Q-stars in extra dimensions

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Abstract

We study q-stars with global and local $U(1)$ symmetry in extra dimensions in asymptotically anti de Sitter or flat spacetime. The behavior of the mass, radius and particle number of the star is quite different in 3 dimensions, but in 5, 6, 8 and 11 dimensions is similar to the behavior in 4.

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1 Introduction

Boson stars are stable field configurations of a massive complex scalar field coupled to gravity, [1, 2, 3]. Quartic self-interactions, [4], and the case of the charged scalar field, [5], have also been investigated. Gravity stabilizes the star, preventing it from decay into free particles.

Another class of boson stars are the non-topological soliton stars. They are stable, even in the absence of gravity. They appear as relativistic generalizations, [7, 8, 9] of non-topological solitons, [6], or q-balls. Q-balls is a subclass of non-topological solitons in Lagrangians with a global $U(1)$, [10], or $SU(3)$, $SO(3)$ symmetry, [11], or with a local $U(1)$ symmetry, [12]. There are q-stars with one or two scalar fields, [14], q-stars with non-abelian symmetries, [15], with fermions and scalars, [16], and with a local $U(1)$ symmetry, [17] in asymptotically flat or anti de Sitter spacetime, [18]. Recently, a great amount of theoretical interest has been focused on anti de Sitter spacetime, in any dimension, due to the close relation between gravitating fields within an anti de Sitter spacetime and a field theory at the boundary of the above spacetime, [20, 21, 22]. Boson (not soliton) stars with negative cosmological constant and in 3, 4, 5 and 6 dimensions are investigated in [23].

The purpose of the present work is to study the formation of q-stars, with global or local $U(1)$ symmetry in 5, 6, 8 and 11 dimensions. We also include results in 4 dimensions for comparison, and in 3 for completeness. We investigate their properties, mass, radius, particle number and the value of the scalar field at the center of the star and study their differences resulting from the different dimensionality or the effect of a negative cosmological constant. We also study the stability of the q-star with respect to gravitational collapse and to fission into free particles.

2 Q-stars in any dimension

We consider a static spherically symmetric metric:

$$ds^2 = -e^\nu dt^2 + e^\lambda d\rho^2 + \rho^2 d\Omega_{D-2}^2,$$

with $g_{tt} = -e^\nu$, $D$ the spacetime dimensionality and $d\Omega_{D-2}^2$ the line element for a $(D-2)$-dimensional unit sphere. The action for a scalar field coupled to gravity in $D$ dimensions is:

$$S_D = -\int_S d^Dx \left[ \frac{\sqrt{-g_D(R - 2\Lambda)}}{16\pi G_D} + L_{\text{matter}} \right] - \frac{1}{8\pi G_D} \int_{\partial S} d^{D-1}x \sqrt{-h}K,$$

(2)
where the second term is the Hawking-Gibbons term, \[13\]. \(S\) is the spacetime region with \(\partial S\) its boundary, \(R\) and \(K\) are the traces of the curvature and \(\Lambda\) stands for the cosmological constant. The matter Lagrangian is:

\[ L_{\text{matter}} = (\partial_\mu \phi)^*(\partial^\nu \phi) - U . \] (3)

The Einstein and Lagrange equations are respectively:

\[ G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} \delta_{\mu\nu} = 8\pi G_D T_{\mu\nu} - \Lambda \delta_{\mu\nu} , \] (4)

\[ \phi_{,\lambda} - \frac{dU}{d|\phi|^2} \phi = 0 , \] (5)

with \(G_{\mu\nu}\) the Einstein tensor and \(T_{\mu\nu}\) the energy-momentum tensor:

\[ T_{\mu\nu} = (\partial_\mu \phi)^*(\partial_\nu \phi) + (\partial_\mu \phi)(\partial_\nu \phi)^* - g_{\mu\nu}[g^{\alpha\beta}(\partial_\alpha \phi)^*(\partial_\beta \phi)] - g_{\mu\nu} U . \] (6)

