MACDONALD POLYNOMIALS OF TYPE $C_n$ WITH ONE-COLUMN DIAGRAMS AND DEFORMED CATALAN NUMBERS

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Abstract. We present an explicit formula for the transition matrix $\mathcal{C}$ from the type $C_n$ Macdonald polynomials $P_{(1^n)}^{(C_n)}(x|b; q, t)$ with one column diagrams, to the type $C_n$ monomial symmetric polynomials $m_{(1^n)}(x)$. The entries of the matrix $\mathcal{C}$ enjoy a set of certain three term recursion relations, which can be regarded as a $(b, q, t)$-deformation of the one for the Catalan triangle or ballot numbers. It is also shown that the $q$-ballot numbers appear as the Kostka polynomials, namely in the transition matrix from the Schur polynomials $P_{(1^n)}^{(C_n)}(x|q; q, q)$ to the Hall-Littlewood polynomials $P_{(1^n)}^{(C_n)}(x|t; 0, t)$.

1. Introduction

The aim of this article is to investigate the transition matrix $\mathcal{C}$, which describes the expansion of the type $C_n$ Macdonald polynomials $\[2\] P_{(1^n)}^{(C_n)} = P_{(1^n)}^{(C_n)}(x|b; q, t)$ with one column diagrams, in terms of the type $C_n$ monomial symmetric polynomials $m_{(1^n)}(x)$. As for our convention of notation, see $[3]$. On this course, we found that certain deformations appear, associated with the Catalan triangle or ballot numbers, and binomial coefficients. We refer the readers to $[S]$ concerning the Catalan triangle numbers, and $[FH]$ and $[A]$ for the $q$-Catalan and $q$-ballot numbers.

Theorem 1.1. Let $n \in \mathbb{Z}_{>0}$. Let $\mathbf{P}^{(n)}$ and $\mathbf{m}^{(n)}$ be the infinite column vectors

$$\begin{align*}
\mathbf{P}^{(n)} &= t^i (P_{(1^n)}^{(C_n)}, \ldots, P_{(1^n)}^{(C_n)}, P_{(1^n)}^{(C_n)}(0, 0, 0, \ldots)), \\
\mathbf{m}^{(n)} &= t^i (m_{(1^n)}, \ldots, m_{(1^n)}, m_{(1^n)}(0, 0, 0, \ldots)).
\end{align*}$$

There exist a unique infinite transition matrix $\mathcal{C} = (\mathcal{C})_{i,j} \in \mathbb{Z}_{\geq 0}$ satisfying the conditions

1. $\mathcal{C}$ is upper triangular, namely $i > j$ implies $\mathcal{C}_{ij} = 0$,
2. $\mathcal{C}$ is even, namely $i + j$ is odd implies $\mathcal{C}_{ij} = 0$,
3. $\mathcal{C}_{ij}$ are rational functions in $b,q$ and $t$ which do not depend on $n$ and we have $\mathbf{P}^{(n)} = \mathcal{C} \mathbf{m}^{(n)}$ for all $n \geq 1$. (stability)

This transition matrix $\mathcal{C}$ is uniquely characterized by the $(b,q,t)$-deformed Catalan triangle type three term recursion relations

$$\begin{align*}
\mathcal{C}_{0,0} &= 1, \\
\mathcal{C}_{i-1,i-1} &= \mathcal{C}_{i,i} & (i = 1, 2, 3, \ldots), \\
f[t] \mathcal{C}_{1,j-1} &= \mathcal{C}_{0,j} & (j = 2, 4, 6, \ldots), \\
f[t^i] \mathcal{C}_{i,j-1} + f[t^{i+1}] \mathcal{C}_{i+1,j-1} &= \mathcal{C}_{i,j} & (i + j : \text{even}, 0 < i < j),
\end{align*}$$

where we have used the notation

$$f[s] = \frac{(1 - 1/s)(1 - t^2/s^2b^2q)(1 + t/sb)(1 + t/sbq)}{(1 - t/s^2b^2q)(1 - t^3/s^2b^2q)}.$$ 

(1.10)
A proof of this is presented in §2.4. The solution to the three term recursion relations (1.6), (1.7), (1.8) and (1.9) for \( C_{i,j} \) given in terms of the function \( f[s] \) is presented in Proposition 7.3.

**Corollary 1.2.** When \( b = q \) and \( t = q \), the Macdonald polynomials become the Schur polynomials \( s_{\lambda}(x) = s_{\lambda}^{(C_n)}(x) \) of type \( C_n \). In this case we have \( f[t^{i+1}] = 1 \) for \( i \geq 0 \), indicating that the recursion relations (1.6)-(1.9) reduces to the ones for the ordinary Catalan triangle (or ballot numbers). Therefore it holds that

\[
s_{\lambda}^{(C_n)}(x) = P_{\lambda}(x|q,q) = \sum_{k=0}^{\lfloor t \rfloor} n - r + 1 \left[ \frac{n - r + 2k}{k} \right] m_{(1r-2k)}(x) ,
\]

where \( \binom{m}{j} = \frac{m(m-1) \cdots (m-j+1)}{j!} \) denote the ordinary binomial coefficient.

**Corollary 1.3.** When \( b = 1 \) and \( t = q \), the Macdonald polynomials become the Schur polynomials \( s_{\lambda}(x) = s_{\lambda}^{(D_n)}(x) \) of type \( D_n \). (See Remark 1.4 below.) In this case we have \( f[t] = 2 \) and \( f[t^{i+1}] = 1 \) for \( i > 0 \), and the recursion relations (1.6)-(1.9) reduces to the ones for (the half of) the ordinary Pascal triangle. We have

\[
s_{\lambda}^{(D_n)}(x) = P_{\lambda}(x|q,t) = \sum_{k=0}^{\lfloor t \rfloor} n - r + 2k \left[ \frac{n - r + 2k}{k} \right] m_{(1r-2k)}(x) .
\]

**Remark 1.4.** To be precise, when \( \ell(\lambda) = n \), the polynomial \( P_{\lambda}^{(C_n)}(x|1;q,t) \) (or \( m_{\lambda} \)) has to be further decomposed in terms of the type \( D_n \) Macdonald (or monomial) polynomials, since the Weyl group is smaller than the one for \( C_n \). Such a decomposition is easy but takes some space for a separate treatment. Therefore throughout in this paper, we do not go in detail in that direction, leaving the extra detail for the readers.

The first few terms of (1.11) and (1.12) read

\[
\begin{pmatrix}
  s_{(1n)}^{(C_n)} & s_{(1n)-1}^{(C_n)} & s_{(1n)-2}^{(C_n)} & s_{(1n)-3}^{(C_n)} \\
  s_{(1n)}^{(D_n)} & s_{(1n)-1}^{(D_n)} & s_{(1n)-2}^{(D_n)} & s_{(1n)-3}^{(D_n)}
\end{pmatrix} =
\begin{pmatrix}
  1 & 1 & 2 & 5 & 14 & \cdots \\
  1 & 1 & 2 & 5 & 14 & 42 \\
  1 & 1 & 2 & 5 & 14 & 42 & 92 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots 
\end{pmatrix}
\begin{pmatrix}
  m_{(1n)}^{(C_n)} \\
  m_{(1n-1)}^{(C_n)} \\
  m_{(1n-2)}^{(C_n)} \\
  m_{(1n-3)}^{(C_n)} \\
  \vdots 
\end{pmatrix}, \quad (1.13)
\]

\[
\begin{pmatrix}
  s_{(1n)}^{(D_n)} & s_{(1n)-1}^{(D_n)} & s_{(1n)-2}^{(D_n)} & s_{(1n)-3}^{(D_n)} \\
  s_{(1n)}^{(D_n)} & s_{(1n)-1}^{(D_n)} & s_{(1n)-2}^{(D_n)} & s_{(1n)-3}^{(D_n)}
\end{pmatrix} =
\begin{pmatrix}
  1 & 1 & 2 & 5 & 14 & \cdots \\
  1 & 1 & 2 & 5 & 14 & 42 \\
  1 & 1 & 2 & 5 & 14 & 42 & 92 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots 
\end{pmatrix}
\begin{pmatrix}
  m_{(1n)}^{(D_n)} \\
  m_{(1n-1)}^{(D_n)} \\
  m_{(1n-2)}^{(D_n)} \\
  m_{(1n-3)}^{(D_n)} \\
  \vdots 
\end{pmatrix}. \quad (1.14)
\]

As an application of our results obtained in this paper, we calculate the transition matrix from the Schur polynomials to the Hall-Littlewood polynomials, namely the Kostka polynomials, associated with one column diagrams.

**Definition 1.5.** Let \( K_{(1r)(1r-2j)}^{(C_n)} \) and \( K_{(1r)(1r-2j)}^{(D_n)} \) be the transition coefficients defined by

\[
s_{(1r)}^{(C_n)}(x) = \sum_{j=0}^{\lfloor \frac{t}{2} \rfloor} K_{(1r)(1r-2j)}^{(C_n)}(t) P_{(1r-2j)}^{(C_n)}(x|t; 0, t), \quad (1.15)
\]
\[
S^{(D_n)}_{(1^r)}(x) = \sum_{j=0}^{[\frac{s}{n}]} K^{(D_n)}_{(1^r)(1^{r-2j})}(t) P^{(D_n)}_{(1^{r-2j})}(x|0, t). \tag{1.16}
\]

**Theorem 1.6.** The \(K^{(C_n)}_{(1^r)(1^{r-2j})}(t)\) and \(K^{(D_n)}_{(1^r)(1^{r-2j})}(t)\) are polynomials in \(t\) with nonnegative integral coefficients. We have
\[
K^{(C_n)}_{(1^r)(1^{r-2j})}(t) = t^{2j} \frac{[n-r+1]_q}{[n-r+j+1]_q} [n-r+2j]_q 
\]
\[
= \begin{bmatrix} n-r+2j \cr j \end{bmatrix}_q - \begin{bmatrix} n-r+2j \cr j-1 \end{bmatrix}_q, \tag{1.17}
\]
\[
K^{(D_n)}_{(1^r)(1^{r-2j})}(t) = t^j \frac{1+t^{n-r}}{1+t^{n-r+2j}} \begin{bmatrix} n-r+2j \cr j \end{bmatrix}_q \tag{1.18}
\]
\[
= t^{n-r+j} \begin{bmatrix} n-r+2j-1 \cr j-1 \end{bmatrix} + t^j \begin{bmatrix} n-r+2j-1 \cr j \end{bmatrix}_q.
\]

Here we have used the notation for the \(q\)-integer \([n]_q\), the \(q\)-factorial \([n]_q!\) and the \(q\)-binomial coefficient \(\binom{m}{j}_q\) as
\[
[n]_q = 1 - q^n, \quad [n]_q! = [1]_q[2]_q \cdots [n]_q, \quad \binom{m}{j}_q = \prod_{k=1}^j \frac{[m-k+1]_q}{[k]_q} = \frac{[m]_q!}{[j]_q![m-j]_q!}. \tag{1.19}
\]

As for our proof of this, see \(\S 8.3\).

**Remark 1.7.** Note that the \(K^{(C_n)}_{(1^r)}(t)\)'s are essentially give by the \(t^2\)-deformed ballot numbers \([A]\) (the case \(n = r\) corresponds to the \(t^2\)-deformation of the Catalan numbers \([F,H]\)), and the \(K^{(C_n)}_{(1^r)}(t)\)'s by a version of \(t\)-deformed binomial numbers.

First few entries of \(K^{(C_n)}_{(1^r)}(t)\) read
\[
\begin{pmatrix}
1 & t^2 & t^4 + t^8 & t^6 + t^{10} + t^{12} + t^{14} + t^{18} \\
1 & t^2 + t^4 & t^4 + t^6 + t^8 + t^{10} + t^{12} & \cdots \\
1 & t^2 + t^4 + t^6 & t^4 + t^6 + 2t^8 + t^{10} + 2t^{12} + t^{14} + t^{16} & \cdots \\
1 & t^2 + t^4 + t^6 + t^8 & \cdots
\end{pmatrix},
\tag{1.20}
\]

and for \(K^{(D_n)}_{(1^r)}(t)\) we have
\[
\begin{pmatrix}
1 & 2t & 2t^2 + 2t^4 + 2t^6 & t^2 + t^3 + t^4 + t^5 + 2t^6 + t^7 + t^8 + t^9 + t^{10} & \cdots \\
1 & t + t^2 + t^3 & t^2 + t^3 + t^4 + t^5 + 2t^6 + t^7 + t^8 + t^9 + t^{10} & \cdots \\
1 & t + 2t^3 + t^5 & t + t^3 + t^4 + t^5 + t^7 & \cdots \\
1 & \cdots
\end{pmatrix}. \tag{1.21}
\]

The present article is organized as follows. In \(\S 2\) several transition formulas obtained in this paper are summarized for the convenience of reading. Then we present a proof of our main result Theorem 1.1. In \(\S 3\) and \(\S 4\) we use Mimachi’s kernel function identity to have a description of the \(BC_n\) Koornwinder polynomials and type \(C_n\) Macdonald polynomials with one column diagrams. The \(\S 5\) and \(\S 6\) are the core of the technical part of this article. In
Bressoud’s matrix inversion is applied to invert the formula for the type $C_n$ Macdonald polynomials with one column diagrams. In [63] the four term relations for $B[s,j]$ and $\tilde{B}[s,j]$ are derived. In [65] is given the basic properties for the transition matrix $C$. In [66] we studied some degenerate cases, including the calculation of the Kostka polynomials. Some conjectures are derived. In [67] polynomials with one column diagrams. In [68] Koornwinder polynomials

$$\text{Definition 2.1.}$$ Define the symmetric Laurent polynomial $P(x|a,b,c,d|q,t)$ be a set of variables. Let $P(x|a,b,c,d|q,t)$ be the Koornwinder polynomial with one column diagram (1r) ($r \in \mathbb{Z}_{\geq 0}$). (See [33] as to our notation.)

**2. Collection of Transition Formulas and Proof of Theorem 1.1**

In this section, we collect several transformation formulas which we need to establish Theorem 1.1 giving brief explanations about our ideas and methods for their derivations.

2.1. **Koornwinder polynomials $P_{(1r)}(x|a,b,c,d|q,t)$ with one column diagrams.** In [FHNS], we studied some explicit formulas for the Koornwinder polynomials [K] with one-row diagrams. The results were interpreted as certain summation over the sets of tableaux of types $C_n$ and $D_n$. While using the same technique as in [FHNS], but replacing the Cauchy type kernel function by the dual-Cauchy type namely Mimachi’s one (as to the kernel functions, see [Mi] and [KNS]), one can study an explicit formula for the Koornwinder polynomials with one column diagrams. Mimachi’s kernel function [Mi] intertwines the action of the Koornwinder operator of type $BC_n$ to the one for $BC_1$ (namely for the Askey-Wilson operator) which in turn act on the Askey-Wilson eigenfunction. To perform the explicit calculations based on this idea, as was in the one-row diagram case, we need the fourfold summation formula for the Askey-Wilson eigenfunction [HNS]. The detail will be given in [33] and [44].

Specializing the parameters of the Koornwinder polynomials, we obtain the Macdonald polynomials of types $C_n$ and $D_n$ with one column diagram. In these particular limits, the fourfold summation (for the Askey-Wilson eigenfunction) reduces to a twofold one. In this way, we have explicit expressions for the Macdonald polynomials of types $C_n$ and $D_n$ with one column diagrams.

Let $n \in \mathbb{Z}_{\geq 0}$ and $x = (x_1, \ldots, x_n)$ be a set of variables. Let $P_{(1r)}(x|a,b,c,d|q,t)$ be the Koornwinder polynomial with one column diagram (1r) ($r \in \mathbb{Z}_{\geq 0}$). (See [33] as to our notation.)

