Loop Equations and the Topological Phase of Multi-Cut Matrix Models

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We study the double scaling limit of $m$KdV type, realized in the two-cut Hermitian matrix model. Building on the work of Periwal and Shevitz and of Nappi, we find an exact solution including all odd scaling operators, in terms of a hierarchy of flows of $2 \times 2$ matrices. We derive from it loop equations which can be expressed as Virasoro constraints on the partition function. We discover a “pure topological” phase of the theory in which all correlation functions are determined by recursion relations. We also examine macroscopic loop amplitudes, which suggest a relation to 2D gravity coupled to dense polymers.

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1. Introduction

Recent studies of two-dimensional Euclidean gravity have yielded a host of surprises. These surprises have come from lattice definitions [1–6], continuum path-integral definitions [7–9], and topological field theory definitions [10–12] of what is presumably the same theory.

Among the surprises from the lattice/matrix-model approach has been the discovery of theories which look very similar to the solved models of two-dimensional gravity coupled to matter, but for which no continuum or topological interpretation has been found. For example, a particularly natural family of matrix models can be derived from the integral over a unitary matrix. These have a double scaling limit whose exact solution is strikingly similar to that of the lattice gravity models, with the KdV hierarchy being replaced by the mKdV hierarchy [13,14]. One is naturally led to ask whether these theories have equally simple world-sheet interpretations.

The simplest way to attack this problem would be to find an interpretation of the perturbation expansion for the theory as a lattice gravity theory, possibly with matter and appropriate weights in the sum over graphs, and directly take the world-sheet continuum limit. This can be difficult since the same continuum theory can be obtained from many different matrix models. The important point is not the type of matrix but rather the structure of the saddle-point eigenvalue distribution near its points of non-analyticity. The mKdV scaling limits come from tuning a coupling to a critical point where two endpoints of the eigenvalue distribution collide. This is generic for a unitary matrix, but is easy to arrange for a Hermitian matrix as well, for example by taking a double-well potential [15,16]. Thus there are many possible graphical formalisms which one might try to interpret. Perturbation theory for the unitary matrix integral can be done by changing variables $U = \exp iH$; the resulting graphs have vertices of arbitrary order as well as fermionic lines coming from a fermionic representation of the Jacobian [17]. We will discuss this expansion further in the conclusions. Alternatively one could sum over the expansions about the various minima of the double-well problem.

It is not obvious how to interpret any of these lattice theories, because we do not know how to take the continuum limit directly on the world-sheet. The simplest lattice
gravity is exceptional since it has the correct degrees of freedom and an everywhere positive measure. Nevertheless, already with non-positive measures there are subtleties in the continuum limit which can reproduce gravity coupled to matter. The measures in the present problem are too complicated for heuristic arguments. All this underscores the importance of developing a more precise understanding of this continuum limit.

Since we cannot deduce the continuum theory directly from the lattice we are forced to compare the structure of the exact solution of the lattice theory with the structure of known continuum theories. With this motivation, we describe below the exact solution of the lattice theory in some detail. In section two we derive from the lattice the complete string and flow equations for two-cut models, including all even and odd perturbations in the potential. In section three we show how these equations can be rewritten as Virasoro-type constraints. This reformulation suggests the existence of a topological phase of the theory, discussed in section four. In section five we make some brief remarks about macroscopic loops in these theories. In the conclusions we attempt to draw some inferences about what the continuum formulation of these models might be.

2. Multicritical Matrix Model Potentials and Physical Observables

In this section we derive the string equation and flow formalism, generalizing the analysis of the two-cut Hermitian matrix model and study the most general scaling potential. The general even multicritical potential resulting in a two-cut eigenvalue distribution was obtained in:

\[ V'_m(\lambda) = k(m)\lambda^{2m+1} \left(1 - \frac{1}{\lambda^2}\right)^{1/2} \bigg|_+ , \quad m = 1, 2, \ldots , \]

where the subscript + means we keep the polynomial part in an expansion about infinity and the constant

\[ k(m) = 2^{2m+1}(m+1)!(m-1)!/(2m-1)! . \]

The universality class of a potential is determined by the behaviour of the saddle-point eigenvalue density \( \rho(\lambda) \) around the critical point. The eigenvalue density corresponding
to $V_m$ vanishes at the critical point $\lambda = 0$ as $\lambda^{2m}$. To obtain the most general scaling behaviour, we look for the potentials resulting in $\rho(\lambda) \sim \lambda^{2m+1}$ for $\lambda \to 0$. Such $\rho(\lambda)$ are not positive definite and we will introduce them through perturbations around some $V_m(\lambda)$, in the spirit of [21]. The required behaviour is

$$
\tilde{\rho}_n(\lambda) = 0 \quad \text{for} \quad |\lambda| > 1,
$$

$$
\tilde{\rho}_n(\lambda) \propto \lambda^{n-1} \quad \text{for} \quad \lambda \to 0,
$$

$$
\propto (1 - \lambda)^{1/2} \quad \text{for} \quad \lambda \to 1^-,
$$

$$
\propto (1 + \lambda)^{1/2} \quad \text{for} \quad \lambda \to -1^+,
$$

$$
\int_{-1}^{1} d\lambda \tilde{\rho}_n(\lambda) = 0.
$$

This is satisfied by

$$
\tilde{\rho}_n(\lambda) = \frac{d}{d\lambda} [\lambda^n (1 - \lambda^2)^{3/2}], \quad n = 1, 2, \ldots
$$

and the corresponding potential is

$$
\tilde{V}_n = v(n) N^{c(n)} \lambda^n (\lambda^2 - 1)^{3/2}|_+,
$$

with $v(n)$ and $c(n)$ constants to be fixed.

