MINIMUM ENERGY COMPENSATION FOR DISCRETE DELAYED SYSTEMS WITH DISTURBANCES

SALMA SOUHAILE* AND LARBI AFIFI
Faculty of Sciences Ain Chock, University Hassan II
B.P.5366, Maarif, Casablanca, Morocco

ABSTRACT. This work is devoted to the remediability problem for a class of discrete delayed systems. We investigate the possibility of reducing the disturbance effect with a convenient choice of the control operator. We give the main properties and characterization results of this concept, according to the delay and the observation. Then, under an appropriate hypothesis, we demonstrate how to find the optimal control which ensures the compensation of a disturbance measured through the observation (measurements, signals,...). The discrete version of the wave equation, as well as the usual actuators and sensors, are examined. Numerical results are also presented.

1. Introduction. Discrete dynamical systems have proved their efficiency in the modeling of evolutionary problems. At a sequence of times, we collect new measurements or observations to complete the description of the dynamical systems. One can find several applications in different fields of discrete dynamical systems such as population growth, engineering or data analysis (see [6], [12]).

Many physical phenomena have inherent delay properties and necessitate the current as well as the historic states. The mathematical model of the discrete delayed system is known as an infinite-dimensional one which is theoretically challenging. One can also find several examples of delay aspects in dynamical systems such as the case of actuators and sensors (used generally as an input and output dual) which usually introduce a delayed response. However, the delay can be benefit in some stability or controllability issues. For more details see [20].

Discrete systems may be discrete naturally (digital or economic systems). They may also derive from the discretization of continuous-time systems.

On the other hand, several dynamical processes require adding a perturbation functional to approach real behavior. It is generally unknown (totally or partially) and detected through the observation (signals, radiations, pollution...). Several studies are devoted to the detection and reconstruction of unknown perturbations (sources) using their corresponding observation (see [2], [11], [19]), still our motivation is to compensate the perturbation effect.

However, the knowledge of the disturbance’s origin (pollution, radiation, infection) is not generally sufficient. One must act by introducing a convenient control term, in order to remedy and compensate the disturbance effect (see [17], [18]).

The compensation problem examined in this work as well as the considered approach, are different. It consists to study the possibility to remedy the disturbance
effect by bringing the corresponding observation to its normal situation state. Under
congenial hypothesis and an appropriate choice of the input operator, we establish
the existence and uniqueness of the optimal (minimum energy) control ensuring the
remediability of the disturbance.

This issue has been studied previously for various systems without delays (see
[1], [2]) and for multi-input delays (see [21]). In this paper, we examine a general
and more realist model with state or input delays.

This work is organized as follows: Section 2 is focused on discrete systems with
state delays. We first give their formulation and the general solution. Then, we
introduce the compensation problem for such systems, we also define the notions of
weak and exact remediability. Finally, we examine the minimum energy problem,
we show how to obtain the optimal control ensuring the exact remediability of a
disturbance. In section 3, we establish an extension to a class of discrete systems
with delayed states and inputs. In section 4, we examine the discretization of
a continuous dynamical system with delay. We present two approaches: exact
and approximative. In section 5, we explore the mathematical results through a
hyperbolic system. The usual case of actuators and sensors is also considered.
Illustrative and numerical results are equally presented.

2. Discrete-time systems with multiple state delays.

2.1. Problem formulation and resolution. We consider a class of systems de-
scribed by the following discrete equation:

\[
\begin{aligned}
x(k + 1) &= \sum_{i=0}^{n} A_i x(k - i) + B u(k) + f(k); \quad 0 \leq k \leq N - 1 \\
x(0) &= 0 \\
\psi &= (x(-n), x(-n + 2), ..., x(-1))
\end{aligned}
\]

where \( x(k) \in X \) is the state element, \( u(k) \in U \) is the input and \( f(k) \in X \) is the
disturbance term. \( k \) is the current step. \( X \) and \( U \) are respectively the state and the
control spaces, assumed to be Hilbert spaces. \( A_i, B \) are linear bounded operators.

The historic states \((x(-n), x(-n + 2), ..., x(-1))\) are generally given NULL.

The system (1) is augmented by the following output:

\[
y(k) = C x(k)
\]

where \( C \in \mathcal{L}(X,Y) \) is the output operator and \( Y \) is the observation space.

Let \((S_k)_{k \geq 0}\) be the sequence of bounded operators given by (see [10], [15], [16]):

\[
S_k = \begin{cases} 
I & \text{for } k = 0 \\
\sum_{i=0}^{k-1} A_i S_{k-1-i} & \text{for } k = 1, ..., N - 1 
\end{cases}
\]

Proposition 1. The general solution of (1) is given by:

\[
x(k + 1) = S_{k+1} x(0) + \sum_{i=0}^{k} S_{k-i} B u(i) + \sum_{i=0}^{k} S_{k-i} f(i) + \Psi_k; \quad 0 \leq k \leq N - 1
\]

where the historic states are defined by:

\[
\Psi_k = \begin{cases} 
\psi_1 & \text{for } k = 0 \\
\sum_{i=0}^{k-1} A_i \Psi_{k-1-i} + \psi_{k+1} & \text{for } k = 1, ..., N - 1
\end{cases}
\]
with: $\psi_k = \sum_{i=k}^{n} A_i x(k - 1 - i)$

Proof. Using system (1), we have:

- For $k = 0$ $x(1) = \sum_{i=0}^{n} A_i x(-i) + Bu(0) + f(0)$
  $= A_0 x(0) + Bu(0) + f(0) + \psi_1$
  $= A_0 S_0 x(0) + S_0 Bu(0) + S_0 f(0) + \psi_1$
  $x(1) = S_1 x(0) + S_0 Bu(0) + S_0 f(0) + \psi_1$

where $S_1 = A_0 S_0$ (3).

- For $k = 1$ $x(2) = \sum_{i=0}^{n} A_i x(1 - i) + Bu(1) + f(1)$
  $= A_0 x(1) + A_1 x(0) + Bu(1) + f(1) + \psi_2$
  $= A_1 S_0 x(0) + A_0 (S_1 x(0) + S_0 Bu(0) + S_0 f(0))$
  $+ S_0 Bu(1) + S_0 f(1) + \psi_2 + A_0 \psi_1$
  $= (A_1 S_0 + A_0 S_1) x(0) + S_1 Bu(0) + S_0 Bu(1)$
  $+ S_1 f(0) + S_0 f(1) + A_0 \psi_1 + \psi_2$
  $x(2) = S_2 x(0) + B(S_1 u(0) + S_0 u(1)) + S_1 f(0)$
  $+ S_0 f(1) + A_0 \psi_1 + \psi_2$

By induction, assume that this results holds for $k \geq N - 2$ i.e.

$$x(k) = S_k x(0) + \sum_{i=0}^{k-1} S_{k-1-i} Bu(i) + \sum_{i=0}^{k-1} S_{k-1-i} f(i) + \Psi_{k-1}$$

