q-Multiple Zeta Functions and q-Multiple Polylogarithms

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Abstract. We shall define the q-analogs of multiple zeta functions and multiple polylogarithms in this paper and study their properties, based on the work of Kaneko et al. and Schlesinger, respectively.

1 Introduction and definitions

Let $0 < q < 1$ and for any positive integer $k$ define its $q$-analog $[k]_q = (1 - q^{k+1})/(1 - q)$. In [5] Kaneko et al. define a function of two complex variables $f_q(s; t) = \sum_{k} q^{kt}/[k]_q^s$ such that the $q$-analog of Riemann zeta function is realized as

$$\zeta_q(s) := f_q(s; s - 1).$$

For any property $P$ let $Z_P$ be the set of integers satisfying $P$. After analytically continuing $f_q(s; t)$ to $\mathbb{C}^2$ as a meromorphic function Kaneko et al. proved the following main result

**Theorem 1.1.** ([5 Prop. 2, Thm. 2]) One can analytically continue $\zeta_q(s)$ to $\mathbb{C} \setminus \{Z_{\leq 0} + \frac{2\pi i}{\log q} Z_{\neq 0}\}$. Moreover, for any $s \neq 1$ one has

$$\lim_{q \uparrow 1} \zeta_q(s) = \zeta(s).$$

They also study the special values of $\zeta_q(s)$ at non-negative integers. In this paper we shall generalize these to the (Euler-Zagier) multiple zeta functions, which are defined as nested generalizations of Riemann zeta function $\zeta(s)$:

$$\zeta(s_1, \ldots, s_d) = \sum_{0 < k_1 < \cdots < k_d} k_1^{-s_1} \cdots k_d^{-s_d}$$

for complex variables $s_1, \ldots, s_d$ satisfying $\sigma_j + \cdots + \sigma_d > d - j + 1$ for all $j = 1, \ldots, d$. Here and in what follows, whenever $s \in \mathbb{C}$ we always write $\sigma = \Re(s)$, the real part of $s$. The analytic continuation of multiple zeta functions has been studied independently in [1] and [11]. We know that $\zeta(s_1, \ldots, s_d)$ can be extended to a meromorphic function on $\mathbb{C}^d \setminus \mathcal{S}_d$ where

$$\mathcal{S}_d = \left\{(s_1, \ldots, s_d) \in \mathbb{C}^d \mid s_d = 1, s_{d-1} + s_d = 2, 1, -2m, \forall m \in \mathbb{Z}_{\geq 0}, \right.$$ 

$$\left. \text{and } s_j + \cdots + s_d \in \mathbb{Z}_{\leq j+2} \forall j \leq d - 2 \right\}$$

To find the $q$-analog of multiple zeta functions we first define an auxiliary function of $2d$ complex variables $s_1, \ldots, s_d, t_1, \ldots, t_d \in \mathbb{C}$

$$f_q(s_1, \ldots, s_d; t_1, \ldots, t_d) = \sum_{0 < k_1 < \cdots < k_d} q^{k_1t_1 + \cdots + k_dt_d}/[k_1]_q^{s_1} \cdots [k_d]_q^{s_d}$$

which converges if $\Re(t_j + \cdots + t_d) > 0$ for all $j = 1, \ldots, d$ (see Prop. [22]). In the next section we are going to analytically continue this function to $\mathbb{C}^{2d}$ as a meromorphic function with explicitly defined poles.

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We now define the $q$-multiple zeta function by specialization of $f_q$:

$$\zeta_q(s_1, \ldots, s_d) := f_q(s_1, \ldots, s_d; s_1 - 1, \ldots, s_d - 1)$$

which will be shown to be the correct $q$-analogue of multiple zeta functions. When $\sigma_j > 1$ for all $j$ we can express this by the series

$$\zeta_q(s_1, \ldots, s_d) = \sum_{0 < k_1 < \cdots < k_d} \frac{q^{k_1(s_1-1)+\cdots+k_d(s_d-1)}}{[k_1]^{s_1} \cdots [k_d]^{s_d}}. \quad (3)$$

Note that when $d = 1$ this is the same as the $q$-analogue of the Riemann zeta function defined in \[5\]. Put

$$\mathcal{S}_d' = \{(s_1, \ldots, s_d) \in \mathbb{C}^d : s_d \in 1 + \frac{2\pi i}{\log q} \mathbb{Z}, \text{ or } s_d \in \mathbb{Z}_{\leq 0} + \frac{2\pi i}{\log q} \mathbb{Z}_{\neq 0}, \quad \text{or } s_j + \cdots + s_d \in \mathbb{Z}_{\leq d-j+1} + \frac{2\pi i}{\log q} \mathbb{Z}, \ j < d \} \supset \mathcal{S}_d.$$

Here the last part in $\mathcal{S}_d'$ is vacuous if $d = 1$. The primary goal of this paper is to prove

**Main Theorem.** The $q$-multiple zeta function $\zeta_q(s_1, \ldots, s_d)$ can be extended to a meromorphic function on $\mathbb{C}^d \setminus \mathcal{S}_d'$ with simple poles along $\mathcal{S}_d'$. Further, for all $(s_1, \ldots, s_d) \in \mathbb{C}^d \setminus \mathcal{S}_d$

$$\lim_{q \uparrow 1} \zeta_q(s_1, \ldots, s_d) = \zeta(s_1, \ldots, s_d).$$

In section 2, we propose a new definition of the $q$-multiple polylogarithms and briefly study their properties. We also review Jackson’s $q$-derivatives and $q$-definite integrals and define $q$-iterated integrals as $q$-analogs of Chen’s iterated integrals.

It is known that there are two kinds of shuffle relations among multiple zeta values (MZV for short). The first one is produced by their power series expansions, the second by using Chen’s iterated integrals. In the last section of this paper we will apply our $q$-iterated integral technique to $q$-multiple polylogarithms in order to study the $q$-shuffle relations of the second kind for $q$-MZV. For simplicity we will only deal with $\zeta_q(m)\zeta_q(n)$ for positive integers $m \neq n$. These relations reduce to the ordinary ones when $q \uparrow 1$. We thank Prof. Kaneko for his questions relating to this part of our study and sending us his offprint upon which the current work is based.

## 2 Analytic continuations of $f_q$ and $\zeta_q$

The purpose of this section is two-fold: first we will use the auxiliary functions $f_q$ introduced in the first section to give a quick analytic continuation of $q$-multiple zeta functions $\zeta_q(s_1, \ldots, s_d)$, though this is not enough to show it’s the right $q$-analogue of the multiple zeta functions. Second, we write down these expressions involving binomial coefficients explicitly which will be used to study special values of $\zeta_q(s_1, \ldots, s_d)$ in section 3.

We need a simple lemma first.

**Lemma 2.1.** Let $k$ be a positive integer. For all $1 > q > 1/2$ we have

$$1 \leq \frac{1 - q^k}{1 - q} < 2k.$$ 

**Proof.** The inequality $1 \leq \frac{1 - q^k}{1 - q}$ is obvious. Let $f(q) = 2k(1 - q) - (1 - q^k)$. We show that $f(q) \geq 0$ for all $0 < q < 1$. Indeed, for all such $q$

$$f'(q) = -2k + kq^{k-1} \leq -k < 0.$$ 

So $f$ is strictly decreasing and the positivity of $f$ follows from $f(1) = 0$. This implies that $2k > (1 - q^k)/(1 - q)$ as desired.\)

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Proposition 2.2. The function

\[ f_q(s_1, \ldots, s_d; t_1, \ldots, t_d) = \sum_{0 < k_1 < \cdots < k_d} \frac{q^{\sum k_i t_i}}{[k_1]^{s_1} \cdots [k_d]^{s_d}} \]

converges if \( \Re(t_j + \cdots + t_d) > 0 \) for all \( j = 1, \ldots, d \). It can be analytically continued to a meromorphic function over \( \mathbb{C}^{2d} \) via the series expansion

\[ f_q(s_1, \ldots, s_d; t_1, \ldots, t_d) = (1 - q)^{\text{wt}(\vec{s})} \sum_{r_1, \ldots, r_d = 0}^{+\infty} \prod_{j=1}^{d} \left( \frac{s_j + r_j - 1}{r_j} \right) \frac{q^{\sum (r_j + t_j)}}{1 - q^{r_j + t_j + \cdots + r_d + t_d}}. \]  

(4)

where \( \text{wt}(\vec{s}) = s_1 + \cdots + s_d \). It has the following (simple) poles: \( t_j + \cdots + t_d \in \mathbb{Z}_{\leq 0} + \frac{2\pi i}{\log q} \mathbb{Z} \).

Proof. Assume \( |\Re(s_j)| < N_j \) and let \( \tau_j = \Re(t_j) \) for all \( j = 1, \ldots, d \). By Lemma 2.2

\[ |f_q(s_1, \ldots, s_d; t_1, \ldots, t_d)| < \sum_{0 < k_1 < \cdots < k_d} \prod_{j=1}^{d} (2k_j)^{N_j} q^{\tau_j}. \]  

(5)

Let \( k = k_d - 1 \) (when \( d = 1 \) take \( k_0 = 0 \), \( n = k_d \), \( N = N_d \), and \( \tau = \tau_d \). Then by root test \( \sum_{n \geq k} n^N q^{n\tau} \) converges and moreover

\[ \sum_{n \geq k} n^N q^{n\tau} = \left( q \frac{d}{dq} \right)^N \sum_{n \geq k} q^{n\tau} = (1 - q)^{-N} f_N(k; q, \tau) q^{k\tau} \]  

(6)

where \( f_N(x; q, \tau) = \sum_{l=0}^{\infty} c_l x^l \) is a polynomial of degree \( N \) whose coefficients depend only on the constants \( N \), \( q \) and \( \tau \). This proves the first part of the lemma when \( d = 1 \). In the general case it follows from (5) and (6) that

\[ |f_q(s_1, \ldots, s_d; t_1, \ldots, t_d)| < \frac{2^{N_1+\cdots+N_d}}{(1 - q^\tau)^N} \sum_{l=0}^{\infty} c_l \sum_{0 < k_1 < \cdots < k_{d-1} < k} \prod_{j=1}^{d-2} (k_j)^{N_j} q^{k_j \tau_j} \]  

\[ \times k_{d-1}^{N_{d-1}+l} q^{k_{d-1} \tau_{d-1} + \tau_d}. \]

Hence the first part of the lemma follows from an easy induction on \( d \).

By binomial expansion \( (1 - x)^{-s} = \sum_{r=0}^{+\infty} \binom{s+r-1}{r} x^r \) we get

\[ f_q(s_1, \ldots, s_d; t_1, \ldots, t_d) = (1 - q)^{\text{wt}(\vec{s})} \sum_{0 < k_1 < \cdots < k_d} \sum_{r_1, \ldots, r_d = 0}^{+\infty} \prod_{j=1}^{d} \left( \frac{s_j + r_j - 1}{r_j} \right) q^{k_j (r_j + t_j)}. \]

As \( 0 < q < 1 \) the series converges absolutely by Stirling’s formula so we can exchange the summations. The proposition follows immediately from the next lemma by taking \( x_j = q^{r_j + t_j} \) for \( j = 1, \ldots, d \). \( \square \)

Lemma 2.3. Let \( x_j \in \mathbb{C} \) such that \( |x_j| < 1 \) for \( j = 1, \ldots, d \). Then

\[ \sum_{0 < k_1 < \cdots < k_d} \prod_{j=1}^{d} x_j^{k_j} = \prod_{j=1}^{d} x_j^{x_j} = \prod_{j=1}^{d} x_j^{1 - x_j} \]

(7)

\[ \prod_{j=1}^{d} x_j^{k_j} = \prod_{j=1}^{d-1} x_j^{k_j} \sum_{k_d > k_{d-1}} x_d^{k_d} = \frac{x_d}{1 - x_d} F_{d-1}(x_1, \ldots, x_{d-2}, x_{d-1} x_d). \]

The lemma follows immediately by induction. \( \square \)
Recall that $\zeta(s_1, \ldots, s_d) = f_q(s_1, \ldots, s_d; s_1 - 1, \ldots, s_d - 1)$. Hence we have the following immediate consequence.

