THE PERIODIC DEFOCUSING ABLOWITZ-LADIK EQUATION
AND THE GEOMETRY OF FLOQUET CMV MATRICES

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Abstract. In this work, we show that the periodic defocusing Ablowitz-Ladik equation can be expressed as an isospectral deformation of Floquet CMV matrices. We then introduce a Poisson Lie group whose underlying group is a loop group and show that the set of Floquet CMV matrices is a Coxeter dressing orbit of this Poisson Lie group. By using the group-theoretic framework, we establish the Liouville integrability of the equation by constructing action-angle variables, we also solve the Hamiltonian equations generated by the commuting flows via Riemann-Hilbert factorization problems.

1. Introduction.

The defocusing Ablowitz-Ladik (AL) equation (a.k.a. defocusing discrete nonlinear Schrödinger equation) is the system of differential-difference equations given by

$$-i\dot{\alpha}_n = \alpha_{n+1} - 2\alpha_n + \alpha_{n-1} - |\alpha_n|^2(\alpha_{n-1} + \alpha_{n+1})$$  \hspace{1cm} (1.1)

where $\alpha_n$ is a sequence of numbers inside the unit disk $\mathbb{D}$. It was introduced by Ablowitz and Ladik in [AL] as a spatial discretization of the defocusing nonlinear Schrödinger equation and since then has been the subject of numerous studies. In particular, much attention has been focused on the inverse scattering method for solving the equation in the two-sided case where the index $n$ ranges over the set of integers. (See [APT] and the references therein.) By contrast, the literature on the periodic problem is relatively sparse. (See, for example, [K], [MEKL], [N], [GHMT].)

In recent years, one of the interesting developments in the arena of the defocusing AL equation has been the connection with the theory of orthogonal polynomials on the circle (OPUC) and the so-called CMV matrices [S2], [N], and our work here is a continuation of this development. Since we are dealing with the periodic defocusing AL equation here, let us begin with a set of Verblunsky coefficients $\{\alpha_j\}_{j=0}^{\infty}$ satisfying the periodicity condition $\alpha_{j+p} = \alpha_j, j = 0, 1, 2, \ldots$. Without loss of generality, we may assume $p$ is even. In [S2], Simon introduced the discriminant $\Delta(z)$ associated with $\{\alpha_j\}_{j=0}^{\infty}$ and together with the second author, they obtained the following involution theorem [S2], [N]

$$\{\Delta(z), \Delta(w)\}_{AL} = 0, \quad \{P, \Delta(z)\}_{AL} = 0$$  \hspace{1cm} (1.2)
by calculating with Wall polynomials. Here \( \{ \cdot, \cdot \}_{AL} \) is the Ablowitz-Ladik Poisson bracket \([KM]\), \([S2]\) and \( P = \prod_{j=0}^{p-1} \rho_j \), where \( \rho_j = \sqrt{1 - |\alpha_j|^2} \). This prompted the search for an integrable system which is related to OPUC in the same way the Toda lattice is related to orthogonal polynomials on the line. As it turned out, the sought-after integrable system is the periodic defocusing AL equation \([N]\). In \([N]\), the Lax equations for the commuting flows were expressed in terms of the extended CMV matrix \( \mathcal{E} \) with periodic Verblunsky coefficients. However, as \( \Delta(z) \) is related to the Floquet CMV matrix \( \mathcal{E}(h) \) (which is a unitary loop with spectral parameter \( h \)) through the characteristic polynomial \( \det(zI - \mathcal{E}(h)) \), it is natural to ask if the same equations can be rewritten as isospectral deformations of \( \mathcal{E}(h) \). As the reader will see in Section 2, this is indeed the case and the result is the point of departure in this work. More precisely, the result not only suggests that the set of \( p \times p \) Floquet CMV matrices should have some Poisson geometric meaning, but also points to the linearization of such flows on geometric objects related to the Jacobi varieties of the underlying spectral curves. Thus our goal in this work is two-fold. First of all, we will link the Floquet CMV matrices to Poisson Lie groups, analogous to what we did in our earlier work on finite CMV matrices. (See \([L1]\) and \([KN]\).) Secondly, by using the group-theoretic framework, we will study the defocusing AL equation with regard to action-angle variables. We will also solve the commuting Hamiltonian flows via Riemann-Hilbert factorization problems. At this juncture, let us mention some earlier works related to the integration of the periodic defocusing AL equation which is part of our second goal here. To start with, it has been known for quite some time that the defocusing AL equation (1.1) can be represented as a Lax system on a lattice (or discrete zero curvature representation), where the Lax operator \( L_j(z) \) associated to site \( j \) of the lattice is given by (see, for example, \([AL1]\), \([AL2]\) and \([FT]\))

\[
L_j(z) = \begin{pmatrix} z & \alpha_j^* \\ \alpha_j & z^{-1} \end{pmatrix}.
\]

Therefore, in the periodic case with period \( p \), the monodromy matrix \( M(z) = L_{p-1}(z) \cdots L_0(z) \) undergoes an isospectral deformation which means that an equation in Lax pair form (and different from the one we are using here) is known for the periodic defocusing AL equation. In \([MEKL]\), a transformation of a natural generalization of (1.3) was discovered and the result was applied in the construction of finite genus solutions of a more general version of the AL equation. In particular, the authors in \([MEKL]\) were able to write down the solution of the initial value problem for the periodic defocusing AL equation itself. On the other hand, from a different direction, the authors in \([GHMT]\) considered a more general version of the AL hierarchy, and discussed the problem of solving the \( r \)-th AL flow when the initial data is the stationary solution of the \( p \)-th equation of the hierarchy. As the reader will see in Section 6 below, our approach in solving the commuting Hamiltonian flows associated with the periodic defocusing AL equation is quite different from those in these earlier works.

The paper is organized as follows. In Section 2, we begin by recalling the notion of CMV matrices, extended CMV matrices and Floquet CMV matrices. Then we
show how to rewrite the Lax equations for $\mathcal{E}$ of the commuting flows associated with the periodic defocusing AL equation as isospectral deformations of $\mathcal{E}(h)$. To prepare for what we need in subsequent sections, we also discuss the structure of the powers of $\mathcal{E}(h)$. In Section 3, we have two main goals. The first goal is to show that the set of $p \times p$ Floquet CMV matrices is a symplectic leaf of a Poisson Lie group whose underlying group is a loop group $\tilde{G}_w^\mathbb{R}$. Indeed, as one would expect from results in [L1] and [KN] concerning the finite dimensional case, the Poisson structure here is also a Sklyanin structure. In fact, it is the Sklyanin structure associated with the Iwasawa decomposition of the loop group $\tilde{G}_w^\mathbb{R}: \tilde{G}_w^\mathbb{R} = \tilde{K}_w \tilde{B}_w$.

(The decomposition was established in [GW].) However, in order to write down this Sklyanin structure $\{\cdot, \cdot\}_{jH}$, we find it necessary to restrict ourselves to a subclass of functions $\mathcal{F}(\tilde{G}_w^\mathbb{R})$ of $C^\infty(\tilde{G}_w^\mathbb{R})$. Fortunately, $\mathcal{F}(\tilde{G}_w^\mathbb{R})$ forms an algebra of functions under ordinary multiplication and is closed under $\{\cdot, \cdot\}_{jH}$. Hence $\{\cdot, \cdot\}_{jH}$ defines a Poisson bracket on $\mathcal{F}(\tilde{G}_w^\mathbb{R})$. Now note that although we are dealing with a restricted class of functions here, it can be checked that the notion of Poisson Lie groups can be extended to this infinite dimensional context in a rigorous way. Moreover, we can check by hand that the symplectic leaves of $(\tilde{G}_w^\mathbb{R}, \{\cdot, \cdot\}_{jH})$ are still given by the orbits of the dressing action. With this preparation, the technique in [L1] can be naturally extended to show that the set of $p \times p$ Floquet CMV matrices is a dressing orbit through a Coxeter element $x_f$ of the affine Weyl group $W_{aff}$. Indeed, the induced Poisson structure on $\tilde{K}_w$ is a loop group analog of the Bruhat Poisson structure in [LW] and [Soi] and we can show that the set of $p \times p$ Floquet CMV matrices is a product of two dimensional orbits. In the rest of the section, our goal is to clarify the relation between the AL bracket and the Sklyanin bracket $\{\cdot, \cdot\}_{jH}$, and to describe the Hamiltonian equations generated by the central functions on $\tilde{G}_w^\mathbb{R}$, thus connecting the group-theoretic framework with the equations in Section 2.

In Section 4, we study the analytical properties of the Bloch solution of $\mathcal{E}u = zu$, which play an important role in subsequent sections. Since $\mathcal{E}$ defines a (pentadiagonal) periodic difference operator, the seminal work of van Moerbeke and Mumford [MM] comes to mind. However, we note that neither $\mathcal{E}$ nor its factors $\mathcal{L}$ and $\mathcal{M}$ in the theta-factorization of $\mathcal{E}$ satisfy the genericity assumption in [MM]. So the analysis in this section is more delicate than the standard case [MM], [AM]. In Section 5, we start with a simple proof of the involution theorem in (1.2), which is possible because of the group-theoretic setup in Section 3. Then we proceed to construct the angle variables. To compute the Poisson brackets between the conserved quantities and the various quantities related to the putative angles, we make use of a device introduced in [DLT]. We would like to point out that in general, such computations could be difficult because they may require detailed information on the asymptotics of the normalized eigenvectors in neighborhoods of the points at infinity of the Riemann surface. In our case, asymptotics beyond the leading order are difficult to get because we are in a non-generic situation, but fortunately we are saved by some special structure. Finally, in Section 6 we solve the commuting Hamiltonian flows via Riemann-Hilbert factorization problems, which again are suggested by the group-theoretic framework. We remark that it is in this very last
section that we find it advantageous to think of our flows on $\mathcal{E}(h)$ as flows on the factors $g^e$ and $g^o(h)$ in the theta-factorization of $\mathcal{E}(h)$. This is precisely the reason why we introduce Lax systems on a period 2 lattice in Section 3.

2. Preliminaries.

In this section, for the convenience of the reader, we begin with some background material on CMV matrices and the involution theorem of Nenciu-Simon. (Good references are [S2] and [S3].) Then we will show how to rewrite the Lax equation in [N] for the periodic defocusing AL equation (in terms of the extended CMV matrix $\mathcal{E}$) as an isospectral deformation of the Floquet CMV matrix $\mathcal{E}(h)$. We will also present a result on the structure of the powers of $\mathcal{E}(h)$ which we will use in Sections 5 and 6.

The CMV matrices are the unitary analogs of Jacobi matrices [S3] and made their debut in the numerical linear algebra literature. (See [B-GE] and in particular [W].) Subsequently, they were rediscovered by Cantero, Moral and Valázquez [CMV] in the context of the theory of orthogonal polynomials on the circle (OPUC). To introduce these objects, let $\mathbb{D} = \{ z \in \mathbb{C} \mid |z| < 1 \}$, and let $d\mu$ be a nontrivial probability measure on $\partial \mathbb{D}$, then one can produce an orthonormal basis of $L^2(\partial \mathbb{D}, d\mu)$ by applying the Gram-Schmidt process to $1, z, z^{-1}, z^2, z^{-2}, \cdots$. As it turns out [CMV], the matrix representation of the operator $f(z) \mapsto z f(z)$ in $L^2(\partial \mathbb{D}, d\mu)$ with respect to this orthonormal basis is the infinite CMV matrix

$$
\mathcal{C} = \begin{pmatrix}
\bar{\alpha}_0 & \rho_0 \bar{\alpha}_1 & \rho_0 \rho_1 & 0 & 0 & \cdots \\
\rho_0 & -\alpha_0 \bar{\alpha}_1 & -\alpha_0 \rho_1 & 0 & 0 & \cdots \\
0 & \rho_1 \bar{\alpha}_2 & -\alpha_1 \bar{\alpha}_2 & \rho_2 \alpha_3 & \rho_2 \rho_3 & \cdots \\
0 & \rho_1 \rho_2 & -\alpha_1 \rho_2 & -\alpha_2 \bar{\alpha}_3 & -\alpha_2 \rho_3 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix} = \mathcal{L} \tilde{\mathcal{M}} \tag{2.1}
$$

where $\alpha_j \in \mathbb{D}$ are the so-called Verblunsky coefficients, $\rho_j = \sqrt{1 - |\alpha_j|^2}$, $j = 0, 1, \cdots$, and where $\mathcal{L} = \text{diag}(\theta_0, \theta_2, \cdots)$, $\tilde{\mathcal{M}} = \text{diag}(1, \theta_1, \theta_3, \cdots)$, with

$$
\theta_j = \left( \begin{array}{c}
\bar{\alpha}_j \\
\rho_j \\
-\alpha_j \\
\end{array} \right), \quad j = 0, 1, \cdots. \tag{2.2}
$$

The factorization on the right hand side of (2.1) is called the theta-factorization and lends itself to generalization. Indeed, if we now have a two-sided sequence $\{\alpha_j\}_{j=-\infty}^{\infty}$ with $\alpha_j \in \mathbb{D}$ for all $j$, then we can define the extended (two-sided) CMV matrix $\mathcal{E}$ by extending $\mathcal{L}$ and $\tilde{\mathcal{M}}$ to doubly-infinite matrices in the obvious way. (Of course, the $1 \times 1$ block will not appear in this extension.) In this work, we are mainly interested in the case where the sequence $\{\alpha_j\}_{j=0}^{\infty}$ of Verblunsky coefficients is periodic of period $p$ and in this context, it is convenient to extend the one-sided sequence to a two-sided sequence $\{\alpha_j\}_{j=-\infty}^{\infty}$ satisfying the periodicity condition $\alpha_{j+p} = \alpha_j$ for all $j \in \mathbb{Z}$. Thus correspondingly, we have an extended CMV matrix with periodic Verblunsky coefficients and such matrices have been
used to formulate the Lax equation for the periodic defocusing AL equation in [N]. Now suppose $E$ is an extended CMV matrix with periodic Verblunsky coefficients with period $p$. Without loss of generality, we will assume from now onwards that $p$ is even (otherwise, we just replace $p$ by $2p$). Note that if $T$ is the operator on $l^\infty(\mathbb{Z})$ defined by $(Tu)_j = u_{j+p}$, then $ET = TE$. Therefore, if for $h \in \partial \mathbb{D}$, we define

\[ X(h) = \{ u \in l^\infty(\mathbb{Z}) \mid Tu = h^{-1}u \}, \]

then the finite dimensional space $X(h)$ is invariant under $E$. A basis of $X(h)$ is given by the vectors

\[ \delta_k = \sum_{j=-\infty}^{\infty} h^{-j} e_{jp+k}, \quad k = 0, \ldots, p-1 \]

where $e_j$ is the vector in $l^\infty(\mathbb{Z})$ with $j$-th component equal to 1 and zeros elsewhere. By definition, the Floquet CMV matrix $E(h)$ is the matrix of $E \mid X(h)$ with respect to the ordered basis $(\delta_0, \ldots, \delta_{p-1})$, i.e.

\[ E\delta_k = \sum_{j=0}^{p-1} (E(h))_{jk} \delta_j, \quad k = 0, \ldots, p-1. \]

From this, it is clear that the matrix of $E^n \mid X(h)$ with respect to the same ordered basis is $E(h)^n$. Thus the entries of $E(h)^n$ are related to those of $E^n$ by the formula

\[ (E(h)^n)_{jk} = \sum_{l \in \mathbb{Z}} h^{-l} (E^n)_{j,k+lp} \]

for $0 \leq j, k \leq p-1$. Finally, the Floquet CMV matrix $E(h)$ also has a theta-factorization $E(h) = g^e g^o(h)$, where

\[ g^e = \text{diag}(\theta_0, \theta_2, \cdots, \theta_{p-2}), \]

and

\[ g^o(h) = \begin{pmatrix}
-\alpha_{p-1} & 0 & \cdots & 0 & \rho_{p-1}h \\
0 & \theta_1 & & & 0 \\
& \ddots & \ddots & \ddots & \ddots \\
0 & & \theta_{p-3} & 0 & \\
\rho_{p-1}h & 0 & \cdots & 0 & \alpha_{p-1}
\end{pmatrix}. \]

This is of course a consequence of the factorization for the corresponding extended CMV matrix $E$. In case we want to emphasize the dependence of $g^e$ and $g^o$ on $\alpha = (\alpha_o, \cdots, \alpha_{p-1}) \in \mathbb{D}^p$, we also write $g^e = g^e(\alpha)$ and $g^o = g^o(\alpha)$.

Another very important notion associated with periodic Verblunsky coefficients is that of the discriminant introduced in [S2]:

\[ \Delta(z) = z^{-p/2} \text{tr} (T_p(z)), \quad z \in \mathbb{C} \setminus \{0\} \]
where
\[ T_p(z) = \frac{1}{\prod_{j=0}^{p-1} \rho_j} \begin{pmatrix} z & -\bar{\alpha}_{p-1} \\ -\alpha_{p-1} z & 1 \end{pmatrix} \cdots \begin{pmatrix} z & -\bar{\alpha}_0 \\ -\alpha_0 z & 1 \end{pmatrix} \] (2.10)
is the transfer matrix. In [S2], by seeking a Poisson bracket on \( \mathbb{D}^p \) so that the modulus \( P = \prod_{j=0}^{p-1} \rho_j \) generates the Aleksandrov flow, the author arrives at the Ablowitz-Ladik bracket (see [KM] for more general versions of this structure)
\[ \{ f_1, f_2 \}_{AL} = 2i \sum_{j=0}^{p-1} \rho_j^2 \left( \frac{\partial f_1}{\partial \alpha_j} \frac{\partial f_2}{\partial \bar{\alpha}_j} - \frac{\partial f_1}{\partial \bar{\alpha}_j} \frac{\partial f_2}{\partial \alpha_j} \right). \] (2.11)

We recall the involution theorem of Nenciu-Simon which was obtained by calculating with Wall polynomials.

**Theorem 2.1** [N], [S2]. For all \( z, w \in \mathbb{C} \setminus \{0\} \),
\[ \{ \Delta(z), \Delta(w) \}_{AL} = 0, \quad \{ P, \Delta(z) \}_{AL} = 0. \] (2.12)

Hence if \( P \cdot \Delta(z) = \sum_{j=-p/2}^{p/2} I_j z^j \), the functions \( P, I_0, \text{Re} I_j, \text{Im} I_j, j = 1, \cdots, p/2 - 1 \) Poisson commute with each other.