We will now insert the q-soliton ansatz writing:

\[ \phi(\vec{\rho}, t) = \sigma(\rho)e^{-i\omega t} . \] (7)

with \(\omega\) the eigen-frequency with which the star rotates within its internal \(U(1)\) space. We define:

\[ A = e^{-\lambda} , \quad B = e^{-\nu} , \] (8)

\[ W \equiv e^{-\nu}(\frac{\partial \phi}{\partial t})^* \left( \frac{\partial \phi}{\partial t} \right) = e^{-\nu}\omega^2\sigma^2 , \]

\[ V \equiv e^{-\lambda}(\frac{\partial \phi}{\partial \rho})^* \left( \frac{\partial \phi}{\partial \rho} \right) = e^{-\lambda}\sigma^2 \] (9)

and rescale:

\[ \tilde{\rho} = \rho m , \quad \tilde{\omega} = \omega/m , \quad \tilde{\phi} = \phi/m^{D-2} , \]

\[ \tilde{U} = U/m^D , \quad \tilde{W} = W/m^D , \quad \tilde{V} = V/m^D . \] (10)

Gravity is important when \(R \sim G_D M(R)\), where \(M(R)\) is the mass within a sphere of radius \(R\). For a q-star \(U \sim W \sim m^D, V \sim \epsilon^2 m^D\) with \(\epsilon\):

\[ \epsilon \equiv \sqrt{8\pi G m^{D-2}} . \]
This is a very small quantity for \( m \) of the order of magnitude of some (hundreds) GeV. Quantities of the same order of magnitude as \( \epsilon \) will be neglected. We also redefine:
\[
\bar{r} = \epsilon \bar{\rho}, \quad \bar{\Lambda} = \frac{\Lambda}{8\pi G m^D}.
\] 
As we will see, \( \bar{\Lambda} \) may vary from zero, down to a \( \sim -1 \) value. This means that the cosmological constant \( \Lambda \) has the same order of magnitude as:
\[
\Lambda \sim -G m^D.
\]
We use a (rescaled) potential admitting q-ball type solutions in the absence of gravity, namely:
\[
\bar{U} = |\bar{\phi}|^2 \left( 1 - |\bar{\phi}|^2 + \frac{1}{3} |\bar{\phi}|^4 \right) = \bar{\sigma}^2 \left( 1 - \bar{\sigma}^2 + \frac{1}{3} \bar{\sigma}^4 \right).
\] 
Dropping form now on the tildes and the \( O(\epsilon) \) quantities form the Lagrange equation we find an analytical solution for the matter scalar:
\[
\sigma = (1 + \omega B^{1/2})^{1/2}.
\]

We will now find the eigenvalue equation for the frequency. Within the interior of the star both the matter end metric fields vary very slowly, because the radial derivative is of order of \( O(\sqrt{\epsilon}) \). Within the star surface, the metric fields vary slowly but the matter scalar rapidly. Dropping the \( O(\epsilon) \) terms from the Lagrange equation within the surface we take:
\[
V + W - U = 0.
\] 
At the inner edge of the surface \( \sigma' \) is zero in order to match the interior with the surface solution. So, at the inner edge of the surface the equality \( W = U \) together with eq. \ref{eq:13} gives:
\[
\omega = \frac{A_{\text{sur}}^{1/2}}{2} = \frac{B_{\text{sur}}^{-1/2}}{2},
\] 
where \( A_{\text{sur}}, B_{\text{sur}} \) denote the value of the metrics at the surface of the star. Eq. \ref{eq:15} is the eigenvalue equation for the frequency of the q-star and has the right limiting value (\( \omega = 1/2 \)) in the absence of gravity (\( A_{\text{sur}} = 1 \), when the potential is given by eq. \ref{eq:12}.

\[4\]
We will now turn to the Einstein equations. With the rescalings and redefinitions of eqs. 8-11 and dropping the $O(\epsilon)$ terms, they take the simple form:

$$\frac{A - 1}{2r^2}(D - 3)(D - 2) + \frac{A'}{2r}(D - 2) = -W - U - \Lambda, \quad (16)$$

$$\frac{A - 1}{2r^2}(D - 3)(D - 2) - (D - 2)\frac{A'B'}{2rB} = W - U - \Lambda. \quad (17)$$

The total mass of the field configuration can be estimated by the $-T_{0}^{0}$ component of the energy-momentum tensor:

$$M = \frac{2\pi^{(D-1)/2}}{\Gamma \left(\frac{D-1}{2}\right)} \int_{0}^{R} dr (U + W)r^{D-2}, \quad (18)$$

with $R$ the star radius, defined by the Schwarzschild condition $A(r) = 1/B(r)$. There is another relation connecting the metrics with the mass trapped within a sphere of radius $\rho$, namely:

$$A(\rho) = 1 - \frac{2Gm_{\rho}}{r^{D-3}} - \frac{2\Lambda r^{2}}{(D - 2)(D - 1)}. \quad (19)$$

