Space-time spectral element method solution for the acoustic wave equation and its dispersion analysis

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(Received 28 October 2016, Accepted for publication 12 May 2017)

Abstract: This paper presents the solution for the continuous space-time spectral element method (CSTSEM) based on Chebyshev polynomials for the acoustic wave equation. Acoustic wave propagation in various dimensions is simulated using quadrilateral, hexahedral, and tesseract elements. The convergence is studied for 1+1-dimensional wave propagation. The extended 2+1- and 3+1-dimensional wave equations are also numerically solved by CSTSEM and the dispersion characteristics are investigated. A fixed-\(\omega\) (\(\omega\) is the angular frequency) method is proposed for computing the dispersion of space-time coupled methods. CSTSEM is verified as a simple, practical isotropic algorithm with low dispersion and high accuracy.

Keywords: Space-time spectral element method, Gauss-Lobatto-Chebyshev nodes, Dispersion, Acoustic wave propagation

PACS number: 43.28.Js, 43.20.–f, 43.20.+g [doi:10.1250/ast.38.303]

1. INTRODUCTION

The spectral element method (SEM) has become increasingly popular for numerically solving partial differential equations due to its high accuracy. It was introduced by Patera in 1984 [1] to approximate the spatial derivative operator for incompressible flow problems by combining the spectral method and finite element method (FEM). It exploits the rapid convergence and high accuracy of spectral methods while retaining the geometric flexibility of low-order FEMs. Numerous semi-discrete spectral element approaches have been used in time-dependent partial differential equations, where the SEM is applied for spatial dimensions, with difference methods widely used in the time domain. This yields unbalanced schemes, i.e., exponential convergence in space and algebraic convergence in time, and when the exact solution is very smooth, the accuracy of the approximate solution is limited by the difference scheme for time. Thus, improvement of the precision of the time discretization is essential.

Space-time FEMs and SEMs have been developed in recent years. Continuous space-time FEMs for wave equations [2–5] and heat conduction problems [6] have been proposed. At the same time, the solution of problems with discontinuities in the displacement gradients have been addressed using discontinuous Galerkin (DG) FEMs [7]. The space-time SEM is of enormous significance, and has been developed for various problems, such as the Burgers equation [8], advection-diffusion problems [4], and second-order hyperbolic equations [9]. Least-squares spectral formulations have also been applied to linear and nonlinear equations [10,11].

The dispersion of numerical methods plays an important role in computational simulations [12]. Dispersion arises because the phase and group velocities vary for waves with different frequencies in the computational mesh [13]. Many methods have been proposed for studying the dispersive properties of various numerical schemes (e.g., finite difference method (FDM), FEM, and SEM) for many problems, such as wave propagation, linear convection diffusion, and Helmholtz equations. Some methods measure dispersion, such as eigenvalue [13–18], wavenumber [12,19–22], angular frequency [23–25], and error derivation [26–29] methods. However, such dispersion analysis considers only spatial discretization. Dispersion analysis for the continuous space-time spectral element method (CSTSEM) is presented here for the first time.

In Sect. 2, the governing equations and computational domains are introduced, and CSTSEM is described in Sect. 3. Some wave propagation problems are solved
numerically in Sect. 4 using CSTSEM. Section 5 introduces the fixed-ω method for dispersion analysis of CSTSEM. Finally, conclusions are presented in Sect. 6.

2. GOVERNING EQUATION

Consider a disturbance propagating in a bounded domain, \( \Omega \), of \( \mathbb{R}^d \), \( d \in \{1, 2, 3\} \) with boundary \( \partial \Omega \) and time region \([0, T]\). The acoustic wave equations for a homogeneous medium are

\[
\frac{\partial^2 u}{\partial t^2} - c_0^2 \sum_{i=1}^{d} \frac{\partial^2 u}{\partial (x^i)^2} = f \quad \text{in} \quad \Omega, \tag{1}
\]

where \( c_0 \) is the local sound wave velocity, \( f \) is the source, and \( t \) is time.