**Theorem 2.4.** The q-multiple zeta function $\zeta_q(s_1, \ldots, s_d)$ can be extended to a meromorphic function with simple poles lying along $\mathbb{C}^d_+$:

$$
\zeta_q(s_1, \ldots, s_d) = (1 - q)^{\text{wt}(\bar{n})} \sum_{r_1, \ldots, r_d=0}^{+\infty} \prod_{j=1}^{d} \left( \frac{\left( s_j + r_j - 1 \right)}{r_j} \right)^{q^{(r_j + s_j - 1)}} \cdot \frac{1}{1 - q^{r_j + s_j + \cdots + r_d + s_d - d + j - 1}}.
$$

To see the effect of taking different specializations of $t_j$ in $f_q$ we define the shifting operators $S_j$ (1 ≤ j ≤ d) on the multiple zeta functions as follows:

$$
S_j \zeta(s_1, \ldots, s_d) = \zeta(s_1, \ldots, s_d) + (1 - q)\zeta(s_1, \ldots, s_j - 1, \ldots, s_d).
$$

It is obvious that these operators are commutative.

**Proposition 2.5.** Let $n_1, \ldots, n_d$ be non-negative integers. Then we have

$$
f_q(s_1, \ldots, s_d; s_1 - 1 - n_1, \ldots, s_d - 1 - n_d)
= S_1^{n_1} \circ \cdots \circ S_d^{n_d} \zeta(s_1, \ldots, s_d) = \sum_{r_1=0}^{n_1} \cdots \sum_{r_d=0}^{n_d} \left( \prod_{j=1}^{d} \left( \frac{n_j}{r_j} \right)^{(1 - q)^{r_j}} \right) \zeta(s_1 - r_1, \ldots, s_d - r_d).
$$

**Proof.** We only sketch the proof in the case $n_1 = \cdots = n_{d-1} = 0$. The general case is completely similar. In the rest of the paper we always let $S$ be the shifting operator on the last variable. Suppose $n_d = n = 1$. Then

$$
f_q(s_1, \ldots, s_d; s_1 - 1, \ldots, s_{d-1} - 1, s_d - 2)
= \sum_{0 < k_1 < \cdots < k_d} \frac{q^{k_1(s_1-1) + \cdots + k_d(s_d-1) + k_d(s_d-2)}}{[k_1]^{s_1} \cdots [k_d]^{s_d}}
= \sum_{0 < k_1 < \cdots < k_d} \frac{q^{k_1(s_1-1) + \cdots + k_d(s_d-1)}}{[k_1]^{s_1} \cdots [k_d-1]^{s_d-1}} \cdot \frac{q^{k_d(s_d-2)}(1 - q^{k_d}) + q^{k_d(s_d-1)}}{[k_d]^{s_d}}
= S \zeta(s_1, \ldots, s_d).
$$

The rest follows easily by induction.

The next corollary answers an implicit question in [5].

**Corollary 2.6.** Let $n$ be a positive integer. The specialization of $t$ in $f_q(s; t)$ to $s - 1 - n$ is

$$
f_q(s; s - 1 - n) = S^n \zeta_q(s) = \sum_{r=0}^{n} \binom{n}{r} (1 - q)^r \zeta_q(s - r).
$$

We observe that one effect of the shifting operator is to bring in more poles. Essentially, $S^n$ shifts all the poles of $\zeta_q(s)$ by $n$ to the right on the complex plane.

### 3 Analytic continuation of multiple zeta functions

Let’s begin with a review of some classical results on Bernoulli polynomials $B_k(x)$ which are defined by the generating function

$$
\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}.
$$

Let $\tilde{B}_k(x)$ be the “periodic Bernoulli polynomial”

$$
\tilde{B}_k(x) = B_k(\{x\}), \quad x \geq 1,
$$
where \( \{x\} \) is the fractional part of \( x \). Then we have \( n \in \mathbb{Z} \setminus \{0\} \) \( B_k(x) = -k! \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{e^{2\pi inx}}{(2\pi in)^k} \) (9)

Recall that the Bernoulli numbers satisfy \( B_k = B_k(1) \) if \( k \geq 2 \) while \( B_0 = 1 \) and \( B_1 = -1/2 \).

**Lemma 3.1.** For every positive integer \( M \geq 2 \) and \( x > 1 \) we have

\[
|\hat{B}_M(x)| \leq \frac{4M!}{(2\pi)^M}.
\]

**Proof.** It follows from the fact that \( \zeta(M) \leq \zeta(2) = \pi^2/6 < 2 \) for \( M \geq 2 \).

We know that one can analytically continue the multiple zeta functions as independently presented in \( \text{[11]} \) and \( \text{[1]} \) by different methods. Moreover, \( \zeta(s_1, \ldots, s_d) \) has singularities on the hyperplanes in \( \mathbb{Q}_d \) defined by \( \mathbb{Q} \). However, neither approach is suitable for our purpose here. So we follow the idea in \( \text{[2]} \) to provide a third approach in the rest of this section. The same idea will also be used to deal with the \( q \)-multiple zeta functions.

Let’s recall the classical Euler-Maclaurin summation formula \( \text{[10}, 7.21\text{]} \). Let \( f(x) \) be any (complex-valued) \( C^\infty \) function on \([1, \infty)\) and let \( m \) and \( M \) be two positive integers. Then we have

\[
\sum_{n=1}^{m} f(n) = \int_{1}^{m} f(x) \, dx + \frac{1}{2} (f(1) + f(m)) + \sum_{r=1}^{M} \frac{B_{r+1}}{(r+1)!} \left( f^{(r)}(m) - f^{(r)}(1) \right) - \frac{(-1)^{M+1}}{(M+1)!} \int_{1}^{m} \hat{B}_{M+1}(x) f^{(M+1)}(x) \, dx.
\]

To simplify our notation, in definition \( \text{[1]} \) we replace \( s_d, k_d-1, k_d \) by \( s, k \) and \( n \), respectively. Taking \( f(x) = 1/x^s \) and \( m = k \) and \( n = \infty \) in \( \text{[10]} \) we have:

\[
\sum_{n>k}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{k} \frac{1}{n^s} = \int_{k}^{\infty} f(x) \, dx - \frac{1}{2} f(k) - \sum_{r=1}^{M} \frac{B_{r+1}}{(r+1)!} \frac{f^{(r)}(k)}{k^{r+1}} - \frac{(-1)^{M+1}}{(M+1)!} \int_{k}^{\infty} \hat{B}_{M+1}(x) f^{(M+1)}(x) \, dx
\]

\[
= \frac{1}{(s-1)k^{s-1}} - \frac{1}{2k^s} + \sum_{r=1}^{M} \frac{B_{r+1}}{(r+1)!} \frac{(s)_r}{k^{s+r}} - \frac{(s)_{M+1}}{(M+1)!} \int_{k}^{\infty} \hat{B}_{M+1}(x) \frac{1}{x^{s+M+1}} \, dx.
\]

Here we have used the fact that \( B_k = 0 \) if \( k \geq 3 \) is odd. By definition \( \text{[1]} \) we have

**Theorem 3.2.** For all \( (s_1, \ldots, s_d) \in \mathbb{C}^d \setminus \mathbb{Q}_d \) and \( M > 1 + |\sigma_d| + |\sigma_{d-1}| \) we have

\[
\zeta(s_1, \ldots, s_d) = \sum_{r=0}^{M+1} \frac{B_r}{r!} (s_d r - 1) \cdot \zeta(s_1, \ldots, s_{d-1} + s_d + r - 1) - \frac{(s_d)_{M+1}}{(M+1)!} \sum_{0<k_1<\ldots<k_d} \frac{1}{k_1^{s_1} \ldots k_{d-1}^{s_{d-1}}} \int_{k_{d-1}}^{\infty} \hat{B}_{M+1}(x) \frac{1}{x^{s_d+M+1}} \, dx.
\]

where we set \( (s)_0 = 1 \) and \( (s)_{-1} = 1/(s-1) \). This provides an analytic continuation of \( \zeta(s_1, \ldots, s_d) \) to \( \mathbb{C}^d \setminus \mathbb{Q}_d \).

**Proof.** We only need to show that the series in \( \text{[11]} \) converges. Lemma \( \text{[3]} \) implies (if \( d = 2 \) then take \( k_0 = 1 \))

\[
\sum_{k_{d-1}=k_{d-2}}^{\infty} \left| \int_{k_{d-1}}^{\infty} \hat{B}_M(x) \frac{1}{x^{M+s_d+1}} \, dx \right| \leq \frac{4M!}{(2\pi)^M (M-|\sigma_d|)} \sum_{k_{d-1}=k_{d-2}}^{\infty} \frac{1}{k_{d-1}^{M-|\sigma_d|-1}}
\]

which converges absolutely whenever \( M > 1 + |\sigma_d| + |\sigma_{d-1}| \).
4 Proof of Main Theorem

Fix \((s_1, \ldots, s_d) \in \mathbb{C}^d\) such that \(s_j + \cdots + s_d < d - j + 1\) for all \(j = 1, \ldots, d\). When \(d = 1\) this is Thm. 1 due to Kaneko et al. We now assume \(d \geq 2\) and proceed by induction. The key is a recursive formula for \(\zeta_q(s_1, \ldots, s_d)\) similar to (11) for \(\zeta(s_1, \ldots, s_d)\). To derive this formula we appeal to the Euler-Maclaurin summation formula (10) again. Hence we set

\[
F(x) = \frac{q^{x(s-1)}}{(1-q^x)^s}
\]
as in [10]. Then
\[
F'(x) = (\log q)q^{x(s-1)} \frac{s-1 + q^x}{(1-q^x)^{s+1}},
\]
\[
F''(x) = (\log q)^2q^{x(s-1)} \frac{s(s+1) - 3s(1-q^x) + (1-q^x)^2}{(1-q^x)^{s+2}}.
\]

In definition (10) we replace \(s_d, k_{d-1}\), and \(k_d\) by \(s, k\) and \(n\), respectively. We now take \(M = 1\), \(f(x) = F(x + k - 1)\) and let \(m \to \infty\) in (11) and get

\[
\sum_{n > k} \frac{q^{n(s-1)}}{|n|^s} = (1-q)^s \left( -F(k) + \sum_{n=1}^{\infty} f(n) \right)
\]
\[
= (1-q)^s \left( \int_k^{\infty} F(x) \, dx - \frac{1}{2} F(k) - \frac{1}{2} \int_k^{\infty} F'(x) \, dx \right)
\]
\[
= \frac{(q-1) \log q}{(s-1) \log k} - \frac{1}{2} \frac{q^{k(s-1)} - 1}{|k|^s} + \frac{1}{12} \frac{q^{k(s-1)}(s + q^k - 1)}{|k|^{s+1}} + \frac{(1-q)^s(\log q)^2}{2} \int_k^{\infty} \frac{B_2(x) q^{x(s-1)}}{(1-q^x)^{s+2}} \right)
\]
because \(B_2(x + k - 1) = B_2(x)\) by periodicity. By the same argument as in (11), setting the incomplete beta integrals

\[b_t(\alpha, \beta) = \int_0^t u^{\alpha-1}(1-u)^{\beta-1} \, du,\]
we can obtain from (11) the following expression for the last term in (12):

\[
\sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{(1-q)^s \log q}{(2\pi i n)^2} \sum_{|nu|=1,0} a_\nu(s) b_{q^k}(s - 1 + \delta n, -s + \nu)
\]
where \(\delta = 2\pi i/\log q\), \(a_{-1}(s) = s(s+1), a_0(s) = -3s, \text{ and } a_1(s) = 1\). Repeatedly applying integration by parts on these incomplete beta integrals we get for \(\nu = \pm 1, 0\) and positive integer \(M \geq 2\)

\[
b_{q^k}(s - 1 + \delta n, -s + \nu) = \sum_{r=1}^{M-1} (-1)^{r-1} \frac{(s + 1 - \nu)_{s+1} q^{k(s+r-2)}}{(s-1 + \delta n)_r (1-q^x)^{s+r-\nu}}
\]
\[
+ (-1)^{M-1} \frac{(s + 1 - \nu)_{s+1} q^{k(s+M-1)} b_{q^k}(s - 2 + M + \delta n, -s - M + 1 + \nu)}{(s-1 + \delta n)_{M-1}}.
\]

