Septic B-spline collocation method for numerical solution of the coupled Burgers’ equations

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ABSTRACT
In this paper, a numerical solution of the coupled Burgers’ equations based on septic B-spline collocation method is presented. The scheme is based on the Crank–Nicolson formulation for time integration and septic B-spline functions for space integration. The method has been shown unconditionally stable by using Von-Neumann technique. The efficiency of this method is demonstrated by applying two test problems. The obtained numerical results are found in a good agreement with the exact solution. This method is efficient, powerful, and economical. It can also applicable to other linear and nonlinear partial differential equations.

1. Introduction
The coupled Burger equations originally derived by Esipov to study the model of polydisperse sedimentation (Esipov, 1995). It’s a simple model of sedimentation or evolution of scaled volume concentrations of two kinds of particles in fluid suspensions and colloids, under the effect of gravity (Nee and Duan, 1998).

The study of Burger equations have an important tasks for the system describes various kinds of physical phenomena, such as a mathematical model of turbulence, traffic, and the approximate theory of flow through a shock wave travelling in viscous fluid (Burger, 1948; Cole, 1951). Therefore many techniques have been proposed to obtain analytical and numerical solutions for one dimensional coupled Burgers equations, for example, a modified extended tanh-function method (Soliman, 2006), adomain decomposition method (Kaya, 2001), variational iteration method (Abdou and Soliman, 2005), and a conjugate filter approach (Wei and Gu, 2002). Esipov has gave numerical solutions and comparisons (Esipov, 1995). The Fourier pseudo-spectral method (Rashid and Ismail, 2009), Chebyshev spectral collocation method (Khaet er et al., 2008), adomin-pade technique (Deghan et al., 2007), fully implicit and Crank-Nicolson schemes (Srivastava et al, 2013; Srivastava et al, 2013). Implicit logarithmic finite-difference method (Srivastava et al, 2014), and collocation of local radial basis functions (Islam et al., 2012). For more about Burgers’ equation see (Bonkile et al., 2018; Lashkarian et al., 2019; Pana et al., 2018; Prakasha et al., 2015; Shi et al., 2017; Wang and Kara, 2018; Karakoc et al., 2014).

Spline functions theory is very active field of approximate theory in partial differential equations. Many researchers have proposed numerical solution for the nonlinear equations including Burger equations, such as, Galerkin B-Spline-collocation method (Bryan et al., 2017), exponential cubic B-spline differential quadrature method (Korkmaz and Akmaz, 2015), trigonometric cubic B-spline differential quadrature method (Korkmaz and Akmaz, 2018), cubic B-spline collocation method (Sharifi and Rashidinia, 2016), B-spline collocation and self-adapting differential evolution (jDE) algorithm (Luo et al., 2018), fourth-order cubic B-spline collocation method (Rohila and Mittal, 2018), cubic B-spline collocation scheme (Mittal and Arora, 2011), non-polynomial spline method (Ali et al., 2015), collocation method with cubic trigonometric B-spline (Raslan et al., 2016), collocation method with quintic B-spline method (Raslan et al., 2017), generalized differential quadrature method (Mokhtari et al., 2011), exponential cubic B-spline finite element method (Ersoy and Dag, 2015), B-spline Differential Quadrature Method (Bashan et al., 2015), and the Galerkin quadratic B-spline finite element method (Kutluay and Ucar, 2013). The septic B-spline approach has been used to establish approximate solutions for several partial differential equations (Ramadan et al., 2005; El-Danaf, 2008; Soliman and Hussien, 2005; Quarteroni et al., 2008; Lashkarian et al., 2019; Pana et al., 2018; Prakasha et al., 2015; Shi et al., 2017; Wang and Kara, 2018; Karakoc et al., 2014).

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2. Septic B-spline collocation method

Consider a mesh \( a = x_0 < x_1, x_2, \ldots, x_n = b \) as a uniform partition of the solution domain \( a \leq x \leq b \), with \( h = x_j - x_{j-1}, j = 1, 2, \ldots, N \). The septic B-spline basis functions \( B_j(x) \) at knots \( x_j \) given as below:

\[
B_j(x) = \frac{1}{h^7} \begin{cases} 
(x-x_{j-4})^7, \\
(x-x_{j-4})^7 - 8(x-x_{j-3})^7, \\
(x-x_{j-4})^7 - 28(x-x_{j-3})^7 + 28(x-x_{j-2})^7, \\
(x-x_{j-4})^7 - 8(x-x_{j-3})^7 + 28(x-x_{j-2})^7 - 56(x-x_{j-1})^7, \\
(x_j-x) - 8(x_{j+3}-x) + 28(x_{j+2}-x) - 56(x_{j+1}-x), \\
(x_{j+4}-x) - 2(x_{j+3}-x) + 28(x_{j+2}-x), \\
(x_{j+4}-x) - 8(x_{j+3}-x) + 28(x_{j+2}-x), \\
0,
\end{cases}
\]

where \( \{B_{-3}, B_{-2}, B_{-1}, B_0, B_1, \ldots, B_{N-1}, B_{N+2}, B_{N+3} \} \) forms a basis over the region \( [a, b] \). Each septic B-spline covers eight elements so that an element is covered by eight septic B-splines.

