SOME NEW NON-UNIMODAL LEVEL ALGEBRAS

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Arthur Jay Weiss

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ADVISOR: George McNinch
Abstract

In 2005, building on his own recent work and that of F. Zanello, A. Iarrobino discovered some constructions that, he conjectured, would yield level algebras with non-unimodal Hilbert functions. This thesis provides proofs of non-unimodality for Iarrobino’s level algebras, as well as for other level algebras that the author has constructed along similar lines.

The key technical contribution is to extend some results published by Iarrobino in 1984. Iarrobino’s results provide insight into some naturally arising vector subspaces of the vector space $\mathbb{R}_d$ of forms of fixed degree in a polynomial ring in several variables. In this thesis, the problem is approached by combinatorial methods and results similar to Iarrobino’s are proved for a different class of vector subspaces of $\mathbb{R}_d$.

The combinatorial methods involve the definition of a new class of matrices called $L$-Matrices, which have useful properties that are inherited by their submatrices. A particular class of square L-Matrices, associated with some specialized partially ordered sets having interesting combinatorial properties, is identified. For this class of L-Matrices, necessary and sufficient conditions are given that they
be nonsingular.

Several larger questions are discussed whose answers are incrementally improved by the knowledge that the new non-unimodal level algebras exist.
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CHAPTER 1

Introduction

In this section, which is intended to provide an overview, we use some technical terms without stopping to provide definitions. The definitions can all be found in later sections.

For over a century, mathematicians have been investigating Hilbert functions of (standard) graded quotients of polynomial rings, and the subject is still a focus of active study. In particular, among the graded quotients are the Gorenstein Artinian graded algebras, which arise in various contexts. R. Stanley defined a generalization of this class, the class of level algebras, that is useful for studying Gorenstein Artinian graded algebras, but is also interesting in its own right.

The general question for Hilbert functions of level algebras, that is, what sequences could be their Hilbert functions, is the subject of the recent paper [GHMS06], whose introduction provides an excellent history of work that has been done in this direction to date. Most of that work proceeds in different directions from what is done in this thesis.

Here, we focus on a property called unimodality that Hilbert functions of level algebras sometimes have. In studying unimodality of level algebras, it is usual to
classify them by codimension and type; and one can ask whether it is possible for a level algebra of some particular codimension and type to be non-unimodal. The following is a summary of the history so far, for which the author is indebted to A. Iarrobino, F. Zanello, and [BI92].

In codimensions 1 and 2, level algebras of all types are necessarily unimodal. The level algebras of codimension 1 are sufficiently simple that this is easy. The investigations in codimension 2 were performed by F. S. Macaulay in [Mac04] and [Mac27], written in the first several decades of the twentieth century.

The next step was in showing that Gorenstein Artinian graded algebras in codimension 3 are necessarily unimodal. This was done by R. Stanley in [Sta77], although it was D. Buchsbaum and D. Eisenbud who first determined the actual Hilbert functions in [BE77]. In [Sta78], Stanley also demonstrated a level algebra in codimension 13 that was not unimodal.

The next progress was accomplished in [BI92] by D. Bernstein and A. Iarrobino, who showed that a non-unimodal Gorenstein Artinian algebra could be found in codimension 5 and in any higher codimension.

Meanwhile, groundwork was being laid for further progress. In particular, we note the work of J. Emsalem and A. Iarrobino in [EI78], which contained some basic concepts underlying the investigation of catalecticants by A. Iarrobino in
[FL84] and differently by R. Froberg and D. Laksov in [FL84]. Investigations of non-unimodality in Gorenstein Artinian graded algebras were conducted in [B94] by M. Boij, and by M. Boij and D. Laksov in [BL94].

In 2005, F. Zanello published the first non-unimodal level algebra in codimension 3 in [Z06]. Its type is 28. Later that year, A. Iarrobino used the same general idea to produce a level algebra in codimension 3 of type 5 that, he conjectured, would prove to be non-unimodal, as well as showing how to perform a similar construction for any type higher than 5. Iarrobino also suggested methods for codimension 4 that, he conjectured, would produce non-unimodal level algebras. It is his construction in codimension 3, as well as some constructions in codimensions 3, 4, and 5 that proceed along lines suggested by his work, that are analyzed in this thesis, and shown to be non-unimodal.

\[
\begin{array}{cccccc}
  r \setminus t & 1 & 2 & 3 & 4 & 5 & \ldots \\
  1 & \text{yes} & \text{yes} & \text{yes} & \text{yes} & \text{yes} & \text{yes} \\
  2 & \text{yes} & \text{yes} & \text{yes} & \text{yes} & \text{yes} & \text{yes} \\
  3 & \text{yes} & ? & ? & ? & \text{no} & \text{no} \\
  4 & ? & ? & \text{no} & \text{no} & \text{no} & \text{no} \\
  5 & \text{no} & \text{no} & \text{no} & \text{no} & \text{no} & \text{no} \\
  \vdots & \text{no} & \text{no} & \text{no} & \text{no} & \text{no} & \text{no} \\
\end{array}
\]

Table 1. Is a Level Algebra of Codimension \( r \) and Type \( t \) Necessarily Unimodal?

As discussed in a later chapter, with a few additional observations we will be able to summarize the current state of knowledge as follows. Necessarily
Unimodal: Codimensions 1 and 2 of all types, codimension 3 of type 1. Non-unimodals exist: Codimension 3, of types 5 and greater; codimension 4, of types 3 and greater; codimension 5 and greater, of all types. Unknown: Codimension 3, types 2, 3, and 4; codimension 4, types 1 and 2.

Among the classes listed as unknown, some useful progress has been made. In particular, we note [IS05].
CHAPTER 2

Algebraic Preliminaries

1. Level Algebras

We fix \( k \), an algebraically closed field of characteristic 0. Throughout this work, it will be implicitly assumed that all our vector spaces are over the field \( k \).

Let \( R \) be the polynomial ring over \( k \) in \( r \) variables: \( R := k[X_1, ..., X_r] \). \( R \) can be written as a direct sum \( R = \bigoplus_{d \geq 0} R_d \), where the subspaces \( R_d \) consist of all homogeneous polynomials (forms) in \( R \) of degree \( d \). For every \( d \), \( R_d \) is a finite-dimensional vector space, of dimension \( \binom{d + r - 1}{r - 1} \). One basis of \( R_d \) consists of all monomials of degree \( d \). By way of notation, let \( D := (d_1, ..., d_r) \) be any \( r \)-tuple of non-negative integers such that \( d_1 + ... + d_r = d \). Then \( D \) determines a monomial \( X^D := X_1^{d_1} \cdots X_r^{d_r} \) of degree \( d \), and monomials of degree \( d \) are indexed by the \( r \)-tuples \( D \). \( D \) is sometimes called a multi-index of dimension \( r \) and degree \( d \).

When considering the monomials \( X^D \) of \( R \), we sometimes use lexicographic ordering, defined as follows. For two different multi-indexes \( C := (c_1, ..., c_r) \) and \( D := (d_1, ..., d_r) \), we say \( C \) comes before \( D \) if, in the leftmost co-ordinate for which \( c_i \neq d_i, c_i > d_i \). In this case, we write \( C > D \). By extension, we place an ordering on the monomials of \( R \): \( X^C > X^D \iff C > D \). In this definition, there is no requirement
that $X_C$ and $X_D$ be monomials of the same degree. When listing all monomials of a fixed degree $d$, to say that they are listed lexicographically means that the listing is according to lexicographical order. For example, if $R = k[X_1, X_2, X_3]$, we list the monomials of degree 2 lexicographically as follows: $X_1^2, X_1X_2, X_1X_3, X_2^2, X_2X_3, X_3^2$.

If $C := (c_1, ..., c_r)$ and $D := (d_1, ..., d_r)$ are two multi-indexes, we define their addition and subtraction coordinate wise. That is, $C + D := (c_1 + d_1, ..., c_r + d_r)$, and $C - D := (c_1 - d_1, ..., c_r - d_r)$.

The direct-sum decomposition of $R = \bigoplus_{d \geq 0} R_d$ makes it a graded $R$-module, graded by total degree, since for non-negative integers $d$ and $e$, $R_dR_e \subseteq R_{d+e}$.

Let $I = \bigoplus_{d \geq 0} I_d \subseteq R$ be a homogeneous ideal of $R$, where $I_d$ consists of all forms in $I$ of degree $d$. We form the quotient ring $A := R / I$, which is a $k$-algebra. The direct-sum decomposition $A = \bigoplus_{d \geq 0} A_d$, where $A_d := R_d / I_d$, makes $A$ both a graded $k$-algebra and a graded $R$-module. In each case, the grading is by total degree.

In considering $R$ or its graded quotients by homogeneous ideals, the only grading we will ever use is the grading by total degree, which is sometimes called standard. From now on, standard grading will always be implicitly assumed.
For any graded quotient $A = R/I$, we say $A$ is Artinian if it is finite-dimensional as a vector space. In this case, we write $A = \bigoplus_{0 \leq d \leq j} A_d$, where $j$ is the largest integer for which $A_j$ is nonzero. In writing such a direct sum decomposition, we will always assume $A_j$ is nonzero unless otherwise stated.

For an Artinian quotient $A = \bigoplus_{0 \leq d \leq j} A_d$, we define $\text{soc}(A)$, the socle of $A$, to be the annihilator of the linear part of $A$: $\text{soc}(A) := \{ a \in A | aA_1 = 0 \}$.

Soc($A$) is easily seen to be a homogeneous ideal of $A$. We remark that $A_j \subseteq \text{soc}(A)$ since $A_j A_1 \subseteq A_{j+1} = 0$, but equality need not hold.

**Example 2.1.** $A := k[X, Y]/(X^2, XY, Y^3) \simeq k \bigoplus kX \bigoplus kY \bigoplus kY^2$,

where we adopt the usual notation that for any $F \in R$, $\overline{F}$ denotes the homomorphic image of $F$ in $A = R/I$. Then $A_j = A_2 = kY^2$, and $\text{soc}(A) = kX \bigoplus kY^2$.

An Artinian quotient $A = \bigoplus_{0 \leq d \leq j} A_d$ is said to be level if $A_j = \text{soc}(A)$, and in this case we call $j$ the socle degree of $A$, and we call $t := \dim_k A_j$ the type of $A$. If the type $t = 1$, we say the level algebra $A$ is Gorenstein.

The following lemma provides an equivalent condition for a graded Artinian quotient $A = R/I$ to be level.
Lemma 2.2. The graded Artinian quotient $R/I = A = \bigoplus_{0 \leq d \leq j} A_d$ is level if and only if

(1) $\text{For all } d < j, F \in R_d - I_d \Rightarrow R_{j-d}F \not\subseteq I_j.$

Proof. Assume (1) holds. Let $\overline{F} \in A_d$ be nonzero, where $F \in R_d - I_d$ and $d < j$. We must show $\overline{F} \not\in \text{soc}(A)$. Reasoning by contradiction, if $\overline{F} \in \text{soc}(A)$, then $A_1\overline{F} = 0$, and $R_1F \subseteq I_{d+1}$, so $R_{j-d-1}R_1F \subseteq I_j$. That is, $R_{j-d}F \subseteq I_j$, contradicting (1).

Conversely, assume that $A$ is level, and let $F \in R_d - I_d$ with $d < j$. To prove (1), it suffices to prove the following statement, and then iterate $j - d$ times:

(2) $\text{For all } d < j, F \in R_d - I_d \Rightarrow R_1F \not\subseteq I_{d+1}.$

To prove (2): Since $\overline{F} \not\in \text{soc}(A)$, there exists some $L \in R_1$ with $\overline{LF} \neq 0$, that is $LF \not\in I_{d+1}$. This shows $R_1F \not\subseteq I_{d+1}$. \qed

Corollary 2.3. If $A = R/I$ is a level algebra of socle degree $j$, then $I$ is determined by $I_j$. More precisely,

For $d < j, I_d = \{ F \in R_d | R_{j-d}F \subseteq I_j \}$. \qed
2. Polynomials as Differential Operators

A good reference for the material in this section is [IK99], Appendix A.

Recalling that $k$ is an algebraically closed field of characteristic 0 and $R := k[X_1, ..., X_r]$ is a polynomial ring in $r$ variables, we define $D := k[x_1, ..., x_r]$, an isomorphic copy of $R$, where the variables $x_i$ are written in lower case to distinguish them from the variables of $R$. To distinguish elements of the two rings, we denote elements of $R$ by uppercase letters $F, G, ...$ and elements of $D$ by lowercase letters $f, g, ...$.

We let $R$ operate on $D$ according to the rule that, for $f \in D$,

$$
X_1 \ast f = \frac{\partial}{\partial x_1} f, \quad X_2 \ast f = \frac{\partial}{\partial x_2} f, \quad X_1 X_2 \ast f = \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} f,
$$

and so on, extended by linearity. We remark that this makes $D$ an $R$-module, with scalar multiplication $Ff := F \ast f$ for $F \in R, f \in D$. We say the elements of $R$ act on $D$ as differential operators. If $F$ is a homogeneous polynomial of degree $e$, we call $F \ast f$ an $e^{th}$ partial derivative of $f$.

$D$ can be written as a direct sum $\bigoplus_{d \geq 0} D_d$, where the submodules $D_d$ consist of all homogeneous polynomials (forms) in $D$ of degree $d$. For any $d$, $D_d$ is a finite-dimensional vector space, of dimension $\binom{d + r - 1}{r - 1}$. One basis consists of all monomials of degree $d$. Analogously with monomials in $R_d$, we adopt the notation
that an \( r \)-tuple \( D := (d_1, \ldots, d_r) \) of non-negative integers such that \( d_1 + \ldots + d_r = d \) determines the monomial \( x^D := x_1^{d_1} \cdots x_r^{d_r} \).

The direct-sum decomposition of \( D = \bigoplus_{d \geq 0} D_d \) does not make \( D \) a graded \( R \)-module, since it obeys a different grading rule:

\[
R_d \ast D_e \subseteq D_{d-e}.
\]

However, if we fix a value of \( d \) and consider what happens when \( R_d \) operates on \( D_d \), we can view \( R_d \) as the dual vector space of \( D_d \). Specifically, we take as basis of \( D_d \) the set of all monomials \( x^D \), where as usual \( D := (d_1, \ldots, d_r) \) and \( d_1 + \ldots + d_r = d \). Then, setting \( D! := d_1! \cdots d_r! \), we evaluate \( X^D \ast x^D = D! \); and for any other monomial \( x^{D'} \in D_d \), we evaluate \( X^D \ast x^{D'} = 0 \). In other words, \( X^D / D! \) is the dual vector to \( x^D \).

Since \( R_d \) is dual to \( D_d \), there is a perfect pairing

\[
(3) \quad R_d \times D_d \rightarrow k
\]

and in this context it makes sense to talk about perpendicular spaces. Specifically, if \( V \subseteq R_d \) is a vector subspace, \( V^\perp := \{ f \in D_d | V \ast f = 0 \} \); and if \( W \subseteq D_d \) is a vector subspace, \( W^\perp := \{ F \in R_d | F \ast W = 0 \} \).
3. Matlis Duality

The material in this section was first considered by F. S. Macaulay in his work on inverse systems in \cite{Mac94}. For a more recent treatment, see \cite{E95} or \cite{G96}.

We use the structure described in the previous section to get an alternative description of what it means for an Artinian graded algebra $A = R/I$ to be a level algebra. We begin with a definition.

If $A = R/I$ is a level algebra of socle degree $j$, then we define $\mathcal{W}_A := I_j^\perp \subseteq D_j$, a vector subspace. That is,

$$\mathcal{W}_A := \{ f \in D_j \mid I_j \ast f = 0 \}.$$

We give another characterization of the vector space $\mathcal{W}_A$.

**Lemma 2.4.** $\mathcal{W}_A = \{ f \in D_j \mid I \ast f = 0 \}$.

**Proof.** Assume $I_j \ast f = 0$, where $f \in D_j$. We must show that $I_d \ast f = 0$ for all $d$. This is surely true when $d > j$, and it is true when $d = j$ by hypothesis. If $d < j$, we argue by contradiction. Suppose, for $F \in I_d$, $F \ast f = g \in D_{j-d}$ and $g \neq 0$. Then, recalling that $R_{j-d}$ is dual to $D_{j-d}$, there is at least one vector $G \in R_{j-d}$ such that $G \ast g = 1$. Then $GF \in I_j$ and $GF \ast f \neq 0$, a contradiction.  \qed
Since $D$ is an $R$-module and $W_A \subseteq D$, it is permissible to consider $Ann_R(W_A)$, easily seen to be a homogeneous ideal of $R$. In fact, this construction just recovers $I$.

**Lemma 2.5.** Let $A = R/I$ be a level algebra. Then $Ann_R(W_A) = I$.

**Proof.** Since $W_A := I^\perp_j$, $I \ast W_A = 0$, so $I \subseteq Ann_R(W_A)$. For the other direction, we must show that, for all $d$, $[Ann_R(W_A)]_d \subseteq I_d$.

For $d > j$, $[Ann_R(W_A)]_d = R_d = I_d$.

For $d = j$, $[Ann_R(W_A)]_j = (I^\perp_j)^\perp = I_j$, the last equality being true because the pairing in (3) is perfect.

For $d < j$, we argue by contradiction. Assume $F \in R_d - I_d$ and $F \in [Ann_R(W_A)]_d$, so that $F \ast W_A = 0$. Then $R_{j-d}F \ast W_A = 0$, and $R_{j-d}F \subseteq [Ann_R(W_A)]_j = I_j$. However, by Lemma 2.2, $R_{j-d}F \not\subseteq I_j$, a contradiction. $\square$

We have defined $W_A$ to be $I^\perp_j$. We now wish to characterize $I^\perp_d$ for values of $d < j$.

**Lemma 2.6.** For $d < j$, $I^\perp_d = R_{j-d} \ast I^\perp_j$

**Proof.** For any $F, G \in R$ and $f \in D$, we have $FG \ast f = F \ast (G \ast f)$. In particular, letting $G$ range through $R_{j-d}$ and $f$ range through $I^\perp_j$, we have

(4) $\quad$ For any $F \in R_d, FR_{j-d} \ast I^\perp_j = F \ast (R_{j-d} \ast I^\perp_j)$. 

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We can equate the set of those $F \in R_d$ for which the left-hand side of (4) equals 0 with the set for which the right-hand side equals 0.

For the left-hand side,

$$\{F \in R_d | FR_{j-d} \ast I_j^\perp = 0\} = \{F \in R_d | FR_j \subseteq (I_j^\perp)^\perp = I_j\} = I_d,$$

the last equality being guaranteed by Corollary 2.3. For the right-hand side,

$$\{F \in R_d | F \ast (R_{j-d} \ast I_j^\perp) = 0\} = (R_{j-d} \ast I_j^\perp)^\perp.$$

Thus $I_d = (R_{j-d} \ast I_j^\perp)^\perp$, so $I_d^\perp = R_{j-d} \ast I_j^\perp$. □

**Corollary 2.7.** If $A = R/I$ is a level algebra, then

$$\dim_k [R/I]_d = \dim_k R_{j-d} \ast I_j^\perp.$$

□

So far we have shown that, given a level algebra $A = R/I$, we can characterize $I$ as the annihilator (in $R$) of a vector subspace $\mathcal{W}_A \subseteq \mathcal{D}_j$. We next turn the question around. Given an arbitrary vector subspace $\mathcal{W} \subseteq \mathcal{D}_j$, we can define

$$I_{\mathcal{W}} := \text{Ann}_R(\mathcal{W}).$$

It is easy to see that $I_{\mathcal{W}}$ is a homogeneous ideal. However, is $R/I_{\mathcal{W}}$ a level algebra?
**Lemma 2.8.** Let \( \mathcal{W} \subseteq D_j \) be a vector subspace. Then \( A_{\mathcal{W}} := R/I_{\mathcal{W}} := R/\text{Ann}_R(\mathcal{W}) \) is a level algebra.

**Proof.** First of all, \( A_{\mathcal{W}} \) is Artinian because, for \( d > j \), \([I_{\mathcal{W}}]_d = R_d\). To show \( A_{\mathcal{W}} \) is level, by Lemma 2.2 it is enough to establish that, given \( d < j \) and \( F \in R_d - [I_{\mathcal{W}}]_d \), we have \( R_{j-d}F \not\subseteq [I_{\mathcal{W}}]_j \). Consider such an \( F \). Since \( F \not\in I_{\mathcal{W}} \), there is some \( f \in \mathcal{W} \) such that \( F * f = g \neq 0 \). \( g \) is a nonzero element of \( D_{j-d} \), so there exists at least one \( G \in R_{j-d} \) such that \( G * g = 1 \). Thus \( GF * f = G * (F * f) \neq 0 \), and \( R_{j-d}F \not\subseteq [I_{\mathcal{W}}]_j \) as required. \( \square \)

We are now ready to state another characterization of level algebras.

**Theorem 2.9.** **Matlis Duality.** Let \( k \) be a field of characteristic 0 and let the elements of \( R := k[X_1, ..., X_r] \) act on \( D := k[x_1, ..., x_r] \) as differential operators. Then the level quotients \( R/I \) of socle degree \( j \) are in bijection with the nonzero vector subspaces \( \mathcal{W} \subseteq D_j \). Specifically, given \( I \), take \( \mathcal{W} = I_j^\perp \); and given \( \mathcal{W} \), take \( I = \text{Ann}_R(\mathcal{W}) \), which is the unique homogeneous ideal with \( I_j = \mathcal{W}^\perp \) such that \( R/I \) is level.

**Proof.** We first remark that if \( \mathcal{W} = \{0\} \), then \( \text{Ann}_R(\mathcal{W}) = R \) and \( R/\text{Ann}_R(\mathcal{W}) \) is the 0-ring, which is not of socle degree \( j \). This explains the stipulation that \( \mathcal{W} \) be nonzero.

We next remark that \([\text{Ann}_R(\mathcal{W})]_j = \{ F \in R_j | F * \mathcal{W} = 0 \} = \mathcal{W}^\perp \). By Lemma 2.8, \( R/\text{Ann}_R(\mathcal{W}) \) is level, and by Corollary 2.3, \( R/\text{Ann}_R(\mathcal{W}) \) is the only level quotient \( R/I \) such that \( I_j = \mathcal{W}^\perp \).
We define the maps \( \alpha(I) := I_j^\perp \) and \( \beta(W) := \Ann_R(W) \). We must show that 
\[ \beta\alpha(I) = I \]
for any homogeneous ideal \( I \) such that \( R/I \) is level of socle degree \( j \), and 
\[ \alpha\beta(W) = W \]
for any nonzero vector subspace \( W \subseteq D_j \). We have

\[ \beta\alpha(I) = \beta(I_j^\perp) = \Ann_R(I_j^\perp) = I, \]

where the last equality follows from Lemma 2.5. Also

\[ \alpha\beta(W) = \alpha(\Ann_R(W)) = [\Ann_R(W)]_j^\perp = W, \]

where the last equality follows because we showed that \( [\Ann_R(W)]_j = W^\perp \). □

**Lemma 2.10.** Let \( W \subseteq D_j \) be a vector subspace. Then

\[ \dim_k [A_W]_d = \dim_k R_{j-d} \ast W. \]

**Proof.** Set \( I = \Ann_R(W) \). Then by Theorem 2.9 \( W = I_j^\perp \). We substitute these values into the formula of Corollary 2.7, which is permitted because, by Lemma 2.8 \( A_W \) is a level algebra. □
CHAPTER 3

Hilbert Functions

1. Definitions and Preliminaries

As we have seen, level algebras are graded Artinian quotients of the form\(A := R/I\), where \(I\) is a homogeneous ideal of \(R := k[X_1, \ldots, X_r]\). In considering a homogeneous ideal \(I\), we will always assume that \(I\) contains no constant or linear polynomials as elements; equivalently, \(I_0 = 0\) and \(I_1 = 0\). The condition that \(I_0 = 0\) ensures that \(A\) is nonzero; the condition that \(I_1 = 0\) is equivalent to saying that \(A\) is not isomorphic (as a graded ring with standard grading) to any quotient of a polynomial ring with fewer than \(r\) variables. With this understanding, we define the codimension of \(A\) to be \(r\), the number of variables.

For a graded Artinian quotient \(A = \bigoplus_{0 \leq d \leq j} A_d\), we define its Hilbert function \(h_A : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}\) as follows: For non-negative integers \(d\), \(h_A(d) := \dim_k A_d\). When we form the level algebra \(A_W := R/\text{Ann}_R(W)\), where \(W\) is a vector subspace of \(D_j\), we may write \(h_W\) instead of \(h_{A_W}\).

Notationally, it is sometimes useful to express the Hilbert function as an \(h\)-vector, which is to say a \((j + 1)\)-tuple of values taken on. In Example 2.1, \(h_A(0) = 1, h_A(1) = 2, h_A(2) = 1\). As an \(h\)-vector, the Hilbert function is written
The Hilbert function turns out to be a useful concept in algebraic geometry. This has been known for many years, but some recent research has extended the applicability of the Hilbert function in some new ways. The details are beyond the scope of this thesis, but we describe the concept, and refer the reader to [BZ06], [Mig05], [Mig06], and [GHMS06] for basic definitions and further details.

The concept is this: one uses the Hilbert function of a graded Artinian quotient $R/I$ and of related algebras to define certain properties of $R/I$, specifically the **Uniform Position Property** (UPP), **Weak Lefschetz Property** (WLP), **Strong Lefschetz Property** (SLP), and **Unimodality**. If a projective scheme is arithmetically Cohen-Macaulay, one can discover some of its geometric properties by asking whether all Artinian reductions of its co-ordinate ring have these properties.

We are interested in the last of these properties, unimodality, as it applies to level algebras. We say the Hilbert function $h_A$ of a graded Artinian quotient $A = \bigoplus_{0 \leq d \leq j} A_d$ is **unimodal** if there is some degree $i$ such that $h_A$ is nondecreasing for values of $d$ between 0 and $i$ (inclusive), and nonincreasing for values of $d$ between $i$ and $j$ (inclusive). Otherwise, we say $h_A$ is **non-unimodal**. By extension, we say $A$ itself is unimodal or non-unimodal.
If one works with level algebras for even a small amount of time, either by hand or using a computer, it becomes immediately evident that, in some sense, non-unimodal Hilbert functions are difficult to find. One might never find one at all, unless armed with some particular strategy of construction. Based on this experience, we pose the following questions:

(1) For $r = 1, 2, 3, \ldots$, is every level algebra of codimension $r$ necessarily unimodal?

(2) If not, what is the lowest possible type $t$ of a non-unimodal level algebra of codimension $r$?

(3) For a given codimension $r$ for which type-$t$ non-unimodals exist, what is the lowest possible socle degree $j$?

For the first question, the answer is yes for $r = 1$ or $2$, no for $r \geq 3$.

For the second question, the most difficult cases are $r = 3$ and $r = 4$. This thesis describes non-unimodal level algebras in codimension 3 of type 5 or more, and non-unimodal level algebras in codimensions 4 and 5, of type 3 or more, and proves that they are in fact non-unimodal. The strategy used in constructing some of them is due A. Iarrobino, who conjectured that they would turn out to be non-unimodal; others involve minor variations on Iarrobino’s strategy of construction. For a more detailed description of the current state of play, please refer back to Chapter 1.
For the third question, very little is known, and the state of knowledge appears to be too rudimentary to attempt a comprehensive theory. We will make a few observations, but prove no results, on this subject in a later section.

2. Splicing

In trying to construct non-unimodal Hilbert functions, one strategy is to build them up out of smaller pieces. We perform our constructions in the polynomial rings \( R \) and \( D \) with \( r \) variables, and we focus on the \( j^{th} \) graded piece \( D_j \) of \( D \). Always, the level algebras constructed will turn out to have codimension \( r \) and socle degree \( j \); so when we choose values for \( r \) and \( j \) we will say we are fixing the codimension and fixing the socle degree.

We start by fixing the codimension \( r \) and the socle degree \( j \). We consider two vector subspaces \( V, W \subseteq D_j \), and for convenience we require that \( V \cap W = \{0\} \), so that \( V + W = V \oplus W \), an internal direct sum. If we know the Hilbert functions \( h_V \) and \( h_W \) of the level algebras \( A_V \) and \( A_W \), it is reasonable to hope that the Hilbert function \( h_{V \oplus W} \) of \( A_{V \oplus W} \) will be related to \( h_V \) and \( h_W \). A first step is provided by the following lemma.

**Lemma 3.1.** Fix codimension \( r \) and socle degree \( j \). Let \( V \) and \( W \) be two vector subspaces of \( D_j \) such that that \( V \cap W = \{0\} \). Then for any \( d \),

\[
\tag{6} h_{V \oplus W}(d) \leq h_V(d) + h_W(d),
\]

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with equality if and only if $R_{j-d} \ast \mathcal{V} \cap R_{j-d} \ast \mathcal{W} = \{0\}$.

**Proof.** From Theorem 2.9 and Corollary 2.7,

\[
\begin{align*}
  h_{\mathcal{V} \oplus \mathcal{W}}(d) &= \dim_k R_{j-d} \ast [\mathcal{V} \oplus \mathcal{W}]. \\
  h_{\mathcal{V}}(d) &= \dim_k R_{j-d} \ast \mathcal{V}. \\
  h_{\mathcal{W}}(d) &= \dim_k R_{j-d} \ast \mathcal{W}.
\end{align*}
\]

The lemma then follows from the observation that

\[
R_{j-d} \ast [\mathcal{V} \oplus \mathcal{W}] = R_{j-d} \ast \mathcal{V} + R_{j-d} \ast \mathcal{W}.
\]

\[\square\]

**Example 3.2.** $r = 3, R = k[X,Y,Z], \mathcal{D} = k[x,y,z], j = 6, \mathcal{V} = \langle x^3y^3 \rangle, \mathcal{W} = \langle x^3z^3 \rangle$.

For $d = 6,5,4$, we have $R_{6-d} \ast \mathcal{V} \cap R_{6-d} \ast \mathcal{W} = \{0\}$ since $y$ divides every element of $R_{6-d} \ast \mathcal{V}$ but divides no nonzero element of $R_{6-d} \ast \mathcal{W}$.

For $d = 3,2,1,0$, we have $R_{6-d} \ast \mathcal{V} \cap R_{6-d} \ast \mathcal{W} \neq \{0\}$, since $x^d$ is in the intersection.

One constraint on the dimension of $R_{j-d} \ast [\mathcal{V} \oplus \mathcal{W}]$ is that it cannot exceed the dimension of $\mathcal{D}_d$, of which it is a subspace:

\[
(7) \quad h_{\mathcal{V} \oplus \mathcal{W}}(d) \leq \dim_k \mathcal{D}_d.
\]

This places an immediate limitation on the choices of $\mathcal{V}$ and $\mathcal{W}$ that give equality.
In [I84], A. Iarrobino proved a result of which the following is a special case. To state the result, we use the word *general* in the sense of algebraic geometry, that is, to say that a statement is true for general \( f \) means that the statement is true for \( f \) lying in some dense Zariski-open subset of \( D_j \), regarded as an affine variety. (See, for example, [Sha94].) We will be more precise about this notion later on.

