Beyond Erdős-Kunen-Mauldin: 
Singular sets with shift-compactness properties 
by 
H. I. Miller†, L. Miller-Van Wieren and A. J. Ostaszewski.

Abstract. The Kestelman-Borwein-Ditor Theorem asserts that a non-negligible subset of $\mathbb{R}$ which is Baire (=has the Baire property, BP) or measurable is shift-compact: it contains some subsequence of any null sequence to within translation by an element of the set. Effective proofs are recognized to yield (i) analogous category and Haar-measure metrizable generalizations for Baire groups and locally compact groups respectively, and (ii) permit under $V = L$ construction of co-analytic shift-compact subsets of $\mathbb{R}$ with singular properties, e.g. being concentrated on $\mathbb{Q}$, the rationals.

Keywords. shift-compactness, semitopological groups, Baire groups, Haar-density topology, Steinhaus-Weil property, Ger-Kuczma classes, finite similarity embeddings, co-analytic sets, sets concentrated on the rationals, Gödel’s Axiom.

Classification: 26A03, 04A15, 02K20, 39B62.

1 Introduction

This paper is a sequel to [MilO] where two of the present authors studied shift-compactness (below), a compactness-like embedding property arising from infinite combinatorics in $\mathbb{R}$, from two points of view: topological (group action yielding dual ways of embedding, and so two ways of asserting the property), and combinatorial (effective embeddings, employing completeness of $\mathbb{R}$, and limitations exemplified by ‘counter-examples’). Here we return to both these themes, motivated principally by the effectiveness theme. First, we show that effectiveness allows completeness to be replaced by category: a semitopological Baire group $X$ will suffice (definitions in §2). Secondly,

†It is with regret that we announce that the first author Harry I. Miller (1939-2018) died on 16 Dec 2018, whilst finishing this paper. His last communication ended with: ‘I only wish I was 10 years younger (make that 15) so I could contribute (and learn) more.”
effectiveness enables ‘counter-examples’ to gain ‘good topological character’: they may be co-analytic under Gödel’s Axiom of Constructibility $V = L$.

In its most useful form and in its simplest context, that of $\mathbb{R}$, the property of shift-compactness of a subset $T$ asserts that some subsequence $\{z_m\}_{m \in M}$ (for an infinite $M \subseteq \mathbb{N}$) of any given null sequence $z_n \to 0$ may be embedded in $T$ under the action of translation. This embedding idea can be traced back to Banach [Ban, Ch. I, Th. 4], but its explicit development goes back to Kestelman [Kes1,2] and to Borwein-Ditor [BorD]. Here, the classically familiar non-negligible sets, both the the Baire non-meagre and the measurable non-null sets, have this property. That is precisely the content of the Kestelman-Borwein-Ditor Theorem, KBD. Because of this, it is often possible to unify category and measure arguments, and so to bring unity to several areas of classical analysis, such as the automatic continuity results in the theory of functional equations (the theorems of Ostrowski and Banach-Mehdi concerning the familiar Cauchy equation, cf. [BinO5], that of Bernstein-Doetsch concerning mid-point convex functions, cf. [BinO6]), and fundamental results in the theory of regularly varying (RV) functions (for instance, Karamata’s Uniform Convergence Theorem – see [BinGT], or Kendall’s Theorem, which characterizes RV sequentially, cf. [BinO11]).

The broader context is that of groups $G$ with some appropriate topological structure acting on metrizable spaces $X$, the embeddings being provided by group action (isometries, or more generally homeomorphisms), including that of a group $X$ acting on itself by translation. So the null sequences now converge either to the identity map on $X$ or to the neutral element of the group $1_X$. Here the category argument can assume primacy, since it subsumes the measure analogue, at least in the locally-compact context provided by Haar measure, by passage to the Haar density topology (under which null sets become meagre, as first observed by Haupt and Pauc [HauP], cf. [Kec, 17.47(iii)] and [BinO7, § 2]). Shift-compactness under group action implies the celebrated ‘Open Mapping Principle’ due to Effros [Eff], cf. [Ost1,3].

A further key to success in unifying several areas of analysis is the Steinhaus-Weil Interior Point Theorem [Ste], [Wei], here regarded as including the Picard-Pettis Theorem [Pic], [Pet] (since it is true both for category and measure), that the neutral element is an interior point of $AA^{-1}$ for $A$ non-negligible (Baire/measurable). In fact, the theorem follows from shift-compactness of $A$: see [BinO5], [BinO9], and Theorem 3(ix) below.

To go beyond the Haar context of Polish groups, one needs to abandon measure invariance, which is prescribed for all measurable sets and all
translations. On the measure side an abelian setting is usually (though not exclusively) preferable and, referring to the family of probability measures, one needs the **Haar-null** sets of Christensen [Chr1,2], where one particular set remains null under one corresponding (probability) measure and all translations; more generally, one may make do with the (left) Haar-null sets of Solecki [Sol] (albeit aided by a localized notion of amenability). On the category side there are their relatives: the **Haar-meagre** sets of Darji [Dar], where the particular set and all its translations have meagre preimages under some one continuous map with a compact metric domain; for background see [Jab]. (These are indeed all meagre.)

But, instead, one may fix one reference measure and then use only **admissible** translations, rather than all translations, and relative quasi-invariance of measure (preservation of nullity, relative to the admissible translations). The canonical example here is a Gaussian measure in a Hilbert space where the admissible translations form the **Cameron-Martin space** [Bog], again a Hilbert space, but under a refinement of the norm. For literature and generalizations, see [BinO10].

The category-measure duality visible above relies on qualitative aspects of measure theory, rather than quantitative, and it is refinement topologies (density topologies) which clarify the transition: see [BinO7].

The dichotomy of category – meagre versus non-meagre sets – has a corresponding dichotomy (in an abelian group) between shift-compact and non-shift-compact sets. The latter have recently been named **null-finite** [BanJ], by analogy with Haar null and Haar meagre, and indeed universally measurable null-finite sets are Haar-null (i.e. non-Haar-null sets are shift-compact, as has been noted independently in [BanJ, Th. 4.1] and [BinO9, Th. 3]). Likewise, null-finite sets that are universally Baire (i.e. pre-images under all continuous maps with a compact metric domain are Baire) are Haar-meagre [BanJ, Th. 3.1]; for further background see [BanGJSJ]. The universal Baire property first arose in mathematical logic: see [FenMW].

In §2 we re-prove KBD in a Baire-space setting. Effective versions are shown in §3 and used later in §5. In §4 we take up the study of singular sets, reviewing some recent results and also adding new ones to the stock of known examples; here they are often constructed by transfinite induction. We typify in §5 Theorem 4 the detailed treatment needed to upgrade the topological character by reference to just one of the results reviewed in §4, Theorem MM, by applying Gödel’s Axiom of Constructibility \( V = L \); the other relevant examples of Theorem 3 are relegated to Theorem 4′ but with
a sketched proof. Our treatment follows in the footsteps of Erdős-Kunen-Mauldin [ErdKM], as in our title, but we take note of the general ‘black-box’ approach recently advanced by Vidnyánszky [Vid] (contemporaneous with our own earlier development, acknowledged in [MilM], for which it was drafted as supporting material). We close in §6 with complements.

2 Kestelman-Borwein-Ditor Theorem: topological setting

There are a number of versions of the KBD and so of its proof, which go back to [Kes1,2], [BorD] – for an account see [BinO5], [BinO4], [BinO3] and [MilO]. This section is dedicated to a proof applicable to the context of a Baire semitopological group (defined below), Theorem 2, based on the proof strategy used in [MilO] to prove KBD in \( \mathbb{R} \). Although on first inspection it may seem that that proof, in constructing inductively a sequence of approximations to a translator, uses completeness of \( \mathbb{R} \), and so is adaptable only to a completely metrizable space, in fact matters are otherwise. The inductive step is sufficiently typical, i.e. unspecific to the preceding step, that it may be applied anywhere in space; so the Baire theorem will carry through the induction ‘to the limit’ at least somewhere (and so almost everywhere, according to the Generic Dichotomy Principle [BinO2]).

We close the section with the statement of another version of the KBD applicable to topological groups, one that is strong enough to imply the celebrated result of Effros [Eff, §2] known as the Open Mapping Principle [Anc], cf. [Ost1,3]. As one would expect this does indeed imply Theorem 2 when specialized to topological groups (see the Theorem from [Ost3] at the end of the section).

We first prove in Theorem 1 a special case of our KBD here, and then deduce the main result as Theorem 2. That is followed by its Haar-measure version, Theorem 2H. The latter closely follows the argument in [MilO, Th. 1M]. However, the present Haar context calls for some extra details.

Below \( z_0 \) is the identity element of the group. Also we recall that a group is said to be semitopological [ArhT] if translation is continuous – so an autohomeomorphism. A space is Baire if it obeys Baire’s Category Theorem; however, a set \( A \) is Baire if it has the Baire property, BP. Then we denote by \( A^q \) the quasi-interior of \( A \), i.e. the largest open set equal to \( A \) modulo a
meagre set.

**Theorem 1.** In a Baire semitopological group \( X \), if \( A \) is co-meagre and \( \{z_n\}_{n \in \mathbb{N}} \to 1_X \) is a null sequence, then for a dense \( \mathcal{G}_\delta \)-set of points \( a \) in \( A \):

\[
\{az_n : n = 0, 1, 2, \ldots \} \subseteq A.
\]

For subsets \( A, B \) of \( X \) below we write \( AB := \{ab : a \in A, b \in B\} \). We abbreviate neighbourhood to \( nhd \) and nowhere dense to \( nwd \). We begin with

**Lemma 1 (Extension of a separation).** In a semitopological group \( X \), for \( f \in X \), finite \( F \subseteq X \), \( L \) nwd, and \( V \) non-empty open, if

\[
(FV) \cap L = \emptyset,
\]
then there is a non-empty open \( V' \subseteq V \) with

\[
((F \cup \{f\})V') \cap L = \emptyset.
\]

**Proof.** Given \( L, f, F \) and open \( V \), as \( fU \) is non-empty and open, choose a non-empty open \( U \subseteq fV \) with \( \emptyset = U \cap L \). Put \( V' := f^{-1}U \subseteq V \); then

\[
(fV') \cap L = U \cap L = \emptyset,
\]
and

\[
(FV') \cap L \subseteq (FV) \cap L = \emptyset.
\]
Since \( (F \cup \{f\})V' = (FV') \cup (fV') \),

\[
((F \cup \{f\})V') \cap L = \emptyset. \quad \Box
\]

**Proof of Theorem 1.** W.l.o.g. the sequence is injective, so in particular (with \( z_0 = 1_X \)) \( Z := \{z_i : i = 0, 1, 2, \ldots \} \) is infinite. We put \( Z_n := \{z_0, z_1, \ldots, z_n\} \). For any finite set of points \( F = \{f_0, f_1, \ldots, f_k\} \), with \( f_0 = z_0 = 1_X \), \( L \) nwd, and \( G \) open and dense in \( X \), for \( 0 \leq i \leq n \) put \( F_i := \{f_0, \ldots, f_i\} \) (so that \( F = F_n \)), and let

\[
W^F_L(G) := \{x \in G : (\exists \text{ open } W_x)[x \in W_x \text{ and } (FW_x) \cap L = \emptyset]\} \subseteq G \setminus L.
\]
Then $W := W^F_L(G)$ is open, since $W_x \subseteq W$ for each $x \in W$. Notice that $V_x := (x^{-1}W_x)$ is a nhd of $1_X$ (since translation is a homeomorphism) with

$$[(Fx)V_x] \cap L = 0,$$

i.e. $V_x$ generates a nhd of the shifted set $Fx$ disjoint from $L$.

