Regular resolution for CNFs with almost bounded one-sided treewidth

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September 1, 2022

Abstract

We introduce a one-sided incidence tree decomposition of a CNF $\phi$. This is a tree decomposition of the incidence graph of $\phi$ where the underlying tree is rooted and the set of bags containing each clause induces a directed path in the tree. The one-sided treewidth is the smallest width of a one-sided incidence tree decomposition.

We consider a class of unsatisfiable CNF $\phi$ that can be turned into one of one sided treewidth at most $k$ by removal of at most $p$ clauses. We show that the size of regular resolution for this class of CNFs is FPT parameterized by $k$ and $p$. The results contributes to understanding the complexity of resolution for CNFs of bounded incidence treewidth, an open problem well known in the areas of proof complexity and knowledge compilation. In particular, the result significantly generalizes all the restricted classes of CNFs of bounded incidence treewidth that are known to admit an FPT sized resolution.

The proof includes an auxiliary result and several new notions that may be of an independent interest.

1 Introduction

It is well known that an unsatisfiable CNF $\phi$ has a resolution proof of unsatisfiability of size FPT in the primal treewidth of $\phi$. On the other hand, it is a well known open problem whether there is an FPT sized resolution parameterized by the incidence treewidth of $\phi$ [7]. The only progress related to this problem we are aware of (besides the FPT upper bound parameterized by the primal treewidth) is establishing that resolution is FPT if parameterized by incidence pathwidth [10]. In this paper we establish an FPT upper bound significantly generalizing the parameterization by both primal treewidth and incidence pathwidth.

First of all, we consider the regular resolution rather than the full-fledged one. Second, we introduce the notion of a one-sided treewidth of a CNF $\phi$.

*The second author would like to thank Stefan Szeider and Friedrich Slivovsky for very useful personal communication.
This is the smallest width of a one-sided tree decomposition of the incidence graph of $\varphi$ meaning that the underlying tree is rooted and the set of bags containing each clause induces a directed path in the tree. Suppose that $\varphi$ has $p$ clauses such that after their removal, the resulting CNF has one-sided treewidth at most $k$. We prove that in this case, unsatisfiability of $\varphi$ can be proved by regular resolution of size $n \cdot 2^{O(k^2 + kp + p \log p)} + n^2 \cdot 2^{O(k^2 + kp)}$. Intuitively speaking the regular resolution is FPT for CNFs whose one-sided treewidth is almost bounded. As we said above, the considered class of CNFs significantly generalizes all the restricted classes of CNFs of bounded incidence treewidth for which an FPT upper bound on resolution is known. The result also carries an important message for researchers attempting to prove an XP lower bound for the general case: the target class of CNFs must be involved in the sense that in any rooted tree decomposition of the incidence graph of a CNF of this class there must be many clauses each of them appearing in bags of more than one branch of the underlying tree.

An important part of the proof is an auxiliary result stating that transitional resolution is FPT for CNFs of bounded one-sided treewidth. The transitional resolution is a novel generalization of regular resolution that may be of an independent interest so let us consider it in more detail. First of all, throughout this paper, we regard regular resolution as a read-once branching program $R$ with clauses associated with the sinks [12]. The semantic of this representation is the following. Let $u$ be a sink of $R$ and $C$ be the clause associated with $u$. Let $A$ be an assignment ’carried’ by a path from the source of $R$ to $u$. Then $A$ must falsify $C$. In the transitional resolution, we set aside a subset $\text{TR}$ of clauses of $\varphi$ and call them transitional clauses. In a transitional resolution, the sinks may be non-transitional and transitional ones. A non-transitional sink is associated with a non-transitional clause $C$ with the same semantic as described above. A transitional sink $u$ is associated with a subset $C$ of $\text{TR}$ and the semantic is as follows. Let $A$ be an assignment carried by a path from the source of $R$ to $u$. Then $A$ falsifies all the clauses of $C$ and there is an extension $A^*$ of $A$ that satisfies the remaining clauses of $\varphi$.

To understand the motivation, let $V$ be a proper subset of $\text{Var}(\varphi)$ the set of variables of $\varphi$. Consider $\varphi_V$, a CNF obtained by projection to $V$ of all the clauses having at least one occurrence of a variable of $V$. Suppose $\varphi_V$ is unsatisfiable. Can we use a regular resolution proof for $\varphi_V$ as a building block for the regular resolution proof for $\varphi$? Yes if it is transitional resolution! The transitional clauses are those that have occurrences of both $V$ and $\text{Var}(\varphi) \setminus V$. Indeed, a non-transitional clause $C$ of $\varphi_V$ is also a clause of $\varphi$. Therefore, a sink labelled with $C$ can also be a sink for a regular resolution for $\varphi$. However, each transitional clause of $\varphi_V$ is a proper subset of a clause of $\varphi$. Therefore, if such a clause is falsified, further reasoning is needed to derive a falsified clause for the whole $\varphi$. The transitional sinks provide all the required information for this reasoning. They serve as ‘interface’ or ‘connection ports’ that allow to ‘plug in a resolution for $\varphi_D$ into a resolution for $\varphi$. This approach is critical for our main result and we believe it will be useful in other contexts.

Let us now state formally our result regarding the transitional resolution.
Let $\varphi$ be an unsatisfiable CNF of one-sided treewidth $k$ and with $p$ transitional clauses. Then there is a transitional resolution proof for $\varphi$ of size $n \cdot 2^{O(kp+\log^2 p)}$. Put it differently, the resolution size remains FPT in $k$ only if $p = O(\log n/\log \log n)$.

Both results of this paper are obtained constructively, that is we demonstrate an algorithm for constructing the required resolution subject to the size upper bound. The transitional resolution is constructed using top-down dynamic programming. This means that we define local DAGs associated with the nodes $t$ of the tree of the underlying tree decomposition, the sinks are associated with sources of local DAGs of children of $t$. This approach is quite standard in knowledge compilation \[8, 13, 9\]. The difficulty of handling transitional clauses is that we need to implement the conjunction in the FBDD style \[2\]. This may potentially lead to exponential explosion. The most non-trivial aspect of the proof is to demonstrate that the explosion is tamed and the required upper bound is indeed achieved.

The regular resolution witnessing the main result is constructed using bottom up dynamic programming: the local DAGs are associated with prefixes of the postorder traversal of the underlying tree. To obtain the desired upper bound, we use the following nice fact. If for each non-leaf node of a rooted binary tree $T$, the number of nodes in the left subtree is greater than or equal to the number of nodes in the right subtree then the subgraph induced by each prefix of the postorder traversal is the union of at most $\lfloor \log n \rfloor + 1$ vertex-disjoint sub-trees of $T$. Bottom-up dynamic programming has been used, example in \[5\] and \[6\]. However, we are not aware of other results utilizing the above combinatorial statement about postorder traversals.

Let us now overview related results that we have not considered so far. Arguably, the best known parameter in the area of proof complexity is the width of a resolution proof. A classical result \[3\] demonstrates that the size of a resolution proof is exponential in its width. It was open for some time whether the resolution size is FPT in its width until it was resolved negatively in \[1\].

The equivalent definition of regular resolution as read-once branching program connects it to the area of knowledge compilation. On the surface, regular resolution is very similar to free binary decision diagram (FBDD): the difference only in the labelling of sinks and the corresponding semantics of source-sink paths. However, regular resolution is in fact much closer to decision decomposable negation normal forms (Decision DNNF) \[13\]: they both can easily handle conjunction of two variable disjoint CNFs. Decision DNNF simply has a decomposable decision gate at its disposal. Regular resolution does not have luxury of using such a gate but instead it has a not less impressive power of forgetting. Indeed if $\varphi = \varphi_1 \land \varphi_2$ is unsatisfiable and $\varphi_1$ is variable-disjoint with $\varphi_2$ then one of them is unsatisfiable. If, say $\varphi_1$ is unsatisfiable then a regular resolution proof for $\varphi_1$ is in fact such a proof for the whole $\varphi$, so that $\varphi_2$ can simply be discarded! On the contrast, FBDDs do not possess the gift of forgetting and have no conjunction gate at their disposal. As a result they cannot even efficiently represent CNFs of bounded primal treewidth \[14\]. It is also interesting to note that it is open whether CNFs of bounded incidence treewidth can be
represented by FPT-sized Decision DNNFs.

We conclude the literature overview by saying that investigation of new graph parameters is currently a bustling research direction. Notable examples include twin-width [4] and several variants of maximum matching width [11], the latter set of parameters is known to be of a significant relevance for knowledge compilation [15].

We conclude the introduction by overviewing the structure of the paper. There are five sections in the main body of the paper and three sections in the appendix. Section 2 is preliminaries. The auxiliary result (Theorem 1) is proved in Section 3. The main result (Theorem 2) is proved in Section 4. The conclusion is provided in Section 5. The proofs of both theorems consist of establishing correctness of the dynamic programming constructions and proving upper bounds on their sizes. While the latter is neat and compact, the former is rather tedious. Therefore, in the main body of the paper we provide only sketches of proofs of Theorems 1 and 2 postponing detailed proofs to the appendix that is structured as follows. Section A provides theorems essentially reducing the correctness proofs to proving that the local DAGs satisfy certain properties. A detailed proof of Theorem 1 is provided in Section B and a detailed proof of Theorem 2 is provided in Section C of the appendix.

2 Preliminaries

2.1 Set of literals, CNFs, and DAGs.

In this paper when we consider a set \( S \) of literals of Boolean variables, we mean that \( S \) is well formed in the sense that it does not contain both positive and negative literals of the same variable. A variable \( x \) occurs in \( S \) if either \( x \in S \) or \( \neg x \in S \). In the former case, we say that \( x \) occurs positively or has a positive assignment in \( S \). In the latter case, we say that \( x \) occurs negatively or has a negative assignment in \( S \).

We use \( \text{Var}(A) \) to denote the set of variables of an object \( A \) which may be a set of literals, a CNF, a graph with sets of variables associated with its vertices (e.g. a tree decomposition) or a branching program.

We consider a CNF \( \varphi \) as a set of clauses and each clause is just a set of literals. Let \( C \) be a clause and let \( S \) be a set of literals. We say that \( S \) satisfies \( C \) if \( C \cap S \neq \emptyset \). If \( S \) does not satisfy \( C \) then \( C \vert_S = C \setminus \neg S \) where \( \neg S = \{ \neg x \mid x \in S \} \). We denote by \( \varphi \vert_S \) the set of all \( C \vert_S \) such that \( C \) is a clause of \( \varphi \) that is not satisfied by \( S \). A CNF \( \varphi' \) is a subCNF of \( \varphi \) if \( \varphi' \subseteq \varphi \) and the subCNF is modular if \( \text{Var}(\varphi) \cap \text{Var}(\varphi \setminus \varphi') = \emptyset \).

For a set \( S \) of literals and a set \( V \) of variables, we denote by \( \text{Proj}(S, V) \) the projection of \( S \) to \( V \) that is \( S' \subseteq S \) such that \( \text{Var}(S') = \text{Var}(S) \cap V \).

The primal graph of a CNF has vertices corresponding to \( \text{Var}(\varphi) \) and two variables are adjacent if they occur in the same clause. The incidence graph of a CNF has vertices corresponding to the variables and clauses of \( \varphi \) and there is an edge between \( x \in \text{Var}(\varphi) \) and \( C \in \varphi \) if \( x \) occurs in \( C \).
In this paper we often consider directed acyclic graphs (DAGs) (including rooted binary trees). For a DAG $Z$ and $u \in V(Z)$, we denote by $Z_u$ the subgraph of $Z$ including $u$ and all the vertices reachable from $u$.

### 2.2 One-sided tree decomposition

A **tree decomposition** $(T, B)$ of a graph $G$ is a pair where $T$ is a tree, $B$ is a set of bags $B(t)$ associated with each node $t \in V(T)$. The bags must obey the rules of (i) union ($\bigcup_{t \in V(T)} B(t) = V(G)$) (ii) containment ($\forall e \in E(G) \exists t \in V(t)e \subseteq B(t)$) and (iii) connectivity (for each vertex $v \in V(G)$, the set of nodes whose bags contain $v$ induces a connected subgraph of $T$). The width of the tree decomposition is the largest size of a bag minus one. The treewidth of a graph is the smallest width of its tree decomposition.

In this paper we will consider tree decompositions for the incidence graph of a CNF $\varphi$ (**incidence tree decompositions of $\varphi$**). In this case, for each node $t$ of the underlying tree, $B(t)$ is partitioned into $\text{Var}(t)$ and $\text{CL}(t)$ respectively corresponding to variables and clauses.

**Definition 1** Let $(T, B)$ be a rooted incidence tree decomposition of $\varphi$. For each clause $C$ of $\varphi$, let $V_C$ denotes the set of nodes of $T$ whose bags contain $C$. Then $(T, B)$ is one-sided if for each $C \in \varphi$, $T[V_C]$ is a directed path.

The one-sided (incidence) treewidth of $\varphi$ is the smallest width of a one-sided incidence tree decomposition of $\varphi$.

It is not hard to see that the one-sided treewidth of $\varphi$ 'dominates' both the incidence pathwidth and the primal treewidth of $\varphi$ in the following sense.

**Proposition 1** The one-sided treewidth of $\varphi$ does not exceed the incidence pathwidth of $\varphi$ and is at most the primal treewidth of $\varphi$ plus one.

**Proof.** An incidence path decomposition can be turned into a one-sided one by making an end vertex of the underlying path the root.

Let $(T, B)$ be a primal tree decomposition of $\varphi$. Apply the following standard transformation. For each non-empty clause $C$, identify a node $t$ of $T$ such that $B(t)$ includes all the variables of $C$. Introduce a new node $t'$ connected to $t$ and make $\text{Var}(C) \cup \{C\}$ the bag of $t'$.

Clearly we obtained an incidence tree decomposition of $\varphi$ of width at most the width $(T, B)$ plus one where each clause $C$ is present in at most one node. Therefore, whatever node of the underlying tree is made the root, the resulting decomposition will be one-sided. ■

**Proposition 2** Let $(T, B)$ be a one-sided incidence tree decomposition of $\varphi$. Then $(T, B)$ can be replaced by a one sided incidence tree decomposition $(T^*, B^*)$ of $\varphi$ with the number of bags linear in $|\varphi| + |\text{Var}(\varphi)|$, whose width is not larger than that of $(T, B)$ and $T^*$ being a binary tree.

**Proof.** First we produce a tree decomposition of a linear size while keeping one-sidedness. This is done is the following two stages.
1. Remove each leaf whose bag is a subset of its parent.

2. For each non-root node \( u \) having one child such that \( B(u) \subseteq B(v) \) where \( v \) is the parent of \( u \), contract \( u \).

Note that the result of the first step is that each leaf is associated with a unique element of \( \varphi \cup Var(\varphi) \). This ensures that the number of leaves is at most \(|\varphi| + |Var(\varphi)|\). The same argument bounds the number of non-root nodes with one child. Taking into account that the number of non-root nodes with two or more children is not greater than the number of leaves and, adding the non-root nodes implies that the total number of nodes is at most \( 3 \times (|\varphi| + |Var(\varphi)|) \).

Clearly, these transformations do not violate the read-onceness.

If \( T \) is binary we are done. Otherwise, let \( u \) be a node of \( T \) with children \( v_1, \ldots, v_q \) such that \( q \geq 3 \). We obtain a one-side incidence tree decomposition \((T', B')\) of \( \varphi \) of width not larger than that of \((T, B)\) by applying the following transformation.

1. Introduce a new node \( u' \).
2. Let \( u', v_q \) be the children of \( u \) and \( v_1, \ldots, v_{q-1} \) the children of \( u' \).

Clearly by repeated application of this transformation we will eventually obtain an underlying tree which is binary. It remains to specify the content of the bags of the nodes involved. The bags of \( u, v_1, \ldots, v_q \) are exactly the same as for \((T, B)\). The bag for \( u' \) is constructed as follows.

1. \( Var(u') = Var(u) \).
2. \( CL(u') = CL(u) \setminus CL(v_q) \).

Let \( C \in CL(V_q) \cap CL(u) \). By the one-sidedness property, \( C \) does not occur in bags of any \( v_1, \ldots, v_{q-1} \). Therefore, the connectedness property regarding \( C \) is preserved. The rest of the required properties of a one-sided incidence tree decomposition are easy to verify by direct inspection.

Note that this transformation does change the number of leaves nor the number of nodes having one child. Hence the counting argument made before the transformation applies and we still have the upper bound of \( 3 \times (|\varphi| + |Var(\varphi)|) \).

The notions of \( Var(t) \) and \( CL(t) \) naturally extend to structures containing several nodes of \( T \). For example, if \( T_1 \) is a subtree \( T \) then \( Var(T_1) = \bigcup_{t \in V(T')} Var(t) \). If \( x \in Var(T_1) \), we may also say that \( x \) is contained in \( T_1 \). Also, if \( T \) is a subset of subtrees of \( T \) then \( Var(T) = \bigcup_{T' \subseteq T} Var(T') \).

### 2.3 Postorder traversals

Let \( T \) be a rooted binary tree. For each non-leaf node \( u \) we identify its left and right children as follows. If \( u \) has only one child \( v \) then \( v \) is considered the left child of \( u \). If \( u \) has two children \( v_1 \) and \( v_2 \) and, say, \( |V(T_{v_1})| > |V(T_{v_2})| \) then \( v_1 \)
is the left child of $u$ and $v_2$ is the right child of $u$. Finally, $|V(T_u)| = |V(T_{v_2})|$ then the left and the right children are assigned in an arbitrary (but fixed) way.

Having defined the left and right children for the non-leaf nodes of $T$, we can define the permutation $\pi_T$ of the nodes of $T$ explored according to the postorder traversal (left subtree of $T$, if any, is recursively traversed then the right subtree, if any, is recursively traversed, then the root is traversed).

Let $\pi$ be a prefix of $\pi_T$. It is not hard to see that $T[\pi_1, \pi_k]$ is the union of vertex disjoint trees $T_u$ (meaning subtrees of $T$ rooted by some vertices $u \in V(T)$). We denote the set of these trees by $\text{Trees}_\pi$.

**Proposition 3** Let $T_1, T_2$ be two distinct elements of $\text{Trees}_\pi$. Then one of them, say, $T_1$ occurs before the other in the following sense: all the nodes of $T_1$ precede in $\pi$ all the nodes of $T_2$.

**Proof.** By induction on $|V(T)|$. The statement is clearly true for $|V(T)| = 1$ so assume that $|V(T)| > 1$.

Let $rt$ be the root of $T$. If $T$ has only one child, the statement is easily seen to hold by the induction assumption. So, assume that $rt$ has two children $t_1$ (the left child) and $t_2$ (the right child). Denote the respective postorder traversals for $T_{t_1}$ and $T_{t_2}$ by $\pi_1$ and $\pi_2$. Clearly $\pi_T = \pi_1 + \pi_2 + rt$. Assume that $rt \in \pi$. Then $T_{rt}$ is the only elements of $\text{Trees}_\pi$ and hence the statement is vacuously true. Next, assume that $t_1 \notin \pi$. It follows that $\pi \subseteq \pi_1$ and hence the statement holds by the induction assumption.

Assume now that $t_1 \in \pi$. This means that $\pi_1$ is a prefix of $\pi$. Let $\pi = \pi_1 + \pi'$. Clearly $T_{t_1} \in \text{Trees}(\pi)$ and it is the largest element of $\text{Trees}_\pi$ according to the order specified in the statement of the proposition. Moreover, $\text{Trees}_\pi \setminus \{T_{t_1}\}$ are all in $\pi'$. The order between them exists by the induction assumption. It remains to say that the order between the trees in $\pi'$ is the same as in $\pi$. ■

Proposition 3 naturally defines a linear order on $\text{Trees}_\pi$. In what follows, if we say that for $T_1, T_2 \in \text{Trees}_\pi$ $T_1 < T_2$, we mean that $T_1$ occurs before $T_2$ in $\pi$ as specified in Proposition 3.

**Definition 2** Let $\pi$ be a proper prefix of $\pi_T$ and let $t$ be its immediate successor (that is, the node immediately following $\pi$ in $\pi_T$). If $t$ is a leaf of $T$, we call $t$ an expanding node. Otherwise $t$ is a contracting node.

**Proposition 4** Let $\pi$ be a prefix of $\pi_T$. Then $|\text{Trees}_\pi| \leq \lceil \log |V(T)| \rceil + 1$.

**Proof.** By induction on $|V(T)|$. The statement is clearly true for $|V(T)| = 1$ so assume that $|V(T)| > 1$.

Let $rt$ be the root of $T$. If $T$ has only one child, the statement is easily seen to hold by the induction assumption. So, assume that $rt$ has two children $t_1$ (the left child) and $t_2$ (the right child). Then the reasoning is done through the following case analysis.

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1Here and in several other places we slightly abuse notation by identifying a sequence with its underlying set. The correct interpretation will always be clear from the context.

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1. \( rt \in \pi \). This is possible only if \( \pi = \pi_T \) But in this case \( |Trees_{\pi}| = 1 \).

2. \( t_1 \not\in \pi \). It follows that \( \pi \subseteq V(T_{t_1}) \) and hence the statement holds regarding \( V(T_{t_1}) \) by the induction assumption and hence holds regarding \( V(T) \).

3. Assume that \( t_1 \in \pi \). Let \( \pi' \) be the subsequence of \( \pi \) consisting of all elements of \( T_{t_2} \). Clearly \( Trees(\pi) = \{T_{t_1} \cup Trees_{T_{t_2} \pi'} \} \) (the subscript \( T_{t_2} \) is used to emphasize that the set is considered w.r.t. \( T_{t_2} \) rather than \( T \). By the induction assumption, \( |Trees(\pi)| \leq 2 + \lfloor \log \frac{|V(T_{t_2})|}{2} \rfloor \). By selection of the left and right children \( |V(T_{t_2})| < \frac{|V(T)|}{2} \) and hence \( \lfloor \log |V(T_{t_2})| \rfloor \leq \lfloor \log |V(T)| \rfloor - 1 \).

Finally, we introduce two more notations. For a set \( T \) of subtrees of \( T \), we denote the set of roots of \( T \) by \( \text{Roots}(T) \). Also, the last tree of \( Trees_{\pi} \) according to the above order is denoted by \( \text{last}(Trees_{\pi}) \).

### 2.4 Regular resolution and \( \varphi \)-based functions

The compressed decision tree defined below is essentially a read-once branching program where the sinks are not labelled and no related semantics is provided. In terms of programming languages, this notion can be thought of as an abstract class. We find this generic notion very handy as we can then easily define several related models by simply specifying labelling of links and providing constraints on source-sink paths.

**Definition 3 (Compressed Decision Tree)** A Compressed Decision Tree (CDT) \( H \) is a DAG with a single source with the following additional properties.

1. Each non-terminal node is associated with a variable.
2. Each non-terminal node \( u \) has exactly two out-neighbours, one of them is labelled with \( \neg x \), the other is labelled with \( x \), where \( x \) is the variable labelling \( u \).
3. Read-onceness property: on each directed path \( P \) of \( H \), no variable occurs twice as a literal labelling an edge of \( P \).

For a directed path \( P \) of \( H \), we denote by \( A(P) \) the set of literals labelling the edges of \( P \). We sometimes refer to \( A(P) \) as the assignment carried by \( P \).

If the underlying graph of \( H \) is a tree then \( H \) is called a decision tree. If \( A \) is the set of all assignments carried by root-leaf paths of \( H \), we sometimes say that \( H \) is a decision tree over \( A \).

**Definition 4 (Regular resolution)** A regular resolution (RR) of a CNF \( \varphi \) is a CDT \( H \) with clauses associated with the sinks such that the following holds.

Let \( u \) be a sink of \( H \), let \( C \) be a clause associated with \( u \) and let \( P \) be a path from the root to \( u \). Then \( A(P) \) falsifies \( C \).
Definition 5 (\(\varphi\)-based functions) Let \(\varphi\) be a CNF. A \(\varphi\)-based function \(f\) is a function with the domain \(\text{dom}(f) \subseteq \varphi\) and, for each \(C \in \text{dom}(f)\), \(f(C) \subseteq C\). We denote by \(\text{range}(f)\) the set of all clauses \(C'\) such that there is a clause \(C \in \text{dom}(f)\) such that \(f(C) = C'\).

We identify two special \(\varphi\)-based functions. The first is the \(C \to ()\) function for each \(C \in \varphi\). where \(\text{dom}(C \to ()) = \{C\}\) and \(C\) is mapped to (\(\nothing\)). The second is the identity function \(1_\varphi\) with \(\text{dom}(1_\varphi) = \varphi\) and for each \(C \in \varphi\), \(1_\varphi(C) = C\).

We also say that a \(\varphi\)-based function \(f\) is unsatisfiable if \(\text{range}(f)\) is unsatisfiable.

Definition 6 1. Let \(f_1\) and \(f_2\) be two \(\varphi\)-based functions such that for each \(C \in \text{dom}(f_1) \cap \text{dom}(f_2)\), \(f_1(C) = f_2(C)\). The union \(f = f_1 \cup f_2\) is a function with \(\text{dom}(f) = \text{dom}(f_1) \cup \text{dom}(f_2)\), for each \(C \in \text{dom}(f_1)\), \(f(C) = f_1(C)\) and for each \(C \in \text{dom}(f_2)\), \(f(C) = f_2(C)\). Functions \(f_1 \cap f_2\) and \(f_1 \setminus f_2\) are defined accordingly.

2. Let \(f\) be a \(\varphi\)-based function and let \(A\) be a set of literals. Let \(C \subseteq \text{dom}(f)\) be the subset of \(\text{dom}(f)\) consisting of all clauses \(C\) such that \(f(C)\) is satisfied by \(A\). Then \(f\vert_A\) is a function whose domain is \(\text{dom}(f) \setminus C\) and for each \(C \in \text{dom}(f\vert_A)\), \(f\vert_A(C) = f(C)\vert_A\).