**Definition 2.1.** Define the symmetric Laurent polynomial $E_r(x)$’s by expanding the generating function $E(x|y)$ as

$$E(x|y) = \prod_{i=1}^{n} (1 - yx_i)(1 - y/x_i) = \sum_{r \geq 0} (-1)^r E_r(x)y^r. \quad (2.1)$$

Note that we have $E_{2n-r}(x) = E_r(x)$ for $0 \leq r \leq n$ and $E_r(x) = 0$ for $r > 2n$.

**Theorem 2.2.** We have the following fourfold summation formula for the $BC_n$ Koornwinder polynomial $P_{(1r)}(x|a,b,c,d|q,t)$ with one column diagram

$$P_{(1r)}(x|a,b,c,d|q,t) = \sum_{k,l,i,j \geq 0} (-1)^{i+j} E_{r-2k-2l-i-j}(x) \hat{c}_e(k,l;t^{n-r-1+i+j})\hat{c}_o(i,j;t^{n-1}), \quad (2.2)$$

Throughout the paper, we use the standard notation (see [GR])

$$\begin{align*}
(z;q)_\infty &= \prod_{k=0}^{\infty} (1 - q^k z), \\
(z)_k &= \frac{(z;q)_\infty}{(q^k z;q)_\infty} \quad (k \in \mathbb{Z}),
\end{align*} \quad (1.22)$$

$$\begin{align*}
(a_1, a_2, \cdots, a_r; q)_k &= (a_1; q)_k(a_2; q)_k \cdots (a_r; q)_k \quad (k \in \mathbb{Z}),
\end{align*} \quad (1.23)$$

$$\begin{align*}
r+1\phi_r \left( \frac{a_1, a_2, \cdots, a_{r+1}}{b_1, b_2, \cdots, b_r}; q, z \right) &= \sum_{n=0}^{\infty} \frac{(a_1, a_2, \cdots, a_{r+1})_n}{(q, b_1, b_2, \cdots, b_r)_n} z^n. \quad (1.24)
\end{align*}$$

$$\begin{align*}
K_{C_n} &= \text{the basic properties for the transition matrix } C. \quad (1.10)
\end{align*}$$

$$\begin{align*}
\text{some degenerate cases, including the calculation of the Kostka polynomials. Some conjectures are presented in [63] concerning the asymptotically free type eigenfunctions for the type } C_n \text{ when } b = t.
\end{align*}$$

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\text{In } \text{§ } 4 \text{ A. HOSHINO AND J. SHIRAISHI}
\end{align*}$$

$$\begin{align*}
\text{collection several transformation formulas which we need to establish Theorem 1.1 giving brief explanations about our ideas and methods for their derivations.}
\end{align*}$$

$$\begin{align*}
\text{2. Collection of Transition Formulas and Proof of Theorem 1.1}
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\text{the Koornwinder polynomials with one column diagrams.}
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where
\[
\tilde{c}_e'(k; l; s) = \frac{(t^2/a^2; t^2)_k (sc^2/t; t^2)_k (s^2c^4/t_2^2; t^2)_k (1/c^2; t)_l (s/t; t)_{2k+l} 1 - st^{2k+2l-1} a_{2k} c_{2l}}{(t^2; t^2)_k (sc^2/t; t^2)_k (s^2a^2c^2/t^2; t^2)_k (t; t)_l (sc^2/t; t)_{2k+l}} a_{2k} c_{2l}, \tag{2.3}
\]
\[
\tilde{c}_a(i; j; s) = -\frac{(-a/b; t)_i (scd/t; t)_i (s; t)_{i+j} (-sac/t; t)_{i+j} (s^2a^2c^2/t^3; t)_{i+j}}{(t; t)_i (-sac/t; t)_i} \frac{(-c/d; t)_j (sab/t; t)_j b^t q^t}{(t; t)_j (-sac/t; t)_j}.
\tag{2.4}
\]

Degenerating Koornwinder’s parameters as \((a, b, c, d) \rightarrow (-b^{1/2}, b^{1/2}, -q^{1/2}b^{1/2}, q^{1/2}b^{1/2})\) we have the Macdonald polynomials of type \(C_n\). (See (3.3))

**Corollary 2.3.** We have the following twofold summation formula for for the Macdonald polynomials \(P^{(C_n)}_{(1^r)}(x|b; q, t)\) of type \(C_n\) with one column diagram \(P^{(C_n)}_{(1^r)}(x|b; q, t)\). Write \(s = t^{n-r+1}\) for simplicity. We have
\[
P^{(C_n)}_{(1^r)}(x|b; q, t) = \sum_{0 \leq k, l \leq r \leq r} \frac{(1/bq; t)_l (s/t; t)_{2k+l} 1 - st^{2k+2l-1} (bq)^t}{(t; t)_l (sbq; t)_{2k+l} 1 - st^{-1}} \frac{(qt; t^2)_k (sbqt; t^2)_k (s^2b^2q^2/t^2; t^2)_k b^k}{(t^2; t^2)_k (sbqt; t^2)_k (s^2b^2q^2/t^2; t^2)_k}.
\tag{2.5}
\]

Note that we have \(P^{(D_n)}_{(1^r)}(x|q, t) = P^{(C_n)}_{(1^r)}(x|1; q, t)\). Hence by setting \(b = 1\), we have a formula for the Macdonald polynomials \(P^{(D_n)}_{(1^r)}(x|q, t)\) of type \(D_n\) with one column diagram.

One can find that the formula (2.5) can be inverted by applying the Bressouel matrix inversion technique \([B, L]\). See \([L]\) as to the detail.

**Theorem 2.4.** We have
\[
E_r(x) = \sum_{0 \leq k, l \leq r} \frac{P^{(C_n)}_{(1^r-2k)}(x|b; q, t) (bq; t)_l (st^{l-1}bq; t)_{l+2k}}{(t; t)_l (st^{l-1}bq; t)_{l+2k}} \times \frac{(1/qt; t^2)_k (s^2t^6-2b^2q^2; t^2)_k (s^2t^6+2b^2q^2; t^2)_k (bq)^t}{(t^2; t^2)_k (s^2t^6-2b^2q^2; t^2)_k (s^2t^6+2b^2q^2; t^2)_k},
\tag{2.6}
\]
where \(s = t^{n-r+1}\).

This is proved in (5.10) of Theorem 5.5.

2.2. **Coefficients** \(B[s, j]\) and \(\tilde{B}[s, j]\). By using the \(q\)-analogue of Bailey’s transformation \([GR \ p.99, (3.10.14)]\), one can rewrite the twofold summations in (2.5) and (2.6) as sums having certain \(\phi_3\) series as their coefficients.

**Definition 2.5.** Let \(B[s, j]\) and \(\tilde{B}[s, j]\) be the rational functions in \(s\) defined by
\[
B[s, j] = (-1)^j s^{-j} \frac{(s^2/t^2; t^2)_j}{(t^2; t^2)_j} 1 - s^2 t^{2j-2} 2 \phi_3 \begin{bmatrix} -sb, -sbq, s^2t^2j-2, t^{2j-2} & -s, -st, s^2b^2q/t \end{bmatrix} \tag{2.7}
\]
\[
\tilde{B}[s, j] = (st^{j+1})^{-j} \frac{(l^2s^2; t^2)_j}{(t^2; t^2)_j} 2 \phi_3 \begin{bmatrix} t^{-2j+2}/sb, t^{-2j+2}/sbq, t^{-2j+2}/s^2, t^{-2j} & t^{2j+2}/s, t^{2j+2}/s^2, t^{2j} \end{bmatrix} \tag{2.8}
\]

**Theorem 2.6.** The formulas (2.5) and (2.6) can be recast as (see Theorem 5.7)
\[
P^{(C_n)}_{(1^r)}(x|b; q, t) = \sum_{j=0}^{|z|} B[z^{n-r+1}, j] E_{r-2j}(x), \tag{2.9}
\]
\[ E_r(x) = \sum_{j=0}^{\lfloor \frac{r}{2} \rfloor} \widetilde{B}[t^{n-r+1}, j] P_{(1^{r-2j})}^{(C_n)}(x) b, q, t. \]  

**Definition 2.7.** Let \( f[s] \) be the function defined in (1.10). Set for simplicity of display that 
\[ F[s, t] = f[s / t^i] = \frac{(1 - t^i / s)(1 - t^i + 2 / sbq)(1 + t^i + 1 / sb)(1 + t^i + 1 / sbq)}{(1 - t^{2i+1} / s^2 b^2 q)(1 - t^{2i+2} / s^2 b^2 q)}. \]  

We summarize the basic properties for the functions \( B[s, i]'s \) and \( \widetilde{B}[s, i]'s. \) See Theorem 6.1 and Proposition 6.2.

**Theorem 2.8.** We have the four term relations 
\[ B[s, i] + F[s, -1]B[st^2, i - 1] = B[st, i] + B[st, i - 1], \]  
\[ \widetilde{B}[s, i] + F[s, 2 - 2i]\widetilde{B}[s, i - 1] = \widetilde{B}[st^{-1}, i] + \widetilde{B}[st, i - 1], \]  
and the inversion relations 
\[ \sum_{k=0}^{i} B[s, k] \widetilde{B}[st^{2k}, i - k] = \delta_{i,0}, \] 
\[ \sum_{k=0}^{i} \widetilde{B}[s, k] B[st^{2k}, i - k] = \delta_{i,0}. \]

**2.3. Coefficients \( C[s, j] \) and Catalan triangle three term relations.** We have (in Lemma 3.3 below) the expansion of \( E_r(x) \) in terms of the monomial symmetric polynomials as 
\[ E_r(x) = \sum_{j=0}^{\lfloor \frac{r}{2} \rfloor} \left[ \binom{n-r+2j}{j} \right] m_{(1^{r-2j})}(x), \]  
where \( \binom{m}{j} \) denotes the ordinary binomial coefficient. In view of this, we are naturally led to the following definition.

**Definition 2.9.** Let \( s \in \mathbb{C}, \) and write \( s = t^m+1 \) for simplicity. Let \( C[s, j] \) be the function in \( s \) defined by 
\[ C[s, j] := \sum_{i=0}^{j} B[s, i] \left[ \binom{m+2j}{j-i} \right]. \]

**Theorem 2.10.** The type \( C_n \) Macdonald polynomial \( P_{(1^r)}^{(C_n)}(x) b, q, t) \) with one column diagram is expanded in terms of the monomial symmetric polynomials as 
\[ P_{(1^r)}^{(C_n)}(x) b, q, t) = \sum_{j=0}^{\lfloor \frac{r}{2} \rfloor} C[1^{r-r+1}, j] m_{(1^{r-2j})}(x). \]  

**Proof.** It follows from (2.9) and (2.16). \( \square \)

**Theorem 2.11.** We have the three term relation (see Proposition 7.1) 
\[ C[s, j] + F[s, -1]C[st^2, j - 1] = C[st, j], \]  
and the specialization formula for \( s = 1, \) i.e. for \( m = -1 \) (see Proposition 7.2) 
\[ C[1, j] = \delta_{j,0}. \]
Definition 2.12. Define the even infinite upper triangular matrix \( C = (C_{ij})_{i,j \in \mathbb{Z}_{\geq 0}} \) by setting the nonzero entries by

\[
C_{r,r+2i} = C[t^{r+1}, i] \quad (r, i \geq 0).
\]

(2.21)

Theorem 2.13. The \( C_{ij} \)'s satisfy the recursion relations in Theorem 2.4.

\[
C_{0,0} = 1,
\]

(2.22)

\[
C_{i-1,i-1} = C_{i,i}
\]

\((i = 1, 2, 3, \ldots),
\)

(2.23)

\[
F[1,-1]C_{1,j-1} = C_{0,j}
\]

\((j = 2, 4, 6, \ldots),
\)

(2.24)

\[
C_{i-1,j-1} + F[t^i,-1]C_{i+1,j-1} = C_{i,j}
\]

\((i + j : \text{even}, \ 0 < i < j).
\)

(2.25)

Proof. We have \( C_{0,0} = 1 \). When \( i + j \) is even and \( 0 \leq i \leq j \), we have \( C_{ij} = C[t^{i+1}, (j-i)/2] \). Therefore from (2.19) we have

\[
C_{i-1,j-1} + F[t^i,-1]C_{i+1,j-1} = C[t^{i},(j-i)/2] + F[t^i,-1]C[t^{i+2},(j-i)/2-1]
\]

\[
= C[t^{i+1}, (j-i)/2] = C_{ij},
\]

(2.26)

giving the three term recursion relation (2.25) for \( 0 < i < j \) in (2.26). When \( 0 < i = j \), noting that \( C_{i+1,i-1} = 0 \) from the upper triangularity, we have (2.23). When \( i = 0 \) and \( j \in 2\mathbb{Z}_{>0} \), we have from (2.20) that \( C_{-1,j-1} = C[1,(j-2)/2] = 0 \), hence (2.24) holds.

\[\square\]

2.4. Proof of Main Theorem. Now we are ready to present a proof of our main theorem. Proof of Theorem 1.7. The transition matrix \( C \) is even and upper triangular. In view of Theorem 2.10 and \( C[t^{n-r+1}, j] = C_{n-r,n-r+2j}, \) we have for any \( n > 0 \) and \( 0 \leq r \leq n \)

\[
P^{(C_n)}(x|b;q,t) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} C_{n-r,n-r+2j} m_{(1^{r-2j})}(x),
\]

(2.27)

indicating the stabilized transition formula (1.5). The three term recursion relation (1.6), (1.7), (1.8) and (1.9) are shown in Theorem 2.11.

\[\square\]

3. Koornwinder’s \( q \)-Difference Operator, Koornwinder Polynomials and Mimachi Kernel Function

We briefly recall some basic properties concerning the Koornwinder polynomials \[K\] and the Mimachi Kernel function identity \[M\].

3.1. Koornwinder’s operator and Mimachi’s Kernel function. Let \((a, b, c, d; q, t)\) be a set of complex parameters. We assume that \(|q| < 1\). Set \(\alpha = (abcd/q)^{1/2}\) for simplicity. Let \(x = (x_1, \ldots, x_n)\) be a set of independent indeterminates. The Weyl group of type \(BC_n\) is denoted by \(W_n(\simeq \mathbb{Z}^n_+ \rtimes \mathfrak{S}_n)\). Let \(\mathbb{C}[x_1^\pm, x_2^\pm, \ldots, x_n^\pm]^{W_n}\) be the ring of \(W_n\)-invariant Laurent polynomials in \(x\). For a partition \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)\) of length \(n\), i.e. \(\lambda_1 \geq \cdots \geq \lambda_n\), we denote by \(m_\lambda = m_\lambda(x)\) the monomial symmetric polynomial being defined as the orbit sums of monomials

\[
m_\lambda = \frac{1}{|\text{Stab}(\lambda)|} \sum_{\mu \in \Phi_{W_n}} \prod_i x_i^{\mu_i},
\]

(3.1)

where \(\text{Stab}(\lambda) = \{s \in W_n \mid s\lambda = \lambda\}\).

Koornwinder’s \(q\)-difference operator \(D_x = D_x(a, b, c, d; q, t)\) \[K\] reads

\[
D_x = \sum_{i=1}^{n} \frac{(1 - ax_i)(1 - bx_i)(1 - cx_i)(1 - dx_i)}{\alpha t^{n-1}(1 - x_i^t)(1 - qx_i^t)} \prod_{j \neq i} \frac{(1 - tx_ix_j)(1 - tx_i/x_j)}{(1 - x_i x_j)(1 - x_i/x_j)} (T_{q,x_i} - 1)
\]

(3.2)
\[
+ \sum_{i=1}^{n} \frac{(1 - a/x_i)(1 - b/x_i)(1 - c/x_i)(1 - d/x_i)}{\alpha t^{n-1}(1 - x_i^2)(1 - q/x_i^2)} \prod_{j \neq i} \frac{(1 - t x_j/x_i)(1 - t/x_j x_i)}{(1 - x_j/x_i)(1 - 1/x_j x_i)} (T^{-1}_{q, x_i} - 1),
\]
where we have used the notation \(T^{-1}_{q, x_i} f(x_1, \ldots, x_i, \ldots, x_n) = f(x_1, \ldots, x^{-1}_i, \ldots, x_n)\).