**Claim 1.** The open set $W$ is dense in $G$.

For non-empty open $U \subseteq G$, define inductively non-empty open sets $V_i$ with $U \supseteq V_0 \supseteq \ldots V_{i-1} \supseteq V_i \supseteq \ldots \supseteq V_n$ such that for $0 \leqslant i \leqslant n$

$$(F_i V_i) \cap L = 0,$$

i.e. for any $x \in V_i$, $V_x^i = x^{-1}V_i$ is a nhd of $1_X$ with

$$[(F_i x)V_x^i] \cap L = 0,$$

so providing a uniform nhd for the shifted set $F_i x$ disjoint from $L$.

* **Basis step.** As $L$ is nw, choose non-empty open $V_0 \subseteq U$ with $\emptyset = V_0 \cap L = (F_0 V_0) \cap L$, as $F_0 = \{z_0\} = \{1_X\}$.

* **Inductive step.** Given $V_{i-1}$, apply Lemma 1 to $L, f_i, F_{i-1}$ and $V_{i-1}$ to choose a non-empty open $V'$ as in the Lemma. Take $V_i := V'$; then $V_i \subseteq V_{i-1}$ and

$$(F_i V_i) \cap L = ((F_{i-1} \cup \{f_i\}) V_i) \cap L = \emptyset.$$

At the conclusion of the induction, for $x \in V_n$ the set $W_x := V_n$ gives

$$(FW_x) \cap L = \emptyset,$$

and so $x \in W \cap U$, proving density of $W$ in $G$. □

**Claim 2.** For $F' = F \cup \{f\}$, and $G = W$, $W^F_L(G)$ is open and dense in $G$.

This is almost a repeat of the inductive step in the proof of Claim 1. For non-empty open $U \subseteq G$, w.l.o.g. we may assume by Claim 1 that $U \subseteq W = W^F_L(G)$, as $W$ is dense open. Consider any $x \in U$. As $x \in W^F_L(G)$, there is an open nhd $W_x$ of $x$ with $(FW_x) \cap L = \emptyset$. W.l.o.g. $W_x \subseteq U$. (Indeed, as $x \in U \cap W_x \subseteq W_x$, we have $[F(U \cap W_x)] \cap L = \emptyset$.)

Apply Lemma 1 to $L, f, F$ and $W_x$ to choose a non-empty open $V' \subseteq W_x$ with

$$(F'V') \cap L = \emptyset.$$

So $y \in W^F_L(G)$, for any $y \in V' \subseteq W_x \subseteq U$ (with $V'$ doing duty for $W_y$). □
Now write $X \setminus A = \bigcup_{n=0}^\infty N_n$ with the $N_n$ increasing and nwd. Put $N_{-1} = \emptyset$, $W_{-1} = X$, and define inductively dense open sets

$$W_{2n} = W_{N_{2n-1}}^Z(W_{2n-1}), \quad W_{2n+1} = W_{N_n}^Z(W_{2n}),$$

so that

$$W_0 = W_0^{\{z_0\}}(X) = X, \quad W_1 = W_0^Z(X), \quad W_2 = W_0^Z(W_1),$$

$$W_3 = W_1^Z(W_1), \quad \text{etc.}$$

Now put

$$H = \bigcap_{n=0}^\infty W_n,$$

which, since $X$ is Baire, is dense. Fix $x \in H$, and $n$; then for $m \geq n$, since $x \in W_{2m+1}$

$$xZ_n \subseteq xZ_m \subseteq X \setminus N_m$$

and so

$$xZ_n \subseteq \bigcap_{m=n}^\infty X \setminus N_m = X \setminus \bigcup_{m=n}^\infty N_m = A,$$

since the sequence $N_n$ is increasing. But $n$ was arbitrary, so

$$xZ \subseteq A. \quad \square$$

We deduce as an easy corollary the category version of the KBD:

**Theorem 2 (Theorem KBD).** In a Baire (metric) semitopological group $X$, if $\{z_n\}_{n \in \mathbb{N}}$ is a null sequence, and $A$ is a non-meagre Baire set, then for some $a \in A$ and some $n = n(a)$

$$\{az_m : m > n\} \subseteq A.$$ 

In fact, the embedding holds for quasi almost all $a \in A$.

**Proof.** W.l.o.g. $A = A^q \setminus M$ with $M \subseteq A^q$ meagre and $A^q$ non-empty. As $X \setminus M$ is co-meagre, by Th. 1 there is a $\mathcal{G}_\delta$-set $H$ of points $x$ in $X \setminus M$ with

$$\{xz_n : n = 0, 1, 2, ...\} \subseteq X \setminus M.$$ 

As $A^q$ is open, we may choose $a \in H \cap A^q \subseteq A^q \setminus M = A$. As $a = \lim_n(az_n)$, by continuity of left translation, there is $n(a)$ with $\{az_n : n > n(a)\} \subseteq A^q$. But then

$$\{az_n : n > n(a)\} \subseteq A^q \cap (X \setminus M) = A.$$ 

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The final conclusion holds by the Generic Dichotomy Theorem [BinO2]. □

Note the following immediate corollary, concerning measurable groups [Hal, §62] equipped with a probability measure \( \mu \) and a ‘differentiation basis’ [Bru], giving rise to a density topology \( D_\mu \) in the sense of Martin [Mar]. (The Haar density topology case, using a differentiation basis provided by [Mue], is discussed in [BinO3, §7]; for background on density topologies see [BinO7].)

**Corollary.** For a topological group \( X \) supporting a density topology \( D_\mu \) generated by a measure \( \mu \) (e.g. Haar measure on a locally compact group): if \( A \) is co-null and \( \{z_n\}_{n \in \mathbb{N}} \) is a null sequence, then for a dense \( G_\delta(D_\mu) \)-set of points \( a \) in \( A \):

\[
\{az_n : n = 0, 1, 2, \ldots \} \subseteq A.
\]

**Proof.** Under the density topology the group is both Baire [BinO7, Prop 4] and semitopological. As the nwd sets are precisely the \( \mu \)-null sets [BinO7, Th. 7.2], Th. 1 applies. □

The next result, which emerges as more demanding, goes beyond a co-null setting.

**Theorem 2H.** Let \( G \) be a locally compact metrizable topological group.

(i) For any convergent sequence \( \{x_n\}_{n \in \mathbb{N}} \) with limit \( x_0 \) and any (right) non-null Haar-measurable set \( T \), there are a left shift \( \theta(x) = cx \) and an infinite set \( \mathbb{M} \subseteq \mathbb{N} \) such that \( \theta(x_0) \in T \) and

\[
\theta(x_m) = cx_m \in T \quad \text{for} \quad m \in \mathbb{M}.
\]

(ii) Moreover, for \( S \) and \( T \) density-open with \( Sx_0 \subseteq T \) the shift may be chosen with \( c \in S \).

**Proof.** Below \(|.|\) denotes a right-invariant Haar measure on \( G \). By the Birkhoff-Kakutani metrization theorem ([Bir], [Kak1], [DieS], or [Ost2]), we may equip \( G \) with a group norm \( ||x|| := d(1_G, x) \) for \( d \) a right-invariant metric.

(i) Let \( T \) be Haar-measurable non-null. By inner regularity of the measure, we may assume that \( T \) is compact and non-null. Applying a left shift to the sequence \( x_n \) if necessary, \( x_0 \) is w.l.o.g. a density point of \( T \). Put \( m(0) := 0, \theta_0 := \text{id} \).
Suppose inductively that \( \theta_n(x) := c_{n,1} \ldots c_1 x \) for some \( c_i \) with \( ||c_i|| \leq 2^{-i} \) for \( i \leq n \), and that an increasing sequence of integers \( m(j) \) for \( j \leq n \) has been selected with each \( u_j := \theta_n(x_{m(j)}) \) a density point of \( T \).

For each \( \varepsilon = 2^{-n} \) we may choose \( U \) a finite union of (left) translates of an open nhd of \( 1_G \) to cover \( T \) with the complement \( E = U \setminus T \) having \( |E| < \varepsilon \). Choose open nhds \( I_j \) with \( u_j \in I_j \subseteq U \), for \( 0 \leq j \leq n \). Let \( \eta := \min_{0 \leq j \leq n} \{ d(u_j, X \setminus I_j), \varepsilon \} \). Since each \( u_j \) is a density point, choose a symmetric open nhd \( V \) round \( 1_G \) such that each \( V_j := Vu_j \subseteq I_j \) has \( |V_j| = |V| \), \( u_j \in V_j \) and \( |V_j \cap T| \geq (1 - \eta)|V| \) for all \( j \leq n \) and \( |V| < \varepsilon \). Choose \( m = m(n+1) \) with \( m > m(n) \) such that \( d(x_m, x_0) < \eta \) and \( u_{n+1} := \theta_n(x_{m(n+1)}) \in V_0 \), both are possible as \( x_0 = \lim_n x_m \) and \( \theta_n(x_0) = u_0 \in V_0 \) and \( \theta_n \) is continuous.

Choose an open interval \( V_{n+1} \subseteq I_0 \) centered on \( u_{n+1} \).

For \( j \leq n \) one has \( |V_j \cap E| < \eta|V| \) as \( V_j \setminus E \subseteq U \setminus E \subseteq T \), and so \( |V_j \setminus E| > (1 - \eta)|V_j| \). Invoking the Haar Density Theorem ([Mue], [Mar]), let \( F \) be a measure-zero set such that \( (V_j \setminus E) \setminus F \) is a density-open subset of \( T \) (all its points are density points) for each \( j < n \).

