CONVERGENCE OF FIXED POINT CONTINUATION ALGORITHMS FOR MATRIX RANK MINIMIZATION

DONALD GOLDFARB∗ AND SHIQIAN MA *

June 18, 2009. Revised Aug. 05, 2009. Revised June 28, 2010.

Abstract. The matrix rank minimization problem has applications in many fields such as system identification, optimal control, low-dimensional embedding etc. As this problem is NP-hard in general, its tightest convex relaxation, the nuclear norm minimization problem is often solved instead. Recently, Ma, Goldfarb and Chen proposed a fixed point continuation algorithm for solving the nuclear norm minimization problem [33]. By incorporating an approximate singular value decomposition technique in this algorithm, the solution to the matrix rank minimization problem is usually obtained. In this paper, we study the convergence/recoverability properties of the fixed point continuation algorithm and its variants for matrix rank minimization. Heuristics for determining the rank of the matrix when its true rank is not known are also proposed. Some of these algorithms are closely related to greedy algorithms in compressed sensing. Numerical results for these algorithms for solving affinely constrained matrix rank minimization problems are reported.

Key words. Matrix Rank Minimization, Matrix Completion, Greedy Algorithm, Fixed Point Method, Restricted Isometry Property, Singular Value Decomposition

1. Introduction. In this paper, we are interested in the affinely constrained matrix rank minimization (MRM) problem, which can be cast as

\[
\min \text{ rank}(X) \\
\text{s.t. } A(X) = b,
\]

where \( X \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^p \) and \( A : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^p \) is a linear map. Without loss of generality, we assume that \( m \leq n \) throughout this paper.

Problem (1.1) has applications in many fields such as system identification [32], optimal control [20, 16, 18], and low-dimensional embedding in Euclidean space [30], etc. For example, consider the problem of designing a low-order discrete-time controller for a plant, so that the step response of the combined controller and plant lies within specified bounds. Suppose the plant impulse response is \( h(t), t = 0, \ldots, N \), the controller impulse response is \( x(t), t = 0, \ldots, N \), and \( u(t) = 1, t = 0, \ldots, N \) is the step input. Then finding a low-order system is equivalent to solving the following problem:

\[
\min \text{ rank}(\mathcal{H}(x)) \\
\text{s.t. } b_l(t) \leq (h \ast x \ast u)(t) \leq b_u(t), t = 0, \ldots, N,
\]

where \( b_l \) and \( b_u \) are given lower and upper bounds on the step response, \( \ast \) denotes the convolution operator, and \( \mathcal{H}(x) \) is the Hankel matrix (see e.g., [17, 39]). (1.2) is just a specific application of (1.1).

A special case of (1.1) is the matrix completion problem:

\[
\min \text{ rank}(X) \\
\text{s.t. } X_{ij} = M_{ij}, \quad \forall (i, j) \in \Omega.
\]

This problem has applications in online recommendation system, collaborate filtering [40, 41], etc. The

∗Department of Industrial Engineering and Operations Research, Columbia University, New York, NY 10027. Email: {goldfarb,sm2756}@columbia.edu . Research supported in part by NSF Grant DMS 09-0712, ONR Grants N00014-03-0514 and N00014-08-I-111, and DOE Grants DE-FG01-92ER-25126 and DE-FG02-08ER-25856.
famous Netflix problem \[37\] is an example of a matrix completion problem. In it users provide ratings to some of the movies in a list of movies. Here \(M_{ij}\) is the rating given to \(j\)-th movie by \(i\)-th user. Since users only rate a limited number of movies in the list, we only know some of the entries of the matrix \(M\). The goal of the Netflix problem is to fill in the missing entries in this matrix. It is commonly believed that only a few factors contribute to people’s tastes in movies. Thus the matrix \(M\) will generally be of low-rank. Finding this low-rank completion to \(M\) is just the matrix completion problem \[1.3\].

If \(X\) is a diagonal matrix, then \[1.4\] becomes the compressed sensing problem \[8, 12\]:

\[
\min \|x\|_0 \quad \text{s.t.} \quad Ax = b,
\]

where \(A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m\), and \(\|x\|_0\), which is called the \(\ell_0\) norm, counts the number of nonzero elements in the vector \(x\). The compressed sensing problem, which is currently of great interest in signal processing, is NP-hard \[35\]. Recent studies in compressed sensing have shown that under certain randomness hypothesis, a limited number of measurements is sufficient to find the optimal solution to \[1.4\] by solving a convex relaxation of it. The convex envelope of the function \(\|x\|_0\) on the set \(\{x \in \mathbb{R}^n : \|x\|_\infty \leq 1\}\) is the \(\ell_1\) norm \(\|x\|_1 := \sum_i |x_i|\) \[22\]. Thus the closest convex relaxation of problem \[1.4\] is the convex problem:

\[
\min \|x\|_1 \quad \text{s.t.} \quad Ax = b.
\]

Many algorithms for solving \[1.4\] and \[1.5\] have been proposed. These include greedy algorithms \[12, 13, 45, 14, 11, 1, 2\] for \[1.4\] and convex optimization algorithms \[7, 19, 21, 25, 46\] for \[1.5\]. See \[10\] for more information on the theory and algorithms for compressed sensing.

The matrix rank minimization problem \[1.1\] is also NP-hard. To get a tractable problem, we can replace rank(\(X\)) by its convex envelope, the nuclear norm of \(X\) \[38\], as proposed by Fazel et al. \[16\]. The nuclear norm of \(X\), denoted by \(\|X\|_*\), is defined as the sum of the nonzero singular values of \(X\); i.e., if the singular values of \(X\) are \(\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > \sigma_{r+1} = \ldots = \sigma_m = 0\), then

\[
\|X\|_* = \sum_{i=1}^r \sigma_i.
\]

Thus, the nuclear norm relaxation of \[1.1\] is:

\[
\min \|X\|_* \quad \text{s.t.} \quad \mathcal{A}(X) = b.
\]

Let \(A\) be the matrix version of \(\mathcal{A}\), i.e., \(\mathcal{A}(X) = A \cdot \text{vec}(X)\), \(\text{vec}(X)\) is the vector obtained by stacking the columns of the matrix \(X\) in a natural order. Recht et al. \[38\] proved that if the entries of \(A\) obey some random distribution and the number of measurements \(p \geq Cr(m + n)\log(mn)\), then with very high probability, most \(m \times n\) matrices of rank \(r\) can be recovered by solving problem \[1.6\], where \(C\) is a positive constant; i.e., an optimal solution to \[1.6\] gives an optimal solution to \[1.1\].

If \(b\) is contaminated by noise, then \[1.6\] should be relaxed to

\[
\min \|X\|_* \quad \text{s.t.} \quad \|\mathcal{A}(X) - b\|_2 \leq \theta,
\]
where \( \theta > 0 \) is the noise level. The Lagrangian version of (1.7) can be written as

\[
\min \mu \| X \|_* + \frac{1}{2} \| A(X) - b \|_2^2,
\]

(1.8)

where \( \mu \) is a Lagrangian multiplier.

Several algorithms have been proposed for solving (1.1) and (1.6). Using the fact that (1.6) is equivalent to the semidefinite programming (SDP) problem

\[
\min_{X,W_1,W_2} \frac{1}{2} (\text{Tr}(W_1) + \text{Tr}(W_2))
\]

s.t.

\[
\begin{bmatrix}
W_1 & X \\
X^\top & W_2
\end{bmatrix} \succeq 0,
\]

(1.9)

\( A(X) = b \),

where \( \text{Tr}(W) \) denotes the sum of the diagonal elements of the square matrix \( W \). Recht, Fazel and Parrilo [38] and Liu and Vandenberghe [32] proposed interior point methods to solve this SDP. However, these interior point methods cannot be used to solve large problems. First-order methods were proposed by Cai, Candès and Shen [4] and Ma, Goldfarb and Chen [33] that can solve very large matrix rank minimization problems efficiently. One of the algorithms in [33], which is called FPCA (Fixed Point Continuation with Approximation SVD), almost always achieves the best recoverability. FPCA can recover \( m \times n \) matrices of rank \( r \) using \( p \) samples even when \( r \) is very close to \( r_{\text{max}} := \max \{ r | r(m + n - r)/p < 1 \} \). \( r_{\text{max}} \) is the largest rank of \( m \times n \) matrices that one can recover with only \( p \) samples. In this paper, we study the convergence/recoverability properties and numerical performance of FPCA and some of its variants. Our main contribution is a weakening of the conditions previously given by Lee and Bresler [28] required for the approximate recovery of a low-rank matrix.

**Notation.** We use \( \mathbb{R}_+^n \) to denote the nonnegative orthant of \( \mathbb{R}^n \). We use \( \mathcal{A}^* \) to denote the adjoint operator of \( \mathcal{A} \). We define the inner product of two matrices \( X \) and \( Y \in \mathbb{R}^{m \times n} \) to be \( \langle X,Y \rangle = \text{Tr}(X^\top Y) = \text{Tr}(Y^\top X) \), and denote by \( \| X \|_F = (\text{Tr}(X^\top X))^{1/2} \) the Frobenius norm of the matrix \( X \) and by \( \| x \|_2 \) the Euclidean norm of the vector \( x \). \( \| X \|_2 \) is the spectral norm of the matrix \( X \), which is defined as the largest singular value of \( X \). Henceforth, we will denote \( \mathcal{A}(X) \) by \( AX \) as this should not cause any confusion. For example, \( \mathcal{A}^*AX := \mathcal{A}^*(AX) \).

**Outline.** The rest of this paper is organized as follows. In Section 2 we review the role that the restricted isometry property plays in the theory of compressed sensing and matrix rank minimization. We also present three propositions from [28] that provide the bases for theoretical results that we give later in the paper. We review the Fixed Point Continuation (FPC) and FPC with Approximate SVD (FPCA) algorithms proposed in [33] in Section 3. We then address the first variant of FPCA, which we call iterative hard thresholding (IHT), and prove convergence results for it in Section 4. Section 5 is devoted to another variant of FPCA, which is called iterative hard thresholding with matrix shrinkage (IHTMS), and convergence results for it. We establish convergence/recoverability properties of FPCAr, a very close variant of FPCA, in Section 6. Some practical issues regarding numerical difficulties and ways to overcome them are discussed in Section 7. Finally, we give some numerical results obtained by applying these algorithms to both randomly created and real matrix rank minimization problems in Section 8.

2. **Restricted Isometry Property.** In compressed sensing and matrix rank minimization, the restricted isometry property (RIP) of the matrix \( A \) or linear operator \( \mathcal{A} \) plays a key role in the relationship
between the original combinatorial problem and its convex relaxation and their optimal solutions.

