A 4D asymptotically flat rotating black hole solution including supertranslation corrections

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Abstract

One of the problems in the current asymptotic symmetry would be to extend the black hole to the rotating one. Therefore, in this paper, we obtain a four-dimensional asymptotically flat rotating black hole solution including the supertranslation corrections.
1 Introduction

Asymptotic symmetry (or BMS symmetry) \[1, 2\] is the diffeomorphism to change the shape of 4D asymptotically flat space time in the range the falloff conditions allow.

An important conjecture in the context of the 4D asymptotic symmetry has been provided in \[3\], which is that the symmetry of the diffeomorphism in the vicinity of the infinite null region should be the direct product of the following two symmetries: 1) Virasoro symmetry and its antiholomorphic part on the conformal sphere and 2) super-translation acting on the retarded/advanced time directions to which these conformal spheres attach.

After this conjecture is provided, the central charges of the asymptotic symmetry have been analyzed \[4, 5\]. On the other hand, the following 2 directions have been investigated and are still ongoing: i) relation between the gravitational S-matrix and the gravitational soft-theorem \[6, 7, 8, 9, 10, 11, 12, 13, 14\] and to get the CFT with the S-matrix equivalent to the gravitational S-matrix in the 4D asymptotically flat space time \[15, 16, 17, 18, 19, 20\] and ii) resolution of the black hole information paradox\(^*\) by the consideration that the evaporated information of the matters would be finally reflected in one of the infinite number of the degenerated space time configurations by the asymptotic symmetry \[51, 52, 53\].

Based on the gravitational memory effect \[54, 55, 56\], some observational studies of the asymptotic symmetry are also ongoing \[59, 60, 61, 62, 63, 64\]. There are variation of the gravitational memory effect: spin memory effect \[65\], color memory effect \[66\], and electromagnetic memory effect \[67, 68, 69\].

The black holes addressed so far in the context of the asymptotic symmetry are always not rotating, and its extension to the rotating one would be one of the important problems in the current context of the asymptotic symmetry. Therefore, we in this paper obtain a rotating black hole solution with the displacement of the supertranslation.

Concretely, we first obtain a rotating supertranslated black hole space time, where the corrections of both the supertranslation and black hole rotation are involved to the linear order. Then using some coordinate transformation rule we obtain from this, we finally obtain a rotating black hole space time solution where the supertranslation corrections are involved to the linear order but the black hole rotation is fully involved.

As for our linear order supertranslation corrections, it is considered it is enough if it is to the linear order for the reason we mention in Sec.\(3\). Therefore, as we fully involve the black hole rotation, our result could be considered to be sufficiently involve the supertranslation corrections and the black hole rotation.

We mention the organization of this paper. In Sec.\(2\) we first obtain a supertranslated

\(^*\)Fig.9 in \[21\] would be a better sketch for the whole view of current ideas for information paradox. For some of review papers, i) for introductory descriptions and comments, \[22, 23, 24, 25, 26, 27\], ii) for reviews qualitatively describing ideas, \[21, 28\], iii) for black hole complementarity and AMPS Firewall \[29, 30, 31\], iv) for fuzz ball and string theory \[32, 33, 34, 35, 36, 37, 38, 39, 40\], v) for remnant \[11, 12, 13\], vi) for page curve and island \[14\], vii) for AdS/CFT \[15, 16, 17, 18\], viii) for ER=EPR \[19\], ix) for the case in CGHS model, \[50, 51, 52\], x) for some other ones, \[53\].
black hole solution with the linear order supertranslation and the linear order black hole rotation, then using the technique obtained there, we finally obtain a supertranslated black hole solution with the linear order supertranslation corrections but the fully involved black hole rotation. In Fig.1, we sketch what we will do in Sec. 2. We mention this in more detail in what follows.

