(Super)conformal many-body quantum mechanics
with extended supersymmetry

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Abstract
We study $\mathcal{N} = 4$ supersymmetric quantum-mechanical many-body systems with $M$ bosonic and $4M$ fermionic degrees of freedom. We also investigate the further restrictions of conformal and superconformal invariance. In particular, we construct conformal $\mathcal{N} = 4$ extensions of the $A_{M-1}$ Calogero models, which for generic values of the coupling constant are not SU$(1,1|2)$ superconformal. This class of models is also extended to arbitrary (even) $\mathcal{N}$. We give both hamiltonian and (classical) lagrangean formulations. In the latter case we use both component and $\mathcal{N} = 4$ superfield formulations.

1 Introduction

Recently there has been increased interest in supersymmetric quantum-mechanical models. Contrary to the situation in higher dimensions, such models have been much less studied. One recent application is to black hole physics \[1,2,3,4\]. A related issue is the still incompletely understood adS$_2$/CFT$_1$ correspondence \[4\]. In the case of black holes, most work has so far been concerned with $\mathcal{N} = 4$ models with $4M$ bosonic and $4M$ fermionic coordinates, and general results for such models have been obtained \[2,4,8\]. The emphasis has been on which (sigma-model) metrics are consistent with supersymmetry and the properties of the resulting geometries. Our focus is slightly different; we discuss models with $M$ bosonic and $4M$ fermionic degrees of freedom, take the metric to be flat and study the constraints on the potential coming from supersymmetry. We also investigate the constraints arising from adding more symmetry such as translational invariance, conformal invariance and superconformal invariance. We do not have any particular application in mind, although $\mathcal{N} = 4$ supersymmetric superconformal Calogero models have been conjectured \[1\] to provide a microscopic description of four-dimensional extremal Reissner-Nordström black holes. The Calogero models \[9\] and their generalisations comprise a particular class of many-body quantum-mechanical models that have been intensely studied over the years. These models have appeared in various areas of theoretical physics, ranging from problems in condensed matter physics to Seiberg-Witten theory. For reviews with extensive lists of references to the early literature, see \[11\] (for

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reviews on the connection to Seiberg-Witten theory, see e.g. [10]. It is well known that
the Calogero systems are intimately connected with the semi-simple Lie algebras. For
every (semi-simple) Lie algebra there is an associated Calogero system. It is perhaps less
widely known that the conditions can be weakened. Recently it has been shown [12, 13]
that one actually does not need a root system associated to a Lie algebra to construct a
Calogero model. It is sufficient to have a root system associated to any finite reflection
group (Coxeter group); only when the root system is crystallographic can one associate it
to a Lie algebra and the Coxeter group is then called the Weyl group. The Calogero sys-
tems are integrable (see e.g. [14, 13] and references therein) and, for the cases with discrete
spectrum, exactly solvable. By exactly solvable we mean the condition that it should be
possible to obtain the eigenfunctions in an “algebraic” way. This has been shown using
various different approaches, see e.g. [15, 16, 17]. An interesting feature of the $A_{M-1}$
Calogero models is that they are translational- and conformal-invariant. The two-particle
case coincides (after removing the centre-of-mass motion) with the model of conformal
mechanics studied in [18]. Supersymmetric extensions of the Calogero models with $\mathcal{N} = 2$
supersymmetry have also been constructed [13, 20, 21, 22]. So far the supersymmetric
models have not had as many applications as the bosonic models. The models constructed
in [13] are also superconformal; the superconformal algebra being $\text{osp}(2|2) \cong \text{su}(1,1|1)$. The
relative motion of the two-particle case was studied before in [23]. In [24, 25, 26]
(see also [27]) an $\mathcal{N} = 4$ superconformal extension of the conformal quantum mechanics
model was constructed (a related development is [28]). The superconformal group in this
case is $\text{SU}(1,1|2)$. This result has not been extended to the many-body case.

In the next section we investigate (using the quantum hamiltonian formalism [29]) the
restrictions of $\mathcal{N} = 4$ supersymmetry, conformal invariance and $\text{SU}(1,1|2)$ superconformal
symmetry. We first discuss the one-particle case and then move on to the many-body case
d and derive general results. We concentrate on the $A_{M-1}$ Calogero models, but our results
are applicable also to other cases. We show that it is possible to construct conformal
$\mathcal{N} = 4$ extensions of the $A_{M-1}$ Calogero models, which are $\text{SU}(1,1|2)$ superconformal only
for a particular value of the coupling constant. Furthermore, we show that (given certain
assumptions) for $M > 2$ and generic values of the coupling constant there are no natural
$\text{SU}(1,1|2)$ superconformal extensions of the $A_{M-1}$ Calogero models. In section 3 we
present a similar discussion employing the language of the classical lagrangean formalism
[30]. We use both superfield and component formulations. We also briefly discuss the
connection between the classical lagrangean approach and the quantum hamiltonian one.
We end with a short discussion of the possible relevance of our results to black hole physics
and some open questions.

2 Quantum hamiltonian formulation

We assume the $\mathcal{N} = 4$ supersymmetry algebra to be of the form

$$
[Q^a, Q^b]_+ = 2\delta^{ab}H, \quad [Q^a, Q_b]_+ = 0, \quad [Q^i, Q^j]_+ = 0,
$$

(2.1)

where $a, b = 1, 2$. In other words, we use a complex formalism. Some of our conclusions
may be altered if the supersymmetry algebra is changed, i.e. if central charges are allowed
or if more general supersymmetry algebras are considered (such as the ones in [31, 8]). In this section we will investigate the restrictions on the potential resulting from requiring an $\mathcal{N} = 4$ symmetry in the form of the supersymmetry algebra (2.1). We will also discuss the restrictions coming from demanding conformal and superconformal invariance.

