Birational classification of moduli spaces of vector bundles over \(\mathbb{P}^2\)

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Abstract

The depth of a vector bundle \(E\) over \(\mathbb{P}^2\) is the largest integer \(h\) such that \([E]/h\) is in the Grothendieck group of coherent sheaves on \(\mathbb{P}^2\) where \([E]\) is the class of \(E\) in this Grothendieck group. We show that a moduli space of vector bundles is birational to a suitable number of \(h\) by \(h\) matrices up to simultaneous conjugacy where \(h\) is the depth of the vector bundles classified by the moduli space. In particular, such a moduli space is a rational variety if \(h \leq 4\) and is stably rational when \(h\) divides 420.

1 Introduction

The purpose of this paper is to study moduli spaces of vector bundles over \(\mathbb{P}^2\) birationally. Particular cases are known to be rational and other cases, for example, the moduli space of vector bundles of rank \(n\), first Chern class 0 and second Chern class \(n\) are known to be birational to two \(n\) by \(n\) matrices up to simultaneous conjugacy. We shall see that this is a general phenomenon, that is, any such moduli space is birational to a suitable number of suitably sized matrices up to simultaneous conjugacy. Our method will be to reduce to a problem for representations of a suitable quiver with relations and then to apply the results and methods of [7]. The results of this paper depend heavily on those of [7] and the reader will be assumed to have some familiarity with this paper.

From the work of Beilinson [1], one knows that the category of vector bundles over \(\mathbb{P}^2\) is derived equivalent to the category of representations of a quiver with relations; this is the quiver with 3 vertices \(u\), \(v\) and \(w\) and 3 arrows from \(u\) to \(v\), which are \(x\), \(y\) and \(z\), and 3 arrows from \(v\) to \(w\) which are \(x'\), \(y'\) and \(z'\) with relations \(xy' = yx'\), \(xz' = zx'\) and \(yz' = zy'\). The path algebra of this quiver with relations is the endomorphism ring of \(\mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-2)\). Thus, roughly speaking, a moduli space of vector bundles over \(\mathbb{P}^2\) is also a moduli space of representations for this quiver with relations. In fact, one can show that the moduli spaces of representations that occur in this way may be taken to parametrise certain rather special representations and using the known results on moduli spaces of representations of quivers it is possible to show that these
moduli spaces are birational to a suitable number of matrices up to simultaneous conjugacy.

The next section introduces the notation that we shall use in this paper. Then section 3 studies particular moduli spaces of representations that include all the examples we shall need. In section 4 we show how to reduce a moduli space of vector bundles over $\mathbb{P}^2$ to one of these moduli spaces of representations.

2 Terminology

We introduce some notation and terminology. The terminology we introduce for representations of a multiplication below is useful since this is a particular case of moduli spaces of vector spaces of dimension $n$ and $R$ in a basis-free way. Let $U$ be a finite dimensional vector space. A representation $R$ of the vector space $U$ is a triple $(R(0), R(1), R(\phi))$ where $R(0)$ and $R(1)$ are finite dimensional vector spaces and $R(\phi): R(0) \otimes U \to R(1)$ is a linear map. Its dimension vector is $\dim R = (\dim R(0) \dim R(1))$. The representations of dimension vector $\alpha = (a \ b)$ are parametrised by the vector space $R(U, \alpha) = \Hom(k^a \otimes V, k^b)$ on which the algebraic group $\Gl_a \times \Gl_b$ acts by change of bases. The orbits correspond to the isomorphism classes of representations. Of course the category of representations of a vector space of dimension $n$ is just the category of representations of a quiver with two vertices and $n$ arrows from the first to the second vertex as one sees by choosing a basis of $U$. We note that $R(\phi)^\vee: R(1)^\vee \to R(0)^\vee \otimes U^\vee$ gives a representation of $U$, $R^\vee$, from the linear map $R^\vee(\phi): R(1)^\vee \otimes U \to R(0)^\vee$.

A multiplication is a quadruple $(U, V, W, f)$ where $U$, $V$ and $W$ are vector spaces and $f: U \otimes V \to W$ is a linear map. We shall usually talk of the multiplication $f$. A representation $R$ of the multiplication $f$ is a sextuple $(R(0), R(1), R(2), R(\phi_{01}), R(\phi_{12}), R(\phi_{02}))$ where each $R(i)$ is a finite dimensional vector space and $R(\phi_{01}): R(0) \otimes U \to R(1)$, $R(\phi_{12}): R(1) \otimes V \to R(2)$ and $R(\phi_{02}): R(0) \otimes W \to R(2)$ are linear maps such that

$$(\phi_{01} \otimes I_V) \phi_{12} = (I_{R(0)} \otimes f) \phi_{02}$$

(1)

as linear maps from $R(0) \otimes U \otimes V$ to $R(2)$. We shall eventually be interested in representations of the multiplication $\sigma: k^3 \otimes k^3 \to S^2(k^3)$ since the results of Beilinson show that the derived category of coherent sheaves on $\mathbb{P}^2$ is equivalent to the derived category of representations of the multiplication $\sigma$. The dimension vector of a representation $R$ of the multiplication $f$ is $\dim R = (\dim R(0) \dim R(1) \dim R(2))$. The representations of dimension vector $\alpha = (a \ b \ c)$ are parametrised by the closed subvariety $R(f, \alpha)$ of $\Hom(k^a \otimes U, k^b) \times \Hom(k^b \otimes V, k^c) \times \Hom(k^a \otimes W, k^c)$ of triples $(\phi_{01}, \phi_{12}, \phi_{02})$ satisfying equation (1). The algebraic group $\Gl_a \times \Gl_b \times \Gl_c$ acts via change of bases on $R(f, \alpha)$ and the orbits correspond to the isomorphism classes of representations of dimension vector $\alpha$ of the multiplication. Vector space duality gives rise to linear maps $R(2)^\vee \otimes V \to R(1)^\vee$, $R(1)^\vee \otimes U \to R(0)^\vee$ and $R(2)^\vee \otimes W \to R(0)^\vee$. 

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and this gives a representation of the multiplication \( \hat{f} : V \otimes U \to W \) obtained from \( f \) by switching \( U \) and \( V \).

\( R(f, \alpha) \) is in general a reducible variety and the description of its components and their orbit spaces in complete generality is not something we shall undertake in this paper. However, in the case arising from vector bundles over \( \mathbb{P}^2 \), we can restrict to components parametrising representations of a fairly nice form for which the moduli space is relatively comprehensible.

Given a dimension vector \( \alpha \) for a multiplication, \( \alpha_l \) will be the dimension vector \((\alpha(0) \; \alpha(1))\) and \( \alpha_r \) will be the dimension vector \((\alpha(1) \; \alpha(2))\). If \( R \) is a representation of the multiplication \( f : U \otimes V \to W \), then \( \alpha_l \) is the dimension vector of the representation of the vector space \( U \) obtained by restriction and \( \alpha_r \) is the dimension vector of the representation of the vector space \( V \) obtained by restriction. Thus we have morphisms from \( R(f, \alpha_l) \) to \( R(U, \alpha_l) \) and to \( R(V, \alpha_r) \).