3. Let \(f\) be a \(\varphi\)-based function \(C \subseteq \text{dom}(f)\). Then the restriction of \(f\) to \(C\) is a function \(f'\) with \(\text{dom}(f') = C\) and for each \(C \in C\), \(f'(C) = f(C)\). We also sometimes refer to \(f'\) as a subfunction of \(f\). Another way to define a restriction is to specify the set of clauses \(C^*\) to be removed from the domain. The resulting function is denoted by \(f \setminus C^*\).

Remark. Let \(f\) be a \(\varphi\)-based function and let \(C \subseteq \text{dom}(f)\). Then \(f(C) = \{f(C)\vert_C\} \subseteq C\).

Definition 7 (Functional regular resolution) A Functional Regular Resolution (FRR) is a CDT \(H\) with \(\varphi\)-based functions associated with (labelling) its root and its sinks so that the following holds. Let \(f^*\) be the function associated with the root of \(H\). Let \(u\) be a sink of \(H\) and let \(f\) be the function associated with \(u\). Then \(\text{dom}(f) \subseteq \text{dom}(f^*)\). Moreover, let \(P\) be a root-\(u\) path. Then for each \(C \in \text{dom}(f)\), \(A(P)\) does not satisfy \(C\) and \(f(C) = f^*(C)\vert_{A(P)}\). \(H\) is a falsifying FRR if each sink of \(H\) is labelled with \(C \to ()\) for some \(C \in \text{dom}(f^*)\).

FRR is a more general structure than RR due to the following easy to establish argument.

Proposition 5 Let \(Z\) be a falsifying FRR for \(1_\varphi\). Then a RR for \(\varphi\) can be obtained from \(Z\) by changing each label \(C \to ()\) of a sink just to \(C\).

In light of the above proposition, we will construct FRRs instead of RRs in the subsequent sections. The benefit of this increased generality is that a composition of FRRs is easier to describe formally that a composition of RRs because in the former case we do not need to change the labels of sinks. Indeed,
suppose that \( R \) is an FRR for a \( \varphi \)-based function \( F \) and let \( u \) be a non-root node. Let \( A \) be an assignment carried by a path from the source of \( R \) to \( u \). Then it can be shown that \( R_u \) is an FRR for \( F|_A \). On the other hand, \( R \) is also an RR for \( \text{range}(F) \) (subject to changing all the sink labels \( C \rightarrow () \) to \( C \). However, in this case \( R_u \) is not an RR for \( \text{range}(F)|_A \) because the labels of the sinks are the clauses of \( \text{range}(F) \) not of \( \text{range}(F)|_A \)!

3 Regular resolution with transitional clauses

Throughout this section \( \varphi \) is an unsatisfiable CNF and \((T,B)\) is a one-sided tree decomposition of the incidence graph of \( \varphi \) having width at most \( k \). In light of Proposition 2 we assume that \( T \) is a binary tree. To define a transitional resolution we need to identify for the given CNF \( \varphi \) a subset \( \text{TR} \) of clauses that we call transitional. We need two auxiliary definitions.

**Definition 8 (Falsifier)** Let \( f \) be a \( \varphi \)-based function and let \( C \subseteq \text{dom}(f) \cap \text{TR} \). An assignment \( A \) is a \( C \)-falsifier (for \( f \)) if the following conditions hold.

1. for each \( C \in C \), \( f(C) \) is falsified by \( A \).
2. There is a witnessing extension \( A' \) of \( A \) such that for each \( C \in \text{dom}(f) \backslash C \), \( A \cup A' \) satisfies \( f(C) \).

**Definition 9 (Transition function)** A \( \psi \)-based function \( f \) is called transitional if \( \text{dom}(f) \subseteq \text{TR} \) and \( \text{range}(f) = \{()\} \). We refer to a transition function \( f \) with \( C = \text{dom}(f) \) by \( \text{trans}_C \).

**Definition 10 (Transitional resolution)** A transitional resolution (TRes) is a CDT \( R \) whose source is associated with a \( \varphi \)-based function \( f \) and the sinks are partitioned into two types of nodes: terminal and transitional ones.

Each terminal node \( u \) is associated with a function \( C \rightarrow () \) for some \( C \in \text{dom}(f) \backslash \text{TR} \) and for each path \( P \) from the source to \( u \), \( A(P) \) falsifies \( f(C) \).

Each transitional node \( u \) is associated with a transitional function \( \text{trans}_C \) with \( C \subseteq \text{dom}(f) \cap \text{TR} \) such that for each path \( P \) from the source to \( u \), \( A(P) \) is a \( C \)-falsifier.

\( R \) is a TRes for \( \varphi \) if \( f = 1_\varphi \).

**Theorem 1** Let \( n = |\text{Var}(\varphi)| + |\varphi| \). Then there is a transitional resolution for \( \varphi \) of size \( n \cdot 2^{O(k \cdot |\text{TR}| + \log^2(|\text{TR}|))} \) in case \( \text{TR} \neq \emptyset \) and \( n \cdot 4^k \) in case \( \text{TR} = \emptyset \). In other words, if \( |\text{TR}| = O(\log n / \log \log n) \), the size if FPT in \( k \) only.

In the next section, we use a TRes as a building block for construction of the FRR witnessing the main result (Theorem 2). Theorem 1 provides an upper bound on the size of the building blocks. In fact, the TRes occurring in the proof of Theorem 2 has its sets of transitional clauses of size at most \( k \), so the full power of Theorem 2 has expressed in its last sentence will not be needed. The
idea of the use of a TRes for construction of FRRs may be of an independent
interest, therefore, we briefly describe the underlying intuition.

Let $V_1, V_2$ be a partition of $\text{Var}(\varphi)$. The idea is to construct first a resolution
for a $\varphi$-based function mapping clauses to their projections to $V_1$ (of course the only
clauses for which these projections are non-empty) and then to somehow
extend the resolution to one for the whole $1_\varphi$.

$\varphi$ can be seen as the disjoint union of sets $C_0, C_1, C_2$, where $C_0$ consists
of clauses whose sets of variables have non-empty intersections with both $V_1$
and $V_2$ and for each $i \in \{1, 2\}$, $C_i = \{C | C \in \varphi, \text{Var}(C) \subseteq V_i\}$. Let $F_1$ be a
$\varphi$-based function with $\text{dom}(F_1) = C_0 \cup C_1$ and for each $C \in \text{dom}(F_1)$, $F_1(C) =
\text{Proj}(C, V_1)$. What is a possible use of $F_1$ for the construction of an FRR for
$1_\varphi$? First, if $F_1$ is satisfiable, then $C_2$ is an unsatisfiable set of clauses, so we
can construct $1_{C_2}$ instead of $1_\varphi$. If $F_1$ is unsatisfiable then the situation is more
complicated.

Let $Z_1$ be an falsifying FRR for $F_1$. Can it be extended to a falsifying FRR
for $1_\varphi$? No, not quite. A sink labelled with $C \rightarrow (\cdot)$ for $C \in C_1$ can indeed
be a sink for a falsifying FRR for $1_\varphi$. But if $C \in C_0$, this sink needs to be
further expanded. However, the information provided by the sink (falsification
of $\text{Proj}(C, V_1)$) is not sufficient for such an expansion. The issue of ‘information
transfer’ is completely resolved by the use of transitional resolution. Indeed,
suppose that $Z_1$ is a transitional resolution for $F_1$ with $C_0$ being the set of
transitional clauses. Then the clauses of $C_0$ only take part in the labels for
transitional sinks and these labels provide sufficient information for their further
expansion. Indeed, let $u$ be such a sink and assume that it is labelled with $\text{trans}_C$
for some $C \subseteq C_0$. This means that each assignment carried by a
source-$u$ path of $Z_1$ can be extended to an assignment over the whole $V_1$
that satisfies all the clauses of $C_0 \cup C_1 \setminus C$ but falsifies the projections to $V_1$ of
the clauses of $C$. Let now $F_2$ be a $\varphi$-based function with domain $C_2 \cup C$ mapping
each clause of the domain to its projection to $V_2$. From $\text{trans}_C$ being a label
of $u$, we conclude that $F_2$ is unsatisfiable. Let $Z_2$ be a falsifying FRR for $F_2$.
Identify the source of $Z_2$ with $u$ and perform the same operation for the rest of
transitional sinks of $Z_1$ (of course, w.r.t. their corresponding labels). It can be
shown that the resulting construction is a falsifying FRR for $1_\varphi$. In the actual
construction in the next section, the expansions of different transitional sinks
are not necessarily vertex disjoint, thus the construction is made more compact.

In fact what we need in the next section is the following corollary of Theorem

\begin{corollary}
Let $f$ be a $\varphi$-based unsatisfiable function. Then there is a transi-
tional resolution of $f$ of size as specified in Theorem 1. Moreover, the set
of variables of this transitional resolution is a subset of $\text{Var}(\text{range}(f))$
\end{corollary}

\begin{proof}
Let $\psi = \text{range}(f)$. First we show that the one-sided treewidth
of $\psi$ cannot be large than $k$. Indeed, $\text{dom}(f) \subseteq \varphi$, therefore the one-sided
treewidth of $\text{dom}(f)$ is at most $k$. Now, replace each $C$ with $f(C)$ (removing
double occurrences in case the mapping is not one to one). Clearly, the tree
each such a function \( f \) satisfies \( f_t \) functions associated with \( t \) nodes. This means that if \( t \) lying in the bags of the ancestors of \( t \) functions. Each such a function is induced by an assignment to the variables lying in the bags of the ancestors of \( t \). Assume first that, in \( R^* \), \( u \) be associated with \( C \rightarrow () \). This means that \( A(P) \) falsifies \( C \). In \( R^* \), \( u \) is associated with \( w(C) \), hence \( A(P) \) falsifies \( f(w(C)) = C \). Next we assume that, in \( R \), \( u \) be associated with \( trans_{C^*} \). We observe that that \( A(P) \) falsifies \( f(C) \) for each \( C \in C^* \) simply because \( f(C) \in C \) by definition of \( C^* \) and \( A(P) \) falsifies all the clauses of \( C \) by definition of \( R \). Next, there is an extension \( A \) of \( A(P) \) satisfies \( f(C') \) for each \( C' \in \psi \setminus C \). Now, we claim that \( A(P) \) satisfies \( f(C) \) for each \( C \in \phi \setminus C^* \) simply because \( f(C) \notin C \) by definition of \( C^* \) as a 'saturated' set. We have thus verified that \( R^* \) is indeed a transitional resolution.

Finally, we note that the proof of Theorem 1 uses only variables of \( I_\psi \) simply by construction of local graphs for the resolution, This means that, by construction \( Var(R^*) \subseteq Var(I_\psi) = Var(range(f)) \).

### 3.1 Proof of Theorem 1: Informal overview

It is instructive to first assume that the number of transitional clauses is zero. In this case, we associate with each node \( t \in V(T) \) an FPT number of \( \phi \)-based functions. Each such a function is induced by an assignment to the variables lying in the bags of the ancestors of \( t \). Then we construct a DAG \( R \) where, for each such a function \( f_t \), there is a node \( u \) such that \( R_u \) is an FRR of \( f_t \). One of these functions is \( I_\phi \) hence \( R_u \), where \( u \) be the node corresponding to \( I_\phi \) is the required FRR.

In order to construct \( R \), we order the nodes of \( T \) so that children occur before their parents. Then we order the functions in an arbitrary order adhering to the order of their respective nodes. This means that if \( t_1 < t_2 \) then all the functions associated with \( t_1 \) are ordered before all the functions associated with \( t_2 \). Let \( f_1, f_2, \ldots, f_q \) be the resulting sequence of functions, the nodes of \( R \) corresponding to them as above are denoted by \( u_1, \ldots, u_q \). For each \( f_i \), we
construct an FPT sized local subgraph $L_i$ whose source is $u_i$ and each sink is either a sink of the whole $R$ or is identified with some $u_j$ for $j < i$.

In order to see how the local subgraphs $L_i$ are created let us see in a greater detail how the functions are defined. In fact what we associate with each node $t \in V(T)$ are not functions but types, that is tuples of the form $tp = (t, CL, S)$ where $C \subseteq CL(t)$ and $S$ is an assignment to $Var(t) \cap Var(p(t))$ where $p(t)$ is the parent of $t$ (of course $S = \emptyset$ if $t$ is the root). Intuitively, $CL$ is the set of clauses satisfied by the assignment to the variables in the bags of the ancestors of $t$ and $S$ is the projection of this assignment to $Var(t) \cap Var(p(t))$. The functions are associated with the types, in particular, the function associated with $\psi$ and $S$ is the projection of this assignment to $Var(t)$.

Why do we discard the other type? The intuition is the following. Imagine an unsatisfiable type such that all clauses in the range of $tp$. The subCNF $(\neg x_1 \lor x_2) \land (\neg x_3 \lor x_4)$ is just $\phi$ of the assignment to the ancestors of $t$ on $CL(T_i)$.

As the number of types per node is at most $2^k$, so the total number of types and hence the number of functions is at most $2^k \cdot |V(T)|$. We consider only unsatisfiable types $tp$, that is those for which $f_{tp}$ is unsatisfiable. A detailed consideration is provided in the next subsection. For now, we consider only a single case that conveys the idea of the design of the local graphs.

Let $t \in V(T)$ be a non-leaf having two children $t_1$ and $t_2$ and let $tp = (t, CL, S)$ be an unsatisfiable type such that all clauses in the range of $f_{tp}$ are non-empty and, in addition, $Var(t) \setminus Var(S) \neq \emptyset$. In this case, the local subgraph $L_{tp}$ corresponding to $tp$ is nothing else then a decision tree over all possible assignments $A$ to $Var(t) \setminus Var(S)$ with the root corresponding to $tp$ and each leaf corresponding to a type of one of the children of $t$. Let us consider more in detail how we choose this type.

Each $i \in \{1, 2\}$, $tp$ and $A$ naturally induces a type $tp_i = (t_i, CL_i, S_i)$, where $S_i = Proj(S \cup A, Var(t) \cap Var(t_i))$ and $CL_i$ consists of $C \cap CL(t_i)$ and, in addition, those clauses of $CL(t_i)$ that are satisfied by $S \cup A$. It is not hard to see that one of $tp_1, tp_2$ must be unsatisfiable. The algorithm arbitrarily chooses an unsatisfiable type $tp_i$ and associates the vertex corresponding to $A$ with $tp_i$. The other type is simply discarded even if it is also unsatisfiable.

Why do we discard the other type? The intuition is the following. Imagine an unsatisfiable CNF $\psi$ consisting of two variable disjoint subCNFs $\psi_1, \psi_2$. Clearly, one of these subCNFs must be unsatisfiable, let it be say $\psi_1$. Then, in order to prove the unsatisfiability of $\psi$ using regular resolution, it is enough to do so for $\psi_1$. The subCNF $\psi_2$ can be discarded even if it is unsatisfiable as well.

**Example 1** Let $\varphi = (x_1 \lor x_2) \land (\neg x_1 \lor x_3) \land (\neg x_1 \lor x_4) \land (\neg x_3 \lor \neg x_4) \land (\neg x_2 \lor x_6) \land (\neg x_5 \lor \neg x_6)$ Consider the incidence tree decomposition $(T, B)$ where $V(T) = \{t, t_1, t_2, t_3, t_4\}$. We set $t$ to be the root with children $t_1$ and $t_2$, $t_3$ being the child of $t_1$ and $t_4$ being the child of $t_2$. Further on, we let $Var(t) = \{x_1, x_2\}$, $CL(t) = \{(x_1 \lor x_2)\}$, $Var(t_1) = \{x_1, x_3, x_4\}$, $CL(t_1) = \{(\neg x_1 \lor x_3), (\neg x_1 \lor x_4)\}$, $Var(t_2) = \{x_2, x_5, x_6\}$, $CL(t_2) = \{(\neg x_2 \lor x_5), (\neg x_2 \lor x_6)\}$, $Var(t_3) = \{x_3, x_4\}$, $CL(t_3) = \{(\neg x_3 \lor \neg x_4)\}$, $Var(t_4) = \{x_5, x_6\}$, $CL(t_4) = \{(\neg x_5 \lor \neg x_6)\}$.

Consider the type $tp = (t, \emptyset, \emptyset)$. It is not hard to see that range($f_{tp}$) is just
φ. Now let \( A = \{x_1, x_2\} \) and consider the types of the children of \( t \) induced by \( t \) and \( A \). These types are \( tp_1 = (t_1, S_1, C_1) \) and \( tp_2 = (t_2, S_2, C_2) \) where \( S_1 = \{x_1\} \) and \( C_1 = \{(-x_1 \lor x_3), (-x_1 \lor x_4)\} \), \( S_2 = \{x_2\} \), \( C_2 = \emptyset \).

It follows that \( \text{dom}(f_{tp_1}) = \{(-x_3 \lor -x_4)\} \) and the clause is mapped to itself. In other words, \( tp_1 \) is a satisfiable type. On the other hand, \( \text{dom}(f_{tp_2}) = \{(-x_2 \lor x_5), (-x_2 \lor x_6), (-x_5 \lor -x_6)\} \) and \( f_{tp_2}(-x_2 \lor x_5) = (x_5) \), \( f_{tp_2}(-x_2 \lor x_6) = (x_6) \), \( f_{tp_2}(-x_5 \lor -x_6) = (-x_5 \lor -x_6) \). Thus range(\( f_{tp_2} \)) = \{\( (x_5), (x_6), (-x_5 \lor -x_6)\)\) and clearly it forms an unsatisfiable CNF thus \( tp_2 \) is an unsatisfiable type. The branch of the local subgraph of \( t \) corresponding to \( S \) corresponds to \( tp_2 \).

Note, however that the assignment \( \{x_1, x_2\} \) induces two unsatisfiable types, one for each child and hence an arbitrary one of them can be chosen as the sink of the local subgraph of \( t \) corresponding to \( \{x_1, x_2\} \).

Let us now discuss the general case where the set \( \text{TR} \) of transitional clauses is not empty. In this case the types are of the form \( tp = (t, CL, S, ET) \) where \( ET \subset \text{TR} \). The idea is that \( ET \) is a subset of transitional clauses that are falsified by assignment to variables located in the bags of the ancestors of \( t \). The function corresponding to \( tp \) is \( f_{tp} \cup \text{trans}_{ET} \) where \( f_{tp} \) (with a slight abuse of notation) is the function corresponding to the shortened type \( (t, CL, S) \) as described above. Like in the case with \( \text{TR} = \emptyset \), the types are ordered and DAGs are gradually built so that for each type \( tp \), the resulting DAG contains a subgraph which is a transitional resolution of \( f_{tp} \cup \text{trans}_{ET} \). Like in the case with \( \text{TR} = \emptyset \), this DAG is constructed by adding a local subgraph for each type in the considered order. Most of the cases of construction of local graphs are easy extensions of the case of \( \text{TR} = \emptyset \). However, there is one case that is conceptually different. To understand it, consider the situation below.

Suppose that a CNF \( \varphi \) is a variable disjoint union of two unsatisfiable CNFs \( \varphi_1 \) and \( \varphi_2 \). Suppose next that both \( \text{TR} \cap \varphi_1 \) and \( \text{TR} \cap \varphi_2 \) are non-empty. Because of the last assumption we cannot use just one of \( \varphi_1 \), \( \varphi_2 \) and to discard the other one. Indeed, suppose we consider only \( \varphi_1 \) and discard \( \varphi_2 \). If all the sinks of the resulting resolution are terminal ones, good and well. But what if we have a transitional sink \( u \)? Consider an assignment \( A \) carried by a path ending at \( u \). Then \( A \) is a falsifier of some \( C \subset \text{TR} \cap \varphi_1 \) w.r.t. \( 1_{\varphi_1} \). However, \( A \) is not such a falsifier w.r.t. the whole \( 1_{\varphi} \) as, by assumption, \( A \) cannot be extended to an assignment satisfying all the clauses of \( \varphi_2 \). Put it differently, a transitional node needs information about all the clauses of \( \text{TR} \) not just part of them.

Thus in the considered situation the resolution must be applied to a conjunction of \( \varphi_1 \) and of \( \varphi_2 \). Obviously, the resolution does not have conjunction gates, so we use the standard trick of putting the resolution for say \( 1_{\varphi_1} \) ‘on top’ of the one for \( 1_{\varphi_2} \). We postpone the nuances of this arrangement to the stage of formal description and discuss here only the aspect of combinatorial explosion. If the DAGs of resolutions for \( 1_{\varphi_1} \) and \( 1_{\varphi_2} \) have already been defined then the ‘on top’ configuration requires creating a separate copy of the resolution of \( 1_{\varphi_1} \).

\(^2\)Note a slight abuse of not using the double brackets for the sake of a better readability.
Thus the local graph corresponding to $1_\varphi$ is of size comparable to the union of all the DAGs that have been created before. We refer to this situation as doubling. To mitigate its effect we, essentially compare the quantities $|\varphi_1 \cap \text{TR}|$ and $|\varphi_2 \cap \text{TR}|$ and put on top the one whose respective quantity is smaller (ties can be broken arbitrarily). This way we ensure that the sequence of nested doublings is of size at most $\log |\text{TR}|$.

### 3.2 Proof of Theorem 1

**Definition 11 (Bag-related type)** A (bag-related) type $tp$ is a quadruple $(t, S, CL, ET)$ where $t \in V(T)$, $S$ is an assignment with $\text{Var}(S) = \text{Var}(t) \cap \text{Val}(t^*)$ where $t^*$ is the parent of $t$ ($\text{Var}(S) = \emptyset$ if $t$ is the root), $CL \subseteq CL(t)$, and $ET \subseteq \text{TR}$. We may refer to the respective components of $tp$ by $t(tp), S(tp), CL(tp), ET(tp)$.

**Definition 12 (Associated functions)** The function $f_{tp}$ associated with $tp$ is a $\varphi$-based function with the domain consisting of the clauses of $CL(T_t)$ that do not belong to $CL \cup ET$ and are not satisfied by $S$. For each $C \in \text{dom}(f_{tp})$, $f_{tp}(C) = \text{Proj}(C, \text{Var}(T_t) \setminus \text{Var}(S))$.

We also denote the function $f_{tp} \cup \text{trans}_{ET}$ by $h_{tp}$

**Definition 13 (Unsatisfiable and basic types)** A type $tp$ is unsatisfiable if $h_{tp}$ is unsatisfiable. An unsatisfiable type $tp$ is basic if either (i) there is a non-transitional clause $C \in \text{dom}(f_{tp})$ such that $f_{tp}(C) = ()$ or (ii) $f_{tp}$ is satisfiable. In case (i), $tp$ is a non-transitional basic type with $C$ being a witnessing clause and in case (ii) it is a transitional one.

**Definition 14 (Extension for a type)** Let $tp$ be a type with $t = t(tp)$. An extension $S'$ of $tp$ is an assignment to $\text{Var}(t) \setminus \text{Var}(S)$. We denote the set of all extensions of $tp$ by $\text{Ext}(tp)$. In particular, if $\text{Var}(t) \setminus \text{Var}(S) = \emptyset$ then $\text{Ext}(tp) = \{\emptyset\}$.

**Definition 15 (Successor of a type)** Let $tp$ be a non-basic unsatisfiable type such that $t = t(tp)$ is not a leaf of $T$. Let $A \in \text{Ext}(tp)$ and let $t_1$ be a child of $t$. The $(t_1, A)$-successor of $tp$ is a type $tp_1$ such that $t(tp_1) = t_1$, $S(tp_1) = \text{Proj}(S \cup A, \text{Var}(t_1))$, $CL(tp_1)$ consists of $CL(tp) \cap CL(t_1)$ and all $C \in CL(t_1)$ such that $C$ is satisfied by $S(tp) \cup A$, $ET(tp_1)$ is the union of $ET(tp)$ and the set clauses $C \in \text{dom}(f_{tp}) \cap \text{TR}$ such that $f_{tp}(C)$ is falsified by $A$.

If $t$ has two children $t_1$ and $t_2$ then each extension $A$ of $tp$ identifies the $(t_1, A)$ and $(t_2, A)$-successors of $tp$. In most cases, during the construction of the transitional resolution, one of them will be chosen as the preferred successor. making the resulting construction similar to the case with $\text{TR} = \emptyset$. However, there is one particular case where the preferred successor cannot be chosen. We define this case through the notion of a special pair.
Definition 16 (Special pair) Let $tp$ be a non-basic unsatisfiable type such that $t = t(tp)$ has two children $t_1$ and $t_2$. Let $A \in \text{Ext}(tp)$. Then $(tp, A)$ is a special pair if (i) both $(t_1, A)$ and $(t_2, A)$-successors of $tp$ are unsatisfiable and nonbasic and (ii) both $\text{CL}(T_{t_1}) \cap \text{TRANS}$ and $\text{CL}(T_{t_2}) \cap \text{TRANS}$ are non-empty.

We also need two concepts related to special pairs: those of the main child and the main successor.

Definition 17 (The main child) Let $t$ be a node of $T$ having two children and let $t_1$ and $t_2$ be the children of $t$. Assume that $|\text{CL}(T_{t_1}) \cap \text{TRANS}| < |\text{CL}(T_{t_2}) \cap \text{TRANS}|$. Then we say that $t_1$ is the main child of $t$. If both intersections are of the same size then we arbitrarily fix a child of $t$ and name it the main child of $t$.

Definition 18 (The main and secondary successors.) Let $(tp, A)$ be a special pair and let $t = t(tp)$. Then clearly, $t$ has two children $t_1$ and $t_2$. Let $tp_1$ and $tp_2$ be the respective $(t_1, A)$ and $(t_2, A)$-successors of $tp$. Assume w.l.o.g. that $t_1$ is the main child of $t$. Then we refer to $tp_1$ and $tp_2$ as, respectively, the main and secondary successors of $(tp, A)$.

Now, we establish a linear order on the union of the set of unsatisfiable non-basic types and the set of special pairs.

Definition 19 Let $NS$ be the set of all unsatisfiable types and let $SP$ be the set of all special pairs. We fix a linear order $\text{ORD}$ satisfying the constraints as specified below.