Set \(\bar{s}' = (s_1, \ldots, s_{d-2})\) if \(d \geq 3\) and \(\bar{s}' = \emptyset\) if \(d = 2\). Putting (11), (12), (13) and (14) together and applying Prop. 2.5 we get

\[
\zeta_q(\bar{s}'', s_{d-1}, s_d) = \frac{(q-1)}{(s-1) \log q} \zeta_q(\bar{s}'', s_{d-1} + s_d - 1) - \frac{1}{2} S \zeta_q(\bar{s}'', s_{d-1} + s_d)
\]
\[
+ \frac{s \log q}{12 q - 1} S^2 \zeta_q(\bar{s}'', s_{d-1} + s_d + 1) + \frac{\log q}{12 S} \zeta_q(\bar{s}'', s_{d-1} + s_d) - \sum_{\nu = \pm 1, 0} (C_\nu + D_\nu).
\]
where $C_\nu$ and $D_\nu$ are the contributions from the sum involving $b_q(\ldots, -s + \nu)$. Explicitly they are computed as follows. Write
\begin{equation}
T(q, s, n, r) = \prod_{j=0}^{r-1} (2\pi i n + (s - 1 + j) \log q)^{-1}.
\end{equation}

Then
\[
C_{-1} = \sum_{r=1}^{M-1} \sum_{n \in \mathbb{Z} \setminus \{0\}} T(q, s, n, r) \left( \frac{\log q}{q-1} \right)^{r+1} (s)_{r+1} \cdot S^3 \zeta_q(s', s_{d-1} + s_d + r + 1),
\]
\[
D_{-1} = - \sum_{0 < k_1 < \ldots < k_{d-2} < k_{d-1}} q^{k_1(s_1-1) + \ldots + k_{d-1}(s_{d-1}-1)} \sum_{n \in \mathbb{Z} \setminus \{0\}} T(q, s, n, M-1) \left( \frac{\log q}{q-1} \right)^{M+1} (s)_{M+1} \cdot \int_{k_{d-1}}^{\infty} e^{2\pi i n x} q^{x(s-2+M)} \left( \frac{1 - q^x}{1-q} \right)^{-s-M-1} dx
\]
\[
= \sum_{0 < k_1 < \ldots < k_{d-2} < k_{d-1}} q^{k_1(s_1-1) + \ldots + k_{d-2}(s_{d-2}-1)} \sum_{k=k_{d-2}+1}^{\infty} R(M, q, k, s_{d-1}, s),
\]
where we replace the index $k_{d-1}$ by $k$. Similarly,
\[
C_0 = 3 \log q \sum_{r=1}^{M-1} \sum_{n \in \mathbb{Z} \setminus \{0\}} T(q, s, n, r) \left( \frac{\log q}{q-1} \right)^{r} (s)_{r} \cdot S^2 \zeta_q(s', s_{d-1} + s_d + r),
\]
\[
D_0 = - 3 \log q \sum_{0 < k_1 < \ldots < k_{d-2} < k_{d-1}} q^{k_1(s_1-1) + \ldots + k_{d-1}(s_{d-1}-1)} \sum_{n \in \mathbb{Z} \setminus \{0\}} T(q, s, n, M-1) \left( \frac{\log q}{q-1} \right)^{M} (s)_{M} \cdot \int_{k_{d-1}}^{\infty} e^{2\pi i n x} q^{x(s-2+M)} \left( \frac{1 - q^x}{1-q} \right)^{-s-M} dx,
\]
and
\[
C_1 = \log q \sum_{r=1}^{M-1} \sum_{n \in \mathbb{Z} \setminus \{0\}} T(q, s, n, r) \left( \frac{\log q}{q-1} \right)^{r-1} (s)_{r-1} \cdot S \zeta_q(s', s_{d-1} + s_d + r - 1),
\]
\[
D_1 = - \log q \sum_{0 < k_1 < \ldots < k_{d-2} < k_{d-1}} q^{k_1(s_1-1) + \ldots + k_{d-1}(s_{d-1}-1)} \sum_{n \in \mathbb{Z} \setminus \{0\}} T(q, s, n, M-1) \left( \frac{\log q}{q-1} \right)^{M-1} (s)_{M-1} \cdot \int_{k_{d-1}}^{\infty} e^{2\pi i n x} q^{x(s-2+M)} \left( \frac{1 - q^x}{1-q} \right)^{-s-M+1} dx.
\]

The crucial step next is to control the summations over $k_{d-1}$ and show that they converge uniformly with respect to $q$. When $0 < q \leq 1/2$ this is clear. The only non-trivial part is when $q \uparrow 1$. So we assume $1/2 < q < 1$. Note that
\[
\lim_{q \uparrow 1} T(q, s, n, r) = \frac{1}{(2\pi i n)^r}, \quad \lim_{q \uparrow 1} \log q = 1.
\]

**Lemma 4.1.** Let $s_d = \sigma + i\tau$. Let $q_0 = \max\{1/2, e^{(6-2e)/\tau}\}$ if $\tau > 0$ and let $q_0 = 1/2$ if $\tau \leq 0$. Then for all $1 > q > q_0$ and positive integer $k$ we have
\[
\left| \frac{\log q}{q - 1} \right| < 2, \quad \text{and} \quad |T(q, s, n, r)| < \frac{1}{(6\pi)^r}.
\]

**Proof.** Let $f(q) = 2(1-q) + \log q$. Then $f'(q) = -2 + 1/q < 0$ whenever $q > q_0$. So $f(q) > f(1) = 0$ whenever $1 > q > q_0$. This implies that $2(1-q) > -\log q$ whence $\log q/(q-1) < 2$. 

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To bound $T(q, s, n, r)$ we consider each of its factors in definition \ref{0}. For each $0 \leq j < r$ we have
\[
|2\pi n + (s - 1 + j) \log q|^2 = \left((\sigma - 1 + j) \log g\right)^2 + (2\pi + \tau \log q)^2 \geq (2\pi + \tau \log q)^2
\]
which is independent of $j$. If $\tau \leq 0$ then clearly $|2\pi n + (s - 1 + j) \log q| > 6n$. If $\tau > 0$ then it follows from $q > e^{(6-2\pi)\tau}$ that
\[
2\pi + \tau \log q > 2\pi n + 6 - 2\tau \geq 6n,
\]
as desired. \hfill \Box

Next we want to bound the integral terms in $D_{-1}$. Let $|\sigma_d| < N$ and $|\sigma_{d-1}| < N'$ for some positive integers $N$ and $N'$. Fix an arbitrary $x > k$ and a positive integer $M > 16 + 2N + 6 \sum_{j=1}^{d-1} (N_j + 1)$. Then
\[
q^{-(M/6-N'-1)} \left| q^{k(s'-1)} x^{(s-2+M)} \left( \frac{1-q^x}{1-q} \right)^{-s-M-1} \right| < q^{x(M-N-2)-kM/6} \left( \frac{1-q}{1-q^x} \right)^{M-N+1}.
\]
Denote by $g(q)$ the right hand side of the above inequality.

**Lemma 4.2.** Let $1/2 < q < 1$. Then $g(q)$ is increasing as a function of $q$ so that
\[
g(q) \leq \lim_{q \uparrow 1} g(q) = \frac{1}{x^M-N+1}.
\]

**Proof.** Taking the logarithmic derivative of $g(q)$ we have
\[
g'(q) = \frac{g(q)}{g(q)} = \frac{x(M-N-2)-kM/6 + (M-N+1)xq^{x-1}(1-q) - (1-q^x)}{(1-q)(1-q^x)}
\]
whose numerator is denoted by $h(q)$. Then
\[
h'(q) = \left( (x+1)q^x - qx^{x-1} - 1 \right) (x(M-N-2) - kM/6)
\]
\[+ (M-N+1) \left( x(q^{x-1} - (x+1)q^x) - 1 + (x+1)q^x \right).
\]
Clearly $h'(1) = 0$ and moreover
\[
q^{2-x} h''(q) = (x(x+1)q - x(x-1))(x(M-N-2) - kM/6)
\]
\[+ (M-N+1) \left( x(x+1)q - x(x+1)q + x(x+1)q \right).
\]
\[= x(x-1)(kM/6 + 3x) + qx(x+1)(M-N+1 - (kM/6 + 3x))
\]
\[= (kM/6 + 3x)(x^2(1-q) - x(1+q)) + qx(x+1)(M-N+1)
\]
\[> qx(x+1)(M-N+1) - x(1+q)(kM/6 + 3x) \quad \text{(since } 1 > q) \]
\[\geq qx^2 \left\{ M-N+1 - \frac{1+q}{q} (3 + M/6) \right\} \quad \text{(since } k < x) \]
\[> qx^2 (M/2 - N - 8) > 0
\]
where we used the fact that if $q > 1/2$ then $(1+q)/q < 3$. This implies that $h'(q)$ is increasing so that $h'(q) < 0$ for all $1 > q > 1/2$ (recall that $h'(1) = 0$). It follows that $h(q)$ is decreasing. But $h(1) = 0$ so we know $h(q) > 0$ for all such $q$. Thus $g'(q) > 0$ and therefore $g(q)$ is increasing. This completes the proof of the lemma. \hfill \Box
We now can bound the innermost sum of $D_{-1}$. From Lemma 2.1, Lemma 1.1 and Lemma 4.2 we have (if $d = 2$ then take $k_0 = 1$)

$$
\sum_{k=1+k_{d-2}}^{\infty} |R(M, q, k, s', s)| < \sum_{k=1+k_{d-2}}^{\infty} \frac{(2k)^N}{4\pi^2 M-1} 2^{M+1}(N)_{M+1} \sum_{k=1+k_{d-2}}^{\infty} q^{k(M/6-N'-1)} \int_k^{\infty} \frac{dx}{x^{M-N+1}} < (M + 1)! \left( \frac{M + N}{M + 1} \right) \sum_{k=1+k_{d-2}}^{\infty} \frac{q^{k(M/6-N'-1)}}{kM-N-N'}
$$

$$
< (M + 1)!(2M)^{M+1} \sum_{k=1+k_{d-2}}^{\infty} q^{k(M/6-N'-1)}
$$

since $2^{N'+M+2}(M + 1) < 4\pi^2 M^{-1}$ and $M - N > 2$. Therefore by Lemma 2.1

$$
|D_{-1}| < (2M)^{2M} \sum_{0 < k_1 < \cdots < k_{d-2}} \left( \prod_{l=1}^{d-2} k_l^{N_q} q^{k_l(-N_l-1)} \right) q^{k_{d-2}(M/6-N_{d-1}-1)}
$$

which converges as proved in Prop. 2.2.

