On absolutely and simply popular rankings

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Abstract Van Zuylen et al. [29] introduced the notion of a popular ranking in a voting context, where each voter submits a strictly-ordered list of all candidates. A popular ranking $\pi$ of the candidates is at least as good as any other ranking $\sigma$ in the following sense: if we compare $\pi$ to $\sigma$, at least half of all voters will always weakly prefer $\pi$. Whether a voter prefers one ranking to another is calculated based on the Kendall distance.

A more traditional definition of popularity—as applied to popular matchings, a well-established topic in computational social choice—is stricter, because it requires at least half of the voters who are not indifferent between $\pi$ and $\sigma$ to prefer $\pi$. In this paper, we derive structural and algorithmic results in both settings, also improving upon the results in [29]. We also point out strong connections to the famous open problem of finding a Kemeny consensus with 3 voters.

Keywords majority rule · Kemeny consensus · complexity · preference aggregation · popular matching

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1 Introduction

A fundamental question in preference aggregation is the following: given a number of voters who rank candidates, can we construct a ranking that expresses the preferences of the entire set of voters as a whole? A common way of evaluating how close the constructed ranking is to a voter’s preferences is the Kendall distance, which measures the pairwise disagreements between two rankings. Among others, a well-known rank aggregation method is the Kemeny ranking method [22], in which the winning ranking minimises the sum of its Kendall distances to the voters’ rankings.

For the preference aggregation problem, van Zuylen et al. [29] introduce a new rank aggregation method called popular ranking, which is also based on the Kendall distance. Each voter can compare two given rankings $\pi$ and $\sigma$, and prefers the one that is closer to her submitted ranking in terms of the Kendall distance. Van Zuylen et al. define $\pi$ to be a winning ranking in an instance if for any ranking $\sigma$, at least half of the voters prefer $\pi$ to $\sigma$ or are indifferent between them. This implies that there is no ranking $\sigma$ such that switching to $\sigma$ from $\pi$ would benefit a majority of all voters.

According to the definition of popularity in [29], even in a situation where exactly half of the voters are indifferent between rankings $\pi$ and $\sigma$, whilst the other half of the voters prefer $\sigma$ to $\pi$, $\sigma$ is not more popular than $\pi$. This example demonstrates how hard it is for the dissatisfied voters to find a ranking that overrules $\pi$—the definition requires them to find a profiting set of voters who build an absolute majority, that is, a majority of all voters for this endeavour.

A straightforward option would be to require only a simple majority, this is, a majority of the non-abstaining voters, to profit from switching to $\sigma$ from $\pi$. Excluding the abstaining voters in a pairwise majority voting rule is common practice [13]. It is also analogous to the classical popularity notion in the matching literature [1, 9, 26]. In this paper, we propose an alternative definition of a popular ranking. We define $\pi$ to be a simply popular ranking if for every ranking $\sigma$, at least half of the non-abstaining voters prefer $\pi$ to $\sigma$. This means that switching from $\pi$ to $\sigma$ would harm at least as many voters as it would benefit.

Related literature. Aggregating voters’ preferences given as ordered lists has been challenging researchers for decades. The most common approach to this problem is to search for a ranking that minimises the sum of the distances to the voters’ rankings. If the Kendall distance [23] is used as the metric on rankings, then this optimality concept corresponds to the Kemeny consensus [22, 24]. Deciding whether a given ranking is a Kemeny consensus is NP-complete, and calculating a Kemeny consensus is NP-hard [6] even if there are only 4 voters [7, 14], or at least 6 of them [2]. Interestingly, the complexity of the problem is open for 3 and 5 voters [2, 7].

Majority voting rules offer another natural way of aggregating voters’ preferences. The earliest reference for this might be from Condorcet [8], who uses
pairwise comparisons to calculate the winning candidate, establishing his famous paradox on the smallest voting instance not admitting any majority winner. The absolute and simple majority voting rules have both been extensively discussed in the setting where the goal is to choose the winning candidate [4, 5]. Vermeule [30] focuses on strategic minorities and demonstrates the effect of the simple majority rule compared to the absolute majority rule based on data from decisions made by the United States Congress. By undertaking a probabilistic analysis, Dougherty and Edward [13] discuss the differences between the two rules. Felsenthal and Machover [25] generalise simple voting games to ternary voting games by adding the possibility to abstain.

The concept of majority voting readily translates to other scenarios, where voters submit preference lists. One such field is the area of matchings under preferences, where popular matchings [18, 1, 26, Chapter 7, 9] serve as a voting-based alternative concept to the well-known notion of stable matchings [3, 17] in two-sided markets. In short, a popular matching $M$ is a simple majority winner among all matchings, because it guarantees that no matter what alternative matching is offered on the market, a weak majority of the non-abstaining agents will opt for $M$.

Besides two-sided matchings, majority voting has also been defined for the house allocation problem [1, 28], the roommates problem [15, 19], spanning trees [10], permutations [29], the ordinal group activity selection problem [11], and very recently, to branchings [21]. The notion of popularity is aligned with simple majority in all these papers, with one exception, namely [29], which defines popularity based on the absolute majority rule.

A part of this work revisits the paper from van Zuylen et al. [29]. They show that a popular ranking—according to their definition of popularity—need not necessarily exist. More precisely, they show that the acyclicity of a structure known as the majority graph is a necessary, but not sufficient condition for the existence of a popular ranking. They also prove that if the majority graph is acyclic, then one can efficiently compute a ranking, which may or may not be popular, but for which the voters have to solve an NP-hard problem to compute a ranking that a majority of them prefer.

Our contribution. We study both the weaker notion of popularity from [29] and the stronger notion of popularity analogous to the one in the matching literature, which excludes abstaining voters. Our most important results are as follows.

1. For at most 5 voters, the two notions are equivalent, but with 6 voters this does not hold anymore.
2. We give a sufficient condition for the two notions to be equivalent for a given ranking $\pi$.
3. In the case of 2 or 3 voters, one can find a popular ranking of either kind and verify the absolute or simple popularity of a given ranking in polynomial time.
4. The problem of verifying the absolute or simple popularity of a given ranking for 4 voters is polynomial-time solvable if and only if it is polynomial-time solvable for 5 voters.

5. If finding a ranking that is more popular in either sense than a given ranking in an instance with 4 (or 5) voters were polynomial-time solvable, then the famously open Kemeny consensus problem for 3 voters would also be polynomial-time solvable.

Structure of the paper. In Section 2, we introduce the necessary definitions and notations used in the following sections. Section 3 deals with the relationship between the two different popularity notions. In Section 4, we study the complexity of the problems of deciding whether a given ranking is absolutely or simply popular with a small number of voters. We also demonstrate the strong connection to the Kemeny consensus problem there. We lay out some problems that still remain to be answered in Section 5.

2 Preliminaries

We start this section with the formal definitions of various standard notions in voting theory in Section 2.1. Then, in Section 2.2, we introduce absolutely and simply popular rankings and the decision problems we will later analyse.

2.1 Rankings

We are given a set $C$ of $m$ candidates and a set $V$ of $n$ voters. A ranking $\pi$ is a permutation of the $m$ candidates. When exhibiting a specific ranking, we will often enclose parts of it in square brackets, e.g. we will write $[1,2],[3,4]$ instead of 1,2,3,4. These brackets can be ignored and are simply used for better readability in instances with specific structural properties. Each voter $v$ has her own ranking denoted by $\pi_v$. An example is depicted in Figure 1. The rank of candidate $a$ in ranking $\pi$ is the position (counting from 1) it appears at in $\pi$, and it is denoted by $\text{rank}_\pi[a]$. We say that voter $v$ prefers candidate $a$ to candidate $b$ if $\text{rank}_\pi[a] < \text{rank}_\pi[b]$. In Figure 1, voter $v_1$ prefers candidate 1 to candidate 2, and $\text{rank}_{\pi_{v_1}}[5] = 6$. A voting instance $\mathcal{I}$ comprises $C$, $V$, and the rankings $\pi_v$ for each $v \in V$. For convenience, we use the set of rankings and $\mathcal{I}$ interchangeably, since the former already contains all information on $C$ and $V$. For candidates $a$ and $b$, $\{a,b\}$ denotes the unordered pair of them, while $(a,b)$ is their ordered pair.

We say that voter $v_1$ (or ranking $\pi_{v_1}$) agrees with voter $v_2$ (or with ranking $\pi_{v_2}$) in the order of two distinct candidates $a$ and $b$ if $v_1$ and $v_2$ either equivocally prefer $a$ to $b$ or they equivocally prefer $b$ to $a$. Otherwise they disagree in the order of $a$ and $b$. The similarity between two rankings can be measured by various metrics defined on permutations. Possibly the most common metric, the Kendall distance [23], is defined below.
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Fig. 1 A voting instance $I$ with 6 voters $v_1,v_2,\ldots,v_6$ and 9 candidates $1,2,\ldots,9$.

**Definition 1** The *Kendall distance* $K(\pi,\sigma)$ between two rankings $\pi$ and $\sigma$ is defined as the number of pairwise disagreements between $\pi$ and $\sigma$, or, formally as

$$K(\pi,\sigma) = |\{(a,b) : \text{rank}_\pi[a] > \text{rank}_\pi[b] \text{ and } \text{rank}_\sigma[a] < \text{rank}_\sigma[b]\}| + |\{(a,b) : \text{rank}_\pi[a] < \text{rank}_\pi[b] \text{ and } \text{rank}_\sigma[a] > \text{rank}_\sigma[b]\}|.$$

The Kendall distance is also called bubble-sort distance, because it is equivalent to the number of pairwise swaps that the bubble sort algorithm \[16\] executes when converting ranking $\pi$ to ranking $\sigma$. To be more precise, let us first define a total order on $1,\ldots,n$ such that under this order, ranking $\sigma$ is sorted in increasing order. We define the *bubble swap path* from a ranking $\pi$ to $\sigma$ as the sequence $\pi_0 := \pi, \pi_1, \ldots, \pi_\ell := \sigma$ of intermediate permutations obtained when sorting $\pi$ using the bubble sort algorithm. We call the change $\pi_i \rightarrow \pi_{i+1}$ a *swap*. Alternatively we denote the swap by the consecutive candidates $a$ and $b$ it interchanges: $(b,a) \rightarrow (a,b)$. We say that a swap $(b,a) \rightarrow (a,b)$ is *good* for voter $v$ if $v$ prefers $a$ to $b$, otherwise this swap is *bad* for $v$. Note that if the swap $\pi_i \rightarrow \pi_{i+1}$ is good for $v$, then $K(\pi_{i+1},\pi_v) = K(\pi_i,\pi_v) - 1$ and, analogously, if the swap is bad for $v$, then $K(\pi_{i+1},\pi_v) = K(\pi_i,\pi_v) + 1$.

