Pair creation rate for $U(1)^2$ black holes

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Abstract

I consider a truncation of low-energy string theory which contains two $U(1)$ gauge fields. After making some general comments on the theory, I describe a previously-obtained instanton for the pair creation of black holes when both gauge fields are non-zero, and obtain the pair creation rate by calculating its action. This calculation agrees qualitatively with the earlier calculation of the pair creation rate for black holes in Einstein-Maxwell theory. That is, the pair creation is strongly suppressed in realizable circumstances, and it reduces to the Schwinger result in the point-particle limit. The pair creation of non-extreme black holes is enhanced over that of extreme black holes by $e^{A_{bh}/4}$. 
1 Introduction

The study of black hole pair creation is of considerable interest for the exploration of quantum gravity. Like black hole evaporation, it represents a truly quantum gravitational process, being classically completely forbidden. At the same time, it is easy to achieve a physical understanding of what is happening; there is a strong analogy with the creation of particle-antiparticle pairs in quantum field theory, as can be seen from the fact that the black hole pair creation rate reduces to the particle-antiparticle rate in the limit of small black holes. If we trust this analogy to particle physics, the pair creation rate should depend on the number of accessible states for the black hole, so we can find out how many states the black holes should have by studying pair creation (although we can’t find out what those states are). Black hole pair creation involves topology change, and this suggests that including the effects of topology change will be important to a proper understanding of quantum gravity.

Because of the topology change, black hole pair creation is studied in the path-integral approach to quantum gravity, by finding a suitable instanton (that is, a solution of the classical equations of motion with Euclidean signature) which describes the transition from the given initial to final data. Most of the work to date has focussed on the pair creation of charged black holes in a background electromagnetic field, both in Einstein-Maxwell theory \[1\] and in a generalisation of this theory which includes a dilaton \[2, 3\], whose action is

\[
I = -\frac{1}{16\pi} \int_M (R - 2\partial^\mu \phi \partial_\mu \phi - e^{-2a\phi} F^2) - \frac{1}{8\pi} \oint_{\partial M} (K - K_0). \tag{1}
\]

The present paper is concerned with a different generalisation of Einstein-Maxwell theory, to include two $U(1)$ gauge fields and a dilaton, called the $U(1)^2$ theory. This is a somewhat more typical example of a low-energy effective theory arising from superstring theory, as the compactification of the extra dimensions will typically give an effective theory with a large number of $U(1)$ gauge fields. As I will argue in Sec. 2, the most appropriate effective action for this theory is \[4\]:

\[
I_{SO(4)} = -\frac{1}{16\pi} \int_M (R - 2\partial^\mu \phi \partial_\mu \phi - (e^{2\phi} \tilde{F}^2 + e^{-2\phi} G^2)) - \frac{1}{8\pi} \oint_{\partial M} (K - K_0), \tag{2}
\]

where $\tilde{F}_{\mu\nu}$ and $G_{\mu\nu}$ are the two $U(1)$ gauge fields. This theory is a consistent truncation of low-energy heterotic string theory \[4\]. One of the advantages of this truncation is that it includes the Einstein-Maxwell theory as a special case, when $\phi = 0$ and $\tilde{F}_{\mu\nu} = G_{\mu\nu}$. That is, Einstein-Maxwell is also a consistent truncation of string theory. The truncation (2) can also be derived from the $SO(4)$ version of $N = 4$ supergravity \[5\]. It also includes the action (1) with $a = 1$, when one of the gauge fields vanishes.

There are two duality symmetries in the $U(1)^2$ theory; one of them is a generalisation of the usual electric-magnetic duality, while the other is trivial on the Einstein-Maxwell solutions. Charged black hole solutions of (2) were found by Gibbons \[6\]. These solutions include the Reissner-Nordström metrics when the two gauge charges are equal, so the Reissner-Nordström solutions correspond to dyonic solutions of this theory. The duality symmetries and the black hole solutions are reviewed in Sec. 2.

An instanton describing pair creation of charged black holes in background fields in this theory was obtained in \[7\], and is reviewed in Sec. 3. This instanton is obtained from a generalisation of the Ernst solution of Einstein-Maxwell theory so that the black holes have two gauge charges and there are two corresponding background fields. The instanton is very
similar to the Ernst instanton, but the presence of two background fields introduces some interesting complications. In particular, the black holes are not spherically symmetric in the extremal limit in this case, unlike the Einstein-Maxwell case [3].

The main aim in this paper is to calculate the pair creation rate given by this instanton, which will allow us to extend the conclusions of [3, 8] to this case. The amplitude for pair creation in the instanton approximation to the path integral is given by $e^{-I}$, where $I$ is the action of the instanton. The pair creation rate will thus be given by $e^{-I_b}$, where $I_b$ is the action of the “bounce”, an instanton–anti-instanton pair. Sec. 4 is thus dedicated to the calculation of the action for the bounce. We find that the pair creation of non-extreme black holes is enhanced over that for extreme black holes by $e^{A_{bh}/4}$, from which we conclude that the non-extreme black holes have $e^{A_{bh}/4}$ more states than the extreme ones. That is, we conclude that the number of states is given by $e^{S_{bh}}$. In the point-particle limit, where the black holes are small on the scale set by the acceleration, the pair creation rate reduces to the Schwinger result. That is, to leading order, the pair creation rate for the black holes is the same as that for particles of the same mass and charges. In summary, the results of this calculation of the pair creation rate are essentially those of the calculation for the Einstein-Maxwell theory in [3, 8]; this might not seem surprising, as the Einstein-Maxwell theory is included as a special case, but there is much more freedom in the $U(1)^2$ theory, so it is a non-trivial result.

2 Properties of the theory

In [4], two actions were given for a low-energy theory with two $U(1)$ gauge fields and a dilaton,

$$I_{SO(4)} = -\frac{1}{16\pi} \int_M (R - 2\partial^\mu \phi \partial_\mu \phi - (e^{2\phi} F^2 + e^{-2\phi} G^2)) - \frac{1}{8\pi} \oint_{\partial M} (K - K_0)$$

(3)

and

$$I_{SU(4)} = -\frac{1}{16\pi} \int_M (R - 2\partial^\mu \phi \partial_\mu \phi - e^{-2\phi} (F^2 + G^2)) - \frac{1}{8\pi} \oint_{\partial M} (K - K_0).$$

(4)

These can be regarded as arising from the $SO(4)$ and $SU(4)$ versions of $N = 4$ supergravity respectively [3]. If we take

$$\tilde{F}_{\mu\nu} = \frac{1}{2} e^{-2\phi} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma},$$

(5)

It is easy to see that (3) and (4) give the same equations of motion, but the values of these two actions are different. To calculate the pair creation rate, we need to know which of these actions we should take.

