to 0} sJ_{F,\alpha}(p,q) \\
B_F(p,q) &= \lim_{\alpha \to 1} sJ_{F,\alpha}(p,q)
\end{align}

(10)

\begin{equation}
sJ_{F,\alpha}(p,q) = \frac{1}{\alpha(1-\alpha)} \left( (1-\alpha)B_F(p, (1-\alpha)p+\alpha q) + \alpha B_F(q, (1-\alpha)p+\alpha q) \right)
\end{equation}

(12)

**Definition 2.** (Definition of the \(g\)-convex set) Let \(S\) be a vector space over the real numbers and \(g\) be an injective function \(g : S \to S\). If \(S\) satisfies the following condition, we call the set \(S\) \("g\)-convex set\).

For all \(p\) and \(q\) in \(S\) and all \(\alpha\) in the interval \((0,1)\), the point \(g^{-1}((1-\alpha)g(p) + \alpha g(q))\) also belongs to \(S\).

**Definition 3.** (Definition of the \(g\)-divergences) Let \(\Omega\) be a \(g\)-convex set and let \(g : \Omega \to \Omega\) be a function which satisfies \(g(p) = g(q) \iff p = q\). Let \(F : \Omega \to \mathbb{R}\) be a strictly convex function. Let \(\alpha\) be a parameter in \((0,1)\).

We define the \(g\)-Bregman divergence, the \(g\)-symmetric Bregman divergence and the (scaled) skew \(g\)-Jensen divergence as follows.

\begin{align}
B^g_F(p,q) &\overset{\text{def}}{=} B_F(g(p),g(q)) \\
B^g_{F,\text{sym}}(p,q) &\overset{\text{def}}{=} B_{F,\text{sym}}(g(p),g(q)) \\
sJ^g_{F,\alpha}(p,q) &\overset{\text{def}}{=} sJ_{F,\alpha}(g(p),g(q))
\end{align}

(13)

(14)

(15)

From the definition of function \(g\), these functions satisfy divergence properties \(D^g(p,q) = 0 \iff g(p) = g(q) \iff p = q\) and \(D^g(p,q) \geq 0\), where \(D^g\) denotes the \(g\)-divergences. When \(g(p)\) is equal to \(p\) for all \(p \in \Omega\), the \(g\)-divergences are consistent with original divergences.

In the following, we assume that \(g\) is a function having an inverse function.

### 2.2. Geometrical properties of the \(g\)-Bregman divergence

In this subsection, we show the \(g\)-Bregman divergence and the symmetric \(g\)-Bregman divergence satisfy some geometrical properties as well as the Bregman and the symmetric Bregman divergence.

First, we show the \(g\)-Bregman divergence and the symmetric \(g\)-Bregman divergence satisfy the linearity, the generalized law of cosines and the generalized parallelogram law.

Then, we show the point which minimize the weighted average of \(g\)-Bregman divergence (\(g\)-Bregman centroids) is equal to the quasi-arithmetic mean \(\overset{\circ}{\frac{1}{1+c_1+c_2}}\).

This property can be used for clustering algorithms such as k-means algorithm \cite{14}.

**Linearity**

For positive constant \(c_1\) and \(c_2\),

\begin{equation}
B^g_{c_1F_1 + c_2F_2}(p,q) = c_1 B^g_{F_1}(p,q) + c_2 B^g_{F_2}(p,q)
\end{equation}

(16)

**Proposition 1.** (Generalized law of cosines)

Let \(\Omega\) be a \(g\)-convex set. For points \(p, q, r \in \Omega\), the following equations hold.

\begin{align}
B^g_F(p,q) &= B^g_F(p,r) + B^g_F(r,q) - \langle g(p) - g(r), \nabla F(g(q)) - \nabla F(g(r)) \rangle \\
B^g_{F,\text{sym}}(p,q) &= B^g_{F,\text{sym}}(p,r) + B^g_{F,\text{sym}}(r,q) \\
-\langle g(p) - g(r), \nabla F(g(q)) - \nabla F(g(r)) \rangle + \langle g(q) - g(r), \nabla F(g(p)) - \nabla F(g(r)) \rangle
\end{align}

(17)

(18)
It is easily proved in the same way as the Bregman divergence by putting \( p' = g(p), \ q' = g(q) \) and \( r' = g(r) \).

