EVALUATION STRATEGY AND MASS BALANCE FOR MAKING DECISION ABOUT THE AMOUNT OF ALUMINUM FLUORIDE ADDITION BASED ON SUPERHEAT DEGREE

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ABSTRACT. The purpose of aluminum fluoride ($\text{AlF}_3$) addition is to adjust the superheat degree (SD) in the aluminum reduction process. Determining the appropriate amount of $\text{AlF}_3$ to add has long been a challenging industrial issue as a result of its inherent complexity. Because of the decreasing number of experienced technicians, the manual addition of $\text{AlF}_3$ is usually inexact, which easily leads to an unstable cell condition. In this paper, an evaluation strategy based on the SD for $\text{AlF}_3$ addition is proposed. An extended naïve Bayesian classifier (ENBC) is designed to estimate the states of SD and its trends that represent the current and potential cell condition respectively, and then the process is graded by evaluating the estimated results based on fuzzy theory. The reduction process is divided into a few situations based on the evaluation grades, and mass balance is introduced to determine the amount of $\text{AlF}_3$ addition in each situation. The results of experiments show that the proposed strategy is feasible, and the effectiveness of $\text{AlF}_3$ addition is improved compared to the existing method. Moreover, automatic $\text{AlF}_3$ addition is promising based on the proposed strategy.

1. Introduction. The $\text{AlF}_3$ concentration of aluminum electrolyte plays an important role in the adjustment of the SD. The SD is the comprehensive embodiment of the heat transfer and mass transfer. Accordingly, it can reflect the condition of an aluminum reduction cell, hereafter referred to as the cell [3,32]. The SD, which will decrease approximately 1.2%-1.5% with an increase in the SD of 10°C, has great potential ‘energy’ to improve the current efficiency [25]. The fluoride materials in a furnace will melt or solidify as a result of changes in the SD. The factors affecting the $\text{AlF}_3$ concentration include two main sources: (a) a loss that is not equal to the $\text{AlF}_3$ addition and (b) the variation of the distribution of the $\text{AlF}_3$ in the cell. The SD can be adjusted by controlling the daily $\text{AlF}_3$ addition, which varies from 0 kg to 50 kg per day per cell. Because of the difficulty of measuring the SD,
it is difficult to make a precise adjustment of the AlF₃ addition. In practice, the adjustment of the SD usually depends on the experience of technologists. However, because of the inherent complexity of the reduction process, fluctuation of the SD, and mass and energy balance coupling, inexperienced technologists have difficulty precisely adjusting the SD [19,31]. Because of this manual AlF₃ addition, an excess or insufficient addition is common in practice. Not only is an excessive addition wasteful, but the SD will be very high. In contrast, the SD will fail to decrease with an insufficient addition. Because of the decreasing number of experienced technologists, an inaccurate AlF₃ addition is inevitable, which may cause a large fluctuation in the furnace. Accordingly, it is expected that an accurate AlF₃ additive dosage could be determined using a scientific strategy.

These problems have attracted the attention of researchers. The research achievements on strategies for AlF₃ addition, which are mainly concerned with controlling the AlF₃ concentration, can mainly be divided into three types. The first was an empirical approach, which depended on understanding the AlF₃ dynamics in cells. For example, references [8,16] studied a control logic based on electrolyte sampling and temperature measurement. However, the AlF₃ concentration was only monitored sporadically by sampling. As demonstrated in [6], a strong correlation existed between the AlF₃ concentration and temperature, which was thoroughly discussed in [24], and further investigation was done in [18]. Both the time lagged sample analysis results and temperature were used for AlF₃ concentration adjustment strategies in their control feedback loop, and these strategies mainly depended on building a logic rule base. In the second type, the AlF₃ feeding amount was calculated as a function of the deviation from a target AlF₃ concentration and/or target temperature [9,21,33]. The side ledge thickness varied with the AlF₃ addition and temperature, which resulted in changes in the AlF₃ concentration. [14] proposed a linear regression model for the AlF₃ addition, which was a function of the target temperature and target AlF₃ concentration. The third type involved an analysis of the AlF₃ evolution, and an AlF₃ addition model was built based on the mass balance and/or energy balance [19], [7, 10, 13]. Consider reference [19] as an example. An AlF₃ concentration control strategy based on an estimation and decoupling technique was proposed using detailed process and plant knowledge.

In contrast to the above three types, this paper proposes a novel strategy based on the characteristics of the reduction process. Because of the complicated nonlinearity and uncertainty of a cell, a precise mechanism model is difficult to obtain. Fuzzy events are usually introduced to describe the practical process. The SD and its trend are applied in the proposed strategy to evaluate the reduction process. However, it is difficult and expensive to obtain the SD and its trend. Therefore, an ENBC is developed to estimate the states of the SD and its trend, which includes the coupling of the classifier attributes as an extension of the naïve Bayesian classifier (NBC). Fuzzy events can be transformed into mathematical formulas based on the fuzzy theory, which can be easily understood and developed. In view of these advantages, the fuzzy logic technique is introduced to evaluate the grade of the cell condition, and the evaluation results are used for the AlF₃ addition. The contributions of this paper are as follows: (1) the states of the SD and its trend are obtained based on the estimation model, (2) a fuzzy evaluation method is proposed using the results of the estimation model for the reduction process, and (3) an AlF₃ feeding amount adjustment strategy integrating with the evaluation method is implemented, which stabilizes the aluminum reduction process.
**Nomenclature**

- \( M(\ast) \): molecular weight (g/mol);
- \( m_{\ast\ast} \): mass (kg);
- \( C_{\ast\ast} \): concentration (wt%);
- \( \vartheta \): the category of an instance;
- \( \omega(\ast) \): mass fraction;
- \( T \): the cell temperature (°C);
- \( A_{sl} \): the project area;
- \( Q_{in} \): energy transferred from electrolyte to side ledge;
- \( \rho \): bulk density of the large alumina particles;
- \( h_{sl} \): heat convection coefficient;
- \( \Delta H_{diss} \): enthalpy change for dissolution;
- \( h_{alu} \): alumina specific heat capacity;
- \( V_{diss,Al_2O_3} \): dissolution rate of single alumina particle.

