A NOTE ON THE BURRIS-WILLARD CONJECTURE

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Abstract. Based on results by Daniľčenko, in 1987 Burris and Willard have conjectured that on any $k$-element domain where $k \geq 3$ it is possible to bicentrically generate every centraliser clone from its $k$-ary part.

Later, for every $k \geq 3$, Snow constructed algebras with a $k$-element carrier set where the minimum arity of the clone of term operations from which the bicentraliser can be generated is at least $(k - 1)^2$, which is larger than $k$ for $k \geq 3$.

We prove that Snow’s examples do not violate the Burris-Willard conjecture nor invalidate the results by Daniľčenko on which the latter is based. We also complement our results with some computational evidence for $k = 3$, obtained by an algorithm to compute a primitive positive definition for a relation in a finitely generated relational clone over a finite set.

1. Introduction

Centraliser clones are collections of homomorphisms of finite powers of algebras into themselves. That is, if $A$ is an algebra and $F$ is the set of fundamental operations of $A$, then the centraliser $F^*$ of $F$ is the set $\bigcup_{n<\omega} \text{Hom}(A^n, A)$. From a categorical perspective, this is a very natural construction that makes sense in every category $\mathcal{C}$ with arbitrary finite powers. If $A$ is an object in such a category $\mathcal{C}$, we call $\bigcup_{n<\omega} \text{Hom}_\mathcal{C}(A^n, A)$ the clone over the object $A$. With this understanding centraliser clones are simply the clones over algebras in the category of algebras of a certain type. If we change the signature of the structures to allow relation symbols (that is, we change the category to relational structures of a certain signature), we obtain clones over some relational structure $\mathcal{A}$ with set of fundamental relations $Q$: $\bigcup_{n<\omega} \text{Hom}(\mathcal{A}^n, \mathcal{A})$. This clone is called the clone $\text{Pol}_\mathcal{A} Q$ of polymorphisms of $Q$ (or just the polymorphism clone of the structure $\mathcal{A}$), and it is well known by results of Bodnarčuk, Kalužnin, Kotov, Romov [1] and Geiger [11] on the classical $\text{Pol}$-$\text{Inv}$ Galois correspondence that every clone on a finite carrier set $A$ arises as a polymorphism clone of some relational structure $\mathcal{A}$.

As every algebraic structure $A$ can also be understood as a relational one (by taking the graphs of the fundamental operations as the fundamental relations), it is clear that the centraliser clones on a given set $A$ form a subcollection of the polymorphism clones on that set. This fact is very closely related to restricting the $\text{Pol}$-$\text{Inv}$ Galois correspondence on the relational side in such a way that the only relations taken into consideration are those which are graphs of a function. This restriction of the preservation relation (underlying $\text{Pol}$-$\text{Inv}$) between functions and relations to functions and function graphs leads to the notion of commutation of functions, which is exactly the homomorphism property between finite powers of algebras that was used above to introduce the concept of centraliser clone. As the Galois correspondence is restricted on one side only (the relational one), there is a connection between the associated Galois closures: the $\text{Pol}$-$\text{Inv}$ closure $\text{Pol}_A \text{Inv}_A F$ of a set of operations $F$ (which for finite $A$ agrees with the generated clone $\langle F \rangle_{O_A}$)

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is weaker than the bicentrical closure $F^{**}$, that is, the double centraliser of $F$, or, equivalently, all functions commuting with all those functions that commute with the functions in $F$. The strength of the bicentrical closure in comparison to $\text{Pol}_A \text{Inv}_A$ manifests itself in the following way: while $\text{Pol}_A \text{Inv}_A F$ closes $F$ against all compositions of $F$-functions with themselves and projections (i.e. one iteratively substitutes functions and variables until nothing new appears), $F^{**}$ computes all functions that are primitive positively definable from the function graphs of $F$ (i.e. one interprets all existentially quantified finite conjunctions of predicates of the form $f(v) = x$ and equality predicates $y = z$ (where $f \in F$, $v$ is a tuple of variables and $x, y, z$ are variables) and among these interpretations selects those relations that are function graphs). Functions whose graphs are constructible via such primitive positive formulae from $F$ have been called \textit{parametrically expressible} through $F$ \cite[p. 26]{13} (in contrast to functions in the clone $\langle F \rangle_{O \Lambda}$ that are \textit{explicitly expressible} via $F$), and also the connection of this construction with the preservation of function graphs and the commutation of operations has first been noted in \cite[p. 27 et seq.]{13}. For this reason centraliser clones have also been studied under the name \textit{parametrically closed classes} (see e.g. \cite{7}) or \textit{primitive positive clones} (e.g. \cite{2}).

It may not seem so at first glance, but the parametrical (primitive positive, bicentral) closure is notably much stronger than closure under substitution. Namely, it has the remarkable consequence that on every finite set $A$ there are only finitely many centraliser clones \cite[Corollary 4, p. 429]{2}, which is in sharp contrast to the situation for polymorphism clones, of which there is a continuum whenever $|A| \geq 3$ \cite{12}. If $F$ is a centraliser clone (i.e. $F^{**} = F$), then $F^{(1)**} \subseteq F^{(2)**} \subseteq \cdots \subseteq F^{(n)**} \subseteq F$ holds for all $n < \omega$ and $\bigcup_{n<\omega} F^{(n)**} = F^{**} = F$. Since there are only finitely many centraliser clones on a given finite set there must be some $n < \omega$ such that for arities larger than $n$ none of the inclusions is strict anymore, that is, $F^{(n)**} = F^{(m)**}$ for all $n \leq m < \omega$. Hence, $F = \bigcup_{1 \leq n} F^{(n)**} = F^{(n)**}$; so there is some arity $n$ such that $F$ is bicentrically generated by its $n$-ary part. Take this $n_F$ to be minimal and then take the maximum over all (finitely many) $n_F$:

$$cdeg(k) := \max \{n_F \mid F = F^{**} \text{ on } A, |A| = k\}.$$ 

We shall refer to this number as the \textit{uniform centraliser degree} for a $k$-element set, since every centraliser clone $F$ on a carrier set of size $k$ satisfies $F = F^{(cdeg(k))**}$.

With the help of Post’s lattice, one can show that $cdeg(2) = 3$. Burris and Willard explain in \cite[p. 429]{2} that $cdeg(k) \leq 4 + k^k - k + k^2$ and they claim that \cite{11} by slightly different methods one can show that any primitive positive clone on a $k$-element set is \textit{bicentrically} generated by its members of arity at most $k^k$, which implies $cdeg(k) \leq k^k$. No written account of the details of this argument has appeared in the literature so far. However, at the end of the sentence cited above Burris and Willard conjecture that $cdeg(k) \leq k$ for every $k \geq 3$. Besides intuition the only support for this conjecture is a series of works by A. F. Daniľčenko on the case $k = 3$ \cite[4, 5, 7, 8], all of these are in Russian, \cite{5} has been translated in \cite{6}; \cite{9} is written in English. As a side note we remark that a $k$-ary example function, stated in \cite[p. 269]{6} for a different proof, can be used to show that $cdeg(k) \geq k$ for $k \geq 3$; so if the Burris-Willard conjecture is true, then it certainly is sharp.

In her thesis \cite[Section 6, p. 125 et seq.]{8} Daniľčenko gives a complete description of all 2986 centraliser clones on the three-element domain. A central step in this process is to identify a set $\Gamma$ of 197 parametrically indecomposable functions \cite[Theorem 4, p. 103]{8} such that every centraliser clone $F$ is the centraliser of a subset of $\Gamma$ \cite[Theorem 5, p. 105]{8}. The maximum arity of functions in $\Gamma$ is three, so Daniľčenko’s theorems imply that $F = F^{(3)**}$ for every centraliser clone on three elements (cf. Proposition 1(a),(d)), that is, $cdeg(3) \leq 3$. The results of Theorems 4...
and 5 of [8], which make the Burris-Willard conjecture true for $k = 3$, are also mentioned in [9, p. 155 et seq.] and [5, Section 5, p. 414 et seq.] ([6, Section 5, p. 279], respectively), but no proofs are given there.

Drastically cut down versions of this work have been published in [3, 4, 5, 7, 9], of which only [6, 9] are accessible without difficulties. Given that the whole thesis comprises 141 pages, these excerpts are rough sketches of the classification at best (sometimes containing mistakes, many but not all of which have been corrected in [8]), and leading experts in the field agree that it is very hard if not impossible to reconstruct the proof of the description of all centralisers on three-element sets from the readily available resources. For example, Theorem 4 of [8] has appeared as part of [7, Proposition 2.2, p. 16] with a proof sketch of less than two pages, while the proof from [8] goes through technical calculations and case distinctions for several pages (however from Propositions 2.2, 2.3 and 2.4 of [7], a proof of Theorem 5 of [8] can be obtained). The chances of understanding might be better using the thesis as a primary source, but for unknown reasons Moldovan librarians seem to be rather reluctant to grant full access to it. In the light of this discussion, Daniľčenko’s classification is a result that one may believe in, but that should not be trusted unconditionally to build further theory on as it remains not easily verifiable at the moment. This of course also casts some doubts on the basis of the Burris-Willard conjecture.

Another possible challenge to the conjecture (and likewise to the correctness of Daniľčenko’s list of parametrically indecomposable functions) is presented by much later results of Snow [16]. In this article the minimum arity needed to generate the bicentraliser clone of a finite algebra from its term operations is investigated, and, under certain assumptions on the algebra, quite satisfactory upper bounds for that number are produced. These sometimes match (or almost match) the number $k$ predicted by the Burris-Willard conjecture, and sometimes even fall below $k$. This is possible (and supports the conjecture) since the bounds given by Snow do not apply to all algebras on a $k$-element set, but only to some specific subclass. Hence, they are not in contradiction with the $k$-ary function from [6, p. 269]. Even more interestingly, in Section 3 of [16] a class of examples of algebras on $k$-element carrier sets is given, for which Snow proves $(k - 1)^2$ to be a lower bound for the minimum arity of term functions from which the bicentraliser can be generated. This number is larger than $k$ whenever $k \geq 3$. Explicitly, when $k = 3$, the lower bound is equal to 4, which means that arity three or less does not suffice to generate the bicentraliser clone of that specific algebra.

In more detail, Snow defines for an algebra $A$ with set $F$ of fundamental operations the number $\text{PPC#}(A) = \min \left\{ n \in \mathbb{N} \mid \langle F \rangle_{O_A}^{**} = \langle F \rangle_{O_A}^{(n)**} \right\}$. This number certainly only depends on the clone $\langle F \rangle_{O_A}$ of term operations of the algebra, hence no generality is lost in simply considering the number

$$
\mu_F := \min \left\{ n \in \mathbb{N} \mid \langle F \rangle_{O_A}^{n**} = \langle F \rangle_{O_A}^{(n)**} \right\} = \min \left\{ n \in \mathbb{N} \mid F^{**} = F^{(n)**} \right\}
$$

associated with clones $F$ on a $k$-element set $A$. If $F$ happens to be a centraliser clone, the definition clearly simplifies to $\mu_F = \min \left\{ n \in \mathbb{N} \mid F = F^{(n)**} \right\} = n_F$, which is bounded above by $\text{cdeg}(k)$. However, if now $F$ is the clone of term operations of the example constructed by Snow, then the lower bound on $\mu_F$ from [16, Theorem 3.1, p. 171] implies the following contradiction

$$
k < (k - 1)^2 \leq \mu_F \leq n_F \leq \text{cdeg}(k) \leq k.
$$

This offers two conclusions: either $?_1$ does not hold, which means that the Burris-Willard conjecture and, in particular, the Daniľčenko classification on three-element domains fail, or $?_2$ is false, which simply means that $F$ is not a centraliser clone.
If we are to believe in Daniľčenko’s theorems, then (for \( k = 3 \)) the latter is the only possible consequence. However, for the reasons mentioned above, it would be desirable to obtain such a conclusion independently of Daniľčenko’s œuvre.

Such is the aim of the present article. We are going to give a proof that the clone \( F \) of term operations of the algebra given in [16, Theorem 3.1, p. 171] is not bicentrically closed and hence poses no threat to the Burris-Willard conjecture. To do this, for every \( k \geq 3 \) we exhibit a \((k-1)\)-ary function in \( F^{**} \) that cannot be obtained by composition of the fundamental operation(s) of Snow’s algebra. In doing so we use the case \( k = 3 \) as a guideline, where we show, for example, that \( F \) and \( F^{**} \) cannot be separated by unary functions, and that the mentioned operation is the only separating binary function.

2. Notation and preliminaries

Throughout we use \( \mathbb{N} = \{0, 1, 2, \ldots \} \) to denote the set of natural numbers, and we write \( \mathbb{N}_+ \) for \( \mathbb{N} \setminus \{0\} \). It will be convenient for us to understand the elements \( n \in \mathbb{N} \) as \( n \)-element sets \( n = \{0, 1, \ldots, n-1\} \) as originally suggested by John von Neumann in its model of natural numbers as finite ordinals.

