Analytical approximations for the oscillators with anti-symmetric quadratic nonlinearity

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Abstract. A second-order ordinary differential equation involving anti-symmetric quadratic nonlinearity changes sign. The behaviour of the oscillators with an anti-symmetric quadratic nonlinearity is assumed to oscillate different in the positive and negative directions. In this reason, Harmonic Balance Method (HBM) cannot be directly applied. The main purpose of the present paper is to propose an analytical approximation technique based on the HBM for obtaining approximate angular frequencies and the corresponding periodic solutions of the oscillators with anti-symmetric quadratic nonlinearity. After applying HBM, a set of complicated nonlinear algebraic equations is found. Analytical approach is not always fruitful for solving such kinds of nonlinear algebraic equations. In this article, two small parameters are found, for which the power series solution produces desired results. Moreover, the amplitude-frequency relationship has also been determined in a novel analytical way. The presented technique gives excellent results as compared with the corresponding numerical results and is better than the existing ones.

1. Introduction
In recent past, the mathematical interpretation of the nonlinear oscillations has received extensive attention in the field of physics, applied mathematics engineering and other discipline. The mathematical model of oscillation of the human eardrum is the quadratic nonlinear oscillator [1]. Researchers have been investigated many analytical approaches to solve nonlinear differential equations. Perturbation technique [2-5] is the versatile technique, whereby the solution is expanded in powers of a small parameter. However, for the strongly nonlinear regime the perturbation technique is almost fail. Several authors employed many other powerful analytical methods (non-perturb) to drive approximate periodic solutions especially for the strongly nonlinear oscillators, include as, the Max-
Min Approach [6], Parameter Expansion Method [7], Variational Iteration Method [8], Amplitude Frequency Formulation [9], Energy Balance Method [10-12], He’s Energy Balance Method [13], Global Residue Harmonic Balance Method [14] and so on.

Cveticanin [15] has used the Jacobi elliptic functions to derive the exact solution of the anti-symmetric quadratic equation. However, the author only assumed that its exact solution is given by an equation which includes the Jacobian elliptic function and five unknown parameters that need to be determined. The author did not solve the nonlinear differential equation. Mickens and Mixon [16] obtained accurate analytical approximate solutions to an anti-symmetric quadratic non-linear oscillator using the generalized harmonic balance method and Chen et. al. [17] using the elliptic perturbation method. Beléndez et. al. approximately solved this nonlinear oscillator using a modified He’s homotopy perturbation method [18] as well as a novel rational harmonic balance approach [19]. Hu [20] has used Harmonic Balance Method to determine an approximate solution of a quadratic nonlinear oscillator; but the method is not a simple one. Hu [20] has obtained two separate harmonic balance solutions respectively for two regions $x > 0$ and $x < 0$. The solution is continuous, but the derivative does not exist when it cuts the axis. Momani et. al. [21] introduced the modified homotopy perturbation method for solving strongly nonlinear oscillators with anti-symmetric quadratic nonlinearity. Recently, Cveticanin et. al. [22] obtained approximate solutions for the oscillators with symmetric and asymmetric quadratic nonlinearity in the form of Jacobi elliptic functions. Mahtab et. al. [23] has also approximated the quadratic nonlinear oscillators using Harmonic balance method. An iteration procedure has been applied to determine approximate frequencies for the Quadratic Nonlinear Oscillator by Haque et. al. [24].

Solving strongly nonlinear systems, the method of harmonic balance [25-32] is another efficient method. As a rule, a set of complex nonlinear algebraic equations are appeared when Harmonic Balance Method is imposed. In this paper, such nonlinear algebraic equations are approximated using a power series solutions by a new small parameter. Moreover, an analytical technique has also been applied to obtain the amplitude-frequency relationship. The approximated results have compared with the existing results and the corresponding numerical solutions (Runge-Kutta fourth order method).

2. The Solution Procedure

A second-order nonlinear differential equation can be considered as

$$y'' + a_0 y = -\varepsilon f(y, y')$$

and the initial conditions $y(0) = A_0$, $y'(0) = 0$, where $f(y, y')$ is a nonlinear function such that $f(-y, -y') = f(y, y')$, $a_0 \geq 0$ and $\varepsilon$ is a constant.