2. Process analyses.

2.1. The Hall-Heroult process. The Hall-Heroult process is used worldwide in the production of aluminum. This process is named after its inventors, Hall and Heroult. Although research on the cell used for this process has been conducted for more than a century, because of its complexity and high nonlinearity, the optimal operation of a cell still poses significant challenges around the world. The fundamental process involves the dissolution of \( Al_2O_3 \) in an electrolyte with a suitable SD, with metallic aluminum produced on the cathode. In the cell, a series of anode carbon blocks are dipped into the electrolyte. The final products are molten aluminum, and a mixed gas of CO, \( CO_2 \), and HF. The primary electrochemical reaction occurs in the electrolyte, as shown in (1).

\[
\frac{1}{2}Al_2O_3(par) + \frac{3}{4}C(sol) \rightarrow Al(liq) + \frac{3}{4}CO_2(gas) \tag{1}
\]

An \( AlF_3 \) addition can lower the liquidus temperature of the electrolyte, which makes it possible to operate at a lower electrolyte temperature. If the concentration of \( Al_2O_3 \) is less than approximately 2 wt%, the anode effect will be more frequent. Because of the loss of \( AlF_3 \), \( AlF_3 \) must be added as required in the reduction process. A schematic of the cell is shown in Fig. 1.

**Figure 1.** Sketch of aluminum reduction cell

In practice, to improve the current efficiency, technicians use data reports, visual information, mechanistic knowledge, and expert knowledge to make correct adjustments for the cell. Because of parameter coupling, it is necessary to synergistically adjust the parameters to achieve a stable condition. The side ledge is mainly comprised of \( Na_3AlF_6 \), with a small amount of \( CaF_2 \) [26], which acts as a buffer when the SD increases or decreases. The heat preservation performance will vary with the thickness of the side ledge. In this study, the cell voltage fluctuated around 4.0 V, and the line current was 400 kA.

**Assumption 1.** The \( AlF_3 \) masses in the side ledge are neglected.
2.2. Roles of AlF$_3$ addition and SD. The molten electrolyte is the carrier for the electrochemical reactions. The AlF$_3$ lowers the liquidus temperature, which allows the electrolyte temperature to be lowered from 1011°C to approximately 940°C. The details are illustrated in Fig. 2.

![Figure 2. Sketch of binary phase diagram of NaF-AlF$_3$](image)

A suitable AlF$_3$ concentration is approximately 4% to 6%, which is believed to increase the current efficiency. The SD has a great influence on the dissolution of the Al$_2$O$_3$. A higher dissolution and viscosity will be obtained with a high SD. However, the side ledge melting with a higher SD will increase the energy consumption. The details are illustrated by (2) [13].

$$Q_{in} = A_{sl} \times h_{sl} \times SD \tag{2}$$

In contrast, the Al$_2$O$_3$ will hardly dissolve with a low SD, as illustrated by (3) [36]. Sludge will be produced on the cathode, which will cause a seriously high fluctuation of the cell voltage. The above phenomena will decrease the cell stability, and an inefficient cell will be developed with no suitable operation over time.

$$V_{diss, Al_2O_3} = \frac{2 \times h \times SD}{\rho \times (h_{alu} (T - T_{alu}) + \Delta H_{diss})} \tag{3}$$

In general, a suitable SD influences the reduction process in two ways. First, it guarantees the appropriate thickness of the side ledge, and stabilizes the furnace shape of the cell. Second, it guarantees the appropriate dissolution of the alumina. The AlF$_3$ addition is the most important factor to adjust the SD. As a consequence, an adjustment strategy for the AlF$_3$ feeding amount is crucial for the aluminum reduction process.

2.3. AlF$_3$ evolution analysis. The AlF$_3$ addition is an essential operation to maintain the cell stability and increase the current efficiency. Therefore, it is important to conduct a detail analysis of the AlF$_3$ evolution. The AlF$_3$ evolution is divided into three types based on its characteristic in practice.

2.3.1. Neutralization. Alumina contains several kinds of impurities, including Na$_2$O and CaO, which neutralize AlF$_3$, and this neutralization influences the reduction process. There are primarily two types of neutralization reactions, as shown in (4) and (5) [14].

$$\frac{3}{4}Na_2O + AlF_3 \rightarrow \frac{1}{2}Na_3AlF_6 + \frac{1}{4}Al_2O_3 \tag{4}$$

$$\frac{3}{4}CaO + \frac{1}{2}AlF_3 \rightarrow \frac{3}{4}CaF_2 + \frac{1}{4}Al_2O_3 \tag{5}$$

According to (4), pure Na$_3$AlF$_6$ is produced by neutralization. Thus, it is necessary to add AlF$_3$ to obtain the desired concentration of AlF$_3$. The amount produced
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by neutralization and the required AlF$_3$ addition are shown in (6) and (7), respectively.

$$m_{Na_3AlF_6, produced} = m_{Al_2O_3} \frac{\omega_{Na_2O} 2M(AlF_3)}{100} \frac{3M(Na_2O)}{3M(CaO)}$$

where $m_{Na_3AlF_6, produced}$ is the mass produced by (4);

$$m_{Na_3AlF_6, produced} + m_{AlF_3, feeding} = m_{neutralization, desired concentration}$$

Based on (4)-(7), the loss amount by neutralization reactions are described by (8).

$$m_{AlF_3, neutralization} = m_{Al_2O_3} \left( \frac{\omega(Na_2O)}{100} \frac{4M(AlF_3)}{3M(Na_2O)} + \frac{\omega(CaO)}{100} \frac{2M(AlF_3)}{3M(CaO)} \right)$$

$$+ \frac{\omega(Na_2O)}{100} \frac{2M(Na_3AlF_6)}{3M(Na_2O)} \frac{C_{AlF_3, desired}}{1 - C_{AlF_3, desired}}$$

where the values of these parameters in (8) were obtained by measurements. Practically, $m_{AlF_3, neutralization}$ varies considerably with the grade of Al$_2$O$_3$ according to the analysis of [19]. Although the cell age also affects the AlF$_3$ addition, reference [29] verified that the amount of neutralization estimated using (8) is close to the actual addition under different cell ages. In order to simplify the neutralization model, three assumptions were made on the premise of a very small influence on the accuracy as follows:

**Assumption 2.** The Al$_2$O$_3$ grade is constant over a period of time.

**Assumption 3.** The impurities are equally distributed in the Al$_2$O$_3$.

**Assumption 4.** The Al$_2$O$_3$ concentration control is considered to be ‘solved’ and the Al$_2$O$_3$ concentration is close to a constant target value.

