Probability in Orthodox Quantum Mechanics: Probability as a Postulate Versus Probability as an Emergent Phenomenon

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Abstract. The role of probability in quantum mechanics is reviewed, with a discussion of the "orthodox" versus the statistical interpretative frameworks, and of a number of related issues. After a brief summary of sources of unease with quantum mechanics, a survey is given of attempts either to give a new interpretive framework assuming quantum mechanics is exact, or to modify quantum mechanics assuming it is a very accurate approximation to a more fundamental theory. This survey focuses particularly on the issue of whether probabilities in quantum mechanics are postulated or emergent.

1 Orthodox Quantum Mechanics and Issues it Raises

Quantum mechanics (QM) is our most successful physical theory, encompassing phenomena as diverse as chemical bonding, the band structure of solids, and the standard model of particle physics. But the probabilistic aspect of quantum mechanics has been a source of unease from the outset. As surveyed by Home [1], the Founding Fathers were divided over this aspect of the theory that they had created: Bohr, Born, and Heisenberg were comfortable with the probabilistic structure of quantum theory, whereas Einstein and Schrödinger had profound reservations. This unease, and division, have continued to the present.

1.1 Postulates of Quantum Mechanics

Let us begin with a review of the postulates of QM, in the arena of a complex Hilbert space, following for the larger part the presentation of Ballentine [2].

- Observables are associated with self-adjoint operators. Thus we have

\[ S = \sum_n s_n P_n, \quad (1) \]

\[ P_n = \sum_a |a, s_n\rangle\langle a, s_n|, \quad (2) \]

with \( S \) an operator, \( s_n \) its eigenvalues, \( P_n \) the corresponding orthogonal projectors, and \( a \) a label that distinguishes degenerate eigenvectors.
• Each state is associated with a density matrix $\rho$, which is self-adjoint, non-negative, and has trace unity,

$$\rho = \rho^\dagger,$$

$$\rho \geq 0, \quad \text{Tr}\rho = 1,$$

so that the spectral decomposition of $\rho$ takes the form

$$\rho = \sum_n \rho_n |\phi_n\rangle\langle\phi_n|,$$

$$0 \leq \rho_n \leq 1, \quad \sum_n \rho_n = 1.$$

• A pure state is associated with an idempotent density matrix, satisfying $\rho^2 = \rho$. This condition, together with the condition of unit trace, implies that there is exactly one nonzero eigenvalue $\rho_n = 1$, with all other eigenvalues $\rho_{n'}$, $n' \neq n$ vanishing, and so for a pure state the spectral decomposition or the density matrix reduces to

$$\rho = |\phi_n\rangle\langle\phi_n|.$$

• The average of an observable $S$ in a general state $\rho$ is given by

$$\langle S \rangle = \text{Tr}\rho S.$$

In particular, for a pure state with $\rho = |\psi\rangle\langle\psi|$, $\langle\psi|\psi\rangle = 1$, we have

$$\langle S \rangle = \langle\psi|S|\psi\rangle.$$

• An observable $S$ can only take the values $s_n$ of its eigenvalues. The probability of finding the eigenvalue $s_n$ in a normalized pure state $|\psi_n\rangle$ is

$$p_n = \sum_a |\langle\psi|a, s_n\rangle|^2.$$

• Within coherent sectors of Hilbert space, superpositions of pure states are pure states, and self-adjoint compositions of observables are observables.

• The dynamics of the density matrix $\rho$ and of a state $\psi$ in Hilbert space is specified by

$$\rho(t) = U\rho(t_0)U^{-1},$$

$$|\psi(t)\rangle = U|\psi(t_0)\rangle.$$

Here $U(t, t_0)$ is a unitary operator, which for small $t - t_0 = \delta t$ takes the form

$$U = \exp \left(-\frac{i}{\hbar}H(t)\delta t\right),$$

defining the (possibly time dependent) system Hamiltonian. This dynamics is termed the “U operation” by Penrose [3].
We finally come to the effect of a measurement. After the measurement of the eigenvalue $s_n$ in a pure state, the new system state is

$$\psi' = \frac{\sum_a |a, s_n\rangle \langle a, s_n| \psi \rangle}{\sum_a |\langle a, s_n| \psi \rangle|^2}.$$  \hspace{1cm} (14)

This equation summarizes what is termed the "R operation" by Penrose.

