ELEMENTARY PROOFS OF IDENTITIES FOR
SCHUR FUNCTIONS AND PLANE PARTITIONS

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To George Andrews on the occasion of his 60th birthday

Abstract. We use elementary methods to prove product formulas for sums of restricted classes of Schur functions. These imply known identities for the generating function for symmetric plane partitions with even column height and for the generating function for symmetric plane partitions with an even number of angles at each level.

1. Introduction

By a plane partition, we mean a finite set, \( P \), of lattice points with positive integer coefficients, \( \{(i, j, k)\} \subseteq \mathbb{N}^3 \), with the property that if \((r, s, t) \in P\) and \(1 \leq i \leq r, \ 1 \leq j \leq s, \ 1 \leq k \leq t\), then \((i, j, k)\) must also be in \( P \). A plane partition is symmetric if \((i, j, k) \in P\) if and only if \((j, i, k) \in P\). The height of stack \((i, j)\) is the largest value of \(k\) for which there exists a point \((i, j, k)\) in the plane partition. A plane partition is column strict if the height of stack \((i, j)\) is strictly less than the height of stack \((i - 1, j)\) whenever \(i \geq 2\) and \((i, j, 1)\) is in the plane partition.

Symmetric plane partitions were studied by P. A. MacMahon [12] who conjectured in 1898 that the generating function for symmetric plane partitions with \(1 \leq i, j \leq n\) and \(1 \leq k \leq m\) is

\[
\prod_{i=1}^{n} \frac{1 - q^{m+2i-1}}{1 - q^{2i-1}} \prod_{1 \leq i < j \leq n} \frac{1 - q^{2(m+i+j-1)}}{1 - q^{2i+j-1}}.
\]

This was proven independently by Andrews [1] and Macdonald [11]. As shown by Andrews [2], this is equivalent to the Bender-Knuth conjecture [3], that the generating function for column strict plane partitions with \(1 \leq i, k \leq n\), \(1 \leq k \leq m\) is

\[
\prod_{1 \leq i \leq j \leq n} \frac{1 - q^{m+i+j-1}}{1 - q^{i+j-1}}.
\]

Both of these generating functions are consequences of the following theorem of Macdonald [11], the first when we set \(x_i = q^{2n-2i+1}\) and the second when we set \(x_i = q^{n+1-i}\).

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Theorem I (Macdonald). For positive integers \(m\) and \(n\),

\[
\sum_{\lambda \subseteq \{m^n\}} s_\lambda(x_1, \ldots, x_n) = \frac{\det(x_j^{x_{m+2n-j}})}{\prod_{i=1}^n (1 - x_i) \prod_{1 \leq i < j \leq n} (x_i x_j - 1)(x_i - x_j)}.
\]

The sum is over all partitions into at most \(n\) parts, each of which is less than or equal to \(m\).

I gave an elementary proof of Macdonald’s identity in [4]. Désarménien [7] and Stembridge [15] found a similar theorem where the sum on the left is over partitions into even parts. Désarménien [8] has also found the generalization in which any number of odd parts are specified.

Theorem II (Désarménien-Stembridge). For positive even integer \(m\) and positive integer \(n\),

\[
\sum_{\lambda \subseteq \{m^n\} \atop \lambda \text{ even}} s_\lambda(x_1, \ldots, x_n) = \frac{\det(x_j^{x_{m+2n-j}})}{\prod_{i=1}^n (1 - x_i^2) \prod_{1 \leq i < j \leq n} (x_i x_j - 1)(x_i - x_j)}.
\]

This theorem has two corollaries that were found by Désarménien and Stembridge and, independently, Proctor [14]. The \(q = 1\) case was first discovered by DeSainte-Catherine and Viennot [6]. The generating function for symmetric plane partitions with \(1 \leq i, j \leq n\) and \(1 \leq k \leq m\) where \(m\) is even and every stack has even height is given by

\[
\prod_{i=1}^n \frac{1 - q^{m+2i}}{1 - q^{2i}} \prod_{1 \leq i < j \leq n} \frac{1 - q^{2(m+i+j)}}{1 - q^{4(i+j)}}.
\]

The generating function for column strict plane partitions with \(1 \leq i, k \leq n\), \(1 \leq j \leq m\) (\(m\) even), and all rows of even length is

\[
\prod_{i=1}^n \frac{1 - q^{m+2i}}{1 - q^{2i}} \prod_{1 \leq i < j \leq n} \frac{1 - q^{m+i+j}}{1 - q^{4+j}}.
\]

Okada [13] has proven the following companion using his minor summation formula. Krattenthaler [10] has used the special orthogonal tableaux of Lakshmibai, Musili, and Seshadri to generalize this result to one in which the number of columns of odd length is specified.