Let $V(a,b) \subseteq V$ be the set of voters who prefer candidate $a$ to $b$, i.e. $V(a,b) = \{v \in V : \text{rank}_\sigma(a) < \text{rank}_\sigma(b)\}$. The *majority graph* belonging to a voting instance is defined as the directed graph which has as vertices the candidates and an arc from candidate $a$ to candidate $b$ if a majority of the voters prefer $a$ to $b$, i.e. $|V(a,b)| > |V(b,a)|$. As mentioned in the introduction, Condorcet observed that the majority graph may contain a directed cycle. This has come to be known as Condorcet paradox \[8\]. A *tournament* is a majority graph that is complete, or, in other words, for every $a$ and $b$ in its voting instance either $|V(a,b)| > |V(b,a)|$ or $|V(a,b)| < |V(b,a)|$ holds. The majority graph of our instance in Figure\[1\] is depicted in Figure\[2\]. As the edges form no cycle, it is an acyclic majority graph, but since it is not a complete graph, it is not a tournament.

Ranking $\pi$ is a *topologically sorted ranking* in $I$ if $\text{rank}_\pi[a] < \text{rank}_\pi[b]$ holds for each pair of candidates $a$ and $b$ with $|V(a,b)| > |V(b,a)|$. Topologically sorted rankings correspond to the graph-theoretical topological sort of the vertices in the majority graph, and thus only exist if the majority graph is acyclic. Acyclic tournaments trivially have a unique topologically sorted
Fig. 2 The majority graph of the instance from Figure 1. Each gray set of 3 candidates denotes a specific subgraph called component. Arcs between these components symbolise the complete set of 9 arcs, between any two vertices from different components.

ranking. A topologically sorted ranking for the instance in Figure 1 with 9 candidates is \( \sigma = [1, 2, 3], [4, 5, 6], [7, 8, 9] \), as can be checked easily.

The Kemeny rank of a ranking \( \pi \) for a given voting instance with voters \( v_1, \ldots, v_n \) is defined as \( \sum_{i=1}^{n} K(\pi, \pi_{v_i}) \). If ranking \( \sigma \) has minimum Kemeny rank over all rankings, then \( \sigma \) is a Kemeny consensus [22]. The following well-known observation [12] will be useful in our proofs.

Observation 1 Each topologically sorted ranking is a Kemeny consensus ranking. For acyclic majority graphs, the set of topologically sorted rankings coincides with the set of Kemeny consensus rankings.

2.2 Preferences over rankings

Voters prefer rankings that are similar to their submitted ranking. More precisely, voter \( v \) prefers ranking \( \sigma \) to ranking \( \pi \) if \( K(\sigma, \pi_v) < K(\pi, \pi_v) \). Analogously, voter \( v \) abstains between \( \pi \) and \( \sigma \) if \( K(\sigma, \pi_v) = K(\pi, \pi_v) \). We will simply call \( v \) an abstaining voter if \( \pi \) and \( \sigma \) are clear from the context.

For example, let \( \sigma_1 = [1, 2, 3], [5, 6, 4], [9, 7, 8] \) and \( \sigma_2 = [1, 2, 3], [4, 5, 6], [7, 8, 9] \). Looking back at Figure 1, clearly \( v_4 \) prefers \( \sigma_2 \) to \( \sigma_1 \), since \( \pi_{v_4} = \sigma_2 \) and \( \pi_{v_4} \neq \sigma_1 \), that is, \( K(\pi_{v_4}, \sigma_2) = 0 < K(\pi_{v_4}, \sigma_1) \). Voter \( v_1 \) in the same instance is an abstaining voter since \( K(\pi_{v_1}, \sigma_2) = 4 = K(\pi_{v_1}, \sigma_1) \).

We now define the two different notions of popularity. The first notion of an absolutely popular ranking corresponds to the popular ranking as defined in [20].

Definition 2 Ranking \( \pi' \) is more popular than ranking \( \pi \) in the absolute sense if \( K(\pi', \pi_v) < K(\pi, \pi_v) \) for an absolute majority of all voters. Ranking \( \pi \) is absolutely popular in \( I \) if no ranking \( \pi' \) is more popular than \( \pi \) in the absolute sense, in other words, if there is no ranking \( \pi' \) such that \( K(\pi', \pi_v) < K(\pi, \pi_v) \) for an absolute majority of all voters in \( I \).

If we consider \( \sigma_3 = [2, 1, 3], [4, 5, 6], [7, 8, 9] \), then in the instance in Figure 1, \( \sigma_2 = [1, 2, 3], [4, 5, 6], [7, 8, 9] \) is more popular than \( \sigma_3 \) in the absolute sense. Notice that \( \sigma_3 \) and \( \sigma_2 \) only differ in their ordering of the pair of candidates.
{1, 2}. So since five out of six voters prefer candidate 1 to candidate 2, they form an absolute majority of all voters who prefer $\sigma_2$ to $\sigma_3$.

This definition requires more than half of the $n$ voters to prefer $\pi'$ to $\pi$ in order to declare $\pi'$ to be more popular than $\pi$. Abstaining voters make it hard to beat $\pi$ in such a pairwise comparison. However, if $\pi'$ only needs to receive more votes than $\pi$ among the voters not abstaining between these two rankings, then it can beat $\pi$. This leads to the notion of simple popularity.

Let $V_{abs}(\pi, \pi')$ be the set of voters who abstain in the vote between $\pi$ and $\pi'$, that is, $v \in V_{abs}(\pi, \pi')$ if and only if $K(\pi_v, \pi) = K(\pi_v, \pi')$. In Figure 1, $V_{abs}(\sigma_1, \sigma_2) = \{v_1, v_2, v_3\}$ with $\sigma_1$ and $\sigma_2$ as before.

**Definition 3** Ranking $\pi'$ is more popular than ranking $\pi$ in the simple sense if $K(\pi', \pi_v) < K(\pi, \pi_v)$ for a majority of the non-abstaining voters $V \setminus V_{abs}(\pi, \pi')$. Ranking $\pi$ is simply popular in $\mathcal{I}$ if no ranking $\pi'$ is more popular than $\pi$ in the simple sense, in other words, if there is no ranking $\pi'$ such that $K(\pi', \pi_v) < K(\pi, \pi_v)$ for a majority of the non-abstaining voters $V \setminus V_{abs}(\pi, \pi')$.

It follows directly from the two definitions above that simply popular rankings are absolutely popular as well, but absolutely popular rankings are not necessarily simply popular. In the instance in Figure 1, $\sigma_1 = [1, 2, 3], [5, 6, 4], [9, 7, 8]$ is more popular than $\sigma_2 = [1, 2, 3], [4, 5, 6], [7, 8, 9]$ in the simple sense, since $v_5$ and $v_6$ prefer $\sigma_1$ to $\sigma_2$, while $v_1, v_2$, and $v_3$ abstain. Notice that $\sigma_1$ is not more popular than $\sigma_2$ in the absolute sense, because two voters do not constitute an absolute majority of all 6 voters, only a majority of the non-abstaining 3 voters.

We now define a natural verification problem arising from the notions of absolutely and simply popular rankings, starting with absolutely popular rankings. We are given an input ranking $\pi$ and ask if there exists a ranking that is preferred to $\pi$ by a majority of voters. In other words, we ask whether $\pi$ is not absolutely popular.

| n-ABSOLUTELY-UNPOPULAR-RANKING-VERIFY (n-AURV) |
|-----------------------------------------------|
| **Input:** A voting instance $\mathcal{I}$ and a ranking $\pi$. |
| **Question:** Does there exist a ranking $\sigma$ preferred by a majority of all voters to $\pi$? |

The variant where we ask whether there exists a ranking $\sigma$ that is preferred to $\pi$ by a majority of the non-abstaining voters $V \setminus V_{abs}(\pi, \sigma)$ is called (n-SURV).

3 Absolutely and simply popular rankings

In this section, we study the relationship between absolutely and simply popular rankings. We first place absolutely and simply popular rankings in the context

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1 The term *absolutely popular* is a shortened version of *absolute-majority popular*, while *simply popular* abbreviates *simple-majority popular*. These abbreviations should be kept in mind when interpreting the observation, given later, that an absolutely popular ranking need not be simply popular in general.
of Kemeny consensus rankings in Section 3.1. Then in Section 3.2 we show that for as few as six voters, the two notions of popularity may not be equivalent and that at the heart of this lies Condorcet’s paradox. Building upon our 6-voter example, in Section 3.3 we derive a sufficient, but not necessary condition for an absolutely popular ranking to be simply popular as well. This condition opens a way to prove in Section 3.4 that for up to five voters the two notions are equivalent.

3.1 A subset of Kemeny consensus rankings

We first revisit two results from [29], established for absolute popularity, and translate them for the notion of simple popularity.

**Lemma 1** If a voting instance $I$ has a majority graph with a directed cycle, then there does not exist a simply popular ranking. If $I$ has an acyclic majority graph, then a topologically sorted ranking is not necessarily simply popular.

**Proof** These two statements hold for absolute popularity [29, Lemmas 2 and 3] and hence also for simple popularity, because simply popular rankings are also absolutely popular by definition. \(\square\)

**Example 1** A voting instance $I$ with 4 voters and 6 candidates. Ranking $\pi_{v4}$ is a Kemeny consensus, but $\sigma = [2,1][4,3],[6,5]$ is more popular than $\pi_{v4}$ in the absolute sense.

\[
\begin{align*}
\pi_{v1} &= [2,1],[4,3],[5,6] \\
\pi_{v2} &= [1,2],[4,3],[6,5] \\
\pi_{v3} &= [2,1],[3,4],[6,5] \\
\pi_{v4} &= [1,2],[3,4],[5,6]
\end{align*}
\]

Each absolutely popular ranking must be topologically sorted, as shown by the proof of [29, Lemma 2]. This results together with Lemma 1 and Observation 1 lead to the following set inclusion relationships on absolutely popular, simply popular, topologically sorted, and Kemeny consensus rankings.

**Observation 2** Simply popular rankings form a subset of absolutely popular rankings, absolutely popular rankings form a subset of topologically sorted rankings, and finally, topologically sorted rankings form a subset of Kemeny consensus rankings. In voting instances with an acyclic majority graph, topologically sorted rankings coincide with Kemeny consensus rankings.