In Sec. 2.1, we start with the rotating black hole space time given in the Boyer-Lindquist (BL) coordinate system. Then, since the expressions of the displaced coordinates by supertranslation are given in the isotropic coordinate system, we first rewrite these BL coordinate system into the Schwarzschild type coordinate system by taking the black hole rotation to the linear order, then rewrite these into the isotropic coordinate system. In Sec. 2.2, we involve the supertranslation corrections in that isotropic coordinate system.

Once we involved the supertranslation corrections, in Sec. 2.3 and 2.4 we back these supertranslated isotropic coordinate system into the Schwarzschild coordinate system, then in Sec. 2.5 back these into the original rotating space time. By this, we obtain a rotating supertranslated black hole space time with linear-order supertranslation corrections and black hole rotation. We can check that it can satisfy Einstein equation.

In the process above, we can get a transformation rule to directly transform just a Schwarzschild to supertranslated Schwarzschild without going through the isotropic coordinate system like the above (which is “C.T. (Eq.(19) and (24))” horizontally written in Fig.1). Then, in Sec. 2.6 using these rules in the first rotating black hole space time given in the BL coordinate system, we obtain a rotating supertranslated black hole solution with the linear-order supertranslation corrections but the fully involved black hole rotation (which is “C.T. (Eq.(19) and (24))” vertically written in Fig.1). We can check that it can satisfy the Einstein equation.

\section{Rotating supertranslated black hole solution}

In this section, we first obtain a 4D slowly rotating supertranslated black hole space time solution. In this process, we can get the transformation rule to directly transform just a Schwarzschild to the supertranslated Schwarzschild. Using it, we finally obtain a 4D supertranslated black hole space time solution with the full black hole rotation.

\subsection{Rewriting BL to isotropic coordinate systems before performing supertranslation}

We start with the 4D rotating black hole space time given by the Boyer-Lindquist (BL) coordinate system,

\begin{equation}
\begin{aligned}
ds^2 &= -(1 - \frac{2mr}{r^2 + a^2 \cos^2 \theta}) dt^2 + \frac{r^2 + a^2 \cos^2 \theta}{r^2 - 2mr + a^2} dr^2 + (r^2 + a^2 \cos^2 \theta) d\theta^2 \\
&+ (r^2 + a^2 + \frac{2mra^2 \sin^2 \theta}{r^2 + a^2 \cos^2 \theta}) \sin^2 \theta d\phi^2 - \frac{4mra \sin \theta}{r^2 + a^2 \cos^2 \theta} dtd\phi.
\end{aligned}
\end{equation}

We will involve the supertranslation corrections in this. Then, for this purpose, let us check the following facts:
Figure 1: This is the sketch for what to do in Sec.2. “C.T.” is the abbreviation of the coordinate transformation. As the point in this sketch, in the process to obtain a slowly rotating one (23), we can get a transformation rule to transform just a Schwarzschild to supertranslated Schwarzschild directly, which is, “C.T. (Eq.(19) and (24))”. Using it in the first rotating black hole space time in BL coordinate system (1), we obtain a supertranslated black hole solution with the fully involved rotation, Eq.(30).

• in [71], one way to involve the supertranslation is given in the non-rotating isotropic coordinate system,

• the non-rotating isotropic coordinate system can be obtained from Schwarzschild coordinate system,

• (11) can be written in the Schwarzschild type coordinate system by taking $a$ to the linear order as

$$ds^2 = -(1 - 2m/r)dt^2 + (1 - 2m/r)^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 + O(a^2),$$  \hspace{1cm} (2)

where $d\varphi = d\phi - 2m a/r^3 dt$.

Hence, let us rewrite (11) to the isotropic coordinate system via the Schwarzschild type coordinate system (2) as

$$ds^2 = -\frac{(1 - m/2\rho)^2}{(1 + m/2\rho)^2} dt^2 + (1 + m/2\rho)^4 (d\rho^2 + \rho^2 (d\theta^2 + \sin \theta d\varphi^2)) + O(a^2).$$ \hspace{1cm} (3)

where $r = \rho(1 + m/2\rho)^2$. In (3), the effect of the black hole rotation is in $d\varphi$. 

