GLOBAL OKOUNKOV BODIES FOR BOTT-SAMUELSON VARIETIES

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Abstract. We use the theory of Mori dream spaces and GIT to prove that the global Okounkov body of a Bott-Samelson variety, with respect to a natural flag of subvarieties, is rational polyhedral. As a corollary, Okounkov bodies of effective line bundles over Schubert varieties are shown to be rational polyhedral. In particular, it follows that the global Okounkov body of a flag variety $G/B$ is rational polyhedral.

As an application, we derive polyhedral expressions for the asymptotics of weight multiplicities in section spaces of line bundles.

1. Introduction

Okounkov bodies were first introduced by A. Okounkov in his famous paper [Oko96] as a tool for studying multiplicities of group representations. The idea is that one should be able to approximate these multiplicities by counting the number of integral points in a certain convex body in $\mathbb{R}^n$. More precisely, the setting is the following. Let $G$ be a complex reductive group which acts as automorphisms on an effective line bundle $L$ over a projective variety $X$, and hence defines a representation on the space of sections $H^0(X, L^k)$ for each integral power, $L^k$, of $L$. Okounkov constructs a convex compact set $\Delta \subseteq \mathbb{R}^n$, where $n = \dim X$, with the property that for each irreducible finite-dimensional representation $V_\lambda$, where $\lambda$—the so-called highest weight—is a parameter, the multiplicity $m_{k\lambda, k} := \dim \text{Hom}_G(V_\lambda, H^0(X, L^k))$ of $V_\lambda$ in $H^0(X, L^k)$ is asymptotically given by the volume of the convex body $\Delta_\lambda := \Delta \cap H_\lambda$, where $H_\lambda \subseteq \mathbb{R}^{n+1}$ is a certain affine subspace, in the following sense:

$$\lim_{k \to \infty} \frac{m_{k\lambda, k}}{k^m} = \text{vol}_m(\Delta_\lambda),$$

where $m$ is the dimension of $\Delta_\lambda$, and the volume on the right hand side denotes the $m$-dimensional Euclidean volume of $\Delta_\lambda$. An approximation of the integral $\text{vol}_m(\Delta_\lambda)$ by Riemann sums yields that the multiplicity $m_{k\lambda, k}$ is asymptotically given by the number of points of the set $\Delta_\lambda \cap \frac{1}{k}\mathbb{Z}^m$.

The construction of the body $\Delta$ is purely geometric and depends on a choice of a flag $Y_\bullet$, $Y_n \subseteq Y_{n-1} \subseteq \cdots \subseteq Y_0 = X$ of irreducible subvarieties of $X$, and the “successive orders of vanishing” of certain invariant sections $s \in H^0(X, L^k)$ along this flag. It was later realized by Kaveh and Khovanskii ([KK09]), and independently by Lazarsfeld and Mustaţă, ([LM09]), that Okounkov’s construction makes sense for more general subseries of the

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section ring $R(X, L)$ of a line bundle over a variety $X$, and that the asymptotics of dimensions of linear series can be studied by counting lattice points of convex compact bodies. For example, the analog of (1) for the complete linear series of a big line bundle $L$ is given by the identity

$$\lim_{k \to \infty} \frac{h^0(X, L^k)}{n!k^n} = \frac{1}{n!} \text{vol}_n(\Delta_{Y_\bullet}(L)),$$

where $\Delta_{Y_\bullet}(L)$ denotes the Okounkov body of the line bundle $L$ with respect to the flag $Y_\bullet$.

The above formula shows in particular that the volume of the Okounkov body is an invariant of the line bundle $L$, and thus does not depend on the choice of the flag $Y_\bullet$. However, the shape of $\Delta_{Y_\bullet}(L)$ depends heavily on the flag, and it is a notoriously hard problem to explicitly describe these bodies, or even to show that they possess some nice properties, such as being polyhedral. A yet more difficult problem is to determine the global Okounkov body $\Delta_{Y_\bullet}(X)$ of a variety $X$ (cf. [LM09]), which is a convex cone in a certain Euclidean space, and no longer depends on a particular line bundle $L$.

Returning to the original motivation by Okounkov of studying multiplicities of representations, there is also another approach to describing multiplicities by counting lattice points in convex bodies, namely Littelmann’s construction of string polytopes ([L98]). The setting here is the following. Let $G$, again, be a complex reductive group, and let $H \subseteq G$ be a maximal torus in $G$. Then any irreducible finite-dimensional $G$-representation $V_\lambda$ admits a basis of weight vectors with respect to $H$, and this basis is parameterized by the integral points in a rational polytope $C^\lambda$, the string polytope of $V_\lambda$. Moreover, the approximative lattice counting problem is even exact here. Since the irreducible representations $V_\lambda$ can be realized as section spaces $H^0(X, L_\lambda)$, where $X = G/B$ for a Borel subgroup $B \subseteq G$, and $L_\lambda$ is a line bundle over $X$, it would be interesting to recover Littelmann’s string polytopes $C^\lambda$ as Okounkov bodies, or at least to construct rational polyhedral Okounkov bodies which describe asymptotic multiplicities of weight spaces.

In the present paper we study both problems described above—namely the Okounkov bodies for complete linear series, and the asymptotics of weight multiplicities—for general Bott-Samelson varieties $Z = Z_w$ (given by a reduced expression $w$ for an element $w$ in the Weyl group of $G$), that is, Bott-Samelson varieties which desingularize some Schubert variety $X_w$ in a flag variety $G/B$.

Using Geometric Invariant Theory (GIT) and the theory of Mori dream spaces, we prove the following theorem.

**Theorem 1.1.** Let $X$ be a Mori dream space and assume that there exists an admissible flag $Y_\bullet : X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_n$ such that

1. $Y_i$ is a Mori dream space for each $0 \leq i \leq n$, and
2. the restriction maps $p_i : N^1(Y_i) \to N^1(Y_{i+1})$ induce surjective maps $\mathcal{F}_D(Y_i) \cap p_i^{-1}(\mathcal{F}_D(Y_{i+1})) \to \mathcal{F}_D(Y_{i+1})$.

Then, the global Okounkov body $\Delta_{Y_\bullet}(X)$ of $X$ with respect to the flag $Y_\bullet$ is a rational polyhedral cone.
In order to apply the above result to Bott-Samelson varieties, we prove the following theorem, where the second part is a direct consequence of the first part and the above theorem.

**Theorem 1.2.** Let $Z = Z_w$ be Bott-Samelson variety defined by a reduced expression $w$ of an element $w$ in the Weyl group of $G$.

(i) The variety $Z$ is a Mori dream space.

(ii) The global Okounkov body $\Delta_{Y_\bullet}(Z)$ of $Z$, with respect to a natural flag $Y_\bullet$ of Bott-Samelson subvarieties of $Z$, is a rational polyhedral cone.

As a consequence, all Okounkov bodies $\Delta_{Y_\bullet}(L)$ of line bundles $L$ over $Z$ are rational polyhedral. Using the desingularization $Z_w \to X_w$ of the Schubert variety $X_w$, we also see that line bundles over Schubert varieties admit rational polyhedral Okounkov bodies.

On the representation-theoretic side, we obtain Okounkov bodies describing weight multiplicities. Indeed, the flag $Y_\bullet$ of subvarieties is $B$-invariant, which allows for the construction of affine subspaces $H_\mu$ mentioned before. We then get the following result on asymptotics of weight multiplicities in a section ring $R(Z, L)$.

**Theorem 1.3.** Let $L$ be an effective line bundle over the Bott-Samelson variety $Z$. Let $H \subseteq B$ be a torus contained in a maximal torus of $G$ lying in $B$, and let $\mu$ be an rational $H$-weight. Then there exists an affine subspace $H_\mu$ (in $\mathbb{R}^{n+1}$, where $n = \dim Z$) such that the asymptotics of the multiplicity function $m_{k\mu,k}$ defined above is given by

$$\lim_{k \to \infty} \frac{m_{k\mu,k}}{k^m} = vol_m(\Delta_{Y_\bullet}(L) \cap H_\mu),$$

where $\Delta_{Y_\bullet}(L)$ is the rational polyhedral Okounkov body of $L$.

If we apply this to the situation when the torus $H$ is a maximal torus, $Z$ is of maximal dimension, and thus admits a birational morphism $f : Z \to G/B$ to the flag variety of $G$, and $L = f^*(L_\lambda)$ is the pull-back of the line bundle $L_\lambda$ over $G/B$, we obtain the following corollary, which can be seen as an analog of Littelmann’s result which describes weight multiplicities using string polytopes.

**Corollary 1.4.** Let $V_\lambda \cong H^0(G/B, L_\lambda)$ be the irreducible $G$-representation of highest weight $\lambda$. If $\mu$ is rational weight, let $m_{k\mu,k}$, for $k\mu$ integral, denote the multiplicity of the weight $k\mu$ in the $G$-module $V_{k\lambda}$. Then there exists an $m \in \mathbb{N}$ such that

$$\lim_{k \to \infty} \frac{m_{k\mu,k}}{k^m} = vol_m(\Delta_{Y_\bullet}(f^*(L_\lambda)) \cap H_\mu).$$

The present paper is organized as follows: we begin by recalling basic facts about Okounkov bodies and Bott-Samelson varieties in sections 2 and 3 respectively. The main result is proved in section 4. Finally, in section 5 the result is applied to Bott-Samelson varieties, furthermore we address there the representation-theoretic consequences.

We work throughout over the complex numbers $\mathbb{C}$ as our base field.

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2. Okounkov bodies

For the convenience of the reader not familiar with the construction of Okounkov bodies we give here a quick overview. For a thorough discussion, we refer the reader to [KK09] and [LM09].