**Theorem 3.3.** With notation as above, let \( \mathcal{V} \) be arbitrary and let \( \mathcal{W} := \langle f \rangle \subseteq D_j \), the one-dimensional subspace generated by the single element \( f \). Then for general \( f \in D_j \),

\[
h_{\mathcal{V} \oplus \mathcal{W}}(d) = \min(h_{\mathcal{V}}(d) + h_{\mathcal{W}}(d), h_{D}(d)).
\]

\[\square\]

In other words, for general \( f \in D_j \), \( h_{\mathcal{V} \oplus \mathcal{W}}(d) \) is as large as it could possibly be, subject to (7).

We will not rely on Theorem 3.3 because we will sometimes want to choose non-general \( f \in D_j \). Instead, we will prove similar-looking results, in contexts where \( \mathcal{V} \), rather than being arbitrary, is required to satisfy specified conditions.

In order to use Theorem 3.3 or anything similar, it is of course desirable to know the Hilbert function \( h_{\mathcal{V}} \). To this end, we quote another theorem of Iarrobino from [I84] (and others in [FL84] and [G78]). For details, see, for example, [IK99].
THEOREM 3.4. With the notation above, let $W := \langle f \rangle \subseteq D_j$. Then for general $f \in D_j$

(8) \[ h_W(d) = \min(\dim_k R_{j-d}, \dim_k D_d). \]

□

We will be proving this theorem (by different methods) and extending it in a later chapter. For now, to see better what is involved, we work an explicit example.

We let $j = 3$ and write $R = k[X, Y, Z], D = k[x, y, z]$.

Any $f \in D_3$ can be written

(9) \[ f = Ax^3 + Bx^2y + Cx^2z + Dxy^2 + Eyz + Fxz^2 + Gy^3 + Hy^2z + Iyz^2 + Jz^3, \]

where $A, B, ..., J \in k$ are the co-ordinates of $f \in D_3$, a 10-dimensional vector space of which the monomials form a basis.

Setting $W := \langle f \rangle$, let us compute $h_W(2)$, which is the same as the dimension of $R_{j-d} * f = R_1 * f = \langle X * f, Y * f, Z * f \rangle$. (Here we have put $d = 2$, so $j - d = 1$.) We compute $X * f, Y * f$, and $Z * f$.

\[ X * f = \frac{\partial f}{\partial x} = 3Ax^2 + 2Bxy + 2Cxz + Dy^2 + Eyz + Fz^2. \]
\[ Y \ast f = \frac{\partial f}{\partial y} = Bx^2 + 2Dxy + Exz + 3Gy^2 + 2Hyz + Iz^2. \]

\[ Z \ast f = \frac{\partial f}{\partial z} =Cx^2 + Exy + 2Fxz + Hy^2 + 2Iyz + 3Jz^2. \]

Note that in writing the three partial derivatives, we have listed the six monomials of degree 2 lexicographically across the page; and we have listed the three monomials of degree 1 (\(X, Y,\) and \(Z\)), which determine which partial derivative is being taken, lexicographically down the page.

To determine the dimension of \(R_1 \ast f = (X \ast f, Y \ast f, Z \ast f)\), one must compute the rank of the \(3 \times 6\) coefficient matrix

\[
\begin{bmatrix}
3A & 2B & 2C & D & E & F \\
B & 2D & E & 3G & 2H & I \\
C & E & 2F & H & 2I & 3J
\end{bmatrix}
\]

Of course, \(3 = \dim_k R_{j-d}\), and \(6 = \dim_k D_d\); and Theorem 3.4 is saying that the coefficient matrix has maximal rank for general \(f \in D_j\).

To generalize the context of Theorem 3.4, we consider what might happen if \(f\) were defined to be, not a member of the whole space \(D_j\), but instead a member of some vector subspace \(W_{\mathcal{M}}\) generated by monomials. For example, let \(W_{\mathcal{M}} := \langle xy^2, xyz, xz^2, y^3, y^2z, yz^2, z^3 \rangle\). Then any member \(f \in W_{\mathcal{M}}\) can be written

\[ f = Dxy^2 + Exyz + Fxz^2 + Gy^3 + Hy^2z + Iyz^2 + Jz^3, \]
where we have retained the same coefficient names for purposes of comparison.

Then

\[ X \ast f = \frac{\partial f}{\partial x} = Dy^2 + Eyz + Fz^2. \]

\[ Y \ast f = \frac{\partial f}{\partial y} = 2Dxy + Exz + 3Gy^2 + 2Hyz + Iz^2. \]

\[ Z \ast f = \frac{\partial f}{\partial z} = Exy + 2Fxz + Hy^2 + 2Iyz + 3Jz^2. \]

The matrix of coefficients is now

\[
\begin{bmatrix}
0 & 0 & D & E & F \\
2D & E & 3G & 2H & I \\
E & 2F & H & 2I & 3J
\end{bmatrix}
\]

Alternatively, we could have observed that (10) is obtained from (9) by substituting \( A = 0, B = 0, C = 0 \), so the new matrix of coefficients is obtained from the old one by making the same set of substitutions (and then deleting the column that consists entirely of zeroes).

As a third example, we modify the previous example. We retain the definition of \( \mathcal{W}_M \), but this time we let \( \mathcal{W} := \langle f_1, f_2 \rangle \) be generated by two vectors of \( \mathcal{W}_M \), denoted

\[ f_1 = D_1xy^2 + E_1xyz + F_1xz^2 + G_1y^3 + H_1y^2z + I_1yz^2 + J_1z^3 \]
and
\[ f_2 = D_2xy^2 + E_2xyz + F_2xz^2 + G_2y^3 + H_2y^2z + I_2yz^2 + J_2z^3. \]

Then, for \( i = 1, 2 \), we have
\[
X \ast f_i = \frac{\partial f_i}{\partial x} = 3A_ix^2 + 2B_ixy + 2C_ixz + D_iy^2 + E_iyz + F_iz^2.
\]
\[
Y \ast f_i = \frac{\partial f_i}{\partial y} = B_ix^2 + 2D_ixy + E_ixz + 3G_iy^2 + 2H_iyz + I_iz^2.
\]
\[
Z \ast f_i = \frac{\partial f_i}{\partial z} = C_ix^2 + E_ixy + 2F_ixz + H_iy^2 + 2I_iyz + 3J_iz^2.
\]

and the matrix of coefficients is now
\[
\begin{bmatrix}
0 & 0 & D_1 & E_1 & F_1 \\
0 & 0 & D_2 & E_2 & F_2 \\
2D_1 & E_1 & 3G_1 & 2H_1 & I_1 \\
2D_2 & E_2 & 3G_2 & 2H_2 & I_2 \\
E_1 & 2F_1 & H_1 & 2I_1 & 3J_1 \\
E_2 & 2F_2 & H_2 & 2I_2 & 3J_2
\end{bmatrix}
\]

We remark that adding another generator of \( \mathcal{W}_M \) created new rows in the
matrix of coefficients without changing the number of columns.

Searching for a generalization of Theorem 3.4 it is logical to look more closely
at matrices of coefficients and ask whether they provide a means for computing
values of the Hilbert function of \( A_\mathcal{W} \).
CHAPTER 4

L-Matrices

1. Definitions and Preliminaries

Recall that $k$ is an algebraically closed field of characteristic 0. Let $B := k[z_1, ..., z_n]$ be a polynomial ring. Then a matrix $U$ is called PV-matrix over $B$ if every nonzero entry of $U$ has the form $\lambda z_i$, where $\lambda$ is a positive integer and $1 \leq i \leq n$.

A variable $z_i$ in a PV-matrix $U = (u_{ij})$ over $k[z_1, ..., z_n]$ is said to move to the left in $U$ if, whenever the variable $z_i$ appears in both $u_{ij_1}$ and $u_{ij_2}$, then $i_1 < i_2 \iff j_1 > j_2$. In more precise language: if $u_{ij_1} = \lambda_1 z_i$ and $u_{ij_2} = \lambda_2 z_i$ with $\lambda_1$ and $\lambda_2$ positive integers, then $i_1 < i_2 \iff j_1 > j_2$.

**Lemma 4.1.** Let $U$ be a PV-matrix in which the variable $z_i$ moves to the left. Then $z_i$ does not appear twice in the same row of $U$ or twice in the same column of $U$. If $z_i$ appears in two distinct rows, its column in the lower row will be to the left of its column in the higher row.

**Proof.** These results follow directly from the definitions. \qed

A PV-matrix $U$ over $k[z_1, ..., z_n]$ in which all variables move to the left is called an L-matrix.
For example, the three coefficient matrices worked as examples in the previous chapter are PV-matrices over \( k[A, B, C, ..., J] \). Since all variables move to the left, they are also L-matrices.

We will be proving several results about the ranks of PV-matrices. The following lemma is crucial to their proofs.

**Lemma 4.2.** Let \( U \) be a PV-matrix over a polynomial ring \( B \) and let \( T \) be a submatrix of \( U \). Then \( T \) is a PV-Matrix over \( B \). Any variable that moves to the left in \( U \) moves to the left in \( T \). If \( U \) is an L-matrix, so is \( T \).

**Proof.** The definitions of PV-matrix and of variables moving to the left put conditions on the entries of \( U \), and it is immediate that \( T \) inherits them from \( U \). □

We remark that it is unusual for some useful property of a class of matrices to be inherited by its submatrices.

**Lemma 4.3.** Let \( U \) be a square \( s \times s \) PV-matrix over \( k[z_1, ..., z_n] \) with block decomposition

\[
\begin{bmatrix}
A & * \\
* & Z
\end{bmatrix}
\]

where \( A \) is a square \( q \times q \) matrix with nonzero determinant (when \( q > 0 \)), \( Z \) is a square \( r \times r \) matrix whose entries on the main diagonal are all nonzero and all contain variables that move to the left in \( U \), and the entries of blocks marked "*" are not restricted. (To be precise, we assume \( r \geq 0, q \geq 0, s := q + r \geq 1 \). Then the determinant \( D(z_1, ..., z_n) \) of \( U \) is a nonzero polynomial.
PROOF. For any non-negative integer $q$, we prove the lemma for matrices $U$ for which $A$ is a $q \times q$ matrix. Let $U = (u_{ij})$, an $s \times s$ matrix. We perform induction on $s$. If $q = 0$, we start the induction with $s = 1$, in which case $U$ has a single nonzero entry, and the determinant must necessarily be nonzero. If $q > 0$, we start the induction with $s = q$, in which case $U = A$, whose determinant is nonzero by hypothesis.

For the induction step, we assume the result proved for $A$ a $q \times q$ matrix and $U$ an $(s-1) \times (s-1)$ matrix, and we prove it for $A$ a $q \times q$ matrix and $U$ an $s \times s$ matrix.

Let $\Sigma_s$ denote the symmetric group on $s$ letters, and recall that

$$D(z_1, \ldots, z_n) := \sum_{\sigma \in \Sigma_s} \text{sgn} (\sigma) \prod_{1 \leq i \leq s} u_{i \sigma(i)}, \tag{11}$$

where as usual $\text{sgn}(\sigma)$ is $+1$ if $\sigma$ is an even permutation, $-1$ if $\sigma$ is an odd permutation.

Let $u_{ss} = \lambda z_k$. Since the variable $z_k$ moves to the left, we claim it can appear only in the entry $u_{ss}$ of $U$: Suppose $z_k$ appears in $u_{ij}$. If $i < s$, then $j > s$, which is impossible; if $j < s$, then $i > s$, which is again impossible.
If we wish to compute those terms of \( D(z_1, ..., z_n) \) in which \( z_k \) appears, we must take, in (11), those \( \sigma \) for which \( \sigma(s) = s \). Collecting these terms together into a polynomial \( P(z_1, ..., z_n) \), and letting \( U' \) denote the \((s-1) \times (s-1)\) submatrix of \( U \) formed by the first \((s-1)\) rows and the first \((s-1)\) columns, we have

\[
P(z_1, ..., z_n) = u_{ss} \sum_{\sigma \in \Sigma_{s-1}} \text{sgn}(\sigma) \prod_{1 \leq i \leq s-1} u_{i\sigma(i)}
\]

\[
= u_{ss} \det(U').
\]

\( \operatorname{Det}(U') \) is nonzero by induction and \( u_{ss} \) by hypothesis. This shows that that \( P(z_1, ..., z_n) \), and hence \( D(z_1, ..., z_n) \), is nonzero. \( \square \)

**Corollary 4.4.** Let \( U \) be a square PV-matrix over \( k[z_1, ..., z_n] \) with block decomposition

\[
\begin{bmatrix}
A & * \\
* & Z
\end{bmatrix}
\]

where \( A \) is a square \( q \times q \) matrix with nonzero determinant (when \( q > 0 \)), \( Z \) is a square \( r \times r \) matrix with nonzero entries on the main diagonal, and the entries of blocks marked “*” are not restricted. We assume \( q + r \geq 1 \). If \( U \) is either (a) an L-matrix or (b) the result of permuting the first \( q \) rows and the first \( q \) columns of an L-matrix, then the determinant \( D(z_1, ..., z_n) \) of \( U \) is nonzero.

**Proof.** For case (a), if \( U \) is an L-matrix, the variables on the main diagonal of \( Z \) move to the left, and the result follows immediately from Lemma 4.3. For case (b), if \( U \) is the result of permuting the first \( q \) rows and the first \( q \) columns of an
L-matrix $U'$, $U'$ is of the form

$$\begin{bmatrix}
A' & \ast \\
\ast & Z
\end{bmatrix}$$

where $\det(A') = \pm \det(A)$, which is nonzero by hypothesis. Thus, by part (a), $\det(U')$ is nonzero. But $\det(U) = \pm \det(U')$. □

**Corollary 4.5.** Let $U$ be a square $s \times s$ PV-matrix of block form

$$\begin{bmatrix}
0 & A' & \ast \\
B' & C' & \ast \\
\ast & \ast & Z'
\end{bmatrix}$$

where 0 denotes a block of zeroes, $A'$ is a square $q \times q$ matrix with nonzero determinant, $B'$ is a square $r \times r$ matrix with nonzero determinant, $Z'$ is a square matrix with nonzero entries on the main diagonal, and the entries of blocks marked “*” are not restricted. If $U$ is either (a) an L-matrix, or (b) the result of permuting the first $q + r$ rows and the first $q + r$ columns of an L-matrix, then $U$ has nonzero determinant.

**Proof.** This follows from Corollary 4.4, taking $A$ to be the submatrix formed by joining the blocks $0, A', B'$ and $C'$. □

2. PV-Matrices as Parameterized Families

Let $U$ be a $q \times r$ PV-matrix over $k[z_1, \ldots, z_n]$ and let $C = (c_1, \ldots, c_n) \in k^n$. Substituting $c_i$ for each $z_i$ in $U$, we obtain the matrix $U(C)$, a matrix with entries in $k$. It is therefore possible to view $U$ as a family $\{U(C)|C \in k^n\}$ of matrices with
entries in $k$. We wish to translate the notion of $U$ having the maximal possible rank $\min(q, r)$ into a statement about the family $\{U(C) | C \in k^n\}$. We use the word \textit{general} in the sense of algebraic geometry: regarding $k^n$ as an affine variety, a statement is true for general $C \in k^n$ if it is true for all $C$ contained in a dense Zariski-open subset of $k^n$.

\textbf{Lemma 4.6.} Let $U$ be a matrix with entries in $k[z_1, \ldots, z_n]$ having maximal rank. Then for general $C \in k^n$, $U(C)$ has maximal rank.

\textbf{Proof.} We must show there is a dense Zariski-open subset of $k^n$ on which $U(C)$ has maximal rank. In fact, we will show that $V := \{C \in k^n | U(C) \text{ has maximal rank}\}$ is itself a dense Zariski-open set.

Since $k$ is algebraically closed, $k^n$ is an irreducible affine variety. Since $k^n$ is irreducible, any non-empty open subset is dense.

Having maximal rank is equivalent to there being at least one maximal square submatrix with nonzero determinant. Let $M_1, \ldots, M_m$ be the finitely many maximal square submatrices of $U$. For $i = 1, \ldots, m$, let $D_i \in k[z_1, \ldots, z_n]$ be the determinant of $M_i$ and let $V_i := \{C \in k^n | D_i(C) \neq 0\}$. Then $V = \bigcup_{1 \leq i \leq m} V_i$, so it is enough to show that each $V_i$ is Zariski-open and that at least one of them is nonempty (hence dense).
\( V_i \) is Zariski-open because it is the complement of the zero-set of \( D_i \), a polynomial in \( k[z_1, ..., z_n] \). By hypothesis, at least one of the \( D_i \) is nonzero, say \( D_{i_0} \). Since \( D_{i_0} \) is a nonzero polynomial and \( k \) is infinite, there is some \( C \in k^n \) such that \( D_{i_0}(C) \neq 0 \). That is, \( C \in V_{i_0} \) and \( V_{i_0} \) is nonempty.

Before leaving this section, we remind the reader of the rule for combining two (and by iteration, finitely many) statements, each of which is true for general \( C \in k^n \).

**Lemma 4.7.** Let \( S_1 \) and \( S_2 \) be two statements, each of which is true for general \( C \in k^n \). Then, for general \( C \in k^n \), \( S_1 \) and \( S_2 \) are simultaneously true.

**Proof.** \( S_1 \) is true on some Zariski-open dense set \( V_1 \); \( S_2 \) is true on some Zariski-open dense set \( V_2 \). \( V_1 \) and \( V_2 \) being dense, \( V_1 \cap V_2 \) is dense as well. That is, \( V_1 \cap V_2 \) is a Zariski-open dense set on which \( S_1 \) and \( S_2 \) are both true. \qed
CHAPTER 5

Combinatorial Preliminaries

1. Partially Ordered Sets

A partially ordered set or poset is a set $S$ together with a binary relation $\succeq$ satisfying the following three properties (See [vLW92]):

(i) (Reflexivity) For any $a \in S$, $a \succeq a$.

(ii) (Transitivity) For any $a, b, c \in S$, if $a \succeq b$ and $b \succeq c$, then $a \succeq c$.

(iii) (Antisymmetry) For any $a, b \in S$, if $a \succeq b$ and $b \succeq a$, then $a = b$.

If, for $a, b \in S$, $a \succeq b$ but $a \neq b$, we write $a \succ b$.

In this work, the only partially ordered sets we will be considering are finite and nonempty. Whenever we use the phrase partially ordered set, we will mean a finite, nonempty partially ordered set.

We say two partially ordered sets $S_1$ and $S_2$ are isomorphic if there exists a bijection $\beta : S_1 \to S_2$ such that, for all $a, b \in S_1$, $a \succeq b \iff \beta(a) \succeq \beta(b)$.

Let $S$ be a partially ordered set and let $T \subseteq S$. We say $T$ is a co-ideal or filter or topset if the following condition is satisfied:

(12) If $a, b \in S, a \in T$, and $b \succeq a$, then $b \in T$. 

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Similarly, let $S$ be a partially ordered set and let $B \subseteq S$. We say $B$ is an *ideal* or *bottomset* if the following condition is satisfied:

(13) If $a, b \in S, b \in B,$ and $b \succeq a$, then $a \in B$.

We collect some properties of topsets and bottomsets into a lemma for future use.

**Lemma 5.1.** Let $S$ be a partially ordered set.

1. $T$ is a topset of $S$ if and only if $B := S - T$ is a bottomset of $S$.
2. If $T_1$ and $T_2$ are topsets of $S$, then $T_1 \cup T_2$ and $T_1 \cap T_2$ are topsets of $S$, and $T_1 \cap T_2$ is a topset of $T_1$.
3. Let $X \subseteq S$ be any subset. Then $T_X := \{ a \in S | \text{for some } x \in X, a \succeq x \}$ is a topset of $S$ and $B_X := \{ a \in S | \text{for some } x \in X, x \succeq a \}$ is a bottomset of $S$.
4. Let $U$ be a topset of $S$ and let $C := S - U$.
   (a) Let $T$ be a topset of $S$. Then $T \cap C$ is a topset of $C$.
   (b) Let $W$ be a topset of $C$. Then
      (i) $W = T_W \cap C$.
      (ii) $\{\text{topsets of } C\} = \{T \cap C | T \text{ is a topset of } S\}$.
      (iii) $U \cup W$ is a topset of $S$.
      (iv) Let $B$ be a bottomset of $U \cup W$. Then $B \cap U$ is a bottomset of $U$, and $W - B$ is a topset of $C$.
   (c) Let $B$ be a bottomset of $S$. Then $U \cap B$ is a bottomset of $U$.
   (d) Let $D$ be a bottomset of $U$. Then
(i) \( D = B_D \cap U \).

(ii) \( \{ \text{bottomsets of } U \} = \{ B \cap U \mid B \text{ is a bottomset of } S \} \).

(5) Recalling that \( S \) is finite by definition, let \( \{ a_1, ..., a_n \} \) be the finite set of all minimal elements of \( S \). For each \( i = i, ..., n \), let \( T_i \) be a topset of \( S \) containing \( a_i \). Then \( S = \bigcup_{1 \leq i \leq n} T_i \).

Proof. (1) Assume \( T \) is a topset. Let \( a, b \in S, b \in B, b \succeq a \). We must show \( a \in B \). Since \( T \) is a topset, \( b \notin T \Rightarrow a \notin T \). That is, \( a \in B \).

Assume \( B \) is a bottomset. Let \( a, b \in S, a \in T, b \succeq a \). We must show \( b \in T \). Since \( B \) is a bottomset, \( a \notin B \Rightarrow b \notin B \). That is, \( b \in T \).

(2) Assume \( b \succeq a \). For \( i = 1, 2 \), \( T_i \) is a topset, so if \( a \in T_i \) then \( b \in T_i \). That is, if \( a \) is a member of both \( T_1 \) and \( T_2 \), so is \( b \); and if \( a \) is a member of \( T_1 \) or \( T_2 \), so is \( b \).

Let \( a \in T_1 \cap T_2, b \in T_1, b \succeq a \). Then \( b \in T_1 \cap T_2 \) since \( T_1 \cap T_2 \) is a topset of \( S \).

(3) Assume \( a \in T_X \) and \( b \succeq a \). We must show \( b \in T_X \). Since \( a \in T_X, a \succeq x \) for some \( x \in X \). That is, \( b \succeq a \succeq x \), and by transitivity \( b \succeq x \), thus \( b \in T_X \).

Assume \( b \in B_X \) and \( b \succeq a \). We must show \( a \in B_X \). Since \( b \in B_X, x \succeq b \) for some \( x \in X \). That is, \( x \succeq b \succeq a \), and by transitivity \( x \succeq a \), thus \( a \in B_X \).

(4)(a) Assume \( a, b \in C, b \succeq a, a \in T \cap C \). We must show \( b \in T \cap C \). But \( b \in T \) because \( T \) is a topset, and \( b \in C \) by hypothesis.
(4)(b)(i) We first show \( W \subseteq T_W \cap C \). \( W \subseteq T_W \) since, by reflexivity, \( w \succeq w \) for all \( w \in W \); and \( W \subseteq C \) by hypothesis.

For the other direction, let \( a \in T_W \cap C \). Since \( a \in T_W \), there is some \( w \in W \) for which \( a \succeq w \). Since \( a \in C \) and \( W \) is a topset of \( C \), this gives \( a \in W \).

(4)(b)(ii) This follows from the previous results. If \( W \) is a topset of \( C \), then \( W = T_W \cap C \) and \( T_W \) is a topset of \( S \). If \( T \) is a topset of \( S \), then \( T \cap C \) is a topset of \( C \).

(4)(b)(iii) Let \( a \in U \cup W, b \in S \), and \( b \succeq a \). We must show \( b \in U \) or \( b \in W \). If \( a \in U \), then since \( U \) is a topset of \( S \), \( b \in U \). If \( a \in W \), then either \( b \in C \), in which case \( b \in W \), since \( W \) is a topset of \( C \); or else \( b \notin C \), in which case \( b \in S - C = U \).

(4)(b)(iv) For the first assertion, let \( b \in B \cap U, a \in U, b \succeq a \). We must show \( a \in B \cap U \), so it is enough to show \( a \in B \). This follows from \( B \) being a bottomset of \( U \cup W \): \( a, b \in U \cup W, b \in B \), and \( b \succeq a \).

For the second assertion, let \( a \in W - B, b \in C, b \succeq a \). We must show \( b \in W - B \). That \( b \in W \) follows from \( W \) being a topset of \( C \): \( a \in W, b \in C, b \succeq a \). That \( b \notin B \) follows from \( B \) being a bottomset of \( U \cup W \): \( a \in U \cup W, b \succeq a, a \notin B \); if \( b \) were an element of \( B \), \( a \) would also have to be an element of \( B \), which it is not.
(4)(c) Assume \(a, b \in U, b \in U \cap B, b \succeq a\). We must show \(a \in U \cap B\). But \(a \in B\) because \(B\) is a bottomset, and \(a \in U\) by hypothesis.

(4)(d)(i) We first show \(D \subseteq B_D \cap U\). \(D \subseteq B_D\) since, by reflexivity, \(d \succeq d\) for all \(d \in D\); and \(D \subseteq U\) by hypothesis.

For the other direction, let \(a \in B_D \cap U\). Since \(a \in B_D\), there is some \(d \in D\) for which \(d \succeq a\). Since \(a \in U\) and \(D\) is a bottomset of \(U\), this gives \(a \in D\).

(4)(d)(ii) This follows from the previous results. If \(D\) is a bottomset of \(U\), then 
\[D = B_D \cap U\] and \(B_D\) is a bottomset of \(S\). If \(B\) is a bottomset of \(S\), then \(B \cap U\) is a bottomset of \(U\).

(5) Let \(a \in S\), which is a finite set. We claim that there is a minimal \(a_i \in S\) such that \(a \succeq a_i\). Assuming the claim, \(a \in T_i\) because \(T_i\) is a topset, and we are done.

To prove the claim: If \(a\) is not minimal, there is some \(b_1 \in S\) such that \(a \succ b_1\); if \(b_1\) is not minimal, there is some \(b_2 \in S\) such that \(b_1 \succ b_2\). Continuing in this manner, we get a chain \(a \succ b_1 \succ b_2 \succ \ldots \succ b_r\), which must stop because \(S\) is finite and (being a partially ordered set) antisymmetric. □

Let \(S\) be a partially ordered set. A real-valued function \(\phi : S \to \mathbb{R}\) is called order-preserving if, for \(a, b \in S\), \(\phi(a) \geq \phi(b)\) whenever \(a \succeq b\).
We investigate the connection between order-preserving functions and topsets. We start by defining two properties that a partially ordered set might or might not have.

We say that a partially ordered set $S$ has the **Topset Positivity Property (TPP)** if $\sum_{a \in T} \phi(a) \geq 0$ for any order-preserving function $\phi$ defined on $S$ such that $\sum_{a \in S} \phi(a) \geq 0$ and any topset $T \subseteq S$. We say that $S$ has the **Topset Average Property (TAP)** if for any order-preserving function $\phi$ defined on $S$ and any nonempty topset $T \subseteq S$, the average of the values of $\phi$ on $T$ is at least as large as the average of the values of $\phi$ on $S$. In symbols, \[ \frac{\sum_{a \in T} \phi(a)}{\#(T)} \geq \frac{\sum_{a \in S} \phi(a)}{\#(S)}. \]

**Example 5.2.** $S = \{a, b\}$, with neither $a \geq b$ nor $b \geq a$.

$S$ has neither TPP nor TAP, as can be verified by considering the order-preserving function $\phi$ with $\phi(a) = 3, \phi(b) = -1$, and the topset $\{b\}$.

**Proposition 5.3.** A partially ordered set $S$ has TAP if and only if it has TPP.

**Proof.** Assume $S$ has TAP and let $\phi$ be an order-preserving function on $S$ such that $\sum_{a \in S} \phi(a) \geq 0$. Let $T \subseteq S$ be a nonempty topset. By TAP, \[ \frac{\sum_{a \in T} \phi(a)}{\#(T)} \sum_{a \in S} \phi(a) \geq 0. \] And if $T$ is empty, $\sum_{a \in T} \phi(a) = 0$. 


For the other direction, assume $S$ has TPP and let $\phi$ be an order-preserving function on $S$. Let $C := \frac{\sum_{a \in S} \phi(a)}{\#(S)}$ and define a new order-preserving function $\psi$ on $S$ by setting $\psi(a) = \phi(a) - C$ for all $a \in S$. We observe that $\sum_{a \in S} \psi(a) = 0$. For any topset $T \subseteq S$, TPP gives $\sum_{a \in T} \psi(a) \geq 0$. So

\[
\sum_{a \in T} \psi(a) \geq 0 = \sum_{a \in S} \psi(a) = \sum_{a \in S} \phi(a) - \sum_{a \in S} C.
\]

\[
\Rightarrow \sum_{a \in T} \phi(a) \geq \sum_{a \in S} \phi(a) - \sum_{a \in S} C.
\]

\[
\Rightarrow \frac{\sum_{a \in T} \phi(a)}{\#(T)} \geq \frac{\sum_{a \in S} \phi(a)}{\#(S)} - C.
\]

\[
\Rightarrow \frac{\sum_{a \in T} \phi(a)}{\#(T)} \geq \frac{\sum_{a \in S} \phi(a)}{\#(S)}.
\]

\[\square\]

2. The Partially Ordered Set $G_Q$

Let $Q = (Q_1, ..., Q_n)$ be an $n$-tuple of non-negative integers. We define the set $G_Q$ as follows:

\[
G_Q := \{ \text{n-tuples } I := (I_1, ..., I_n) \mid 0 \leq I_i \leq Q_i \text{ for } i = 1, ..., n \}.
\]

We call $n$ the dimension of $Q$ or of $G_Q$.

An element of $G_Q$ will often be called a multi-index. The multi-index $(0, ..., 0)$ consisting of all zeroes comes up frequently, and we sometimes refer to it as 0. We
give $G_Q$ a partial ordering as follows:

\[(15) \quad I \succeq J \iff I_i \leq J_i \text{ for } i = 1, \ldots, n.\]

For example, for any $I \in G_Q$, $0 \succeq I \succeq Q$. We note that the partial ordering defined here is not at all the same as lexicographic ordering, which is also defined on sets of $n$-tuples.

As a first step, we consider some linearly ordered subsets of $G_Q$. Specifically, for some co-ordinate $j$, we fix the values of all co-ordinates of $I = (I_1, \ldots, I_n)$ except the $j^{th}$ to be $I_i = \lambda_i$. We say that the $Q_j + 1$-element set

\[(16) \quad X_{\lambda} := \{(\lambda_1, \ldots, \lambda_{j-1}, I_j, \lambda_{j+1}, \ldots \lambda_n) \mid 0 \leq I_j \leq Q_j\}\]

is a one-parameter subset of $G_Q$.