3
2.2 Performing supertranslation in isotropic coordinate system

Isotropic coordinates after supertranslation are given as

\[ x_s = (\rho - C) \sin \theta \cos \varphi + \sin \varphi \csc \theta \partial_\varphi C - \cos \theta \cos \varphi \partial_\theta C, \]

\[ y_s = (\rho - C) \sin \theta \sin \varphi \csc \theta \partial_\varphi C - \cos \theta \sin \varphi \partial_\theta C, \]

\[ z_s = (\rho - C) \cos \theta + \cos \theta \cos \varphi \partial_\varphi C, \]

and \( t_s = t \), where the function \( C \) is the Goldstone boson field for supertranslation [57].

We take \( C \) as follows:

\[ C = m \varepsilon Y_2^0. \]  

We list the 3 points in (4) and (5) in what follows:

- \( \varepsilon \) is some infinitesimal dimensionless parameter, which we attach to measure the order of our supertranslation corrections in our analysis. We perform our analysis to the linear order with regard to \( \varepsilon \).

- \( m \) is that in (3), which we involve to have \( C \) have the dimension same with \( \rho \) (where \( G/c^2 \) is 1 in this paper).

- Why we consider \( Y_2^0 \) is that \( Y_2^0 \) is considered to be the most dominant mode in the quasi-normal modes (ringdown waves) emitted from the process that a soft-hairy black hole is formed [70].

Using (4) with (5), we can write \( \rho_s \) and one part in (3) after the supertranslation as

\[ \rho_s = (x_s^2 + y_s^2 + z_s^2)^{1/2} = \rho - \frac{\varepsilon}{8} \sqrt{\frac{5}{\pi}} m(3 \cos(2\theta) + 1) + \mathcal{O}(\varepsilon^2) \equiv \rho + \delta \rho, \]

\[ d\rho_s^2 + \rho_s^2(d\theta_s^2 + \sin^2 \varphi_s d\varphi_s^2) = dx_s^2 + dy_s^2 + dz_s^2. \]

With these, we write (3) to the linear order with regard to \( \varepsilon \) as

\[ ds^2 = \frac{(1 - m/2\rho_s)^2}{(1 + m/2\rho_s)^2} dt^2 + (1 + m/2\rho_s)^4(dx_s^2 + dy_s^2 + dz_s^2) + \mathcal{O}(\varepsilon) + \mathcal{O}(a^2) \]

\[ \equiv g_{tt} dt^2 + g_{\rho\rho} d\rho^2 + g_{\theta\theta} d\theta^2 + g_{\varphi\varphi} d\varphi^2, \]

where

- \( g_{tt} = \frac{(m - 2\rho)^2}{(m + 2\rho)^2} - \frac{5}{\pi} \varepsilon m^2 \frac{(m - 2\rho)}{(m + 2\rho)^3} (3 \cos(2\theta) + 1) + \mathcal{O}(\varepsilon^2), \)

- \( g_{\rho\rho} = \frac{(m + 2\rho)^4}{16 \rho^4} + \frac{5}{\pi} \varepsilon m^2 \frac{(m + 2\rho)^3}{32 \rho^5} (3 \cos(2\theta) + 1) + \mathcal{O}(\varepsilon^2), \)

- \( g_{\theta\theta} = \frac{(m + 2\rho)^4}{16 \rho^2} + \frac{5}{\pi} \varepsilon m (m + 2\rho)^3 \frac{3 \cos(2\theta)(5m + 6\rho) + m - 2\rho}{64 \rho^3} + \mathcal{O}(\varepsilon^2), \)

\[ ^{†} \text{Author has heard this from Vitor Cardoso in [70].} \]
\[ g_{\varphi s} = \frac{(m + 2\rho)^4}{16\rho^2} \sin^2 \theta + \sqrt{\frac{5}{\pi}} \varepsilon m \left( \frac{(m + 2\rho)^3}{64\rho^3} \sin^2 \theta \cos(2\theta)(9m + 6\rho) + 7m + 10\rho \right). \]

As for \( d\varphi \) above, as can be seen by looking at \( d\varphi \) given under (2), under the supertranslation, we can see \( d\varphi \) is also displaced. Once formally denoting it as \( d\varphi_s \) in (8), we will proceed with the calculation. We treat it explicitly in Sec.2.5.