If we consider the Einstein-Maxwell case, where $\tilde{F}_{\mu\nu} = G_{\mu\nu} = \frac{1}{\sqrt{2}} F_{\mu\nu}$ (say), and $\phi = 0$, [3] reduces to the usual Einstein-Maxwell action, with Maxwell field $F_{\mu\nu}$, while in (4), the two gauge field terms cancel. I therefore think that, since we use the Einstein-Maxwell action in the calculation of the pair creation rate in the Einstein-Maxwell case, we should use (3) to calculate the pair creation rate in this case. In [3], where the pair creation instanton was obtained, the solutions were written in terms of $F$ and $G$. Since I will use (3) to calculate the pair creation rate, I will instead write them here in terms of $\tilde{F}$ and $G$.

One interesting feature of the $U(1)^2$ theory is that it has two distinct duality symmetries. The equations of motion of this theory are invariant under a duality transformation,

$$F_{\mu\nu} \rightarrow \tilde{F}_{\mu\nu} \equiv \frac{1}{2} e^{-2\phi} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma},$$

(6)
\[ G_{\mu\nu} \rightarrow \tilde{G}_{\mu\nu} \equiv \frac{1}{2} e^{-2\phi} \epsilon_{\mu\nu\rho\sigma} G^{\rho\sigma}, \phi \rightarrow -\phi, \] (7)

which is analogous to the ordinary electric-magnetic duality transformation of Einstein-Maxwell theory. The equations of motion and the action (4) are also invariant under the interchange of the two gauge fields, \( F_{\mu\nu} \leftrightarrow G_{\mu\nu} \). If we combine these, we find that the equations of motion and the action (3) are invariant under the “duality”

\[ F_{\mu\nu} \rightarrow \tilde{G}_{\mu\nu}, G_{\mu\nu} \rightarrow \tilde{F}_{\mu\nu}, \phi \rightarrow -\phi. \] (8)

If we think of \( \tilde{F} \) as the field variable rather than \( F \), this “duality” just interchanges the two gauge fields and reverses the sign of the dilaton.

On the Einstein-Maxwell solutions, for which \( \tilde{F}_{\mu\nu} = G_{\mu\nu} \) and \( \phi = 0 \), (8) is a trivial transformation. In general, we will consider solutions for which the transformation (8) just corresponds to an interchange of the parameters of the solution. These solutions will be said to have a manifest duality symmetry. The action (3) is invariant under this manifest duality symmetry.

The charged black hole solutions of the \( U(1)^2 \) theory are (5):

\[ ds^2 = -\lambda dt^2 + \lambda^{-1} dr^2 + R^2 d\Omega, \] (9)

\[ e^{2\phi} = e^{2\phi_0} \frac{r + \Sigma}{r - \Sigma}, \] (10)

\[ \tilde{F} = Q e^{-\phi_0} \sin \theta d\theta \wedge d\varphi, \quad G = P e^{\phi_0} \sin \theta d\theta \wedge d\varphi, \] (11)

where

\[ \lambda = \frac{(r - r_+)(r - r_-)}{R^2}, \quad R^2 = r^2 - \Sigma^2, \] (12)

and (3)

\[ r_\pm = M \pm \sqrt{M^2 + \Sigma^2 - P^2 - Q^2}, \quad \Sigma = \frac{P^2 - Q^2}{2M}. \] (13)

There is a curvature singularity at \( r = |\Sigma| \). The physical degrees of freedom are \( P, Q, M \) and \( \phi_0 \); \( M \) is the mass of the black hole, and \( e^{-\phi_0} Q \) and \( e^{\phi_0} P \) are its gauge charges. Note that both the gauge fields are magnetic, when we write the solutions this way. One can also obtain a solution with two electric fields, but I will restrict attention to the magnetic case. We could keep the asymptotic value of the dilaton \( \phi_0 \) as a free parameter, but I will instead fix it by requiring that the dilaton match to an appropriate background value at infinity. The solution has a manifest duality symmetry, as the solution is unchanged when

\[ \tilde{F} \leftrightarrow G, \phi \leftrightarrow -\phi, \] (14)

and

\[ Q \leftrightarrow P, \Sigma \leftrightarrow -\Sigma, \phi_0 \leftrightarrow -\phi_0. \] (15)

3 The pair creation instanton

The pair creation of black holes is described by an instanton, that is, a solution of the classical equations of motion with Euclidean signature, which acts as a saddle-point in the path integral. The solution which gives the instanton in the \( U(1)^2 \) theory, which I will refer to as the \( U(1)^2 \)
Ernst solution, was obtained in [7]. It is a generalisation of the Ernst solution of Einstein-Maxwell theory [10]. Like the Ernst solution, it describes a pair of oppositely-charged black holes undergoing uniform acceleration under the influence of background electromagnetic fields. It asymptotically approaches an analogue of the Melvin solution [11], which describes the background fields, which I will refer to as the $U(1)^2$ Melvin solution.

The $U(1)^2$ Melvin solution is

$$ds^2 = \Lambda \Psi [-dt^2 + d\rho^2 + dz^2] + \frac{\rho^2 d\varphi^2}{\Lambda \Psi},$$

$$e^{-2\phi} = \frac{\Lambda}{\Psi}, A_\varphi = -\frac{\tilde{B}_M \rho^2}{2\Lambda}, B_\varphi = -\frac{\tilde{E}_M \rho^2}{2\Psi},$$

$$G_{\mu\nu} = \partial_{[\mu} A_{\nu]}, \tilde{F}_{\mu\nu} = \partial_{[\mu} B_{\nu]},$$

$$\Lambda = 1 + \frac{1}{2} \tilde{B}_M^2 \rho^2, \Psi = 1 + \frac{1}{2} \tilde{E}_M^2 \rho^2.$$ (19)

This solution has a manifest duality symmetry under

$$\tilde{F} \leftrightarrow G, \phi \leftrightarrow -\phi, \text{ and } \tilde{B}_M \leftrightarrow \tilde{E}_M.$$ (20)

It represents a pair of magnetic fields which are essentially uniform near the axis $\rho = 0$, with field strengths given by $\tilde{E}_M$ and $\tilde{B}_M$. The fields depart from uniformity away from the axis because the field energy curves the spacetime. However, in practice we cannot construct such strong fields, so the physically interesting part of this solution is the region near the axis.