This result, when combined with the proof that the functions \( P, I_0, \text{Re} I_j, \text{Im} I_j, j = 1, \cdots, p/2 - 1 \) are functionally independent on an open dense subset of \( \mathbb{D}^p \) [S2], shows that any of the functions in the above list generates a completely integrable Hamiltonian system. To write down the Lax pairs, the author in [N] actually considered a different, but equivalent set of commuting Hamiltonians, which are constructed from the real and imaginary parts of
\[ K_n = \frac{1}{n} \sum_{k=0}^{p-1} (\mathcal{E}^n)_{kk}, \quad 1 \leq n \leq p/2 - 1, \]
together with \( K_{p/2} \) and \( P \). Indeed, an easy computation shows that
\[ \{ \text{Re}(K_1), \alpha_j \}_{AL} = i\rho_j^2 (\alpha_{j-1} + \alpha_{j+1}), \quad \{ \text{log}(P), \alpha_j \}_{AL} = i\alpha_j \] (2.13)
for all \( 0 \leq j \leq p - 1 \). Hence the periodic defocusing AL equation is generated by the Hamiltonian \( \text{Re}(K_1) - 2 \log(P) \).

To relate the \( K_n \)'s to the Floquet CMV matrix, and to the coefficients of \( P \cdot \Delta(z) \), first recall that [S2]
\[ \det(zI - \mathcal{E}(h)) = \left( \prod_{j=0}^{p-1} \rho_j \right) z^{\frac{n}{2}} [\Delta(z) - (h + h^{-1})]. \] (2.14)

In view of (2.13), it is clear we must consider the structure of the powers of \( \mathcal{E} \). We will skip the proof of the following result which can be established by induction on \( n \).
Lemma 2.2. For \( n \geq 1 \), \((E^n)_{j,k}\) is identically zero if one of the following holds:
(a) \( |j - k| \geq 2n + 1 \), or
(b) \( j - k = 2n \), where \( j \) and \( k \) are both even, or
(c) \( j - k = -2n \) where \( j \) and \( k \) are both odd.

As a consequence of this result, note that for \( n \leq \frac{p}{2} - 1 \) and for \( 0 \leq k \leq p - 1 \), we have
\[
(E(h)^n)_{k,k} = (E^n \delta_k)_k = \sum_{q \in \mathbb{Z}} h^{-q}(E^n)_{k,qp+k} = (E^n)_{k,k}
\]
Since \( |k - (pq + k)| \geq p \geq 2n + 2 \) for all \( q \neq 0 \). Hence it follows that (cf. (5.4))
\[
K_n = \frac{1}{n} \oint_{|h|=1} \text{tr}(E(h)^n) \frac{dh}{2\pi ih}, \quad n = 1, \cdots, p/2 - 1.
\]

From the relation in (2.14), we see that the integral in (2.16) for \( n = p/2 \) are also relevant, as this is related to \( I_0 \) and \( P \). But by an induction argument similar to the proof of Lemma 2.2 above, we can show that
\[
(E^m)_{k,k+2m} = \prod_{j=k}^{k+2m-1} \rho_j \quad \text{if } k \text{ is even}
\]
and
\[
(E^m)_{k,k-2m} = \prod_{j=k-2m}^{k-1} \rho_j \quad \text{if } k \text{ is odd}.
\]
Since \( P = \prod_{j=k}^{k+p-1} \rho_j \) for any \( k \in \mathbb{Z} \), it follows that
\[
(E(h)^{p/2})_{k,k} = \begin{cases} (E^{p/2})_{k,k} + h^{-1} \cdot P & \text{if } k \text{ is even} \\ (E^{p/2})_{k,k} + h \cdot P & \text{if } k \text{ is odd} \end{cases}
\]
and therefore
\[
\text{tr}(E(h)^{p/2}) = \text{tr}(E^{p/2}) + \frac{p}{2}(h + h^{-1})P.
\]
Thus it follows from (2.20) and (2.15) that
\[
K_{p/2} = \frac{2}{p} \oint_{|h|=1} \text{tr}(E(h)^{p/2}) \frac{dh}{2\pi ih}
\]
and so the set of Hamiltonians in Theorem 2.1 is equivalent to \( P, \text{Re} K_j, \text{Im} K_j, j = 1, \cdots, p/2 - 1, K_{p/2} \).

The next result gives the Lax equations of the Hamiltonian systems generated by the above set of functions and is the central result of [N]. We will use the following notation: for an infinite two-sided matrix \( A \), \( \Pi_u(A) = \frac{1}{2}A_0 + A_+ \), where \( A_+ \) is the upper triangular part of \( A \) and \( A_0 \) is the diagonal part.
Theorem 2.3. For $1 \leq n \leq p/2$, 
(a) the Hamiltonian equation generated by $\Re K_n$ can be expressed as 
\[ \dot{\mathcal{E}} = [\mathcal{E}, i\Pi_u(\mathcal{E}^n) + i((\Pi_u(\mathcal{E}^n))^*)], \] 
(2.22) 
(b) the Hamiltonian equation generated by $\Im K_n$ can be expressed as 
\[ \dot{\mathcal{E}} = [\mathcal{E}, \Pi_u(\mathcal{E}^n) - ((\Pi_u(\mathcal{E}^n))^*)]. \] 
(2.23)

Our next goal is to compute the evolution of $\mathcal{E}(h)$ under (2.22) and (2.23). In order to do this, we have to establish the following result.

Proposition 2.4. Let $n \leq p/2 - 1$, then the structure of $\mathcal{E}(h)^n$ as a Laurent polynomial in $h$ is given by 
\[ \mathcal{E}(h)^n = A_0(n) + hA_1(n) + h^{-1}A_{-1}(n), \] 
(2.24) 
where $A_1(n)$ is strictly upper triangular and $A_{-1}(n)$ is strictly lower triangular.

Proof. In order to prove (2.24), let $0 \leq j, k \leq p - 1$ be two indices. Then as in (2.15), we have 
\[ (\mathcal{E}(h)^n)_{jk} = \sum_{q \in \mathbb{Z}} h^{-q}(\mathcal{E}^n)_{j,k+pq}. \] 
(2.25)

Now, note that for $|q| \geq 2$ and any $j, k$, we have the inequality 
\[ |j - (k + pq)| \geq 2p - |j - k| \geq 2p - (p - 1) = p + 1 \geq 2n + 1 \]
and therefore $(\mathcal{E}^n)_{j,k+pq} \equiv 0$. Hence formula (2.24) holds for some matrices $A_{\pm 1}(n)$.

To find the structure of the matrices $A_{\pm 1}(n)$, note that for $j \geq k$, we have $|j - (k - p)| = p + (j - k) \geq p \geq 2n + 1$ and so it follows from Lemma 2.2 that $(\mathcal{E}^n)_{j,k-p} \equiv 0$. Consequently, 
\[ (\mathcal{E}(h)^n)_{jk} = (\mathcal{E}^n)_{jk} + h^{-1}(\mathcal{E}^n)_{j,k+p} \quad \text{for } j \geq k. \] 
(2.26)

A similar argument shows that 
\[ (\mathcal{E}(h)^n)_{jk} = (\mathcal{E}^n)_{jk} + h(\mathcal{E}^n)_{j,k-p} \quad \text{for } j \leq k. \] 
(2.27)

These last two equations then establish our claim about the triangularity of $A_{\pm 1}(n)$.

In fact, we find that 
\[ (A_0(n))_{jk} = (\mathcal{E}^n)_{jk} \quad \text{for all } 0 \leq j, k \leq p - 1, \] 
(2.28) 
\[ (A_1(n))_{jk} = \begin{cases} 
(\mathcal{E}^n)_{j,k-p} & \text{if } j < k; \\
0 & \text{if } j \geq k,
\end{cases} \] 
(2.29) 
and 
\[ (A_{-1}(n))_{jk} = \begin{cases} 
(\mathcal{E}^n)_{j,k+p} & \text{if } j > k; \\
0 & \text{if } j \leq k.
\end{cases} \] 
(2.30)

This completes the proof. \qed

We are now ready to give the result alluded to above which is the point of departure in this work. We will make use of the projections $\Pi_E$ and $\Pi_{\mathcal{T}^w}$ introduced in Section 3 below. (See the second paragraph of Section 3 and (3.12).)
Proposition 2.5. (a) If $E$ evolves according to (2.22), then
$$\dot{E}(h) = [E(h), \Pi_{\tilde{w}}(iE(h)^n)].$$
(2.31)

(b) If $E$ evolves according to (2.23), then
$$\dot{E}(h) = [E(h), \Pi_{\tilde{w}}(E(h)^n)].$$
(2.32)

Proof. Let $Q_n(h)$ be the matrix of $\Pi_u(E^n)|X(h)$ with respect to the ordered basis $(\delta_0, \cdots, \delta_{p-1})$. To establish (2.31) and (2.32), it suffices to prove that
$$Q_n(h) = \frac{1}{2}(A_0(n))_0 + (A_0(n))_+ + h^{-1}A_{-1}(n),$$
(2.33)
where $A_0(n)$ and $A_{-1}(n)$ are the matrices in (2.24), and where $(A_0(n))_+$ and $(A_0(n))_0$ are respectively the upper triangular part and diagonal part of $A_0(n)$. But by a direct calculation and making use of Lemma 2.2, we find
$$E^n_+ \delta_k = h^{-1} \sum_{j=0}^{p-1} (E^n)_{j-p,k} \delta_j + \sum_{j=0}^{k-1} (E^n)_{jk} \delta_j.$$  
(2.34)
Hence from (2.28) and (2.30), we conclude that the matrix of $(E^n)_+|X(h)$ with respect to the ordered basis $(\delta_0, \cdots, \delta_{p-1})$ is given by $h^{-1}A_{-1}(n) + A_0(n)_+$. On the other hand, it is easy to see that the matrix of $(E^n)_0|X(h)$ with respect to the same basis is $(A_0(n))_0$. Hence (2.33) follows. $\square$

We close this section with an important remark about the periodic defocusing AL equation itself. Namely, if $\dot{\alpha}_j = i\alpha_j^2(\alpha_{j-1} + \alpha_{j+1})$, or equivalently, $\dot{E}(h) = [E(h), \Pi_{\tilde{w}}(iE(h))]$, then $\beta_j(t) = e^{-2i\alpha_j(t)}$ satisfies the periodic defocusing AL equation and vice versa. Thus in order to solve the periodic defocusing AL equation, it suffices to solve $\dot{E}(h) = [E(h), \Pi_{\tilde{w}}(iE(h))]$, which is generated by the Hamiltonian $\text{Re} K_1 = -\text{Re} I_{p/2-1}(E(h))$. (Compare (2.16) and (5.4).) We will solve the Hamiltonian equations generated by the conserved quantities in Theorem 2.1 in Section 6 below.

3. Floquet CMV matrices, dressing orbits and Hamiltonian flows.

The first goal of this section is to show that the set of Floquet CMV matrices is a symplectic leaf of a Poisson Lie group whose underlying group is a loop group. Indeed, by following the method of investigation in [L1], we will show that there exist symplectic leaves of a more elementary nature in terms of which we can describe the collection of Floquet CMV matrices.

As explained in Section 2 above, we can assume that $p$ is even. Let $G^\mathbb{R}$ be $GL(p, \mathbb{C})$ considered as a real Lie group, and let $K$ and $B$ be respectively the unitary
group $U(p)$ and the lower triangular group of $p \times p$ matrices with positive diagonal entries. It is well-known that $G_\mathbb{R}$ admits the Iwasawa decomposition $G_\mathbb{R} = KB$. On the Lie algebra level, this corresponds to $g_\mathbb{R} = \mathfrak{k} \oplus \mathfrak{b}$ (with associated projections $\Pi_\mathfrak{k}, \Pi_\mathfrak{b}$), where $g_\mathbb{R}$, $\mathfrak{k}$ and $\mathfrak{b}$ are, respectively, the Lie algebras of $G_\mathbb{R}$, $K$ and $B$. Later on in the section, we will also need the maximal torus $T$ of $K$, consisting of unitary diagonal matrices.

Let $\tilde{G}_\mathbb{R} = C^\infty(S^1, G_\mathbb{R})$ be the smooth loop group with the $C^\infty$ topology. $\tilde{G}_\mathbb{R}$ is a Fréchet Lie group with the Lie algebra $\tilde{g}_\mathbb{R} = C^\infty(S^1, g_\mathbb{R})$. We will use the following nondegenerate ad-invariant pairing on $\tilde{g}_\mathbb{R}$:

$$(X, Y) = \text{Im} \oint_{|h|=1} \text{tr}(X(h)Y(h)) \frac{dh}{2\pi ih}. \quad (3.1)$$

As the reader will see, this choice is critical for what we have in mind.

Following [GW], choose a symmetric weight function $w : \mathbb{Z} \rightarrow \mathbb{R}_+$, which is rapidly increasing in the sense that

$$\lim_{n \rightarrow \infty} w(n)n^{-s} = \infty, \quad \forall s > 0. \quad (3.2)$$

Also, assume that $w$ is of non-analytic type:

$$\lim_{n \rightarrow \infty} w(n)^{1/n} = 1. \quad (3.3)$$

For $X \in \tilde{g}_\mathbb{R}$ given by $X(h) = \sum_{j=-\infty}^{\infty} X_j h^j$, we define

$$\|X\|_w = \sum_{j=-\infty}^{\infty} \|X_j\|w(j), \quad (3.4)$$

where $\| \cdot \|$ is a norm on $g_\mathbb{R}$. Also, set

$$(P_+ X)(h) = \sum_{j>0} X_j h^j, \quad (P_- X)(h) = \sum_{j<0} X_j h^j, \quad (P_0 X)(h) = X_0. \quad (3.5)$$

Consider the Banach Lie group

$$\tilde{G}_w^\mathbb{R} = \{ g \in \tilde{G}_\mathbb{R} \mid \|g\|_w < \infty \} \quad (3.6)$$

with Lie algebra

$$\tilde{g}_w^\mathbb{R} = \{ X \in \tilde{g}_\mathbb{R} \mid \|X\|_w < \infty \}. \quad (3.7)$$

From [GW], we have the Iwasawa decomposition for the loop group $\tilde{G}_w^\mathbb{R}$ and its Lie algebra

$$\tilde{G}_w^\mathbb{R} = \tilde{K}_w \cdot \tilde{B}_w, \quad \tilde{g}_w^\mathbb{R} = \tilde{\mathfrak{k}}_w \oplus \tilde{\mathfrak{b}}_w \quad (3.8)$$

where

$$\tilde{K}_w = \{ g \in \tilde{G}_w^\mathbb{R} \mid g^* g = I \}, \quad \tilde{B}_w = \{ g \in \tilde{G}_w^\mathbb{R} \mid P_- g = 0, P_0 g \in B \} \quad (3.9)$$
are Banach Lie subgroups of $\tilde{G}_w^R$ and $\tilde{t}_w, \tilde{b}_w$ are their respective Lie algebras. Denote by $k: \tilde{G}_w^R \rightarrow \tilde{K}_w$, $b: \tilde{G}_w^R \rightarrow \tilde{B}_w$, the analytic maps defined by the factorization $g = k(g)b(g)^{-1}, g \in \tilde{G}_w^R$. Also, denote by $\Pi_{\tilde{t}_w}$ and $\Pi_{\tilde{b}_w}$, the projection maps relative to the splitting $\tilde{g}_w^R = \tilde{t}_w \oplus \tilde{b}_w$. Then from standard classical r-matrix theory [STS1], [STS2],

$$J^2 = \Pi_{\tilde{t}_w} - \Pi_{\tilde{b}_w}$$

(3.10)

is a solution of the modified Yang-Baxter equation (mYBE). Hence we can equip $\tilde{g}_w^R$ with the $J^2$-bracket

$$[X,Y]_{J^2} = \frac{1}{2}([J^2X,Y] + [X,J^2Y]).$$

(3.11)

We will denote the vector space $\tilde{g}_w^R$ equipped with the Lie bracket $[\cdot,\cdot]_{J^2}$ by $(\tilde{g}_w^R)_{J^2}$. In fact, it is easy to check from (3.11) that $(\tilde{g}_w^R)_{J^2} = \tilde{t}_w \oplus \tilde{b}_w$ (Lie algebra antidirect sum). Note that explicitly, the projection maps $\Pi_{\tilde{t}_w}, \Pi_{\tilde{b}_w}$ are given by the formulas

$$\Pi_{\tilde{t}_w} X = P_+ X + \Pi_{\tilde{t}_w}X_0 - (P_- X)^*,
\Pi_{\tilde{b}_w} X = P_- X + \Pi_{\tilde{b}_w}X_0 + (P_- X)^*,$$

(3.12)

where $P_\pm$ are defined in (3.5).

In order to introduce the Poisson structure on $\tilde{G}_w^R$, it is necessary to restrict ourselves to a subclass of functions in $C^\infty(\tilde{G}_w^R)$. We say that a function $\varphi \in C^\infty(\tilde{G}_w^R)$ is smooth at $g \in \tilde{G}_w^R$ iff there exists $D\varphi(g) \in \tilde{g}_w^R$ (called the right gradient of $\varphi$ at $g$) such that

$$\left.\frac{d}{dt}\right|_{t=0} \varphi(e^{tx}g) = (D\varphi(g), X), \quad X \in \tilde{g}_w^R$$

(3.13)

where $\langle \cdot, \cdot \rangle$ is the pairing in (3.1). If $\varphi \in C^\infty(\tilde{G}_w^R)$ is smooth at $g$ for all $g \in \tilde{G}_w^R$, then we say it is smooth on $\tilde{G}_w^R$. Note that the nondegeneracy of $\langle \cdot, \cdot \rangle$ implies that the map

$$i: \tilde{g}_w^R \rightarrow (\tilde{g}_w^R)^*, \quad X \mapsto \langle X, \cdot \rangle$$

(3.14)

is an isomorphism onto a subspace of $(\tilde{g}_w^R)^*$ which we will call the smooth part of $(\tilde{g}_w^R)^*$. Thus $\varphi \in C^\infty(\tilde{G}_w^R)$ is smooth at $g$ iff $T_e^*\varphi(g) d\varphi(g)$ is in the smooth part of $(\tilde{g}_w^R)^*$ and we can define the left gradient of such a function at $g$ by

$$\left.\frac{d}{dt}\right|_{t=0} \varphi(ge^{tx}) = (D'\varphi(g), X), \quad X \in \tilde{g}_w^R.$$ 

(3.15)

For each $g \in \tilde{G}_w^R$, we will denote the collection of all smooth functions at $g$ by $\mathcal{F}_g(\tilde{G}_w^R)$ and we set $\mathcal{F}(\tilde{G}_w^R) = \cap_{g \in \tilde{G}_w^R} \mathcal{F}_g(\tilde{G}_w^R)$. With the above considerations, it is easy to check that $\mathcal{F}(\tilde{G}_w^R)$ is non-empty and forms an algebra under ordinary multiplication of functions.
Proposition 3.1. (a) For \( \varphi, \psi \in \mathcal{F}(\tilde{G}_w^R) \) and \( g \in \tilde{G}_w^R \), define
\[
\{ \varphi, \psi \}_{J^*} (g) = \frac{1}{2} (J^*(D'\varphi(g)), D'\psi(g)) - \frac{1}{2} (J^*(D\varphi(g)), D\psi(g)).
\] (3.16)

Then \( \{ \varphi, \psi \}_{J^*} \in \mathcal{F}(\tilde{G}_w^R) \) and hence \( \{ \cdot, \cdot \}_{J^*} \) defines a Poisson bracket on \( \mathcal{F}(\tilde{G}_w^R) \).

