On Quasilinear Perpendicular Diffusion

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Received / Accepted

Abstract. Quasilinear perpendicular diffusion of charged particles in fluctuating electromagnetic fields is the focus of this paper. A general transport parameter for perpendicular diffusion is presented being valid for an arbitrary turbulence geometry and a plasma wave dispersion relation varying arbitrarily in wavevector. The new diffusion coefficient is evaluated in detail for slab turbulence geometry for two special cases: (1) Alfvénic turbulence and (2) dynamical magnetic turbulence. Furthermore, perpendicular diffusion in 2D geometry is considered for a purely dynamical magnetic turbulence. The derivations and numerical calculations presented here cast serious doubts on the applicability of quasilinear theory for perpendicular diffusion. Furthermore, they emphasize that nonlinear effects play a crucial role in the context of perpendicular diffusion.

Key words. cosmic rays – diffusion – plasmas – turbulence

1. Introduction and Motivation

The influence of a turbulence on the spatial transport of charged particles plays a key role in a variety of heliospheric and astrophysical scenarios. The knowledge of the properties of the underlying turbulence is crucial for understanding the three-dimensional (anisotropic) diffusive particle transport in a collisionless, turbulent and magnetized plasma such as the solar wind or the interstellar medium. Of particular interest are the transport coefficients describing particle diffusion perpendicular to an ambient magnetic field.

In spite of its long-standing importance in space and astrophysics, perpendicular diffusion has been an unsolved puzzle during many decades and a variety of studies have been carried out to achieve closure and to pin it down at a theoretical level. Models have been proposed using hard-sphere scattering in a magnetized plasma (Gleeson 1969) and extended version of it developed on the basis of the Boltzmann equation (Jones 1990), but they are inapplicable to space plasmas where (electro)magnetic fluctuations trigger particle scattering. Other models are based on the field line random walk (FLRW) limit emerging, for slab turbulence geometry, from quasilinear theory (QLT; Jokipii 1966; Jokipii & Parker 1969), which has been considered and used in subsequent studies (e.g., Forman et al. 1974; Forman 1977; Bieber & Matthaeus 1997). Although FLRW provides for a physically appealing picture, it has been shown recently that its applicability is questionable for particle transport in certain turbulence geometries, particularly those with at least one ignorable coordinate being the case for slab geometry (Jokipii et al. 1993). Furthermore, numerical simulations, taking into account the magnetic nature of a turbulence, have shown that FLRW fails to explain perpendicular transport of low-energy particles (Giacalone & Jokipii 1999; Mace et al. 2000), nor is it clear that it actually accounts for the cross-field transport of charged particles.

While perpendicular and parallel particle transport processes are necessarily distinct from one another in QLT, it has been argued for some time that parallel particle scattering can reduce perpendicular diffusion to subdiffusive levels if the turbulence reveals slab geometry only (see, e.g., Urch 1977; Köta & Jokipii 2000; Qin et al. 2002a). However, when the turbulent magnetic field has sufficient structure normal to the mean magnetic field, subdiffusion as seen in pure slab turbulence can be overcome and diffusion is recovered (see Qin et al. 2002a).

Recently, Matthaeus et al. (2003) proposed a promising model for perpendicular diffusion, also referred to as nonlinear guiding center (NLGC) model. The model takes into account different turbulence geometries and parallel diffusion while the particle gyrocenter follows magnetic field lines. Based on their model and its comparison with numerical simulations, it became clear that nonlinear effects play a crucial role for a more realistic understanding of perpendicular diffusion.

Although more advanced models such as the NLGC approach have been developed, it is nevertheless instructive to provide a rigorous treatment of perpendicular diffusion in QLT. Related calculations by Shalchi & Schlickeiser...
In view of this lack, it is desirable to derive a general QLT Fokker-Planck coefficient for an electromagnetic turbulence with arbitrary geometry. This and the detailed evaluation of the new coefficient for the limit of slab and 2D geometry is the purpose of this paper.

The structure of this paper is as follows: Section 2 introduces the governing quasilinear equations of motions for test particles in fluctuating electromagnetic fields, and a general Fokker-Planck coefficient for a plasma wave turbulence with arbitrary geometry and arbitrary wave dispersion relation is derived. The limit of a slab turbulence geometry is considered in detail in section 3. There, perpendicular diffusion coefficients are presented for (1) an Alfvénic turbulence and (2) a purely magnetic turbulence. The limit of 2D turbulence geometry is considered in section 4. Numerical calculations and conclusions for both the slab and 2D geometry are presented in section 5.

2. Quasilinear Derivation of the General Diffusion Coefficient

An often used standard approach for the evaluation of spatial diffusion coefficients is to calculate them from an ensemble of particle trajectories. To do so, the Taylor-Green-Kubo (TGK) formula is often applied. For the diffusion coefficient in x-direction, the TGL formula (Kubo 1957) reads

\[ \kappa_{xx} = \int_{0}^{t} d\xi <v_x(0)v_x(\xi)> \]  

in the limit \(t \to \infty\), where \(v_x\) is the x-component of the particle velocity. The brackets \(\langle \ldots \rangle\) denote an ensemble average over the relevant two-time distribution of particles. An analogous expression holds for the diffusion coefficient in y-direction. For a large coherence time \(\xi\), the second-order velocity correlation function \(\langle v_x(0)v_x(\xi)\rangle\) must go to zero, and the integral in Eq. (1) approaches a constant value for \(t \to \infty\). Following the argumentation done by Kubo (see also Kôta & Jokipii 2000), equation (1) results from

\[ <\Delta x^2> = \left( \int_{0}^{t} d\xi v_x(\xi) \right)^2 = 2 \int_{0}^{t} d\xi (t - \xi) <v_x(0)v_x(\xi)> \]  

where \( <\Delta x^2> \) is the mean square displacement of the particle position in time \(t\). For \(t\) large compared to \(\xi\), Eq. (2) leads to \( <\Delta x^2> = 2\kappa_{xx}t \), with the diffusion coefficient \(\kappa_{xx}\) given by equation (1).

The use of the TGK formula is somewhat critical for situations where particle transport reveals rather an anomalous diffusion process than normal Markovian diffusion. In this case, the standard Fokker-Planck coefficient (anomalous transport law)

\[ \kappa_{xx} = \frac{<\Delta x^2>}{2t} \]  

applies, where the mean square displacement scales more general as \(<\Delta x^2> \propto t^\gamma\). Depending on the exponent \(\gamma\), different diffusion processes can be taken into account: \(\gamma = 1\) for the diffusive regime, corresponding to a Gaussian random walk of particles; \(\gamma = 2\) in the superdiffusive regime, i.e. strictly scatter-free propagation of the particles, and \(\gamma < 1\) (e.g. \(\gamma = 1/2\)) as shown by Qin et al. 2002b and Kota & Jokipii 2000 in the case of particle trapping (subdiffusion or compound diffusion). For \(\gamma = 1\) (normal Markovian diffusion), the anomalous transport law (3) is equivalent to the TGK formula (1) for the large \(t\) limit (see also Bieber & Matthaeus 1997).

In the context of QLT, the quasilinear perpendicular diffusion coefficients \(\kappa_{XX}\) and \(\kappa_{YY}\) can be written as (Schlickeiser 2002 Eqs. [12.3.25] and [12.3.26])

\[ \kappa_{XX} = \frac{1}{2} \int_{-1}^{1} d\mu D_{XX} ; \quad \kappa_{YY} = \frac{1}{2} \int_{-1}^{1} d\mu D_{YY} \]  

where \(\mu = v_\parallel/v\) is the pitch-angle of a particle having the velocity component \(v_\parallel\) along the ordered magnetic field \(B_0\). The subscripts \(X\) and \(Y\) denote guiding center coordinates, and \(D_{XX}\) and \(D_{YY}\) are Fokker-Planck coefficients (cf. Schlickeiser 2002 Eq. [12.1.29]) of the form

\[ D_{XX} = \Re \int_{0}^{\infty} d\xi <\dot{X}(t)\dot{X}(t + \xi)> ; \quad D_{YY} = \Re \int_{0}^{\infty} d\xi <\dot{Y}(t)\dot{Y}(t + \xi)> \]
They represent the interaction of a particle with electromagnetic fluctuations. The relations as given in equation (6) correspond to the TGK formula (4) used earlier by Bieber & Matthaeus (1997) (see their Eq. [2]). In QLT, however, one has to perform an additional average with respect to μ. If the fluctuations are statistically homogeneous in space and time t, the velocity correlation functions in equation (5) depend only on ξ. This can simply be taken into account in the Fokker-Planck coefficients (6) by setting t = 0, and the quasilinear relations (5) then correspond, apart from the μ-integration, to the TGK formula (4).

On the basis of Eq. (5), QLT does not allow to consider subdiffusion or superdiffusion, since the derivations of the relations (5) are based on the assumption, that t is larger than the coherence time ξ (see Schlickeiser 2002, Eq. [12.1.17] and the comments following it). However, it was shown recently by Kóta & Jokipii (2000) that the Kubo formalism does not necessarily contradict compound (sub)diffusion if modifications are applied. The consideration of anomalous particle diffusion (γ ≠ 1), particularly of subdiffusion and an associated modification of the Kubo formalism, is beyond the purview of this paper and normal diffusion is simply assumed by applying the TGK approach.

2.1. Equations of Motion

The determination of $D_{XX}$ and $D_{YY}$ requires the knowledge of the equations of motion. According to Schlickeiser (2002) (see his Eqs. [12.1.9d] and [12.1.9e]), the perpendicular components of the fluctuating force fields can be written as

$$g_X = \dot{X}(t) = -v \cos(\phi(t)) \sqrt{1 - \mu} \delta B_y \frac{c}{B_0} \left( \delta E_y + \frac{v}{c} \delta B_x \right)$$

$$g_Y = \dot{Y}(t) = -v \sin(\phi(t)) \sqrt{1 - \mu} \delta B_x \frac{c}{B_0} \left( \delta E_x - \frac{v}{c} \delta B_y \right)$$

(6)

(7)

where $\phi$ denotes the gyrophase of the particle. Note that the Cartesian components of the fluctuating electromagnetic field, i.e. $\delta B_{x,y,\parallel}$ and $\delta E_{x,y,\parallel}$, are used and not the helical description, i.e. left- and right-hand polarized fields.