To consider space-time discretization, suppose there is a domain \( \Omega = \Omega_0 \times [0, T] \subset \mathbb{R}^{d+1} \). Let \( x^0 = t \) represent time and \( \vec{x} = (x^0, \vec{x}) \) denote a point \( \vec{x} \in \mathbb{R}^{d+1} \). We reformulate the acoustic equation in the space-time framework as

\[
\nabla \cdot (A \nabla u) = f, \tag{2}
\]

where,

\[
A = \begin{bmatrix} 1 & 0 \\ 0 & D \end{bmatrix},
\]

\[
D = -c_0^2 I, \tag{3}
\]

where \( I \) is a \( d \times d \) unit diagonal matrix and \( \nabla = ((\partial / \partial x^0, \partial / \partial x^1, \ldots, \partial / \partial x^d)) \) is the gradient operator in \( \mathbb{R}^{d+1} \).

The boundary conditions are

\[
\begin{align*}
\frac{\partial u}{\partial n} &= 0 \quad \text{on} \quad \Omega_0, \\
\frac{\partial u}{\partial n} &= u^1 \quad \text{on} \quad \Omega_0, \\
u &= g^0 \quad \text{on} \quad \Gamma_D, \\
\frac{\partial u}{\partial n} \vec{n} &= g^N, \quad i = 1 \sim d \quad \text{on} \quad \Gamma_N,
\end{align*}
\]

where \( u^0 \) is the initial displacement and \( u^1 \) is the initial velocity of the initial disturbance; \( \Gamma_D \) is the Dirichlet boundary and \( \Gamma_N \) is the Neumann boundary. No boundary conditions are imposed on \( \Omega_T \).

3. CONTINUOUS SPACE-TIME SPECTRAL ELEMENT METHOD

First, the Galerkin formulation of Eq. (1) can be derived by multiplying Eq. (1) with an arbitrary test function, \( v \), and integrating by parts over the space-time domain, \( \Omega \), then finding \( u \in U \) such that

\[
\int_{\Gamma_N} (A \nabla u) \cdot \nabla v d\Gamma - \int_{\Omega} A \nabla u \cdot \nabla v d\Omega = \int_{\Gamma_N} f v d\Gamma, \quad \forall v \in V, \tag{7}
\]

where \( V = \{ v \in H_\Omega^1 \mid v|_{\partial \Omega \\cap \Gamma_N} = 0 \} \),
\[
U = \{ u \in H_\Omega^1 \mid u|_{\partial \Omega \\cap \Gamma_D} = g^D, u|_{\Omega_T \cap \partial \Omega} = g^N, u|_{\Gamma_N} = u^0 \}. \tag{8}
\]

The SEM is used for spatial and temporal discretization. The computational domain, \( \Omega \), is divided into \( ND \) non-overlapping elements, where \( N_i \) is the number of elements in the \( x^i \) direction. Each spectral element is mapped into a standard element \([-1, 1]^{d+1}\) by

\[
\xi^i = \frac{2}{L_i} (x^i - x_i^0) - 1 \quad \text{or} \quad x^i = \frac{L_i}{2} (\xi^i + 1) + x_i^0, \tag{9}
\]

where \( L_i = x_i^{m+1} - x_i^m \) is the length of the element in the \( x^i \) direction. In the standard element, the Chebyshev spectral element approximation is written with space-time basis functions as

\[
\vec{u} = \sum_{\rho=0}^{N} \cdots \sum_{\rho=0}^{N} \sum_{i=0}^{d} u_{j}^{\rho, \cdots, \rho} \prod_{i=0}^{d} h_{j}^{\rho}(\xi^{i}), \tag{11}
\]

where \( N \) is the order of the Chebyshev polynomial in the \( x^i \) direction and \( h_j^{\rho}(\xi^{i}) \) are the basis functions, which are identically zero outside the element and are Lagrangian interpolants satisfying

\[
h_{j}^{\rho}(\xi^{i}) = \delta_{jk}, \tag{12}
\]

within the elements, where \( \delta_{jk} \) is the Kronecker delta. The interpolation points \( \xi_k^i \) in each standard element are the extreme points of the Chebyshev polynomials,

\[
\xi_k^i = -\cos \left( \frac{k\pi}{N} \right), \quad k = 0, 1, \cdots, N. \tag{13}
\]