**Theorem 1. (Generalized parallelogram law)**

Let \( \Omega \) be a \( g \)-convex set. When points \( p, q, r, s \in \Omega \) satisfy \( g(p) + g(r) = g(q) + g(s) \), the following equations holds.

\[
B_{F,\text{sym}}^g(p,q) + B_{F,\text{sym}}^g(q,r) + B_{F,\text{sym}}^g(r,s) + B_{F,\text{sym}}^g(s,p) = B_{F,\text{sym}}^g(p,r) + B_{F,\text{sym}}^g(q,s)
\]

(19)

The left hand side is the sum of four sides of rectangle \( pqrs \) and the right hand side is the sum of diagonal lines of rectangle \( pqrs \).

**Proof.** It has been shown that the Bregman divergence satisfies the four point identity(see equation (2.3) in [20]).

\[
\langle \nabla F(p) - \nabla F(q), p - q \rangle = B_F(p,q) + B_F(p,s) - B_F(p,r) - B_F(q,s)
\]

(20)

By putting \( p' = g(p), \ q' = g(q), \ r' = g(r) \) and \( s' = g(s) \), we can prove the \( g \)-Bregman divergence satisfies the similar four point identity.

\[
\langle \nabla F(g(r)) - \nabla F(g(s)), g(p) - g(q) \rangle = B_{F,\text{sym}}^g(p,q) + B_{F,\text{sym}}^g(p,s) - B_{F,\text{sym}}^g(p,r) - B_{F,\text{sym}}^g(q,s)
\]

(21)

By combining the assumption and the definition of the symmetric \( g \)-Bregman divergence (8) and (14), we have

\[
- B_{F,\text{sym}}^g(r,s) = B_{F,\text{sym}}^g(p,r) + B_{F,\text{sym}}^g(p,s) - B_{F,\text{sym}}^g(p,r) - B_{F,\text{sym}}^g(q,s).
\]

(22)

Exchanging \( p \leftrightarrow r \) and \( q \leftrightarrow s \) in (22) and taking the sum with (22), the result follows. This theorem can be also proved in the same way as mentioned in the paper [13].

**Definition 4. (Definition of the multivariate skew \( g \)-Jensen divergence)**

Let \( \Omega \) be a \( g \)-convex set and let \( g : \Omega \to \Omega \) be a function which satisfies \( g(p) = g(q) \iff p = q \). Let \( F : \Omega \to \mathbb{R} \) be a strictly convex function. Let \( p_\nu(\nu = 1,2,3 \cdots, N) \) be points in \( \Omega \). Let \( \alpha_\nu \geq 0(\nu = 1,2,3 \cdots, N) \) be parameters which satisfy \( \sum_{\nu=1}^N \alpha_\nu = 1 \).

We define the multivariate skew \( g \)-Jensen divergence as follows.

\[
J_{F,\alpha}^g(p_1,p_2,\cdots,p_N) \overset{\text{def}}{=} \sum_{\nu=1}^N \alpha_\nu F(g(p_\nu)) - F\left(\sum_{\nu=1}^N \alpha_\nu g(p_\nu)\right),
\]

(23)

where \( \alpha \) denotes a vector \((\alpha_1,\alpha_2,\cdots,\alpha_N)\).

**Theorem 2. (The centroids of the \( g \)-Bregman divergence)**

Let \( \Omega \) be a \( g \)-convex set. Let \( p_\nu(\nu = 1,2,3 \cdots, N) \) and \( q \) are points in \( \Omega \). Let \( \alpha_\nu \geq 0(\nu = 1,2,3 \cdots, N) \) be parameters which satisfy \( \sum_{\nu=1}^N \alpha_\nu = 1 \).