The definition of the RIP for matrix rank minimization is:

**Definition 2.1.** The linear operator \( A : \mathbb{R}^{m \times n} \to \mathbb{R}^p \) is said to satisfy the Restricted Isometry Property with the restricted isometry constant \( \delta_r(A) \) if \( \delta_r(A) \) is the minimum constant that satisfies

\[
(1 - \delta_r(A)) \| X \|_2^2 \leq \| AX \|_2^2 \leq (1 + \delta_r(A)) \| X \|_2^2,
\]

for all \( X \in \mathbb{R}^{m \times n} \) with \( \text{rank}(X) \leq r \). \( \delta_r(A) \) is called the RIP constant. Note that \( \delta_s \leq \delta_t \), if \( s \leq t \).

The RIP concept and the RIP constant \( \delta_r(A) \) play a central role in the theoretical developments of this paper. We first note that if the operator \( A \) has a nontrivial kernel, i.e., there exists \( X \in \mathbb{R}^{m \times n} \) such that \( AX = 0 \) and \( X \neq 0 \), then \( \delta_n(A) = 1 \). Second, since the rank of a matrix \( r \) must be an integer number, the smallest possible value of the restricted isometry constant is \( \delta_1(A) \). Finally, if we represent \( A \) in the coordinate form \( (AX)_i = \text{Tr}(A_iX), i = 1, \ldots, p \), then \( \delta_r(A) \) is related to the joint kernel of the matrices \( A_i \). For example, if there exists a matrix \( X \in \mathbb{R}^{m \times n} \) with rank \( r \) such that \( A_iX = 0, i = 1, \ldots, p \), then \( \delta_r(A) = 1 \). Our results in this paper do not apply to such a pathological case.

For matrix rank minimization (1.1), Recht et al. [38] proved the following results.

**Theorem 2.2** (Theorem 3.3 in [38]). Suppose that \( \text{rank}(X) \leq r, r \geq 1 \) and \( \delta_{2r}(A) < 0.1 \). Then (1.1) and (1.6) have the same optimal solution.

**Theorem 2.3** (Theorem 4.2 in [38]). Fix \( \delta \in (0, 1) \). If \( A : \mathbb{R}^{m \times n} \to \mathbb{R}^p \) is a nearly isometric random map (see Definition 4.1 in [38]), then for every \( 1 \leq r \leq m \), there exist constants \( c_0, c_1 > 0 \) depending only on \( \delta \) such that, with probability at least \( 1 - \exp(-c_1p) \), \( \delta_r(A) \leq \delta \) whenever \( p \geq c_0r(m + n) \log(mn) \).

Theorems 2.2 and 2.3 indicate that if \( A \) is a nearly isometric random map, then with very high probability, \( A \) will satisfy the RIP with a small RIP constant and thus we can solve (1.1) by solving its convex relaxation (1.6). As an example, if \( A \) is the matrix version of the operator \( A \), and its entries \( A_{ij} \) are independent, identically distributed (i.i.d.) Gaussian, i.e., \( A_{ij} \sim \mathcal{N}(0, 1/p) \), then \( A \) is a nearly isometric random map. For other nearly isometric random maps, see [38].

In Section 3, we will show empirically that when the entries of \( A \) are i.i.d. Gaussian, the algorithms proposed in this paper can solve the matrix rank minimization problem (1.1) very well.

It is worth noticing that the linear map \( A \) in the matrix completion problem (1.3) does not satisfy the RIP. A counter example is given in [9]. For theories and algorithms for matrix completion problems, we refer to [0, 1, ..., 20]. For theories and algorithms for matrix completion problems, we refer to [0, 1, ..., 20].

In our proofs of the convergence of FPCA variants, we need \( A \) to satisfy the RIP. Before we describe some properties of the RIP that we will use in our proofs, we need the following definitions.

**Definition 2.4 (Orthonormal basis of a subspace).** Given a set of rank-one matrices \( \Psi = \{ \psi_1, \ldots, \psi_r \} \), there exists a set of orthonormal matrices \( \Gamma = \{ \gamma_1, \ldots, \gamma_s \} \), i.e., \( \langle \gamma_i, \gamma_j \rangle = 0 \), for \( i \neq j \) and \( \| \gamma_i \|_F = 1 \) for all \( i \), such that \( \text{span}(\Gamma) = \text{span}(\Psi) \). We call \( \Gamma \) an orthonormal basis for the subspace \( \text{span}(\Psi) \). We use \( P_{\Gamma}X \) to denote the projection of \( X \) onto the subspace \( \text{span}(\Gamma) \). Note that \( P_{\Gamma}X = P_{\Psi}X \) and \( \text{rank}(P_{\Gamma}X) \leq r, \forall X \in \mathbb{R}^{m \times n} \).

**Definition 2.5 (SVD basis of a matrix).** Assume that the rank-\( r \) matrix \( X_r \) has the singular value decomposition \( X_r = \sum_{i=1}^{r} \sigma_i u_i v_i^\top \). \( \Gamma := \{ u_1v_1^\top, u_2v_2^\top, \ldots, u_r v_r^\top \} \) is called an SVD basis for the matrix \( X_r \). Note that elements in \( \Gamma \) are orthogonal unit-norm rank-one matrices.
We now list some important properties of linear operators that satisfy RIP.

**Proposition 2.6.** Suppose that the linear operator \( A : \mathbb{R}^{m \times n} \to \mathbb{R}^p \) satisfies the RIP with constant \( \delta_r(A) \). Let \( \Psi \) be an arbitrary orthonormal subset of \( \mathbb{R}^{m \times n} \) such that \( \text{rank}(P_\Psi X) \leq r, \forall X \in \mathbb{R}^{m \times n} \). Then, for all \( b \in \mathbb{R}^p \) and \( X \in \mathbb{R}^{m \times n} \), the following properties hold:

\[
\begin{align*}
\| P_\Psi A^* b \|_F & \leq \sqrt{1 + \delta_r(A)} \| b \|_2 \\
(1 - \delta_r(A)) \| P_\Psi X \|_F & \leq \| P_\Psi A^* A P_\Psi X \|_F \leq (1 + \delta_r(A)) \| P_\Psi X \|_F
\end{align*}
\]

**Proposition 2.7.** Suppose that the linear operator \( A : \mathbb{R}^{m \times n} \to \mathbb{R}^p \) satisfies the RIP with constant \( \delta_r(A) \). Let \( \Psi, \Psi' \) be arbitrary orthonormal subsets of \( \mathbb{R}^{m \times n} \) such that \( \text{rank}(P_{\Psi \cup \Psi'} X) \leq r \), for any \( X \in \mathbb{R}^{m \times n} \). Then the following inequality holds:

\[
\| P_\Psi A^*(I - P_\Psi) X \|_F \leq \delta_r(A) \| (I - P_\Psi) X \|_F, \forall X \in \text{span}(\Psi').
\]

**Proposition 2.8.** If a linear map \( A : \mathbb{R}^{m \times n} \to \mathbb{R}^p \) satisfies

\[
\|AX\|_2 \leq (1 + \delta_r(A)) \|X\|_2, \forall X \in \mathbb{R}^{m \times n}, \text{rank}(X) \leq r,
\]

then

\[
\|AX\|_2 \leq \sqrt{1 + \delta_r(A)} \left( \|X\|_F + \frac{1}{\sqrt{r}} \|X\|_* \right), \forall X \in \mathbb{R}^{m \times n}.
\]

The proofs of Propositions 2.6, 2.7 and 2.8 are given in the Appendix.

### 3. FPC Revisited.

To describe FPC and FPCA and its variants, we need the following definitions.

**Definition 3.1.** Assume that the singular value decomposition of the matrix \( X \in \mathbb{R}^{m \times n} \) is given by \( X = \sum_{i=1}^m \sigma_i u_i v_i^\top \) with \( \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m \). Then the best rank-\( r \) approximation \( R_r(X) \) to the matrix \( X \) is defined as

\[
R_r(X) = \sum_{i=1}^r \sigma_i u_i v_i^\top.
\]

\( R_r : \mathbb{R}^{m \times n} \to \mathbb{R}^{m \times n} \) is also called the hard thresholding/shrinkage operator with threshold \( r \).

**Definition 3.2.** Assume the SVD of the matrix \( X \) is given by \( X = UD_\sigma V^\top \). For \( \nu > 0 \), the matrix shrinkage operator \( S_\nu(X) \) is defined as:

\[
S_\nu(X) = UD_\sigma ((\sigma - \nu)_+) V^\top,
\]

where \( a_+ \) is defined as \( a_+ \) := \( \max(a, 0) \). This matrix shrinkage operator is also called the soft shrinkage operator with threshold \( \nu \).

FPC, whose development was motivated by the work on \( \ell_1 \) regularized problems in [21], is based on applying an operator splitting technique to the optimality conditions for (1.5). Note that \( X^* \) is the optimal

---

\(^1\)Propositions 2.6 and 2.8 were first proposed by Lee and Bresler without proof in [28]. Proofs of Propositions 2.6 and 2.8 were provided later in [29].
solution to (1.8) if and only if

\[ 0 \in \mu \partial \| X^* \|_* + g(X^*), \]

where \( g(X^*) = A^*(AX^* - b) \) is the gradient of the least squares term \( \frac{1}{2} \| AX^* - b \|_2^2 \), and \( \partial \| X^* \|_* \) is the subgradient of the nuclear norm \( \| X^* \|_* \) of \( X^* \). According to [3], the subgradient of \( \| X \|_* \) is given by

\[ \partial \| X \|_* = \{ UV^T + W : U^TW = 0, WV = 0, \| W \| \leq 1 \}, \]

where the SVD of \( X \) is given by \( X = U \text{Diag}(\sigma)V^\top \), \( U \in \mathbb{R}^{m \times r} \), \( V \in \mathbb{R}^{n \times r} \), \( \sigma \in \mathbb{R}^{r^+} \).

Now based on the optimality condition (3.1), we can develop a fixed point iterative scheme for solving (1.8) by adopting an operator splitting technique. Note that (3.1) is equivalent to

\[ 0 \in \tau \mu \partial \| X^* \|_* + X^* - (X^* - \tau g(X^*)) \]

for any \( \tau > 0 \). If we let

\[ Y^* = X^* - \tau g(X^*), \]

then (3.3) can be rewritten as

\[ 0 \in \tau \mu \partial \| X^* \|_* + X^* - Y^*, \]

i.e., \( X^* \) is the optimal solution to

\[ \min_{X \in \mathbb{R}^{m \times n}} \tau \mu \| X \|_* + \frac{1}{2} \| X - Y^* \|_F^2. \]

It is known that \( S_{\tau \mu}(Y^*) \) gives the optimal solution to (3.5) [33]. Hence, the following fixed point iterative scheme can be given for solving (1.8):

\[ \begin{cases} 
Y^{k+1} = X^k - \tau g(X^k) \\
X^{k+1} = S_{\tau \mu}(Y^{k+1}).
\end{cases} \]

The following convergence result is proved in [33].

