Active Spherical Model

Harukuni Ikeda
Department of Physics, Gakushuin University, 1-5-1 Mejiro, Toshima-ku, Tokyo 171-8588, Japan
(Dated: September 13, 2022)

The spherical model is a popular solvable model and has been applied to describe several critical phenomena such as the ferromagnetic transition, Bose-Einstein condensation, spin-glass transition, glass transition, jamming transition, and so on. Motivated by recent developments of active matter, here we consider the spherical model driven by the Ornstein–Uhlenbeck type self-propulsion force with persistent time $\tau_p$. We show that the model exhibits the Ising universality for finite $\tau_p$. On the contrary, the model exhibits the random field Ising universality in the limit $\tau_p \to \infty$.

I. INTRODUCTION

In this work, we extend the spherical model to describe non-equilibrium critical phenomena in the steady state. In particular, we have in mind the so-called Motility-Induced Phase Separation (MIPS) of active matter: the self-propelled particles spontaneously aggregate on increasing the motility, eventually leading to a phase separation [1]. One of the fundamental questions is to which universality class MIPS belongs. Extensive numerical simulations showed that the MIPS has the Ising universality class [2,3], while a different universality is also reported [4]. The field theoretical studies support the Ising universality [5,7], where the self-propulsion force on the $\phi^4$ field theory gives rise to only irrelevant terms in the case of the bulk phase separation, while a different universality appears in the case of the micro phase separation [5]. Here we study the effect of the self-propulsion force on the Ising universality in a different way by considering an exactly solvable model.

There are several solvable models to describe non-equilibrium critical phenomena in the steady state, such as the celebrated Kuramoto model [8], asymmetric simple exclusion process [9], zero-range process [10], and so on [11,12]. However, most of those results are obtained in the mean-field limit or one dimension, and it is not obvious how to generalize the results to arbitrary spatial dimensions. The spherical model is one of the few models that can be solved in any dimension [13] and several settings [14–20], even out of equilibrium [21–25]. Therefore, the spherical model would be the most promising candidate to describe the non-equilibrium critical phenomena in steady states, such as MIPS.

Here we consider the spherical model driven by a self-propulsion force with persistent time $\tau_p$ [26,27]. We show that for finite $\tau_p$, the model indeed has the Ising universality, while in the limit $\tau_p \to \infty$, the model exhibits the random field Ising universality.

The structure of the paper is as follows. In Sec. II, we review some known results for the spherical model through the analysis of a simplified version of the spherical model originally proposed by Berlin and Kac [13]. In particular, we argue that the condensation transition of the model can be identified with the underlying ferromagnetic transition [13]. In Sec. III, we consider the SDM driven by an active noise with the persistent time $\tau_p$. We show that the model exhibits the Ising universality for finite $\tau_p$. In Sec. IV, we investigate the model in the limit $\tau_p \to \infty$. In Sec. V, we briefly discuss the effect of correlated noise. Finally, in Sec. VI we conclude the work.

II. SPHERICAL DEBYE MODEL

Here we first review some known results for the spherical model through the analysis of a simplified version of the spherical model originally proposed by Berlin and Kac [13]. We show that the model exhibits the condensation transition at a critical temperature $T_c$, and the transition has the same universality as the ferromagnetic transition [13,14].

Let us consider the following quadratic interaction potential:

$$V_N = \frac{x \cdot W \cdot x}{2} + \frac{\mu}{2} (x \cdot x - N),$$

where $x = \{x_1, \ldots, x_N\}$ denotes the state vector, and $W$ is a $N \times N$ matrix representing the nearest neighbor interaction in a $d$ dimensional lattice. $\mu$ denotes the Lagrange multiplier to impose the following constraint:

$$\sum_{i=1}^{N} \langle x_i^2 \rangle = N,$$

where the braket denotes the thermal average by the Maxwell-Boltzmann distribution at temperature $T$. To analyze the model, one can expand $V_N$ by the normal modes:

$$V_N = \sum_{i=1}^{N} \frac{\omega_i^2 + \mu}{2} \langle x_i^2 \rangle - \frac{N\mu}{2},$$

where $\omega_i$ denotes the frequency of the $i$-th mode. We will order $\omega_i$ as

$$\omega_1 < \omega_2 < \cdots < \omega_N.$$
Since, an orthogonal transformation does not change the inner product, the spherical constraint is written as

