RAMANUJAN’S CUBIC TRANSFORMATION INEQUALITIES FOR ZERO-BALANCED HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. In this paper, a generalization of Ramanujan’s cubic transformation, in the form of an inequality, is proved for zero-balanced Gaussian hypergeometric function \( F(a, b; a + b; x) \), \( a, b > 0 \).

1. Introduction. For real numbers \( a, b \) and \( c \) with \( c \neq 0, -1, -2, \ldots \), the Gaussian hypergeometric function is defined by

\[
F(a, b; c; x) = 2 F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{x^n}{n!},
\]

for \( x \in (-1, 1) \), where \((a, n)\) denotes the shifted factorial function \((a, n) = a(a+1)(a+2)(a+3) \cdots (a+n-1)\) for \( n = 1, 2, \ldots \), and \((a, 0) = 1\) for \( a \neq 0 \). Also, \( F(a, b; c; x) \) is called zero-balanced if \( c = a + b \).

It is well known that \( F(a, b; c; x) \) has many important applications in various fields of the mathematical and natural sciences [4, 7], and many classes of special function in mathematical physics are particular cases of this function [8]. For an extensive list of \( F(a, b; c; x) \), see [1, 2, 3, 9].

As a special case of the Gaussian hypergeometric function, for \( r \in (0, 1) \), Legendre’s complete elliptic integrals of the first kind are defined by

\[
K(r) = \int_0^{\pi/2} (1 - r^2 \sin^2 \theta)^{-1/2} d\theta = \frac{\pi}{2} F\left( \frac{1}{2}; \frac{1}{2}; 1; r^2 \right).
\]
Some of the most important properties of the elliptic integrals $K(r)$ are the Landen identities:

$$K\left(\frac{2\sqrt{r}}{1+r}\right) = (1 + r)K(r),$$
$$K\left(\frac{1-r}{1+r}\right) = \frac{1+r}{2}K(\sqrt{1-r^2}),$$

namely,

$$F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{4r}{(1+r)^2}\right) = (1 + r)F\left(\frac{1}{2}, \frac{1}{2}; 1; r^2\right), \quad (1.2)$$
$$F\left(\frac{1}{2}, \frac{1}{2}; 1; \left(\frac{1-r}{1+r}\right)^2\right) = \frac{1+r}{2}F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - r^2\right). \quad (1.3)$$

For zero-balanced Gaussian hypergeometric functions $F(a, b; a + b; x)$, $a, b > 0$, Simić and Vuorinen [10] determined the maximal region of the $ab$ plane where equations (1.2) and (1.3) turn on respective inequalities valid for each $x \in (0, 1)$.

As is known to all, Ramanujan’s cubic transformation is defined as

$$F\left(\frac{1}{3}, \frac{2}{3}; 1; \left(1 - \left(\frac{1-r}{1+2r}\right)^3\right)\right) = (1 + 2r)F\left(\frac{1}{3}, \frac{2}{3}; 1; r^3\right), \quad (1.4)$$
$$F\left(\frac{1}{3}, \frac{2}{3}; 1; \left(\frac{1-r}{1+2r}\right)^3\right) = \frac{1+2r}{3}F\left(\frac{1}{3}, \frac{2}{3}; 1; 1 - r^3\right). \quad (1.5)$$

Inspired by the ideas of Simić and Vuorinen [10], we find the maximal region of the $ab$ plane for $F(a, b; a + b; x)$, $a, b > 0$, where equations (1.4) and (1.5) turn on respective inequalities valid for each $x \in (0, 1)$.

The following asymptotic formulas for the zero-balanced hypergeometric function (see [5, 6]) will be used in this paper.

$$F(a, b; a + b; r) \sim -\frac{1}{B(a, b)} \log(1 - r) \quad (1.6)$$

and

$$B(a, b)F(a, b; a + b; r) + \log(1 - r) = R(a, b) + O((1 - r) \log(1 - r)), \quad (1.7)$$
as $r$ tends to 1, where

\begin{equation}
B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z + w)}, \quad \text{Re } z > 0, \text{ Re } w > 0
\end{equation}

is the classical beta function,

\begin{equation}
R(a, b) = -\Psi(a) - \Psi(b) - 2\gamma, \quad R\left(\frac{1}{3}, \frac{2}{3}\right) = \log 27,
\end{equation}

\begin{equation}
\Psi(z) = \frac{d}{dz}(\log \Gamma(z)) = \frac{\Gamma'(z)}{\Gamma(z)}, \quad \text{Re } z > 0,
\end{equation}

and $\gamma$ is the Euler-Mascheroni constant.