One of the central concepts for this paper are functions, such as \( f: A \to B \) and \( g: B \to C \), and we use a left-to-right notation for composition. That is, \( g \circ f: A \to C \) sends any \( x \in A \) to \( g(f(x)) \). The set of all functions from \( A \) to \( B \) is written as \( B^A \). Moreover, if \( f \in B^A \) and \( U \subseteq V \) and \( V \subseteq B \) we denote by \( f[U] = \{ f(x) \mid x \in U \} \) the image of \( U \) under \( f \) and by \( f^{-1}[V] = \{ x \in A \mid f(x) \in V \} \) the preimage of \( V \) under \( f \). We also use the symbol \( \text{im} f \) to denote the full image \( f[A] \) of \( f \). All these notational conventions will apply in particular to tuples \( \mathbf{x} \in A^n , n \in \mathbb{N} \), that we formally understand as maps \( \mathbf{x} : \{0, \ldots, n-1\} \to A \). This does, of course, not preclude us from using a different indexing for the entries of \( \mathbf{x} = (x_1, \ldots, x_n) \), if that seems more handy. So, e.g., we have \( \text{im} \mathbf{x} = \{x_1, \ldots, x_n\} \) and \( f \circ \mathbf{x} = (f(x_1), \ldots, f(x_n)) \in B^n \).

Notably, we are interested in functions of the form \( f: A^n \to A \) that we call \( n \)-ary operations on \( A \). All such operations form the set \( A^{A^n} \), and if we let the parameter \( n \) vary in \( \mathbb{N}_+ \), then we obtain the set \( O_A = \bigcup_{0 < n < \omega} A^{A^n} \) of all finitary (non-nullary) operations over \( A \). If \( F \subseteq O_A \) is any set of finitary operations, we denote by \( F^{(n)} := A^{A^n} \cap F \) its \( n \)-ary part. In particular, \( O_A^{(n)} = A^{A^n} \). Some specific \( n \)-ary operations will be needed: for \( a \in A \) we denote the constant \( n \)-ary function with value \( a \) by \( e_a^{(n)}: A^n \to A \). Moreover, if \( n \in \mathbb{N} \) and \( 1 \leq i \leq n \) we call \( e_i^{(n)}: A^n \to A \), given by \( e_i^{(n)}(x_1, \ldots, x_n) := x_i \) for all \( (x_1, \ldots, x_n) \in A^n \), the \( i \)-th \( n \)-variable projection on \( A \). Collecting all projections on \( A \) in one set, we obtain \( J_A = \{ e_i^{(n)} \mid 1 \leq i \leq n, n \in \mathbb{N} \} \).

We call a set \( F \subseteq O_A \) a (concrete) clone on \( A \) if \( J_A \subseteq F \) and if \( F \) is closed under composition, i.e., whenever \( m, n \in \mathbb{N} \) and \( f \in F^{(m)} \), \( g_1, \ldots, g_n \in F^{(m)} \), then also the composition \( f \circ (g_1, \ldots, g_n) \), given by \( (f \circ (g_1, \ldots, g_n))(\mathbf{x}) := f(g_1(\mathbf{x}), \ldots, g_n(\mathbf{x})) \) for any \( \mathbf{x} \in A^m \), belongs to the set \( F \). All sets of operations that were named ‘clone’ in the introduction are indeed clones in this sense (except for the fact that they were allowed to contain nullary operations, which we want to exclude to avoid unnecessary technicalities). Clones are closed under intersections, and hence for any set \( G \subseteq O_A \) there is a least clone \( F \) under inclusion with the property \( G \subseteq F \). This clone \( F \) is called the clone generated by \( G \) and is denoted as \( \langle G \rangle_{O_A} \). It is computed by adding all projections to \( G \) and then closing under composition, that is, by forming all term operations (of any positive arity) over the algebra \( \langle A; G \rangle \).

A function \( f \in O_A^{(n)} \) preserves a relation \( \varrho \subseteq A^n \) (with \( m, n \in \mathbb{N} \)) if for every \( \mathbf{r} = (r_1, \ldots, r_n) \in \varrho^n \) the tuple \( f \circ \mathbf{r} := (f(r_1(i), \ldots, r_n(i)))_{1 \leq i \leq m} \) belongs to \( \varrho \). For
a set \( Q \) of finitary relations, the set \( \text{Pol}_A Q \) of polymorphisms of \( Q \) consists of all functions preserving all relations belonging to \( Q \). Every polymorphism set is a clone. Dually, for a set \( F \subseteq O_A \), the set \( \text{Inv}_A F \) contains all invariant relations of \( F \), that is, all relations being preserved by all functions in \( F \).

For the convenience of the reader we now give a perhaps more accessible characterisation of the (non-nullary part of the) centraliser \( F^+ \) of some set of operations \( F \subseteq O_A \), which was already defined at the beginning of the introduction. A function \( g: A^m \rightarrow A \) belongs to the centraliser \( F^+ \) (commutes with all functions from \( F \)) if for every function \( f \in F \) the following holds (where \( n \) is the arity of \( f \)): for every matrix \( X \in A^{m \times n} \) applying \( g \) to the \( m \)-tuple obtained from applying \( f \) to the rows of \( X \) gives the same result as evaluating \( f \) on the \( n \)-tuple obtained from applying \( g \) to the columns of the matrix. In symbols:

\[
g((f((x_{ij}))_{i \in m}))_{j \in n} = f((g((x_{ij}))_{i \in m}))_{j \in n}
\]

has to hold for all \((x_{ij})_{i \in m \times j \in n} \in A^{m \times n} \) (and all \( f \in F \)). A brief moment of reflection shows that this condition is the same as saying that \( g: \langle A; F^* \rangle \rightarrow \langle A; F \rangle \) is a homomorphism. A yet different way of saying this is that \( g \) is a polymorphism of \( A = \langle A; F^* \rangle \), that is, \( g \in \text{Pol}_A F^* \) preserves all relations \( F^* = \{ (x, f(x)) | x \in A^n \} \subseteq A^{n+1} \) of all functions \( f \in F \) of any arity \( n \in \mathbb{N} \). From this, it is again clear that \( F^+ \) always must be a clone. On the other hand, it is obvious from the matrix formulation that centralisation is a symmetric condition: for all \( F, G \subseteq O_A \) we have \( G \subseteq F^+ \) if and only if \( F \subseteq G^+ \). Hence, we see that

\[
F^+ = \{ g \in O_A | g \in F^* \} = \{ g \in O_A | F \subseteq \{ g \}^* \} = \{ g \in O_A | \langle F \rangle_{O_A} \subseteq \{ g \}^* \} = \{ g \in O_A | \langle F \rangle_{O_A} \} = \langle F \rangle_{O_A}^*
\]

for every \( F \subseteq O_A \), so the centraliser of a whole clone is not smaller than the centraliser of its generators. Since the clone constructed in Snow’s paper is given in terms of a single generator function, we can thus study its centraliser as the set of all operations commuting with this one generating function.

In the introduction the uniform centraliser degree was defined as the least arity \( n \) such that every centraliser clone \( F \) on a given finite set can be bicentrically generated as \( F = F^{(n)*} \). The following result shows that the search for this number is likewise a search for an arity \( n \) such that every centraliser clone is a centraliser of a set of functions of arity at most \( n \).

**Proposition 1.** For any carrier set \( A \) and an integer \( n \in \mathbb{N} \) the following facts are equivalent:

(a) For every centraliser clone \( F \) we have \( F = F^{(n)**} \).
(b) For every centraliser clone \( F \) we have \( F^{(n)*} = F^* \).
(c) For every centraliser clone \( F \) we have \( F^{(n)*} = F \).
(d) For every centraliser clone \( F \) there is some \( G \subseteq \bigcup_{\ell \leq n} O_A^{(\ell)} \) such that \( F = G^* \).
(e) For every centraliser clone \( F \) there is some \( G \subseteq O_A^{(n)} \) such that \( F = G^* \).
(f) For every set \( F \subseteq O_A \) we have \( F^{(n)*} = F^{**} \).
(g) For every centraliser clone \( F \) we have \( F^{(n)*} = F^* \).

**Proof.** If (a) holds and \( F \) is a centraliser clone, then \( F^* = F^{(n)*} = F^{(n)*} \), so (b) is true. If (b) holds, then \( F = F^* = F^{(n)**} \) for any centraliser clone \( F \), so (a) \( \Leftrightarrow \) (b).

Suppose now that (a), and thus (b), hold. Letting \( G := F^{(n)*} \) for a centraliser clone \( F \), we have \( F = F^{(n)**} = G^* \) from (a). Applying now (b) to the centraliser \( G \) gives \( F = G^* = G^{(n)*} = F^{(n)*} \), so (a) implies (c).

From (c) we get (e) by letting \( G = F^{(n)*} \subseteq O_A^{(n)} \), and (e) directly gives (d).
Now, suppose that (d) holds for $F$ with functions $G$ of arity at most $n$. Since we have excluded nullary operations, this implies that $G \subseteq (G^{(n)}_{O_A})_{O_A}$, so we obtain $G^* \supseteq (G^{(n)}_{O_A})_{O_A} = (G^{(n)}_{O_A})^* = (G^*)_{O_A} = G^*$, which means that $F = G^* = H^*$ where $H := (G^{(n)}_{O_A})_{O_A}$. Thus (e) $\Leftrightarrow$ (d).

From (e), for every $F \subseteq O_A$, we can express the bicentraliser $F^{**} = G^*$ with some $G \subseteq O_A$. Clearly, $G \subseteq G^{**} = F^*$, so $G \subseteq F^{*(n)} \subseteq F^*$. Therefore, we obtain $F^{**} = G^* \supseteq F^{*(n)*} \supseteq F^{**}$, i.e. (f). The latter entails (g) as a special case, for every centraliser clone $F$ satisfies $F^{**} = F$. Moreover, (g) directly gives (e) by letting $G := F^{*(n)} \subseteq O_A$.

It remains to show that (g) implies (a). Namely, for a centraliser clone $F$, applying (g) to $G = F^*$, we get $G = G^{*(n)*} = F^{*(n)*} = F^{(n)*}$, so $F^{(n)**} = G^* = F$.

\[\Box\]

**Remark 2.** A closer inspection of the proof of Proposition 1 reveals that for an individual centraliser clone $F$ the conditions in statements (a) and (b) are equivalent without the universal quantifier. The same holds for the facts (d), (e) and (g).

Let us now assume that $F$ denotes the clone constructed by Snow in [16]. It is our aim to show that there is a separating function $f \in F^{**} \setminus F$. Since the clone $F$ is given in [16] as $F = (\{T\})_{O_A}$ by means of a generating function $T$, once we have selected an $n$-ary candidate function $f$, it is not too hard to show that $f \notin F$. One simply has to describe the $n$-ary term operations of $T$ and to show that $f$ is not among them. The harder part is to choose a suitable function $f \in F^{**}$: by the definition of the bicentraliser one first has to understand the whole set $F^*$ in order to calculate $F^{**}$. As $F^*$ contains functions of all arities this task may require infinitely many steps. Admittedly, there is an upper bound on the arities that have to be considered, but this bound is connected to $\deg(|A|)$ (see (a) $\Leftrightarrow$ (f) in Proposition 1) and hence under current knowledge the number of steps is at least exponentially big.

As a way out of this dilemma, we can however consider upper approximations of $F^{**}$. Namely, if we cut down the centraliser at some arity $\ell$, then $F^{*(\ell)*} \supseteq F^{**}$. The smaller $\ell$ the coarser these approximations are, but also the easier it becomes to describe $F^{*(\ell)}$. In the subsequent section we shall employ a strategy, where we always start with the least interesting arity $\ell = 1$; it turns out that this already produces good results by ruling out many functions that cannot belong to $F^{**}$.

To obtain more information about $F^{*(\ell)}$ for some fixed $\ell$, it will be important to derive as many necessary conditions as possible to help to narrow down the possible candidate functions in the centraliser. This is done by observing that any $g \in F^* = Pol_A F^*$ belongs to $Pol_A Inv_A Pol_A F^*$ and thus has to preserve all relations in the relational clone $Inv_A Pol_A F^*$ generated by the graphs of the functions from $F$. This set contains all relations that can be defined via primitive positive formulæ from $F^* = \{f^* \mid f \in F\}$, and among these there are a few notorious candidates: the image, the set of fixed points and the kernel of any function $f \in F^{(n)}$:

\[
\begin{align*}
\text{im}(f) &= \{ z \in A \mid \exists x_1, \ldots, x_n \in A : z = f(x_1, \ldots, x_n) \}, \\
\text{fix}(f) &= \{ z \in A \mid f(z, \ldots, z) = z \}, \\
\text{ker}(f) &= \{(x_1, \ldots, x_{2n}) \in A^{2n} \mid \exists z \in A : f(x_1, \ldots, x_n) = z = f(x_{n+1}, \ldots, x_{2n}) \}.
\end{align*}
\]

To make this more concrete, we now give the generating function $T \in O_A^{(n^2)}$ for the clone $F = (\{T\})_{O_A}$ where $\mu_F \geq n^2$ on $A = \{0, \ldots, n\}$, $n \geq 2$ (see p. 172 of [16]):

\[
T(x_1, \ldots, x_{n}, x_{21}, \ldots, x_{2n}, \ldots, x_{n1}, \ldots, x_{nn}) = 1 \text{ if } x_{ij} = i \text{ for all } i, j \in \{1, \ldots, n\}
\]
or \( x_{ij} = j \) for all \( i, j \in \{1, \ldots, n\} \), and it is zero for all other arguments. Hence, \( \text{im}(T) = \{0, 1\} \), \( \text{fix}(T) = \{0\} \) and \( \ker(T) \) identifies \((1, 2, \ldots, n, \ldots, 1, 2, \ldots, n)\) with \((1, 1, \ldots, 1, n, n, \ldots, n)\) in one block, and all other \( n^2 \)-tuples in a second block.

