Sources of symmetric potentials

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Introduction and main result

Laboratory experiments on gravitation are usually performed with objects of constant density, so that the analysis of the forces concerns only the geometry of their shape. In an ideal experiment, the shapes of the constituent parts will be optimised to meet certain mathematical criteria, which ensure that the experiment has maximal sensitivity.

Using this idea, the author suggested an experiment to determine the departure of the gravitational force from Newton’s force law [1]. The geometrical problem which has to be solved is to find two shapes which differ significantly, but have the same Newtonian potential. Essentially, the experiment determines whether the two objects are distinguishable by their gravitational force. Here, we consider the case when one of them is a round ball. The result, Theorem 1, establishes a fact which appeared in numerical simulations, that the second object has to have a hole in it.

Consider the Laplace equation in $\mathbb{R}^n$, $n \geq 2$, and let the surface area of a unit sphere in $\mathbb{R}^n$ be $\omega_n$. The fundamental solution

$$ K(x, \xi) = \begin{cases} \frac{|x - \xi|^{2-n}}{(2-n)\omega_n} & \text{for } n > 2, \\ \frac{\log |x - \xi|}{2\pi} & \text{for } n = 2 \end{cases} $$

(1)

can be regarded as the potential due to a point source. Let $\Omega$ be a connected open bounded domain of $\mathbb{R}^n$, with a smooth boundary. Define the Newtonian potential of $\Omega$ to be

$$ \phi(\xi) = \int_\Omega K(x, \xi) \, dx $$

(2)
As is well known this defines a $C^1$ function of $\mathbb{R}^n$ which is $C^\infty$ on $\mathbb{R}^n \setminus \partial \Omega$ and satisfies

$$
\triangle \phi = \begin{cases} 
1 & \text{on } \Omega \\
0 & \text{on } \mathbb{R}^n \setminus \Omega 
\end{cases}
$$

(3)

If $\Omega$ is a solid round ball, or a set of concentric shells, then on the outside of the set $\Omega$, i.e., on the component $\Sigma$ of $\mathbb{R}^n \setminus \Omega$ which is connected to infinity, $\phi(\xi)$ is proportional to $K(x, \xi)$, with $x$ the centre of symmetry.

Which other sets $\Omega$ share this property? In [1], a picture of an approximate solution appears, which does not possess spherical symmetry. I coined the term ‘monopole’ for such sets, on account of the vanishing of all other multipole moments. The term ‘centrobaric’ has also been used to denote a more general case of such solutions to Poisson’s equation, where a variable positive density appears in (3)[2]. The main result presented here is that the hole which is to be seen in the interior of the example of [1] is a necessary feature of all asymmetric solutions, in the following manner.

**Theorem 1** Let $\Omega$ be a connected open bounded domain of $\mathbb{R}^n$, $n \geq 2$, with a smooth boundary, such that $\Sigma = \mathbb{R}^n \setminus \Omega$ is connected. Let $\phi$ be its Newtonian potential, and suppose that $\phi$ has the exact form of a point mass on $\Sigma$, i.e.,

$$
\phi(\xi) = M K(0, \xi)
$$

(4)

for a constant $M$. Then $\Omega$ is a solid round ball, $|x| < \text{const.}$

The proof proceeds in two parts, first establishing a bound for the interior potential, then by applying a rotational version of a method due to Serrin [3]. Serrin’s problem was to consider $\triangle \phi = 1$ on a bounded domain with $\phi = 0$ and $\partial \phi / \partial n = \text{const.}$ on the boundary, and ask for the possible shapes for the domain. Serrin investigated the solutions to this free boundary problem by exploiting the invariance of Poisson’s equation, the source term and the boundary condition by reflection in a plane. The plane was adjusted so that a reflected portion of the boundary touched the boundary itself. In our case, there are not enough reflection planes which respect the form (1) of the potential on the outside of $\Omega$. The appropriate symmetries to use are rotations and homotheties (linear scalings) about the origin. Thus one could shrink and rotate the figure so that its boundary touched the original boundary at a point. Lemma 3 is a slightly modified version of this idea, where the second figure is replaced by a ball.
Details
Let us assume that \( \Omega, \phi, \Sigma \) are defined as in Theorem 1. We are interested in comparing the interior potential with the point mass potential (4).

**Lemma 2** Let \( \phi \) obey equation (4) everywhere on \( \Sigma \), which, for the purposes of this lemma need not be connected. Then it follows that

\[
\phi(\xi) > MK(0, \xi)
\]

for \( \xi \in \Omega \setminus \{0\} \), and that \( \Omega \) is star-shaped about 0.