**Lemma 1.1.** (see [10, Lemma 1.1]). Suppose that the power series

\[ f(x) = \sum_{n=0}^{\infty} a_n x^n \]

and

\[ g(x) = \sum_{n=0}^{\infty} b_n x^n \]

have the radius of convergence $r > 0$ and $a_n, b_n > 0$ for all $n \in \{0, 1, 2, \ldots\}$. Let $h(x) = f(x)/g(x)$. Then

(i) if the sequence $\{a_n/b_n\}_{n=0}^{\infty}$ is (strictly) increasing (decreasing), then $h(x)$ is also (strictly) increasing (decreasing) on $(0, r)$;

(ii) if the sequence $\{a_n/b_n\}$ is (strictly) increasing (decreasing) for $0 < n \leq n_0$ and (strictly) decreasing (increasing) for $n > n_0$, then there exists an $x_0 \in (0, r)$ such that $h(x)$ is (strictly) increasing (decreasing) on $(0, x_0)$ and (strictly) decreasing (increasing) on $(x_0, r)$.

**2. Main results.** For convenience, we first introduce the following regions in $\{(a, b) \in \mathbb{R}^2 | a > 0, b > 0\}$ (see Figure 1):

\[ D_1 = \left\{ (a, b) | a, b > 0, ab \leq \frac{2}{9}, ab - \frac{2}{9}(a + b) \leq 0 \right\}, \]

\[ D_2 = \left\{ (a, b) | a, b > 0, ab < \frac{2}{9}, ab - \frac{2}{9}(a + b) > 0 \right\}, \]
D_3 = \{(a, b) | a, b > 0, ab \geq \frac{2}{9}, ab - \frac{2}{9} (a + b) \geq 0\},

D_4 = \{(a, b) | a, b > 0, ab > \frac{2}{9}, ab - \frac{2}{9} (a + b) < 0\},

D_5 = \{(a, b) | a, b > 0, a + b \leq 1, ab - \frac{2}{9} (a + b) \leq 0\},

D_6 = \{(a, b) | a, b > 0, a + b \geq 1, ab - \frac{2}{9} (a + b) \geq 0\}.

Clearly, D_1 \cup D_2 \cup D_3 \cup D_4 = \{(a, b) \in \mathbb{R}^2 | a > 0, b > 0\}, D_5 \subset D_1 and D_6 \subset D_3.

**Theorem 2.1.** If \((a, b) \in D_1\), then the inequality

\[
F\left( a, b; \frac{9r(1 + r + r^2)}{(1 + 2r)^3} \right) \leq (1 + 2r) F(a, b; a + b; r^3)
\]
holds for all \( r \in (0, 1) \). Also, if \( (a, b) \in D_3 \), then the reversed inequality
\[
F(a, b; a + b; \frac{9r(1 + r + r^2)}{(1 + 2r)^3}) \geq (1 + 2r)F(a, b; a + b; r^3)
\]
takes place for each \( r \in (0, 1) \), with equality in each instance if and only if \( (a, b) = (1/3, 2/3) \) or \( (a, b) = (2/3, 1/3) \).

In the remaining region \( (a, b) \in D_2 \cup D_4 \), neither of the above inequalities holds for each \( r \in (0, 1) \).

**Theorem 2.2.** If \( (a, b) \in D_1 \), then the double inequality
\[
1 \leq \frac{(1 + 2r)F(a, b; a + b; r^3)}{F(a, b; a + b; 9r(1 + r + r^2)/(1 + 2r)^3)} \leq \frac{\sqrt{3}B(a, b)}{2\pi}
\]
holds for all \( r \in (0, 1) \). And, if \( (a, b) \in D_3 \), then inequality (2.3) is reversed,
\[
\frac{\sqrt{3}B(a, b)}{2\pi} \leq \frac{(1 + 2r)F(a, b; a + b; r^3)}{F(a, b; a + b; 9r(1 + r + r^2)/(1 + 2r)^3)} \leq 1.
\]
Moreover, both bounds in inequalities (2.3) and (2.4) are sharp and each equality is reached for \( a = 1/3 \) and \( b = 2/3 \), or \( a = 2/3 \) and \( b = 1/3 \).

