Abstract

Consider a sequence of data points $X_1, \ldots, X_n$ whose underlying mean $\theta^* \in \mathbb{R}^n$ is piecewise constant of at most $k^*$ pieces. This paper establishes sharp nonasymptotic risk bounds for the least squares estimator (LSE) on estimating $\theta^*$. The main results are twofold. First, when there is no additional shape constraint assumed, we reveal a new phase transition for the risk of LSE: As $k^*$ increases from 2 to higher, the rate changes from $\log \log n$ to $k^* \log(en/k^*)$. Secondly, when $\theta^*$ is further assumed to be nondecreasing, we show the rate is improved to be $k^* \log \log(16n/k^*)$ over $2 \leq k^* \leq n$. These bounds are sharp in the sense that they match the minimax lower bounds of the studied problems (without sacrificing any logarithmic factor). They complement their counterpart in the change-point detection literature and fill some notable gaps in recent discoveries relating isotonic regression to piecewise constant models. The techniques developed in the proofs, which are built on Levy’s partial sum and Doob’s martingale theory, are of independent interest and may have potential applications to the study of some other shape-constrained regression problems.

Keywords: Piecewise constant models; least squares estimator; reduced isotonic regression; risk bounds; law of the iterated logarithm; Levy’s maximal inequality; Doob’s maximal inequality.

1 Introduction

Consider an observed sequence $X = (X_1, \ldots, X_n)^T$ of independent entries and an unknown underlying mean $\theta^* = (\theta^*_1, \ldots, \theta^*_n)^T \in \mathbb{R}^n$. This paper studies, for $\theta^*$, the following two models of piecewise constant structure:

1. The model $\Theta_k$ that consists of all piecewise constant sequences of at most $k$ pieces;
2. The model $\Theta_k^\uparrow$ that consists of all sequences in $\Theta_k$ that are also nondecreasing.
Piecewise constant models are benchmark models in change-point analysis [Csörgő and Horváth, 1997] and shape-constrained regression [Groeneboom and Jongbloed, 2014] literature. They have proven their usefulness in various applications. Specifically, the model $\Theta_k$ forms the foundation in the change-point detection literature, and a variety of theory and methods have been built surrounding it. The model $\Theta_k^\uparrow$ adds monotonicity, a condition met in many real problems [Silvapulle and Sen, 2011] and motivating research in shape-constrained regression.

Assume $\theta^*$ belongs to either of the following two classes, $\theta^* \in \Theta_{k^*}$ or $\theta^* \in \Theta_{k^*}^\uparrow$, for some unknown $k^* \in [n]$ (where $[n]$ denotes the set of all integers from 1 to $n$). Our focus is on estimating $\theta^*$, based on the sole observation $X$, using the possibly misspecified least squares estimator (LSE):

$$\hat{\theta}(\Theta) = \arg\min_{\theta \in \Theta} \|X - \theta\|^2. \quad (1)$$

Here $\|\cdot\|$ represents the Euclidean norm, and we have either $\Theta = \Theta_k$ or $\Theta = \Theta_k^\uparrow$ for some $k \in [n]$, which is not necessarily equal to $k^*$. If the program (1) has multiple global optima, $\hat{\theta}(\Theta)$ is defined to be an arbitrary one. We evaluate the estimation error using the squared $L_2$ risk $E\|\hat{\theta}(\Theta) - \theta^*\|^2$, which we will refer to as the risk of $\hat{\theta}(\Theta)$.

In this paper, we establish sharp risk bounds for the LSE in the two models under consideration. That is, they match the corresponding minimax lower bounds when $k = k^*$ up to a constant factor not depending on $\theta^*$ or $k^*$.

### 1.1 Piecewise constant model: upper and minimax lower bounds

We formally state $\Theta_k$ as follows:

$$\Theta_k = \{\theta \in \mathbb{R}^n : \text{there exist } \{a_j\}_{j=0}^k \text{ and } \{\mu_j\}_{j=1}^k \text{ such that } 0 = a_0 \leq a_1 \leq ... \leq a_k = n, \text{ and } \theta_i = \mu_j \text{ for all } i \in (a_{j-1} : a_j)\}, \quad (2)$$

where for any two integers $a < b$, $(a : b]$ denotes all integers from $a + 1$ to $b$.

For presenting the result, we first introduce some necessary notation. For any real number $x \in \mathbb{R}$, define $\lfloor x \rfloor$ to be the largest integer no greater than $x$. Define the space of $\ell$-sparse vectors to be

$$S_\ell := \{\theta \in \mathbb{R}^n : \sum_{i=1}^n \mathbb{I}(\theta_i \neq 0) \leq \ell\} \quad \text{for } \ell \in [n],$$

where $\mathbb{I}(\cdot)$ stands for the indicator function. It is worthwhile to mention, if $k$ is larger than 2, $\Theta_k$ contains $S_{\lfloor (k-1)/2 \rfloor}$, the space of $\lfloor (k-1)/2 \rfloor$-sparse vectors. In addition, same as $S_\ell$, $\Theta_k$ is a non-convex set.

Suppose $\theta^* \in \Theta_{k^*}$ and denote

$$\hat{\theta}(\Theta_k) = \arg\min_{\theta \in \Theta_k} \|X - \theta\|^2$$

to be the possibly misspecified LSE for some $k \in [n]$. We are now ready to state our main result regarding the piecewise constant model. In Theorem 2.1, under the assumption that
\(X \sim N(\theta^*, \sigma^2 I_n)\) with \(I_n\) denoting the \(n\) by \(n\) identity matrix, we develop uniform upper bounds of the risk in terms of \((n, k^*, k, \sigma)\):

\[
\sup_{\theta^* \in \Theta_{k^*}} \mathbb{E}\|\hat{\theta}(\Theta_k) - \theta^*\|^2 \leq \begin{cases} 
C\sigma^2, & k = k^* = 1, \\
C\sigma^2 \log \log n, & k = k^* = 2, \\
C\sigma^2 k \log(en/k), & k \geq k^*.
\end{cases}
\]

Here \(C > 0\) represents a universal constant. Our risk bounds, when \(k = k^*\), further prove to be sharp up to a constant in the minimax sense:

\[
\inf_{\hat{\theta}} \sup_{\theta^* \in \Theta_k} \mathbb{E}\|\hat{\theta} - \theta^*\|^2 \\
\asymp \sigma^2 \left\{1 + \log \log n \cdot 1(k = 2) + k \log(en/k) \cdot 1(k > 2)\right\},
\]

where the infimum is taken over all measurable functions of \(X\).

There has been a huge literature on deciphering the piecewise constant model \(\Theta_k\). A good survey of recent advances is given in Fryzlewicz [2014]. Here we mention three of them that are most relevant.

(i) Csörgő and Horváth [1997] studied the case \(\theta^* \in \Theta_2\) (there referred to as the “at most one change-point (AMOC)” model). Let

\[
\delta = \max_{i \in [n-1]} |\theta_{i+1}^* - \theta_i^*|
\]

represent the “signal gap” in the sequence \((\theta_1^*, \ldots, \theta_n^*)^T\). They showed that, for consistent recovery of the change-point, when \(\sigma^2\) is a constant, it is sufficient to require

\[
\lim_{n \to \infty} \delta^2 \min \left\{a_1^*, n - a_1^*\right\} \log \log n = \infty,
\]

where \(a_1^*\) represents the true change-point location. Various results also suggest that a signal weaker than this order could be too weak to be detectable (see, for example, comments on Page 1563 of Hao et al. [2013]).

(ii) Yao [1993] and Arias-Castro et al. [2005] studied a multiple change-points setting. Specially, they were focused on the case that \(\theta^* \in \Theta_3\) with an additional constraint that \(\mu_1^* = \mu_3^*\) (there referred to as the “epidemic alternative”). Even for this simple setting, when \(\sigma^2\) is a constant, Arias-Castro et al. [2005] showed that no method can reliably detect the change-points if

\[
\delta^2 (a_2^* - a_1^*) < c \log n,
\]

for some sufficiently small constant \(c > 0\). Recall that the numbers \(\mu_1^*, \mu_3^*, a_1^*,\) and \(a_2^*\) are defined in (2). The above inequality gives a benchmark for consistent detection of signals, and draws a “\(\log n\) v.s. \(\log \log n\)” comparison viewing (6) and (5).

(iii) When \(3 \leq k \leq n^1-\epsilon\) with an arbitrarily small \(\epsilon \in (0, 1)\), the minimax rate is \(\sigma^2 k \log n\). Boysen et al. [2009] showed this rate by solving the following penalized least squares optimization

\[
\min_{\theta \in \mathbb{R}^n} \left\{\|X - \theta\|^2 + \gamma n \sum_{j=2}^n 1(\theta_j \neq \theta_{j-1})\right\},
\]
with an appropriate choice of $\gamma_n$. See also Friedrich et al. [2008] for a computationally efficient algorithm and Li et al. [2016] for related risk bounds.

Equations (3) and (4) together reveal a new phase transition for the piecewise constant model when the number of pieces $k = 1, 2$, or larger than 2. It is shown that the corresponding risk bound (scaled by $\sigma^2$) increases from 1, $\log \log n$, to $k \log(en/k)$. Though the focus is on estimation instead of model selection/testing, it is clear that the jump from $\log \log n$ to $\log n$-type rate of convergence in (3) is very similar to the findings in (5) and (6). Later in Remark 3.1, we will illustrate the connections in more details. Specifically, in Proposition 3.1 we will show, by checking the derivation of our minimax lower bound, we could also rigorously determine the “region of detectability” in the AMOC model, and hence close another gap.

Equations (3) and (4) are new even though there has been a huge literature in studying the piecewise constant (step-function or stairwise-function) model. The inequality that is closest to ours was derived in Donoho and Johnstone [1994], which asserts that

$$\inf \sup_{\hat{\theta}, \theta^* \in S_k} \|\hat{\theta} - \theta^*\|^2 \asymp k \log(en/k).$$

Observing that $S_{(k-1)/2} \subset \Theta_k$, our results are naturally connected to theirs for the case $k > 2$. On the other hand, to the best of our knowledge, when $k = 2$, the $\log \log n$-type risk bound is for the first time unveiled in the literature.

1.2 Isotonic piecewise constant model: upper and minimax lower bounds

In the second part of the paper, we study the case that $\theta^*$ is both piecewise constant and nondecreasing. The later condition is also called “isotonic” in the literature. Estimation of $\theta^*$ under this condition has been one of the most popular and successful directions in the shape-constrained regression literature. General discussions on relevant methods and theory are in Robertson et al. [1988], Groeneboom and Wellner [1992], Silvapulle and Sen [2011], and Groeneboom and Jongbloed [2014], to name just a few.

To present our new bounds on the isotonic problem, some additional notation is necessary. We first formally state the isotonic piecewise constant model as follows:

$$\Theta_k^\uparrow = \{ \theta \in \mathbb{R}^n : \text{there exist } \{a_j\}_{j=0}^k \text{ and } \{\mu_j\}_{j=1}^k \text{ such that}$$

$$0 = a_0 \leq a_1 \leq \ldots \leq a_k = n,$$

$$\mu_1 \leq \mu_2 \leq \ldots \leq \mu_k, \text{ and } \theta_i = \mu_j \text{ for all } i \in (a_{j-1} : a_j) \}. \}

Define the possibly misspecified LSE to be:

$$\hat{\theta}(\Theta_k^\uparrow) = \arg\min_{\theta \in \Theta_k^\uparrow} \|X - \theta\|^2.$$

For $\theta^* \in \Theta_k^\uparrow$, there exist $0 = a_0 \leq a_1 \leq \ldots \leq a_{k^*} = n$ such that $\theta^*$ takes constant values on the intervals $\{ (a_{j-1} : a_j) \}_{j=1}^{k^*}$. If there are multiple choices for such $\{a_j\}_{j \in [k^*]}$, pick any one of them. We use

$$n_j := a_j - a_{j-1}$$

to denote the block size for the $j$th interval for each $j \in [k^*]$. For any number $a \in \mathbb{R}$, let $\lceil a \rceil$
denote the smallest integer that is no smaller than $a$.

Next, we define as follows two parameters that play important roles in our analysis:

$$m = \sum_{j=1}^{k^*} \lceil 1 + \log_2 n_j \rceil \quad \text{and} \quad \bar{k} = \min\{k, m\}.$$  \hfill (7)

Of note, $m$ is a parameter measuring the “effective size” of the model. In fact, by examining Example 2.2 of Chatterjee et al. [2015], one can easily check that $m$ is the “statistical dimension” [Meyer and Woodroofe, 2000; Amelunxen et al., 2014] of the isotonic piecewise constant model. On the other hand, later we will see $\bar{k}$ is a parameter measuring the impact of the tuning parameter $k$ on the estimation accuracy.

Now we state the main result. Assume $X \sim N(\theta^*, \sigma^2 I_n)$ with $\theta^* \in \Theta_{k^*}^\uparrow$. In Theorem 2.2, our main risk bound shows

$$\sup_{\theta^* \in \Theta_{k^*}^\uparrow} E\|\hat{\theta}(\Theta_k^\uparrow) - \theta^*\|^2 \leq \begin{cases} C\sigma^2, & k = k^* = 1, \\ C\sigma^2(k^* + \bar{k} \log(em/\bar{k})), & k \geq k^*, \end{cases}$$ \hfill (8)

for some universal constant $C > 0$.

The above inequality gives rise to many corollaries, and helps answer a number of questions raised in the literature. Three noteworthy implications are in order.

(i) We first compare the risk bound derived in this paper to some of the most related ones in the literature. Consider the set of all nondecreasing sequences:

$$\mathcal{M} := \Theta_{k^*}^\uparrow = \left\{ \theta \in \mathbb{R}^n : \theta_1 \leq \theta_2 \leq \cdots \leq \theta_n \right\},$$

and the LSE:

$$\hat{\theta}(\mathcal{M}) = \arg\min_{\theta \in \mathcal{M}} \|X - \theta\|^2.$$  

Define $V(\theta) = \theta_n - \theta_1$ to be the total variation of any $\theta \in \mathcal{M}$. Previous results [Meyer and Woodroofe, 2000; Zhang, 2002] have shown that, for some universal constant $C > 0$,

$$\sup_{\{\theta^* \in \mathcal{M} : V(\theta^*) \leq V^*\}} E\|\hat{\theta}(\mathcal{M}) - \theta^*\|^2 \leq C\sigma^2\left\{ \log(en) + n^{1/3}(V^*/\sigma)^{2/3} \right\}. $$ \hfill (9)

The above risk bound is later re-examined with a different set of structures imposed. In particular, Chatterjee et al. [2015] disclosed the relation between $\hat{\theta}(\mathcal{M})$ and $\Theta_{k^*}^\uparrow$, proving

$$\sup_{\theta^* \in \Theta_{k^*}^\uparrow} E\|\hat{\theta}(\mathcal{M}) - \theta^*\|^2 \leq C\sigma^2 \cdot k^* \log(en/k^*), $$ \hfill (10)

in Corollary 2.2, we show that (10) is an implication of (8) by setting $k = n$. Accordingly, our risk bound could be considered as an extension to that in Chatterjee et al. [2015]. Moreover, our inequality (8) gives a series of risk bounds for $\hat{\theta}(\Theta_k^\uparrow)$ for $k \in [k^* : n]$. In particular, for $k = k^*$, Corollary 2.1 will show that our risk bound renders

$$\sup_{\theta^* \in \Theta_{k^*}^\uparrow} E\|\hat{\theta}(\Theta_k^\uparrow) - \theta^*\|^2 \leq C\sigma^2 \cdot k^* \log \log(16n/k^*), $$ \hfill (11)

and thus reveals another “log log $n$ v.s. log $n$” rate of convergence phenomenon.

(ii) We then reveal the minimax rate of $\Theta_{k}^\uparrow$ in view of (11). Global and local minimax
lower bounds for the isotonic piecewise constant model have been provided in the literature. For example, Bellec and Tsybakov [2015] showed
\[
\inf_{\hat{\theta}} \sup_{\theta^* \in \Theta^+_k} \mathbb{E} \|\hat{\theta} - \theta^*\|_2^2 \geq c \sigma^2 \cdot k
\]
for some universal constant \(c > 0\). In a related problem, Chatterjee et al. [2015] proved a local minimax lower bound; see Theorem 5.4 therein for a detailed description.