To a graded linear series \( W_* \) on a normal projective variety \( X \) of dimension \( n \) we want to assign a convex subset of \( \mathbb{R}^n \) carrying information on \( W_* \). In practice, more often than not, \( W_* \) will be the complete graded linear series \( \bigoplus_k H^0(X, \mathcal{O}_X(kD)) \) corresponding to an effective divisor \( D \). However, at times it is convenient to work with an arbitrary linear series associated to some divisor \( D \), i.e., a series \( W_* = \{ W_k \} \) of subspaces \( W_k \subseteq H^0(X, \mathcal{O}_X(kD)) \) satisfying the condition \( W_k \cdot W_l \subseteq W_{k+l} \). The construction will depend on the choice of a valuation-like function

\[
\nu : \bigcup_{k \geq 0} W_k \setminus \{ 0 \} \to \mathbb{Z}^n.
\]

Instead of recounting the conditions on \( \nu \), we describe a certain type of valuation which automatically satisfies these conditions. To this end, let

\[
Y_* : X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_n
\]

be a flag of irreducible subvarieties such that \( \text{codim}_X(Y_i) = i \) and such that \( Y_i \) is a smooth point of each \( Y_i \). Then, for a section \( s \in W_k \subseteq H^0(X, \mathcal{O}_X(kD)) \), we set \( \nu_1(s) := \text{ord}_{Y_1}(s) \). If we choose a local equation for \( Y_1 \), we obtain a unique section \( s_1 \in H^0(Y_1, \mathcal{O}_{Y_1}(D - \nu_1(s)iY_1)) \) not vanishing identically along \( Y_1 \), and thus determining a section \( s_1 \in H^0(Y_1, \mathcal{O}_{Y_1}(D - \nu_1(s)iY_1)) \). We then set \( \nu_2(s) = \text{ord}_{Y_2}(s_1) \), and proceed as before to obtain the valuation vector \( \nu_Y(s) = (\nu_1(s), \ldots, \nu_n(s)) \).

One then defines the valuation semi-group of \( W_* \) with respect to \( Y_* \) as

\[
\Gamma_{Y_*}(W_*):=\left\{(\nu_{Y_*}(s),m)\in \mathbb{Z}^{n+1} \mid 0 \neq s \in W_m\right\}.
\]

Furthermore, we define the Okounkov body of \( W_* \) as

\[
\Delta_{Y_*}(W_*):=\Sigma(\Gamma_{Y_*}(W_*)) \cap (\mathbb{R}^n \times \{ 1 \})
\]

where \( \Sigma(\Gamma_{Y_*}(W_*)) \) denotes the closed convex cone in \( \mathbb{R}^{n+1} \) spanned by \( \Gamma_{Y_*}(W_*)) \).

If \( W_* \) is the complete graded linear series of a divisor \( D \), we write \( \Delta_{Y_*}(D) \) for the Okounkov body of \( W_* \). In this case, by [LM09 Theorem 2.3], we have the important identity

\[
\text{vol}_{\mathbb{R}^n}(\Delta_{Y_*}(D)) = \frac{1}{n!} \text{vol}_X(D),
\]

showing in particular, that the volume of the body \( \Delta_{Y_*}(D) \) is independent of the choice of the flag \( Y_* \). Another important observation made in [LM09] is that even the shape of the Okounkov body \( \Delta_{Y_*}(D) \) for a divisor \( D \) only depends on the numerical equivalence class of \( D \). It is therefore a natural question how the bodies \( \Delta_{Y_*}(D) \) change as \( |D| \) varies in the Néron-Severi

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vector space $N^1(X)_\mathbb{R}$. An answer to this question is given in [LM09, Theorem 4.5] by proving the existence of the global Okounkov body: there exists a closed convex cone $\Delta_{Y^*}(X) \subset \mathbb{R}^n \times N^1(X)_\mathbb{R}$ such that for each big divisor $D$ the fiber of the second projection over $[D]$ is exactly $\Delta_{Y^*}(D)$.

The concrete determination or even the description of geometric properties of Okounkov bodies associated to some graded linear series is extremely difficult in general. As is to be expected this will be even more true of the global Okounkov body of a given variety. In particular, it is an intriguing question under which conditions on $X$ it is possible to pick a flag such that the corresponding global Okounkov body is rational polyhedral. Already in [LM09] this was shown to be possible for toric varieties. Based on this evidence it is conjectured to work also for any Mori dream space $X$. In [SS14], we introduce a possible technique to prove rational polyhedrality of global Okounkov bodies by constructing so-called Minkowski bases on $X$. Work in progress hints at the feasibility of this approach for any Mori dream space. In this paper however, we use a more direct strategy to prove the rational polyhedrality of global Okounkov bodies with respect to a natural choice of flag for a special class of Mori dream spaces, namely Bott-Samelson varieties, which we introduce in the following section.

Let us make a small remark on how the construction by Littelmann mentioned in the introduction compares to Okounkov bodies. Littelmann’s string polytopes are constructed by purely algebraic and combinatorial means, notably using quantum enveloping algebras of Lie algebras, and the result thus only shows formal analogies with the outcome of Okounkov’s approach. However, since - by the Borel-Weil theorem - every irreducible $G$-module $V_\lambda$ can be realized as the space of sections $H^0(X,L_\lambda)$ of a line bundle $L_\lambda$ over a flag variety $X := G/B$, where $B$ is a Borel subgroup of $G$, Okounkov’s approach makes sense for the study of asymptotics of weight spaces in the section ring $R(X,L_\lambda)$. For the approach to work, the flag $Y^*$ should consist of $H$-invariant subvarieties. A natural candidate for such a flag would then be a flag of Schubert varieties, and indeed this approach was taken by Kaveh in [K11]. For technical reasons, notably for having a flag of Cartier divisors, Kaveh passed to a Bott-Samelson resolution $Z \to X$ of $X$, pulled back $L_\lambda$ to $Z$, and replaced the flag $Y^*$ by a flag $Z^*$ of (translations of) Bott-Samelson subvarieties of $Z$. The main result in [K11] is that Littelmann’s string bases can be interpreted in terms of a $\mathbb{Z}^n$-valued valuation on the function field $\mathbb{C}(Z)$ of $Z$, depending on the flag $Z^*$. This valuation, however, differs from those introduced by Okounkov: whereas orders of vanishing of a regular function $f$ are described in local coordinates $x_1, \ldots, x_n$ by the smallest monomial term of $f(x) = \sum_{a \in \mathbb{N}^n} c_a x^a$, with respect to some ordering of the variables $x_1, \ldots, x_n$, Kaveh’s valuation is locally defined by the highest monomial term. In geometric language, this valuation thus tells how often $f$ can be differentiated in the various directions defined by the $x_i$ in the given order. It still remains an open problem to interpret Littelmann’s string polytopes as Okounkov bodies, or indeed, more generally,
to construct rational polyhedral Okounkov bodies for line bundles over flag varieties using some $H$-invariant flag $Y_\bullet$.

3. Bott-Samelson varieties

Let us recall the basics of Bott-Samelson varieties, following [LT04].

Let $G$ be a connected complex reductive group, let $B \subseteq G$ be a Borel subgroup, and let $W$ be the Weyl group of $G$. If $s_i \in W$ is a simple reflection, let $P_i$ denote the associated minimal parabolic subgroup containing $B$. Then the quotient space $P_i/B$ is isomorphic to $\mathbb{P}^1$. For a sequence $w = (s_1, \ldots, s_n)$ (where the $s_i$ are not necessarily distinct), let $P_w := P_1 \times \cdots \times P_n$ be the product of the corresponding parabolic subgroups, and consider the right action of $B^n$ on $P_w$ given by

$$(p_1, \ldots, p_n)(b_1, \ldots, b_n) := (p_1 b_1, b_1^{-1} p_2 b_2, b_2^{-1} p_3 b_3, \ldots, b_{n-1}^{-1} p_n b_n).$$

The Bott-Samelson variety $Z_w$ is the quotient

$$Z_w := P_w / B^n.$$

An alternative description of this quotient can be given as follows. Suppose that $X$ and $Y$ are two varieties, such that $X$ is equipped with a right action and $Y$ with a left action of $B$. Consider the right action of $B$ on the product given by

$$(x, y).b := (xb, b^{-1}y), \quad (x, y) \in X \times Y, \quad b \in B,$$

and let $X \times_B Y := (X \times Y)/B$ denote the quotient space. Then the map $X \times_B Y \rightarrow X/B$, $[(x, y)] \mapsto xB$ exhibits $X \times_B Y$ as a fiber bundle over $X/B$ and with fiber $Y$. Now, we can alternatively describe $Z_w$ as

$$Z_w = (P_1 \times_B \cdots \times_B P_n)/B,$$

where $B$ acts on the right on $P_1 \times_B \cdots \times_B P_n$ by

$$[(p_1, \ldots, p_n)].b := [(p_1, \ldots, p_{n-1}, p_n b)], \quad (p_1, \ldots, p_n) \in P_w, \quad b \in B.$$

As a consequence, using the fact that each quotient $P_i/B$ is isomorphic to $\mathbb{P}^1$, $Z_w$ is given as an iteration of $\mathbb{P}^1$-bundles. To describe this structure in more detail, let, for $j \in \{1, \ldots, n\}$, $w[j]$ denote the truncated sequence $(s_1, \ldots, s_{n-j})$, and let $Z_{w[j]} := P_{w[j]}/B^{n-j}$ denote the associated Bott-Samelson variety. Then the projections $P_w \rightarrow P_{w[j]}$ are $B^n$-equivariant, where $B^n$ acts on $P_{w[j]}$ by the factor $B^{n-j}$, and thus induce a projections $\pi_{w[j]} : Z_w \rightarrow Z_{w[j]}$, which can be factorized as a sequence of $\mathbb{P}^1$-fibrations

$$\pi_{w[1]} : Z_w \rightarrow Z_{w[1]} \rightarrow \cdots \rightarrow Z_{w[j]}.$$