2.3.2. Emission and recycling. A mass of particles together with HF volatilize from the cell in a high temperature environment include particulate AlF$_3$, NaAlF$_4$, CaF$_2$, Na$_3$AlF$_6$, and Al$_2$O$_3$. The concentrations of CaF$_2$, Na$_3$AlF$_6$, AlF$_3$, and Al$_2$O$_3$ are considered to be equal to the electrolyte. The HF is produced by a hydrolysis reaction, and the reaction equation is as shown in (9) [10].

$$\frac{1}{2}NaAlF_4 + \frac{1}{2}H_2O \rightarrow \frac{1}{6}Na_3AlF_6 + HF$$

In addition, an aspirator is used for exhaust collection to reduce exhaust air pollution. The unstable NaAlF$_4$ resolves into Na$_5$Al$_3$F$_{14}$ and HF with a temperature decrease, as shown in (10).

$$\frac{5}{2}NaAlF_4 \rightarrow \frac{1}{2}Na_5Al_3F_{14} + AlF_3$$

Then, with the help of a dry scrubber, the exhaust is purified with Al$_2$O$_3$, as shown in (11), and the purified AlF$_3$ is added to the cell.

$$\frac{1}{2}Al_2O_3 + 3HF \rightarrow \frac{3}{2}H_2O + AlF_3$$

Practically, according to (9)-(11), the total amount of AlF$_3$ emission is close to the result of the following equation:

$$m_{AlF_3, emission} = \frac{F_{emission}}{3M(F)} \frac{M(AlF_3)}{3M(Al)} m_{Al day}$$

(12)
where \( m_{\text{AlF}_3,\text{emission}} \) (kg/day) is the amount of AlF\(_3\) emission in one day; \( m_{\text{Al,day}} \) (ton Al/day) is the amount of aluminum produced in one day; and \( F_{\text{emission}} \) (kg F/ton Al) is the amount of fluorine emission per ton of aluminum, which is calculated as shown in (13).

\[
F_{\text{emission}} = F_{\text{VP}} + F_{\text{EP}} + F_{\text{GP}}
\]  

(13)

where (kg F/ton Al) is the volatilized electrolyte, and \( F_{\text{EP}} \) (kg F/ton Al) is the particulate fluoride obtained by entrainment, which are both volatilized particles. \( F_{\text{GP}} \) (kg F/ton Al) is the gaseous fluorine formed by the hydrolysis of the volatilized electrolyte. The models of \( F_{\text{VP}}, F_{\text{EP}}, \) and \( F_{\text{GP}} \) can be found in [26]. In addition, reference [19] verified that the variation in the emission from a cell is small under different electrolyte temperatures and different AlF\(_3\) concentrations. Therefore, they are assumed to be constant emissions.

2.4. Difficulties and solution of AlF\(_3\) addition.

2.4.1. Difficulties with AlF\(_3\) addition. Like most industrial processes, the aluminum reduction process interacts with both the internal and external environments, as illustrated in Fig. 3. However, it is also different from most industrial processes. The electrolyte temperature is very high, even reaching values greater than 945°C [34]. Complex electrochemical reactions occur in the electrolyte, which is a strong magnetic and corrosion environment.

Three aspects of the external disturbance and internal environmental influence on the aluminum reduction process:

(1) An external disturbance such as an anode change/\( E_{\text{AC}} \), aluminum tapping/\( E_{\text{AT}} \), or beam raising/\( E_{\text{BR}} \) will disturb the energy equilibrium, and the AlF\(_3\) concentration varies with the energy equilibrium.

(2) The side ledge/\( I_{\text{SL}} \) and sludge/\( I_{\text{S}} \) on the cathode will melt or solidify with the variation of the SD, which also affects the AlF\(_3\) concentration. However, because of high temperature, the degree of influence is difficult to determine.

(3) AlF\(_3\) is added to a cell with a high temperature and low SD. However, because of the time-lag/\( I_{\text{TL}} \), the temperature will not decrease immediately. Over time, the side ledge will become thinner with an increase in the SD, resulting in a heat loss. Because of the thin side ledge, the AlF\(_3\) concentration will decrease, with more solidified electrolyte melting into the cell. After that, the electrolyte temperature will slightly recover. The details are shown in Fig. 4.

Correspondingly, the difficulties with the AlF\(_3\) addition that arise in practice include the following:

(a) Disturbance: the external operations and environment will have an influence on the AlF\(_3\) concentration that is difficult to determine.
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2.4.2. Solution for making decision about amount of AlF$_3$ addition. Based on the above analysis of the aluminum reduction process, role of the SD, and difficulties with the AlF$_3$ addition, this paper proposes a novel strategy for the AlF$_3$ addition. The solution mainly consists of two stages. First, the grade of the cell condition is evaluated based on the ENBC with the SD and its trend. The selection of attributes for the ENBC depends on the experience of technologists. The SD is the decision variable and is classified into five states according to the experience of the technologists, just like the trend of the SD. Afterward, the classification result is used for evaluating the grade of the cell based on the fuzzy logic theory. Second, the AlF$_3$ feeding amount decision is based on the mass balance of the AlF$_3$, which contains two feeding modes, one for maintaining the feeding mode and the other for adjusting the feeding mode, as illustrated in Fig. 5.

3. SD-based cell condition evaluation. In practice, technicians describe the cell condition with the state of the SD. The state of the SD reveals the current cell
Table 1. Eight characteristic parameters.

| Parameter     | Ab.   | Value       | Role analysis                                                                 |
|---------------|-------|-------------|-------------------------------------------------------------------------------|
| Aluminum level| AL    | 20-23 cm    | The height of the molten aluminum. A higher AL leads to greater heat loss, and vice versa. A suitable AL can stabilize the cell voltage. |
| Molecular ratio| MR    | 2.64-3.0    | This affects the dissolution of the alumina in the electrolyte, with a higher MR leading to a lower SD, and vice versa. |
| Electrolyte level| EL    | 23-28 cm    | This stabilize the thermal balance of the cell. Thus, the thermal balance is robust with a suitable EL. |
| Waving        | WA    | 0-20 mv     | A strong low-frequency noise may be due to insufficient energy intake for the cell. |
| Vibration     | VI    | 0-50 mv     | VI is an indicator of the stability of the cell. A greater VI is more likely for a cold cell. |
| Under/over number ratio| UO    | 0.75-1      | The UO is the ratio between the under and over feeding times. A smaller UO is more likely for a cold cell, and vice versa. |
| Tapping amount| TA    | 2.9-3.05 ton| The TA has a great influence on the energy balance. A greater TA is more likely for a hot cell, and vice versa. |
| Electrolyte temperature| ET    | 955-965°C   | This affects the entire operation condition of the cell. A higher temperature is more likely for a hot cell, and vice versa. |

condition, and its trend indicates the underlying future condition. However, it is expensive and difficult to detect the SD, and its state and trend are also difficult to evaluate. As a consequence, the ENBC is proposed to evaluate the cell condition in this section.

3.1. **State evaluation of SD and its trend based on ENBC.** Eight attributes of the ENBC are selected by investigating and consulting experienced technologists and relevant studies [4], including those dealing with the vibration, waving, tapping amount, electrolyte level, aluminum level, under/over number ratio, mole ratio, and electrolyte temperature, which are gathered in Table 1 together with their corresponding roles. The parameters AL, EL, TA, and ET are measured once a day, and the values of WA, VI, UO, and CV are daily averages. However, MR is measured once every two days, with the averages of the closest two days used for the missing values.