Theorem III (Okada). For positive integer \(m\) and positive even integer \(n\),

\[
\sum_{\lambda' \subseteq \{m^n\} \atop \lambda' \text{ even}} s_\lambda(x_1, \ldots, x_n) = \frac{1}{2} \frac{\det(x_j^{x_{m+2n-1-j}} + x_j^{m+2n-1-j})}{\prod_{1 \leq i < j \leq n} (x_i x_j - 1)(x_i - x_j)},
\]

where \(\lambda'\) is the partition conjugate to \(\lambda\). In other words the sum on the left is over partitions with even column lengths.

This has the following corollary when \(x_i = q^{2n-2i+1}\).
Corollary. The generating function for symmetric plane partitions, \(1 \leq i, j \leq n\) (\(n\) even) and \(1 \leq k \leq m\), such that for each \(k\) there are an even number of lattice points of the form \((i, i, k)\) is given by

\[
\frac{1}{2} \left( \prod_{i=0}^{n-1} (1 - q^{m+2i}) + \prod_{i=0}^{n-1} (1 + q^{m+2i}) \right) \prod_{1 \leq i < j \leq n} \frac{1 - q^{2(m+i+j-2)}}{1 - q^{2(i+j-1)}}.
\]

The generating functions that are derived from Theorems I, II, and III have particularly nice formulations. We define \(B(n, n, m) = \{(i, j, k) \mid 1 \leq i, j \leq n, 1 \leq k \leq m\}\) and \(B(n, n, m)/S_2\) to be the set of orbits of \(B(n, n, m)\) under transposition of the first two coordinates. For \(\eta \in B(n, n, m)/S_2\), we define \(\text{Ht}(\eta) = i + j + k - 2\) where \((i, j, k)\) is any one element of \(\eta\). An orbit counting generating function is the sum in which each plane partition is weighted by \(q\) to the number of orbits.

The generating function for symmetric plane partitions in \(B(n, n, m)\) is given by

\[
\prod_{\eta \in B(n, n, m)/S_2} \frac{1 - q^{|\eta|(1+\text{Ht}(\eta))}}{1 - q^{|\eta|\text{Ht}(\eta)}}.
\]

The orbit counting generating function for symmetric plane partitions in \(B(n, n, m)\) is given by

\[
\prod_{\eta \in B(n, n, m)/S_2} \frac{1 - q^{1+\text{Ht}(\eta)}}{1 - q^{\text{Ht}(\eta)}}.
\]

The generating function for symmetric plane partitions with even stack height in \(B(n, n, m)\) (\(m\) even) is given by

\[
\prod_{\eta \in B(n, n, m)/S_2} \frac{1 - q^{2+\text{Ht}(\eta)}}{1 - q^{1+\text{Ht}(\eta)}}.
\]

The orbit counting generating function for symmetric plane partitions with even stack height in \(B(n, n, m)\) (\(m\) even) is given by

\[
\prod_{\eta \in B(n, n, m)/S_2} \frac{1 - q^{2\text{Ht}(\eta)}}{1 - q^{1+\text{Ht}(\eta)}}.
\]

The generating function for symmetric plane partitions in \(B(n, n, m)\) (\(n\) even) such that for each \(k\), \(1 \leq k \leq m\), there are an even number of points of the form \((i, i, k)\) is given by

\[
\prod_{\eta \in B(n, n, m-1)/S_2} \frac{1 - q^{\text{Ht}(\eta)}}{1 - q^{\text{Ht}(\eta)}} \sum_{S \subseteq \{(i, i, m) \mid 1 \leq i \leq n\}} \prod_{\eta \in S} q^{\text{Ht}(\eta)}.
\]

There is a formula given by Krattenthaler (equation(7.15) in [10]) for the corresponding orbit counting generating function. It is not as readily stated in terms of orbits.