Eventually, after we have found a suitable candidate function \( f \notin F = \langle \{T\} \rangle_{\mathcal{O}_A} \), upper approximations \( F^{(\ell)} \) will not any more be enough to prove that \( f \notin F^{**} \) (unless we use an exponentially high value for \( \ell \), cf. Proposition 1(a),(f)). Instead, we can apply a Galois theoretic trick. Namely, \( f \in F^{**} \) if and only if \( F^{*} \subseteq \{f^{*}\} = \text{Pol}_A\{f^{*}\} \), which is equivalent to \( f^{*} \in \text{Inv}_A F^{*} = \text{Inv}_A \text{Pol}_A F^{**} \). As the carrier set is finite, this means that the graph of \( f \) must belong to the relational clone generated from the graphs of functions in \( F \), i.e., that it is primitive positively definable from those graphs. Finding a primitive positive formula, which does the job, requires some creativity, and we will try our best to give some intuition how it can be found in the case where \( |A| = 3 \). For the general case \( |A| = k \geq 3 \) we shall only state the generalisation of the respective formula and verify that it suffices to define the graph of a \((k - 1)\)-ary function that does not belong to \( F \).

3. Separating a clone from its bicentraliser

For the remainder of the paper we let \( A = \{0, 1, \ldots, k - 1\} \) where \( k \geq 3 \), and we consider the clone \( F = \langle \{T\} \rangle_{\mathcal{O}_A} \) constructed by Snow in [16, Section 3]. For the definition of the \((k - 1)^2\)-ary generating function \( T \), see the end of the preceding section.

It is our task to identify some arity \( n \) and some \( n \)-ary operation \( f \in O^{(n)}_A \) such that \( f \in F^{**} = \{T\}^{**} \), but \( f \notin F = \langle \{T\} \rangle_{\mathcal{O}_A} \). In order to avoid a combinatorial explosion of the structure of the involved clones, it is of course desirable to keep the arity \( n \) as low as possible. Hence, we shall start with a description of \( \langle \{T\} \rangle_{\mathcal{O}_A}^{(n)} \) for \( n < k - 1 \). Then, using the method of upper approximations, we shall show that it is impossible to find a separating \( f \in \{T\}^{**} \) of such a low arity. So the next step will be to consider \( n = k - 1 \). Here, we will first study the case \( k = 3 \), where we can show that there is a unique function of arity \( n = k - 1 = 2 \), for which we can prove \( f \in \{T\}^{**} \), but \( f \notin \langle \{T\} \rangle_{\mathcal{O}_A}^{(2)} \). Subsequently, we shall demonstrate that the construction of this particular \( f \) (and the proof of \( f \in \{T\}^{**} \)) can be generalised to any \( k \geq 3 \).

**Lemma 3.** For any \( k \geq 3 \) we have \( \langle \{T\} \rangle_{\mathcal{O}_A}^{(n)} = J_A^{(n)} \cup \{c_0^{(n)}\} \) for all \( 1 \leq n < k - 1 \) where \( c_0^{(n)} \) denotes the \( n \)-ary constant zero function.

**Proof.** We have \( c_0^{(n)} = T \circ (e_1^{(n)}, \ldots, e_1^{(n)}) \) since \( T \) maps every constant tuple to 0. Thus the mentioned functions belong to the \( n \)-ary part of \( \langle \{T\} \rangle_{\mathcal{O}_A} \). Moreover, the given set is a subalgebra of \( \langle A; T \rangle^A \); namely every composition of \( T \) involving only (some of) the \( n \) projections is also \( c_0^{(n)} \) as \( T \) maps every tuple with at most \( n < k - 1 \) distinct entries to zero. \( \square \)

To describe \( \{T\}^{**(n)} \) for \( 0 < n < k - 1 \), we shall study lower approximations of \( \{T\}^* \). We begin by cutting the arity at the level \( \ell = 1 \).
Lemma 4. For $A = \{0, \ldots, k-1\}$ of size $k \geq 3$ we have

$$\{ T \}^{s(1)} = \{ \text{id}_A \} \cup \left\{ f \in O_A^{(1)} \mid f(0) = f(1) = 0 \right\}.$$

Proof. Let us fix $f \in O_A^{(1)}$, commuting with $T$. Since $f \in \text{Pol}_A(\text{fix}(T))$, we have $f(0) = 0$. Moreover, since $f$ preserves the image of $T$, we must have $f(1) \in \{0,1\}$. If $f(1) = 0$, we are done. Otherwise, if $f(1) = 1$, we shall show that $f = \text{id}_A$.

Namely, since $f$ and $T$ commute, we have

$$1 = f(1) = f(T(1, \ldots, 1, 2, \ldots, 2, \ldots, k-1, \ldots, k-1))$$

$$= T(f(1), \ldots, f(1), f(2), \ldots, f(2), \ldots, f(k-1), \ldots, f(k-1)),$$

which implies that

$$(f(1), \ldots, f(1), f(2), \ldots, f(2), \ldots, f(k-1), \ldots, f(k-1))$$

$$\in T^{-1}([1]) \setminus \{(1,2,\ldots,2,\ldots,k-1,\ldots,k-1)\}$$

$$= \{(1,1,2,\ldots,2,\ldots,k-1,\ldots,k-1)\},$$

whence clearly $f(x) = x$ for all $0 < x < k$, i.e. $f = \text{id}_A$.

Conversely, we prove that every $f \in O_A^{(1)}$ with $f(0) = f(1) = 0$ commutes with $T$.

Assume, for a contradiction, that for some $x \in A^{(k-1)^2}$ we had $T(f \circ x) = 1$; then $\{1, \ldots, k-1\} = \text{im } f \circ x \subseteq \text{im } f$, so $f$ would be surjective, and, by finiteness of $A$, bijective. This would contradict $f(0) = f(1) = 0$, so for every $x \in A^{(k-1)^2}$ we have $T(f \circ x) = 0 = f(0) = f(1) = f(T(x))$, since $T(x) \in \{0,1\}$. Thus $f \in \{ T \}^*$.

Corollary 5. For $A = \{0, \ldots, k-1\}$ of cardinality $k \geq 3$, we have the inclusion

$$\{ T \}^{s(1)} \supseteq \{ u_{j,a} \mid a \in A \land j \in A \setminus \{0,1\} \},$$

where $u_{j,a}$ is given by the rule

$$u_{j,a}(x) = \begin{cases} 
    a & \text{ if } x = j, \\
    0 & \text{ otherwise.} 
\end{cases}$$

The binary part of the centraliser already becomes rather obscure in the general case. So we only give a description for the case $k = 3$ (which can certainly also be verified by a brute-force enumeration using a computer).

Lemma 6. For $A = \{0,1,2\}$ the set $\{ T \}^{s(2)}$ contains the following 65 functions

$$\{ T \}^{s(2)} = \left\{ e_1^{(2)}, e_2^{(2)} \right\} \cup \{ z_a \mid a \in \{0,1,2\} \}$$

$$\cup \bigcup_{c\in\{1,2\}} \left\{ f_{a,x} \mid a \in \{0,c\} \land x \in \{0,c\} \land \{(0,0,0,0)\} \right\},$$

given by the following tables$^2$:

| $z_a(x)\setminus y$ | 0 | 1 | 2 |
|---------------------|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 |
| 2 | 0 | 0 | a |

$^1$For $k = 3$ the correctness of this lemma can be checked with the Z3-solver [10, 15] using the ancillary file unaryfunccommutingT.z3. It can also be seen from the file Tcent1.txt produced by the function findallunaries() from the ancillary file commutationTs.cpp.

$^2$The correctness of this lemma (and its proof) can be checked with the Z3-solver [10, 15] using the ancillary file binaryfunccommutingT.z3. The completeness of the list of 65 operations can also be verified with the function findallbinaries() from the ancillary file commutationTs.cpp, resulting in the file Tcent2.txt.
Proof. Using a case distinction, one can verify that every function $f \in \{T\}^{*^{(2)}}$ must be among the ones mentioned in the lemma.

1. Assume $f(1, 1) = 1$. We can show that $f(2, 2) = 2$ and \{f(1, 2), f(2, 1)\} = \{1, 2\}. Namely, $f \in \{T\}^*$ implies
\[
1 = f(1, 1) = f(T(1, 1, 2, 2), T(1, 2, 1, 2))
= T(f(1, 1), f(1, 2), f(2, 1), f(2, 2)) = T(1, f(1, 2), f(2, 1), f(2, 2)),
\]
which is only possible if $f(2, 2) = 2$ and \{f(1, 2), f(2, 1)\} = \{1, 2\}.

1.1. Assume $f(1, 2) = 1$ and $f(2, 1) = 2$. It follows that $f = e_1^{(2)}$. In fact, our assumption $f \in \{T\}^*$ implies
\[
f(1, 0) = f(T(1, 1, 2, 2), T(1, 1, 1, 2))
= T(f(1, 1), f(1, 1), f(2, 1), f(2, 2)) = T(1, 1, 2, 2) = 1,
\]
moreover
\[
f(0, 1) = f(T(1, 1, 1, 2), T(1, 1, 2, 2))
= T(f(1, 1), f(1, 1), f(1, 2), f(2, 2)) = T(1, 1, 1, 2) = 0,
\]
and
\[
1 = f(1, 0) = f(T(1, 2, 1, 2), T(1, 0, 1, 2))
= T(f(1, 1), f(2, 0), f(1, 1), f(2, 2)) = T(1, f(2, 0), 1, 2),
\]
which is only possible if $f(2, 0) = 2$. Finally, we have
\[
0 = f(0, 1) = f(T(1, 0, 1, 2), T(1, 2, 1, 2))
= T(f(1, 1), f(0, 2), f(1, 1), f(2, 2)) = T(1, f(0, 2), 1, 2),
\]
which means $f(0, 2) \neq 2$, and
\[
0 = f(0, 0) = f(T(0, 2, 1, 2), T(2, 2, 1, 2))
= T(f(0, 2), f(2, 2), f(1, 1), f(2, 2)) = T(f(0, 2), 2, 1, 2),
\]
which gives $f(0, 2) \neq 1$. Thus $f(0, 2) \in A \setminus \{1, 2\} = \{0\}$.

1.2. Assume $f(1, 2) = 2$ and $f(2, 1) = 1$. It follows that $f = e_2^{(2)}$ by a dual argument. In fact, $f \in \{T\}^*$ implies
\[
f(1, 0) = f(T(1, 1, 2, 2), T(1, 1, 1, 2))
= T(f(1, 1), f(1, 1), f(2, 1), f(2, 2)) = T(1, 1, 1, 2) = 0,
\]
moreover
\[
f(0, 1) = f(T(1, 1, 1, 2), T(1, 1, 2, 2))
= T(f(1, 1), f(1, 1), f(1, 2), f(2, 2)) = T(1, 1, 2, 2) = 1,
\]
and
\[
1 = f(0, 1) = f(T(1, 0, 1, 2), T(1, 2, 1, 2))
= T(f(1, 1), f(0, 2), f(1, 1), f(2, 2)) = T(1, f(0, 2), 1, 2),
\]
which is only possible if $f(0, 2) = 2$. Finally, we have
\[
0 = f(1, 0) = f(T(1, 2, 1, 2), T(1, 0, 1, 2))
= T(f(1, 1), f(2, 0), f(1, 1), f(2, 2)) = T(1, f(2, 0), 1, 2),
\]
which means \( f(2, 0) \neq 2 \), and
\[
0 = f(0, 0) = f(T(2, 2, 1, 2, T(0, 2, 1, 2))
\]
\[
= T(f(2, 0), f(2, 2), f(1, 1), f(2, 2)) = T(f(2, 0), 2, 1, 2),
\]
which gives \( f(2, 0) \neq 1 \). Thus \( f(2, 0) \in A \setminus \{1, 2\} = \{0\}. 

2. Now assume that \( f(1, 1) \neq 1 \). Since \( f \in \text{Pol}_A \text{im}(T) \), we must have \( f(1, 1) = 0 \). We can show that \( f(0, 1) = 0 = f(1, 0) \). In point of fact, we have
\[
f(0, 1) = f(T(1, 1, 0, 0), T(1, 1, 2, 2))
\]
\[
= T(f(1, 1), f(1, 1), f(0, 2), f(0, 2)) = T(0, 0, f(0, 2), f(0, 2)) = 0,
\]
and for \( f(1, 0) = 0 \) we argue by swapping the arguments of \( f \).