For the further treatment of equations (6) and (7), a standard perturbation method is applied. To do so, it is convenient to replace in the Fourier transform of the irregular electromagnetic field the true particle orbit $x(t)$ by an unperturbed particle orbit, yielding

$$\delta B(x, t) = \int d^3k \delta B(k, t) e^{i\mathbf{k} \cdot \mathbf{x}(t) - \omega t} = \sum_{n=\pm \infty} \int d^3k \delta B(k, t) J_n(W) \exp \left[ m[\psi - \phi(t)] + i k_\parallel v_\parallel t \right]$$

(8)

and an analogous expression for $\delta E$. The quantity $J_n(W)$ is a Bessel function of the first kind and order $n$. The particle gyrophase for an unperturbed orbit is given by $\phi(t) = \phi_0 - \Omega t$, where the random variable $\phi_0$ denotes the initial gyrophase of the particle. Furthermore, the abbreviation $W = k_\perp R_L \sqrt{1 - \mu^2}$ is introduced, where $R_L = v/\Omega$ is the Larmor radius. The relativistic gyrofrequency is given by $\Omega = q B_0 / (\gamma m c)$ with $m$ being the mass and $q$ the charge of the particle, $\gamma$ is the Lorentz factor. The angle $\psi$ results from the wavenumber representation $k_x = k_\perp \cos \psi$ and $k_y = k_\perp \sin \psi$. With equation (8), the equations of motion (6) and (7) can be manipulated to become

$$\dot{X}(t) = \frac{v}{B_0} \sum_{n=\pm \infty} \int d^3k \exp \left[ m[\psi - \phi(t)] + i k_\parallel v_\parallel t \right]$$

(9)

$$\times \left\{ -\frac{\sqrt{1 - \mu^2}}{2} \left[ J_{n+1}(W) e^{i\psi} + e^{-i\psi} J_{n-1}(W) \right] \delta B_\parallel \frac{c}{v} J_n(W) \left( \delta E_y + \frac{v}{c} \delta B_x \right) \right\}$$

$$\dot{Y}(t) = \frac{v}{B_0} \sum_{n=\pm \infty} \int d^3k \exp \left[ m[\psi - \phi(t)] + i k_\parallel v_\parallel t \right]$$

(10)

$$\times \left\{ -\frac{\sqrt{1 - \mu^2}}{2} \left[ J_{n+1}(W) e^{i\psi} - e^{-i\psi} J_{n-1}(W) \right] \delta B_\parallel \frac{c}{v} J_n(W) \left( \delta E_x - \frac{v}{c} \delta B_y \right) \right\}$$

For the evaluation of equation (9), it is convenient to consider now the nature of the electromagnetic turbulence. Here, the “wave viewpoint” is used by assuming that the turbulence can be represented by a superposition of $N$ individual plasma wave modes, so that

$$\delta B(k, t) = \sum_{j=1}^{N} \delta B^j(k) \exp(-i\omega_j t) \quad ; \quad \delta E(k, t) = \sum_{j=1}^{N} \delta E^j(k) \exp(-i\omega_j t)$$

(11)
Here, $\omega_j(k) = \omega_{j,R}(k) + i\Gamma_j(k)$ is a complex dispersion relation of wave mode $j$, where $\omega_{j,R}$ is the real frequency of the mode. The imaginary part, $\Gamma_j(k) \leq 0$, represents dissipation of turbulent energy due to plasma wave damping.

Restricting the considerations to transverse fluctuations, i.e. $\delta E^y \cdot k = 0$, and using Faraday’s law, the turbulent electric field can easily be expressed by the corresponding magnetic counterparts, yielding

$$\delta E^y = \frac{\omega_j}{ck^2} \left( \delta B^y_{||} k_{||} - \delta B^y_{\perp} k_{\perp} \right) \quad \text{and} \quad \delta E^y = \frac{\omega_j}{ck^2} \left( \delta B^x_{||} k_{\perp} - \delta B^x_{\perp} k_{||} \right)$$

(12)

Having expressed the electric by the magnetic field, it is also convenient to introduce now the Bessel function identities

$$J_{n-1}(W) + J_{n+1}(W) = \frac{2n}{W} J_n(W) \quad \text{and} \quad J_{n-1}(W) - J_{n+1}(W) = 2J'_n(W)$$

(13)

where the prime denotes the derivation with respect to $W$. With equations (12) and (13), the equations of motion (9) and (10) can readily be rearranged, and one arrives at

$$\dot{X}(t) = -\frac{v}{B_0} \sum_{j} \sum_{n=-\infty}^{\infty} \int d^3k \exp \left[ i(\psi - \phi(t)) + i(k_{||} v_{||} - \omega_{j} t) \right] \times \left\{ J_n(W) \left[ a \frac{k_y}{k_{\perp}} \delta B^y_{||} + b \delta B^y_{\perp} \right] \right. $$

$$- \left. t \sqrt{1 - \mu^2} \frac{k_x}{k_{\perp}} \delta B^x_{||} J'_n(W) \right\}$$

(14)

$$\dot{Y}(t) = -\frac{v}{B_0} \sum_{j=1}^{N} \sum_{n=-\infty}^{\infty} \int d^3k \exp \left[ i(\psi - \phi(t)) + i(k_{||} v_{||} - \omega_{j} t) \right] \times \left\{ J_n(W) \left[ a \frac{k_y}{k_{\perp}} \delta B^y_{||} + b \delta B^y_{\perp} \right] \right. $$

$$+ \left. t \sqrt{1 - \mu^2} \frac{k_x}{k_{\perp}} \delta B^x_{||} J'_n(W) \right\}$$

(15)

where the following complex functions have been introduced:

$$a = \frac{n}{W} \sqrt{1 - \mu^2} - \frac{\omega_j k_{||}}{v k^2} \quad ; \quad b = \frac{\omega_j k_{||}}{v k^2} - \mu$$

(16)

### 2.2. Velocity Correlation Functions

Having determined the equations of motion, one can now proceed to calculate the second-order velocity correlation functions $< \dot{X}(t) \dot{X}^*(t + \xi) >$ and $< \dot{Y}(t) \dot{Y}^*(t + \xi) >$ entering equation (5). The procedure for the calculation is relatively lengthy, but can be carried out with simple algebra. For the sake of generality, the explicit time dependence of the correlation functions is included. At the end of this section, the limit $t = 0$ is considered to obtain $< \dot{X}(0) \dot{X}^*(\xi) >$ and $< \dot{Y}(0) \dot{Y}^*(\xi) >$, required for statistically homogeneous conditions.

The calculations for both velocity correlation functions are analogous, and the calculations are, therefore, restricted to $< \dot{X}(t) \dot{X}^*(t + \xi) >$. Multiplying equation (13) with its conjugated leads to

$$\dot{X}(t) \dot{X}^*(t + \xi) = \frac{v^2}{B_0^2} \sum_{j} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \int d^3k \int d^3k' \exp(\chi)$$

$$\times \left\{ J_n(W) J_m(W') \left[ a \delta B^y_{||} \cdot (\delta B^y_{||} + \delta B^y_{\perp}) + b \delta B^y_{\perp} \cdot (\delta B^y_{||} + \delta B^y_{\perp}) \right] \right. $$

$$+ \left. (1 - \mu^2) \frac{k_x}{k_{\perp}} \left[ \delta B^x_{||} \cdot (\delta B^y_{||} + \delta B^y_{\perp}) + \delta B^x_{\perp} \cdot (\delta B^y_{||} + \delta B^y_{\perp}) \right] \right\}$$

(17)

where the abbreviation

$$\chi = i(n\psi - m\bar{\psi}) - i(n - m)\phi(t) + iv_j(k_{||} - \bar{k}_{||})t + i(\bar{\omega}_j - \omega_j)t - i(\bar{k}_{||} v_{||} - k_{||} \bar{v}_{||} + m\Omega)\xi$$

(18)
The uncorrelated state implies \( \alpha \) and \( \beta \) for Cartesian coordinates, the ensemble averages of the magnetic field fluctuations can also be written as

\[
< \delta B_\alpha \delta B_\beta^* > = \delta (k - k') P_{\alpha \beta}^j (k)
\]

The next step consists of the assumption that Fourier components at different wave vectors are uncorrelated. Introducing the subscripts \( \alpha \) and \( \beta \) for Cartesian coordinates, the ensemble averages of the magnetic field fluctuations can also be written as

\[
< \delta B_\alpha \delta B_\beta^* > = \delta (k - k') P_{\alpha \beta}^j (k)
\]

The uncorrelated state implies \( \psi = \overline{\psi} \), \( W = \overline{W} \) and \( \omega_j = \overline{\omega}_j \), and the velocity correlation function, equation \((20)\), reduces to

\[
< \dot{X}(t) \dot{X}^*(t + \xi) > = \frac{\nu^2}{B_0} \sum_j \sum_{n=-\infty}^{\infty} \int d^3 k \int d^3 \overline{k} \exp(\chi)
\]

where \( \chi = m(\psi - \overline{\psi}) + w_j (k|| - \overline{k||}) t + i(\overline{\omega}_j - \omega_j) t - i(k|| v|| - \overline{k|| v||} + n\Omega) \xi 

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\]

where \( \chi = m(\psi - \overline{\psi}) + w_j (k|| - \overline{k||}) t + i(\overline{\omega}_j - \omega_j) t - i(k|| v|| - \overline{k|| v||} + n\Omega) \xi 

The next step consists of the assumption that Fourier components at different wave vectors are uncorrelated. Introducing the subscripts \( \alpha \) and \( \beta \) for Cartesian coordinates, the ensemble averages of the magnetic field fluctuations can also be written as

\[
< \delta B_\alpha \delta B_\beta^* > = \delta (k - k') P_{\alpha \beta}^j (k)
\]
The advantage of having applied the Bessel function identities to the equations of motion is now obvious. Both velocity correlation functions and, therefore, \( \kappa_{XY} \) and \( \kappa_{YY} \) are expressed as a sum of three terms. Each contribution includes either \( J_2^2(W) \), \( J_n(W)J'_n(W) \) or \( [J'_n(W)]^2 \). Each term is accompanied by a specific factor which contains the components of the magnetic correlation tensor \( P_{\alpha\beta}^j(k) \) and, therefore, the complete information about the turbulence geometry.