(b) The Hamiltonian equation of motion generated by \( \varphi \in \mathcal{F}(\tilde{G}_w^R) \) is given by
\[
\dot{g} = g (\Pi_{\tilde{\mathfrak{g}}_w} (D'\varphi(g))) - (\Pi_{\tilde{\mathfrak{g}}_w} (D\varphi(g))) g
= (\Pi_{\tilde{\mathfrak{g}}_w} (D\varphi(g))) g - g (\Pi_{\tilde{\mathfrak{g}}_w} (D'\varphi(g))).
\] (3.17)

Proof. (a) A straightforward calculation shows that for \( \varphi, \psi \in \mathcal{F}(\tilde{G}_w^R) \), \( D \{ \varphi, \psi \}_{J^*} (g) \) exists for each \( g \in \tilde{G}_w^R \) and is given by
\[
D \{ \varphi, \psi \}_{J^*} (g) = Ad_g [D'\varphi(g), D'\psi(g)]_{J^*} + Ad_g \frac{d}{ds} \big|_{s=0} D'\varphi(e^{s\eta(g)}D\varphi(g))
- Ad_g \frac{d}{ds} \big|_{s=0} D'\psi(e^{s\eta(g)}D\varphi(g)),
\] (3.18)

where
\[
\eta(g) = \frac{1}{2} Ad_g \circ J^* \circ Ad_g - 1 - \frac{1}{2} J^*.
\] (3.19)

This shows \( \{ \varphi, \psi \}_{J^*} \in \mathcal{F}(\tilde{G}_w^R) \). To prove the second half of (a), first note that \( \text{Tr}(X(h)Y(h)) \in \mathbb{R} \) for \( X, Y \in \tilde{\mathfrak{g}}_w \). From this, it follows that \( \langle X, Y \rangle = 0 \). Consequently, \( \tilde{\mathfrak{g}}_w \) is an isotropic subalgebra of \( \tilde{\mathfrak{g}}_w^R \) relative to the pairing \( \langle \cdot, \cdot \rangle \). On the other hand, if \( X, Y \in \tilde{\mathfrak{g}}_w \), we have
\[
\int_{|h|=1} \text{tr}(X(h)Y(h)) \frac{dh}{2\pi i} = \text{tr}(X_0Y_0) \in \mathbb{R}
\]
because \( X_0, Y_0 \) are lower triangular with real diagonal entries. \( \tilde{\mathfrak{g}}_w \) is also an isotropic subalgebra of \( \tilde{\mathfrak{g}}_w^R \). Combining these two facts, we can now conclude that \( J^* \) is skew-symmetric relative to \( \langle \cdot, \cdot \rangle \). Finally, the Jacobi identity and the derivation property now follow from standard calculations in [STS2] which work without change in our infinite dimensional context.

(b) This derivation of the Lax equation from the Poisson structure is standard.

Note that although we are dealing with a restricted class of functions, the notion of Poisson submanifolds can be defined analogously to the standard case. Now it is easy to check that if \( \varphi \in \mathcal{F}(\tilde{G}_w^R) \), then so are \( \varphi \circ r_g \) and \( \varphi \circ l_g \) for all \( g \in \tilde{G}_w^R \), where \( r_g \) and \( l_g \) denote right and left translation by \( g \), respectively. Hence the notion of Poisson Lie group can be extended to this infinite dimensional context and \( (\tilde{G}_w^R, \{ \cdot, \cdot \}_{J^*}) \) is a coboundary Poisson Lie group. On the infinitesimal level, we will call \((\tilde{\mathfrak{g}}_w^R, \{ \cdot, \cdot \}_{J^*})\) the tangent Lie bialgebra of \( (\tilde{G}_w^R, \{ \cdot, \cdot \}_{J^*}) \) as the map \( \rho : \tilde{\mathfrak{g}}_w^R \to L(\tilde{\mathfrak{g}}_w^R, \tilde{\mathfrak{g}}_w^R) \) (\( L(\tilde{\mathfrak{g}}_w^R, \tilde{\mathfrak{g}}_w^R) \) is the space of linear maps on \( \tilde{\mathfrak{g}}_w^R \)) given by \( \rho(X) = \frac{1}{2} (ad_X \circ J^* - J^* \circ ad_X) \) and satisfying the relation \( [Y,Z]_{J^*} = [Z, \rho(X)(Y)] \) is a 1-coboundary (and hence a 1-cocycle) with respect to the adjoint representation. Thus in speaking of a Lie bialgebra here, the underlying vector spaces of the pair of Lie algebras involved are only required to be in duality with respect to \( \langle \cdot, \cdot \rangle \) and this is what we will continue to do.
Corollary 3.2. (a) $\tilde{K}_w$ is a Poisson Lie subgroup of $(\tilde{G}^R_w, \{\cdot, \cdot\}_{J^1})$ in the sense that $\tilde{K}_w$ is a Lie subgroup of $\tilde{G}^R_w$ which is also a Poisson submanifold of $(\tilde{G}^R_w, \{\cdot, \cdot\}_{J^1})$. Moreover, the tangent Lie bialgebra $(\tilde{\mathfrak{g}}^R_w, (\tilde{\mathfrak{g}}^R_w, J^1) / (\tilde{\mathfrak{k}}^R_w))$ of $\tilde{K}_w$ (where $\tilde{\mathfrak{k}}^R_w$ is defined relative to $(\cdot, \cdot)$) is isomorphic to $(\mathfrak{t}_w, \mathfrak{b}_w)$ where $\mathfrak{b}_w$ is $\mathfrak{b}$ but equipped with the $-$ bracket.

(b) The underlying group of the Poisson group $(\tilde{G}^R_w)_{J^1}$ dual to $(\tilde{G}^R_w, \{\cdot, \cdot\}_{J^1})$ consists of $\tilde{G}^R_w$ equipped with the multiplication

$$g * h = k(g)hb(g)^{-1}.$$ (3.20)

Proof. (a) To show that $\tilde{K}_w$ is a Poisson Lie subgroup, it is enough to check that $\tilde{K}_w$ is a Poisson submanifold of $(\tilde{G}^R_w, \{\cdot, \cdot\}_{J^1})$ and this can be done by using the expression for the Hamiltonian vector field in Proposition 3.1(b). To show that the tangent Lie bialgebra $(\tilde{\mathfrak{g}}^R_w, (\tilde{\mathfrak{g}}^R_w, J^1) / (\tilde{\mathfrak{k}}^R_w))$ of $\tilde{K}_w$ is isomorphic to $(\mathfrak{t}_w, \mathfrak{b}_w)$, first note that $\tilde{\mathfrak{t}}_w = \tilde{\mathfrak{t}}_w$. Hence we have $X + \tilde{\mathfrak{t}}^R_w = \tilde{\Pi}_{\mathfrak{b}_w} X + \tilde{\mathfrak{t}}_w$ for all $X \in (\tilde{\mathfrak{g}}^R_w)_{J^1}$. Therefore, the induced Lie bracket on $(\tilde{\mathfrak{g}}^R_w)_{J^1} / (\tilde{\mathfrak{k}}^R_w)$ is given by

$$[X + \tilde{\mathfrak{t}}_w, Y + \tilde{\mathfrak{t}}_w] = [\tilde{\Pi}_{\mathfrak{b}_w} X, \tilde{\Pi}_{\mathfrak{b}_w} Y]_{J^1} + \tilde{\mathfrak{t}}_w$$

Consequently, the map $X + \tilde{\mathfrak{t}}_w \mapsto \tilde{\Pi}_{\mathfrak{b}_w} X$ is an isomorphism, when $\mathfrak{b}_w$ is equipped with the $-$ bracket.

(b) The formula is a consequence of the fact that $\tilde{\mathfrak{g}}^R_w = \tilde{\mathfrak{t}}_w \ominus \mathfrak{b}_w$ and can be verified easily.

Remark 3.3. In view of the second half of Corollary 3.2 (a), the induced structure on $\tilde{K}_w$ is the loop group analog of the Bruhat Poisson structure in [LW] and [Soi].

We next turn to the description of the symplectic leaves of the Sklyanin structure in (3.16). Unfortunately, we cannot assume the general results in [STS2] and [LW] apply to our case without some verification, because the analysis in these works is for finite dimensional Poisson Lie groups. In this regard, let us also remark that as far as we know, the integrability of the characteristic distribution of a Poisson structure on an infinite dimensional manifold is by no means automatic because an analog of the Stefan-Sussmann result [St, Su] is not available. In the following, we will check things by hand. So let us define

$$S_x = \{X_\varphi(x) \mid \varphi \in \mathcal{F}_x(\tilde{G}^R_w)\}$$ (3.21)
and the infinitesimal generator of this action corresponding to $\xi \in \tilde g_w^\mathbb{R}$ is the vector field $\xi_{\tilde G_w^\mathbb{R}}$ on $\tilde G_w^\mathbb{R}$, where
\begin{equation}
\xi_{\tilde G_w^\mathbb{R}}(x) = \frac{1}{2} xJ^i(x^{-1}\xi x) - \frac{1}{2} J^i(\xi)x. \quad (3.23)
\end{equation}
For each $x \in \tilde G_w^\mathbb{R}$, let $F_x$ be the subspace of $T_x\tilde G_w^\mathbb{R}$ spanned by the vectors $\xi_{\tilde G_w^\mathbb{R}}(x)$. Then $F = \bigcup_{x \in \tilde G_w^\mathbb{R}} F_x$ is an integrable generalized distribution whose leaves are the orbits of the dressing action $\Phi$.

**Proposition 3.4.** For each $x \in \tilde G_w^\mathbb{R}$, $T_x\mathcal{O} = S_x$, where $\mathcal{O}$ is the orbit of the dressing action containing $x$. Hence the characteristic distribution $S$ is integrable and the leaves of this distribution are given by the orbits of the dressing action $\Phi$.

**Proof.** From (3.7),
\begin{equation}
X_\varphi(x) = \frac{1}{2} xJ^i(x^{-1}D\varphi(x)x) - \frac{1}{2} J^i(D\varphi(x))x. \quad (3.24)
\end{equation}
To show that $S_x = F_x$, it suffices to show that for each $\xi \in \tilde g_w^\mathbb{R}$, there exists $\varphi \in F_x (\tilde g_w^\mathbb{R})$ such that $D\varphi(x) = \xi$. To do so, we use the fact that the exponential map $\exp : \tilde g_w^\mathbb{R} \rightarrow \tilde G_w^\mathbb{R}$ is a diffeomorphism of a neighborhood $U$ of $0$ onto a neighborhood $V$ of the identity element of $\tilde G_w^\mathbb{R}$ [GW]. Clearly, $Vx$ is a neighborhood of $x$ and in this neighborhood, define $\varphi_\xi(g) = (\xi, \log(gx^{-1}))$ and extend this to a function $\varphi_\xi \in C^\infty(\tilde G_w^\mathbb{R})$. Then $\varphi_\xi \in F_x (\tilde G_w^\mathbb{R})$ with $D\varphi_\xi(x) = \xi$. This completes the proof. □

Let $\mathcal{O}$ be a dressing orbit, as in the proposition above. We next show that $\mathcal{O}$ is a symplectic leaf of $(\tilde G_w^\mathbb{R}, \{\cdot, \cdot\}_{J^i})$, i.e., there exists a weak (resp. strong) symplectic form (see, for example, [OR] for such matters) $\omega_\mathcal{O}$ on $\mathcal{O}$ consistent with the Poisson bracket $\{\cdot, \cdot\}_{J^i}$ in the case when $\mathcal{O}$ is infinite (resp. finite) dimensional. For this purpose, let $(\tilde g_w^\mathbb{R})^*_S$ denote the smooth part of $(\tilde g_w^\mathbb{R})^*$ and let $(T^*\tilde G_w^\mathbb{R})_S = \bigcup_{g \in \tilde G_w^\mathbb{R}} T^*g\tau_{g-1}(\tilde g_w^\mathbb{R})^*_S$. Also, denote by $\pi^i : (T^*\tilde G_w^\mathbb{R})_S \rightarrow T\tilde G_w^\mathbb{R}$ the bundle map corresponding to $\{\cdot, \cdot\}_{J^i}$ and let $j$ be the left inverse of $i : \tilde g_w^\mathbb{R} \rightarrow (\tilde g_w^\mathbb{R})^*$. For each $x \in \mathcal{O}$, we define a skew-symmetric bilinear form $\omega_x$ on $S_x$ by the formula
\begin{align}
\omega_x(\pi^i(x)(\alpha), \pi^j(x)(\beta)) \\
= (T_x \tau_{x^{-1}}\pi^i(x)(\alpha), j(T^*_x \tau_{x}\beta)) \\
= -(j(T^*_x \tau_{x}\alpha), T_x \tau_{x^{-1}}\pi^j(x)(\beta)).
\end{align}
Clearly, the value of the above expression depends only on the values of $\pi^i(x)(\alpha)$ and $\pi^j(x)(\beta)$. Thus $\omega_x$ is a well-defined skew-symmetric bilinear form on $S_x$. Now suppose $\omega_x(\pi^i(x)(\alpha), \pi^j(x)(\beta)) = 0$ for all $\pi^i(x)(\beta)$. Then from (3.25) and the nondegeneracy of $\langle \cdot, \cdot \rangle$, it follows that $j(T^*_x \tau_{x}\alpha) = 0$ which in turn implies $\pi^j(x)(\alpha) = 0$. So this establishes the nondegeneracy of $\omega_x$. Thus there exists a 2-form $\omega_\mathcal{O}$ on $\mathcal{O}$ such that $\omega_\mathcal{O}(x) = \omega_x$ for each $x \in \mathcal{O}$. Now the argument that $\omega_\mathcal{O}$ is differentiable and closed follows as in the finite dimensional case in [Ko]. Consequently, $\omega_\mathcal{O}$ defines a Poisson structure $\{\cdot, \cdot\}_{\mathcal{O}}$ on $\mathcal{O}$ and we have
\begin{equation}
\{\varphi, \psi\}_{J^i} \mid \mathcal{O} = \{\varphi \mid \mathcal{O}, \psi \mid \mathcal{O}\}_{\mathcal{O}} \quad (3.26)
\end{equation}
for all $\varphi, \psi \in F(\tilde G_w^\mathbb{R})$. Hence we have established the following result.
Proposition 3.5. The symplectic leaves of \((\tilde{G}_w^\mathbb{R},\{\cdot,\cdot\}_{J^1})\) are given by the orbits of the dressing action \(\Phi\).

In our next step, we will make the connection between certain symplectic leaves of \((\tilde{G}_w^\mathbb{R},\{\cdot,\cdot\}_{J^1})\) (these are also those of the Poisson Lie subgroup \(\tilde{K}_w\)) and the Floquet CMV matrices. We begin with some notations. As in [L1], we will denote by \(g^e\) any \(p \times p\) block diagonal matrices with \(2 \times 2\) diagonal blocks of the form

\[
\begin{pmatrix}
\bar{\alpha} & \rho \\
\rho & -\alpha
\end{pmatrix}, \quad \text{with } \alpha \in \mathbb{D} \quad \text{and} \quad \rho = \sqrt{1 - |\alpha|^2}.
\]

(3.27)

We will denote the collection of such matrices by \(T^e\). On the other hand, we will denote by \(g^o(h)\) any loops in \(\tilde{K}_w\) of the form

\[
\begin{pmatrix}
\bar{\alpha} & 0 & \cdots & 0 & \rho h \\
0 & 0 & \ddots & \vdots & \vdots \\
\ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & -\alpha
\end{pmatrix},
\]

(3.28)

where \(\alpha \in \mathbb{D}\), \(\rho = \sqrt{1 - |\alpha|^2}\) and \(\tilde{b}\) is a \((p - 2) \times (p - 2)\) block diagonal matrix with \(2 \times 2\) blocks of the same kind as in \(g^e\). We will denote the collection of such unitary loops by \(T^o\). Clearly, for given \(g^e \in T^e\) and \(g^o \in T^o\), the product \(g^e g^o(h)\) is a Floquet CMV matrix. Indeed, the map

\[
m | T^e \times T^o : T^e \times T^o \to \{p \times p \text{ Floquet CMV matrices}\}
\]

(3.29)

\[
(g^e, g^o) \mapsto g^e g^o
\]

is a diffeomorphism, where \(m : \tilde{K}_w \times \tilde{K}_w \to \tilde{K}_w\) is the multiplication map of the Poisson Lie subgroup \(\tilde{K}_w\). Finally, we will denote the dressing orbit through

\[
x \in \tilde{G}_w^\mathbb{R} \text{ by } O_x.
\]

In analogy to formula (2.21) in [L1], we introduce the following special Floquet CMV matrix

\[
x_f(h) = x_f^e x_f^o(h)
\]

(3.30)

corresponding to

\[
\underline{\alpha} = (0,0,\cdots,0).
\]

(3.31)

In other words,

\[
x_f^e = \text{diag}(w^*,w^*,\cdots)
\]

(3.32)

and

\[
x_f^o(h) = \begin{pmatrix}
0 & 0 & \cdots & 0 & h \\
0 & w^* & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & w^* & 0 \\
h^{-1} & 0 & \cdots & 0 & 0
\end{pmatrix}
\]

(3.33)
where

\[
wx^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\] (3.34)

These matrices are elements of the affine Weyl group \(W_{aff} = W \times \hat{T} [PS]\), where \(W = N(T)/T\) is the Weyl group of \(K\), and \(\hat{T}\) is the lattice of homomorphisms \(S^1 \to T\). Indeed, if \(\mathfrak{h}\) is the Cartan subalgebra of \(\mathfrak{g}\) consisting of diagonal matrices, \(\lambda_i - \lambda_j, i \neq j\) are the roots corresponding to the pair \((\mathfrak{g}, \mathfrak{h})\), then in terms of the simple roots \(\alpha_i = \lambda_i - \lambda_{i+1}, i = 1, \ldots, p - 1\) and the highest root \(\theta = \lambda_1 - \lambda_p\), we have

\[
x_f^e = w_{\alpha_1} w_{\alpha_2} \cdots w_{\alpha_p - 1},
\] (3.35)

and

\[
x_f^e(h) = w_{\alpha_2} w_{\alpha_4} \cdots w_{\alpha_{p-2}} \exp(h(E_{00} - E_{p-1,p-1})) w_\theta.
\] (3.36)

Here for each \(j = 1, \ldots, p - 1\), \(w_{\alpha_j} = \text{diag}(I_{j-1}, w^*, I_{p-j-1})\) is the element in \(W\) which corresponds to the simple reflection \(s_{\alpha_j}\), while

\[
w_\theta = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & & & & 0 \\ \vdots & I_{p-2} & \vdots \\ 0 & & & & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}
\] (3.37)

is the element in \(W\) which corresponds to the reflection \(s_\theta\). Finally, the element \(\exp(h(E_{00} - E_{p-1,p-1}))\) is in \(\hat{T}\), where \(E_{jj}\) denote the diagonal matrix with a 1 in the \((j,j)\) position and zeros elsewhere. But now recall that there is an additional element \(\alpha_0 = \delta - \theta\) in the simple system of affine roots in addition to those given by the extensions of the \(\alpha_j's, j = 1, \ldots, p - 1\). (see, for example, [Mac] for details on affine Lie algebras). With \(\alpha_0\), we can interpret the product \(w_{\alpha_0} = \exp(h(E_{00} - E_{p-1,p-1})) w_\theta\) as corresponding to \(s_\alpha\). Therefore, by (3.35) and (3.36), we conclude that the CMV matrix \(x_f(h)\) introduced above is a Coxeter element of the affine Weyl group. With this background, we are now ready to accomplish our first goal of this section.