**Lemma 5.4.** Let $\phi$ be an order-preserving function on $G_Q$, let $X$ be a one-parameter subset of $G_Q$, and let $T$ be a nonempty topset of $X$ (under the partial order inherited from $G_Q$). Then the average value of $\phi$ on $T$ is at least as great as the average value of $\phi$ on $X$. In symbols, $\sum_{I \in T} \phi(I)/\#(T) \geq \sum_{I \in X} \phi(I)/\#(X)$.

**Proof.** This is an immediate consequence of the definition of order-preserving: restricting from $X$ to $T$ removes the smallest values of $\phi(I)$. \(\square\)

We next prove a computational lemma.
Lemma 5.5. Let \( \phi \) be an order-preserving function defined on \( \mathcal{G}_Q \), let \( P = (P_1, \ldots, P_n) \in \mathcal{G}_Q \), and let \( 0 \leq j \leq n \). Consider the two sums:

\[
S_1 = \sum_{I_1 \leq Q_1} \cdots \sum_{I_{j+1} \leq Q_{j+1}} \sum_{I_j \leq Q_j} \sum_{I_{j-1} \leq P_{j-1}} \cdots \sum_{I_1 \leq P_1} \phi(I)
\]

and

\[
S_2 = \sum_{I_1 \leq Q_1} \cdots \sum_{I_{j+1} \leq Q_{j+1}} \sum_{I_j \leq P_j} \sum_{I_{j-1} \leq P_{j-1}} \cdots \sum_{I_1 \leq P_1} \phi(I).
\]

Then there is a positive constant \( \mu \) such that \( \mu S_2 \geq S_1 \). In particular, if \( S_1 \geq 0 \), then \( S_2 \geq 0 \).

Proof. We note that the two sums are taken over the same set of values of \( I_1, \ldots, I_{j-1}, I_{j+1}, \ldots, I_n \). Only the values of \( I_j \) are different in the two sums. The set of multi-indexes \( I \) over which the first sum is taken can be subdivided into one-parameter subsets \( X_\lambda := \{(\lambda_1, \ldots, \lambda_{j-1}, I_j, \lambda_{j+1}, \ldots, \lambda_n) \mid 0 \leq I_j \leq Q_j\} \), one for each choice of \( \lambda := (\lambda_1, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_n) \); and then the second sum can be subdivided into subsets \( T_\lambda \subseteq X_\lambda \), where \( T_\lambda := \{(\lambda_1, \ldots, \lambda_{j-1}, I_j, \lambda_{j+1}, \ldots, \lambda_n) \mid 0 \leq I_j \leq P_j\} \). For each \( \lambda \), we note that \( T_\lambda \) is a topset of \( X_\lambda \) (under the partial order inherited from \( \mathcal{G}_Q \)), that there are \( Q_j + 1 \) elements in \( X_\lambda \), and that there are \( P_j + 1 \) elements in \( T_\lambda \).

By Lemma 5.4 for each \( \lambda \) we have

\[
\frac{Q_j + 1}{P_j + 1} \sum_{I \in T_\lambda} \phi(I) \geq \sum_{I \in X_\lambda} \phi(I),
\]
and summing over all values of $\lambda$,

\[
\frac{Q_j + 1}{P_j + 1} \sum_{\lambda} \sum_{I \in T_\lambda} \phi(I) \geq \sum_{\lambda} \sum_{I \in X_\lambda} \phi(I)
\]

or

\[
\frac{Q_j + 1}{P_j + 1} S_2 \geq S_1.
\]

The proof is completed by setting $\mu := \frac{Q_j + 1}{P_j + 1}$.

Next we state a result about topsets of $\mathcal{G}_Q$ that have the form $\mathcal{G}_P$, for some element $P = (P_1, ..., P_n) \in \mathcal{G}_Q$.

**Proposition 5.6.** Let $P = (P_1, ..., P_n) \in \mathcal{G}_Q$ and let $\phi$ be an order-preserving function on $\mathcal{G}_Q$ such that

\[
\sum_{I \in \mathcal{G}_Q} \phi(I) \geq 0.
\]

Then

\[
\sum_{I \in \mathcal{G}_P} \phi(I) \geq 0.
\]

**Proof.** The first inequality can be rewritten

\[
\sum_{I_n \leq Q_n} \ldots \sum_{I_1 \leq Q_1} \phi(I) \geq 0
\]

and the second inequality can be written

\[
\sum_{I_n \leq P_n} \ldots \sum_{I_1 \leq P_1} \phi(I) \geq 0.
\]

The first is transformed into the second by $n$ iterations of Lemma 5.5.\]
We next prove an extension of the previous proposition.

**Proposition 5.7.** The partially ordered set $G_Q$ has TPP. Equivalently, for any topset $T \subseteq G_Q$ and any order-preserving function $\phi$ on $G_Q$ such that

$$\sum_{I \in G_Q} \phi(I) \geq 0,$$

then

$$\sum_{I \in T} \phi(I) \geq 0.$$

**Proof.** We proceed by induction on $n$, the dimension of $G_Q$. To start the induction, we must show that TPP holds for $G_Q$ of dimension 1, so we assume $Q := (Q_1)$. Any nonempty topset $T$ has the form $\{(I_1)|0 \leq I_1 \leq P_1\} = G_p$ for some 1-tuple $P := (P_1)$. So the dimension-1 case follows from Proposition 5.6.

For the induction step, we assume the proposition proved for $G_Q$ of dimension $n - 1$, and prove it for dimension $n$. We first deal with the special case that $Q_n = 0$. In this case, $G_Q = G_{(Q_1,\ldots,Q_{n-1},0)}$ is isomorphic to $G_{(Q_1,\ldots,Q_{n-1})}$ by the bijection $(I_1,\ldots,I_{n-1},0) \leftrightarrow (I_1,\ldots,I_{n-1})$. Since $G_{(Q_1,\ldots,Q_{n-1})}$ has dimension $n - 1$, the result follows by the induction hypothesis.

To prove the result for arbitrary $Q$ of dimension $n$, assuming it to be true for lower dimensions, we perform another induction. As the induction step, we assume that the proposition is true for all $P = (P_1,\ldots,P_n)$ for which $P_1 \leq Q_1$, 44
..., $P_n \leq Q_n$, and at least one inequality is strict; and we prove the proposition for $(Q_1, ..., Q_n)$. To start the induction, we note that the result is immediate for $Q = (0, ..., 0)$. Also, we have already dealt with the special case that $Q_n = 0$, so in proving the induction step we are entitled to assume that $Q_n \geq 1$.

To prove the induction step, we assume that $\phi$ is an order-preserving function on $G_Q$ such that $\sum_{I \in G_Q} \phi(I) \geq 0$, and that $T \subseteq S$ is a topset. Our goal is to show that $\sum_{I \in T} \phi(I) \geq 0$.

We make several definitions. Let $H_0 := G_{(Q_0, ..., Q_{n-1}, 0)} = \{(I_1, ..., I_{n-1}, 0)\} \subseteq G_Q$. Let $H_1 := \{(I_1, ..., I_{n-1}, 1)\} \subseteq G_Q$. We observe that $\#(H_0) = \#(H_1)$. We observe that $\#(H_0 \cap T) \geq \#(H_1 \cap T)$ since if $(I_1, ..., I_{n-1}, 1)$ is a member of the topset $T$ then $(I_1, ..., I_{n-1}, 0)$ is also a member.

Let $G' := G_Q - H_0$. We observe that $G'$ is isomorphic to $G_{(Q_1, ..., Q_{n-1}, Q_n - 1)}$ by the bijection $(I_1, ..., I_{n-1}, I_n) \leftrightarrow (I_1, ..., I_{n-1}, I_n - 1)$. We let $T' := T - H_0$ and observe that $T'$ is a topset of $G'$ by Lemma 5.1 (4)(a).

We let $C := \frac{\sum_{I \in H_0} \phi(I)}{\#(H_0)}$, and observe that $C \geq 0$ by Proposition 5.6. We define a new order-preserving function $\psi$ on $G'$ according to the rule: if $I \in H_1$, $\psi(I) := \phi(I) + C$; otherwise, $\psi(I) := \phi(I)$. To verify that $\psi$ is order-preserving, let $I, J \in G'$ and let $I \preceq J$. If $I \in H_1$, then $\psi(I) = \phi(I) + C \geq \phi(J) + C \geq \psi(J)$. If $I \notin H_1$, then
\( J \notin H_1 \) (since \( I_n > 1 \) and \( I \succeq J \)) so \( \psi(I) = \phi(I) \geq \phi(J) = \psi(J) \). We observe that

\[
\sum_{I \in G'} \psi(I) = \sum_{I \in G'} \phi(I) + \#(H_1)C \\
= \sum_{I \in G'} \phi(I) + \#(H_1) \frac{\sum_{I \in H_0} \phi(I)}{\#(H_0)} \\
= \sum_{I \in G'} \phi(I) + \sum_{I \in H_0} \phi(I) \\
= \sum_{I \in G_Q} \phi(I) \geq 0,
\]

the last inequality being true by hypothesis.

We now use the induction hypothesis, applied to \( G', T', \) and \( \psi \), which is applicable because of the previous computation. We deduce that \( 0 \leq \sum_{I \in T'} \psi(I) \).

We observe, for use in the next paragraph:

\[
(17) \quad 0 \leq \sum_{I \in T'} \psi(I) = \sum_{I \in T'} \phi(I) + \#(T \cap H_1)C = \sum_{I \in T'} \phi(I) + \#(T \cap H_1) \frac{\sum_{I \in H_0} \phi(I)}{\#(H_0)}.
\]

Recall that our goal is to show that \( \sum_{I \in T} \phi(I) \geq 0 \). We have

\[
\sum_{I \in T} \phi(I) = \sum_{I \in T'} \phi(I) + \sum_{I \in T \cap H_0} \phi(I) \\
\geq \sum_{I \in T'} \phi(I) + \frac{\#(T \cap H_0)}{\#(H_0)} \sum_{I \in H_0} \phi(I),
\]

the last inequality being demonstrated as follows:

(a) By the special case, \( H_0 = G_{(Q_0, \ldots, Q_{n-1}, 0)} \) has TPP.
(b) $T \cap H_0$ is a topset of $H_0$ by Lemma 5.1(2).

(c) TPP $\Rightarrow$ TAP by Proposition 5.3.

Continuing the computation,

$$\sum_{I \in T} \phi(I) \geq \sum_{I \in T'} \phi(I) + \frac{\#(T \cap H_0)}{\#(H_0)} \sum_{I \in H_0} \phi(I)$$

$$= \sum_{I \in T'} \phi(I) + \frac{\#(T \cap H_0)}{\#(H_0)} \sum_{I \in H_0} \phi(I)$$

$$\geq \sum_{I \in T'} \phi(I) + \frac{\#(T \cap H_1)}{\#(H_0)} \sum_{I \in H_0} \phi(I) \geq 0,$$

the last inequality having been established as (17). □

3. Block L-Matrices Associated to $G_Q$

Recall that the elements of $G_Q$ have two different orderings on them. There is lexicographic order, denoted $I \geq J$, and the partial order, denoted $I \geq J$.

We say that an L-Matrix $U$ has $G_Q$ pattern if

1. For each $I \in G_Q$, there exist non-negative integers $r_I$ and $c_I$ such that $U$ has block form, with one $r_I \times c_I$ block $B_{IJ}$ corresponding to each ordered pair $(I, J)$ of elements of $G_Q$.
2. The block-row indices $I$ occur in lexicographic order.
3. The block-column indices $J$ occur in reverse lexicographic order.
4. All entries in the block $B_{IJ}$ are nonzero if $I \geq J$ and all entries are zero if $I \not\geq J$. 

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Recalling that in lexicographic order \( Q := (Q_1, ..., Q_n) \) comes first and \( 0 := (0, ..., 0) \) comes last, an L-matrix with \( G_Q \) pattern decomposes into blocks as follows:

\[
\begin{bmatrix}
B_{Q0} & \ldots & B_{QQ} \\
 \ldots & \ldots & \ldots \\
 \ldots & B_{1J} & \ldots \\
B_{00} & \ldots & B_{0Q}
\end{bmatrix}
\]

Recall that the size of \( B_{1J} \) is \( r_I \times c_J \). If we wish to include this information along with the matrix, we will do it as follows, with the understanding that the \( r_I \)'s and \( c_J \)'s are not matrix entries:

\[
\begin{bmatrix}
| & c_0 & c_J & c_Q |
\hline
r_Q & B_{Q0} & \ldots & B_{QQ} \\
 \ldots & \ldots & \ldots & \ldots \\
r_I & \ldots & B_{1J} & \ldots \\
r_0 & B_{00} & \ldots & B_{0Q}
\end{bmatrix}
\]

We will be interested in determining necessary and sufficient conditions that an L-Matrix with \( G_Q \) pattern have nonzero determinant.
Let $U$ be an L-Matrix with $G_Q$ pattern. We wish to define the notion of a superblock of $U$. Let $I, J \subseteq G_Q$ be subsets. Then the $IJ$ superblock of $U$ is the submatrix composed of all blocks $B_{IJ}$ such that $I \in I$ and $J \in J$. A superblock of zeroes is a superblock of $U$ composed entirely of blocks of zeroes. A maximal superblock of zeroes is a superblock of zeroes that is not properly contained in any larger superblock of zeroes.

**Example 5.8.** Let $n = 2$, $Q = (Q_1, Q_2) = (1, 1)$. $G_Q = \{(1, 1), (1, 0), (0, 1), (0, 0)\}$. Abusing notation, $G_Q = \{11, 10, 01, 00\}$.

Let us analyze an L-Matrix $U$ with $G_Q$ pattern, for the choice of $Q$ in Example 5.8. According to the definition of $G_Q$ pattern, the blocks $B_{IJ}$ are composed entirely of zeroes if $I \preceq J$ and contain no zeroes if $I \succeq J$. The pairs $(I, J)$ for which $I \not\preceq J$ are:

$$(11, 00), (11, 01), (11, 10), (10, 00), (10, 01), (01, 00), (01, 10).$$

Therefore $U$ has the following block form, where an asterisk denotes a block that contains only nonzero entries:
The four maximal superblocks of zeroes can be demonstrated by placing spaces into the diagram, as follows, and perhaps permuting the rows and columns of $U$ (as was done to demonstrate the third maximal superblock).

$$
\begin{array}{cccc}
  & c_{00} & c_{01} & c_{10} & c_{11} \\
 r_{11} & 0 & 0 & 0 & * \\
r_{10} & 0 & 0 & * & * \\
r_{01} & 0 & * & 0 & * \\
r_{00} & * & * & * & * \\
\end{array}
$$

$$
\begin{array}{cccc}
  & c_{00} & c_{01} & c_{10} & c_{11} \\
 r_{11} & 0 & 0 & 0 & * \\
r_{10} & 0 & 0 & * & * \\
r_{01} & 0 & * & 0 & * \\
r_{00} & * & * & * & * \\
\end{array}
$$

$$
\begin{array}{cccc}
  & c_{00} & c_{10} & c_{01} & c_{11} \\
 r_{11} & 0 & 0 & 0 & * \\
r_{01} & 0 & 0 & * & * \\
r_{10} & 0 & * & 0 & * \\
r_{00} & * & * & * & * \\
\end{array}
$$
Lemma 5.9. Let \( U \) be an L-Matrix with \( G_Q \) pattern. Then its maximal superblocks \( Y_T \) of zeroes are determined by the proper nonempty topsets \( T \subseteq G_Q \): the block \( Y_T \) contains precisely those blocks \( B_{IJ} \) such that \( I \notin T \) and \( J \in T \).

Proof. We first show that any such superblock contains only zeroes. By the definition of \( G_Q \) pattern, it is enough to show that, for any block \( B_{IJ} \) in the superblock, \( I \not\geq J \). But \( J \) is an element of the topset \( T \) and \( I \) is not, so \( I \not\geq J \).

We next show that, given a superblock \( Y \) of zeroes, it is a subsuperblock of one of the \( Y_T \)'s. Since all its constituent blocks \( B_{IJ} \) are 0, we always have \( I \not\geq J \). So if we consider the set \( T = \{ K \in G_Q | K \succeq J \text{ where some } B_{IJ} \text{ is in } Y \} \), then \( T \) is a topset by Lemma 5.1 (3), and \( Y \) is a subsuperblock of \( Y_T \). \( \square \)

In the matrix \( U \) with \( G_Q \) pattern discussed in connection with Example 5.8, we see that the four maximal superblocks of zeroes correspond, respectively, to the four nonempty proper subsets of \( G_Q \): \( \{00,01,10\}, \{00,01\}, \{00,10\}, \{00\} \).
For an L-matrix $U$ with $G_Q$ pattern, with block dimensions $r_I$ and $c_I$ as above, we define, for each $I \in G_Q$, the excess $A_I := r_I - c_I$. Despite the name, there is no requirement that $r_I \geq c_I$, and $A_I$ can certainly be a negative number.

**Lemma 5.10.** Let $U$ be an L-matrix with $G_Q$ pattern and excesses $A_I$. Then $U$ is a square matrix if and only if $\sum_{I \in G_Q} A_I = 0$.

**Proof.** The condition is equivalent to $\sum_{I \in G_Q} r_I = \sum_{I \in G_Q} c_I$. □

**Theorem 5.11.** Let $U$ be a square L-matrix with $G_Q$ pattern and excesses $A_I$. Then the following are equivalent:

(1) $\det(U) \neq 0$.

(2) For any nonempty proper topset $T \subseteq G_Q$, $\sum_{I \in T} A_I \geq 0$.

(3) For any nonempty topset $T \subseteq G_Q - \{Q\}$, $\sum_{I \in T} A_I \geq 0$.

(4) For any nonempty proper bottomset $B \subseteq G_Q$, $\sum_{I \in B} A_I \leq 0$.

(5) For any nonempty bottomset $B \subseteq G_Q - \{0\}$, $\sum_{I \in B} A_I \leq 0$.

**Proof.** To see that (2) and (3) are equivalent, we observe that a proper topset of $G_Q$ is the same thing as a topset of $G_Q - \{Q\}$, since the only topset of $G_Q$ containing $Q$ is $G_Q$ itself. Similarly, to see that (4) and (5) are equivalent, we observe that a proper bottomset of $G_Q$ is the same thing as a bottomset of $G_Q - \{0\}$, since the only bottomset of $G_Q$ containing 0 is $G_Q$ itself.

To see that (2) and (4) are equivalent, we first observe that

$\{B \subseteq G_Q | B \text{ is a bottomset} \} = \{G_Q - T \subseteq G_Q | T \text{ is a topset} \}$. (See Lemma 5.1(1).) By
Lemma 5.10 \( \sum_{I \in G_{Q-T}} A_I \leq 0 \iff \sum_{I \in T} A_I \geq 0. \)

To see that \( (1) \implies (2) \), let \( T \) be a nonempty proper topset and let \( Y_T \) be the corresponding maximal superblock of zeroes. Recall that \( Y_T \) consists of blocks \( B_{IJ} \) such that \( I \notin T \) and \( J \in T \). Then some matrix \( U' \), formed by (perhaps) permuting some of the rows and columns of \( U \), has a decomposition into four superblocks as follows:

\[
\begin{bmatrix}
0 & A \\
B & Z
\end{bmatrix}
\]

where 0 represents \( Y_T \). Assume \( \det(U) \neq 0 \). Then \( \det(U') \neq 0 \), since \( U' \) was formed by permuting rows of columns of \( U \). So the first \( \sum_{I \in T} c_I \) columns must be linearly independent, which means the rank of the superblock \( B \) must be at least \( \sum_{I \in T} c_I \). This implies \( \sum_{I \in T} r_I \geq \sum_{I \in T} c_I \), that is, \( \sum_{I \in T} A_I \geq 0. \)

To prove that \( (2) \implies (1) \) takes several pages and constitutes the remainder of this chapter. We proceed by induction on the size \( s \times s \) of \( U \). We fix a value of \( Q \), which remains unchanged throughout the induction.

We start the induction with \( s = 1 \), which is to say \( U \) has a single entry \( u \). For \( U \) to have a nonzero determinant, \( u \) must be nonzero. We observe that, for some choice of multi-indexes \( I_0 \) and \( J_0 \), \( u \) constitutes the \( B_{I_0J_0} \) block of \( U \). That is,
\( r_{I_0} = c_{J_0} = 1 \), and these are the only nonzero \( r_I \) and \( c_J \). In particular, if \( I_0 \neq J_0 \), the only nonzero values of \( A_I := r_I - c_I \) are \( A_{I_0} = 1 \) and \( A_{J_0} = -1 \).

To prove the case \( s = 1 \): we assume, for every nonempty proper topset \( T \subseteq G_Q \), that \( \sum_{I \in T} A_I \geq 0 \); and our goal is to show that the \( B_{I_0J_0} \) block has a nonzero entry. Equivalently, since \( U \) has \( G_Q \) pattern, we must show that \( I_0 \succeq J_0 \).

If \( I_0 = J_0 \) or \( J_0 = Q \), this is immediate. Otherwise, we form the nonempty proper topset \( T_{\{J_0\}} := \{ I \in G_Q | I \succeq J_0 \} \) (See Lemma 5.1(3)). Since \( \sum_{I \in T_{\{J_0\}}} A_I \geq 0 \) and \( J_0 \in T_{\{J_0\}} \), it must be that \( I_0 \in T_{\{J_0\}} \). That is, \( I_0 \succeq J_0 \).

For the induction step, we assume that the theorem has been proved for \( 1, ..., s - 1 \), and we let \( U \) be an \( s \times s \) matrix such that (2) holds. We must show that the determinant of \( U \) is nonzero. To do this, having Corollary 4.5 in mind, we choose an arbitrary nonempty proper topset \( S \subseteq G_Q \) (switching to \( S \) in order to reserve the letter \( T \) for future use) and we set \( C = G_Q - S \). Then we permute the rows and columns of \( U \), if necessary, so that \( Y_S \), the maximal superblock of zeroes associated to \( S \), is in the upper left. We call this permuted matrix \( U_S \), and look at its decomposition into four superblocks.

\[
\begin{pmatrix}
\sum_{I \in S} c_I & \sum_{I \in C} c_I \\
\sum_{I \in C} r_I & 0 & A \\
\sum_{I \in S} r_I & B & Z
\end{pmatrix}
\]
We observe that it was not necessary to permute the last \( r_0 \) rows or the last \( c_Q \) columns, since automatically \( 0 \preceq Q \), which forces the \( B_{0Q} \) block (which is contained in \( Z \)) to consist of nonzero entries. Since we are assuming that (2) holds, we know that \( A \) has at least as many columns as rows, and \( B \) has at least as many rows as columns. We let \( A' \) denote the leftmost maximal square submatrix of \( A \), and let \( B' \) denote the uppermost maximal square submatrix of \( B \). With this notation, the decomposition of \( U_S \) can be rewritten

\[
\begin{bmatrix}
0 & A' & * \\
B' & C' & * \\
* & * & Z'
\end{bmatrix}
\]

This rewriting did not involve any further permuting of the rows and columns, so we may be sure that the last \( r_0 \) rows and the last \( c_Q \) columns have never been permuted from the original \( U \).

In order to show that \( U \) has nonzero determinant, it is enough to show that, for some choice of \( S \), \( U_S \) has nonzero determinant. According to Corollary 4.5, we can show that \( U_S \) has nonzero determinant by showing that (a1) \( A' \) has nonzero determinant, (b1) \( B' \) has nonzero determinant, and (c) the block \( Z' \) lies entirely within the \( B_{0Q} \) block, which guarantees that its rows and columns have not been
permuted and that its entries (and hence its main diagonal entries) are all nonzero.

In order to show this, we fix some notations. We recall that both $A'$ and $B'$ were formed by first permuting some of the rows and columns of $U$, and then deleting some of the rows and columns of the permuted matrix. We observe that the result would have been the same if we had first deleted the appropriate rows and columns of $U$ and then suitably permuted the rows and columns of what remained. With regard to this equivalent alternative construction of $A'$ and $B'$, we define $A'_0$ and $B'_0$ to be the submatrices of $U$ that were formed by deletion of rows and columns, and were subsequently altered by permutations of rows and columns to form, respectively, $A'$ and $B'$. We remark that $A'_0$ and $B'_0$, being submatrices of the L-matrix $U$, are themselves L-matrices; and that, in order to show that $A'$ and $B'$ have nonzero determinant, it is enough to show that $A'_0$ and $B'_0$ have nonzero determinant.

In order to use induction, we need to view $A'_0$ and $B'_0$ as L-matrices with $G_Q$ pattern. To do this, we say that an entry is in the $B_{IJ}$ block of $A'_0$ or $B'_0$ if it was in the $B_{IJ}$ block of $U$. Thus, if we use primes to denote block dimensions and excesses in $A'_0$ (viz. $r'_I, c'_I, A'_I$) and double primes to denote block dimensions and excesses in $B'_0$ (viz. $r''_I, c''_I, A''_I$),

1. For all $I \in G_Q$, $r'_I \leq r_I, r''_I \leq r_I, c'_I \leq c_I, c''_I \leq c_I$.
2. For all $I \in S, 0 = r'_I = c'_I = A'_I$, and $c''_I = c_I$.
3. For all $I \in C, 0 = r''_I = c''_I = A''_I$, and $r'_I = r_I$. 

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With these notations, we repeat the previous decomposition of $U_S$, this time including some of the block dimensions.

\[
\begin{bmatrix}
\sum_{I \in S} c_I & \sum_{I \in C} c'_I \\
\sum_{I \in C} r_I & 0 & A' & * \\
\sum_{I \in S} r''_I & B' & C' & * \\
* & * & * & Z'
\end{bmatrix}
\]

With these notations, we can break statement (c) above into two parts:

Statement (a2):

\[
\sum_{I \in C} c'_I \geq \sum_{I \in C - \{Q\}} c_I.
\]

and Statement (b2):

\[
\sum_{I \in S} r''_I \geq \sum_{I \in S - \{0\}} r_I.
\]

To state (a1) with the new notations, we use the induction hypothesis to obtain a set of conditions that $A'_0$ have nonzero determinant:

For all nonempty topsets $T \subseteq G_Q - \{Q\}$, $\sum_{I \in T} A'_I \geq 0$.

By (ii), this is equivalent to:

For all nonempty topsets $T \subseteq G_Q - \{Q\}$, $\sum_{I \in T \cap C} A'_I \geq 0$. 

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By Lemma 5.1(4)(b)(ii), this is equivalent to:

(18) For all nonempty topsets $T \subseteq C - \{Q\}$, $\sum_{I \in T} A'_I \geq 0$.

Also, we recall (a2):

$$\sum_{I \in C} c'_I \geq \sum_{I \in C - \{Q\}} c_I$$

which, since $A'$ is square, is equivalent to

$$\sum_{I \in C} r_I \geq \sum_{I \in C - \{Q\}} c_I$$

or

(19) $r_Q + \sum_{I \in C - \{Q\}} A_I \geq 0$.

We now introduce a condition on $U$ that, we claim, implies both (18) and (19):

(20) For all nonempty topsets $T \subseteq C - \{Q\}$, $\sum_{I \in T} A_I \geq 0$.

To see that (20) implies (18), let $T$ be a nonempty topset of $C - \{Q\}$. Then

$$0 \leq \sum_{I \in T} A_I = \sum_{I \in T} r_I - \sum_{I \in T} c_I = \sum_{I \in T} r'_I - \sum_{I \in T} c'_I \leq \sum_{I \in T} r'_I - \sum_{I \in T} c'_I = \sum_{I \in T} A'_I.$$
To see that (20) implies (19), apply (20) to $C - \{Q\}$ (which is a topset of itself):

$$0 \leq \sum_{I \in C - \{Q\}} A_I \leq r_Q + \sum_{I \in C - \{Q\}} A_I.$$

Summarizing the progress so far, we have shown that the induction step will follow if we can demonstrate a maximal superblock $Y_S$ for which (a1), (a2), (b1), and (b2) hold. We have shown that (a1) and (a2) hold if $S$ satisfies (20). We now proceed in a completely analogous fashion to establish another condition on $S$ that will ensure (b1) and (b2) hold.

To state (b1) with the new notations, we argue similarly, using the induction hypothesis to obtain a set of sufficient conditions that $B'_0$ have nonzero determinant:

For all nonempty bottomsets $B \subseteq G_Q - \{0\}$, $\sum_{I \in B} A''_I \leq 0$.

By (iii), this is equivalent to:

For all nonempty bottomsets $B \subseteq G_Q - \{0\}$, $\sum_{I \in B \cap S} A''_I \leq 0$.

By Lemma 5.1(4)(d)(ii), this is equivalent to:

$$\sum_{I \in S} r''_I \geq \sum_{I \in S - \{0\}} r_I$$

Also, we recall (b2):
which, since $B'$ is square, is equivalent to

$$\sum_{I \in S} c_I \geq \sum_{I \in S - \{0\}} r_I$$

or

(22) $$\sum_{I \in S - \{0\}} A_I - c_0 \leq 0.$$

We now introduce a condition on $U$ that, we claim, implies both (21) and (22):

(23) For all nonempty bottomsets $B \subseteq S - \{0\}$, $\sum_{I \in B} A_I \leq 0$.

To see that (23) implies (21), let $B$ be a bottomset of $S - \{0\}$. Then

$$0 \geq \sum_{I \in B} A_I = \sum_{I \in B} r_I - \sum_{I \in B} c_I = \sum_{I \in B} r_I - \sum_{I \in B} c''_I$$

$$\sum_{I \in B} r''_I - \sum_{I \in B} c''_I = \sum_{I \in B} A''_I.$$

To see that (23) implies (22), apply (23) to $S - \{0\}$ (which is a bottomset of itself):

$$0 \geq \sum_{I \in S - \{0\}} A_I \geq \sum_{I \in S - \{0\}} A_I - c_0.$$
Again summarizing the progress so far, we have shown that the induction step will hold if we can guarantee the existence of a nonempty topset $S$ of $G_Q - \{Q\}$ for which (20) and (23) are true.

Our method of proof is algorithmic. We start with a candidate for $S$, namely $\{0\}$, for which (23) is (vacuously) true, but for which (20) may not be true. We describe a procedure for replacing the current candidate for $S$ by another candidate topset that properly contains it, a process that must stop because it can be carried out only a finite number of times. We then show that (i) the procedure results in a new candidate that also satisfies (23), and (ii) if the process cannot be continued, the current candidate satisfies (20) as well. Establishing (i) and (ii) suffices to prove the induction step, and the theorem.