However, the existing minimax lower bounds fail to match the corresponding upper bounds, and thus leave gaps for improvement. In Theorem 3.1, we fill these gaps by providing a minimax lower bound that exactly matches the upper bound in (11) without sacrificing any logarithmic factor. Specifically, we establish, universally over \(k \in [n]\),
\[
\inf_{\hat{\theta}} \sup_{\theta^* \in \Theta^+_k} \mathbb{E} \|\hat{\theta} - \theta^*\|_2 \asymp \sigma^2 \left\{1 + k \log \log (16n/k) \cdot 1(k \geq 2)\right\}. \tag{12}
\]
The above equation then provides an accurate picture of the studied problem.

(iii) Lastly, we reveal the necessity of the Gaussian assumption on the bound (8). There has been a vast amount of work discussing the robustness of shape-constrained regression estimators to heavy-tailed distributions. For the isotonic regression estimator \(\hat{\theta}(M)\), Donoho [1990] first derived a bound similar to (9) for the Gaussian distributed data. Birgé and Massart [1993] and Wang and Chen [1996] improved his result and proved the cube-root convergence for the isotonic regression without requiring normality. Later, Zhang [2002] and Chatterjee et al. [2015] derived (9) and (10), and also insightfully pointed out that Gaussian or other ‘light-tailed” assumptions are not crucial there.

In Section 4, we obtain the same bound (8) for both heavy-tailed and non-independent-and-identically-distributed (non-i.i.d.) error sequences provided that the \((2 + \epsilon)\)th moment exists for all margins of the error. This should also be compared with the bound (3), of which Proposition 4.1 will show it is impossible to relax the (sub-)Gaussian assumption in a minimax sense. Thus it is clear that the extra isotonic constraint helps estimation. See Remark 4.1 for detailed discussions.

In the end, we give a brief description of the proof techniques we used and developed in this paper. In contrast to the common metric entropy argument [Groeneboom and Jongbloed, 2014], our risk bounds are derived using Levy’s partial sum and Doob’s martingale theory. For this, our analysis heavily relies on Lemma 2.1, a new “optimization-type” inequality for the LSE \(\hat{\theta}(\Theta^+_k)\). We then repeatedly exploit a sequence splitting technique in handling the partial sums, which intrinsically comes from the proof of the law of the iterated logarithm (see Section 7.5 in Chung [2000] for example).

1.3 Notation

Let \(Z\) and \(R\) be the sets of integers and real numbers. For any \(a, b \in R\), write \(a \land b = \min\{a, b\}\) and \(a \lor b = \max\{a, b\}\). For an arbitrary vector \(\theta = (\theta_1, \ldots, \theta_n)^T \in R^n\) and an index set...
$J \subset [n]$, we denote $\theta_J$ to be the sub-vector of $\theta$ with entries indexed by $J$,
\[ \|\theta\| = \left( \sum_{i=1}^{n} \theta_i^2 \right)^{1/2}, \quad \text{and} \quad \|\theta\|_J = \left( \sum_{i \in J} \theta_i^2 \right)^{1/2}. \]

Let $\bar{\theta}_{(a:b)} = \frac{1}{b-a} \sum_{i=a+1}^{b} \theta_i$ represent the sample mean across the sequence $\theta_{(a:b)}$. For any real value $a$ and positive integer $n$, define
\[ \{a\}^n = (a, a, \ldots, a)^T. \]

For any sets of vectors $\Theta_1 \subset \mathbb{R}^{n_1}, \ldots, \Theta_m \subset \mathbb{R}^{n_m}$, denote
\[ \bigotimes_{\ell=1}^{m} \Theta_\ell = \left\{ \theta = (\theta_{(1)}^T, \ldots, \theta_{(m)}^T)^T \in \mathbb{R}^{\sum_{i=1}^{n_i} n_i} : \theta_{(\ell)} \in \Theta_\ell \right\}. \]

Throughout the paper, let $c, C, c_1, C_1, c_2, C_2, \ldots$ be generic universal constants whose actual values vary at different places. For any two positive data sequences $\{a_n, n = 1, 2, \ldots\}$ and $\{b_n, n = 1, 2, \ldots\}$, we write $a_n \asymp b_n$ if there exist constants $c, C > 0$ such that $ca_n \leq b_n \leq Ca_n$ for all $n$ from natural numbers.

### 1.4 Paper organization

The rest of the paper is organized as follows. In Section 2 we present our main risk bounds for the possibly misspecified LSEs $\hat{\theta}(\Theta_k)$ and $\hat{\theta}(\Theta_k^I)$, tailored for the piecewise constant and isotonic piecewise constant models. Section 3 gives the corresponding minimax lower bounds and proves the sharpness of the derived risk bounds. In Section 4 we extend the risk bounds in Section 2 to heavy-tailed and non-i.i.d. error sequence. Adaptive estimation of the mean is provided in Section 5. Section 6 contains all proofs.

### 2 Risk bounds

This section presents the inequalities (3) and (8) under the Gaussian assumption. The possible relaxation of the Gaussian assumption will be discussed in Section 4.

We first rigorously state the inequality (3).

**Theorem 2.1.** Consider $X \sim N(\theta^*, \sigma^2 I_n)$ with $\theta^* \in \Theta_{k^*}$, and let $\hat{\theta}(\Theta_k)$ be defined through (1) with $\Theta = \Theta_k$ for some $k \geq k^*$. Then, there exists some universal constant $C > 0$, such that
\[
\mathbb{E} \|\hat{\theta}(\Theta_k) - \theta^*\|^2 \leq \begin{cases} 
C\sigma^2, & k = k^* = 1, \\
C\sigma^2 \log \log n, & k = k^* = 2, \\
C\sigma^2 k \log(en/k), & k \geq k^*.
\end{cases}
\]

Discussions of Theorem 2.1 are in order. First of all, consider the non-misspecified case $k = k^*$. Theorem 2.1 shows that the error bound for the class $\Theta_k$ scales differently for $k$ in different regimes. While the rate has a unified expression $\sigma^2 k \log(en/k)$ for $k \geq 3$, the logarithmic dependence on $n$ can be improved for $k = 1$ and $k = 2$. 

7
Secondly, the $\sigma^2 \log \log n$ rate we obtain for the space $\Theta_2$ is interesting. The intuition behind the $\sigma^2 \log \log n$ error is that there is at most one change-point in the model $\Theta_2$, rendering that either the “starting” or “ending” point is fixed for the two constant-valued sequences of $\hat{\theta}(\Theta_2)$. The iterated logarithmic error behavior is then driven by the bound

$$E \left( \max_{1 \leq \ell \leq n} \frac{1}{\sqrt{\ell}} \sum_{i=1}^{\ell} |Z_i|^2 \right) \leq C \sigma^2 \log \log n,$$

(13)

where $Z = X - \theta^* \sim N(0, \sigma^2 I_n)$ is the error vector. Of note, it is known that (13) is a direct consequence of the celebrated Levy’s maximal inequality for the partial sums.

Lastly, as has been pointed out in Introduction, the phase transition phenomenon revealed in Theorem 2.1 is consistent to the common belief in the change-point analysis literature [Csörgő and Horváth, 1997; Arias-Castro et al., 2005]. That is, detection of one change-point is much easier than detection of many. However, to the best of our knowledge, the formulation (3) and its derivation are new.

We then proceed to study the possibly misspecified LSE $\hat{\theta}(\Theta_{k^*}^\uparrow)$ over the isotonic piecewise constant model. The following theorem formally states the bound (8). Recall the numbers $\widetilde{k}$ and $m$ defined in (7).

**Theorem 2.2.** Consider $X \sim N(\theta^*, \sigma^2 I_n)$ with $\theta^* \in \Theta_{k^*}^\uparrow$, and let $\hat{\theta}(\Theta_{k}^\uparrow)$ be defined through (1) with $\Theta = \Theta_{k}^\uparrow$ for some $k \geq k^*$. Then, there exists some universal constant $C > 0$, such that

$$E \|\hat{\theta}(\Theta_{k}^\uparrow) - \theta^*\|^2 \leq \begin{cases} C\sigma^2, & k = k^* = 1, \\ C\sigma^2 \left\{ k^* + \widetilde{k} \log(e m/\widetilde{k}) \right\}, & k \geq k^*, \end{cases}$$

where the parameters $m$ and $\widetilde{k}$ are defined in (7).

Theorem 2.2 shows that the error bound for the space $\Theta_{k}^\uparrow$ can be improved over that of $\Theta_k$ because of the extra monotonic constraint. To better understand the rate of Theorem 2.2, we give a further upper bound as a corollary.

**Corollary 2.1.** Under the setting of Theorem 2.2, we further let $k = k^*$. Then, there exists some universal $C > 0$, such that

$$E \|\hat{\theta}(\Theta_{k^*}^\uparrow) - \theta^*\|^2 \leq \begin{cases} C\sigma^2, & k^* = 1, \\ C\sigma^2 k^* \log \log(16n/k^*), & k^* \geq 2. \end{cases}$$

Compared with the rate of Theorem 2.1, the extra monotonic constraint improves the logarithmic dependence on $n$ for the space $\Theta_k$ with $k \geq 3$ to an iterated logarithmic dependence on $n$ for $\Theta_{k^*}^\uparrow$. With this extra monotonic constraint, we are able to apply a similar bound as (13) in the error analysis within each of the $k^*$ blocks. As has been discussed in Introduction, this improves previous error bounds for the space $\Theta_{k^*}^\uparrow$ in the literature [Chatterjee et al., 2015; Bellec, 2015]. An interesting problem to be investigated in the future is whether error bounds of other shape-constrained classes [Han and Wellner, 2016; Kim et al., 2016] can be improved.
When \( k = n \), \( \hat{\theta}(\Theta^*_n) \) is the classic isotonic regression estimator. In this case, Theorem 2.2 is specialized to the following error bound. Interestingly, this is a special case of an error bound for isotonic regression in Chatterjee et al. [2015].

**Corollary 2.2.** Under the setting of Theorem 2.2, we further let \( k = n \). Then, there exist some universal constants \( C, C_1 > 0 \), such that

\[
E\|\hat{\theta}(\Theta^*_n) - \theta^*\|^2 \leq C\sigma^2 \sum_{j=1}^{k^*} [1 + \log n_j] \leq C_1 \sigma^2 k^* \log(en/k^*).
\]

The rate of \( \hat{\theta}(\Theta^*_n) \) is not optimal compared with that given by \( \hat{\theta}(\Theta^{\uparrow}_k) \) in Corollary 2.1. Moreover, this suboptimal rate is intrinsic to \( \hat{\theta}(\Theta^*_n) \), as is shown by the next proposition.

**Proposition 2.1.** There exists a universal constant \( C > 0 \), such that

\[
\sup_{\theta^* \in \Theta^*_k} E\|\hat{\theta}(\Theta^*_n) - \theta^*\|^2 \geq C\sigma^2 k^* \log(en/k^*).
\]

**Remark 2.1.** In the previous works, Schell and Singh [1997] and many others have argued that isotonic regression using \( \hat{\theta}(\Theta^*_n) \) could over-fit the data or produce estimators of too many steps. Instead, they advocated using \( \hat{\theta}(\Theta^{\uparrow}_k) \), which performs isotonic regression, yet restricts the number of pieces in the estimate, an idea that could be traced back to W.D. Fisher [Fisher, 1958] and is referred to as reduced isotonic (or monotonic) regression in the literature. Our work, however, shows that even with \( k = n \), the monotonic constraint is able to adaptively achieve a rate as nearly good as that of Corollary 2.1 for \( k = k^* \) when \( k \) is large enough. On the other hand, when \( k \) is substantially small, the advantage of \( \hat{\theta}(\Theta^{\uparrow}_k) \) over \( \hat{\theta}(\Theta^*_n) \) is large by comparing Corollaries 2.1, 2.2, and Proposition 2.1.

The proof of Theorem 2.2 relies on a nontrivial sequence splitting strategy that has been historically established for analyzing the diverging rate of the partial sums, and in particular, for proving the LIL. However, for fully utilizing this strategy, an important property of the LSE has to be disclosed first.

In detail, according to the definition of \( \hat{\theta}(\Theta^{\uparrow}_k) \), there exists some \( \hat{k} \leq k \), and \( \{\hat{a}_j\}_{j=0}^\hat{k} \) such that \( 0 = \hat{a}_0 < \hat{a}_1 < ... < \hat{a}_\hat{k} = n \) and \( \hat{\theta}(\Theta^{\uparrow}_k) \) takes distinct constants on the intervals \( \{\hat{a}_{j-1} : \hat{a}_j\}_{j=1}^\hat{k} \). The following optimization-type lemma establishes a property regarding these estimated change-points.

**Lemma 2.1.** For each \( j \in [\hat{k}] \), \( \{\hat{\theta}(\Theta^{\uparrow}_k)\}_i = \overline{X}_{(\hat{a}_{j-1} : \hat{a}_j)} \) for all \( i \in (\hat{a}_{j-1} : \hat{a}_j) \). Moreover, for each \( j \in [\hat{k}] \), \( \hat{\theta}(\Theta^{\uparrow}_k)_{\hat{a}_t} \leq \overline{X}_{(\hat{a}_j : t]} \) for all integers \( t > \hat{a}_j \) (symmetrically, for each \( j \in [\hat{k}] \), \( \hat{\theta}(\Theta^{\uparrow}_k)_{\hat{a}_t} \geq \overline{X}_{(t : \hat{a}_{j-1}]} \) for all integers \( 0 < t < \hat{a}_{j-1} \).

The lemma allows us to establish the relation between the estimated change-points and the true change-points in \( \theta^* \) by setting \( t \) as one of the true change-points. Later, by comparing the proof of Theorem 2.2 to that of Theorem 2.1, one can easily tell that Lemma 2.1 is the sole reason why we can improve the rate from \( \log n \) to \( \log \log n \) for \( k^* \geq 3 \). The proof of the lemma is given in Section 6.3.
3 Minimax lower bounds

This section presents (4) and (12) using the minimax optimality framework based on the worst-case analysis. We start by noting that, for \( k = k^* \), the results of Theorem 2.1 and Corollary 2.1 can be made uniform over the two parameter spaces. In particular, they lead to the following two minimax upper bounds,

\[
\inf_{\hat{\theta}} \sup_{\theta^* \in \Theta_k} \mathbb{E} \| \hat{\theta} - \theta^* \|^2 \leq C\sigma^2 M(k) \quad \text{and} \quad \inf_{\hat{\theta}} \sup_{\theta^* \in \Theta^+_k} \mathbb{E} \| \hat{\theta} - \theta^* \|^2 \leq C\sigma^2 M^+(k),
\]

for the rate functions defined as

\[
M(k) = \begin{cases} 
1, & k = 1, \\
\log \log n, & k = 2, \\
k \log(e n/k), & k \geq 3,
\end{cases}
\]

and

\[
M^+(k) = \begin{cases} 
1, & k = 1, \\
\log \log(16 n/k), & k \geq 2.
\end{cases}
\]

As was argued in Introduction, it is important to answer whether the minimax upper bounds are indeed sharp for the two parameter spaces. The answer is affirmative. Specifically, the minimax lower bounds of the two spaces \( \Theta_k \) and \( \Theta^+_k \) are given in the following theorem.

**Theorem 3.1.** There exists some universal constant \( c > 0 \), such that

\[
\inf_{\hat{\theta}} \sup_{\theta^* \in \Theta_k} \mathbb{E} \| \hat{\theta} - \theta^* \|^2 \geq c\sigma^2 M(k) \quad (14)
\]

and

\[
\inf_{\hat{\theta}} \sup_{\theta^* \in \Theta^+_k} \mathbb{E} \| \hat{\theta} - \theta^* \|^2 \geq c\sigma^2 M^+(k), \quad (15)
\]

where the infimum is taken over all measurable functions of \( X \) and the expectation is taken under which \( X \sim N(\theta^*, \sigma^2 I_n) \).

Combining the upper and lower bounds, we can state the minimax rates of the two spaces.

**Corollary 3.1.** For the two parameter spaces \( \Theta_k \) and \( \Theta^+_k \), their minimax rates are given by

\[
\inf_{\hat{\theta}} \sup_{\theta^* \in \Theta_k} \mathbb{E} \| \hat{\theta} - \theta^* \|^2 \asymp \sigma^2 M(k) \quad \text{and} \quad \inf_{\hat{\theta}} \sup_{\theta^* \in \Theta^+_k} \mathbb{E} \| \hat{\theta} - \theta^* \|^2 \asymp \sigma^2 M^+(k).
\]

For the first time, we are able to show by Corollary 3.1 that the minimax rates for \( \Theta_k \) and \( \Theta^+_k \) are different as long as \( k \geq 3 \). The subtle difference is between a logarithmic factor and an iterated logarithmic factor in the rates.

