A REMARK ON THE CRITICAL EXPONENT FOR THE SEMILINEAR DAMPED WAVE EQUATION ON THE HALF-SPACE

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Abstract. In this short notice, we prove the non-existence of global solutions to the semilinear damped wave equation on the half-space, and we determine the critical exponent for any space dimension.

1. Introduction

Let $n \geq 1$ be an integer and let $\mathbb{R}_+^n$ be the $n$-dimensional half-space, namely,
\[
\mathbb{R}_+^n = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n ; x_n > 0 \} \quad (n \geq 2), \quad \mathbb{R}_+ = (0, \infty) \quad (n = 1).
\]

We consider the initial-boundary value problem for the semilinear damped wave equation on the half-space:
\[
\begin{cases}
u_{tt} - \Delta \nu + \nu_t = |\nu|^p & t > 0, x \in \mathbb{R}_+^n, \\
u(t, x) = 0, & t > 0, x \in \partial \mathbb{R}_+^n, \\
u(0, x) = u_0(x), \nu_t(0, x) = u_1(x), & x \in \mathbb{R}_+^n.
\end{cases}
\]

(1.1)

Here, $\nu$ is a real-valued unknown function and $u_0, u_1$ are given initial data.

Our aim is to show the non-existence of global solutions and determine the critical exponent for any space dimension. Here, the critical exponent stands for the threshold of the exponent of the nonlinearity for the global existence and the finite time blow-up of solution with small data.

For the semilinear heat equation $\nu_t - \Delta \nu = \nu^p$ on the whole space, Fujita [1] discovered that if $p > p_F(n) := 1 + 2/n$, then the unique global solution exists for every small positive initial data, while the local solution blows up in finite time for any positive data if $1 < p < p_F(n)$. Namely, the critical exponent of the semilinear heat equation on the whole space is given by $p_F(n)$, which is so-called Fujita’s critical exponent. Later on, Hayakawa [3] and Kobayashi, Shirao and Tanaka [8] proved that the case $p = p_F(n)$ belongs to the blow-up region. Moreover, the initial-boundary value problem of the semilinear heat equation on the half space $\mathbb{R}_+^k \times \mathbb{R}_+^{n-k}$ was studied by [9][10][11][12] and they determined the critical exponent as $p = p_F(n + k)$.

The critical exponent for the semilinear damped wave equation on the whole space was studied by many authors and it is determined as $p = p_F(n)$. We refer the reader to [13][14] and the references therein.

Ikehata [5][6][7] studied the semilinear damped wave equation on the half-space (1.1) and proved that if $p_F(n+1) < p < \infty$ ($n = 1, 2$), $p_F(n+1) < p \leq \frac{n}{n-2}$ ($n \geq 3$), $(u_0, u_1) \in H_0^1(\mathbb{R}_+^n) \times L^2(\mathbb{R}_+^n)$ have compact support in $\mathbb{R}_+^n$ and $\|\nabla u_0\|_{L^2} + \|u_1\|_{L^2}$

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is sufficiently small, then the problem (1.1) admits a unique global solution. When $n = 1$, Nishihara and Zhao [13] proved the blow-up of solutions when $1 < p \leq p_F(2)$, namely, the critical exponent of (1.1) on the half-line is determined as $p = p_F(2)$. However, there is no blow-up result for (1.1) when $n \geq 2$.

In this paper, we prove the non-existence of global classical solutions for (1.1) for all $n \geq 1$, and we determine the critical exponent of (1.1) as $p_F(n + 1)$.

**Theorem 1.1.** Let $1 < p \leq p_F(n + 1) = 1 + \frac{2}{n+1}$. We assume that the initial data satisfy $x_n u_0, x_n u_1 \in L^1(\mathbb{R}^n_+)$ and

\[
(1.2) \quad \int_{\mathbb{R}^n_+} x_n(u_0(x) + u_1(x)) \, dx > 0
\]

(when $n = 1$, we interpret $x_n = x$). Then, there is no global classical solution to (1.1).

