Velocity Curves for Stars in Disk Galaxies: A case for Nearly Newtonian Dynamics

M. D. Maia[∗], A. J. S. Capistrano[†], and D. Muller[‡]

Universidade de Brasília, Instituto de Física, Brasília, DF 70919-970,
[∗]maia@unb.br, [†]yishaicapistrano@yahoo.com.br, [‡]muller@fis.unb.br

The dark matter constraint imposed by the recent WMAP experiment on gravitational theories is analyzed. Using the non-linearity of the vacuum Einstein’s equations, it is shown that when the slow motion condition is applied to the geodesic equations, the resulting nearly Newtonian gravitational field describes nearly flat velocity curves for rotating stars in the vicinity of thin disk galaxies.

I. INTRODUCTION

The discrepancy between the observed rotation velocity curves for stars in a spiral galaxy and the theoretical prediction from Newtonian mechanics is a long standing problem in modern astrophysics. The velocity measurements are based on the Tully-Fisher relation between the mass of the galaxy and the width of the 21-cm line of hydrogen emissions, suggesting that a larger galaxy mass would increase the rotation rate. Since these velocities are much smaller than the speed of light \( c \), in principle they should be described by Newton’s gravitational theory. However, as shown in the example of Figure 1 for the galaxy NGC3198, the theoretical Newtonian curve agrees with the experimental one only at the galaxy’s nucleus [1]. For larger distances, the observed curve becomes almost horizontal, separating from the theoretical Newtonian curve which drops rapidly with \( r \). Such pattern is observed in most spiral galaxies and galaxy clusters [2, 3].

![Figure 1: The Observed Rotation velocity curve (error bars) compared with the predicted Newtonian curve of the NGC3198.](image)

The most common explanation for this problem was originally proposed by Zwicky in 1933 [4]. Accordingly, a certain quantity of dark matter, invisible with respect to the electromagnetic radiation spectrum, should be added to each galaxy. Such matter can in principle be composed of ordinary baryonic matter, like planets distributed in a spherical halo orbiting the galaxy itself, far away from the stars [5]. These have been observed with the help of the gravitational microlensing effects, but only in very small amounts, far beyond the required quantity to correct the velocity curves. In the cosmological scale, dark matter seems to be consistent with the standard FRW model [6], but only recently the cosmic microwave radiation data analysis from the WMAP experiment indicated that most of the estimated 22% dark matter content of the universe must be of non-baryonic nature. More specifically, the analysis of the power spectrum indicates that a theory of gravity based essentially on the properties of baryonic matter would produce a lower third peak [8]. Therefore, either some exotic particles must be considered [7], or else an adequate gravitational theory should be devised.

In principle that constraint does not exclude non-linear theories like general relativity. However, general relativity has such strong commitments with its Newtonian limit, that it makes it difficult to explain the rotation velocity curves. The usual argument goes as follows: The velocity curves for stars in a spiral galaxy derived from Newton’s theory...
agree with the observed curves only at the galaxy’s core (FIG. 1.), precisely where the space-time curvature produced by Einstein’s gravity would be more pronounced. Beyond that region, the gravitational field becomes sufficiently weak to be taken over by its Newtonian limit. Over the time, this has motivated research on many alternative gravitational theories, which we separate into two main categories:

(i) Modifications of Newtonian Gravity
According to this proposal, Newton’s gravitational theory should be modified so as to correctly describe the velocity curves. The first thought is of course the post-Newtonian approximations of general relativity, regarded as corrections to Newtonian theory. However, a simple exercise shows that in a second order parametric post-Newtonian approximation, the corrections term in the velocity curves decay with $1/r^3$, not improving substantially the velocity curves [10]. On the other hand, post-Newtonian cosmology (see e.g. [11]) does not meet the WMAP power spectrum constraint.

Other modifications of the Newtonian theory have been considered [3, 12, 13]. Among these, MOND has received a substantial attention and it has been backed by a theory in which Poisson’s equation for the Newtonian gravitational field is replaced by an equation like [14]

$$<\nabla, \mu(\nabla\varphi)\nabla\varphi> = 4\pi G \rho$$

where $\mu(x)$ is a function to be adjusted to the specific type of galaxy and $a_0$ is a constant acceleration. For example, in a spherically symmetric distribution of matter, it is suggested that $\mu(x) = x/\sqrt{x^2-1}$, producing the following correction for Newtonian potential

$$\varphi = \sqrt{a_0 GM \ln r}$$

This theory has shown good agreement with most known spiral galaxies, but there are reports suggesting that it may be constrained by galaxy clusters [13, 10]. Finally, in spite of being essentially a local gravitational theory, its global effect on the composition of the total energy of the universe does not meet the power spectrum constraint [8].

