Howe duality and dichotomy for exceptional theta correspondences

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Abstract We study three exceptional theta correspondences for $p$-adic groups, where one member of the dual pair is the exceptional group $G_2$. We prove the Howe duality conjecture for these dual pairs and a dichotomy theorem, and determine explicitly the theta lifts of all non-cuspidal representations.

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1 Introduction

Let $F$ be a non-archimedean local field of characteristic 0 and residue characteristic $p$. In this paper, we study the local theta correspondence furnished by the following diagram of dual pairs:
where $D$ denotes a cubic division $F$-algebra, so that $PD^\times$ is the unique inner form of PGL$_3$. More precisely, one has the dual pairs

$$
\begin{align*}
\{ & (\text{PGL}_3 \times \mathbb{Z}/2\mathbb{Z}) \times G_2 \subset E_6 \rtimes \mathbb{Z}/2\mathbb{Z}, \\
 & PD^\times \times G_2 \subset E_6^D, \\
 & G_2 \times \text{PGSp}_6 \subset E_7,
\end{align*}
$$

where the exceptional groups of type $E$ are all of adjoint type. In each of the three cases, the centralizer of $G_2$ is a group $H_J = \text{Aut}(J)$, where $J$ is a Freudenthal-Jordan algebra of degree 3. One can thus consider the restriction of the minimal representation $\Pi$ (see [7] or [23]) of $E$ to the relevant dual pair and obtain a local theta correspondence.

More precisely, if $\pi \in \text{Irr}(G_2)$ is an irreducible smooth representation of $G_2$, then the maximal $\pi$-isotypic quotient of $\Pi$

$$
\Pi / \cap_{\phi \in \text{Hom}_{G_2}(\Pi, \pi)} \ker(\phi)
$$

can be expressed as $\pi \otimes \Theta(\pi)$ for some smooth representation $\Theta(\pi)$ of $H_J$ [27, Lemme 2.III.4]. The representation $\Theta(\pi)$ is called the big theta lift of $\pi$, and its maximal semi-simple quotient (cosocle) is denoted $\theta(\pi)$. We say that $\pi$ has nonzero theta lift to $H_J$ if $\Theta(\pi) \neq 0$, or equivalently $\text{Hom}_{G_2}(\Pi, \pi) \neq 0$. Similarly, one can consider the theta lift from $H_J$ to $G_2$ and have the analogous notions.

The first main result of this paper is the following dichotomy theorem (for the statement in the theory of classical theta correspondences, see [17,21] and [39]):

**Theorem 1.1** Let $\pi \in \text{Irr}(G_2(F))$. Then $\pi$ has nonzero theta lift to exactly one of $PD^\times$ or PGSp$_6(F)$.

The group $\text{PGL}_3 \rtimes \mathbb{Z}/2\mathbb{Z}$ is not featured in the dichotomy theorem, but it is needed for some finer aspects of the theta correspondences. For example, every irreducible discrete series representation of $G_2$ lifts to a discrete series representation of precisely one of the three groups. After the above dichotomy theorem, we consider the problem of understanding these theta correspondences more precisely. These local theta correspondences have all been studied to some extent by Maggard and Savin [25], Gross and Savin [5,6,15], Gan [2], Savin [33], and Savin and Weissman [34]. Though various neat results were obtained in the various cases, they fall short of determining the theta correspondences completely. One of the main results of this paper is the completion of the analysis begun in these papers.

The main difficulty in studying these exceptional theta correspondences is that, unlike the classical theta correspondence, one does not know a priori the
analog of the Howe duality conjecture. Namely, one does not know that \( \Theta(\pi) \) has finite length with unique irreducible quotient (that is, \( \theta(\pi) \) is irreducible if \( \Theta(\pi) \) is nonzero). In this paper, we show that the analog of the Howe duality conjecture holds for these dual pairs. To summarize, we have:

**Theorem 1.2** The Howe duality conjecture holds for the three dual pairs considered here. Namely, for \( \pi_1, \pi_2 \in \text{Irr}(G_2(F)) \), \( \Theta(\pi_i) \) has finite length and

\[
\dim \text{Hom}_{HJ}(\theta(\pi_1), \theta(\pi_2)) \leq \dim \text{Hom}_{G_2}(\pi_1, \pi_2).
\]

Likewise, for \( \tau \in \text{Irr}(H_J) \), \( \Theta(\tau) \) has finite length with unique irreducible quotient (if nonzero).

More precisely, we have:

(i) The theta correspondence for \( PD^\times \times G_2 \) defines an injective map

\[
\theta_D : \text{Irr}^\square(PD^\times) \hookrightarrow \text{Irr}(G_2(F)),
\]

where \( \text{Irr}^\square(PD^\times) \subset \text{Irr}(PD^\times) \) is the subset of representations which have nonzero theta lift to \( G_2 \). If \( p \neq 3 \), then \( \text{Irr}^\square(PD^\times) = \text{Irr}(PD^\times) \), so that one has an injective map:

\[
\theta_D : \text{Irr}(PD^\times) \hookrightarrow \text{Irr}(G_2(F))
\]

(ii) The theta correspondence for \( \text{PGL}_3(F) \rtimes \mathbb{Z}/2\mathbb{Z} \times G_2 \) defines an injective map

\[
\theta_B : \text{Irr}^\square(\text{PGL}_3(F) \rtimes \mathbb{Z}/2\mathbb{Z}) \hookrightarrow \text{Irr}(G_2(F)),
\]

where \( \text{Irr}^\square(\text{PGL}_3(F) \rtimes \mathbb{Z}/2\mathbb{Z}) \subset \text{Irr}(\text{PGL}_3(F) \rtimes \mathbb{Z}/2\mathbb{Z}) \) is the subset of representations which have nonzero theta lift to \( G_2 \). Moreover, one can determine the subset \( \text{Irr}^\square(\text{PGL}_3(F) \rtimes \mathbb{Z}/2\mathbb{Z}) \) explicitly, and the image of \( \theta_B \) is disjoint from that of \( \theta_D \) by the dichotomy theorem.

(iii) The theta correspondence for \( G_2 \times \text{PGSp}_6 \) defines an injection

\[
\theta : \text{Irr}(G_2(F)) \setminus \text{Im}(\theta_D) \hookrightarrow \text{Irr}(\text{PGSp}_6(F)).
\]

For an irreducible representation \( \tau \) of \( PD^\times \), the non-vanishing of \( \theta_D(\tau) \) is equivalent to the existence of non-zero vectors in \( \tau \) fixed by a maximal torus in \( PD^\times \). The existence of such vectors has been checked by Lonka and Tandon [24, Thm. 2.4] in the tame case, where \( p \neq 3 \). Thus, if \( p = 3 \), we do not know that all irreducible representations of \( PD^\times \) lift to \( G_2 \) (though one certainly expects this to hold), but the lift is still one-to-one on the subset of those representations that have nonzero lift.
In fact, for the three dual pairs, we determine the theta lift of all non-supercuspidal representations of $G_2$, and the lift of supercuspidal representations whose lift is not supercuspidal. The detailed statements are in the main text, and we simply state the following qualitative result here:

**Theorem 1.3** The three theta correspondences satisfy the following properties:

(i) The correspondences preserve tempered representations.

(ii) Any discrete series representation of $G_2$ lifts to a discrete series representation of precisely one of the three groups.

(iii) The correspondences are functorial for non-tempered representations.

The main motivation for showing the results of this paper is the application to the local Langlands correspondence for the exceptional group $G_2$. A proof of the local Langlands conjecture for $G_2$ is given in our followup work [10].

We would now like to explain the general idea and strategy for proving the Howe duality theorem. We begin with a discussion of the statement:

(a) $\Theta(\pi)$ has finite length.

This finiteness result is fundamental and it was shown by Kudla [20] for the classical theta correspondence. The main tools used are his computation of the Jacquet modules of the Weil representation (relative to maximal parabolic subgroups of the two members of the dual pair) and his exploitation of the doubling see–saw identity. One key consequence of the finite length of $\Theta(\pi)$ is

(b) If $\Theta(\pi) \neq 0$, then it has an irreducible quotient.

For the dual pairs considered in this paper, we will in fact first prove statement (b) and then use it with other inputs to show (a).

Let us elaborate on this slightly subtle point and our strategy of proof. By Bernstein’s decomposition, we may decompose

$$\Theta(\pi) = \Theta(\pi)_c \oplus \Theta(\pi)_{nc}$$

as the sum of its cuspidal part and non-cuspidal part. If $\Theta(\pi)_c$ is nonzero, then it certainly has an irreducible quotient, since it is semisimple. On the other hand, we shall show using Jacquet module computations that

(c) $\Theta(\pi)_{nc}$ has finite length and hence has an irreducible quotient if it is nonzero.

The necessary Jacquet module computations are already available in the literature [25,33] when $H_J = PD^\times$ or $\text{PGL}_3 \rtimes \mathbb{Z}/2\mathbb{Z}$ and are partially available [15,25] for $H_J = \text{PGSp}_6$. In Sect. 13, we complete the remaining Jacquet
module computations. We stress that the material in Sect. 13 is independent of the rest of the paper and could have been discussed earlier in the paper; we have refrained from doing so, as the computations are rather technical. Consequences of the results of Sect. 13 are then discussed in Sect. 14.

In any case, we show statement (b) by showing (c) via Jacquet modules; the proof is written in Sect. 14.2. The statement (b) is used in the proof of the dichotomy theorem (i.e. Theorem 1.1) for tempered representations in Sect. 6. For nontempered representations, the Jacquet module computations (of [15,25] and Sects. 13–14) will tell us everything about their theta lifts.

For the statement (a) (finite length of $\Theta_1(\pi)$), it remains to show that $\Theta_1(\pi)$ is of finite length. We shall show this together with the Howe duality conjecture, by showing that $\Theta_1(\pi)$ is either irreducible or 0. This part of the argument may be considered the analog of the doubling see–saw argument, though one would legitimately question what that means in the setting of exceptional dual pairs.

It will be instructive to first recall the argument for a classical dual pair $\text{Sp}(W) \times \text{O}(V)$, where $W$ is a symplectic space and $V$ a quadratic space (see [26] and [12]). The Howe duality theorem was shown by examining the so-called doubling see–saw diagram:

\[
\begin{array}{ccc}
\text{O}(V^\square) & \rightarrow & \text{Sp}(W) \times \text{Sp}(W) \\
\downarrow & & \downarrow \\
\text{O}(V) \times \text{O}(V) & \rightarrow & \text{Sp}(W)^\Delta
\end{array}
\]

where $V^\square = V + V^-$ is the doubled quadratic space. Starting from $\pi, \pi' \in \text{Irr}(\text{O}(V))$, the resulting see–saw identity gives

\[
\dim \text{Hom}_{\text{Sp}(W)}(\theta(\pi'), \theta(\pi)) \leq \dim \text{Hom}_{\text{O}(V) \times \text{O}(V)}(\Theta(1), \pi' \otimes \pi^\vee).
\]

where $\Theta(1)$ is the big theta lift of the trivial representation of $\text{Sp}(W)^\Delta$ to $\text{O}(V^\square)$. By the local analog of the Siegel-Weil formula, one identifies $\Theta(1)$ with a submodule of a certain degenerate principal series representation $I$ on $\text{O}(V^\square)$. This implies that, for $\pi$ outside a small family of representations,

\[
\dim \text{Hom}_{\text{O}(V) \times \text{O}(V)}(\Theta(1), \pi' \otimes \pi^\vee) \subset \dim \text{Hom}_{\text{O}(V) \times \text{O}(V)}(I, \pi' \otimes \pi^\vee).
\]

Using Mackey theory, one can analyze the latter space and show that, for $\pi$ outside another small family of representations,

\[
\dim \text{Hom}_{\text{O}(V) \times \text{O}(V)}(I, \pi' \otimes \pi^\vee) \leq \dim \text{Hom}_{\text{O}(V)}(\pi', \pi).
\]
Taken together, one obtains the desired inequality
\[ \dim \text{Hom}_{\text{Sp}(W)}(\theta(\pi'), \theta(\pi)) \leq \dim \text{Hom}_O(V)(\pi', \pi) \]
for \( \pi \) outside a small family of representations. For this small family of representations, one needs to do a separate argument.

Now for the exceptional dual pairs \( G \times H \) studied in this paper, there is no analog of the doubling see-saw; this is ultimately tied to the sporadic nature of the geometry underlying exceptional groups. There is thus no direct analog of the above argument. However, the above argument is a particular manifestation of a general principle:

*Theta correspondence typically relates or transfers a period on \( G \) to a period on \( H \).*

More precisely, given a subgroup \( G_1 \subset G \) and \( \pi \in \text{Irr}(G) \), we may consider the space \( \text{Hom}_{G_1}(\pi, \chi) \) for some one-dimensional character \( \chi \) of \( G_1 \). Let us call this Hom space the \( G_1 \)-period for \( \pi \). Now assume that \( \pi \) is a quotient of \( \Theta(\tau) \) for some \( \tau \in \text{Irr}(H) \). Then one typically obtains a statement of the form
\[ \text{\( G_1 \)-period of} \ \pi \subset \text{\( G_1 \)-period of} \ \Theta(\tau) \cong \text{\( H_1 \)-period of} \ \tau^\vee \]
for some subgroup \( H_1 \) of \( H \). Now one can turn the table around. For an irreducible quotient \( \tau \) of \( \Theta(\pi) \), one can consider the \( G_1 \)-period of \( \Theta(\tau^\vee) \), which has \( \pi^\vee \) as an irreducible quotient. One typically gets a statement
\[ \text{\( H_1 \)-period of} \ \tau^\vee \subset \text{\( H_1 \)-period of} \ \Theta(\pi^\vee) \cong \text{\( G_2 \)-period of} \ \pi \]
for some subgroup \( G_2 \) of \( G \).

Iterating this process, one obtains a family of periods relative to subgroups \( G_i \subset G \) and \( H_i \subset H \) such that
\[ \text{\( G_i \)-period of} \ \pi \subset \text{\( G_i \)-period of} \ \Theta(\tau) \cong \text{\( H_i \)-period of} \ \tau^\vee, \]
and
\[ \text{\( H_i \)-period of} \ \tau^\vee \subset \text{\( H_i \)-period of} \ \Theta(\pi^\vee) \cong \text{\( G_{i+1} \)-period of} \ \pi, \]
thus leading to a chain of containment of periods of \( \pi \) and \( \tau \). One may call this a game of ping-pong with periods. Now an (empirical) observation is that the subgroups \( G_i \) and \( H_i \) become more and more reductive (as \( i \) increases) and one ultimately obtains a reductive period. When that happens, the next iteration will result in a see-saw diagram analogous to that in the classical case above and the consideration of an appropriate degenerate principal series representation.
Now the miracle is that a Mackey theory argument with this degenerate principal series representation then returns us the initial $G_1$-period! In other words, for some $i > 1$, one has $G_i = G_1$, and this allows one to complete the chain of containment of periods into a cycle. In particular, if one of these period spaces is finite-dimensional, then this cycle of containment is a cycle of equalities. This is the key step in our proof of the Howe duality theorem for the dual pairs treated here. We shall play this game of ping-pong with periods on two occasions, in Sect. 6 and Sect. 12. This seems to us to be a rather robust method for proving the Howe duality conjecture and should be applicable to other exceptional dual pairs, though the precise details will undoubtedly be different in each case.

We finish the introduction by presenting the key case of this period ping-pong for this paper. Let $G$ be the exceptional group of type $G_2$, and $H = \text{Aut}(J)$, for a Freudenthal-Jordan algebra $J$. The group $G$ has two conjugacy classes of maximal parabolic subgroups, one of which is the Heisenberg parabolic subgroup whose unipotent radical $N$ is a 5-dimensional Heisenberg group. The conjugacy classes of generic characters $\psi_E : N \to \mathbb{C}^\times$ are parameterized by cubic étale algebras $E$ over $F$. For such an $E$, fix an embedding $i : E \to J$ (if it exists). Then we have a see–saw of dual pairs (in $E_6$ or $E_7$)

$$G_E := \text{Spin}_8^E \quad \text{Aut}(J) = H$$

$$G \quad \text{Aut}(i : E \to J) =: H_{J,E}$$

With $\Pi$ the minimal representation, a twisted Jacquet module computation gives $\Pi_{N,\psi_E} \cong \text{ind}^H_{H_{E,J}}(1)$. This implies a chain of containments

$$\text{Hom}_N(\pi, \psi_E) \subset \text{Hom}_N(\Theta(\tau), \psi_E) \cong \text{Hom}_{N \times H}(\Pi, \psi_E \otimes \tau) \cong \text{Hom}_{H_{E,J}}(\tau^\vee, 1)$$

where the first is a natural inclusion, since $\pi$ is a quotient of $\Theta(\tau)$, and the last follows by the twisted Jacquet module computation and Frobenius reciprocity. Now, in order to do the next step, we need to compute $H_{E,J}$-coinvariants of $\Pi_{N,\psi_E}$. This was done in our paper [9], where it was shown that $\Pi_{H_{E,J}}$ is a submodule of a degenerate principal series representation $I_E$ of $\text{Spin}_8^E$. The miracle here is that, as a $G$-module, $I_E$ contains $\text{ind}_N^G(\psi_E)$ as a large $G$-submodule. Thus, the see-saw identity associated to the above see-saw diagram gives the next chain of containments is

$$\text{Hom}_{H_{E,J}}(\tau^\vee, 1) \subset \text{Hom}_{H_{E,J}}(\Theta(\pi^\vee), 1) \cong \text{Hom}_N(\pi, \psi_E),$$
that is, we arrive where we started.

To be honest, just as in the classical case, this last step will hold for representations of \( G \) outside a small family, roughly those that are in the quotient of \( \text{IE} \) by \( \text{ind}_N^G(\psi_E) \). One can characterize this exceptional family precisely, but we prefer not to do it, and simply observe that tempered irreducible representations of \( G \) do not lie in this exceptional family. (As mentioned earlier, the theta lifts of nontempered representations can be explicitly determined using the Jacquet module computations of [15,25] and Sect. 13–14.) Thus, with that caveat in mind, we conclude that

\[
\text{Hom}_N(\pi, \psi_E) \cong \text{Hom}_N(\Theta(\tau), \psi_E)
\]

for all \( E \), and this implies that \( \pi = \Theta(\tau) \). This is the argument which replaces the doubling see-saw argument in classical theta correspondence.

Finally, let us remark that many of the arguments in our paper work over nonarchimedean local fields of characteristic \( p > 0 \) as well, at least when \( p \) is not too small (say \( p \neq 2, 3 \)). However, many of the prior results we rely on were only written in the context of characteristic 0 local fields. An example is the construction of the minimal representation itself. Hence, though our arguments should in principle work for positive characteristic local fields, many details require careful verification.

# 2 The group \( G_2 \)

We begin by introducing the algebraic group \( G_2 \) over \( F \).

## 2.1 Octonion algebra

Let \( \mathbb{O} \) be the split octonion algebra over \( F \). Thus, \( \mathbb{O} \) is an 8-dimensional non-associative and non-commutative \( F \)-algebra. It comes equipped with a conjugation map \( x \mapsto \bar{x} \) with associated norm \( N(x) = x \cdot \overline{x} = \overline{x} \cdot x \) and trace \( \text{Tr}(x) = x + \overline{x} \). Moreover, \( N : \mathbb{O} \rightarrow F \) is a nondegenerate quadratic form.

Every element \( x \) of \( \mathbb{O} \) is a zero of its characteristic polynomial \( t^2 - \text{Tr}(x)t + N(x) \). A nonzero element \( x \in \mathbb{O} \) is said to be of rank 1 if \( N(x) = 0 \). Otherwise it is of rank 2, in which case the subalgebra \( F[x] \) of \( \mathbb{O} \) generated by \( x \) over \( F \) is isomorphic to the separable quadratic \( F \)-algebra \( F[t]/(t^2 - \text{Tr}(x)t + N(x)) \). We denote by \( \mathbb{O}_0 \) the 7-dimensional subspace of trace 0 elements in \( \mathbb{O} \).

## 2.2 Automorphism group

The group \( G_2 \) is the automorphism group of the \( F \)-algebra \( \mathbb{O} \). It is a split simple linear algebraic group of rank 2 which is both simply connected and
adjoint. If we fix a maximal torus $T$ contained in a Borel subgroup $B$, then we obtain a system of simple roots $\{\alpha, \beta\}$ of $G_2$ relative to $(T, B)$, with $\alpha$ short and $\beta$ long. The resulting root system is given by the following diagram.

The highest root is $\beta_0 = 3\alpha + 2\beta$.

### 2.3 Maximal torus

Following Muić, we will fix the isomorphism $T \cong \mathbb{G}_m^2$ by

$$t \mapsto ((2\alpha + \beta)(t), (\alpha + \beta)(t)).$$

Any pair of characters $(\chi_1, \chi_2)$ of $F^\times$ thus define a character $\chi_1 \times \chi_2$ of $T$ by composition with the above isomorphism.

### 2.4 Parabolic subgroups

Up to conjugation, $G_2$ has 2 maximal parabolic subgroups which may be described as follows. Let $V_1 \subset V_2 \subset \mathbb{O}_0$ be subspaces of dimension 1 and 2 respectively on which the octonion multiplication is identically zero. Let $P$ and $Q$ be the stabilizers of $V_2$ and $V_1$ respectively. Then $P = MN$ and $Q = LU$ are the two maximal parabolic subgroups of $G_2$. Moreover, their intersection $B = P \cap Q$ is a Borel subgroup of $G_2$.

The Levi factor $M$ of $P$ is given by

$$M \cong \text{GL}(V_2) \cong \text{GL}_2$$
The isomorphism of $M$ with $\text{GL}_2$ can be fixed so that the modulus character of $M$ is $\delta_P = |\text{det}|^3$. Its unipotent radical $N$ is a 5-dimensional Heisenberg group with 1-dimensional center $Z = U_{\beta_0}$. The action of $M$ on $N/Z$ is isomorphic to $\text{Sym}^3(F^2) \otimes \text{det}^{-1}$. Moreover, the generic $M(F)$-orbits on $N(F)/Z(F)$ is naturally parametrized by the set of isomorphism classes of separable cubic $F$-algebras.

The Levi factor $L$ of $Q$ is given by

$$L \cong \text{GL}(V_3/V_1) \cong \text{GL}_2$$

where

$$V_3 = \{x \in \mathcal{O}_0 : x \cdot y = 0 \text{ for all } y \in V_1\}.$$ 

The isomorphism of $L$ with $\text{GL}_2$ can be fixed so that the modulus character of $L$ is $\delta_Q = |\text{det}|^5$. The unipotent radical $U$ is a 5-dimensional 3-step nilpotent group:

$$U = U_0 \supset U_1 \supset U_2 \supset U_3 = \{1\}$$

such that

$$U_0/U_1 = U_{\alpha} \times U_{\alpha + \beta}, \quad U_1/U_2 \cong U_{2\alpha + \beta} \quad U_2/U_3 = U_{\beta_0} \times U_{\beta_0 - \beta}.$$ 

As representations of $L$, one has

$$U_0/U_1 \cong F^2, \quad U_1/U_2 \cong \text{det}, \quad U_2/U_3 \cong F^2 \otimes \text{det}.$$ 

### 2.5 The subgroup $\text{SL}_3$

The subgroup of $G_2$ generated by the long root subgroups is isomorphic to $\text{SL}_3$. The normaliser of $\text{SL}_3$ in $G_2$ is a semidirect product $\text{SL}_3 \rtimes \mathbb{Z}/2\mathbb{Z}$, with the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ acting on $\text{SL}_3$ as a pinned outer automorphism. The subgroup $\text{SL}_3$ is the pointwise stabilizer of a quadratic subalgebra of $\mathcal{O}$ which is isomorphic to $F \times F$, whereas the setwise stabilizer of such a subalgebra is $\text{SL}_3 \rtimes \mathbb{Z}/2\mathbb{Z}$.

More generally, given a subalgebra of $\mathcal{O}$ which is isomorphic to a quadratic field extension of $F$, the pointwise stabilizer of this subalgebra is isomorphic to the quasi-split special unitary group $\text{SU}_3^K$; the setwise stabilizer of this subalgebra is $\text{SU}_3^K \rtimes \mathbb{Z}/2\mathbb{Z}$. 

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2.6 The dual group

The Langlands dual group of $G_2$ is the complex Lie group $G_2(\mathbb{C})$. In particular, one has the subgroups

$$\text{SO}_3(\mathbb{C}) \subset \text{SL}_3(\mathbb{C}) \subset \text{SL}_3(\mathbb{C}) \rtimes \mathbb{Z}/2\mathbb{Z} \subset G_2(\mathbb{C}).$$

The centralizer of $\text{SL}_3(\mathbb{C})$ in $G_2(\mathbb{C})$ is $\mu_3$, and the centralizer of $\text{SO}_3(\mathbb{C})$ in $G_2(\mathbb{C})$ is $S_3 = \mu_3 \rtimes \mathbb{Z}/2\mathbb{Z}$. Let $\text{SL}_{2,l}(\mathbb{C})$ be a long root $\text{SL}_2$. Then the centralizer of $\text{SL}_{2,l}(\mathbb{C})$ in $G_2(\mathbb{C})$ is $\text{SL}_{2,s}(\mathbb{C})$, a short root $\text{SL}_2$, and vice versa. Thus we also have the subgroup

$$\text{SL}_{2,l}(\mathbb{C}) \times \mu_2 \text{SL}_{2,s}(\mathbb{C}) \cong \text{SO}_4(\mathbb{C}) \subset G_2(\mathbb{C}).$$

2.7 Nilpotent orbits

Recall that the geometric nilpotent orbits (i.e. nilpotent orbits over $\overline{F}$) of a simple group $G$ defined over $F$ form a partially ordered set, where $O_1 \leq O_2$ if $O_1$ is contained in the Zariski closure of $O_2$. Determining the $G(F)$-orbits in the set of $F$-points of each of these orbits is an exercise in Galois cohomology. More precisely, if $\varphi : \text{SL}_2 \to G$ is a map that corresponds to a nilpotent orbit $O$ by the Jacobson-Morozov theorem, then the $G(F)$-orbits of $F$-points in $O$ are parametrized by

$$\text{Ker} \left( H^1(F, C_\varphi) \to H^1(F, G) \right),$$

where $C_\varphi$ is the centralizer of $\varphi(\text{SL}_2)$ in $G$.

For the group $G_2$, the geometric nilpotent orbits form a chain

$$\{0\} \leq O_l \leq O_s \leq O_{sr} \leq O_{reg},$$

where $O_l$ and $O_s$ are orbits of non-zero elements in long and short root spaces, that is $\varphi(\text{SL}_2) = \text{SL}_{2,l}$ and $\varphi(\text{SL}_2) = \text{SL}_{2,s}$ respectively, while $O_{sr}$ is the subregular orbit, with $\varphi(\text{SL}_2) = \text{SO}_3 \subset \text{SL}_3$, and $O_{reg}$ is the regular nilpotent orbit. The centralizers of the respective $\varphi(\text{SL}_2)$ are

$$G_2, \text{SL}_{2,s}, \text{SL}_{2,l}, S_3, \text{ and } 1.$$
3 Representations of $G_2$

In this section, we state some facts for the representations of $G_2(F)$. In particular, we shall describe all non-supercuspidal representations. The results in this section are sourced from Muić [29, Thm. 3.1, Props 4.1, 4.2, 4.3 and 4.4, Thm 5.3] and organized for our purpose.