**Theorem 3.6.** (a) \(O_{x_f^e} = \tilde{K}_w \cap \tilde{B}_w x_f^e \tilde{B}_w = \mathcal{T}^e\).

(b) \(O_{x_f^o} = \tilde{K}_w \cap \tilde{B}_w x_f^o \tilde{B}_w = \mathcal{T}^o\).

(c) Equip \(\tilde{K}_w \times \tilde{K}_w\) with the product structure, then \(O_{x_f^e} \times O_{x_f^o}\) is a symplectic leaf of \(\tilde{K}_w \times \tilde{K}_w\). Moreover, the collection of \(p \times p\) Floquet CMV matrices is the image of \(O_{x_f^e} \times O_{x_f^o}\) under the Poisson automorphism \(m | O_{x_f^e} \times O_{x_f^o} : O_{x_f^e} \times O_{x_f^o} \to \{p \times p\text{ Floquet CMV matrices}\}\), where \(m\) is the multiplication map of \(\tilde{K}_w\). Hence \(\{p \times p\text{ Floquet CMV matrices}\} = O_{x_f} = \tilde{K}_w \cap \tilde{B}_w x_f \tilde{B}_w\), a Coxeter dressing orbit.

**Proof.** (a) Take an arbitrary element

\[
a = k(g)^{-1} x_f^e k((x_f^e)^{-1} g x_f^e)\]

\[
= b(g)^{-1} x_f^o b((x_f^o)^{-1} g x_f^o)
\] (3.38)
in the dressing orbit through $x_0^*$. From the first line of the above expression, it is clear that $a$ is unitary. On the other hand, it follows from the second line of the above expression that $P_.a = 0$ and that $P_0 a$ is block lower triangular with $2 \times 2$ blocks on the diagonal. Moreover, from the fact that the elements in $B$ have positive diagonal entries, it follows that each of the $2 \times 2$ blocks on the main diagonal of $P_0 a$ has the following properties: (i) the entry in the upper right hand corner is positive, (ii) the determinant is negative (since $\det(w^*) = -1$). But $P_.a = 0$ implies $P_.a^{-1} = 0$. As $P_.a^{-1} = P_.a^* = (P_+.a)^*$, we conclude that $P_.a = 0$ and hence $a = P_0 a$. But then $(a^*)^{-1} = ((P_0 a)^*)^{-1}$ is upper block triangular with diagonal blocks having the same properties. Since $a = (a^*)^{-1}$, it follows that $a$ must be block diagonal, i.e.,

$$a = \text{diag}(\phi_0, \phi_2, \ldots, \phi_{p-2})$$  \hfill (3.39)

where for each $j$, $\phi_{2j}$ is a unitary matrix with a positive entry in the upper right hand corner and whose determinant is $-1$. Consequently, $\phi_{2j}$ must be of the form

$$\phi_{2j} = \begin{pmatrix} \bar{\alpha}_{2j} & \rho_{2j} \\ \rho_{2j} & -\bar{\alpha}_{2j} \end{pmatrix}$$  \hfill (3.40)

for some $\alpha_{2j} \in \mathbb{D}$, where $\rho_{2j} = (1 - |\alpha_{2j}|^2)^{\frac{1}{2}}$. Hence we have shown that $\mathcal{O}_{x_0^*} \subset T^e$. The reverse inclusion $T^e \subset \mathcal{O}_{x_0^*}$ follows exactly as in the proof of the corresponding assertion in Theorem 2.4 (a) of [L1].

(b) Take an arbitrary element

$$b = k(g)^{-1}x_0^*k((x_0^*)^{-1}gx_0^*)$$

$$= b(g)^{-1}x_0^*b((x_0^*)^{-1}gx_0^*)$$  \hfill (3.41)

in the dressing orbit through $x_0^*$. From the first line of the above expression, $b$ is unitary. On the other hand, it follows from the second line of the same expression that

$$b(h) = \sum_{j=-\infty}^{\infty} b_j h^j.$$  \hfill (3.42)

Now,

$$b^{-1} = (b((x_0^*)^{-1}gx_p^*))^{-1}(x_0^*)^* b(g)$$  \hfill (3.43)

since $x_0^*$ is unitary. From this, it follows that

$$(P_+(h^{-1}b))^* = P_-(((\bar{h}b)^*) = P_-(hb^{-1}) = 0.$$  \hfill (3.44)

Therefore, when we combine this with (3.42), we conclude that

$$b(h) = b_{-1} h^{-1} + b_0 + b_1 h.$$  \hfill (3.45)
Considering the coefficient of $h^{-1}$ in the second line of (3.41), we see that

$$b_{-1} = \left( b_0 \right)^{-1} (x_0^\gamma - 1) \left( b_0 ((x_0^\alpha)^{-1} g x_0^\alpha) \right)^{-1},$$

as the first and last factors in the second line of (3.39) only contain nonnegative powers of $h$. Since $(x_0^\gamma)^{-1}$ has only one nonzero entry in its bottom left corner, and it is multiplied by lower triangular matrices with positive diagonal entries, we see that all the entries of $b_{-1}$ are zero, except $(b_{-1})_{p-1,0} = \gamma > 0$.

Further note that, if $x(h)$ is unitary for all $h \in S^1$, then an element $y(h) \in O_x$ if and only if $y(h)^* \in O_x^*$. Indeed,

$$y(h) = k(g)^{-1} x k(x^{-1} g x) \iff y(h)^* = k(g')^{-1} x^* k(x g' x^*),$$

where $g' = x^{-1} g x$. In our case, $(x_0^\alpha)^* = x_0^{\alpha^*}$, and hence

$$b(h)^* = b_0^* h^{-1} + b_0^* + b_{-1}^* h \in O_{x_0^\alpha}.$$ 

So we see, by the above argument applied to $b_{1}^*$, that $b_{1}$ must have only one nonzero entry, in its upper right-hand corner: $(b_{1})_{0,0-p} = \delta > 0$.

It remains to understand $b_0$. Recall that $b(h)$ is unitary and expand the right-hand side of $1 = b(h)^* \cdot b(h) = b(h) \cdot b(h)^*$ in powers of $h$. Given the structure of the matrices $b_{\pm 1}$, it is immediate that the coefficients of $h^{\pm 2}$ are both zero. The coefficients of $h^{\pm 1}$ must also be zero. This translates into

$$b_1^* b_0 + b_0^* b_{-1} = 0$$

and

$$b_{-1}^* b_0 + b_0^* b_1 = 0.$$ 

Taking into account the shape of $b_{\pm 1}$, and writing down explicitly these relations leads to the fact that

$$0 - \text{th row of } b_0 = (\eta, 0, \ldots, 0), \quad (3.46)$$

$$(p - 1) - \text{th row of } b_0 = (0, \ldots, 0, \nu), \quad (3.47)$$

where

$$\delta \eta + \gamma \nu = 0. \quad (3.48)$$

Therefore, when we combine our analysis above on $b_{\pm 1}$, together with (3.46) and (3.47), our conclusion is that $b(h)$ looks like

$$b(h) = \begin{pmatrix}
\eta & 0 & \cdots & 0 & \delta h \\\n0 & \ddots & \ddots & \ddots \\
& \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 0 & \nu
\end{pmatrix}, \quad (3.49)$$
where \( \tilde{b} \) is a \((p-2) \times (p-2)\) matrix which is independent of \( h \). But a unitary matrix can only be a direct sum of unitary matrices, and \( b(h) \) is unitary. Unitarity of the \( 2 \times 2 \) piece at \( h = 1 \), together with the fact that \( \gamma, \delta > 0 \), means that

\[
\begin{pmatrix}
\eta & \delta h \\
\gamma h^{-1} & \nu
\end{pmatrix} = \begin{pmatrix}
-\alpha_{p-1} & \rho_{p-1} h \\
\rho_{p-1} h^{-1} & \alpha_{p-1}
\end{pmatrix}
\]  \hspace{1cm} (3.50)

for some complex number \( \alpha_{p-1} \in \mathbb{D} \) and \( \rho_{p-1} = \sqrt{1 - |\alpha_{p-1}|^2} \).

The last step requires that we understand the shape of \( \tilde{b} \). This is achieved by writing the coefficient of \( h_0 \) in \( b(h) \) and the second line of (3.41). Again using the fact that the coefficients of \( h_0 \) in the external factors on the right-hand side of (3.41) are lower triangular with positive diagonal entries, and keeping track of all the (potentially) nonzero entries, we obtain that \( \tilde{b} \) is a unitary matrix which is block-lower triangular, each \( 2 \times 2 \) diagonal block having a positive entry in its upper right-hand corner. Hence the same argument as in part (a) of this theorem shows that

\[
\tilde{b} = \text{diag}(\phi_1, \phi_3, \ldots),
\]  \hspace{1cm} (3.51)

where

\[
\phi_k = \begin{pmatrix}
\bar{\alpha}_k & \rho_k \\
\rho_k & -\alpha_k
\end{pmatrix}.
\]  \hspace{1cm} (3.52)

Inserting this into (3.49) and using (3.50) shows that \( b \in T^o \).

The proof is finished once we prove that \( T^o \subset \mathcal{O}_x \). This is done as in the previous cases. More precisely, if \( b = \theta_1 \oplus \theta_3 \oplus \cdots \oplus \theta_{p-1}(h) \) is an element of \( T^o \), choose a matrix \( g = l_1 \oplus \cdots \oplus l_{p-3} \oplus l_{p-1}(h) \), where, for \( 0 \leq j \leq (p-4)/2 \),

\[
l_{2j+1} = \begin{pmatrix}
\rho_{2j+1} & 0 \\
-\alpha_{2j+1} & 1
\end{pmatrix}
\]  \hspace{1cm} (3.53)

is a \( 2 \times 2 \) block situated between the rows and columns \( 2j + 1 \) and \( 2j + 2 \), and

\[
l_{p-1}(h) = \begin{pmatrix}
1 & \bar{\alpha}_{p-1} h \\
0 & \rho_{p-1}
\end{pmatrix}.
\]  \hspace{1cm} (3.54)

Note that \( l_{2j+1} \) are all independent of \( h \) and lower triangular with positive diagonal entries. Further note that

\[
l_{p-1}(h) = \begin{pmatrix}
1 & 0 \\
0 & \rho_{p-1}
\end{pmatrix} + h \begin{pmatrix}
0 & \bar{\alpha}_{p-1} \\
0 & 0
\end{pmatrix}.
\]  \hspace{1cm} (3.55)

In particular, this implies that \( g \in \tilde{B}_w \) and hence

\[
k(g)^{-1} x^o_j \Gamma((x^o_j)^{-1} g x^o_j) = k(g x^o_j).
\]  \hspace{1cm} (3.56)

Furthermore, we have the factorizations

\[
l_{2j+1} w^* = \theta_{2j+1} \begin{pmatrix}
\rho_{2j+1} & 0 \\
-\alpha_{2j+1} & 1
\end{pmatrix}
\]  \hspace{1cm} (3.57)
for $0 \leq j \leq (p - 4)/2$, and

$$l_{p-1}(h) \cdot \begin{pmatrix} 0 & h \\ h^{-1} & 0 \end{pmatrix} = \theta_{p-1}(h) \cdot \begin{pmatrix} 1 & \alpha_{p-1}h \\ 0 & \rho_{p-1} \end{pmatrix}. \tag{3.58}$$

In other words $g \cdot x_j^o = b \cdot \tilde{g}$, where $\tilde{g}$ is in $\tilde{B}_w$ for the same reason as $g$ is. We therefore conclude that

$$k(g)^{-1}x_j^o k((x_p^o)^{-1}gx_j^o) = k(gx_j^o) = b \tag{3.59}$$

is indeed an element of $O_{x_j^o}$.

(c) It is easy to see that $O_{x_p} = O_{x_j^o} \cdot O_{x_j^o}$. The rest of the assertion is clear from what we have already done. \qed

**Remark 3.7.** By essentially following the same argument, we can also establish the following fact:

$$\{ p \times p \text{ Floquet CMV matrices} \} = O_{w_{\alpha_1}} \cdots O_{w_{\alpha_{p-1}}} \cdot O_{w_{\alpha_{p-2}}} \cdots O_{w_{\alpha_0}}, \tag{3.60}$$

where each orbit on the right hand side is two-dimensional.

In the next result, we clarify the relation between the Ablowitz-Ladik bracket in (2.11) and the Sklyanin bracket $\{ \cdot, \cdot \}^{J}$.  

**Theorem 3.8.** The map

$$\mathbb{D}^p \to \tilde{G}_w^R,$$

$$\alpha = (\alpha_0, \cdots , \alpha_{p-1}) \mapsto \mathcal{E}(h) = g^e(\alpha)g^o(\alpha)(h) \tag{3.61}$$

is a Poisson embedding, when $\mathbb{D}^p$ is equipped with the Ablowitz-Ladik bracket, and $\tilde{G}_w^R$ is equipped with Sklyanin structure $\{ \cdot, \cdot \}^{J}$.

**Proof.** As the multiplication map of $\tilde{G}_w^R$ is a Poisson map, it is enough to show that the map $\alpha \mapsto (g^e(\alpha), g^o(\alpha))$ is Poisson, when $\tilde{G}_w^R \times \tilde{G}_w^R$ is equipped with the product structure. For this purpose, denote by $E_{ab}$ the $p \times p$ matrix whose $(a,b)$ entry is equal to 1 and whose other entries are zero. For $j$ even, $j \in \{ 0, \cdots, p-1 \}$ and for $l$ odd, $l \in \{ 0, \cdots, p-1 \}$, introduce the following functions on $\tilde{G}_w^R \times \tilde{G}_w^R$:

$$F_j(A, B) = \operatorname{Im} \int_{|h|=1} \operatorname{tr} (A(h)E_{jj}) \frac{dh}{2\pi i h}, \quad G_j(A, B) = \operatorname{Re} \int_{|h|=1} \operatorname{tr} (A(h)E_{jj}) \frac{dh}{2\pi i h},$$

$$F_l(A, B) = \operatorname{Im} \int_{|h|=1} \operatorname{tr} (B(h)E_{ll}) \frac{dh}{2\pi i h}, \quad G_l(A, B) = \operatorname{Re} \int_{|h|=1} \operatorname{tr} (B(h)E_{ll}) \frac{dh}{2\pi i h}. \tag{3.62}$$

Then we have

$$F_j(g^e, g^o) = -\operatorname{Im} \alpha_j, \quad G_j(g^e, g^o) = \operatorname{Re} \alpha_j,$$

$$F_l(g^e, g^o) = -\operatorname{Im} \alpha_l, \quad G_l(g^e, g^o) = \operatorname{Re} \alpha_l. \tag{3.63}$$
In view of this, it suffices to compute the Poisson brackets of these functions at \((g^e, g^o) \in \mathcal{G}^e_w \times \mathcal{G}^o_w\). For a function \(\varphi\) on \(\mathcal{G}^e_w \times \mathcal{G}^o_w\) which is smooth in both variables, we denote by \(D_i \varphi(A, B)\) (resp. \(D'_i \varphi(A, B)\)) its right gradient (resp. left gradient) with respect to the \(i\)-th variable, \(i = 1, 2\). Then we have

\[
\begin{align*}
D_1 F_j(g^e, g^o) &= g^e E_{jj} = \alpha_j E_{jj} + \rho_j E_{j+1,j}, \\
D'_1 F_j(g^e, g^o) &= E_{jj} g^e = \alpha_j E_{jj} + \rho_j E_{j,j+1}, \\
D_2 F_j(g^e, g^o) &= 0, \quad D'_2 F_j(g^e, g^o) = 0,
\end{align*}
\tag{3.64}
\]

and similarly,

\[
\begin{align*}
D_2 F_i(g^e, g^o) &= g^o(h) E_{ii} = \bar{\alpha}_i E_{ii} + \rho_i E_{i+1,i}, \\
D'_2 F_i(g^e, g^o) &= E_{ii} g^o(h) = \bar{\alpha}_i E_{ii} + \rho_i E_{i,i+1}, \quad l \text{ odd}, 1 \leq l \leq p - 3, \\
D_2 F_p-1(g^e, g^o) &= g^o(h) E_{p-1,p-1} = \rho_{p-1} h E_{p-1} \cdot \rho_{p-1} E_{p-1}, \\
D'_2 F_p-1(g^e, g^o) &= E_{p-1,p-1} g^o(h) = \rho_{p-1} h^{-1} E_{p-1,0} + \rho_{p-1} E_{p-1}, \\
D_1 F_i(g^e, g^o) &= 0, \quad D'_1 F_i(g^e, g^o) = 0, \quad l \text{ odd}, 1 \leq l \leq p - 1.
\end{align*}
\tag{3.65}
\]

On the other hand, it is clear that

\[
\begin{align*}
D_1 G_j(g^e, g^o) &= i D_1 F_j(g^e, g^o), \quad D'_1 G_j(g^e, g^o) = i D'_1 F_j(g^e, g^o), \\
D_2 G_j(g^e, g^o) &= 0, \quad D'_2 G_j(g^e, g^o) = 0,
\end{align*}
\tag{3.66}
\]

and similarly,

\[
\begin{align*}
D_2 G_i(g^e, g^o) &= i D_2 F_i(g^e, g^o), \quad D'_2 G_i(g^e, g^o) = i D'_2 F_i(g^e, g^o) \\
D_1 G_i(g^e, g^o) &= 0, \quad D'_1 G_i(g^e, g^o) = 0.
\end{align*}
\tag{3.67}
\]

In what follows, the indices \(j\) and \(k\) are even, while the indices \(l\) and \(m\) are odd, and all indices are from \(\{0, \cdots, p - 1\}\). Let \(\{\cdot, \cdot\}_*\) denote the product structure on \(\mathcal{G}^e_w \times \mathcal{G}^o_w\), then it is immediate from the definition of \(\{\cdot, \cdot\}_*\) that

\[
\{F_j, F_i\}_* = 0, \quad \{G_j, G_i\}_* = 0, \quad \{F_j, G_i\}_* = 0, \quad \{G_j, F_i\}_* = 0.
\tag{3.68}
\]

Now, from (3.64), \(D_1 F_j(g^e, g^o)\) and \(D'_1 F_j(g^e, g^o)\) are constant loops, so it follows from (3.12) and equation (2.6) of [L1] that

\[
\begin{align*}
J^\sharp(D_1 F_j(g^e, g^o)) &= -\alpha_j E_{jj} - \rho_j E_{j+1,j}, \\
J^\sharp(D'_1 F_j(g^e, g^o)) &= \rho_j E_{j,j+1} - 2\rho_j E_{j+1,j} - \alpha_j E_{jj}.
\end{align*}
\tag{3.69}
\]