(ii) Modifications of General Relativity
There are just too many ideas on how to modify general relativity to correct the velocity curves based on a variety of suppositions. Here we just list some of these: (1) Add a scalar field to Einstein’s equation, in such a way that the scalar-tensor theory corrects the Newtonian limit [17]; (2) Modify the concept of time in general relativity, so that the Newtonian limit of the theory differs from the original Newton’s’ theory [18, 19]; (3) Add a cosmological constant with the appropriate sign (depending on which side of the equation it is placed) [20]; (4) Include higher order curvature terms in the gravitational variational principle as a means to increase the local gravitational pull on galaxies [21]; (5) Several brane-world models and variants have been considered, in the hope that the additional degree of freedom would explain the rotation curves. [22, 23, 24, 25, 26, 27, 28].

The purpose of this paper is to show that when the slow motion condition $v \ll c$ is applied to the geodesic equations only, then the self interacting vacuum gravitational field produced by a disk galaxy, contributes to a nearly Newtonian motion of a star in the galactic plane, with nearly flat velocity curves.

This is justified first by the fact that the geodesic equations are derived from Einstein’s equations, but in the Newtonian limit the equations of motion corresponds to a separate postulate [3]. Therefore, when the gravitational field of a galaxy acting upon a free falling star is sufficiently weak, then the slow motion condition $v \ll c$ applies to Einstein’s equations and the only remaining option is the Newton’s law of motion. On the other hand, admitting that the free falling star gets in a region where the gravitational field is beyond the Newtonian limit, then the condition $v \ll c$ still applies to the geodesic equations, but not necessarily on Einstein’s equations. In fact, the geodesic equations depend only linearly in the connection, while Einstein’s equations depend quadratically in the same connection. Therefore, the effect of the condition $v \ll c$ in the connection, becomes less restrictive on the geodesic equation than in Einstein’s equations. The result is that in that region a nearly Newtonian gravity prevails.

II. NEARLY NEWTONIAN GRAVITY

Consider a slowly free falling star, $v \ll c$, in a region of the space-time, where the pull of the gravitational field on the particle is initially weak:

$$g_{\mu\nu} = \eta_{\mu\nu} + \delta h_{\mu\nu}, \quad \delta h^2_{\mu\nu} \ll \delta h_{\mu\nu},$$  \hspace{1cm} (1)
where $\delta h_{\mu\nu}$ is not parameterized by $v/c$. Under these conditions Newtonian coordinates, can be applied, so that the three spatial components of the geodesic equation become (Here we follow essentially the derivation in [29])

$$\frac{d^2x^i}{dt^2} = -\Gamma^i_{j4} \frac{dx^j}{dt} - 2\Gamma^i_{j4} \frac{dx^j}{dt} = -\Gamma^i_{44} = -\frac{1}{2} \delta h^{i}_{44}$$

(2)

where $t$ denotes the Newtonian time. Therefore, we may apply a Newton’s-like equations of motion for a scalar field $\phi$ defined by

$$\frac{d^2x^i}{dt^2} = -\frac{\partial \phi}{\partial x^i}$$

(3)

Notice that $\phi$ is not necessarily the Newtonian potential because the $v << c$ condition was not applied to Einstein’s equations. Comparing the above expression with (2), we obtain

$$\frac{\partial \phi}{\partial x^i} = -\frac{1}{2} \frac{\partial \delta h^{44}}{\partial x^i}$$

(4)

As the particle continues its free fall, while maintaining the slow motion, the gravitational field continuously builds up by small increments as

$$g_{\mu\nu} \approx \eta_{\mu\nu} + \delta h_{\mu\nu} + (\delta h_{\mu\nu})^2 + \cdots$$

Actually, there is no reason to stop this process, so that (4) can be integrated for all perturbations of the Minkowski metric, up to a finite $h_{\mu\nu}$, leading to

$$\phi = -\frac{1}{2} \int_0^{\delta h^{44}} d(\delta h_{44}) = -\frac{1}{2}(1 + g_{44})$$

(5)

This nearly Newtonian gravitational potential is characterized by an exact solution of Einstein’s equations, with the non-linear effects present in the component $g_{44}$ [29].