The Koornwinder polynomial \(P_\lambda(x) = P_\lambda(x[a, b, c, d|q, t]) \in \mathbb{C}[x^{\pm 1}_1, \ldots, x^{\pm 1}_n] W_n\) is uniquely characterized by the conditions

\[(a) \quad P_\lambda(x) = m_\lambda(x) + \text{lower terms w.r.t the dominance ordering}, \quad (3.3)
\]

\[(b) \quad D_x P_\lambda = d_\lambda P_\lambda. \quad (3.4)
\]

The eigenvalue \(d_\lambda\) is explicitly written as

\[d_\lambda = \sum_{j=1}^{n} \langle abcdq^{-1}t^{2n-2j}q^j \rangle \langle q^j \rangle = \sum_{j=1}^{n} \langle \alpha t^{n-j}q^j; \alpha t^{n-j} \rangle, \quad (3.5)
\]

where we used the notations \(\langle x \rangle = x^{1/2} - x^{-1/2}\) and \(\langle x; y \rangle = \langle xy \rangle x/y = x + x^{-1} - y - y^{-1}\) for simplicity of display.

**Definition 3.1.** Define the involution \(\tilde{*}\) of the parameters by

\[
\tilde{\alpha} = \alpha, \quad \tilde{b} = b, \quad \tilde{c} = c, \quad \tilde{d} = d, \quad \tilde{q} = t, \quad \tilde{t} = q. \quad (3.6)
\]

We write \(\tilde{D}_x = D_x(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}|\tilde{q}, \tilde{t})\) and \(\tilde{P}_\lambda(x) = P_\lambda(x[\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}|\tilde{q}, \tilde{t}]\) for short.

**Theorem 3.2** ([Mi] Lemma 3.2). Let \(n\) and \(m\) be positive integers, and let \(x = (x_1, \ldots, x_n)\), \(y = (y_1, \ldots, y_m)\) be two sets of independent indeterminates. Mimachi’s kernel function

\[
\Psi(x; y) = (y_1 y_2 \cdots y_m)^{-n} \prod_{i=1}^{n} \prod_{j=1}^{m} (1 - y_j x_i)(1 - y_j x_i), \quad (3.7)
\]

enjoys the kernel function identity

\[
\langle t \rangle D_x \Psi(x; y) + \langle q \rangle D_y \Psi(x; y) = \langle abcdq^{-1}t^{2n-2j}q^j \rangle \langle q^j \rangle \langle abcdt^{n-1}q^{m-1} \rangle \Psi(x; y). \quad (3.8)
\]

When we apply Mimachi’s kernel function, the following Lemmas will be used. Recall that the generating function \(E(x|y)\) is introduced in Definition 2.1

\[
E(x|y) = \prod_{i=1}^{n} (1 - y x_i)(1 - y x_i) = \sum_{r \geq 0} (-1)^r E_r(x)y^r. \quad (3.9)
\]

**Lemma 3.3.** We have

\[
E_r(x) = \sum_{k=0}^{\left[\frac{r}{2}\right]} \binom{n-r+2k}{k} m_{(1 r-2k)}(x), \quad (3.10)
\]

where \(\binom{m}{j}\) denotes the ordinary binomial coefficient.

**Proof.** For an integer \(s\) satisfying \(0 \leq s \leq n\), we can find that the coefficient of the monomial \(x_1 x_2 \cdots x_s\) in \(E(x|y) = \prod_{i=1}^{n} (1 - (x_i + 1)/x_i) y + y^2\) is \((-1)^s y^s(1 + y^2)^{n-s}\). Hence we have

\[
E(x|y) = \sum_{s=0}^{n} \sum_{k=0}^{n-s} m_{(1 r)}(x)(-1)^s y^{s+2k} \binom{n-s}{k}
\]

\[
= \sum_{r=0}^{n} (-1)^r y^r \sum_{k=0}^{\left[\frac{r}{2}\right]} \binom{n-r+2k}{k} m_{(1 r-2k)}(x). \quad (3.11)
\]

\[\square\]
Lemma 3.4. Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition satisfying the condition $\lambda_1 \leq n$. We have

$$\prod_{i=1}^{m} E_{\lambda_i}(x) = m_{\lambda}(x) + \text{ lower terms.}$$  \hspace{1cm} (3.12)

**Proof.** Note that for any partitions $\lambda$ and $\mu$, we have $m_{\lambda}\mu = m_{\lambda+\mu} + \text{ lower terms.}$ By using Lemma 3.3 we have $E_{\nu}(x) = m_{(\nu)} + \text{ lower terms.}$ Hence we have (3.12). \hfill $\square$

3.2. Asymptotically free eigenfunction $f(x; s)$ for $D_x$ and reproduction formula. Let $s = (s_1, \ldots, s_n) \in \mathbb{C}^n$ be a set of complex parameters. It is convenient to parametrize $s$ by using another set of parameters $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ as $s_i = t^{-n+i}q^{-\lambda_i}$ ($i = 1, \ldots, n$). We use the shorthand notation $x^{-\lambda} = \prod x_i^{-\lambda_i}$. Let $f(x; s) \in x^{-\lambda} \mathbb{C}[[x_1/x_2, \ldots, x_{n-1}/x_n, x_n]]$ be the infinite series satisfying the conditions

$$f(x; s) = x^{-\lambda} \sum_{\beta \in Q^+} c_{\beta}(s)x^\beta, \quad c_0(s) = 1,$$  \hspace{1cm} (3.13)

$$D_x f(x; s) = \sum_{i=1}^{n} \langle \alpha s_i^{-1}; \alpha t^{n-i} \rangle f(x; s),$$  \hspace{1cm} (3.14)

where $Q^+$ denotes the positive octant of the root lattice of type $BC_n$. To be more explicit, corresponding to the simple roots $\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_3, \ldots, \alpha_{n-1} = \epsilon_{n-1} - \epsilon_n, \alpha_n = \epsilon_n$, we have $x^{\alpha_1} = x_1/x_2, \ldots, x^{\alpha_{n-1}} = x_{n-1}/x_n, x^{\alpha_n} = x_n$. Assuming the genericity of the eigenvalue, one can show that the $f(x; s)$ is determined uniquely.

**Definition 3.5.** The adjoint $D_x^*$ of $D_x$ is defined to be

$$D_x^* = \sum_{i=1}^{n} (T_{x_i}^{-1} - 1) \frac{(1 - a x_i)(1 - b x_i)(1 - c x_i)(1 - d x_i)}{\alpha t^{n-1}(1 - x_i^2)(1 - q x_i^2)} \prod_{j \neq i} (1 - t x_j x_i)(1 - t x_j/x_i)$$

$$+ \sum_{i=1}^{n} (T_{x_i}^{-1} - 1) \frac{(1 - a x_i)(1 - b x_i)(1 - c x_i)(1 - d x_i)}{\alpha t^{n-1}(1 - x_i^2)(1 - q x_i^2)} \prod_{j \neq i} (1 - t x_i x_j)(1 - t x_i/x_j).$$

**Definition 3.6.** Denote by $V(x)$ the Weyl denominator of type $BC_n$

$$V(x) = \prod_{k=1}^{n} x_k^{-n+k-1} \prod_{i=1}^{n} (1 - x_i^2) \prod_{1 \leq i < j \leq n} (1 - x_i x_j)(1 - x_i/x_j).$$  \hspace{1cm} (3.16)

**Definition 3.7.** Define the involution $\overline{\phi}$ of the parameters by

$$\overline{\alpha} = q/a, \quad \overline{\beta} = q/b, \quad \overline{\tau} = q/c, \quad \overline{d} = q/d, \quad \overline{\tau} = q, \quad \overline{t} = q/t.$$  \hspace{1cm} (3.17)

Write for simplicity the composition of the involutions as $\hat{\phi} = \overline{\phi}$, namely we have

$$a = t/a, \quad b = t/b, \quad c = t/c, \quad d = t/d, \quad q = t, \quad \overline{t} = t/q.$$  \hspace{1cm} (3.18)

**Proposition 3.8.** ([HNS] Proposition 6.2). We have

$$V(x)^{-1}D_x^* V(x) - \overline{D}_x = \sum_{j=1}^{n} \langle \overline{\alpha} t^{n-j}; \alpha t^{n-j} \rangle.$$  \hspace{1cm} (3.19)

**Theorem 3.9.** Let $n \geq m$ be positive integers, and $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_m)$ be sets of independent indeterminates. Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition satisfying $t(\lambda) \leq m$ and $\lambda_1 \leq n$. Set

$$s_i = \hat{t}^{-m+i}q^{-\lambda_m+1-i+m+1-i+n} \quad (1 \leq i \leq m).$$  \hspace{1cm} (3.20)
Let \( \hat{f}(y; s) \) be the formal series in \( y \) uniquely characterized by \( \hat{c}_0(s) = 1 \) and

\[
\hat{f}(y; s) = \prod_{i=1}^{m} y_i^{-\lambda_{m+1-i}+m+1-i+n} \sum_{\beta \in Q^+} \hat{c}_\beta(s)y^\beta,
\]

(3.21)

\[
\hat{D}_y \hat{f}(y; s) = \sum_{i=1}^{m} (\hat{\alpha}s_i^{-1}; \hat{\alpha}t^{m-i}) \hat{f}(y; s).
\]

(3.22)

Then we have

\[
P_{\chi}(x|a, b, c, d|q, t) = (-1)^{\lambda} |\Psi(x; y)V(y)\hat{f}(y; s)|_{1,y},
\]

(3.23)

where the notation \([\cdots]_{1,y}\) denotes the constant term in \( y \), and \( \chi' \) is the conjugate diagram of \( \lambda \).

**Proof.** Firstly, we show that the product \( \Psi(x; y)V(y)\hat{f}(y; s) \) has a non vanishing constant term in \( y \). Write

\[
\prod_{i=1}^{m} (1 - y_i^2) \cdot \prod_{1 \leq i < j \leq m} (1 - y_i y_j)(1 - y_i/y_j) \cdot \sum_{\beta \in Q^+} \hat{c}_\beta(s)y^\beta = \sum_{\beta \in Q^+} \hat{c}'_\beta(s)y^\beta,
\]

(3.24)

for short. Noting that we have \( \ell(\chi') \leq n \) from the assumption \( \lambda_1 \leq n \), we have

\[
[P_{\chi}(x|a, b, c, d|q, t)]_{1,y} = \left[ \prod_{i=1}^{m} y_i^{-\lambda_{m+1-i}+m+1-i+n} \sum_{\beta \in Q^+} \hat{c}_\beta(s)y^\beta \right]_{1,y}
\]

\[
= (-1)^{\lambda} \sum_{\beta = \sum k_i \alpha_i \in Q^+} (-1)^{\beta} \sum_{i=1}^{m} \hat{c}'_\beta(s)E_{\lambda_{m-k_i}(x)} \prod_{i=2}^{m} E_{\lambda_{m+1-i+k_i-k_i}(x)}
\]

\[
= (-1)^{\lambda} m_{\chi'}(x) + \text{lower order terms} \neq 0.
\]

In the last step, we have used the Lemma 3.4.

Next, we can show that the constant term satisfies the eigenvalue equation as

\[
(D_x - \frac{\langle m \rangle \langle q^m \rangle \langle abcdt^{m-1}q^{m-1} \rangle}{\langle t \rangle}) \left[ \Psi(x; y)V(y)\hat{f}(y; s) \right]_{1,y}
\]

\[
= \left[ -\frac{\langle q \rangle}{\langle t \rangle} \hat{D}_y \Psi(x; y) V(y) \hat{f}(y; s) \right]_{1,y}
\]

\[
= -\frac{\langle q \rangle}{\langle t \rangle} \left[ \Psi(x; y) \left( \hat{D}_y V(y) \hat{f}(y; s) \right) \right]_{1,y}
\]

\[
= -\frac{\langle q \rangle}{\langle t \rangle} \left[ \Psi(x; y)V(y) \left( \hat{D}_y + \sum_{i=1}^{m} (\hat{\alpha}t^{m-i}; \hat{\alpha}t^{m-i}) \right) \hat{f}(y; s) \right]_{1,y}
\]

\[
= -\frac{\langle q \rangle}{\langle t \rangle} \left( \sum_{i=1}^{m} (\hat{\alpha}s_i^{-1}; \hat{\alpha}t^{m-i}) + (\hat{\alpha}t^{m-i}; \hat{\alpha}t^{m-i}) \right) \left[ \Psi(x; y)V(y)\hat{f}(y; s) \right]_{1,y}
\]

\[
= -\frac{\langle q \rangle}{\langle t \rangle} \left( \sum_{i=1}^{m} (\hat{\alpha}s_i^{-1}; \hat{\alpha}t^{m-i}) \right) \left[ \Psi(x; y)V(y)\hat{f}(y; s) \right]_{1,y}.
\]

Here we have used Theorem 3.2 Proposition 3.8 and the property \( \langle x; y \rangle + \langle y; z \rangle = \langle x; z \rangle \).

To check that the eigenvalue is the desired one, we prepare some lemmas.
Lemma 3.10. The eigenvalue of the Koornwinder polynomial $P_{\lambda}(x)$ can be recast as

$$
\sum_{i=1}^{m} \sum_{j=i+1}^{\lambda_i} \langle \alpha t^{-i} q^j; \alpha t^{-i} \rangle
= \frac{1}{(t)} \langle q^m \rangle \langle \lambda_i \lambda_{i+1} \rangle \langle \alpha^2 q t^{2m-1} \rangle
= \frac{1}{(t)} \langle q^m \rangle \left( \alpha q^{m/2} t^{-1/2+n} + \alpha^{-1} q^{-m/2} t^{1/2-n} \right) - \frac{1}{(t)} \sum_{i=1}^{m} \left( \alpha q^{1/2-l^{-1/2+n} - \lambda_i} + \alpha^{-1} q^{-l^{-1/2+n} + \lambda_i} \right).
$$

(3.27)

Lemma 3.11. We have

$$
\frac{1}{(t)} \langle t^n \rangle \langle q^m \rangle \langle abcd t^{-1} q^{m-1} \rangle + \sum_{i=1}^{m} \frac{q}{(t)} \left( \alpha q^{1/2+m-l^{-1/2} + \alpha^{-1} q^{-1/2-m+l^{1/2}} \rangle \right) \frac{1}{(t)} \langle q^m \rangle \left( \alpha q^{m/2} t^{-1/2+n} + \alpha^{-1} q^{-m/2} t^{1/2-n} \right)
$$

(3.28)

Using Lemmas 3.10 and 3.11 and by noting $\tilde{\alpha} = \alpha q^{1/2} t^{-1/2}$, $\tilde{\alpha} = \alpha^{-1} q^{-1/2} \tilde{t}^{1/2}$ and $\tilde{\alpha} = \alpha^{1} q^{-1/2} \tilde{t}^{1/2}$, we can show that

$$
\sum_{i=1}^{m} \sum_{j=i+1}^{\lambda_i} \langle \alpha t^{-i} q^j; \alpha t^{-i} \rangle
= \frac{1}{(t)} \langle t^n \rangle \langle q^m \rangle \langle abcd t^{-1} q^{m-1} \rangle - \frac{1}{(t)} \sum_{i=1}^{m} \left( \tilde{\alpha} q^{-1/2-l^{-1/2+n} - \lambda_i} + \alpha^{-1} q^{-1/2-m+l^{1/2}} \right)
$$

(3.29)

Therefore we have

$$
\left[ \Psi(x; y)V(y) \tilde{f}(y; s) \right]_{1, y} = (-1)^{\lambda} m_{\chi}(x) + \text{lower order terms},
$$

(3.30)

$$
D_x \left[ \Psi(x; y)V(y) \tilde{f}(y; s) \right]_{1, y} = \sum_{i=1}^{m} \langle \alpha t^{-i} q^j; \alpha t^{-i} \rangle \left[ \Psi(x; y)V(y) \tilde{f}(y; s) \right]_{1, y},
$$

(3.31)

thereby proving $P_{\chi}(x | a, b, c, d | q, t) = (-1)^{\lambda} [\Psi(x; y)V(y) \tilde{f}(y; s)]_{1, y}$. Therefore, we have

3.3. Macdonald polynomials of types $C_n$ and $D_n$. We consider some degenerations of the Koornwinder polynomials to the Macdonald polynomials. As for the details, we refer the readers to [K] and [M].