Then, the following inequality holds.

\[
\sum_{\nu=1}^N \alpha_\nu B_{F,\text{sym}}^g(p_\nu,q) \geq J_{F,\alpha}^g(p_1,p_2,\cdots,p_N)
\]

(24)

Equality holds if and only if

\[
q = g^{-1}\left(\sum_{\nu=1}^N \alpha_\nu g(p_\nu)\right)
\]

(25)

Because the function \( g \) is injective by the definition of the \( g \)-Bregman divergence, this value is the quasi-arithmetic mean. For example,

\[
g(p) = \begin{cases} 
  p & (q: \text{the arithmetic mean}) \\
  \ln p & (q: \text{the geometric mean}) \\
  p^{-1} & (q: \text{the harmonic mean}) \\
  p^r & (q: \text{the power mean}).
\end{cases}
\]

(26a) (26b) (26c) (26d)
Generalized Bregman and Jensen Divergences Which Include Some \( F \)-Divergences

**Proof.** We can prove this theorem in the same way as Theorem 3.1 in [17]. By the definition of the \( g \)-Bregman divergence, we have

\[
\sum_{\nu=1}^{N} \alpha_{\nu} B_{p}^{g}(p_{\nu}, q) = \sum_{\nu=1}^{N} \alpha_{\nu} \left( F(g(p_{\nu})) - F(g(q)) - \langle g(p_{\nu}) - g(q), \nabla F(g(q)) \rangle \right)
\]  

(27)

Let \( g(p') \triangleq \sum_{\nu=1}^{N} \alpha_{\nu} g(p_{\nu}) \). The RHS of (27) yields

\[
\left( \sum_{\nu=1}^{N} \alpha_{\nu} F(g(p_{\nu})) - F(g(p')) \right) + \left( F(g(p')) - F(g(q)) \right) - \sum_{\nu=1}^{N} \alpha_{\nu} (g(p_{\nu}) - g(q), \nabla F(g(q)))
\]

(28)

\[
= \left( \sum_{\nu=1}^{N} \alpha_{\nu} F(g(p_{\nu})) - F(g(p')) \right) + \left( F(g(p')) - F(g(q)) \right) - \sum_{\nu=1}^{N} \alpha_{\nu} (g(p_{\nu}) - g(q), \nabla F(g(q)))
\]

\[
= \sum_{\nu=1}^{N} \alpha_{\nu} F(g(p_{\nu})) - F\left( \sum_{\nu=1}^{N} \alpha_{\nu} g(p_{\nu}) \right) \geq J_{F, \alpha}^{g}(p_1, p_2, \cdots, p_N).
\]

Because \( B_{p}^{g}(p', q) \) equal to zero if and only if \( q = p' = g^{-1}\left( \sum_{\nu=1}^{N} \alpha_{\nu} g(p_{\nu}) \right) \), we prove the theorem.

**Proposition 2.** For the \( g \)-Bregman divergence, the following equation holds.

\[
B_{p}^{g}(q, p) = B_{p}^{\hat{g}}(p, q),
\]

(29)

where \( F^{\star}(x) \triangleq \sup_{x} \{ \langle x, x \rangle - F(x) \} \) denotes the Legendre convex conjugate and \( \hat{g} \triangleq \nabla F \circ g \).

**Proof.** For the Bregman divergence, the equation \( B_{p}(q', p') = B_{p}^{\hat{g}}(p', \nabla F(q'), \nabla F(q')) \) holds [16, 17]. By putting \( p' = g(p) \) and \( q' = g(q) \), the result follows.

**Corollary 1.** Let \( \Omega \) be a \( g \)-convex set. Let \( p_{\nu} (\nu = 1, 2, 3 \cdots, N) \) and \( q \) are points in \( \Omega \). Let \( \alpha_{\nu} \geq 0 (\nu = 1, 2, 3 \cdots, N) \) be parameters which satisfy \( \sum_{\nu=1}^{N} \alpha_{\nu} = 1 \).