In order to understand the implications of (5) to the dark matter problem, suppose that we have the Schwarzschild’s solution of Einstein’s equations written in the usual spherical coordinates, so that $g_{44} = -(1 - 2M/r)$. In this case, $\phi$ coincides with Newton’s gravitational potential $\phi = -M/r$ for a spherically symmetric gravitational source with mass $M$. If, this particular potential is applied to describe the motion of a star in a spiral galaxy corresponding to a spherically symmetric “visible mass” $M$, it does not describe correctly the rotation curves outside the galaxy nucleus, regardless of how strong the Schwarzschild field may be. On the other hand, if the star is close to the galaxy nucleus, then it will feel the pull of a spherically symmetric gravitational field which coincides with the above Newtonian potential $-M/r$. This coincidence explains why the two curves in Fig.1 agree at the galaxy’s nucleus. For any other solution of Einstein’s equations which is not diffeomorphic to the Schwarzschild’s solution, (5) will produce a different near Newtonian dynamics. Our understanding is that the nearly Newtonian potential (4) carries a symmetry dependent non-linear effects contained in Einstein’s equations through the component $g_{44}$, as it will be exemplified in the next section.

It is relevant to distinguish the present application of (5) from a solution of the same problem using full general relativity as in [30, 31]. Here, besides having lost general covariance as a consequence of the slow motion condition, only one component of metric has a direct contribution to the motion. In the following section we show that the velocity curves derived from (5) for a vacuum gravitational field are compatible with the observed curves, using an exact solution of the vacuum Einstein’s equations corresponding to a disk galaxy.

### III. VELOCITY CURVES NEAR A DISK GALAXY

As a simple model for a disk galaxy we may consider a cylinder such that its height $h_0$ is much smaller than its radius $r_0$. The line element produced by such object can be derived from the Weyl cylindrically symmetric metric, expressed in cylindrical coordinates $(r, z, \theta)$ as [33]

$$dS^2 = e^{2(\lambda-\sigma)} dr^2 + r^2 e^{-2\sigma} d\phi^2 + e^{2(\lambda-\sigma)} dz^2 - e^{2\sigma} dt^2$$

(6)
where $\lambda = \lambda(r, z)$ and $\sigma = \sigma(r, z)$. The exterior gravitational field outside the cylinder, is given by vacuum Einstein’s equations:

\begin{align*}
-\lambda_{,r} + r\sigma_{,r}^2 - r\sigma_{,z}^2 &= 0 \quad (7) \\
-\sigma_{,r} - r\sigma_{,rr} - r\sigma_{,zz} &= 0 \quad (8) \\
\lambda_{,rr} + \lambda_{,zz} + \sigma_{,r}^2 + \sigma_{,z}^2 &= 0 \quad (9) \\
2\sigma_{,r}\sigma_{,z} &= \lambda_{,z} \quad (10)
\end{align*}

To the above metric we apply the thin disk condition

\[ z \in [-h_0/2, h_0/2], \quad \text{for } r \in [0, r_0], \quad h_0 << r_0 \quad (11)\]

In this case we may expand the functions $\sigma(r, z)$ and $\lambda(r, z)$ around $z = 0$, obtaining

\begin{align*}
\sigma(r, z) &= \sigma(r, 0) + za(r) + \cdots \\
\lambda(r, z) &= \lambda(r, 0) + zb(r) + \cdots
\end{align*}

where we have denoted

\[ a(r) = \left. \frac{\partial \sigma(r, z)}{\partial z} \right|_{z=0} \quad \text{and} \quad b(r) = \left. \frac{\partial \lambda(r, z)}{\partial z} \right|_{z=0} \quad (12)\]

The thin disk condition (11) implies that the above expansion can be truncated to the linear terms in $z$. Therefore, replacing $\sigma_{zz} = 0$ and $\lambda_{zz} = 0$ in (8) and (10), they become simple ordinary differential equations on $\sigma$, with general solution

\[ \sigma(r, z) = \frac{K_0}{2} \ln r + c_2(z) \]

where we have denoted

\[ K_0 = \frac{b(r)}{a(r)} \quad (13)\]

and where $c_2(z)$ is an $r$-integration constant. Derivation of $\sigma$ with respect to $z$ gives $c_2(z) = a(r)z + c_0$, but since $c_2$ does not depend on $r$, it follows that $a(r) = a_0$ = constant. By similar arguments we find that $b(r) = b_0 = \text{constant}$, so that $K_0$ is also a constant. Replacing these results in (7) and (9), together with $\lambda_{zz} = 0$, we also obtain ordinary equations for $\lambda(r, z)$. Therefore, the solution of Einstein’s equations for the thin Weyl disk is

\begin{align*}
\sigma(r, z) &= \frac{K_0}{2} \ln r + a_0 z + c_0 \quad (14) \\
\lambda(r, z) &= \frac{K_0^2}{2} \ln r - a_0 r^2 + b_0 z + d_0 \quad (15)
\end{align*}

where $d_0$ is another integration constant.

