Abstract

Two-dimensional gas dynamics equations in mass Lagrangian coordinates are studied in this paper. The equations describing these flows are reduced to two Euler-Lagrange equations. Using group classification and Noether’s theorem, conservation laws are obtained. Their counterparts in Eulerian coordinates are given. Among these counterparts there are new conservation laws.

Keywords: Lagrangian Coordinates, Gas dynamics equations, Conservation Law, Lie group, Noether’s theorem

1. Introduction

1.1. Lagrangian fluid dynamics

Physical phenomena in continuum mechanics are modeled in two distinct ways. The typical approach for fluid dynamics uses Eulerian coordinates, where the system describes fluid motion at fixed locations. Velocities, density and other properties of fluid particles in the Eulerian description are considered to be functions of time and of fixed space coordinates. In contrast, in the Lagrangian description, particles are identified by the positions which they occupy at some initial time. Typically, Lagrangian coordinates are not applied in the description of fluid motion. However, in some special contexts the Lagrangian description is indeed useful in solving certain problems. Analytically, both coordinate systems are capable of producing exact solutions of fluid flows, including discontinuous flows. They are regarded as equivalent to another, except that the Lagrangian system gives more information: ‘in practice such description is often too detailed and complicated, but it is always implied in formulating physical laws’ [1].

There is extensive literature on Lagrangian fluid dynamics (see, for example, [2, 3, 4] and references therein). These studies can be roughly separated out into two groups: the analysis of equations in the gas dynamics variables in mass Lagrangian coordinates [5, 4] and analysis of Euler-Lagrange equations obtained by varying the Lagrangian in mass Lagrangian coordinates [3, 2, 6, 7]. In the present paper the Euler-Lagrange equations in mass Lagrangian coordinates of the gas dynamics equations are studied.

1.2. Symmetries and conservation laws

Symmetries have always attracted the attention of scientists. One of the tools for studying symmetries is the group analysis method [8, 9, 10, 11, 12], which is a basic method for constructing exact solutions of partial differential equations. The group properties of the gas dynamics equations in Eulerian coordinates were studied in [8,13]. Extensive group analysis of the one-dimensional gas dynamics equations in mass Lagrangian coordinates was given in [14,15]. Here the results of [16] should also be mentioned, where nonlocal conservation laws of the one-dimensional gas dynamics equations in mass Lagrangian coordinates were found. The authors of [3, 2] analyzed the Euler-Lagrange equations corresponding to the one-dimensional gas dynamics equations in mass Lagrangian coordinates and extensions of the known conservation laws were derived. These conservation laws correspond to special forms of the entropy. The group nature of these conservation laws is given in [17], where a complete group classification of these Euler-Lagrange equations is presented. Notice that the hyperbolic shallow water equations are equivalent to the isentropic gas dynamics equations of a polytropic gas with \( \gamma = 2 \). The complete group analysis of the Euler-Lagrange equations of the one-dimensional gas dynamics equations of
isentropic flows of a polytropic gas with \( \gamma = 2 \) was given in [6]. Group properties of the two-dimensional shallow water equations describing flows over a bed which is rotating with position-dependent Coriolis parameter in Lagrangian coordinates are studied in [18]. Conservation laws were constructed by using a Lagrangian of the form presented in [19].

As mentioned above, besides assisting with the construction of exact solutions, the knowledge of an admitted Lie group allows one to derive conservation laws. Conservation laws provide information on the basic properties of solutions of differential equations, and they are also needed in the analyses of stability and global behavior of solutions. Noether’s theorem [20] is the tool which relates symmetries and conservation laws. However, the application of Noether’s theorem depends on the following condition: that the differential equations under consideration can be rewritten as Euler-Lagrange equations with appropriate Lagrangian. Among approaches which try to overcome this limitation one can mention here the approaches developed in [21, 22, 23, 24].

1.3. Noether’s theorem

We begin with the background related to the application of symmetries for constructing conservation laws.[3] Let

\[
X = \xi^i \frac{\partial}{\partial x^i} + \eta^k \frac{\partial}{\partial u^k} + \eta^k \frac{\partial}{\partial u^k} + \ldots
\]

be a Lie-Bäcklund operator, and \( F = F(x, u, p) \). Here \( x = (x^1, x^2, \ldots, x^n) \) are the independent variables, \( u = (u^1, u^2, \ldots, u^m) \) are the dependent variables, and \( p \) denotes their derivatives \( u^1_{0}, \ldots, u^m_{0} \) of finite order.

Noether’s theorem is based on two identities. The first identity is called the Noether identity [25],

\[
XF + FD_i \xi^i = W^k \frac{\delta F}{\delta u^k} + D_i (\mathcal{N}^i F) \tag{1}
\]

where

\[
\frac{\delta}{\delta u^k} = \frac{\partial}{\partial u^k} + \sum_{s=1}^{n} (-1)^s D_{i_1} \ldots D_{i_s} \frac{\partial}{\partial u^k_{i_1 \ldots i_s}}, \quad (k = 1, 2, \ldots, m), \tag{2}
\]

are variational derivatives,

\[
W^k = \eta^k - \xi^k u^k_i, \quad (k = 1, 2, \ldots, m),
\]

and

\[
\mathcal{N}^i F = \xi^i F + W^k \frac{\delta F}{\delta u^k_i} + \sum_{s=1}^{n} D_{i_1} \ldots D_{i_s} (W^k_{i_s}) \frac{\delta F}{\delta u^k_{i_1 \ldots i_s}}, \quad (i = 1, 2, \ldots, n).
\]

Here the variational derivatives \( \frac{\delta F}{\delta u^k_{i_1 \ldots i_s}} \) are obtained from [2] by replacing \( u^k \) with the corresponding derivative \( u^k_{i_1 \ldots i_s} \). The second identity is [9, 20]:

\[
\frac{\delta}{\delta u^j} \left( XF + FD_i \xi^i - D_i B^i \right) = X \left( \frac{\delta F}{\delta u^j} \right) + \frac{\delta F}{\delta u^k} \left( \frac{\partial \eta^k}{\partial u^j} - \frac{\partial \xi^i}{\partial u^j} u^k_i + \delta_{ij} L_j \xi^i \right), \quad (j = 1, 2, \ldots, m). \tag{3}
\]

**Theorem** (Noether). Suppose, a generator

\[
X = \xi^i (x, u) \frac{\partial}{\partial x^i} + \eta^k (x, u) \frac{\partial}{\partial u^k}
\]

satisfies the equation

\[
XL + LD_i \xi^i = D_i B^i. \tag{4}
\]

Then the generator \( X \) is an admitted symmetry of the system of Euler-Lagrange equations

\[
\frac{\delta L}{\delta u^k} = 0, \quad (k = 1, 2, \ldots, m), \tag{5}
\]

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[1][Therein one can find more details and references.