**Proof:** Note first that (4) implies that 0 \( \in \Omega \). Consider the function

\[
\psi(\xi) = \phi(\xi) - MK(0, \xi)
\]

on \( \mathbb{R}^n \setminus 0 \). Then \( \psi = 0 \) and \( \nabla \psi = 0 \) on \( \partial \Omega \). Let \( r = |\xi| \) denote the radial coordinate in spherical coordinates. The function \( r\partial \psi / \partial r \) defined on the set \( \overline{\Omega} \setminus \{0\} \) has some interesting properties; considering its behaviour is a mathematical counterpart to decomposing the source \( \Omega \) into concentric shells. One has that \( r\partial \psi / \partial r \in C^0(\overline{\Omega} \setminus \{0\}) \cap C^2(\Omega \setminus \{0\}) \), \( \triangle (r\partial \psi / \partial r) = 2 \) on \( \Omega \setminus \{0\} \), \( r\partial \psi / \partial r = 0 \) on \( \partial \Omega \), and

\[
r \frac{\partial \psi}{\partial r} = r \left( \frac{\partial \phi}{\partial r} - \frac{M}{\omega_n} r^{1-n} \right) < 0
\]

for \( r \) sufficiently close to 0, \( 0 < r < \varepsilon \), say. Thus by applying the maximum principle to \( \Omega \) with the ball of radius \( \varepsilon \) removed, the inequality (7) extends to the whole of \( \Omega \setminus \{0\} \).

Now consider the values of \( \psi \) along a straight line segment joining \( p \in \Omega \) to 0. The inequality (7) implies that the whole of the segment must lie in \( \Omega \) and that \( \psi > 0 \) in \( \Omega \).

**Lemma 3** Let \( \phi \) obey (4) on \( \Sigma = \mathbb{R}^n \setminus \Omega \), for an \( \Omega \) as described in Theorem 1. If, in addition, (5) is obeyed on \( \Omega \setminus \{0\} \), then the boundary, \( \partial \Omega \), is a sphere, \( |x| = \text{const.} \)

**Proof:** Suppose \( \partial \Omega \) is not a sphere, so that the maximum and minimum distance of its points from the origin satisfy \( r_{\text{max}} > r_{\text{min}} > 0 \). Let \( m \in \partial \Omega \) be a point at which a minimum distance is attained. The open ball \( B \) of radius \( r_{\text{min}} \) centred at 0 has boundary \( \partial B \) which touches \( \partial \Omega \) at \( m \). Denote its Newtonian potential by \( \phi_B \), and set

\[
u(\xi) = \frac{M}{\text{vol}(B)} \phi_B(\xi) - \phi(\xi).
\]

The scaling has been chosen so that \( u = 0 \) on \( \Sigma \). Now, \( B \subset \Omega \) and \( \text{vol}(B) < M = \text{vol}(\Omega) \). Thus \( \triangle u = M/\text{vol}(B) - 1 > 0 \) on \( B \), \( u = 0 \) on \( \partial \Omega \cap \partial B \), and \( u < 0 \) on \( \Omega \cap \partial B \), by virtue of (5). Using the maximum principle, \( u < 0 \) on \( B \). As is common to Serrin’s method, there is a point \( m \) on the boundary at which \( u = \partial u / \partial r = 0 \). The Hopf lemma (see [4]) applied to a small ball \( C \subset B \), \( m \in \overline{C} \), yields a contradiction, unless \( u \equiv 0 \), in which case \( \partial \Omega \) and \( \partial B \) coincide. This contradicts the supposition \( r_{\text{max}} > r_{\text{min}} \), and so the boundary is a sphere.

**Proof of Theorem 1** This follows from lemmas 2 and 3.

Asymmetric monopoles with holes (i.e., \( \Sigma \) not connected) escape Theorem 1 because (4) is not obeyed in the interior holes. In fact, the conclusion of Lemma 2 means that it is fruitless to enforce (4) on an interior hole, because then \( \Omega \) cannot have any holes.
References

[1] J.W. Barrett: The Asymmetric Monopole and non-Newtonian Forces. Nature 341, 131-132 (1989)

[2] W.D. MacMillan: Theory of the Potential. New York, 1930

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[4] B. Gidas, Wei-Ming Ni and L. Nirenberg: Symmetry and Related Properties via the Maximum Principle. Comm. Math. Phys. 68, 209-243 (1979)