For $r \in NS \cup SP$, the type $tp(r)$ is defined as follows. If $r = tp$ then $tp(r) = tp$ and if $r = (tp, A)$ then $tp(r) = tp$. Let $r_1, r_2 \in NS \cup SP$, let $tp_1 = tp(r_1)$, $tp_2 = tp(r_2)$, and let $t_i = t(tp_i)$ for $i \in \{1, 2\}$. Then $r_1 < r_2$ is mandatory in the following two cases

1. $t_1$ is a descendant of $t_2$.

2. $t_1 = t_2$ but $r_1$ is a special pair while $r_2$ is a type.

The above definition is somewhat tedious because of the need to consider types and special pairs within the same set. The idea, however, is simple. We first look at the nodes corresponding to $r_1$ and $r_2$ and if $t_1$ is an ancestor of $t_2$ in $T$ then $r_1$ is ahead of $r_2$. In the other case, $r_1$ and $r_2$ are associated with the same type. In this case $r_1$ must precede $r_2$ if $r_1$ is a special pair while $r_2$ is not. Otherwise, the ordering is arbitrary.

Definition 20 (Local subgraphs for elements of $\text{ORD}$) We define a graph $R$ as the union of local graphs $L_r$ for each $r \in \text{ORD}$. The graphs $L_r$ are defined inductively along $\text{ORD}$. In particular, when we define $L_r$ for some $r \in \text{ORD}$, we assume that for all $r' < r$, $L_{r'}$ have been defined. The set of vertices of each $L_r$ is disjoint with $\bigcup_{r' < r} V(L_{r'})$ except some sinks that may be identified with the sources of previous $L_{r'}$.
Assume first that \( r = \text{tp} \). \( L_{\text{tp}} \) is a single source DAG whose source is \( u_{\text{tp}} \). If \( \text{Var}(t) \setminus \text{Var}(S) = \emptyset \) Then \( u_{\text{tp}} \) is the only node of \( L_{\text{tp}} \) and the assignment \( A = A(u_{\text{tp}}) \) corresponding to it is \( \emptyset \). Otherwise, \( L_{\text{tp}} \) is a decision tree (with the root \( u_{\text{tp}} \)). Let \( u \) be a sink of the \( L_{\text{tp}} \). The assignment \( A = A(u) \) corresponding to \( u \) is \( A(P) \) where \( P \) is the \( u_{\text{tp}} \) to \( u \) path of \( L_{\text{tp}} \).

Now we make a decision regarding the sinks \( u \) of \( L_{\text{tp}} \). The decision is specified in the list below. Each item in the list specifies a condition and the decision made in case this condition holds. For each item but the first one, we assume that the conditions of the previous items do not hold.

We denote \( A(u) \) by \( A \). The children of \( t \) (if any) are denoted by \( t_1 \) and \( t_2 \), if \( t \) has only one child then it is \( t_1 \). Their respective successors of \( t \) are denoted by \( \text{tp}_1 \) and \( \text{tp}_2 \).

1. \( A \) falsifies \( f_{\text{tp}_1}(C) \) for some \( C \subseteq \text{dom}(f_{\text{tp}_1}) \setminus \text{TR} \). In this case \( u \) is labelled with \( C \rightarrow () \). If there are several such \( C \) choose one arbitrarily.
2. \( A \) falsifies \( f_{\text{tp}_1}(C) \) for \( C \subseteq \text{dom}(f_{\text{tp}_1}) \cap \text{TR} \) while \( (f_{\text{tp}_1}|A) \setminus C \) is satisfiable. In this case, \( u \) is labelled with \( \text{trans}_{\text{ET}(\text{tp}_1)} \cup C \).
3. \((\text{tp}, A)\) is a special pair. Then \( u \) is identified with the source of \( L_{(\text{tp}, A)} \).
4. \( t \) has only one child. Then \( u \) is identified with the source of \( L_{\text{tp}_1} \).
5. Satisfiable sibling case: \( t \) has two children and, say \( f_{\text{tp}_1} \) is satisfiable. Then \( u \) is identified with the source of \( L_{\text{tp}_2} \).
6. Non-transitional child case: \( t \) has two children and, say \( \text{dom}(f_{\text{tp}_1}) \cap \text{TR} = \emptyset \). Then \( u \) is identified with the source of \( L_{\text{tp}_1} \).

In the last three cases, \( u \) is identified with the source of the local graph of one successor of \( \text{tp} \) through \( A \). We refer to this successor as the preferred successor of \( \text{tp} \) through \( A \).

Now, we assume that \( r = (\text{tp}, A) \), where \((\text{tp}, A)\) is a special pair. Let \( t = t(\text{tp}), t_1, t_2 \) be the children of \( t \), \( \text{tp}_1, \text{tp}_2 \) be the respective successors through \( A \) with \( \text{tp}_1 \) being the main successor. Let \( Q \) be a transitional resolution for \( h_{\text{tp}_1} \) such that \( \text{Var}(Q) \subseteq \text{Var} \text{range}(h_{\text{tp}_1}) \). Then \( L_{(\text{tp}, A)} \) is obtained from \( Q \) by the following modification of the transitional sinks. Let \( u \) be a sink of \( Q \) labelled with \( \text{trans}_C \). Then, in \( L_{(\text{tp}, A)} \), \( u \) is identified with the source of \( L_{\text{tp}_1} \) where \( \text{ET}(\text{tp}_1) = \text{ET}(\text{tp}_2) \cup C \) and the rest of the components remain the same.

**Remark 1** Note that if \( t = t(\text{tp}) \) is a leaf then any extension of \( \text{tp} \) assigns all the variables of \( \text{range}(f_{\text{tp}}) \), therefore, the first or the second case must occur.

Note also that for the definition to be valid, whenever a sink of a local graph is identified with the source of local graph of another type, it needs to be established that this type is an element of \( \text{ORD} \). In the correctness proof in the appendix, we show that this is indeed the case.
Proof sketch of Theorem 1. The proof consists of two stages. First we establish correctness of the construction as per Definition 20 then we establish an upper bound on its size.

For the correctness, we let $R = \bigcup_{r \in ORD} L_r$. Then we show that for each $tp \in N_S$, $R_{utp}$ is a transitional resolution for $h_{tp}$ and for each $(tp, A) \in SP_r$, $R_{(tp, A)}$ is a transitional resolution for $h_{tp|A}$. In particular, this is so for the starting type $st = (rt, \emptyset, \emptyset, \emptyset)$, where $rt$ is the root of $T$. We observe that $h_{st} = 1_\varphi$. Hence, we conclude that $R_{st}$ is the required transitional resolution for $\varphi$. The proof uses the machinery developed in Section A of the appendix enabling a 'piecewise' construction of a transitional resolution. In particular, given a sequence of 'local' CDTs satisfying certain criteria, it is shown that their union is a transitional resolution for a certain function. Because of this framework, in the actual proof of Theorem 1 (Section B of the Appendix), we only need to show that the graphs $L_{tp}$ and $L_{(tp, A)}$ satisfy the required criteria.

The proof is mostly straightforward checking of conditions, the most interesting aspect is the use the properties of tree decompositions and one-sidedness.

To upper-bound the size of $R$, we essentially, upper bound the sum of sizes of local graphs. We observe an immediate difficulty: there is uncertainty in the definition of the local graph $L_{(tp, A)}$. Definition 20 says that is is a transitional resolution for $h_{tp_1}$ (where $tp_1$ is the main successor of $tp$ through $A$) subject to a certain variable constraint, but, beyond this, no specification is provided and hence a size upper bound is not clear. We know, however that $R_{utp_1}$ is a transitional resolution for $h_{tp_1}$. So we can simply take a copy of $R_{utp_1}$ as $L_{(tp, A)}$. This, however, immediately creates another problem: we take a 'whole' graph $R_{utp_1}$ to serve as a local graph, the total of the sizes essentially 'doubles' and this may lead to exponential explosion. Demonstration that this exponential explosion is controlled by parameters requires a more careful view.

First of all, we observe that the number of nodes $t$ such that $t = t(tp)$ and $(tp, A)$ is a special pair is at most $|TR|$. This means that the number $sp$ of special pairs is upper bounded by a function of $k$ and $|TR|$. Next, we define a rank of an element of $ORD$. For this, we first say that if a sink $L_{r_1}$ is identified with $u_{r_2}$ then $r_2$ is a child of $r_1$. The notion of a child naturally leads to the notion of a descendant. With this in mind the rank is defined as follows. The rank of $tp$ is the maximum over ranks of its descendants (if $tp$ has no descendants then the rank is zero). The rank of a special pair $(tp, A)$ is the rank of $tp_1$ plus one where $tp_1$ is the main successor of $tp$ through $A$.

We prove that the rank of an element of $ORD$ is at most $max(0, \log(|TR|))$. Then we show that the local graph size for a special pair of rank $i$ is upper bounded by $n \cdot sp^i \cdot 2^{O(k+|TR|)}$ where $n = |Var(\varphi)| + |\varphi|$. We conclude that, since the rank is upper bounded by $\log(|TR|)$ and the number of elements of $ORD$ is upper bounded by $2^{O(k+|TR|)}$, the required upper bound follows. ■
4 The main result

Theorem 2 Let $\varphi$ be a CNF, $\text{LONG} \subseteq \varphi$ to which we refer as long clauses and $k > 0$ be an integer. Assume that $\varphi \setminus \text{LONG}$ has a one-sided tree decomposition $(T, B)$ of the incidence graph of width at most $k$ (due to Proposition 2, we assume that $T$ is a binary tree). Then there is a resolution of $\varphi$ of size $(n + |\text{LONG}|)^{O(k^2 + k \cdot |\text{LONG}|)} \cdot n \cdot 2^{O(k + |\text{LONG}|)}$.

In the rest of this section, we introduce definitions towards the proof of Theorem 2 and provide a sketch of the proof. The complete proof is available in Section C of the Appendix. Formal statements in this section alternate with fragments of informal discussion. For a reader interested in the formal reasoning only, the informal discussion can be safely omitted.

Similarly to the proof of Theorem 1, the resolution is constructed in a piece-wise manner as a union of local graphs of types. However, the types are defined in a different manner. In particular, while for the proof of Theorem 1, the types were associated with nodes of the underlying search tree, in the considered case, the types are associated with prefixes of $\pi_T$. Before we continue the discussion, let us define the types and the corresponding classification of variables.

Definition 21 (Type) A type $\text{tp}$ is a quadruple $(\pi, \text{map}, \text{CN}, \text{RA})$ where $\pi$ is a (possibly empty) prefix of $\pi_T$ and $\text{map}$ is function from $\text{LONG}$ to $\text{Trees}_{\pi} \cup \{\text{none}\}$. To define the remaining two components, we need to have a more detailed look at the domain and range of $\text{map}$. So, let $\text{LS}$ be the subset of $\text{LONG}$ that are not mapped to $\text{none}$ and let $M_T = \text{map}(\text{LS})$. Then $\text{CN}$ is a subset of $\text{CL}(	ext{Roots}(M_T))$ and $\text{RA}$ is an assignment to $\text{Var}(	ext{Roots}(M_T))$.

When it may be not clear from the context which type the above structures are related to, we prove the name of the type in the brackets, for example, $\pi(\text{tp})$, $M_T(\text{tp})$ and so on.

Definition 22 (Inner, fixed, and outer variables) Let $\text{tp}$ be a type. We call $\text{Var}(M_T)$ the inner variables of $\text{tp}$ and denote the set by $\text{Var}_{\text{in}}$. Let $x \in \text{Var}(\pi) \setminus \text{Var}_{\text{in}}$. We say that $x$ is fixed if there is a long clause $C$ such that $x \in \text{Var}(C)$ and one of the following holds: (i) $\text{map}(C) = \text{none}$ or $\text{map}(C) \in \text{Trees}_{\pi}$ and $M_T(x) < \text{map}(C)$ where $M_T(x)$ is the smallest tree $T'$ of $\text{Trees}_{\pi}$ such that $x \in \text{Var}(T')$. We call $C$ a witnessing clause of $x$. If there are several clauses matching the definition, pick an arbitrary one. We denote by $\text{Var}_{\text{fix}}$ the set of all fixed variables. For each $x \in \text{Var}_{\text{fix}}$, the fixed assignment of $x$ is the literal opposite to its occurrence in the witnessing clause $C$. We denote by $\text{FA}$ the set of all the fixed literals. Finally the variables of $\varphi \setminus (\text{Var}_{\text{fix}} \cup \text{Var}_{\text{in}})$ are called outer variables and their set is denoted by $\text{Var}_{\text{out}}$.

In order to interpret the above definitions, we need to note that, in the resulting resolution, each type $\text{tp}$ (reachable from the source) is associated with a node $u_{\text{tp}}$. The types describe invariant properties of the assignments $A$ carried by the paths from the source to $u_{\text{tp}}$. For example, the long clauses satisfied by $A$ are $\text{LS}(\text{tp})$. In this case, $\text{map}(C)$ is the smallest tree $T'$ of $\text{Trees}(\pi)$ such that
some nuances related to definition of sinks. The case when $t$ node. If

Definition 25 (Basic, final, and unsatisfiable types.) A type $tp$ is unsatisfiable if $F_{tp}$ is unsatisfiable. An type $tp$ is basic if there is $C \in \text{dom}(F_{tp})$ such that $F_{tp}(C) = \emptyset$. A type $tp$ is final if $\pi(tp) = \pi_T$.

Now we start defining local subgraphs. We will consider only non-basic types. One useful observation is that for a long clause $C \in \text{dom}(F_{tp})$, $Var(F_{tp}(C))$ does not intersect with $Var(\pi)$. Indeed, by definition, $\text{map}(C) = \emptyset$. Therefore, any $x \in (Var(\pi) \setminus Var_{in}) \cap Var(C)$ must belong to $Var_{fix}$ and cannot be in $Var_{out}$. This immediately simplifies construction of local graphs for the final types. Indeed, if $tp$ is final then $Var_{out}(tp) \subseteq Var(\pi)$. Since we assumed that $tp$ is non-basic $dom(F_{tp})$ does not contain long clauses. In other words, $F_{tp}$ is a $\varphi \setminus \text{LONG}$-based function and hence there is an FPT size falsifying FRR for $F_{tp}$ by Corollary 1.

For a non-final type $tp$, the construction of the local subgraph depends on whether the immediate successor $t$ of $\pi(tp)$ is an expanding or an contracting node. If $t$ is expanding then the local graph is effectively a decision tree with some nuances related to definition of sinks. The case when $t$ is contracting is more involved. In order to informally discuss it, we need to introduce several additional definitions.

Definition 24 (The type function) For a type $tp$, $F_{tp}$ is a $\varphi$-based function. For each $C \in \text{dom}(F_{tp})$ $F_{tp}(C) = \text{Proj}(C, V_{out})$.

The domain of $F_{tp}$ $C \in \varphi$ such that one of the following three conditions hold: (i) $C \in \text{LONG} \setminus \text{LS}$; (ii) $C \in \text{CN}$; (iii) $C \notin \text{CL}(\text{Roots}(MT))$ and $\text{Var}(C) \cap \text{Var}_{out} \neq \emptyset$ and $C$ is not satisfied by $RA \cup FA$.

Thus the assignment $A$ as above does not assign $\text{Var}_{out}(tp)$. $\text{Var}_{fix}(tp)$ occur in $A$ with assignment $FA$, $\text{Var}(\text{Roots}(MT))$ occur with assignment $RA$ but what about the rest of the variables of $Var_{in}$? To understand this, note that variables of $Var_{in}$ 'communicate' with the rest of the variables through the variables of the roots of $MT$ and through the clauses having variables both inside $Var_{in}$ and outside $Var_{in}$ (let us call them 'connecting' clauses for the sake of the argument). In particular let $B_1$ and $B_2$ be two assignments to $Var_{in}$ that do not falsify any clause, have the same projection to $Var(\text{Roots}(bfMT))$, and satisfy the same set of connecting clauses. Then it can be shown that $\varphi|_{B_1} = \varphi|_{B_2}$. The meaning of the component $CN$ of $tp$ is exactly the set of connecting clauses that are not satisfied by $A$. So, the assignment to the roots of $MT$ plus $CN$ do provide a complete description and, subject to this invariant, the assignment to the rest of the variables of $Var_{in}$ may be arbitrary!

Definition 23 (The type function) For a type $tp$, $F_{tp}$ is a $\varphi$-based function. For each $C \in \text{dom}(F_{tp})$ $F_{tp}(C) = \text{Proj}(C, V_{out})$.

The domain of $F_{tp}$ $C \in \varphi$ such that one of the following three conditions hold: (i) $C \in \text{LONG} \setminus \text{LS}$; (ii) $C \in \text{CN}$; (iii) $C \notin \text{CL}(\text{Roots}(MT))$ and $\text{Var}(C) \cap \text{Var}_{out} \neq \emptyset$ and $C$ is not satisfied by $RA \cup FA$.

$Var(T') \cap Var(A \cap C) \neq \emptyset$ or, to put it differently, the minimal tree containing a variable whose occurrence in $A$ satisfies $C$. Then $MT$ is the set of such minimal trees. To continue the discussion, let us first define the function of the type.
Definition 26 Let $tp$ be a non-basic non-final unsatisfiable type and let $t$ be the immediate successor of $\pi(tp)$. We say that $t$ is established if $t$ is a contracting node and at least one element of $MT(tp)$ is a subtree of $last(Trees_{\pi(tp)+t})$.

Definition 27 With the data as in Definition 26, a variable $x \in Var(t) \cap Var_{out}$ is determining if there is a clause $C \in LONG \setminus LS$ such that $x \in Var(C)$. We denote by $Det(tp)$ the set of all determining variables, omitting the brackets if the type is clear from the description.

Definition 28 Let $tp$ be a non-basic non-final unsatisfiable type and $t$ be the immediate successor of $\pi(tp)$. The dome $D_{tp}$ of $tp$ is defined as follows.

- If $t$ is established then $D_{tp}$ is a single node if $Var(t) \cap Var_{out} = \emptyset$. Otherwise, $D_{tp}$ is a decision tree over all the assignments to $Var(t) \cap Var_{out}$.

- If $t$ is non-established then $D_{tp}$ is a single node if $Det = \emptyset$. Otherwise, $D_{tp}$ is constructed in two stages. On the first stage, we define $D^*$ as a decision tree over all assignments of $Det$. If $Var(t) \cap Var_{out} = Det$ then $D_{tp} = D^*$. Otherwise, for each sink $u$ of $D^*$ such that the assignment $A$ carried by the path from the root of $D^*$ to $u$ satisfies at least one $C \in LONG \setminus LS$, make $u$ the root of a decision tree over all the assignments to $(Var(t) \cap Var_{out}) \setminus Det$.

Definition 29 Let $tp$ be a non-basic non-final unsatisfiable type and $t$ be the immediate successor of $\pi(tp)$. Assume that $t$ is contracting and let $A$ be an assignment to $Var(t) \cap Var_{out}$ that does not falsify any clause in $range(F_{tp})$. Then we say that $A$ is potentially complex if one of the following two conditions holds.

- $t$ is established.
- $A$ satisfies a clause $C \in LONG \setminus LS$.

The local subgraph of any type $tp$ of $ORD$ includes the dome $D_{tp}$ plus subgraphs whose sources are sinks of $D_{tp}$. Why is not a a sink $u$ of $D(tp)$ not necessarily a sink of the whole local subgraph $L_{tp}$ of $tp$? This may happen when the assignment $A$ carried by the path from the source of $D_{tp}$ to $u$ is potentially complex. In this case $T_t$ is going to be a minimal tree in any successor $tp'$ of $tp$ identified with a sink of $L_{tp}$ reachable from $u$, meaning that all the variables of $T_t$ will be inner for $tp'$. Yet, $T_t$ may include variables of $Var_{out}(tp)$ that are not yet assigned after assigning the variables of $A$. These are precisely the variables taken care by $[L_{tp}]_u$. To see how this is done, we need one more definition.

Definition 30 [Filling function and its transitional resolution] Let $tp$ be a non-basic non-final unsatisfiable type and $t$ be the immediate successor of $\pi(tp)$. Let $A$ be a potentially complex assignment. For the sake of brevity, let us denote $last(Trees_{\pi(tp)+t})$ by $T^*$. Denote the set $Var_{out}(tp) \cap Var(T^*) \setminus Var(t)$ by $Var_{free}$ and refer to them as free variables.
The filling function $F^*$ ($\text{tp}$ and $A$ may be added in brackets if not clear from the context) is a $\varphi$-based function such that for each $C \in \text{dom}(F^*)$, $F^*(C) = \text{Proj}(C, \text{Var}_{f\text{ree}})$. The domain of $F^*$ consists of all clauses $C$ of $\text{dom}(F_{\text{tp}})$ such that $C$ is not satisfied by $A$ and $\text{Var}(F(C)) \cap \text{Var}_{f\text{ree}} \neq \emptyset$.

If range($F^*$) is unsatisfiable, let $R^*$ be a transitional resolution of $F^*$ with the transitional clauses being $\text{dom}(F^*) \cap \text{CL}(t)$ and with the extra constraint that $\text{Var}(R^*) \subseteq \text{Var}_{f\text{ree}}(\text{tp})$.

Continuing the informal discussion, we introduce a subgraph of $L_{\text{tp}}$ with $u$ being the source if the corresponding filling function $F^*$ is unsatisfiable. This subgraph is nothing but the transitional resolution $R^*$ for $F^*$ as defined above. Earlier types of ORD are identified with the transitional sinks of $R^*$. The precise construction is provided in the definition below. It is worth noting that due to the same reason as for the final types, $\text{dom}(F^*)$ does not include long clauses, hence $R^*$ can be of FPT size in $k$.

**Definition 31** The local graph $L_{\text{tp}}$ for each unsatisfiable type $\text{tp}$ is constructed recursively along ORD. All the nodes of $L_{\text{tp}}$, except sinks, are unique for $\text{tp}$ in the sense that they do not occur in the earlier types. The sinks may be identified with sources of local graphs for earlier types.

If $\text{tp}$ is final then $L_{\text{tp}}$ is a falsifying FRR for $F_{\text{tp}}$. In the rest of the definition, we assume that $\text{tp}$ is non-basic and non-final.

The dome $D_{\text{tp}}$ is a subgraph of $L_{\text{tp}}$ and the source of $D_{\text{tp}}$ is the source of $L_{\text{tp}}$. $L_{\text{tp}}$ is obtained from $D_{\text{tp}}$ by processing of each sink and making one of the following three decisions (i) associating the sink with some $C \rightarrow ()$ or (ii) identifying the sink with the source of the local graph of an earlier type or (iii) deciding that the considered sink $u$ of $D_{\text{tp}}$ is not a sink of $L_{\text{tp}}$ and constructing the graph $[L_{\text{tp}}]_u$. The detailed construction is specified below.

Let $t$ be the immediate successor of $\pi(\text{tp})$. Let $u$ be a sink of $D_{\text{tp}}$. Then $A(u)$, the assignment corresponding to $u$ is $\emptyset$ if $u$ is the only node of $D_{\text{tp}}$. Otherwise, $D_{\text{tp}}$ is a decision tree and $u$ is its leaf. In this case, $A(u)$ is $A(P)$ where $P$ is the root-$u$ path of $D_{\text{tp}}$.

So, let $u$ be the considered sink of $D_{\text{tp}}$. Suppose that $A(u)$ falsifies $F_{\text{tp}}(C)$ for some $C \in \text{dom}(F_{\text{tp}})$. Then $u$ is associated with $C \rightarrow ()$.

Otherwise, we consider separately the cases where $t$ is an expanding node and where $t$ is a contracting node. Suppose first that $t$ is an expanding node. If $A(u)$ does not satisfy any clause of LONG $\setminus$ LS then $u$ is identified with source of $L_{\text{tp}}'$ where $\text{tp}' = (\pi(\text{tp}) + t, \text{map}, CN', RA)$ where $CN'$ is obtained from $CN$ by removal of clauses that are satisfied by $A(u)$. Otherwise, $u$ is identified with the source of $L_{\text{tp}}'$ where $\text{tp}' = (\pi(\text{tp}) + t, \text{map}', CN^*, RA')$ where $\text{map}'$ is obtained from $\text{map}$ by replacing $C \rightarrow \text{none}$ with $C \rightarrow \text{last}(\text{Trees}_{\pi(\text{tp})})$ for each $C \in \text{LONG} \setminus \text{LS}$ that is satisfied by $A(u)$. $CN^*$ obtained from $CN$ by removal of all clauses that are satisfied by $A(u)$ and adding those clauses of $\text{dom}(F_{\text{tp}}) \cap \text{CL}(t)$ that are not satisfied by $A(u)$. Finally $RA' = RA \cup A(u) \cap \text{Proj}(F_{\text{tp}}(\text{tp}), \text{Var}(t))$.

Assume now that $t$ is a contracting node. If $A(u)$ is not potentially complex then $u$ is associated with the type $(\pi(\text{tp}) + t, \text{map}, CN', RA)$ where $CN'$ is obtained from $CN$ by removal of clauses that are satisfied by $A(u)$. 


The most interesting case though is where $A(u)$ is potentially complex. For this case we need to introduce additional notations.

Let $CN_1$ be the subset of $CN$ that are not satisfied by $A(u)$. Let $Trees^* = \langle MT(tp) \setminus Trees_{\pi}^* \setminus \{T_i\} \rangle$. Let $CN_2$ be the subset of $\text{dom}(F_{tp}) \setminus (\text{LONG} \cup \text{CL(\text{Roots}(MT)))}$ consisting of all clauses $C$ such that $C$ is not satisfied by $A(u)$ and $C \in \text{CL}(t)$ and $\text{Var}(F_{tp}(C)) \cap \text{Var}_{\text{free}}(tp) = \emptyset$.

Let $\text{map}'$ be a function from $\text{LONG}$ to $Trees_{\pi}^*$ obtained from $\text{map}$ as follows.

- For each $C \in \text{LONG}$ such that $\text{map}(C) = \text{none}$ and $C$ is satisfied by $A(u)$, $\text{map}'(C) = T_i$.
- For each $C \in \text{LONG}$ such that $\text{map}(C) \in Trees_{\pi}^*$, $\text{map}'(C) = T_i$.

Let $RA' = \text{Proj}(RA, \text{Var}(\text{Roots}(Trees^*))) \cup \text{Proj}(FA, \text{Var}(t)) \cup A(u)$.