$m_{\rho}$ is straightforward connected to the total mass $M$:

$$M = \frac{D - 2}{8\pi} \frac{2\pi^{(D-1)/2}}{\Gamma \left(\frac{D-1}{2}\right)} m_{\rho}, \quad \rho \to \infty, \quad (20)$$

which, with our rescalings, takes the form:

$$M = (D - 2) \frac{\pi^{(D-1)/2}}{\Gamma \left(\frac{D-1}{2}\right)} r^{D-3} \left(1 - A(r) - \frac{2\Lambda r^{2}}{(D - 2)(D - 1)}\right), \quad r \to \infty. \quad (21)$$

There is a Noether current:

$$j^{\mu} = ig^{\mu\nu}(\phi \partial_{\nu} \phi^{*} - \phi^{*} \partial_{\nu} \phi) \quad (22)$$

which gives a conserved Noether charge:

$$N = \int_{0}^{\infty} d^{D-1}x \sqrt{-g_{\mu\nu}} j^{\mu} = \frac{4\pi^{(D-1)/2}}{\Gamma \left(\frac{D-1}{2}\right)} \int_{0}^{R} dr \omega \sigma^{2} r^{D-2} \sqrt{\frac{B}{A}}. \quad (23)$$

The star owes its stability to the conserved Noether charge. The field configuration is stable with respect to fission into free particles when the star mass is less than the total energy of the free particles with the same charge.
Figure 1: The radius of a q-star as a function of its frequency in an asymptotically flat spacetime. The numbers within the figures denote the spacetime dimensionality. Dashed lines correspond to charged q-stars with $e = 1$. Figures 1-7 refer to asymptotically flat spacetime. We do not consider charged q-stars in $2 + 1$ dimensions. When lowering $\omega$ for a global $U(1)$ symmetry or $\theta_{\text{sur}}$ for the local case, or, equivalently, $A_{\text{sur}}$ for any case, the star radius is in generally larger, because stronger surface gravity is generated by larger solitons. Charged stars have larger radii due to the electrostatic repulsion between the different parts of the star.

The Einstein equations have no analytical solution within the star, except for the $D = 3$, $\Lambda = 0$ case:

$$A(r) = 1 - \frac{3}{4} r^2, \quad B(r) = \frac{1}{4\omega^2}, \quad (24)$$

$$M = \pi(1 - 4\omega^2), \quad N = 4\pi(1 - 2\omega). \quad (25)$$

Every quantity (total mass, e.t.c.) is $D$-dependent according to the eqs. 10-11, but we use one figure for different dimensions for brevity.

3 The case of local symmetry

We will now investigate the case of a q-star with a local $U(1)$ symmetry, i.e. a charged q-star. Charged q-balls, [12], charged q-stars, [17] and boson (not solitonic) stars with local $U(1)$ symmetry, [5], [13], have been considered. The
Figure 2: The value of the scalar field at the center of the q-star as a function of its frequency. Dashed lines correspond to charged q-stars with \( e = 1 \). In 2 + 1 dimensions eqs. 13, 15 and 24 give \( \sigma(r)^2 = 1.5 \).

Figure 3: The mass of a q-star with global symmetry as a function of its frequency.
Figure 4: The mass of a charged q-star as a function of $\theta_{\text{sur}}$ with $e = 1$.

Figure 5: The particle number of a q-star with global symmetry as a function of its frequency.
Figure 6: The particle number of a charged q-star with $e = 1$ as a function of $\theta_{\text{sur}}$.

Figure 7: The mass of a q-star with global symmetry as a function of its radius.
difference between the global case is the form that the matter Lagrangian takes:

\[ L_{\text{matter}} = g^{\mu\nu}(D_{\mu}\phi)^{*}(D_{\nu}\phi) - U - \frac{1}{4}F_{\mu\nu}F_{\mu\nu} , \]  

(26)

with:

\[ F_{\mu\nu} \equiv \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} , \quad D_{\mu}\phi \equiv \partial_{\mu}\phi - ieA_{\mu} . \]

\( e \) is the charge or field strength and \( A_\mu \) is the gauge field. The energy-momentum tensor is:

\[ T_{\mu\nu} = (D_{\mu}\phi)^{*}(D_{\mu}\phi) + (D_{\mu}\phi)(D_{\mu}\phi)^{*} - g_{\mu\nu}[g^{\alpha\beta}(D_{\alpha}\phi)^{*}(D_{\beta}\phi)] \\
- g_{\mu\nu}U - \frac{1}{4}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} + g^{\alpha\beta}F_{\nu\alpha}F_{\nu\beta} , \]  