1.2 Interpretive Framework

While everyone agrees that the above postulates provide a practical set of rules for making predictions in quantum mechanics, and that these predictions to date have always agreed with experiment, there is a dichotomy when it comes to giving an interpretive framework for the rules.

- On the one hand, we have the "orthodox" interpretation, as given, for example, in the text of Mandl [4]. This asserts that the state $|\psi\rangle$ gives a complete description of an individual system, and that (14) corresponds to "reduction" of the individual’s state vector.

- On the other hand, we have the "statistical" interpretation, as discussed, for example, in the review of Ballentine [2], according to which the state $|\psi\rangle$ describes certain statistical properties of an ensemble of similarly prepared systems. According to this interpretation, (14) corresponds to the preparation of a new ensemble by a measurement. There may be, or there may not be, hidden variables that specify a complete, nonstatistical interpretation of individual systems: the statistical interpretation is agnostic with respect to this issue.

1.3 Why the "R" Operation is Needed

To see why the "R" operation is needed, let us demonstrate that the measurement process cannot be represented by a deterministic, unitary evolution on a closed system. Let us consider a Stern-Gerlach experiment, in which an initial spin eigenstate $|\psi\rangle$ with eigenvalue 1/2 along the $x$ axis is separated, by means of a magnetic field that is inhomogeneous along the $z$ axis, into orthonormal states $|\psi_\uparrow\rangle$, $|\psi_\downarrow\rangle$ that have respective spin eigenvalues 1/2, $-1/2$ along the $z$ axis. Thus, at the detector one sees either $|\psi_\uparrow\rangle$ or $|\psi_\downarrow\rangle$, with

$$\langle \psi_\downarrow | \psi_\uparrow \rangle = 0.$$ \hspace{1cm} (15)

Let us suppose that these final states evolved from the initial state by the deterministic unitary "U" process, which would imply that

$$|\psi_\uparrow\rangle = U |\psi\rangle,$$ \hspace{1cm} (16)

$$|\psi_\downarrow\rangle = U |\psi\rangle.$$ \hspace{1cm} (17)

This would imply

$$0 = \langle \psi_\downarrow | \psi_\uparrow \rangle = \langle \psi | U^\dagger U |\psi\rangle = \langle \psi | \psi \rangle = 1,$$ \hspace{1cm} (18)
which is a contradiction. Hence, the measurement process, in which an initial coherent superposition of states leads to a definite but unpredictable outcome, cannot be described by a deterministic unitary time evolution. Thus measurement involves a physical process, which we have called the “R” process, that is distinct from the deterministic unitary “U” process that governs the unobserved evolution of the quantum system.

However, the “R” process does not have to be nonunitary; one can have for the $i$th atom going through the apparatus the evolution

$$|\psi_i\rangle = U_i |\psi\rangle,$$

(19)

with $U_i$ a unitary evolution that is different for each $i$. This is possible because any path through Hilbert space connecting two normalized pure states can be described by a succession of infinitesimal unitary transformations. To prove this, it suffices to consider the infinitesimal segment $|\psi\rangle \rightarrow |\psi\rangle + |d\psi\rangle$, with $\langle \psi| d\psi \rangle = 0$. If we take

$$U = 1 + |d\psi\rangle \langle \psi| - |\psi\rangle \langle d\psi|,$$

(20)

$$U^\dagger = 1 - |d\psi\rangle \langle \psi| + |\psi\rangle \langle d\psi|,$$

(21)

then we have $U^\dagger U = UU^\dagger = 1$ up to an error of second order, and $U|\psi\rangle = |\psi\rangle + |d\psi\rangle$, as needed. So it is perfectly possible for the “R” process to be described by a stochastic unitary process, constructed from a sequence of random or partially random infinitesimal unitary transformations, and we will see in Sect. 3.1 examples of how this can be accomplished.