By way of notation, let $S$ represent the current candidate topset, which is known to satisfy (23), and let $S'$ represent the next candidate. That is, we start with $S = \{0\}$. If we have a current $S$, the procedure to find $S'$ is as follows: Among all nonempty topsets of $G_Q - S - \{Q\}$, pick a smallest one $X$ (smallest by inclusion) such that $\sum_{I \in X} A_I < 0$. We set $S' := S \cup X$.

We note that if no such $X$ exists, $S$ satisfies (20), since for all nonempty topsets $T \subseteq G_Q - \{Q\} - S$, we have $\sum_{I \in T} A_I \geq 0$, and $G_Q - S - \{Q\} = C - \{Q\}$. This
establishes (ii).

If $X$ can be found, we note that $S' := S \cup X$ is a topset of $G_Q$ by Lemma 5.1 (4)(b)(iii). We note that $S'$ is nonempty because $0 \in S$ and proper because $Q \notin S$ and $Q \notin X$. We note that, by construction, $S$ and $X$ are disjoint.

We must show that $S' := S \cup X$ satisfies (23). That is, for any bottomset $B \subseteq (S \cup X) - \{0\} = (S - \{0\}) \cup X$, we must show that $\sum_{I \in B} A_I \leq 0$. For this, it is enough establish the following two inequalities:

\begin{equation}
\sum_{I \in B \cap (S - \{0\})} A_I \leq 0
\end{equation}

and

\begin{equation}
\sum_{I \in B \cap X} A_I \leq 0.
\end{equation}

Statement (24) follows immediately from the fact that (23) holds for $S$, once one has verified that $B \cap (S - \{0\})$ is a bottomset of $S - \{0\}$, which follows from Lemma 5.1 (4)(b)(iv).

Statement (25) follows from the following two inequalities:
(26) \[ \sum_{I \in X} A_I < 0 \]

and

(27) \[ \sum_{I \in X - B} A_I \geq 0 \]

since

\[ \sum_{I \in B \cap X} A_I = \sum_{I \in X} A_I - \sum_{I \in X - B} A_I. \]

Statement (26) is true by construction. Statement (27) is certainly true if \( X - B \) is empty. If \( X - B \) is nonempty it is a topset of \( G_Q - S - \{Q\} \) by Lemma 5.1 (4)(b)(iv), and then (27) follows by construction, because \( X - B \) is topset of \( G_Q - S - \{Q\} \) that is smaller (by inclusion) than \( X \).  \( \square \)
CHAPTER 6

Coefficient Matrices

1. Coefficient Matrices of $R_{j-d} \ast \mathcal{E}(C)$

We fix a codimension $r$ and a socle degree $j$. We consider a vector subspace $\mathcal{E} \subseteq \mathcal{D}_j$. We fix a degree $d \leq j$ and we wish to consider $\dim_k R_{j-d} \ast \mathcal{E}$. By way of notation, we make the convention that $e := j - d$, $j = e + d$. Also, whenever we wish to specify that generators $f_1, ..., f_s$ of $\mathcal{E}$ are to be taken from a particular vector subspace $\mathcal{W} \subseteq \mathcal{D}_j$, we will simply write $\mathcal{E} := \langle f_1, ..., f_s \rangle \subseteq \mathcal{W}$.

Let $\mathcal{M}$ be a set of multi-indexes of degree $j$, and define $\mathcal{W}_\mathcal{M} := \langle \{ x^J | J \in \mathcal{M} \} \rangle$. That is, $\mathcal{W}_\mathcal{M}$ is a vector subspace of $\mathcal{D}_j$ generated by monomials. We let $\mathcal{E} := \langle f_1, ..., f_s \rangle \subseteq \mathcal{W}_\mathcal{M}$. For each generator $f_i$, we write $f_i = \sum_{J \in \mathcal{M}} z_{ij} x^J$, where each $z_{ij} \in k$.

We will always adopt the point of view that the $z_{ij}$’s are allowed to vary. Specifying a value $c_{ij} = z_{ij}$ for each of them, or equivalently specifying an element $C := (..., c_{ij}, ...) \in k^{\#(\mathcal{M})}$, determines a particular subspace $\mathcal{E}(C) = \langle f_1(C), ..., f_s(C) \rangle \subseteq \mathcal{W}_\mathcal{M} \subseteq \mathcal{D}_j$. When we later define the matrices $U'$ and $U$ with coefficients in $k[[z_{ij}]]$, $U'(C)$ and $U(C)$ will similarly be specific matrices with coefficients in $k$. In other words, from now on we will view the $f_i$’s, $E$, $U'$, $U$.
and \(U\) (written without the “\(C\)”) as functions whose domain is the irreducible affine variety \(k^{s\#(\mathcal{M})}\). We will consider the images of these functions as families of vectors, vector subspaces, or matrices, parameterized by elements \(C \in k^{s\#(\mathcal{M})}\). Whenever we wish to indicate a specific element of a family, we will use the notation with the “\(C\)”, sometimes without explicitly mentioning the \(f_i\)’s or the \(z_{ij}\)’s.

**Lemma 6.1.** For all \(C \in k^{s\#(\mathcal{M})}\), \(R_e \ast \mathcal{E}(C)\) is generated as a vector space by
\[
\{X^E \ast f_i(C) | X^E \text{ is a monomial of degree } e \text{ and } 1 \leq i \leq s\}.
\]

**Proof.** Any element of \(R_e \ast \mathcal{E}(C)\) can be written
\[
(\sum a_E X^E) \ast (\sum b_i f_i(C)) = \sum a_E b_i (X^E \ast f_i(C)).
\]

**Lemma 6.2.** Let \(X^E\) be a monomial of degree \(e\) and \(f_i = \sum_{J \in \mathcal{M}} z_{ij} x^J\) as above. Then
\[
X^E \ast f_i = \sum_{J_1, E \geq J_1, J_2, E \in \mathcal{M}} n_{ij} z_{ij} x^{J_1 - E} = \sum_{D, D+E = J, E \in \mathcal{M}} n_{ij} z_{ij} x^D,
\]
where the \(n_{ij}\)’s are positive integers.

**Proof.** This follows immediately from the definition of the operation \(\ast\) as partial differentiation.

We wish to translate the problem of determining \(\dim_k(R_e \ast \mathcal{E}(C))\) into the language of matrices. To this end, we define a matrix \(U'\), the *uncropped coefficient matrix of \(e^{th}\) partial derivatives of \(\mathcal{E}\), or simply the \(e^{th}\) *uncropped matrix of \(\mathcal{E}\), as follows. (We use the prime to distinguish it from the \(e^{th}\) cropped matrix \(U\), to be defined
The matrix $U'$ has rows indexed by ordered pairs $(E, i)$ where $E$ is a multi-index of degree $e$ and $i \in \{1, ..., s\}$. The rows are ordered according to the rule that $(E_1, i_1)$ comes before $(E_2, i_2)$ if $E_1 > E_2$ (in lexicographic order) or if $E_1 = E_2$ and $i_1 < i_2$. The columns of $U'$ are indexed by multi-indexes $D$ of degree $d$, where $D_1$ comes before $D_2$ if $D_1 > D_2$ (in lexicographic order). The entry of $U'$ in the $((E, i), D)$ position is $n_{ij}z_{ij}$ if $J := D + E \in \mathcal{M}$ and 0 otherwise.

**Lemma 6.3.** Let the vector subspace $E := \langle f_1, ..., f_s \rangle \subseteq W_M \subseteq D_j$ be defined as above and let $U'$ be its $e^{th}$ uncropped matrix. Then $U'$ is an L-matrix over $k[\{z_{ij} | i \in \{1, ..., s\} and J \in \mathcal{M}\}]$.

**Proof.** By construction, the entries of $U'$ are either 0 or positive integer multiples of some $z_{ij}$. So it remains to show that every $z_{ij}$ moves to the left.

Assume $z_{ij}$ is the variable in two different locations $((E_1, i_1), D_1)$ and $((E_2, i_2), D_2)$. We first note that $i_1 = i_2 = i$, since that is the only way (by the definition of $U'$) that the variable $z_{ij}$ can appear at all. Since the order of the rows $(E_1, i)$ and $(E_2, i)$ in $U'$ is determined by lexicographical order of $E_1$ and $E_2$, and the order of the columns is determined by lexicographical order of $D_1$ and $D_2$, we must show $E_1 > E_2$ if and only if $D_1 < D_2$. 


By way of notation, let $E_1 := (\ldots, e_{1i}, \ldots); E_2 := (\ldots, e_{2i}, \ldots); D_1 := (\ldots, d_{1i}, \ldots); D_2 := (\ldots, d_{2i}, \ldots)$. From the definition of $U'$, $D_1 + E_1 = J = D_2 + E_2$. So for each coordinate $i$, $e_{1i} - e_{2i} = d_{2i} - d_{1i}$. Then

$$E_1 > E_2 \iff e_{1i} = e_{2i} \text{ for } i = 1, \ldots, m - 1 \text{ and } e_{1m} > e_{2m} \text{ (for some } m)$$

$$\iff d_{1i} = d_{2i} \text{ for } i = 1, \ldots, m - 1 \text{ and } d_{1m} < d_{2m} \iff D_1 < D_2.$$

□

**Lemma 6.4.** Let the family of vector subspaces $\mathcal{E} := \langle f_1, \ldots, f_s \rangle \subseteq W_M \subseteq D_j$ be defined as above and let $U'$ be its $e^{th}$ uncropped matrix. Then for all $C \in k^{e(M)}$, $\text{rank}(U'(C)) = \text{dim}_k(R_e * \mathcal{E}(C))$.

**Proof.** We combine previous results concerning the vector space $D_{d'}$, of which $\{x^D\}$ is a basis and $R_e * \mathcal{E}(C)$ is a vector subspace. $R_e * \mathcal{E}(C)$ is generated by the vectors $X_E * f_i(C)$, so to find its dimension we express each generator as a linear combination of basis vectors and determine the rank of the matrix of coefficients. Since $X_E * f_i(C) = \sum_{D; D + E = J \in M} n_{ij} e_{ij} x^D$, its matrix of coefficients is $U'(C)$. □

2. **Coefficient Matrices for Constrained Subspaces of $D_j$**

As before, we assume that the codimension $r$ and socle degree $j$ have been fixed. We now fix a nonnegative number $n \leq r$ and a multi-index $Q := (Q_1, \ldots, Q_n)$, where $0 \leq Q_i \leq j$ for $i = 1, \ldots, n$. We say an $r$-tuple $I := (I_1, \ldots, I_r)$ of non-negative integers, of any degree $d \leq j$, is *constrained by Q* if $I_i \leq Q_i$ for $i = 1, \ldots, n$. We say that a monomial $X^I$ or $x^I$ is *constrained by Q* if $I$ is constrained by $Q$. We
define $\mathcal{M}_Q(d)$ to be the set of all multi-indexes of degree $d$ that are constrained by $Q$. In particular, $\mathcal{M}_Q(j)$ is a set of multi-indexes of degree $j$, so we can consider $\mathcal{W}_{\mathcal{M}_Q(j)} \subseteq D_j$. By way of notation, we will always write $m_Q(d)$ for $\#(\mathcal{M}_Q(d))$. As in the previous section, we fix degree $d$ and write $e = j - d$.

**Lemma 6.5.** Let $\mathcal{E} = \langle f_1, ..., f_s \rangle \subseteq \mathcal{W}_{\mathcal{M}_Q(j)} \subseteq D_j$ be a family of vector subspaces and let $U'$ be its $e^{th}$ uncropped matrix. If $E \notin \mathcal{M}_Q(e)$ is a multi-index of degree $e$ and $1 \leq i \leq s$, the $(E, i)$ row of $U'$ consists entirely of zeroes. If $D \notin \mathcal{M}_Q(d)$ is a multi-index of degree $d$, the $D$ column of $U'$ consists entirely of zeroes.

**Proof.** For the ($(E, i), D$) entry to be nonzero, we need $E + D = J \in \mathcal{M}_Q(j)$. That is, writing $D = (d_1, ..., d_r)$ and $E = (e_1, ..., e_r)$, we must have $d_i + e_i \leq Q_i$ for $i = 1, ..., n$. This implies $d_i \leq Q_i$ and $e_i \leq Q_i$ for $i = 1, ..., n$. Equivalently, $D \in \mathcal{M}_Q(d)$ and $E \in \mathcal{M}_Q(e)$. □

Since our interest in $U'$ stems from our desire to compute its rank, we lose nothing by deleting rows and columns that consist entirely of zeroes. We define $U$, the cropped coefficient matrix of $e^{th}$ partial derivatives of $\mathcal{E}$, or simply the $e^{th}$ cropped matrix of $\mathcal{E}$, to be the submatrix of $U'$ obtained by taking only those $((E, i), D)$ entries for which $D \in \mathcal{M}_Q(d)$ and $E \in \mathcal{M}_Q(e)$. More precisely,

**Definition 6.6.** For a fixed choice of $r, j, Q, d, e := j - d$, and $s$, the $e^{th}$ cropped matrix of $\mathcal{E}$ is defined as follows:

The rows are indexed by pairs $(E, i)$ where $E \in \mathcal{M}_Q(e)$ and $i \in \{1, ..., s\}$, ordered by the rule that $(E_1, i_1)$ comes before $(E_2, i_2)$ if $E_1 > E_2$ (in lexicographic
order) or if \( E_1 = E_2 \) and \( i_1 < i_2 \). The columns are indexed by elements \( D \in \mathcal{M}_Q(d) \), ordered by the rule that \( D_1 \) comes before \( D_2 \) if \( D_1 > D_2 \) (in lexicographic order).

Writing

\[
X^E \ast f_i = \sum_{D; D+E=J \in \mathcal{M}_Q(j)} n_{ij} z_{ij} x^D,
\]

the entry of \( U \) in the \(((E, i), D)\) position is \( n_{ij} z_{ij} = n_{i(D+E)} z_{i(D+E)} \) if \( J \in \mathcal{M}_Q(j) \) and 0 otherwise.

**Corollary 6.7.** Let the family of vector subspaces \( \mathcal{E} = \langle f_1, \ldots, f_s \rangle \subseteq W_{\mathcal{M}_Q(j)} \subseteq \mathcal{D}_j \) be defined as above, let \( U' \) be its \( e^{th} \) uncropped matrix, and let \( U \) be its \( e^{th} \) cropped matrix. Then \( \text{rank}(U) = \text{rank}(U') \).

**Proof.** Removing rows and columns of zeroes does not affect the rank of a matrix. \( \square \)

**Corollary 6.8.** For \( C \in k^{sm_Q(j)} \), let \( \mathcal{E}(C) = \langle f_1(C), \ldots, f_s(C) \rangle \subseteq W_{\mathcal{M}_Q(j)} \subseteq \mathcal{D}_j \), let \( U'(C) \) be its \( e^{th} \) uncropped matrix, and let \( U(C) \) be its \( e^{th} \) cropped matrix. Then \( \text{rank}(U(C)) = \text{rank}(U'(C)) = \dim_k R_e \ast \mathcal{E}(C) \).

**Proof.** Again, removing rows and columns of zeroes does not affect the rank of a matrix. The second equality simply repeats the statement of Lemma 6.4. \( \square \)

**Lemma 6.9.** Let the family of vector subspaces \( \mathcal{E} := \langle f_1, \ldots, f_s \rangle \subseteq W_{\mathcal{M}_Q(j)} \subseteq \mathcal{D}_j \) be defined as above and let \( U \) be the \( e^{th} \) cropped matrix of \( \mathcal{E} \). Then \( U \) is an L-matrix over \( k[\{z_{ij} | i \in \{1, \ldots, s\} \text{ and } J \in \mathcal{M}_Q(j)\}] \).
PROOF. By Lemma 6.3, the uncropped matrix is an L-matrix over $k[\{z_{ij} \mid i \in \{1, \ldots, s\} \text{ and } J \in \mathcal{M}_Q(j)\}]$. By Lemma 4.2, $U$ is as well, since it is defined to be a submatrix of the uncropped matrix. \qed

The matrix $U$ is an L-Matrix, and we describe a scheme for subdividing it into blocks $B_{IJ}$, where $I, J \in G_Q$, that makes $U$ an L-Matrix with $G_Q$ pattern.

Given a multi-index $I \in G_Q$, we must designate $r_I$ row indices $(E, i)$ and $c_I$ column indices $D$ to associate with $I$. Writing $I = (I_1, \ldots, I_n)$, $D = (d_1, \ldots, d_r)$ and $E = (e_1, \ldots, e_r)$, we associate with $I$ those row indices $(E, i)$ for which $E = (I_1, \ldots, I_n, e_{n+1}, \ldots, e_r)$, and we associate with $I$ those column indices $D$ for which $D = (Q_1 - I_1, \ldots, Q_n - I_n, d_{n+1}, \ldots, d_r)$. We call this assignment of rows and columns the standard assignment.

**Lemma 6.10.** Let the family of vector subspaces $\mathcal{E} := \langle f_1, \ldots, f_s \rangle \subseteq \mathcal{W}_{\mathcal{M}_Q(j)} \subseteq \mathcal{D}_j$ be defined as above and let $U$ be the $e^{th}$ cropped matrix of $\mathcal{E}$. Then the standard assignment makes $U$ an L-matrix with $G_Q$ pattern.

PROOF. We must verify the following statements.

(i) Every row and every column of $U$ is associated to a unique $I$.

(ii) The standard assignment subdivides $U$ into blocks. That is, for any multi-index $I \in G_Q$, all rows associated to $I$ are consecutive in $U$, and all columns associated to $I$ are consecutive in $U$.

(iii) The block-row indices $I$ occur in lexicographic order, and the block-column indices $I$ occur in reverse lexicographic order.
(iv) The entries in the block $B_{IJ}$ are nonzero if $I \succeq J$ and 0 otherwise.

For (i), consider a row $(E, i)$ of $U$ and write $E = (e_1, ..., e_r)$. Then the only possible candidate for $I$ is $(e_1, ..., e_n)$, and we must verify that it is an element of $G_Q$. Since $U$ is the cropped matrix, $E \in M_Q(e)$, which implies $e_i \leq Q_i$ for each $i$; hence $I \in G_Q$.

Similarly, let $D$ be a column of $U$ and write $D = (d_1, ..., d_r)$. Then the only possible candidate for $I$ is $(Q_1 - d_1, ..., Q_n - d_n)$, and we must verify that it is an element of $G_Q$. This is true because $D \in M_Q(d)$, so $d_1 \leq Q_1, ..., d_n \leq Q_n$, which is to say $0 \leq Q_1 - d_1 \leq Q_1, ..., 0 \leq Q_n - d_n \leq Q_n$.

For (ii), recall that the ordering of the rows and columns of $U$ is given in Definition 6.6. Assume that $(E_1, i_1)$ and $(E_3, i_3)$ are two row indices associated to $I$, and that $(E_2, i_2)$ comes between them. Write $E_1 = (I_1, ..., I_n, ...), E_2 = (e_1, ..., e_n, ...), E_3 = (I_1, ..., I_n, ...)$. We must show that $e_i = I_i$ for $i = 1, ..., n$, and we argue by contradiction. If not, let $m$ be the first co-ordinate in which this is not so. By the rule for ordering the rows of $U$, $E_1 \succeq E_2 \succeq E_3$, so $I_m \succeq e_m \succeq I_m$, which is impossible if $e_m \neq I_m$.

The argument for columns is similar. Assume that $D_1$ and $D_3$ are two column indices associated to $I$, and that $D_2$ comes between them. Write $D_1 = (Q_1 - I_1, ..., Q_n - I_n, ...), D_2 = (d_1, ..., d_n, ...), D_3 = (Q_1 - I_1, ..., Q_n - I_n, ...)$. We must show that $d_i = Q_i - I_i$ for $i = 1, ..., n$, and we argue by contradiction. If not, let $m$ be the first co-ordinate in which this is not so. By the rule for ordering the columns of $U$,
\[D_1 \geq D_2 \geq D_3,\text{ so } Q_m - I_m \geq d_m \geq Q_m - I_m,\] which is impossible if \(d_m \neq Q_m - I_m\).

For (iii), let row index \((E_1, i_1)\) associated to \(I = (I_1, ..., I_n) \in \mathcal{G}_Q\) come before row index \((E_2, i_2)\) associated to \(J = (J_1, ..., J_n) \in \mathcal{G}_Q\). Then \(E_1 \geq E_2\), or as expanded by co-ordinates, \((I_1, ..., I_n, ...) \geq (J_1, ..., J_n, ...)\). That is, either \(I_i = J_i\) for \(i = 1, ..., n\), or else, for some \(m\), \(I_i = J_i\) for \(i = 1, ..., m - 1\) and \(I_m > J_m\); equivalently, \(I \geq J\).

The argument for columns is similar. Let column index \(D_1\) associated to \(I = (I_1, ..., I_n) \in \mathcal{G}_Q\) come before column index \(D_2\) associated to \(J = (J_1, ..., J_n) \in \mathcal{G}_Q\). Then \(D_1 > D_2\), or as expanded by co-ordinates, \((Q_1 - I_1, ..., Q_n - I_n, ...) \geq (Q_1 - J_1, ..., Q_n - J_n, ...)\). That is, either \(Q_i - I_i = Q_i - J_i\) for \(i = 1, ..., n\), or else, for some \(m\), \(Q_i - I_i = Q_i - J_i\) for \(i = 1, ..., m - 1\) and \(Q_m - I_m > Q_m - J_m\). So either \(I_i = J_i\) for \(i = 1, ..., n\) or \(I_i = J_i\) for \(i = 1, ..., m - 1\) and \(J_m > I_m\); equivalently, \(J \geq I\).

For (iv), we recall from the definition of \(U\) that the \(((E, i), D)\) entry is nonzero if and only if \(E + D = J \in \mathcal{M}_Q(j)\). So if \((E, i)\) is associated to \(I = (I_1, ..., I_n)\) and \(D\) is associated to \(J = (J_1, ..., J_n)\), the condition for being nonzero becomes \(I_i + (Q_i - J_i) \leq Q_i\) for \(i = 1, ..., n\), or equivalently, \(I_i \leq J_i\) for \(i = 1, ..., n\). But that is the definition of \(I \geq J\). \(\square\)

**Proposition 6.11.** Let the family of vector subspaces \(\mathcal{E} := \langle f_1, ..., f_s \rangle \subseteq \mathcal{W}_{\mathcal{M}_Q(j)} \subseteq \mathcal{D}_j\) be defined as above and let \(U\) be the \(e^{th}\) cropped matrix of \(\mathcal{E}\). Then, with the standard assignment, the block dimensions \(r_I\) and \(c_I\) of \(I = (I_1, ..., I_n) \in \mathcal{G}_Q\) are given as follows.
Setting \( p = I_1 + \ldots + I_n \) and \( q = Q_1 + \ldots + Q_n \), we have:

\[(28) \quad \text{If } p \leq e, \text{ then } r_I = s \left( \frac{e - p + r - n - 1}{r - n - 1} \right); \text{ otherwise, } r_I = 0.\]

\[(29) \quad \text{If } q - p \leq d, \text{ then } c_I = \left( \frac{d - (q - p) + r - n - 1}{r - n - 1} \right); \text{ otherwise, } c_I = 0.\]

**Proof.** To find \( r_I \), we count the number of ways of forming multi-indexes \((E, i)\) associated to \( I \). Writing \( E = (I_1, \ldots, I_n, e_{n+1}, \ldots, e_r) \), and recalling that \( E \) must be of degree \( e \), we see immediately that this is impossible unless \( e \geq I_1 + \ldots + I_n = p \).

In this case, we must assign non-negative integer values of \( e_{n+1}, \ldots, e_r \) that bring the total degree up to \( e \). Equivalently, we must count the number of monomials of degree \( e - p \) in \( r - n \) variables (and then multiply by \( s \) to account for all possible choices of \( i \)). As is well-known, there are \( \binom{t + u - 1}{u - 1} \) monomials of degree \( t \) in \( u \) variables, so \( r_I = s \left( \frac{e - p + r - n - 1}{r - n - 1} \right) \) when \( p \leq e \).

Similarly, to find \( c_I \) we count the number of ways of forming multi-indexes \( D \) associated to \( I \). Writing \( D = (Q_1 - I_1, \ldots, Q_n - I_n, d_{n+1}, \ldots, d_r) \), and recalling that \( D \) must be of degree \( d \), we see immediately that this is impossible unless \( d \geq (Q_1 - I_1) + \ldots + (Q_n - I_n) = q - p \). In this case, we must assign non-negative integer values of \( d_{n+1}, \ldots, d_r \) that bring the total degree up to \( d \). That is, we must count the number of monomials of degree \( d - (q - p) \) in \( r - n \) variables. This gives \( c_I = \left( \frac{d - (q - p) + r - n - 1}{r - n - 1} \right) \) when \( d \geq q - p \). \( \square \)
**Corollary 6.12.** Let the family of vector subspaces
\[ \mathcal{E} := \langle f_1, \ldots, f_s \rangle \subseteq \mathcal{W}_{\mathcal{M}(j)} \subseteq \mathcal{D}_j \] be defined as above and let \( U \) be the \( e^{th} \) cropped matrix of \( \mathcal{E} \). Under the standard assignment, denote the block dimensions \( r_I \) and \( c_I \). If \( I, J \in \mathcal{G}_Q \) and \( I \succeq J \), then \( r_I \geq r_J \) and \( c_I \leq c_J \). In particular, both \( r_I \) and the excess \( A_I = r_I - c_I \) are order-preserving functions on \( \mathcal{G}_Q \).

**Proof.** This is a consequence of the formulas in Proposition 6.11. Write \( I = (I_1, \ldots, I_n) \), \( J = (J_1, \ldots, J_n) \), \( p_I = I_1 + \ldots + I_n \), \( p_J = J_1 + \ldots + J_n \). We remark that if \( I \succeq J \), then the definition of partial order gives \( p_I \leq p_J \) and \( q - p_I \geq q - p_J \).

To see that \( r_I \) is order-preserving, assume that \( I \succeq J \). If \( e < p_I \leq p_J \), then \( r_I = r_J = 0 \) and \( r_I \geq r_J \) as required. If \( p_I \leq e < p_J \), then \( r_I = s \left( \frac{e - p_I + r - n - 1}{r - n - 1} \right) \) and \( r_J = 0 \), and again \( r_I \geq r_J \). Finally, if \( p_I \leq p_J \leq e \), then \( r_I = s \left( \frac{e - p_I + r - n - 1}{r - n - 1} \right) \) and \( r_J = s \left( \frac{e - p_J + r - n - 1}{r - n - 1} \right) \). Since \( p_I \leq p_J \), this gives \( r_I \geq r_J \).

The argument for \( c_I \) is similar. If \( q - p_I \geq q - p_J > d \), then \( c_I = c_J = 0 \) and \( c_I \leq c_J \) as required. If \( q - p_I > d \geq q - p_J \), then \( c_I = 0 \) and \( c_J = s \left( \frac{d - (q - p_J) + r - n - 1}{r - n - 1} \right) \), and again \( c_I \leq c_J \). Finally, if \( d \geq q - p_I \geq q - p_J \), then \( c_I = \left( \frac{d - (q - p_I) + r - n - 1}{r - n - 1} \right) \) and \( c_J = \left( \frac{d - (q - p_J) + r - n - 1}{r - n - 1} \right) \). Since \( p_I \leq p_J \), this gives \( c_I \leq c_J \). □

**Theorem 6.13.** Let the family of vector subspaces \( \mathcal{E} := \langle f_1, \ldots, f_s \rangle \subseteq \mathcal{W}_{\mathcal{M}(j)} \subseteq \mathcal{D}_j \) be defined as above and let \( U \) be the \( e^{th} \) cropped matrix of \( \mathcal{E} \). If \( U \) has at least as many rows as columns, then it has maximal rank.

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PROOF. We use the standard assignment to regard $U$ as an L-matrix with $G_Q$ pattern. Since it has at least as many rows as columns,

$$0 \leq \sum_{I \in G_Q} r_I - \sum_{I \in G_Q} c_I = \sum_{I \in G_Q} A_I.$$ 

By Corollary 6.12, $A_I$ is an order-preserving function on $G_Q$, so according to Proposition 5.7:

(30) For any topset $T \subseteq G_Q$, $\sum_{I \in T} A_I \geq 0$.

By Theorem 5.11 this would settle the matter, if only $U$ were square. So our goal is to show that we can delete rows, one at a time, in such a way that, at every stage, (30) remains true for the new values of $A_I$ corresponding to the submatrix (still with $G_Q$ pattern) formed by deleting the row.

At each stage, we consider the subset $G' \subseteq G_Q$, consisting of all multi-indexes $I$ for which $r_I$ remains nonzero. When we start out, $G'$ is a topset, because, by Corollary 6.12 $r_I$ is order-preserving: given $I, J \in G_Q$ such that $r_I > 0$ and $J \succeq I$, we have $r_J \geq r_I > 0$ and $J \in G'$. When we delete a row, we always choose a row associated with an $I$ that is minimal in $G'$, and claim that $G'$ remains a topset: If before the deletion, $r_I > 1$, $G'$ is unchanged. If before the deletion, $r_I = 1$, the deletion will remove $I$ from $G'$, and the result will remain a topset. (See Lemma 5.1 (3), setting $X := G' - \{I\}$.)
So assume at some stage that we have deleted some number of rows from $U$, each time diminishing $r_1$ by 1 for some minimal $I \in G'$, and that (30) remains true. If we are not yet done, by Lemma 5.10 it must be that $\sum_{I \in G'} A_I > 0$, that is, the inequality is strict. We seek to find a minimal multi-index $I \in G'$ with the property that any topset $T \subseteq G'$ that contains $I$ has $\sum_{I \in T} A_I > 0$. If such an $I$ exists, we can delete a row associated to $I$ (thus diminishing $r_I$, and therefore also $A_I$, by 1) and (30) will remain true. The assertion is that such a minimal multi-index $I$ can always be found.

To prove the assertion, we argue by contradiction. Assume there are $m$ minimal elements $I_1, \ldots, I_m$ of $G'$ and that each $I_i$ lies in a topset $T_i$ for which $\sum_{I \in T_i} A_I = 0$. We claim $\sum_{I \in T_1 \cup \ldots \cup T_m} A_I = 0$. But by Lemma 5.1(5), $T_1 \cup \ldots \cup T_m = G'$, and we are assuming $\sum_{I \in G'} A_I > 0$. This contradiction proves the theorem, once the claim is established.

To establish the claim, we prove by induction on $p$ the statement that
$$\sum_{I \in T_1 \cup \ldots \cup T_p} A_I = 0.$$  For $p = 1$, this is true because we have assumed $\sum_{I \in T_1} A_I = 0$. For the induction step, assume $\sum_{I \in T_1 \cup \ldots \cup T_{p-1}} A_I = 0$. Write $X := T_1 \cup \ldots \cup T_{p-1}$, and observe that $X$ is a topset by Lemma 5.1(2). Then $X \cap T_p$ and $X \cup T_p$ are also topsets, again by Lemma 5.1(2), and
$$\sum_{I \in X \cap T_p} A_I + \sum_{I \in X \cup T_p} A_I = \sum_{I \in X} A_I + \sum_{I \in T_p} A_I = 0 + 0 = 0.$$
This forces $\sum_{I \in X \cup T} A_I = 0$, since both terms on the left are non-negative. But of course $X \cup T = T_1 \cup \ldots \cup T_p$. \hfill \Box

We collect several results together into one theorem.