In section 2, we shall warm up to the proof of Theorems II and III with a general result that includes the limiting cases of Theorems I, II, and III. It was first proved by Ishikawa and Wakayama [9] using Okada’s minor-summation formula of Pfaffians.
Theorem IV (Ishikawa and Wakayama). For any positive integer $n$, we have that

$$
\sum_{\lambda} f_\lambda(t,v)s_\lambda(x_1, \ldots, x_n) = \prod_{i=1}^{n} \frac{1}{(1-tx_i)(1-vx_i)} \prod_{1\leq i<j\leq n} \frac{1}{1-x_ix_j},
$$

where we let $a_j$ be the number of columns of length $j$ in $\lambda$ (equivalently, the number of parts of size $j$ in $\lambda'$) and

$$
f_\lambda(t,v) = \prod_{j \text{ odd}} \frac{v^{a_j+1} - t^{a_j+1}}{v - t} \prod_{j \text{ even}} \frac{1 - (tv)^{a_j+1}}{1 - tv}.
$$

We note that

$$
f(0,1) = 1,
\begin{align*}
&f(1,-1) = \begin{cases} 
0 & \text{if any } a_j \text{ is odd,} \\
1 & \text{otherwise,}
\end{cases} \\
&f(0,0) = \begin{cases} 
0 & \text{if any } a_j \text{ is positive for any odd } j, \\
1 & \text{otherwise.}
\end{cases}
\end{align*}
$$

Theorem IV implies the following Littlewood formulas ([11], examples 4 and 5 in section I.5):

$$
\begin{align*}
\sum_{\lambda} s_\lambda(x_1, \ldots, x_n) &= \prod_{i=1}^{n} \frac{1}{1-x_i} \prod_{1\leq i<j\leq n} \frac{1}{1-x_ix_j}, \\
\sum_{\lambda \text{ even}} s_\lambda(x_1, \ldots, x_n) &= \prod_{i=1}^{n} \frac{1}{1-x_i^2} \prod_{1\leq i<j\leq n} \frac{1}{1-x_ix_j}, \\
\sum_{\lambda' \text{ even}} s_\lambda(x_1, \ldots, x_n) &= \prod_{1\leq i<j\leq n} \frac{1}{1-x_ix_j}.
\end{align*}
$$

In section 3, we shall give the proof of Theorem III as well as a new proof of Theorem II. Section 4 will show the derivation of the generating function for symmetric plane partitions with an even number of lattice points of the form $(i,i,k)$ for each $k$. With the exceptions of Lemmas 1 and 2, the results presented in this paper are not new. The proofs, however, are considerably simpler than those that have been given before.

2. Proof of Theorem IV

Lemma 1. For any positive integer $n$ we have that

$$
\begin{align*}
&x_1 \cdots x_n \sum_{k=1}^{n} x_k^{-1}(1-tx_k)(1-vx_k) \prod_{i=1 \atop x_k}^{n} \frac{1-x_ix_k}{x_i - x_k} \\
&= \begin{cases} 
(1-tx_1 \cdots x_n)(1-vx_1 \cdots x_n), & \text{if } n \text{ is odd}, \\
(1-x_1 \cdots x_n)(1-tvx_1 \cdot x_n), & \text{if } n \text{ is even.}
\end{cases}
\end{align*}
$$
**Proof:** This lemma is correct for \( n = 1 \). We assume that it is correct with \( n - 1 \) variables and proceed by induction. If we multiply both sides of equation (2.1) by \( \prod_{1 \leq i < j \leq n} (x_i - x_j) \), each side is an alternating polynomial in \( x_1, \ldots, x_n \). It follows that both sides of equation (2.1) are symmetric polynomials that are quadratic in each of the variables \( x_1 \) through \( x_n \). We only need to show that they agree at three values of \( x_1 \). Both polynomials are 1 when \( x_1 = 0 \). When \( x_1 = t^{-1} \), the polynomial on the left is equal to