Setting the parameters as $(a, b, c, d, q, t) \rightarrow (-b^{1/2}, ab^{1/2}, -q^{1/2} b^{1/2}, q^{1/2} ab^{1/2}, q, t)$ in the Koornwinder polynomial $P_{\chi}(x)$, we obtain the Macdonald polynomials of type $(BC_n, C_n)$

$$
P_{\chi}(BC_n, C_n)(x | a, b; q, t) = P_{\lambda}(x) - b^{1/2}, ab^{1/2}, -q^{1/2} b^{1/2}, q^{1/2} ab^{1/2}, q, t).
$$

(3.32)
Namely, setting
\[ D^{(BC_n,C_n)}_x = \sum_{\sigma_1,\ldots,\sigma_n=\pm 1} \prod_{i=1}^n \frac{(1 - ab^{1/2} x_i^{\sigma_i})(1 + b^{1/2} x_i^{\sigma_i})}{1 - x_i^{2\sigma_i}} \prod_{1 \leq i < j \leq n} \frac{1 - t x_i^{\sigma_i} x_j^{\sigma_j}}{1 - x_i^{\sigma_i} x_j^{\sigma_j}} T_{q^{\sigma_1/2,x_1}} \cdots T_{q^{\sigma_n/2,x_n}}, \] (3.33)
we have
\[ P^{(BC_n,C_n)}_\lambda(x) = m_\lambda + \text{lower terms}, \] (3.34)
\[ D^{(BC_n,C_n)}_x P^{(BC_n,C_n)}_\lambda(x) = (ab)^{n/2} q^{n(n-1)/4} \sum_{\sigma_1,\ldots,\sigma_n=\pm 1} s_1^{\sigma_1} \cdots s_n^{\sigma_n} D^{(BC_n,C_n)}_x, \] (3.35)
where \( s_i = abt^{n-i} q^{\lambda_i} \).

The special case \( a = 1 \) is called the Macdonald polynomials of type \( C_n \)
\[ P^{(C_n)}_\lambda(x|b; q, t) = P^{(BC_n,C_n)}_\lambda(x|1, b; q, t). \] (3.36)
Setting \( b = 1 \) further, we have the Macdonald polynomial of type \( D_n \)
\[ P^{(D_n)}_\lambda(x|q, t) = P^{(BC_n,C_n)}_\lambda(x|1; q, t). \] (3.37)

Note that the application of the twist \( \hat{\sigma} \) on \((-b^{1/2}, ab^{1/2}, -q^{1/2}b^{1/2}, q^{1/2}ab^{1/2}, q, t)\) gives
\[ (-t/b^{1/2}, t/ab^{1/2}, -t/q^{1/2}b^{1/2}, t/q^{1/2}ab^{1/2}, t/t/q). \] (3.38)

4. KOORNWINDER POLYNOMIAL WITH ONE COLUMN DIAGRAM

When we apply Theorem 3.9 for the simplest case \( m = 1 \), namely when we plug the \( BC_1 \) asymptotically free eigenfunction \( \tilde{f}(y; s) \) into the formula (3.23), we have the Koornwinder polynomials \( P^{(1)}_\lambda(x) \) with one column diagrams. To execute the explicit calculation based on this, we need to recall the fourfold series expansion of the Askey-Wilson polynomials [HNS].

Let \( D \) denote the Askey-Wilson \( q \)-difference operator [AW]
\begin{align*}
D &= \frac{(1 - ax)(1 - bx)(1 - cx)(1 - dx)}{(1 - x^2)(1 - qx^2)} (T_{q^{x+1}} - 1) \\
&\quad + \frac{(1 - a/x)(1 - b/x)(1 - c/x)(1 - d/x)}{(1 - 1/x^2)(1 - q/x^2)} (T_{q^{-1}x} - 1).
\end{align*}
(4.1)

Let \( s \in \mathbb{C} \) be a parameter. Introduce \( \lambda \) satisfying \( s = q^{-\lambda} \). Then we have \( T_{q,s,x^-\lambda} = sx^-\lambda \).

Let \( f(x; s) = f(x; s|a, b, c, d|q) \) be a formal series in \( x \)
\[ f(x; s) = x^{-\lambda} \sum_{n \geq 0} c_n x^n, \quad c_0 \neq 0, \] (4.2)
satisfying the \( q \)-difference equation
\[ D f(x; s) = \left( s + \frac{abcd}{qs} - 1 - \frac{abcd}{q} \right) f(x; s). \] (4.3)

With the normalization \( c_0 = 1 \), (4.3) determines the coefficients \( c_n = c_n(s|a, b, c, d|q) \) uniquely as rational functions in \( a, b, c, d, q \) and \( s \). We call \( f(x; s) = f(x; s|a, b, c, d|q) \) the asymptotically free eigenfunction associated with the Askey-Wilson operator \( D \).

**Definition 4.1** ([HNS] Definition 3.1). Set
\[ \Phi(x; s|a, b, c, d|q) = \sum_{k,l,m,n \geq 0} c_e(k, l; q^{m+n+1}s|a, c|q)c_o(m, n; s|a, b, c, d|q)x^{2k+2l+m+n}, \] (4.4)
where
\[ c_e(k, l; s) = \frac{(qa^2/c^2; q^2)_k(q^3 s/c^2; q^2)_k(q^2 s^2/c^4; q^2)_k(q^2 x^2/a^2)_k}{(q^2; q^2)_k(q s/c^2; q^2)_k(q^2 s^2/a^2; q^2)_k(q^2 x^2/a^2)_k}, \]
Lemma 5.1. \[ (\text{Lemma 5.1.}) \]

Lemma 4.3 \[ (\text{Lemma 5.1.}) \]

\[ (\text{Theorem 1.2, Proposition 4.3}) \]

Theorem 4.2 \[ (\text{Theorem 1.2, Proposition 4.3}) \]

\[ (\text{Theorem 1.2, Proposition 4.3}) \]

\[ (\text{Theorem 1.2, Proposition 4.3}) \]

\[ (\text{Theorem 1.2, Proposition 4.3}) \]

\[ (\text{Theorem 1.2, Proposition 4.3}) \]

The asymptotically free eigenfunction \( f(x; s) \) associated with the Askey-Wilson operator \( D \) is expressed as the following fourfold summation

\[ f(x; s) = x^{-\lambda} \Phi(x; s|a, b, c, d|q). \]  

(4.7)

Note that for \( BC_1 \) case we have \( V(x) = x^{-1}(1 - x^2) \). It has been studied that the product \( (1 - x^2) \Phi(x; s|a, b, c, d|q) \) can be studied easily.

Lemma 4.3 \[ (\text{HNS} \text{ Lemma 5.1.}) \]

We have

\[ (1 - x^2) \sum_{k,l \geq 0} c_e(k, l; s)x^{2k+2l} = \sum_{k,l \geq 0} c'_e(k, l; s|a, c|q)x^{2k+2l}, \]

(4.8)

where

\[ c'_e(k, l; s|a, c|q) = \frac{(qa^2/c^2; q^2)_{k}(q^3 s/c^2; q^2)_{k}(q^2 s^2/c^2; q^2)_{k}(q^2 x^2/a^2)_{k}}{(q^2; q^2)_{k}(q^3 s/c^2; q^2)_{k}(q^2 s^2/c^2; q^2)_{k}} \]

\[ \times \frac{(c^2/q^2; q)_l(s/q; q)_{2k+1} - q^{2k+2l+1}s}{(q^2 x^2/c^2)^{l}}. \]

(4.9)

4.1. Koornwinder polynomial with one column diagram \( P_{(1')}(|a, b, c, d|q, t) \). We move on to the proof of Theorem 2.2. Recall that \( n \) is a positive integer, \( x = (x_1, \ldots, x_n) \) is a set of variables, and \( P_{(1')}(|a, b, c, d|q, t) \) denotes the Koornwinder polynomial with one column diagram \((1')\).

**Proof of Theorem 2.2** We consider the following special case of Theorem 3.9 above

\[ x = (x_1, \ldots, x_n) \quad (n \in \mathbb{Z}_{>0}), \]

(4.10)

\[ y = (y_1) \quad (m = 1), \]

(4.11)

\[ \Psi(x; y) = y^{-n} \prod_{i=1}^{n}(1 - yx_i)(1 - y/x_i) = y^{-n} \sum_{r \geq 0} (-1)^r E_r(x)y^r, \]

(4.12)

\[ \lambda = (r) \quad (r \in \mathbb{Z}_{\geq 0} \text{ and } r \leq n), \]

(4.13)

\[ s = (s_1) = t^{n-r+1}, \]

(4.14)

\[ V(y) = y^{-1}(1 - y^2), \]

(4.15)

\[ \hat{f}(y; s) = y^{-r+1+n} \Phi(y; s) = y^{-r+1+n} \sum_{k,l,j \geq 0} c_e(k, l; t^{i+j}s) c_o(i, j; s) y^{2k+2l+i+j}, \]

(4.16)

where

\[ c_e(k, l; s) = c_e(k, l; s|t/a, t/c|t), \]

(4.17)

\[ c_o(i, j; s) = c_o(i, j; s|t/a, t/b, t/c, t/d|t). \]

(4.18)

Then calculating the constant term in \( y \), we have

\[ \left[ \Psi(x; y)V(y)\hat{f}(y; s) \right]_{1,y} \]

(4.19)
\[
\begin{align*}
&= \left[ \sum_{r \geq 0} (-1)^r E_r(x) y^{-n+r} \right] y^{-1} (1 - y^2) \left( \sum_{k, l, i, j \geq 0} \tilde{c}_e(k, l; i, j) s \tilde{c}_o(i, j; s) y^{2k+2l+i+j} \right) \\
&= (-1)^r \sum_{k, l, i, j \geq 0} (-1)^{i-j} E_{r-2k-2l-i-j}(x) \tilde{c}_e'(k, l; i, j) s \tilde{c}_o(i, j; s) y^{2k+2l+i+j} \\
&= (-1)^r P^{(1)}[x|a, b, c, d|q, t].
\end{align*}
\]

where \( \tilde{c}_e'(k, l; s) = c_e(k, l; s|t/a, t/c|t) \) (see (4.9) above). This proves Theorem 2.2 \( \square \)

4.2. Macdonald polynomial \( P^{(C_n)}(x|b, q, t) \) of type \( C_n \) with one column diagram. Let \( P^{(C_n)}(x|b, q, t) \) be the Macdonald polynomial of type \( C_n \). (As for the notation, see § 3.3.) In view of (3.38), we need \( \Phi(x; s|a, b, c, d|q, t) \) written for the parameters

\[
(-t/b^1/2, t/b^1/2, -t/q^1/2b^1/2, t/q^1/2b^1/2),
\]

Note this case, we have the simplification of the coefficient as \( \tilde{c}_o(m, n; s) = \delta_{m,0} \delta_{n,0} \). Hence we have a twofold summation formula for the \( \Phi(x; s) \). By Lemma 4.3 we have

\[
(1 - y^2)^2 \Phi(y; s) = (1 - y^2) \Phi(y; s) - t/b^1/2, t/b^1/2, -t/q^1/2b^1/2, t/q^1/2b^1/2 | t
\]

Write \( s = t^{n-r+1} \) for simplicity. Plugging (4.21) in (4.19), we have

\[
P^{(C_n)}(x|b, q, t) = \sum_{0 \leq 2k+2l \leq r} E_{r-2k-2l}(x) \frac{(1/|b|; t)_1(s/t; t)_{2k+1} - st^{2k+2l-1}}{(t; t)_{2k+1} - 1 - st^{-1}} (bq)^l
\]

\[
\times \frac{(qt^2)_k(sqbq/t; t^2)_k(s^2b^2q^2/t^2; t^2)_k}{(t^2; t^2)_k(sqbq/t; t^2)_k(s^2b^2q^2/t^2; t^2)_k} E_{r-2k-2l}(x)
\]

This proves the formula in Corollary 2.3.

5. Transition Matrices \( B[s], \tilde{B}[s] \) and Bressoud’s Matrix Inversion

5.1. Bressoud’s Matrix Inversion. An infinite-dimensional matrix \( (f_{ij})_{i,j \in \mathbb{Z}_{\geq 0}} \) is said to be lower-triangular if \( f_{ij} = 0 \) unless \( i \geq j \). Two infinite-dimensional lower-triangular matrices \( (f_{ij})_{i,j \in \mathbb{Z}_{\geq 0}} \) and \( (g_{ij})_{i,j \in \mathbb{Z}_{\geq 0}} \) are said to be mutually inverse if \( \sum_{i,j \geq k} f_{ij} g_{jk} = \delta_{i,k} \). A matrix \( (f_{ij})_{i,j \in \mathbb{Z}_{\geq 0}} \) is said to be even if the following parity condition holds: \( i+j \) is odd implies \( f_{ij} = 0 \).

Proposition 5.1 ([B], p.1, Theorem, [L], p.5, Corollary). Let \( M[u, v; x, y; q] \) be the infinite even lower-triangular matrix with nonzero entries given by

\[
M_{r, r-2i}[u, v; x, y; q] = y^i v^i \frac{(x/y; q)_i (uq^{-2i}; q)_i}{(q; q)_i (uxq^{r-1}; q)_i (uyq^{r-1}; q)_i}.
\]

for \( r, i \in \mathbb{Z}_{\geq 0}, i \leq \left\lfloor \frac{r}{2} \right\rfloor \). The we have

\[
M[u, v; x, y; q] M[u, v; y, z; q] = M[u, v; x, z; q].
\]

In particular, \( M[u, v; x, y; q] \) and \( M[u, v; y, x; q] \) are mutually inverse.

If two infinite matrices \( (f_{ij}) \) and \( (g_{ij}) \) are mutually inverse, then the conjugated ones \( (f_{ij} d_i/d_j) \) and \( (g_{ij} d_i/d_j) \) are also mutually inverse for any sequence \( (d_i) \) with nonzero entries.
Definition 5.2. Set
\[ d_r = \frac{(t^2v^{1/2}; t)_r}{(u^{1/2}; t)_r} (u^{1/4}/v^{3/4})^r. \] (5.3)

Let \( \tilde{M}[u, v; x, y; t] \) denotes the conjugation of the matrix \( M[u, v; x, y; t^2] \) by the \((d_r)\) with entries
\[ \tilde{M}_{r,-2l}[u, v; x, y; t] = M_{r,-2l}[u, v; x, y; t^2] \times d_r/d_{r-2l} \]
\[ = \frac{(x/y; t^2)_l}{(t^2; t^2)_l} \frac{(v^{1/2}t^{-2l+2}; t^2)_2l}{(u^{1/2}t^{2l-2}; t^2)_2l} \frac{(u^{2r-4l}; t^2)_2l}{(u^{2r-4l}; t^2)_2l} (yu^{1/2}/v^{1/2})^l. \] (5.4)

5.2. Transition Matrices \( \mathcal{B}[s] \) and \( \tilde{\mathcal{B}}[s] \).

Definition 5.3. Let \( \mathcal{B}[s] \) and \( \tilde{\mathcal{B}}[s] \) be even lower triangular matrices defined by
\[ \mathcal{B}[s] = \tilde{M}[t^2/s^2b^2q^2, 1/s^2t^4; q/t, 1/t^2; t] M[1/s, t; 1/bq, 1; t], \] (5.5)
\[ \tilde{\mathcal{B}}[s] = M[1/s, t; 1, 1/bq; t] \tilde{M}[t^2/s^2b^2q^2, 1/s^2t^4; 1/t^2, q/t; t]. \] (5.6)

Proposition 5.4. The \( \mathcal{B}[s] \) and \( \tilde{\mathcal{B}}[s] \) are mutually inverse.