Moreover, if \( \{1, 2\} \not\subseteq \{f(0, 2), f(2, 0), f(1, 2), f(2, 1)\} \), then \( f \notin \{T\}^* \). Indeed, if there are \( x, y \in \{0, 1\} \) such that
(a) \( f(x, 2) = 1, f(y, 2) = 2 \), then
\[
f(T(2, 2, 2, 2), T(x, x, y, y)) = f(0, z) = 0
\]
\[
\neq 1 = T(1, 1, 2, 2) = T(f(2, x), f(2, x), f(y, y), f(2, y)),
\]
where \( z \in \{0, 1\} \).

(b) \( f(x, 2) = 1, f(y, 2) = 2 \), then we argue with swapped arguments for \( f \).

(c) \( f(2, x) = 1, f(y, 2) = 2 \), then
\[
f(T(2, 2, y, y), T(x, x, 2, 2)) = f(0, z) = 0
\]
\[
\neq 1 = T(1, 1, 2, 2) = T(f(2, x), f(2, x), f(y, y), f(2, y)),
\]
where \( z \in \{0, 1\} \).

(d) \( f(x, 2) = 1, f(2, y) = 2 \), then we argue with swapped arguments for \( f \).

Hence, we know that \( \{1, 2\} \not\subseteq \{f(0, 2), f(2, 0), f(1, 2), f(2, 1)\} \) for \( f \in \{T\}^* \).

2.1. Suppose that \( f(2, 2) = 0 \). There is nothing left to prove: we already have
\[
f \in \{z_0\} \cup \bigcup_{e \in \{1, 2\}} \{f_{0,x} | x \in \{0, c\}^4 \setminus \{0\}\}.
\]

2.2. Suppose that \( f(2, 2) = c \in \{1, 2\} \) and let \( d \) be such that \( \{c, d\} = \{1, 2\} \).

We prove that \( d \notin \{f(0, 2), f(2, 0), f(1, 2), f(2, 1)\} \), as otherwise \( f \notin \{T\}^* \).

This demonstrates that \( \{f(0, 2), f(2, 0), f(1, 2), f(2, 1)\} \subseteq \{0, c\} \), so we have
\[
f \in \{z_c\} \cup \bigcup_{e \in \{1, 2\}} \{f_{e,x} | x \in \{0, c\}^4 \setminus \{0\}\}.
\]

For a contradiction suppose that there is some argument \( x \in \{0, 1\} \) such that \( f(x, 2) = d \). Then for some \( z \in \{0, 1\} \) we have
\[
f(T(x, x, 2, 2), T(2, 2, 2, 2)) = f(z, 0) = 0
\]
\[
\neq 1 = T(d, d, c, c) = T(f(x, 2), f(x, 2), f(2, 2), f(2, 2)),
\]
when \( (c, d) = (2, 1) \), and
\[
f(T(2, 2, x, x), T(2, 2, 2, 2)) = f(z, 0) = 0
\]
\[
\neq 1 = T(c, c, d, d) = T(f(2, 2), f(2, 2), f(x, 2), f(x, 2)),
\]
when \( (c, d) = (1, 2) \). In the case where \( f(2, x) = d \) for some \( x \in \{0, 1\} \) we argue similarly, by swapping the arguments of \( f \).

For the converse inclusion, we have to check that all mentioned functions commute with \( T \). So let \( g = z_a \) for some \( a \in A \) or \( g = f_{a,(b,c,d,e)} \) and consider \( x_1, \ldots, x_4, y_1, \ldots, y_4 \in A \) to verify that \( g \) commutes with \( T \). Put \( u := T(x_1, \ldots, x_4) \) and \( v := T(y_1, \ldots, y_4) \). Since \( \{u, v\} \subseteq \text{im}(T)^2 = \{0, 1\}^2 \), we have \( g(u, v) = 0 \). On the other hand, the values \( w_i := g(x_i, y_i) \) for \( 1 \leq i \leq 4 \) belong to \( \text{im}(g) \subseteq \{0, a, b, c, d, e\} \). If at least one of them equals 0, then \( T(w_1, \ldots, w_4) = 0 \) as needed. Otherwise, all of
them belong to \( \{a, b, c, d, e\} \setminus \{0\} \). If \( g = z_a \), then they are all equal to \( a \) and we thus have \( T(w_1, \ldots, w_4) = 0 \), too. In the case that \( g = f_{a,b,c,d,e} \), we know from the definition of \( g \) that \( \{a, b, c, d, e\} \subseteq \{0, j\} \) for some \( j \in \{1, 2\} \). Thus, \( w_1 = \cdots = w_4 = j \), and again \( T(w_1, \ldots, w_4) = 0 \). In any case, we have shown \( g \in \{T\}^* \).

Next, with the help of the coarse approximations from Lemma 4, we observe that the bicentraliser of \( T \) only contains functions that are close to being conservative and have many congruences.

**Lemma 7.** For \( A = \{0, \ldots, k - 1\} \) of size \( k \geq 3 \) we have

\[
\langle\{T\}\rangle_{\text{OA}}^n \subseteq \{T\}^* \subseteq \{T\}^{(1)*}
\]

\[
\subseteq \text{Pol}_A\{U \subseteq A \mid 0 \subseteq U\} \cap \text{Pol}_A\{\theta \in \text{Eq}(A) \mid (0, 1) \in \theta\},
\]

where \( \text{Eq}(A) \) denotes the set of all equivalence relations on \( A \).

**Proof.** It is clear that \( \{T\}^{(1)*} \subseteq \text{Pol}_A\{\ker(f) \mid f \in \{T\}^{(1)}\} \) since the image of a function is primitive positively definable from its graph. If \( 0 \subseteq U \subseteq A \), then \( U \) contains \( t < k - 1 \) elements distinct from \( 0 \). According to the description of the functions in \( \{T\}^{(1)} \) given in Lemma 4, there is some \( f \in \{T\}^{(1)} \) whose image is \( U \).

Likewise we have \( \{T\}^{(1)*} \subseteq \text{Pol}_A\{\ker(f) \mid f \in \{T\}^{(1)}\} \) since the kernel of a function is primitive positively definable from its graph. Any partition of \( A \) having a class containing the set \( \{0, 1\} \) can again be realised as the kernel of a function \( f \in \{T\}^{(1)} \) since the value \( f(x) \) can be chosen arbitrarily for every \( x \in A \setminus \{0, 1\} \).

Based on this lemma we can show that the \( n \)-ary part of the bicentraliser of \( T \) is not bigger than \( \langle\{T\}\rangle_{\text{OA}}^{(n)} \) when \( n < k - 1 \).

**Lemma 8.** For \( k = |A| \geq 3 \) we have \( \{T\}^{(n)} = \{T\}^{(1 SWITCH}\} = \mathcal{J}_A^{(n)} \cup \mathcal{J}_0^{(n)} \) for all \( 1 \leq n < k - 1 \) where \( \mathcal{J}_0^{(n)} \) denotes the \( n \)-ary constant zero function.

**Proof.** We shall prove that \( \{T\}^{(1)*}(n) \subseteq \mathcal{J}_A^{(n)} \cup \mathcal{J}_0^{(n)} \) by Lemma 7 we know that \( f \in \text{Pol}_A\{0, 2, \ldots, n + 1\} \), so we obtain \( b := f(2, \ldots, n + 1) \in \{0, 2, \ldots, n + 1\} \). Now for any \( (a_1, \ldots, a_n) \in A^n \) we consider the unary map \( u \) sending \( j \mapsto a_{j-1} \) for \( 2 \leq j \leq n + 1 \) and \( j \mapsto 0 \) otherwise. Since \( u \in \{T\}^{(1)} \) by Lemma 4, we have \( f \in \{u\}^* \) and thus

\[
f(a_1, \ldots, a_n) = f(u(2), \ldots, u(n + 1)) = u(f(2, \ldots, n + 1)) = u(b).
\]

If \( b = 0 \), then \( f(a_1, \ldots, a_n) = u(b) = 0 \), so \( f = \mathcal{J}_0^{(n)} \). If \( b \neq 0 \), then it follows that \( 2 \leq b \leq n + 1 \). Thus, we have \( f(a_1, \ldots, a_n) = u(b) = a_{b-1} \) for all \( (a_1, \ldots, a_n) \in A^n \), which shows that \( f = \mathcal{J}_b^{(n)} \).

According to Lemmata 3 and 8, it is impossible to find \( f \in \{T\}^{(n)} \setminus \langle\{T\}\rangle_{\text{OA}}^{(n)} \) for \( n < k - 1 \) where \( k = |A| \). Next, we thus turn our attention to \( n = k - 1 \), where we will first describe \( \langle\{T\}\rangle_{\text{OA}}^{(k-1)^*} \) beside projections the \( (k - 1) \)-ary part of \( \langle\{T\}\rangle_{\text{OA}}^{(k-1)} \) contains only functions being zero everywhere with a possible exception in only one argument tuple which may be sent to one. After that we shall focus for a while on
the case $k = 3$ to develop the right ideas in connection with $\{T\}^{**(k-1)} = \{T\}^{**(2)}$, which can eventually be generalised to any $k \geq 3$.

**Lemma 9.** Given a set $A$ of cardinality $k \geq 3$, put $n = k - 1$. We then have $\langle\{T\}\rangle_{O_A}^{(n)} = J_A^{(n)} \cup \{e_0^{(n)}\} \cup F$ where $F \subseteq O_A^{(n)}$ is the set of $n$-ary functions in $\langle\{T\}\rangle_{O_A}$ which map exactly one $n$-tuple to 1 and everything else to 0.

**Proof.** We have $e_0^{(n)} = T \circ (e_1^{(n)}, \ldots, e_i^{(n)}) \in \langle\{T\}\rangle_{O_A}^{(n)}$ as in Lemma 3, so the inclusion $G := J_A^{(n)} \cup \{e_0^{(n)}\} \cup F \subseteq \langle\{T\}\rangle_{O_A}^{(n)}$ is clear. For the opposite inclusion, we prove that $G$ is a subuniverse of $\langle A; T \rangle^A$. The first step is to check that any variable identification of $T$ with at most $n$ variables ends up in $F \cup \{e_0^{(n)}\}$.

Let $i: \{1, \ldots, n^2\} \rightarrow \{1, \ldots, n\}$, $j \mapsto i_j$ be a map describing an $n$-variable identification $f = T \circ (e_1^{(n)}, \ldots, e_i^{(n)})$ of $T$. Clearly, $\text{im}(f) \subseteq \text{im}(T) = \{0, 1\}$, so every tuple that is not mapped to one by $f$ will be sent to zero. To obtain a contradiction, let us assume that $|f^{-1}(\{1\})| \geq 2$. So there are tuples $x \neq y \in A^n$ such that $T(x_{i_1}, \ldots, x_{i_{n^2}}) = 1 = T(y_{i_1}, \ldots, y_{i_{n^2}})$. The preimage $T^{-1}(\{1\})$ contains only two tuples, and these mention $n$ distinct elements. To obtain one of them in the form $(x_{i_1}, \ldots, x_{i_{n^2}})$ or $(y_{i_1}, \ldots, y_{i_{n^2}})$ one has to use at least $n$ distinct variable indices, so the map $i$ has to be surjective. It is therefore impossible that the distinct tuples $x$ and $y$ produce the same tuple $(x_{i_1}, \ldots, x_{i_{n^2}}) = (y_{i_1}, \ldots, y_{i_{n^2}}) \in T^{-1}(\{1\})$.

This means, one of them, say $x$, gives $(x_{i_1}, \ldots, x_{i_{n^2}}) = (1, n, 1, \ldots, 1, \ldots, n)$, from which it follows that $i_1, \ldots, i_n$ are all distinct (so $\{i_1, \ldots, i_n\} = \{1, \ldots, n\}$); the other one however produces $(y_{i_1}, \ldots, y_{i_{n^2}}) = (1, 1, 2, \ldots, 2, n, \ldots, n)$. This implies that $\{y_{i_1}, \ldots, y_{i_n}\} = \{y_{i_{n+1}}, \ldots, y_{i_{n^2}}\} = \{1\}$, so $\{y_{i_1}, \ldots, y_{i_{n^2}}\} = \{1\}$, which is a contradiction for $n \geq 2$.

To prove that $G$ is closed under application of $T$, we take functions $f_1, \ldots, f_{n^2}$ from $G$ and show that $f = T \circ (f_1, \ldots, f_{n^2}) \in G$. If $f_1, \ldots, f_{n^2} \in J_A$, then the composition is a variable identification of $T$ that belongs to $F \cup \{e_0^{(n)}\} \subseteq G$.

Otherwise, suppose that (for some $1 \leq j \leq n^2$) $f_j$ is a non-projection in $F \cup \{e_0^{(n)}\}$. For every $x \in A^n$ with possibly one exception we have $f_j(x) = 0$. So for all those arguments $x \in A^n$, the $j$-th component of $(f_1(x), \ldots, f_{n^2}(x))$ contains a zero, whence this tuple is mapped to zero by $T$. Consequently, $f(x) = 0$ for all but possibly one $x \in A^n$, so $f \in F \cup \{e_0^{(n)}\} \subseteq G$. \hfill $\square$

For three-element domains we obtain a more specific result.