### 2.1 Preliminaries: one-body models

The supercharges are $Q_a$ and their hermitean conjugates. We sometimes use the notation $Q \equiv Q_1$ and $\tilde{Q} \equiv Q_2$ to reduce the number of indices. The discussion below of the one-body case is in part a review (of [25, 26]), but it is presented in such a way as to facilitate the extension to the many-particle case to be discussed later. We will denote the bosonic coordinate by $x$ and use the concrete realisation $\theta, \tilde{\theta}, \partial_{\theta}$ and $\partial_{\tilde{\theta}}$ for the fermionic coordinates. On general grounds the supercharges can be taken to be of the form:

$$Q = \theta \left( p - i W^{[0]}(x) - i W^{[1]}(x) \tilde{\theta} \frac{\partial}{\partial \theta} \right), \quad \tilde{Q} = \tilde{\theta} \left( p - i W^{[0]}(x) - i W^{[1]}(x) \theta \frac{\partial}{\partial \theta} \right),$$

(2.2)

Together with their hermitean conjugates (using $\theta^\dagger = \frac{\partial}{\partial \theta}$ and $(\xi \zeta)^\dagger = \zeta^\dagger \xi^\dagger$). The super-symmetry algebra (2.1) is satisfied if the following equation is satisfied ($\partial \equiv \frac{d}{dx}$)

$$2\partial W^{[0]} - 2 W^{[0]} W^{[1]} + \partial W^{[1]} - W^{[1]} W^{[1]} = 0.$$

(2.3)

The hamiltonian then becomes

$$H = \frac{1}{2} p^2 + \frac{1}{2} (W^{[0]})^2 - \frac{1}{2} \partial W^{[0]} + \partial W^{[0]} [\theta \frac{\partial}{\partial \theta} + \tilde{\theta} \frac{\partial}{\partial \tilde{\theta}}] + \partial W^{[1]} \theta \frac{\partial}{\partial \theta} \frac{\partial}{\partial \tilde{\theta}}$$

(2.4)

We will now restrict ourselves to conformal models. Such models satisfy $[D, H] = \imath H$, where $D = -\frac{1}{4} [x, p]_+$ [18]. To regain the OSp(2|2) superconformal mechanics of [23] when restricting to the $\mathcal{N} = 2$ sub-sector, we have to set $W^{[0]} = \frac{4}{x}$. Somewhat surprisingly, for this choice of $W^{[0]}$ there are two solutions to the constraint (2.3) preserving conformal invariance, namely: $W^{[1]} = -\frac{1}{x}$ and $W^{[1]} = -\frac{2x}{x}$. Thus, there are two different conformal $\mathcal{N} = 4$ supersymmetrisations of conformal mechanics (or equivalently, of the relative motion of the $A_1$ Calogero model). The corresponding hamiltonians are

$$H_1 = \frac{1}{2} p^2 + \frac{1}{2x^2} \left( \nu^2 + \nu - 2 \nu [\theta \frac{\partial}{\partial \theta} + \tilde{\theta} \frac{\partial}{\partial \tilde{\theta}}] + 2 \theta \frac{\partial}{\partial \theta} \frac{\partial}{\partial \tilde{\theta}} \right),$$

$$H_2 = \frac{1}{2} p^2 + \frac{1}{2x^2} \left( \nu^2 + \nu - 2 \nu [\theta \frac{\partial}{\partial \theta} + \tilde{\theta} \frac{\partial}{\partial \tilde{\theta}}] + 4 \nu \theta \frac{\partial}{\partial \theta} \frac{\partial}{\partial \tilde{\theta}} \right)$$

$$= \frac{1}{2} p^2 + \frac{\nu (\nu + [\theta, \partial_{\theta}] [\tilde{\theta}, \partial_{\tilde{\theta}}])}{x^2}.$$  

(2.5)

Although both models in (2.3) are conformal, only the first has su(1,1|2) as its superconformal algebra for generic $\nu$. This can be seen by making a general Ansatz for the generators of special supersymmetries $S, \tilde{S}$ and their hermitean conjugates. The su(1,1|2) superconformal algebra (see appendix) is satisfied if $S = \theta x, \tilde{S} = \tilde{\theta} x$ and $x W^{[1]} = -1$. The other generators of su(1,1|2) are then given by

$$J_1 = -\frac{1}{2} (\theta \partial_{\tilde{\theta}} + \tilde{\theta} \partial_{\theta}), \quad J_2 = -\frac{1}{2} (\theta \partial_{\tilde{\theta}} - \tilde{\theta} \partial_{\theta}), \quad J_3 = \frac{1}{2} (\tilde{\theta} \partial_{\tilde{\theta}} - \theta \partial_{\theta}).$$

(2.6)
Furthermore, the central element \( T \) is given by \( T = xW[0] - \frac{1}{2} \). Thus, there is a unique SU(1,1|2) conformally invariant model. For this model \( T = \nu - \frac{1}{2} \). The second model above is SU(1,1|2) superconformal only when \( \nu = \frac{1}{2} \), in which case it coincides with a special case of the first model (with \( T = 0 \)). For other values of \( \nu \) the conformal and supersymmetry generators belong to some other superconformal algebra. The free theory is not SU(1,1|2) superconformal. The above SU(1,1|2) superconformal model is, however, “on-shell”-dual to a free \( \mathcal{N} = 4 \) theory with a complex bosonic coordinate (2 real ones) \([24]\). Let us also mention that for the first model above there is a simple extension to \( \mathcal{N} \) \([26]\), i.e arbitrary number of supersymmetries (there is no restriction on the number of supersymmetries in one dimension since there is no notion of spin). The arbitrary-\( \mathcal{N} \) models have the following supercharges \([26]\).

\[
Q_a = \theta_a(p - i \frac{\nu}{x} + i \frac{1}{x} \sum_{c \neq a} \theta_c \partial_{\theta_c}), \quad Q^i a = \partial_{\theta_a}(p + i \frac{\nu}{x} - i \frac{1}{x} \sum_{c \neq a} \theta_c \partial_{\theta_c}), \quad (2.7)
\]

where \( a, c = 1, \ldots, \frac{N}{2} \). These models are also superconformal; the superconformal algebra being su(1,1|\( \frac{N}{2} \)) (see appendix). The other generators of su(1,1|\( \frac{N}{2} \)) are given by: \( S_a = \theta_a x, \) \( S^i a = \partial_{\theta_a} x, \) and

\[
J_a^b = \theta_a \partial_{\theta_b} (a \neq b), \quad J_a^a = \theta_a \partial_{\theta_a} - \frac{2}{N} \sum_c \theta_c \partial_{\theta_c}, \quad U = \frac{1}{4} \sum_c \theta_c \partial_{\theta_c}. \quad (2.8)
\]