Then, the following inequality holds.

\[
\sum_{\nu=1}^{N} \alpha_{\nu} B_{p}^{g}(q, p_{\nu}) \geq J_{F, \alpha}^{g}(p_1, p_2, \cdots, p_N).
\]

(30)

Equality holds if and only if

\[
q = \hat{g}^{-1}\left( \sum_{\nu=1}^{N} \alpha_{\nu} g(p_{\nu}) \right)
\]

(31)

**Proof.** From Proposition 2, we obtain

\[
\sum_{\nu=1}^{N} \alpha_{\nu} B_{p}^{g}(q, p_{\nu}) = \sum_{\nu=1}^{N} \alpha_{\nu} B_{p}^{\hat{g}}(p_{\nu}, q)
\]

(32)

Applying Theorem 2 to this equation, the result follows.

From Theorem 2 and Corollary 1, we conclude the point which minimize the weighted average of the \( g \)-Bregman divergence is unique.
2.3. Relation between the \( g \)-Bregman and skew \( g \)-Jensen divergence. In this subsection, we derive inequality between the \( g \)-Bregman and the skew \( g \)-Jensen divergence. The following equations also hold for the \( g \)-Bregman and the skew \( g \)-Jensen divergence.

\[
B_{F}^{g}(q,p) = \lim_{\alpha \downarrow 0} sJ_{F,\alpha}^{g}(p,q) \tag{33}
\]
\[
B_{F}^{g}(p,q) = \lim_{\alpha \uparrow 1} sJ_{F,\alpha}^{g}(p,q) \tag{34}
\]

It is easily proved by putting \( p' = g(p) \) and \( q' = g(q) \) in (10) and (11).

Then, we show some lemmas.

**Lemma 1.** Let \( \Omega \) be a \( g \)-convex set and let \( p, q \) be points \( p, q \in \Omega \). For a parameter \( \alpha \in (0,1) \) and a point \( r \in \Omega \) which satisfies \( g(r) = (1-\alpha)g(p) + \alpha g(q) \), the \( g \)-Bregman divergence can be expressed as follows.

\[
sJ_{F,\alpha}^{g}(p,q) \defeq \frac{1}{\alpha(1-\alpha)} \left( (1-\alpha)B_{F}^{g}(p,r) + \alpha B_{F}^{g}(q,r) \right) \tag{35}
\]

It is easily proved in the same way as equation (12) by putting \( p' = g(p) \) and \( q' = g(q) \).

**Lemma 2.** Let \( \Omega \) be a \( g \)-convex set and let \( p, q \) be points \( p, q \in \Omega \). For a parameter \( \alpha \in (0,1) \) and a point \( r \in \Omega \) which satisfies \( g(r) = (1-\alpha)g(p) + \alpha g(q) \), the following equations hold.

\[
B_{F}^{g}(p,q) = B_{F}^{g}(p,r) + B_{F}^{g}(r,q) + \frac{\alpha}{1-\alpha} B_{F,\sym}^{g}(r,q) \tag{36}
\]
\[
B_{F}^{g}(q,p) = B_{F}^{g}(r,p) + B_{F}^{g}(q,r) + \frac{1}{\alpha} B_{F,\sym}^{g}(r,p) \tag{37}
\]

**Proof.** We first prove (36). By the generalized law of cosines (17), we have

\[
B_{F}^{g}(p,q) = B_{F}^{g}(p,r) + B_{F}^{g}(r,q) + \langle g(r) - g(p) + \nabla F(g(q)) - \nabla F(g(r)) \rangle \tag{38}
\]

From the assumption, the equation

\[
g(r) - g(p) = \frac{\alpha}{1-\alpha} (g(q) - g(r)) \tag{39}
\]

holds. By substituting (39) to (38) and using (14), the result follows. By exchanging \( p \) and \( q \) in (38), we can prove (37) in the same way.

