STABILITY OF RAREFACTION WAVE FOR THE
COMPRESSIBLE NON-ISENTPIC
NAVIER-STOKES-MAXWELL EQUATIONS

HUANCHENG YAO
School of Mathematics, South China University of Technology
Guangzhou 510641, China

HAIYAN YIN
School of Mathematical Sciences, Huaqiao University
Quanzhou 362021, China

CHANGJIANG ZHU*
School of Mathematics, South China University of Technology
Guangzhou 510641, China

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Abstract. We study the large-time asymptotic behavior of solutions toward
the rarefaction wave of the compressible non-isentropic Navier-Stokes equations
coupling with Maxwell equations under some small perturbations of initial data
and also under the assumption that the dielectric constant is bounded. For
that, the dissipative structure of this hyperbolic-parabolic system is studied to
include the effect of the electromagnetic field into the viscous fluid and turns
out to be more complicated than that in the simpler compressible Navier-Stokes
system. The proof of the main result is based on the elementary $L^2$ energy
methods.

1. Introduction.

1.1. The problem. Plasma dynamics is a field of studying flow problems of elec-
trically conducting fluids. The scope of plasma dynamics is very broad. A complete
analysis in this field consists of the study of the gasdynamic field, the electro-
magnetic field and the radiation field simultaneously [28, 27]. In this paper, we consider
the flow of an electrically conducting fluid in the presence of an electromagnetic
field. Since the flow and the electromagnetic field are closely connected with each
other, the governing system in the non-isentropic case consists of the hydrodynam-
ical equations and the electromagnetic ones; they are the laws of conservation of

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* Corresponding author.
mass, momentum and energy, Maxwell’s law, and the law of conservation of electric charge (cf. [11, 14, 31]):

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\rho \left(\partial_t u + u \cdot \nabla u\right) + \nabla p &= \mu \Delta u + (\lambda + \mu') \nabla (\text{div } u) + \rho_e E + J \times B, \\
\frac{\partial \rho_e}{\partial \theta} (\partial_t \theta + u \cdot \nabla \theta) + \frac{1}{\theta} \text{div } u &= \text{div}(\kappa \nabla \theta) + N(u) + (J - \rho_e u) \cdot (E + u \times B), \\
\varepsilon \partial_t E - \frac{1}{\mu_0} \text{curl } B + J &= 0, \\
\partial_t B + \text{curl } E &= 0, \\
\partial_t \rho_e + \text{div } J &= 0, \\
\varepsilon \text{div } E &= \rho_e, \\
\text{div } B &= 0,
\end{align*}
\]

(1.1)

where \((x, t) \in \mathbb{R}^3 \times \mathbb{R}^+\). Here, \(\rho(x, t) > 0\) denotes the mass density; \(u = (u_1, u_2, u_3) \in \mathbb{R}^3\) is the fluid velocity; \(\theta(x, t) > 0\) is the absolute temperature; \(E = (E_1, E_2, E_3) \in \mathbb{R}^3\) and \(B = (B_1, B_2, B_3) \in \mathbb{R}^3\) denote the electric field and the magnetic field, respectively. \(\rho_e(x, t)\) is the electric charge density. The magnetic permeability \(\mu_0\) and the heat conductivity coefficient \(\kappa\) are assumed to be positive constants. In addition, \(\lambda\) and \(\mu'\) are the viscosity coefficients of the fluid which satisfy \(\mu' > 0\) and \(2\mu' + 3\lambda > 0\). And \(\varepsilon > 0\) is the dielectric constant. The pressure \(p\) and the internal energy \(e\) are expressed by the equations of states. We will focus on only polytropic fluids for the sake of simplicity throughout this paper, namely

\[
p = R \rho \theta, \quad e = \frac{R}{\gamma - 1} \theta, \quad (1.2)
\]

where the gas constant \(R > 0\) and the adiabatic exponent \(\gamma > 1\). \(N(u)\) denotes the viscous dissipation function

\[
N(u) = \sum_{i,j=1}^{3} \frac{\mu'}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2 + \lambda (\text{div } u)^2. \quad (1.3)
\]

What’s more, the electric current density \(J\) can be expressed by Ohm’s law

\[
J = \rho_e u + \sigma (E + u \times B),
\]

where \(\sigma > 0\) denotes the electric conductivity coefficient.

For the original nonlinear system, there is quite limited mathematical progress since as pointed out by Kawashima in [14], the system (1.1) is neither symmetric hyperbolic nor strictly hyperbolic. This means that the classical local well-posedness theorem (cf. [13]), cannot be directly applied to the system (1.1). Due to these difficulties for the original nonlinear system (1.1), some simplified models are derived according to the actual physical application. As it was pointed out by Imai [11], the assumption that the electric charge density \(\rho_e \simeq 0\) is physically very good for the study of plasmas. Here, we need to recognize that the quasi-neutrality assumption \(\rho_e \simeq 0\) is different from the assumption of exact neutrality \(\rho_e = 0\) since the latter would lead to the superfluous condition \(\text{div } E = 0\). According to this quasi-neutrality assumption, we can eliminate the terms involving \(\rho_e\) in the system...
(1.1) and derive the following simplified system (cf. [31]):

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho \mathbf{u}) &= 0, \\
\rho (\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) + \nabla \rho &= \mu' \Delta \mathbf{u} + (\lambda + \mu') \nabla (\text{div} \mathbf{u}) + \mathbf{J} \times \mathbf{B}, \\
\rho \frac{\gamma - 1}{\gamma} (\partial_t \theta + \mathbf{u} \cdot \nabla \theta) + p \text{div} \mathbf{u} &= \text{div} (\kappa \nabla \theta) + \mathcal{N} (\mathbf{u}) + \mathbf{J} \cdot (\mathbf{E} + \mathbf{u} \times \mathbf{B}), \\
\varepsilon \partial_t \mathbf{E} - \frac{1}{\mu_0} \text{curl} \mathbf{B} + \mathbf{J} &= 0, \\
\partial_t \mathbf{B} + \text{curl} \mathbf{E} &= 0, \quad \text{div} \mathbf{B} = 0,
\end{align*}
\]

with the electric current density \( \mathbf{J} \) given by

\[ \mathbf{J} = \sigma (\mathbf{E} + \mathbf{u} \times \mathbf{B}). \]

We are interested in the motion of one fluid described by the compressible Navier-Stokes equations coupled with the Maxwell equations through the Lorentz force, namely system (1.4). For convenience of presentation, we call (1.4) the Navier-Stokes-Maxwell system. Notice that the same terminology was used in Duan [2] and Masmoudi [24] but for the different modeling system.

Let us recall some known results about Navier-Stokes-Maxwell equations. There have been some research on the existence and large-time behavior of solutions to compressible Navier-Stokes-Maxwell equations. In [17, 18], Kawashima and Shizuta established the global existence of smooth solutions for small data and studied its zero dielectric constant limit in the whole space \( \mathbb{R}^2 \). Later, Jiang and Li [12] studied the zero dielectric constant limit and obtained the convergence of the system (1.4) to the full compressible magnetohydrodynamic equations in the torus \( \mathbb{T}^3 \). Recently, Xu [31] studied the large-time behavior of the classical solution toward the steady state with the strictly positive constant pressure and entropy, the vanishing velocity and electromagnetic field; and also obtained the time-decay estimates under small initial perturbations in regular Sobolev space. For the 1-D non-isentropic Navier-Stokes-Maxwell equations, Fan and Hu [4] proved the uniform estimates with respect to the dielectric constant and the global-in-time existence in a bounded interval without vacuum. What’s more, Fan and Ou [5] proved the similar result in which the pressure and the internal energy include radiative source term and also the heat conductivity coefficient is different from [4].

As far as we know, for the Navier-Stokes-Maxwell equations, there are few results about the large-time behavior of the solution toward some non-constant states, especially hyperbolic elementary waves, which is our target in this paper, except that Luo, Yao and Zhu [22] proved the stability of rarefaction wave for the compressible isentropic Navier-Stokes-Maxwell equations under small \( H^1 \)-initial perturbations. For this target, we shall restrict ourselves to the one-dimensional non-isentropic motion. This means that the flow is uniform in the \( x_2, x_3 \)-axis, i.e., all the quantities \((\rho, \mathbf{u}, \theta, \mathbf{E}, \mathbf{B})\) appearing in (1.4) are independent of the second and the third component of space variable \((x_1, x_2, x_3)\) (Below \( x_1 \) will be denoted by \( x \)). We also take \( \mu_0 = \sigma = 1 \) and denote \( (\lambda + 2 \mu') \) by \( \mu \) for the simplicity of notation. Motivated by [4, 5], we consider the following interesting case:

\[ \mathbf{u} = (u, 0, 0), \quad \mathbf{E} = (0, 0, E), \quad \mathbf{B} = (0, b, 0). \]
In this situation, the non-isentropic system (1.4) reduces to the following 1-D case:

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
\rho (u_t + uu_x) + p_x &= \mu u_{xx} - (E + ub)b, \\
\frac{\rho R}{\gamma - 1} (\theta_t + u_x \theta_x) + pu_x &= \kappa \theta_{xx} + \mu u_x^2 + (E + ub)^2, \\
\varepsilon E_t - b_x + E + ub &= 0, \\
b_t - E_x &= 0.
\end{align*}
\]  

(1.5)

Initial data for system (1.5) are given by

\[
(\rho, u, \theta, E, b)(x, 0) = (\rho_0, u_0, \theta_0, E_0, b_0)(x), \quad x \in \mathbb{R}.
\]  

(1.6)

The initial data at both far fields \(x = \pm \infty\) are assumed to be constants, namely

\[
\lim_{x \to \pm \infty} (\rho_0, u_0, \theta_0, E_0, b_0)(x) = (\rho_{\pm}, u_{\pm}, \theta_{\pm}, 0, 0).
\]  

(1.7)

In this paper, for the 1-D compressible non-isentropic Navier-Stokes-Maxwell equations, we are interested in the asymptotical stability of the rarefaction wave to the Cauchy problem (1.5), (1.6) and (1.7) for the large time behavior. The main idea is to generalize some known results of the viscous fluid, particularly about the global existence and large time behavior of classical solutions near hyperbolic elementary waves, in order to include the effects of the electromagnetic field. In fact, there are extensive studies concerning those results for the Navier-Stokes equations in the context of gas dynamics. It is well known that the large-time behavior of solutions to the Cauchy problem for Navier-Stokes equations can converge to the Riemann solutions for the corresponding Euler equations. As for the Riemann problem, we know that under a proper assumption on the flux function, there are three kinds of elementary wave solutions: shock wave, rarefaction wave and contact discontinuity, and then the Riemann solution generally forms a multi-wave pattern given by a various linear combination of these three elementary waves (cf. [25, 30]). Here, we mention several works related to Navier-Stokes equations: [19, 15, 20] for the asymptotic stability of shock wave; [26, 16, 21] for the asymptotic stability of rarefaction wave; [7, 9] for the asymptotic stability of contact discontinuity; [6, 8] for the asymptotic stability of combination of viscous contact wave with rarefaction waves. In addition, some results about the asymptotic stability of rarefaction waves were also shown for more complex models, and we refer interested readers to [3, 29, 23] for reference.