3.1 Representations of $GL_2$

Since the maximal parabolic subgroups of $G_2$ have $GL_2$ as Levi factors and we will be considering parabolic induction, let us begin by setting up some notations for representations of $GL_2(F)$. If $\chi_1$ and $\chi_2$ are two characters of $F^\times$, then $\chi_1 \times \chi_2$ denotes the parabolically induced representation of $GL_2(F)$ constructed from the character $\chi_1 \otimes \chi_2$ of the diagonal split torus. This induced representation is irreducible unless $\chi_1/\chi_2 = |−|^{±1}$, in which case it is non-semisimple of length 2. In particular, for a character $\chi$ of $F^\times$, one has

$$0 \rightarrow st_\chi \rightarrow \chi |−|^{1/2} \times \chi |−|^{−1/2} \rightarrow \chi \circ \text{det} \rightarrow 0,$$

where $\chi \circ \text{det}$ is a 1-dimensional character of $GL_2(F)$ and $st_\chi$ is a discrete series representation. If $\chi = 1$ is trivial, we will simply write $st_\chi$ as $st$: this is the Steinberg representation. For nontrivial $\chi$, one has $st_\chi \cong st \otimes (\chi \circ \text{det})$ and we call $st_\chi$ a twisted Steinberg representation.

3.2 Principal series representations for $P$

We first consider the principal series representations for the Heisenberg parabolic subgroup $P = MN$, where $M \cong GL_2$. Let $\tau$ be an irreducible representation of $M$ with central character $\omega_\tau$ and set

$$I_P(\tau) = \text{Ind}^{G_2}_{P^\tau} \tau \text{ and } I_P(s, \tau) = \text{Ind}^{G_2}_{P^\tau}(|\det|^s \cdot \tau)$$

if we need to consider a family of induced representations. If $I_P(s, \tau)$ is a standard module, we will denote its unique Langlands quotient by $J_P(s, \tau)$. Now we have:

**Proposition 3.1** (i) If $\tau$ is a unitary supercuspidal representation, then $I_P(s, \tau)$ is reducible if and only if $\tau^{\vee} \cong \tau$ (so $\omega_\tau^2 = 1$) and one of the following holds:

- $\omega_\tau = 1$ and $s = 1/2$, in which case there is a non-split short exact sequence of length 2,

$$0 \rightarrow \delta_P(\tau) \rightarrow I_P(1/2, \tau) \rightarrow J_P(1/2, \tau) \rightarrow 0,$$


where $\delta_P(\tau)$ is a generic discrete series representation.

- $\omega_\tau \neq 1$ and $s = 0$, in which case

$$I_P(\tau) = I_P(\tau)_{\text{gen}} \oplus I_P(\tau)_{\text{deg}}$$

where $I_P(\tau)_{\text{gen}}$ is generic.

(ii) If $\tau = \text{st}_\chi$ is a twisted Steinberg representation, then $I_P(s, \tau)$ is irreducible except for the following cases:

- $\chi = 1$ and $s = \pm 3/2$ or $\pm 1/2$, in which case one has:

$$0 \longrightarrow \text{St}_{G_2} \longrightarrow I_P(3/2, \text{st}) \longrightarrow J_P(3/2, \text{st}) \longrightarrow 0,$$

with $\text{St}_{G_2}$ the Steinberg representation. On the other hand, $I_P(1/2, \text{st})$ has length 3, with a unique irreducible submodule $\pi_{\text{gen}}[1]$ which is a generic discrete series representation, a unique irreducible Langlands quotient $J_P(1/2, \text{st})$ and a subquotient $J_Q(1/2, \text{st})$.

- $\chi^2 = 1$ but $\chi \neq 1$ and $s = \pm 1/2$, in which case one has:

$$0 \longrightarrow \pi_{\text{gen}}[\chi] \longrightarrow I_P(1/2, \text{st}_\chi) \longrightarrow J_P(1/2, \text{st}_\chi) \longrightarrow 0,$$

where $\pi_{\text{gen}}[\chi]$ is a generic discrete series representation.

- $\chi^3 = 1$ but $\chi \neq 1$ and $s = \pm 1/2$, in which case one has:

$$0 \longrightarrow \pi_{\text{gen}}[\chi] \longrightarrow I_P(1/2, \text{st}_\chi) \longrightarrow J_P(1/2, \text{st}_\chi) \longrightarrow 0,$$

where $\pi_{\text{gen}}[\chi] \cong \pi_{\text{gen}}[\chi^{-1}]$ is a generic discrete series representation.

(iii) If $\tau = \chi$ is 1-dimensional unitary, then $I_P(s, \tau)$ is irreducible except in the following cases:

- $\chi = 1$ and $s = \pm 1/2$ or $\pm 3/2$, in which case one has:

$$0 \longrightarrow J_Q(5/2, \text{st}) \longrightarrow I_P(3/2, 1) \longrightarrow 1_{G_2} \longrightarrow 0,$$

whereas $I_P(1/2, 1)$ is of length 3, with a unique irreducible submodule $\pi_{\text{deg}}[1]$ which is a nongeneric discrete series representation, a unique irreducible quotient $J_Q(1, \pi(1, 1))$ and a subquotient $J_Q(1/2, \text{st})$.

- $\chi^2 = 1$ but $\chi \neq 1$ and $s = \pm 1/2$, in which case one has:

$$0 \longrightarrow J_Q(1/2, \text{st}_\chi) \longrightarrow I_P(1/2, \chi) \longrightarrow J_Q(1, \pi(1, \chi)) \longrightarrow 0.$$

- $\chi^3 = 1$ but $\chi \neq 1$ and $s = \pm 1/2$, in which case one has:

$$0 \longrightarrow J_P(1/2, \text{st}_\chi^{-1}) \longrightarrow I_P(1/2, \chi) \longrightarrow J_Q(1, \pi(\chi, \chi^{-1})) \longrightarrow 0.$$
3.3 Principal series representations for $Q$

Now we consider the principal series representations for the 3-step parabolic subgroup $Q = LU$, where $L \cong \text{GL}_2$. Let $\tau$ be an irreducible unitary representation of $L$ with $L$-parameter $\phi_\tau$ and set

$$I_Q(\tau) = \text{Ind}^{G^2}_Q \tau \text{ and } I_Q(s, \tau) = \text{Ind}^{G^2}_Q |\det|^s \cdot \tau$$

if we need to consider a family of induced representations. As before, we let $J_Q(s, \tau)$ denote the unique Langlands quotient of $I_Q(s, \tau)$ if the latter is a standard module. Then we have:

**Proposition 3.2** (i) If $\tau$ is unitary supercuspidal, then $I_Q(s, \tau)$ is reducible if and only if $\tau^\vee \cong \tau$ (so $\omega_\tau^2 = 1$) and one of the following holds:

- $\omega_\tau = 1$ and $s = \pm 1/2$, in which case one has:
  $$0 \longrightarrow \delta_Q(\tau) \longrightarrow I_Q(1/2, \tau) \longrightarrow J_Q(1/2, \tau) \longrightarrow 0,$$
  where $\delta_Q(\tau)$ is a generic discrete series representation.

- $\omega_\tau \neq 1$ (so $\tau$ is dihedral), $\text{Im}(\phi_\tau) = S_3$ (the symmetric group on 3 letters, regarded as a subgroup of $\text{GL}_2(\mathbb{C})$) and $s = \pm 1$, in which case one has:
  $$0 \longrightarrow \pi_{\text{gen}}[\tau] \longrightarrow I_Q(1, \tau) \longrightarrow J_Q(1, \tau) \longrightarrow 0,$$
  where $\pi_{\text{gen}}[\tau]$ is a generic discrete series representation.

- $\omega_\tau \neq 1$, $\text{Im}(\phi_\tau) \neq S_3$ (the symmetric group on 3 letters, regarded as a subgroup of $\text{GL}_2(\mathbb{C})$) and $s = 0$, in which case one has:
  $$I_Q(\tau) = I_Q(\tau)_{\text{gen}} \oplus I_Q(\tau)_{\text{deg}}$$
  where $I_Q(\tau)_{\text{gen}}$ is generic.

(ii) If $\tau = \text{st}_\chi$ is a twisted Steinberg representation, the $I_Q(s, \tau)$ is irreducible except for the following cases:

- $\chi = 1$ and $s = \pm 5/2$ or $\pm 1/2$, in which case one has
  $$0 \longrightarrow \text{St}_{G^2} \longrightarrow I_Q(5/2, \text{st}) \longrightarrow J_Q(5/2, \text{st}) \longrightarrow 0,$$
  and
  $$0 \longrightarrow \pi_{\text{gen}}[1] \oplus \pi_{\text{deg}}[1] \longrightarrow I_Q(1/2, \text{st}) \longrightarrow J_Q(1/2, \text{st}) \longrightarrow 0.$$

Here $\pi_{\text{gen}}[1]$ is the generic discrete series representation already defined in Proposition 3.1(ii) (first bullet point) and $\pi_{\text{deg}}[1]$ is the
nongeneric discrete series representation already defined in Proposition 3.1(iii) (first bullet point).
• $\chi_2 = 1$ but $\chi \neq 1$ and $s = \pm 1/2$, in which case one has:

$$0 \longrightarrow \pi_{\text{gen}}[\chi] \longrightarrow I_Q(1/2, \text{st}_\chi) \longrightarrow J_Q(1/2, \text{st}_\chi) \longrightarrow 0.$$  

Here, $\pi_{\text{gen}}[\chi]$ is the generic discrete series representation defined in Proposition 3.1(ii) (second bullet point).

(iii) If $\tau = \chi$ is 1-dimensional unitary, then $I_Q(s, \tau)$ is irreducible except in the following cases:
• $\chi = 1$ and $s = \pm 1/2$ or $\pm 5/2$, in which case one has:

$$0 \longrightarrow J_P(3/2, \text{st}) \longrightarrow I_Q(5/2, 1) \longrightarrow 1_{G_2} \longrightarrow 0,$$

whereas $I_Q(1/2, 1)$ is of length 3, with unique irreducible submodule $J_Q(1/2, \text{st})$, a unique irreducible quotient $J_Q(1, \pi(1, 1))$ and subquotient $J_P(1/2, \text{st})$.
• $\chi^2 = 1$ but $\chi \neq 1$ and $s = \pm 1/2$, in which case one has:

$$0 \longrightarrow J_P(1/2, \text{st}_\chi) \longrightarrow I_Q(1/2, \chi) \longrightarrow J_Q(1, \pi(1, \chi)) \longrightarrow 0.$$  

3.4 Principal series representations for $B$

We now consider the principal series representations induced from the Borel subgroup $B$. More precisely, suppose that $\pi$ is a Langlands quotient of a standard module

$$I(s_1, s_2, \chi_1, \chi_2) \twoheadrightarrow \pi$$

with

$$s_1 \geq s_2 \geq 0$$

and $\chi_i$ unitary characters of $F^\times$. Here, recall the convention about characters of $T$ which we have fixed in Sect. 2.3. Then

$$\pi \hookrightarrow I(-s_1, -s_2, \chi_1^{-1}, \chi_2^{-1}) = I_P(\pi(\chi_1^{-1} |-s_1, \chi_2^{-1} |-s_2)).$$

Now the representation $\pi(\chi_1^{-1} |-s_1, \chi_2^{-1} |-s_2)$ of $M \cong GL_2$ is reducible if and only if

$$\chi_2/\chi_1 \cdot |-s_2-s_1 = |-^{-1}, \text{ i.e. } \chi_1 = \chi_2 \text{ and } s_1 = s_2 + 1 \geq 1,$$
in which case one has
\[ \pi \leftrightarrow I_{\rho}(-s + \frac{1}{2}, \chi_1^{-1}), \quad \text{with } s \geq 1. \]

There is another, convenient, way to bookkeep the principal series \( \text{Ind}^{G_2}_B(\chi) \). Let \( \beta_1, \beta_2, \beta_3 \) be three long roots such that \( \beta_1 + \beta_2 + \beta_3 = 0 \). This triple is unique up to the action of the Weyl group of \( G_2 \). Then the corresponding co-roots \( \beta_i^\vee : F^\times \to T \) generate \( T \), in particular, the character \( \chi \) defines three characters of \( F^\times \) by \( \chi_i = \chi \circ \beta_i^\vee \) (and is determined by them). Clearly, these characters satisfy \( \chi_1 \cdot \chi_2 \cdot \chi_3 = 1 \).

**Proposition 3.3** The induced representation \( \text{Ind}^{G_2}_B(\chi) \) is irreducible unless one of the following two conditions hold:

- \( \chi_i = | \cdot |^{\pm 1} \) for some \( i \) or \( \chi_i / \chi_j = | \cdot |^{\pm 1} \) for a pair \( i \neq j \).
- The three characters \( \chi_i \) are quadratic, non-trivial and pairwise different.

Then
\[ \text{Ind}^{G_2}_B(\chi) = \text{Ind}^{G_2}_B(\chi)_{\text{gen}} \oplus \text{Ind}^{G_2}_B(\chi)_{\text{deg}}, \]

where \( \text{Ind}^{G_2}_B(\chi)_{\text{gen}} \) is generic.

### 3.5 Conjectural L-packets of \( G_2 \)

The above results allow one to give an enumeration of the non-cuspidal representations of \( G_2 \). Using the desiderata of the conjectural local Langlands correspondence (LLC) for \( G_2 \), we explain how one can assign L-parameters to the noncuspidal representations of \( G_2 \), and hence partition them into L-packets. Recall that an L-parameter of \( G_2 \) is an admissible homomorphism
\[ \varphi : WD_F \longrightarrow G_2^\vee = G_2(\mathbb{C}) \]
of the Weil-Deligne group \( WD_F = W_F \times SL_2(\mathbb{C}) \) to the dual group \( G_2(\mathbb{C}) \), taken up to conjugacy by \( G_2(\mathbb{C}) \). Let
\[ A_\varphi = \pi_0(Z_{G_2}(\varphi)) \]
be the associated component group of \( \varphi \). Then one expects that there should be an L-packet
\[ \Pi_\varphi = \{ \pi(\rho) : \rho \in \hat{A}_\varphi \} \subset \text{Irr}(G_2) \]
associated to each $\varphi$, whose members are indexed by the characters of $A_{\varphi}$, such that

$$\text{Irr}(G_2) = \bigcup_{\varphi} \Pi_{\varphi}.$$  

The non-tempered irreducible representations of $G_2$ are uniquely realized as Langlands quotients of standard modules, so have the form $J_P(\tau), J_Q(\tau)$ or $J_B(\chi)$. The Levi factors of the parabolic subgroups $P, Q$ and $B$ are isomorphic to $\text{GL}_2$ and $\text{GL}_1 \times \text{GL}_1$. Since the LLC for these groups are known, one can assign L-parameters to the nontempered representations. For example, if $\pi = J_P(\tau)$, and $\varphi_{\tau} : WD_F \to M^\vee = \text{GL}_2(\mathbb{C})$ is the L-parameter of $\tau$, then the L-parameter of $\pi = J_P(\tau)$ is the composite

$$\varphi_{\pi} : WD_F \to M^\vee \hookrightarrow G^\vee_2 = G_2(\mathbb{C}).$$

Since the L-packets on the Levi subgroups are singletons, we see also that the nontempered L-packets of $G_2$ are singletons, and $A_{\varphi_{\pi}}$ is correspondingly trivial.

In other words, the non-tempered irreducible representations of $G_2$ are naturally parametrized by the nontempered L-parameters of $G_2$; these are the L-parameters $\varphi$ such that $\varphi(W_F)$ is unbounded. In the following, we will use this partial LLC to describe the effect of the various theta correspondences on nontempered representations.

By the same token, since irreducible tempered representations which are not square-integrable are uniquely realized as summands of principal series representations induced from unitary square-integrable representations of Levi factors, one can attach L-parameters to these tempered (but not square-integrable) representations of $G_2$. The resulting L-parameters $\varphi$ have the property that $\varphi(W_F)$ is bounded but $\varphi(WD_F)$ is contained in a proper Levi subgroup. The size of such a tempered L-packet now depends on the number of irreducible summands in the corresponding parabolically induced representations. From the results recalled in this section, one sees that the size of a tempered L-packet $\Pi_{\varphi}$ is 1 or 2. One can verify that this is the same as the size of $A_{\varphi}$. Moreover, in each tempered L-packet, there is a unique generic representation, and this is assigned to the trivial character of $A_{\varphi}$. Thus, the LLC for tempered non-discrete series representations of $G_2$ is also known, and we may refer to this partial LLC for describing these representations.

Hence, the main issue with the LLC for $G_2$ comes down to the classification of the square-integrable or discrete series representations by discrete series L-parameters; these are the L-parameters $\varphi$ which do not factor through any proper Levi subgroup, or equivalently whose centralizer $C_{\varphi} = Z_{G_2}(\varphi)$ is finite. Guided by the desiderata of the LLC, we can now describe the various
families of discrete series L-parameters, according to \( \varphi(\text{SL}_2) \), and list all non-supercuspidal members.

(1) \( \varphi(\text{SL}_2) \) is the principal \( \text{SL}_2 \). Then \( A_{\varphi} = 1 \) and the packet consists of the Steinberg representation:

\[
\Pi_{\varphi} = \{ \text{St}_{G_2} \}
\]

(2) \( \varphi(\text{SL}_2) = \text{SO}_3 \subseteq \text{SL}_3 \subseteq G_2 \); this is the subregular \( \text{SL}_2 \). The centralizer of \( \text{SO}_3 \) in \( G_2 \) is the finite symmetric group \( S_3 \), so that \( \varphi \) gives by restriction a map \( \phi : W_F \to S_3 \). There are four cases to discuss:

- \( \phi(W_F) = 1 \). Then \( A_{\varphi} = S_3 \). Let \( 1, r, \epsilon \) be the three irreducible representations of \( S_3 \): the trivial, 2-dimensional and the sign character respectively. Then

\[
\Pi_{\varphi} = \{ \pi(1) = \pi_{\text{gen}}[1], \pi(r) = \pi_{\text{deg}}[1], \pi(\epsilon) = \pi_{\text{sc}}[1] \},
\]

where \( \pi_{\text{gen}}[1] \) is defined in Proposition 3.1(ii) (first bullet point) and \( \pi_{\text{deg}}[1] \) is given in Proposition 3.1(iii) (first b.p.). The representation \( \pi(\epsilon) \) is a depth 0 supercuspidal representation induced from a cuspidal unipotent representation of \( G_2(\mathbb{F}_q) \), inflated to a hyperspecial maximal compact group [18]. The cuspidal unipotent representation is denoted in the literature by \( G_2[1] \) and hence our notation \( \pi_{\text{sc}}[1] \).

- \( \phi(W_F) = \mu_2 \). Then, by the local class field theory, \( \phi \) defines a quadratic character \( \chi \) of \( F^\times \). Let \( 1 \) and \( -1 \) denote the trivial and non-trivial characters of \( A_{\varphi} = \mu_2 \). Then

\[
\Pi_{\varphi} = \{ \pi(1) = \pi_{\text{gen}}[\chi], \pi(-1) \},
\]

where \( \pi_{\text{gen}}[\chi] \) is as defined in Proposition 3.1(ii) (second b.p.). If the character \( \chi \) is unramified, then \( \pi(-1) = \pi_{\text{sc}}[-1] \) is a depth 0 supercuspidal representation. It is induced from a cuspidal unipotent representation of \( G_2(\mathbb{F}_q) \), denoted by \( G_2[-1] \), inflated to a hyperspecial maximal compact group.

- \( \phi(W_F) = \mu_3 \). Then, by local class field theory, \( \phi \) defines a cubic character \( \chi \) of \( F^\times \). Let \( 1, \omega \) and \( \omega^2 \) denote the characters of \( A_{\varphi} = \mu_3 \). Then

\[
\Pi_{\varphi} = \{ \pi(1) = \pi_{\text{gen}}[\chi], \pi(\omega), \pi(\omega^2) \},
\]

where \( \pi_{\text{gen}}[\chi] \) is as defined in Proposition 3.1(ii) (third b.p.). If the character \( \chi \) is unramified, then \( \pi(\omega) = \pi_{\text{sc}}[\omega] \) and \( \pi(\omega^2) = \pi_{\text{sc}}[\omega^2] \) are induced from a cuspidal unipotent representations of \( G_2(\mathbb{F}_q) \), denoted
by $G_2[\omega]$ and $G_2[\omega^2]$, inflated to a hyperspecial maximal compact group.

- $\phi(W_F) = S_3$. Then $r \circ \phi$ corresponds to a supercuspidal representation $\tau$ of $GL_2$ (where we recall that $r$ denotes the two-dimensional irreducible representation of $S_3$). In this case $A_\varphi$ is trivial and

$$\Pi_\varphi = \{\pi_{\text{gen}}[\tau]\},$$

where $\pi_{\text{gen}}[\tau]$ is as defined in Proposition 3.2(i) (second b.p.).

3. $\varphi(SL_2) = SL_{2,s}$, a short root $SL_2$. The centralizer of $SL_{2,s}$ in $G_2$ is $SL_{2,l}$, a long root $SL_2$. Then $\varphi$ gives, by restriction, a map from the Weil group $\phi : W_F \to SL_{2,l}$, that corresponds to supercuspidal representation $\tau$ of $GL_2$ with the trivial central character (and hence $\tau \cong \tau^\vee$). In this case $A_\varphi = \mu_2$, and

$$\Pi_\varphi = \{\pi(1) = \delta_P(\tau), \pi(-1)\},$$

where $\delta_P(\tau)$ is as defined in Proposition 3.1(i) (first b.p.) and $\pi(-1)$ is supercuspidal.

4. $\varphi(SL_2) = SL_{2,l}$, a long root $SL_2$. The centralizer of $SL_{2,l}$ in $G_2$ is $SL_{2,s}$, a short root $SL_2$. Then $\varphi$ gives, by restriction, a map from the Weil group $\phi : W_F \to SL_{2,s}$, that corresponds to supercuspidal representation $\tau$ of $GL_2$ with the trivial central character (and hence $\tau \cong \tau^\vee$). In this case $A_\varphi = \mu_2$, and

$$\Pi_\varphi = \{\pi(1) = \delta_Q(\tau), \pi(-1)\},$$

where $\delta_Q(\tau)$ is as defined in Proposition 3.2(i) (first b.p.) and $\pi(-1)$ is supercuspidal.

5. $\varphi(SL_2) = 1$. Then $\varphi : W_F \to G_2(\mathbb{C})$ gives rise to an L-packet consisting entirely of supercuspidal representations of $G_2$.

There has been some work towards the above conjectural LLC for $G_2$, most notably [16,34]. At the moment, we simply wish to point out that all the noncuspidal discrete series representations are fully accounted for by the above classification scheme.

### 3.6 Local Fourier coefficients

It will be useful to consider the twisted Jacquet modules of a representation $\pi$ of $G_2$ along the unipotent radical $N$ of $P$. The $M$-orbits of 1-dimensional characters of $N$ are naturally indexed by cubic $F$-algebras, with the generic orbits corresponding to étale cubic $F$-algebras. For any such cubic $F$-algebra
$E$, we shall write $\psi_E$ for a character of $N$ in the corresponding $M$-orbit. Then one may consider $\pi_{N,\psi_E}$. In particular, we note:

**Proposition 3.4** For any irreducible, infinite dimensional representation $\pi$ of $G_2$, there exists an étale cubic $F$-algebra $E$ such that $\pi_{N,\psi_E} \neq 0$. Moreover, if $\pi$ is degenerate, then $\pi_{N,\psi_E}$ is finite-dimensional for any étale $E$.

**Proof** Wave front sets of irreducible representations of $G_2$ are supported on special orbits, that is, $\{0\}$, $O_{sr}$ and $O_{reg}$, see [19,23]. Thus, if $\pi$ is degenerate (not Whittaker generic), and not the trivial representation, its wave-front set is supported on subregular nilpotent orbits. If $O_E$ is in the wave-front set of $\pi$ then $\pi_{N,\psi_E}$ is non-zero and finite-dimensional, by the main result of [28,41].

Assume now that $\pi$ is generic. The restriction of a Whittaker character to $U$ is a character $\psi_\alpha$ supported on the simple root space $U_\alpha$. Hence $\pi_{U,\psi_\alpha} \neq 0$. Let $N'$ be obtained by adding $U_\beta$ to $U$ and removing $U_\alpha+\beta$, so that $N'$ is conjugate to $N$ (by the simple Weyl reflection $w_\beta$). Abusing notation, let $\psi_\alpha$ be the character of $N'$ supported on the simple root space $U_\alpha \subset N'$. Now we claim that there is an isomorphism (a root exchange)

\[ \text{Hom}_{U}(\pi, \psi_\alpha) \cong \text{Hom}_{N'}(\pi, \psi_\alpha), \]

which sends $\ell$ on the LHS to an element $\ell'$ on the RHS defined by the convergent integral

\[ \ell'(v) = \int_{U_\beta} \ell(\pi(u) \cdot v) \, du. \]

Conversely, we can recover $\ell$ from $\ell'$ by integrating over $U_\alpha+\beta$.

Assuming the claim, let $U'$ be the conjugate of $U$ by $w_\alpha+\beta$. Let $Z'$ be two-dimensional center of $U'$. Observe that $[U', U']/Z' \cong U_\alpha$. Since $[U', U'] \subset N'$, it follows that $\pi_{[U', U'], \psi_\alpha} \neq 0$. But this means that the Fourier–Jacobi functor of $\pi$ with respect to the 3-step unipotent $U'$ is non-trivial, and $\pi_{N,\psi_E} \neq 0$ for some $E = F + K$, by [19, Proposition 6.1].

To justify the root exchange argument in the claim, we observe that $U_\beta$ and $U_\alpha+\beta$ generate a Heisenberg group with center $U_\alpha$, modulo higher order commutators. More precisely, consider the group

\[ V' = U \cdot U_\beta = N' \cdot U_\alpha+\beta, \]

which is a maximal unipotent subgroup of $G_2$ and hence conjugate to $V$ (by the simple reflection $w_\beta$). If we consider the lower central series of the unipotent group $V'$:

\[ V' \supset [V', V'] = V'_1 \supset V'_2 = [V', V'_1] \supset V'_3 \supset \{1\}, \]
then \( V/ V'_2 \) is the Heisenberg group in question with center \( V'_1/ V'_2 = U_a \). Note moreover that the elements \( \ell \) and \( \ell' \) in the two Hom spaces in the claim both factors through \( \pi_{V'_2} \) (which is a module for the Heisenberg group \( V'/ V'_2 \)). With this observation, the justification of the claim is given by the following lemma, included as a convenience to the reader.

**Lemma 3.5** Let \( H \) be a Heisenberg group. Let \( \pi \) be a smooth \( H \)-module. Let \( X \) and \( Y \) be two maximal abelian subgroups of \( H \). Let \( \psi_X \) and \( \psi_Y \) be characters of \( X \) and \( Y \), agreeing on the intersection \( X \cap Y \), and non-trivial on the center of \( H \). Then we have an isomorphism \( \text{Hom}_X(\pi, \psi_X) \cong \text{Hom}_Y(\pi, \psi_Y) \), \( \ell \mapsto \ell' \), defined by

\[
\ell'(v) = \int_{Y/ X \cap Y} \ell(\pi(y)v) \, dy.
\]

**Proof** By the Frobenius reciprocity, we have

\[
\text{Hom}_X(\pi, \psi_X) \cong \text{Hom}_H(\pi, \text{Ind}^H_X \psi_X)
\]

and

\[
\text{Hom}_Y(\pi, \psi_Y) \cong \text{Hom}_H(\pi, \text{Ind}^H_Y \psi_Y).
\]

We also have an isomorphism \( \text{Ind}^H_X \psi_X \cong \text{Ind}^H_Y \psi_Y \), where every \( f \in \text{Ind}^H_X \psi_X \) goes to \( f' \in \text{Ind}^H_Y \psi_Y \) defined by

\[
f'(h) = \int_{Y/ X \cap Y} f(yh) \overline{\psi_Y(y)} \, dy.
\]

This integral is convergent, in fact, it is a finite sum. The lemma follows by combining this isomorphism with the two Frobenius reciprocity isomorphisms.

\(\square\)

### 4 Exceptional dual pairs

In this section, we briefly describe the dual pairs which intervene in this paper and some structural results which will be important in the study of the associated theta correspondences.