[2][The reader is referred to [22] for the details on symmetries and conservation laws.

[3][For the sake of simplicity this identity is presented in the case, where a function \( F(x, u, p) \) does not depend on derivatives of order 2 and higher, and the coefficients do not depend on derivatives \( \xi^i = \xi^i (x, u), \eta^k = \eta^k (x, u) \). Identity [3] is valid in more general cases.
and the vector

\[(N^1 \mathcal{L} - B^1, N^2 \mathcal{L} - B^2, \ldots, N^n \mathcal{L} - B^n)\]

is a conserved vector.

In the case \(B = (B^1, B^2, \ldots, B^n) = 0\), the symmetry \(X\) is called a variational symmetry, otherwise for \(B \neq 0\) the symmetry \(X\) is called a divergent one.

1.4. Objectives of the present paper

The present paper is focused on the two-dimensional Euler-Lagrange gas dynamics equations of a polytropic gas. Its objective is to make group classification of the Euler-Lagrange equations with respect to the function of entropy, and to construct conservation laws by applying Noether’s theorem.

2. Gas dynamics equations of a polytropic gas

In this section the gas dynamics equations of a polytropic gas in mass Lagrangian coordinates are considered. The two-dimensional Euler-Lagrange equations are obtained by using a variational approach.

2.1. Gas dynamics equations in Eulerian coordinates

The gas dynamics equations of a polytropic gas are

\[
\rho_t + u \rho_x + v \rho_y + \rho (u_t + u u_x + v u_y) = 0, \\
\rho (u_t + uu_x + vu_y) + p_x = 0, \\
\rho (v_t + uv_x + vv_y) + p_y = 0, \\
S_t + u S_x + v S_y = 0,
\]

where \(\rho\) is the density, \((u, v)\) is the velocity, \(p = S \rho \gamma\), and \(S\) is a function depending on the entropy \(\tilde{S}\) of the polytropic gas. The function \(S\) is related with the entropy \(\tilde{S}_0\) of a polytropic gas as follows \[27\]

\[S = \frac{R (S - \tilde{S}_0)}{c_v},\]

where \(R\) is the gas constant, \(c_v\) is the dimensionless specific heat capacity at constant volume, and \(\tilde{S}_0\) is constant.

2.2. Lagrangian coordinates

The relations

\[x = \varphi_1(t, \xi, \eta), \quad y = \varphi_2(t, \xi, \eta)\]

between the Lagrangian \((t, \xi, \eta)\) and Eulerian \((t, x, y)\) coordinates are defined by the equations

\[\varphi_1(t, \xi, \eta) = u(t, \varphi_1(t, \xi, \eta), \varphi_2(t, \xi, \eta)), \quad \varphi_2(t, \xi, \eta) = v(t, \varphi_1(t, \xi, \eta), \varphi_2(t, \xi, \eta)).\]

In Lagrangian coordinates the entropy is an arbitrary function of \(\xi\) and \(\eta\). The conservation law of mass provides the relation \[1\]:

\[\rho = \frac{\rho_0}{\varphi_1 \varphi_2 \eta - \varphi_1 \eta \varphi_2},\]

where \(\rho_0 = \rho_0(\xi, \eta) > 0\) is the function of integration.

Applying the change

\[\xi = h_1(\xi, \eta), \quad \eta = h_2(\xi, \eta),\]

one finds that

\[\varphi_1 \varphi_2 \eta - \varphi_1 \eta \varphi_2 = (\varphi_1 \varphi_2 \eta - \varphi_1 \eta \varphi_2)(h_1 \xi h_2 \eta - h_1 \eta h_2 \xi).\]

Hence, choosing \(h_1(\xi, \eta)\) and \(h_2(\xi, \eta)\) such that

\[h_1 \xi h_2 \eta - h_1 \eta h_2 \xi = \rho_0,\]
one derives that
\[ \rho(t, \xi, \eta) = J^{-1}(t, \xi, \eta), \]
where \( J = \varphi_{1\xi} \varphi_{2\eta} - \varphi_{1\eta} \varphi_{2\xi} \neq 0 \). Following the one-dimensional case, the coordinates \((t, \xi, \eta)\) are called the mass Lagrangian coordinates. The gas dynamics equations (6)-(9) in the mass Lagrangian coordinates with the gas dynamics variables become [4]:

\[
\left( \frac{1}{\rho} \right)_t = \ddot{u}\varphi_{2\eta} - \varphi_{1\eta} \ddot{v} - (\ddot{u}_\eta \varphi_{2\xi} - \varphi_{1\xi} \ddot{v}_\eta),
\]
\[
\ddot{u} + \varphi_{2\xi} \ddot{p}_\xi - \varphi_{2\eta} \ddot{p}_\eta = 0,
\]
\[
\ddot{v} - \varphi_{1\eta} \ddot{p}_\xi + \varphi_{1\xi} \ddot{p}_\eta = 0,
\]
\[
\dot{S}_t = 0,
\]
\[
\varphi_{1t} = \ddot{u}, \quad \varphi_{2t} = \ddot{v}.
\]

The gas dynamics variable \( \tilde{f}(t, \xi, \eta) \) in mass Lagrangian coordinates and in Eulerian coordinates \( f(t, x, y) \) are related by the formulae
\[
\tilde{f}(t, \xi, \eta) = f(t, \varphi_1(t, \xi, \eta), \varphi_2(t, \xi, \eta)).
\]
As there is no ambiguity, the sign \( \tilde{\ } \) is further omitted. Notice that in the one-dimensional case one assumes that \( \varphi_1 = \varphi_1(t, \xi), \quad \varphi_2 = \eta, \)

the system of equations (16), (17) is closed, and it becomes the classical one-dimensional gas dynamics equations in mass Lagrangian coordinates [5]:
\[
\left( \frac{1}{\rho} \right)_t = u_\xi, \quad u_t + p_\xi = 0, \quad S_t = 0.
\]