Figure 3 illustrates these relations. In voting instances with a cyclic majority graph, Kemeny consensus rankings offer a preference aggregation method by relaxing the definition of topologically sorted rankings. Absolutely and simply popular rankings do exactly the opposite: they restrict the set of topologically sorted rankings in instances with an acyclic majority graph, in order to serve the welfare of the majority to an even larger degree than topologically sorted
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do. Absolute and simple popularity are desirable robustness properties of a ranking. However, the set of absolutely popular rankings may be empty even if a topologically sorted ranking exists [29]. Analogous to Condorcet consistency, in which a Condorcet winner is chosen whenever it exists, one could require to always choose a simply popular ranking if one exists, or, failing that, an absolutely popular ranking if that exists.

Fig. 3 The hierarchy of various optimality notions for rankings as a Venn diagram. A Kemeny consensus ranking always exists, but its subset of topologically sorted rankings might be empty.

3.2 Difference for 6 voters

Before presenting our example with $n = 6$, we present a useful technical lemma. Let $\{C_1, \ldots, C_k\}$ be a partition of $C$ into $k$ sets. We say that a ranking $\pi$ preserves $\{C_1, \ldots, C_k\}$ if each voter prefers $a$ to $b$ whenever $a \in C_i$ and $b \in C_j$ for some $i < j$.

**Lemma 2** Let $\{C_1, \ldots, C_k\}$ be a partition of $C$ and $\pi$ be a ranking that preserves $\{C_1, \ldots, C_k\}$. If a subset of voters $V' \subseteq V$ prefers a ranking $\sigma$ to $\pi$, then there exists a ranking $\zeta$ such that $\zeta$ preserves $\{C_1, \ldots, C_k\}$ and all voters in $V'$ prefer $\zeta$ to $\pi$.

Let $\tau_i$ be a permutation of the candidates in $C_i$. We denote by $\tau_1\tau_2\ldots\tau_n$ the permutation of the candidates $\bigcup_{i=1}^{m} C_i = C$, in which each candidate in $C_i$ is preferred to each candidate in $C_j$ whenever $i < j$, and candidates within a set $C_i$ are ranked according to $\tau_i$.

**Proof** Let $\zeta_i$ be the permutation on $C_i$ that orders candidates in $C_i$ according to their rank in $\sigma$, that is, $\text{rank}_{\zeta_i}[a] < \text{rank}_{\zeta_i}[b]$ if $\text{rank}_{\sigma}[a] < \text{rank}_{\sigma}[b]$. Let $\zeta := \zeta_1\zeta_2\ldots\zeta_k$. So $\zeta$ preserves $\{C_1, \ldots, C_k\}$. Suppose $v \in V$ prefers $a$ to $b$, and $\zeta$ orders $b$ before $a$. Then $a, b \in C_i$ for some $1 \leq i \leq m$. Since the relative order of candidates in $C_i$ is the same in $\zeta$ and $\sigma$ by construction, $\sigma$ also orders $b$ before $a$. In particular, every pair of candidates that contributes 1 to $K(\zeta, \pi_v)$
also contributes 1 to $K(\sigma, \pi_v)$. We conclude $K(\zeta, \pi_v) \leq K(\sigma, \pi_v)$, as desired.

**Theorem 3** There exists a voting instance with six voters that has an absolutely popular ranking which is not simply popular.

**Proof** We prove this statement for the voting instance $I$ from Figure 1.

**Claim 1** $\sigma_2 = [1, 2, 3], [4, 5, 6], [7, 8, 9]$ is absolutely popular.

**Proof** Let $C_1 = \{1, 2, 3\}, C_2 = \{4, 5, 6\}, C_3 = \{7, 8, 9\}$. Note that all voters preserve $\{C_1, C_2, C_3\}$. By Lemma 2 in order to check if $\sigma_2$ is absolutely popular it suffices to generate the $6^3$ rankings that preserve $\{C_1, C_2, C_3\}$, and compare each of them to $\sigma_2$. We checked all of these rankings using a program code, which is available under https://github.com/SonjaKrai/PopularRankingsExampleCheck.

**Claim 2** $\sigma_2 = [1, 2, 3], [4, 5, 6], [7, 8, 9]$ is not simply popular.

**Proof** Ranking $\sigma_1 = [1, 2, 3], [5, 6, 4], [9, 7, 8]$ is more popular than $\sigma_2$ in the simple sense, because voters $v_1, v_2, v_3$ abstain, voter $v_4$ prefers $\sigma_2$ to $\sigma_1$, and finally, voters $v_5$ and $v_6$ prefer $\sigma_1$ to $\sigma_2$. Appendix 6.1 contains the detailed calculations behind this.

This finishes the proof of our theorem.

For the sake of completeness, we remark that the voting instance $I$ from Figure 1 has 4 further absolutely popular rankings, each of which is simply popular as well. For an instance that admits an absolutely popular ranking but does not admit a simply popular ranking, please consult Appendix 6.2.

### 3.3 A conditional equivalence

We now present a sufficient condition under which simple popularity follows from absolute popularity. We start by showing that for a ranking $\pi$ that is not simply popular, there is a condition under which we can compute a ranking that is also more popular than $\pi$ in the absolute sense.

**Theorem 4** If $\sigma_1$ is more popular than $\pi$ in the simple sense and the majority graph of the voters in $V_{\text{abs}}(\sigma_1, \pi)$ is acyclic, then there is a ranking $\sigma_2$ that is more popular than $\pi$ in the absolute sense. Such a $\sigma_2$ can be computed in polynomial time.

**Proof** The theorem immediately follows from Observation 2 if $\pi$ is not a topologically sorted ranking. In case there is a topologically sorted ranking, then it will serve as $\sigma_2$, otherwise $\sigma_2$ can be constructed by the algorithm developed for this purpose in [29, Lemma 2]. Furthermore, if $V_{\text{abs}}(\sigma_1, \pi) = \emptyset$, then $\sigma_1$ is preferred to $\pi$ by a majority of all voters, and thus the theorem is proved by taking $\sigma_2 := \sigma_1$. 
From here on we therefore assume that $\pi$ is topologically sorted and that $V_{\text{abs}}(\sigma_1, \pi) \neq \emptyset$. Let $D(\sigma_1, \pi) \neq \emptyset$ be the set of pairs of candidates in the order of which $\sigma_1$ and $\pi$ disagree.

**Claim 3** If $v \in V_{\text{abs}}(\sigma_1, \pi)$, then $v$ agrees with $\pi$ on the order of exactly half of the pairs in $D(\sigma_1, \pi)$, and disagrees on the other half. In particular, $|D(\sigma_1, \pi)|$ is even.

*Proof* Any pair of candidates $\{a, b\}$ such that $\sigma_1$ and $\pi$ agree on the order $(a, b)$, adds 1 to both $K(\pi_v, \sigma_1)$ and $K(\pi_v, \pi)$ if $\pi_v$ has them in the relative order $(b, a)$, and adds 0 to both otherwise. So to be impartial, i.e. to have $K(\pi_v, \sigma_1) = K(\pi_v, \pi)$, voter $v \in V_{\text{abs}}(\sigma_1, \pi)$ must agree with $\sigma_1$ on the order of exactly half of the pairs in $D(\sigma_1, \pi)$, and disagree on the other half. Since $V_{\text{abs}}(\sigma_1, \pi) \neq \emptyset$ can be assumed, $|D(\sigma_1, \pi)|$ must be even. $\square$

**Claim 4** There exist consecutive candidates $(a^*, b^*)$ in $\sigma_1$ such that at least half of the voters in $V_{\text{abs}}(\sigma_1, \pi)$ prefer $b^*$ to $a^*$.

*Proof* Assume the contrary, i.e. that for any two consecutive candidates in $\sigma_1$, a majority of voters in $V_{\text{abs}}(\sigma_1, \pi)$ agree with $\sigma_1$. For an arbitrary pair of candidates $\{a, b\}$, at least half of the voters in $V_{\text{abs}}(\sigma_1, \pi)$ must then also agree with their order in $\sigma_1$, as otherwise this would imply a directed cycle in their majority graph, which is acyclic by assumption.

Since $\sigma_1 \neq \pi$, there is some ordered pair $(a^*, b^*)$ that is consecutive in $\sigma_1$ and $b^*$ is ordered somewhere before $a^*$ in $\pi$, that is $\text{rank}_\pi[b^*] < \text{rank}_\pi[a^*]$. Note that $(a^*, b^*) \in D(\sigma_1, \pi)$. In particular, by our starting assumption, a majority of voters in $V_{\text{abs}}(\sigma_1, \pi)$ agrees with $\sigma_1$ on the order $(a^*, b^*)$ and hence disagrees with $\pi$.

We now introduce the indicator variable $I_{v, \pi, \{a, b\}}$, which is set to 1 if $\pi_v$ and $\pi$ disagree on the order of candidates $a$ and $b$, and it is set to 0 otherwise. We sum up the disagreements of voters in $V_{\text{abs}}(\sigma_1, \pi)$ with $\pi$ over pairs in $D(\sigma_1, \pi)$ and obtain a contradiction.

\[
|V_{\text{abs}}(\sigma_1, \pi)| \frac{|D(\sigma_1, \pi)|}{2} = \sum_{\{a, b\} \in D(\sigma_1, \pi)} \sum_{v \in V_{\text{abs}}(\sigma_1, \pi)} I_{v, \pi, \{a, b\}} 
\]  

\[
> \sum_{\{a, b\} \in D(\sigma_1, \pi)} \frac{|V_{\text{abs}}(\sigma_1, \pi)|}{2} = |V_{\text{abs}}(\sigma_1, \pi)| \frac{|D(\sigma_1, \pi)|}{2} 
\]  

The right-hand side of Line 1 is a formulation of disagreements in terms of the indicator variable. Due to Claim 3 the number of disagreements that abstaining voters have with $\pi$ is exactly half of $|D(\sigma_1, \pi)|$, expressed on the left-hand side of Line 1. The inequality in Line 2 follows, because for all pairs in $D(\sigma_1, \pi)$, at least half of the abstaining voters disagree with $\pi$, and, additionally, there exists a pair $\{a^*, b^*\}$ such that a majority of voters in $V_{\text{abs}}(\sigma_1, \pi)$ disagree with $\pi$, as we proved above. Finally, reordering the terms leads back to the same formula as on the left-hand side in Line 1, creating a contradiction. $\square$
The pair of candidates \((a^*, b^*)\) in Claim 4 leads us to a suitable ranking \(\sigma_2\). Let \(\sigma_2\) be the permutation we get from \(\sigma_1\) by the swap \((a^*, b^*) \rightarrow (b^*, a^*)\). In Claims 5 and 6 we will show that two groups of voters prefer this \(\sigma_2\) to \(\pi\), and that these two groups constitute a majority of all voters.

**Claim 5** All voters \(v \in V_{abs}(\sigma_1, \pi)\) who prefer \(b^*\) to \(a^*\) also prefer \(\sigma_2\) to \(\pi\).

