SHARP LOWER BOUNDS FOR THE FIRST EIGENVALUES OF
THE BI-LAPLACE OPERATOR

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ABSTRACT. We obtain sharp lower bounds for the first eigenvalue of four types
of eigenvalue problem defined by the bi-Laplace operator on compact manifolds
with boundary and determine all the eigenvalues and the corresponding eigen-
functions of a Wentzell-type bi-Laplace problem on Euclidean balls.

1. INTRODUCTION AND THE MAIN RESULTS

The classical Lichnerowicz-Obata theorem is stated as follows (Cf. [14, 30, 34]):

Theorem A (Lichnerowicz-Obata). Let $M$ be an $n$-dimensional complete Rie-
mannian manifold with Ricci curvature bounded below by $(n - 1)k > 0$, then the
first non-zero eigenvalue of the Laplacian of $M$ satisfies

$$\eta \geq nk,$$

and equality holds if and only if $M$ is isometric Euclidean $n$-sphere of radius $1/\sqrt{k}$.

When $M$ has nonnegative Ricci curvature, Li-Yau [29] proved that

$$\eta_1 \geq \frac{\pi^2}{2D^2},$$

where $D$ is the diameter of $M$. Later, Zhong-Yang improved this estimate to an
optimal lower bound [45]:

$$\lambda \geq \frac{\pi^2}{D^2}.$$

Moreover, Hang-Wang [21] showed that if the equality in (1.3) holds then $M$ must
be isometric to a circle of radius $D/\pi$.

On the other hand, in 1977, Reilly obtained a sharp lower for the first Dirichlet
eigenvalue of the Laplacian of compact manifolds with boundary and positive Ricci
curvature [37].

Theorem B (Reilly). Let $M$ be an $n(\geq 2)$-dimensional compact Riemannian
manifold with Ricci curvature bounded below by $(n - 1)\kappa > 0$ and boundary. If
the mean curvature of $\partial M$ is nonnegative then the first Dirichlet eigenvalue of the
Laplacian of $M$ satisfies

$$\lambda_1 \geq nk,$$

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and equality holds if and only if $M$ is isometric to an $n$-dimensional Euclidean hemisphere of radius $1/\sqrt{\kappa}$.

For the Neumann boundary case a similar result has been proven by Escobar [17] and Xia [43] independently. They showed that if $M$ is an $n(\geq 2)$-dimensional compact Riemannian manifold with Ricci curvature bounded below by $(n-1)\kappa > 0$ and convex boundary, then the first nonzero eigenvalue of the Laplace of $M$ with Neumann boundary condition satisfies

$$\mu_1 \geq n\kappa,$$

and equality holds if and only if $M$ is isometric to and $n$-dimensional Euclidean hemisphere of curvature $\kappa$.

A natural question related to the above results is to study similar rigidity phenomenon for other eigenvalue problems on Riemannian manifolds. In the first part of this paper, we consider two kinds of eigenvalue problem of the bi-Laplace operator on compact Riemannian manifolds with boundary and obtain sharp lower bound for the first eigenvalue of them. The point of interest is that no assumption on the boundary is made.

**Theorem 1.1.** Let $M$ be an $n(\geq 2)$-dimensional compact Riemannian manifold with boundary. Assume that the Ricci curvature of $M$ is bounded below by $(n-1)\kappa \geq 0$. Denote by $\lambda_1$ the first Dirichlet eigenvalue of the Laplacian of $M$ and $\Gamma_1$, the first eigenvalue of the following problem:

$$\begin{cases}
\Delta^2 u = \Gamma u & \text{in } M, \\
u = \frac{\partial^2 u}{\partial \nu^2} = 0 & \text{on } \partial M.
\end{cases}$$

Then

$$\Gamma_1 \geq \lambda_1 \left(\frac{\lambda_1}{n} + (n-1)\kappa\right),$$

with equality holding if and only if $\kappa > 0$ and $M$ is isometric to an $n$-dimensional Euclidean hemisphere of radius $1/\sqrt{\kappa}$.

**Theorem 1.2.** Let $M$ be an $n(\geq 2)$-dimensional compact Riemannian manifold with boundary. Assume that the Ricci curvature of $M$ is bounded below by $(n-1)\kappa$. Denote by $\lambda_1$ the first Dirichlet eigenvalue of the Laplacian of $M$ and $\Lambda_1$ the first eigenvalue of the problem

$$\begin{cases}
\Delta^2 u = -\Lambda \Delta u & \text{in } M, \\
u = \frac{\partial^2 u}{\partial \nu^2} = 0 & \text{on } \partial M.
\end{cases}$$

Then

$$\Lambda_1 \geq \frac{\lambda_1}{n} + (n-1)\kappa,$$

with equality holding if and only if $\kappa > 0$ and $M$ is isometric to an $n$-dimensional Euclidean hemisphere of radius $1/\sqrt{\kappa}$.