Therefore, on using (3.69) and (3.64), we find for \(j \neq k\) that

\[
\begin{align*}
\text{tr} \ (J^\sharp(D'_1 F_j(g^e, g^o))D'_1 F_k(g^e, g^o))
&= (D'_1 F_k(g^e, g^o))_{kk} (J^\sharp(D'_1 F_j(g^e, g^o)))_{kk} + (D'_1 F_k(g^e, g^o))_{k,k+1} (J^\sharp(D'_1 F_j(g^e, g^o)))_{k+1,k}
= 0,
\end{align*}
\tag{3.70}
\]
and similarly,

\[
\text{tr } \left( J^2(D_1 F_j(g^e, g^o))D_1 F_k(g^e, g^o) \right) \\
= \langle D_1 F_k(g^e, g^o) \rangle_{kk} (J^2(D_1 F_j(g^e, g^o)))_{kk} + \langle D_1 F_k(g^e, g^o) \rangle_{k+1,k} (J^2(D_1 F_j(g^e, g^o)))_{k,k+1} = 0.
\]

Thus it follows from (3.70), (3.71) and (3.66) that

\[
\{F_j, F_k\}_*(g^e, g^o) = 0, \quad \{G_j, G_k\}_*(g^e, g^o) = 0, \quad \{F_j, G_k\}_*(g^e, g^o) = 0, \quad \text{for } j \neq k.
\]

In a similar fashion, by using (3.64), (3.66) and (3.69), we find

\[
\{F_j, G_j\}_*(g^e, g^o) = \frac{1}{2} \text{tr} (i J^2(D_1 F_j(g^e, g^o)))D_1 F_j(g^e, g^o) - \frac{1}{2} \text{tr} (i J^2(D_1 F_j(g^e, g^o)))D_1 F_j(g^e, g^o) = -\rho_j^2.
\]

Analogously, for the odd indices \( l, m \), with \( l, m \in \{0, \cdots, p-3\} \), we have

\[
\{F_l, F_m\}_*(g^e, g^o) = 0, \quad \{G_l, G_m\}_*(g^e, g^o) = 0, \quad \{F_l, G_m\}_*(g^e, g^o) = 0, \quad \text{for } l \neq m.
\]

Also,

\[
\{F_l, G_l\}_*(g^e, g^o) = -\rho_l^2.
\]

Next, we consider brackets with the quantities \( F_{p-1} \) and \( G_{p-1} \). To this end, we note the formulas

\[
J^2(D_2 F_{p-1}(g^e, g^o)) = -\alpha_{p-1} E_{p-1,p-1} - \rho_{p-1} h E_{0,p-1},
\]

\[
J^2(D_2' F_{p-1}(g^e, g^o)) = \rho_{p-1} h E_{0,p-1} - \alpha_{p-1} E_{p-1,p-1} - 2\rho_{p-1} h E_{0,p-1}.
\]

Therefore, if \( l \) is odd, \( l \leq p-3 \), a calculation similar to (3.70) and (3.71) above shows that

\[
\{F_{p-1}, F_l\}_*(g^e, g^o) = 0, \quad \{G_{p-1}, G_l\}_*(g^e, g^o) = 0, \quad \{F_{p-1}, G_l\}_*(g^e, g^o) = 0.
\]

Now, by (3.76), (3.67) and (3.65), we have

\[
\text{tr } \left( J^2(D_2 F_{p-1}(g^e, g^o))D_2 G_{p-1}(g^e, g^o) \right) = -i|\alpha_{p-1}|^2,
\]

while

\[
\text{tr } \left( J^2(D_2' F_{p-1}(g^e, g^o))D_2' G_{p-1}(g^e, g^o) \right) = -2i\rho_{p-1}^2 - i|\alpha_{p-1}|^2.
\]

Consequently,

\[
\{F_{p-1}, G_{p-1}\}_*(g^e, g^o) = -\rho_{p-1}^2.
\]
Assembling the calculations, we conclude that for all $a, b \in \{0, \cdots, p - 1\}$, we have
\[
\{F_a, F_b\}_\ast (g^e, g^o) = 0, \quad \{G_a, G_b\}_\ast (g^e, g^o) = 0, \quad \{F_a, G_b\}_\ast (g^e, g^o) = 0, \quad \text{if } a \neq b,
\]
\[
\{F_a, G_a\}_\ast (g^e, g^o) = -\rho_a^2.
\]
(3.81)

So finally, we obtain the following Poisson bracket relations
\[
\{G_a - iG_a, G_b - iF_b\}_\ast (g^e, g^o) = 0, \quad \{G_a + iF_a, G_b + iF_b\}_\ast (g^e, g^o) = 0, \quad \text{if } a \neq b,
\]
\[
\{G_a - iF_a, G_b + iF_b\}_\ast (g^e, g^o) = 2i\delta_{ab}\rho_a^2.
\]
(3.82)
as desired.

Finally we describe the Hamiltonian equations generated by central functions on $\tilde{G}_w^R$ in the above framework. We also introduce the kind of equations which we will need to use in Section 6 below.

**Proposition 3.9.** (a) The Hamiltonian equation of motion generated by a central function $\varphi$ on $\tilde{G}_w^R$ is given by the Lax equation
\[
\dot{g} = g \left( \Pi_{\tilde{t}_w}(D\varphi(g)) \right) - \left( \Pi_{\tilde{t}_w}(D\varphi(g)) \right) g
= (\Pi_{\tilde{t}_w}(D\varphi(g))) g - g (\Pi_{\tilde{t}_w}(D\varphi(g)))
\]
(3.83)

(b) Consider $(\tilde{G}_w^R, \{\cdot, \cdot\}_{\tilde{f}})$ and equip the group $\tilde{G}_w^R \times \tilde{G}_w^R$ with the product Poisson structure. If $\varphi$ is a central function on $\tilde{G}_w^R$, then the Lax system
\[
\dot{g}_1 = g_1 \left( \Pi_{\tilde{t}_w}(D\varphi(g_1 g_2)) \right) - \left( \Pi_{\tilde{t}_w}(D\varphi(g_1 g_2)) \right) g_1
= (\Pi_{\tilde{t}_w}(D\varphi(g_1 g_2))) g_1 - g_1 (\Pi_{\tilde{t}_w}(D\varphi(g_1 g_2)))
\]
\[
\dot{g}_2 = g_2 \left( \Pi_{\tilde{t}_w}(D\varphi(g_1 g_2)) \right) - \left( \Pi_{\tilde{t}_w}(D\varphi(g_1 g_2)) \right) g_2
= (\Pi_{\tilde{t}_w}(D\varphi(g_1 g_2))) g_2 - g_2 (\Pi_{\tilde{t}_w}(D\varphi(g_1 g_2)))
\]
(3.84)
is the Hamiltonian equation on $\tilde{G}_w^R \times \tilde{G}_w^R$ generated by $H_\varphi(g_1, g_2) = \varphi(g_1 g_2)$. Moreover, under the Hamiltonian flow defined by (3.84), $g = g_1 g_2$ evolves according to (3.83).

(c) If $k_i(t), b_i(t), i = 1, 2$ are the solutions of the factorization problems
\[
e^{t D\varphi(g_1(0) g_2(0))} = k_1(t) b_1^{-1}(t), \quad e^{t D\varphi(g_2(0) g_1(0))} = k_2(t) b_2^{-1}(t),
\]
(3.85)
where $k_i(t) \in \tilde{K}_w$, $b_i(t) \in \tilde{B}_w$, then the flow defined by (3.84) is given by
\[
g_1(t) = k_1(t)^{-1} g_1(0) k_2(t) \quad \text{and} \quad g_2(t) = k_2(t)^{-1} g_2(0) k_1(t)
\]
(3.86)

To conclude this section, we remark that equations of the type in (3.84) are a special case of so-called Lax systems on a periodic lattice or difference Lax equations and we refer the reader to [STS2] and [LP] for the general theory. In Section 6 below, we will show how to solve the factorization problems for the flows generated by the commuting integrals of the periodic defocusing Ablowitz-Ladik equation by means of Riemann theta functions associated with a hyperelliptic curve.
4. Analytical properties of the Bloch solution.

For any $z \in \mathbb{C}$, consider the equation

$$\mathcal{E}u = zu, \quad (4.1)$$

where $\mathcal{E}$ is the extended CMV matrix with periodic Verblunsky coefficients with period $p$, as in Section 2. Since $\mathcal{E}$ admits a $\theta$-factorization $\mathcal{E} = LM$ [S2], as in the one-sided case, it follows that (4.1) is equivalent to

$$\mathcal{M}u = L^*u. \quad (4.2)$$

In terms of the components of $u$ and the entries of $L$ and $M$, (4.2) gives the three-term recurrence relations

$$\rho_{2j-1}u_{2j-1} - \alpha_{2j-1}u_{2j} = z(\alpha_{2j}u_{2j} + \rho_{2j}u_{2j+1})$$
$$\alpha_{2j+1}u_{2j+1} + \rho_{2j+1}u_{2j+2} = z(\rho_{2j}u_{2j} - \alpha_{2j}u_{2j+1}) \quad (4.3)$$

for all $j \in \mathbb{Z}$. Due to the equivalent form in (4.2), the space of solutions of (4.1) is two dimensional. Indeed, it is clear from (4.3) that for given values of $u_{-1}$ and $u_0$, we can determine all other values of $u_n$ by recursion. For our analysis, we will fix a basis with the following initial conditions:

$$\phi_{-1}(z) = 1, \quad \phi_0(z) = 0, \quad \psi_{-1}(z) = 0, \quad \psi_0(z) = 1. \quad (4.4)$$

By using the first relation in (4.3) corresponding to $j = 0$, we have

$$\phi_1(z) = \frac{\rho_1}{z \rho_0}. \quad (4.5)$$

In general, an easy induction using (4.3) gives the following result.

**Proposition 4.1.** For all $j \geq 1$,

$$\phi_{2j}(z) = -\frac{\alpha_0 \rho_{p-1}}{\rho_0 \cdots \rho_{2j-1}} z^{j-1} \cdots - \frac{\alpha_{2j-1} \rho_{p-1}}{\rho_0 \cdots \rho_{2j-1}} \frac{1}{z^j},$$
$$\phi_{2j+1}(z) = \frac{\alpha_0 \alpha_2 \rho_{p-1}}{\rho_0 \cdots \rho_{2j}} z^{j-1} \cdots + \frac{\rho_{p-1}}{\rho_0 \cdots \rho_{2j}} \frac{1}{z^{j+1}}. \quad (4.6)$$

In particular,

$$\phi_{p-1}(z) = \frac{\alpha_0 \alpha_{p-2} \rho_{p-1}}{\rho_0 \cdots \rho_{p-2}} z^{p/2-2} \cdots + \frac{\rho_{p-1}}{\rho_0 \cdots \rho_{p-2}} \frac{1}{z^{p/2}}. \quad (4.7)$$

Similarly, we have

$$\psi_1(z) = -\frac{\alpha_0}{\rho_0} - \frac{\alpha_{p-1}}{\rho_0} \frac{1}{z}, \quad (4.8)$$

and by induction, we obtain the following analog of Proposition 4.1.
Proposition 4.2. For all \( j \geq 1 \),
\[
\psi_{2j}(z) = \frac{z^j}{\rho_0 \cdots \rho_{2j-1}} + \cdot \cdot \cdot + \frac{\bar{\alpha}_{2j-1} \alpha_{p-1}}{\rho_0 \cdots \rho_{2j-1}} \frac{1}{z^j},
\]
\[
\psi_{2j+1}(z) = -\frac{\alpha_{2j}}{\rho_0 \cdots \rho_{2j}} z^j - \cdot \cdot \cdot - \frac{\alpha_{p-1}}{\rho_0 \cdots \rho_{2j}} \frac{1}{z^{j+1}}.
\]

In particular,
\[
\psi_{p-1}(z) = -\frac{\alpha_{p-2}}{\rho_0 \cdots \rho_{p-2}} z^{p/2} - \cdot \cdot \cdot - \frac{\alpha_{p-1}}{\rho_0 \cdots \rho_{p-2}} \frac{1}{z^{p/2}},
\]
and
\[
\psi_{p}(z) = \frac{z^{p/2}}{\rho_0 \cdots \rho_{p-1}} + \cdot \cdot \cdot + \frac{|\alpha_{p-1}|^2}{\rho_0 \cdots \rho_{p-1}} \frac{1}{z^{p/2}}.
\]

Now by the periodicity of the Verblunsky coefficients, we have
\[
(\phi_{j+p}(z) \quad \psi_{j+p}(z)) = (\phi_{j}(z) \quad \psi_{j}(z)) M(z)
\]
for all \( j \), where
\[
M(z) = \begin{pmatrix} \phi_{p-1}(z) & \psi_{p-1}(z) \\ \phi_{p}(z) & \psi_{p}(z) \end{pmatrix}
\]
is the monodromy matrix.

Proposition 4.3. For all \( z \), \( \det M(z) = 1 \).

Proof. Let \( W_j(z) = \rho_j(\phi_j(z) \psi_{j+1}(z) - \phi_{j+1}(z) \psi_j(z)) \). Then from the first relation in (4.3), we have
\[
W_{2j-1}(z) = -z W_{2j}(z)
\]
for all \( j \). Similarly, from the second relation in (4.3), we find that
\[
W_{2j+1}(z) = -z W_{2j}(z)
\]
for all \( j \). As the right hand sides of (4.14) and (4.15) are equal, it follows that \( W_{2j-1}(z) \) is independent of \( j \) and consequently \( W_{p-1}(z) = W_{-1}(z) \) from which the assertion follows. \( \square \)

From Proposition 4.3, the eigenvalues of the monodromy matrix (i.e., the Floquet multipliers) are the roots of the characteristic polynomial
\[
h^2 - \text{tr} M(z) h + 1 = 0.
\]
If \( T : l^\infty(\mathbb{Z}) \rightarrow l^\infty(\mathbb{Z}) \) denote the shift operator defined by \( (Tu)_j = u_{j+p} \), then the unique solution of the problem
\[
Ef = zf, \quad Tf = h^{-1}f, \quad f_{p-1} = 1
\]
is called the Bloch solution and finding this solution is equivalent to considering the spectrum of the corresponding Floquet CMV matrix \( \mathcal{E}(h) \),

\[
\mathcal{E}(h) \hat{v} = z \hat{v}
\]  

(4.18)

where

\[
\hat{v} = \begin{pmatrix}
  f_0 \\
  \vdots \\
  f_{p-2}
\end{pmatrix}.
\]  

(4.19)

Hence the ordered pair \((z, h)\) in (4.17) must obey the equation

\[
\det(zI - \mathcal{E}(h)) = z^{p/2} \prod_{j=0}^{p-1} \rho_j \left( \begin{array}{cc}
  z & -\alpha_{p-1} \\
  -\alpha_{p-1}z & 1
\end{array} \right) \cdots \left( \begin{array}{cc}
  z & -\alpha_0 \\
  -\alpha_0z & 1
\end{array} \right) = 0,
\]  

(4.20)

where the discriminant \( \Delta(z) \) is related to the transfer matrix

\[
T_p(z) = \frac{1}{\prod_{j=0}^{p-1} \rho_j} \left( \begin{array}{cc}
  z & -\alpha_{p-1} \\
  -\alpha_{p-1}z & 1
\end{array} \right) \cdots \left( \begin{array}{cc}
  z & -\alpha_0 \\
  -\alpha_0z & 1
\end{array} \right)
\]  

(4.21)

by the formula [S2]

\[
\Delta(z) = z^{-p/2} \text{tr } T_p(z).
\]  

(4.22)

By comparing (4.16) and (4.20), we therefore conclude that \( \text{tr } M(z) = \Delta(z) \) and this relates the multiplier curve and the spectral curve. We will make the genericity assumption

\[(GA)_1 \text{ the roots of } P(z) = \left( \prod_{i=0}^{p-1} \rho_i \right)^2 \left( \Delta(z)^2 z^p - 4z^p \right) = \prod_{i=1}^{2p} (z - \lambda_i) \text{ are distinct.}\]

Then it is straightforward to check that the affine curve as defined by the equation

\[
I(h, z) := h \cdot \det(zI - \mathcal{E}(h)) = 0
\]  

(4.23)

is smooth with branch points located at \( \lambda_1, \ldots, \lambda_{2p} \). We will denote by \( C \) the hyperelliptic Riemann surface of genus \( g = p-1 \) corresponding to this affine curve. In order to find the divisor structure of \( f_j, j = 0, \ldots, p-2 \) on \( C \), let

\[
(z) = Q_+ + Q_- - P_+ - P_-
\]  

(4.24)

where \( P_+, Q_+ \) are on the + sheet and \( P_-, Q_- \) are on the - sheet of \( C \). (The \( \pm \) sheets correspond to the choice of sign in front of the radical in the first line of (4.25).) Solving for \( h \) in terms of \( z \) from (4.20), we find

\[
h(z) = \frac{z^{p/2} \Delta(z) \pm \sqrt{\Delta(z)^2 z^p - 4z^p}}{2z^{p/2}}
\]  

\[
= \frac{z^{p/2} \Delta(z) \pm z^{p/2} \Delta(z) \left[ 1 - \frac{2z^p}{z^{p/2} \Delta(z) z} + \cdots \right]}{2z^{p/2}} \quad \text{for } z \text{ near } \infty,
\]  

(4.25)

\[
= \frac{1}{\prod_{j=0}^{p-1} \rho_j} + \cdots \quad \text{for } z \text{ near } \infty \text{ on the + sheet},
\]

\[
= \frac{1}{\prod_{j=0}^{p-1} \rho_j} + \cdots \quad \text{for } z \text{ near } \infty \text{ on the - sheet}.
\]
On the other hand, it is clear from (5.1) that $P z^{p/2} \Delta(z) \sim 1$ as $z \to 0$. Therefore, it follows from the first line of (4.25) that $h(z) \sim \frac{1}{P z^{p/2}}$ as $z \to 0$ on the $+$ sheet, while $h(z) \sim P z^{p/2}$ as $z \to 0$ on the $-$ sheet. Thus we have

$$(h) = -\frac{p}{2} P_+ + \frac{p}{2} P_- - \frac{p}{2} Q_+ + \frac{p}{2} Q_-.$$  \hspace{1cm} (4.26)

**Proposition 4.4.** For each $0 \leq j \leq p - 2$,

$$f_j(P) = h(P) \phi_j(z(P)) + \frac{1 - h(P) \phi_{p-1}(z(P))}{\psi_{p-1}(z(P))} \psi_j(z(P)), \quad P \in C.$$  \hspace{1cm} (4.27)

**Proof.** Since $\phi$ and $\psi$ form a basis of solutions of the equation $\mathcal{E} u = zu$, we must have $f_j = c_1 \phi_j + c_2 \psi_j$ for some constants $c_1$ and $c_2$. Putting $j = -1$ and 0 in the above expression and using the initial conditions in (4.4), we find $f_j = h \phi_j + f_0 \psi_j$. As the vector with components $f$ and $f_0$ is an eigenvector of the monodromy matrix $M(z)$ with eigenvalue $h^{-1}$, we find that $\phi_{p-1} h + \psi_{p-1} f_0 = 1$. Solving for $f_0$ from this expression, we obtain the desired expression for $f_j$. $\square$

We next make the following assumption.