The $U(1)^2$ Ernst solution is

$$ds^2 = \frac{\Lambda \Psi}{A^2(x-y)^2} [F(x)(G(y))dt^2 - G^{-1}(y)dy^2]$$

$$+ F(y)G^{-1}(x)dx^2 + \frac{F(y)G(x)}{\Lambda \Psi A^2(x-y)^2} d\varphi^2,$$

$$e^{-2\phi} = e^{-2\phi_0} \frac{\Lambda}{\Psi} \left( 1 + \Sigma A y \right) \left( 1 - \Sigma A x \right),$$

$$A_\varphi = -\frac{e^{\phi_0}}{B\Lambda} \left( 1 + \frac{B\beta x}{1 - \Sigma A x} \right) + k, \quad (23)$$

$$B_\varphi = -\frac{e^{\phi_0}}{E\Psi} \left( 1 + \frac{E\alpha x}{1 + \Sigma A x} \right) + k', \quad (24)$$

$$G_{\mu\nu} = \partial_{[\mu} A_{\nu]}, \tilde{F}_{\mu\nu} = \partial_{[\mu} B_{\nu]},$$

where

$$\Lambda = \left( 1 + \frac{B\beta x}{1 - \Sigma A x} \right)^2$$

$$+ \frac{B^2 (1 - x^2 - r_+ Ax^3)(1 + r_- Ax)(1 - \Sigma A y)^2}{2A^2(x-y)^2(1 - \Sigma A x)^2}.$$ (26)
\[ \Psi = \left( 1 + \frac{E_0} {1 + \Sigma A y} \right)^2 + \frac{E^2 (1 - x^2 - r_+ A x^3)(1 + r_- A x)(1 + \Sigma A y)} {2 A^2 (x - y)^2 (1 + \Sigma A x)^2}, \]  

\[ F(\xi) = 1 - \Sigma^2 A^2 \xi^2, \]  

\[ G(\xi) = \frac{(1 - \xi^2 - r_+ A \xi^3)(1 + r_- A \xi)} {(1 - \Sigma^2 A^2 \xi^4)}, \]  

and

\[ \alpha^2 = \frac{1}{2} (r_+ - \Sigma)(r_- - \Sigma) + \frac{1}{2} A^2 \Sigma^3 (r_- - \Sigma) \]
\[ = Q^2 + \frac{1}{2} A^2 \Sigma^3 (r_- - \Sigma), \]

\[ \beta^2 = \frac{1}{2} (r_+ + \Sigma)(r_- + \Sigma) - \frac{1}{2} A^2 \Sigma^3 (r_- + \Sigma) \]
\[ = P^2 - \frac{1}{2} A^2 \Sigma^3 (r_- + \Sigma). \]

As we will see below, this solution represents a pair of oppositely-charged black holes accelerating away from each other in a background field, although the coordinate system used here only includes one of the black holes. The black holes carry two magnetic gauge charges, and the background consists of two magnetic fields, which reduce to the fields in the \( U(1)^2 \) Melvin solution if we go to infinity along the axis of symmetry. The constants \( \phi_0, k, \) and \( k' \) will be chosen so that the solution at infinity agrees with \( \Psi \).

For \( r_+ A < 2/(3\sqrt{3}) \), the function \( G(\xi) \) has four real roots, which I denote in ascending order by \( \xi_1, \xi_2, \xi_3, \xi_4 \). It is convenient to define another function \( H(\xi) = G(\xi) F(\xi) \), so that I may write

\[ H(\xi) = -(r_+ A)(r_- A)(\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3)(\xi - \xi_4). \]

I restrict the parameters so that \( \xi_1 = -1/r_- A \) and \( \xi_1 \leq \xi_2 \leq \xi_3 < \xi_4 \). The surface \( y = \xi_0 = -1/|\Sigma| A \) is singular; this is the singular surface inside the black hole, that is, the singular surface at \( r = |\Sigma| \) in the black hole solutions \( \bar{\Psi} \). As \( r_- \geq |\Sigma|, \xi_1 \geq \xi_0 \). The surfaces \( y = \xi_1, y = \xi_2 \) are the inner and outer black hole horizons, and \( y = \xi_3 \) is the acceleration horizon for an observer comoving with the black hole. The coordinates \( (x, \varphi) \) are angular coordinates which cover two-spheres around the black hole, except when \( y = \xi_3 \). So that the metric has the appropriate signature, \( x \) is restricted to the range \( \xi_3 \leq x \leq \xi_4 \) in which \( G(x) \) is positive. At \( x = \{\xi_3, \xi_4\} \), the norm of \( \partial/\partial \varphi \) vanishes, so these points are interpreted as the poles of the two-spheres; that is, the axis of symmetry is \( x = \xi_3, \xi_4 \), with \( x = \xi_3 \) pointing at infinity, and \( x = \xi_4 \) pointing at the other black hole. There is a divergence in the metric at \( x = y \), which is interpreted as the point at infinity, so \( y \) is restricted to the range \( \xi_0 < y < x \).

Spatial infinity is reached only along the axis, that is, when \( y = x = \xi_3 \), and null or timelike infinity when \( y = x \neq \xi_3 \).

This solution has a manifest duality symmetry under

\[ \bar{F} \leftrightarrow G, \; \phi \leftrightarrow -\phi, \]  

\[ (33) \]
and

\[ Q \leftrightarrow P, \Sigma \leftrightarrow -\Sigma, B \leftrightarrow E, k \leftrightarrow k', \phi_0 \leftrightarrow -\phi_0. \tag{34} \]

As in the Ernst solution \cite{10}, the background fields provide the force necessary to accelerate the black holes. To eliminate the nodal singularities in this metric at \( x = \xi_3 \) and \( x = \xi_4 \) simultaneously, \( A \) must be chosen so that

\[ G'(\xi_3)\Lambda(\xi_4)\Psi(\xi_4) = -G'(\xi_4)\Lambda(\xi_3)\Psi(\xi_3) \tag{35} \]

and we must take \( \Delta \varphi = 4\pi L^2/G'(\xi_3) \), where I have introduced \( L^2 = \Lambda(\xi_3)\Psi(\xi_3) \). In the limit \( r_+ A \ll 1 \), (35) reduces to Newton’s law, \( MA \approx BP + EQ \), and in general it determines the acceleration of the black holes in terms of the other parameters.

