String Theory and ALE Instantons

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ABSTRACT

We show that the classical equations of motion of the low-energy effective field theory describing the massless modes of the heterotic (or type I) string admit two classes of supersymmetric self–dual backgrounds. The first class, which was already considered in the literature, consists of solutions with a (conformally) flat metric coupled to axionic instantons. The second includes Asymptotically Locally Euclidean (ALE) gravitational instantonic backgrounds coupled to gauge instantons through the so–called “standard embedding”.

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Asymptotically Locally Euclidean (ALE) Instantons have played a fundamental role in Euclidean Quantum (Super)-Gravity [1, 2]. Furthermore, it is hoped that they may be responsible for the formation of a gravitino condensate which may trigger dynamical supersymmetry (SUSY) breaking [4, 5, 6].

In this talk, after reviewing the role of fermionic condensates in the problem of SUSY breaking, we will discuss how to promote gravitational ALE instantons to full-fledged solutions of the heterotic string equations of motion and we will study the geometrical properties of the simplest among them, an Eguchi-Hanson instanton coupled to a gauge instanton through the “standard embedding”. This investigation is propaedeutical to a saddle-point evaluation of instanton dominated Green functions in the low-energy effective supergravity arising from the heterotic string compactified to D=4 [7].

Let us first turn to a brief discussion of the role of fermionic condensates in SUSY breaking. As is well-known, a globally SUSY theory has SUSY vacua with zero energy. The Hamiltonian of the theory can in fact be written as a quadratic form of the SUSY charges,  \( Q \). A signal of SUSY breaking is thus provided by a non zero vacuum expectation value (v.e.v.)

\[
\langle 0 | \delta X | 0 \rangle = \langle 0 | \{ Q, X \} | 0 \rangle \neq 0
\]

that can be interpreted as the charge \( Q \) not annihilating the vacuum.

Let us introduce chiral and vector superfields (in the Wess-Zumino gauge)

\[
\begin{align*}
\Phi & = z + \sqrt{2} \theta \chi + \theta^2 F \\
V & = -\theta \sigma_\mu \bar{\theta} A^\mu + i \theta^2 \bar{\theta} \lambda - i \bar{\theta}^2 \theta \lambda + \frac{1}{2} \theta^2 \bar{\theta}^2 D
\end{align*}
\]

Under a SUSY transformation with parameter \( \epsilon \), the fermions transform as

\[
\begin{align*}
\delta \chi & = \sqrt{2} \epsilon F + i \sqrt{2} \epsilon \sigma^\mu \partial_\mu z \\
\delta \lambda & = \epsilon \sigma^{\mu \nu} F_{\mu \nu} + \epsilon D
\end{align*}
\]
Using (3) we find

\[ \langle 0 | \delta \psi | 0 \rangle \approx \langle 0 | \{ Q, \psi \} | 0 \rangle, \quad \langle 0 | \delta \lambda | 0 \rangle \approx \langle 0 | \{ Q, \lambda \} | 0 \rangle \]  

(4)
as the v.e.v’s of all the other fields (and of their derivatives) contained in (3) are supposed to vanish. SUSY is thus broken if and only if the auxiliary fields get a non zero v.e.v. The fermionic partner of the auxiliary field that develops a v.e.v. is the massless Goldstone fermion (goldstino) of broken SUSY. The scalar potential of the theory is

\[ V = FF^* + \frac{1}{2} D^2 \]

and a non-vanishing v.e.v for the auxiliary fields implies a vacuum energy different from zero in agreement with the form of the Hamiltonian of a SUSY theory.

For a local SUSY theory the matter is a little more complicated. The auxiliary fields which are the partners of spin \( \frac{1}{2} \) chiral and gauge fermions are

\[ F_i = e^{-\frac{G}{2}} (G^{-1})^j_i G_j + \frac{1}{4} f_{abk} (G^{-1})^k_i \lambda^a \lambda^b \\
- (G^{-1})^k_i G^j_k \chi_j \chi_l - \frac{1}{2} \chi_i (G_j \chi^j), \]

\[ D_a = i \Re f^{-1}_{ab} (-q G^i_T z^j_i + \frac{1}{2} i f^{i}_{bc} \lambda^i \lambda^c) \\
- \frac{1}{2} i f^{i}_{bc} \lambda^i \lambda^c) - \frac{1}{2} \lambda_a (G^i \chi_i) \]  