2.3. Variational approach
Choosing the Lagrangian
\[
L = \frac{\varphi_{1t}^2 + \varphi_{2t}^2}{2} - \frac{1}{\gamma - 1} J^{1-\gamma} S,
\]
one derives the Euler-Lagrange equations \( \frac{\delta L}{\delta \varphi_1} = 0 \) and \( \frac{\delta L}{\delta \varphi_2} = 0 \):

\[
J^{\gamma} \varphi_{1tt} + S_\xi \varphi_{2tt} - S_{\eta\xi} \varphi_{2t} + \gamma J^{-1} S (\varphi_{2\eta}(\varphi_{1\xi} \varphi_{2\eta} - \varphi_{1\eta} \varphi_{2\xi})
+ \varphi_{2\xi}(\varphi_{1\xi} \varphi_{2\eta} - \varphi_{1\eta} \varphi_{2\xi}) + 2 \varphi_{2\xi} \varphi_{2\eta} \varphi_{1\eta} - (\varphi_{1\xi} \varphi_{2\eta} + \varphi_{1\eta} \varphi_{2\xi}) \varphi_{2\eta}) = 0,
\]
\[
J^{\gamma} \varphi_{2tt} - S_\xi \varphi_{1tt} + S_{\eta\xi} \varphi_{1t} + \gamma SJ^{-1} (\varphi_{1\eta}(\varphi_{2\eta} \varphi_{1\xi} - \varphi_{2\xi} \varphi_{1\eta})
+ \varphi_{1\xi}(\varphi_{2\eta} \varphi_{1\xi} - \varphi_{2\xi} \varphi_{1\eta}) + 2 \varphi_{1\xi} \varphi_{2\eta} \varphi_{1\eta} - (\varphi_{1\xi} \varphi_{2\eta} + \varphi_{1\eta} \varphi_{2\xi}) \varphi_{2\eta}) = 0.
\]

One can show that equations (16), (17) are reduced to the equations of linear momentum (7) and (8). Corresponding changes of derivatives are obtained from the identities
\[
\varphi_{1t}(t, \xi, \eta) = u(t, \varphi_1(t, \xi, \eta), \varphi_2(t, \xi, \eta)), \quad \varphi_{2t}(t, \xi, \eta) = v(t, \varphi_1(t, \xi, \eta), \varphi_2(t, \xi, \eta)),
\]
\[
J(t, \xi, \eta) = \rho^{-1}(t, \varphi_1(t, \xi, \eta), \varphi_2(t, \xi, \eta)).
\]

For example,
\[
\varphi_{1t} = \varphi_{1\xi} u_x + \varphi_{2\xi} u_y, \quad \varphi_{2t} = \varphi_{1\eta} u_x + \varphi_{2\eta} u_y,
\]
\[
\varphi_{1\eta} = \varphi_{1\eta} u_x + \varphi_{2\eta} u_y, \quad \varphi_{2\eta} = \varphi_{1\eta} u_x + \varphi_{2\eta} u_y,
\]
\[
\varphi_{1tt} = u_t + u u_x + v u_y, \quad \varphi_{2tt} = v_t + u u_x + v u_y,
\]
\[
S_\xi = \varphi_{1\xi} S_x + \varphi_{2\xi} S_y, \quad S_\eta = \varphi_{1\eta} S_x + \varphi_{2\eta} S_y,
\]
\[
\varphi_{1tt} = u_t + u u_x + v u_y, \quad \varphi_{2tt} = v_t + u u_x + v u_y,
\]
\[
J_\xi = -\rho^{-2}(\varphi_{1\xi} \rho_x + \varphi_{2\xi} \rho_y), \quad J_\eta = -\rho^{-2}(\varphi_{1\eta} \rho_x + \varphi_{2\eta} \rho_y),
\]
\[
J_t = -\rho^{-2}(\rho_t + u \rho_x + v \rho_y).
\]

In the present paper equations (16), (17) are called the gas dynamics equations in mass Lagrangian coordinates, whereas equations (10)-(13) are called the gas dynamics equations in the gas dynamics variables in mass Lagrangian coordinates.
2.4. Relations between conserved vectors

As there is the change

\[ D_c T = \varphi_{1c} D_x T + \varphi_{2c} D_y T, \quad D_n T = \varphi_{1n} D_x T + \varphi_{2n} D_y T, \]

one has

\[ D_c T = J \left( D_x \left( \frac{\varphi_{1c}}{J} \right) + D_y \left( \frac{\varphi_{2c}}{J} \right) \right), \quad D_n T = J \left( D_x \left( \frac{\varphi_{1n}}{J} \right) + D_y \left( \frac{\varphi_{2n}}{J} \right) \right). \]

Because of the following relations

\[
D_i T^i + D_j T^j + D_n T^n = D_i (\rho J T^i) + D_j T^j + D_n T^n \\
= T^i \frac{\partial T^i}{\partial t} + J D_i (\rho T^i) + D_j T^j + \rho T_i
\]

\[
+ \varphi_{1c} D_x T^x + \varphi_{2c} D_y T^y + \varphi_{1n} D_x T^n + \varphi_{2n} D_y T^n
\]

\[
= \rho T^i \frac{\partial T^i}{\partial t} + J (D_i (\rho T^i) + \rho T_i T^i) + D_j (\rho T^j) + D_y (\rho T^y) + \varphi_{1c} D_x T^x + \varphi_{2c} D_y T^y + \varphi_{1n} D_x T^n + \varphi_{2n} D_y T^n
\]

\[
= J (D_i (\rho T^i) + D_x (\rho T^j) + D_y (\rho T^j) + \varphi_{1c} D_x T^x + \varphi_{2c} D_y T^y + \varphi_{1n} D_x T^n + \varphi_{2n} D_y T^n)
\]

one finds that a conserved vector \((T^i, T^j, T^n)\) in Lagrangian coordinates reduces to the conserved vector \((^c T^i, ^c T^j, ^c T^n)\) in Eulerian coordinates, where

\[
^c T^i = \rho T^i, \quad ^c T^j = \rho T^j + \frac{\varphi_{1c}}{J} T^x + \frac{\varphi_{2c}}{J} T^y, \quad ^c T^n = \rho T^n + \frac{\varphi_{1n}}{J} T^x + \frac{\varphi_{2n}}{J} T^y.
\]

3. Group classification of equations \([16, 17]\)

The group classification consists of two steps \([3]\). For the first step in the group classification one needs to find an equivalence Lie group which can be used for simplification of the arbitrary elements contained in the equations studied. These simplifications are used in analysis of classifying equations, a general solution of which provides admitted Lie groups and representations of simplified arbitrary elements.

