STEIN COMPLEMENTS IN COMPACT KÄHLER MANIFOLDS

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Abstract. Given a projective or compact Kähler manifold $X$ and a (smooth) hypersurface $Y$, we study conditions under which $X \setminus Y$ could be Stein. We apply this in particular to the case when $X$ is the projectivization of the so-called canonical extension of the tangent bundle $T_M$ of a projective manifold $M$ with $Y$ being the projectivization of $T_M$ itself.

1. Introduction

1.A. Motivation. Let $M$ be a compact Kähler manifold, and let $\alpha \in H^1(M, \Omega_M)$ be a Kähler class. This defines an extension of vector bundles,

$$0 \to O_M \to V^\alpha \to T_M \to 0$$

and therefore an embedding $\mathbb{P}(T_M) \subset \mathbb{P}(V^\alpha)$. We then consider the complement

$$Z_M := Z_M^\alpha := \mathbb{P}(V^\alpha) \setminus \mathbb{P}(T_M).$$

Following Greb and Wong [GW20] we call $Z_M$ a canonical extension of $M$. They showed that $Z_M$ does contain any compact subvarieties and it is Stein if $M$ is a torus or has a Kähler metric of non-negative holomorphic bisectional curvature. In view of their results one is tempted to make the following conjecture; already formulated as a question in [GW20],

1.1. Conjecture. Let $M$ be a compact Kähler manifold, and let $Z_M$ be a canonical extension defined by some Kähler class on $M$. Then the following holds:

- $Z_M$ is Stein if and only if $T_M$ is nef.
- $Z_M$ is affine if and only if $T_M$ is nef and big.

Let us first focus on the affine version of the conjecture. We will use the basepoint-free theorem to show one implication:

1.2. Theorem. Let $M$ be a compact Kähler manifold such that $T_M$ is nef and big. Then $Z_M$ is affine for any Kähler class $M$.

For the other implication observe first that if $Y := \mathbb{P}(T_M) \subset \mathbb{P}(V^\alpha) =: X$ is the embedding defined above, the normal bundle $N_{Y/X}$ identifies to the tautological bundle $O_{\mathbb{P}(T_M)}(1)$. Let us recall that Greb and Wong [GW20] Prop.4.2 used Goodman’s theorem [Goo69] to show that if $X$ is projective and $X \setminus Y$ is affine, the

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normal bundle $N_{Y/X}$ is big. Thus the other implication would essentially reduce to a conjecture of Goodman:

**1.3. Conjecture.** [Har70, II.V.2] Let $X$ be a compact algebraic manifold, and let $Y \subset X$ be a prime divisor. If $X \setminus Y$ is affine, then $N_{Y/X}$ is nef.

The case where $X$ is a surface is of course well-known, see Proposition 3.4: if $X \setminus Y$ is affine (or more generally Stein), then $N_{Y/X}$ is nef, i.e., $Y^2 \geq 0$. We construct however a threefold that is a counterexample to Conjecture 1.3 (but not to Conjecture 1.1):

**1.4. Proposition.** There exists a smooth projective threefold $X$ containing a smooth connected hypersurface $Y$ such that $X \setminus Y$ is affine and $N_{Y/X}$ is not nef.

In this paper we prove a weak version of Goodman’s conjecture. We then show that in the geometric setting of Conjecture 1.1 one can obtain much stronger results.

**1.B. General Stein complements.** Let $X$ be a compact complex Kähler manifold, and let $Y \subset X$ be an irreducible hypersurface. Generalising Conjecture 1.1 to this setup we consider the following problem:

**1.5. Problem.** Let $X$ be a compact Kähler manifold, and let $Y \subset X$ be an irreducible (smooth) hypersurface. Give necessary and sufficient criteria such $X \setminus Y$ is affine or Stein.

Our first main result is a necessary condition for smooth hypersurfaces:

**1.6. Theorem.** Let $X$ be a compact Kähler manifold of dimension $n$. Let $Y \subset X$ be a smooth hypersurface such that $X \setminus Y$ is Stein. Then the normal bundle $N_{Y/X}$ is pseudoeffective.

The restriction to smooth hypersurfaces is due to the fact that our main technical ingredient, a result of Kosarew and the second named author relating analytic and algebraic functions on the complement, see Proposition 2.10, is only known in the smooth case. We then combine this result with Yang’s recent partial converse Andreotti-Grauert theorem [Yan19, Thm.1.5]. Proposition 1.4 shows that Theorem 1.6 is essentially optimal: in our example the normal bundle $N_{Y/X}$ is pseudoeffective, but not nef in codimension one. In view of these results it is clear that the properties of complements are of a birational nature, therefore we study in Subsection 3.D how Mori theory can be used to obtain a better understanding of the situation.

**1.C. Canonical extensions.** As a direct consequence of Theorem 1.6 one obtains:

**1.7. Corollary.** Let $M$ be a compact Kähler manifold. Assume that $Z_M$ is Stein for some Kähler class. Then the tangent bundle $T_M$ is pseudoeffective.

While Proposition 1.4 shows that Theorem 1.6 can’t be improved in general, it seems possible that Conjecture 1.1 has a positive answer: in fact, as already mentioned, Greb and Wong observed that $Z_M$ does not contain subvarieties of positive dimension [GW20, Prop. 2.7]. Thus, $Z_M$ is Stein if and only $Z_M$ is holomorphically convex. Using classical results about the holomorphic convexity of certain complements, we show in Theorem 4.5 that the Stein property leads to strong restrictions on divisorial Mori contractions. In low dimension this already gives a rather complete picture:
1.8. Corollary. Let $M$ be a compact Kähler manifold of dimension at most three. Assume that $Z_M$ is Stein for some Kähler class. Then $M$ does not admit a birational Mori contraction. Thus, either $K_M$ is nef or $M$ is a Mori fibre space.

The scope of our technique is not necessarily limited to divisorial Mori contractions:

1.9. Proposition. Let $M$ be a smooth compact Kähler fourfold. Assume that $Z_M$ is Stein for some Kähler class. Then $M$ does not admit a small Mori contraction.

In a different direction we can use stability arguments to exploit the property that the tangent bundle is pseudoeffective:

1.10. Proposition. Let $M$ be a projective manifold of dimension $n$ such that $T_M$ is pseudoeffective. If $M$ is not uniruled, there exists a decomposition $T_M \cong \mathcal{F} \oplus \mathcal{G}$, where $\mathcal{F} \neq 0$ and $\mathcal{G}$ are integrable subbundles such that $c_1(\mathcal{F}) = 0$.

In particular, by [Pet11 Cor.11] the manifold $M$ is not of general type and $\Omega_M$ is not generically ample. In view of this result and [HP19 Thm.1.6] we expect the following to be true:

1.11. Conjecture. Let $M$ be a compact Kähler manifold that is not uniruled. Assume that $Z_M$ is Stein for some Kähler class. Then $M$ is an étale quotient of a torus.

This statement would of course be a consequence of the first part of Conjecture 1.1. Conjecture 1.11 is obviously related to a conjecture of Pereira and Touzet on the algebraic integrability of foliations with trivial first Chern class [PT13 6.5]. We will use some recent results from foliation theory to confirm it for projective manifolds of low dimension:

1.12. Theorem. Let $M$ be a projective manifold of dimension at most three that is not uniruled. Assume that $Z_M$ is Stein for some Kähler class. Then $M$ is an étale quotient of an abelian variety.

For uniruled manifolds the situation is more complicated, see the discussion in Section 5. For surfaces we can summarise our results as follows:

1.13. Theorem. Let $M$ be a smooth projective surface. Assume that $Z_M$ is Stein for some Kähler class. Then one of the following holds:

- $M$ is an étale quotient of a torus;
- $M$ is rational homogeneous, i.e., $M = \mathbb{P}_2$ or $M = \mathbb{P}_1 \times \mathbb{P}_1$.
- $M$ is a ruled surface over a curve of genus $B$ at least one. In case $g(B) \geq 2$, $M$ is given by a semi-stable vector bundle.

The canonical extension $Z_M$ is affine if and only if $M = \mathbb{P}_2$ or $M = \mathbb{P}_1 \times \mathbb{P}_1$.

We expect actually more to be true. First, we should always have $g(B) \leq 1$. If $M$ is a ruled surface over an elliptic curve, given by a vector bundle $\mathcal{E}$, then $\mathcal{E}$ should be semi-stable; the converse does hold. Most of the arguments in the proof of Theorem 1.13 are valid for Kähler surfaces, but there are additional difficulties arising from non-projective elliptic bundles over curves of higher genus (cf. Remark 5.2).
1.D. Future directions. While we have seen above that the existence of a canonical extension $Z_M$ leads to numerous restrictions on the geometry of $M$, it is quite difficult to construct non-trivial examples of projective manifolds such that $Z_M$ is Stein or even affine. Theorem 1.2 should have the following extension which is affirmed positively in dimension two by Theorem 1.13.

1.14. Conjecture. Let $M$ be a compact Kähler manifold with nef tangent bundle $T_M$. Then $Z_M$ is affine if and only if $M$ is Fano.

Theorem 1.6 depends on Proposition 2.10 whose proof is highly transcendental. In Section 6 we present a weaker version with a simple purely algebraic proof. This version is almost sufficient to prove Theorem 1.13 with some K3-surfaces $M$ left over, namely when $\rho(M) \geq 2$ or when $M$ does not contain a nodal rational curve. It also leads to the following

1.15. Conjecture. Let $M$ be a simply connected projective manifold of dimension $n$ with $K_M \cong \mathcal{O}_M$. Then the tautological class $\zeta_M$ on $\mathbb{P}(T_M)$ is not generically nef, i.e.,
$$\zeta_M \cdot H_1 \cdot \ldots \cdot H_{2n-2} < 0$$
for some ample divisors $H_j$ on $\mathbb{P}(T_M)$.

It is known that $\zeta_M$ is not pseudoeffective, [HP19, Thm.1.6], but the difference between pseudoeffectivity and generic nefness is quite significant.

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2. Preliminaries

We work over the complex numbers, for general definitions we refer to [Har77, KKS83]. Complex spaces and varieties will always be supposed to be irreducible and reduced.

We use the terminology of [Deb01, KM98] for birational geometry and notions from the minimal model program. We follow [Laz04] for algebraic notions of positivity, in particular the terminology of $\mathbb{Q}$-twisted bundles as explained in [Laz04, Ch.6.2]. For positivity notions in the analytic setting, we refer to [Dem12].

2.1. Notation. Let $X$ be a normal projective variety, and by $N_1(X)$ the $\mathbb{R}$-vector space of curves modulo numerical equivalence.

We denote by $\overline{\text{Mov}}(X) \subset N_1(X)$ the closed cone spanned by (classes of) movable curves. By [BDPP13], this is the dual cone to the pseudoeffective cone.

We denote by $\overline{\text{CI}}(X) \subset N_1(X)$ the closed cone spanned by (classes of) general complete intersection curves $C = H_1 \cap \ldots \cap H_n$ of very ample divisors $H_j$. If the $H_j$ have sufficiently large degree and if $C$ is general, then $C$ is called a MR-curve (MR for Mehta-Ramanathan).

2.2. Notation. Let $M$ be a complex space, and let $V \to M$ be a vector bundle over $M$. We denote by $\pi : \mathbb{P}(V) \to M$ the projectivisation of $V$ and by $\zeta_{\mathbb{P}(V)} := c_1(\mathcal{O}_{\mathbb{P}(V)}(1))$ the tautological class on $\mathbb{P}(V)$. 

When the situation is clear, we will use the simplified notation \( \zeta := \zeta_{\mathbb{P}(V)} \). Since the terminology varies in the literature, let us recall:

**2.3. Definition.** Let \( M \) be a normal compact complex space, and let \( V \to M \) be a vector bundle. We say that \( V \) is pseudoeffective if the tautological class \( \zeta_{\mathbb{P}(V)} \) is pseudoeffective. The vector bundle is big if \( \zeta_{\mathbb{P}(V)} \) is big.

**2.4. Definition.**

- Let \( X \) be a normal projective variety. An \( \mathbb{R} \)-divisor class on \( X \) is generically nef, if it is non-negative on \( \overline{\mathcal{T}}(X) \subset N_1(X) \).
- Let \( M \) be a normal projective variety, and let \( V \to M \) be a vector bundle. The vector bundle \( V \) is pseudoeffective if the tautological class \( \zeta_{\mathbb{P}(V)} \) is pseudoeffective.
- Let \( M \) be a normal projective variety, and let \( V \to M \) be a vector bundle. The vector bundle \( V \) is big if \( \zeta_{\mathbb{P}(V)} \) is big.