Now, each $\mathbb{P}^1$-bundle admits a natural section as follows. Let $w(j) := (s_1, s_j, \ldots, s_n)$, so that $P_{w(j)}$ embeds naturally as a subgroup of $P_w$. The embedding $\sigma^0_{w,j} : P_{w(j)} \hookrightarrow P_w$ is $B^{n-1}$-equivariant, and thus induces an embedding

$$\sigma_{w,j} : Z_{w(j)} \hookrightarrow Z_w$$

of $Z_{w(j)}$ as a divisor in $Z_w$ such that the divisors $Z_{w(j)}$, $j = 1, \ldots, n$ intersect transversely in a point. In particular, $\sigma_{w,1} : Z_{w(n)} \cong Z_{w[1]} \rightarrow Z_w$ defines a section of the $\mathbb{P}^1$-bundle $\pi_{w[1]} Z_w \rightarrow Z_{w[1]}$, and identifies $Z_{w[1]}$ with a
In particular, if \( Z \) is effective if and only if \( \{y \} \) is an ample divisor on \( Y \) to the restriction to \( Z \). Then \( \pi \) admits a divisor \( m_1 Z_{w(1)} + \cdots + m_n Z_{w(n)} \), \( m_1, \ldots, m_n \in \mathbb{Z} \), is effective if and only if \( m_1, \ldots, m_n \geq 0 \) (cf. [LT04, Prop. 3.5]). The basis \( \{Z_{w(1)}\}, \ldots, Z_{w(n)} \} \) is called the effective basis for \( \text{Pic}(Z_w) \). Notice that, since \( Z_{w(n)} \) defines a section of the bundle \( \pi_{w[1]} \), the restricted divisors
\[
(2) \quad Z_{w(1)}', Z_{w(n)}', \ldots, Z_{w(n-1)}', Z_{w(n)}
\]
form the effective basis for \( Z_{w[1]} \cong Z_{w(n)} \).

Now assume that \( w \) is a reduced sequence. Recall that the product map
\[
P_w : Z_w \to G, (p_1, \ldots, p_n) \mapsto p_1 \cdots p_n
\]
defines a morphism
\[
p_w : Z_w \to Y_{\overline{w}} := B\overline{w}B
\]
into the Schubert subvariety \( Y_{\overline{w}} \) of the flag variety \( G/B \), and that this morphism is in fact birational. Moreover, it is \( B \)-equivariant with respect to the left action of \( B \) on \( Z_w \) defined by
\[
b [(p_1, \ldots, p_n)] := [(bp_1, p_2, \ldots, p_n)], \quad (p_1, \ldots, p_n) \in P_w, \quad b \in B.
\]
In particular, if \( \overline{w} \) is the longest element of the Weyl group, \( p_{\overline{w}} \) defines a birational map \( Z_{\overline{w}} \to G/B \).

The following theorem is the crucial result for our application of the theory from the following section.

**Theorem 3.1.** Let \( G \) be a complex reductive group with Weyl group \( W \), and let \( Z = Z_w \) be a Bott-Samelson variety defined by a reduced sequence \( w \) of simple reflections. Then \( Z \) admits a divisor \( \Delta \) such that \( (Z, \Delta) \) is a log Fano pair. In particular, \( Z \) is a Mori dream space.

**Proof.** Let \( Y = G/B \), and let \( D_\rho \) be the divisor on \( Y_{\overline{w}} \) which corresponds to the restriction to \( Y_{\overline{w}} \) of the square root of the anticanonical bundle of \( Y \). Then \( D_\rho \) is an ample divisor on \( Y_{\overline{w}} \), so that \( p_{\overline{w}}^*(D_\rho) \) is a nef divisor on \( Z \).

In order to facilitate the notation, let \( \{D_1, \ldots, D_n\} \) be the basis of effective divisors for \( \text{Pic}(Z) \). Now choose integers \( a_1, \ldots, a_n > 0 \) so that \( \sum_{i=1}^n a_i D_i \) is an ample divisor. Then, for every \( N > 0 \), the divisor \( p_{\overline{w}}^*(-D_\rho) - \sum_{i=1}^n a_i/N D_i \) is anti-ample. Now let \( N \in \mathbb{N} \) be so big that \( a_i/N < 1 \) for every \( i \), and put
\[
\Delta := \sum_{i=1}^n (1 - a_i/N) D_i.
\]

If \( K_Z \) is the canonical divisor of \( Z \), we then have that
\[
K_Z + \Delta = \pi^*(-D_\rho) - \sum_{i=1}^n D_i + \sum_{i=1}^n (1 - a_i/N) D_i = \pi^* L_\rho - \sum_{i=1}^n (a_i/N) D_i
\]
(cf. [LT04 Lemma 5.1]) is anti-ample. Since $Z$ is nonsingular, and all subsets of the set of smooth divisors $\{D_1, \ldots, D_n\}$ intersect transversely and smoothly, the pair $(Z, \Delta)$ thus defines a log Fano pair. □

**Remark 3.2.** In the context of the above theorem it is worth mentioning an analogous result by Anderson and Stapledon ([AS14]) on the log Fano property of Schubert varieties.

4. **Good flags on Mori dream spaces**

In this section we prove the main theorem of this paper. The main objective is to establish conditions on a flag on a Mori dream space, such that its global Okounkov body is rational polyhedral.

First let us recall that a Mori dream space $X$ is a normal $\mathbb{Q}$-factorial variety such that $\text{Pic}(X) \cong N^1(X)_{\mathbb{Q}}$ and with a Cox ring $\text{Cox}(X)$ which is a finitely generated $\mathbb{C}$-algebra. We make heavy use of the theory of Mori dream spaces developed by Hu and Keel in [HK00] and we refer the reader to this beautiful paper for a detailed investigation of Mori dream spaces.

Note that for any effective divisor $D$ on a Mori dream space $X$, the ring of sections $R(X, D) := \bigoplus_{k \geq 0} H^0(X, \mathcal{O}_X(kD))$ is finitely generated, so we obtain a natural rational map $f_D : X \dashrightarrow \text{Proj}(R(X, D))$ which is regular outside the stable base locus of $D$. One obtains an equivalence relation of effective divisors as follows: two effective divisors $D$ and $D'$ are *Mori-equivalent* if up to isomorphism they yield the same rational maps. Hu and Keel prove ([HK00 Proposition 1.11]) that there are only finitely many equivalence classes, indexed by contracting rational maps $f : X \dashrightarrow X'$ and that the closure $\Sigma_f$ of a maximal dimensional equivalence class can be described as the closed convex cone spanned by the $f$-exceptional rays together with the face $f^*(\text{Nef}(X'))$ of the moving cone. These subcones $\Sigma_f$, which decompose the pseudo-effective cone $\overline{\text{Eff}}(X)$, are in the remainder of this paper following [HK00] referred to as *Mori-chambers*.

Let then $X$ be a Mori dream space, and let $R$ be the Cox ring of $X$. Let $\{D_1, \ldots, D_r\}$ be a basis of integral divisors for the torsion-free part of the Picard group $\text{Pic}(X)$ of $X$ such that the pseudo-effective cone $\overline{\text{Eff}}(X)$ of $X$ is contained in the closed convex cone generated by the $D_i$. Without loss of generality, we can assume that $R$ is the section ring defined by the divisors $D_1, \ldots, D_r$, i.e., that

$$R = \bigoplus_{(m_1, \ldots, m_r) \in \mathbb{N}^r} H^0(X, \mathcal{O}_X(m_1D_1 + \cdots + m_rD_r)).$$

Let $T^r := \text{Hom}(\mathbb{N}^r, \mathbb{C}^*) \cong (\mathbb{C}^*)^r$ be the $r$-dimensional complex torus. The character group $\hat{T}$ of $T^r$ is isomorphic to $\mathbb{Z}^r$ and there is a natural action of $T^r$ on $R$ by graded automorphisms of rings by letting $T^r$ operate on the homogeneous piece

$$R_{(m_1, \ldots, m_r)} := H^0(X, \mathcal{O}_X(m_1D_1 + \cdots + m_rD_r))$$
by the character
\begin{equation}
\chi_{m_1,\ldots,m_r} : T^r \longrightarrow \mathbb{C}^*, \quad (t_1,\ldots,t_r) \mapsto t_1^{m_1} \cdots t_r^{m_r}, \quad (t_1,\ldots,t_r) \in (\mathbb{C}^*)^r.
\end{equation}
By [HK00, Proposition 2.9], we can describe $X$ as the GIT quotient
\[ X = \text{Spec}(R)^{ss}(\chi)/T^r, \]
where $\chi \in \widehat{T}$ is a character of $T^r$ corresponding to an ample line bundle over $X$, and $\text{Spec}(R)^{ss}(\chi)$ denotes the set of semi-stable points in $\text{Spec}(R)$ with respect to the character $\chi$.

If $Y \subseteq X$ is a prime Cartier divisor, $Y = V(s_Y)$, for some $s_Y \in H^0(X, \mathcal{O}_X(m_1D_1 + \cdots + m_rD_r))$, and some $(m_1,\ldots,m_r) \in \mathbb{N}^r$, the section $s_Y$ thus defines a homogeneous element in the $\mathbb{N}^r$-graded ring $R$. The ideal
\[ R_1 := (s_Y) := R \cdot s_Y \]
is then a homogeneous prime ideal of $R$. Hence, the ring $R/R_1$ is equipped with an $\mathbb{N}^r$-grading and an induced action of $T^r$ by graded automorphisms of rings which makes the natural surjection
\[ \varphi : R \longrightarrow R/R_1 \]
$T^r$-equivariant. The action of $T^r$ on the rings $R$ and $R_1$ induces actions on the respective prime spectra $\text{Spec}(R)$ and $\text{Spec}(R/R_1)$, and the embedding
\[ ^{\sigma}\varphi : \text{Spec}(R/R_1) \longrightarrow \text{Spec}(R) \]
is $T^r$-equivariant with respect to these $T^r$ actions, i.e., $\text{Spec}(R/R_1)$ is a closed $T^r$-invariant subvariety of $\text{Spec}(R)$.