\[
(2.2) \quad t^{-1} x_2 \cdots x_n \sum_{k=2}^{n} x_k^{-1} (1 - t x_k) (1 - v x_k) \prod_{i \neq k}^{n} \frac{1 - x_i x_k}{x_i - x_k} \frac{1 - t^{-1} x_k}{t^{-1} - x_k} = x_2 \cdots x_n \sum_{k=2}^{n} x_k^{-1} (1 - t^{-1} x_k) (1 - v x_k) \prod_{i \neq k}^{n} \frac{1 - x_i x_k}{x_i - x_k} = \left\{ \begin{array}{ll}
(1 - x_2 \cdots x_n) (1 - t^{-1} v x_2 \cdots x_n), & \text{if } n \text{ is odd}, \\
(1 - t^{-1} x_2 \cdots x_n) (1 - v x_2 \cdots x_n), & \text{if } n \text{ is even}.
\end{array} \right.
\]

Similarly, the two polynomials agree at \( x_1 = v^{-1} \). \( \square \)

**Lemma 2.** For even positive integer \( n \) we have that

\[
(2.3) \quad (x_1 \cdots x_n)^2 \sum_{k=1}^{n} \sum_{i \neq k}^{n} x_k^{-2} x_i^{-1} \prod_{i \neq k}^{n} \frac{1 - x_i x_k}{x_i - x_k} \prod_{i \neq k, l}^{n} \frac{1 - x_i x_l}{x_i - x_l} = 1 - x_1 \cdots x_n.
\]

**Proof:** This follows from lemma 1 with \( t = v = 0 \), summing first over \( l \) and then over \( k \). \( \square \)

**Proof of Theorem IV:** When \( n = 1 \), the left side of equation (1.4) is

\[
\sum_{k=0}^{\infty} \frac{v^{k+1} - t^{k+1}}{v - t} x^k = \frac{1}{(1 - vx)(1 - tx)}.
\]

We proceed by induction and assume that the equation is valid for \( n - 1 \) variables. We rewrite equation (1.4) as

\[
(2.3) \quad \sum_{\lambda} \sum_{\sigma \in S_n} (-1)^{i} f_\lambda(t, v) \prod_{i=1}^{n} \lambda^x_{\sigma(i)} + n - \sigma(i) (1 - t x_i) (1 - v x_i) \prod_{1 \leq i < j \leq n} (1 - x_i x_j)
\]

where \( I(\sigma) \) is the inversion number of \( \sigma \). We shall prove that the left side is equal to the Vandermonde determinant.

We take the double summation and first sum over the possible values of \( \lambda_n \) and \( k = \sigma^{-1}(n) \). We let \( \tau \) be the restriction of \( \sigma \) to \( \{1, \ldots, n\} \backslash \{k\} \). If we subtract \( \lambda_n \) from each of the parts in \( \lambda \), we are left with a partition, \( \mu \), into at most \( n - 1 \) parts.
We have that $f_\lambda(t, v) = c_\lambda f_\mu(t, v)$ where $c_\lambda$ is $(v^{\lambda_n+1} - t^{\lambda_n+1})/(v - t)$ if $n$ is odd, $(1 - (vt)^{\lambda_n+1})/(1 - vt)$ if $n$ is even. The left side of equation (2.3) is equal to

$$\sum_{\lambda_n=0}^{\infty} \sum_{k=1}^{n} (-1)^{n-k} x_k^{-1} (1 - tx_k)(1 - vx_k)c_{\lambda_n}(x_1 \cdots x_n)^{\lambda_n+1} \prod_{i=1 \atop i \neq k}^{n} (1 - x_ix_k) \times \sum_{\mu, \tau} (-1)^{\tau} f_\mu(t, v) \prod_{i=1 \atop i \neq k}^{n} x_i^{\mu(i) + n - 1 - \tau(i)} (1 - tx_i)(1 - vx_i) \prod_{1 \leq i < j \leq n \atop i, j \neq k} (1 - x_ix_j).$$

We use our induction hypothesis to rewrite this as

$$\prod_{1 \leq i < j \leq n} (x_i - x_j) \sum_{\lambda_n=0}^{\infty} \sum_{k=1}^{n} x_k^{-1} (1 - tx_k)(1 - vx_k)c_{\lambda_n}(x_1 \cdots x_n)^{\lambda_n+1} \prod_{i=1 \atop i \neq k}^{n} \frac{1 - x_ix_k}{x_i - x_k}.$$ 

By Lemma 1, the double sum is equal to 1. \qed

3. PROOF OF THEOREMS II AND III

The proofs of Theorems II and III are similar in structure to the proof of Theorem IV, just more complicated in detail.