For any \( c \), note that \( cu_{n+1} \) is a density point of \( T \cap V_0 \) iff \( c \) is a density point of \( T' := (T \cap V_0)u_{n+1}^{-1} \), as the measure is right-invariant. Again by the Haar Density Theorem, off a null subset \( N \) of \( T' \) all its members are density points. In what follows we ensure that \( c \notin N \).

Choose \( c_{n+1} \in V_j \setminus (N \cup (E \cup F)u_j^{-1}) \) with \( ||c_{n+1}|| < \varepsilon \) such that \( c_{n+1}u_j \in V_j \setminus E_0 \subseteq T \) and \( c_{n+1}u_j \) is a density point of \( T \), for each \( j \leq n+1 \).

Set \( \theta_{n+1}(x) := c_{n+1} \theta_n(x) \); then, for each \( j \leq n+1 \), \( \theta_{n+1}(x_{m(j)}) \) is a density point of \( T \) in \( T \).

Moreover, \( s_n := (c_n \ldots c_1) \) converges, to \( s \) say, as \( d(c_{n+1} \ldots c_1, c_n \ldots c_1) = d(c_{n+1}1_G, c_{n+1}1_G) \), by right-invariance of \( d \), and so \( \{ s_n \}_{n \in \mathbb{N}} \) is a Cauchy sequence in a locally compact nhd of \( 1_G \). Take \( \theta(x) := sx \); then, for each \( j \), as \( T \) is compact, \( \theta(x_{m(j)}) = \lim_n \theta_n(x_{m(j)}) \in T \). Also \( \lim_j \theta(x_{m(j)}) = \theta(x_0) \in T \), as \( x_0 = \lim_n x_m \).

(ii) This now follows quite easily. Specialize the sequence arising in the proof above to a null sequence \( z_n \to z_0 = 1_G \) and replace \( T \) by \( S \) to obtain \( \theta(z_0) = s1_G \in S \) and \( s \) is an infinite set of \( m \), in \( \mathbb{M}_s \) say.

Returning to a general sequence \( x_n \) with limit \( x_0 \), put \( z_n := x_nx_0^{-1} \). Then, as before, for some \( s \in S \) and some infinite set \( \mathbb{M}_s \), one has \( sz_m \in S \) for \( m \in \mathbb{M}_s \). But then \( sz_m x_0 = sx_m \in Sx_0 \subseteq T \) for \( m \in \mathbb{M}_s \), as asserted. \( \Box \)

Remark. A locally compact metrizable group, being topologically complete, is completely metrizable [Enge, 4.3.26]. More generally, it is a theorem of Loy
and of Christensen that a topological Baire group which is analytic is in fact Polish – see e.g. [TopH, Th. 2.3.6].

We close the section by recalling the promised ‘strong’ version of KBD that is applicable to topological groups.

**Definition** [Ost3], cf. [Pet]. For $G$ a metrizable group, say that the group action $\varphi : G \times X \to X$ is a Nikodym action (or that it has the Nikodym property) if for every non-empty open neighbourhood $U$ of $1_G$ and every $x \in X$ the set $Ux = \varphi_x(U) := \varphi(x, U)$ contains a non-meagre Baire set.

**Example.** For $G$ a semitopological group acting on itself: $\varphi_x(u) = \varphi(x, u) = ux$; so $\varphi$ is separately continuous and $\varphi_x$ is an autohomeomorphism. So $Ux$ is open, for any open $U$. In particular, if $G$ in Theorem 2 is a topological Baire group acting on itself, that action has the Nikodym property, so the following result implies the conclusion of Theorem 2.

**Shift-compactness Theorem** [Ost3]. For $T$ a Baire non-meagre subset of a metric space $X$ and $G$ a group, Baire under a right-invariant metric, and with separately continuous and transitive Nikodym action on $X$:

for every convergent sequence $\{x_n\}_{n \in \mathbb{N}}$ with limit $x_0$ and any Baire non-meagre $A \subseteq G$ with $1_G \in A'$ and $A'x \cap T' \neq \emptyset$, there are $\alpha \in A$ and an integer $N$ such that $\alpha x_0 \in T$ and

$$\{\alpha(x_n) : n > N\} \subseteq T.$$  

3 **KBD: effective version**

In this section we give in Theorem 1E an effective version of KBD. Our treatment below of coding overlaps with that of the corresponding sections of the contemporaneous paper [BinO11].

In what follows, we will need to distinguish between (general) sets of reals, and ‘nice’ sets which can be defined by a suitable (effective) coding so that an individual set is coded by a single real. For background here, see e.g. the monograph Kechris [Kec, Ch.V] on the analytical hierarchy (note [Kec, V.40B] on classical v. effective descriptive set theory), [Rog2, Part 4] and our recent survey [BinO8]. For a deeper analysis of coding see [Solo, II.1.1, 25-33]; a minimal amount is in [FenN, § 2, p. 93].
We begin with a short introduction to this topic in the next sub-section (on preliminaries), which the expert reader can omit. The non-expert reader may also take ‘coding on trust’, observing the basic case of an open set \( W \subseteq \mathbb{R} \) which may be coded by first enumerating (effectively) the rational intervals as \( \{I_n\}_{n \in \mathbb{N}} \) and then coding \( W \) up as the binary real which is the indicator function \( 1_M \) of the subset \( M := \{n : I_n \subseteq W\} \), and thus omit §3.1.

We defer further discussion of some of the finer points to the Appendix of this paper.

3.1 Preliminaries on coding

We work in the space \( I \) of irrationals, interpreted as the non-recurring binary sequences \( x : \mathbb{N} \to \{0, 1\} \); here \( x \) may also be viewed as the indicator function of a subset of \( \mathbb{N} \) and thereby as a real number code for that subset. We may identify \( x \in I \) with the sequence \( \{x_n\}_{n \in \mathbb{N}} \), where \( x_n \) is the \( n \)th projection of \( x \). With \( x \) viewed as (a code for) a subset of \( \mathbb{N} \), \( x_n \) is a code for \( x \cap \{1 \cdot 2^n, 3 \cdot 2^n, 5 \cdot 2^n, \ldots\} \).

The proof of Theorem 3 in §4 below relies on the ability to refer to various subsets of the real line in terms of real numbers; in particular, an open set \( G \), closed set \( F \), and \( \mathcal{G}_0 \)-set \( H \) may be coded by \( a \in I \) via one of

\[
G(a) := \bigcup_{n \in a} I_n, \quad F(a) = I \setminus G(a), \quad H(a) := \bigcap_{n \in \mathbb{N}} G(a_n),
\]

where as above \( \{I_n\}_{n \in \mathbb{N}} \) enumerates (constructively) all the rational-ended intervals and \( a_n \) is the \( n \)th projection of \( a \). (Evidently, one must separately code which of the three displayed equations is to be chosen.) Coding clarifies in what form the property of ‘membership in \( G(a) \)’, or \( F(a) \) etc., is expressible as an (arithmetic) predicate in the language of set theory (below); indeed,

\[
x \in G(a) \text{ iff } (\exists n \in \mathbb{N})[(n \in a) \& (x \in I_n)], \quad x \in F(a) \text{ iff } (\forall n \in \mathbb{N})[(n \in a) \& (x \notin I_n)].
\]

Both predicates here use an arithmetic quantifier (ranging over \( \mathbb{N} \), the type 0 objects) while its matrix (the part without quantifiers, delimited here by square brackets) refers to elementary relations. The first is said to be \( \Sigma^0_1(a) \) : this identifies a single existential quantification over type 0 objects, and the presence of \( a \); its complement is \( \Pi^0_1(a) \), with \( \Pi \) for the universal quantifier. One may suppress the explicit mention of \( a \) by use of bold-face symbols \( \Sigma^0_1 \) and \( \Pi^0_1 \), which imply the need for a parameter. By contrast,

\[
x \in H(a) \text{ iff } (\forall n \in \mathbb{N})(\exists m \in \mathbb{N})[(m \in a_n) \& (x \in I_m)],
\]
which is $\Pi^0_2$ because there is a universal quantifier leading the alternating pair of quantifiers. Similar conventions govern analytic quantifiers (ranging over $\mathbb{R}$, the type 1 objects): the superscript here changes to a 1.

Codes in $\mathbb{R}$ known as ‘notations’ are also needed for the countable ordinals. This is somewhat tedious, so omitted here. (One may start with the indicator function of $\mathbb{N}$ as a code for $\omega$ as an order type.)

We make use of the language of set theory, $LST$: its (first-order) formulae, needed here and again in §4, are written using: free variables, symbols denoting constants, the relation of membership, the usual connectives, negation, and quantifiers ranging over sets. This enables us to recall the constructible hierarchy $\langle L_\alpha : \alpha \in \text{On} \rangle$, in which, for ordinal $\alpha$, the sets $L_\alpha$ are obtained by iterating transfinitely the operation which defines $L_{\beta+1}$ as the family of those subsets of $L_\beta$ that are definable by the first-order formulas of $LST$. Here they are allowed to refer to a finite string of elements of $L_\beta$ and all their quantifiers range over $L_\beta$– see e.g. [Sac, 9.2.III], [Dev], or [BinO8, §2]. The class $L := \bigcup \{ L_\alpha : \alpha \in \text{On} \}$ comprising all the constructible sets has a canonical well-ordering $<_L$ (defined by transfinite induction using an effective listing of all predicates).

### 3.2 KBD: a version effective in the codes

We develop a version of KBD suitable for work in $\mathbb{R}$. We begin by demonstrating that, for a null sequence $\{z_n\}_{n \in \mathbb{N}}$ coded by a $z \in I$ (with $z_n$ as its $n^{th}$ projection) and for a target $G_\delta$-set coded by $s \in I$, the relevant translator $t$ may be constructed effectively (recursively) in $z$ and $s$. This guarantees that when the $G_\delta$-set has code $s$ in $L_\alpha$, then such a translator is in $L_{\alpha+\omega}$ (for $L_\alpha$ point-definable, as in the preamble in §5 to the proof of Theorem 4). The corresponding $G_\delta$-sets/codes form the family $G_\alpha$ defined below. (Later on we will also require the sets $L_\alpha$ to be models of the axiom system $\text{ZF}^-$ ($\text{ZF}$ less the Power Axiom); we note that if $L_\alpha$ is point-definable, then so is $L_{\alpha+\omega}$ – this is proved in [EngMS].)

**Definitions.**

1. Following [MiO], say that the group of translations $Tr(\mathbb{R}^d)$ strongly $L_\alpha$-separates points from a family $\mathcal{F}$ of closed nowhere dense sets in $\mathbb{R}^d$ if for each $p \in L_\alpha$ and $F \in \mathcal{F}$ and arbitrarily small $q \in \mathbb{Q}_+$ there is $H \subseteq (-q, q)$ with code in $L_\alpha$ such that $h_c(p) := p + c \notin F$ for every $c \in H$.