Two remarks are in order.

**Remark 3.1.** Inequality (14) is strongly related to the problem of determining the “region of detectability” (ROD) in the change-point detection literature. On one hand, when there are multiple change-points, the ROD has been established in Arias-Castro et al. [2005], where
these authors showed that in various settings a signal strength of the order at least $\sqrt{\log n/n}$ is necessary for consistent detection. A gap exists when there is only one change-point. Inequality (14) helps close this gap. As a matter of fact, through a line-by-line follow of the proof of (14) for the case $k = 2$, combined with Le Cam’s minimax testing framework [Yu, 1997], it is straightforward to prove the following corollary. It shows that it is impossible to differentiate the one-piece model from a two-piece one when the signal gap is of the order smaller than $(\log \log n/n)^{1/2}$. In comparison, consistent testing of signal existence when the signal gap is of a comparable order has already been established [Csörgő and Horváth, 1997].

Proposition 3.1. Let $E_\theta$ stand for the expectation induced by $N(\theta, \sigma^2 I_n)$. Define the following parameter space:

$$\Theta_2(c) := \left\{ \theta \in \Theta : (\mu_2 - \mu_1)^2 \cdot (a_1 \wedge (n - a_1)) > c\sigma^2 \log \log n \right\},$$

where $\mu_1, \mu_2, a_1, a_2$ are defined in (2). We then have, for some small enough universal constant $c > 0$,

$$\inf_{0 \leq \phi \leq 1} \left\{ \sup_{\theta \in \Theta_1} E_\theta \phi + \sup_{\theta \in \Theta_2(c)} E_\theta (1 - \phi) \right\} \geq c_1,$$

where $c_1$ is another universal constant in $(0, 1)$.

Proposition 3.1 complements Theorem 2.3 in Arias-Castro et al. [2005], and both results together give a clear picture of the ROD when one or multiple change-points are present.

Remark 3.2. When $k > 2$, the proof of (14) is one-line by noticing that $S_{\lfloor (k-1)/2 \rfloor} \subset \Theta_k$ and using the well-known normal mean estimation lower bound established in Donoho and Johnstone [1994]. The same argument was also employed in Arias-Castro et al. [2005].

The bound (14) when $k = 2$ is much more interesting. It is tempting to consider the following simple “balanced” parameter set that mimicks the corresponding one in Donoho and Johnstone [1994]:

$$\Theta_\Delta := \left\{ \theta \in \Theta_2 : \theta_1, \ldots, \theta_i = \Delta, \theta_{i+1} = \cdots = \theta_n = 0, \text{ for some } i \in [n] \right\}.$$ 

Here “balanced” means that the signal gap $\Delta$ does not change with the change-point position. However, one could show that, for any $\Delta \in \mathbb{R}$, the minimax rate over the set $\Theta_\Delta$ is of the order at most 1. Instead, our analysis is based on an imbalanced parameter set, where the signal gap varies with the change-point position. The construction is given in Section 6.2.

## 4 Extension to heavy-tailed and non-i.i.d. error

Theorem 2.1 and Theorem 2.2 both assume Gaussian error vectors. In this section, we study whether the Gaussian-like error assumption can be relaxed. While the answer is negative for $\Theta_k$, we are able to show the same error bound in Theorem 2.2 holds for $\Theta_k^\uparrow$ under a $(2 + \epsilon)$-moment condition.

We first present the negative result for $\Theta_k$. Consider the observation $X = \theta^* + Z \in \mathbb{R}^n$. We assume i.i.d. error variables $Z_1, ..., Z_n \sim p_\gamma$, where the density function is specified as

$$p_\gamma(x) \propto \exp(-|x|^\gamma), \quad (16)$$

11
for some $\gamma \in (0, 2]$. When $\gamma = 2$, we recover the Gaussian-like (sub-Gaussian) error. For $\gamma \in (0, 2)$, we get a heavier tail than the Gaussian one. The following simple example shows that the Gaussian-like tail assumption cannot be relaxed.

**Proposition 4.1.** Consider the error distribution (16) for some $\gamma \in (0, 2]$. For the space $\Theta_3$, we have the lower bound,

$$\inf_{\hat{\theta}} \sup_{\theta^* \in \Theta_3} \mathbb{E}\|\hat{\theta} - \theta^*\|^2 \geq c\left(\log n\right)^{2/\gamma},$$

for some universal constant $c > 0$. Since the desired minimax rate for $\Theta_3$ is $\log n$ when $\sigma^2$ is a constant, Proposition 4.1 implies that the minimax rate under the Gaussian assumption cannot be achieved unless $\gamma = 2$. In other words, a Gaussian-like tail is necessary for the result of Theorem 2.1. On the other hand, the relaxation to sub-Gaussian error distributions can be easily obtained. We omit the details in the paper.

For the space $\Theta_k^\dagger$, the extra monotonic constraint allows us to relax the Gaussian assumption as weak as a $(2 + \epsilon)$-moment assumption. Consider the observation

$$X = \theta^* + Z,$$

where we assume the error variables $\{Z_i\}_{i=1}^n$ are independent with zero mean and satisfy

$$\begin{cases} \max_{1 \leq i \leq n} \mathbb{E}\left|Z_i/\sigma\right|^{2+\epsilon} \leq C, & \text{not identically distributed } Z_i, \\ \mathbb{E}(Z_i^2/\sigma) \log(e + Z_i^2/\sigma^2) \leq C, & \text{identically distributed } Z_i, \end{cases}$$

for some number $\sigma > 0$, an arbitrarily small constant $\epsilon \in (0, 1)$, and some universal constant $C > 0$. Note that the Gaussian error $Z \sim N(0, \sigma^2 I_n)$ considered earlier in this section is a special case of (17). The following theorem proves that (8) applies to the case (17).

**Theorem 4.1.** Assume the moment condition (17). Consider $\theta^* \in \Theta_k^\dagger$, and let $\hat{\theta}$ be defined through (1) with $\Theta = \Theta_k^\dagger$ for some $k \geq k^*$. Then, there exists some universal constant $C > 0$, such that

$$\mathbb{E}\|\hat{\theta} - \theta^*\|^2 \leq \begin{cases} C\sigma^2, & k = k^* = 1, \\ C\sigma^2\left\{k^* + \bar{k}\log(em/\bar{k})\right\}, & k \geq k^*. \end{cases}$$

**Remark 4.1.** The comparison between Proposition 4.1 and Theorem 4.1 shows that, when isotonic constraint exists, it can not only help improve the estimation accuracy, but allows for much heavier-tailed distributions than the Gaussian. This phenomenon has been discovered in the literature; see, for example, Wang and Chen [1996], Zhang [2002], and Chatterjee et al. [2015]. By checking our proofs, it is immediate that such a relaxation is possible by the piecewise constant structure of the (reduced) isotonic regression estimates. The high level reason is that (13) could hold under the moment condition (17), while an analogous bound

$$\mathbb{E}\left(\max_{1 \leq i \leq n} Z_i^2\right) \leq C\sigma^2 \log n$$

holds only for sub-Gaussian tails. Technically speaking, such a relaxation is possible by employing a truncation argument on Doob’s martingale theory. This cannot be done with
the Gaussian-width and entropy-type arguments that were used in van de Geer [1990] and Chatterjee [2014].

5 Adaptive estimation

The minimax rates presented in Section 2 and Section 3 assume the knowledge of the number of blocks $k^*$ the true signal has. Using the aggregation technique developed in Leung and Barron [2006], we show the same minimax rates are achievable without the knowledge of $k^*$.

We consider Gaussian observation $X \sim N(\theta^*, \sigma^2 I_n)$ with a known $\sigma^2$. With the knowledge of $\sigma^2$, we can create an independent error vector $W \sim N(0, \sigma^2 I_n)$. Then, define $U = X + W$ and $V = X - W$. It is easy to see that $U$ and $V$ are independent and both follow the distribution $N(\theta^*, 2\sigma^2 I_n)$. The general strategy is to use $U$ to form primitive estimators with various choices of $k$, and then use $V$ to aggregate these estimators. The Gaussian error assumption and the knowledge of $\sigma^2$ allow us to create two identically distributed observations. Whether adaptation is possible to more general distributions under unknown variance situation needs further investigation.

We apply the following aggregation procedure. First, solve the constrained least squares optimization (1) using the data $U$. That is, for each $k \in K = \{1, 2, 4, 8, ..., 2^\lfloor \log_2 n \rfloor, n\}$, define

$$\hat{\theta}_k^U = \arg\min_{\theta \in \Theta} \|U - \theta\|^2,$$

for either $\Theta = \Theta_k$ or $\Theta = \Theta_k^\perp$. Denote the simplex on $K$ by $\Lambda_K = \{\{\lambda_k\}_{k \in K} : \lambda_k \geq 0, \sum_{k \in K} \lambda_k = 1\}$. The second step is to use $V$ to pick an element from $\Lambda_K$ for aggregation. For two elements $\lambda, \pi \in K$, the Kullback-Leibler divergence is defined as

$$D(\lambda||\pi) = \sum_{k \in K} \lambda_k \log(\lambda_k / \pi_k).$$

Define

$$\hat{\lambda}^V = \arg\min_{\lambda \in \Lambda_K} \left\{ \sum_{k \in K} \lambda_k \|V - \hat{\theta}_k^U\|^2 + 8\sigma^2 D(\lambda||\pi) \right\},$$

where the vector $\pi$ is specified as

$$\pi_k = \begin{cases} \frac{1}{2}, & k = 1, \\ \frac{1}{2^{|\lfloor \log_2 n \rfloor - 1|}}, & k \in K \setminus \{1\}. \end{cases}$$

Our final aggregated estimator is

$$\tilde{\theta} = \sum_{k \in K} \hat{\lambda}_k^V \hat{\theta}_k^U.$$
by $E\|\hat{\theta}_k^U - \theta^*\|^2$, we are able to achieve the rates in Theorem 2.1 and Theorem 2.2 adaptively over all $k^*$.

**Proposition 5.1.** Consider $\theta^* \in \Theta_{k^*}$, and let $\tilde{\theta}$ be defined through the above procedure with $\Theta = \Theta_k$ for $k \in \mathcal{K}$. Then, there exists some universal constant $C > 0$, such that

$$E\|\tilde{\theta} - \theta^*\|^2 \leq \begin{cases} C\sigma^2, & k^* = 1, \\ C\sigma^2 \log \log n, & k^* = 2, \\ C\sigma^2 k^* \log(en/k^*), & k^* \geq 3. \end{cases}$$

**Proposition 5.2.** Consider $\theta^* \in \Theta_{k^*}^\uparrow$, and let $\tilde{\theta}$ be defined through the above procedure with $\Theta = \Theta_k$ for $k \in \mathcal{K}$. Then, there exists some universal constant $C > 0$, such that

$$E\|\tilde{\theta} - \theta^*\|^2 \leq \begin{cases} C\sigma^2, & k^* = 1, \\ C\sigma^2 k^* \log(em/k^*), & k^* \geq 2, \end{cases}$$

where $m$ is defined in (7).

### 6 Proofs

This section contains the proofs of all results in this paper. In the sequel, we omit the symbols $\Theta_k$ and $\Theta_k^\uparrow$ in the LSEs when no confusion could be made.

#### 6.1 Proofs of upper bounds

In this section, we state the proofs of Theorem 2.1 and Theorem 4.1. Since Theorem 2.2 is a special case of Theorem 4.1, we do not need to state its proof.

We first state the proof of Theorem 4.1.

**Proof of Theorem 4.1.** When $k = k^* = 1$, $\hat{\theta}$ is a constant vector with each entry taking $\overline{X}_{(0:n)}$. Therefore, $E\|\hat{\theta} - \theta^*\|^2 \leq C_1 \sigma^2$.

Consider the case $k \geq k^*$. For $\theta^* \in \Theta_{k^*}$, there exist integers $\{a_j\}_{j=0}^{k^*}$, such that $0 = a_0 \leq a_1 \leq \ldots \leq a_{k^*} = n$ and $\theta^*$ takes constants on the intervals $\{(a_{j-1} : a_j)\}_{j=1}^{k^*}$. We are going to derive a bound for $\|\hat{\theta} - \theta^*\|^2$. Since $\|\theta - \theta^*\|^2 = \sum_{j=1}^{k^*} \|\hat{\theta} - \theta^*\|_{(a_{j-1} : a_j)}^2$, we bound each $\|\hat{\theta} - \theta^*\|_{(a_{j-1} : a_j)}^2$, respectively. From now on, we use generic notation $a$ and $b$ for $a_{j-1}$ and $a_j$. There is some $\mu \in \mathbb{R}$ such that $\theta^*_i = \mu$ for all $i \in (a : b)$. Define

$$j_1 = \min \{j \in [n]: \hat{a}_j > a\} \quad \text{and} \quad j_2 = \max \{j \in [n]: \hat{a}_j \leq b\}. \quad (18)$$

We consider the following two cases.
Case 1: \( j_1 \leq j_2 \). By Lemma 2.1, \( \hat{\theta}_i = \mathbf{X}(\bar{a}_{j-1};\bar{a}_j) \) for all \( i \in (\bar{a}_{j-1}: \bar{a}_j) \). We have

\[
\|X - \hat{\theta}\|_{(a:b)}^2 - \|X - \theta^*\|_{(a:b)}^2 = \sum_{i=a+1}^{\bar{a}_{j_1}} \left\{ (X_i - \hat{\theta}_i)^2 - (X_i - \theta_i^*)^2 \right\} + \sum_{i=\bar{a}_{j_2}+1}^b \left\{ (X_i - \hat{\theta}_i)^2 - (X_i - \theta_i^*)^2 \right\} \\
+ \sum_{j=j_1+1}^{j_2} \sum_{i=\bar{a}_{j-1}+1}^{\bar{a}_j} \left\{ (X_i - \hat{\theta}_i)^2 - (X_i - \theta_i^*)^2 \right\} \\
= \sum_{i=a+1}^{\bar{a}_{j_1}} \left\{ (X_i - \hat{\theta}_{\bar{a}_{j_1}})^2 - (X_i - \mu)^2 \right\} + \sum_{i=\bar{a}_{j_2}+1}^b \left\{ (X_i - \hat{\theta}_{\bar{a}_{j_2}+1})^2 - (X_i - \mu)^2 \right\} \\
+ \sum_{j=j_1+1}^{j_2} \sum_{i=\bar{a}_{j-1}+1}^{\bar{a}_j} \left\{ (X_i - \mathbf{X}(\bar{a}_{j-1};\bar{a}_j))^2 - (X_i - \mu)^2 \right\} \\
= (\bar{a}_{j_1} - a) \left( \mathbf{X}(a;\bar{a}_{j_1}) - \hat{\theta}_{\bar{a}_{j_1}} \right)^2 - (\mathbf{X}(a;\bar{a}_{j_1}) - \mu)^2 + (b - \bar{a}_{j_2}) \left( \mathbf{X}(\bar{a}_{j_2};b) - \hat{\theta}_{\bar{a}_{j_2}+1} \right)^2 - (\mathbf{X}(\bar{a}_{j_2};b) - \mu)^2 \\
- \sum_{j=j_1+1}^{j_2} (\bar{a}_j - \bar{a}_{j-1}) (\mathbf{X}(\bar{a}_{j-1};\bar{a}_j) - \mu)^2.
\]

Therefore, we can deduce a useful equality

\[
(\bar{a}_{j_1} - a) \left( \mathbf{X}(a;\bar{a}_{j_1}) - \hat{\theta}_{\bar{a}_{j_1}} \right)^2 + (b - \bar{a}_{j_2}) \left( \mathbf{X}(\bar{a}_{j_2};b) - \hat{\theta}_{\bar{a}_{j_2}+1} \right)^2 \\
= (\bar{a}_{j_1} - a) \left( \mathbf{X}(a;\bar{a}_{j_1}) - \mu \right)^2 + (b - \bar{a}_{j_2}) \left( \mathbf{X}(\bar{a}_{j_2};b) - \mu \right)^2 \\
+ \sum_{j=j_1+1}^{j_2} (\bar{a}_j - \bar{a}_{j-1}) (\mathbf{X}(\bar{a}_{j-1};\bar{a}_j) - \mu)^2 + \|X - \hat{\theta}\|_{(a:b)}^2 - \|X - \theta^*\|_{(a:b)}^2.
\]