$$\sum_{i=1}^{N} \langle u_i^2 \rangle = N. \tag{5}$$

The precise value of $\omega_i$ depends on the details of $W$, but the vibrational properties of a $d$ dimensional lattice would eventually be dominated by the phonon modes on a large enough scale. So we assume that the distribution of $\omega_i$, $D(\omega) = N^{-1} \sum_{i=1}^{N} \delta(\omega - \omega_i)$, is given by the Debye density of states \[28\]:

$$D(\omega) = \begin{cases} d \omega D C D d \omega^{-1} & \omega \in [0, \omega_D], \\ 0 & \text{otherwise}, \end{cases} \tag{6}$$

where $\omega_D$ denotes the Debye frequency. $D(\omega)$ is normalized so that $\int_0^{\infty} D(\omega)d\omega = 1$. To simplify the notation, hereafter we set $\omega_D = 1$. We call the model defined by Eqs. \[3\], \[4\], and \[6\] as the spherical Debye model (SDM).

In this section, we study the model in equilibrium at temperature $T$. From the equipartition theorem \[29\], we get

$$\langle u_i^2 \rangle = \frac{k_B T}{\omega_i^2 + \mu}, \tag{7}$$

where $k_B$ denotes the Boltzmann constant. To simplify the notation, hereafter we set $k_B = 1$. Since $\langle u_i^2 \rangle \leq N$, $\mu$ should satisfy the following condition

$$\mu \geq - \min_i \omega_i^2 = 0. \tag{8}$$

The value of $\mu$ is to be determined by the spherical constraint

$$1 = \frac{1}{N} \sum_{i=1}^{N} \frac{T}{\omega_i^2 + \mu}. \tag{9}$$

In the limit $N \to \infty$, we expect that the summation can be replaced with an integral:

$$1 = TF(\mu), \quad F(\mu) = \int d\omega \frac{D(\omega)}{\omega^2 + \mu}. \tag{10}$$

$F(\mu)$ is a decreasing function of $\mu$ and is maximal in the limit $\mu \to +0$:

$$\lim_{\mu \to +0} F(\mu) = \begin{cases} +\infty & (d \leq 2), \\ 1/T_c & (d > 2). \end{cases}, \tag{11}$$

where

$$T_c = \frac{d-2}{2}. \tag{12}$$

When $d > 2$ and $T < T_c$,

$$TF(\mu) \leq TF(0) < 1, \tag{13}$$

implying that $TF(\mu) = 1$ has no solution. This is the signature of the condensation to the lowest eigenmode $\omega_1 \[14 \[30\]$. Below $T_c$, we should separate the first and other terms in Eq. \[9\] to replace the summation with an integral, as in the case of the Bose-Einstein condensation \[29 \[30\].

$$\frac{1}{N} \sum_{i=1}^{N} \frac{T}{\omega_i^2 + \mu} = \langle u_1^2 \rangle \frac{1}{N} + 1 \sum_{i=2}^{N} \frac{T}{\omega_i^2 + \mu} = \langle u_1^2 \rangle \frac{1}{N} + TF(\mu). \tag{14}$$

Substituting it back into Eq. \[9\], we get for $T < T_c$

$$\langle u_1^2 \rangle = 1 - TF(0) = 1 - \frac{T}{T_c}. \tag{15}$$

In the case of the spherical model of a ferromagnet \[13\], the condensation transition is identified with the ferromagnetic phase transition, and Eq. \[15\] corresponds to the square of the magnetization $\langle u_1^2 \rangle / N \sim m^2$, which leads to the well-known scaling behavior $m \sim (1 - T/T_c)^{1/2} \[13 \[14 \[30\].$

The detailed analysis of Eq. \[10\] reveals that on approaching $T_c$ from above, $\mu$ behaves as follows (see Appendix. A):

$$\mu \sim \begin{cases} (T - T_c)^{d-2} & d \in (2, 4), \\ (T - T_c)^{1} & d > 4. \end{cases} \tag{16}$$