**Proof** If \(v \in V_{abs}(\sigma_1, \pi)\) prefers \(b^*\) to \(a^*\), then the following holds.

\[
K(\pi_v, \sigma_2) = K(\pi_v, \sigma_1) - 1
\]

\[< K(\pi_v, \sigma_1) = K(\pi_v, \pi)
\]

Line 3 is true, because \(\sigma_2\) is obtained from \(\sigma_1\) by performing a swap that is good for voter \(v\). The equality in Line 4 holds since \(v\) abstains between \(\sigma_1\) and \(\pi\). The inequality in Line 4 proves the claim. \(\square\)

The second group of voters we investigate consists of voters who prefer \(\sigma_1\) to \(\pi\). This group by assumption makes up a majority of the non-abstaining voters \(V \setminus V_{abs}(\sigma_1, \pi)\). Let voter \(v\) belong to this group.

**Claim 6** If \(K(\pi_v, \sigma_1) < K(\pi_v, \pi)\) for voter \(v \in V \setminus V_{abs}(\sigma_1, \pi)\), then \(K(\pi_v, \sigma_1) + 2 \leq K(\pi_v, \pi)\).

**Proof** Only the pairs in \(D(\sigma_1, \pi)\) contribute differently to \(K(\pi_v, \sigma_1)\) and to \(K(\pi_v, \pi)\). In particular, a pair in \(D(\sigma_1, \pi)\) adds 1 to either \(K(\pi_v, \sigma_1)\) or \(K(\pi_v, \pi)\), and 0 to the other. If \(k\) is the number of pairs on whose order \(\sigma_1\) and \(\pi\) agree, but \(\pi_v\) disagrees, then by the previous argument, \(K(\pi_v, \sigma_1) + K(\pi_v, \pi) = |D(\sigma_1, \pi)| + 2k\), which is even by Claim 3. Since \(K(\pi_v, \sigma_1) < K(\pi_v, \pi)\) by assumption and their sum \(|D(\sigma_1, \pi)| + 2k\) is even, \(K(\pi_v, \sigma_1) + 2 \leq K(\pi_v, \pi)\) must hold. \(\square\)

Since a swap of consecutive candidates in a ranking can increase the distance to any other ranking by at most one, Claim 6 implies that for any voter \(v\) who prefers \(\sigma_1\) to \(\pi\), the following holds.

\[
K(\pi_v, \sigma_2) \leq K(\pi_v, \sigma_1) + 1 < K(\pi_v, \pi)
\]

We conclude that voters who prefer \(\sigma_1\) to \(\pi\) also prefer \(\sigma_2\) to \(\pi\).

Claims 4 and 5 show that at least half of the voters in \(V_{abs}(\sigma_1, \pi)\) prefer \(\sigma_2\) to \(\pi\), and Claim 6 proves that more than half of the voters outside of \(V_{abs}(\sigma_1, \pi)\) prefer \(\sigma_2\) to \(\pi\). The two sets of voters thus constitute a majority of all voters who prefer \(\sigma_2\) to \(\pi\). \(\square\)

By rephrasing Theorem 4, we get the following.

**Corollary 1** If ranking \(\pi\) is absolutely popular, and for any ranking \(\sigma\), \(V_{abs}(\sigma, \pi)\) has an acyclic majority graph, then \(\pi\) is also simply popular. Furthermore, if \(V_{abs}(\sigma, \pi)\) has an acyclic majority graph for each absolutely popular ranking \(\pi\) and any ranking \(\sigma\) in an instance \(I\), then absolutely and simply popular rankings in \(I\) coincide.
The following example demonstrates that if $\sigma_1$ is more popular than $\sigma_2$ in the simple sense, but the majority graph of the voters in $V_{\text{abs}}(\sigma_1, \sigma_2)$ is cyclic (contradicting our assumption in Theorem 4), then $\sigma_2$ might be absolutely popular. The calculations supporting the example can be found in Appendix 6.1.

**Example 2** Consider the instance described in Figure 1. The voters abstaining between $\sigma_1 = [1, 2, 3], [5, 6, 4], [9, 7, 8]$ and $\sigma_2 = [1, 2, 3], [4, 5, 6], [7, 8, 9]$ have a cyclic majority graph. In this instance, $\sigma_2$ is the unique absolutely popular ranking but is not simply popular, since $\sigma_1$ is more popular than $\sigma_2$ in the simple sense. Notice that for both $\sigma_1$ and $\sigma_2$, for every pair of consecutive candidates in each, a majority of the abstaining voters agree with the order of these candidates.

### 3.4 At most 5 voters

From Theorem 4 we can deduce that for a small number of voters, the two notions of popularity are equivalent. This is due to the fact that we need at least 3 abstaining voters in order for $V_{\text{abs}}(\sigma, \pi)$ to have a cyclic majority graph.

**Theorem 5** A ranking $\sigma$ is absolutely popular in a voting instance $\mathcal{I}$ with at most five voters if and only if it is simply popular in $\mathcal{I}$.

**Proof** From Observation 2 we know that simply popular rankings are also absolutely popular, which allows us to concentrate only on one direction of the statement, namely that if a ranking is not simply popular, then it also cannot be absolutely popular. Let us assume that ranking $\pi$ is not simply popular. By definition there exists a ranking $\sigma$ that is preferred to $\pi$ by a majority of non-abstaining voters $V \setminus V_{\text{abs}}(\sigma, \pi) \neq \emptyset$.

If at least one voter prefers $\pi$ to $\sigma$, then at least two voters must prefer $\sigma$ to $\pi$, and the remaining at most two abstaining voters can only form an acyclic majority graph. From this follows by Theorem 4 that $\pi$ is not absolutely popular.

On the other hand, if no voter prefers $\pi$ to $\sigma$, then $K(\pi_v, \sigma) \leq K(\pi_v, \pi)$ holds for all voters. For $V \setminus V_{\text{abs}}(\sigma, \pi) \neq \emptyset$, by assumption there is a voter $v^*$ who prefers $\sigma$ to $\pi$, that is, for whom $K(\pi_{v^*}, \sigma) < K(\pi_{v^*}, \pi)$. We thus have $\sum_{v=1}^{n} K(\pi_{v}, \sigma) < \sum_{v=1}^{n} K(\pi_{v}, \pi)$. This means that $\pi$ is not a Kemeny consensus and by Observation 2 $\pi$ is not absolutely popular. $\square$

### 4 On the complexity of $n$-aurv and $n$-surv

In this section, we analyse the complexity of the verification problems for the two popularity notions. We prepare for this by giving a sufficient condition for absolute popularity in Section 4.1. This condition will then be used in Section 4.2 where we prove the polynomial solvability of both problems in the case of $n \leq 3$. For $4 \leq n \leq 5$, we reach the same conclusion in Section 4.3.
however, only for special instances. Then, in Section 4.4, NP-hardness is proved for \( n = 6 \). We finish by drawing attention to the connection with the complexity of the famous Kemeny consensus problem in Section 4.5. This explains why we only succeeded to prove polynomial-time solvability for special cases of \( 4 \leq n \leq 5 \) — if either one of our problems is in \( \mathbb{P} \) for \( n = 4 \) or \( n = 5 \) in the general case, then one can find a Kemeny consensus for 3 voters in polynomial time. We reach these results via both Karp and Turing reductions.

4.1 A sufficient condition for \( n \)-aurv

We call a ranking \( \pi \) \( c \)-sorted for \( 0 < c \leq 1 \) if for all pairs of candidates \( \{a, b\} \) with \( \text{rank}_c[a] < \text{rank}_c[b] \), at least a \( c \)-fraction of the voters prefers \( a \) to \( b \). A topologically sorted ranking \( \pi \) is by definition \( \frac{1}{2} \)-sorted. In [29], van Zuylen et al. show that even a topologically sorted ranking is not necessarily absolutely popular. Here we ask for which constant \( \frac{1}{2} < c \leq 1 \) does this negative result change to a positive result, guaranteeing a yes answer for \( n \)-aurv.

**Theorem 6** \( c = \frac{3}{4} \) is the smallest constant \( c \) for which the following holds: If a voting instance \( \mathcal{I} \) admits a \( c \)-sorted ranking \( \pi \), then \( \pi \) is absolutely popular.

**Proof** We first show that any \( \frac{3}{4} \)-sorted ranking is absolutely popular.

**Claim 7** Let \( \mathcal{I} \) be a voting instance and \( \pi \) be a ranking. If for every pair of candidates \( \{a, b\} \) with \( \text{rank}_c[a] < \text{rank}_c[b] \), at least a \( \frac{3}{4} \)-fraction of the voters agree with the order \( (a, b) \), then \( \pi \) is absolutely popular.

**Proof** Let \( \sigma \) be any ranking different from \( \pi \). We will show that there is no voter set of cardinality \( \lfloor \frac{n}{2} \rfloor + 1 \), in which every voter prefers \( \sigma \) to \( \pi \). Let \( V' \) be an arbitrary set of \( \lfloor \frac{n}{2} \rfloor + 1 \) voters. On the bubble swap path \( \pi_0 := \pi, \pi_1, \ldots, \pi_k := \sigma \) from \( \pi \) to \( \sigma \), each swap \( \pi_i \rightarrow \pi_{i+1} \) is bad for at least

\[
\sum_{v \in V'} K(\pi_{i+1}, \pi_v) - K(\pi_i, \pi_v) \geq 0.
\]

In particular, not every \( v \in V' \) can prefer \( \sigma \) to \( \pi \). Since \( V' \) was an arbitrary set of size \( \lfloor \frac{n}{2} \rfloor + 1 \), no voter set of at least this size can prefer \( \sigma \) to \( \pi \), and thus, \( \pi \) must be absolutely popular. \( \square \)

We now show that in fact \( c = \frac{3}{4} \) is tight, meaning that for any \( c < \frac{3}{4} \), we can construct an instance \( \mathcal{I} \) and a \( c \)-sorted ranking \( \pi \) such that \( \pi \) is not absolutely popular in \( \mathcal{I} \).
Claim 8 For arbitrary \( c < \frac{3}{4} \) there exists a voting instance \( \mathcal{I} \) and a topologically sorted, but not absolutely popular ranking \( \pi \) in it, such that a \( c \)-fraction of the voters agree with \( (a, b) \) for every pair of candidates \( \{a, b\} \) with \( \text{rank}_\pi[a] < \text{rank}_\pi[b] \).