(5)

In the above formula \((z_i, \chi_i)\) are the components of a chiral matter multiplet and \(i\) is a gauge group index, \(T^\alpha_j\) is the group generator and \(q\) the gauge coupling constant. \(f^{i}_{\alpha \beta} = \frac{\partial f^{i}_{\alpha \beta}}{\partial z^i}, G^i = \frac{\partial G}{\partial z^i}\) where \(f\) is a coupling function depending on the chiral field \(z\) and \(G\) is the Kähler potential. From (5) we see that the auxiliary fields develop a v.e.v. if the fermion bilinears do. This should convince the reader of the importance of the study of fermionic correlators in order to see whether or not they can develop a v.e.v. when non-perturbative instanton effects are taken into account.
In the case of globally SUSY theories a detailed study of a possible breaking mechanism of SUSY via gauge instantons was carried out in [3] with rather encouraging results. The case of locally supersymmetric theories was addressed to in [6, 4], where it was argued that, in supergravity, gravitino condensation due to gravitational instantons can trigger the breaking of SUSY. An important step forward in this line of arguments was made in [5], where, in the case of $N = 1$ supergravity, the gravitino field strength condensate, $< \psi_{ab} \psi^{ab} >$, was indeed computed and shown to be finite and space–time independent. The calculation was done by performing a saddle–point approximation around the non trivial classical solution of the theory represented by the Eguchi–Hanson gravitational instanton [2].

The idea behind this kind of approach is the hope of being able to infer SUSY breaking by putting non trivial condensates in relation with some anomalous SUSY transformation. In globally SUSY theories in flat space [3] such a relation exists and it is called the Konishi anomaly equation [8]. For a Super Yang-Mills theory coupled to chiral scalar matter multiplets, $\Phi^i$, belonging to the representations $R^i$ of the gauge group, the relevant anomalous commutator reads

$$\{ \tilde{Q}, \bar{\chi}_i z^j \} = \frac{q^2 c_i}{32\pi^2} \lambda \lambda \delta_i^j + z^j \frac{dW}{dz_i}$$

(6)

where $i, j$ are flavour indices, $W$ is the superpotential, $\tilde{Q}$ is a SUSY charge, $q$ the gauge coupling constant, $\lambda$ is the gaugino, $z^i$ and $\chi^i$ are components of the chiral multiplet, $\Phi^i$, and $c_i$ is the index of the representation $R^i$. The anomaly equation (6) lies in the same supermultiplet as the chiral anomaly, i.e. it is a partner of the anomalous divergence equation of the $R$-symmetry current of the theory. The discovery that (some of) the condensates appearing in (6) acquire a non-zero vacuum expectation value allowed to derive detailed information on the degeneracy pattern of the vacuum state manifold and, in some cases (SUSY theories with no flat directions and chiral matter in suitably choosen representations of the gauge group) even to conclude that SUSY is dynamically broken by non–perturbative instanton effects.
To construct a similar argument in supergravity one has to start with the anomalous divergence of the $R$-symmetry chiral current, $j^\mu$

$$D_\mu j^\mu = -\frac{1}{384\pi^2} R_{abcd} \tilde{R}^{abcd}$$

where the tilde stands for the duality operation. Since $R_{abcd} \tilde{R}^{abcd}$ is the top component of a chiral multiplet, which has $\psi_{ab}\psi^{ab}$ as lowest component [9], the analogue of the anomalous SUSY transformation (6) is

$$\{\bar{Q}, \bar{\lambda}\phi\} = \frac{\kappa^2}{384\pi^2} \psi_{ab}\psi^{ab}$$

where $\kappa$ is the gravitational coupling constant, $\psi^{ab}$ is the gravitino field–strength and $\lambda$ and $\phi$ are components of a chiral matter multiplet (for the sake of definiteness they may be respectively taken to be the dilatino and the dilaton). The appearance of the gravitino field–strength bilinear in the right–hand–side of (8) led correctly the authors of [5] to compute the expectation value of $\psi_{ab}\psi^{ab}$, instead of $\psi_\mu\psi^\mu$, as suggested in [6].

Many of the computations we will present in this paper are very much in the same line of thought of instanton calculus in globally SUSY theories [3], where, as we said, the simultaneous presence of globally SUSY and of classical instantonic solutions conspire to give exact space–time independent constant results for certain correlators. The calculations we present here are propaedeutical to extend instanton calculus to locally supersymmetric (i.e. to supergravity) theories [10, 7].