**Lemma 3.** Let \( \Omega \) be a \( g \)-convex set and let \( p, q \) be points \( p, q \in \Omega \). For a parameter \( \alpha \in (0,1) \) and a point \( r \in \Omega \) which satisfies \( g(r) = (1-\alpha)g(p) + \alpha g(q) \), the following equation holds.

\[
B_{F,\sym}^{g}(p,q) = \frac{1}{\alpha} B_{F,\sym}^{g}(p,r) + \frac{1}{1-\alpha} B_{F,\sym}^{g}(r,q) \tag{40}
\]

**Proof.** Taking the sum of (36) and (37), the result follows.

**Theorem 3.** (Bregman-Jensen inequality)

Let \( \Omega \) be a \( g \)-convex set and let \( p, q \) be points \( p, q \in \Omega \). For a parameter \( \alpha \in (0,1) \), the skew \( g \)-Jensen divergence and the symmetric \( g \)-Bregman divergence, the following inequality holds.

\[
B_{F,\sym}^{g}(p,q) \geq sJ_{F,\alpha}^{g}(p,q) \tag{41}
\]
Proof.
Let \( r \in \Omega \) be a point which satisfies \( g(r) = (1-\alpha)g(p) + \alpha g(q) \). From (35) and using \( B^g_{F}(p, q) \leq B^g_{F, \text{sym}}(p, q) \), we have
\[
sJ^g_{F, \alpha}(p, q) = \frac{1}{\alpha} B^g_{F}(p, r) + \frac{1}{1-\alpha} B^g_{F}(q, r) \leq \frac{1}{\alpha} B^g_{F, \text{sym}}(p, r) + \frac{1}{1-\alpha} B^g_{F, \text{sym}}(q, r). \tag{42}
\]
Using Lemma \( \ref{lemma:4} \) we have
\[
sJ^g_{F, \alpha}(p, q) \leq \frac{1}{\alpha} B^g_{F, \text{sym}}(p, r) + \frac{1}{1-\alpha} B^g_{F, \text{sym}}(r, q) = B^g_{F, \text{sym}}(p, q) \tag{43}
\]

Corollary 2. (Bregman-Jensen inequality for parallelograms)
Let \( \Omega \) be a \( g \)-convex set and let \( p, q, r, s \in \Omega \) be points which satisfy \( g(p) + g(r) = g(q) + g(s) \) (The points \( p, q, r, s \) are vertices of "parallelogram").

Let \( \alpha = \left( \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right) \).

Then, the following inequality holds.
\[
\frac{1}{8} \left( B^g_{F, \text{sym}}(p, q) + B^g_{F, \text{sym}}(q, r) + B^g_{F, \text{sym}}(r, s) + B^g_{F, \text{sym}}(s, p) \right) \geq J^g_{F, \alpha}(p, q, r, s) \tag{44}
\]

Proof. Let \( c \in \Omega \) be a point which satisfies \( g(c) = \frac{1}{2}(g(p) + g(r)) = \frac{1}{2}(g(q) + g(s)) \). From Theorem \( \ref{thm:3} \) we have
\[
B^g_{F, \text{sym}}(p, r) \geq sJ^g_{F, \frac{1}{2}}(p, r) = 4 \left( \frac{1}{2} F(g(p)) + \frac{1}{2} F(g(r)) \right) - F(g(c)) \tag{45}
\]
\[
B^g_{F, \text{sym}}(q, s) \geq sJ^g_{F, \frac{1}{2}}(q, s) = 4 \left( \frac{1}{2} F(g(q)) + \frac{1}{2} F(g(s)) \right) - F(g(c)) \tag{46}
\]
Taking the sum of (45) and (46) and applying Theorem \( \ref{thm:4} \) we have
\[
B^g_{F, \text{sym}}(p, q) + B^g_{F, \text{sym}}(q, r) + B^g_{F, \text{sym}}(r, s) + B^g_{F, \text{sym}}(s, p) \tag{47}
\]
\[
\geq 2 \left( F(g(p)) + F(g(q)) + F(g(r)) + F(g(s)) \right) - 8F(g(c))
\]
By the assumption, we get \( g(c) = \frac{1}{2}(g(p) + g(q) + g(r) + g(s)) \). By the definition of the multivariate skew \( g \)-Jensen divergence (Definition \( \ref{def:4} \)), the result follows.