Proof We create voting instance \( \mathcal{I} \) with \( 4j \) voters and \( 4j + 2 \) candidates. Let \( c = \frac{3}{4} - \epsilon \) for some \( \epsilon > 0 \). Next, choose \( j \) large enough such that \( (\frac{3}{4} + \epsilon)|V| \geq j + 1 \), which is equivalent to \( j \geq \frac{1}{\epsilon} \). We create a set \( V_1 \) of \( 2j - 1 \) voters, each of whom has ranking \( \pi = [1, 2], [3, 4], \ldots, [4j + 1, 4j + 2] \). Note that we group the candidates into \( 2j + 1 \) blocks of 2 candidates each, in the form \([a, b] \). The set of the remaining \( 2j + 1 \) voters is called \( V_2 \), and their set of rankings is depicted in Table 1. For informal intuition, the first ranking orders the first \( j + 1 \) blocks \([a, b] \) in decreasing order and the remaining \( j \) blocks in increasing order. With each new row, the \( j + 1 \) decreasingly ordered blocks are shifted by 1 to the right. When the end of the row is reached, the \( j + 1 \) decreasingly ordered blocks are wrapped around to continue at the beginning of the row. This construction is similar to the one in [29, Lemma 3].

| \[2, 1\] | \[4, 3\] | \ldots | \[2, j - 1\] | \[2, j + 2, j + 1\] | \[2, j + 3, j + 2\] | \[2, j + 5, j + 4\] | \[2, j + 5, j + 6\] | \ldots | \[4, j + 1, j + 2\] |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| \[1, 2\] | \[4, 3\] | \ldots | \[2, j - 1\] | \[2, j + 2, j + 1\] | \[2, j + 3, j + 2\] | \[2, j + 5, j + 4\] | \[2, j + 5, j + 6\] | \ldots | \[4, j + 1, j + 2\] |
| \[2, j\] | \[4, 3\] | \ldots | \[2, j - 1\] | \[2, j + 2, j + 1\] | \[2, j + 3, j + 2\] | \[2, j + 5, j + 4\] | \[2, j + 5, j + 6\] | \ldots | \[4, j + 1, j + 2\] |

Table 1 The blocks in decreasing order are highlighted.

For each \((2i - 1, 2i)\) for \( 1 \leq i \leq 2j + 1 \), every \( v \in V_1 \) agrees with this pair and also exactly \( j \) of the voters in \( V_2 \) agree with it. In total, \( 3j - 1 \) voters thus agree with this pair. This gives the following fraction of all voters.

\[
\frac{3j - 1}{4j} = \frac{3}{4} - \frac{1}{4j} > \frac{3}{4} - \epsilon
\]

It is trivial to see for all other pairs of voters \([a, b]\) that if \( a < b \) then all voters prefer \( a \) to \( b \). Hence in particular \( \pi \) is \((\frac{3}{4} - \epsilon)\)-sorted.

We now show that ranking \( \pi = [2, 1], [4, 3], \ldots, [4j + 1, 4j + 2] \) is preferred to \( \pi \) by a majority of all voters, namely by all \( 2j + 1 \) voters in \( V_2 \). Each voter in \( V_2 \) has exactly \( j + 1 \) blocks with decreasing order and \( j \) blocks with increasing order. Thus, each of them would rather have all pairs in decreasing order than all pairs in increasing order.

This finishes the proof of our theorem.

As an aside, following the proof from Theorem 3, it is natural to ask about the existence of \( \epsilon \)-popular rankings: rankings that are preferred to all other rankings by some \( \epsilon \)-fraction of voters. The following theorem (whose proof can be found
in Appendix 6.3) shows that for any $\varepsilon > 0$, there is a voting instance that does not admit any $\varepsilon$-popular ranking.

**Theorem 7** There is a voting instance with $n$ voters and $n$ candidates such that for any ranking $\pi$, there exists another ranking $\sigma$ such that $n - 1$ voters prefer $\sigma$ to $\pi$ and only one voter prefers $\pi$ to $\sigma$.

4.2 Polynomial-time solvability for $n \leq 3$

Since we have shown in Theorem 5 that absolute and simple popularity are equivalent notions for $n \leq 5$, we will refer to them as popularity if $n \leq 5$ in a voting instance $I$.

We first show that for $n \leq 3$, the problems $n$-aurv and $n$-surv are polynomial-time solvable. We establish this by proving that for at most three voters, the set of topologically sorted rankings coincides with the set of popular rankings. Since verifying whether a given ranking is topologically sorted can be carried out in polynomial time, $n$-aurv and $n$-surv turn out to be polynomial-time solvable for $n = 2, 3$.

**Lemma 3** In a voting instance $I$ with $n = 2$, a ranking is popular if and only if it is topologically sorted.

**Proof** From Observation 2 we know that all popular rankings must be topologically sorted. Let $D(\pi_{v_1}, \pi_{v_2})$ be the set of pairs of candidates $\{a, b\}$ that $\pi_{v_1}$ and $\pi_{v_2}$ order differently. Consider any ranking $\pi$. Each pair $\{a, b\} \in D(\pi_{v_1}, \pi_{v_2})$ adds 1 to either $K(\pi_{v_1}, \pi)$ or $K(\pi_{v_2}, \pi)$, and each pair $\{a, b\} \notin D$ can at best add 0 to both and at worst 1 to both $K(\pi_{v_1}, \pi)$ and $K(\pi_{v_2}, \pi)$. From this follows that

$$|D(\pi_{v_1}, \pi_{v_2})| \leq K(\pi_{v_1}, \pi) + K(\pi_{v_2}, \pi).$$

If $\sigma$ is a topologically sorted ranking, then by definition there is no pair of candidates $\{a, b\}$ that adds 1 to both $K(\pi_{v_1}, \sigma)$ and $K(\pi_{v_2}, \sigma)$. Thus, $K(\pi_{v_1}, \pi) + K(\pi_{v_2}, \sigma) = |D(\pi_{v_1}, \pi_{v_2})|$. If $\pi$ is preferred to $\sigma$ by a majority, then both voters prefer $\pi$ to $\sigma$, which leads to

$$|D(\pi_{v_1}, \pi_{v_2})| \leq K(\pi_{v_1}, \pi) + K(\pi_{v_2}, \pi) < K(\pi_{v_1}, \pi) + K(\pi_{v_2}, \sigma) = |D(\pi_{v_1}, \pi_{v_2})|. \tag{5}$$

Since this is a contradiction, $\sigma$ must be absolutely popular. By Theorem 5, $\sigma$ is also simply popular. \qed

**Lemma 4** In a voting instance $I$ with $n = 3$, a ranking is popular if and only if it is topologically sorted.

**Proof** Just as for the $n = 2$ case, Observation 2 implies here as well that all popular rankings must be topologically sorted. Let $\pi$ be a topologically sorted ranking in $I$. Note that whenever $\text{rank}_\pi[a] < \text{rank}_\pi[b]$ holds for candidates
a and b, at least half of the voters, that is, at least two voters prefer a to b. 
So for any two voters, at least one of them prefers a to b, implying that π is also 
topologically sorted for any two of the three voters. In particular, π is absolutely popular for any two voters by Lemma 3, showing that there is no ranking that they both prefer to π. Hence π must be absolutely popular in I and by Theorem 5 also simply popular.

Using the fact that verifying whether a given ranking is topologically sorted 
can be done in polynomial time, we derive the following result.

**Theorem 8** For $n \leq 3$, n-aurv and n-surv are polynomial-time solvable.

### 4.3 Equivalence of the cases $n = 4$ and $n = 5$

If we have 4 or 5 voters, it turns out a topologically sorted ranking may not be 
absolutely or simply popular anymore. In the case of an acyclic tournament as 
the majority graph, finding and verifying a popular ranking are both polynomial-
time solvable. We further show that 4-aurv (4-surv) in general is polynomial-
time solvable if and only if 5-aurv (5-surv) is.

**Lemma 5** If a voting instance $I$ with $n = 4$ has an acyclic tournament as 
its majority graph, then the unique topologically sorted ranking is the unique 
popular ranking.

**Proof** Since the majority graph of $I$ is a tournament, the unique topologically 
sorted ranking $\pi$ is $\frac{1}{4}$-sorted. The lemma then follows from Theorem 6 and 
Observation 2. □

**Lemma 6** In a voting instance $I$ with $n = 4$ and an acyclic majority graph, 
if $\pi$ is popular in at least one of the instances formed by three of the voters, 
then $\pi$ is popular in $I$.

**Proof** Let $\pi$ be absolutely popular in the instance formed by $v_1, v_2,$ and $v_3$. 
Then by definition there exists no ranking $\sigma$ that is preferred by two of $v_1, v_2, v_3$, 
as this would contradict the fact that $\pi$ is absolutely popular in the instance 
formed by voters $v_1, v_2, v_3$. In particular, there does not exist a ranking $\sigma$ 
preferred by a majority of $v_1, v_2, v_3, v_4$, as this would require at least 3 voters 
and hence at least 2 of $v_1, v_2, v_3$. We conclude that $\pi$ is absolutely popular in $I$. 
By Theorem 5, it follows that $\pi$ is simply popular as well. □

We now present an example in which there is ranking $\pi$ that is not topolog-
ically sorted such that $\pi$ is more popular than a topologically sorted ranking.

**Observation 9** In a voting instance $I$ with $n = 4$ and an acyclic majority 
graph, a ranking that is not topologically sorted can be absolutely (and simply) 
more popular than a topologically sorted ranking.
Proof Consider the instance $\mathcal{I}$ below, with $n = 4$ and $m = 10$.

\[
\begin{align*}
\pi_{v_1} &= [1, 2], [3, 4], [5, 6], [7, 8], [9, 10] \\
\pi_{v_2} &= [1, 2], [4, 3], [6, 5], [7, 8], [10, 9] \\
\pi_{v_3} &= [1, 2], [4, 3], [6, 5], [8, 7], [9, 10] \\
\pi_{v_4} &= [2, 1], [3, 4], [5, 6], [8, 7], [10, 9]
\end{align*}
\]

It is easy to verify that $\pi_{v_1}$ is a topological sorted ranking of $\mathcal{I}$. However, $\sigma = [2, 1], [4, 3], [6, 5], [8, 7], [10, 9]$ is preferred by $v_2$, $v_3$, and $v_4$ to $\pi$, since $K(\pi_{v_i}, \pi_{v_1}) = 3$ and $K(\pi_{v_i}, \sigma) = 2$ for $2 \leq i \leq 4$. Since a majority of voters prefer candidate 1 to candidate 2, $\sigma$ is not topologically sorted. For the sake of completeness, we remark that the topologically sorted ranking $[1, 2], [4, 3], [6, 5], [8, 7], [10, 9]$ is popular. $\square$

We now discuss a strongly related decision problem called $n$-ALL-CLOSER-RANKING, which will come useful when establishing results for the cases $n = 4$ and $n = 5$. For a voting instance with $n$ voters and a given ranking $\pi$, we ask whether there is a ranking that all the voters prefer to $\pi$.