**Corollary 2.3.** For \( r \in (0, 1) \), and \( (a, b) \in D_1 \), one has
\[
\frac{2\pi}{\sqrt{3}B(a, b)} \frac{1}{F(a, b; a + b; r^3)} \leq F\left(a, b; a + b; \frac{9r(1 + r + r^2)}{(1 + 2r)^3}\right) < F\left(a, b; a + b; \frac{9r(1 + r + r^2)}{(1 + 2r)^3}\right) < 3F(a, b; a + b; r^3).
\]
In the region \( (a, b) \in D_3 \), one has
\[
F(a, b; a + b; r^3) < F\left(a, b; a + b; \frac{9r(1 + r + r^2)}{(1 + 2r)^3}\right) < \frac{6\pi}{\sqrt{3}B(a, b)}F(a, b; a + b; r^3).
\]

**Theorem 2.4.** Let \( B = B(a, b) \) and \( R = R(a, b) \) be defined as in (1.8) and (1.9), respectively. Then for \( (a, b) \in D_5 \), inequality
\[
0 \leq (1 + 2r)F(a; b; a + b; r^3) - F(a; b; a + b; \frac{9r(1 + r + r^2)}{(1 + 2r)^3}) \leq \frac{2(R - \log 27)}{B}
\]
holds for all \( r \in (0, 1) \). Also, for \((a, b) \in D_6\),
\[
0 \leq F(a; b; a + b; \frac{9r(1 + r + r^2)}{(1 + 2r)^3}) - (1 + 2r)F(a; b; a + b; r^3) \leq \frac{2(\log 27 - R)}{B}.
\]

**Theorem 2.5.**

(i) For \((a, b) \in D_1\) and each \( x \in (0, 1) \), one has
\[
\frac{1}{3} \leq \frac{F(a, b; a + b; ((1 - x)/(1 + 2x))^3)}{(1 + 2x)F(a, b; a + b; 1 - x^3)} \leq \frac{\sqrt{3}B(a, b)}{6\pi}.
\]

(ii) For \((a, b) \in D_3\) and each \( x \in (0, 1) \), one has
\[
\frac{\sqrt{3}B(a, b)}{6\pi} \leq \frac{F(a, b; a + b; ((1 - x)/(1 + 2x))^3)}{(1 + 2x)F(a, b; a + b; 1 - x^3)} \leq \frac{1}{3}.
\]

(iii) For \((a, b) \in D_5\) and each \( x \in (0, 1) \), we have
\[
(1 + 2x)F(a, b; a + b; 1 - x^3) \leq 3F(a, b; a + b; \left(\frac{1 - x}{1 + 2x}\right)^3) \leq (1 + 2x) \left[F(a, b; a + b; 1 - x^3) + \frac{2(R(a, b) - \log 27)}{B(a, b)}\right].
\]

(iv) For \((a, b) \in D_6\) and each \( x \in (0, 1) \), we have
\[
0 \leq (1 + 2x)F(a, b; a + b; 1 - x^3) - 3F(a, b; a + b; \left(\frac{1 - x}{1 + 2x}\right)^3) \leq \frac{2(1 + 2x)(\log 27 - R(a, b))}{B(a, b)}.
\]
3. Proofs of theorems. In order to prove our main results, we introduce several symbols. Throughout this section, we let
\[ F(x) = F(a, b; a + b; x), \quad G(x) = F(a, b; a + b + 1; x), \]
where \(a, b > 0\) with \((a, b) \neq (1/3, 2/3)\) and \((a, b) \neq (2/3, 1/3)\), and
\[ F^*(x) = F\left(\frac{1}{3}, \frac{2}{3}; 1; x\right), \quad G^*(x) = F\left(\frac{1}{3}, \frac{2}{3}; 2; x\right). \]

Lemma 3.1.