1.4 Micro Versus Macro

In QM, the probability is the squared modulus of the probability amplitude. Probability amplitudes superimpose coherently, and between measurements evolve in time by the “U” process. Thus, in the microscopic realm:

- One sees coherent superpositions.
- Amplitudes evolve through deterministic, unitary evolution.

On the other hand, in the macroscopic realm:

- One does not see coherent superpositions, e.g., of dead and alive cats.
- Measurements involve the “R” process, which is not deterministic unitary.

This dichotomy leaves us with the following questions (for more detailed discussions, see Penrose [3] and Leggett [5]):

- Where is the dividing line between “micro” and “macro”?  
- What is responsible for it?
1.5 Postulated Versus Emergent Probability

A unique feature of orthodox QM is that it is the only probabilistic theory where the probabilities are postulated ab initio, and are not emergent from unobserved, deterministic phenomena at a deeper level. A typical theory where probabilities are emergent is statistical mechanics. In statistical mechanics one starts from a probability postulate of uniform phase space occupation. This assumption, and the related concept of an equilibrium ensemble (which reflects the implications of conserved quantities for the phase space occupation), is consistent because of Liouville’s theorem, which implies that a uniform distribution is preserved in time by the dynamics.

However, these probabilistic statements are not the end of the story in statistical mechanics. There are underlying laws – the equations of classical molecular dynamics – which are deterministic; no probabilities enter into their formulation. These laws lead, by a process that is still not completely understood (as reflected in discussions of ergodicity at this Conference), to an effectively uniform phase space distribution for systems that are sufficiently complex. Thus, the probabilistic theory of statistical mechanics is emergent from the deterministic theory of classical mechanics.

1.6 Recapitulation

To sum up, there are a number of sources of unease about QM:

- There is no predictive description of individuals.
- There is a micro-macro divide of unclear origin.
- There is a probabilistic structure that is postulated rather than emergent.

But QM works! Many subtle and remarkable predictions of QM are experimentally verified in many different physical arenas. Thus either

- (A) QM is exact, but to deal with the sources of unease it needs reinterpretation at the conceptual level (although no modification of the standard postulates is needed to use QM as a practical computational and predictive tool).
- (B) QM is not exact, but rather is a very accurate asymptotic approximation to a deeper level theory.

I do not believe that it is just a matter of taste which of these possibilities is chosen, because the distinction between (A) and (B) is relevant to the issue of Planck scale unification of the forces and the particle spectrum. If QM changes, it may profoundly influence the ground rules for unifying gravity with the other interactions.

2 Reinterpretations of Quantum Mechanics Assuming it is Exact

Let us now review four differing approaches based on premise (A), that QM is exactly correct but in need of reinterpretation. Our focus in each case will be on the extent to which the probabilistic structure is postulated or is emergent.
2.1 Everett’s “Many Worlds” Interpretation

In the “Many Worlds” interpretation introduced by Everett [6] and discussed in further detail in the articles collected in [7], there is no state vector reduction; instead, there is only Schrödinger evolution of the wave function of the entire universe. To describe $N$ successive measurements in this interpretation requires an $N$-fold tensor product wave function.

Probability is not emergent, but rather is postulated in the Everett picture. Everett introduces a measure on the coefficients of the final superposition resulting from $N$ successive measurements, which as $N \to \infty$ behaves mathematically like the usual QM probability rule. There is a logical jump (or an implicit assumption – this is still a matter of debate) in going from the Everett measure on tensor product coefficients to statements about the relative frequencies of experimental outcomes.

