LOCAL-TO-GLOBAL EXTENSIONS OF $\mathcal{D}$-MODULES IN
POSITIVE CHARACTERISTIC

LARS KINDLER

Abstract. In [Kat87], Katz defines the notion of a special flat connec-
tion on $\mathbb{P}_C^1 \setminus \{0, \infty\}$, and he shows that the functor which restricts a flat connection to the punctured disc around the point at infinity gives rise to an equivalence between the category of special flat connections on $\mathbb{P}_C^1 \setminus \{0, \infty\}$ and the category of differential modules on $\mathbb{C}(\!(t)\!)$.

In this article, we prove the corresponding statement over an algebraically closed field $k$ of positive characteristic. The role of flat connections is played by vector bundles carrying an action of the (full) ring of differential operators. Such objects are also called stratified bundles. The formal local variant on the field of Laurent series $k(\!(t)\!)$ is called iterated differential module. We define the notion of a special stratified bundle on $\mathbb{P}_{k}^1 \setminus \{0, \infty\}$, and show that restriction to the punctured disc around the point at infinity induces an equivalence between the category of special stratified bundles and iterated differential modules on $k(\!(t)\!)$. This extends one of the main results of [Kat86], and has several interesting consequences which extend well-known statements about étale fundamental groups to higher dimensional fundamental groups.

Let $\mathbb{G}_{m,C} := \mathbb{P}_C^1 \setminus \{0, \infty\}$, and let $\text{Conn}(\mathbb{G}_{m,C})$ be the category of vector bundles on $\mathbb{G}_{m,C}$ equipped with a flat connection. Consider $\text{Spec} \mathbb{C}(\!(t)\!)$ as the punctured disc around $\infty$, and write $DM(\mathbb{C}(\!(t)\!))$ for the category of differential modules on $\mathbb{C}(\!(t)\!)$. Restriction induces a functor $\text{Conn}(\mathbb{G}_{m,C}) \to DM(\mathbb{C}(\!(t)\!))$. In [Kat87], Katz defines a full subcategory

$$\text{Conn}^{\text{special}}(\mathbb{G}_{m,C}) \subseteq \text{Conn}(\mathbb{G}_{m,C})$$

of special flat connections, with the property that the restriction functor

$$\text{Conn}^{\text{special}}(\mathbb{G}_{m,C}) \to DM(\mathbb{C}(\!(t)\!)) \quad (1)$$

is an equivalence. It follows that the category of differential modules on $\mathbb{C}(\!(t)\!)$ can be equipped with a $\mathbb{C}$-valued fiber functor. The main result of this article is a generalization of Katz’ equivalence $(1)$ to positive characteristic.

Fix an algebraically closed field $k$ of positive characteristic $p$. If $X$ is a smooth $k$-variety, then we write $\text{Strat}(X)$ for the category of stratified bundles on $X$, i.e. for the category of $\mathcal{O}_X$-coherent $\mathcal{D}_X/k$-modules, where $\mathcal{D}_X/k$ is the sheaf of differential operators on $X$ relative to $k$, as defined in [EGA4, §16]. In many respects, the behavior of these objects is very similar to the behavior of vector bundles with flat connections on smooth complex varieties, see e.g. [Gie75, EM10, Esn13, Kin12]. In particular,

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1 The notion of a stratification goes back to [Gro6]. On a smooth $k$-variety $X$, the datum of a stratification on an $\mathcal{O}_X$-module $E$ is equivalent to an action of $\mathcal{D}_X/k$ on $E$ ([BO78 Ch. 2]) and if $E$ is $\mathcal{O}_X$-coherent, then it is locally free ([SR72 p.314]).
Moreover, the notion of a special stratified bundle on \(G\) is a full subcategory with its natural \(k\)-action which acts on powers of \(t\) via the formula
\[
\delta_t^{(n)}(t^r) = \binom{r}{n} t^r.
\] (2)

Abusing notation slightly, we will write \(\text{Strat}(k((t)))\) for the category of iterated differential modules on \(k((t))\), and also say “stratified bundle on \(k((t))\)” (even though \(k((t))\) is not a \(k\)-algebra of finite type). The category \(\text{Strat}(k((t)))\) is a \(k\)-linear, abelian, rigid tensor category with unit 1 := \(k((t))\) with its natural \(k[\delta_t^{(n)}][n \geq 0]\)-action given by (2). One has two restriction functors \(\text{Strat}(G_m) \to \text{Strat}(k((t)))\), and \(\text{Strat}(G_m) \to \text{Strat}(k((t^{-1})))\) arising from the inclusions \(k[t^{\pm 1}] \subseteq k((t))\), \(t \mapsto t\) and \(k[t^{\pm 1}] \subseteq k((t^{-1}))\), \(t \mapsto t^{-1}\).

**Convention 0.1.** For notational ease, the roles of 0 and \(\infty\) in the following will be opposite to their roles in [Kat87, Kat86]. In other words, will always consider the polynomial ring \(k[t]\) as a subring of \(k[t^{\pm 1}]\) via the inclusion defined by mapping \(t \mapsto t\). Similarly, we will consider \(k[t^{\pm 1}]\) as a subring of both \(k((t))\) and \(k((t^{-1}))\) via the inclusions given by \(t \mapsto t\). From now on, we also simply write \(G_m := G_{m,k}\).

We can now formulate the first main result of this article.

**Theorem A.** There is a full subcategory \(\text{Strat}^{\text{special,}\infty}(G_m) \subseteq \text{Strat}(G_m)\), such that the restriction functor
\[
\text{res} : \text{Strat}(G_m) \to \text{Strat}(k((t))), E \mapsto E|_{k((t))} := E \otimes_{k[t^{\pm 1}]} k((t)),
\]
given by the inclusion \(k[t^{\pm 1}] \subseteq k((t))\), induces an equivalence
\[
\text{Strat}^{\text{special,}\infty}(G_m) \xrightarrow{\sim} \text{Strat}(k((t))).
\]

Moreover, the notion of a special stratified bundle on \(G_m\) extends the notion of a special covering of \([Kat86]\). see Proposition 3.2.

Just as in [Kat87, Cor. 2.4.12], we obtain from Theorem A the following consequence.

**Corollary 0.2.** There exists an exact \(\otimes\)-functor \(\omega : \text{Strat}(k((t))) \to \text{Vect}_k\), where \(\text{Vect}_k\) denotes the category of finite dimensional \(k\)-vector spaces.

As the last part of the theorem suggests, Theorem A is closely related to the main theorem of [Kat86, Part 1]. Exchanging the roles of 0 and \(\infty\) in Theorem A we obtain a full subcategory \(\text{Strat}^{\text{special,}0}(G_m)\), such that restriction induces an equivalence
\[
\text{Strat}^{\text{special,}0}(G_m) \xrightarrow{\sim} \text{Strat}(k((t^{-1}))).
\] (3)

Let \(\text{Cov}(G_m)\) denote the category of finite étale coverings of \(G_m\). In [Kat86], Katz and Gabber define a subcategory \(\text{Cov}^{\text{special}}(G_m)\) of special coverings
of $\mathbb{G}_m$, such that the restriction functor $\text{Cov}(\mathbb{G}_m) \to \text{Cov}(k((t^{-1})))$ induces an equivalence

$$\text{Cov}^{\text{special}}(\mathbb{G}_m) \xrightarrow{\sim} \text{Cov}(k((t^{-1}))).$$

(4)

For an étale covering $f : X \to \mathbb{G}_m$ the $\mathcal{O}_X$-module $f_*\mathcal{O}_X$ is naturally a stratified bundle on $\mathbb{G}_m$; we obtain a faithful functor $\text{Cov}(\mathbb{G}_m) \to \text{Strat}(\mathbb{G}_m)$. Theorem A claims that $f$ is special in the sense of [Kat86] if and only if $f_*\mathcal{O}_X \in \text{Strat}^{\text{special,0}}(\mathbb{G}_m)$. This means that the equivalence (3) restricts to (4).

The structure of this article is as follows. In Section 1 we establish some facts about $R^1\lim\leftarrow$ of a projective system of non-abelian groups. These will be the main tools in the proof of Theorem A, as certain sets of isomorphism classes related to stratified bundles can be naturally described as $R^1\lim\leftarrow$ of such projective systems. We explain these identifications in Section 2. In Section 3 we define the notion of a special stratified bundle. The proof of Theorem A is given in Section 4. Finally, in Section 5 we present applications of Theorem A and the methods used in its proof; we conclude this introduction with a brief summary of these applications.

Definition 0.3. If $X$ is a smooth, finite type $k$-scheme, and $\omega : \text{Strat}(X) \to \text{Vect}_k^{\text{f}}$ a fiber functor, then we write $\pi_1^{\text{Strat}}(X,\omega)$ for the group scheme attached to $\omega$ via the Tannaka formalism; it is a reduced ([dS07]), affine $k$-group scheme. If $X = \text{Spec} R$ is affine, we also write $\pi_1^{\text{Strat}}(R,\omega)$ instead of $\pi_1^{\text{Strat}}(\text{Spec} R,\omega)$. If $R = k((t))$, we denote by $\pi_1^{\text{Strat}}(k((t)),\omega)$ the affine $k$-group scheme attached to the category $\text{Strat}(k((t)))$, even though $k((t))$ is not a finite type $k$-algebra.

In the same vein we write $\pi_1^{\text{unip}}(X,\omega)$ (resp. $\pi_1^{\text{rs}}(X,\omega)$) for the affine group schemes associated with the full subcategory of $\text{Strat}(X)$ with objects the unipotent (resp. regular singular) stratified bundles.

The first application is a straightforward consequence of Corollary 0.2.

Theorem B. Fix a fiber functor $\omega : \text{Strat}(k((t))) \to \text{Vect}_k^{\text{f}}$. There is a short exact sequence of reduced affine $k$-group schemes

$$1 \to P(\omega) \to \pi_1^{\text{Strat}}(k((t)),\omega) \to \pi_1^{\text{rs}}(k((t)),\omega) \to 1.$$  

(5)

The group scheme $P(\omega)$ is unipotent.

Theorem B should be seen as a generalization of the usual short exact sequence of profinite groups

$$1 \to P \to \text{Gal}(k((t)))^{\text{ep}}/k((t)) \to \hat{\mathbb{Z}}(\rho') \to 1.$$  

(6)

with the wild ramification group $P$ the unique pro-$p$-Sylow subgroup of $\text{Gal}(k((t)))^{\text{ep}}/k((t))$. In fact, (6) can be obtained up to inner automorphism from (5) via profinite completion.

Theorem C. Let $x \in \mathbb{G}_m$ be a closed point, and denote by $\omega_x : \text{Strat}^{\text{unip}}(A^1) \to \text{Vect}_k^{\text{f}}$ the associated fiber functor. By Theorem A, this induces a neutral
fiber functor for \( \text{Strat}^{\text{unip}}(k((t))) \), which we also denote by \( \omega_x \). The inclusion \( k[t] \subseteq k((t^{-1})) \) induces an isomorphism

\[
\pi_1^{\text{unip}}(k((t^{-1})),\omega_x) \xrightarrow{\cong} \pi_1^{\text{unip}}(A_k^1,\omega_x).
\]

There are statements analogous to Theorem C in various contexts: In particular, see [Kat86, Prop. 1.4.2] for the case of the maximal pro-\( p \) quotient of the étale fundamental group, and [Unv10] for the case of “purely irregular” connections over the complex numbers.

We also obtain a coproduct formula analogous to [Kat86, p. 98] and [Unv10, Thm. 2.10]:

**Theorem D.** Let \( x \in G_m \) be a closed point and \( \omega_x : \text{Strat}^{\text{unip}}(G_m) \rightarrow \text{Vect}_k \) the associated fiber functor. By Theorem A, \( \omega_x \) induces fiber functors on \( \text{Strat}^{\text{unip}}(k((t))) \) and \( \text{Strat}^{\text{unip}}(k((t^{-1}))) \), which we also denote by \( \omega_x \). The inclusions \( k[t^\pm 1] \subseteq k((t)) \) and \( k[t^\pm 1] \subseteq k((t^{-1})) \) induce an isomorphism

\[
\pi_1^{\text{unip}}(k((t)),\omega_x) \ast^{\text{unip}} \pi_1^{\text{unip}}(k((t^{-1})),\omega_x) \xrightarrow{\cong} \pi_1^{\text{unip}}(G_m,\omega_x)
\]

where \( \ast^{\text{unip}} \) denotes the coproduct in the category of unipotent affine \( k \)-group schemes, see Definition 5.3.

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1. **Derived inverse limits of nonabelian groups**

   We first recall the some notions from “homological algebra” for noncommutative groups.

   **Definition 1.1** ([BK72, IX.2]). If \( G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} \ldots \) is a projective system of groups, then we define the pointed set \( \prod_{n \in \mathbb{N}} G_n \) as follows: The product \( \prod_{n \in \mathbb{N}} G_n \) acts on itself from the left via

   \[
   (g_1, \ldots, g_n, \ldots) \cdot (x_1, \ldots, x_n, \ldots) := (g_1 x_1 f_1(g_2)^{-1}, g_2 x_2 f_2(g_3)^{-1}, \ldots),
   \]

   and \( \prod_{n \in \mathbb{N}} G_n \) is the set of orbits of this action, pointed by the orbit containing \( 1 \in \prod_n G_n \). This reduces to the usual definition of \( \prod_{n \in \mathbb{N}} G_n \) if the \( G_n \) are abelian.