We notice that the 1-D Navier-Stokes-Maxwell system (1.5) is a combination of compressible non-isentropic Navier-Stokes equations and Maxwell equations. Actually when we ignore the effect of the electric field and the magnetic field, the system (1.5) reduces to the classical Navier-Stokes equations in Eulerian coordinates. Therefore, motivated by the relationship between the compressible non-isentropic Navier-Stokes-Maxwell equations and Navier-Stokes equations, we provisionally assume the initial data of the electric field and the magnetic field at both far fields \(x = \pm \infty\) are zero, respectively, that is to say,

\[
\lim_{x \to \pm \infty} (\rho_0, u_0, \theta_0, E_0, b_0)(x) = (\rho_{\pm}, u_{\pm}, \theta_{\pm}, 0, 0).
\]  

(1.8)

We focus on the global solution in time of the Cauchy problem (1.5), (1.6), (1.8) and their large-time behavior toward rarefaction wave in the relations with the spatial asymptotic states \((\rho_{\pm}, u_{\pm}, \theta_{\pm}, 0, 0)\). And the more challenging case for \(E_- \neq E_+\) and \(b_- \neq b_+\) is left for study in future.
Here, we briefly state the main difficulties for our problem and review some key analytical techniques. First of all, the key step is to obtain the zero-order energy estimates; see Lemma 2.1 and Lemma 2.2. Unlike the Navier-Stokes equations, we have to deal with the additional terms which include the electric field $E$ and the magnetic field $b$. The estimate (2.4) shows that the electromagnetic field satisfying the Maxwell system indeed has some time-space integrability property weaker than the fluid components, whereas its highest-order spatial derivative is not time-space integrable, which becomes a typical feature of this kind of system with the regularity-loss property. Secondly, different from the derivation of zero-order estimate for the Navier-Stokes equations with heat conduction where all the inhomogeneous terms contain at least one derivative of the solution, here we have inhomogeneous terms like $-\int_0^t \int_{\mathbb{R}} (E + \psi b + u^r b) \psi b \, dx \, d\tau$ and $\int_0^t \int_{\mathbb{R}} (E + \psi b + u^r b)^2 \frac{\partial}{\partial x} \, dx \, d\tau$ due to the appearance of the electromagnetic fields. In the absence of time-space integrable good terms of $\psi$ and $b$ for the equations, it is difficult to deal with the above two terms directly, see (2.18). Compared with the vanishing velocity steady state case of [31] and the zero velocity boundary condition case of [4], the main difference is that we have to deal with the additional non-null term such as $-\int_0^t \int_{\mathbb{R}} u^r \psi b^2 \, dx \, d\tau$ in (2.18), which is generated by the rarefaction wave. To overcome these difficulties, based on the structure of the Maxwell equations, our idea is to package extra terms to produce a compound time-space integrable good term $\int_0^t \int_{\mathbb{R}} (E + \psi b + u^r b)^2 \, dx \, d\tau$, which is crucial to achieve the zero-order energy estimates and will be beneficial to high-order energy estimates. Thirdly, in order to absorb some nonlinear bad terms by the single or the compound good term associated with the electromagnetic fields, we require a technical condition (1.29) that dielectric constant $\varepsilon$ is bounded for some positive constants $\tilde{C}$ (depending only on $|u_+|$ and $\theta_-$), see estimates (2.23)-(2.26), (2.31) in Lemma 2.1 and estimates (2.40)-(2.43) in Lemma 2.2. We would like to mention that Huang-Liu in [10] consider the stability of rarefaction wave for a macroscopic model derived from the Vlasov-Maxwell-Boltzmann system. Here we should point out that, except for the similar dissipative term $E + ub$, the model we consider in this paper is obviously different from that in [10]. In contrast to the requirement of smallness of the profile $|u^r(x, t)|$ imposed in the proof of [10], in this paper, we require $\varepsilon \cdot \max\{|u_-|, |u_+|\}$ to be small instead, which can not only contain but also extend the condition of [10] in some extent, see Remark 1 for details.

**Notation.** Throughout this paper, $C$ denotes some universal positive constants which are independent of $x$ and $t$ and may vary from line to line. $\| \cdot \|_{L^p}$ stands the $L^p$-norm on the Lebesgue space $L^p(\mathbb{R}) (1 \leq p \leq \infty)$. For the sake of convenience, we always denote $\| \cdot \| = \| \cdot \|_{L^2}$. What’s more, $H^k$ will be used to denote the usual Sobolev space $W^{k,2}(\mathbb{R})(k \in \mathbb{Z}_+)$ with respect to variable $x$.

### 1.2. Rarefaction wave and smooth approximate profile

By employing asymptotic analysis arguments with the setting of $E = b = 0$ for the large-time behavior, the asymptotic behavior of solutions for the Cauchy problem (1.5), (1.6), (1.8) is expected to be determined by the following Riemann problem for Euler system:

$$
\begin{cases}
\rho_t + (\rho u) x = 0, \\
\rho (u_t + uu_x) + px = 0, \\
\rho \frac{R}{\gamma - 1} (\theta_t + u \theta_x) + pu_x = 0,
\end{cases}
$$

(1.9)
with initial data
\[ (\rho, u, \theta)(x, 0) = \begin{cases} 
(\rho_-, u_-, \theta_-), & x < 0, \\
(\rho_+, u_+, \theta_+), & x > 0.
\end{cases} \] (1.10)

As is customary in thermodynamics, after giving any two of the thermodynamics variables \( \rho, \theta, p, e \) and \( S \), we can obtain the remaining three variables. The second law of thermodynamics asserts that
\[ \theta \, dS = de + p \, d\frac{1}{\rho}, \] (1.11)
where \( S \) is the specific entropy. (1.11) and the equation of state (1.2) imply that
\[ p = p(\rho, S) = k e^{\frac{-\gamma}{\gamma-1} S} \rho^{\gamma}, \] (1.12)
where \( k \) is a positive constant. By applying (1.9) and (1.9) together with (1.12), we can transform (1.9) into
\[ S_t + uS_x = 0. \] (1.13)

By replacing (1.9) by (1.13), we can rewrite the system (1.9) in terms of \( (\rho, u, S) \) as follows:
\[
\begin{cases} 
\rho_t + (\rho u)_x = 0, \\
\rho(u_t + uu_x) + p_x = 0, \\
S_t + uS_x = 0.
\end{cases} \] (1.14)

As in [30] pointed out, the transformed system may be used for the purposes of deciding eigenvalues, genuinely nonlinearity and Riemann invariants. There are three characteristic families corresponding to the eigenvalues:
\[ \lambda_1 = u - c, \quad \lambda_2 = u, \quad \lambda_3 = u + c, \]
with corresponding right eigenvectors:
\[
\begin{pmatrix} \rho, -c, 0 \end{pmatrix}^T, \quad \begin{pmatrix} p, 0, -p \rho \end{pmatrix}^T, \quad \begin{pmatrix} \rho, c, 0 \end{pmatrix}^T,
\]
where the sound speed \( c = c(\rho, S) = \sqrt{\frac{2 \gamma}{\rho}} \) and \( (\cdot)^T \) denotes the transpose of a row vector. The three pairs of Riemann invariants associated with these eigenvectors can be taken as
\[
\begin{cases} S, u + \frac{2}{\gamma - 1} c \end{cases}, \quad \{ u, p \}, \quad \begin{cases} S, u - \frac{2}{\gamma - 1} c \end{cases}.
\]
This implies the first and third characteristic fields are genuinely nonlinear and the second field is linearly degenerate. The basic theory of hyperbolic systems of conservation laws (for example, see Ref. [30]) shows that for any given constants \( (\rho_-, u_-, \theta_-) \) with \( \rho_- > 0, \theta_- > 0 \), there exists a suitable neighborhood \( \Omega(\rho_-, u_-, \theta_-) \) of \( (\rho_-, u_-, \theta_-) \) such that for any \( (\rho_+, u_+, \theta_+) \in \Omega(\rho_-, u_-, \theta_-) \), the Riemann problem (1.9) and (1.10) admits a solution consisting of the basic wave patterns.