If I set \( r_+ = r_- = 0 \), (21) becomes

\[ ds^2 = \frac{\Lambda \Psi}{A^2(x-y)^2}[(1-y^2)dt^2 - (1-y^2)^{-1}dy^2 + (1-x^2)^{-1}dx^2] + \frac{1-x^2}{\Lambda \Psi A^2(x-y)^2}d\varphi^2, \tag{36} \]

where

\[ \Lambda = 1 + \frac{1}{2}B^2 \left( \frac{1-x^2}{A^2(x-y)^2} \right), \tag{37} \]

and

\[ \Psi = 1 + \frac{1}{2}E^2 \left( \frac{1-x^2}{A^2(x-y)^2} \right). \tag{38} \]

This is just the \( U(1)^2 \) Melvin solution \cite{16} in non-standard coordinates \cite{7}. That is, the \( U(1)^2 \) Melvin solution is a special case of the \( U(1)^2 \) Ernst solution, where the black hole parameters are set to zero.

The \( U(1)^2 \) Ernst solution \cite{21} also approaches \cite{16} at large spacelike distances, that is, when we go to infinity along the axis. Spatial infinity corresponds to \( x, y \rightarrow \xi_3 \), and in this limit it is convenient to use the change of coordinates given in \cite{3},

\[ x - \xi_3 = \frac{4F(\xi_3)L^2}{G'(\xi_3)A^2} \frac{\rho^2}{(\rho^2 + \zeta^2)^2}, \tag{39} \]

\[ \xi_3 - y = \frac{4F(\xi_3)L^2}{G'(\xi_3)A^2} \frac{\zeta^2}{(\rho^2 + \zeta^2)^2}, \tag{40} \]

\[ t = \frac{2\eta}{G'(\xi_3)}, \varphi = \frac{2L^2 \varphi}{G'(\xi_3)}. \tag{41} \]

For large \( \rho^2 + \zeta^2 \), the \( U(1)^2 \) Ernst solution in these coordinates reduces to

\[ ds^2 \rightarrow \tilde{\Lambda} \tilde{\Psi}(-\zeta^2 d\eta^2 + d\zeta^2 + d\rho^2) + \frac{\rho^2 d\varphi^2}{\tilde{\Lambda} \tilde{\Psi}}, \tag{42} \]

where

\[ \tilde{\Lambda} = (1 + \frac{1}{2}B^2 \rho^2) \text{ with } \tilde{B}^2_E = \frac{B^2 G'(\xi_3)}{4L^2 \Lambda(\xi_3)} \tag{43} \]

\footnote{Note that \( \Lambda(\xi_i) \equiv \Lambda(x = \xi_i) \) and \( \Psi(\xi_i) \equiv \Psi(x = \xi_i) \) are constants.}
and

\[ \tilde{\Psi} = (1 + \frac{1}{2} \tilde{E}_E^2 \rho^2) \text{ with } \tilde{E}_E^2 = \frac{E^2 G'(\xi_3)}{4L^2 \Psi(\xi_3)}. \]  

(44)

If I now set \( \ell = \zeta \sinh \eta, z = \zeta \cosh \eta \), we once again regain (19). For large \( \rho^2 + \zeta^2 \), the dilaton and gauge fields tend to

\[ e^{-2\phi} \to e^{-2\phi_0} \frac{\Lambda(\xi_3) \hat{\Lambda}}{\Psi(\xi_3) \Psi}, \]

\[ A_\varphi \to e^{\phi_0} \frac{\Psi(\xi_3)^{1/2} \hat{B}_\rho^2}{\Lambda(\xi_3)^{1/2} 2\Lambda}, B_\varphi \to e^{-\phi_0} \frac{\Lambda(\xi_3)^{1/2} \hat{E}_\rho^2}{\Psi(\xi_3)^{1/2} 2\Psi}. \]

(45)

(46)

so if I set \( e^{2\phi_0} = \Lambda(\xi_3)/\Psi(\xi_3) \), I recover (17) in this limit. I will take this to define \( \phi_0 \) in general. Thus, I recover the \( U(1)^2 \) Melvin solution at large spacelike distances, and this allows me to identify the physical strength of the background fields in the Ernst solution as \( \tilde{E}_E \) and \( \tilde{B}_E \).

We can also calculate the physical charges on the black hole by integrating the field tensors over two-spheres surrounding the black holes. We find

\[ \hat{P} = \frac{1}{4\pi} \int G = \frac{\Lambda(\xi_3) \Psi(\xi_3)^{3/2}}{G'(\xi_3) \Lambda(\xi_4)^{1/2}} \frac{\beta(\xi_4 - \xi_3)}{(1 - \Sigma A\xi_4)(1 - \Sigma A\xi_3)} \]

(47)

and

\[ \hat{Q} = \frac{1}{4\pi} \int \tilde{F} = \frac{\Psi(\xi_3) \Lambda(\xi_3)^{3/2}}{G'(\xi_3) \Psi(\xi_4)^{1/2}} \frac{\alpha(\xi_4 - \xi_3)}{(1 + \Sigma A\xi_4)(1 + \Sigma A\xi_3)}. \]

(48)

where \( \alpha \) and \( \beta \) are given by (30,31).

The solution (21) describes two black holes accelerating away from each other, propelled by the background fields. Now we take the Euclidean section obtained by taking \( \tau = it \) in (21). Half the Euclidean section gives an instanton describing black hole pair production [13, 14]. There are three possible instantons: one describing pair production of non-extreme black holes, one describing pair production of extreme black holes, with \( \xi_1 = \xi_2 \), and another special case when \( \xi_2 = \xi_3 \). We will not consider this last here, as it does not describe black hole pair production (see [17] for more details of this case).