2. Denote by $G_\alpha$ the family of sets $G$ open (in $\mathbb{R}^d$) with $\mathbb{Q} \subseteq G$ possessing a code in $L_\alpha$, and by $\mathcal{F}_\alpha$ the complements of sets in $G_\alpha$. 

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3. \(B_{\varepsilon}(x)\) denotes the ball centred at \(x\) of radius \(\varepsilon\).

**Proposition 1 (Strong Separation, cf. [MilO, Prop. 1]).** For \(\mathbb{R}^d\) and \(Tr(\mathbb{R}^d)\) both equipped with the Euclidean topology, the group \(Tr(\mathbb{R}^d)\) strongly \(L_\alpha\)-separates points of \(L_\alpha\) and the closed nowhere dense sets of \(\mathcal{F}_\alpha\).

**Proof.** Let \(q_i\) be an effective enumeration of \(\mathbb{Q}\). For \(0 < q \in \mathbb{Q}\), if \(p \in L_\alpha\) and \(F = \mathbb{R} \setminus G\) with \(G\) open and coded in \(L_\alpha\), choose the first \(q_i \in B_q(p)\) and thereafter the first pair \((q_{L(i)}, q_{R(i)})\) with \(q_{L(i)} < q_i < q_{R(i)}\) such that \(I := (q_{L(i)}, q_{R(i)}) \subseteq G \cap B_q(p)\). Then \(H := I - p \subseteq (-q, q)\) has code in \(L_\alpha\), and, for \(c = m - p \in H\) with \(m \in I\), \(|c| = |m - p| < q\) and \(p + c = m \in G = \mathbb{R} \setminus F\).

\(\Box\)

The next result follows, as it is an inductive construction applying Prop. 1 at each inductive stage.

**Proposition 2 (Finitary Euclidean Strong Separation, cf. [MilO, Prop. 2]).** With \(F \in \mathcal{F}_\alpha\) as above, let \(U\) be Euclidean open with code in \(L_\alpha\) and \(u_i \in U\) for \(i \leq n\) with \(u_i \in L_\alpha\). Then, for each \(\varepsilon > 0\), in \(B_{\varepsilon}(0)\) there is a neighbourhood of \(c\)-shifts \(x \to x + c\) with code in \(L_\alpha\) such that \(u_i + c \in U\) and \(u_i + c \notin F\) for each \(i \leq n\).

**Proof.** Proceed exactly as in [MilO, Prop. 2], using Prop. 1 here in place of Prop. 1 there. \(\Box\)

**Theorem 1E (cf. [MilO, Th. 1E]).** For the real line under the Euclidean topology, given \(y \in \mathbb{N}^\mathbb{N} \cap L_\alpha\) coding a convergent sequence \(\{y_n\}_{n \in \mathbb{N}} \to y_0\), and \(x \in \mathbb{N}^\mathbb{N} \cap L_\alpha\) such that the set \(T = \bigcap G_n\) with \(G_n\) coded by \(x_n\) and \(G_n \in \mathcal{G}_\alpha\), there are a \(c\)-shift \(h(x) = x + c\) and an integer \(M\) such that \(h(x_0) \in T\) and

\[h(y_m) = y_m + c \in T\text{ for }m > M,\]

and \(c\) has a code in \(L_{\alpha + \omega}\).

**Proof.** Again proceed exactly as in [MilO, Th. 1E], using Prop. 2 here in place of Prop. 2 there. \(\Box\)
4 Singular sets

In [MilO] the first and third authors studied extensions of the Kestelman-Borwein-Ditor theorem from the perspective of group action, on the one hand, and certain limitations (exemplified by ‘singular’ sets) of the infinite combinatorics involved, on the other. The latter included ‘wild’ 2-place-function actions in place of group actions [MilO, Th. 8], and examples of non-shift-compactness (existence, for a given closed nwfd set $A$, of a monotonic null sequence with $\{d_n\}_{n\in\mathbb{N}}$ so that for each $x$, $x + d_n \notin A$ infinitely often, and an example, under the Axiom of Choice, of a non-measurable $A$ with $x + (1/n) \notin A$ for all $n$).

Here, in similar spirit, we offer further examples of singular behaviour. We recall a particular result needed quite soon. (Below $d(A) := A - A = \{a - a' : a, a' \in A\}$; for $\mathcal{E}mb$ see the Definitions hereunder.) In the theorem below, the first assumption needed for (i) may be regarded as a topological variant of Martin’s Axiom (MA) : see [MilO], [BinO8, §6b]; in (ii) $W$ is concentrated on $\mathbb{Q}$ [Rog1, §2.3], and such sets are of (strong) measure zero: see §6.3.

**Theorem MO ([MilO, Th. 9]).** Assume that the union of fewer than $\mathfrak{c}$ many sets that are meagre and null is itself meagre and null, then there exists $A \in \mathcal{E}mb$ with $d(A) = \mathbb{R}$ such that:

(i) $(\forall x \in A) \ x + (1/n) \notin A$ for all $n$ with at most one exception;

(ii) assuming the Continuum Hypothesis, $\text{CH}$: $A \setminus W$ is countable for open $W$ with $\mathbb{Q} \subseteq W$.

For set-theoretic background we refer to [BinO8]. We now recall a few of the classes used to study the adequacy of sets in sustaining (topologically) ‘good behaviour’ – notions of adequate size or largeness. These are known as *gauges*. We compare some of these to help introduce ‘strange sets’, outside the bounds of the usual standard classification of the size of a set. Our interest here rests on the following families of subsets of $\mathbb{R}$. (Below two subsets are *similar* if they are images under some (injective) affine function: $f(x) = mx + c$ with $m \neq 0$.)

**Definitions.** Put:

- $\mathcal{L}^+ := \{A : \lambda(A) > 0\}$ with $\lambda$ Lebesgue measure on $\mathbb{R}$ and $\lambda^*$ (below) its outer measure;
- $\mathcal{B}a^+ := \{A : A$ is second category and has the Baire property$\}$;
- $\mathcal{S}W := \{A : d(A) \text{ contains an interval}\}$;
\( \mathcal{E}_{mb} := \{ A : \text{for each finite set } F, A \text{ contains a set } \tilde{F} \text{ similar to } F \} \);
\( \mathcal{SC} := \{ A : A \text{ is shift-compact} \} \);
\( \mathcal{BC} = \{ A : \text{if } f : \mathbb{R} \to \mathbb{R} \text{ is additive and bounded above on } A, \)
\( \text{then } f \text{ is linear (i.e. continuous) } \} \).

The last case above is the Ger-Kuczma class \( \mathfrak{B} \) of [Kuc,§9,10] (cf. [Bino1]), but with so many families in play our chosen notation above gives more of a mnemonic (as with \( \mathcal{E}_{mb} \) for embedding). The condition ‘bounded from above’ in \( \mathcal{BC} \) can be replaced by ‘bounded from below’, as \( -f \) is additive whenever \( f \) is. Note, however, that replacing either of these by just ‘bounded’, yields a larger class, the Ger-Kuczma class \( \mathfrak{C} \) – see [Kuc, Th. 9.1.1]. There is in principle a third Ger-Kuczma class, denoted \( \mathfrak{A} \) in [Kuc], analogous to \( \mathfrak{B} \) but referring to mid-point convex functions; however, it emerges that \( \mathfrak{A} = \mathfrak{B} \) [Kuc, Th. 10.2.2].

The first two classes are thoroughly studied in [Oxt] and it is well known that \( \mathcal{L}^+ \cup \mathcal{B}a^+ \subseteq \mathcal{SW} \cap \mathcal{E}_{mb} \) (for \( \mathcal{SW} \) this is the Steinhaus-Weil Theorem, [Oxt, Th. 4.8], cf. [Bino9]; for \( \mathcal{E}_{mb} \) see [Kel] which gives a brief survey, cf. [Sve] a much earlier survey including the related ‘Erdős similarity problem’). Furthermore, \( \mathcal{L}^+ \cup \mathcal{B}a^+ \subseteq \mathcal{SC} \cap \mathcal{BC} \) is also well-known (for \( \mathcal{BC} \) see \( \mathfrak{B} \) in [Kuc, Th. 9.3.3] and for \( \mathcal{SC} \) see [Bino5] – though this goes back to Kestelman [Kes1] and Borwein and Ditor [BorD]). Note that if \( d(A) = A - A \) contains an interval, then \( A + A \) need not: see [CrnGH] for an example of a compact subset \( S \) such that \( d(S) = S - S \) contains an interval, but \( S + S \) has measure zero.

We recall some recent results and offer some new examples along similar lines. Firstly,

**Theorem MM** ([MilM]).
(i) There exists a shift-compact set that is concentrated on \( \mathbb{Q} \), the rationals.
(ii) There exists a non-measurable shift-compact set.

These two results appeared recently in [MilM]. We study the possible topological character of the set in (i) under Gödel’s axiom \( V = L \) in §5. We continue with some new examples accompanied by earlier results which they complement. Thus the example in (ii) below is simpler than that in [Kuc, Th. 9.3.4]. We review the effective nature of the constructions here in Lemma 2 in §5, where we study these examples in the light of \( V = L \). Below \( \mathfrak{c} \) denotes the
cardinality of the continuum, treated here as an initial ordinal, as is common in set theory [Jec], [Kun], cf. [BinO8].

Theorem 3.
(i) $SW \not\subseteq BC$ and $BC \not\subseteq SW$;
(ii) $Emb \not\subseteq SC$, $SC \not\subseteq Emb$;
(iii) There exists a set $A \subseteq [0,1]$, with $\lambda^*(A) = 1$ such that $A \not\in Emb$;
(iv) There exists a set $A \subseteq [0,1]$, $A \cap I$ second category for each closed interval $I \subseteq [0,1]$ such that $A \not\in Emb$;
(v) There exists a set $A \subseteq [0,1]$, with $\lambda^*(A) = 1$ such that $A \not\in BC$;
(vi) There exists a set $A \subseteq [0,1]$, $A \cap I$ second category for each closed interval $I \subseteq [0,1]$ such that $A \not\in BC$;
(vii) There exists a non-measurable set $A$, $A \in BC$;
(viii) There exists $A \in (SW \cap Emb) \setminus SC$;
(ix) $SC \subset SW$, $SC \subset BC$.