(i) The function \(f(r) = F(r)/F^*(r)\) is strictly decreasing in \((0, 1)\) on \(D_1\), and strictly increasing in \((0, 1)\) on \(D_3\). Moreover, if \((a, b) \in D_2\) (\(D_3\), respectively), then there exists \(r_0\) (\(r_0^*\), respectively) such that \(f(r)\) is strictly increasing (decreasing, respectively) in \((0, r_0)\) ((0, \(r_0^*)\), respectively), and strictly decreasing (increasing, respectively) in \((r_0, 1)\) ((\(r_0^*, 1)\), respectively).

(ii) The function \(g(r) = G(r)/G^*(r)\) is strictly decreasing in \((0, 1)\) on \(D_5\) and strictly increasing in \((0, 1)\) on \(D_6\).

Proof. For part (i), denote by \(A_n = (a, n)(b, n)/[(a + b, n)!]\) and \(A_n^* = (1/3, n)(2/3, n)/[(n)!]^2\), then
\[
(3.1) \quad f(r) = \frac{F(r)}{F^*(r)} = \frac{\sum_{n=0}^{\infty} A_n r^n}{\sum_{n=0}^{\infty} A_n^* r^n}.
\]

Note that the monotonicity of \(\{A_n/A_n^*\}\) depends on the sign of
\[
(3.2) \quad H_n = \left(ab - \frac{2}{9}\right)n + ab - \frac{2}{9}(a + b).
\]

We divide the proof into four cases.

Case 1. \((a, b) \in D_1\). Then equation (3.2) implies \(H_n < 0\) for \(n = 0, 1, 2, \ldots\), and \(f(r)\) is strictly decreasing on \((0, 1)\) by equation (3.1) and Lemma 1.1.

Case 2. \((a, b) \in D_3\). Then equation (3.2) implies \(H_n > 0\) for \(n = 0, 1, 2, \ldots\), and \(f(r)\) is strictly increasing on \((0, 1)\) by equation (3.1) and Lemma 1.1.
Case 3. \((a, b) \in D_2\). Then from equation (3.2) we conclude that the sequence \(\{A_n/A^*_n\}\) is increasing and then decreasing. By equation (3.1) and Lemma 1.1 (ii), there exists \(r_0 \in (0, 1)\) such that \(f(r)\) is strictly increasing on \((0, r_0)\) and strictly decreasing on \((r_0, 1)\).

Case 4. \((a, b) \in D_4\). Then from equation (3.2) we know that the sequence \(\{A_n/A^*_n\}\) is decreasing and then increasing. By equation (3.1) and Lemma 1.1 (ii), there exists \(r^*_0 \in (0, 1)\) such that \(f(r)\) is strictly decreasing on \((0, r^*_0)\) and strictly increasing on \((r^*_0, 1)\).

For part (ii), denote \(B_n = (a, n)(b, n)/[(a + b + 1)n!]\) and \(B^*_n = (1/3, n)(2/3, n)/[(2, n)(n)!]\). Then

\[
(3.3) \quad g(r) = \frac{G(r)}{G^*(r)} = \frac{\sum_{n=0}^{\infty} B_n r^n}{\sum_{n=0}^{\infty} B^*_n r^n}.
\]

Note that the monotonicity of \(\{B_n/B^*_n\}\) depends on the sign of

\[
(3.4) \quad H^*_n = \left( a + b + ab - \frac{11}{9} \right) n + \frac{2}{9} (9ab - a - b - 1).
\]

We divide the proof into two cases.

Case A. \((a, b) \in D_5\). Then \(a + b + ab - 11/9 \leq 11(a + b)/9 - 11/9 \leq 0\) and \(9ab - a - b - 1 = 9ab - 2(a + b) + (a + b) - 1 \leq 0\). Thus, \(H^*_n < 0\) for \(n = 0, 1, 2, \ldots\) (because \((a, b) \neq (1/3, 2/3)\) and \((a, b) \neq (2/3, 1/3)\)) by equation (3.4). Therefore, \(g(r)\) is strictly decreasing in \((0, 1)\) by equation (3.3) and Lemma 1.1 (i).

Case B. \((a, b) \in D_6\). Then \(a + b + ab - 11/9 \geq 11(a + b)/9 - 11/9 \geq 0\) and \(9ab - a - b - 1 = 9ab - 2(a + b) + (a + b) - 1 \geq 0\). Thus, \(H^*_n > 0\) for \(n = 0, 1, 2, \ldots\) by equation (3.4). Therefore, \(g(r)\) is strictly increasing in \((0, 1)\) by equation (3.3) and Lemma 1.1 (i).