Practically, technologists make a judgment about the state of the SD based after analyzing these parameters synthetically. A schematic diagram of this manual judgment is shown in Fig. 6. The SD trend is represented by dSD, which reflects the changes in the parameters AL, EL, TA, ET, WA, VI, UO, CV, and MR. The dSD judgment is shown in Fig. 7.

The Bayes theory is introduced to build a mathematical model to estimate the state of the SD. A specific kind of Bayesian classifier known as the NBC was proposed by Hart and Duda in 1973 [4], and its attributes are conditionally independent [20]. Suppose that the decision attribute categories are \( s_1, s_2, \ldots, s_c \), which indicate instances categorized into classes. In this way, any instance in
$X = (x_1, x_2, \cdots, x_N)$ will be denoted as a $d$-dimensional vector.

$$x_i = \{x_{i1}, x_{i2}, \cdots, x_{id}\} \quad (1 \leq i \leq N)$$

where $N$ is the size of instance $x$. According to the Bayes theory, the category $\vartheta$ of a new instance is determined based on the posterior probability, which is illustrated by the following (14) [12].

$$\vartheta = \arg \max_{k=1, 2, \cdots, c} \left\{ P(s_k | x_i) \right\}$$

$$= \arg \max_{k=1, 2, \cdots, c} \left\{ \frac{P(s_k)P(x_i | s_k)}{P(x_i)} \right\}$$

$$\Leftrightarrow \left\{ P(s_k)P(x_i | s_k) \right\}$$

(14)

where $P(s_k)$ is the prior probability of the $i^{th}$ category, which is estimated by the frequency of instances of the $k^{th}$ category, i.e., $P(s_k) = n_k / N$, $n_k$ denotes the size of the $k^{th}$ category, and $N = \sum_{k=1}^{c} n_k$ denotes the size of dataset $x$. In addition, $P(x_i | s_k)$ represents the class-conditional probability between $x$ and $s_k$.

With the conditional independence assumption, the category $\vartheta$ of a new instance is determined by the following (15).

$$\vartheta = \arg \max_{k=1, 2, \cdots, c} \left\{ P(s_k)P(x_{i1}, x_{i2}, \cdots, x_{id} | s_k) \right\}$$

$$= \arg \max_{k=1, 2, \cdots, c} \left\{ \frac{n_k}{N} \prod_{j=1}^{d} P(x_{ij} | s_k) \right\}$$

(15)

where $P(x_{ij} | s_k)$ is crucial for determining the category of a new instance. The estimation methods for $P(x_{ij} | s_k)$ mainly include parametric estimations such as histogram computation, $k$-nearest neighbor estimation, maximum likelihood estimation, and Bayesian estimation [2]-[11]. These methods depend on the assumption that attributes are conditionally independent.

However, in many practical engineering applications, the relationships between attributes fail to satisfy the fundamental assumption of independence. For example, the characteristic parameters AL and VI are correlative. The horizontal magnetic field force of molten aluminum increases with a lower AL, resulting in an increasing in VI, as illustrated by Fig. 6. Therefore, the relationships between the eight characteristic parameters do not satisfy the assumption of conditional independence.
3.1.1. Joint PDF estimation for SD and its trend. In this study, the ENBC was used for the classification of new instance based on the joint probability density function (PDF) estimation method. The multivariate kernel function was used to build the joint PDF model. Therefore, the ENBC can avoid the limitation of the fundamental assumption of the NBC.

Parzen window is a well-known method that provides a consistent and asymptotic approach for approximately estimating the PDF. The target for the error between this function and the unobservable underlying PDF can be as small as possible. Based on minimizing the mean integrated squared error (MISE) between the true PDF and estimated PDF, the estimation of the aforementioned method aims to find the optimal bandwidth for the Parzen window, which depends on the following equation.

\[
\hat{f}_k(x) = \frac{1}{Nh^d} \sum_{i=1}^{N} K\left(\frac{x-x_i}{h}\right)
= \frac{1}{Nh^d} \sum_{i=1}^{N} \left[K\left(\frac{x_1-x_{i1}}{h}, \frac{x_2-x_{i2}}{h}, \ldots, \frac{x_d-x_{id}}{h}\right)\right]
\] (16)

where \(x_i = (x_{i1}, x_{i2}, \ldots, x_{id})\) is a d-dimensional dataset, \(K(*)\) denotes a multivariate kernel function, and \(h\) is the bandwidth of \(K(*)\). The multivariate Gaussian kernel is one of the most common kernels, as shown by (17).

\[
K(x) = \frac{1}{(\sqrt{2\pi})^d} e^{-\frac{x x^T}{2}}
\] (17)

where \(x^T\) denotes the transposition of vector \(x\).

3.1.2. Optimal bandwidth selection. As is well known, the accuracy of the bandwidth selection is the most important factor for the estimation performance of the Parzen window method. The existing contributions indicate that the estimated error between the true PDF and estimated PDF would be minimum with an appropriate selection of \(h\), which turns the problem into one for finding the optimal \(h\) for \(K(*)\). Some studies have been conducted on bandwidth selection for univariate kernel density estimation. However, because of computational difficulties, there have been fewer contributions to bandwidth selection for multivariate data. The integrated squared error (ISE) and MISE are usually used as criteria when seeking an optimal bandwidth matrix for a Parzen window. In this paper, we employ MISE as the error criterion, which is illustrated by (18).

\[
\text{MISE}(h) = E\left(\int_{R^d} \left[\hat{f}_h(x) - f(x)\right]^2 \, dx\right)
= \int \text{var}\left(\hat{f}_h(x)\right) \, dx + \int \text{bias}^2\left(\hat{f}_h(x)\right) \, dx
\] (18)

where \(\text{var}\left(\hat{f}_h(x)\right) = E\left\{\hat{f}_h(x) - E\left[\hat{f}_h(x)\right]\right\}^2\), and \(\text{bias}\left(\hat{f}_h(x)\right) = E\left(\hat{f}_h(x)\right) - f_h(x)\).

We can obtain the derivation of bias \(\left(\hat{f}_h(x)\right)\) as shown in (19).

\[
\text{bias}\left(\hat{f}_h(x)\right) = \frac{1}{2} h^2 \hat{f}(x) \int zz^T K(z) \, dz - h \hat{f}(x) \int z K(z) \, dz + o(h^2) \int z \, dz
\] (19)
where $z = (x - y)/h$.