The scaling ansatz appropriate to the potential $V_m$ is

$$
R_n = r_c + (-1)^n a^{1/m} f(x) + \cdots,
$$

$$
S_n = s_c + (-1)^n a^{1/m} g(x) + \cdots,
$$

$$
\frac{n}{N} = 1 - a^2 (x - \mu), \quad Na^{2+1/m} = 1.
$$

$s_c$ is of the order of odd perturbations, which are supressed by a power of $N$. The orthonormal polynomials $\mathcal{P}_n(\lambda)$ tend to two scaling functions, depending on parity:

$$
(-1)^n \mathcal{P}_{2n}(a^{1/m} \lambda) = a^{1/2m} p_+(x, \lambda),
$$

$$
(-1)^n \mathcal{P}_{2n+1}(a^{1/m} \lambda) = a^{1/2m} p_-(x - a^{1/m}, \lambda).
$$

From (2.3) and (2.4) it follows that the multiplication by the eigenvalue $\lambda \equiv a^{1/m} \tilde{\lambda}$ is represented on $\Psi = \begin{pmatrix} p_+ \\ p_- \end{pmatrix}$ by a $2 \times 2$ matrix (after a rescaling of $f$ and $g$ and a rotation of $\Psi$):

$$
\tilde{\lambda} J_3 \Psi = (D + q) \Psi,
$$

(2.5)
where \( D \equiv d/dx, q = f(x)\mathcal{J}_1 + g(x)i\mathcal{J}_2, \) and \( \mathcal{J}_i \) are matrices satisfying \([\mathcal{J}_i, \mathcal{J}_j] = i\varepsilon_{ijk}\mathcal{J}_k.\) Finally, redefining \( \Psi \to \mathcal{J}_3 \Psi \) we obtain (dropping tilde from \( \tilde{\lambda} \))

\[
\lambda \Psi = 4(D + q)\mathcal{J}_3 \Psi \equiv A \Psi.
\]

We will need the resolvent

\[
R \equiv \frac{1}{(D + q)\mathcal{J}_3 - \lambda}.
\]

In their work on the resolvents of matrix differential operators [22] Gelfand and Dikii show that \( R \) satisfies

\[
\mathcal{J}_3 R' = [\mathcal{J}_3 R, q - \lambda \mathcal{J}_3],
\] (2.6)

and that it has an asymptotic expansion

\[
R(x, \lambda) = \sum_{k=0}^{\infty} R_k \lambda^{-k}.
\]

From (2.4) it is clear that, up to constants, the most general expansion of \( R \) is

\[
\mathcal{J}_3 R = \sum_{k=0}^{\infty} \text{grad} \, \mathcal{H}_k \lambda^{-k},
\] (2.7)

where \( \text{grad} \, \mathcal{H}_k \equiv -\mathcal{J}_1 G_k - i\mathcal{J}_2 F_k + \mathcal{J}_3 H_k. \) Plugging (2.7) into (2.6) results in the recursion relations determining \( G_k, F_k, H_k. \) Setting \( G_{-1} = F_{-1} = 0, \, H_{-1} = 1, \) we obtain

\[
F_{k+1} = G_k' + g H_k,
G_{k+1} = F_k' + f H_k,
H_k' = gG_k - f F_k,
\] (2.8)

for all \( k \geq 0. \) The first few are

\[
\begin{align*}
F_0 &= g, & G_0 &= f, & H_0 &= 0, \\
F_1 &= f', & G_1 &= g', & H_1 &= \frac{1}{2}(g^2 - f^2), \\
F_2 &= g'' + \frac{1}{2}g(g^2 - f^2), & G_2 &= f'' + \frac{1}{2}f(g^2 - f^2), & H_2 &= gf' - fg', \\
F_3 &= f''' + \frac{3}{2}f'(g^2 - f^2), & G_3 &= g''' + \frac{3}{2}g'(g^2 - f^2), \\
H_3 &= gg'' - \frac{1}{2}(g')^2 - f f'' + \frac{1}{2}(f')^2 + \frac{3}{8}(g^2 - f^2)^2.
\end{align*}
\]
This determines the expansion of the resolvent
\[ R = \sum_{k=0}^{\infty} (-iJ_2G_{k-1} - J_1F_{k-1} + H_{k-1}/4)\lambda^{-k}. \]

There is an \(SO(1,1)\) symmetry in our system coming from possible choices in the original definition of \(q\). Observables will be invariant under this symmetry.

A related derivation of (2.8) would start from two Hamiltonian structures defined in [23] for the system with the Lax operator
\[ L = D + q - \lambda J_3. \] (2.10)

For the flow in the coupling \(t_k\), with dimension \(k\), their compatibility gives
\[ \frac{d}{dt_k}q = [\text{grad} \, \hat{H}_k, J_3] = [\text{grad} \, \hat{H}_{k-1}, D + q], \] (2.11)
which is easily seen to result in (2.8).