3. Solution of coupled burger equations

The coupled Burger equations is given by

\[
\begin{align*}
\frac{u_{t}}{k} - u_{xx} + k_{1}uu_{x} + k_{2}(uv)_{x} &= 0, \quad \text{Equation (2)} \\
v_{t} - v_{xx} + k_{1}v_{x} + k_{3}(uv)_{x} &= 0, \quad \text{Equation (3)}
\end{align*}
\]

with the boundary conditions:

\[
\begin{align*}
u(a, t) &= \alpha_1, u(b, t) = \alpha_2, \\
v(a, t) &= \beta_1, v(b, t) = \beta_2, \\
v_x(a, t) &= 0, v_x(b, t) = 0, \\
u_x(a, t) &= 0, u_x(b, t) = 0, \quad a \leq x \leq b, 0 \leq t \leq T
\end{align*}
\]

and initial conditions:

\[
\begin{align*}
u(x, 0) &= \alpha_1, \quad \text{Equation (5)} \\
u(x, 0) &= \alpha_2
\end{align*}
\]

To apply the proposed method, the time derivative was discretized by forward finite difference approximation and using Crank-Nicolson approach to equations (2) and (3), which obtained:

\[
\begin{align*}
\frac{u^{n+1} - u^{n}}{k} \left( \frac{u_{xx}^{n+1} + u_{xx}^{n}}{2} + k_{1} \left( u_{x}^{n+1} + u_{x}^{n} \right) \right) + k_{2} \left( \frac{(uv)_{x}^{n+1} + (uv)_{x}^{n}}{2} \right) &= 0, \\
\frac{v^{n+1} - v^{n}}{k} \left( \frac{v_{xx}^{n+1} + v_{xx}^{n}}{2} + k_{3} \left( v_{x}^{n+1} + v_{x}^{n} \right) \right) &= 0,
\end{align*}
\]

where \( k = \Delta t \) is the time step. The nonlinear terms in equations (6) and (7) are linearized using the form

\[
\begin{align*}
\left( uu_{x} \right)^{n+1} + \left( uu_{x} \right)^{n} &= 0, \\
\left( uu_{x} \right)^{n+1} + \left( uu_{x} \right)^{n} &= 0, \\
\left( uu_{x} \right)^{n+1} + \left( uu_{x} \right)^{n} &= 0,
\end{align*}
\]

given by Rubin and Graves ((Rubin and Graves, 1975). Then the nonlinear terms are approximated as the below:

\[
\begin{align*}
\left( uu_{x} \right)^{n+1} &= uu_{x}^{n+1} + uu_{x}^{n}, \\
\left( uu_{x} \right)^{n+1} &= uu_{x}^{n+1} + uu_{x}^{n}, \\
\left( uu_{x} \right)^{n+1} &= uu_{x}^{n+1} + uu_{x}^{n},
\end{align*}
\]

by approximating \( u(x, t) \) and \( v(x, t) \) by using septic B-spline functions \( B_j(x) \) and the time dependent parameters \( \delta_j(t) \) and \( \sigma_j(t) \), for \( U(x, t) \) and \( V(x, t) \) respectively, so the approximate solution written as:

\[
\begin{align*}
U_n(x, t) &= \sum_{j=-3}^{N+3} \delta_j(t) B_j(x), \\
V_n(x, t) &= \sum_{j=-3}^{N+3} \sigma_j(t) B_j(x)
\end{align*}
\]

using approximate function (9) and septic B-spline functions (1), the approximate values at the knots of \( U(x, t) \) and \( V(x, t) \) and their derivatives up to second order are determined in terms of the time parameters \( \delta_j(t) \) and \( \sigma_j(t) \) respectively, as:
\[ U_j = U(x_j) = \delta_{j-3} + 120\delta_{j-2} + 1191\delta_{j-1} + 2416\delta_j + 1191\delta_{j+1} + 120\delta_{j+2} + \delta_{j+3}, \]

\[ U'_j = U'(x_j) = \frac{7}{h}(-\delta_{j-3} - 56\delta_{j-2} - 245\delta_{j-1} + 245\delta_{j+1} + 56\delta_{j+2} + \delta_{j+3}), \]

\[ U''_j = U''(x_j) = \frac{42}{h^2}(\delta_{j-3} - 24\delta_{j-2} + 15\delta_{j-1} - 80\delta_j + 15\sigma_{j+1} + 24\sigma_{j+2} + \sigma_{j+3}), \]

\[ V_j = V(x_j) = \sigma_{j-3} + 120\sigma_{j-2} + 1191\sigma_{j-1} + 2416\sigma_j + 1191\sigma_{j+1} + 120\sigma_{j+2} + \sigma_{j+3}, \]

\[ V'_j = V'(x_j) = \frac{7}{h}(-\sigma_{j-3} - 56\sigma_{j-2} - 245\sigma_{j-1} + 245\sigma_{j+1} + 56\sigma_{j+2} + \sigma_{j+3}), \]

\[ V''_j = V''(x_j) = \frac{42}{h^2}(\sigma_{j-3} - 24\sigma_{j-2} + 15\sigma_{j-1} - 80\sigma_j + 15\sigma_{j+1} + 24\sigma_{j+2} + \sigma_{j+3}). \]