As is well known, $\int zK(z)dz = 0$ and $\int K(z)dz = 1$ for the multivariate Gaussian kernel. We obtain (20) by combining the above two integrals and (19).

$$\text{bias} \left( \hat{f}_h(x) \right) = \frac{1}{2} h^2 \hat{f}(x) \int zz^T K(z)dz + o(h^2)$$  \hspace{1cm} (20)

In addition to $E \left\{ 2\hat{f}_h(x)E \left[ \hat{f}_h(x) \right] \right\} = 2 \left\{ E \left[ \hat{f}_h(x) \right] \right\}^2$, we get the result of $\text{var} \left( \hat{f}_h(x) \right)$ as follows:

$$\text{var} \left( \hat{f}_h(x) \right) = \left[ \frac{1}{N h^d} \int \bar{f}(x) \int K(z)dz - \int \bar{f}(x) \int zK(z)dz \right]$$
$$\quad + \frac{1}{2} h^2 \int \hat{f}(x) \int zz^T K(z)dz + \frac{1}{N h^d} o(h^2) + o(N^{-1})$$  \hspace{1cm} (21)

With $\frac{1}{N h^d} \left[ h \hat{f}(x) \int zK(z)dz \right] = o(N^{-1})$, and $\frac{1}{N h^d} \left[ \frac{1}{2} h^2 \hat{f}(x) \int zz^T K(z)dz \right] = o(N^{-1})$, $\text{var} \left( \hat{f}_h(x) \right)$ is simplified as (22).

$$\text{var} \left( \hat{f}_h(x) \right) = \frac{1}{N h^d} \int \bar{f}(x) \int K(z)dz + o \left( N^{-1} h^{-d} \right)$$  \hspace{1cm} (22)

Combining (20) and (22), $\text{MISE}(h)$ is simplified as follows.

$$\text{MISE}(h) = \frac{1}{4} h^4 \left[ \int z z^T K(z)dz \right] \left\{ \int \left[ \bar{f}(x) \right]^2 dx \right\} + \frac{1}{N h^d} \left[ \int K(z)dz \right] \left[ \int f(x)dx \right]$$
$$\quad + o \left( \left( N^{-1} h^{-1} \right) + h^4 \right)$$  \hspace{1cm} (23)

When $o \left( \left( N^{-1} h^{-1} \right) + h^4 \right)$ is omitted, we get the asymptotic mean integrated square error (AMISE). The AMISE equation is as follows.

$$\text{AMISE}(h) = \frac{1}{4} h^4 \left[ \mu_2(K) \right]^2 R \left( \bar{f} \right) + \frac{1}{N h^d} R(K)$$  \hspace{1cm} (24)

where $\mu_2(K) = \int zz^T K(z)dz$, $R(K) = \int K(z)dz$ and $R \left( \bar{f} \right) = \int \left[ \bar{f}(x) \right]^2 dx$.

In order to find the optimal bandwidth, there exist many optimization methods [27, 30]. In this paper, the method in [28] are introduced to minimize $\text{AMISE}(h)$, and make the derivative of $\text{AMISE}(h)$ be zero, which is shown as follows.

$$\frac{d \text{AMISE}(h)}{dh} = \frac{d}{dh} \left\{ \frac{1}{4} h^4 \left[ \mu_2(K) \right]^2 R \left( \bar{f} \right) + \frac{1}{N h^d} R(K) \right\}$$
$$= h^3 \left[ \mu_2(k) \right]^2 R \left( \bar{f} \right) - \frac{R(K)d}{N h^{d+1}} = 0$$  \hspace{1cm} (25)

The solution of (25) is as follows.

$$h_{\text{optimal}} = \left[ \frac{R(K)d}{\left[ \mu_2(K) \right]^2 R(\bar{f}) \left( \bar{f} \right)} \right]^{\frac{1}{d+1}}$$  \hspace{1cm} (26)

where

$$R(K) = \frac{1}{(\sqrt{2\pi})^{2d}} \prod_{j=1}^{d} \int e^{-z_j^2} dx_j = (4\pi)^{-\frac{d}{2}}$$  \hspace{1cm} (27)
\[
\mu_2(K) = \frac{1}{(\sqrt{2\pi})^{2d}} \sum_{j=1}^{d} \int x_j^2 e^{-x_j^2} \, dx_j = 1 \tag{28}
\]

\[
R(\hat{f}) = 2 \sum_{j=1}^{d} \frac{1}{\sigma_j^2} + \left( \sum_{j=1}^{d} \frac{1}{\sigma_j^2} \right)^2 \left/ \left[ 4 \left( \sqrt{2\pi} \right)^d \prod_{j=1}^{d} \sigma_j \right] \right. \\
= (4\pi)^{-d/2} \left| \sum \right|^{-\frac{1}{2}} \left\{ 2\text{tr} \left( \sum^{-2} \right) + \text{tr}^2 \left( \sum^{-1} \right) \right\} / 4 \tag{29}
\]

where \( \sum = \text{diag} \left( \sigma_1^2, \sigma_2^2, \cdots, \sigma_d^2 \right) \), \( \sigma_j \) is the standard deviation of \( x_{1j}, x_{1j}, \cdots, x_{Nj} \), and \( \text{tr}(\cdot) \) is the trace of a matrix.

Combining (26), (27), (28), and (29), we get the optimal bandwidth \( h_{\text{optimal}} \).

\[
h_{\text{optimal}} = \left( 4d \left/ \left( N \left| \sum \right|^{-\frac{1}{2}} \left\{ 2\text{tr} \left( \sum^{-2} \right) + \text{tr}^2 \left( \sum^{-1} \right) \right\} \right) \right. \right)^{1/4} \tag{30}
\]

3.1.3. Estimation model for state of SD and its trend. The NBC is invalid for the state estimation of SD and its trend because of the dependence between the attributes. Because of this limitation, ENBC is used to determine the states of SD and its trend corresponding to a new instance. The estimation model is as follows.

\[
\vartheta_k = \arg \max_{k=1,2,\ldots,c} \left\{ \frac{1}{N h_k^d} \sum_{i=1}^{N} K \left( \frac{x_1 - x_{1i}}{h_k}, \cdots, \frac{x_d - x_{di}}{h_k} \right) \right\} \tag{31}
\]

where \( k \) represents the number of classifications for SD and its trend, as well as the \( k^{th} \) category samples. SD\(_k\) refers to the state of SD. The optimal bandwidth is

\[
h_k = \left( \frac{4d}{n_k \left| \sum \right|^{-\frac{1}{2}} \left\{ 2\text{tr} \left( \sum^{-2} \right) + \text{tr}^2 \left( \sum^{-1} \right) \right\} } \right)^{1/4} \tag{32}
\]

3.2. State estimation of SD and its trend. In the aluminum reduction process, technicians usually employ fuzzy events to describe the variable states. Very low, Little low, Normal, Little high, and Very high are used as the linguistic variables to describe the SD states, and High negative, Low negative, Zero, Low positive, and High positive are introduced to describe the states of SD trend. As shown in [5], the membership represents the probability that an instance belongs to a specific cluster. The corresponding membership of a fuzzy event is defined as follows.