2.2 The Histories Approach

The so-called “Histories” approach has been extensively developed recently by Griffiths, Omnès, and Gell-Mann and Hartle (for a review and references see [8], and for a semipopular account see [9]). The histories approach takes as a given that QM is a stochastic theory; probability is introduced as a postulate, and is not emergent. The basic objects in the histories approach are time-dependent projectors $E_k(t_k)$ associated with properties occurring in a history. The probability of a history is then postulated to be given by

$$p = \text{Tr}[E_n(t_n) \ldots E_1(t_1) \rho E_1(t_1) \ldots E_n(t_n)]$$

with $\rho$ the density matrix at the initial time. This definition can be shown to lead, under appropriate circumstances, to all of the expected properties of probabilities. In this interpretation, state vector reduction appears only in the statistical interpretation sense discussed above, as a rule for relating the density matrix after a measurement to the density matrix before the measurement.

In both the “Many Worlds” and the “Histories” interpretations, there is by definition no concept of the “individual”. We shall now discuss two other currently studied interpretations of QM that enlarge the mathematical framework to give an “individual”.

2.3 Bohmian Mechanics

There has been a recent revival of interest in Bohmian mechanics (see [10] for a technical account and references, and [11] for a semipopular account). In Bohmian mechanics, in addition to the Schrödinger equation for the $N$-body wave function $\psi(q_1, \ldots, q_N, t)$,

$$\frac{i\hbar}{\partial t} \frac{\partial \psi}{\partial t} = -\sum_{k=1}^{N} \frac{\hbar^2}{2m_k} \nabla^2_{q_k} \psi + V\psi,$$

(23)
the mathematical framework is enlarged by introducing hidden “particles” moving in configuration space with coordinates $Q_k$ and velocities

$$v_k = \frac{dQ_k}{dt} = \frac{\hbar}{m_k} \text{Im} \nabla Q_k \log \psi(Q_1, \ldots, Q_k, t).$$

The state of the “individual” is then specified by giving both the wave function and the coordinates $Q_k$ of the hidden particles. A probability postulate is introduced, that the probability distribution on configuration space obeys $p = |\psi|^2$ at some initial time $t_0$. The Bohmian equations given above then imply that this remains true for all times subsequent to $t_0$; the logic here resembles the use of the Liouville theorem in statistical mechanics. Unlike statistical mechanics, Bohmian mechanics has no underlying molecular dynamics-like layer, so the probabilities are not prima facie emergent. We note, however, that in [10] arguments are given (and are further discussed at this Conference) that the Bohmian probability postulate follows from considerations of “typicality” of initial configurations (in distinction to the ergodicity arguments used in attempts to derive the postulates of statistical mechanics from the equations of molecular dynamics).

### 2.4 The Ax-Kochen Proposal

Recently Ax and Kochen [12] have extended the mathematical framework of QM in a different way to encompass the “individual”. They identify the ray with the ensemble, and the ray representative, i.e. the $U(1)$ phase associated with a particular state vector, with the individual. They then give a mathematical construction to specify a unique physical state from knowledge of the toroid of phases. They introduce a probability assumption, that the a priori distribution of phases is uniform, and then show that, by their construction, this implies that the probabilities of outcomes obey the usual QM rule. Thus, probability in the Ax-Kochen interpretation is not emergent, but their probabilistic postulate is arguably weaker than that in standard QM or in Bohmian QM.

### 2.5 Are Interpretations of Quantum Mechanics Falsifiable?

We conclude this brief survey of alternative interpretations of an assumed exact quantum mechanics by posing the question, can the interpretations given above be falsified? By construction, the four interpretations described above are designed to agree with the predictions of standard QM. Clearly, if an interpretation could be shown to differ in some prediction from that of QM, and if this difference in predictions were resolved experimentally in favor of standard QM (in the way that the Bell inequalities have been tested and favor QM), then the interpretation would be falsified. But suppose that an interpretation makes empirical predictions that, within the domain in which the rules of QM apply, are without exception indistinguishable from the predictions of QM. Then is it possible, in principle, to falsify that interpretation?
The answer, I believe, may be “yes”, because none of the interpretations described above gives a quantitative account of the micro-macro divide – that is, when do we, and when don’t we, expect to see coherent superpositions? To the extent that this becomes an experimentally answerable question, and to the extent that one can get corresponding predictions from the interpretations sketched above, one might be able to distinguish between different interpretive frameworks for an exact QM.