Exactly the same argument applies to the integral terms in $D_0$ and $D_1$, which we leave to the interested readers. These convergence results imply two things. First we can show by induction on $d$ that (13) gives rise to an analytic continuation of $\zeta(s_1, \ldots, s_d)$ as a meromorphic function on $\C^d \setminus \Gamma_d$. Second, also by induction on $d$, we can conclude that its legitimate to take the limit $q \uparrow 1$ inside the sums of $C_n$ and $D_n$ to get (note that $\lim_{q \uparrow 1} S^n \zeta(s) = \zeta(s)$ for any $s \in \C^{d-1} \setminus \Gamma_d'$ and any positive integer $n$)

$$
\lim_{q \uparrow 1} \zeta(s', s_{d-1}, s_d) = \frac{1}{s-1} \zeta(s', s_{d-1} + s_d - 1) - \frac{1}{2} \zeta(s', s_{d-1} + s_d + 1) - \sum_{r=1}^{M-1} \sum_{n \in \Z \setminus \{0\}} \frac{1}{(2\pi i n)^{r+2}} (s)_{r+1} \cdot \zeta(s', s_{d-1} + s_d + r + 1) + \sum_{0 < k_1 < \cdots < k_{d-1}} \frac{1}{k_1 \cdots k_{d-1}} \sum_{n \in \Z \setminus \{0\}} \frac{(s)_{M+1}}{(2\pi i n)^{M+1}} \int_{k_{d-1}}^{\infty} e^{2\pi i n x} x^{s-M-1} dx
$$

\begin{align*}
&= \sum_{r=0}^{M+1} \frac{B_r}{r!} (s)_{r+1} \cdot \zeta(s_1, \ldots, s_{d-1} + s_r + 1) - \frac{1}{(M + 1)!} \sum_{0 < k_1 < \cdots < k_{d-1}} \frac{1}{k_1 \cdots k_{d-1}} \int_{k_{d-1}}^{\infty} B_{M+1}(x) \frac{(s)_{M+1}}{x^{s+M+1}} dx
\end{align*}

by (13) and its specialization with $x = 1$

$$
\sum_{n \in \Z \setminus \{0\}} \frac{1}{(2\pi i n)^{r+2}} = -\frac{B_{r+2}(1)}{(r + 2)!} = -\frac{B_{r+2}}{(r + 2)!}.
$$

The main theorem now mostly follows from Thm. 5.2 The poles at $s_d = m - \frac{2\pi i}{\log q} n$ are given by the first term in formula (13) when $m = 1$ and $n = 0$ and by the terms $T(q, s_d, n, r)$ as defined in (10) if $m \leq 1$ and $n \neq 0$. The location of the other poles are obtained by induction using those poles of the $q$-Riemann zeta function presented in Thm. 1.1 for the initial step. This completes the proof of our main theorem.

5 Series $q$-shuffle relations

The classical multiple zeta functions satisfy shuffle relations originating from their series representations. For example,

$$
\zeta(s_1) \zeta(s_2) = \zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2).
$$

(17)
In general we can define a shuffle operation $\ast$ on finite words so that for any complex numbers $a$, $b$ (regarded as letters) and words $w_1$ and $w_2$ of complex numbers

$$aw_1 \ast bw_2 = a(w_1 \ast bw_2) + b(aw_1 \ast w_2) + (a + b)(w_1 \ast w_2)$$

so that treating complex variables as words we have

$$\zeta(w_1)\zeta(w_2) = \zeta(w_1 \ast w_2).$$

To generalize these relations we first define shifting operators on words of complex variables:

$$(s_1, \ldots, s_{i-1}, \mathcal{S}(s_i), s_{i+1}, \ldots, s_d) = \mathcal{S}_i(s_1, \ldots, s_d)$$

and set $\zeta(\mathcal{S}(s_1, \ldots, s_d)) = \mathcal{S}_i \zeta(s_1, \ldots, s_d)$. Now we define the $q$-shuffle operator $*_q$ on words by

$$aw_1 *_q bw_2 = a(w_1 *_q bw_2) + b(aw_1 *_q w_2) + S(a + b)(w_1 *_q w_2).$$

**Theorem 5.1.** For any two words of complex variables $w_1$ and $w_2$ we have

$$\zeta_q(w_1)\zeta_q(w_2) = \zeta_q(w_1 *_q w_2).$$

**Proof.** Induction on the length of $w_1w_2$. 

For example

$$\zeta_q(s_1)\zeta_q(s_2) = \zeta_q(s_1, s_2) + \zeta_q(s_2, s_1) + \zeta_q(s_1 + s_2) + (q - 1)\zeta_q(s_1 + s_2 - 1).$$

We can recover the shuffle relation of (17) by taking $\lim_{q \uparrow 1}$. 

### 6 Special values of $\zeta_q(s_1, s_2)$

For integers $n_1, \ldots, n_d$ we set

$$\zeta_q(n_1, \ldots, n_d) = \lim_{s_1 \rightarrow n_1} \cdots \lim_{s_d \rightarrow n_d} \zeta_q(s_1, \ldots, s_d),$$

$$\zeta_q^R(n_1, \ldots, n_d) = \lim_{s_d \rightarrow n_d} \cdots \lim_{s_1 \rightarrow n_1} \zeta_q(s_1, \ldots, s_d)$$

if the limits exist. Interesting phenomena occur already in the case $d = 2$ and these should be generalized to arbitrary depth. By Thm. 2.4 we get

$$\zeta_q(s_1, s_2) = (1 - q)^{s_1 + s_2} \sum_{r_1, r_2 = 0}^{+\infty} \binom{s_1 + r_1 - 1}{r_1} \binom{s_2 + r_2 - 1}{r_2} q^{2s_2 + 2r_2 + s_1 + r_1 - 3} (1 - q^{s_2 + r_2 - 1})(1 - q^{s_2 + r_2 + s_1 + r_1 - 2})$$

$$+ \frac{q^{2s_2 + s_1 - 3}}{s_2 q^{s_2 + s_1 - 1} (1 - q^{s_2 - 1})(1 - q^{s_2 + s_1 - 2})} + \frac{s_1 q^{s_2 + s_1 - 2}}{s_1 s_2 q^{s_2 + s_1}} + \frac{1}{s_1 (s_1 + 1) q^{2s_2 + s_1 - 1}} + \frac{s_2 (s_2 + 1) q^{2s_2 + s_1 + 1}}{2(1 - q^{s_2 + 1})(1 - q^{s_2 + s_1})} + \cdots$$

Clearly we have

$$\zeta_q(0, 0) = \lim_{s_1 \rightarrow 0} \lim_{s_2 \rightarrow 0} \zeta_q(s_1, s_2) = \frac{1}{(q^2 - 1)(q - 1)} - \frac{3}{2(q - 1) \log q} + \frac{1}{\log^2 q},$$

$$\zeta_q^R(0, 0) = \lim_{s_2 \rightarrow 0} \lim_{s_1 \rightarrow 0} \zeta_q(s_1, s_2) = \frac{1}{(q^2 - 1)(q - 1)} - \frac{1}{(q - 1) \log q} + \frac{q}{2(q - 1) \log q}.$$
It is not too hard to find that
\[
\lim_{q \to 1} \zeta_q(0,0) = \frac{1}{3}, \quad \lim_{q \to 1} \zeta_q^R(0,0) = \frac{5}{12}.
\]

This is consistent with what we found in [11] by using generalized functions (distributions). See also equations (22) below. In [11] we further showed that near (0,0) the double zeta function has the following asymptotic expansion:

\[
\zeta(s_1, s_2) = \frac{4s_1 + 5s_2}{12(s_1 + s_2)} + R(s_1, s_2)
\]

where \( R(s_1, s_2) \) is analytic at (0,0) and \( \lim_{(s_1, s_2) \to (0,0)} R(s_1, s_2) = 0. \)

Let \( n, k \) be two non-negative integers, and \( m = k - n - 2. \) We now consider the double zeta function around \((s_1, s_2) = (-m, -n)\) which has the following expression by Thm. 3.2:

\[
\zeta(-m, -n) = \lim_{s_1 \to -m} \lim_{s_2 \to -n} \zeta(s_1, s_2)
\]

while it has possibly nontrivial contribution for \( \zeta^R(-m, -n) \) since

\[
\lim_{s_2 \to -n} \lim_{s_1 \to -m} (s_2 + n)\zeta(s_2 + s_1 + k - 1) = 1.
\]

We get

| \(k, n\) | pole, residue | indeterminacy, \(\zeta = \zeta^R(n + 2 - k, -n)\) |
|---|---|---|
| \(k = 0\) | \(-1/(n + 1)\) | NO |
| \(k = 1\) | \(-1/2\) | NO |
| \(2 \nmid k, 3 \leq k \leq n + 1\) | NO | \(B_{k-1}/2(k-1) + \sum_{r=1}^{n+1} \frac{B_r}{r} \left( \frac{n}{r-1} \right) \zeta(r + 1 - k)\) |
| \(2 \nmid k, k = n + 2\) | NO | \(B_{k-1}/(k-1)\) |
| \(2 \nmid k, k > n + 2\) | NO | \(B_{k-1}/2(k-1)\) |
| \(2 \nmid k, 2 \leq k \leq n + 1\) | \((-1)^{k+1} \frac{B_k}{k} \left( \frac{n}{k-1} \right)\) | NO |

Table 1: Poles and indeterminacy of double zeta function.

When \(m \geq 0\) and \(2|k\) the values of \(\zeta(-m, -n)\) and \(\zeta^R(-m, -n)\) are different in general:

\[
\zeta(-m, -n) = \frac{B_k}{k(n + 1)} + \sum_{r=1}^{n+1} \frac{B_r}{r} \left( \frac{n}{r-1} \right) \frac{B_{k-r}}{k-r}
\]

\(\zeta^R(-m, -n) = \zeta(-m, -n) + \left(-1\right)^n \frac{B_k}{k!} n! (k - n - 2)!. \)

Note that the term corresponding to \(r = 1\) is non-zero if and only if \(k = 2\) (and \(n = 0\)). From this observation we again recover that

\[
\zeta(0,0) = \frac{1}{3}, \quad \zeta^R(0,0) = \frac{5}{12}.
\]

We now consider the \(q\)-double zeta function.
Theorem 6.1. Let \( k, n \) be two non-negative integers, and \( m = k - n - 2 \). If \( m \leq -1 \) then the \( q \)-double zeta function \( \zeta_q(s_1, s_2) \) has a pole at \((-m, -n)\) with residue given by:

\[
\text{Res}_{(s_1, s_2)=(-m, -n)} \zeta_q(s_1, s_2) = \frac{\sum_{r=0}^{k} (-1)^r \binom{n+1-r}{k-r} \binom{n}{r} / (q^{n+1-r} - 1)}{-(1-q)^{2-k}(\log q)^{-1}} = \begin{cases} 
\sum_{r=0}^{k} (-1)^r \binom{n+1-r}{k-r} \binom{n}{r} / (q^{n+1-r} - 1) & \text{if } k \leq n, \\
\sum_{r=0}^{k} (-1)^r \binom{n}{r} / (q^{n+1-r} - 1) - \frac{(-1)^n}{(n+1)\log q} & \text{if } k = n + 1.
\end{cases}
\]

Proof. Use Thm. 6.1.

Corollary 6.2. Let \( n \) be a non-negative integer. Then

\[
\text{Res}_{(s_1, s_2)=(1,-n)} \zeta_q(s_1, s_2) = \frac{q-1}{\log q} \zeta_q(-n)
\]

and

\[
\lim_{q \uparrow 1} \zeta_q(-n) = -\frac{B_{n+1}}{n+1} = \text{Res}_{(s_1, s_2)=(1,-n)} \zeta_q(s_1, s_2).
\]

Proof. Equation (24) follows from the case \( m = -1 \) in the above theorem and (6) [5]:

\[
\zeta_q(-n) = (1-q)^{-n} \left\{ \sum_{r=0}^{n} (-1)^r \binom{n}{r} / (q^{n+1-r} - 1) - \frac{(-1)^n}{(n+1)\log q} \right\}.
\]

The first equality in (24) is [5] Thm. 1 and the second equality follows from Table I.