Contrary to globally supersymmetric Yang-Mills theories, however, supergravity is not renormalizable. Strictly speaking, this puts the entire subject of instanton calculus in supergravity on a rather shaky basis. On this problem we would like to take the point of view (as also suggested in [5]) that supergravity theories should be considered as low energy limits of string theories, which are expected not to suffer from these deficiencies. Thus to the order to which supergravity theories are formally renormalizable (i.e. generically up to two loops) results from perturbative
and non–perturbative (instanton) calculations should be considered as the limiting values of the corresponding exact string results.

There are two more reasons to pursue this philosophy we would like to mention here. The first has to do with the fact that effective field theories appear at the moment the only arena where non–perturbative aspects of string theories can be studied. The second is that, exploiting the relation between bosonic zero–modes and instanton “collective coordinates”, explicit computations of the former may prove to be a useful starting point in the investigation of the structure of the instanton moduli space over non-compact ALE manifolds. Except for the purely gravitational sector, this subject is as yet poorly understood.

We now give a brief outline of our results, referring the interest reader to our original paper [7] for more details.

Non trivial supersymmetric solutions of the lowest order (in $\alpha'$) equations of motion may be found setting to zero the fermion fields together with their supersymmetric variations. A supersymmetric ansatz for the solution is given by [11, 12, 13]:

\[
F_{\mu\nu} = \tilde{F}_{\mu\nu} \quad H_{\mu\nu\rho} = \sqrt{G} \varepsilon_{\mu\nu\rho\sigma} \partial_{\sigma} \phi \quad G_{\mu\nu} = e^{2\phi} \hat{g}_{\mu\nu}
\]

(9)

where $F_{\mu\nu}$ is the field-strength of the gauge fields $A_{\mu}$, $H_{\mu\nu\rho}$ is the (modified) field-strength of the antisymmetric tensor $B_{\mu\nu}$ and $\hat{g}_{\mu\nu}$ is a self-dual metric. The generalized spin-connection with torsion $\Omega_{\mu\pm}^{ab} = \omega_{\mu}^{ab} \pm H_{\mu}^{ab}$ deriving from (9) is self-dual. (9) must be supplemented with the Bianchi identity: $dH = \alpha' \{ tr R(\Omega_-) \wedge R(\Omega_-) - tr F(A) \wedge F(A) \}$. To simplify matters it is convenient to impose the “standard embedding” of the gauge connection in the $SU(2)$ spin group: $A = \Omega_-$. There are two options. The first has been considered in [11, 12, 11] and leads to conformally flat axionic instantons. We would like to concentrate on the other, i.e. $A_{\mu}^i = \frac{1}{2} \eta_{ab} \omega_{\mu}^{ab} = \frac{1}{2} \eta_{ab} \hat{\omega}_{\mu}^{ab}$, whose consistency requires a constant dilaton and a vanishing torsion [7]. In this case, (9) is completely specified by the choice of a self-dual metric $\hat{g}_{\mu\nu}$.
An interesting class of self-dual metrics is given by the Gibbons-Hawking multi-center (GHMC) ansatz [see, e.g. 2]:

$$ds^2 = V^{-1}(\bar{x})(d\tau + \bar{\omega} \cdot d\bar{x})^2 + V(\bar{x})d\bar{x} \cdot d\bar{x}$$  \hspace{1cm} (10)$$

with $\bar{\nabla}V = \bar{\nabla} \times \bar{\omega}$ and $V(\bar{x}) = \varepsilon^{\alpha\beta} + 2m \sum_{i=1}^{k+1} \frac{1}{|x-x_i|}$. The choice $\varepsilon^{\alpha\beta} = 0, m = \frac{1}{2}$ corresponds to ALE metrics. ALE manifolds are smooth resolutions of singular varieties in $\mathbb{C}^3$ and are completely classified in terms of the kleinian subgroups $\Gamma$ of $SU(2)$ [see, e.g. 2]. ALE instantons are non-compact Ricci-flat hyperkähler manifolds of $SU(2)/\Gamma$ holonomy and deserve to be considered as non-compact Calabi-Yau manifold of complex dimension two [14].