Assume first that $F^*$ is satisfiable. Then $u$ is identified with the source of $L_{tp'}$ where $tp' = (\pi + t, \text{map}', CN_1 \cup CN_2, RA')$.

Finally assume that $F^*$ is not satisfiable. Let $R^*$ be the transitional resolution for $F^*$ as per Definition 10. Identify $u$ with the source of $R^*$. Then for each transitional sink $v$ of $R^*$ do the following. Let $\text{trans}_C$ be the function associated with $v$. Then, identify $v$ with the source of $L_{tp''}$, where $tp'' = (\pi + t, \text{map}', CN_1 \cup CN_2 \cup C, RA')$

**Proof sketch of Theorem 2.** We start the proof from defining a well-formed type. Let $x$ be a fixed variable for $tp$. Recall that for $x$ we define the witnessing clause $C$ for $x$ and choose $C$ arbitrarily if there are several candidates. The type $tp$ is well-formed if $x$ has the same occurrence in all the candidates, hence it does not matter which candidate we choose. We did not introduce the definition in the main body of the paper to improve readability. The proof of Theorem 2 is applied not to $\text{ORD}$ but rather to its subsequence $\text{ORD}^*$ consisting of all the well-formed types.

Let $R$ be the union of all the local subgraphs of the elements of $\text{ORD}^*$. For each $tp \in \text{ORD}^*$, let $u_{tp}$ be the source of $L_{tp}$. We prove that each $R_{u_{tp}}$ is a falsifying FRR for $F_{tp}$. The proof uses Theorem 3 provided in Appendix A. In order to apply the theorem, we need to demonstrate for each $tp \in \text{ORD}^*$ that $L_{tp}$ is valid in the following sense. First, whenever a sink $u$ of $L_{tp}$ is labelled with $C \rightarrow ()$ for each path $P$ from $u_{tp}$ to $u$, $A(P)$ falsifies $F_{tp}(C)$. Second, whenever a sink $u$ of $L_{tp}$ is identified with an earlier type $tp'$ then for each path $P$ of $L_{tp}$ from $u_{tp}$ to $u$, $F_{tp'}$ is a subfunction of $F_{tp}[A(P)]$ and $tp' \in \text{ORD}^*$ (that is $tp'$ is well-formed, non-basic, an unsatisfiable). Next, we observe that there is one particular type $stp = (\emptyset, \emptyset, \emptyset, \emptyset)$ (the first $\emptyset$ denotes the empty prefix) such that $F_{tp} = 1_{\emptyset}$. (That is, $R_{u_{stp}}$ is, in fact a regular resolution for $\varphi$ by Proposition 5.) This completes the correctness proof and it remains to establish the upper bound on the construction size.

The upper bound can be obtained by multiplying an upper bound on the size of a local subgraph by the number of types. The most critical part in assessing the local subgraph size is that it may include a falsifying FRR. However, as we
explained before, the resolution is always an FPT size in $k$ since the domain of the corresponding function does not contain long clauses. For the number of types, the number of possible first components is $O(n)$. With the first component fixed, by Proposition 4 the number of possible second components is $O(\log n|\text{LONG}|)$, which is well known to be FPT. With the first two components fixed, the number of possible third and forth components is at most $2^k|\text{LONG}|$. ■

5 Conclusion

We have proved that regular resolution is FPT for CNFs whose one-sided treewidth is almost bounded. We have also demonstrated how a resolution under a more restricted parameterization can serve as a building block for the construction towards the main result, increasing by one the degree of $n$. We believe that using this approach FPT algorithms can be defined for more and more general parameters at the price of higher and higher degree of the polynomial dependence on $n$. The main question however is: can we reach this way the general case of bounded incidence treewidth? We believe that the answer is no and that an XP lower bound needs to be sought at least in the case of regular resolution.

We believe that the first step in the design of hard instances should be understanding the properties of the underlying hypergraphs that make the instances hard. Such properties would be most convenient to investigate if the CNFs were monotone. In the context of resolution, this is clearly impossible since the CNFs must be unsatisfiable (we may, of course, allow empty clauses but the resulting class would hardly be interesting). We think that the right approach is to consider a closely related model of Decision DNNFs representing monotone CNFs of bounded incidence treewidth. As mentioned in the introduction, this model has a lot in common with the regular resolution. Therefore, we believe that understanding the complexity of the former on CNFs of bounded incidence treewidth will provide an important insight for the latter.

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A Assembling a resolution from small fragments

Definition 32 Let $f_1, \ldots, f_m$ be $\varphi$-based functions. For each $1 \leq i \leq m$ we define a CDT $L_i$ with source $u_i$ satisfying the following conditions.

- Each sink of $L_i$ is labelled with $C \rightarrow ()$ or, otherwise, is identified with the source of $L_j$ for some $j < i$, (Of course the latter condition is possible only if $i > 1$).
• For each $i$ and each $v \in V(L_i)$, except those that are identified with roots of earlier $L_j$, $v \notin \bigcup_{j=1}^{i-1} V(L_j)$.

• Let $u$ be a sink of $L_i$ and let $P$ be a path from $u_i$ to $u$. Then if $u$ is associated with $C \rightarrow ()$ then $C \in \text{dom}(f_i)$ and $f_i(C)$ is falsified by $A(P)$. Otherwise, $u$ is associated with the source of $L_j$ for some $j < i$. In this case $f_j$ is a subfunction of $f_i|_{A(P)}$.

We call $L_1, \ldots, L_m$ a falsifying sequence.

Definition 33 With data as in the previous definition we say that $f_j$ is a child of $f_i$ is $L_i$ has a sink identified with the root of $L_j$ (clearly $j < i$). Let $i_1, \ldots, i_r$ be a sequence such that for each $1 \leq k < r - 1$ $f_{i_k}$ is a child of $f_{i_{k+1}}$. We then say that $f_{i_1}$ is a descendant of $f_{i_r}$. The falsifying sequence is read-once if for each $i, j$ such that $F_j$ is a descendant of $F_i$, $\text{Var}(L_i) \cap \text{Var}(L_j) = \emptyset$.

Theorem 3 Let $L_1, \ldots, L_m$ be a falsifying read-once sequence for the respective $\varphi$-based functions $f_1, \ldots f_m$. Let $R = \bigcup_{i=1}^m L_i$. Let $u_1, \ldots, u_m$ be the respective sources of $L_1, \ldots L_m$. Then each $R_{u_i}$ is a falsifying FRR for $f_i$.

Proof.

Claim 1 Let $D_1$ and $D_2$ be DAGs with labels on their edges. Suppose that $V(D_1) \cap V(D_2) \subseteq \text{sinks}(D_2)$. Then the following statements hold.

1. $D = D_1 \cup D_2$ is a DAG.
2. $D[V(D_1)] = D_1$.
3. $\text{sinks}(D) = \text{sinks}(D_1) \cup (\text{sinks}(D_2) \setminus V(D_1))$.

Proof. The second statement is immediate by definition as $D$ does not add nor removes edges between vertices of $D_1$ (note that there are no edges between sinks). That is $D[V(D_1)]$ is a DAG and also $D[V(D_2) \setminus V(D_1)]$ but there is no path from the former to the latter. This proves the first statement. Finally, for the third statement, $\text{sinks}(D_1)$ are sinks of $D$ because of the second statement and $\text{sinks}(D_2) \setminus V(D_1)$ are sinks of $D$ simply by construction. The rest of the vertices are not sinks either in $D_1$ or in $D_2$ so, obviously, they remain non-sinks in $D$. □

For $1 \leq i \leq m$, let $R^i = \bigcup_{1 \leq j \leq i} L_i$. Using inductively the above claim we make the following list of observations that will be referred by their numbers in the rest of the proof.

1. Each $R^i$ is a DAG all sinks of which are labelled with $C \rightarrow ()$. Apply inductively the first and the third statements of the above claim.
2. For each $v \in D(R)$ that is not a sink, $v$ has two outgoing edges one labelled with the positive and one labelled with the negative literal of the same variable. Let $i$ be the smallest index such that $v \in V(R^i)$. Note that if $v$ is...
a sink in \( R^i \), then, by inductive application of the second statement of the claim, starting from \( R^i \) and up to \( R^m = R \), we observe that \( v \) remains a sink in \( R \). Hence \( v \) is not a sink in \( R^i \). By definition \( v \in V(L_i) \). Moreover, if \( i > 1 \) then \( v \) is not a vertex of \( R^{i-1} \) by the minimality assumption. Hence, as \( v \) is not a sink of \( R^i \), it follows from the first statement of the claim that \( v \) is not a sink of \( L_i \). But then \( v \) has two outgoing edges as specified in the statement of the current item. Again applying part 2 of the claim inductively until \( R = R^m \), we observe that this situation with two outgoing edges is preserved in \( R \).

3. For each \( 1 \leq i \leq m \), \( R^i_{u_i} = R^i_{u_i} \) for all \( j \geq i \). Again, starting from \( R^i_{u_i} \), apply inductively the second part of the claim.

Now, we need to prove that each \( R_{u_i} \) is read-once. For this purpose, we prove that for each path \( P \) in \( R \) from \( u_i \) to a sink \( P = P_1 + \cdots + P_r \) where each \( P_j \) is a source sink path in one of \( L_1, \ldots, L_m \). The proof is by induction on \( i \). By the last item in the above list, it is enough to consider \( R^i_{u_i} \). \( R^i_{u_i} = L_1 \), so we are done by construction. Assume now that \( i > 1 \). Then \( P \) has a prefix \( P' \) which is a source-sink path of \( L_i \). If \( P = P' \), we are done. Otherwise, \( P = P' + P'' \), where \( P'' \) starts from \( u_j \) for some \( j < i \). By the induction assumption, \( j \) satisfies the requirement. Hence, the desired concatenation of subpaths for \( P'' \) plus \( P' \) at the beginning provides the desired concatenation of paths for \( P \). Now, consider the concatenation \( P = P_1 + \cdots + P_r \) as above. Each \( P_j \) is read-once by definition so, if there is a repetition, there are some \( P_j \) and \( P_k \), \( j < k \) such that \( \text{Var}(A(P_j)) \cap \text{Var}(A(P_k)) \) intersect. But \( P_j \) is a path of some \( L_j' \) and \( P_k \) is a path of some \( L_k' \) such that \( k' < j' \) and \( f_{k'} \) is a descendant of \( f_{j'} \). By assumption, the variables of \( L_j' \) and \( L_{k'} \) must be disjoint, a contradiction.

It follows from the combination of the first three items the above list and the read-onceness claim that each \( R_{u_i} \) is a falsifying FRR. It remains to prove that each \( R_{u_i} \) has this capacity for \( f_i \). We proceed by induction on \( 1, \ldots, m \). For \( i = 1 \), as we have seen above, \( R_{u_i} = L_1 \), so the statement holds by construction. Assume now that \( i > 1 \). Let \( P \) be a source-sink path of \( R_{u_i} \). Let \( C \rightarrow (\) \) be the label of the final node of \( P \). As verified above \( P \) has a prefix \( P' \) that is a source-sink path of \( L_i \). Assume first that \( u = u_j \) for some \( j < i \). Then \( P = P' + P'' \) and, by the induction assumption \( C \in \text{dom}(f_j) \) and \( A(P'') \) falsifies \( f_j(C) \). By construction, \( f_j \) is a subfunction of \( f_i|_{A(P')} \). We conclude that \( C \in \text{dom}(f_i) \) and \( A(P) \) falsifies \( f_j(C) \). If \( u \) is not an earlier \( u_j \) then the same conclusion follows by construction.

In the rest of this section, \( \text{TR} \) is the set of transitional clauses of \( \varphi \).

**Definition 34** Let \( f \) be a \( \varphi \)-based function. We denote by \( \text{empty}(f) \) the subset of \( \text{dom}(f) \) consisting of all clauses \( C \) such that \( f(C) = (\) \). We say that \( f \) is interesting if the following two conditions hold.

- \( \text{empty}(f) \subseteq \text{TR} \).
- \( f \setminus \text{empty}(f) \) is unsatisfiable.
**Definition 35** Let $f$ be an interesting $\varphi$-based function. Let $A$ be an assignment and suppose that $f|_A$ is also an interesting $\varphi$-based function. Let $g$ be an interesting subfunction of $f|_A$. We say that $g$ is a good function for $f, A$ if one of the following conditions is true.

1. $\text{dom}(g) \cap \text{TR} \subseteq \emptyset$.
2. $(f|_A) \setminus g$ is satisfiable and $\text{Var}(\text{range}(g)) \cap \text{Var}(\text{range}(f|_A \setminus g)) = \emptyset$.

**Proposition 6** $f|_A$ is a good function for $f, A$.

**Lemma 1** Suppose that $g$ is a good function for $f, A$. Let $S$ be a $C$ falsifier for $g$ for some $C \subseteq \text{TR}$ such that $\text{Var}(S)$ is disjoint with both $\text{Var}(A)$ and $\text{Var}(\text{range}(f|_A \setminus g))$. Then $S \cup A$ is a $C$-falsifier for $f$.

**Proof.** If the first condition of Definition 35 holds then the statement holds in a vacuous way because no such a falsifier exists. Indeed by assumption $C \subseteq \emptyset$, hence an extension of $S$ must satisfy $g(\text{dom}(g) \setminus \emptyset)$ in contradiction to the definition of an interesting function. So, we assume that the second case holds.

Let $S_1$ be an extension of $S$ satisfying $g(\text{dom}(g) \setminus C)$ and let $S_2$ be a satisfying assignment to $\text{range}(f|_A \setminus g)$. We may assume w.l.o.g. that $\text{Var}(S) \setminus S_1 \subseteq \text{Var}(\text{range}(g))$ and that $\text{Var}(S_2) \subseteq \text{Var}(\text{range}(f|_A \setminus g))$. By assumption $\text{Var}(S_1), \text{Var}(S_2)$, and $\text{Var}(A)$ are mutually disjoint. Therefore $S^* = S_1 \cup S_2 \cup A$ is a well formed set of literals. We claim that $S^*$ satisfies $f(C)$ for all $C \in \text{dom}(f) \setminus C$. Indeed, if $C \in \text{dom}(f) \setminus \text{dom}(f|_A)$ then $f(C)$ is satisfied by $A$. If $C \in \text{dom}(g)$ then $g(C)$ and hence $f(C)$ is satisfied by $S_1$. It remains to assume that $C \in \text{dom}(f|_A \setminus g)$. But then $f(C)$ is satisfied by $S_2$. ■

**Definition 36** Let $f_1, \ldots, f_m$ be interesting $\varphi$-based functions. For each $1 \leq i \leq m$ we define a CDT $L_i$ with source $u_i$ satisfying the following conditions.

- Each sink of $L_i$ is labelled with $C \rightarrow ()$ or $\text{trans}_C$ or, otherwise, is identified with the source of $L_j$ for some $j < i$. (Of course the latter condition is possible only if $i > 1$).
- For each $i$ and each $v \in V(L_i)$, except those that are identified with roots of earlier $L_j$, $v \notin \bigcup_{j=1}^{i-1} V(L_j)$.
- Let $u$ be a sink of $L_i$ and let $P$ be a path from $u_i$ to $u$. Then we consider the following subcases.
  - If $u$ is associated with $C \rightarrow ()$ then $C \in \text{dom}(f_i)$ and $f_i(C)$ is falsified by $A(P)$.
  - if $u$ is associated with $\text{trans}_C$ then $A(P)$ is a $C$-falsifier for $f_i$
  - Otherwise, $u$ is associated with the source of $L_j$ for some $j < i$. In this case $f_j$ is a good function for $f_i, A(P)$.

We call $L_1, \ldots, L_m$ a transitional falsifying sequence.
We define children and descendants analogously to the non-transitional case. With this in mind, we are now in a position to state a version of Theorem 3 for the transitional case.

**Theorem 4** Let $L_1, \ldots, L_m$ be a transitional falsifying sequence for the respective $\varphi$-based functions $f_1, \ldots, f_m$. Let $R = \bigcup_{i=1}^m L_i$. Let $u_1, \ldots, u_m$ be the respective sources of $L_1, \ldots, L_m$. Then each $R_{u_i}$ is a DAG with $\text{Var}(R_{u_i})$ being the union of $\text{Var}(L_i)$ and all $\text{Var}(L_j)$ such that $f_j$ is a descendant of $f_i$ and sinks labelled by either $C \rightarrow ()$ or $\text{trans}_C$. Also, let $R^i = L_1 \cup \cdots \cup L_i$. Then, for each $j \geq i$, $R^i_{u_i} = R^i_{u_j}$.

Suppose that two additional conditions take place for each $R_{u_i}$

1. Each path of $R_{u_i}$ is read-once.
2. Let $P$ be a source-sink path of $L_i$ and suppose that the final vertex of $P$ is $u_j$ for some $j < i$. Then $\text{Var}(R_{u_i})$ is disjoint with $\text{Var}(\text{range}(f_i|_{A(P) \setminus f_j}))$.

Then $R_{u_i}$ is a transitional resolution for $f_i$.

**Proof.** Claim 1 applies for this proof. The three observations in the proof of Theorem 3 also hold but with the statement for the first one is slightly modified as specified below.

1. Each $R^i$ is a DAG all sinks of which are labelled either with $C \rightarrow ()$ or with $\text{trans}_C$. Apply inductively the first and the third statements of Claim 1.
2. For each $v \in D(R)$ that is not a sink, $v$ has two outgoing edges one labelled with the positive and one labelled with the negative literal of the same variable.
3. For each $1 \leq i \leq m$, $R^i_{u_i} = R^j_{u_j}$ for all $j \geq i$.

These observations already prove the part of the theorem that does not take into account the additional conditions. With these conditions in mind, let $P$ be a source-sink path of $R_{u_i}$. Due to read-oneness condition, $A(P)$ is a well formed set of literals. We need to prove that (i) if the final vertex of $P$ is labelled with $C \rightarrow ()$ then $C \in \text{dom}(f_i)$ and $A(P)$ falsifies $f_i(C)$ and (ii) if the final vertex of $P$ is associated with $\text{trans}_C$ then $C \subseteq \text{dom}(f_i)$ and $A(P)$ is a $C$-falsifier for $f_i$.

The proof of (i) is analogous to the corresponding proof in Theorem 3 so we concentrate on proving (ii). If $P$ is a source-sink path of $L_i$ the statement follows by construction. Let $P'$ be the prefix of $P$ that is a source-sink path of $L_i$. Then the final vertex of $P'$ is $u_j$ for some $j < i$. Let $P''$ be the suffix of $P$ starting from $u_j$. By the induction assumption (the induction basis is correct by construction), $C \subseteq \text{dom}(f_j)$ and $A(P'')$ is a $C$-falsifier for $f_j$. Recall that by construction, $f_j$ is a good function for $f_i$, $A(P')$ and that $\text{Var}(P'')$ is disjoint with $\text{Var}(P')$ (due to read-oneness of $P$) and with $\text{Var}(\text{range}(f_i|_{A(P) \setminus f_j}))$ (by construction). Therefore, all the premises of Lemma 1 are satisfied and we conclude that $A(P)$ is a $C$-falsifier for $f_i$. ■
B Proof of Theorem 1

Lemma 2 Let \( tp \in ORD, A = A(P) \) where \( P \) is a source-sink path of \( L_{tp} \). Assume that \( f_{tp|A} \) is an interesting function. Further on, we assume that \( t = t(tp) \) is not a leaf and \( tp_1 \) and \( tp_2 \) be successors of \( tp \) through \( A \) (in case \( t \) has only one child, let \( tp_1 \) be the only successor of \( tp \) through \( A \)). Then the following statements hold.

1. \( f_{tp|A} \setminus empty(f_{tp|A}) \) is the disjoint union of \( f_{tp_1}, f_{tp_2} \). In case \( t \) has only one child, \( f_{tp|A} \setminus empty(f_{tp|A}) = f_{tp_1} \).

2. At least one of \( f_{tp_1}, f_{tp_2} \) is unsatisfiable.

3. \( h_{tp|A} \) is the disjoint union of \( f_{tp_1}, f_{tp_2} \), and \( trans_{ET(tp)} \setminus empty(f_{tp|A}) \).

4. For each \( i \in \{1, 2\}, h_{tp_i} \) is the disjoint union of \( f_{tp_i} \) and \( trans_{ET(tp)} \setminus empty(f_{tp|A}) \).

The last two cases naturally adapted to the situation where \( t \) has only one child. In particular, \( f_{tp_2} \) is not part of the union in the third case and the definition holds only for \( i = 1 \) in the fourth case.

Proof. First of all observe that by construction, \( empty(f_{tp|A}) \) is a subset of \( ET(tp_i) \) for each \( i \in \{1, 2\} \). Therefore, to establish that \( f_{tp} \) is a subfunction of \( f_{tp|A} \setminus empty(f) \), it is enough to show that \( f_{tp} \) is a subfunction of \( f_{tp|A} \).

First of all, we need to show that for each \( C \in dom(f_{tp}) \cap dom(f_{tp|A}), f_{tp}(C) = f_{tp|A}(C) \).

Let \( C \in dom(f_{tp}) \cap dom(f_{tp|A}) \). By definition, \( C \in CL(T_i) \). Hence, by one-sidedness of \( (T, B) \), \( C \notin CL(T_{i-1}) \). In particular, this implies that \( f_{tp}(C) = Proj(C, Var(T_i) \setminus Var(S)) = Proj(C, (Var(T_i) \cup Var(t)) \setminus Var(S)) \) (the rest of the variables are present only in \( T_{i-1} \), so \( C \) would need to be present in one of the bags of the subtree due to the containment property). Then \( f_{tp}(C) = Proj(C, Var(T_i) \setminus Var(t))(Var(S) \cup Var(A)) \). As \( Var(S) \cup Var(A) = Var(t) \), we conclude that \( Var(T_i) \cup Var(t) \setminus (Var(S) \cup Var(A)) = Var(T_i) \setminus Var(t) \) and hence \( f_{tp}(C) = f_{tp|A}(C) \).

Now, let \( C \in dom(f_{tp}) \). Then \( C \in CL(T_i) \subseteq CL(T_i) \). There are three reasons why \( C \) may not belong to \( dom(f_{tp}) \) we will observe that none of these reasons holds. The first reason is that \( C \in CL(tp) \). This means that \( C \in CL(t) \). Since \( C \in CL(T_i) \), the connectivity condition of tree decompositions, \( C \in CL(T_i) \). That is \( C \in CL(tp) \cap CL(T_i) \subseteq CL(tp_i) \), a contradiction.

The next reason is that \( C \in ET(tp) \). However, in this case, \( C \in ET(tp_i) \) in contradiction to \( C \in dom(f_{tp}) \).

Finally, it may be that \( C \) is satisfied by \( S \). Then \( C \) is satisfied by \( S \cup A \). If \( C \in CL(T_i) \) then, by definition, \( C \in CL(tp_i) \), a contradiction. Otherwise, let \( x \in Var(S \cup A) \) be a variable whose occurrence in \( S \cup A \) satisfies \( C \). Since \( C \notin CL(T_i) \), \( C \) cannot occur in any bag outside \( T_i \) by the treewidth connectivity condition. Then the containment condition implies that \( x \) must also occur in a bag inside \( T_i \). Since \( x \) is occurring in a bag outside \( T_i \) (namely in the bag of
In particular, $x$ occurs in $\text{Proj}(S \cup A, \text{Var}(t)) = S(tp_1)$ in contradiction to $C \in \text{dom}(f_{tp_1})$.

Thus we conclude that $C \in \text{dom}(f_{tp})$. It remains to be seen that $C$ is not satisfied by $A$. But in this case $C$ is satisfied by $S \cup A$ and we apply the reasoning of the previous paragraph.

Now, we assume that $C \in \text{dom}(f_{tp_1}) \setminus \text{empty}(f)$ and that $C \in CL(T_{t_1})$. We claim that $C \in \text{dom}(f_{tp_2})$. Indeed, there may be two reasons why this is not so. One such a reason is that $C \notin \text{dom}(f_{tp_2})$ and if $C$ is satisfied by $A$ then it is not contained in $\text{dom}(f_{tp_1})$, a contradiction in both cases. The other reason that may cause $C$ to not be in $\text{dom}(f_{tp_1})$ is that $C$ is satisfied by $S(tp_1)$. But, by definition, $S(tp_1)$ is a subset of $S \cup A$, so the last argument applies.

Next, we observe that if $t$ has two children then $\text{dom}(f_{tp_1})$ and $\text{dom}(f_{tp_2})$ are disjoint. Indeed, for a clause $C$ to be in both domains, it must be present in both $CL(T_{t_1})$ and $CL(T_{t_2})$ which is impossible due to one-sidedness.

Finally, we observe that $f_{tp_1}|A \setminus \text{empty}(f_{tp_1}|A)$ does not have any clauses but $\text{dom}(f_{tp_2}) \cup \text{dom}(f_{tp_2})$. Indeed, suppose that such a clause $C$ exists. By the proven above $C \notin CL(T_{t_1}) \cup CL(T_{t_2})$. By the containment condition, $\text{Var}(C)$ cannot intersect with $\text{Var}(T_{t_1}) \setminus \text{Var}(t)$. It follows that all the variables of $f_{tp_1}(C)$ are assigned by $S \cup A$, in fact by $A$ by definition of $f_{tp}$. Then $f_{tp_2}$ is either satisfied by $A$ (contradiction to $C \in \text{dom}(f_{tp_1})$) or falsified by $A$ (contradiction to $C \notin \text{empty}(f_{tp_1}|A)$). So we have proved the first statement for the case of two children. The proof applies for the case of one child simply by observing that $\text{dom}(f_{tp_1}|A) \cap CL(T_{t_2}) = \emptyset$.

The second statement follows immediately if $t$ has only one child. For the case of two children, we observe that $\text{Var}(\text{range}(f_{tp_1})) \cap \text{Var}(\text{range}(f_{tp_2})) = \emptyset$. Indeed, by definition of the type functions $\text{Var}(\text{range}(f_{tp_1})) \subseteq \text{Var}(T_{t_1}) \setminus \text{Var}(t)$ and it is a well known property of tree decompositions that $\text{Var}(T_{t_1}) \cap \text{Var}(T_{t_2}) \subseteq \text{Var}(t)$ (for otherwise the connectivity condition does not hold).