Then, using equation (1) we have:

$$x(k + 1) = \sum_{i=0}^{n} A_i x(k - i) + Bu(k) + f(k)$$

$$= A_k x(0) + \sum_{i=0}^{k-1} A_i x(k - i) + \sum_{i=0}^{k-1} A_i x(k - i) + Bu(k) + f(k)$$

$$= A_k x(0) + \sum_{i=0}^{k-1} A_i x(k - i) + \sum_{i=0}^{k-1} A_i \left( S_{k-i} x(0) + \sum_{j=0}^{k-i-1} S_{k-i-1-j} Bu(j) \right)$$

$$+ \sum_{i=0}^{k-1} A_i \left( \sum_{j=0}^{k-i-1} S_{k-i-1-j} f(j) + \Psi_{k-1-i} \right) + Bu(k) + f(k)$$

$$= \sum_{i=0}^{k-1} A_i S_{k-i} x(0) + S_0 A_k x(0) + \sum_{i=k}^{n} A_i x(k - 1 - i) + \sum_{i=0}^{k-1} A_i \Psi_{k-1-i}$$

$$+ \sum_{i=0}^{k-1} A_i \left( \sum_{j=0}^{k-i-1} S_{k-i-1-j} Bu(j) \right) + S_0 Bu(k)$$

$$+ \sum_{i=0}^{k-1} A_i \left( \sum_{j=0}^{k-i-1} S_{k-i-1-j} f(j) \right) + S_0 f(k)$$
Using (3) and (5), we get:

\[ x(k + 1) = S_{k+1}x(0) + \sum_{i=0}^{k} S_{k-i}Bu(i) + \sum_{i=0}^{k} S_{k-i}f(i) + \Psi_k \]

Then the general solution is given by (4).

Let us note that the historic states \( \Psi_k \) are generally NULL.

At step \( N \), the observation is given by:

\[ y_{u,f}(N) = CS_Nx(0) + \sum_{i=0}^{N-1} CS_{N-1-i}Bu(i) + \sum_{i=0}^{N-1} CS_{N-1-i}f(i) + C\Psi_{N-1} \]

In the normal case without disturbance and control, the observation becomes:

\[ y_{0,0}(N) = CS_Nx(0) + C\Psi_{N-1} \]

2.2. Compensation problem or remediability. The remediability or the compensation problem consists to bring back at the last step \( N \), the measured observation \( y_{u,f}(N) \) to the normal observation \( y_{0,0}(N) \) i.e.:

\[ \sum_{i=0}^{N-1} CS_{N-1-i}Bu(i) + \sum_{i=0}^{N-1} CS_{N-1-i}f(i) = 0 \]

Let us consider the following operators:

\[ H_N : U^N \to X \]

\[ u \to H_Nu = \sum_{i=0}^{N-1} S_{N-1-i} Bu(i) \]

and

\[ \bar{H}_N : X^N \to X \]

\[ f \to \bar{H}_Nf = \sum_{i=0}^{N-1} S_{N-1-i} f(i) \]

This leads to (8) becomes:

\[ CH_Nu + C\bar{H}_Nf = 0 \]

We consider the following definitions:

**Definition 2.1.** The system (1) augmented with the output equation (2), or simply (1)+(2) is said to be:

i): exactly remediable, if for any disturbance \( f \in X^N \), there exists a control \( u \in U^N \) such that:

\[ CH_Nu + C\bar{H}_Nf = 0 \]

ii): weakly remediable, if for any \( f \in X^N \) and \( \epsilon > 0 \), there exists \( u \in U^N \) such that:

\[ \|CH_Nu + C\bar{H}_Nf\| < \epsilon \]

The characterization of the exact and the weak remediability are given in the following proposition.

**Proposition 2.** The following properties are equivalent:

1. (1)+(2) is exactly remediable.
2. \( \text{Im}(CH_N) \subset \text{Im}(C\bar{H}_N) \)
3. There exists $\gamma > 0$ such that:

$$\|S_{N-1}^* C^* \theta\|_{X'} \leq \gamma \|B^* S_{N-1}^* C^* \theta\|_{U'}; \quad \forall \theta \in Y'$$

- Concerning the weak remediability, there is equivalence among:
  1. $(1)+(2)$ is weakly remediable.
  2. $\text{Im}(CH_N) \subset \text{Im}(\bar{C}H_N)$
  3. $\ker(H_N^* C^*) = \ker(\bar{H}_N C^*)$ (11)

The proofs are similar to those established in the finite time case for continuous linear systems with delays (see [2]).

**Remark 1.** Let us note the following:

- The exact remediability implies the weak one. The converse is true if $\text{Im}(CH_N)$ is a finite dimension space.
- The exact (weak) controllability implies the exact (weak) remediability. The converse is not generally true. The concept of remediability is then weaker and more flexible than that of controllability.
- In the case of linear systems, the concept of remediability do not depend on the state delay and also on the initial state $x(0)$, then we can consider $x(0) = 0$.

The problem now consists to find the optimal control ensuring the exact remediablity of a disturbance. This problem is examined in the following section.

### 2.3. Minimum energy problem.

We study in this section the problem of exact remediability with the following minimal energy control:

$$\min_{u \in \mathcal{C}} J(u)$$

(12)

where

$$J(u) = \|CH_N u + C\bar{H}_N f\|_Y^2 + \|u\|^2_{U_N}$$

and

$$\mathcal{C} = \{ u \in U_N, \text{u satisfies (10)} \}$$

Otherwise, for the corresponding observation $CH_N f \in Y$ which is usually known, does a control $u \in U_N$ such that the equality (10) is verified? If $u$ exists, is it optimal? i.e. $u$ is the solution of (12).

**Remark 2.** Let us note that if one consider the cost function:

$$J_{\alpha}(u) = \|CH_N u + C\bar{H}_N f\|_Y^2 + \alpha \|u\|^2_{U_N}$$

$J_{\alpha}$ is obviously more general. However, the considered problem concerns the exact compensation with minimum energy, then we have for $u \in \mathcal{C}$:

$$CH_N u + C\bar{H}_N f = 0$$

Consequently

$$J_{\alpha}(u) = \alpha \|u\|^2_{U_N}$$

Since $\alpha > 0$, the minimization of $J_{\alpha}$ in $\mathcal{C}$ is equivalent to that of the cost function (12).
For its resolution, we will use an extension of the Hilbert Uniqueness Method (H.U.M.) introduced by Lions in [13]. We define hereafter a semi-norm on the observation space \( Y \):

\[
\| \theta \|_F = \| H_N^* C^* \theta \|_{(U^N)^\prime} = \left( \sum_{k=0}^{N-1} \| B^* S_{N-1-k}^* C^* \theta \|_U^2 \right)^{\frac{1}{2}}
\]

where \((U^N)^\prime\) is the dual space of \( U^N \).

We assume that \( \| . \|_F \) is a norm. Let us note the following:

**Remark 3.**

- Sufficient condition: If (1) + (2) is weakly controllable, i.e. \( \ker(H_N^*) = \{0\} \) and \( \ker(C^*) = \{0\} \) (which is true in the case of sensors), then \( \| . \|_F \) is a norm.
- On the other hand, if \( \| . \|_F \) is a norm, then (1) + (2) is weakly remediable. This derives from Proposition 2.
- Conversely, if \( \ker(\tilde{H}_N^* C^*) = \{0\} \), then \( \| . \|_F \) is a norm if and only if (1) + (2) is weakly remediable. This derives from (11).