Now we are ready to give a bound for \( \|\hat{\theta} - \theta^*\|_{(a:b)}^2 \). That is,

\[
\|\hat{\theta} - \theta^*\|_{(a:b)}^2 \\
= (\bar{a}_{j_1} - a) \left( \hat{\theta}_{\bar{a}_{j_1}} - \mu \right)^2 + (b - \bar{a}_{j_2}) \left( \hat{\theta}_{\bar{a}_{j_2}+1} - \mu \right)^2 + \sum_{j=j_1+1}^{j_2} (\bar{a}_j - \bar{a}_{j-1}) (\mathbf{X}(\bar{a}_{j-1};\bar{a}_j) - \mu)^2 \\
\leq 2(\bar{a}_{j_1} - a) \left( \mathbf{X}(a;\bar{a}_{j_1}) - \mu \right)^2 + 2(b - \bar{a}_{j_2}) \left( \mathbf{X}(\bar{a}_{j_2};b) - \mu \right)^2 + \sum_{j=j_1+1}^{j_2} (\bar{a}_j - \bar{a}_{j-1}) (\mathbf{X}(\bar{a}_{j-1};\bar{a}_j) - \mu)^2 \\
+ 2(\bar{a}_{j_1} - a) \left( \hat{\theta}_{\bar{a}_{j_1}} - \mathbf{X}(a;\bar{a}_{j_1}) \right)^2 + 2(b - \bar{a}_{j_2}) \left( \hat{\theta}_{\bar{a}_{j_2}+1} - \mathbf{X}(\bar{a}_{j_2};b) \right)^2 \\
= 4(\bar{a}_{j_1} - a) \left( \mathbf{X}(a;\bar{a}_{j_1}) - \mu \right)^2 + 4(b - \bar{a}_{j_2}) \left( \mathbf{X}(\bar{a}_{j_2};b) - \mu \right)^2 + 3 \sum_{j=j_1+1}^{j_2} (\bar{a}_j - \bar{a}_{j-1}) (\mathbf{X}(\bar{a}_{j-1};\bar{a}_j) - \mu)^2 \\
+ 2 \left( \|X - \hat{\theta}\|_{(a:b)}^2 - \|X - \theta^*\|_{(a:b)}^2 \right).
\]
where the last inequality is by applying (19). For $j \in (j_1 : j_2]$, the result of Lemma 2.1 implies
\[ |\bar{X}_{[\tilde{a}_{j-1}:\tilde{a}_j]} - \mu| \leq |\bar{X}_{(a:\tilde{a}_j]} - \mu| \lor |\bar{X}_{(\tilde{a}_{j-1}:b]} - \mu|. \]

We therefore deduce the bound
\[
|\hat{\theta} - \theta^*|_{(a:b]}^2 \\
\leq 4(\tilde{a}_{j_1} - a)(\bar{X}_{(a:\tilde{a}_{j_1}]} - \mu)^2 + 4(b - \tilde{a}_{j_2})(\bar{X}_{(\tilde{a}_{j_2}:b]} - \mu)^2 \\
+ 3 \sum_{j=j_1+1}^{j_2} (\tilde{a}_j - \tilde{a}_{j-1})(\bar{X}_{(a:\tilde{a}_j]} - \mu)^2 + 3 \sum_{j=j_1+1}^{j_2} (\tilde{a}_j - \tilde{a}_{j-1})(\bar{X}_{(\tilde{a}_{j-1}:b]} - \mu)^2 \\
+ 2 \left( \|X - \hat{\theta}\|_{(a:b]}^2 - \|X - \theta^*\|_{(a:b]}^2 \right). \tag{21}
\]

**Case 2:** $j_1 > j_2$. In this case, $\tilde{\theta}$ is a constant in $(a:b]$. Then,
\[
|\hat{\theta} - \theta^*|_{(a:b]}^2 = (b - a)(\hat{\theta}_b - \mu)^2 \\
\leq 2(b - a)(\hat{\theta}_b - \bar{X}_{(a:b]} - \mu)^2 + 2(b - a)(\bar{X}_{(a:b]} - \mu)^2 \\
= 2(b - a)(\bar{X}_{(a:b]} - \mu)^2 + 2 \sum_{i=a+1}^b (X_i - \hat{\theta}_i)^2 - 2 \sum_{i=a+1}^b (X_i - \bar{X}_{(a:b]})^2 \\
= 4(b - a)(\bar{X}_{(a:b]} - \mu)^2 + 2 \sum_{i=a+1}^b (X_i - \hat{\theta}_i)^2 - 2 \sum_{i=a+1}^b (X_i - \mu)^2. \tag{22}
\]

In other words, we have the inequality
\[
|\hat{\theta} - \theta^*|_{(a:b]}^2 \leq 4(b - a)(\bar{X}_{(a:b]} - \mu)^2 + 2 \left( \|X - \hat{\theta}\|_{(a:b]}^2 - \|X - \theta^*\|_{(a:b]}^2 \right). \tag{23}
\]

Combining the two inequalities (21) and (23) in both cases, we have
\[
|\hat{\theta} - \theta^*|_{(a:b]}^2 \\
\leq 4(\tilde{a}_{j_1} - a)(\bar{X}_{(a:\tilde{a}_{j_1}]} - \mu)^2 + 4(b - \tilde{a}_{j_2})(\bar{X}_{(\tilde{a}_{j_2}:b]} - \mu)^2 + 4(b - a)(\bar{X}_{(a:b]} - \mu)^2 \\
+ 3 \sum_{j=j_1+1}^{j_2} (\tilde{a}_j - \tilde{a}_{j-1})(\bar{X}_{(a:\tilde{a}_j]} - \mu)^2 + 3 \sum_{j=j_1+1}^{j_2} (\tilde{a}_j - \tilde{a}_{j-1})(\bar{X}_{(\tilde{a}_{j-1}:b]} - \mu)^2 \\
+ 2 \left( \|X - \hat{\theta}\|_{(a:b]}^2 - \|X - \theta^*\|_{(a:b]}^2 \right). \tag{24}
\]

Define the error vector $Z = X - \theta^*$. Then, define random variables
\[
\xi_+(a, b, \ell) = 2^\ell \max \left\{ |Z_{(a:b]}|^2 : a + 2^\ell - 1 \leq t \leq b \land (a + 2^\ell - 1) \right\}, \tag{25}
\]
\[
\delta_+(a, b, \ell) = \max_{a \leq \tilde{a}_j \leq b} \left\{ a + 2^\ell - 1 \leq \tilde{a}_j \leq b \land (a + 2^\ell - 1) \right\}, \tag{26}
\]
\[
\xi_-(a, b, \ell) = 2^\ell \max \left\{ |Z_{(t:b]}|^2 : (a + 1) \lor (b + 2 - 2^\ell) \leq t \leq b + 1 - 2^\ell \right\}, \tag{27}
\]
\[
\delta_-(a, b, \ell) = \max_{a \leq \tilde{a}_j \leq b} \left\{ (a + 1) \lor (b + 2 - 2^\ell) \leq \tilde{a}_j \leq b + 1 - 2^\ell \right\}. \tag{28}
\]
Using these definitions, we can further bound (24) by
\[
\|\hat{\theta} - \theta^*\|_{(a:b)}^2 \leq 4 \sum_{\ell \geq 1 : a + 2^{\ell-1} \leq b} \delta_+(a, b, \ell) \xi_+(a, b, \ell) + 4 \sum_{\ell \geq 1 : a \leq b - 2^{\ell-1}} \delta_-(a, b, \ell) \xi_-(a, b, \ell) 
\text{(29)}
\]
\[
+4(b - a)Z_{(a:b)}^2 + 2\left(\|X - \hat{\theta}\|_{(a:b)}^2 - \|X - \theta^*\|_{(a:b)}^2\right).
\]

The inequality (29) holds for all \((a : b) = (a_{j-1} : a_j)\). By summing up over \(j \in [k^*]\), we obtain the following bound,
\[
\|\hat{\theta} - \theta^*\|^2 \leq 4 \sum_{j=1}^{k^*} \sum_{\ell \geq 1 : a_{j-1} + 2^{\ell-1} \leq a_j} \delta_+(a_{j-1}, a_j, \ell) \xi_+(a_{j-1}, a_j, \ell)
\text{(30)}
\]
\[
+4 \sum_{j=1}^{k^*} \sum_{\ell \geq 1 : a_{j-1} \leq a_j - 2^{\ell-1}} \delta_-(a_{j-1}, a_j, \ell) \xi_-(a_{j-1}, a_j, \ell)
\text{(31)}
\]
\[
+4 \sum_{j=1}^{k^*} (a_j - a_{j-1}) Z_{(a_{j-1}:a_j)}^2.
\text{(32)}
\]

Note that we have dropped the term \(2 \sum_{j=1}^{k^*} \left(\|X - \hat{\theta}\|_{(a_{j-1}:a_j)}^2 - \|X - \theta^*\|_{(a_{j-1}:a_j)}^2\right)\) because \(\|X - \hat{\theta}\|^2 - \|X - \theta^*\|^2 \leq 0\) according to the definition of \(\hat{\theta}\) and the fact that \(\theta^* \in \Theta_{k^*} \subset \Theta_k^+\).

The following lemma gives bounds for the two terms (30) and (31). Its proof is given in Section 6.3.

**Lemma 6.1.** Consider \(m\) and \(\bar{k}\) defined in (7). There exists a universal constant \(C > 0\), such that
\[
\sum_{j=1}^{k^*} \sum_{\ell \geq 1 : a_{j-1} + 2^{\ell-1} \leq a_j} \mathbb{E} \delta_+(a_{j-1}, a_j, \ell) \xi_+(a_{j-1}, a_j, \ell) \leq C \sigma^2 \left\{k^* + \bar{k} \log(em/\bar{k})\right\},
\]
\[
\sum_{j=1}^{k^*} \sum_{\ell \geq 1 : a_{j-1} \leq a_j - 2^{\ell-1}} \mathbb{E} \delta_-(a_{j-1}, a_j, \ell) \xi_-(a_{j-1}, a_j, \ell) \leq C \sigma^2 \left\{k^* + \bar{k} \log(em/\bar{k})\right\}.
\]

Using Lemma 6.1, we have
\[
\mathbb{E}\|\hat{\theta} - \theta^*\|^2 \leq 8C \sigma^2 \left\{k^* + \bar{k} \log(em/\bar{k})\right\} + 4 \sum_{j=1}^{k^*} (a_j - a_{j-1}) \mathbb{E} Z_{(a_{j-1}:a_j)}^2.
\]

Note that
\[
\sum_{j=1}^{k^*} (a_j - a_{j-1}) \mathbb{E} Z_{(a_{j-1}:a_j)}^2 \leq C_2 \sigma^2 k^*,
\]
and the proof is complete.

Now we state the proof of Theorem 2.1. Note that similar arguments that we have already used in the proof of Theorem 4.1 will be omitted.
Proof of Theorem 2.1. When \( k = k^* = 1 \), it is trivial that \( \mathbb{E}\|\hat{\theta} - \theta^*\|^2 \leq C_1\sigma^2 \).

When \( k = k^* = 2 \), we consider the vector \( \theta^* \) that takes constants \( \mu_1 \) and \( \mu_2 \) on two intervals \( (0 : a) \) and \( (a : n) \), respectively. Moreover, the estimator \( \hat{\theta} \) takes constants on two intervals \( (0 : \hat{a}) \) and \( (\hat{a} : n) \). When \( \hat{a} < a \), using the same argument for deriving the bounds (20) and (22), we get

\[
||\hat{\theta} - \theta^*||^2_{[0:a]} \leq 4\hat{a}(X_{(0:\hat{a})} - \mu_1)^2 + 4(a - \hat{a})(X_{(\hat{a}:a)} - \mu_1)^2 + 2\left(||X - \hat{\theta}||^2_{[0:a]} - ||X - \theta^*||^2_{[0:a]}\right),
\]

and

\[
||\hat{\theta} - \theta^*||^2_{(a:n)} \leq 4(n - a)(X_{(a:n)} - \mu_2)^2 + 2\left(||X - \hat{\theta}||^2_{(a:n)} - ||X - \theta^*||^2_{(a:n)}\right).
\]

Therefore, since \( ||\hat{\theta} - \theta^*||^2 = ||\hat{\theta} - \theta^*||^2_{[0:a]} + ||\hat{\theta} - \theta^*||^2_{(a:n)} \) and \( ||X - \hat{\theta}||^2 \leq ||X - \theta^*||^2 \), we have

\[
||\hat{\theta} - \theta^*||^2 \leq 4\hat{a}(X_{(0:\hat{a})} - \mu_1)^2 + 4(a - \hat{a})(X_{(\hat{a}:a)} - \mu_1)^2 + 4(n - a)(X_{(a:n)} - \mu_2)^2. \tag{33}
\]

When \( \hat{a} > a \), a similar argument leads to

\[
||\hat{\theta} - \theta^*||^2 \leq 4a(X_{(0:a)} - \mu_1)^2 + 4(a - \hat{a})(X_{(a:\hat{a})} - \mu_2)^2 + 4(n - \hat{a})(X_{(\hat{a}:n)} - \mu_2)^2. \tag{34}
\]

Combining the two bounds (33) and (34) for the two cases, and using Lemma 6.1 with \( k = k^* = 2 \), we obtain the conclusion \( \mathbb{E}\|\hat{\theta} - \theta^*\|^2 \leq C_2\sigma^2 \log \log n \).

For general \( k \geq k^* \), according to the definition of \( \hat{\theta} \), there exists some \( \hat{k} \leq k \), and \( \{\hat{a}_j\}_{j=0}^{\hat{k}} \), such that \( 0 = \hat{a}_0 < \hat{a}_1 < ... < \hat{a}_{\hat{k}} = n \) and \( \hat{\theta} \) takes constants on the intervals \( \{\hat{a}_j : \hat{a}_{j-1}\}_{j=1}^{\hat{k}} \). For \( \theta^* \), there exist integers \( \{a_j\}_{j=0}^{k^*} \), such that \( 0 = a_0 \leq a_1 \leq ... \leq a_{k^*} = n \) and \( \theta^* \) takes constants on the intervals \( \{a_j : a_{j-1}\}_{j=1}^{k^*} \). We are going to derive a bound for \( ||\hat{\theta} - \theta^*||^2 \). Note that \( ||\theta - \theta^*||^2 = \sum_{j=1}^{k^*} ||\hat{\theta} - \theta^*||^2_{(a_{j-1}:a_j)} \), and we are going to bound each \( ||\hat{\theta} - \theta^*||^2_{(a_{j-1}:a_j)} \), respectively.