The Lagrange interpolation functions \( h_j^{\rho}(\xi) \) can be expressed as

\[
h_{j}^{\rho}(\xi) = \frac{2}{N \cdot c_{j}^{\rho}} \sum_{m=0}^{N} T_{m}(\xi)T_{m}(\xi), \tag{14}
\]

where \( T_{m}(\xi) = \cos(m \arccos(\xi)) \), \(-1 \leq \xi \leq 1\) are Chebyshev polynomials; and \( c_{j}^{\rho} \) are coefficients given by

\[
c_{j}^{\rho} = \begin{cases} 2, & g = 0 \text{ or } N \leq N \leq 1, \\ 1, & 1 \leq g \leq N - 1. \end{cases} \tag{15}
\]

By definition, all Lagrange polynomials are equal to 1 or 0 at Gauss-Lobatto-Chebyshev points.
The element matrix equation is obtained as

\[ K^{e} = -A_{jk\ell}B_{jk\ell} + c_{0}^{2}B_{jk\ell}A_{jk\ell} \]

and the source term \( F^{e} = f_{j\ell} \cdot \prod_{i=0}^{d} B_{jk\ell} \), where

\[ J_{q} = \begin{cases} 0, & m+n \text{ odd} \\ \frac{m-n}{2} (J_{m-n/2} - J_{m+n/2}), & m+n \text{ even} \end{cases} \]

and

\[ b_{mn} = \int_{-1}^{1} T_{m}(\xi)T_{n}(\xi) d\xi \]

where

\[ a_{mn} = \begin{cases} 0, & m+n \text{ odd} \\ \frac{m-n}{2} (J_{m-n/2} - J_{m+n/2}), & m+n \text{ even} \end{cases} \]

and

\[ \varepsilon_{i} \] is the permutation symbol.

Next we substitute the trial function (Eq. (11)), the test function

\[ \tilde{u} = \sum_{k=0}^{N_{x}} \sum_{k'=0}^{N_{x}} \left( \prod_{i=0}^{d} h_{i}(\xi) \right), \]

and the source term

\[ \tilde{f} = \sum_{p=0}^{N_{x}} \sum_{p'=0}^{N_{x}} \left( \prod_{i=0}^{d} h_{i}(\xi) \right) \]

into Eq. (7). The element matrix equation is obtained as

\[ K^{e} u = F^{e}, \]

with the element stiffness matrix \( K^{e} \) and element loading vector \( F^{e} \), as listed in Table 1, where

\[ A_{jk\ell} = \int_{-1}^{1} \frac{\partial h_{j}(\xi)}{\partial x^{i}} \frac{\partial h_{k}(\xi)}{\partial x^{i}} \frac{\partial h_{\ell}(\xi)}{\partial x^{i}} d\xi^{i} \]

\[ B_{jk\ell} = \int h_{j}(\xi) h_{k}(\xi) h_{\ell}(\xi) d\xi^{i} \]

and

\[ e_{mn} = 1 \]

Table 1: Element stiffness matrix and loading vector for Eq. (18) at various numbers of dimensions.