A n-th order periodic solution of Eq. (1) can be supposed as

$$y = A_n(\rho \cos(\omega t) + u \cos(3\omega t) + v \cos(5\omega t) + w \cos(7\omega t) + z \cos(9\omega t) + \cdots),$$

where $A_n$, $\rho$ and $\omega$ are constants. If $\rho = 1 - u - v - \cdots$, the solution Eq. (2) readily satisfies the initial condition given in Eq. (1).

Substituting Eq. (2) into Eq. (1) and expanding $f(y, y')$ in a Fourier series, it can be converted to an algebraic identity as follows

$$A_n[\rho(\omega_0^2 - \omega^2) \cos(\omega t) + u(\omega_0^2 - 9\omega^2) \cos(3\omega t) + \cdots] = -\varepsilon[F_1(A_n, u, \cdots) \cos(\omega t) + F_3(A_n, u, \cdots) \cos(3\omega t) + \cdots]$$(3)

By comparing the coefficients of equal harmonics of Eq. (3), the following nonlinear algebraic equations can be obtained as

$$\rho(\omega_0^2 - \omega^2) = -\varepsilon F_1, \quad u(\omega_0^2 - 9\omega^2) = -\varepsilon F_3, \quad v(\omega_0^2 - 25\omega^2) = -\varepsilon F_5, \quad \cdots$$

With the help of the first equation, $\omega^2$ is eliminated from all the rest of Eq. (4). Thus Eq. (4) takes the following form

$$\rho \omega_0^2 = \rho \omega_0^2 + \varepsilon F_1, \quad 8\omega_0^2 u \rho = \varepsilon(\rho F_1 - 9u F_3), \quad 24\omega_0^2 v \rho = \varepsilon(\rho F_3 - 25v F_5), \quad \cdots$$

Substituting $\rho = 1 - u - v - \cdots$, and simplification, second-, third- equations of Eq. (5) can be transformed into
$$u = G_1(\omega_0, \varepsilon, A_0, u, v, \ldots, \lambda_0), \quad v = G_2(\omega_0, \varepsilon, A_0, u, v, \ldots, \lambda_0), \cdots,$$

where $G_1, G_2, \cdots$ exclude respectively the linear terms of $u, v, \cdots$.

Whatever the values of $\varepsilon, \omega_0$ and $A_0$ there exists a parameter $\lambda_0(\varepsilon, \omega_0, A_0) \ll 1$, such that $u, v, \cdots$ are expandable in following power series in terms of $\lambda_0$ as

$$u = U_1 \lambda_0 + U_2 \lambda_0^2 + \cdots, \quad v = V_1 \lambda_0 + V_2 \lambda_0^2 + \cdots, \quad \cdots$$

where $U_1, U_2, \ldots, V_1, V_2, \ldots$ are constants.

Finally, substituting the values of $u, v, \cdots$ from Eq. (7) into the first equation of Eq. (5), the angular frequency $\omega$ is determined. This completes the determination of all related functions for the proposed n-th order periodic solution as given in Eq. (2).

### 3. Example

#### 3.1 Differential equation with anti-symmetric quadratic nonlinearity

Let us consider the following equation which has investigated in Momani et. al. [21] is

$$y'' + 2y + y^2 = 0 \quad \text{and the initial conditions } \{y(0) = a_0, \ y'(0) = 0\}. \quad (8)$$

As we know, an asymmetric behavior of the nonlinear oscillator is different in positive and negative directions. An asymmetric limit zone $[-b, a]$, for positive $a$ and negative $b$ has been considered where the system is assumed to oscillate. Both $y = a$ and $y = -b$ represent the turning points in which $\dot{y} = 0$, $a$ and $b$ are an unknown amplitude to be determined.

The method is applicable to determine approximate solutions of various differential equations whose nonlinear function satisfies the relation, $f(-y, -y') = -f(y, y')$. However, the method is also useful when nonlinear function satisfies the relation, $f(-y, -y') = f(y, y')$ (quadratic nonlinearities as well as to some nonlinear problems with mixed parity).