Corollary 6.3. Let \( k, n \) be two non-negative integers, \( m = k - n - 2 \leq -2 \). Then

\[
\lim_{q \uparrow 1} \text{Res}_{(s_1, s_2)=(-m,-n)} \zeta_q(s_1, s_2) = \text{Res}_{(s_1, s_2)=(-m,-n)} \zeta_q(s_1, s_2).
\]

Proof. By Thm. 6.1 and Table I we only need to prove

\[
\text{Res}_{(s_1, s_2)=(-m,-n)} \zeta_q(s_1, s_2) = \frac{1}{(q^{n+1-r} - 1)} = e^{(n+1-r)\log q} - 1 = \sum_{l=0}^{\infty} \frac{B_l}{l!} (n+1-r)\log q)^{l-1}.
\]

First by generating function of the Bernoulli numbers

\[
\frac{1}{q^{n+1-r} - 1} = e^{(n+1-r)\log q} - 1 = \sum_{l=0}^{\infty} \frac{B_l}{l!} (n+1-r)\log q)^{l-1}.
\]

Plugging this into the left hand side of equation (26), replacing \( 1-q \) by \( -\log q \), and exchanging the summation we get

\[
(-\log q)^{1-k} \sum_{r=0}^{k} (-1)^r \binom{n+1-r}{k-r} \binom{n}{r} / (q^{n+1-r} - 1) = (-1)^{1-k} \sum_{l=0}^{\infty} \frac{B_l}{l!} (-\log q)^{1-k} \sum_{r=0}^{k} (-1)^r \binom{n+1-r}{k-r} \binom{n}{r} (n+1-r)^{l-1}.
\]
Then the inner sum over \( r \) is the coefficient of \( x^k \) of the following polynomial

\[
f_l(x) = \sum_{k=0}^{n+1} \sum_{r=0}^{k} (-1)^r \binom{n+1-r}{k-r} \binom{n}{r} (n+1-r)^{l-1} x^k
\]

\[
= \sum_{r=0}^{n} (-1)^r \binom{n}{r} (n+1-r)^{l-1} \sum_{k=r}^{n+1} \binom{n+1-r}{k-r} x^k
\]

\[
=(x+1)^{n+1} \sum_{r=0}^{n} (-y)^r \binom{n}{r} (n+1-r)^{l-1},
\]

where \( y = x/(x+1) \). When \( l = 0 \) this expression becomes

\[
f_0(x) = \frac{(x+1)^{n+1}}{n+1} \sum_{r=0}^{n} (-y)^r \binom{n+1}{r} = \frac{(x+1)^{n+1}}{n+1} \left( (1-y)^{n+1} - (-y)^{n+1} \right) = \frac{1}{n+1} \left[ 1 - (-x)^{n+1} \right].
\]

Note that \( k \leq n \) we see the coefficient of \( x^k \) in \( f_0(x) \) is 0 if \( k > 0 \) and it’s \( 1/(n+1) \) if \( k = 0 \). If \( k = 0 \) then only the constant term \( -1/(n+1) \) in \( f_0(x) \) remains when \( q \uparrow 1 \) which proves the corollary in this case. So we can assume \( l, k > 0 \). Then

\[
f_l(x) = \left. \left( \frac{d}{dz} \right)^{l-1} \left\{ (x+1)^{n+1} \sum_{r=0}^{n} (-y)^r \binom{n}{r} z^{n+1-r} \right\} \right|_{z=1}
\]

\[
= \left. \left( \frac{d}{dz} \right)^{l-1} \left\{ (x+1)^{n+1} z(z-y)^n \right\} \right|_{z=1}.
\]

Note that highest degree term in \( f_l(x) \) is contained in

\[
(x+1)^{n+1} \left. \left( \frac{d}{dz} \right)^{l-1} (z-y)^n \right|_{z=1}
\]

\[
=n(n-1) \cdots (n-l+2)(x+1)^{n+1}(1-y)^{n-l+1}
\]

\[
=n(n-1) \cdots (n-l+2)(x+1)^{l}.
\]

If \( l = 1 \) one can easily modify this to get just \( x+1 \). If \( l < k \) then the coefficient of \( x^k \) in \( f_l(x) \) is 0. If \( l = k \) it is equal to

\[
n(n-1) \cdots (n-k+2) = (k-1)! \left\{ \binom{n}{k-1} \right\}.
\]

The last express is valid even for \( k = l = 1 \). Thus the range of \( l \) in the outer sum of \( f_l(x) \) starts from \( k \). Moreover, the first term of \( f_l(x) \) is

\[
(-1)^{1-k} \frac{B_k}{k} \binom{n}{k-1}
\]

as desired. This completes the proof of the corollary.

\[\square\]

**Proposition 6.4.** Let \( k, n \) be two non-negative integers such that \( n \geq k \) and \( k \) is even. Let \( m = k - n - 2 \). Then

\[
\text{Res}_{(s_1, s_2) = (-m, -n)} \zeta_q(s_1, s_2) = \frac{-f(q)(q-1)/\log q}{D(q)}, \quad D(q) = \prod_{j=n+1-k}^{n+1} F(q, j)^{\epsilon_j}
\]

where \( f(q) \in \mathbb{Z}[q] \) is a palindrome with leading coefficient \( \binom{n}{k} \), \( F(q, j) \in \mathbb{Z}[q] \) is a factor of \( (q^3 - 1)/(q-1) \), \( \epsilon = 0 \) or 1, and \( \text{deg}_q D(q) = n + \text{deg}_q f(q) \), such that

\[
\lim_{q \uparrow 1} \text{Res}_{(s_1, s_2) = (-m, -n)} \zeta_q(s_1, s_2) = \text{Res}_{(s_1, s_2) = (-m, -n)} \zeta(s_1, s_2).
\]
**Proof.** The computational proof is left as an exercise for the interested readers. \qed

**Example 6.5.** By Thm. 6.1 we find with the help of Maple

$$\lim_{q \to q^1} \text{Res}_{(s_1, s_2) = (4, -4)} \zeta_q(s_1, s_2) = \frac{-2q^3(3q^2 + 4q + 3)(q - 1)/\log q}{P_1(q, 2)P_1(q, 3)P_1(q, 4)}.$$ 

where $P_a(q, m) = \sum_{j=0}^{m} q^{aj}$. Moreover we can check that

$$\lim_{q \to q^1} \text{Res}_{(s_1, s_2) = (4, -4)} \zeta_q(s_1, s_2) = \text{Res}_{(s_1, s_2) = (4, -4)} \zeta(s_1, s_2) = -\frac{1}{3}$$

by Table 1 with $k = 2$ and $n = 4$.

**Example 6.6.** From Thm. 6.1 we get

$$\lim_{q \to q^1} \text{Res}_{(s_1, s_2) = (6, -8)} \zeta_q(s_1, s_2) = \frac{-14q^5g(q)(q - 1)/\log q}{P_1(q, 4)P_1(q, 5)P_1(q, 6)P_2(q, 2)P_3(q, 2)}$$

where $g(q)$ is a polynomial in $q$ of degree 14 satisfying

$$q^{14}f(1/q) = f(q) = 5q^{14} + 6q^{13} + 8q^{12} + 7q^{11} - q^{10} - 20q^9 - 30q^8 - 34q^7 - \cdots.$$ 

Then we can compute with Maple

$$\lim_{q \to q^1} \text{Res}_{(s_1, s_2) = (6, -8)} \zeta_q(s_1, s_2) = \text{Res}_{(s_1, s_2) = (6, -8)} \zeta(s_1, s_2) = \frac{7}{15}$$

by Table 1 with $k = 4$ and $n = 8$.

**Example 6.7.** Consider the point $(s_1, s_2) = (5, -9)$. We have

$$\lim_{q \to q^1} \text{Res}_{(s_1, s_2) = (5, -9)} \zeta_q(s_1, s_2) = \frac{-42q^4g(q)(q - 1)/\log q}{P_1(q, 4)P_1(q, 6)P_1(q, 7)P_2(q, 2)P_3(q, 2)A(q, 4)}$$

where $A(q, m) = \sum_{j=0}^{m} (-1)^j q^j$ and $g(q)$ is a polynomial in $q$ of degree 18 satisfying

$$q^{18}g(1/q) = g(q) = 2q^{18} - q^{17} - 7q^{15} - 11q^{14} - 16q^{13} - 4q^{12} + 9q^{11} + 28q^{10} + 30q^9 + \cdots$$

so that we again have the equality

$$\lim_{q \to q^1} \text{Res}_{(s_1, s_2) = (5, -9)} \zeta_q(s_1, s_2) = \text{Res}_{(s_1, s_2) = (5, -9)} \zeta(s_1, s_2) = -\frac{1}{2}$$

by Table 1 with $k = 6$ and $n = 9$.

**Theorem 6.8.** Let $m, n$ be two non-negative integers and $k = m + n + 2$. Then $\zeta_q(s_1, s_2)$ has indeterminacy at $(-m, -n)$ such that

$$\zeta_q(-m, -n) = (1 - q)^{2-k} \left\{ \frac{(-1)^k}{(m+1)(n+1)(\log q)^2} + \sum_{r=0}^{m} \frac{(-1)^{r+n+1}}{(n+1)\log q - m} \left( \begin{array}{c} m \\ r \end{array} \right) q^{n+1-r} - 1 \right\}$$

$$\quad \quad \quad \quad + \sum_{r=0}^{n} \frac{(-1)^{r+m+1} m!(n+1-r)!}{\log q} \left( \begin{array}{c} n \\ k-r \end{array} \right) \left( \begin{array}{c} n \\ r \end{array} \right) q^{n+1-r} - 1$$

$$\quad \quad \quad \quad + \sum_{r_1=0}^{m} \sum_{r_2=0}^{n} \frac{(-1)^{r_1+r_2}}{m!} \left( \begin{array}{c} m \\ r_1 \end{array} \right) \left( \begin{array}{c} n \\ r_2 \end{array} \right) \frac{1}{q^{n+1-r_2} - 1} \frac{1}{q^{k-r_1-r_2} - 1} \right\}.$$
and
\[\zeta_q^R(-m, -n) = (1 - q)^{2-k}\left\{\sum_{r=0}^{m} \frac{(-1)^{r+n+1}}{(n+1)\log q} \binom{m}{r} q^{m+1-r} - 1\right\} + \sum_{r=0}^{m} \frac{(-1)^{r+n}}{\log q} \binom{k-n-2}{r} n!(m+1-r)! \frac{q^{m+1-r}}{(k-r)!} \frac{1}{q^{m+1-r} - 1} + \sum_{r_1=0}^{m} \sum_{r_2=0}^{n} (-1)^{r_1+r_2} \binom{m}{r_1} \binom{n}{r_2} q^{n+1-r_2} - 1 \frac{1}{q^{k-r_1-r_2} - 1}\].