If \( A \) is an associative algebra over \( F \), then \( A^+ \) will denote the underlying Jordan algebra, that is, \( A \) with the Jordan multiplication \( a \circ b = \frac{1}{2}(ab + ba) \).
4.1 The group $M_J$

Let $J$ be a Freudenthal–Jordan $F$-algebra [22, §37 and §38]. The algebra $J$ comes equipped with a cubic norm form $N_J$, and we let

$$M_J = \{ g \in \text{GL}(J) : N_J \circ g = N_J \}.$$  

It contains the automorphism group $\text{Aut}(J)$ as a subgroup. Now we consider the $F$-vector space $g_J = \mathfrak{sl}_3 \oplus \text{Lie}(M_J) \oplus (F^3 \otimes J) \oplus (F^3 \otimes J)^*$

Then $g_J$ can be given the structure of a simple exceptional Lie algebra (see, for example, [7]). We have the following cases of interest:

| $\dim J$ | $g_J$ |
|----------|-------|
| 1        | $G_2$ |
| 3        | $D_4$ |
| 9        | $E_6$ |
| 15       | $E_7$ |

We observe:

- If $\dim J = 3$, then $J$ is a cubic étale $F$-algebra $E$.
- If $\dim J = 9$, then $J$ corresponds to a pair $(B_K, \iota)$ where $B_K$ is a central simple algebra over an étale quadratic $F$-algebra $K$ and $\iota$ is an involution of the second kind. Thus, $J = B_K^\iota$ is the subspace of $\iota$-symmetric elements. If $K = F^2$, then $J = B \oplus B$ for a central simple algebra $B$ over $F$, $\iota$ permutes two summands, and $J = B^\perp$. The split version is when $B = \mathbb{M}_3$, the algebra of $3 \times 3$ matrices, and $\text{Aut}(\mathbb{M}_3^+) = \text{PGL}_3 \rtimes \mathbb{Z}/2\mathbb{Z}$.
- If $\dim J = 15$, then $J$ is $H_3(B_F)$ is the space of all $3 \times 3$ hermitian-symmetric matrices, where $B_F$ is a quaternion algebra over $F$. The split version is when $B_F = \mathbb{M}_2$.

Let $G_J$ be the identity component of $\text{Aut}(g_J)$. If $\dim J = 9$ then

$$1 \longrightarrow G_J \longrightarrow \text{Aut}(g_J) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1.$$

This short exact sequence may not be split in general.

4.2 Dual pair $G_2 \times \text{Aut}(J)$

We can now describe some dual pairs in $G_J$ or rather in $\text{Aut}(g_J)$. It will be easier to do this on the level of Lie algebras.

The centralizer of $\text{Aut}(J)$ in $g_J$ is

$$\mathfrak{sl}_3 \oplus F^3 \otimes 1_J \oplus (F^3 \otimes 1_J)^*$$
which one recognizes to be \( \mathfrak{g}_F \) (i.e. taking \( J = F \)). Thus this is a Lie subalgebra of type \( G_2 \), and we have a dual pair

\[
G_2 \times \text{Aut}(J) \subset \text{Aut}(G_J).
\]

If \( \dim J = 9 \), we recall that \( \text{Aut}(J) \) sits in a short exact sequence

\[
1 \longrightarrow \text{Aut}(J)^0 \longrightarrow \text{Aut}(J) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1.
\]

If \( J \) is associated to a pair \((B_K, \tau)\), then \( \text{Aut}(J)^0 = PGU(B_K, \tau) \) is an adjoint group of type \( A_2 \).

### 4.3 Dual pair \( \text{Aut}(i : E \to J) \times G_E \)

Now we fix an embedding \( i : E \longrightarrow J \), where \( E \) is a cubic étale \( F \)-algebra. We have the subgroup

\[
\text{Aut}(i : E \to J) \subset \text{Aut}(J).
\]

If \( \dim J = 9 \), a detailed description of this group is in [8]. Its identity component is a 2-dimensional torus. The centralizer of \( \text{Aut}(i : E \to J) \) in \( \mathfrak{g}_J \) contains

\[
\mathfrak{g}_E = \mathfrak{sl}_3 \oplus \mathfrak{t}_E \oplus F^3 \otimes E \oplus (F^3 \otimes E)^*, \]

where \( E \hookrightarrow J \) via \( i \) and \( \mathfrak{t}_E \cong E^0 \) is the toral Lie subalgebra of trace 0 elements in \( E \). This Lie algebra is isomorphic to \( \text{Lie}(G_E) \) (where \( G_E \) is the simply connected quasi-split group \( \text{Spin}^E_{16} \)), and we have the dual pair

\[
\text{Aut}(i : E \to J) \times G_E \longrightarrow \text{Aut}(G_J).
\]

Note that this map is not injective.

### 4.4 A see–saw diagram

The two dual pairs we described above fit together into a see–saw diagram:

\[
\begin{align*}
G_E := \text{Spin}^E_{16} & \quad \text{Aut}(J) =: H_J \\
G_2 & \quad \text{Aut}(i : E \to J) =: H_{J,E}
\end{align*}
\]
in $\text{Aut}(G_J)$. The various $J$’s of interest in this paper, and the corresponding groups $H_J = \text{Aut}(J)$ and $H_{J,E} = \text{Aut}(i : E \to J)$ are given in the table below.

| $J$ | $D^+$ | $\mathbb{M}_3^+$ | $H_3(\mathbb{M}_2)$ |
|-----|-------|----------------|-------------------|
| $H_J$ | $PD^\times$ | $\text{PGL}_3 \rtimes \mathbb{Z}/2\mathbb{Z}$ | $\text{PGSp}_6$ |
| $H_{J,E}$ | $PE^\times$ | $PE^\times \rtimes \mathbb{Z}/2\mathbb{Z}$ | $\text{Res}_{E/F} \text{SL}_2 / \mu_2$ |

Here, note that $D^+$ denotes the Jordan algebra associated to a cubic division $F$-algebra $D$, in which case $E$ is necessarily a field.

5 The see–saw argument

In this section, we shall consider the see–saw identity arising from the see-saw diagram (4.1) and pursue some of its consequences.

5.1 See–saw identity

Suppose that $\pi \in \text{Irr}(G_2)$. Then we have the see–saw identity associated with the see-saw (4.1):

$$\text{Hom}_{H_{J,E}}(\Theta(\pi), \mathbb{C}) \cong \text{Hom}_{G_2}(R_J(E), \pi) \quad (5.1)$$

where

$$R_J(E) := \Theta(1)$$

is the big theta lift of the trivial representation of $H_{J,E}$. To make use of this see–saw identity, we need to understand the representation $R_J(E)$ of $\text{Spin}^E_{8}$. This has been studied in [9] and we recall the relevant results there.

5.2 Degenerate principal series of $\text{Spin}^E_{8}$

Let $P_E = M_E \cdot N_E \subset \text{Spin}^E_{8}$ be the Heisenberg parabolic subgroup, so that its Levi factor is

$$M_E \cong \text{GL}_2(E)^{\text{det}} = \{ g \in \text{GL}_2(E) : \det(g) \in F^\times \}.$$ 

Then the determinant map defines an algebraic character $M_E \to \mathbb{G}_m$ which is a basis element of $\text{Hom}(M_E, \mathbb{G}_m)$. We may consider the degenerate principal series representation

$$I_E(s) = \text{Ind}_{P_E}^{\text{Spin}^E_{8}} | \det |^s.$$
In [36] and [9, Cor. 12.11, Thm. 17.6, Thm 18.1, Prop. 18.5 and Prop. 18.6], the module structure of this family of degenerate principal series representations has been determined. In particular, we have:

**Proposition 5.2**

$$R_J(E) \hookrightarrow I_E(s_J),$$

where

$$s_J = \begin{cases} -1/2, & \text{if } J = D^+ \text{ or } M_3^+; \\ 1/2, & \text{if } J = H_3(M_2). \end{cases}$$

The representation $I_E(1/2)$ has length 3 when $E$ is a field and has length 2 otherwise. More precisely, it has a unique irreducible submodule $V$ with quotient isomorphic to $R_{M_3}(E) \oplus R_D(E)$ (where $R_D(E)$ is interpreted to be 0 when $E$ is not a field). Indeed, one has the short exact sequence:

$$0 \longrightarrow R_{H_3(M_2)}(E) \longrightarrow I_E(1/2) \longrightarrow R_D(E) \longrightarrow 0.$$ 

and

$$0 \longrightarrow V \longrightarrow R_{H_3(M_2)}(E) \longrightarrow R_{M_3}(E) \longrightarrow 0.$$ 

In particular, when $E$ is not a field, $I_E(1/2) = R_{H_3(M_2)}(E)$.

As a consequence of the above discussion, we see that it is useful to understand the Hom space

$$\text{Hom}_{G_2}(I_E(s), \pi) \quad \text{for } \pi \in \text{Irr}(G_2).$$

We shall study this in two ways.

**5.3 Vanishing of an Ext$^1$**

In view of the proposition, we see that there is an exact sequence

$$0 \longrightarrow \text{Hom}_{G_2}(R_D(E), \pi) \longrightarrow \text{Hom}_{G_2}(I_E(1/2), \pi) \longrightarrow \text{Hom}_{G_2}(R_{H_3(M_2)}(E), \pi) \quad \text{Ext}^1_{G_2}(R_D(E), \pi).$$

Now we have the following useful fact:
**Proposition 5.3** Assume that $E$ is a field. If $\pi \in \text{Irr}(G_2)$ is tempered or has cuspidal support different from $\pi_{\text{deg}[1]}$, then

$$\text{Ext}^1_{G_2}(R_D(E), \pi) = 0,$$

so that one has a short exact sequence

$$0 \rightarrow \text{Hom}_{G_2}(R_D(E), \pi) \rightarrow \text{Hom}_{G_2}(I_E(1/2), \pi) \rightarrow \text{Hom}_{G_2}(R_{H(M)}(E), \pi) \rightarrow 0.$$

**Proof** One needs to understand $R_D(E)$ as a representation of $G_2$, and this is essentially done in [33, Conj. 4.1 and §6], where the dual pair correspondence for $PD \times G_2$ was studied. We shall recall the results of [33] in greater detail later on. At this point, we simply note that as a representation of $G_2$, $R_D(E)$ is the direct sum of a supercuspidal representation (of infinite length) and the irreducible discrete series representation $\pi_{\text{deg}[1]}$, which is a constituent of $I_Q(1/2, \text{st})$. From this, the vanishing of $\text{Ext}^1_{G_2}(R_D(E), \pi)$ for those $\pi$ with different cuspidal support from $\pi_{\text{deg}[1]}$ follows immediately. On the other hand, if $\pi$ is tempered, then one also has $\text{Ext}^1(\pi_{\text{deg}[1]}, \pi) = 0$ since discrete series representations are projective in the category of tempered representations. $\Box$

**5.4 $I_E(s)$ as $G_2$-module**

On the other hand, we may understand the restriction of $I_E(s)$ to $G_2$ using Mackey theory. The following is a key technical result:

**Proposition 5.4** As a representation of $G_2$, $I_E(s)$ admits an equivariant filtration

$$0 \subset I_0 \subset I_1 \subset I_2 \subset I_3 \subset I_4$$

with successive quotients described as follows:

- $I_0 \cong \text{ind}^{G_2}_N \tilde{\psi}_E$;
- $J_1 := I_1/I_0 \cong I_P(\frac{s}{2} + \frac{1}{4}, C_c^\infty(\text{PGL}_2))$.
- $J_2 := I_2/I_1 \cong m_E \cdot I_P(\frac{s}{2} + \frac{1}{4}, \text{ind}^{\text{PGL}_2}_N \psi)$
- $J_3 := I_3/I_2 \cong m_E \cdot I_Q(s + 1)$.
- $J_4 := I_4/I_3 \cong I_P(s + 1)$.

Here,

$$m_E = \begin{cases} 
3, & \text{if } E = F^3; \\
1, & \text{if } E = F \times K; \\
0, & \text{if } E \text{ is a field.} 
\end{cases}$$
The proposition implies that one has a short exact sequence
\[ 0 \longrightarrow \text{ind}^G_N \tilde{\psi}_E \longrightarrow I_E(s) \longrightarrow \Sigma_E(s) \longrightarrow 0, \]
from which one deduces an exact sequence:
\[ 0 \longrightarrow \text{Hom}_G(\Sigma_E(s), \pi) \longrightarrow \text{Hom}_G(I_E(s), \pi) \longrightarrow \text{Hom}_N(\pi^\vee, \psi_E) \longrightarrow \text{Ext}^1_G(\Sigma_E(s), \pi). \]

We now specialize to \( s = 1/2 \), where we need to be more precise.

**Proposition 5.5** Suppose that \( \pi \in \text{Irr}(G_2) \) is tempered or has cuspidal support along \( Q \). Then
\[ \text{Hom}_G(\Sigma_E(1/2), \pi) \cong \text{Hom}_N(\pi^\vee, \psi_E). \]

**Proof** We need to prove
\[ \text{Hom}_G(\Sigma_E(1/2), \pi) = 0 = \text{Ext}^1_G(\Sigma_E(1/2), \pi). \]
To that end, it suffices to prove the following lemma:

**Lemma 5.6** For all \( i \) and \( j \), \( \text{Ext}^i_G(J_j, \pi) = 0 \), with \( \pi \) as in Proposition 5.5.

**Proof** Consider \( J_1 \) firstly. By the Frobenius reciprocity,
\[ \text{Ext}^i_G(J_1, \pi) = \text{Ext}^i_M(|\det|^{1/2} \cdot C_c^\infty(\text{PGL}_2), r_{\mathcal{F}}(\pi)). \]
Since \( \pi \) is tempered, the center of \( M = \text{GL}_2 \) acts on \( R_{\mathcal{F}}(\pi) \) by characters \( \chi \)
such that \( |\chi(z)| = |z|^t \) where \( t \leq 0 \). On the other hand, the center of \( M \) acts on \( |\det|^{1/2} \cdot C_c^\infty(\text{PGL}_2) \) by \( |z| \). Thus the right hand side is 0. The other cases are dealt with in the same way.

This completes the proof of Proposition 5.5.

\[ \square \]

6 Dichotomy

The goal of this section is to prove the following theorem:

**Theorem 6.1** For any representation \( \pi \in \text{Irr}(G_2) \), \( \pi \) has nonzero theta lift to exactly one of \( PD^x \) or PGSp6.

To prove this dichotomy theorem, we need some preliminary results which are consequences of the computation of the Jacquet modules of the minimal representation \( \Pi_J \) with respect to the various maximal parabolic subgroups of \( G_2 \) and \( H_J \). The required Jacquet module computations were carried out in

\[ \square \]
[33, Prop. 5.1] when \( H_J = PD^\times \) and in [25, Thm. 4.3 and Thm. 7.6] when \( H_J = \text{PGL}_3 \) (see also [6, Prop. 4 and Prop. 6]). For \( H_J = \text{PGSp}_6 \), the Jacquet module computations for some parabolic subgroups were carried out in [25, Thm. 5.3 and Thm 7.6]. The remaining ones will be done in Sect. 13 and some implications of these computations are discussed in Sect. 14. We note that Sect. 13 is a self-contained section independent of the rest of this paper. Hence, we first record some results from Sect. 13-14 and the earlier references [6,25,33] that we will use.

6.1 Consequences of Jacquet module computations

We first note:

**Lemma 6.2** Consider the theta correspondence for \( G_2 \times H_J \) for the 3 cases of \( J \).

(i) Let \( \pi \in \text{Irr}(G_2) \) and write

\[
\Theta_J(\pi) = \Theta_J(\pi)_c \oplus \Theta_J(\pi)_{nc}
\]

as a sum of its cuspidal and noncuspidal components. Then \( \Theta_J(\pi)_{nc} \) has finite length. In particular, if \( \Theta_J(\pi) \neq 0 \), then it has an irreducible quotient.

(ii) Likewise, let \( \tau \in \text{Irr}(H_J) \) and write

\[
\Theta_J(\tau) = \Theta_J(\tau)_c \oplus \Theta_J(\tau)_{nc}.
\]

Then \( \Theta_J(\tau)_{nc} \) has finite length. In particular, if \( \Theta_J(\tau) \neq 0 \), then it has an irreducible quotient.

**Proof** We consider the 3 cases of \( H_J \) in turn:

- The case of \( J = H_3(\mathbb{M}_2) \) is shown in Sect. 14, based on the results of Sect. 13. As we remarked above, the results of Sects. 13 and 14 are independent of the rest of the paper.
- The case of \( J = D^+ \) follows from results of [33, §6], proving [33, Conjecture 4.1(3)].
- For \( J = \mathbb{M}_3^+ \), (ii) follows from [6, Prop. 7, Cor. 9(i), first paragraph of proof of Thm. 14 and last paragraph of §9]. The proof of (i) is analogous to that for the case \( J = H_3(\mathbb{M}_2) \), which we describe in Sect. 14.2, and uses the Jacquet module computation for \( \text{PGL}_3 \) given in [6, Prop. 4] and Proposition 14.2. \( \square \)

In fact, the Jacquet module computations allow one to determine the theta lift of non-tempered representations explicitly (see Theorem 14.1). We simply note the following here:
Lemma 6.3 (i) Let $\pi \in \text{Irr}(G_2)$ and $\tau \in \text{Irr}(H_J)$ be such that $\pi \otimes \tau$ is a quotient of $\Pi_J$. Then

$$\pi \text{ is tempered } \iff \tau \text{ is tempered}.$$ (ii) Let $\pi \in \text{Irr}(G_2)$ be non-tempered. Then $\pi$ has nonzero theta lifting to $\text{PGSp}_6$.

Proof For $H_J = PD^\times$ or $\text{PGL}_3 \rtimes \mathbb{Z}/2\mathbb{Z}$, the desired results have been verified in [33, §6] and [6, Cor. 9(i) and proof of Prop. 10] respectively. For the case when $H_J = \text{PGSp}_6$, this is shown in Theorem 14.1 in Sect. 14. □

6.2 Reduction to non-generic tempered case

With the above inputs in place, we can now begin the proof of the dichotomy theorem. We note:

- The dichotomy theorem holds for nontempered $\pi$. Indeed, if $\pi$ is non-tempered, then Lemma 6.3(ii) says that $\pi$ has nonzero theta lift to $\text{PGSp}_6$, whereas [33] shows that $\pi$ has zero theta lift to $PD^\times$.
- The dichotomy theorem holds for generic $\pi$. Indeed, it was shown in [6, Cor. 20] that a generic $\pi$ has nonzero theta lift to $\text{PGSp}_6$ (see also Cor. 11.2 below) and it was shown in [33] that $\pi$ has zero theta lift to $PD^\times$.

Thus, to prove the dichotomy theorem, it remains to deal with non-generic tempered $\pi$.

6.3 Weak dichotomy

We first prove that a non-generic tempered $\pi$ has nonzero theta lift to one of $PD^\times$ or $\text{PGSp}_6$. Since $\pi$ is non-generic and infinite-dimensional, there exists an étale cubic $F$-algebra $E$ such that $\text{Hom}_N(\pi^\vee, \psi_E) \neq 0$. By Proposition 5.5, we have an isomorphism

$$\text{Hom}_{G_2}(I_E(1/2), \pi) \cong \text{Hom}_N(\pi^\vee, \psi_E) \neq 0.$$ This implies, by Proposition 5.3, that

$$\text{Hom}_{G_2}(R_D(E), \pi) \neq 0 \text{ or } \text{Hom}_{G_2}(R_{H_3(M_2)}(E), \pi) \neq 0.$$ By the see–saw identity (5.1), we deduce that

$$\text{Hom}_{H_E}(\Theta_J(\pi), \mathbb{C}) \neq 0.$$
for $J = D^+$ or $H_3(M_2)$. In particular, $\Theta_J(\pi) \neq 0$ for $J = D^+$ or $H_3(M_2)$. We have thus verified that $\pi$ has nonzero theta lift to at least one of $PD^\times$ or $\text{PGSp}_6$.

6.4 Curious chain of containments

It remains to show that a nongeneric tempered $\pi$ cannot lift to both $PD^\times$ and $\text{PGSp}_6$. Let $\bar{\pi}$ be the complex conjugate of $\pi$. If $\pi$ is unitarizable (e.g. if $\pi$ is tempered), then $\bar{\pi} \cong \pi^\vee$. Thus

$$\pi_{\psi_E} \cong \bar{\pi}_{\psi_E} \cong (\pi^\vee)_{\psi_E},$$

where, in the second isomorphism, we assume that $\pi$ is unitarizable. Since the minimal representation $\Pi_J$ used in this paper is defined over $\mathbb{R}$, we have a canonical isomorphism $\bar{\Pi}_J \cong \Pi_J$. It follows that $\Theta_J(\bar{\pi})$ is the complex conjugate of $\Theta_J(\pi)$.

We shall make use of the curious chain of containment given in the following lemma; this is the first instance of the game of ping-pong with periods discussed at the end of the introduction.

**Lemma 6.4** Let $\pi \in \text{Irr}(G_2)$ be tempered. For $J = D^+$, $M_3^+$ or $H_3(M_2)$, let $\tau \in \text{Irr}(H_J)$ be tempered and such that

$$\text{Hom}_{G_2 \times H_J}(\Pi_J, \pi \boxtimes \tau) \neq 0.$$

Then we have the following natural inclusions

$$\text{Hom}_{N}(\pi, \psi_E) \subseteq \text{Hom}_{N}(\Theta(\tau), \psi_E) \cong \text{Hom}_{H_J,E}(\tau^\vee, \mathbb{C})$$

$$\subseteq \text{Hom}_{H_J,E}(\Theta(\tau^\vee), \mathbb{C}) \cong \text{Hom}_{G_2}(R_J(E), \tau^\vee).$$

If one of these spaces is finite-dimensional, then all inclusions are isomorphisms.

**Proof** The first inclusion arises from $\Theta(\tau) \rightarrow \pi$. The second follows from

$$\text{Hom}_{N}(\Theta(\tau), \psi_E) \cong \text{Hom}_{H_J}((\Pi_J)_N, \psi_E, \tau)$$

combined with (see [15, Lemma 2.9, Pg 213])

$$(\Pi_J)_N, \psi_E \cong \text{ind}_{H_J,E}^1(1)$$

and the Frobenius reciprocity. For the third, observe that $\Theta_J(\bar{\pi})$ is the complex conjugate of $\Theta_J(\pi)$. Since $\bar{\pi} \cong \pi^\vee$ and $\bar{\tau} \cong \tau^\vee$ and we have $\Theta(\pi^\vee) \rightarrow \tau^\vee$. The fourth is the see–saw identity (5.1).
If any of the spaces is finite-dimensional, then $\text{Hom}_N(\pi, \psi_E)$ is finite-dimensional. If this space is finite dimensional then, since $\pi$ is tempered, by Propositions 5.3 and 5.5, one has

$$\dim \text{Hom}_{G_2}(R_J(E), \pi^\vee) \leq \dim \text{Hom}_{G_2}(I_E(1/2), \pi^\vee) = \dim \text{Hom}_N(\pi, \psi_E).$$

(6.5)

It follows that all spaces have the same dimension and the lemma is proved. \hfill \Box

6.5 Conclusion of proof

Using the lemma, we can now conclude the proof of Theorem 6.1.

Assume $\pi$ is tempered nongeneric and has nonzero theta lift to $PD^\times$. Since $PD^\times$ is compact, one can find $\tau \in \text{Irr}(PD^\times)$ such that $\tau$ is an irreducible quotient of $\Theta_D(\pi)$. Moreover $\tau$ is tempered. Choose $E$ so that $\text{Hom}_N(\pi, \psi_E) \neq 0$. We may now apply Lemma 6.4 with the chosen $\tau$ and $E$ to deduce that

$$d := \dim \text{Hom}_{G_2}(I_E(1/2), \pi) = \dim \text{Hom}_{G_2}(R_D(E), \pi) = \dim \text{Hom}_N(\pi, \psi_E) \neq 0$$

Similarly, if $\pi$ has nonzero theta lift to $\text{PGSp}_6$, then we may find a tempered $\tau \in \text{Irr}(\text{PGSp}_6)$ such that $\tau$ is an irreducible quotient of $\Theta(\pi)$ (by Lemma 6.2 and Lemma 6.3(i)). With $E$ as above, an application of Lemma 6.4 shows that

$$d = \dim \text{Hom}_{G_2}(I_E(1/2), \pi) = \dim \text{Hom}_{G_2}(R_H(\mathcal{M}_2)(E), \pi) = \dim \text{Hom}_N(\pi, \psi_E) \neq 0$$

Moreover, since all these dimensions are finite, one deduces by Proposition 5.3 that

$$d = \dim \text{Hom}_{G_2}(I_E(1/2), \pi) = \dim \text{Hom}_{G_2}(R_D(E), \pi) + \dim \text{Hom}_{G_2}(R_{H_3}(\mathcal{M}_2)(E), \pi) = 2d.$$

This gives the desired contradiction and completes the proof of Theorem 6.1.

6.6 Uniqueness results

As further applications of Lemma 6.4, we may now derive two multiplicity one statements which will play a key role in the reminder of the paper. These statements are the first steps towards the proof of the Howe duality theorem.
Proposition 6.6 Let $\tau \in \text{Irr}(H_J)$ be tempered. Let $\pi \in \text{Irr}(G_2)$ be a tempered, non-generic quotient of $\Theta_J(\tau)$. Then $\Theta_J(\tau) \cong \pi$.

Proof Since $\pi$ is non-generic, for every $E$, the space $\text{Hom}_N(\pi, \psi_E)$ is finite-dimensional. By Lemma 6.4, $\text{Hom}_N(\Theta_J(\tau), \psi_E) = \text{Hom}_N(\pi, \psi_E)$, for every $E$. Thus, by Proposition 3.4, the kernel of the projection $\Theta_J(\sigma) \to \pi$ has trivial action of $G_2$. But this submodule would split off, giving a trivial representation as a quotient of $\Theta_J(\sigma)$. This contradicts Lemma 6.3. \hfill $\square$

Proposition 6.7 Let $\pi \in \text{Irr}(G_2)$ be tempered and non-generic. Then $\Theta_J(\pi)$ cannot have two tempered irreducible quotients. In particular, the cuspidal representation $\Theta_J(\pi)_c$ is irreducible or 0.

Proof Let $\tau_1, \tau_2 \in \text{Irr}(H_J)$, irreducible tempered, such that $\Theta_J(\pi) \to \tau_1 \oplus \tau_2$. Since $\pi$ is non-generic, there exists $E$ such that $d = \dim \text{Hom}_N(\pi^\vee, \psi_E)$ is finite and non-zero. By Lemma 6.4, applied to $\pi^\vee, \tau_1^\vee$ and then to $\pi^\vee, \tau_2^\vee$,

$$d = \dim \text{Hom}_{H_J,E}(\tau_1, \mathbb{C}) = \dim \text{Hom}_{H_J,E}(\Theta_J(\pi), \mathbb{C}) = \dim \text{Hom}_{H_J,E}(\tau_2, \mathbb{C}).$$

Since $\tau_1 \oplus \tau_2$ is a quotient of $\Theta(\pi)$,

$$d = \dim \text{Hom}_{H_J,E}(\Theta_J(\pi), \mathbb{C}) \geq \dim \text{Hom}_{H_J,E}(\tau_1, \mathbb{C}) + \dim \text{Hom}_{H_J,E}(\tau_2, \mathbb{C}) = 2d,$$

a contradiction. \hfill $\square$

Combining Propositions 6.6 and 6.7 with Lemmas 6.2(i) and 6.3(i), we deduce the following corollary which may be considered as a first step towards the Howe duality theorem.