Let \( C \) be an irreducible component of \( R(f, \alpha) \) such that the morphism to \( R(U, \alpha) \) is dominant; then we shall say that \( C \) is a left general component and that a general representation in the component \( C \) is left general. The term right general is defined in a similar way using the morphism to \( R(V, \alpha_r) \). Later results about left general representations have analogous results for right general representations since duality will carry the one to the other.

Representations of a vector space or of a multiplication are special cases of the more general notion of representations of a quiver or of a quiver with relations as one sees by choosing bases for the vector spaces \( U \), \( V \) and \( W \).

We refer the reader to section 2 of [7] for the terminology we shall use for representations of a quiver.

A relation on a quiver \( Q \) is a linear combination of paths \( r = \sum_{i=1}^{n} \lambda_i p_i \) such that the initial and terminal vertex of the path \( p_i \) are all equal. Given a representation \( R \) of the quiver, we may extend our notation by defining \( R(p) = R(a_1) \ldots R(a_n) \) if \( p = a_1 \ldots a_n \) and \( R(r) = \sum_{i=1}^{n} \lambda_i R(p_i) \). Given a set of relations \( I \), a representation \( R \) of the quiver with relations \((Q, I)\) is a representation of the quiver \( Q \) such that \( R(r) = 0 \) for every \( r \in I \). The category \( \text{Rep}(Q, I) \) is the full subcategory of \( \text{Rep}(Q) \) whose objects are the representations of the quiver with relations \((Q, I)\).

A family \( \mathcal{R} \) of representations of dimension vector \( \alpha \) of the quiver with relations \((Q, I)\) over an algebraic variety \( X \) is a collection of vector bundles \( R(v) \) for each vertex \( v \) and homomorphisms of vector bundles \( \mathcal{R}(a) : R(ia) \to R(ta) \) for each arrow \( a \) such that for each relation \( r \), \( \mathcal{R}(r) = 0 \). Given a point \( p \in X \) there is an associated representation \( \mathcal{R}_p \), the fibre of \( \mathcal{R} \) above the point \( p \). When \( X \) is an irreducible algebraic variety we shall say that the family is irreducible and in this case we shall say that a representation \( R \) is of type \( \mathcal{R} \) if there exists a point \( p \) such that \( R \cong \mathcal{R}_p \). Again when \( \mathcal{R} \) is an irreducible family, we shall say that a general representation of type \( \mathcal{R} \) has property \( P \) if there exists a dense open subvariety \( O \) of \( X \) such that for all \( p \in O \), \( \mathcal{R}_p \) has property \( p \).

Given a dimension vector \( \alpha \), the vector space \( R(Q, \alpha) = \bigoplus_{a \in A} \alpha(ia) k^{\alpha(ta)} \) parametrises the representations of the quiver of dimension vector \( \alpha \) and this carries a family \( \mathcal{R} \) of representations of the quiver which we shall refer to as the canonical family. If \( V \) is a locally closed subvariety of \( R(Q, \alpha) \), then the
There is a closed subvariety $R(Q, I, \alpha)$, the representation space of dimension vector $\alpha$ for the quiver with relations $(Q, I)$ consisting of those points $p$ such that $R_p(r) = 0$ for every relation $r$ in $I$. In general this is a reducible algebraic variety.

The algebraic group $Gl_\alpha = \times_v Gl_{\alpha(v)}$ acts on this family compatibly with an action of $P Gl_\alpha = Gl_\alpha/k^*$ on $R(Q, I, \alpha)$ and the orbits of $P Gl_\alpha$ on $R(Q, I, \alpha)$ correspond to the isomorphism classes of representations of dimension vector $\alpha$. In addition, if $p \in R(Q, \alpha)$, the stabiliser of $p$ in $Gl_\alpha$ acts on the representation defined by the point $p$ as the units of its endomorphism ring.

Let $X$ be an algebraic variety on which the algebraic group $G$ acts. Let

$$1 \to k^* \to \tilde{G} \to G \to 1$$

be a short exact sequence of algebraic groups. Let $E$ be a vector bundle over $X$ on which $\tilde{G}$ acts compatibly with the action of $G$. Then $k^*$ acts on the fibres of $E$ and if this action is via the character $\phi_v(\lambda) = \lambda^w$ then we shall say that $E$ is a $\tilde{G}$ vector bundle of weight $w$. A morphism of $\tilde{G}$ vector bundles of weight $w$ is a morphism of vector bundles that is also $\tilde{G}$ equivariant. We see that if $R$ is a canonical family of representations of dimension vector $\alpha$ then each $R(v)$ is a $Gl_\alpha$ vector bundle of weight 1.

Two irreducible families of representations of the quiver with relations $(Q, I)$, $R$ over $X$ and $S$ over $Y$, are said to be birationally representation equivalent if there exist open dense subvarieties $O \subset X$ and $O' \subset Y$ such that for all $p \in O$, there exists $q \in Y$ such that $R_p \cong S_q$ and for all $q' \in O'$ there exists $p \in X$ such that $R_p \cong S_q$. We shall say that the family $S$ birationally contains the family $R$ if there exists an open subvariety $O \subset X$ such that for all $p \in O$, there exists $q \in Y$ such that $R_p \cong S_q$. We shall say that the family $R$ is birationally constant if there exists a dense open subvariety $O$ of $X$ such that $R_p \cong R_q$ for all $p$ and $q$ in $O$.

We shall say that a family $R$ of representations of dimension vector $\alpha$ is reducible to matrix normal form of type $h$ if it is birationally representation equivalent to a family $S$ over an irreducible algebraic variety $X$ on which $P Gl_h$ acts such that $X$ is $P Gl_h$ birational to $M_t(k)^t$ for some non-negative integer $t$ where $P Gl_h$ acts by conjugation on each factor, and each $S(v)$ is a $Gl_h$ vector bundle of weight 1 so that the action of the stabiliser in $Gl_h$ of a point $p$ in $X$ acts on $S_p$ as the unit group of the endomorphism ring of $S_p$. It is said to be reducible to matrix normal form if in addition $h = hcf_v(\alpha_v)$. A family $S$ with these properties will be called an $R$-standard family.

### 3 Left general components

The purpose of this section is to show that a moduli space of left general representations of a multiplication is birational to a suitable number of matrices up to simultaneous conjugacy when there is a left general representation with trivial endomorphism ring. We shall actually prove a more general result about
families of representations of a quiver with relations which may be of future use. We begin by showing that there is only one left general component of given dimension vector.

**Lemma 3.1.** Let $\alpha$ be a dimension vector for the multiplication $f$. There is a unique left general component of dimension vector $\alpha$.

**Proof.** Let $\alpha = (a\ b\ c)$ and let $f: U \otimes V \to W$ be the multiplication. Let $O$ be the open subvariety of $R(U,(a\ b))$ of points such that the linear map $I \otimes f \oplus -R_p(\phi) \otimes I: k^a \otimes U \otimes V \to k^a \otimes W \oplus k^b \otimes V$ has maximal rank. Then $O$ carries a vector bundle $E$ whose fibre above the point $p$ is simply the cokernel of $I \otimes f \oplus -R_p(\phi) \otimes I$. We consider the vector bundle $E^\vee \otimes k^c$ which is an irreducible variety and has a morphism to $R(f,(a\ b\ c))$ whose image must contain an open dense subvariety of any left general component since its image contains every point $q$ such that the left restriction of $R_q$ is isomorphic to a representation of the form $R_p$ for $p \in O$. Its image is irreducible and consequently must lie in one of these components so it follows that there must be precisely one left general component and the image of $E^\vee \otimes k^c$ must lie in this unique component.

