A strategy to compute the b-quark mass with non-perturbative accuracy

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We describe a strategy for a non-perturbative computation of the b-quark mass to leading order in $1/m$ in the Heavy Quark Effective Theory (HQET). The approach avoids the perturbative subtraction of power law divergencies, and the continuum limit may be taken. First numerical results in the quenched approximation demonstrate the potential of the method with a preliminary result $m_{b}^{\text{MS}}(4\text{ GeV}) = 4.56(2)(7)$ GeV. In principle, the idea may also be applied to the matching of composite operators or the computation of $1/m$ corrections in HQET.

1. THE PROBLEM

Since the b-quark is heavier than presently achievable inverse lattice spacings, $1/a$, effective theories (HQET, NRQCD, ...) are used for its numerical treatment. These theories are based on some large mass expansion and hence do not have a chiral symmetry to protect the quark mass from additive renormalization. The relation

\[ Z(m_{b}^{\text{bare}} + \delta m) = m_{b} \] (1)

between the bare quark mass $m_{b}^{\text{bare}}$ and a renormalized b-quark mass, $m_{b}$, therefore contains an additive term, $\delta m$. For dimensional reasons it is linearly divergent,

\[ \delta m = \frac{1}{a} f(g_{0}) = \frac{1}{a} (f_{1}g_{0}^{2} + f_{2}g_{0}^{4} + \ldots) \] (2)

\[ g_{0} \to 0 \quad \Lambda_{\text{QCD}} e^{1/(2b_{0}g_{0}^{2})} (f_{1}g_{0}^{2} + \ldots). \] (3)

A perturbative approximation to $f(g_{0})$ results in a truncation error in eq. (1) which diverges in the $g_{0} \to 0$ limit. The continuum limit cannot be taken. Available results for the b-quark mass in the static approximation \[ m_{b}^{\text{bare}} \] are limited by this fact and consequently are obtained at lattice spacing $a \approx 0.1 \text{ fm}$.

While the numerics may be improved by the computation of higher order $f_{i}$, a non-perturbative strategy to compute or avoid $\delta m$ is needed to solve the problem.

2. STRATEGY

We treat the b-quark field in static approximation with action $S_{h} = \sum \psi_{h} \nabla_{0}^{*} \psi_{h}$ (note that $\delta m$ is not included in the action and consult for any unexplained notation) and consider a timeslice correlation function $C_{\text{stat}}(x_{0})$ of some interpolating field for the B-meson. Its large time decay $C_{\text{stat}}(x_{0}) \sim B \exp(-E x_{0})$ defines the static bare binding energy $E$, related to the B-meson mass, $m_{B}$, via

\[ m_{B} = E + m_{b}^{\text{bare}} + O(1/m_{b}). \] (4)

2.1. Matching

In order to replace $m_{b}^{\text{bare}}$ by the renormalized quark mass we consider another such relation, which we may take to be the condition matching the QCD quark mass to the one of HQET. It reads

\[ \Gamma_{\text{rel}}(L, M, g_{0}) = \Gamma_{\text{stat}}(L, g_{0}) + m_{b}^{\text{bare}}(M, g_{0}) + O(1/M) \] (5)

with

\[ \Gamma_{\text{stat}}(L, g_{0}) = -\frac{1}{2}(\partial_{0} + \partial_{0}^{*}) \ln[C_{\text{stat}}(x_{0})]_{x_{0}=L} + \frac{1}{2} \] (6)

and $\Gamma_{\text{rel}}(L, M, g_{0})$ defined in the same way but for a relativistic quark with renormalization group invariant (RGI) quark mass, $M$. Its relation to the bare PCAC mass and the bare mass in the Lagrangian of the $O(a)$-improved theory is known non-perturbatively. At this stage $L$ may be any finite length scale.
In principle the function $m^\text{bare}_b(M, g_0)$ can be obtained by evaluating eq. (8) for some range of $M$. It may then be inserted into eq. (9) and solved for $M$ to obtain the RGI b-quark mass $M_b$.

In practice, not much has been achieved yet because an implementation of eq. (5) is not possible: the small lattice spacings necessary for the relativistic theory are not available.

### 2.2. The use of finite volume

The important idea is to consider $C(x_0)$ to be a correlation function in finite volume of linear extent $L$, which may assume several values $L = L_i$. We then have (suppressing the argument $g_0$)

$$m_B = E - \Gamma_{\text{stat}}(L_0) + \Gamma_{\text{rel}}(L_0, M_b)$$

$$= \Delta E_a + \Delta E_b + \Gamma_{\text{rel}}(L_0, M_b),$$

$$\Delta E_a = E - \Gamma_{\text{stat}}(L_n),$$

$$\Delta E_b = \Gamma_{\text{stat}}(L_n) - \Gamma_{\text{stat}}(L_0).$$

In both energy differences $\Delta E_i$, defined in the static theory, the unknown $m^\text{bare}_b$ (and thereby $\delta m$, too) cancels, and the continuum limit of

$$\Omega_a = -L_0\Delta E_a, \quad \Omega_b = -L_0\Delta E_b$$

exists. Choosing furthermore $L_0$ such that $L_0M_b \gg 1$ and $M_b a \ll 1$ can be achieved at the same time, the function