We use the following notations in what follows:

- \( g_{\mu\nu} \): metrics of the supertranslated isotropic coordinate system
- \( j_{\mu\nu} \): metrics of the supertranslated Schwarzschild coordinate system
- \( J_{MN} \): metrics of the rotating supertranslated BL coordinate system with linear-order \( a \)
- \( K_{MN} \): metrics of the rotating supertranslated BL coordinate system with fully involved \( a \)

where \( \mu, \nu = t, \rho, \theta, \varphi \) and \( M, N = t, \rho, \theta, \phi \). In addition, the coordinate with the lower index ‘s’ (e.g. ‘\( \varphi_s \)’ ) means the displaced coordinate by the supertranslation.

### 2.3 Rewriting from supertranslated isotropic to supertranslated Schwarzschild coordinate systems. 1

Since we have obtained the metrics in the supertranslated isometric coordinate system, we will rewrite these to the following Schwarzschild coordinate system:

\[ ds^2 = -(1 - 2\mu(\rho)/r)dt^2 + (1 - 2\mu(\rho)/r)^{-1}dr^2 + j_{\theta\theta}d\theta^2 + j_{\varphi\varphi}d\varphi^2_s. \] (11)

In what follows, we obtain 1) a relation between \( r \) and \( \rho \), and 2) \( \mu(\rho) \), by solving the following relations:

- \(- (1 - 2\mu(\rho)/r) = g_{tt}, \) (12a)
- \( \frac{1}{1 - 2\mu(\rho)/r} \left( \frac{dr}{d\rho} \right)^2 = g_{\rho\rho}, \) (12b)

where \( g_{tt} \) and \( g_{\rho\rho} \) above are given in (8).

We can obtain the \( r \) satisfying (12a) to \( \varepsilon^1 \)-order as

\[ r = \frac{\mu(\rho)(m + 2\rho)^2}{4m\rho} + \varepsilon \sqrt{\frac{5}{\pi}} (3\cos(2\theta) + 1)\mu(\rho)(m^2 - 4\rho^2) + O(\varepsilon^2). \] (13)

Let us obtain the \( \mu(\rho) \). For this, look at (12b), then plugging (13) into the \( r \) in (12b), solve it regarding \( \mu(\rho) \) order by order to \( \varepsilon^1 \)-order. As a result we can obtain

\[ \mu(\rho) = m + \frac{c_1\rho\varepsilon}{(m + 2\rho)^2} + O(\varepsilon^2), \] (14)

where we took the integral constant at \( \varepsilon^0 \)-order so that \( \varepsilon^0 \)-order becomes \( m \). \( c_1 \) is the integral constant at \( \varepsilon^1 \)-order, which we put as

\[ c_1 = 0. \] (15)
As a result, $\mu(\rho)$ is given just $m$. We denote $\mu(\rho)$ as $\mu$ in what follows.

Now that we have obtained the relation between $\rho$ and $r$ in the form “$r = \cdots$” as in (13), using it we can rewrite the supertranslated Schwarzschild to the supertranslated isotropic coordinate system to $\varepsilon^1$-order as

$$
\begin{align*}
&ds^2 = -(1 - 2\mu/r)dt^2 + (1 - 2\mu/r)^{-1}dr^2 + j_{\theta\theta}d\theta^2 + j_{\varphi\varphi}s^2 d\varphi^2 \\
&\quad \to g_{tt}dt^2 + g_{\rho\rho}d\rho^2 + \left(\frac{1}{1 - \frac{2\mu}{r}}\left(\frac{\partial r}{\partial \theta}\right)^2\right)d\theta^2 + \frac{2}{1 - \frac{2\mu}{r}}\frac{\partial r}{\partial \theta}d\rho d\theta + g_{\varphi\varphi}s^2 d\varphi^2.
\end{align*}
$$

(16)

However what is needed for us is rewriting from the supertranslated isotropic to the supertranslated Schwarzschild coordinate system. Using what we have obtained in this subsection, we obtain it in the next subsection.