Proof. It follows from Bressoud’s matrix inversion [5.2]. \(\square\)

Theorem 5.5. We have
\[ P^{(C_n)}_{(1^r)}(x|b, q, t) = \sum_{k=0}^{\frac{1}{2}m} \mathcal{B}_{r,-2k}[t^m] E_{r-2k}(x), \] (5.7)
\[ E_r(x) = \sum_{k=0}^{\frac{1}{2}m} \tilde{\mathcal{B}}_{r,-2k}[t^m] P^{(C_n)}_{(1^r-2k)}(x|b, q, t). \] (5.8)

Writing the coefficients explicitly, these read
\[ P^{(C_n)}_{(1^r)}(x|b, q, t) = \sum_{k=0}^{\frac{1}{2}m} \sum_{l=0}^{2k} E_{r-2k-2l}(x) (1/bq; t)_l (st^{2k-1}; t)_l (st^{2k}; t)_l (sbqt^{2k}; t)_l \]
\[ \times (st^{2l}; t)_l (sbqt^{2l}; t)_l (s^{2k}b^2q^2/t^2; t^2)_k (s; t)_k b^k \]
\[ \times (1/qt; t^2)_k (s^2t^{4l-2}b^2q^2; t^2)_k (s^2t^{4l+2k-2}b^2q^2; t^2)_k \]
\[ \times (2^{l+k}; t^2)_l (s^2t^{4l+2k-3}b^2q^2; t^2)_k (s^2t^{4l+2k-3}b^2q^2; t^2)_k b^k, \] (5.9)
\[ E_r(x) = \sum_{l=0}^{\frac{1}{2}m} \sum_{k=0}^{2l} \mathcal{B}^{(C_n)}_{1^r-2l-2k}(x|b, q, t) (bq/t)_l (st^{2l}; t)_{l+2k} \]
\[ \times (st^{2l+1}; t)_{l+2k} (s^2t^{4l+2k-1}b^2q^2; t^2)_k (s^2t^{4l+2k-1}b^2q^2; t^2)_k \]
\[ \times (bq/t)^k, \] (5.10)
where \( s = t^{r-1} \). In particular form (5.10), we have Theorem 2.4.

Proof. Clearly, (5.22) and (5.9) are the same. We show that (5.9) and (5.7) are the same. By (5.1) and (5.3) are the same. We have
\[ (1/bq; t)_l (s^{2k-1}; t)_l (s^{2k}; t)_l (sbqt^{2k}; t)_l (s^{2k-1}; t)_l (sbqt^{2k}; t)_l (s^{-2k-2l+1}; t)_l \]
\[ = M_{r,-2k-2l-2l}[t^{-n}, t, 1/bq, 1; t], \]
and
\[ (qt; t^2)_k (sbqt^{2k}; t^2)_k (s^{2k}b^2q^2/t^2; t^2)_k (s; t)^{2k} b^k \]
\[ = M_{r,-2k-2l-2l}[t^{-n}, t, 1/bq, 1; t], \] (5.12)
\[ (q t; t^2)_k \frac{(s^{-1}t^{-2k+1}; t)_2k}{(t^2; t^2)_k (s^{-1}t^{-2k+2}/bq; t)_2k} \frac{(s^{-2}t^{-4k+4}/b^2q^2; t^2)_2k}{(s^{-2}t^{-4k+3}/b^2q^2; t^2)_2k} (t/bq)^k \]
\[ = \tilde{M}_{r,r-2k}[t^{-2n+2}/b^2q^2, t^{-2n-4}, q/t, 1/t^2; t]. \]

As for (5.8) and (5.10), we have
\[ \tilde{M}_{r,r-2k-2}[t^{-2n+2}/b^2q^2, t^{-2n-4}, 1/t^2, q/t; t] \]
\[ = (1/qt; t^2)_k \frac{(s^{-1}t^{-2l-2k+1}; t)_2k}{(t^2; t^2)_k (s^{-1}t^{-2l-2k+2}/bq; t)_2k} \frac{(s^{-2}t^{-4l-4k+4}/b^2q^2; t^2)_2k}{(s^{-2}t^{-4l-4k+5}/b^2q^2; t^2)_2k} (t/bq)^l \times \]
\[ \frac{(bq/t)_l}{(t/t)_l} \frac{(st^l/t)_l}{(t^2; t^2)_l} \frac{(1/qt; t^2)_k}{(t^2; t^2)_k} \frac{(s^2t^{4l-2k}b^2q^2; t^2)_k}{(s^2t^{4l-2k-3}b^2q^2; t^2)_k} (bqt)^k. \]

\[ \square \]

### 5.3. Entries of \( B[s] \) and \( \tilde{B}[s] \) in terms of \( 4\phi_3 \) series.

Recall that we have defined \( B[s, j] \) and \( \tilde{B}[s, j] \) in Definition 2.3 as
\[ B[s, j] = (-1)^j s^{-j} \frac{(s^2/t^2; t^2)_j}{(t^2; t^2)_j} \frac{1 - s^2t^{2j-2}}{1 - st^{-2}} 4\phi_3 \left[ -sb/t, -sbq/t, s^2t^{2j-2}, t^{-2j} ; s, st, s^2b^2q/t \right], \]
\[ \tilde{B}[s, j] = (st^{-j}-1) \frac{(st^2; t^2)_j}{(t^2; t^2)_j} \frac{1 - st^{-2j-1}}{1 - st^{-2}} 4\phi_3 \left[ -t^{-2j+3}/sbq, -t^{-2j+3}/sb, t^{-2j+2}/s^2, t^{-2j} ; t^{-2j+3}/s, t^{-2j+2}/s, t^{-4j+5}/b^2q^2 \right]. \]

**Proposition 5.6.** By applying the Sears transformation \([GR, p.49, (2.10.4)]\), we have
\[ B[s, j] = (1/j) st^{-j} s^{-j} \frac{(s^2/t^2; t^2)_j}{(t^2; t^2)_j} \frac{1 - st^{-2j-1}}{1 - st^{-2}} 4\phi_3 \left[ -sb/t, -sbq/t, s^2t^{2j-2}, t^{-2j} ; s, st, s^2b^2q/t \right], \]
\[ \tilde{B}[s, j] = t^j (st^{-j}-1) \frac{(st^2; t^2)_j}{(t^2; t^2)_j} \frac{1 - st^{-2j-1}}{1 - st^{-2}} 4\phi_3 \left[ -t^{-2j+3}/sbq, -t^{-2j+3}/sb, t^{-2j+2}/s^2, t^{-2j} ; t^{-2j+3}/s, t^{-2j+2}/s, t^{-4j+5}/b^2q^2 \right]. \]

**Theorem 5.7.** We have
\[ B_{r,r-2k}[s] = B[st^{-r+1}, t], \]
\[ \tilde{B}_{r,r-2k}[s] = \tilde{B}[st^{-r+1}, t]. \]

**Proof.** Recall that the bicasic hypergeometric series \( \Phi \) (see \([GR, p.99, (3.9.1)]\)) is defined by
\[ \Phi \left[ \frac{a_1, \ldots, a_{r+1}; c_1, \ldots, c_s; p; z}{b_1, \ldots, b_r; d_1, \ldots, d_s; q; t} \right] = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_{r+1}; q)_n (c_1, \ldots, c_s; p)_n}{(q, b_1, \ldots, b_r; q)_n (d_1, \ldots, d_s; p)_n} z^n. \]

We use the \( q \)-analogue of Bailey’s transformation \([GR, p.99, (3.10.14)]\):
\[ \Phi \left[ \frac{a^2, at^2, -at^2, b^2, c^2; -at/w, t^{-i}}{a, -a, a^2t^2/b^2, b^2t^2/c^2; w, -at^{i+1}; t^2; t;} \right] = \frac{(-at, at^2/w, w/at; t)_i}{(t, at/w, w; t)_i} \frac{1}{\Phi_4} \left[ \frac{at, at^2, b^2t^2/b^2c^2, a^2t^2/w^2, t^{-2i}}{a^2t^2/b^2, b^2t^2/c^2, at^{i-1}/w, at^{-3}/w; t^2, t^2} \right]. \]

When \( a = c^2 \), (5.22) becomes
\[ \Phi \left[ \frac{a^2, -at^2, b^2; -at/w, t^{-i}}{-a, a^2t^2/b^2; w, -at^{i+1}; t^2, t; w/t^2} \right] = \frac{1}{\Phi_4} \left[ \right]. \]
we can prove (5.20) as

\[
\frac{(-at, at^2/w, w/at; t)_i}{(-\theta, at^2/w, w; t)_i} \Phi_3 \left[ \frac{at, at^2/b^2, a^2t^2/w^2, t^{-2i}}{a^2t^2/b^2, at^{-i}/w, at^{3-i}/w; t^2} \right].
\]

Replacing the parameters in (5.23) as

\[
(a, w, b^2) \rightarrow (-sbq/t, bqt^{-i+1}, qt),
\]

we can prove (5.19) as

\[
\mathcal{B}_{r, r - 2i}[st^{-1}]
\]

\[
= \sum_{k=0}^{i} \frac{(1/bq; t)_i (st^{i-k}; t)_i}{(t; t)_i (sbq; t)_i} (st^{i-k}; q; t)_i \Phi_i \left[ \frac{qt, sbqt; s^2b^2q^2/t^2; t^2}{sbqt/t, sbqt; t^2} \right] (1/bq; t)_i (s/t; t)_i \Phi_i \left[ -sb, -sbq, s^2t^{2i+2}t^{-2i} - st, st, s^2b^2q/t; t^2, t^2 \right]
\]

When \( at = c^2 \), (5.22) becomes

\[
\Phi \left[ \frac{a^2, at^2, -at^2, b^2 : -at/w, t^{-i}}{a, -a, at^2/b^2 : w, -at^{i+1}}; i ; t^2, \frac{wt^i}{b^2} \right] = \Phi \left[ \frac{-at^2/w, w/at; t}_i (-t, at^2/w, w; t)_i}{at^2/b^2, at^2/w, at^3/w; t^2, t^2} \right] \Phi_3 \left[ \frac{at^2/b^2, a^2t^2/w^2, t^{-2i}}{a^2t^2/b^2, at^{-i}/w, at^{3-i}/w; t^2, t^2} \right]
\]

Replacing the parameters in (5.26) as

\[
(a, w, b^2) \rightarrow (-t^{-2i+1}/sbq, t^{-i+1}/bq, 1/qt),
\]

we can prove (5.20) as

\[
\mathcal{B}_{r, r - 2i}[st^{-1}]
\]

\[
= \sum_{k=0}^{i} \frac{(bq, t^{-2i+2k+1}/s; t)_i (t/bq)_i}{(t, t^{-2i+2k+2}/sbq; t)_i} \Phi_i \left[ \frac{t^{-2i+1}/s; t}_i (t^{-4i+2}/s^2b^2q^2; t^2)_k}{(t^2, t^2)_k} \right] \Phi_i \left[ \frac{at^2/b^2, a^2t^2/w^2, t^{-2i}}{a^2t^2/b^2, at^{-i}/w, at^{3-i}/w; t^2, t^2} \right].
\]
We can write

\[ \times \Phi \left[ \frac{1}{gt}, t^{-4i+2}/s^2b^2q^2, t^{-2i+1}/sbq, -t^{-2i+3}/sbq : t^{-i+1}/s, t^{-i} \right] \\
= \frac{(s/t, s, t)i}{(t^2, s^2b^2q, t^{-2i+1}/sbq, -t^{-2i+3}/sbq)} (t/s)^i t^{-i(i-1)} \Phi_3 \left[ t^{-2i+3}/sbq, -t^{-2i+3}/sbq, t^{-2i+2}/s^2, t^{-2i} \right] \\
= \tilde{B}[s, i]. \]

6. Four term relations for \( B[s, i] \) and \( \tilde{B}[s, i] \)

6.1. Four term relations. Recall that we have defined \( f[s] \) in (1.10) and have introduced the notation \( F[s, l] \) in (2.11) as

\[ F[s, l] = f[s/t^l] = \frac{(1 - t^l/s)(1 - t^{l+2}/sb^2q)(1 + t^{l+1}/sb)(1 + t^{l+1}/sbq)}{(1 - t^{2l+1}/s^2b^2q)(1 - t^{2l+3}/s^2b^2q)}. \]  

Theorem 6.1. We have

\[ B[s, i] + F[s, -1]B[st^2, i - 1] = B[st, i] + B[st, i - 1], \]  

\[ \tilde{B}[s, i] + F[s, 2 - 2i]B[s, i - 1] = \tilde{B}[st^{-1}, i] + \tilde{B}[st, i - 1]. \]

Proof. For simplicity of display, we write

\[ b[s, i] := (-1)^i s^{-1} (s^2/t^2)^i \frac{1 - s^2t^{4i-2}}{1 - s^2t^{2i-2}}, \]

\[ 4\Phi_3[s, i, k] := \frac{(-sb, -sbq, s^2/2^{2i-2}, t^{-2i}; t^2_k)}{(t^2, s, -s, -st, s^2b^2q t^{-1}; t^2_k)}. \]  

Then we have \( B[s, i] = b[s, i] \sum_{k=0}^i 4\Phi_3[s, i, k] \). For (6.2), we need to show

\[ \sum_{k=0}^i 4\Phi_3[s, i, k] + F[s, -1] \left( \frac{b[st^2, i - 1]}{b[s, i]} \right) \sum_{k=0}^{i-1} 4\Phi_3[st^2, i - 1, k] \]

\[ = \left( \frac{b[st, i]}{b[s, i]} \right) \sum_{k=0}^i 4\Phi_3[st, i, k] + \left( \frac{b[st, i - 1]}{b[s, i]} \right) \sum_{k=0}^{i-1} 4\Phi_3[st, i - 1, k]. \]

Firstly, we have

LHS of (6.3)

\[ = 1 + \sum_{k=1}^i \left( 4\Phi_3[s, i, k] + F[s, -1] \left( \frac{b[s, i - 1]}{b[s, i]} \right) 4\Phi_3[st^2, i - 1, k - 1] \right) \]

\[ = 1 + \sum_{k=1}^i \frac{(-sb, -sbq, st^2, s^2t^{2i-2}, t^{-2i}; t^2_k)_{k-1}}{(t^2, s, -s, -st, s^2b^2q t^{-1}; t^2_k)_{k}} \]

\[ = 5\Phi_4 \left[ -sb, -sbq, st^2, s^2t^{2i-2}, t^{-2i} \right. \]

\[ \left. s, -s, -st, s^2b^2q t^{-1}; t^2, t^2 \right]. \]