**Lemma 10.** For $A = \{0, 1, 2\}$ we have $\langle\{T\}\rangle_{O_A}^{(2)} = \{e_1^{(2)}, e_2^{(2)}, e_0^{(2)}, \delta_{(1,2)}, \delta_{(2,1)}\}$, where $e_0^{(2)}$ is the constant zero function and $\delta_a(x) = 1$ if $x = a$ and $\delta_a(x) = 0$ otherwise.

**Proof.** It is easy to see that the listed binary functions belong to the clone, namely $e_0^{(2)} = T \circ (e_1^{(2)}, e_1^{(2)}, e_1^{(2)}, e_1^{(2)}),$

$\delta_{(1,2)} = T \circ (e_1^{(2)}, e_1^{(2)}, e_1^{(2)}, e_1^{(2)}) = T \circ (e_1^{(2)}, e_1^{(2)}, e_2^{(2)}, e_2^{(2)}),$

$\delta_{(2,1)} = T \circ (e_1^{(2)}, e_1^{(2)}, e_1^{(2)}, e_1^{(2)}) = T \circ (e_2^{(2)}, e_2^{(2)}, e_1^{(2)}, e_1^{(2)}).$

It is not hard to verify that the given subset is a subuniverse of $\langle A; T \rangle^A$. Any $T$-composition involving only projections except for the ones shown to yield $\delta_{(1,2)}$
or \(\delta_{(2,1)}\) produces \(c_{(2)}^{(2)}\). Any composition involving \(c_{(2)}^{(2)}\), or just the \(\delta_a\) functions yields again the constant map \(c_0^{(2)}\). Therefore, only compositions involving the \(\delta_a\) functions and projections have to be checked. If all four of them are substituted into \(T\) (in any order), the result is \(c_0^{(2)}\). If only one projection (and possibly some non-projections) are substituted, then in most cases, the result is \(c_0^{(2)}\), and for a few substitutions it is one of the \(\delta_a\) functions. If both projections and only one of the \(\delta_a\) functions are substituted, the result is either the substituted function \(\delta_a\) or \(c_0^{(2)}\). \(\square\)

With the aim of finding separating binary functions in \(\{T\}^{**}\) for \(|A|=3\), we collect some properties of binary operations in upper approximations of \(\{T\}^{**}\).

**Lemma 11.** Let \(A = \{0, 1, 2\}\) and \(g \in \{T\}^{(1)+2}\), then the following implications hold:

\[\begin{align*}
(a) & \quad g(1,2) = 2 \implies \forall a \in A: g(0,a) = a. \\
(b) & \quad g(2,1) = 2 \implies \forall a \in A: g(a,0) = a. \\
(c) & \quad g(1,2) \in \{0,1\} \implies \forall a \in A: g(0,a) = 0. \\
(d) & \quad g(2,1) \in \{0,1\} \implies \forall a \in A: g(a,0) = 0.
\end{align*}\]

**Proof.** By Corollary 5 we have \(g \in \{u_{2,a} \mid a \in A\}^*\). This implies for all \(a \in A\) that \(a = u_{2,a}(2) = u_{2,a}(g(1,2)) = g(u_{2,a}(1), u_{2,a}(2)) = g(0,a)\) provided \(g(1,2) = 2\). A symmetric argument works for \(g(2,1) = 2\). Similarly, if \(g(1,2) \in \{0,1\}\), then \(0 = u_{2,a}(g(1,2)) = g(u_{2,a}(1), u_{2,a}(2)) = g(0,a)\) for all \(a \in A\), and symmetrically, if \(g(2,1) \in \{0,1\}\). \(\square\)

Not very surprisingly, \(\{T\}^{(1)+2}\) does not encode enough information about \(\{T\}^+\) to determine functions in \(\{T\}^{**}\) sufficiently well. However, using the description of \(\{T\}^{(2)+2}\) available for \(|A|=3\) from Lemma 6, we are able to derive a more promising result: for \(|A|=3\) there is a unique binary function in \(\{T\}^{(2)+2} \setminus \langle\{T\}\rangle_{O_A}\). This function might—and although we do not know it yet at this point, it actually will—serve to distinguish \(\{T\}^+\) and \(\langle\{T\}\rangle_{O_A}\).

**Lemma 12.** For \(A = \{0, 1, 2\}\) we have\(^{3}\) \(\{T\}^{(2)+2} = \langle\{T\}\rangle_{O_A}^2 \cup \{f\}\) where for all \(x, y \in A\)

\[f(x, y) = \begin{cases} 1 & \text{if } \{x, y\} = \{1, 2\}, \\ 0 & \text{else.} \end{cases}\]

**Proof.** The proof is by a systematic case distinction. Let \(g \in \{T\}^{(2)+2}\), which implies that \(g \in \{T\}^{+2}_{\{T\}^+} = \langle\{T\}^{(2)+2}\rangle_{O_A} \subseteq \langle\{T\}^{+2}\rangle_{O_A}\) since \(\{T\}^{+2} \subseteq \langle\{T\}^{+2}\rangle_{O_A}\). Hence, we can apply the implications from Lemma 11 to \(g\).

Assume \(g(1,2) = 2\). It follows by Lemma 11 that \(g(0,a) = a\) for all \(a \in A\). Our goal is to show that \(g = c_2^{(2)}\). For a contradiction, suppose that \(g(2,1) = 2\). Since \(g \in \{z_1\}^+\) by Lemma 6, we obtain

\[1 = z_1(2,2) = z_1(g(1,2), g(2,1)) = g(z_1(1,2), z_1(2,1)) = g(0,0),\]

in contradiction to \(g(0,0) = 0\) derived above. Hence \(g(2,1) \in \{0,1\}\). Using again Lemma 11, this implies \(g(a,0) = 0\) for all \(a \in A\).

\(^3\)The correctness of this lemma can be checked with the Z3-solver [10, 15] using the ancillary file \texttt{func_Te2c2.z3}.
Again, for a contradiction, we suppose that $g(2, 1) = 0$. Since $g \in \{f_{0, (1, 1, 1, 0)}\}^*$ by Lemma 6, we get

\[ 1 = f_{0, (1, 1, 1, 0)}(2, 0) = f_{0, (1, 1, 1, 0)}(g(1, 2), g(2, 1)) \\
= g(f_{0, (1, 1, 1, 0)}(1, 2), f_{0, (1, 1, 1, 0)}(2, 1)) = g(1, 0), \]

which contradicts $g(1, 0) = 0$.

Hence $g(2, 1) = 1$. Then, since $g \in \{f_{0, (c, c, c, c)}\}^*$ for $c \in \{1, 2\}$ by Lemma 6, we get

\[ c = f_{0, (c, c, c, c)}(2, 1) = f_{0, (c, c, c, c)}(g(1, 2), g(2, 1)) \\
= g(f_{0, (c, c, c, c)}(1, 2), f_{0, (c, c, c, c)}(2, 1)) = g(c, c), \]

which shows that $g = e_2^{(2)}$. Note that a symmetric argument shows that the assumption $g(2, 1) = 2$ implies $g = e_1^{(2)}$.

Assume $\{g(1, 2), g(2, 1)\} \subseteq \{0, 1\}$. By Lemma 11 we get that $g(0, a) = g(a, 0) = 0$ for all $a \in A$. Clearly, $h = g \circ (\text{id}_A, \text{id}_A) \in \{T\}^*_{\{1\}} \subseteq \{T\}^*_{\{2\}}$ is trivially true and one verifies that, indeed, $h \in \{T\}^*_{\{2\}}$. We postpone the latter until Lemma 14, where we shall show more generally that even $f \in \{T\}^*_{\{2\}}$. Alternatively, one may ask a computer to check that $f$ commutes with all the 65 functions given in Lemma 6, immediately giving a positive answer.  

So far, for the binary operation $f$ exhibited in Lemma 12 we do not know whether it actually belongs to $\{T\}^*_{\{2\}}$ as we have only worked with upper approximations of this bicentraliser, not with $\{T\}^*_{\{2\}}$ itself.

**Remark 13.** Without much more ingenuity but some additional computational effort, it is possible to show that the unique binary operation $f$ from Lemma 12 belongs to $\{T\}^*_{\{3\}}$, which is even closer to $\{T\}^*_{\{2\}}$.

To do this one needs to enumerate $\{T\}^*_{\{3\}}$. Since $\{T\}^*$ is a clone, for every ternary $g \in \{T\}^*$ each of its identification minors $g \circ (e_1^{(2)}, e_1^{(2)}, e_2^{(2)})$, $g \circ (e_1^{(2)}), e_2^{(2)}, e_1^{(2)})$ and $g \circ (e_2^{(2)}, e_1^{(2)}, e_1^{(2)})$ must also belong to the same clone, i.e. to $\{T\}^*_{\{2\}}$. However, the latter set has been completely described in Lemma 6 above, it contains precisely 65 functions. Thus, the behaviour of $g$ on tuples of the form $(x, y)$, and $g$ on tuples of the form $(x, x, x)$ has to coincide with one of these 65 functions, likewise, the results on tuples of the form $(y, y, x)$ and of the form $(y, x, x)$ are determined by one of these functions, respectively. Moreover, on the three tuples of the form $(x, x, x)$, the three binary operations from $\{T\}^*_{\{2\}}$ have to prescribe non-contradictory values. Therefore, except for the six tuples that are permutations of $(0, 1, 2)$ the values of $g$ are determined by

\footnote{This can, for example, be done with the Z3-solver [10, 15] using the ancillary file func_Tc2c2_x3.}
one of at most $65^3$ choices. Altogether no more than $65^3 \cdot 3^6 = 200\,201\,625$ ternary functions have to be considered.

This can be done by a computer, resulting in a list\(^5\) of exactly $1\,048\,578$ functions belonging to $\{T\}^{*(3)}$. Again for each of these ternary operations it is readily verified by a computer that they commute\(^6\) with the binary operation $f$ given in Lemma 12. Consequently, by a complete case distinction, we have indeed that $f \in \{T\}^{*(3)}$. Together with Lemma 10, this proves $f \in \{T\}^{*(3)} \setminus \{(T)\}_{0,1}^{(2)}$ for $A = \{0, 1, 2\}$.

It is not a suitable strategy to continue indefinitely with individual verifications that the unique binary operation $f$ from Lemma 12 belongs to and more accurate upper approximations $\{T\}^{*(k)}$, $k \to \infty$, of $\{T\}^{**}$. Instead we need a more creative Galois theoretic argument to be sure that $f \in \{T\}^{**}$. This confirmation is given in the following lemma in the form of a primitive positive definition. As it turns out, the argument used there for $k = |A| = 3$ and the definition of $f$ from Lemma 12 can then be generalised to any $k \geq 3$, see Theorem 15. However, we think it is instructive to first show where the idea for the theorem originates from.

Lemma 14. The binary function $f \in \{T\}^{*(2)**}$ defined in Lemma 12 indeed belongs to $\{T\}^{**}$ for its graph is definable by a primitive positive formula\(^7\) over $A = \{0, 1, 2\}$ involving only the graph of $T$:

\[
\{(x_2, x_3, x_5) \in A^3 \mid f(x_2, x_3) = x_5\} = \{(x_2, x_3, x_5) \in A^3 \mid \exists x_4 \in A: (x_2, x_3, x_5) \in \ker(T) \land (x_2, x_3, x_4, x_2) \in \ker(T) \land T(x_2, x_3, x_4) = x_5 \land T(x_2, x_3, x_2) = u \land T(x_3, x_2, x_3) = v \land T(x_1, x_3, x_4, x_2) = v\}
\]

Proof. The idea how to construct the graph of $f$ is by considering the full graph of $T$, that is, the relation

\[
\{(x_1, x_2, x_3, x_4, x_5) \in A^5 \mid T(x_1, x_2, x_3, x_4) = x_5\},
\]

and to project it to the second, third and fifth coordinate. This is motivated by the fact that $T$ sends only two arguments, $(1, 1, 2, 2)$ and $(1, 2, 1, 2)$, to one and every other quadruple to zero, and the middle two components of the two mentioned quadruples coincide with those pairs that are mapped to one by $f$. Of course, such a projection will not result in a function graph, but it almost does. The pairs $(1, 2)$ and $(2, 1)$ will be assigned two values each: the value one (as desired for $f$) and an erroneous value zero caused by some other quadruples $(x_1, x_2, x_3, x_4)$ with the same middle component $(1, 2)$ or $(2, 1)$. Hence, the goal is to remove those quadruples from the relation before projecting. There are 16 disturbing argument tuples in the

\(^{5}\)This list can be computed using the function `findallternaries_optimised()` from the ancillary file `computationTs.cpp`, and it is given in the file `Tcent3_sorted.txt`.