For the second model in (2.5) a similar extension to arbitrary (even) \( \mathcal{N} \) can be constructed by taking the supercharges to be of the form

\[
Q_a = \theta_a(p - iW[\frac{N-2}{2}](x) \prod_{c \neq a} [\theta_c, \frac{\partial}{\partial \theta_c}]), \quad Q^i a = \partial_{\theta_a}(p + iW[\frac{N-2}{2}](x) \prod_{c \neq a} [\theta_c, \frac{\partial}{\partial \theta_c}]), \quad (2.9)
\]

where \( a, c = 1, \ldots, \frac{N}{2} \) (we use a slightly different normalisation for \( W[1] \) than before). The corresponding hamiltonian is obtained from \([Q_a, Q^i b]_+ = 2 \delta_a^b H \) and becomes

\[
H = \frac{1}{2} p^2 + \frac{1}{2} (W[\frac{N-2}{2}])^2 + \frac{1}{2} \partial W[\frac{N-2}{2}] \prod_c [\theta_c, \frac{\partial}{\partial \theta_c}]. \quad (2.10)
\]

In particular, for \( W[\frac{N-2}{2}] = -\frac{\nu}{x} \) we get

\[
H = \frac{1}{2} p^2 + \frac{1}{2} \frac{\nu (\nu + \frac{N-2}{2})}{x^2}. \quad (2.11)
\]

Notice that \((\prod_c [\theta_c, \frac{\partial}{\partial \theta_c}])^2 = 1.\)

2.2 Extension to many-body models

We will now discuss the extension of the above results to the many-body case. The coordinates are \( x_i, \theta_i, \tilde{\theta}_i, \partial_{\theta_i} \) and \( \partial_{\tilde{\theta}_i} \), where \( i = 1, \ldots, M \). Here, and throughout the
paper, we will assume that the hamiltonians are invariant under permutations of the coordinates. We take the supercharge $Q$ to be of the form

$$Q = \sum_j \theta_j \left( p_j - i W_j^0(x_k) - i \sum_{nm} W_{jm}^1(x_k) \tilde{\theta}_n \frac{\partial}{\partial \theta^m} \right),$$

(2.12)

with a similar expression for $\tilde{Q}$. This is not the most general choice, but it is a natural extension of the supercharges used to construct $\mathcal{N} = 2$ models [19, 32, 21, 22]. The supersymmetry algebra (2.1) is satisfied if the following conditions are fulfilled

$$W_i^0 = \partial_i W^0, \quad W_{ijk} = \partial_i \partial_j \partial_k W^1,$$

$$\sum_l W_{l[n}^1 W_{m]l}^1 = 0,$$

$$\partial_j \partial_k \tilde{W}^0 = \sum_l W_{ijk} \partial_l \tilde{W}^0, \quad (\tilde{W}^0 := W^0 + \frac{1}{2} \sum_n \partial_n \partial_n W^1),$$

(2.13)

and the hamiltonian is then given by

$$H = \frac{1}{2} \sum_i [p_i^2 + (\partial_i W^0)^2 - \partial_i^2 W^0] + \sum_{ij} (\theta_i \partial_j \tilde{\theta} + \tilde{\theta}_i \partial_j \theta) \partial_i \partial_j W^0$$

$$+ \sum_{ijm} \theta_i \partial_j \tilde{\theta}_m \partial_i \partial_j \partial_m W^1.$$

(2.14)

One solution to the last constraint in (2.13) is the trivial one: $\tilde{W}^0 = 0$, i.e. $W^0 = -\frac{1}{2} \sum_n \partial_n \partial_n W^1$. This provides a possible way to construct $\mathcal{N} = 4$ extensions of known $\mathcal{N} = 2$ models, e.g. the Calogero models. We would like to stress that one also has to check that the other conditions in (2.13) hold. With $\tilde{W}^0 = 0$, the supercharge $Q$ takes the form

$$Q = \sum_j \theta_j \left( p_j - \frac{i}{2} \sum_{nm} \partial_j \partial_n \partial_m W^1 [\tilde{\theta}_n, \frac{\partial}{\partial \theta^m}] \right),$$

(2.15)

with a similar expression for $\tilde{Q}$. The $A_{M-1}$ Calogero models have $W^0 = \frac{\nu}{2} \sum_{i \neq j} \ln |x_i - x_j|$. If we set

$$W^1 = -\frac{\nu}{2} \sum_{i \neq j} \left((x_i - x_j)^2 \ln |x_i - x_j| - \frac{3}{2} (x_i - x_j)^2 \right),$$

(2.16)

it can readily be checked that all conditions in (2.13) are fulfilled. The resulting hamiltonian becomes

$$H = \frac{1}{2} \sum_i p_i^2 + \sum_{i < j} \frac{\nu (\nu + K_{ij})}{(x_i - x_j)^2},$$

(2.17)

where $K_{ij} = \frac{1}{4} [(\theta_i - \theta_j), (\partial_{\tilde{\theta}_i} - \partial_{\tilde{\theta}_j})] [(\tilde{\theta}_i - \tilde{\theta}_j), (\partial_{\theta_i} - \partial_{\theta_j})]$. The operator $K_{ij}$ is an exchange operator satisfying: $\theta_i K_{ij} = K_{ij} \theta_j$, $\tilde{\theta}_i K_{ij} = K_{ij} \tilde{\theta}_j$, $K_{ij}^2 = 1$ and $K_{ij} K_{jk} = K_{jk} K_{ji} = K_{ki} K_{ij}$. The above models are closely related to the general models in [33] (see also [15]) and should hence be integrable. Notice that $K_{ij}$ only acts on the fermionic coordinates whereas the operators in [33, 15] also act on the bosonic coordinates; it is however easy
to extend the above models to this more general setting. The models (2.17) can be straightforwardly extended to arbitrary $\mathcal{N}$. The supercharges take the form

$$ Q_a = \sum_i \theta_a^i \left( p_i + i \frac{\nu}{2^{\frac{N}{2}} - 1} \sum_{c \neq a} \prod_{c=1}^{\frac{N}{2}} \left[ \left( \theta_c^i - \theta_c^m \right), \left( \partial_{\theta_c^i} - \partial_{\theta_c^m} \right) \right] \frac{x^i - x^m}{x_i - x_m} \right), $$