Let also remark that for a large class of systems such discrete versions of continuous systems\(^1\), we have (see [2] and the references therein) \( \ker(\tilde{H}_N^* C^*) = \ker(C^*) \). Consequently, if \( \ker(C^*) = \{0\} \) and (1) + (2) is weakly remediable, we have

\[
\ker(H_N^* C^*) = \ker(\tilde{H}_N^* C^*) = \{0\}
\]

then \( \| . \|_F \) is a norm and hence there is equivalence.

Let

\[ F = Y^{\| \|_F} \]

be the completion of the space \( Y \) with respect the norm \( \| . \|_F \). \( F \) is a Hilbert space with the inner product:

\[
\langle \theta, \delta \rangle = \sum_{i=0}^{N-1} \langle B^* S_{N-1-i}^* C^* \theta, B^* S_{N-1-i}^* C^* \delta \rangle; \quad \theta, \delta \in F
\]

where \( B^* \), \( S_{N-1-i}^* \) and \( C^* \) are respectively the adjoint operators of \( B \), \( S_{N-1-i} \) and \( C \). Now, let us consider the following operator \( \Lambda_N \) defined on the space \( Y' \) (\( Y' \) is identified to its dual \( Y \)) as follows:

\[
\Lambda_N \theta = C H_N H_N^* C^* \theta = \sum_{i=0}^{N-1} C S_{N-1-i} B B^* S_{N-1-i}^* C^* \theta
\]

where \( H_N^* \) is the adjoint operator of \( H_N \). \( \Lambda_N \) admits a unique extension as an isomorphism \( F \rightarrow F' \) such that:

\[
\langle \Lambda_N \theta, \sigma \rangle_Y = \langle \theta, \sigma \rangle_F; \quad \theta, \sigma \in F
\]

and

\[
\| \Lambda_N \theta \|_{F'} = \| \theta \|_F; \quad \text{for any } \theta \in F
\]

\( F' \) is the dual space \( F \). The proof is similar to that exercising dynamical systems without delays [1].

\(^1\)See section 4.
Proposition 3. If the observation $CH_N f \in F'$, then there exists a unique $\theta_N \in F$ such that $\Lambda_N \theta_N = -CH_N f$, and the control:

$$u^* = H_N^* C^* \theta_N$$

$$= \sum_{i=0}^{N-1} B^* S_{N-1-i}^* C^* \theta_N$$

satisfies the condition (10) and also is optimal with:

$$\|u^*\|_{U_N} = \|\theta_N\|_{F'}$$

Proof. Since $CH_N f \in F'$ and $\Lambda_N$ is an isomorphism $F \to F'$, there exists a unique $\theta_N \in F$ such that $\Lambda_N \theta_N = -CH_N f$, i.e.

$$\sum_{i=0}^{N-1} CS_{N-1-i} BB^* S_{N-1-i}^* C^* \theta = -CH_N f$$

Consequently, the control $u^*$ satisfies $CH u^* = -CH_N f$, i.e.

$$CH u^* + CH_N f = 0$$

and hence the exact remediability condition (10). The set $C$ is then non-empty ($u^* \in C$). Moreover, it is closed and convex, we deduce that the quadratic function $J$ admits a unique minimum $w \in C$ characterized by

$$\langle w, v - w \rangle \geq 0; \quad \forall v \in C$$

For $v \in C$, we have

$$\langle w, v - w \rangle = \sum_{i=0}^{N-1} (B^* S_{N-1-i}^* C^* \theta, v - w)$$

$$= \langle \theta, \sum_{i=0}^{N-1} CS_{N-1-i} B v - \sum_{i=0}^{N-1} CS_{N-1-i} B w \rangle$$

$$= \langle \theta, CH_N v - CH_N w \rangle = 0$$

Since $u^*$ and $w$ are in $C$ and using the fact that $w$ is unique, we deduce that $u^* = w$ and $u^*$ is optimal.

3. Discrete-time systems with multiple states and inputs delays. In this case, we examine a more general situation where there are also input delays. Contrarily to the previous situation where the problem of remediability do not depend on the states delays, the concept of remediability depends on the control delays.

3.1. Preliminaries. We consider the following discrete system:

$$x(k + 1) = \sum_{i=0}^{n} A_i x(k - i) + \sum_{j=0}^{m} B_j u(k - j) + f(k)$$

$$0 \leq k \leq N - 1$$

$$x(0) = 0$$

$$\psi = (x(-n), x(-n + 2), ..., x(-1))$$

$$\xi = (u(-m + 1), u(-m + 2), ..., u(0))$$

where for $j = 0, ..., m; B_j \in \mathcal{L}(U, X)$ is a bounded input operator and $\xi$ defines the historic inputs.
System (16) is augmented with the following observation:
\[ y(k) = Cx(k) \] (17)

We define hereafter the bounded operators (see [16] and [10]):

\[ D_k = \begin{cases} 
S_0B_0 & \text{if } k = 0 \\
\sum_{i=0}^{k} S_{k-i}B_i & \text{if } k = 1, \ldots, N-1 
\end{cases} \] (18)

The historic inputs are given by:
\[ \Theta_k = \begin{cases} 
\xi_1 & \text{for } k = 0 \\
\sum_{j=0}^{k-1} B_j \Theta_{k-1-j} + \xi_{k+1} & \text{for } k = 1, \ldots, N-1 
\end{cases} \] (19)

where \( \xi_k = \sum_{j=k}^{n} B_j u(k-1-j) \). We have the following result:

**Proposition 4.** The general solution of (16) is given by:
\[ x(k+1) = S_{k+1}x(0) + \sum_{j=0}^{k} D_{k-j}u(j) + \sum_{i=0}^{k} S_{k-i}f(i) + \Psi_k + \Theta_k; \quad 0 \leq k \leq N-1 \] (20)

**Proof.**

- For \( k = 0 \)
  \[ x(1) = \sum_{i=0}^{n} A_i x(-i) + \sum_{j=0}^{m} B_j u(-j) + f(0) \]
  \[ = A_0 x(0) + B_0 u(0) + f(0) + \psi_1 + \xi_1 \]
  \[ = S_1 x(0) + D_0 u(0) + S_0 f(0) + \psi_1 + \xi_1 \]

- For \( k = 1 \)
  \[ x(2) = \sum_{i=0}^{n} A_i x(1-i) + \sum_{j=0}^{m} B_j u(1-j) + f(1) \]
  \[ = A_0 x(1) + A_1 x(0) + B_0 u(1) + B_1 u(0) \]
  \[ + f(1) + \psi_2 + \xi_2 \]
  \[ = (A_1 S_0 + A_0 S_1) x(0) + (A_0 B_0 S_0 + B_1 S_0) u(0) + B_0 S_0 u(1) + A_0 S_0 f(0) + S_0 f(1) \]
  \[ + A_0 \xi_1 + \xi_2 + A_0 \psi_1 + \psi_2 \]
  \[ = S_2 x(0) + D_1 u(0) + D_0 u(1) \]
  \[ + S_1 f(0) + S_0 f(1) + \Psi_1 + \Theta_1 \]