Before proceeding we mention a beautiful result of Cieselski-Rosenblatt [CieR, Th. 12] that the Erdős and Kakutani [ErdK] set

$$C_{EK} := \{\sum_{k=2}^{\infty} a_k/k! : a_k \in \{0,1,2,\ldots, k-2\}\},$$

which is a compact perfect set of measure zero, is shift-compact. It was already known [EleS] (cf. [EleT]) that for every perfect set $P \subseteq \mathbb{R}$ there is $x \in \mathbb{R}$ with $C_{ES} \cap (x + P)$ uncountable. For further literature on this and related matters see [BarLS]. Notice also that $C$, the excluded middle-thirds Cantor set in $[0,1]$, is compact, and $\lambda(C) = 0$, but $C \in SW$ ($d(C) = [-1,1]$ and $C + C = [0,2]$) and hence $C \in BC$. Also $C \not\in Emb$.

Proof of Theorem 3.
Proof of (i). Let $f$ be any discontinuous additive function on $\mathbb{R}$ (for examples see e.g. [Kuc, §5.2]). Put $A = \{x : f(x) \leq 0\}$; then, as $f$ is bounded from above on $A$ but not continuous (linear), $A \not\in BC$. However, since $f$ is additive and $0 \in A$ it is immediate that $d(A) = \mathbb{R}$. (If $f(x) > 0$, then $x = 0 - (-x) \in d(A)$). So $A \in SW$. Therefore $SW \not\subseteq BC$.

For the second part, take a Hamel basis $H = \{h_\alpha\}_{\alpha < \mathfrak{c}}$, where $\mathfrak{c}$ the cardinality of the continuum as above, and let

$$A := \{qh_\alpha : q \in \mathbb{Q}, q \neq 0, \alpha < \mathfrak{c}\}.$$
Then $d(A)$ consists of all $q_1 h_{\alpha_1} - q_2 h_{\alpha_2}$ with $q_1, q_2 \neq 0$. Fix three distinct $\alpha_1, \alpha_2, \alpha_3$. Then

$$q_1 h_{\alpha_1} + q_2 h_{\alpha_2} + q_3 h_{\alpha_3} \notin d(A)$$

whenever $q_1 \neq 0$, $q_2 \neq 0$, $q_3 \neq 0$, and these numbers are dense in $\mathbb{R}$. Hence $d(A)$ contains no interval. However, $A \in BC$: if $f$ is additive and bounded above on $A$, then $f(h_{\alpha}) = 0$ for every $\alpha$, so $f = 0$, and so is vacuously linear (continuous). □ (i)

Proof of (ii). This falls into two parts.

Part 1.

We will construct a set $B \in \mathcal{E}mb \setminus \mathcal{S}C$ by transfinite induction of length $c$. Let $F_{\alpha}$, $\alpha < c$ denote all finite sets of real numbers. Set $B_0 = F_0$. Let

$$A_1 := \{ b \pm \frac{1}{n} : b \in B_0, n \in \mathbb{N} \}.$$

Clearly there exists $\tilde{F}_1$ similar to $F_1$, such that $\tilde{F}_1 \cap (A_1 \cup B_0) = \emptyset$. Set $B_1 = \tilde{F}_1 \cup B_0$.

Now for some $\alpha < c$, suppose we have constructed $\langle B_{\beta}, \beta < \alpha \rangle$, so that, for each $\beta < \alpha$, $B_\beta = \tilde{F}_\beta \cup \bigcup_{\gamma < \beta} B_\gamma$, with $\tilde{F}_\beta$ similar to $F_\beta$ and with $\tilde{F}_\beta \cap (A_{\beta} \cup \bigcup_{\gamma < \beta} B_{\gamma}) = \emptyset$, where

$$A_\beta = \{ b \pm \frac{1}{n} : b \in \bigcup_{\gamma < \beta} B_\gamma, n \in \mathbb{N} \},$$

and each $B_\beta$ (from construction) has cardinality less than or equal to that of $\beta$ (when $\beta$ is infinite). Let

$$A_\alpha = \{ b \pm \frac{1}{n} : b \in \bigcup_{\beta < \alpha} B_\beta, n \in \mathbb{N} \}.$$

From the inductive hypothesis, $S_\alpha := A_\alpha \cup \bigcup_{\beta < \alpha} B_\beta$ has cardinality less than or equal to that of $\alpha$, and thus less than $c$. So there exists $\tilde{F}_\alpha$ similar to $F_\alpha$ such that $\tilde{F}_\alpha \cap S_\alpha = \emptyset$. (Consider the similarity $f(t) = at$ with $a \notin S_\alpha f^{-1}$ for $f \in F_\alpha$.) Now define $B_\alpha = \tilde{F}_\alpha \cup \bigcup_{\beta < \alpha} B_\beta$. If we set $B := \bigcup_{\alpha < c} B_\alpha$, it is routine to verify that $B \in \mathcal{E}mb \setminus \mathcal{S}C$.

Part 2.

We will construct a set $B \in \mathcal{S}C \setminus \mathcal{E}mb$ by transfinite induction, ensuring that it contains no subset similar to $\{1, 2, 3\}$.

Arrange all the null-sequences in a transfinite sequence $\langle \{ x_n^\alpha \} : \alpha < c \rangle$.
Set $B_0 = \{b_0\} \cup \{x_{0,k,0}^0 : n_{k,0} \in \mathbb{N}\}$, where $b_0 = 0$ and $\{x_{0,k,0}^0\}_{n_{k,0} \in \mathbb{N}}$ is a subsequence of $\{x_n^0\}_{n \in \mathbb{N}}$ so that $B_0$ contains no set similar to the set $\{1, 2, 3\}$.

Now suppose that for some $\alpha < \mathfrak{c}$ we have chosen $\langle B_{\beta} : \beta < \alpha \rangle$ to satisfy

$$B_{\beta} = \bigcup_{\gamma < \beta} B_{\gamma} \cup \{b_{\beta}\} \cup \{x_{n_{k,\beta}}^\beta : n_{k,\beta} \in \mathbb{N}\},$$

where $\{x_{n_{k,\beta}}^\beta\}_{n_{k,\beta} \in \mathbb{N}}$ is a subsequence of $\{x_n^\beta\}_{n \in \mathbb{N}}$, with $\beta$ a real number such that $B_{\beta}$ contains no set similar to the set $\{1, 2, 3\}$. Clearly $\bigcup_{\beta < \alpha} B_{\beta}$ has less than $\mathfrak{c}$ elements, so it is easy to verify that we can choose $b_{\alpha}$ and $\{x_{n_{k,\alpha}}^\alpha\}_{n_{k,\alpha} \in \mathbb{N}}$ a subsequence of $\{x_n^\alpha\}_{n \in \mathbb{N}}$, so that

$$B_{\alpha} = \bigcup_{\beta < \alpha} B_{\beta} \cup \{b_{\alpha}\} \cup \{x_{n_{k,\alpha}}^\alpha : n_{k,\alpha} \in \mathbb{N}\}$$

contains no set similar to $\{1, 2, 3\}$.

Finally set $B = \bigcup_{\alpha < \mathfrak{c}} B_{\alpha}$. Then $B$ is shift-compact, as $0 \in B$ and each null sequence contains a subsequence in $B$, and further $B \notin \mathcal{Em}b$. \(\square\) (ii)

**Proof of (iii).** $A \subseteq [0, 1]$ satisfies $\lambda^*(A) = 1$ iff $A \cap F \neq \emptyset$ for every closed subset $F$ of $[0, 1]$ of positive measure. Let $\langle F_{\alpha} : \alpha < \mathfrak{c} \rangle$ enumerate the closed subsets of $[0, 1]$ of positive measure. By transfinite induction, we can construct $A = \{x_{\alpha} : \alpha < \mathfrak{c}\}$ by picking $x_{\alpha} \in F_{\alpha}$ at each step $\alpha < \mathfrak{c}$ in such a way that $\{x_{\beta} : \beta \leq \alpha\}$ contains no subset similar to $\{1, 2, 3\}$. This is possible since at each step $\alpha < \mathfrak{c}$ we have less than $\mathfrak{c}$ excluded values for the choice of $x_{\alpha}$, and $F_{\alpha}$ has cardinality $\mathfrak{c}$. \(\square\) (iii)

**Proof of (iv).** First notice that for $A \subseteq [0, 1]$, (A) and (B) below are equivalent:

(A) $A \cap F \neq \emptyset$, $\forall F \subseteq [0, 1]$ with $F$ closed and second category.

(B) $A \cap I$ is second category $\forall I \subseteq [0, 1]$, with $I$ a closed interval.

Let $\langle F_{\alpha} : \alpha < \mathfrak{c} \rangle$ enumerate the collection of second-category closed subsets of $[0, 1]$. Again, by transfinite induction, we can construct $A = \{x_{\alpha} : \alpha < \mathfrak{c}\}$ by picking $x_{\alpha} \in F_{\alpha}$ at each step $\alpha < \mathfrak{c}$ in such a way that $\{x_{\beta} : \beta \leq \alpha\}$ contains no subset similar to $\{1, 2, 3\}$. Then for $A$, (A), or equivalently (B), holds and $A \notin \mathcal{Em}b$. \(\square\) (iv)

**Proof of (v).** Treating $\mathbb{R}$ as a vector space over $\mathbb{Q}$ and with $H$ a Hamel basis, as above, take $A := \text{Lin}_\mathbb{Q}(H \setminus \{h_0\})$ the vector subspace generated by $H \setminus \{h_0\}$ of co-dimension 1. Then the additive function $f$ generated by taking
\( f(h_0) = 1 \) and \( f(h) = 0 \) for \( h \in H \setminus \{h_0\} \) is discontinuous, and bounded on \( A \). So \( A \notin BC \). Also \( \lambda^*(A) = 1 \). \( \square \) (v)

**Proof of (vi).** Suppose \( A \) is the same as in the proof of (v) so that \( A \notin BC \). We will show that \( A \cap I \) is of second category for every closed interval \( I \subseteq [0, 1] \).

Let \( I \) be given. Let \( T = \frac{I}{2} \) with the same centre as \( I \) and half the length. Now
\[
T = \bigcup_{r \in \mathbb{Q}} [(A + rh_0) \cap T] =: \bigcup_{r \in \mathbb{Q}} T_r.
\]
Since \( T \) is of second category, at least one \( T_r \) is of second category, \( T_r \) say. That is, \((A + rh_0) \cap I\) is of second category.