(2.18)

where $a, c = 1, \ldots, \frac{N}{2}$. The Hamiltonian has the same form as in (2.17), but with $K_{ij}$ given by $K_{ij} = \frac{1}{2^{\frac{N}{2}}} \sum_{c=1}^{\frac{N}{2}} \left[ \left( \theta_c^i - \theta_c^j \right), \left( \partial_{\theta_c^i} - \partial_{\theta_c^j} \right) \right]$.

The $\mathcal{N} = 4$ models just constructed are conformal, but as we shall see next the superconformal algebra is not su(1,1|2). We now turn to the question of what restrictions follow from demanding SU(1,1|2) superconformal invariance. With $S = \sum_i \theta_i x_i$ and $\bar{S} = \sum_i \bar{\theta}_i x_i$, the superconformal algebra is satisfied if

$$ \sum_i x_i W^{[1]}_{ijk} = -\delta_{jk}, $$

(2.19)

and $\sum_i x_i \partial_i W^{[0]} = \text{const}$ (the latter condition follows from $[D, W^{[0]}_i] = -i W^{[0]}_i$, i.e. from conformal invariance). The other generators are then given by

$$ J_1 = -\frac{1}{2} \sum_i (\theta_i \partial_{\theta_i} + \bar{\theta}_i \partial_{\bar{\theta}_i}), \quad J_2 = -\frac{i}{2} \sum_i (\theta_i \partial_{\theta_i} - \bar{\theta}_i \partial_{\bar{\theta}_i}), \quad J_3 = \frac{1}{2} \sum_i (\theta_i \partial_{\theta_i} - \theta_i \partial_{\bar{\theta}_i} + \bar{\theta}_i \partial_{\theta_i}) $$

(2.20)

and $T = \sum_i x_i \partial_i W^{[0]} - \frac{\mathcal{N}}{4}$. The restriction (2.19) on $W^{[1]}_{ijk}$, show that the models (2.17) are not SU(1,1|2) superconformal.

Another issue is translational invariance. The condition for superconformal invariance (2.19) is not consistent with translational invariance of $W^{[1]}$ (since that would imply $\sum_i \partial_i W^{[1]} = 0$). However, after extracting from $W^{[1]}_{ijk}$ the non-translational-invariant centre-of-mass part $W^{[1]}_{ijk} = \frac{1}{M} W^{[1]}_{ij}$ (where $X = \sum_i x_i$) the remaining relative part can be taken to be translational-invariant and the superconformal condition is replaced by $\sum_i x_i W^{[1]}_{ijk} = -\delta_{jk} + \frac{1}{M}$. With this modification of the models in (2.17), they become SU(1,1|2) superconformal for certain exotic values of the coupling constant $\nu$, namely when $\nu = \frac{1}{M}$. We will now address the question of whether there exist SU(1,1|2) superconformal extensions of the $A_{M-1}$ Calogero models for generic values of the coupling constant $\nu$. If we assume that $W^{[0]}$ has an overall parameter (as in the case of the Calogero models) then (if demand conformal invariance and discard the above solution) the last equation in (2.13) decouples into two equations: $\partial_i \partial_j W^{[0]} = \sum_i W^{[1]}_{ij} \partial_i W^{[0]}$, and $\partial_i \sum_n W^{[1]}_{ijn} = \sum_{i,n} W^{[1]}_{ij} W^{[1]}_{inn}$. Notice that the latter equation is consistent with (2.19) and conformal invariance. When $W^{[0]}$ has the Calogero form $W^{[0]} = \frac{1}{2} \sum_{n \neq m} \ln |x_n - x_m|$ one can show that there are no solutions to the coupled set of equations (2.13), (2.19) for $M = 3$; we believe that this continues to be true for higher $M$ (it can be shown for all $M > 2$ that for generic $\nu$ there is no solution with two-body interaction forces only). Thus, we conclude that for $M > 2$ there is no natural candidate for an $\mathcal{N} = 4$ SU(1,1|2) superconformal $A_{M-1}$ Calogero model which has the proper $\mathcal{N} = 2$ limit. We would like to stress that this conclusion depends on the particular (but natural) choice of supercharges (2.12).
Is it possible to find other SU(1,1|2) superconformal models? For simplicity let us discuss the $M = 3$ (three-particle) case in more detail. There is actually no solution to the set of constraints given above for any $W^{[0]}$ with an overall parameter and two-body interactions only, so the conditions have to be weakened. One has to allow for a $\nu$-independent part in $W^{[0]}$ and/or higher-body interactions if one is to be able to satisfy the constraints. At least for the $M = 3$ case it turns out that it is not sufficient to introduce higher-body interactions, so we will therefore allow for a $\nu$-independent part in $W^{[0]}$. At this point we recall that there is another three-particle translational-invariant (bosonic) Calogero model (besides the $A_2$ one) namely the model associated to the $G_2$ Lie algebra \[34\]. The Hamiltonian is

$$H_{G_2} = \frac{1}{2} \sum_i p_i^2 + \sum_{i < j} \frac{\nu_1 (\nu_1 - 1)}{(x_i - x_j)^2} + 3 \sum_{\substack{j<k \atop i \neq j \neq k}} \frac{\nu_2 (\nu_2 - 1)}{(2x_i - x_j - x_k)^2}. \quad (2.21)$$