**Theorem 6.14.** Let the family of vector subspaces $\mathcal{E} := \langle f_1, \ldots, f_s \rangle \subseteq \mathcal{W}_{M_Q(j)} \subseteq \mathcal{D}_j$ be defined as above and let $U$ be the $e^{th}$ cropped matrix of $\mathcal{E}$. If $U$ has at least as many rows as columns, or more generally if $U$ has maximal rank, then for general $C \in k^{sm_Q(j)}$, $h_{\mathcal{E}(C)}(d) = \text{rank}(U) = \dim_k R_e \ast \mathcal{E}(C)$.

**Proof.** By Theorem 6.13, $U$ having at least as many rows as columns guarantees that $U$ has maximal rank.

In any event, $U$ is an L-matrix by Lemma 6.9. By Lemma 4.6, $U(C)$ has maximal rank = $\text{rank}(U)$ for general $C \in k^{sm_Q(j)}$, which is the same as $\dim_k R_e \ast \mathcal{E}(C)$ by Corollary 6.8. Finally, by Lemma 2.10, this is the same as $h_{\mathcal{E}(C)}(d)$. \hfill \Box

**Corollary 6.15.** Let the family of vector subspaces $\mathcal{E} := \langle f_1, \ldots, f_s \rangle \subseteq \mathcal{W}_{M_Q(j)} \subseteq \mathcal{D}_j$ be defined as above, where $s \leq m_Q(j)$. Then for general $C \in k^{sm_Q(j)}$, $\dim_k \mathcal{E}(C) = s$.

**Proof.** Let $N := m_Q(j)$ and let $\mathcal{E}' := \langle f_1, \ldots, f_N \rangle \subseteq \mathcal{W}_{M_Q(j)} \subseteq \mathcal{D}_j$. Then setting $e = 0$, the $e^{th}$ cropped matrix $U$ of $\mathcal{E}'$ is $N \times N$, and we apply Theorem 6.14. We find that, for general $C \in k^{N^2}$, $\dim_k \mathcal{E}'(C) := \dim_k R_0 \ast \mathcal{E}'(C) = \text{rank}(U) = N$; thus $f_1(C), \ldots, f_N(C)$ are linearly independent, and perforce $f_1(C), \ldots, f_s(C)$ are also linearly independent. Let $V \subseteq k^{N^2}$ be the Zariski-open dense set on which
\( f_1(C), ..., f_N(C) \) are linearly independent. Then, as a subset of \( k^{sN} \), \( V \cap k^{sN} \) is Zariski-open; and it is nonempty, thus dense. Equivalently, for general \( C \in k^{sN} \), \( f_1(C), ..., f_s(C) \) are linearly independent, and \( \dim_k E(C) = s. \)

\[ \square \]

3. Intersections of Subspaces of \( D_j \)

For the theorem proved in this section, we select a notation that will be convenient later. We define, as above, a family of vector subspaces \( \mathcal{F} := \langle g_1, ..., g_u \rangle \subseteq \mathcal{W}_{\mathcal{M}_Q(j)} \subseteq D_j \). We let \( U \) be the \( e^{th} \) cropped matrix of \( \mathcal{F} \) and let \( T \) be a submatrix of \( U \). Recalling that the columns of \( U \) are indexed by \( \mathcal{M}_Q(d) \), let \( \mathcal{M}_T \subseteq \mathcal{M}_Q(d) \) be the set of all column indices in \( T \), and let \( \mathcal{M}_{T^c} \subseteq \mathcal{M}_Q(d) \) be the set of all column indices not in \( T \). We define, for use in this section and in later sections,

\begin{equation}
\mathcal{V}_{\mathcal{M}_Q(d)} := \langle \{ x^D | D \in \mathcal{M}_Q(d) \} \rangle \subseteq D_d.
\end{equation}

We remark a peculiarity of the notation, namely, that \( \mathcal{V}_{\mathcal{M}_Q(j)} \) is the same as \( \mathcal{W}_{\mathcal{M}_Q(j)} \). The difference in notation highlights the distinction that \( \mathcal{W}_{\mathcal{M}_Q(j)} \) is a vector subspace of \( D_j \), whereas \( \mathcal{V}_{\mathcal{M}_Q(d)} \) is a vector subspace of \( D_d \). We also define

\[ \mathcal{V}_{\mathcal{M}_T} := \langle \{ x^D | D \in \mathcal{M}_T \} \rangle \subseteq D_d. \]

and

\begin{equation}
\mathcal{V}_{\mathcal{M}_{T^c}} := \langle \{ x^D | D \in \mathcal{M}_{T^c} \} \rangle \subseteq D_d.
\end{equation}
Theorem 6.16. Let the family of vector subspaces \( \mathcal{F} := \langle g_1, ..., g_u \rangle \subseteq \mathcal{W}_{MQ(i)} \subseteq \mathcal{D} \) be defined as above and let \( U \) be the \( e \)th coefficient matrix of \( \mathcal{F} \). Assume that \( U \) is of dimension \( p \times q \) and that \( T \) is a \( t \times t \) square submatrix of \( U \) whose determinant is nonzero. Let \( \mathcal{M}' \) be the set of all row indices \((E, i)\) of \( T \). Let \( \mathcal{W} \subseteq \mathcal{D}_d \) be a vector subspace such that \((R_e \ast \mathcal{F}(C)) \cap \mathcal{W} \subseteq \mathcal{V}_{MQ TC} \) for all \( C \in k_{MQ(i)} \). Then

(i) For general \( C \in k_{MQ(i)} \), \( \langle \{X^E \ast g_i(C)| (E, i) \in \mathcal{M}'\} \rangle \cap \mathcal{V}_{MQ TC} = \{0\} \).

(ii) If \( q \geq p = t \), then for general \( C \in k_{MQ(i)} \), \( (R_e \ast \mathcal{F}(C)) \cap \mathcal{V}_{MQ TC} = \{0\} \) and \( (R_e \ast \mathcal{F}(C)) \cap \mathcal{W} = \{0\} \).

Proof. We can express \( \mathcal{V}_{MQ(d)} \) as an internal direct sum:

\[
\mathcal{V}_{MQ(d)} = \mathcal{V}_{MT} \bigoplus \mathcal{V}_{MTc}.
\]

If we now focus on a particular \( X^E \ast g_i(C) \in \mathcal{V}_{MQ(d)} \subseteq \mathcal{D}_d \), we have

\[
X^E \ast g_i(C) = \sum_{D \in MQ(d)} u_{(E, i)D}(C)x^D
\]

\[
= \sum_{D \in MT} u_{(E, i)D}(C)x^D + \sum_{D \in MTc} u_{(E, i)D}(C)x^D
\]

\[
\subseteq \mathcal{V}_{MT} \bigoplus \mathcal{V}_{MTc},
\]

and we observe from Definition 6.6 that \( u_{(E, i)D} \) is the entry of \( U \) appearing in the \(((E, i), D)\) position. For a linear combination \( \sum_{(E, i) \in \mathcal{M}'} a_{Ei}X^E \ast g_i(C) \), we have
\[
\sum_{(E,i) \in \mathcal{M}'} a_{Ei} X^E \ast g_i(C) = \sum_{(E,i) \in \mathcal{M}'} a_{Ei} \sum_{D \in \mathcal{M}_Q(d)} u_{(E,i)D}(C) x^D
\]

\[
= \sum_{(E,i) \in \mathcal{M}'} a_{Ei} \sum_{D \in \mathcal{M}_T} u_{(E,i)D}(C) x^D + \sum_{(E,i) \in \mathcal{M}'} a_{Ei} \sum_{D \in \mathcal{M}_T} u_{(E,i)D}(C) x^D
\]

\[
\subseteq \mathcal{V}_{\mathcal{M}_T} \bigoplus \mathcal{V}_{\mathcal{M}_{Tc}}.
\]

The non-vanishing of \( \det(T) \) (as a polynomial in the coefficients \( z_{ij} \) of the \( g_i \)'s), being a Zariski-open condition, guarantees that the row vectors
\[
\{ \sum_{D \in \mathcal{M}_T} u_{(E,i)D}(C) x^D | (E,i) \in \mathcal{M}' \}
\]
of \( T \) are linearly independent for general \( C \in k^{\sum_{Q(j)}} \). For such a choice of \( C \), the linear combination
\[
\sum_{(E,i) \in \mathcal{M}'} a_{Ei} \sum_{D \in \mathcal{M}_T} u_{(E,i)D}(C) x^D \in \mathcal{V}_{\mathcal{M}_T}
\]
is never 0 unless all of the \( a_{Ei} \)'s are 0, in which case \( \sum_{(E,i) \in \mathcal{M}'} a_{Ei} X^E \ast g_i(C) \) = 0. This gives \( \{ \{ X^E \ast g_i(C) | (E,i) \in \mathcal{M}' \} \} \cap \mathcal{V}_{\mathcal{M}_{Tc}} = \{0\} \), which proves (i).

For (ii), we are considering the special case that \( q \geq p = t \), which is to say that \( U \) has at least as many columns as rows, and that \( T \) is a maximal square submatrix. In this case, \( \mathcal{M}' \) comprises all the rows of \( U \), which by construction are indexed by \( \{ (E,i) | E \in \mathcal{M}_Q(e) \} \). By Lemma 6.1, \( R_e \ast \mathcal{F}(C) = \langle \{ X^E \ast g_i(C) | E \text{ is of degree } e \} \rangle \), and we have seen in Lemma 6.5 that nothing is lost by considering only those \( E \) that lie in \( \mathcal{M}_Q(e) \). Thus \( R_e \ast \mathcal{F}(C) = \langle \{ X^E \ast g_i(C) | (E,i) \in \mathcal{M}' \} \rangle \), and the first statement of (ii) follows from (i).
Finally, for general $C \in k^{umQ(j)}$, we have assumed $(R_e \ast F(C)) \cap W \subseteq \mathcal{V}_{\mathcal{M}_T}$, and of course $(R_e \ast F(C)) \cap W \subseteq R_e \ast \mathcal{F}(C)$. Thus

$$(R_e \ast \mathcal{F}(C)) \cap W \subseteq (R_e \ast \mathcal{F}(C)) \cap \mathcal{V}_{\mathcal{M}_T} = \{0\}.$$
CHAPTER 7

Special Cases for Interesting Choices of \( Q \)

In this chapter we examine some special cases that result from particular choices of constraints \( Q := (Q_1, ..., Q_n) \), some of which will be used later to construct non-unimodal level algebras. In the first three sections of this chapter, the following outline will be followed. We assume that a choice of codimension \( r \) and socle degree \( j \) has been made, and we state the constraint \( Q := (Q_1, ..., Q_n) \) that is to be studied in the section. We assume that a degree \( d \) has been chosen and we set \( e = j - d \). We study the situation that some number \( u \) of polynomials \( g_1, ..., g_u \) have been selected from \( D_j \) to generate a vector subspace \( \langle g_1, ..., g_u \rangle \subseteq D_j \), subject to the condition that all monomials appearing in these generators are to be constrained by \( Q \). That is, we consider the family of subspaces \( \mathcal{F} := \langle g_1, ..., g_u \rangle \subseteq \mathcal{W}_{\mathcal{M}_Q(j)} \subseteq \mathcal{D}_j \), parameterized by elements \( C' = (..., c'_{ij}, ...) \in k^{um_Q(j)} \).

We remark that the choice of notation has been influenced by context: we will be applying these results in a context where we have already defined, for some other constraint \( P \), a family \( \mathcal{E} := \langle f_1, ..., f_s \rangle \subseteq \mathcal{W}_{\mathcal{M}_P(j)} \subseteq \mathcal{D}_j \), parameterized by elements \( C = (..., c_{ij}, ...) \in k^{sm_P(j)} \). We let \( \mathcal{U} \) denote the \( e^{th} \) cropped matrix of \( \mathcal{F} \).

1. Absence of Constraints

If we set \( n = r \) and \( Q_1 = ... = Q_r = j \), there are no actual constraints imposed. For any degree \( d \), we have \( \mathcal{V}_{\mathcal{M}_Q(d)} = D_d \) (recalling the definition in (31)); and in
particular $\mathcal{W}_{\mathcal{M}_Q(j)} = D_j$. We can easily count $m_Q(d) := \#(\mathcal{M}_Q(d))$ as the number of monomials of degree $d$ in $r$ variables, namely, $\binom{d + r - 1}{r - 1}$.

With the results obtained so far, we are in a position to prove one of the two theorems of A. Iarrobino quoted earlier, although we now restate it slightly different language.

**Theorem 7.1.** Consider the family of vector subspaces $\mathcal{F} := \langle g_1 \rangle \subseteq D_j$. Then for general $C' \in k^{m_Q(j)}$

\[ h_{\mathcal{F}(C')}(d) = \min(\dim_k R_{j-d}, \dim_k D_d). \]

**Proof.** Given $d$, we set $e := j - d$ and we construct $U$, the $e^{th}$ cropped matrix of $\mathcal{F}$, which by Lemma 6.9 is an L-matrix. We remark that all entries of $U$ are nonzero, since for any two multi-indexes $E$ of degree $e$ and $D$ of degree $d$, $E + D \in \mathcal{M}_Q(j)$. Thus, by Lemma 4.3 every square submatrix of $U$ has nonzero determinant, and $U$ has maximal rank. That is, its rank is either $m_Q(e) = \dim_k R_e$, the number of rows, or $m_Q(d) = \dim_k D_d$, the number of columns, whichever is smaller; equivalently, the rank is $\min(\dim_k R_{j-d}, \dim_k D_d)$. By Theorem 6.14, for general $C' \in k^{m_Q(j)}$, this is the same as $h_{\mathcal{F}(C')}(d)$.

\[ \square \]
2. $k[x_1,...,x_m]_j$

In this section we choose $m$ such that $r \geq m \geq 3$, and consider the constraint $Q := (j,...,j,0,...,0)$ of dimension $r$, in which the first $m$ constraints are $j$ and the remaining $r - m$ constraints are 0. We do not exclude the possibility that $m = r$.

**Proposition 7.2.** Let $r = n \geq m \geq 3$, fix a socle degree $j$, and consider the constraint $Q := (j,...,j,0,...,0)$ in which the first $m$ constraints are $j$ and the remaining $r - m$ constraints are 0. Let $p := m - 1$. Fix a degree $d$, and let $e := j - d$. Let $U$ be the $e$th cropped matrix of the family of vector subspaces $\mathcal{F} := \langle g_1,...,g_u \rangle \subseteq \mathcal{W}_{\mathcal{M}(j)} = k[x_1,...,x_m] \subseteq D_j$, where $u$ is chosen such that $u \left( \begin{array}{c} e + p \\ p \end{array} \right) \leq \left( \begin{array}{c} d + p \\ p \end{array} \right)$. Then

(i) $U$ is a $u \left( \begin{array}{c} e + p \\ p \end{array} \right) \times \left( \begin{array}{c} d + p \\ p \end{array} \right)$ matrix with at least as many columns as rows, all of whose entries are nonzero.

(ii) $U$ is of maximal rank $u \left( \begin{array}{c} e + p \\ p \end{array} \right)$. For general $C' \in k^{um_Q(j)}$, $\dim_k \mathbb{R}^\ast F(C') = u \left( \begin{array}{c} e + p \\ p \end{array} \right)$.

If also $\mathcal{W} \subseteq D_d$ is a vector subspace for which $\mathcal{W} \cap k[x_1,...,x_m]_d \subseteq \mathcal{Z}$, where $\mathcal{Z} \subseteq k[x_1,...,x_m]_d$ is a vector subspace, generated by monomials, such that

$\dim_k \mathcal{Z} \leq \left( \begin{array}{c} d + p \\ p \end{array} \right) - u \left( \begin{array}{c} e + p \\ p \end{array} \right)$, then

(iii) For general $C' \in k^{um_Q(j)}$, $\mathcal{W} \cap (\mathbb{R}^\ast F(C')) = \{0\}$.

**Proof.** Since $Q_1 = ... = Q_m = j$, no effective constraint is placed on the first $m$ variables $x_1,...,x_m$. Since $Q_{m+1} = ... = Q_n = 0$, no other variables are allowed to appear at all. So for any degree $d$, $\mathcal{W}_{\mathcal{M}(d)}$ (defined in (81)) is spanned by all monomials (of degree $d$) in which no variables other than $x_1,...,x_m$ appear,
of which there are \( \binom{d+p}{p} \). If we regard \( k[x_1, ..., x_m]_d \) as a vector subspace of \( D_d \), we have \( V_{\mathcal{M}_Q(d)} = k[x_1, ..., x_m]_d \).

To show (i): By definition, \( U \) has \( um_Q(e) = u \binom{e+p}{p} \) rows and \( m_Q(d) = \binom{d+p}{p} \) columns. Since \( u \binom{e+p}{p} \leq \binom{d+p}{p} \), \( U \) has at least as many columns as rows. By construction, the \(((E, i), D)\) row has a nonzero entry whenever \( E + D \in \mathcal{M}_Q(j) \); this always happens because if the only nonzero co-ordinates of \( E \) and \( D \) occur among the first \( m \), the same is true for \( E + D \).

For (ii), we apply Lemma 6.9 to show \( U \) is an L-matrix, and then Lemma 4.3 to show \( U \) has maximal rank. In fact, since all entries of \( U \) are nonzero, any maximal square submatrix of \( U \) has nonzero determinant, and \( U \) has maximal rank. This rank is of course \( u \binom{e+p}{p} \), the number of rows.

For (iii), we recall the definition of \( V_{\mathcal{M}_{Tc}} \) from (32) and we seek to apply Theorem 6.16(ii). In order to do so, we must find a maximal square submatrix \( T \) of \( U \) whose determinant is nonzero and whose columns are indexed by multi-indexes \( D \in \mathcal{M}_Q(d) \) for which the monomial \( x^D \in k[x_1, ..., x_m]_d \) is not among the monomial generators of \( Z \). This would ensure that \( Z \subseteq V_{\mathcal{M}_{Tc}} \). Also, the hypothesis that \( W \cap k[x_1, ..., x_m]_d \subseteq Z \) ensures that, for all \( C' \in k^{um_Q(j)} \), \( W \cap R_e \ast \mathcal{F}(C') \subseteq Z \subseteq V_{\mathcal{M}_{Tc}} \). Thus, provided a suitable \( T \) can be found, the conditions of Theorem 6.16 are met, and we conclude that \( W \cap (R_e \ast \mathcal{F}(C')) = \{0\} \).
for general $C' \in k^{u_{MQ}(j)}$.

Finding $T$ is easy: create $T$ from any $u\left(\begin{array}{c} e+p \\ p \end{array}\right)$ columns corresponding to indices $D$ for which $x^D$ is not among the generators of $Z$. This can be done because we have assumed $u\left(\begin{array}{c} e+p \\ p \end{array}\right) \leq \left(\begin{array}{c} d+p \\ p \end{array}\right) - \dim_k Z$. As remarked above, any maximal square submatrix of $U$ has nonzero determinant, so in particular $\det(T)$ is nonzero.

□

3. $Q = (1)$

In this section, we consider the case that a single variable $x_1$ is constrained so that if it appears in any term, it does so with exponent 1.

**Proposition 7.3.** Let $n = 1$ and let $Q := (1)$ for some socle degree $j$. Fix a degree $d$, let $e := j - d$, and assume $d \geq e \geq 1$. Let $U$ be the $e^{th}$ cropped matrix of $F := \langle g_1 \rangle \subseteq W_{M_Q(j)} \subseteq D_j$. Then

(i) $U$ is a block matrix of the form

\[
\begin{bmatrix}
  c_0 & c_1 \\
  r_1 & 0 & A \\
  r_0 & B & C
\end{bmatrix}
\]
where 0 denotes a block of zeroes and blocks A, B, and C consist entirely of nonzero entries. The dimensions of the blocks are

\[
\begin{align*}
  r_1 &= \binom{(e - 1) + (r - 2)}{r - 2}, \\
  r_0 &= \binom{e + (r - 2)}{r - 2}, \\
  c_0 &= \binom{(d - 1) + (r - 2)}{r - 2}, \\
  c_1 &= \binom{d + (r - 2)}{r - 2}.
\end{align*}
\]

(ii) \( U \) has maximal rank.

If also \( W \subseteq D_d \) is a vector subspace for which \( W \cap V_{M_Q(d)} \subseteq Z \), where \( Z \subseteq V_{M_Q(d)} \) is a vector subspace generated by \( 2c \) monomials, in \( c \) of which \( x_1 \) appears (with exponent 1) and in \( c \) of which \( x_1 \) does not appear, then

(iii) \( W \cap (R_e \ast F(C')) = \{0\} \) for general \( C' \in k^{m_Q(j)} \) if both

\[
(33) \quad m_Q(d) - 2c \geq m_Q(e)
\]

and

\[
(34) \quad \binom{d + (r - 2)}{r - 2} - c \geq \binom{(e - 1) + (r - 2)}{r - 2}.
\]

PROOF. For (i), we apply Lemma \([6,10]\) to establish that \( U \) is an L-matrix with \( G_Q \) pattern. Since \( G_Q = \{(0), (1)\} \), which for simplicity we will write as \( \{0, 1\} \), the order of rows, which must be lexicographic, is 1 then 0, and the order of columns,
which must be reverse lexicographic, is 0 then 1. The entries of block $B_{1j}$ are zero if and only if $I \not\supset J$, that is, only when $I = 1$ and $J = 0$.

To establish the dimensions of the blocks, we use the formulas in Proposition 6.11 with $s = 1, p_0 = 0, p_1 = 1$, and $q = 1$.

For (ii): From the formulas in (i) and the hypothesis that $d \geq e$, we see that $U$ has at least as many columns as rows, so it suffices to show that the rightmost square submatrix $T$ has nonzero determinant. If $T$ contains no entries from the 0 block, its entries are all nonzero, and $\det(T)$ is nonzero by Lemma 4.3. Otherwise, $T$ contains blocks $A$ and $C$ in their entirety, and has the form

\[
\begin{bmatrix}
  c'_0 & c_1 \\
r_1 & 0' & A \\
r_0 & B' & C
\end{bmatrix}
\]

By Theorem 5.11, $T$ will be nonsingular if, for every nonempty proper bottom-set $B \subseteq G_Q, \sum_{I \in B} A_I \leq 0$, where $A_I$ is the excess $r_I - c_I$. Since $G_Q = \{0, 1\}$, its only nonempty proper bottomset is $\{1\}$, so the condition reduces to $A_1 \leq 0$, that is, $r_1 \leq c_1$. This last condition follows from the formulas in (i) because we have assumed $d \geq e$.

For (iii), we apply Theorem 6.16(ii). To do so, we must construct a square submatrix $T$ of $U$ with nonzero determinant, such that the $2c$ monomial generators
of \( Z \) lie in \( \mathcal{V}_{Tc} \). Assuming this, the hypothesis that \( \mathcal{W} \cap \mathcal{V}_{M_Q(d)} \subseteq \mathcal{Z} \) ensures that \( \mathcal{W} \cap R_e \ast \mathcal{F}(C') \subseteq \mathcal{Z} \subseteq \mathcal{V}_{M_{Tc}} \) for all \( C' \in k^{m_Q(j)} \). Thus, provided a suitable \( T \) can be found, the conditions of Theorem 6.16 are met, and we conclude that \( \mathcal{W} \cap (R_e \ast \mathcal{F}(C')) = \{0\} \) for general \( C' \in k^{m_Q(j)} \).

So we ask under what circumstances a suitable submatrix \( T \) of \( U \) can be found. One requirement is that \( U \) have enough columns so that, when \( 2c \) of them are not used, there are still enough columns left to form a square \( m_Q(e) \times m_Q(e) \) submatrix \( T \). That is, we require \( m_Q(d) - 2c \geq m_Q(e) \). Assuming this, we must still ask whether we can find a suitable submatrix \( T \) whose determinant is nonzero. To this end, we delete from \( U \) the \( 2c \) columns corresponding to the generators of \( Z \), to obtain a submatrix \( U_0 \) of the form

\[
\begin{bmatrix}
c_0 - c & c_1 - c \\
\hline
r_1 & 0' & A' \\
r_0 & B' & C'
\end{bmatrix}
\]

and we argue as in part (ii): If the rightmost square submatrix \( T_0 \) of \( U_0 \) has no entries from the \( 0' \) block, \( \det(T_0) \) is nonzero by Lemma 4.3. Otherwise, the condition from Theorem 5.11 is that \( c_1 - c \geq r_1 \), or equivalently

\[
\binom{d + (r - 2)}{r - 2} - c \geq \binom{(e - 1) + (r - 2)}{r - 2}.
\]

The following special case will be of interest later.
**Corollary 7.4.** Let \( r = 4, n = 1 \) and \( Q := (1) \) for some socle degree \( j \). Fix a degree \( d \), let \( e := j - d \), and assume \( d \geq e \geq 1 \). Let \( U \) be the \( e \)th cropped matrix of the family of vector subspaces \( \mathcal{F} := \langle g_1 \rangle \subseteq W_{\mathcal{M}_Q(j)} \subseteq D_j \). Then

(i) \( U \) has maximal rank \( (e + 1)^2 \).

(ii) For general \( C' \in k^m_{Q(j)} \), \( \dim_k R_e \ast \mathcal{F}(C') = (e + 1)^2 \).

**Proof.** We use Proposition 7.3 for the case that \( r = 4 \).

For (i): \( U \) is of maximal rank, which is the number of rows. Substituting \( r = 4 \) into the formulas for the number of rows in each block gives

\[
r_1 + r_0 = \binom{e + 1}{2} + \binom{e + 2}{2} = \frac{e(e + 1)}{2} + \frac{(e + 2)(e + 1)}{2} = (2e + 2)\frac{(e + 1)}{2} = (e + 1)^2.
\]

For (ii), \( U(C') \) has maximal rank for general \( C' \in k^m_{Q(j)} \) by Lemma 4.6 and this rank is \( \dim_k R_e \ast \mathcal{F}(C') \) by Corollary 6.8.

\[\square\]

4. **Essentially n-fold-constrained**

In this section, we find it convenient to assume that exactly \( n \geq 1 \) of the variables are constrained to have less than the full range of exponents. In
In this case, we will say that, for any degree $d$, the multi-indexes in $M_Q(d)$ and their corresponding monomials are \textit{essentially $n$-fold-constrained}, or simply \textit{$n$-fold-constrained}.

We wish to compute $m_Q(d)$, or equivalently the dimension of the vector space generated by monomials of degree $d$ constrained by $Q$. For the purposes of this computation, we may as well assume that the constrained variables are listed first; that is, we assume $Q$ is a constraint of dimension $n \geq 1$ where $Q_i < j$ for $i = 1, ..., n$.

The following lemma makes this precise.

\textbf{Lemma 7.5.} Fix codimension $r$ and socle degree $j$, and let $Q := (Q_1, ..., Q_m)$ be a constraint of dimension $m$, such that $Q_{i_1}, ..., Q_{i_n}$ are all strictly less than $j$ and the rest of the $Q_i$'s are equal to $j$. Let $P := (Q_{i_1}, ..., Q_{i_m}, j, ..., j) := (Q_{\sigma(1)}, ..., Q_{\sigma(m)})$ be another constraint of dimension $m$ whose entries are related to those of $Q$ via some permutation $\sigma$ of $\{1, ..., m\}$, such that the entries equal to $j$ all come last. Then for any degree $d$, $m_Q(d) = m_P(d)$.

\textbf{Proof.} We define a function $b_{\sigma}$, evidently a bijection, from the set of all $r$-tuples of degree $d$ to itself, induced by $\sigma$, as follows. If $D := (d_1, ..., d_r)$, then $b_{\sigma}(D) := (d_{\sigma(1)}, ..., d_{\sigma(m)}, d_{m+1}, ..., d_r)$.

We now claim that a multi-index $D := (d_1, ..., d_r)$ of degree $d$ lies in $M_Q(d)$ if and only if the multi-index $b_{\sigma}(D)$ lies in $M_P(d)$: the first condition is that, for each $i = 1, ..., m$, $d_i \leq Q_i$; the second is that, for each $i = 1, ..., m$, $d_{\sigma(i)} \leq P_i = Q_{\sigma(i)}$. 


which is the same as the first condition because \( \{1, \ldots, m\} = \{\sigma(1), \ldots, \sigma(m)\} \).

Since \( b_{r'} \) is a bijection, its restriction to \( \mathcal{M}_Q(d) \) is a bijection from \( \mathcal{M}_Q(d) \) to its image \( \mathcal{M}_P(d) \). □

**Lemma 7.6.** Fix codimension \( r \) and socle degree \( j \), and let \( Q := (Q_1, \ldots, Q_n) \) be a constraint on \( r \)-tuples that fails to constrain the value of at least one coordinate. Then the function \( m_Q(d) \) is a non-decreasing function of \( d \) for \( 0 \leq d \leq j \).

**Proof.** We must show that if \( d_1 < d_2 \), then \( m_Q(d_1) := \#(\mathcal{M}_Q(d_1)) \leq m_Q(d_2) := \#(\mathcal{M}_Q(d_2)) \). Let the value of the \( i \)th coordinate not be constrained. Then for any element \( I = (I_1, \ldots, I_{i-1}, I_i, I_{i+1}, \ldots, I_r) \in \mathcal{M}_Q(d_1) \), there is a corresponding element \( (I_1, \ldots, I_{i-1}, I_i + (d_2 - d_1), I_{i+1}, \ldots, I_r) \in \mathcal{M}_Q(d_2) \). □

Getting closed formulas of a simple form will sometimes not be possible for some values of \( d \), but we will typically find that patterns emerge when \( d \) is large enough. We classify the results by \( n \), the number of constraints. In order to state the results more concisely, we define \( q := Q_1 + \ldots + Q_n \) and \( a_i := Q_i + 1 \) for \( i = 1, \ldots, n \). We denote a multi-index of degree \( d \) constrained by \( Q \) as \( D := (d_1, \ldots, d_r) \).

### 4.1. \( r \)-fold-constrained.

**Proposition 7.7.** When multi-indexes are \( r \)-fold constrained by \( Q \), \( m_Q(d) = 0 \) for \( d > q \).
PROOF. The largest possible degree of any multi-index is $q := Q_1 + ... + Q_r$. So for $d > q$, no multi-indexes are possible. □

4.2. $(r-1)$-fold-constrained.

PROPOSITION 7.8. When multi-indexes are $(r-1)$-fold constrained by $Q$, $m_Q(d) = a_1 a_2 \cdots a_{r-1}$ for $d \geq q$.