Take \( x \in T_r \). Since \( A \) is dense in \( \mathbb{R} \), \( rh_0 \) can be written as \( rh_0 = a + \epsilon \) with \( a \in A \) and \( |\epsilon| < \frac{|I|}{5} \), and hence \( x = a + \epsilon \) for some \( a \in A \). So \( T_r \subseteq (A + \epsilon) \cap T \), and so \((A + \epsilon) \cap I\) is of second category. This implies that \( A \cap I \) is of second category, being a translate by \(-\epsilon\) of the set \((A + \epsilon) \cap T\), completing the proof. \( \square \) (vi)

**Proof of (vii).** Take \( B \subseteq (0, 1) \), with \( B \) non-measurable. Then \( A = B \cup [1, 2] \) is automatically non-measurable, and in \( SC \), and so in \( BC \), by Darboux’s theorem (see e.g. [BinO5]). \( \square \) (vii)

**Proof of (viii).** Let \( A \) be the set of Theorem MO above (constructed in the proof of Theorem 9 in [MilO]). Then \( A \in SW \cap E \text{mb} \). We will show \( A \) is not shift-compact. Suppose otherwise, and consider the null sequence \((-1/n)\).

We show that
\[
A \cap \bigcap_{k=1}^{\infty} \left(A - \frac{1}{n_k}\right) = \emptyset,
\]
for every subsequence \((-1/n_k)\), so contradicting that \( A \) is shift-compact. So suppose the intersection above is non-empty for some subsequence \((-1/n_k)\).

Then, as \( A \) is assumed shift-compact, there exist \( a \in A \) such that \( a + (1/n_k) \in A \) for all \( k \), which is impossible by Th. MO(i) (i.e. (c) in Theorem 9 of [MilO]). \( \square \) (viii)

**Proof of (ix).** It is a corollary of earlier parts, already proved, that these two inclusions are proper: thus the first being proper follows from (viii). Both \( \subseteq \)-inclusions are well-known: see [BinO1, Th.1] for the first and [BinO5] for the second. For completeness, we recall the inclusion proofs here, as they are short (and needed together below).

Suppose \( A \in SC \). We claim that \( [0, \delta] \subseteq d(A) \) for some \( \delta > 0 \). Otherwise, there exists a null sequence \( y = \{y_n\}_{n \in \mathbb{N}} \), with \( y_n \notin d(A) \) for each \( n \in \mathbb{N} \). Since \( A \in SC \), there exists a subsequence \( \{y_{n_k}\}_{n_k \in \mathbb{N}} \), and \( a \in A \) such that
a + y_{n_k} ∈ A for all k. Hence \( y_{n_k} = (a + y_{n_k}) - a \in d(A) \) for all k, a contradiction. Thus \( SC ⊆ SW \).

Now we show \( SC ⊆ BC \). Suppose otherwise. Then there is a shift-compact set \( A \notin BC \) and an additive function \( f \) on \( R \) that is discontinuous but bounded above on \( A \). By Darboux’s theorem, there exists a null sequence \( \{ y_n \}_{n \in \mathbb{N}} \) with \( f(y_n) \to \infty \) (as otherwise \( f \) is locally bounded at 0, and so continuous again by Darboux’s theorem). Since \( A \) is shift-compact, there exists \( a \in A \) and a subsequence \( \{ y_{n_k} \}_{n_k \in \mathbb{N}} \) such that \( a + y_{n_k} \in A \) for all \( k \in \mathbb{N} \). Then \( f(a + y_{n_k}) = f(a) + f(y_{n_k}) \to \infty \), a contradiction since \( f \) is bounded from above on \( A \).

By the earlier part of this proof, \( SC \subseteq SW \cap BC \); but, as in (i), \( BC \not\subseteq SW \), so again this is a proper inclusion. \( \square \) (ix)

An alternative example for (viii) is provided by [MilM], cf. Th. MM (ii) above. See also § 6.2 below on Sierpiński sets. We stress that the inclusions mentioned in the Theorem 3 above are all proper, as shown in the proofs.

## 5 Singular sets of good character

In this section we reconsider an earlier counter-example and show that under \( V = L \) it will have good character: it will be co-analytic.

We recall from §1 that a subset \( T \) of the reals is shift-compact if for any null sequence \( z_n \to 0 \) there is \( t \in T \) such that \( t + z_m \in T \) for infinitely many \( m \). We refer to \( t \) as a ‘translator into \( T \) for \( z \).

Recall that a set \( S \) is concentrated on the rationals \( \mathbb{Q} \) if it is uncountable and for every open set \( W \supseteq \mathbb{Q} \) the set \( S \setminus W \) is countable [Rog1, §2.3]. Such a set is of strong measure zero. Under the assumption that less than \( \mathfrak{c} \) meagre sets have meagre union, the first two authors have shown in [MilM] (cf. Th. MM in §4 above) that there is a set concentrated on \( \mathbb{Q} \) which is shift-compact. To discuss a refinement of this result involving effective aspects, we recall the (effective) analytical hierarchy of predicates in the language of set theory (i.e. with the non-logical symbol \( \in \), cf. §3.1) concerned with numbers (members of \( \omega \)) and ‘reals’ (represented by number sequence in \( \omega^\omega \)). See e.g. [Rog2, Part 4] or [BinO8, §8 The syntax of analysis] for background. Write these with all quantifiers \( \forall x \) and \( \exists x \) (ranging over reals \( x \)) at the front, followed by an arithmetical predicate (this can be done assuming the Axiom of Dependent Choices, DC); then list and name the predicates according to the
starting quantifier and the number alternations (between ∀ and ∃, universal and existential) binding all the variables. Thus, as above in §3.1, a (lightface) $\Sigma^1_1$ predicate has just one existential and $\Pi^1_1$ has just one universal; $\Sigma^1_2$ has the $\exists \forall$ format, etc. If a free variable parameter $x \in (0,1)$ is allowed in the predicate (with $x$, regarded via its binary expansion as a function with domain $\mathbb{N}$, not necessarily effectively defined), this is recognized by bold-face lettering, yielding a hierarchy that is ‘relativized’ in the parameter (permitting relative effectiveness [Kec, V.40B]). Here $\Sigma^1_1$ corresponds to classical analytic sets and $\Pi^1_1$ to the co-analytic sets: an arbitrary open set in the line can be coded by a not necessarily recursive sequence of the rational-ended basic open intervals it contains.

The first two authors’ result amends a classical construction of a concentrated set using transfinite induction – so that, as first noted by Kuratowski [Kur] – under $V = L$ such a set would be $\Delta^1_2 = \Sigma^1_2 \cap \Pi^1_2$. In fact, under $V = L$, as [ErdKM, Th. 13] have shown, with careful monitoring of the effectiveness of constructions, a set $S$ concentrated on $\mathbb{Q}$ can be constructed which is $\Pi^1_1$. The underlying reason for the character improvement is that their construction is based on combinatorial analysis that is suitably ‘effective’.

We will similarly demonstrate an effective construction of a translator for a null sequence $z$ into any dense $G^\delta$ set $T$, when $T = \bigcap_n G_n$ with each $G_n$ open and containing $\mathbb{Q}$. This uses an effective enumeration of $\mathbb{Q}$ and the fairly recent constructive proof of shift-compactness [MilO]. We regard this as a geometric counterpart to the more combinatorial argument of [ErdKM], establishing the following

**Theorem 4.** Under $V = L$, there is a $\Pi^1_1$ subset of the reals which is concentrated on $\mathbb{Q}$ and is shift-compact.

The result is not altogether surprising. In the language of Turing reducibility (below), Vidnyánszky [Vid] captures the general procedure of adapting a construction of a set $S$ in a Polish space by transfinite induction under the assumption $V = L$ to yield a coanalytic version $C$ of $S$ in the following formulation, a result implied by $V = L$:

**Theorem V** ([Vid, Th. 1.3]). Assume $V=L$. For $B$ an uncountable Borel subset of an arbitrary Polish space, if

(i) $F$ is a co-analytic subset of $M^\leq \omega \times B \times M$ with $M \in \{R^n, 2^\omega, \wp(\omega), \omega^\omega\}$, and
(ii) for all \( A \in M^{\leq \omega} \), \( p \in B \), the vertical section \( F(A, p) \subseteq M \) of \( F \) is (upwards) cofinal in the ordering \( \preceq_T \) of Turing-reducibility then there exists a co-analytic set \( C \) that is 'compatible with \( F \').

Here \( M^{\leq \omega} \) denotes the countable subsets of \( M \), and we recall that \( x \preceq_T y \) for \( x, y \in M \) (read: '\( x \) is Turing reducible to \( y \)'), if \( x \) can be effectively computed from \( y \) (more exactly: there exists a Turing machine which computes \( x \) from the input \( y \)).

Rather than apply Th. V, which shadows [ErdKM], we have ourselves shadowed [ErdKM] in the preamble to the proof of Theorem 4 in an exposition of the tools from logic – which we hope analysts will find congenial – thereby clarifying the nub of the result. We rely on specified background from an analyst-friendly source: [BinO8].

The proof of Theorem 4 is given below, as indicated. We preface that now with a discussion of its salient features, in particular on its reliance on Kleene's theorem below (which gives a circumstance when an existential quantifier can be converted to a universal one).

**Proof of Theorem 4 preamble: proof strategy.**

We need to refer to the (metamathematical – 'external' to the discourse in the language) semantic relation \( \models \) of satisfaction/truth (below), due to Tarski (see [Tar1,2], cf. [BelS, Ch. 3 §2], cf. [BinO8]), which is read as 'models', or informally as ‘thinks’ (adopting a common enough anthropomorphic stance). A formula \( \varphi \) of LST with free variables \( x, y, \ldots, z \) may be interpreted in the structure \( \mathcal{M} := \langle M, \in_M \rangle \) (with \( \in_M \) now a binary set relation on the set \( M \)) for a given assignment \( a, b, \ldots, c \) in \( M \) for these free variables, and one writes

\[
\mathcal{M} \models \varphi(x, y, \ldots, z)[a, b, \ldots, c],
\]

if the property holds; this requires an induction on the syntactic complexity of the formula starting with the atomic formulas (for instance, the atomic case \( x \in y \) is interpreted under the assignment \( a, b \) as holding iff \( a \in_M b \)).

We recall also that Skolemization of a formula of LST, say \( \phi(\bar{x}, \bar{\tau}) \) with \( \bar{x} \) a finite list of free variables and \( \bar{\tau} \) a finite list of ordinals, is the elimination of all quantifiers by

(i) replacing existential quantifiers with functions appointing a ‘witness’ of an asserted existence (from among the available instances, assuming any exist), and
(ii) making free the variables previously bound by universal quantifiers (for which see [Hod, Ch. 3, p. 71], cf. [ManW, p. 87]).