**Proof of Theorem II:** We verify that this theorem is correct for $n = 1$ and proceed by induction on the number of variables. We shall prove this theorem in the form

$$\sum_{\lambda \leq \{mn\}} \det(x_1^{\lambda_1} + \cdots + j) \prod_{i=1}^{n} (1 - x_i^2) \prod_{1 \leq i < j \leq n} (x_i - x_j) = \sum_{\sigma \in S_n} \sum_{S \subseteq \{1, \ldots, n\}} (-1)^{\lambda(S)} \prod_{i \in S} x_i^{m+2n-1 - \sigma(i)} \prod_{i \notin S} x_i^{\sigma(i)-1},$$

where $m$ is an even integer.

As in the proof of Theorem IV, we expand the left side as a sum over partitions, $\lambda$, and permutations, $\sigma$. We then sum separately over $\lambda_n$ which must now be even, $\lambda_n = 2t$, and over $k = \sigma^{-1}(n)$, leaving $\mu$, the partition obtained from $\lambda$ when $\lambda_n$ is subtracted from each part, and $\tau$, the restriction of $\sigma$ to $\{1, \ldots, n\} \setminus \{k\}$. We then apply our induction hypothesis. The left side of equation (3.1) becomes

$$\sum_{t=0}^{m/2} \sum_{k=1}^{n} (-1)^{n-k} x_k^{-1} (1 - x_k^2)(x_1 \cdots x_n)^{2t+1} \prod_{i=1 \atop i \neq k}^{n} (x_i x_k - 1) \times \sum_{\mu, \tau} (-1)^{\tau} x_1^{\mu(i) + n - 1 - \tau(i)} (1 - x_1^2) \prod_{1 \leq i < j \leq n \atop i, j \neq k} (x_i x_j - 1)$$

$$= \sum_{t=0}^{m/2} \sum_{k=1}^{n} (-1)^{n-k} x_k^{-1} (1 - x_k^2)(x_1 \cdots x_n)^{2t+1} \prod_{i=1 \atop i \neq k}^{n} (x_i x_k - 1) \times \sum_{\sigma \in S_{n-1}} \sum_{S \subseteq \{1, \ldots, n\} \setminus \{k\}} (-1)^{\lambda(S)} \prod_{i \in S} x_i^{m-2t+2n-1 - \sigma(i)} \prod_{i \notin S} x_i^{\sigma(i)-1},$$

where $m$ is an even integer.
where $S_{n-1}$ is the set of 1–1 mappings from $\{1, \ldots , n\} \setminus \{k\}$ to $\{1, \ldots , n-1\}$ and $S$ is the complement of $S$ in $\{1, \ldots , n\} \setminus \{k\}$.

We simplify this summation and then sum over $\sigma \in S_{n-1}$ and over $t$. The left side of equation (3.1) is equal to

$$
\sum_{i=0}^{m/2} \sum_{k=1}^{n} \sum_{\sigma \in S_{n-1}} \sum_{S \subseteq \{1, \ldots , n\} \setminus \{k\}} (-1)^{n-k+|S|} x_k^{-1} (1 - x_k^2) \prod_{i \in S} x_i^{2t+1} \prod_{i \notin S} (x_i x_k - 1)
$$

$$
\times \prod_{i \in S} x_i^{m+2n-\sigma(i)} \prod_{i \notin S} x_i^{\sigma(i)-1}
$$

$$
= \sum_{k=1}^{n} \sum_{S \subseteq \{1, \ldots , n\} \setminus \{k\}} (-1)^{n-k+|S|} x_k^{-1} (1 - x_k^2) \prod_{i \notin S} x_i^{m+2} \prod_{i \in S} x_i^{m+2} \prod_{1 \leq i < j \leq n} (x_i^{e_i} - x_j^{e_j}),
$$

where $e_i$ is -1 if $i \in S$ and +1 if $i \notin S$.