From (13) we obtain $g_{44} = -e^{2\sigma} = -e^{2\frac{K_0}{a_0}} \ln r e^{2a_0} e^{2c_0}$. Therefore, for an object moving in the disk plane $z = 0$, we obtain

\[ \phi(r) = -\frac{1}{2}(1 + g_{44})|_{z=0} = -\frac{1}{2}(1 - e^{2c_0} r K_0) \quad (16)\]

In analogy with the Schwarzschild solution, the integration constant $e^{2c_0}$ may be interpreted as a mass, with the difference that here we cannot compare it with the same Newtonian mass. However, we may assume that this constant is proportional to the visible mass $M_v$ of a disk-like galaxy (in units $G=\text{c}=1$): $e^{2c_0} = 2\beta_0 M_v$, where $\beta_0$ is a mass scale factor. It is even possible to interpret the factor $\beta_0$ as something to do with the observed baryonic dark matter, but then we would require a correlation between the visible and dark matter in each galaxy.

The tangent velocity of a star as a function of the distance to the center of the galaxy can now be derived from the Newtonian-like equations of motion (3), using the potential (5). Taking the force acting upon a star of unit mass with tangent velocity $v = \omega r$, $\omega = \text{constant}$ to be $\vec{F} = \frac{v^2}{r^2} \hat{r}$ and comparing with the force generated by (5) $\vec{F} = -\frac{\partial \phi}{\partial r}$, we obtain, $v^2 = r \frac{\partial \phi}{\partial r}$, so that for the considered disk we obtain (in units $G=\text{c}=1$)

\[ v(r) = \sqrt{|\beta_0 M_v K_0 r K_0|} \quad (17)\]
The values of $K_0$ given by (13) are determined by the coefficients of $z$ in the thin Weyl disk metric. Interestingly, $K_0$ is present even in the plane $z=0$, a subtle consequence of the non-linearity of the vacuum Einstein’s equations.

In the thin disk case, the Newtonian velocity can be recovered for $K_0 = -1$ and $\beta_0 = 1$. Since this particular value does not contribute to the rotation curves outside the galaxy core, the value $K_0 = -1$ must be ruled out for disk shaped galaxies. On the other hand, when $K_0 = +1$, the velocity expression (17) does not correspond to any observed rotation curve. We conclude that $|K_0|$ must be smaller than one.

Figure 2 shows the velocities calculated with (17) for a few known examples. Since the stars are supposed to be at the rim of each disk galaxy with radius $r_0$, the origins of each curve were shifted, replacing $r$ by $r - r_0$, so that the shown curves start at the estimated $r_0$ for each galaxy. The values of $K_0$ were determined by comparing the observed average velocity $< v_0 >$ with the calculated velocity for each galaxy. In the given examples all values of $K_0$ are positive but different, so that the curves have slightly different slopes. The values of $\beta_0$ do not affect the shapes of the curves and were estimated for each galaxy from the known top speed in each case.

FIG. 2: Velocity curves from (17), for some examples. Distances are in Kpc and velocities in Km/sec:
(a) The continuous red line represents a simulation of the Newtonian curve for the Sun in the Milky Way. 
(b) Dotted red line is the Milky Way, for $\beta_0 = 1$, $K_0 = 0.08$, $M_v = 1 \times 10^{11} \times M_\odot$ and $r_0 = 5$ (at the Sun).
(c) Magenta is NGC3198 for $\beta_0 = 1$, $K_0 = 0.068$, $M_v = 6 \times 10^{11} \times M_\odot$ and $r_0 = 5$.
(d) Green is NGC3949 for $\beta_0 = 15.8$, $K_0 = 0.13$, $M_v = 2.5 \times 10^9 \times M_\odot$ and $r_0 = 5$.
(e) Blue is NGC3877 for $\beta_0 = 20$, $K_0 = 0.18$, $M_v = 1.1 \times 10^9 \times M_\odot$ and $r_0 = 3$.

For comparison purposes we include below the observed plots (error bars) for NGC3877 and NGC3949:

FIG. 3: Observed rotation curves for NGC3877 and NGC3949
The slow motion of objects in general relativity is described by the nearly Newtonian potential, obtained by imposing \( v \ll c \) in the geodesic equations only, while leaving Einstein's equations and the geodesic deviation equations intact. The result is the nearly Newtonian gravity, something in between general relativity and Newtonian theory, characterized essentially by \( g_{44} \). The existence of such potential follows from the fact that in general relativity the equations of motion are a consequence of the non-linearity of Einstein's equations, making a contrast with Newtonian gravity, where the equation of motion is postulated separately from the field equations.