\(^{6}\)This verification can be carried out using the function `readTcent3("Tcent3_sorted.txt")` from the ancillary file `computationTs.cpp` and confirms once more the concluding sentence in the proof of Lemma 12.

\(^{7}\)The correctness of this formula has been checked with the Z3-solver \cite{10,15}, see the script `checkformulaforbinfunc.z3` available as an ancillary file.
Theorem 15. Let

\[
\{ (u, a, b, v) \mid \{a, b\} = \{1, 2\}, u, v \in A \} \setminus \{(1, 1, 2, 2), (1, 2, 1, 2)\}.
\]

They need to be removed by imposing additional conditions that have to be satisfied by the quadruples \((1, 1, 2, 2)\) and \((1, 2, 1, 2)\) since we have to ensure that these are kept in the relation.

It turns out that this is possible by imposing just two additional requirements involving the kernel of \(T\). The kernel is an equivalence relation on quadruples that we interpret as an octonary relation on \(A\), and it partitions \(A^4\) into two classes: \(\{(1, 2, 1, 2), (1, 1, 2, 2)\}\) and the complement \(B\) of this set in \(A^4\). In particular \(B\) includes all tuples containing a zero or three ones or three twos or a two in the first position or a one in the last position. Using this observation it is easy to verify that the following two sets jointly (i.e. their intersection) exclude all 16 undesired quadruples. So these two sets represent the restrictions that we are going to apply to the graph of \(T\):

\[
\{ (x_1, \ldots, x_4) \in A^4 \mid T(x_2, x_3, x_2, x_3) = T(x_1, x_2, x_4, x_3) \} = A^4 \setminus \{ \begin{array}{c}
0001112222 \\
1111121111 \\
2222222222 \\
0120110112
\end{array} \}
\]

\[
\{ (x_1, \ldots, x_4) \in A^4 \mid T(x_3, x_2, x_3, x_2) = T(x_1, x_3, x_4, x_2) \} = A^4 \setminus \{ \begin{array}{c}
0001112222 \\
2222222222 \\
1111112111 \\
0120110112
\end{array} \}
\]

Both sets also exclude the tuple \((1, 2, 2, 1)\), but this is not harmful, as there are sufficiently many other quadruples left having \((2, 2)\) as their middle component, for example \((0, 2, 2, 0)\).

As the arity of \(T\) is \((k - 1)^2\) where \(k = |A|\), it is perhaps helpful to arrange the arguments of \(T\) in a \((k - 1) \times (k - 1)\)-square. Expressing the primitive positive formula from Lemma 14 using such \((2 \times 2)\)-squares then yields

\[
\exists x_1, x_4 \in \{0, 1, 2\} : T(\frac{x_1}{x_2}, \frac{x_3}{x_4}) = x_5 \land T(\frac{x_1}{x_3}, \frac{x_2}{x_4}) = T(\frac{x_1}{x_3}, \frac{x_2}{x_4}) \land T(\frac{x_1}{x_3}, \frac{x_2}{x_4}) = T(\frac{x_1}{x_3}, \frac{x_2}{x_4}).
\]

This kind of interpretation is key for the understanding of the following main result.

Theorem 15. Let \(A = \{0, \ldots, k - 1\}\) where \(k \geq 3\) and put \(n = k - 1\). Let the function \(f : A^n \rightarrow A\) be defined by

\[
f(x) = \begin{cases} 
1 & \text{if } x \in \{\uparrow, \downarrow\}, \\
0 & \text{else,}
\end{cases}
\]

where \(\uparrow = (1, \ldots, n)\) and \(\downarrow = (n, \ldots, 1)\). The graph of \(f\) can be defined by a primitive positive formula using the graph of \(T\) as follows:

\[
\{ (\langle, y) \in A^k \mid f(\langle) = y \} = \begin{cases} 
\{ (\langle, y) \in A^k \mid \exists x_{ij} \in A_{1 \leq i, j \leq n} : \begin{array}{c}
\forall i+j \neq k : T(\rightarrow_1, \rightarrow_2, \ldots, \rightarrow_n) = y \\
T(\langle, \rightarrow_1, \rightarrow_2, \ldots, \rightarrow_n) = T(\rightarrow_1, \rightarrow_2, \ldots, \rightarrow_n) \\
T(\langle, \rightarrow_1, \rightarrow_2, \ldots, \rightarrow_n) = T(\rightarrow_1, \rightarrow_2, \ldots, \rightarrow_n)
\end{array} \\
\text{\(T(\rightarrow_1, \rightarrow_2, \ldots, \rightarrow_n) = y \land\)} \\
\text{\(T(\langle, \rightarrow_1, \rightarrow_2, \ldots, \rightarrow_n) = u \land\)} \\
\text{\(T(\langle, \rightarrow_1, \rightarrow_2, \ldots, \rightarrow_n) = v \land\)} \\
\text{\(T(\downarrow_1, \downarrow_2, \ldots, \downarrow_n) = u\)} \\
\text{\(T(\downarrow_1, \downarrow_2, \ldots, \downarrow_n) = v\)) \}
\end{cases}
\]
where the arrows represent the following sequences of variables for $1 \leq i \leq n$:

\[
\begin{align*}
\rightarrow &= x_{1,n}, x_{2,n-1}, \ldots, x_{n-1,2}, x_{n,1} \\
\leftarrow &= x_{n,1}, x_{n-1,2}, \ldots, x_{2,n-1}, x_{1,n} \\
\downarrow &= x_{i,1}, \ldots, x_{i,n} \\
\uparrow &= x_{n,i}, \ldots, x_{1,i}
\end{align*}
\]

Proof. We imagine the $n^2$ variables of $T$ arranged in a square as follows

\[
\begin{array}{c}
\square = \begin{cases} \\
\end{cases} \\
\vdots \\
\end{array}
\begin{array}{c}
x_{1,1}, \ldots, x_{1,n} \\
x_{n,1}, \ldots, x_{n,n}
\end{array}
\]

which we feed row-wise into $T$, that is, as a notational convention we identify $\square$ with $\rightarrow_1, \ldots, \rightarrow_n$ and thus stipulate $T(\square) := T(\rightarrow_1, \ldots, \rightarrow_n) = T(x_{1,1}, \ldots, x_{n,n})$. Reversing this line of thought, we can as well start with some square $\square$ of variables, feed its elements into $f$ in some order (indicated, for instance, by certain arrows) and then interpret this sequence of variables as rows of a new square. For example, given $\square$, the value $T(\downarrow_1, \ldots, \downarrow_n)$ is the result of $T$ applied to a square whose rows are the columns of $\square$; so we apply $T$ to the transposed $\square$. Subsequently, we shall often consider sequences as squares where the rows are connected to the ordering of the given sequence and the meaning of columns, diagonals etc. is tied to this particular square interpretation.

Two squares play a special role for $T$, namely those where $T$ outputs 1. First, we have $T(p_1) = 1$ where $p_1$ is given by $\rightarrow_i = (i, \ldots, i)$ for all $1 \leq i \leq n$ (that is, $\downarrow_i = \uparrow$ for all $1 \leq i \leq n$ and also $\nearrow = \nwarrow$). Second we have $T(\uparrow, \ldots, \uparrow) = 1$, and we denote the square all of whose rows $\rightarrow_i$ are $\uparrow$ by $p_2$ (this means $\downarrow_i = (i, \ldots, i)$ for all $1 \leq i \leq n$ and $\nearrow = \nwarrow$).

With the square interpretation in mind we form the set

\[
\theta = \left\{ (\square, y) \in A^{n^2+1} \mid \begin{array}{l}
T(\square) = y \\
T(\nearrow, \nearrow, \ldots, \nearrow) = T(\rightarrow_1, \leftarrow_2, \ldots, \leftarrow_n) \\
T(\nwarrow, \nwarrow, \ldots, \nwarrow) = T(\downarrow_1, \uparrow_2, \ldots, \uparrow_n)
\end{array} \right\}
\]

and then project it to the diagonal $\nearrow$ and the last coordinate $y$, representing the image value of $T$. To show that this projection coincides with the graph of $f$, we shall prove the following statements:

(i) For every $(\square, y) \in \theta$ where $\nearrow = \nwarrow$, it follows $y = 1$. This means that $\nearrow = \nwarrow$ implies $\square = p_1$.

(ii) For every $(\square, y) \in \theta$ where $\nearrow = \uparrow$, it follows $y = 1$. This means that $\nearrow = \uparrow$ implies that $\square = p_2$.

(iii) For every $x \in A^n \setminus \{\uparrow, \downarrow\}$ there is some $\square$ such that $(\square, 0) \in \theta$ and $\nearrow = x$.

Moreover, $(p_1, 1), (p_2, 1) \in \theta$.

Now, if $(\square, y) \in \theta$ then $y \in \text{im}(T) = \{0, 1\}$. If $y = 1$, then $\square = p_1$ or $\square = p_2$, whence $\nearrow = \uparrow$ or $\nearrow = \nwarrow$ and both $(\uparrow, 1), (\nwarrow, 1) \in f^*$. If $y = 0$, then $\square \neq p_1$, so statement (i) yields $\nearrow \neq \uparrow$; similarly, $\square \neq p_2$ and so $\nearrow \neq \nwarrow$ by statement (ii). Hence in each case we have $(\nearrow, y) \in f^*$ which shows that the projection of $\theta$ is a subset of the graph of $f$. Conversely, statement (iii) shows that the full graph of $f$ is obtainable as a projection of $\theta$.

We proceed with the proof of the three statements.
(i) If \((\bar{a}, y) \in \theta\) and \(\not\vdash = \uparrow\), then \(1 = T(\bar{a}, \uparrow, \ldots, \uparrow) = T(\rightarrow_1, \leftarrow_2, \ldots, \leftarrow_n)\). This means \((\rightarrow_1, \leftarrow_2, \ldots, \leftarrow_n) \in \{p_1, p_2\}\). Because \(\not\vdash = \uparrow\), \(x_{11} = 1\) and \(x_{n1} = n\), so the \(n\)-th column of \((\rightarrow_1, \leftarrow_2, \ldots, \leftarrow_n)\) is not constant and hence the latter cannot be equal to \(p_2\). Thus it is \(p_1\) and therefore also \(\square = p_1\).

(ii) If \((\bar{a}, y) \in \theta\) and \(\not\vdash = \downarrow\), then reading backwards we have \(\not\vdash = \uparrow\), and therefore \(1 = T(\bar{a}, \downarrow, \ldots, \downarrow) = T(\rightarrow_1, \leftarrow_2, \ldots, \leftarrow_n)\), whence \((\downarrow_1, \uparrow_2, \ldots, \uparrow_n) \in \{p_1, p_2\}\). As \(\not\vdash = \downarrow\), we have \(x_{1n} = n\) and \(x_{n1} = 1\), so the \(n\)-th column of \((\downarrow_1, \uparrow_2, \ldots, \uparrow_n)\) is not constant (recall that, by our convention, these tuples are fed as rows into \(T\)). This means that \((\downarrow_1, \uparrow_2, \ldots, \uparrow_n)\) must have constant rows (be equal to \(p_1\)), so \(\downarrow_1 = (1, \ldots, 1)\), and \(\downarrow_i = (i, \ldots, i)\) for \(2 \leq i \leq n\). This means that \(\square\) has constant columns with values \(1, \ldots, n\), which means that \(\square = p_2\).

(iii) First we check that \((p_1, 1) \in \theta\). Clearly, \(T(p_1) = 1\). For \(p_1\) we have \(\not\vdash = \uparrow\) and \(\not\vdash = \downarrow\), so \(T(\uparrow_1, \ldots, \uparrow_n) = 1 = T(p_1) = T(\rightarrow_1, \leftarrow_2, \ldots, \leftarrow_n)\) holds as \(p_1\) has constant rows, and \(T(\downarrow_1, \ldots, \downarrow_n) = 0 = T(\uparrow_1, \downarrow_2, \ldots, \downarrow_n)\) is true, as well.

Next we verify that \((p_2, 1) \in \theta\). Again, \(T(p_2) = 1\). This time we have \(\not\vdash = \uparrow\) and \(\not\vdash = \downarrow\), so \(T(\uparrow_1, \ldots, \uparrow_n) = 0 = T(\uparrow_1, \downarrow_2, \ldots, \downarrow_n) = T(\rightarrow_1, \leftarrow_2, \ldots, \leftarrow_n)\). Furthermore, \(T(\uparrow_1, \ldots, \uparrow_n) = 1 = T(p_1) = T(\uparrow_1, \uparrow_2, \ldots, \uparrow_n)\) because the columns of \(p_2\) have constant values \(1, \ldots, n\).