2.4 Rewriting from supertranslated isotropic to supertranslated Schwarzschild coordinate systems. 2

We will obtain the relation between $\rho$ and $r$ in the form “$\rho = \cdots$” to $\varepsilon^1$-order to rewrite (16) in the opposite direction. For this, there are two ways: 1) to solve (12b) or 2) to solve (13). We can confirm that the same $\rho$ can be obtained from either of them.

Writing what we do, plugging $\mu(\rho)$ in (14) into the $\mu$ in (13), then expanding it to $\varepsilon^1$-order, we can obtain $\rho$ order by order. As a result, we obtain as

$$
\rho(\pm) = \frac{1}{2}\left(\pm\sqrt{r(r - 2m)} - m + r\right) + \frac{1}{8}\sqrt{\frac{5}{\pi}}\varepsilon^{m(3\cos(2\theta) + 1)} + O(\varepsilon^2),
$$

(17)

We discard $\rho^(-)$ and adopt $\rho^+$ from the large $r$ situation. We denote $\rho^+$ just as $\rho$ in what follows.

Using $\rho$ in (17) and $\mu$ in (14), we can rewrite the supertranslated isotropic coordinate system to the supertranslated Schwarzschild coordinate system to $\varepsilon^1$-order as

$$
\begin{align*}
&ds^2 = g_{tt}dt^2 + g_{\rho\rho}d\rho^2 + g_{\theta\theta}d\theta^2 + g_{\varphi\varphi}s^2 d\varphi^2 \\
&\quad \to -(1 - \frac{2\mu}{r})dt^2 + \left(1 - \frac{2\mu}{r}\right)^{-1}dr^2 + \left(g_{\theta\theta} + g_{\rho\rho}\left(\frac{\partial \rho}{\partial \theta}\right)^2\right)d\theta^2 + 2g_{\rho\rho}\frac{\partial \rho}{\partial \theta}d\rho d\theta \\
&\quad + j_{\varphi\varphi}s^2 d\varphi^2,
\end{align*}
$$

(18)

where $j_{\mu\nu}$ are given in (8). Then denoting (18) as

$$
\begin{align*}
&[18] \equiv j_{tt}dt^2 + j_{rr}dr^2 + j_{\theta\theta}d\theta^2 + 2j_{\theta\rho}d\theta d\rho + j_{\varphi\varphi}s^2 d\varphi^2,
\end{align*}
$$

(19)

where

- $j_{tt} = -(1 - \frac{2m}{r}) + O(\varepsilon^2)$,
- $j_{rr} = (1 - \frac{2m}{r})^{-1} + O(\varepsilon^2)$,
\[ j_{\theta\theta} = r^2 + \frac{3\sqrt{\frac{2}{5}} m \cos(2\theta)(\sqrt{r(r - 2m)} + r)^4}{2(\sqrt{r(r - 2m)} - m + r)^3} \varepsilon + O(\varepsilon^2), \]

\[ j_{r\theta} = -\frac{3\sqrt{\frac{2}{5}} m \sin(2\theta)(r - \sqrt{r(r - 2m)})^4}{8\sqrt{r(r - 2m)}(\sqrt{r(r - 2m)} - m + r)^3} \varepsilon + O(\varepsilon^2), \]

\[ j_{\phi\phi_s} = r^2 \sin^2 \theta + \frac{3\sqrt{\frac{2}{5}} m \sin^2(2\theta)(\sqrt{r(r - 2m)} + r)^4}{8(\sqrt{r(r - 2m)} - m + r)^3} \varepsilon + O(\varepsilon^2). \]

### 2.5 A rotating supertranslated black hole solution with linear-order \( \varepsilon \) and \( a \)

In this subsection, we obtain a slowly rotating supertranslated black hole solution with the linear order \( \varepsilon \) and \( a \) by backing the coordinate \( \varphi_s \) to \( \phi \) to include the black hole rotation.