Assume that the second statement does not hold. For each $i \in \{1, 2\}$, let $S_i$ be an assignment satisfying $\text{range}(f_{tp_i})$. Clearly, we may assume that $\text{Var}(S_i) \subseteq \text{Var}(\text{range}(f_{tp_i}))$ for each $i \in \{1, 2\}$. But then $\text{Var}(S_1) \cap \text{Var}(S_2) = \emptyset$ and hence $S = S_1 \cup S_2$ is a well formed set of literals. By the first statement $S$ satisfies $f_{tp_1} \setminus \text{empty}(f_{tp_1}|A)$ in contradiction to our assumption about unsatisfiability of the latter. This proves the second statement.

The third statement follows from the combination of the first statement and the observation that $h_{tp_1}|A = f_{tp_1}|A \cup \text{trans}_{ET}(tp)$. For the fourth statement, recall that, by definition each $h_{tp_i}$ is the union of $f_{tp_i}$ and $\text{trans}_{ET}(tp_i)$ and that $ET(tp_1) = ET(tp) \cup (\text{empty}(f_{tp_1}|A) \cap \text{TR})$. However, by assumption $\text{empty}(f_{tp_1}|A) \subseteq \text{TR}$, so the fourth statement follows.

**Lemma 3** Let $tp \in ORD$ and let $A = A(P)$ where $P$ is a source-sink path of $L_{tp}$. Assume that for each non-transitional clause $C \in \text{dom}(h_{tp_i})$, $h_{tp}(C)$ is not falsified by $A$ and that $h_{tp_1}|A \setminus \text{empty}(h_{tp_1}|A)$ is not satisfiable. (In other words,
we assume that the conditions of the first two items in the list in Definition 20 do not hold.)

Then the following statements hold.

1. If \((tp, A)\) is a special pair then \(h_{tp}|A\) is a good function for \(h_{tp}, A\).

2. Otherwise, let \(tp'\) be the preferred successor of \(tp\) through \(A\). Then \(tp' \in ORD\) (meaning that the definition of \(L_{tp}\) is valid) and \(h_{tp'}\) is a good function for \(h_{tp}, A\).

Proof. It is not hard to see that for each \(tp \in ORD\), \(h_{tp}\) is an interesting function. Also, \(h_{tp}|A\) is an interesting function by assumption of the lemma. This means that the premises of the definition of a good function are satisfied.

The first statement of the lemma is immediate from Proposition 6. For the second statement, observe that \(empty(f_{tp}|A) \subseteq empty(h_{tp}|A)\) and that \(f_{tp}|A \setminus empty(f_{tp}|A) = h_{tp}|A \setminus empty(h_{tp}|A)\). It follows that \(f_{tp}|A\) is an interesting function and, in particular, that all the premises of Lemma 2 have been met.

It follows from the first statement of Lemma 2 that \(f_{tp'}\) does not map any clause in its domain to \(\()\). It also follows from the definition of \(L_{tp}\) and from the second statement of Lemma 2 that \(f_{tp'}\) is unsatisfiable. Thus we conclude that \(tp'\) is a non-basic unsatisfiable type (implying validity of definition of \(L_{tp}\) in Definition 20) and \(h_{tp'}\) is a good function by the second item of Definition 35. In the non-transitional child case \(dom(f_{tp'} = dom(h_{tp}) \setminus empty(h_{tp})\) does not intersect with \(TR\) thus \(h_{tp'}\) is a good function by the first item of Definition 35. Finally, in the single child case, the last two cases of Lemma 2 imply that \(h_{tp'} = h_{tp}|A\) thus, so the lemma follows from Proposition 6.

Lemma 4 Let \((tp, A) \in ORD\). Let \(P\) be a source-sink path \(L_{(tp, A)}\) and let \(u\) be the final vertex of \(P\). Then the following statements hold.

- Suppose that \(u\) is associated with \(C \rightarrow ()\). Then \(C \in dom(h_{tp}|A)\) and \(A(P)\) falsifies \(h_{tp}|A(C)\).

- Suppose that \(u\) is associated with a source of \(L_{tp'}\). Then \(tp^*\) is non-basic unsatisfiable type (implying validity of definition of \(L_{(tp, A)}\) in Definition 20) and \(h_{tp^*}\) is a good subfunction for \(h_{tp}|A, A(P)\).

Proof. Assume first that \(u\) is associated with \(C \rightarrow ()\). Combining the third and fourth statements of Lemma 2 we observe that \(h_{tp^*}\) is a subfunction of \(h_{tp}|A\).
By definition of a transitional resolution, \( C \in \text{dom}(\text{htp}_1) \) and \( A(P) \) falsifies \( \text{htp}_1(C) \). Hence the same holds if \( \text{htp}_1 \) is replaced by \( \text{htp}_{A} \).

Let us now assume that \( u \) is identified with the source of some \( L_{\text{tp}^*} \). By construction and the fourth statement of Lemma \(\ref{corollary1} \) \( A(P) \) is a \( C \)-falsifier for \( \text{htp}_1 \), where \( C \) is the union of \( \text{ET}(\text{tp}_1) \) and some clauses of \( \text{dom}(\text{f}_{\text{tp}_1}) \). Applying again the fourth statement of Lemma \(\ref{corollary1} \) we observe that \( \text{ET}(\text{tp}_1) = \text{ET}(\text{tp}_2) \), hence by construction, \( \text{ET}(\text{tp}^*) = C \). By the third statement of Lemma \(\ref{corollary1} \) \( \text{dom}(\text{f}_{\text{tp}_1}) \cap \text{dom}(\text{f}_{\text{tp}_2}) = \emptyset \), hence by construction \( f_{\text{tp}^*} = f_{\text{tp}_2} \). In particular, it follows from \( \text{tp}_2 \in \text{ORD} \) that \( \text{tp}^* \in \text{ORD} \) and hence \( \text{htp}^* \) is an interesting function. Thus it remains to be shown that \( \text{htp}^* \) is a subfunction of \( \text{htp}|_{A \cup A(P)} \) and that the range of \( \text{htp}|_{A \cup A(P)} \setminus \text{htp}^* \) is satisfiable.

Let \( C \in \text{dom}(\text{htp}^*) \). Assume first that \( C \in \text{dom}(f_{\text{tp}_2}) \). Note that, \( \text{Var}(\text{range}(f_{\text{tp}_2}) \cap \text{Var}(\text{range}(f_{\text{tp}_1})) = \emptyset \) (see the proof of the second statement of Lemma \(\ref{corollary1} \) for an explanation). On the other hand, by construction, \( \text{Var}(A(P)) \subseteq \text{Var}(\text{range}(f_{\text{tp}_1})) \). It follows that \( C \in \text{dom}(\text{htp}|_{A \cup A(P)}) \) and, moreover, that \( \text{htp}|_{A \cup A(P)}(C) = \text{htp}^*(C) \). Next, if \( C \in C \), then \( A(P) \) falsifies \( \text{htp}|_{A(P)} \). Hence \( C \in \text{dom}(\text{htp}|_{A \cup A(P)}) \) and \( () = \text{htp}^*(C) = \text{htp}|_{A \cup A(P)}(C) \). Thus we have shown that \( \text{htp}^* \) is a subfunction of \( \text{htp}|_{A \cup A(P)} \).

Let \( C^* = \text{dom}(\text{htp}|_{A \cup A(P)}) \setminus \text{dom}(\text{htp}^*) \). We need to show that \( \text{htp}|_{A \cup A(P)}(C^*) = \text{htp}|_{A}(C^*)|_{A(P)} \) is satisfiable. It follows from the previous paragraph that \( C^* \) is the subset of \( \text{dom}(\text{htp}_1) \setminus C \) that are not satisfied by \( A(P) \). As by Lemma \(\ref{corollary1} \) \( \text{htp}_1 \) is a subfunction of \( \text{htp}_1(A) \), we in fact need to show that \( \text{htp}_1(C^*)|_{A(P)} \). But this is immediate since \( A(P) \) is a \( C \)-falsifier for \( \text{htp}_1 \).

\( \blacksquare \)

**Corollary 2** Let \( \text{ORD} = \{a_1, \ldots, a_m\} \) Let \( g_1, \ldots, g_m \) be \( \varphi \)-based functions and let \( L_1, \ldots, L_m \) be graphs such that such that \( g_i = \text{htp}_1 \) and \( L_i = L_{\text{tp}_A} \) if \( a_i = \text{tp}_A \) and \( g_i = \text{htp}_1 \). Then \( L_1, \ldots, L_m \) is a transitional falsifying sequence.

**Proof.** Immediate from the combination Lemma \(\ref{corollary1} \) and Lemma \(\ref{corollary2} \) \( \blacksquare \)

**Theorem 5** With the data as in Corollary \(\ref{corollary2} \) let \( R = L_1 \cup \cdots \cup L_m \). Then the following statements hold.

- For each \( \text{tp} \in \text{ORD} \), let \( u_{\text{tp}} \) be the source of \( L_{\text{tp}} \). Then \( R_u_{\text{tp}} \) is a transitional resolution for \( \text{htp}_1 \) and \( \text{Var}(R_u_{\text{tp}}) \subseteq \text{Var}(T) \setminus \text{Var}(p(t)) \) where \( t = t(\text{tp}) \) and \( p(t) \) is the parent of \( t \) in \( T \) (if \( t \) is the root of \( t \), consider \( \text{Var}(p(t)) = \emptyset \)).

- For each \( (\text{tp}, A) \in \text{ORD} \), let \( u_{(\text{tp}, A)} \) be the source of \( L_{(\text{tp}, A)} \). Then \( R_u_{(\text{tp}, A)} \) is a transitional resolution for \( \text{htp}_1 \) and \( \text{Var}(R_u_{(\text{tp}, A)}) \subseteq \text{Var}(T) \setminus \text{Var}(t) \).

**Proof.** First of all, let us introduce one additional notation: for \( 1 \leq j \leq m \), \( R^j_i = L_1 \cup \cdots \cup L_j \)

**Corollary 2** gives us the possibility to apply Theorem \(\ref{corollary2} \) without the read-oneness assumption. From this application, we obtain the following two statements.
1. For each $1 \leq i \leq m$, $R_{u_i}$ is a DAG with all the sinks labelled by either $\mathcal{C} \to ()$ or $\text{trans}_\mathcal{C}$. where $u_i$ is the source of $L_i$

2. For each $1 \leq i \leq m$ and each $i \leq j \leq m$, $R_{u_i}^j = R_{u_i}^j$.

**Claim 2**
1. For each $\mathbf{tp} \in \mathcal{ORD}$, $\text{Var}(R_{\mathbf{utp}}) \subseteq \text{Var}(T_i) \setminus \text{Var}(p(t))$.

2. For each $(\mathbf{tp}, A) \in \mathcal{ORD}$, $\text{Var}(R_{\mathbf{utp}, A}) \subseteq \text{Var}(T_i) \setminus \text{Var}(t)$.

3. Each $R_{u_i}$ is read-once.

**Proof.** The proof is by induction on $1 \leq i \leq m$. The above reasoning allows us to prove the claim for graphs $R_{u_i}$ only. Assume first that $i = 1$. Then $a_1 = \mathbf{tp}$ and $u_1 = u_{\mathbf{tp}}$. Moreover, $R_{u_{\mathbf{tp}}}^1 = L_{\mathbf{tp}}$ and the claim follows by construction.

Assume now that $i > 1$. Assume first that $a_i = \mathbf{tp}$. Let $P$ be a source-sink path of $R_{u_{\mathbf{tp}}}^i$. We need to prove that $P$ is read-once and that $\text{Var}(P) \subseteq \text{Var}(T_i) \setminus \text{Var}(p(t))$ where $t = t(\mathbf{tp})$. By construction, $P$ has a prefix $P'$ which is a source-sink path of $L_{\mathbf{tp}}$. If $P = P'$ then we are done by definition of $L_{\mathbf{tp}}$. Otherwise, the final node of $P'$ is $u_j$ for some $j < i$. Let $P''$ be the suffix of $P$ starting from $u_j$. We will prove that $P''$ is read-once, $\text{Var}(P'') \subseteq \text{Var}(T_i) \setminus \text{Var}(t)$ and hence does not intersect with $\text{Var}(P') \subseteq \text{Var}(t) \setminus \text{Var}(p(t))$. As a minor subtlety, note that this will imply that $\text{Var}(P'') \subseteq \text{Var}(T_i) \setminus \text{Var}(p(t))$ since, by the treewidth connectivity condition, $\text{Var}(p(t)) \cap \text{Var}(T_i) \subseteq \text{Var}(t)$.

Path $P''$ is a source-sink path of $R_{u_j}$, hence it is a source-sink path of $R_{u_j}$ and hence, in turn, it is read-once by the induction assumption. For the constraint on the set of variables, assume first that $a_j = (\mathbf{tp}, A)$. Let $P_{\mathbf{tp}, A}$ be the successor of $P_{\mathbf{tp}}$ through $A$ and $t' = t(\mathbf{tp})$ is a child of $t$. Hence $\text{Var}(P'') \subseteq \text{Var}(T_{t'}) \setminus \text{Var}(t) \subseteq \text{Var}(T_i) \setminus \text{Var}(t)$, the first subset relation follows by the induction assumption. Otherwise, $u_j = (\mathbf{tp}, A)$ but then $\text{Var}(P'') \subseteq \text{Var}(T_i) \setminus \text{Var}(t)$ is immediate by the induction assumption.

It remains to assume that $a_i = (\mathbf{tp}, A)$. Let $\mathbf{tp}_1$ and $\mathbf{tp}_2$ be the successors of $\mathbf{tp}$ through $A$, $t = t(\mathbf{tp})$, $t_1 = t(\mathbf{tp}_1)$ and $t_2 = t(\mathbf{tp}_2)$ are the children of $t$. We assume w.l.o.g. that $\mathbf{tp}_1$ is the main successor of $\mathbf{tp}$ through $A$.

As in the previous case, let $P$ be a source-sink path of $R_{(\mathbf{tp}, A)}^i$. We need to prove that $P$ is read-once and that $\text{Var}(P) \subseteq \text{Var}(T_i) \setminus \text{Var}(t)$. By construction, $P$ has a prefix $P'$ which is a source-sink path of $L_{(\mathbf{tp}, A)}$. By construction, $P'$ is read-once and $\text{Var}(P') \subseteq \text{Var}(T_{t_i}) \setminus \text{Var}(t)$. Therefore, if $P = P'$, we are done. Otherwise, the final node of $P'$ is $u_j$ for some $j < i$. Let $P''$ be the suffix of $P$ starting at $u_j$. Then $P''$ is a source-sink path of $R_{u_j}^i$ and hence a source-sink path of $R_{u_j}^i$. By construction, $a_j = \mathbf{tp}^*$ such that $t(\mathbf{tp}^*) = t_2$. It follows from the induction assumption that $P''$ is read-once and $\text{Var}(P'') \subseteq \text{Var}(T_{t_2}) \setminus \text{Var}(t) \subseteq \text{Var}(T_i) \setminus \text{Var}(t)$. To conclude that the whole $P$ is read-once, we observe that the connectedness condition implies that $\text{Var}(T_{t_1}) \cap \text{Var}(T_{t_2}) \subseteq \text{Var}(t)$ implying that $\text{Var}(P') \cap \text{Var}(P'') = \emptyset$ thus confirming the read-onceness of $P$. □

**Claim 3** Let $a_i \in \mathcal{ORD}$. Let $P$ be a source-sink path of $L_i$, let $u$ be the final vertex of $P$ and assume that $u$ is the source of $L_j$ for $j < i$. Then $\text{Var}(R_{u_i})$ is disjoint from $\text{Var}(\text{range}(g_i|_{A(P)} \setminus g_j))$. 

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Lemma 5 For each $t \in V(T)$, $\text{split}(t) \leq \max(0, \text{trans}(t) - 1)$

Proof. The proof is by induction from the leaves to the root of $t$. If $t$ is a leaf then both $\text{split}(t)$ and $\text{trans}(t)$ are zeroes, so the inequality clearly holds.
Suppose that \( t \) is not a leaf. If \( \text{split}(t) = 0 \) then the inequality again clearly holds, so we assume that \( \text{split}(t) > 0 \). Assume first that \( t \) is not a splitting node. Then \( t \) has a child \( t_1 \) such that \( CL(T_t \setminus t) \cap \text{TR} = CL(T_{t_1}) \cap \text{TR} \). It follows that all the splitting nodes of \( T_t \) are in \( T_{t_1} \). Consequently, \( \text{split}(t_1) = \text{split}(t) > 0 \) and hence, by the induction assumption, \( \text{split}(t_1) \leq \text{trans}(t_1) - 1 \). As \( \text{trans}(t_1) \leq |CL(T_{t_1}) \cap \text{TR}| = \text{trans}(t) \), the statement follows.

It remains to assume that \( t \) is a splitting node. Let \( t_1 \) and \( t_2 \) be the children of \( t \). For \( i \in \{1, 2\} \) let \( C_i = CL(T_{t_i}) \cap \text{TR} \). Due to one-sidedness \( C_1 \cap C_2 = \emptyset \). In particular, it follows that

1. \( \text{trans}(t) \geq 2 \), thus immediately implying the statement for the case where both \( \text{split}(t_1) \) and \( \text{split}(t_2) \) are zeroes.

2. For each \( i \in \{1, 2\} \), \( \text{trans}(t) \geq \text{trans}(t_i) + 1 \). This means that, if say \( \text{split}(t_1) > 0 \) while \( \text{split}(t_2) = 0 \), then taking into account the induction assumption, \( \text{split}(t) = \text{split}(t_1) + 1 \leq \text{trans}(t_1) - 1 + 1 \leq \text{trans}(t) - 1 \).

3. \( \text{trans}(t) \geq \text{trans}(t_1) + \text{trans}(t_2) \). In particular, if both \( \text{split}(t_1) > 0 \) and \( \text{split}(t_2) > 0 \) hold then \( \text{split}(t) = \text{split}(t_1) + \text{split}(t_2) + 1 \leq (\text{trans}(t_1) - 1) + (\text{trans}(t_2) - 1) + 1 \leq \text{trans}(t) - 1 \), the first equality accounts 1 for \( t \), the first inequality follows from the induction assumption.

\begin{definition}
In this definition we introduce three notions of a rank.

- **Ranks for types and special pairs** We define the rank \( \text{rank}(x) \) of each element \( x \) of \( \text{ORD} \) under assumption that the ranks have been defined for all the smaller elements. Let \( x = \text{tp} \). If \( t(\text{tp}) \) is a leaf then \( \text{rank}(x) = 0 \). Otherwise, \( \text{rank}(x) = \min_{y \in \text{cl}(x)} \text{rank}(y) \). If \( x = (\text{tp}, A) \), let \( \text{tp}_1 \) be the main successor of \( \text{tp} \) through \( A \). Then \( \text{rank}(x) = \text{rank}(\text{tp}_1) + 1 \).

- **Ranks for the nodes of \( T \).** For a node \( t \), the main rank \( \text{mrank}(t) \), and the splitting rank \( \text{srank}(t) \) are defined recursively from the leaves to the root as follows. If \( t \) is a leaf then both \( \text{mranks}(t) \) and \( \text{srank}(t) \) are zeroes. If \( t \) is not a splitting node then \( \text{srank}(t) = 0 \) and \( \text{mrank}(t) \) is the maximum over all \( \text{mrank}(t') \) such that \( t' < t \) (descendant of \( t \) in \( T \)). If \( t \) is a splitting node, let \( t_1 \) be the main child of \( t \). Then \( \text{srank}(t) = \text{mrank}(t_1) + 1 \) and \( \text{mrank}(t) \) is the maximum between \( \text{srank}(t) \) and all \( \text{mrank}(t') \) such that \( t' < t \).

\end{definition}

\begin{lemma}
For each \( x \in \text{ORD} \), the following two statements hold.

1. If \( x = \text{tp} \) then \( \text{rank}(x) \leq \text{mrank}(t(\text{tp})) \).

2. If \( x = (\text{tp}, A) \) then \( \text{rank}(x) \leq \text{srank}(t(\text{tp})) \).

\end{lemma}

\begin{proof}
By induction from the leaves to the root of \( T \). Let \( x \in \text{ORD} \) be such that \( t = t(\text{tp}(x)) \) is a leaf. Since \( t \) is not a splitting node, \( x \) is not a special

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pair. By definition, \( \text{rank}(x) = 0 \) and hence the statement holds because the \( \text{mrank} \) function is non-negative.

Assume now that \( x = \text{tp} \) such that \( t = t(\text{tp}) \) is not a splitting node. By definition, there is \( y <_d x \) such that \( \text{rank}(x) = \text{rank}(y) \). As \( t \) is not a splitting node, \( y \) cannot be a special pair with type equal \( \text{tp} \). Let \( \text{tp}' = \text{tp}(y) \) Then \( t' = t(\text{tp}) \) is a descendant of \( t \) in \( T \).

This means that either \( y = \text{tp}' \) or \( y \) is a special pair with the type equal \( \text{tp}' \). It is not hard to observe that in the latter case \( \text{tp}' \) is also a descendant of \( \text{tp} \) (as the only parent of a special pair is its type). Furthermore, in the latter case \( \text{rank}(\text{tp}') \geq \text{rank}(y) \). Therefore, we may assume that \( y = \text{tp}' \). Then, by the induction assumption and definition of \( \text{mrank} \), \( \text{rank}(\text{tp}) = \text{rank}(\text{tp}') \leq \text{mrank}(t') \leq \text{mrank}(t) \).

Assume now that \( x = (\text{tp}, A) \). Let \( t = t(\text{tp}) \) and let \( t_1 \) be the main child of \( t \). Let \( \text{tp}_1 \) be the \( (t_1, A) \)-successor. Then, by definition of \( \text{rank} \), and the induction assumption \( \text{rank}(x) \leq \text{rank}(\text{tp}_1) + 1 \leq \text{mrank}(t_1) + 1 = \text{srank}(1) \).

Finally, we assume that \( x = \text{tp} \) such that \( t = t(\text{tp}) \) is a splitting node. Let \( y <_d x \) be an element of \( \text{ORD} \) such that \( \text{rank}(x) = \text{rank}(y) \). If \( y \) is a special pair \( (\text{tp}, A) \) then, by the induction assumption, \( \text{rank}(x) \leq \text{srank}(t) \leq \text{mrank}(t) \). Otherwise, we apply the same argumentation as for the case where \( t \) is not a splitting node.

Lemma 7 For each \( t \in V(T) \), \( \text{mrank}(t) \leq \max(0, \lceil \log(\text{trans}(t)) \rceil) \).

Proof. By induction from the leaves to the root of \( t \). If \( t \) is a leaf then \( \text{mrank}(t) = 0 \) so the statement clearly holds.

Suppose that \( t \) is not a leaf. If \( \text{mrank}(t) = 0 \) then the inequality again clearly holds, so we assume that \( \text{mrank}(t) > 0 \). Assume first that \( t \) is not a splitting node. Then \( t \) has a child \( t_1 \) such that \( \text{CL}(T_i \setminus t) \cap \text{TR} = \text{CL}(T_i) \cap \text{TR} \). Consequently, \( \text{mrank}(t) = \text{mrank}(t_1) > 0 \) and hence, by the induction assumption, \( \text{mrank}(t_1) \leq \lceil \log(\text{trans}(t_1)) \rceil \). As \( \text{trans}(t_1) \leq |\text{CL}(T_i) \cap \text{TR}| = \text{trans}(t) \), the statement follows.

It remains to assume that \( t \) is a splitting node. Let \( t_1 \) and \( t_2 \) be the children of \( t \) with \( t_1 \) being the main child. As \( \text{mrank} \) is monotone non-decreasing on the direction from the leaves to the root, \( \text{mrank}(t) = \max(\text{mrank}(t_1) + 1, \text{mrank}(t_2)) \)

For \( i \in \{1, 2\} \) let \( C_i = \text{CL}(T_i) \cap \text{TR} \). Due to one sidedness \( C_1 \cap C_2 = \emptyset \). In particular, it follows that \( \text{trans}(t) \geq 2 \) thus immediately implying the statement in case both \( \text{mrank}(t_1) \) and \( \text{mrank}(t_2) \) are zeroes. Otherwise, assume first that \( \text{mrank}(t) = \text{mrank}(t_2) \). Then \( \text{mrank}(t_2) \geq \text{mrank}(t_1) + 1 \geq 0 \) and the argumentation in the previous paragraph applies.

It remains to assume that \( \text{mrank}(t) = \text{mrank}(t_1) + 1 \). If \( \text{mrank}(t_1) = 0 \), the statement is immediate since \( \text{trans}(t) \geq 2 \), so we assume that \( \text{mrank}(t_1) > 0 \) By definition of the main child, \( |C_1| \leq \text{trans}(t)/2 \). Further on, \( \text{trans}(t_1) \leq |C_1| \). It follows that \( \lceil \log(\text{trans}(t_1)) \rceil \leq \lceil \log(\text{trans}(t)) \rceil - 1 \). By the induction assumption, \( \text{mrank}(t_1) \leq \lceil \log(\text{trans}(t)) \rceil - 1 \). Hence, the statement holds. ■

Recall, that by definition, \( \text{ORD} \) is the disjoint union of the set \( \mathcal{N}S \) of types and the set \( \mathcal{SP} \) of special pairs. Let \( ns = |\mathcal{N}S| \) and \( sp = |\mathcal{SP}| \). We observe
that the definitions of local graphs $L_{tp}$ and especially $L_{(tp,A)}$ are not completely deterministic in the sense that there may be several possible graphs meeting the requirements of the respective definitions. In the next lemma, we claim that these graphs can be such that a good upper bound can be claimed on the size of the respective $R$-graphs.

**Lemma 8** There are local graphs for the elements of $ORD$ matching their respective definitions so that the following size upper bounds are observed.