**Theorem 3.3 (Theorem 4 in [33])**. Assume \( \tau \in (0, 2/\lambda_{max}(A^*A)) \), where \( \lambda_{max}(A^*A) \) denotes the largest eigenvalue of \( A^*A \). The sequence \( \{X^k\} \) generated by the fixed point iterations (3.6) converges to some \( X^* \in X^* \), where \( X^* \) is the optimal set of problem (1.8).

Note that in every iteration of (3.6), an SVD has to be computed to perform the matrix shrinkage operation, which is very expensive. Consequently, FPCA uses an approximate SVD to replace the whole SVD, i.e., it computes only a rank-\( r \) approximation to \( Y^{k+1} \). Note that there are many ways to get a rank-\( r \) approximation to \( Y^{k+1} \). Here we assume that the best rank-\( r \) approximation \( R_r(Y^{k+1}) \) is used. In Section 7, we discuss a Monte Carlo method to approximately compute \( R_r(Y^{k+1}) \) since computing \( R_r(Y^{k+1}) \) exactly is still expensive if \( r \) is not very small and the matrices are large. By adopting a continuation strategy for the parameter \( \mu \) in (3.6), we arrive at the following FPCA algorithm (Algorithm 1) as proposed in [33].

We can see that FPCA adopts three techniques, hard thresholding, soft shrinkage and continuation. These three techniques have different properties which, when combined, produce a very robust and efficient
**Algorithm 1:** Fixed Point Continuation with Approximate SVD for MRM (FPCA)

Initialization: Set \( X := X^0 \).

for \( \mu = \mu_1, \mu_2, \ldots, \mu_L = \bar{\mu} \) do

while not converged do

\[ Y := X - \tau A^*(AX - b). \]

choose \( r \).

\[ X := S_{\tau \mu}(R_r(Y)). \]

algorithm with great recoverability properties. By using only one or two of these three techniques, we get different variants of FPCA. We will study two of these variants, Iterative Hard Thresholding (IHT) and Iterative Hard Thresholding with soft Matrix Shrinkage (IHTMS) in Sections 4 and 5 respectively, and FPCA with given rank \( r \) (FPCAr) in Section 6.

In the following sections, we assume that the rank \( r \) of the optimal solution is given and we compute the best rank-\( r \) approximation to \( Y \) in each iteration. In Section 7, we give a heuristic for choosing \( r \) in each iteration if \( r \) is unknown and use the fast Monte Carlo algorithm proposed in [15] to compute a rank-\( r \) approximation to \( Y \).

4. Iterative Hard Thresholding. In this section, we study a variant of FPCA that we call Iterative Hard Thresholding (IHT) because of its similarity to the algorithm in [2] for compressed sensing.

If in FPCA, we assume that the rank \( r \) is given, we do not do any continuation or soft shrinkage, and always choose the stepsize \( \tau \) equal to one, then FPCA becomes Algorithm 2 (IHT). At each iteration of IHT, we first perform a gradient step \( Y^{k+1} := X^k - A^*(AX^k - b) \), and then apply hard thresholding to the singular values of \( Y^{k+1} \), i.e., we only keep the largest \( r \) singular values of \( Y^{k+1} \), to get \( X^{k+1} \).

**Algorithm 2:** Iterative Hard Thresholding (IHT)

Initialization: Given \( X^0, r \).

for \( k = 0, 1, \ldots \) do

\[ Y^{k+1} := X^k - A^*(AX^k - b). \]

\[ X^{k+1} := R_r(Y^{k+1}). \]

As previously mentioned, IHT is closely related to an algorithm proposed by Blumensath and Davies [2] for compressed sensing. Their algorithm for solving (1.4) performs the following iterative scheme:

\[
\begin{align*}
  y^{k+1} &= x^k - \tau g(x^k) \\
  x^{k+1} &= H_r(y^{k+1}),
\end{align*}
\]

(4.1)

where \( g(x^k) = A^T(Ax^k - b) \), \( H_r(y) \) is the hard thresholding operator that sets all but the largest (in magnitude) \( r \) elements of \( y \) to zero. Clearly, IHT for matrix rank minimization and compressed sensing are the same except that the shrinkage operator in the matrix case is applied to the singular values while in the compressed sensing case it is applied to the solution vector.

To prove the convergence/recoverability properties of IHT for matrix rank minimization, we need the following lemma.

**Lemma 4.1.** Suppose \( X := R_r(Y) \) is the best rank-\( r \) approximation to the matrix \( Y \), and \( \Gamma \) is an SVD
basis of X. Then for any rank-r matrix X_r and SVD basis \Gamma_r of X_r, we have

\begin{equation}
\|P_B X - P_B Y\|_F \leq \|P_B X_r - P_B Y\|_F,
\end{equation}

where B is any orthonormal set of matrices satisfying span(\Gamma \cup \Gamma_r) \subseteq span(B).

**Proof.** Since X is the best rank-r approximation to Y and rank(X_r) = r, \(\|X - Y\|_F \leq \|X_r - Y\|_F\).

Hence,

\[\|P_B(X - Y)\|_F^2 + \|(I - P_B)(X - Y)\|_F^2 \leq \|P_B(X_r - Y)\|_F^2 + \|(I - P_B)(X_r - Y)\|_F^2.\]

Since \((I - P_B)X = 0\) and \((I - P_B)X_r = 0\), this reduces to (12).

For IHT, we have the following convergence results, whose proofs essentially follow those given by Blumensath and Davies \[2\] for IHT for compressed sensing. Our first result considers the case where the desired solution \(X_r\) satisfies a perturbed linear system of equations \(AX + e = b\).

**Theorem 4.2.** Suppose that \(b = AX_r + e\), where \(X_r\) is a rank-r matrix, and \(A\) has the RIP with \(\delta_{3r}(A) < 1/\sqrt{32}\). Then, at iteration \(k\), IHT will recover an approximation \(X^k\) satisfying

\begin{equation}
\|X_r - X^k\|_F \leq 2^{-k} \|X_r - X^0\|_F + 4.34\|e\|_2.
\end{equation}

Furthermore, after at most \(k^* := \lceil \log_2 \left( \|X_r - X^0\|_F / \|e\|_2 \right) \rceil\) iterations, IHT estimates \(X_r\) with accuracy

\begin{equation}
\|X_r - X^{k^*}\|_F \leq 5.34\|e\|_2.
\end{equation}

**Proof.** Let \(\Gamma_r\) and \(\Gamma^k\) denote SVD bases of \(X_r\) and \(X^k\), respectively, and \(B_k\) denote an orthonormal basis of the subspace \(\text{span}(\Gamma_r \cup \Gamma^k)\). Let \(Z^k := X_r - X^k\) denote the residual at iteration \(k\). Since \(P_{B_{k+1}}X_r = X_r\) and \(P_{B_{k+1}}X^{k+1} = X^{k+1}\), it follows first from the triangle inequality and then from Lemma 4.1 that

\begin{equation}
\|X_r - X^{k+1}\|_F \leq \|P_{B_{k+1}}X_r - P_{B_{k+1}}Y^{k+1}\|_F + \|P_{B_{k+1}}X^{k+1} - P_{B_{k+1}}Y^{k+1}\|_F
\end{equation}

\[\leq 2 \|P_{B_{k+1}}X_r - P_{B_{k+1}}Y^{k+1}\|_F.\]

Using the fact that \(b = AX_r + e\), \(Y^{k+1} = X^k - A^*(AX^k - AX_r - e) = X^k + A^*(AZ^k + e)\). Hence, from (14),

\[\|X_r - X^{k+1}\|_F \leq 2 \|P_{B_{k+1}}X_r - P_{B_{k+1}}X^k\|_F + \|P_{B_{k+1}}X^k - P_{B_{k+1}}Y^{k+1}\|_F
\]

\[\leq 2 \|P_{B_{k+1}}X_r - P_{B_{k+1}}X^k - P_{B_{k+1}}A^*A(P_{B_{k+1}}Z^k + (I - P_{B_{k+1}})Z^k) - P_{B_{k+1}}A^*e\|_F
\]

\[\leq 2 \|P_{B_{k+1}}Z^k - P_{B_{k+1}}A^*A(P_{B_{k+1}}Z^k + (I - P_{B_{k+1}})Z^k)\|_F + \|P_{B_{k+1}}A^*e\|_F
\]

\[\leq 2 \|P_{B_{k+1}}A^*A(P_{B_{k+1}}Z^k) - P_{B_{k+1}}Z^k\|_F + \|P_{B_{k+1}}A^*e\|_F
\]

Since \(\text{rank}(P_{B_{k+1}}) \leq 2r, \forall X \in \mathbb{R}^{m \times n}\), by applying (22) in Proposition 2.6, we get,

\[\|P_{B_{k+1}}A^*e\|_F \leq \sqrt{1 + \delta_{2r}(A)}\|e\|_2.\]

Since \(P_{\Psi}P_{\Psi} = P_{\Psi}\), it follows from (23) in Proposition 2.6 that the eigenvalues of the linear operator \(P_{\Psi}A^*AP_{\Psi}\) are in the interval \([1 - \delta_r(A), 1 + \delta_r(A)]\). Letting \(\Psi = B_{k+1}\), it follows that the eigenvalues of
\(P_{B_{k+1}}A^*AP_{B_{k+1}}\) lie in the interval \([1 - \delta_{2r}(A), 1 + \delta_{2r}(A)]\). Hence the eigenvalues of \(I - P_{B_{k+1}}A^*AP_{B_{k+1}}\) are bounded above by \(\delta_{2r}(A)\) and it follows that

\[
\|(I - P_{B_{k+1}}A^*AP_{B_{k+1}})P_{B_{k+1}}Z^k\|_F \leq \delta_{2r}(A) \|P_{B_{k+1}}Z^k\|_F.
\]

Also, since \(P_{B_k}Z^k = Z^k, Z^k \in \text{span}(B_k)\) and \(\text{rank}(P_{B_k} \cup B_{k+1}X) \leq 3r, \forall X \in \mathbb{R}^{m \times n}\), by applying Proposition 2.7 we get

\[
\|P_{B_{k+1}}A^*(I - P_{B_{k+1}})Z^k\|_F \leq \delta_{3r}(A) \|(I - P_{B_{k+1}})Z^k\|_F.
\]

Thus, since \(\delta_{2r}(A) \leq \delta_{3r}(A)\),

\[
\|X_r - X^{k+1}\|_F \leq 2\delta_{2r}(A) \|P_{B_{k+1}}Z^k\|_F + 2\delta_{3r}(A) \|(I - P_{B_{k+1}})Z^k\|_F + 2\sqrt{1 + \delta_{2r}(A)}\|e\|_2
\]

\[
\leq 2\sqrt{2}\delta_{3r}(A) \|Z^k\|_F + 2\sqrt{1 + \delta_{3r}(A)}\|e\|_2.
\]

By assumption, \(\delta_{3r}(A) < 1/\sqrt{32}\); hence we have

\[
(4.6) \quad \|Z^{k+1}\|_F < \frac{1}{2} \|Z^k\|_F + 2.17\|e\|_2.
\]

Iterating this inequality, we get (4.8).