First, when supertranslation is performed, \( \rho \) and \( r \) get displaced for that. The displaced \( \rho \) has been already obtained in (6). On the other hand, the displaced \( r \) can be obtained as

\[ r_s = \rho_s(\rho)(1 + m/2\rho_s(\rho))^2 \text{eq.}(17) = r + O(\varepsilon^2) \equiv r + \delta r, \]

where we have written the correction part as \( \delta r \) formally, but \( \delta r = 0 + O(\varepsilon^2) \).

Then, \( d\varphi \) can be obtained as

\[ d\varphi_s = d\phi - \frac{2ma}{(r + \delta r)^3} dt \equiv d\phi + \Theta_r dt, \]

\[ = d\phi - \frac{2ma}{(\rho_s(1 + m/2\rho_s)^2)^3} dt \equiv d\phi + \Theta_\rho dt, \]

where \( \delta r \) is defined in (20), the top and below are written in terms of \( r \) and \( \rho \) respectively, and

\[ \Theta_r \equiv -\frac{2a}{r^3} + O(\varepsilon^2), \]

\[ \Theta_\rho \equiv -\frac{128am\rho^3}{(m + 2\rho)^6} + \frac{48\sqrt{\frac{2}{5}}a\varepsilon m^2 \rho^2(3\cos(2\theta) + 1)(m - 2\rho)}{(m + 2\rho)^7} + O(\varepsilon^2). \]

Plugging (21) into (18), we can get a slowly rotating black hole with the displacement of the supertranslation to linear order as

\[ ds^2 = J_{tt} dt^2 + J_{rr} dr^2 + J_{\theta\theta} d\theta^2 + J_{\phi\phi} d\phi^2 + 2J_{r\theta} drd\theta + O(a^2) + O(\varepsilon^2), \]
where \( J_{MN} \) are given using \( J_u \) and \( \Theta \) as

\[
J_{MN} = \begin{pmatrix}
J_u & 0 & 0 & j_{\varphi \varphi}, \Theta_r \\
0 & j_{rr} & j_{r\theta} & 0 \\
0 & j_{r\theta} & j_{\theta \theta} & 0 \\
j_{\varphi \varphi}, \Theta_r & 0 & 0 & j_{\varphi \varphi},
\end{pmatrix}
+ \mathcal{O}(a^2) + \mathcal{O}(\varepsilon^2).
\]

- \( J_u = -1 + 2m/r, \)
- \( J_{t\varphi} = -1 + 2m/r - \frac{3\sqrt{2}m^2 \sin^2(2\theta)(\sqrt{r(r-2m)} + r)^4}{4r^3(\sqrt{r(r-2m)} - m + r)^3} \varepsilon, \)
- \( J_{rr} = (1 - 2m/r)^{-1}, \)
- \( J_{r\theta} = -\frac{3\sqrt{2}m \sin(2\theta)(\sqrt{r(r-2m)} + r)^4}{8\sqrt{r(r-2m)}(\sqrt{r(r-2m)} - m + r)^3} \varepsilon, \)
- \( J_{\theta \theta} = r^2 + \frac{3\sqrt{2}m \cos(2\theta)(\sqrt{r(r-2m)} + r)^4}{2(\sqrt{r(r-2m)} - m + r)^3} \varepsilon, \)
- \( J_{\varphi \varphi} = r^2 \sin^2 \theta + \frac{3\sqrt{2}m \sin^2(2\theta)(\sqrt{r(r-2m)} + r)^4}{8(\sqrt{r(r-2m)} - m + r)^3} \varepsilon. \)

We can check these can satisfy Einstein equation to \( \varepsilon^1 \)- and \( a^{1}\)-orders.