Below $T_c$, $\mu$ is written as

$$\mu = \frac{1}{\langle u_1^2 \rangle} = \frac{1}{N} \frac{T_c}{T_c - T}, \tag{17}$$

which vanishes in the thermodynamic limit $N \to \infty$. The Lagrange multiplier $\mu$ of the spherical model plays a similar role as the chemical potential of the ideal Bose gas, in that it fixes the mean value of an extensive quantity. Indeed, both quantities have the same critical exponent \[14 \[29 \[30\]. Since $\langle u_1^2 \rangle = 1/\mu$, we get

$$\langle u_1^2 \rangle \sim \begin{cases} (T - T_c)^{-\frac{d-2}{2}} & d \in (2, 4), \\ (T - T_c)^{-1} & d > 4. \end{cases}, \tag{18}$$

which can be identified with the susceptibility $\chi \sim m^2$ of the spherical ferromagnetic model for $T > T_c \[14 \[30\]. Using the equipartition theorem, the mean-value of the interaction potential Eq. \[3\] is calculated as follows

$$u = \frac{V_N}{N} = T - \mu = \begin{cases} T - \mu & (T > T_c), \\ T & (T \leq T_c). \end{cases} \tag{19}$$

The specific heat for $d \in (2, 4)$ is

$$C = \frac{du}{dT} = \begin{cases} (T - T_c)^{\frac{d-2}{2}} & (T > T_c) \\ \text{const} & (T < T_c). \end{cases}, \tag{20}$$
and for \( d > 4 \), the critical exponent is zero, i.e., \( C \) changes discontinuously at \( T = T_c \). The result is again consistent with the spherical ferromagnetic model \([13]\). One can also argue the scaling of the correlation length by assuming the linear relation \( \omega_i = c q_i \) between the frequency \( \omega_i \) and wave number \( q_i \). \([14, 31]\).

In summary, the SDM, which consists of \( N \) non-interacting oscillators Eq. \([6]\), has the same universality as the spherical model of a ferromagnet. It is worth mentioning that the phase behavior and the critical exponent of \( \mu \) can be deduced from only the information of the second moment of \( u_i \). This allows us to draw the phase diagram even for a non-equilibrium version of the model where the steady-state distribution is in general not known. Furthermore, the critical exponent of the Lagrange multiplier \( \mu \), which controls the other critical exponents, is also calculated from \( u_i^2 \). This allows us to discuss the lower and upper critical dimensions, and the universality class.

### III. Active Spherical Model

Now we consider a non-equilibrium model. We consider the following equation \([26]\):

\[
\frac{du_i(t)}{dt} = -\frac{\partial V_i}{\partial u_i(t)} + f_i(t) = - (\mu + \omega_i^2) u_i(t) + f_i(t), \tag{21}
\]

where \( f_i(t) \) denotes the self-propulsion force. The time evolution of \( f_i(t) \) is given by the Ornstein-Uhlenbeck process \([27]\):

\[
\tau_p \dot{f}_i(t) = - f_i(t) + \sqrt{2T} \eta_i(t), \tag{22}
\]

where \( \tau_p \) denotes the persistent time, \( T \) denotes the strength of the noise, and \( \eta_i(t) \) denotes the white noise satisfying the following condition

\[
\langle \eta_i(t) \rangle = 0, \quad \langle \eta_i(t) \eta_j(t') \rangle = \delta_{ij} \delta(t - t'). \tag{23}
\]

Eq. \(22\) can be directly integrated as follows:

\[
f_i(t) = \frac{\sqrt{2T}}{\tau_p} \int_{-\infty}^{t} dt' e^{-(t-t')/\tau_p} \xi_i(t'). \tag{24}
\]

Then, we get

\[
\langle f_i(t) f_j(t') \rangle = \frac{2T}{\tau_p^2} \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t'} dt_2 e^{-(2t-t_1-t_2)/\tau_p} \times \delta_{ij} \delta(t_1 - t_2) = \delta_{ij} \frac{T}{\tau_p} e^{-|t-t'|/\tau_p}. \tag{25}
\]

In the limit \( \tau_p \to 0 \), \( \langle f_i(t) f_j(t') \rangle \to 2T \delta_{ij} \delta(t - t') \), implying that \( f_i(t) \) reduces to the thermal white noise at temperature \( T \). \([32]\). Now we assume that the system is in the steady state at \( t = -\infty \) so that the Lagrange multiplier \( \mu \) does not depend on time. Then, one can easily integrate Eq. \(21\) as follows:

\[
u_i(t) = \int_{-\infty}^{t} dt' e^{-(\mu + \omega_i^2)(t-t')} f_i(t'). \tag{26}
\]