To ensure that the recovery accuracy \(\|Z^k\|_F \leq 5.34\|e\|_2\), it is sufficient to let \(2^{-k} \|X_r - X^0\|_F \leq \|e\|_2\), which implies that when \(k \geq k^* := \lceil\log_2 (\|X_r - X^0\|_F / \|e\|_2)\rceil\), 4.3 holds. \(\square\)

For an arbitrary matrix \(X\), we have the following result.

**Theorem 4.3.** Suppose that \(b = AX + e\), where \(X\) is an arbitrary matrix, and \(A\) has the RIP with \(\delta_{3r}(A) < 1/\sqrt{32}\). Let \(X_r\) be the best rank-\(r\) approximation to \(X\). Then, at iteration \(k\), IHT will recover an approximation \(X^k\) satisfying

\[
(4.7) \quad \|X - X^k\|_F \leq 2^{-k} \|X_r - X^0\|_F + 5.71 \tilde{\varepsilon}_r,
\]

where

\[
(4.8) \quad \tilde{\varepsilon}_r = \|X - X_r\|_F + \frac{1}{\sqrt{r}} \|X - X_r\|_\ast + \|e\|_2,
\]

is called the unrecoverable energy (see [30]). Furthermore, after at most \(k^* := \lceil\log_2 (\|X_r - X^0\|_F / \tilde{\varepsilon}_r)\rceil\) iterations, IHT estimates \(X\) with accuracy

\[
(4.9) \quad \|X - X^{k^*}\|_F \leq 6.71 \tilde{\varepsilon}_r.
\]

**Proof.** From Theorem 4.2 with \(\varepsilon = A(X - X_r) + e\) instead of \(e\), we have

\[
\|X_r - X^k\|_F = \|Z^k\|_F \leq 2^{-k} \|X_r - X^0\|_F + 4.34\|\tilde{\varepsilon}\|_2.
\]
By Proposition 2.8 we know that

$$\|\tilde{e}\|_2 \leq \|A(X - X_r)\|_F + \|e\|_2 \leq \sqrt{1 + \delta_r(A)} \left(\|X - X_r\|_F + \frac{1}{\sqrt{r}} \|X - X_r\|_*\right) + \|e\|_2.$$  

Thus we have from the triangle inequality and (4.8)

$$\|X - X^k\|_F \leq \|X_r - X^k\|_F + \|X - X_r\|_F$$

$$\leq 2^{-k} \|X_r - X^0\|_F + 4.34\|\tilde{e}\|_2 + \|X - X_r\|_F$$

$$\leq 2^{-k} \|X_r - X^0\|_F + (4.34\sqrt{1 + \delta_r(A)} + 1)\tilde{\epsilon}_r$$

$$\leq 2^{-k} \|X_r - X^0\|_F + 5.71\tilde{\epsilon}_r.$$  

This proves (4.7).

Furthermore, if we want $\|X - X^k\|_F \leq 6.71\tilde{\epsilon}_r$, it is sufficient to require $2^{-k} \|X_r - X^0\|_F \leq \tilde{\epsilon}_r$. Therefore, after at most $k^* := \lceil\log_2 (\|X_r - X^0\|_F / \tilde{\epsilon}_r)\rceil$ iterations, (5.1) holds.

Similar bounds on the RIP constant for an approximate recovery were obtained by Lee and Bresler [28, 27] for affinely constrained matrix rank minimization and by Lee and Bresler for ellipsoidally constrained matrix rank minimization [29]. The results in Theorems 4.2 and 4.3 improve the previous results for affinely constrained matrix rank minimization in [28, 27]. Specifically, Theorems 4.2 and 4.3 require the RIP constant $\delta_{3r} < 1/\sqrt{32} \approx 0.1768$, while the result in [28, 27] requires $\delta_{3r} \leq 0.04$. The IHT algorithm for matrix rank minimization has also been independently studied by Meka, Jain and Dhillon in [34], who have obtained very different results than those in Theorems 4.2 and 4.3.

5. Iterative Hard Thresholding with Matrix Shrinkage. We study another variant of FPCA in this section. If in each iteration of IHT, we perform matrix shrinkage to $R_r(Y)$ with fixed thresholding $\mu > 0$, we get the following algorithm (Algorithm 3), which we call Iterative Hard Thresholding with Matrix Shrinkage (IHTMS). Note that $S_{\mu}(R_r(Y)) = R_r(S_{\mu}(Y))$, 4.7 and $Y$.

Algorithm 3: Iterative Hard Thresholding with Matrix Shrinkage (IHTMS)

Initialization: Given $X^0, \mu$ and $r$.

for $k = 0, 1, \ldots$ do

- $Y^{k+1} := X^k - A^* (AX^k - b)$.
- $X^{k+1} := R_r(S_{\mu}(Y^{k+1}))$.

For IHTMS, we have the following convergence results.

Theorem 5.1. Suppose that $b = AX_r + e$, where $X_r$ is a rank-$r$ matrix, and $\mathcal{A}$ has the RIP with $\delta_{3r}(\mathcal{A}) < 1/\sqrt{32}$. Then, at iteration $k$, IHTMS will recover an approximation $X^k$ satisfying

$$\|X_r - X^k\|_F \leq 2^{-k} \|X_r - X^0\|_F + 4.34\|e\|_2 + 4\mu\sqrt{m}. \quad (5.1)$$

Furthermore, after at most $k^* := \lceil\log_2 (\|X_r - X^0\|_F / (\|e\|_2 + \mu\sqrt{m}))\rceil$ iterations, IHTMS estimates $X_r$ with accuracy

$$\|X_r - X^{k^*}\|_F \leq 5.34\|e\|_2 + 5\mu\sqrt{m}. \quad (5.2)$$
Proof. Using the same notation as in the proof of Theorem 4.2 we know that $P_{B_{k+1}}X_r = X_r$ and $P_{B_{k+1}}X^{k+1} = X^{k+1}$. Using the triangle inequality we get,

$$\|X_r - X^{k+1}\|_F \leq \|P_{B_{k+1}}X_r - P_{B_{k+1}}Y^{k+1}\|_F$$

(5.3)

$$+ \|P_{B_{k+1}}X^{k+1} - P_{B_{k+1}}S_\mu(Y^{k+1})\|_F$$

$$+ \|P_{B_{k+1}}S_\mu(Y^{k+1}) - P_{B_{k+1}}Y^{k+1}\|_F.$$  

Since $X^{k+1}$ is the best rank-$r$ approximation to $S_\mu(Y^{k+1})$, by applying Lemma 4.1 we get

$$\|P_{B_{k+1}}X^{k+1} - P_{B_{k+1}}S_\mu(Y^{k+1})\|_F \leq \|P_{B_{k+1}}X_r - P_{B_{k+1}}S_\mu(Y^{k+1})\|_F$$

(5.4)

$$\leq \|P_{B_{k+1}}X_r - P_{B_{k+1}}Y^{k+1}\|_F$$

$$+ \|P_{B_{k+1}}S_\mu(Y^{k+1}) - P_{B_{k+1}}Y^{k+1}\|_F.$$  

Therefore, by combining (5.3), (5.4) and noticing that

$$\|P_{B_{k+1}}S_\mu(Y^{k+1}) - P_{B_{k+1}}Y^{k+1}\|_F \leq \|S_\mu(Y^{k+1}) - Y^{k+1}\|_F \leq \mu \sqrt{m},$$

we have

$$\|X_r - X^{k+1}\|_F \leq 2 \|P_{B_{k+1}}X_r - P_{B_{k+1}}Y^{k+1}\|_F + 2 \mu \sqrt{m}.$$  

Using an identical argument following (5.3) in the proof of Theorem 4.2 we get

$$\|X_r - X^{k+1}\|_F \leq 2 \sqrt{2} \delta_{3r}(A) \|Z^k\|_F + 2 \sqrt{1 + \delta_{3r}(A)} \|e\|_2 + 2 \mu \sqrt{m}.$$  

Now since $\delta_{3r}(A) < 1/\sqrt{32}$, we have

$$\|Z^{k+1}\|_F \leq \frac{1}{2} \|Z^k\|_F + 2.17 \|e\|_2 + 2 \mu \sqrt{m},$$

which implies that (5.1) holds. Hence (5.2) holds if $k^* := \lceil \log_2 \left( \|X_r - X^0\|_F / (\|e\|_2 + \mu \sqrt{m}) \right) \rceil$. \hfill \Box

For arbitrary $X$, we have the following results.

Theorem 5.2. Suppose that $b = AX + e$, where $X$ is an arbitrary matrix, and $A$ has the RIP with $\delta_{3r}(A) < 1/\sqrt{32}$. Let $X_r$ be the best rank-$r$ approximation to $X$. Then, at iteration $k$, IHTMS will recover an approximation $X^k$ satisfying

$$\|X - X^k\|_F \leq 2^{-k} \|X_r - X^0\|_F + 5.71 \hat{\epsilon}_r + 4 \mu \sqrt{m},$$

(5.5)

where $\hat{\epsilon}_r$ is defined by (4.8). Furthermore, after at most $k^* := \lceil \log_2 \left( \|X_r - X^0\|_F / (\hat{\epsilon}_r + \mu \sqrt{m}) \right) \rceil$ iterations, IHTMS estimates $X$ with accuracy

$$\|X - X^{k^*}\|_F \leq 6.71 \hat{\epsilon}_r + 5 \mu \sqrt{m}.$$  

(5.6)

Proof. The proof of (5.5) is identical to the proof of (4.7) in Theorem 4.3 except that (5.1) is used instead of (4.3). It also immediately follows from (5.5) that (5.6) holds for $k^* := \lceil \log_2 \left( \|X_r - X^0\|_F / (\hat{\epsilon}_r + \mu \sqrt{m}) \right) \rceil$. \hfill \Box
6. FPCA with Given Rank $r$. In this section, we study the FPCA when rank $r$ is known and the stepsize is always chosen as $\tau = 1$. This is equivalent to applying a continuation strategy to $\mu$ in IHTMS. We call this algorithm FPCAr (see Algorithm 4 below). The parameter $\eta$, determines the rate of reduction of the consecutive $\mu_j$ in continuation, i.e.,

\[
\mu_{j+1} = \max\{\mu_j, \eta\}, j = 1, \ldots, L - 1
\]

For FPCAr, we have the following convergence results.