**Remark.** If \( X \) is a smooth projective surface, the cone \( \overline{\mathcal{T}}(X) \) coincides with the nef cone, so a divisor class on \( X \) is generically nef if and only if it is pseudoeffective.

Let \( V \) be a generically nef vector bundle on a higher-dimensional manifold. Then \( \zeta_{\mathbb{P}(V)} \) might not be generically nef, even if \( V \) is semistable for any polarisation (cf. Proposition 6.3 for an example).

**2.5. Proposition.** Let \( C \) be a smooth compact curve, and let \( V \) be a semistable vector bundle on \( C \). Assume that \( \zeta_{\mathbb{P}(V)} \) is generically nef. Then \( V \) is nef.

**Proof.** Let \( F \) be a fiber of \( \mathbb{P}(V) \to C \) and \( \ell \) a line in \( F \). Let \( r \) be the rank of \( V \) and set \( \mu = c_1(V) \). By [Ful11, 1.2], the cone \( \overline{\mathcal{N}(\mathbb{P}(V))} \) is spanned by \( (\zeta - \mu F)^{-1} \) and \( \ell \). Since \( V \) is semistable, the class \( \zeta - \mu F \) is nef, hence

\[
\overline{\mathcal{N}(\mathbb{P}(V))} = \overline{\mathcal{T}(\mathbb{P}(V))}.
\]

Since \( \zeta \) is generically nef, \( \zeta \) is therefore nef, hence \( V \) is nef. \( \square \)

**Remark.** Proposition 2.5 is wrong if the bundle is not stable; in fact \( \zeta_{\mathbb{P}(V)} \) might not be pseudoeffective. For example, consider

\[
V = \mathcal{O}_{\mathbb{P}_1}(-1) \oplus \mathcal{O}_{\mathbb{P}_1}(-2) \oplus \mathcal{O}_{\mathbb{P}_1}(-3)
\]
on \( C = \mathbb{P}_1 \). Let \( \zeta = \zeta_{\mathbb{P}(V)} \) and \( F \) a fiber of \( \mathbb{P}(V) \to C \). Then the nef cone of \( \mathbb{P}(V) \) is generated by \( F \) and \( \zeta + 3F \). Since

\[
\zeta 
cdot F^2 = 0 = \zeta 
cdot (\zeta + 3F)^2, \quad \zeta 
cdot F \cdot (\zeta + 3F) = 1,
\]

the tautological class \( \zeta \) is generically nef.

In the example above \( V^* \) is ample although \( \zeta_{\mathbb{P}(V)} \) is generically nef. However we have the following:

**2.6. Lemma.** Let \( V \) be a vector bundle over a projective variety \( M \) such that \( \zeta_{\mathbb{P}(V)} \) is pseudoeffective. Then \( V^* \) is not ample.

**Proof.** If \( V^* \) is ample there exists an ample \( \mathbb{Q} \)-divisor \( A \) such that the twist \( V^* \otimes A^* \) is still ample. In particular, by [Laz04, Ex.6.1.4] one has \( H^0(M, S^d(V \otimes A)) = 0 \) for all sufficiently divisible \( d \in \mathbb{N} \). Yet this implies that \( \kappa(\mathbb{P}(V), \zeta_{\mathbb{P}(V)} + \pi^*A) = -\infty \), hence \( \zeta_{\mathbb{P}(V)} \) is not pseudoeffective. \( \square \)
2.7. Definition. Let $X$ be a projective manifold, and let $Y \subset X$ be an analytic set. Then we define local cohomology groups $H^n_Y(X, F)$ and $H^n_Y(X, F)$ to obtain long exact sequences
\[ \ldots \to H^{n-1}(X \setminus Y, F)_{\text{alg}} \to H^n_Y(X, F) \to H^n(X \setminus Y, F)_{\text{alg}} \to \ldots \]
for algebraic cohomology and
\[ \ldots \to H^{n-1}(X \setminus Y, F)_{\text{an}} \to H^n_Y(X, F) \to H^n(X \setminus Y, F)_{\text{an}} \to \ldots \]
in the analytic case, see [Har70], [KP90].

2.8. Remark. We may extend the previous definition of $H^n_Y(X, F)$ to the case of compact complex manifolds as in [KP90 Sect.1]. In this case $H^n(X \setminus Y, F)_{\text{alg}}$ has to be substituted by a cohomology group of moderate growth, $H^n(X \setminus Y, F)_{\text{mod}}$. In particular, $H^n(X \setminus Y, \mathcal{O}_{X\setminus Y})_{\text{mod}}$ is the space of those holomorphic functions on $X \setminus Y$ extending to meromorphic functions on $X$.

2.9. Definition. Let $X$ be a compact complex manifold of dimension $n$, and let $\mathcal{L}$ be a holomorphic line bundle on $X$.

a) The bundle $\mathcal{L}$ is $q$-positive if there exists a metric on $\mathcal{L}$ whose curvature has at least $n-q$ positive eigenvalues.

b) Suppose that $X$ is projective. Then $\mathcal{L}$ is $q$-ample if the following holds. Given any coherent sheaf $F$ on $X$, there is a number $k_0$, depending on $F$, such that
\[ H^j(X, F \otimes \mathcal{L}^\otimes k) = 0 \]
for $k \geq k_0$ and $j \geq q+1$.

It was conjectured ("the converse Andreotti-Grauert conjecture") that on a smooth projective $n$-fold, $(n-1)$-ample line bundles are $(n-1)$-positive, [DPS96]. This is now established by Yang [Yan19]. In dimension 2 this has previously been proved by Matsumura [Mat13 Thm.1.3]. Notice that there are counterexamples to the converse Andreotti-Grauert theorem in case $\frac{n}{2} - 1 < q < n - 1$, see [Ott12 Thm.10.3]. Of course, in case $q = 0$ the converse Andreotti-Grauert theorem is a classical result of Kodaira.

2.10. Proposition. Let $X$ be a compact complex manifold of dimension $n$, and let $Y \subset X$ be a smooth hypersurface. Assume that the conormal bundle $N^*_Y/X$ is $(n-2)$-positive. Let $F$ be a locally free sheaf on $X$. Then the canonical morphism
\[ H^1_Y(X, F) \to H^1_Y(X, F) \]
is bijective. Equivalently, the canonical morphism
\[ H^0(X \setminus Y, F)_{\text{mod}} \to H^0(X \setminus Y, F)_{\text{an}} \]
is bijective. In case $X$ is algebraic, equivalently
\[ H^0(X \setminus Y, F)_{\text{alg}} \to H^0(X \setminus Y, F)_{\text{an}} \]
is bijective.

Proof. This is a special case of [KP90 Thm. 2.5] (note that $N^*_Y/X$ is $(n-2)$-positive if and only if $N_Y/X$ is $(n-1)$-convex in the notation of [Ko82], [Sc73]).

Remark. Proposition 2.10 remains true for singular hypersurfaces $Y$, if one uses the notion of $(n-1)$-convexity of the normal bundle $N_Y/X$, see [Ko82 2.8].
3. General Stein complements

3.A. General results. We first recall

3.1. Definition. [GR79, IV, §1, Defn.1] Let \( X \) be a complex space. We say that \( X \) is Stein if for every coherent analytic sheaf \( \mathcal{F} \) on \( X \) one has

\[
H^q(X, \mathcal{F}) = 0 \quad \forall \ q \geq 1
\]

It is well-known [GR79, V, §2, Thm.3] that a complex space \( X \) is Stein if and only if \( X \) is holomorphically convex and does not contain any compact analytic subspaces of positive dimension. We will use the holomorphic convexity of a Stein space via the following characterisation

3.2. Theorem. [GR79, IV, §2, Thm.4, Thm.12] A complex space \( X \) is holomorphically convex if and only if for every discrete infinite subset \( D \subset X \) there exists a holomorphic function \( f \in \mathcal{O}(X) \) such that \( f|_D \) is unbounded.

3.3. Lemma. [GR79, p.156] Let \( X \) be a compact complex manifold, and let \( Y \subset X \) be a compact hypersurface such that \( X \setminus Y \) is Stein. Then the restriction maps

\[
H^q(X, Z) \to H^q(Y, Z)
\]

are isomorphisms for \( q < n - 1 \). In particular, if \( n \geq 4 \), and \( Y \) is normal, then the restriction map

\[
N^1(X) \to N^1(Y)
\]

is an isomorphism.

In dimension two, Problem 1.5 has clearly a positive solution:

3.4. Proposition. Let \( X \) be a smooth complex surface, and let \( Y \subset X \) be an irreducible compact curve such that \( X \setminus Y \) is holomorphically convex. Then the normal bundle \( N_{Y/X} \) is nef.

Proof. If \( Y^2 < 0 \), then by Grauert’s criterion \( Y \) can be blown down by a bimeromorphic morphism \( X \to X' \) onto a normal surface \( X' \). Hence all holomorphic functions on \( X \setminus Y \) would extend to \( X' \) by Riemann’s extension theorem on \( X' \). Thus \( X \setminus Y \) is not holomorphically convex. \( \square \)

3.5. Remark. [KP90, Thm.5.3] Let \( X \) be a relatively minimal projective surface, and let \( Y \subset X \) be an elliptic curve such that \( X \setminus Y \) is Stein and \( N_{Y/X} \) is not ample. Then \( X \simeq \mathbb{P}(V) \), where \( V \) is the non-split extension

\[
0 \to \mathcal{O}_C \to V \to \mathcal{O}_C \to 0
\]

over an elliptic curve and \( Y = \mathbb{P}(\mathcal{O}_C) \).

Proposition 3.4 yields the following

3.6. Lemma. Let \( X \) be a projective manifold of dimension \( n \), and let \( Y \subset X \) be an irreducible hypersurface such that \( X \setminus Y \) is holomorphically convex. Then

\[
Y^2 \cdot H_1 \cdot \ldots \cdot H_{n-2} \geq 0
\]

for all nef divisors \( H_j \) on \( X \).
Proof. We may assume without loss of generality that the divisors $H_j$ are very ample and general. Then the surface $S := H_1 \cap \ldots \cap H_{n-2}$ is smooth, and $C = S \cap Y$ is an irreducible curve. By Proposition 3.4, one has

$$(C^2)_S = Y^2 \cdot H_1 \cdot \ldots \cdot H_{n-2} \geq 0.$$ 

□

3.7. Corollary. In the situation of Problem 1.5, if the restriction map of the nef cones

$$\text{Nef}(X) \to \text{Nef}(Y)$$

surjective, then $N_{Y/X}$ is generically nef.

We can now prove our first main result:

Proof of Theorem 1.6. We argue by contradiction and assume that $N_{Y/X}$ is not pseudoeffective. By [Yan19, Thm.1.5] this is equivalent to assuming that $N_{Y/X}^*$ is $(\dim Y - 1)$-positive, hence $(n - 2)$-positive.

Since $X \setminus Y$ is Stein, so we can find holomorphic functions $f_1, \ldots, f_N$ on $X \setminus Y$ yielding a closed embedding $X \setminus Y \hookrightarrow \mathbb{C}^N$. By Proposition 2.10 the functions $f_j$ have moderate growth, so they extend to $X$ (cf. Remark 2.8). Hence $X$ is at least Moishezon. Since $X$ is assumed to be Kähler, it is projective by Moishezon’s theorem. Thus by GAGA (or by the second part of Proposition 2.10) the functions $f_j$ are algebraic and the embedding is actually an algebraic embedding $X \setminus Y \hookrightarrow \mathbb{A}^N$. Hence $X \setminus Y$ is affine and $N_{Y/X}$ is big by [GW20, Prop.4.2]. Thus we have reached a contradiction. □

3.8. Setup. Let $B$ be a projective manifold, and let $F$ be a vector bundle on $B$. Fix a cohomology class $0 \neq \zeta \in H^1(B, F^*)$. Then $\zeta$ defines a non-split extension

$$0 \to \mathcal{O}_B \to V \to \mathcal{F} \to 0.$$ 

We set

$$X = \mathbb{P}(V); \ Y = \mathbb{P}(F).$$

and denote by $\pi : \mathbb{P}(V) \to B$ the natural map. We set further $\zeta_X = c_1(\mathcal{O}_{\mathbb{P}(V)}(1))$ and $\zeta_Y = c_1(\mathcal{O}_{\mathbb{P}(F)}(1))$

3.9. Lemma. In the situation of Setup 3.8 assume furthermore that $\dim B = 1$. Then $X \setminus Y$ does not contain compact subvarieties of positive dimension.