Assume that \( x(k) \) holds for \( 1 \leq k \leq N - 2 \) i.e.
\[ x(k) = S_k x(0) + \sum_{j=0}^{k-1} D_{k-1-j}u(j) + \sum_{i=0}^{k-1} S_{k-1-i}f(i) + \Psi_{k-1} + \Theta_{k-1} \]
Using (18) and (19), we obtain:

\[ x(k + 1) = A_k x(0) + \sum_{i=0}^{n-1} A_i x(0) + \sum_{i=0}^{k-1} A_i x(k - i) + \sum_{i=0}^{k-1} B_i u(k - i) + f(k) \]

According to system (16), we have:

\[ x(k + 1) = A_k x(0) + \sum_{i=0}^{n-1} A_i x(0) + \sum_{i=0}^{k-1} A_i x(k - i) + \sum_{i=0}^{k-1} B_i u(k - i) + f(k) \]

\[ x(k + 1) = A_k x(0) + \sum_{i=0}^{n-1} A_i x(0) + \sum_{i=0}^{k-1} A_i x(k - i) + \sum_{i=0}^{k-1} B_i u(k - i) + f(k) \]

3.2. Remediability problem. As in the previous sections, we try to remedy any disturbance through the observation measurements and with a convenient choice of control terms. The remediability condition can be formulated as follows:

For \( f = (f_0, \ldots, f_{N-1}) \in X^N \), does a control \( u = (u_0, \ldots, u_{N-1}) \in U^N \) verifying the following condition:

\[ \sum_{j=0}^{N-1} CD_{N-1-j} u(j) + \sum_{i=0}^{N-1} CS_{N-1-i} f(i) = 0 \]
We consider the corresponding operators:

\[ R_N : U^N \rightarrow X \]
\[ u \rightarrow R_N u = \sum_{i=0}^{N-1} D_{N-1-i} u(i) \]

and

\[ \bar{R}_N : X^N \rightarrow X \]
\[ f \rightarrow \bar{R}_N f = \sum_{i=0}^{N-1} S_{N-1-i} f(i) \]

the condition (22) becomes:

\[ CR_N u + C\bar{R}_N f = 0 \quad (23) \]

By replacing in Def.(2.1) the linear operators \( H_N \) and \( \bar{H}_N \) by \( R_N \) and \( \bar{R}_N \) respectively, the remediability definition is similar.

Let us note that the corresponding problem depends only on the observation term \( C\bar{R}_N f \) which is usually known in linear systems. The knowledge of \( f \) is not really necessary.

Using an analogous approach and with the same notations \( \Lambda_N \) and \( \theta_N \), we show by the same that the optimal control ensuring the condition (10) is given by the following equation:

\[ u^* = R_N^* C^* \theta_N \]
\[ = \sum_{i=0}^{N-1} B^* D_{N-1-i}^* C^* (\Lambda_N)^{-1} (- \sum_{j=0}^{N-1} S_{N-1-j} f(j)) \quad (24) \]

In the following section, we examine the case of discrete versions of continuous systems.

4. Case of discrete versions of continuous time dynamical systems with multiple input delays. In this section, we examine the case of discrete versions of continuous time linear systems with delays. We present two different approaches. The first one is more precise and more practice. The second seems simple, but it is not sufficiently efficient.

4.1. Exact approach. We consider the dynamical system described by the following equation:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + \sum_{i=0}^{n} B_i u_i(t - h_i) + f(t); \quad 0 < t < T \\
u_i(\alpha_i) &= \vartheta_i(\alpha_i); \quad \alpha_i \in [-h_i, 0] \\
x(0) &= x_0 \in X
\end{align*}
\]

where \( X \) is the state space, \( B_i \in \mathcal{L}(U, X) \) are input operators with the controls \( u_i \in L^2(-h_i, T; U) \) for \( i = 1, .., n \). \( h_0 = 0 < h_1 < ... < h_n \) are constant delays with the historic data \( \vartheta_i(.) \). \( f \) is a perturbation term. \( X \) and \( U \) are respectively the state and the control spaces, assumed to be Hilbert spaces.

We assume that \( A \) generates a strongly continuous semigroup \( (S(t))_{t \geq 0} \). We define hereafter the left-shift semigroup describing the input solution (for more details, one can see [4]):

\[
\begin{align*}
Q &:= \frac{\partial}{\partial \alpha} \\
D(Q) &:= \vartheta_i \in W^{1,2}(-h_i, 0; U) : \vartheta_i(0) = 0
\end{align*}
\]
\( (Q, D(Q)) \) generates a strongly continuous semigroup:

\[
\Psi_i(t) \vartheta_i = \begin{cases} 
0 & \text{if } t - h_i \geq 0 \\
\vartheta_i(t - h_i) & \text{if } t - h_i \leq 0
\end{cases}
\] (26)

We consider the linear operator (see [22]):

\[
\Phi_i(t) u_i = \begin{cases} 
u_i(t - h_i) & \text{if } t - h_i \geq 0 \\
0 & \text{if } t - h_i \leq 0
\end{cases}
\] (27)

Its adjoint is defined by:

\[
\Phi_i^*(t) u_i = \begin{cases} 
u_i(t + h_i) & \text{if } t + h_i \geq 0 \\
0 & \text{if } t + h_i \leq 0
\end{cases}
\]

The delayed inputs may be expressed as follows ([7]-[9]):

\[
\sum_{i=1}^{n} B_i u_i(t - h_i) = \sum_{i=1}^{n} B_i \Phi_i(t) u_i + \sum_{i=1}^{n} B_i \Psi_i(t) \vartheta_i = B \Phi(t) u + B \Psi(t) \vartheta
\] (28)

where \( B = (B_1, B_2, ..., B_n) \), \( u = (u_1, u_2, ..., u_n) \), \( \vartheta = (\vartheta_1, \vartheta_2, ..., \vartheta_n) \) and using the notations for simplicity sake:

\[
v_i = \Phi(t) u \equiv \begin{pmatrix} \Phi_1(t) u_1 \\ \Phi_2(t) u_2 \\ \vdots \\ \Phi_n(t) u_n \end{pmatrix}; \quad w_i = \Psi(t) \vartheta \equiv \begin{pmatrix} \Psi_1(t) \vartheta_1 \\ \Psi_2(t) \vartheta_2 \\ \vdots \\ \Psi_n(t) \vartheta_n \end{pmatrix}
\]

then the system (25) becomes:

\[
\begin{cases}
\dot{x}(t) = A x(t) + B v_t + B w_t + f(t); \quad 0 < t < T \\
x(0) = x_0 \in X
\end{cases}
\] (29)

The solution is given by:

\[
x(t) = S(t) x_0 + \int_0^t S(t - s) B v_s ds + \int_0^t S(t - s) B w_s ds + \int_0^t S(t - s) f(s) ds
\] (30)

**Discretization:** For \( N \in \mathbb{N}^* \) (\( N \) sufficiently large), let

\[
\tau = \frac{T}{N} \quad \text{and} \quad I_k = [t_k, t_{k+1}] \quad \text{where} \quad t_i = i \tau
\]

For \( 0 \leq k \leq N - 1 \), we note \( u_k \) (respectively \( f_k \)) the restriction of \( u \) (respectively \( f \)). We define also to the time interval \( I_k \)

\[
x_k \equiv x(k \tau)
\]

- For \( k = 1 \), we have

\[
x_1 \equiv x(t_1) = S(\tau) x_0 + \int_0^\tau S(\tau - s) B v_s ds + \int_0^\tau S(\tau - s) B w_s ds + \int_0^\tau S(\tau - s) f(s) ds
\]
We have:
\[
\int_0^\tau S(\tau - s)B\Psi(s)\vartheta ds + \int_0^\tau S(\tau - s)f(s)ds
\]
\[
= S_0x_0 + B\bar{v}_0 + B\varpi_0 + F_0
\]
where \(S_0 = S(\tau), B\bar{v}_0 = \int_0^\tau S(\tau - s)Bv_s ds, B\varpi_0 = \int_0^\tau S(\tau - s)Bw_s ds\) and \(F_0 = \int_0^\tau S(\tau - s)f_0(s)ds\).