**Algorithm 4**: FPCA with given rank $r$ (FPCAr)

| Input |
|-------|
| $X^{(0)}_r$, $r$, $\mu_1 > \mu_2 \ldots > \mu_L = \bar{\mu}$. |

for $j = 1, \ldots, L$ do

for $k = 0, 1, \ldots$, until convergence do

\[
\begin{aligned}
Y_{(j)}^{k+1} &= X_{(j)}^k - A^* (AX_{(j)}^k - b), \\
X_{(j)}^{k+1} &= S_\mu \left( R_{r} \left( Y_{(j)}^{k+1} \right) \right).
\end{aligned}
\]

Set $X^{(0)}_{(j+1)} = X_{(j)}^{k+1}$.

Output: $X^* := X^{(0)}_{(L+1)}$.

**Theorem 6.1.** Suppose that $b = AX_r + e$, where $X_r$ is a rank-$r$ matrix, and $A$ has the RIP with $\delta_{3r}(A) < 1/\sqrt{32}$. Also, suppose in FPCAr, after $K_j$ iterations with fixed $\mu = \mu_j$, we obtain a solution $X_{(j)}^{(K_j)}$ that is then set to the initial point $X^{(0)}_{(j+1)}$ for the next continuation subproblem $\mu = \mu_{j+1}$. Then FPCAr will recover an approximation $X_{(L)}^{(K_L)}$ that satisfies

\[
\left\| X_r - X_{(L)}^{(K_L)} \right\|_F \leq \left( 2^{-\sum_{i=1}^{K_j} K_i} \right) \left\| X_r - X^{(0)}_r \right\|_F + \left( \sum_{j=2}^{L} 2^{-\sum_{i=j}^{L} K_i} + 1 \right) 4.34 \|e\|_2 \]

\[
+ \left( \sum_{j=2}^{L} \left( 2^{-\sum_{i=j}^{L} K_i} \right) \mu_{j-1} + \mu_L \right) 4\sqrt{m}.
\]

**Proof.** For $X_{(1)}^{(K_1)}$, which is obtained by setting $\mu = \mu_1$ in the first $K_1$ iterations, we get from Theorem 6.1 that if $\delta_{3r}(A) < 1/\sqrt{32}$,

\[
\left\| X_r - X_{(1)}^{(K_1)} \right\|_F \leq \left( 2^{-K_1} \right) \left\| X_r - X^{(0)}_r \right\|_F + 4.34 \|e\|_2 + 4\mu_1\sqrt{m}.
\]

Then from iteration $K_1 + 1$ to $K_1 + K_2$, we fix $\mu = \mu_2$. Again by Theorem 6.1 we get

\[
\left\| X_r - X_{(2)}^{(K_2)} \right\|_F \leq \left( 2^{-K_2} \right) \left\| X_r - X_{(1)}^{(K_1)} \right\|_F + 4.34 \|e\|_2 + 4\mu_2\sqrt{m}.
\]

By substituting (6.3) into (6.4), we get

\[
\left\| X_r - X_{(2)}^{(K_2)} \right\|_F \leq \left( 2^{-(K_1+K_2)} \right) \left\| X_r - X^{(0)}_r \right\|_F + \left( 2^{-K_2} + 1 \right) 4.34 \|e\|_2
\]

\[
+ \left( 2^{-K_2} + 4\mu_1\sqrt{m} + 4\mu_2\sqrt{m} \right).
\]
Repeating this procedure we can get the following result for $X_{(L)}^{(K_L)}$.

$$\left\| X_r - X_{(L)}^{(K_L)} \right\|_F \leq \left( 2 - \sum_{i=1}^{L} K_i \right) \left\| X_r - X^0 \right\|_F + \left( \sum_{j=2}^{L} 2^{-\sum_{i=j}^{L} K_i} + 1 \right) 4.34 \| \varepsilon \|_2$$

(6.6)

$$+ \left( \sum_{j=2}^{L} \left( 2^{-\sum_{i=j}^{L} K_i} \right) \mu_{j-1} + \mu_L \right) 4\sqrt{m}.$$ 

So as long as $\mu_L$ is small and $K_L$ is large, the recovery error will be very small. $\blacksquare$

For arbitrary $X$, we have the following convergence results.

**Theorem 6.2.** Suppose that $b = AX + e$, where $X$ is an arbitrary matrix. Let $X_r$ be the best rank-$r$ approximation to $X$. With the same notation and under the same conditions as in Theorem 6.1, FPCA will recover an approximation $X_{(L)}^{(K_L)}$ that satisfies

$$\left\| X - X_{(L)}^{(K_L)} \right\|_F \leq \left( 2 - \sum_{i=1}^{L} K_i \right) \left\| X_r - X^0 \right\|_F + \left( \sum_{j=2}^{L} 2^{-\sum_{i=j}^{L} K_i} + 1 \right) 4.71 \tilde{\varepsilon}_r$$

(6.7)

$$+ \left( \sum_{j=2}^{L} \left( 2^{-\sum_{i=j}^{L} K_i} \right) \mu_{j-1} + \mu_L \right) 4\sqrt{m},$$

where $\tilde{\varepsilon}_r$ is defined by (4.8).

**Proof.** We skip the proof here since it is similar to the proof of Theorem 6.1 $\blacksquare$

7. **Practical Issues.** In practice, the rank $r$ of the optimal solution is usually unknown in practice. Thus, in every iteration, we need to determine $r$ appropriately. We propose some heuristics for doing this here. We start with $r := r_{\text{max}}$. So $X^1$ is a rank-$r_{\text{max}}$ matrix. For the $k$-th iteration ($k \geq 2$), $r$ is chosen as the number of singular values of $X^{k-1}$ which are greater than $\varepsilon_s \sigma_1^{k-1}$, where $\sigma_1^{k-1}$ is the largest singular value of $X^{k-1}$ and $\varepsilon_s \in (0, 1)$ is a given tolerance. Sometimes the given tolerance truncates too many of the singular values, so we need to increase $r$ occasionally. One way to do this is to increase $r$ by 1 whenever the Frobenius norm of gradient $g$ increased by more than 10 times. We tested this heuristic for determining $r$ extensively. It enables our algorithms to achieve very good recoverability and appears to be very robust. For many examples, our algorithms can recover matrices whose rank is almost $r_{\text{max}}$ with a limited number of measurements.

Another issue in practice is concerned with the SVD computation. Note that in IHT, IHTMS and FPCA, we need to compute the best rank-$r$ approximation to $Y^{k+1}$ at every iteration. This can be very expensive even if we use a state-of-the-art code like PROPACK [20], especially when the rank of the matrix is relatively large. Therefore, we used instead the Monte Carlo algorithm LinearTimeSVD proposed in [15] to approximate the best rank-$r$ approximation. For a given matrix $A \in \mathbb{R}^{m \times n}$, and parameters $\varepsilon_s, k_s \in \mathbb{Z}^+$ with $1 \leq k_s \leq c_s \leq n$ and $\{p_i\}_{i=1}^{n}$, $p_i \geq 0$, $\sum_{i=1}^{n} p_i = 1$, this algorithm returns an approximation to the largest $k_s$ singular values and corresponding left singular vectors of the matrix $A$ in $O(m + n)$ time. The LinearTimeSVD Algorithm, which we found to be much faster than PROPACK, is outlined below in Algorithm 6. The outputs $\sigma_t(C), t = 1, \ldots, k_s$ are approximations to the largest $k_s$ singular values and
approximate SVDs. Our numerical experiments that our algorithms are very robust and do not rely on the accuracy of the other choices of \( p \) was reduced without significantly degrading the accuracy. There are many ways to choose the probabilities \( k \) of \( C \) above and provide comparisons with the SDP solver SDPT3 \([43]\). We use IHTr, IHTMSr, FPCAr to denote algorithms in which the rank \( r \) is specified, and IHT, IHTMS, FPCA to denote those in which \( r \) is determined by the heuristics described in Section 7. We tested all of the algorithms IHT, IHTMS, FPCA and IHTr, IHTMSr, FPCAr on both randomly created and realistic matrix rank minimization problems \((1.1)\). IHTr,

\[
H^{(t)}_k, t = 1, \ldots, k \text{ are approximations to the corresponding left singular vectors of the matrix } A. \text{ Thus, the SVD of } A \text{ is approximated by}
\]

\[
A \approx A_k := H_k \cdot \text{Diag}(\sigma(C)) \cdot (A^\top H_k \cdot \text{Diag}(1/\sigma(C)))^\top.
\]

Drineas et al. \([15]\) prove that with high probability, the following estimate holds for both \( \xi = 2 \) and \( \xi = F \):

\[
(7.1) \quad \|A - A_k\|_\xi^2 \leq \min_{D: \text{rank}(D) \leq k_s} \|A - D\|_\xi^2 + \text{poly}(k_s, 1/c_s)\|A\|_F^2,
\]

where \( \text{poly}(k_s, 1/c_s) \) is a polynomial in \( k_s \) and \( 1/c_s \). Thus, \( A_k \) is an approximation to the best rank-\( k_s \) approximation to \( A \).

**Algorithm 5** Linear Time Approximate SVD Algorithm \([15]\)

Input : \( A \in \mathbb{R}^{m \times n}, c_s, k_s \in \mathbb{Z}^+ \) s.t. \( 1 \leq k_s \leq c_s \leq n \), \( \{p_i\}_{i=1}^n \) s.t. \( p_i \geq 0 \), \( \sum_{i=1}^n p_i = 1 \).

Output: \( H_k \in \mathbb{R}^{m \times k_s} \) and \( \sigma_t(C), t = 1, \ldots, k_s \).

for \( t = 1, \ldots, c_s \) do

- Pick \( i_t \in 1, \ldots, n \) with \( \text{Pr}[i_t = \alpha] = p_\alpha, \alpha = 1, \ldots, n \).
- Set \( C^{(t)} = A^{(i_t)} / \sqrt{c_p p_i} \).
- Compute \( C^{(t)}^\top C \) and its SVD; say \( \sigma_t(C) = \sum_{i=1}^{c_s} \sigma_i^2(C) y_i^t y_i^\top \).
- Compute \( h^t = C y^t / \sigma_t(C) \) for \( t = 1, \ldots, k_s \).
- Return \( H_k \), where \( H_k^{(t)} = h^t \), and \( \sigma_t(C), t = 1, \ldots, k_s \).