In this paper, we only consider the stability of 1-rarefaction wave \( (\rho^R, u^R, \theta^R)(x, t) \) and the case of 3-rarefaction wave is analogous. Now we construct the 1-rarefaction wave for the Riemann problem of Euler system (1.9) and (1.10). Using (1.12) and the first equation of (1.2) gives \( \theta \) in terms of \( S \) and \( \rho \) that
\[ \theta = \frac{k}{R} e^{\frac{-1}{\gamma-1} S} \rho^{\gamma-1}. \] (1.15)

Note that entropy \( S \) takes constant along 1-rarefaction wave since the Riemann invariants are constant in a rarefaction wave. Therefore, in the rarefaction wave \( \theta \)
can be determined in terms of $\rho$ by (1.15) in the way that $S$ takes the constant value $S_\pm$. Here $S_\pm$ satisfies $\theta_\pm = \frac{k}{\rho_+} e^{\frac{2}{\gamma-1} S_\pm \rho_\pm^{-1}}$. Motivated by the above computations, the 1-rarefaction wave $(\rho^R, u^R)(x, t)$ is expected to be determined by the Riemann problem for Euler system

$$
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
\rho (u_t + uu_x) + k e^{\frac{2}{\gamma-1} S \rho^\gamma} &= 0,
\end{align*}
$$

(1.16)

with initial data given by

$$(\rho, u)(x, 0) = \begin{cases} 
(\rho_-, u_-), & x < 0, \\
(\rho_+, u_+), & x > 0.
\end{cases}$$

(1.17)

The Riemann problem (1.16), (1.17) has two characteristics (see [30, 1]) $\lambda_1$ and $\lambda_2$ (which we relabel), i.e.,

$$
\lambda_1(\rho, u) = u - \sqrt{k\gamma e^{\frac{2}{\gamma-1} S \rho^\gamma}} - 1,
$$

(1.18)

which gives rise to the 1-rarefaction wave curve

$$
R_1(\rho_-, u_-) = \left\{ \left( \rho, u \right) \left| u = u_- - \int_{\rho_-}^{\rho} \sqrt{k\gamma e^{\frac{2}{\gamma-1} S z^\gamma}} - 1 \right\} d\rho, \rho_+ > \rho, \ u_+ < u \right\}.
$$

(1.19)

Hence constant states $(\rho_\pm, u_\pm)$ satisfy

$$
u_+ = u_- - \int_{\rho_-}^{\rho_+} \sqrt{k\gamma e^{\frac{2}{\gamma-1} S z^\gamma}} - 1 d\rho, \ 0 < \rho_+ < \rho_-, \ u_- < u_+,
$$

(1.20)

and the Riemann problem (1.16), (1.17) admits the 1-rarefaction wave solution $(\rho^R, u^R)$ as follows:

$$
\begin{align*}
u^R (\frac{x}{t}) &= u_- - \int_{\rho_-}^{\rho_+} \sqrt{k\gamma e^{\frac{2}{\gamma-1} S z^\gamma}} - 1 d\rho, \\
\lambda_1(\rho_- u_-) &= \begin{cases} 
\lambda_1(\rho_-, u_-), & x < \lambda_1(\rho_-, u_-) t, \\
\lambda_1(\rho_-, u_-) t & \lambda_1(\rho_-, u_-) t \leq x \leq \lambda_1(\rho_+, u_+) t, \\
\lambda_1(\rho_+, u_+) & x > \lambda_1(\rho_+, u_+) t.
\end{cases}
\end{align*}
$$

(1.21)

In light of (1.15) and $\theta_\pm = \frac{k}{\rho_+} e^{\frac{2}{\gamma-1} S_\pm \rho_\pm^{-1}}$, one can define the corresponding profile of $\theta$ as follows:

$$
\theta^R \left( \frac{x}{t} \right) = \theta_\pm - \left( \frac{\rho^R \left( \frac{x}{t} \right)}{\theta^R \left( \frac{x}{t} \right)} \right) \left( \rho \left( \frac{x}{t} \right) \right)^\gamma.
$$

(1.22)

as long as the compatibility condition for the far-field data that

$$
\frac{\theta_+}{\theta_-} = \left( \frac{\rho_+}{\rho_-} \right)^\gamma
$$

(1.23)

holds.

Since the rarefaction wave $(\rho^R, u^R, \theta^R)(x, t)$ is not smooth enough, it is convenient to construct its smooth approximation. Motivated by [26], we define the
smooth rarefaction wave \((\rho^r, u^r, \theta^r)(x, t)\) as
\[
\begin{aligned}
\lambda_1(\rho^r, u^r) &= w(x, 1 + t), \quad \lambda_1(\rho\pm, u\pm) = w\pm,

u^r &= u_\pm - \int_{\rho\pm}^{\rho^r} \sqrt{k\gamma e^{\frac{\nu - 1}{\nu} S \gamma - 3}} \, dz,

\theta^r &= \theta_\pm \rho_\pm^-(\gamma - 1)(\rho^r)^{\gamma - 1},
\end{aligned}
\] (1.24)
where \(w(x, t)\) is the solution of Cauchy problem for the typical Burgers equation
\[
\begin{aligned}
&w_t + w\,w_x = 0, \\
&w(x, 0) = \frac{w_+ + w_-}{2} + \frac{w_+ - w_-}{2} \cdot \frac{e^x - e^{-x}}{e^x + e^{-x}}.
\end{aligned}
\] (1.25)
And we have the following lemma (see [26]):

**Lemma 1.1.** Set \(\delta_r = w_+ - w_-\) for \(w_- < w_+\). Then the Cauchy problem (1.25) has a unique smooth solution \(w(x, t)\) satisfying

(i) \(w_x > 0, w_- < w(x, t) < w_+, \) for \(x \in \mathbb{R}\) and \(t \geq 0\).

(ii) For any \(1 \leq p \leq +\infty\), there exists a constant \(C_p\) depending on \(p\) such that for any \(t > 0\),
\[
\|w_x\|_{L^p} \leq C_p \min\{\delta_r, \delta_r^\frac{1}{p}, t^{-1}\},
\]
\[
\|w_{xx}\|_{L^p} \leq C_p \min\{\delta_r, t^{-1}\}.
\]

(iii) \(\lim_{t \to +\infty} \sup_{x \in \mathbb{R}} |w(x, t) - w^R(x/|t|)| = 0\), where \(w^R(x/|t|)\) is Riemann solution of Burgers equation (1.25) with Riemann initial data \(w(x, 0) = w_-\), if \(x < 0\) and \(w(x, 0) = w_+\), if \(x > 0\).

It is easy to verify that \((\rho^r, u^r, \theta^r)\) satisfies (1.9). Due to Lemma 1.1 and (1.24), \((\rho^r, u^r, \theta^r)\) has the following properties (cf. [26]):

**Lemma 1.2.** Let \(\delta = |\rho_+ - \rho_-| + |u_+ - u_-| + |\theta_+ - \theta_-|\) be the wave strength. Then the smooth approximate profile \((\rho^r, u^r, \theta^r)(x, t)\) which is defined by (1.24) satisfies

(i) \(u_x^r > 0, \ |(\rho_x^r, \theta_x^r)| \leq Cu_x^r, \ 0 < \rho_+ < \rho^r(x, t) < \rho_-, \ 0 < \theta_+ < \theta^r(x, t) < \theta_-, \) for \(x \in \mathbb{R}\) and any \(t \geq 0\).

(ii) For any \(1 \leq q \leq +\infty\), there exists a constant \(C_q\) depending on \(q, \rho_\pm, \theta_\pm\) and \(|u_\pm|\) such that for \(t > 0\),
\[
\|(\rho^r_x, u^r_x, \theta^r_x)\|_{L^q} \leq C_q \min\{\delta, \delta^\frac{1}{q}(1 + t)^{-1}\},
\] (1.26)
\[
\|(\rho^r_{xx}, u^r_{xx}, \theta^r_{xx})\|_{L^q} \leq C_q \min\{\delta, (1 + t)^{-1}\}.
\] (1.27)

(iii) \(\lim_{t \to +\infty} \sup_{x \in \mathbb{R}} |(\rho^r, u^r, \theta^r)(x, t) - (\rho^R, u^R, \theta^R)(x/|t|)| = 0\). (1.28)

### 1.3. Main results.

From the above preparation, we can state the main result of this paper as follows.

**Theorem 1.3.** Assume that the initial data satisfy (1.8), (1.20), (1.23), and the dielectric constant \(\varepsilon\) satisfies
\[
0 < \varepsilon < \bar{C}
\] (1.29)
for some positive constants \(\bar{C}\) (depending only on \(|u_\pm|\) and \(\theta_-\)). There exist two small positive constants \(\varepsilon_1\) and \(\delta_1\) which are independent of \(T\), such that if \(0 < \delta < \delta_1\) and the initial data satisfy
\[
\|(\rho_0(x) - \rho^r(x, 0), u_0(x) - u^r(x, 0), \theta_0(x) - \theta^r(x, 0), E_0(x), b_0(x))\|_{H^1}^2 \leq \varepsilon_1,
\] (1.30)
then the Cauchy problem (1.5), (1.6), (1.8) admits a unique global solution \((\rho, u, \theta, E, b)\) satisfying
\[
\sup_{t > 0} \| (\rho - \rho^r, u - u^r, \theta - \theta^r, E, b) (\cdot, t) \|_{H^1}^2 \leq C_0 \varepsilon_1. \tag{1.31}
\]
Moreover, the solution \((\rho, u, \theta, E, b)(x, t)\) tends time-asymptotically to the rarification wave in the sense that
\[
\lim_{t \to +\infty} \sup_{x \in \mathbb{R}} \| (\rho, u, \theta, E, b)(x, t) - (\rho^R, u^R, \theta^R, 0, 0)(x/t) \| = 0. \tag{1.32}
\]

Remark 1. From (2.43), we can take
\[
\bar{C} = \frac{1}{32 \left( \sqrt{\gamma R \theta_{-}} + \max\{|u_-|, |u_+|\} \right) \cdot \max\{|u_-|, |u_+|\}}. \tag{1.33}
\]
Then for each given \(\varepsilon\) satisfying the condition (1.29), our system (1.5) is definitely well-defined. On the one hand, when we take \(\max\{|u_-|, |u_+|\}\) suitably small, the dielectric constant \(\varepsilon\) can be large enough, which can be seen from conditions (1.29) and (1.33). This fact can relax the requirement of smallness of \(\varepsilon\) in [22]. On the other hand, notice that [10] requires \(|u^r|\) to be small enough, and it can been derived from \(|u^r| \leq \max\{|u_-|, |u_+|\}\) by employing (1.19). Hence conditions (1.29) and (1.33) together can relax the restriction on \(\max\{|u_-|, |u_+|\}\) as long as the dielectric constant \(\varepsilon\) is suitably small. Moreover, an interesting problem occurs, that is how to remove the technical condition (1.29) in future.