- For \(k = 2\),
\[
x_2 \equiv x(t_2) = S(2\tau)x_0 + \int_0^{2\tau} S(2\tau - s)Bv_s ds
\]
\[
+ \int_0^{2\tau} S(2\tau - s)Bw_s ds + \int_0^{2\tau} S(2\tau - s)f(s)ds
\]
\[
x(t_2) = S_0^2x_0 + S_0\int_0^\tau S(\tau - s)B\Psi(s)uds + \int_0^\tau S(\tau - r)B\Psi(\tau + r)udr
\]
\[
+ S_0\int_0^\tau S(\tau - s)B\Psi(s)\vartheta ds + \int_0^\tau S(\tau - r)B\Psi(\tau + r)\vartheta dr
\]
\[
+ S_0\int_0^\tau S(\tau - s)f(s)ds + \int_0^\tau S(\tau - r)f(\tau + r)dr
\]
\[
x(t_2) = S_0^2x_0 + S_0B\bar{v}_0 + B\bar{v}_1 + S_0B\varpi_0 + B\varpi_1 + S_0F_0 + F_1
\]

- For \(1 \leq k \leq N - 2\), assume that:
\[
x_k = S_0^kx_0 + \sum_{i=0}^{k-1} S_0^{k-1-i}B\bar{v}_i + \sum_{i=0}^{k-1} S_0^{k-1-i}B\varpi_i + \sum_{i=0}^{k-1} S_0^{k-1-i}F_i
\]

We have:
\[
x_{k+1} \equiv x((k + 1)\tau) = S((k + 1)\tau)x_0 + \int_0^{(k+1)\tau} S((k + 1)\tau - s)B\Psi(s)uds
\]
\[
+ \int_0^{(k+1)\tau} S((k + 1)\tau - s)B\Psi(s)\vartheta ds
\]
\[
+ \int_0^{(k+1)\tau} S((k + 1)\tau - s)f(s)ds
\]
\[
= S_0^{k+1}x_0 + \sum_{i=0}^k S_0^{(k+1)-i}S((k + 1)\tau - s)B\Phi(s)uds
\]
\[
+ \sum_{i=0}^k S_0^{(k+1)-i}S((k + 1)\tau - s)B\Phi(s)\vartheta ds
\]
\[
+ \sum_{i=0}^k S_0^{(k+1)-i}S((k + 1)\tau - s)f(s)ds
\]
\[
= S_0^{k+1}x_0 + \sum_{i=0}^k \int_0^\tau S((k + 1)\tau - (r + i\tau))B\Phi(r + i\tau)udr
\]
and

\[ x_{k+1} = S_0^{k+1} x_0 + \sum_{i=0}^{k} S_0^{k-i} \int_0^\tau S(\tau - r) B \Phi(r + i\tau) \vartheta dr \]

\[ + \sum_{i=0}^{k} S_0^{k-i} \int_0^\tau S(\tau - r) B \Psi(r + i\tau) \vartheta dr \quad \text{for} \quad 0 \leq k \leq N - 1. \]

**Remark 4.** Concerning the state, the control and the disturbance.

**4.2. Approximative approach.** Under convenient regularity conditions \(^2\), one can also consider the following approximative discretization method:

\[ \dot{x}(t) \simeq \frac{1}{\tau} (x(t + \tau) - x(t)) \]

then:

\[ x(t + \tau) - x(t) \simeq \tau \left[ Ax(t) + \sum_{i=1}^{n} B_i u_i(t - h_i) + f(t) \right] \]

Let us note \( t = k\tau, x_k = x(k\tau), \bar{u}_{k,i} = u_i(k\tau - h_i) \) and \( f_k = f(k\tau) \) for \( 0 \leq k \leq N - 1. \)

We have:

\[ x_{k+1} \simeq x_k + \tau \left[ Ax_k + \sum_{i=1}^{n} B_i \bar{u}_k + f_k \right] \]

\[ = x_k + \tau Ax_k + \tau \sum_{i=1}^{n} B_i \bar{u}_k + \tau f_k \]

\[ x_{k+1} = A x_k + B \bar{u}_k + F_k \]

where \( \mathcal{A} = I_d + \tau A, \mathcal{B} = \tau (B_1, B_2, ..., B_n), \bar{u}_k = (\bar{u}_{k,1}, ..., \bar{u}_{k,n})^{tr} \) and \( F_k = \tau f_k. \)

**Proposition 5.** The discrete version of system (25) is given by:

\[ \begin{align*}
    x_{k+1} &= S_0^{k+1} x_0 + \sum_{i=0}^{k} S_0^{k-i} B \bar{u}_i + \sum_{i=0}^{k} S_0^{k-i} B \bar{w}_i + \sum_{i=0}^{k} S_0^{k-i} F_i \\
    x_0 &\in X
\end{align*} \tag{31} \]

then we have the following result.
• Contrary to the previous approximation (exact), the operator \( \Phi = I_d + \tau A \) is not necessary bounded if \( A \) isn’t.
• To have a good approximation, \( \tau \) must be very small.

We examine hereafter an application to an hyperbolic equation using the first approach. The system is assumed to be excited by actuators and the output is given by sensors.

5. Application to an hyperbolic system.

5.1. Considered system and resolution. We consider hereafter a perturbed hyperbolic systems with multiple input delays:

\[
\begin{cases}
\frac{\partial^2 x}{\partial t^2}(\xi, t) = Ax(\xi, t) + \sum_{i=0}^{p} B_i u_i(t - h_i) + f(\xi, t) \quad \Omega \times ]0, T[ \\
x(\xi, 0) = 0; \quad \frac{\partial x}{\partial t}(\xi, 0) = 0 \quad \Omega \\
x(\eta, t) = 0; \quad \frac{\partial}{\partial \Omega} \times ]0, T[ \\
u_i(\alpha_i) = \partial_i(\alpha_i); \quad \alpha_i \in [-h_i, 0]
\end{cases}
\]  

(32)

augmented with the output equation:

\[
y(t) = \begin{pmatrix} C_1 x(\cdot, t) \\ \frac{\partial x}{\partial t}(\cdot, t) \end{pmatrix} = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}
\]

(33)

We consider without loss of generality the case where \( A \) is the Laplacian operator. However, one can consider a second order elliptic, self-adjoint and coercive operator. We assume that \( B_i \in L(L(U, L^2(\Omega))) \), \( C_1 \in L(L^2(\Omega), Y_1) \), \( C_2 \in L(L^2(\Omega), Y_2) \) and \( f \in L^2(0, T; L^2(\Omega)) \). \( U \) is a control space, \( Y_1 \) and \( Y_2 \) are two observation spaces \((U, Y_1) \) and \((U, Y_2) \) are Hilbert spaces). \( X = L^2(\Omega) \) is the state space, \( \Omega \) is an open and bounded subset of \( \mathbb{R}^n \) \((n \geq 1) \) with a sufficiently regular boundary \( \partial \Omega \).