Proof. Since $X$ is projective it suffices to show that $X \setminus Y$ does not contain compact irreducible curves. Arguing by contradiction, let $f : C \to X$ be a morphism from a smooth compact curve such that $f(C) \subset X \setminus Y$. Then the composition $\pi \circ f$ is a finite map. Let $\tau \in H^0(B, F^* \otimes \omega_B)$ be the Serre dual of $\zeta$. Then

$$0 \neq f^*(\tau) \in H^0(C, f^* F^* \otimes \omega_B) \subset H^0(C, f^* F^* \otimes \omega_C).$$

Since the Serre dual of $f^*(\tau)$ is $f^*(\zeta)$, we obtain

$$0 \neq f^*(\zeta) \in H^1(C, f^*(\mathcal{F})).$$

Thus, up to making a base change, we may assume that $C$ is a section of $X \to B$ that is disjoint from $Y = \mathbb{P}(F)$. Hence the exact sequence defined by $\zeta$ splits, a contradiction. □
We can now given the example that proves Proposition [1.4]

3.10. Example. In the situation of Setup [3.8] let $B = \mathbb{P}_1$ and $\mathcal{F} = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Since the extension class $\zeta$ is not zero, one has

$$V \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

Let $C_0 \subset X$ be the section corresponding to the quotient $\mathcal{F} \to \mathcal{O}_{\mathbb{P}^1}(-1)$. Then the fibrewise projection from $C_0 \subset X$ defines a rational map

$$\psi : X \dashrightarrow \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}).$$

More precisely, let $\pi : \hat{X} \to X$ be the blow-up of $X$ along $C_0$ and denote by $E$ the exceptional divisor. Then $\hat{X}$ has a structure of $\mathbb{P}^1$-bundle $\varphi : \hat{X} \to \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2})$ such that the exceptional divisor $E$ is a section. Moreover if $\zeta_Q$ is the tautological class on $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2})$, one has

$$\pi^*(\zeta_X) - E = \varphi^*\zeta_Q.$$

Now observe that the strict transform $\hat{Y} \simeq Y$ is an element of the linear system $|\pi^*(\zeta_X)|$, since $Y \in |\zeta_X|$ contains the curve $C_0$. The isomorphism $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}) \simeq \mathbb{P}^1 \times \mathbb{P}^1$ identifies the class $\zeta_Q$ to $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)$, so we see that $\hat{Y} = \varphi^*\ell$ with $\ell$ a smooth conic.

One has $\hat{X} \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-2,-1))$, and $E$ corresponds to the quotient $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-2,-1))$. Thus $\hat{X} \setminus E$ is the total space of the line bundle $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-2,-1)$ and

$$\hat{X} \setminus (\hat{Y} \cup E) \to (\mathbb{P}^1 \times \mathbb{P}^1) \setminus \ell$$

is a line bundle over the affine surface $Q_0 := (\mathbb{P}^1 \times \mathbb{P}^1) \setminus \ell$. Hence

$$X \setminus Y \simeq \hat{X} \setminus (\hat{Y} \cup E)$$

is affine: if $s_1, \ldots, s_n$ are algebraic sections that generate the line bundle on $Q_0$, they define an algebraic embedding of the total space into $Q_0 \times \mathbb{C}^n$. But the normal bundle $N_{Y/X} = \zeta_Y$ is not nef, since $\zeta_Y \cdot C_0 = -1$.

3.11. Example. Let $B$ be an elliptic curve and $\mathcal{L}$ a line bundle of negative degree. Let

$$0 \to \mathcal{O}_B \to \mathcal{F} \to \mathcal{O}_B \to 0$$

be the non-split extension. Note that $\mathbb{P}(\mathcal{F}) \setminus \mathbb{P}(\mathcal{O}_B) \simeq \mathbb{C}^* \times \mathbb{C}^*$ is Stein (but not affine); this is Serre’s famous example [Har70, §6.3], [Nee88, §7]. Note next that $\mathbb{P}(\mathcal{O}_B \oplus \mathcal{L}) \setminus \mathbb{P}(\mathcal{L})$ is not Stein, since $\mathbb{P}(\mathcal{L})$ has negative normal bundle. However

$$\mathbb{P}(\mathcal{F} \oplus \mathcal{L}) \setminus \mathbb{P}(\mathcal{O}_B \oplus \mathcal{L})$$

is Stein. In fact, we have a projection

$$\mathbb{P}(\mathcal{F} \oplus \mathcal{L}) \setminus \mathbb{P}(\mathcal{L}) \to \mathbb{P}(\mathcal{F})$$

which restricts to a morphism

$$\Phi : \mathbb{P}(\mathcal{F} \oplus \mathcal{L}) \setminus \mathbb{P}(\mathcal{O}_B \oplus \mathcal{L}) \to \mathbb{P}(\mathcal{F}) \setminus \mathbb{P}(\mathcal{O}_B) \simeq \mathbb{C}^* \times \mathbb{C}^*.$$

Since $\Phi$ is an affine $\mathbb{C}$-bundle over $\mathbb{C}^* \times \mathbb{C}^*$, the complement $\mathbb{P}(\mathcal{F} \oplus \mathcal{L}) \setminus \mathbb{P}(\mathcal{O}_B \oplus \mathcal{L})$ is Stein by a result of Mok [Mok82].
3.C. **Criteria for affineness.** We next give criteria when $X \setminus Y$ is affine. This will be used later in the case of canonical extensions.

**3.12. Proposition.** Let $X$ be a normal compact analytic space, and let $Y \subset X$ be an irreducible hypersurface such that some multiple $mY$ is Cartier and the line bundle $\mathcal{O}_X(mY)$ is semiample. Assume that $X \setminus Y$ does not contain any positive-dimensional compact subvarieties.

Then $X \setminus Y$ is affine. Moreover the divisor $Y$ is big and some multiple defines a bimeromorphic map $\varphi : X \to Z$, whose exceptional locus is strictly contained in $Y$ and there exists a $\mathbb{Q}$-Cartier Weil divisor $Y' \subset Z$ such that $Y = \varphi^{-1}(Y')$.

A similar statement is shown in the algebraic setting in [Har70, Ch.II,Prop.2.2].

**Proof.** The sheaf $\mathcal{O}_X(mY)$ being generated by global sections, denote by $\varphi : X \to Z$ the associated morphism with connected fibers, so that $\mathcal{O}_X(mY) = \varphi^*(\mathcal{A})$ for some very ample line bundle $\mathcal{A}$ on $Z$.

In particular the divisor $mY = \varphi^*(D)$ for some effective Cartier divisor $D \in |\mathcal{A}|$. Let now $z \in Z$ a point such that the fibre $\varphi^{-1}(z)$ has positive dimension. Then $z \in D$ since otherwise $X \setminus Y$ contains a compact subvariety of positive dimension. Thus $\varphi$ is bimeromorphic, in particular it is Moishezon by Chow’s lemma [Hir75 Cor.2] and all the fibres of positive dimension are covered by irreducible curves. Thus if the inclusion $\varphi^{-1}(z) \cap Y \subset Y$ is strict, we can find irreducible curves $C \subset \varphi^{-1}(z)$ that meet $Y$ and such that $C \not\subset Y$. Yet this implies $Y \cdot C > 0$, contradicting the fact that $C$ is contracted by $\varphi$. Thus we see that the exceptional locus of $\varphi$ is contained in $Y$.

Since $mY$ is the pull-back of an ample divisor, it is not contracted by $\varphi$. Thus the exceptional locus is strictly contained in $Y$. Since $mY = \varphi^*D$ we obtain that $D = mY'$ where $Y' = \varphi(Y)$. Since $D$ is Cartier, the divisor $Y'$ is $\mathbb{Q}$-Cartier.

**3.13. Remark.** If $H^1(X, \mathcal{O}_X) = 0$ and the normal bundle $N_{Y/X}$ is generated by global sections, then $Y$ is semiample and Proposition 3.12 applies.

Without assuming $H^1(X, \mathcal{O}_X) = 0$, this is false: in fact, let $A$ be an abelian variety and let

$$0 \to \mathcal{O}_A \to V \to T_A \to 0,$$

be the canonical extension of $T_A$ given by some Kähler class (see Section 4). Set $X = \mathbb{P}(V)$ and $Y = \mathbb{P}(T_A)$. Then by [GW20 Prop.2.13] the complement $X \setminus Y$ is isomorphic to $(\mathbb{C}^*)^{2n}$, hence Stein. Yet the normal bundle $N_{Y/X} = \zeta_{\mathbb{P}(T_A)}$ is globally generated, but not big.

In special situations, the semiampleness assumption in Proposition 3.12 can be checked. We consider right away a singular setting.

**3.14. Proposition.** Let $X$ be a $\mathbb{Q}$-factorial projective klt variety, $Y \subset X$ an irreducible hypersurface. Assume that $Y$ is nef and $aY - K_X$ is nef and big for some $a > 0$. If $X \setminus Y$ does not contain positive-dimensional compact subvarieties, then $X \setminus Y$ is affine.

**Proof.** By the base point free theorem, the divisor $Y$ is semiample. Now Proposition 3.12 applies.

Note that the assumption that $aY - K_X$ is nef and big for some $a > 0$ holds if
• $Y$ is nef and big and $-K_X$ is nef; or
• $Y$ is nef $-K_X$ is nef and big (i.e. $X$ is weakly Fano).

3.D. Use of Mori theory. In the situation of Problem 1.5 assume that $X$ is projective and the divisor $Y$ is not nef (equivalently, the normal bundle $N_{Y/X}$ is not nef). We will use Mori theory to analyze this case.

3.15. Proposition. Let $X$ be a $\mathbb{Q}$-factorial projective klt variety, and let $Y \subset X$ be an irreducible hypersurface such that $X \setminus Y$ is Stein. Suppose that there is an extremal ray $R \in \overline{NE}(X)$ with $Y \cdot R < 0$ and $K_X \cdot R \leq 0$ or with $Y \cdot R = 0$ and $K_X \cdot R < 0$. Then the contraction

$$\varphi : X \to X'$$

defined by the $(K_X + \epsilon Y)$-negative ray $R$ is small with exceptional locus properly contained in $Y$. Furthermore there exists a flip

$$X \dashrightarrow X^+$$
such that $X \setminus Y \simeq X^+ \setminus Y^+$, where $Y^+$ is the strict transform of $Y$ in $X^+$.

Proof. For small positive $\epsilon$, the pair $(X, \epsilon Y)$ is klt and $(K_X + \epsilon Y) \cdot R < 0$. The existence of $\varphi$ then follows from the contraction theorem. Since $Y \cdot R \leq 0$ and since $X \setminus Y$ is Stein, every curve contracted by $\varphi$ is contained in $Y$. Thus the exceptional locus $E$ is contained in $Y$, and $X' \setminus \varphi(Y) \simeq X \setminus Y$ is Stein. By [GR79, V, Thm. 4] this implies that $\varphi(Y) \subset X'$ has codimension one, in particular $Y$ is not contracted by $\varphi$. Thus the contraction $\varphi$ is small.

By [BCHM10] the flip of $\varphi$ exists, and we denote by $\varphi^+ : X^+ \to X'$ the induced map, which is small. Since $X' \setminus \varphi(Y)$ is $\mathbb{Q}$-factorial, the image of the $\varphi^+$-exceptional locus is contained in $\varphi(Y)$. Hence if $Y^+ \subset X^+$ is the strict transform of $Y$, then the exceptional locus $E^+$ is contained in $Y^+$.

□

3.16. Corollary. Let $X$ be a $\mathbb{Q}$-factorial projective klt variety, and let $Y \subset X$ be an irreducible hypersurface such that $X \setminus Y$ is Stein. Suppose that one of the following conditions is satisfied:

a) $\dim X \leq 3$;
b) $\dim X = 4$ and $(X, \epsilon Y)$ is canonical for some small positive $\epsilon$;
c) $\dim X$ is arbitrary and any sequence of (log) flips on $X$ terminates.

Then there exists a finite sequence of (log) flips

$$X \dashrightarrow X$$

with strict transform $\tilde{Y} \subset \tilde{X}$ such that the following holds.

a) $\tilde{X} \setminus \tilde{Y} \simeq X \setminus Y$ is Stein;
b) for any extremal ray $\tilde{R} \in \overline{NE}(\tilde{X})$ with $\tilde{Y} \cdot \tilde{R} < 0$, we have $K_{\tilde{X}} \cdot \tilde{R} > 0$;
c) for any extremal ray $\tilde{R} \in \overline{NE}(\tilde{X})$ with $\tilde{Y} \cdot \tilde{R} = 0$, we have $K_{\tilde{X}} \cdot \tilde{R} \geq 0$.

Proof. The statement follows from the repeated application of Proposition 3.15 having in mind the termination of the procedure in dimension four due to our extra assumption, [Kaw92], [Fuj04], [Fuj05].