Remark 2. For the compressible non-isentropic Navier-Stokes-Maxwell equations, the stability of wave patterns, namely, shock wave, rarefaction wave, contact discontinuity, stationary wave and their compositions can also be taken into account, which will be studied by the authors in future.

The rest of the paper is organized as follows. We construct a perturbation equations and make a priori estimates in Section 2. The proof of Theorem 1.3 is concluded in Section 3.

2. Uniform a priori estimates. To prove Theorem 1.3, we use elementary energy method. Define the perturbation as
\[
(\phi, \psi, \zeta) = (\rho - \rho^r, u - u^r, \theta - \theta^r).
\]
It is easy to derive that \((\phi, \psi, \zeta, E, b)\) satisfy
\[
\begin{cases}
\phi_t + u \phi_x + u \phi_x + \rho \psi_x + \rho \psi_x \psi = 0, \\
\rho (\psi_t + u \psi_x) + p_x = \mu \psi_{xx} + \mu u^r_{xx} + \frac{\rho}{\rho^r} p_x^r - \rho u_x^r \psi - (E + \psi b + u^r b) b, \\
\rho \left( \frac{R}{\gamma - 1} (\zeta_t + \psi \zeta_x) + p \psi_x = \kappa \zeta_{xx} + \kappa \theta^r_{xx} + \mu (\psi_x + u_x^r)^2 - \frac{\rho}{\gamma - 1} (\psi \theta^r_x + u^r \zeta_x) \\
\varepsilon E_x - b_x + E + \psi b + u^r b = 0, \\
\theta_t - E_x = 0,
\end{cases}
\tag{2.1}
\]
with initial data for \(x \in \mathbb{R}\),
\[
(\phi_0, \psi_0, \zeta_0, E_0, b_0)(x) := (\rho_0(x) - \rho^r(x, 0), u_0(x) - u^r(x, 0), \theta_0(x) - \theta^r(x, 0), E_0(x), b_0(x)). \tag{2.2}
\]
For any $T > 0$, we define a function space $X(0, T)$ as

$$X(0, T) = \left\{ (\phi, \psi, \zeta, E, b) \mid (\phi, \psi, \zeta, E, b) \in L^\infty((0, T); H^1), \ (E_x, b_x) \in L^2((0, T); L^2) \right\}. $$

The local existence of solutions to the reformulated Cauchy problem (2.1), (2.2) can be established by the standard iteration argument. In order to prove Theorem 1.3 for brevity, we only devote ourselves to obtaining the global-in-time a priori estimates as follows.

**Proposition 1.** (A priori estimates) Suppose that all the conditions in Theorem 1.3 hold. Let $(\phi, \psi, \zeta, E, b) \in X(0, T)$ be a smooth solution to the problem (2.1), (2.2) on $0 \leq t \leq T$ for $T > 0$. There exist some suitably small positive constants $\delta_0$ and $\varepsilon_0$ such that if $\delta < \delta_0$,

$$\sup_{0 \leq t \leq T} \|(\phi, \psi, \zeta, E, b)(t)\|_{H^1} \leq \varepsilon_0, \quad (2.3)$$

then the Cauchy problem (2.1), (2.2) has a unique global solution $(\phi, \psi, \zeta, E, b)(x, t)$ satisfying

$$\sup_{0 \leq t \leq T} \|(\phi, \psi, \zeta, \sqrt{\varepsilon} E, b)(t)\|_{H^1}^2 + \int_0^T \|\sqrt{\varepsilon} \phi_x, \psi_x, \zeta_x, E_x, b_x\|^2 \, dt$$

$$+ \int_0^T \|\phi, \psi, \zeta_x, E_x, b_x\|^2 \, dt + \int_0^T \|E + \psi b + u^* b\|^2 \, dt$$

$$\leq C\|(\phi_0, \psi_0, \zeta_0, E_0, b_0)\|_{H^1}^2 + C\delta^\frac{3}{2}. \quad (2.4)$$

It is easy to deduce

$$\|(\phi, \psi, \zeta, E, b)\|_{L^\infty} \leq \sqrt{2}\varepsilon_0, \quad (2.5)$$

where we have used (2.3) and the following Sobolev inequality

$$\|f\|_{L^\infty} \leq \sqrt{2}\|\|\frac{1}{2}f\|_{H^1}\|_{\frac{1}{2}}, \quad \text{for } f(x) \in H^1(\mathbb{R}). \quad (2.6)$$

By (2.5) and the smallness of $\varepsilon_0$, we can also get

$$0 < \frac{1}{2} \min\{\rho_-, \rho_+\} \leq \rho = \phi + \rho^* \leq \frac{3}{2} \max\{\rho_-, \rho_+\}, \quad (2.7)$$

$$0 < \frac{1}{2} \min\{\theta_-, \theta_+\} \leq \theta = \zeta + \theta^* \leq \frac{3}{2} \max\{\theta_-, \theta_+\}, \quad (2.8)$$

$$|u| \leq |\psi| + |u^*| \leq \frac{3}{2} \max\{|u_-|, |u_+|\}, \quad (2.9)$$

which will be frequently used in the sequel.

Proposition 1 is a conclusion of several lemmas below. We first give the zero-order energy estimate.

**Lemma 2.1.** Suppose that all the conditions in Theorem 1.3 hold. For all $0 < t < T$, there exists a constant $\bar{C}$ depending only on $|u_\pm|$ and $\theta_-$ such that if $0 < \varepsilon < \bar{C}$, then

$$\|(\phi, \psi, \zeta, \sqrt{\varepsilon} E, b)\|^2 + \int_0^t \|(\psi_x, \zeta_x)\|^2 \, d\tau$$

$$+ \int_0^t \|\sqrt{\varepsilon} \phi_x, \psi_x, \zeta_x, E_x, b_x\|^2 \, d\tau + \int_0^t \|E + \psi b + u^* b\|^2 \, d\tau$$
\[
\eta(x,t) = \frac{\rho}{2} \psi^2 + R \rho \theta^r \Phi \left( \frac{\rho^r}{\rho} \right) + \frac{R}{\gamma - 1} \rho \theta^r \Phi \left( \frac{\theta}{\theta^r} \right).
\]

Direct calculations give rise to
\[
\eta_h(x,t) + \mu \psi_x^2 + \kappa \zeta_x^2 + Q + H_x = G + \frac{\zeta}{\theta} (E + \psi b + u^r b)^2 - (E + \psi b + u^r b) \psi b,
\]
where
\[
Q = u_x^r \left( \rho \psi^2 + (\gamma - 1) R \rho \theta^r \Phi \left( \frac{\rho^r}{\rho} \right) + R \rho \theta^r \Phi \left( \frac{\theta}{\theta^r} \right) \right) + \frac{R}{\gamma - 1} \rho \theta^r \nu \zeta - R \theta_x^r \psi \phi,
\]
\[
H = R \rho \theta^r u \Phi \left( \frac{\rho^r}{\rho} \right) + \frac{R}{\gamma - 1} \rho \theta^r \mu \Phi \left( \frac{\theta}{\theta^r} \right) - \mu \psi \psi_x + \frac{1}{2} \rho \psi \psi^2 + (p - p^r) \psi - \kappa \frac{\zeta_x}{\theta},
\]
and
\[
G = R \rho \theta^r \Phi \left( \frac{\rho^r}{\rho} \right) + \frac{R}{\gamma - 1} \rho \psi \theta_x^r \Phi \left( \frac{\theta}{\theta^r} \right) + \mu \psi u_x^r + \kappa \frac{\zeta_x \theta_x^r}{\theta^2} + \frac{\zeta}{\theta} \left( \kappa \theta_x^r + \mu (\psi_x + u_x^r)^2 \right).
\]

In what follows, we first treat \( Q \). The estimates for the higher order term \( G \) will be given later on. We observe from \( \Phi(1) = \Phi'(1) = 0, \Phi''(1) = 1 \) and \( \Phi''(s) = -\frac{2}{s^2} \) that
\[
\Phi \left( \frac{\rho^r}{\rho} \right) = \frac{1}{2} \frac{\phi^2}{\rho^2} + O(1) \phi^3, \quad \Phi \left( \frac{\theta}{\theta^r} \right) = \frac{1}{2} \frac{\zeta^2}{(\theta^r)^2} + O(1) \zeta^3,
\]
as \(|\phi|\) and \(|\zeta|\) can be sufficiently small. Moreover, we get from (1.24) that
\[
\theta_x^r = - \frac{\gamma - 1}{\gamma} R^{-\frac{1}{2}} \left( \theta \right)^{-\frac{1}{2}} u_x^r.
\]
Recall \( u_x^r(x,t) > 0 \) for \( x \in \mathbb{R} \) and all \( t \geq 0 \). From (2.12) and (2.13), one can claim that
\[
Q = Q_1 + O(1) u_x^r \phi^3 + O(1) u_x^r \zeta^3,
\]
with
\[
Q_1 = u_x^r \left[ \rho \psi^2 + \frac{1}{2} (\gamma - 1) R \rho^{-1} \theta^r \phi^2 + \frac{1}{2} R \rho (\theta^r)^{-1} \zeta^2 
\right.
- \left. \gamma^{-\frac{1}{2}} R^{\frac{1}{2}} (\theta^r)^{-\frac{1}{2}} \rho \psi \zeta + (\gamma - 1) \gamma^{-\frac{1}{2}} R^{\frac{1}{2}} (\theta^r)^{\frac{1}{2}} \phi \psi \right].
\]