We shall need to be able to recognise that a representation of a multiplication lies in the left general component for its dimension vector.

**Lemma 3.2.** Let $f: U \otimes V \to W$ be a multiplication such that $f$ is surjective. Let $K$ be the kernel of $f$. Let $R$ be a representations of the multiplication $f$ such that the composition of the linear maps from $R(0) \otimes K$ to $R(0) \otimes U \otimes V$ to $R(1) \otimes V$ is injective. Then $R$ lies in the left general component of representations of its dimension vector.

**Proof.** The injectivity of this linear map implies that the map considered in the proof of the previous lemma has maximal rank.

In order to prove the main theorem of this section we shall need to summarise the information we already have on representations of a vector space.

We consider representations of the vector space $U$ of dimension vector $(a\ b)$. Let $g = \text{hcf}(a,b)$. After choosing a basis of $U$, we see that we are simply considering representations of a generalised Kronecker quiver, that is, a quiver $Q(u)$ where $u = \dim U$ which has two vertices $v$ and $w$ and $u$ arrows from $v$ to $w$. We shall be interested in the structure of general representations of these quivers. If $u = 1$, nothing difficult happens. For a dimension vector $(a\ b)$, if $a < b$, the general representation is isomorphic to $(k\ k)^a \oplus (0\ k)^{b-a}$; if $a = b$, the general representation is isomorphic to $(k\ k)^a$ and if $a > b$ the general representation is isomorphic to $(k\ 0)^{a-b} \oplus (k\ k)^b$. For $u > 1$, we need to introduce some terminology. The results in this case which are described below may be found on page 159 in [3]. The projective indecomposable representations of $U$ are $P_0 = (0\ k)$ and $P_1 = (k\ U)$ whilst the injective indecomposable representations are $I_0 = (k\ 0)$ and $I_1 = (U^\vee\ k)$ which are simply the dual representations to the projective representations. We assume that we have constructed representations $\{P_i : i = 0 \to t\}$ and we have shown that $\text{Hom}(P_i, P_{i+1}) \cong U$ if $i$ is even.
whilst \( \text{Hom}(P_i, P_{i+1}) \cong U^\vee \) if \( i \) is odd. Then if \( t \) is odd, there is a canonical homomorphism from \( P_{t-1} \) to \( U^\vee \otimes P(t) \) and we define \( P_{t+1} \) to be the cokernel whilst if \( t \) is even, there is a canonical homomorphism from \( P_{t-1} \) to \( U \otimes P \) whose cokernel we define to be \( P_{t+1} \). One may check that the inductive hypothesis has been extended and thus we have defined representations \( P_n \) for all integers \( n \). These representations are called the preprojective representations of \( U \). We define \( I_n = P_n^\vee \) and these are the preinjective representations. In the case where \( u = 2 \), \( P_i \cong (S^{i-1}(U) \otimes S^i(U)) \) for \( i > 0 \) where the linear map from \( S^{i-1}(U) \otimes U \) to \( S^i(U) \) is the obvious one. These representations are important to us because they allow us to describe the general representations of arbitrary dimension vector for a vector space \( U \).

We first state the results for \( u = 2 \). We consider the dimension vector \((a, b)\). If \( a \neq b \), then \( GL_a \times GL_b \) has an open orbit in \( R(U, (a, b)) \) and the corresponding representation will be called \( G(a, b) \). If \( a < b \) then for some integer \( m \), \( \frac{m - 1}{m} \leq \frac{a}{b} < \frac{m + 1}{m} \). If \( \frac{a}{b} = \frac{m - 1}{m} \) then \( G(a, b) \) is isomorphic to \( P_{m - 1}^{a/m} \) and clearly \( \frac{b}{a} = \text{hcf}(a, b) \). Otherwise \( G(a, b) \) is isomorphic to \( P_{m - 1}^c \oplus P_m^d \) where \( c \) and \( d \) are non-zero integers such that \( \text{hcf}(c, d) = \text{hcf}(a, b) \). If \( a > b \) then duality leads to the same picture using projective representations. Thus if \( \frac{a}{b} = \frac{m - 1}{m} \) then \( G(a, b) \cong I_{m - 1}^{a/m} \) where \( \frac{m}{m + 1} = \text{hcf}(a, b) \) and if \( \frac{m}{m + 1} > \frac{b}{a} > \frac{m - 1}{m} \) then \( G(a, b) \) is isomorphic to \( I_{m - 1}^c \oplus I_m^d \) where \( \text{hcf}(c, d) = \text{hcf}(a, b) \). If \( a = b \), then \( GL_a \times GL_b \) does not have an open orbit on \( R(U, (a, b)) \), however, this case is essentially one \( a \) by \( a \) matrix up to simultaneous conjugacy as we shall now see. There is a family of representations of the vector space \( U \) on the algebraic variety \( X = M_a(k) \) on which \( PGL_a \) acts by conjugation defined as follows. Let \( \{u_1, u_2\} \) be a basis of \( U \); let \( R(0) = R(1) = k^a \times X \) and \( R_p(\phi)(v_1 \otimes u_1 + v_2 \otimes u_2) = v_1 + v_2 \phi \) (recall that \( p \in X = M_a(k) \)). Note that this is a reduction to matrix normal form for the dimension vector \((a, a)\).

This leaves the case where \( u > 2 \). If \( (a, b) \) is a Schur root then \((a, b)\) is reducible to matrix normal form. Otherwise \( GL_a \times GL_b \) has an open orbit on \( R(U, (a, b)) \) and the corresponding representation \( G(a, b) \) is isomorphic to \( P_m^c \oplus P_m^d \) for suitable integers \( m, c \) and \( d \) when \( a < b \) and is isomorphic to \( I_m^c \oplus I_m^d \) when \( a > b \). Further \( \text{hcf}(c, d) = \text{hcf}(a, b) \) which is not demonstrated in these theorems but may be checked quickly by use of the reflection functors to reduce to the case where \( m = 0 \) where it is clear.

In all the cases where \( G(a, b) \) is defined and is isomorphic to either \( P_m^c \) or \( I_m^c \), the dimension vector is reducible to matrix normal form trivially; so \( X \) is a point on which \( PGL_g \) acts trivially; the family is simply the representation \( P_m^c \) which has endomorphism ring \( M_g(k) \) and \( GL_g \) acts on the family via its action as the group of automorphisms of \( P_m^c \).