| Dimension | Element stiffness matrix | Element loading vector |
|-----------|--------------------------|------------------------|
| 1-D       | \( K^{e} = -A_{jk\ell}B_{jk\ell} + c_{0}^{2}B_{jk\ell}A_{jk\ell} \) | \( F^{e} = f_{j\ell} \cdot \prod_{i=0}^{d} B_{jk\ell} \) |
| 2-D       | \( K^{e} = -A_{jk\ell}B_{jk\ell} + \frac{c_{0}^{2}}{2} \sum_{i=1}^{N_{x}} A_{jk\ell} \sum_{k=1}^{N_{x}} B_{jk\ell} \delta_{i} = 0 \) | \( F^{e} = f_{j\ell} \cdot \prod_{i=0}^{d} B_{jk\ell} \) |
| 3-D       | \( K^{e} = -A_{jk\ell}B_{jk\ell} + \frac{c_{0}^{2}}{3} \sum_{i=1}^{N_{x}} A_{jk\ell} \sum_{k=1}^{N_{x}} B_{jk\ell} \sum_{m=1}^{N_{x}} B_{jkm} \) | \( F^{e} = f_{j\ell} \cdot \prod_{i=0}^{d} B_{jk\ell} \) |

4. NUMERICAL RESULTS AND DISCUSSION

The absolute error is \( error = u - \tilde{u} \), where \( u \) is the analytical solution, \( \tilde{u} \) is the numerical solution, and the norm of the absolute error is \( \|error\|_{1} = \sum_{i=1}^{NG} |error_{i}| \), \( \|error\|_{2} = (\sum_{i=1}^{NG} error_{i}^{2})^{1/2} \), \( \|error\|_{\infty} = \max_{1 \leq i \leq NG} |error_{i}| \), where \( i \) is the index of \( error \) and \( NG \) is the total number of unknowns.

4.1. 1+1 Dimensions

The governing equation of one-dimensional (1-D) propagation is

\[ \frac{\partial^{2} u}{\partial t^{2}} - c_{0}^{2} \frac{\partial^{2} u}{\partial x^{2}} = f, \]

where \( c_{0} = 340.29 \text{ m/s} \) and the boundary conditions are

\[ u(0, \xi) = \sin\left(\frac{5\pi \xi}{2}\right), \]

\[ u(t, 0) = u(t, 0.4) = 0. \]

The source term is \( f = 25\pi^{2}(c_{0}^{2} - 1) \sin(5\pi \xi/2) \cos(5\pi t/2) \). The computational domain is \( \Omega = [0, 0.4] \times [0, 0.4] \). \( N_{l} := N_{0} \) and \( N_{m} := N_{1} \) are the numbers of elements and \( N_{t} := N_{0} \) and \( N_{x} := N_{1} \) are the orders of the Chebyshev polynomial in the \( t \) and \( x \) directions, respectively.

In Fig. 1, the logarithmic error is defined as \( \log_{10}[|error|] \). The legends of Fig. 1 are composed in the mode of \( "Nm \times N_{x}, N_{l} \times N_{t}.\) CSTSEM convergence is shown in Fig. 1 for various polynomial degrees. Spatial grids have more effect on numerical accuracy than time grids. The convergence rate as a function of the polynomial degree is shown. Generally, the logarithmic error decays as \( \alpha^{N}(0 < \alpha \leq 1) \), where

\[ \alpha = \begin{cases} 1, & N = N_{l} \\ 0 < \alpha < 1, & N = N_{m}, N_{x}, N_{t} \end{cases} \]

which affects the rate of convergence. This shows that the CSTSEM is an \( h-p \) numerical method.

Figures 1(a) and 1(b) show that the numerical precision increases and is exponentially convergent with increasing polynomial degree. However, until the polynomial degree

\[ \begin{cases} 0, & \text{for } m+n \text{ odd} \\ \frac{1}{1-(m+n/2)}, & \text{for } m+n \text{ even} \end{cases} \]
numerical accuracy is independent of \( N_l \), and Fig. 1(d) shows that fewer elements in spatial dimensions produce a loss of accuracy when \( N_m \) exceeds a threshold.