3 Theories Where Quantum Mechanics is Modified

Let us turn now to approaches based on premise (B), that QM is a very accurate approximation to a deeper level theory. We will first discuss phenomenological approaches based on this premise, and then turn to attempts at a more fundamental theory.

3.1 Phenomenological Modifications: Stochastic Models

As we discussed in Subsect. 1.3, although the “R” process cannot be described by a deterministic unitary evolution, it is perfectly admissible for it to be described by a unitary evolution that differs for each individual measurement act, and in particular by a stochastic unitary evolution. Considerable effort has gone over the past two decades into attempts to unify the “U” process and the “R” process into a single dynamical rule, by formulating phenomenological modifications of the Schrödinger equation in which the “individual” is described by a stochastic unitary evolution of a pure state. The physical motivation for such modifications is that if quantum theory is an approximation to physics at a deeper level, there could be small fluctuation or “Brownian motion” corrections to this physics, which determine the outcomes for individual systems.

The natural mathematical language for formulating stochastic modifications of the Schrödinger equation is the Itô stochastic calculus, which is basically a differential calculus version of the theory of Gaussian random variables. (For a clear exposition of the Itô rules, see Gardiner [13].) One introduces the stochastic Itô differential \( dW_t \), which obeys the rules

\[
(dW_t)^2 = dt, \quad dW_t dt = 0;
\]  

(25)

thus \( dW_t \) is a fluctuating variable with magnitude \((dt)^\frac{1}{2}\), and as is familiar from the theory of path integrals, quantities of order \( dt \) are retained while those of order \((dt)^\frac{3}{2}\) are dropped. Let us now consider the following equivalent stochastic evolutions (introduced at various times, and in various forms, by Diósi; Ghirardi, Rimini, and Weber; Gisin; Hughston; Pearle; and Percival – for references, see [14] and [15]). Letting \(|z\rangle\) be a pure state, and \(\rho = |z\rangle\langle z|/\langle z|z\rangle\) be the corresponding density matrix, we can write a stochastic pure state evolution

\[
d|z\rangle = (\alpha dt + \beta dW_t)|z\rangle,
\]

(26)
\[ \alpha = -iH - \frac{1}{8} \sigma^2 [A - \langle A \rangle]^2, \quad \beta = \frac{1}{2} \sigma [A - \langle A \rangle], \]  
\[ A = A^\dagger, \quad \langle A \rangle = \langle z | A | z \rangle / \langle z | z \rangle, \]  
(27)

or the equivalent [15] density matrix evolution

\[ d\rho = -i[H,\rho]dt - \frac{1}{8} \sigma^2 [A, [A, \rho]]dt + \frac{1}{2} \sigma [\rho, [\rho, A]]dW_t. \]  
(29)

Here we have taken units with \( \hbar = 1 \), \( \sigma \) is a numerical parameter which governs the strength of the stochastic process that modifies the standard Schrödinger dynamics, and one can generalize the above equations by replacing \( A \rightarrow A^j \), \( dW_t \rightarrow dW^j_t \) and including a sum over \( j \) in each term. Letting \( E[\rho] \) denote the stochastic expectation of \( \rho \) with respect to \( dW_t \) (not the same as the quantum expectation \( \langle \rho \rangle \)), the evolution of \( \rho \) implies the following Lindblad type evolution of \( E[\rho] \),

\[ \frac{dE[\rho]}{dt} = -i[H, E[\rho]] - \frac{1}{8} \sigma^2 [A, [A, E[\rho]]]. \]  
(30)

Let us now ask [15], when does this equation admit stationary solutions \( E[\rho]_S \), for which

\[ \frac{dE[\rho]_S}{dt} = 0 = -i[H, E[\rho]_S] - \frac{1}{8} \sigma^2 [A, [A, E[\rho]_S]] \]  
(31)