   **Remark 1.2.** The general results about derived inverse limits of projective systems of arbitrary groups will follow from the following construction, due to Ogus ([Har75]): Equip \( \mathbb{N} \) with the topology in which the open sets are \( \emptyset, \mathbb{N} \) and \( [1,n], n \geq 1 \). Then the category of projective systems of groups is equivalent to the category of sheaves of groups on this topological space. We can hence apply the theory of [Gr74, Ch. 3]. In fact, if we write \( G \) for the sheaf on \( \mathbb{N} \) associated with the projective system \( \{ G_n \} \), then it is not difficult to functorially identify the pointed set \( \prod_{n \in \mathbb{N}} G_n \) with the pointed set \( H^1(\mathbb{N}, G) \), which by definition is the pointed set of isomorphism classes of \( G \)-torsors in the category of sheaves on \( \mathbb{N} \).

   **Lemma 1.3.** We list some properties of the functor \( \prod_{n \in \mathbb{N}} G_n \):
a) If $G_1 \xleftarrow{f_1} G_2 \xleftarrow{f_2} \ldots$ is a projective system of groups with surjective transition morphisms, then $\varprojlim_n G_n = 0$.

b) If

$$1 \to \{A_n\} \to \{B_n\} \to \{C_n\} \to 1$$

is a short exact sequence of projective systems of groups, then there is a functorial exact sequence of pointed sets

$$1 \to \varprojlim A_n \to \varprojlim B_n \to \varprojlim C_n \to 1.$$ 

Moreover, the map $\varprojlim C_n \to \varprojlim A_n$ extends to a functorial action of $\varprojlim C_n$ on $\varprojlim A_n$, which induces an injection

$$\left(\varprojlim_n C_n\right) \setminus \varprojlim A_n \hookrightarrow \varprojlim B_n.$$ 

c) If in b) the groups $A_n$ are central subgroups of $B_n$, then the map

$$\varprojlim A_n \to \varprojlim B_n$$

extends to a functorial action of the abelian group $\varprojlim A_n$ on the pointed set $\varprojlim B_n$, which induces a bijection

$$\left(\varprojlim A_n\right) \setminus \varprojlim B_n \to \varprojlim C_n.$$ 

Proof. From the definition of $\varprojlim G_n$ one immediately verifies a). For b) one either translates [Gir71, III.3.2] using Remark [1.2] or one looks in [BK72, IX.2]. Statement c) follows from [Gir71, Prop. III.3.4.5]. For the reader’s convenience we check it by hand: Of course $\prod_n A_n$ acts on $\prod_n B_n$, and the orbits of this action are precisely the fibers of $\prod_n B_n \to \prod_n C_n$. The centrality of $A_n$ in $B_n$ guarantees that this action descends to the quotient map $\varprojlim A_n \to \varprojlim B_n$. This also shows that the orbits of the action are precisely the fibers of $\varprojlim B_n \to \varprojlim C_n$, which completes the proof of c). □

**Definition 1.4.** For notational convenience, we say that a short exact sequence of pointed sets

$$1 \to A \to B \to C \to 1$$

is **strongly exact**\(^{2}\) if the following hold:

- $A$ is an abelian group.
- $A \to B$ is injective and extends to an action of $A$ on $B$, with orbits precisely the fibers of $B \to C$, i.e. such that one gets an induced bijection $A \backslash B \to C$.

\(^2\)This is nonstandard terminology.
A morphism of strongly exact short exact sequences is a morphism of short exact sequences of pointed sets

\[
\begin{array}{ccc}
1 & \rightarrow & A \\
\downarrow f & & \downarrow g \\
1 & \rightarrow & A'
\end{array}
\quad
\begin{array}{ccc}
\rightarrow & B & \rightarrow C \\
\downarrow g & & \downarrow h \\
\rightarrow & B' & \rightarrow C'
\end{array}
\quad
\begin{array}{c}
\rightarrow 1
\end{array}
\]

such that \( f \) is a homomorphism of groups and such that \( g \) is compatible with the \( A \)-action on \( B \) and the \( A' \)-action on \( B' \), i.e. \( g(ab) = f(a)g(b) \) for \( a \in A, b \in B \).

We will need a weak version of the Snake lemma for strongly exact short exact sequences of pointed sets:

**Lemma 1.5.** Let

\[
\begin{array}{ccc}
1 & \rightarrow & G \\
\downarrow i & & \downarrow j \\
1 & \rightarrow & M \\
\downarrow f & & \downarrow g \\
1 & \rightarrow & N \\
\downarrow h & & \downarrow k \\
1 & \rightarrow & N'
\end{array}
\]

be a morphism of strongly exact (Definition 1.4) short exact sequences of pointed sets.

a) If \( f \) is surjective, then \( i \) and \( j \) induce a strongly exact short exact sequence of pointed sets

\[
\begin{array}{c}
1 \\
\rightarrow \ker(f) \quad \xrightarrow{j_{\ker(f)}} \ker(g) \quad \xrightarrow{j_{\ker(g)}} \ker(h) \quad \rightarrow 1.
\end{array}
\]  \hspace{1cm} (7)

b) If \( f \) and \( h \) are surjective, then \( g \) is surjective.

**Proof.** For [a] it is clear that the sequence (7) is exact on the left and in the middle. We check that \( \ker(g) \rightarrow \ker(h) \) is surjective. Let \( x \in \ker(h) \) and let \( y \in M \) be a lift of \( x \). Then \( j'g(y) = 1 \), so \( g(y) \in M' \). Let \( y' \in G \) be a lift of \( g(y) \). Then \( y'^{-1} \cdot y \in \ker(g) \) is a lift of \( x \).

Moreover, \( \ker(f) \) still is an abelian group, and the action of \( G \) on \( M \) induces an action of \( \ker(f) \) on \( M \). It is straightforward to check that this action stabilizes \( \ker(g) \). If \( m, m' \in \ker(g) \) and \( x \in \ker(f) \), such that \( m = xm' \), then \( j(m) = j(xm') \). Conversely, if \( j(m) = j(m') \), then there exists \( x \in G \) such that \( m = xm' \). But this means \( 1 = g(m) = g(xm') = f(x)g(m') = f(x), \) so \( x \in \ker(f) \), and \( m \) and \( m' \) are in the same \( \ker(f) \)-orbit. This means that the fibers of \( j_{\ker(g)} \) are precisely the orbits of the \( \ker(f) \)-action on \( \ker(g) \).

This finishes the proof of [a].

For [b], let \( m' \in M' \). By the surjectivity of \( h \) and \( j \), there exists \( m \in M \), such that \( hj(m) = j'(m') \). Then \( g(m) \) and \( m' \) lie in the same \( G' \)-orbit of \( M' \), say \( m' = x'g(m) \) for some \( x' \in G' \). There exists \( x \in G \) with \( f(x) = x' \), and \( g(xm) = f(x)g(m) = x'g(m) = m' \), so \( g \) is surjective. \( \square \)

2. Stratified bundles and \( R^1\lim \)

We continue to denote by \( k \) an algebraically closed field of positive characteristic \( p \).
Definition 2.1. For an integer $r > 0$, we write $B_r \subseteq \text{GL}_r$ for the smooth affine $k$-group scheme, such that for a $k$-algebra $R$ the group $B_r(R)$ is the group of invertible upper triangular $r \times r$-matrices, and $U_r \subseteq B_r$ for the smooth $k$-subgroup scheme given by the unipotent upper triangular matrices. Note that we have a split short exact sequence of group schemes

$$1 \to U_r \to B_r \to \mathbb{G}_m^r \to 1$$

where the map $B_r \to \mathbb{G}_m^r$ is defined by sending an upper triangular matrix to the associated diagonal matrix.

If $R$ is a $k$-algebra we write $R^{p^n}$ for the subring of $p^n$-th powers in $R$, i.e. for the image of the map $R \to R$. If is $G$ a group scheme, we write $\{G(R^{p^n})\}_n$ for the projective system with transition maps the homomorphisms $G(R^{p^{n+1}}) \to G(R^{p^n})$ induced by the inclusion $R^{p^{n+1}} \to R^{p^n}$. To simplify notation, we write $R^1 \lim_n G(R^{p^n}) := R^1 \lim_n \{G(R^{p^n})\}_n$.

Proposition 2.2. Let $R = k[[t^{\pm 1}]]$, $k((t))$ or $k((t^{-1}))$.

a) The pointed set of isomorphism classes of stratified bundles of rank $r$ on $R$ can be identified with

$$R^1 \lim_n \text{GL}_r(R^{p^n}).$$

b) The pointed set of isomorphism classes of composition series of stratified bundles

$$0 \subseteq E_1 \subseteq E_2 \subseteq \ldots \subseteq E_r$$

with rank $E_i/E_{i-1} = 1$ (resp. with $E_i/E_{i-1} \cong R$ as stratified bundles) can be identified with

$$R^1 \lim_n B_r(R^{p^n}) \quad \text{resp.} \quad R^1 \lim_n U_r(R^{p^n}).$$

c) The functorial long exact sequence obtained by applying $\lim$ to the split short exact sequence of pointed sets

$$1 \to \{U_r(R^{p^n})\}_n \to \{B_r(R^{p^n})\}_n \to \{\mathbb{G}_m^r(R^{p^n})\}_n \to 1$$

induces a short exact sequence of pointed sets

$$1 \to R^1 \lim_n U_r(R^{p^n}) \to R^1 \lim_n B_r(R^{p^n}) \to R^1 \lim_n \mathbb{G}_m^r(R^{p^n}) \to 1$$

with $R^1 \lim_n U_r(R^{p^n}) \to R^1 \lim_n B_r(R^{p^n})$ injective. The canonical section of the projection $B_r \to \mathbb{G}_m^r$ induces a section $s_R$ of the sequence (9).

d) The splitting $s_R$ identifies $R^1 \lim \mathbb{G}_m^r(R^{p^n})$ with the set of isomorphism classes of composition series (9) such that $E \cong \bigoplus_{i=1}^r E_{i+1}/E_i$.

e) The above identifications and the section $s_R$ are functorial with respect to the inclusions $k[[t^{\pm 1}]] \subseteq k((t))$ and $k[[t^{\pm 1}]] \subseteq k((t^{-1}))$.

Proof. a) A stratified bundle $E$ on $R$ can be described as a sequence of pairs

$$(E^{(n)}, \sigma_n)_{n \geq 0}$$
where $E^{(n)}$ is a locally free finite rank module on $\mathbb{P}^n$ and $\sigma_n$ an $\mathbb{P}^n$-linear isomorphism

$$\sigma_n : \mathbb{P}^n \otimes \mathbb{P}^{n+1} E^{(n+1)} \xrightarrow{\cong} E^{(n)},$$

see [Gie75, Thm. 1.3]. If $E' := (E'^{(n)}, \sigma'_n)_{n \geq 0}$ is a second stratified bundle, then a morphism of stratified bundles $E \to E'$ is a sequence of $\mathbb{P}^n$-linear morphisms $\psi_n : E^{(n)} \to E'^{(n)}$, such that the diagrams

$$\begin{array}{ccc}
\mathbb{P}^n \otimes \mathbb{P}^{n+1} E^{(n+1)} & \xrightarrow{\sigma_n} & E^{(n)} \\
1 \otimes \psi_{n+1} & \downarrow & \psi_n \\
\mathbb{P}^n \otimes \mathbb{P}^{n+1} E'^{(n+1)} & \xrightarrow{\sigma'_n} & E'^{(n)}
\end{array}$$

(10)

commute. Since in our case all vector bundles on $\mathbb{P}$ are free, we may choose a basis for each $E^{(n)}$, so the $\sigma_n$ give rise to an element $(\sigma_n) \in \prod_{n \geq 0} \text{GL}_r(\mathbb{P}^n)$, if $r = \text{rank } E^{(n)}$. If $E'$ is a second stratified bundle, and $\psi = (\psi_n) : E \to E'$ an isomorphism, then we see from (10) that $\sigma_n = \psi_n^{-1} \sigma'_n \psi_{n+1}$. This shows that $[(\sigma_n)] = [(\sigma'_n)] \in R^1 \lim_n \text{GL}_r(\mathbb{P}^n)$, so the class $[(\sigma_n)]$ is independent of the choices and $R^1 \lim_n \text{GL}_r(\mathbb{P}^n)$ is the pointed set of isomorphism classes of rank $r$ stratified bundles on $\mathbb{P}$.

b) Let $E$ and $E'$ be stratified bundles on $\mathbb{P}$, with composition series

$$0 \subseteq E_1 \subseteq E_2 \subseteq \cdots \subseteq E_r = E, \quad 0 \subseteq E'_1 \subseteq E'_2 \subseteq \cdots \subseteq E'_r = E'$$

such that $\text{rank } E_{i+1}/E_i = \text{rank } E'_{i+1}/E'_i = 1$ (resp. such that $E_{i+1}/E_i \cong E'_{i+1}/E'_i$ is the trivial stratified bundle of rank 1). Fixing bases of $E^{(n)}, E'^{(n)}$, adapted to the composition series, the same reasoning as in [n] shows that an isomorphism of the composition series corresponds to a sequence of elements $\psi_n \in B_r(\mathbb{P}^n)$ (resp. $\psi_n \in U_r(\mathbb{P}^n)$).