$$\Omega_{\text{rel}}(z) \equiv L_0\Gamma_{\text{rel}}(L_0, M), \quad z = L_0M,$$

may be evaluated (in the relativistic theory) and extrapolated to the continuum limit. Then $M_b$ can be determined by solving

$$\Omega_{\text{rel}}(L_0M_b) = \Omega_a + \Omega_b + \Omega_c,$$

with the experimental input (spin averaged mass)

$$\Omega_c = L_0m_{B_s} = L_0 \times 5405 \text{ MeV}.$$}

On the one hand, to treat the relativistic b-quark, $L_0$ may not be too large and on the other hand, to be able to compute $\Delta E_a$, $L_n$ may not be too small. To bridge this gap, a step scaling function

$$\sigma_T(u) = \lim_{a/L \to 0} \Sigma_T(u, a/L),$$

$$\Sigma_T(u, a/L) = 2L \left[ \Gamma_{\text{stat}}(2L) - \Gamma_{\text{stat}}(L) \right]_{u = \bar{g}^2(L)}$$

depending on a coupling, $\bar{g}^2(L)$, renormalized at scale $L$, is introduced in the spirit of [4]. For $L_i = 2^iL_0, u_i = \bar{g}^2(L_i)$, one then has

$$\Omega_b = -\sum_{i=0}^{n-1} 2^{-i-1} \sigma_T(u_i).$$

We now demonstrate the accessibility of the continuum limit with preliminary

### 3. RESULTS

in the quenched approximation. We choose Schrödinger functional (SF) b.c.’s and specifically

$$C(x_0) = f_\Lambda(x_0) = \frac{1}{\beta},$$

$\theta = 0.5, T = L$ and vanishing background field; see [3] for the definition of $f_\Lambda$ in the static approximation and with a relativistic b-quark. $\bar{g}^2$ is taken to be the standard SF coupling [3]; we set $L_0 = 1.436r_0/4 \approx 0.2 \text{ fm}$ and the light quark mass to zero, except in the large volume B-meson correlation function where, in accordance with eq. (14), it is set to the strange quark mass [7]. All quantities are $O(a)$-improved.

After continuum extrapolation of $\Sigma_T$ and a slight interpolation of the function $\sigma(u)$ (see Fig. 3) we estimate ($n = 2$)

$$\Omega_b = 0.10(1).$$

The binding energy $E$ is computed in a $(1.5 \text{ fm})^3 \times 2.3 \text{ fm}$ box with SF b.c.’s and a new technique to suppress excited states [8]. From the plateau in Fig. 2 we obtain

$$\Omega_a = -0.369(6) \text{ at } \beta \equiv 6/g_0^2 = 6.0.$$
Finally, the missing piece of the puzzle, $\Omega_{\text{rel}}(z)$, is plotted for various $L/a$- and $z$-values in Fig. 3, where also the solution of eq. (13) is illustrated. Setting $r_0 = 0.5$ fm, we end up with

$$M_b = 7.01(3)(10) \text{ GeV,}$$

$$m_b^{\text{MS}}(4 \text{ GeV}) = 4.56(2)(7) \text{ GeV,}$$

where the second, presently dominating uncertainty originates from the renormalization factors and improvement coefficients needed to fix $M$ in the relativistic theory.

4. COMMENTS, GENERALIZATIONS

Our computation of $M_b$ is valid up to (relative) $1/M_b$ corrections. Generically these are of the form $\mu/M_b$, with $\mu \approx 0.5 \text{ GeV}$, a typical QCD scale. However, also explicitly introduced external scales may enter. In our strategy only the scale $L_0$ where the matching is performed appears. We therefore do not choose $L_0$ much smaller but keep $1/L_0 \approx 1 \text{ GeV}$.

Since $\Omega_a$, eq. (19), has only been evaluated at one value of $a$, our present estimate does not yet reflect a final result in the full continuum limit. Nevertheless, it is already useful to compare this with earlier results which were obtained at similar lattice spacings but perturbative $\delta m$. Within the present errors we agree with them [1]. Our results still need a more detailed analysis of discretization effects also in $\Omega_b, \Omega_{\text{rel}}$. The uncertainties of improvement coefficients and renormalization constants in the range of $g_0$ relevant for Fig. 3 are presently contributing the dominant 70 MeV error in eq. (21). It can and should be reduced.

A relevant issue not mentioned in the literature of lattice determinations of $m_b$ is the effect of the scale ambiguity in the quenched approximation (see e.g. sect. 6 in [6]). We found that a 10% ambiguity in the scale roughly corresponds to a 100 MeV error in the quark mass.

A theoretically interesting point is that in our strategy power law divergences in the effective theory are removed by a non-perturbative matching to relativistic QCD in finite volume. The same strategy may e.g. be applied to the renormalization of the static axial current and to $1/M_b$ corrections in HQET. Numerical experiments are needed to test the practicability of such generalizations, which open up exciting possibilities for applications of HQET without perturbative (in the QCD-coupling) truncation errors.

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