The delay input is defined similarly to (28) by:

\[
\sum_{i=1}^{p} B_i u_i(t - h_i) = B\Phi(t)u + B\Psi(t)\vartheta
\]

\[
= Bv_t + Bw_t
\]

where \( \Phi \) and \( \Psi \) are defined in section (4.1). Taking into account the zero initial control \( w_t = 0 \) for \( t \in [-h_i, 0] \), the considered system (32) is equivalent to:

\[
\begin{cases}
\dot{z}(t) = Az(t) + Lv_t + F(t); \quad 0 < t < T \\
z(0) = 0
\end{cases}
\]

(34)

The corresponding space is \( Z = H^1(\Omega) \times L^2(\Omega) \), with the inner product:

\[
< \psi, \psi' > \equiv \langle (-A)^{1/2} \psi_1(\xi), (-A)^{1/2} \psi'_1(\xi) \rangle_{L^2(\Omega)} + \langle \psi_2(\xi), \psi'_2(\xi) \rangle_{L^2(\Omega)}
\]

for \( \psi = (\psi_1, \psi_2) \in Z \) and \( \psi' = (\psi'_1, \psi'_2) \in Z \).

The state \( z \) is defined by \( z = \begin{pmatrix} x \\ \dot{x} \end{pmatrix} \in D(A) \) and \( A \) is the operator defined by (see [3]):

\[
A = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}
\]

(35)

with \( D(A) = D(A) \times H^1_0(\Omega) \). The adjoint operator \( A^* \) of \( A \) is given by \( A^* = -A \).
The input operator $L$ is defined as follows $L = \begin{pmatrix} 0 \\ B \end{pmatrix} \in \mathcal{L}(U, Z)$ and its adjoint $L^* = \begin{pmatrix} 0 \\ B^* \end{pmatrix}$. Also we define the disturbance term by:

$$ F(t) = \begin{pmatrix} 0 \\ f(t) \end{pmatrix} \in Z $$

The system (34) is augmented by the following output:

$$ y(t) = Cz(t) = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \end{pmatrix} \in Y = Y_1 \times Y_2 \quad (36) $$

The operator $(A, D(A))$ generates a strongly continuous semigroup $(S(t))_{t \geq 0}$ defined by:

$$ S(t) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \sum_{n=1}^{+\infty} \int_{\Omega} \phi_{nm} \cdot \varphi_n(t) \cos(\sqrt{-\lambda_n} t) \\ \sqrt{-\lambda_n} \sum_{n=1}^{+\infty} \int_{\Omega} \phi_{nm} \cdot \varphi_n(t) \sin(\sqrt{-\lambda_n} t) \\ \sum_{n=1}^{+\infty} \int_{\Omega} \phi_{nm} \cdot \varphi_n(t) \cos(\sqrt{-\lambda_n} t) \varphi_{nm} \end{pmatrix} \quad (37) $$

$(\varphi_{nm})_{n,m}$ is a complete system of eigenfunctions of $A$, corresponding to the eigenvalues $\lambda_n$ such that $0 > \lambda_1 > \lambda_2 > \ldots > \lambda_n > \ldots$.

The state of (34) is then given by:

$$ z(t) = S(t)z(0) + \int_0^t S(t-s)L\Phi(s)ds + \int_0^t S(t-s)F(s)ds \quad (38) $$

and according to section 4.1, we have:

$$ z_{k+1} = S_{k+1}z_0 + \sum_{i=0}^{k} S^{k-i}L_i\Phi_i + \sum_{i=0}^{k} S^{k-i}F_i $$

where $L_i\Phi_i = \int_0^\tau S(\tau - r)\mathcal{L}(r + i\tau)\Phi_i$ and $F_i = \int_0^\tau S(\tau - r)F(r + i\tau)\Phi_i$.

5.2. Case of actuators and sensors. To describe the input/output process, we develop in this part the corresponding example using the usual concept of actuators and sensors. Technically, the actuator is used as mechanism to excite the system (generator of energy), while a sensor detects the measurements (detector device).

We consider the case where the system (34) is exited by $p$ zone actuators $(\Omega_i, g_i)_{1 \leq i \leq P}$. $D_i$ is the zone where the actuator $(D_i, g_i)$ is active and $g_i$ is its spacial distribution. In this case, the operator $B$ is as follows ([5]):

$$ B : \mathbb{R}^p \rightarrow L^2(\Omega) $$

$$ u(t) \rightarrow \sum_{i=1}^{p} g_i u_i(t) \quad (39) $$

Concerning the output of the system, we assume that the observation is given by two sensors $(D_1, r_1)$ and $(D_2, r_2)$ according to (36), then ([1], [5]):

$$ Cz = \begin{pmatrix} C_1 z_1 \\ C_2 z_2 \end{pmatrix} $$
where
\[ C_1 z_1 = \langle r_1, z_1 \rangle ; \quad C_2 z_2 = \langle r_2, z_2 \rangle \]
The adjoint operators are defined by:
\[ C^*_1 \theta_1 = \theta_1 r_1; \quad C^*_2 \theta_2 = \theta_2 r_2 \]
then the output of (34) is given by:
\[ y_{v,f}(k) = CS_0^k x_0 + \sum_{i=0}^{k-1} CS_0^{k-1-i} L_i v_s + \sum_{i=0}^{k-1} CS_0^{k-1-i} F_i \] (40)
Using (37) and (39), the output becomes:
\[
y_{v,f}(k) = \begin{cases} 
\sum_{n=1}^{\infty} \sum_{m=1}^{r_n} \sum_{i=0}^{k-1} \int_0^\tau \left( \langle gv_r + f(r + i\tau), \varphi_{nm} \rangle \sin(\sqrt{-\lambda_n}(\tau(k - i) - r)) \right) \, dr \\
\times \langle r_1, \varphi_{nm} \rangle \\
\sum_{n=1}^{\infty} \sum_{m=1}^{r_n} \sum_{i=0}^{k-1} \int_0^\tau \left( \langle gv_r + f(r + i\tau), \varphi_{nm} \rangle \cos(\sqrt{-\lambda_n}(\tau(k - i) - r)) \right) \, dr \\
\times \langle r_2, \varphi_{nm} \rangle 
\end{cases}
\] (41)
where \( g = (g_1, g_2, ..., g_p) \) and \( v_t \) is defined in Section 4.1.
As mentioned in Section 2.2, the compensation problem or remediability, consists to bring back at the final step \( N \), the measured observation \( y_{v,f} \) to the normal observation \( y_{0,0} \), i.e.:
\[ \sum_{i=0}^{k} CS_0^{k-i} L_i v_s + \sum_{i=0}^{k} CS_0^{k-i} F_i = 0 \] (42)

**Remark 5.**

- For a finite number of sensors, the exact and weak remediability are equivalent.