Furthermore, note that the velocity correlation functions reveal an explicit dependence on time \( t \). This is a consequence of the dissipation of turbulent energy due to plasma wave damping. To demonstrate this, equation is considered again but, for simplicity, without the concept of a superposition of different wave modes, so that \( \delta \mathbf{B}(k, t) = \delta \mathbf{B}(k) \exp(-i\omega t) \) where \( \omega = \omega_R + i\Gamma \) with the dissipation rate \( \Gamma \leq 0 \). The corresponding turbulent energy can then be expressed as

\[
< \delta \mathbf{B}(k, t)\delta \mathbf{B}^*(k, t + \xi) > = P(k, t, \xi) = < \delta \mathbf{B}(k)\delta \mathbf{B}^*(k) > e^{(i\omega_R + \Gamma)\xi}e^{2\Gamma t} = P(k, \xi)e^{2\Gamma t}
\]  

(25)

yielding, for a dissipative system, the well known relation

\[
\frac{dP(k, t, \xi)}{dt} = 2\Gamma P(k, t, \xi)
\]

(26)

where \( 2\Gamma \) is the rate of energy dissipation due to collisionless plasma wave damping. According to the TGK formula, one has to evaluate the velocity correlation functions for the condition \( t = 0 \), e.g. \( < X(0)X^*(\xi) > \), since the fluctuations are assumed to be homogeneous in space and time, and the variation in \( t \) vanishes. Hence, the calculations presented below do not take into account the contribution \( 2\Gamma, t \). However, such a contribution may be of interest for a more general treatment, whether in QLT or in an extendet version of the NLGC model, which is not considered here.

### 2.3. Fokker-Planck Coefficients

Having determined the velocity correlation functions in the previous section, one can now proceed and evaluate the Fokker-Planck coefficients as given by the relations in equation with \( t = 0 \) in the velocity correlation functions. Upon substituting the expressions and , one obtains

\[
D_{XX} = \frac{v^2}{B_0^2} \sum_j \sum_{n=-\infty}^{\infty} \Re \int d^3k R_j \left[ J_n^2(W)F^j_X + iJ_n(W)J'_n(W)G^j_X + [J'_n(W)]^2 H^j_X \right]
\]

(27)

and an analogous expression for \( D_{YY} \), but including different auxiliary functions \( F^j_Y, G^j_Y \) and \( H^j_Y \). The latter and their corresponding counterparts for \( D_{XX} \) read as follows:

\[
F^j_X = aa^* \frac{k^2_x}{k^2_\perp} P^j_{\parallel\parallel} + bb^* P^j_{xx} + \frac{k_x}{k_\perp} \left[ ab^* P^j_{\parallel x} + ba^* P^j_{x\parallel} \right]
\]

(28)

\[
F^j_Y = aa^* \frac{k^2_y}{k^2_\perp} P^j_{\parallel\parallel} + bb^* P^j_{yy} + \frac{k_y}{k_\perp} \left[ ab^* P^j_{\parallel y} + ba^* P^j_{y\parallel} \right]
\]

(29)

\[
G^j_X = \sqrt{1 - \mu^2} \frac{k_y}{k_\perp} \left[ (a-a^*) \frac{k_x}{k_\perp} P^j_{\parallel\parallel} + bP^j_{xx} - b^* P^j_{x\parallel} \right]
\]

(30)

\[
G^j_Y = -\sqrt{1 - \mu^2} \frac{k_x}{k_\perp} \left[ (a-a^*) \frac{k_y}{k_\perp} P^j_{\parallel\parallel} + bP^j_{yy} - b^* P^j_{y\parallel} \right]
\]

(31)

\[
H^j_X = (1 - \mu^2) \frac{k^2_y}{k^2_\perp} P^j_{\parallel\parallel}
\]

(32)

\[
H^j_Y = (1 - \mu^2) \frac{k^2_x}{k^2_\perp} P^j_{\parallel\parallel}
\]

(33)
The integration with respect to $\xi$ leads to the complex resonance function,

$$R_j = \int_0^\infty d\xi \exp \left[ -i(k_\parallel v_\parallel - \omega_{j,R} + n\Omega)\xi + \Gamma_j\xi \right]$$

$$= \frac{\Gamma_j + i(k_\parallel v_\parallel - \omega_{j,R} + n\Omega)}{\Gamma_j^2 + (k_\parallel v_\parallel - \omega_{j,R} + n\Omega)^2}$$

which describes interactions of the particles with the plasma wave turbulence (remember that $\Gamma_j \leq 0$).

The coefficients $D_{XX}$ and $D_{YY}$, as given by equation (27), can also be summarized to yield a net coefficient, i.e. $D_\perp = D_{XX} + D_{YY}$. According to equation (1), one can then define a net perpendicular diffusion coefficient

$$\kappa_\perp = \kappa_{XX} + \kappa_{YY} = \frac{1}{2} \int_{-1}^{+1} d\mu D_\perp$$

with

$$D_\perp = \frac{v^2}{D_0} \sum_{j} \sum_{n=-\infty}^{\infty} \Re \int d^3k R_j \left[ J_n^2(W)F_{\perp} + iJ_n(W)J_n'(W)G_{\perp}' + [J_n'(W)]^2 H_{\perp} \right]$$

The auxiliary functions $F_{\perp}^j$, $G_{\perp}'^j$ and $H_{\perp}^j$ are then a simple superposition of the corresponding functions given by equations (28) to (32), resulting in

$$F_{\perp} = aa^* P_{\parallel\parallel} + bb^* (P_{xx} + P_{yy}) + \frac{ab^*}{k_\perp} \left[ k_x P_{\parallel x} + k_y P_{\parallel y} \right] + \frac{ba^*}{k_\perp} \left[ k_x P_{\perp x} + k_y P_{\perp y} \right]$$

$$G_{\perp}' = \sqrt{1 - \mu^2} \left[ b \left( k_y P_{\perp x} - k_x P_{\perp y} \right) - b^* \left( k_y P_{\perp x} + k_x P_{\perp y} \right) \right]$$

$$H_{\perp}^j = (1 - \mu^2) P_{\parallel\parallel}^j$$

The coefficient (36), one of the main results of this paper and presented here in this general form for the first time, enables one to calculate the perpendicular diffusion coefficient (35) for an arbitrary turbulence geometry. Moreover, it allows to evaluate $\kappa_\perp$ for a turbulence consisting of transverse wave modes with dispersion relations depending arbitrarily on wavevector.

Whether for numerical or analytical calculations, a further treatment of its auxiliary functions (27) to (32) requires a certain representation for $P_{\alpha\beta}^j$. In view of the structures of equations (34) to (38), it is clear that different representations for $P_{\alpha\beta}^j$ will alter the underlying mathematical and physical structure of $D_\perp$ and, therefore, $\kappa_\perp$.

Here, a representation is chosen commonly used in the literature. Following, e.g., Lerche & Schlickeiser (2001), the nine components of $P_{\alpha\beta}^j$ can be expressed as

$$P_{\alpha\beta}^j(k_\perp, k_\parallel) = A^j(k_\perp, k_\parallel) \left[ \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2} + i \sigma^j (k_\perp, k_\parallel) \epsilon_{\alpha\beta\nu} \frac{k_\nu}{k} \right]$$

where the real quantity $\sigma^j$ denotes the magnetic helicity, $\delta_{\alpha\beta}$ is Kronecker’s delta and $\epsilon_{\alpha\beta\nu}$ is the Levi-Civita tensor, and $A^j$ is the wave power spectrum. With the representation (39) for $P_{\alpha\beta}^j$, one can now proceed with the evaluation of equation (36). For this, the appropriate components of (39) are substituted into equations (37), (38) and (39). Making use of equation (10), one arrives at

$$\frac{F_{\perp}^j}{A^j} = \frac{(a k_\perp - b k_\parallel)^2}{k^2} + b^2 + 2\frac{\omega_j}{v} \left( 1 - \frac{\omega_{j,R}}{\omega_j} \right) \left[ \frac{n}{W} k_\perp \sqrt{1 - \mu^2} + 2 \mu k_\parallel \right]$$

$$\frac{G_{\perp}'^j}{A^j} = -2i \sigma^j \sqrt{1 - \mu^2} k_\perp k \left[ b \frac{\omega_j}{v} - \frac{b}{v} \left( 1 - \frac{\omega_{j,R}}{\omega_j} \right) \right]$$

$$\frac{H_{\perp}^j}{A^j} = (1 - \mu^2) \frac{k_\perp^2}{k^2}$$
where \( a \) and \( b \) are now real functions, i.e.
\[
a = \frac{n}{W} \sqrt{1 - \mu^2} - \frac{|\omega_j| k_\perp}{v k^2} ; \quad b = \frac{|\omega_j| k_\parallel}{v k^2} - \mu
\]
(44)
with \(|\omega_j|^2 = \omega_j \omega_j^*\). The magnetic helicity \( \sigma^j \) enters the coefficient \( D_\perp \) only via \( G^j_\perp \), which reveals a purely imaginary character, whereas \( F^j_\perp \) and \( H^j_\parallel \) are real fields. An evaluation of equation (46) involves, therefore, only the real part of the resonance function \( \mathbb{R} \), i.e.
\[
\Re \mathcal{R}_j = -\frac{\Gamma_j}{\Gamma^2_j + (k_\parallel v_\parallel - \omega_{j,R} + n\Omega)^2}
\]
(45)
which is a positive-definite entry in the net diffusion coefficient \( \kappa_\perp \), since \( \Gamma_j \leq 0 \).