Corollary 6.8 Let $\pi \in \text{Irr}(G_2)$ be tempered and non-generic. Then $\Theta_J(\pi)$ has finite length. If $\Theta_J(\pi) \neq 0$, then it has a unique irreducible quotient $\theta(\pi)$ and $\theta(\pi)$ is tempered. Moreover, for $\pi_1, \pi_2 \in \text{Irr}(G_2)$ tempered and non-generic,

$$0 \neq \theta(\pi_1) \cong \theta(\pi_2) \implies \pi_1 \cong \pi_2.$$

Proof Writing $\Theta_J(\pi) = \Theta_J(\pi)_c \oplus \Theta_J(\pi)_{nc}$, Proposition 6.7 tells us that $\Theta_J(\pi)_c$ is irreducible or 0, whereas Lemma 6.2(i) tells us that $\Theta_J(\pi)_{nc}$ has finite length. Hence $\Theta_J(\pi)$ has finite length, so that its cosocle $\theta_J(\pi)$ is a finite sum of irreducible representations. Moreover, Lemma 6.3(i) says that $\theta_J(\pi)$ is tempered, and Proposition 6.7 then shows the irreducibility of $\theta_J(\pi)$ if it is nonzero. The final implication now follows by Proposition 6.6. \hfill $\square$

In the rest of the paper, we shall examine each of the 3 dual pairs $G_2 \times H_J$ in turn and complete the proof of the Howe duality conjecture.
7 Theta correspondence for $PD^\times \times G_2$

In this section, we discuss the theta correspondence for the dual pair $PD^\times \times G_2$. A preliminary study of this dual pair correspondence has been carried out by the second author in [33]. We first recall the results established there.

Let $\Pi_D$ be the minimal representation of $PD^\times \times G_2$. Then we have

$$\Pi_D = \bigoplus_{\tau \in \text{Irr}(PD^\times)} \tau \boxtimes \Theta(\tau).$$

The following was shown in [33, §6]:

**Proposition 7.1** (i) If $\tau = 1$ is the trivial representation of $PD^\times$, then

$$\Theta(1) = \pi_{\text{deg}}[1],$$

the unipotent discrete series representation introduced in Proposition 3.1(iii) (first bullet point).

(ii) If $\tau$ is not the trivial representation, then $\Theta(\tau)$ is nongeneric supercuspidal of finite length (possibly zero).

(iii) If $\tau = \chi$ is a nontrivial unramified cubic character, then

$$\Theta(\chi) = \pi_{\text{sc}}[\chi]\quad\text{and}\quad\Theta(\chi^2) = \pi_{\text{sc}}[\chi^2]$$

the two depth 0 supercuspidal representations introduced in Sect. 3.5 (2) (third bullet point).

(iv) For each cubic field extension $E/F$,

$$\text{Hom}_N(\Theta(\tau), \psi_E) \cong \text{Hom}_{PE^\times}(\tau, \mathbb{C}).$$

We can now easily extend the above results. More precisely,

**Theorem 7.2** (i) For any $\tau \in \text{Irr}(PD^\times)$, $\Theta(\tau)$ is an irreducible representation of $G_2$ if it is nonzero.

(ii) If $\tau_1, \tau_2 \in \text{Irr}(PD^\times)$ are such that $\Theta(\tau_1) \cong \Theta(\tau_2) \neq 0$, then $\tau_1 \cong \tau_2$.

(iii) If $p \neq 3$, then the map $\tau \mapsto \Theta(\tau)$ defines an injection

$$\text{Irr}(PD^\times) \hookrightarrow \text{Irr}(G_2).$$

Hence, the Howe duality theorem holds for $PD^\times \times G_2$, so that

$$\dim \text{Hom}_{G_2}(\theta(\tau_1), \theta(\tau_2)) \leq \dim \text{Hom}_{PD^\times}(\tau_1, \tau_2)$$

for any $\tau_1, \tau_2 \in \text{Irr}(PD^\times)$. In particular, for any $\pi \in \text{Irr}(G_2)$, the representation $\Theta(\pi)$ of $PD^\times$ is irreducible or zero.
Proof The first two parts follow from Propositions 6.6 and 6.7. As for (iii) we use

\[ \text{Hom}_N(\Theta(\tau), \psi_E) \cong \text{Hom}_{PE \times}(\tau, \mathbb{C}), \]

so it suffices to show that there exists a field \( E \) such that \( \text{Hom}_{PE \times}(\tau, \mathbb{C}) \neq 0 \).

If \( p \neq 3 \), this was proved for all irreducible \( \tau \) by [24, Thm. 2.4]. \( \square \)

8 Theta correspondence for \( (\text{PGL}_3 \rtimes \mathbb{Z}/2\mathbb{Z}) \times G_2 \)

In this section, we consider the theta correspondence for the dual pair \( (\text{PGL}_3 \rtimes \mathbb{Z}/2\mathbb{Z}) \times G_2 \) and prove various results analogous to those in the last section. In fact, the theta correspondence for \( \text{PGL}_3 \times G_2 \) was almost completely studied in [6]. But the treatment there ignores the outer automorphism group of \( \text{PGL}_3 \); this is akin to working with special orthogonal groups instead of orthogonal groups in classical theta correspondence and is of course undesirable. Thus, we shall complete the results of [6] in their natural setting here. We extend the minimal representation of \( E_6 \) to \( E_6 \rtimes \mathbb{Z}/2\mathbb{Z} \) so that \( \mathbb{Z}/2\mathbb{Z} \) fixes the spherical vector.

8.1 Representations of \( H = \text{PGL}_3 \rtimes \mathbb{Z}/2\mathbb{Z} \)

We realize \( \mathbb{Z}/2\mathbb{Z} \), acting on \( \text{GL}_3 \) as a pinned automorphism preserving the standard pinning, i.e. acting via

\[ A \mapsto w_0 \cdot \top A^{-1} \cdot w_0^{-1} \quad \text{with} \quad w_0 = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \]

Let \( U \subset \text{GL}_3 \) be the maximal unipotent subgroup of upper triangular matrices and let \( \psi \) be a \( \mathbb{Z}/2\mathbb{Z} \)-invariant Whittaker character of \( U \). Then \( \psi \) extends to two characters of \( U \rtimes \mathbb{Z}/2\mathbb{Z} \). Let \( \psi \otimes 1 \) be the extension such that \( \mathbb{Z}/2\mathbb{Z} \) acts trivially, and let \( \psi \otimes \text{sign} \) be the other extension. If \( \tau \in \text{Irr}(\text{PGL}_3) \), then there are two possibilities:

- if \( \tau^\vee \not\cong \tau \), then

\[ \tau^+ := \text{Ind}_{\text{PGL}_3}^H \tau \cong \text{Ind}_{\text{PGL}_3}^H \tau^\vee \]

is irreducible. If \( \tau \) is generic then

\[ \dim \text{Hom}_{U \rtimes \mathbb{Z}/2\mathbb{Z}}(\tau^+, \psi \otimes 1) = \dim \text{Hom}_{U \rtimes \mathbb{Z}/2\mathbb{Z}}(\tau^+, \psi \otimes \text{sign}) = 1. \]
• If $\tau \cong \tau^\vee$, then $\tau$ has two extensions to $H$, which differ from each other by twisting with the unique quadratic character sign: $H \rightarrow \langle \pm 1 \rangle$ of $H$. When $\tau$ is generic (for example when $\tau$ is tempered), we let $\tau^+$ denote the unique extension of $\tau$ such that

$$\dim \text{Hom}_{U \rtimes \mathbb{Z}/2\mathbb{Z}}(\tau^+, \psi \otimes 1) = 1,$$

and let $\tau^-$ denote the other extension. The only nongeneric and self-dual representations of $\text{PGL}_3$ are the trivial representation and Langlands quotients $J_B(\mu)$, where $B = TU$ is the normalizer of $U$ and

$$\mu = \chi_1 | -1/2 \otimes 1 \otimes \chi_2 | -1/2$$

is a character of $T$ such that $\chi_2^2 = 1$. In this case, $\theta(\tau)$ is irreducible by [6, Thm. 11 and Cor. 13], and we define $\tau^+$ by setting $\theta(\tau^+) = \theta(\tau)$ and $\theta(\tau^-) = 0$. Observe that $1^+$ is the trivial representation of $H$.

It follows from the above discussion that any irreducible representation of $H$ is self-dual.

### 8.2 Whittaker models

The following lemma summarizes some basic computations.

**Lemma 8.1** Let $\Pi$ be the minimal representation of split $E_6 \times \mathbb{Z}/2\mathbb{Z}$.

(i) Let $\psi_V : V \rightarrow \mathbb{C}^\times$ be a Whittaker character for $G_2$ (so $V$ is a maximal unipotent subgroup of $G_2$). Then

$$\Pi V, \psi_V \cong \text{ind}_{U \rtimes \mathbb{Z}/2\mathbb{Z}}^H \psi \otimes 1.$$

In particular, for any $\tau^\epsilon \in \text{Irr}(H)$,

$$\text{Hom}_V(\Theta(\tau^\epsilon), \psi_V) \cong \text{Hom}_{U \rtimes \mathbb{Z}/2\mathbb{Z}}(\tau^\epsilon, \psi \otimes 1).$$

(ii) For any étale cubic $F$-algebra, we have:

$$\text{Hom}_N(\Theta(\tau^\epsilon), \psi_E) \cong \text{Hom}_{PE \rtimes \mathbb{Z}/2\mathbb{Z}}(\tau^\epsilon, \mathbb{C}).$$

### 8.3 Our earlier results

The following is a simple combination of the results of [6, Thms. 11, 14, 15 and 21] and the previous discussion:
Theorem 8.2 For $\tau \in \text{Irr}(\text{PGL}_3)$, let $\phi_\tau : WDF \longrightarrow \text{SL}_3(\mathbb{C})$ denote the L-parameter of $\tau$. If $\tau$ is non-supercuspidal, then $\Theta(\tau)$ has finite length. If $\tau \not\cong \tau^\vee$ is supercuspidal, then $\Theta(\tau)$ is irreducible supercuspidal. (This covers all $\tau \in \text{Irr}(\text{PGL}_3)$ if $p \neq 2$). In these cases, set $\theta(\tau^\epsilon)$ to be the maximal semisimple quotient of $\Theta(\tau^\epsilon)$ for $\epsilon = \pm$.

More precisely, we have:

(i) If $\tau \neq \tau^\vee$, then $\theta(\tau^+)$ is irreducible and nonzero. If $\tau$ is generic, or supercuspidal, or a discrete series representation, or tempered, so is $\theta(\tau^+)$. When $\tau$ is not supercuspidal, then $\theta(\tau^+)$ is not supercuspidal and its L-parameter is obtained by composing $\phi_\tau$ with the inclusion $\text{SL}_3(\mathbb{C}) \subset G_2(\mathbb{C})$.

(ii) If $\tau = \tau^\vee$ and the parameter $\phi_\tau$ contains the trivial representation, then $\theta(\tau^-) = 0$ and $\theta(\tau^+)$ is nonzero irreducible, non-discrete-series and its L-parameter is obtained by composing $\phi_\tau$ with the inclusion $\text{SL}_3(\mathbb{C}) \subset G_2(\mathbb{C})$.

(iii) If $\tau = \tau^\vee$ and the parameter $\phi_\tau$ does not contain the trivial representation, then we have the following cases:

$\bullet$ $\tau = \text{St}$, the Steinberg representation. Then

$$\theta(\text{St}^+)) \oplus \theta(\text{St}^-) = \pi_{\text{gen}}[1] \oplus \pi_{\text{sc}}[1]$$

where $\pi_{\text{gen}}[1]$ is the generic discrete series representation introduced in Proposition 3.1(ii) and $\pi_{\text{sc}}[1]$ is the depth 0 supercuspidal representation introduced in Sect. 3.5 (2).

$\bullet$ $\tau$ is a tempered representation induced from a supercuspidal representation $\sigma \cong \sigma^\vee$ of $\text{GL}_2$ with a non-trivial central character. Then

$$\theta(\tau^+) \oplus \theta(\tau^-) = \text{Ind}_{P}^{G_2}(\sigma) = \text{Ind}_{P}^{G_2}(\sigma)_{\text{gen}} \oplus \text{Ind}_{P}^{G_2}(\sigma)_{\text{deg}}$$

$\bullet$ $\tau$ is a tempered principal series induced from a triple of non-trivial quadratic characters $(\chi_1, \chi_2, \chi_3)$ such that $\chi_1 \cdot \chi_2 \cdot \chi_3 = 1$ then

$$\theta(\tau^+) \oplus \theta(\tau^-) = \text{Ind}_{B}^{G_2}(\chi) = \text{Ind}_{B}^{G_2}(\chi)_{\text{gen}} \oplus \text{Ind}_{B}^{G_2}(\chi)_{\text{deg}}$$

where $\chi$ is the quadratic character of $T$ determined by $(\chi_1, \chi_2, \chi_3)$ as in Sect. 3.4.

$\bullet$ $\tau$ is a self-dual supercuspidal representation (so $p = 2$). Then $\Theta(\tau^\epsilon)$ is supercuspidal and

$$\Theta(\tau^+) \oplus \Theta(\tau^-) = \pi_{\text{gen}} \oplus \pi_{\text{deg}}$$

where $\pi_{\text{gen}}$ is a generic irreducible supercuspidal representation, while $\pi_{\text{deg}}$ is a non-generic supercuspidal representation of unknown length.
Observe that the only case for which we do not know that $\Theta(\tau^\epsilon)$ has finite length (and hence $\theta(\tau^\epsilon)$ is defined) is when $\tau$ is a self-dual supercuspidal representation (so $p = 2$). In this case, however, the last bullet point states that $\Theta(\tau^\epsilon)$ is supercuspidal and hence semisimple. Hence, even in this exceptional case, we may set $\theta(\tau^\epsilon) = \Theta(\tau^\epsilon)$. Moreover, observe that if $\tau^\epsilon$ is nontempered, then $\theta(\tau^\epsilon)$ is irreducible nontempered and is completely determined by Theorem 8.2. On the other hand, when $\tau^\epsilon$ is tempered, then so is every irreducible summand of $\theta(\tau^\epsilon)$. In particular, the results highlighted in Lemmas 6.2 and 6.3 hold in this case.

In the rest of the section, we shall complete the results above by completing the unresolved parts of the theorem.

### 8.4 A miracle of Oberwolfach

Let $\tau \in \text{Irr}(\text{PGL}_3)$ be a self-dual supercuspidal representation. The goal here is to show that $\Theta(\tau^-) \neq 0$. Let $Q = LU$ be the 3-step maximal parabolic subgroup of $G_2$. Recall that the group $U$ has the 3-step filtration

$$U \supset [U, U] \supset Z_U$$

where $Z_U$ is the 2-dimensional center of $U$ and $U/Z_U$ is a 3-dimensional Heisenberg group with the center $[U, U]/Z_U$. Let $\psi$ be a non-trivial character of $[U, U]$, trivial on $Z_U$. Then $\Pi_{[U, U], \psi}$ is naturally a $(\text{PGL}_3 \rtimes \mathbb{Z}/2\mathbb{Z}) \times \text{SL}_2$-module, where $\text{SL}_2 = [L, L]$. In order to describe $\Pi_{[U, U], \psi}$, we need some additional notation.

Consider the action of the group $\text{GL}_2 \rtimes \mathbb{Z}/2\mathbb{Z}$ on $\mathbb{M}_2$, the space of $2 \times 2$ matrices, with elements in $\text{GL}_2(F)$ acting by conjugation and the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ acting via:

$$X \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot X^\top \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$ 

This action preserves the determinant (quadratic) form on $M_2(F)$ and descends to the quotient group

$$\text{PGL}_2(F) \times \mathbb{Z}/2\mathbb{Z} \cong \{\pm 1\} \times \text{SO}_3 = \text{O}_3.$$ 

On the space $C_c(\mathbb{M}_2(F))$, we have a Weil representation of $\text{O}_3 \times \text{SL}_2$, which we may regard as a representation of $\text{GL}_2(F) \rtimes \mathbb{Z}/2\mathbb{Z}$. Then the following lemma follows by a standard computation:
Lemma 8.3 We have an isomorphism of \((\operatorname{PGL}_3 \rtimes \mathbb{Z}/2\mathbb{Z}) \times \operatorname{SL}_2\)-modules:

\[
\Pi_{[U,U],\psi} \cong \operatorname{ind}_{\operatorname{GL}_2 \rtimes \mathbb{Z}/2\mathbb{Z}}^{\operatorname{PGL}_3 \rtimes \mathbb{Z}/2\mathbb{Z}}(C_c(\mathbb{M}_2(F))),
\]

where \(\operatorname{GL}_2\) is embedded in \(\operatorname{PGL}_3\) via

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ 1 & c \\ d \end{pmatrix}.
\]

Using the lemma, we can now prove:

Proposition 8.4 Let \(\tau \in \operatorname{Irr}(\operatorname{PGL}_3)\) be a self-dual supercuspidal representation. Then \(\Theta(\tau^-) \neq 0\).

Proof It suffices to show that \(\tau^-\) is a quotient of \(\Pi_{[U,U],\psi}\), in fact we shall show that \(\tau^-\) is a quotient of \(\operatorname{SL}_2\)-coinvaraints of \(\Pi_{[U,U],\psi}\). Decompose \(\mathbb{M}_2(F) = \mathbb{M}_2^0(F) \oplus F\), where \(\mathbb{M}_2^0(F)\) is the subspace of trace 0 elements and \(F\) is the center. Accordingly, the Weil representation of \(\operatorname{O}_3 \times \widetilde{\operatorname{SL}}_2\) on \(C_c(\mathbb{M}_2(F))\) decomposes as a tensor product

\[
C_c(\mathbb{M}_2(F)) = C_c(\mathbb{M}_2^0(F)) \otimes C_c(F),
\]

where \(\operatorname{O}_3\) acts trivially on \(C_c(F)\) and \(\widetilde{\operatorname{SL}}_2\) acts by the Weil representations \(\rho_{\psi}\). Recall that as an \(\widetilde{\operatorname{SL}}_2\)-module, \(\rho_{\psi}\) decomposes as a sum

\[
\rho_{\psi} = \rho_{\psi}^+ \oplus \rho_{\psi}^-
\]

of even and odd Weil representations. Let \(\Theta(\rho_{\psi}^+)\) and \(\Theta(\rho_{\psi}^-)\) be the theta lifts of their contragredients to \(\operatorname{O}_3\), via the Weil representation on \(C_c(\mathbb{M}_2^0(F))\) with respect to \(\psi\). Thus the \(\operatorname{SL}_2\)-coinvariant of \(\Pi_{[U,U],\psi}\) is given by

\[
\operatorname{ind}_{\operatorname{GL}_2 \times \mathbb{Z}/2\mathbb{Z}}^{\operatorname{PGL}_3 \times \mathbb{Z}/2\mathbb{Z}}(\Theta(\rho_{\psi}^+)) \oplus \operatorname{ind}_{\operatorname{GL}_2 \times \mathbb{Z}/2\mathbb{Z}}^{\operatorname{PGL}_3 \times \mathbb{Z}/2\mathbb{Z}}(\Theta(\rho_{\psi}^-)).
\]

Let \(\operatorname{st}\) be the Steinberg representation of \(\operatorname{SO}(3) \cong \operatorname{PGL}_2\). We extend \(\operatorname{st}\) to two representations \(\operatorname{st}^+\) and \(\operatorname{st}^-\) of \(\operatorname{O}(3)\) by letting \(-1 \in \operatorname{O}(3)\) act by 1 and \(-1\) respectively. Then \(\Theta(\rho_{\psi}^-) \cong \operatorname{st}^-\) while \(\Theta(\rho_{\psi}^+)\) is the principal series representation with the trivial representation as a quotient and \(\operatorname{st}^+\) as a submodule. Since \(\tau^-\) is a supercuspidal representation, it suffices to show that it is a quotient of

\[
\operatorname{ind}_{\operatorname{GL}_2 \times \mathbb{Z}/2\mathbb{Z}}^{\operatorname{PGL}_3 \times \mathbb{Z}/2\mathbb{Z}}(\operatorname{st}^+) \oplus \operatorname{ind}_{\operatorname{GL}_2 \times \mathbb{Z}/2\mathbb{Z}}^{\operatorname{PGL}_3 \times \mathbb{Z}/2\mathbb{Z}}(\operatorname{st}^-).
\]
It is known that any generic representation of $GL_2$, in particular $st$, is a quotient of $\tau$. Hence either $st^+$ or $st^-$ is a quotient of $\tau^-$. Now the proposition follows from Frobenius reciprocity.

\[ \boxdot \]

### 8.5 Main result

We shall now strengthen Theorem 8.2.

**Theorem 8.5** For any $\tau \in \text{Irr}(PGL_3)$, let $\phi_\tau : WD_F \rightarrow SL_3(\mathbb{C})$ denote the $L$-parameter of $\tau$.

(i) The representation $\Theta(\tau^\epsilon)$ is zero if and only if $\phi_\tau$ contains the trivial representation (so $\tau \cong \tau^\vee$) and $\epsilon = -$.

(ii) For any $\epsilon = \pm$, $\Theta(\tau^\epsilon)$ has finite length with unique irreducible quotient $\theta(\tau^\epsilon)$ (if it is nonzero).

(iii) $\theta(\tau^\epsilon)$ is generic if and only if $\tau$ is generic and $\epsilon = +$.

(iv) Suppose that $\theta(\tau^\epsilon) \neq 0$. If $\tau$ is a discrete series (resp. tempered) representation, so is $\theta(\tau^\epsilon)$. Moreover, $\theta(\tau^\epsilon)$ is supercuspidal if and only if $\tau$ is supercuspidal or $\tau^\epsilon = St$.

(v) If $\theta(\tau_1^{\epsilon_1}) \cong \theta(\tau_2^{\epsilon_2}) \neq 0$, then $\tau_1^{\epsilon_1} = \tau_2^{\epsilon_2}$.

In particular, the Howe duality theorem holds for the dual pair $(PGL_3 \rtimes \mathbb{Z}/2\mathbb{Z}) \times G_2$:

$$\dim \text{Hom}_{G_2}(\theta(\tau_1^{\epsilon_1}), \theta(\tau_2^{\epsilon_2})) \leq \dim \text{Hom}_{PGL_3 \rtimes \mathbb{Z}/2\mathbb{Z}}(\tau_1^{\epsilon_1}, \tau_2^{\epsilon_2})$$

for any $\tau_1^{\epsilon_1}, \tau_2^{\epsilon_2} \in \text{Irr}(PGL_3 \rtimes \mathbb{Z}/2\mathbb{Z})$. Moreover, for $\pi \in \text{Irr}(G_2)$, $\Theta(\pi)$ is a finite length representation of $PGL_3$ with a unique irreducible quotient (if nonzero).

**Proof** (i) From Theorem 8.2, it remains to show that $\Theta(\tau^\epsilon)$ is nonzero for those representations $\tau$ as in Theorem 8.2(iii) and any $\epsilon = \pm$. Consider first the Steinberg representation. Recall that $\pi_{\text{gen}}[1]$ is generic while $\pi_{\text{sc}}[1]$ is not. It follows, from Lemma 8.1 part (ii), that $\pi_{\text{gen}}[1]$ is a summand of $\theta(St^+)$. Furthermore, by Proposition 6.6, $\pi_{\text{sc}}[1]$ cannot be a summand of $\theta(St^+)$. Hence

$$\theta(St^+) = \pi_{\text{gen}}[1] \text{ and } \theta(St^-) = \pi_{\text{sc}}[1].$$

The same argument works in the other three cases to show that $\theta(\tau^\epsilon)$ is the generic $G_2$ summand and $\theta(\tau^-)$ is the degenerate summand. Moreover, in the last case of Theorem 8.2, where $\tau$ is a self-dual supercuspidal representation (so $p = 2$), we deduce by Proposition 6.6 again that $\pi_{\text{deg}}$ is nonzero irreducible.

(ii) This follows from Theorem 8.2 and the irreducibility of $\pi_{\text{deg}}$ in the proof of (i) above.
(iii) and (iv): These summarize what we already know from Theorem 8.2.

(v) Suppose that
\[ \pi := \theta(\tau_{1}^{\epsilon_1}) \cong\theta(\tau_{2}^{\epsilon_2}) \neq 0. \]

If \( \pi \) is non-supercuspidal, then \( \tau_1 \) and \( \tau_2 \) are both non-supercuspidal. The desired equality \( \tau_1^{\epsilon_1} \cong \tau_2^{\epsilon_2} \) follows from the results of [6, Thms. 11, 14 and 15] and our new understanding in (i) (which determines \( \theta(\tau^\epsilon) \) for those \( \tau \) in Theorem 8.2(iii))

Suppose that \( \pi \) is supercuspidal. Then \( \tau_{i}^{\epsilon_i} \) is either supercuspidal or St\(^{-}\), in which case both \( \tau_1 \) and \( \tau_2 \) are generic discrete series representations. By (iii), we deduce that \( \epsilon_1 = \epsilon_2 \). Hence, it remains to show that \( \tau_1 = \tau_2 \) or \( \tau_2^\vee \). We now consider the following two cases:

(a) Suppose \( \tau_1 \not\cong \tau_2^\vee \) and \( \tau_2 \not\cong \tau_1^\vee \). Then \( \epsilon_1 = \epsilon_2 = + \) and \( \pi \) is an irreducible generic supercuspidal representation.

Consider, for \( i = 1 \) or \( 2 \), the induced representation \( \text{Ind}_{P_3}^{\text{PGSp}_6}(\tau_i) \), where \( P_3 \) is the Siegel parabolic subgroup. Its normalized Jacquet functor with respect to \( P_3 \) is \( \tau_i \oplus \tau_i^\vee \). Since \( \tau_i \neq \tau_i^\vee \), it follows that \( \text{Ind}_{P_3}^{\text{PGSp}_6}(\tau_i) \) is an irreducible generic tempered representation.

By the computation of the Jacquet module of the minimal representation \( \Pi' \) of \( G_2 \times \text{PGSp}_6 \) along \( P_3 \) given in [25, Thm. 5.3], we deduce that \( \pi \otimes \text{Ind}_{P_3}^{\text{PGSp}_6}(\tau_i) \) is an irreducible quotient of \( \Pi' \). By [6, Prop. 19 and Cor. 20], a generic representation of \( G_2 \) cannot lift to two different generic representations of PGSp\(_6\). Hence, we must have
\[ \text{Ind}_{P_3}^{\text{PGSp}_6}(\tau_1) \cong \text{Ind}_{P_3}^{\text{PGSp}_6}(\tau_2). \]

By consideration of the Jacquet modules with respect to \( P_3 \), we see that \( \tau_2 \cong \tau_1 \) or \( \tau_1^\vee \), as desired.

(b) Assume now that \( \tau_1 = \tau_1^\vee \). In this case, we know that
\[ \theta(\tau_1^+) = \pi_{\text{gen}} \quad \text{and} \quad \theta(\tau_1^-) = \pi_{\text{deg}}. \]

Moreover, the tempered representation \( \text{Ind}_{P_3}^{\text{PGSp}_6}(\tau_1) \) is the sum of two representations, one of which is generic and the other degenerate (see Proposition 10.3(i)). By the Jacquet module of \( \Pi' \) again, we see that both \( \pi_{\text{gen}} \) and \( \pi_{\text{deg}} \) lifts to irreducible summands of \( \text{Ind}_{P_3}^{\text{PGSp}_6}(\tau_1) \). Moreover, \( \pi_{\text{deg}} \) cannot lift to a generic representation of PGSp\(_6\) and hence must lift to the degenerate summand [6, Prop. 19]. By Proposition 6.6, it follows
that $\pi_{\text{gen}}$ cannot lift to the degenerate summand and thus must lift to the generic summand.

Now suppose that $\epsilon_1 = \epsilon_2 = +$, so that $\pi = \theta(\tau_1^+) = \theta(\tau_2^+)$ is generic. Then as before, we see that $\pi$ lifts to the generic summand of $\text{Ind}_{P_3}^{\text{PGSp}_6}(\tau_i)$ (regardless of whether $\tau_2$ is self-dual or not). By Jacquet module consideration, we see that $\tau_1 \cong \tau_2$. On the other hand, if $\epsilon_1 = \epsilon_2 = -$, so that $\pi = \theta(\tau_1^-) = \theta(\tau_2^-)$ is nongeneric, then Proposition 6.7 implies that the nongeneric summand of $\text{Ind}_{P_3}^{\text{PGSp}_6}(\tau_1)$ is contained in $\text{Ind}_{P_3}^{\text{PGSp}_6}(\tau_2)$. Again, Jacquet module considerations show that $\tau_1 \cong \tau_2$.

The inequality at the end of the theorem is simply a restatement of (v).