3.1. Equivalence transformations

The group classification of equations depends on the representations of the arbitrary elements \([3]\). As the function \(S(\xi, \eta)\) is an arbitrary element of equations \([16, 17]\), the group classification has to be made with respect to it.

The generator of the equivalence transformations is considered in the form

\[
X^c = \xi^t \frac{\partial}{\partial t} + \xi^x \frac{\partial}{\partial x} + \xi^y \frac{\partial}{\partial y} + \zeta_1 \frac{\partial}{\partial \varphi_1} + \zeta_2 \frac{\partial}{\partial \varphi_2} + \zeta_S \frac{\partial}{\partial S},
\]

where all coefficients depend on \((t, \xi, \eta, \varphi_1, \varphi_2, S)\). Calculations show that a basis of the Lie algebra corresponding to the equivalence Lie group consists of the generators

\[
X_1 = \partial_{\varphi_1}, \quad X_2 = \partial_{\varphi_2}, \quad X_3 = t \partial_{\varphi_1}, \quad X_4 = t \partial_{\varphi_2}, \quad X_5 = \varphi_2 \partial_{\varphi_1} - \varphi_1 \partial_{\varphi_2}, \quad X_6 = \partial_t,
\]

\[
X_7 = \varphi_1 \partial_{\varphi_1} + \varphi_2 \partial_{\varphi_2} + 2 \gamma S \partial_S, \quad X_8 = \xi \partial_t + \eta \partial_\eta + 2(1 - \gamma) S \partial_S,
\]

\[
X_9 = t \partial_t - 2 S \partial_S, \quad X_\psi = - \psi_S \partial_\xi + \psi \partial_\eta,
\]

where \(\psi(\xi, \eta)\) is an arbitrary function. For \(\gamma = 2\) there is an additional set of equivalence transformations corresponding to the generator

\[
X_{10} = t (t \partial_t + \varphi_1 \partial_{\varphi_1} + \varphi_2 \partial_{\varphi_2}).
\]
3.2. Classifying equation

The admitted Lie group is derived by solving the determining equations

\[ \dot{X} E_i |_{E=0} = 0, \quad (i = 1, 2), \]  

where \( E_i \) is the left-hand side of equations (16), (17), \( |E = 0 \) means that the expressions \( X E_i \) are considered on the manifold defined by equations (16), (17), and the generator \( \dot{X} \) is the prolongation of the generator

\[ X = \zeta^l \frac{\partial}{\partial t} + \zeta^\xi \frac{\partial}{\partial \xi} + \zeta^\eta \frac{\partial}{\partial \eta} + \zeta^{\varphi_1} \frac{\partial}{\partial \varphi_1} + \zeta^{\varphi_2} \frac{\partial}{\partial \varphi_2}, \]

where all coefficients depend on \((t, \xi, \eta, \varphi_1, \varphi_2)\). Notice that the function \( S(\xi, \eta) \) is considered to be given.

Calculations show that the admitted generator has the form

\[ X = \sum_{j=1}^{10} c_i X_j + X_h, \]

and the classifying equation is

\[ h_\xi S_\eta - (h_\eta - 2\gamma c_9 \xi) S_\xi = 2\gamma c_10 S, \quad (19) \]

where \( c_7(\gamma - 2) = 0 \), and

\[ X_1 = \frac{\partial}{\partial \varphi_1}, \quad X_2 = \frac{\partial}{\partial \varphi_2}, \quad X_3 = t \frac{\partial}{\partial \varphi_1}, \quad X_4 = t \frac{\partial}{\partial \varphi_2}, \]

\[ X_5 = \varphi_2 \frac{\partial}{\partial \varphi_1} - \varphi_1 \frac{\partial}{\partial \varphi_2}, \quad X_6 = \frac{\partial}{\partial t}, \]

\[ X_7 = t \left( \frac{\partial}{\partial t} + \varphi_2 \frac{\partial}{\partial \varphi_1} + \varphi_1 \frac{\partial}{\partial \varphi_2} \right), \quad X_8 = \gamma t \frac{\partial}{\partial t} + \varphi_1 \frac{\partial}{\partial \varphi_1} + \varphi_2 \frac{\partial}{\partial \varphi_2}, \]

\[ X_9 = 2\gamma \xi \frac{\partial}{\partial \xi} + (\gamma - 1)(\varphi_1 \frac{\partial}{\partial \varphi_1} + \varphi_2 \frac{\partial}{\partial \varphi_2}), \quad X_{10} = \varphi_1 \frac{\partial}{\partial \varphi_1} + \varphi_2 \frac{\partial}{\partial \varphi_2}, \]

\[ X_h = -h_\eta \frac{\partial}{\partial \xi} + h_\xi \frac{\partial}{\partial \eta}. \]

The analysis of the classifying equation is split into two cases: isentropic and nonisentropic flows.

4. Isentropic flows

In this case \( S_\xi = 0, \ S_\eta = 0 \), and one derives from the classifying equation that

\[ c_{10} = 0. \]

Thus, for general choice of \( \gamma \), the admitted Lie algebra is infinite dimensional: \( L_8 \oplus \{ X_h \} \), where a basis of \( L_8 \) consists of the generators \( X_j \) (\( j = 1, 2, ..., 9; \ j \neq 7 \)), and \( h(\xi, \eta) \) is an arbitrary function. For \( \gamma = 2 \) the basis of the finite part \( L_8 \) is extended by the generator \( X_7 \).