Elementary computations give
\[
Q_1 = u_x^r \left[ \phi, \psi, \zeta \right] M \left( \phi, \psi, \zeta \right)^T,
\]
and the \( 3 \times 3 \) real symmetric matrix \( M \) is given by
\[
\begin{pmatrix}
a_{11} & a_{12} & 0 \\
a_{21} & a_{22} & a_{23} \\
0 & a_{32} & a_{33}
\end{pmatrix},
\]
with
\[
a_{11} = \frac{1}{2} (\gamma - 1) R \rho^{-1} \theta^r, \quad a_{22} = \rho, \quad a_{33} = \frac{1}{2} R \rho (\theta^r)^{-1},
\]
\[
a_{12} = a_{21} = \frac{1}{2} (\gamma - 1) \gamma^{-\frac{1}{2}} R^{\frac{1}{2}} (\theta^r)^{\frac{1}{2}}, \quad a_{23} = a_{32} = -\frac{1}{2} \gamma^{-\frac{1}{2}} R^{\frac{1}{2}} (\theta^r)^{-\frac{1}{2}} \rho.
\]
We can compute all the leading principal minors of $M$ as follows:

$$\Delta_{11} = a_{11} = \frac{1}{2}(\gamma - 1)R\rho^{-1}\theta^r > 0,$$

$$\Delta_{22} = a_{11}a_{22} - a_{12}a_{21} = \frac{1}{2}(\gamma - 1)R\theta^r \left(1 - \frac{1}{2}(\gamma - 1)\gamma^{-1}\right) = \frac{1}{4}(\gamma - 1)R\theta^r \left(1 + \frac{1}{\gamma}\right) > 0,$$

$$\Delta_{33} = a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} = \frac{1}{4}(\gamma - 1)R^2\rho - \frac{(\gamma - 1)\gamma - 1}{8\gamma}R^2\rho = \frac{1}{4}(\gamma - 1)R^2\rho \left(1 - \frac{\gamma - 1}{2\gamma} - \frac{1}{2\gamma}\right) = \frac{1}{8}(\gamma - 1)R^2\rho > 0.$$  

This indicates that matrix $M$ is positive definite. So there is a constant $\lambda_0 > 0$ such that

$$Q_1 \geq \lambda_0 u_x^r(\phi^2 + \psi^2 + \zeta^2).$$

Using the smallness of $|\phi|$ and $|\zeta|$, together with (2.14), we can get

$$Q \geq Cu_x^r(\phi^2 + \psi^2 + \zeta^2). \quad (2.15)$$

We now have by substituting (2.15) into (2.11) and integrating the resultant equation with respect to $x$ over $\mathbb{R}$ that

$$\frac{d}{dt} \int_{\mathbb{R}} \eta(x, t) \, dx + \mu \int_{\mathbb{R}} \psi_x^2 \, dx + \kappa \int_{\mathbb{R}} \psi_{xx} \, dx + C \int_{\mathbb{R}} u_x^r(\phi^2 + \psi^2 + \zeta^2) \, dx \leq \int_{\mathbb{R}} G \, dx + \int_{\mathbb{R}} (E + \psi b + u^r b)\zeta_x \, dx - \int_{\mathbb{R}} (E + \psi b + u^r b)\eta \, dx. \quad (2.16)$$

Motivated by the treatment of similar terms for the Navier-Stokes equations, we continue to estimate the higher order terms of $\int_{\mathbb{R}} G \, dx$. By applying Sobolev and Young inequalities, and using Lemma 1.2, it is direct to obtain the following estimate:

$$\int_{\mathbb{R}} G \, dx = \int_{\mathbb{R}} R\theta^r \phi^2 \Phi \left(\frac{\theta^r}{\rho}\right) \, dx + \int_{\mathbb{R}} R\theta^r \phi^2 \Phi \left(\frac{\theta^r}{\rho}\right) \, dx + \int_{\mathbb{R}} \mu\psi u_{xx} \, dx$$

$$+ \int_{\mathbb{R}} \kappa \zeta_x \theta_x \, dx + \int_{\mathbb{R}} \kappa \theta_{xx} + \mu(\psi_x + u_{xx}^r) \, dx$$

$$\leq C \int_{\mathbb{R}} |\psi| u_x^r(\phi^2 + \psi^2) \, dx + C \int_{\mathbb{R}} (|u_{xx}^r| |\psi| + |\theta_{xx}^r| |\zeta|) \, dx$$

$$+ C \int_{\mathbb{R}} (\zeta_x^2 + u_{xx}^r) \, dx + C \int_{\mathbb{R}} (\zeta_x^2 + u_{xx}^r) \, dx$$

$$\leq C(\varepsilon_0 + \eta) ||\sqrt{u_x^r}(\phi, \zeta)||^2 + C(\varepsilon_0^2 + \delta) ||(\psi, \zeta)||^2 + C\delta^\delta (1 + t)^{-\frac{\delta}{2}}. \quad (2.17)$$

By plugging (2.17) into (2.16) and choosing $\delta$, $\varepsilon_0$ and $\eta$ suitably small, we obtain

$$\frac{d}{dt} \int_{\mathbb{R}} \eta(x, t) \, dx + C \int_{\mathbb{R}} \psi_x^2 \, dx + C \int_{\mathbb{R}} \zeta_x^2 \, dx + C \int_{\mathbb{R}} u_x^r(\phi^2 + \psi^2 + \zeta^2) \, dx$$

$$\leq C\delta^\delta (1 + t)^{-\frac{\delta}{2}} + C \int_{\mathbb{R}} (E + \psi b + u^r b)\zeta_x \, dx - \int_{\mathbb{R}} (E + \psi b + u^r b)\eta \, dx. \quad (2.18)$$

We next try to treat the second and the third term in the right-hand side of (2.18). Multiplying (2.1a) by $E$ and (2.1b) by $b$ respectively, summing them up and integrating the resulting equation with respect to $x$ over $\mathbb{R}$ gives

$$\frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}} (\varepsilon E^2 + b^2) \, dx + \int_{\mathbb{R}} (E + \psi b + u^r b) E \, dx = 0. \quad (2.19)$$
Moreover, multiplying (2.1) by \( u^*b \), integrating it with respect to \( x \) and applying integration by parts, together with (2.1), we have

\[
\frac{d}{dt} \int_R \varepsilon Ebu^* \, dx + \frac{1}{2} \int_R u^*_x (\varepsilon E^2 + b^2) \, dx + \int_R (E + \psi b + u^*b) u^*b \, dx = \int_R \varepsilon Ebu^* \, dx. \tag{2.20}
\]

Elementary calculations give rise to

\[
u^*_x = -u^* u^*_x - k e^e \frac{1}{\varepsilon} \left[ (\rho')^\gamma \right] = -u^* u^*_x + \sqrt{\frac{\gamma p'}{p'}} u^*_x = \left( \sqrt{\gamma R\theta} - u^* \right) u^*_x
\]

\[
\leq \left( \sqrt{\gamma R\theta} + \max \{|u_-|, |u_+|\} \right) u^*_x := \left( \sqrt{\gamma R\theta} + \beta \right) u^*_x, \tag{2.21}
\]

where \( \beta = \max \{|u_-|, |u_+|\} \). In addition,

\[
\int_R \varepsilon Ebu^* \, dx \tag{2.22}
\]

\[
= \int_R \varepsilon (E + ub - ub) bu^*_x \, dx = -\int_R \varepsilon ub^2 u^*_x \, dx + \int_R \varepsilon (E + ub) bu^*_x \, dx =: I_1 + I_2,
\]

together with (2.21), by applying Sobolev inequality, Cauchy inequality, (2.5), (2.9) and (1.26) in Lemma 1.2, we can deduce

\[
|I_1| \leq \frac{3}{2} \left( \sqrt{\gamma R\theta} + \beta \right) \varepsilon \int_R u^*_x b^2 \, dx \tag{2.23}
\]

and

\[
|I_2| \leq \left( \sqrt{\gamma R\theta} + \beta \right) \varepsilon \int_R u^*_x |E + \psi b + u^*b| |b| \, dx
\]

\[
\leq \left( \sqrt{\gamma R\theta} + \beta \right) \varepsilon ||b||_{L^\infty} \|u^*_x\| \|E + \psi b + u^*b\|
\]

\[
\leq C \left( \sqrt{\gamma R\theta} + \beta \right) \varepsilon \|b\|_2 \|b_x\| \frac{1}{2} \varepsilon \delta^2 \left( 1 + t \right)^{-\frac{1}{2}} \|E + \psi b + u^*b\|
\]

\[
\leq C \left( \sqrt{\gamma R\theta} + \beta \right) \varepsilon \frac{1}{2} \varepsilon \delta^2 \left[ (1 + t)^{-2} + \|b_x\|^2 + \|E + \psi b + u^*b\|^2 \right]. \tag{2.24}
\]

By plugging (2.23), (2.24) into (2.22) and taking

\[
\left( \sqrt{\gamma R\theta} + \beta \right) \varepsilon \frac{1}{2} \varepsilon < \left( \sqrt{\gamma R\theta} + \beta \right) \varepsilon < \frac{1}{6}, \tag{2.25}
\]

we can get

\[
\left| \int_R \varepsilon Ebu^* \, dx \right| \leq \frac{1}{4} \int_R u^*_x b^2 \, dx + C \delta^2 \left[ (1 + t)^{-2} + \|b_x\|^2 + \|E + \psi b + u^*b\|^2 \right]. \tag{2.26}
\]

Then inserting (2.26) into (2.20) gives

\[
\frac{d}{dt} \int_R \varepsilon Ebu^* \, dx + \frac{1}{2} \int_R u^*_x (\varepsilon E^2 + b^2) \, dx + \int_R (E + \psi b + u^*b) u^*b \, dx
\]

\[
\leq C \delta^2 \left[ (1 + t)^{-2} + \|b_x\|^2 + \|E + \psi b + u^*b\|^2 \right]. \tag{2.27}
\]

