Cosmic acceleration and crossing of $w = -1$ barrier in non-local Cubic Superstring Field Theory model

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Abstract: We show that the late time rolling of the Cubic Superstring Field Theory (CSSFT) non-local tachyon in the FRW Universe leads to a cosmic acceleration with a periodic crossing of the $w = -1$ barrier. An asymptotic solution for the tachyon and Hubble parameter by linearizing the non-local equations of motion is constructed explicitly. For a small Hubble parameter the period of oscillations is a number entirely defined by the parameters of the CSSFT action.

Keywords: Cosmology of Theories beyond the SM, String Field Theory
1. Introduction

The combined analysis of the type Ia supernovae, galaxy clusters measurements and WMAP (Wilkinson Microwave Anisotropy Probe) data brings out clearly an evidence of the accelerated expansion of the Universe [1]-[5]. The cosmological acceleration strongly indicates that the present day Universe is dominated by a smoothly distributed slowly varying Dark Energy (DE). Recent results of WMAP [6] together with Ia supernovae data give a strong support that the present time DE state parameter is close to $-1$:

$$w = -0.97^{+0.07}_{-0.09}$$

or without an a priori assumption that the Universe is flat and together with large-scale structure and supernovae data $w = -1.06^{+0.13}_{-0.08}$.

From a theoretical point of view the above mentioned domain of $w$ covers three essentially different cases. The first case, $w > -1$, is achieved in quintessence models [7, 8] containing an extra light scalar field which is not in the Standard Model set of fields [9]. The second case, $w = -1$, is the cosmological constant [10, 11]. The third case, $w < -1$, is called a “phantom” one and can be realized by a scalar field with a ghost (phantom) kinetic term. In this case all natural energy conditions are violated and there are problems of an instability at classical and quantum levels [12, 13].

Since experimental data do not contradict with a possibility $w < -1$ and moreover a direct search strategy to test inequality $w < -1$ has been proposed [14] a study of such models attracts a lot of attention. Some projects [15] explore whether $w$ varies with the time or an exact constant. Varying $w$ obviously corresponds to a dynamical model of DE (see [16] for a review) which generally speaking includes a scalar field$^1$.

$^1$Modified models of GR also generate an effective scalar field (see for example [17] and refs. therein).
A possible way to evade the instability problem for models with \( w < -1 \) is to yield a phantom model as an effective one, arising from a more fundamental theory without a negative kinetic term. In this paper we develop in more details a cosmological SFT tachyon model \([18]\). The model is based on an SFT formulation of a fermionic NSR string with the GSO– sector \([19]\). In this model a scalar field is the open string tachyon, which describes according to the Sen’s conjecture \([20]\) a dynamical transition of a non-BPS D-brane\(^2\) to a stable vacuum (see \([22]\) for review). Since the concerned model is a string theory limit, all stability issues are related to a stability of a VSFT (a Vacuum String Field Theory, i.e. the SFT in a true vacuum) and one has to discuss only an application of this limit to a full string theory. There are general arguments \([23]\) that there does not exist a local scalar field model for a phantom Universe without an UV pathology. In a recent paper \([24]\) it has been proposed a phantom model without UV pathology in which a vector field is used.

The scalar model we investigate in this paper is a nonlocal one. Our goal is a construction of an analytic solution to linearized Friedmann equations in this model at large times. A characteristic feature of this model in the flat background is a presence of a rolling tachyon solution \([25, 26, 27]\). This property persuades us to consider a fermionic string instead of bosonic one where such a solution does not exist \([28, 29]\). However one might expect an existence of a rolling solution in bosonic strings in a non-flat case. The dynamics of a non-local tachyon scalar field on a cosmological background in the Hamilton-Jacobi formalism is studied in \([30]\).

We find that during the late time evolution in the FRW Universe the tachyon goes to its minimum oscillating with an exponentially decreasing amplitude. This is similar to the flat case. Consequently the Hubble parameter goes to a constant, and the state and deceleration parameters go to \(-1\) all oscillating around their asymptotic values. The DE state parameter \( w \) crosses the phantom divide \( w = -1 \) during an evolution.

Models with a crossing of the \( w = -1 \) barrier are also a subject of recent studies. Simplest models include two fields (one phantom and one usual field, see \([31, 32, 33]\) and refs. therein). General \( \kappa \)-essence models \([34]\) can have both \( w < -1 \) and \( w \geq -1 \) but a dynamical transition from the region \( w \geq -1 \) to the region \( w < -1 \) or vice versa is forbidden under general assumptions \([35]\) and is possible only under special conditions \([36]\).

In our case a non-locality provides a crossing of the \( w = -1 \) barrier in spite of the presence of only one scalar field. Hence the Universe exhibits an acceleration but because of oscillations quintessence and phantom phases change one each other with time.