(10)

by substituting the approximate solution for \( U, V \) and its derivatives from equations (10), (6) and (7) yields the following difference equations with the unknowns \( \delta_j(t) \) and \( \sigma_j(t) \):

\[ A_1\delta_{j-3}^n + A_2\delta_{j-2}^n + A_3\delta_{j-1}^n + A_4\delta_j^n + A_5\delta_{j+1}^n + A_6\delta_{j+2}^n + \delta_{j+3}^n = A_{10}\delta_{j-3} + A_{11}\delta_{j-2} + A_{12}\delta_{j-1} + A_{13}\delta_j + A_{14}\delta_{j+1} + A_{15}\delta_{j+2} + \delta_{j+3}, \]

\[ B_1\sigma_{j-3}^n + B_2\sigma_{j-2}^n + B_3\sigma_{j-1}^n + B_4\sigma_j^n + B_5\sigma_{j+1}^n + B_6\sigma_{j+2}^n + \sigma_{j+3}^n = B_{10}\sigma_{j-3} + B_{11}\sigma_{j-2} + B_{12}\sigma_{j-1} + B_{13}\sigma_j + B_{14}\sigma_{j+1} + B_{15}\sigma_{j+2} + \sigma_{j+3}, \]

where

\[ A_1 = -\frac{21k}{h^2} + 7kk_jz_1 + \frac{7kk_jz_2}{2h} + \frac{7kk_jz_3}{2h} - \frac{7kk_jz_4}{2h}, \]

\[ A_2 = 120 - \frac{50k}{h^2} + \frac{392kk_jz_1}{2h} + \frac{840kk_jz_2}{2h} + \frac{840kzk_3}{2h} + \frac{196kkzk_4}{2h}, \]

\[ A_3 = 1191 - \frac{315k}{h^2} + 1715kk_jz_1 + \frac{8337kk_kz_2}{2h}, \]

\[ A_4 = 2416 + \frac{160k}{h^2} + \frac{1691kkzk_1}{2h} + \frac{1691kkzk_2}{2h}, \]

\[ A_5 = 1191 - \frac{315k}{h^2} + 1715kk_jz_1 + \frac{8337kk_kz_2}{2h}, \]

\[ A_6 = 120 - \frac{50k}{h^2} + \frac{392kk_jz_1}{2h} + \frac{840kk_jz_2}{2h} + \frac{840kzk_3}{2h} + \frac{196kkzk_4}{2h}, \]

\[ A_7 = 1 - \frac{21k}{h^2} + 7kk_jz_1 + \frac{7kk_jz_2}{2h} + \frac{7kk_jz_3}{2h} + \frac{7kk_jz_4}{2h}, \]

\[ A_8 = 120 - \frac{50k}{h^2} + \frac{392kk_jz_1}{2h} + \frac{840kk_jz_2}{2h} + \frac{840kzk_3}{2h} + \frac{196kkzk_4}{2h}, \]

\[ A_9 = -\frac{1715kk}{2h} + \frac{8337kk_kz_1}{2h} + \frac{1691kkzk_2}{2h}, \]

\[ A_{10} = \frac{1691kkzk_1}{2h} + \frac{1691kkzk_2}{2h}, \]

\[ A_{11} = \frac{1691kkzk_1}{2h} + \frac{1691kkzk_2}{2h}, \]

\[ A_{12} = \frac{1691kkzk_1}{2h} + \frac{1691kkzk_2}{2h}, \]

\[ A_{13} = \frac{1691kkzk_1}{2h} + \frac{1691kkzk_2}{2h}, \]

\[ A_{14} = \frac{1691kkzk_1}{2h} + \frac{1691kkzk_2}{2h}, \]

\[ A_{15} = 1 + \frac{42k}{2h^2}, \]

\[ A_{16} = 120 + \frac{50k}{h^2}, \]

\[ A_{17} = 1191 + \frac{315k}{h^2}, \]

\[ B_1 = 1 - \frac{21k}{h^2} + \frac{7kk_jz_1}{2h} + \frac{7kk_jz_2}{2h} + \frac{7kk_jz_3}{2h} - \frac{7kk_jz_4}{2h}, \]

\[ B_2 = 120 - \frac{50k}{h^2} + \frac{392kk_jz_1}{2h} + \frac{840kk_jz_2}{2h} + \frac{840kzk_3}{2h} + \frac{196kkzk_4}{2h}, \]

\[ B_3 = 1191 - \frac{315k}{h^2} + \frac{1715kk_jz_1}{2h} + \frac{8337kk_kz_2}{2h}, \]

\[ B_4 = 2416 + \frac{160k}{h^2} + \frac{1691kkzk_1}{2h} + \frac{1691kkzk_2}{2h}, \]

\[ B_5 = 1191 - \frac{315k}{h^2} + \frac{1715kk_jz_1}{2h} + \frac{8337kk_kz_2}{2h}, \]

\[ B_6 = 120 - \frac{50k}{h^2} + \frac{392kk_jz_1}{2h} + \frac{840kk_jz_2}{2h} + \frac{840kzk_3}{2h} + \frac{196kkzk_4}{2h}, \]

\[ B_7 = 1 - \frac{21k}{h^2} + \frac{7kk_jz_1}{2h} + \frac{7kk_jz_2}{2h} + \frac{7kk_jz_3}{2h} - \frac{7kk_jz_4}{2h}, \]