3. Examples
We show some examples of the \( g \)-divergences.

We denote by "GBD" the \( g \)-Bregman divergence, by "SGBD" the symmetric \( g \)-Bregman divergence, "SGJD" by the skew \( g \)-Jensen divergence and by "BJ-inequality" the Bregman-Jensen inequality (11).

A parameter \( \alpha \) is in \((0,1)\) except for example 3) and we use a parameter \( \beta \in (0,1) \) instead of \( \alpha \) in example 3). The variables \( p_i \) and \( q_i \) are non-negative real numbers for all \( i \).

From example 1) to 6), we show the case that GBD belong to a class of \( f \)-divergence and example 7) is the case of information geometry [3,5] and statistical mechanics.

1) KL-divergence, Jeffreys divergence, JS-divergence

- Generating functions: \( F(p) = \sum_i p_i \ln p_i, \ g(p) = p \)
- GBD: generalized KL-divergence \( KL(p||q) \overset{\text{def}}{=} \sum_i (-p_i + q_i + p_i \ln \frac{p_i}{q_i}) \)
• SGBD: Jeffreys divergence (J-divergence) \[11\] \[ J(p, q) \overset{\text{def}}{=} \sum_i (p_i - q_i) \ln \left( \frac{p_i}{q_i} \right) \]

• SGJD: scaled skew JS-divergence
\[
\frac{1}{\alpha(1-\alpha)} JS_\alpha(p\|q) \overset{\text{def}}{=} \frac{1}{\alpha(1-\alpha)} \left( (1 - \alpha) KL(p\|(1 - \alpha)p + \alpha q) + \alpha KL(q\|(1 - \alpha)p + \alpha q) \right)
\]

• BJ-inequality: generalized Lin’s inequality \[\alpha(1 - \alpha)J(p, q) \geq JS_\alpha(p\|q)\]. When \(\alpha\) is equal to \(\frac{1}{2}\), this inequality is consistent with Lin’s inequality \[13\].

2) KL-divergence, Jeffreys divergence, \(\alpha\)-divergence

• Generating functions: \(F(p) = \sum_i \exp(p_i), \ g(p) = \ln p\)

• GBD: generalized reverse KL-divergence \(KL(q\|p) \overset{\text{def}}{=} \sum_i (p_i - q_i + q_i \ln \left( \frac{q_i}{p_i} \right))\)

• SGBD: J-divergence \(J(p, q) \overset{\text{def}}{=} \sum_i (p_i - q_i) \ln \left( \frac{p_i}{q_i} \right)\)

• SGJD: generalized \(\alpha\)-divergence \(D_{1 - \alpha}(p\|q)\),
\[
\text{where } D_\alpha(p\|q) \overset{\text{def}}{=} \sum_i \frac{1}{\alpha(1 - \alpha)} (p_i^\alpha q_i^{1 - \alpha} - \alpha p_i - (1 - \alpha) q_i)
\]

• BJ-inequality: \(J(p, q) \geq D_\alpha(p\|q)\)

3) \(\alpha\)-divergence

• Generating functions: \(F(p) = \frac{1}{1 - \alpha} \sum_i p_i^\alpha, \ g(p) = p^\alpha\) for \(\alpha \in \mathbb{R} \setminus \{0, 1\}\)

• GBD: generalized \(\alpha\)-divergence \(D_\alpha(p\|q)\)

• SGBD: symmetric \(\alpha\)-divergence \(D_{\alpha, \text{sym}}(p, q) \overset{\text{def}}{=} D_\alpha(p\|q) + D_\alpha(q\|p)\)