The paper is organized as follows. In Section 2 we setup the cosmological model which is an approximation of the CSSFT describing a non-BPS brane within the level truncation scheme in the FRW Universe. In Section 3 we present details of the tachyon dynamics in the flat case at large times where an approximation linear in fluctuations around a non-perturbative vacuum is valid. We compare this linear approximation with a numeric solution to full equations. In Section 4 we study the tachyon dynamics in the FRW background again using a linear approximation to the Friedmann equations and find out that the obtained solution describes an accelerating Universe. In Section 5 we

\(^{2}\)DE models based on brane-world scenarios are presented in \([21]\).
discuss cosmological consequences of the obtained results and point out further directions of studying this type of models.

2. Setup

An action for the tachyon in the CSSFT \[37, 38\] in the flat background\(^3\) when fields up to zero mass are taken into account is found to be \[19, 25\]

\[
S_{\text{SFT}} = \frac{1}{g_o^2 \alpha'^2} \int dx \left( u^2(x) - \frac{\alpha'}{2} \eta^{\mu \nu} \partial_\mu \phi(x) \partial_\nu \phi(x) + \frac{1}{4} \phi^2(x) + \frac{e^{2\lambda}}{3} \phi^2(x) \tilde{u}(x) \right)
\]

where \(\phi(x)\) is the tachyon field, \(u(x)\) is an auxiliary field,

\[
\tilde{\phi} = e^{\alpha' \lambda \Box} \phi,
\]

and \(\lambda = - \log \frac{4}{3\sqrt{3}} \approx 0.2616\). \(\eta\) is the flat Minkowskian metric, \(\Box = \eta^{\mu \nu} \partial_\mu \partial_\nu\). For simplicity we will use hereafter \(\alpha' = 1\) units in which all fields, coordinates and the coupling constant \(g_o\) are dimensionless.

An auxiliary field \(u(x)\) can be integrated out to yield

\[
S_{\text{tach}} = \frac{1}{g_o^2} \int dx \left( -\frac{\xi^2}{2} \eta^{\mu \nu} \partial_\mu \phi(x) \partial_\nu \phi(x) + \frac{1}{4} \phi^2(x) - \frac{1}{4} \left( e^{\lambda \Box} \phi \right)^4(x) \right).
\]

A reasonable but an ad hoc assumption\(^4\) that \(u\) has no the tilde simplifies the last term in this action. Namely, under this assumption and a rescaling \(x \to 2\sqrt{\lambda}x\), \(\phi \to \frac{3}{\sqrt{2}} e^{-2\lambda} \phi\), and \(g_o \to 12\lambda e^{-2\lambda} g_o\) the action for the tachyon becomes

\[
S_{\text{tach, approx}} = \frac{1}{g_o^2} \int dx \left( -\xi^2 \eta^{\mu \nu} \partial_\mu \phi(x) \partial_\nu \phi(x) + \frac{1}{2} \phi^2(x) - \frac{1}{4} \left( e^{\frac{\xi}{4} \Box} \phi \right)^4(x) \right) \quad (2.1)
\]

where \(\xi^2 = \frac{1}{4\lambda} \approx 0.9556\). The last term in this action contains an infinite number of derivatives. Just due to this nonlocal factor a novel behavior in a dynamics of the tachyon field appears \[25\].

Cosmological scenarios with our Universe to be considered as a D3-brane embedded in 10-dimensional space-time was proposed in \[18\] and a dynamics of this brane is given by the following covariant version of action (2.1) in a non-flat space

\[
S = \int dx \sqrt{-g} \left( \frac{R}{2\kappa^2} + \frac{1}{g_o^2} \left( -\frac{\xi^2}{2} g^{\mu \nu} \partial_\mu \phi(x) \partial_\nu \phi(x) + \frac{1}{2} \phi^2(x) - \frac{1}{4} \Phi^4(x) - \Lambda \right) \right) \quad (2.2)
\]

where

\[
\Phi = e^{\frac{\xi}{4} \Box} \phi, \quad \Box_g = \frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu \nu} \partial_\nu.
\]

\(^3\)We always use the signature \((-+, +, +, +, \ldots)\).

\(^4\)As it was shown in \[24, 26\] this assumption can be applied to study a special class of rolling solutions and related questions.
Here $g$ is the metric, $\kappa$ is a gravitational coupling constant and we choose such units that it is dimensionless, $\Lambda$ is a constant.

In the present analysis we focus on the four dimensional Universe with the spatially flat FRW metric which can be written as

$$g_{\mu\nu} = \text{diag}(-1, a^2, a^2, a^2)$$

with $a = a(t)$ being a space homogeneous scale factor. In this particular case $\Box$ is expressed through $a$ as

$$\Box_g = -\partial_t^2 - 3H\partial_t + \frac{1}{a^2}\partial_x^2,$$

where $H \equiv \dot{a}/a$ is the Hubble parameter and the dot denotes the time derivative.