\[ B_8 = \frac{7kk_jz_1}{2h} + \frac{7kk_jz_2}{2h}, \]

\[ B_9 = \frac{7kk_jz_1}{2h} + \frac{7kk_jz_2}{2h}, \]

\[ B_{10} = \frac{7kk_jz_1}{2h} + \frac{7kk_jz_2}{2h}, \]

\[ B_{11} = \frac{7kk_jz_1}{2h} + \frac{7kk_jz_2}{2h}, \]

\[ B_{12} = \frac{7kk_jz_1}{2h} + \frac{7kk_jz_2}{2h}, \]

\[ B_{13} = \frac{7kk_jz_1}{2h} + \frac{7kk_jz_2}{2h}, \]

\[ B_{14} = \frac{7kk_jz_1}{2h} + \frac{7kk_jz_2}{2h}, \]

\[ B_{15} = 1 + \frac{42k}{2h^2}, \]

\[ B_{16} = 120 + \frac{50k}{h^2}, \]

\[ B_{17} = 1191 + \frac{315k}{h^2}, \]

\[ B_{18} = 2416 - \frac{160k}{h^2}. \]
The systems in the equations (11) and (12) consists of $2N + 2$ equation in $2N + 14$ unknowns. To get a unique solution to the systems, 12 additional constraints are required. These are obtained from the boundary conditions (4). Application the boundary conditions enables us to eliminate the parameters $\delta_{0}^{0}, \delta_{0}^{0}, \delta_{0}^{0}, \delta_{0}^{0}, \delta_{0}^{0}, \delta_{0}^{0}, \delta_{0}^{0}, \delta_{0}^{0}$ from the system. Thus, we have a system of dimension $(2N + 2) \times (2N + 2)$, which is the septa-diagonal system that can be solved by any algorithm.

4. Initial values

To solve the system, we apply the initial conditions to determine: $\left(\sigma_{0}^{0}, \sigma_{0}^{0}, \sigma_{0}^{0}, \sigma_{0}^{0}, \sigma_{0}^{0}, \sigma_{0}^{0}, \sigma_{0}^{0}, \sigma_{0}^{0}\right)$ and $\left(\sigma_{0}^{0}, \sigma_{0}^{0}, \sigma_{0}^{0}, \sigma_{0}^{0}, \sigma_{0}^{0}, \sigma_{0}^{0}, \sigma_{0}^{0}, \sigma_{0}^{0}\right)$.

When $t = 0$, equation (9) becomes:

$$
\begin{align*}
U_{N}^{0}(x, 0) &= \sum_{j=3}^{N+1} \delta_{0}^{0} B_{j}(x), \\
V_{N}^{0}(x, 0) &= \sum_{j=3}^{N+1} \sigma_{0}^{0} B_{j}(x),
\end{align*}
$$

The approximate solution must satisfy the following:

(i) It must agree with the initial conditions at the knots.

(ii) The derivatives of the approximate initial condition agree with the exact initial conditions at both ends of the range.

The initial conditions and the derivatives at the boundaries are used as below:

$$
(\mathcal{U})'(x_{0}, 0) = \frac{7}{h} \left(-\delta_{3-3} - 56\delta_{3-2} - 245\delta_{3-1} + 245\delta_{3-1} + 56\delta_{2} + \delta_{3}\right) = f'(x_{0}),
$$

$$
(\mathcal{U})''(x_{0}, 0) = \frac{42}{h^{2}} \left(\delta_{3-2} - 24\delta_{3-2} + 15\delta_{3-1} - 80\delta_{0} + 15\delta_{1} + 24\delta_{2} + \delta_{3}\right) = f''(x_{0}),
$$

$$
(\mathcal{U})'''(x_{0}, 0) = \frac{210}{h^{3}} \left(-\delta_{3-3} - 8\delta_{3-2} + 19\delta_{3-1} - 19\delta_{1} + 8\delta_{2} + \delta_{3}\right) = f'''(x_{0}),
$$

$$
(\mathcal{U})''(x_{0}, 0) = \delta_{j-3} + 120\delta_{j-2} + 1191\delta_{j-1} + 2416\delta_{j} + 1191\delta_{j+1} + 120\delta_{j+2} + \delta_{j+3},
$$

$$
(\mathcal{V})''(x_{0}, 0) = \sigma_{j-3} + 120\sigma_{j-2} + 1191\sigma_{j-1} + 2416\sigma_{j} + 1191\sigma_{j+1} + 120\sigma_{j+2} + \sigma_{j+3}.
$$

which is a septa-diagonal system for unknown initial values $\delta_{0}^{0}$ and $\sigma_{0}^{0}$ of order $(2N + 2)$, after eliminating the values of $\delta_{0}^{0}$ and $\sigma_{0}^{0}$. This system can be solved by any algorithm. Once the initial vectors of parameters have been calculated, the numerical solution of coupled Burger equations $U$ and $V$ can be determined from the time evaluation of the vectors $\delta_{0}^{0}$, and $\sigma_{0}^{0}$ by using the recurrence relations:

$$
(\mathcal{U})(x_{n}, t_{n}) = \delta_{j-3} + 120\delta_{j-2} + 1191\delta_{j-1} + 2416\delta_{j} + 1191\delta_{j+1} + 120\delta_{j+2} + \delta_{j+3},
$$

$$
(\mathcal{V})(x_{n}, t_{n}) = \sigma_{j-3} + 120\sigma_{j-2} + 1191\sigma_{j-1} + 2416\sigma_{j} + 1191\sigma_{j+1} + 120\sigma_{j+2} + \sigma_{j+3}.
$$