The two coupling constants $\nu_1$ and $\nu_2$ can be chosen independently. This Hamiltonian has all the nice properties of the Calogero models, such as integrability and exact solvability \[35\]. Using reasoning similar to the one used in the $A_2$ case one can show that there does not exist any SU(1,1|2) superconformal $\mathcal{N} = 4$ extension when the $Q_a$'s are of the form \[2.12\] and the two coupling constants are unrelated. Choosing the centre-of-mass part of $W_{ijk}^{[1]}$ as before, allowing for a linear relation between $\nu_1$ and $\nu_2$, and choosing

$$W_{rel}^{[1]} = \beta_1 \sum_{i < j} (x_i - x_j)^2 \ln |x_i - x_j| + \beta_2 \sum_{j < k} (2x_i - x_j - x_k)^2 \ln |2x_i - x_j - x_k|,$$

$$W^{[0]} = \alpha_1 \sum_{i < j} \ln |x_i - x_j| + \alpha_2 \sum_{j < k} \ln |2x_i - x_j - x_k|, \quad (2.22)$$

we have found the following SU(1,1|2) superconformal extensions of the $G_2$ model. The following different choices for the parameters $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ are possible: $(-\frac{1}{6}, \frac{1}{12}, -\frac{1}{12})$, $(\nu_1, -\frac{1}{6}, \frac{1}{6}, \frac{1}{36})$, or $(\frac{1}{3} - \nu, \nu, \frac{1}{2} - \frac{1}{6}, -\frac{5}{6})$. The last case is more trivial than the others since it has $W^{[0]} = 0$. The corresponding potential is

$$V_{rel}^{[1]} = \frac{1}{x_{ij}^2} \left[ \alpha_1 (\alpha_1 + 1) - \alpha_1 (\theta_{ij} \partial_{\theta_{ij}} + \tilde{\theta}_{ij} \partial_{\tilde{\theta}_{ij}}) - 2\beta_1 (\theta_{ij} \partial_{\theta_{ij}} \tilde{\theta}_{ij} \partial_{\tilde{\theta}_{ij}}) \right]$$

$$+ \sum_{\substack{j<k \atop i \neq j \neq k}} \frac{1}{x_{ijk}^2} \left[ 3\alpha_2 (\alpha_2 + 1) - \alpha_2 (\theta_{ijk} \partial_{\theta_{ijk}} + \tilde{\theta}_{ijk} \partial_{\tilde{\theta}_{ijk}}) - 2\beta_2 (\theta_{ijk} \partial_{\theta_{ijk}} \tilde{\theta}_{ijk} \partial_{\tilde{\theta}_{ijk}}) \right], \quad (2.23)$$

where $x_{ij} = x_i - x_j$, $\theta_{ij} = \theta_i - \theta_j$, and $\partial_{\theta_{ij}} = \partial_{\theta_i} - \partial_{\theta_j}$; $x_{ijk} = 2x_i - x_j - x_k$, $\theta_{ijk} = 2\theta_i - \theta_j - \theta_k$, and $\partial_{\theta_{ijk}} = 2\partial_{\theta_i} - \partial_{\theta_j} - \partial_{\theta_k}$.

For $M = 4$ there also exists a translational-invariant (bosonic) "Calogero" model, which in general has two- and four-body interactions \[36\]

$$H_4 = \frac{1}{2} \sum_i p_i^2 + \sum_{i < j} \frac{\nu_1 (\nu_1 - 1)}{(x_i - x_j)^2} + 2 \sum_{\substack{i < j, k < l \atop i \neq j \neq k \neq l}} \frac{\nu_2 (\nu_2 - 1)}{(x_i + x_j - x_k - x_l)^2}. \quad (2.24)$$
A similar analysis for this case leads to the following SU(1,1) superconformal solution with four-body interactions only

\[ W^{[1]}_{\text{rel}} = -\frac{1}{8} \sum_{i < j, k < l \atop i \neq j \neq k \neq l} (x_i + x_j - x_k - x_l)^2 \ln |x_i + x_j - x_k - x_l| , \]

\[ W^{[0]} = \nu \sum_{i < j, k < l \atop i \neq j \neq k \neq l} \ln |x_i + x_j - x_k - x_l| , \]

\[ V_{\text{rel}} = \sum_{i < j, k < l \atop i \neq j \neq k \neq l} \frac{1}{16} [2\nu(\nu + 1) - \nu(\theta_{ijkl}\partial\theta_{ijkl} + \bar{\theta}_{ijkl}\bar{\partial}\bar{\theta}_{ijkl}) + \frac{1}{4} \theta_{ijkl}\partial\theta_{ijkl} + \bar{\theta}_{ijkl}\bar{\partial}\bar{\theta}_{ijkl}] . \]

where, \( x_{ijkl} = x_i + x_j - x_k - x_l \), \( \theta_{ijkl} = \theta_i + \theta_j - \theta_k - \theta_l \), and \( \partial_{\theta_{ijkl}} = \partial_{\theta_i} + \partial_{\theta_j} - \partial_{\theta_k} - \partial_{\theta_l} \). One could continue this analysis to higher \( M \) and try to find interesting solutions. One restriction one could impose is that the bosonic part should have special properties, such as e.g. integrability. Perhaps it is possible to turn things around and use supersymmetry considerations to construct interesting bosonic models.

### 3 Classical lagrangean treatment

In this section we perform a study of \( \mathcal{N} = 4 \) models similar to the one in section 2 but from a (classical) lagrangean perspective.