The Friedmann equations for the space homogeneous tachyon field have the form \cite{18}

$$3H^2 = \frac{\kappa^2}{g_0^2} \left( \frac{\ell^2}{2} \dot{\phi}^2 - \frac{1}{2} \phi^2 + \frac{1}{4} \Phi^4 + \mathcal{E}_1 + \mathcal{E}_2 + \Lambda \right), \quad (2.3a)$$

$$\dot{H} = \frac{\kappa^2}{g_0^2} \left( -\frac{\ell^2}{2} \dot{\phi}^2 - \mathcal{E}_2 \right) \quad (2.3b)$$

where

$$\mathcal{E}_1 = -\frac{1}{8} \int_0^1 ds (e^{s\hat{\mathcal{D}}^3}\Phi^3) \mathcal{D} e^{-\frac{1}{4}\hat{\mathcal{D}}^3} \Phi, \quad \mathcal{E}_2 = -\frac{1}{8} \int_0^1 ds (\partial_t e^{s\hat{\mathcal{D}}^3}\Phi^3) \partial_t e^{-\frac{1}{4}\hat{\mathcal{D}}^3} \Phi$$

with

$$\Phi = e^{\frac{1}{4}\hat{\mathcal{D}}^3} \phi, \quad \mathcal{D} = -\partial_t^2 - 3H(t)\partial_t.$$  

The equation of motion for the tachyon is

$$(\ell^2 \mathcal{D} + 1) e^{-\frac{1}{4}\hat{\mathcal{D}}^3} \Phi = \Phi^3. \quad (2.4)$$

The latter equation is in fact the continuity equation for the cosmic fluid. To see this explicitly the following operator relation is useful

$$\lambda \int_0^1 ds \left( (e^{s\lambda \Box_g} \square_g \phi) P e^{(1-s)\lambda \Box_g} \psi - (e^{s\lambda \Box_g} \phi) \square_g P e^{(1-s)\lambda \Box_g} \psi \right) =$$

$$= (e^{\lambda \Box_g} \phi) P \psi - \phi P e^{\lambda \Box_g} \psi - \lambda \int_0^1 ds (e^{s\lambda \Box_g} \phi) \pi e^{(1-s)\lambda \Box_g} \psi$$

where $P$ and $\pi$ are operators satisfying $[\Box_g, P] = \pi$.

Equations \textbf{2.3} are complicated because of the presence of an infinite number of derivatives and a non-flat metric. Before we are going to present a construction of an asymptotic solution to these equations to gain an insight into the problem of dealing with an infinite number of derivatives we shortly review the results of an analysis of action \textbf{2.1} in the flat space \textbf{25, 26, 27}.
3. A Late time Rolling Tachyon in the flat background

Approximate action for the tachyon (2.1) in the flat background was already studied in [25, 26, 27] and we will present the results relevant for the sequel. The equation of motion for space homogeneous configurations of the tachyon field is found to be

\[ (-\xi^2 \partial_t^2 + 1) e^{\frac{1}{4} \partial_t^2} \Phi(t) = \Phi(t)^3. \]  

(3.1)

where \( \Phi = e^{\frac{1}{8} \partial_t^2} \phi \). The operator \( e^{s \partial_t^2} \) for positive \( s \) can be rewritten in an integral form and this allows further to implement an iteration procedure which is capable to find a numeric solution to the problem.

If we are interesting in the Rolling Tachyon solution then we have in mind the following picture. The tachyon field starts from the origin (may be with a non-zero velocity), rolls down to the minimum of the tachyon potential \( \Phi \) and eventually stops in the minimum. In our notation the minima are located at \( \Phi_0 = \pm 1 \). For \( \xi^2 \neq 0 \) and \( \xi^2 < \xi_{cr} \approx 1.38 \) there are damping fluctuations near the minimum [26]. Let us note that in our case \( \xi^2 < \xi_{cr}^2 \). To analyze the late time behavior one can linearize equation (3.1) as \( \Phi = \Phi_0 - \delta \Phi \) keeping only liner in \( \delta \Phi \) terms. A substitution yields the following equation for \( \delta \Phi \)

\[ (-\xi^2 \partial_t^2 + 1) e^{\frac{1}{4} \partial_t^2} \delta \Phi = 3 \delta \Phi. \]  

(3.2)

At this point we want to mention two facts about this equation. First, if one omits the non-local exponential operator then one arrives to an equation for the harmonic oscillator

\[ (\xi^2 \partial_t^2 + 2) \delta \Phi = 0 \]

which does not give the desired late time behavior. Second, if we expand the non-local operator and keep only the constant and second derivative terms the situation depends on the value of \( \xi^2 \) and for \( \xi^2 > \frac{1}{4} \) (which is our case) remains qualitatively the same. Using that \( e^{\partial_t^2} e^{-mt} = e^{m^2t} e^{-mt} \) one can try to find a solution to equation (3.2) of the form \( \delta \Phi(t) = Ce^{-mt} \) where for \( m \) we have the following characteristic equation

\[ (-\xi^2 m^2 + 1) e^{\frac{1}{4} m^2} = 3. \]  

(3.3)

This is exactly equation (3.27) in [18] with a redefinition \( m_{our} = im \).

Note, that this solution is valid only for late times. One expects thereby that the non-local operator plays a crucial role in the asymptotic regime. The latter characteristic equation for \( m \) is solved in Appendix A. Here only the results are used. A general solution for \( \delta \Phi \) is an infinite sum

\[ \delta \Phi(t) = \sum_k (A_k e^{-m_k t} + B_k e^{m_k t}) \]

because infinitely many roots \( m_k^2 \) exist. This reflects the presence of an infinite number of derivatives. The latter means an infinite number of initial conditions which however are not

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5 See also [25, 26] for a cubic action.