**Proof.** Use Thm. \[2.4\]

Similar to Cor. \[6.2\] and Cor. \[6.3\] we have

**Corollary 6.9.** Let \(m\) and \(n\) be two non-negative integers. Then
\[
\lim_{q \uparrow 1} \zeta_q(-m, -n) = \zeta(s_1, s_2), \quad \lim_{q \uparrow 1} \zeta_q^R(-m, -n) = \zeta^R(s_1, s_2).
\]

**Proof.** Set \(k = m + n + 2\). We consider \(\zeta_q^R(-m, -n)\) first. From Thm. \[6.8\] we get
\[
\zeta_q^R(-m, -n) = (q - 1)^{-k} \left(\frac{q - 1}{\log q}\right)^2 (A + B + C),
\]
where
\[
A = \sum_{j=0}^{\infty} \frac{B_j}{j!} (\log q)^j \sum_{r=0}^{m} \frac{(-1)^{m+r+1}}{n+1} \binom{m}{r} (m+1-r)^{j-1},
\]
\[
B = \sum_{i=0}^{\infty} \frac{B_i}{i!} (\log q)^i \sum_{r=0}^{m} \frac{(-1)^{r+m+i}}{m} n!(m+1-r)! \frac{1}{(k-r)!} (m+1-r)^{i-1},
\]
\[
C = \sum_{i,j=0}^{\infty} \frac{B_i B_j}{i! j! (n+1)} \sum_{r_1=0}^{m} \sum_{r_2=0}^{n} (-1)^{k+r_1+r_2} \binom{m}{r_1} \binom{n+1}{r_2} (n+1-r_2)^{i-1} (k-r_1-r_2)^{j-1}.
\]

We first compute \(B\) as follows. Write
\[
B = \sum_{i=0}^{\infty} \frac{B_i}{i!} (\log q)^i W(m, n, i),
\]
where
\[
W(m, n, i) = \sum_{r=0}^{m} (-1)^{r+k+n+i} \frac{m!n!}{r!(k-r)!} (m+1-r)^i.
\]
If \(i = 0\) then we can prove by decreasing induction on \(n\) that
\[
W(m, n, 0) = \sum_{r=0}^{m} (-1)^{r+k+n} \frac{m!n!}{r!(k-r)!} = \frac{1}{k(n+1)}.
\]
This is trivial if \(n = k - 2\). Suppose \[30\] is true for \(n \geq 1\) then we have
\[
W(m, n-1, 0) = -\frac{k-n-1}{n} W(m, n, 0) + \frac{(n-1)!}{(n+1)!} = \frac{1}{kn}
\]
as desired.

Similarly, we can compute \(C\) as follows. Put
\[
C = \sum_{i,j=0}^{\infty} \frac{B_i B_j}{i! j!(n+1)} \sum_{r_1=0}^{m} \sum_{r_2=0}^{n} (-1)^{k+r_1+r_2} \binom{m}{r_1} \binom{n+1}{r_2} (n+1-r_2)^{i-1} (k-r_1-r_2)^{j-1}.
\]
We now change the upper limit of \( r_2 \) from \( n \) to \( n + 1 \) in the above. The extra terms correspond to those by setting \( i = 0 \) and \( r_2 = n + 1 \), which produce exactly \( A \). Therefore,

\[
C = \sum_{i,j=0}^{\infty} \frac{B_i B_j (\log q)^{i+j}}{i! j!} V(m, n, i, j) - A
\]

where

\[
V(m, n, i, j) = \sum_{r_1=0}^{m} \sum_{r_2=0}^{n+1} \frac{(-1)^{k+r_1+r_2}}{n+1} \binom{m}{r_1} \binom{n+1}{r_2} (n+1-r_2)^i (k-r_1-r_2)^{j-1}.
\]

For \( j \geq 1 \) we have

\[
V(m, n, i, j) = \left( x \frac{d}{dx} \right)^{j-1} \left\{ \left( y \frac{d}{dy} \right)^i \left\{ \sum_{r_1=0}^{m} \sum_{r_2=0}^{n+1} \frac{(-1)^{k+r_1+r_2}}{n+1} \binom{m}{r_1} \binom{n+1}{r_2} y^{n+1-r_2} x^{k-r_1-r_2} \right\} \right\} \bigg|_{y=1}^{x=1} \]

\[
= \frac{(-1)^k}{n+1} \left( x \frac{d}{dx} \right)^{j-1} \left\{ \left( y \frac{d}{dy} \right)^i \left\{ x (xy-1)^{n+1} (x-1)^{k-n-2} \right\} \right\} \bigg|_{y=1}^{x=1} \]

\[
= \begin{cases} 
0 & \text{if } i+j < k, \\
\frac{(-1)^k n!}{(n+1-i)!} (k-i-1)! & \text{if } i+j = k, i \leq n+1.
\end{cases}
\]

We have used the fact that if \( i+j = k \) and \( i > n+1 \) then \( j < k-n-1 \) and by exchanging the two operators \( x(d/dx) \) and \( y(d/dy) \) we can easily show that (31) is zero. So if \( l < k \) the total contribution to the coefficient of \( (\log q)^l \) from \( V(m, n, i, j) \) with \( j > 0 \) is trivial and if \( l = k \) it is equal to

\[
\sum_{i=0}^{n+1} B_i B_{k-i} (-1)^k n!(k-i-1)! = \begin{cases} 
B_{k-1} & \text{if } 2 \nmid k, \\
\frac{B_k}{k(n+1)} + \sum_{i=1}^{n+1} \frac{B_i B_{k-i}}{i(k-i)} \binom{n}{i-1} & \text{if } 2 \mid k,
\end{cases}
\]

because \( k \geq n+2 \geq 2 \) and \( B_k = 0 \) if \( k \) is odd.

To deal with \( V(m, n, i, 0) \) note that equation (31) still makes sense if we interpret the operator \( x(d/dx)^{-1} \) as follows:

\[
\left( x \frac{d}{dx} \right)^{-1} \left\{ F(x) \right\} \bigg|_{x=1} = \int_0^1 \frac{F(x)}{x} \, dx
\]

whenever \( F(0) = 0 \). Thus we get

\[
V(m, n, i, 0) = \frac{(-1)^k}{n+1} \left( y \frac{d}{dy} \right)^i \left\{ \int_0^1 (xy-1)^{n+1} (x-1)^m \, dx \right\} \bigg|_{y=1}.
\]

Therefore if \( i = 0 \) then we get

\[
V(m, n, 0, 0) = \frac{(-1)^k}{n+1} \int_0^1 (x-1)^{k-1} \, dx = -\frac{1}{k(n+1)} = -W(m, n, 0)
\]

\[
(33)
\]
from equation (30). If \( i \geq 1 \) then integrating by parts we get

\[
\int_0^1 (xy - 1)^{n+1}(x-1)^m \, dx
\]

\[
= \frac{(xy - 1)^{n+2}}{y(n+2)}(x-1)^m \bigg|_0^1 - \frac{m}{y(n+2)} \int_0^1 (xy - 1)^{n+2}(x-1)^{m-1} \, dx
\]

\[
= \cdots \cdots
\]

\[
= (-1)^{k+1} \left( \frac{1}{y(n+2)} - \frac{m}{y^2(n+2)(n+3)} + \cdots + (-1)^{m-1} \frac{m!(n+1)!}{y^m(m+n+1)!} \right) + \cdots
\]

\[
= (-1)^k \frac{m!(n+1)!}{k!} \frac{1}{y^{m+1}} + \sum_{r=0}^{m} (-1)^{r+k+1} \frac{m!(n+1)!}{r!(m-r)!(n+2+r)!}.
\]

It follows from changing the index \( r \) to \( m-r \) that

\[
V(m, n, i, 0) = (-1)^{k+n} \frac{m!n!}{k!} \left( \frac{y}{d} \right)^i \left\{ (y-1)^k \right\} \bigg|_{y=1} + \sum_{r=0}^{m} (-1)^{r+n+i+1} \frac{m!n!}{r!(m-r)!(k-n-2)!} (m+1-r)^i
\]

\[
= \begin{cases} (-1)^{k+1} W(m, n, 0) & \text{if } 0 < i < k, \\ (-1)^{k+1} W(m, n, 0) + (-1)^{k+n} n! (k-n-2)! & \text{if } i = k. \end{cases}
\]

(34)

Thus when \( 0 < i < k \) and \( k \) is even we have \( V(m, n, i, 0) = -W(m, n, i) \). It follows from \( 32 \), \( 33 \) and \( 34 \) that

\[
\lim_{q \uparrow 1} \zeta_q^R(-m, -n) = \zeta^R(s_1, s_2)
\]

since \( B_k = 0 \) if \( k > 2 \) is odd.

Let’s turn to prove the first equality in (30). By Thm. 6.8 we have

\[
\zeta_q(-m, -n) = (q-1)^{-k} \left( \frac{q-1}{\log q} \right)^2 (D + A + E + C)
\]

where \( A \) and \( C \) are as above and

\[
D = \frac{1}{(m+1)(n+1)}, \quad E = \sum_{i=0}^{\infty} \frac{B_i}{i!} (\log q)^i U(m, n, i),
\]

where

\[
U(m, n, i) = -\sum_{r=0}^{n} (-1)^{r+n} \frac{m!n!}{r!(m-r)!} (n+1-r)^i.
\]

Hence

\[
U(m, n, 0) = W(n, m, 0) = \frac{-1}{k(m+1)} = \frac{-1}{k(k-n-1)} = \frac{1}{k(n+1)} - D.
\]

(35)

We only need to show that

\[
U(m, n, i) = \begin{cases} W(m, n, i) & \text{if } 0 < i < k, \\ W(m, n, i) - (-1)^n n! (k-n-2)! & \text{if } i = k. \end{cases}
\]

(36)
It is well known that special values of the multiple zeta function ζ_q can be evaluated at positive integers k and n. For example, it’s easy to compute directly that

\[
\Res_{(s_1,s_2)=(-3,2)} \zeta_q(s_1,s_2) = -\frac{1}{(1-q) \log q} \sum_{r=0}^{3} (-1)^r \binom{3}{r} (r+1) \frac{q^{r+1}}{1-q^{r+1}} = -\frac{q(q-1)^2}{(q+1)(q^2+1)(q^2+q+1) \log q},
\]

which can be obtained also by the shuffle relation \[18\] and the expression

\[
\Res_{(s_1,s_2)=(2,-3)} \zeta_q(s_1,s_2) = \frac{q(q-1)^2}{(q+1)(q^2+1)(q^2+q+1) \log q}
\]

by taking \(k_n = n = 3\) in Thm. \[16\]. Thus \((-3,2)\) is a simple pole of \(\zeta_q(s_1,s_2)\). On the other hand \(\zeta(s_1,s_2)\) does not have a pole along \(s_1 + s_2 = -1\). Indeed we find that

\[
\lim_{q\to 1} \Res_{(s_1,s_2)=(-3,2)} \{ \zeta_q(s_1,s_2) \} = 0.
\]

7 \(q\)-multiple polylogarithms

It is well known that special values of the multiple zeta function \(\zeta(s_1,\ldots,s_d)\) at positive integers \((n_1,\ldots,n_d)\) can be regarded as single-valued version of multiple polylogarithm \(L_{n_1,\ldots,n_d}(z_1,\ldots,z_d)\) evaluated at \(z_1 = \cdots = z_d = 1\). For \(|z_j| < 1\) these functions can be defined as

\[
L_{n_1,\ldots,n_d}(z_1,\ldots,z_d) = \sum_{0 < k_1 < \cdots < k_d} \frac{z_1^{k_1} \cdots z_d^{k_d}}{k_1^{n_1} \cdots k_d^{n_d}}.
\]

By Chen’s iterated integral

\[
(-1)^d \int_0^1 \frac{dt_1}{t_1-a_1} \circ \left( \frac{dt_1}{t_1} \right)^{(n_1-1)} \circ \cdots \circ \frac{dt_d}{t_d-a_d} \circ \left( \frac{dt_d}{t_d} \right)^{(n_d-1)},
\] (37)

where \(a_j = 1/\prod_{i=j}^d z_i\) for all \(j = 1,\ldots,d\), we can obtain the analytic continuation of this function as a multi-valued function on \(\mathbb{C}^d \setminus \mathcal{D}_d\) where

\[
\mathcal{D}_d = \{(z_1,\ldots,z_d) \in \mathbb{C}^d : \prod_{i=j}^d z_i = 1, \; j = 1,\ldots,d\}.
\]
When $|z_j|<1$ we define its $q$-analog ($0<q<1$) by
\[
Li_{q,n_1,...,n_d}(z_1,...,z_d) = \sum_{0<k_1<...<k_d} \frac{z_1^{k_1}...z_d^{k_d}}{[k_1]^{n_1}...[k_d]^{n_d}}.
\]

Clearly when $q \uparrow 1$ we recover the ordinary multiple polylogarithm. Moreover, the special value of $q$-multiple zeta function at positive integers
\[
\zeta_q(n_1,...,n_d) = Li_{q,n_1,...,n_d}(q^{n_1-1},...,q^{n_d-1}).
\]

Note that our definition of the $q$-multiple polylogarithms is different from that of [8]. In case of logarithm and dilogarithm our definitions are different from that of [9]. We want to convince the readers that ours are also good analogs of the ordinary ones.