Let us first consider the non-extreme or wormhole instantons, \( \text{i.e.}, \xi_1 < \xi_2 < \xi_3 \). In the Euclidean section, we must restrict \( y \) to \( \xi_2 \leq y \leq \xi_3 \) to obtain a positive definite metric, and \((y, \tau)\) are now also coordinates on a two-sphere, except when \( x = \xi_3 \). We must impose another condition on the parameters to eliminate the possible conical singularities at the black hole horizon \( y = \xi_2 \) and the acceleration horizon \( y = \xi_3 \) simultaneously. Namely, the period of \( \tau \) must be taken to be \( \Delta \tau = 4\pi/|G'(\xi_2)| \), and we must set

\[ |G'(\xi_2)| = |G'(\xi_3)|, \]

(49)

where \( G(\xi) \) is given by (29). This condition is satisfied by setting

\[ \frac{(\xi_2^2 - \xi_0^2)}{(\xi_3^2 - \xi_0^2)} \left( \frac{\xi_3 - \xi_1}{\xi_2 - \xi_1} \right) = \frac{\xi_4 - \xi_2}{\xi_4 - \xi_3}. \]

(50)

This condition provides a further restriction on the black hole parameters, which may be thought of as determining the mass of the black hole in terms of its charges. More precisely, we can solve it for \( r_- A \) in terms of \( r_+ A \) and \( \Sigma A \). The whole Euclidean section is a bounce,
that is, an instanton–anti-instanton pair joined along a spacelike slice. The topology of the bounce is \( S^2 \times S^2 - \{ pt \} \), where the removed point is \( x = y = \xi_3 \).

For the extremal instantons, when \( \xi_1 = \xi_2 \), we must take \( \Delta \tau = 4\pi/|G'(\xi_3)| \) to ensure regularity at the acceleration horizon. The black hole event horizon is at infinite distance in all spatial directions, so we do not have to worry about a conical singularity there. The range of \( y \) in the Euclidean section is now \( \xi_2 < y \leq \xi_3 \), so that \((y, \tau)\) are now polar coordinates on an \( R^2 \), except when \( x = \xi_3 \). The extremal bounce has topology \( S^2 \times R^2 - \{ pt \} \), and the instanton can be interpreted as creating a pair of extremal black holes, with infinitely long throats.

However, I have found that, unlike the case with one \( U(1) \) gauge field \([3]\), the extremal solutions do not become spherically symmetric near the event horizon, and therefore do not approach the static black hole solutions at this internal infinity. This can be most easily seen by computing the intrinsic curvature scalar \( ^2R \) for the black hole horizon itself when the black holes are extremal, and calculating its numerical values at some typical horizon positions. I will omit the rather unilluminating formula for \( ^2R \), and simply state that one finds that the curvature is larger at the poles than at the equator of the two-sphere. Since the horizon is not a round two-sphere, the solution cannot be spherically symmetric. The point of this is that, unlike the case with one gauge field, even the extremal black holes are accelerating in some sense. It would be interesting to see if this could be extended to a Kaluza-Klein theory with two gauge fields, as in the usual Kaluza-Klein theory there is a well-defined sense in which the extremal black holes move on geodesics, and are thus not accelerating.

Another difference that it is worth highlighting is that, even once the no-strut condition \((B5)\) and either \((D5)\) or \( \xi_1 = \xi_2 \) have been satisfied, there are still four parameters in the solution, the two charges \( \hat{Q} \) and \( \hat{P} \) of the black hole and the background field strengths \( \hat{B}_E \) and \( \hat{E}_E \). This means we have a lot more freedom than in the Einstein-Maxwell case, where we only had two parameters once the regularity constraints were satisfied. In particular, if \( \hat{Q} \) and \( \hat{P} \) have opposite signs, it is possible to take large values of \( \hat{B}_E \) and \( \hat{E}_E \) without producing very large accelerations. This implies that, unlike the case with one \( U(1) \) gauge field \([3]\), there does not seem to be any universal bound on \( \hat{Q}\hat{E}_E \) or \( \hat{P}\hat{B}_E \).

### 4 The pair creation rate

Having described the pair creation instanton, I now turn to the calculation of the pair creation rate. The principal results are that the pair creation rate for non-extreme black holes is enhanced over that for extreme black holes by \( e^{A_{bh}/4} \) (as in the Einstein-Maxwell case \([3]\)), the pair creation rate is always suppressed, and it reduces to the Schwinger result in the limit of small black holes. The \( U(1)^2 \) Ernst metric reduces to the Ernst metric when \( \phi = 0 \), and to the dilaton Ernst metric when either \( \tilde{F} \) or \( G \) vanishes, and so I can check the calculation by showing that it agrees with the results of \([8, 3]\) in these cases.

The amplitude for pair creation in a background field is given by the path integral

\[
\Psi = \int d[g] d[A] d[B] e^{-I},
\]

where the action \( I \) in the path integral is the action \([2]\), and the integral is over all metrics and gauge fields which interpolate between the background fields at infinity and a spacelike slice which contains the pair of black holes. If there is an appropriate instanton, we assume that \( \Psi \) will be approximately \( \Psi \approx e^{-I} \), where \( I \) is now the action of the instanton. The pair creation
rate $\Gamma$ is given by the modulus squared of this amplitude, so it will be approximately $\Gamma \approx e^{-I_b}$, where $I_b$ is the action of the bounce. For the pair creation of black holes, the Euclidean sections of the solutions discussed in Sec. 3 are the bounces, so the calculation of the pair creation rates reduces to the problem of the calculation of the actions of these bounces.

The simplest way to evaluate the action is by a Hamiltonian decomposition, following the techniques given in [14]. Since the solutions we are interested in are stationary, if the Euclidean section was of the form $\Sigma \times S^1$, where the $S^1$ factor represents the time direction, the action would just be given by $I = \beta H$, where $H$ is the Hamiltonian and $\beta = \Delta \tau$ is the period in imaginary time. However, the time-translation Killing vector has fixed points at the black hole event horizon and the acceleration horizon, so by doing this we have neglected a contribution from a neighbourhood of each horizon. Including the contributions from these corners, the total Euclidean action is (in the non-extreme case)

$$I = \beta H - \frac{1}{4} (\Delta A + A_{bh}),$$

(52)

where $A_{bh}$ is the area of the black hole horizon, and $\Delta A$ is the difference in area of the acceleration horizon between the solution and the background [14, 8]. In the extreme case, the term proportional to $A_{bh}$ is absent, as the black hole event horizon is not part of the Euclidean section. The Hamiltonian $H$, which is only defined with respect to the background spacetime, can be expressed as [14]

$$H = \int_{\Sigma} N H - \frac{1}{8\pi} \int_{S^\infty} N (2K - 2K_0),$$

(53)

where $N$ is the lapse, $H$ is the Hamiltonian constraint, $2K$ is the trace of the two-dimensional extrinsic curvature of the boundary near infinity, and $2K_0$ is the analogous quantity for the background spacetime. On solutions, the constraint vanishes, and so the only non-zero contribution comes from the gravitational surface term.