### 4.2. 2+1 Dimensions

For 2+1 dimensions, referring to [4] we modify the analytical solution to

\[
u = e^{-(x - \cos t)^2 - (y + \sin t)^2},
\]

whose boundary conditions are

\[
u(0, x, y) = e^{-(x-1)^2 - y^2},
\]
\[
u(t, -2, y) = e^{-(2 + \cos t)^2 - (y + \sin t)^2},
\]
\[
u(t, 2, y) = e^{-(2 - \cos t)^2 - (y + \sin t)^2},
\]
\[
u(t, x, -2) = e^{-(x - \cos t)^2 - (2 - \sin t)^2},
\]
\[
u(t, x, 2) = e^{-(x - \cos t)^2 - (2+\sin t)^2}.
\]

Also, the source term in Eq. (2) is

\[
f = 2e^{-(x - \cos t)^2 - (y + \sin t)^2} [\sin t - x \cos t]
\]
\[+ 2(\sin t + y \cos t)^2 - 2c_1^2(x - \cos t)^2 - 2c_2^2(y + \sin t)^2 + 2c_3^2.
\]

The domain \( \Omega = [-2, 2] \times [-2, 2] \times [0, 2] \) was divided into \( 3 \times 3 \times 3 \) elements with fourth-order Chebyshev polynomials in each element. The total number of nodes was 2197. The pressure contours for the \( x = 0, y = 0 \), and \( t = 1 \) slices are shown in Fig. 2. With the fourth-order Chebyshev polynomials, the proposed CSTSEM obtains third-order accuracy and provides excellent agreement between the numerical and analytical solutions.

### 4.3. 3+1 Dimensions

For 3+1 dimensions, the analytical solution is \( u = \sin(5\pi x/2) \sin(5\pi y/2) \sin(5\pi z/2) \cos(5\pi t/2) \). The boundary conditions are

\[
u(0, x, y, z) = \sin \left( \frac{5\pi x}{2} \right) \sin \left( \frac{5\pi y}{2} \right) \sin \left( \frac{5\pi z}{2} \right),
\]
\[
u(t, 0, y, z) = \nu(t, 0.4, y, z) = 0,
\]
\[
u(t, x, 0, z) = \nu(t, x, 0.4, z) = 0,
\]
\[
u(t, x, y, 0) = \nu(t, x, y, 0.4) = 0.
\]

and the source term is

\[
f = \frac{25}{4} \pi^2 (3c_0^2 - 1) \sin \left( \frac{5\pi x}{2} \right) \sin \left( \frac{5\pi y}{2} \right) \sin \left( \frac{5\pi z}{2} \right) \cos \left( \frac{5\pi t}{2} \right),
\]

The computational domain, \( \Omega = [0, 0.4]^4 \), was divided into \( 2 \times 2 \times 2 \times 2 \) tesseract elements [30] with third-order Chebyshev polynomials in each element. The total number

![Fig. 1 Convergence about the computational parameters.](image)

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of nodes was 10,000 and the numerical results at $t = 0.05$, 0.25 s and 0.4 s are shown in Fig. 3. The 3-D datasets shown are time slices from the 4-D mesh. Again, we find excellent agreement between the numerical and analytical solutions.

5. DISPERSION OF THE CONTINUOUS SPACE-TIME SPECTRAL ELEMENT METHOD

5.1. Fixed-$\omega$ Method for Dispersion

For the exact traveling-wave solution of Eq. (1),

$$u(x, t) = R\exp[i(k \cdot x - \omega t)],$$

(41)

the exact dispersion is $\omega |k| = c_0$, where $\omega$ and $k$ are the angular frequency and wave number, respectively. Suppose the numerical solution of a method is of the same form [17],

$$\tilde{u}(\tilde{x}, t) = R\exp[i(\tilde{k} \cdot \tilde{x} - \tilde{\omega} t)],$$

(42)

where $\tilde{\omega}$ and $\tilde{k}$ define the phase velocity, $c_p = \omega |\tilde{k}|$. The phase velocity dispersion is a first-order effect and the group velocity dispersion is a second-order effect. We will
only discuss the phase velocity dispersion. The phase velocity error, i.e., dispersion, is

\[ \text{Dispersion} = \frac{\omega}{|k|c_0}, \]

where \( |k| \neq \frac{\omega}{c_0} \).