PROOF. All but one of the variables are constrained, and the last is allowed to vary. Thus, for any degree $d \geq q$, any choice of $(d_1, ..., d_{r-1})$ can be augmented by $d_r$ in a unique way to create a monomial of degree $d$. Since each $d_i$ can be chosen in $Q_i + 1 = a_i$ ways, we have $m_Q(d) = a_1 a_2 \cdots a_{r-1}$. □

4.3. $(r-2)$-fold-constrained.

PROPOSITION 7.9. Let $r \geq 3$ and let multi-indexes be $(r-2)$-fold constrained by $Q$. Let $S_n = \sum_{1 \leq i \leq n} a_i$ and $P_n = \prod_{1 \leq i \leq n} a_i$. Then for $d \geq q$, $m_Q(d) = \frac{P_n(2d - S_n + r)}{2}$.

PROOF. We proceed by induction on $r$, starting with $r = 3, n = 1$. For the initial case, we are counting multi-indexes $D = (d_1, d_2, d_3)$ such that $d_1 \leq Q_1$. For any choice of $d_1$, we can complete $D$ by choosing values of $d_2$ and $d_3$ whose sum is $d - d_1$, and there are $\binom{(d-d_1)+(2-1)}{2-1} = d - d_1 + 1$ ways to do it. So for
\[ d \geq q = Q_1, \]

\[
m_Q(d) = \sum_{0 \leq d_1 \leq Q_1} (d - d_1 + 1)
\]
\[
= \sum_{0 \leq d_1 \leq Q_1} (d + 1) - \sum_{0 \leq d_1 \leq Q_1} d_1
\]
\[
= a_1(d + 1) - a_1(a_1 - 1)/2
\]
\[
= \frac{a_1(2d - a_1 + 3)}{2}.
\]

In the induction step, we assume the proposition has been proved for \( r - 1 \) and we prove it for \( r \). For a fixed choice of \( d_n \), we ask how many choices of \( (d_1, \ldots, d_{n-1}, d_{n+1}, d_r) \) are permissible. This amounts to asking the value of \( m_{Q'}(d - d_n) \), for dimension \( r' = r - 1 \) and constraint \( Q' = (Q_1, \ldots, Q_{n-1}) \), which we claim satisfies the hypothesis of the proposition: \( r' \)-tuples are \( (r' - 2) \)-fold constrained by \( Q' \); and we have \( d_1 + \ldots + d_{n-1} \geq Q_1 + \ldots + Q_{n-1} \) since \( d \geq q \).
and $d_n \leq Q_n$. Applying the induction hypothesis,

$$m_Q(d) = \sum_{0 \leq d_n \leq Q_n} m_{Q'}(d - d_n)$$

$$= \sum_{0 \leq d_n \leq Q_n} \frac{P_{n-1}[2(d - d_n) - S_{n-1} + (r - 1)]}{2}$$

$$= a_n \frac{P_{n-1}[2d - S_{n-1} + (r - 1)]}{2} - a_n \frac{1}{2} \sum_{0 \leq d_n \leq Q_n} d_n$$

$$= a_n \frac{P_{n-1}[2d - S_{n-1} + (r - 1)]}{2} - a_n \frac{a_n(a_n - 1)}{2}$$

$$= a_n \frac{P_{n-1}[2d - S_{n-1} + (r - 1) - (a_n - 1)]}{2}$$

$$= a_n \frac{P_{n-1}[2d - S_n + r]}{2}$$

$$= \frac{P_n[2d - S_n + r]}{2}.$$

□

For the cases $r = 3$ and $r = 4$, we will need to know $\#(M_Q(d))$ for smaller values of $d$ than those covered by Proposition 7.9.

**Proposition 7.10.** Let $r = 3$ and let multi-indexes be once-constrained by $Q = (Q_1)$. Then

(i) $m_Q(d) = \frac{a_1(2d - a_1 + 3)}{2}$ for $d \geq q - 1 = a_1 - 2$.

(ii) $m_Q(d) = \frac{a_1(2d - a_1 + 3)}{2} + 1$ for $d = q - 2 = a_1 - 3$.

(iii) $m_Q(d) = \binom{d + 2}{2}$ for $d < a_1$. 

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PROOF. We start with the formulas from Proposition \ref{prop:7.9}

\[
\frac{a_1(2d - a_1 + 3)}{2} = m_Q(d) = \sum_{0 \leq d_1 \leq Q_1} (d - d_1 + 1) \quad \text{when } d \geq Q_1.
\]

Viewing the left- and right-hand sides of this equation as polynomials in \(d\), we see they agree for the infinitely many integer values of \(d\) such that \(d \geq Q_1\), and hence must agree for all \(d\).

We remark that, for values of \(d < Q_1\), a valid expression for \(m_Q(d)\) can be obtained as before, by summing terms of the form \(d - d_1 + 1\), for a suitable range of values of \(d_1\). We now investigate how to do this for \(d = Q_1 - 1\) and \(d = Q_1 - 2\).

For \(d = Q_1 - 1\), the summation ends with \(d_1 = Q_1 - 1\). Equivalently, one can take the previous summation and subtract the term for \(d_1 = Q_1\). That is,

\[
m_Q(d) = \sum_{0 \leq d_1 \leq Q_1 - 1} (d - d_1 + 1)
= \sum_{0 \leq d_1 \leq Q_1} (d - d_1 + 1) - [(Q_1 - 1) - Q_1 + 1]
= \frac{a_1(2d - a_1 + 3)}{2} - [(Q_1 - 1) - Q_1 + 1]
= \frac{a_1(2d - a_1 + 3)}{2}.
\]
For $d = Q_1 - 2$, the terms with $d_1$ having the values $Q_1$ and $Q_1 - 1$ must be omitted. That is,

$$m_Q(d) = \sum_{0 \leq d_1 \leq Q_1 - 2} (d - d_1 + 1) = \sum_{0 \leq d_1 \leq Q_1} (d - d_1 + 1) - [(Q_1 - 2) - Q_1 + 1] - [(Q_1 - 2) - (Q_1 - 1) + 1] = a_1(2d - a_1 + 3) - [(Q_1 - 2) - Q_1 + 1] - [(Q_1 - 2) - (Q_1 - 1) + 1] = a_1(2d - a_1 + 3) + 1.$$

When $d < a_1$, we have $d \leq Q_1$. So all of the $\binom{d + 2}{2}$ multi-indexes of degree $d$ satisfy the condition of being constrained by $(Q_1)$. Equivalently, $m_Q(d) = \binom{d + 2}{2}$. □

**Proposition 7.11.** Let $r = 4$ and let multi-indexes be twice-constrained by $Q = (Q_1, Q_2)$ Then

(i) $m_Q(d) = \frac{a_1a_2(2d - a_1 - a_2 + 4)}{2}$ for $d \geq q - 1 = a_1 + a_2 - 3$.

(ii) $m_Q(d) = \frac{a_1a_2(2d - a_1 - a_2 + 4)}{2} + 1$ for $d = q - 2 = a_1 + a_2 - 4$.

(iii) $m_Q(d) = \binom{d + 3}{3}$ for $d < \min(a_1, a_2)$.

**Proof.** The formula for $d \geq q$ is given by Proposition 7.9. An alternative formula is derived as follows, in a manner similar to the codimension 3 case. We are counting multi-indexes $D = (d_1, d_2, d_3, d_4)$ such that $d_1 \leq Q_1$ and $d_2 \leq Q_2$. For any choice of values of $d_1$ and $d_2$, we can complete $D$ by choosing values of $d_3$ and $d_4$ whose sum is $d - d_1 - d_2$, and there are $\binom{(d - d_1 - d_2) + (2 - 1)}{2 - 1} = \binom{d - d_1 - d_2 + 1}{1}$.
$d - d_1 - d_2 + 1$ ways to do it. So for $d \geq q = Q_1 + Q_2$,

$$m_Q(d) = \sum_{0 \leq d_1 \leq Q_1} \sum_{0 \leq d_2 \leq Q_2} (d - d_1 - d_2 + 1).$$

As in the previous theorem, we can regard the left- and right-hand sides of

$$\frac{a_1a_2(2d - a_1 - a_2 + 4)}{2} = m_Q(d) = \sum_{0 \leq d_1 \leq Q_1} \sum_{0 \leq d_2 \leq Q_2} (d - d_1 - d_2 + 1)$$

as an identity in $d$. And again, the same argument justifies the right-hand side as an expression for $m_Q(d)$ when $d = q - 1$ or $q - 2$, except that the range of summation must change.

For $d = q - 1$, the term with $d_1 = Q_1$ and $d_2 = Q_2$ must be omitted. That is,

$$m_Q(d) = \sum_{0 \leq d_1 \leq Q_1} \sum_{0 \leq d_2 \leq Q_2} (d - d_1 - d_2 + 1) - [(Q_1 + Q_2 - 1) - Q_1 - Q_2 + 1]$$

$$= \frac{a_1a_2(2d - a_1 - a_2 + 4)}{2} - [(Q_1 + Q_2 - 1) - Q_1 - Q_2 + 1]$$

$$= \frac{a_1a_2(2d - a_1 - a_2 + 4)}{2}.$$

For $d = q - 2$, there are three terms that must be omitted: those with $d_1 = Q_1$ and $d_2 = Q_2$; with $d_1 = Q_1$ and $d_2 = Q_2 - 1$; and with $d_1 = Q_1 - 1$ and $d_2 = Q_2$; that is,
\[ m_Q(d) = \sum_{0 \leq d_1 \leq Q_1} \sum_{0 \leq d_2 \leq Q_2} (d - d_1 - d_2 + 1) - [(Q_1 + Q_2 - 2) - Q_1 - Q_2 + 1] \\
- [(Q_1 + Q_2 - 2) - Q_1 - (Q_2 - 1) + 1] - [(Q_1 + Q_2 - 2) - (Q_1 - 1) - Q_2 + 1] \\
= \frac{a_1 a_2 (2d - a_1 - a_2 + 4)}{2} - [(Q_1 + Q_2 - 2) - Q_1 - Q_2 + 1] \\
- [(Q_1 + Q_2 - 2) - Q_1 - (Q_2 - 1) + 1] - [(Q_1 + Q_2 - 2) - (Q_1 - 1) - Q_2 + 1] \\
= \frac{a_1 a_2 (2d - a_1 - a_2 + 4)}{2} + 1. \]

When \( d < \min(a_1, a_2) \), we have \( d \leq Q_1 \) and \( d \leq Q_2 \). So all of the \( \binom{d+3}{3} \) multi-indexes of degree \( d \) satisfy the condition of being constrained by \((Q_1, Q_2)\). Equivalently, \( m_Q(d) = \binom{d+3}{3} \). \( \square \)

4.4. \((r - 3)\)-fold-constrained. For \((r - 3)\)-fold-constrained monomials, a closed-form expression would be complicated. We give a formula involving summations, and then obtain closed-form expressions for the cases that \( r \) is 4 or 5.

**Proposition 7.12.** Let \( r \geq 4 \) and let multi-indexes be \((r - 3)\)-fold constrained by \( Q \). Then for \( d \geq q \),

\[ m_Q(d) = \sum_{0 \leq d_1 \leq Q_1} \ldots \sum_{0 \leq d_{r-3} \leq Q_{r-3}} \binom{d - d_1 - \ldots - d_{r-3} + 2}{2}. \]

**Proof.** Once again, the approach is similar to the previous cases. We are counting multi-indexes \( D = (d_1, ..., d_r) \) such that \( d_1 \leq Q_1, ..., d_{r-3} \leq Q_{r-3} \). For any
choice of values of $d_1, ..., d_{r-3}$, we can complete $D$ by choosing values of $d_{r-2}, d_{r-1},$ and $d_r$ whose sum is $d - d_1 - ... - d_{r-3}$, and there are
\[
\left(\frac{(d - d_1 - ... - d_{r-3} + (3 - 1)}{3 - 1}\right)\text{ ways to do it.}
\]

\[\square\]

**Corollary 7.13.** Let $r = 4$ and let multi-indexes be once constrained by $Q$. Then for $d \geq q = Q_1$,
\[
m_Q(d) = \frac{a_1}{2} [d^2 + (4 - a_1)d + \frac{a_1^2 - 6a_1 + 11}{3}].
\]

**Proof.** We apply the formula from Proposition 7.12.

\[
m_Q(d) = \sum_{0 \leq d_1 \leq Q_1} \left(\frac{d - d_1 + 2}{2}\right)
\]
\[
= \sum_{0 \leq d_1 \leq Q_1} \frac{(d + 2 - d_1)(d + 1 - d_1)}{2}
\]
\[
= \frac{1}{2} \left[ \sum_{0 \leq d_1 \leq Q_1} (d^2 + 3d + 2) - (2d + 3) \sum_{0 \leq d_1 \leq Q_1} d_1 + \sum_{0 \leq d_1 \leq Q_1} d_1^2 \right]
\]
\[
= \frac{1}{2} [a_1(d^2 + 3d + 2) - (2d + 3) \frac{a_1(a_1 - 1)}{2} + \frac{a_1(a_1 - 1)(2a_1 - 1)}{6}]
\]
\[
= \frac{a_1}{2} [d^2 + (4 - a_1)d + \frac{12 - 9a_1 + 9 + 2a_1^2 - 3a_1 + 3}{6}]
\]
\[
= \frac{a_1}{2} [d^2 + (4 - a_1)d + \frac{a_1^2 - 6a_1 + 11}{3}]
\]
\[\square\]
We remark that substituting $a_1 = 2$ gives a confirmation of the formula in Corollary 7.4.

**Corollary 7.14.** Let $r = 5$ and let multi-indexes be twice constrained by $Q$. Then for $d \geq q = Q_1 + Q_2$,

$$m_Q(d) = \frac{a_1a_2}{2}[d^2 + (5 - a_1 - a_2)d + \frac{2(a_1^2 + a_2^2) - 15(a_1 + a_2) + 3a_1a_2 + 35}{6}].$$

**Proof.** We apply the formula from Proposition 7.12. The computation is similar in nature to that of the previous corollary. We omit the details. \[\square\]
CHAPTER 8

Construction of New Non-Unimodal Level Algebras

In this chapter we construct several families of non-unimodal level algebras. As was mentioned in an earlier chapter, some of the algebras described here were described and conjectured to be non-unimodal by A. Iarrobino in 2005 following lines suggested by F. Zanello in [Z06], and all of them use the same general framework of Iarrobino and Zanello. What is new here is that we prove these algebras to be non-unimodal.

1. Overview of the Construction of Level Algebras

In this section we co-ordinate earlier results and describe our framework for constructing new non-unimodal level algebras. We review some notation from previous sections, establish some new notation, and give an overall description of the process. This section is meant to be a qualitative overview. The quantitative statements that ensure non-unimodality are proved in later sections.

We let $k$ be an algebraically closed field of characteristic 0 and define $R := k[X_1, ..., X_r]$ and $D := k[x_1, ..., x_r]$. As previously described, the elements of $R$ act as differential operators on $D$. Specifically, we will let $r$ take the value 3, 4, or 5, and it will be convenient to reduce the number of subscripts by defining $X := X_1, Y :=$
$X_2, Z := X_3, W := X_4,$ and $V := X_5$; also $x := x_1, y := x_2, z := x_3, w := x_4,$ and $v := x_5$.

We fix a positive integer $j$ that will become the socle degree of the level algebra of codimension $r$ being constructed. For any constraint $K := (K_1, ..., K_n)$ of dimension $n \leq r$ such that $0 \leq K_i \leq j$ for $i = 1, ..., n$, we let $\mathcal{M}_K(d)$ denote the set of multi-indexes of dimension $r$ and degree $d$ constrained by $K$. We define $m_K(d) := \#(\mathcal{M}_K(d))$. We define $W_{\mathcal{M}_K(j)}$ to be the vector subspace of $D_j$ spanned by all monomials $x^J$ such that $J \in \mathcal{M}_K(j)$. For a degree $d \leq j$, we define $V_{\mathcal{M}_K(d)}$ to be the vector subspace of $D_d$ spanned by all monomials $x^D$ such that $D \in \mathcal{M}_K(d)$.

We consider two constraints $P := (P_1, ..., P_{n_1})$ of dimension $n_1$ and $Q := (Q_1, ..., Q_{n_2})$ of dimension $n_2$, and use them as follows.

We choose a positive integer $s$ and specify a family of vector subspaces $\mathcal{E} := \langle f_1, ..., f_s \rangle \subseteq W_{\mathcal{M}_P(j)} \subseteq D_j$ parameterized by elements of the irreducible affine variety $k^{smP(j)}$, where such an element represents a choice $C$ of coefficients for the polynomials $f_1, ..., f_s$. We construct, for general $C \in k^{smP(j)}$, the graded level algebra $A_{\mathcal{E}(C)} := R / \text{Ann}_R(\mathcal{E}(C))$. We remark that our previous discussion of Matlis Duality (in Chapter 2, section 3) motivates this construction, and in particular that Theorem 2.9 guarantees $A_{\mathcal{E}(C)}$ is level.
If \( s \) is sufficiently large, then for general \( C \) the Hilbert function \( h_{E(C)} \) of \( A_{E(C)} \) is computed, according to Theorem 6.14, by the rule 
\[
h_{E(C)}(d) = \dim_k R_e * E(C) = \text{rank}(U) = m_P(d),
\]
where \( e := j - d \) and \( U \) is the \( e^{th} \) cropped matrix of \( E \).

We will always choose \( P \) so that the monomials are \((r - 2)\)-fold constrained. That is, for \( r = 3 \), \( P := (P_1) \), where for technical reasons we require that \( P_1 \geq 3 \). For \( r = 4 \), \( P := (P_1, P_2) \) or \( (j, P_2, P_3) \), where we require respectively that \( P_2 \geq P_1 \geq 2 \) or \( P_3 \geq P_2 \geq 2 \). For \( r = 5 \), \( P := (P_1, P_2, P_3) \), where we require that \( P_3 \geq P_2 \geq P_1 \geq 2 \).

To compute values of \( m_P(d) \), we rely on Propositions 7.9, 7.10, and 7.11 which deal with monomials that are \((r - 2)\)-fold constrained. In these propositions, the definition is made that \( a_i := P_i + 1 \). However, again to reduce the number of subscripts, we define \( a := a_1, b := a_2, c := a_3 \). Also, we define \( \Delta \) to be either \( a, ab, \) or \( abc \), according to whether \( r = 3, 4, \) or 5.

We next perform a similar construction to specify another family of vector subspaces of \( D_j \). This time we choose a positive integer \( u \), which for the examples here will always be either 1 or 2, and specify a family of vector subspaces \( F := \langle g_1, ..., g_u \rangle \subseteq W_{MQ(j)} \subseteq D_j \) parameterized by elements of the irreducible affine variety \( k^{umQ(j)} \), where such an element represents a choice \( C' \) of coefficients for the polynomials \( g_1, ..., g_u \). Then we construct, for general \( C' \in k^{umQ(j)} \), the graded level algebra \( A_{F(C')} := R/Ann_R(F(C')) \).
The specific constraint $Q$ varies according to the family of non-unimodals being constructed, as follows. For $r = 3$, $Q := (j, j, j)$. For $r = 4$, $Q := (j, j, j, 0)$, (1), or $(j, j, j, j)$. For $r = 5$, $Q := (j, j, j, 0, 0)$.

For these choices of constraints, the Hilbert function $h_{F(C')}$ of $A_{F(C')}$ is computed according to Proposition 7.2 or Corollary 7.4.

Having constructed families of vector subspaces $E$ and $F$, we can construct the family $E + F$ and consider the family of level algebras $A_{E+ F}$ with corresponding Hilbert functions $h_{E+ F}$. For general $C$ and $C'$, we will prove that $E(C) + F(C') = E(C) \oplus F(C')$ and that $h_{E(C) \oplus F(C')}$ is non-unimodal.

We establish some terminology for discussing non-unimodality of a Hilbert function $h$.

**Definition 8.1.** The terms single drop, double drop, initial degree, final degree, and critical range are defined as follows.

If for some degree $i$, $h(i) > h(i+1) < h(i+2)$, we say that $h$ has a single drop with initial degree $i$ and final degree $i_f := i + 2$. If for some degree $i$, $h(i) > \max\{h(i+1), h(i+2)\} < h(i+3)$ we say that $h$ has a double drop with initial degree $i$ and final degree $i_f := i + 3$. In this chapter, we will use the variables $i$ and $i_f$ to represent the candidates for the initial and final degrees of a single or double drop.
We say that a degree $d$ is in the critical range if $i \leq d \leq i_f$.

To establish, for general $C \in k^{smp}(j)$ and $C' \in k^{unq}(j)$, that $h_{E(C) \oplus F(C')}$ exhibits a single drop with initial degree $i$, our method will be to show that

$$h_{E(C)}(i) + h_{F(C')}(i) > h_{E(C)}(i + 1) + h_{F(C')}(i + 1) < h_{E(C)}(i + 2) + h_{F(C')}(i + 2)$$

and then to establish that $h_{E(C) \oplus F(C')}(d) = h_{E(C)}(d) + h_{F(C')}(d)$ for $d = i, i + 1, i + 2$; similarly for a double drop. To this end, we define differences

$$\Delta_d = h_{E(C)}(d + 1) - h_{E(C)}(d) \text{ and } \delta_d = h_{F(C')}(d + 1) - h_{F(C')}(d).$$

**Lemma 8.2.** Assume, for some degree $d$, that

$$h_{E(C)} \oplus F(C')(d) = h_{E(C)}(d) + h_{F(C')}(d)$$

and that

$$h_{E(C)} \oplus F(C')(d + 1) = h_{E(C)}(d + 1) + h_{F(C')}(d + 1).$$

Then

$$h_{E(C)} \oplus F(C')(d + 1) = h_{E(C)} \oplus F(C')(d) + \Delta_d + \delta_d.$$

**Proof.** This follows immediately from the definitions. \qed

Finally, we will need to establish, for appropriate values of $d$, that the Hilbert functions $h_{E(C)}$ and $h_{F(C')}$ do indeed add as desired. To this end, we will be using Lemma 3.1 together with Propositions 7.2 and 7.3.
2. Computations by Computer

Direct computation is difficult with polynomials of high degree having many terms; instead, we use the [Macaulay2] computer program. We set the field $k$ equal to $\mathbb{Z}/32749\mathbb{Z}$, a large finite field (as suggested in [E01]); the finiteness permits rapid calculations and gives access to some special applications that are implemented only for finite fields. Following suggestions of A. Iarrobino, to simulate the selection of general members of a vector space we first generate pseudorandom scalars on the computer; for a fixed basis, we then use these scalars as coefficients to produce members of the vector space; and we hope that this procedure does in fact approximate the selection of general members. We use the command “fromDual($\mathcal{W}$)” to compute $R/\text{Ann}(\mathcal{W})$ for a vector subspace $\mathcal{W} \subseteq \mathcal{D}$.

Computers are useful for comprehension and they sometimes provide persuasive plausibility arguments. But the proofs of non-unimodality given here are entirely independent of computer results.

3. Six Families of Level Algebras, together with Computer-Calculated Hilbert Functions

In this section we define six parameterized families $A_{\mathcal{E}+\mathcal{F}}$ of level algebras according to the program of the previous section. That is, for each choice of parameters we obtain a family of algebras. We will show, in a later section, that each choice of parameters yields a family of algebras that are non-unimodal for general $C$ and $C'$. In this section, we confine ourselves to definitions, examples,
and display of computer results.

For each parameterized family, one of the parameters is $i$, which denotes the initial degree of the single or double drop that (we will subsequently prove) occurs in the Hilbert function. It is not necessary to specify the final degree $i_f$ as another parameter, because its value can be calculated from $i$, once we make the claim that all algebras in families $F_2, G_2,$ and $G_3$ have a single drop (so that $i_f = i + 2$) and all algebras in families $F_1, G_1,$ and $H_1$ have a double drop (so that $i_f = i + 3$).

For each family we specify that the parameter $s$, the number of vectors generating $\mathcal{E}$, be $h_{\mathcal{E}}$-sufficient, by which we mean that the $(j - i_f)^{th}$ cropped matrix of $\mathcal{E}$ should have at least as many rows as columns, a condition motivated by Theorem 6.14. For now, we do not state the precise values of $s$ that are $h_{\mathcal{E}}$-sufficient, postponing the discussion until Lemma 8.19. The number $u$ of vectors generating $\mathcal{F}$ is not a parameter, since it is fixed within each family. The type of the resulting level algebra is then $\min(s, \dim_k \mathcal{M}_p(j)) + u$.

When displaying computer results, we will simplify notation by writing $h$ instead of $h_{\mathcal{E}(C)} \oplus \mathcal{F}(C')$.

**Definition 8.3.** The family $F_1(a, i, s)$ is obtained by setting $r = 3$, $j = i + a$, $P = (a - 1)$, $Q = (j, j, j)$, $u = 1$. We require that $a \geq 4$, that $i \geq 2a$, and that $s$
EXAMPLE $F_1(21, 42, 4)$: Let $r = 3, t = 4 + 1 = 5, R = k[X, Y, Z], \mathcal{D} = k[x, y, z]$. We set socle degree $j = 63$. We define constraints $P := (20)$ with $n_1 = 1$ and $Q := (63, 63, 63)$ with $n_2 = 3$. We define the vector space $\mathcal{E}(C)$ as the span of 4 general members of $\mathcal{W}_{M_P(63)}; \mathcal{F}(C')$ as the span of one general member of $\mathcal{W}_{M_Q(63)} = \mathcal{D}_{63}$. Then $F_1(21, 42, 4)(C, C') := R/\text{Ann}(\mathcal{E}(C) \oplus \mathcal{F}(C'))$. According to Macaulay2:

$$h(42) = 946; \ h(43) = 945; \ h(44) = 945; \ h(45) = 946.$$ 

**Definition 8.4.** The family $F_2(a, i, s)$ is obtained by setting $r = 3$, $j = i + (a - 1)/2, P = (a - 1), Q = (j, j, j), u = 2$. We require that $a \geq 7$ be odd, that

$$i \geq \frac{2a - 3 + \sqrt{2a^2 + 8a + 7}}{2},$$

and that $s$ be $h_\mathcal{E}$-sufficient.

EXAMPLE $F_2(21, 36, 14)$: Let $r = 3, t = 14 + 2 = 16, R = k[X, Y, Z], \mathcal{D} = k[x, y, z]$. We set socle degree $j = 46$. We define constraints $P := (20)$ with $n_1 = 1$ and $Q := (46, 46, 46)$ with $n_2 = 3$. We define the vector space $\mathcal{E}(C)$ as the span of 14 general members of $\mathcal{W}_{M_P(46)}; \mathcal{F}(C')$ as the span of two general members of $\mathcal{W}_{M_Q(46)} = \mathcal{D}_{46}$. Then $F_1(21, 36, 14)(C, C') := R/\text{Ann}(\mathcal{E}(C) \oplus \mathcal{F}(C'))$. According to Macaulay2:

$$h(36) = 699; \ h(37) = 698; \ h(38) = 699.$$
**Definition 8.5.** The family $G_1(a, b, i, s)$ is obtained by setting $r = 4$, $j = i + ab$, $P = (a - 1, b - 1)$, $Q = (j, j, j, 0)$, $u = 1$. We require that $b \geq a \geq 2$, that $i \geq ab + 1$, and that $s$ be $h_E$-sufficient.

**Example $G_1(3, 4, 13, 2)$:** Let $r = 4$, $t = 2 + 1 = 3$, $R = k[X, Y, Z, W]$, $\mathcal{D} = k[x, y, z, w]$. We set socle degree $j = 25$. We define constraints $P := (2, 3)$ with $n_1 = 2$ and $Q := (25, 25, 25, 0)$ with $n_2 = 4$. We define the vector space $\mathcal{E}(C)$ as the span of two general members of $\mathcal{W}_{M_P(25)}$; $\mathcal{F}(C')$ as the span of one general member of $\mathcal{W}_{M_Q(25)} = k[x, y, z]_{25}$. Then $G_1(3, 4, 13, 2)(C, C') := R/\text{Ann}(\mathcal{E}(C) \oplus \mathcal{F}(C'))$. According to Macaulay2:

$$h(13) = 229; \quad h(14) = 228; \quad h(15) = 228; \quad h(16) = 229.$$  

**Definition 8.6.** The family $G_2(a, b, i, s)$ is obtained by setting $r = 4$, $j = i + ab/2$, $P = (j, a - 1, b - 1)$, $Q = (1)$, $u = 1$. We require that $b \geq a \geq 2$, that $ab$ be even, that $i \geq ab/2 + 2$, and that $s$ be $h_E$-sufficient.

**Example $G_2(4, 6, 14, 2)$:** Let $r = 4$, $t = 2 + 1 = 3$, $R = k[X, Y, Z, W]$, $\mathcal{D} = k[x, y, z, w]$. We set socle degree $j = 26$. We define constraints $P := (26, 3, 5)$ with $n_1 = 3$ and $Q := (1)$ with $n_2 = 1$. We define the vector space $\mathcal{E}(C)$ as the span of two general members of $\mathcal{W}_{M_P(26)}$; $\mathcal{F}(C')$ as the span of one general member of $\mathcal{W}_{M_Q(26)}$. Then $G_2(4, 6, 14, 2)(C, C') := R/\text{Ann}(\mathcal{E}(C) \oplus \mathcal{F}(C'))$. According to Macaulay2:

$$h(14) = 433; \quad h(15) = 432; \quad h(16) = 433.$$
DEFINITION 8.7. The family $G_3(a,b,i,s)$ is obtained by setting $r = 4; j = i + m$, where $m$ is the largest integer such that $\binom{m+1}{2} < ab$; $P = (a-1,b-1)$; $Q = (j,j,j,j)$; $u = 1$. We require that $b \geq a \geq 2$, that $ab$ not be equal to a binomial coefficient of the form $\binom{N}{2}$, that $i \geq a + b - 3$; that
\begin{equation}
\frac{ab[2i - a - b + 4]}{2} + \binom{m+3}{3} \leq \binom{i+3}{3};
\end{equation}
and that $s$ be $h_\mathcal{E}$-sufficient.

EXAMPLE $G_3(4,4,8,7)$: Let $r = 4$, $t = 7 + 1 = 8$, $R = k[X,Y,Z,W]$, $\mathcal{D} = k[x,y,z,w]$. We set socle degree $j = 13$. We define constraints $P := (3,3)$ with $n_1 = 2$ and $Q := (13,13,13,13)$ with $n_2 = 4$. We define the vector space $\mathcal{E}(C)$ as the span of 7 general members of $\mathcal{W}_{M_p(13)}$; $\mathcal{F}(C')$ as the span of one general member of $\mathcal{W}_{M_Q(13)} = \mathcal{D}_{13}$. Then $G_3(4,4,8,7)(C,C') := R/Ann(\mathcal{E}(C) \oplus \mathcal{F}(C'))$.