**$n$-ALL-CLOSER-RANKING**

**Input:** A voting instance $\mathcal{I}$ and a ranking $\pi$.

**Question:** Does there exist a ranking $\sigma$ preferred by each of the $n$ voters to $\pi$?

The next theorem reveals some features of this problem.

**Theorem 10** In a voting instance $\mathcal{I}$ with $n = 3$ and an acyclic majority graph, we can decide in polynomial time if there exists a ranking preferred by all voters to a given ranking $\pi$ and if it does, output it.

**Proof** Our first two technical observations apply for the case of one and two voters, respectively.

**Observation 11** If $K(\sigma, \pi_v) > 0$ for a voter $v$ and a ranking $\sigma$, then there exists a swap in $\sigma$ that is good for $v$.

**Proof** Suppose there is no swap that is good for $v$. So $\sigma$ is a topologically sorted ranking for $v$, and for one voter this means $\pi_v = \sigma$, i.e. $K(\sigma, \pi_v) = 0$. $\square$

**Observation 12** If there is no swap in ranking $\pi$ that is good for both $v_1$ and $v_2$, then there is no ranking $\sigma$ preferred to $\pi$ by both $v_1$ and $v_2$.

**Proof** Since there is no swap that is good for both $v_1$ and $v_2$, $\pi$ is a topologically sorted ranking for $v_1$ and $v_2$. By Lemma 3 this means that $\pi$ is absolutely popular, so there exists no ranking preferred by a majority, that is, preferred by both voters. $\square$

We are now ready to proceed to the case of 3 voters. First we compute the unique topologically sorted ranking $\sigma$ in $\mathcal{I}$ in polynomial time [20]. We distinguish four cases, based on how many of the three voters prefer $\sigma$ to $\pi$, which can be checked in polynomial time.
1. If $\sigma$ is preferred to $\pi$ by all 3 voters, then we are done.
2. Suppose that two of the voters, $v_1$ and $v_2$, prefer $\sigma$ to $\pi$. Suppose that $v_1$ and $v_2$ are distances $d_1 = K(\pi, \pi_{v_1}) - K(\sigma, \pi_{v_1})$ and $d_2 = K(\pi, \pi_{v_2}) - K(\sigma, \pi_{v_2})$ closer to $\sigma$ than to $\pi$, respectively, and that $v_3$ is further away from it by distance $d_3 = K(\sigma, \pi_{v_3}) - K(\pi, \pi_{v_3})$. Notice that we have $d_1 \geq 0$ for all $1 \leq i \leq 3$. Without loss of generality we can further assume that $d_1 \leq d_2$. Let $\pi_0 := \pi, \ldots, \pi_k := \sigma$ be the bubble sort swap path from $\pi$ to $\sigma$.

**Claim 9** For the above defined distances, $d_1 - d_3 \geq 2$ and similarly, $d_2 - d_3 \geq 2$ hold.

**Proof** Firstly, no swap is bad for both $v_1$ and $v_3$, since every swap $\pi_i \to \pi_{i+1}$ that is bad for $v_3$ must be good for both $v_2$ and $v_1$, because $\sigma$ is the topologically sorted ranking of $I$. Secondly, there is at least one swap that is good for both $v_1$ and $v_3$. If there was no swap that is good for both $v_1$ and $v_3$, then there cannot exist a ranking preferred to $\pi$ by all voters, since there cannot exist a ranking preferred by both $v_1$ and $v_2$ by Observation 12. Good swaps for voters $v_1, v_2$, and $v_3$ add 1 to $d_1$ and $d_2$, and subtract 1 from $d_3$, respectively. It follows that $d_1 - d_3 \geq 2$ and since $d_2 \geq d_1$, the same argument also implies $d_2 - d_3 \geq 2$. \( \square \)

We are now ready to transform $\sigma$ to a ranking that is preferred by all 3 voters to $\pi$.

**Procedure**

Let $\sigma_0 = \sigma$. In the $i$th round we search for a swap in $\sigma_{i-1}$ that is good for $v_3$ and if found, perform the swap to obtain $\sigma_i$ for $i \geq 1$. Otherwise we output an error message. We stop the procedure in round $i = d_3 + 1$ and output $\sigma_{d_3+1}$.

**Claim 10** If the process terminates outputting $\sigma_{d_3+1}$, then $v_1, v_2,$ and $v_3$ prefer $\sigma_{d_3+1}$ to $\pi$. Otherwise, there does not exist a ranking preferred by all voters to $\pi$.

**Proof** The process terminates before reaching $\sigma_{d_3+1}$ if and only if $K(\sigma_j, \pi_{v_3}) = 0$ for some integer $j < d_3 + 1$, otherwise by Observation 11 we can find a swap that is good for $v_3$. Again by Observation 11 $K(\pi_{v_3}, \sigma) \leq d_3$. Since $K(\pi_{v_3}, \sigma) - K(\pi_{v_3}, \pi) = d_3$, $K(\pi_{v_3}, \pi) = 0$ i.e. $\pi_{v_3} = \pi$. So clearly there cannot exist a ranking preferred by all voters, including $v_3$, to $\pi$.

If we successfully obtain $\sigma_{d_3+1}$, it will be at most $d_3 + 1$ swaps away from $\sigma$. So $\sigma_{d_3+1}$ is closer to $\pi_{v_1}$ than $\pi$ by

$$d'_1 := K(\pi, \pi_{v_1}) - K(\sigma_{d_3+1}, \pi_{v_1}) \geq K(\pi, \pi_{v_1}) - K(\sigma, \pi_{v_1}) - (d_3 + 1) = d_1 - d_3 - 1 > 0.$$ 

Similarly $d'_2 > 0$, that is, $\pi_{v_2}$ is closer to $\sigma_{d_3+1}$ than to $\pi$.

Also, by construction

$$d'_3 := K(\pi, \pi_{v_3}) - K(\sigma_{d_3+1}, \pi_{v_3}) = K(\pi, \pi_{v_3}) - K(\sigma, \pi_{v_3}) + K(\sigma, \pi_{v_3}) - K(\sigma_{d_3+1}, \pi_{v_3}) = -d_3 + d_3 + 1 = 1.$$
So πₙ is closer to σₙ+₁ than to π. That is, all of v₁, v₂ and v₃ prefer to σₙ+₁ to π. □

3. Suppose that only one voter, v₁ prefers σ to π. We show that this case cannot occur. There exists a bubble sort swap on the path from π to σ that is good for both v₂ and v₃, else there cannot be a ranking preferred by all voters by Observation 12. Since every bubble sort swap is good for at least one of v₂ and v₃, without loss of generality, let v₂ be the voter for whom at least half of the bubble sort swaps are good. This means that v₂ has more good swaps on the path than bad swaps, i.e. v₂ also prefers σ to π, a contradiction to v₁ being the only voter who prefers σ to π.

4. Finally, suppose that no voter prefers σ to π, i.e. K(πᵱᵢ, σ) ≥ K(πᵱᵢ, π) for all 1 ≤ i ≤ 3. Since σ is topologically sorted and hence a Kemeny consensus (see Observation 1), ∑₁≤i≤₃ K(πᵱᵢ, σ) ≤ ∑₁≤i≤₃ K(πᵱᵢ, π) holds. From these two inequalities follows that K(πᵱᵢ, σ) = K(πᵱᵢ, π) for all 1 ≤ i ≤ 3. That is, π is also a Kemeny consensus, hence there does not exist a ranking preferred to π by all voters.

Having discussed all 4 cases, we now can output a ranking preferred by all the voters to a given ranking π—or a proof for its non-existence—in polynomial time. □

Lemma 7 For n ≥ 3, if at least one of (2n – 1)-SURV, (2n – 1)-AURV, and (2n – 2)-AURV is polynomial-time solvable, then n-ALL-CLOSER-RANKING is polynomial-time solvable.

Proof Assume first that (2n – 2)-AURV ∈ P holds. Consider an instance of n-ALL-CLOSER-RANKING with input ranking π and voters v₁, . . . , vₙ. To this instance of n-ALL-CLOSER-RANKING we construct the following instance of (2n – 2)-AURV. We copy π as the given input ranking and create 2n – 2 voters, n – 2 of them with ranking π and the other n voters corresponding to v₁, . . . , vₙ. Since voters with ranking π clearly prefer π to any other ranking, if there exists a ranking preferred by a majority (at least n) of the 2n – 2 voters, then these n voters must be v₁, . . . , vₙ. If a ranking is preferred by v₁, . . . , vₙ to π, then it is a solution to n-ALL-CLOSER-RANKING. Hence there is a ranking σ preferred by v₁, . . . , vₙ to π if and only if σ is a solution to n-ALL-CLOSER-RANKING. For (2n – 1)-AURV and (2n – 1)-SURV we simply add another voter with ranking π, and otherwise keep the proof intact. □

Lemma 8 Let n ≥ 3 be a constant. If n-ALL-CLOSER-RANKING is polynomial-time solvable, then (2n – 1)-AURV and (2n – 2)-AURV are both polynomial-time solvable.

Proof If n-ALL-CLOSER-RANKING has a polynomial-time algorithm A, then we can solve (2n – 2)-AURV by applying A to each of the \( \binom{2n-2}{n} \) voter groups of size n, which itself is a polynomial-time procedure if n is a constant. If one of the calls to A returns yes, return yes, else return no. It is easy to see that this procedure returns yes if and only if there is some group of n voters that prefers
another ranking i.e. if and only if there is a majority that prefers another ranking. A similar argument can be applied for \((2n - 1)\)-\textit{aurv}. □

An immediate consequence of Lemmas 7 and 8 is the following result.

\textbf{Theorem 13} \(4\)-\textit{aurv} (resp. \(4\)-\textit{surv}) is polynomial-time solvable if and only if \(5\)-\textit{aurv} (resp. \(5\)-\textit{surv}) is polynomial-time solvable.

\textit{Proof} With \(n = 3\) in Lemma 8, \(5\)-\textit{aurv} is \(\in\ P\) implies \(3\)-\textit{all-closer-ranking} \(\in P\). Due to Lemma 7, \(3\)-\textit{all-closer-ranking} \(\in P\) implies \(4\)-\textit{aurv} \(\in P\). By a similar argument \(4\)-\textit{aurv} \(\in P\) implies \(5\)-\textit{aurv} \(\in P\). By Theorem 5, an analogous result holds for \(4\)-\textit{surv} and \(5\)-\textit{surv}. □

4.4 \textit{NP}-hardness for \(n = 6\)

We now improve upon the \textit{NP}-hardness result of [29, Theorem 4] on the search version of \(7\)-\textit{aurv} from 7 voters to 6 voters, and also extend it to simply popular rankings with 6 or 7 voters.