**Definition 1.** The posterior probability \( P(\vartheta_k | x_i) \) of the corresponding target category (i.e., the aforementioned fuzzy event \( \vartheta_k \)) of sample \( x_i \) is defined as fuzzy membership \( \mu_{\vartheta_k}(SD) \), which is represented as (32)

\[
\mu_{\vartheta_k}(SD) = P(\vartheta_k | x_i) \tag{32}
\]

where \( k = 1, 2, \ldots, 5 \), \( \vartheta_k \) represents the \( k^{th} \) fuzzy event, and \( i = 1, 2, \ldots, N \), where \( i \) denotes the \( i^{th} \) group sample.

The fuzzy membership indicates the maximum degree of the SD and its trend belonging to the \( k^{th} \) fuzzy event. The posterior probability is exactly the maximum probability of the SD and its trend subordinating to the fuzzy event. Accordingly, Definition 1 is reasonable.
Table 2. Fuzzy numbers definitions for SD and its trends.

| Label | Meaning | Membership | Label | Meaning | Membership |
|-------|---------|------------|-------|---------|------------|
| VL    | Very low | $\mu(VL) = P(VL | x_i)$ | HN    | High negative | $\mu(HN) = P(HN | \Delta x_i)$ |
| LL    | Little low | $\mu(LL) = P(LL | x_i)$ | LN    | Low negative | $\mu(LN) = P(LN | \Delta x_i)$ |
| N     | Normal   | $\mu(N) = P(N | x_i)$ | Z     | zero     | $\mu(N) = P(N | \Delta x_i)$ |
| LH    | Little high | $\mu(LP) = P(LP | x_i)$ | LP    | Low positive | $\mu(LP) = P(LP | \Delta x_i)$ |
| VH    | Very high | $\mu(LH) = P(LH | x_i)$ | HP    | High positive | $\mu(HP) = P(HP | \Delta x_i)$ |

Introducing fuzzy events in the practical process is suitable and meaningful. The fuzzy membership is the posterior probability calculated based on the ENBC model. In order to make the evaluation more accessible and intuitive, the fuzzy events are labeled with linguistic variables, in accordance with their corresponding physical meanings. SD values ranging from 8°C to 12°C are considered to be located in the normal state, abbreviated N. A lower equilibrium point is associated with a lower SD. Therefore, when the value of SD is less than 8°C, its fuzzy state is labeled as Little low (LL) or Very low (VL). Similarly, the SD values in a higher range are divided into Little high (LL) and Very high (VL). The SD trend reveals the underlying variation of SD. It may return to a normal cell condition with a sharp trend for SD from an abnormal state. In contrast, it may also cause the normal state to be unstable in the next period. Therefore, the SD trend also has a great influence on the evaluation of the cell condition. The fuzzy membership of the SD trend is defined according to its maximum posterior probability. When the SD may skip or drop from one state to the next higher or lower state in the next period, its trend is defined as Low positive or Low negative, respectively; when the SD skips or drops from the normal state to the highest or lowest state, its trend is considered to be High positive or High negative, respectively. Table 2 presents the membership, labels, and corresponding physical meanings for SD and its trends.

The fuzzy events and memberships obtained in the previous sections are applied to evaluate the aluminum reduction process. The SD trend indicates the possible future state of SD, and the potential foreground is thus considered in the proposed evaluation process. When the current SD is normal, and the foreground of SD is also satisfactory, the cell condition is considered to be stable. When the current SD is normal but there is a sharp trend that causes a cloudy foreground, the reduction process is unstable, as expected. In contrast, when SD is less than 8°C or more than 12°C but it has an ascending or descending trend that can cause it to return to the normal state, the reduction process condition is not as bad as it seems to be. Based on the previously mentioned method, the specific evaluation standard, presented as reference rules for the aluminum reduction process, is shown in Fig. 8. With the Max-Prod operator, the evaluation fuzzy events are obtained by following Larsen’s fuzzy inference method. Based on the centroid defuzzification method, the fuzzy events are converted into specific output values. This method is applied to compute a crisp value by finding the center of gravity of the membership function.
The output crisp value is obtained using the following equation.

\[ g^* = \frac{\sum_{i=1}^{n} \mu(g_i) g_i}{\sum_{i=1}^{n} \mu(g_i)} \]

where \( g^* \) is the evaluation grade of the aluminum reduction process; when \( g^* \) is negative, it reveals a cold cell condition and vice versa. A value that is closer to zero indicates that the cell condition is currently normal.

4. Mode analysis with AlF\(_3\) added. In practice, the AlF\(_3\) addition depends on experiential knowledge. Practically, an abnormal condition is difficult to adjust to normal without rich knowledge and on-line measurements. On the basis of the knowledge of authentic experts on aluminum reduction, the golden principle for a cell is that the furnace control is the most important factor, and the temperature control is considered for emphasis. According to the above principle, there should be as little external interference with a cell as possible. Therefore, the decision strategy for AlF\(_3\) addition includes the following two modes: the AlF\(_3\) addition maintenance mode and adjustment mode.

As shown in Fig. 5, a strategy for the AlF\(_3\) addition is proposed based on the process evaluation results to promote cell stability in this section. The strategy includes an evaluation unit for the practical process, and a mode selection unit for the AlF\(_3\) addition. The selection unit produces a selection mode signal based on the analysis result of the evaluation unit. Then, the corresponding AlF\(_3\) addition mode is activated by the signal. Finally, the AlF\(_3\) addition is set by the maintenance mode and adjustment mode.

Because of changes in the energy balance, the AlF\(_3\) concentration varies with the side ledge variation, resulting in changes in the energy loss and required reduction, as shown in Fig. 9. As a cell condition, the amount of side ledge melting into the cell decreases the insulation performance of the side ledge. The energy dissipation capacity of the cell in the target state will be greater than that in the current state, and the AlF\(_3\) content will be as follows.

\[ m_{\text{AlF}_3,\text{goal}} = f ([T_{\text{ele,goal}}, C_{\text{Al}_2\text{O}_3,\text{goal}}, D_{\text{sl,goal}}]) \] (33)

where \( m_{\text{AlF}_3,\text{goal}} \), \( T_{\text{ele,goal}} \), \( C_{\text{Al}_2\text{O}_3,\text{goal}} \), \( D_{\text{sl,goal}} \) are the mass of AlF\(_3\), electrolyte temperature, content of Al\(_2\)O\(_3\), and thickness of the side ledge in the target state, respectively.