To calculate this surface term, we need to introduce a boundary near infinity, and calculate its extrinsic curvature in the instanton and the background solution. To ensure that the boundary used in both calculations is the same, I need to match the intrinsic features of the boundary; that is, the induced metric, the gauge field, and the value of the dilaton on the boundary.

I take the boundary in the $U(1)^2$ Ernst solution to be

$$x = \xi_3 + \epsilon_E \chi, \quad y = \xi_3 + \epsilon_E (\chi - 1),$$

(54)

where $0 \leq \chi \leq 1$, and make the coordinate transformations

$$\varphi = \frac{2L^2}{G'(\xi_3)} \varphi', \quad t = \frac{2}{G'(\xi_3)} t',$$

(55)

and I assume that the boundary in the $U(1)^2$ Melvin solution lies at

$$x = -1 + \epsilon_M \chi [1 + \epsilon_E f(\chi)],$$

(56)

$$y = -1 + \epsilon_M (\chi - 1) [1 + \epsilon_E g(\chi)]$$

(57)

in the accelerated coordinate system.[36]. Other choices for the boundary in the $U(1)^2$ Melvin solution may be possible, but this is the only choice that I have been able to explicitly carry
out. We evaluate all quantities to second nontrivial order in $\epsilon_E$, as higher-order terms will not affect the result in the limit $\epsilon_E \to 0$. For the $U(1)^2$ Ernst metric, the induced metric on the boundary is

$$\begin{align} (2) d^2 s^2 &= \frac{L^2 F(\xi_3)}{A^2 \epsilon_E G'(\xi_3)} \left\{ -\frac{\lambda \psi d \lambda^2}{\chi (\chi - 1)} \left[ 1 + \epsilon_E (2\chi - 1) \frac{F'(\xi_3)}{F(\xi_3)} \right] + \frac{4 \chi}{\lambda \psi} \left[ 1 + \epsilon_E \frac{H''(\xi_3)}{2H'(\xi_3)} - \epsilon_E \frac{F'(\xi_3)}{F(\xi_3)} \right] d\phi^2 \right\}, \end{align}$$

where

$$\lambda = 1 + \frac{2L^2 \hat{B}_E^2 F(\xi_3) \chi}{A^2 \epsilon_E G'(\xi_3)} \left( 1 + \frac{1}{2} \epsilon_E \frac{H''(\xi_3)}{H'(\xi_3)} + \frac{2\Sigma A \epsilon_E}{1 - A \xi_3} \right)$$

and

$$\psi = 1 + \frac{2L^2 \hat{E}_E^2 F(\xi_3) \chi}{A^2 \epsilon_E G'(\xi_3)} \left( 1 + \frac{1}{2} \epsilon_E \frac{H''(\xi_3)}{H'(\xi_3)} - \frac{2\Sigma A \epsilon_E}{1 + A \xi_3} \right).$$

The gauge potentials on the boundary for the $U(1)^2$ Ernst solution are

$$A_{\phi'} = \frac{L e^{\phi_0}}{\Lambda(\xi_3) B_E} \left[ 1 - \frac{A^2 \epsilon_E G'(\xi_3)}{2L^2 F(\xi_3) B^2_E \chi} \right]$$

and

$$B_{\phi'} = \frac{L e^{-\phi_0}}{\Psi(\xi_3) \hat{E}_E} \left[ 1 - \frac{A^2 \epsilon_E G'(\xi_3)}{2L^2 F(\xi_3) \hat{E}_E^2 \chi} \right].$$

The dilaton at the boundary is

$$e^{-2\phi} = e^{-2\phi_0} \frac{\Lambda(\xi_3) \lambda}{\Psi(\xi_3) \psi} \left( 1 - \frac{2\Sigma A \epsilon_E}{1 - \Sigma^2 A^2 \xi_3^2} \right).$$

For the $U(1)^2$ Melvin solution, the induced metric on the boundary is

$$\begin{align} (2) d^2 s^2 &= \frac{-\Lambda \Psi}{2\chi (\chi - 1) A^2 \epsilon_M} \left\{ 1 - \epsilon_E (\chi - 1) f(\chi) + \epsilon_E \chi g(\chi) \right\} d\chi^2 d\phi^2 \\
&\quad - 2\epsilon_E \chi (\chi - 1) [f'(\chi) - g'(\chi)] - 2\epsilon_E [\chi f(\chi) - (\chi - 1) g(\chi)] \right\} d\phi^2, \end{align}$$

where

$$\Lambda = 1 + \frac{\hat{B}_M^2 \chi}{A^2 \epsilon_M} \left\{ 1 - \frac{1}{2} \epsilon_M \chi + \epsilon_E f(\chi) \right\}$$

and

$$\Psi = 1 + \frac{\hat{E}_M^2 \chi}{A^2 \epsilon_M} \left\{ 1 - \frac{1}{2} \epsilon_M \chi + \epsilon_E f(\chi) \right\}$$

and

$$- 2\epsilon_E [\chi f(\chi) - (\chi - 1) g(\chi)].$$
The gauge potentials on the boundary in $U(1)^2$ Melvin are

$$A_\phi = \frac{1}{B_M} \left[ 1 - \frac{\hat{A}^2 \epsilon_M}{B_M^2 \chi} \right]$$

and

$$B_\phi = \frac{1}{E_M} \left[ 1 - \frac{\hat{A}^2 \epsilon_M}{E_M^2 \chi} \right],$$

and the dilaton at the boundary in $U(1)^2$ Melvin is

$$e^{-2\phi} = \frac{\Lambda}{\Psi},$$

where $\Lambda$ is given by (65) and $\Psi$ is given by (66).