6 The potential is equivalent in terms of both \( \phi \) and \( \Phi \) since \( \phi = \Phi \) for zero momentum.
arbitrary (see [28] for a discussion on this point). Moreover in the full an action asymptotic behavior is subject to initial conditions in the origin. In other words considering the full non-linear equation and imposing initial conditions in the beginning of the evolution one arrives to a specific asymptotic configuration. We assume in our case a solution converging to an asymptotic value exists. For the definition of $m_k$ given in Appendix a vanishing condition imposes $B_k = 0$ for all $k$ and a reality condition for $\delta \Phi$ is formulated as $A_k = A^*_{-1-k}$. Therefore, the most general real vanishing solution to equation (3.2) is

$$\delta \Phi(t) = \sum_{k \geq 0} (A_k e^{-m_k t} + A^*_k e^{-m^*_k t}).$$  \hspace{1cm} (3.4)

Hence one yields a real function of time which oscillates with an exponentially decreasing amplitude. The function $\Phi(t)$ in turn goes to 1 oscillating with the decreasing amplitude near the asymptotic value. Recall that the solution constructed is valid only for late times. Let us also note that in a UV region, $k \geq 1$, one has to take into account string modes next to leading ones and this is beyond of our approximation.

The main (i.e. the most slowly vanishing) contribution in (3.4) is given by $k = 0$ and can be represented as

$$\delta \phi(t) = Ce^{-rt} \sin(\nu t + \varphi) \quad \text{where} \quad r \approx 1.1365, \nu \approx 1.7051$$  \hspace{1cm} (3.5)

where we passed to $\delta \phi = e^{\frac{1}{2} \alpha^2 t} \delta \Phi$. All other $k$ will give a faster convergence and will play a role of small corrections. We find that a rather simple choice of constants $C \approx 1$ and $\varphi \approx 2$ in (3.5) matches the rolling solution presented in Fig. 1a. Figures Fig. 1b and Fig. 1c contain corresponding plots.

**Figure 1:** a. Solution to equation (3.1): $\Phi$ (red line), $\phi$ (green line), $\dot{\phi}$ (blue line) for $\xi^2 = 0.9556$; b. An approximation to this solution $1 - \delta \phi$ given by (3.5) (blue line); c. The same plots as in b in a different scale.

Note that for $\xi^2 = 0$ (this case corresponds to a $p$-adic string [39, 40] with $p = 3$) equation (3.2) is simplified drastically and we have

$$m^2_k = 4(\log 3 + 2\pi k i)$$

where again different branches may be considered. The principal branch is $k = 0$ and it corresponds to the rolling solution [25, 26, 27]. Other $k$ do not match the rolling solution. They will give oscillations which were not present in a numeric analysis. Absence of such an oscillating behavior is related to initial conditions in the origin. One can also say that $m_k$ for $k = 1, 2, \ldots$ are in a UV region beyond of our approximation.
The main conclusion of this Section is that the non-local operator $e^{s\partial_t^2}$ for positive $s$ acts like a friction. For $\xi^2 = 0$ the late time behavior of the tachyon is just a smooth rolling with a monotonically vanishing velocity to the minimum of the potential. This is not surprising mathematically but looks rather strange physically. Term with $\xi^2$ in the l.h.s. of equation (3.2) accelerates the tachyon but for actual $\xi^2$ a friction becomes stronger and the tachyon eventually stops in the minimum.

Another point of view is that for the considered $\xi^2$ time intervals when the kinetic energy on space homogeneous configurations is negatively defined dominate providing a phantom behavior for the tachyon field. This justifies a phantom approximation used in [41].

4. A Late time Rolling Tachyon in the FRW Universe

We consider action (2.2) for the tachyon coupled to the gravity. We are going to employ an analog of the asymptotic expansion used above in Section 3. To this end we have to expand $\phi = \phi_0 - \delta\phi$ (and accordingly $\Phi = \phi_0 - \delta\Phi$, where $\delta\Phi = e^{\frac{1}{4}D}\delta\phi$) in Friedmann equations (2.3). In our notations $\phi_0 = \pm 1$ and the resulting equations read

$$3H^2 = \frac{\kappa^2}{g_0^2} \left( \frac{\xi^2}{2} \dot{\delta\phi}^2 - \frac{1}{2} \delta\phi^2 + \frac{3}{2} \delta\Phi^2 + \delta\varepsilon_1 + \delta\varepsilon_2 + \Lambda_0 \right),$$

$$\dot{H} = \frac{\kappa^2}{g_0^2} \left( -\frac{\xi^2}{2} \delta\phi^2 - \delta\varepsilon_2 \right)$$

where

$$\delta\varepsilon_1 = -\frac{3}{8} \int_0^1 ds e^{\frac{1}{4}sD} \delta\Phi \dot{D} e^{-\frac{1}{4}sD} \delta\phi, \quad \delta\varepsilon_2 = -\frac{3}{8} \int_0^1 ds \dot{\delta\phi} e^{\frac{1}{4}sD} \delta\Phi \partial_t e^{-\frac{1}{4}sD} \delta\phi$$