With a similar calculation as Eq. \(25\), the second moment of \( u_i(t) \) in the steady state is calculated as follows \([27]\):

\[
\langle u_i(t)^2 \rangle = \frac{T}{(\mu + \omega_i^2) \left[ 1 + \tau_p(\mu + \omega_i^2) \right]}. \tag{27}
\]

This implies that the equipartition theorem Eq. \(7\) does not hold for \( \tau_p > 0 \), as a consequence of the violation of the detailed balance \([27]\). Some readers may wonder how \( f_i(t) \) behaves in real space. In fact, the properties of \( f_i(t) \) do not depend on the choice of the coordinate system, since a transformation by an orthogonal matrix \( O \) preserves the second moment of \( f_i(t) \):

\[
\langle (Of(t))i(OF(t'))j \rangle = \sum_{nm} O_{in} O_{jm} \langle f_n(t) f_m(t') \rangle = \frac{T}{\tau_p} \delta_{ij} e^{-|t_1-t_2|/\tau_p}. \tag{28}
\]

The value of \( \mu \) is to be determined by the spherical constraint: \( 1 = \frac{1}{N} \sum_{i=1}^{N} \langle u_i^2 \rangle \). For \( T > T_c \), the summation can be replaced with an integral:

\[
1 = TF(\mu), \quad F(\mu) = \int d\omega D(\omega) \frac{1}{(\mu + \omega^2) \left[ 1 + \tau_p(\mu + \omega^2) \right]}. \tag{29}
\]

For finite \( \tau_p \) and for \( d > 2 \), \( F(\mu) \) converges to a finite value, implying that the condensation transition occurs at the critical temperature

\[
T_c = \frac{1}{F(0)}. \tag{30}
\]

As shown in Fig. \([1]\) \( T_c \) increases on increasing \( \tau_p \), suggesting that the motility facilitates the phase transition, as in the case of the MIPS \([1]\). Below \( T_c \), the condensation to the first mode occurs as in the case of the equilibrium model, see Eq. \([15]\). The critical behavior for \( \mu \ll 1 \) is governed by the small \( \omega \) behavior of the integrand in Eq. \(29\):

\[
D(\omega) \sim \frac{D(\omega)}{(\mu + \omega^2)} \left[ 1 + \tau_p(\mu + \omega^2) \right] \sim \frac{D(\omega)}{(\mu + \omega^2)}. \tag{31}
\]

which is the same as that of the original model Eq. \(10\). Therefore, the critical exponent is unchanged from the original one.
FIG. 1. Phase diagram of the active spherical model. The vertical axis is rescaled by the critical temperature in equilibrium $T_\text{eq}$. The solid line denotes the transition line.

IV. EXTREME ACTIVE MATTER

Recently, the extreme active matter, which corresponds to the limit $\tau_p \to \infty$, has attracted much attention [33-35]. So we here consider the corresponding limit. For this purpose, we introduce a scaled variable:

$$\tilde{T} = \frac{T}{\tau_p}. \quad (32)$$

Then, in the limit $\tau_p \to \infty$, Eq. (27) reduces to

$$\langle u_i^2 \rangle \to \frac{\tilde{T}}{(\mu + \omega_i)^2}. \quad (33)$$

Repeating the same arguments as in the previous sections, we obtain the self-consistent equation for $\mu$:

$$1 = \tilde{T} F(\mu), \quad F(\mu) = \int d\omega \frac{D(\omega)}{(\omega^2 + \mu)^2}. \quad (34)$$

In the limit $\mu \to 0$, we get

$$\lim_{\mu \to 0} F(\mu) = \begin{cases} \infty & (d \leq 4) \\ d/(d-4) & (d > 4) \end{cases}, \quad (35)$$

implying that the lower critical dimension is $d_u = 4$. Below $T < T_c$, the condensation to the first mode occurs:

$$\langle u_i^2 \rangle = \frac{1}{N} - \frac{T}{T_c}, \quad (36)$$

where

$$T_c = F(0)^{-1} = \frac{d-4}{d}. \quad (37)$$

The detailed analysis of Eq. (34) leads to

$$\mu = \begin{cases} (T - T_c)^{\frac{1}{d}} & d \in (4, 6) \\ (T - T_c)^{\frac{1}{d-4}} & d > 6, \end{cases} \quad (38)$$

$$\langle u_i^2 \rangle = \begin{cases} (T - T_c)^{-\frac{1}{d}} & d \in (4, 6) \\ (T - T_c)^{-2} & d > 6, \end{cases} \quad (39)$$

implying that the upper critical dimension is $d_u = 6$, see Appendix [A]. The above results are consistent with the spherical model with random field [36]. This would be a reasonable result because in the large $\tau_p$ limit, the self-propulsion force is permanently frozen and may be identified with a random field, see also Appendix [B].