$(GA)_2 \quad \alpha_j \neq 0$ for $j = 0, \cdots, p - 1$.

Note that in particular, $\alpha_{p-2} \neq 0$ and so the degree of the polynomial $z^{p/2} \psi_{p-1}(z)$ is $p-1$. The roots of this polynomial will be denoted by $z_k, k = 1, \cdots, p$. (In general, the $z_k$'s are not necessarily all distinct.)

**Proposition 4.5.** For each $0 \leq j \leq p - 2$, $f_j(P)$ is a single-valued meromorphic function on the Riemann surface $C$. On the finite part of $C$ away from $Q_+$, $f_j(P)$ has at worst poles at the points $P_k = (\phi_{p-1}(z_k), z_k)$, where $\psi_{p-1}(z_k) = 0$ for $k = 1, \cdots, p - 1$. Moreover, the $z_k$'s coincide with the eigenvalues of the Dirichlet problem

$$\mathcal{E} u = z u, \quad u_{-1} = 0, u_{p-1} = 0.$$  \hspace{1cm} (4.28)

Equivalently, the $z_k$'s are the zeros of the equation

$$\det (z (g^*)^* - g^o(h)) = 0,$$  \hspace{1cm} (4.29)

where $(g^*)^*$ and $g^o(h)$ are $(p-1) \times (p-1)$ matrices obtained from $(g^*)^*$ and $g^o(h)$ by removing their last row and last column.

**Proof.** For each $j$, it follows from (4.27) that $f_j(P)$ is meromorphic on $C$. Let us consider a point $z_k$ where $\psi_{p-1}(z_k) = 0$. From (4.13), we see that at such a point, the Floquet multipliers are given by $\phi_{p-1}(z_k)$ and $\phi_{p-1}(z_k)^{-1}$ and hence $(\phi_{p-1}(z_k), z_k)$ and $(\phi_{p-1}(z_k)^{-1}, z_k)$ are points on $C$. Clearly, $1 - h \phi_{p-1}(z)$ vanishes at $(\phi_{p-1}(z_k)^{-1}, z_k)$. Therefore, provided that $\phi_{p-1}(z_k) \neq \pm 1$ or equivalently, $\Delta(z_k) \neq \pm 2$, $f_j(P)$ has a pole at $P_k = (\phi_{p-1}(z_k), z_k)$. Since $\psi_{-1} = 0$, we see that
the solutions of $\psi_{p-1}(z) = 0$ coincide with the eigenvalues of the Dirichlet problem in (4.28). On the other hand, observe that if $\psi_{p-1}(z_k) = 0$, then
\[
\text{diag}(-\alpha_{p-1}, \theta_1, \cdots, \theta_{p-3}) \begin{pmatrix} \psi_0(z_k) \\ \vdots \\ \psi_{p-2}(z_k) \end{pmatrix} = z_k \text{diag}(\theta^*_0, \cdots, \theta^*_{p-4}, \alpha_{p-2}) \begin{pmatrix} \psi_0(z_k) \\ \vdots \\ \psi_{p-2}(z_k) \end{pmatrix}
\]
from the connection with (4.28) where $\psi_0 = 1$. As the matrix on the left hand side of the above formula is $\hat{g}^0(h)$, while the one on the right hand side is $\hat{g}^e$, the last assertion in the proposition follows.

Remark 4.6. The Dirichlet eigenvalues above should not be confused with the Dirichlet data in Chapter 11 of Simon [S2]. While the latter quantities always reside on the unit circle and the number of such quantities is equal to $p$, it is not the case for the $z_k$'s, as is evident from the relation
\[
\prod_{k=1}^{p-1} z_k = -\frac{\alpha_{p-1}}{\alpha_{p-2}}.
\]
In what follows, we will denote by $h^\pm(z)$ (resp. $f^\pm_j(z)$) the values of the function $h(P)$ (resp. $f_j(P)$) on the $\pm$ sheets of the Riemann surface.

Proposition 4.7. For $j = 0, \cdots, p/2 - 1$, we have
\[
f^{-2j}(z) \sim -\frac{\left(\prod_{i=2j}^{p-2} \rho_i\right)}{\alpha_{p-2}} z^{-(p/2 - j - 1)},
\]
\[
f^{-2j+1}(z) \sim \frac{\alpha_{2j}}{\alpha_{p-2}} \left(\prod_{i=2j+1}^{p-2} \rho_i\right) z^{-(p/2 - j - 1)}
\]
as $z \to \infty$. Hence $f_{2j}(P)$ and $f_{2j+1}(P)$ have zeros of order $p/2 - j - 1$ at $P_-$.

Proof. Consider first the even case. By using (4.25) and Propositions 4.1 and 4.2, we have
\[
\frac{1 - h^-(z)\phi_{p-1}(z)}{\psi_{p-1}(z)} \psi_{2j}(z) \sim -\frac{\left(\prod_{i=2j}^{p-2} \rho_i\right)}{\alpha_{p-2}} \frac{1}{z^{p/2 - j - 1}}
\]
as $z \to \infty$. Similarily,
\[
h^-(z)\phi_{2j}(z) \sim -\frac{\bar{\alpha}_0 \rho_{p-1} \left(\prod_{i=2j}^{p-1} \rho_i\right)}{z^{p/2 - j + 1}}
\]
as $z \to \infty$. Therefore, on comparing (4.31) and (4.32), the assertion for the even case follows. We will skip the details for the odd case as it proceeds in the same way.

□
To investigate the behaviour of $f_j^+(z)$ as $z \to \infty$, we will first establish an identity for the product $f_j^+(z)f_j^-(z)$. To this end, observe that

$$
\frac{1 - h^+\phi_{p-1}(z)}{\psi_{p-1}(z)} = \frac{1}{2\psi_{p-1}(z)} \left[ (2 - \phi_{p-1}(z)(\phi_{p-1}(z) + \psi_p(z))) + \phi_{p-1}(z)\sqrt{\Delta(z)^2 - 4} \right]. \tag{4.33}
$$

Therefore, by a direct multiplication and using (4.31), we find

$$
f_j^+(P)f_j^-(P) = \frac{-\psi_j(\phi_j\phi_{p-1} + \psi_j\psi_p + \phi_j\psi_j\psi_{p-1})}{\psi_{p-1}^j}
= \frac{-\psi_j\phi_j + \psi_j\psi_j\psi_{j+1} + \psi_{p-1}^j\psi_{j+1}^j}{\psi_{p-1}^j}
\tag{4.34}
$$

where on the right hand side, we have omitted the variable $z$ throughout. Note that in going from the first line of (4.34) to the second line, we have used (4.12).

Our next task is to interpret the numerator of the right hand side of (4.34), which is necessary because performing a direct asymptotic analysis of this quantity by using Propositions 4.1 and 4.2 proves to be difficult. That this is so is due to the degeneracy of the tridiagonal matrices $L$ and $M$. (Note that neither $E$ nor its factors $L$ and $M$ satisfy the genericity assumption in [MM].) For this purpose, we introduce for each $0 \leq j \leq p - 2$ the shifted matrix $E^{[j]}$ whose $(k,l)$ entry is given by

$$
(E^{[j]})_{kl} = E_{k+j,l+j}. \tag{4.35}
$$

On the other hand, let $\psi^{[j]}$ denote the solution of

$$
E^{[j]}u = zu, \quad u_{-1} = 0, u_0 = 1. \tag{4.36}
$$

**Proposition 4.8.** For each $0 \leq j \leq p - 2$,

$$
f_j^+(z)f_j^-(z) = \frac{B_{j+1}(z)\psi_{p-1}^{[j+1]}(z)}{\psi_{p-1}(z)}, \tag{4.37}
$$

where

$$
B_{j+1}(z) = \det \begin{pmatrix} \phi_j(z) & \psi_j(z) \\ \phi_{j+1}(z) & \psi_{j+1}(z) \end{pmatrix}
= \begin{cases} \frac{\rho_{p-1}}{\rho_j}, & \text{for } j \text{ odd} \\ -\frac{\rho_{p-1}}{\rho_j}, & \text{for } j \text{ even}. \end{cases} \tag{4.38}
$$

**Proof.** From the definition of $E^{[j]}$ and $\psi^{[j]}$, it is clear that

$$
\psi^{[j]}_k(z) = c_1(z)\phi_{k+j}(z) + c_2(z)\psi_{k+j}(z) \tag{4.39}
$$
for some \(c_1(z)\) and \(c_2(z)\). By imposing the initial conditions in (4.36), we find that

\[
\begin{align*}
    c_1(z) &= -\frac{\psi_{j-1}(z)}{B_j(z)}, \\
    c_2(z) &= \frac{\phi_{j-1}(z)}{B_j(z)}.
\end{align*}
\]

Therefore, on substituting into (4.39), we obtain

\[
B_j(z)\psi^{[j]}_k(z) = -\psi_{j-1}(z)\phi_{k+j}(z) + \phi_{j-1}(z)\psi_{k+j}(z).
\]

Hence (4.37) follows from (4.34) if we replace \(j\) by \(j+1\) and let \(k = p-1\) in (4.41). To complete the proof, it remains to calculate \(B_{j+1}(z)\). Here we make use of the quantity \(W_j(z)\) introduced in the proof of Proposition 4.3 which is related to \(B_{j+1}(z)\) by the relation \(W_j(z) = \rho_j B_{j+1}(z)\). From the proof of Proposition 4.3 (see relations (4.14) and (4.15)), we learn that \(W_{2j-1}(z)\) is independent of the value of \(j\), and the same holds true for \(W_{2j}(z)\). Consequently, we have

\[
\rho_j B_{j+1}(z) = \rho_{j-2} B_{j-1}(z).
\]

From this, we find

\[
\rho_j B_{j+1}(z) = \begin{cases} 
\rho_{-1} B_0(z), & \text{for } j \text{ odd}, \\
\rho_0 B_1(z), & \text{for } j \text{ even}, \\
\rho_{p-1}, & \text{for } j \text{ odd}, \\
-\frac{\rho_1}{z}, & \text{for } j \text{ even}.
\end{cases}
\]

This completes the proof. \(\square\)

In our next result, we will analyze \(\psi^{[j]}(z)\).

**Proposition 4.9.** (a) For \(0 \leq j \leq p/2 - 1\) and \(0 \leq k \leq p/2 - 1\),

\[
\psi^{[2j]}_{2k}(z) = \frac{z^k}{\rho_{2j} \cdots \rho_{2j+k}} + \cdots + \frac{\bar{\alpha}_{2j+2k-1} \alpha_{2j-1}}{\rho_{2j} \cdots \rho_{2j+k-1} z^k},
\]

\[
\psi^{[2j]}_{2k+1}(z) = -\frac{\alpha_{2j+2k}}{\rho_{2j} \cdots \rho_{2j+2k} z^k} + \cdots - \frac{\alpha_{2j-1}}{\rho_{2j} \cdots \rho_{2j+k} z^{k+1}}.
\]

(b) For \(0 \leq j \leq p/2 - 1\) and \(0 \leq k \leq p/2 - 1\),

\[
\psi^{[2j+1]}_{2k}(z) = \frac{\bar{\alpha}_{2j+2k} \alpha_{2j+2k-1}}{\rho_{2j+1} \cdots \rho_{2j+2k+1} z^k} + \cdots + \frac{1}{\rho_{2j+1} \cdots \rho_{2j+2k} z^k},
\]

\[
\psi^{[2j+1]}_{2k+1}(z) = -\frac{\bar{\alpha}_{2j}}{\rho_{2j+1} \cdots \rho_{2j+2k+1} z^{k+1}} + \cdots - \frac{\alpha_{2j+2k+1}}{\rho_{2j+1} \cdots \rho_{2j+2k} z^k}.
\]

In particular,

\[
\psi^{[2j]}_{p-1}(z) = -\frac{\alpha_{2j-2p+1} \rho_{2j-1}}{\prod_{i=0}^{p-1} \rho_i} \cdot \cdots - \frac{\alpha_{2j-1} \rho_{2j-1}}{\prod_{i=0}^{p-1} \rho_i} \cdot \frac{1}{z^{p/2}}.
\]
and
\[ \psi_{p-1}^{[2j+1]}(z) = -\frac{\bar{\alpha}_{2j} \rho_{2j}}{\prod_{i=0}^{p-1} \rho_i} z^{p/2} - \ldots - \frac{\bar{\alpha}_{2j-1} \rho_{2j}}{\prod_{i=0}^{p-1} \rho_i} z^{p/2-1}. \]  
(4.47)

**Proof.** For the equation \( E[i] u = zu \) associated with the shifted matrix \( E[i] \), the recurrence relations in (4.3) have to be replaced by
\[
\begin{align*}
\rho_{2k-1} u_{2k-1-i} - \alpha_{2k-1} u_{2k-i} &= z(\alpha_{2k} u_{2k-i} + \rho_{2k} u_{2k+1-i}) \\
\bar{\alpha}_{2k+1} u_{2k+1-i} + \rho_{2k+1} u_{2k+2-i} &= z(\rho_{2k} u_{2k-i} - \bar{\alpha}_{2k} u_{2k+1-i}).
\end{align*}
\]  
(4.48)

For \( i = 2j \), we can therefore obtain (4.44) from (4.9) by shifting the indices. For \( i = 2j + 1 \), we have to solve (4.48) and an inductive argument leads to (4.45). \( \square \)

**Proposition 4.10.** For \( j = 0, \ldots, p/2 - 1 \), we have
\[
\begin{align*}
f^+_{2j}(z) &\sim -\frac{\bar{\alpha}_{2j}}{\alpha_{p-2}} z^{p/2-j-1}, \\
f^+_{2j+1}(z) &\sim \frac{1}{\prod_{i=0}^{p-2} \rho_i} z^{p/2-j-1}
\end{align*}
\]  
(4.49)
as \( z \to \infty \). Hence \( f_{2j}(P) \) and \( f_{2j+1}(P) \) have poles of order \( p/2 - j - 1 \) at \( P_+ \).

**Proof.** From (4.46), (4.47), and Proposition 4.7, we have
\[
\begin{align*}
f^+_{2j}(z) f^+_{2j}(z) &\sim -\frac{\bar{\alpha}_{2j}}{\alpha_{p-2}}, \\
f^+_{2j+1}(z) f^-_{2j+1}(z) &\sim \frac{\alpha_{2j}}{\alpha_{p-2}},
\end{align*}
\]  
(4.50)
as \( z \to \infty \). Therefore the assertion follows from (4.50) and (4.30). \( \square \)

We next investigate the behaviour of \( f^\pm_j(z) \) as \( z \to 0 \).

**Proposition 4.11.** For \( j = 0, \ldots, p/2 - 1 \), we have
\[
\begin{align*}
f^-_{2j}(z) &\sim -\bar{\alpha}_{2j-1} \left( \prod_{i=2j}^{p-2} \rho_i \right) z^{p/2-j}, \\
f^-_{2j+1}(z) &\sim \left( \prod_{i=2j+1}^{p-2} \rho_i \right) z^{p/2-j-1}
\end{align*}
\]  
(4.51)
as \( z \to 0 \). Hence at \( Q_- \), \( f_{2j}(P) \) has a zero of order \( p/2 - j \), while \( f_{2j+1}(P) \) has a zero of order \( p/2 - j - 1 \).

**Proof.** By (4.25), Proposition 4.1 and 4.2, we have
\[
\frac{1 - h^-(z) \phi_{p-1}(z)}{\psi_{p-1}(z)} \psi_{2j}(z) \sim -|\alpha_{p-1}|^2 \bar{\alpha}_{2j-1} \left( \prod_{i=2j}^{p-2} \rho_i \right) z^{p/2-j}
\]  
(4.52)
as \( z \to 0 \). On the other hand,

\[
h^-(z)\phi_{2j}(z) \sim -\bar{\alpha}_{2j-1}\rho_{p-1}^2 \left( \prod_{i=2j}^{p-2} \rho_i \right) z^{b/2-j}.
\] (4.53)

On combining (4.31) and (4.32) and simplify, we obtain the first relation in (4.50). The other relation in (4.51) follows in the same way. □

**Proposition 4.12.** For \( j = 0, \cdots, p/2 - 1 \), we have

\[
f_{2j}^+(z) \sim \frac{1}{\alpha_{p-1}} \left( \prod_{i=2j}^{p-2} \rho_i \right) z^{-(p/2-j)},
\]

\[
f_{2j+1}^+(z) \sim \frac{\alpha_{2j+1}}{\alpha_{p-1} \left( \prod_{i=2j+1}^{p-2} \rho_i \right)} z^{-(p/2-j - 1)}
\] (4.54)

as \( z \to 0 \). Hence at \( Q_+ \), \( f_{2j}^+(P) \) has a pole of order \( p/2 - j \) while \( f_{2j+1}^+(P) \) has a pole of order \( p/2 - j - 1 \).

**Proof.** From (4.46), (4.47) and Proposition 4.7, we find

\[
f_{2j}^+(z)f_{2j}^-(z) \sim -\frac{\bar{\alpha}_{2j-1}}{\alpha_{p-1}}, \quad f_{2j+1}^+(z)f_{2j+1}^-(z) \sim \frac{\alpha_{2j+1}}{\alpha_{p-1}}
\] (4.55)

as \( z \to 0 \). Therefore the assertion follows from (4.55) and (4.51). □

Combining Propositions 4.5, 4.7, 4.10, 4.11 and 4.12, we obtain the main result of the section.

**Theorem 4.13.** For \( 0 \leq j \leq p/2 - 1 \),

\[
(f_{2j}) \geq -D - \left( \frac{p}{2} - j - 1 \right) P_+ + \left( \frac{p}{2} - j - 1 \right) P_- - \left( \frac{p}{2} - j \right) Q_+ + \left( \frac{p}{2} - j \right) Q_-,
\]

\[
(f_{2j+1}) \geq -D - \left( \frac{p}{2} - j - 1 \right) P_+ + \left( \frac{p}{2} - j - 1 \right) P_- - \left( \frac{p}{2} - j - 1 \right) Q_+ + \left( \frac{p}{2} - j - 1 \right) Q_-,
\] (4.56)

where \( D = \sum_{k=1}^{p-1} P_k \).