The claim follows as before.

c) Since $\lim_n B_r(\mathbb{P}^n) = B_r(k)$ and $\lim_n \mathcal{G}_m(\mathbb{P}^n) = \mathcal{G}_m(k)$, we obtain the short exact sequence (9). The map $R^1 \lim_n U_r(\mathbb{P}^n) \to R^1 \lim_n B_r(\mathbb{P}^n)$ is injective by Lemma 1.3 [b]. The section $s_R$ is induced by the functoriality of $R^1 \lim$.

d) This is clear: The composition series corresponding to the image of $s_R$ are split.

e) This is also clear due to the functoriality of $R^1 \lim$.

□

Next, we briefly recall the notion of regular singularity and the classification of stratified line bundles on $\mathcal{G}_m$ and $k((t))$ based on results of [MvdP03].

**Definition 2.3.**

a) Let $E$ be a stratified bundle on $\text{Spec } k((t))$. Following [Gie75, Sec. 3], we say that $E$ is regular singular if there exists a free $k[[t]]$-submodule $\mathcal{E}$ of $E$, such that $\dim E = \text{rank } \mathcal{E}$ and such that $\mathcal{E}$ is stable under the action of the differential operators $\delta^{(n)}_t$, $n \geq 0$, given by $\delta^{(n)}_t(t^n) = \binom{n}{t} t^n$. 

8
Proposition 2.4. As before let \( \mathbb{G}_m = \mathbb{P}_k^1 \setminus \{0, \infty\} \). Fix a coordinate \( t \) such that \( \mathbb{G}_m = \text{Spec}(k[t^{\pm 1}]) \subseteq \text{Spec} k[t] = \mathbb{P}_k^1 \setminus \{0\} \). Then \( E \) is said to be \textit{regular singular} at \( 0 \) if there exists a free \( k[t] \)-submodule \( \mathcal{E} \subseteq E \) such that \( \text{rank} \mathcal{E} = \text{rank} E \) and such that \( \mathcal{E} \) is stable under the action of the operators \( \delta^i_t \), \( n \geq 0 \).

This is easily seen to be equivalent to saying that \( E \otimes_{\mathcal{O}_{\mathbb{G}_m}} k((t)) \) is regular singular in the sense of [5] where we identify \( k((t)) \cong \text{Frac} \mathcal{O}_{\mathbb{G}_m,0} \).

The notion of \textit{regular singularity} at \( \infty \in \mathbb{P}_k^1 \) is defined analogously.

We say that \( E \) is \textit{regular singular}, if it is regular singular both at \( 0 \) and \( \infty \).

If \( R = k[t^{\pm 1}] \) or \( k((t)) \), and \( \alpha \in \mathbb{Z}_p \), then define \( \mathcal{O}_R(\alpha) \) to be the stratification on the free \( R \)-module of rank 1 given by \( \delta_t^i(1) = \binom{\alpha}{n} \).

Here \( \delta_t^i \) is the differential operator acting via

\[
\delta_t^i(t^r) = \binom{r}{n} t^r.
\]

The following proposition is essentially contained in [MvdP03, 4.2].

**Proposition 2.4.** As before let \( k \) be an algebraically closed field, \( R = k[t^{\pm 1}] \) or \( k((t)) \).

a) If \( \alpha \in \mathbb{Z}_p \), then \( \mathcal{O}_R(\alpha) \) is isomorphic to the trivial stratified bundle if and only if \( \alpha \in \mathbb{Z} \).

b) For \( \alpha, \beta \in \mathbb{Z}_p \) we have

\[
\text{hom}_{\text{Strat}(R)}(\mathcal{O}_R(\alpha), \mathcal{O}_R(\beta)) = \{0\} \cup \text{Isom}_{\text{Strat}(R)}(\mathcal{O}_R(\alpha), \mathcal{O}_R(\beta)) = \begin{cases} k & \text{if } \alpha - \beta \in \mathbb{Z} \\ 0 & \text{else.} \end{cases}
\]

c) \( \mathcal{O}_{k((t))}(\alpha) \) is regular singular and \( \mathcal{O}_{k[t^{\pm 1}])(\alpha) \) is regular singular at \( 0 \) and \( \infty \).

d) The pointed set of isomorphism classes of stratified bundles of rank 1 is an abelian group with multiplication given by the tensor product. It is isomorphic to the abelian group

\[
\mathbb{G}_m(R^{p^n}) \cong \mathbb{Z}_p/\mathbb{Z}.
\]

The class of \( \mathcal{O}_R(\alpha) \) is represented by \( (t^{\alpha n}p^n)_n \in \prod_n \mathbb{G}_m(R^{p^n}) \), if \( \alpha = \sum_{n \geq 0} \alpha np^n \), \( \alpha_n \in [0, p) \).

In particular, every rank 1 stratified bundle on \( R \) is isomorphic to \( \mathcal{O}_R(\alpha) \) for some \( \alpha \).

e) The inclusion \( k[t^{\pm 1}] \to k((t)) \) induces a commutative diagram

\[
\begin{array}{ccc}
\mathbb{G}_m(k[t^{\pm 1}]p^n) & \to & \mathbb{G}_m(k((t))p^n) \\
\downarrow \quad & & \downarrow \\
\mathbb{Z}_p/\mathbb{Z} & \xrightarrow{\text{id}} & \mathbb{Z}_p/\mathbb{Z}.
\end{array}
\]

The inclusion \( k[t^{\pm 1}] \to k((t^{-1})) \) induces \( -\text{id} \) on \( \mathbb{Z}_p/\mathbb{Z} \).
f) The inclusion $k[t^{±1}] \rightarrow k((t))$, induces an equivalence on the full subcategories of rank 1 stratified bundles

$$\text{Strat}^{\text{rank} 1}(G_m) \cong \text{Strat}^{\text{rank} 1}(k((t))).$$

**Proof.**

a) Let $\alpha \in \mathbb{Z}_p$ and consider $\mathcal{O}_R(\alpha)$. If $\alpha \in \mathbb{Z}$, then it is not difficult to check that $\delta_t^{(n)}(t^{-\alpha} \cdot 1) = 0$ for every $n \geq 1$. It follows that $\mathcal{O}_R(\alpha)$ contains a trivial subobject, and hence is trivial. Conversely, if $\mathcal{O}_R(\alpha) \cong 1_R$, then $\mathcal{O}_R(\alpha)$ contains a horizontal element, i.e. there exists $f \in R$, such that

$$0 = \delta_t^{(n)}(f \cdot 1) = \sum_{a+b=n, a, b \geq 0} \delta_t^{(n)}(f) \binom{a}{b}$$

If $m$ denotes the pole order of $f$, then it follows that for every $n \geq 0$

$$\binom{\alpha - m}{n} = \sum_{a+b=n, a, b \geq 0} \binom{-m}{a} \binom{\alpha}{b} = 0$$

which means that $\alpha = m$.

b) Clearly, any nonzero morphism between rank 1 stratified bundles must be an isomorphism, as every such morphism is locally split. Since

$$\text{Hom}_{\text{Strat}(R)}(\mathcal{O}_R(\alpha), \mathcal{O}_R(\beta)) = \text{Hom}_{\text{Strat}(R)}(\mathcal{O}_R, \mathcal{O}_R(\beta - \alpha))$$

the claim follows from part a).

c) This is clear by definition: If $1 \in \mathcal{O}_{k[t^{±1}]}(\alpha)$ is a basis element such that $\delta_t^{(n)}(1) = \binom{\alpha}{n}$, then $1 \cdot k[t]$ is a $\delta_t^{(n)}$-stable lattice at 0, and $1 \cdot k[[t^{-1}]]$ is a $\delta_t^{(-1)}$-stable lattice at $\infty$. More generally, see [Gie75, Lemma 3.12].

d) By Proposition 2.2, the pointed set of stratified bundles of rank 1 can be identified with $R^1 \lim_{\longrightarrow} G_m(\mathcal{O}_{R^n})$. Since $G_m$ is abelian, this pointed set carries the structure of an abelian group.

For $\alpha := \sum_{n \geq 0} \alpha_n p^n$, $\alpha_n \in [0, p] \cap \mathbb{Z}$, the class of the rank 1 stratified bundle $L := \mathcal{O}_R(\alpha)$ in $R^1 \lim_{\longrightarrow} G_m(\mathcal{O}_{R^n})$ is the class $[(t^{\alpha_n}p^n)]$. Indeed, if we identify $L$ with a sequence $(L^{(n)}, \sigma_n)$ as in the proof of Proposition 2.2, then $L^{(n)}$ is given by

$$L^{(n)} = \{ x \in L | \delta_t^{(p^m)}(x) = 0, 0 \leq m < n \},$$

so if $e$ is a basis of $L^{(0)}$, then $t^{\alpha_0-\alpha_1 p^{-1} - \cdots - \alpha_n p^{-n}} e$ is a basis for $L_{n+1}$, and the isomorphism $\sigma_n : \mathcal{O}_{R^n} \otimes_{\mathcal{O}_{R^{n+1}}} L^{(n+1)} \cong L^{(n)}$ is multiplication by $t^{\alpha_n p^n}$.

On the other hand, every $f \in (\mathcal{O}_{R^n})^\times$ can be uniquely written as

$$t^{\alpha_n p^n} \lambda_n u_n,$$

where $\alpha_n \in \mathbb{Z}$, $\lambda_n \in k^\times$ and $u_n \in \left(1 + t^{p^n} k[t^{p^n}]\right)$ if $R = k((t))$, or $u_n = 1$ if $R = k[t^{±1}]$. This induces a surjective homomorphism $G_m(\mathcal{O}_{R^n})/k^\times \rightarrow p^n \mathbb{Z}$, $f \mapsto p^n \lambda_n$, which is injective if $R = k[t^{±1}]$, and
has kernel $1 + tk[[t]]$ if $R = k((t))$. We get a morphism of short exact sequences of projective systems

$$
1 \to \{ \mathbb{G}_m(R^p)/k^x \}_n \to \mathbb{G}_m(R)/k^x \to \{ \mathbb{G}_m(R)/\mathbb{G}_m(R^p) \}_n \to 1
$$

$$
0 \to \{ p^n\mathbb{Z} \}_n \to \mathbb{Z} \to \{ \mathbb{Z}/p^n \}_n \to 0
$$

where the middle terms are constant projective systems. Accordingly, we get a morphism of short exact sequences

$$
1 \to \mathbb{G}_m(R)/k^x \to \lim_{\leftarrow n} \mathbb{G}_m(R)/\mathbb{G}_m(R^p) \to \mathbb{R}_1 \lim_{\leftarrow n} \mathbb{G}_m(R^p)/k^x \to \mathbb{1}
$$

$$
0 \to \mathbb{Z} \to \mathbb{Z}_p/\mathbb{Z} \to 0
$$

It is not difficult to see that $\ker(f) = \ker(\hat{f})$: If $R = k[[t]]^{t+1}$, this is clear; if $R = k((t))$, then the multiplicative group $\ker(f) = 1 + tk[[t]]$ is already $p$-adically complete. Moreover, the projective system $(1 + tk[[t]])/(1 + tp^n k[[t^n]])$ has surjective transition morphisms, so $\hat{f}$ is surjective. We conclude that the induced map

$$
\mathbb{R}_1 \lim_{\leftarrow n} \mathbb{G}_m(R^p) \cong \mathbb{R}_1 \lim_{\leftarrow n} \mathbb{G}_m(R^p)/k^x \to \mathbb{Z}_p/\mathbb{Z}
$$

is an isomorphism of abelian groups. In particular, every stratified line bundle is isomorphic to $\mathcal{O}_R(\alpha)$ for some $\alpha \in \mathbb{Z}_p$.

Since $\mathcal{O}_R(\alpha) \otimes \mathcal{O}_R(\beta) \cong \mathcal{O}_R(\alpha + \beta)$, we see that the group structure of $\mathbb{R}_1 \lim_{\leftarrow n} \mathbb{G}_m(R^p)$ agrees with the group structure defined by the tensor product on the set of isomorphism classes of stratified line bundles.

e) This was already contained in the proof of [d].

f) This follows from [e] and [b].

\[ \square \]

3. Special stratified bundles

As before, $k$ is an algebraically closed field of characteristic $p > 0$, and $\mathbb{G}_m := \mathbb{P}^1_k \setminus \{0, \infty\}$.

**Definition 3.1.** We denote by $\text{Strat}_{\text{special, } \infty}^{\text{special, } 0}(\mathbb{G}_m)$ (resp. $\text{Strat}_{\text{special, } 0}^{\text{special, } 0}(\mathbb{G}_m)$) the full subcategory of $\text{Strat}(\mathbb{G}_m)$ with objects the stratified bundles $E$ which are regular singular at $\infty$ (resp. 0), and which admit a filtration $(E_i)_{i \geq 0}$, such that $E_{i+1}/E_i$ has rank 1. Note that $E_{i+1}/E_i \cong \mathcal{O}_{\mathbb{G}_m}(\alpha)$ for some $\alpha \in \mathbb{Z}_p$ according to Proposition 2.3. In particular, $E_{i+1}/E_i$ is regular singular. For the notion of regular singularity, see Definition 2.3 or, in a more general context, Ginsburg [Gir75, §3.] and [Kin12].