Nextly, by setting

\[ 4\Phi_3[s, i, k] := \frac{(t^{-2i}, s^2t^{2i-2}, -sbq/t, -sb/t; t^2_k)_{k-1}}{(t^2, s^2b^2q/t, -s, -st; t^2_k)}, \]

we can write

\[ \sum_{k=0}^i 4\Phi_3[s, i, k] = \frac{1 + s^{-1}t}{1 + s^{-1}t^{-2i+1} t^{-i}} \sum_{k=0}^i 4\Phi_3[s, i, k]. \]
Hence we have

\[
\text{RHS of (6.6)} = \sum_{k=0}^{i} \left( \left( \frac{b[st, i]}{b[s, i]} \right) \frac{1 + s^{-1}t^{-2i}}{1 + s^{-1}t^{-2i-2}t^{-2i}l^{2k}} \tilde{\phi}_3[st, i, k] \right.
+ \left. \frac{b[st, i - 1]}{b[s, i]} \frac{1 + s^{-1}t^{-2i}}{1 + s^{-1}t^{-2i+2}t^{-2i}l^{2k}} \tilde{\phi}_3[st, i - 1, k] \right)
\]

\[
= \sum_{k=0}^{i} \left( -sb, -sbq, st^2, s^2t^{2i-2}, t^{-2i}, t^2 \right) \frac{1}{(t^2, s, -s, -st, s^2b^2qt, t^2)_k} t^{2k}
= 5\phi_4 \left[ -sb, -sbq, st^2, s^2t^{2i-2}, t^{-2i}, s, -s, -st, s^2b^2qt ; t^2, t^2 \right] = \text{LHS of (6.6)}.
\]

Now we turn to (6.3). Set

\[
\tilde{b}[s, i] := \frac{(st^{-1}, s; t)_2 t^{i-i(i-1)}}{(t^2, s^2; t^2)_i},
\]

\[
\tilde{\phi}_3[s, i, k] := \frac{(-s^{-1}t^{-2i+3} + bq, -s^{-1}t^{-2i+3} + b, -s^{-1}t^{-2i+2}, t^{-2i}, t^2)}{(s^2t^{-4i+5} + b^2q, -s^{-1}t^{-2i+2}, -s^{-1}t^{-2i+3}, t^2)} t^{2k},
\]

\[
\text{for simplicity. Then we can write}
\]

\[
\tilde{B}[s, i] = \tilde{b}[s, i] \sum_{k=0}^{i} \tilde{\phi}_3[s, i, k],
\]

\[
\sum_{k=0}^{i} \tilde{\phi}_3[s, i, k] = \frac{1 + s^{-1}t^{-2i+1}}{1 + s^{-1}t} t^{i} \sum_{k=0}^{i} \tilde{\phi}_3[s, i, k].
\]

We shall show

\[
\sum_{k=0}^{i} \tilde{\phi}_3[s, i, k] + F[s, 2 - 2i] \left( \frac{b[i, s - 1]}{b[i, s]} \right) \sum_{k=0}^{i-1} \tilde{\phi}_3[s, i - 1, k]
= \left( \frac{\tilde{b}[st^{-1}, i]}{b[s, i]} \right) \sum_{k=0}^{i} \tilde{\phi}_3[st^{-1}, i, k] + \left( \frac{\tilde{b}[st, i - 1]}{b[s, i]} \right) \sum_{k=0}^{i-1} \tilde{\phi}_3[st, i - 1, k].
\]

We have

\[
\text{LHS of (6.16)}
= 1 + \sum_{k=1}^{i} \left( \tilde{\phi}_3[s, i, k] + F[s, 2 - 2i] \left( \frac{b[s, i - 1]}{b[s, i]} \right) \tilde{\phi}_3[s, i - 1, k] - 1 \right)
\]

\[
= 1 + \sum_{k=1}^{i} \left( \frac{(t^{-2i}, s^2t^{-2i+2} - s^{-1}t^{-2i+3} + bq, -s^{-1}t^{-2i+3} + b, s^{-1}t^{-2i+3}, t^2)}{(t^2, s^2t^{-4i+7} + b^2q, -s^{-1}t^{-2i+2}, s^{-1}t^{-2i+3}, t^{-2i}, t^2)} t^{2k} \right)
\]

\[
= 5\phi_4 \left[ t^{-2i}, s^2t^{-2i+2} - s^{-1}t^{-2i+3} + bq, s^{-1}t^{-2i+3}, s^{-1}t^{-2i+3}, s^{-1}t^{-2i+3} ; t^2, t^2 \right].
\]

On the other hand, we have

\[
\text{RHS of (6.16)}
\]
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6.2. Another proof of Theorem 5.4. As an application of the four term relations \( B[s, i] \) and \( \tilde{B}[s, i] \), we present another proof of Theorem 5.4 providing an amusing complementary argument based on the Bressoud matrix inversion.

**Proposition 6.2.** The four terms relations in Theorem 6.1 imply that

\[
\sum_{k=0}^{i} B[s, k] \tilde{B}[st^{2k}, i - k] = \delta_{i,0},
\]

\[
\sum_{k=0}^{i} \tilde{B}[s, k] B[st^{2k}, i - k] = \delta_{i,0},
\]

hence, that the matrices \( B[s] \) and \( \tilde{B}[s] \) are mutually inverse.

**Proof.** Set

\[
\text{LHS}[s, i] := \sum_{k=0}^{i} B[s, k] \tilde{B}[st^{2k}, i - k],
\]

for simplicity.

First we show that for \( i \geq 0 \) we have the difference equation

\[
\text{LHS}[s, i] - \text{LHS}[s/t, i] = 0.
\]

We prove this by induction. The case \( i = 0 \) is clearly correct. Suppose that it is valid for \( i - 1 \). Then we have

\[
\text{LHS}[s, i] - \text{LHS}[s/t, i] = \sum_{k=0}^{i} B[s, k] \left( \tilde{B}[st^{2k-1}, i - k] - F[s, 2 - 2i] \tilde{B}[st^{2k}, i - k - 1] \right) + \sum_{k=0}^{i} (B[s, k] - F[s, 0] B[st, k - 1] + B[s, k - 1]) \tilde{B}[st^{2k-1}, i - k] = - F[s, 2 - 2i] \text{LHS}[s, i - 1] + F[s, 0] \text{LHS}[st, i - 1] = 0.
\]

By definition \( \text{LHS}[s, i] \) is a rational function in \( s \), and it satisfies the difference equation (6.22). Therefore, \( \text{LHS}[s, i] \) must be a constant. We have \( \text{LHS}[s, 0] = 1 \). Then we can check that for \( i > 0 \) \( \text{LHS}[1, i] = 0 \) (hence \( \text{LHS}[s, i] = 0 \)) by using the following Lemma as

\[
\text{LHS}[s, i] = \sum_{k=0}^{i} B[1, k] \tilde{B}[t^{2k}, i - k] = B[1, 0] \tilde{B}[1, i] - B[1, 1] \tilde{B}[1, i - 1] = 0.
\]
Lemma 6.3. We have

\[ B[1, j] = \begin{cases} 
  1 & (j = 0) \\
  -1 & (j = 1) \\
  0 & (j > 1)
\end{cases} \] \quad (6.24)

\[ \tilde{B}[1, i] - \tilde{B}[t^2, i - 1] = 0 \quad (i > 1). \] \quad (6.25)

Proof. The (6.24) follows from the definition of \( B[s, i] \). By noting

\[ \tilde{b}[s, i] \tilde{\phi}_3[s, i, k] - \tilde{b}[t^2, i - 1] \tilde{\phi}_3[t^2, i - 1, k] \] \quad (6.26)

\[ = s^{-i-t(i-1)} \frac{(t^{2i+2} s^2; t^2)_i}{(t^2; t^2)_i} \left( 1 - s \right) \frac{(-t^{-2i+2} s b_t; t^2)_k}{(-t^{-2i+2}/s^2; t^2)_k} \frac{(t^{-2i}/s^2; t^2)_k}{(t^{-2i}/s; t^2)_k} \frac{(t^{-2i}/s; t^2)_k}{(t^{-2i}/s; t^2)_k} \frac{(-t^{-2i}/s; t^{-2i}/t^2)_k}{(-t^{-2i}/s; t^{-2i}/t^2)_k}, \]

where we used the notation (6.11) and (6.12), we have (6.25) from the identity

\[ \tilde{B}[s, i] - \tilde{B}[t^2, i - 1] \] \quad (6.27)

\[ = s^{-i-t(i-1)} \frac{(t^{2i+2} s^2; t^2)_i}{(t^2; t^2)_i} \left( 1 - s \right) \frac{(-t^{-2i+2} s b_t; t^2)_k}{(-t^{-2i+2}/s^2; t^2)_k} \frac{(t^{-2i}/s^2; t^2)_k}{(t^{-2i}/s; t^2)_k} \frac{(t^{-2i}/s; t^2)_k}{(t^{-2i}/s; t^2)_k} \frac{(-t^{-2i}/s; t^{-2i}/t^2)_k}{(-t^{-2i}/s; t^{-2i}/t^2)_k}. \]

\[ \Box \]

7. Transition Matrix \( C \) and \((b, q, t)\)-Deformation of Catalan triangle three term recursion relations

7.1. Coefficient \( C[s, j] \). Recall that in Definition 2.9 we have defined the function \( C[s, j] \) as

\[ C[s, j] := \sum_{i=0}^{j} B[s, i] \left[ \begin{array}{c} m + 2j \\ j - i \end{array} \right], \] \quad (7.1)

where \( s = t^{m+1} \) and \( \left[ \begin{array}{c} m \\ j \end{array} \right] \) denotes the ordinary binomial coefficient. Then (5.7), (5.19), and (7.1) imply (Theorem 2.10)

\[ F_{(1^r)}^{(C_n)} (x|b; q, t) = \sum_{k=0}^{r} C[t^{m-r+1}, k] m_{(1^r-2k)}(x). \] \quad (7.2)

7.2. Deformed Catalan triangle recursion relations.

Proposition 7.1. We have the three term relation

\[ C[s, j] + F[s, -1] C[st^2, j - 1] = C[st, j]. \] \quad (7.3)

Proof. We have

\[ C[s, j] + F[s, -1] C[st^2, j - 1] \]

\[ = \sum_{i=0}^{j} B[s, i] \left[ \begin{array}{c} m + 2j \\ j - i \end{array} \right] + \sum_{i=0}^{j-1} F[s, -1] B[st^2, i] \left[ \begin{array}{c} m + 2j \\ j - 1 - i \end{array} \right] \]

\[ = \left[ \begin{array}{c} m + 2j \\ j \end{array} \right] + \sum_{i=1}^{j} (B[st, i] + B[st, i - 1]) \left[ \begin{array}{c} m + 2j \\ j - i \end{array} \right] \]

\[ = \left[ \begin{array}{c} m + 2j \\ j \end{array} \right] + m + 2j + B[st, j] + \sum_{i=1}^{j-1} B[st, i] \left( \left[ \begin{array}{c} m + 2j \\ j - i \end{array} \right] + \left[ \begin{array}{c} m + 2j \\ j - i - 1 \end{array} \right] \right). \] \quad (7.4)
\[
\sum_{i=0}^{j} B[st, i] \binom{m + 1 + 2j}{j - i} = C[st, j].
\]

Proposition 7.2. We have
\[
C[1, j] = \delta_{j, 0}. \tag{7.5}
\]
Hence the three term relation (7.3) for \( s = 1 \) reads
\[
F[1, -1]C[t^2, j - 1] = C[t, j]. \tag{7.6}
\]

Proof. We have \( C[1, 0] = 1 \). From Lemma 6.3, we have for \( j > 0 \)
\[
C[1, j] = B[1, 0] \begin{bmatrix} -1 + 2j \\ j \end{bmatrix} + B[1, 1] \begin{bmatrix} -1 + 2j \\ j - 1 \end{bmatrix} = \begin{bmatrix} -1 + 2j \\ j - 1 \end{bmatrix} = 0.
\]

\[\square\]

7.3. Solution to the Deformed Catalan triangle recursion relations.

Proposition 7.3. We have \( C[t^{r+1}, 0] = 1 \) for \( r \in \mathbb{Z}_{\geq 0} \), and for \( i \in \mathbb{Z}_{\geq 0}, r \in \mathbb{Z}_{\geq 0} \) we have
\[
C[t^{r+1}, i] = \sum_{(d_1, \ldots, d_i) \in P[r, i]} F[t^{r+1}, d_1] F[t^{r+1}, d_2] \cdots F[t^{r+1}, d_i], \tag{7.7}
\]
where \( P[r, i] \) denotes the finite set defined by
\[
P[r, i] = \{(d_1, d_2, \ldots, d_i) \in \mathbb{Z}^i | 0 \leq d_1 \leq r, d_k - 1 \leq d_{k+1} \leq r \text{ for } 1 \leq k < i \}. \tag{7.8}
\]
We prepare some lemmas.

Lemma 7.4. For \( r \in \mathbb{Z}_{\geq 0} \), we have
\[
C[t^{r+1}, i + 1] = \sum_{k=0}^{r} F[t^k, -1] C[t^{k+2}, i]. \tag{7.9}
\]
The case \( r = 0 \) holds since \( C[t, i + 1] = F[1, -1]C[t^2, i] \). Then we can show the induction step as
\[
C[t^{r+1}, i + 1] = C[t^{r+1}, i + 1] + F[t^{r+1}, -1] C[t^{r+3}, i]. \tag{7.10}
\]
\[
= \sum_{k=0}^{r} F[t^k, -1] C[t^{k+2}, i] + F[t^{r+1}, -1] C[t^{r+3}, i] = \sum_{k=0}^{r+1} F[t^k, -1] C[t^{k+2}, i].
\]
\[\square\]

Lemma 7.5. We have
\[
P[r, i + 1] \tag{7.11}
\]
\[
= \{(d, d_1, d_2, \ldots, d_i) \in \mathbb{Z}^i | 0 \leq d_1 \leq r, (d_1 - d + 1, \ldots, d_i - d + 1) \in P[r - d + 1, i + 1] \}.
\]

Proof of Proposition 7.3. We prove (7.8) by induction on \( i \). It holds for \( i = 0 \), since we have \( C[t^{r+1}, 0] = 1 \) \( (r \in \mathbb{Z}_{\geq 0}) \). The induction step is shown as follows. Lemmas 7.4 and 7.5 and the induction hypothesis give us
\[
C[t^{r+1}, i + 1] = \sum_{k=0}^{r} F[t^k, -1] C[t^{k+2}, i] = \sum_{d=0}^{r} F[t^{r+1}, d] C[t^{r-d+2}, i]
\]
\[
= \sum_{d=0}^{r} F[t^{r+1}, d] \sum_{(d_1, \ldots, d_i) \in P[r - d + 1, i]} F[t^{r+1}, d_1] F[t^{r+1}, d_2] \cdots F[t^{r+1}, d_i]. \tag{7.12}
\]
8. Some Degenerations of Macdonald Polynomials of Types $C_n$ and $D_n$ with One Column Diagrams and bv Polynomials

This section is devoted to the study of several degenerations of our formulas for the Macdonald polynomial $P_{(r)}^{(r)}(x|b;q,t)$.

8.1. Some degenerations of $B[s,j]$ and $\tilde{B}[s,j]$.

8.1.1. $C_n$ case.