We are now ready to start determining the observables. We will calculate the 2-point functions of the puncture operator \(P\) with the scaling operators \(\sigma_n\) corresponding to the perturbations by \(\tilde{V}_n\). We will follow the presentation of [21], beginning with the one-point function
\[ \frac{\partial F}{\partial t_n} = v(n)Nc(n) \oint_{\mathcal{C}} \frac{dy}{2\pi i} y^n (y^2 - 1)^{3/2} |_+ tr \left( \frac{1}{y - A} \Pi_N \right), \] (2.12)
where
\[ (\Pi_N)_{ij} = \begin{cases} \delta_{ij} & \text{if } 0 \leq i, j \leq N - 1, \\ 0 & \text{otherwise}, \end{cases} \]
and \(\mathcal{C}\) surrounds all of the eigenvalues of \(A\). By deforming \(\mathcal{C}\) to a circle at infinity one can see that the subscript + in (2.12) can be dropped. To obtain a two-point function, we study a variation \(\delta\partial F/\partial t_n\) and use [21]:
\[ \delta tr \left( \frac{1}{y - A} \Pi_N \right) = \frac{1}{2} tr \left( \frac{1}{y - A} [S(\delta V(A)), \Pi_N] \right), \]
\[ S(B)_{ij} = \varepsilon(i - j)B_{ij}, \]
\[ \varepsilon(j) = \text{sign of } j, \quad j \neq 0, \]
\[ = 0, \quad j = 0. \]
We choose the simplest even operator as the puncture operator. The corresponding variation of the potential is \( \delta V(A) = N^{c_p} A^2 \), and we obtain (to leading order in \( N \)):

\[
\frac{\partial^2 F}{\partial x \partial t_n} = \frac{v(n)}{2} N^{c_p+c(n)} \int_C \frac{dy}{2\pi i} y^n (y^2 - 1)^{3/2} tr \left( \frac{1}{y-A} [S(A^2), \Pi_N] \right),
\]

\[
= v(n) r_c N^{c_p+c(n)} \int_C \frac{dy}{2\pi i} y^n (y^2 - 1)^{3/2} \left[ \left( \frac{1}{y-A} \right)_{N,N-2} + \left( \frac{1}{y-A} \right)_{N-1,N+1} \right].
\]

The scaled quantities are

\[
y = a^{1/m} \lambda, \\
A = a^{1/m}4(D+q)J_3, \\
|2n\rangle = (-1)^n a^{1/2m} |x, +\rangle, \\
|2n+1\rangle = (-1)^n a^{1/2m} |x - a^{1/m}, -\rangle,
\]

and (2.13) becomes (again to leading order)

\[
\frac{\partial^2 F}{\partial x \partial t_n} = v(n) \frac{r_c}{4} a^{(n+1)/m} N^{c_p+c(n)} \int_C \frac{d\lambda}{2\pi i} \lambda^n \left( \langle x, +| R|x, +\rangle + \langle x, -| R|x, -\rangle \right)_{x=\mu},
\]

\[
= v(n) \frac{r_c}{2} a^{(n+1)/m} N^{c_p+c(n)} H_n(\mu).
\]

The simplest non-trivial case in this formalism is \( m = 1 \) potential (2.1) with simplest odd perturbation \[20\]

\[
V = V_1 + Naetr(\phi).
\]

(Note that, in the context of the unitary matrix model as 2D lattice Yang-Mills theory this coupling corresponds to a discretized version of \( \log \det U \mod 2\pi \), and hence corresponds to a lattice version of the theta angle \[24\].) Using (2.3) with \( s_c = -ae/16 \) in the recursion relations gives string equations\[4\]

\[
0 = x f + e g' + 2(f'' + \frac{1}{2} f(g^2 - f^2)),
\]

\[
0 = x g + e f' + 2(g'' + \frac{1}{2} g(g^2 - f^2)).
\]

\[1\] To obtain these string equations we rescale \( e, f \) and \( g \) and shift \( x \). In order to compare with (2.3) the rescalings of \( f \) and \( g \) have to be the same as the ones used to obtain (2.3).
The specific heat is
\[ \langle PP \rangle = -\frac{x}{4} + \frac{e^2}{48} + \frac{1}{4} (g^2 - f^2). \] (2.18)

The first two terms are non-universal and the universal piece is given by \( H_{1/2} \). Therefore, we can identify \( x \equiv t_1, \sigma_1 \equiv P \). This identification shows that, in order to obtain a finite result in (2.15), we need \( c(n) = n/(2m + 1) \). Finally, for later convenience, we choose \( u(n) = 1/r_c \) for every \( n \). In conclusion, in the scaling limit of the two-cut matrix model we have obtained the following observables
\[ \frac{\partial^2 F}{\partial x \partial t_n} \equiv \langle P\sigma_n \rangle = \frac{1}{\tau} H_n. \] (2.19)

The odd flows \( F_{2k} \) and \( G_{2k} \) have terms with no derivatives. However, an insertion of an odd operator is still suppressed by \( 1/N \).

To derive the string equations we will use
\[ n = R_n^{1/2} \int V' P_n P_{n-1} = R_n^{1/2} \int_C \frac{dy}{2\pi i} V'(y) \left( \frac{1}{y-A} \right)_{n,n-1}, \] (2.20)

\[ V' = Nk(m)y^{2m}(y^2-1)^{1/2}|_++ \sum_{k=1}^{2m} t_k \frac{1}{r_c} N^{2m+1} y^{k-1}(y^2-1)^{3/2}|_+ + \cdots, \] (2.21)

In (2.21) \( + \cdots \) denotes terms of higher order in \( a \) upon introduction of scaling variables through \( n = N(1-a^2x) \) and (2.14).