Finally, consider some \(x \in A^n \setminus \{\hat{x}, \hat{y}\}\) and \(\not\vdash = x\) and having zeros everywhere else. All rows of \((\not\vdash, \ldots, \not\vdash)\) and of \((\not\vdash, \ldots, \not\vdash)\) are identical, so none of these two squares is \(p_1\). If one of these were \(p_2\), then \(x = \not\vdash = \uparrow\) or \(\not\vdash = \downarrow\), which would mean \(x = \not\vdash = \hat{y}\). Both options are excluded by the choice of \(x\). Since neither of these two squares is \(p_1\) or \(p_2\), we have \(T(\not\vdash, \ldots, \not\vdash) = 0 = T(\not\vdash, \ldots, \not\vdash)\). As \(\square\) has zeros outside the \(\not\vdash\)-diagonal, it follows that also \((\rightarrow_1, \leftarrow_2, \ldots, \leftarrow_n)\) and \((\downarrow_1, \uparrow_2, \ldots, \uparrow_n)\) have zeros somewhere and are hence mapped to zero by \(T\). Thus \(\square\) satisfies the two conditions regarding the kernel of \(T\). As \(\square\) contains zeros, we also have \(T(\square) = 0 = y\), concluding the argument. \(\square\)

As a corollary we obtain that the example algebras \(\langle A; T \rangle\) constructed by Snow in [16] do not generate centraliser clones as term operations and are thus no counter-example to the Burris-Willard conjecture or to Daničenko’s results.

**Corollary 16.** For every carrier \(A\) of cardinality \(k \geq 3\) the \((k - 1)\)-ary function \(f\) defined in Theorem 15 satisfies \(f \in \{T\}^* \setminus \langle\{T\}\rangle_{O_A}\).

**Proof.** By Theorem 15 we have \(f \in \{T\}^*\); since \(f\) is \((k - 1)\)-ary, it cannot belong to the clone generated by \(T\) as it is maps two distinct tuples to one and is not a projection (cf. Lemma 9). \(\square\)

### 4. Some computational remarks

We conclude with a few comments on computational aspects related to verifying that for \(A = \{0, 1, 2\}\), the simplest case in question, the binary function \(f \in \{T\}^{(2)\to(2)} \setminus \langle\{T\}\rangle^{(2)}\) found in Lemma 12 actually belongs to \(\{T\}^*\).

The first possibility is based on trusting the classification results shown by Daničenko in [8, Theorems 4, 5, pp. 103, 105]. Using the equivalence of statements (i) and (l) in Proposition 1, these theorems imply that \(\{T\}^* = \{T\}^{(2)\to(2)}\), which contains \(f\) by the calculations described in Remark 13. Believing in Daničenko’s thesis obviously does not render Theorem 15 obsolete, as the latter also covers the cases where \(|A| > 3\).

The second option we would like to discuss is whether it is feasible to compute a primitive positive formula over \(T^*\) that allows to define \(f^*\). The formula shown in
Lemma 14 uses five \(T^\bullet\)-atoms and four existentially quantified variables. Of course, these bounds are not known beforehand, and even if they were, simply trying to produce all formulae with \(\ell = 1, 2, 3, \ldots\ \) \(T^\bullet\)-atoms and trying to find a \(3\)-variable projection that gives \(f^\bullet\) becomes unwieldy very quickly. Indeed, before even dealing with projections, there are \(\kappa^6\) possible variable substitutions, where \(\kappa := \ell \cdot \ar(T^\bullet)\) for a primitive positive formula with \(\ell\) atoms of type \(T^\bullet\) and at most \(\kappa\) variables. More concretely, to get the formula from Lemma 14, we would have \(\ell = 5\) and \(\ar(T^\bullet) = 5\), so \(\kappa = 25\), and \(25^{25} \approx 10^{35}\) substitutions are currently too many to check in a reasonable amount of time.

However, if \(f \in \{T\}^\bullet\), then \(f^\bullet \in \Inv_A \Pol_A\{T^\bullet\}\), and there is a more systematic method to compute a primitive positive formula for a relation \(\varrho_0 \in \Inv_A F\) on a finite set \(A\) where \(F = \Pol_A\{\varrho_1, \ldots, \varrho_t\},\ t \in \mathbb{N}\). It comes from an algorithm to compute \(F^{(n)}\) interpreted as a relation \(\Gamma_F(\chi_n)\) of arity \(|A|^n\), which is given in [14, 4.2.5., p. 100 et seq.], combined with the proof of the second part of the main theorem on the Pol-Inv Galois connection, showing that any \(\varrho_0 \in \Inv_A \Pol_A Q\) belongs to the relational clone generated by \(Q\), as it is primitive positively definable from \(\Gamma_F(\chi_n)\) where \(F = \Pol_A Q\) and \(n = |\varrho_0|\) (cf. [14, 1.2.2. Lemma, p. 53 et seq.]).

The following is slightly more general than what is described in [14, 4.2.5] for we can deal with finitely many describing relations \(\varrho_1, \ldots, \varrho_t\) for the polymorphism clone \(F\), while only one is used in [14]. Taking \(Q = \{\varrho_1 \times \cdots \times \varrho_t\}\) as a singleton in [14] is inefficient from a computational point of view, so we give a proof of this not very original modification. Additionally, we allow for a generating system \(\gamma_0\) of the relation \(\varrho_0\) for which a formula is sought (although this is somehow implicit in [14, 4.2.5] as \(\Gamma_F(\chi_n)\) is generated by the \(n\)-element subrelation \(\chi_n\)).

**Proposition 17.** Assume \(Q := \{\varrho_\ell \mid 1 \leq \ell \leq t\},\ F := \Pol_A Q,\ \varrho_0 \in \Inv_A F\) where \(\varrho_\ell \subseteq A^{m_\ell}\) for \(0 \leq \ell \leq t\). Let \(\gamma_0 \subseteq \varrho_0\) with \(n := |\gamma_0|\) be a generating system of \(\varrho_0\), that is, \(\varrho_0 = (\gamma_0|_{(A,F)^{m_0}}\). There is \(m' \leq m_0\) and \(\gamma \subseteq \varrho \in \Inv_A^{(m')} F\) and \(\alpha : m_0 \rightarrow m'\) such that \(\gamma_0 = \{x \circ \alpha \mid x \in \gamma\}\) where \(\gamma\) does not have any duplicate coordinates. If we imagine the tuples in \(\gamma_0\) written as columns of an \((m_0 \times n)\)-matrix, then the distinct rows of this matrix are precisely the rows of the matrix whose columns form the tuples of \(\gamma\). Some of these \(m'\) rows will be found as rows of a relation \(\mu \subseteq A^L\) with \(|\mu| = n\) defined below. For notational simplicity we choose \(\alpha\) such that the rows \(1, \ldots, m\) have this property and \(p := m' - m \geq 0\).

The matrix representation of the relation \(\mu\) has \(n\) columns (tuples) and \(L\) rows \((z_i)_{0 \leq i < L}\) where \(L = \sum_{\ell=1}^L m_\ell\) with \(s_\ell = |\varrho_\ell|\). Let the columns of \(\mu\) arise by stacking on top of each other all possible submatrices of \(\varrho_1\) with \(n\) columns, followed by all possible submatrices of \(\varrho_2\), and so forth, finishing with all submatrices obtained by choosing \(n\) of the \(s_\ell\) columns of \(\varrho_\ell\). Thus \(\mu \subseteq \pi := \varrho_1^{s_1} \times \cdots \times \varrho_L^{s_L}\).

Define the kernel relation \(\varepsilon := \{(i,j) \in L^2 \mid z_i = z_j\}\) and identify variables in \(\pi\) accordingly with \(\delta_\varepsilon = \{x \in A^L \mid \forall (i,j) \in \varepsilon: x_i = x_j\}\). This gives \(\sigma := \pi \cap \delta_\varepsilon\) having the same row kernel as \(\mu\). By finding the first \(m\) rows of \(\gamma\) among the rows of \(\mu\), we find a projection \(pr\) to an \(m\)-element set of indices such that \(\gamma \subseteq pr(\mu) \times A^p\). It follows that \(\varrho = pr(\sigma) \times A^p = pr\left((\varrho_1^{s_1} \times \cdots \times \varrho_L^{s_L}) \cap \delta_\varepsilon\right) \times A^p\).

**Proof.** To show that \(\varrho \subseteq pr(\sigma) \times A^p\) we note that \(pr(\sigma) \times A^p \in \Inv_A F\) since for every \(1 \leq \ell \leq t\) we have \(\varrho_\ell \subseteq \varrho \in Q \subseteq \Inv_A F\). Moreover, \(\varrho\) is a projection of \(\varrho_0\) in the same way as \(\gamma\) is a projection of \(\gamma_0\), and since \(\varrho_0 = (\gamma_0|_{(A,F)^{m_0}}\), we have \(\varrho = (\gamma)|_{(A,F)^{m'}}\). Due to \(\gamma \subseteq pr(\mu) \times A^p \subseteq pr(\sigma) \times A^p\), the generating set \(\gamma\) is a subset of the invariant \(pr(\sigma) \times A^p\), and so is the generated invariant \(\varrho\).

For the converse inclusion we take a tuple \(x \in \sigma\) and some \(a \in A^p\) and denote by \(y\) the tuple obtained from \(x\) by projection to the \(m\) indices that relate the first \(m\) rows
\(v_1, \ldots, v_n\) of \(\gamma\) to a certain section of \(x\). Since \(x \in \sigma \subseteq \delta_x\), this tuple defines a function \(f_x : B \rightarrow A\) where \(B := \{ z_i \mid 0 \leq i < L \} \supseteq C := \{ v_i \mid 1 \leq i \leq m \}\) by finding for any \(z \in B\) some \(0 \leq i < L\) such that \(z = z_i\) and letting \(f_x(z) := x_i\) (the choice of \(i < L\) is inconsequential as \(x \in \delta_x\)). The remaining rows \(v_i\) with \(m < i \leq m + p\) do not belong to \(B\) by the choice of \(m\) and \(p\). This means, \((y,a)\) defines a function \(f_{y,a} : \{ v_i \mid 1 \leq i \leq m \} \rightarrow A\), where \(f_{x,C} = f_{y,a}|_C\). Moreover, it is possible to extend \(f_{y,a}\) to a globally defined function \(f : A^n \rightarrow A\) such that \(f|_{\{v_i\mid 1 \leq i \leq m'\}} = f_{y,a}\) and \(f|_B = f_x\) without contradictory value assignments. We pick one particular such \(f\), no matter which one, and we show below that \(f \in F = Pol_A Q\). By the hypothesis of the proposition, \(\varrho\) belongs to Inv\(_A F\), so \(f\) preserves \(\varrho\). Thus, applying \(f\) to the tuples in \(\gamma\), gives \((y,a) = (f_{y,a}(v_i))_{1 \leq i \leq m'} = (f(v_i))_{1 \leq i \leq m'} \in \varrho\) as needed.

It remains to argue that \(f \in Pol_A Q\). Hence, take any \(1 \leq l \leq t\) and any matrix of \(n\) columns taken from \(\varrho\). By the construction of \(\mu\) there are \(m_t\) consecutive indices \(0 \leq i, i + 1, \ldots, i + m_t - 1 < L\) such that the rows of this matrix are \(z_{i+1}, \ldots, z_{i+m_t-1}\). Now \((f(z_{i+m_t}))_{0 \leq \nu < m_t} = (f_x(z_{i+m_t}))_{0 \leq \nu < m_t}\), and this tuple is in \(\varrho\) because \(f_x\) is defined via \(x \in \sigma = \pi \cap \delta_x\).

The expression \(\varrho = pr\left((\varphi_1^{y_1} \times \cdots \times \varphi_n^{y_n}) \cap \delta_x\right) \times A^p\) in Proposition 17 gives a primitive positive definition of \(\varrho\) in terms of \(\varphi_1, \ldots, \varphi_t\). Duplicating variables as indicated by \(\alpha\), one can then give a primitive positive formula for the original relation \(\varrho_0\). The inclusion \(\varrho \subseteq pr\left((\varphi_1^{y_1} \times \cdots \times \varphi_n^{y_n}) \cap \delta_x\right) \times A^p\) holds in any case, regardless of the assumption that \(\varrho \in \text{Inv}_A F\). It can be seen from the proof of Proposition 17 that the latter condition is only needed for the opposite inclusion.

That is, the formula computed by the following algorithm will always be satisfied by all tuples from \(\varrho\) (or \(\varrho_0\)), but if the containment \(\varrho \in \text{Inv}_A F\) is only suspected but not known in advance, then one needs to check afterwards that the tuples satisfying the generated primitive positive formula really belong to \(\varrho\) (or to \(\varrho_0\), respectively).

**Algorithm 18.** Compute a primitive positive definition\(^8\)
(Pseudocode is given on page 22, line numbers in the description refer to this code.)

