On a problem of Terence Tao

Xue-Gong Sun

School of Sciences, HuaiHai Institute of Technology,
Lian Yun Gang 222005, P. R. China
Email: xgsunlyg@163.com

Abstract. In this paper, we solve a problem of Terence Tao. We prove that for any $K \geq 2$ and sufficiently large $N$, the number of primes $p$ between $N$ and $(1 + \frac{1}{K})N$ such that $|kp + ja^i + l|$ is composite for all $1 \leq a, |j|, k \leq K$, $1 \leq i \leq K \log N$ and $l$ in any set $L_N \subseteq \{-KN, \cdots, KN\}$ of cardinality $K$ with $ja^i + l \neq 0$ is at least $C_K \frac{N}{\log N}$, where $C_K > 0$ depending only on $K$.

Key Words: powers of $a$; primes; Selberg’s sieve method.

2000 Mathematics subject classifications: 11A07, 11B25, 11P32.

1. Introduction

Let $p$ be a prime and $n$ be a nonnegative integer. In 1934, Romanoff [12] proved that the set of positive odd integers which can be expressed in the form $2^n + p$ has a positive proportion in the set of all positive odd numbers. In 1950, van der Corput [4] proved that there are a positive proportion odd integers not of the form $2^n + p$. In the same year, using covering congruences, Erdős [5] proved that there is an infinite arithmetic progression of positive odd integers each of which has no representation of the form $2^n + p$. In 1975, Cohen and Selfridge [3] proved that there exist infinitely many odd numbers which are neither the sum nor the difference of a power of two and a prime power.

Recently, using Selberg’s sieve method, Tao [16] proved that for any $K \geq 2$ and sufficiently large $N$, the number of primes $p$ between $N$ and $(1 + \frac{1}{K})N$ such that $|kp \pm ja^i|$ is composite for all $1 \leq a, j, k \leq K$ and $1 \leq i \leq K \log N$, is at least $C_K \frac{N}{\log N}$, where $C_K$ is a constant depending only on $K$.

On the other hand, Tao [16] posed the following problem:

For any $K \geq 2$ and sufficiently large $N$, the number of primes $p$ between $N$ and $(1 + \frac{1}{K})N$ such that $|kp + ja^i + l|$ is composite for all $1 \leq a, |j|, k \leq K$, $1 \leq i \leq K \log N$ and $l$ in some set $L = L_N \subseteq \{-KN, \cdots, KN\}$ of cardinality at most $K$ is at least $C_K \frac{N}{\log N}$, where $C_K$ is a constant depending only on $K$.

---

1Supported by the National Natural Science Foundation of China( 11471017) and the Natural Science Foundation of HuaiHai Institute of Technology(KQ10002).
Using Tao’s idea, in this paper we shall solve the above Tao’s problem. More precisely, we establish

**Theorem 1.** For any \( K \geq 2 \) and sufficiently large \( N \), the number of primes \( p \) between \( N \) and \( (1 + \frac{1}{K})N \) such that \( |kp + ja^i + l| \) is composite for all \( 1 \leq a, |j|, k \leq K, 1 \leq i \leq K \log N \) and \( l \) in any set \( L_N \subseteq \{-KN, \cdots, KN\} \) of cardinality \( K \) with \( ja^i + l \neq 0 \) is at least \( C_K \frac{N}{\log N} \), where \( C_K > 0 \) depending only on \( K \).

**Remark 1.** Let \( p(K) \) be a prime with \( p(K) > K \). In Theorem 1, we can take \( L_N = \{p(K), \cdots, Kp(K)\} \). Moreover, we can take \( L_N = \{K! + 1, \cdots, (2K - 1)! + 1\} \).

**Remark 2.** From Theorem 1, we know that for any \( K \geq 2 \) and sufficiently large \( N \), the number of primes \( p \) between \( N \) and \( (1 + \frac{1}{K})N \) such that \( |kp + ja^i + l| \) is composite for all \( 2 \leq k \leq K \), \( 1 \leq a, |j| \leq K \), \( 1 \leq i \leq K \log N \) and \( l \) in any set \( L_N \subseteq \{-KN, \cdots, KN\} \) of cardinality \( K \) is at least \( C_K \frac{N}{\log N} \), where \( C_K > 0 \) depending only on \( K \).

2. **Proofs**

In this paper, \( p, q, p_{i,j}, q_{i,j} \) are all primes, and the implied constants in \( \ll, \gg \) are all absolute.