The eigenvalue problems (1.6) and (1.8) should be compared with the clamped plate problem and the buckling problem, respectively. The later two ones are as follows:

$$\begin{cases}
\Delta^2 u = \Gamma u & \text{in } M, \\
u = \frac{\partial^2 u}{\partial \nu^2} = 0 & \text{on } \partial M.
\end{cases}$$

(1.10)
Theorems (1.1) and (1.2) show that the first eigenvalue of (1.6) (or (1.8)) is closely related to the first Dirichlet eigenvalue of the Laplacian. Many important results have been obtained for the eigenvalues of the problems (1.10) and (1.11) (cf. [2], [3], [4], [5], [10], [11], [12], [13], [22], [23], [24], [33], [39], [40], [41], etc). It would be interesting to characterize the hemisphere by the first non-zero eigenvalue of the problem (1.10) or (1.11).

In the second part of this paper, we study Steklov eigenvalue problems of the bi-Laplace operator. The eigenvalue problems we are interested in are as follows:

\begin{align}
\Delta^2 u = 0 & \quad \text{in } M, \\
u = \Delta u - p \frac{\partial u}{\partial \nu} = 0 & \quad \text{on } \partial M, 
\end{align}

(1.12)

\begin{align}
\Delta^2 u = 0 & \quad \text{in } M, \\
u = \Delta u - q \frac{\partial u}{\partial \nu} = 0 & \quad \text{on } \partial M, 
\end{align}

(1.13)

\begin{align}
\Delta^2 u = 0 & \quad \text{in } M, \\
\frac{\partial u}{\partial \nu} = \frac{\partial (\Delta u)}{\partial \nu} + \xi u & = 0 \quad \text{on } \partial M, 
\end{align}

(1.14)

\begin{align}
\Delta^2 u = 0 & \quad \text{in } M, \\
\frac{\partial u}{\partial \nu} = \frac{\partial (\Delta u)}{\partial \nu} + \beta \Delta u + \varsigma u & = 0 \quad \text{on } \partial M, 
\end{align}

(1.15)

where \(\beta\) is a nonnegative constant and \(\Delta\) is the Laplace operator with respect to the induced metric of \(\partial M\).

Elliptic problems with parameters in the boundary conditions are called Steklov problems from their first appearance in [38]. Problem (1.12) was considered by Kuttler [25] and Payne [35] who studied the isoperimetric properties of the first eigenvalue \(p_1\) which is the sharp constant for \(L^2\) a priori estimates for solutions of the (second order) Laplace equation under nonhomogeneous Dirichlet boundary conditions (cf. [26, 27, 28]). The whole spectrum of (1.12) was studied in [18, 31, 32] where one can also find a physical interpretation of \(p_1\). We refer to [7, 8, 9, 19, 36, 42] for some further developments about \(p_1\). One can see that \(p_1\) is positive and given by

\begin{equation}
p_1 = \min_{w|\partial M = 0, w \neq \text{const.}} \frac{\int_M (\Delta w)^2}{\int_{\partial M} (\frac{\partial w}{\partial \nu})^2}.
\end{equation}

The problem (1.13) is a natural Steklov problem and is equivalent to (1.12) when the mean curvature of \(\partial M\) is constant. We have a sharp relation between the first eigenvalues of (1.12) and (1.13).

**Theorem 1.3.** Let \(M\) be an \(n\)-dimensional compact Riemannian manifold with boundary and non-negative Ricci curvature. Denote by \(p_1\) and \(q_1\) the first eigenvalue of the problems (1.12) and (1.13), respectively. Then we have

\begin{equation}
q_1 \geq \frac{p_1}{n},
\end{equation}

with equality holding if and only if \(M\) is isometric to a ball in \(\mathbb{R}^n\).

The next result is a sharp lower bound for \(p_1\).
Theorem 1.4. Let $M$ be a compact Riemannian manifold with Ricci curvature bounded below by $-(n-1)\kappa$ for some constant $\kappa \geq 0$. Denote by $\lambda_1$ the first Dirichlet eigenvalue of the Laplacian of $M$ and let $p_1$ be the first eigenvalue of the problems (1.12). Suppose that the mean curvature of $\partial M$ is bounded below by a positive constant $c$. Then we have

$$p_1 \geq \frac{nc\lambda_1}{n\kappa + \lambda_1}$$

with equality holding if and only if $\kappa = 0$ and $M$ is isometric to a Euclidean $n$-ball of radius $1/c$.

The problem (1.14) was first studied in [27] where some estimates for the first non-zero eigenvalue $\xi_1$ were obtained. When $M$ is an Euclidean ball, all the eigenvalues of the problem (1.13) have been recently obtained in [44]. Also, the authors proved an isoperimetric upper bound for $\xi_1$ when $M$ is a bounded domain in $\mathbb{R}^n$.