### 2.6 A rotating supertranslated black hole solution with linear-order \( \varepsilon \) but full \( a \)

In this subsection, we obtain a rotating supertranslated black hole solution with linear-order \( \varepsilon \) but full \( a \). For this purpose, first of all, let us obtain \( \theta_s \) and \( \varphi_s \) in terms of \( z \). For this purpose, we use the stereographic map, \( z_a = e^{i\varphi_s} \cot \theta_a / 2 \), then \( \theta_a \) and \( \varphi_a \) can be written in terms of \( z_a \) as \( \theta_a = 2 \cot^{-1} |z_a|, \varphi_a = \frac{i}{2} \ln \bar{z}_a z_a^{-1} \). The expression of \( z_a \) in terms of \( z \) is given in (26) in [71] as

\[
\bar{z}_a = \frac{(z \bar{z} - 1)(\rho - C) + (z \bar{z} + 1)(\rho_s - \bar{z} \partial_z C - z \partial_z C)}{2z(\rho - C) + (z \bar{z} + 1)(\bar{z} \partial_z C - z \partial_z C)},
\]

where \( C \) and \( \rho_s \) are [51] and [20] in this study.

We plug \( \rho_s \) in [20] and \( C \) in [51] in the \( z_a \) above. Here, these \( \rho_s \) and \( C \) are given in terms of \( z \) by writing \( \theta \) and \( \varphi \) in these in terms of \( z \) using the stereographic map. Then,

\[\text{[71]}\]
we can obtain $\theta_s$ and $\varphi_s$ as

$$
\theta_s = \theta + \frac{3m}{2(r + \sqrt{(r-2m)r-m})} \sqrt{\frac{5}{\pi}} \sin(2\theta) + \mathcal{O}(\varepsilon^2) \equiv \theta + \delta \theta, \quad (25)
$$

$$
\varphi_s = \varphi + \mathcal{O}(\varepsilon^3), \quad (26)
$$

where no corrections regarding $\varepsilon$ at $\varepsilon^1$- and $\varepsilon^2$-orders in $\varphi_s$. From (26) and (20), we can see

$$
\phi_s = \phi + \mathcal{O}(\varepsilon^3). \quad (27)
$$

Then, in the previous section, we have obtained $J_{MN}$ from (2) via the isotropic coordinate system (3), however now we can directly obtain $J_{MN}$ from (2) by replacing $r$ and $\theta$ in (2) as

$$
(r, \theta, \varphi) \rightarrow (r_s, \theta_s, \varphi_s), \quad (28)
$$

where $r_s$ is given in (20).

Then, by performing the replacement

$$
(r, \theta, \phi) \rightarrow (r_s, \theta_s, \phi_s) \quad (29)
$$

in (25), we can obtain a rotating black hole space time with the supertranslation corrections of $\varepsilon$ to first-order but the fully involved $a$ as