**Lemma 1** [13]. Let \( x \geq 2 \). Then
\[
\log \log x < \sum_{p \leq x} \frac{1}{p} < \log \log x + 1.
\]

**Lemma 2** [13]. Let \( x \geq 59 \). Then
\[
\frac{x}{\log x} (1 + \frac{1}{2 \log x}) < \pi(x) < \frac{x}{\log x} (1 + \frac{3}{2 \log x}).
\]

By the Brun’s theorem, we get

**Lemma 3.** There exists a constant \( A \) such that
\[
\sum_{p, mp+1, \text{primes}} \frac{1}{p} \leq A
\]
for every \( 1 < m \leq M \), where the constant \( A \) depending only on \( M \).

**Lemma 4.** For any \( 2 \leq a \leq K \) and \( M > 12K^3 \), there exists a set \( P_a \) of some primes \( p_{a,t} \) in \( [\exp \exp((2^a - 1)A + 1)M), \exp \exp((2^a - 1)(A + 1)M)) \) with the following properties:

(1) For any \( p_{a,t} \), there exists a prime \( q_{p_{a,t}} \) such that \( a^{p_{a,t}} \equiv 1 \pmod{q_{p_{a,t}}} \) and \( q_{p_{a,t}} \geq M p_{a,t} \).

(2) For each \( a \), we have \( \sum_{t} \frac{1}{p_{a,t}} \in [M - 3, M] \).

**Proof.** For \( a = 2 \), let \( P_2^* \) be the set of primes in the interval \( [\exp \exp((A+1)M), \exp \exp(3(A+1)M)] \) satisfying that \( mp + 1 \) is composite for every \( 1 \leq m \leq M \).

By Lemma 1 and Lemma 3, we have
\[
\sum_{p \leq P^*} \frac{1}{p} \geq \sum_{m=M}^{\exp((A+1)M) \leq \exp(3(A+1)M)} \frac{1}{p} - \sum_{m=1}^{\exp((A+1)M) \leq \exp(3(A+1)M), \text{primes}} \frac{1}{p} \\
\geq 3(A+1)M - (A+1)M - 2 - MA \\
\geq 2M - 2.
\]

So, we can find a set \( P_2 \) in \( P^*_2 \) with \( \sum_{t} \frac{1}{p_{2,t}} \in [M - 3, M] \). Let \( q_{p_{2,t}} \) be the largest prime factor \( 2^{p_{2,t}} - 1 \). We know all \( q_{p_{2,t}} \) are distinct. By the Fermat’s little theorem, we know that \( p_{2,t} \) divides \( q_{p_{2,t}} - 1 \). On the other hand, we know that \( mp_{2,t} + 1 \) is composite for every \( 1 \leq m \leq M \). Thus, we get \( q_{p_{2,t}} \geqMp_{2,t} \).

Now, suppose that \( a > 2 \) and we have chosen disjoint finite sets of primes \( P_2, \ldots, P_{a-1} \) with the stated properties.

Let \( P_a^* \) be the set of primes in the interval \([\exp((2^a - 1) - 1)(A+1)M, \exp((2^a - 1)(A+1)M))\) satisfying that \( mp + 1 \) is composite for every \( 1 \leq m \leq M \).

Let \( \omega_a = \prod_{2 \leq i < a} \prod_{t} (q_{p_{i,t}} - 1) \). Note \( q_{p_{i,t}} | p_{i,t} - 1 \), by Lemma 2, we get

\[
\frac{\log(\omega_a)}{\log a} \\
\leq \sum_{2 \leq i < a} \sum_{t} \frac{\log(i^{p_{i,t}} - 1)}{\log a} \\
\leq \sum_{2 \leq i < a} \sum_{t} p_{i,t} \frac{\log i}{\log a} \\
\leq \sum_{2 \leq i < a} \sum_{t} p_{i,t} \\
\leq \sum_{p \leq \exp((2^{a-1} - 1)(A+1)M)} p \\
\leq 2 \exp(-(2^{a-1} - 1)(A+1)M) \exp(2 \exp((2^{a-1} - 1)(A+1)M)).
\]

Thus, we have \( \Omega(\omega_a) \leq \exp(2 \exp((2^{a-1} - 1)(A+1)M)) \).

Moreover, by Lemma 2, there exists at least \( \exp(2 \exp((2^{a-1} - 1)(A+1)M)) \) primes in the interval \([1, \exp(4 \exp((2^{a-1} - 1)(A+1)M))]\).