The Rayleigh-Ritz formula for $\xi_1$ is:

$$\xi_1 = \min_{0 \neq u \in H^2(M)} \frac{\int_M (\Delta u)^2}{\int_{\partial M} u^2},$$

where $\Delta u = 0$ in $M$, $\beta \Delta u + \partial_\nu u = \lambda u$ on $\partial M$.

The problem (1.15) is a so called Wentzell problem for the bi-laplace operator which is motivated by (1.14) and the following Wentzell-Laplace problem:

$$\begin{cases} 
\Delta u = 0 & \text{in } M, \\
-\beta \Delta u + \partial_\nu u = \lambda u & \text{on } \partial M,
\end{cases}$$

where $\beta$ is a given non-negative number. The problem (1.20) has been studied recently, in [16], [44], etc.

The first non-zero eigenvalue of (1.15) can be characterized as

$$\varsigma_{1,\beta} = \min_{0 \neq u \in H^2(M)} \frac{\int_M (\Delta u)^2 + \beta \int_{\partial M} |\nabla u|^2}{\int_{\partial M} u^2},$$

From (1.19), one can see that if $\beta > 0$, $\xi_1$ is the first non-zero eigenvalue of the Steklov problem (1.13) and $\lambda_1$ the first non-zero eigenvalue of the Laplacian of $\partial M$, then we have

$$\varsigma_{1,\beta} \geq \xi_1 + \beta \lambda_1,$$

with equality holding if and only if any eigenfunction $f$ corresponding to $\varsigma_{1,\beta}$ is an eigenfunction corresponding to $\xi_1$ and $f|_{\partial M}$ is an eigenfunction corresponding to $\lambda_1$.

For each $k = 0, 1, \cdots$, let $\mathcal{D}_k$ be the space of harmonic homogeneous polynomials in $\mathbb{R}^n$ of degree $k$ and denote by $\mu_k$ the dimension of $\mathcal{D}_k$. We refer to [6] for the basic properties of $\mathcal{D}_k$ and $\mu_k$. In particular, we have

$$\mathcal{D}_0 = \text{span}\{1\}, \quad \mu_0 = 1,$$

$$\mathcal{D}_1 = \text{span}\{x_i, i = 1, \cdots, n\}, \quad \mu_1 = n,$$

$$\mathcal{D}_2 = \text{span}\{x_ix_j, x_i^2 - x_h, 1 \leq i < j \leq n, h = 2, \cdots, n\}, \quad \mu_2 = \frac{n^2 + n - 2}{2}.$$
In the next result we determine explicitly all the eigenvalues of (1.15) and the corresponding eigenfunctions when \( M = \mathbb{B} \) (the unit ball with center at the origin in \( \mathbb{R}^n \)).

**Theorem 1.5.** If \( n \geq 2 \) and \( M = \mathbb{B} \), then we have

1. the eigenvalues of (1.15) are
   \[
   \varsigma_k = k^2(n + 2k) + \beta k(k + n - 2), \quad k = 0, 1, 2, \ldots
   \]
   \[
   \mathbf{E}_k = \{-2w_k + k(|x|^2 - 1)w_k \mid w_k \in D_k\}. \quad (1.23)
   \]

Our last result is a lower bound for \( \varsigma_1 \).

**Theorem 1.6.** Let \( M \) be an \( n \)-dimensional compact Riemannian manifold with boundary and Ricci curvature bounded below by \( -(n - 1)\kappa \), \( \kappa \geq 0 \). Assume that the principal curvatures of \( \partial M \) are bounded below by a positive constant \( c \) and denote by \( \varsigma_1 \) the first eigenvalue of the problems and (1.15). Then we have

\[
\varsigma_1 > \frac{n\mu_1}{(n - 1)(\mu_1 + n\kappa)} + \beta \lambda_1,
\]

where \( \mu_1 \) and \( \lambda_1 \) are the first nonzero Neumann eigenvalue of the Laplacian of \( M \) and the first nonzero eigenvalue of the Laplacian of \( \partial M \), respectively.

2. **Proof of Theorems 1.1 and 1.2**

In this section, we shall prove Theorems 1.1 and 1.2. Before doing this, we first recall Reilly’s formula which will be used later. Let \( M \) be an \( n \)-dimensional compact manifold with boundary. We will often write \( \langle \cdot, \cdot \rangle \) the Riemannian metric on \( M \) as well as that induced on \( \partial M \). Let \( \nabla \) and \( \Delta \) be the connection and the Laplacian on \( M \), respectively. Let \( \nu \) be the unit outward normal vector of \( \partial M \). The shape operator of \( \partial M \) is given by \( S(X) = \nabla_X \nu \) and the second fundamental form of \( \partial M \) is defined as \( II(X, Y) = \langle S(X), Y \rangle \), here \( X, Y \in T\partial M \). The eigenvalues of \( S \) are called the principal curvatures of \( \partial M \) and the mean curvature \( H \) of \( \partial M \) is given by \( H = \frac{1}{n-1} \text{tr} S \), here \( \text{tr} S \) denotes the trace of \( S \). For a smooth function \( f \) defined on \( M \), the following identity holds \cite{37} if \( h = \frac{\partial f}{\partial \nu} \big|_{\partial M} \), \( z = f \big|_{\partial M} \) and \( \text{Ric} \) denotes the Ricci tensor of \( M \):