V. CORRELATED NOISE

The analysis in the previous sections revealed that the phase transition does not occur for $d \leq d_u = 2$ when $\tau_p < \infty$ and for $d \leq d_u = 4$ when $\tau_p = \infty$. However, there is some numerical evidence of the phase transitions in $d = 2$ [2, 3]. A possible ingredient of the phase transition in low $d$ is a long-range correlation of $f_i$, which originates from the hydrodynamic interaction [37], elastic interaction [38], or something else [39, 40]. So, here we briefly discuss the effects of the correlated noise. For this purpose, we modify Eq. (25) as follows:

$$\langle f_i(t) f_j(t') \rangle = \delta_{ij} \frac{T_i}{\tau_p} e^{-|t-t'|/\tau_p}, \quad (40)$$

where $T_i$ depends on the frequency $\omega_i$ (or equivalently wave number $q_i = c^{-1} \omega_i$). The long-range correlation would suppress the fluctuation of $f_i$ on a large scale, i.e., small $q_i = c^{-1} \omega_i$. To express this effect, we assume a power-law function:

$$T_i = T \omega_i^{p_i}. \quad (41)$$

Now the self-consistent equation for $\mu$ Eq. (29) is modified as

$$1 = TF(\mu), \quad F(\mu) = \int_0^1 d\omega \frac{\omega^{d+p-1}}{(\mu + \omega^2)^2 [1 + \tau_p (\mu + \omega^2)]}. \quad (42)$$

From the above expression, one can see that the correlated noise Eq. (41) effectively increases the spatial dimension from $d$ to $d+p$. Therefore, the upper and lower critical dimensions are

$$d_l = \begin{cases} 2 - p & (\tau_p < \infty) \\ 4 & (\tau_p = \infty) \end{cases}, \quad (43)$$

$$d_u = \begin{cases} 4 - p & (\tau_p < \infty) \\ 6 & (\tau_p = \infty) \end{cases}. \quad (44)$$
Therefore, the \( \omega_i \) (or \( q_i \)) dependence of the noise Eq. (41) indeed can cause the phase transition below the lower critical dimension. However in reality, the noise should be determined self-consistently as a consequence of the complex interaction between the system and environment [41]. Further studies for this problem would be beneficial.

VI. SUMMARY AND DISCUSSIONS

In summary, we investigated the effects of the active noise on the spherical model. For this purpose, we first introduced and analyzed the spherical Debye model, which is a simplified version of the spherical model for a ferromagnet. The model is so simple that one can discuss the phase behavior solely from the information of the second-order moment at the steady state, even without complete knowledge of the steady state distribution. The simplicity also allows us to introduce an active noise: the noise produced by the Ornstein–Uhlenbeck process with the persistent time \( \tau_p \). We found that for a finite value of \( \tau_p \), the model has the same universality as the original model, while in the limit \( \tau_p \rightarrow \infty \), the universality reduces to that of the random field Ising model.

In this work, we only investigated the static properties of the model. Of course, what is more interesting is the dynamic behaviors such as aging dynamics [22, 42], effective temperature [21, 24], entropy production [43, 44], and so on. The current model may allow us to derive a full dynamical solution as done for \( p \)-spin spherical models [21, 22]. Further studies on this problem would be beneficial.

ACKNOWLEDGMENTS

This project has received JSPS KAKENHI Grant Numbers 21K20355.

Appendix A: Critical exponent

To determine \( \mu \), one should solve the following self-consistent equation:

\[
1 = T F(\mu) \equiv T A \int_0^{\omega_{\mu}^d-1} d\omega \frac{\omega_{\mu}^{d-1}}{(\omega_{\mu}^2 + \mu)^n}.
\]  

(A1)

where \( A \) is a constant, \( n = 1 \) for \( \tau_p < \infty \) and \( n = 2 \) for \( \tau_p = \infty \). If \( d - 2n > 0 \), \( F(0) \) is finite, implying that there exists the condensation transition. So, the lower critical dimension is

\[
d_u = 2n.
\]

(A2)