Similarly, write $\text{Strat}_{\text{unip, } \infty}^{\text{unip, } 0}(\mathbb{G}_m)$ (resp. $\text{Strat}_{\text{unip, } 0}^{\text{unip, } 0}(\mathbb{G}_m)$) for the full subcategory of $\text{Strat}(\mathbb{G}_m)$ with objects unipotent stratified bundles which are regular singular at $\infty$ (resp. 0). Here a stratified bundle $E$ is called unipotent if it admits a filtration $(E_i)_{i \geq 0}$, such that $E_{i+1}/E_i \cong \mathcal{O}_{\mathbb{G}_m}$ as stratified bundles.
The category \( \text{Strat}^{\text{special}}(\mathbb{G}_m) \) is a direct generalization of the category of special coverings of \([\text{Kat}86, 1.3]\): Let \( f : Y \to \mathbb{G}_m \) be a finite étale covering, and for simplicity assume that \( f \) is Galois with Galois group \( G \). The covering \( f \) is called special, if it is tame at 0, and if the finite group \( G \) contains a unique (hence normal) \( p \)-Sylow subgroup \( P \). The quotient \( G/P \) corresponds to a tame Galois covering of \( \mathbb{G}_m \) and hence is cyclic of order prime to \( p \). If \( f \) is not Galois, then by definition \( f \) is special if it is the disjoint union of connected coverings, whose Galois closures are special. The push-forward \( f_* \mathcal{O}_Y \) is naturally a stratified bundle, and if \( f \) is connected, its monodromy group is the finite constant \( k \)-group scheme associated with the Galois group of its Galois closure ([SR72, Ch. VI, 1.2.4.1]).

**Proposition 3.2.** In the notations from the previous paragraph, the covering \( f \) is special if and only if \( f_* \mathcal{O}_Y \in \text{Strat}^{\text{special}}(\mathbb{G}_m) \).

**Proof.** Without loss of generality, we may assume \( f \) to be Galois. By [Kin12, Thm. 6.1], \( f \) is tame at 0 if and only if the stratified bundle \( f_* \mathcal{O}_Y \) is regular singular at 0. The condition on \( G \) is equivalent to the condition that every irreducible representation of \( G \) on finite dimensional \( k \)-vector spaces has rank 1. Indeed, if \( f \) is special and \( V \neq 0 \) a representation of \( G \), then \( V^P \neq 0 \), since \( P \) is a \( p \)-group, and \( V^P \) is a \( G \)-representation as \( P \) is normal in \( G \). If \( V \) is irreducible, then \( V^P = V \), so \( V \) comes from a representation of the cyclic group \( G/P \); the irreducible representations of \( G/P \) are all of rank 1.

Conversely, if the irreducible representations of \( G \) all have rank 1, then \( G \) can be realized as a closed subgroup of the group of upper triangular matrices of some rank. Its unique \( p \)-Sylow subgroup is the subgroup of unipotent matrices. Thus \( f \) is special in the sense of [Kat86] if and only if \( f_* \mathcal{O}_Y \in \text{Strat}^{\text{special}}(\mathbb{G}_m) \). \( \square \)

**Remark 3.3.**

a) The categories \( \text{Strat}^{\text{special}, \infty}(\mathbb{G}_m) \) and \( \text{Strat}^{\text{special}, 0}(\mathbb{G}_m) \) are strictly full subtannakian categories of \( \text{Strat}(\mathbb{G}_m) \). They are not stable under taking extensions, as the property of being regular singular at \( \infty \) (resp. 0) is not stable under taking extensions. For example, let \( f : \mathbb{G}_m \to \mathbb{G}_m \) be the Artin-Schreier covering given by \( u^p - u = \ell^{-1} \). The rank 2 subbundle of \( f_* \mathcal{O}_{\mathbb{G}_m} \) spanned by 1 and \( \ell \) is a stratified subbundle which is regular singular at \( \infty \) and not regular singular at 0. It is easily seen to be an extension of two trivial stratified bundles of rank 1.

b) A stratified bundle \( E \) on \( \mathbb{G}_m \) which lies in both \( \text{Strat}^{\text{unip}, 0}(\mathbb{G}_m) \) and \( \text{Strat}^{\text{unip}, \infty}(\mathbb{G}_m) \) is trivial. Indeed, such an \( E \) is regular singular with exponents \( 0 \in \mathbb{Z}_p/\mathbb{Z} \), so by [Kin12, Cor. 5.4] it extends to a stratified bundle on \( \mathbb{P}_k \) and hence is trivial ([Gie75, Thm. 2.2]).

c) The condition that a special stratified bundle admits a filtration with graded pieces of rank 1 is forced on us, as this property holds for all stratified bundles on \( k((t)) \) by [MvdP03, Prop. 6.3].

This is a striking difference to the situation in characteristic 0. The category \( \text{Strat}(k((t))) \) is at the same time simpler and more complicated than the category \( \text{DM}(\mathbb{C}((t))) \) of differential modules on \( \mathbb{C}((t)) \): On the one hand, it is not true that every irreducible differential module on \( \mathbb{C}((t)) \) is of rank 1, but on the other hand, if \( L_1, L_2 \) are...
differential modules of rank 1, then \( \text{Ext}^1_{\text{DM}(\mathbb{C}(t))}(L_1, L_2) = 0 \) unless \( L_1 \cong L_2 \). The nontrivial irreducible objects of \( \text{Strat}(k((t))) \) have rank 1, but if \( L_1, L_2 \) are such objects, then the group \( \text{Ext}^1_{\text{Strat}(k((t)))}(L_1, L_2) \) can be nonzero even if \( L_1 \neq L_2 \).

4. Proof of Theorem [A]

We first show that the restriction functor

\[
\text{res} : \text{Strat}^{\text{special, } \infty}(\mathbb{G}_m) \rightarrow \text{Strat}(k((t))), E \mapsto E|_{k((t))}
\]

is fully faithful.

**Proposition 4.1.** The functor (11) is fully faithful.

**Proof.** \( \text{res} \) is fully faithful by [DM82, Prop. 1.19].

To see that \( \text{res} \) is full, it suffices to show that for every stratified bundle \( E \in \text{Strat}^{\text{special, } \infty}(\mathbb{G}_m) \), every morphism \( 1_{k((t))} \rightarrow E|_{k((t))} \) lifts to a morphism \( 1_{\mathbb{G}_m} \rightarrow E \). We induct on the rank of \( E \). If \( E \) has rank 1, then we can invoke Proposition 2.4. Assume that we have proved the statement for all bundles of rank \( r = \text{rank} E \), and fix a morphism \( \varphi : 1_{k((t))} \rightarrow E|_{k((t))} \). By assumption, \( E \) has a subobject \( E' \) of rank 1. If \( \varphi \) factors through \( E'|_{k((t))} \), then we are done by induction. If not, then the composition \( 1_{k((t))} \xrightarrow{\varphi} E|_{k((t))} \rightarrow (E/E')|_{k((t))} \) is nonzero, and by induction it can be lifted to a nonzero morphism \( 1_{\mathbb{G}_m} \rightarrow E/E' \). Let \( e_1, \ldots, e_r \) be a basis of \( E \) such that \( \delta_i^{(p^n)}(e_i) \in \langle e_1, \ldots, e_r \rangle \) for every \( i, n \geq 0 \), and such that \( E' = \langle e_r \rangle = \mathcal{O}_{\mathbb{G}_m}(\alpha) \).

Then

\[
\varphi(1) = g_1 e_1 + \ldots + g_r e_r \in E|_{k((t))}
\]

is horizontal with \( g_1, \ldots, g_r \in k((t)) \), and we have shown that \( g_1, \ldots, g_{r-1} \in k[t^{\pm 1}] \). It remains to prove that \( g_r \in k[t^{\pm 1}] \). For \( n \geq 0 \) we compute

\[
0 = \delta_i^{(p^n)} \left( \sum_{i=1}^{r-1} g_i e_i \right) = \sum_{i=1}^{r-1} \delta_i^{(p^n)}(g_i e_i) + \sum_{\alpha+b=p^n, a \geq 0} \delta_i^{(\alpha)}(g_r) \left( \frac{\alpha}{b} \right) e_r.
\]

From this we see that for every \( n \geq 0 \)

\[
\sum_{\alpha+b=p^n, a \geq 0} \delta_i^{(\alpha)}(g_r) \left( \frac{\alpha}{b} \right) \in k[t^{\pm 1}],
\]

because \( g_i \in k[t^{\pm 1}] \) for \( i < r \), and \( \delta_i^{(\alpha)}(e_i) \in E \subseteq E|_{k((t))} \). This means we can define

\[
d(n) := \deg \left( \sum_{\alpha+b=p^n, a \geq 0} \delta_i^{(\alpha)}(g_r) \left( \frac{\alpha}{b} \right) \right).
\]

Since \( E \) is regular singular at \( \infty \), i.e. with respect to \( k[t^{\pm 1}] \subseteq k((t^{-1})) \), \( d(n) \) is bounded from above, say by \( M \in \mathbb{N} \). Writing \( g_r = \sum_{i=-N}^{\infty} g_{ir} t^i \), \( g_{ir} \in k \), for all \( n \geq 0 \) we get

\[
\sum_{\alpha+b=p^n, a \geq 0} \delta_i^{(\alpha)}(g_r) \left( \frac{\alpha}{b} \right) = \sum_{i=-N}^{M} g_{ir} \left( \frac{\alpha + i}{p^n} \right) t^i.
\]
In particular, if \( i > M \) and \( g_{ir} \neq 0 \), then \( (\alpha+i)^n = 0 \) for all \( n \geq 0 \), so \( \alpha = -i \).
This shows that \( g_{ir} = 0 \) for \( i \gg 0 \), so \( g_r \) is a Laurent polynomial. \( \square \)

To prove that the restriction functor is essentially surjective, we first describe the pointed set of isomorphism classes of composition series of length \( r \) in Strat\textsuperscript{special,\( \infty \)}(\( \mathbb{G}_m \)) using \( R^1 \lim \).

**Proposition 4.2.** The inclusion \( k[t^{\pm 1}] \hookrightarrow k((t^{-1})) \) induces a commutative diagram

\[
\begin{array}{ccc}
R^1 \lim_n B_r(k((t^{-1}))^{p^n}) & \xrightarrow{\varphi} & R^1 \lim_n \mathbb{G}^r_m(k((t^{-1}))^{p^n}) \\
\downarrow s_{k(t^{-1})} & & \downarrow s_{k[t^{\pm 1}]}
\end{array}
\]

The set of isomorphism classes of composition series of length \( r \geq 1 \) in Strat\textsuperscript{special,\( \infty \)}(\( \mathbb{G}_m \)) is given by

\[ \varphi^{-1}(\text{im}(s_{k(t^{-1}))})) \]

and writing

\[ K_r := \ker \left( R^1 \lim_n U_r(k[t^{\pm 1}]^{p^n}) \to R^1 \lim_n U_r(k((t^{-1}))^{p^n}) \right) \]

the short exact sequence

\[ 1 \to R^1 \lim_n U_r(k[t^{\pm 1}]^{p^n}) \to R^1 \lim_n B_r(k[t^{\pm 1}]^{p^n}) \to R^1 \lim_n \mathbb{G}^r_m(k[t^{\pm 1}]^{p^n}) \to 1 \]

from Proposition 2.2 restricts to a short exact sequence of pointed sets

\[ 1 \to K_r \to \varphi^{-1}(\text{im}(s_{k(t^{-1}))})) \to R^1 \lim_n \mathbb{G}^r_m(k[t^{\pm 1}]^{p^n}) \to 1 \quad (14) \]

with section \( s_{k[t^{\pm 1}]} \).

**Proof.** This follows almost entirely from Proposition 2.2. The description of the preimage \( \varphi^{-1}(\text{im}(s_{k(t^{-1}))})) \) follows from the fact that a stratified bundle \( E \) on \( \mathbb{G}_m \) is regular singular at \( \infty \) if and only if \( E \otimes k((t^{-1})) \) is a direct sum of rank 1 objects, see [Gie75, Thm. 3.3] or [MvdP03, Prop. 6.1], which by Proposition 2.2(d) means that its class maps to \( \text{im}(s_{k(t^{-1}))}) \). For the sequence (14), note that \( K_r = R^1 \lim_n U_r(k[t^{\pm 1}]^{p^n}) \cap \varphi^{-1}(\text{im}(s_{k(t^{-1}))})) \). \( \square \)

Theorem A now follows from the following proposition.

**Proposition 4.3.** With the notation from Proposition 4.2, the inclusion \( k[t^{\pm 1}] \hookrightarrow k((t)) \) induces a surjection of pointed sets

\[ \varphi : \varphi^{-1}(\text{im}(s_{k((t^{-1}))})) \to R^1 \lim_n B_r(k((t))^{p^n}). \quad (15) \]

In particular, the restriction functor Strat\textsuperscript{special,\( \infty \)}(\( \mathbb{G}_m \)) \( \to \text{Strat}(k((t))) \) is essentially surjective.