From now on, we use generic notation \( a \) and \( b \) for \( a_{j-1} \) and \( a_j \). There is some \( \mu \in \mathbb{R} \) such that \( \theta^*_i = \mu \) for all \( i \in (a : b) \). Define \( j_1 \) and \( j_2 \) as in (18). Using the same argument for deriving the bounds (20) and (22), we get

\[
||\hat{\theta} - \theta^*||^2_{(a:b)}
\]

\[
\leq 4(\hat{a}_{j_1} - a)(X_{(a:\hat{a}_{j_1})} - \mu)^2 + 4(b - \hat{a}_{j_2})(X_{(\hat{a}_{j_2}:b)} - \mu)^2 + 3\sum_{j=j_1+1}^{j_2} (\hat{a}_j - \hat{a}_{j-1})(X_{(\hat{a}_{j-1}:\hat{a}_j)} - \mu)^2
\]

\[
+ 2\left(||X - \hat{\theta}||^2_{(a:b)} - ||X - \theta^*||^2_{(a:b)}\right),
\]

when \( j_1 \leq j_2 \), and

\[
||\hat{\theta} - \theta^*||^2_{(a:b)} \leq 4(b - a)(X_{(a:b)} - \mu)^2 + 2\left(||X - \hat{\theta}||^2_{(a:b)} - ||X - \theta^*||^2_{(a:b)}\right),
\]

when \( j_2 > j_2 \). Combining the two cases, we have

\[
\leq 4(\hat{a}_{j_1} - a)(X_{(a:\hat{a}_{j_1})} - \mu)^2 + 4(b - \hat{a}_{j_2})(X_{(\hat{a}_{j_2}:b)} - \mu)^2 + 3\sum_{j=j_1+1}^{j_2} (\hat{a}_j - \hat{a}_{j-1})(X_{(\hat{a}_{j-1}:\hat{a}_j)} - \mu)^2
\]

\[
+ 4(b - a)(X_{(a:b)} - \mu)^2 + 2\left(||X - \hat{\theta}||^2_{(a:b)} - ||X - \theta^*||^2_{(a:b)}\right).
\]
Replacing $a$ and $b$ by $a_{j-1}$ and $a_j$ and summing over $j \in [k^*]$, we obtain the following bound,

$$
||\hat{\theta} - \theta^*||^2 
\leq 4 \sum_{j=1}^{k^*} \left( \sum_{\ell \geq 1: a_{j-1} + 2^\ell - 1 \leq a_j} \delta_+(a_{j-1}, a_j, \ell) \xi_+(a_{j-1}, a_j, \ell) \right) \tag{35}
+ 4 \sum_{j=1}^{k^*} \left( \sum_{\ell \geq 1: a_{j-1} \leq a_j - 2^\ell} \delta_-(a_{j-1}, a_j, \ell) \xi_-(a_{j-1}, a_j, \ell) \right) \tag{36}
+ 4 \sum_{j=1}^{k^*} (a_j - a_{j-1}) \mathcal{Z}_j^2(\bar{a}_{j-1}; a_j) \tag{37}
+ 3 \sum_{j=1}^{\hat{k}} (\hat{a}_j - \bar{a}_{j-1}) \mathcal{Z}_j^2(\bar{a}_{j-1}; \bar{a}_j), \tag{38}
$$

with the notation $\delta_+(a, b, \ell)$, $\xi_+(a, b, \ell)$, $\delta_-(a, b, \ell)$ and $\xi_-(a, b, \ell)$ defined in (25)-(28). Note that we have dropped the term $2 \sum_{j=1}^{k^*} \left( \|X - \hat{\theta}\|^2_{[a_{j-1}: a_j]} - \|X - \theta^*\|^2_{[a_{j-1}: a_j]} \right)$ because $\|X - \hat{\theta}\|^2 - \|X - \theta^*\|^2 \leq 0$ according to the definition of $\hat{\theta}$ and the fact that $\theta^* \in \Theta_{k^*} \subset \Theta_k$. According to the proof of Theorem 4.1, we can use Lemma 6.2 to bound the expectation of the first three terms (35)-(37) by $C_3 \sigma^2 (k^* + k \log(\kappa \hat{k}))$. Thus, it is sufficient to give a bound for the expectation of (38). This is done through the following lemma.

**Lemma 6.2.** There exists a universal constant $C > 0$, such that

$$
\mathbb{E} \left[ \sum_{j=1}^{\hat{k}} (\hat{a}_j - \bar{a}_{j-1}) \mathcal{Z}_j^2(\bar{a}_{j-1}; \bar{a}_j) \right] \leq C \sigma^2 \kappa \log(em/\hat{k}).
$$

The proof of Lemma 6.2 is given in Section 6.3. Combining the bounds for (35)-(38), we complete the proof. \square

### 6.2 Proofs of lower bounds

This section is devoted to proving the lower bounds, including Proposition 2.1, Theorem 3.1, Proposition 3.1, and Proposition 4.1.

**Proof of Proposition 2.1.** Without loss of generality, consider the case when $n/k^*$ is an integer. Then, $[n] = \bigcup_{j=1}^{k^*} C_j$, where $C_j$ is the $j$th consecutive interval with cardinality $n/k^*$. Then, we take $\theta^* \in \Theta^*_k$, with $\theta^*_i = \mu_j$ if $i \in C_j$. Use the notation $\mathcal{H}_n = \{\theta \in \mathbb{R}^n : \theta_1 \leq ... \leq \theta_n\}$. Then, as long as $\mu_1, ..., \mu_k$ are sufficiently separated,

$$
\min_{\theta \in \mathcal{H}_n} \sum_{i=1}^n (X_i - \theta_i)^2 = \sum_{j=1}^{k^*} \min_{\theta \in \mathcal{H}_{n/k^*}} \sum_{i \in C_j} (X_i - \theta_i)^2,
$$

with high probability. This high-probability event is denoted as $E$. We take $\mu_j = \kappa j$ for some $\kappa > 0$. Then, as $\kappa \to \infty$, $\mathbb{P}(E^c)$ converges to 0. In other words, $\mathbb{P}(E^c)$ is arbitrarily small for
sufficiently large \( \kappa \). We have

\[
\mathbb{E}\|\hat{\theta} - \theta^*\|^2 \geq \sum_{j=1}^{k^*} \mathbb{E}\|\hat{\theta}_{Cj} - \theta^*_{Cj}\|^2 - \mathbb{E}\|\hat{\theta} - \theta^*\|^2 I_{Ec}.
\]

Since \( \mathbb{E}\|\hat{\theta} - \theta^*\|^2 I_{Ec} \leq \sqrt{\mathbb{E}\|\hat{\theta} - \theta^*\|^4 P(Ec)} \) is arbitrarily small for sufficiently large \( \kappa \), the term \( \mathbb{E}\|\hat{\theta} - \theta^*\|^2 I_{Ec} \) can be neglected. It is sufficient to give a lower bound for \( \sum_{j=1}^{k^*} \mathbb{E}\|\hat{\theta}_{Cj} - \theta^*_{Cj}\|^2 \). Note that

\[
\sum_{j=1}^{k^*} \mathbb{E}\|\hat{\theta}_{Cj} - \theta^*_{Cj}\|^2 = \sum_{j=1}^{k^*} \mathbb{E}\|\Pi_{H_{n/k^*}^C} Z_{Cj}\|^2,
\]

where \( \Pi_{H_{n/k^*}^C} \) is the projection operator onto the space \( H_{n/k^*} \). By Amelunxen et al. [2014], \( \|\Pi_{H_{n/k^*}^C} Z_{Cj}\|^2 \geq C \log(en/k^*) \), which leads to the desired result. \( \square \)

We continue to state the proofs of other results. The main tool we will use is Fano’s lemma. For any probability measures \( P, Q \), define the Kullback-Leibler divergence to be

\[
D(P||Q) = \int \left( \log \frac{dP}{dQ} \right) dP.
\]

The Fano’s lemma is stated as follows. The version we present here is by Yu [1997].

**Proposition 6.1.** Let \( (\Theta, \rho) \) be a metric space and \( \{P_\theta : \theta \in \Theta\} \) be a collection of probability measures. For any totally bounded \( T \subset \Theta \), define the Kullback-Leibler diameter by

\[
d_{KL}(T) = \sup_{\theta, \theta' \in T} D(P_\theta||P_{\theta'}).
\]

Then

\[
\inf_{\hat{\theta}} \sup_{\theta, \theta' \in \Theta} P_\theta \left[ \rho^2 \left( \hat{\theta}(X), \theta \right) \geq \frac{\epsilon^2}{4} \right] \geq 1 - \frac{d_{KL}(T) + \log 2}{\log \mathcal{M}(\epsilon, T, \rho)},
\]

for any \( \epsilon > 0 \), where \( \mathcal{M}(\epsilon, T, \rho) \) stands for the packing number of \( T \) with radius \( \epsilon \) with respect to the metric \( \rho \).

We will divide the proof of Theorem 3.1 into two parts. The first part derives the lower bound for \( \Theta_k \), and the second part derives the lower bound for \( \Theta^*_k \).

**Proof of Theorem 3.1: Part 1.** We only need to deal with the case when \( n > C \) for a sufficiently large constant, since when \( n \leq C \), the rate is a constant and the conclusion automatically holds.

When \( k = 1 \), the standard lower bound argument for the one-dimensional normal mean problem [Lehmann and Casella, 2006] applies here, and we get the desired rate.

The case \( k = 2 \) is the most interesting one. For each \( \ell \in [\lceil \log_2 n \rceil] \), construct the vector \( \theta_\ell \in \mathbb{R}^n \) by filling the last \( \lceil n2^{-\ell} \rceil \) entries with \( \sqrt{\alpha \sigma^2 2^{\ell} \log \log_2 n/n} \) and the remaining entries
0. It is easy to see that $\theta_\ell \in \Theta_2$ for all $\ell \in [\lceil \log_2 n \rceil]$. For any $k < \ell$, we have
\[
\|\theta_\ell - \theta_k\|^2 \geq \lceil n2^{-\ell}\rceil \left( \frac{\alpha \sigma^2 2^\ell \log_2 n}{n} - \frac{\alpha \sigma^2 2^k \log_2 n}{n} \right)^2 \\
\geq \alpha \sigma^2 \log_2 n \left( 1 - 2^{\frac{k-\ell}{2}} \right)^2 \\
\geq \frac{\alpha \sigma^2}{20} \log_2 n.
\]
Therefore,
\[
\log M \left( \sqrt{\frac{\alpha \sigma^2}{20} \log_2 n}, T, \|\cdot\| \right) \geq \log \log_2 n, \quad (40)
\]
where $T := \{ \theta_\ell : \ell \in [\lceil \log_2 n \rceil] \}$. Moreover, since $\|\theta_\ell\|^2 \leq 3\alpha \sigma^2 \log_2 n$ for all $\ell \in [\lceil \log_2 n \rceil]$, we have
\[
d_{\text{KL}}(T) = \max_{\hat{\theta}, \theta \in T} \frac{1}{2\sigma^2} \|\hat{\theta} - \theta\|^2 \leq 6\alpha \log_2 n. \quad (41)
\]
Combining (39), (40), and (41), we have
\[
\inf_{\hat{\theta}} \sup_{\theta \in \Theta_2} \mathbb{P} \left( \|\hat{\theta} - \theta\|^2 \geq \frac{\alpha \sigma^2}{80} \log_2 n \right) \geq 1 - \frac{6\alpha \log_2 n + \log 2}{\log_2 n} \geq c.
\]
with $\alpha = 1/60$ and a sufficiently small value $c > 0$. Thus, with an application of Markov’s inequality, we obtain the desired minimax lower bound in expectation.

When $k \geq 3$, the problem is reduced to finding the minimax lower bound for a sparse normal mean estimation problem. Define the space of sparse vectors
\[
S_\ell = \left\{ \theta \in \mathbb{R}^n : \sum_{i=1}^n 1\{\theta_i \neq 0\} \leq \ell \right\}.
\]
Then, we observe that $S_{\lfloor \frac{k+1}{2} \rfloor} \subset \Theta_k$. This leads to the argument
\[
\inf_{\hat{\theta}} \sup_{\theta \in \Theta_k} \|\hat{\theta} - \theta\|^2 \geq \inf_{\hat{\theta}} \sup_{\theta \in S_{\lfloor \frac{k+1}{2} \rfloor}} \|\hat{\theta} - \theta\|^2 \geq C_1 k \log(en/k),
\]
where the last inequality above is given by Donoho and Johnstone [1994]. The proof is complete.

Proof of Theorem 3.1: Part 2. When $k = 1$, the lower bound is trivial. We state the proof for $k \geq 2$. The same space $T$ constructed in the proof of Part 1 is also a subset of $\Theta_1^\dagger$. Therefore, using the same argument, we have
\[
\inf_{\hat{\theta}} \sup_{\theta \in \Theta_2^\dagger} \mathbb{E} \|\hat{\theta} - \theta\|^2 \geq c \log \log n. \quad (42)
\]
Now we derive the lower bound for $k \geq 3$.

We first consider the case $n > C$, $k > C$ and $n/k > C$ for some sufficiently large constant $C > 0$. Define the space $\Theta_2^\dagger(\bar{n}, a, b) \subset \mathbb{R}^{\bar{n}}$ to be the class of vectors of length $\bar{n}$ that have two
non-decreasing pieces taking values $a$ and $b$ respectively. Then, construct the following space
\[ \tilde{T} = \bigtimes_{\ell=1}^{\left\lceil \frac{k}{2} \right\rceil} \tilde{T}_\ell. \]
where for $1 \leq \ell \leq \left\lceil \frac{k}{2} \right\rceil - 1$, we define
\[ \tilde{T}_\ell = \Theta_2^f \left\{ \left\lceil \frac{2n}{k} \right\rceil \left( 2\ell - 2 \right) \sqrt{2\alpha \sigma^2 \log \log n}, (2\ell - 1) \sqrt{2\alpha \sigma^2 \log \log n} \right\}, \]
and
\[ \tilde{T}_{\left\lceil \frac{k}{2} \right\rceil} = \left\{ k \sqrt{2\alpha \sigma^2 \log \log n} \left( \left\lceil \frac{2n}{k} \right\rceil - \left\lfloor \frac{2n}{k} \right\rfloor \right) \right\}^{n-\left\lceil \frac{2n}{k} \right\rceil \left( \left\lceil \frac{k}{2} \right\rceil - 1 \right)} \].
Observe that $\tilde{T} \subset \Theta_k^\perp$. Thus,
\[ \inf_{\hat{\theta}} \sup_{\theta \in \Theta_k^\perp} \mathbb{E} \| \hat{\theta} - \theta \|^2 \geq \inf_{\hat{\theta}} \sup_{\theta \in \tilde{T}} \mathbb{E} \| \hat{\theta} - \theta \|^2 \]
\[ = \inf_{\hat{\theta} = (\hat{\eta}_1, \ldots, \hat{\eta}_{k/2})} \sum_{\ell=1}^{\left\lceil \frac{k}{2} \right\rceil} \sup_{\eta_{\ell} \in \tilde{T}_\ell} \mathbb{E} \| \hat{\eta}_{\ell} - \eta_{\ell} \|^2 \]
\[ \geq \sum_{\ell=1}^{\left\lceil \frac{k}{2} \right\rceil - 1} \inf_{\hat{\eta}_{\ell} \in \tilde{T}_\ell} \mathbb{E} \| \hat{\eta}_{\ell} - \eta_{\ell} \|^2 \]
\[ \geq c_1 \left( \left\lceil \frac{k}{2} \right\rceil - 1 \right) \log \log \left\lceil \frac{2n}{k} \right\rceil \]
\[ \geq c_2 k \log \log \left( \frac{16n}{k} \right). \]
where the equality (43) is by taking advantage of the separable structure and a sufficiency argument, and the inequality (44) is by the same argument that we use to derive (42).