We want to replace the \( y \)-integral along \( C \), a contour surrounding all of the eigenvalues of \( A \), by an integral over \( \lambda \), i.e. over infinitesimally small \( y \)'s. An obstacle to passing to a \( \lambda \)-integral comes from the \( A \)-eigenvalues outside of the critical region. Their contributions will not have scaling dependence on coupling constants, but can be large \( (O(N)) \) nonetheless.

In order to be able to neglect consistently eigenvalues of \( A \) at large \( y \), instead of scaling (2.20), we will scale its derivative with respect to \( t_l, l \neq 1 \). Namely, the \( t_l \) dependence of \( V \) appears only for \( y \) very small \( (O(a^{1/m})) \), while for \( y \sim O(1) \) it is suppressed by a power of \( (large) N \). The result is
\[ 0 = \frac{\delta}{\delta t_l} \int_0^\infty \frac{d\lambda}{2\pi i} \sum_k k t_k \lambda^{k-1}(\mp) \langle x, \pm | r=0 \sum_{r=0}^\infty (-i J_2 G_{r-1} - J_1 F_{r-1} + H_{r-1}/4) \lambda^{-r} | x, \mp \rangle, \] (2.22)

\( ^2 \) The factor of \( Na \) in front of \( tr(\phi) \) in (2.16) is precisely \( N^{2/3} \) appearing in \( \tilde{V}_2 \). We use the fact that the scaling limit of a \textit{generic} odd (even) potential like \( tr(\phi) (tr(\phi^2)) \) is given by \( \tilde{V}_2 (\tilde{V}_1) \).
where the upper (lower) sign corresponds to $n$ even (odd). After doing the integral and taking the 1, 2 (2, 1) element of the resulting matrix, we have

\[ 0 = \sum_{k \geq 1} k t_k G_{k-1} \equiv S_g, \]

\[ 0 = \sum_{k \geq 1} k t_k F_{k-1} \equiv S_f. \]  \hfill (2.23)

The integration constant, which is independent of all $t_l, l \neq 1$, can be seen to be zero by comparison with (2.17). (It was shown in [25] that by modifying the action to include boundaries, or “quark” degrees of freedom one can introduce a nonzero constant on the left hand side of (2.23).)

The string equation generalizes (2.17). Moreover, it may be equivalently written as a flatness condition, generalizing that in [10]:

\[ \left[ L, \frac{d}{d\lambda} - M \right] = 0 \] \hfill (2.24)

where

\[ M = \sum_{k \geq 1} k t_k M_{k-1}, \]

\[ M_k = \text{grad} \hat{H}_{k-1} + \lambda \text{grad} \hat{H}_{k-2} + \cdots + \lambda^{k-1} \text{grad} \hat{H}_0 + \lambda^k J_3, \] \hfill (2.25)

and $M_0 = J_3$. $M$ is chosen to make the left hand side of (2.24) independent of $\lambda$, as described in [23].

The flatness condition (2.24) can be interpreted as the isomonodromic deformation condition for the monodromy defined by a solution of

\[ \left( \frac{d}{d\lambda} - M \right) \Psi = 0. \] \hfill (2.26)

Furthermore, the partition function is again the isomonodromic $\tau$-function. The proof of these statements follows the steps already outlined in [26, 16].

3 The form of recursion relations useful in the proof is (2.8) with the third equation replaced by

\[ H_{k+1} = \frac{1}{2} \sum_{r=0}^{k} (F_r F_{k-r} - G_r G_{k-r} - \hat{H}_r \hat{H}_{k-r}). \]
to the string equations chosen by the matrix model will in general lead to nontrivial Stokes data. As described in [26] this implies that the $\tau$ function does not correspond to a point in the Segal-Wilson Grassmannian, although it does correspond to a point in the Sato Grassmannian.

3. Loop Equations.

The solutions of one-matrix models satisfy Schwinger-Dyson equations which can be derived by variation of the functional integral [27]. These have a double scaling limit [28,29] which in the KdV-type systems can be rewritten as Virasoro constraints on the partition function [29,30].

Since the same Schwinger-Dyson equations apply (before taking the continuum limit) to the two-cut Hermitian matrix model, and since the unitary matrix model has analogous Schwinger-Dyson equations, one would expect continuum loop equations to exist in this case as well.

Previously we had derived a subset of these loop equations (with two non-zero couplings) by directly scaling the matrix model Schwinger-Dyson equations [31]. Here we derive the complete loop equations from the string equation/mKdV formalism. The argument is similar to that used in [30] for the KdV theories. Using the flow equations, the string equation can be reformulated as a differential constraint on the partition function as a function of the couplings. In principle we can then derive a series of constraints as follows: the same recursion relation which takes the $n$’th flow to the $n + 1$’st can be applied to the $n$’th constraint to find a new constraint.

Since the constraint operators all annihilate the partition function, they form a left ideal. Therefore commutators of constraints are also constraints. Guided by our expectation that the constraints form a Virasoro algebra, we will find a minimal set of constraints which generate this algebra. The completeness of the resulting algebra will then follow if we can argue that there is a unique solution of the resulting constraints. This will be true in a certain sense to be discussed below.

The scaling operator corresponding to the coupling $t_n$ will be called $\sigma_n$. $\sigma_1$ will also be called $P$, though one should probably think of the $P$ as standing for “primary” and not
“puncture” for reasons explained below. The string equation is now $S_f = S_g = 0$, and our constraints will be derived from combinations of these.