Note that (15) is a priori injective, according to Proposition 4.4.
An obvious approach would be to use \( \text{(14)} \) together with some variant of a “5-lemma” to reduce to the unipotent case: According to Proposition \( \text{2.2} \), the inclusion \( k[t^{\pm 1}] \subseteq k((t)) \) gives rise to a morphism of short exact sequences of pointed sets

\[
\begin{array}{ccc}
1 & \longrightarrow & K_r \\
\downarrow & & \downarrow \varphi^{-1}(\text{im } s_{k((t^{-1}))}) \\
1 & \longrightarrow & \text{R}^1 \lim_{\leftarrow n} G^r_n(k[t^{\pm 1}]^p) \\
\end{array}
\]

compatible with the sections \( s_{k[t^{\pm 1}]} \) and \( s_{k((t))} \). The surjectivity of the left vertical arrow is the statement corresponding to Proposition \( \text{4.3} \) for unipotent bundles. The vertical arrow on the right is a bijection by Proposition \( \text{2.4} \). Now if we had some form of 5-lemma applicable to this situation, the proof of Proposition \( \text{4.3} \) would be reduced to the unipotent case. Unfortunately, I do not know of such a 5-lemma. The problem is that if \( c \in \text{R}^1 \lim_{\leftarrow n} G^r_n(k[t^{\pm 1}]^p) \cong (\mathbb{Z}/p\mathbb{Z})^r \), then a priori the fiber of \( \varphi^{-1}(\text{im } s_{k((t^{-1}))}) \) over \( c \) is different from the fiber over 1, and since we do not have a uniform structure, there does not seem to be a way to relate these fibers.

Instead we have to take a more involved path. Fix \( c \in (\mathbb{Z}/p\mathbb{Z})^r \cong \text{R}^1 \lim_{\leftarrow n} G^r_n(k[t^{\pm 1}]^p) \) and let \( \gamma := (\gamma_n)_n \in \prod_n G^r_n(k[t^{\pm 1}]^p) \) be a representative of \( c \). We “twist” the projective systems by \( \gamma \): Let \( R = k[t^{\pm 1}] \) or \( k((t)) \) and let \( \{B_r(R^p)\}^\gamma \) denote the projective system with groups \( B_r(R^p) \) and transition morphisms

\[ u_{n,\gamma} : B_r(R^{p_{n+1}}) \to B_r(R^p), \quad u_{n,\gamma}(A_{n+1}) = s_R(\gamma_n^{-1})A_{n+1}s_R(\gamma_n), \]

where \( s_R : G^r_n(R) \to B_r(R^p) \) is the section of \( B_r(R) \to G^r_n(R) \), attaching to an ordered \( r \)-tuple of units the corresponding diagonal matrix. Write \( \{U_r(R^p)\}^\gamma_n \) for the induced projective system. We get a short exact sequence of projective systems

\[
1 \to \{U_r(R^p)\}^\gamma_n \to \{B_r(R^p)\}^\gamma_n \to \{G^r_n(R^p)\}_n \to 1,
\]

where the twist on the right hand term is trivial, as \( G^r_n(R^p) \) is abelian. We write \( \lim^\gamma \) and \( \text{R}^1 \lim^\gamma \) for the limit and derived limit of the twisted systems. Note that the section \( s_R \) induces a section of \( \{B_r(R^p)\}^\gamma_n \to \{G^r_n(R^p)\}_n \), so we obtain a short exact sequence of pointed sets

\[
1 \to \text{R}^1 \lim_{\leftarrow n} U_r(R^p) \to \text{R}^1 \lim_{\leftarrow n} B_r(R^p) \to \text{R}^1 \lim_{\leftarrow n} G^r_n(R^p) \to 1
\]

by Lemma \( \text{1.3} \). Moreover, there is a canonical bijection

\[ \text{R}^1 \lim_{\leftarrow n} B_r(R^p) \cong \text{R}^1 \lim_{\leftarrow n} B_r(R^p) \]

mapping a class \([[(b_n)]\] to \([[(b_n)s_R(\gamma_n))]\). It is straight-forward to check that this is well-defined. We get a commutative diagram with bijective vertical
arrows and exact rows:

\[
1 \rightarrow R^1 \lim_n U_r(R^{p^n}) \hookrightarrow R^1 \lim_n B_r(R^{p^n}) \twoheadrightarrow R^1 \lim_n G_r^\gamma(R^{p^n}) \rightarrow 1
\]

This way we can identify the fiber of \( R^1 \lim_n B_r(R^{p^n}) \rightarrow R^1 \lim_n G_r^\gamma(R^{p^n}) \) over \( c^{-1} \) with \( R^1 \lim_n \gamma U_r(R^{p^n}) \).

Coming back to the sequence (14), we see that we can identify the fiber of

\[
\varphi^{-1}(\im s_k((t^{-1})^{1})) \rightarrow R^1 \lim_n G_r^\gamma(k[t^{\pm 1}]^{p^n})
\]

over \( c^{-1} \) with

\[
K_r^\gamma := \ker \left( R^1 \lim_n \gamma U_r(k[t^{\pm 1}]^{p^n}) \rightarrow R^1 \lim_n \gamma U_r(k((t^{-1})^{1})^{p^n}) \right)
\]

and our objective is to prove the following lemma.

**Lemma 4.4.** The inclusion \( k[t^{\pm 1}] \rightarrow k((t)) \) induces a surjection

\[
K_r^\gamma \twoheadrightarrow R^1 \lim_n \gamma U_r(k((t))^{p^n}),
\]

for all \( c \).

If the lemma holds, the discussion preceding it implies that the map

\[
\varphi^{-1}(\im s_k((t^{-1})^{1})) \rightarrow R^1 \lim_n B_r(R^{p^n})
\]

is surjective.

**Proof.** To show that (17) is surjective we induct on the rank \( r \geq 2 \); the case \( r = 2 \) is Lemma 4.3 below.

Let \( r > 2 \) and assume that we know that (17) is surjective for all ranks \( < r \). For \( i = 1, \ldots, r - 1 \) let \( G_i \subseteq U_r \) be the normal closed subgroup of \( U_r \) given by matrices of the form

\[
\begin{pmatrix}
\id & a_1 \\
& \vdots \\
& a_i \\
& 0 \\
0 & \cdots & 0 & 1
\end{pmatrix}
\]

and \( G_0 = 0 \). Then the groups \( G_i \) are also normal and closed in \( B_r \). Note that \( G_i \cong \mathbb{G}_a^i \), and one easily checks that \( G_i/G_{i-1} \) is a central subgroup of \( U_r/G_{i-1} \). Now let \( R = k[t^{\pm 1}] \) or \( R = k((t)) \). Then

\[
(U_r/G_i)(R) = U_r(R)/G_i(R) \quad \text{and} \quad (G_i/G_{i-1})(R) = G_i(R)/G_{i-1}(R).
\]

Write \( \{ G_i(R^{p^n}) \}^\gamma \) for the induced projective system. Note that this is well defined, as conjugation by a diagonal matrix fixes \( G_i \subseteq B_r \), and also note that we have

\[
\{ G_i(R^{p^n}) \}^\gamma / \{ G_{i-1}(R^{p^n}) \}^\gamma = \{ (G_i/G_{i-1}) (R^{p^n}) \}^\gamma,
\]

(16)
We only have to check that the third arrow is surjective. For everywhere both right hand sides are meaningful, again because $G$ by conjugation with diagonal matrices.

We claim that for every $i < r$, one obtains a short exact sequence

\[
\begin{align*}
1 \to \lim\limits_{\leftarrow n} (G_i/G_{i-1})(R^p) & \to \lim\limits_{\leftarrow n} (U_i/G_{i-1})(R^p) \to \lim\limits_{\leftarrow n} (U_i/G_i)(R^p) \to 1 \\
& \to 1 \to \lim\limits_{\leftarrow n} (G_{r-1}/G_i)(R^p) \to \lim\limits_{\leftarrow n} (U_{r-1}/G_i)(R^p) \to \lim\limits_{\leftarrow n} (U_r/G_i)(R^p) \to 1.
\end{align*}
\]

We only have to check that the third arrow is surjective. For every $i$ the projections $G_{r-1}/G_i \to G_r/G_i$ and $U_r/G_{i-1} \to U_r/G_{r-1} = U_{r-1}$ have a section. It follows that for every $i$ we have a commutative diagram

\[
\begin{align*}
1 \to \lim\limits_{\leftarrow n} (G_{r-1}/G_i)(R^p) & \to \lim\limits_{\leftarrow n} (U_{r-1}/G_i)(R^p) \to \lim\limits_{\leftarrow n} (U_r/G_i)(R^p) \to 1 \\
1 \to \lim\limits_{\leftarrow n} (G_{r-1}/G_i)(R^p) & \to \lim\limits_{\leftarrow n} (U_{r-1}/G_i)(R^p) \to \lim\limits_{\leftarrow n} (U_r/G_i)(R^p) \to 1.
\end{align*}
\]

We see that the middle vertical arrow is surjective, as claimed.

Since $G_i/G_{i-1}$ is central in $U_i/G_{i-1}$, Lemma 1.3 shows that we obtain from (18) a strongly exact (Definition 1.4) short exact sequence of pointed sets

\[
\begin{align*}
1 \to R^1 \lim\limits_{\leftarrow n} (G_i/G_{i-1})(R^p) & \to R^1 \lim\limits_{\leftarrow n} (U_i/G_{i-1})(R^p) \to R^1 \lim\limits_{\leftarrow n} (U_i/G_i)(R^p) \to 1
\end{align*}
\]

If we write

\[
J^\gamma_{i-1} := \ker \left( R^1 \lim\limits_{\leftarrow n} (G_i/G_{i-1})(k[t^1]R^p) \to R^1 \lim\limits_{\leftarrow n} (G_i/G_{i-1})(k((t^{-1})R^p) \right)
\]

\[
K^\gamma_{r,i-1} := \ker \left( R^1 \lim\limits_{\leftarrow n} (U_r/G_{i-1})(k[t^1]R^p) \to R^1 \lim\limits_{\leftarrow n} (U_r/G_{i-1})(k((t^{-1})R^p) \right),
\]

then $K^\gamma_{0} = K^\gamma_{r}$, and $K^\gamma_{r,i-1}$ in the notation from (16). Lemma 1.5 shows that we have strongly exact short exact sequences of pointed sets

\[
\begin{align*}
1 \to J^\gamma_{i-1} & \to K^\gamma_{r,i-1} \to K^\gamma_{r,i} \to 1
\end{align*}
\]

together with morphisms

\[
\begin{align*}
1 \to J^\gamma_{i-1} & \to K^\gamma_{r,i-1} \to K^\gamma_{r,i} \to 1 \\
1 \to R^1 \lim\limits_{\leftarrow n} (G_i/G_{i-1})(k((t)R^p)) & \to R^1 \lim\limits_{\leftarrow n} (U_i/G_{i-1})(k((t)R^p)) \to R^1 \lim\limits_{\leftarrow n} (U_i/G_i)(k((t)R^p) \to 1
\end{align*}
\]

Since $G_i/G_{i-1} = \mathbb{G}_a$, Lemma 1.5 below shows that $f_{i-1}$ is a bijection for $i = 1, \ldots, r$. By induction hypothesis we know that

\[
g_{r-1} : K^\gamma_{r,r-1} = K^\gamma_{r-1} \to R^1 \lim\limits_{\leftarrow n} (U_{r-1}(k((t^p)))
\]

is surjective. Thus, applying Lemma 1.5 repeatedly, it follows that $g_0$ is surjective. But $g_0$ is precisely the morphism from (17), which completes the proof, up to verification of Lemma 1.5 below.

---

3 Here we slightly abuse notation by writing $\lim\limits_{\leftarrow n} U_{r-1}(R^p)$ for the inverse limit of the projective system $\{U_{r-1}(R^p)\}$ twisted by the image of $\gamma$ under the projection $\Pi_n G_m^\gamma(R^p) \to \Pi_n G_m^{r-1}(R^p)$ onto the first $r-1$ factors.
Lemma 4.5. Identify $G_a$ with $U_2$, and consider $G_m^2$ as the group of invertible diagonal rank 2 matrices. Every class $c \in R^1 \lim_{n} G_m^2 (k[t^{±1}]^{p^n})$ has a representative $γ \in \prod_n G_m^2 (k[t^{±1}]^{p^n})$, such that the inclusion $k[t^{±1}] \hookrightarrow k((t))$ induces a bijection

$$\ker \left( R^1 \lim_{n} U_2 (k[t^{±1}]^{p^n}) \to R^1 \lim_{n} U_2 (k((t^{-1}))^{p^n}) \right) \cong R^1 \lim_{n} U_2 (k((t))^{p^n}).$$

Proof. If $c$ has a representative $γ = (γ_1, γ_2) \in \prod_n G_m (k[t^{±1}]^{p^n})$, then $γ_1 = γ_2$, then twisting the projective system $\{ U_2 (k[t^{±1}]^{p^n}) \}_{n}$ by $γ$ has no effect, as matrices of the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \lambda \in R$$

are central in $B_2 (R)$. This is a particular case of the fact that $\text{Ext}^1 (L, L) \cong \text{Ext}^1 (1, 1)$ if $L$ is a rank 1 object in a Tannaka category.