• SGJD: \(\frac{1}{\beta(1 - \beta)(1 - \alpha)} \sum_i \left( (1 - \beta) p_i + \beta q_i - \left( (1 - \beta) p_i^\alpha + \beta q_i^\alpha \right)^{\frac{1}{\alpha}} \right)\)

• BJ-inequality: \(D_{\alpha, \text{sym}}(p, q) \geq \frac{1}{\beta(1 - \beta)(1 - \alpha)} \sum_i \left( (1 - \beta) p_i + \beta q_i - \left( (1 - \beta) p_i^\alpha + \beta q_i^\alpha \right)^{\frac{1}{\alpha}} \right)\)

By using the equations \(\frac{1}{2}D_{\chi^2, 1}(p\|q) = H^2(p, q)\), \(2D_{\chi^2}(p\|q) = D_{\chi^2, p}(p\|q)\) and \(2D_{\chi^2, N}(p\|q)\) [9], we obtain example 4) to 6).

4) Hellinger distance

• Generating functions: \(F(p) = \sum_i p_i^2, \ g(p) = \sqrt{p}\)

• GBD: Hellinger distance \(H^2(p, q) \overset{\text{def}}{=} \sum_i (\sqrt{p_i} - \sqrt{q_i})^2\)

• SGBD: Hellinger distance \(2H^2(p, q)\)

• SGJD: Hellinger distance \(H^2(p, q)\)
In the dually flat space $\mathbb{M}_q$ functions called potential $\psi$.

Generalized Bregman and Jensen divergences which include some $F$-divergences.

In the dually flat space, the same equations and inequality hold when the distribution function belongs to exponential families. Because the manifold of the exponential family is equal to the skew Bhattacharyya distance [6, 16], and for the mixture families, $\text{SGJD:}$ canonical divergence.

BJ-inequality: $D_{\chi^2,p}(p,q) \geq \frac{2}{\alpha(1-\alpha)} \sum_i ((1-\alpha)p_i + \alpha q_i - \frac{p_i q_i}{(1-\alpha)q_i + \alpha p_i})$

5) Pearson $\chi$-square divergence

- Generating functions: $F(p) = -2 \sum_i \sqrt{p_i}, g(p) = p^2$
- GBD: Pearson $\chi$-square divergence $D_{\chi^2,p}(q,p) \overset{\text{def}}{=} \sum_i \frac{(p_i-q_i)^2}{q_i}$
- SGBD: symmetric $\chi$-square divergence $D_{\chi^2,\text{sym}}(p,q) \overset{\text{def}}{=} D_{\chi^2}(p\|q) + D_{\chi^2}(q\|p)$
- SGJD: $\frac{2}{\alpha(1-\alpha)} \sum_i ((1-\alpha)p_i - \alpha q_i + \sqrt{(1-\alpha)p_i^2 + \alpha q_i^2})$
- BJ-inequality: $D_{\chi^2,\text{sym}}(p,q) \geq \frac{2}{\alpha(1-\alpha)} \sum_i ((1-\alpha)p_i - \alpha q_i + \sqrt{(1-\alpha)p_i^2 + \alpha q_i^2})$

6) Neyman $\chi$-square divergence

- Generating functions: $F(p) = \sum_i p_i^{-1}, g(p) = p^{-1}$
- GBD: Neyman $\chi$-square divergence $D_{\chi^2,N}(q,p) \overset{\text{def}}{=} \sum_i \frac{(p_i-q_i)^2}{p_i}$
- SGBD: symmetric $\chi$-square divergence $D_{\chi^2,\text{sym}}(p,q)$
- SGJD: $\frac{1}{\alpha(1-\alpha)} \sum_i ((1-\alpha)p_i + \alpha q_i - \frac{p_i q_i}{(1-\alpha)q_i + \alpha p_i})$
- BJ-inequality: $D_{\chi^2,\text{sym}}(p,q) \geq \frac{1}{\alpha(1-\alpha)} \sum_i ((1-\alpha)p_i + \alpha q_i - \frac{p_i q_i}{(1-\alpha)q_i + \alpha p_i})$

7) Information geometry and statistical mechanics

In the dually flat space $M$, there exist two affine coordinates $\theta^i, \eta_i$ and two convex functions called potential $\psi(\theta), \phi(\eta)$. For the points $P, Q \in M$, we put $p_i = \theta^i(P), q_i = \theta^i(Q)$ or $p_i = \eta_i(P), q_i = \eta_i(Q)$.