**Corollary 4.14.** For \( 0 \leq j \leq p/2 - 1 \),

\[
(f_{2j+p}) \geq -D + (j + 1)P_+ - (j + 1)P_- + jQ_+ - jQ_-,
\]

\[
(f_{2j+1+p}) \geq -D + (j + 1)P_+ - (j + 1)P_- + (j + 1)Q_+ - (j + 1)Q_-.
\] (4.57)

**Proof.** This follows from (4.51), the relation \( f_{k+p} = h^{-1}f_k \) for all \( k \) and (4.26). □

To close, we present the following result which is essential in Section 6 below.
Proposition 4.15. For each $0 \leq j \leq p/2 - 1$, the divisors

$$U_1^j = D + \left( \frac{p}{2} - j - 1 \right) P_+ + \left( \frac{p}{2} - j - 1 \right) Q_+ - \left( \frac{p}{2} - j - 1 \right) P_- - \left( \frac{p}{2} - j - 1 \right) Q_-$$

and

$$U_2^j = D + \left( \frac{p}{2} - j - 1 \right) P_+ + \left( \frac{p}{2} - j \right) Q_+ - \left( \frac{p}{2} - j - 1 \right) P_- - \left( \frac{p}{2} - j \right) Q_-$$

are general, i.e.,

$$\dim L(U_1^j) = \dim L(U_2^j) = 1$$

(4.58)

where for a divisor $U$ on $C$,

$$L(U) = \{ \text{meromorphic function } \phi \mid (\phi) \geq -U \}.$$ 

Proof. We will adopt an adaptation of [MM] to our situation. For a divisor $U$ on $C$, denote by $\Omega(U)$ the set of meromorphic 1-forms $\omega$ on $C$ such that $(\omega) \geq U$. Take $k$ and $k'$ such that $k + k' > g - 2 = p - 3$, then $D + kP_- + k'Q_-$ has degree $g + k + k' > 2g - 2$. If $\omega \geq D + kP_- + k'Q_-$, then $\omega$ must be 0 since a holomorphic 1-form can have at most $2g - 2$ zeros. Thus $\dim \Omega(D + kP_- + k'Q_-) = 0$. By Riemann-Roch, it follows that $\dim L(D + kP_- + k'Q_-) = k + k' + 1 > g - 1$. For concreteness, take $k = k' = \frac{p}{2}$, and we claim that

$$L(D + (j - 1)P_- + (j - 1)Q_-) \subset L(D + jP_- + (j - 1)Q_-)$$

$$\subset L(D + jP_- + jQ_-), \quad 1 \leq j \leq \frac{p}{2}. \quad (4.59)$$

To establish this claim, we just have to observe that by Corollary 4.14 above, we have $f_{2(j-1)+1} \in L(D + (j - 1)P_- + (j - 1)Q_-)$, but $f_{2(j-1)+1} \notin L(D + jP_- + (j - 1)Q_-)$. Similarly, $f_{2(j-1)+p} \in L(D + jP_- + (j - 1)Q_-)$, but $f_{2(j-1)+p} \notin L(D + jP_- + jQ_-)$. Thus it follows from $\dim L(D + \frac{p}{2}P_- + \frac{p}{2}Q_-) = p + 1$ and the claim in (4.58) that $\dim L(D) = 1$.

We next show that $\dim L(D + Q_+ - Q_-) = 1$. Here we use the fact that allowing an extra pole increases $\dim L(U)$ by at most one. Thus we have

$$1 \leq \dim L(D + Q_+ - Q_-) \leq \dim L(D - Q_-) + 1. \quad (4.60)$$

But $L(D - Q_-) \subset L(D)$, as $f_{p-1} = 1 \in L(D)$ but $f_{p-1} \notin L(D - Q_-)$. Hence we conclude from $\dim L(D) = 1$ that $\dim L(D - Q_-) = 0$. Consequently, it follows from (4.54) that $\dim L(D + Q_+ - Q_-) = 1$. Based on this, we can establish $\dim L(D + Q_+ - Q_- + P_+ - P_-) = 1$ from the inequality

$$1 \leq \dim L(D + Q_+ - Q_- + P_+ - P_-) \leq \dim L(D + Q_+ - Q_- - P_-) + 1 \quad (4.61)$$

and the observation that $L(D + Q_+ - Q_- - P_-) \subset L(D + Q_+ - Q_-)$. (This follows because $f_{p-2} \in L(D + Q_+ - Q_-)$ but $f_{p-2} \notin L(D + Q_+ - Q_- - P_-)$.) Proceed inductively, we have the assertion.

$\square$
5. Action-angle variables.

As we saw in Section 2 above, the periodic Ablowitz-Ladik equation can be expressed in Lax pair form with Lax operator given by \( \mathcal{E}(h) \). Therefore, the characteristic polynomial \( \det(zI - \mathcal{E}(h)) \) is invariant under the Hamiltonian flow and provides us with a collection of conserved quantities. From [S2], we have

\[
\det(zI - \mathcal{E}(h)) = \left( \prod_{j=0}^{p-1} \rho_j \right) z^{p/2} \left[ \Delta(z) - (h + h^{-1}) \right]
\]

(5.1)

where the functions \( I_j \) as defined in the second line of (5.1) are such that

\[
I_{p/2} = I_{-p/2} = 1, \quad \bar{I}_j = I_{-j}, \quad j = 0, \cdots, p/2 - 1.
\]

Moreover, they are all polynomials in the \( \alpha_j \)'s, their conjugates, and \( P = \prod_{j=0}^{p-1} \rho_j \), \( j = 0, \cdots, p - 1 \).

By using the fact that the collection of \( p \times p \) Floquet CMV matrices is a symplectic leaf of the Sklyanin bracket \( \{ \cdot, \cdot \}_{J^\#} \), we begin by reproving the involution theorem in [N] and [S2].

**Theorem 5.1.** The functions \( P, I_0, \text{Re} I_j, \text{Im} I_j, j = 1, \cdots, p/2 - 1 \) provide a collection of \( p \) conserved quantities in involution for the Ablowitz-Ladik equation.

**Proof.** Write

\[
\det(zI - \mathcal{E}(h)) = \sum_{r=0}^{p} E_r(\mathcal{E}(h))z^{p-r}.
\]

(5.3)

Then up to signs, the \( E_r \)'s are the elementary symmetric functions. From (5.1) and (5.3), we find that

\[
I_j(\mathcal{E}) = \oint_{|h|=1} E_{p/2-j}(\mathcal{E}(h)) \frac{dh}{2\pi i}, \quad j = 0, \cdots, p/2 - 1,
\]

(5.4)

\[
P(\mathcal{E}) = -\oint_{|h|=1} E_{p/2}(\mathcal{E}(h)) \frac{dh}{2\pi i}.
\]

Hence the functions \( P, I_0, \text{Re} I_j, \text{Im} I_j, j = 1, \cdots, p/2 - 1 \) are the pullbacks of central functions on \( \tilde{G}^R_w \) to the 2p dimensional dressing orbit consisting of \( p \times p \) Floquet CMV matrices. Consequently, the assertion follows from the abstract involution theorem in [STS2] and Theorem 3.6. \( \square \)

**Remark 5.2.** (a) For readers who are not familiar with the abstract involution theorem in [STS2], let us remark that its demonstration is a one line proof making
use of the fact that for a central function, its left gradient is the same as its right gradient. (See the expression for the Poisson bracket \( \{ \cdot, \cdot \} \) in (3.16) above for our case.)

(b) Note that we are using the symbol \( E \) as a shorthand for the unitary loop \( E(\cdot) \) in (5.4) above and we will henceforth continue to use the symbol with this meaning. Since we will not be using the extended CMV matrices in what follows, this should not cause any confusion.

Thus the number of commuting integrals as provided by the quantities in the above theorem is exactly equal to one half the dimension of the phase space. In the rest of the section, we will construct the variables (essentially) conjugate to these actions. As in Section 4, we denote by \( C \) the hyperelliptic Riemann surface of genus \( g = p - 1 \) corresponding to the affine curve \( I(h, z) = h \det(zI - E(h)) = 0 \). On \( C \), we introduce the holomorphic 1-forms

\[
\xi_k = \frac{z^{k-1}}{\partial I/\partial h} \, dz, \quad k = 1, \ldots, g = p - 1.
\]  

We also introduce the meromorphic 1-form

\[
\xi^m = (h + h^{-1})^{p/2 - 1} \frac{\partial I/\partial h}{\partial h} \, dz
\]

with poles at \( P_{\pm} \) and \( Q_{\pm} \). Pick a fix point \( P_0 \) on the finite part of \( C \) and put \( D_0 = gP_0 \). Then for \( E(h) \) satisfying the genericity assumptions \((GA)_1\) and \((GA)_2\), we define

\[
\phi_k(E) = \int_{D_0}^{D} \xi_k, \quad k = 1, \ldots, g,
\]

\[
\nu(E) = \int_{D_0}^{D} \xi^m,
\]

where \( D = \sum_{k=1}^g P_k \) is the divisor of poles of \( f_j, 0 \leq j \leq g - 1 \) in the finite part of \( C \). Note that in the definition of \( \nu(E) \), the paths of integration going from the points of \( D_0 \) to the points of \( D \) must avoid the points \( P_{\pm} \). These multi-valued variables are well defined because the points in \( D_0 \) and \( D \) are in the finite part of \( C \). On the other hand, the multi-valuedness can be resolved in the standard way and we will not try to get into the details here. (See, for example, [DLT] and [L2].)

To compute the Poisson brackets between the conserved quantities in Theorem 5.1 and the variables in (5.7), we will make use of a device in [DLT] (which has also proved to be successful in [L2]) which will allow us to simplify the calculation. In the following, we will deal with \( E(h) \) for \( h \) not necessarily on the unit circle. Note that in this general case, we have the relation

\[
E(h)E(\bar{h}^{-1})^* = E(\bar{h}^{-1})^*E(h) = I, \quad h \in \mathbb{C} \setminus \{0\}
\]

which can be checked by using the fact that \( E(h) \) is unitary for \( h \in \partial \mathbb{D} \). For our purpose, we pick a fixed \( h_0 \in (-1, 1) \setminus \{0\}, \ z_0 \in \partial \mathbb{D} \) such that \( (h_0, z_0) \) is not on \( C \).
and define
\[
H_{h_0,z_0}(\mathcal{E}) = \text{Re} \log \det (z_0 I - \mathcal{E}(h_0)),
\]
\[
J_{h_0,z_0}(\mathcal{E}) = \text{Im} \log \det (z_0 I - \mathcal{E}(h_0)).
\] (5.9)

As the reader will see in the calculation which follow, this choice of $h_0$ and $z_0$ is critical.

**Lemma 5.3.** (a) The Hamiltonian equation generated by $H_{h_0,z_0}(\mathcal{E})$ is given by the equation
\[
\dot{\mathcal{E}}(h) = [\mathcal{E}(h), B(h)]
\] (5.10)
where
\[
B(h) = (i(z_0 I - \mathcal{E}(h_0))^{-1}\mathcal{E}(h_0))_0 + i \text{Im} \left( (i(z_0 I - \mathcal{E}(h_0))^{-1}\mathcal{E}(h_0))_0 \right)
- \left( (i(z_0 - \mathcal{E}(h_0))^{-1}\mathcal{E}(h_0))_0 \right)^* + iz_0(z_0 I - \mathcal{E}(h_0^{-1}))^{-1} \cdot \frac{1}{1 - hh_0}.
\] (5.11)

(b) The Hamiltonian equation generated by $J_{h_0,z_0}(\mathcal{E})$ is given by the equation
\[
\dot{\mathcal{E}}(h) = [\mathcal{E}(h), C(h)]
\] (5.12)
where
\[
C(h) = ((z_0 I - \mathcal{E}(h_0))^{-1}\mathcal{E}(h_0))_0 + i \text{Im} \left( (z_0 I - \mathcal{E}(h_0))^{-1}\mathcal{E}(h_0) \right)_0 
- \left( (z_0 I - \mathcal{E}(h_0))^{-1}\mathcal{E}(h_0) \right)^* - z_0(z_0 I - \mathcal{E}(h_0^{-1}))^{-1} \cdot \frac{1}{1 - hh_0}.
\] (5.13)

**Proof.** (a) If we let $\phi(g) = \text{Re} \log \det(z_0 I - g(h_0))$ for $g \in \tilde{G}_{w}$, then from the second equation in (3.60) and a direct calculation, we find that the Hamiltonian equation generated by $H_{h_0,z_0}$ is given by (5.10), where
\[
B(h) = \Pi_{b,w} \left( i(z_0 I - \mathcal{E}(h_0))^{-1}\mathcal{E}(h_0) \right)_0 \frac{1}{h - h_0}
= (i(z_0 I - \mathcal{E}(h_0))^{-1}\mathcal{E}(h_0))_0 + i \text{Im} \left( (i(z_0 I - \mathcal{E}(h_0))^{-1}\mathcal{E}(h_0))_0 \right)
- \left( (i(z_0 I - \mathcal{E}(h_0))^{-1}\mathcal{E}(h_0))_0 \right)^* + \left( i(z_0 I - \mathcal{E}(h_0))^{-1}\mathcal{E}(h_0) \cdot \frac{h}{h - h_0} \right)^*.
\] (5.14)

Note that in going from the first line of (5.14) to the second line, we have used (3.12), together with the formula for $\Pi_{b}$ from [L1]. In the next step of the calculation, we will try to rewrite the last term in the above expression in the desired form, and it is here that the choice of $h_0$ and $z_0$ is important. To wit, by using (5.8), we have
\[
\left( i(z_0 I - \mathcal{E}(h_0))^{-1}\mathcal{E}(h_0) \cdot \frac{h}{h - h_0} \right)^*
= \left( i(z_0 \mathcal{E}(h_0^{-1})^* - I)^{-1} \cdot \frac{h}{h - h_0} \right)^*
= -i z_0 \mathcal{E}(h_0^{-1}) - I^{-1} \cdot \frac{h}{h - h_0}
= iz_0(z_0 I - \mathcal{E}(h_0^{-1}))^{-1} \cdot \frac{1}{1 - hh_0}.
\] (5.15)
Hence (5.11) follows from (5.14) and (5.15).
(b) The proof is similar to (a) and so we will skip the details.

In our next two results, we will denote the Poisson bracket on the set of $p \times p$ Floquet CMV matrices induced from $\{.,\}_H$ simply by $\{.,\}.$

**Proposition 5.4.** For $j, k = 1, \ldots, p/2 - 1$, we have the following Poisson bracket relations

\[
\begin{align*}
(a) \quad & \left\{ I_0, -\text{Im}\left(\frac{\phi_k + \phi_p}{2}\right) \right\} (\mathcal{E}) = 1, \left\{ I_0, -\text{Im}\left(\frac{\phi_k + \phi_{p-1}}{2}\right) \right\} (\mathcal{E}) = 0, \\
& \left\{ I_0, -\text{Re}\left(\frac{\phi_k - \phi_{p-1}}{2}\right) \right\} (\mathcal{E}) = 0; \\
(b) \quad & \left\{ \text{Re} I_j, -\text{Im}\left(\frac{\phi_k + \phi_p}{2}\right) \right\} (\mathcal{E}) = 0, \left\{ \text{Re} I_j, -\text{Im}\left(\frac{\phi_k + \phi_{p-1}}{2}\right) \right\} (\mathcal{E}) = \delta_{j,p/2-k}, \\
& \left\{ \text{Re} I_j, -\text{Re}\left(\frac{\phi_k - \phi_{p-1}}{2}\right) \right\} (\mathcal{E}) = 0; \\
(c) \quad & \left\{ \text{Im} I_j, -\text{Im}\left(\frac{\phi_k + \phi_p}{2}\right) \right\} (\mathcal{E}) = 0, \left\{ \text{Im} I_j, -\text{Im}\left(\frac{\phi_k + \phi_{p-1}}{2}\right) \right\} (\mathcal{E}) = 0, \\
& \left\{ \text{Im} I_j, -\text{Re}\left(\frac{\phi_k - \phi_{p-1}}{2}\right) \right\} (\mathcal{E}) = \delta_{j,p/2-k}; \\
(d) \quad & \left\{ P, -\text{Im}\left(\frac{\phi_k + \phi_p}{2}\right) \right\} (\mathcal{E}) = 0, \left\{ P, -\text{Im}\left(\frac{\phi_k + \phi_{p-1}}{2}\right) \right\} (\mathcal{E}) = 0, \\
& \left\{ P, -\text{Re}\left(\frac{\phi_k - \phi_{p-1}}{2}\right) \right\} (\mathcal{E}) = 0.
\end{align*}
\]

**Proof.** In order to compute $\{ H_{h_{0,z_0},\phi_k} \}(\mathcal{E})$, it suffices to evaluate it on an open dense set consisting of Floquet CMV matrices $\mathcal{E}(h)$ for which

(a) the points $P_j$ of the divisor $D$ are distinct,
(b) supp $D \cap \{ \frac{\partial}{\partial h} = 0 \} = \emptyset,$
(c) $\{ Q_j \}_{j=1}^p \cap \text{supp} D = \emptyset$, where $Q_j = (h_0^{-1}, z_j(h_0)) \in C$, $j = 1, \ldots, p,$
(d) $\{ Q_j \}_{j=1}^p \cap \{ \frac{\partial}{\partial h} = 0 \} = \emptyset.$

So suppose $\mathcal{E}(h)$ satisfies (a)-(d) above. Then in the neighborhood of each $P_j$, we can take $z$ to be the local coordinate and express $h$ in terms of $z$. Thus in particular, $h_j = h(z_j)$. Let $\mathcal{E}(t)$ be the Hamiltonian flow generated by $H_{h_{0,z_0}}$ and let $D(t) = \sum_{j=1}^{p-1} P_j(t)$ (where $t$ is small) be the divisor of poles in the finite part of $C$ of the corresponding eigenvector with last component normalized to 1, $P_j(t) = (h_j(t), z_j(t))$. Then

\[
\{ H_{h_{0,z_0},\phi_k} \}(\mathcal{E}) = \frac{d}{dt} \bigg|_{t=0} \phi_k(\mathcal{E}(t)) = \sum_{j=1}^{p-1} \frac{z_j^{k-1}}{\partial h_j(h_j, z_j)} \left. \frac{dz_j(t)}{dt} \right|_{t=0}. \tag{5.16}
\]

To compute the rate of change of $z_j(t)$ at $t = 0$, consider an eigenvector $f(z,t)$ for $z$ in a neighborhood of $z_j$ such that $(e_{p-1}, f(z_j(t), t)) = 0$ for small values of $t$. Differentiate this relation with respect to $t$ at $t = 0$, we obtain

\[
\frac{dz_j(t)}{dt} \bigg|_{t=0} = \frac{(e_{p-1}, B(h_j)f(z_j, 0))}{(e_{p-1}, \frac{\partial f}{\partial z}(z_j, 0))}.
\]
Therefore, on substituting this expression into (5.16), we find

\[
\{ H_{h_0, z_0}, \phi_k \}(\mathcal{E}) = \sum_{j=1}^{p-1} \frac{z_j^{k-1}}{\frac{\partial}{\partial z_j}(h_j, z_j)} \frac{(e_{p-1}, B(h_j)f(z_j, 0))}{e_{p-1}, \frac{\partial}{\partial z_j}(z_j, 0)}.
\]

(5.17)