I fix the remaining coordinate freedom by taking

$$\hat{A}^2 = - G'(\xi_3)^2 L^2 F'(\xi_3) H''(\xi_3) A_2,$$

and write

$$e^{\phi_0} = \frac{\Lambda(\xi_3)^{1/2}}{\Psi(\xi_3)^{1/2}} (1 - \gamma \epsilon_E), \quad \hat{B}_M = \hat{B}_E (1 + \alpha \epsilon_E), \quad \hat{E}_M = \hat{E}_E (1 + \beta \epsilon_E).$$

I then find that the intrinsic metric, gauge potentials and dilaton on the boundary can all be matched by taking

$$\epsilon_M = - \frac{H''(\xi_3)}{H'(\xi_3)} \epsilon_E,$$

$$f(\chi) = \frac{F'(\xi_3)}{F(\xi_3)} (4 \chi - 3), \quad g(\chi) = \frac{F'(\xi_3)}{F(\xi_3)} (4 \chi - 1),$$

and

$$\gamma = \alpha = - \beta = \frac{\Sigma A}{1 - \Sigma^2 A^2 \xi_3^2}.$$

Note that the lapse function is also matched by these conditions. For the $U(1)^2$ Ernst metric, the lapse function at the boundary is given by

$$N = \left[ \frac{4 L^2 F'(\xi_3)(1 - \chi \lambda \psi)}{A^2 \epsilon_E G'(\xi_3)} \right]^{1/2} \left[ 1 + \frac{1}{4} \epsilon_E (\chi - 1) \frac{H''(\xi_3)}{H'(\xi_3)} + \frac{1}{2} \epsilon_E \frac{F'(\xi_3)}{F(\xi_3)} \right],$$

While the lapse function for the $U(1)^2$ Melvin metric is

$$N = \left[ \frac{2(1 - \chi) \Lambda \Psi}{A^2 \epsilon_M} \right]^{1/2} \left\{ 1 - \frac{1}{4} \epsilon_M (\chi - 1) + \frac{1}{2} \epsilon_E g(\chi) 
- \epsilon_E [\chi f(\chi) - (\chi - 1) g(\chi)] \right\},$$

so we see that the matching conditions (70-74) make (75) and (76) equal as well.
The extrinsic curvature of this boundary embedded in the $U(1)^2$ Ernst solution is

$$2K = \frac{A\epsilon_E^{1/2}G'(\xi_3)^{1/2}}{LF(\xi_3)^{1/2}\lambda\psi} \left[ 1 + \frac{1}{4} \epsilon_E \frac{H''(\xi_3)}{H'(\xi_3)} (4\chi - 3) \right. \\
- \left. \frac{1}{2} \epsilon_E \frac{F'(\xi_3)}{F(\xi_3)} (4\chi - 3) \right],$$

(77)

while the extrinsic curvature of the boundary embedded in the $U(1)^2$ Melvin solution is

$$2K_0 = \frac{\bar{A}\epsilon_M^{1/2} \sqrt{2}}{\Lambda\psi} \left[ 1 - \frac{1}{4} \epsilon_M (4\chi - 3) \right. \\
- \left. \frac{1}{2} \epsilon_E \frac{F'(\xi_3)}{F(\xi_3)} (24\chi - 13) \right].$$

(78)

Using the matching conditions (70-74), one may now evaluate

$$2K - 2K_0 = \frac{5A\epsilon_E^{3/2}G'(\xi_3)^{1/2}F'(\xi_3)}{LF(\xi_3)^{1/2}\lambda\psi} \frac{F'(\xi_3)}{F(\xi_3)} (2\chi - 1).$$

(79)

Therefore, taking the limit $\epsilon_E \to 0$, the Hamiltonian is

$$H_E = -\frac{1}{4} \int_0^1 d\chi N \sqrt{h} (2K - 2K_0) = -\frac{5L^2F'(\xi_3)}{A^2G'(\xi_3)} \int_0^1 d\chi (2\chi - 1) = 0.$$ 

(80)

The action is thus given by

$$I = -\frac{1}{4} (\Delta A + A_{bh})$$

(81)

when the black holes are non-extremal, and by

$$I = -\frac{1}{4} \Delta A$$

(82)

if the black holes are extremal. Note that the action in the non-extreme case is less than the action in the extreme case by $-\frac{1}{4} A_{bh}$, and thus the pair creation rate for non-extreme black holes in enhanced by $e^{A_{bh}/4}$ over that for extreme black holes. A natural interpretation of this result is that the non-extreme black holes have $e^{A_{bh}/4}$ more states than the extreme ones, as in the Einstein-Maxwell case [8]. As the difference in entropy between the non-extreme and extreme solutions is also $\frac{1}{4} A_{bh}$, this suggests that the entropy is a reliable guide to the number of states, that is, the number of states of a black hole $\sim e^{S_{bh}}$.

I now proceed to calculate the right hand side of (81) and (82). The area of the black hole horizon is

$$A_{bh} = \int_{y=\xi_2} \sqrt{g_{xx}g_{\phi\phi}} dx d\phi = \frac{4\pi F(\xi_2) L^2}{A^2G'(\xi_3)} \frac{(\xi_4 - \xi_3)}{(\xi_3 - \xi_2)(\xi_4 - \xi_2)}.$$ 

(83)

Again, in the calculation of the difference in area of the acceleration horizon, I need to introduce a boundary, in this case a circle, at large distances, and match the intrinsic features of this boundary. The area of the acceleration horizon in the $U(1)^2$ Ernst spacetime up to a large circle at $x = \xi_3 + \epsilon_E$ is

$$A_E = \int_{y=\xi_3} \sqrt{g_{xx}g_{\phi\phi}} dx d\phi = -\frac{4\pi L^2 F(\xi_3)}{A^2G'(\xi_3)(\xi_4 - \xi_3)} + \pi \rho_E^2,$$ 

(84)
where $\rho_E^2 = 4F(\xi_3)L^2/[G'(\xi_3)A^2\epsilon_E]$. The area of the acceleration horizon in the $U(1)^2$ Melvin spacetime inside a circle at $\rho = \rho_M$ is $A_M = \pi \rho_M^2$.