When \( d > 2n + 2 \), \( F(\mu) \) can be expanded as

\[
\frac{1}{T} = F(0) + \mu F'(0) + \cdots
\]

\[
= \frac{1}{T_c} + \mu F'(0) + \cdots,
\]  

(A3)

leading to

\[
\mu \sim (T - T_c)^{1 - d}.
\]  

(A4)

On the contrary, if \( d \in (2n, 2n + 2) \), \( F'(\mu) \) for small \( \mu \) behaves as

\[
F'(\mu) \sim \mu^{\frac{d-2n}{2}},
\]  

(A5)

implying

\[
F(\mu) - F(0) = \int_0^\mu d\mu' F'(\mu') \sim \mu^{\frac{d-2n}{2}},
\]  

(A6)

leading to

\[
\frac{1}{T} = F(\mu) = \frac{1}{T_c} - B \mu^{\frac{d-2n}{2}},
\]

(A7)

where \( B \) is a constant. Therefore, the scaling of \( \mu \) for \( \mu \ll 1 \) is

\[
\mu \sim (T - T_c)^{\frac{1}{n}}.
\]  

(A8)

A logarithmic correction may appear when \( d = 2n + 2 \) [31]. The above results imply that the upper critical dimension is

\[
d_u = 2n + 2.
\]

(A9)

Appendix B: Spherical model with random field

We here consider the spherical model with a random field [30, 45]:

\[
V_N = \sum_{i=1}^N \lambda_i + \frac{\mu}{2} u_i^2 - \frac{N \mu}{2} + \sum_{i=1}^N h_i u_i,
\]

(B1)

where \( h_i \) is an i.i.d random variable of zero mean and variance \( \Delta \). In equilibrium at temperature \( T \), we get [45]

\[
\langle u_i^2 \rangle = \frac{T}{\omega_i^2 + \mu} + \frac{\Delta}{(\omega_i^2 + \mu)^2},
\]

(B2)

where the overline denotes the average for \( h_i \), and \( \Delta = \overline{h_i^2} \). The Lagrange multiplier \( \mu \) is to be determined by the spherical constraint:

\[
1 = \frac{1}{N} \sum_{i=1}^N \left[ \frac{T}{\omega_i^2 + \mu} + \frac{\Delta}{(\omega_i^2 + \mu)^2} \right].
\]

(B3)
As mentioned in the main text, the scaling for $\mu \ll 1$ is determined by the small $\omega_i$ behavior of $\langle u_i^2 \rangle$, where the second term in Eq. (B2) gives a dominant contribution:

$$\langle u_i^2 \rangle \approx \frac{\Delta}{(\omega_i^2 + \mu^2)^2}. \quad \text{(B4)}$$

This agrees with the active spherical model with $\tau_p \to \infty$, Eq. (33). Therefore, the active spherical model with $\tau_p \to \infty$ can be identified with the spherical model with random field of variance $\Delta = \tilde{T}$.