(27)

and the equation of motion for the matter scalar field is:

\[ \left[ \frac{1}{\sqrt{-g}}D_{\mu}(\sqrt{-g}g^{|\mu\nu|}D_{\nu}) - \frac{dU}{d|\phi|^2} \right] \phi = 0 . \]  

(28)

In order to realize a static metric we choose \( A_\mu = (A_0, 0, 0, 0) \), i.e.: we eliminate any magnetic fields. It is very useful to define a new quantity:

\[ \theta = \omega + eA_0 , \]  

(29)

which is rescaled in the same way as the frequency. We use the same ansatz for the scalar field as in the case of global symmetry and rescale as in eqs. 10, 11. The rescaling for the charge is similar to the one used in charged boson stars:

\[ \tilde{e} = e\epsilon^{-1} . \]  

(30)

Dropping the \( O(\epsilon) \) quantities and the tildes from the Lagrange equation for the scalar field, we find for the interior:

\[ \sigma = (1 + \theta B^{1/2})^{1/2} . \]  

(31)

Repeating the discussion of eqs. 14, 15 we find:

\[ \theta_{\text{sur}} = \frac{A_{\text{sur}}^{1/2}}{2} = \frac{B_{\text{sur}}^{-1/2}}{2} . \]  

(32)

The new dynamical quantity \( \theta \) plays now the role of the frequency and eq. 32 is the eigenvalue equation for this “frequency”. 
The Einstein equations take now the form:

\[
\frac{A - 1}{2r^2} (D - 3)(D - 2) + \frac{A'}{2r} (D - 2) = -W - U - \frac{\theta'^2}{2e^2} - \Lambda ,
\]

\[
\frac{A - 1}{2r^2} (D - 3)(D - 2) - (D - 2) \frac{A B'}{2r B} = W - U - \frac{\theta'^2}{2e^2} - \Lambda ,
\]

and the equation of motion for the gauge field:

\[
\theta'' + \left[ \frac{D - 2}{r} + \frac{1}{2} \left( \frac{A'}{A} + \frac{B'}{B} \right) \right] \theta' - \frac{2e^2 \sigma^2 \theta}{A} = 0 .
\]

At the center of the star the electric field is zero, so \( \theta'(0) = 0 \). The absence of electric fields at infinity leads to \( \theta(\infty) = \omega \). We numerically solve the coupled system of equations 33-35. For the exterior of the star we find the analytical solution:

\[
\theta' = \frac{\Gamma \left( \frac{D-1}{2} \right)}{2\pi^{D-1/2}} e^2 Q ,
\]

and the contribution of the exterior to the total energy is:

\[
M_{\text{ext.}} = \frac{e^2 Q}{2} \frac{\Gamma \left( \frac{D-1}{2} \right)}{2\pi^{D-1/2}} \int_{R}^{\infty} dr \frac{1}{r^{D-2}} ,
\]

with \( R \) the radius of the star.

For a charged q-star, the Noether current is:

\[
j^\mu = ig^{\mu\nu} (\phi D_\nu \phi^* - \phi^* D_\nu \phi) ,
\]

and the corresponding particle number:

\[
N = \frac{4\pi^{(D-1)/2}}{\Gamma \left( \frac{D-1}{2} \right)} \int_{0}^{R} dr \theta \sigma^2 r^{D-2} \sqrt{\frac{B}{A}} .
\]

The total electric charge is \( Q = eN \).

## 4 Conclusions

We find stable with respect to fission into free particles q-star-type field configurations in 3, 4, 5, 6, 8 and 11 dimensions. We investigate the overall phase space. The independent parameters are the frequency, \( \omega \) or \( \theta_{\text{sur}} \), the
Figure 8: The radius of a q-star as a function of the cosmological constant. In figures 8-14 we take $A_{\text{sur}} = 0.81$, equivalently $\omega = \theta_{\text{sur}} = 0.45$.

Figure 9: The value of the scalar field at the center of the star as a function of the cosmological constant.
Figure 10: The mass of a q-star with global symmetry as a function of the cosmological constant.

Figure 11: The mass of a charged q-star with $\epsilon = 1$ as a function of the cosmological constant.
Figure 12: The particle number of a q-star with global symmetry as a function of the cosmological constant.