We reverse the order of summation so that we first sum over all proper subsets of $\{1, \ldots , n\}$ and then over all $k \notin S$. For each $i \in S$, we rewrite $x_i x_k - 1$ as $-x_i (x_i^{e_i} - x_k^{e_k})$ if $i < k$, and rewrite it as $x_i (x_i^{e_i} - x_k^{e_i})$ if $i > k$. The left side of equation (3.1) has become

$$
(-1)^{m/2} \sum_{S \subseteq \{1, \ldots , n\}} (-1)^{|S|} \prod_{i \in S} x_i^{m+2n} \frac{1 - \prod_{i \notin S} x_i^{m+2}}{1 - \prod_{i \in S} x_i^2} \prod_{1 \leq i < j \leq n} (x_i^{e_i} - x_j^{e_j})
$$

$$
\times \prod_{i \notin S} x_i \sum_{k \notin S} x_k^{-1} (1 - x_k^2) \prod_{i \notin S} \frac{1 - x_i x_k}{x_i - x_k}.
$$

By Lemma 1, the second line is equal to $1 - \prod_{i \notin S} x_i^2$ which cancels with the factor in the denominator. We now expand the Vandermonde product. The left side of equation (3.1) is equal to

$$
\sum_{S \subseteq \{1, \ldots , n\}} \sum_{\sigma \in S_n} (-1)^{\tau(\sigma)+|S|} \prod_{i \in S} x_i^{m+2n+1-\sigma(i)} \prod_{i \notin S} x_i^{\sigma(i)-1}
$$

$$
- \sum_{S \subseteq \{1, \ldots , n\}} \sum_{\sigma \in S_n} (-1)^{\tau(\sigma)+|S|} \prod_{i \in S} x_i^{m+2n+1-\sigma(i)} \prod_{i \notin S} x_i^{m+\sigma(i)+1}.
$$

We use the fact that

$$
\sum_{S \subseteq \{1, \ldots , n\}} \sum_{\sigma \in S_n} (-1)^{\tau(\sigma)+|S|} \prod_{i \in S} x_i^{m+2n+1-\sigma(i)} \prod_{i \notin S} x_i^{m+\sigma(i)+1}
$$

$$
= \det \left( x_i^{m+j+1} - x_i^{m+2n+1-j} \right) = 0,
$$

to replace

$$
- \sum_{S \subseteq \{1, \ldots , n\}} \sum_{\sigma \in S_n} (-1)^{\tau(\sigma)+|S|} \prod_{i \in S} x_i^{m+2n+1-\sigma(i)} \prod_{i \notin S} x_i^{m+\sigma(i)+1}
$$
by
\[ \sum_{\sigma \in S_n} (-1)^{I(\sigma)+n} \prod_{i=1}^{m} x_i^{m+2n+1-\sigma(i)}. \]
This puts the left side of equation (3.1) in the desired form. □

**Proof of Theorem III:** We begin by rewriting the identity to be proven as

(3.2) \[ \sum_{\lambda \subseteq (m^n), \lambda \text{ even}}^{\infty} \sum_{\sigma \in S_n} (-1)^{I(\sigma)} \prod_{i=1}^{n} x_i^{\lambda(i)+n-\sigma(i)} \prod_{1 \leq i < j \leq n} (x_i x_j - 1) \]

\[ = \sum_{\sigma \in S_n} \sum_{S \subseteq \{1, \ldots, n\}, |S| \text{ even}} (-1)^{I(\sigma)} \prod_{i \in S} x_i^{m+2n-\sigma(i)} \prod_{i \notin S} x_i^{\sigma(i)-1}. \]

We again proceed by induction. For this theorem, we need to identify both \(\sigma^{-1}(n)\) and \(\sigma^{-1}(n-1)\). We form \(\mu\) by subtracting \(\lambda_n\) from each part. Since each column has even length, \(\mu\) has at most \(n-2\) parts. The left side of equation (3.2) is equal to

\[ \sum_{\lambda_n = 0}^{m} \sum_{1 \leq k < l \leq n} (-1)^{n-k+n-l} (x_k^{-2} x_l^{-1} - x_k^{-1} x_l^{-2}) (x_1 \cdots x_n)^{\lambda_n+2} \]

\[ \times (x_k x_l - 1) \prod_{i \notin \{k, l\}} (x_i x_{\mu(i)} x_i x_l - 1) \]

\[ \times \sum_{\mu, \tau} (-1)^{I(\tau)} \prod_{i=1}^{n} x_i^{\mu(i)+n-2-\tau(i)} \prod_{1 \leq i < j \leq n} (x_i x_j - 1). \]