1. Let $tp \in NS$. Then $|R_{ns}| \leq 2^{k+1} \cdot ns \cdot \sum_{i=0}^{r} sp^i$ where $r$ is the rank of $tp$.

2. Let $(tp,A) \in SP$. Then $|L_{(tp,A)}| \leq 2^{k+1} \cdot ns \cdot \sum_{i=0}^{r-1} sp^i$, where $r$ is the rank of $(tp,A)$.

**Proof.** It is not hard to infer from the combination of Corollary 2 and Lemma 1 that for each $a_i \in ORD$, $R_{ns}$ is the union of $L_i$ and all the $L_j$ such that $a_j$ is a descendant of $a_i$. Therefore, for $a_i = tp$, instead of $|R_{ns}|$, we can use $\sum_{a_i \leq a_i} |L_i|$.

Note also that for each $tp \in NS$ we can assume that $|L_{tp}| \leq 2^{k+1}$ simply because $L_{tp}$ is a decision tree over at most $k$ variables.

The proof is by induction along $ORD$. Consider first the smallest element of $ORD$. This element is some $tp$. In fact, we make a more general assumption that $rank(tp) = 0$. In this case, all the successors of $tp$ are elements of $NS$. Hence, taking into account the reasoning in the first two paragraphs, $|R_{tp}| \leq ns \cdot 2^{k+1}$. In particular, the statement of the lemma holds for this case.

Consider now an element $x \in ORD$ under assumption that the statement of the lemma holds for all the smaller elements of $ORD$. Assume first that $x = (tp,A)$. Let $tp_1$ be the main successor of $tp$, $t = t(tp)$ and $t_1 = t(tp_1)$.

By construction, $L_{(tp,A)}$ is a transitional resolution for $h_{tp_1}$ under additional constraint that its set of variables must be a subset of $Var(T_{t_1}) \setminus Var(t)$. It turns out that a good upper bound on such a transitional resolution follows from the induction assumption. Indeed, $tp_1 \in ORD$ and precedes $(tp,A)$. By Theorem 5, $R_{tp_1}$ is a transitional resolution for $h_{tp_1}$ with the required constraint on its set of variables, hence any upper bound on its size can be used to upper-bound the size of $L_{(tp,A)}$. By definition, of the rank of $(tp,A)$ the rank of $tp_1$ is $r-1$.

Therefore, by the induction assumption, $|R_{ns}| \leq 2^{k+1} \cdot ns \cdot \sum_{i=0}^{r-1} sp^i$ thus confirming the lemma for the considered case.

It remains to assume that $x \in NS$, that is $x = tp$. Then we can upper bound the size of $R_{ns}$ by the total size of $ns$ local graphs of size at most $2^k$ each and at most $sp$ special pairs of rank at most $r$ each (since these special pairs must be descendants of $tp$). Applying the induction assumption to the special pairs, observe that the sizes of their respective local graphs are at most $2^{k+1} \cdot ns \cdot (\sum_{i=1}^{r-1} sp^i + 1)$. Thus $|R_{ns}| \leq 2^{k+1} \cdot ns + sp \cdot (2^{k+1} \cdot ns \cdot \sum_{i=0}^{r-1} sp^i = 2^{k+1} \cdot ns \cdot \sum_{i=0}^{r-1} sp^i$. □

**Proof of Theorem 1.** Let $rt$ be the root of $T$. Consider the type $tp = (rt, 0, 0, 0)$. It is not hard to see that $h_{tp} = 1_x$. It follows that $tp \in ORD$ and
hence, by Theorem 5 $R_{\text{top}}$ is a transitional resolution for $\varphi$ and $\text{TR}$. Thus we need to demonstrate that the required upper bound holds for $|R_{\text{top}}|$. For this purpose we employ Lemma 8 and conclude that $|R_{\text{top}}| \leq 2^{k+1} \cdot n_s \cdot \sum_{i=1}^{t} sp^i$, where $r$ is the rank of $\text{tp}$. So, we need to upper-bound, $n_s, sp$, and $r$.

In order to upper-bound $n_s$, let us compute for the given fixed $t$ the number of types $\text{tp}$ such that $t = t(\text{tp})$. It is not hard to see that the number of such types is at most the product of the possible elements of the second, third and fourth components, that is $2^{\text{Var}(t)} \cdot 2^{|\text{Cl}(t)|} \cdot 2^{\text{TR}}$. Taking into account that $|\text{Var}(t)| + |\text{Cl}(t)| \leq k$, we conclude that $2^{\text{Var}(t)} \cdot 2^{|\text{Cl}(t)|} \cdot 2^{\text{TR}} \leq 2^k \cdot 2^{\text{TR}}$. Because of our assumption about $(T, B)$, $|V(t)| \leq O(n)$ where $n = |\text{Var}(\varphi)| + |\varphi|$. Thus $n_s = O(n \cdot 2^k \cdot 2^{\text{TR}})$.

In order to upper-bound $sp$, we again fix $t \in V(t)$ and upper-bound the number of $(\text{tp}, A)$ such that $t = t(\text{tp})$. Taking into account that $\text{Var}(A) \subseteq \text{Var}(t)$, we obtain an upper bound of $4^k \cdot 2^{\text{TR}}$. Next, we observe that if $t = t(\text{tp})$ and $(\text{tp}, A)$ is a special pair then $t$ is a splitting node. This means that $sp$ is at most the number of splitting nodes of $T$ multiplied by $4^k \cdot 2^{\text{TR}}$.

Applying Lemma 5 and taking into account that $\text{trans}(t) \leq |\text{TR}|$ for each $t \in V(T)$, we conclude that $sp \leq |\text{TR}| \cdot 4^k \cdot 2^{\text{TR}}$.

In order to upper bound $r$, we combine the inequality $\text{trans}(t) \leq |\text{TR}|$ obtained in the previous paragraph and Lemmas 5 and 7 to observe that $r \leq \max(0, \log(|\text{TR}|))$. Substituting the data in the formula in the end of the first paragraph of this proof and performing elementary transformations, we conclude that assuming that $|\text{TR} \neq \emptyset$, a transitional resolution for $\varphi$ can be upper bounded by $n \cdot 2^{O(k \log(|\text{TR}|) + \log^2(|\text{TR}|))}$. If $\text{TR} = \emptyset$ then the upper bound becomes $O(4^k)$.

\section*{C Proof of Theorem 2}

We first prove two propositions concerning postorder traversals.

\begin{proposition}
Let $\pi$ be a proper prefix of $\pi_T$ and let $t$ be its immediate successor. Then the following statements hold.

1. If $t$ is an expanding node then $\text{Trees}_{\pi+t} = \text{Trees}_\pi \cup \{T_i\}$ and $T_i$ is the last element of $\text{Trees}_{\pi+t}$. The order between the remaining elements is the same as in $\text{Trees}_\pi$.

2. Assume that $t$ is a contracting node. Then $\text{Trees}_\pi$ are the set of trees rooted by the children of $t$. Then $\text{Trees}_\pi$ are the last elements of $\text{Trees}(\pi)$, $\text{Trees}_{\pi+t} = (\text{Trees}_\pi \setminus \text{Trees}_\pi') \cup \{T_i\}$, $T_i$ is the last node of $\text{Trees}_{\pi+t}$, and the order of the rest of the elements is preserved as in $\text{Trees}_\pi$.

\end{proposition}

\begin{proof}

By induction on $|V(T)|$. The statement is clearly true for $|V(T)| = 1$ so assume that $|V(T)| > 1$.

Let $rt$ be the root of $T$. Assume that $t = rt$. Then clearly $t$ is a contracting node and $\text{Trees}_\pi = \text{Trees}_\pi'$. It is not hard to see that $\text{Trees} (\pi + t) = \{T_{rt}\}$, hence the statement holds for this case.

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Let \( t_1 \) be the left child of \( rt \) and let \( \pi_1 \) be the postorder traversal of \( T_{t_1} \). Assume that \( t \in T_{t_1} \). Then, by the induction assumption, the statement is correct regarding \( \pi_1 \). Clearly, it remains correct if \( \pi \) is used instead of \( \pi_1 \). It remains to assume that \( rt \) has two children. Let \( t_2 \) be the right child of \( rt \) and let \( \pi_2 \) be the postorder traversal of \( T_{t_2} \). Since the previous cases do not hold, \( \pi = \pi_1 + \pi' \). Moreover, \( t \) is the immediate successor of \( \pi' \) in \( \pi_2 \). By the induction assumption, the statement holds regarding \( \pi' \) and \( t \) w.r.t. \( \pi_2 \). As \( \text{Trees}_{\pi + t} \) (and their order) are the same w.r.t. \( \pi_2 \) and \( \pi_T \) and \( T_{t_1} \) is located before all the elements of \( \text{Trees}_{\pi' + t} \), the statement holds regarding \( \pi \) and \( t \). ■

**Proposition 9** Let \( \pi \) be a proper prefix of \( \pi_T \) and let \( t \) be the immediate successor of \( \pi \). Let \( x \in \text{Var}(\pi) \).

1. Assume that \( t \) is an expanding node. Then \( MT_{\pi}(x) = MT_{\pi + t}(x) \).

2. Assume that \( t \) is a contracting node. If \( MT_{\pi}(x) \notin \text{Trees}'_\pi \) then \( MT_{\pi}(x) = MT_{\pi + t}(x) \). Otherwise, \( MT_{\pi + t}(x) = T_t \).

**Proof.** Assume that \( t \) is an expanding node. Let \( T' = MT_{\pi}(x) \). By Proposition \( \text{S} \) \( T' \in \text{Trees}_{\pi + t} \). Suppose that \( T'' \in \text{Trees}_{\pi + t} \) such that \( x \in \text{Var}(T'') \) and \( T'' < T' \). Again by Proposition \( \text{S} \) \( T'' \in \text{Trees}_\pi \) and \( T'' < T' \) in contradiction to the minimality of \( T'' \).

Assume now that \( t \) is a contracting node with \( T' \) being as above. If \( T' \notin \text{Trees}'_\pi \) then \( MT_{\pi}(x) = MT_{\pi + t}(x) \) by the same argument as in the previous paragraph. Otherwise, by Proposition \( \text{S} \) \( x \in \text{Var}(T_t) \). Suppose there is \( T'' \in \text{Trees}_{\pi + t} \) such that \( x \in \text{Var}(T'') \) and \( T'' < T_t \). By Proposition \( \text{S} \) \( T'' \in \text{Trees}_\pi \) and \( T'' \) is smaller than each of \( \text{Trees}'_\pi \), a contradiction to the minimality of \( T'' \). ■

When we consider the types of \( \mathcal{ORD} \), we in fact concentrate on specific types that are called well-formed. We define them below.

**Definition 40** Let \( x \in \text{Var}_p(x) \). We denote by \( W_{tp}(x) \) the set of witnessing clauses of \( x \) as per Definition \( \text{A} \). \( W_{tp}(x) \) is the disjoint union of two sets \( W_{tp}^0(x) \) and \( W_{tp}^1(x) \) where \( W_{tp}^0(x) \) consists of the clauses satisfying part (i) of definition of a witnessing clause as per Definition \( \text{A} \), and \( W_{tp}^1(x) \) are those satisfying part (ii).

**Definition 41** Continuing on Definition \( \text{A} \) we say that \( x \) is consistent if it has the same occurrence in all the clauses of \( W_{tp}(x) \). Put it differently, \( x \) is consistent if it either occurs positively in all the clauses of \( W_{tp}(x) \) or occurs negatively in all of them. We say that \( tp \) is well-formed if all the fixed variables are consistent.

The rest of the proof is divided into five subsections. In the first four subsections we prove that sinks of local graphs of non-basic non-final types \( tp \) satisfy the validity conditions in terms of Theorem \( \text{??} \). Each subsection is devoted to a specific condition concerning the kind of the immediate successor \( t \) of \( \pi(tp) \) and the assignment \( A(u) \) of the considered sink. The actual proof, gathering all facts together, is provided in the fifth subsection.
C.1 Expanding $t$, $A(u)$ does not satisfy new long clauses

Lemma 9 Let $\mathbf{tp} = (\pi, \text{map}, \text{CN}, \text{RA})$ be a well-formed non-basic non-final unsatisfiable type and let $t$ be the immediate successor of $\pi$. Assume that $t$ is an expanding node. Let $u$ be a sink of $L_{\mathbf{tp}}$ such that $A(u)$ does not satisfy any clauses of $\text{LONG} \setminus \text{LS}$. Let $\mathbf{tp}' = (\pi(\mathbf{tp}) + t, \text{map}, \text{CN}', \text{RA})$ where $\text{CN}'$ is obtained from $\text{CN}$ by removal of clauses that are satisfied by $A(u)$. Then the following statements hold.

1. For each $x \in \text{Var}_{\text{fix}}(\mathbf{tp}) \cap \text{Var}_{\text{fix}}(\mathbf{tp}')$, $W_{\mathbf{tp}}(x) = W_{\mathbf{tp}'}(x)$.
2. $\text{Var}_{\text{fix}}(\mathbf{tp}')$ is the disjoint union of $\text{Var}_{\text{fix}}(\mathbf{tp})$ and $\text{Det}(\mathbf{tp})$.
3. For each $x \in \text{Det}(\mathbf{tp})$, $W_{\mathbf{tp}'}(x) = W_{\mathbf{tp}}^0(x)$.
4. $\text{Var}(A(u)) = \text{Det}(\mathbf{tp})$.
5. $\text{FA}(\mathbf{tp}') = \text{FA}(\mathbf{tp}) \cup A(u)$.
6. $\text{Var}_{\text{out}}(\mathbf{tp})$ is the disjoint union of $\text{Var}_{\text{out}}(\mathbf{tp}')$ and $\text{Det}(\mathbf{tp})$.

Proof.

Claim 4 Let $x \in \text{Var}_{\text{fix}}(\mathbf{tp})$. Then $x \in \text{Var}_{\text{fix}}(\mathbf{tp}')$ and $W_{\mathbf{tp}}(x) \subseteq W_{\mathbf{tp}'}(x)$.

Proof. First of all, it follows directly from definition of $\mathbf{tp}'$ that $\text{Var}_{\text{in}}(\mathbf{tp}) = \text{Var}_{\text{in}}(\mathbf{tp}')$. Then, $x \in \text{Var}(\pi + t) \setminus \text{Var}_{\text{in}}(\mathbf{tp}')$.

Let $C \in W_{\mathbf{tp}}(x)$. Assume first that $C \in W_{\mathbf{tp}}^0(x)$. By definition of $\mathbf{tp}'$, $[\text{map}(\mathbf{tp}')](C) = \text{none}$. It follows that $x \in \text{Var}_{\text{fix}}(\mathbf{tp}')$ and that $C \in W_{\mathbf{tp}'}(x)$. Assume now that $C \in W_{\mathbf{tp}}^1(x)$. By definition of $\mathbf{tp}'$, $[\text{map}(\mathbf{tp})](C) = [\text{map}(\mathbf{tp}')](C)$. By Proposition 3 $\text{MT}_{\pi+t}(x) = \text{MT}_{\pi}(x)$. As $\text{MT}_{\pi}(x) < [\text{map}(\mathbf{tp})](C)$ (due to $C$ being witnessing for $x$ in $\mathbf{tp}$), we conclude that $\text{MT}_{\pi+t}(x) < [\text{map}(\mathbf{tp}')](C)$ thus confirming that $x \in \text{Var}_{\text{fix}}(\mathbf{tp}')$ and that $C \in W_{\mathbf{tp}'}(x)$. \□

The proof of the following claim is analogous to the proof of Claim 4 with $\mathbf{tp}$ and $\mathbf{tp}'$ exchanging their roles.

Claim 5 Let $x \in \text{Var}_{\text{fix}}(\mathbf{tp}') \cap \text{Var}(\pi)$. Then $x \in \text{Var}_{\text{fix}}(\mathbf{tp})$ and $W_{\mathbf{tp}'}(x) \subseteq W_{\mathbf{tp}}(x)$.

Taking into account that $\text{Var}_{\text{fix}}(\mathbf{tp}) \subseteq \text{Var}(\pi)$ by definition, Claim 4 in fact shows that $\text{Var}_{\text{fix}}(\mathbf{tp}) \subseteq \text{Var}_{\text{fix}}(\mathbf{tp}') \cap \text{Var}(\pi)$. Hence, combination of Claim 4 and 5 imply that these two sets are equal and the witnessing sets of each variable are the same w.r.t $\mathbf{tp}$ and $\mathbf{tp}'$. This implies the first statement of the lemma.

Let now $x \in \text{Var}_{\text{fix}}(\mathbf{tp}') \setminus \text{Var}(\pi)$. Then $\text{MT}_{\pi+t}(x) = T_t$, the largest tree of $\text{Trees}(\pi + t)$. Hence $W_{\mathbf{tp}'}^1(x) = \emptyset$, implying that $W_{\mathbf{tp}'}(x) = W_{\mathbf{tp}}^0(x)$ and, in particular (since $x \in \text{Var}_{\text{fix}}(\mathbf{tp}')$ must be witnessed by at least one clause), that $W_{\mathbf{tp}'}^0(x) \neq \emptyset$. 41
Let $C \in W_{tp}^0(x)$. Then $C \in \text{LONG} \setminus \text{LS}(tp') = \text{LONG} \setminus \text{LS}(tp)$. As all the variables outside of $Var(\pi)$ belong to $Var_{out}(tp)$, $x \in Var(t) \cap Var_{out}(tp)$.

Hence $x \in \text{Det}(tp)$.

Conversely, let $x \in \text{Det}(tp)$. This means that there is $C \in \text{LONG} \setminus \text{LS}(tp') = \text{LONG} \setminus \text{LS}(tp)$, such that $x \in Var(C)$. This already implies that $x \in Var_{fix}(tp')$. It remains to show that $x \notin Var(\pi)$. But if it were so then $x$ would belong to $Var_{fix}(tp)$ causing a contradiction since $x \in Var_{out}(tp)$ by definition of $\text{Det}(tp)$. Thus the second and the third statements are settled.

For the fourth statement, note that, by construction, $\text{Det}(tp) \subseteq Var(A(u))$.

If $A(u)$ assigned any extra variables, it would have satisfied a clause of $\text{LONG} \setminus \text{LS}(tp)$ which is not the case.

For the fifth statement, remains to show that for each $x \in Var_{fix}(tp)$, the occurrences of $x$ in $FA(tp)$ and $FA(tp')$ are the same and that for each $x \in \text{Det}(tp)$, the occurrences of $x$ in $A(u)$ and in $FA(tp')$ are the same.

In the former case, the desired statement follows from $W_{tp}(x) = W_{tp'}(x)$. In the latter case, the assignment of $x$ in $A(u)$ is precisely the one to not satisfy any witnessing clause of $x$, hence precisely the assignment of $x$ in $FA(tp')$.

For the last statement, note that $Var_{out}(tp') = Var(\pi) \setminus (Var_{in}(tp') \cup Var_{fix}(tp')) = Var(\pi) \setminus (Var_{in}(tp) \cup Var_{fix}(tp) \cup \text{Det}(tp)) = (Var(\pi) \setminus (Var_{in}(tp) \cup Var_{fix}(tp))) \setminus \text{Det}(tp) = \text{Var}(tp) \setminus \text{Det}(tp)$.

**Theorem 6** Let $tp = (\pi, map, CN, RA)$ be a well-formed non-basic non-final unsatisfiable type and let $t$ be the immediate successor of $\pi$. Assume that $t$ is an expanding node. Let $u$ be a sink of $L_{tp}$ such that $A(u)$ does not satisfy any clauses of $\text{LONG} \setminus \text{LS}$. Then the following statements hold.

1. If $u$ is a labelled with $C \rightarrow ()$ then $C \in \text{dom}(F_{tp})$, $C$ is not satisfied by $A(u)$ and $F(C)|_{A(u)} = ()$.

2. If $u$ is identified with the source of $L_{tp'}$, then $tp'$ is a well-formed non-basic unsatisfiable type and $F_{tp'} = F_{tp}|_{A(u)}$.

**Proof.** The first statement is immediate by construction, so we prove the second statement. Assume the premise of the statement regarding $u$. By construction, $tp' = (\pi(tp) + t, map, CN', RA)$ where $CN'$ is obtained from $CN$ by removal of clauses that are satisfied by $A(u)$. In order to show that $tp'$ is a well-formed type, we demonstrate that each variable $x \in Var_{fix}(tp')$ is consistent. If $x \in Var_{fix}(tp') \cap Var_{fix}(tp)$ then the consistency of $x$ follows from the well-formity of $tp$ by the first statement of Lemma. Otherwise, by the second statement of Lemma, $x \in \text{Det}_{tp}$. For the sake of contradiction, assume that there are two clauses of $W_{tp'}(x)$ such that $x$ occurs positively, say in $C_1$ and negatively in $C_2$. Then the assignment of $x$ in $A(u)$ must satisfy one of the clauses. However, by the third statement of Lemma, $C_1, C_2 \in \text{LONG} \setminus \text{LS}(tp)$ and $A(u)$ does not satisfy any of these clauses by assumption.

Let $C \in \text{dom}(F_{tp}|_{A(u)}) \cap \text{dom}(F_{tp'})$. Then $F_{tp}(C)|_{A(u)} = \text{Proj}(C, Var_{out}(tp))|_{A(u)} = \text{Proj}(C, Var_{out}(tp) \setminus Var(A(u))) = \text{Proj}(C, Var_{out}(tp')) = F_{tp'}(C)$. The penultimate equality follows form the combination of statements 4 and 6 of
Lemma 9 It follows that it is enough to show that $\text{dom}(F_{tp'}) = \text{dom}(F_{tp}|_A)$. Since $\text{LS}(tp) = \text{LS}(tp')$, $\text{dom}(F_{tp}) \cap \text{LONG} = \text{LONG} \setminus \text{LS}(tp) = \text{LONG} \setminus \text{LS}(tp') = \text{dom}(F_{tp'}) \cap \text{LONG}$. Since, by assumption, $A(u)$ does not satisfy any clause of $\text{LONG} \setminus \text{LS}(tp)$, we conclude that $\text{dom}(F_{tp}|_{A(u)}) \cap \text{LONG} = \text{dom}(F_{tp'}) \cap \text{LONG}$.

Next, recall that $\text{MT}(tp) = \text{MT}(tp')$ and let $CL^* = CL(\text{Roots}(\text{MT}(tp)))$. By definition, $\text{dom}(F_{tp'}) \cap CL^* = CN'$ while $\text{dom}(F_{tp}) \cap CL^* = CN$. This means that $\text{dom}(F_{tp}|_{A(u)}) \cap CL^*$ is the subset of $CN$ consisting of the clauses that are not satisfied by $A(u)$. But, by definition, this is exactly $CN'$!

Now, let $C \in \text{dom}(F_{tp'}) \setminus (\text{LONG} \cup CL^*)$. This means that $\text{Var}(C) \cap \text{Var}_{out}(tp') \neq \emptyset$ and $C$ is not satisfied by $RA(tp') \cup FA(tp')$. By construction and Lemma 9 $\text{Var}(C) \cap \text{Var}_{out}(\text{tp}) \neq \emptyset$, $C$ is not satisfied by $RA(tp) \cup FA(tp)$ and also not satisfied by $A(u)$. We conclude that $C \in \text{dom}(F_{tp}|_{A(u)})$. Conversely, suppose that $C \in \text{dom}(F_{tp}|_{A(u)})$. This means that $C$ is not satisfied by $RA(tp) \cup FA(tp) \cup A(u) = RA(tp') \cup FA(tp')$ It remains to see that $\text{Var}(C) \cap \text{Var}_{out}(tp') \neq \emptyset$. By Lemma 9 the opposite can happen only if $F_{tp'}(C)|_{A(u)} = \emptyset$. But, in this case, the premise of the first statement is satisfied that is $u$ is a falsifying sink of $L_{tp}$, which is not the case by assumption.

It remains to be added that, based on the previous paragraph, the unsatisfiability of $tp''$ follows from the unsatisfiability of $tp$ since this implies that $F_{tp}|_{A(u)}$ is unsatisfiable. Also $tp''$ because $A(u)$ does not falsify any $F_{tp}(C)$ for any $C \in \text{dom}(F_{tp})$ simply by assumption.

\[ \square \]

\section*{C.2 Expanding $t$, $A(u)$ satisfies new long clauses}

Now, we are going to consider the case where $t$ is an expanding node and $A(u)$ satisfies some clauses mapped to $none$ by $map$. Throughout the consideration, $tp' = (\pi(tp)+t, map', CN^*, RA')$ where $map'$ is obtained from $map$ by replacing $C \rightarrow none$ by $C \rightarrow \text{last}(Trees_{\pi(tp)+t})$ for each $C \in \text{LONG} \setminus \text{LS}$ that is satisfied by $A(u)$. $CN^*$ is obtained from $CN$ by removal of all clauses that are satisfied by $A(u)$ and adding those clauses of $\text{dom}(F_{tp}) \cap CL(t)$ that are not satisfied by $A(u)$. Finally $RA' = RA \cup A(u) \cup Proj(FA(tp), Var(t))$.

Lemma 10 The following statements hold:

1. $\text{Var}_{fix}(tp') = \text{Var}_{fix}(tp) \setminus \text{Var}(t)$.
2. For each $x \in \text{Var}_{fix}(tp')$, $W_{tp}(x) = W_{tp'}(x)$.

Proof. Let $x \in \text{Var}_{fix}(tp) \setminus \text{Var}(t)$. Let $C \in W_{tp}(x)$. If $C \in W_{tp}(x)$ then $map(C) \in Trees_{\pi}$. By construction, $map(C) = map(C)$. By Proposition 9 $MT_{\pi+t}(x) = MT_{\pi}(x) < map(C) = map'(C)$.

Assume now that $C \in W_{tp}(x)$. If $C$ is not satisfied by $A(u)$ then $map'(C) = none$ by construction and we are done since $C \in W_{tp'}(x)$. Finally, assume that $C$ is satisfied by $A(u)$. Let $y \in \text{Var}(A(u))$ be a variable whose occurrence in $A(u)$ satisfies $C$. Then $MT_{\pi+t}(y) = T_{\pi}$. Indeed, otherwise, $y \in \text{Var}(\pi)$ and
hence, by construction, \( y \) cannot belong to \( \text{Var}_{\text{out}}(tp) \), a contradiction. As \( x \in \text{Var}(\pi) \). \( MT_{\pi+1}(x) < \text{map}'(x) \) meaning that \( x \in W_{tp'}(x) \).