□

These statements can be made more explicit in a number of special cases:
3.17. **Theorem.** Let \( X \) be a Gorenstein Fano threefold or a \( \mathbb{Q} \)-Fano variety of any dimension without a small contraction. If \( Y \subset X \) is an irreducible hypersurface such that \( X \setminus Y \) is Stein, then \( Y \) is ample. In particular, \( X \setminus Y \) is affine.

**Proof.** By [Cut88] we know that Gorenstein Fano threefolds do not support small contractions of extremal rays. By Proposition 3.15 we obtain that \( Y \cdot R > 0 \) for all \( K_X \)-negative extremal rays. Since \( X \) is Fano, the cone theorem now implies that \( Y \) is ample. \( \square \)

3.18. **Proposition.** Let \( X \) be a projective \( \mathbb{Q} \)-factorial klt variety with \( K_X \equiv 0 \). Let \( Y \subset X \) be an irreducible hypersurface such that \( X \setminus Y \) is Stein. Suppose that one of the following conditions is satisfied:

- a) \( \dim X \leq 3 \);
- b) \( \dim X = 4 \) and \( (X, \epsilon Y) \) is canonical for some small positive \( \epsilon \);
- c) \( \dim X \) is arbitrary and \( Y \) is big.

Then there exists a birational rational map \( f : X \to \tilde{X} \) with the following properties.

- a) \( f \) is a sequence of flops; the exceptional locus being contained in \( Y \);
- b) \( \tilde{X} \) is a \( \mathbb{Q} \)-factorial klt variety with \( K_{\tilde{X}} \equiv 0 \);
- c) let \( \tilde{Y} \) denote the (possibly singular) strict transform of \( Y \) in \( \tilde{X} \), then \( \tilde{Y} \) is nef.

Moreover, if \( \dim X \leq 3 \) or \( Y \) is big, then \( \tilde{Y} \) is semiample and \( X \setminus Y \) is affine.

**Proof.** The existence of the birational model is given by Corollary 3.16 (note that if \( Y \) is big, we may run a \( (K_X + \epsilon Y) \)-MMP by [BCHM10]).

If \( \dim X = 3 \) it follows from log abundance [KMM94, KMM04] that \( \tilde{Y} \) is semiample. If \( \dim X \) is arbitrary and \( Y \) is big, semiampleness is guaranteed from the basepoint-free theorem. Hence \( X^+ \setminus Y^+ \simeq X \setminus Y \) is affine by Corollary 3.12. \( \square \)

In dimension three we can prove more, at least in a smooth setting.

3.19. **Theorem.** Let \( X \) be a smooth projective threefold, \( Y \subset X \) a smooth hypersurface such that \( X \setminus Y \) is Stein. Let \( z \in \overline{NE}(X) \) with \( K_X \cdot z < 0 \). Then \( Y \cdot z \geq 0 \).

**Proof.** Suppose to the contrary that \( Y \cdot z < 0 \). Then there exists an irreducible curve \( C \) such that \( K_X \cdot C < 0 \) and \( Y \cdot C < 0 \). Since \( C \subset Y \), the adjunction formula yields \( K_Y \cdot C \leq -2 \). Since \( Y \cdot C < 0 \) and \( N_Y/X \) is pseudoeffective by Theorem 1.6, the curve \( C \) is in the negative part of the Zariski decomposition of the divisor \( N_Y/X \). In particular we obtain \( \deg N_C/Y < 0 \). Hence the adjunction formula on \( Y \) yields \( \deg \omega_Y \leq K_Y \cdot C < -2 \). Yet it is well-known that \( \deg \omega_Y \geq -2 \) for any irreducible curve. \( \square \)

4. **The canonical extension**

We will now consider the special case of the canonical extension of \( \mathbb{P}(TM) \). In this case the geometric interpretation of the normal bundle allows to prove stronger results.
4.A. The smooth case.

4.1. Setup. Let $M$ be a compact Kähler manifold, and let $\alpha$ be a Kähler class on $M$. Then the class $\alpha \in H^1(M, \Omega_M) \simeq \text{Ext}^1(O_M, T_M)$ defines a non-split extension
\[
0 \to O_M \to V^\alpha \to T_M \to 0.
\]
Denote by $\pi : \mathbb{P}(V^\alpha) \to M$ and $\pi_M : \mathbb{P}(T_M) \to M$ the projectivisations, and by $\zeta_V$ (resp. $\zeta_M$) the tautological class on $\mathbb{P}(V^\alpha)$ (resp. $\mathbb{P}(T_M)$).

The exact sequence (1) determines an embedding
\[
\mathbb{P}(T_M) \subset \mathbb{P}(V^\alpha)
\]
such that $[\mathbb{P}(T_M)] = \zeta_V$ and we denote
\[
Z^2_M := \mathbb{P}(V^\alpha) \setminus \mathbb{P}(T_M).
\]
When $\alpha$ is fixed, we will simply write $V = V^\alpha$ and $Z_M = Z^2_M$.

Proof of Theorem 1.2. Since $T_M$ is nef and big, $\mathbb{P}(T_M)$ is a nef and big divisor on $\mathbb{P}(V)$. Moreover, since $\text{det} V = -K_M$, the anticanonical class $-K_{\mathbb{P}(V)}$ is nef and big. Hence Proposition 3.14 applies. □

Theorem 1.2 applies in particular to rational-homogeneous manifolds, reproving a part of [GW20, 3.4]. It is however conjectured, [CP91], $T_M$ is big and nef if and only $M$ is rational homogeneous.

In the general homogenous case we may state

4.2. Corollary. Let $M$ be a homogeneous compact Kähler manifold. Then $Z_M$ is Stein for any Kähler class $\alpha$ on $M$.

Proof. By the theorem of Borel-Remmert, $M \simeq N \times T$ with $N$ rational homogeneous, in particular Fano, and $T$ a torus. Hence $Z_M \simeq Z_N \times Z_A$ is Stein by [GW20, 2.10, 3.4]. □

4.3. Definition. Let $M$ be a normal complex space and $E$ a non-zero torsion free sheaf on $M$. We denote by $\mathbb{P}(E)$ the unique irreducible component of $\pi : \mathbb{P}(E) \to M$ that dominates $M$. 

4.B. The singular case and contractions of extremal rays. In order to understand the birational geometry of canonical extensions we have to generalise the setup to mildly singular spaces.

Let $M$ be a normal $\mathbb{Q}$-factorial complex space with klt singularities, and let $\alpha \in N^1(M)$ be a $(1,1)$-class with local potentials (cf. [HP16] for definitions). By [GS21] we can associate to $\alpha$ a cohomology class in $H^1(M, \Omega_M)$ which we also denote by $\alpha$. Since we have $H^1(M, \Omega_M) \simeq \text{Ext}^1(O_M, \Omega_M)$ the class $\alpha$ defines an extension
\[
0 \to \Omega_M \to W^\alpha_M \to O_M \to 0
\]
that is locally splittable. Thus dualising yields a locally splittable exact sequence (2)
\[
0 \to O_M \to V^\alpha_M \to T_M \to 0.
\]
The surjection $V^\alpha_M \to T_M$ induces an inclusion $\mathbb{P}(T_M) \subset \mathbb{P}(V^\alpha_M)$. If $M$ is singular, the complex spaces $\mathbb{P}(T_M)$ and $\mathbb{P}(V^\alpha_M)$ might be reducible, so we refine the construction:

4.3. Definition. Let $M$ be a normal complex space and $E$ a non-zero torsion free sheaf on $M$. We denote by $\mathbb{P}(E)$ the unique irreducible component of $\pi : \mathbb{P}(E) \to M$ that dominates $M$. 

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4.4. Definition. Let $M$ be a normal $\mathbb{Q}$-factorial complex space with klt singularities, and let $\alpha \in N^1(M)$ be a $(1,1)$-class. Set $Z_M^\alpha := \mathbb{P}'(V_M^\alpha) \setminus \mathbb{P}'(T_M)$. We call
\[ \pi : Z_M^\alpha \to M \]
the canonical extension associated to the $(1,1)$-class $\alpha$.

Note that contrary to the Setup 4.1, for technical reasons we do not assume that the class $\alpha$ is Kähler. The goal of this section is to show the following result:

4.5. Theorem. Let $M$ be a normal $\mathbb{Q}$-factorial compact Kähler space with klt singularities. Let $\varphi : M \to N$ be a bimeromorphic morphism contracting an irreducible divisor $E$ onto an analytic set $B$. Assume that $N$ is klt, the higher direct images $R^j \varphi_* \mathcal{O}_M$ vanish and that $\rho(M/N) = 1$. Assume furthermore one of the following

- $B$ is not contained in $N_{\text{sing}}$;
- $\dim M \geq 3$ and $B$ is an isolated hypersurface singularity of $N$;
- $B$ is an irreducible component of $N_{\text{sing}}$ and $N$ has quotient singularities in a general point of $B$.

Then for any Kähler class $\alpha$, the canonical extension $Z_\alpha$ is not Stein.

Remark. The technical assumption that $N$ is klt, that the higher direct images $R^j \varphi_* \mathcal{O}_M$ vanish and that $\rho(M/N) = 1$ is of course satisfied if $\varphi$ is the contraction of an extremal ray in $\overline{NE}(M)$ such that $-K_M$ is $\varphi$-nef.

4.6. Corollary. Let $M$ be a smooth compact Kähler surface. Assume that $M$ contains a smooth rational curve $C \simeq \mathbb{P}^1$ such that $C^2 < 0$. Then for any Kähler class $\alpha$, the canonical extension $Z_\alpha$ is not Stein.

Proof. By Grauert’s criterion the curve $C$ can be contracted onto a point by a bimeromorphic morphism $\varphi : M \to N$. Since $C \simeq \mathbb{P}^1$, one has $R^1 \varphi_* \mathcal{O}_M = 0$ and the complex surface $N$ is klt [Ko97, Ex.3.8]. Hence $N$ has at most a quotient singularity in $\varphi(C)$ [KM98, Prop.4.18] and we may apply Theorem 4.5. \hfill \Box

4.7. Setup. In the situation of Theorem 4.5, fix a Kähler class $\alpha$ on $M$. Then there exists a unique $\lambda \in \mathbb{R}$ such that $\alpha - \lambda E$ is $\varphi$-numerically trivial, so by [HP16, Lemma 3.3] there exists a unique class $\beta \in N^1(N)$ such that $\alpha - \lambda E = \varphi^* \beta$.

Denote by $\pi : Z_M^\alpha \to M$ the canonical extension associated to $\alpha$, and by $\psi : Z_N^\beta \to N$ the canonical extension associated to the class $\beta$.

4.8. Lemma. In the situation of Setup 4.7, assume that $Z_M^\alpha$ is Stein. Then $Z_N^\beta \setminus \psi^{-1}(B)$ is Stein.

Proof. Since $M \setminus E \simeq N \setminus B$ and the restriction of the class of $E$ to $M \setminus E$ is trivial, it follows from the construction of $\beta$ that
\[ Z_M^\alpha \setminus \pi^{-1}(E) \simeq Z_N^\beta \setminus \psi^{-1}(B). \]

Since $E$ is a prime divisor and $M$ is $\mathbb{Q}$-factorial, there exists a $m \in \mathbb{N}$ such that $mE$ is Cartier. In particular $\pi^* m E$ is Cartier, hence by [GR79, V, Thm.5] the complex space $Z_M^\alpha \setminus \pi^{-1}(E)$ is Stein. \hfill \Box

We start by proving a rather technical lemma which will be useful for the proof of Theorem 4.5.
4.9. Lemma. Let $N$ be a normal $\mathbb{Q}$-factorial Kähler space with klt singularities. Assume that there exists an irreducible component $B \subset N_{\text{sing}}$ such that $N$ has quotient singularities at a general point of $B$. Let

$$\beta: \begin{array}{c}
0 \to \mathcal{O}_N \to V \to Q \to 0 
\end{array}$$

be an extension of torsion-free coherent sheaves that are locally free in $N_{\text{non}}$.

We say that $(\ast)$ has the extension property near $B$ if the following holds: let $y \in B$ be a general point, and let $N' \subset N$ be an analytic neighbourhood of $y$ such that there exists a quasi-étale cover $p: \tilde{N} \to N'$ such that $\tilde{N}$ is smooth. Denote by $R \subset \tilde{N}$ the preimage of the singular locus $B \cap N'$. Then the exact sequence

$$0 \to \mathcal{O}_{\tilde{N} \setminus R} \to p^*(V \otimes \mathcal{O}_{N' \setminus B}) \to p^*(Q \otimes \mathcal{O}_{N' \setminus B}) \to 0$$

extends to an exact sequence of vector bundles

$$0 \to \mathcal{O}_{\tilde{N}} \to V_{\tilde{N}} \to Q_{\tilde{N}} \to 0.$$ 

Assume now that the exact sequence $(\ast)$ has the extension property near $B$, and consider the complex space

$$Z := \mathbb{P}(V) \setminus \mathbb{P}(Q).$$

Then $Z \setminus \psi^{-1}(B)$ is not holomorphically convex, where $\psi: Z \to N$ is the natural map.