$$
ds^2 = K_{MN}dx^M dx^N + \mathcal{O}(\varepsilon^2),
$$

where

- $K_{tt} = -1 + \frac{2m a r}{a^2 \cos^2 \theta + r^2} - \frac{3\sqrt{\frac{5}{\pi}} a^2 mr \sin^2(2\theta)}{(r + \sqrt{(r-2)r-1})(a^2 \cos^2 \theta + r^2)^2} \varepsilon,$
- $K_{t\phi} = - \frac{2m a r \sin^2(\theta)}{a^2 \cos^2 \theta + r^2} + \frac{3\sqrt{\frac{5}{\pi}} a m r (a^2 + r^2) \sin^2(2\theta)}{(r + \sqrt{(r-2)r-1})(a^2 \cos^2 \theta + r^2)^2} \varepsilon,$
- $K_{rr} = \frac{a^2 \cos^2 \theta + r^2}{a^2 - 2m r + r^2} + \frac{3\sqrt{\frac{5}{\pi}} a^2 \sin^2(2\theta)}{2(r + \sqrt{(r-2)r-1})(a^2 + r(r-2m))} \varepsilon,$
- $K_{r\theta} = - \frac{3\sqrt{\frac{5}{\pi}} \sin(2\theta)(a^2 \cos^2 \theta + r^2)}{2\sqrt{(r-2)r}(r + \sqrt{(r-2)r-1})} \varepsilon,$
- $K_{\theta\theta} = a^2 \cos^2 \theta + r^2 + \frac{3\sqrt{\frac{5}{\pi}}(3 \cos(4\theta) + 1) + 4(a^2 + 2r^2) \cos(2\theta))}{4(r + \sqrt{(r-2)r-1})} \varepsilon,$
- $K_{r\theta} = - \frac{3\sqrt{\frac{5}{\pi}} \sin(2\theta)(a^2 \cos^2 \theta + r^2)}{2\sqrt{(r-2)r}(r + \sqrt{(r-2)r-1})} \varepsilon,$
\[ K_{\phi\phi} = \sin^2 \theta \left( \frac{2a^2 mr \sin^2 \theta}{a^2 \cos^2 \theta + r^2} + a^2 + r^2 \right) \]

\[ + \frac{3 \sqrt{5}}{16} \sin^2 (2\theta) \frac{1}{3a^6 + a^4 r(10m + 11r)} + 16a^2 r^3 (m + r) + a^2 (a^2 + r(r - 2m)) (a^2 \cos(4\theta) + 4(a^2 + 2r^2) \cos(2\theta)) + 8r^6) \varepsilon. \]

We can check that these can satisfy the Einstein equation to \( \varepsilon^1 \)-order for arbitrary \( a \). We can also check that these can agree with the slowly rotating metrics (23) if expanding these to the linear order with regard to \( a \).

### 3 Summary and Comment

In this study, a rotating supertranslated black hole solution has been obtained as in (30). However, since (30) is not obtained by solving Einstein equation it is not be the general solution. However, since (30) can satisfy the Einstein equation to the linear order with regard to the supertranslation corrections for arbitrary \( a \). Therefore, (30) is the metrics of a rotating supertranslated black hole space time solution.

As for our linear order supertranslation corrections, it would be considered it is enough. This is because the original Lie derivatives to define the diffeomorphism of the supertranslation are defined to the linear order of the vector field’s coefficients [57]. Therefore, since the supertranslation is originally defined to the linear order, if one can involve the supertranslation corrections to the liner order, it would be considered it is enough (conversely, it would be non-meaning if one performed the analysis more than quadratic order with regard to the supertranslation).

Then, we have involved the supertranslation corrections to the linear order and have fully involved the black hole rotation in (30). Therefore, (30) could be considered to have sufficiently involved the supertranslation corrections and the black hole rotation.

In the current context of the asymptotic symmetry, the extension to the rotating black hole is one of the problems, and what has been obtained in this study would be a solution to that.

The future work of this study is of course to obtain some general rotating supertranslated black hole solution by solving the Einstein equation.

In the actual analysis, to obtain the rotating solution, one of ideas author has tried is to use Newman-Janis (NJ) algorithm [73, 74]. Concretely, author has followed the way given in Eq.(6) in [75]. However, it could not work in our metrics.

Concisely, its reason is that there are \( r \) in the metrics, and the powers of some of them are given by odd numbers. As a result, the imaginary numbers appear in the metrics if we follow Eq.(6) in [75].

Author has also tried the idea to obtain the rotating solution by using (4) in the Kerr-Schild (KS) form. However it has not gone well for the situation that when author
transform the metrics obtained from the KS form into the ones in the BL coordinate system, these cannot agree to the ones originally obtained in the BL coordinate system.

Essential point in this problem would be that (4) could be available in the isotropic coordinate system and perhaps cannot be adapted in the KS form.

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