**Lemma 4.1.** If $\psi \in \widehat{T}$ is a character of $T^r$ such that the unstable locus $\text{Spec}(R)^{us}(\psi)$ is of codimension at least two, then the inclusion $\text{Spec}(R/R_1) \longrightarrow \text{Spec}(R)$ induces an embedding
\[ \text{Spec}(R/R_1)^{ss}(\psi)/T^r \longrightarrow \text{Spec}(R)^{ss}(\psi)/T^r \]
of GIT quotients. Moreover, if every semi-stable point in $\text{Spec}(R)$ is stable, i.e., $\text{Spec}(R)^{ss}(\psi) = \text{Spec}(R)^{s}(\psi)$, then the same holds for $\text{Spec}(R/R_1)$.

**Proof.** Since $s_Y \in R$ is not the zero divisor, we have that $\dim \text{Spec}(R/R_1) = \dim \text{Spec}(R) - 1$. Hence, by the assumption on the codimension of $\text{Spec}(R)^{us}(\psi)$, the intersection $\text{Spec}(R/R_1) \cap \text{Spec}(R)^{ss}(\psi)$ is a nonempty closed $T^r$-invariant subvariety of $\text{Spec}(R)^{ss}(\psi)$. We now claim that
\[ \text{Spec}(R/R_1)^{ss}(\psi) = \text{Spec}(R/R_1) \cap \text{Spec}(R)^{ss}(\psi). \]
In order to prove this claim, suppose first that $x \in \text{Spec}(R/R_1)^{ss}(\psi)$ is a (closed) semistable point. Then $x$ is a maximal ideal $m_x$ in $R/R_1$, and viewing the structure sheaf $\mathcal{O}_{\text{Spec}(R/R_1)}$ as a sheaf of $\mathbb{C}$-valued functions on the set of maximal ideals in $\text{Spec}(R/R_1)$, the semi-stability of $x$ means that there exists a $k \in \mathbb{N}$ and an element $b \in (R/R_1)^{\psi,k}$ with $b(x) \neq 0$. By the surjectivity and the $T^r$-equivariance of $\varphi$, there exists an element $a \in R^{\psi,k}$ with $\varphi(b) = a$. Since $\text{Spec}(R/R_1)$ is a closed subvariety of $\text{Spec}(R)$, the
point \( x \) is also a closed point in \( \text{Spec}(R) \), so that we can evaluate \( a \) in \( x \). Here
\[ a(x) = \varphi(a)(x) = b(x) \neq 0. \]
Hence, \( x \) is also semi-stable in \( \text{Spec}(R) \). This proves the inclusion
\[ \text{Spec}(R/R_1)^{ss}(\psi) \subseteq \text{Spec}(R)^{ss}(\psi). \]

On the other hand, if \( x \in \text{Spec}(R/R_1) \cap \text{Spec}(R)^{ss}(\psi) \), there exists a \( k \in \mathbb{N} \) and an \( a \in R_{\psi k} \) so that \( a(x) \neq 0 \). Then \( b := \varphi(a) \in (R/R_1)_{\psi k} \), and, since \( x \in \text{Spec}(R/R_1) \), \( b(x) = \varphi(a)(x) = a(x) \neq 0 \). Thus, the inclusion \( \text{Spec}(R/R_1) \cap \text{Spec}(R)^{ss}(\psi) \subseteq \text{Spec}(R/r_1)(\psi) \) holds.

If \( x, x' \in \text{Spec}(R/R_1)^{ss}(\psi) \) are two points, then, since \( \text{Spec}(R/R_1)^{ss}(\psi) \) is a closed \( T^r \)-invariant subvariety of \( \text{Spec}(R)^{ss}(\psi) \), their orbit closures \( T^r \cdot x \) and \( T^r \cdot x' \) intersect in \( \text{Spec}(R/R_1)^{ss}(\psi) \) if and only they intersect in \( \text{Spec}(R)^{ss}(\psi) \). Hence, the inclusion morphism \( \text{Spec}(R/R_1)^{ss}(\psi) \subseteq \text{Spec}(R)^{ss}(\psi) \) induces a well-defined embedding
\[ \text{Spec}(R/R_1)^{ss}(\psi)/T^r \subseteq \text{Spec}(R)^{ss}(\psi)/T^r \]
of quotients. This proves the first part of the lemma.

Now assume that \( \text{Spec}(R)^{ss}(\psi) = \text{Spec}(R)^{ss}(\psi) \), and let \( x \in \text{Spec}(R/R_1)^{ss}(\psi) \).
By the above, \( x \) is then a semi-stable, and hence stable, point of \( \text{Spec}(R) \), so that the orbit \( T^r \cdot x \) is closed in \( \text{Spec}(R)^{ss}(\psi) \). Since the orbit lies in \( \text{Spec}(R/R_1)^{ss}(\psi) \), it is, again since \( \text{Spec}(R/R_1)^{ss}(\psi) \) is a closed subvariety of \( \text{Spec}(R)^{ss}(\psi) \), also closed in \( \text{Spec}(R)^{ss}(\psi) \). Finally, the stabilizer \( (T^r)_x \) is finite since \( x \) is a stable point in \( \text{Spec}(R) \). This shows that \( x \in \text{Spec}(R/R_1)^{ss}(\psi) \), and this finishes the proof of the lemma.

Note that the last part of the lemma shows the quotient \( \text{Spec}(R/R_1)^{ss}(\psi)/T^r \) to be geometric since so is the original quotient \( \text{Spec}(R)^{ss}(\psi)/T^r \). Since for geometric quotients the dimension equals the difference of the dimensions of the affine variety and the torus, and the image of \( \text{Spec}(R/R_1) \) in \( \text{Spec}(R) \) equals the hyperplane \( V(s_Y) \), the image of the embedding
\[ \text{Spec}(R/R_1)^{ss}(\psi)/T^r \to \text{Spec}(R)^{ss}(\psi)/T^r \]
defines a prime divisor in \( X \) on which the section \( s_Y \) vanishes. Since both are irreducible, this divisor and \( Y \) must agree.

Let us recall that a change of character in GIT induces a rational map of GIT quotients: let \( X = V^{ss}(\chi')/T^r \), and \( X' = V^{ss}(\chi)/T^r \) for a fixed affine variety equipped with an action of \( T^r \). Then \( x, x' \in V \) are \( T^r \)-equivalent in \( U_{\chi} = V^{ss}(\chi) \) if \( \overline{T^r x} = \overline{T^r x'} \) contains \( x' \) and vice versa. In case arbitrary \( x \) and \( x' \) are equivalent in \( U_{\chi'} \) if they are equivalent in \( U_{\chi} \) then mapping each \( x \) to its \( T_{\chi'} \)-equivalence class gives a morphism
\[ V^{ss}(\chi)/T^r \to X' = V^{ss}(\chi')/T^r. \]
In case we can reverse the above implication, we obtain a morphism in the opposite direction. If neither implication holds then by [DHS98], there are faces \( F_j \) of GIT-chambers between \( \chi \) and \( \chi' \) which admit in the same way
as described above birational morphism from \( V^{ss}(\chi)/T^r \) and \( V^{ss}(\chi')/T^r \) to \( V^{ss}(F_j)/T^r \), yielding a birational mapping
\[
V^{ss}(\chi)/T^r \dashrightarrow X' = V^{ss}(\chi')/T^r.
\]

**Proposition 4.2.** Let \( f : X \dashrightarrow X' \) be an SQM given by GIT, i.e., there exists a character \( \chi' \) such that \( X' = \text{Spec}(R)^{ss}(\chi')/T^r \) and \( f \) corresponds to the change of character. Suppose furthermore that every effective divisor on the GIT-embedded subvariety \( Y \) has finitely generated section ring, and that the linear map \( p : N^1(X)_R \rightarrow N^1(Y)_R \) restricts to a surjective map
\[
\overline{\text{Eff}}(X) \cap p^{-1}(\overline{\text{Eff}}(Y)) \rightarrow \overline{\text{Eff}}(Y).
\]
Then \( f \) restricts to a birational map
\[
f_Y : Y \dashrightarrow Y' := \text{Spec}(R/R_1)^{ss}(\chi')/T^r
\]
ton an a subvariety of \( X' \) with
\[
\text{codim exc}(f_Y^{-1}) \geq 2.
\]

**Proof.** Let \( D \) be the divisor on \( X \) corresponding to the character \( \chi' \). We first recall that \( X' = \text{Proj}(\bigoplus_{k \geq 0} R_k\chi') \) and that the birational map \( f \) is defined on the open subset \( O := X \setminus \mathbb{B}(D) \) of \( X \) by
\[
f(x) := \bigoplus_{k \geq 0} \ker ev_x^k, \quad x \in O,
\]
where
\[
ev_x^k : H^0(X, \mathcal{O}_X(kD)) \rightarrow \mathcal{O}_X(kD)_x/m_x\mathcal{O}_X(kD)_x
\]
is the evaluation map at \( x \), and \( m_x \subseteq \mathcal{O}_{X,x} \) is the maximal ideal of the local ring \( \mathcal{O}_{X,x} \). The fact that the exceptional locus of \( f \) is of codimension at least two shows that \( D \) is a movable divisor.