According to Macaulay2:
\begin{align*}
h(8) &= 152; \quad h(9) = 147; \quad h_4(10) = 148.
\end{align*}

DEFINITION 8.8. The family $H_1(a,b,c,i,s)$ is obtained by setting $r = 5, j = i + abc, P = (a-1,b-1,c-1)$, $Q = (j,j,j,0,0), u = 1$. We require that $c \geq b \geq a \geq 2$, that $i \geq abc$, and that $s$ be $h_\mathcal{E}$-sufficient.

EXAMPLE $H_1(2,2,3,12,2)$: Let $r = 5$, $t = 2 + 1 = 3$, $R = k[X,Y,Z,W,V]$, $\mathcal{D} = k[x,y,z,w,v]$. We set socle degree $j = 24$. We define constraints $P := (1,1,2)$ with $n_1 = 3$ and $Q := (24,24,24,0,0)$ with $n_2 = 5$. We define the vector
space $E(C)$ as the span of two general members of $\mathcal{W}_{M_{24}}$; $F(C')$ as the span of one general member of $\mathcal{W}_{M_{Q(24)}} = k[x, y, z]_{24}$. Then $H_1(2, 2, 3, 12, 2)(C, C') := R/\text{Ann}(E(C) \oplus F(C'))$. According to Macaulay2:

\[ h(12) = 223; \quad h(13) = 222; \quad h(14) = 222; \quad h(15) = 223. \]

Before entering a discussion of the six families defined here, we stop to check that they are all nonempty.

**Proposition 8.9.** For any of the families $F_1(a, i, s), F_2(a, i, s), G_1(a, b, i, s), G_2(a, b, i, s), G_3(a, b, i, s), H_1(a, b, c, i, s)$, it is possible to find values of the parameters that satisfy the requirements set forth in their definitions.

**Proof.** For each of the families, it is immediate that all parameters except $s$ can be chosen consistent with the requirements of the definitions of the families. So it is enough to show that, for any such choice, $s := m_p(j)$ is $h_E$-sufficient.

Setting $d = i_f$ and $e = j - i_f$, the size of the $(j - i_f)^{th}$ cropped matrix is $sm_p(e) \times m_p(d)$. To verify it has at least as many rows as columns when $s := m_p(j)$, we observe

\[ sm_p(e) = m_p(j)m_p(e) \geq m_p(j) \geq m_p(d), \]

the last inequality following from Lemma 7.6. □
4. Formulas for $h_{\mathcal{E}(C)}(d)$ and $\Delta_d$

Recall that in definition 8.1 we have defined degrees $d$ to lie in the critical range if $i \leq d \leq i_f$, where $i$ and $i_f$ are the initial and final degrees of a proposed single or double drop; and we have specified that $F_2, G_2$ and $G_3$ are candidates for having a single drop with initial degree $i$, whereas $F_1, G_1,$ and $H_1$ are candidates for having a double drop with initial degree $i$. Also recall that $\Delta$ was defined to be $a, ab$, or $abc$, depending on whether the codimension is 3, 4, or 5.

**Lemma 8.10.** For any of the families $F_1(a, i, s), F_2(a, i, s), G_1(a, b, i, s), G_2(a, b, i, s), G_3(a, b, i, s), H_1(a, b, c, i, s)$, the values of $m_P(d)$ are given as follows for degrees $d$ in the critical range.

- **For $F_1$ and $F_2$**:
  $$m_P(d) = \Delta(2d - a + 3)/2.$$  

- **For $G_1$, $G_2$, $G_3$**:
  $$m_P(d) = \Delta(2d - a - b + 4)/2.$$  

- **For $H_1$**:
  $$m_P(d) = \Delta(2d - a - b - c + 5)/2.$$  

**Proof.** Propositions 7.10, 7.11, and 7.9 yield the formulas above, provided we verify that $d$ is large enough.

For $r = 3$, the condition is that $d \geq a - 2$. This is true for $F_1$, since $d \geq i \geq 2a \geq a - 2$; and for $F_2$, since $d \geq i \geq \frac{2a - 3 + \sqrt{2a^2 + 8a + 7}}{2} \geq a \geq a - 2$. 

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For $r = 4$, the condition is that $d \geq a + b - 3$. Recall that for $G_1$, $b \geq a \geq 2$ and $i \geq ab + 1$; for $G_2$, $b \geq a \geq 2$ and $i \geq ab/2 + 2$. In either case, we use the fact that, for $a \geq 2$ and $b \geq 2$, $ab/2 \geq a + b - 2$. For $G_1$, $d \geq i \geq ab + 1 \geq 2a + 2b - 3 \geq a + b - 3$. For $G_2$, $d \geq i \geq ab/2 + 2 \geq (a + b - 2) + 2 \geq a + b - 3$. For $G_3$, $d \geq i \geq a + b - 3$, where the last inequality was required to hold in the definition of $G_3$.

For $r = 5$, the condition is that $d \geq a + b + c - 3$. For $H_1$, $c \geq b \geq a \geq 2$ and $i \geq abc$, so $d \geq i \geq abc \geq a + b + c \geq a + b + c - 3$. □

**Proposition 8.11.** For any of the families $F_1(a, i, s), F_2(a, i, s), G_1(a, b, i, s), G_2(a, b, i, s), G_3(a, b, i, s), H_1(a, b, c, i, s)$, let $d$ lie in the critical range. Then for general $C$,

[For $F_1$ and $F_2$] : $h_{E(C)}(d) = \Delta(2d - a + 3)/2$.

[For $G_1$, $G_2$, $G_3$] : $h_{E(C)}(d) = \Delta(2d - a - b + 4)/2$.

[For $H_1$] : $h_{E(C)}(d) = \Delta(2d - a - b - c + 5)/2$.

**Proof.** Recall that the hypothesis that $s$ is $h_E$-sufficient means that the $(j - i_f)^{th}$ cropped matrix of $E$ has at least as many rows as columns. This matrix has $sm_p(j - i_f)$ rows and $m_p(i_f)$ columns. We observe that, for any $d$ in the critical range, the $(j - d)^{th}$ cropped matrix of $E$ also has at least as many rows as columns, since by Lemma 7.6, it has $sm_p(j - d) \geq sm_p(j - i_f)$ rows and $m_p(d) \leq m_p(i_f)$ columns. So for all values of $d$ in the critical range, Theorem 6.14 applies, and for general $C$ we
have \( h_{\mathcal{E}(C)}(d) = m_P(d) \), the rank of the \( e^{th} \) cropped matrix \( U \) of \( \mathcal{E} \), or equivalently the number of columns in \( U \).

\[ \square \]

**COROLLARY 8.12.** For any of the families \( F_1(a, i, s), F_2(a, i, s), G_1(a, b, i, s), G_2(a, b, i, s), G_3(a, b, i, s), H_1(a, b, c, i, s), \) and for \( d := i, \ldots, i_f - 1, \Delta_d = \Delta \) for general \( C \).

**PROOF.** Recalling that \( \Delta_d := h_{\mathcal{E}(C)}(d + 1) - h_{\mathcal{E}(C)}(d) \), we obtain values for \( h_{\mathcal{E}(C)}(d + 1) \) (for general \( C \)) and \( h_{\mathcal{E}(C)}(d) \) (for general \( C \)) from Proposition 8.11. Since these formulas both hold for general \( C \), Lemma 4.7 guarantees that they hold simultaneously for general \( C \), so subtracting them gives a formula for their difference that holds for general \( C \).

\[ \square \]

5. **Formulas for** \( h_{\mathcal{F}(C')}(d) \) **and** \( \delta_d \)

**PROPOSITION 8.13.** For any of the families \( F_1(a, i, s), F_2(a, i, s), G_1(a, b, i, s), G_2(a, b, i, s), G_3(a, b, i, s), H_1(a, b, c, i, s), \) let \( d \) lie in the critical range. Then for general \( C' \), the following formulas for \( h_{\mathcal{F}(C')}(d) \) apply. (Recall that \( e := j - d \).)

\[
\begin{align*}
\text{[For } F_1, G_1, H_1 \text{]} : \quad h_{\mathcal{F}(C')}(d) &= \left(\frac{e + 2}{2}\right). \\
\text{[For } F_2 \text{] :} \quad h_{\mathcal{F}(C')}(d) &= 2 \left(\frac{e + 2}{2}\right). \\
\text{[For } G_2 \text{] :} \quad h_{\mathcal{F}(C')}(d) &= (e + 1)^2. \\
\text{[For } G_3 \text{] :} \quad h_{\mathcal{F}(C')}(d) &= \left(\frac{e + 3}{3}\right). 
\end{align*}
\]
PROOF. For all of the families except $G_2$, we apply Proposition 7.2 which requires us to verify that $u \binom{e+2}{2} \leq \binom{d+2}{2}$. We consider each family in turn.

For $F_1$, $u = 1$ and $e \leq a \leq 2a \leq i \leq d$.

For $G_1$, $u = 1$ and $e \leq ab \leq ab + 1 \leq i \leq d$.

For $H_1$, $u = 1$ and $e \leq abc \leq i \leq d$.

For $G_3$, $u = 1$ and $e \leq m \leq i \leq d$, where $m \leq i$ follows immediately from (36).

For $F_2$, $u = 2$ and

$$2 \binom{e+2}{2} \leq 2 \binom{(a - 1)/2 + 2}{2}$$

$$= 2 \binom{(a + 3)/2}{2} \binom{(a + 1)/2}{2}$$

$$= \binom{(a + 3)(a + 1)}{4}$$

$$= \frac{a^2 + 4a + 3}{4}$$

$$\leq \frac{a^2 + 3a + 2}{2}$$

$$= \binom{a + 2}{2} \leq \binom{i + 2}{2} \leq \binom{d + 2}{2},$$

where $a \leq i$ follows immediately from (35).
For $G_2$, we apply Proposition 7.3 which requires that $e \leq d$. We have
\[ e \leq \frac{ab}{2} \leq \frac{ab}{2} + 2 \leq i \leq d. \]

\[ \square \]

**Corollary 8.14.** For the families $F_1(a, i, s), F_2(a, i, s), G_1(a, b, i, s), G_2(a, b, i, s)$, $G_3(a, b, i, s), H_1(a, b, c, i, s)$, for $d := i, \ldots, i_f - 1$, and for general $C'$ the formulas for $\delta_d$ are as follows.

\[ [\text{For } F_1, \ G_1, \ H_1] : \ \delta_d = -(e + 1). \]
\[ [\text{For } F_2] : \ \delta_d = -2(e + 1). \]
\[ [\text{For } G_2] : \ \delta_d = -(2e + 1). \]
\[ [\text{For } G_3] : \ \delta_d = -\binom{e + 2}{2}. \]

**Proof.** Recalling that $\delta_d := h_{F(C')}(d + 1) - h_{F(C')}(d)$, we obtain values for $h_{F(C')}(d + 1)$ (for general $C'$) and $h_{F(C')}(d)$ (for general $C'$) from Proposition 8.13. Since these formulas both hold for general $C'$, Lemma 4.7 guarantees that they hold simultaneously for general $C'$, so subtracting them gives a formula for their difference that holds for general $C'$. In performing the subtraction, we use the following well-known formula for binomial coefficients.

\[ \binom{N}{M} = \binom{N - 1}{M} + \binom{N - 1}{M - 1}. \]

\[ \square \]
**6. Computing $h_{E(C)} \oplus F(C')(d)$**

**Theorem 8.15.** For any of the families $F_1(a, i, s), F_2(a, i, s), G_1(a, b, i, s), G_2(a, b, i, s), G_3(a, b, i, s), H_1(a, b, c, i, s)$, for general $C$ and $C'$, we have

(i) $E(C) \cap F(C') = \{0\}$ and

(ii) $h_{E(C)} \oplus F(C')(d) = h_{E(C)}(d) + h_{F(C')}(d)$, simultaneously for all degrees $d$ in the critical range.

**Proof.** Since we must verify (ii) for only finitely many values of $d$, by Lemma 4.7 it is enough to verify it separately for each degree $d$. If (i) has been established, to verify (ii) it is enough, by Lemma 3.1, to show that

$(37)$ For general $C$ and $C'$, $R_e \ast E(C) \cap R_e \ast F(C') = \{0\}$.

We remark that $(37)$ also implies (i), since the existence of a nonzero polynomial $f \in E(C) \cap F(C')$ would imply the existence of a nonzero $e^{th}$ partial derivative of $f$, which would lie in $R_e \ast E(C) \cap R_e \ast F(C')$. Thus, to prove the theorem, it is enough to prove $(37)$.

To show $(37)$, for all families except $G_2$, we apply Proposition 7.2 as follows. Let $W := V_{M_p(d)}$ (defined in (31)). We observe that, for each family other than $G_2$, $V_{M_p(d)} = k[x_1, ..., x_m]_d$, where $m$ has the value 3 or 4. Let $Z := V_{M_p(d)} \cap V_{M_Q(d)}$. To use Proposition 7.2 with these values of $W$ and $Z$, we must verify that

$(38) \quad \dim_k Z \leq \left( \begin{array}{c} d+p \\ p \end{array} \right) - u \left( \begin{array}{c} e+p \\ p \end{array} \right)$,
where \( p := m - 1 \). Assuming this verification has been done, we conclude from part (iii) of Proposition 7.2 that, for general \( C' \), \( \mathcal{W} \cap R_e \ast \mathcal{F}(C') = \{0\} \). Since \( R_e \ast \mathcal{E}(C) \subseteq \mathcal{W} \), we conclude that \( R_e \ast \mathcal{E}(C) \cap R_e \ast \mathcal{F}(C') = \{0\} \), as required.

Before proceeding to the numerical verifications of (38), we consider the family \( G_2 \), and apply Proposition 7.3 to verify (37) as follows. As with the other five families, we again let \( \mathcal{W} := \mathcal{V}_{M_p(d)} \) and \( \mathcal{Z} := \mathcal{V}_{M_p(d)} \cap \mathcal{V}_{M_Q(d)} \). To use Proposition 7.3 with these values of \( \mathcal{W} \) and \( \mathcal{Z} \), we must verify (33) and (34). Assuming these verifications have been done, we proceed as before, concluding from part (iii) of Proposition 7.3 that, for general \( C' \), \( \mathcal{W} \cap R_e \ast \mathcal{F}(C') = \{0\} \). Again, since \( R_e \ast \mathcal{E}(C) \subseteq \mathcal{W} \), we conclude that \( R_e \ast \mathcal{E}(C) \cap R_e \ast \mathcal{F}(C') = \{0\} \), as required. We now proceed to the verifications of (38), (33), and (34).

For \( F_1 \),

\[
\mathcal{Z} := \mathcal{V}_{M_p(d)} \cap \mathcal{V}_{M_Q(d)} \\
= \mathcal{V}_{M_p(d)} \cap D_d \\
= \mathcal{V}_{M_p(d)}.
\]

From Proposition 8.10 \( \dim_k \mathcal{Z} = \frac{a(2d - a + 3)}{2} \), so to use Proposition 7.2 we must verify that

\[
\frac{a(2d - a + 3)}{2} \leq \binom{d + 2}{2} - \binom{e + 2}{2},
\]
or equivalently that

\[
\binom{d+2}{2} - \frac{a(2d - a + 3)}{2} - \binom{e+2}{2} \geq 0.
\]

We are considering values of \(d \geq 2a\) and \(e \leq a\), so

\[
\binom{d+2}{2} - \frac{a(2d - a + 3)}{2} - \binom{e+2}{2} =
\]

\[
[(d^2 + 3d + 2) - (2ad - a^2 + 3a) - (a^2 + 3a + 2)]/2 =
\]

\[
[d(d - 2a + 3) - 6a]/2 \geq
\]

\[
[2a(2a - 2a + 3) - 6a]/2 = 0.
\]

For \(F_2\), again

\[
\mathcal{Z} := \mathcal{V}_{\mathcal{M}_P(d)} \cap \mathcal{V}_{\mathcal{M}_Q(d)}
\]

\[
= \mathcal{V}_{\mathcal{M}_P(d)} \cap \mathcal{D}_d
\]

\[
= \mathcal{V}_{\mathcal{M}_P(d)},
\]

and again \(\dim_k \mathcal{Z} = \frac{a(2d - a + 3)}{2}\).
To use Proposition \textit{7.2} we must verify that

\[
\frac{a(2d - a + 3)}{2} \leq \binom{d + 2}{2} - 2\binom{e + 2}{2},
\]

or equivalently that

\[
\binom{d + 2}{2} - \frac{a(2d - a + 3)}{2} - 2\binom{e + 2}{2} \geq 0.
\]

We are considering values of \(e \leq (a - 1)/2\), and \(d \geq i\), so

\[
\left(\frac{d + 2}{2}\right) - \frac{a(2d - a + 3)}{2} - 2\binom{e + 2}{2} \geq
\]

\[
\left(\frac{d + 2}{2}\right) - \frac{a(2d - a + 3)}{2} - 2\left(\frac{(a - 1)/2 + 2}{2}\right) =
\]

\[
[(d^2 + 3d + 2) - (2ad - a^2 + 3a) - 2((a - 1)^2/4 + 3(a - 1)/2 + 2)]/2 =
\]

\[
[(d^2 + 3d + 2) - (2ad - a^2 + 3a) - ((a^2 - 2a + 1)/2 + 3(a - 1) + 4)]/2 =
\]

\[
[d(d - 2a + 3) + a^2/2 - 5a + 1/2]/2 =
\]

\[
[d(d - 2a + 3) + \frac{a^2 + 1}{2} - 5a]/2 \geq
\]

\[
i(i - 2a + 3) + \frac{a^2 + 1}{2} - 5a]/2 =
\]

\[
i^2 + (3 - 2a)i + (\frac{a^2 + 1}{2} - 5a)]/2.
\]
We must demonstrate that the expression within square brackets is always non-negative. We recall that, in defining the family $F_2$, we have required that

$$i \geq \frac{2a - 3 + \sqrt{2a^2 + 8a + 7}}{2}.$$

Using the quadratic formula to solve the quadratic inequality

$$i^2 + (3 - 2a)i + \left(\frac{a^2 + 1}{2} - 5a\right) \geq 0,$$

and noting that we are only interested in positive values of $i$ as solutions, we have:

$$i \geq \frac{2a - 3 + \sqrt{(4a^2 - 12a + 9) - (2a^2 + 2 - 20a)}}{2},$$

$$i \geq \frac{2a - 3 + \sqrt{2a^2 + 8a + 7}}{2},$$

which has been assumed true for the family $F_2$.

For $G_1$,

$$Z := \mathcal{V}_{M_P(d)} \cap \mathcal{V}_{M_Q(d)}$$

$$= \mathcal{V}_{M_P(d)} \cap k[x, y, z]_d,$$

or equivalently the vector subspace of $k[x, y, z]_d$ spanned by monomials constrained by $P := (a - 1, b - 1)$. Its dimension is $ab$ by Proposition 7.8 since $d \geq ab \geq a + b > (a - 1) + (b - 1)$. We must verify that
\[ ab \leq \binom{d+2}{2} - \binom{e+2}{2}, \]

or equivalently that

\[ \binom{d+2}{2} - \binom{e+2}{2} - ab \geq 0. \]

We are considering values of \( d \geq ab + 1 \) and \( e \leq ab \), so

\[ \binom{ab+3}{2} - \binom{ab+2}{2} - ab = \]

\[ [(a^2b^2 + 5ab + 6) - (a^2b^2 + 3ab + 2) - 2ab]/2 = \]

\[ [4]/2 \geq 0. \]

For \( G_3 \),

\[ \mathcal{Z} := \mathcal{V}_{M_P(d)} \cap \mathcal{V}_{M_Q(d)} \]

\[ = \mathcal{V}_{M_P(d)} \cap \mathcal{D}_d \]

\[ = \mathcal{V}_{M_P(d)}. \]
By Proposition 8.10, the dimension of $Z$ is $\frac{ab[2d - a - b + 4]}{2}$. To use Proposition 7.2, we must verify for $d = i, i + 1, i + 2$ that

$$\frac{ab[2d - a - b + 4]}{2} + \binom{j - d + 3}{3} \leq \binom{d + 3}{3}.$$ 

For the case that $d = i, e = j - d = m$, this is just (36). Moving to $d = i + 1, e = j - d = m - 1$, the first term on the left increases by $ab$, the second term decreases, and the term on the right increases by

$$\binom{i + 4}{3} - \binom{i + 3}{3} = \binom{i + 3}{2} \geq \binom{m + 3}{2} \geq \binom{m + 2}{2} \geq ab,$$

so the required inequality holds for the case $d = i + 1, e = j - d = m - 1$. A similar computation establishes the inequality for $d = i + 2, e = j - d = m - 2$.

For $H_1$,

$$Z := \mathcal{V}_{M_p(d)} \cap \mathcal{V}_{M_Q(d)}$$

$$= \mathcal{V}_{M_p(d)} \cap k[x, y, z]_d,$$
or equivalently the vector subspace of $k[x, y, z]_d$ spanned by monomials constrained by $P := (a - 1, b - 1, c - 1)$. Its dimension is 0 by Proposition 7.7 since $d \geq abc \geq a + b + c > (a - 1) + (b - 1) + (c - 1)$. We must verify that

$$0 = \dim_k \mathcal{Z} \leq \binom{d + 2}{2} - \binom{e + 2}{2},$$

or equivalently that

$$\binom{d + 2}{2} - \binom{e + 2}{2} \geq 0.$$

Since we are considering values of $d \geq abc$ and $e \leq abc$, this is immediate.

For $G_2$, we use Proposition 7.3 taking $\mathcal{Z} := \mathcal{V}_{\mathcal{M}_P(d)} \cap \mathcal{V}_{\mathcal{M}_Q(d)}$, which is the vector subspace of $D_d$ constrained by $K := (1, a - 1, b - 1)$. Its dimension is $2ab$ by Proposition 7.8 since $d \geq ab/2 + 2 \geq (a + b - 2) + 2 \geq 1 + (a - 1) + (b - 1)$. Looking further, we can see that exactly $ab$ of the generators do not contain the variable $x$ by again applying Proposition 7.8 this time to the constraint $K' := (0, a - 1, b - 1)$. So to apply Proposition 7.3 with this choice of $\mathcal{Z}$, we use $ab$ for the parameter $c$ of that proposition.

According to Proposition 7.3 (iii), there are two inequalities to verify for $d$ in the critical range. The first is that
or equivalently that

$$m_P(d) - m_P(e) - 2c \geq 0.$$  

We are considering values of $d \geq ab/2 + 2$ and $e \leq ab/2$, with $c = ab$, so

$$m_P(d) - m_P(e) - 2c =$$

$$(d + 1)^2 - (e + 1)^2 - 2ab \geq$$

$$(ab/2 + 3)^2 - (ab/2 + 1)^2 - 2ab =$$

$$(a^2b^2/4 + 3ab + 9) - (a^2b^2/4 + ab + 1) - 2ab =$$

$$8 > 0.$$  

The second inequality to be verified is that

$$\binom{d + (r - 2)}{r - 2} - c \geq \binom{e - 1 + (r - 2)}{r - 2},$$
or equivalently that
\[
\left( \frac{d + (r - 2)}{r - 2} \right) - \left( \frac{(e - 1) + (r - 2)}{r - 2} \right) - c \geq 0.
\]

Once again we are considering values of \( d \geq ab/2 + 2 \) and \( e \leq ab/2 \), with \( c = ab \) and \( r = 4 \), so
\[
\left( \frac{d + (r - 2)}{r - 2} \right) - \left( \frac{(e - 1) + (r - 2)}{r - 2} \right) - c =
\left( \frac{d + 2}{2} \right) - \left( \frac{e + 1}{2} \right) - ab \geq
\left( \frac{ab/2 + 4}{2} \right) - \left( \frac{ab/2 + 1}{2} \right) - ab =
\left( \frac{ab}{2} + 4 \right) \left( \frac{ab}{2} + 3 \right) - \left( \frac{ab}{2} + 1 \right) \left( \frac{ab}{2} \right) - 2ab =
\frac{[ab/2 + 4(ab/2 + 3) - (ab/2 + 1)(ab/2) - 2ab]}{2} =
\frac{[(a^2b^2/4 + 7ab/2 + 12) - (a^2b^2/4 + ab/2) - 2ab]}{2} =
\frac{[ab + 12]}{2} > 0.
\]

\[\square\]

7. Proof of Non-Unimodality

**Theorem 8.16.** For general \( C \) and \( C' \), all of the families \( F_1(a, i, s), F_2(a, i, s), G_1(a, b, i, s), G_2(a, b, i, s), G_3(a, b, i, s), H_1(a, b, c, i, s) \), are non-unimodal. The Hilbert functions of \( F_2, G_2, \) and \( G_3 \) have single drops with initial degree \( i \). The Hilbert functions of \( F_1, G_1, \) and \( H_1 \) have double drops with initial degree \( i \).
PROOF. By Theorem 8.15 we are entitled to use Lemma 8.2, in which we use the values of $\Delta_d$ and $\delta_d$ given in Corollaries 8.12 and 8.14. We let $h := h_{\mathcal{E}(C) \oplus \mathcal{F}(C')}$. For each family, our approach is to determine relationships between consecutive values of $h(d)$ that demonstrate the non-unimodality of $h$ in the critical range.

For $F_2$, we have $j - i = (a - 1)/2$, so

$$h(i + 1) = h(i) + \Delta_i + \delta_i$$

$$= h(i) + a - 2[(a - 1)/2 + 1]$$

$$= h(i) - 1,$$

and

$$h(i + 2) = h(i + 1) + \Delta_{i+1} + \delta_{i+1}$$

$$= h(i + 1) + a - 2([(a - 1)/2 - 1] + 1)$$

$$= h(i + 1) + 1.$$

For $G_2$, we have $j - i = ab/2$, so

$$h(i + 1) = h(i) + \Delta_i + \delta_i$$

$$= h(i) + ab - [2(ab/2) + 1]$$

$$= h(i) - 1,$$
and

\[ h(i + 2) = h(i + 1) + \Delta_{i+1} + \delta_{i+1} \]

\[ = h(i + 1) + \Delta_{i+1} + \delta_{i+1} + \Delta_{i+1} + \delta_{i+1} \]

\[ = h(i + 1) + 2\Delta_{i+1} + 2\delta_{i+1} \]

\[ = h(i + 1) + 2(\Delta_{i+1} + \delta_{i+1}) \]

\[ = h(i + 1) + 2(ab - [2(ab/2 - 1) + 1]) \]

\[ = h(i + 1) + 2(ab - [2ab/2 - 1] + 1) \]

\[ = h(i + 1) + ab - (2ab/2 - 1) + 1 \]

\[ = h(i + 1) + 1. \]

For \( G_3 \), we have \( j - i = m \), where by definition

\[ \binom{m+1}{2} < ab < \binom{m+2}{2}. \]

We have

\[ h(i + 1) = h(i) + \Delta_i + \delta_i \]

\[ = h(i) + ab - \binom{m+2}{2} \]

\[ < h(i), \]

and

\[ h(i + 2) = h(i + 1) + \Delta_{i+1} + \delta_{i+1} \]

\[ = h(i + 1) + ab - \binom{(m-1)+2}{2} \]

\[ = h(i + 1) + ab - \binom{m+1}{2} \]

\[ > h(i + 1). \]
For $F_1, G_1$, and $H_1$, we have $j - i = \Delta$, so

\[
h(i + 1) = h(i) + \Delta_i + \delta_i
\]

\[
= h(i) + \Delta - (\Delta + 1)
\]

\[
= h(i) - 1,
\]

and

\[
h(i + 2) = h(i + 1) + \Delta_{i+1} + \delta_{i+1}
\]

\[
= h(i + 1) + \Delta - ([\Delta - 1] + 1)
\]

\[
= h(i + 1),
\]

and

\[
h(i + 3) = h(i + 2) + \Delta_{i+2} + \delta_{i+2}
\]

\[
= h(i) + \Delta - ([\Delta - 2] + 1)
\]

\[
= h(i + 1) + 1.
\]

□
8. Computation of Types

**Lemma 8.17.** For any of the families $F_1(a, i, s), F_2(a, i, s), G_1(a, b, i, s), G_2(a, b, i, s), G_3(a, b, i, s), H_1(a, b, c, i, s)$, let $s$ be chosen such that $s \leq m_P(j)$. Then for general $C$ and $C'$, the type of $A_{E(C) \oplus F(C')}$ is $s + u$.

**Proof.** By Corollary 6.15 for general $C$ and $C'$, the dimension of $E(C)$ is $s$ and the dimension of $F(C')$ is $u$. By Theorem 8.15, $E(C) \cap F(C') = \{0\}$.

Among the defining parameters, we will call $a, b, c$ the $P$-parameters, since they define the constraint $P$.

**Lemma 8.18.** For any of the families $F_1(a, i, s), F_2(a, i, s), G_1(a, b, i, s), G_2(a, b, i, s), G_3(a, b, i, s), H_1(a, b, c, i, s)$, for any fixed choice of the $P$-parameters, the value of $j - i_f$ is constant, given by the following formulas:

- For $F_1, G_1,$ and $H_1$: $\Delta - 3$.
- For $F_2$: $(a - 1)/2 - 2$.
- For $G_2$: $ab/2 - 2$.
- For $G_3$: $m - 2$, where $m$ is the greatest integer such that $\left(\begin{array}{c} m + 1 \\ 2 \end{array}\right) < ab$.

**Proof.** From the definitions, the value of $j - i$ is $\Delta$ for $F_1, G_1,$ and $H_1$; $(a - 1)/2$ for $F_2$; $ab/2$ for $G_2$; and the greatest integer $m$ such that $\left(\begin{array}{c} m + 1 \\ 2 \end{array}\right) < ab$ for $G_3$. For $F_1, G_1,$ and $H_1$, which have double drops, $i_f = i + 3$; for the others, which have single drops, $i_f = i + 2$. 

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We define the \emph{ceiling function} \(\text{ceil}(x)\) as follows. For any non-negative real number \(x\), \(\text{ceil}(x)\) is the the smallest integer \(c\) such that \(c \geq x\).

\textbf{Lemma 8.19.} For any of the families \(F_1(a, i, s), F_2(a, i, s), G_1(a, b, i, s), G_2(a, b, i, s), G_3(a, b, i, s), H_1(a, b, c, i, s)\), for general \(C\) and \(C'\), the requirement that \(s\) is \(h_{E}\)-sufficient is equivalent to the condition that \(s \geq \text{ceil} \left( \frac{m_p(i_f)}{m_p(j - i_f)} \right)\). For any fixed choice of the \(P\)-parameters, this formula attains the smallest possible value when \(i\) is as small as permitted by the definition of the family.

\textbf{Proof.} The definition of \(h_{E}\)-sufficient is that the \((j - i_f)\)th cropped matrix of \(E\) has at least as many rows as columns. It has \(sm_p(j - i_f)\) rows and \(m_p(i_f)\) columns, so the condition is that \(sm_p(j - i_f) \geq m_p(i_f)\), or equivalently that \(s \geq \frac{m_p(i_f)}{m_p(j - i_f)}\). We introduce the ceiling function because \(s\) must be an integer.