Our proof is based on the test function method by Zhang [15]. To apply it to the half-space, we employ the technique by Geng, Yang and Lai [2]. Namely, we use the test function having the form $x_n \psi R(t, x)$, where $\psi_R(t, x)$ is a test function supported on the rectangle $\{ (t, x) \in [0, \infty) \times \mathbb{R}^n : t \leq R^2, |x| \leq R (j = 1, \ldots, n) \}$.

### 2. Proof of Theorem 1.1

We suppose that the global classical solution $u$ of the problem (1.1) exists and derive the contradiction. Let $\psi \in C_0^\infty([0, \infty) \times \mathbb{R}^n_+)$ be a test function. Using the integration by parts, we compute

\[
(2.1) \quad \int_0^\infty \int_{\mathbb{R}^n_+} |u|^p \psi \, dx \, dt = \int_0^\infty \int_{\mathbb{R}^n_+} (u_{tt} - \Delta u + u_t) \psi \, dx \, dt
\]

\[
= \int_0^\infty \int_{\mathbb{R}^{n-1}_+} \partial_{x_n} u(t, x', 0) \psi(t, x', 0) \, dx' \, dt
\]

\[
+ \int_0^\infty \int_{\mathbb{R}^n_+} u(\psi_{tt} - \Delta \psi - \psi_t) \, dx \, dt
\]

\[
- \int_{\mathbb{R}^n_+} ((u_0(x) + u_1(x)) \psi(0, x) - u_0(x) \psi_t(0, x)) \, dx,
\]

where we used the notation $x' = (x_1, \ldots, x_{n-1})$. Now, we choose the test function $\psi$ as follows. Let $\eta(t) \in C_0^\infty([0, \infty))$ be a non-increasing function satisfying

\[
\eta(t) = 1 \quad (t \in [0, 1/2]), \quad \eta(t) = 0 \quad (t \in [1, \infty)).
\]

We also define $\phi \in C_0^\infty(\mathbb{R}^n)$ by $\phi(x) := \eta(|x_1|) \eta(|x_2|) \cdots \eta(|x_n|)$. Let $R > 0$ be a parameter and let $\psi_R(t, x) := \phi(x/R) \eta(t/R^2)$. We denote the rectangle $D_R := \{ x \in \mathbb{R}^n : |x_1| \leq R, \ldots, |x_n| \leq R \}$ and we put $D_R^l := D_R \setminus D_{R/2}$. Then, it is obvious that $\text{supp} (\partial_{x_j} \phi(\cdot / R)) \subset D_R \setminus D_{R/2}$. With the above notations, we choose our test function as $\psi(t, x) = x_n \psi_R(t, x)^l$ with sufficiently large integer $l$.

Let

\[
I_R := \int_0^\infty \int_{\mathbb{R}^n_+} |u|^p x_n \psi_R^l \, dx \, dt.
\]
We note that our choice of test function implies \( x_n\psi_R(t,x)\big|_{x_n=0} = 0 \) and \( x_n\partial_t(\psi_R(t,x))\big|_{t=0} = 0 \). Moreover, by the assumption (1.2), we see that there exists \( R_0 > 0 \) such that

\[
\int_{\mathbb{R}^n_+} ((u_0(x) + u_1(x))x_n\psi_R(0,x)\big) dx > 0
\]

holds for \( R \geq R_0 \). Therefore, we deduce from (2.1) that

\[
I_R \leq \int_0^\infty \int_{\mathbb{R}^n_+} u(\partial_t^2 (x_n\psi_R) - \Delta (x_n\psi_R) - \partial_t (x_n\psi_R)) dxdt
\]

\[
= : K_1 + K_2 + K_3
\]

for \( R \geq R_0 \). We estimate \( K_1, K_2 \) and \( K_3 \) individually. First, for \( K_1 \), we apply the Hölder inequality to obtain

\[
K_1 \leq CR^{-4} \left( \int_{R^2/2}^{R^2} \int_{D^+_R} |u|^p x_n\psi_R dxdt \right)^{1/p} \left( \int_{R^2/2}^{R^2} \int_{D^+_R} x_n dxdt \right)^{1/p'}
\]

\[
\leq CR^{-4+(n+3)/p'} \tilde{I}_R^{1/p},
\]

where \( p' \) stands for the Hölder conjugate of \( p \) and

\[
\tilde{I}_R := \int_{R^2/2}^{R^2} \int_{D^+_R} |u|^p x_n\psi_R dxdt.
\]