Proof of Theorem 2.1. Let \(x = x(r) = r^3\) and \(y = y(r) = 9r(1 + r + r^2)/(1 + 2r)^3\). Then simple computation leads to \(0 < x < y < 1\) for \(0 < r < 1\). Using Lemma 3.1 (i), we get \(f(x) > f(y)\) on \(D_1\), and \(f(x) < f(y)\) on \(D_3\).

For \((a, b) \in D_1\), by equation (1.4), one has

\[
\frac{F(r^3)}{F^*(r^3)} > \frac{F(y)}{F^*(y)},
\]
$$F(y) < \frac{F^*(y)}{F^*(r^3)} F(r^3) = (1 + 2r)F(r^3).$$

Thus, equation (2.1) follows.

Inequality (2.2) is obtained analogously. The remaining conclusions easily follow from Lemma 3.1 (i). □

**Proof of Theorem 2.2.** Let $f(r)$ be defined as in Lemma 3.1 (i), then $f(r)$ is strictly decreasing on $D_1$. Then asymptotic formula (1.6) leads to

$$1 = \lim_{r \to 0^+} \frac{F(r)}{F^*(r)} > \frac{F(r)}{F^*(r)} > \lim_{r \to 1^-} \frac{F(r)}{F^*(r)}$$

$$= \frac{B(1/3, 2/3)}{B(a, b)} = \frac{2\sqrt{3}\pi}{3B(a, b)}$$

and

$$\frac{\sqrt{3}B(a, b)}{2\pi} \frac{1}{F^*(y(r))} > \frac{1}{F(y(r))} \implies \frac{\sqrt{3}B(a, b) F^*(x(r))}{2\pi} > \frac{F(x(r))}{F(y(r))}.$$}

Thus, inequality (2.3) is clear.

Inequality (2.4) valid on $D_3$ can be proved similarly. □

**Lemma 3.2.** The function

$$J(r) = (1 + 2r^{1/3})F(a, b; a + b; r)$$

$$- F\left(a, b; a + b; \frac{9r^{1/3}(1 + r^{1/3} + r^{2/3})}{(1 + 2r^{1/3})^3}\right)$$

is strictly increasing in $(0, 1)$ on $D_5$ and strictly decreasing in $(0, 1)$ on $D_6$.

**Proof.** Let $z = 9r^{1/3}(1 + r^{1/3} + r^{2/3})/(1 + 2r^{1/3})^3$. Then

$$1 - z = \frac{(1 - r^{1/3})^3}{(1 + 2r^{1/3})^3}, \quad \frac{dz}{dr} = \frac{3(1 - r^{1/3})^2}{r^{2/3}(1 + 2r^{1/3})^4}.$$}

Note that

$$(1 - x)F(a + 1, b + 1; a + b + 1; x) = F(a, b; a + b + 1; x).$$
Differentiating $J(r)$ gives

\begin{equation}
(3.5)
r^{2/3}(1 - r^{1/3})J'(r) = \frac{2}{3} (1 - r^{1/3}) F(a, b; a + b; r) \\
+ \frac{ab}{a + b} \frac{r^{2/3}(1 + 2r^{1/3})(1 - r^{1/3})}{1 - r} \\
\times F(a, b; a + b + 1; r) - \frac{3ab}{(a + b)(1 + 2r^{1/3})} F(a, b; a + b + 1; z) \\
= \frac{2}{3} (1 - r^{1/3}) F(r) + \frac{ab}{a + b} \frac{r^{2/3}(1 + 2r^{1/3})(1 - r^{1/3})}{1 - r} G(r) \\
- \frac{3ab}{(a + b)(1 + 2r^{1/3})} G(z).
\end{equation}

On the other hand, differentiating the Ramanujan cubic transformation, we get

\begin{equation}
(3.6)
\frac{2}{3} \frac{G^*(z)}{1 + 2r^{1/3}} = \frac{2}{3} (1 - r^{1/3}) F^*(r) \\
+ \frac{2}{9} \frac{r^{2/3}(1 + 2r^{1/3})(1 - r^{1/3})}{1 - r} G^*(r).
\end{equation}