#### 3.1 One-body models

In [24] (see also [27]) an SU(1,1|2) superconformal mechanics model was constructed. The action is most succinctly written in \( \mathcal{N} = 4 \) superspace. Our superspace conventions coincide with those of [24] and are as follows: \( D_a = \frac{\partial}{\partial \eta_a} + i\bar{\eta}_a\partial_t \), \( D^a = -\frac{\partial}{\partial \eta_a} - i\bar{\eta}^a\partial_t \), and \( \{D_a, D^b\} = -2i\delta_a^b\partial_t \). Indices are raised and lowered with \( \epsilon_{ab} \) and its inverse \( \epsilon^{ab} \) (\( \epsilon_{ab}\epsilon^{bc} = \delta_a^c \)). To reduce the number of indices we will sometimes suppress contracted indices with the understanding that the first index should be in a “natural” position. The action given in [24] was constructed in terms of a real superfield with components \( \phi = x \), \( D_a\phi = i\psi_a \), \( D^a\phi = -i\bar{\psi}^a \) and \( [D_{(a}, D_{b)}]\phi = F_{ab} \), where \( | \) as usual is shorthand for \( |_{\eta_a=0,\eta^a=0} \). Since the representation corresponding to the real superfield \( \phi \) is not irreducible one has to constrain the superfield. The following constraints were used in [24]:

\[ D^2\phi = -\frac{1}{\phi} D\phi D\phi , \]

\[ D^2\phi = -\frac{1}{\phi} D\phi D\phi \] and \( [D_a, D^a]\phi \equiv [D, D]\phi = -\frac{2}{\phi} D\phi D\phi + \frac{4}{\phi} \). These constraints are the one-dimensional analogue of the constraints for the four-dimensional tensor multiplet [37].

The superspace action is

\[ S = \frac{1}{8} \int dt D_a D^a \bar{D}_b \bar{D}_b \left( -\frac{1}{2} \phi^2 \ln |\phi| \right) . \]

After passing to components and eliminating the auxiliary field \( F_{ab} \) one obtains the action

\[ S = \frac{1}{2} \int dt \left[ \dot{x}^2 - i\bar{\psi}\dot{\psi} + i\bar{\psi}\dot{\psi} - \frac{(\nu + \bar{\psi}\psi)^2}{x^2} \right] . \]
For completeness we now briefly describe how to pass to the hamiltonian form and then to quantum mechanics. The classical hamiltonian is \( H_c = xp + p_\psi \bar{\psi} + p_{\bar{\psi}} \bar{\psi} - L \), where \( p = \frac{\delta L}{\delta \dot{x}} \), \( p_\psi = \frac{\delta L}{\delta \dot{\psi}} \), and \( p_{\bar{\psi}} = \frac{\delta L}{\delta \dot{\bar{\psi}}} \). are the conjugate momenta (fermionic variational derivatives act from the left). The canonical Poisson brackets are \( \{p, x\} = -1 \), \( \{p_\psi, \psi\} = \{p_{\bar{\psi}}, \bar{\psi}\} = -\delta_a^b \), and \( \{p_\psi, \bar{\psi}\} = \delta_a^b \). Using standard methods to deal with the second class constraints \( \Upsilon_a = p_\psi - \frac{i}{2} \bar{\psi} \approx 0 \) and \( \bar{\Upsilon}^a = p_{\bar{\psi}} - \frac{i}{2} \bar{\psi} \approx 0 \), lead to the Dirac brackets \( \{\psi_a, \bar{\psi}^b\}^* = i \delta_a^b \). The Noether charges associated to the supersymmetry invariance of the action are \( Q_a \) and \( \bar{Q}^a \). At this point we deviate from the particular model discussed so far and assume the supercharges to be of the more general form

\[
Q_a = \psi_a \left( p - i w[0](x) \right) - i w[1](x) \bar{\psi}_a \bar{\psi}_b , \quad \bar{Q}^a = \bar{\psi}^a \left( p + i w[0](x) \right) + i w[1](x) \psi_a \bar{\psi}_b .
\]  

(3.3)

In order for \( \{Q_a, \bar{Q}^b\}^* = 2i \delta_a^b H_c \) to be satisfied, the following condition has to be fulfilled: \( \partial w[0] = w[0] w[1] \), and \( H_c \) is then determined to be

\[
H_c = \frac{1}{2} p^2 + \frac{1}{2} (w[0])^2 + \partial w[0] \bar{\psi}^c \psi_c + \frac{1}{2} \partial w[1] (\bar{\psi}^c \psi_c)^2 .
\]  

(3.4)

In the conformal case both \( w[0] \) and \( w[1] \) are proportional to \( \frac{1}{x} \) and the equation \( \partial w[0] = w[0] w[1] \) has two solutions corresponding to the two solutions found in the quantum case: \( w[0] = \nu, \quad w[1] = -\frac{1}{x} \) and \( w[0] = 0, \quad w[1] = -\frac{2\nu}{x} \), where \( \nu \) is a constant. The next step is to pass to the quantum theory using the usual rule \( \{\ldots\} \rightarrow i[\ldots] \). Since \( [\psi_a, \psi^b]_+ = \delta_a^b \), the fermions can be realised as \( \psi_a = \theta_a \) and \( \bar{\psi}^a = -\frac{\partial}{\partial \theta^a} \). One has to deal with ordering ambiguities in the supercharges (such ambiguities are absent for \( N = 2 \) systems). Requiring that the supercharges still come in hermitean-conjugate pairs after quantisation (which guarantees that the hamiltonian is hermitean) and that the supersymmetry algebra is still satisfied i.e. \( [Q_a, \bar{Q}^b]_+ = 2 \delta_a^b H \), fixes the ordering ambiguities. We then regain the supercharges and hamiltonian given earlier in (2.2) and (2.3).

What about superspace formulations for the models corresponding to the more general supercharges \( [3.3] \)? For instance, for the other conformal model, the superspace action is (when \( \nu \neq \frac{1}{2} \))

\[
S = \frac{1}{8} \int dt D_\alpha D^\alpha \bar{D}^\alpha \bar{D}_\beta \frac{1}{1 - 2\nu} [-\frac{1}{2} \phi^2] ,
\]  

(3.5)

and the constraints are: \( D^2 \phi = -\frac{2\nu}{\phi} D\phi \bar{D} \phi, \quad \bar{D}^2 \phi = -\frac{2\nu}{\phi} \bar{D} \phi \bar{D} \phi \) and \( [D, \bar{D}] \phi = -\frac{4\nu}{\phi} D\phi \bar{D} \phi \). In components the action becomes

\[
S = \frac{1}{2} \int dt \left[ x^2 - i \bar{\psi} \dot{\psi} + i \dot{\bar{\psi}} \bar{\psi} - \frac{2\nu}{x^2} (\psi \bar{\psi})^2 \right] .
\]  

(3.6)