It follows that we may assume that the representative $γ$ of $c$ has the form $γ = (1, γ_2), γ_2 = (γ_2, n) \in \prod_n G_m (k[t^{±1}]^{p^n})$. Moreover by Proposition 2.4 [4] we may assume that $γ_2, n \in G_m (k[t^{±1}]^{p^n})$ has the form $γ_2, n = t^{α_2} p^n$, with $α_2 \in [0, p)$. Write $α = \sum_{i=0}^{p} α_i p^i, α_i \in [0, p-1)$, for the corresponding $p$-adic integer. If $α ∈ Z$, then we may assume $γ_2, n = 1$ for all $n$, Proposition 2.4 [4].

If we identify $G_a$ with $U_2$ via

$$G_m (R) \ni a \mapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in U_2 (R),$$

then the transition morphisms $U_2 (k((t)))^{p^{n+1}} \to U_2 (k((t)))^{p^n}$ in the projective system $\{ U_2 (k((t)))^{p^n} \}$ are $a \mapsto t^{α_2} p^n a$.

To state the main technical ingredient we need the following definition.

Definition 4.6. We consider an element $f$ of the additive group $k^Z$ as a formal series

$$f = \sum_{i ∈ Z} f_i t^i, f_i ∈ k.$$ Define $\text{supp}(f) := \{ i ∈ Z | f_i ≠ 0 \}$. If $U ⊆ Z$ is a subset, then we write $f(U) := \sum_{i ∈ U} f_i t^i$. If $Z = \bigcup_{j ∈ Z} U_j$ with $U_j$ pairwise disjoint subsets, then

$$f = \sum_{j ∈ Z} f(U_j).$$

Now if $R = k[t^{±1}], k((t))$ or $k((t^{-1}))$, we can consider the additive group underlying $R$ as subgroup of $k^Z$, and the notations above apply to elements $f ∈ R$. In particular, the additive group $k((t))$ (resp. $k((t^{-1}))$, resp. $k[t^{±1}]$) is the subgroup of $k^Z$ consisting of formal series $f$ with $\text{supp}(f)$ bounded from below (resp. from above, resp. from above and below).

Lemma 4.7. Let $R = k[t^{±1}], k((t))$ or $k((t^{-1}))$, and $a \in R^1 \lim_{n} (U_2 (R^{p^n}))$, where $γ \in \prod_{n≥0} G_m^2 (R^{p^n})$ is of the form $(1, t^{α_2} p^n)_{n}$, with $α_2 ∈ [0, p)$ and $α := \sum_{n≥0} α_n p^n$ either $= 0$ or $∉ Z$.

a) The class $a$ has a representative $(a_n)_{n} ∈ \prod_{n≥0} G_a (R^{p^n})$, satisfying

$$\text{supp}(a_n) ⊆ p^{n} Z \setminus (α_n p^n + p^{n+1} Z).$$ (19)
b) If \( R = k((t)) \), and if \((a_{n})_{n}\) satisfies (19), then the class of \((a_{n})_{n}\) is trivial, if and only if \( a_{n} \in k[t] \) for \( n > 1 \).

c) If \( R = k([t^{-1}]) \), and if \((a_{n})_{n}\) satisfies (19), then the class of \((a_{n})_{n}\) is trivial, if and only if \( a_{n} \in t^{-1}k[t^{-1}] \) for \( n > 1 \).

d) If \( R = k[t^{\pm 1}] \), and if \((a_{n})_{n}\) satisfies (19), then the following are equivalent:

i. The class of \((a_{n})_{n}\) is trivial.

ii. \((a_{n})_{n}\) is trivial in \( R^{1} \lim_{\longleftarrow n} U_{2}(k((t))p^{n}) \) and \( R^{1} \lim_{\longleftarrow n} U_{2}(k((t^{-1}))p^{n}) \).

iii. \( a_{n} = 0 \) for \( n > 1 \).

Proof.  

a) An element \( f \in R^{n} \) can be uniquely written as sum

\[
f = f(p^n \in (\alpha_{n}p^n + p^{n+1}Z)) + f(\alpha_{n}p^n + p^{n+1}Z).
\]

See Definition 1 for this notation. To increase legibility we define

\[
f' := f(p^n \in (\alpha_{n}p^n + p^{n+1}Z)), \quad f'' := t^{-\alpha_{n}p^n} f(\alpha_{n}p^n + p^{n+1}Z).
\]

Let \((\tilde{a}_{n})_{n} \in \prod_{n \geq 1} G_{n}(R^{p^{n}})\) be a representative of \( a \). Define \( y_{0} := 0 \), and inductively

\[
y_{n+1} := \frac{y_{n}''}{\tilde{a}_{n}''} - \frac{\tilde{a}_{n}''}{\tilde{a}_{n}'} \in R^{p^{n+1}}.
\]

Then we check that

\[
y_{n} + (\alpha_{n}' - y_{n}' - t^{\alpha_{n}p^{n}} y_{n+1}) = (y_{n} - y_{n}') + \alpha_{n}' - t^{\alpha_{n}p^{n}} y_{n+1}
\]

\[
= t^{\alpha_{n}p^{n}} y_{n}' + \alpha_{n}' - t^{\alpha_{n}p^{n}} y_{n}' + t^{\alpha_{n}p^{n}} \tilde{a}_{n}'' = \tilde{a}_{n}^{n}
\]

so \((\tilde{a}_{n})_{n}\) is equivalent to \((a_{n})_{n} := (\alpha_{n}' - y_{n}')_{n}\), and \(\text{supp}(a_{n}) \subseteq p^{n}Z \cup (\alpha_{n}p^{n} + p^{n+1}Z)\).

b) Let \((a_{n})_{n}\) satisfy (19). First assume that \( a_{n} \in k[t] \) for \( n > 0 \). To show that the class of \((a_{n})_{n}\) is trivial, it suffices to show that the sums

\[
y_{m} := a_{m} + \sum_{n > m} a_{n} t^{\alpha_{n}p^{m} + \ldots + \alpha_{n-1}p^{n-1}}, \quad m \geq 0
\]

exist in \( k((t)) \). Indeed, if the sums (20) exist, then \( a_{n} = y_{n} - t^{\alpha_{n}p^{n}} y_{n+1} \), so the class of \((a_{n})_{n}\) is trivial. To see that the sums exist, note that

\[
\text{supp}(a_{n} t^{\alpha_{1} + \ldots + \alpha_{n-1}p^{n-1}}) 
\subseteq (\alpha_{1}p + \ldots + \alpha_{n-1}p^{n-1} + p^{n}Z) \setminus (\alpha_{1}p + \ldots + \alpha_{n}p^{n} + p^{n+1}Z).
\]

This shows that the sum (20) exists in \( k^{Z} \), first with \( m = 0 \), and then for all \( m \geq 0 \). Since \( a_{n} \in k[t] \) for large \( n \), it follows that the sum (20) exists in \( k((t)) \).

Conversely, assume that the class of \((a_{n})_{n}\) is trivial, i.e. that there exist \( y_{n} \in k((t))p^{n} \), such that \( a_{n} = y_{n} - t^{\alpha_{n}p^{n}} y_{n+1} \). There exists \( N > 0 \), such that for all \( n \geq N \),

\[
\text{supp}(y_{n}) \cap (\alpha_{0} + \alpha_{1}p + \ldots + \alpha_{n}p^{n} - Z_{>0}p^{n+1}) = \emptyset,
\]

where \((\alpha_{0} + \alpha_{1}p + \ldots + \alpha_{n}p^{n} - Z_{>0}p^{n+1})\) denotes the set of integers \( m \), such that \( \frac{m - (\alpha_{0} + \ldots + \alpha_{n}p^{n})}{p^{n+1}} \in Z_{<0} \). Indeed, just take \( N \) such that
$\alpha_0 + \alpha_1 p + \ldots + \alpha_N p^N - p^{N+1}$ is smaller than $\text{ord}(y_n)$. Such an $N$ exists, as the sequence $\alpha_1 p + \ldots + \alpha_N p^N - p^{N+1}$ is decreasing and unbounded, because by our assumptions $\alpha = 0$ or $\alpha \notin \mathbb{Z}$ (note that e.g. for $\alpha = -1$ there would be a problem here).

Summing up the equations $a_n = y_n - t^{\alpha_n p^n} y_{n+1}$, we obtain

$$\sum_{i=0}^n a_i t^{\alpha_0 + \ldots + \alpha_i - 1} p^{i-1} = y_0 - t^{\alpha_0 + \ldots + \alpha_n p^n} y_{n+1}.$$ 

By construction

$$\text{supp}\left(\sum_{i=0}^n a_i t^{\alpha_0 + \ldots + \alpha_i - 1} p^{i-1}\right) \cap (\alpha_0 + \ldots + \alpha_n p^n + p^{n+1} \mathbb{Z}) = \emptyset.$$ 

Hence, if $n > N$, then $y_n \in k[[t]]$, and also $a_n = y_n - t^{\alpha_n p^n} y_{n+1} \in k[[t]]$, which is what we wanted to show.

c) The argument is very similar to the one from the previous part. As before, assume that $(a_n)_n \in \prod_n \mathbb{Z}_\alpha(k((t^{-1}))^{p^n})$ satisfies (15). If the class of $(a_n)_n$ is trivial, then there exist $y_n \in k((t^{-1}))^{p^n}$, such that

$$a_n = y_n - t^{\alpha_n p^n} y_{n+1}.$$ 

Summing up we see that

$$\sum_{i=0}^n a_i t^{\alpha_0 + \ldots + \alpha_i - 1} p^{i-1} = y_0 - t^{\alpha_0 + \ldots + \alpha_n p^n} y_{n+1}$$

Again we have

$$\text{supp}\left(\sum_{i=0}^n a_i t^{\alpha_0 + \ldots + \alpha_i - 1} p^{i-1}\right) \cap (\alpha_0 + \ldots + \alpha_n p^n + p^{n+1} \mathbb{Z}) = \emptyset.$$ 

Hence, if $N$ is such that

$$\text{supp}(y_0) \cap (\alpha_0 + \ldots + \alpha_n p^n + \mathbb{Z} \alpha 0^{p^n+1}) = \emptyset$$

for $n \geq N$, then $y_n \in t^{-1} k[[t^{-1}]]$ for $n > N$, which shows that $a_n \in t^{-1} k[[t^{-1}]]$ for $n > N$. Such an $N$ exists by the same reasoning as in (b).

Conversely, if $a_n \in t^{-1} k[[t^{-1}]]$ for $n \gg 0$, then the sums (20) exist by the same argument as in (b) which shows that the class of $(a_n)_n$ is trivial. We remark that in contrast to the case of $k((t))$, for this argument to work, it is important that $a_n$ does not have a constant term for $n \gg 0$, as can be seen, e.g., by considering the case $\alpha = 0$.

d) If $a_n \in k[[t^{\pm 1}]]$ is such that $a_n = 0$ for $n \gg 1$, then the sums (20) exist, so the class of $(a_n)_n$ is trivial in $R^1 \lim_{\rightarrow \nu} U_2(k[[t^{\pm 1}]]^{p^n})$. Hence we have (10) $\Rightarrow$ (b). We clearly also have (b) $\Rightarrow$ (11).

Finally, if the class of $(a_n)_n$ is trivial over both $k((t))$ and $k((t^{-1}))$, then for $n \gg 0$ we know from (b) and (c) that

$$a_n \in k[[t^{\pm 1}]] \cap k[t] \cap t^{-1} k[t^{-1}] = \{0\},$$

so we are done. □
Returning to the proof of Lemma 4.3, let \( a \in R_1 \lim_n^\gamma G_a(k((t))^{p^n}) \) be a class represented by \( (a_n)_n \in \prod_n G_a(k((t))^{p^n}) \) satisfying the support condition \([19]\). Every class \( a \) has such a representative by Lemma 4.7.

For an element \( f \in k((t)) \) (or \( k((t^{-1})) \) or \( k[t^{\pm 1}] \)), we write \( f = f^{>0} + f^{<0} \), where \( \text{supp}(f^{>0}) \subseteq \mathbb{Z}_{>0} \) and \( \text{supp}(f^{<0}) \subseteq \mathbb{Z}_{<0} \).

We see that \( (a_n)_n = (a^{>0}_n)_n + (a^{<0}_n)_n \), and by Lemma 4.7, the class of \( (a^{<0}_n)_n \) in \( R_1 \lim_n G_a(k((t))^{p^n}) \) is trivial. It follows that the map

\[
R_1 \lim_n^\gamma G_a(k[t^{\pm 1}]^{p^n}) \to R_1 \lim_n^\gamma G_a(k((t))^{p^n})
\]

induced by the inclusion \( k[t^{\pm 1}] \to k((t)) \) is surjective.