and $\Lambda_0 = -\frac{1}{4} + \Lambda$. The equation of motion for the tachyon field $\phi$ (2.4) after expansion has the form

$$(\xi^2\partial_t^2 + 1)e^{-\frac{1}{4}D}\delta\Phi = 3\delta\Phi.$$  

At this point we assume that in this approximation only the constant term $H_0$ in an expansion of $H$ survives in the latter equation. A validity of this assumption will become clear after a solution will be constructed. A value of $H_0$ can be determined from equation (4.1a). However, even in this case an analytic solution to equation (1.2) in a closed form is not achievable so far. Instead of this we are looking for a solution constructed from eigenfunctions of the operator $D_0 = -\partial_t^2 - 3H_0\partial_t$. They have the following form

$$\delta\Phi = A e^{-\bar{m}+t} + B e^{-\bar{m}-t} \text{ where } \bar{m}_{\pm} = \frac{3}{2} H_0 \pm \sqrt{\frac{9H_0^2}{4} + m^2}.$$

To become a solution to equation (4.2) with a constant $H = H_0$ a parameter $m$ should be determined by means of transcendental equation (3.3) already appeared in the flat case. A general solution to equation (4.2) is

$$\delta\Phi = \sum_k (A_k e^{-\bar{m}+kt} + B_k e^{-\bar{m}-kt}).$$
An analysis of this solution goes in an analogy with the flat case considered in Section 3. For $\bar{m}_{\pm k}$ we have

$$\bar{m}_{\pm k} = \bar{r}_{\pm k} + i\bar{\nu}_k, \quad \bar{r}_{\pm k} = \frac{3}{2} H_0 \pm \frac{\beta_k}{\bar{v}}, \quad \bar{\nu}_k = \text{sign}(\beta_k) \frac{\bar{v}}{2},$$

where a notation $m_k^2 = \alpha_k + i \beta_k$ is used. However, in contrary with the flat case a selection of vanishing branches is not so obvious. It depends on a particular value of $H_0$. Notice, that $\bar{\nu}_k$ is always non-zero, i.e. a solution is oscillating for any $k$. Fortunately, the symmetry $\bar{m}_{\pm k} = \bar{m}_k^{(-1-k)}$ persists. Thus a reality condition for $\delta \Phi$ is $A_k = A_{-1-k}^*$ and $B_k = B_{-1-k}^*$, and the most general real vanishing solution becomes

$$\delta \Phi = \sum_{k \geq 0} (A_k e^{-\bar{m}_{+k} t} + A_k^* e^{-\bar{m}_{+k}^* t}) + \sum_{k \geq 0} (B_k e^{-\bar{m}_{-k} t} + B_k^* e^{-\bar{m}_{-k}^* t})$$

(4.3)

where the prime means that only $k$ for which the behavior is vanishing should be taken into account.

Let us spend few lines discussing a question about the main (i.e. most slowly vanishing) contribution. A parameter $H_0$ complicates the story since $H_0 > 0$. Thus $k = 0$ is not necessary the main contribution. Moreover, for specific values of $H_0$ and $k$ both plus and minus in $\bar{r}_k$ may give a positive number. In this case a corresponding $B_k$ term is allowed and this is new compared to the flat case. To illustrate the situation $\bar{m}_k$ are drawn in Fig. 3 for different values of $H_0$. On these plots each point corresponds to a specific $k$.

![Figure 2: $\bar{m}_{\pm k}$ for $H_0 = 0$ (left), $H_0 = \frac{1}{2}$ (middle), and $H_0 = 4$ (right). Circles correspond to the plus and crosses to the minus sign.](image)

For $k \geq 0$ the imaginary part is positive and grows with $k$. For $k \leq -1$ the imaginary part is negative and grows in absolute value with $|k|$. Circles correspond to the plus and crosses to the minus sign in $\bar{m}_{\pm k}$. Points with smallest positive real parts form the main contribution and one sees that a selection of such points is not so transparent. However, for the sequel we will need only to state that one can make such a selection. Thus for the main contribution in (4.3) we have

$$\delta \phi = C e^{-\bar{m} t} + C^* e^{-\bar{m}^* t}$$

(4.4)
where we have passed to $\delta \phi = e^{-\frac{1}{2}D} \delta \Phi$ and an index $\pm k$ in $m_{\pm k}$ is omitted since a specific choice of $k$ and a branch of the square root is made. Also for some values of $H_0$ it is possible that two close points, say $k$ and $k+1$ for $k > 0$ will have equal and smallest positive real parts. In this case they will only differ by a frequency of oscillations which will be suppressed with an equal exponential factor. However, for simplicity we will not consider this situation. Substituting solution (4.4) into Friedmann equation (4.1b) one yields after an integration using relation (3.3)

$$H = H_0 + \frac{\kappa^2}{g_o^2} \left( \frac{C^2 \bar{m}}{16} e^{-2\bar{m}t} \left( 1 + 4\xi^2 - \xi^2 m^2 \right) + \frac{C^*}{16} e^{-2\bar{m}^*t} \left( 1 + 4\xi^2 - \xi^2 m^* \right) \right)$$