If we only consider the numerical algorithm, i.e., we ignore the initial boundary conditions and the loading vector, then

\[ KU = 0. \]

In an unbounded mesh, as shown in Fig. 4, the transformation of physical space and spectral space is

\[ x'_{m,i,j} = x_m \pm \frac{L_x}{2} (\xi_i + 1), \quad i = 0, \ldots, d. \]

Combining Eq. (42) with Eq. (45), each element of Eq. (44) is shown in Table 2, where \( N_t := N_{x,0}, N_x := N_{x,1}, \)

![Fig. 3 Solutions for 3-D space wave equation. Left: numerical solution; right: numerical error.](image1)

(a) \( t = 0.05s \)

(b) \( t = 0.25s \)

(c) \( t = 0.4s \)
\[ \sum_{a=-1}^{0} \sum_{b=-1}^{0} \sum_{c=0}^{N_x} \sum_{d=0}^{N_y} K^a_{x+a,N_x} K^{b}_{y+b,N_y} N_x d N_y \quad \text{for } N_x d N_y (-1)^d t \]

\[ e^{\left(-\frac{i\pi k}{2}\left(\frac{N_x}{N_y}\right)\right)} = 0 \]

*\( \bar{k} \) is the wave number [31]. \( k = |\bar{k}| \) is a scalar quantity in one dimension, \( \bar{k} = \bar{k}(\cos \theta, \sin \theta) \) in two dimensions and \( \bar{k} = \bar{k}(\cos \theta \cos \phi, \sin \theta \cos \phi, \sin \phi) \) in three dimensions.

\[ \omega \Delta t \begin{array}{cccc}
0.62832 & 1.00039 & 1.00039 & 1.00039 \\
1.25664 & 1.00089 & 1.00089 & 1.00089 \\
1.57080 & 0.99730 & 0.99730 & 0.99730 \\
1.72788 & — & 0.99413 & 0.99414 \\
1.88496 & 0.98989 & 0.98987 & 0.98989 \\
2.19911 & 0.97772 & 0.97760 & 0.97772 \\
2.51327 & 0.96010 & 0.95971 & 0.96010 \\
2.82743 & 0.93657 & 0.93557 & 0.93657 \\
2.98451 & — & 0.92251 & 0.92251 \\
3.14159 & 0.90690 & 0.90690 & 0.90690 \\
6.12611 & 0.48933 & 0.48933 & 0.48933 
\end{array} \]
\( Ny := N_x \), and \( Nz := N_y \).

For an arbitrary \( \omega \) [ Note: The text mentions \( \omega \), but the symbol is not defined here. ] [ Equation (43)], we find the CSTSEM dispersion. Thus, the fixed-\( \omega \) method for CSTSEM is simple and practical.

### 5.2. \( \omega \)-Independence

Figures 5–7 show that CSTSEM has almost the same dispersion in 1+1-, 2+1-, and 3+1-dimensional wave equations. The dispersion is not affected by spatial parameters, but rather by \( N_t \) for a given \( N_t \) by \( \omega \Delta t \).

Figures 6(a) and 7(a) show that CSTSEM is isotropic in the spatial directions. Dispersion is greatest for \( N_t = 2 \), so \( N_t = 2 \) was chosen to study the \( \omega \)-dependence. Table 3 shows that when \( \omega \Delta t \leq \pi/2 \), the dispersion in 1+1, 2+1, and 3+1 dimensions is identical and dispersion errors are below 0.27%.

### 5.3. Dispersion Analysis

Because the dispersion is identical for 1+1, 2+1, and 3+1 dimensions when \( \omega \Delta t \leq \pi/2 \), the analysis may be conducted in 1+1 dimensions, without loss of generality. Dispersions for the 1+1-dimensional wave equation are shown in Fig. 8. As expected, the dispersion decreases for larger \( N_t \) or smaller \( \Delta t \) and is always very low.