We start with

$$0 = gS_g - fS_f$$

$$= \sum_k kt_k \langle \sigma_{k-1} PP \rangle$$

$$= D^2 \sum_{k \geq 2} kt_k \langle \sigma_{k-1} \rangle,$$

which we interpret as the second derivative of an $L_{-1}$ constraint. The explicit dependence on $t_1$ has cancelled out.

The recursion relations imply that $gG_{k+1} - fF_{k+1} = gF_k' - fG_k'$, so we can get another equation from

$$0 = gS_f' - fS_g'$$

$$= \sum_k kt_k \langle \sigma_k PP \rangle + (g^2 - f^2)$$

$$= \sum_k kt_k \langle \sigma_k PP \rangle + 2\langle PP \rangle$$

$$= D^2 \sum_k kt_k \langle \sigma_k \rangle$$

the second derivative of an $L_0$ constraint.

We can continue in this vein using the recursions to construct $t_k d/dt_{k+n}$ in terms of differential polynomials of the string equations. We will skip $L_1$ (which is similar) and proceed to the $L_2$ constraint, since it will generate the complete algebra. Using the recursions three times, we have

$$I_2 = \sum_k kt_k \langle \sigma_{k+2} PP \rangle$$

$$= \frac{1}{2} \sum_k ft_k (gG_{k+2} - fF_{k+2})$$

$$= \frac{1}{2} \sum_k ft_k (gF_{k+1}' - fG_{k+1}')$$

$$= \frac{1}{2} \sum_k ft_k (gG_k'' - fF_k'' + (g^2 - f^2)H_k' + (gg' - ff')H_k).$$

$$= \frac{1}{2} \sum_k ft_k (gF_{k-1}''' - fG_{k-1}''' + X_{k-1} + (g^2 - f^2)H_k' + (gg' - ff')H_k).$$
with $X_{k-1} = g(fH_{k-1})'' - f(gH_{k-1})''$. The $X$ piece is identically zero by the $L_{-1}$ constraint (3.1) and its derivatives:

$$D^n \sum_{k \geq 2} kt_k \langle \sigma_{k-1} \rangle = \sum_k kt_k D^{n-1} H_{k-1}. \quad (3.4)$$

Using $S'' = \sum_{k \geq 2}okuska F''_{k-1} + 3g'' = 0$ (resp. for $g$), this is

$$I_2 = -\frac{3}{2} (gg'' - ff'') + 2 \sum_k kt_k (2\langle PP \rangle \langle \sigma_k PP \rangle + \langle PPP \rangle \langle \sigma_k P \rangle) \quad (3.5)$$

and using derivatives of the $L_0$ constraint

$$I_2 = -8\langle PP \rangle^2 - 2\langle PPP \rangle \langle P \rangle - \frac{3}{2} (gg'' - ff''). \quad (3.6)$$

The other terms we expect in an $L_2$ constraint are

$$\langle PPPP \rangle = \frac{1}{2} (gg'' - ff'' + (g')^2 - (f')^2), \quad (3.7)$$

$$\langle \sigma_3 P \rangle = \frac{1}{2} (gg'' - ff'' - \frac{1}{2}(g')^2 + \frac{1}{2}(f')^2 + \frac{3}{8}(g^2 - f^2)^2). \quad (3.8)$$

These combine nicely to give

$$0 = \sum_k kt_k \langle \sigma_{k+2} PP \rangle + 2\langle \sigma_3 P \rangle + \langle PPPP \rangle + 2\langle PP \rangle^2 + 2\langle PPP \rangle \langle P \rangle. \quad (3.9)$$

Defining $Z = e^F$, this is the second derivative of the expected constraint $L_2 Z = 0$ with

$$L_2 = \sum_{k \geq 1} kt_k \frac{\partial}{\partial t_{k+2}} + \frac{\partial^2}{\partial t_1^2}. \quad (3.10)$$

Note that to get to (3.5) we assumed that the undifferentiated $L_{-1}$ and $L_0$ constraints were satisfied, i.e. that they contained no integration constant. Under this assumption, we can now write the result as Virasoro constraints for a two-dimensional free massless boson. Define

$$\partial \phi(z) \equiv \sum_{k \geq 1} k z^{k-1} t_k + \sum_{k \geq 1} z^{-1-k} \frac{\partial}{\partial t_k}, \quad (3.11)$$

an untwisted boson; then the constraints are the modes $L_n$ for $n \geq -1$ of the usual untwisted stress-tensor $(\partial \phi)^2$ with zero ground-state energy.
The integration constants in the constraints must be compatible with the Virasoro algebra, otherwise the commutator of two constraints would generate new constraints which would force $Z$ to be trivial. An obvious possibility would be to include the zero mode of the boson as well,

$$\partial \phi(z) \equiv \sum_{k \geq 1} k z^{k-1} t_k + z^{-1} q + \sum_{k \geq 1} z^{-1-k} \frac{\partial}{\partial t_k}.$$  (3.12)

$q$ would be a new parameter of the theory, not associated with any operators or flows. In the “usual” phase of the theory with all odd couplings set to zero, it is clear that $q = 0$, since all the other terms in the $L_{-1}$ constraint are zero, but we will find a use for non-zero values shortly.