So, we get

\[
\sum_{p \mid \omega_a} \frac{1}{p} \leq \sum_{p \leq \exp(4 \exp((2^{a-1} - 1)(A+1)M))} \frac{1}{p} \leq (2^{a-1} - 1)(A+1)M + \log 4 + 1.
\]
By Lemma 1 and Lemma 3, we have

\[
\sum_{p \in P, p \nmid \omega_a} {1 \over p} \geq \sum_{m=M}^{\exp \exp((2^a - 1)(A+1)M) \leq \exp \exp((2^a - 1)(A+1)M)} {1 \over p} - \sum_{m=1}^{m=M} \sum_{\exp \exp((2^a - 1)(A+1)M) \leq p < \exp \exp((2^a - 1)(A+1)M)} {1 \over p} - \sum_{p \mid \omega_a} {1 \over p} \geq (2^a - 1)(A + 1)M - 1 - (2^a - 1)(A + 1)M - MA - (2^a - 1)(A + 1)M - \log 4 - 1 \geq M - 3.
\]

Since \( p_{a,t} \mid q_{p_{a,t}} - 1 \) and \( p_{a,t} \nmid \omega_a \), we know all these \( q_{p_{a,t}} \) are distinct.

Similar to \( a = 2 \), we can choose a set \( P_a \) in \( P_a^* \) with the stated properties.

This completes the proof of Lemma 4.

**Proof of Theorem 1.** Let

\[
R = \{(j, k, l) : 1 \leq |j|, k \leq K, l \in L_N\}
\]

and take \( p \) and \( q_p \) as in Lemma 4.

By \( \sum_{p \in P_a} {1 \over p} \in [M - 3, M] \), we may partition \( P_a = \bigcup_{j, k, l} P_{a, j, k, l} \) in such a way that

\[
\sum_{p \in P_{a, j, k, l}} {1 \over p} \in \left[ {M \over 4K^3}, {M \over 3K^3} \right].
\]

Let \( W \) be the quantity \( W = \prod_p q_p \). For \( p \in P_{a, j, k, l} \), let \( I(a, j, k, l) \) be the smallest integer \( i \geq 0 \) such that \( ja^i + l \equiv 0 \pmod{q_p} \), we know that \( I(a, j, k, l) = 0, 1 \).

By the Chinese remainder theorem, we can take \( (b, W) = 1 \) satisfying

\[
k + ja^i + l \equiv 0 \pmod{q_p}
\]

for every \( p \in P_{a, j, k, l} \), \( 2 \leq a \leq K \), and \( (j, k, l) \in R \).

Let

\[
Q = \#\{N \leq m \leq (1 + K^{-1})N : m \equiv b \pmod{W}, m \text{ prime}, \text{ but } km + ja^i + l \text{ composite for all } 1 \leq i < K \log N, 1 \leq a \leq K, (j, k, l) \in R\},
\]

\[
Q_N = \#\{N \leq m \leq (1 + K^{-1})N : m \equiv b \pmod{W}, m \text{ prime}\},
\]

4
and

\[ Q_{N,a,i,j,k,l} = \# \left\{ N \leq m \leq (1 + K^{-1})N : m \equiv b \pmod{W}, m, \mid km + ja^i + l \mid, \text{primes} \right\} . \]

Similar to the proof of Theorem 1.2 \[10\], we get

\[ Q \geq Q_N - \sum_{a=2}^{K} \sum_{1 \leq i < K} \sum_{(j,k,l) \in R} Q_{N,a,i,j,k,l} - \sum_{(j,k,l) \in R} Q_{N,1,1,j,k,l} - O(\log N). \]

From the prime number theory in arithmetic progressions, we have

\[ Q_N \geq c_1 \frac{N}{W \log N} \prod_{q \mid W} (1 - \frac{1}{q}). \]

Let \( P^* = \{ p : K < p < N^{\frac{1}{2}}, (p, W) = 1 \} \).