Proof. We argue by contradiction and assume that $Z \setminus \psi^{-1}(B)$ is holomorphically convex. Fix a general point $y \in B$. Since $(\ast)$ has the extension property near $B$, we can consider the quasi-étale cover $p: \tilde{N} \to N'$ and the exact sequence of vector bundles

$$0 \to \mathcal{O}_{\tilde{N}} \to V_{\tilde{N}} \to Q_{\tilde{N}} \to 0.$$ 

appearing in the definition. The difference $Z_{\tilde{N}} := \mathbb{P}(V_{\tilde{N}}) \setminus \mathbb{P}(Q_{\tilde{N}})$ is an affine bundle $\tilde{\psi}: Z_{\tilde{N}} \to \tilde{N}$. Choose a discrete sequence $(z_n)_{n \in \mathbb{N}}$ of points in $Z_{\tilde{N}} \setminus \tilde{\psi}^{-1}(R)$ converging to a point $z_\infty \in \tilde{\psi}^{-1}(R)$. Since $p$ is étale in the complement of $B \cap N'$, we have a natural map

$$\tau: Z_{\tilde{N}} \setminus \tilde{\psi}^{-1}(R) \to Z \setminus \psi^{-1}(B)$$

that is finite onto its image. The sequence $(\tau(z_n))_{n \in \mathbb{N}}$ is discrete in $Z \setminus \psi^{-1}(B)$, so by [GR79, IV, §2, Thm.12] there exists a holomorphic function $f$ on $Z \setminus \psi^{-1}(B)$ such that $\lim_{n \in \mathbb{N}} |f(z_n)| = \infty$. Thus $\tau^*f$ is a holomorphic function on $Z_{\tilde{N}} \setminus \tilde{\psi}^{-1}(R)$ that is unbounded near $z_\infty$. Yet, since $Z_{\tilde{N}} \to \tilde{N}$ is equidimensional and $R \subset \tilde{N}$ has codimension at least two, the preimage $\tilde{\psi}^{-1}(R) \subset Z_{\tilde{N}}$ has codimension at least two. Thus $\tau^*f$ extends to $Z_{\tilde{N}}$ by Hartog’s theorem, in particular it is bounded near $z_\infty$, a contradiction. □

Proof of Theorem 4.5. By Lemma 4.8 it is sufficient to show that $Z_\beta \setminus \psi^{-1}(B)$ is not Stein.

If $N$ is smooth in the general point of $B$, the fibration $Z_\beta \to N$ is smooth in an analytic neighbourhood of a general point $y \in B$. Since $B \subset N$ has codimension at least two, its preimage $\psi^{-1}(B)$ has codimension at least two in any point mapping onto $y$. By [GR79, V, Thm.4] this implies that $Z_\beta \setminus \psi^{-1}(B)$ is not Stein.

If $E$ is contracted onto an isolated hypersurface singularity, observe first that since the extension defined by $\beta$ is locally splittable, the fibre of $\psi^{-1}(B)$ is not empty. Indeed locally near the point $B \subset N$ we have $V_\beta \simeq \mathcal{O}_N \oplus T_N$, so the inclusion
$P'(T_N) \subset P'(V_\beta)$ is strict over $B$. The fibres of $P(V_\beta) \to N$ are linear projective spaces [AT82 §2], and by Lemma 4.10 below, the fibre over the point $B$ has dimension $\dim M + 1$. Thus $\psi^{-1}(B)$ has dimension $\dim M + 1$. Since $\dim M \geq 3$, we see that $\psi^{-1}(B) \subset Z_\beta$ has codimension at least two, so we conclude again by [GR79 V, Thm.4].

Finally in the third case, by Lemma 4.9 it is sufficient to show that the exact sequence

$$(*) \quad 0 \to O_N \to V_\beta \to T_N \to 0$$

has the extension property. For a general point $y \in B$, consider a quasi-étale map as in Lemma 4.9 and note that we can choose $N' \simeq \tilde{N}/G$ where $G$ is a finite group acting on $\tilde{N}$. Since $H^1(N, \Omega_N) \simeq H^1(\tilde{N}, \Omega_{\tilde{N}})^G$, the extension class $\beta|_N$ corresponds to a class $\tilde{\beta} \in H^1(\tilde{N}, \Omega_{\tilde{N}})^G$. Then $\tilde{\beta}$ defines an exact sequence of vector bundles

$$0 \to O_{\tilde{N}} \to V_{\tilde{\beta}} \to T_{\tilde{N}} \to 0.$$

Since $p|_{\tilde{N}\setminus R}$ is étale, this exact sequence extends the pull-back of $(*)$. \hfill $\square$

**4.10. Lemma.** Let $0 \in N \subset \mathbb{C}^{n+1}$ be a local isolated hypersurface singularity. Then

$$T_{N,0}/m_0T_{N,0} \simeq m_0/m_0^2 \simeq \mathbb{C}^{n+1},$$

where $m_0 \subset O_{N,0}$ is the maximal ideal.

**Proof.** Dualizing the cotangent sequence

$$0 \to O_N \simeq N^*_N/\mathbb{C}^{n+1} \to (\Omega_{\mathbb{C}^{n+1}})|_N \to \Omega_N \to 0$$

yields

$$0 \to T_N \to (T_{\mathbb{C}^{n+1}})|_N \to J \to 0,$$

where $J$ is the ideal sheaf of $\{0\}$ in $N$, so that $J_0 = m_0$. Tensoring with $O_N/J$, i.e., restricting to $\{0\}$, gives an exact sequence

$$\text{Tor}^1((T_{\mathbb{C}^{n+1}})|_N, O_N/J) \to \text{Tor}^1(J, O_N/J) \to T_N/J \cdot T_N \to (T_{\mathbb{C}^{n+1}})|_N \cdot J/J^2 \to 0.$$

The first Tor group vanishes, since $T_{\mathbb{C}^{n+1}}$ is locally free. Since $\mu$ is an isomorphism, it follows

$$\text{Tor}^1(J, O_N/J) \simeq T_N/J \cdot T_N.$$ 

In order to compute this Tor group, we restrict - in the same way as before - the ideal sheaf sequence for $J \subset O_N$ to the point $0$ to obtain an isomorphism

$$\text{Tor}^1(J, O_N/J) \simeq J/J^2,$$

which gives our claim. \hfill $\square$

**Proof of Corollary 1.8.** Arguing by contradiction we assume that $M$ supports a birational Mori contraction $\varphi : M \to N$. Since Mori’s arguments [Mor82] are local near the exceptional locus, we can apply the results in the Kähler case: the contraction is divisorial, and if $N$ is not smooth the divisor $E$ is contracted onto a point. Moreover, in the latter case the point $\varphi(E)$ is a quotient or hypersurface singularity. In all of these cases Theorem 4.5 gives a contradiction.

Since $M$ admits no birational contraction, it is a minimal model or Mori fibre space ([Mor82] or [HP16, HP15] in the Kähler case). \hfill $\square$
Proof of Proposition 1.9. Arguing by contradiction we assume that $M$ admits a contraction of a $K_M$-negative extremal ray $\varphi : M \to N$ that is small. Then by [Kaw89, Thm.1.1], we know that the exceptional locus is a disjoint union of two-dimensional projective spaces $\mathbb{P}^2$ such that $N_{\mathbb{P}^2/M} \simeq \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$. Thus the flip can be constructed as follows: denote by $\psi : \Gamma \to M$ the blowup of the exceptional locus. Then each exceptional divisor is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^1$ and one can contract them onto $\mathbb{P}^1$ by a bimeromorphic morphism $\psi^+ : \Gamma \to M^+$. In particular $M^+$ is also smooth, and we denote by $\varphi^+ : M^+ \to N$ the $K_{M^+}$-positive contraction.

Let now $\alpha$ be a Kähler class on $M$ such that $\pi : \mathbb{P}^2_M \to M$ is Stein. We set $\alpha^+ := (\psi^*)_\ast \psi^\ast \alpha \in H^1(M^+, \Omega_{M^+})$ and denote by $\pi^+ : \mathbb{P}^2_{M^+} \to M^+$ the extension defined by this class. Note that, as in Setup 4.7, we do not assume that $\alpha$ is a Kähler class, we will only use that $\mathbb{P}^2_{M^+}$ is an affine bundle defined by the $(1, 1)$-class $\alpha^+$. Since the restriction of the classes $\alpha$ and $\alpha^+$ to $M \setminus \text{Exc}(\varphi) \simeq M^+ \setminus \text{Exc}(\varphi^+)$ coincide, we have a natural isomorphism of affine bundles

$$
\tau_0 : \mathbb{P}^2_{M^+} \setminus \text{Exc}(\varphi^+) \to \mathbb{P}^2_M \setminus \text{Exc}(\varphi).
$$

Since $(\pi^+)_\ast \text{Exc}(\varphi^+) \subset \mathbb{P}^2_{M^+}$ has codimension more than two and $\mathbb{P}^2_M$ is Stein, we know by [AS60, Thm.2] that $\tau_0$ extends to a holomorphic map

$$
\tau : \mathbb{P}^2_{M^+} \to \mathbb{P}^2_M.
$$

Yet such a map can’t exist: fix $u \in \text{Exc}(\varphi^+)$ a point, and let $U \subset M^+$ be a small analytic neighbourhood such that the affine bundle $\pi^+ : \mathbb{P}^2_{M^+} \to M^+$ admits a section $s : U \to \mathbb{P}^2_{M^+}$. Since $\tau_0$ is an isomorphism of affine bundles, we see that the restriction of $\pi \circ \tau \circ s : U \to M$ to $U \setminus \text{Exc}(\varphi^+)$ is the identity. Thus $\pi \circ \tau \circ s$ extends the identity over the exceptional locus, hence the natural map $M^+ \setminus \text{Exc}(\varphi^+) \to M \setminus \text{Exc}(\varphi)$ extends to a holomorphic map $M^+ \to M$, a contradiction. $\square$

4.C. Proof of Theorem 1.12. We start by proving (a slightly more precise version of) Proposition 1.10.

4.11. Proposition. Let $M$ be a projective manifold of dimension $n$ such that $T_M$ is pseudoeffective. If $M$ is not uniruled, there exists exists a decomposition

$$
T_M \simeq \mathcal{F} \oplus \mathcal{G},
$$

where $\mathcal{F} \neq 0$ and $\mathcal{G}$ are integrable subbundles such that $c_1(\mathcal{F}) = 0$. In particular, by [Pet11, Cor.11] the manifold $M$ is not of general type and $\Omega_M$ is not generically ample.

Moreover the decomposition can be chosen such that if $H$ is a polarisation on $M$, and $C$ a MR-general curve associated to $H$, then $\mathcal{G}|_C$ is antiample.

The proof is a rather straightforward extension of [Pet11], based on [LPT18].

\footnote{The statement is for projective fourfolds, but the proof is local around the image of the exceptional locus, so it also works in the Kähler case.}
Proof. Fix a polarisation $H$ and a MR-general curve $C$ associated to $H$. Since $T_M$ is pseudoeffective, the restriction $T_M|C$ is pseudoeffective. Hence $\Omega_M|C$ cannot be ample by Lemma 2.6. In particular $\Omega_M$ is not generically ample with respect to $H$. Now we follow the proof of [Pet11, Prop.2.]: let $G^* \subset \Omega_M|C$ be the maximal ample subbundle, and set $F^* := \Omega_M|C/G^*$. Since $\mu(G^*) > 0$ and $\mu(F^*) = 0$, we see that $G^*$ appears in the Harder-Narasimhan filtration of $\Omega_M|C$. Since $C$ is MR-general, by the theorem of Mehta-Ramanathan, there exists a saturated subsheaf $G^* \subset \Omega_M$ such that $G^* = G^*|C$. We set $\mathcal{F} := \Omega_M/G^*$. By [BDPP13] the determinant of $\mathcal{F}$ is pseudoeffective. Yet $c_1(\mathcal{F}) \cdot H^{n-1} = \mu(\mathcal{F}) = 0$, so $c_1(\mathcal{F}) = 0$. Thus $\mathcal{F} \subset T_M$ is a foliation with $c_1(\mathcal{F}) = 0$. Since $K_M$ is pseudoeffective, we know by [LPT18, Thm.5.2] that $\mathcal{F}$ is integrable and the inclusion $\mathcal{F} \subset T_M$ splits, inducing a decomposition into integrable subbundles $T_M \simeq \mathcal{F} \oplus G$. Finally if for some polarisation $H'$, the restriction $G|C'$ is not ample, then we can repeat the argument above: since $\mathcal{F}|C'$ is numerically trivial, we obtain a maximal ample subbundle $(\mathcal{G}')^* \subset G|C'$ leading to a decomposition $T_M \simeq \mathcal{F}' \oplus \mathcal{G}'$ with $rk \mathcal{F} < rk \mathcal{F}'$. Since $rk T_M$ is finite this process stops after finitely many steps. □

4.12. Proposition. Let $M$ be a non-uniruled projective manifold of dimension $n$ such that $T_M$ is pseudoeffective. Let $T_M \simeq \mathcal{F} \oplus \mathcal{G}$ be the decomposition into integrable subbundles given by Proposition 4.11. If the dual $\mathcal{F}^*$ is not pseudoeffective, then $\mathcal{F}$ is pseudoeffective. Moreover, up to replacing $M$ by a finite étale cover, one has $M \simeq A \times B$ with $A$ an abelian variety of dimension $d > 0$.