Moreover, since \( a^{<0}_n \in t^{-1}k[t^{-1}] \), the class of \( (a^{<0}_n)_n \) in \( R_1 \lim_n^\gamma G_a(k[t^{\pm 1}]^{p^n}) \) maps to the trivial class in \( R_1 \lim_n^\gamma G_a(k((t^{-1}))^{p^n}) \), so the map

\[
\ker \left( R_1 \lim_n^\gamma U_2(k[t^{\pm 1}]^{p^n}) \to R_1 \lim_n^\gamma U_2(k((t^{-1}))^{p^n}) \right) \to R_1 \lim_n^\gamma U_2(k((t))^{p^n})
\]

is surjective. By Lemma 4.7(3) it is also injective, so the proof is complete. \( \square \)

5. Applications

In this section we write \( \text{Vecf}_k \) for the category of finite dimensional \( k \)-vector spaces, and if \( G \) is an affine \( k \)-group scheme, we write \( \text{Repf}_k G \) for the category of \( k \)-linear representations of \( G \) on finite dimensional \( k \)-vector spaces.

First we prove Theorem 13.

Proof of Theorem 13 We first show that every object of \( \text{Repf}_k P(\omega) \) is a successive extension of trivial representations of rank 1, i.e. that \( P(\omega) \) is unipotent. If \( : P(\omega) \to \pi_1^{\text{Strat}}(k((t))), \omega \) denotes the inclusion, then every object of \( \text{Repf}_k P(\omega) \) is a subquotient of \( i^*E \) for some \( E \in \text{Strat}(k((t))) \) by [DM82] Prop. 2.21. As noted in Remark 3.3(3) every object \( E \) of \( \text{Strat}(k((t))) \) is a successive extension of rank 1 objects, so it suffices to prove that \( i^*E \) is trivial, if \( \text{rank} E = 1 \). In this case, according to Proposition 2.21, \( E \) is isomorphic to \( O_{k((t))}(\alpha) \) for some \( \alpha \in \mathbb{Q} \) and hence lies in \( \text{Strat}^{rs}(k((t))) \subseteq \text{Strat}(k((t))) \), so \( i^*E \) is trivial. We have proved that \( P(\omega) \) is unipotent.

For the reducedness of \( P(\omega) \) one shows just like in [1807], that the relative Frobenius homomorphism

\[
\pi_1^{\text{Strat}}(k((t))) \to \pi_1^{\text{Strat}}(k((t)))^{(1)}
\]

is an isomorphism. Indeed, if \( F_{/k} \) denotes the relative Frobenius for \( k((t)) \), then \( F_{/k}^* \) is an equivalence

\[
\text{Strat}(k((t))) \to \text{Strat}(k((t)))^{(1)}.
\]

As in loc. cit. this is translated into the fact that \( (21) \) is an isomorphism.

The same is true for \( \pi_1^{rs}(k((t))) \), as \( F_{/k}^* \) restricts to an equivalence

\[
\text{Strat}^{rs}(k((t))) \to \text{Strat}^{rs}(k((t)))^{(1)}.
\]

Thus the relative Frobenius \( P(\omega) \to P(\omega)^{(1)} \) also is an isomorphism which implies that \( P(\omega) \) is reduced. \( \square \)
Next we prove Theorem \(\text{C}\).

**Proof of Theorem C.** It follows from Theorem \(\text{A}\) that the inclusion \(k[t^\pm 1] \subseteq k((t^{-1}))\) induces an equivalence

\[
\text{Strat}^{\text{unip}}(G_m) \xrightarrow{\cong} \text{Strat}^{\text{unip}}(k((t^{-1}))).
\]

On the other hand, the inclusion \(k[t] \subseteq k[t^\pm 1]\) also induces an equivalence

\[
\text{Strat}^{\text{unip}}(\mathbb{A}^1_k) \xrightarrow{\cong} \text{Strat}^{\text{unip}}(G_m).
\]

(22) Indeed, if \(E\) is a stratified bundle on \(G_m\), which is regular singular along 0, then one assigns to it *exponents along 0*, which are elements of \(\mathbb{Z}_p/\mathbb{Z}\), see [Gie75, Sec. 3], [Kin12]. In fact, if \(E\) is regular singular at 0, then \(E \otimes k((t)) \cong \bigotimes_{i=1}^{\text{rank}} E_{k((t))}(\alpha_i)\), with \(\alpha_i \in \mathbb{Z}_p\), and the classes of \(\alpha_i\) in \(\mathbb{Z}_p/\mathbb{Z}\) are the exponents of \(E\) along 0. If \(E\) is unipotent, \(\alpha_i \equiv 0 \mod \mathbb{Z}\). By [Kin12, Cor. 5.4] this implies that \(E\) extends to a stratified bundle \(\overline{E}\) on \(\mathbb{A}^1_k\). The stratified bundle \(\overline{E}\) is unipotent, as the restriction functor induces an equivalence \((\overline{E})_\emptyset \cong \langle E \rangle_\emptyset\) by [Kin12, Lem. 2.5]. This means that (22) is essentially surjective, but by *loc. cit.* it is also fully faithful. The proof is complete. \(\square\)

Before we can prove Theorem \(\text{D}\) we briefly discuss the coproduct of two affine \(k\)-group schemes.

**Definition 5.1** ([Unv10]). Let \(G_1, G_2\) be two affine \(k\)-group schemes. Define the category \(\mathcal{C}(G_1, G_2)\) as follows: Objects are triples \((\rho_1, \rho_2, V)\) with \(V\) a finite dimensional \(k\)-vector space, and \(\rho_i : G_i \to GL(V), i = 1, 2\), representations. A morphism \((\rho_1, \rho_2, V) \to (\rho_1', \rho_2', V')\) is a morphism of \(k\)-vector spaces \(\varphi : V \to V'\), such that for every \(k\)-algebra \(R\), and every \(g_i \in G_i(R)\), the diagram

\[
\begin{array}{ccc}
V \otimes_k R & \xrightarrow{\rho_i(g_i)} & V \otimes_k R \\
\varphi \otimes \text{id} & & \varphi \otimes \text{id} \\
V' \otimes_k R & \xrightarrow{\rho_i'(g_i)} & V' \otimes_k R
\end{array}
\]

commutes for \(i = 1, 2\).

The following facts are probably folklore, but the author was not able to find a reference.

**Proposition 5.2.** Let \(k\) be an algebraically closed field, and \(G_1, G_2\), two affine \(k\)-group schemes.

a) The forgetful functor \(\omega : \mathcal{C}(G_1, G_2) \to \text{Vect}_k\), \(\omega(\rho_1, \rho_2, V) = V\), makes \(\mathcal{C}(G_1, G_2)\) into a neutral tannakian category. We denote the corresponding affine \(k\)-group scheme by \(G_1 \ast_{\text{alg}} G_2\).

b) The forgetful functors \(\tau_i : \mathcal{C}(G_1, G_2) \to \text{Rep}_k G_i\), which are defined on objects by \(\tau_i((\rho_1, \rho_2, V)) = \rho_i\), induce closed immersions \(j_i : G_i \to G_1 \ast_{\text{alg}} G_2, i = 1, 2\), which make \(G_1 \ast_{\text{alg}} G_2\) a coproduct in the category of affine \(k\)-group schemes.
c) Let $G$ be an affine $k$-group scheme and $j_i^i : G_i \to G$ closed immersions. Assume that there exists no closed proper subgroup scheme of $G$, containing $j_1^i (G_1)$ and $j_2^i (G_2)$. Then the unique map $\gamma : G_1 \ast_{\text{alg}} G_2 \to G$ induced by the $j_i^i$ is faithfully flat.

d) If $G_1$ and $G_2$ are reduced, then so is $G_1 \ast_{\text{alg}} G_2$.

Proof. 

a) This is clear.

b) The $\tau_i$ are $\otimes$-functors, and if $\omega_i : \text{Repf} G_i \to \text{Vect}_k$ is the forgetful functor, then $\omega = \omega_i \tau_i$, so $\tau_i$ induces a morphism of group schemes $j_i : G_i \to G_1 \ast_{\text{alg}} G_2$. The $\tau_i$ admit sections: The functor $\sigma_i : \text{Repf} G_i \to \mathcal{C}(G_1, G_2)$, assigning to a representation $\rho_1 : G_1 \to \text{GL}(V)$ the triple $(\rho_1, G_2 \otimes \text{triv} \to \text{GL}(V), V)$, where triv means the trivial representation of $G_2$ on $V$, has the property that $\tau_i \sigma_i$ is the identity functor on $\text{Repf} G_1$. Analogously we define $\sigma_2$. This shows that $\tau_i$ is essentially surjective, and in particular that the corresponding morphism of group schemes $j_i : G_i \to G_1 \ast_{\text{alg}} G_2$ is a closed immersion for $i = 1, 2$.

To see that $G_1 \ast_{\text{alg}} G_2$ together with $j_1, j_2$ is a coproduct in the category of affine $k$-group schemes, let $G$ be an affine $k$-group scheme and $j_i^i : G_i \to G$ morphisms. Given a representation $\rho : G \to \text{GL}(V)$, we assign to it the triple $(j_i^i \rho, j_i^i, V)$. Clearly this extends to a functor $\text{Repf} G \to \mathcal{C}(G_1, G_2)$, such that $\text{Repf} G \to \mathcal{C}(G_1, G_2) \overset{\omega}{\to} \text{Vect}_k$ is the forgetful functor. Hence we obtain a morphism of group schemes $\gamma : G_1 \ast_{\text{alg}} G_2 \to G$, such that the diagram

\[
\begin{array}{c}
G_1 \quad \xrightarrow{j_1^i} \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
G_1 \ast_{\text{alg}} G_2 \quad \xrightarrow{j_i^i} \quad G \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
G_2 \quad \xrightarrow{j_2^i} \\
\end{array}
\]

commutes. Moreover, $\gamma$ is unique with this property, as the functor $\text{Repf} G \to \mathcal{C}(G_1, G_2)$ is uniquely determined by $j_1, j_2, j_1^i, j_2^i$.

c) To prove that $\gamma : G_1 \ast_{\text{alg}} G_2 \to G$ is faithfully flat, it suffices to show that for every representation $\rho$ of $G$, the functor

$$
\text{Repf} G \overset{\gamma}{\to} \mathcal{C}(G_1, G_2)
$$

restricts to an equivalence

$$(\rho)_\otimes \xrightarrow{\sim} (\gamma \rho)_\otimes.$$ 

Let $G(\rho)$ be the affine group scheme corresponding to the Tannaka category $\langle \rho \rangle_\otimes$ and the forgetful functor $\langle \rho \rangle_\otimes \subseteq \text{Repf} G \to \text{Vect}_k$. We obtain a faithfully flat map $G \to G(\rho)$. It suffices to show that the composition $G_1 \ast_{\text{alg}} G_2 \to G \to G(\rho)$ is faithfully flat. If $H$ denotes the image of $G_1 \ast_{\text{alg}} G_2$ in $G(\rho)$, then $H \times_{G(\rho)} G \to G$ is a closed affine subgroup scheme containing the image of $G_1 \ast_{\text{alg}} G_2$, so by assumption $H \times_{G(\rho)} G \to G$ is an isomorphism. By faithfully flat
Thus write words: There exists a stratified bundle $G = G(\rho)$. Since $G_1 \ast_{\text{alg}} G_2 \to H = G(\rho)$ is faithfully flat we are done.

d) We have to show that every algebraic quotient $G$ of $G_1 \ast_{\text{alg}} G_2$ is reduced. Let $H$ be the smallest closed subgroup scheme of $G$ containing the images of $G_1$ and $G_2$, which are smooth by assumption. Hence $H$ is reduced. But by $\square$ we know that $H = G$.

**Definition 5.3.** If $G_1, G_2$ are $k$-group schemes, we write $G_1 \ast_{\text{unip}} G_2$ for the maximal unipotent quotient of $G_1 \ast_{\text{alg}} G_2$. We have

$$\text{Rep}_k(G_1 \ast_{\text{unip}} G_2) = C^{\text{unip}}(G_1, G_2) \subseteq C(G_1, G_2).$$

the full subcategory of triples $(\rho_1, \rho_2, V)$, such that the closed subgroup generated by the images of $\rho_1, \rho_2$ is unipotent.

We will also need the following easy fact.

**Lemma 5.4.** Let $G$ be a smooth unipotent algebraic group over $k$, and $H \subseteq G$ a proper closed subgroup. Then there exists a normal proper subgroup $H' \triangleleft G$ containing $H$.

**Proof.** If $\dim G = 0$, then $G$ is the constant $k$-group scheme attached to a finite $p$-group because $k$ is algebraically closed, and $H$ corresponds to an abstract subgroup. In this situation it is not difficult to check that $H \subseteq G$ is contained in a subgroup $H' \subseteq G$ of index $p$. But in a finite $p$-group one also has $H' \subseteq N_G(H')$, so $H'$ is normal in $G$.