\[
(2.1) \quad \int_M \left( (\Delta f)^2 - |\nabla^2 f|^2 - \text{Ric}(\nabla f, \nabla f) \right) = \int_{\partial M} \left( ((n - 1)Hh + 2\Sigma z)h + II(\nabla z, \nabla z) \right).
\]

Here \( \nabla^2 f \) is the Hessian of \( f \); \( \Sigma \) and \( \nabla \) represent the Laplacian and the gradient on \( \partial M \) with respect to the induced metric on \( \partial M \), respectively.

**Proof of Theorem 1.1.** Let \( f \) be an eigenfunction of the problem (1.3) corresponding to the first eigenvalue \( \Gamma_1 \). That is,

\[
\begin{aligned}
\Delta^2 f = \Gamma_1 f & \quad \text{in} \quad M, \\
\frac{\partial^2 f}{\partial \nu^2} = 0 & \quad \text{on} \quad \partial M.
\end{aligned} \quad (2.2)
\]
It follows from the divergence theorem that
\[
\Gamma_1 \int_M f^2 = \int_M f \Delta^2 f
\]
\[
= - \int_M \langle \nabla f, \nabla (\Delta f) \rangle
\]
\[
= \int_M (\Delta f)^2 - \int_{\partial M} h \Delta f,
\]
where \( h = \frac{\partial f}{\partial \nu} \bigg|_{\partial M} \). Since \( f|_{\partial M} = \frac{\partial^2 u}{\partial \nu^2} \bigg|_{\partial M} = 0 \), we have
\[
\Delta f|_{\partial M} = (n-1)Hh.
\]
(2.4)

From Reilly’s formula, we infer
\[
\int_M ((\Delta f)^2 - |\nabla^2 f|^2) = \int_M \text{Ric}(\nabla f, \nabla f) + (n-1) \int_{\partial M} Hh^2.
\]
(2.5)

Combining (2.3)-(2.5) and using \( \text{Ric}(\nabla f, \nabla f) \geq (n-1)\kappa |\nabla f|^2 \), we get
\[
\Gamma_1 = \frac{\int_M (|\nabla^2 f|^2 + \text{Ric}(\nabla f, \nabla f))}{\int_M f^2} \geq \frac{\int_M (|\nabla^2 f|^2 + (n-1)\kappa |\nabla f|^2)}{\int_M f^2}.
\]
(2.6)

The Schwarz inequality implies that
\[
|\nabla^2 f|^2 \geq \frac{1}{n} (\Delta f)^2,
\]
with equality holding if and only if
\[
\nabla^2 f = \frac{\Delta f}{n} (\_).
\]
(2.7)

Therefore, we have
\[
\Gamma_1 \geq \frac{\int_M \left( \frac{1}{n} (\Delta f)^2 + (n-1)\kappa |\nabla f|^2 \right)}{\int_M f^2},
\]
with equality holding if and only if (2.8) holds and
\[
\text{Ric}(\nabla f, \nabla f) = (n-1)\kappa |\nabla f|^2 \text{ on } M.
\]

On the other hand, since \( f \) is not a zero function which vanishes on \( \partial M \), we know that
\[
\int_M (\Delta f)^2 \geq \lambda_1 \int_M |\nabla f|^2 \geq \lambda_1^2 \int_M f^2
\]
with equality holding if and only if \( f \) is a first eigenfunction of the Dirichlet Laplacian of \( M \). Thus we conclude that
\[
\Gamma_1 \geq \lambda_1 \left( \frac{\lambda_1}{n} + (n-1)\kappa \right).
\]
(2.11)

Assume now that
\[
\Gamma_1 = \lambda_1 \left( \frac{\lambda_1}{n} + (n-1)\kappa \right).
\]
(2.12)
In this case, (2.10) should take equality sign which implies that \( f \) is a first eigenfunction corresponding to the first eigenvalue \( \lambda_1 \) of the Dirichlet Laplacian of \( M \). Consequently, we have

\[
\Delta f = -\lambda_1 f \quad \text{in } M
\]

