Determination of paleotemperature for the Elbrus glacier based on the inverse problem solution

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Abstract. The surface temperature reconstruction for the Elbrus Plato is built. It takes into account new data on the annual layers in the glacier. The dendrochronological data are used as an additional information that allows us to improve accuracy of calculations and stability of the solutions.

1. Introduction
Previous surface temperature reconstructions for the Elbrus Plato had big discrepancies between each other [1]. It was connected with absence of data on the advection rate in glacier. The new dating of the annual glacier layers [2] allows us to find out the proper solution.

2. Mathematical model
The underground temperature distribution is mainly determined by two types of processes [3]. The first is the surface temperature changes and the second is the heat flux from the Earth that is subjected to the long-time geological processes. The mathematical statement of the inverse problem consists of the thermal conductivity equation that takes into account the vertical advection term, the initial condition, the boundary condition at the bottom of glacier and the re-determination condition. The measured-temperature-depth profile is used as the re-determination condition, \( \chi(z) \), where \( z \) is vertical coordinate. Then the inverse problem to find the temperature in the past is the solution of the following one-dimensional problem:

\[
\rho(z)C(z)T_t = (k(z)T_z)_z - \rho(z)C(z)w(z)T_z, \quad (t, z) \in Q \equiv [0, t_f] \times [0, H],
\]
\[
T(z, 0) = U(z), \quad z \in [0, H],
\]
\[
T(0, t) = U_s + \mu(t), \quad t \in [0, t_f],
\]
\[
-k(H)T_z(H, t) = q, \quad t \in [0, t_f],
\]
\[
T(z, t_f) = \chi(z), \quad z \in [0, H],
\]

where \( H \) is the ice sheet thickness, \( \rho(z), C(z), \) and \( k(z) \) are the density, specific heat, and thermal conductivity of ice, \( w(z) \) is the vertical ice velocity, \( q \) is the geothermal heat flux, \( U(z) \) is the steady-state temperature profile associated with this flux. \( U_s \) is the initial temperature on the surface, which characterizes the average temperature that was on the surface in the past before
the beginning of sharp temperature variations on the surface. \( \mu(t) \) is temperature variations on the surface in time with respect to its initial value \( U_x \) from the moment \( t = 0 \) (\( \mu(0) = 0 \)) to the time of measurements of the borehole temperature profile \( t_f \).

The solution of the direct problem (1) can be represented in the form of the operator relation
\[
T(z, t_f) = R\{\mu(t)\}.
\]
In terms of the measured temperature profile \( \chi(z) \), the solution of the inverse problem can be represented in the form \( \mu(t) = R^{-1}\{\chi(z)\} \). This equation has no exact solution for the element \( \chi(z) \) on the set of the histories of surface temperature variations \( F(\mu(t) \in F) \), because the measured temperature profile \( \chi(z) \) includes temperature perturbations owing to which \( \chi(z) \notin G \), where \( G = RF \) is the set of images of the mapping specified by the operator \( R \). These temperature perturbations appear because measurement errors exist and the mathematical model does not include all the possible processes affecting the temperature distribution in the glacier. Moreover, the operator \( R \) is not mutually continuous; i.e., the solution of the inverse problem is unstable with respect to “small” variations in the profile \( T(z, t_f) \) on the set of images \( G \). Therefore, the problem of the reconstruction of variations in the surface temperature from the borehole temperature measurements is an ill-posed problem.

The Tikhonov method is applied to determine the past surface temperatures [4]. The Tikhonov regularization method is the determination of the boundary temperature minimizing a smoothing functional consisting of the difference and stabilizer:
\[
\Psi = \frac{1}{2} \int_0^H [R\{\mu(t)\} - \chi(z)]^2 dz + \alpha \Omega\{\mu(t)\},
\]
where \( \alpha \) is the regularization parameter matched with the accuracy of the input data. The functional \( \Omega\{\mu(t)\} \) is called stabilizing functional or stabilizer:
\[
\Omega\{\mu(t)\} = \int_0^{t_f} \sum_{j=0}^r q_j \left( \frac{d^j \mu(t)}{dt^j} \right)^2 dt.
\]
Where \( r \) is the stabilizer order, \( q_j \geq 0 \), and \( q_r > 0 \). The procedure of the minimization of the smoothing functional \( \Psi \) can be performed by means of the gradient method and is an iteration procedure. The iteration procedure is carried out until the functional \( \Psi \) reaches a minimum with a given accuracy, which corresponds to the optimal solution of the inverse problem.

3. Reconstruction

The temperature reconstruction for the Elbrus glacier is based on the measured temperature profile in the borehole taking into account new data on the annual layers dating [2]. We assume that the reconstructed temperature corresponds to the temperature at the depth of the activity layer equal approximately to 10m. The temperature changes at this depth arise due to long-term temperature changes at the surface. The seasonal temperature variations do not penetrate at this depth. New data on the annual layers dating allows us to determine the advection rate as a function of depth. The advection rate is approximated by quadratic parabola so that it is equal to zero at the bottom. The other parameters were determined early [2]. The steady-state temperature in the borehole and the geothermal heat flux can be found out by eq. (1). As a rule the borehole temperature near the glacier bottom is very close to the steady-state. Thus, the steady-state temperature profile and the geothermal flux were found out \( Q = 0.3 \text{ Wt/m}^2 \). We assume that the geothermal flux is not changed for the time of consideration. The calculated steady-state temperature profile is shown in Fig. 1.

Let us look for the surface temperature by the finite set of trigonometric Fourier series
\[
\mu(t) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\left(\frac{2\pi t}{T_n}\right) + b_n \sin\left(\frac{2\pi t}{T_n}\right)
\]
(4)
Figure 1. The temperature profiles in the borehole: measured (1), steady-state (2) and calculated (3) by the reconstructed surface temperature.

Figure 2. Tree-ring chronology (a); wavelet power spectrum (b) and global wavelet (c).

The periods of the Fourier series terms are defined by the indirect data based on the high resolution climate sources for the studied region. It is the tree-ring chronology (Fig. 2, a). Such approach enhances the accuracy of the reconstruction. We use the wavelet analysis to retrieve the characteristic periods. The wavelet analysis results are shown in Fig. 2 (b,c). This is spectrum of the wavelet coefficients and the global spectrum of energy. The analysis shows that there are periods of $\sim 264$, $\sim 165$, $\sim 78$, $\sim 41$, $\sim 27$ and $\sim 19$ years.

The reconstruction for this region is shown in Fig. 3. The temperature profile in the
Figure 3. Past surface temperature reconstruction.

borehole corresponding to the determined surface temperature histore is shown in Fig. 1. The reconstruction correlates enough well with the meteo-data of the Teberda station situated near the glacier. The correlation coefficient is 0.65.

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References
[1] Mikhalenko et al. 2011 Stratigraphic structure and thermal regime of the infiltration layer at the Elbrus west plate, in Extreme natural phenomena and disasters, vol. 2: Uranium geology, geo-ecology, glaciology, Moscow: IFZ RAN, 180–188
[2] Mikhalenko et al. 2015 Investigation of a deep ice core from the Elbrus western plateau, the Caucasus, Russia. The Cryosphere 9, 2253–2270
[3] V.M. Kotlyakov et al. 2004 Deep drilling of glaciers in Eurasian Arctic as a source of paleoclimatic records. Quaternary Science Reviews 23, 1371–1390
[4] O.V. Nagornov and S.A. Tyuflin 2017 Inverse Problem for Paleo-Temperature Reconstruction Based on the Tree-Ring Width and Glacier-Borehole Data. Lecture Notes in Computer Science 10187, 508–516