Now, for \( k \geq 0 \), let \( S_k \subseteq H^0(Y, \mathcal{O}_Y(kD \cdot Y)) \) be the image of the restriction map
\[
H^0(X, \mathcal{O}_X(kD)) \rightarrow H^0(Y, \mathcal{O}_Y(kD \cdot Y)).
\]
By the construction of \( Y \), we clearly have \( S_k = (R/R_1)_k\chi' \), and since \( D \) is movable, the graded ring \( S := \bigoplus_{k \geq 0} S_k \) is nonzero. Moreover, the assumption on the restricted map \([4]\) show that linear map \( p : N^1(X)_R \rightarrow N^1(Y)_R \) is surjective, and that no open neighborhood of \([D] \) in \( \overline{\text{Eff}}(X) \) can be mapped into the boundary of the pseudo-effective cone \( \overline{\text{Eff}}(Y) \). By, if necessary, replacing \( D \) by a big divisor in the Mori chamber of \( D \), which thus defines the same birational map \( f \), we may assume that \( D \) is big, so that \( D \cdot Y \) is a big divisor on \( Y \).

The section ring \( \bigoplus_{k \geq 0} R_{k\chi'} \) of \( D \) is finitely generated, and by replacing \( \chi' \) by a sufficiently big multiple, if necessary, we may assume that it is generated in degree one, and that the section ring of the restricted divisor \( D \cdot Y \) is also generated in degree one.

The restriction \( f_Y \) is now defined on \( O \cap Y = Y \setminus Bs(S) \) and given by
\[
f_Y(y) := \bigoplus_{k \geq 0} \ker ev_{S,y}^k, \quad y \in U,
\]
for the evaluation maps
\[
ev_{S,y}^k : S_k \rightarrow \mathcal{O}_Y(kD \cdot Y)_y/m_y\mathcal{O}_Y(kD \cdot Y)_y.
\]
On the other hand, the big divisor $D \cdot Y$ also defines a birational map

$$g : Y \dasharrow Z := \text{Proj}(\bigoplus_{k \geq 0} H^0(Y, \mathcal{O}_Y(kD \cdot Y))),$$

defined on the open subset $U_D := Y \setminus Bs(D \cdot Y)$ of $Y$ by

$$g(y) := \bigoplus_{k \geq 0} ev^k_y, \quad y \in U_D,$$

for the evaluation maps

$$ev^k_y : H^0(Y, \mathcal{O}_Y(kD \cdot Y)) \longrightarrow \mathcal{O}_Y(kD \cdot Y)_y/\mathfrak{m}_y\mathcal{O}_Y(kD \cdot Y)_y, \quad y \in Y.$$  

The evaluation maps $ev^k_{S,y} : S_k \longrightarrow \mathcal{O}_Y(kD \cdot Y)_y/\mathfrak{m}_y\mathcal{O}_Y(kD \cdot Y)_y, \quad y \in Y,$

are clearly the restrictions of the evaluation maps $ev^k_y$ to the subring $S$.

Now, there is a natural rational map

$$h : Z \dasharrow \text{Proj}(S)$$

defined by the inclusion of graded rings $S \subseteq \bigoplus_{k \geq 0} H^0(Y, \mathcal{O}_Y(kD \cdot Y))$. The map $h$ is defined on the set $V$ of all homogeneous relevant prime ideals not containing $S$ by

$$h(p) := p \cap S, \quad p \in V.$$  

Since the ring $\bigoplus_k R_{kD'}$ is finitely generated, this also holds for its image $S$, so that $V$ is indeed an open subset of $Z$. We now claim that $h : V \longrightarrow Y'$ is surjective as a morphism of varieties, i.e., that its image contains all closed points of $Y'$. Indeed, if $q \subseteq S$ is any relevant homogeneous prime ideal, there is a $\xi_1 \in S_1$ not contained in $q$. We extend $\{\xi_1\}$ to a basis $\{\xi_1, \ldots, \xi_m\}$ for the $\mathbb{C}$-vector space $H^0(Y, \mathcal{O}_Y(D \cdot Y))$, and let $I \subseteq \bigoplus_k H^0(Y, \mathcal{O}_Y(kD \cdot Y))$ be the (homogeneous) ideal generated by $q$ and $\{\xi_2, \ldots, \xi_m\}$. It now follows from the fact that both $S$ and $\bigoplus_{k \geq 0} H^0(Y, \mathcal{O}_Y(kD \cdot Y))$ are generated in degree one, that $\xi_1 \notin I$, so that $I$ is a relevant homogeneous ideal. Then $I$ is contained in a maximal homogeneous relevant ideal $m \subseteq \bigoplus_{k \geq 0} H^0(Y, \mathcal{O}_Y(kD \cdot Y))$. Since $m$ is relevant, and $\xi_2, \ldots, \xi_m \in m$, we have $\xi_1 \notin m$. Then $S \cap m$ is a homogeneous prime ideal in $S$, and it is relevant since $\xi_1 \notin m$. Clearly $q \subseteq S \cap m$, and if $q$ is a maximal relevant homogeneous ideal, equality holds, i.e., $h(m) = q$. This shows that the image of $h$ contains all closed points of $Y'$, so that $h$ is surjective as a morphism of varieties $V \longrightarrow Y'$.

Now, on the open subset $U \subseteq U_D$, we have the factorization

$$f_Y = h \circ g,$$

giving rise to a commutative diagram

$$\begin{array}{ccc}
X & \xrightarrow{f} & X' \\
\downarrow & & \downarrow \\
Y & \xrightarrow{g} & Z & \xrightarrow{h} & Y'
\end{array}$$

Here the rational map $g$ is a contraction since the section ring of the divisor $D \cdot Y$ is finitely generated ([HK00], Lemma 1.6), and hence the exceptional locus of $g^{-1}$ is of codimension at least two ([C12], Remark 2.2]). Moreover, we have isomorphisms $H^0(Z, \mathcal{O}_Z(k)) \cong H^0(Y, \mathcal{O}_Y(kD \cdot Y))$ for every $k \in \mathbb{N}$.
(CT12 Prop. 2.4), so that we have an identification of $Bs(S) \cap U_D$ with the corresponding stable base locus $V(S_1) \cap g(U_D) \subseteq Z$. Since $h$ is defined outside of $V(S_1)$, the composed morphism $f_Y : U \rightarrow h(g(U))$ is thus defined. Furthermore, its injectivity implies that $h$ is also injective away from $exc(g^{-1})$. In particular, $h$ becomes an isomorphism $h : V \backslash exc(g^{-1}) \rightarrow Y' \backslash h(exc(g^{-1}) \cap V)$. The exceptional locus of $f_Y^{-1}$ is thus a subset of $h(excg^{-1} \cap V)$, and hence of codimension at least two.

\[ \square \]

**Proposition 4.3.** Let $X$ be a Mori dream space, and let $Y_\bullet$ be an admissible flag of normal subvarieties such that $Y := Y_1$ is embedded by GIT as in the previous proposition, and let $Y_1^\bullet$ denote the admissible flag of subvarieties of $Y$.

Let $f : X \rightarrow X'$ be a SQM, and let $f_Y : Y \rightarrow Y'$ be the induced birational morphism. Assume that the exceptional locus of $f_Y^{-1}$ is of codimension at least two in $Y'$.

If $D'$ is a nef divisor on $X'$ and $D := f^*(D')$ is the corresponding divisor on $X$, let

$$ R : \bigoplus_k H^0(X, \mathcal{O}_X(kD)) \rightarrow \bigoplus_k H^0(Y, \mathcal{O}_Y(kD \cdot Y)) $$

be the restriction map. Then, the identity

$$ \Delta_{Y_1}(im R) = \Delta_{Y_1^\bullet}(D \cdot Y) $$

of Okounkov bodies holds.

**Proof.** We have a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & X' \\
\uparrow & & \uparrow \\
Y & \xrightarrow{f_Y} & Y',
\end{array}
$$

where the vertical arrows denote the respective inclusion morphisms and $f_Y$ is a contracting birational map by assumption. In particular, there are open subsets $U \subseteq Y$ and $W \subseteq Y'$ with $\text{codim}_{Y'}(Y' \backslash W) \geq 2$ such that $f_Y : U \rightarrow W$ is an isomorphism. Then we can pull back Cartier-divisors on $Y'$, and push forward Cartier-divisors on $Y$.

Let us first assume that $D'$ is ample. Denote the restricted divisors $D \cdot Y$ and $D' \cdot Y'$ by $D_Y$ and $D'_{Y'}$, respectively. As a first step, we prove that the difference $D_Y - f^*(D'_{Y'})$ is supported on the exceptional locus $exc(f_Y)$. We thus need to verify that the push-forward $f_{Y_\bullet}(D_Y)$ is exactly $D'_{Y'}$. Let therefore $t'$ be a section in $H^0(X', \mathcal{O}_{X'}(D'))$ whose set of zeros intersects the open set $W$. The section $t'$ corresponds to a section $t \in H^0(X, \mathcal{O}_X(D))$, which agrees with $t' \circ f$ on the subset $V \equiv V'$ where $f$ is an isomorphism.

Now, on the open sets $U \subseteq Y$ and $W \subseteq Y'$ with $U \cong W$, we obtain identifications $f(Z(t) \cap U) = Z(t') \cap W$ of the sets of zeros. By definition, $f(Z(T))$ is in the class of the push-forward divisor of $D_Y$, so $f_\bullet(D_Y) = D'_{Y'}$. 

\[ \square \]
At least in case $Y'$ is also normal we can use the above to prove the restriction maps

$$R_k : H^0(X, \mathcal{O}_X(kD)) \to H^0(Y, \mathcal{O}_Y(kD_Y))$$

in fact become surjective for $k$ large enough. Obviously, this implies the asserted identity of Okounkov bodies.