It then follows that

\[
\Delta^2 f = -\lambda_1 \Delta f = \lambda_1^2 f \quad \text{in } M
\]

which, combining with (2.2) and (2.12), gives \( \lambda_1 = nk > 0 \). Since \( f \) is an eigenfunction corresponding to the first Dirichlet eigenvalue of the Laplacian, it is well known that \( f \) does not change sign in \( M \). Let us assume that \( f \) is negative in the interior of \( M \). From \( \Delta f = -\lambda_1 f \geq 0 \) on \( M \) and \( f|_{\partial M} = 0 \), we know from the maximum principle and the Hopf lemma [20] that

\[
\frac{\partial f}{\partial \nu} > 0 \quad \text{on } \partial M.
\]

It then follows from \( 0 = \Delta f|_{\partial M} \) and (2.14) that the mean curvature of \( \partial M \) vanishes. Thus one can use Reilly theorem as stated before to conclude that \( M \) is isometric to an \( n \)-dimensional hemisphere of curvature \( \kappa \). This completes the proof of Theorem 1.1.

\[\square\]

**Proof of Theorem 1.2.** The discussions are similar to those in the proof of Theorem 1.1. For the sake of completeness, we include it. Let \( g \) be the eigenfunction of the problem (1.8) corresponding to the first eigenvalue \( \Lambda_1 \):

\[
\begin{cases}
\Delta^2 g = -\Lambda_1 \Delta g \quad \text{in } M, \\
g = \frac{\partial^2 g}{\partial \nu^2} = 0 \quad \text{on } \partial M.
\end{cases}
\]

Multiplying the first equation of (2.16) by \( g \) and integrating on \( M \), we have from divergence theorem that

\[
\Lambda_1 \int_M |\nabla g|^2 = \int_M (\Delta g)^2 - \int_{\partial M} s \Delta g,
\]

where \( s = \frac{\partial g}{\partial \nu}|_{\partial M} \). Also, we have

\[
\Delta g|_{\partial M} = (n - 1)Hs.
\]

Hence

\[
\Lambda_1 = \frac{\int_M (\Delta g)^2 - (n - 1) \int_{\partial M} H s^2}{\int_M |\nabla g|^2},
\]

which, combining with Reilly’s formula gives

\[
\Lambda_1 = \frac{\int_M (|\nabla^2 g|^2 + \text{Ric}(\nabla g, \nabla g))}{\int_M |\nabla g|^2} \geq \frac{\int_M \left( \frac{(|\Delta g|^2}{n} + (n - 1)\kappa |\nabla g|^2 \right)}{\int_M |\nabla g|^2}.
\]
We also have
\[ (2.21) \quad \int_M (\Delta g)^2 \geq \lambda_1 \int_M |\nabla g|^2, \]
with equality holding if and only if \( g \) is an eigenfunction corresponding to \( \lambda_1 \). Therefore, we have
\[ (2.22) \quad \Lambda_1 \geq \frac{\lambda_1}{n} + (n - 1)\kappa. \]
If the equality occurs in (2.22), then
\[ (2.23) \quad \Delta g = -\lambda_1 g \quad \text{on} \quad M, \]
and so
\[ (2.24) \quad \Delta^2 g = -\lambda_1 \Delta g. \]
It then follows that
\[ (2.25) \quad \Lambda_1 = \lambda_1 = \frac{\lambda_1}{n} + (n - 1)\kappa, \]
which gives \( 0 < \lambda_1 = n\kappa \). As in the proof of Theorem 1.1, we can deduce that the mean curvature of \( \partial M \) vanishes and by Reilly’s theorem that \( M \) is isometric to an \( n \)-dimensional Euclidean hemisphere of curvature \( \kappa \). 

\[ \square \]

3. Proof of Theorems 1.3-1.6

In this section, we shall prove Theorems 1.3-1.6.

Proof of Theorem 1.3. Let \( w \) be an eigenfunction corresponding to the first eigenvalue \( \eta_1 \) of the problem (1.13):
\[ \left\{ \begin{array}{l}
\Delta^2 w = 0 \quad \text{in} \quad M, \\
w = \frac{\partial^2 w}{\partial \nu^2} - \eta_1 \frac{\partial w}{\partial \nu} = 0 \quad \text{on} \quad \partial M.
\end{array} \right. \]
Note that \( w \) is not a constant since \( w|_{\partial M} = 0 \). Set \( \eta = \frac{\partial w}{\partial \nu}|_{\partial M} \); then \( \eta \neq 0 \). Otherwise, we would deduce from
\[ w|_{\partial M} = \nabla w|_{\partial M} = \left. \frac{\partial^2 w}{\partial \nu^2} \right|_{\partial M} = 0 \]
that \( \Delta w|_{\partial M} = 0 \) and so \( \Delta w = 0 \) on \( M \) by the maximum principle, which in turn implies that \( w = 0 \). This is a contradiction.