Thus we have a useful dichotomy; either the dimension vector \((a, b)\) is reducible to matrix normal form or else \( GL_a \times GL_b \) has an open orbit on \( R(U, (a, b)) \) and the corresponding representation \( G(a, b) \) is isomorphic to either \( P_m^c \oplus P_m^d \) for positive integers \( c \) and \( d \) and non-negative integer \( m \) or else to \( I_m^c \oplus I_m^d \) with the same conditions on \( c, d \) and \( m \). In these cases, \( \text{hcf}(c, d) = \text{hcf}(a, b) \). Further, \( P_m^c \oplus P_{m + 1}^d \) has a unique subrepresentation of dimension vector \( \dim P_{m + 1}^d \) whilst
such that the restriction of is reducible to matrix normal form. Then we may regard as a diagonal embedding. Each arrow a vertex determines a homomorphism from \( \text{Gl} \) a morphism of algebraic varieties from \( X \) and has the property that the pullback of the canonical family on the image of \( \alpha \) such that \( \beta \) acts trivially on \( \text{Gl}_h \) is isomorphic as \( \text{Gl}_h \) a suitable dimension vector \( \dim \text{Gl}_h = (d e) \) where \( \text{hcf}(d, e) = a' \).

For the next result we shall regard \( \text{Gl}_h \) as a subgroup of \( \text{Gl}_\alpha \) via the diagonal embedding in each factor \( \text{Gl}_{\alpha(v)} \) and \( \text{PGL}_h \) as the corresponding subgroup of \( P\text{GL}_\alpha \).

**Lemma 3.3.** Let \( C \) be an irreducible component of \( R(Q, I, \alpha) \) and let \( \mathcal{R} \) be the canonical family of representations of dimension vector \( \alpha \) on \( C \). Assume that \( \mathcal{R} \) is reducible to matrix normal form. Then \( C \) has a \( \text{PGL}_h \) equivariant subvariety \( Y \) such that the restriction of \( \mathcal{R} \) to \( Y \) is a standard \( \mathcal{R} \)-family and therefore \( C \) is \( \text{PGL}_h \) birational to \( M_h(k)^s \times \text{PGL}_h \text{GL}_{\alpha} \) for some integer \( s \). In particular, a moduli space of representations of type \( C \) is birational to a suitable number of matrices up to simultaneous conjugacy.

**Proof.** Let \( S \) be a \( \mathcal{R} \)-standard family over the algebraic variety \( X \). Then for each vertex \( v \), \( S(v) \) is a vector bundle of weight 1 for \( \text{Gl}_h \) of rank \( \alpha(v) = h \beta(v) \) for a suitable dimension vector \( \beta \). Then by the local isomorphism theorem, lemma 3.1 of \[4\], there exists an open \( \text{PGL}_h \) equivariant subvariety \( X' \) of \( X \) such that the restriction of \( S(v) \) to \( X' \) is isomorphic as \( \text{Gl}_h \) vector bundle to \( k^{\beta(v)} \otimes k^h \times X' \) where \( \text{Gl}_h \) acts trivially on \( k^{\beta(v)} \) and diagonally on \( k^h \times X' \). We choose bases for each \( k^{\beta(v)} \) and a basis of \( k^h \) which give bases for each \( k^{\beta(v)} \otimes k^h \cong k^{\alpha(v)} \) and determines a homomorphism from \( \text{Gl}_h \) to \( \text{Gl}_{\alpha(v)} \) and hence to \( \text{Gl}_\alpha \) which we may regard as a diagonal embedding. Each arrow \( a \) determines a morphism of vector bundles from \( k^{\beta(v)} \otimes k^h \times X' \) to \( k^{\beta(v)} \otimes k^h \times X' \) and hence determines a morphism of algebraic varieties from \( X' \) to \( \alpha(v) \) and hence we have a morphism of algebraic varieties from \( X' \) to \( R(Q, \alpha) \) which is \( \text{PGL}_h \) equivariant and has the property that the pullback of the canonical family on the image of the morphism is \( S \). It follows that the image actually lies in \( R(Q, I, \alpha) \) and in fact must lie in the component \( C \). Since the stabiliser in \( \text{Gl}_h \) of a point \( p \) in \( X' \) is isomorphic to the units of \( \text{End}(S_p) \) and so is the stabiliser in \( \text{Gl}_\alpha \) of its image \( q \) in \( C \), we deduce that the morphism from \( X' \) to \( C \) is injective; indeed that the morphism from \( X' \times \text{PGL}_h \text{GL}_\alpha \) to \( C \) is injective and this latter morphism is also dominant since \( S \) is birationally representation equivalent to \( \mathcal{R} \). \( \square \)

Given irreducible families \( \mathcal{R} \) over \( X \) and \( S \) over \( Y \), the two functions on \( X \times Y \) that assign to the point \( (p, q) \) the values \( \text{hom}(\mathcal{R}_p, S_q) \) and \( \text{ext}(\mathcal{R}_p, S_q) \) are upper semicontinuous and consequently there exists an open dense subvariety \( O \subset X \times Y \) where these functions are constant and minimal; these minimal
values we shall call $\text{hom}(\mathcal{R}, S)$ and $\text{ext}(\mathcal{R}, S)$. If $X$ is a point and $R$ is the corresponding representation, we write $\text{hom}(R, S)$ and $\text{ext}(R, S)$; $\text{hom}(\mathcal{R}, S)$ and $\text{ext}(\mathcal{R}, S)$ are defined similarly for a representation $S$.

Let $O$ be the dense open subvariety of $X \times Y$ consisting of points $(p, q)$ where $\text{ext}(\mathcal{R}_p, S_q) = \text{ext}(\mathcal{R}, S)$. Then there is a vector bundle $E(\mathcal{R}, S)$ over $O$ whose fibre above the point $(p, q)$ is $\text{Ext}(\mathcal{R}_p, S_q)$ and there is a family of representations over $E(\mathcal{R}, S)$ of extensions of representations of type $\mathcal{R}$ on representations of type $S$. We shall call this family the extension family of $\mathcal{R}$ on $S$, $E(\mathcal{R}, S)$. If $Y$ is just a point then $S$ is just a representation $S$ and we shall refer to the extension family of $\mathcal{R}$ on $S$, $E(\mathcal{R}, S)$; similarly we define the extension family of $R$ on $S$ for a representation $R$.

Let $S$ be a representation such that $\text{End}(S) = k$ and $\text{Ext}(S, S) = 0$. We shall say that a representation $R$ has an $S$-socle if the natural map from $\text{Hom}(S, R) \otimes S$ to $R$ is injective and the image will be called the $S$-socle. If $\text{Hom}(S, R) = 0$ we shall say that $R$ is left $S$-free. When $R$ has an $S$-socle, $T$, then it is clear that $R/T$ is left $S$-free (apply $\text{Hom}(S, -)$ to the short exact sequence $0 \to T \to R \to R/T \to 0$). Dually we say that $R$ has an $S$-top if the natural map from $R$ to $\text{Hom}(R, S)^\vee \otimes S$ is surjective; if $K$ is the kernel of this homomorphism we say that $R/K$ is the $S$-top of $R$; again it follows that $\text{Hom}(K, S) = 0$ and we define a representation $R$ to be right $S$-free if $\text{Hom}(R, S) = 0$.