It should also be possible to derive these loop equations directly by double scaling the finite $N$ Schwinger-Dyson equations. In particular, the theory of the boson (3.12) should emerge from the formulation of the lattice theory using second quantized fermions. Unfortunately, the currently available derivations of this “field theory on the spectral curve,” are merely heuristic.

4. A Topological Phase

In the series of multicritical models including pure gravity, described by the string equation

$$x = t_1 u + t_2 (u'' + 3u^2) + \ldots,$$  (4.1)

the simplest point is not pure gravity but rather the “pure topological” theory with $t_1 = 1$ and $t_n = 0$ for $n > 1$. This theory has a conserved charge (“ghost number”) with a distinct background charge for each genus surface, so a specific correlation function can be non-zero for at most one genus. In particular $\langle \sigma_0 \sigma_0 \sigma_0 \rangle = u' = 1$ is non-zero only on the sphere.

With purely even couplings, there is no analogous model in the mKdV hierarchy, but with odd couplings there is a precisely analogous model with $t_2 = 1$ and $t_n = 0$ for $n > 2$. The string equation is then

$$x g + f' = 0$$
$$x f + g' = 0.$$  (4.2)
with solution
\[ g + f = C_1 e^{-x^2/2} \]
\[ g - f = C_2 e^{x^2/2}. \] (4.3)

\( C_1 \) and \( C_2 \) are constants of integration. Although these solutions are transcendental, physical quantities are not: for example,
\[ \langle \sigma_1 \sigma_1 \rangle = \frac{1}{2} (g^2 - f^2) = \frac{1}{2} C_1 C_2, \] (4.4)

so this model appears to have \( \gamma_{\text{string}} = 0 \). (The Baker functions, which are Airy functions in topological gravity, are parabolic cylinder functions in this theory.)

The \( SO(1,1) \) symmetry insures that all correlation functions in this model are polynomial in \( x \). Assign the combinations \( f + g \) and \( f - g \) \( SO(1,1) \) charge 1 and \(-1\); then correlation functions must be neutral. All charge-neutral polynomials in these functions are constant, while appearances of \( d/dx \) will produce positive powers of \( x \).

The structure of the topological phase is seen most clearly in the loop equations. Just as for topological gravity, expanding the loop equations around these couplings gives a series of recursion relations, one for each operator in the theory, expressing an operator insertion in terms of correlators of lower total degree. For example, the \( L_{-1} \) constraint gives us
\[ \langle \sigma_1 \prod_i \sigma_{n_i} \rangle = -\frac{1}{2} \sum_i n_i \langle \sigma_{n_i-1} \prod_{j \neq i} \sigma_{n_j} \rangle \] (4.5)
(where \( \sigma_0 \equiv 0 \)) for all cases except one:
\[ \langle \sigma_1 \sigma_1 \rangle = -\frac{q}{2}. \] (4.6)

To get non-trivial answers in the topological phase, we must take the parameter \( q = -C_1 C_2 \) non-zero.

Just as for the KdV models, the existence of a phase in which all operator insertions satisfy recursion relations, means that the expansion of the partition function to all orders is uniquely determined by the constraints. They can therefore have only one analytic solution. Unlike the KdV models, since the parameter \( q \) is not a coupling of the model, we cannot flow from this topological phase to the phases with \( q = 0 \). This is probably
connected with the difficulty of defining this pure topological phase in the matrix model – our odd perturbations of the potential (2.2) really only make sense as perturbations of even potentials. On the other hand the structure of the pure topological phase seems very clear from the solution, so we will not let this stop us from studying it.

The ghost number conservation law follows most directly from the loop equations. If we write them in terms of our boson $\partial \phi(z)$, they are homogeneous in $z$, so the ghost number assignment of each operator and coupling must be just the associated power of $z$ in a mode expansion, up to a possible constant shift. We can use this constant shift to make the ghost number of one coupling of our choice zero. If we choose $t_2$, setting $t_2 = 1$ will not break ghost number conservation.

For topological gravity this is the end of the story. The special role of $t_2$ in this theory is that it is the only choice of non-zero coupling for which all operator insertions have an associated recursion relation. In the present case, both $t_2$ and $q$ must be set non-zero, and they have different ghost number, $q$ having that of “$t_0$”. A simple example of a correlation function which shows that $q$ must be assigned ghost number is

$$\langle \sigma_3 \sigma_4 \rangle = -48\lambda^5 qx - 144\lambda q^3 x + 168\lambda^3 q^2 x^3 - 24\lambda^5 q x^5$$

(4.7)

where $\lambda$ counts derivatives (powers of $1/N$).

We can maintain a ghost number conservation law if we take $q$ non-zero but treat it as the coupling of a new operator in the theory. If the operator $\sigma_n$ is given ghost number $2 - n$, then $q$ will couple to an operator $Q$ with ghost number 2. Now the basic correlation function on the sphere is

$$\langle \sigma_1 \sigma_1 Q \rangle = -\frac{1}{2}.$$  

(4.8)

$Q$ is an unusual operator in that it does not have descendants. The ghost number counting would suggest that it could be renamed $\sigma_0$ and that the other operators are its descendants, but this is misleading. It clearly appears in a very different way in the string equation; in the loop equations, it is different in that its conjugate $d/dq$ does not appear. The necessity to include this operator is the most serious difference with the topological gravity coupled to matter of [10].
5. Macroscopic Loops for Unitary-Matrix Models

It has recently been shown that the study of macroscopic loop amplitudes [32] is particularly well-suited for the comparison of continuum-Liouville and matrix model formulations of 2D quantum gravity [33] [34]. In particular, the Wheeler-DeWitt equation plays a central role in making such comparisons. In this appendix we derive analogous macroscopic loop amplitudes for the unitary-matrix model, and show that they too obey a Wheeler-DeWitt-like equation. This equation gives a hint about the proper worldsheet interpretation of these theories.