Finally, given $\pi \in \text{Irr}(G_2)$, we write

$$\Theta(\pi) = \Theta(\pi)_c \oplus \Theta(\pi)_{nc}$$

as a sum of its cuspidal and noncuspidal component. As we noted in Lemma 6.2, the results of [6] imply that $\Theta(\pi)_{nc}$ has finite length. The result in (v) shows that $\Theta(\pi)$ has a unique irreducible quotient if it is nonzero, implying in particular that $\Theta(\pi)_c$ is either 0 or irreducible, and hence $\Theta(\pi)$ has finite length.

$\square$

9 The group PGSp$_6$

Before discussing the last dual pair $G_2 \times \text{PGSp}_6$, we need to devote the next few sections to a discussion of the structure and representations of PGSp$_6$, as well as certain particular periods on $G_2$ and PGSp$_6$.

Let $e_1, \ldots, e_6$ be the standard basis of $F^6$, where we have a symplectic form defined by

$$\omega(e_1, e_6) = \omega(e_2, e_5) = \omega(e_3, e_4) = 1$$

and all other $\omega(e_i, e_j) = 0$ with $i < j$. Let GSp$_6$ be the group of linear transformations $g$ of $F^6$, such that for some $v(g) \in F^\times$

$$\omega(gv, gw) = v(g) \cdot \omega(v, w)$$

for all $v, w \in F^6$. Then $v : \text{GSp}_6 \to F^\times$ is the similitude character.

Let $\tilde{P}_1, \tilde{P}_2$ and $\tilde{P}_3$ be maximal parabolic subgroups of GSp$_6$ defined as the stabilizers of subspaces

$$\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \langle e_1, e_2, e_3 \rangle$$
respectively. For \( i = 1, 2, 3 \), let \( P_i \subseteq \text{PGSp}_6 \) be the quotient of \( \tilde{P_i} \) by the center of \( \text{GSp}_6 \). The group \( \text{PGSp}_6 \) acts faithfully on \( J = \wedge^2 F^6 \otimes \nu^{-1} \), and we shall (partially) describe how the parabolic subgroups act on this module.

The group \( \text{PGSp}_6 \) can be explicitly described in terms of its action on \( J \) as follows. Let \( x_{ij} = e_i \wedge e_j \in J \) for \( i \neq j \). On \( J \), we have a natural trilinear form \( (x, y, z) \)

\[
\wedge^2 F^6 \times \wedge^2 F^6 \times \wedge^2 F^6 \rightarrow \wedge^6 F^6 \cong F.
\]

The group of linear transformations of \( J \) preserving this form is \( \text{SL}_6/\mu_2 \) and \( \text{PGSp}_6 = \text{Sp}_6/\mu_2 \) is the subgroup fixing \( e = x_{16} + x_{25} + x_{34} \).

The Levi factor \( M_3 \) of \( P_3 \), as an algebraic group, is isomorphic to \( \text{GL}_3/\mu_2 \). Observe that group acts faithfully on \( \wedge^2 F^3 \), and since the latter is a three dimensional vector space, this action gives an isomorphism \( \text{GL}_3/\mu_2 \cong \text{GL}_3 \). Thus we have an identification

\[
M_3 = \text{GL}(\langle x_{12}, x_{13}, x_{23} \rangle).
\]

Under this identification, the maximal torus is given by diagonal matrices \( (t_1, t_2, t_3) \). The three simple co-roots of \( \text{PGSp}_6 \) are, respectively,

\[
\alpha_1^\vee(t) = (1, t, t^{-1}), \quad \alpha_2^\vee(t) = (t, t^{-1}, 1), \quad \alpha_3^\vee(t) = (1, t, t).
\]

An unramified character \( \chi \) of the maximal torus is given by a triple of complex numbers \( (s_1, s_2, s_3) \)

\[
\chi(t_1, t_2, t_3) = |t_1|^{s_1} |t_2|^{s_2} |t_3|^{s_3}.
\]

The Weyl group action on the characters is somewhat different in this picture. The simple reflections corresponding to the first two roots \( \alpha_1 \) and \( \alpha_2 \) are the usual permutations of entries of \( (s_1, s_2, s_3) \), however, the simple reflection corresponding to the third simple root \( \alpha_3 \) is given by

\[
(s_1, s_2, s_3) \mapsto (s_1 + s_2 + s_3, -s_3, -s_2).
\]

Thus the root hyperplanes are \( s_i - s_j = 0 \) and \( s_i + s_j = 0 \) for short and long roots, respectively. This looks like a \( D_3 \) root system; however, the Weyl-invariant quadratic form in this case is

\[
q(s_1, s_2, s_3) = s_1^2 + s_2^2 + s_3^3 - \frac{1}{4} (s_1 + s_2 + s_3)^2.
\]
rather than the usual dot product, and with this form, we have a realization of the $C_3$ root system with simple roots

$$\alpha_1 = (0, 1, -1), \quad \alpha_2 = (1, -1, 0), \quad \alpha_3 = (0, 2, 2).$$

This somewhat unconventional description of the $C_3$ root system is a source of potential confusion, as one has the tendency to confound it with the more familiar description of the root system of $Sp_6$, but what we have done here is definitely the natural way to set things up for $PGSp_6$.

The character $\chi$ is in the positive chamber if for every positive root $\alpha$, $\chi(\alpha^\vee(t)) = |t|^s$ for some $s \in \mathbb{C}$ such that $\Re(s) > 0$ (the real part). One checks that $\chi$ is positive if

$$\Re(s_1) > \Re(s_2) > |\Re(s_3)|.$$

The modulus character of $M_3 \cong GL_3$ is

$$\delta_{P_3}(m) = |\det(m)|^2.$$

It follows that the Levi factor $M_{13}$ of $P_{13} = P_1 \cap P_3$ is

$$M_{13} = GL(\langle x_{12}, x_{13} \rangle) \times GL(\langle x_{23} \rangle).$$

The group $P_{13}$ is the stabilizer of the space $V_2 = \langle x_{12}, x_{13} \rangle$.

Consider now the group $P_2$ and its Levi factor $M_2$. The standard Levi factor of $\tilde{P}_2$ is $GL_2 \times GL_2$ where the first $GL_2$ acts on $\langle e_1, e_2 \rangle$ in the standard way, fixes $\langle e_3, e_4 \rangle$ and acts by transpose-inverse on $\langle e_5, e_6 \rangle$. The second $GL_2$ acts on $\langle e_3, e_4 \rangle$ in the standard way, by det on $\langle e_1, e_2 \rangle$ and fixes $\langle e_5, e_6 \rangle$. The group $P_2$ is the stabilizer of the singular line $V_1 = \langle x_{12} \rangle$, and the Levi factor $M_2$ acts faithfully on the 4-dimensional subspace

$$V_4 = \langle x_{13}, x_{23}, x_{14}, x_{24} \rangle$$

preserving the quadratic form $x \mapsto (x, x, x_{56})$. If we identify $x = ax_{14} + bx_{13} + cx_{24} + d_{23}$ with the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then $(x, x, x_{56}) = 2\det(x)$. Thus, with $V_4$ identified with the set of $2 \times 2$ matrices, we have

$$M_2 \cong GL_2 \times GL_2/GL_1^\vee$$

where $GL_1^\vee = \{(t, t^{-1}) : t \in GL_1\}$.
so that \((\alpha, \beta) \in M_2\) acts on \(x \in V_4\) by \(x \mapsto \alpha x \bar{\beta}\) where \(\bar{\beta}\) is the transpose of \(\beta\). The element \((\alpha, \beta)\) acts on the line \(\langle x_{12} \rangle\) by \(\det(\alpha \beta)\). The modulus character is

\[
\delta_{P_2}((\alpha, \beta)) = |\det(\alpha \beta)|^5.
\]

This sets up the necessary notation to discuss the representations of \(\text{PGSp}_6\).

10 Representations of \(\text{PGSp}_6\)

In this section, we list some irreducible non-supercuspidal representations of \(\text{PGSp}_6(F)\), relevant to this work. Observe that the local Langlands correspondence is known for the Levi factors of all proper parabolic subgroups of \(\text{PGSp}_6(F)\) (by Gan and Takeda [11] for \(M_1 \cong \text{GSp}_4\)). Thus, following Shahidi [37], reducibility points of generalized principal series can be computed using \(L\)-functions of Langlands parameters.

10.1 Principal series representations for \(P_2\)

We first consider certain principal series representations for the parabolic subgroup \(P_2 = M_2 N_2\), where \(M_2 \cong \text{GL}_2 \times \text{GL}_2/\text{GL}_1\). Let \(\tau\) be an irreducible representation of \(\text{GL}_2(F)\) with \(L\)-parameter \(\phi_\tau\) and central character \(\omega_\tau\). Set

\[
I_2(\tau \otimes \tau) = \text{Ind}_{P_2}^{\text{PGSp}_6} \tau \otimes \tau
\]

\[
I_2(s, \tau \otimes \tau) = \text{Ind}_{P_2}^{\text{PGSp}_6} (|\det|^s \tau) \otimes (|\det|^s \tau)
\]

if we need to consider a family of induced representations. Then we have:

**Proposition 10.1** (i) If \(\tau \in \text{Irr}(\text{GL}_2(F))\) is unitary supercuspidal, then \(I_2(s, \tau \otimes \tau)\) is reducible if and only if \(\tau^\vee \cong \tau\) (so \(\omega_\tau^2 = 1\)) and one of the following holds:

- \(\omega_\tau = 1\) and \(s = \pm 1/2\), in which case one has:

\[
0 \longrightarrow \delta_2(\tau) \longrightarrow I_2(1/2, \tau \otimes \tau) \longrightarrow J_2(1/2, \tau \otimes \tau) \longrightarrow 0,
\]

where \(\delta_2(\tau)\) is a generic discrete series representation.

- \(\omega_\tau \neq 1\) (so \(\tau\) is dihedral), \(\text{Im}(\phi_\tau) = S_3\) (the symmetric group on 3 letters, regarded as a subgroup of \(\text{GL}_2(\mathbb{C})\)) and \(s = \pm 1\), in which case one has:

\[
0 \longrightarrow \sigma_{\text{gen}}[\tau] \longrightarrow I_2(1, \tau \otimes \tau) \longrightarrow J_2(1, \tau \otimes \tau) \longrightarrow 0,
\]

where \(\sigma_{\text{gen}}[\tau]\) is a generic discrete series representation.
• \( \omega_\tau \neq 1, \text{Im}(\phi_\tau) \neq S_3 \) (the symmetric group on 3 letters, regarded as a subgroup of \( \text{GL}_2(\mathbb{C}) \)) and \( s = 0 \), in which case one has:

\[
I_2(\tau \otimes \tau) = I_2(\tau \otimes \tau)_{\text{gen}} \oplus I_2(\tau \otimes \tau)_{\text{deg}}
\]

where \( I_2(\tau \otimes \tau)_{\text{gen}} \) is generic.

(ii) If \( \tau = \text{st}_\chi \) is a twisted Steinberg representation, then \( I_2(s, \tau \otimes \tau) \) is irreducible except for the following cases:

- \( \chi = 1 \) and \( s = \pm 5/2 \) or \( \pm 1/2 \), in which case one has:

\[
0 \longrightarrow \text{St}_{\text{PGSp}_6} \longrightarrow I_2(5/2, \text{st} \otimes \text{st}) \longrightarrow J_2(5/2, \text{st} \otimes \text{st}) \longrightarrow 0,
\]

and

\[
0 \longrightarrow \text{Ind}_{P_3}^{\text{PGSp}_6}(\text{St})_{\text{gen}} \longrightarrow I_2(1/2, \text{st} \otimes \text{st}) \longrightarrow J_2(1/2, \text{st} \otimes \text{st}) \longrightarrow 0,
\]

where \( \text{Ind}_{P_3}^{\text{PGSp}_6}(\text{St})_{\text{gen}} \) is the generic summand of \( \text{Ind}_{P_3}^{\text{PGSp}_6}(\text{St}) \).

- \( \chi^2 = 1 \) but \( \chi \neq 1 \) and \( s = \pm 1/2 \), in which case one has:

\[
0 \longrightarrow \sigma_{\text{gen}}[\chi] \longrightarrow I_2(1/2, \text{st}_\chi \otimes \text{st}_\chi) \longrightarrow J_2(1/2, \text{st}_\chi \otimes \text{st}_\chi) \longrightarrow 0,
\]

where \( \sigma_{\text{gen}}[\chi] \) is a generic discrete series representation.

### 10.2 Principal series representations for \( P_{13} \)

Now we consider certain principal series representations for the parabolic subgroup \( P_{13} = M_{13}N_{13} \), where \( M_{13} \cong \text{GL}_2 \times \text{GL}_1 \). Let \( \tau \) be an irreducible representation of \( \text{GL}_2 \) with the central character \( \omega_\tau \) and \( \text{L-parameter} \) \( \phi_\tau \). Set

\[
I_{13}(\tau \otimes 1) = \text{Ind}_{P_{13}}^{\text{PGSp}_6} \tau \otimes 1 \quad \text{and} \quad I_{13}(s, \tau \otimes 1) = \text{Ind}_{P_{13}}^{\text{PGSp}_6}(|\det|^s \tau \otimes 1)
\]

if we need to consider a family of induced representations. In the more familiar language of representations of \( \text{Sp}_6 \), the restriction of \( I_{13}(s, \tau \otimes 1) \) to \( \text{Sp}_6 \) is a principal series induced from \( |\cdot|^{2s} \omega_\tau \otimes |\det|^s \tau \). In particular, if \( \tau \) is unitary tempered and \( s > 0 \), this is a standard module. We have:

**Proposition 10.2** If \( \tau \in \text{Irr}(\text{GL}_2(F)) \) is unitary supercuspidal, then \( I_{13}(s, \tau \otimes 1) \) is reducible if and only if \( \tau^\vee \cong \tau \) (so \( \omega_\tau^2 = 1 \)) and one of the following holds:

- \( \omega_\tau = 1 \) and \( s = \pm 1/2 \), in which case \( I_{13}(1/2, \tau \otimes 1) \) has length 4 and has a unique irreducible submodule \( \delta_{13}(\tau) \), which is a generic discrete series representation.
• \( \omega_{\tau} \neq 1 \) and \( s = 0 \), in which case one has:

\[
I_{13}(\tau \otimes 1) = I_{13}(\tau \otimes 1)_{\text{gen}} \oplus I_{13}(\tau \otimes 1)_{\text{deg}}
\]

where \( I_{13}(\tau \otimes 1)_{\text{gen}} \) is generic.

10.3 Principal series representations for \( P_3 \)

Now we consider certain principal series representations for the parabolic subgroup \( P_3 = M_3 N_3 \), where \( M_3 \cong \text{GL}_3 \). Let \( \tau \) be an irreducible representation of \( \text{GL}_3 \). We set

\[
I_3(\tau) = \text{Ind}_{P_3}^{\text{PGSp}_6} \tau.
\]

**Proposition 10.3** (i) Assume that \( \tau \) is discrete series representation with trivial central character. Then we have two cases:

• If \( \tau \neq \tau^\vee \) then

\[
I_3(\tau) \cong I_3(\tau^\vee)
\]

is irreducible.

• If \( \tau \cong \tau^\vee \) then

\[
I_3(\tau) = I_3(\tau)_{\text{gen}} \oplus I_3(\tau)_{\text{deg}}
\]

where \( I_3(\tau)_{\text{gen}} \) is generic.

(ii) Let \( \chi_1, \chi_2, \chi_3 \) be three characters of \( F^\times \) such that \( \chi_1 \cdot \chi_2 \cdot \chi_3 = 1 \), and let \( \tau = \tau(\chi_1, \chi_2, \chi_3) \) be the associated principal series representation of \( \text{GL}_3(F) \) (which is possibly reducible). Then the induced representation \( I_3(\tau) \) is irreducible unless one of the following two conditions hold:

• \( \chi_i = | \cdot |^{\pm 1} \) for some \( i \) or \( \chi_i / \chi_j = | \cdot |^{\pm 1} \) for a pair \( i \neq j \).

• The three characters \( \chi_i \) are quadratic, non-trivial and pairwise different. Then

\[
I_3(\tau) = I_3(\tau)_{\text{gen}} \oplus I_3(\tau)_{\text{deg}},
\]

where \( I_3(\tau)_{\text{gen}} \) is generic.

**Proof** These are some old results for representations of \( \text{GSp}_6(F) \) translated to our setting. Following [40, S3], the Levi subgroup of the Siegel parabolic subgroup in \( \text{GSp}_6(F) \) is isomorphic to \( \text{GL}_3(F) \times \text{GL}_1(F) \) such that, under this isomorphism, the center of \( \text{GSp}_6(F) \) corresponds to the image of the map.
Howe duality and dichotomy…

\[ \Delta : \GL_1(F) \to \GL_3(F) \times \GL_1(F) \] defined by \( \lambda \mapsto (\lambda, \lambda^2) \) where \( \lambda \in F^\times \).

It follows that

\[ M_3(F) \cong (\GL_3(F) \times \GL_1(F))/\Delta \GL_1(F), \]

a rather awkward description of a group isomorphic to \( \GL_3(F) \). However, by our identification \( M_3 \cong \GL_3 \), the above isomorphism is given by the map \( g \mapsto (g, \det g) \).

Keeping in mind that our \( \tau \) have the trivial central character, it follows that our \( I_3(\tau) \) is given by \( \tau \ctimes 1 \) in the notation of Tadić.

With this translation, we now treat each part of the proposition in turn:

(i) By [38], the representation \( \tau \ctimes 1 \) reduces if and only if \( \tau \cong \tau^\vee \) and the exterior square \( L \)-function of \( \tau \) has no pole at \( s = 0 \). In [38], however, the result is stated for representations of \( \Sp_6(F) \). The variant of that result for representations \( \tau \ctimes \chi \) of \( \GSp_6(F) \) involves a twist by the character \( \chi \).

Since, in our case, \( \chi = 1 \) we have what we stated. Now observe that the exterior square \( L \)-function of \( \tau \) is the same as the standard \( L \)-function of \( \tau^\vee \) (a simple observation that the exterior square of the standard representation of \( \SL_3(\mathbb{C}) \) is the dual of the standard representation). Since the standard \( L \)-function of a square integrable representation of \( \PGL_3(F) \) has no pole at \( 0 \), it follows that \( I_3(\tau) \) reduces if and only if \( \tau \cong \tau^\vee \).

(ii) This is [40, Example 7.7 and Theorem 7.9]. Observe that the condition \( \chi_i \chi_j = |\cdot|^{\pm 1} \) is redundant since \( \chi_1 \chi_2 \chi_3 = 1 \).

Remark Observe that the description of reducibility points in Proposition 10.3 (ii) matches perfectly those for \( G_2 \) in Proposition 3.3. In fact these, and many other reducibility results for induced representations of \( \PGSp_6(F) \) stated in this section, can be derived using the theta lifting from \( G_2 \), see [14] for an illustration of this idea.

10.4 Principal series representations for \( P_1 \)

Now we consider certain principal series representations for the parabolic subgroup \( P_1 = M_1 N_1 \), where \( M_1 \cong \GSp_4 \). Let \( \tau \) be an irreducible representation of \( \GSp_4 \). We set

\[ I_1(\tau) = \Ind_{P_1}^{\PGSp_6(\tau)} \text{ and } I_1(s, \tau) = \Ind_{P_1}^{\PGSp_6(\nu)|\nu|^s \tau}, \]

where \( \nu \) is the similitude character of \( \GSp_4 \). Let \( \tau \) be an irreducible supercuspidal representation of \( \GSp_4(F) \) with trivial central character. Let \( \varphi_\tau : WD_F \to \Spin_5 \cong \Sp(4) \) be its Langlands parameter [11].

Proposition 10.4 Assume that \( \tau \) is a supercuspidal representation of \( \GSp_4(F) \) with trivial central character such that the parameter \( \text{std} \circ \varphi_\tau \) contains the
trivial representation, where std denotes the 5-dimensional standard representation of Spin$_5$. Then $I_1(s, \tau)$ is reducible if and only if $s = \pm 1/2$, in which case one has:

$$0 \longrightarrow \delta_1(\tau) \longrightarrow I_1(1/2, \tau) \longrightarrow J_1(1/2, \tau) \longrightarrow 0,$$

where $\delta_1(\tau)$ is a discrete series representation.

11 Fourier–Jacobi and Shalika periods

In this section, we introduce and study a Fourier–Jacobi-type model for the group $G_2$ and a Shalika period for PGSp$_6$. These are some of the periods that will appear when we consider a game of ping-pong with periods for the dual pair $G_2 \times \text{PGSp}_6$, as discussed at the end of the introduction.

11.1 Whittaker periods

We begin by recalling the following results about Whittaker periods from [6, Prop. 19 and Cor. 20], see also the appendix of [16].

**Proposition 11.1** Let $\Pi$ be the minimal representation of $E_7$ and let $(V', \psi_{V'})$ be a Whittaker datum for PGSp$_6$ (so $V'$ is a maximal unipotent subgroup and $\psi_{V'}$ a generic character of $V'$). Then we have an isomorphism of $G_2$-modules:

$$\Pi_{V', \psi_{V'}} \cong \text{ind}^{G_2}_{V} \psi_{V}$$

where $(V, \psi_{V})$ is a Whittaker datum for $G_2$.

**Corollary 11.2** (i) If $\pi \in \text{Irr}(G_2)$ is generic and $\Theta(\pi)$ is its big theta lift to PGSp$_6$, then

$$\text{dim Hom}_{V'}(\Theta(\pi), \psi_{V'}) = 1$$

so that $\Theta(\pi)$ contains a unique irreducible generic subquotient and thus is nonzero.

(ii) If $\pi \in \text{Irr}(G_2)$ is non-generic and $\tau \in \text{Irr}(\text{PGSp}_6)$ is generic, then

$$\text{Hom}_{G_2 \times \text{PGSp}_6}(\Pi, \pi \otimes \tau) = 0.$$

11.2 Fourier–Jacobi period of $G_2$

Let $Q = LU$ be the 3-step maximal parabolic subgroup of $G_2$. Recall that $[L, L] \cong \text{SL}_2$ corresponds to the long simple root $\beta$. Thus $V = U_\beta U$ is
the unipotent radical of the standard Borel subgroup of $G_2$. If we set $J = [L, L]U$, then the quotient of $J$ by the two-dimensional center $Z_U$ of $U$ is the Jacobi group with one-dimensional central subgroup $[U, U]/Z_U \cong U_{2\alpha+\beta}$. Fix a non-trivial additive character $\psi$ of $U_{2\alpha+\beta} \cong F$. Let $\rho_\psi$ be the unique irreducible representation of the 2-fold cover $\tilde{J}$, trivial on $Z_U$, such that $U_{2\alpha+\beta}$ acts by $\psi$. If $\sigma$ is a genuine representation of $\tilde{SL}_2$, we have a representation of $J$ on $\sigma \otimes \rho_\psi$. For $\pi \in \text{Irr}(G_2)$, the Fourier–Jacobi period of $\pi$ with respect to $\sigma$ is the space

$$\text{Hom}_J(\pi, \sigma \otimes \rho_\psi) \cong \text{Hom}_{G_2}(\pi, \text{Ind}_{\tilde{J}}^G \sigma \otimes \rho_\psi).$$

The character $\psi$ defines a Weil index, that is, a function $\chi_\psi : F^\times \to \mathbb{C}^\times$ with values in roots of one such that $\chi_\psi(a)\chi_\psi(b) = (a, b)\chi_\psi(ab)$ where $(a, b)$ is the Hilbert symbol. Let $\chi$ be a smooth character of $F^\times$. Let $I_\psi(s)$ be a principal series representation of $\tilde{SL}_2$ obtained by inducing $\chi_\psi|\cdot|^s$, via normalized induction. We can fix our data so that for $s = 1/2$, there is a short exact sequence:

$$0 \to \text{St}_\psi \to I_\psi(1/2) \to \rho_\psi^+ \to 0,$$

where $\rho_\psi^+$ is the even Weil representation, i.e. a summand of $\rho_\psi$. The contragredient of $I_\psi(s)$ is $I_\psi(-s)$.

**Proposition 11.3** As a representation of $G_2$, $\text{ind}_J^{G_2}(I_\psi(s) \otimes \rho_\psi)$ admits an equivariant filtration

$$0 \subset I_0 \subset I_1$$

with successive quotients described as follows:

- $I_0 \cong \text{ind}_V^{G_2} \psi_V$;
- $J_1 := I_1/I_0 \cong \text{Ind}_P^N \rho_\psi$)

**Proof** Let $B = T_\beta U_\beta \subset [L, L] \cong SL_2$ be a Borel subgroup, where $T_\beta$ is the one-dimensional torus, the image of the simple coroot $\beta^\vee : \text{GL}_1 \to T$. Observe that we have an isomorphism of $J$-modules

$$I_\psi(s) \otimes \rho_\psi \cong \text{ind}_B^J (\chi_\psi|\cdot|^{s+1} \cdot \rho_\psi),$$

where $f \otimes v \in I_\psi(s) \otimes \rho_\psi$ is mapped to a function on $J$ given by $g \mapsto f(g) \cdot \rho_\psi(g)(v)$. (Here $f$ is inflated from $\tilde{SL}_2$ to $\tilde{J}$.) The later induction is not normalized.
Let $N$ be the unipotent radical of the maximal parabolic $P$. Let $\psi_{2\alpha+\beta}$ be a character of $N$ nontrivial only on the root space $U_{2\alpha+\beta} \subset N$. Then $\rho_\psi$, restricted to $\widetilde{BU}$, is induced from a character of $\widetilde{T}_\beta N$ equal to $\chi_\psi \cdot | |^{1/2}$ on $T_\beta$ and $\psi_{2\alpha+\beta}$ on $N$. Using transitivity of induction, and $\chi_\psi \cdot \chi_\psi = 1$, it follows that

$$I_\psi(s) \otimes \rho_\psi \cong \text{ind}_{T_\beta N}^G(| \cdot |^{s+3/2} \cdot \psi_{2\alpha+\beta}),$$

and hence

$$\text{ind}_{T_\beta N}^G(I_\psi(s) \otimes \rho_\psi) \cong \text{ind}_{T_\beta N}^G(| \cdot |^{s+3/2} \cdot \psi_{2\alpha+\beta}).$$

The next step requires the technique of root exchange, as in the proof of Proposition 3.4. Let $U'$, a conjugate of $U$, be obtained by adding $U_{-\alpha}$ to $N$ and removing $U_{3\alpha+\beta}$ from $N$. The root exchange is an isomorphism

$$\text{ind}_{T_\beta N}^G(| \cdot |^{s+3/2} \cdot \psi_{2\alpha+\beta}) \cong \text{ind}_{T_\beta U'}^G(| \cdot |^{s+5/2} \cdot \psi_{2\alpha+\beta}),$$

$f \mapsto f'$ given by

$$f'(g) = \int_{U_{-\alpha}} f(ug) \, du,$$

where, abusing notation, $\psi_{2\alpha+\beta}$ is also viewed a character of $U'$ supported on the root space $U_{2\alpha+\beta}$.