(4.5)

where an integration constant is chosen according to equation (4.1a) to be $H_0 = \frac{\kappa g_o}{3} \sqrt{\Lambda_0}$ and a constant $C$ is the same as in (4.4). Further one can check that equation (4.1a) is satisfied up to $\delta \phi^4$ terms which are beyond of our approximation. For the succeeding analysis it is convenient to represent $\phi$ and $H$ as follows

$$\phi = 1 - C e^{-\bar{m}t} \sin(\bar{\nu}t + \bar{\varphi}) \quad \text{and} \quad H = \frac{\kappa}{g_o} \sqrt{\Lambda_0} - \frac{\kappa^2}{g_o^2} C_H e^{-2\bar{m}t} \sin(2\bar{\nu}t + \varphi_H)$$

(4.6)

where $C$ and $C_H$ are arbitrary (integration) constants,

$$C_H = \frac{\bar{C}^2}{32} \sqrt{\bar{m} \bar{m}^*} \sqrt{(1 + 4\xi^2)^2 + \xi^4 m^2 m^* - \xi^2 (1 + 4\xi^2)(m^2 + m^*^2)},$$

and $\varphi_H = 2\bar{\varphi} + \frac{\pi}{2} + \arctan \left( \frac{\bar{m}^* - \bar{m}}{\bar{m} + \bar{m}^*} \right) - \arctan \left( \frac{\xi^2 (m^2 - m^*^2)}{2 + 8\xi^2 - \xi^2 (m^2 + m^*^2)} \right)$.

In a special case $\xi^2 = 0$ which corresponds to a $p$-adic string in the FRW Universe the story is much simpler because eigenvalues $\bar{m}_{\pm k}$ can be expressed in elementary functions since $m_k^2 = 4(\log 3 + 2\pi ki)$. Moreover, $\bar{m}_{\pm 0}$ are pure real and $\bar{m}_{+0}$ corresponds to a smooth approach of the tachyon field $\phi$ to the asymptotic value $\phi_0 = 1$ without oscillations. Expressions for $\delta \phi$ and $H$ in this case are as follows

$$\delta \phi = \bar{C} e^{-\bar{m}t} \quad \text{and} \quad H = \frac{\kappa}{g_o} \sqrt{\Lambda_0} - \frac{\kappa^2}{g_o^2} C_H e^{-2\bar{m}t}$$

(4.7)

where $\bar{m} = \frac{3}{2} H_0 + \sqrt{\frac{9H_0^2}{4} + 4 \log 3}$, $C_H = \frac{C^2 \bar{m}}{10}$ and $H_0$ is the same as in (4.3). Again one can check that equation (4.1a) is satisfied up to $\delta \phi^4$ terms.

Plots representing the tachyon field evolution both for $\xi^2 \approx 0.9556$ and $\xi^2 = 0$ are shown in Fig. 3.

5. Cosmological consequences and further directions

The obtained asymptotic solutions for the tachyon and Hubble parameter (4.6) lead to a number of interesting cosmological properties.
First, we mention that $H$ tells us about the time point $t_0$ after which the solution is reasonable. Indeed, for $H$ to be strictly positive we have to require $\sqrt{\Lambda_0/3} > \frac{\kappa}{g_0}CHe^{-2\bar{r}t}$. Solving this inequality w.r.t. the time we have $t > t_0 = -\frac{1}{2\bar{r}} \log \left( \frac{\kappa}{g_0} \sqrt{\Lambda_0/3} \right)$. This is the characteristic time in question.

Then one sees that during the late time evolution of the tachyon field the Hubble parameter goes to a constant. $\dot{H}$ obviously vanishes and moreover, both $H$ and $\dot{H}$ do oscillate. The state parameter $w$ also has an oscillating time behavior and goes asymptotically to $-1$. An explicit expression for it is of the form

$$w = -1 - \frac{2}{3} \frac{\dot{H}}{H^2} = -1 + \frac{2}{3} CHe^{-2\bar{r}t} \frac{-2\bar{r} \sin(2\bar{\nu}t + \varphi_H) + 2\bar{\nu} \cos(2\bar{\nu}t + \varphi_H)}{\left( \sqrt{\frac{\Lambda_0}{3}} - \frac{\kappa}{g_0} CHe^{-2\bar{r}t} \sin(2\bar{\nu}t + \varphi_H) \right)^2}. \quad (5.1)$$

It is very interesting that $w$ crosses the phantom divide $w = -1$ during the evolution. Such a crossing is forbidden in single field cosmological models with a local action. In our model we see that a non-locality breaks this restriction. The deceleration parameter $q$ behaves very similar to $w$ because its expression through $H$ is very close to $w$

$$q = -1 - \frac{\dot{H}}{H^2} = -1 + CHe^{-2\bar{r}t} \frac{-2\bar{r} \sin(2\bar{\nu}t + \varphi_H) + 2\bar{\nu} \cos(2\bar{\nu}t + \varphi_H)}{\left( \sqrt{\frac{\Lambda_0}{3}} - \frac{\kappa}{g_0} CHe^{-2\bar{r}t} \sin(2\bar{\nu}t + \varphi_H) \right)^2}. \quad (5.2)$$

Hence the Universe exhibits an acceleration but because of oscillations quintessence and phantom phases change one each other with the time.