We apply our induction hypothesis to the inner sum and then sum over \(\lambda_n\) and \(\sigma \in S_{n-2}\), the set of 1–1 mappings from \(\{1, \ldots, n\}\{k, l\}\) to \(\{1, \ldots, n-2\}\). The left side of equation (3.2) becomes

\[ \sum_{1 \leq k < l \leq n} \sum_{S \subseteq \{1, \ldots, n\}\{k, l\}, |S| \text{ even}} (-1)^{k+l} (x_k^{-2} x_l^{-1} - x_k^{-1} x_l^{-2}) \frac{1 - \prod_{i \notin S} x_i^{m+1}}{1 - \prod_{i \notin S} x_i} \]

\[ \times (x_k x_l - 1) \prod_{i \notin \{k, l\}} (x_i x_{\mu(i)} x_i x_l - 1) \]

\[ \times \prod_{i \notin S} x_i^{2} \prod_{i \in S} x_i^{m+2n-4} (-1)^{n-2} \prod_{i \notin \{k, l\}} (x_i^{e_i} - x_j^{e_j}). \]

Again we have \(e_i = -1\) if \(i \in S\) and \(+1\) if \(i \notin S\).

For \(i \in S\), we rewrite \((x_i x_k - 1)(x_i x_l - 1)\) as \(x_i^{2} (x_i^{e_i} - x_k^{e_i})(x_i^{e_i} - x_l^{e_i})\). We then interchange the sum on \(S\), proper subsets of \(\{1, \ldots, n\}\) with even cardinality, and the sum on \(k\) and \(l\) which now must lie in the complement of \(S\). The left side of
equation (3.2) is equal to

\[ (-1)^{\frac{n}{2}} \sum_{\substack{S \subseteq \{1, \ldots, n\} \atop |S| \text{ even}}} \prod_{i \in S} x_i^{m+2n-2} \frac{1 - \prod_{i \notin S} x_i^{m+1}}{1 - \prod_{i \notin S} x_i^{m+1}} \prod_{1 \leq i < j \leq n} (x_i^{x_i} - x_j^{x_j}) \times \prod_{i \in S} x_i^2 \sum_{1 \leq k < l \leq n} \frac{(x_k^{-2} x_l^{-1} - x_k^{-1} x_l^{-2}) x_k x_l - 1}{x_k - x_l} \prod_{i \notin S \atop i \neq k, l} (1 - x_i x_k)(1 - x_i x_l) \frac{1}{(x_i - x_k)(x_i - x_l)}. \]

The second line of this expression is equal to

\[ \prod_{i \in S} x_i^2 \sum_{1 \leq k < l \leq n} x_k^{-2} x_l^{-1} \prod_{i \notin S \atop i \neq k, l} \frac{1 - x_i x_k}{x_i - x_k} \prod_{i \notin S \atop i \neq k, l} \frac{1 - x_i x_l}{x_i - x_l}. \]

By Lemma 2, this is equal to \(1 - \prod_{i \notin S} x_i\), which cancels with the factor in the denominator.

As in the proof of Theorem II, we replace the Vandermonde product by the sum over permutations. The left side of equation (3.2) becomes

\[ \sum_{\substack{S \subseteq \{1, \ldots, n\} \atop |S| \text{ even}}} \sum_{\sigma \in S_n} (-1)^{\text{I}(\sigma)} \prod_{i \in S} x_i^{m+2n-\sigma(i)-1} \prod_{i \notin S} x_i^{\sigma(i)-1} \]

\[ - \sum_{\substack{S \subseteq \{1, \ldots, n\} \atop |S| \text{ even}}} \sum_{\sigma \in S_n} (-1)^{\text{I}(\sigma)} \prod_{i \in S} x_i^{m+2n-\sigma(i)-1} \prod_{i \notin S} x_i^{m+\sigma(i)}. \]