In particular, when the nearly Newtonian potential is derived from a vacuum solution of Einstein's equations, the slow motion of a test particle or a falling star is affected by the self-interaction of the gravitational field, so that in principle there are no baryons involved.

The loss of general covariance imposed by \( v \ll c \) means that the symmetry of the gravitational field solution of the vacuum Einstein's equations play a significant role in the velocity curves derived from (5), which is interpreted as a consequence of the non-linearity of Einstein's equations. In this respect, it should be noted that the Weyl cylindrical solution can be transformed to the Schwarzschild's solution by a diffeomorphism. However, we cannot apply such transformation here because the diffeomorphism invariance has been lost. On the other hand, the solutions of Einstein's equations with a symmetry that resembles the gravitational field of a galaxy will describe velocity curves which are closer to the observed ones. This was exemplified by taking the Weyl solution with the format of a thin disk, as a model for a disk galaxy. In this case the velocity curves are remarkably close to the experimental curves.

Clearly, a thin Weyl disk is a very poor mathematical model for a spiral galaxy. A more realistic model would be given by a static oblate spheroid, which can also be derived by a coordinate transformation of the Weyl metric. Work on this is still in progress.

[1] T. S. Van Albada, and R. Sancisi, Phil. Trans. R. Soc. London, A 320, 447 (1986)
[2] Y. Sufue, The Astrophysical Journal 458, 120, (1996), astro-ph/0010595
[3] J.R. Bownstein & J.W. Moffat, Astrophys.J.636, (2006), astro-ph/0506370
[4] F. Zwicky, Helv. Phys. Acta, 6, 110 (1933)
[5] E. W. Kolb & M. S. Turner, The Early Universe, Addison - Wesley, (1994)
[6] D. V. Ahluwalia-Khalilova
[7] J. R. Primack, SLAC Beam Line 31N3, 50, (2001), astro-ph/0112336
[8] D.N. Spergel et al, astro-ph/0603449
[9] L. Infeld & J. Plebański, Motion and Relativity, Pergamon Press (1960).
[10] F.A. Gomes da Silva, The Rotation Curves in Espiral Galaxies and the Post-Newtonian Approximation of General Relativity, Instituto de Física, Universidade de Brasilia, a dissertation (in Portuguese), (2000).
[11] P. Szelees, General Relativity and Gravitation 32, 1025, (2000),
[12] P. D. Mannheim, astro-ph/9511045
[13] M. Milgrom, The Astrophysical Journal 270, pag. 365, Ibid pag. 371, Ibid, pag. 384 (1983)
[14] J. Bekenstein and M. Milgrom, The Astrophysical Journal 286, 7, (1984)
[15] E. Pointecouteau and J. Silk, astro-ph/0505017
[16] A. Shirata et al, astro-ph/0501366
[17] S. Fay, Astron. Astrophys. 413, 799, (2004), gr-qc/0402103
[18] S. Behar and M. Carneli, Int. Jour. Theor. Phys. 39, 1397, (2000), astro-ph/9907244
[19] J. C. Hartnett, gr-qc/0407082
[20] S. E. Whitehouse & G. V. Kranios, astro-ph/9911485
[21] S. Capozziello et al Phys.Lett. A326, 292, (2004), gr-qc/0404114 Also astro-ph/0411114
[22] M. K. Mak and T. Harko, gr-qc/0404104
[23] D. N. Vollick, Gen. Rel & Grav. 34, 471, (2002), hep-th/0005033
[24] T. Nihei et al, hep-ph/0409219
[25] N. Okada, and O. Seto, hep-ph/0407092
[26] K. Ichiki et al, Phys. Rev. D66, 023514, (2002), astro-ph/0210052
[27] J. E. Lidsay et al, astro-oh/011292
[28] J. A. R. Cembranos et al, hep-ph/0406076
[29] C. Misner, K.S. Thorne & J. A. Wheeler, Gravitation, W.H. Freeman & co. 1st ed. p. 412 ff.
[30] D. Garfinkle, Class.Quant.Grav.23, 1391,(2006). gr-qc/0511082
[31] F.I. Cooperstock & S. Tieu, astro-ph/0507619
[32] M. Persic, P. Salucci & F. Stel, Mon. Not. R. Astron. Soc. 281, 27, (1996).
[33] H. Weyl, Ann. Phys. 54, 117 (1917).
[34] N. Rosen, Rev. Mod. Phys. 21, 503, (1949)
[35] D. M. Zipoy, Jour. Math. Phys. 7, 1137 (1966)