By the Selberg’s sieve method, we have

\[
\begin{align*}
Q_{N,a,i,j,k,l} &\leq \# \left\{ N \leq m \leq (1 + K^{-1})N : m \equiv b \pmod{W}, m, \mid km + ja^i + l \mid, \text{primes} \right\} \\
&\leq \# \left\{ 1 \leq r \leq (1 + K^{-1}) \frac{N}{W} + 1 : Wr + b > N^{\frac{1}{2}}, \mid kWr + kb + ja^i + l \mid > N^{\frac{1}{2}}, \text{primes} \right\} + 2N^{\frac{1}{2}} \\
&\leq \# \left\{ 1 \leq r \leq (1 + K^{-1}) \frac{N}{W} + 1 : ((Wr+b)(kWr+kb+ja^i+l), p)=1, p \in P^* \right\} + 2N^{\frac{1}{2}} \\
&\ll \frac{N}{W} \prod_{p|ja^i+l, p \in P^*} \left( 1 - \frac{2}{p} \right) \prod_{p|ja^i+l, p \in P^*} \left( 1 - \frac{1}{p} \right) \\
&\ll \frac{N}{W} \prod_{p \in P^*} \left( 1 - \frac{2}{p} \right) \prod_{p|ja^i+l, p \in P^*} \left( 1 - \frac{1}{p} \right) \left( 1 - \frac{2}{p} \right)^{-1} \\
&\ll \frac{N}{W \log^2 N} \prod_{3 \leq p \leq K} \left( 1 - \frac{2}{p} \right)^{-1} \prod_{K < q | W} \left( 1 - \frac{2}{q} \right)^{-1} \prod_{K < q | ja^i+l, p | W} \left( 1 + \frac{1}{p} \right) \\
&\ll \frac{N}{W \log^2 N} \prod_{3 \leq p \leq K} \left( 1 - \frac{2}{p} \right)^{-1} \prod_{K < q | W} \left( 1 - \frac{2}{q} \right)^{-1} \prod_{K < q | ja^i+l, p | W} \left( 1 + \frac{1}{p} \right). 
\end{align*}
\]

Now suppose that \( 2 \leq a \leq K, (j,k,l) \in R \).

Note that if \( i \equiv I(a,j,k,l) \pmod{p} \) for some \( p \in P_{a,j,k,l} \), then \( q_p | km + ja^i + l \), so \( q_p = \mid km + ja^i + l \mid \).

Thus, we have

\[
\sum_{1 \leq i \leq K \log N, i \equiv I(a,j,k,l) \pmod{p}} Q_{N,a,i,j,k,l} \ll \log N
\]

for some \( p \in P_{a,j,k,l} \).
Let $e_{a,j,l}(d)$ denote the smallest positive integer $i$ such that $ja^i + l \equiv 0 \pmod{d}$. Since $p | d \Rightarrow K < p$, we know that $d | ja^i + l$ if and only if $e_{a,j,l}(d) | i$.

Let

$$E(x) = \sum_{0 < k \leq x} \frac{\sum_{\mu^2(d) = 1, e_{a,j,k,l}(d) = k, p | d \Rightarrow K < p} \omega(d)}{d}.$$

Similar to the proof of Lemma 7.8 [7], we have

$$E(x) \ll \log^2 x.$$

By partial summation, we have

$$\sum_{0 < k \leq x} \frac{\omega(d)}{d} \ll 1.$$

So, we get

$$\sum_{\mu^2(d) = 1, p | d \Rightarrow K < p} \frac{\omega(d)}{de_{a,j,k,l}(d)} = \sum_{0 < k} \frac{1}{k} \sum_{\mu^2(d) = 1, e_{a,j,k,l}(d) = k, p | d \Rightarrow K < p} \omega(d) \ll 1.$$