Although the potential has no “bosonic” part, the quantum potential has such a part, which, as we have seen, arises from ordering ambiguities. The actions for the models with supercharges \( [3.3] \) can also be written in superspace; the general construction will be given in the next subsection.

\[\text{When } \nu = \frac{1}{2} \text{ the model is a special case of the SU}(1, 1/2) superconformal one and is described by the action } [3.3] \text{ and its associated constraints with } \nu = 0.\]
3.2 Many-body models

The above results will now be extended to many-body systems. In this section we use the Einstein summation convention: repeated indices are summed. The construction involves two functions, \( w_1(x_t) \) and \( w_{ijk}(x_t) \), which are assumed to satisfy the following constraints:

\[
\begin{align*}
 w_1(x_t) &= \partial_t w_1(x_t), \\
 w_{ijk}(x_t) &= \partial_i \partial_j \partial_k w_{ijk}, \\
 w_{ijkl} w_{ijkl} &= 0, \\
 \partial_i \partial_j w_1(x_t) &= \partial_i w_1(x_t) w_{ij}. 
\end{align*}
\]  
(3.7)

The following action

\[
S = \frac{1}{2} \int dt [\dot{x}_i \dot{x}^i - i \bar{\psi}_i \dot{\psi}^i + i \bar{\psi}_i \dot{\psi}^i - (\partial_i w_1(x_t))^2 + 2 \partial_i \partial_j w_1(x_t) \psi^i \bar{\psi}^j - \partial_i \partial_j \partial_k w_1(x_t)(\psi^i \bar{\psi}^j)(\bar{\psi}^k \psi^l)],
\]  
(3.8)

is supersymmetric (see below) if the constraints (3.7) hold. The associated supersymmetry Noether charges are

\[
Q_a = \psi^i (p_i - i \partial_t w_1(x_t)) \psi^j (\bar{\psi}^j (\bar{\psi}^i) - \bar{\psi}^i (\bar{\psi}^j)), \\
Q^a = \bar{\psi}^i (p_i + i \partial_t w_1(x_t)) \psi^j (\bar{\psi}^j (\bar{\psi}^i) - \bar{\psi}^i (\bar{\psi}^j)),
\]  
(3.9)

and satisfy \( \{Q_a, \bar{Q}^b\}^*_+ = 2i \delta^b_a H_c \), where \( H_c \) is the classical hamiltonian associated to the lagrangean which can be read off from (3.8).

The conformal \( A_{M-1} \mathcal{N} = 4 \) Calogero models corresponding to the ones constructed in the quantum case (cf. (2.17)) have \( w_1(x_t) = 0 \), which means that classically they have no bosonic potential, however, after passing to quantum mechanics a bosonic potential is generated as a result of ordering ambiguities.

The models (3.8) can also be written in superspace. To this end we introduce \( M \) real superfields \( \phi_i \) with components \( \phi_i^| = x_i, D_a \phi_i^| = i \psi^i_a, \bar{D}^b \phi_i^| = -i \bar{\psi}^b_i \) and \( [D_a, \bar{D}_b] \phi_i^| = F^i_{ab} \), while the other components have to be constrained. We introduce the following constraints:

\[
\begin{align*}
D^2 \phi_i &= w_{ijk}^{[1]}(\phi_i) D \phi^j D \phi^k, \\
\bar{D}^2 \phi_i &= w_{ijk}^{[1]}(\phi_i) \bar{D} \phi^j \bar{D} \phi^k, \\
[D, \bar{D}] \phi_i &= 2 w_{ijk}^{[1]}(\phi_i) D \phi^j \bar{D} \phi^k - 4 w_1^{[0]}(\phi_i). 
\end{align*}
\]  
(3.10)

In this context the constraints (3.7) can be viewed as consistency conditions for the superspace constraints (3.11). We take the superspace action to be of the form

\[
S = \frac{1}{8} \int dt D_a D^a \bar{D}_b \bar{D}^b A(\phi_i),
\]  
(3.11)

where the scalar functional \( A \) is assumed to satisfy the equation

\[
\partial_i \partial_j A + w_{ijk}^{[1]} \partial_k A = -\delta_{ij}.
\]  
(3.12)

The rationale for this choice is that it implies \( \bar{D}^2 A = -\bar{D} \phi_i \bar{D} \phi^i \). The component action can then easily be obtained using the constraints (3.7), (3.11), with the result (3.8). The superspace equation of motion is

\[
[D_a, \bar{D}_b] \phi_i = -2 w_{ijk}^{[1]} D_a \phi^j \bar{D}_b \phi^k,
\]  
(3.13)
and it can be shown that it reproduces the component equations of motion derived from (3.8), after the auxiliary fields \(F_{ab}\) are eliminated.

The requirement (in addition to conformal invariance) for SU(1,1|2) superconformal invariance is \(x^j w_{ij}^{[1]} = -\delta_{ij}\). The question arises if/how \(A\) is related to \(w^{[1]}\) and \(w^{[0]}\). One possibility is that \(A = w^{[1]}\). This choice is consistent with the condition for SU(1,1|2) superconformal invariance (and in fact implies it if the matrix \(M_{ij} = \partial_i \partial_j w^{[1]} + \delta_{ij}\) is invertible and the theory is conformal). Of the many-body models constructed before in section 2, only the classical four-body model corresponding to (2.25) satisfies this constraint.

From the lagrangean read off from (3.8) or from the Dirac bracket of the Noether charges (3.9), one can obtain the classical hamiltonian and then pass to quantum mechanics to regain the results obtained in the previous section. The seeming difference between the classical quantities and the quantum ones result from ordering ambiguities.

4 Discussion

Although we only constructed \(\mathcal{N} \geq 4\) extensions of the \(A_{M-1}\) Calogero models, it should also be possible to extend the results to the Calogero models based on the other root systems. Another question is whether it is possible to extend the results to the super-Sutherland models \([32, 21, 22]\). A more uniform formulation along the lines of the one in \([22]\) would also be desirable. In particular, the integrability properties merit further investigation. The results on exact solvability \([15, 16, 21, 17]\) are expected to hold also for the supersymmetric extensions (of the models with discrete spectrum).