**Proposition 8.1.** When $b = t$, we have

$$B[s,j] = \frac{(1/qt; t^2)_j(s^2/t^2; t^2)_j}{(t^2; t^2)_j^3} \frac{1 - s^2 t^{4j - 2}}{1 - s^2 t^{-2}} \phi_2 \left[ -sqt, s^2 t^{2j - 2}, t^{-2j}; t^2, t^2 \right]$$

**(8.1)**

\[B[s,j] = \frac{(1/qt; t^2)_j(s^2/t^2; t^2)_j}{(t^2; t^2)_j^3} \frac{1 - s^2 t^{4j - 2}}{1 - s^2 t^{-2}} \phi_2 \left[ -sqt, s^2 t^{2j - 2}, t^{-2j}; t^2, t^2 \right] \]

\[\text{and} \]

$$\tilde{B}[s,j] = \frac{(1/qt; t^2)_j(s^2/t^2; t^2)_j}{(t^2; t^2)_j^3} \frac{1 - s^2 t^{4j - 2}}{1 - s^2 t^{-2}} \phi_2 \left[ -sqt, s^2 t^{2j - 2}, t^{-2j}; t^2, t^2 \right]$$

**(8.2)**

\[\tilde{B}[s,j] = \frac{(1/qt; t^2)_j(s^2/t^2; t^2)_j}{(t^2; t^2)_j^3} \frac{1 - s^2 t^{4j - 2}}{1 - s^2 t^{-2}} \phi_2 \left[ -sqt, s^2 t^{2j - 2}, t^{-2j}; t^2, t^2 \right] \]

**Proof.** Setting $b = t$, we have by the Saalschütz expansion formula [GR, p.17, (1.7.2)] that

$$B[s,j] = (-1)^j s^{-j} \frac{(s^2/t^2; t^2)_j}{(t^2; t^2)_j} \frac{1 - s^2 t^{4j - 2}}{1 - s^2 t^{-2}} \phi_2 \left[ -sqt, s^2 t^{2j - 2}, t^{-2j}; t^2, t^2 \right]$$

**(8.3)**

\[B[s,j] = (-1)^j s^{-j} \frac{(s^2/t^2; t^2)_j}{(t^2; t^2)_j} \frac{1 - s^2 t^{4j - 2}}{1 - s^2 t^{-2}} \phi_2 \left[ -sqt, s^2 t^{2j - 2}, t^{-2j}; t^2, t^2 \right] \]

and

$$\tilde{B}[s,j] = \frac{(1/qt; t^2)_j(s^2/t^2; t^2)_j}{(t^2; t^2)_j^3} \frac{1 - s^2 t^{4j - 2}}{1 - s^2 t^{-2}} \phi_2 \left[ -sqt, s^2 t^{2j - 2}, t^{-2j}; t^2, t^2 \right]$$

**(8.4)**

Corollary 8.2. When $b = q, t = q$, we have

$$B[s,j] = \begin{cases} 
1 & (j = 0) \\
-1 & (j = 1) \\
0 & (j > 1)
\end{cases}$$

**(8.5)**

\[B[s,j] = \begin{cases} 
1 & (j = 0) \\
-1 & (j = 1) \\
0 & (j > 1)
\end{cases} \]

Corollary 8.3. Let $m \in \mathbb{C}$. When $b = t, q = 0$, we have

$$B[m^n + 1, j] = (-1)^j \phi_2 \left[ \frac{m + 2j}{m} \right]_{i^2} \left[ \frac{m + j - 1}{j} \right]_{i^2}$$

**(8.7)**

\[B[m^n + 1, j] = (-1)^j \phi_2 \left[ \frac{m + 2j}{m} \right]_{i^2} \left[ \frac{m + j - 1}{j} \right]_{i^2} \]

\[B[m^n + 1, j] = \left[ \frac{m + 2j}{j} \right]_{i^2} \]

**(8.8)**

\[B[m^n + 1, j] = \left[ \frac{m + 2j}{j} \right]_{i^2} \]
8.1.2. $D_n$ case.

**Proposition 8.4.** If $b = 1$, we have

\[
B[s, j] = \frac{(t/q; t^2)_j (s^2/t^2; t^2)_j 1 - st^{2j-1}}{(t^2; t^2)_j (s^2q/t; t^2)_j 1 - s/t} q^j, \tag{8.9}
\]

\[
\tilde{B}[s, j] = \frac{(q/t; t^2)_j (s^2t^2_j; t^2)_j 1 + s/t}{(t^2; t^2)_j (s^2qt^{2j-3}; t^2)_j 1 + st^{2j-1}} p^j. \tag{8.10}
\]

**Proof.** When $b = 1$, we have

\[
B[s, j] = (-1)^j s^{-j} (s^2/t^2; t^2)_j 1 - s t^{2j-2} \frac{-s q, s t^{2j-2}, t^{2j}}{1 - s t^{-2} 3 \phi_2^{(t^2; t^2)_j}} 1 - s^{2j-2} q^j t^2 \tag{8.11}
\]

\[
= (-1)^j s^{-j} (s^2/t^2; t^2)_j 1 - s t^{2j-2} (t/q, t^2)_j (-t^{2j+3}/s; t^2)_j 1 - s t^{2j-2} (s t^{2j+3}/q^2; t^2)_j
\]

\[
= (t/q; t^2)_j (s^2/t^2; t^2)_j 1 - s t^{2j-1} 1 - s/t q^j,
\]

and

\[
\tilde{B}[s, j] = t^j (st^{-1}) 1 - j (s^2t^2; t^2)_j 1 + st^{-1} \frac{-t^{2j+3}/s, t^{-2j+2}/s^2, t^{2j}}{1 + st^{2j-2} \phi_2^{(t^2; t^2)_j}} 1 + st^{2j-2} t^j
\]

\[
= t^j (st^{-1}) 1 - j (t^2; t^2)_j 1 + st^{-1} (q/t; t^2)_j (-s; t^2)_j 1 + st^{2j-2} (s^2qt^{2j-3}; t^2)_j
\]

\[
= (q/t; t^2)_j (s^2t^2_j; t^2)_j 1 + s/t q^j (t^2; t^2)_j (s^2qt^{2j-3}; t^2)_j 1 + st^{2j-1} t^j.
\]

\[
\square
\]

**Corollary 8.5.** When $b = 1$, $t = q$, we have

\[
B[s, j] = \delta_{j,0}, \tag{8.13}
\]

\[
\tilde{B}[s, j] = \delta_{j,0}. \tag{8.14}
\]

**Corollary 8.6.** Let $m \in \mathbb{C}$. When $b = 1$, $q = 0$, we have

\[
B[t^{m+1}, j] = (-1)^j t^j \frac{m + 2j}{m + 2j} \left[ \begin{array}{c} m + j - 1 \\ j \end{array} \right] e^j, \tag{8.15}
\]

\[
\tilde{B}[t^{m+1}, j] = t^j \frac{1 + t^m}{1 + t^{m+2j}} \left[ \begin{array}{c} m + 2j \\ j \end{array} \right] e^j. \tag{8.16}
\]

8.2. Explicit formulas for $P^{(C_n)}(x|t; q, t)$ and $P^{(D_n)}(x|q, t)$. Using the formulas obtained in the previous subsection, we give some explicit transition formulas for the polynomials $P^{(C_n)}(x|t; q, t)$ and $P^{(D_n)}(x|q, t) = P^{(C_n)}(x|1; q, t)$.

**Theorem 8.7.** We have

\[
P^{(C_n)}(x|t; q, t) = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \frac{1/q^j t^{2n-2r} (t^2)^j (t^{2n-2r+4j})}{(t^2; t^2)_j (qt^{2n+2r+3}; t^2)_j} (qt)^j E_{r-2j}(x), \tag{8.17}
\]

\[
E_r(x) = \sum_{j=0}^{\lfloor \frac{r}{2} \rfloor} \frac{(q^j t^2)^j (t^{2n-2r+2j+2} t^2)^j (t^{2n-2r+2j+2}; t^2)^j}{(t^2; t^2)_j (qt^{2n+2r+1}; t^2)_j} P^{(C_n)}(x|t; q, t), \tag{8.18}
\]

\[
P^{(D_n)}(x|q, t) = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \frac{1/q^j t^{2n-2r} (t^2)^j (t^{2n-2r+4j})}{(t^2; t^2)_j (qt^{2n+2r+3}; t^2)_j} (qt)^j E_{r-2j}(x), \tag{8.19}
\]
$$E_r(x) = \sum_{j=0}^{\left\lfloor \frac{r}{2} \right\rfloor} \left( \frac{q/t}{t^2} \right)_j (t^{2n-2r+2j+2}, t^2)_j \frac{1+t^{n-r}}{(t^2; t^2)_j} P^{(D_n)}_{(1^r-2j)}(x|q,t). \quad (8.20)$$

**Corollary 8.8.** Setting $t = q$, we have the formula for the Schur polynomials $s^{(C_n)}_{(1^r)}(x) = P^{(C_n)}_{(1^r)}(x|q; q, q)$ and $s^{(D_n)}_{(1^r)}(x) = P^{(D_n)}_{(1^r)}(x|q, q)$

\[ s^{(C_n)}_{(1^r)}(x) = E_r(x) - E_{r-2}(x) \quad (8.21) \]

\[ E_r(x) = \sum_{j=0}^{\left\lfloor \frac{r}{2} \right\rfloor} s^{(C_n)}_{(1^r-2j)}(x), \quad (8.22) \]

\[ s^{(D_n)}_{(1^r)}(x) = E_r(x). \quad (8.23) \]

Hence, from Lemma 3.3, we have

\[ s^{(C_n)}_{(1^r)}(x) = \sum_{j=0}^{\left\lfloor \frac{r}{2} \right\rfloor} \left( \left\lfloor \frac{n-r+2j}{j} \right\rfloor - \left\lfloor \frac{n-r}{j-1} \right\rfloor \right) m_{(1^r-2j)}(x) \quad (8.24) \]

\[ = \sum_{j=0}^{\left\lfloor \frac{r}{2} \right\rfloor} \frac{n-r+1}{n-r+j+1} \left\lfloor \frac{n-r+2j}{j} \right\rfloor m_{(1^r-2j)}(x), \]

\[ s^{(D_n)}_{(1^r)}(x) = \sum_{j=0}^{\left\lfloor \frac{r}{2} \right\rfloor} \left\lfloor \frac{n-r+2j}{j} \right\rfloor m_{(1^r-2j)}(x). \quad (8.25) \]

### 8.3. Hall-Littlewood polynomials $P^{(C_n)}_{(1^r)}(x|b; q, t)$ and Kostka polynomials

Using the transition formulas we have established, we can study the Kostka polynomials associated one column diagrams for types $C_n$ and $D_n$. Setting $b = t, q = 0$ for $C_n$ (or $b = 1, q = 0$ for $D_n$) in $P^{(C_n)}_{(1^r)}(x|b; q, t)$, we have the type $C_n$ (or type $D_n$) Hall-Littlewood polynomials with one column diagrams.

**Theorem 8.9.** We have

\[ P^{(C_n)}_{(1^r)}(x|b; 0, t) = \sum_{j=0}^{\left\lfloor \frac{r}{2} \right\rfloor} (-1)^j t^j (j-1) \left\lfloor \frac{n-r+2j}{j} \right\rfloor \left[ \frac{n-r+j-1}{j} \right] E_{r-2j}(x), \quad (8.26) \]

\[ E_r(x) = \sum_{j=0}^{\left\lfloor \frac{r}{2} \right\rfloor} \left\lfloor \frac{n-r+2j}{j} \right\rfloor \left[ \frac{n-r+j-1}{j} \right] \frac{1+t^{n-r}}{\left[ \frac{n-r}{j} \right]_t^2} P^{(C_n)}_{(1^r-2j)}(x|b; 0, t), \quad (8.27) \]

\[ P^{(D_n)}_{(1^r)}(x|0, t) = \sum_{j=0}^{\left\lfloor \frac{r}{2} \right\rfloor} (-1)^j t^j \left[ \frac{n-r+2j}{j} \right] \left[ \frac{n-r+j-1}{j} \right] E_{r-2j}(x), \quad (8.28) \]

\[ E_r(x) = \sum_{j=0}^{\left\lfloor \frac{r}{2} \right\rfloor} t^j \left[ \frac{1+t^{n-r}}{1+t^{n-r+2j}} \right] \left[ \frac{n-r+2j}{j} \right] \left[ \frac{n-r+j-1}{j} \right] \frac{1+t^{n-r}}{\left[ \frac{n-r}{j} \right]_t^2} P^{(D_n)}_{(1^r-2j)}(x|0, t). \quad (8.29) \]

Then, applying the the formulas for the Schur polynomials in Corollary 8.8, we can calculate the Kostka polynomials (i.e. the transition coefficients from the Schur polynomials to the Hall-Littlewood polynomials) of types $C_n$ and $D_n$ associated with one column diagrams as follows.
Theorem 8.10. We have
\[ s_{(1^r)}^{(C_n)}(x) = \sum_{j=0}^{\lfloor \frac{x}{2} \rfloor} t^j \frac{m-r+1}{m-r+j+1} \left[ n-r+2j \right]_{\ell^2} P_{(1^r)}^{(C_n)}(x;0,t) \]
\[ = \sum_{j=0}^{\lfloor \frac{x}{2} \rfloor} \left( \left[ n-r+2j \right]_{\ell^2} - \left[ n-r+2j \right]_{\ell^2} \right) P_{(1^r)}^{(C_n)}(x;0,t), \]
\[ s_{(1^r)}^{(D_n)}(x) = \sum_{j=0}^{\lfloor \frac{x}{2} \rfloor} t^j \frac{1+t^{n-r+2j}}{1+t^{n-r+2j}} \left[ n-r+2j \right]_{\ell^2} P_{(1^r)}^{(D_n)}(x;0,t) \]
\[ = \sum_{j=0}^{\lfloor \frac{x}{2} \rfloor} \left( t^{n-r+j} \left[ n-r+2j-1 \right]_{\ell^2} + t^j \left[ n-r+2j-1 \right]_{\ell^2} \right) P_{(1^r)}^{(D_n)}(x;0,t). \]

Hence we have Theorem 2.4.

Remark 8.11. The expansion coefficient of (8.31) (times \( t^{-2j} \)) is identified with the \( q \)-ballot (when \( m = 0 \), \( q \)-Catalan) number \( \left[ \begin{array}{c} m+2j \\ j \end{array} \right]_{q} \)
\[ q^{-j} \left( \left[ m+2j \right]_{q} - \left[ m+2j \right]_{q} \right) = \frac{[m+1]_{q}}{[m+j+1]_{q}} \left[ m+2j \right]_{q}, \]
by the replacement \( m \to n-r, q \to t^2 \). The case \( m = 0 \) gives us the \( q \)-Catalan number. It is known that the \( q \)-Catalan or \( q \)-ballot number is a polynomial in \( q \) with positive integral coefficients (see \[ A \] and \[ HH \]).

The expansion coefficient of (8.31) is identified with the following version of the \( q \)-binomial number
\[ q^j \frac{1+q^{m-2j}}{1+q^{m}} \left[ m \right]_{q} = q^{-j} \left[ m-1 \right]_{q} + q^j \left[ m-1 \right]_{q}, \]
by the replacement \( m \to n-r+2j, q \to t \). Note that this is also a polynomial in \( q \) with positive integral coefficients.

9. Some Conjectures about Macdonald Polynomials of Type \( C_n \)

9.1. Asymptotically free eigenfunctions for the Macdonald operator of type \( A_{n-1} \).
First we recall some facts about the asymptotically free eigenfunctions for the case \( A_{n-1} \). Let \( n \in \mathbb{Z}_{>0} \), and \( q, t \in \mathbb{C} \) be generic parameters. Let \( x = (x_1, \ldots, x_n) \) be a set of independent indeterminates. Macdonald’s difference operator of type \( A_{n-1} \) is defined by
\[ D^{(A_{n-1})} = \sum_{i=1}^{n} \prod_{j \neq i} \frac{t^{x_i - x_j}}{x_i - x_j} T_{q,x_i}. \] (9.1)

For a partition \( \lambda \) with \( \ell(\lambda) \leq n \), the Macdonald symmetric polynomial \( P_{\lambda}(x;q,t) \in \mathbb{C}[x_1, \ldots, x_n]^{S_n} \) exists uniquely characterized by the conditions:
\[ P_{\lambda} = m_{\lambda} + \sum_{\mu \subset \lambda} u_{\lambda \mu} m_{\mu}, \]
\[ D^{(A_{n-1})} P_{\lambda} = \sum_{i=1}^{n} q^{\lambda_i} t^{n-i} \cdot P_{\lambda}. \] (9.3)
Let $s_1, s_2, \ldots, s_n \in \mathbb{C}$ be a set of complex variables. Let $M^{(n)}$ be the set of strict upper triangular matrices with entries in $\mathbb{Z}_{\geq 0}$, namely for $\theta^{(n)} = (\theta^{(n)}_{ij})_{i,j \in \mathbb{Z}_{\geq 0}} \in M^{(n)}$ $i \geq j$ implies $\theta^{(n)}_{ij} = 0$.