According to Section 2.3, the optimal control ensuring the condition (42) at the last step \( N \), is given by:
\[ u^* = \sum_{i=0}^{N-1} L_i^* (S_0^{N-1-i})^* C^* \theta_f \] (43)
where \( u^* = (u_1^*, u_2^*, ..., u_p^*) \) and \( \theta_f = \begin{pmatrix} \theta_{f_1} \\ \theta_{f_2} \end{pmatrix} \), or equivalently:
\[ u_j^* = \sum_{i=0}^{N-1} \sum_{n=1}^{r_n} \sum_{m=1}^{r_m} \int_0^\tau \left[ \sqrt{-\lambda_n} \langle C^*_1 \theta_{f_1}, \varphi_{nm} \rangle \sin(\sqrt{-\lambda_n}(\tau(N - i) - r - h_j)) \cdot B^* \varphi_{nm} \\
- \langle C^*_2 \theta_{f_2}, \varphi_{nm} \rangle \cos(\sqrt{-\lambda_n}(\tau(N - i) - r - h_j)) \right] \, dr \] (44)
for \( j = 1, ..., p \).

**5.3. Case of a one dimension system.** In the case of a one dimension system with \( \Omega = [0, 1] \), the eigenvalues of \( A \) are simple \( (r_n = 1) \) and given by \( \lambda_n = -n^2 \pi^2 \), \( n \geq 1 \) and the eigenfunctions are defined by \( \varphi_n(\xi) = \sqrt{2} \sin(n\pi\xi) \). The input is given by two actuators \( (g_1, g_2) \).
For \( \theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \), the operator \( \Lambda_N \) is given by:

\[
\Lambda_N \theta = \sum_{i=0}^{N-1} CS_0^{N-1-i} \mathcal{L}_i (S_0^{N-1-i})^* C^* \\
= \sum_{i=0}^{N-1} \int_0^\tau CS(\tau(N-i) - r) \mathcal{L}(r+i\tau) \Phi^*(r+i\tau) \mathcal{L}^* (\tau(N-i) - r) C^* \, dr \\
= \sum_{i=0}^{N-1} \sum_{n=1}^{\infty} \int_0^\tau \left( \frac{\sin^2(\pi \alpha_i)}{\cos(\pi \alpha_i) \sin(\pi \alpha_i)} \cos^2(\pi \alpha_i) \right) \\
\times \left( c_1 + c_2 \right. \\
\left. + c_3 + c_4 \right) \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \, dr
\]

where

\[
\alpha_i = \tau(N-i) - r \\
c_1 = \langle g_1, \varphi_n \rangle^2 \langle r_1, \varphi_n \rangle^2 \\
c_2 = \langle g_2, \varphi_n \rangle^2 \langle r_1, \varphi_n \rangle^2 \\
c_3 = \frac{1}{\pi} \langle g_1, \varphi_n \rangle^2 \langle r_1, \varphi_n \rangle \langle r_2, \varphi_n \rangle \\
c_4 = \frac{1}{\pi} \langle g_2, \varphi_n \rangle^2 \langle r_1, \varphi_n \rangle \langle r_2, \varphi_n \rangle \\
c_5 = \pi \langle g_1, \varphi_n \rangle^2 \langle r_1, \varphi_n \rangle \langle r_2, \varphi_n \rangle \\
c_6 = \pi \langle g_2, \varphi_n \rangle^2 \langle r_1, \varphi_n \rangle \langle r_2, \varphi_n \rangle \\
c_7 = \langle g_1, \varphi_n \rangle^2 \langle r_2, \varphi_n \rangle^2 \\
c_8 = \langle g_2, \varphi_n \rangle^2 \langle r_2, \varphi_n \rangle^2
\]

We give hereafter, the optimal control \( u^* = (u_1^*, u_2^*) \) ensuring the remediability condition (42):

\[
u^* = \sum_{j=0}^{N-1} \sum_{n=1}^{\infty} \int_0^\tau \begin{pmatrix} 
\frac{n\pi \langle r_1, \varphi_n \rangle \langle g_1, \varphi_n \rangle \sin\left(n\pi(\tau(N-i) - r - h_1)\right)}{\cos\left(n\pi(\tau(N-i) - r - h_1)\right)} \theta_1 \\
+ \frac{n\pi \langle r_2, \varphi_n \rangle \langle g_1, \varphi_n \rangle \cos\left(n\pi(\tau(N-i) - r - h_1)\right)}{\sin\left(n\pi(\tau(N-i) - r - h_1)\right)} \theta_2 \\
\frac{n\pi \langle r_1, \varphi_n \rangle \langle g_2, \varphi_n \rangle \sin\left(n\pi(\tau(N-i) - r - h_2)\right)}{\cos\left(n\pi(\tau(N-i) - r - h_2)\right)} \theta_1 \\
+ \frac{n\pi \langle r_2, \varphi_n \rangle \langle g_2, \varphi_n \rangle \cos\left(n\pi(\tau(N-i) - r - h_2)\right)}{\sin\left(n\pi(\tau(N-i) - r - h_2)\right)} \theta_2 \\
\end{pmatrix} \, dr
\]

where \( \theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \) is given by:

\[
\theta = -(\Lambda_N)^{-1} \left( \sum_{j=0}^{N-1} CS_0^{N-1-j} \mathcal{F}_j \right) \\
= -(\Lambda_N)^{-1} \sum_{j=0}^{N-1} \int_0^\tau CS(\tau(N-j) - r) \mathcal{F}(r+i\tau) \, dr
\]

### 5.4. Numerical results.

For simplification purposes, we consider without loss of generality, the case where \( g_1 = g_2 = r_1 = r_2 = \varphi_1 \). According to Remark (1), the initial state is considered null \( x_0 = 0 \), then \( y_{0,0} = 0 \). The disturbance term is given as follows:

\[
\mathcal{F} = \begin{pmatrix} 0 \\ f(\xi, I) \end{pmatrix}
\]

where \( f(\xi, I) = \rho_1(\xi)k(I) \); \( I = 0, ..., N-1 \); For example, one can consider the perturbation term \( \rho_1(\xi) = \sqrt{2} \sin(\pi \xi) \) and \( k(I) = \frac{\sin(I)}{I} \) and the delays \( h_1 = 0.1 \) and \( h_2 = 0.2 \). Then without loss of generality the free observation (without control)
is given by:

\[ y_f(N - 1) = \sum_{i=0}^{N-1} \int_0^\tau \left( \frac{1}{\pi} \sin(\pi(N - i) - r) \frac{\sin(r + i \tau)}{r + i \tau} \cos(\pi(N - i) - r) \frac{\sin(r + i \tau)}{r + i \tau} \right) dr \]

and

\[ \theta_f = -(\Lambda_N)^{-1} y_f(N - 1) \]

with \( \theta_f = \left( \theta_{f_1} \theta_{f_2} \right) \). For \( n = 1 \), the optimal control ensures the exact compensation of the disturbance, i.e. brings back the final observation \( y_{u,f}(N - 1) \) to the normal one which is zero (since \( z_0 = 0 \)) is given by: Then the optimal control for \( n = 1 \), is defined as follows:

\[
 u^* = \sum_{i=0}^{N-1} \int_0^\tau \left( \frac{\pi \sin(\pi(N - i) - r - h_1)}{\pi \sin(\pi(N - i) - r - h_2)} \right) \cos(\pi(N - i) - r - h_2) \cos(\pi(N - i) - r - h_2) dr \times \left( \frac{\theta_1}{\theta_2} \right)
\]

(46)

**Remark 6.** The optimal control depends linearly on \( \theta_f \), and hence on \( y_f \) which is usually known in linear systems. The knowledge of \( f \) is not really necessary.