The scale factor $a$ is related to $H$ as $H = \ddot{a}/a$ and can be readily found to be

$$a = a_0 e^{\int H dt} = a_0 \exp \left( \frac{\kappa}{g_0} \sqrt{\frac{\Lambda_0}{3}} t + \frac{\kappa^2}{g_0^2} CHe^{-2\bar{r}t} \frac{-2\bar{r} \sin(2\bar{\nu}t + \varphi_H) + 2\bar{\nu} \cos(2\bar{\nu}t + \varphi_H)}{\bar{r}^2 + \bar{\nu}^2} \right). \quad (5.3)$$

where $a_0$ is an integration constant. For late times it has a simple exponential approximation

$$a = a_0 e^{\kappa g_0 \sqrt{\frac{\Lambda_0}{3}} t}.$$  

To plot $H$ and $w$ we specify $H_0$ (or $\Lambda_0$) to be so small that the $\bar{r}$ and $\bar{\nu}$ in (1.6) are $\bar{r}_{+0}$ and $\bar{\nu}_{+0}$ respectively. This corresponds to the middle plot in Fig. 2. The explained above behavior of $H$, $w$, $q$, and $a$ is visualized in Fig. 4. We point out that in spite of the presence of only one scalar field we observe similarities in the behavior of cosmological quantities with the two-field model analyzed in [31]. These are crossing of the phantom
divide by \( w \) and the qualitative form of plots for \( H, w, \) and \( q \). An order of magnitude of \( \Lambda_0 \) comes from an analysis performed in [41].

For \( \xi_2 = 0 \) one readily gets using (4.7) for the state and deceleration parameters

\[
 w = -1 - \frac{2}{3} \frac{2C_H \bar{m}e^{-2\bar{m}t}}{\left( \sqrt{\frac{\Lambda_0}{3}} - \frac{\kappa g_o}{g_o} C_H e^{-2\bar{m}t} \right)^2}, \quad q = -1 - \frac{2C_H \bar{m}e^{-2\bar{m}t}}{\left( \sqrt{\frac{\Lambda_0}{3}} - \frac{\kappa g_o}{g_o} C_H e^{-2\bar{m}t} \right)^2}. \tag{5.4}
\]

We see that \( w \) is always less than \(-1\) in this case and a crossing of the phantom divide does not occur. Thus the tachyon behaves purely like a phantom for \( \xi_2 = 0 \). The scale factor \( a \) in this case is found to be

\[
 a = a_0 e^{\int H dt} = a_0 \exp \left( \frac{\kappa}{g_o} \sqrt{\frac{\Lambda_0}{3}} + \frac{\kappa^2 C_H}{g_o^2 2 \bar{m}} e^{-2\bar{m}t} \right) \xrightarrow{t \to \infty} a_0 e^{\frac{\kappa g_o}{2 \bar{m}}} \sqrt{\frac{\Lambda_0}{3}} \tag{5.5}
\]

where \( a_0 \) is an integration constant. Plots for \( H, w, q, \) and \( a \) in the case \( \xi_2 = 0 \) are presented in Fig. 5. The behavior of the model in this case is very close to a single

\[
\text{Figure 5: From left to right } H, w, q, \text{ and } a \text{ for } \kappa = g_o = \bar{C} = a_0 = 1 \text{ and } \Lambda_0 = 3 \cdot 10^{-21} \text{ in the case } \xi_2 = 0.
\]

Remarkably, that both for \( \xi_2 \approx 0.9556 \) and \( \xi_2 = 0 \) the Big Rip singularity problem is avoided because \( w \) exhibits a non-trivial time dependence and consequently is not a constant less than \(-1\).

Constructed solution reveals many properties expected from the cosmological point of view giving rise to an interest to a further investigation of this model as well as models coming from the SFT in general. Apart from many other possible directions of developing this type of models we would like to mention ones which are seen as the most important.
It would be very interesting and at the same time very difficult to construct a solution to full non-linear equations (2.3). It seems that if it can be done it will be a numeric solution. However, even a numeric approach faces a difficulty of dealing with an exponential of a Beltrami-Laplace operator in a curved background. The main technical problem is that a Beltrami-Laplace operator itself contains an unknown function $H$ to be determined while solving the equations.

A complete analysis of a stability of the obtained solution is also of great importance. A related issue which is an inclusion of other cosmic fluids and especially the Cold Dark Matter (CDM) forming about 1/3 of the present day Universe and an investigation of a dynamics of such a coupled model and its stability would be very important (see [42] as an example of a coupling of one phantom field to the CDM).

Also it would be interesting to find an exact solution to full equations with infinitely many derivatives in curved background probably by adding extra terms to the action. This is in accord with an analysis performed in [11, 13] where a small in terms of coupling constants correction to the potential makes the problem analytically solvable. An existence of an analytic solution provides a possibility of a qualitative analysis without a strong support of numeric methods.