We now observe that

\[ \sum_{\substack{S \subseteq \{1, \ldots, n\} \atop |S| \text{ even}}} \sum_{\sigma \in S_n} (-1)^{\text{I}(\sigma)} \prod_{i \in S} x_i^{m+2n-\sigma(i)-1} \prod_{i \notin S} x_i^{m+\sigma(i)} = 0. \]

This is true because if we interchange the inverse images of \(n\) and \(n-1\) and change whether or not each inverse image is in \(S\), then we change the sign of the inversion number but do not change the monomial. As a result, we have that

\[ - \sum_{\substack{S \subseteq \{1, \ldots, n\} \atop |S| \text{ even}}} \sum_{\sigma \in S_n} (-1)^{\text{I}(\sigma)} \prod_{i \in S} x_i^{m+2n-\sigma(i)-1} \prod_{i \notin S} x_i^{m+\sigma(i)} \]

\[ = \sum_{\sigma \in S_n} (-1)^{\text{I}(\sigma)} \prod_{i=1}^{n} x_i^{m+2n-\sigma(i)-1} \]

The left side of equation (3.2) is equal to the desired sum. \(\square\)

4. Consequence and Question

If we set \(x_i = q^{2n-2i+1}\) in Theorem III, the left side of equation (1.3) becomes the generating function for symmetric plane partitions with \(1 \leq i, j \leq n, 1 \leq k \leq m\), such that for each \(k\) there are an even number of lattice points of the form \((i, i, k)\).

The right side of equation (1.3) becomes

\[ \prod_{i=1}^{n} q^{(2n-2i+1)(m+2n-2)/2} \sum_{\sigma \in S_n} \sum_{\substack{S \subseteq \{1, \ldots, n\} \atop |S| \text{ even}}} (-1)^{\text{I}(\sigma)} \prod_{i \in S} q^{(2n+m-2\sigma(i))(2i-2n-1)/2} \]

\[ \times \prod_{1 \leq i < j \leq n} (q^{2n-2i+1} - q^{2n-2j+1})^{-1} (q^{4n-2i-2j+1} - 1)^{-1}, \]
where $\epsilon_i = -1$ if $i \in S$ and $+1$ if $i \not\in S$.

We combine the $B_n$ form of the Weyl denominator formula,

$$\sum_{\sigma \in S_n} \sum_{S \subseteq \{1, \ldots, n\}} (-1)^{|\sigma|+|S|} \prod_{i=1}^{n} x_{\sigma(i)}^{\epsilon_i(2i-2n-1)/2}$$

$$= \prod_{i=1}^{n} x_i^{(1-2n)/2} (1-x_i) \prod_{1 \leq i < j \leq n} (x_i - x_j)(x_i x_j - 1),$$

and the identity obtained when each $x_i$ is replaced by $-x_i$:

$$\sum_{\sigma \in S_n} \sum_{S \subseteq \{1, \ldots, n\}} (-1)^{|\sigma|} \prod_{i=1}^{n} x_{\sigma(i)}^{\epsilon_i(2i-2n-1)/2}$$

$$= \prod_{i=1}^{n} x_i^{(1-2n)/2} (1+x_i) \prod_{1 \leq i < j \leq n} (x_i - x_j)(x_i x_j - 1),$$

to derive the result that we need:

$$\sum_{\sigma \in S_n} \sum_{S \subseteq \{1, \ldots, n\} \text{ even}} (-1)^{|\sigma|} \prod_{i=1}^{n} x_{\sigma(i)}^{\epsilon_i(2i-2n-1)/2}$$

$$= \frac{1}{2} \prod_{i=1}^{n} x_i^{(1-2n)/2} \left( \prod_{i=1}^{n} (1-x_i) + \prod_{i=1}^{n} (1+x_i) \right) \prod_{1 \leq i < j \leq n} (x_i - x_j)(x_i x_j - 1).$$

The corollary now follows directly.

It would be of interest to find the analogous formula for

$$\sum_{\lambda \subseteq \{m^n\}} f_\lambda(t, v)s_\lambda(x_1, \ldots, x_n),$$

although the form of it will certainly be much more complicated than anything presented here.

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