From \( w|_{\partial M} = 0 \), one gets from divergence theorem that
\[ (3.2) \quad \int_M \left( \nabla w, \nabla (\Delta w) \right) = -\int_M w \Delta^2 w = 0. \]
Thus
\[ (3.3) \quad \int_{\partial M} \Delta w \frac{\partial w}{\partial \nu} = \int_M \left( \nabla (\Delta w), \nabla w \right) + \int_M (\Delta w)^2 = \int_M (\Delta w)^2. \]
Since
\begin{equation}
\Delta w|_{\partial M} = \frac{\partial^2 w}{\partial \nu^2} + (n-1)H \frac{\partial w}{\partial \nu},
\end{equation}
we conclude that
\begin{equation}
q_1 = \frac{\int_M (\Delta w)^2 - (n-1) \int_{\partial M} H \eta^2}{\int_{\partial M} \eta^2},
\end{equation}
which, combining with Reilly’s formula, gives
\begin{equation}
q_1 = \frac{\int_M (\Delta w)^2 \text{ Ric}(\nabla w, \nabla w)}{\int_{\partial M} \eta^2} \geq \frac{\int_M \nabla^2 w}{\int_{\partial M} \eta^2}.
\end{equation}
The Schwarz inequality implies that
\begin{equation}|
\nabla^2 w| \geq \frac{1}{n} (\Delta w)^2,
\end{equation}
with equality holding if and only if
\begin{equation}\nabla^2 w = \frac{\Delta w}{n} (\cdot, \cdot).
\end{equation}
Therefore, we have
\begin{equation}
q_1 \geq \frac{1}{n} \int_M (\Delta w)^2 \int_{\partial M} \eta^2.
\end{equation}
On the other hand, we have from the variational characterization of $p_1$ (cf. (1.16)) that
\begin{equation}
\int_M (\Delta w)^2 \int_{\partial M} \eta^2 \geq p_1
\end{equation}
with equality holding if and only if $w$ is an eigenfunction corresponding to $p_1$. Combining (3.9) and (3.10), we get
\begin{equation}
q_1 \geq \frac{p_1}{n}.
\end{equation}
This proves the first part of Theorem 1.3.

Assume now that $q_1 = p_1/n$. In this case, we know from the above proof that
\begin{equation}
\nabla^2 w = \frac{\Delta w}{n} (\cdot, \cdot)
\end{equation}
holds on $M$ and $w$ is an eigenfunction corresponding to $q_1$. Hence
\begin{equation}
\Delta w = p_1 \eta \text{ on } \partial M.
\end{equation}
Take an orthornormal frame $\{e_1, \cdots, e_{n-1}, e_n\}$ on $M$ such that when restricted to $\partial M$, $e_n = \nu$. From $0 = \nabla^2 w(e_i, e_n)$, $i = 1, \cdots, n-1$, and $w|_{\partial M} = 0$, we conclude that $\eta = b_0 = \text{ const.} \neq 0$. It then follows from the harmonicity of $\Delta w$ and (3.12) that
\begin{equation}
\Delta w = p_1 b_0 = nq_1 b_0 \text{ on } M.
\end{equation}
Setting
\begin{equation}
\rho = -\frac{w}{p_1 b_0},
\end{equation}
we have
\begin{equation}
\nabla^2 \rho = -\frac{1}{n} (\cdot, \cdot) \text{ on } M.
Taking the covariant derivative of (3.15), we get
\[ \nabla^3 \rho = 0 \]
and from the Ricci identity,
\[ R(X,Y) \nabla \rho = 0, \]
for any tangent vectors \( X, Y \) on \( M \), where \( R \) is the curvature tensor of \( M \). By the maximum principle \( \rho \) attains its maximum at some point \( x_0 \) in the interior of \( M \). Let \( r \) be the distance function to \( x_0 \); then from (3.15) it follows that
\[ \nabla \rho = -\frac{1}{n} r \frac{\partial}{\partial r} \]
(3.17)
Using (3.16), (3.17), Cartan’s theorem (cf. [15]) and \( \rho|_{\partial M} = 0 \), we conclude that \( M \) is a ball in \( \mathbb{R}^n \) whose center is \( x_0 \), and
\[ \rho(x) = \frac{1}{2n} \left( r_0^2 - |x - x_0| \right) \]
in \( M \), here \( r_0 \) is the radius of the ball. Thus,
\[ w(x) = \frac{p_1b_0}{2n} (|x - x_0| - r_0^2). \]
This completes the proof of Theorem 1.3