Conversely, let \( x \in \text{Var}_{fix}(tp') \). By construction, \( x \in \text{Var}(\pi) \setminus \text{Var}(t) \). Let \( C \in W_{tp}(x) \). If \( C \in W_0^{tp}(x) \) meaning that \( \text{map}'(C) = \text{none} \) then, clearly \( C \in W_0^{tp}(x) \) and hence \( x \in \text{Var}_{fix}(tp) \). If \( C \in W_1^{tp}(x) \) then we consider two subcases. The first is if \( \text{map}(C) \neq \text{none} \). In this case \( MT_{\pi+1}(x) = MT_{\pi}(x) < \text{map}(C) = \text{map}'(C) \), the first statement follows from Proposition 9. Hence \( C \in W_{tp}(x) \) and hence \( x \in V_{\text{fix}}(tp) \). In the second case, \( \text{map}(C) = \text{none} \) and hence \( C \in W_0^{tp}(x) \) and again \( x \in \text{Var}_{fix}(tp) \).

\[ \text{Lemma 11} \] The following two statements hold.

\[ \bullet \ \text{Var}_{\text{out}}(tp) = \text{Var}_{\text{out}}(tp') \cup \text{Var}(A(u)) \]

\[ \bullet \ \text{FA}(tp') \cup RA(tp') = FA(tp) \cup RA(tp) \cup A(u) \]

**Proof.** By the first statement of Lemma 11, \( \text{Var}_{fix}(tp') = \text{Var}_{fix}(tp) \setminus \text{Var}(t) \). On the other hand, by construction, \( \text{Var}_{in}(tp') \) is the disjoint union of \( \text{Var}_{in}(tp) \setminus \text{Var}(t), \text{Var}_{fix}(tp) \cap \text{Var}(t), \text{Var}_{in}(tp) \cap \text{Var}(t) \) and \( \text{Var}(A(u)) \). By re-grouping the items, we obtain \( \text{Var}_{fix}(tp') \cup \text{Var}_{in}(tp') = \text{Var}_{fix}(tp) \cup \text{Var}_{in}(tp) \cup \text{Var}(A(u)) \). The first statement now follows by taking the complement of both sides.

It follows from Lemma 11 that \( \text{FA}(tp') = \text{Proj}(\text{FA}(tp), \text{Var}_{fix}(tp) \setminus \text{Var}(t)) \). Next, by construction, \( RA(tp') \) is the disjoint union of \( RA(tp) \), \( \text{Proj}(\text{FA}(tp), \text{Var}(t)) \) and \( A(u) \). Clearly, their union is the right-hand part of the second statement.

\[ \text{Theorem 7} \] Let \( tp = (\pi, \text{map}, CN, RA) \) be a well-formed non-basic non-final unsatisfiable type and let \( t \) be the immediate successor of \( \pi \). Assume that \( t \) is an expanding node. Let \( u \) be a sink of \( L_{tp} \) such that \( A(u) \) satisfies clauses of \( \text{LONG} \setminus \text{LS} \). Then the following statements hold.

1. If \( u \) is a labelled with \( C \rightarrow () \) then \( C \in \text{dom}(F_{tp}) \), \( C \) is not satisfied by \( A(u) \) and \( F(C)|_{A(u)} = () \).

2. If \( u \) is identified with the source of \( L_{tp'} \) then \( tp' \) is a well-formed non-basic unsatisfiable type and \( F_{tp'} = F_{tp}|_{A(u)} \).

**Proof.** The first statement is immediate by construction, so we prove the second statement. Assume the premise of the statement regarding \( u \). By construction, \( tp' = (\pi(tp) + t, \text{map}, CN^*, RA) \) is as defined in the paragraph preceding Lemma 11.

By the first statement of Lemma 11, each \( x \in \text{Var}_{fix}(tp') \) is a variable of \( \text{Var}_{fix}(tp) \) and hence the consistency of \( x \) follows from the second statement of Lemma 11 and the consistency of \( x \) for \( tp \). We conclude that \( tp' \) is well-formed.

Let \( C \in \text{dom}(F_{tp}|_{A(u)}) \setminus \text{dom}(F_{tp'}) \) then \( F_{tp}|_{A(u)}(C) = \text{Proj}(C, \text{Var}_{out})(A(u)) = \text{Proj}(C, \text{Var}_{out}(tp') \setminus A(u)) = \text{Proj}(C, \text{Var}_{out}(tp')) = F_{tp'}(C) \), the penultimate
inequality follows from Lemma \[10\]. It follows that it is sufficient to establish that 
\[ \text{dom}(F_{A(u)}) = \text{dom}(F_{tp'}). \]

Let \( C \in \text{dom}(F_{tp'}|_{A(u)}) \). If \( C \in \text{LS}(tp) \) then, taking into account that \( C \) is not satisfied by \( A(u), C \in \text{LS}(tp') \) by construction (and hence, by definition, belongs to \( \text{dom}(F_{tp'}) \)).

Assume that \( C \in CN \). Then as \( C \) is not satisfied by \( A(u), C \in CN' \) (and, again, belongs to \( \text{dom}(F_{tp'}) \)).

Finally, let \( C \notin \text{LS}(tp) \cup CN \). If \( C \in CL(t) \) then \( C \in CN' \) since it is not satisfied by \( A(u) \). Otherwise, by definition of \( F_{tp'} \), \( C \) is not satisfied by \( FA(tp') \cup RA(tp) \). Hence, by the second statement of Lemma \[11\], \( C \) is not satisfied by \( FA(tp') \cup RA(tp') \) and hence belongs to \( \text{dom}(F_{tp'}) \) in the role of a clause outside of \( \text{LS}(tp') \cup CN' \).

Conversely, assume that \( C \in \text{dom}(F_{tp'}) \). If \( C \in \text{LS}(tp') \) then, by definition \( C \in \text{LS}(tp) \) and not satisfied by \( A(u) \) and hence belongs to \( \text{dom}(F_{tp'}|_{A(u)}) \). Assume that \( C \in CN' \). If \( C \in CN \) then, taking into account that \( C \) is not satisfied by \( A(u) \), we conclude that \( C \in \text{dom}(F_{tp'}|_{A(u)}) \). Otherwise the only way for \( C \) to get into \( CN' \) is if \( C \in \text{dom}(F_{tp}) \) and not satisfied by \( A(u) \).

Finally, if \( C \notin \text{LS}(tp) \cup CN' \) then, by definition of \( tp' \), \( C \) is not satisfied by \( FA(tp') \cup RA(tp') = FA(tp) \cup RA(tp) \cup A(u) \) by Lemma \[10\]. By definition of \( tp, C \notin \text{LS}(tp) \cup CN \). Therefore \( C \in \text{dom}(F_{tp'|_{A(u)}) \). Finally, we note that \( tp' \) being non-basic and unsatisfiable follows from the same argument as in the proof of Theorem \[6\].

\[ ]

C.3 Contracting \( t, A(u) \) not potentially complex

Lemma 12 Let \( tp = (\pi, map, CN, RA) \) be a well-formed non-basic non-final unsatisfiable type and let \( t \) be the immediate successor of \( \pi \). Assume that \( t \) is a contracting node. Let \( u \) be a sink of \( D_{tp} \) such that \( A(u) \) is not potentially complex. Let \( tp' = (\pi(tp) + t, map, CN', RA) \) where \( CN' \) is obtained from \( CN \) by removal of clauses that are satisfied by \( A(u) \). Then the following statements hold.

1. For each \( x \in \text{Var}_{fix}(tp) \cap \text{Var}_{fix}(tp') \), \( W_{tp}(x) = W_{tp'}(x) \).
2. \( \text{Var}_{fix}(tp') \) is the disjoint union of \( \text{Var}_{fix}(tp) \) and \( \text{Det}(tp) \).
3. For each \( x \in \text{Det}(tp), W_{tp'}(x) = W_{tp}^0(x) \).
4. \( \text{Var}(A(u)) = \text{Det}(tp) \).
5. \( FA(tp') = FA(tp) \cup A(u) \).
6. \( \text{Var}_{out}(tp) \) is the disjoint union of \( \text{Var}_{out}(tp') \) and \( \text{Det}(tp) \).

Proof. Unlike in the case of \( t \) being an expanding note, in the considered case the trees of \( Trees'_u \) disappear and we need to verify that \( tp' \) is valid in the sense that none of the trees in the range of \( map \) belongs to \( Trees'_u \). However, if it was so then \( A(u) \) would be potentially complex. So, we conclude that
Claim 6. $\text{MT}(\pi) \subseteq \text{Trees}_\pi \cap \text{Trees}_{\pi+t}$.

Claim 7. Let $x \in \text{Var}_{fijz}(\text{tp})$. Then $x \in \text{Var}_{fijz}(\text{tp}')$ and $W_{\text{tp}}(x) \subseteq W_{\text{tp}'}(x)$.

Proof. First of all, it follows directly from definition of $\text{tp}'$ that $\text{Var}_{in}(\text{tp}) = \text{Var}_{in}(\text{tp}')$. Then, $x \in \text{Var}(\pi + t) \setminus \text{Var}_{in}(\text{tp}').$

Let $C \in W_{\text{tp}}(x)$. Assume first that $C \in W_{\text{tp}'}^0(x)$. By definition of $\text{tp}'$, $[\text{map}(\text{tp}')](C) = \text{none}$. It follows that $x \in \text{Var}_{fijz}(\text{tp}')$ and that $C \in W_{\text{tp}'}(x)$.

Assume now that $C \in W_{\text{tp}}^1(x)$. As $MT_{\pi}(x) < [\text{map}(\text{tp}')](C) = [\text{map}(\text{tp})](C) \in \text{Trees}_\pi \setminus \text{Trees}_t$, by Claim 6 $MT_{\pi}(x) \in \text{Trees}_\pi \setminus \text{Trees}_t$ by Proposition 8.

By Proposition 9 $MT_{\pi+t}(x) = MT_{\pi}(x)$. As $[\text{map}(\text{tp}')]([\text{map}(\text{tp})])(C)$, $MT_{\pi+t}(x) < [\text{map}(\text{tp}')](C)$ by another application of Proposition 8, thus implying that $x \in W_{\text{tp}'}^1(x)$. □

Claim 8. Let $x \in \text{Var}_{fijz}(\text{tp}') \cap \text{Var}(\pi)$. Then $x \in \text{Var}_{fijz}(\text{tp})$ and $W_{\text{tp}'}(x) \subseteq W_{\text{tp}}(x)$ (note that the second part implies the first one).

Proof. Let $C \in W_{\text{tp}'}(x)$. Suppose that $C \in W_{\text{tp}}^0(x)$. This means that $[\text{map}(\text{tp})](C) = \text{none}$ and, in particular, that $C \in W_{\text{tp}'}^0(x)$.

Assume that $C \in W_{\text{tp}}^1(x)$. This means that $[\text{map}(\text{tp}')](C) \neq \text{none}$. By Claim 6 $[\text{map}(\text{tp}')](C) \in \text{Trees}_\pi \cap \text{Trees}_{\pi+t}$. By Proposition 8 $MT_{\pi+t}(x) \in \text{Trees}_\pi \cap \text{Trees}_{\pi+t}$. By Proposition 9 $MT_{\pi+t}(x) = MT_{\pi}(x)$ and one more application of Proposition 8 implies that $MT_{\pi}(x) < [\text{map}(\text{tp}')](C)$. We conclude that $C \in W_{\text{tp}'}^1(x)$. □

Taking into account that $\text{Var}_{fijz}(\text{tp}) \subseteq \text{Var}(\pi)$ by definition, Claim 7 in fact shows that $\text{Var}_{fijz}(\text{tp}) \subseteq \text{Var}_{fijz}(\text{tp}') \cap \text{Var}(\pi)$. Hence, combination of Claim 7 and 8 imply that these two sets are equal and the witnessing sets of each variable are the same w.r.t $\text{tp}$ and $\text{tp}'$. This implies the first statement of the lemma.

Let now $x \in \text{Var}_{fijz}(\text{tp}') \setminus \text{Var}(\pi)$. Then $MT_{\pi+t}(x) = T_t$, the largest tree of $\text{Trees}_{\pi+t}$. Hence $W_{\text{tp}'}(x) = \emptyset$, implying that $W_{\text{tp}}(x) = W_{\text{tp}'}^0(x)$ and, in particular (since $x \in \text{Var}_{fijz}(\text{tp}')$ must be witnessed by at least one clause), that $W_{\text{tp}'}^0(x) \neq \emptyset$.

Let $C \in W_{\text{tp}'}^0(x)$. Then $C \in \text{LONG} \setminus \text{LS}(\text{tp}') = \text{LONG} \setminus \text{LS}(\text{tp})$. As all the variables outside of $\text{Var}(\pi)$ belong to $\text{Var}_{out}(\text{tp})$, $x \in \text{Var}(t) \cap \text{Var}_{out}(\text{tp})$. Hence $x \in \text{Det}(\text{tp})$.

Conversely, let $x \in \text{Det}(\text{tp})$. This means that there is $C \in \text{LONG} \setminus \text{LS}(\text{tp}) = \text{LONG} \setminus \text{LS}(\text{tp}')$, such that $x \in \text{Var}(C)$. This already implies that $x \in \text{Var}_{fijz}(\text{tp}')$. It remains to show that $x \notin \text{Var}(\pi)$. But if it were so then $x$ would belong to $\text{Var}_{fijz}(\text{tp})$ causing a contradiction since $x \in \text{Var}_{out}(\text{tp})$ by definition of $\text{Det}(\text{tp})$. Thus the second and the third statements are settled.

For the fourth statement, note that, by construction, $\text{Det}(\text{tp}) \subseteq \text{Var}(A(u))$. If $A(u)$ assigned any extra variables, it would have satisfied a clause of $\text{LONG} \setminus \text{LS}(\text{tp})$ which is not the case.

For the fifth statement, remains to show that for each $x \in \text{Var}_{fijz}(\text{tp})$, the occurrences of $x$ in $FA(\text{tp})$ and $FA(\text{tp}')$ are the same and that for each
\( x \in \text{Det}(\text{tp}) \), the occurrences of \( x \) in \( A(u) \) and in \( FA(\text{tp}') \) are the same. In the former case, the desired statement follows from \( W_{\text{tp}}(x) = W_{\text{tp}'}(x) \). In the latter case, the assignment of \( x \) in \( A(u) \) is precisely the one to not satisfy any witnessing clause of \( x \), hence precisely the assignment of \( x \) in \( FA(\text{tp}') \).

For the last statement, note that \( \text{Var}_{\text{out}}(\text{tp}') = \text{Var}(\varphi) \setminus (\text{Var}_{\text{in}}(\text{tp}') \cup \text{Var}_{\text{fix}}(\text{tp}')) = \text{Var}(\varphi) \setminus (\text{Var}_{\text{in}}(\text{tp}) \cup \text{Var}_{\text{fix}}(\text{tp})) \setminus \text{Det}(\text{tp}) = \text{Var}_{\text{out}}(\text{tp}) \setminus \text{Det}(\text{tp}) \). □

**Theorem 8** Let \( \text{tp} = (\pi, \text{map}, CN, RA) \) be a well-formed non-basic non-final unsatisfiable type and let \( t \) be the immediate successor of \( \pi \). Assume that \( t \) is a contracting node. Let \( u \) be a sink of \( D_{\text{tp}} \) such that \( A(u) \) is not potentially complex. Then the following statements hold.

1. If \( u \) is a labelled with \( C \rightarrow () \) then \( C \in \text{dom}(F_{\text{tp}}) \), \( C \) is not satisfied by \( A(u) \) and \( F(C)|_{A(u)} = () \).
2. If \( u \) is identified with the source of \( L_{\text{tp}}' \) then \( \text{tp}' \) is a well-formed non-basic unsatisfiable type and \( F_{\text{tp}}' = F_{\text{tp}}|_{A(u)} \).

**Proof.** Analogous to Theorem 6 with Lemma 12 used instead of Lemma 9. □

### C.4 Contracting \( t \), \( A(u) \) potentially complex

In this subsection we assume that \( \text{tp} = (\pi, \text{map}, CN, RA) \) is a well-formed non-basic non-final unsatisfiable type, that \( t \), the immediate successor of \( \pi \) is a contracting node, and that \( A(u) \) is potentially complex.

Let \( CN_1 \) be the subset of \( CN \) that are not satisfied by \( A(u) \). Let \( \text{Trees}^* = (\text{MT}(\text{tp}) \setminus \text{Trees}_z') \cup \{T_1\} \).

**Lemma 13** \( CN_1 \subseteq \text{CL}(\text{Roots}(\text{Trees}^*)) \).

**Proof.** Let \( C \in CN_1 \). If there is \( T' \in \text{MT}(\text{tp}) \setminus \text{Trees}_z' \) such that \( C \in \text{CL}(\text{Root}(T')) \) then we are done. Otherwise, \( C \in \text{CL}(\text{Roots}(\text{Trees}_z')) \). We show that in this case, \( C \in \text{CL}(t) \). First note that, by assumption, \( C \in \text{CL}(t_0) \) for some child \( t_0 \) of \( t \) such that \( T_{t_0} \subseteq \text{MT}(\text{tp}) \). Next, as \( \text{tp} \) is non-basic, \( F_{\text{tp}}(C) \neq () \). This means that there is \( x \in \text{Var}_{\text{out}}(\text{tp}) \) such that \( x \in \text{Var}(C) \).

That is, by the edge containment property of tree decompositions, there is a node \( t_i \) of \( T \) such that \( x \in \text{Var}(t_1) \) and \( C \in \text{CL}(t_1) \). Now \( \text{Var}(T_{t_0}) \subseteq \text{Var}_{\text{in}}(\text{tp}) \) and hence \( t_i \notin V(T_{t_0}) \). By the connectivity property, \( C \) is contained in the bag of the parent of \( t_0 \) which is \( t \). □

Let \( CN_2 \) be the subset of \( \text{dom}(F_{\text{tp}}) \setminus (\text{LONG} \cup \text{CL}(\text{Roots}(\text{MT}))) \) consisting of all clauses \( C \) such that \( C \) is not satisfied by \( A(u) \) and \( C \in \text{CL}(t) \) and \( \text{Var}(F_{\text{tp}}(C)) \cap \text{Var}_{\text{free}}(\text{tp}) = \emptyset \).

Let \( \text{map}' \) be a function from \( \text{LONG} \) to \( \text{Trees}_{\pi+t} \) obtained from \( \text{map} \) as follows.

- For each \( C \in \text{LONG} \) such that \( \text{map}(C) = \text{none} \) and \( C \) is satisfied by \( A(u) \), \( \text{map}'(C) = T_{t} \).
• For each $C \in \text{LONG}$ such that $map(C) \in \text{Trees}_\pi'$, $map'(C) = T_i$.

Let $RA' = \text{Proj}(RA, \text{Var}(\text{Roots}(\text{Trees}^*) )) \cup \text{Proj}(FA, \text{Var}(t)) \cup A(u)$. Let $tp' = (\pi + t, map', CN_1 \cup CN_2, RA')$. It follows from definitions of $map'$ and a potentially complex assignment that $\text{dom}(map') = \text{Trees}^*$. Moreover, Lemma 13 implies that all the elements of $CN_1 \cup CN_2$ occur in the bags of the roots of $MT(tp')$ and hence the type is valid in this sense. Note that in the previous cases this validity was obvious by construction.

**Lemma 14** \( Var_{fix}(tp') = Var_{fix}(tp) \setminus Var(T_i) \) and for each \( x \in Var_{fix}(tp') \), \( W_{tp}(x) = W_{tp'}(x) \).

**Proof.**

Let \( x \in Var_{fix}(tp) \setminus Var(T_i) \). We are going to show that for \( W_{tp}(x) \subseteq W_{tp'}(x) \). Clearly, this will imply that \( x \in Var_{fix}(tp') \).

It follows from Proposition 9 that $MT_\pi(x) \in \text{Trees}_\pi \cap \text{Trees}_{\pi+t}$ and hence, by Proposition 9, $MT_\pi(x) = MT_{\pi+t}(x)$.

Let \( C \in W_{tp}(x) \). Assume first that \( C \in W_{tp}^0(x) \). If $map'(C) = \text{none}$ then $C \in W_{tp}^1(x)$ and we are done. Otherwise, $map'(C) = T_t$. In light of the previous paragraph plus another application of Proposition 9, $MT_{\pi+t}(x) < map'(C)$, hence $C \in W_{tp'}^1(x)$.

Assume now that \( C \in W_{tp}^1(x) \). If $map(C) \notin \text{Trees}_\pi'$ then $map'(C) = map(C)$ through another application of Proposition 9. $MT_\pi(x) < map(C)$ implies that $MT_{\pi+t}(x) < map'(C)$ and hence $C \in W_{tp'}^1(x)$. If $map(C) \in \text{Trees}_\pi'$ then $map'(C) = T_t$. Hence, by the second paragraph and another application of Proposition 9, $MT_{\pi+t}(x) < map'(C)$. Again, it follows that $C \in W_{tp'}^1(x)$.

Conversely, assume that $x \in Var_{fix}(tp')$. As $Var(T_i) \subseteq Var_{in}(tp)$, $x \notin Var(T_i)$. It thus remains to show that $W_{tp}(x) \subseteq W_{tp'}(x)$ thus implying that $x \in Var_{fix}(tp)$.

Let \( C \in W_{tp}(x) \). If \( C \in W_{tp}^0(x) \) then $map'(C) = \text{none}$ and hence $map(C) = \text{none}$ thus implying that $C \in W_{tp}^0(x)$. Assume that $C \in W_{tp}^1(x)$. If $map(C) = \text{none}$ then $C \in W_{tp}^0(x)$. Otherwise, note that $MT_{\pi+t}(x) < map(C)$ implies by Proposition 9 that $MT_{\pi+t} < T_t$ and hence $MT_\pi(x) = MT_{\pi+t}(x)$ by Proposition 9. If $map'(C) = map(C)$ then $MT_\pi(x) < map(C)$ by Proposition 9 and we are done. Otherwise, $map'(C) \notin \text{Trees}_\pi'$ which implies from $MT_\pi(x) = MT_{\pi+t}$ that $MT_\pi(x) \notin Trees'_x$. Hence, by Proposition 9, $MT_\pi(x) < map(C)$ and hence $C \in W_{tp'}^1(x)$ and we are done.

**Lemma 15** \( Var_{out}(tp) = Var_{out}(tp') \cup Var_{free}(tp) \cup Var(A(u)) \).

**Proof.** By construction, $Var_{in}(tp') = Var_{in}(tp) \cup Var(T_i)$. Combining with Lemma 14 we observe that $Var_{fix}(tp') \cup Var_{in}(tp') = Var_{in}(tp) \cup Var_{fix}(tp) \cup Var(T_i)$. Consequently, $Var_{out}(tp') = Var_{out}(tp) \setminus Var(T_i)$. This is the same as to say that $Var_{out}(tp') = Var_{out}(tp) \setminus (Var_{out}(tp) \cap Var(T_i))$. However, by construction, $Var_{out}(tp) \cap Var(T_i) = Var(A(u)) \cup Var_{free}(tp)$.
Lemma 16 Then for each \( C \in \text{dom}(F_{t^p}) \), \( \text{Var}(F_{t^p}(C)) \cap \text{Var}_{\text{free}}(t^p) = \emptyset \).

Proof. Let \( C \in \text{dom}(F_{t^p}) \). Assume fist that \( C \in \text{LONG}. \) Then \( \text{map}'(C) = \text{none} \) and hence \( \text{map}(C) = \text{none} \). Let \( x \in \text{Var}_{\text{free}}(t^p) \). Then (as \( x \notin \text{Var}(t) \)), \( x \in \text{Var}(\pi) \). If \( x \in \text{Var}(C) \) then \( x \in \text{Var}_{\text{fix}}(t^p) \) by definition in contradiction to the definition of \( \text{Var}_{\text{free}}(t^p) \) being a subset of \( \text{Var}_{\text{out}}(t^p) \).

Assume next that \( C \in CN(t^p') \) which is, by definition \( CN_1 \cup CN_2 \). If \( C \in CN_2 \) then \( F_{t^p}(C) \) does not have joint variables with \( \text{Var}_{\text{free}}(t^p) \) simply by definition. If \( C \in CN_1 \) then we need to use the definition of one-sided decomposition. In particular, let \( x \in \text{Var}_{\text{free}}(t^p) \). This means that there is a child \( t_0 \) of \( t \) such that \( x \in \text{Var}(T_{t_0}) \). Since \( x \notin \text{Var}(t) \) by definition, \( x \) is not present in any bag outside of \( T_{t_0} \), \( x \in \text{Var}(C) \) implies \( C \) is present in a bag of \( T_{t_0} \) by the containment property. To this end note that \( T_{t_0} \in Trees_n \setminus MT(t^p) \) simply because \( \text{Var}(T_{t_0}) \) contains a variable of \( Var_{\text{out}}(t^p) \). On the other hand, \( C \in CL(t_1) \) where \( t_1 \) is a root of a tree of \( MT(t^p) \). Now \( V(T_{t_0}) \) does not contain ancestors nor descendants of \( t_1 \) hence \( C \notin CL(T_{t_0}) \) by definition of one-sided tree decomposition.

Finally, let \( C \in \text{dom}(F_{t^p}) \setminus (\text{LONG} \cup CL(\text{Roots}(MT(t^p)))) \). Reusing the previous paragraph, we note that if \( x \in \text{Var}(C) \) then \( C \in CL(T_{t_0}) \). On the other hand, by definition of \( F_{t^p} \), there is \( y \in Var_{\text{out}}(t^p) \) such that \( y \in \text{Var}(C) \). So, by the containment condition \( C \) must be present in a bag containing \( y \). On the other hand, \( \text{Var}(T_{t_0}) \subseteq Var_{\text{in}}(t^p) \) so \( y \) is not present in a bags of \( T_{t_0} \). It follows that \( C \) must be contained in a bag outside of \( T_{t_0} \). By the connectivity condition, this means that \( C \) must be present in the bag of the parent of \( t_0 \) which is \( t \), in contradiction to the definition of \( C \).