Figure 8 is in good agreement with the results obtained in Sect. 5.2. Although temporal discretization affects the dispersion, spatial discretization has no effect. On the other hand, Fig. 1 (see Sect. 4.1.1) shows that spatial discretization has more effect on numerical errors than does temporal discretization. This apparent discrepancy is because temporal discretization induces dispersion, whereas spatial discretization induces dissipation. Thus, CSTSEM minimizes dissipation and is consequently more stable.
6. CONCLUSIONS

We investigated the continuous space-time spectral element method (CSTSEM) based on Gauss-Lobatto-Chebyshev quadrature formulas. Using the numerical algorithm, we presented examples in 1+1, 2+1, and 3+1 dimensions. Spatial discretization affects numerical accuracy more than temporal discretization. Numerical accuracy is exponentially convergent until \( N_x, N_y, N_z \) reach a threshold. In other words, CSTSEM is an \( h-p \) method. To analyze the dispersive property of this method, we proposed a simple and practical dispersion analysis procedure for CSTSEM using the acoustic wave equations. For the acoustic wave equations, the fixed-\( \omega \) method is provided to study CSTSEM dispersion. It is found that CSTSEM is isotropic in spatial dimensions and that the parameters in temporal discretization introduce dispersion. CSTSEM is a natural extension of spectral element methods in that it deals with both time and space operators in the same way, and is capable of solving wave propagation problems with low dispersion and high accuracy. Several improvements to CSTSEM remain to be explored. The proposed CSTSEM will simplify further study on the wave motion in fluids and the noise induced in fluid and flow problems.

REFERENCES

[1] A. T. Patera, “A spectral element method for fluid dynamics: Laminar flow in a channel expansion,” *J. Comput. Phys.*, 54, 468–488 (1984).
[2] D. A. French and T. E. Peterson, “A continuous space-time finite element method for the wave equation,” *Math. Comput.*, 65, 491–506 (1996).
[3] M. N. Guddati and J. L. Tassoulas, “Space-time finite elements for the analysis of transient wave propagation in unbounded layered media,” *Int. J. Solids Struct.*, 36, 4699–4723 (1999).
A. Matthew and K. J. Han, “Numerical approach to space-time finite elements for the wave equation,” *J. Comput. Phys.*, **226**, 466–476 (2007).

G. M. Hulbert and T. J. R. Hughes, “Space-time finite element methods for second-order hyperbolic equations,” *Comput. Methods Appl. Mech. Eng.*, **84**, 327–348 (1990).

C. K. Kim, “On the numerical computation for solving the two-dimensional parabolic equations by space-time finite element method,” *JSME. Int. J. Ser. B*, **44**, 434–438 (2001).

H. Huang and F. Costanzo, “On the use of space-time finite elements in the solution of elasto-dynamic problems with strain discontinuities,” *Comput. Methods Appl. Mech. Eng.*, **116**, 5315–5343 (2002).

B. Y. Piniias, E. Moses, U. Zrahia and A. L. Yarin, “Space-time spectral element methods for one-dimensional nonlinear advection-diffusion problems,” *J. Comput. Phys.*, **72**, 62–74 (1995).

U. Zrahia and B. Y. Piniias, “Space-time spectral element method for solution of second-order hyperbolic equations,” *Comput. Methods Appl. Mech. Eng.*, **116**, 135–146 (1994).

B. D. Maerschalck and M. I. Gerritsma, “The use of Chebyshev polynomials in the space-time least-squares spectral element method,” *Numer. Algorithms*, **38**, 172–196 (2005).

J. P. Pontaza and J. N. Reddy, “Space-time coupled spectral/hp least-squares finite element formulation for the incompressible Navier-Stokes equations,” *J. Comput. Phys.*, **197**, 418–459 (2004).

M. Ainsworth, P. Monk and W. Muniz, “Dispersive and dissipative properties of discontinuous Galerkin finite element methods for the second-order wave equation,” *J. Sci. Comput.*, **27**, 5–40 (2006).

T. Liu, M. K. Sen, T. Hu, J. D. D. Basabe and L. Li, “Dispersion analysis of the spectral element method using a triangular mesh,” *Wave Motion*, **49**, 474–483 (2012).