Proof. Since $\mathcal{F}^*$ is not pseudoeffective we know by [Dru18, Prop.8.4] that the foliation $\mathcal{F}$ has algebraic leaves. Thus by [LPT18, Thm.5.8] we know that, up to replacing $M$ by a finite étale cover, the decomposition $T_M \simeq \mathcal{F} \oplus \mathcal{G}$ is induced by a product structure $M \simeq N \times Z$ such that $\mathcal{F} = T_{M/Z}$ and $\mathcal{G} = T_{M/N}$. The property of having pseudoeffective tangent bundle is preserved under étale cover [HP20, Prop.4.4], so we now argue as in the proof of [HP19, Thm.1.6] that $\mathcal{F}$ or $\mathcal{G}$ is pseudoeffective. Since $\mathcal{G}$ is generically anti-ample for a polarisation, it is not pseudoeffective. Thus $\mathcal{F}$ is pseudoeffective, this shows the first statement. For the second statement note that the restriction $\mathcal{F}|_{N \times p} \simeq T_N$ is pseudoeffective and that $c_1(N) = 0$. By [HP19, Thm.1.6] this implies that the Beauville-Bogomolov decomposition of $N$ contains an abelian factor. Thus, up to replacing $N \times Z$ by an étale cover, we can assume that $N$ is an abelian variety. □
Proof of Theorem 1.12. We start by observing that if the manifold $M$ admits an étale cover by a product $A \times B \to M$, we know by [GW20, Lemma 2.10] that the canonical complex extension $Z_A$ and $Z_B$ are Stein. In this case the statement thus follows by induction on the dimension.

Consider now the decomposition $T_M \simeq F \oplus G$ given by Proposition 4.11. If $\text{rk} F = 3$, then $c_1(M) = c_1(F) = 0$ the statement follows from the Beauville-Bogomolov decomposition and [HP19 Thm.1.6]. If $\text{rk} F < 3$ the manifold is a product after étale cover by [PT13 Thm.D] and [Tou08 Thm.1.2], so the induction hypothesis applies. □

5. Surfaces

In this section we discuss canonical extensions on surfaces. We recall the setup: let $M$ be a compact Kähler surface, and $\alpha$ a Kähler class on $M$. Let

$$0 \to \mathcal{O}_M \to V \to T_M \to 0$$

the extension defined by $\alpha$, and set

$$Z_M := \mathbb{P}(V) \setminus \mathbb{P}(T_M).$$

As before we denote by $\pi_M : \mathbb{P}(T_M) \to M$ and $\pi : \mathbb{P}(V) \to M$ the projectivisations. The following result is a first step towards Theorem 1.13.

5.1. Proposition. Let $M$ be a smooth projective surface. Assume that $Z_M$ is Stein for some Kähler class. Then $M$ is either an étale quotient of a torus, the projective plane $\mathbb{P}^2$, a quadric $\mathbb{P}^1 \times \mathbb{P}^1$ or a ruled surface $f : M \to B$ over a curve of genus $g(B) \geq 1$.

Proof. If $M$ is not uniruled, we conclude with Theorem 1.12. If $M$ is uniruled, it is a Mori fibre space by Corollary 1.8. Thus if $M \neq \mathbb{P}^2$, it is a ruled surface $f : M \to B$. If the curve $B$ is isomorphic to $\mathbb{P}^1$ and $M$ is not a quadric, the Hirzebruch surface $M$ contains a rational curve $C$ such that $C^2 < 0$, in contradiction to Corollary 4.6. □

5.2. Remark. If $M$ is a compact Kähler surface, the arguments developed in the preceding sections show that $M$ is either an étale quotient of a torus, a Mori fibre space or a minimal surface of Kodaira dimension one. In the last case, using that $T_M$ is pseudoeffective by Corollary 1.7 the techniques from [HP20] allow to prove that, up to taking an étale cover, the surface $M$ is an elliptic bundle over a curve of genus at least two. If $M$ is not projective such an elliptic bundle does not trivialise after base change, so we can’t use the technique from Theorem 1.12 to eliminate this case.

5.3. Lemma. Let $M$ be a compact Kähler manifold. Assume that $Z_M$ is Stein for some Kähler class. For every irreducible curve $C \subset M$, the restriction $T_M|_C$ is pseudoeffective.
Proof. Let
\[ 0 \to \mathcal{O}_M \to V \to T_M \to 0 \]
be the canonical extension such that \( Z_M \) is Stein. Then the closed subset \( \mathbb{P}(V_C) \setminus \mathbb{P}(T_M|C) \) is Stein. Let \( \nu : \tilde{C} \to C \) be the normalisation. Then
\[ \mathbb{P}(\nu^*V_C) \setminus \mathbb{P}(\nu^*T_M|C) \to \mathbb{P}(V_C) \setminus \mathbb{P}(T_M|C) \]
is finite, so \( \mathbb{P}(\nu^*V_C) \setminus \mathbb{P}(\nu^*T_M|C) \) is Stein [GR79, V, §1, Thm.1 d)]. By Theorem 1.6 the normal bundle of \( \mathbb{P}(\nu^*T_M|C) \) in \( \mathbb{P}(\nu^*V_C) \) is pseudoeffective. Thus the normal bundle is exactly the tautological class of \( T_M|C \), hence \( T_M|C \) is not pseudoeffective. \( \square \)

We can now prove an analogue of Corollary 4.6 for curves of higher genus:

5.4. Corollary. Let \( M \) be a compact Kähler surface. Assume that \( M \) contains a smooth curve \( C \) of genus at least two such that \( C^2 < 0 \). Then for any Kähler class \( \alpha \), the canonical extension \( Z_M \) is not Stein.

Proof. Assume that \( Z_M \) is Stein. By Lemma 5.3 we know that the restriction \( T_M|C \) is pseudoeffective, so by Lemma 2.6 its dual \( \Omega_M|C \) is not ample.

Yet since \( g(C) \geq 2 \) the normal bundle sequence
\[ 0 \to T_C \to T_M|C \to N_{C/M} \to 0 \]
tells us that \( \Omega_M|C \) is ample, a contradiction. \( \square \)

5.B. Split tangent bundle. We have seen in Proposition 1.10 that manifolds with split tangent bundle appear naturally in the analysis of the canonical extension. This situation can be studied through a technical lemma:

5.5. Lemma. Let \( M \) be a compact Kähler manifold such that \( T_M \simeq F \oplus G \). Let \( \alpha \) be a Kähler class on \( M \) and write
\[ \alpha = \alpha_1 + \alpha_2 \]
with \( \alpha_1 \in H^1(M, F^*) \) and \( \alpha_2 \in H^1(M, G^*) \) according to the splitting. Let
\[ 0 \to \mathcal{O}_M \to V_F \to F \to 0 \]
\[ 0 \to \mathcal{O}_M \to V_G \to G \to 0 \]
be the extensions defined by these cohomology classes, and set
\[ Z_F := \mathbb{P}(V_F) \setminus \mathbb{P}(F), \quad Z_G := \mathbb{P}(V_G) \setminus \mathbb{P}(G). \]
Let \( Z_M = Z_M^0 \) be the canonical extension. Then we have an isomorphism
\[ Z_M \simeq Z_F \times_M Z_G, \]
so in fact
\[ Z_M \simeq (\mathbb{P}(V_F) \times_M \mathbb{P}(V_G)) \setminus (\tau_F^{-1}(\mathbb{P}(F)) \cup \tau_G^{-1}(\mathbb{P}(G))) \]
where \( \tau_F \) and \( \tau_G \) are the natural projections on the factors of the fibre product \( \mathbb{P}(V_F) \times_M \mathbb{P}(V_G) \).

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Proof. We have an extension
\[ 0 \to \mathcal{O}_M \to V_F \oplus V_G \to V_M \to 0 \]
as given in the following diagram

\[
\begin{array}{c}
0 \to \mathcal{O}_M \xrightarrow{f \oplus f} \mathcal{O}_M \oplus \mathcal{O}_M \xrightarrow{\alpha} 0 \\
0 \to \mathcal{O}_M \to V_F \oplus V_G \to V_M \to 0 \\
\mathcal{F} \oplus \mathcal{G} \xrightarrow{\sim} T_M \to 0
\end{array}
\]

Hence
\[ \mathbb{P}(V_M) \subset \mathbb{P}(V_F \oplus V_G). \]

Further, we have a morphism
\[ \Phi : Z_F \times_M Z_G \to \mathbb{P}(V_F \oplus V_G), \]
as follows. Let \( x \in M, u \in (Z_F)_x \) and \( v \in (Z_G)_x \). Then \((u, v)\) is mapped to \([u \oplus v]\).

Checking on the fibers over \( x \), we see that \( \Phi \) maps \( Z_F \times_M Z_G \) isomorphically onto \( Z_M \). \( \square \)

For ruled surfaces this can be made more precise:

**5.6. Lemma.** Let \( f : M = \mathbb{P}(E) \to B \) be a ruled surface where \( E \) is a rank two vector bundle that is normalised in the sense of [Har77, V, Ch.2]. Assume that the tangent bundle splits such that \( T_M \simeq T_{M/B} \oplus p^*(T_B) \). Let \( \alpha \) be a Kähler class on \( M \) and write \( \alpha = \alpha_1 + \alpha_2 \) with \( \alpha_1 \in H^1(M, \Omega_{M/B}) \) and \( \alpha_2 \in H^1(M, p^*(\Omega_B)) \) according to the splitting. Then by Lemma 5.5 we have
\[ Z_M \simeq Z_{M/B} \times_B Z_B, \]
and
\[ Z_{M/B} \simeq \mathbb{P}(f^*E) \setminus \mathbb{P}(\zeta_M), \]
where \( \zeta_M \) is the tautological bundle on \( \mathbb{P}(E) \).

Proof. The class \( \alpha_1 \in H^1(M, \Omega_{M/B}) \) defines an extension
\[ 0 \to \mathcal{O}_M \to V_{M/B} \to T_{M/B} \to 0. \]
Since \( \text{Ext}^1(T_{M/B}, \mathcal{O}_M) = H^1(M, K_{M/B}) \simeq \mathbb{C} \), it follows from the relative Euler sequence that
\[ V_{M/B} \simeq \zeta_M \otimes f^*(\mathcal{E}^*). \]
Thus, using \( \mathcal{E}^* \simeq \mathcal{E} \otimes \det \mathcal{E}^* \), we obtain \( \mathbb{P}(V_{M/B}) \simeq \mathbb{P}(f^*(\mathcal{E})). \) Moreover since \( T_{M/B} \simeq 2\zeta_M \otimes f^* \det \mathcal{E}^* \), this isomorphism maps \( \mathbb{P}(T_{M/B}) \) onto \( \mathbb{P}(\zeta_M). \) \( \square \)

We can finally apply these argument to Serre’s example [Har70 §6.3], [Nee88 §7]:

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5.7. Proposition. Let $B$ be an elliptic curve and
\[0 \to \mathcal{O}_B \to \mathcal{E} \to \mathcal{O}_B \to 0\]
a non split extension. Set $f : M := \mathbb{P}(\mathcal{E}) \to B$ and choose any Kähler class on $M$. Then $Z_M$ is Stein.