This process yields an ‘equi-satisfiable’ (equivalent under \( \models \)) quantifier-free formula \( \Phi(\vec{x}, \vec{\tau}, \vec{y}, \vec{f}) \), involving a further finite list of free variables \( \vec{y} \) and finite list of function symbols \( \vec{f} \) (the Skolem functions for \( \phi \)) arising from the Skolemization, such that

\[
\exists \vec{f} \forall \vec{y}[\phi \rightarrow \Phi \land \psi]
\]

is a theorem of predicate logic (suppressing here the various lists \( \vec{x}, \vec{\tau}, \vec{y}, \vec{f} \)); here \( \psi \) is a certain (known) sentence such that if \( M \) is a transitive set and \( \psi \) holds in \( M \), then \( M \) is an \( L_\alpha \).

The structure \( \langle L_\alpha, \in \rangle \) can be equipped with canonical Skolem functions through always appointing ‘witnesses’ as above that are earliest under the well-ordering \( <_L \) of §3.1 above. Say that \( L_\alpha \) is point-definable if its Skolem hull (smallest set including \( L_\alpha \) and closed under the iteration of all its canonical Skolem functions) is isomorphic to \( L_\alpha \). (Such \( L_\alpha \) exist for unboundedly many \( \alpha \) in \( \omega_1 \) – for proof see [EngMS, Proof of Th. 2.6].) Performing the canonical Skolemization of \( L_\alpha \), one may define a relation \( E_\omega \) on \( \omega \), recursive in the set of all (first-order) sentences true in \( L_\alpha \) (known as the ‘theory of \( L_\alpha \), denoted \( Th(L_\alpha) \)) such that \( (\omega, E_\omega) \approx (L_\alpha, \in) \), where \( \approx \) denotes isomorphism (see [ManW, p. 87]). Bearing in mind its definability, \( E_\omega \in L_{\alpha+3} \), since \( Th(L_\alpha) \in L_{\alpha+2} \).

Consequent on the effective combinatorics used in the transfinite inductive construction in [ErdKM], membership of the singular set \( S \) constructed there can be expressed by a formula, denoted \( S(.) \) (with one free variable), in such a way that if \( x = x_\alpha \in L \) is selected inductively by reference to a point-definable \( (L_\alpha, \in) \) and to the ordering \( <_L \), then one constructs, recursively in \( x \) and in \( Th(L_\alpha) \), a countable set \( M \) and a relation \( E_M \) on \( M \) such that \( (M, E_M) \models S(x) \) (i.e., the sentence \( S(x) \) holds in the structure \( M \)). Taking \( z \) to code \( Th(L_\alpha) \), a real \( \mu \) may be constructed from \( z \) to code the set \( M \) and the relation \( E_M \) on \( M \); when done effectively the real \( \mu \) is called recursive in \( z \). Indeed, \( (M, E_M) \) may be constructed to be isomorphic to \( (L_{\alpha+\omega}, \in) \), cf. [EngMS, Th. 2.6, p. 209].

To verify the \( \Pi^1_1 \) character of the set \( S \), [ErdKM] relies on Kleene’s theorem from recursion theory (for which see e.g. [Sac, Lemma 3.1.III] and the formal proof below) that the existential quantifier over the ‘reals recursive in \( z \)’ (and, more generally, to reals in the set \( HYP(z) \) that are ‘hyperarithmetic in \( z \)’) may in fact be rendered as a universal quantifier ranging over all the reals.
(See [Sac, Lemma 3.1.III], or [ManW, 4.19]; note that there are countably many reals hyperarithmetic in \( z \).) Now the satisfaction relation \( 'M \models S(x)' \) when applied to countable models \( M \) is \( \Delta^1_1 \) as a predicate involving the real number \( \mu \) coding \( M \), as above (see e.g. [ManW, 1.20]), so it is in particular \( \Pi^1_1 \). Now \( x \in S \) iff 

\[ \exists \mu \in HYP(x)[\mu \approx (L_{\alpha+\omega}, \in) \& \mu \models S(x)], \]

which is \( \Pi^1_1 \) in \( x \) (so \( \Pi^1_1 \)) by Kleene’s theorem. The main task in the formal proof Theorem 4 below is analogous: to convert the informal statement “\( x \in X \)” to a formula \( S(x) \), the idea being to recover it, as in [ErdKM] above, from the (somewhat circuitous) definition:

\[ x \in X \iff \exists M \in HYP(x)[M \approx L_{\alpha+\omega} \& M \models ("x \in X")]. \]

Once this is done, one may ostensibly again apply Kleene’s theorem, but needs to check that the satisfaction clause (the last clause in the display above) does not degrade the descriptive character of the entire contents of the square brackets. One needs the final clause to be \( \Pi^1_1 \). However, the satisfaction relation \( M \models P(x) \) arising here is defined (by induction on the complexity of the predicate) only for predicates \( P(x) \) written in \( LST \) subject to the restriction that constants involved in \( P(x) \) (including \( x \) itself) may name only elements of \( M \). (This ensures that these constants have interpretations in \( M \); in particular, \( M \) needs to contain \( x \)).

**Proof of Theorem 4: Formal proof.** Assuming \( V = L \), we have \( I \subseteq L_{\omega_1} \).

For \( \alpha < \omega_1 \), let \( L_\alpha \) be point-definable (as in the preamble above). Select a dense subset \( D \subseteq I \) such that some \( d \in I \) is its recursive enumeration \( d = \{d_n\}_{n \in \mathbb{N}} \), with \( d_n \) the \( n^{th} \) projection of \( d \). Put \( G := \{x \in I : G(x) \supseteq D\} \). For \( x \in I \cap L_\alpha \) with \( G(x) \) containing \( D \), the set \( I \setminus G(x) \) is nowhere dense and so

\[ M_\alpha := \bigcup_{x \in L_\alpha \cap G}(I \setminus G(x)) \]

is meagre, as \( L_\alpha \) is countable. Put \( B_\alpha := I \setminus M_\alpha \). As there are countably many null sequences in \( L_\alpha \), there is \( t \in I \setminus L_\alpha \) such that:

(a) \( t_n \in B_\alpha \) for each \( n \), and

(b) for each null sequence \( z \in I \cap L_\alpha \) there is \( m = m(z) \) and \( N = N(z) \in \mathbb{N} \) such that \( t_m + z_n \in B_\alpha \) for \( n > N(z) \). As above such a \( t \) lies in \( L_{\alpha+\omega} \).

Proceed as in [ErdKM], and define \( X \) to be the set of all \( x \in I \) such that there exist:

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(i) a limit ordinal $\alpha$ such that $L_\alpha$ is point-definable and satisfies $ZF^{-}$, 
(ii) $E \subseteq \omega \times \omega$ recursive in $x$ such that $(\omega, E)$ is isomorphic with $(L_\alpha, \in)$, 
(iii) $x$ is the first element of $I \setminus L_\alpha$ satisfying (i) and (ii) and (a) and (b) above.

As in the preamble we are to apply Kleene’s theorem that, for arithmetic $A(n, f)$ (here $n$ ranges over $\mathbb{N}$ and $f$ over $\mathbb{N}^\mathbb{N}$), the predicate $(\exists f \in HY)^P A(n, f)$ is $\Pi^1_1$ – see e.g. [Sac, Lemma 3.1.III]. (In fact, the Spector-Gandy Theorem [Sac, Th. 3.5] asserts that this format characterizes $\Pi^1_1$.)

We need to verify that the defining clauses (a) and (b) and (i)-(iii) are satisfied in the model $(\omega, E)$.

To this end, we note that when the satisfaction relation $|=\,$ is restricted to a $\Sigma^1_1$ predicate $P(x)$, it is a $\Pi^1_1$ relation (in $x$) – see [Sac, Lemma 4.5.III]. Alternatively, for the relation to be $\Pi^1_1$ the predicate $P(x)$ needs to be a ranked one, i.e. an ordinal bound $\alpha < \omega^L_1$ must be placed on the ranges of the analytic quantifiers and on the free variables appearing in $P(x)$. (Here $\omega^L_1$ denotes the ordinal recognized in $L$ as the first uncountable; it is in fact countable – see e.g. [Dra, §8.4] or [BinO8, §5.2].)

With this in mind, we check that the defining clauses (a) and (b) and (i)-(iii) are ranked. Conditions (i) and (ii) are manifestly ranked, as will be (iii) provided also (a) and (b) are. For (a) one has

$$y \in B_\alpha \iff y \notin M_\alpha \iff (\forall z \in L_\alpha \cap I)[z \in G \to y \in G(z)],$$

and whilst this is $\Pi^1_1$ (rather than $\Sigma^1_1$) the quantifier is bounded to $L_\alpha$; so this is actually of ambiguous class $\Delta^1_1$, i.e. both $\Pi^1_1$ and $\Sigma^1_1$ in the codes (‘notations’) for $\alpha$. For (b) note that

$$(\forall z \in Null \cap L_\alpha)[\exists m \exists k \forall n > k[x_m - z_n \notin M_\alpha]],$$

where $Null$ stands for the set of null sequences (see the Appendix), and this is again $\Pi^1_1$, but nevertheless the quantifier is bounded to $L_\alpha$, so is again $\Delta^1_1$ in the codes for $\alpha$. □

Theorem 4 above offers a co-analytic version (under $V = L$) of the example of Th. MM(i), but not of (ii), as co-analytic sets are measurable. Co-analytic versions may likewise be obtained for the examples of Theorem 3 above (again except for the non-measurable example of (vii)). This is a consequence of the effective nature of the constructions used:

**Lemma 2.** (a) Take $S \subseteq \mathbb{R}$ countable; then all but at most countably many affine transformations $f(t) := at + b$ map any finite set $F$ to the complement of $S$, and in particular:
(i) with $b \notin B := S + S - S$ countable, $f(\{1, 2, 3\}) \not\subseteq S$;
(ii) if further $a \notin T \cap (T/2) \cap (T/3)$ with $T := S - B$ countable, then $f(\{1, 2, 3\}) \subseteq \mathbb{R}\setminus S$.

(b) For non-meagre closed $F$ there is a closed nwd set $N$ with $F = N \cup G(\{n : I_n \subseteq F\})$.

Proof (a): We consider the case $F = \{1, 2, 3\}$, as typical since the generalization is just a tedious exercise in linear algebra. As $f(\{1, 2, 3\}) = \{a + b, 2a + b, 3a + b\}$, if (i) fails, then $b := f(1) + f(2) - f(3) \in S + S - S$; from here (ii) is immediate as $f(i) \in S$ iff $a \in (S - B)/i$. $\square_{(a)}$

(b) $G := \bigcup\{I_n : I_n \subseteq F\}$ is the interior of $F$ and so $F \setminus G$ is closed and nwd. (This decomposition also appears in [BinO11, Th. 6M(b)] $\square_{(b)}$.