As the last step, let $V' = U'U_{-3\alpha-\beta}$. Then $V'$ is the unipotent radical of a Borel subgroup of $G_2$ such that the simple roots are $2\alpha + \beta$ and $-3\alpha - \beta$, and the highest root is $\beta$. Consider

$$\text{ind}_{T_\beta U'}^G(| \cdot |^{s+5/2} \cdot \psi_{2\alpha+\beta}) \cong C_c(U_{-3\alpha-\beta}).$$

We can analyze this module using the Fourier transform on $C_c(U_{-3\alpha-\beta})$. This gives an exact sequence of $T_\beta V'$-modules

$$0 \rightarrow \text{ind}_{T_\beta V'}^{T_\beta U'}(\psi_{V'}) \rightarrow \text{ind}_{T_\beta U'}^{T_\beta V'}(| \cdot |^{s+5/2} \cdot \psi_{2\alpha+\beta}) \rightarrow | \cdot |^{s+7/2} \cdot \psi_{2\alpha+\beta} \rightarrow 0,$$

where $\psi_{V'}$ is a Whittaker character of $V'$ and, in the last term, $\psi_{2\alpha+\beta}$ a character of $V'$ supported on the root space $U_{2\alpha+\beta}$. The lemma follows by induction in stages, giving $\text{ind}_{V'}^G \psi_{V'}$ as a submodule and the claimed quotient, after taking into account the relevant normalization for the parabolic $P'$ with unipotent radical $N'$ such that $V' = N'U_{2\alpha+\beta}$. \hfill \Box
Corollary 11.4 Let $\pi \in \operatorname{Irr}(G_2)$ be generic and tempered. If $s > -1/2$, then

$$\operatorname{Hom}_{G_2}(\operatorname{ind}_{J}^{G_2}(I_{\psi}(s) \otimes \rho_{\psi}), \pi) \cong \mathbb{C}.$$\]

Proof We need to show that we can avoid the top piece of the filtration in Proposition 11.3. By the Frobenius reciprocity,

$$\operatorname{Ext}^i_{G_2}(IP(\frac{s}{2} + \frac{1}{4}, \operatorname{ind}_{N}^{\text{PGL}_2} \psi), \pi) \cong \operatorname{Ext}^i_{M}(|\det|^{s+1/2} \cdot \operatorname{ind}_{N}^{\text{PGL}_2} \psi, r_{\bar{\rho}}(\pi)).$$

Since $\pi$ is tempered, the center of $M \cong \text{GL}_2$ acts on $r_{\bar{\rho}}(\pi)$ by characters $\chi$ such that $|\chi| = |\cdot|^t$ with $t \leq 0$. Since $s > -1/2$, all Ext groups vanish and $\pi$ is a quotient of $\operatorname{Hom}_{G_2}(\operatorname{ind}_{J}^{G_2}(I_{\psi}(s) \otimes \rho_{\psi})$ since it is generic. \qed

11.3 Shalika period on PGSp\(_6\)

We shall now discuss a Shalika period on PGSp\(_6\).

Recall the maximal parabolic subgroup $P_2 = M_2 N_2$ of PGSp\(_6\), with identifications of the Levi factor $M_2 \cong (\text{GL}_2 \times \text{GL}_2)/\text{GL}_1^V$ and of the maximal abelian quotient $N_2/[N_2, N_2]$ of the unipotent radical $N_2$ with $\mathbb{M}_2$, the space of $2 \times 2$ matrices. With these identifications, let $\psi_2$ be a character of $N_2(F)$ obtained by composing the trace on $\mathbb{M}_2(F)$ with a non-trivial additive character of $F$. Then the stabilizer of $\psi_2$ in the Levi group $M_2$ is the diagonally embedded $\text{PGL}_2^\Delta$. The Shalika subgroup of PGSp\(_6\) is the semi-direct product

$$S = \text{PGL}_2^\Delta \ltimes N_2$$

and the Shalika character $\psi_S$ is the character $\psi_2$ extended to $S(F)$ (trivially on $\text{PGL}_2(F)$). For any smooth representation $\pi$ of PGSp\(_6(F)\), the Shalika period of $\pi$ is the coinvariant space $\pi_{S, \psi_S}$.

This Shalika period has already been exploited in [34]. Indeed, the following was shown in [34, Lemma 4.5]:

Proposition 11.5 Let $\Pi$ be the minimal representation of $E_7$ and $(V, \psi_V)$ a Whittaker datum for $G_2$. Then

$$\Pi_{V, \psi_V} \cong \operatorname{ind}_{S}^{\text{PGSp}_6} \psi_S$$

as PGSp\(_6\)-modules.
11.4 Shalika period of $\Pi$

We now consider the minimal representation $\Pi$ of the dual pair $G_2 \times \text{PGSp}_6$ and determine its Shalika period $\Pi_{S,\psi_S}$ as a representation of $G_2(F)$. To describe the answer, we need to introduce some more notations.

The group $\text{PGL}_2$ acts by conjugation on $\mathbb{M}_2$ preserving the determinant (quadratic) form. As we saw in §8.4, there is a Weil representation of $\text{PGL}_2 \times \text{SL}_2$ on $C_c(\mathbb{M}_2(F))$ which decomposes as a tensor product

$$C_c(\mathbb{M}_2(F)) = C_c(\mathbb{M}_2^\circ(F)) \otimes \rho_\psi,$$

where $\mathbb{M}_2^\circ(F)$ is the space of trace zero matrices. We view $C_c(\mathbb{M}_2^\circ(F)) \otimes \rho_\psi$ as a representation of the group $J = [L, L]U \subset G_2$ introduced in Sect. 11.2, where the first factor is a representation of $\tilde{\text{SL}}_2$ and $\rho_\psi$ is the irreducible representation of $\tilde{J}$ introduced in §11.2. With the group $\text{PGL}_2$ acting trivially on $\rho_\psi$, we see that $C_c(\mathbb{M}_2(F))$ becomes a representation of $\text{PGL}_2 \times J$.

We are now ready to compute $\Pi_{S,\psi_S}$. Firstly, we need $\Pi_{N_2,\psi_2}$. This is a twisted variant of $\Pi_{N_2}$, given by Proposition 13.7, and computed along the same lines. In fact, since the character $\psi_2$ is generic, instead of a filtration we end up with a single term:

$$\Pi_{N_2,\psi_2} \cong \text{ind}^{G_2}_J(C_c(\mathbb{M}_2^\circ(F)) \otimes \rho_\psi)$$

as $G_2 \times \text{PGL}_2$-modules. It remains to compute the $\text{PGL}_2$-coinvariants of the right hand side. We need the following:

**Lemma 11.6** Let $H \subset G$ and $L$ be three $p$-adic groups. Let $W$ be a smooth $H \times L$-module, and $\tau$ an irreducible representation of $L$. Let $\Theta(\tau) \otimes \tau$ be the maximal $\tau$-isotypic quotient of $W$. If $\text{Ext}^1_L(\tau, \tau) = 0$ then

$$\text{ind}^G_H \Theta(\tau) \otimes \tau$$

is the maximal $\tau$-isotypic quotient of $\text{ind}^G_H W$. Here $\text{ind}$ stands for induction with compact support.

**Proof** Since $\text{Ext}^1_L(\tau, \tau) = 0$, the kernel of the projection of $W$ on $\Theta(\tau) \otimes \tau$ does not have $\tau$ as a quotient. Thus, it suffices to prove that if $\text{Hom}_L(W, \tau) = 0$, then $\text{Hom}_L(\text{ind}^G_H W, \tau) = 0$. We shall prove that

$$\text{Hom}_L((\text{ind}^G_H W)^K, \tau) = 0$$

for any open compact subgroup $K$ of $G$. Write $G = \bigcup_{i \in I} H g_i K$ where $I$ is an index set, and set $K_i = H \cap g_i K g_i^{-1}$ for every $i \in I$. Then, as an $L$-module,
(\text{ind}_H^G W)^K \) is a direct sum of \( W^{K_i} \). Since \( W^{K_i} \) is a direct summand of the \( L \)-module \( W \), it follows that \( \text{Hom}_L(W^{K_i}, \tau) = 0 \), and this proves the lemma. \( \square \)

We apply Lemma 11.6 taking \( H \subset G \) to be \( J \subset G_2 \) and \( L = \text{PGL}_2 \). Since \( \text{Ext}^1_{\text{PGL}_2}(1, 1) = 0 \), the lemma implies that computing \( \text{PGL}_2 \)-coinvariants of \( \Pi_{N_2, \psi_2} \) boils down to computing the \( \text{PGL}_2 \)-coinvariant of \( C_\psi(M_\psi^\circ(F)) \), where it is the full degenerate principal series \( I_\psi(1/2) \). We have shown:

**Proposition 11.7** As a representation of \( G_2(F) \), one has

\[
\Pi_{S, \psi_S} \cong (\Pi_{N_2, \psi_2})_{\text{PGL}_2} \cong \text{ind}_J^{G_2} (I_\psi(1/2) \otimes \rho_\psi).
\]

12 Howe duality for \( G_2 \times \text{PGSp}_6 \): tempered case

After the preparation of the previous 3 sections, we are now in a position to begin our study of the theta correspondence for the dual pair \( G_2 \times \text{PGSp}_6 \). In this section, we shall show the Howe duality theorem for tempered representations of \( G_2 \). The key is to show the analog of Propositions 6.6 and 6.7 for generic representations of \( G_2 \). This will rely on another curious chain of containments given in the following lemma, which comes from a consideration of a game of ping-pong with periods.

**Lemma 12.1** Let \( \Pi \) be the minimal representation of \( E_7 \). Let \( \pi \in \text{Irr}(G_2) \) be tempered and let \( \psi_V : V \to \mathbb{C}^\times \) be a Whittaker character for \( G_2 \). Let \( H = \text{PGSp}_6 \) and \( \tau \in \text{Irr}(H) \) be tempered such that

\[
\text{Hom}_{G_2 \times H}(\Pi, \pi \boxtimes \tau) \neq 0.
\]

Then we have the following natural inclusions

\[
\text{Hom}_V(\pi, \psi_V) \subseteq \text{Hom}_V(\Theta(\tau), \psi_V) \cong \text{Hom}_S(\tau^\vee, \bar{\psi}_S)
\]

\[
\subseteq \text{Hom}_S(\Theta(\pi^\vee), \bar{\psi}_S) \cong \text{Hom}_{G_2}(\text{ind}_j^{G_2} I_\psi(1/2) \otimes \rho_\psi, \pi^\vee).
\]

If \( \pi \) is generic, then all of these spaces are one-dimensional.

**Proof** We examine each containment in turn:

- The first inclusion arises from the surjection \( \Theta(\tau) \twoheadrightarrow \pi \).
- The second follows from the identity

\[
\text{Hom}_V(\Theta(\tau), \psi_V) \cong \text{Hom}_{V \times H}(\Pi, \psi_V \boxtimes \tau) \cong \text{Hom}_H(\Pi_{V, \psi_V}, \tau)
\]
combined with Proposition 11.5 (i.e. [34, Lemma 4.5]):

$$\Pi_{V,\psi_V} \cong \text{ind}_S^H \psi_S$$

and the Frobenius reciprocity.

- For the third, observe that $\Theta(\bar{\pi})$ is the complex conjugate of $\Theta(\pi)$. Since $\bar{\pi} \cong \pi^\vee$ and $\bar{\tau} \cong \tau^\vee$, we have $\Theta(\pi^\vee) \rightharpoonup \tau^\vee$.

- The fourth follows from the identity,

$$\text{Hom}_S(\Theta(\pi^\vee), \bar{\psi}_S) \cong \text{Hom}_{S \times G_2}(\Pi, \bar{\psi}_S \boxtimes \pi^\vee) \cong \text{Hom}_{G_2}(\Pi_S, \bar{\psi}_S, \pi^\vee)$$

combined with Proposition 11.7:

$$\Pi_{S,\bar{\psi}_S} \cong \text{ind}_{G_2}^G I_\psi(1/2) \otimes \rho_{\bar{\psi}}$$

and Frobenius reciprocity.

If $\pi$ is generic, then the first and the last spaces are one-dimensional, with the latter by Corollary 11.4 applied to $s = 1/2$. Hence, all spaces in the chain are one-dimensional.

We can now obtain the following two propositions as consequences of Lemma 12.1.

**Proposition 12.2** Let $\tau \in \text{Irr}(\text{PGSp}_6)$ be tempered. Then $\Theta(\tau)$ cannot have two irreducible tempered and generic quotients.

**Proof** Let $\pi_1, \pi_2 \in \text{Irr}(G_2)$ be tempered and generic such that $\Theta(\tau) \rightharpoonup \pi_1 \oplus \pi_2$. Then

$$\dim \Theta(\tau)_{V,\psi_V} \geq 2.$$ 

On the other hand, $\dim \Theta(\tau)_{V,\psi_V} = 1$ by Lemma 12.1, which is a contradiction.

**Remark** This proposition is proved in [34] for generic supercuspidal representations using the uniqueness of Shalika functional (shown in [34] for supercuspidal representations), but the proof of this uniqueness given there is difficult. The proof here is based on Corollary 11.4.

**Proposition 12.3** Let $\pi \in \text{Irr}(G_2)$ be tempered and generic. Then $\Theta(\pi)$ cannot have two tempered irreducible quotients. In particular, its cuspidal component $\Theta(\pi)_c$ is irreducible or 0.
Proof Let \( \tau_1, \tau_2 \in \text{Irr}(\text{PGSp}_6) \) be irreducible tempered and such that \( \Theta(\pi) \to \tau_1 \oplus \tau_2 \). By Lemma 12.1, applied to \( \pi \setminus \tau_1 \) and then to \( \pi \setminus \tau_2 \), one has:

\[
1 = \dim \text{Hom}_S(\tau_1, \psi_S) = \dim \text{Hom}_S(\Theta(\pi), \psi_S) = \dim \text{Hom}_S(\tau_2, \psi_S).
\]

Since \( \tau_1 \oplus \tau_2 \) is a quotient of \( \Theta(\pi) \),

\[
1 = \dim \text{Hom}_S(\Theta(\pi), \psi_S) \geq \dim \text{Hom}_S(\tau_1, \psi_S) + \dim \text{Hom}_S(\tau_2, \psi_S) = 2,
\]

which is a contradiction. \( \Box \)

Combining Propositions 12.2 and 12.3 with the results of Sect. 6, we can now show the Howe duality theorem for tempered representations of \( G_2 \):

**Theorem 12.4** Let \( \pi \in \text{Irr}(G_2) \) be tempered and consider its big theta lift \( \Theta(\pi) \) on \( \text{PGSp}_6 \). Then

(i) \( \Theta(\pi) \) has finite length and a unique irreducible quotient \( \theta(\pi) \) (if nonzero), which is tempered.

(ii) Moreover, for tempered \( \pi_1, \pi_2 \in \text{Irr}(G_2) \),

\[
0 \neq \theta(\pi_1) \cong \theta(\pi_2) \implies \pi_1 \cong \pi_2.
\]

**Proof** (i) We have seen (i) for non-generic \( \pi \) in Corollary 6.8. The proof for generic \( \pi \) is the same, using Lemmas 6.2(i) and 6.3(ii), as well as Proposition 12.3.

(ii) If one of \( \pi_1 \) or \( \pi_2 \) is nongeneric, then the desired result follows by Proposition 6.6. If \( \pi_1 \) and \( \pi_2 \) are both generic, then the desired result follows by Proposition 12.2. \( \Box \)

We also point out the following corollary:

**Corollary 12.5** Let \( \pi \in \text{Irr}(G_2) \) be generic, supercuspidal and not a theta lift from \( \text{PGL}_3 \). Then \( \Theta(\pi) \) is generic, supercuspidal and irreducible.

**Proof** By [34], we have known that \( \Theta(\pi) \) is generic and supercuspidal (hence tempered and semisimple), but now we know by Proposition 12.3 that it is also irreducible. \( \Box \)

### 13 Jacquet modules

The purpose of this section is to compute various Jacquet modules of the minimal representation of \( E_7 \) with respect to the maximal parabolic subgroups of \( G_2 \) and \( \text{PGSp}_6 \). We note that the results of this section are entirely...
self-contained, and do not depend on any prior results in this paper. As consequences of the results here, we deduce Lemmas 6.2 and 6.3 for the dual pair $G_2 \times \text{PGSp}_6$. Indeed, we shall determine in Theorem 14.1 the theta lifts of all non-tempered representations of $G_2$ and $\text{PGSp}_6$ precisely.

13.1 Jacquet functors for $G_2$

Recall that $P = MN$ and $Q = LU$ are the two maximal parabolic subgroups of $G_2$ as before, in standard position relative to a maximal split torus $T$ in $G_2$ and a choice of positive roots, so that $P \cap Q$ is a Borel subgroup. In particular, their Lie algebras arise from $\mathbb{Z}$-gradings given by two fundamental co-characters. Since $G_2$ is contained in $E_7$, as a member of the dual pair, the two co-characters give $\mathbb{Z}$-gradings of the Lie algebra of $E_7$, defining parabolic subgroups $P = MN$ and $Q = LU$ of $E_7$, whose intersections with $G_2$ are $P$ and $Q$, respectively. The Lie types of the Levi factors $M$ and $L$ are $D_6$ and $A_1 \times A_5$, as explained in [5]. In the rest of the paper, we shall use the following $P$ and $Q$ modules:

- The group $P$ stabilizes a 2-dimensional space of trace 0 octonions, and hence has a quotient isomorphic to $GL_2$. Then $C_c(GL_2)$ is a $P$-module obtained by inflating the left regular representation of $GL_2$ on $C_c(GL_2)$.
- The group $Q$ stabilizes a 1-dimensional space of trace 0 octonions, and hence has a quotient isomorphic to $GL_1$. Then $C_c(GL_1)$ is a $Q$-module obtained by inflating the left regular representation of $GL_1$ on $C_c(GL_1)$.
- Let $\bar{B}$ be the group of lower-triangular matrices in $GL_2$. Then $\bar{B}$ acts on $C_c(GL_1)$ by right translation by the $(1, 1)$ matrix entry of $g \in \bar{B}$.

We identify $M \cong GL_2$ such that the action of $M$ on $N/[[N, N]]$ is the symmetric cube of the standard representation of $GL_2$ twisted by determinant inverse. In particular, a scalar matrix $(z, z)$ in $GL_2$ acts by $z$. We have [25, Theorem 6.1],

**Proposition 13.1** Let $H = \text{PGSp}_6$. As a $GL_2 \times H$-module, $r_P(\Pi)$ (the normalized Jacquet functor) has a filtration with three successive subquotients (top to bottom):

1. $\delta_P^{-1/2} \cdot \Pi^N_N = \Pi_{D_6} \cdot |\det|^{1/2} \oplus \Pi_\emptyset \cdot |\det|^{3/2}$.
2. $\text{Ind}_{B \times P_2}^{GL_2 \times H}(\delta \cdot C_c(GL_1))$.
3. $\text{Ind}^H_{P_{13}} C_c(GL_2)$.

Here, note that:

- In (1), the center of $M \cong GL_2$ acts trivially on both $\Pi_{D_6}$ and $\Pi_\emptyset$, the minimal and the trivial representation of the Levi $M$. 

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– In (2), $\delta = | \cdot |^{-1/2} \times | \cdot |$ is a character of the group $\bar{B}$ of lower triangular matrices in $GL_2$.

For the computation of $r_Q(\Pi)$, we first make some preparations. Let $W$ be the Weil representation for the similitude dual pair $GL_2 \times GSO_4$; see [32] where theta correspondences for similitude groups are treated in detail. Observe that $GSO_4 \cong (GL_2 \times GL_2)/GL^\vee_2$, with the isomorphism realized by latter acting on the space $M_2(F)$ of $2 \times 2$ matrices by left and right multiplication and the quadratic form given by the determinant. We identify the first factor $GL_2$ with $L$ so that the action of $L$ on $U/[U, U]$ is the standard representation of $GL_2$. The irreducible quotients of $W$ are $\pi^\vee \otimes \pi \otimes \pi$, where $\pi$ is an irreducible representation of $GL_2$. We need a slight refinement of this to obtain the big theta lifts.

**Lemma 13.2** Consider the similitude theta correspondence for the dual pair $GL_2 \times GSO_4$ on $W$. Let $\pi$ be an irreducible generic representation of $GL_2$. Then $\Theta(\pi^\vee) = \pi \otimes \pi$ and $\Theta(\pi \otimes \pi) = \pi^\vee$.

**Proof** Let $V \cong F$ be a maximal unipotent subgroup of $GL_2$ and $\psi$ a non-trivial character of $V$. We shall use the fact that $W_{V, \psi} = C_c(GL_2)$, the regular representation of $GL_2$, where $V$ is in any of the three $GL_2$. Also, when we view $C_c(GL_2)$ as giving a theta correspondence between two $GL_2$, the big theta lift of any irreducible $\pi$ is either $\pi$ or $\pi^\vee$, depending on identifications or convention [1, Lemma 2.4].

We have $\Theta(\pi \otimes \pi) \otimes (\pi \otimes \pi)$ as a quotient of $W$. Applying the functor of $(V, \psi)$-coinvariants, with $V$ sitting in one of $GL_2$ factors of $GSO_4$, we conclude that $\Theta(\pi \otimes \pi) \otimes \pi$ is a quotient of the regular representation of $GL_2$. This implies that $\Theta(\pi \otimes \pi) \cong \pi^\vee$, as desired.

In the other direction, let $\pi_1 \otimes \pi_2$ be a quotient (if any) of the kernel of $\Theta(\pi^\vee) \to \pi \otimes \pi$. If $\pi_1$ or $\pi_2$ is generic, then we can take $(V, \psi)$ twisted co-invariants of $\pi^\vee \otimes \Theta(\pi^\vee)$, for the corresponding $GL_2$, and obtain a contradiction to the fact that, for the regular representation, the big theta lift of $\pi^\vee$ is $\pi$. Thus both $\pi_1$ and $\pi_2$ are one-dimensional. By the Kunneth formula [31]

$$\text{Ext}_{GL_2 \times GL_2}^1(\pi_1 \otimes \pi_2, \pi \otimes \pi) = \bigoplus_{i+j=1} \text{Ext}_{GL_2}^i(\pi_1, \pi) \otimes \text{Ext}_{GL_2}^j(\pi_2, \pi)$$

and this clearly vanishes since $\text{Hom}_{GL_2}(\pi_1, \pi) = 0$, for $i = 1, 2$. Thus $\pi_1 \otimes \pi_2$ is a quotient of $\Theta(\pi^\vee)$. Hence $\pi^\vee \otimes \pi_1 \otimes \pi_2$ is an irreducible quotient of $W$, contradicting the fact that all irreducible quotients are of the form $\pi^\vee \otimes \pi \otimes \pi$.

**Proposition 13.3** Let $H = PGSp_6$. As a $GL_2 \times H$-module, $r_Q(\Pi)$ (the normalized Jacquet functor) has a filtration with three successive subquotients (top to bottom):
(1) \( \delta^{-1/2} \cdot \Pi_{ Ud} = \Pi_{ A_5} \cdot |\det|^{3/2} \oplus \Pi_{ A_1} \cdot |\det|^2 \).

(2) \( \text{Ind}_{B \times P_2}^{GL_2 \times H} (\delta \cdot C_c(\text{GL}_1)) \).

(3) \( \text{Ind}_{H}^{P_2} W \).

- In (1), the center of \( L \cong \text{GL}_2 \) acts trivially on both \( \Pi_{ A_5} \) and \( \Pi_{ A_1} \), the minimal and a principal series representation of the two factors of \( L \).
- In (2) \( \delta = |\cdot|^{1/2} \times |\cdot| \) is a character of the group \( \bar{B} \) of lower triangular matrices in \( \text{GL}_2 \).

Proof This proposition is entirely similar to [5, Prop. 6.8], which treated the case of non-split form of \( H \), except the character \( \delta \) was not determined there. This is done as follows. For a generic character \( \chi \) of \( \text{GL}_2 \), the representations \( I_Q(\chi) \) and \( I_2(\chi \otimes \chi) \) are both irreducible and \( I_Q(\chi) \otimes I_2(\chi \otimes \chi) \) is a quotient of \( \Pi \); this follows from the bottom factor (3) of the filtration. Hence \( r_Q(I_Q(\chi)) \otimes I_2(\chi \otimes \chi) \) is a quotient of \( r_{P_2}(\Pi) \). Now determining \( \delta \) is an easy exercise using \( r_Q(I_Q(\chi)) \). \( \square \)

13.2 Non-tempered representations

We enumerate the non tempered irreducible representations of \( G_2 \) using the discussion from Sect. 3. Let \( P = MN \) and \( Q = LU \) be the two maximal parabolic subgroups in \( G_2 \) as before. Their Levi groups are isomorphic to \( \text{GL}_2 \). Let \( \tau \) be a representation of \( \text{GL}_2 \), and let \( I_P(\tau) \) and \( I_Q(\tau) \) be the corresponding normalized induced representations of \( G_2 \). Irreducible, non-tempered representations of \( G_2 \) are described as follows, where \( \tau \) is irreducible, and \( \omega_\tau \) is the central character of \( \tau \).

(a) Unique irreducible quotient of \( I_Q(\tau) \) where \( \tau \) is an unramified twist of a tempered representation such that \( |\omega_\tau| = |\cdot|^s \) for some \( s > 0 \).

(b) Unique irreducible quotient of \( I_P(\tau) \) where \( \tau \) is an unramified twist of a tempered representation such that \( |\omega_\tau| = |\cdot|^s \) for some \( s > 0 \).

(c) Unique quotient of \( I_P(\tau) \) where \( \tau \) is the unique quotient of a representation induced from an ordered pair of characters \( \chi_1, \chi_2 \) such that \( |\chi_1| = |\cdot|^{s_1}, |\chi_2| = |\cdot|^{s_2} \) where \( s_1 > s_2 > 0 \).

In (a) and (b), \( I_Q(\tau) \) and \( I_P(\tau) \) are standard modules, while in (c), \( I_P(\tau) \) is a quotient of a standard module associated to the minimal parabolic \( P \cap Q \).

In any case, each of these induced representations has a unique irreducible quotient which we denote by \( J_Q(\tau) \) in (a) and by \( J_P(\tau) \) in (b) and (c). These representations \( J_Q(\tau) \) and \( J_P(\tau) \) exhaust the irreducible non tempered representations of \( G_2 \).

We also enumerate some relevant non tempered representations of \( \text{PGSp}_6 \). Let \( P_i = M_i N_i, i = 1, 2, 3 \) be the three maximal parabolic subgroups of
PGSp₆. Let $I_i(\sigma)$ denote the representation of PGSp₆ obtained by normalized parabolic induction from $P_i$, and let $I_{jk}(\sigma)$ denote the representation of PGSp₆ obtained by normalized parabolic induction from $P_j \cap P_k$. We shall consider the following non-tempered representations of PGSp₆, corresponding to the cases (a), (b) and (c) above:

(a') If $\tau$ is an irreducible representation of $L = \text{GL}_2$ satisfying the conditions of (a) above, let $\sigma = \tau \otimes \tau$ be a representation of $M_2 \cong \text{GL}_2 \times \text{GL}_2 / \text{GL}_1 \cong \text{GSO}_4$. Then $I_2(\sigma)$ is a standard module, with unique irreducible quotient $J_2(\sigma) = J_2(\tau \otimes \tau)$.

(b') If $\tau$ is an irreducible representation of $M = \text{GL}_2$, satisfying the conditions of (b) above, let $\sigma = \tau \otimes 1$ be a representation of $M_1 \cap M_3 \cong \text{GL}_2 \times \text{GL}_1$. Then $I_{13}(\sigma)$ is a standard module with unique irreducible quotient $J_{13}(\sigma) = J_{13}(\tau \otimes 1)$.

(c') If $\tau$ is an irreducible representation of $M = \text{GL}_2$, satisfying the conditions of (c) above, let $\sigma = \tau \otimes 1$ be a representation of $M_1 \cap M_3 \cong \text{GL}_2 \times \text{GL}_1$. Then $I_{13}(\sigma)$ is a quotient of a standard module associated to the Borel subgroup, Hence, it has a unique irreducible quotient which we denote by $J_{13}(\sigma) = J_{13}(\tau \otimes 1)$.

### 13.3 Theta lifts from $G_2$

Now the following lemma attempts to compute the theta lifts of the above non-tempered representations of $G_2$ to PGSp₆.

**Lemma 13.4** Let $\pi \in \text{Irr}(G_2)$ be non-tempered.

- If $\pi \subset I_Q(\tau^\vee)$ where $\tau$ is as in (a) above, then $\Theta(\pi)$ is a quotient of $I_2(\tau \otimes \tau)$ and hence has finite length. Moreover, $\Theta(I_2(\tau \otimes \tau)) \neq 0$ where $I_2(\tau \otimes \tau)$ is the unique irreducible quotient of $I_2(\tau \otimes \tau)$.

- If $\pi \subset I_P(\tau^\vee)$ where $\tau$ is as in (b) and (c) above, then $\Theta(\pi)$ is a quotient of $I_{13}(\tau \otimes 1)$ and hence has finite length. Moreover, $\Theta(I_{13}(\tau \otimes 1)) \neq 0$ where $I_{13}(\tau \otimes 1)$ is the unique irreducible quotient of $I_{13}(\tau \otimes 1)$.