Proof of Theorem 1.4 Let \( \phi \) be an eigenfunction corresponding to the first eigenvalue \( p_1 \) of the problem (1.12), that is
\[ \begin{cases} \Delta^2 \phi = 0 \text{ in } M, \\ \phi = \Delta \phi - p_1 \frac{\partial \phi}{\partial \nu} = 0 \text{ on } \partial M. \end{cases} \]
(3.18)
Set \( \eta = \frac{\partial \phi}{\partial \nu} \bigg|_{\partial M} \); then
\[ q_1 = \frac{\int_M (\Delta \phi)^2}{\int_{\partial M} \eta^2}. \]
(3.19)
Substituting \( \phi \) into Reilly’s formula, we have
\[ \int_M \left\{ (\Delta \phi)^2 - |\nabla^2 \phi|^2 \right\} = \int_M \text{Ric}(\nabla \phi, \nabla \phi) + \int_{\partial M} (n-1)H \eta^2 \]
\[ \geq -(n-1)\kappa \int_M |\nabla \phi|^2 + (n-1)c \int_{\partial M} \eta^2 \]
\[ \geq -\frac{(n-1)\kappa}{\lambda_1} \int_M (\Delta \phi)^2 + (n-1)c \int_{\partial M} \eta^2, \]
where, in the last step above, we have used the fact that \( \kappa \geq 0 \) and
\[ \int_M |\nabla \phi|^2 \leq \frac{1}{\lambda_1} \int_M (\Delta \phi)^2. \]
(3.21)
The Schwarz inequality implies that
\[ |\nabla^2 \phi|^2 \geq \frac{1}{n} (\Delta \phi)^2 \]
with equality holding if and only if
\[ \nabla^2 \phi = \frac{\Delta \phi}{n}. \]
(3.22)
Combining (3.19), (3.20) and (3.22), we have
\[ p_1 \geq \frac{n \lambda_1}{n \kappa + \lambda_1}, \]
(3.24)
If the equality sign holds in (3.24), then the inequalities (3.20) and (3.22) must take equality sign and, in particular, (3.23) holds on $M$. By using the same arguments as in the proof of Theorem 1.3, we conclude that $\Delta \phi$ is a nonzero constant and $M$ is isometric to a ball in $\mathbb{R}^n$ which, in turn, implies that $\kappa = 0$. □

**Proof of Theorem 1.5.** Let $u$ be an eigenfunction of the problem (1.15) with $M = B$ corresponding to an eigenvalue $\varsigma$. From the bi-harmonicity of $u$ we can two uniquely determined harmonic functions $g, h$ on $B$, such that

$$u(x) = g(x) + |x|^2 h(x). \tag{3.25}$$

Let us denote by $E$ the Euler operator defined by

$$Ef(x) := \sum_{i=1}^n x_i \frac{\partial f}{\partial x_i}(x). \tag{3.26}$$

We will use the notation $E^l f = E(E^{l-1} f), l = 1, 2, \ldots$. Observe that $E$ is related to the exterior normal derivative by the relation

$$\frac{\partial f}{\partial \nu} \bigg|_{\partial B} = Ef \bigg|_{\partial B}, \tag{3.27}$$

and if $f$ is harmonic then so is $Ef$. Setting $-2w = g + h$, we have $u(x) = -2w(x) + (|x|^2 - 1)h(x)$. From

$$0 = \partial_\nu u \bigg|_{\partial B} = (-2\partial_\nu w + 2h) \bigg|_{\partial B} = (-2Ew + 2h) \bigg|_{\partial B}$$

and the harmonicity of $-2Ew + 2h$, we know that $-2Ew + 2h = 0$ on $B$. Hence

$$u(x) = -2w(x) + (|x|^2 - 1)Ew(x). \tag{3.28}$$

We have

$$\Delta u = 2nEw + 4E^2 w, \tag{3.29}$$

and so

$$\partial_\nu (\Delta u) \bigg|_{\partial B} = (2nE^2 w + 4E^3 w) \bigg|_{\partial B}. \tag{3.30}$$

If $z \in \partial B$, we have from $\partial_\nu u(z) = 0$ that

$$\Delta u(z) = \nabla^2 u(\nu, \nu)(z) = \nabla^2 u(z) + E^2 u(z). \tag{3.31}$$

For any $x \in B$, a simple calculation gives

$$Eu(x) = (|x|^2 - 1)(2Ew + E^2 w)(x) \tag{3.32}$$

and

$$E^2 u(x) = 2|x|^2(2Ew + E^2 w)(x) + (|x|^2 - 1)(2E^2 w + E^3 w)(x). \tag{3.33}$$

Thus

$$E^2 u(y) = 2(2Ew + E^2 w)(y), \quad \forall y \in \partial B, \tag{3.34}$$

which, combining with (3.29) and (3.31), gives

$$\Delta u(y) = ((2n - 4)Ew + 2E^2 w)(y), \quad \forall y \in \partial B. \tag{3.35}$$