We use the solution of the form of Eq. (2) in both regions where $y > 0$ and $y < 0$. In this regard, we usually match the solution satisfying the conditions, $y_1(\phi) = y_2(\phi), \ y_1'(\phi) = y_2'(\phi)$. In article [20], such type of approximate solution (only first approximation) was obtained satisfying the first condition. Since the second condition was ignored, the solution was not clearly matched at the point considered. But the higher approximation gradually coincides with the numerical solution.

For the region $y > 0$, a second approximate solution of Eq. (8), can be considered as

$$y_1(t) = a_0(\rho \cos(\omega_0 t) + u_1 \cos(3\omega_0 t)) \quad (9).$$

Let us consider $y_1'$, can be expanded in a Fourier series as

$$b_1 \cos(\omega_0 t) + b_3 \cos(3\omega_0 t) + \cdots. \quad (10)$$

Herein $b_1, b_3, \ldots$ are evaluated as

$$b_1 = \frac{8a_0^2}{105\pi}(35 - 56u_1 + 48u_1^2),$$

$$b_3 = \frac{8a_0^2}{315\pi}(21 + 120u_1 - 176u_1^2). \quad (11)$$

Now substituting $\rho = 1 - u_1$, using Eq. (9)-(11) into Eq. (8) and then equating the coefficients of $\cos(\omega_0 t)$ and $\cos(3\omega_0 t)$, the following equations can be obtained as

$$-(1 - u_1)\omega_0^2 + 2 - 2u_1 + \frac{8a_0}{105\pi}(35 - 56u_1 + 48u_1^2) = 0,$$

and
\[-9u_\omega^2 + 2u_\omega + \frac{8a_0}{315\pi}(21 + 120u_\omega - 176u_\omega^2) = 0.\] (13)

After simplification, Eq. (12) can be expressed into another form as

\[\omega^2 = 2 + \frac{8a_0}{3\pi} - \frac{8a_0u_\omega}{5\pi} + \frac{72a_0u^2_\omega}{35\pi} + \frac{72a_0u^4_\omega}{35\pi} + \cdots.\] (14)

By elimination of \(u_\omega\) from Eq. (13), with the help of Eq. (14), the equation of \(u_\omega\) can be expressed as

\[u_\omega = \lambda_0 \left(1 + \frac{30\pi u^2_\omega}{a_0} + \frac{1216u^2_\omega}{21} - \frac{160u^8_\omega}{3}\right), \quad \text{where} \quad \lambda_0 = \frac{7a_0}{210\pi + 282a_0}.\] (15)

Therefore, the power series solution of Eq. (15) can be obtained in terms of \(\lambda_0\) as

\[u_\omega = \lambda_0 + \left(\frac{1216}{21} + \frac{30\pi}{a_0}\right)\lambda_0^3 - \frac{160}{3}\lambda_0^4 + \cdots.\] (16)

Now substituting the value of \(u_\omega\) from Eq. (16) into Eq. (14), the relation between \(\omega_\omega\) and \(\lambda_\omega\) is determined and then the approximate angular frequency approximated as

\[\omega_\omega = \sqrt{2 + \frac{8a_0}{3\pi} - \frac{8a_0}{5\pi} \lambda_\omega + \frac{72a_0}{35\pi} \lambda_\omega^3 - 48\lambda_\omega^4 - \frac{9728a_0}{105\pi} \lambda_\omega^5 + \cdots.}\] (17)

Therefore, Eq. (9) represents the second approximate solution of Eq. (8) where \(\omega_\omega\) and \(u_\omega\) are respectively given by Eq. (17) and (16). Now for the region \(y < 0\), the second approximate solution of Eq. (9) can be written as

\[y = b_0(\rho \cos(\omega_\omega t) + u_\omega \cos(3\omega_\omega t)),\] (18)

where \(b_0\) is determined as

\[b_0 = (3 + 2a_0 - 3\sqrt{1 - 4a_0(1 + a_0)/3})/4.\] (19)