---

[1] M. E. Cates and J. Tailleur, Annu. Rev. Condens. Matter Phys. 6, 219 (2015).
[2] B. Partridge and C. F. Lee, Phys. Rev. Lett. 123, 068002 (2019).
[3] C. Maggi, M. Paoluzzi, A. Crisanti, E. Zaccarelli, and N. Gnan, Soft Matter 17, 3807 (2021).
[4] J. T. Siebert, F. Dittrich, F. Schmid, K. Binder, T. Speck, and P. Virnau, Phys. Rev. E 98, 030601 (2018).
[5] F. Caballero, C. Nardini, and M. E. Cates, Journal of Statistical Mechanics: Theory and Experiment 2018, 123208 (2018).
[6] M. E. Cates, arXiv preprint arXiv:1904.01330 (2019).
[7] C. Maggi, N. Gnan, M. Paoluzzi, E. Zaccarelli, and A. Crisanti, Communications Physics 5, 1 (2022).
[8] J. A. Acebrón, L. L. Bonilla, C. J. P. Vicente, F. Ritort, and R. Spigler, Reviews of modern physics 77, 137 (2005).
[9] B. Derrida, Physics Reports 301, 65 (1998).
[10] M. R. Evans and T. Hanney, Journal of Physics A: Mathematical and General 38, R195 (2005).
[11] J. Tailleur and M. E. Cates, Phys. Rev. Lett. 100, 218103 (2008).
[12] T. Arnoux de Pirey, G. Lozano, and F. van Wijland, Phys. Rev. Lett. 123, 260602 (2019).
[13] T. H. Berlin and M. Kac, Physical Review 86, 821 (1952).
[14] J. Gunton and M. Buckingham, Physical Review 166, 152 (1968).
[15] J. M. Kosterlitz, D. J. Thouless, and R. C. Jones, Phys. Rev. Lett. 36, 1217 (1976).
[16] A. Crisanti and H.-J. Sommers, Zeitschrift für Physik B Condensed Matter 87, 341 (1992).
[17] T. Vojta, Phys. Rev. B 53, 710 (1996).
[18] S. Franz and G. Parisi, Journal of Physics A: Mathematical and Theoretical 49, 145001 (2016).
[19] S. Franz, G. Parisi, M. Sevelev, P. Urbani, and F. Zamponi, SciPost Phys. 2, 019 (2017).
[20] H. Casasola, C. A. Herniaski, P. R. S. Gomes, and P. F. Bienzobaz, Phys. Rev. E 104, 034131 (2021).
[21] L. F. Cugliandolo and J. Kurchan, Physical Review Letters 71, 173 (1993).
[22] L. F. Cugliandolo and D. S. Dean, Journal of Physics A: Mathematical and General 28, 4213 (1995).
[23] M. Henkel, H. Hinrichsen, S. Lübeck, and M. Pleimling, Non-equilibrium phase transitions, Vol. 1 (Springer, 2008).
[24] L. Berthier and J. Kurchan, Nature Physics 9, 310 (2013).
[25] D. Barbier, L. F. Cugliandolo, G. S. Lozano, and N. Nessi, arXiv preprint arXiv:2204.03081 (2022).
[26] B. ten Hagen, S. van Teeffelen, and H. Löwen, Journal of Physics: Condensed Matter 23, 194119 (2011).
[27] G. Szamel, Phys. Rev. E 90, 012111 (2014).
[28] C. Kittel and P. McEuen, Introduction to solid state physics (John Wiley & Sons, 2018).
[29] W. Greiner, L. Neise, and H. Stöcker, Thermodynamics and statistical mechanics (Springer Science & Business Media, 2012).
[30] A. Crisanti, A. Sarracino, and M. Zannetti, Phys. Rev. Research 1, 023022 (2019).
[31] H. Nishimori and G. Ortiz, Elements of phase transitions and critical phenomena (Oup Oxford, 2010).
[32] R. Zwanzig, Nonequilibrium statistical mechanics (Oxford university press, 2001).
[33] R. Mandal, P. J. Bhuyan, P. Chaudhuri, C. Dasgupta, and M. Rao, Nature communications 11, 1 (2020).
[34] L. Caprini, U. M. B. Marconi, C. Maggi, M. Paoluzzi, and A. Puglisi, Phys. Rev. Research 2, 023231 (2020).
[35] P. K. Morse, S. Roy, E. Agoritsas, E. Stanifer, E. I. Corwin, and M. L. Manning, Proceedings of the National Academy of Sciences 118, e2019909118 (2021).
[36] T. Vojta and M. Schreiber, Phys. Rev. B 53, 8211 (1996).
[37] L. D. Landau and E. M. Lifshitz, Fluid Mechanics: Landau and Lifshitz: Course of Theoretical Physics, Volume 6, Vol. 6 (Elsevier, 2013).
[38] L. D. Landau, E. M. Lifshitz, E. M. Lifshitz, A. M. Kosevich, and L. P. Pitaevskii, Theory of elasticity: volume 7, Vol. 7 (Elsevier, 1986).
[39] G. Szamel and E. Fleuren, Europhysics Letters 133, 60002 (2021).
[40] Y. Kuroda, H. Matsuyama, T. Kawasaki, and K. Miyazaki, arXiv preprint arXiv:2202.04436 (2022).
[41] F. Sagués, J. M. Sancho, and J. García-Ojalvo, Rev. Mod. Phys. 79, 829 (2007).
[42] M. Henkel, arXiv preprint arXiv:2201.06448 (2022).
[43] F. Caballero and M. E. Cates, Phys. Rev. Lett. 124, 240604 (2020).
[44] M. Paoluzzi, Phys. Rev. E 105, 044139 (2022).
[45] H. Ikeda, arXiv preprint arXiv:2208.08162 (2022).