Secondly, we consider the rest of settings. When $n \leq C$, the rate is a constant and the result automatically holds. When $3 \leq k \leq C$, the rate $\log \log n$ is immediately a lower bound by the fact that $\Theta_2^\perp \subset \Theta_k^\perp$. When $n/k \leq C$, we have $\Theta_{n/C}^\perp \subset \Theta_k^\perp$. Therefore,
\[ \inf_{\hat{\theta}} \sup_{\theta \in \Theta_{n/C}^\perp} \mathbb{E} \| \hat{\theta} - \theta \|^2 \geq \inf_{\hat{\theta}} \sup_{\theta \in \Theta_{n/C}^\perp} \mathbb{E} \| \hat{\theta} - \theta \|^2 \geq c_3 n. \]
Hence, the proof is complete. \qed

Proof of Proposition 3.1. Recall the notation $\mathbb{E}_\theta$ that stands for the expectation associated with the probability measure $\mathbb{P}_\theta = N(\theta, \sigma^2 I_n)$. We consider the alternative set of parameters $\mathcal{F}(\rho)$ that contains vectors $\{ \theta_{\ell} \in \Theta_2, \ell \in \left\lceil \log_2 n \right\rceil \}$ that fill the last $\left\lceil n2^{-\ell} \right\rceil$ entries with $\rho \sigma \sqrt{2^\ell \log \log n} / n$ and the rest 0. Let $\mu_\rho$ be the uniform measure on $\mathcal{F}(\rho)$ and $\rho$ be some sufficiently small constant. We use the notation $\mathbb{P}_{\mu_\rho} = \int \mathbb{P}_\theta d\mu_\rho$ and $\mathbb{E}_{\mu_\rho}$ for its expectation.
Using Le Cam’s method [Yu, 1997], we have
\[
\inf_{0 \leq \phi \leq 1} \left\{ \sup_{\theta \in \Theta_1} \mathbb{E}_\theta \phi + \sup_{\theta \in \Theta_2(c)} \mathbb{E}_\theta (1 - \phi) \right\}
\geq \inf_{0 \leq \phi \leq 1} \left\{ E_0 \phi + E_{\mu_\rho}(1 - \phi) \right\}
\geq 1 - \frac{1}{2} \left\{ E_0 L^2_{\mu_\rho}(Y) - 1 \right\}^{1/2},
\]
where we set \( L_{\mu_\rho}(y) := \frac{dp_{\mu_\rho}}{dp_{\mu_0}}(y) \). The rest of this proof shows \( E_0 L^2_{\mu_\rho}(Y) = 1 + o(1) \) as \( n \to 0 \). To this end, we calculate
\[
L_{\mu_\rho}(y) = \frac{1}{[\log_2 n]^2} \sum_{\theta \in F(\rho)} \exp \left( \frac{2\theta^T y - ||\theta||^2}{2\sigma^2} \right),
\]
yielding
\[
E_0 L^2_{\mu_\rho}(Y) = \frac{1}{[\log_2 n]^2} \sum_{\theta_1, \theta_2 \in F(\rho)} \left( \frac{\theta_1^T \theta_2}{2\sigma^2} \right)
= \frac{1}{[\log_2 n]^2} \sum_{j=1}^{[\log_2 n]} \sum_{k=1}^{[\log_2 n]} \exp \left( \rho^2 2^{(j+k)/2} - 1 \log \log_2 n/n \cdot \lfloor n 2^{-\max(j,k)} \rfloor \right)
= \frac{1}{q^2} \sum_{j=1}^{q} \sum_{k=1}^{q} (q^2)^{2 - |j-k|/2 - 1} (1 + o(1)),
\]
where \( q := [\log_2 n] \). We then truncate the array \( \{ (j, k) : 1 \leq j, k \leq q \} \) to two parts: \( T_1 := \{ (j, k) : |j - k| \leq 2 \log_2 q \} \) and \( T_2 := \{ (j, k) : |j - k| > 2 \log_2 q \} \). It is immediate that \( |T_1| \asymp (\log_2 q)^2 \) and \( |T_2| = q^2(1 + o(1)) \). Then
\[
\frac{1}{q^2} \sum_{j=1}^{q} \sum_{k=1}^{q} (q^2)^{2 - |j-k|/2 - 1} = \frac{1}{q^2} \sum_{(j, k) \in T_1} (q^2)^{2 - |j-k|/2 - 1} + \frac{1}{q^2} \sum_{(j, k) \in T_2} (q^2)^{2 - |j-k|/2 - 1},
\]
with
\[
A_1 \leq C \frac{(\log_2 q)^2}{q^2} q^{2/2} = o(1),
\]
and each element in \( A_2 \) satisfying
\[
1 \leq (q^2)^{2 - |j-k|/2 - 1} \leq (q^2)^{1/2} = 1 + o(1).
\]
This yields \( E_0 L^2_{\mu_\rho}(Y) = 1 + o(1) \) and hence completes the proof. \( \square \)

Finally, we give the proof of Proposition 4.1. This requires the following result to bound the Kullback-Leibler divergence.

Lemma 6.3. Consider the density function \( p_{\gamma, a}(x) \propto \exp(-|x-a|^\gamma) \) for some \( \gamma \in (0, 2] \) and
$a \in \mathbb{R}$. Then, there exists some universal constant $C > 0$, such that

$$D(p_{\gamma,a}||p_{\gamma,b}) \leq \begin{cases} C|a - b|^{\gamma}, & \gamma \in (0, 1], \\ C(|a - b| + |a - b|^{\gamma}), & \gamma \in (1, 2]. \end{cases}$$

The proof of Lemma 6.3 is given in Section 6.3.

Proof of Proposition 4.1. Let $e_j$ be the $j$th canonical vector of $\mathbb{R}^n$. That is, the entries of $e_j$ are all 0 except that the $j$th entry is 1. Construct the space $T = \{\alpha(\log n)^{1/\gamma}e_j\}_{j=1}^n$. It is easy to see that $T \subset \Theta$. For any $\theta, \theta' \in T$, we have $\|\theta - \theta\|^{2} = 2\alpha(\log n)^{2/\gamma}$. Therefore,

$$\log \mathcal{M}\left(\sqrt{2}\alpha(\log n)^{1/\gamma}, T, \|\cdot\|\right) \geq \log n.$$

Moreover, using Lemma 6.3, we have

$$\max_{\theta, \theta' \in T} D(\mathbb{P}_\theta || \mathbb{P}_{\theta'}) \leq C_1(\alpha + \alpha\gamma) \log n.$$

Using Fano’s inequality (41), we have

$$\inf_{\theta} \sup_{\theta' \in \Theta} \mathbb{P}\left(\|\hat{\theta} - \theta\|^2 \geq 2\alpha^2(\log n)^{2/\gamma}\right) \geq 1 - \frac{C_1(\alpha + \alpha\gamma) \log n + \log 2}{\log n} \geq c,$$

as long as we choose a small enough $\alpha$. Thus, with an application of Markov’s inequality, the proof is complete. \qed

6.3 Proofs of Auxiliary Results

This section collects the proofs of Lemma 2.1, Lemma 6.1, Lemma 6.2, and Lemma 6.3.

Proof of Lemma 2.1. The first conclusion that $\hat{\theta}_i = \mathcal{X}(\hat{a}_{j-1}: \hat{a}_j)$ for all $i \in (\hat{a}_{j-1}: \hat{a}_j)$ is a direct consequence of the minimax formula for isotonic regression [Silvapulle and Sen, 2011]. Since the proofs of the next two conclusions are similar, we only state that of the first one of them.

We use a contradiction argument in the proof. That is, we show, if there exists an $t > \hat{a}_j$ such that $\hat{\theta}_{\hat{a}_j} > \mathcal{X}(\hat{a}_j: t)$, then $\hat{\theta}$ does not attain the global minimum of the objective function. To this end, we construct a new estimator

$$\hat{\theta}' = \arg\min_{\theta' \in \Theta'} \|X - \theta'\|^2.$$

The space $\Theta'$ is defined to be the class of vectors that take nondecreasing constants on the intervals $(\hat{a}_0: \hat{a}_1], ..., (\hat{a}_{j-2}: \hat{a}_{j-1}], (\hat{a}_{j-1}: t], (t: \hat{a}_{j'}], (\hat{a}_{j'}: \hat{a}_{j'+1}], ..., (\hat{a}_{k-1}: \hat{a}_k]$, where $j' = \min\{j \in [\hat{k}]: \hat{a}_j > t\}$. Our goal is to show $\|X - \hat{\theta}'\|^2 < \|X - \hat{\theta}\|^2$ under the condition $\hat{\theta}_{\hat{a}_j} > \mathcal{X}(\hat{a}_j: t)$ for some $t > \hat{a}_j$.

Using the minimax formula for isotonic regression [Silvapulle and Sen, 2011], there exists an integer $s$, such that $\hat{\theta}'$ is constant over $(\hat{a}_{j-s}: t]$, and $\hat{\theta}'$ takes the same values as $\hat{\theta}$ over $(0: \hat{a}_{j-s}]$. Moreover, the condition $\mathcal{X}(\hat{a}_{j-1}: \hat{a}_j) = \hat{\theta}_{\hat{a}_j} > \mathcal{X}(\hat{a}_j: t)$ implies that

$$\max\left\{\mathcal{X}(\hat{a}_0: \hat{a}_1], ..., \mathcal{X}(\hat{a}_{j-1}: \hat{a}_j], \mathcal{X}(\hat{a}_j: t]\right\} < \min\left\{\mathcal{X}(t: \hat{a}_{j'}], \mathcal{X}(\hat{a}_{j'}: \hat{a}_{j'+1}], ..., \mathcal{X}(\hat{a}_{k-1}: \hat{a}_k]\right\}.$$
Therefore, we have the property
\[
\min_{\theta \in \Theta} \|X - \theta\|^2 = \min_{\theta \in \Theta'[0:t]} \|X - \theta\|^2_{[0:t]} + \min_{\theta \in \Theta'[t:n]} \|X - \theta\|^2_{[t:n]},
\]
where the space \( \Theta'[0:t] \) is defined to be the class of sub-vectors \( \theta_{[0:t]} \) that take nondecreasing constants on the intervals \([\tilde{a}_0 : \tilde{a}_1], ..., [\tilde{a}_{j-2} : \tilde{a}_{j-1}], [\tilde{a}_{j-1} : t]\), and the space \( \Theta'[t:n] \) is defined in a similar way.

Using these properties of \( \tilde{\theta}' \), we have
\[
\|X - \tilde{\theta}'\|^2 = \sum_{i=1}^{\tilde{a}_{j-s}} (X_i - \tilde{\theta})^2 + \sum_{i=\tilde{a}_{j-s}+1}^{t} (X_i - \tilde{\theta})^2 + \sum_{i=t+1}^{n} (X_i - \tilde{\theta})^2
\]
\[
= \sum_{i=1}^{\tilde{a}_{j-s}} (X_i - \tilde{\theta})^2 + \min_{\theta \in \Theta'[t:n]} \|X - \theta\|^2_{[t:n]}
\]
\[
+ \sum_{\ell=j-s+1}^{j} \sum_{i=\tilde{a}_{j-s}+1}^{\tilde{a}_j} (X_i - X_{(\tilde{a}_{j-s}, \tilde{a}_j)})^2 + \sum_{i=\tilde{a}_j+1}^{t} (X_i - X_{(\tilde{a}_{j-s}, t)})^2
\]
\[
+ \sum_{\ell=j-s+1}^{j} (X_{(\tilde{a}_{j-s}, \tilde{a}_j)})^2 - (X_{(\tilde{a}_{j-s}, t)})^2.
\]
Moreover, using the properties of \( \tilde{\theta} \), we have
\[
\|X - \tilde{\theta}\|^2 = \sum_{i=1}^{\tilde{a}_{j-s}} (X_i - \tilde{\theta})^2 + \sum_{i=\tilde{a}_{j-s}+1}^{t} (X_i - \tilde{\theta})^2 + \sum_{i=t+1}^{n} (X_i - \tilde{\theta})^2
\]
\[
= \sum_{i=1}^{\tilde{a}_{j-s}} (X_i - \tilde{\theta})^2 + \|X - \tilde{\theta}\|^2_{[t:n]}
\]
\[
+ \sum_{\ell=j-s+1}^{j} \sum_{i=\tilde{a}_{j-s}+1}^{\tilde{a}_j} (X_i - X_{(\tilde{a}_{j-s}, \tilde{a}_j)})^2 + \sum_{i=\tilde{a}_j+1}^{t} (X_i - X_{(\tilde{a}_{j-s}, t)})^2
\]
\[
+ \sum_{i=\tilde{a}_j+1}^{t} (X_{(\tilde{a}_{j-s}, t)})^2 - (X_{(\tilde{a}_{j-s}, t)})^2.
\]
Hence, to show \( \|X - \tilde{\theta}'\|^2 < \|X - \tilde{\theta}\|^2 \), it is equivalent to showing
\[
\min_{\theta \in \Theta'[t:n]} \|X - \theta\|^2_{[t:n]} + \sum_{\ell=j-s+1}^{j} (\tilde{\alpha}_{\ell} - \tilde{\alpha}_{\ell-1})^2 (X_{(\tilde{a}_{j-s}, \tilde{a}_j)})^2 - (X_{(\tilde{a}_{j-s}, t)})^2 + (t - \tilde{\alpha}_j) (X_{(\tilde{a}_j, t)})^2
\]
\[
< \|X - \tilde{\theta}\|^2_{[t:n]} + \sum_{i=\tilde{a}_j+1}^{t} (X_{(\tilde{a}_j, t)})^2 - (X_{(\tilde{a}_j, t)})^2.
\]
Since \( \tilde{\theta}_{[t:n]} \in \Theta'[t:n] \), we have \( \min_{\theta \in \Theta'[t:n]} \|X - \theta\|^2_{[t:n]} \leq \|X - \tilde{\theta}\|^2_{[t:n]} \). Moreover, since for all
\( i \in (\tilde{a}_j : t], \tilde{\theta}_i \geq \underline{X}_{(\tilde{a}_j, \tilde{a}_{j+1})} \geq \underline{X}_{(\tilde{a}_j, t]} \geq \underline{X}_{(\tilde{a}_j, \tilde{a}_j)}, \) we have
\[
\sum_{i=\tilde{a}_j+1}^{t} (\underline{X}_{(\tilde{a}_j, t]} - \tilde{\theta}_i)^2 > (t - \tilde{a}_j)(\underline{X}_{(\tilde{a}_j, t]} - \underline{X}_{(\tilde{a}_j-1, \tilde{a}_j)})^2.
\]
Thus, it is sufficient to show
\[
\sum_{j=0}^{s} (\tilde{a}_j - \tilde{a}_{j-1})(\underline{X}_{(\tilde{a}_j, \tilde{a}_{j-1})} - \underline{X}_{(\tilde{a}_j, \tilde{a}_{j-1})})^2 + (t - \tilde{a}_j)(\underline{X}_{(\tilde{a}_j, t]} - \underline{X}_{(\tilde{a}_j-1, \tilde{a}_j)})^2
\]
\[
\leq (t - \tilde{a}_j)(\underline{X}_{(\tilde{a}_j, t]} - \underline{X}_{(\tilde{a}_j-1, \tilde{a}_j)})^2.
\]
We introduce the notation \( Y_0 = \underline{X}_{(\tilde{a}_j, t]} \), \( \lambda_0 = (t - \tilde{a}_j), Y_h = \underline{X}_{(\tilde{a}_j-h, \tilde{a}_j-h+1)} \) for all \( h \in [s] \), and \( \lambda_h = (\tilde{a}_j-h+1 - \tilde{a}_j-h) \) for all \( h \in [s] \). Thus, we have
\[
\underline{X}_{(\tilde{a}_j-s, t]} = \frac{\sum_{j=0}^{s} \lambda_h Y_h}{\sum_{h=0}^{s} \lambda_h},
\]
and it is equivalent to show
\[
\sum_{h=0}^{s} \lambda_h \left( Y_h - \frac{\sum_{h=0}^{s} \lambda_h Y_h}{\sum_{h=0}^{s} \lambda_h} \right)^2 \leq \lambda_0 (Y_1 - Y_0)^2.
\]
The definitions of \( s \) and \( \{Y_h\}_{h=0}^{s} \) together with the minimax formula for isotonic regression [Silvapulle and Sen, 2011] imply the condition
\[
Y_1 > Y_2 > \ldots > Y_s \geq \underline{\gamma}, \tag{45}
\]
where \( \underline{\gamma} = \frac{\sum_{h=0}^{s} \lambda_h Y_h}{\sum_{h=0}^{s} \lambda_h} \). Define the function
\[
L(Y_1, \ldots, Y_s) = \sum_{h=0}^{s} \lambda_h \left( Y_h - \underline{\gamma} \right)^2,
\]
where we treat \( Y_0 \) and \( \{\lambda_h\}_{h=0}^{s} \) as fixed. Then, using (45) for each \( h \in [s] \), we have
\[
\frac{\partial L(Y_1, \ldots, Y_s)}{\partial Y_h} = 2\lambda_k(Y_k - \underline{\gamma}) \geq 0.
\]
Hence,
\[
L(Y_1, \ldots, Y_s) \leq L(Y_1, \ldots, Y_1) = \sum_{h=0}^{s} \lambda_h \lambda_0 (Y_1 - Y_0)^2 \leq \lambda_0 (Y_1 - Y_0)^2.
\]
The proof is complete. \( \square \)

To prove Lemma 6.1, we need the following two famous maximal inequalities. The versions we present here are Corollary II.1.6 in Revuz and Yor [1999] and Theorem 1.1.1 in De la Peña and Giné [2012].