Now let $\mathcal{R}$ be an irreducible family over the algebraic variety $X$. Let $O$ be the dense open subvariety on which $\text{hom}(S, \mathcal{R}_p) = c = \text{hom}(S, \mathcal{R})$. Assume that there is a point $p$ in $O$ such that $\mathcal{R}_p$ has an $S$-socle and so the natural map from $\text{Hom}(S, \mathcal{R}_p) \otimes S$ to $\mathcal{R}_p$ is injective. Then a general representation of type $\mathcal{R}$ must have an $S$-socle which is isomorphic to $S^c$. Then on a suitable dense open subvariety $U$ of $X$, we have the associated left $S$-free family, $\mathcal{R}'$, where the representation $\mathcal{R}'_p$ is $\mathcal{R}_p/T_p$ where $T_p \cong S^c$ is the $S$-socle of $\mathcal{R}_p$. The family $\mathcal{R}$ is birationally contained in $\mathcal{E}(\mathcal{R}', S^c)$ and we shall say that $\mathcal{R}$ is left general with respect to $S$ if they are birationally representation equivalent. We note that if $\mathcal{R}$ is the canonical family on a component $C$ of $R(Q, I, \alpha)$ it is forced to be left general with respect to $S$ whenever a general representation of type $C$ has an $S$-socle.

Dually, a general representation of type $\mathcal{R}$ may have an $S$-top isomorphic to $S^d$; then we have a family on $X$, $\mathcal{R}'$, which we call the associated right $S$-free family, where $\mathcal{R}'_p$ is the kernel of the surjection from $\mathcal{R}_p$ onto $S^d$ and we define $\mathcal{R}$ to be right general with respect to $S$ if $\mathcal{R}$ and $\mathcal{E}(S^d, \mathcal{R}'')$ are birationally representation equivalent.

The following lemma is our main reduction.

**Lemma 3.4.** Let $\mathcal{R}$ be an irreducible family of representations of dimension vector $\alpha$ of the quiver with relations $(Q, I)$ over the algebraic variety $X$ such that a general representation of type $\mathcal{R}$ has trivial endomorphism ring. Let $S$ be a representation such that $\text{End}(S) = k$ and $\text{Ext}(S, S) = 0$. Assume that a general representation of type $\mathcal{R}$ has an $S$-socle and that $\mathcal{R}$ is left general with respect to $S$. Let $c = \text{hom}(S, \mathcal{R})$. Assume that the associated left $S$-free family is reducible to matrix normal form of type $g$. Then $\mathcal{R}$ is reducible to matrix
normal form of type $\text{hcf}(g, c)$. Similarly, if a general representation of type $\mathcal{R}$ has an $S$-top, $\mathcal{R}$ is right general with respect to $S$, the associated right $S$-free family is reducible to matrix normal form of type $h$ and $\hom(\mathcal{R}, S) = d$ then $\mathcal{R}$ is reducible to matrix normal form of type $\text{hcf}(h, d)$.

Proof. This argument essentially occurs as a special case of the proof of theorem 6.1 in [7] to which the reader should refer for greater detail.

We deal only with the first case since the second case has the same proof. Let $\mathcal{R}'$ be the associated left $S$-free family on $X$ (after replacing $X$ by an open dense subvariety). Let $\mathcal{S}$ be an $\mathcal{R}'$-standard family over $Y$ where $Y$ is $PGL_g$ equivariant to a dense open subvariety of $M_g(k)^n$ for some integer $s \geq 0$. Let $E = E(\mathcal{S}, S^c)$ and let $\mathcal{E} = \mathcal{E}(\mathcal{S}, S^c)$ be the extension family of $\mathcal{S}$ on $S^c$. Then since we assume that $\mathcal{R}$ is left general with respect to $S$, $\mathcal{R}$ and $\mathcal{E}$ are birationally representation equivalent.

Let $\text{ext}(\mathcal{S}, S) = t$. After shrinking $Y$ a little we may assume that for all $p \in Y$, $\text{ext}(\mathcal{S}_p, S) = t$. Let $\beta = (g, c)$ be a dimension vector for the quiver $Q'$ which has two vertices $v$ and $w$, $s$ arrows from $v$ to itself and $t$ arrows from $v$ to $w$. Then $PGL_\beta$ acts on $E$ whilst $GL_\beta$ acts on $\mathcal{E}$ so that $\mathcal{E}(v)$ is a $GL_\beta$ vector bundle of weight 1 and $E$ is $PGL_\beta$ birational to $R(Q', \beta)$.

Since $\mathcal{R}$ and $\mathcal{E}$ are birationally representation equivalent, a general representation of type $\mathcal{E}$ has an $S$-socle which must coincide with its obvious subrepresentation isomorphic to $S^c$. We therefore pass to a dense open $PGL_\beta$ subvariety of $E$ where this is true. Then the orbits of $PGL_\beta$ correspond to the isomorphism classes of representations in the restriction of $\mathcal{E}$ to this subvariety. Since a general representation of type $\mathcal{R}$ and hence of type $\mathcal{E}$ has trivial endomorphism ring, it follows that $PGL_\beta$ has trivial stabilisers generically on $E$ and hence $\beta$ is a Schur root for the quiver $Q'$. Thus the main result of [8], theorem 6.3, allows us to conclude that $\beta$ is reducible to matrix normal form and hence by lemma 3.3 there exists a $PGL_m$ equivariant subvariety $Z$ of $R(Q', \beta)$ where $m = \text{hcf}(g, c)$ such that the restriction of the canonical family on $R(Q', \beta)$ to $Z$ is standard and $ZPGL_\beta$ is a dense subvariety of $R(Q', \beta)$. Since $E$ and $R(Q', \beta)$ are $PGL_\beta$ birational, there is a corresponding $PGL_m$ equivariant subvariety $Z'$ of $E$ and the restriction of $\mathcal{E}$ to $Z'$ is what we want.

Now let $S$ and $S_1$ be representations such that $\text{End}(S_1) = k = \text{End}(S)$, $\text{Ext}(S_1, S_1) = 0 = \text{Ext}(S, S)$ and also $\text{Hom}(S, S_1) = 0 = \text{Hom}(S_1, S) = \text{Ext}(S, S_1)$. Let $E(S, S_1)$ be the full subcategory of representations that contains $S$ and $S_1$ and is closed under extensions; we call the pair $(S, S_1)$ a Kronecker reduction pair. Then following Ringel [8] we have the following lemma.

Lemma 3.5. Let $S$ and $S_1$ be a Kronecker reduction pair. Then every representation in $E(S, S_1)$, the full subcategory of representations that contain $S$ and $S_1$ and is closed under extensions, has an $S$-socle such that the factor is isomorphic to $S_V^n$ for some integer $n$. Let $t = \text{ext}(S_1, S)$ and let

$$0 \to S^t \to S' \to S_1 \to 0$$
be the canonical extension of $S_1$ on $S^t$. Then $\text{Ext}(S \oplus S', M) = 0$ for all $M$ in $E(S, S_1)$, and $\text{Hom}(S \oplus S', \cdot)$ induces a natural equivalence with the category of representations of the $t$th Kronecker quiver whose inverse is given by $\otimes(S \oplus S')$. This functor also induces isomorphisms on $\text{Ext}$ groups.