The macroscopic loop operator is defined to be $\Psi^\dagger e^{-\ell L}\Psi$ in the fermionic formulation, where $\Psi$ is a two-component fermi field (obtained from scaling the even and odd indexed orthogonal polynomials) and $L$ is the Lax operator representing multiplication by $\lambda$. The one-point function is therefore

$$\langle w(\ell) \rangle = \int_t^\infty dx \langle x | \text{tr} \left\{ e^{-\ell \left( \begin{array}{cc} D & -f - g \\ f - g & -D \end{array} \right) } \right\} |x \rangle$$  (5.1)

We obtain the genus zero approximation by evaluating the matrix element with the WKB approximation to get

$$\langle w(\ell) \rangle = \int_t^\infty dx \int_{-\infty}^\infty dp \text{tr} \left\{ e^{-\ell \left( \begin{array}{cc} ip & -f - g \\ f - g & -ip \end{array} \right) } \right\}$$

$$= \int_t^\infty dx \int_{-\infty}^\infty dp \left( e^{-\ell \sqrt{g^2 - f^2 - p^2}} + e^{\ell \sqrt{g^2 - f^2 - p^2}} \right)$$  (5.2)

$$= 2 \int_t^\infty dx \int_{-\infty}^\infty dp \cos(\ell \sqrt{p^2 - H_1})$$

$$= -\pi \int_t^\infty dx h_1^{1/2} J_1(\ell h_1^{1/2})$$

where $h_1 \equiv -H_1$ is positive (for large $x$) and $J_1$ is a Bessel function. Using the flow equations, specialized to the case of genus zero, we therefore obtain the wavefunctions of the even flow operators:

$$\langle \sigma_{2k-1} w(\ell) \rangle = -\pi \ell \int_{h_1}^\infty dy y^{k-1} J_0(\ell y^{1/2})$$  (5.3)
The operators for the odd flows have zero wavefunction at tree level. The integral in (5.3) only converges in a distributional sense. We may imagine regularizing the original macroscopic loop operator so as to obtain an additional factor of $e^{-\varepsilon\sqrt{y}}$ in (5.3). It is then easy to show that (5.3) and subsequent formulae below possess a finite unambiguous limit as $\varepsilon \to 0^+$. In particular, using the identities $\frac{2\nu}{\pi} J_{\nu}(z) = J_{\nu+1}(z) + J_{\nu-1}(z)$ and

$$\lim_{\varepsilon \to 0^+} \int_{1}^{\infty} dy e^{-\varepsilon\sqrt{y}} (y - 1)^{k-1} J_0(a\sqrt{y}) = \Gamma(k)2^k(-a)^{-k} J_k(a)$$

we may prove (as in [33]) the existence of a linear, upper-triangular, analytic change of operators $\sigma_\nu \to \hat{\sigma}_\nu$ which have the wavefunctions:

$$\psi_\nu(\ell) \equiv \langle \hat{\sigma}_\nu w(\ell) \rangle = h^{\nu/2}_1 J_\nu(\ell h^{1/2}_1)$$

for $\nu = 2k - 1$, with $k = 1, 2, ...$.

The wavefunctions of the $\hat{\sigma}$ operators satisfy the Bessel equation, which, in the one-cut phase has been interpreted as the Wheeler-DeWitt equation for gravity coupled to conformal matter in the minisuperspace approximation. Assuming a similar interpretation holds in this case we see that the Bessel equation should be interpreted as

$$\left[\left(\ell \frac{\partial}{\partial \ell}\right)^2 - \frac{\mu}{\gamma^4} \ell^2 - \frac{8}{\gamma^2}\left(\frac{Q^2}{8} + \Delta - 1\right)\right] \psi_\nu = 0$$

where $\gamma, Q, \mu$ are parameters in the Liouville theory, following the conventions of [33], and $\Delta$ is the dimension of the matter field being dressed. It follows that the cosmological constant must be considered to be negative. Accordingly, for a convergent path integral the worldsheet should have a Minkowskian signature [36]. Moreover, it follows from (5.6) that the central charge and a subset of dimensions of the matter theory must satisfy

$$\nu^2 = \frac{8}{\gamma^2}\left(\frac{1 - c}{24} + \Delta\right)$$

$$\gamma^2 = \frac{1}{6}(13 - c - \sqrt{(1 - c)(25 - c)})$$

where $\nu$ is any odd integer.