4.1. Conservation laws in Lagrangian coordinates

For a symmetry to be divergent (variational) one needs to require that

\[ c_9 = \frac{(\gamma - 2)}{2(2\gamma - 1)} c_8. \quad (20) \]

For a divergent symmetry one obtains

\[ b_1 = c_4 \varphi_2 + \frac{1}{\gamma(\gamma - 1)} c_7 (\varphi_1^2 + \varphi_2^2) + c_4 \varphi_1 \quad b_2 = 0, \quad b_3 = 0. \]

This gives that each of the generators \( X_j \) (\( j = 1, 2, 3, ..., 7 \)), \( X_h \) and

\[ \dot{X}_8 = X_8 + \frac{(\gamma - 2)}{2(2\gamma - 1)} X_9 \]
provides a conservation law. These conservation laws have the conserved vector coordinates:

\[ T_1^i = \varphi_{1t}, \quad T_1^j = S\varphi_{2y}J^{-\gamma}, \quad T_1^0 = -S\varphi_{2z}J^{-\gamma}; \]
\[ T_2^j = \varphi_{2t}, \quad T_2^k = -S\varphi_{1y}J^{-\gamma}, \quad T_2^0 = S\varphi_{1z}J^{-\gamma}; \]
\[ T_3^j = \varphi_{1t} - \varphi_1, \quad T_3^k = tS\varphi_{2y}J^{-\gamma}, \quad T_3^0 = -tS\varphi_{2z}J^{-\gamma}; \]
\[ T_4^j = \varphi_{2t} - \varphi_2, \quad T_4^k = -tS\varphi_{1y}J^{-\gamma}, \quad T_4^0 = tS\varphi_{1z}J^{-\gamma}; \]
\[ T_5^j = \varphi_{1t} - \varphi_1 - 2\varphi_{2y}J^{-\gamma}, \quad T_5^k = (\varphi_{1y} + \varphi_{2y}J^{-\gamma}), \quad T_5^0 = -S(\varphi_{1y} + \varphi_{2y}J^{-\gamma}); \]
\[ T_6^j = \varphi_{1t}^2 + \varphi_{2t}^2 + \frac{2}{\gamma - 1}SJ^{-\gamma}, \quad T_6^k = 2(\varphi_{1t}\varphi_{2t} - \varphi_{1y}\varphi_{2y})SJ^{-\gamma}; \]
\[ T_6^0 = 2(\varphi_{1y}\varphi_{2z} - \varphi_{1z}\varphi_{2y})SJ^{-\gamma}; \]
\[ \hat{T}_8^i = 2(1 - \gamma)(\gamma - 2)\xi(\varphi_{1t}\varphi_{1\xi} + \varphi_{2t}\varphi_{2\xi}) + 2(1 - 2\gamma)SJ^{-\gamma} \]
\[ -t(2\gamma - 1)(\varphi_{1t}^2 + \varphi_{2t}^2) + (\gamma - 1)(\varphi_{1t}\varphi_{1y} + \varphi_{2t}\varphi_{2y}), \]
\[ \hat{T}_8^j = (\gamma - 2)\xi((\gamma - 1)(\varphi_{1t}^2 + \varphi_{2t}^2) - 2\gamma SJ^{-\gamma}) \]
\[ +SJ^{-\gamma}(\gamma - 1)(2(2\gamma - 1)t(\varphi_{1y}\varphi_{2t} - \varphi_{1t}\varphi_{2y}) + (\gamma + 1)(\varphi_{1y}\varphi_{2y} - \varphi_{1t}\varphi_{2y})), \]
\[ \hat{T}_8^k = SJ^{-\gamma}(\gamma - 1)(2(2\gamma - 1)t(\varphi_{1y}\varphi_{2t} - \varphi_{1t}\varphi_{2y}) + (\gamma + 1)(\varphi_{1y}\varphi_{2y} - \varphi_{1t}\varphi_{2y})); \]
\[ T_8^j = 2(1 - \gamma)^{-1}((\varphi_{1t}\varphi_{1y} + \varphi_{2t}\varphi_{2y})h\eta - (\varphi_{1t}\varphi_{1y} + \varphi_{2t}\varphi_{2y})h\xi), \]
\[ T_8^k = -h\eta((\gamma - 1)(\varphi_{1t}^2 + \varphi_{2t}^2) - 2\gamma SJ^{-\gamma}), \]
\[ T_8^0 = h\xi((\gamma - 1)(\varphi_{1t}^2 + \varphi_{2t}^2) - 2\gamma SJ^{-\gamma}). \]

Here and further on the subscripts relate to the corresponding generator providing the conservation law.

In the case where \( \gamma = 2 \) there is one more conservation law in Lagrangian coordinates which corresponds to the generator \( X_7 \):

\[ T_7^j = ((\varphi_{2} - \varphi_{2t})^2 + (\varphi_{1} - \varphi_{1t})^2 + 2SJ^{-1}t^2), \quad T_7^k = 2tSJ^{-2}((\varphi_{2} - \varphi_{2t})\varphi_{1y} - (\varphi_{1} - \varphi_{1t})\varphi_{2y}), \]
\[ T_7^0 = 2tSJ^{-2}((\varphi_{1} - \varphi_{1t})\varphi_{2z} - (\varphi_{2} - \varphi_{2t})\varphi_{1z}). \]

### 4.2. Conservation laws in Eulerian coordinates

First of all, we should mention the conservation law of mass which is valid for isentropic as well as nonisentropic flows,

\[ \rho_t + (\rho u)_x + (\rho v)_y = 0. \]