3. Diffusion Coefficients for Slab Geometry

Although is has been found that the solar wind turbulence is often dominated by its 2-D modes and has only a small fraction (say \( \sim 20\% \)) of its energy in the slab contribution (e.g., Matthaeus et al. 1990; Bieber et al. 1994), it is nevertheless instructive and desirable to have a solid treatment of quasilinear perpendicular diffusion in pure slab geometry. For slab geometry, the wave power spectrum can be given by
\[
A^j = g^j(k_\parallel) \frac{\delta(k_\perp)}{k_\perp}
\]
(46)
To examine \( D_\perp \) for the slab geometry limit, the following approximation process is applied to equation (46): the argument \( W = k_\perp v_\perp \sqrt{1 - \mu^2} \) of all Bessel functions is assumed to be much less than unity, since \( k_\perp \rightarrow 0 \). Then, \( k_\perp \) is small compared to \( k_\parallel \), but not equal to zero. This leads to a “quasi”-slab model. Finally, the limit \( k_\perp = 0 \) is considered.

3.1. Fokker-Planck Coefficient for Slab Geometry

To start with the approximation, it is convenient to rewrite in equation (46) the \( n \)-summation. Introducing, for illustrative purposes, an arbitrary function \( I(n) \), one gets
\[
\sum_{n=-\infty}^{\infty} I(n) = I(0) + \sum_{n=1}^{\infty} I(-n) + I(n) = \sum_{r=\pm 1} \sum_{n=0}^{\infty} I(rn) + \sum_{n=0}^{\infty} \sum_{r=\pm 1} I(rn)
\]
(47)
where the last step indicates that the sequence of summation may not be switched. Based on equation (47), the coefficient (46) can be written as
\[
D_\perp = \frac{\nu^2}{B_0^2} \sum_{j=\pm 1} \sum_{r=\pm 1} \sum_{n=0}^{\infty} \int d^3k \mathcal{R}_j(rn)
\]
\[
\times \left[ J^j_n(W) F^j_\perp(rn) + iJ_n(W) J'_n(W) G^j_\perp(rn) + \left[ J'_n(W) \right]^2 H^j_\perp(rn) \right]
\]
(48)
The argument occurring at the resonance function and the auxiliary functions indicates that one has to change in the corresponding equations the quantity \( n \) to \( rn \). The Bessel functions are not affected by the new sum index \( r \). For slab geometry, the asymptotic
\[
J_n(W) \sim \frac{1}{\Gamma(1+n)} \left( \frac{W}{2} \right)^n
\]
(49)
is used, representing the Bessel function for small arguments \( W \rightarrow 0 \), with \( \Gamma(x) \) being the Gamma function. Substitution of expression (19) into the coefficient (48) results in
\[
D_\perp = \frac{\nu^2}{B_0^2} \sum_{j=\pm 1} \sum_{r=\pm 1} \sum_{n=0}^{\infty} \int d^3k \frac{\mathcal{R}_j(rn)}{\Gamma^2(1+n)} \left( \frac{W}{2} \right)^{2n} \left[ F^j_\perp(rn) + iG^j_\perp(rn) \frac{n}{W} + H^j_\perp \frac{n^2}{W^2} \right]
\]
(50)
A further treatment of equation \((50)\) requires a closer inspection of equations \((51), (52)\), and \((53)\). They can be represented as polynomials of different orders in the ratio \(n/W\): the field \(F_{\perp}^1\) is a second-order polynomial in \(n/W\), whereas \(G_{\perp}^1\) and \(H_{\perp}^1\) are polynomials of zeroth-order. The term in the brackets of equation \((50)\) then yields

\[
F_{\perp}^1 (rn) + i G_{\perp}^1 (rn) \frac{n}{W} + H_{\perp}^1 \frac{n^2}{W^2} = A^1 (k_{\perp}, k_{\parallel}) \left[ \alpha_1 \left( \frac{n}{W} \right)^2 + \alpha_2 \left( \frac{n}{W} \right) + \alpha_3 \right]
\]

where the coefficients \(\alpha_1, \alpha_2\) and \(\alpha_3\) read

\[
\alpha_1 = 2(1 - \mu^2) \frac{k_{\perp}^2}{k_{\parallel}^2}
\]

\[
\alpha_2 = \sqrt{2} \alpha_1 \left[ \frac{(k_{\parallel} v_{\perp} - 2 \omega_{j,R} + |\omega_j|) r}{v k} + \sigma^j \left( \frac{\omega_{j,R} k_{\parallel}}{v k^2} - \mu \right) \right]
\]

\[
\alpha_3 = \left( \frac{|\omega_j| k_{\parallel} + \mu}{v k^2} \right)^2 - 4 \mu \omega_{j,R} + \frac{(k_{\parallel} v_{\perp} - \omega_{j,R})^2}{v^2 k_{\parallel}^2}
\]

With equations \((51)\) to \((54)\), the perpendicular diffusion coefficient \((50)\) can be rearranged to become \(D_{\perp} = D_{\perp,1} + D_{\perp,2} + D_{\perp,3}\), where

\[
D_{\perp,1} = \frac{v^2}{4B_0^2} \sum_{j=\pm 1} \sum_{r=\pm 1} \int dk_{\parallel}^3 \alpha_1 A^1 (k_{\perp}, k_{\parallel}) \sum_{n=0}^\infty \frac{n^2 \mathcal{R}_j(rn)}{\Gamma^2(1+n)} \left( \frac{W}{2} \right)^{2n-2}
\]

\[
D_{\perp,2} = \frac{v^2}{2B_0^2} \sum_{j=\pm 1} \sum_{r=\pm 1} \int dk_{\parallel}^3 \alpha_2 A^1 (k_{\perp}, k_{\parallel}) \sum_{n=0}^\infty \frac{n \mathcal{R}_j(rn)}{\Gamma^2(1+n)} \left( \frac{W}{2} \right)^{2n-1}
\]

\[
D_{\perp,3} = \frac{v^2}{B_0^2} \sum_{j=\pm 1} \sum_{r=\pm 1} \int dk_{\parallel}^3 \alpha_3 A^1 (k_{\perp}, k_{\parallel}) \sum_{n=0}^\infty \frac{\mathcal{R}_j(rn)}{\Gamma^2(1+n)} \left( \frac{W}{2} \right)^{2n}
\]

For the limit of pure slab geometry, the \(n\)-summations in the integrals are subjected to a closer inspection. The \(n=0\) contribution vanishes in any case for \(D_{\perp,1}\) and \(D_{\perp,2}\). For \(W \propto k_{\perp} = 0\), it is clear that only \(n=1\) can contribute to the sum appearing in \(D_{\perp,1}\). Concerning \(D_{\perp,2}\), all contributions vanish, and one obtains

\[
\sum_{n=0}^\infty \frac{n^2 \mathcal{R}_j(rn)}{\Gamma^2(1+n)} \left( \frac{W}{2} \right)^{2n-2} = \mathcal{R}_j(r) \quad ; \quad \sum_{n=0}^\infty \frac{n \mathcal{R}_j(rn)}{\Gamma^2(1+n)} \left( \frac{W}{2} \right)^{2n-1} = 0
\]

The evaluation of the sum in equation \((57)\) leads to

\[
\sum_{n=0}^\infty \frac{\mathcal{R}_j(rn)}{\Gamma^2(1+n)} \left( \frac{W}{2} \right)^{2n} = \mathcal{R}_j(n=0)
\]

The second relation in equation \((53)\) implies \(D_{\perp,2}=0\). Consequently, the magnetic helicity \(\sigma^j\) does not affect QLT perpendicular diffusion of particles in a slab plasma wave turbulence. Substitution of the left-handed expression of equation \((53)\) into \((50)\), and making use of \((52)\) and the wave power spectrum \((46)\), yields

\[
D_{\perp,1}^{\text{lab}} = \frac{\pi v^2}{B_0^2} (1 - \mu^2) \sum_{j=\pm 1} \sum_{r=\pm 1} \int dk_{\parallel} \gamma^0 (k_{\parallel}) \mathcal{R}_j(r) \int dk_\perp \frac{\delta(k_{\perp}) k_{\perp}^2}{k_{\parallel}^2 + k_\perp^2} = 0
\]

Apparently, \(D_{\perp,3}\) is the only nonvanishing contribution for slab geometry. Rearranging some terms in \(\alpha_3\), equation \((54)\), one arrives at

\[
D_{\perp,3}^{\text{lab}} = D_{\perp,3} = \frac{2\pi v^2}{B_0^2} \sum_{j=\pm 1} \int dk_{\parallel} \gamma^0 (k_{\parallel}) \mathcal{R}_j(n=0) \times \left[ \frac{(k_{\parallel} v_{\perp} - \omega_{j,R})^2}{v^2 k_{\parallel}^2} + \frac{\Gamma_j^2 (k_{\parallel})}{v^2 k_{\parallel}^2} + 2 \frac{\mu v_{\perp}}{v k_{\parallel}} (|\omega_j| - \omega_{j,R}) \right]
\]

Equation \((61)\) is the general FPC for perpendicular diffusion of particles in a slab turbulence consisting of transverse, damped plasma waves. The wave power spectrum \(\gamma^0\), the real frequency \(\omega_{j,R}\) of the plasma wave mode and the corresponding dissipation rate \(\Gamma_j\) depend arbitrarily on \(k_{\parallel}\). Equation \((61)\) allows to derive a general diffusion coefficient \(\kappa_{\perp}^{\text{lab}}\) not only for the plasma wave viewpoint, but also for the so-called dynamical magnetic turbulence approach. Perpendicular diffusion coefficients for slab geometry and for these two approaches are derived in the next two sections.
3.2. Plasma Wave Turbulence Approach

After having considered the slab limit of the general Fokker-Planck coefficient \( \kappa_{\text{lab}} \) in some detail, the corresponding diffusion coefficient \( \kappa_{\text{lab}} \) is derived in this section. For this, the pitch-angle integration as given by equation (36) is performed and then, if necessary, the integration in wavenumber space. Substitution of equation (61) into (36) results in

\[
\kappa_{\text{lab}}^{\perp} = \frac{1}{2} \int_{-1}^{+1} d\mu D_{\mu}^{\perp} = -\frac{\pi v^2}{B_0^2} \sum_{j=\pm 1} \int dk_\parallel g^j \Gamma_j [I_1 + I_2 + I_3]
\]

where the functions \( I_1, I_2 \) and \( I_3 \) are pitch-angle integrals. Furthermore, the resonance function (45) has been used.