Lemma 17 \( F_{t^p} \) is a subfunction of \( F_{t^p}|_{A(u)} \).

Proof. First of all, let \( C \in \text{dom}(F_{t^p}) \cap \text{dom}(F_{t^p}|_{A(u)}) \). By combination of Lemmas 15 and 16 \( \text{Proj}(C, Var_{\text{out}}(t^p)) = \text{Proj}(C, Var_{\text{out}}(t^p') \cup \text{Var}(A(u))) \). This means that \( F_{t^p}|_{A(u)}(C) = \text{Proj}(C, Var_{\text{out}}(t^p')) = F_{t^p'}(C) \). It thus remains to show that \( \text{dom}(F_{t^p}) \subseteq \text{dom}(F_{t^p}|_{A(u)}) \).

So, let \( C \in \text{dom}(F_{t^p}) \). If \( C \in \text{LONG} \) then \( map'(C) = \text{none} \). By definition of \( map' \) this means \( map(C) = \text{none} \) (and hence \( C \in \text{dom}(F_{t^p}) \)) and \( C \) is not satisfied by \( A(u) \). If \( C \in CN_1 \cup CN_2 \) then it is immediate from the definition of these sets that \( C \in \text{dom}(F_{t^p}) \) and that is not satisfied by \( A(u) \).

It thus remains to assume that \( C \in \text{dom}(F_{t^p}) \setminus (\text{LONG} \cup CL(\text{Roots}(MT(t^p)))) \).

One necessary condition for that is that \( C \) is not satisfied by \( FA(t^p') \cup RA(t^p) \).

It follows from Lemma 14 and the definition of \( RA' \) that \( FA(t^p') = \text{Proj}(FA, Var_{\text{fix}}(t^p) \setminus \text{Var}(T_1)) \) and \( RA(t^p') = \text{Proj}(RA(t^p), Var(\text{Roots}(Trees^*))) \cup \text{Proj}(FA(t^p), \text{Var}(t)) \cup A(u) \). From this we already conclude that \( C \) is not satisfied by \( A(u) \), so we only need to show that \( C \in \text{dom}(F_{t^p}) \).

The other necessary condition for \( C \in \text{dom}(F_{t^p}) \) is that there is \( y \in Var_{\text{out}}(t^p) \) such that \( y \in \text{Var}(C) \). Since \( Var_{\text{out}}(t^p) \subseteq Var_{\text{out}}(t^p) \) by Lemma 14 we know now that \( \text{Var}(C) \cap Var_{\text{out}}(t^p) \neq \emptyset \). It thus remains to verify that \( C \) is not satisfied by \( FA(t^p) \cup RA(t^p) \). More precisely, in light of the previous
paragraph we need to prove that $C$ is not satisfied by $(FA(tp) \cup RA(tp)) \setminus (FA(tp') \cup RA(tp'))$. Taking into account the reasoning in the previous paragraph, this amounts to showing that $C$ is not satisfied by $Proj(FA(tp) \cup RA(tp), Var(T_t) \setminus Var(t))$. Assume the opposite. This means that there is $x \in Var(T_t) \setminus Var(t)$ such that $x \in Var(C)$. As $x \notin Var(t)$, by the connectivity condition, $x$ cannot appear in any bag outside of $T_t$. This means that $C$ itself must appear in a bag inside $T_t$ by the containment condition of tree decompositions. On the other hand, as $Var(T_t) \subseteq Var_{in}(tp')$, $y$ cannot appear in any bag inside $T_t$. This means that by the containment condition, $C$ must appear in a bag outside $T_t$. By the connectivity condition this means that $C \in CL(t)$ contradicting the definition of $C$. \hfill $\blacksquare$

**Lemma 18** Assume that $F^*$ (the filling function) is satisfiable. Then $F_{tp'}$ is not satisfiable.

**Proof.** Assume the opposite and let $S_1$ be a satisfying assignment of $range(F_{tp'})$. Also, let $S_2$ be a satisfying assignment of $range(F^*)$. Clearly, we may assume that $Var(S_1) \subseteq Var(range(F_{tp'}))$ and $Var(S_2) \subseteq Var(range(F^*))$. According to Lemma 16, $Var(range(F_{tp'})) \cap Var(range(F^*)) = \emptyset$. Therefore, $S = S_1 \cup S_2$ is a well formed set of literals.

By definition of $F^*$ for each $C \in dom(F^*)$, $Var(C) \cap Var_{free}(tp) \neq \emptyset$. Therefore, by Lemma 16 $dom(F_{tp'}) \cup dom(F^*) = \emptyset$. Hence, we can consider the function $F' = F_{tp'} \cup F^*$. The set $S$ as in the previous paragraph satisfies $range(F')$. We will establish a contradiction by showing that $F'$ is in fact unsatisfiable.

In order to do this, we need the following claim.

**Claim 9** $dom(F_{tp'|A(u)}) = dom(F')$.

Assume that the claim holds. Then for each $C \in dom(F')$, $F'(C) \subseteq F_{tp'|A(u)}(C)$. Indeed, for $C \in dom(F_{tp'})$ this follows from Lemma 17. For $C \in dom(F^*)$, $F^*(C) = Proj(C, Var_{free}(tp)) \subseteq Proj(C, Var_{out}(tp) \setminus Var(A(u))) = F_{tp'|A(u)}(C)$, the containment relation follows from Theorem 15. It thus follows that $S$ is a satisfying assignment for $range(F_{tp'|A(u)})$. However, this is a contradiction: $F_{tp}|A(u)$ simply because $F_{tp}$ is unsatisfiable by assumption. It thus remains to prove the claim.

In light of Lemma 17 it is enough to show that each $C \in dom(F_{tp}|A(u))$ such that $Var(C) \cap Var_{free}(tp) = \emptyset$ is contained in $dom(F_{tp'})$. Assume first that $C \in LONG$. Then from map$(C) = none$ and $C$ not being satisfied by $A(u)$, it follows that map$(C) = none$ and hence $C \in dom(F_{tp'})$. If $C \in CN$ then, since $C$ is not satisfied by $A(u)$, $C \in CN_1$.

It remains to assume that $C \in dom(F_{tp}) \setminus (LONG \cup CN)$. If $C \in Var(t)$ then $C \in CN_2$. Otherwise, applying the same argument as in the first paragraph of Lemma 17, we observe that $Var(F_{tp}|A(u)) \subseteq Var_{out}(tp')$. As we assumed that $A(u)$ does not falsify $F_{tp}(C)$, $Var(F_{tp}|A(u)) \cap Var_{out}(tp') \neq \emptyset$. In this remains to show that $C$ is not satisfied by $FA(tp') \cup RA(tp')$. We note that $FA(tp') \cup RA(tp') \subseteq FA(tp) \cup RA(tp) \cup A(u)$ (see the third paragraph of the
proof of Lemma [17] for a detailed reasoning). Now \( C \) is clearly not satisfied by \( A(u) \) and since, \( C \in \text{dom}(F_{tp}) \setminus (\text{LONG} \cup CN) \), \( C \) is not satisfied by \( FA(tp) \cup RA(tp) \) simply by definition of \( F_{tp} \). 

Now we are considering the case where \( F^* \) is not satisfiable. The reasoning is similar to the case where \( F^* \) is satisfiable. So, rather than reproving everything, we will refer to relevant proofs or their fragments wherever appropriate.

Let \( TR^* \) be the subset of \( \text{dom}(F^*) \) consisting of all the clauses \( C \) such that \( Var(F_{tp}(C)) \setminus Var(T_i) \neq \emptyset \). Let \( R^* \) be a transitional resolution for \( F^* \) with \( TR^* \) being the set of transitional clauses.

**Lemma 19** \( TR^* \subseteq CL(t) \).

**Proof.** Let \( C \in TR^* \). By definition of \( F^* \), there is a variable \( x \in Var(C) \) such that \( x \in Var(T_i) \) but does not occur in a bag outside \( T_i \). This means that \( C \in CL(T_i) \) by the edge connectivity property. On the other hand, by definition of \( TR^* \), there is a variable \( y \in Var(C) \setminus Var(T_i) \). Because of the containment condition, \( C \) must occur in a bag outside \( T_i \). We conclude that, by the connectivity condition, \( C \in CL(t) \). 

**Lemma 20** Let \( u \) be a non-transitional sink of \( R^* \) labelled with \( C \rightarrow () \). Let \( A \) be an assignment carried by a source-\( u \) path of \( R^* \). Then \( F_{tp}(C)|_{A(u) \cup A} = () \).

**Proof.** In other words, we need to show that \( Var(F_{tp}(C)) \subseteq Var(A(u)) \cup Var(A) \). By definition of \( F_{tp} \) and a non-transitional clause of \( F^* \), this is the same as to show that

\[
Var(C) \cup Var_{out}(tp) \cap Var(T_i) \subseteq Var(A(u)) \cup Var(A).
\]

Now, \( Var_{out}(tp) \cap Var(T_i) = Var(A(u)) \cup Var_{free}(tp) \). That is, \( Var(C) \cap Var_{out}(tp) \cap Var(T_i) = (Var(C) \cap Var(A(u))) \cup (Var(C) \cap Var_{free}(tp)) \). As \( A \) falsifies \( F^*(C) = Var(C) \cap Var_{free}(tp) \), we are done.

Let \( C \subseteq TR^* \) and \( A \) be a \( C \)-falsifier for \( F^* \) with \( Var(A) \subseteq Var_{free}(tp) \). Let \( tp'' = (\pi + t, map', CN_1 \cup CN_2 \cup C, RA') \). It follows from the combination of Lemmas [18] and [19] that \( CN_1 \cup CN_2 \cup C \) all occur in bags of \( Roots(MT(tp'')) \), hence \( tp'' \) is valid in this sense.

The following theorem is proved analogously to Lemma [14].

**Lemma 21** \( Var_{fix}(tp'') = Var_{fix}(tp) \setminus Var(T_i) \) and for each \( x \in Var_{fix}(tp') \), \( W_{tp}(x) = W_{tp'}(x) \).

The following lemma is proved analogously to Lemma [15] with Lemma [21] used instead of Lemma [14].

**Lemma 22** \( Var_{out}(tp) = Var_{out}(tp'') \cup Var_{free}(tp) \cup Var(A(u)) \)

**Lemma 23** Then for each \( C \in \text{dom}(F_{tp'}) \setminus C \), \( Var(C) \cap Var_{free}(tp) = \emptyset \).

**Proof.** Let \( C \in \text{dom}(F_{tp'}) \). The subsequent reasoning is analogous to that of Lemma [16] with Lemma [21] and Lemma [22] used instead of Lemma [14] and Lemma [15] respectively.
Lemma 24 $F_{tp'}$ is a subfunction of $F_{tp | A(u) \cup A}$.

Proof. Let $F_1$ be the restriction of $F_{tp'}$ to $dom(F_{tp'}) \setminus C$ and $F_2$ be the restriction of $F_{tp'}$ to $C$.

It follows from Lemma 16 that $F_1$ being a subfunction of $F_{tp | A(u) \cup A}$ is equivalent to $F_1$ being a subfunction of $F_{tp | A(u)}$. This can be proved analogously to Lemma 17 with $F_1$ used instead of $F_{tp'}$ and Lemmas 21, 22, and 23 being used instead of Lemmas 14, 15, and 16 respectively.

So, it remains to show that $F_2$ is a subfunction of $F_{tp | A(u) \cup A}$. By definition of $C, C \in dom(F_{tp})$ and not satisfied by $A(u) \cup A$. So, it remains to show that $F_2(C) = F_{tp | A(u) \cup A}(C)$. This is the same as to show that $\text{Var}(F_2(C)) = \text{Var}(F_{tp | A(u) \cup A}(C))$. Now, $\text{Var}(F_2(C)) = \text{Var}(C) \cap \text{Var}_\text{free}(tp''')$, which is by Lemma 21 (Var(C) $\cap$ Var_out(tp)) $\setminus$ ((Var(C) $\cap$ Var(A(u))) $\cup$ (Var(C) $\cap$ Var_free(tp))). On the other hand, the only difference in the representation of $\text{Var}(F_{tp | A(u) \cup A}(C))$ is that $\text{Var}(C) \cap \text{Var}(A)$ is used instead of $\text{Var}(C) \cap \text{Var}_\text{free}(tp)$. The former is a superset of the latter since $A$ falsifies $F^*(C)$ by assumption. On the other hand $\text{Var}(A) \subseteq \text{Var}_\text{free}(tp)$, therefore, $\text{Var}(C) \cap \text{Var}(A) = \text{Var}(C) \cap \text{Var}(A) \cap \text{Var}_\text{free}(tp)$ meaning that the containment in the opposite direction holds as well.

Lemma 25 Then $F_{tp'}$ is not satisfiable.

Proof. Assume the opposite. Let $S_1$ be a satisfying assignment for range($F_{tp'}$). Also, let $F_0$ be the restriction of $F^*$ to $dom(F^*) \setminus C$ and let $S_2$ be a satisfying assignment for range($F_0$) existing by definition of $A$. Clearly, we assume that $\text{Var}(S_1) \subseteq \text{Var}_\text{out}(tp''')$ and that $\text{Var}(S_2) \subseteq \text{Var}_\text{free}(tp')$. That is, $\text{Var}(S_1) \cap \text{Var}(S_2) = \emptyset$ and hence $S = S_1 \cup S_2$ is a well formed set of literals.

It follows from Lemma 23 that $\text{dom}(F_{tp'}) \cap \text{dom}(F_0) = \emptyset$, therefore, we can consider a function $F^* = F_{tp'} \cup F_0$. Clearly, $S$ satisfies range($F^*$). We are going to demonstrate that $\text{dom}(F^*) = \text{dom}(F_{tp | A(u)})$ and for each $C \in dom(F^*)$, $F^*(C) \subseteq F_{tp | A(u)}$. This will imply that $S$ satisfies range($F_{tp | A(u)}$), a contradiction to the assumption that $F_{tp | A(u)}$.

In light of Lemma 24 it is enough to show the following.

- $\text{dom}(F_0) \subseteq \text{dom}(F_{tp | A(u)})$. Just follows from the definition of $F^*$.

- For each $C \in \text{dom}(F_0)$, $F_0(C) \subseteq F_{tp | A(u)}(C)$. In fact this is proved for the whole $F^*$ in the proof of Lemma 18.

- $\text{dom}(F_{tp'}) \cup \text{dom}(F_0) = \text{dom}(F_{tp | A(u)})$. This is the claim in the proof of Lemma 18 (with Lemma 24 used instead of Lemma 17).

Theorem 9 Let $tp = (\pi, \text{map}, CN, RA)$ be a well-formed non-basic non-final unconvertible type and let $t$ be the immediate successor of $\pi$. Assume that $t$ is a contracting node. Let $u$ be a sink of $D_{tp}$ such that $A(u)$ is potentially complex. Let $F^*$ be the filling function.
If $F^*$ is satisfiable then, $u$ is identified with the source of $L_{tp}$, where $tp'$ is non-basic well-formed unsatisfiable type. Moreover, $F_{tp'}$ is a subfunction of $F_{tp}|_{A(u)}$.

If $F^*$ is unsatisfiable then, by construction, $u$ is the source of a transitional resolution of $F^*$ with $TR^*$ being the set of transitional clauses. Let $v$ be a sink of $R^*$. Let $P$ be an arbitrary path from $u$ to $v$. Suppose that $v$ is associated with $C \rightarrow (\emptyset)$ for some non-transitional clause $C$. Then $F_{tp}(C)|_{A(u) \cap A(P)} = (\emptyset)$. Otherwise, $v$ is a transitional link associated with $trans_C$. In this case $v$ is identified with the source of $L_{tp'}$, where $tp''$ is a well-formed non-basic unsatisfiable type. Moreover, $F_{tp''}$ is a subfunction of $F_{tp}|_{A(u) \cup A(P)}$.

Proof. If $F^*$ is satisfiable then the identification of $u$ with the source of $L_{tp}$ follows by construction. It follows from Lemma 14 that $tp'$ is well-formed. Further on, it follows from Lemma 17 that $F_{tp'}$ is a subfunction of $F_{tp}|_{A(u)}$. As $A(u)$ does not falsify any clause of $range(F_{tp})$, we conclude that $tp'$ is a non-basic type. Finally, it follows from Lemma 18 that $tp'$ is a non-satisfiable type.

Assume now that $F^*$ is not satisfiable. If $v$ is a non-transitional sink then the related statement follows from Lemma 20. If $v$ is transitional sink then it is identified with the source of $L_{tp'}$ by construction. It follows from Lemma 21 that $tp''$ is well formed. Next, it follows from Lemma 22 that $F_{tp''}$ is a subfunction of $F_{tp}|_{A(u) \cup A}$. The argument that $tp''$ is non-basic is now a little bit more complicated. As $A(u)$ is potentially complex, $A(u)$ does not falsify any clause of $range(F_{tp})$ but there is also extra effect by $A$. For each $C \in dom(F_{tp''}) \setminus C$ this does not matter because by Lemma 23 $Var(C) \cap Var(A) = \emptyset$ (recall that $Var(A) \subseteq Var_{free}(tp)$ by assumption). On the other hand, for each $C \in C$, $Var(C) \cap Var_{out}(tp'') \neq \emptyset$ and hence $F_{tp}(C)$ is not falsified by $A(u) \cup A$ either. Finally, it follows from Lemma 25 that $tp''$ is unsatisfiable.

C.5 Combining things together

Theorem 10 Let $ORD^*$ be a subsequence of $ORD$ consisting of all the non-basic well-formed unsatisfiable types. Let $R = \bigcup_{tp \in ORD^*} L_{tp}$. For each $tp \in ORD^*$ let $u_{tp}$ be the source of $L_{tp}$. Then for each $tp \in ORD^*$ $R_{u_{tp}}$ is a falsifying $FRR$.

Proof. Let $ORD^* = (tp_1, \ldots, tp_m)$. First, we observe that $F_{tp_1}, \ldots, F_{tp_m}$ is a falsifying sequence of functions. In order to do this, we need to demonstrate that each $L_{tp_i}$ satisfies the conditions of Definition 6. If $tp_i$ is final, this follows from the definition. Otherwise, this follows from Theorems 6, 7, 8, 9, 53, 54, 55, 56, 57, 58, 59, and 60. Let $L_{tp_1}, \ldots, L_{tp_m} \subseteq F_{tp_1}, \ldots, F_{tp_m}$. Let us prove that the sequence is read-once.

Let us say that a type $tp'$ is a descendant of type $tp$ (both types belong to $ORD^*$) if $tp'$ is a descendant of $tp$ as per the definition before Theorem 10. Let $Var^*(tp)$ be the union of $Var(L_{tp})$ and all the sets $Var(L_{tp'})$ such that $tp'$ is a descendant of $tp$.

Claim 10 $Var^*(tp) \subseteq Var_{out}(tp)$.
Proof. By induction on $\text{ORD}^*$. For $\mathbf{tp}_1$, $\text{Var}^*(\mathbf{tp}_1) = \text{Var}(L_{\mathbf{tp}_1})$ and $\text{Var}(L_{\mathbf{tp}_1}) \subseteq \text{Var}_{\text{out}}(\mathbf{tp}_1)$ simply by construction.

Now consider $\mathbf{tp}_i$ for $i > 1$. Then $\text{Var}^*(\mathbf{tp}_i)$ is the union of $\text{Var}(L_{\mathbf{tp}_i})$, which is a subset of $\text{Var}_{\text{out}}(\mathbf{tp}_i)$ by construction and the union of $\text{Var}^*(\mathbf{tp}_j)$ for children $\mathbf{tp}_j$ of $\mathbf{tp}_i$. By the induction assumption, $\text{Var}^*(\mathbf{tp}_j) \subseteq \text{Var}_{\text{out}}(\mathbf{tp}_j)$. It follows that $\text{Var}_{\text{out}}(\mathbf{tp}_j) \subseteq \text{Var}_{\text{out}}(\mathbf{tp}_i)$ from the combination of the following statements:

- the last statement of Lemma 9;
- the first statement of Lemma 11;
- the last statement of Lemma 12;
- Lemma 15 and Lemma 22.

Now, let $\mathbf{tp}'$ be a descendant of $\mathbf{tp}$. This means that there is a child $\mathbf{tp}''$ of $\mathbf{tp}$ such that either $\mathbf{tp}' = \mathbf{tp}''$ or $\mathbf{tp}'$ is a descendant of $\mathbf{tp}''$. In any case, $\text{Var}(L_{\mathbf{tp}'}) \subseteq \text{Var}^*(\mathbf{tp}'')$ and hence, by the claim $\text{Var}(L_{\mathbf{tp}'}) \subseteq \text{Var}_{\text{out}}(\mathbf{tp}'')$. The combination of statements provided in the end of the proof of the claim implies that $\text{Var}_{\text{out}}(\mathbf{tp}'') \subseteq \text{Var}_{\text{out}}(\mathbf{tp}) \setminus \text{Var}(L_{\mathbf{tp}})$. It follows that $\text{Var}(L_{\mathbf{tp}}) \cap \text{Var}(L_{\mathbf{tp}'}) = \emptyset$ as required. □

Proof of Theorem 2. Assume that $\varphi$ does not contain empty clauses for otherwise, the statement is trivial. Let $\mathbf{st} = (\emptyset, nm, \emptyset, \emptyset)$ be a type where $\emptyset$ at the first position denotes the empty prefix of $\pi_T$ and $nm$ maps every clause to none. We call $\mathbf{st}$ the starting type. We claim that $F_{\mathbf{st}} = 1_{\varphi}$. Indeed, as $\pi(\mathbf{st})$ is empty, $\text{Var}_{\text{fix}}(\mathbf{st}) = \text{Var}_{\text{in}}(\mathbf{st}) = \emptyset$. That is, $\text{Var}_{\text{out}}(\mathbf{tp}) = \text{Var}(\varphi)$. Thus for each $C \in \text{dom}(f_{\mathbf{st}})$, $f_{\mathbf{st}}(C) = C$. It remains to show that $\text{dom}(F_{\mathbf{st}}) = \varphi$. As $nm$ maps all the long clauses to none, they all belong to $\text{dom}(F_{\mathbf{st}})$. As for a clause $C \in \varphi \setminus \text{LONG}$, direct inspection of the last condition of the type function definition shows that $C \in \text{dom}(F_{\mathbf{st}})$. It follows from Theorem 10 that $R_{\text{out}}$ is an FRR for $\varphi$.

Now, we need to upper-bound the size of $R_{\text{out}}$. Let us denote $|\text{Var}(\varphi)| + |\varphi|$ by $n$. Clearly, $R_{\text{out}}$ is the union of $L_{\text{out}}$ and the local graphs of all the descendants of $\mathbf{st}$ in the sense defined in the proof of Theorem 10. We are going to upper bound the size of each local graph and the number of local subgraphs in the union.

Consider first the local graph for a final type $\mathbf{tp}$, by definition, $L_{\mathbf{tp}}$ is a falsifying FRR for $F_{\mathbf{tp}}$, so it might look like a recursion. However, the situation is much simpler. In particular, observe that $\text{dom}(F_{\mathbf{tp}}) \cap \text{LONG} = \emptyset$. Indeed, suppose a long clause $C$ belongs to $\text{dom}(F_{\mathbf{tp}})$. This means that $|\text{map}(\mathbf{tp})|(C) = \text{none}$. Consequently, by definition, $\text{Var}(\pi(\mathbf{tp})) \cap \text{Var}(C) \subseteq \text{Var}_{\text{in}}(\mathbf{tp}) \cup \text{Var}_{\text{fix}}(\mathbf{tp})$. As $\mathbf{tp}$ is final, it follows that $\text{Var}(C) \cap \text{Var}_{\text{out}}(\mathbf{tp}) = \emptyset$ in contradiction to $\mathbf{tp}$ being non-basic. As $F_{\mathbf{tp}}$ is in fact $\varphi \setminus \text{LONG}$, Corollary 1 is applicable. As there are no transitional clauses, the upper bound for $L_{\mathbf{tp}}$ is at most $n \cdot 4^k$. 54
Each local graph of a non-final type consists of a dome of size at most $2^k$ plus at most $2^k$ (at most one per leaf transitional resolutions). By Lemma 19 the number of transitional clauses for such a resolution is at most $k$. Therefore, by Corollary 1 the size of each resolution is $n \cdot 2^{O(k^2)}$. We conclude that the size of each local subgraph is $n \cdot 2^{O(k^2)}$.

In order to calculate the number of local graphs in the union we calculate the number of respective types. A type consists of four components. The number of possible first components is the number of bags of $T$ plus one. As the number of bags is $O(n)$, this is an upper bound on the number of the first components as well. For the number of second components, recall that by Proposition 4 (taking into account that the number of bags is $O(n)$), $|Trees_\pi|$ is $O(\log n)$ for each prefix $\pi$ of $\pi_T$. Therefore, with the first component being fixed, each clause of $\text{LONG}$ can be mapped to $O(\log n)$ different values. Thus the number of possible maps is $O(\log n^{|\text{LONG}|})$ which is well known to be upper-bounded by $O(n + |\text{LONG}|^{|\text{LONG}|})$.

For the last two components, observe that for each type $tp$, $\text{MT}(tp)$ partitions a subset of $\text{LONG}$. Therefore $|\text{MT}(tp)| \leq |\text{LONG}|$. By definition, $CN(tp)$ is a subset of bags of the roots of $\text{MT}(tp)$ and $RA(tp)$ assigns the variables in the bags of the roots of $\text{MT}(tp)$. As the $\text{MT}$ set of trees is completely determined by the first two components, with these components being fixed, the number of possible third components as well as the number of possible fourth components is $2^{O(k \cdot |\text{LONG}|)}$ (recall that the size of each bag is at most $k$). We conclude that the total size of $R_{\text{START}}$ is $(n + |\text{LONG}|^{|\text{LONG}|}) \cdot n \cdot 2^{O(k^2 + k \cdot |\text{LONG}|)}$. ■