We can mimic the method in section 2 to get the analytic continuation of $Li_{q,n_1,...,n_d}(z_1,...,z_d)$.

**Theorem 7.1.** The $q$-multiple polylogarithm function $Li_{q,n_1,...,n_d}(z_1,...,z_d)$ converges if $|z_j|<1$ for all $j=1,...,d$. It can be analytically continued to a multi-valued function over $\mathbb{C}^d \setminus \mathcal{D}_{qd}$ via the series expansion
\[
Li_{q,n_1,...,n_d}(z_1,...,z_d) = (1-q)^{n_1+...+n_d} \sum_{r_1,...,r_d=0}^{\infty} \prod_{j=1}^{d} \left( \frac{(n_j + r_j - 1)}{r_j} \right) \frac{z_j^r q^{rj}}{1 - (z_j q)^{r+...+r_d}}.
\]

**Proof.** The first part of the lemma is obvious. Let’s concentrate on the analytic continuation. By binomial expansion $(1-x)^{-n} = \sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r$ we get
\[
Li_{q,n_1,...,n_d}(z_1,...,z_d) = (1-q)^{n_1+...+n_d} \sum_{0<k_1<...<k_d} \prod_{j=1}^{d} \left( \frac{(n_j + r_j - 1)}{r_j} \right) (z_j q)^{k_j}.
\]

As $0<q<1$ the series converges absolutely by Stirling’s formula so we can exchange the summations. The theorem follows immediately from Lemma 2.3 by taking $x_j = z_j q^{r_j}$. 

However, this analytic continuation is not suitable for comparing with its ordinary counterpart. The we define Jackson’s $q$-differential operator (cf. [3]) by
\[
D_{q,z}f(z) = \frac{f(z) - f(qz)}{(1-q)z}.
\]

**Lemma 7.2.** Let $d,n_1,...,n_d$ be positive integers. If $n_j \geq 2$ then we have
\[
D_{q,z}Li_{q,n_1,...,n_d}(z_1,...,z_d) = \frac{1}{z_j} Li_{q,n_1,...,n_j-1,...,n_d}(z_1,...,z_d);
\]
if $d \geq 2$ and $n_j = 1$ then
\[
D_{q,z}Li_{q,n_1,...,n_d}(z_1,...,z_d) = \frac{1}{1-z_j} Li_{q,n_1,...,n_j-1,...,n_d}(z_1,...,z_j-1,z_j,...,z_d)
- \frac{1}{z_j(1-z_j)} Li_{q,n_1,...,n_j,...,n_d}(z_1,...,z_j z_{j+1},...,z_d).
\]

Here the second term does not appear if $j=d$. If $d = n_1 = 1$ then
\[
D_{q,z}Li_{q,1}(z) = \frac{1}{1-z}.
\]

**Proof.** Clear.
The same properties listed in the lemma are satisfied by the ordinary multiple polylogarithms. We note that the first equation in [35 Lemma 1] is valid only for \( n_j \geq 2 \).

Recall that for any continuous function \( f(x) \) on \([a, b]\) Jackson's \( q \)-integral (cf. [2]) is defined by

\[
\int_a^b f(x) \, dq_x := \sum_{i=0}^{\infty} f(a + q^i(b - a))(q^i - q^{i+1})(b - a).
\]

Then for every \( x \in [a, b] \) we have

\[
\int_a^x D_{q,t} f(t) \, dq_t = f(x) - f(a), \quad \text{and} \quad \lim_{q \uparrow 1} \int_a^x f(t) \, dq_t = \int_a^x f(t) \, dt.
\]

**Remark 7.3.** Note that in general \( \int_a^b f(x) \, dq_x + \int_b^c f(x) \, dq_x \neq \int_a^c f(x) \, dq_x \).

Similar to Chen'siterated integrals we can define the \( q \)-iterated integrals as follows:

\[
\int_a^b \frac{dq_{t_1}}{t_1 - a_1} \cdot \ldots \cdot \frac{dq_{t_r}}{t_r - a_r} := \int_a^b \left( \int_a^{t_1} \cdots \int_a^{t_r} \left( \int_a^{t_{r-1}} \frac{dq_{t_r}}{t_r - a_r} \right) \frac{dq_{t_{r-1}}}{t_{r-1} - a_{r-1}} \ldots \frac{dq_{t_2}}{t_2 - a_2} \right) \frac{dq_{t_1}}{t_1 - a_1}.
\]

Define

\[
\mathcal{D}_{q,d} := \{ (z_1, \ldots, z_d) \in \mathbb{C}^d : \prod_{i=j}^d z_i = q^{-m}, \ m \in \mathbb{Z}_{\geq 0}, \ j = 1, \ldots, d \}.
\]

**Corollary 7.4.** We can analytically continue \( Li_{q,n_1,\ldots,n_d}(z_1, \ldots, z_d) \) to \( \mathbb{C}^d \setminus \mathcal{D}_{q,d} \) by the \( q \)-iterated integral

\[
(-1)^d \int_0^1 \frac{dq_{t_1}}{t_1 - a_1} \circ \left( \frac{dq_{t_2}}{t_2 - a_2} \circ \ldots \circ \frac{dq_{t_d}}{t_d - a_d} \circ \left( \frac{dq_{t_1}}{t_1 - a_1} \circ \left( \frac{dq_{t_2}}{t_2 - a_2} \circ \ldots \circ \frac{dq_{t_d}}{t_d - a_d} \right) \right) \right) = \lim_{q \uparrow 1} Li_{q,n_1,\ldots,n_d}(z_1, \ldots, z_d),
\]

where \( a_j = 1/\prod_{i=j}^d z_i \) for all \( j = 1, \ldots, d \). Further, for all \( (z_1, \ldots, z_d) \in \mathbb{C}^d \) such that \( \prod_{i=j}^d z_i \notin [1, +\infty) \) for \( 1 \leq j \leq d \) we have

\[
\lim_{q \uparrow 1} Li_{q,n_1,\ldots,n_d}(z_1, \ldots, z_d) = Li_{n_1,\ldots,n_d}(z_1, \ldots, z_d),
\]

where \( Li_{n_1,\ldots,n_d}(z_1, \ldots, z_d) \) is defined by [37] with the path being the straight line segment from 0 to 1 in \( \mathbb{C}^1 \).

**Proof.** It follows from Lemma [36] and equation [39]. The singular set \( \mathcal{D}_{q,d} \) is determined by Thm. [37] so that for each \( j = 1, \ldots, d \) the function \( 1/(t - a_j) \) is continuous on \([0, 1]\).  

\[
\square
\]

### 8 Iterated integral \( q \)-shuffle relations

In section 5 we encountered some \( q \)-shuffle relations of \( q \)-multiple zeta functions. Classically, multiple zeta values satisfy another kind of shuffle relation coming from their representations by Chen’s iterated integrals. In our setting we have seen that special values of \( q \)-multiple zeta functions can be also represented by \( q \)-iterated integrals. In this last section we would like to study the shuffle relations related to these \( q \)-iterated integrals. We shall see that they’re more involved than their ordinary counterparts. We start by writing

\[
\text{Shff}(u_1 \circ \cdots \circ u_r, u_{r+1} \circ \cdots \circ u_{r+s}) = \sum_{\sigma} u_{\sigma(1)} \circ \cdots \circ u_{\sigma(r+s)},
\]

where \( \sigma \) runs through all the permutations of \( \{1, \ldots, r + s\} \) such that \( \sigma^{-1}(a) < \sigma^{-1}(b) \) whenever \( 1 \leq a < b \leq r \) or \( r + 1 \leq a < b \leq r + s \). For any expressions \( F_i \) we put

\[
\bigcup_{i=1}^r F_i = F_1 \circ \cdots \circ F_r
\]
Lemma 8.1. Let \( u_i = d_q t / (t - a_i) \) and \( v_j = d_q t / (t - b_j) \) where \( |a_i|, |b_j| \leq 1 \) for all \( 1 \leq i \leq r \) and \( 1 \leq j \leq s \). Let \( a \) be any positive number. Then

\[
\int_0^a u_1 \circ \cdots \circ u_r \cdot \int_0^a v_1 \circ \cdots \circ v_s = \int_0^a \text{Shfl}(u_1 \circ \cdots \circ u_r, v_1 \circ \cdots \circ v_s) + \sum_{c=1}^{\min(r,s)} (q-1)^c.
\]

\[
\cdot \sum_{1 \leq i_0 < \cdots < i_c \leq r} \sum_{1 \leq j_0 < \cdots < j_c \leq s} \int_0^a c+1 \cdot \text{Shfl}(u_{i_0+1} \circ \cdots \circ u_{i_0-1}, v_{j_0+1} \circ \cdots \circ v_{j_0-1}) \circ (u_{i_0}, v_{j_0}), \tag{40}
\]

where \( i_0 = j_0 = 0, i_{c+1} = r + 1, j_{c+1} = s + 1, (u_{r+1}, v_{s+1}) = 1 \), and for all \( i, j \)

\[
\langle u_i, v_j \rangle = \frac{t d_q t}{(t-a_i)(t-b_j)} = \begin{cases} \frac{1}{b_i-a_i}, & \text{if } a_i \neq b_j, \\ \frac{1}{b_i-a_i}, & \text{if } a_i = b_j = b. 
\end{cases}
\]

Proof. This can be proved by induction on \( r+s \). We only want to mention that the key formula is

\[
D_{q;x}\{f(x)g(x)\} = \{D_{q;x}\{f(x)\}g(x) + f(x)\{D_{q;x}g(x)\} + x(q-1)\{D_{q;x}f(x)\}\{D_{q;x}g(x)\}\}
\]

\[
\square
\]

We will say the term \( \langle u_i, v_j \rangle \) is a \emph{collapse} in the shuffle. The lemma roughly says that \( q \)-iterated shuffle relations is different from those produced by Chen’s iterated integrals because collapses may occur. The number of collapses is at most \( \min(r,s) \).