By combining (2.18), (2.19) and (2.27), then choosing \( \delta \) suitably small, we arrive at

\[
\frac{d}{dt} \int_R \left[ \eta (x, t) + \left( \frac{1}{2} \varepsilon E^2 + \frac{1}{2} b^2 + \varepsilon Ebu^* \right) \right] \, dx
\]

\[
+ C \| (\psi_x, \zeta_x, E, E + \psi b + u^*b) \|^2 + C \| \sqrt{u_x^2 (\phi, \psi, \zeta, \sqrt{\varepsilon} E, b) \|}^2
\]

\[
\leq C \delta^2 (1 + t)^{-\frac{1}{2}} + C \delta^2 \|b_x\|^2. \tag{2.28}
\]
Proof. Firstly, taking the derivative of (2.1) with respect to $t$ and using (2.29), we can obtain
\begin{align*}
&\frac{\partial}{\partial t} \eta(x, t) = \frac{1}{2} \frac{\partial^2}{\partial x^2} + R \rho \Phi\left(\frac{\rho}{\rho}\right) + \frac{\partial^2}{\partial x^2} + \frac{R}{\gamma - 1} \rho \Phi\left(\frac{\theta}{\theta}\right) \\
&= \frac{1}{2} \rho \psi^2 + \frac{1}{2} R \rho^{-1} \xi_1^2 \rho_1^2 + \frac{1}{2} (\gamma - 1) R \rho (\theta - \xi_2^2 \rho^2) \geq C(\phi^2 + \psi^2 + \xi_2^2),
\end{align*}
where $\xi_1$ takes value between $\rho$ and $\rho_1$; $\xi_2$ takes value between $\theta$ and $\theta^\prime$. By integrating (2.28) with respect to $t$ and using (2.29), we can obtain
\begin{align*}
&\int_t^\infty \left( \frac{\partial}{\partial t} \eta(x, t) + \rho \frac{\partial}{\partial x} \eta(x, t) \right) \mathrm{d}t + C \int_t^\infty \left( \frac{\partial}{\partial x} \left( \sqrt{\xi} \eta \right) \right) \mathrm{d}t \leq C \int_t^\infty \left( \frac{\partial}{\partial x} \left( \sqrt{\xi} \eta \right) \right) \mathrm{d}t + C \int_t^\infty \left( \frac{\partial}{\partial x} \left( \sqrt{\xi} \eta \right) \right) \mathrm{d}t.
\end{align*}
Using Cauchy inequality to the last term of (2.30) gives
\begin{align}
\left| \int_t^\infty \frac{\partial}{\partial x} \left( \sqrt{\xi} \eta \right) \mathrm{d}t \right| \leq \sqrt{\int_t^\infty \left( \frac{\partial}{\partial x} \left( \sqrt{\xi} \eta \right) \right) \mathrm{d}t} + \sqrt{\int_t^\infty \left( \frac{\partial}{\partial x} \left( \sqrt{\xi} \eta \right) \right) \mathrm{d}t}.
\end{align}
Note that (2.25) implies $\beta^2 < \frac{1}{6}$. Hence (2.31) and (2.30) conclude (2.10). The proof of Lemma 2.1 is completed.

To complete the zero-order energy estimate, we continue to control $\int_0^\infty \|b_x\|^2 \mathrm{d}t$ as the following lemma.

**Lemma 2.2.** Suppose that all the conditions in Theorem 1.3 hold. For all $0 < t < T$, there exists a constant $C$ depending only on $|u_\pm|$ and $\theta_-$ such that if $0 < \varepsilon < C$, then
\begin{align}
\|(\sqrt{\xi} E_x, b_x)\|^2 + \int_0^t \left( (E_x, b_x) \right) \mathrm{d}t \leq C \|(\phi_0, \psi_0, \zeta_0)\|^2 + C \|(E_0, b_0)\|^2_{H^1} + C \delta^\frac{1}{2}.
\end{align}

**Proof:** Firstly, taking the derivative of (2.1) with respect to $x$ and multiplying it by $E_x$, then integrating the resulting equality with respect to $x$, we obtain
\begin{align}
\frac{d}{dt} \int_\mathbb{R} \left( \varepsilon E_x^2 + b_x^2 \right) \mathrm{d}x - \int_\mathbb{R} b_x E_x \mathrm{d}x + \int_\mathbb{R} E_x^2 \mathrm{d}x + \int_\mathbb{R} (\psi) E_x \mathrm{d}x + \int_\mathbb{R} (\psi^\prime) E_x \mathrm{d}x = 0.
\end{align}
Secondly, taking the derivative of (2.1) with respect to $x$ and multiplying $b_x$, then integrating the resulting equality with respect to $x$, we get
\begin{align}
\frac{d}{dt} \int_\mathbb{R} b_x^2 \mathrm{d}x - \int_\mathbb{R} E_x b_x \mathrm{d}x = 0.
\end{align}
Combining (2.33) and (2.34), we obtain
\begin{align}
\frac{d}{dt} \int_\mathbb{R} \left( \varepsilon E_x^2 + b_x^2 \right) \mathrm{d}x + \int_\mathbb{R} E_x^2 \mathrm{d}x = - \int_\mathbb{R} (\psi) E_x \mathrm{d}x - \int_\mathbb{R} (\psi^\prime) E_x \mathrm{d}x.
\end{align}
By using (2.5) and the decay rate of the smooth rarefaction wave as (1.26), together with Cauchy inequality, we can derive that
\begin{align}
\frac{d}{dt} \int_\mathbb{R} \left( \varepsilon E_x^2 + b_x^2 \right) \mathrm{d}x + \int_\mathbb{R} E_x^2 \mathrm{d}x \\
\leq \int_\mathbb{R} \left( |\psi b E_x| + |u^\prime b E_x| \right) \mathrm{d}x + \int_\mathbb{R} (|\psi| + |u^\prime|) |b_x E_x| \mathrm{d}x.
\end{align}
\[ \leq 4\eta\|E_x\|^2 + \frac{1}{4\eta} \left( \|b\|^2_{L^\infty} + \|\psi_x\|^2 + \|u_x^\tau\|_{L^\infty} \sqrt{\|u_x^\tau|b\|^2} + \|\psi\|^2_{L^\infty} \|b_x\|^2 + \beta^2 \|b_x\|^2 \right) \]
\[ \leq \frac{1}{2} \|E_x\|^2 + C \left( \epsilon_0^2 \|\psi_x\|^2 + \delta \|u_x^\tau|b\|^2 \right) + \left( 4\epsilon_0^2 + 2\beta^2 \right) \|b_x\|^2 \]
\[ \leq \frac{1}{2} \|E_x\|^2 + C\beta^2 \|\psi_x, \sqrt{u_x^\tau}b\|^2 + 3\beta^2 \|b_x\|^2, \]  
(2.36)

where in the third inequality we have taken \( \eta = \frac{1}{4} \) and in the last inequality we have chosen \( \epsilon_0^2 \leq \frac{1}{4} \beta^2 \) and \( \delta \leq \frac{1}{4} \beta^2 \). Hence (2.36) gives that

\[ \frac{d}{dt} \| (\sqrt{\epsilon}E_x, b_x) \|^2 + \|E_x\|^2 \leq C\beta^2 \|\psi_x, \sqrt{u_x^\tau}b\|^2 + 6\beta^2 \|b_x\|^2. \]  
(2.37)

Now we only need to estimate the second term on the right-hand side of (2.37). Multiplying (2.1)_4 by \(-b\), integrating the resulting equation with respect to \( x \) over \( \mathbb{R} \) and applying (2.1)_5, we can easily deduce

\[ -\frac{d}{dt} \int_\mathbb{R} \epsilon Eb_x \, dx + \int_\mathbb{R} b_x^2 \, dx \]
\[ = -\int_\mathbb{R} \epsilon Eb_{bx} \, dx + \int_\mathbb{R} (E + \psi + u^\tau b) b_x \, dx = \int_\mathbb{R} \epsilon E_x^2 \, dx + \int_\mathbb{R} (E + \psi + u^\tau b) b_x \, dx \]
\[ \leq \int_\mathbb{R} \epsilon E_x^2 \, dx + \frac{1}{2} \int_\mathbb{R} (E + \psi + u^\tau b)^2 \, dx + \frac{1}{2} \int_\mathbb{R} b_x^2 \, dx, \]  
(2.38)
i.e.,

\[ -\frac{d}{dt} \int \epsilon Eb_x \, dx + \|b_x\|^2 \leq 2\epsilon \|E_x\|^2 + \|E + \psi + u^\tau b\|^2. \]  
(2.39)

Multiplying (2.39) by \( 8\beta^2 \) and summing it to (2.37), then integrating the resulting equation with respect to \( t \), we obtain

\[ \| (\sqrt{\epsilon}E_x, b_x) \|^2 + \frac{1}{2} \int_0^t \|E_x\|^2 \, d\tau + 2\beta^2 \int_0^t \|b_x\|^2 \, d\tau \]
\[ \leq C \| (E_0, E_0x, b_0x) \|^2 + C\beta^2 \int_0^t \|\psi_x, \sqrt{u_x^\tau}b\|^2 \, d\tau \]
\[ + 8\beta^2 \int_0^t \|E + \psi + u^\tau b\|^2 \, d\tau + 16\beta^2 \int_\mathbb{R} \epsilon Eb_x \, dx, \]  
(2.40)

where we have taken \( 1 - 16\beta^2 \epsilon \geq \frac{1}{2} \), i.e., \( \beta^2 \epsilon \leq \frac{1}{32} \). By employing Cauchy inequality, we can deduce

\[ 16\beta^2 \int_\mathbb{R} \epsilon Eb_x \, dx \leq \frac{1}{2} \int_\mathbb{R} b_x^2 \, dx + \frac{1}{2} 16\beta^4 \epsilon \int_\mathbb{R} \epsilon E^2 \, dx \leq \frac{1}{2} \int_\mathbb{R} b_x^2 \, dx + 4\beta^2 \int_\mathbb{R} \epsilon E^2 \, dx, \]  
(2.41)

where in the last inequality we have used \( \beta^2 \epsilon \leq \frac{1}{32} \). Taking (2.41) into (2.40), together with (2.10), and choosing \( \delta \) suitably small gives

\[ \| (\sqrt{\epsilon}E_x, b_x) \|^2 + \int_0^t \|E_x\|^2 \, d\tau + \beta^2 \int_0^t \|b_x\|^2 \, d\tau \]
\[ \leq C \| (E_0, E_0x, b_0x) \|^2 + C\beta^2 \left[ \| (\psi_0, \phi_0, \zeta_0, E_0, b_0) \|^2 + \delta \right]. \]  
(2.42)