**Proof.** If $M$ and $N$ are representations for which we have short exact sequences

$$
0 \rightarrow S^{m_1} \rightarrow M \rightarrow S^{n_1} \rightarrow 0
$$

$$
0 \rightarrow S^{m_2} \rightarrow N \rightarrow S^{n_2} \rightarrow 0
$$

and a short exact sequence

$$
0 \rightarrow M \rightarrow L \rightarrow N \rightarrow 0
$$

then it is clear that the induced extension of the subrepresentation $S^{m_2}$ over $M$ splits; thus $L$ has a subrepresentation $T$ isomorphic to $S^{m_1 + m_2}$ such that the factor is isomorphic to $S^{n_1 + n_2}$ and $T$ must be the $S$-socle of $L$. Thus by induction all objects in $E(S, S_1)$ have the required structure.

By construction, $\text{Ext}(S', S) = 0$ and clearly $\text{Ext}(S', S_1) = 0$; therefore $\text{Ext}(S \oplus S', M) = 0$ for all $M$ in $E(S, S_1)$ as required. Now let $M$ be any object in $E(S, S_1)$; so there is a short exact sequence

$$
0 \rightarrow S^{m} \rightarrow M \rightarrow S^{n} \rightarrow 0.
$$

Then $\text{hom}(S, M) = m$ and since $\text{ext}(S', S) = 0$, $\text{hom}(S', M) = n$ so that $\text{Hom}(S \oplus S', M)$ is a representation of dimension vector $(n, m)$ of the $t$th Kronecker quiver. In fact, the natural homomorphism from $\text{Hom}(S, M) \otimes S \oplus \text{Hom}(S', M) \otimes S$ to $M$ is surjective with kernel isomorphic to $S^s$ for some integer $s$. Using this short exact sequence it is a simple matter to check that the two functors are mutually inverse and that $\text{Ext}(M, N)$ is preserved by this functor.

Let $\mathcal{R}$ be some irreducible family of representations of the quiver with relations $(Q, I)$ over the algebraic variety $X$. We shall say that $\mathcal{R}$ has a Kronecker reduction of type $(S, S_1)$ to the dimension vector $(a, b)$ for the $t$th Kronecker quiver if there exists a Kronecker reduction pair $(S, S_1)$ where $\text{ext}(S_1, S) = t$ such that a general representation of type $\mathcal{R}$, $R$, has a subrepresentation isomorphic to $S^b$ with factor isomorphic to $S^a$ for suitable integers $b$ and $a$ (so a general representation of type $\mathcal{R}$ has an $S$-socle) and $\mathcal{R}$ is left general with respect to $S$. Note that this is a self-dual condition since being left general with respect to $S$ is equivalent to being right general with respect to $S_1$.

We note the following consequence of lemma 3.5.

**Lemma 3.6.** Let $\mathcal{R}$ be an irreducible family of representations of dimension vector $\alpha$ of the quiver with relations $(Q, I)$ over the algebraic variety $X$. Assume that $\mathcal{R}$ has a Kronecker reduction of type $(S, S_1)$ to the dimension vector $(a, b)$ for the $t$th Kronecker quiver. Then $\mathcal{R}$ is reducible to matrix normal form of type $\text{hcf}(a, b)$ if and only if $(a, b)$ is reducible to matrix normal...
form for the \( t \)th Kronecker quiver. When \( \mathcal{R} \) is not reducible to matrix normal form then \( \mathcal{R} \) is a birationally constant family and the general representation of type \( \mathcal{R} \) is isomorphic to \( T_0^a \oplus T_1^b \) for representations \( T_0 \) and \( T_1 \) such that \( \text{hom}(T_0, T_1) = \text{ext}(T_0, T_1) = \text{ext}(T_1, T_0) \) and \( \text{hom}(T_1, T_0) = t \) where \( a' \) and \( b' \) are positive integers such that \( \text{hcf}(a', b') = \text{hcf}(a, b) \).

**Proof.** If \((a, b)\) is reducible to matrix normal form for the \( t \)th Kronecker quiver we take the family of representations of dimension vector \((a, b)\) for the \( t \)th Kronecker quiver and apply the functor \( \otimes(S \oplus S') \) considered in lemma 3.3. This gives a family of representations for the quiver with relations \((Q, I)\) that is birationally equivalent to \( \mathcal{R} \) since a general representation of type \( \mathcal{R} \) lies in \( E(S, S_1) \) and \( \mathcal{R} \) is left general with respect to \( S \) and therefore shows that \( \mathcal{R} \) is reducible to matrix normal form of type \( \text{hcf}(a, b) \). In the remaining case where \((a, b)\) is not reducible to matrix normal form then a general representation of dimension vector \((a, b)\) for the \( t \)th Kronecker quiver is isomorphic to \( T_0^{a'} \oplus T_1^{b'} \) where \( \text{hom}(R_0, R_1) = \text{ext}(R_0, R_1) = \text{ext}(R_1, R_0) \) and \( \text{hom}(R_1, R_0) = t \) and for positive integers \( a' \) and \( b' \) such that \( \text{hcf}(a', b') = \text{hcf}(a, b) \). Using the facts that a general representation of type \( \mathcal{R} \) lies in \( E(S, S_1) \) and that \( \mathcal{R} \) is left general with respect to \( S \), we deduce that \( \mathcal{R} \) is birationally constant and the general representation of type \( \mathcal{R} \) is isomorphic to \( T_0^{a''} \oplus T_1^{b''} \) where \( T_i \cong R_i \otimes (S \oplus S') \) and the representations \( T_0 \) and \( T_1 \) and the integers \( a' \) and \( b' \) satisfy the conditions of the lemma.

Let \( S_0, S_1 \) and \( S_2 \) be three representations such that \( \text{End}(S_i) = k \) and \( \text{Ext}(S_i, S_i) = 0 \) for \( i = 0, 1 \) and \( 2 \), and \( \text{Hom}(S_i, S_j) = 0 = \text{Ext}(S_i, S_j) \) for \( 0 \leq i < j \leq 2 \). Let \( \mathcal{R} \) be an irreducible family of representations of the quiver with relations \((Q, I)\). We shall say that \( \mathcal{R} \) has a two-step Kronecker reduction of type \((S_0, S_1, S_2)\) if a general representation of type \( \mathcal{R} \) has an \( S_0 \)-socle, \( \mathcal{R} \) is left general with respect to \( S_0 \), and the associated left \( S_0 \)-free family has a Kronecker reduction of type \((S_1, S_2)\). Dually, we shall say that \( \mathcal{R} \) has a two-step Kronecker coreduction of type \((S_0, S_1, S_2)\) if a general representation of type \( \mathcal{R} \) has an \( S_2 \)-top, \( \mathcal{R} \) is right general with respect to \( S_2 \) and the associated right \( S_2 \)-free family has a Kronecker reduction of type \((S_0, S_1)\).

**Lemma 3.7.** Assume that the irreducible family \( \mathcal{R} \) of representations of the quiver with relations \((Q, I)\) over the algebraic variety \( X \) has a two-step Kronecker reduction of type \((S_0, S_1, S_2)\) where \( \text{hom}(S_0, \mathcal{R}) = c \), \( \text{hom}(S_1, \mathcal{R}') = b \) for the associated left \( S_0 \)-free family and \( \text{hom}(\mathcal{R}', S_2) = a \). Assume that a general representation of type \( \mathcal{R} \) has trivial endomorphism ring. Then \( \mathcal{R} \) is reducible to matrix normal form of type \( \text{hcf}(a, b, c) \).