Theorem 4'. Under $V = L$, there is a $\Pi^1_1$ subset of the reals that is shift-compact but not in $\mathcal{E}mb$, and likewise there is a $\Pi^1_1$ subset in $\mathcal{E}mb$ which is not shift-compact. Indeed, under $V = L$, all the examples in the proof of Theorem 3, save for 3(vii), have $\Pi^1_1$ versions.

Proof of Theorem 4'.
(i) Here by [MilA2] (cf. [Vid]) there is a co-analytic Hamel basis $H$, and so the set $A = \bigcup_{q \in \mathbb{Q}} qH$ is also co-analytic and in $\mathcal{B}C$, but not shift-compact, since it fails to have the Steinhaus-Weil property.

(ii) Part 1. The finite subsets of $\mathbb{R}$ are in an effective 1-1 correspondence with $\mathbb{R}$ and effective choices of affine similarities may be made on the basis of Lemma 2(a).

Part 2. This follows from Lemma 2(a).

(iii) Each $F_\alpha$ may be coded, as in §3.1, by its complement $G(a) := [0, 1]\setminus F_\alpha$ with $|G(a)| < 1$. The latter property is arithmetical, being equivalent to the existence of a rational $q < 1$ with $|\bigcup_{m \in F} G(a(m))| < q$, for all finite $F \subseteq \mathbb{N}$.

(iv) By Lemma 2(b), each set $F_\alpha$ may be expressed in the form $G_\alpha \cup N_\alpha$ with $N_\alpha$ closed nwd and $G_\alpha$ coded as $G(a)$ where $a(n) = 1$ iff $I_n \subseteq F_\alpha$. We may thus pick $x \in G(a)$ avoiding both $N_\alpha$ (as in Th. 1E above) and the further countable set generated by the choices made earlier in the transfinite induction.

(v) $A$ is co-analytic as in (i).

(vi) This refers to the same set as in (iii).

(vii) Co-analytic sets are measurable [Rog2, Th. 2.9.2] [Kec, 29.7].

(viii) This is covered by Th. 4.

(ix) This refers back to (i). $\square$
6 Complements: related singular sets

1. Luzin sets. Recall that a Luzin set $L$ is an uncountable set which meets every meagre set in at most a countable set. A Luzin set does not have BP. If $L$ were Baire it would be non-meagre as $L$ is uncountable. But then $L$ is co-meagre on some rational-ended interval $I$, so w.l.o.g. is a dense $G_δ$ on $I$, and so contains an uncountable meagre set, a contradiction.

Hence $L$ cannot be analytic or co-analytic. This means that the $V = L$ construction of §5 above cannot be improved to yield a Luzin set.

Marczewski observed in 1938 that a set $L$ is Luzin iff $L$ is uncountable and is concentrated on every countable dense set. (Clear, since $L$ is Luzin iff for every dense open $G$, $L \setminus G$ is countable.) As such, $L$ is of strong measure zero (SMZ). (Being of measure zero, it is Lebesgue measurable.)

The first two authors’ example in Th. MM(i) of §4 may be made Luzin, so despite being SMZ it is shift-compact.

2. Sierpiński sets. Recall that a Sierpiński set is an uncountable set which meets every measure zero set in at most a countable set. It is known from work of (Szpiłrajn-)Marczewski and Kuratowski that a Sierpiński set $S$ is not only meagre, but in fact perfectly meagre (i.e. $S \cap P$ is meagre in $P$ for any perfect set $P$) – see e.g. A. Miller’s survey article for a proof [MilA1, Th. 4.1 and 5.2].

If $S$ were measurable, then it would be of positive measure, as $S$ is uncountable. So $S$ then contains a compact subset of positive measure, inside which there exists an uncountable set of measure zero – just repeat the construction of the Cantor set. This contradicts the defining property of $S$. So $S$ is not measurable.

In fact its complement, $\mathbb{R} \setminus S$, also non-measurable, is shift-compact by virtue of being co-meagre. As this is thematic, we give a direct proof based on KBD of the following.

Proposition 3. If $S$ is a Sierpiński set and $z_n \to 0$ is null, then for quasi all $t$ one has $t + z_n \subseteq \mathbb{R} \setminus S$ for all $n$.

Proof. Choose $H$ a dense $G_δ$ of zero measure containing the points $z_n$. As $S$ is a Sierpiński set, $D := S \cap H$ is countable, and $S \subseteq D \cup (\mathbb{R} \setminus H)$. Now $T := (\mathbb{R} \setminus H) \cup D$ is meagre, so $(\mathbb{R} \setminus T) = H \setminus D$ is co-meagre. By KBD, for quasi all $t \in H \setminus D$ one has $t + z_n \subseteq H \setminus D \subseteq \mathbb{R} \setminus S$ for all $n$. □

The result above also follows from a stronger result of Jasiński and Weiss
[JasW], concerning shifting a null $\mathcal{F}_\sigma$ (‘measure zero’ null) rather than a null sequence, and from Carlson [Car], who also studies associated $\sigma$-ideals. See §6.7 below.

3. Characterization of Strong Measure Zero sets (SMZ).

A set $X$ is of strong measure zero if for each sequence $\{\delta_n\}_{n\in\mathbb{N}}$ with each $\delta_n > 0$ there is a corresponding sequence of intervals $\{I_n\}_{n\in\mathbb{N}}$ covering $X$ with each $I_n$ of length at most $\delta_n$. Such sets $X$ are characterized by the property that, for each meagre set $H$, there is $x$ with $X \cap (x + H) = \emptyset$. See [MilA1, Th. 3.5].

Carlson [Car, Th. 2.1] shows that under $\text{MA}_\kappa$ (Martin’s Axiom at $\kappa < \mathfrak{c}$) these sets are closed under unions of size $\kappa$. In particular, this is so for countable unions. He also shows that no perfect set can be covered by such a countable union (in both $\mathbb{R}$ and in the Cantor space).

In this context we recall the contrasting property [EleS] (cf. [EleT]) of the Erdős-Kakutani set $C_{ES}$ of § 4 that, for every perfect set $P$, there is $x \in \mathbb{R}$ with $C_{ES} \cap (x + P)$ uncountable. Compare also §§ 6.4 and 6.7 below.

For further characterizations of SMZ see [GalMS]. We mention one such which is thematic for the present context. Here the target sets $T$ for embeddings are dense $\mathcal{G}_\delta$-sets. Embeddings which are performed simultaneously in any neighbourhood by a perfect subset of any such $T$ of a fixed set $Z$ into $T$ characterize those sets $Z$ that are strongly measure zero. Since any countable set is strongly of measure zero this result includes ‘simultaneous embeddings’ of a null sequence.

4. Strongly meagre (strong first category). By analogy with SMZ, a set $X$ is strong first category if for any measure-zero set $N$ there is $t$ with $X \cap (t + N) = \emptyset$. See [BarS].

5. Consistency results. Laver has proved in [Lav1,2] that it is consistent that every strong measure zero set is countable. Carlson [Car] shows that likewise it is consistent that every strong first category set is countable.

6. Luzin/Sierpiński sets versus SMZ. Every Luzin set has strong measure zero – see 1 above (this is (Szpiilrajn-)Marczewski’s observation). Bartoszyński and Judah [BartJ, Th. 2] show that, under the continuum hypothesis CH, every Sierpiński set is a union of at most two SMZ sets.

7. Carlson’s $\sigma$-ideals. Extending the SMZ idea, Carlson [Car, Th. 5.7] proves that each of the following families of sets forms a $\sigma$-ideal:

(i) those sets $X$ with the property that for every a meagre set $M$ there is $t$ such that $X \cap (t + M) = \emptyset$;
(ii) those sets $X$ with the property that for every a null $\mathcal{F}_\sigma$ set $H$ there is
such that \( X \cap (t + H) = \emptyset \). (Equivalently, for every \( G_\delta \) set \( G \) of full measure (=co-null), \( X \) is covered by some translate of \( G \).)

8. *Effective versions and coding.* The proof of Theorem 4 above relied on the ability to refer to various subsets of the real line, especially open sets, in terms of ‘codes’. Our canonical sources there were [Kec, Ch.V] on the analytical hierarchy (and the note [Kec, V.40B] on classical versus effective descriptive set theory), and our recent survey [BinO8], and for coding the wide-ranging use in [Sol, II.1.1, 25-33] and the much more minimal amount in [FenN, § 2, p. 93].

Appendix. We begin with some notation.

Let \( \{I_n\}_{n \in \mathbb{N}} \) enumerate (constructively) all the rational-ended intervals, with \( I_n = (l_n, r_n) \). Write \( \mathbb{M} \) for the odd natural numbers; for \( a \subseteq \mathbb{N} \) we may extract an \( n \)th canonical subset of \( a \) and also an open set naturally ‘coded’ by \( a \) by setting:

\[
a(n) = a \cap \{2^n m : m \in \mathbb{M}\}, \quad G(a) := \bigcup_{n \in a} I_n.
\]

We identify \( a \subseteq \mathbb{N} \) with the real number in \( \{0, 1\}^\mathbb{N} \) whose binary expansion is the indicator function of \( a \). Thus \( \{a : m \in a\} \) is open (being the set of reals with \( m \)-th binary digit \( =1 \)).

**Examples.** 1. Say that \( z \in I \) represents a null sequence, briefly \( \text{Null}(z) \), if for each \( k \) there is \( n \) so that \( x_m|k = 0_k \) for all \( m \geq n \) (so \( z_n \to 0 \)). Thus

\[
\text{Null}(z) \iff \forall k \exists l (\forall n \geq l)(\forall m)[|z(2^n(2m+1))| < 1/k].
\]

2. Let \( D := \{d_n : n \in \mathbb{N}\} \) enumerate effectively a subset dense in \( I \). By abuse of notation, say that \( x \) contains \( D \) when \( G(x) \supseteq D \), i.e. for each \( n \) there is \( m \) with \( d_n \in \varphi(x(m)) \). We denote the set of such \( x \) by \( \mathcal{G} \). Since

\[
x \in \mathcal{G} \iff (\forall n \in \mathbb{N})(\exists m \in \mathbb{N})(\exists k \in \mathbb{N})[d_n \in \varphi(k) \text{ and } k = x(m)],
\]

this is an arithmetic relation which is (light-faced) \( \Pi^0_2 \).
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Faculty of Engineering and Natural Sciences/Mathematics, International University of Sarajevo, 71000 Sarajevo, Bosnia-Herzegovina; harrymiller609@yahoo.com
Faculty of Engineering and Natural Sciences/Mathematics, International University of Sarajevo, 71000 Sarajevo, Bosnia-Herzegovina; lmiller@ius.edu.ba
Mathematics Department, London School of Economics, Houghton Street, London WC2A 2AE; A.J.Ostaszewski@lse.ac.uk