\textbf{Theorem 14} The search versions of \(6\)-\textit{aurv}, \(6\)-\textit{surv}, and \(7\)-\textit{surv} are all \textit{NP}-hard.

\textit{Proof} We modify the proof from [29, Theorem 4], which shows that the search version of \(7\)-\textit{aurv} is \textit{NP}-hard. In that proof, 4 of the 7 voters have rankings \(\pi_1, \pi_2, \pi_3, \pi_4\), respectively, and the remaining 3 voters have ranking \(L(\sigma)\). The authors (implicitly) prove that it is \textit{NP}-hard to construct a ranking \(\zeta\) that all 4 voters with rankings \(\pi_1, \pi_2, \pi_3, \pi_4\) prefer to \(L(\sigma)\). We will first show that \(6\)-\textit{aurv} and \(7\)-\textit{surv} both lead to this \textit{NP}-hard problem.

We start with \(7\)-\textit{surv}. For any ranking \(\zeta \neq L(\sigma)\), the 3 voters with lists \(L(\sigma)\) must prefer \(L(\sigma)\) to \(\zeta\). In order for \(\zeta\) to be more popular than \(L(\sigma)\) in the simple sense, ranking \(\zeta\) must be preferred to \(L(\sigma)\) by more than 3 voters. This happens if and only if all 4 voters with rankings \(\pi_1, \pi_2, \pi_3, \pi_4\) prefer \(\zeta\) to \(L(\sigma)\).

For \(6\)-\textit{aurv}, we have two voters with lists \(L(\sigma)\) instead of three. The same reduction holds as for \(7\)-\textit{surv}, since a majority of all 6 voters, that is, the 4 voters with rankings \(\pi_1, \pi_2, \pi_3, \pi_4\) must prefer \(\zeta\) to \(L(\sigma)\).

We keep the same instance for \(6\)-\textit{surv}. Now only two voters prefer \(L(\sigma)\) to \(\zeta\), and thus \(\zeta\) is more popular than \(L(\sigma)\) in the simple sense if and only if at least 3 of the remaining 4 voters prefer \(\zeta\) to \(L(\sigma)\), and none of these four voters prefer \(L(\sigma)\) to \(\zeta\). Even though it is not directly observed by van Zuylen et al., their \textit{NP}-hardness proof carries over to this case without modification. □

4.5 The relationship with the Kemeny consensus

In this last technical section, we show that if the search version of any of \(4\)-\textit{aurv}/\(5\)-\textit{aurv} (\(4\)-\textit{surv}/\(5\)-\textit{surv}) is polynomial-time solvable, then so is the well-known Kemeny consensus problem for 3 voters, whose complexity is
currently unknown. Consider the following problem: we are given a ranking \( \pi \) as well as three voters’ rankings \( \pi_{v_1}, \pi_{v_2}, \pi_{v_3} \). Our task is to output a ranking \( \sigma \) that has smaller Kemeny rank than \( \pi \), or reports that none exists. We call this search problem \textit{n}-\text{smaller-Kemeny-rank}.

\textbf{Theorem 15} A Kemeny consensus for \( n \) voters can be computed in polynomial time if and only if \textit{n}-\text{smaller-Kemeny-rank} is in \( \mathcal{P} \).

\textit{Proof} Assume that \textit{n}-\text{smaller-Kemeny-rank} has a polynomial-time algorithm \( A \). We simply choose an arbitrary ranking \( \pi_1 \) for the Kemeny consensus problem and apply \( A \) to find \( \pi_2 \) with smaller Kemeny rank than \( \pi_1 \), and continue this way until we have found a Kemeny consensus. The number of calls to \( A \) can be naively bounded by \( n^{m(m-1)/2} \), which is the maximum Kemeny rank of a ranking. Similarly, if we can find a Kemeny consensus for \( n \) voters in polynomial time, then we can check if it has smaller Kemeny rank than \( \pi \) in the input of the \textit{n}-\text{smaller-Kemeny-rank} problem. \( \square \)

By an argument similar to the one in [29, Theorem 5], we prove the following result on the complexity of \textit{3}-\text{smaller-Kemeny-rank}.

\textbf{Theorem 16} If \textit{3}-\text{smaller-Kemeny-rank} is \( \mathcal{NP} \)-hard, then the search version of \textit{3}-\text{all-closer-ranking} is also \( \mathcal{NP} \)-hard.

\textit{Proof} We create an instance \( I' \) of \textit{3}-\text{all-closer-ranking} as follows. The three voters in \( I' \) will be \( v'_1, v'_2, \) and \( v'_3 \). We create a set of 3m candidates as \( C' = C^1 \cup C^2 \cup C^3 \), where \( C^j = \{ c^j_r : 1 \leq r \leq n \} \) for each \( 1 \leq j \leq 3 \). Intuitively, \( C' \) consists of three distinguishable copies of \( C \). Given any ranking \( \sigma \) of the \( m \) candidates in \( I \), let \( \sigma^i \) be the ranking obtained from \( \sigma \) by replacing each candidate \( c_r \) by \( c^i_r \), preserving the original order in \( \sigma \). Let \( \pi' \) be a permutation of the candidates in \( C' \). We denote by \( \pi^1 \pi^2 \ldots \pi^m \) the permutation of the candidates, in which each candidate in \( C^i \) is preferred to each candidate in \( C^j \) whenever \( i < j \), and candidates within a set \( C^i \) are ranked according to \( \pi^i \). Now we form the preference lists of the voters in \( I' \) as follows.

\[
\begin{align*}
\pi_{v'_1} &= \pi^1_{v_1} \pi^2_{v_2} \pi^3_{v_3} \\
\pi_{v'_2} &= \pi^1_{v_2} \pi^2_{v_3} \pi^3_{v_1} \\
\pi_{v'_3} &= \pi^1_{v_3} \pi^2_{v_1} \pi^3_{v_2}
\end{align*}
\]

Finally we create ranking \( \pi' = \pi^1 \pi^2 \pi^3 \) for the input of \textit{3}-\text{all-closer-ranking}, besides \( I' \).
Claim 11 Ranking $\sigma$ in $I$ has a smaller Kemeny rank than $\pi$ if and only if its counterpart in $I'$, ranking $\sigma'$ is preferred by all of $v'_1, v'_2,$ and $v'_3$ to $\pi'$.

Proof Suppose first that $\sigma$ has a smaller Kemeny rank than $\pi$ in $I$, that is, $\sum_{p=1}^{3} K(\sigma_{vp}, \sigma) < \sum_{p=1}^{3} K(\pi_{vp}, \pi)$. Let $\sigma' = \sigma^1\sigma^2\sigma^3$. Note that for each $1 \leq i \leq 3,$

$$K(\sigma_{vp}, \sigma') = \sum_{p=1}^{3} K(\sigma_{vp}, \sigma) < \sum_{p=1}^{3} K(\pi_{vp}, \pi) = K(\sigma_{vp}, \pi').$$

So indeed each $v'_j$ for $1 \leq i \leq 3$ prefers $\sigma'$ to $\pi'$.

For the converse direction, we first informally summarise the argument. We will argue that if there is a ranking $\sigma'$ in $I'$ preferred to $\pi'$ by all voters, then we can extract a ranking $\sigma$ in $I$ with smaller Kemeny rank than $\pi$ in two steps. First of all, by a previous lemma, we can break up $\sigma'$ into three different rankings, each defined on a different candidate set. Secondly, one of these rankings translated back to our instance $I$ will be a ranking with a smaller Kemeny rank than $\pi$, as we will argue using the averaging principle.

This argument relies on every $\pi_{vi}, 1 \leq i \leq 3$, appearing once in each “column” of $I'$, hence justifying the cyclic shift used in $I'$.

Suppose that $\sigma'$ is preferred to $\pi'$ by all three voters $v'_1, v'_2, v'_3$. By Lemma 2 we can assume that $\sigma'$ preserves $\{C^1, C^2, C^3\}$. So we can also assume that $\sigma' = \zeta_1^i \zeta_2^j \zeta_3^k$, where $\zeta_j^i$ is a ranking of the candidates in $C^j$ for $1 \leq j \leq 3$, that is, these are three different rankings. We let $\zeta_j^i$ be the ranking that is obtained from $\zeta_j^i$ by replacing candidate $c'_i$ with candidate $c'_r$ for $1 \leq i, n \leq 3$ and $1 \leq r \leq n$, preserving the original order in $\zeta_j^i$. Let $\tau_j = c_1^i c_2^j c_3^k$, so that intuitively, we copy the same ranking three times, on different candidate sets.

We will show that for some $1 \leq j \leq 3$, $\tau_j$ is also preferred to $\pi'$ by all the voters $v'_1, v'_2, v'_3$.

Notice that $\sum_{j=1}^{3} K(\tau_j, \pi_{vi}) = \sum_{j=1}^{3} K(\tau_j, \pi_{vi})$. Since $K(\sigma', \pi_{vi}) < K(\pi', \pi_{vi})$ for all $1 \leq i \leq 3$ and $K(\pi', \pi_{vi}) = K(\pi', \pi_{vi}) = K(\pi', \pi_{vi})$, it follows that

$$\sum_{j=1}^{3} K(\tau_j, \pi_{vi}) = \sum_{i=1}^{3} K(\sigma', \pi_{vi}) < \sum_{i=1}^{3} K(\pi', \pi_{vi}) = 3K(\pi', \pi_{vi}).$$

So there must exist an index $j$, $1 \leq j \leq 3$, such that $K(\tau_j, \pi_{vi}) < K(\pi', \pi_{vi})$.

But then

$$\sum_{i=1}^{3} K(\zeta_j^i, \pi_{vi}) = \sum_{i=1}^{3} K(\zeta_j^i, \pi_{vi}) = K(\tau_j, \pi_{vi}) <$$

$$K(\pi', \pi_{vi}) = \sum_{i=1}^{3} K(\pi, \pi_{vi}),$$

which means that $\zeta_j^i$ has smaller Kemeny rank than $\pi$, and from this, $\tau_j$ is preferred to $\pi'$ by all voters as desired. \qed
This finishes the proof of our theorem. □

We observe that a slight tweak to the above proofs lets us show NP-hardness for 4 problems related to 3-ALL-CLOSER-RANKING.

**Observation 17** If finding a ranking with a smaller Kemeny rank than a given ranking \( \pi \) for 3 voters is NP-hard, then finding a ranking \( \zeta \) that exactly one / at least one / exactly two / at least two of the 3 voters prefer \( \pi \), while no voter prefers \( \pi \) to \( \zeta \) is also NP-hard.