The pitch-angle integrals can be solved analytically, yielding

\[
I_1 = \frac{2}{v^3 k^3} \int_{-B^+}^{+B^-} dx \frac{x^2}{T^2_j + x^2} = \frac{2 \Gamma_j}{v^3 k^3} \left[ \arctan \left( \frac{B_+}{\Gamma_j} \right) + \arctan \left( \frac{B_-}{\Gamma_j} \right) \right]
\]

\[
I_2 = \frac{\Gamma_j^2}{v^3 k^3} \int_{-B^+}^{+B^-} dx \frac{1}{T^2_j + x^2} = \frac{\Gamma_j}{v^3 k^3} \left[ \arctan \left( \frac{B_+}{\Gamma_j} \right) + \arctan \left( \frac{B_-}{\Gamma_j} \right) \right]
\]

\[
I_3 = \frac{2 \mid \omega_j \mid - \omega_{j,R}}{v^3 k^3} \int_{-B^+}^{+B^-} dx \frac{x + \omega_{j,R}}{T^2_j + x^2} = \left( \frac{\mid \omega_j \mid - \omega_{j,R}}{v^3 k^3} \right) \left\{ \ln \left( B^2_+ + \Gamma_j^2 \right) - \ln \left( B^2_+ + \Gamma_j^2 \right) \right\}

+ \frac{\omega_{j,R}}{\Gamma_j} \left[ \arctan \left( \frac{B_+}{\Gamma_j} \right) + \arctan \left( \frac{B_-}{\Gamma_j} \right) \right]
\]

where the integration boundaries are given by \( B_\pm = k_\parallel v \pm \omega_{j,R} \). Concerning the term \( \mid \omega_j \mid - \omega_{j,R} = - \omega_{j,R} \left( 1 + \Gamma_j^2 / \omega_{j,R}^2 \right)^{1/2} \) in equation (63), it is obvious that \( I_3 \) becomes negligible for the condition \( \Gamma_j \ll \omega_{j,R} \).

This is an often used assumption within quasilinear theory to derive the dissipation rates of plasma wave modes, and the approximation \( I_3 \approx 0 \) is considered as reasonable for further progress.

With equations (63) and (64), the quasilinear perpendicular diffusion coefficient for a plasma wave turbulence with slab geometry reads

\[
\kappa_{\text{lab}}^{\perp} = \frac{2 \pi}{B_0^2} \sum_{j=\pm 1} \int dk_\parallel g^j \kappa_{\parallel}^{-2} \Gamma_j \left\{ 4 - \frac{\Gamma_j}{v k_\parallel} \left[ \arctan \left( \frac{B_+}{\Gamma_j} \right) + \arctan \left( \frac{B_-}{\Gamma_j} \right) \right] \right\}
\]

where it was assumed that \( g^j (k_\parallel) = g^j (\mid k_\parallel \mid), \Gamma_j (k_\parallel) = \Gamma_j (\mid k_\parallel \mid) \) and \( \omega_{j,R} (k_\parallel) = \omega_{j,R} (\mid k_\parallel \mid) \). Furthermore, the fact was used that \( \arctan (x) \) is an odd function in \( x \). As a consequence of this, the dissipation rate \( \Gamma_j \) is now given as a positive quantity, meaning that \( \Gamma_j \geq 0 \) in equation (66).

Further evaluation of the wavenumber integral in equation (66) requires specific wavenumber variations of \( g^j, \omega_{j,R} \) and \( \Gamma_j \). Especially the dissipation rate \( \Gamma_j \) can usually be given as a quite complicated function not only in \( k_\parallel \), but also in the so-called plasma \( \beta_{p,e} = 8 \pi n_{p,e} e B T_{p,e} / B_0^2 \). In the latter, the subscripts refer to protons and electrons as plasma constituents having the corresponding temperatures \( T_{p,e} \) and the number densities \( n_{p,e} \). The temperatures of the plasma electrons and protons can differ quite significantly, giving rise to instabilities (e.g., Gary 1993). It is well known that such a plasma configuration with different temperatures of the plasma components can then result in a variety of wave modes with substantially different \( \omega_{j,R} \) and dissipation rates \( \Gamma_j \).

For instance, consider the Alfvén wave mode with real frequency \( \omega_{j,R} = j \nu A k_\parallel \). A typical representation for the damping rate of this wave mode can be derived by using a quasilinear Vlasov theory code (e.g., Gary 1993). Doing this, one obtains

\[
\Gamma_{j,\parallel} (k_\parallel) = 0.60 \beta_p^{0.36} \Omega_P \left( \frac{k_\parallel c}{\omega_P} \right)^{1.54} \exp \left( -\frac{0.32 \omega_P^2}{\beta_p^{0.65} k_\parallel^2 c^2} \right)
\]

where \( \omega_P \) and \( \Omega_P \) are the plasma frequency and the plasma proton gyrofrequency, respectively (Stawicki et al. 2001). In view of (67), it is clear that the wavenumber integration in equation (66) has to be solved numerically. This is beyond
the scope of this paper, and, for illustrative purposes, a simplified version of the damping rate is demanded. For this, the exponential expression in Eq. (67) is roughly neglected. Furthermore, it is assumed that \( \beta_R \) is such that the exponent of the power law is equal to unity. The two crude simplifications lead to

\[
\Gamma_j(k_{||}) = \Gamma_{j,0} v_A k_{||}
\]  (68)

where \( \Gamma_{j,0} \) is a constant. Substitution into (66) leads to

\[
\kappa_{\perp}^{slab} = \frac{4\pi v_{\eta}}{B_0^2} \left( 4 - \eta \left[ \arctan \left( \frac{1 + v_A / v}{\eta} \right) + \arctan \left( \frac{1 - v_A / v}{\eta} \right) \right] \right) \int_0^\infty dk_{\parallel} \frac{g(k_{||})}{k_{||}}
\]  (69)

where it was assumed that \( \Gamma_{\perp,0} = \Gamma_{\parallel,0} = \Gamma_0 \) and \( g^+ = g^- = g \). Furthermore, the quantity \( \eta = \Gamma_0 v_A / v \) has been introduced. Apparently, the expression in the curly brackets of (66) can be shifted in front of the wavenumber integral. It has to be stressed that this is a consequence of the crude approximations made to simplify the dissipation rate (67): \( \Gamma_j \) is linear in \( k_{||} \) and, therefore, wipes out the wavenumber dependence of \( \omega_{j,R} \).

Taking into account that \( v_A / v \ll 1 \) and neglecting the corresponding terms in equation (69), one obtains

\[
\kappa_{\perp}^{slab} = \frac{8\pi v_{\eta}}{B_0^2} \left( 2 - \eta \arctan (\eta^{-1}) \right) \int_0^\infty dk_{||} \frac{g(k_{||})}{k_{||}}
\]  (70)

A closer inspection of equation (70), and its general version, equation (66), results in the following important observation: \( \kappa_{\perp}^{slab} \) vanishes for a dissipationless plasma wave turbulence, i.e. \( \Gamma_j = 0 \). Consequently, for a static turbulence, charged particles can not move in perpendicular direction and are trapped to a line of force of the background magnetic field.

### 3.2.1. Dissipationless Alfvénic Turbulence

\[
\mathcal{R}_j(n = 0) = \pi \delta (k_{||} v_{\mu} - j v_A k_{||})
\]  (71)

where the Alfvén dispersion relations \( \omega_{j,R} = j v_A k_{||} \) was used. For \( \Gamma_j = 0 \), the second and third term in the brackets of equation (66) vanish in any case. The first term remains as the only one for the following two derivations.

Derivation I: In order to obtain \( \kappa_{\perp}^{slab} \), the wavenumber integration is first performed, and then the \( \mu \)-integration. Making use of equation (71), one obtains for the FPC (61) the expression

\[
D_{\perp}^{slab} = \frac{4\pi^2}{B_0^2} \sum_{j = \pm 1} \int dk_{||} g^j(k_{||}) \delta(k_{||}) |v_{\mu} - j v_A| = \frac{4\pi^2}{B_0^2} \sum_{j = \pm 1} g^j(0) |v_{\mu} - j v_A|
\]  (72)

The argument of the wave power spectrum \( g^j \) indicates that it has to be evaluated for \( k_{||} = 0 \). Equation (72) then yields

\[
\kappa_{\perp}^{slab} = \frac{1}{2} \int_{-1}^{+1} d\mu D_{\perp}^{slab} = 2\pi^2 v \left[ \frac{g^+(0) + g^-(0)}{B_0^2} \right] \int_0^1 d\mu \left[ |\mu - j v_A / v| + |\mu + j v_A / v| \right]
\]  (73)

The integrand can be approximated by \( 2 v_{\mu} / v \) and \( 2 \mu \) for the intervals \( 0 \leq \mu \leq v_{\mu} / v \) and \( v_{\mu} / v \leq \mu \leq 1 \), respectively, and one arrives at

\[
\kappa_{\perp}^{slab} = 2\pi^2 v \left[ \frac{g^+(0) + g^-(0)}{B_0^2} \right] \left( 1 + \frac{v_{\mu}^2}{v^2} \right) \approx \frac{\pi v L}{2} \left( \frac{\delta B(0)}{B_0} \right)^2
\]  (74)

In the last step of equation (74) it was made use of \( g^+(0) + g^-(0) = \delta B^2(0) L / 4\pi \), with \( L \) being a scale length providing for the right normalization. The neglect of \( v_{\mu}^2 / v^2 \ll 1 \) corresponds to a vanishing irregular electric field and, therefore, results in a purely magnetic turbulence. Equation (74) agrees with earlier quasilinear results interpreted as non-resonant field line random walk in a magnetostatic turbulence (Jokipii 1966; Forman 1974).