Note that in this algorithm, we compute an exact SVD of a smaller matrix \( C^\top C \in \mathbb{R}^{c_s \times c_s} \). Thus, \( c_s \) determines the speed of this algorithm. If we choose a large \( c_s \), we need more time to compute the SVD of \( C^\top C \). However, the larger \( c_s \) is, the more likely are the \( \sigma_t(C), t = 1, \ldots, k_s \) to be close to the largest \( k_s \) singular values of the matrix \( A \) since the second term in the right hand side of \((7.1)\) is smaller. In our numerical experiments, we found that we could choose a relatively small \( c_s \) so that the computational time was reduced without significantly degrading the accuracy. There are many ways to choose the probabilities \( p_i \). In our numerical experiments in Section 8 we used the simplest one, i.e., we set all \( p_i \) equal to \( 1/n \). For other choices of \( p_i \), see \([15]\) and the references therein.

Although PROPACK is more accurate than this Monte Carlo method (Algorithm 5), we observed from our numerical experiments that our algorithms are very robust and do not rely on the accuracy of the approximate SVDs.

In the \( j \)-th inner iteration in FPCA we solve problem \((1.8)\) for a fixed \( \mu = \mu_j \); and stop when

\[
(7.2) \quad \frac{\|X^{k+1} - X^k\|_F}{\max\{1,\|X^k\|_F\}} < xtol,
\]

where \( xtol \) is a small positive number. We then decrease \( \mu \) and go to the next inner iteration.

Algorithms IHT and IHTMS are terminated when \((7.2)\) holds.

**8. Numerical Experiments.** In this section, we present numerical results for the algorithms discussed above and provide comparisons with the SDP solver SDPT3 \([43]\). We use IHTr, IHTMSr, FPCAr to denote algorithms in which the rank \( r \) is specified, and IHT, IHTMS, FPCA to denote those in which \( r \) is determined by the heuristics described in Section 7. We tested all of the algorithms IHT, IHTMS, FPCA and IHTr, IHTMSr, FPCAr on both randomly created and realistic matrix rank minimization problems \((1.1)\). IHTr,
IHT, IHTMSr and IHTMS were terminated when \( (7.2) \) holds. FPCAr and FPCA were terminated when both \( (7.2) \) holds and \( \mu_k = \bar{\mu} \). All numerical experiments were run in MATLAB 7.3.0 on a Dell Precision 670 workstation with an Intel xeon(TM) 3.4GHZ CPU and 6GB of RAM. All CPU times reported in this section are in seconds.

### 8.1. Randomly Created Test Problems

We tested some randomly created problems to illustrate the recoverability/convergence properties of our algorithms. The random test problems (1.1) were created in the following manner. We first generated random matrices \( M_L \in \mathbb{R}^{m \times r} \) and \( M_R \in \mathbb{R}^{n \times r} \) with i.i.d. Gaussian entries \( \sim N(0, 1) \) and then set \( M = M_L M_R^\top \). We then created a matrix \( A \in \mathbb{R}^{p \times mn} \) with i.i.d. Gaussian entries \( A_{ij} \sim N(0, 1/p) \). Finally, the observation \( b \) was set equal to \( b := \text{Vec}(M) \). We use \( SR = p/(mn) \), i.e., the number of measurements divided by the number of entries of the matrix, to denote the sampling ratio. We also list \( FR = r(m + n - r)/p \), i.e. the dimension of the set of rank \( r \) matrices divided by the number of measurements, in the tables. Note that if \( FR > 1 \), then there is always an infinite number of matrices with rank \( r \) satisfying the \( p \) linear constraints, so we cannot hope to recover the matrix in this situation. We also report the relative error

\[
\text{rel.err.} := \frac{\| X_{opt} - M \|_F}{\| M \|_F}
\]

to indicate the closeness of \( X_{opt} \) to \( M \), where \( X_{opt} \) is the optimal solution to (1.1) produced by our algorithms. We declared \( M \) to be recovered if the relative error was less than \( 10^{-3} \). We solved 10 randomly created matrix rank minimization problems for each set of \( (m, n, p, r) \). We used \( NS \) to denote the number of matrices that were recovered successfully. The average time and average relative error of the successfully solved problems are also reported.

The parameters used in the algorithms are summarized in Table 8.1.

| parameter | value | description |
|-----------|-------|-------------|
| \( \bar{\mu} \) | \( 10^{-8} \) | parameter in Algorithms 1 and 4 |
| \( \eta_\mu \) | 0.25 | parameter in (6.1) |
| \( \epsilon_s \) | 0.01 | parameter in LinearTimeSVD |
| \( c_s \) | \( 2r_{\text{max}} - 2 \) | parameter in LinearTimeSVD |
| \( p_i \) | \( 1/n, \forall i \) | parameter in LinearTimeSVD |
| \( xtol \) | \( 10^{-6} \) | parameter in (7.2) |

We first compare the solvers discussed above that specify the rank \( r \) with the SDP solver SDPT3 [43]. The results for a set of small problems with \( m = n = 60 \), with 20 percent sampling (i.e., \( SR = 0.2 \) and \( p = 720 \)) and different ranks are presented in Table 8.2. Note that for this set of parameters \( (m, n, p) \), the largest rank that satisfies \( FR < 1 \) is \( r_{\text{max}} = 6 \).

From Table 8.2 we can see that the performance of our methods is very robust. They are much faster and their abilities to recover the matrices are much better than SDPT3. For ranks less than or equal to 5, which is almost the largest rank guaranteeing \( FR < 1 \), IHT, IHTMSr and FPCAr can recover all randomly generated matrices with a relative error of the order of \( 1e^{-5} \). However, SDPT3 can only recover all matrices with a rank equal to 1 or 2. When the rank \( r \) increases to 3, SDPT3 can only recover 3 of the 10 matrices. When the rank \( r \) increases to 4 or 5, none of the 10 matrices can be recovered by SDPT3.

To verify the theoretical results in Sections 4, 5 and 6, we plotted the approximation errors between \( X^k \)
Table 8.2
Comparison between IHTr, IHTMSr and FPCAr with SDPT3

| Prob | SDPT3 | IHTr | IHTMSr | FPCAr |
|------|-------|------|--------|-------|
| r    | FR    | NS   | time   | rel.err. | NS   | time   | rel.err. | NS   | time   | rel.err. |
| 1    | 0.17  | 10   | 122.93 | 2.31e-10 | 10 | 2.60  | 1.67e-05 | 10 | 2.59  | 1.67e-05 | 10 | 4.63  | 9.00e-06 |
| 2    | 0.33  | 10   | 124.26 | 3.46e-09 | 10 | 4.97  | 1.99e-05 | 10 | 4.98  | 2.11e-05 | 10 | 6.06  | 1.51e-05 |
| 3    | 0.49  | 3    | 149.74 | 2.84e-07 | 10 | 10.04 | 2.38e-05 | 10 | 9.95  | 2.27e-05 | 10 | 10.64 | 2.35e-05 |
| 4    | 0.64  | 0    | —     | —       | 10 | 22.99 | 2.88e-05 | 10 | 22.72 | 3.05e-05 | 10 | 23.29 | 2.93e-05 |
| 5    | 0.80  | 0    | —     | —       | 10 | 75.86 | 3.89e-05 | 10 | 84.13 | 3.95e-05 | 10 | 79.46 | 3.94e-05 |

Fig. 8.1. Approximation error versus the iteration number for a problem where the rank equaled 2

and $X^*$ versus the iteration number for algorithms IHTr, IHTMSr and FPCAr in Figure 8.1 for the results on a particular matrix of rank 2. From this figure, we can see that $\log \|X^k - X^*\|_F$ is approximately a linear function of the iteration number $k$. This implies that our theoretical results in Sections 4, 5 and 6 approximately hold in practice.

For the same set of test problems, Tables 8.3, 8.4, and 8.5 present comparisons of IHTr versus IHT, IHTMSr versus IHTMS and FPCAr versus FPCA.

Table 8.3
Comparison between IHTr and IHT

| Prob | IHTr | IHT |
|------|------|-----|
| r    | FR   | NS  | time | rel.err. |
|      |      |     |      |          |
| 1    | 0.17 | 10  | 2.60 | 1.67e-05 |
| 2    | 0.33 | 10  | 4.97 | 1.99e-05 |
| 3    | 0.49 | 10  | 10.04| 2.38e-05 |
| 4    | 0.64 | 10  | 22.99| 2.88e-05 |
| 5    | 0.80 | 10  | 75.86| 3.89e-05 |

From these tables we see that by using our heuristics for determining the rank $r$ at every iteration, algorithms IHT, IHTMS and FPCA perform similarly to algorithms IHTr, IHTMSr and FPCAr which make
Table 8.4
Comparison between IHTMS$_r$ and IHTMS

| Prob | IHTMS$_r$ | IHTMS |
|------|-----------|-------|
|      | r | FR | NS | time | rel.err. | r | NS | time | rel.err. |
| 1    | 1.07 | 10 | 2.59 | 1.67e-05 | 10 | 3.98 | 1.77e-05 |
| 2    | 0.33 | 10 | 4.98 | 2.11e-05 | 10 | 6.95 | 2.04e-05 |
| 3    | 0.49 | 10 | 9.95 | 2.27e-05 | 10 | 12.65 | 2.30e-05 |
| 4    | 0.64 | 10 | 22.72 | 3.05e-05 | 10 | 27.12 | 2.86e-05 |
| 5    | 0.80 | 10 | 84.13 | 3.95e-05 | 10 | 94.13 | 4.10e-05 |

Table 8.5
Comparison between FPCAr and FPCA

| Prob | FPCAr | FPCA |
|------|-------|------|
|      | r | FR | NS | time | rel.err. | r | NS | time | rel.err. |
| 1    | 1.07 | 10 | 4.63 | 9.00e-06 | 10 | 4.66 | 8.88e-06 |
| 2    | 0.33 | 10 | 6.06 | 1.51e-05 | 10 | 6.15 | 1.55e-05 |
| 3    | 0.49 | 10 | 10.64 | 2.35e-05 | 10 | 11.50 | 2.24e-05 |
| 4    | 0.64 | 10 | 23.29 | 2.93e-05 | 10 | 25.66 | 2.88e-05 |
| 5    | 0.80 | 10 | 79.46 | 3.94e-05 | 10 | 83.91 | 3.87e-05 |

use of knowledge of the true rank $r$. Specifically, algorithms IHT, IHTMS and FPCA are capable of recovering low-rank matrices very well even when we do not know their rank.