If $\dim G > 0$, and if $Z$ is the center of $G$, then $\dim Z > 0$ by [Hum75, Ch. 17, Ex. 5]. Thus $\dim G/Z < \dim G$, and $H/(Z \cap H) \subseteq G/Z$. By induction there exists a proper normal subgroup $H' \triangleleft G$ containing $H/(Z \cap H)$. If $H' \triangleleft G$ is the preimage of $H'$, then $H \subseteq H'$ and $H'$ is normal in $G$. $\square$

We are now ready to prove Theorem 4. Let $E$ be a unipotent stratified bundle on $\mathbb{G}_m$, and write $G(E, x)$ for associated monodromy group, which is a quotient of $\pi_1^{\text{unip}}(\mathbb{G}_m, x)$. Note that $G(E, x)$ is smooth by [AS07]. Let $H$ denote the image of $\pi_1^{\text{unip}}(k((t)), \omega_x) \ast_{\text{unip}} \pi_1^{\text{unip}}(k((t^{-1})), \omega_x)$ in $G(E, x)$, which is also smooth by Proposition 5.2. Since $G(E, x)$ is unipotent, if $H \neq G(E, x)$, then by Lemma 5.4 there exists a proper normal subgroup $H' \triangleleft G(E, x)$ containing $H$. In other words: There exists a stratified bundle $E'$ on $\mathbb{G}_m$, with monodromy group $G(E, x)/H' \neq \{1\}$, which is trivial on $k((t))$ and $k((t^{-1}))$. This is impossible. Thus $H = G(E, x)$, and it follows from Proposition 5.2 that the map

$$\pi_1^{\text{unip}}(k((t)), \omega_x) \ast_{\text{unip}} \pi_1^{\text{unip}}(k((t^{-1})), \omega_x) \to \pi_1^{\text{unip}}(\mathbb{G}_m, \omega_x)$$

is faithfully flat. To show that it is an isomorphism, it thus remains to show that the functor

$$\text{Strat}^{\text{unip}}(\mathbb{G}_m) \to C^{\text{unip}}(\pi_1^{\text{unip}}(k((t)), \omega_x), \pi_1^{\text{unip}}(k((t^{-1})), \omega_x)),$$

given by $E \mapsto (E|_{k((t))}, E|_{k((t^{-1})}))$ is essentially surjective. But this is a consequence of the following lemma. $\square$
Lemma 5.5. Consider the two maps

\[
R^i \lim_n U_r(k((t))^p^n) \xleftarrow{\varphi} R^i \lim_n U_r(k[t^{\pm 1}]^p^n) \xrightarrow{\varphi} R^i \lim_n U_r(k((t^{-1}))^p^n)
\]

induced by the inclusions \(k[t^{\pm 1}] \to k((t)), k[t^{\pm 1}] \to k((t^{-1}))\) (see Convention \([U\])\). The map of pointed sets

\[
(\varphi^+_r, \varphi^-_r): R^i \lim_n U_r(k[t^{\pm 1}]^p^n) \to R^i \lim_n U_r(k((t))^p^n) \times R^i \lim_n U_r(k((t^{-1}))^p^n)
\]

is surjective.

Proof. We first prove the statement for \(r = 2\) and then induct on \(r\).

Let \(a \in R^1 \lim_n G_a(k((t))^p^n)\) and \(b \in R^1 \lim_n G_a(k((t^{-1}))^p^n)\) be classes represented by \((a_n)_n \in \prod_{n \geq 0} G_a(k((t))^p^n)\) and \((b_n)_n \in \prod_{n \geq 0} G_a(k((t^{-1}))^p^n)\). By Lemma 4.4 we know that the classes of \((a_n)_n\) and \((b_n)_n\) are trivial, where \(a_0^\geq := a_n^{(\geq 0)}\) and \(a_0^< := a_n^{(< 0)}\) in the notation from Definition 4.3 and similarly for \(b_n\). Thus we may assume that \(a_n \in k[t^{-1}]^p^n\), and \(b_n \in k[t]^{p^n}\) for every \(n \geq 1\). Then \(a_n + b_n \in k[t^{\pm 1}]^p^n\), and \(\varphi^+_r\left(\{(a_n + b_n)_n\}\right) = a, \varphi^-_r\left(\{(a_n + b_n)_n\}\right) = b\), so \((\varphi^+_r, \varphi^-_r)\) is surjective.

Now let \(r > 2\), and assume that the lemma is proved for all ranks \(< r\). Again let \(G_i \leq U_r\) denote the normal subgroups given by matrices of the form

\[
\begin{pmatrix}
  a_1 \\
  \vdots \\
  a_i \\
  0 \\
  \vdots \\
  0 \\
  1
\end{pmatrix}
\]

As in the proof of Lemma 4.3, we obtain morphisms of strongly exact (Definition 1.4) short exact sequence of pointed sets

\[
1 \to R^1 \lim_n (G_{r-1}/G_{r-2})(k((t))^p^n) \xrightarrow{\varphi} R^1 \lim_n (U_r/G_{r-2})(k((t))^p^n) \xrightarrow{\varphi} R^1 \lim_n U_{r-1}(k((t))^p^n) \to 1
\]

We sloppily also write \(\varphi^+_r, \varphi^-_r\) for the middle vertical maps in this diagram.

Let \(c_+ \in R^1 \lim_n (U_r/G_{r-2})(k((t))^p^n)\) and \(c_- \in R^1 \lim_n (U_r/G_{r-2})(k((t^{-1}))^p^n)\), and write \(\tilde{c}_+\) (resp. \(\tilde{c}_-\)) for the image of \(c_+\) in \(R^1 \lim_n U_{r-1}(k((t))^p^n)\) (resp. for the image of \(c_-\) in \(R^1 \lim_n U_{r-1}(k((t^{-1}))^p^n)\)). By induction hypothesis, there exists a class \(\tilde{c} \in R^1 \lim_n U_{r-1}(k[t^{\pm 1}]^p^n)\), such that \((\varphi^+_r, \varphi^-_r)(\tilde{c}) = (\tilde{c}_+, \tilde{c}_-)\).

Recall that \(G_{r-1}/G_{r-2} \cong G_a\) and that Lemma 4.3 shows that the group

\[
R^1 \lim_n (G_{r-1}/G_{r-2})(k((t))^p^n)
\]

acts on \(R^1 \lim_n (U_r/G_{r-2})(k((t))^p^n)\), such that the orbits are precisely the fibers of the projection to \(R^1 \lim_n U_{r-1}(k((t))^p^n)\), and similarly for \(k[t^{\pm 1}]\) and \(k((t^{-1}))\).
If \( c \in \prod_n (U_r/G_{r-2})(k[t^{±1}])^{p^n} \) is any lift of \( \hat{c} \), then \( \varphi_r^+(c), \varphi_r^{-}(c) \) both map to \( \hat{c}_+ \). Similarly, \( \varphi_r^{-}(c) \) and \( c_- \) both map to \( \hat{c}_- \). This means there exist elements

\[
g_+ \in \prod_n (G_{r-1}/G_{r-2})(k((t))^{p^n}), g_- \in \prod_n (G_{r-1}/G_{r-2})(k((t^{-1}))^{p^n}),
\]

such that \( g_+ + c_+ = \varphi_r^+(c) \) and \( g_- + c_- = \varphi_r^{-}(c) \).

Since \( G_{r-1}/G_{r-2} \cong U_2 \), by induction there exists a class

\[
g \in \prod_n (G_{r-1}/G_{r-2})(k[t^{±1}]),
\]

mapping to \( g_- \) and \( g_+ \). Since \( \varphi_r^+ \) and \( \varphi_r^{-} \) are equivariant with respect to the action of \( \prod_n (G_{r-1}/G_{r-2})(k[t^{±1}])^{p^n} \), it follows that \( \varphi_r^+(c-g) = c_+ \) and \( \varphi_r^{-}(c-g) = c_- \).

We have proved that the map

\[
\prod_n (U_r/G_{r-2})(k[t^{±1}]) \to \prod_n (U_r/G_{r-2})(k((t))^{p^n}) \times \prod_n (U_r/G_{r-2})(k((t^{-1}))^{p^n})
\]

is surjective.

Now we apply the above argument inductively to the short exact sequences

\[
1 \to G_{r-i}/G_{r-i-1} \to U_r/G_{r-i-1} \to U_r/G_{r-i} \to 1.
\]

Eventually we arrive at

\[
1 \to G_1 \to U_r \to U_r/G_1 \to 1
\]

which completes the proof. \( \square \)

Finally, we show that one can recover from Theorem 5.6 the result of Katz-Gabber (bottom of p. 98) stating that the morphism

\[
\pi_1^\text{ét}(k((t)))^{(p)} \star_{(p)} \pi_1^\text{ét}(k((t^{-1})))^{(p)} \to \pi_1^\text{ét}(G_m)^{(p)}
\]

induced by compatible choices of base points, is an isomorphism. Here \((-)^{(p)}\) denotes the maximal pro-\(p\)-quotient, and \(*_{(p)}\) the coproduct in the category of pro-\(p\)-groups. For the construction of the coproduct in the category of pro-\(p\)-groups, see [RZ00, Sec. 9.1].

**Proposition 5.6.** Recall that \( k \) is an algebraically closed field of positive characteristic \( p \). Let \( G, G_1, G_2 \) be affine, reduced, unipotent \( k \)-group schemes.

a) The profinite completion \( \hat{G} \) of \( G \) is a pro-\( p \)-group. By profinite completion we mean the inverse limit over all finite (hence constant) quotients of \( G \).

b) The profinite completion of \( G_1 \star \text{unip} G_2 \) is the pro-constant group scheme associated with the coproduct \( \hat{G}_1 \star_{(p)} \hat{G}_2 \) in the category of pro-\( p \)-groups.

**Proof.**

a) This is clear, as finite group of unipotent matrices is a finite \( p \)-group.

b) The profinite completion of \( G_1 \star \text{unip} G_2 \) can be described as the affine \( k \)-group scheme associated with the full subcategory \( C_{\text{finite}}(G_1, G_2) \) of \( C_{\text{unip}}(G_1, G_2) \) (see Definition 5.3) whose objects have a finite monodromy group, i.e. the full subcategory of triples \((\rho_1, \rho_2, V)\), such
that the closed subgroup $H$ of $\GL(V)$ generated by the images of $\rho_1$ and $\rho_2$ is a finite $p$-group. In particular $\rho_i : G_i \to \GL(V)$ factors uniquely through the profinite completion of $G_i$, for $i = 1, 2$. This means that we get a fully faithful functor
\[
C^{\text{finite}}(G_1, G_2) \to \text{Rep}^{\text{cont}}_k \left( \hat{G}_1 *_{(p)} \hat{G}_2 \right),
\]
where continuous means with respect to the profinite topology and the discrete topology on $k$, i.e. if $G$ is a profinite group, then a representation $G \to \GL_k(V)$ is continuous if and only if the image of $G$ is finite. The functor (23) is also essentially surjective: A continuous representation $\rho : \hat{G}_1 *_{(p)} \hat{G}_2 \to \GL(V)$ gives rise to a pair of representations $\rho_i : G_i \to \hat{G}_i \to \GL(V)$, $i = 1, 2$, such that the images of $\rho_1$ and $\rho_2$ generate a finite subgroup of $\GL(V)$, which is a finite $p$-group, as $\hat{G}_1 *_{(p)} \hat{G}_2$ is a pro-$p$-group. Thus the triple $(\rho_1, \rho_2, V)$ is an object of $C^{\text{finite}}(G_1, G_2)$ and the proof is complete. 

By Theorem \[\blacksquare\] the inclusion $k[t] \subseteq k((t^{-1}))$ induces an isomorphism
\[
\pi_1^{\text{unip}}(k((t^{-1})), \omega_x) \cong \pi_1^{\text{unip}}(\hat{A}_k, \omega_x).
\]
Applying Proposition 5.6 to the isomorphism in Theorem D we obtain an isomorphism
\[
\pi_1^{\text{et}}(\mathbb{P}_k^1 \setminus \{0\}, x)^{(p)} *_{(p)} \pi_1^{\text{et}}(\hat{A}_k, x)^{(p)} \cong \pi_1^{\text{et}}(\mathbb{G}_m, x)^{(p)}.
\]
Choosing algebraic closures of $k((t))$ and $k((t^{-1}))$, we get two geometric points $x_0 : \Spec k((t)) \to \mathbb{G}_m$ and $x_\infty : \Spec k((t^{-1})) \to \mathbb{G}_m$. Fixing isomorphisms ("chemins") from $x$ to $x_0$ and $x_\infty$, we obtain an isomorphism
\[
\Gal(k((t))/k((t)))^{(p)} *_{(p)} \Gal(k((t^{-1}))/k((t^{-1})))^{(p)} \cong \pi_1^{\text{et}}(\mathbb{G}_m, x)^{(p)},
\]
which is the isomorphism Katz and Gabber construct in [Kat86] p. 98, except that they work with strict henselizations instead of completions.

**Remark 5.7.** The isomorphism (24) constructed in [Kat86] contradicts [Ked08, Thm. 2.30]. Indeed, a consequence of loc. cit. would be that the free pro-$p$-group $\pi_1(\mathbb{G}_m, x)^{(p)}$ is the direct product
\[
\pi_1(\hat{A}_k, x)^{(p)} \times \pi_1(\mathbb{P}_k^1 \setminus \{0\}, x)^{(p)}.
\]
But it is not hard to check that such a direct product is not a free pro-$p$-group by computing
\[
H^2(\pi_1(\hat{A}_k, x)^{(p)} \times \pi_1(\mathbb{P}_k^1 \setminus \{0\}, x)^{(p)}, \mathbb{F}_p) \neq 0.
\]
For a correction, see [Ked13].

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Lars Kindler, Freie Universität Berlin, Mathematisches Institut, Arnimallee 3, 14195 Berlin, Germany
E-mail address: kindler@math.fu-berlin.de