Another important question is a consideration of closed string scalar fields, the closed string tachyon and dilaton, as well as their coupling to the open string tachyon. This problem is related to finding of lump solutions and a development in the flat background was started in [13]. Similar lump solutions have been constructed in [44]. An extension to the FRW Universe using a linearization of equations of motion developed in the present paper would be an interesting analysis of the role of closed string excitations.

In spirit of a selected role of vector fields in a construction of local phantom models without an UV pathology [24] it would be very interesting to incorporate a string vector field that has in the SFT approach a non-local interaction in a study of a nonlocal rolling tachyon dynamics.

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A. A Solution to the characteristic equation

The equation in question is the following

$$
(−ξ^2 m^2 + 1) e^{\frac{i}{4}m^2} = 3.
$$

See [13] for a discussion on the closed string tachyon and dilaton condensation.
It is a transcendental one and a solution to it can be represented analytically as
\[
m_k^2 = \frac{1}{\xi^2} + 4W_k \left( -\frac{3}{4\xi^2} e^{-\frac{1}{4\xi^2}} \right) = 4\lambda + 4W_k \left( -3\lambda e^{-\lambda} \right)
\]
where \( W_k \) is the \( k \)-th branch of Lambert \( W \) function satisfying a relation \( W(x)e^{W(x)} = x \). It has infinitely many branches distinguished as \( W_k \). A branch analytic at 0 is referred to as a principal one. There is exactly one such a branch and \( k = 0 \) is assigned to it. The branch cut dividing \( W_0 \) and \( W_{\pm 1} \) is the part of the real axis \((-\infty, -1/e)\) and \(-1/e\) is the branch point. The branch cut dividing all other neighbor branches is the negative real semi-axis and 0 is the branch point. For the principal branch \( W_0(x) \) the image of the \((-\infty, -1/e)\) interval is \(-\beta \cot \beta + i\beta\), where \( \beta \in (0, \pi] \). For the branch \( W_{-1}(x) \) the image of the \((-\infty, -1/e)\) interval is \(-\beta \cot \beta + i\beta\), where \( \beta \in (-\pi, 0] \). For all other branches \( W_k(x) \) the image of the \((-\infty, 0)\) interval is \(-\beta \cot \beta + i\beta\), where \( \beta \in (2k\pi, (2k+1)\pi] \) if \( k > 0 \) and \( \beta \in ((2k+1)\pi, (2k+2)\pi] \) if \( k < -1 \). For our purposes we are interested in the argument \( x = -3\lambda e^{-\lambda} \approx -0.6042 < -1/e \), so the above properties are relevant. To understand the dependence \(-\beta \cot \beta + i\beta\) we are going to consider the equation
\[
ye^y = x
\]
for \( x < -1/e \). Assuming a complex solution \( y = \alpha + i\beta \) we have
\[
x = (\alpha + i\beta)e^{\alpha + i\beta} = \sqrt{\alpha^2 + \beta^2} e^{\alpha \left( \arctan \frac{\beta}{\alpha} + \beta \right)}.
\]
Since \( x \) is a negative real number we have to require a relation \( \arctan \frac{\beta}{\alpha} + \beta = (2k + 1)\pi \) which yields \( \alpha = -\beta \cot \beta \) where a branch of the cotangent is the origin of branches of \( W \) function. Using this expression for \( \alpha \) we can write \( x = -\frac{\beta}{\sin \beta} e^{-\beta \cot \beta} \). Here the dependence on \( \beta \) is manifestly odd. From the point of view of \( W \) function the change \( \beta \rightarrow -\beta \) is the change of branches \( k \rightarrow 1 - k \). This means the relations
\[
\text{Re} W_k(x) = \text{Re} W_{-1-k}(x), \quad \text{Im} W_k(x) = -\text{Im} W_{-1-k}(x)
\]
hold for \( x < -1/e \). Moreover absolute values of real and imaginary parts of \( W_k(x) \) grow when \( k \) grows for \( k \geq 0 \) or \( k \) decreases for \( k \leq -1 \). The latter is not evident and can be derived from the series expansion of \( W \) function.

For our value of \( \xi^2 \) all roots are complex with non-zero real and imaginary parts. Denoting \( m_k^2 = \alpha_k + i\beta_k \) we have
\[
m_k = r_k + i\nu_k, \quad r_k = \frac{|\beta_k|}{v}, \quad \nu_k = \text{sign}(\beta_k) \frac{1}{2} v, \quad v = \sqrt{-2\alpha_k + 2\sqrt{\alpha_k^2 + \beta_k^2}}
\]
where for all square roots the arithmetic branch is chosen. In this case \( \alpha_k > 0 \). Another root is \( m_k = -(r_k + i\nu_k) \). Using the symmetry \( \alpha_k = \alpha_{-1-k}, \beta_k = -\beta_{-1-k} \) we have for our definition of \( m_k \) that \( m_k = m_{-1-k}^* \). Also for our value of \( \xi^2 \) all \( \alpha_k < 0 \).
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