**Proof** Let $\Pi$ be the minimal representation, and $\pi \in \text{Irr}(G_2)$. We shall use the fact that

$$\Theta(\pi)^* \cong \text{Hom}_{G_2}(\Pi, \pi)$$

as non-smooth $H = \text{PGSp}_6$-modules, where the former is the linear dual of $\Theta(\pi)$. Assume that $\pi \subset I_Q(\tau^\vee)$. Then

$$\Theta(\pi)^* = \text{Hom}_{G_2}(\Pi, \pi) \subset \text{Hom}_{G_2}(\Pi, I_Q(\tau^\vee)) \cong \text{Hom}_{L}(r_Q(\Pi), \tau^\vee).$$

Now we shall use the filtration of $r_Q(\Pi)$ from Proposition 13.3.
Let $\Pi_1, \Pi_2$ and $\Pi_3$ denote the three subquotients in the same order. Observe that $\text{Ext}^1_L(\Pi_1, \tau^\vee)$ are trivial from the central character considerations, since the central character of $\tau^\vee$ is a negative power of $|z|$. Hence we have a long exact sequence

$$0 \rightarrow \text{Hom}_L(\Pi_2, \tau^\vee) \rightarrow \text{Hom}_L(r_Q(\Pi), \tau^\vee) \rightarrow \text{Hom}_L(\Pi_3, \tau^\vee) \rightarrow \text{Ext}^1_L(\Pi_2, \tau^\vee).$$

Since $\Pi_2$ is induced from $\bar{B}$, by the second adjointness,

$$\text{Ext}^i_L(\Pi_2, \tau^\vee) \cong \text{Ext}^i_T(\text{Ind}^{H}_{P_2}(\delta \cdot C_c(\text{GL}_1)), r_B(\tau^\vee)),$$

where $T = \text{GL}_1 \times \text{GL}_1$, the maximal torus in $B$. Observe that the action of the second $\text{GL}_1$ on $\text{Ind}^{H}_{P_2}(\delta \cdot C_c(\text{GL}_1))$ is $|\cdot|$, and this is different from the action on $r_B(\tau^\vee)$ by our assumption on $\tau$. Hence $\text{Ext}^i_L(\Pi_2, \tau^\vee) = 0$ for all $i$, and we can conclude that

$$\text{Hom}_L(r_Q(\Pi), \tau^\vee) \cong \text{Hom}_L(\Pi_3, \tau^\vee) \cong \text{Hom}_L(\text{Ind}^H_{P_3} W, \tau^\vee),$$

where, for the second isomorphism, we have simply substituted the explicit expression for $\Pi_3$ given in Proposition 13.3. By [3, Lemma 9.4], the maximal $\tau^\vee$ isotypic quotient of $\text{Ind}^{H}_{P_2} W$ is $(\text{Ind}^H_{P_2} \Theta(\tau^\vee)) \otimes \tau^\vee$ where $\Theta(\tau^\vee)$ is the big theta lift for the similitude theta correspondence on $W$. Since $\tau$ is generic, Lemma 13.2 shows that $\Theta(\tau^\vee) = \tau \otimes \tau$ and it follows that

$$\text{Hom}_L(\Pi_3, \tau^\vee) \cong I_2(\tau \otimes \tau)^*.$$

Hence $\Theta(\pi)^* \subset I_2(\tau \otimes \tau)^*$, and $\Theta(\pi)^\vee \subset I_2(\tau \otimes \tau)^\vee$ by taking smooth vectors. Thus $\Theta(\pi)$ is a quotient of $I_2(\tau \otimes \tau)$. Observe that we have proved in the process that $I_2(\tau \otimes \tau)$ is a quotient of $r_Q(\Pi)$, so that $\Theta(I_2(\tau \otimes \tau)) \neq 0$. This establishes the first bullet. The proof of the second is completely analogous. $\square$

13.4 Jacquet functors for PGSp$_6$

Recall that in PGSp$_6$, we have fixed three standard maximal parabolic subgroups $P_1$, $P_2$ and $P_3$. They correspond to $\mathbb{Z}$-gradings of the Lie algebra of PGSp$_6$ given by three fundamental co-characters. The action of each of these three co-characters gives a $\mathbb{Z}$-grading of the Lie algebra of $E_7$, and these gradings define three parabolic subgroups $P_1$, $P_2$ and $P_3$ of $E_7$. To recognize these parabolic subgroups, perhaps it is easiest to proceed as follows. Observe that the $E_7$ Dynkin diagram contains a unique $D_4$ subdiagram. We embed $G_2$ into $D_4$. The centralizer of $G_2$ in the split, adjoint $E_7$ is PGSp$_6$. Let $P$ be the parabolic subgroup of $E_7$, whose Levi factor has the type $D_4$. This parabolic

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is contained in precisely three maximal parabolic subgroups denoted by \( P_1 \), \( P_2 \) and \( P_3 \), whose Levi factor types are, \( D_6 \), \( A_1 \times D_5 \) and \( E_6 \), respectively. The intersection of \( P_i \) and PGSp\(_6\) is \( P_i \), for each \( i \). We write \( P_i = M_i N_i \) and \( P_i = M_i N_i \) for the Levi decompositions for these parabolic subgroups.

**Case \( P_3 \)** This is treated in [25, §5], and we summarize the results as follows. The unipotent subgroups of \( P_3 \) and \( P_3 \) are abelian, \( M_3 \cong GL_3 \) and the modular character is

\[
\delta_{P_3}(m) = |\det(m)|^2.
\]

Let \( \mathbb{O}_0 \) denote the space of trace 0 elements in the octonion algebra \( \mathbb{O} \). On the space \( \mathbb{O}_0^3 \), we have the natural diagonal action \((x, y, z) \mapsto (gx, gy, gz)\) of \( g \in G_2 \) and the row-vector action \((x, y, z) \mapsto (x, y, z)m^{-1}\) of \( m \in GL_3 \). Let \( \Omega \subset \mathbb{O}_0^3 \) be the set of all nonzero \((x, y, z)\) such that the linear subspace \( \langle x, y, z \rangle \subset \mathbb{O}_0 \) is a null-space for octonion multiplication, i.e. the product of any two elements in the space is 0. Such non-zero null-spaces in \( \mathbb{O}_0 \) are of dimension 1 or 2.

We have an exact sequence of \( G_2 \times GL_3 \)-modules

\[
0 \to C_c(\Omega) \to \Pi_{N_3} \to \Pi_{N_3} \to 0,
\]

where \((g, m) \in G_2 \times GL_3\) acts on \( f \in C_c(\Omega)\) by

\[
((g, m) \cdot f)(x, y, z) = |\det(m)|^2 \cdot f((g^{-1}x, g^{-1}y, g^{-1}z)m).
\]

The group \( G_2 \times GL_3 \) acts on \( \Omega \) with two orbits \( \Omega_1 \) and \( \Omega_2 \), where \( \Omega_i \) is the subset of triples \((x, y, z)\) such that \( \langle x, y, z \rangle \) has dimension \( i \). Thus \( C_c(\Omega) \) has a filtration with \( C_c(\Omega_2) \) as a submodule and \( C_c(\Omega_1) \) as a quotient. Each of these can be explicitly described as \( G_2 \times GL_3 \)-modules.

In order to state the result, let \( Q_1 \) and \( Q_2 \) be the maximal parabolic subgroups of \( GL_3 \) stabilizing subspaces consisting of row vectors \((*, 0, 0)\) and \((*, *, 0)\), respectively. Observe that these are block lower-triangular groups with Levi factors isomorphic to \( GL_1 \times GL_2 \) and \( GL_2 \times GL_1 \), respectively. Their modular characters are

\[
\delta_{Q_1}(g_1, g_2) = |g_1|^{-2} \cdot |\det(g_2)| \quad \text{and} \quad \delta_{Q_2}(g_2, g_1) = |\det(g_2)|^{-1} \cdot |g_1|^2.
\]

Recall that \( r_{P_3}(\Pi) = \delta_{P_3}^{-1/2} \cdot \Pi_{N_3} \) is the normalized Jacquet module. Then:

**Proposition 13.5** As a \( G_2 \times GL_3 \)-module, \( r_{P_3}(\Pi) \) has a filtration with three successive subquotients (from top to bottom):

1. \( \delta_{P_3}^{-1/2} \cdot \Pi_{N_3} = \Pi_{E_6} \oplus \Pi_\emptyset \cdot |\det| \).
(2) \( \text{Ind}^{G_2 \times \text{GL}_3}_{Q \times Q_1}(\delta \cdot C_c(\text{GL}_1)) \).
(3) \( \text{Ind}^{G_2 \times \text{GL}_3}_{P \times Q_2}(C_c(\text{GL}_2)) \).

Here, note that:

- In (1), the center of \( M_3 \cong \text{GL}_3 \) acts trivially on both \( \Pi_{E_6} \) and \( \Pi_{\phi} \), the minimal and the trivial representation of the Levi \( \mathcal{M}_3 \).
- In (2), \( \delta(g_1, g_2) = |g_1|^{-1/2} \times |\det(g_2)|^{1/2} \) is a character of \( Q_1 \).
- For \( i = 1, 2 \), \( Q_i \) acts on \( C_c(\text{GL}_i) \) by right translations via the factor \( \text{GL}_i \).

Case \( P_1 \)
This case is not in the literature; however, it is similar to the computation of the Jacquet module of the minimal representation of \( E_8 \) with respect to a maximal parabolic subgroup of \( F_4 \) in [35, §5]. The unipotent radical subgroups of \( P_1 \) and \( P_1 \) are Heisenberg groups with \( M_1 \cong \text{GSp}_4 \). Let \( \nu \) be the similitude character of \( \text{GSp}_4 \). The modulus character of \( M_1 \) is \( \delta_{P_1}(m) = |\nu(m)|^3 \).

Recall that \( \mathbb{O}_0 \) is the space of trace 0 octonions. On \( \mathbb{O}_0^4 \), we have the row-vector action

\[
(x, y, x', y') \mapsto (x, y, x', y')m^{-1} \quad \text{of } m \in \text{GSp}_4
\]

preserving the form \( \mathbb{O}_0^4 \rightarrow \wedge^2 \mathbb{O}_0 \) given by

\[
(x, y, x', y') \mapsto x \wedge x' + y \wedge y'.
\]

Let \( \Omega \subset \mathbb{O}_0^4 \) be the set of all nonzero \((x, y, x', y')\) such that the linear subspace \( \langle x, y, x', y' \rangle \subset \mathbb{O}_0 \) is a null-space for octonion multiplication and \( x \wedge x' + y \wedge y' = 0 \). We have an exact sequence of \( G_2 \times \text{GSp}_4 \)-modules

\[
0 \rightarrow C_c(\Omega) \rightarrow \Pi_{N_1} \rightarrow \Pi_{N_1} \rightarrow 0,
\]

where \((g, m) \in G_2 \times \text{GSp}_4 \) acts on \( f \in C_c(\Omega) \) by

\[
((g, m) \cdot f)(x, y, x', y')) = |\nu(m)|^3 \cdot f((g^{-1}x, g^{-1}y, g^{-1}x', g^{-1}y')m).
\]

Now the group \( G_2 \times \text{GSp}_4 \) acts on \( \Omega \) with two orbits \( \Omega_1 \) and \( \Omega_2 \), where \( \Omega_i \) is the subset of quadruples \((x, y, x', y')\) such that \( \langle x, y, x', y' \rangle \) has dimension \( i \). Thus \( C_c(\Omega_i) \) has a filtration with \( C_c(\Omega_2) \) as a submodule and \( C_c(\Omega_1) \) as a quotient. Each of these can be explicitly described as \( G_2 \times \text{GSp}_4 \)-modules.

In order to state the result, let \( Q_1 \) and \( Q_2 \) be the maximal parabolic subgroups of \( \text{GSp}_4 \) stabilizing subspaces consisting of row vectors \((*, 0, 0, 0)\) and \((*, *, 0, 0)\), respectively. Let \( L_1 \cong \text{GL}_1 \times \text{GL}_2 \) be the Levi subgroup of \( Q_1 \).
such that \((g_1, g_2) \in \text{GL}_1 \times \text{GL}_2\) acts on the quadruples, after rearranging the order, by
\[
(x, x', y, y') \mapsto (xg_1^{-1}, x'g_1 \det(g_2)^{-1}, (y, y')g_2^{-1}).
\]
Let \(L_2 \cong \text{GL}_2 \times \text{GL}_1\) be the Levi subgroup of \(Q_2\) such that \((g_2, g_1) \in \text{GL}_2 \times \text{GL}_1\) acts on the quadruples by
\[
(x, y, x', y') \mapsto ((x, y)g_2^{-1}, (x', y')g_1^{-1}g_2^\top).
\]
The similitude character \(\nu\), restricted to \(L_1\) and \(L_2\), is given by
\[
\nu(g_1, g_2) = \det(g_2)\quad \text{and} \quad \nu(g_2, g_1) = g_1,
\]
respectively, and the modulus characters are
\[
\delta_{Q_1}(g_1, g_2) = |g_1|^{-4} \cdot |\det(g_2)|^2\quad \text{and} \quad \delta_{Q_2}(g_2, g_1) = |\det(g_2)|^{-3} \cdot |g_1|^3.
\]
Recalling that \(r_{P_1}(\Pi) = \delta_{P_1}^{-1/2} \cdot \Pi_{N_1}\) is a normalized Jacquet module, we have:

**Proposition 13.6** As a \(G_2 \times \text{GSp}_4\)-module, \(r_{P_1}(\Pi)\) has a filtration with three successive subquotients (from top to bottom):

1. \(\delta_{P_1}^{-1/2} \cdot \Pi_{N_1} = \Pi_{D_6} \cdot |v|^{1/2} \oplus \Pi_{\emptyset} \cdot |v|^{3/2}\).
2. \(\text{Ind}_{Q_1 \times Q_2}^{G_2 \times \text{GSp}_4}(\delta \cdot C_c(\text{GL}_1))\).
3. \(\text{Ind}_{P \times Q_2}^{G_2 \times \text{GSp}_4}(C_c(\text{GL}_2))\).

Here, note that
- In (1), the center of \(M_1 \cong \text{GSp}_4\) acts trivially on both \(\Pi_{D_6}\) and \(\Pi_{\emptyset}\), the minimal and the trivial representation of the Levi \(M_1\).
- In (2), \(\delta(g_1, g_2) = |g_1|^{-1/2} \times |\det(g_2)|^{1/2}\), a character of \(Q_1\).
- For \(i = 1, 2\), \(Q_i\) acts on \(C_c(\text{GL}_i)\) by right translations via the factor \(\text{GL}_i\).

**Case \(P_2\)** A variant of this case can be found in [5] for the non-split form of PGSp_6. However, for the split case considered in this paper, the Jacquet module filtration contains an additional “middle” term.

The unipotent radical subgroups \(N_2 \subset P_2\) and \(\mathcal{N}_2 \subset \mathcal{P}_2\) are two-step nilpotent subgroups. Let \(Z_2 \subset N_2\) be the center of \(N_2\). We now explain how the kernel of the natural projection \(\Pi_{\mathcal{Z}_2} \rightarrow \Pi_{N_2}\) contributes to \(\Pi_{N_2}\). We have
\[
0 \rightarrow C_c(\omega) \rightarrow \Pi_{\mathcal{Z}_2} \rightarrow \Pi_{N_2} \rightarrow 0,
\]
where \(\omega\) is the \(M_2\)-highest weight orbit in
\[
\mathcal{N}_2/\mathcal{Z}_2 \cong \emptyset \otimes M_2(F) = M_2(\emptyset),
\]
where $\tilde{N}_2$ is the unipotent group opposite to $N_2$, and $M_2(F)$ is the set of two-by-two matrices. (In the non-split case $M_2(F)$ is replaced by a division algebra, so $\omega$ is empty; see the discussion on [5, Pg. 137].) Recall that the type of $M_2$ is $D_5 \times A_1$ and $\tilde{N}_2/\tilde{Z}_2 \cong F^{16} \otimes F^2$ where $F^{16}$ is a spin-module of $D_5$.

In the above isomorphism we assume that $A_1$ acts from the right on $M_2(\mathbb{O})$, and columns are vectors in the spin-module. Thus $\omega$ is the set of non-zero matrices

$$\begin{pmatrix} x & x' \\ y & y' \end{pmatrix},$$

where the two columns are linearly dependent over $F$ and each column (if non-zero) is a highest weight vector in the spin-module. Let $\Omega_1$ be the subset of $\omega$ such that $x, x', y, y'$ (of an element in $\Omega_1$) generate a nil-subalgebra. The group $G_2 \times M_2$ acts on $\Omega_1$ with two orbits $\Omega_1^1$ and $\Omega_1^2$, where $\Omega_1^i$ consists of elements such that $\langle x, x', y, y' \rangle$ has dimension $i$. Thus $C_c(\Omega_1)$, as a $G_2 \times M_2$-module, has $C_c(\Omega_1^2)$ as a submodule and $C_c(\Omega_1^1)$ as quotient.

**Proposition 13.7** As a $G_2 \times (GL_2 \times GL_2)/GL_1^\vee$-module, $r_{P_2}(\Pi)$ has a filtration with four successive sub quotients:

1. $\delta_{P_2}^{-1/2} \cdot \Pi_{\tilde{N}_2} = \Pi_{D_5} \cdot |\det|^{1/2} \oplus \Pi_{A_1} \cdot |\det|^{3/2}$.
2. $\text{Ind}_{Q \times (B \times B)/GL_1^\vee}^{G_2 \times (GL_2 \times GL_2)/GL_1^\vee} (\delta \cdot C_c(GL_1))$.
3. $\text{Ind}_{P \times B}^{G_2 \times GL_2} (C_c(GL_2))$.
4. $\text{Ind}_{Q}^{G_2 \times GL_2} W$.

*Here, note that:

In (1), the second $SL_2 \subset M_2$ acts trivially on the first summand, and the first $SL_2 \subset M_2$ acts trivially on the second summand. The center of $M_2$
acts trivially on both $\Pi_D$ and $\Pi_A$, the minimal and a principal series representation of the two factors of $M_3$.

- In (2) $\delta = |\cdot|^{1/2} \times |\cdot|$ on each $B$.
- In (3), $\bar{B}$ is the subgroup of the second factor $GL_2$ of $M_2$. It acts on $C_c(GL_2)$ by right translation by the scalar given by the $(1, 1)$ matrix entry. The first factor $GL_2$ of $M_2$ acts by right translations on $C_c(GL_2)$.
- In (4), $W$ is the Weil representation of $GL_2 \times (GL_2 \times GL_2)/GL_1^{\vee} \cong GL_2 \times GSO_4$.

This proposition is a combination of [5, Proposition 8.1], which accounts for the bottom piece of the filtration (4), and the above discussion. The pieces (2) and (3) are the spaces of functions $C_c(\Omega_1)$ and $C_c(\Omega_2)$, respectively. This also assumes that we have explicated the character $\delta'$ appearing in the action on $C_c(\Omega)$. To that end, observe that (3) (or any unknown twist) gives a correspondence of generic principal series representations of $G_2$ and PGSp$_6$ that has to be compatible with the one in Lemma 13.4, and this determines $\delta'$ uniquely.

13.5 Theta lifts from PGSp$_6$

Using Propositions 13.5, 13.6 and 13.7, we can now prove the following analog of Lemma 13.4.

Lemma 13.8 Let $\sigma \in \text{Irr}(\text{PGSp}_6)$ be non-tempered. Then $\Theta(\sigma) = 0$ unless $\sigma$ is as described in Lemma 13.4. More precisely,

- If $\sigma \subseteq I_2(\tau^{\vee} \otimes \tau^{\vee})$, then $\Theta(\sigma)$ is a quotient of $I_Q(\tau)$ and hence has finite length. Moreover, $\Theta(J_Q(\tau)) \neq 0$ where $J_Q(\tau)$ is the unique irreducible quotient of $I_Q(\tau)$.
- If $\sigma \subseteq I_{13}(\tau^{\vee} \otimes 1)$, then $\Theta(\sigma)$ is a quotient of $I_P(\tau)$, and hence has finite length. Moreover, $\Theta(J_P(\tau)) \neq 0$ where $J_P(\tau)$ is the unique irreducible quotient of $I_P(\tau)$.

Proof We set $H = \text{PGSp}_6$. Assume that $\sigma$ is a Langlands quotient of a standard module for the maximal parabolic $P_2$. Then $\sigma \subseteq I_2(\tau_1^{\vee} \otimes \tau_2^{\vee})$ where $\tau_1$ and $\tau_2$ have the same central character and are both tempered representations of $GL_2$ twisted by a positive power of $|\det|$. Then

$$\text{Hom}_H(\Pi, \sigma) \subseteq \text{Hom}_H(\Pi, I_2(\tau_1^{\vee} \otimes \tau_2^{\vee})) \cong \text{Hom}_{M_2}(r_{P_2}(\Pi), \tau_1^{\vee} \otimes \tau_2^{\vee}).$$

Let $\Pi_i, i = 1, 2, 3, 4$ be the subquotients of $r_{P_2}(\Pi)$ as in Proposition 13.7, in the same order. We claim that

$$\text{Hom}_{M_2}(r_{P_2}(\Pi), \tau_1^{\vee} \otimes \tau_2^{\vee}) \cong \text{Hom}_{M_2}(\Pi_4, \tau_1^{\vee} \otimes \tau_2^{\vee}).$$
Assume this claim for a moment. Then

$$\text{Hom}_{M_2}(r_{P_2}(\Pi), \tau_1^\vee \otimes \tau_2^\vee) \cong \text{Hom}_{M_2}(\Pi_4, \tau_1^\vee \otimes \tau_2^\vee)$$

$$\cong \text{Hom}_{M_2}(\text{Ind}^{G_2}_Q W, \tau_1^\vee \otimes \tau_2^\vee)$$

where $W$ is the Weil representation of $GL_3$. This implies that $\Theta(\sigma) = 0$ unless $\tau_1 \cong \tau_2$, and if we denote this representation as $\tau$, then $\Theta(\sigma)$ is a non-zero quotient of the standard module $I_Q(\tau)$. In order to prove the claim, we need to show that $\text{Ext}^n_{M_2}(\Pi_i, \tau_1^\vee \otimes \tau_2^\vee) = 0$ for all $n$ and $i < 4$. Consider $i = 3$. Then, using the (second) Frobenius reciprocity for induction from $\bar{B}$ to $GL_2$, we have

$$\text{Ext}^i_{M_2}(\text{Ind}^{G_2\times GL_2}_P(C_c(GL_2)), \tau_1^\vee \otimes \tau_2^\vee)$$

$$\cong \text{Ext}^i_{T\times GL_2}(\text{Ind}^{G_2}_P(C_c(GL_2)), r_B(\tau_1^\vee) \otimes \tau_2^\vee)$$

where $T \cong GL_1 \times GL_1$ is the torus of diagonal matrices in $GL_2$. Now recall that the second $GL_1$ acts trivially on $\text{Ind}^{G_2}_P(C_c(GL_2))$. On the other hand, since $\tau_1$ is tempered with a positive twist of $|\det|$, the second $GL_1$ acts on $r_B(\tau_1^\vee)$ with characters $\chi$ such that $|\chi|$ is a negative power of absolute value. This proves the vanishing for $i = 3$. The other two cases are just as easy or even easier: for $i = 1$ vanishing follows from central character considerations, and for $i = 2$ using Frobenius reciprocity where it suffices that either $\tau_1$ or $\tau_2$ is twist of a tempered representation by a positive power of $|\det|$.

Now assume that $\sigma$ is a Langlands quotient of a standard module for the parabolic $P_{23} = P_2 \cap P_3$. Then, by induction in stages, we get that $\sigma \subseteq I_2(\tau_1^\vee \otimes \tau_2^\vee)$ where $\tau_1$ is a twist of a tempered representation by a positive power of $|\det|$. This is enough to show that $\text{Hom}_{M_2}(\Pi_i, \tau_1^\vee \otimes \tau_2^\vee) = 0$ for $i < 4$. Thus, if $\Theta(\sigma) \neq 0$ then $\text{Hom}_{M_2}(\Pi_4, \tau_1^\vee \otimes \tau_2^\vee) \neq 0$. This implies that $\tau_1 \cong \tau_2$, contradicting that $\sigma$ is a Langlands quotient of a standard module for the parabolic $P_{23}$. Hence $\Theta(\sigma) = 0$.

If $\sigma$ is a Langlands quotient of a standard module for the parabolic $P_{12} = P_1 \cap P_2$ then, by induction in stages, we get that $\sigma \subseteq I_2(\tau_1^\vee \otimes \tau_2^\vee)$ where now $\tau_2$ is a twist of a tempered representation by a positive power of $|\det|$. In this case, $\text{Hom}_{M_2}(\Pi_i, \tau_1^\vee \otimes \tau_2^\vee) = 0$ for $i \neq 3$ by repeating the above arguments. For $i = 3$ we have

$$\text{Hom}_{M_2}(\text{Ind}^{G_2\times GL_2}_P(C_c(GL_2)), \tau_1^\vee \otimes \tau_2^\vee)$$

$$\cong \text{Hom}_{T\times GL_2}(\text{Ind}^{G_2}_P(C_c(GL_2)), r_B(\tau_1^\vee) \otimes \tau_2^\vee)$$
and the last space is isomorphic to

$$\text{Hom}_{T \times \text{GL}_2}(\text{Ind}_P^{G_2}(\tau_2) \otimes \tau_2^\vee, r_B(\tau_1^\vee) \otimes \tau_2^\vee).$$

Recall that $T = \text{GL}_1 \times \text{GL}_1$ and the second $\text{GL}_1$ acts trivially on $\text{Ind}_P^{G_2}(C_c(\text{GL}_2))$ and hence on its quotient $\text{Ind}_P^{G_2}(\tau_2) \otimes \tau_2^\vee$. The first $\text{GL}_1$ acts on this space by the central character of $\tau_2$, which is equal to the central character of $\tau_1$, hence it is a nontrivial character, say $\chi$. Hence the above Hom space, if non-zero, is non-trivial if and only if $\chi \otimes 1$ is an exponent of $\tau_1$, and then it is isomorphic to

$$\text{Hom}_{\text{GL}_2}(\text{Ind}_P^{G_2}(\tau_2) \otimes \tau_2^\vee, \tau_2^\vee) \cong \text{Hom}(\text{Ind}_P^{G_2}(\tau_2), \mathbb{C}) = I_P(\tau_2^*).$$

Summarizing, $\Theta(\sigma) \neq 0$ implies that $\Theta(\sigma)$ is a quotient of $I_P(\tau_2)$. It follows that $J_P(\tau_2) \otimes \sigma$ is a quotient of $\Pi$, where $J_P(\tau_2)$ is the unique irreducible quotient of $I_P(\tau_2)$. But, by Lemma 13.4, $J_P(\tau_2)$ does not lift to $\sigma$. This is a contradiction, hence $\Theta(\sigma) = 0$.

The remaining non-tempered representations of $H$ (associated to standard modules induced from $P_{123}$, $P_{13}$, $P_1$ or $P_3$) are easily dealt with using $r_{P_1}(\Pi)$ and $r_{P_3}(\Pi)$. We leave details to the reader. \qed

14 Consequences of Jacquet module computations

We can now draw some definitive consequences of the Jacquet module computations of the previous section. In particular, we shall determine the theta lift of nontempered representations explicitly, and also complete the proofs of Lemmas 6.2 and 6.3 for the dual pair $G_2 \times \text{PGSp}_6$.

14.1 Lift of nontempered representations

Taken together, Lemmas 13.4 and 13.8 allow us to determine the theta lift of nontempered representations explicitly:

**Theorem 14.1** We have:

(a) $\Theta(J_Q(\tau))$ is a nonzero quotient of $I_2(\tau \otimes \tau)$ and hence has finite length with unique irreducible quotient $J_2(\tau \otimes \tau)$. Likewise, $\Theta(J_2(\tau \otimes \tau))$ is a nonzero quotient of $I_Q(\tau)$ and hence has finite length with unique irreducible quotient $J_Q(\tau)$.