Now I need to match the proper length of the boundary, the integral of both gauge potentials around the boundary, and the value of $\phi$ at the boundary. The proper length of the boundary in the $U(1)^2$ Ernst solution is

$$l_E = \frac{4\pi}{E_E B_E \rho_E} \left[ 1 - \frac{F(\xi_3)L^2}{G'(\xi_3)A^2} \frac{H''(\xi_3)}{H'(\xi_3)} \frac{1}{\rho_E^2} + \frac{2F(\xi_3)L^2}{G'(\xi_3)A^2} \frac{F'(\xi_3)}{\rho_E^3} \right]$$

(85)

while the proper length of the boundary in the $U(1)^2$ Melvin solution is

$$l_M = \frac{4\pi}{E_M B_M \rho_M} \left( 1 - \frac{1}{E_M^2 \rho_M^2} - \frac{1}{B_M^2 \rho_M^2} \right).$$

(86)

The integral of the gauge potentials around the boundary are, in the $U(1)^2$ Ernst solution,

$$\oint A_\varphi d\varphi = \frac{2\pi}{B_E} e^{\phi_0} \Psi(\xi_3)^{1/2} \left( 1 - \frac{2}{B_E^2 \rho_E^2} \right)$$

(87)

and

$$\oint B_\varphi d\varphi = \frac{2\pi}{E_E} e^{-\phi_0} \Lambda(\xi_3)^{1/2} \left( 1 - \frac{2}{E_E^2 \rho_E^2} \right).$$

(88)

while in the $U(1)^2$ Melvin solution, they are

$$\oint A_\varphi d\varphi = \frac{2\pi}{B_M} \left( 1 - \frac{2}{B_M^2 \rho_M^2} \right)$$

(89)

and

$$\oint B_\varphi d\varphi = \frac{2\pi}{E_M} \left( 1 - \frac{2}{E_M^2 \rho_M^2} \right).$$

(90)

The dilaton at the boundary is

$$e^{-2\phi} = e^{-2\phi_0} \frac{\Lambda(\xi_3) B_E^2}{\Psi(\xi_3) E_E^2} \left[ 1 + \frac{2\Sigma A}{1 - \Sigma^2 A^2 \xi_3^2} \frac{4F(\xi_3)L^2}{G'(\xi_3)A^2 \rho_E^2} \right]$$

(91)

in the $U(1)^2$ Ernst solution, and

$$e^{-2\phi} = \frac{\tilde B_M^2}{E_M^2} \left( 1 + \frac{2}{B_M^2 \rho_M^2} - \frac{2}{E_M^2 \rho_M^2} \right)$$

(92)

in the $U(1)^2$ Melvin solution. Now $e^{\phi_0}$, $B_M$ and $E_M$ are given by (71) and (74), and we may see that we can match the proper length of the boundary, the integrals of the gauge fields and the dilaton if we also take

$$\rho_M = \rho_E \left( 1 + \frac{1}{\rho_E^2} \frac{F(\xi_3)L^2}{G'(\xi_3)A^2} \frac{H''(\xi_3)}{H'(\xi_3)} - \frac{2F'(\xi_3)}{F(\xi_3)} \right).$$

(93)
This implies that the difference in horizon area is

\[ \Delta A = -\frac{4\pi L^2 F(\xi_3)}{G'(\xi_3) A^2} \left[ \frac{1}{\xi_4 - \xi_3} + \frac{H''(\xi_3)}{2H'(\xi_3)} - \frac{F'(\xi_3)}{F(\xi_3)} \right] \]

\[ = -\frac{4\pi L^2 F(\xi_3)}{G'(\xi_3) A^2} \left[ \frac{(\xi_2 - \xi_1)}{(\xi_3 - \xi_2)(\xi_3 - \xi_1)} + \frac{2}{(\xi_3 - \xi_1)} - \frac{F'(\xi_3)}{F(\xi_3)} \right]. \]

For the extreme case, we therefore have

\[ -\frac{1}{4} \Delta A = \frac{2\pi L^2 F(\xi_3)}{G'(\xi_3) A^2} \left[ \frac{1}{\xi_3 - \xi_1} - \frac{F'(\xi_3)}{2F(\xi_3)} \right], \]

while for the non-extreme case, we have

\[ -\frac{1}{4} (\Delta A + A_{bh}) = \frac{\pi L^2 F(\xi_3)}{G'(\xi_3) A^2} \left[ \frac{2}{\xi_3 - \xi_1} - \frac{F'(\xi_3)}{F(\xi_3)} \right] \]

\[ + \frac{(\xi_2 - \xi_1)}{(\xi_3 - \xi_2)(\xi_3 - \xi_1)} - \frac{F(\xi_2)(\xi_4 - \xi_3)}{F(\xi_3)(\xi_3 - \xi_2)(\xi_4 - \xi_2)}, \]

\[ = \frac{2\pi L^2 F(\xi_3)}{G'(\xi_3) A^2} \left[ \frac{1}{\xi_3 - \xi_1} - \frac{F'(\xi_3)}{2F(\xi_3)} \right], \]

where I have used the instanton condition (50) to cancel the last two terms. Thus, I deduce that the action must be

\[ I_b = \frac{2\pi L^2 F(\xi_3)}{G'(\xi_3) A^2} \left[ \frac{1}{\xi_3 - \xi_1} - \frac{F'(\xi_3)}{2F(\xi_3)} \right] \]

in both cases. This answer agrees with the action of the Ernst solution found in [1, 8] when \( \Sigma = 0 \) (which implies \( F(\xi) = 1 \)), as it should. It also reduces to the answer for the action of the dilaton Ernst solution found in [3, 8] when either \( Q = 0 \) or \( P = 0 \). Thus, this result is consistent with the previously-obtained results.

The point-particle limit is \( r_+ A \ll 1 \), as the black hole becomes small on the scale set by the acceleration in this limit. In this limit, both the extreme and non-extreme black holes satisfy \( r_+ \approx r_- \). When \( r_+ A \ll 1 \), the action reduces to

\[ I_b \approx \frac{\pi r_-}{A} \approx \frac{\pi M^2}{BP + EQ}, \]

where I have used Newton’s law in the second step. The pair creation rate is \( e^{-I_b} \), so we recover the Schwinger result (generalised to the case of two gauge fields) in this limit, as we would expect. That is, we find that small black holes are pair created at the same rate (to leading order) as we would expect for some hypothetical particles carrying the same mass and charges. In particular, the pair creation rate will be very small for realistic fields, as we must have \( M > M_{pl} \) for this semi-classical approximation to be valid. Because of the number of parameters involved, it is difficult to say anything more about the general behaviour of this action, but the qualitative agreement with [8] is remarkable, given the much more complicated nature of this solution, and the presence of twice as many free parameters.
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