In the current state of the cell, the AlF\(_3\) content is as follows.

\[ m_{\text{AlF}_3,\text{current}} = C_{\text{AlF}_3,\text{current}} * m_{\text{ele,\text{current}}} \] (34)
where \( m_{\text{AlF}_3,\text{current}} \), \( C_{\text{AlF}_3,\text{current}} \), and \( m_{\text{elec, current}} \) are the mass of the \( \text{AlF}_3 \), content of the \( \text{AlF}_3 \), and mass of the electrolyte in the current state, respectively.

The mass balance proposed in this paper is as follows [19].

\[
\dot{m}_{\text{AlF}_3} = -\dot{m}_{\text{AlF}_3,\text{loss}} + \dot{m}_{\text{AlF}_3,\text{addition}}
\]

where \( \dot{m}_{\text{AlF}_3} \), \( \dot{m}_{\text{AlF}_3,\text{loss}} \), and \( \dot{m}_{\text{AlF}_3,\text{addition}} \) are the change rate of \( \text{AlF}_3 \), and the \( \text{AlF}_3 \) leaving and entering the cell, respectively.

In a period of time, the amount of added \( \text{AlF}_3 \) is divided into two parts, including \( \int t \dot{m}_{\text{AlF}_3,\text{loss}} \) and \( \Delta m_{\text{AlF}_3,\Delta\text{energy}} \) from the current state to the target state, which are shown as follows:

\[
m_{\text{AlF}_3,\text{addition}} = m_{\text{AlF}_3,\text{goal}} - m_{\text{AlF}_3,\text{current}} = \int t \dot{m}_{\text{AlF}_3,\text{loss}} + \Delta m_{\text{AlF}_3,\Delta\text{energy}}
\]

where \( \Delta m_{\text{AlF}_3,\Delta\text{energy}} \) is the amount of \( \text{AlF}_3 \) added as a result of the variation in the energy input, and \( \int t \dot{m}_{\text{AlF}_3,\text{loss}} \) is caused by losing \( \text{AlF}_3 \).

In order to determine the mode of the \( \text{AlF}_3 \) addition, two sets of expert rules are extracted from experienced technicians. One is used for the \( \text{AlF}_3 \) addition maintenance mode, and the other one is used for the \( \text{AlF}_3 \) addition adjustment mode. The general expression of the expert rules for the maintenance mode is as follows.

\[ R_1: \text{if } g^* \in [G^{nd}, G^{nu}], \text{then } m_{\text{AlF}_3,\text{addition}} = m_{\text{loss}} \]

where \( g^* \) is the grade of the cell condition, and \( G^{nd}, G^{nu} \) are the upper and lower bounds of the grade for the normal cell condition, respectively.

The expert rule for the \( \text{AlF}_3 \) addition adjustment mode is shown in \( R_2: \)

\[ R_2: \text{if } g^* \notin [G^{nd}, G^{nu}], \text{then } m_{\text{AlF}_3,\text{addition}} = m_{\text{loss}} + m_{\text{adjustment}} \]

where the adjustment amount \( m_{\text{adjustment}} \) can be positive or negative.

a) \( \text{AlF}_3 \) addition maintenance mode

Based on the investigations of authoritative aluminum reduction technicians, an excessive or insufficient \( \text{AlF}_3 \) addition is considered external interference, which has a significant influence on maintaining the structured profile of the furnace. The maintenance mode can maintain the structure by adding \( \text{AlF}_3 \). To determine the \( \text{AlF}_3 \) addition maintenance mode, it is necessary to observe the process evaluation grade. An evaluation grade closer to zero means a normal condition for the practical process, and the \( \text{AlF}_3 \) additive amount is equal to the amount of the loss over a
previous period. The AlF$_3$ additive amount of the maintenance mode is found as follows:

$$m_{\text{AlF}_3,\text{addition}} = m_{\text{AlF}_3,\text{loss}} = m_{\text{AlF}_3,\text{emission}} + m_{\text{AlF}_3,\text{neutralization}}$$ (37)

where $m_{\text{AlF}_3,\text{addition}} = m_{\text{AlF}_3,\text{loss}}$ is the total amount of the AlF$_3$ addition per day for the normal cell condition.

b) AlF$_3$ addition adjustment mode

In an abnormal cell condition, i.e., where $g^*$ deviates from $[G^{nd}, G^{nu}]$, the adjustment mode for the AlF$_3$ addition is activated. In this case, it is necessary to decrease or increase the addition amount according to the cell condition. Consequently, the abnormal cell condition will be adjusted by the above measures. In order to determine the addition amount of AlF$_3$, the parameters $k_0$ and $k_1$ are introduced in the adjustment mode, and the addition amount is found as follows.

$$m_{\text{AlF}_3,\text{addition}} = m_{\text{AlF}_3,\text{adjustment}} + m_{\text{loss}} = g^* \cdot k_1 \cdot m_{\text{loss}} + k_0 \cdot m_{\text{loss}}$$ (38)

where $m_{\text{AlF}_3,\text{addition}} = m_{\text{AlF}_3,\text{adjustment}} + m_{\text{loss}}$ is the total amount of the AlF$_3$ addition per day for the unstable cell condition; $k_1$=-0.51 and $k_0$=1 are used for computing the adjustment amount for the added AlF$_3$, which are determined by experienced technicians.

5. Result and discussion. In order to verify the feasibility and ability of the proposed strategy, the results of an experimental study of AlF$_3$ addition in the aluminum reduction process are discussed in this section.

5.1. ENBC model verification. We first conducted an experimental comparison between ENBC and NBC, to evaluate the feasibility and ability of ENBC using the UCI data sets seeds and banknote. The results based on ENBC were compared with the results of NBC in this experiment, which are shown in Fig. 10 and Fig. 11. In these figures, ‘actual-index’ is the right class, and ‘comput-index’ represents the estimated class.
Table 3. Detailed statistical results.

| Data set names | Number of instances | Accuracy rate |
|----------------|---------------------|---------------|
|                | training | test | attributes | ENBC  | NBC  |
| seeds          | 135      | 75   | 7          | 0.9733 | 0.8933 |
| banknote       | 1297     | 75   | 5          | 0.9467 | 0.8267 |

Table 4. Labels for each data group.

| Labels | Index | Parameters |
|--------|-------|------------|
|        |       | MR (cm)    | EL (mv) | WA (mv) | VI (mv) | UO kg | TA | ET (°C) | AL (cm) |
| VL     | 1     | 3.05       | 33      | 537     | 799     | 0.76  | 3023 | 959     | 22.0   |
| LL     | 2     | 2.98       | 32      | 89      | 114     | 0.52  | 2988 | 960     | 23.0   |
| N      | 3     | 2.79       | 23      | 6       | 10      | 0.40  | 2811 | 974     | 25.5   |
| LH     | 4     | 2.87       | 31      | 2       | 6       | 1.18  | 2787 | 976     | 21.0   |
| VH     | 5     | 2.52       | 25      | 3       | 5       | 0.33  | 2896 | 985     | 24.0   |

As shown in the above figures, eight groups of seeds were classified incorrectly based on NBC, whereas only two groups were classified incorrectly based on ENBC. For the banknote data set, fourteen groups were classified incorrectly based on NBC, but only four groups were classified incorrectly based on ENBC. The detailed statistical results are listed in Table 3.