- Generating functions: $F(p) = \psi(\theta) \text{ or } F(p) = \phi(\eta), g(p) = p$
- GBD: canonical divergence $D(P\|Q) \overset{\text{def}}{=} \phi(\eta(Q)) + \psi(\theta(P)) - \sum_i \eta_i(Q)\theta^i(P)$
- SGBD: $D_A(P,Q) \overset{\text{def}}{=} \sum_i (\eta_i(Q) - \eta_i(P))(\theta^i(Q) - \theta^i(P))$
- SGJD: $sJ_{\psi,\alpha}(\theta(P), \theta(Q)) \text{ or } sJ_{\phi,\alpha}(\eta(P), \eta(Q))$
- BJ-inequality: $D_A(P,Q) \geq sJ_{\psi,\alpha}(\theta(P), \theta(Q)) \text{ or } D_A(P,Q) \geq sJ_{\phi,\alpha}(\eta(P), \eta(Q))$

For the exponential or mixture families, GBD is equal to the KL-divergence and SGBD is equal to the J-divergence. For the exponential families, SGJD $sJ_{\psi,\alpha}(\theta(P), \theta(Q))$ is equal to the skew Bhattacharyya distance [6,16], and for the mixture families, SGJD $sJ_{\phi,\alpha}(\eta(P), \eta(Q))$ is equal to the skew-JS divergence (see [13] in detail).

In the case of canonical ensemble in statistical mechanics, the probability distribution function belongs to exponential families. Because the manifold of the exponential family is a dually flat space, the same equations and inequality hold.
for
\[ \theta = -\beta \equiv -\frac{1}{k_B T} \] (48)
\[ \eta = U \] (49)
\[ \psi = -\beta F \] (50)
\[ \phi = -\frac{1}{k_B} S, \] (51)
where \( k_B \) is the Boltzmann constant, \( T \) is temperature, \( U \) is the internal energy, \( S \) is the entropy and \( F \) is the Helmholtz free energy.

Defining a parameter \( \alpha \) as a index of the state which satisfy \( \frac{1}{T_\alpha} = (1 - \alpha) \frac{1}{T_0} + \alpha \frac{1}{T_1} \) or \( U_\alpha = (1 - \alpha) U_0 + \alpha U_1 \), BJ-inequality can be written as follows.
\[
\alpha(1 - \alpha) \left( \frac{T_1 - T_0}{T_0 T_1} \right) (U_1 - U_0) \geq \frac{F_\alpha}{T_\alpha} - (1 - \alpha) \frac{F_0}{T_0} - \alpha \frac{F_1}{T_1} \tag{52}
\]
\[
\alpha(1 - \alpha) \left( \frac{T_1 - T_0}{T_0 T_1} \right) (U_1 - U_0) \geq S_\alpha - (1 - \alpha) S_0 - \alpha S_1 \tag{53}
\]

4. Conclusion

We have introduced the \( g \)-Bregman divergence and the skew \( g \)-Jensen divergence by extending the definitions of the Bregman and the skew Jensen divergence respectively.

First, we have shown the geometrical properties of the \( g \)-Bregman divergence and the symmetric \( g \)-Bregman divergence.

Then, we have derived an inequality between the symmetric \( g \)-Bregman divergence and the skew \( g \)-Jensen divergence.

Finally, we have shown they include divergences which belong to a class of \( f \)-divergence.

If the relationship between these three classes of divergence are studied in more detail, it is expected to proceed applications to various fields including machine learning.

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