For any of the families, once we have fixed a choice of \(P\)-parameters, the lemma follows if we show that \(\frac{m_p(i_f)}{m_p(j - i_f)}\) is a non-decreasing function of \(i\). Since the ceiling function is nondecreasing, it is enough to show that \(\frac{m_p(i_f)}{m_p(j - i_f)}\) is non-decreasing as a function of \(i\), and by the previous lemma it is enough that \(m_p(i_f)\) be non-decreasing as a function of \(i\). By Lemma 7.6 it is enough that \(i_f\) be a non-decreasing function of \(i\). But \(i_f := i + 2\) or \(i + 3\), depending on whether the family has a single drop or a double drop.

\textbf{Lemma 8.20.} For any of the families \(F_1(a, i, s), F_2(a, i, s), G_1(a, b, i, s), G_2(a, b, i, s), G_3(a, b, i, s), H_1(a, b, c, i, s)\), for any fixed choice of the \(P\)-parameters and \(i\),
let \( s = \text{ceil} \left( \frac{m_p(i_f)}{m_p(j - i_f)} \right) \). Then for general \( C \) and \( C' \), the type of the algebra so obtained is \( u + s \).

**Proof.** By Lemma 8.19 the specified value of \( s \) yields an algebra in the family. To verify the formula for the type, by Lemma 8.17 we must verify that \( s \leq m_p(j) \). We have

\[
\text{ceil} \left( \frac{m_p(i_f)}{m_p(j - i_f)} \right) \leq \text{ceil}(m_p(i_f)) = m_p(i_f) \leq m_p(j),
\]

where the last inequality follows from Lemma 7.6. \( \square \)

Up to this point, we have stated results that emphasized the similarities between the families. Now, we focus on the particulars of each family in turn.

**Theorem 8.21.** In the family \( F_1(a, i, s) \), we have

(i) For a fixed choice of \( a \) and \( i \), the smallest possible type \( t = 1 + s \) is achieved by taking \( s = \text{ceil} \frac{a(2i - a + 9)}{a^2 - 3a + 2} \).

(ii) For a fixed choice of \( a \), the smallest possible type \( t = 1 + s \) is achieved by taking \( i = 2a \), \( s = \text{ceil} \frac{a(3a + 9)}{a^2 - 3a + 2} \). With these choices, the type is greater than 5 for \( a \leq 20 \), and the type is exactly 5 for \( a \geq 21 \).

(iii) For any choice of \( a \), \( i \), and \( s \), the type is at least 5.
PROOF. For (i), we combine the various lemmas in this section with Lemma 7.10.

\[ s = \text{cei} \left( \frac{m_P(i_f)}{m_P(j - i_f)} \right) \]

For (ii), we observe that \( s \) is an increasing function of \( i \), and we substitute the smallest permissible value \( i = 2a \) to obtain \( t = 1 + s = 1 + \text{cei} \left( \frac{3a + 9}{a^2 - 3a + 2} \right) \). We observe that the fraction \( \alpha := \frac{3a^2 + 9a}{a^2 - 3a + 2} \) is a decreasing function of \( a \) that is always strictly greater than 3. Evaluating for \( a = 20 \) gives \( \alpha = \frac{1380}{342} > 4 \), and for \( a = 21 \), \( \alpha = \frac{1512}{380} < 4 \).

Part (iii) follows immediately from part (ii). \qed

THEOREM 8.22. In the family \( F_2(a, i, s) \), we have

(i) For a fixed choice of \( a \) and \( i \), the smallest possible type \( t = 2 + s \) is achieved by taking \( s = \text{cei} \left( \frac{4a(2i - a + 7)}{(a - 1)(a - 3)} \right) \).

(ii) For a fixed choice of \( a \), the smallest possible type \( t = 2 + s \) is achieved by taking \( i = M := \text{cei} \left( \frac{2a - 3 + \sqrt{2a^2 + 8a + 7}}{2} \right), s = \text{cei} \left( \frac{4a(2M - a + 7)}{(a - 1)(a - 3)} \right) \). With these
choices, the type is 12 for $a = 205$, $a = 209$, and $a \geq 213$. The corresponding values of $M$ are 350, 357, and 364. For any other value of $a$, the type is greater than 12.

(iii) For any choice of $a$, $i$, and $s$, the type is at least 12.

PROOF. For (i), we combine the various lemmas in this section with Lemma 7.10 To evaluate $m_P((a - 1)/2 - 2)$, we observe $(a - 1)/2 - 2 \leq a.$

\[
s = \left\lceil \frac{m_P(i_f)}{m_P(j - i_f)} \right\rceil = \left\lceil \frac{m_P(i + 2)}{m_P((a - 1)/2 - 2)} \right\rceil = \left\lceil \frac{a[2(i + 2) - a + 3]/2}{((a - 1)/2 - 2 + 2)/2} \right\rceil = \left\lceil \frac{a[2i - a + 7]/2}{((a - 1)/2)/2} \right\rceil = \left\lceil \frac{a[2i - a + 7]/2}{[(a - 1)/2][(a - 1)/2 - 1]/2} \right\rceil = \left\lceil \frac{a(2i - a + 7)}{(a - 1)(a - 3)/4} \right\rceil = \left\lceil \frac{4a(2i - a + 7)}{(a - 1)(a - 3)} \right\rceil.
\]
For (ii), we observe that $s$ is a non-decreasing function of $i$, and we substitute the smallest permissible value $i = M$ to obtain

\[
s = \ceiling{\frac{4a(2M - a + 7)}{(a - 1)(a - 3)}}
\]

\[
= \ceiling{\frac{4a(2 \ceiling{\frac{2a - 3 + \sqrt{2a^2 + 8a + 7}}{2}} - a + 7)}{(a - 1)(a - 3)}}.
\]

Because of the effect of the inner ceiling function, we do not claim that $s$ is non-increasing as a function of $a$. In fact, using a computer, we calculated $s$ for integer values of $a$ up to 220, and found that $s = 10$ for $a = 198, 202, 205, 206, 208, 209,$ and 210, and for all values of $a \geq 212$; and that $s > 10$ for all other values of $a$.

However, the formula for $s$ is sandwiched between

\[
L(a) := \ceiling{\frac{4a(2(\frac{2a - 3 + \sqrt{2a^2 + 8a + 7}}{2}) - a + 7)}{(a - 1)(a - 3)}}
\]

and

\[
U(a) := \ceiling{\frac{4a(2(\frac{2a - 3 + \sqrt{2a^2 + 8a + 7}}{2} + 1) - a + 7)}{(a - 1)(a - 3)}}
\]

which are both nonincreasing as functions of $a$ and which both approach the value $\ceiling{4(2 + \sqrt{2} - 1)} = \ceiling{4 + 4\sqrt{2}} = 10$ as a limit for large values of $a$. Since $L(220) = U(220) = 10$, it must be that $s = 10$ for $a \geq 220$.

Our construction of $F_2$ requires $a$ to be odd, so we have type 2 + $s = 12$ for $a = 205$ and 209, and for any odd $a \geq 213$. For $a = 205$, $M = 350, j = 452$. For
THEOREM 8.23. In the family $G_1(a, b, i, s)$, we have

(i) For a fixed choice of $a$, $b$, and $i$, the smallest possible type $t = 1 + s$ is achieved by taking $s = \lceil \frac{2i - a - b + 10}{2ab - a - b - 2} \rceil$.

(ii) For a fixed choice of $a$ and $b$, the smallest possible type $t = 1 + s$ is achieved by taking $i = ab + 1$, $s = \lceil \frac{2ab - a - b + 12}{2ab - a - b - 2} \rceil$. With these choices, the type is 3 for $(a, b)$ if and only if either $a \geq 3$, $b \geq 4$ or $a \geq 2$, $b \geq 6$; the lowest values of $(a, b)$ for which the type is 4 are $(2, 4)$ and $(3, 3)$.

(iii) For any choice of $a$, $b$, $i$, and $s$, the type is at least 3.

PROOF. For (i), we combine the various lemmas in this section with Lemma 7.11 Then to evaluate $m_P(ab - 3)$, we observe $ab - 3 \geq a + b - 3$.

\[
s = \lceil \frac{m_P(i_f)}{m_P(j - i_f)} \rceil
= \lceil \frac{m_P(i + 3)}{m_P(ab - 3)} \rceil
= \lceil \frac{ab[2(i + 3) - a - b + 4]/2}{ab[2(ab - 3) - a - b + 4]/2} \rceil
= \lceil \frac{2i - a - b + 10}{2ab - a - b - 2} \rceil.
\]

For (ii), we observe that $s$ is an increasing function of $i$, and we substitute the smallest permissible value $i = ab + 1$ to obtain $t = 1 + s = 1 +
We observe that the fraction \( \alpha := \frac{2ab - a - b + 12}{2ab - a - b - 2} \) is a decreasing function, separately in \( a \) and \( b \), that is always strictly greater than 1.

Evaluating \( \alpha \) for the relevant values of \((a, b)\), we have:

\[(a, b) = (2, 3) : \alpha = \frac{19}{5}, \ 4 > \alpha > 3.\]

\[(a, b) = (2, 4) : \alpha = \frac{22}{8}, \ 3 > \alpha > 2.\]

\[(a, b) = (2, 5) : \alpha = \frac{25}{11}, \ 3 > \alpha > 2.\]

\[(a, b) = (3, 3) : \alpha = \frac{24}{10}, \ 3 > \alpha > 2.\]

\[(a, b) = (3, 4) : \alpha = \frac{29}{15}, \ 2 > \alpha > 1.\]

\[(a, b) = (2, 6) : \alpha = \frac{28}{14}, \ 2 = \alpha > 1.\]

Part (iii) follows immediately from part (ii).

\[\Box\]

**Theorem 8.24.** In the family \( G_2(a, b, i, s) \), we have
(i) For a fixed choice of $a$, $b$, and $i$, the smallest possible type $t = 1 + s$ is achieved by taking

$$s = \left\lceil \frac{ab[2i - a - b + 8]}{ab[a - b] + 2} \right\rceil \text{ if } a = 2.$$  
$$s = \left\lceil \frac{2i - a - b + 8}{ab - a - b} \right\rceil \text{ if } a > 2.$$

(ii) For a fixed choice of $a$ and $b$, the smallest possible type $t = 1 + s$ is achieved by taking $i = \frac{ab}{2} + 2$, in which case

$$s = \left\lceil \frac{ab[ab - a - b + 12]}{ab[a - b] + 2} \right\rceil \text{ if } a = 2.$$  
$$s = \left\lceil \frac{ab - a - b + 12}{ab - a - b} \right\rceil \text{ if } a > 2.$$

With these choices, the type is 3 for $(a, b)$ if and only if either $a \geq 2$, $b \geq 14$ or $a \geq 3$, $b \geq 8$ or $a \geq 4$, $b \geq 6$. The smallest values of $(a, b)$ for which the type is 4 are $(2,8)$, $(3,6)$, and $(4,4)$.

(iii) For any choice of $a$, $b$, and $i$, the type is at least 3.

PROOF. For (i), we combine the various lemmas in this section with Lemma 7.11. To evaluate $m_P(ab/2 - 2)$, we observe that if $a = 2$, $ab/2 - 2 = b - 2 = a + b - 4$; but if $a > 2$, $ab/2 - 2 = a(b - 2)/2 + a - 2 > (b - 2) + a - 2 = a + b - 4$. 

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If $a = 2$,

\[
    s = \text{ceil} \frac{m_p(i_f)}{m_p(j - i_f)}
    = \text{ceil} \frac{m_p(i + 2)}{m_p(ab/2 - 2)}
    = \text{ceil} \frac{ab[2(i + 2) - a - b + 4]/2}{ab[2(ab/2 - 2) - a - b + 4]/2 + 1}
    = \text{ceil} \frac{ab[2i - a - b + 8]}{ab[a - a - b] + 2}.
\]

If $a > 2$,

\[
    s = \text{ceil} \frac{m_p(i_f)}{m_p(j - i_f)}
    = \text{ceil} \frac{m_p(i + 2)}{m_p(ab/2 - 2)}
    = \text{ceil} \frac{ab[2(i + 2) - a - b + 4]/2}{ab[2(ab/2 - 2) - a - b + 4]/2}
    = \text{ceil} \frac{2i - a - b + 8}{ab - a - b}.
\]

For (ii), we observe that $s$ is an increasing function of $i$, and we substitute the smallest permissible value $i = ab/2 + 2$ to obtain, for $a = 2$, $t = 1 + s = 1 + \text{ceil} \frac{ab(ab - a - b + 12)}{ab(ab - a - b) + 2}$; and for $a > 2$, $t = 1 + s = 1 + \text{ceil} \frac{ab - a - b + 12}{ab - a - b}$. We let $\alpha := \frac{ab(ab - a - b + 12)}{ab(ab - a - b) + 2}$ and $\beta := \frac{ab - a - b + 12}{ab - a - b}$. We observe that, for all values of $a$ and $b$, $\alpha > 1$ and $\beta > 1$. In addition, $\alpha$ and $\beta$ are both decreasing functions, separately in $a$ and $b$. Evaluating $\alpha$ and $\beta$ for the relevant values of $(a, b)$ (recalling
that either $a$ or $b$ must be even), we have:

$(a, b) = (2, 6) : \alpha = 192/50, \ 4 > \alpha > 3.$

$(a, b) = (2, 7) : \alpha = 238/72, \ 4 > \alpha > 3.$

$(a, b) = (3, 4) : \beta = 17/5, \ 4 > \beta > 3.$

$(a, b) = (2, 8) : \alpha = 288/98, \ 3 > \alpha > 2.$

$(a, b) = (2, 13) : \alpha = 598/288, \ 3 > \alpha > 2.$

$(a, b) = (3, 6) : \beta = 21/9, \ 3 > \beta > 2.$

$(a, b) = (4, 4) : \beta = 20/8, \ 3 > \beta > 2.$

$(a, b) = (4, 5) : \beta = 23/11, \ 3 > \beta > 2.$

$(a, b) = (2, 14) : \alpha = 672/338, \ 2 > \alpha > 1.$

$(a, b) = (3, 8) : \beta = 25/13, \ 2 > \beta > 1.$

$(a, b) = (4, 6) : \beta = 26/14, \ 2 > \beta > 1.$

Part (iii) follows immediately from part (ii). \hfill \square

**Theorem 8.25.** In the family $H_1(a, b, c, i, s)$

(i) For a fixed choice of $a, b, c, \text{and} \ i$, the smallest possible type $t = 1 + s$ is achieved by taking $s = \left\lceil \frac{2i - a - b - c + 11}{2abc - a - b - c - 1} \right\rceil$. 

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(ii) For a fixed choice of $a$, $b$, and $c$, the smallest possible type $t = 1 + s$ is achieved by taking $i = abc$, $s = \lceil \frac{2abc - a - b - c + 11}{2abc - a - b - c - 1} \rceil$. With these choices, the type is 3 unless $(a, b, c) = (2, 2, 2)$, in which case the type is 4.

(iii) For any choice of $a$, $b$, $i$, and $s$, the type is at least 3.

PROOF. For (i), we combine the various lemmas in this section with Lemma 7.9. To evaluate $m_P(abc - 3)$, we observe $abc - 3 \geq a + b + c - 3 = (a - 1) + (b - 1) + (c - 1)$.

\[ s = \lceil \frac{m_P(i_f)}{m_P(j - i_f)} \rceil = \lceil \frac{m_P(i + 3)}{m_P(abc - 3)} \rceil = \lceil \frac{ab[2(i + 3) - a - b - c + 5]/2}{ab[2(abc - 3) - a - b - c + 5]/2} \rceil = \lceil \frac{2i - a - b - c + 11}{2abc - a - b - c - 1} \rceil \]

For (ii), we observe that $s$ is an increasing function of $i$, and we substitute the smallest permissible value $i = abc$ to obtain

\[ t = 1 + s = 1 + \lceil \frac{2abc - a - b - c + 11}{2abc - a - b - c - 1} \rceil. \]

We observe that the fraction

\[ \alpha := \frac{2abc - a - b - c + 11}{2abc - a - b - c - 1} \]

is a decreasing function, separately in $a$, $b$, and $c$, that is always strictly greater than 1. Evaluating for the relevant values of $(a, b, c)$, we have:
(a, b, c) = (2, 2, 2) : \alpha = \frac{21}{9}, \ 3 > \alpha > 2.

(a, b, c) = (2, 2, 3) : \alpha = \frac{28}{16}, \ 2 > \alpha > 1.

Part (iii) follows immediately from part (ii).

For the family \(G_3(a, b, i, s)\), we do not attempt to state a theorem with closed-form solutions, similar to those we have proved for the other five families. The most serious obstacle is finding a closed-form solution of (36), which is cubic in the variable \(i\). Instead, we content ourselves with demonstrating the method that led to choosing the example \(G_3(4, 4, 8, 7)\) above.

We begin with some suitable choice of \(a\) and \(b\), in this case \(a = b = 4\), in which case \(m = 5\) because

\[
\binom{5 + 1}{2} = 15 < ab = 16 < 21 = \binom{5 + 2}{2}.
\]

Then (36) becomes

\[
(16i - 32) + 56 \leq (i + 3)(i + 2)(i + 1)/6.
\]

This is false for \(i = 7\) since 136 > 120, but true for \(i = 8\), since 152 \leq 165, so we must choose \(i \geq 8\); and the additional requirement that \(i \geq a + b - 3 = 5\)
imposes no further condition. If we choose \( i = 8 \), with the intention of obtaining the smallest possible type for this choice of \( a \) and \( b \), the condition on \( s \) (using Proposition 7.11) is that

\[
s \geq \frac{m_P(i + 2)}{m_P(m - 2)} = \frac{m_P(10)}{m_P(3)} = \frac{16(20 - 4 - 4 + 4)/2}{\binom{3 + 3}{3}} = \frac{128}{20} = 6.4,
\]

so we must take \( s \geq 7 \).
CHAPTER 9

Further Remarks

1. For Which Codimensions and Types are Non-Unimodal Level Algebras Possible?

We now return to a question raised in Chapter 1, at which time we were not yet ready to provide justification for the answer given: for a specified codimension $r$ and type $t$, must level algebras necessarily be unimodal?

**Proposition 9.1.** In codimension 3, there exist non-unimodal level algebras for any type 5 or greater. In codimensions 4 and 5, there exist non-unimodal level algebras for any type 3 or greater.

**Proof.** For codimension 3, we let $t \geq 5$ and describe a procedure for finding a member of $F_1(a,i,s)$ of type $t$. We choose $a \geq 21$ such that $m_P(3a) \geq t$, that is, $\frac{a(5a + 3)}{2} \geq t$. We apply Theorem 8.16 for $F_1$, with $i = 2a$ and $s = t - 1$. That is, we let $E(C) := \langle f_1(C), \ldots, f_s(C) \rangle$, where $f_1(C), \ldots, f_s(C)$ are general elements of $W_{M_P(3a)} \subseteq D_{3a}$, and we let $F(C') := \langle g_1(C') \rangle$, where $g_1(C')$ is a general element of $D_{3a}$. To ensure that $A_{E(C)} \oplus F(C')$ is non-unimodal of type $1 + s = 1 + (t - 1) = t$ for general $C$ and $C'$, we must check that the parameters $a, i, s$ are permissible. It is immediate that $a \geq 4$ and $i \geq 2a$, and by Lemma 8.19 we must check that $s \geq m_P(2a + 3)/m_P(a - 3)$. But, having assumed that $a \geq 21$, we know from Theorem 8.21 that $4 \geq m_P(2a + 3)/m_P(a - 3)$, and we have also assumed that...
Thus the values of the parameters \(a, i,\) and \(s\) are permissible, so \(A_{\mathcal{E}(C)} \oplus \mathcal{F}(C')\) is indeed a member of the family \(F_1\), hence non-unimodal by Theorem 8.16.

Analogous constructions are available in codimensions 4 and 5. For codimension 4, we let \(t \geq 3\) and we find a member of \(G_1(a, b, i, s)\) of given type \(t\). We choose \(b \geq a \geq 4\) such that \(m_p(2ab + 1) \geq t\), that is, \(ab(4ab - a - b + 6) \geq 2t\), and we again apply Theorem 8.16 this time for \(G_1\), with \(i = ab + 1\) and \(s = t - 1\). That is, we let \(\mathcal{E}(C) := \langle f_1(C), ..., f_s(C) \rangle\), where \(f_1(C), ..., f_s(C)\) are general elements of \(W_{M_p(2ab+1)} \subseteq D_{2ab+1}\), and we let \(\mathcal{F}(C') := \langle g_1(C') \rangle\), where \(g_1(C')\) is a general element of \(k[x, y, z]_{2ab+1} \subseteq D_{2ab+1}\). To ensure that \(A_{\mathcal{E}(C)} \oplus \mathcal{F}(C')\) is non-unimodal of type \(1 + s = 1 + (t - 1) = t\) for general \(C\) and \(C'\), we must check that the parameters \(a, b, i, s\) are permissible. It is immediate that \(b \geq a \geq 2\) and \(i \geq ab + 1\), and by Lemma 8.19 we must check that \(s \geq m_p(ab + 4)/m_p(ab - 3)\).

But, having assumed that \(b \geq a \geq 4\), we know from Theorem 8.23 that \(2 \geq m_p(ab + 4)/m_p(ab - 3)\), and we have also assumed that \(s := t - 1 \geq 2\). Thus the values of the parameters \(a, b, i,\) and \(s\) are permissible, so \(A_{\mathcal{E}(C)} \oplus \mathcal{F}(C')\) is indeed a member of the family \(G_1\), hence non-unimodal by Theorem 8.16.

For codimension 5, we let \(t \geq 3\) and we find a member of \(H_1(a, b, c, i, s)\) of given type \(t\). We choose \(c \geq b \geq a \geq 3\) such that \(m_p(2abc) \geq t\), that is, \(abc(4abc - a - b - c + 5) \geq 2t\), and we again apply Theorem 8.16 this time for \(H_1\), with \(i = abc\) and \(s = t - 1\). That is, we let \(\mathcal{E}(C) := \langle f_1(C), ..., f_s(C) \rangle\), where
$f_1(C), \ldots, f_s(C)$ are general elements of $W_{M_p(2abc)} \subseteq D_{2abc}$, and we let $F(C') := \langle g_1(C') \rangle$, where $g_1(C')$ is a general element of $k[x, y, z]_{2abc} \subseteq D_{2abc}$. To ensure that $A_{\mathcal{E}(C)} \oplus F(C')$ is non-unimodal of type $1 + s = 1 + (t - 1) = t$ for general $C$ and $C'$, we must check that the parameters $a, b, c, i, s$ are permissible. It is immediate that $c \geq b \geq a \geq 2$ and $i \geq abc$, and by Lemma 8.19 we must check that $s \geq m_P(abc + 3)/m_P(abc - 3)$. But, having assumed that $c \geq b \geq a \geq 3$, we know from Theorem 8.25 that $2 \geq m_P(abc + 3)/m_P(abc - 3)$, and we have also assumed that $s := t - 1 \geq 2$. Thus the values of the parameters $a, b, c, i, s$ are permissible, so $A_{\mathcal{E}(C)} \oplus F(C')$ is indeed a member of the family $H_1$, hence non-unimodal by Theorem 8.16.

\[ \blacksquare \]

**Proposition 9.2.** Let $W = \langle f_1, \ldots, f_t \rangle \subseteq k[x_1, \ldots, x_r]_j$ be a nonzero vector subspace of dimension $t$. Define $V = \langle f_1, \ldots, f_t, x_{r+1}^j \rangle \subseteq k[x_1, \ldots, x_r, x_{r+1}]_j$. Then $A_W$ is a level algebra of codimension $r$, type $t$, and socle degree $j$; and $A_V$ is a level algebra of codimension $r + 1$, type $t + 1$, and socle degree $j$. For $d = 1, \ldots, j$, $h_V(d) = h_W(d) + 1$.

**Proof.** By Theorem 2.9, $A_W$ and $A_V$ are level algebras of the stated codimension and socle degree, and the types are, by construction, the dimensions respectively of $W$ and $V$. We write

\[ V = \langle f_1, \ldots, f_t, x_{r+1}^j \rangle \]

\[ = \langle f_1, \ldots, f_t \rangle \bigoplus \langle x_{r+1}^j \rangle \]

\[ = W \bigoplus \langle x_{r+1}^j \rangle \]
and observe that \( R_{j-d} \ast \langle x_{r+1}^j \rangle = \langle x_{r+1}^d \rangle \subseteq D_d \) is a vector space of dimension 1 whose intersection with \( R_{j-d} \ast \mathcal{W} \) is \( \{0\} \). By Lemmas 2.10 and 3.1,

\[
h_V(d) = h_W(d) + h_{\langle x_{r+1}^j \rangle}(d)
\]

\[
= h_W(d) + 1.
\]

\[\square\]

**Proposition 9.3.** If there exists a non-unimodal level algebra \( A \) of codimension \( r \), type \( t \), and socle degree \( j \), then there exists a non-unimodal level algebra \( A' \) of codimension \( r + 1 \), type \( t + 1 \), and socle degree \( j \).

**Proof.** This follows immediately from the construction of the previous proposition. Writing \( A = A_W \), take \( A' = A_V \). \[\square\]

We next cite a result of D. Bernstein from [BI92], which we restate in our own notation.

**Theorem 9.4.** Let \( r = 5, R = k[X_1, ..., X_5], D = k[x_1, ..., x_5] \). Consider the family of vector subspaces \( \mathcal{W} = \langle x_4 f + x_5 g \rangle \subseteq D_{16} \), where \( f, g \in k[x_1, x_2, x_3]_{15} \). Then for general \( f \) and \( g \), \( A_W \) is a level algebra of type 1 and socle degree 16, with Hilbert function \( (1,5,12,22,35,51,70,91,90,91,70,51,35,22,12,5,1) \). That is, \( A_W \) is a non-unimodal Gorenstein algebra.

\[\square\]
PROPOSITION 9.5. Given integers $r \geq 5$ and $t \geq 1$, there exists a non-unimodal level algebra of codimension $r$ and type $t$.

PROOF. By Proposition 9.3, it is enough to demonstrate a non-unimodal level algebra (a) when $r = 5$, for any $t$, and (b) when $t = 1$, for any $r$.

By virtue of the Bernstein example and Proposition 9.1, to establish (a) it only remains to consider the case $r = 5, t = 2$. For this, we modify the Bernstein example so that $\mathcal{V} = \langle x_4f + x_5g, x_5^{16} \rangle$. Then, for $d \geq 2$,

$$h_{\mathcal{V}}(d) = h_{\langle x_4f + x_5g \rangle}(d) + h_{\langle x_5^{16} \rangle}(d)$$

$$= h_{\mathcal{V}}(d) + 1.$$

for exactly the same reasons as in the proof of Proposition 9.2. So $A_{\mathcal{V}}$ is non-unimodal.

To establish (b) for codimension $r > 5$, we modify the Bernstein example in a different way. This time, we let $\mathcal{V} = \langle x_4f + x_5g + x_6^{16} + ... + x_r^{16} \rangle$, and then for $d = 2, ..., j - 1$,
\[ h_Y(d) = \dim_k R_e \ast (x_4 f + x_5 g + x_6^{16} + ... + x_r^{16}) \]
\[ = \dim_k R_e \ast (x_4 f + x_5 g) + \dim_k R_e \ast (x_6^{16} + ... + x_r^{16}) \]
\[ = h_Y(d) + (r - 5), \]

once again by Lemma 3.1 and Lemma 2.10. So, again, \( A_Y \) is non-unimodal. \( \square \)

We remark that Propositions 9.1 and 9.5 were used in Chapter 1 to list the codimensions and types for which non-unimodal level algebras are known to exist.

2. Minimal Socle Degree

For non-unimodal level algebras of given codimension \( r \) and type \( t \), we have no actual methods for determining what the lowest possible socle degree \( j \) might be. We make several remarks about interesting cases.

For \( r = 3, t = 5 \), the only known non-unimodals come from the family \( F_1(a, i, s) \). For these, we see from Theorem 8.21 that it is possible to achieve type 5 only for \( a \geq 21 \). We have \( j = i + a \geq 2a + a = 3a \), so the smallest known socle degree is 63.

For \( r = 4, t = 3 \), we must consider the families \( G_1(a, b, i, s) \) and \( G_2(a, b, i, s) \). (Logically, we ought also to consider \( G_3(a, b, i, s) \), but this family is not known to yield algebras of type 3). By arguments analogous to the one in the previous
paragraph, we see from Theorems 8.23 and 8.24 that we must check $G_1(2, 6, 13, 2)$, $G_1(3, 4, 13, 2)$, $G_2(2, 14, 16, 2)$, $G_2(3, 8, 14, 2)$, and $G_2(4, 6, 14, 2)$, of respective socle degrees $j = 25, 25, 30, 26, 26$. Thus $j = 25$ represents the smallest known socle degree, arising from family $G_1$.

For $r = 5, t = 3$, the lowest known socle degree does not result from family $H_1$, which can do no better than $H_1(2, 2, 3, 12, 2)$, of socle degree 24. Instead, we can modify the Bernstein example $W = \langle x_4f + x_5g \rangle$, discussed earlier, of a non-unimodal Gorenstein of codimension 5. We let $V = \langle x_4f + x_5g, x_3^{16}, x_4x_5^{15} \rangle$. Then, for $d \geq 2$,

$$h_V(d) = h_{\langle x_4f + x_5g \rangle}(d) + h_{\langle x_3^{16}, x_4x_5^{15} \rangle}(d)$$

$$= h_W(d) + 2.$$ 

So we can find socle degree $j = 16$.

For $r = 5, t = 4$, there are several possible sources to consider. $H_1(2, 2, 2, 8, 3)$ gives $j = 16$. We could try modifying a type-3 member of family $H_1$ by adding a generator, but the socle degree would then be at least 24. We could try using Proposition 9.2, applied to a type-3 non-unimodal of codimension 4, but the socle degree would be at least 25. Finally, we could consider another modification of the
Bernstein example: we let $V = \langle x_{4}f + x_{5}g, x_{5}^{16}, x_{4}x_{3}^{15}, x_{4}^{2}x_{3}^{14} \rangle$. Then, for $d \geq 2$,

\[
\begin{align*}
    h_{V}(d) &= h_{\langle x_{4}f + x_{5}g \rangle}(d) + h_{\langle x_{5}^{16}, x_{4}x_{3}^{15}, x_{4}^{2}x_{3}^{14} \rangle}(d) \\
    &= h_{V}(d) + 3.
\end{align*}
\]

This is another example of socle degree $j = 16$. With its single drop, this is certainly different from $H_{1}(2, 2, 2, 8, 3)$, which has a double drop.
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