Derivation II: The second derivation is based on the fact, that one can easily switch the order of integration: first, the \( \mu \)-integration is performed, and then, if necessary, the integration over \( k_{||} \). Substitution of the resonance function (71)
into Eq. (61) results in

\[ D_{\perp}^{\text{lab}} = \frac{4\pi^2}{B_0^2} \sum_{j=\pm 1} \int_{-\infty}^{\infty} dk_\parallel g(j)(k_\parallel)k_\parallel^{-2}\delta(k_\parallel v\mu - jv_A k_\parallel)(k_\parallel v\mu - jv_A k_\parallel)^2 \]

\[ = \frac{8\pi^2 v}{B_0^2} \sum_{j=\pm 1} \int_{0}^{\infty} dk_\parallel g(j)(k_\parallel)k_\parallel^{-1}\delta(\mu - jv_A/v)(\mu - jv_A/v)^2 \] (75)

Applying now the pitch-angle integration, according to Eq. (35), one obtains

\[ \kappa_{\perp}^{\text{lab}} = \frac{4\pi^2 v}{B_0^2} \sum_{j=\pm 1} \int_{0}^{\infty} dk_\parallel g(j)(k_\parallel)k_\parallel^{-1} \int_{-1}^{+1} d\mu \delta(\mu - jv_A/v)(\mu - jv_A/v)^2 = 0 \] (76)

Therefore, the perpendicular diffusion coefficient vanishes for a dissipationless turbulence. As expected, the result then agrees with the general diffusion coefficient (66).

### 3.3. Dynamical Magnetic Turbulence Approach

In this section, derivations of quasilinear perpendicular diffusion coefficients for the so-called dynamical magnetic turbulence (DMT) model, introduced by Bieber et al. (1994), are presented. Within the context of DMT, fluctuations are assumed to be purely magnetic. To take into account the dynamical behavior of such purely magnetic fluctuations, Bieber et al. (1994) defined two models: the damping as well as the random sweeping model. It is relatively straightforward to derive \( \kappa_{\perp}^{\text{lab}} \) on the basis of the more general FPC (31) for these two models, and both are considered in turn. The limit of vanishing turbulent electric fields can easily be achieved by setting \( \omega_{j,R} = 0 \) and \( \Gamma_j = 0 \) in Eq. (61), initially derived for the plasma wave viewpoint. The only modification concerns the real part of the resonance function (34). Furthermore, since the concept of a superposition of individual wave modes does not apply anymore, the \( j \)-nomenclature is dropped.

#### 3.3.1. Damping Model

For the damping model, Bieber et al. (1994) suggested a dynamical behavior of the turbulent energy being of the form

\[ \langle \delta B(k,t)\delta B^*(k,t + \xi) \rangle = P(k) \exp(-\nu_c \xi) \] (77)

and the resonance function (34) has to be modified in this respect. What has to be done is to set \( \Gamma_j = 0 \) in the upper part of Eq. (34), and then multiply the exponential expression with \( \exp(-\nu_c \xi) \). The contributions resulting from the unperturbed particle orbit still hold. Having in mind that one has to take the real part of equation (34) (see Eq. (44) and the comments following it), the resonance function for the damping model reads

\[ \Re = \Re \int_{0}^{\infty} d\xi \exp[-i(k_\parallel v_\parallel + n\Omega)\xi - \nu_c \xi] = \frac{\nu_c}{\nu_c^2 + (k_\parallel v_\parallel + n\Omega)^2} \] (78)

Following Bieber et al. (1994), the rate for turbulent decorrelation, \( \nu_c \), is assumed to be

\[ \nu_c = \alpha_m v_A k_\parallel \] (79)

where the parameter \( 0 \leq \alpha_m \leq 1 \) allows adjustment of the strength of the dynamical effects. The case \( \alpha_m = 0 \) then represents the magnetostatic limit, \( \alpha_m = 1 \) describes a strongly dynamical magnetic turbulence. With the decorrelation rate (79) and the resonance function (78), one readily obtains for Eq. (61) the expression

\[ D_{\perp}^{\text{lab}} = \frac{4\pi v^2}{B_0^2} \int_{-\infty}^{\infty} dk_\parallel g(k_\parallel)\mu^2 \Re(n = 0) = \frac{8\pi v\zeta}{B_0^2} \frac{\mu^2}{\zeta^2 + \mu^2\zeta^2} \int_{0}^{\infty} dk_\parallel g(k_\parallel/k_\parallel) \] (80)

where \( \zeta = \alpha_m v_A/v \). Furthermore, \( g(k_\parallel) = g(|k_\parallel|) \). With equation (34), one obtains

\[ \kappa_{\perp}^{\text{lab}} = \frac{1}{2} \int_{-1}^{+1} d\mu D_{\perp}^{\text{lab}} = \frac{8\pi v\zeta}{B_0^2} \int_{0}^{\infty} dk_\parallel g(k_\parallel/k_\parallel) \int_{0}^{1} d\mu \frac{\mu^2}{1 + \mu^2\zeta^2} \] (81)
where the $\mu$-integration can be carried out analytically, yielding

$$\kappa_{\perp}^{slab} = \frac{8\pi v_c}{B_0^2} \left[ 1 - \zeta \arctan \left( \zeta^{-1} \right) \right] \int_0^\infty dk_\| \frac{g(k_\|)}{k_\|} \right]$$

(82)

Obviously, for the limit $\alpha_m = 0$ (magnetostatic limit), $\kappa_{\perp}^{slab}$ vanishes, implying that the particle remains tied to the background magnetic field. Shalchi & Schlickeiser (2004) came to the same conclusion.

### 3.3.2. Random Sweeping Model

For the random sweeping model, Bieber et al. (1994) used a Gaussian dependence for the turbulence decay, i.e.

$$< \delta B(\mathbf{k}, t) \delta B^*(\mathbf{k}, t + \xi) > = P(\mathbf{k}) \exp(-\nu_c^2 \xi^2)$$

(83)

This changes the governing resonance function (84) significantly, and one obtains

$$\mathcal{R} = \int_0^\infty d\xi \exp \left[ -i(k_\| v_\| + n\Omega)\xi - \nu_c^2 \xi^2 \right] = \frac{\sqrt{\pi}}{2\nu_c} \exp \left( -\frac{(k_\| v_\| + n\Omega\alpha)^2}{4\nu_c^2} \right)$$

(84)

Making use of equation (81) and setting $\omega_j = \Gamma_j = 0$, one arrives at

$$\kappa_{\perp}^{slab} = \int_{-1}^{+1} d\mu D_{\perp}^{slab} = \frac{4\pi^3/2}{B_0^2} \int_0^\infty dk_\| \frac{g(k_\|)}{\nu_c(k_\|)} \int_0^1 d\mu^2 \exp \left( -\frac{k_\|^2 v_c^2 \mu^2}{4\nu_c^2(k_\|)} \right)$$

(85)

The pitch-angle integral can be solved again analytically, resulting in

$$\kappa_{\perp}^{slab} = \frac{8\pi^3/2}{B_0^2 v} \int_0^\infty dk_\| \frac{g(k_\|)}{k_\|^2 \nu_c^2(k_\|)} \left[ \sqrt{\pi} \text{Erf} \left( \frac{k_\| v}{2\nu_c(k_\|)} \right) - \frac{k_\| v}{\nu_c(k_\|)} \exp \left( -\frac{k_\|^2 v^2}{4\nu_c^2(k_\|)} \right) \right]$$

(86)

where Erf$(x)$ is the Error function. So far, equation (86) is the general representation of the perpendicular diffusion coefficient in a slab turbulence obeying the random sweeping model. Upon using for the decorrelation rate the same dependence on $k_\|$ as for the damping model, i.e. $\nu_c = \alpha m v_A k_\|$, one obtains

$$\kappa_{\perp}^{slab} = \frac{8\pi^3/2 v_c^2}{B_0^2} \int_0^\infty dk_\| \frac{g(k_\|)}{k_\|^2 \nu_c^2(k_\|)} \left[ \sqrt{\pi} \text{Erf} \left( \frac{\zeta^2}{4} \right) - \frac{\zeta^2}{2} \exp \left( -\frac{\zeta^2}{4} \right) \right]$$

(87)

Again, $\kappa_{\perp}^{slab}$ vanishes for the magnetostatic limit $\alpha_m \to 0$.

### 4. Diffusion Coefficient for 2D Geometry

The previous section offered detailed insight into the slab limit of the general FPC (80) and its associated transport parameter (85). Perpendicular diffusion coefficients are presented for the slab limit for both the plasma wave viewpoint and the dynamical magnetic turbulence approach. In this section, the evaluation of the perpendicular diffusion coefficient for the 2D contribution of the turbulence is presented. For simplicity, the calculations are restricted to the damping model of the DMT description (see also Section 3.3), and the more general case of a plasma wave turbulence is left as an exercise to the interested reader. The calculations presented in this section generalize the derivation presented by Shalchi & Schlickeiser (2004), since their approach is restricted to a simplified magnetic power spectrum and turbulence decorrelation rate.

For 2D turbulence, wavevectors are perpendicular to the mean magnetic field, and the wave power spectrum can be given by

$$A^j(k_\perp, k_\|) = g(k_\|) \frac{\delta(k_\|)}{k_\|}$$

(88)

Upon substituting the expression (88) into equation (86) and making use of the Bessel function identities (10), the FPC for 2D geometry can be cast into the form

$$D_{\perp}^{2D} = \frac{2\pi v_c^2}{B_0^2} \sum_{n=-\infty}^{\infty} \int_0^\infty dk_\| \frac{g(k_\|)}{\nu_c^2(k_\|)} \left[ \frac{1 - \mu^2}{2} \left( J_{n+1}^2(W) + J_{n-1}^2(W) \right) + \mu^2 J_n^2(W) \right]$$

(89)
Fig. 1. Numerical solutions of Eq. (91) representing the behavior of $I(\xi, z)$ for three different values of $z$

where $\sigma = 0$ was assumed. The corresponding resonance function has already been inserted with $\nu_c(k_\perp)$ being the decorrelation rate in normal direction (cf. Eq. (78)). According to equation (35), one has to perform the $\mu$-integration to obtain the corresponding diffusion coefficient $\kappa_\perp^{2D}$. The integration can still be carried out analytically, and the detailed calculations are presented in Appendix A. There, it is shown that the diffusion coefficient for 2D geometry reads

$$\kappa_\perp^{2D} = \frac{\pi v R_L}{B_0^2} \int_0^\infty dk_\perp g(k_\perp) I(\xi, z)$$

(90)

with

$$I(\xi, z) = \int_0^{\pi/2} d\theta \frac{\cosh(2\xi \theta)}{\xi^3 \sinh(z \pi)} \cos^2(2\xi \cos \theta) \sin(2\xi \cos \theta) - 2\xi \cos \theta \cos(2\xi \cos \theta)$$

(91)

where the abbreviations $\xi = k_\perp R_L$ and $z = \nu_c/\Omega$ are introduced. Furthermore, $R_L = v/\Omega$ is the Larmor radius. Equation (90) is valid for a power spectrum and a decorrelation rate varying arbitrarily in wavenumber $k_\perp$. The integral representation (91) results from the $\mu$-integration and has to be evaluated for further progress. Unfortunately, an analytical solution for this integral does not exist, and any progress requires numerical treatment. In order to obtain some insight into the behavior of the function $I(\xi, z)$, Figure 1 shows numerical solutions of Eq. (91) as function of $\xi$ for three different values of $z$. For illustrative purposes, $z$ is assumed to be constant in $k_\perp$.