(b) $\Theta(J_P(\tau))$ is a nonzero quotient of $I_{13}(\tau \otimes 1)$ and hence has finite length with unique irreducible quotient $J_{13}(\tau \otimes 1)$. Likewise, $\Theta(J_{13}(\tau \otimes 1))$ is a nonzero quotient of $I_P(\tau)$ and hence has finite length with unique irreducible quotient $J_P(\tau)$. \qed
(c) For all other nontempered $\sigma \in \text{Irr}(\text{PGSp}_6)$ different from those in (a) and (b), $\Theta(\sigma) = 0$.

In particular, if $\pi \otimes \sigma \in \text{Irr}(G_2 \times \text{PGSp}_6)$ is such that $\pi \otimes \tau$ is a quotient of the minimal representation $\Pi$, then

$$\pi \text{ nontempered } \iff \sigma \text{ nontempered}.$$

Hence, we have shown Lemma 6.2 for nontempered representations and also Lemma 6.3.

14.2 Finiteness of $\Theta(\pi)_{nc}$

To complete the proof of Lemma 6.2, we need to show that for tempered $\pi \in \text{Irr}(G_2)$ and $\sigma \in \text{Irr}(\text{PGSp}_6)$, the noncuspidal components $\Theta(\pi)_{nc}$ and $\Theta(\sigma)_{nc}$ are of finite length.

To show that $\Theta(\pi)_{nc}$ has finite length, it suffices to show that for each maximal parabolic subgroup $P_i = M_i N_i$ (with $1 \leq i \leq 3$) of $\text{PGSp}_6$, the Jacquet module $J_{P_i}(\Theta(\pi))$ has finite length as an $M_i$-module. In other words, we need to show that the multiplicity space of the maximal $\pi$-isotypic quotient of $r_{P_i}(\Pi)$ has finite length as an $M_i$-module.

We have described in Propositions 13.5, 13.6 and 13.7 an equivariant filtration of $r_{P_i}(\Pi)$ as an $G_2 \times M_i$-module and described the successive quotients. It suffices to show that, for each of these successive quotients $\Sigma$, the multiplicity space of the $\pi$-isotypic quotient of $\Sigma$ has finite length. We shall explain how this can be shown, depending on whether $\Sigma$ is a top piece of the filtration or not. The difference lies in the fact that the top piece of the filtration involves a minimal representation of a smaller group $M_i$ and hence one needs to consider theta correspondence in lower rank situations. When $\Sigma$ is not the top piece of the filtration, the finite length of the multiplicity space of the maximal $\pi$-isotypic quotient of $\Sigma$ as an $M_i$-module follows readily from the explicit description of $\Sigma$. We give two examples as illustration:

- Consider the case of $P_3 = \text{GL}_3 \cdot N_3$. The bottom piece of the filtration in Proposition 13.5 is

$$\Sigma = \text{Ind}_{P \times Q_2}^{G_2 \times \text{GL}_3} C_c(\text{GL}_2).$$

Then for $\pi \in \text{Irr}(G_2)$,

$$\Theta_{\Sigma}(\pi)^* := \text{Hom}_{G_2}(\Sigma, \pi) \cong \text{Hom}_M \left( \text{Ind}_{M \times Q_2}^{M \times \text{GL}_3} C_c(\text{GL}_2), r_{\bar{P}}(\pi) \right)$$

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where \( M \cong \text{GL}_2 \). Now \( r_{\rho}(\pi) \) is a finite length \( M \)-module and for any of its irreducible subquotient \( \sigma \),

\[
\text{Hom}_M \left( \text{Ind}_{M \times Q_2}^{M \times \text{GL}_3} C_c(\text{GL}_2), \sigma \right) \cong \left( \text{Ind}_{Q_2}^{\text{GL}_3} \sigma^\vee \right)^* 
\]

using the fact that the maximal \( \sigma \)-isotypic quotient of the regular representation \( C_c(\text{GL}_2) \) is of the form \( \sigma^\vee \otimes \sigma \). On taking smooth vectors (which is a left exact functor), we see that \( \Theta_\Sigma(\pi)^\vee \) has a finite filtration whose successive quotients are submodules of \( \text{Ind}_{Q_2}^{\text{GL}_3} \sigma^\vee \) for some irreducible \( \sigma \). In particular, \( \Theta_\Sigma(\pi) \) has finite length.

- Consider the case of \( P_2 = M_2 \cdot N_2 \) with \( M_2 = (\text{GL}_2 \times \text{GL}_2)/\text{GL}_1^\vee \cong \text{GSO}_4 \).

The bottom piece of the filtration in Proposition 13.7 is

\[
\Sigma = \text{Ind}_{Q \times M_2}^{G_2 \times M_2} W,
\]

where \( W \) is the Weil representation for \( \text{GL}_2 \times \text{GSO}_4 \). Then for \( \pi \in \text{Irr}(G_2) \),

\[
\Theta_\Sigma(\pi)^* := \text{Hom}_{G_2}(\Sigma, \pi) \cong \text{Hom}_{L \times M_2}(W, r_Q(\pi)).
\]

Now \( r_Q(\pi) \) has finite length as \( L \)-module (where \( L \cong \text{GL}_2 \)) and if \( \sigma \) is an irreducible subquotient, \( \text{Hom}_{L \times M_2}(W, \sigma) = \Theta_W(\sigma)^* \) where \( \Theta_W(\sigma) \) is the big theta lift of \( \sigma \in \text{Irr}(\text{GL}_2) \) to \( \text{GSO}_4 \), which has finite length by the Howe duality theorem for classical (similitude) theta correspondence. From this, one deduces as above that \( \Theta_\Sigma(\pi) \) has finite length as an \( M_2 \)-module.

Now let’s consider the case when \( \Sigma \) is the top piece of the filtration. From Propositions 13.5, 13.6 and 13.7, we see that we need to consider the following theta correspondences in lower rank:

- \( G_2 \times \text{PGL}_3 \) in \( E_6 \): for this case, the finite length of the big theta lift has been verified in Theorem 8.5.
- \( G_2 \times \text{SO}_3 \subset \text{SO}_{10} \) or \( G_2 \times \text{SO}_5 \subset \text{SO}_{12} \); we shall now treat these two cases together in the following proposition.

**Proposition 14.2** Let \( \Pi_n \) be the minimal representation of \( \text{SO}(2n) \) for \( n = 5 \) or 6. Then for tempered \( \pi \in \text{Irr}(G_2) \), \( \Theta_n(\pi) \) is a finite length \( H_n \)-module where \( H_n = \text{SO}_{2n-7} \).

**Proof** We shall use the fact that the minimal representation of \( \text{SO}_{2n} \) (\( n = 5 \) or 6) is the big theta lift of the trivial representation of \( \text{SL}_2 \) (see [42, Prop. 8.4] for the irreducibility of this big theta lift) and then appeal to the see-saw
identity arising from the see-saw diagram:

\[
\begin{array}{c}
\tilde{SL}_2 \times \tilde{SL}_2 \\
\downarrow \\
\tilde{SL}_2 \\
\downarrow \\
G_2 \times SO_{2n-7}
\end{array}
\begin{array}{c}
SO_{2n}
\end{array}
\]

From this, we see that

\[
\Theta_n(\pi)^* = \text{Hom}_{G_2}(\Pi_n, \pi) = \text{Hom}_{\tilde{SL}_2}(\Omega_{2n-7, \psi} \otimes \tilde{\Theta}_\psi(\pi), \mathbb{C})
\]

\[
\cong \text{Hom}_{\tilde{SL}_2}(\Omega_{2n-7, \psi}, \tilde{\Theta}_\psi(\pi))
\]

as $H_n$-modules, where

- $\Omega_{2n-7, \psi}$ is the Weil representation of $\tilde{SL}_2 \times SO_{2n-7}$ (with respect to a nontrivial additive character $\psi$ of $F$);
- $\tilde{\Theta}_\psi(\pi)$ denotes the big $\psi$-theta lift of $\pi$ to $\tilde{SL}_2$, with respect to the Weil representation $\Omega_\psi$ of $\tilde{SL}_2 \times SO_7 \supset \tilde{SL}_2 \times G_2$.

We see in particular that if $\Theta_n(\pi)$ is nonzero, then $\pi$ has nonzero $\psi$-theta lift to $\tilde{SL}_2$. Moreover, it remains now to show that $\tilde{\Theta}_\psi(\pi)$ has finite length as an $\tilde{SL}_2$-module; the desired result would then follow from this and the Howe duality theorem for $\tilde{SL}_2 \times H_n$.

Now the theta correspondence for $\tilde{SL}_2 \times G_2$ has been completely determined in [3], though the finiteness of $\tilde{\Theta}_\psi(\pi)$ was not formally stated there. Let us see how this finiteness can be deduced from [3].

As before, let us write $\tilde{\Theta}_\psi(\pi) = \tilde{\Theta}(\pi)_c \oplus \tilde{\Theta}(\pi)_{nc}$ as a sum of its cuspidal and noncuspidal component. To show that $\tilde{\Theta}(\pi)_{nc}$ has finite length as a $\tilde{SL}_2$-module, it suffices to show that the Jacquet module of $\tilde{\Theta}(\pi)$ with respect to a Borel subgroup $B = \tilde{T} \cdot N$ of $\tilde{SL}_2$ has finite length as a $\tilde{T}$-module. Now [3, Prop. 8.1] gives a short exact sequence of $G_2$-modules:

\[
0 \longrightarrow \text{Ind}_{Q}^{G_2} C_c(GL_1) \longrightarrow \Omega_N \longrightarrow \mathbb{C} \longrightarrow 0
\]

where the action of $L \cong GL_2$ on $GL_1$ is via det. The finite length of $\Theta(\pi)_N$ follows from this via a similar argument as above, by examining $\text{Hom}_{G_2}(\Omega_N, \pi)$.

It remains to show that $\tilde{\Theta}(\pi)_c$ has finite length. In fact, it was shown in [3, Thm. 9.1(c) and (d)] that for genuine supercuspidal representations $\sigma_1 \not\cong \sigma_2$ of $\tilde{SL}_2$, one has $\tilde{\Theta}_\psi(\sigma_1) \not\cong \tilde{\Theta}_\psi(\sigma_2)$. In other words, $\tilde{\Theta}(\pi)_c$ is irreducible or 0. This shows that $\tilde{\Theta}_\psi(\pi)$ has finite length.
14.3 Finiteness of $\Theta(\sigma)_{nc}$

For tempered $\sigma \in \operatorname{Irr}(\mathrm{PGSp}_6)$, the finite length of $\Theta(\sigma)_{nc}$ as a $G_2$-module is shown in the same way, using Propositions 13.1 and 13.3. We leave the details to the reader and only consider the top pieces in the filtration of the two Jacquet modules.

- For the maximal parabolic subgroup $P$, we have to consider the theta correspondence for $\mathrm{PGSp}_6 \times \mathrm{PGL}_2$ with respect to the minimal representation $\Pi_6$ of $\mathrm{PGSO}_{12}$. For the purpose of showing finiteness, there is no harm in working with $\mathrm{Sp}_6 \times \mathrm{SL}_2$. Hence, the theta correspondence in question arises as follows. If $V_2$ and $V_6$ denote the 2-dimensional and 6-dimensional symplectic vector spaces, then we are considering the map

$$\mathrm{Sp}(V_2) \times \mathrm{Sp}(V_6) \longrightarrow \mathrm{SO}(V_2 \otimes V_6)$$

and pulling back the minimal representation $\Pi_6$ of $\mathrm{SO}(V_2 \otimes V_6)$. As before, we shall use the fact that this minimal representation is the big theta lift of the trivial representation of $\mathrm{SL}_2$. More precisely, let $V'_2$ be another symplectic space of dimension 2, then we have the map

$$\mathrm{Sp}(V'_2) \times \mathrm{Sp}(V_2) \times \mathrm{Sp}(V_6) \longrightarrow \mathrm{Sp}(V'_2) \otimes \mathrm{SO}(V_2 \otimes V_6) \longrightarrow \mathrm{Sp}(V'_2 \otimes V_2 \otimes V_6).$$

Given the Weil representation $\Omega$ of $\mathrm{Sp}(V'_2) \times \mathrm{SO}(V_2 \otimes V_6)$ and $\sigma \in \operatorname{Irr}(\mathrm{Sp}(V_6))$, we have

$$\Theta(\sigma)^* \cong \operatorname{Hom}_{\mathrm{Sp}(V_6)}(\Pi_6, \sigma) \cong \operatorname{Hom}_{\mathrm{Sp}(V'_2) \times \mathrm{Sp}(V_6)}(\Omega, 1_{\mathrm{Sp}(V'_2)} \otimes \sigma) \cong \operatorname{Hom}(\Theta'(\sigma), 1_{\mathrm{Sp}(V'_2)})$$

where $\Theta'(\sigma)$ is the big theta lift of $\sigma$ to $\mathrm{SO}(V'_2 \otimes V_2)$. Note that there is a natural isogeny

$$\mathrm{Sp}(V'_2) \times \mathrm{Sp}(V_2) \longrightarrow \mathrm{SO}(V'_2 \otimes V_2)$$

whose image is of finite index. Hence, by the classical Howe duality theorem, $\Theta'(\sigma)$ is a finite length representation of $\mathrm{Sp}(V'_2) \times \mathrm{Sp}(V_2)$. This implies that $\Theta(\sigma)$ has finite length.

- For the maximal parabolic subgroup $Q$, we need to consider the restriction of $\Pi_{A_5}$, a minimal representation of $\mathrm{SL}_6$ to $\mathrm{Sp}_6$. Note that $\Pi_{A_5}$ is a degenerate principal series representation induced from a maximal parabolic subgroup which stabilizes a line in the standard representation. Since $\mathrm{Sp}_6$ acts transitively on such lines, we see that the restriction of $\Pi_{A_5}$ to $\mathrm{Sp}_6$ is simply a degenerate principal series representation of $\mathrm{Sp}_6$. This implies the desired finiteness.
We have thus completed the proofs of Lemmas 6.2 and 6.3.

15 Howe duality for $G_2 \times \text{PGSp}_6$: general case

Finally, by combining Theorem 12.4 and Theorem 14.1, we can establish the Howe duality theorem for $G_2 \times \text{PGSp}_6$.

**Theorem 15.1** Let $\pi \in \text{Irr}(G_2)$.

(i) $\Theta(\pi)$ is nonzero if and only if $\pi$ has zero theta lift to $PD^\times$.

(ii) If $\Theta(\pi) \neq 0$, then $\Theta(\pi)$ is a finite length representation of $\text{PGSp}_6$ with a unique irreducible quotient $\theta(\pi)$.

(iii) For $\pi_1, \pi_2 \in \text{Irr}(G_2)$,

$$\theta(\pi_1) \cong \theta(\pi_2) \neq 0 \implies \pi_1 \cong \pi_2.$$  

(iv) If $\Theta(\pi) \neq 0$, then $\theta(\pi)$ is tempered if and only if $\pi$ is tempered.

(v) If $\pi$ is non-tempered, then $\theta(\pi)$ is nonzero and the $L$-parameter of $\theta(\pi)$ is obtained from that of $\pi$ by composing with the natural inclusion $G_2(\mathbb{C}) \subset \text{Spin}_7(\mathbb{C})$.

15.1 Explicit correspondence

We can in fact determine the theta lift $\theta(\pi)$ explicitly if $\pi$ is tempered and noncuspidal. Indeed, we may also determine $\theta(\pi)$ for those tempered $\pi$ which has nonzero theta lift to $\text{PGL}_3$. To achieve this, we shall use the following four facts:

- If $\pi$ does not appear in the correspondence with $PD^\times$, then $\theta(\pi) \neq 0$ (Theorem 15.1(i)).
- If $\pi$ is tempered and $\theta(\pi) \neq 0$, then $\theta(\pi)$ is irreducible and tempered (Theorem 15.1(iv)).
- If $\pi$ is nongeneric, then $\theta(\pi)$ is nongeneric (Corollary 11.2(ii)).
- The cuspidal support of $\theta(\pi)$ can be computed (from the Jacquet module computations of Sect. 13).

More precisely we have:

**Theorem 15.2** Let $\pi$ be an irreducible tempered representation of $G_2$. Assume that $\pi$ is a lift of a (necessarily tempered) representation $\tau$ of $\text{PGL}_3$, that is, $\pi = \theta_B(\tau^\epsilon)$ for some $\epsilon = \pm$. Then we have the following:

(i) If $\tau \not\cong \tau^\vee$, then $\theta(\pi) \cong I_3(\tau) \cong I_3(\tau^\vee) \in \text{Irr}(\text{PGSp}_6)$.

(ii) If $\tau \cong \tau^\vee$ and the parameter of $\tau$ contains a trivial summand, then $\theta(\pi) \cong I_3(\tau)$.  

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(iii) If \( \tau \cong \tau^\vee \) and the parameter of \( \tau \) does not contain a trivial summand, then \( \pi \) is one of the two representations \( \pi_{\text{gen}} = \theta_B(\tau^+) \) and \( \pi_{\text{deg}} = \theta_B(\tau^-) \). In this case \( I_3(\tau) = I_3(\tau)_{\text{gen}} \oplus I_3(\tau)_{\text{deg}} \),

\[
\theta(\pi_{\text{gen}}) = I_3(\tau)_{\text{gen}} \quad \text{and} \quad \theta(\pi_{\text{deg}}) = I_3(\tau)_{\text{deg}}.
\]

We now deal with the remaining tempered representations of \( G_2 \). Non-supercuspidal representations are mostly constituents of the principal series \( I_Q(\tau) \) where \( \tau \) is a discrete series representation. These representations lift to constituents of the principal series \( I_2(\tau \otimes \tau) \). More precisely, we have:

**Theorem 15.3** Let \( \pi \) be an irreducible tempered representation of \( G_2 \) which is not a lift from \( \text{PGL}_3 \). Then we have the following:

(i) Let \( \tau \) be a unitary discrete series representation of \( \text{GL}_2 \). Then \( I_Q(\tau) \) is irreducible if and only if \( I_2(\tau \otimes \tau) \) is irreducible. We have:

- If \( I_Q(\tau) \) is irreducible, then
  \[
  \theta(I_Q(\tau)) \cong I_2(\tau \otimes \tau).
  \]

- If \( I_Q(\tau) \) is reducible, then
  \[
  \theta(I_Q(\tau)_{\text{gen}}) \cong I_2(\tau \otimes \tau)_{\text{gen}} \quad \text{and} \quad \theta(I_Q(\tau)_{\text{deg}}) \cong I_2(\tau \otimes \tau)_{\text{deg}}.
  \]

(ii) Assume that \( \tau \cong \tau^\vee \) is a supercuspidal representation of \( \text{GL}_2 \) with the trivial central character. Let \( \delta_Q(\tau) \) and \( \delta_P(\tau) \) be the square integrable constituents of \( I_Q(1/2, \tau) \) and \( I_P(1/2, \tau) \). Then

\[
\theta(\delta_Q(\tau)) \cong \delta_2(\tau) \quad \text{and} \quad \theta(\delta_P(\tau)) \cong \delta_{13}(\tau)
\]

where \( \delta_2(\tau) \) and \( \delta_{13}(\tau) \) are the square integrable constituents of \( I_2(1/2, \tau \otimes \tau) \) and \( I_{13}(1/2, \tau \otimes 1) \).

(iii) Assume that \( \tau \cong \tau^\vee \) is a supercuspidal representation of \( \text{GL}_2 \) whose Langlands parameter has the image \( S_3 \). Recall that \( I_Q(1, \tau) \) has a square integrable constituent denoted by \( \pi_{\text{gen}}[\tau] \). Then

\[
\theta(\pi_{\text{gen}}[\tau]) \cong \sigma_{\text{gen}}[\tau],
\]

where \( \sigma_{\text{gen}}[\tau] \) is the square integrable constituent of \( I_2(1, \tau \otimes \tau) \).

(iv) Assume that \( \chi^2 = 1 \) and \( \chi \neq 1 \). Recall that \( I_Q(1/2, \text{st}_\chi) \) has a square integrable constituent denoted by \( \pi_{\text{gen}}[\chi] \). Then

\[
\theta(\pi_{\text{gen}}[\chi]) \cong \sigma_{\text{gen}}[\chi],
\]

where \( \sigma_{\text{gen}}[\chi] \) is the square integrable constituent of \( I_2(1/2, \text{st}_\chi \otimes \text{st}_\chi) \).
(v) Steinberg lifts to Steinberg:

$$\theta(\text{St}_{G_2}) = \text{St}_{\text{PGSp}_6}.$$

Finally we need to deal with supercuspidal representations. In view of Theorem 15.1(i) and Theorem 15.2, we only need to consider those supercuspidal representations which do not lift to $\text{PGL}_3$ or $PD^\times$. We first introduce a thin family of supercuspidal representations of $G_2$, namely those which participate in the theta correspondence for $\widetilde{\text{SL}}_2 \times G_2$. We have already encountered this theta correspondence in the proof of Proposition 14.2. As mentioned there, this theta correspondence has been studied in detailed in [3].

We first introduce some notation. For each cuspidal representation $\rho$ of $\text{PGL}_2 \cong \text{SO}_3$, let $JL(\rho)$ be its Jacquet-Langlands lift to the anisotropic inner form $PB^\times = \text{SO}_3^\times$ (where $B$ is the quaternion division algebra) and let $\sigma_\rho$ be the $\psi$-theta lift of $JL(\rho)$ to $\widetilde{\text{SL}}_2$ (where $\psi$ is a fixed nontrivial additive character of $F$ and the theta lift is induced by the Weil representation $\omega_\psi$ associated to $\psi$). Then $\sigma_\rho$ is an irreducible supercuspidal genuine representation of $\widetilde{\text{SL}}_2$. Consider now the $\psi$-theta lift

$$\pi_\rho := \theta(\sigma_\rho) \in \text{Irr}(G_2)$$

of $\sigma_\rho$ from $\widetilde{\text{SL}}_2$ to $G_2$. Now we recall some results from [3, Thm. 9.1]:

**Lemma 15.4** With the above notations, we have:

(i) The representation $\pi_\rho$ is nonzero irreducible supercuspidal. Moreover, $\Theta(\pi_\rho) = \sigma_\rho$ under the theta correspondence for $\widetilde{\text{SL}}_2 \times G_2$.

(ii) The map $\rho \mapsto \pi_\rho$ is an injective map from the set $\text{Irr}_{sc}(\text{PGL}_2)$ of supercuspidal representations of $\text{PGL}_2$ to $\text{Irr}_{sc}(G_2)$.

(iii) Any $\pi \in \text{Irr}_{sc}(G_2)$ which lifts to $\widetilde{\text{SL}}_2$ but not $\text{PGL}_3$ or $PD^\times$ is of the form $\pi_\rho$ for some $\rho \in \text{Irr}_{sc}(\text{PGL}_2)$.

For $\sigma_\rho \in \text{Irr}(\widetilde{\text{SL}}_2)$ as above, we may also consider its $\psi$-theta lift from $\widetilde{\text{SL}}_2$ to $\text{SO}_5$ and set

$$\tau_\rho = \Theta(\sigma_\rho) = \theta(\sigma_\rho) \in \text{Irr}(\text{SO}_5).$$

Then $\tau_\rho$ is a nongeneric supercuspidal representation of $\text{SO}_5$ belonging to a so-called Saito–Kurokawa A-packet. The representations $\pi_\rho$ and $\tau_\rho$ are related as follows:

**Lemma 15.5** Consider the restriction of the minimal representation of $\text{SO}_{12}$ to $G_2 \times \text{SO}_5$. Then for $\rho \in \text{Irr}_{sc}(\text{PGL}_2)$,

$$\Theta(\pi_\rho) = \tau_\rho.$$
Proof We shall use the see-saw diagram in the proof of Proposition 14.2. The ensuing see-saw identity (14.3) and Lemma 15.4(i) give:

$$\Theta(\pi_\rho)^* \cong \text{Hom}_{\widetilde{\text{SL}}_2}(\Omega_5, \sigma_\rho) = \tau_\rho^*.$$ 

Hence $\Theta(\pi_\rho) = \tau_\rho$. 

Now we have:

**Proposition 15.6** Let $\pi$ be an irreducible supercuspidal representation of $G_2$ that is not a lift from $\text{PGL}_3$ or $PD^\times$. Then we have the following two possibilities:

- If $\pi = \pi_\rho$ for some $\rho \in \text{Irr}_{\text{sc}}(\text{PGL}_2)$ (as in Lemma 15.4), then $$\theta(\pi_\rho) = \delta_1(\tau_\rho),$$

where $\delta_1(\tau_\rho)$ is the (nongeneric) square integrable subquotient of $I_1(1/2, \tau_\rho)$ given in Proposition 10.4.

- If $\pi$ is not of the above form, then $\theta(\pi)$ is supercuspidal.

Proof Let $\Pi$ be the minimal representation of $E_7$. Recall that $r_{P_i}$ is the normalized Jacquet functor with respect to the maximal parabolic $P_i$ in $\text{PGSp}_6$. Then $\pi \otimes r_{P_1}(\theta(\pi))$ is a quotient of $r_{P_1}(\Pi)$.

By the assumption that $\pi$ does not lift to $\text{PGL}_3$, it follows that $r_{P_i}(\theta(\pi)) = 0$ for $i = 2, 3$. Thus either $r_{P_1}(\theta(\pi)) = 0$, in which case $\theta(\pi)$ is supercuspidal, or $r_{P_1}(\theta(\pi))$ is a supercuspidal representation of the Levi factor $L_1 = \text{GSp}_4$. In fact, from Proposition 13.6, it follows that $r_{P_1}(\theta(\pi)) = \tau \otimes |v|^{1/2}$ where $\tau$ is a (possibly reducible) supercuspidal representation of $\text{PGSp}_4 \cong \text{SO}_5$ such that $\pi \otimes \tau$ appears as a quotient of the minimal representation of $\text{SO}_{12}$. By the see-saw in the proof of Proposition 14.2, we see that $\pi$ must have nonzero theta lift to $\widetilde{\text{SL}}_2$ and hence is of the form $\pi_\rho$ for some $\rho \in \text{Irr}(\text{PGL}_2)$ by Lemma 15.4(iii). Then Lemma 15.5 implies that $\tau = \tau_\rho$. By Frobenius reciprocity and the fact that $\theta(\pi_\rho)$ is tempered, we see that $\theta(\pi_\rho) = \delta_1(\tau_\rho)$, as desired. 

As a consequence of the explicit results in this section, we have:

**Corollary 15.7** If $\pi \in \text{Irr}(G_2)$ is a discrete series representation which does not lift to $\text{PGL}_3$ or $PD^\times$, then $\theta(\pi)$ is an irreducible discrete series representation of $\text{PGSp}_6$. As a result, any discrete series representation of $G_2$ lifts to a discrete series of exactly one of $PD^\times$, $\text{PGL}_3$ or $\text{PGSp}_6$. That lift is Whittaker generic if and only if $\pi$ is.

Finally, we have the following consequence, proving a case of a conjecture of Prasad [30, Remark 4, page 624].
Corollary 15.8 Every $\pi \in \text{Irr}(G_2)$ that lifts to $\text{PGSp}_6$ is self dual. In particular, every Whittaker generic irreducible representation of $G_2(F)$ is self-dual.

Proof By inspection, it suffices to prove for tempered $\pi$. Then $\theta(\pi)$ is also tempered. Recall that the complex conjugate of a tempered irreducible representation is isomorphic to its dual. Furthermore, since theta lift commutes with taking complex conjugates, we have

$$\theta(\pi^\vee) \cong \theta(\bar{\pi}) \cong \theta(\bar{\pi}) \cong \theta(\pi).$$

As irreducible representations of $\text{PGSp}_6$ are self dual [27, page 91], it follows that $\theta(\pi)^\vee \cong \theta(\pi)$ and $\pi \cong \pi^\vee$, since the theta correspondence is one to one.

\[\square\]

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