We obtain the following numerical results which illustrate the previous developments. To simplify the notations, let us note \( y_{u,f} \) the observation corresponding to the control \( u \) and the disturbance \( f \). Hence \( y_{0,f} \) (respectively \( y_{0,0} \)) represents the observation with \( u = 0 \) (respectively the normal observation, i.e. with \( u = 0 \) and \( f = 0 \)).

![Figure 1](image)

**Figure 1.** Control observation for \( N = 10 \)

In figures 1 and 2 corresponding to \( N = 10 \) and \( N = 20 \) respectively, we present the evolution of the observations \( y_{u,f} \) and \( y_{0,f} \) with \( \tau = 0.1 \). In each case, we note
that the optimal control (45) leads at the final step $N$, the observation $y_{u,f}$ (blue curve) to the normal observation $y_{0,0} = 0$ as mentioned in (5).

On the other hand, in both cases, the perturbed and non controlled ($u = 0$) observation $y_{0,f}$ is increasing and moves away from the normal one (red curves in Fig.1 and Fig.2).

The obtained results concern a particular choice of $f$. However, the mathematical developments provide a credible design for any other disturbance.

6. Conclusion. As shown above, the main purpose consists to find the optimal control ensuring the compensation of a disturbance via the output equation. To give more general mathematical models, the results are developed for a class of discrete delay systems and also for discretized continuous systems with input delays.

As an application, we considered a one dimension hyperbolic system with two actuators and two sensors. Illustrative numerical results are also presented.

As noted before, the knowledge of the disturbance function is not necessary in the linear case. The developments are based on the corresponding observation which is generally known. Concerning the optimal control ensuring the remediability condition, the approach is based on an extension of Hilbert Uniqueness Method. We examined also the usual case of actuators and sensors as an input and output process.

The considered approach can be extended to other classes of systems and other similar problems. More general situations require further developments such as the cases of nonlinear and stochastic systems or other types of delays.

REFERENCES

[1] L. Affi, M. Bahadi, A. Chafiai and A. El Mizane, Asymptotic compensation in discrete distributed systems: Analysis, approximations and simulations, *Applied Mathematical Sciences*, 2 (2008), 99–137.

[2] L. Affi and A. El Jai, *Systèmes Distribués Perturbés*, Presses Universitaires de Perpignan (french), 2015.
[3] R. F. Curtain and A. J. Pritchard, *Infinite Dimensional Linear Systems Theory*, Lecture Notes in Control and Information Sciences, 8. Springer-Verlag, Berlin-New York, 1978.

[4] O. Diekmann, S. A. van Gils, S. M. Verduyn Lunel and H.-O. Walther, *Delay Equations: Functional-, Complex-, and Nonlinear Analysis*, Applied Mathematical Sciences, 110. Springer-Verlag, New York, 1995.

[5] A. El Jaï and A. J. Pritchard, *Sensors and actuators in distributed systems*, International Journal of Control, 46 (1987), 1139–1153.

[6] E. Fridman, *Introduction to Time-Delay Systems: Analysis and Control*, Systems & Control: Foundations & Applications, Birkhäuser/Springer, Cham, 2014.

[7] S. Hadd and A. Idrissi, *Regular linear systems governed by systems with state, input and output delays*, IMA Journal of Mathematical Control and Information, 22 (2005), 423–439.

[8] S. Hadd, *An evolution equation approach to nonautonomous linear systems with state, input, and output delays*, SIAM Journal on Control and Optimization, 45 (2006), 246–272.

[9] S. Hadd and Q.-C. Zhong, *On feedback stabilizability of linear systems with state and input delays in Banach spaces*, IEEE Transactions on Automatic Control, 54 (2009), 438–451.

[10] H. Shi, G. M. Xie and W. G. Luo, *Controllability of linear discrete time systems with both delayed states and delayed inputs*, Abstract and Applied Analysis, (2013), Art. ID 975461, 5 pp.

[11] V. Isakov, *Inverse Source Problems*, Mathematical Surveys and Monographs, 34. American Mathematical Society, Providence, RI, 1990.

[12] H. Kwakernaak and R. Sivan, *Linear Optimal Control Systems*, Wiley-Interscience, New York-London-Sydney, 1972.

[13] J.-L. Lions, *Contrôle Optimal de Systèmes Gouvernés par des Équations aux Dérivées Partielles*, Dunod, Paris, Gauthier-Villars, Paris, 1968.

[14] M. Naim, F. Lahmidi, A. Namir and M. Rachik, *On the output controllability of positive discrete linear delay systems*, Abstract and Applied Analysis, (2017), Art. ID 3651271, 12 pp.

[15] V. N. Phat and T. C. Dieu, *Constrained controllability of linear discrete nonstationary systems in banach spaces*, SIAM J. Control Optim., 30 (1992), 1311–1318.

[16] V. N. Phat, *Controllability of discrete-time systems with multiple delays on controls and states*, International Journal of Control, 49 (1989), 1645–1654.

[17] R. Rabah and M. Malabare, *Structure at infinity revisited for delay systems*, IEEE-SMC-IMACS Multiconference, Symposium on Robotics and Cybernetics (CESA’96). Symposium in Modelling, Analysis and Simulation, (1996), 87–90. https://hal.archives-ouvertes.fr/hal-01466183

[18] R. Rabah and M. Malabare, *Weak structure at infinity and row-by-row decoupling for linear delay systems*, Kybernetika, 40 (2004), 181–195.

[19] M. Rachik, M. Lhous and A. Tridane, *Controllability and Optimal Control Problem for Linear Time-varying Discrete Distributed Systems*, Systems Analysis Modelling Simulation, 43 (2003), 137–164.

[20] J.-P. Richard, *Time-delay systems: An overview of some recent advances and open problems*, Automatica J. IFAC, 39 (2003), 1667–1694.

[21] S. Souhaile and L. Afifi, *Cheap compensation in disturbed linear dynamical systems with multi-input delays* International Journal of Dynamics and Control. https://doi.org/10.1007/s40435-018-00505-6.

[22] M. Tucsnak and G. Weiss, *Observation and Control for Operator Semigroups*, Birkhäuser Advanced Texts: Basler Lehrbücher, Birkhäuser Verlag, Basel, 2009.

Received October 2018; 1st revision January 2019; 2nd revision January 2019.

E-mail address: souhaile.salma@gmail.com
E-mail address: larbi.afifi@gmail.com