The latter conserved vectors \((\hat{T}_i^i, \hat{T}_j^j, \hat{T}_k^k)\), \(i = 1, 2, ..., 8\) have the following forms in Eulerian coordinates

\[ \hat{e}T_1^i = \rho u, \quad T_1^j = \rho u^2 + \rho^\gamma S, \quad T_1^0 = \rho u; \]
\[ \hat{e}T_2^j = \rho v, \quad T_2^k = \rho uv, \quad T_2^0 = \rho v^2 + \rho^\gamma S; \]
\[ \hat{e}T_3^j = \rho(tu - x), \quad T_3^k = \rho(uu - x) + t\rho^\gamma S, \quad T_3^0 = \rho(uu - x); \]
\[ \hat{e}T_4^j = \rho(tv - y), \quad T_4^k = \rho(tv - y) + t\rho^\gamma S; \]
\[ \hat{e}T_5^j = \rho(uy - vx), \quad T_5^k = \rho(uy - vx) + y\rho^\gamma S, \quad T_5^0 = \rho(uy - vx); \]
\[ \hat{e}T_6^j = \rho\frac{u^2 + v^2}{2} + \frac{1}{\gamma - 1}\rho^\gamma S, \quad T_6^k = \rho\left(\frac{u^2 + v^2}{2} + \frac{1}{\gamma - 1}\rho^\gamma S\right), \]
\[ T_6^0 = v\left(\frac{u^2 + v^2}{2} + \frac{1}{\gamma - 1}\rho^\gamma S\right). \]

The function\(^4\) \(h(\xi, \eta)\) becomes a function \(h(t, x, y)\) which only satisfies the equation

\[ h_t + uh_x + vh_y = 0. \quad (21) \]
The corresponding conservation law is
\[ \epsilon T_h^i = 2(\gamma - 1)(uh_y - vhx), \quad T_h^x = (\gamma - 1)((u^2 - v^2)h_y - 2uvhx) + 2\gamma \rho^{\gamma - 1}h_y S, \]
\[ T_h^y = (\gamma - 1)((u^2 - v^2)hx + 2uvh_y) - 2\gamma \rho^{\gamma - 1}hx S. \]
(22)

The conserved vector \((\tilde{T}_8^i, \tilde{T}_8^x, \tilde{T}_8^y)\) cannot be represented in Eulerian coordinates because of the presence of \(\xi\). Also notice that for \(\gamma = 2\) the term containing \(\xi\) vanishes.

For \(\gamma = 2\) there are two more conserved vectors: the conserved vector \((T_7^i, T_7^x, T_7^y)\) becomes
\[ \epsilon T_7^i = \rho \{2t(ux + vy) - (x^2 + y^2) - t^2(u^2 + v^2 + 2\rho S)\}, \]
\[ T_7^x = \rho \{u(2t(ux + vy) - (x^2 + y^2) - (u^2 + v^2)t^2) - 2t\rho S(2tu - x)\}, \]
\[ T_7^y = \rho \{v(2t(ux + vy) - (x^2 + y^2) - (u^2 + v^2)t^2) - 2t\rho S(2tv - y)\}, \]
and, as noted above, for \(\gamma = 2\) the term with \(\xi\) vanishes in the conserved vector \((\tilde{T}_8^i, \tilde{T}_8^x, \tilde{T}_8^y)\), and the conserved vector \((\tilde{T}_8^i, \tilde{T}_8^x, \tilde{T}_8^y)\) also has its representation in Eulerian coordinates:
\[ \epsilon T_8^i = \rho (t(u^2 + v^2) - (ux + vy) + 2t\rho S), \quad T_8^x = \rho (u(t(u^2 + v^2) - (ux + vy)) + (4tu - x)\rho S), \]
\[ T_8^y = \rho (v(t(u^2 + v^2) - (ux + vy)) + (4tv - y)\rho S). \]

4.3. Discussion

First of all we note that for deriving relation \([20]\) it was important to consider generator \(X\) in its general form: not analyzing condition \((1)\) for each specific generator.

The conserved vectors \((T_1^i, T_1^x, T_1^y)\), \((T_2^i, T_2^x, T_2^y)\) are well-known \([28]\): the vectors \((T_1^i, T_1^x, T_1^y)\) and \((T_2^i, T_2^x, T_2^y)\) give conservation laws of linear momentum; the vectors \((T_3^i, T_3^x, T_3^y)\) and \((T_4^i, T_4^x, T_4^y)\) correspond to the conservation law of angular momentum motion of the center of mass; the vector \((T_5^i, T_5^x, T_5^y)\) corresponds to the conservation law of energy; the vectors \((T_7^i, T_7^x, T_7^y)\) and \((T_8^i, T_8^x, T_8^y)\) are extensions of the classical conservation laws for a polytropic gas with \(\gamma = 2\) derived in \([28]\).

The conserved vector \((T_8^i, T_8^x, T_8^y)\) with the Lagrangian invariant \(h(t, x, y)\) (satisfying equation \([21]\)) provides a new conservation law. In contrast to \([16]\) this conservation law is local. It should be also noted that this conservation law is naturally derived: its counterpart in Lagrangian coordinates was derived directly using Noether’s theorem without any additional constructions.

Remark. The gas dynamics equations \([10]-[14]\) for an isentropic flow and \(\gamma = 2\), coincide with the hyperbolic shallow water equations. Group properties of the shallow water equations describing flows over a bed which is rotating with position-dependent Coriolis parameter in Lagrangian coordinates \([3]\) are studied in \([18]\).

Conservation laws were constructed by using a Lagrangian of the form presented in \([19]\). In the present paper the admitted Lie group is wider, and we also obtained more conservation laws. This can be explained by the presence of nonzero Coriolis parameter and different Lagrangian used in \([18]\).

5. Nonisentropic flows

5.1. Admitted Lie algebra

Choosing the functions \(\psi_1(\xi, \eta), \psi_2(\xi, \eta)\) and \(\psi_0(\xi, \eta)\) satisfying the equations
\[ \psi_{1\eta}S_\xi - (\psi_{1\xi}S_\eta + \xi S_\eta) = 0, \]
\[ S_\xi \psi_{2\eta} - S_\eta \psi_{2\xi} + 2S = 0, \]
\[ \psi_{0\eta}S_\xi - \psi_{0\xi}S_\eta = 0, \]
(23)
one finds the general solution of the classifying equation \([19]\),
\[ h = \psi_0 + \gamma(2c_9\psi_1 + c_{10}\psi_2). \]
Notice that the general solution of \([23]\) is \(\psi_0 = F(S)\), where the function \(F\) is an arbitrary function.