Assume therefore that $Y'$ is normal. Since $D_Y$ is equivalent to the sum of a pullback and some exceptional divisor $E$, for all large enough $k$ we have an isomorphism

$$H^0(Y, \mathcal{O}_{Y'}(kD_Y)) \cong H^0(Y, \mathcal{O}_Y(kf_Y^*(D_{Y'}') + kE)) \cong H^0(Y', \mathcal{O}_{Y'}(kD_{Y'}')),$$

with the second equality coming from the facts that every component of $kD_Y$ supported on $\text{exc}(f_Y)$ is a component of the fixed part, and that $f_Y$ is a birational contraction between normal projective varieties.

On the other hand, since $D'$ is assumed to be ample, by replacing $D'$ with a sufficiently big multiple, if necessary, we may assume that all restriction maps

$$R'_k : H^0(X', \mathcal{O}_{X'}(kD')) \to H^0(Y', \mathcal{O}_{Y'}(kD_{Y'}')),$$

are surjective. Furthermore, we have an isomorphism

$$f_* : H^0(X, \mathcal{O}_X(D)) \to H^0(X', \mathcal{O}_{X'}(D')),$$

since $f$ is a SMQ. Together, this shows that the restriction $H^0(X, \mathcal{O}_X(kD)) \to H^0(Y, \mathcal{O}_Y(kD_Y))$ must also be surjective for large enough $k$, so that the linear series $\text{im}R$ and $\bigoplus_k H^0(Y, \mathcal{O}_Y(kD_Y))$ agree “eventually”. In particular, the Okounkov bodies of these series coincide,

$$\Delta_{Y_2}(\text{im} R) = \Delta_{Y'_1}(D \cdot Y).$$

Now, if $Y'$ is not normal, we pass to the normalization $\pi : \widetilde{Y'} \to Y'$. Since $Y$ is normal and $\text{codim}(\text{exc}(f_Y^{-1})) \geq 2$, the non-normal locus of $Y'$ is of codimension at least two, and hence $\pi$ is a small modification, making $f_Y := \pi^{-1} \circ f_Y$ a birational contraction between normal varieties. In particular, for any divisor $F$ on $\widetilde{Y'}$, the pull-back map

$$\widetilde{f}_{Y'}^* : H^0(\widetilde{Y'}, \mathcal{O}_{\widetilde{Y'}}(F)) \to H^0(X, \mathcal{O}_X(\widetilde{f}_{Y'}^*(F)))$$

is an isomorphism. Hence, by putting

$$F := \pi^*(D_{Y'}'),$$

we obtain the identity

$$\text{(5)} \quad \text{vol}(\bigoplus_{k=0}^{\infty} H^0(Y, \mathcal{O}_Y(kD \cdot Y))) = \text{vol}(\bigoplus_{k=0}^{\infty} H^0(\widetilde{Y'}, \mathcal{O}_{\widetilde{Y'}}(kF)))$$

of volumes of the corresponding linear series. Moreover, as above, by replacing $D'$ with a sufficiently big multiple, we may assume that the restriction maps

$$R'_k : H^0(X', \mathcal{O}_{X'}(kD')) \to H^0(Y', \mathcal{O}_{Y'}(kD_{Y'}')),$$

are surjective. Unfortunately, since $Y'$ need not be normal, we do not have isomorphisms between sections of Cartier divisors on $Y'$, $H^0(Y', \mathcal{O}_{Y'}(D_Y'))$.
and sections of the pullback on $Y$, $H^0(Y, \mathcal{O}_Y(f_Y^*(D'_Y)))$. However, as before, we can write a section $s$ of $\mathcal{O}_Y(kD_Y)$ as a product $t \cdot s_E$, where $t \in H^0(Y, \mathcal{O}_Y(f_Y^*(kD'_Y))))$, for a fixed section $s_E$ of the fixed divisorial components of $D_Y$ contained in the exceptional locus $\text{exc}(f_Y)$. On the other hand, the isomorphism of line bundles

$$ \mathcal{O}_Y(f_Y^*(kD'_Y))) \mid_{U} \cong \mathcal{O}_Y(kD_Y) \mid_{U} $$

over the open set $U \subseteq Y$ is given by multiplication by the non-vanishing section $s_E \mid_{U} \in H^0(U, \mathcal{O}_U(kE) \mid_{U})$. Hence, if $\xi \in H^0(X, \mathcal{O}_X(kD))$ is a global section of $\mathcal{O}_X(kD)$, and $\xi' \in H^0(X', \mathcal{O}_{X'}(kD'))$ is the corresponding section under the isomorphism $f^*: H^0(X', \mathcal{O}_{X'}(kD')) \to H^0(X, \mathcal{O}_X(kD))$, the identity

$$ R_k(\xi) = s_E^k \cdot f_Y^*(R'_k(\xi')) $$

of sections of $\mathcal{O}_Y(kD_Y)$ holds since it holds over the open subset $U \subseteq Y$ by the identification of line bundles $\mathcal{O}_Y(f_Y^*(kD'_Y))) \mid_{U} \cong \mathcal{O}_Y(kD_Y) \mid_{U}$. Thus, from this, and the surjectivity of $R'_k$, it follows that the section $s = s_E^k \cdot t \in H^0(Y, \mathcal{O}_Y(kD_Y))$ lies in the image of the map $R_k$ if and only if the factor $t$ is the pull-back by $f_Y^*$ of a section $t' \in H^0(Y', \mathcal{O}_{Y'}(kD'_Y)))$. Thus, we have an isomorphism

$$ f_Y^*(\bigoplus_{k=0}^\infty H^0(Y', \mathcal{O}_{Y'}(kD'_Y))) \cong \text{im} (R), \quad t' \mapsto s_E^k \cdot f_Y^*(t'), $$

of linear subs serie of $\bigoplus_{k=0}^\infty H^0(Y, \mathcal{O}_Y(kD_Y))$, yielding the obvious identities of volumes

$$ \text{vol}(\text{im} (R)) = \text{vol}(f_Y^*(\bigoplus_{k=0}^\infty H^0(Y', \mathcal{O}_{Y'}(kD'_Y)))) $$

$$ = \text{vol}(\bigoplus_{k=0}^\infty H^0(Y', \mathcal{O}_{Y'}(kD'_Y))) $$

of linear series. Finally, by birational invariance of volume ([La04, Prop. 2.2.43]),

$$ \text{vol}(D'_Y) = \text{vol}(F). $$

This equality, together with \[\text{(5)}\], now yields

$$ \text{vol}(D_Y) = \text{vol}(\text{im} (R)). $$

Since $D$ is big, it follows that the Okounkov bodies of the linear series $\text{im} (R)$ and $\bigoplus_{k=0}^\infty H^0(Y, \mathcal{O}_Y(kD_Y))$ have the same volume ([LM09, Theorem 2.13, Lemma 2.16]), and since the former is contained in the latter, they have to coincide;

$$ \Delta_{Y, \mathcal{O}_Y} \left( \text{im} (R) \right) = \Delta_{Y, \mathcal{O}_Y}(D \cdot Y). $$

This proves the claim in case $D'$ is ample.

If $D'$ is merely nef, write $[D']$ as a limit $[D'] = \lim_{i \to \infty} [D'_i]$ of numerical equivalence classes of ample divisors $D'_i, i \in \mathbb{N}$. Put $D_i := f^*D_i$. Then $[D] = \lim_{i \to \infty} [D_i]$. Now, let $a_i \in \Delta_{Y, \mathcal{O}_Y}(D \cdot Y)$. Then $(a_i, [D \cdot Y]) \in \Delta_{Y, \mathcal{O}_Y}(Y)$. Now choose points $a_i \in \Delta_{Y, \mathcal{O}_Y}(Y \cdot D_i)$ so that $(a_i, [D \cdot Y]) = \lim_{i \to \infty} (a_i, [D_i \cdot Y])$.\]
By the above, the identity (7) hold when $D$ is replaced by $D_i$, $i \in \mathbb{N}$. Hence, by [LM09, Theorem 4.26],

\[((0, a_i), [D_i]) \in \Delta_{Y_•}(X)\]

for each $i$, so that

\[((0, a), [D]) = \lim_{i \to \infty} ((0, a_i), [D_i]) \in \Delta_{Y_•}(X),\]

i.e., $(0, a) \in \Delta_{Y_•}(D)$. This shows that the identity (7) holds for an arbitrary nef divisor $D'$ on $X'$.

\[\square\]

In order to apply the above proposition to obtain information on the structure of the global Okounkov body of a Mori dream space, we need the following construction formulated in a more general context. Here $Y_•$ can be any admissible flag on a normal projective variety $X$.

If $s \in H^n(X, \mathcal{O}_X(m_1D_1 + \cdots + m_nD_n))$ is a section which does not vanish on $Y_1$, so that $v(s) = (v_1(s), \ldots, v_n(s))$ with $v_1(s) = 0$, then the restriction of $s$ to $Y_1$ defines a section of the line bundle $\mathcal{O}_{Y_1}(D \cdot Y_1)$ over $Y_1$ with value $\nu^1(s) = (\nu_2(s), \ldots, \nu_n(s))$ with respect to the truncated flag

\[(8)\]

\[Y_n \subseteq \cdots \subseteq Y_1\]

on $Y_1$.