The limit $\xi \ll 1$, implying $R_L \ll k_\perp^{-1}$, leads to an instructive, analytical solution for equation (91). To show this, small argument approximations for the circular functions are used, i.e.

$$\cos(2\xi \cos \theta) \approx 1; \quad \sin(2\xi \cos \theta) \approx 2\xi \cos \theta$$

(92)

and inserted into $I(\xi, z)$. Partial integration then results in

$$I(\xi \ll 1, z) = \frac{z}{1 + z^2}$$

(93)

Consequently, one obtains

$$\kappa_\perp^{2D} = \frac{2\pi v R_L}{B_0^2} \int_0^\infty dk_\perp g(k_\perp) \frac{(\tau_c \Omega)}{1 + (\tau_c \Omega)^2}$$

(94)

for the limit $R_L \ll k_\perp^{-1}$, where the relation $\nu_c = \tau_c^{-1}$ has been used. An eyecatching feature of equation (94) is the term including the dimensionless product $\tau_c \Omega$. It is formally the same as those derived by Forman et al. (1974) for field line random walk in a slab geometry and, more recently, by Bieber & Matthaeus (1997).

For the magnetostatic limit, $\nu_c \to 0$ or $z = 0$, it can be shown easily that $\kappa_\perp^{2D} \to \infty$, by using small argument approximations for the hyperbolic functions. However, for slab geometry, the diffusion coefficients approach zero for $\nu_c \to 0$ (see section 3).
B is the 2D contribution (80%), which is believed to be consistent with solar wind observations (Bieber et al. 1994). The rate \( \nu_{\delta B} \) turbulent energy, for \( \kappa \) the bend-over scales, denoted by \( \lambda \) turbulent magnetic field. For static conditions, the NLGC theory provides for a nonvanishing, \( QLT \) can not explain their simulations, in contrast to the NLGC theory, where the transverse complexity of the finite diffusion coefficient (see Matthaeus et al. 2003, Eq. [7] with \( \gamma \)).

\[ \nu_{\delta B} = \frac{1}{\pi} \frac{1}{\kappa} \delta B^2 \frac{\delta B}{B} \]

For the numerical computation, the power spectra

\[
g(k_{||}) = C(q)\lambda_{slab} \delta B^2_{slab}(1 + k_{||}^2 \lambda_{slab}^2)^{-q} \tag{95}
\]

and

\[
g(k_{\perp}) = C(q)\lambda_{2D} \delta B^2_{2D}(1 + k_{\perp}^2 \lambda_{2D}^2)^{-q} \tag{96}
\]

are used for the slab and 2D contribution, respectively. Here, \( C(q) = (2\sqrt{\pi})^{-1} \Gamma(q)/\Gamma(q - 1/2) \), with \( q = 5/6 \) being the spectral index. For simplicity, the latter is assumed to be equal for slab and 2D geometry. The remaining parameter in equations (95) and (96) are the energy densities in slab and 2D fluctuations, \( \delta B^2_{slab} \) and \( \delta B^2_{2D} \), respectively, and the bend-over scales, denoted by \( \lambda_{slab} \) and \( \lambda_{2D} \). They are proportional to the respective turbulence correlation lengths \( l_{slab} \) and \( l_{2D} \).

Concerning the correlation lengths, is assumed that \( l_{2D} = l_{slab}/10 = 10^9 \) m. Furthermore, it is used that the net turbulent energy, \( \delta B^2 = \delta B^2_{slab} + \delta B^2_{2D} \), has only a small fraction in slab turbulent energy (20%) and is dominated by the 2D contribution (80%), which is believed to be consistent with solar wind observations (Bieber et al. 1994). The background magnetic field \( B_0 \) is given by \( 4 \cdot 10^{-5} \) G, and the ratio \( \delta B/B_0 \) is chosen to be 0.2. For the decorrelation rate \( \nu_c \), expression (79) is employed (with \( \alpha_m = 1 \) and \( v_A = 50 \) km s\(^{-1} \)) for both the slab and the 2D contribution. However, in general, different parallel and perpendicular decorrelation processes might govern particle diffusion.

Numerical results are shown in Figure 2. The dotted line gives the computations for \( \kappa_{slab} \), equation (82). Solutions for \( \kappa^2 \), Eq. (82), are visualized by the dashed and solid curves for electrons (\( e^- \)) and protons (\( p \)), respectively. At a glance, for the parameter used here, quasilinear perpendicular diffusion is governed by the transverse structure of the turbulent magnetic field.

However, in view of recent numerical results by Giacalone & Jokipii (1999) (see their Fig. 7 for composite turbulence) and Matthaeus et al. (2003) for a fully three-dimensional and static turbulence, one becomes aware of the fact that QLT can not explain their simulations, in contrast to the NLGC theory, where the transverse complexity of the turbulent magnetic field plays a crucial role. For static conditions, the NLGC theory provides for a nonvanishing, finite diffusion coefficient (see Matthaeus et al. 2003, Eq. [7] with \( \gamma(k) = 0 \)). This is not the case in QLT, as it is shown in section 4 and 5 for slab and 2D turbulence geometry, respectively, since \( \kappa_{slab} \) vanishes and \( \kappa_{2D} \to \infty \) for the magnetostatic case. This implies for a fully three-dimensional turbulence \( \kappa_{\perp} = \kappa_{slab} + \kappa_{2D} \to \infty \). QLT can, therefore, not provide for an adequate description for perpendicular diffusion, at least for static turbulence. As shown above, QLT leads to a nonvanishing, finite diffusion coefficient only if the turbulence is non-static. In QLT, parallel and perpendicular diffusion processes are considered as being independent, and the success of the NLGC model is based on the nonlinear coupling of these two processes. It becomes clear that such nonlinear effects are crucial for a more realistic understanding of spatial particle diffusion and other processes such as particle drift.

Fig. 2. Numerical solutions of equation (82) and (90) representing the diffusion coefficients \( \kappa^2 \), as functions of the particle kinetic energy. The calculations were performed for protons (\( p \)) and electrons (\( e^- \)).
Another issue concerns the transport of charged particles in turbulent fields having at least one ignorable coordinate. It was shown by Jokipii et al. (1993) and Jones et al. (1998) that perpendicular particle diffusion is then suppressed and that a particle is tied to the background magnetic field. This result implies that it is not possible to get cross-field diffusion for neither slab nor 2D turbulence geometry. The numerical simulations by Qin et al. (2002) were performed for a “quasi”-slab, magnetostatic turbulence, leading to the insight that perpendicular diffusion is indeed suppressed for such a simplified geometry. The particle transport is then subdiffusive and not a Markovian diffusion process. This can not be taken into account by neither the NLGC model nor the QLT calculations presented here, since both are based on the (Markovian) diffusion approximation. However, though normal diffusion is explicitly assumed by using the TGK formula (8), the calculations in section 3 show that the QLT perpendicular diffusion coefficient approaches zero, based on the (Markovian) diffusion approximation. However, through normal diffusion is explicitly assumed by using the TGK formula (8), the calculations in section 3 show that the QLT perpendicular diffusion coefficient approaches zero, both for the plasma wave and the dynamical magnetic turbulence viewpoint, if slab fluctuations are static. Giacalone & Jokipii (1994) have shown that cross-field diffusion is also suppressed for a static, two-dimensional turbulence. This is in agreement with Jokipii et al. (1993) and Jones et al. (1998). In stark contrast to this is the QLT result for a two-dimensional turbulence presented in section 4. There, it is shown that \( \kappa_{\perp,2D} \rightarrow \infty \) for static conditions.

The calculations presented here cast serious doubts on the applicability of QLT for turbulence topologies different from slab geometry. This implies that QLT is inapplicable for obtaining conclusions on the nature of interplanetary magnetic fluctuations, being in contradiction to the statement made by Shalchi & Schlickeiser (2004).

Besides being valid only for weak turbulence (\( \delta B/B_0 \ll 1 \)) and unperturbed particles orbits, the list of limitations of QLT can be extended by the result that it fails for non-slab geometries, at least for a static turbulence. Concerning the dynamical behavior of the turbulence, Appendix A describes briefly a modification of the QLT results given above, yielding a nonvanishing and finite diffusion coefficient for an “intrinsic” static turbulence. The crude approach used there might probably be of more academic interest, since it is based on the speculation that particles alter the dynamical behavior of their scattering agent governing their diffusive transport. However, it emphasizes the importance of nonlinear contributions for perpendicular diffusion.

**Acknowledgements.** The author thanks R. A. Burger, J. Minnie and M. S. Potgieter for valuable conversations. Partial support by the South African National Research Foundation (NRF) and the Deutsche Forschungsgemeinschaft (DFG), SFB 591, is acknowledged.