**Proof** We only need to argue why the converse direction still holds with the weaker assumption in Theorem 16. Note that in the proof of Theorem 16, Inequality 6 still holds if only one / at least one / exactly two / at least two of the three voters is non-abstaining and prefers \( \sigma' \) to \( \pi' \), while the other voters abstain. □

**Corollary 2** If finding a ranking that is more popular than a given ranking \( \pi \) in an instance of 4-AURV, 4-SURV, 5-AURV, or 5-SURV were polynomial-time solvable, then we could find a Kemeny consensus for 3 voters in polynomial time.

**Proof** This proof is illustrated in Figure 4. By Lemma 7, if the search version of 4-AURV or 5-AURV is in P, then the search version of 3-ALL-CLOSER-RANKING is also in P. Now, if the latter is true, then by Theorem 16, 3-SMALLER-KEMENY-RANK is also in P. This would finally imply that finding a Kemeny consensus for 3 voters is in P, by Theorem 15. An analogous result holds for 4-SURV and 5-SURV by Theorem 5. □

![Diagram](image)

**Fig. 4** If any of 4-AURV, 5-AURV, 4-SURV, and 5-SURV is solvable in polynomial time, then via the depicted implications, finding a Kemeny consensus for 3 voters is \( \in P \).

## 5 Summary and open questions

We studied absolutely popular rankings, defined in [29], and also introduced the notion of simply popular rankings analogous to the concept of popularity
found in the matching literature which ignores abstaining voters. Then we showed that a ranking \( \pi \) is absolutely popular if and only if it is simply popular assuming that the majority graph of the abstaining voters between \( \pi \) and any other ranking \( \sigma \) is acyclic. Using this result we also proved that the two notions of popularity are equivalent for voting instances with at most five voters. For instances with six voters, however, we showed that this equivalence does not hold anymore.

We found the smallest constant \( c \), for which any \( c \)-sorted ranking for an instance is absolutely popular. For two or three voters, a topologically sorted ranking turned out always to be popular with respect to both of the popularity notions. For four voters this also holds as long as the majority graph of the voters is a tournament, but it does not hold in general. We explained that the problem of deciding whether there exists a ranking \( \sigma \) that is preferred to a given ranking \( \pi \) by simple or absolute a majority of voters for instances with four of five voters boils down to the problem of deciding for three voters whether there is a ranking \( \pi \) by simple or absolute a majority of voters for instances with four of five voters boils down to the problem of deciding for three voters whether there is a ranking \( \pi \) they all prefer to \( \sigma \). This problem, we showed, is polynomial-time solvable if the majority graph of the three voters is acyclic, but open in general. Importantly, if it were polynomial-time solvable, this would imply the polynomial-time solvability of the well-known Kemeny consensus problem for three voters, whose complexity is currently open.

The study of popular rankings can be extended into various directions. We now list some open problems that our work poses, starting with a question already raised by van Zuylen et al. \[29\], which we made some progress on.

1. Determine the complexity of deciding whether an absolutely or simply popular ranking exists for an instance with arbitrary \( n \). Our Lemmas \[3\] and \[5\] show that for at most 3 voters and for 4 voters with an acyclic tournament as the majority graph, the existence of absolutely/simply popular rankings can be checked efficiently. Besides this, Theorem \[6\] gives a sufficient condition for the existence of an absolutely popular ranking for instances with arbitrary \( n \).

2. Determine the complexity of \textsc{3-all-closer-ranking}.

3. Construct an example showing that for any \( n > 5 \), the two notions of popularity are not equivalent. Theorem \[6\] might prove to be helpful here.

Finally, popular rankings can be defined and studied in instances where ties in the rankings are allowed, or the rankings are not necessarily complete. Besides the Kendall distance, other metrics on rankings can also be applied, such as the Spearman distance.

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6 Appendix

6.1 Calculations for Example 2

We remind the reader that the voter’s rankings are as follows.

\[
\pi_{v_1} = [1, 2, 3], [6, 4, 5], [8, 9, 7] \\
\pi_{v_2} = [2, 3, 1], [4, 5, 6], [9, 7, 8] \\
\pi_{v_3} = [3, 1, 2], [5, 6, 4], [7, 8, 9] \\
\pi_{v_4} = [1, 2, 3], [4, 5, 6], [7, 8, 9] = \sigma_2 \\
\pi_{v_5} = [1, 2, 3], [5, 4, 6], [9, 7, 8] \\
\pi_{v_6} = [1, 2, 3], [5, 6, 4], [7, 9, 8]
\]

Consider the two rankings of the candidates \(\sigma_1 = [1, 2, 3], [5, 6, 4], [9, 7, 8]\) and \(\sigma_2 = [1, 2, 3], [4, 5, 6], [7, 8, 9]\). Below we discuss the roles of the voters and we justify them with calculations and observations. Trivially, \(v_4\) prefers \(\sigma_2\) to \(\sigma_1\).

Voters \(v_1, v_2, v_3\) abstain in the vote between \(\sigma_2\) and \(\sigma_1\). We first discuss the three impartial voters and justify why they indeed are impartial between \(\sigma_1\) and \(\sigma_2\). Note that each of \(v_1, v_2, v_3\) agrees on one triple with \(\sigma_1\), and on one triple with \(\sigma_2\). For the remaining one or two triples, each of these three voters agrees with neither the corresponding triple in \(\sigma_1\), nor the one in \(\sigma_2\), but instead have distance 2 to both of these. For example, voter \(v_1\) agrees on the first triple with both \(\sigma_1\) and \(\sigma_2\), but agrees on the other two triples with neither of them, and \(K([8, 9, 7], [9, 7, 8]) = K([8, 9, 7], [7, 8, 9]) = 2\) and \(K([6, 4, 5], [4, 5, 6]) = K([6, 4, 5], [5, 6, 4]) = 2\). Voter \(v_2\) agrees with \(\sigma_2\) on the second triple, with \(\sigma_1\) on the third triple, while the distances to those triples she disagrees with are \(K([4, 5, 6], [5, 6, 4]) = K([9, 7, 8], [7, 8, 9]) = 2\). Since \(\sigma_1\) and \(\sigma_2\) agree on the first triple, these both have the same distance to the first triple in \(\pi_{v_2}\), \(K([2, 3, 1], [1, 2, 3]) = 2\). This can be checked similarly for voter \(v_3\).

Notice that there are three directed cycles in the majority graph of these three voters, one for candidates 1, 2, 3, one for candidates 4, 5, 6 and one for candidates 7, 8, 9. So the abstaining voters have a cyclic majority graph, justifying the condition in Theorem 4.

Voters preferring \(\sigma_1\) to \(\sigma_2\). Finally, we argue that \(v_5\) and \(v_6\) prefer \(\sigma_1\) to \(\sigma_2\). Both of them agree with the first triple with both of \(\sigma_1\) and \(\sigma_2\), as well with one other triple in \(\sigma_2\), being 2 swaps away from this triple in \(\sigma_1\). The remaining triple is distance 2 from both the corresponding triples in \(\sigma_1\) and \(\sigma_2\) away.

Absolutely but not simply popular ranking. By Theorem 3, \(\sigma_2\) is absolutely popular, and by the above argument it is not simply popular, since \(\sigma_1\) is more popular than \(\sigma_2\) in the simple sense. This example also justifies the acyclicity condition for the majority graph of the abstaining voters in Theorem 4.
6.2 An instance admitting an absolutely popular ranking, but no simply popular ranking

\[
\pi_{v_1} = [1, 2, 3], [6, 4, 5], [8, 9, 7]
\]
\[
\pi_{v_2} = [2, 3, 1], [4, 5, 6], [9, 7, 8]
\]
\[
\pi_{v_3} = [3, 1, 2], [5, 6, 4], [7, 8, 9]
\]
\[
\pi_{v_4} = [1, 2, 3], [4, 5, 6], [8, 9, 7]
\]
\[
\pi_{v_5} = [1, 2, 3], [5, 6, 4], [7, 8, 9]
\]
\[
\pi_{v_6} = [2, 3, 1], [4, 5, 6], [7, 8, 9]
\]
\[
\pi_{v_7} = [2, 3, 1], [5, 6, 4], [8, 9, 7]
\]
\[
\pi_{v_8} = [2, 3, 1], [5, 6, 4], [8, 9, 7]
\]

Our computer simulations (available under https://github.com/SonjaKrai/PopularRankingsExampleCheck) confirmed that the unique absolutely popular ranking is \(\pi_{v_7} = \pi_{v_8} = [2, 3, 1], [5, 6, 4], [8, 9, 7]\). Ranking \([1, 2, 3], [4, 5, 6], [7, 8, 9]\) is more popular than \(\pi_{v_7}\) in the simple sense.

6.3 Proof of Theorem 7

Proof Consider the extended Condorcet paradox, i.e. \(n\) voters with rankings of \(n\) candidates as follows:

\[
\pi_{v_1} = 1, 2, \ldots, n-1, n
\]
\[
\pi_{v_2} = 2, 3, \ldots, n, 1
\]
\[\vdots\]
\[
\pi_{v_n} = n, 1, \ldots, n-2, n-1
\]

Let \(\pi\) be an arbitrary ranking of the \(n\) candidates. We will show that there is a ranking \(\sigma\) preferred by \(n-1\) of the voters. Exactly \(n-1\) voters of the above instance agree with each ordered pair in the set \(A := \{(1, 2), (2, 3), \ldots (n-1, n), (n, 1)\}\). We also know that there must be at least one ordered pair in \(A\) with which \(\pi\) disagrees. Let \((a, b)\) be such a pair.

Let \(\sigma\) be the ranking obtained from \(\pi\) by swapping \(a\) and \(b\). Let \(c\) be a candidate ranked between \(a\) and \(b\) in \(\pi\). Each voter \(v\) who prefers \(b\) to \(a\) must also prefer \(c\) to \(a\) or \(b\) to \(c\). So together the pairs \((a, c)\) and \((b, c)\) add at least one to \(K(\pi_v, \pi)\) and at most one to \(K(\pi_v, \sigma)\). Pair \((a, b)\) adds 1 to \(K(\pi_v, \pi)\) and 0 to \(K(\pi_v, \sigma)\). Since \(\sigma\) ranks \(b\) higher than \(a\), it must also prefer \(c\) to \(a\) or \(b\) to \(c\). This implies that at most one of the pairs \((a, c)\) and \((b, c)\) adds one to \(K(\pi_v, \sigma)\). From definition of \(\sigma\) follows that \(K(\pi_v, \sigma) < K(\pi_v, \pi)\). \(\square\)