5. Stability analysis of the method

The stability analysis based on the von Neumann concept in which the growth factor of a typical Fourier mode defined as:

$$
\begin{align*}
\delta_{n} &= A e^{i\phi}, \\
\sigma_{n} &= B e^{i\phi}, \\
g &= \frac{\xi_{n+1}}{\xi_{n}},
\end{align*}
$$

where $A, B$ are the harmonics amplitude, $i$ is the imaginary unit, $\phi = kh$, $k$ is the mode number, $h$ is the element size, and $g$ is the amplification factor of the schemes.

The non-linear terms in the scheme are linearized by assuming the nonlinear terms as a constant $\lambda_{1}$ and $\lambda_{2}$ respectively. At $x_{j}$, the equations (11) and (12) can be rewritten as:
\[
\begin{align*}
& a_{1j}^{\alpha_{j}+1} + a_{2j}^{\alpha_{j}+1} + a_{3j}^{\alpha_{j}+1} + a_{4j}^{\alpha_{j}+1} + a_{5j}^{\alpha_{j}+1} + a_{6j}^{\alpha_{j}+1} \\
& + a_{7j}^{\alpha_{j}+1} - a_{8j}^{\alpha_{j}+1} - a_{9j}^{\alpha_{j}+1} - a_{10j}^{\alpha_{j}+1} - a_{11j}^{\alpha_{j}+1} = \\
& a_{12j}^{\alpha_{j}+1} + a_{13j}^{\alpha_{j}+1} + a_{14j}^{\alpha_{j}+1} + a_{15j}^{\alpha_{j}+1} + a_{16j}^{\alpha_{j}+1} + a_{17j}^{\alpha_{j}+1} \\
& + a_{18j}^{\alpha_{j}+1} - a_{9j}^{\alpha_{j}+1} - a_{10j}^{\alpha_{j}+1} - a_{11j}^{\alpha_{j}+1} = 0, \\
& a_{11j}^{\alpha_{j}+1} \quad \text{(14)}
\end{align*}
\]

where

\[
\begin{align*}
& a_{1} = 1 - \frac{21k}{h^2} - \frac{7kk_{1}}{2h} - \frac{7kk_{3}}{2h}, \\
& a_{2} = 120 - \frac{24k}{2h^2} - \frac{756kk_{1}}{2h} - \frac{756kk_{2}}{2h}, \\
& a_{3} = 119 - \frac{15k}{2h^2} - \frac{245kk_{1}}{2h} - \frac{245kk_{2}}{2h}, \\
& a_{4} = 2416 + \frac{42}{2h^2} - \frac{15k}{2h^2} - \frac{756kk_{1}}{2h} - \frac{756kk_{2}}{2h}, \\
& a_{5} = 119 - \frac{15k}{2h^2} - \frac{245kk_{1}}{2h} - \frac{245kk_{2}}{2h}, \\
& a_{6} = 120 - \frac{24k}{2h^2} - \frac{756kk_{1}}{2h} - \frac{756kk_{2}}{2h}, \\
& a_{7} = 1 - \frac{21k}{h^2} - \frac{7kk_{1}}{2h} - \frac{7kk_{2}}{2h}, \\
& a_{8} = \frac{7kk_{2}}{2h}, \\
& a_{11} = 1 + \frac{21k}{h^2} + \frac{7kk_{2}}{2h} + \frac{7kk_{2}}{2h}, \\
& a_{12} = 120 + \frac{42k}{2h^2} - \frac{24k}{2h^2} - \frac{756kk_{1}}{2h} - \frac{756kk_{2}}{2h}, \\
& a_{13} = 1191 - \frac{15k}{2h^2} + \frac{245kk_{1}}{2h} + \frac{245kk_{2}}{2h}, \\
& a_{14} = 2416 - \frac{42}{2h^2}, \\
& a_{15} = 1191 + \frac{15k}{2h^2} - \frac{245kk_{1}}{2h} - \frac{245kk_{2}}{2h}, \\
& a_{16} = 120 - \frac{24k}{2h^2} - \frac{756kk_{1}}{2h} - \frac{756kk_{2}}{2h}, \\
& a_{17} = 1 - \frac{21k}{h^2} - \frac{7kk_{1}}{2h} - \frac{7kk_{2}}{2h}, \\
& b_{1j}^{\alpha_{j}+1} + b_{2j}^{\alpha_{j}+1} + b_{3j}^{\alpha_{j}+1} + b_{4j}^{\alpha_{j}+1} + b_{5j}^{\alpha_{j}+1} + b_{6j}^{\alpha_{j}+1} \\
& + b_{7j}^{\alpha_{j}+1} + b_{8j}^{\alpha_{j}+1} + b_{9j}^{\alpha_{j}+1} + b_{10j}^{\alpha_{j}+1} + b_{11j}^{\alpha_{j}+1} \\
& + b_{12j}^{\alpha_{j}+1} + b_{13j}^{\alpha_{j}+1} + b_{14j}^{\alpha_{j}+1} + b_{15j}^{\alpha_{j}+1} + b_{16j}^{\alpha_{j}+1} + b_{17j}^{\alpha_{j}+1} \\
& + b_{18j}^{\alpha_{j}+1} - b_{9j}^{\alpha_{j}+1} - b_{10j}^{\alpha_{j}+1} - b_{11j}^{\alpha_{j}+1} = 0, \\
& b_{11}^{\alpha_{j}+1} \quad \text{(15)}
\end{align*}
\]