**Proof.** We shall proceed by induction on \( a + b + c \). The associated left \( S_0 \)-free family, \( \mathcal{R}' \), has a Kronecker reduction of type \((S_1, S_2)\) to the dimension vector \((a, b)\) for the \( t \)th Kronecker quiver \( Q(t) \) for some integer \( t \). Now suppose that the dimension vector \((a, b)\) is reducible to matrix normal form. Then by lemma 3.3, \( \mathcal{R}' \), is reducible to matrix normal form of type \( \text{hcf}(a, b) \) and so, by lemma 3.3, \( \mathcal{R} \) is reducible to matrix normal form of type \( \text{hcf}(a, b, c) \) as required.
If the dimension vector \( (a\ b) \) is not reducible to matrix normal form then, by lemma 3.9, a general representation of type \( R' \) is isomorphic to \( T_1^{a'} \oplus T_2^{b'} \) where \( \text{hom}(T_1, T_2) = 0 = \text{ext}(T_1, T_2) = \text{ext}(T_2, T_1) \) and \( \text{hom}(T_2, T_1) = 0 \). Therefore a moduli space of left general hom, by induction.

\[
0 \rightarrow T_0^c \rightarrow R \rightarrow T_1^{a'} \oplus T_2^{b'} \rightarrow 0
\]  

(2)

Then since \( T_1 \) and \( T_2 \) are representations that lie in \( E(S_1, S_2) \), it follows that \( \text{hom}(T_0, T_i) = 0 = \text{ext}(T_0, T_i) \) for \( i = 1, 2 \), since \( \text{hom}(T_0, S_i) = 0 = \text{ext}(T_0, S_i) \) for \( i = 1, 2 \). It also follows that \( \text{hom}(T_1, T_0) = 0 \) from equation (2) since a general representation of type \( R \) has trivial endomorphism ring and \( \text{ext}(T_1, T_0) \neq 0 \) for the same reason.

Since \( \text{hom}(T_1, T_2) = 0 \) for \( i = 0, 1 \), it follows that \( \text{hom}(R, T_0) = b' \) and \( R \) has a \( T_2 \)-top. Let \( K \) be the kernel of the homomorphism from \( R \) onto \( T_2^{b'} \). We have a short exact sequence

\[
0 \rightarrow T_0^c \rightarrow K \rightarrow T_1^{a'} \rightarrow 0.
\]

Therefore the linear map from \( \text{Ext}(T_1^{a'}, T_0^c) \) to \( \text{Ext}(T_2^{b'}, K) \) is surjective and \( \text{Ext}(T_2^{b'}, T_0^c) \) is a summand of \( \text{Ext}(T_1^{a'} \oplus T_2^{b'}, T_0^c) \). Therefore, \( R \) is right general with respect to \( T_2 \) because \( R \) is left general with respect to \( T_0 \) and so an open subvariety of the extensions of \( T_1^{a'} \oplus T_2^{b'} \) on \( T_0^c \) occur in the family \( R \).

Further, the associated right \( T_2 \)-free family, \( R'' \) has a Kronecker reduction of type \( (T_0, T_1) \). The only thing remaining to check is that the family \( R'' \) is left general with respect to \( T_0 \) but this follows because \( \text{Ext}(T_1^{a'}, T_0^c) \) is a summand of \( \text{Ext}(T_1^{a'} \oplus T_2^{b'}, T_0^c) \) and the family \( R \) is left general with respect to \( T_0 \).

Thus we have shown that the family \( R \) has a Kronecker coreduction of type \( (T_0, T_1, T_2) \) and if \( a' + b' < a + b \) then \( a' + b' + c < a + b + c \) and we are done by induction.

If \( a' + b' = a + b \), we noted above that \( \text{ext}(T_1, T_0) \neq 0 \) and so when we perform the same argument again for this coreduction the numbers will drop this time so again we are done by induction.

\[\square\]

**Theorem 3.8.** Let \( \alpha = (a\ b\ c) \) be a dimension vector for the multiplication \( f \) such that there is a left general representation of dimension vector \( \alpha \) with trivial endomorphism ring. Then the canonical family on the left general component is reducible to matrix normal form. Therefore a moduli space of left general representations of this dimension vector is birational to a suitable number of \( h \) by \( h \) matrices up to simultaneous conjugacy where \( h = \text{hcf}(a, b, c) \).

**Proof.** Let \( R \) be the canonical family on the left general component \( C \) of representations of dimension vector \( \alpha \). Let \( S_0 = (0\ 0\ k), S_1 = (0\ k\ 0) \) and \( S_2 = (k\ 0\ 0) \). Then \( R \) has a two-step Kronecker reduction of type \( (S_0, S_1, S_2) \) and so by lemma 3.7, \( R \) is reducible to matrix normal form.

\[\square\]
4 The main result

By the work of Beilinson [1], the derived category of coherent sheaves on \( \mathbb{P}^2 \) is equivalent to the derived category of representations of the multiplication \( \sigma : k^3 \otimes k^3 \to S^2(k^3) \). We shall identify these two derived categories using this equivalence. Thus we have a triangulated category \( D \) which has two subcategories \( \text{Coh}(\mathbb{P}^2) \), the category of coherent sheaves on \( \mathbb{P}^2 \), and \( \text{Rep}(\sigma) \), the category of representations of the multiplication \( \sigma \). We shall make the identification of the two derived categories in such a way that \( O = (k^3 \otimes S^2(k^3)) = P(0) \), \( O(-1) = (0 k^3) = P(1) \) and \( O(-2) = (0 0 k) = P(2) \). The representations \( P(0) \), \( P(1) \) and \( P(2) \) are the indecomposable projective representations of the multiplication \( \sigma \). We shall say that an object of \( D \) is a representation if it lies in \( \text{Rep}(\sigma) \) and that it is a sheaf if it lies in \( \text{Coh}(\mathbb{P}^2) \). Thus we can ask the question whether a sheaf is a representation and vice versa. One direction is clear; a sheaf \( S \) is a representation if and only if \( H^i(S(j)) = 0 \) for \( i = 1, 2 \) and \( j = 0, 1 \) and \( 2 \).

Lemma 4.1. Let \( R = (R(0) \ R(1) \ R(2)) \) be a representation of the multiplication \( \sigma \). Let

\[
0 \to R(0) \otimes \wedge^2(k^3) \otimes P(2) \to R(0) \otimes k^3 \otimes P(1) \oplus R(1) \otimes k^3 \otimes P(2)
\]

\[
\to \bigoplus_{i=1}^3 R(i) \otimes P(i) \to R \to 0
\]

be a projective resolution of \( R \). Then \( R \) is a sheaf if and only if the complex of sheaves

\[
0 \to R(0) \otimes \wedge^2(k^3) \otimes O(-2) \to R(0) \otimes k^3 \otimes O(-1) \oplus R(1) \otimes k^3 \otimes O(-2)
\]

\[
\to \bigoplus_{i=1}^3 R(i) \otimes O(-i) \to 0
\]

has homology only at the penultimate term.