It is sufficient to conclude (2.32) from (2.42). Due to (2.25) and \( \beta^2 \epsilon \leq \frac{1}{32} \), we can take

\[ \bar{C} = \min \left\{ \frac{1}{32} \left( \sqrt{\gamma R^2 \beta^2 + \beta} \right), \frac{1}{32 \beta^2} \right\}. \]
we can rewrite (2.1)

\[ \mu \text{tion by} \]

Secondly, differentiating (2.1) with respect to \( x \), we have

\[ \text{Multiplying the above equation by} \quad \phi \]

Inspired by the work of [26], firstly, due to

Suppose that all the conditions in Theorem 1.3 hold. For all \( t \), there exists a constant \( \bar{C} \) depending only on \( |u_\pm| \) and \( \theta_- \) such that if \( 0 < \varepsilon < \bar{C} \), then

\[
\|\phi_x\|^2 + \int_0^t \|\phi_x\|^2 d\tau + \int_0^t \|u^x_{\phi_x}\|^2 d\tau \\
\leq C \varepsilon_0 \int_0^t \|\psi_{xx}\|^2 d\tau + C\| (\phi_0, \phi_0 x, \psi_0, \zeta_0, E_0, b_0) \|^2 + C \delta^\varepsilon. \quad (2.44)
\]

Proof. Inspired by the work of [26], firstly, due to

\[
p_x - p^*_x = (R \rho \theta - R \rho^\theta \theta_x + (R \rho^\theta \theta - R \rho^\theta \theta^\tau)_x = (R \rho \theta)_x + (R \rho^\theta \zeta)_x \\
= (R \rho \theta^\tau x + (R \rho \zeta)_x x + (R \rho \theta^\tau) x = R \phi_x \theta^\tau + R \phi \theta^\tau_x + R \phi^\tau_x + R \phi \zeta + R \phi \zeta_x + R \phi \zeta^\tau + R \phi \zeta^\tau_x,
\]

we can rewrite (2.1) as

\[
\rho (\psi_x + u \psi_x) + (R \phi_x \theta^\tau + R \phi \theta^\tau_x + R \phi \zeta + R \phi \zeta_x + R \phi \zeta^\tau + R \phi \zeta^\tau_x) \\
= \mu \psi_{xx} + \mu u_{xx} + \frac{\phi}{\rho^2} p_x - \rho u_x \psi - (E + \psi b + u^\tau b) b. \quad (2.46)
\]

Multiplying the above equation by \( \frac{\phi_x}{\rho} \), then integrating the resulting equality with respect to \( x \) over \( \mathbb{R} \), after some elementary computations, we can obtain

\[
\frac{d}{dt} \int_{\mathbb{R}} \phi_x \psi \, dx + \int_{\mathbb{R}} \frac{R \theta^\tau}{\rho} \phi_x^2 \, dx \\
= \int_{\mathbb{R}} u^x \phi_x \psi \, dx + \int_{\mathbb{R}} \rho \psi_x^2 \, dx + \int_{\mathbb{R}} \rho^\tau_x \psi_x \, dx - \int_{\mathbb{R}} \frac{R \phi_x \theta^\tau \phi_x}{\rho} \, dx \\
- \int_{\mathbb{R}} \frac{R \phi \theta^\tau_x \phi_x}{\rho} \, dx - \int_{\mathbb{R}} \frac{R \phi \zeta \phi_x}{\rho} \, dx - \int_{\mathbb{R}} \frac{R \phi \zeta_x \phi_x}{\rho} \, dx \\
- \int_{\mathbb{R}} \frac{R \phi \zeta^\tau \phi_x}{\rho} \, dx + \int_{\mathbb{R}} \frac{\phi \phi_x^2}{\rho^2} \, dx - \int_{\mathbb{R}} \frac{u^x \phi_x^2}{\rho} \, dx - \int_{\mathbb{R}} \frac{(E + \psi b + u^\tau b) b}{\rho} \phi_x \, dx. \quad (2.47)
\]

Secondly, differentiating (2.1) with respect to \( x \) and multiplying the resulting equation by \( \frac{\phi_x^2}{\rho^2} \), then integrating the resulting equality with respect to \( x \) over \( \mathbb{R} \), after some elementary calculations, we can get

\[
\frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}} \frac{\phi_x^2}{\rho^2} \, dx + \frac{1}{2} \int_{\mathbb{R}} \frac{\mu u^x_x \phi_x^2}{\rho^2} \, dx \\
= - \int_{\mathbb{R}} \frac{\mu u^x_x \phi \phi_x}{\rho^2} \, dx - \int_{\mathbb{R}} \frac{\mu \phi_x^x \phi \phi_x}{\rho^2} \, dx - 2 \int_{\mathbb{R}} \frac{\mu \phi_x^x \phi \phi_x}{\rho^2} \, dx \\
- \frac{1}{2} \int_{\mathbb{R}} \frac{\phi_x^x \psi_x^x}{\rho^2} \, dx - \int_{\mathbb{R}} \frac{\psi_x^x \phi_x}{\rho} \, dx. \quad (2.48)
\]
The summation of (2.47) and (2.48) further implies

\[
\frac{d}{dt} \int_{\mathbb{R}} \left( \frac{\mu}{2} \phi_x^2 + \phi_x \psi \right) \, dx + \int_{\mathbb{R}} \frac{R\phi_x}{\rho} \, dx + \frac{1}{2} \int_{\mathbb{R}} \mu u_x \phi_x^2 \, dx
\]

\[
= - \int_{\mathbb{R}} \mu \frac{u_x \phi_x}{\rho^2} \, dx - \int_{\mathbb{R}} \frac{\rho_x \psi \phi_x}{\rho^2} \, dx - 2 \int_{\mathbb{R}} \frac{\rho_x \phi_x \psi_x}{\rho^2} \, dx - \frac{1}{2} \int_{\mathbb{R}} \mu \phi_x^2 \psi_x \, dx
\]

\[
+ \int_{\mathbb{R}} u_x \phi_x \, dx + \int_{\mathbb{R}} \rho \psi_x^2 \, dx + \int_{\mathbb{R}} \rho_x \psi_x \, dx - \int_{\mathbb{R}} \frac{R \phi_x}{\rho} \, dx - \int_{\mathbb{R}} \frac{R \phi_x^2}{\rho} \, dx
\]

\[
- \int_{\mathbb{R}} \frac{R \phi_x}{\rho} \, dx - \int_{\mathbb{R}} \frac{R \rho_x \phi_x}{\rho} \, dx - \int_{\mathbb{R}} \frac{R \rho_x \phi_x}{\rho} \, dx + \int_{\mathbb{R}} \frac{u_x \phi_x}{\rho} \, dx
\]

\[
+ \int_{\mathbb{R}} \frac{\phi_x \phi_x}{\rho} \, dx - \int_{\mathbb{R}} u_x \psi_x \, dx - \int_{\mathbb{R}} (E + \psi b + u^\tau b) \phi_x \, dx =: \sum_{i=1}^{16} J_i, \quad (2.49)
\]

where \( J_i \) \((1 \leq i \leq 16)\) denote the corresponding terms on the right-hand side of (2.49). Applying Sobolev inequality, Young inequality, (2.5) and using Lemma 1.2, one has

\[ J_1 + J_2 + J_{13} \leq C \delta^2 (1 + t)^{-\frac{3}{2}} + \eta \| \phi_x \|_2^2, \]

\[ J_3 + J_{10} + J_{12} \leq C (\delta + \varepsilon_0 + \eta) \| \phi_x \|_2^2 + C \| (\psi_x, \zeta_x) \|_2^2, \]

\[ J_6 + J_9 \leq C \varepsilon_0 \| \phi_x \|_2^2 + C \| \psi_x \|_2^2, \]

\[ J_5 + J_7 + J_8 + J_{11} + J_{14} + J_{15} \leq C \eta \| \phi_x \|_2^2 + C \| u_x (\phi, \psi, \zeta) \|_2^2 + C \| \psi_x \|_2^2, \]

\[ J_{16} \leq C \eta \| \phi_x \|_2^2 + \| E + \psi b + u^\tau b \|_2^2, \]

and

\[ J_4 \leq C \| \psi_x \|_{L^\infty} \| \phi_x \|_2^2 \leq C \| \psi_x \|_2 \| \psi_x \|_2 \| \phi_x \|_2 \]

\[ \leq C \varepsilon_0 \| \psi_x \|_2 + \| \psi_xx \|_2 \| \phi_x \|_2 \leq C \varepsilon_0 \| \phi_x \|_2^2 + C \varepsilon_0 \| \psi_xx \|_2^2 + C \| \psi_x \|_2^2. \]

Inserting the above estimations for \( J_i \) \((1 \leq i \leq 16)\) into (2.49) and choosing \( \delta, \varepsilon_0 \) and \( \eta \) suitably small, and then integrating the resulting inequality with respect to \( t \), together with (2.10) and (2.32), we arrive at (2.44). The proof of Lemma 2.3 is completed. \( \square \)

To control the term \( \int_0^t \| \psi_xx \|^2 d\tau \) on the right-hand side of (2.44) and the energy \( \| \psi_x \|^2 \), we need the following lemma.