Let $g(r)$ be defined as in Lemma 3.1 (ii), then $g(r)$ is strictly decreasing in $(0, 1)$ on $D_5$. Then from $0 < r < z < 1$ we get $g(r) > g(z)$, namely,

\begin{equation}
(3.7)
G(z) < \frac{G^*(z)}{G^*(r)} G(r).
\end{equation}

Equations (3.5) and (3.6) together with inequality (3.7) yield

\begin{equation}
r^{2/3}(1 - r^{1/3})J'(r) \\
> \frac{2}{3} (1 - r^{1/3}) F(r) \\
+ \frac{ab}{a + b} \frac{r^{2/3}(1 + 2r^{1/3})(1 - r^{1/3})}{1 - r} G(r) - \frac{3ab}{(a + b)(1 + 2r^{1/3})} \frac{G^*(z)}{G^*(r)} G(r) \\
= \frac{2}{3} (1 - r^{1/3}) F(r) + \frac{ab}{a + b} \frac{r^{2/3}(1 + 2r^{1/3})(1 - r^{1/3})}{1 - r} G(r)
\end{equation}
\[- \frac{3ab}{(a+b)(1+2r^{1/3})} \times \left( (1-r^{1/3})(1+2r^{1/3}) \frac{F^*(r)}{G^*(r)} + \frac{1}{3} \frac{r^{2/3}(1+2r^{1/3})^2(1-r^{1/3})}{1-r} \right) G(r) \]

\[= \frac{2}{3}(1-r^{1/3})F(r) - \frac{3ab}{(a+b)(1-r^{1/3})} \frac{F^*(r)}{G^*(r)} G(r) \]

\[= \frac{2}{3}(1-r^{1/3}) \left[ F(r) - \frac{9ab}{2(a+b)} \frac{F^*(r)}{G^*(r)} G(r) \right]. \]

Note that

\[\frac{F'(r)}{F^*(r)} = \frac{9ab}{2(a+b)} \frac{G(r)}{G^*(r)}.\]

Thus,

\[\frac{3}{2} r^{2/3} J'(r) > F(r) - \frac{F'(r)}{F^*(r)} F^*(r) = \frac{F^2(r)}{F^*(r)} \left( \frac{F^*(r)}{F(r)} \right)' . \]

It follows from Lemma 3.1 (i) and $D_5 \subset D_1$ that $(F^*(r)/F(r))' \geq 0$ on $D_5$. Hence, $J'(r) > 0$ and $J(r)$ is strictly increasing in $(0, 1)$ on $D_5$.

Since $g(r)$ is strictly increasing in $(0, 1)$ on $D_6$, we have $g(z) > g(r)$, namely,

\[G(z) > \frac{G^*(z)}{G^*(r)} G(r).\]

Making use of a similar argument, one has

\[\frac{3}{2} r^{2/3} J'(r) < \frac{F^2(r)}{F^*(r)} \left( \frac{F^*(r)}{F(r)} \right)' < 0, \]

since $f(r) = F(r)/F^*(r)$ is strictly increasing in $(0, 1)$ on $D_6 \subset D_3$. Hence, $J(r)$ is strictly decreasing in $(0, 1)$ on $D_6$. \qed

**Proof of Theorem 2.4.** From Lemma 3.2, we clearly see that

\[\lim_{r \to 0^+} J(r) < J(r) < \lim_{r \to 1^-} J(r), \quad \text{on } D_5,\]

and

\[\lim_{r \to 1^-} J(r) < J(r) < \lim_{r \to 0^+} J(r), \quad \text{on } D_6.\]
Clearly, $\lim_{r \to 0^+} J(r) = 0$, and, by equation (1.7), we have

$$\lim_{r \to 1^-} J(r) = \lim_{r \to 1^-} \frac{3R(a,b)-3\log(1-r)-(R(a,b)-3\log((1-r^{1/3})/((1+2r^{1/3}))))+o(1)}{B(a,b)}$$

$$= \frac{2(R(a,b) - \log 27)}{B(a,b)}.$$

The assertion of Theorem 2.4 follows. $\square$

Proof of Theorem 2.5. Theorem 2.5 follows from Theorems 2.2 and 2.4 with $x = (1 - r)/(1 + 2r) \in (0, 1)$. $\square$

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