It is convenient to introduce
\[ c_{10} = \frac{\gamma - 2}{2}c_8 + \tilde{c}_{10}. \]
Then the admitted generator of equations (16), (17) has the form

\[ X = \sum_{j=1}^{7} c_j X_j + c_8 \dot{X}_8 + c_9 \dot{X}_9 + \dot{c}_{10} \dot{X}_{10} + X_{\psi_0}, \]

where \( c_7 (\gamma - 2) = 0 \), and

\[
\begin{align*}
X_1 &= \frac{\partial}{\partial \varphi_1}, & X_2 &= \frac{\partial}{\partial \varphi_2}, & X_3 &= t \frac{\partial}{\partial \varphi_1}, & X_4 &= t \frac{\partial}{\partial \varphi_2}, \\
X_5 &= \varphi_2 \frac{\partial}{\partial \varphi_1} - \varphi_1 \frac{\partial}{\partial \varphi_2}, & X_6 &= \frac{\partial}{\partial t}, & X_7 &= t \left( \frac{\partial}{\partial t} + \varphi_1 \frac{\partial}{\partial \varphi_1} + \varphi_2 \frac{\partial}{\partial \varphi_2} \right), \\
\dot{X}_8 &= \gamma \left( \frac{2t}{\partial t} - (\gamma - 2) \left( \psi_{2t} \frac{\partial}{\partial \xi} - \psi_{2\xi} \frac{\partial}{\partial \eta} \right) + \varphi_1 \frac{\partial}{\partial \varphi_1} + \varphi_2 \frac{\partial}{\partial \varphi_2} \right), \\
\dot{X}_9 &= 2\gamma \left( (\xi - \psi_{1\eta}) \frac{\partial}{\partial \xi} + \psi_{1\xi} \frac{\partial}{\partial \eta} \right) + (\gamma - 1) \left( \varphi_1 \frac{\partial}{\partial \varphi_1} + \varphi_2 \frac{\partial}{\partial \varphi_2} \right), \\
\dot{X}_{10} &= \gamma \left( -\psi_{2\eta} \frac{\partial}{\partial \xi} + \psi_{2\xi} \frac{\partial}{\partial \eta} \right) + \varphi_1 \frac{\partial}{\partial \varphi_1} + \varphi_2 \frac{\partial}{\partial \varphi_2}, \\
X_F &= F' \left( S_{\xi} \frac{\partial}{\partial \eta} - S_{\eta} \frac{\partial}{\partial \xi} \right).
\end{align*}
\]

5.2. Conservation laws in mass Lagrangian coordinates

The condition for the generator \( X \) to be divergent (variational) gives that

\[ c_9 = -\frac{1}{2\gamma - 1} \dot{c}_{10} \]

and

\[ b_1 = c_3 \varphi_1 + c_4 \varphi_2 + \frac{c_7}{2(\gamma - 1)} (\varphi_1^2 + \varphi_2^2), \quad b_2 = 0, \quad b_3 = 0, \]

\[ \dot{X}_9 = \dot{X}_{10} - \frac{1}{2\gamma - 1} \dot{X}_9 = \frac{\gamma}{2\gamma - 1} \left( (2(\psi_{1\eta} - \xi) - (\gamma - 1)(\psi_{2\eta}) \frac{\partial}{\partial \xi} + (2\psi_{1\xi} + (\gamma - 1)(\psi_{2\xi}) \frac{\partial}{\partial \eta} + \varphi_1 \frac{\partial}{\partial \varphi_1} + \varphi_2 \frac{\partial}{\partial \varphi_2} \right). \]

The conserved vectors for \((T_i^t, T_i^\xi, T_i^\eta)\), \((i = 1, 2, ..., 6)\) are the same as in the isentropic case. The remaining conserved vectors are

\[
T_8^t = (\gamma - 2) \left( (\varphi_1 \varphi_2 \xi + \varphi_2 t \varphi_2 \xi) \psi_{2\eta} - (\varphi_1 \varphi_1 \eta + \varphi_2 t \varphi_2 \eta) \psi_{2\xi} \right) + \psi_1 \varphi_{1t} t \varphi_{2t} - \frac{2}{\gamma - 1} t J^{1-\gamma} S,
\]

\[
T_8^\xi = ((\varphi_1 - 2 \varphi_{1t}) \varphi_2 \eta - (\varphi_2 - 2 \varphi_{2t}) \varphi_1 \eta) S J^{1-\gamma} - \frac{\gamma - 2}{2} \psi_{2\eta} \left( \varphi_{1t}^2 + \varphi_{2t}^2 - \frac{2\gamma}{\gamma - 1} J^{1-\gamma} S \right),
\]

\[
T_8^\eta = -((\varphi_1 - 2 \varphi_{1t}) \varphi_2 \xi - (\varphi_2 - 2 \varphi_{2t}) \varphi_1 \xi) S J^{1-\gamma} + \frac{\gamma - 2}{2} \psi_{2\xi} \left( \varphi_{1t}^2 + \varphi_{2t}^2 - \frac{2\gamma}{\gamma - 1} J^{1-\gamma} S \right);
\]

\[
T_9^t = -((2\gamma - 1) \psi_{2\xi} - 2 \psi_{1\xi}) \left( \varphi_{1t} \varphi_{1\eta} + \varphi_{2t} \varphi_{2\eta} \right) + (2\gamma - 1) \psi_{2\eta} + (\varphi_1 \varphi_{1t} + \varphi_2 \varphi_{2t}),
\]

\[
T_9^\xi = \frac{1}{2} \left( (2\psi_{1\xi} - \xi - (2\gamma - 1) \psi_{2\xi}) \left( \varphi_{1t}^2 + \varphi_{2t}^2 - \frac{2\gamma}{\gamma - 1} J^{1-\gamma} S \right) + 2 J^{-\gamma} S(\varphi_{1\varphi_{2\eta} - \varphi_{1\varphi_{2\xi}}},
\]

\[
T_9^\eta = \frac{1}{2} \left( (2\gamma - 1) \psi_{2\xi} - 2 \psi_{1\xi} \left( \varphi_{1t}^2 + \varphi_{2t}^2 - \frac{2\gamma}{\gamma - 1} J^{1-\gamma} S \right) - 2 J^{-\gamma} S(\varphi_{1\varphi_{2\xi} - \varphi_{1\varphi_{2\xi}}},
\]