**Lemma 6.4** (Doob’s maximal inequality). Given a martingale \( \{M_i, i = 1, 2, \ldots\} \) and a scalar \( p > 1 \), we have for any \( n \geq 1 \),
\[
\left\{ \mathbb{E} \left( \max_{1 \leq i \leq n} |M_i|^p \right) \right\}^{1/p} \leq \frac{p}{p-1} \left( \mathbb{E} |M_n|^p \right)^{1/p}.
\]
Lemma 6.5 (Levy’s maximal inequality). Given \( n \) independent symmetric random variables \( X_1, \ldots, X_n \), we have for any \( x > 0 \),
\[
\Pr\left( \max_{1 \leq k \leq n} \sum_{i=1}^{k} X_i > x \right) \leq 2 \Pr\left( \sum_{i=1}^{n} X_i > x \right).
\]

Proof of Lemma 6.1. Since the proofs of the two inequalities are the same, we only state the proof of the first one. For \( a + 2^\ell - 1 \leq t \wedge b \), we observe that
\[
|Z_{(a:t \wedge b)}|^2 \leq 2|Z_{(a:(a+2^\ell-1)]}|^2 + 2 \left( \frac{1}{2^\ell-1} \right)^2 (t \wedge b - a - 2^\ell - 1)^2 |Z_{((a+2^\ell-1):t \wedge b]}|^2.
\]
This leads to the inequality
\[
\xi_+(a, b, \ell) \leq 2^{\ell+1} |Z_{(a:(a+2^\ell-1)]}|^2 + 2 \xi_+(a, b, \ell),
\]
where
\[
\xi_+(a, b, \ell) = 2^{-\ell} \max \left\{ (t \wedge b - a - 2^\ell - 1)^2 |Z_{(a+(2^\ell-1):t \wedge b]}|^2 : a + 2^\ell - 1 < t \leq b \wedge (a + 2^\ell - 1) \right\},
\]
and by convention the summation over an empty set is set to be 0. Therefore, it is sufficient to bound the sum of
\[
2 \sum_{j=1}^{k^*} \sum_{\ell \geq 1 : \ell^j - 1 + 2^{\ell-1} \leq a_j} \mathbb{E} \delta_+(a_{j-1}, a_j, \ell) \xi_+(a_{j-1}, a_j, \ell)
\]
and
\[
2 \sum_{j=1}^{k^*} \sum_{\ell \geq 1 : \ell^j - 1 + 2^{\ell-1} \leq a_j} \mathbb{E} \delta_+(a_{j-1}, a_j, \ell) 2^\ell |Z_{(a_{j-1}:(a_{j-1}+2^\ell-1) \wedge a_j]}|^2,
\]
Bounding (46). The proof consists of a symmetrization argument and a two-layer truncation argument. We first proceed with the symmetrization argument. Let \( \bar{Z} \) be an independent copy of the error vector \( Z \). Then, using Jensen’s inequality, we have
\[
\mathbb{E} \delta_+(a, b, \ell) \xi_+(a, b, \ell)
\]
\[
= \mathbb{E} \left\{ \delta_+(a, b, \ell) 2^{-\ell} \max_{t \in \mathcal{C}(a, b, \ell)} \left| \sum_{i=a+2^{\ell-1}+1}^{t} \mathbb{E}(Z_i - \bar{Z}_i | Z) \right|^2 \right\}
\]
\[
\leq \mathbb{E} \left\{ \delta_+(a, b, \ell) 2^{-\ell} \max_{t \in \mathcal{C}(a, b, \ell)} \left| \sum_{i=a+2^{\ell-1}+1}^{t} (Z_i - \bar{Z}_i) \right|^2 \right\}
\]
\[
= 2 \mathbb{E} \left\{ \delta_+(a, b, \ell) 2^{-\ell} \max_{t \in \mathcal{C}(a, b, \ell)} \left| \sum_{i=a+2^{\ell-1}+1}^{t} Y_i \right|^2 \right\},
\]
where we have used the notation
\[
\mathcal{C}(a, b, \ell) = \left\{ t : a + 2^{\ell-1} < t \leq b \wedge (a + 2^{\ell} - 1) \right\}
\]
and \( Y_i = 2^{-1/2}(Z_i - \bar{Z}_i) \). The symmetry of \( Y_i \) and moment condition (17) imply \( \mathbb{E} Y_i = 0 \).
Now we use a truncation argument to split each \( Y_i \) into two parts. That is, \( Y_i = Y_i^* + Y_i'' \).
where
\[ Y_{i\ell}' = Y_i \mathbb{1}\{Y_i^2 \leq \sigma^2 2^\ell / \ell\}, \quad \text{and} \quad Y_{i\ell}'' = Y_i \mathbb{1}\{Y_i^2 > \sigma^2 2^\ell / \ell\}. \]

By the symmetry of \(Y_i\), we have \(\mathbb{E}Y_{i\ell}' = \mathbb{E}Y_{i\ell}'' = 0\). It is easy to see the bound
\[ \mathbb{E}\delta_+(a, b, \ell) \xi_+(a, b, \ell) \leq 4\mathbb{E}\left(\delta_+(a, b, \ell) 2^{-\ell} \max_{t \in \mathcal{C}(a, b, \ell)} \left| \sum_{i=a+2^\ell-1+1}^t Y_{i\ell}' \right|^2 \right) \]
\[ + 4\mathbb{E}\left(\delta_+(a, b, \ell) 2^{-\ell} \max_{t \in \mathcal{C}(a, b, \ell)} \left| \sum_{i=a+2^\ell-1+1}^t Y_{i\ell}'' \right|^2 \right), \]
and we will bound the two terms (48) and (49) separately. We first give a bound for (49).
\[
\mathbb{E}\left(\delta_+(a, b, \ell) 2^{-\ell} \max_{t \in \mathcal{C}(a, b, \ell)} \left| \sum_{i=a+2^\ell-1+1}^t Y_{i\ell}'' \right|^2 \right)
\leq \mathbb{E}\left(2^{-\ell} \max_{t \in \mathcal{C}(a, b, \ell)} \left| \sum_{i=a+2^\ell-1+1}^t Y_{i\ell}'' \right|^2 \right)
\leq 4\mathbb{E}\left(2^{-\ell} \left| \sum_{i=a+2^\ell-1+1}^t Y_{i\ell}'' \right|^2 \right)
\leq 4 \times 2^{-\ell} \sum_{i=a+2^\ell-1+1}^{a+2^\ell-1} \mathbb{E}(Y_{i\ell}'')^2
\leq 4 \times 2^{-\ell} \sum_{i=a+2^\ell-1+1}^{a+2^\ell-1} \left(\mathbb{E}|Y_i|^{2+\epsilon}\right)^{2/(2+\epsilon)} \mathbb{P}\left(Y_i^2 > \sigma^2 2^\ell / \ell\right)^{\epsilon/(2+\epsilon)}
\leq 4 \times 2^{-\ell} \sum_{i=a+2^\ell-1+1}^{a+2^\ell-1} \mathbb{E}|Y_i|^{2+\epsilon} \left(\frac{\ell}{\sigma^2 2^\ell}\right)^{\epsilon/2}
\leq C_1 \sigma^2 \left(\frac{\ell}{2^\ell}\right)^{\epsilon/2}.
\]
We have used Doob’s maximal inequality (Lemma 6.4) to derive (50). The equality (51) is because of the fact \(\mathbb{E}Y_{i\ell}'' = 0\). Finally, we have used Hölder’s inequality and Markov’s inequality to derive (52) and (53), respectively. When \(Z_i\) are identically distributed, \(Y_i\) are
identically distributed and

\[
\sum_{\ell \geq 1} 4 \times 2^{-\ell} \sum_{i=a+2^{\ell-1}+1}^{a+2^{\ell}-1} \mathbb{E}(Y''_{i\ell})^2 \\
\leq 2 \sum_{\ell=1}^{\infty} \mathbb{E}|Y|^2 \mathbb{I}\{Y^2 > \sigma^2 2^\ell/\ell\} \\
\leq 2 \mathbb{E}|Y|^2 \sum_{\ell=1}^{\infty} \mathbb{I}\{(C_1/2) \log(e + Y^2/\sigma^2) \geq \ell\} \\
\leq C_1 \mathbb{E}|Y|^2 \log(e + Y^2/\sigma^2).
\]

Next, we are going to derive a bound for (48). For simplicity, we use the notation

\[
\eta(a, b, \ell) = 2^{-\ell} \max_{t \in C(a,b,\ell)} \left| \sum_{i=a+2^{\ell-1}+1}^{t} Y'_{i\ell} \right|^2.
\]

Notice that \(\{\eta(a_j-1, a_j, \ell)\}_{j,\ell}\) are independent across all \(j\) and \(\ell\). We first show that \(\sqrt{\eta(a, b, \ell)}\) has a mixed-type sub-Gaussian and sub-exponential tail. For any \(x > 0\), we have

\[
\mathbb{P}\left\{\sqrt{\eta(a, b, \ell)} > \sigma x \right\} \leq \mathbb{P}\left(2^{-\ell/2} \max_{t \in C(a,b,\ell)} \left| \sum_{i=a+2^{\ell-1}+1}^{t} Y'_{i\ell} \right| > \sigma x \right) \\
\leq 2 \mathbb{P}\left(2^{-\ell/2} \left| \sum_{i=a+2^{\ell-1}+1}^{a+2^{\ell}-1} Y'_{i\ell} \right| > \sigma x \right) \\
\leq 4 \exp\left( -C_2 \min\{x^2, \sqrt{x}\} \right),
\]

where we have used Levy’s maximal inequality (Lemma 6.5) and Bernstein’s inequality to derive (55) and (56), respectively. This motivates another truncation argument on \(\eta(a, b, \ell)\). That is, we consider the split \(\eta(a, b, \ell) = \eta'(a, b, \ell) + \eta''(a, b, \ell)\), where

\[
\eta'(a, b, \ell) = \eta(a, b, \ell) \mathbb{I}\{\eta(a, b, \ell) \leq \sigma^2 2^\ell\} \quad \text{and} \quad \eta''(a, b, \ell) = \eta(a, b, \ell) \mathbb{I}\{\eta(a, b, \ell) > \sigma^2 2^\ell\}.
\]

We first give a bound for \(\mathbb{E}\{\delta_+(a, b, \ell) \eta''(a, b, \ell)\}\):

\[
\mathbb{E}\{\delta_+(a, b, \ell) \eta''(a, b, \ell)\} \leq \mathbb{E}\eta''(a, b, \ell) \\
\leq \left\{\mathbb{E}\eta^2(a, b, \ell)\right\}^{1/2} \mathbb{P}\left\{\eta(a, b, \ell) > \sigma^2 2^\ell\right\}^{1/2} \\
\leq C_3 \sigma^2 \exp\left( -C_2 2^\ell \right),
\]

where the last inequality above is obtained by integrating the tail

\[
\mathbb{E}\eta^2(a, b, \ell) = \sigma^4 \int_0^{\infty} \mathbb{P}\left\{\sqrt{\eta(a, b, \ell)} > \sigma u^{1/4}\right\} du
\]

using the tail bound (56). The term \(\mathbb{E}\{\delta_+(a, b, \ell) \eta'(a, b, \ell)\}\) will be analyzed in the end.
Combining all the bounds above, we have
\[ \mathbb{E} \delta_+(a, b, \ell) \cdot \tilde{\xi}_+(a, b, \ell) \leq 4C_1 \sigma^2 \left( \frac{\ell}{2} \right)^{\ell/2} + 4C_3 \sigma^2 \exp \left( -C_2 \ell \right) + 4\mathbb{E} \left\{ \delta_+(a, b, \ell) \eta'(a, b, \ell) \right\} . \] (57)

Replacing \( a \) and \( b \) in (57) by \( a_{j-1} \) and \( a_j \), and summing up over \( l \) and \( j \), we have
\[
\sum_{j=1}^{k^*} \sum_{\ell \geq 1: a_{j-1} + 2^{\ell-1} \leq a_j} \mathbb{E} \delta_+(a_{j-1}, a_j, \ell) \cdot \tilde{\xi}_+(a_{j-1}, a_j, \ell)
\leq 4C_1 k^* \sigma^2 \sum_{\ell} \left( \frac{\ell}{2} \right)^{\ell/2} + 4C_3 k^* \sigma^2 \sum_{\ell} \exp \left( -C_2 \ell \right)
\]
\[ + 4 \sum_{j=1}^{k^*} \sum_{\ell \geq 1: a_{j-1} + 2^{\ell-1} \leq a_j} \mathbb{E} \left\{ \delta_+(a_{j-1}, a_j, \ell) \eta'(a_{j-1}, a_j, \ell) \right\} \]
\[
\leq C_4 k^* \sigma^2 + 4 \sum_{j=1}^{k^*} \sum_{\ell \geq 1: a_{j-1} + 2^{\ell-1} \leq a_j} \mathbb{E} \left\{ \delta_+(a_{j-1}, a_j, \ell) \eta'(a_{j-1}, a_j, \ell) \right\} . \] (58)

When \( Z_i \) are identically distributed, we are allowed to replace the term \( \sum_{\ell}(\ell/2)^{\ell/2} \) in the above inequality by \( C_1 \) in view of (54). We omit the proof for identically distributed \( Z_i \) in the sequel as its difference only involves another application of the above argument.

Finally, it suffices to give a bound for the second term in (58). We shorthand \( \eta'(a_{j-1}, a_j, \ell) \) by \( \eta_{j,\ell} \). Observe that
\[
\sum_{j=1}^{k^*} \sum_{\ell \geq 1: a_{j-1} + 2^{\ell-1} \leq a_j} \delta_+(a_{j-1}, a_j, \ell) \eta'(a_{j-1}, a_j, \ell)
\leq \max \left\{ \sum_{j=1}^{k^*} \sum_{\ell=1}^{[1 + \log_2 n_j]} \delta_{j,\ell} \eta_{j,\ell} : \delta_{j,\ell} \in \{0, 1\}, \sum_{j, \ell} \delta_{j,\ell} \leq \bar{k} \right\}, \] (60)

where \( n_j = a_j - a_{j-1} \) and \( \bar{k} = \min \{k, m\} \) with \( m = \sum_{j=1}^{k^*} \lfloor 1 + \log_2 n_j \rfloor \). Equation (59) leads to a union bound argument. That is, for any \( x > 0 \), we have
\[
\mathbb{P} \left\{ \sum_{j=1}^{k^*} \sum_{\ell \geq 1: a_{j-1} + 2^{\ell-1} \leq a_j} \delta_+(a_{j-1}, a_j, \ell) \eta'(a_{j-1}, a_j, \ell) > x \sigma^2 \right\}
\leq \sum_{\{\delta_{j,\ell}: \delta_{j,\ell} \in \{0, 1\}, \sum_{j, \ell} \delta_{j,\ell} \leq \bar{k}\}} \mathbb{P} \left( \sum_{j, \ell} \delta_{j,\ell} \eta_{j,\ell} > x \sigma^2 \right)
\leq \sum_{\{\delta_{j,\ell}: \delta_{j,\ell} \in \{0, 1\}, \sum_{j, \ell} \delta_{j,\ell} \leq \bar{k}\}} \exp \left( -C_2 x/2 \right) \prod_{j, \ell} \mathbb{E} \exp \left( \frac{C_2 \delta_{j,\ell} \eta_{j,\ell}}{2 \sigma^2} \right)
\leq \left( \frac{m}{\bar{k}} \right) \exp \left( -C_2 x/2 + \bar{k} \log 5 \right). \] (61)
To derive (61), note that for $\delta_{j\ell} = 1$, we have
\[
\mathbb{E}\exp\left(\frac{C_2\eta_{j\ell}}{2\sigma^2}\right) = \int_0^\infty \mathbb{P}\left\{\exp\left(\frac{C_2\eta_{j\ell}}{2\sigma^2}\right) > u\right\} du
\]
\[
\leq 1 + \int_1^\infty \mathbb{P}\left\{\sqrt{\eta_{j\ell}} > \sigma \sqrt{\frac{2\log u}{C_2}}\right\} du
\]
\[
= 1 + \int_1^{e^{C_2/2}} \mathbb{P}\left\{\sqrt{\eta(a_{j-1}, a_j, \ell)} > \sigma \sqrt{\frac{2\log u}{C_2}}\right\} du
\]
\[
\leq 1 + 4 \int_1^\infty \exp\left[-C_2 \min\left\{\frac{2\log u}{C_2}, \sqrt{\frac{2\ell \log u}{C_2}}\right\}\right] du
\]
\[
\leq 1 + 4 \int_1^\infty u^{-2} du = 5,
\] (62)
where (62) is an application of the tail bound (56). The tail bound (62) allows us to integrate out the tail and bound the expectation. That is,
\[
\sum_{j=1}^{k^*} \sum_{\ell \geq 1: a_{j-1} + 2^{\ell-1} \leq a_j} \mathbb{E}\left\{\delta_+^{(a_{j-1}, a_j, \ell)}\eta_1^{(a_{j-1}, a_j, \ell)}\right\} \leq C_5 \sigma^2 \bar{k} \log(\em/\bar{k}).
\]
In view of (58), we have
\[
\sum_{j=1}^{k^*} \sum_{\ell \geq 1: a_{j-1} + 2^{\ell-1} \leq a_j} \mathbb{E}\delta_+^{(a_{j-1}, a_j, \ell)} \bar{Z}_+^{(a_{j-1}, a_j, \ell)} \leq C_4 \sigma^2 \bar{k} + C_3 \sigma^2 \bar{k} \log(\em/\bar{k}).
\]
This gives the desired bound for (46).