Substituting (13) into the difference equation (14), yields

\[
g = \frac{X_{1} - iY}{X_{2} + iY}, \quad \text{(16)}
\]

where

\[
\begin{align*}
x_{1} &= A \left( 2 + \frac{42k}{h^2} \right) \cos \phi + \left( 240 + \frac{42k}{h^2} \right) \cos 2\phi + \left( 2382 + \frac{42k}{h^2} \right) \cos 3\phi \\
& + \left( 2416 - \frac{42k}{h^2} \right) \cos 4\phi, \\
x_{2} &= A \left( 2 - \frac{42k}{h^2} \right) \cos \phi + \left( 240 - \frac{42k}{h^2} \right) \cos 2\phi + \left( 2382 - \frac{42k}{h^2} \right) \cos 3\phi \\
& + \left( 2416 + \frac{42k}{h^2} \right) \cos 4\phi
\end{align*}
\]

and

\[
\begin{align*}
& B \left( \frac{7kk_{2}}{h} \right) \sin \phi + \left( \frac{2(56kk_{2})}{h} \right) \sin 2\phi + \left( \frac{7(245kk_{2})}{h} \right) \sin 3\phi \\
& + \left( \frac{2416}{h} \right) \sin 4\phi, \\
& Y = \frac{7k}{h} \left( k_{2} \right) \sin \phi + \left( \frac{56kk_{2}}{h} \right) \sin 2\phi
\end{align*}
\]

Similarly, substituting (13) into the difference equation (15), results:

\[
g = \frac{X_{3} - iY_{1}}{X_{4} + iY_{1}}, \quad \text{(17)}
\]

where
and
\[ \begin{align*}
X_3 & = \begin{bmatrix}
\left( 2 + \frac{42\beta}{\tau} \right) \cos 3\phi + \left( 120 + \frac{42\beta}{\tau} \right) \cos 2\phi + \left( 2382 + \frac{42\beta}{\tau} \right) \cos \phi \\
2416 - 3 \frac{80\beta}{\tau}
\end{bmatrix}, \\
X_4 & = \begin{bmatrix}
\left( 2 + \frac{42\beta}{\tau} \right) \cos 3\phi + \left( 120 - \frac{42\beta}{\tau} \right) \cos 2\phi + \left( 2382 - \frac{42\beta}{\tau} \right) \cos \phi \\
2416 + 21 \frac{80\beta}{\tau}
\end{bmatrix}.
\end{align*} 
\]

From equations (16) and (17), |g| ≤ 1 hence the scheme is unconditionally stable.

6. Numerical tests and results of coupled burgers’ equations

The performance of the proposed method was tested by using two numerical examples, in this section \(L_2\) and \(L_\infty\) error norms obtained by the following formulas:

\[ \begin{align*}
L_2 &= \|u^{\text{exact}} - u^{\text{num}}\|_2 = \sqrt{\frac{1}{N} \sum_{j=0}^{N-1} |u_j^{\text{exact}} - u_j^{\text{num}}|^2}, \\
L_\infty &= \|u^{\text{exact}} - u^{\text{num}}\|_\infty = \max \{|u_j^{\text{exact}} - u_j^{\text{num}}| \}.
\end{align*} \]

**Test problem (1):**

Numerical solution of coupled Burgers’ equations (2) and (3) is calculated for \(k_1 = -2, k_2 = k_3 = 1\) which leads to (2) and (3) as:

\[ \begin{align*}
&u_t - u_{xx} - 2uu_x + (uv)_x = 0, \\
v_t - v_{xx} - 2vv_x + (vu)_x = 0,
\end{align*} \]

with the following initial and boundary conditions:

\[ u(x, 0) = v(x, 0) = \sin(x), \quad -\pi \leq x \leq \pi. \]

and

\[ \begin{align*}
&u(-\pi, t) = u(\pi, t) = 0, \quad 0 \leq t \leq T, \\
v(-\pi, t) = v(\pi, t) = 0, \quad 0 \leq t \leq T.
\end{align*} \]

The exact solution is

\[ u(x, t) = v(x, t) = e^{-t} \sin(x), \quad -\pi \leq x \leq \pi, \quad 0 \leq t \leq T. \]

In the first computation, \(L_2\) and \(L_\infty\) error norms at \(t = 0.1, k = 0.001\) was computed with various values of \(\Delta t\). The corresponding results are presented in Table 1. Second computation, \(L_2\) and \(L_\infty\) error norms at time level \(t = 1, N = 200\) with decreasing values of \(\Delta t\) was calculated, the results are showed in Table 2. The results of both computations are the same for \(u(x, t)\) and \(v(x, t)\), because of the symmetric initial and boundary conditions. Furthermore, comparison of the numerical results of the problem (1) with the results

obtained from Raslan et al. (2016) for \(N = 50, k = 0.01, k_1 = -2, k_2 = k_3 = 1\) with different time \(t\) has been studied. The results are presented in Table 3.