Another issue worth studying is supersymmetry breaking (extending the results in \([38]\)). One could also investigate more general models, e.g. by introducing a non-trivial metric so that \(H = \frac{1}{2} g^{ij} p_i p_j + V(x_i)\). It may be interesting to try and lift the more general superspace constraints \((1.10)\) for \(\phi_i\) to four dimensions, which might lead to a generalisation of the result in \([37]\) to many fields.

It was conjectured by Gibbons and Townsend \([1]\) that an \(\mathcal{N} = 4\) SU(1,1|2) superconformal extension of the \(A_{M-1}\) Calogero models could provide a microscopic description of an extremal \(d = 4\) Reissner-Nordström black hole. This conjecture was partly based on the observation \([39, 27]\) that the radial motion of a super-particle in the near-horizon limit of a large-mass extremal \(d = 4\) Reissner-Nordström black hole is described by the SU(1,1|2) superconformal mechanics model of \([24]\). A related issue is the quantum mechanics of \(M\) slowly moving extremal black holes in four dimensions. This multi-black hole mechanics should be described in terms of \(3M\) bosonic and \(4M\) fermionic degrees of freedom (just as the multi-black holes in five dimensions discussed in \([3]\) are described in terms of \(4M\) bosonic and \(4M\) fermionic coordinates). Thus, it would seem that models with \(M\) bosonic and \(4M\) fermionic coordinates are perhaps more naturally connected with two-dimensional black holes. However, the near-horizon geometry of an extremal \(d = 4\) Reissner-Nordström black hole is adS\(_2 \times S_2\) and there is a natural “angular/radial” split. Hence, it is not excluded that models with \(M\) bosonic and \(4M\) fermionic coordinates may provide a microscopic description of \(d = 4\) black holes (this is in the spirit of the adS/CFT correspondence). Such a many-body model is expected to have an SU(1,1|2)
superconformal symmetry. For generic values of the coupling constant, we have not been able to construct an $\mathcal{N} = 4$ extension of the $A_{M-1}$ Calogero models with an SU(1,1|2) symmetry. The only possible way around this result is to change the supercharges. Another possibility is that another generalisation of the one-body case is needed, however, without further input it is not clear which assumptions should be made to pinpoint such a model. One criterion one could use [1] is that when all coordinates but one are small, then the model should reduce to the one-body SU(1,1|2) model. Even if it turns that there is no direct connection between the models considered in this paper and black hole physics, they may still be valuable as toy models.

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A The su(1,1|\(N/2\)) algebras

The su(1,1|2) Lie superalgebra generators comprise the odd elements $Q_a$, $S_a$ and their hermitean conjugates, together with the even hermitean generators $J_A$, $K$, $D$, and $H$. A list of supercommutators sufficient to specify the algebra completely is:

\[
\begin{align*}
[Q_a, Q_b]_+ & = 2\delta_a^b H, \\
[Q_a, Q_b]_+ & = 0, \\
[S_a, S_b]_+ & = 2\delta_a^b K, \\
[S_a, S_b]_+ & = 0, \\
[J_A, Q_b] & = -\frac{1}{2}(\sigma^A)_a^b Q_c, \\
[J_A, S_a] & = -\frac{1}{2}(\sigma^A)_a^c S_c, \\
[Q^T, D] & = \frac{i}{2} Q^a, \\
[S^T, D] & = -\frac{i}{2} S^a, \\
[H, D] & = iH, \\
[J_A, J_B] & = i\epsilon_{ABC} J_C, \\
[K, Q^T] & = iS^a, \\
[H, S^T] & = -iQ^a.
\end{align*}
\]

(A.1)

Here $a, b = 1, 2; A, B, C = 1, 2, 3$, and the $2 \times 2$ matrices $(\sigma^A)_a^b$ are generators of su(2) and are in this paper taken to be the usual Pauli matrices satisfying the relation $[\sigma_A, \sigma_B] = 2\epsilon_{ABC}\sigma_C$. In (A.1) $T$ is a central element which can be removed, however, such a central extension is present for some of the models considered in this paper.

The generators of the Lie superalgebra su(1,1|\(N/2\)) comprise the odd elements $Q_a$, $S_a$ and their hermitean conjugates, together with the even generators $J_a^b$, $K$, $D$, and $H$. The supercommutators of su(1,1|\(N/2\)) are the same as in (A.1) with the following differences. Now the indices take the values $a, b = 1, \ldots, N/2$ and the $\sigma^A$'s are replaced by $\frac{N}{2} \times \frac{N}{2}$ matrix generators of su(\(N/2\)). In this paper we use the following realisation of these generators: $(\lambda_a^b)_c = -2\delta_a^c (\delta^d_b - \delta^b_d)$ (when $a \neq b$) and $(\lambda_a^b)_c = -2(\delta_a^d \delta^b_c - \frac{1}{2} \delta^b_d)$. They satisfy $[\lambda_a^b, \lambda_c^d] = 2(\delta_a^d \lambda^b_c - \delta^b_d \lambda_a^c)$. Furthermore, the anticommutation relations between the
$Q$’s and the $S$’s are replaced by

\[
[Q_a, S_b^{\dagger}] = \frac{i}{2} (\lambda^d_a b J_d^c + 2i \delta^b_a U - 2 \delta^b_a D - i \delta^b_a T),
\]

\[
[S_a, Q_b^{\dagger}] = -\frac{i}{2} (\lambda^c_a b J_c^d - 2i \delta^b_a U - 2 \delta^b_a D + i \delta^b_a T).
\]  

(A.2)

The commutation relations involving the additional $u(1)$ generator $U$ are

\[
[U, S_a] = \frac{1}{2} S_a, \quad [U, S_b^{\dagger}] = -\frac{1}{2} S_b^{\dagger},
\]

\[
[U, Q_a] = \frac{1}{2} Q_a, \quad [U, Q_b^{\dagger}] = -\frac{1}{2} Q_b^{\dagger},
\]  

(A.3)

and finally, $[J_a^b, J_c^d] = (\delta^c_b J_a^d - \delta^d_a J_c^b)$.

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