**Appendix A: Derivation of \( \kappa_{\perp,2D}^2 \)**

To derive the quasilinear diffusion coefficient \( \kappa_{\perp,2D}^2 \), equation (90), for 2D turbulence geometry, one can rearrange some terms in the corresponding Fokker-Planck coefficient (89), yielding

\[
D_{\perp,2D} = \frac{2\pi R_i^2}{B_0^2} \int_0^\infty dk_\perp g(k_\perp) \nu_c \left[ \frac{1 - \mu^2}{2} \sum_{n=-\infty}^\infty J_{n+1}^2(W) + \frac{J_{n-1}^2(W)}{z^2 + n^2} + \mu^2 \sum_{n=-\infty}^\infty \frac{J_n^2(W)}{z^2 + n^2} \right] (A.1)
\]

where \( z = \nu_c/\Omega \). This corresponds to Eq. (54) of Shalchi & Schlickeiser (2004), but note that equation (A.1) is derived from the general coefficient (89). The two terms including the \( n \)-sum can be subjected to a further treatment. Following Shalchi & Schlickeiser (2004), the formulas

\[
J_{n-1}^2(W) = \frac{2}{\pi} (-1)^{n-1} \int_0^{\pi/2} d\theta J_0(2W \cos \theta) \cos(2\theta[n - 1]) \quad (A.2)
\]

and

\[
J_{n+1}^2(W) = \frac{2}{\pi} (-1)^{n+1} \int_0^{\pi/2} d\theta J_0(2W \cos \theta) \cos(2\theta[n + 1]) \quad (A.3)
\]

can be used (see Gradshteyn & Ryzhik, 1966), so that

\[
J_{n+1}^2(W) + J_{n-1}^2(W) = \frac{4}{\pi} (-1)^{(n+1)} \int_0^{\pi/2} d\theta J_0(2W \cos \theta) \cos(2\theta n) \cos(2\theta) \quad (A.4)
\]

The first term then reads

\[
\frac{1}{2} \sum_{n=-\infty}^\infty \frac{J_{n+1}^2(W) + J_{n-1}^2(W)}{z^2 + n^2} = \frac{2}{z \sinh(\pi z)} \int_0^{\pi/2} d\theta J_0(2W \cos \theta) \cosh(2\theta z) \quad (A.5)
\]
where the following relation (Gradshteyn & Ryzhik 1966) is used:

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{z^2 + n^2} \cos(2\theta n) = \frac{\pi}{2z} \mathrm{cosh}(2\theta z) - \frac{1}{2z^2} \tag{A.6}
\]

The second term can be treated in a similar manner, resulting in

\[
\sum_{n=-\infty}^{\infty} \frac{J_n^2(W)}{z^2 + n^2} = \frac{2}{z \sinh(\pi z)} \int_{0}^{\pi/2} d\theta J_0(2W \cos \theta) \cosh(2\theta z) \tag{A.7}
\]

In order to solve the remaining \(\theta\)-integrations approximately, Shalchi & Schlickeiser (2004) used simplified expressions for the power spectrum and the turbulence correlation timescale. This is not done here. Instead, the integration with respect to \(\mu\) is first performed. To do so, equations (A.5) and (A.7) are first inserted into (A.1). Making use of equation (A.6), one obtains

\[
\kappa_{\perp}^{2D} = \frac{4\pi v R_L}{B_0^2} \int_{0}^{\infty} dk_\perp \frac{g(k_\perp)}{\sinh(\pi z)} \int_{0}^{\pi/2} d\theta \cosh(2\theta z) I_{\mu}(\theta, \zeta) \tag{A.8}
\]

with

\[
I_{\mu}(\theta, \zeta) = \frac{1}{\mu} \int_{0}^{\mu} d\mu J_0(2\zeta \cos \theta \sqrt{1 - \mu^2})[1 - 2(1 - \mu^2) \cos^2 \theta] \tag{A.9}
\]

where \(W = k_\perp R_L \sqrt{1 - \mu^2} = \zeta \sqrt{1 - \mu^2}\) is used. The \(\mu\)-integration can be solved analytically (Gradshteyn & Ryzhik 1966) to obtain

\[
I_{\mu}(\theta, \zeta) = (1 - 2 \cos^2 \theta) \int_{0}^{\pi/2} d\phi \sin \phi J_0(2\zeta \cos \theta \sin \phi) + 2 \cos^2 \theta \int_{0}^{\pi/2} d\phi \sin \phi \cos^2 \theta J_0(2\zeta \cos \theta \sin \phi)
\]

\[
= (1 - 2 \cos^2 \theta) \frac{\Gamma(1/2)}{\sqrt{4\zeta \cos \theta}} J_0(2\zeta \cos \theta) + \cos^2 \theta \frac{\Gamma(3/2)}{(\zeta \cos \theta)^{3/2}} J_2(2\zeta \cos \theta)
\]

\[
= \frac{1}{4\zeta \cos \theta} \left[ (1 - 2\zeta^2 \cos(2\theta)) \sin(2\zeta \cos \theta) - 2\zeta \cos \theta \cos(2\zeta \cos \theta) \right] \tag{A.10}
\]

where \(\Gamma(x)\) and \(J_{n}^{\pm}(x)\) denote the Gamma function and spherical Bessel functions of the first kind, respectively. With this, the diffusion coefficient (A.8) can be written as

\[
\kappa_{\perp}^{2D} = \frac{\pi v R_L}{B_0^2} \int_{0}^{\infty} dk_\perp g(k_\perp) I(\zeta, z) \tag{A.11}
\]

with the function \(I(\zeta, z)\) given by equation (A.10).

**Appendix B: Modification of QLT Calculations**

In view of the failure of QLT in explaining perpendicular diffusion for a static turbulence, the question arises whether the quasilinear calculations presented above can be modified to obtain a nonvanishing, finite diffusion coefficient. For this, the general perpendicular diffusion coefficient is rewritten, and the DMT viewpoint (damping model) is used. Upon substituting the general Fokker-Planck coefficient (\ref{eq:kappa}) into \(\kappa_{\perp}\), Eq. (33), and using the DMT resonance function (\ref{eq:DMT}), one obtains

\[
\kappa_{\perp} = \frac{v^2}{2B_0^2} \sum_{n=-\infty}^{\infty} \int_{-1}^{+1} d\mu \int d^3 k \frac{\nu_e}{\nu_e^2 + (k_\parallel v_\parallel + n\Omega)^2} U(k, \mu, n) \tag{B.1}
\]

where \(\nu_e = \tau_e^{-1}\) denotes again the turbulence decorrelation rate. Furthermore,

\[
U(k, \mu, n) = \frac{A(k)}{k^2} \left[ ((k_\parallel v_\parallel + n\Omega)^2v^{-2} + \mu^2 k^2) J_0^2(W) + (1 - \mu^2)k_\perp^2 [J_n^2(W)]^2 \right] \tag{B.2}
\]
Here, Eqs. (41), (42) and (43) were evaluated for $\omega_{j,R} = 0$ and $\Gamma_j = 0$ (DMT approach) and inserted into equation (36). It was assumed that $\sigma = 0$.

Speculated that a number of possible effects may contribute to the decorrelation of magnetic fluctuations. Particles diffuse in parallel and perpendicular direction and interact with the turbulence and, therefore, have probably an influence on the dynamics of their scattering agent governing their diffusive transport. Proceeding, it is assumed that the additional “diffusive feedback” of the particles on the turbulence is given by $\kappa_{||}$, the parallel diffusion coefficient, and by $\kappa_\perp$ itself. This leads for the net turbulence decorrelation rate to the Ansatz

$$
\nu_c = \sum_i \nu_{c,i} = \gamma + \kappa_{||} k_{||}^2 + \kappa_\perp k_\perp^2 + \ldots
$$

(B.3)

where $\gamma(k)$ now represents the intrinsic turbulence decorrelation rate. The dots denote other processes such as particle drift and stochastic acceleration which might also influence turbulent decorrelation and, therefore, perpendicular diffusion. In what follows, equation (B.1) can be written as

$$
\kappa_\perp = \frac{v^2}{2B_0^2} \sum_{n=-\infty}^{\infty} \int_{-1}^{1} \frac{d\mu}{d^3k} \frac{U(k, \mu, n)}{(v/\lambda_{||}) \Lambda + \kappa_{||} k_{||}^2 + \kappa_\perp k_\perp^2 + \gamma}
$$

(B.4)

where the following auxiliary function is introduced:

$$
\Lambda(\mu, n) = \frac{3(\mu + n/(R_L \kappa_\perp))^2}{1 + (\kappa_\perp/\kappa_{||})(k_{||}/k_\perp)^2 + \gamma/\kappa_{||} k_{||}^2}
$$

(B.5)

It contains the QLT limit of an unperturbed particle orbit. Here, the relation $\kappa_{||} = \lambda_{||} v/3$ has been used, with $\lambda_{||}$ being the mean free path for parallel scattering. Apart from the terms appearing in $\Lambda$, equation (B.4) reveals the same nonlinear structure as the recent NLGC result by Matthaeus et al. (2003) (see their Eq. [7]).

Going to the last extrem, it is assumed that particles move only forward and backward to the mean magnetic field. This can be taken into account in equation (B.4) by considering the cases $\mu = \pm 1$ in terms of a Dirac delta distribution. Since $W = 0$ for $\mu = \pm 1$, the Bessel functions in Eq. (B.2) are nonvanishing only for $n = 0$. The evaluation then yields

$$
\kappa_\perp = \frac{v^2}{2B_0^2} \int d^3k \frac{A(k) (1 + 2k_{||}^2/k_\perp^2)/(1 + k_{||}^2/k_\perp^2)}{(v/\lambda_{||}) \Lambda + \kappa_{||} k_{||}^2 + \kappa_\perp k_\perp^2 + \gamma}
$$

$$
\approx \frac{v^2}{2B_0^2} \int d^3k \frac{A(k)}{(v/\lambda_{||}) + \kappa_{||} k_{||}^2 + \kappa_\perp k_\perp^2 + \gamma}
$$

(B.6)

where, for the last step, the denominator is subjected to a rough approximation and $\Lambda(\mu = \pm 1, n = 0) := 1$ is used. A closer inspection of Eq. (B.6) leads to the result that the diffusion coefficient is nonvanishing and finite for $\gamma = 0$. Apart from constants, Eq. (B.6) corresponds to the NLGC result by Matthaeus et al. (2003). However, one should keep in mind that the derivation of the diffusion coefficient (B.6) is based on a speculative approach and rough assumptions, and that it is presented here for illustrative purposes only.

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