For a finite set $F_1, \ldots, F_r$ of movable divisors on $X$, let

\[\Gamma(F_1, \ldots, F_r) \subseteq \text{Mov}(X)\]

be the semigroup generated by the divisors $F_1, \ldots, F_r$, and let

\[C(F_1, \ldots, F_r) \subseteq \text{Mov}(X)\]

be the cone generated by $F_1, \ldots, F_r$. Define the semigroups

\[S(F_1, \ldots, F_r) := \{(\nu(s), [D]) \in \mathbb{N}_0^n \times \Gamma(F_1, \ldots, F_r) \mid s \in H^n(X, \mathcal{O}_X(D)), D \in \Gamma(F_1, \ldots, F_r), v_1(s) = 0\}\]

and

\[S_1(F_1, \ldots, F_r) := \{((\nu^1(s), [D \cdot Y_1]) \in \mathbb{N}_0^{n-1} \times N^1(Y_1)_{\mathbb{R}} \mid [D] \in \Gamma(F_1, \ldots, F_r), s \in H^n(Y_1, \mathcal{O}_{Y_1}(D \cdot Y_1))\},\]

as well as the morphism of semigroups

\[q_0 : S \to S_1, \quad q_0(\nu(s), [D]) := (\nu^1(s), [D \cdot Y_1]),\]

which extends to the linear map

\[(9)\]

\[q : \mathbb{R}^n \oplus V(F_1, \ldots, F_r) \to \mathbb{R}^{n-1} \oplus N^1(Y_1)_{\mathbb{R}},\]

\[(x_1, \ldots, x_n), [D]) \mapsto ((x_2, \ldots, x_n), [D \cdot Y_1]),\]

where $V(F_1, \ldots, F_r) \subseteq N^1(X)_{\mathbb{R}}$ is the $\mathbb{R}$-vector space generated by the numerical equivalence classes $[F_1], \ldots, [F_r]$. Furthermore, we denote by $C(S(F_1, \ldots, F_r))$ and $C(S_1(F_1, \ldots, F_r))$ the closed convex cones in $\mathbb{R}^n \times N^1(X)_{\mathbb{R}}$ and $\mathbb{R}^{n-1} \times N^1(Y_1)_{\mathbb{R}}$ spanned by the semigroups $S(F_1, \ldots, F_r)$ and $S_1(F_1, \ldots, F_r)$, respectively.

We now recall that for a Mori dream space $X$ the pseudo-effective cone $\overline{\text{Eff}}(X)$ is the union of finitely many Mori chambers, $\Sigma_1, \ldots, \Sigma_m$, where
each Mori chamber $\Sigma_j$ is the convex hull of finitely many integral divisors $D^j_1, \ldots, D^j_{\ell_j}$. More concretely, by [HK00 Proposition 1.1], the chambers are in correspondence to contracting birational maps $f_j : X \to X_j$ with image a Mori dream space, and are given as the convex cone spanned by $f_j^* (\text{Nef}(X_j))$ together with the rays spanning the exceptional locus $\text{exc}(f_j)$. The corresponding decomposition of a divisor $D \in \Sigma_j$ is exactly its decomposition into its fixed and movable parts. We can thus reorder the divisors spanning each chamber in such a way that the first $n_j$ of them are movable and the remaining ones are fixed. Let $\sigma^j_i \in H^0(X, \mathcal{O}_X(N^j_i))$ be the defining section of $D^j_i$, for $j = 1, \ldots, m$, $i = n_j + 1, \ldots, \ell_j$.

**Corollary 4.4.** Let $Y_\bullet$ be an admissible flag on a Mori dream space $X$ such that the conditions of Proposition 4.2 are satisfied, and let $D_1, \ldots, D_r$ be the movable generators of a Mori chamber $\Sigma$. Then we have the identity

\begin{equation}
C(S(D_1, \ldots, D_r))
= q^{-1}(C(S_1(D_1, \ldots, D_r)) \cap \Delta_{Y_\bullet}(X) \cap \{0\} \times \mathbb{R}^{n-1}_{\geq 0} \times C(D_1, \ldots, D_r)).
\end{equation}

**Proof.** This follows from the previous proposition together with the fact that there exists a SQM $\pi : X \to X'$ such that each divisor in the cone $C(D_1, \ldots, D_r)$ is a pullback by $\pi$ of a nef divisor on $X'$ ([HK00 Proposition 1.11(3)]).

We can now prove the following theorem which will—together with identity (10)—enable us to inductively infer information on the shape of global Okounkov bodies of certain Mori dream spaces.

**Theorem 4.5.** Suppose in the above situation that each of the cones $C(S(D^j_1, \ldots, D^j_{\ell_j}))$ is rational polyhedral with generators given by vectors $w^j_1, \ldots, w^j_{\ell_j}$. Then the global Okounkov body $\Delta_{Y_\bullet}(X)$ is the cone generated by the vectors

\begin{equation}
(\nu(s_{Y_1}, [Y_1]), (\nu(\sigma^j_i), [D^j_i]), w^j_h,
\end{equation}

for $j = 1, \ldots, m$, $i = n_j + 1, \ldots, \ell_j$, $h = 1, \ldots, r_j$.

**Proof.** Let $E$ be an effective integral divisor on $X$, and let $s \in H^0(X, \mathcal{O}_X(E))$ be a nonzero section of $E$. Let $\nu_1(s) = a$. Then, $\zeta := s/s_{Y_1}$, where $s_{Y_1} \in H^0(X, \mathcal{O}_X(Y_1))$ is the defining section of $Y_1$, is a section of $\mathcal{O}_X(E - aY_1)$ which vanishes to order 0 along $Y_1$. Now, let $\Sigma = \text{conv}\{D_1, \ldots, D_{\ell}\}$ be a Mori chamber such that $E - aY_1 \in \Sigma$, and with generators ordered so that $D_1, \ldots, D_r$ are the movable generators. Let $E - aY_1 = P + N$ be the corresponding decomposition of $E - aY_1$ into its movable part $P$ and fixed part $N$. Choose $M \in \mathbb{N}$ large enough such that all the divisors $MP = c_1D_1 + \cdots + c_{\ell}D_{\ell}$, $MN = c_{r+1}D_{r+1} + \cdots + c_{\ell}D_{\ell}$, where $c_1, \ldots, c_{\ell} \in \mathbb{N}_0$, and all $c_iD_i$ are integral divisors. Let $\sigma_i \in H^0(X, \mathcal{O}_X(D_i))$ be the defining section of $N_i$, $i = r + 1, \ldots, \ell$. The section $\zeta^M \in H^0(X, \mathcal{O}_X(m(E - aY_1)))$ now decomposes uniquely as a product

$$\zeta^M = \eta \sigma,$$
where $\eta \in H^0(X, \mathcal{O}_X(c_1 D_1 + \cdots + c_r D_r))$, and $\sigma = \sigma^{c_{r+1}} \cdots \sigma^{c_t}$. Since $\nu_1(\zeta) = 0$, we also have $\nu_1(\eta) = 0$. Now, by assumption we have integral generators $w_1, \ldots, w_k \in \mathbb{R}^n \times \mathcal{E}ff(X)$ for the cone $C(S(D_1, \ldots, D_r))$, so that $(\nu(s), MP) = s_1 w_1 + \cdots + s_k w_k$, for some $s_1, \ldots, s_k \geq 0$. Hence,

$$
(\nu(s), E) = a(\nu(s_1), Y_1) + \frac{c_{r+1}}{M}(\nu(\sigma_{r+1}), [D_{r+1}]) + \cdots + \frac{c_t}{M}(\nu(\sigma_t), [D_t])
$$

$$
+ \frac{s_1}{M} w_1 + \cdots + \frac{s_k}{M} w_k.
$$

It follows that $\Delta Y_1(X)$ lies in the closed convex cone generated by the vectors $\mathbf{11}$. Since all these vectors clearly belong to $\Delta Y_1(X)$, this finishes the proof.

We are now in the position to prove the main result of this paper. Let us first define what we mean by a good flag on a Mori dream space.

**Definition 4.6.** Let $X$ be a Mori dream space. An admissible flag $Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_n$ is good, if

1. $Y_i$ is a Mori dream space for each $0 \leq i \leq n$, and
2. the restriction maps $p_i : N^1(Y_i)_{\mathbb{R}} \rightarrow N^1(Y_{i+1})_{\mathbb{R}}$ induce surjective maps $\mathcal{E}ff(Y_i) \cap p_i^{-1}(\mathcal{E}ff(Y_{i+1})) \rightarrow \mathcal{E}ff(Y_{i+1})$.

Our main theorem now follows from the above results.

**Theorem 4.7.** If there exists a good flag $Y_*$ on a Mori dream space $X$, then $\Delta Y_*(X)$ is rational polyhedral.

**Proof.** We prove the theorem by induction over $n$. Every Mori dream curve is isomorphic to $\mathbb{P}^1$, which for any choice of flag (i.e., choice of a point) has rational polyhedral global Okounkov body, namely the cone in $\mathbb{R}^2$ spanned by the points $(0, 1)$ and $(1, 1)$.

For the inductive step assume that $\Delta Y_*(Y_1)$ is rational polyhedral. By Theorem 4.5 what we need to prove is that for any Mori chamber $\Sigma$ in $\mathcal{E}ff(X)$ the set of movable generators $D_1, \ldots, D_r$ of $\Sigma$ yield a rational polyhedral cone $C(S(D_1, \ldots, D_r))$.

Since $Y_*$ is a good flag, the assumptions of Proposition 4.2 are satisfied for $Y_1 \subset X$, so we can apply Corollary 4.4. Since the linear map $q$ (cf. 9) is defined over $\mathbb{Z}$, equality 10 implies that $C(S(D_1, \ldots, D_r))$ is rational polyhedral if $C(S(D_1, \ldots, D_r))$ is. Now the rational polyhedrality of $C(S(D_1, \ldots, D_r))$ follows from the rational polyhedrality of $\Delta Y_*(Y_1)$ since

$$
C(S(D_1, \ldots, D_r)) = pr_2^{-1}(\Gamma(D_1 \cdot Y_1, \ldots, D_r \cdot Y_1)) \cap \Delta Y_*(Y_1),
$$

where $\Gamma(D_1 \cdot Y_1, \ldots, D_r \cdot Y_1) \subset \mathcal{E}ff(Y_1)$ is the convex cone generated by the numerical equivalence classes of the divisors $D_1 \cdot Y_1, \ldots, D_r \cdot Y_1$ on $Y_1$.