Choosing $r$ is crucial in algorithms IHTr, IHTMSr and FPCAr as it is in greedy algorithms for matrix rank minimization and compressed sensing. In Table 8.6 we present results on how the choice of $r$ affects the performance of algorithms IHTr, IHTMSr and FPCAr when the true rank of the matrix is not known. In Table 8.6 the true rank is 3 and the results for choices of the rank from 1 to 6 are presented. The rows labeled IHT, IHTMS and FPCA present the results for these algorithms which use the heuristics in Section 7 to determine the rank $r$. From Table 8.6 we see that if we specify a rank that is smaller than the true rank, then all of the algorithms IHTr, IHTMSr and FPCAr are unable to successfully recover the matrices (i.e., the relative error is greater than 1e-3). Specifically, since for the problems tested the true rank of the matrix was 3, the algorithms failed when $r$ was chosen to be either 1 or 2. If the chosen rank is slightly greater than the true rank (i.e., the rank was chosen to be 4 or 5), all the three algorithms IHTr, IHTMSr and FPCAr still worked. However, the relative errors and times were much worse than those produced by the heuristics based solvers IHT, IHTMS and FPCA. When the chosen rank was too large (i.e., was chosen to be 6), IHTr, IHTMSr and FPCAr were only able to recover the matrices in 4, 1 and 3 out of 10 problems, respectively. However, IHT, IHTMS and FPCA always recovered the matrices.

8.2. A Video Compression Problem. We tested the performance of our algorithms on a video problem. By stacking each frame of the video as a column of a large matrix, we get a matrix $M$ whose $j$-th column corresponds to the $j$-th frame of the video. Due to the correlation between consecutive frames of the video matrix $M$ is expected to be of low-rank. Hence we should be able to recover the video by only taking a limited number of measurements. The video used in our experiment was downloaded from the website [http://media.xiph.org/video/derf](http://media.xiph.org/video/derf). The original colored video consisted of 300 frames where each frame was an image stored in the RGB format, which was a $144 \times 176 \times 3$ cube. Since it was too large for our use, we preprocessed the video in the following way. We first converted each frame from RGB format into grayscale image, so each frame was a $144 \times 176$ matrix. We then used only that portion of each frame
Table 8.6
Comparison when the given rank is different from the true rank of 3

| Given rank | NS | time | rel.err. |
|------------|----|------|----------|
| HTr        |    |      |          |
| 1          | 0  | —    | —        |
| 2          | 0  | —    | —        |
| 3          | 10 | 10.04| 2.38e-05 |
| 4          | 10 | 21.42| 3.42e-05 |
| 5          | 10 | 63.53| 5.51e-05 |
| 6          | 4  | 109.00| 4.44e-04 |
| IHT        | 10 | 13.27| 2.32e-05 |

| IHTSr       |    |      |          |
| 1          | 0  | —    | —        |
| 2          | 0  | —    | —        |
| 3          | 10 | 9.95 | 2.27e-05 |
| 4          | 10 | 22.53| 3.40e-05 |
| 5          | 10 | 67.89| 5.93e-05 |
| 6          | 1  | 116.62| 6.04e-04 |
| IHTS       | 10 | 12.65| 2.30e-05 |

| FPCAr       |    |      |          |
| 1          | 0  | —    | —        |
| 2          | 0  | —    | —        |
| 3          | 10 | 10.64| 2.35e-05 |
| 4          | 10 | 21.26| 3.46e-05 |
| 5          | 10 | 63.67| 5.99e-05 |
| 6          | 3  | 108.02| 4.04e-04 |
| FPCA       | 10 | 11.50| 2.24e-05 |

corresponding to a 39 × 47 submatrix of pixels in the center of each frame. We took the first 20 frames in the video. Consequently, the matrix $M$ has $m = 1833$ rows and $n = 20$ columns. We then created a Gaussian sampling matrix $A \in \mathbb{R}^{p \times (mn)}$ as in Section 8.1 with $p = 1833 \times 20 \times 0.4 = 14664$ rows (i.e., we used sampling ratio $SR = 0.4$) and computed $b = \text{vec}(M) \in \mathbb{R}^p$. This $14664 \times 36660$ matrix $A$ was close to the size limit of what could be created by calling the MATLAB function $A = \text{randn}(p, mn)$ on our computer. Although the matrix $M$ is expected to be of low-rank, it is only approximately of low-rank. Therefore, besides comparing the recovered matrices with the original matrix $M$, we also compared them with the best rank-5 approximation of $M$. Since the relative error of the best rank-5 approximation of $M$ is $2.33e-2$, we cannot expect to get a more accurate solution. Therefore, we set $xtol$ equal to 0.002 for this problem. The results of our numerical tests are reported in Table 8.7. The relative errors and CPU times are averaged over 5 runs. We do not report any results for SDPT3, because the problem is far too large to be solved by an SDP solver. Note that the relative error of the best rank-5 approximation of $M$ is $2.33e-2$, thus from Table 8.7 we see that our algorithms were able to recover the matrix $M$ very well and achieve an order of accuracy as the best rank-5 approximation.

We show three frames of video in Figure 8.2. The three images in the first column correspond to the frames in the original video. The images in the second column correspond to the frames in the rank-5 approximation matrix of the video. The images in the third column correspond to the frames in the matrix recovered by FPCA. The other five solvers recovered images that were very similar visually to FPCA so we
Table 8.7
Results on recovery of compressed video

| Solvers | rank | rel.err. | time |
|---------|------|----------|------|
| IHTr    | 5    | 6.87e-2  | 645  |
| IHT     | 5    | 9.76e-2  | 949  |
| IHTMSr  | 5    | 6.72e-2  | 688  |
| IHTMS   | 5    | 9.69e-2  | 804  |
| FPCAr   | 5    | 5.10e-2  | 514  |
| FPCA    | 5    | 5.17e-2  | 1296 |

Fig. 8.2. Comparison of frames 4, 12 and 18 of (a) the original video, (b) the best rank-5 approximation and (c) the matrix recovered by FPCA

do not show them here. From Figure 8.2 we see that FPCA recovers the video very well by taking only 40% of the measurements.

Acknowledgement. We would like to thank two anonymous referees for insightful comments that greatly improved the presentation of the paper. We would also like to thank Dr. Thomas Blumensath for pointing out an error in an earlier version of this paper.

Appendix. Here we give proofs of Propositions 2.6, 2.7 and 2.8.

Proof of Proposition 2.6

Proof. We prove (2.2) first. Since for any $X \in \mathbb{R}^{m \times n}$, $\text{rank}(P_X X) \leq r$, we have

$$
|\langle X, P_X A^* b \rangle| = |\langle A P_X X, b \rangle| 
\leq \| A P_X X \|_2 \| b \|_2 
\leq \sqrt{1 + \delta_r(A)} \| P_X X \|_F \| b \|_2 
\leq \sqrt{1 + \delta_r(A)} \| X \|_F \| b \|_2.
$$
Thus
\[ \|P_\Psi A^* b\|_F = \max_{\|X\|_F = 1} |\langle X, P_\Psi A^* b \rangle| \leq \sqrt{1 + \delta_r(A)} \|b\|_2. \]

To prove (A-1), note that by the RIP,
\[ (1 - \delta_r(A)) \|P_\Psi X\|_F^2 \leq \|AP_\Psi X\|_F^2 \leq (1 + \delta_r(A)) \|P_\Psi X\|_F^2, \]
which means the eigenvalues of \(P_\Psi A^* A P_\Psi\) restricted to \(\text{span}(\Psi)\) are in the interval \([1 - \delta_r(A), 1 + \delta_r(A)]\). Thus (2.3) holds. \(\square\)

**Proof of Proposition 2.7** First, we will prove
\[ (\text{A-1}) \quad |\langle A(I - P_\Psi)X, AP_\Psi Y \rangle| \leq \delta_r(A) \|A(I - P_\Psi) X\|_F \|P_\Psi Y\|_F, \forall Y \in \mathbb{R}^{m \times n}, X \in \text{span}(\Psi'). \]

(A-1) holds obviously if \((I - P_\Psi)X = 0\) or \(P_\Psi Y = 0\). Thus we can assume \((I - P_\Psi)X \neq 0\) and \(P_\Psi Y \neq 0\).
Define \(\hat{X} = \frac{(I - P_\Psi)X}{\|I - P_\Psi\|_F}\) and \(\hat{Y} = \frac{P_\Psi Y}{\|P_\Psi\|_F}\); then we have \(\|\hat{X}\|_F = 1\), \(\|\hat{Y}\|_F = 1\) and \(\langle \hat{X}, \hat{Y} \rangle = 0\). Since \(\hat{X} \in \text{span}(\Psi \cup \Psi')\) and \(\hat{Y} \in \text{span}(\Psi)\), we have rank \((\hat{X} + \hat{Y}) \leq r\) and rank \((\hat{X} - \hat{Y}) \leq r\). Hence by RIP,
\[
2(1 - \delta_r(A)) = (1 - \delta_r(A)) \left\| \hat{X} + \hat{Y} \right\|_F^2 \leq \left\| A\hat{X} + A\hat{Y} \right\|_2^2 \\
\leq (1 + \delta_r(A)) \left\| \hat{X} + \hat{Y} \right\|_F^2 = 2(1 + \delta_r(A)).
\]
and
\[
2(1 - \delta_r(A)) = (1 - \delta_r(A)) \left\| \hat{X} - \hat{Y} \right\|_F^2 \leq \left\| A\hat{X} - A\hat{Y} \right\|_2^2 \\
\leq (1 + \delta_r(A)) \left\| \hat{X} - \hat{Y} \right\|_F^2 = 2(1 + \delta_r(A)).
\]
Therefore we have
\[ \langle A\hat{X}, A\hat{Y} \rangle = \frac{\left\| A\hat{X} + A\hat{Y} \right\|_2^2 - \left\| A\hat{X} - A\hat{Y} \right\|_2^2}{4} \leq \delta_r(A) \]
and
\[ -\langle A\hat{X}, A\hat{Y} \rangle = \frac{\left\| A\hat{X} - A\hat{Y} \right\|_2^2 - \left\| A\hat{X} + A\hat{Y} \right\|_2^2}{4} \leq \delta_r(A). \]
Thus, \(|\langle A\hat{X}, A\hat{Y} \rangle| \leq \delta_r(A)\) and (A-1) holds.

Finally we have, for any \(X \in \text{span}(\Psi')\),
\[ \|P_\Psi A^* A(I - P_\Psi)X\|_F = \max_{\|Y\|_F = 1} |\langle P_\Psi A^* A(I - P_\Psi)X, Y \rangle| \\
= \max_{\|Y\|_F = 1} |\langle A(I - P_\Psi)X, AP_\Psi Y \rangle| \\
\leq \delta_r(A) \|((I - P_\Psi)X\|_F), \]
\[ 20 \]
i.e., \(2.4\) holds, which completes the proof. \(\Box\)

**Proof of Proposition 2.8** This proof essentially follows that given by Needell and Tropp in [36].

**Proof.** Let \(B^s := \{ X \in \mathbb{R}^{m \times n} : \text{rank}(X) = s, \| X \|_F \leq 1 \} \) be the unit ball of rank-\(s\) matrices in \(\mathbb{R}^{m \times n}\).