By the Cauchy-Schwarz inequality and the Selberg’s sieve method, we know

\[
\sum_{1 \leq i < K \log N, p \in \mathcal{P}_{a,j,k,l} \Rightarrow p | (i - I(a,j,k,l)) (K \log N)^{1/2}} (1 + \frac{1}{p}) \prod_{1 \leq i < K \log N, p \in \mathcal{P}_{a,j,k,l} \Rightarrow p | ja^i + l} \prod_{1 \leq i < K \log N, \mu^2(d) = 1, p | d \Rightarrow K < p} \frac{2\omega(d)}{d} \ll 1.
\]
So, we have
\[
\sum_{1 \leq i \leq K \log N; i \not\equiv j(a, j, k, l) \pmod{p}} Q_{N, a; i, j, k, l} (1 - \frac{2}{p})^{-1} \prod_{q \mid W} (1 - \frac{2}{q})^{-1} \prod_{p \mid P, j, k, l} (1 - \frac{1}{p})^\frac{1}{2},
\]
\[
\ll \frac{KN}{W \log N} \prod_{3 \leq p \leq K} (1 - \frac{2}{p})^{-1} \prod_{q \mid W} (1 - \frac{2}{q})^{-1} \prod_{p \mid P, j, k, l} (1 - \frac{1}{p})^\frac{1}{2},
\]
Thus, we get
\[
\sum_{1 \leq i \leq K \log N} Q_{N, a; i, j, k, l} \ll \frac{KN}{W \log N} \prod_{3 \leq p \leq K} (1 - \frac{2}{p})^{-1} \prod_{q \mid W} (1 - \frac{2}{q})^{-1} \prod_{p \mid P, j, k, l} (1 - \frac{1}{p})^\frac{1}{2} + \log N.
\]
By Lemma 1, we get
\[
\begin{align*}
Q & \geq c_1 \frac{N}{W \log N} \prod_{q \mid W} (1 - \frac{1}{q})^{-1} - c_2 \frac{KN}{W \log N} \prod_{3 \leq p \leq K} (1 - \frac{2}{p})^{-1} \prod_{q \mid W} (1 - \frac{2}{q})^{-1} \sum_{a=2}^{K} \sum_{(j, k, l) \in R} \prod_{p \mid P, j, k, l} (1 - \frac{1}{p})^\frac{1}{2} \\
& \geq c_1 \frac{N}{W \log N} \prod_{q \mid W} (1 - \frac{1}{q})^{-1} (1 - c_4) \prod_{3 \leq p \leq K} (1 - \frac{2}{p})^{-1} \prod_{q \mid W} (1 - \frac{1}{q})^{-1} \sum_{a=2}^{K} \sum_{(j, k, l) \in R} \prod_{p \mid P, j, k, l} (1 - \frac{1}{p})^\frac{1}{2} \\
& \geq c_1 \frac{N}{W \log N} \prod_{q \mid W} (1 - \frac{1}{q})^{-1} (1 - c_5 (\log K)^2) \sum_{a=2}^{K} \sum_{(j, k, l) \in R} \prod_{p \mid P, j, k, l} (1 - \frac{1}{2p}) \\
& \geq c_1 \frac{N}{W \log N} \prod_{q \mid W} (1 - \frac{1}{q})^{-1} (1 - c_5 (\log K)^2 K^4 \exp(K - \frac{M}{8K^3})).
\end{align*}
\]
Taking \( M > \max\{12K^3, 8K^4 + 8K^3 \log(4c_5 (\log K)^2 K^4)\} \), we get
\[
Q \geq C \frac{N}{W \log N} \prod_{q \mid W} (1 - \frac{1}{q})^{-1},
\]
where the constant \( C \) is absolute.

This completes the proof of the Theorem 1.

**Acknowledgement.** I am grateful to the referee for his/her useful suggestions on this paper. In October 2009, Professor Hao Pan in Nanjing gave a talk to introduce his work "On the number of distinct prime factors of \( nj + a^b k \)." I would like to thank Professor Hao Pan for his wonderful talk which attract my interest to this topic.
References

[1] Y. G. Chen and X. G. Sun, *On Romanoff’s constant*, J. Number Theory 106(2004), 275-284.

[2] Y. G. Chen, R. Feng and N. Templier, *Fermat numbers and integers of the form $a^k + a^l + p^\alpha$*, Acta Arith. 135 (2008), 51-61.

[3] F. Cohen and J. L. Selfridge, *Note every number is the sum or difference of two prime powers*, Math. Comput. 29(1975), 79-81.

[4] R. Crocker, *On the sum of a prime and two powers of two*, Pacific J. Math. 36(1971), 103-107.

[5] P. Erdős, *On integers of the form $2^r + p$ and some related problems*, Summa Brasil. Math. 2(1950), 113-123.

[6] H. Halberstam and H. E. Richert, *Sieve Methods*, Academic Press Inc. (London) Ltd., 1974.

[7] M. B. Nathanson, *Additive Number Theory, The Classical Bases*, Springer, New York, 1996.

[8] H. Pan, *On the integers not of the form $p + 2^a + 2^b$*, Acta Arith. 148 (2011), no. 1, 55-61.

[9] H. Pan, *On the number of distinct prime factors of $nj + a^hk$*, Monatsh math. 175(2014), 293-305.

[10] J. Pintz, *A note on Romanov’s constant*, Acta Math. Hungar. 112(2006), 1-14.

[11] A. de Polignac, *Recherches nouvelles sur les nombres premiers*, C. R. Acad. Sci. Paris Math. 29(1849), 397-401, 738-739.

[12] N. P. Romanoff, *Über einige Sätze der additiven Zahlentheorie*, Math. Ann. 109(1934), 668-678.

[13] J. Barkley Rosser and Lowell Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois Journal of Mathematics, 6(1962), 64-94.

[14] Z. W. Sun and M. H. Le, *Integers not of the form $c(2^a + 2^b) + p^\alpha$*, Acta Arith. 99 (2001), 183-190.
[15] X. G. Sun and L. X. Dai, *Chen’s conjecture and its generalization*, Chin. Ann. Math. 34B (2013), 957-962.

[16] T. Tao, *A remark on primality testing and decimal expansion*, J. Austr. Math. Soc. 3(2011), 405-413.

[17] P. Z. Yuan, *Integers not of the form $c(2^a + 2^b) + p^a$*, Acta Arith. 115 (2004), 23-28.