**Lemma 2.4.** Suppose that all the conditions in Theorem 1.3 hold. For all \( 0 < t < T \), there exists a constant \( \bar{C} \) depending only on \( |u_\pm| \) and \( \theta_- \) such that if \( 0 < \varepsilon < \bar{C} \), then

\[ \| \psi_x \|^2 + \int_0^t \| \psi_xx \|^2 d\tau \leq C \| (\phi_0, \phi_{0x}, \psi_0, \psi_{0x}, \zeta_0, E_0, b_0) \|^2 + C \delta^{\frac{3}{2}}. \quad (2.50) \]

**Proof.** Multiplying (2.46) by \( \frac{\psi_xx}{\rho} \) and integrating the resulting equality with respect to \( x \), we have

\[
\int_{\mathbb{R}} \frac{\mu \psi_xx^2}{\rho} \, dx = \int_{\mathbb{R}} \psi_t \psi_xx \, dx + \int_{\mathbb{R}} u \psi_x \psi_xx \, dx \]

\[
+ \int_{\mathbb{R}} \left( R \phi_x \theta^r + R \phi_{x} \psi_x \phi_x + R \phi_x \zeta \phi_x + R \phi_x \psi_x \phi_x + R \phi_x \zeta \phi_x + R \phi_x \psi_x \phi_x \right) \frac{\psi_xx}{\rho} \, dx \quad (2.51)
\]
Proof. In a similar way as can be done for the Navier-Stokes equations, multiplying one can directly obtain (2.53). Applying Sobolev inequality, Young inequality, (2.5) and using Lemma 1.2, one has

\[ J_{17} = \int_{\mathbb{R}} \psi J \psi_{xx} \, dx = -\int_{\mathbb{R}} \psi \tau \psi_{x} \, dx = -\frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}} \psi_{x}^{2} \, dx, \]

\[ J_{18} = \int_{\mathbb{R}} \omega \psi_{x} \psi_{xx} \, dx \leq \eta \|\psi_{xx}\|^{2} + C\|\psi_{x}\|^{2}, \]

\[ J_{19} \leq \eta \|\psi_{xx}\|^{2} + C\|\langle \phi_{x}, \zeta_{x}, \sqrt{u^{2}}(\phi, \zeta)\rangle\|^{2}, \]

and

\[ J_{20} \leq \eta \|\psi_{xx}\|^{2} + C\|\sqrt{u^{2}}(\phi, \psi)\|^{2} + C\|E + \psi b + u^{r} b\|^{2} + C\delta^{\frac{1}{2}}(1 + t)^{-\frac{1}{2}}. \]

Taking the above estimations for \( J_{i} \) (17 \leq i \leq 20) into (2.51) and integrating the resulting inequality with respect to \( t \), together with (2.10), (2.32) and (2.44), and then choosing \( \eta \) and \( \varepsilon_{0} \) suitably small, we arrive at (2.50). The proof of Lemma 2.4 is completed.

Finally, we control the energy \( \|\zeta_{x}\|^{2} \).

**Lemma 2.5.** Suppose that all the conditions in Theorem 1.3 hold. For all \( 0 < t < T \), there exists a constant \( C \) depending only on \( |u_{\pm}| \) and \( \theta_{\pm} \) such that if \( 0 < \varepsilon < C \), then

\[ \|\zeta_{x}\|^{2} + \int_{0}^{t} \|\zeta_{xx}\|^{2} \, d\tau \leq C\|\langle \phi_{0}, \psi_{0}, \zeta_{0}\rangle\|_{H^{1}}^{2} + C\|\langle E_{0}, b_{0}\rangle\|^{2} + C\delta^{\frac{1}{2}}. \quad (2.52) \]

**Proof.** In a similar way as can be done for the Navier-Stokes equations, multiplying (2.1) by \( -\frac{\zeta_{x}}{\rho} \) and integrating the resulting equality with respect to \( x \), we get

\[
\frac{d}{dt} \int_{\mathbb{R}} R \frac{\zeta_{x}^{2}}{\gamma - 1} \, dx + \int_{\mathbb{R}} \kappa \zeta_{xx} \, dx = \int_{\mathbb{R}} \psi \zeta_{xx} \, dx + \int_{\mathbb{R}} \psi \zeta_{xx} \, dx - \int_{\mathbb{R}} \mu \zeta_{xx} \zeta_{xx} \, dx - \int_{\mathbb{R}} \mu \zeta_{xx} \, dx - \int_{\mathbb{R}} (E + \psi b + u^{r} b)^{2} \zeta_{xx} \, dx
\]

\[
+ \int_{\mathbb{R}} \psi \zeta_{xx} \, dx + \int_{\mathbb{R}} \psi \zeta_{xx} \, dx - \int_{\mathbb{R}} (E + \psi b + u^{r} b)^{2} \zeta_{xx} \, dx
\]

\[ = \sum_{i=1}^{27} J_{i}, \quad (2.53) \]

where \( J_{i} \) (21 \leq i \leq 27) denote the corresponding terms on the right-hand side of (2.53). Applying Sobolev inequality, Young inequality, (2.5) and using Lemma 1.2, one can directly obtain

\[ J_{21} + J_{22} + J_{23} + J_{25} + J_{26} \leq C \eta \|\zeta_{x}\|^{2} + C\|\langle \zeta_{x}, \zeta_{x}\rangle\|^{2} + C\sqrt{u^{2}}(\phi, \psi, \zeta)\|^{2} + C\delta^{\frac{1}{2}}(1 + t)^{-\frac{1}{2}}, \]

\[ J_{24} \leq C \int_{\mathbb{R}} \psi_{x}^{2} \zeta_{xx} \, dx + C \int_{\mathbb{R}} (u_{x}^{2})^{2} \zeta_{xx} \, dx \leq C \|\psi_{x}\|_{L^{\infty}} \|\zeta_{xx}\|^{2} + \eta \|\zeta_{xx}\|^{2} + C\|u_{x}^{2}\|_{L^{4}}^{4} \leq C \|\psi_{x}\|^{2} \|\zeta_{xx}\|^{2} + \eta \|\zeta_{xx}\|^{2} + C\delta(1 + t)^{-3} \leq C \|\psi_{x}\|^{2} \|\zeta_{xx}\|^{2} + \eta \|\zeta_{xx}\|^{2} + C\delta(1 + t)^{-3} \leq C \|\psi_{x}\|^{2} \|\zeta_{xx}\|^{2} + \eta \|\zeta_{xx}\|^{2} + C\delta(1 + t)^{-3} \]

(2.52) is obtained.

\( \square \)
Choosing $\varepsilon$, one can verify that the a priori estimate (2.4) follows from the standard continuation argument based on the local existence and stability assumption

\[ \int_0^\infty \left( \| \phi_x \| + \| \psi_x \| + \| \zeta_x \| + \eta \| \zeta_x \| + C \delta (1 + t)^{-3} \right) dt < \infty, \]

and

\[ J_{27} \leq \| E + \psi b + u^* b \|_{L^\infty} \| E + \psi b + u^* b \| \cdot \| \zeta_x \| \leq \varepsilon_0 \| \zeta_x \|^2 + C \| E + \psi b + u^* b \|^2. \]

Taking the above estimations for $J_i$ $(21 \leq i \leq 27)$ into (2.53) gives

\[
\begin{align*}
\frac{d}{dt} \frac{1}{2} & \int_R \frac{R}{\gamma - 1} \zeta^2 dx + \int_R \frac{\kappa \zeta^2}{\rho} dx \\
& \leq C(\varepsilon_0 + \eta) \| \zeta_x \|^2 + C \| (\psi_x, \psi_{xx}, \zeta_x) \|^2 + C \| \sqrt{u^*_x} (\phi, \psi, \zeta) \| \\
& \quad + C \| E + \psi b + u^* b \|^2 + C \delta^2 (1 + t)^{-\frac{3}{2}}. \\
& \text{(2.54)}
\end{align*}
\]

Choosing $\varepsilon, \eta$ small enough and integrating (2.54) with respect to $t$, together with (2.10), (2.32) and (2.50), we get (2.52). The proof of Lemma 2.5 is completed.

**Proof of Proposition 1**: We combine Lemma 2.1–Lemma 2.5 together, then choose $\delta$ and $\varepsilon_0$ small enough to finish the proof of Proposition 1.

3. **Global existence and large time behavior.** We are now in a position to complete the following.

**Proof of Theorem 1.3.** By the a priori estimate (2.4), there exists a positive constant $C_0$ such that

\[ \| (\phi, \psi, \zeta, E, b) \|^2_{H^1} \leq C_0 \| (\phi_0, \psi_0, \zeta_0, E_0, b_0) \|^2_{H^1} + \delta \bar{\pi}. \]

(3.1)

Notice that $\delta > 0$ is the wave strength independent of $\varepsilon_0$. By letting both $\delta > 0$ and perturbations of initial data be small enough, we can close the a priori assumption (2.3). The global existence of the solution to the Cauchy problem (2.1), (2.2) then follows from the standard continuation argument based on the local existence and the a priori estimate (2.4). Moreover, (3.1) implies (1.31). For the large time behavior in (1.32), one can verify that

\[ \lim_{t \to +\infty} \| (\phi_x, \psi_x, \zeta_x, E_x, b_x)(t) \|^2 = 0. \]

(3.2)

Now, we need to prove (3.2). Notice that, for $0 < \varepsilon < \bar{C}$, estimate (2.4) together with the equations (2.1) implies that

\[ \int_0^\infty \left( \| \phi_x \| + \| \psi_x \| + \| \zeta_x \| + \| E_x \| + \| b_x \| \right) \right| dt < \infty, \]

which as well as (2.4) easily leads to (3.2). Then using the Sobolev inequality (2.6) and (3.2) together with Lemma 1.2 (iii) implies (1.32). This ends the proof of Theorem 1.3.

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E-mail address: mayaohch@mail.scut.edu.cn
E-mail address: byyin@hqu.edu.cn
E-mail address: machjzhu@scut.edu.cn