Similarly, by using

\[
\Delta (x_n\psi_R) = lR^{-2}x_n \left( \phi \left( \frac{x}{R} \right) \right)^{l-1} \left( \Delta \phi \left( \frac{x}{R} \right) \right) + (l-1) \phi \left( \frac{x}{R} \right)^{l-2} \left( \nabla \phi \left( \frac{x}{R} \right) \right) \left( \frac{t}{R^2} \right) + 2lR^{-1} \phi \left( \frac{x}{R} \right)^{l-1} \left( \partial_{x_n} \phi \right) \left( \frac{x}{R} \right) \eta \left( \frac{t}{R^2} \right),
\]

we estimate \( K_2 \) as

\[
K_2 \leq CR^{-2} \left( \int_0^{R^2} \int_{D^+_R \setminus D^+_{R/2}} |u|^p x_n\psi_R dxdt \right)^{1/p} \left( \int_0^{R^2} \int_{D^+_R \setminus D^+_{R/2}} x_n dxdt \right)^{1/p'} + CR^{-1} \left( \int_0^{R^2} \int_{D^+_R \setminus D^+_{R/2}} |u|^p x_n\psi_R dxdt \right)^{1/p} \left( \int_0^{R^2} \int_{D^+_R \setminus \{x_n > R/2\}} x_n^{-p'/p} dxdt \right)^{1/p'}
\]

\[
\leq CR^{-2+(n+3)/p'} \tilde{I}_R^{1/p'},
\]

where

\[
\tilde{I}_R = \int_0^{R^2} \int_{D^+_R \setminus D^+_{R/2}} |u|^p x_n\psi_R dxdt
\]

and we note that \( (\partial_{x_n} \phi)(x/R) = 0 \) on the set \( \{x_n \leq R/2\} \). The term \( K_3 \) can be estimated in the same way as \( K_1 \) and we have

\[
K_3 \leq CR^{-2+(n+3)/p'} \tilde{I}_R^{1/p'}.
\]

Combining the estimates above, we deduce

\[
(2.2) \quad I_R \leq C(R^{-4+(n+3)/p'} \tilde{I}_R^{1/p} + R^{-2+(n+3)/p'} \tilde{I}_R^{1/p} + R^{-2+(n+3)/p'} \tilde{I}_R^{1/p}).
\]
In particular, using $\hat{I}_R \leq I_R$ and $\tilde{I}_R \leq I_R$, we have
\begin{equation}
I_R \leq C(R^{-4+(n+3)/p'} + R^{-2+(n+3)/p'})I_R^{1/p}.
\end{equation}
When $1 < p < p_F(n+1)$, letting $R \to \infty$, we see that $I_R \to 0$, which implies $u \equiv 0$. However, this contradicts $(u_0, u_1) \not\equiv 0$.

On the other hand, when $p = p_F(n+1)$, we have $-2 + (n+3)/p' = 0$ and hence, we see from (2.3) that $I_R \leq C$ with a constant $C$ independent of $R$. Thus, letting $R \to \infty$, we have $\|u\|_{L^1(0, \infty) \times \mathbb{R}^n_+}$ noting this and the integral region of $\hat{I}_R$ and $\tilde{I}_R$, we also deduce
\begin{equation}
\lim_{R \to \infty} (\hat{I}_R + \tilde{I}_R) = 0.
\end{equation}
This and (2.2) imply
\begin{equation}
I_R \leq C(\hat{I}_R^{1/p} + \tilde{I}_R^{1/p}) \to 0 \quad (R \to \infty),
\end{equation}
and hence, $u \equiv 0$. This again contradicts $(u_0, u_1) \not\equiv 0$ and completes the proof.

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