Define the convex hull of the unit norm matrices with rank at most \(r\) as:

\[
S := \text{conv} \left\{ \bigcup_{s \leq r} B^s \right\} \subset \mathbb{R}^{m \times n}.
\]

By (2.5), we know that the operator norm

\[
\|A\|_{s \rightarrow 2} = \max_{X \in S} \|AX\|_2 \leq \sqrt{1 + \delta_r(A)}.
\]

Define another convex set

\[
K := \{ X \in \mathbb{R}^{m \times n} : \|X\|_F + \frac{1}{\sqrt{r}} \|X\|_* \leq 1 \} \subset \mathbb{R}^{m \times n},
\]

and consider the operator norm

\[
\|A\|_{K \rightarrow 2} = \max_{X \in K} \|AX\|_2.
\]

The content of the proposition is the claim that \(K \subset S\).

Choose a matrix \(X \in K\) with SVD \(X = U \text{Diag}(\sigma)V^\top\). Let \(I_0\) index the \(r\) largest components of \(\sigma\), breaking ties lexicographically. Let \(I_1\) index the next largest \(r\) components, and so forth. Note that the final block \(I_J\) may have fewer than \(r\) components. We may assume that \(\sigma|_{I_j}\) is nonzero for each \(j\). This partition induces a decomposition

\[
X = U[\text{Diag}(\sigma|_{I_0}) + \sum_{j=1}^J \text{Diag}(\sigma|_{I_j})]V^\top = \lambda_0 Y_0 + \sum_{j=1}^J \lambda_j Y_j,
\]

where \(\lambda_j = \|U \text{Diag}(\sigma|_{I_j})V^\top\|_F\) and \(Y_j = \lambda_j^{-1} U \text{Diag}(\sigma|_{I_j})V^\top\). By construction, each matrix \(Y_j\) belongs to \(S\) because it’s rank is at most \(r\) and it has unit Frobenius norm. We will prove that \(\sum_j \lambda_j \leq 1\), which implies that \(X\) can be expressed as a convex combination of matrices from the set \(S\). So \(X \in S\) and \(K \subset S\).

Fix \(j\) in the range \(\{1, 2, \ldots, J\}\). It follows that \(\sigma|_{I_j}\) contains at most \(r\) elements and \(\sigma|_{I_{j-1}}\) contains exactly \(r\) elements. Therefore,

\[
\lambda_j = \|\sigma|_{I_j}\|_2 \leq \sqrt{r} \|\sigma|_{I_j}\|_\infty \leq \sqrt{r} \frac{1}{\sqrt{r}} \|\sigma|_{I_{j-1}}\|_1.
\]

Summing these relations, we obtain,

\[
\sum_{j=1}^J \lambda_j \leq \frac{1}{\sqrt{r}} \|\sigma|_{I_{j-1}}\|_1 \leq \frac{1}{\sqrt{r}} \|X\|_*.
\]
It is obvious that $\lambda_0 = \|\sigma|_{l_0}\|_2 \leq \|X\|_F$. We now conclude that

$$\sum_{j=0}^{J} \lambda_j \leq \|X\|_F + \frac{1}{\sqrt{r}}\|X\|_* \leq 1$$

because $X \in K$. This implies that $X \in S$ and $K \subset S$, and thus completes the proof. □

REFERENCES

[1] T. Blumensath and M. E. Davies. Gradient pursuits. IEEE Transactions on Signal Processing, 56(6):2370–2382, 2008.
[2] T. Blumensath and M. E. Davies. Iterative hard thresholding for compressed sensing. Applied and Computational Harmonic Analysis, 27(3):265–274, 2009.
[3] J. M. Borwein and A. S. Lewis. Convex Analysis and Nonlinear Optimization. Springer-Verlag, 2003.
[4] J. Cai, E. J. Candes, and Z. Shen. A singular value thresholding algorithm for matrix completion. SIAM J. on Optimization, 20(4):1956–1982, 2010.
[5] E. J. Candes and Y. Plan. Matrix completion with noise. Proceedings of the IEEE, 2009.
[6] E. J. Candes and B. Recht. Exact matrix completion via convex optimization. Foundations of Computational Mathematics, 9:717–772, 2009.
[7] E. J. Candes and J. Romberg. $\ell_1$-MAGIC: Recovery of sparse signals via convex programming. Technical report, Caltech, 2005.
[8] E. J. Candes, J. Romberg, and T. Tao. Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information. IEEE Transactions on Information Theory, 52:489–509, 2006.
[9] E. J. Candes and T. Tao. The power of convex relaxation: near-optimal matrix completion. IEEE Trans. Inform. Theory, 56(5):2053–2080, 2009.
[10] Rice compressed sensing website. http://dsp.rice.edu/cs.
[11] W. Dai and O. Milenkovic. Subspace pursuit for compressive sensing signal reconstruction. IEEE Trans. on Information Theory, 55(5):2230–2249, 2009.
[12] D. Donoho. Compressed sensing. IEEE Transactions on Information Theory, 52:1289–1306, 2006.
[13] D. Donoho, Y. Tsaig, I. Drori, and J.-C. Starck. Sparse solution of underdetermined linear equations by stagewise orthogonal matching pursuit. Submitted to IEEE Transactions on Information Theory, 2006.
[14] D. L. Donoho and Y. Tsaig. Fast solution of $\ell_1$-norm minimization problems when the solution may be sparse. Technical report, Department of Statistics, Stanford University, 2006.
[15] P. Drineas, R. Kannan, and M. W. Mahoney. Fast Monte Carlo algorithms for matrices ii: Computing low-rank approximations to a matrix. SIAM J. Computing, 36:158–183, 2006.
[16] M. Fazel, H. Hindi, and S. Boyd. A rank minimization heuristic with application to minimum order system approximation. In Proceedings of the American Control Conference, volume 6, pages 4734–4739, 2001.
[17] M. Fazel, H. Hindi, and S. Boyd. Log-det heuristic for matrix rank minimization with applications to Hankel and Euclidean distance matrices. In Proceedings of the American Control Conference, pages 2156–2162, 2003.
[18] M. Fazel, H. Hindi, and S. Boyd. Rank minimization and applications in system theory. In American Control Conference, pages 3273–3278, 2004.
[19] M. A. T. Figueiredo, R. D. Nowak, and S. J. Wright. Gradient projection for sparse reconstruction: Application to compressed sensing and other inverse problems. IEEE Journal on Selected Topics in Signal Processing, 1(4), 2007.
[20] L. El Ghaoui and P. Gahinet. Rank minimization under LMI constraints: A framework for output feedback problems. In Proceedings of the European Control Conference, 1993.
[21] E. T. Hale, W. Yin, and Y. Zhang. Fixed-point continuation for $\ell_1$-minimization: Methodology and convergence. SIAM Journal on Optimization, 19(3):1107–1130, 2008.
[22] J.-B. Hiriart-Urruty and C. Lemaréchal. Convex Analysis and Minimization Algorithms II: Advanced Theory and Bundle Methods. Springer-Verlag, New York, 1993.
[23] R. H. Keshavan, A. Montanari, and S. Oh. Matrix completion from a few entries. Preprint available at http://arxiv.org/abs/0901.3150, 2009.
[24] R. H. Keshavan, A. Montanari, and S. Oh. Matrix completion from noisy entries. arxiv:0906.2027, 2009.
[25] S. J. Kim, K. Koh, M. Lustig, S. Boyd, and D. Gorinevsky. A method for large-scale $\ell_1$-regularized least-squares. IEEE Journal on Selected Topics in Signal Processing, 4(1):606–617, 2007.
[26] R. M. Larsen. PROPACK - software for large and sparse SVD calculations. Available from http://sun.stanford.edu/~rmunk/PROPACK.

[27] K. Lee and Y. Bresler. ADMIRA: atomic decomposition for minimum rank approximation. ArXiv preprint: arXiv:0905.0044, 2009.

[28] K. Lee and Y. Bresler. Efficient and guaranteed rank minimization by atomic decomposition. preprint, available at arXiv: 0901.1898v1, 2009.

[29] K. Lee and Y. Bresler. Guaranteed minimum rank approximation from linear observations by nuclear norm minimization with an ellipsoidal constraint. Arxiv preprint arXiv:0903.4742, 2009.

[30] N. Linial, E. London, and Y. Rabinovich. The geometry of graphs and some of its algorithmic applications. Combinatorica, 15:215–245, 1995.

[31] Y. Liu, D. Sun, and K.-C. Toh. An implementable proximal point algorithmic framework for nuclear norm minimization. preprint, National University of Singapore, 2009.

[32] Z. Liu and L. Vandenberghe. Interior-point method for nuclear norm approximation with application to system identification. SIAM Journal on Matrix Analysis and Applications, 31(3):1235–1256, 2009.

[33] S. Ma, D. Goldfarb, and L. Chen. Fixed point and Bregman iterative methods for matrix rank minimization. To appear in Mathematical Programming Series A, 2009. (published online: 23 September 2009).

[34] Raghu Meka, Prateek Jain, and Inderjit S. Dhillon. Guaranteed rank minimization via singular value projection. Arxiv preprint, available at http://arxiv.org/abs/0909.5457, 2009.

[35] B. K. Natarajan. Sparse approximate solutions to linear systems. SIAM Journal on Computing, 24:227–234, 1995.

[36] D. Needell and J. A. Tropp. CoSaMP: Iterative signal recovery from incomplete and inaccurate samples. Applied and Computational Harmonic Analysis, 26:301–321, 2009.

[37] Netflix prize website. http://www.netflixprize.com/.

[38] B. Recht, M. Fazel, and P. Parrilo. Guaranteed minimum rank solutions of matrix equations via nuclear norm minimization. To appear in SIAM Review, 2007.

[39] E. Sontag. Mathematical Control theory. Springer-Verlag, New York, 1998.

[40] N. Srebro. Learning with Matrix Factorizations. PhD thesis, Massachusetts Institute of Technology, 2004.

[41] N. Srebro and T. Jaakkola. Weighted low-rank approximations. In Proceedings of the Twentieth International Conference on Machine Learning (ICML-2003), 2003.

[42] R. Tibshirani. Regression shrinkage and selection via the lasso. Journal Royal Statistical Society B, 58:267–288, 1996.

[43] K.-C. Toh, M. J. Todd, and R. H. Tütüncü. SDPT3 - a Matlab software package for semidefinite programming. Optimization Methods and Software, 11:545–581, 1999.

[44] K.-C. Toh and S. Yun. An accelerated proximal gradient algorithm for nuclear norm regularized least squares problems. preprint, National University of Singapore, 2009.

[45] J. Tropp. Just relax: Convex programming methods for identifying sparse signals. IEEE Transactions on Information Theory, 51:1030–1051, 2006.

[46] E. van den Berg and M. P. Friedlander. Probing the Pareto frontier for basis pursuit solutions. SIAM J. on Scientific Computing, 31(2):890–912, 2008.