G. Gabard and E. J. Bramley, “A full discrete dispersion analysis of time-domain simulations of acoustic liners with flow,” *J. Comput. Phys.*, **273**, 310–326 (2014).

J. D. D. Basabe and M. K. Sen, “Grid dispersion and stability criteria of some common finite element methods for acoustic and elastic wave equations,” *Geophysics*, **72**, 81–95 (2007).

W. Daukscher and A. F. Emery, “Accuracy in modeling the acoustic wave equation with Chebyshev spectral finite elements,” *Finite Elem. Anal. Des.*, **26**, 115–128 (1997).

H. Yang, F. Li and J. Qiu, “Dispersion and dissipation errors of two fully discrete discontinuous Galerkin methods,” *J. Sci. Comput.*, **55**, 552–574 (2012).

G. Gassner and D. A. Kopriva, “A comparison of the dispersion and dissipation errors of Gauss and Gauss-Lobatto discontinuous Galerkin spectral element methods,” *SIAM. J. Sci. Comput.*, **33**, 2560–2579 (2011).

I. Harari, “Finite element dispersion of cylindrical and spherical acoustic waves,” *Comput. Methods Appl. Mech. Eng.*, **190**, 2533–2542 (2001).

Y. Liu and M. K. Sen, “Time-space domain dispersion-relation-based finite-difference method with arbitrary even-order accuracy for the 2D acoustic wave equation,” *J. Comput. Phys.*, **232**, 327–345 (2013).

S. Suleau and Ph. Bouillard, “One-dimensional dispersion analysis for the element-free Galerkin method for the Helmholtz equation,” *Int. J. Numer. Methods Eng.*, **47**, 1169–1188 (2000).

A. Deraemaeker, I. Babuška and P. Bouillard, “Dispersion and pollution of the FEM solution for the helmholtz equation in one, two and three dimensions,” *Int. J. Numer. Methods Eng.*, **46**, 471–499 (1999).

F. I. Zysberman, P. M. Gauzelleno and J. E. Santos, “Dispersion analysis of a non-conforming finite element method for the dispersion analysis of the spectral element method using a triangular mesh,” *Wave Motion*, **49**, 474–483 (2012).
Helmholtz and elastodynamic equations,” *Int. J. Numer. Methods Eng.*, **58**, 1381–1395 (2003).

[24] F. I. Zyserman and P. M. Gauzellino, “Dispersion analysis of a nonconforming finite element method for the three-dimensional scalar and elastic wave equations,” *Finite Elem. Anal. Des.*, **41**, 1309–1326 (2005).

[25] T. Okuzono, T. Otsuru, R. Tomiku and N. Okamoto, “A finite-element method using dispersion reduced spline elements for room acoustics simulation,” *Appl. Acoust.*, **79**, 1–8 (2014).

[26] M. Ainsworth and H. A. Wajid, “Dispersive and dissipative behavior of the spectral element method,” *SIAM. J. Numer. Anal.*, **47**, 3910–3937 (2009).

[27] A. Sei and W. Symes, “Dispersion analysis of numerical wave propagation and its computational consequences,” *J. Sci. Comput.*, **10**, 1–26 (1995).

[28] E. Johnson, “Dispersion properties of simple 3D finite elements,” *Commun. Numer. Methods Eng.*, **11**, 97–103 (1995).

[29] J. D. D. Basabe, M. K. Sen and M. F. Wheeler, “The interior penalty discontinuous Galerkin method for elastic wave propagation: Grid dispersion,” *Geophys. J. Int.*, **175**, 83–93 (2008).

[30] E. W. Weisstein, “Tesseract: From MathWorld—A Wolfram Web Resource,” http://mathworld.wolfram.com/Tesseract.html (accessed 2015-10-06).

[31] G. Seriani and S. P. Oliveira, “Dispersion analysis of spectral element methods for elastic wave propagation,” *Wave Motion*, **45**, 729–744 (2008).