**Input:** finitary relations \(\varphi_1 \subseteq A^{m_1}, \ldots, \varphi_t \subseteq A^{m_t}\) defining \(F := Pol_A Q\) where \(Q = \{ \varphi_i \mid 1 \leq i \leq t \}\)

a generating system \(\gamma_0 \subseteq A^{m_0}\) for a relation \(\varrho_0 = (\gamma_0)(A,F)^{m_0} \in \text{Inv}_A F\)

**Output:** a primitive positive formula describing \(\varrho_0\) in terms of \(\varphi_1, \ldots, \varphi_t\)

**Description:** We assume that \(\gamma_0 = (r_1, \ldots, r_n)\), the tuples of which we represent as a matrix with columns \(r_1, \ldots, r_n\) and rows \(v_1, \ldots, v_{m_0}\). We first define a map \(\alpha : \{1, \ldots, m_0\} \rightarrow \{1, \ldots, m'\}\) to a transversal of the equivalence relation \(\{ (i,j) \in \{1, \ldots, m_0\}^2 \mid v_i = v_j \}\) (lines 1–9). For this we iterate over all rows, and, if \(v_j\) has been seen previously among \(v_1, \ldots, v_{j-1}\),

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\(^8\)An implementation is available in the file `ppdefinitions.cpp`, which can be compiled using `compile.sh`, resulting in an executable `getppformula`. This executable expects a file `input.txt`, the formatting of which is explained in `input_template.txt`, which can also be used as `input.txt`. After a successful run the programme will produce files `ppoutput.out`, an ascii text file containing the computed primitive positive formula, and `checkppoutput.z3`, a script to verify the correctness of the formula using the Z3 theorem prover [10, 15].

There are two caveats with the implementation added as ancillary file to this submission: first, the initial preprocessing step turning \(\gamma_0\) into \(\gamma\) has not been implemented. Hence, `ppdefinitions.cpp` expects a goal relation \(\gamma\) (relation 5 in `input.txt`) without duplicate coordinates. If \(\gamma_0\) has duplicate rows, the initial massaging and the final adjustment of the formula by duplicating the respective variables has to be done by hand. Second, it is possible to use a proper generating set \(\gamma_0 \subseteq \gamma_0\) in the input (provided it does not contain duplicate rows), but then in the output file `checkppoutput.z3` the goal relation \(\gamma\) has to be completed manually with all tuples from \(\varrho_0\), since the closure \((\gamma_0)(A,F)^{m_0}\) is not computed.
we assign to \( \alpha(j) \) the same index \( \iota(v_j) \) as previously, and if \( v_j \) is a fresh row, we assign to \( \alpha(j) \) the least index \( \iota(v_j) \) not used before (lines 4–9). When this is finished, \( \gamma_0 = \{(x_{\alpha(1)}, \ldots, x_{\alpha(m_0)}) \mid (x_1, \ldots, x_{m'}) \in \gamma \} \) where \( \gamma \subseteq A^{m'} \) is a projection of \( \gamma_0 \) to its distinct rows, and \( m' \) is the last used value of \( t \).

Next we iterate over all \( 1 \leq \ell \leq t \) and for each relation \( g_\ell \) we iteratively extend the set \( \mathcal{L}_\ell \) of \( g_\ell \)-atoms for the final formula, starting from \( \mathcal{L}_\ell = \emptyset \) (lines 10–13). We iterate over the rows \( z_1, \ldots, z_{m_\ell} \) of all possible matrices with \( n \) columns chosen from \( g_\ell \) (lines 14–16). For any of these matrices we construct an \( m_\ell \)-tuple \( a \) of variable symbols (lines 17–24), which will represent a \( g_\ell \)-atom and will be added to \( \mathcal{L}_\ell \) if it is not already present in the list of atoms (lines 25–26). The atoms have to be constructed in such a way that any two identical rows occurring within all possible matrices get the same variable symbol. This ensures that the variable identification represented in Proposition 17 by intersection with \( \delta_\ell \) takes place. Moreover, if a row in the matrices occurs as a row of \( \gamma \) (or equivalently of \( \gamma_0 \)), then the corresponding variable is not going to be existentially quantified, while all others are. This takes care of the projection in the formula for \( \varrho \) from Proposition 17.

In more detail, if a row \( z_j \) with \( 1 \leq j \leq m_\ell \) has not occurred previously (line 17), we have to define its variable symbol \( u(z_j) \). If \( z_j \in \{v_1, \ldots, v_{m_0}\} \), that is, \( z_j \) is among the rows of \( \gamma_0 \), we use the variable \( u(z_j) := x_{i(z_j)} \) (lines 19–20). Otherwise, the fresh row \( z_j \) needs to be projected away by existential quantification, and we use a different symbol \( u(z_j) := y_k \) where \( k > 0 \) is the least previously unused index for existentially quantified variables (lines 21–23). Regardless of whether \( z_j \) is fresh or not, we define the \( j \)-th entry of the current atom \( a \) as \( a(j) := u(z_j) \) (line 24). Only if the resulting string \( a = \langle a(1), \ldots, a(m_\ell) \rangle \notin \mathcal{L}_\ell \), that is, if \( a \) is a new atom, it will be added to \( \mathcal{L}_\ell \) (lines 25–26).

After all iterations, we state that all variables \( x_1, \ldots, x_i \) occurring in \( \{x_{\alpha(1)}, \ldots, x_{\alpha(m_0)}\} \) come from the base set \( A \), we existentially quantify all variables \( y_1, \ldots, y_k \) and write out (line 27) a long conjunction over all relations \( g_1, \ldots, g_\ell \) and over all \( g_\ell \)-atoms \( a \in \mathcal{L}_\ell \) (cf. the direct product in the formula for \( \varrho \) in Proposition 17).

**Example 19.** In the case discussed in this section, we have \( A = \{0, 1, 2\} \), \( t = 1 \), \( Q = \{T^*\} \), \( g_0 = f^* \), \( m_0 = 3 \) and \( m_1 = 5 \). Moreover, \( F = \text{Pol}_A Q = \{T^*\} \). As the size \( s_1 \) of \( T^* \) is \( |A|^4 = 81 \), it is crucial for the applicability of Algorithm 18 to find a small generating system \( \gamma_0 \) of \( f^* \) with respect to \( A^3 \) where \( A = \langle A; \{T^*\} \rangle \).

Given \( |\gamma_0| = n \), the algorithm has to iterate over \( s_1 = 81^n \) matrices and thus over \( m_1 \cdot s_1 = 5 \cdot 81^n \) rows. Experiments show that if we blindly took \( \gamma_0 = f^* \), i.e., \( n = |A|^2 = 9 \), the algorithm would need more than eighteen thousand years to finish, perhaps less by a factor of ten if run on a computer much faster than the author’s. Fortunately, the number \( n \) can be reduced significantly to a value far below 9.

Indeed, listing the tuples of \( f^* \) as columns, we have

\[
\begin{align*}
f^* &= \{(0, 0), (0, 0), (\frac{1}{2}, 0), (\frac{1}{2}, 0), (\frac{1}{2}, 0), (\frac{1}{2}, 0), (\frac{1}{2}, 0)\} \\
&= \langle \{\frac{1}{2}, \frac{1}{2}\} \rangle \subseteq A^3.
\end{align*}
\]

To see this, we can take advantage of the unary operations \( u_{2,a} \in \{T\}^{s(1)} \) with \( a \in A \), described in Corollary 5, and \( f_{0,(2,2,2,2),2} \in \{T\}^{s(2)} \) from Lemma 6. Namely,
Compute a primitive positive definition

Input: finitary relations \( \varrho_1 \subseteq A^{n_1}, \ldots, \varrho_t \subseteq A^{n_t} \)

\[ F := \text{Pol}_A Q \text{ where } Q = \{ \varrho_\ell \mid \ell \leq \ell \leq t \} \]

\[ \text{generating system } \gamma_0 \subseteq A^{n_0} \text{ for a relation } \varrho_0 = (\gamma_0)_{(A,F)^{n_0}} \in \text{Inv}_A F \]

\[ \text{where } \gamma_0 = \{ r_1, \ldots, r_n \}, \text{ i.e., } |\gamma_0| \leq n \]

written as a matrix \((r_1, \ldots, r_n) = \left( \begin{array}{c} v_1 \\ \vdots \\ v_{n_0} \end{array} \right)\) with rows \( v_j \in A^n \)

Output: a primitive positive presentation of \( \varrho_0 \) in terms of \( \varrho_1, \ldots, \varrho_t \)

1 begin
2 \( i \leftarrow 0 ; \) // initialise index for distinct rows of \( \gamma_0 \)
3 \( D_0 \leftarrow \emptyset ; \) // initialise domain of distinct rows of \( \gamma_0 \)
4 forall \( 1 \leq j \leq m_0 \) do
5 \quad if \( v_j \notin D_0 \) then
6 \quad \quad \( D_0 \leftarrow D_0 \cup \{ v_j \} ; \)
7 \quad \quad \( i \leftarrow i + 1 ; \)
8 \quad \quad \( \iota(v_j) \leftarrow i ; \)
9 \quad \quad \( \alpha(j) \leftarrow \iota(v_j) \)
10 \quad // Now \( D_0 = \{ v_1, \ldots, v_{m_0} \} \)
11 \( k \leftarrow 0 ; \) // initialise index for \( \exists \)-quantified variables
12 \( D \leftarrow \emptyset ; \)
13 \quad // initialise domain of distinct rows from submatrices
14 \quad of \( \varrho_1, \ldots, \varrho_t \) to define \( u: D \rightarrow \{ x_1, \ldots, x_{|D_0|} \} \cup \{ y_1, \ldots, y_k \} \)
15 forall \( 1 \leq \ell \leq t \) do
16 \quad \( \mathcal{L}_\ell \leftarrow \emptyset ; \) // initialise list of atoms pertaining to \( \varrho_\ell \)
17 forall \( c: n \rightarrow \varrho_\ell \) do
18 \quad Form a matrix \((c_0, \ldots, c_{n-1}) = \left( \begin{array}{c} z_1 \\ \vdots \\ z_{m_\ell} \end{array} \right)\) with rows \( z_j \in A^n ; \)
19 \quad // Iterate over its rows and form a possibly new atom \( a \)
20 forall \( 1 \leq j \leq m_\ell \) do
21 \quad if \( z_j \notin D \) then // A previously unseen row \( z_j \) appears.
22 \quad \quad \( D \leftarrow D \cup \{ z_j \} ; \)
23 \quad \quad \( \text{if } z_j \in D_0 \) then // It is a row of \( \gamma_0 \).
24 \quad \quad \quad \( u(z_j) \leftarrow x_{i(z_j)} \)
25 \quad \quad \text{else}
26 \quad \quad \quad \( k \leftarrow k + 1 ; \)
27 \quad \quad \quad \( u(z_j) \leftarrow y_k \)
28 \quad \quad \( a(j) \leftarrow u(z_j) ; \) // extend current atom with the appropriate variable symbol
29 \quad \text{if } a = \{ a(1), \ldots, a(m_\ell) \} \notin \mathcal{L}_\ell \text{ then } // \text{If it is really new...}
30 \quad \quad \mathcal{L}_\ell \leftarrow \mathcal{L}_\ell \cup \{ a \} ; // \ldots \text{add current atom } a \text{ to the list.}
31 return String \( \varrho_0 = \{ \{ x_{\alpha(1)}, \ldots, x_{\alpha(m_0)} \} \mid x_1, \ldots, x_t \in A \land \exists y_1 \cdots \exists y_k : \bigwedge_{1 \leq \ell \leq t} \bigwedge_{a \in \mathcal{L}_\ell} \varrho_\ell(a) \} \)
for \( a \in \{0, 1, 2\} \), we have

\[
\begin{align*}
    u_{2,a}(1) &= 0 & u_{2,a}(2) &= a & f_{0,2,2,2}(1, 2) &= 2 & u_{2,1}(2) &= 1 \\
    u_{2,a}(2) &= a & u_{2,a}(1) &= 0 & f_{0,2,2,2}(2, 1) &= 2 & u_{2,1}(2) &= 1 \\
    u_{2,a}(1) &= 0, & u_{2,a}(1) &= 0, & f_{0,2,2,2}(1, 1) &= 0, & u_{2,1}(0) &= 0.
\end{align*}
\]

Hence, we can use the 2-element generating set \( \gamma_0 = \{(1, 2), (2, 1, 1)\} \subseteq f^* \) and thus we only have to enumerate \( 5 \cdot 81^2 = 32,805 \) rows. This can be done in a fraction of a second\(^9\) and results in a primitive positive formula\(^10\) with 6 existentially quantified variables and 6,561 \( T^* \)-atoms, the correctness of which can be verified by a sat-solver in a few minutes\(^11\).

We conclude that it is possible to computationally find a proof that the graph of \( f \) is primitive positively definable from \( T^* \) for \( A = \{0, 1, 2\} \). However, the resulting formula is not suitable for a generalisation to larger carrier sets as the one from Lemma 14 was.

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\(^{9}\)After compilation the programme \texttt{ppdefinitions.cpp} may be run on \texttt{input_2generated.txt} copied to \texttt{input.txt}. The mentioned files can be found in the ancillary directory of this submission.

\(^{10}\)Running \texttt{ppdefinitions.cpp} on \texttt{input_2generated.txt} (see the ancillary directory) produces the content of \texttt{ppoutput_2generated.z3} and \texttt{checkppoutput_2generated.z3}, which both contain the resulting primitive positive formula (as plain text and in SMT-LIB2.0-syntex).

\(^{11}\)This can, for example, be done with the Z3 theorem prover \cite{10, 15} using the ancillary file \texttt{checkppoutput_2generated.z3}. This file also contains the computed primitive positive formula for \( f^* \) expressed in the SMT-LIB2.0-format.
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