The classification results show that ENBC is more efficient than NBC in the classification of the data sets, which are multivariate and have interrelated attributes. For the state estimation of SD, another experiment to verify the effectiveness of ENBC was conducted, and verified using a total of 82 groups of industrial data. The data consisted of nearly three months of data from a 400 KA aluminum reduction cell. In this experiment, the industrial data sets were tagged with labels corresponding to the SD states, including VL, LL, N, LH, and VH, according to experts in the aluminum reduction industry, as illustrated by Table 4.

As shown in Fig. 12, for the multivariate and dependent parameters of the aluminum reduction cell, the accuracy rates are 0.5610 and 0.8659 based on NBC and ENBC, respectively. Obviously, ENBC is significantly more efficient than NBC for the SD state estimation. Accordingly, using ENBC is valid for evaluating the state of SD.

Figure 12. Classification result-based NBC and ENBC with data set of cell.
5.2. Simulation results with proposed strategy. To observe the effectiveness of the proposed method, a better operating performance of the electrolytic cell was selected from the aluminum reduction workshop, and suitable AlF$_3$ additions were made by very experienced technicians. Under the better AlF$_3$ addition, the evaluation grades were shown in Fig. 13 with existing SD measured off-line, which demonstrated that the evaluation grade varies within the red line, i.e. $[-0.5, 0.5]$ ⇒ $[G^{nd}, G^{nu}]$.

The lower vibration, waving and mean cell voltage are symbols of better operations of AlF$_3$ addition by consulting technicians or experts. Corresponding to lower vibration, waving and MV, the truth values of aforementioned eight parameters are selected for validation. The verification test was based on two months of data, which were sampled from the practical reduction process, as shown in Fig. 14. We first gave an evaluation grade using the proposed strategy for the above case, as shown in Fig. 15. This figure shows the evaluation grade under a relatively normal condition, except for the 27$^{th}$ (27, -0.6567), 33$^{rd}$ (33, -0.7978), and 34$^{th}$ days (34, -0.5615). Fig. 14(e) reveals that the mole ratio increases from the 24$^{th}$ day to the 29$^{th}$ day, reaching a higher level. In Fig. 14(f), the electrolyte temperature decreases from the 26$^{th}$ day to the 28$^{th}$ day, dropping to a lower level. However, sharp changes occur on the 28$^{th}$ and 29$^{th}$ days. The mole ratio decreases after the 29$^{th}$ day, and the electrolyte temperature increases after the 28$^{th}$ day. Accordingly, the evaluation grade is outside of the normal situation on the 27$^{th}$ day. The evaluation grades on the 33$^{rd}$ day and 34$^{th}$ day are the same as that on the 27$^{th}$ day. Because the cell is very complex, the existence of the above abnormal situations in the practical process is allowable.

After further experiments, we found the actual amount of AlF$_3$ addition by experienced technicians in the practical process. In order to verify the superiority of the proposed strategy, the existing strategy for AlF$_3$ addition is compared with the proposed strategy. The existing strategy uses a linear programming model of the electrolyte temperature, molecular ratio, and tapping amount [33]. Based on the analysis of section IV we can see that the amount of AlF$_3$ added is just equal to the amount of the emission and neutralization, which aims to stabilize the furnace shape of the cell and improve the current efficiency. As shown in Fig. 16, the amount of AlF$_3$ addition is equal to the amount of AlF$_3$ emission and neutralization in the normal situation of the practical process. We should comply with the AlF$_3$ addition maintenance mode in the above normal situation. However, because of the abnormal situation involving the evaluation grades on the 27$^{th}$, 33$^{rd}$, and 34$^{th}$ days, it is unreasonable for the amount of AlF$_3$ addition to the cell to be equal to the amount of emission and neutralization, and we should follow the AlF$_3$ addition adjustment mode. The amount of AlF$_3$ addition includes the amount of emission and neutralization, and also contains the amount of adjustment. In the aluminum reduction process, because of the inherent feeding pattern of the cell, the total feeding amount of the AlF$_3$ addition to cell is divided into batches, with 1.8 kg added once to the cell by one feeder. The total feeding times of all the AlF$_3$ feeders are computed based on each algorithm as shown in Fig. 16. MAPE is 5.8989% based on the proposed strategy, and it is 9.1793% based on naive Bayesian network and mass balance. However, it is 12.0254% based on linear programming model which is greater than based on our proposed strategy.

In order to observe the comparison results more clearly, the error feeding times are illustrated in Fig. 17. Based on the proposed evaluation strategy, the maximum
error of the feeding times is 1.786 on the 47th day, while it is 3.3093 based on the linear programming model on the 16th day. The average error of the feeding times based on the proposed strategy is 0.6744, which is less than once a day in the actual production. However, the average error based on the linear programming model is much larger than that based on the proposed strategy. Accordingly, the proposed evaluation strategy is more effective than the existing strategy. To some extent, the proposed evaluation strategy is able to help workers make the right decisions about the amount of AlF₃ addition to the cell, and there is a potential application.
6. Conclusion and future work. It is difficult to make an accurate judgment about the AlF$_3$ addition amount because of the constraints of the current inspection equipment and the severe coupling qualities of the cell. In the practical process, the AlF$_3$ addition amount is decided manually. However, the decision is always unreasonable when technicians are inexperienced. In this study, an ENBC model was established to estimate the states of the SD and its trends under the limitations of online detection. The estimated results were used to describe the states of the SD and its trend, and posterior probabilities were considered to determine the fuzzy membership of each state. The fuzzy theory was used to evaluate grades that represented the integrated situation of a cell combining the current situation and underlying situation. Based on the evaluation method, a strategy for determining the AlF$_3$ additive amount was designed. In this strategy, two kinds of AlF$_3$ additive amount decision modes were set up for various situations. Experimental results showed that the proposed strategy can simulate the addition of experienced technicians, and is also more effective than the existing methods. Moreover, the
proposed strategy interferes with the cell as little as possible, which will improve the current efficiency and process stability. Not only can it automatically determine an accurate feeding amount using the existing parameter values, but it will also reduce the production costs. The proposed strategy has promising applications for the aluminum reduction process. However, there remain some problems that need to be addressed, such as the feeding pattern. At present, timed feeding is used for the reduction process. Thus, there is an urgent need to study feeding interval optimization.

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