The non-linear $\sigma$-model describing the propagation of the heterotic string on ALE instantons with the standard embedding is left-right symmetric and admits $N = (4,4)$ supersymmetry [15]. The corresponding $\beta$-functions vanish to all orders [15] thus the lowest order ALE solutions receive no radiative corrections in $\alpha'$. Indeed, at the singular point of the moduli space where ALE instantons coincide with algebraic varieties, string propagation is governed by a $\mathbb{C}^2/\Gamma$ orbifold conformal field theory [14].

We now turn to describe some geometrical properties of the heterotic solution based on the EH (Eguchi-Hanson) instanton [see, e.g. 2]:

$$ds^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu = (\frac{r}{u})^2 dr^2 + r^2(\sigma_x^2 + \sigma_y^2) + u^2 \sigma_z^2$$  \hspace{1cm} (11)$$

To remove the bolt singularity at $r = a$, we change the radial variable to $u = r \sqrt{1 - (\frac{a}{r})^4}$ and identify antipodal points. The EH instanton has an $S^3/Z_2$ boundary and admits an $SU(2)_R \otimes U(1)_L$ isometry group. Its Euler characteristic $\chi$ is two and its Hirzebruch signature $\tau = b_2^+ - b_2^-$ is one. Exploiting the global right-handed supersymmetry generated by the covariantly constant spinor $\bar{\epsilon}$, the two (left-handed) gravitino zero-modes on EH may be expressed in terms of the closed self-dual two-form [6, 5]. These zero-modes are generated by the broken global left-handed supersymmetry with parameter $\eta = u\eta_0 \ (\eta_0 = i\sigma_0 \bar{\epsilon})$: $\psi_\mu = D_\mu \eta - \frac{1}{4}\sigma_\mu \bar{\psi} \eta$.
The three zero-modes of the metric can be obtained performing a further supersymmetry transformation:

\[ h_{\mu \nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu - \frac{1}{2} (\nabla : \xi) g_{\mu \nu} \]

with \( \xi_\mu = \eta \sigma_\mu \bar{\epsilon} \).

A suitable interpretation of the infinitesimal diffeomorphisms \( \xi^\mu \) allows to relate them to the lack of invariance of the EH metric under dilatation and the two rotations in the coset \( SU(2)_L/U(1)_L \). No dilatino zero-modes are expected on EH since the index of the Dirac operator (for gauge singlets) is zero [see, e.g., 2]. However, the presence of “charged” spinor zero-modes is guaranteed by a non-vanishing value of the index of the Dirac operator coupled to the gauge bundle \( V \). Embedding the instantonic \( SU(2) \) in the “hidden” \( E(8) \) only the gauginos will be affected. With respect to the subgroup, \( SU(2) \otimes E(7) \), the adjoint of \( E(8) \) breaks as: 248 = (3, 1) \( \oplus \) (2, 56) \( \oplus \) (1, 133). The general formula for the index of the Dirac operator, \( \bar{\mathcal{D}}_V \), coupled to a vector bundle \( V \) may be found in [see, e.g., 2].

Performing the necessary manipulations, we find:

\[ \text{ind}(\bar{\mathcal{D}}_{\mathcal{D}_2}, M, \partial M) = -1 = \tau \]

and

\[ \text{ind}(\bar{\mathcal{D}}_{\mathcal{D}_3}, M, \partial M) = -6. \]

Since \( \bar{\mathcal{D}}_V \) has no right-handed zero-modes, one (normalizable) left-handed gaugino zero-mode is expected for each of the 56 doublets and six for the triplet. The explicit expression of the gaugino zero-modes in the doublet can be easily found by exploiting the explicit form of the normalizable harmonic self-dual two-form on the EH manifold. The six triplet zero-modes may be found by explicitly solving the Dirac equation, but we haven’t been able to provide a geometrical interpretation [7]. Thanks to the global right-handed supersymmetry, each zero-mode of the gaugino generates two zero-modes for the gauge fields.

As is well known the bosonic zero-modes are related to the collective coordinate of the heterotic EH background. Thanks to the cancellation of the non-zero-mode functional determinants, the evaluation of the relevant correlation functions reduces to an integration of zero-modes of fermi fields over the finite-dimensional moduli space of the heterotic EH instanton. In [7] the saddle-point approximation is performed and the consistency of the result with Ward identities of a properly defined global supersymmetry is checked. The interpretation in terms of condensates as well as the relation to topological amplitudes in string theory and to topological field theories [16] is under investigation [7].
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