The graphical illustrations are presented in Figure 1(A, B) for computed solutions of \(u(x, t)\) and \(v(x, t)\) at \(k_1 = -2, k_2 = k_3 = 1, N = 200, \Delta t = 0.001, t = 0.05, 1\). Figure 2(A, B) represented solutions of \(u(x, t)\), and \(v(x, t)\) for \(k_1 = -2, k_2 = k_3 = 1, N = 200\) and \(\Delta t = k = 0.001\) at \(t = 0.05, 1\). Figures 3(A, B) illustated solutions of \(u(x, t)\) and \(v(x, t)\) for \(k_1 = -2, k_2 = k_3 = 1, N = 200\) and \(\Delta t = k = 0.001\) at \(t = 0.1\). The profiles of \(u(x, t)\) and \(v(x, t)\) at \(\Delta t = k = 0.001, N = 200\) and various values of \(k_1, k_2\) and \(k_3\) are plotted at different time steps and are showed in Figures 4–6(A and B).

**Test problem (2):**

Numerical solutions of considered coupled Burgers’ equations have been obtained for \(k_1 = 2\) with different values of \(k_2\) and \(k_3\) at different time levels. In this situation the exact solution is

\[ \begin{align*}
u(x, t) &= a_0 - 2A \frac{2k_2 - 1}{4k_3 k_5 - 1} \tanh(A(x - 2At)), \\
v(x, t) &= a_0 \frac{2k_3 - 1}{2k_2 - 1} - 2A \frac{2k_2 - 1}{4k_3 k_5 - 1} \tanh(A(x - 2At)).
\end{align*} \]

The initial and boundary conditions are taken from the exact solution given below:

\[ \begin{align*}
u(x, 0) &= a_0 - 2A \frac{2k_2 - 1}{4k_3 k_5 - 1} \tanh(A(0)), \\
v(x, 0) &= a_0 \frac{2k_3 - 1}{2k_2 - 1} - 2A \frac{2k_2 - 1}{4k_3 k_5 - 1} \tanh(A(x)).
\end{align*} \]
where \( a_0 = 0.05 \) and \( A = \frac{1}{2} \left[ \frac{a_0 (k_2 - 1)}{k_3 - 1} \right] \). The numerical solutions for \( u(x, t) \) and \( v(x, t) \) have been computed for the domain \( x \in [-10, 10], k = 0.01 \) and number of partitions \( N = 10, N = 50, N = 100 \) and \( N = 200 \). \( L_2 \) and \( L_\infty \) norms have been presented in Table 4 for \( t = 1, k_1 = 2, k_2 = 0.1 \) and \( k_3 = 0.3 \). Tables 5 and 6 include comparison of our numerical results of problem (2) with results obtained from (Raslan et al., 2016) for the variables \( u(x, t) \) and \( v(x, t) \) with \( a_0 = 0.05 \), \( N = 16 \), \( k = 0.01 \) at different time and different values of \( k_2, k_3 \). Tables 7 and 8, contain comparison of our numerical results of problem (2) with results obtained from (Raslan et al., 2016) for the variables \( u(x, t) \) and \( v(x, t) \) with \( a_0 = 0.05 \), \( N = 21 \), \( k = 0.01 \) at different values of \( k_2, k_3 \) and \( t \).

The corresponding graphical illustrations are presented in (Figures 7(A, B)), computed approximation solutions of \( u(x, t) \) and \( v(x, t) \) for \( k_1 = 2, k_2 = 0.1 \),
Figure 4. Approximate solutions of $u$ in (A) and $v$ in (B) for $k_1 = -2$, $k_2 = 1$, $k_3 = 8$, $N = 200$ and $\Delta t = k = 0.001$ at $t = 0, 0.05, 0.1$.

Figure 5. Approximate solutions of $u$ in (A) and $v$ in (B) for $k_1 = -2, k_2 = 8, k_3 = 1, N = 200$ and $\Delta t = k = 0.001$ at $t = 0, 0.05, 0.1$.

Figure 6. Approximate solutions of $u$ in (A) and $v$ in (B) for $k_1 = 2, k_2 = 1, k_3 = 1, N = 200$ and $\Delta t = k = 0.001$ at $t = 0, 0.05, 0.1$.