Proof. Since $M$ is almost homogeneous the vector field on $B$ lifts to $M$, so we have a splitting $T_M \simeq T_{M/B} \oplus \mathcal{O}_M$. Consider the commutative diagram
\[\begin{array}{ccc}
M \times_B Z_B & \overset{p}{\longrightarrow} & M \\
\downarrow\sigma & & \downarrow f \\
Z_B & \underset{q}{\longrightarrow} & B.
\end{array}\]
Then by Lemma 5.6 we have
\[Z_M \simeq \mathbb{P}(\rho^*f^*\mathcal{E}) \setminus \mathbb{P}(\rho^*(\zeta_M)).\]
Since $g(B) = 1$, the space $Z_B$ is Stein, so $H^1(Z_B, \mathcal{O}_{Z_B}) = 0$. Thus the exact sequence
\[0 \to \mathcal{O}_{Z_B} \to q^*(\mathcal{E}) \to \mathcal{O}_{Z_B} \to 0\]
splits. Hence
\[M \times_B Z_B = \mathbb{P}(q^*(\mathcal{E})) \simeq \mathbb{P}_1 \times Z_B.\]
Further,
\[\rho^*\pi^*(\mathcal{E}) = \sigma^*q^*(\mathcal{E}) \simeq \mathcal{O}^{\mathbb{P}^2}_{M \times_B Z_B}\]
so that
\[Z_M \simeq (\mathbb{P}_1 \times \mathbb{P}_1 \times Z_B) \setminus \mathbb{P}(\rho^*(-K_M)).\]
Since $\rho^*(-K_M) = p_1^*(\mathcal{O}_{\mathbb{P}_1})$, where $p_1 : M \times_B Z_B \to \mathbb{P}_1 \times Z_B$ is the projection, the divisor $\mathbb{P}(-K_M)$ cuts out in $(\mathbb{P}_1 \times \mathbb{P}_1) \times Z_B \to Z_B$ in every fiber the same quadric hence $Z_M \simeq Q \times Z_B$ with an affine quadric $Q$. Thus $Z_M$ is Stein. □

5.8. Corollary. Let $B$ be an elliptic curve, $\mathcal{E}$ a semistable vector bundle of rank two on $B$ and $M = \mathbb{P}(\mathcal{E})$. Then $Z_M$ is Stein for any Kähler class on $M$.

Proof. Since $\mathcal{E}$ is semistable, we may assume that $\mathcal{E}$ fits into an exact sequence
\[0 \to \mathcal{O}_B \to \mathcal{E} \to \mathcal{L} \to 0,\]
where $\mathcal{L}$ has degree 0 or 1. If $\mathcal{L}$ has degree 0, then either $\mathcal{L}$ is torsion, and we are in a product situation after finite étale cover, or we are in the Serre example, which is settled by Proposition 5.7. If $\mathcal{L}$ has degree one, we may perform an étale cover of degree two and land in the Serre example. Hence $Z_M$ is Stein in all cases.

\[\Box\]

Proof of Theorem 1.13. Assume first that $Z_M$ is Stein. By Proposition 5.1 we are left to discuss the case of ruled surfaces $f : M \simeq \mathbb{P}(\mathcal{E}) \to B$ over a curve of genus at least one. If $g(B) \geq 2$ we know by Corollary 5.4 that $M$ does not contain a curve with negative self-intersection. Yet by [Har77, V, Prop.2.20] this implies that the invariant $e$ of the normalised vector bundle $\mathcal{E}$ is at most 0. But by definition of $e$, see [Har77, V, Prop.2.8], this means that $\mathcal{E}$ is semistable.

Assume now that $Z_M$ is even affine. Then the tangent bundle $T_M$ is big by [GW20, Prop.4.20], so $M$ is obviously not an étale quotient of a torus. By the first part of
the statement we are left to exclude the case of ruled surfaces $f : M \simeq \mathbb{P}(\mathcal{E}) \to B$ over a curve of genus at least one. If $g(B) \geq 2$ we know that $\mathcal{E}$ is semistable, so $T_{M/B}$ is nef, but not big. Since $f^*T_B$ has negative degree, the tangent sequence shows that

$$H^0(M, S^l T_M) \simeq H^0(M, S^l T_{M/B})$$

for every $l \in \mathbb{N}$. Thus $T_M$ is not big. So suppose $g(B) = 1$. If $\mathcal{E}$ is semistable, then $T_M$ is nef. If $T_M$ would be big, then $c_1^2(M) > c_2(M)$, which is absurd. If $\mathcal{E}$ is unstable, i.e., its invariant $e > 0$, we have $\mathcal{E} \simeq O_B \oplus L$ with $\deg L < 0$. Let $B_0 \subset M$ be the section corresponding to the negative quotient $\mathbb{P}(L)$. Then we have $T_M|_{B_0} \simeq O_B \oplus L$, so the $T_M|_{B_0}$ is not big. If $Z_M$ was affine, its closed subset $Z_M|_{B_0}$ would also be affine. Yet by [GW20, Prop.4.2] this implies that $T_M|_{B_0}$ is big, a contradiction.

5.9. Remarks. While Theorem 1.13 gives the expected characterisation of a fffineness of the canonical extension, the picture is less complete for the Stein property. Let us explain why the remaining cases are more subtle than one might expect:

1) Let $B$ be an elliptic curve, and let $\mathcal{E} \simeq O_B \oplus L$ be a rank two vector bundle with $\deg L < 0$. Then the restriction $T_M|_{C}$ is nef unless $C$ is the section $B_0 \subset M$ corresponding to the negative quotient $\mathbb{P}(L)$. Making an elementary transformation in the sense of [Mar73] along $B_0$ one can construct a rank two subbundle $K \subset T_M$ that is nef and big.

2) Let $B$ be a smooth curve of genus $g \geq 2$ and $\mathcal{E}$ a normalized locally free sheaf of rank two given by the exact sequence

$$0 \to O_B \to \mathcal{E} \to \mathcal{L} \to 0.$$ 

Assume that the invariant $e$ is strictly negative, i.e., $\mathcal{E}$ is stable.

Let $f : M := \mathbb{P}(\mathcal{E}) \to B$ be the ruled surface, and denote by $\eta$ the extension class corresponding to

$$0 \to T_{M/B} \to T_M \to f^*T_B \to 0.$$ 

Denoting by $\mathcal{F}$ the sheaf of traceless endomorphisms, the extension class $\eta$ lives in

$$H^1(X, T_{M/B} \otimes p^* K_B) = H^1(B, \mathcal{F} \otimes K_B) = H^0(B, \mathcal{F}) = 0.$$ 

hence $\eta = 0$. Thus we have $T_M \simeq T_{M/B} \oplus f^*T_B$, so this situation fits into the framework of Lemma 5.6. Thus it might be a surprise that we were not able to prove that $Z_M$ is not Stein. Note however that in this case the subsheaf $T_{M/B} \subset T_M$ is nef, so again $T_M$ is rather positive.

6. The algebraic approach

We start giving an algebraic proof of the following result, which is weaker than Theorem 1.6 but has a more elementary proof.

6.1. Theorem. Let $M$ be a projective manifold of dimension $n$. Assume that $Z_M$ is Stein for some Kähler class. Then $\zeta_M$ is generically nef:

$$\zeta_M \cdot H_1 \cdot \ldots \cdot H_{2n-2} \geq 0$$

for all ample divisors $H_j$ on $\mathbb{P}(T_M)$. 23
Proof of Theorem 6.1. We will use the terminology of $\mathbb{Q}$-twisted bundles as explained in [Laz04, Ch.6.2]. By Corollary 6.7, it suffices to show that
\[
\text{Nef}(\mathbb{P}(V)) \to \text{Nef}(\mathbb{P}(T_M))
\]
is surjective. So let $L$ be a nef divisor class on $\mathbb{P}(T_M)$, then we can write
\[
L = a\zeta_M \otimes \pi_M^*\mathcal{A}
\]
with $a \in \mathbb{N}$ and $\mathcal{A}$ a divisor class on $M$. We may assume $a > 0$, otherwise there is nothing to prove.

If $M = \mathbb{P}^n$, the class $\zeta\nu$ is ample, so we can assume $M \neq \mathbb{P}^n$. We claim that $\mathcal{A}$ is nef. This implies the statement: $a\zeta_M + \pi_M^*(\mathcal{A})$ being nef is equivalent saying that the $\mathbb{Q}$-bundle $T_M \otimes \frac{1}{a} \mathcal{A}$ is nef. Thus we have an extension of nef $\mathbb{Q}$-vector bundles
\[
0 \to \mathcal{A} \to V \otimes \frac{1}{a} \mathcal{A} \to T_M \otimes \frac{1}{a} \mathcal{A} \to 0,
\]
hence $V \otimes \frac{1}{a} \mathcal{A}$ is nef [Laz04, Thm.6.2.12,(ii),(v)].

Proof of the claim. By the cone theorem, it suffices to show that $\mathcal{A} \cdot C \geq 0$ for either $C$ a curve generating a $K_M$-negative extremal ray and for curves $C$ with $K_M \cdot C \geq 0$.

- If $K_M \cdot C \geq 0$, the vector bundle $T_M|_C$ cannot be ample. If $T_M|_C$ is not nef, then $\deg A|_C > 0$ by Hartshorne’s criterion [Laz04, Thm.6.4.15]. If $T_M|_C$ is nef, then $c_1(T_M|_C) \leq 0$ implies that $T_M|_C$ is numerically flat. Thus $\deg A|_C \geq 0$.

- If $C$ is an extremal rational curve with $0 < -K_M \cdot C \leq n$, denote by $\nu : \mathbb{P}^1 \to M$ its normalisation. Then $\nu^*T_M$ has a trivial quotient since it contains the image of $\mathcal{O}_{\mathbb{P}^1}(2) \to \nu^*T_M$. Thus again $\deg \nu^*A = \deg A|_C \geq 0$.

- If $C$ is an extremal rational curve with $-K_M \cdot C = n + 1$, then $M = \mathbb{P}^n$ by CMSB02, which we excluded.

\[ \square \]

6.2. Corollary. Let $M$ be a projective manifold. Assume $Z_M$ Stein and that $\mathcal{O}(\mathbb{P}(T_M)) = \mathcal{O}(\mathbb{P}(M)).$ Then $\zeta_M$ is pseudoeffective.

Proof. By Theorem 6.1 the class $\zeta_M$ is generically nef. By our assumption and BDPPT13, the class $\zeta_M$ is then pseudoeffective. \[ \square \]

We give some first evidence in favour of Conjecture 1.15

6.3. Proposition. Let $M$ be a projective K3 surface with $\rho(M) = 1$ Then $\zeta_M$ is not generically nef.

By Theorem 6.1 this shows that $Z_M$ is never Stein for K3 surfaces $M$ with $\rho(M) = 1$.

Proof. Let $H$ be the ample generator the Picard group, and write $H^2 = d$. Let $a$ be the unique positive real number such that $\zeta_M + a\pi_M^*H$ is nef but not ample. Then $\zeta_M$ is generically nef precisely when
\[
\zeta_M \cdot (\zeta + a\pi_M^*H)^2 \geq 0.
\]
Now $\zeta_M \cdot (\zeta + a\pi_M^*H)^2 = -24 + a^2d$. If $d > 2$, then $a^2d < 24$ by GO20 Prop.3.2, thus
\[
\zeta_M \cdot (\zeta + a\pi_M^*H)^2 < 0.
\]
In case $d = 2$, $a = 3$ by [GO20] Sect. 4.1, so the inequality holds, too, and $\zeta_M$ is not generically nef.

Finally, we present a completely different approach for K3-surfaces $M$ containing a nodal curve to show that $Z_M$ is not Stein. We start with some preparations.

6.4. Lemma. Let $M$ be a smooth projective surface, and $T$ a rank two vector bundle on $M$. Let $C \subset \mathbb{P}(T)$ be a smooth curve that is not a fibre of $\pi : \mathbb{P}(T) \to M$. Then there exists a smooth surface $S \subset \mathbb{P}(T)$ such that $C \subset S$ and $\pi|_S$ is finite.

Proof. We can assume without loss of generality that $T$ is ample, and denote by $\zeta$ the tautological class of $\mathbb{P}(T)$. By [BS95, Cor.1.7.5] we know that for $t \gg 0$ a general element $S \in |I_C \otimes O_{\mathbb{P}(T)}(t\zeta)|$ is smooth, so we are left to show that $\pi|_S$ is finite. Using the fact the $t$-relative Segre embedding embeds the fibres of $\mathbb{P}(T)$ as linearly non-degenerate curves in $\mathbb{P}(S^tT)$, this is equivalent to showing that there exists a global section of $\nu_*|_S(I_C \otimes O_{\mathbb{P}(T)}(t\zeta))$ that does not vanish in any point of $S$.