We can now use (5.5)(5.7) to learn about the continuum theory. Let us assume that the structure of wavefunctions (5.5) is exactly analogous to that described in [33]. In particular,
we assume that the multicritical phases of the matrix model defined by \( k = 1, 2, 3, \ldots \) are characterized by the identification of the wavefunction of the cosmological constant with \( \psi_{\nu_0} \) for \( \nu_0 = 2k - 1 \). Plugging into (5.7) we find a series of matter theories with central charges

\[
c = 1 - \frac{6\nu_0^2}{\nu_0 + 1}
\]  

(5.8)

Again using (5.7) we discover that the \( \psi_{\nu} \) are wavefunctions of operators whose flatspace dimensions are

\[
\Delta_{\nu} = \frac{\nu^2 - \nu_0^2}{4(\nu_0 + 1)}
\]  

(5.9)

The above critical exponents are consistent with the identification of the matter theory as a special case of the \( O(n) \) model [37]. In particular, the \( O(n) \) model for \( n = -2\cos\pi g \) is thought [38] to have a hierarchy of multicritical points, for a fixed \( n \), with central charges

\[
c = 1 - \frac{6(g - 1)^2}{g}
\]  

(5.10)

Moreover, the spectrum of the model includes operators with weights given by [38]

\[
\Delta_{\nu} = \frac{\nu^2 - (g - 1)^2}{4g}
\]  

(5.11)

for \( \nu \in \mathbb{Z} \). The operator algebra respects the \( \mathbb{Z}_2 \) grading defined by the parity of \( \nu \). It is possible to identify the \( \mathbb{Z}_2 \) eigenspaces as Ramond and Neveu-Schwarz sectors of a conformal field theory [40]. In view of these results [38] [39] [40] we may identify \( g = \nu_0 + 1 \), an even integer, so that \( n = -2 \). Only the “Ramond operators” with \( \nu \) odd have nonvanishing wavefunctions at genus zero, but it is natural to identify all the \( \sigma_{\nu} \) as the gravitationally dressed versions of the \( O(n = -2) \) operators with dimensions (5.11).

6. No Conclusions

In the previous sections we have described in detail the solution of the unitary-matrix model, including string equations, flows, Virasoro constraints and macroscopic loop amplitudes. Returning to the questions mentioned in the introduction, we can now make some comparisons with candidate continuum formulations. We will discuss three candidate theories below. Unfortunately, in all three cases the available evidence remains inconclusive.
The first candidate is 2D supergravity. The multicritical points would correspond to supergravity coupled to the $(2, 4k)$ superminimal matter theories, as already discussed in [31]. Evidence for this is (1) the structure of even and odd scaling operators and their dimensions match the $(2, 4k)$ series of super-minimal matter; (2) correlators of even (Neveu-Schwarz) operators agree with those of bosonic gravity at tree level [11]; (3) there is no space-time supersymmetry or Grassmann couplings, as one might expect for supergravity coupled to low $c$ minimal models with no possibility of GSO projection.

Clearly the above is not very compelling evidence. One would like to have, for example, tree level Ramond correlators or an understanding of the super-WdW equation to make a more definite statement. Higher genus correlators could also settle the question but problems associated with sums over spin structures make these difficult to calculate. In particular, the calculation of the one-loop partition function from the continuum theory could be revealing. The contribution for the even spin structures has been carefully determined in [42]. There is a further point which is rather difficult to reconcile with this interpretation. The two-cut model in one dimension is the same to all orders in perturbation theory as the one-cut model [13] [14]. This would be rather surprising for $\hat{c} = 1$ matter coupled to supergravity. Nevertheless, the supergravity interpretation has not yet been definitely ruled out.

The second candidate is dense polymers or the $O(n = -2)$ fermionic model, (these two being intimately related non-unitary matter theories with $c = -2$ [39]) coupled to Lorentzian metric gravity. The principle evidence for this was described in section five and follows from assuming that the macroscopic loop amplitudes satisfy the Wheeler-DeWitt equation of 2D gravity. A further hint that this is the correct interpretation comes from examining in detail the perturbation series of the simplest unitary-matrix model using the action in [17]. The Feynman perturbation series can be written as a sum over surfaces together with a sum over self-avoiding loop configurations (the fermion loops) on those surfaces. Unfortunately, the Boltzman weights for the loops are similar to, but not precisely equal to those of the $O(n)$ model and it is not clear if the difference is important at criticality. Moreover, in this interpretation it is also rather mysterious that the partition functions at every genus should be positive [13]. Finally, the $\mathbb{Z}_2$-odd loops should be
calculated and compared with the NS sector of the polymer problem.

The third candidate is a candidate for the topological phase discussed in section four. The similarity of the structure of the topological phase to the KdV-type systems strongly suggests that the theory has a topological interpretation. Ghost number counting rules out the obvious possibility that it is $Osp(2|1)$ topological gravity. Another idea follows from recalling the identification of $t_2$ with the “Yang-Mills $\theta$ angle” described below (2.16). In the continuum this operator becomes simply $\int \text{tr} F$ which could serve as an action for a topological Yang-Mills matter theory, which could then be coupled to topological gravity \cite{15,16,17}. However, there are some troubles with any candidate topological matter theory. If we have a theory of topological gravity coupled to topological matter, then, as shown by Dijkgraaf and Witten there must be a “puncture operator” $P$ which satisfies a “puncture equation” expressing an insertion of $P$ in terms of correlation functions of lower degree. The natural candidate for the puncture equation would be the $L_{-1}$ constraint. However, this interpretation would make all operators $\sigma_n$ descendants of a single primary, $\sigma_1$, which seems incompatible with the uniqueness of the known TFT with a single primary. In addition, ghost number conservation required us to introduce a new operator $Q$ with no descendants, which does not fit at all into the topological gravity framework.

Although it is possible that a different interpretation of the model exists, which is more compatible with the original topological gravity framework, we did not find such a description and suspect that it does not exist. If this is true, clearly it would be very interesting to find the correct world-sheet topological description.

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