Multiplying by \( E[\rho]_S \), taking the trace, and using cyclic permutation under the trace, which implies that

\[ \text{Tr}E[\rho]_S [H, E[\rho]_S] = \text{Tr}H [E[\rho]_S, E[\rho]_S] = 0, \]  
(32)

we get the condition

\[ 0 = -\frac{1}{8} \sigma^2 \text{Tr}E[\rho]_S [A, [A, E[\rho]_S]] \]  
(33)

\[ = -\frac{1}{8} \sigma^2 \text{Tr}A [E[\rho]_S][A, E[\rho]_S]^\dagger. \]  
(34)

Since the argument of the final trace is positive semidefinite, it must vanish, and so we learn by reference to the evolution equation for \( E[\rho]_S \) that we must have

\[ [A, E[\rho]_S] = 0, \quad [H, E[\rho]_S] = 0, \]  
(35)

in other words, a stationary value \( E[\rho]_S \) must commute with both the Hamiltonian \( H \) and with the operator \( A \) which drives the dissipative process.

Various cases are possible, depending on the choice of \( A \):

- For an energy driven process, with \( A = H \), the stationary value \( E[\rho]_S \) can be any function of \( H \). One can then prove [15] with no approximations that, when all energy eigenstates are nondegenerate, in the limit of large times, \( \rho \) approaches an energy eigenstate projector \( |e\rangle\langle e| \), with each such projector
occurring as the outcome of the stochastic process with the corresponding probability \( P_e = \text{Tr}_e \rho(0) |e\rangle \langle e| \), with \( \rho(0) \) the initial time density matrix. Correspondingly, the stochastic expectation of \( \rho \), which is what we customarily term the density matrix, evolves from a pure state density matrix \( E[\rho(0)] = \rho(0) \) to the mixed state density matrix \( E[\rho]_S = \sum_e P_e |e\rangle \langle e| \). Thus, for an energy driven process (or more generally, processes in which there are several \( A^j \) which all commute with the Hamiltonian), the QM probability rule is emergent from the phenomenological stochastic dynamics.

As discussed in [14], if one assumes that the parameter \( \sigma \) and the corresponding stochastic process originate from Planck scale physics, one gets the estimate (again in units with \( \hbar = 1 \)) \( \sigma \sim M_{\text{Planck}}^{-\frac{1}{2}} \), which implies a characteristic state vector reduction time scale

\[
t_R \sim \left( \frac{2.8 \text{MeV}}{\Delta E} \right)^2 \text{sec } ,
\]

with \( \Delta E \) the energy dispersion of the initial state. An important question that remains to be answered is whether this estimate gives a satisfactory phenomenology for state vector reduction in all cases, when the characteristic \( \Delta E \) arising from environmental interaction effects is assumed. That is, does the predicted micro-macro divide always occur in the right place?

- For a localization process (the Ghirardi, Pearle, Rimini form of the original Ghirardi, Rimini, Weber idea; for references see [14], [15]), one takes \( A \) to be an operator that produces a Gaussian localization, or one uses multiple \( A^j \) corresponding to many such localizations. Since the kinetic term \( p^2/(2M) \) in the Hamiltonian \( H \) does not commute with \( x \), and thus does not commute with a localizing operator \( A \), there now is no stationary limit unless the usual Schrödinger evolution term in the stochastic Schrödinger equation is neglected. In this approximation, in which only the stochastic terms are kept, one gets similar results to those found in the energy driven case, with \( H \) eigenstates replaced now by \( A \) eigenstates.

For the localizing case, an important issue that remains to be addressed is whether the phenomenological theory can be made relativistic. This may be a more severe problem than in the energy driven case because, while the Hamiltonian operator \( H \) of Schrödinger dynamics appears in a similar role in quantum field theory, the coordinate \( x \) appears in quantum field theory as a label for degrees of freedom, rather than as an operator.