Figure 13: The particle number of a charged q-star with $e = 1$ as a function of the cosmological constant.
cosmological constant and the field strength, \( e \). The star radius, the particle number, the total electric charge, \( eN \), and the mass of the star are derivative quantities. The star shows a similar behavior in \( D \geq 4 \) dimensions. Its behavior in \( 2 + 1 \) seems to be quite different. Charged q-stars in \( 2 + 1 \) dimensions are not considered. The numbers within the figures denote the spacetime dimensionality. Dashed lines correspond to charged q-stars with \( e = 1 \). Figures 1-7 refer to asymptotically flat spacetime and figures 8-14 refer to asymptotically anti de Sitter spacetime. All the field configurations depicted in our figures are stable with respect to fission into free particles.

In figures 1-7 the frequency, \( \omega \) or \( \theta_{\text{sur}} \) for the charged case, is the independent parameter. We start from \( \omega = \theta_{\text{sur}} = 1/2 \), equivalently \( A_{\text{sur}} = 1 \), which corresponds to the absence of gravity (q-ball limit). We gradually decrease the frequency down to a minimum value, determined by the soliton stability demand. For the case of global symmetry, the smaller (or critical) frequency corresponds to the last stable field configuration with respect to gravitational collapse. (We also include a short range of frequencies smaller than the critical one.) In figure 7 the last stable configuration with respect to gravitational collapse corresponds to the top of the \( M = M(R) \) curve. For charged q-stars the lowest frequency corresponds to \( M/N = 1 \). Below this frequency the star mass is larger than the mass of the free particles with the same charge, and the star decays into free particles. No gravitational collapse can happen when \( e \neq 0 \). In general, charged q-stars have larger radii and masses but a smaller \( \phi(0) \), due to the electrostatic repulsion between the different parts of

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{cosmological_constant.png}
\caption{The mass and the particle number of a q-star in 2+1 dimensions as a function of the cosmological constant.}
\end{figure}
the star. They avoid for the same reason gravitational collapse, because for $A_{\text{sur}} \to 0$, equivalently $\theta_{\text{sur}} \to 0$, the electrostatic repulsion makes the fission into free particles energetically favorable.

Figures 8-14 refer to spacetime with negative cosmological constant. The independent parameter is the cosmological constant $\Lambda$. We start from $\Lambda = 0$ and gradually decrease its value. We interrupt calculations when $\phi(0) \to \infty$ for $e = 0$ and when $M/N$ tends to unity for a charged q-star.

The behavior of the star parameters, $R$, $M$, $N$ and $\phi(0)$, for $4, 5$ and $6$ dimensions as a function of the cosmological constant is similar. The situation is absolutely different in $2 + 1$ dimensions, as we see from figure 14. For small values of the cosmological constant both $N$ and $M$ increase with the increase in absolute values of $\Lambda$. Below $\Lambda \simeq -1.6$ the energy is approximately constant but the particle number decreases slowly and for $\Lambda \simeq -10$ equals to the energy. Below that value of the cosmological constant the star is unstable, decaying to free particles.

In four or more spacetime dimensions the total energy, particle number and radius increases initially with the increase of the cosmological constant in absolute values, but below a certain value of the cosmological constant, these parameters decrease. Qualitatively, negative cosmological constant reflects a competitive effect to gravity attraction. So, the soliton mass and consequently the particle number and radius increase, in order the field configuration to be stable against this “negative” gravity implied by the negative cosmological constant. But, when $\Lambda$ exceeds a certain value, no additional energy amount can deserve the soliton stability, if this is too extended. So, when $\Lambda$ exceeds this certain value, the star shrinks, and, consequently, its energy and charge decreases. Also, the value of the scalar field at the center of the soliton increases rapidly for the same reason. This happens because gravity in $D > 3$ dimensions is a long-range force. The two independent Einstein equations in static, spherically symmetric systems contain the $(A - 1)(D - 3)(D - 2)/(2r^2)$ term. On the other hand, in $2 + 1$ dimensions no such competition between “negative” gravity from the cosmological constant and “positive” gravity from the energy-momentum density happens, because gravity in $2 + 1$ dimensions is not a propagating long-range force. Also, the above mentioned term is absent. These are the reasons for the similarities in the behavior of the soliton parameters in $D > 3$ dimensions and the differences in $2 + 1$ dimensions.

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