Table 4. $L_2$-norm and $L_\infty$-norm for $t = 1, k = 0.01$ at different values of $N, k_1 = 2, k_2 = 0.1$ and $k_3 = 0.3$.

| $N$ | $L_2$-norm | $L_\infty$-norm | $L_2$-norm | $L_\infty$-norm |
|-----|-------------|-----------------|-------------|-----------------|
| 10  | 5.10277E-5  | 6.57098E-6      | 3.00479E-5  | 5.06719E-6      |
| 50  | 4.02338E-5  | 7.45513E-6      | 3.10494E-5  | 5.16722E-6      |
| 100 | 4.41231E-5  | 7.45741E-6      | 3.13501E-5  | 5.19233E-6      |
| 200 | 4.47578E-5  | 7.73936E-6      | 3.23412E-5  | 5.23444E-6      |

Table 5. Comparison between numerical results of problem (2) and results obtained from (Raslan et al., 2016) for the variable $u$ with $a_0 = 0.05, N = 16, k = 0.01$.

| $t$  | $k_2$ | $k_3$ | $L_2$-norm | $L_\infty$-norm | $L_\infty$-norm |
|------|-------|-------|-------------|-----------------|-----------------|
| 0.5  | 0.1   | 0.3   | 2.23482E-5  | 5.13242E-6      | 4.43465E-5      |
| 0.3  | 0.3   | 0.3   | 3.56457E-5  | 7.42341E-6      | 6.41587E-5      |
| 1.0  | 0.1   | 0.3   | 4.67791E-5  | 9.12314E-6      | 8.44084E-5      |
| 0.3  | 0.3   | 0.3   | 6.34821E-5  | 1.01244E-5      | 1.19154E-4      |
Table 6. Comparison between numerical results of problem (2) and results obtained from (Raslan et al., 2016) for the variable \( v \) with \( \alpha_0 = 0.05, N = 16, k = 0.01 \).

| \( t \) | \( k_2 \) | \( k_3 \) | \( L_2 \)- norm | \( L_\infty \)- norm | Raslan et al., 2016 |
|-------|-------|-------|-------------|-------------|------------------|
| 0.5   | 0.1   | 0.30  | 4.34977E-5  | 1.32354E-6  | 2.34474E-5 |
| 0.3   | 0.30  | 3.56457E-5 | 7.42341E-6  | 6.43587E-5  |
| 1.0   | 0.1   | 0.30  | 3.13458E-5  | 4.42166E-5  | 1.19154E-4 |
| 0.3   | 0.30  | 6.34821E-5 | 1.01294E-5  | 1.19154E-4  |

Moreover, for problem (2), \( L_2 \) and \( L_\infty \) norms have been computed at the domain \( x \in [0,1] \). \( k = 0.01 \) and \( k_1 = 2, k_2 = 0.1 \) and \( k_3 = 0.3 \) and showed in Table 9 for \( t = 1 \) with different values of \( N \).

Table 7. Comparison between numerical results of problem (2) and results obtained from (Raslan et al., 2016) for the variable \( u \) with \( \alpha_0 = 0.05, N = 21, k = 0.01 \).

| \( t \) | \( k_2 \) | \( k_3 \) | \( L_2 \)- norm | \( L_\infty \)- norm | Raslan et al., 2016 |
|-------|-------|-------|-------------|-------------|------------------|
| 0.5   | 0.1   | 0.30  | 1.01532E-5  | 3.45212E-6  | 4.33232E-5 |
| 0.3   | 0.30  | 3.18276E-5 | 7.14311E-6  | 6.10213E-5  |
| 1.0   | 0.1   | 0.30  | 1.92315E-5  | 7.34832E-5  | 8.16821E-5 |
| 0.3   | 0.30  | 3.39489E-5 | 2.72331E-5  | 1.77123E-4  |

Table 8. Comparison between numerical results of problem (2) and results obtained from (Raslan et al., 2016) for the variable \( v \) with \( \alpha_0 = 0.05, N = 21, k = 0.01 \).

| \( t \) | \( k_2 \) | \( k_3 \) | \( L_2 \)- norm | \( L_\infty \)- norm | Raslan et al., 2016 |
|-------|-------|-------|-------------|-------------|------------------|
| 0.5   | 0.1   | 0.30  | 6.28163E-6  | 3.19278E-6  | 4.15257E-5 |
| 0.3   | 0.30  | 3.18276E-5 | 7.14311E-6  | 6.10213E-5  |
| 1.0   | 0.1   | 0.30  | 2.34719E-5  | 4.15257E-5  | 4.42146E-5 |
| 0.3   | 0.30  | 5.39486E-5 | 1.01294E-5  | 1.19154E-4  |

(\( \alpha_0 \) is the initial condition and \( k, k_1, k_2, k_3 \) are the relaxation factors.)

Figure 7. Approximate solutions of \( u(x,t) \) (A) and \( v(x,t) \) (B) for \( k_1 = 2, k_2 = 1, k_3 = 0.3, N = 200 \) and \( \Delta t = k = 0.01 \) at \( t = 0, 0.5, 1 \).

\( k_3 = 0.3, N = 200 \) and \( \Delta t = k = 0.01 \) at \( t = 0, 0.5, 1 \), \( x \in [0,1] \).

7. Conclusions

In this paper, a numerical scheme for the nonlinear coupled Burger’s equations has been proposed using a collocation method based on septic B-spline functions. The proposed method is unconditionally stable. The method has been evaluated by two test problems. The accuracy of the method has been measured by computing \( L_2 \) and \( L_\infty \) error norms. The obtained numerical results are quite satisfactory and comparable with the analytic solution and better than the obtained numerical results in (Raslan et al., 2016). Based on the stability and accuracy of the proposed method, the method can be extended to solve various linear and nonlinear partial differential equations.

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No potential conflict of interest was reported by the authors.

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