**Bounding (47).** For any $\ell \geq 1$ such that $a_{j-1} + 2^{\ell-1} \leq a_j$, we have
\[
2^\ell |\bar{Z}_{(a_{j-1}:(a_{j-1}+2^{\ell-1})]}|^2
\]
\[
\leq 2^\ell \left(\frac{3}{8} |\bar{Z}_{(a_{j-1}:(a_{j-1}+2^{\ell-2})]}|^2 + \frac{3}{4} |\bar{Z}_{((a_{j-1}+2^{\ell-2}):(a_{j-1}+2^{\ell-1})]}|^2\right)
\]
\[
\leq 2^\ell \left(\frac{9}{64} |\bar{Z}_{(a_{j-1}:(a_{j-1}+2^{\ell-3})]}|^2 + \frac{9}{32} |\bar{Z}_{((a_{j-1}+2^{\ell-3}):(a_{j-1}+2^{\ell-2})]}|^2 + \frac{3}{4} |\bar{Z}_{((a_{j-1}+2^{\ell-2}):(a_{j-1}+2^{\ell-1})]}|^2\right)
\]
\[
\leq 2^\ell \frac{3}{4} \sum_{h=0}^{\ell-1} \left(\frac{3}{8}\right)^{\ell-1-h} |\bar{Z}_{((a_{j-1}+2^{h-1}):(a_{j-1}+2^{h})]}|^2.
\]
We introduce the notation
\[
u_{\ell h} = 2^{h-1} |\bar{Z}_{((a_{j-1}+2^{h-1}):(a_{j-1}+2^{h})]}|^2,
\]
Notice that $\{\nu_{\ell h}\}_{j,h}$ are independent across all $j$ and $h$. Then, we have
\[
2^\ell |\bar{Z}_{(a_{j-1}:(a_{j-1}+2^{\ell-1})]}|^2 \leq 4 \sum_{h=0}^{\ell-1} \left(\frac{3}{4}\right)^{\ell-h} \nu_{\ell h}.
\]
Therefore, (47) can be bounded by
\[
8 \sum_{j=1}^{k^*} \sum_{\ell \geq 1: a_{j-1} + 2^{\ell-1} \leq a_j} \mathbb{E}\delta_+^{(a_{j-1}, a_j, \ell)} \sum_{h=0}^{\ell-1} \left(\frac{3}{4}\right)^{\ell-h} \nu_{\ell h}.
\]
A similar double truncation argument that is used to drive (57) also gives
\[
\mathbb{E} \delta_+ (a_{j+1}, a_j, \ell) u_{jh} \leq C_9 \sigma^2 \left( \frac{h}{2k} \right)^{\ell/2} + C_7 \sigma^2 \exp (-C_8 h) + 4 \mathbb{E} \left( \delta_+ (a_{j-1}, a_j, \ell) u'_{jh} \right),
\]
where the random variable \(u'_{jh}\) satisfies \(\mathbb{E} \exp \left( \frac{tu'_{jh}}{\sigma^2} \right) \leq e^{ct} \) for all \(0 < t < c\) for some small constant \(c > 0\), and is independent across \(j\) and \(h\). Summing up over \(j, \ell, h\), we get
\[
8 \sum_{j=1}^{k^*} \sum_{\ell=1: a_{j-1} + 2^{j-1} \leq a_j} \sum_{h=0}^{\ell-1} \left( \frac{3}{4} \right)^{\ell-h} \left( C_9 \sigma^2 \left( \frac{h}{2k} \right)^{\ell/2} + C_7 \sigma^2 \exp (-C_8 h) \right) \leq C_9 \sigma^2 k^*,
\]
and
\[
32 \sum_{j=1}^{k^*} \sum_{\ell=1: a_{j-1} + 2^{j-1} \leq a_j} \mathbb{E} \delta_+ (a_{j-1}, a_j, \ell) \sum_{h=0}^{\ell-1} \left( \frac{3}{4} \right)^{\ell-h} u_{jh} \leq 32 \max \left\{ k^* \sum_{j=1}^{k^*} \sum_{\ell=1: a_{j-1} + 2^{j-1} \leq a_j} \sum_{h=0}^{\ell-1} \left( \frac{3}{4} \right)^{\ell-h} \delta_j \delta_{\ell} u'_{jh} : \delta_j, \delta_{\ell} \in \{0, 1\}, \sum_{j, \ell} \delta_j \delta_{\ell} \leq k \right\}.
\]
For every \(\{\delta_j, \delta_{\ell}\}\),
\[
\mathbb{P} \left( \sum_{j=1}^{k^*} \sum_{\ell=1: a_{j-1} + 2^{j-1} \leq a_j} \sum_{h=0}^{\ell-1} \left( \frac{3}{4} \right)^{\ell-h} \delta_j \delta_{\ell} u'_{jh} > \sigma^2 x \right) \leq e^{-\lambda x} \mathbb{E} \exp \left( \lambda \sum_{j, \ell, h} \left( \frac{3}{4} \right)^{\ell-h} \delta_j \delta_{\ell} u'_{jh} / \sigma^2 \right) = e^{-\lambda x} \prod_{j, h} \mathbb{E} \exp \left( \lambda \sum_{\ell} \left( \frac{3}{4} \right)^{\ell-h} \delta_j \delta_{\ell} u'_{jh} / \sigma^2 \right) \leq e^{-\lambda x} \exp \left( c' \lambda \sum_{j, \ell, h} \left( \frac{3}{4} \right)^{\ell-h} \delta_j \delta_{\ell} \right) \leq e^{-\lambda x} \exp \left( c_1 \lambda \sum_{j, \ell} \delta_j \delta_{\ell} \right) \leq \exp \left( -\lambda x + c_1 \lambda \tilde{k} \right),
\]
where \(\lambda\) is chosen to be a constant so that \(\lambda \sum_{\ell} \left( \frac{3}{4} \right)^{\ell-h} < c\) for all \(h\). Therefore, a union bound argument leads to
\[
\mathbb{E} \max \left\{ \sum_{j=1}^{k^*} \sum_{\ell=1: a_{j-1} + 2^{j-1} \leq a_j} \sum_{h=0}^{\ell-1} \left( \frac{3}{4} \right)^{\ell-h} \delta_j \delta_{\ell} u'_{jh} : \delta_j, \delta_{\ell} \in \{0, 1\}, \sum_{j, \ell} \delta_j \delta_{\ell} \leq \tilde{k} \right\} \leq C'' \sigma^2 \tilde{k} \log (em/\tilde{k}).
\]
Combining the bounds, we obtain \(C'_1 \sigma^2 k^* + C'_2 \sigma^2 \tilde{k} \log (em/\tilde{k})\) as an upper bound for (47). The proof is thus complete. \qed
Proof of Lemma 6.2. Since
\[ \sum_{j=1}^{\hat{k}} (\hat{a}_j - \hat{a}_{j-1})Z^2_{(\hat{a}_{j-1}; \hat{a}_j)} \]
\[ \leq \max \left\{ \sum_{j=1}^{k} (a_j - a_{j-1})Z^2_{(a_{j-1}; a_j)} : 0 = a_0 \leq a_1 \leq \ldots \leq a_{k-1} \leq a_k = n \right\}, \]
we are going to use a union bound argument. For each fixed choice of \( a_j \) for any \( x > 0 \), we are going to use a union bound argument. For each fixed choice of \( a_j \), \( \sigma^{-2} \sum_{j=1}^{k} (a_j - a_{j-1})Z^2_{(a_{j-1}; a_j)} \) is stochastically no greater than a \( \chi^2_k \) random variable. Therefore, using Lemma 1 of Laurent and Massart [2000], we have
\[ \mathbb{P} \left\{ \sigma^{-2} \sum_{j=1}^{k} (a_j - a_{j-1})Z^2_{(a_{j-1}; a_j)} > 2k + 3x \right\} \leq \exp(-x), \]
for any \( x > 0 \). By a union bound argument, we have
\[ \mathbb{P} \left\{ \sigma^{-2} \sum_{j=1}^{\hat{k}} (\hat{a}_j - \hat{a}_{j-1})Z^2_{(\hat{a}_{j-1}; \hat{a}_j)} > 2k + 3x \right\} \]
\[ \leq \sum_{\{a_j\} : 0 = a_0 \leq \ldots \leq a_k = n} \mathbb{P} \left\{ \sigma^{-2} \sum_{j=1}^{k} (a_j - a_{j-1})Z^2_{(a_{j-1}; a_j)} > 2k + 3x \right\} \]
\[ \leq \exp \left( c(k \log k - x) \right). \]
Integrating over this tail bound, we get
\[ \mathbb{E} \sum_{j=1}^{\hat{k}} (\hat{a}_j - \hat{a}_{j-1})Z^2_{(\hat{a}_{j-1}; \hat{a}_j)} \leq C \sigma^2 k \log(en/k), \]
and the proof is complete. \( \Box \)

Proof of Lemma 6.3. For \( \gamma \in (0, 1] \), we have
\[ D(p_{\gamma,a}||p_{\gamma,b}) \leq \int p_{\gamma,a}(x) \left| |x - a|^{\gamma} - |x - b|^{\gamma} \right| dx \leq |a - b|^{\gamma} \int p_{\gamma,a}(x) dx = |a - b|^{\gamma}, \]
where we have used the inequality \( |x + y|^{\gamma} \leq |x|^\gamma + |y|^\gamma \) for \( \gamma \in (0, 1] \). For \( \gamma \in (1, 2] \), we write \( \beta = \gamma - 1 \in (0, 1] \). For the function \( f(\Delta) = |x + \Delta|^{\gamma} \), its absolute derivative is \( |f'(\Delta)| = \gamma |x + \Delta|^{\beta} \). Then, \( f(\Delta) = f(0) + f'(\xi)\Delta \), where \( \xi \) is a scalar between 0 and \( \Delta \). This leads to the inequality
\[ \left| |x + \Delta|^{\gamma} - |x|^{\gamma} \right| \leq \gamma |\Delta| |x + \xi|^{\beta} \leq \gamma |\Delta| (|x|^{\beta} + |\xi|^{\beta}) \leq \gamma |\Delta| |x|^{\beta} + \gamma |\Delta|^{\gamma}. \]
Using (63), with $\Delta = a - b$, we have

\[
D(p_{\gamma,a}||p_{\gamma,b}) \leq \int p_{\gamma,a}(x)|x - a|^\gamma - |x - b|^\gamma dx \\
= \int p_{\gamma,0}(x)|x|^\gamma - |x + a|^\gamma dx \\
\leq \gamma \int p_{\gamma,0}(x)|x|^\beta dx|a - b| + \gamma \int p_{\gamma,0}(x)dx|a - b|^\gamma \\
\leq C(\parallel a - b \parallel + |a - b|^\gamma).
\]

Hence, the proof is complete. \qed

**Acknowledgements**

The authors thank Qiyang Han for carefully reading the manuscript and offering many helpful suggestions.

**References**

Amelunxen, D., Lotz, M., McCoy, M. B., and Tropp, J. A. (2014). Living on the edge: Phase transitions in convex programs with random data. *Information and Inference*, 3(3):224–294.

Arias-Castro, E., Donoho, D. L., and Huo, X. (2005). Near-optimal detection of geometric objects by fast multiscale methods. *IEEE Transactions on Information Theory*, 51(7):2402–2425.

Bellec, P. C. (2015). Sharp oracle inequalities for least squares estimators in shape restricted regression. *arXiv preprint arXiv:1510.08029*.

Bellec, P. C. and Tsybakov, A. B. (2015). Sharp oracle bounds for monotone and convex regression through aggregation. *Journal of Machine Learning Research*, 16:1879–1892.

Birgé, L. and Massart, P. (1993). Rates of convergence for minimum contrast estimators. *Probability Theory and Related Fields*, 97(1-2):113–150.

Boysen, L., Kempe, A., Liebscher, V., Munk, A., and Wittich, O. (2009). Consistencies and rates of convergence of jump-penalized least squares estimators. *The Annals of Statistics*, 37(1):157–183.

Chatterjee, S. (2014). A new perspective on least squares under convex constraint. *The Annals of Statistics*, 42(6):2340–2381.

Chatterjee, S., Guntuboyina, A., and Sen, B. (2015). On risk bounds in isotonic and other shape restricted regression problems. *The Annals of Statistics*, 43(4):1774–1800.

Chung, K. L. (2000). *A Course in Probability Theory*. Academic Press.
Csörgö, M. and Horváth, L. (1997). *Limit Theorems in Change-point Analysis*. John Wiley and Sons.

De la Pena, V. and Giné, E. (2012). *Decoupling: From Dependence to Independence*. Springer.

Donoho, D. L. (1990). Gel’fand n-widths and the method of least squares. Technical report, Stanford University.

Donoho, D. L. and Johnstone, I. M. (1994). Minimax risk over $\ell_p$-balls for $\ell_p$-error. *Probability Theory and Related Fields*, 99(2):277–303.

Fisher, W. D. (1958). On grouping for maximum homogeneity. *Journal of the American statistical Association*, 53(284):789–798.

Friedrich, F., Kempe, A., Liebscher, V., and Winkler, G. (2008). Complexity penalized M-estimation: fast computation. *Journal of Computational and Graphical Statistics*, 17(1):201–224.

Fryzlewicz, P. (2014). Wild binary segmentation for multiple change-point detection. *The Annals of Statistics*, 42(6):2243–2281.

Groeneboom, P. and Jongbloed, G. (2014). *Nonparametric Estimation under Shape Constraints*, volume 38. Cambridge University Press.

Groeneboom, P. and Wellner, J. A. (1992). *Information Bounds and Nonparametric Maximum Likelihood Estimation*, volume 19. Springer.

Han, Q. and Wellner, J. A. (2016). Multivariate convex regression: global risk bounds and adaptation. *arXiv preprint arXiv:1601.06844*.

Hao, N., Niu, Y. S., and Zhang, H. (2013). Multiple change-point detection via a screening and ranking algorithm. *Statistica Sinica*, 23(4):1553–1572.

Kim, A. K., Guntuboyina, A., and Samworth, R. J. (2016). Adaptation in log-concave density estimation. *arXiv preprint arXiv:1609.00861*.

Laurent, B. and Massart, P. (2000). Adaptive estimation of a quadratic functional by model selection. *The Annals of Statistics*, 28(5):1302–1338.

Lehmann, E. L. and Casella, G. (2006). *Theory of Point Estimation*. Springer.

Leung, G. and Barron, A. R. (2006). Information theory and mixing least-squares regressions. *IEEE Transactions on Information Theory*, 52(8):3396–3410.

Li, H., Munk, A., and Sieling, H. (2016). FDR-control in multiscale change-point segmentation. *Electronic Journal of Statistics*, 10(1):918–959.

Meyer, M. and Woodroofe, M. (2000). On the degrees of freedom in shape-restricted regression. *The Annals of Statistics*, 28(4):1083–1104.
Revuz, D. and Yor, M. (1999). *Continuous Martingales and Brownian Motion (3rd Edition)*, volume 293. Springer.

Robertson, T., Wright, E., and Dykstra, R. (1988). *Order Restricted Statistical Inference*. Wiley.

Schell, M. J. and Singh, B. (1997). The reduced monotonic regression method. *Journal of the American Statistical Association*, 92(437):128–135.

Silvapulle, M. J. and Sen, P. K. (2011). *Constrained Statistical Inference: Order, Inequality, and Shape Constraints*, volume 912. John Wiley and Sons.

van de Geer, S. (1990). Estimating a regression function. *The Annals of Statistics*, 18(2):907–924.

Wang, Y. and Chen, K. (1996). The $L_2$ risk of an isotonic estimate. *Communications in Statistics - Theory and Methods*, 25(2):281–294.

Yao, Q. (1993). Tests for change-points with epidemic alternatives. *Biometrika*, 80(1):179–191.

Yu, B. (1997). Assouad, Fano, and Le Cam. In *Festschrift for Lucien Le Cam*, pages 423–435. Springer.

Zhang, C.-H. (2002). Risk bounds in isotonic regression. *The Annals of Statistics*, 30(2):528–555.