**Remark 4.8.** It should be noted that the above result does not hold for general admissible flags of subvarieties of a Mori dream space. Indeed, [KLM12] Example 3.4] shows that $X := \mathbb{P}^2 \times \mathbb{P}^2$ can be equipped with an admissible flag $Y_*$ such that the Okounkov body $\Delta Y_*(D)$, where $D$ is a divisor in the linear series $\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(3, 1)$, is not polyhedral. The flag $Y_*$ here is of course not a good flag; it is not even a flag of Mori dream spaces: if $Y_1 := \mathbb{P}^2 \times E$, [
where $E \subseteq \mathbb{P}^2$ is a general elliptic curve, were a Mori dream space, then its image $E$ under the second projection would also be a Mori dream space by [Oka11, Theorem 1.1]. However, $\mathbb{P}^1$ is the only Mori dream curve (cf. [C12, p. 6]).

5. **Okounkov bodies on Bott-Samelson varieties**

In this section we apply the general results from the previous section to Bott-Samelson varieties.

5.1. **Global Okounkov bodies.** On a Bott-Samelson variety $Z_w$ defined by a reduced sequence $w$ let $Y_\bullet$ be the natural admissible flag of subvarieties of $Z_w$ defined by

$$Y_k := \bigcap_{j=n-k+1}^{n-1} Z_w(j), \quad k = 0, \ldots, n$$

where we put $Z_w(n+1) := Z_w$.

**Theorem 5.1.** Let $Z_w$ be a Bott-Samelson variety defined by a reduced sequence $w$. Then the global Okounkov body $\Delta_{Y_\bullet}$ is rational polyhedral.

**Proof.** By Theorem 3.1, the variety $Z_w$ is a Mori dream space. We prove that $Y_\bullet$ is a good flag, from which the assertion follows by Theorem 4.7.

Consider the fibration $\pi_w[1] : Z_w \to Z_w[1]$. Since the sequence $w(n)$ is also reduced, the Bott-Samelson variety $Y_1 = Z_w(n) \cong Z_w[1]$ is a Mori dream space. Repeating this argument shows that all the $Y_i$ are also Mori dream spaces.

On the other hand, the fact that the restricted divisors (2) form the effective basis for $\text{Pic}(Z_w[1])$ shows that $Z_w$ and the divisor $Y_1$ satisfy the assumptions of Proposition 4.2. The same argument applies when the pair $(Z_w,Y_1)$ is replaced by the pair $(Y_k,Y_{k+1})$, for $k = 1, \ldots, n-1$, and hence the flag $Y_\bullet$ is a good flag. This finishes the proof. □

We can now also show that the Okounkov bodies of effective line bundles over Schubert varieties, with respect to a natural valuation-like function, are rational polyhedral. Indeed, Schubert varieties have rational singularities, so that the projection morphism $p_w : Z_w \to Y_\bullet$ satisfies the property $(p_w)_* \mathcal{O}_{Z_w} = \mathcal{O}_{Y_\bullet}$ (cf. [B, Section 2.2]). Hence, for any effective line bundle $L$ on $Y_\bullet$, we have

$$H^0(Y_\bullet, L) \cong H^0(Z_w, p_w^* L).$$

(12)

Let now

$$\nu : \text{Cox}(Z_w)_h \setminus \{0\} \to \mathbb{N}_0^n,$$

where $\text{Cox}(Z_w)_h$ denotes the set of homogeneous elements in the Cox ring of $Z_w$ with respect to the effective basis, be the valuation-like function defined by the flag $Y_\bullet$, and let

$$\nu_L : \bigcup_{k \geq 0} H^0(Y_\bullet, L^k) \setminus \{0\} \to \mathbb{N}_0^n$$

be the valuation-like function naturally defined by the isomorphisms (12) (for all powers $L^k$) and restriction of $\nu$. Then, the Okounkov body $\Delta_{\nu_L}(L)$
Corollary 5.2. Let \( L \) be an effective line bundle over the Schubert variety \( Y_{\mathfrak{w}} \) of \( G/B \). Then, the Okounkov body \( \Delta_{Y_{\mathfrak{w}}}(L) \) defined by the natural valuation-like function \( v \) defined by the flag \( Y_{\mathfrak{w}} \) in \( Z_{\mathfrak{w}} \) is a rational polytope.

If \( Y_{\mathfrak{w}} = G/B \) is a flag variety, the Picard group \( \text{Pic}(G/B) \) has an effective basis, namely the line bundles \( L_i = G \times_{\omega_i} C \) defined by the fundamental weights \( \omega_i, i = 1, \ldots, r \), with respect to a choice of simple roots for the root system of \( G \). Let \( \Sigma \subseteq \text{Eff}(Z_{\mathfrak{w}}) \) be the closed convex cone generated by the divisors of the line bundles \( p^*_w L_i, i = 1, \ldots, r \). By the isomorphisms \( \{12\} \) we now have

\[
\Delta_{Y_{\mathfrak{w}}}(G/B) \cong p_w^{-1}(\Sigma) \cap \Delta_{Y_{\mathfrak{w}}}(Z_{\mathfrak{w}}).
\]

Since the cone \( \Sigma \) is finitely generated, the cone on the right hand side is rational polyhedral, so that we have proved the following corollary.

Corollary 5.3. The global Okounkov body \( \Delta_{Y_{\mathfrak{w}}}(G/B) \) of the flag variety \( G/B \), with respect to the valuation defined by the flag \( Y_{\mathfrak{w}} \) of subvarieties of \( Z_{\mathfrak{w}} \), is a rational polyhedral cone.

5.2. Weight multiplicities. We now turn our attention to the action of a torus \( H \subseteq B \), contained in a maximal torus of \( G \) lying in \( B \), on the section ring \( R(D) := \bigoplus_{k \geq 0} H^0(Z_{\mathfrak{w}}, \mathcal{O}_{Z_{\mathfrak{w}}}(kD)) \) of an effective divisor \( D \) on \( Z_{\mathfrak{w}} \). Recall that each section space \( H^0(Z_{\mathfrak{w}}, \mathcal{O}_{Z_{\mathfrak{w}}}(kD)) \) carries a representation of \( B \) given by the action of \( B \) as automorphisms of the line bundle \( \mathcal{O}_{Z_{\mathfrak{w}}}(kD) \) (cf. \( \{LT04\} \)). Moreover, the flag \( Y_{\mathfrak{w}} \) consists of \( B \)-invariant subvarieties of \( Z_{\mathfrak{w}} \), so that the valuation-like function

\[
\nu_D : \bigcup_{k \geq 0} H^0(Z_{\mathfrak{w}}, \mathcal{O}_{Z_{\mathfrak{w}}}(kD)) \setminus \{0\} \to \mathbb{N}_0\]

is \( B \)-invariant, i.e., the identity \( \nu(b,s) = \nu(s) \) holds for any non-zero section \( s \in H^0(Z_{\mathfrak{w}}, \mathcal{O}_{Z_{\mathfrak{w}}}(kD)) \), and \( b \in B \). Hence, there is a well-defined projection

\[
q : \Delta_{Y_{\mathfrak{w}}}(D) \to \Pi_D
\]

onto the weight polytope (cf. \( \{H86\} \)) of the section ring \( R(D) \) for the action of the torus \( H \) (cf. \( \{O10\} \) \( \{K10\} \)). If \( \mathfrak{h} = \text{Lie}(H) \) is the Lie algebra of \( H \), and the \( \mu \in \Pi_D \subseteq \mathfrak{h} \) is a rational point in the interior of the weight polytope, we then have that the asymptotics of the weight spaces \( W_{k\mu} \subseteq H^0(Z_{\mathfrak{w}}, \mathcal{O}_{Z_{\mathfrak{w}}}(kD)) \) are given by

\[
\lim_{k \to \infty} \frac{\dim W_{k\mu}}{k^{d-r}} = \text{vol}_{d-r}(q^{-1}(\mu) \cap \Delta_{Y_{\mathfrak{w}}}(D)),
\]

where \( r \) is the dimension of the moment polytope \( \Pi_D \), \( d \) is the dimension of the Okounkov body \( \Delta_{Y_{\mathfrak{w}}}(D) \) (and which equals the Iitaka dimension of the line bundle \( \mathcal{O}_{Z_{\mathfrak{w}}}(D) \)), and the right hand side denotes the \((d-r)\)-dimensional Lebesgue measure of the slice \( q^{-1}(\lambda) \cap \Delta_{Y_{\mathfrak{w}}}(D) \) of the Okounkov body \( \Delta_{Y_{\mathfrak{w}}}(D) \). We thus get the following result, saying the the asymptotics of weight spaces are given by polyhedral expressions.
Corollary 5.4. For any effective divisor $D$ on $Z_w$, and rational point $\mu \in \Pi_D$ in the interior of $\Pi_D$, the asymptotic multiplicity

$$\lim_{k \to \infty} \frac{\dim W_{k\mu}}{k^{d-r}}$$

is the volume of a rational polytope. As a consequence, the same holds for the weight spaces

$$W_{k\mu} \subseteq H^0(Y_w, L^k)$$

for an effective line bundle $L$ over a Schubert variety $Y_w$.

Proof. We only need to prove the second claim about Schubert varieties. Here we notice that the projection morphism $p_w : Z_w \to Y_w$ is $B$-equivariant, and so in particular $H$-invariant. Hence, the isomorphisms (12) for the powers $L^k$ are $H$-equivariant, so that the claim thus follows from the first part about Bott-Samelson varieties. $\square$

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