### 3.2 Fundamental Modifications

Of course, even though the phenomenological stochastic Schrödinger equations discussed above give exactly (in the energy driven case) or approximately (in the localizing case) an emergent QM probability rule, there is still a probabilistic postulate in the form of the appearance of the Itô differential \( dW_t \). Since these equations have the characteristic form expected for the dynamics of open
quantum systems, it is natural to ask whether they are simply the Brownian motion description of some underlying dynamics. Specifically, can one achieve a fully emergent probabilistic structure at the QM level from a pre-quantum dynamics that is not probabilistic?

Two approaches of this type have been discussed in the literature:

- In [16] 't Hooft has proposed that quantum states are the equilibrium limit orbits or Poincaré cycles of an underlying chaotic, dissipative, deterministic theory.
- In [17] we have proposed as a possible pre-quantum dynamics a "generalized quantum" or "trace" dynamics, obtained by setting up a generalized classical dynamics of noncommuting phase space variables \( \{q_r\}, \{p_r\} \) with no a priori commutativity properties beyond cyclic permutation inside a trace.

One can show that, with an approximation similar to assuming a large hierarchy between the pre-quantum and the QM energy scales, that by an equipartition argument the canonical commutation relations of QM are an emergent property of the statistical mechanics of this system.

Both of these proposed approaches to an emergent probability structure in QM are at present programmatic, and significant open questions remain: Can one construct an effective wave function and Schrödinger equation from the pre-quantum dynamics? What do the leading fluctuation corrections look like, and are they the mechanism responsible for state vector reduction? Can one use them to make contact with the phenomenological stochastic extensions of the Schrödinger equation discussed above? Affirmative answers to these questions would yield the probabilistic structure of QM as an emergent phenomenon, in close analogy with the origins, in the underlying deterministic layer of molecular dynamics, of the probabilistic structure of statistical mechanics. Failure, after sufficient effort, to construct such a pre-quantum underpinning for QM would support the view that QM is exact, in need perhaps only of a modified interpretation.

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References

1. D. Home: *Conceptual Foundations of Quantum Mechanics*, Chapt. 8 (Plenum, New York 1997)
2. L. E. Ballentine, Rev. Mod. Phys. 42, 358 (1970)
3. R. Penrose: *The Emperor’s New Mind* (Oxford University Press, New York 1989)
4. F. Mandl: *Quantum Mechanics*, 2nd. edn., Chapt. 4, Sec. 16, pp. 69-70 (Butterworth, London 1957)
5. A. J. Leggett, Suppl. Prog. Theor. Phys. 69, 80 (1980)
6. H. Everett, III, Rev. Mod. Phys. 29, 454 (1957)
7. B. S. DeWitt and N. Graham: The Many-Worlds Interpretation of Quantum Mechanics (Princeton University Press, Princeton 1973)
8. R. Omnès, Rev. Mod. Phys. 64, 339 (1992)
9. R. B. Griffiths and R. Omnès, Physics Today 52, No. 8, Part 1, 26 (1999)
10. D. Dürr, S. Goldstein, and N. Zanghí, J. Stat. Phys. 67, 843 (1992)
11. S. Goldstein, Physics Today 51, No. 3, 42 (1998) 51, No. 4, 38 (1998)
12. J. Ax and S. Kochen, ‘Extension of Quantum Mechanics to Individual Systems’, quant-ph/9909077
13. C. W. Gardiner: Handbook of Stochastic Methods, Chapt. 4 (Springer, Berlin 1990)
14. L. P. Hughston, Proc. Roy. Soc. Lond. A 452, 953 (1996)
15. S. L. Adler and L. P. Horwitz, ‘Structure and Properties of Hughston’s Stochastic Extension of the Schrödinger Equation’, quant-ph/9909026
16. G ’t Hooft, J. Stat. Phys. 53, 323 (1988) ‘Quantum Gravity as a Dissipative Deterministic System’, gr-qc/9903084 ‘Determinism and Dissipation in Quantum Gravity’, hep-th/0003005
17. S. L. Adler, Nucl. Phys. B 415, 195 (1994) S. L. Adler and A. C. Millard, Nucl. Phys. B 473, 199 (1996) S. L. Adler and A. Kempf, J. Math. Phys. 39, 5083 (1998)