**Definition 9.1.** For $n \geq 1$, define recursively the rational functions $c_n(\theta^{(n)}; s_1, \ldots, s_n; q, t) \in \mathbb{Q}(q, t, s_1, \ldots, s_n)$ by $c_1(-; s_1; q, t) = 1$, and

$$c_n(\theta^{(n)}; s_1, \ldots, s_n; q, t) = c_{n-1}(\theta^{(n-1)}; q^{-\theta_{1,n} s_1}, \ldots, q^{-\theta_{n-1,n} s_{n-1}}; q, t) \prod_{1 \leq i < j \leq n} \frac{(ts_{j+1}/s_i; q)_{\theta_{i,n}} (q^{-\theta_{j,n} s_j} / ts_i; q)_{\theta_{j,n}}}{(qs_{j+1}/s_i; q)_{\theta_{i,n}} (q^{-\theta_{j,n} s_j} / s_i; q)_{\theta_{j,n}}}.$$  \hfill (9.4)

**Definition 9.2.** Set

$$\varphi^{(A_{n-1})}(s|x) = \sum_{\theta^{(n)} \in M^{(n)}} c_n(\theta^{(n)}; s_1, \ldots, s_n; q, t) \prod_{1 \leq i < j \leq n} \frac{x_j}{x_i}^{\theta_{ij}}.$$ \hfill (9.5)

**Theorem 9.3.** Write $s_i = t^{n-i} q^\lambda_i$ ($1 \leq i \leq n$) for simplicity. We have

$$D^{(A_{n-1})} x^\lambda \varphi^{(A_{n-1})}(s|x) = (s_1 + \cdots + s_n) x^\lambda \varphi^{(A_{n-1})}(s|x),$$ \hfill (9.6)

where $x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$. When $\lambda$ is a partition with $\ell(\lambda) \leq n$, we have

$$x^\lambda \varphi^{(A_{n-1})}(s|x) = P_\lambda(x).$$ \hfill (9.7)

**Remark 9.4.** We have the decomposition of the series $\varphi^{(A_{n-1})}$ in terms of the $\varphi^{(A_{n-2})}$ series as

$$\varphi^{(A_{n-1})}(s_1, \ldots, s_n|x_1, \ldots, x_n) = \sum_{\theta^{(n)} \in M^{(n)}} \varphi^{(A_{n-2})}(q^{-\theta_{1,n} s_1}, \ldots, q^{-\theta_{n-1,n} s_{n-1}}|x_1, \ldots, x_{n-1}) \times \prod_{1 \leq i < j \leq n} \frac{(ts_{j+1}/s_i; q)_{\theta_{i,n}} (q^{-\theta_{j,n} s_j} / ts_i; q)_{\theta_{j,n}}}{(qs_{j+1}/s_i; q)_{\theta_{i,n}} (q^{-\theta_{j,n} s_j} / s_i; q)_{\theta_{j,n}}} \prod_{i=1}^{n-1} \frac{x_n}{x_i}^{\theta_{in}}.$$ \hfill (9.8)

9.2. Asymptotically free eigenfunction of type $C_n$. Let $n \in \mathbb{Z}_{>0}$. Let $x = (x_1, \ldots, x_n)$ and $(s_1, \ldots, s_n)$ be a pair of variables. Let $D^{(C_n)}_x = D_x(-t^{1/2}, t^{1/2} - q^{1/2} t^{1/2}, q^{1/2} t^{1/2}|q, t)$ be the $BC_n$ Koornwinder operator degenerated to the $C_n$ case.

**Definition 9.5.** Set

$$s_i = t^{n-i+1} q^{\lambda_i} \quad (1 \leq i \leq n),$$ \hfill (9.9)

and write $x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ for simplicity. We define the asymptotically free eigenfunction $x^\lambda \varphi^{(C_n)}(s|x)$ of type $C_n$ by

$$\varphi^{(C_n)}(s|x) = \varphi^{(C_n)}(s_1, \ldots, s_n|x_1, \ldots, x_n) = \sum_{k_1, \ldots, k_n \geq 0} c_{k_1, \ldots, k_n} s_1, \ldots, s_n; q, t \left(\frac{x_2}{x_1}\right)^{k_1} \cdots \left(\frac{x_n}{x_{n-1}}\right)^{k_{n-1}} \left(\frac{1}{x_n}\right)^{k_n},$$ \hfill (9.10)

where

$$D^{(C_n)}_x x^\lambda \varphi^{(C_n)}(s|x) = \varepsilon^{(C_n)}(s) x^\lambda \varphi^{(C_n)}(s|x),$$ \hfill (9.11)

and

$$\varepsilon^{(C_n)}(s) = \sum_{i=1}^{n} s_i + s_i^{-1} - t^i - t^{-i}.$$ \hfill (9.12)
9.3. \(C_2\) case.

**Definition 9.6.** Let \(x_1, x_2, s_1, s_2\) be variables. Set

\[
\psi^{(C_2)}(s_1, s_2|x_1, x_2) = \sum_{\theta_{12}, \mu_{12}, \rho_1, \rho_2; s_1, s_2; q, t} c^{(C_2)}(\theta_{12}, \mu_{12}, \rho_1, \rho_2; s_1, s_2; q, t) \left( \frac{x_2}{x_1} \right)^{\theta_{12}} \left( \frac{1}{x_1 x_2} \right)^{\mu_{12}} \left( \frac{1}{x_1^2} \right)^{\rho_1} \left( \frac{1}{x_2^2} \right)^{\rho_2},
\]

where

\[
c^{(C_2)}(\theta_{12}, \mu_{12}, \rho_1, \rho_2; s_1, s_2; q, t) = (t)_{\theta_{12}} (ts_2/s_1)_{\theta_{12}} (q/t)^{\theta_{12}} (t/s_1 s_2)_{\mu_{12}} (q/t)^{\mu_{12}} \times (q/s_2/s_1)_{\theta_{12}} (q/s_1 s_2)_{\mu_{12}} (q/s_2)_{\mu_{12}} (q^{-\theta_{12}} s_2/s_1)_{\mu_{12}} (q^{\theta_{12}} q/s_1)_{\mu_{12}} \times (q/s_1)_{\rho_1} (q^{\theta_{12}+\mu_{12}} q t/s_1)_{\rho_1} (q/t)_{\rho_1} (q/t)_{\rho_1} (q^{-\theta_{12}+\mu_{12}} t/s_2)_{\rho_2} (q/t)_{\rho_2} (q^{\theta_{12}+\mu_{12}} q/s_2)_{\rho_2} (q/t)_{\rho_2}.
\]

**Conjecture 9.7.** We have \(\psi^{(C_2)}(s_1, s_2|x_1, x_2) = \varphi^{(C_2)}(s_1, s_2|x_1, x_2)\). Namely, setting \(s_1 = t^2 q^{\lambda_1}, s_2 = tq^{\lambda_2}, x^1 = x_1^{\lambda_1} x_2^{\lambda_2}, \) we have

\[
D_x^{(C_2)} x^\lambda \psi^{(C_2)}(s|x) = \varepsilon^{(C_2)}(s) x^\lambda \psi^{(C_2)}(s|x).
\]

When \(\lambda = (\lambda_1, \lambda_2)\) is a partition, we have

\[
x^\lambda \psi^{(C_2)}(s|x) = P^{(C_2)}(x|t; q, t).
\]

9.4. \(C_3\) case with rectangular diagrams. We can study the decomposition of the \(C_3\) Macdonald polynomials \(P^{(C_3)}_\lambda(x|t; q, t)\) in terms of the \(C_2\) Macdonald polynomials. It seems that such a decomposition becomes rather simple when we consider the case of a rectangular diagram consisting of three equal rows \(\lambda = (\lambda_3, \lambda_3, \lambda_3)\).

**Definition 9.8.** Let \(\lambda_3 \in \mathbb{C}\) and set

\[
s_1 = t^2 s_3, \quad s_2 = t s_3, \quad s_3 = tq^{\lambda_3}.
\]

Define

\[
\psi^{(C_3),\text{rect}}(s_3|x_1, x_2, x_3) = \sum_{\mu_{13}, \rho_1 \geq 0} (t)_{\mu_{13}} \frac{1}{(s_1^3)_{\mu_{13}}} \left( \frac{q/t}{s_3^2} \right)^{\mu_{13}} \left( \frac{q}{s_3} \right)^{\mu_{13}} \left( \frac{1}{s_3^2} \right)^{\rho_1} \times \left( \frac{1}{x_1 x_3} \right)^{\mu_{13}} \left( \frac{x_1^2}{x_3^2} \right)^{\rho_1} \varphi^{(C_2)}(s_3, q^{-\mu_{13}} s_3|x_2, x_3).
\]

**Conjecture 9.9.** We have \(\psi^{(C_3),\text{rect}}(s_3|x_1, x_2, x_3) = \varphi^{(C_3)}(t^2 s_3, ts_3, s_3|x_1, x_2, x_3)\). Namely, setting \(x^1 = (x_1 x_2 x_3)^{\lambda_3}\), we have

\[
D_x^{(C_3)} x^\lambda \varphi^{(C_3),\text{rect}}(s_3|x_1, x_2, x_3) = \varepsilon^{(C_3)}(s) x^\lambda \varphi^{(C_3),\text{rect}}(s_3|x_1, x_2, x_3),
\]

where \(s_1, s_2, s_3\) are as given in [9.18]. When \(\lambda_3\) is a nonnegative integer, we have

\[
x^\lambda \varphi^{(C_3),\text{rect}}(s_3|x_1, x_2, x_3) = P^{(C_3)}_{\lambda_3, \lambda_3, \lambda_3}(x|t; q, t).
\]
9.5. Folding of $A_{2n-1}$ eigenfunctions and decomposition with respect to $C_n$ eigenfunctions.

Definition 9.10. Let $n \in \mathbb{Z}_{>0}$. Let $x = (x_1, \ldots, x_n)$ and
\[ s = (s_1, \ldots, s_n), \quad s_i = t^{n-i+1}q^{i}, \quad (1 \leq i \leq n), \]
be a pair of variables. Define the folded series $\tilde{\varphi}^{(A_{2n-1})}(s|x) = \tilde{\varphi}^{(A_{2n-1})}(s_1, \ldots, s_n|x_1, \ldots, x_n)$ by
\[
\tilde{\varphi}^{(A_{2n-1})}(s|x) = \varphi^{(A_{2n-1})}(t^{n-1}s_1, \ldots, t^{n-1}s_n, t^{n-2}, \ldots, t, 1|x_1, \ldots, x_n, x_n^{-1}, \ldots, x_1^{-1}).
\]

Proposition 9.11. When $n = 1$, we have
\[
\tilde{\varphi}^{(A_1)}(s|x) = \varphi^{(A_1)}(s_1, 1|x_1, x_1^{-1}) = \sum_{\theta \geq 0} (t; q)_{\theta} (ts_1; q)_{\theta} (q/t)^{\theta} s_{x_1 2^\theta}.
\]
Hence we have $\tilde{\varphi}^{(A_1)}(s|x) = \varphi^{(C_1)}(s_1|x_1)$.

We calculated the decomposition of the folded eigenfunctions $\tilde{\varphi}^{(A_{2n-1})}(s|x)$ with respect to the $C_n$ series $\varphi^{(C_n)}(s|x)$ for the cases $n = 2$ and $3$ by using Mathematica.

Conjecture 9.12. We have
\[
\tilde{\varphi}^{(A_1)}(s_1, s_2|x_1, x_2) = \sum_{\mu_{12} \geq 0} e_2(s_1, s_2; \mu_{12}) \left( \frac{1}{x_1 x_2} \right)^{\mu_{12}} \varphi^{(C_2)}(q^{-}\mu_{12}s_1, q^{-}\mu_{12}s_2|x_1, x_2),
\]
where
\[
e_2(s_1, s_2; \mu_{12}) = \frac{(t/s_1)_{\mu_{12}} (t/s_2)_{\mu_{12}} (t)_{\mu_{12}} (q^{\mu_{12}}q/ts_1s_2)_{\mu_{12}} (q/t)^{\mu_{12}}}{(q/s_1)_{\mu_{12}} (q/s_2)_{\mu_{12}} (q)_{\mu_{12}} (q^{\mu_{12}}/s_1s_2)_{\mu_{12}} (q/t)^{\mu_{12}}}. \tag{9.27}
\]

Conjecture 9.13. We have
\[
\tilde{\varphi}^{(A_3)}(s_1, s_2, s_3|x_1, x_2, x_3) = \sum_{\mu_{12}, \mu_{13}, \mu_{23} \geq 0} c_3(s_1, s_2, s_3; \mu_{12}, \mu_{13}, \mu_{23}) \times \left( \frac{1}{x_1 x_2} \right)^{\mu_{12}} \left( \frac{1}{x_1 x_3} \right)^{\mu_{13}} \left( \frac{1}{x_2 x_3} \right)^{\mu_{23}} \varphi^{(C_3)}(q^{-}\mu_{12}-\mu_{13}s_1, q^{-}\mu_{12}-\mu_{23}s_2, q^{-}\mu_{13}-\mu_{23}s_3|x_1, x_2, x_3),
\]
where
\[
c_3(s_1, s_2, s_3; \mu_{12}, \mu_{13}, \mu_{23}) = \frac{(t/s_1)_{\mu_{12}+\mu_{13}} (t/s_2)_{\mu_{12}+\mu_{23}} (t/s_3)_{\mu_{13}+\mu_{23}}}{(q/s_1)_{\mu_{12}+\mu_{13}} (q/s_2)_{\mu_{12}+\mu_{23}} (q/s_3)_{\mu_{13}+\mu_{23}}} \times \frac{\mu_{12}}{\mu_{13} \mu_{23}} (q^{\mu_{12}+\mu_{13}+\mu_{23}}q/ts_1s_2)_{\mu_{12}} (q/t)^{\mu_{12}} \times (q/s_1)_{\mu_{13}} (q^{\mu_{12}+\mu_{13}+\mu_{23}}/s_1s_2)_{\mu_{12}} (q)_{\mu_{12}} (q^{\mu_{12}}/s_1s_2)_{\mu_{12}} \times (t/s_3/s_1)_{\mu_{12}} (q^{-\mu_{23}}q/s_3/ts_1)_{\mu_{12}} (t/s_3/s_2)_{\mu_{12}} (q^{-\mu_{13}}q/s_3/ts_2)_{\mu_{12}} \times (q/s_3/s_1)_{\mu_{12}} (q^{-\mu_{23}}q/s_3/s_1)_{\mu_{12}} (q/s_3/s_2)_{\mu_{12}} (q^{-\mu_{13}}q/s_3/s_2)_{\mu_{12}} \times (t/s_2/s_1)_{\mu_{13}} (q^{\mu_{12}+\mu_{13}+\mu_{23}}q/ts_1s_3)_{\mu_{13}} (q/t)^{\mu_{13}} \times (q/s_2/s_1)_{\mu_{13}} (q^{-\mu_{23}}q/s_2/ts_1)_{\mu_{13}} \times (q/s_2/s_1)_{\mu_{13}} (q^{-\mu_{23}}q/s_2/s_1)_{\mu_{13}} \times (t/s_2/s_1)_{\mu_{23}} (q^{\mu_{12}+\mu_{13}+\mu_{23}}q/ts_2s_3)_{\mu_{23}} (q/t)^{\mu_{23}}. \tag{9.29}
\]
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