For $t \gg 0$ the sheaf $I_C \otimes O_{\mathbb{P}(T)}(t\zeta)$ is globally generated, so its direct image $\nu_*|_S(I_C \otimes O_{\mathbb{P}(T)}(t\zeta))$ is globally generated. Let $V$ be its space of global sections, then the morphism

$$V \otimes O_M(\nu(C)) \to S^tT \otimes O_M(\nu(C))$$

is surjective. Since $t > 1$ the vector bundle $S^tT$ has rank at least three, so by [Har77, II, Ex.8.2] a general element of $V$ defines a section of $S^tT \otimes O_M(\nu(C))$ that does not vanish on any point of $M \setminus \nu(C)$. Thus we are left to show the statement over $\nu(C)$: denote by $\nu : C \to S$ the restriction of $\nu$ to $C$, and by $\tilde{\nu} : \mathbb{P}(\nu^*T) \to \mathbb{P}(T)$ the natural map. Let $\tilde{C} \subset \mathbb{P}(\nu^*T)$ be the natural section such that $\tilde{\nu}(\tilde{C}) = C$. Since $\zeta$ is ample we know that $\nu_*|_S(I_C) \otimes O_{\mathbb{P}(T)}(t\zeta) \simeq I_C \otimes O_{\mathbb{P}(T)}(t\zeta)$ is globally generated for $t \gg 0$, in particular $O_{\mathbb{P}(\nu^*V)}(-\tilde{C}) \otimes \tilde{\nu}^*O_{\mathbb{P}(T)}(t\zeta)$ is ample and generated by pull-backs of global sections for $t \gg 0$. Thus a general element of the linear subsystem of $[\tilde{\nu}^*|_S\zeta - \tilde{C}]$ generated by the pull-backs is a smooth and irreducible curve $R$. Thus the intersection of $S$ with $\pi^{-1}(\nu(C))$ has exactly two irreducible components, the curve $C$ and $\tilde{\nu}(R)$, none of them is a fibre. □

6.5. Lemma. Let $U$ be a smooth analytic surface that contains a rational curve $\mathbb{P}^1 \simeq C \subset U$ such that $C^2 := -k \leq -1$ and $U$ is a deformation retract of $C^3$. For every $l \in \mathbb{Z}$, we denote by $O_U(l)$ the unique line bundle such that $O_U(l) \otimes O_C \simeq O_{\mathbb{P}^1}(l)$. Then the following holds:

- One has an exact sequence

$$0 \to O_U(-k) \to T_U \to O_U(2) \to 0.$$  

- If $k \leq -3$, let $T$ be a vector bundle on $U$ such that $\det T \simeq O_U$ and $T \otimes O_C \simeq O_{\mathbb{P}^1}(2) \oplus O_{\mathbb{P}^1}(-2)$. Then one has

$$T \simeq O_U(2) \oplus O_U(-2).$$

\[3\text{In particular one has } \operatorname{Pic}(U) \simeq \operatorname{Pic}(C) \simeq \mathbb{Z} \]
Proof. For the first statement we observe that $U$ is isomorphic to a neighbourhood of the negative section in the Hirzebruch surface $\psi : \mathbb{F}_k \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-k)) \rightarrow \mathbb{P}^1$. The exact sequence is then obtained as the restriction of the tangent sequence

$$0 \rightarrow T_{\mathbb{P}^1/\mathbb{P}^1} \rightarrow T_{\mathbb{F}_k} \rightarrow \psi^* T_{\mathbb{P}^1} \rightarrow 0$$

to this neighbourhood.

For the second statement, let $C = \mathbb{P}^1$ and consider the $l-$th infinitesimal neighborhood $C_l$ of $C$ in $U$. Then the extensions of $T|_{C_l}$ to $C_{l+1}$ are parametrized by

$$H^1(C, (T_{|_{C}}^* \otimes T_{|_{C}} \otimes S_k N_{C/U}^*),$$

see e.g. [Pet81]. This group equals

$$H^1(C, (\mathcal{O}_C(4) \otimes \mathcal{O}_C(-4)) \otimes \mathcal{O}_C(3k)) = 0.$$ 

Hence $T|_{C}$ has a unique extension to all infinitesimal neighborhoods $C_k$, namely $\mathcal{O}_{C_k}(2) \oplus \mathcal{O}_{C_k}(-2)$. Thus by [Pet81], there exists an open neighborhood $V \subset U$ of $C$ such that

$$T|_{V} \simeq \mathcal{O}_V(2) \oplus \mathcal{O}_V(-2).$$

□

6.6. Theorem. Let $M$ be a projective $K3$-surface containing a nodal rational curve. Then $Z_M$ is not Stein.

Proof. We argue by contradiction and assume that $Z_M$ is Stein.

Choose a nodal rational curve $f : \mathbb{P}^1 \rightarrow M$. Since $f$ is immersive, we have a canonical surjection $f^* \Omega_M \rightarrow \Omega_{\mathbb{P}^1}$. Since the image of $f$ is nodal, the induced map $\tilde{f} : \mathbb{P}^1 \rightarrow \mathbb{P}(T_M)$ is embedding, and we denote by $C$ its image. By Lemma 6.4 there exists a surface $S \subset \mathbb{P}(T_M)$ be a surface containing $C$ such that the morphism $g := \pi|_S : S \rightarrow M$ is finite. The pull-back of the canonical extension gives an exact sequence of vector bundles

$$0 \rightarrow \mathcal{O}_S \rightarrow g^* V \rightarrow g^* T_M \rightarrow 0$$

Since $g$ is finite and $\mathbb{P}(V) \setminus \mathbb{P}(T_M)$ is Stein, the space $\mathbb{P}(g^* V) \setminus \mathbb{P}(g^* T_M)$ is also Stein [GR79, V, §1, Thm.1 d)].

Since the restriction of $g$ to $C$ is the normalisation of $f(\mathbb{P}^1)$ one has

$$g^* T_M \otimes \mathcal{O}_C \simeq T_C \otimes \mathcal{O}_C(-2) \simeq \mathcal{O}_C(2) \otimes \mathcal{O}_C(-2).$$

By the ramification formula one has $K_S \cdot C \geq 0$, hence $-k := C^2 \leq -2$ with equality if and only if $g$ is étale in a neighbourhood of $C$. Thus the smooth rational curve $C$ is contractible, and we denote by $\varphi : S \rightarrow S'$ its contraction. We claim that the exact sequence (4) has the extension property near $C$, by Lemma 4.9 this gives the desired contradiction.

If $C^2 = -2$ the tangent map $T_S \rightarrow g^* T_M$ is an isomorphism near $C$, so the exact sequence (4) identifies to the canonical extension of $S$ induces by the Kähler class $g^* \alpha$. In this case the extension property follows as in the proof of the last case of Theorem 4.5 so for the rest of the proof we focus on the (more difficult) case $k \leq -3$. 
Let $U \subset S$ be an analytic neighbourhood of $C$ that is a deformation retract of $C$. Then using the notation of Lemma 6.5 one has $g^*T_M \cong \mathcal{O}_U(2) \oplus \mathcal{O}_U(-2)$. Observe that

$$\text{Ext}^1(\mathcal{O}_U(2) \oplus \mathcal{O}_U(-2), \mathcal{O}_U) \cong H^1(U, \mathcal{O}_U(-2) \oplus \mathcal{O}_U(+2)) \cong H^1(\mathbb{P}^1, \omega_{\mathbb{P}^1})$$

where in the last step we used that the inclusion $\mathcal{O}_U(2) \to g^*T_M$ restricts to the natural inclusion $T_{\mathbb{P}^1} \to f^*T_M$ on $C$. Thus the restriction of (4) to $U$ is isomorphic to an exact sequence

$$0 \to \mathcal{O}_U \to E_U \oplus \mathcal{O}_U(-2) \to \mathcal{O}_U(2) \oplus \mathcal{O}_U(-2) \to 0$$

where the extension

$$0 \to \mathcal{O}_U \to E_U \to \mathcal{O}_U(2) \to 0$$

restricts to the Euler sequence on $C$.

Denote by $\varphi_U : U \to U'$ the restriction of the contraction to $U$. Since $U \to U'$ contracts a smooth rational curve onto a point, the surface $U'$ has a Hirzebruch-Jung singularity in $\varphi(C)$, in particular it is a quotient singularity.

Since the line bundle $\mathcal{O}_U(-2)$ has negative degree on the exceptional divisor $C$, we have a chain of inclusion $\mathcal{O}_U(C) \cong \mathcal{O}_U(-k) \to \mathcal{O}_U(-2) \to \mathcal{O}_U$. Pushing forward to $U'$ we obtain $\mathcal{O}_{U'} \to (\varphi_U)_*\mathcal{O}_U(-2) \to \mathcal{O}_{U'}$, so $(\varphi_U)_*\mathcal{O}_U(-2) \cong \mathcal{O}_{U'}$. Since $R^1(\varphi_U)_*\mathcal{O}_U = 0$ The push-forward of the exact sequence (5) is the exact sequence

$$0 \to \mathcal{O}_{U'} \to (\varphi_U)_*\mathcal{E}_U \oplus \mathcal{O}_{U'} \to (\varphi_U)_*\mathcal{O}_U(2) \oplus \mathcal{O}_{U'} \to 0.$$

Thus it is sufficient to check the extension property in Lemma 4.9 for the sequence (6). The idea is now to reduce to the case of a canonical extension on $U$: by Lemma 6.5 we have an exact sequence

$$0 \to \mathcal{O}_U(-k) \to T_U \to \mathcal{O}_U(2) \to 0.$$

Since $H^1(U, \mathcal{O}_U(k)) = 0$ the long exact sequence in cohomology shows that

$$\text{Ext}^1(T_U, \mathcal{O}_U) \cong \text{Ext}^1(\mathcal{O}_U(2), \mathcal{O}_U) \cong H^1(\mathbb{P}^1, \omega_{\mathbb{P}^1}).$$

Thus if

$$0 \to \mathcal{O}_U \to V_U \to T_U \to 0$$

is the canonical extension defined by $g^*\alpha|_U$, the morphism $\mathcal{O}_U(-k) \to T_U$ lifts to a morphism $\beta_U : \mathcal{O}_U(-k) \to V_U$ such that the exact sequence

$$0 \to \mathcal{O}_U \to V_U / \mathcal{O}_U(-k) \to T_U / \mathcal{O}_U(-k) \cong \mathcal{O}_U(2) \to 0$$

is isomorphic to (6).
Let now \( p : \tilde{U} \to U' \) be the quasi-étale cover as in Lemma 4.9. Pushing down to \( U' \), and taking the reflexive pull-back to \( \tilde{U} \) we obtain a commutative diagram:

\[
\begin{matrix}
& & & 0 & & & \\
& & & \downarrow & & & \\
& & & \mathcal{O}_{\tilde{U}} & & & \\
& & & \downarrow & & & \\
0 & \rightarrow & \mathcal{O}_{U} & \rightarrow & p^*\mathcal{O}_{(\varphi U)} & \rightarrow & \mathcal{O}_{U}(2) & \rightarrow & 0 \\
& & & \downarrow i_{V_U} & & & \\
& & & \mathcal{O}_{U} & \rightarrow & p^*\mathcal{O}_{(\varphi U)} & \rightarrow & \mathcal{O}_{U}(2) & \rightarrow & 0 \\
& & & \downarrow & & & \\
& & & \tilde{U} & \rightarrow & \tilde{U} & \rightarrow & 0 \\
\end{matrix}
\]

Since the extension property holds for (7), we already know that the morphism \( i_{V_U} \) is injective in every point, and we want to see that the same holds for \( i_{E_U} \). Now observe that since \( U \to U' \) is a functorial resolution of \( U' \), one has \( (\varphi U)_*T_U \simeq T_{U'} \).

In particular the exact sequence

\[
0 \rightarrow \mathcal{O}_U \rightarrow (\varphi U)_*V_U \rightarrow (\varphi U)_*T_U \rightarrow 0
\]

is locally split and the image of \( \mathcal{O}_{U'} \simeq (\varphi U)_*\mathcal{O}_U(-k) \rightarrow (\varphi U)_*V_U \) is contained in the image of the splitting map. This property is preserved under the reflexive pull-back to \( \tilde{U} \), i.e. for every point of \( \tilde{U} \) the image of the injection \( i_{V_U} \) is not contained the image of \( j \). Since \( i_{E_U} \) is obtained from \( i_{V_U} \) by taking the quotient by \( j_*\mathcal{O}_{U} \), it is injective in every point.

□

**Remark.** Let \( M \) be a projective K3 surface with Picard number \( \rho(M) \geq 5 \). Then \( M \) admits an elliptic fibration [Huy16, Ch.11, Prop.1.3]. If \( Z_M \) is Stein, Theorem 6.6 in connection with the non-existence of \((-2)\)-curves shows that all the singular fibres are cuspidal elliptic curves. Looking at the Weierstraß model of the associated Jacobian fibration and using [Mir89, p.40] one can then show that the elliptic fibration is isotrivial with general fibre the elliptic curve with \( J \)-invariant 0. Thus Theorem 6.6 covers almost all the K3 surfaces with \( \rho(M) \geq 5 \).

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