\[
T_F^t = F' \left( \varphi_{1t} \varphi_{1\xi} + \varphi_{2t} \varphi_{2\xi} \right) S_{\eta} - (\varphi_{1t} \varphi_{1\eta} + \varphi_{2t} \varphi_{2\eta}) S_{\xi},
\]

\[
T_F^\xi = -\frac{1}{2} F' S_{\eta} (\varphi_{1t}^2 + \varphi_{2t}^2 - \frac{2\gamma}{\gamma - 1} J^{1-\gamma} S), \quad T_F^\eta = \frac{1}{2} F' S_{\xi} (\varphi_{1t}^2 + \varphi_{2t}^2 - \frac{2\gamma}{\gamma - 1} J^{1-\gamma} S).
\]
In the case $\gamma = 2$ there is one more conserved vector corresponding to the generator $X_7$. This conserved vector is the vector $(T^i_7, T^x_7, T^y_7)$ as in the isentropic case for $\gamma = 2$:

$$
T^i_7 = ((\varphi_2 - \varphi_{2t}t)^2 + (\varphi_1 - \varphi_{1t}t)^2 + 2SJ^{-1}t^2),
$$

$$
T^x_7 = 2tSJ^{-2}((\varphi_2 - \varphi_{2t}t)\varphi_{1n} - (\varphi_1 - \varphi_{1t}t)\varphi_{2n}),
$$

$$
T^y_7 = 2tSJ^{-2}((\varphi_1 - \varphi_{1t}t)\varphi_{2c} - (\varphi_2 - \varphi_{2t}t)\varphi_{1c}).
$$

5.3. Conservation laws in Eulerian coordinates

Representations of the conserved vectors $(T^i_1, T^x_1, T^y_1)$, $(i = 1, 2, ..., 7)$ are of similar forms $(\epsilon T^i_1, T^x_1, T^y_1)$ as in the isentropic case. The other conserved vectors in Eulerian coordinates have the form

$$
\epsilon T^i_8 = (\gamma - 2) (\psi_2u - \psi_2v) - \left(\rho(t(u^2 + v^2) - (ux + vy)) + \frac{2}{\gamma - 1}t\rho^\gamma S\right),
$$

$$
T^x_8 = -\left(\rho^\gamma S(\frac{2}{\gamma - 1}tu - x) + \rho u(t(u^2 + v^2) - (ux + vy))\right)
$$

$$
+ \frac{2}{\gamma - 2}(\psi_2u(u^2 - v^2 + \frac{2}{\gamma - 1}\rho^\gamma - 1S) - 2\psi_2uv),
$$

$$
T^y_8 = -\left(\rho^\gamma S(\frac{2}{\gamma - 1}tv - y) + \rho v(t(u^2 + v^2) - (ux + vy))\right)
$$

$$
+ \frac{2}{\gamma - 2}(\psi_2x(u^2 - v^2 + \frac{2}{\gamma - 1}\rho^\gamma - 1S) + 2\psi_2uv)
$$

$$
\epsilon T^i_9 = 2F'((S_yu - S_xv), T^x_9 = F'((u^2 - v^2)S_y - 2S_xuv + \frac{2}{\gamma - 1}\rho^\gamma - 1SS_y),
$$

$$
T^y_9 = F'((u^2 - v^2)S_x + 2S_yuv - \frac{2}{\gamma - 1}\rho^\gamma - 1SS_x).
$$

Here the function $\psi_2(t, x, y)$ satisfies the equations

$$
\psi_2t + u\psi_2x + v\psi_2y = 0, \quad \psi_2xS_y - \psi_2yS_x = 2\rho S.
$$

(24)

The conserved vector $(T^i_9, T^x_9, T^y_9)$ has no a representation in Eulerian coordinates.

5.4. Discussion

Continuing the discussions presented in the previous section, the conserved vectors $(\epsilon T^i_8, T^x_8, T^y_8)$ and $(\epsilon T^i_9, T^x_9, T^y_9)$ provide new conservation laws. The conserved vector $(\epsilon T^i_9, T^x_9, T^y_9)$ is similar to the vector $(\epsilon T^i_7, T^x_7, T^y_7)$ studied for isentropic flows.

The conserved vector $(\epsilon T^i_8, T^x_8, T^y_8)$ contains the function $\psi_2(t, x, y)$ which satisfies the equations (24). The overdetermined system of the gas dynamics equations with (24) is involutive. This leads to a similar idea as in [16] to extend an original set of the dependent variables, and to seek for new conservation laws containing the added variables. However, in the present cases the conservation laws are naturally derived: their counterparts in Lagrangian coordinates were derived directly using Noether’s theorem without any additional assumptions.

6. Conclusion

New conservation laws of two-dimensional gas dynamics equations of a polytropic gas are found in the present paper. The conservation laws are derived using corresponding Lagrangian and Noether’s theorem. For constructing conservation laws we used the complete group classification of the Euler-Lagrange equations [16], [17]. In contrast to the one-dimensional case of the Lagrangian gas dynamics equations [17], where $\varphi_{1n} = 0$ and $\varphi_2 = \eta$, the group classification only separates out the gas dynamics equations [16], [17] on analysis of isentropic and nonisentropic flows, and the admitted Lie group found has no constraints on the entropy $\eta$. The complete set of admitted generators allowed us to use Noether’s theorem for deriving conservation laws in mass Lagrangian coordinates. Their corresponding counterparts in Eulerian coordinates were also constructed. Some of these counterparts contain Lagrangian invariants. Moreover, the conserved vector $(\epsilon T^i_7, T^x_7, T^y_7)$ contains the function $\psi_2(t, x, y)$ which satisfies an overdetermined system of equations. One can show that the overdetermined system of equations consisting of these equations and the gas dynamics equations is involutive. Applying the group analysis method to this overdetermined system the admitted Lie group of the gas dynamics can be extended.

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*See also [3].*
The involutiveness of this system also gives a similar idea as in [16]: to extend an original set of the dependent variables, and to seek for new conservation laws containing the added variables. However, in the present paper the conservation laws are naturally derived: their counterparts in Lagrangian coordinates were derived directly using Noether’s theorem without any additional assumptions.

It should be also noted that there are conservation laws which have no their counterpart in Eulerian coordinates.

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