Substituting (3.25), (3.30) and (3.33) into the equation:

$$\partial_\nu (\Delta u) + \beta \Delta u + \varsigma u = 0 \quad \text{on} \quad \partial B,$$
we infer
\begin{equation}
0 = (2nE^2w + 4E^3w + \beta((2n - 4)Ew + 2E^2w) - 2\varsigma w)_{\partial B}.
\end{equation}
It follows that
\begin{equation}
nE^2w + 2E^3w + \beta((n - 2)Ew + E^2w) = \varsigma w \quad \text{on} \quad B.
\end{equation}
Since \( w \) is harmonic on \( B \), there exist \( p_m \in D_m \) such that
\begin{equation}
w(x) = \sum_{m=0}^{\infty} p_m(x)
\end{equation}
for all \( x \in B \), the series converging absolutely and uniformly on compact subsets of \( B \) (cf. [6]). Substituting (3.38) into (3.37) and using \( E p_m = mp_m \), we infer
\begin{equation}
2m^3 + nm^2 + \beta(n^2 + (n - 2)m) = \varsigma, \quad m = 0, 1, \ldots
\end{equation}
which shows that all but one of the \( p'_m \)'s are zeros. Consequently, the eigenvalues are
\begin{equation}
\varsigma_k = k^2(n + 2k) + \beta k(k + n - 2), \quad k = 0, 1, \ldots
\end{equation}
the multiplicity of \( \varsigma_k \) is the dimension of \( D_k \) and the eigenspace corresponding to \( \varsigma_k \) is
\begin{equation}
E_k = \{-2w_k + k(|x|^2 - 1)w_k \mid w_k \in D_k\}.
\end{equation}
This completes the proof of Theorem 1.5. \( \square \)

\textbf{Proof of Theorem 1.6} From (1.22), we only need to show that the first non-zero eigenvalue \( \xi_1 \) of the problem (1.14) satisfies
\begin{equation}
\xi_1 > \frac{nc\lambda_1 \mu_1}{(n - 1)(\mu_1 + n\kappa)}.
\end{equation}
Let \( f \) be an eigenfunction corresponding \( \xi_1 \):
\begin{equation}
\begin{cases}
\Delta^2 f = 0 \quad \text{in} \quad M, \\
\frac{\partial}{\partial \nu} f = \frac{\partial}{\partial \nu} \Delta f + \xi_1 f = 0 \quad \text{on} \quad \partial M.
\end{cases}
\end{equation}
Set \( z = f|_{\partial M} \); then \( z \neq 0 \) and
\begin{equation}
\xi_1 = \frac{\int_M (\Delta f)^2}{\int_{\partial M} z^2}.
\end{equation}
Substituting \( f \) into Reilly's formula, we have
\begin{equation}
\int_M \{ (\Delta f)^2 - |\nabla^2 f|^2 \} = \int_M \operatorname{Ric}(\nabla f, \nabla f) + \int_{\partial M} II(\nabla z, \nabla z)
\geq -(n - 1) \kappa \int_M |\nabla f|^2 + c \int_{\partial M} |\nabla z|^2.
\end{equation}
Since \( \partial \nu f|_{\partial M} = 0 \), we have
\begin{equation}
\int_M (\Delta f)^2 \geq \mu_1 \int_M |\nabla f|^2.
\end{equation}
It follows from (3.38) that \( \int_{\partial M} z = 0 \) and so we have from the Poincaré inequality that
\begin{equation}
\int_{\partial M} |\nabla z|^2 \geq \lambda_1 \int_{\partial M} z^2.
\end{equation}
The Schwarz inequality implies that

\[ |\nabla^2 f|^2 \geq \frac{1}{n}(\Delta f)^2 \]

with equality holding if and only if \( \nabla^2 f = \frac{\Delta f}{n} \). Combining (3.44)-(3.48), we get

\[ \xi \geq \frac{nc\lambda_1\mu_1}{(n-1)(\mu_1 + n\kappa)}. \]

Let us show by contradiction that the equality in (3.49) can’t occur. In fact, if take equality sign, then we must have

\[ \nabla^2 f = \frac{\Delta f}{n} \text{ on } M. \]

Thus for a tangent vector field \( X \) of \( \partial M \), we have from (3.50) and \( \partial \nu f|_{\partial M} = 0 \) that

\[ 0 = \nabla^2 f(\nu, X) = X\nu f - (\nabla_X \nu)f = -\langle \nabla_X \nu, \nabla z \rangle. \]

In particular, we have

\[ II(\nabla z, \nabla z) = 0. \]

This is impossible since \( II \geq cI \) and \( z \) is not constant. This finishes the proof Theorem 1.6. \( \square \)

An interesting question related to the above proof is to find a sharp lower bound for \( \xi_1 \). We propose a

**Conjecture.** Let \( M \) be an \( n \)-dimensional compact Riemannian manifold with boundary and nonnegative Ricci curvature. Assume that the principal curvatures of \( \partial M \) are bounded below by a positive constant \( c \) and denote by \( \lambda_1 \) the first nonzero eigenvalue of the Laplacian of \( \partial M \). Then the first nonzero eigenvalue of the problem (1.14) satisfies

\[ \xi_1 \geq \frac{(n + 2)c\lambda_1}{n - 1} \]

with equality holding if and only if \( M \) is isometric to an \( n \)-dimensional Euclidean ball of radius \( 1/c \).

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