From Eq. (19), it has been shown that \(a_0 \leq 0.5\) and \(b_0 \leq 1,\) (see [20] for details). Without repeating the solution process, \(u_\omega\) and \(\omega_\omega\) are respectively obtained from Eqs. (16)-(17), replacing \(a_0\) by \(-b_0\). Therefore, it becomes

\[u_\omega = \lambda_\omega + \left(\frac{1216}{21} - \frac{30\pi}{b_0}\right)\lambda_\omega^3 - \frac{160}{3}\lambda_\omega^4 + \cdots, \quad \lambda_\omega = \frac{-7b_0}{210\pi - 282b_0},\] (20)

and

\[\omega_\omega = \sqrt{2 + \frac{8b_0}{3\pi} + \frac{8b_0}{5\pi} \lambda_\omega - \frac{72b_0}{35\pi} \lambda_\omega^3 - 48\lambda_\omega^4 + \frac{9728b_0}{105\pi} \lambda_\omega^5 + \cdots}.\] (21)

4. Results and Discussions

The approximated solutions have been compared with the existing solutions obtained by homotopic perturbation method (HPM) and modified MHBM [21], and the available exact solutions which are illustrated in table 1 for initial oscillation amplitude \(a_0 = 0.1\). The comparison between the results obtained by present study and the corresponding numerical solutions are shown in figure 1 to figure 2 for the initial oscillation amplitude \(a_0 = 0.2\) and \(a_0 = 0.3\) respectively. It is observed from all tables and figures that the results obtained by modified harmonic balance method (MHBM) are in good agreement with the corresponding numerical solutions compare to those obtained by Momani et al. [21] applying homotopy perturbation method. Moreover, using the MHBM the angular frequency as well as the period of oscillatory problem can be calculated for different values of initial amplitudes.

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5. Conclusion
In this paper, an analytical approximate technique based on the Harmonic Balance Method has been introduced to determine approximate angular frequencies and the corresponding periodic solutions of the oscillators with anti-symmetric quadratic nonlinearity. The solution procedure is straightforward and simple as compared with the existing methods. The approximated solutions show a good agreement with its exact ones and much better than the existing results. All of these allow us to conclude that the method presented in this article for solving the oscillators with anti-symmetric quadratic nonlinearity can be considered as an efficient alternative of the already published methods.

Table 1. The comparison the approximated results with existing solutions and the corresponding numerical solutions for the initial oscillation amplitude \( a_0 = 0.1 \).

| \( t \) | \( y_{(21)}^{\text{HPM}} \) | \( y_{(23)}^{\text{HPM}} \) | \( y_{(\text{Numerical})} \) | \( y_{(\text{Presented})}^{\text{HBM}} \) |
|---|---|---|---|---|
| 0  | 0.1 | 0.1 | 0.1 | 0.1 |
| 0.5 | 0.07492588 | 0.07492588 | 0.07492584 | 0.07606976 |
| 1  | 0.01261291 | 0.01262114 | 0.01260204 | 0.01574689 |
| 1.5 | -0.05542130 | -0.05564197 | -0.05599434 | -0.05208194 |
| 2  | -0.08949363 | -0.09641338 | -0.09857369 | -0.09488080 |
| 2.5 | -0.02186309 | -0.08941601 | -0.09605590 | -0.09218266 |
| 3  | 0.34944163 | -0.03758556 | -0.04954852 | -0.04547820 |
| 3.5 | 1.63139818 | 0.03306331 | 0.02002292 | 0.02285490 |
| 4  | 5.37207116 | 0.08689346 | 0.07971480 | 0.08009427 |
| 4.5 | 14.98507704 | 0.09750473 | 0.09971271 | 0.09873981 |
| 5  | 37.20789062 | 0.05990474 | 0.06971159 | 0.07017312 |

Note: In table 1, \( y_{(21)}^{\text{HPM}} \) and \( y_{(23)}^{\text{HPM}} \) respectively denote approximate solutions determined previously in Momani et. al. [21] by using Homotopy Perturbation Method and Modified Homotopy Perturbation Method. \( y_{(\text{Presented})}^{\text{HBM}} \) represents the approximate solutions obtained in the presented technique. \( y_{(\text{Numerical})} \) indicates the numerical solutions (Runge-Kutta fourth order method) which has considered to be the exact solutions.
Figure 1. Comparison between the approximated solutions (represented by blue dashed line) and the corresponding numerical solutions (represented by red solid line) for the initial amplitude $a_0 = 0.2$.

Figure 2. Comparison between the approximated solutions (represented by blue dashed line) and the corresponding numerical solutions (represented by red solid line) for the initial amplitude.

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