Proof. Once stated, this result is also clear. In the derived category \( D \), the object \( R \) is equivalent to the complex

\[
0 \to R(0) \otimes \wedge^2(k^3) \otimes P(2) \to R(0) \otimes k^3 \otimes P(1) \oplus R(1) \otimes k^3 \otimes P(2)
\]

\[
\to \bigoplus_{i=1}^3 R(i) \otimes P(i) \to 0
\]

which after our identification is the complex of sheaves

\[
0 \to R(0) \otimes \wedge^2(k^3) \otimes O(-2) \to R(0) \otimes k^3 \otimes O(-1) \oplus R(1) \otimes k^3 \otimes O(-2)
\]

\[
\to \bigoplus_{i=1}^3 R(i) \otimes O(-i) \to 0
\]

and this itself is a sheaf if and only if it has homology only at the penultimate term as required. \( \square \)

This last lemma gives us the following result which allows us to identify at least birationally families of sheaves and families of representations.
Lemma 4.2. Let $\mathcal{R}$ be a family of representations of the multiplication $\sigma$ over the irreducible algebraic variety $X$. Assume that there exists a point $p$ such that $\mathcal{R}_p$ is a sheaf. Then there exists an open subvariety $O$ of $X$ such that for all $p \in O$, $\mathcal{R}_p$ is a sheaf. Similarly, if $\mathcal{S}$ is a family of sheaves on the irreducible algebraic variety $Y$ and there exists a point $q$ such that $\mathcal{S}_q$ is a representation then there exists an open subvariety $O'$ of $Y$ such that for all $q \in O'$, $\mathcal{S}_q$ is a representation.

Proof. Consider the complex of sheaves on $X \times \mathbb{P}^2$

$$0 \to \mathcal{R}(0) \otimes S^2(k^3) \otimes \mathcal{O}(-2) \to \mathcal{R}(0) \otimes k^3 \otimes \mathcal{O}(-1) \oplus \mathcal{R}(1) \otimes k^3 \otimes \mathcal{O}(-2) \to \oplus_{i=1}^{3} \mathcal{R}(i) \otimes \mathcal{O}(-i) \to 0.$$ 

Let $Z$ be the support of the homology of this complex except at the penultimate term. Then $Z$ is closed and so is its image in $X$; therefore the complement of the image of $Z$ in $X$ is open and it is the set of points $p$ where $\mathcal{R}_p$ is a sheaf.

In the second case, the vanishing of $H^i(\mathcal{S}_q(j))$ is an open condition so the result follows. 

We define the depth of a vector bundle $E$ over $\mathbb{P}^2$ to be the largest integer $h$ such that $[E]/h$ is in the Grothendieck group $K_0(\text{Coh}(\mathbb{P}^2))$ of $\text{Coh}(\mathbb{P}^2)$ where $[E]$ is the class of $E$ in this Grothendieck group. The Grothendieck group of the derived category $D$ coincides with $K_0(\text{Coh}(\mathbb{P}^2))$ and also with $K_0(\text{Rep}(\sigma))$; hence if $E$ is actually a representation of dimension vector $(a \ b \ c)$ it follows that the depth of $E$ is $\text{hcf}(a, b, c)$.

A sheaf $E$ on $\mathbb{P}^2$ is said to have natural cohomology if for all integers $j$ at most one of $H^i(E(j))$ is non-zero for $i = 0, 1$ and $2$. This definition is important to us since general vector bundles have this property by [3] and [4] where it comes in the form that slope-semistable sheaves are prioritary and prioritary sheaves have natural cohomology. This allows an easy reduction to left general representations.

Theorem 4.3. A moduli space of sheaves on $\mathbb{P}^2$ such that the general sheaf has natural cohomology is birational to a moduli space of left general representations of the multiplication $\sigma$: $k^3 \otimes k^3 \to S^2(k^3)$ and hence is birational to a suitable number of $h$ by $h$ matrices up to simultaneous conjugacy where $h$ is the depth of every sheaf classified by the moduli space. In particular this holds for a moduli space of vector bundles.

Proof. After tensoring by a suitable line bundle we may assume for a general $[S]$ in the moduli space that $S$ is a representation but $S(-1)$ is not; that is one of $H^1(S(-1))$ and $H^2(S(-1))$ is non-zero. Since $S$ has natural cohomology, it follows that $H^0(S(-1)) = 0$; it also follows that $H^2(S(-1)) = 0$ since by Serre duality this is dual to $\text{Hom}(S, \mathcal{O}(-2))$ and since $S$ is a representation any such homomorphism is split surjective. So we consider the exact complex of sheaves on $\mathbb{P}^2$

$$0 \to \Lambda^3(k^3) \otimes \mathcal{O}(-2) \to \Lambda^2(k^3) \otimes \mathcal{O}(-1) \to k^3 \otimes \mathcal{O} \to \mathcal{O}(1) \to 0.$$ 

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from which we deduce that for a sheaf \( S \) that is also a representation the groups \( H^i(S(-1)) \) are the homology of the complex

\[
0 \to k^3 \otimes H^0(S) \to \Lambda^2(k^3) \otimes H^0(S(1)) \to \Lambda^3(k^3) \otimes H^0(S(2)) \to 0
\]

and this complex is exact except at the middle term. In particular, after tensoring with \( \Lambda^3(k^3) \) we see that the linear map from \( \Lambda^2(k^3) \otimes H^0(S) \) to \( k^3 \otimes H^0(S(1)) \) is injective. Let \( a = \dim H^0(S) \), \( b = \dim H^0(S(1)) \) and \( c = \dim H^0(S(2)) \). Then \( S \) considered as a representation has dimension vector \( \alpha = (a, b, c) \) and the injectivity of this linear map shows by lemma 4.2 that \( S \) is a representation in the left general component. Therefore by lemma 4.2 our moduli space of sheaves is birational to an orbit space for \( PGL_\alpha \) on the left general component of representations of dimension vector \( \alpha \) but we know that the canonical family on this component is reducible to matrix normal form by theorem 3.8. Therefore our moduli space is birational to a suitable number of \( h \) by \( h \) matrices up to simultaneous conjugacy where \( h = \text{hcf}(a, b, c) \) is the depth of \( S \).

Since the canonical family of left general representations of dimension vector \( (a, b, c) \) considered in the proof of this theorem is reducible to matrix normal form it follows that if we have a moduli space \( M \) of sheaves whose general member has natural cohomology then there is a family of sheaves in \( M \) over an algebraic variety \( X \) on which \( PGL_h \) acts so that \( X \) is \( PGL_h \) birational to \( M_h(k)^s \) for some integer \( s \) and the morphism from \( X \) to \( M \) is dominant and is the orbit map.

It is perhaps worth stating the rationality results that follow from this theorem.

**Theorem 4.4.** A moduli space of sheaves on \( \mathbb{P}^2 \) such that the general sheaf has natural cohomology and depth \( n \) is rational when \( n = 1, 2, 3 \) or 4. If \( 4 < n \) and \( n \) divides 420 then the moduli space is stably rational. If \( n \) is square-free then the moduli space is retract rational.

**Proof.** This follows from the known results on matrices up to simultaneous conjugacy. A good summary of the known results may be found in [4].

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