Lemma 8 implies that if \( m, n \geq 2 \) are different then

\[
\zeta_q(m)\zeta_q(n) = \int_0^1 \frac{d_q t}{t-q^{-m}} \circ \left( \frac{d_q t}{t} \right)^{\zeta_q(m-1)} \cdot \int_0^1 \frac{d_q t}{t-q^{1-n}} \circ \left( \frac{d_q t}{t} \right)^{\zeta_q(n-1)}
= A_q(m,n) + A_q(m,n) + B_q(m,n), \tag{41}
\]

where

\[
A_q(m,n) = \sum_{a=0}^{m-1} \sum_{c=0}^{\min(a,n)} E(a, n; c) \int_0^1 \frac{d_q t}{t-q^{-m}} \circ \left( \frac{d_q t}{t} \right)^{\zeta_q(m-1-a)} \circ \frac{d_q t}{t-q^{1-n}} \circ \left( \frac{d_q t}{t} \right)^{\zeta_q(n-1-c)},
\]

\[
B_q(m,n) = (q-1) \sum_{c=0}^{\min(m,n)-1} E(m-1, n, 1-c) \int_0^1 \frac{t d_q t}{(t-q^{-m})(t-q^{1-n})} \circ \left( \frac{d_q t}{t} \right)^{\zeta_q(m+n-2-c)}.
\]

Here the coefficient \( E(r, s; c) \) represents \( (q-1)^c \) times the numbers of terms in shuffle of \( u_1 \circ \cdots \circ u_r \) and \( v_1 \circ \cdots \circ v_s \) with \( c \) collapses (see \ref{40}). It is not hard to see that \( E(r, s; 0) = (r+s) \). Thus

\[
E(r, s; c) = (q-1)^c \sum_{1 \leq i_1 < \cdots < i_c \leq r} \prod_{a=1}^{c+1} (i_a + j_a - i_{a-1} - j_{a-1} - 2).
\]

We want to convert the expressions in \ref{41} into something that is close to linear combinations of multiple zeta functions. Since \( m \neq n \) we get

\[
\int_0^1 \frac{t d_q t}{(t-q^{-m})(t-q^{1-n})} \circ \left( \frac{d_q t}{t} \right)^{\zeta_q(m+n-2-c)}
= \int_0^1 \left( \frac{1}{q^{m-n} t - q^{-m}} + \frac{1}{1 - q^{m-n} t - q^{1-n}} \right) \circ \left( \frac{d_q t}{t} \right)^{\zeta_q(m+n-2-c)}
= \frac{1}{q^{m-n} - 1} \sum_{k=1}^{\infty} \frac{q^{(m-1)k}}{[k]^{m+n-1-c}} + \frac{1}{q^{m-n} - 1} \sum_{k=1}^{\infty} \frac{q^{(n-1)k}}{[k]^{m+n-1-c}}.
\]

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Proposition 8.2. For any positive integers \( m, n \) we have

\[
B_q(m, n) = (q - 1) \sum_{c=0}^{\min(m,n)-1} E(m-1, n-1; c) \cdot \\
\left( \frac{1}{q^{m-n}} S^{n-1-c} \zeta_q(m + n - 1 - c) + \frac{1}{q^{m-n}} S^{m-1-c} \zeta_q(m + n - 1 - c) \right).
\]

Proof. It follows from Cor. 2.6.

To handle \( A_q(m, n) \) we need to evaluate

\[
L_{q,m-\alpha,\beta}(q^{m-n}, q^{n-1}) = \int_0^1 \frac{d_q t}{t-q^{1-m}} \circ \left( \frac{d_q t}{t} \right)^{\alpha(m-\alpha-1)} \circ \frac{d_q t}{t-q^{1-n}} \circ \left( \frac{dq t}{t} \right)^{\beta(1-\beta)}
\]

where \( 0 \leq \alpha \leq m-1 \) and \( \max(n, \alpha) \leq \beta \leq n + \alpha \). By Cor. 2.6 we get

\[
L_{q,m-\alpha,\beta}(q^{m-n}, q^{n-1}) = \sum_{i=0}^{\beta-n} \begin{pmatrix} \beta-n \\ i \end{pmatrix} (1-q)^i L_{q,m-\alpha,\beta-i}(q^{m-n}, q^{\beta-i-1}). \tag{42}
\]

So we need to evaluate

\[
L_{q,m-\alpha,\gamma}(q^{m-n}, q^{\gamma-1}) = \sum_{1 \leq k < l} \frac{q^{l(m-n)k}}{|k|^{m-\alpha}} \frac{q^{l(\gamma-1)l}}{|l|^{\gamma}}.
\]

If \( n > \alpha \) then by Cor. 2.6 we get

\[
L_{q,m-\alpha,\gamma}(q^{m-n}, q^{\gamma-1}) = \sum_{j=0}^{n-\alpha-1} \begin{pmatrix} n-\alpha-1 \\ j \end{pmatrix} (1-q)^j \zeta_q(m-\alpha-j, \gamma). \tag{43}
\]

By taking \( \alpha = a, \beta = n + a - c \) and \( \gamma = \beta - i \) in (42) and (43) we get

Proposition 8.3. If \( n > m \) then

\[
A_q(m, n) = \sum_{a=0}^{m-1} \sum_{c=0}^{\min(a,n) - a - c} \sum_{i=0}^{n-a-1} \sum_{j=0}^{n-a-1} E(a, n; c) \left( \begin{pmatrix} a-c \\ i \end{pmatrix} (1-q)^i \left( \begin{pmatrix} n-a-1 \\ j \end{pmatrix} \right) \right) \zeta_q(m-a-j, n+a-c-i).
\]

We now consider the case \( n < m \). For \( j > 0 \) define

\[
T^j \zeta_q(\gamma) = \sum_{l \geq 1} \frac{q^l - q^{1l}}{1 - q^l} \frac{q^{(\gamma-1)l}}{|l|^{\gamma}}.
\]

Set \( T^0 \zeta_q(\gamma) = \lim_{j \to 0} T^j \zeta_q(\gamma) = \sum_{l \geq 1} (l-1)q^{(\gamma-1)l}/|l|^{\gamma} \). To evaluate these we need

Lemma 8.4. For any \( \epsilon \geq 0 \) we have

\[
L_{q,\gamma}(q^{\epsilon+\gamma}) = \sum_{l \geq 1} \frac{q^{(\epsilon+\gamma)l}}{|l|^{\gamma}} = \sum_{i=0}^{\gamma-2} \frac{(i+\epsilon)}{e} (q^{\gamma+i} \zeta_q(\gamma-i) + (q-1)\gamma^{-2} \sum_{l \geq 1} \frac{q^l(q^{(\epsilon+1)l})}{|l|^{\gamma}} - 1).
\]

Proof. Set \( a(\epsilon, \gamma) = L_{q,\gamma}(q^{\epsilon+\gamma}) \). If \( \epsilon = 0 \) we have

\[
a(0, \gamma) = L_{q,\gamma}(q^{\gamma}) = \sum_{i=0}^{\gamma-2} (q^{\gamma+i} \zeta_q(\gamma-i) + (q-1)\gamma^{-1} \sum_{l \geq 1} \frac{q^l}{|l|^2}.
\]

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which confirms the lemma in this case. If \( e \geq 1 \) we have

\[
a(e, \gamma) = \sum_{j=0}^{\gamma-2} (q-1)^j a(e-1, \gamma-j) + (q-1)^{\gamma-1} \sum_{l \geq 1} q^{(e+1)l} / [l] \\
= \sum_{j=0}^{\gamma-2} (q-1)^j \sum_{i=0}^{\gamma-j-2} (q-1)^i \left( \frac{i+e-1}{e-1} \right) \zeta_q(\gamma-i-j) + (q-1)^{\gamma-1} \sum_{l \geq 1} q^{(e+1)l} / [l] \\
= \sum_{\sigma=0}^{\gamma-2} (q-1)^\sigma \left( \frac{\sigma+e}{e} \right) \zeta_q(\gamma-\sigma) + (q-1)^{\gamma-2} \sum_{l \geq 1} q^{(e+1)l} / [l]^2
\]

by induction and the combinatorial identity

\[
\sum_{i=0}^{\sigma} \left( \frac{i+e-1}{e-1} \right) = \left( \frac{\sigma+e}{e} \right).
\]

(44)

\[\square\]

**Corollary 8.5.** For all \( j \geq 1 \) we have

\[
T^j \zeta_q(\gamma) = -\zeta_q(\gamma) + \sum_{i=1}^{\gamma-2} \frac{(q-1)^j}{q^j-1} \left( i+j-1 \right) \zeta_q(\gamma-i) + (q-1)^{\gamma-2} \sum_{l \geq 1} q^{(e+1)l} / [l]^2.
\]

Now we fix \( \gamma \) and for all \( s \geq r \geq 1 \) we set

\[
X_\gamma(r, s) := X(r, s) := L_{\gamma q r, \gamma}(q^r, q^{\gamma-1}).
\]

**Lemma 8.6.** For every \( e \geq 0 \) we have

\[
X_\gamma(r, r+e) = \sum_{i=0}^{r-1} (q-1)^i \left( \frac{i+e}{e} \right) \zeta_q(r-i, \gamma) + (q-1)^r \sum_{j=0}^{e} T^j \zeta_q(\gamma).
\]

**Proof.** Clearly

\[
X(r, r) = \zeta_q(r, \gamma) + \sum_{1 \leq k < l} q^{(r-1)k} \frac{q^{k-1} q^{(\gamma-1)l}}{[l]^2} \\
= \zeta_q(r, \gamma) + (q-1)X(r-1, r-1) \\
= \sum_{i=0}^{r-1} (q-1)^i \zeta_q(r-i, \gamma) + (q-1)^r \sum_{l \geq 1} \frac{(l-1)q^{(\gamma-1)l}}{[l]^2}.
\]

And for \( e \geq 1 \)

\[
X(r, r+e) = X(r, r+e-1) + (q-1)X(r-1, r+e-1) \\
= \sum_{i=0}^{r-1} (q-1)^i X(r-i, r+e-1-i) + (q-1)^r \sum_{1 \leq k < l} q^{ek} \frac{q^{(\gamma-1)l}}{[l]^2} \\
= \sum_{i=0}^{r-1} (q-1)^i X(r-i, r+e-1-i) + (q-1)^r \sum_{l \geq 1} q^e - q^{el} \frac{q^{(\gamma-1)l}}{[l]^2} \\
= \sum_{i=0}^{r-1} (q-1)^i \left( \sum_{j=0}^{r-i-1} \left( \frac{j+e-1}{e-1} \right) \zeta_q(r-i-j, \gamma) \right) + (q-1)^r \sum_{j=0}^{e} T^j \zeta_q(\gamma)
\]

by induction. Let \( \sigma = i + j \) in the first sum of the last expression. Then the lemma follows from (44). \[\square\]
Define \( \xi_q(j) = \sum_{i \geq 1} q^{(i+1)j}/[i]^2 \). We now have

**Proposition 8.7.** If \( n \leq m \) then

\[
A_q(m, n) = \sum_{a=0}^{m-1} \sum_{c=0}^{\min(a,n)-a-1} E(a, n; c) \binom{a-c}{i} (1-q)^{i+j} \binom{n-a-1}{j} \zeta_q(m-a-j, n+a-c-i)
\]

where

\[
B_q(m, n) = \sum_{a=n}^{m-1} \sum_{c=0}^{\min(a,n)-a-1} E(a, n; c) \binom{a-c}{i} (1-q)^{i} X_{n+a-c-i}(m-a, m-n),
\]

where

\[
X_{\gamma}(r, s) = \sum_{i=0}^{r-1} (q-1)^i \binom{i+s-r}{s-r} \zeta_q(r-i, \gamma) + (q-1)^r \sum_{j=1}^{s-r} \sum_{i=0}^{\gamma-2} \frac{(q-1)^i}{q^j-1} \binom{i+j-1}{j-1} \zeta_q(\gamma-i)
\]

\[
+ (q-1)^r \sum_{j=0}^{s-r} \frac{(q-1)^{\gamma-2}}{q^j-1} (\zeta_q(j)-\zeta_q(2)) -(q-1)^r (s-r+1) \zeta_q(\gamma).
\]

**Proof.** This is clear from equations 42 and 43, Lemma 8.6 and Cor. 8.5. 

Putting everything together we arrive at

**Theorem 8.8.** Let \( m \neq n \) be two positive integers no less than 2. Then

\[
\zeta_q(m) \zeta_q(n) = A_q(m, n) + A_q(n, m) + B_q(m, n),
\]

where \( A_q(m, n) \) is given by Prop. S.3 and Prop. S.7 and \( B_q(m, n) \) is given by Prop. S.2.

It is not hard to see that when \( q \uparrow 1 \) we recover the ordinary shuffle relations of the MZVs originally produced by using Chen’s iterated integrals. The only unpleasant terms in our \( q \)-analog are given by \( \xi_q(j) \) which is closely related to \( \zeta_q(2) \).

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