Now, on using the fact that the last column of a strictly lower triangular matrix is the zero vector, and \((e_{p-1}, \text{Re}(i(z_0I - \mathcal{E}(h_0))^{-1}\mathcal{E}(h_0))af(z_j, 0)) = 0\), it follows from (5.11) that

\[
(e_{p-1}, B(h_j)f(z_j, 0)) = \left(\frac{e_{p-1}}{1 - h_j h} \right) f(z_j, 0).
\]

(5.18)

Consequently, when we substitute (5.18) into (5.17), the result is

\[
\{ H_{h_0, z_0}, \phi_k \}(\mathcal{E}) = i \sum_{j=1}^{p-1} \text{Res}_{P_j} F_k + i \sum_{j=1}^{p-1} \text{Res}_{P_j} G_k,
\]

(5.19)

where

\[
F_k = (e_{p-1}, (z_0I - \mathcal{E}(h_0))^{-1}\mathcal{E}(h_0)\hat{\nu}(z, 0)) \xi_k,
\]

\[
G_k = (e_{p-1}, (z_0I - \mathcal{E}(h_0^{-1}))^{-1}\hat{\nu}(z, 0)) \frac{\xi_k}{1 - h_0 h}
\]

are meromorphic 1-forms on \(C\). By a similar calculation, we also have

\[
\{ J_{h_0, z_0}, \phi_k \}(\mathcal{E}) = \sum_{j=1}^{p-1} \text{Res}_{P_j} F_k - \sum_{j=1}^{p-1} \text{Res}_{P_j} G_k.
\]

(5.20)

(5.21)

Now from [S2], we know that for \(|z_0| = 1, \Delta(z_0) \in \mathbb{R}\), and so this implies \(J_{h_0, z_0}\) is a constant independent of \(\mathcal{E}(h)\). Consequently, \(\{ J_{h_0, z_0}, \phi_k \}(\mathcal{E}) = 0\) and hence it follows from (5.21) that \(\sum_{j=1}^{p-1} \text{Res}_{P_j} F_k = \sum_{j=1}^{p-1} \text{Res}_{P_j} G_k\). Consequently,

\[
\{ H_{h_0, z_0}, \phi_k \}(\mathcal{E}) = 2i \sum_{j=1}^{p-1} \text{Res}_{P_j} G_k.
\]

(5.22)

Consider the meromorphic 1-form \(G_k\). Since \(\left(\frac{\partial}{\partial h}\right) \sim \pm 1\) as \(z \to 0\), it follows that \(\xi_k \sim z^{k-1} \text{dz} \) as \(z \to 0\). On the other hand, we have

\[
\frac{1}{1 - h_0 h^+} \sim \frac{P}{h_0} z^{p/2}, \quad \frac{1}{1 - h_0 h^-} \sim 1 \quad \text{as} \quad z \to 0.
\]

(5.23)

Since the \(f_j\)'s are analytic at \(Q_-\), it follows that \(G_k\) has no poles at \(Q_-\). At \(Q_+\), the most singular component of \(\hat{\nu}(z, 0)\) is \(f_0^+(z) \sim \frac{1}{\alpha_{p-1}(\Pi_{i=0}^\rho_i)} z^{-p/2}\). Hence it follows from the asymptotics of \(\xi_k^+, f_0^+(z)\) and (5.23) that

\[
G_k^+ \sim \text{const.} \cdot z^{k-1} \quad \text{as} \quad z \to 0.
\]

(5.24)
So again $G_k$ has no poles at $Q_+$. Thus $G_k$ has poles only at $D$ and at $Q_j$, $j = 1, \ldots, p$ as we can similarly check that there are no poles at $P_\pm$. Hence by the residue theorem and (5.22),

$$\{ H_{ho, z_0}, \phi_k \}(\mathcal{E}) = -2i \sum_{j=1}^{p} \text{Res}_{Q_j} G_k. \quad (5.25)$$

To simplify notation, let $b_j^{(k)} = \text{Res}_{Q_j} G_k$ and put $b^{(k)} = \sum_{j=1}^{p} b_j^{(k)}$. We first calculate $b_j^{(k)}$. We have

$$b_j^{(k)} = -\frac{1}{h_0} \cdot \frac{z_0}{z_0 - z_j(h_0)} \cdot \frac{z_j(h_0)^{k-1}}{\frac{\partial}{\partial z}(h_0^{-1}, z_j(h_0))} \quad (5.26)$$

Therefore,

$$b^{(k)} = \sum_{j=1}^{p} \frac{1}{h_0} \cdot \frac{z_0}{z_0 - z_j(h_0)} \cdot \frac{z_j(h_0)^{k-1}}{\frac{\partial}{\partial z}(h_0^{-1}, z_j(h_0))}$$

$$= z_0 \frac{\lim_{R \to \infty} \int_{|z|=R} \frac{z^{k-1}}{(z_0 - z) \cdot I(h_0^{-1}, z)} \, dz}{h_0 I(h_0^{-1}, z_0)} = \frac{z_0}{z_0 \det (z_0 I - \mathcal{E}(h_0))} \quad (5.27)$$

and so finally we conclude that

$$\{ H_{ho, z_0}, \phi_k \}(\mathcal{E}) = -2ib^{(k)} = \frac{-2iz_0^k}{\det (z_0 I - \mathcal{E}(h_0))}. \quad (5.28)$$

But on the other hand, it follows from $\{ J_{ho, z_0}, \phi_k \}(\mathcal{E}) = 0$ and (5.1) that

$$\{ H_{ho, z_0}, \phi_k \}(\mathcal{E}) = \{ H_{ho, z_0} + iJ_{ho, z_0}, \phi_k \}(\mathcal{E})$$

$$= \frac{1}{\det (z_0 I - \mathcal{E}(h_0))} \left( \sum_{j=-p/2}^{p/2} \{ I_{j}, \phi_k \}(\mathcal{E})z_0^{j+p/2} - z_0^{p/2}(h_0 + h_0^{-1})\{ P, \phi_k \}(\mathcal{E}) \right). \quad (5.29)$$

By equating (5.28) and (5.29), we conclude that

$$-2iz_0^k = \sum_{j=p/2}^{p/2} \{ I_{j}, \phi_k \}(\mathcal{E})z_0^{j+p/2} - z_0^{p/2}(h_0 + h_0^{-1})\{ P, \phi_k \}(\mathcal{E}), \quad 1 \leq k \leq p - 1. \quad (5.30)$$
We now divide into three cases.

Case 1: \( k \geq p/2 + 1 \)

In this case, all brackets are zero except for \( \{ I_{k-p/2}, \phi_k \}(\mathcal{E}) = -2i \), i.e.,

\[
\begin{align*}
\{ I_j, \phi_k \}(\mathcal{E}) &= -2i \delta_{j-k,p/2}, & \{ I_j, \phi_k \}(\mathcal{E}) &= 0, 1 \leq j \leq p/2 - 1, \\
\{ P, \phi_k \}(\mathcal{E}) &= 0, & \{ I_0, \phi_k \}(\mathcal{E}) &= 0.
\end{align*}
\]

Hence

\[
\begin{align*}
\{ \text{Re} I_j, \phi_k \}(\mathcal{E}) &= -i \delta_{j,k-p/2}, & \{ \text{Im} I_j, \phi_k \}(\mathcal{E}) &= -\delta_{j,k-p/2}, 1 \leq j \leq p/2 - 1, \\
\{ P, \phi_k \}(\mathcal{E}) &= 0, & \{ I_0, \phi_k \}(\mathcal{E}) &= 0.
\end{align*}
\]  

(5.31)

Case 2: \( k \leq p/2 - 1 \)

In this case, we have

\[
\begin{align*}
\{ \bar{I}_j, \phi_k \}(\mathcal{E}) &= -2i \delta_{j,p/2-k}, & \{ I_j, \phi_k \}(\mathcal{E}) &= 0, 1 \leq j \leq p/2 - 1, \\
\{ P, \phi_k \}(\mathcal{E}) &= 0, & \{ I_0, \phi_k \}(\mathcal{E}) &= 0,
\end{align*}
\]

which implies

\[
\begin{align*}
\{ \text{Re} I_j, \phi_k \}(\mathcal{E}) &= -i \delta_{j,p/2-k}, & \{ \text{Im} I_j, \phi_k \}(\mathcal{E}) &= \delta_{j,p/2-k}, 1 \leq j \leq p/2 - 1, \\
\{ P, \phi_k \}(\mathcal{E}) &= 0, & \{ I_0, \phi_k \}(\mathcal{E}) &= 0.
\end{align*}
\]  

(5.32)

Case 3: \( k = p/2 \)

In this case, all brackets are zero except for

\[
-2i = \{ I_0, \phi_{p/2} \}(\mathcal{E}) - (h_0 + h_0^{-1}) \{ P, \phi_{p/2} \}(\mathcal{E}).
\]

(5.33)

Therefore,

\[
\begin{align*}
\{ \text{Re} I_j, \phi_{p/2} \}(\mathcal{E}) &= 0, & \{ \text{Im} I_j, \phi_{p/2} \}(\mathcal{E}) &= 0, 1 \leq j \leq p/2 - 1, \\
\{ P, \phi_{p/2} \}(\mathcal{E}) &= 0, & \{ I_0, \phi_{p/2} \}(\mathcal{E}) &= -2i.
\end{align*}
\]  

(5.34)

(5.35)

(5.36)

\[\square\]

**Proposition 5.5.** With \( \nu \) defined as in relation (5.7), we have the following Poisson bracket relation:

\[
\left\{ P, \text{Im} \left( \frac{\nu}{4} \right) \right\}(\mathcal{E}) = 1.
\]

Proof. It suffices to compute the Poisson brackets \( \{ H_{h_0,z_0}, \nu \}(\mathcal{E}) \) and \( \{ J_{h_0,z_0}, \nu \}(\mathcal{E}) \) on an open dense set consisting of Floquet CMV matrices \( \mathcal{E}(h) \) which satisfy conditions (a)-(d) in Proposition 5.4. Indeed, by following the same method of calculation, we find

\[
\{ H_{h_0,z_0}, \nu \}(\mathcal{E}) = i \sum_{j=1}^{p-1} \text{Res}_{P_j} F + i \sum_{j=1}^{p-1} \text{Res}_{P_j} G,
\]

(5.37)
and
\[
\{ J_{h_0, z_0, \nu} \}(\mathcal{E}) = \sum_{j=1}^{p-1} \text{Res}_{P_j} F - \sum_{j=1}^{p-1} \text{Res}_{P_j} G = 0, \tag{5.38}
\]
where
\[
F = \left( e_{p-1}, (z_0 I - \mathcal{E}(h_0))^{-1} \mathcal{E}(h_0) \tilde{v}(z, 0) \right) \xi^m,
\]
\[
G = \left( e_{p-1}, z_0 (z_0 I - \mathcal{E}(h_0^{-1}))^{-1} \tilde{v}(z, 0) \right) \frac{\xi^m}{1 - h_0 h}.
\tag{5.39}
\]
Thus
\[
\{ H_{h_0, z_0, \nu} \}(\mathcal{E}) = 2i \sum_{j=1}^{p-1} \text{Res}_{P_j} G. \tag{5.40}
\]
Consider the meromorphic 1-form \( G \). As \( z \to 0 \), it follows from the asymptotics of \( h^\pm \) (see Section 4) and \( (\frac{\partial}{\partial h})^\pm \) that
\[
(\xi^m)^\pm \sim \mp \frac{1}{Pz} \, dz. \tag{5.41}
\]
Since the \( f_j \)'s are analytic at \( Q^- \), it follows from (5.23) and (5.41) that \( G \) has a simple pole at \( Q^- \). Indeed, it follows from Proposition 4.11 that
\[
\text{Res}_{Q^-} G = \frac{z_0 \left( (z_0 I - \mathcal{E}(h_0^{-1}))^{-1} \right)_{p-1, p-1}}{P}. \tag{5.42}
\]
In a similar way, it follows from (5.23), (5.41) and Proposition 4.12 that \( G \) also has a simple pole at \( Q^+ \) and
\[
\text{Res}_{Q^+} G = \frac{z_0 \left( (z_0 I - \mathcal{E}(h_0^{-1}))^{-1} \right)_{p-1, 0}}{P} \cdot \frac{\rho_{p-1}}{h_0 \alpha_{p-1}}. \tag{5.43}
\]
Now consider the two points \( P_\pm \) at infinity and let \( u = \frac{1}{z} \) be the local coordinate. Then from (5.23), (5.41) and Proposition 4.10, we have
\[
f_{2j}^+(z, 0) \xi^m \sim \frac{\alpha_{2j}^{-}}{h_0 \prod_{i=2j}^{p-2} \rho_i} u^j \, du \quad \text{as} \quad u \to 0, \tag{5.44}
\]
while
\[
f_{2j+1}^+(z, 0) \xi^m \sim - \frac{1}{h_0 \prod_{i=2j+1}^{p-2} \rho_i} u^j \, du \quad \text{as} \quad u \to 0. \tag{5.45}
\]
From (5.44) and (5.45), we conclude that \( G \) is analytic at \( P_+ \). Similarly, by making use of Proposition 4.7 and (5.23), (5.41), we find that \( G \) has a simple pole at \( P^- \) with
\[
\text{Res}_{P^-} G = \frac{z_0 \left( (z_0 I - \mathcal{E}(h_0^{-1}))^{-1} \right)_{p-1, p-2}}{P} \cdot \frac{\rho_{p-2}}{\alpha_{p-2}} \cdot \frac{z_0 \left( (z_0 I - \mathcal{E}(h_0^{-1}))^{-1} \right)_{p-1, p-1}}{P}. \tag{5.46}
\]
Since $G$ obviously has poles at the points of $D$ and at the points $Q_j, j = 1, \cdots, p$, it follows by the residue theorem, (5.40), (5.42), (5.43) and (5.46) that

$$
\{ H_{h_0, z_0, \nu} \}_{(E)} = -2i \sum_{j=1}^{p} \text{Res}_{Q_j} G \frac{2iz_0}{P} \left[ \left( (z_0I - E(h_0^{-1}))^{-1} \right)_{p-1,0} \cdot \frac{\rho_{p-1}}{h_0 \alpha_{p-1}} \right. \\
+ \left. \left( (z_0I - E(h_0^{-1}))^{-1} \right)_{p-1,p-2} \cdot \frac{\rho_{p-2}}{\alpha_{p-2}} \right].
$$

To compute the second term on the right-hand side of (5.47), we introduce the following ad-hoc notation: if $A$ is a $p \times p$ matrix and $0 \leq j_1 \leq \cdots \leq j_l \leq p - 1$, $0 \leq k_1 \leq \cdots \leq k_l \leq p - 1$ are two sets of indices, with $l \geq 1$, then we denote by $A[j_1, \ldots, j_l; k_1, \ldots, k_l]$ the submatrix obtained from $A$ by deleting rows $j_1, \ldots, j_l$ and columns $k_1, \ldots, k_l$. For simplicity of notation, let

$$M = z_0I - E(h_0^{-1}).$$

Then the terms appearing on the right-hand side of (5.47) can be identified as

$$(M^{-1})_{p-1,0} = \frac{\det(M[0, p-1])}{\det(M)} \quad \text{and} \quad (M^{-1})_{p-1,p-2} = \frac{\det(M[p-2, p-1])}{\det(M)}.$$

To proceed, we expand both $\det(M[0, p-1])$ and $\det(M[p-2, p-1])$ along their 0th columns, which makes all the minors appearing the calculation $h_0$-independent, and so allows us to separate the $h_0$-dependent terms from the $h_0$-independent ones. This leads to

$$\frac{\rho_{p-1}}{h_0 \alpha_{p-1}} \cdot (M^{-1})_{p-1,0} + \frac{\rho_{p-2}}{\alpha_{p-2}} \cdot (M^{-1})_{p-1,p-2} = m_1 + m_2,$$

where the $h_0$-dependent term is

$$m_1 = \frac{1}{\det(M)} \left( \frac{1}{h_0} \cdot \rho_0 \rho_{p-1} \det(M[0, 1; 0, p-1]) \\
+ h_0 \cdot \rho_{p-2} \rho_{p-1} \det(M[p-2, p-1; 0, p-1]) \right),$$

and $m_2$ is $h_0$-independent. While the minors appearing in $m_2$ have a structure which cannot be easily simplified, the minors appearing in $m_1$ can be computed explicitely.

Indeed, for each $0 \leq j \leq \frac{p}{2} - 1$, consider the $2 \times 2$ blocks

$$A_j = \begin{pmatrix} -\rho_{2j-1} \bar{\alpha}_{2j} & z_0 + \alpha_{2j-1} \bar{\alpha}_{2j} \\ -\rho_{2j-1} \alpha_{2j} & \alpha_{2j-1} \rho_{2j} \end{pmatrix} \quad \text{and} \quad \bar{A}_j = \begin{pmatrix} -\rho_{2j} \bar{\alpha}_{2j+1} & -\rho_{2j} \rho_{2j+1} \\ -\rho_{2j} \alpha_{2j} & \alpha_{2j} \rho_{2j+1} \end{pmatrix}.$$

Note $A_j$ and $\bar{A}_j$ are, respectively, the left and right “halves” of the $2 \times 4$ blocks appearing in the extended matrix $z_0I - E$. In particular, direct investigation shows
that $M[0, 1; 0, p − 1]$ is a block bi-diagonal matrix, having the blocks $A_1, \ldots, A_{p−1}$ on the diagonal and $\tilde{A}_1, \ldots, \tilde{A}_{p−2}$ on the upper diagonal, while $M[p−2, p−1; 0, p−1]$ has $\tilde{A}_0, \ldots, \tilde{A}_{p−2}$ on the diagonal and $A_1, \ldots, A_{p−2}$ on the lower diagonal. This implies that

$$\det(M[0, 1; 0, p−1]) = \prod_{j=1}^{p−1} \det(A_j) = \prod_{j=1}^{p−1} z_0 \rho_{2j−1} \rho_{2j} = z_0^{p−1} \prod_{j=1}^{p−3} \rho_j,$$

and

$$\det(M[p−2, p−1; 0, p−1]) = \prod_{j=0}^{p−2} \det(\tilde{A}_j) = \prod_{j=0}^{p−2} z_0 \rho_{2j} \rho_{2j+1} = z_0^{p−1} \prod_{j=0}^{p−3} \rho_j.$$

Plugging these two expressions in (5.49) leads to a very simple expression:

$$m_1 = \frac{z_0^{p−1} \cdot P}{\det(z_0 I − E(h_0^{−1}))} \cdot \left(h_0 + \frac{1}{h_0}\right). \quad (5.50)$$

On the other hand, proceeding exactly as in the proof of Proposition 5.4, we find that

$$\sum_{j=1}^{P} \text{Res}_{Q_j, G} = \frac{z_0^{p/2} \left(h_0 + \frac{1}{h_0}\right)}{\det(z_0 I − E(h_0))}. \quad (5.51)$$

By combining (5.48), (5.50), and (5.51) into (5.47), and using (5.38), we conclude that

$$\{H_{h_0, z_0} + i J_{h_0, z_0}, \nu\}(E) = \frac{1}{\det(z_0 I − E(h_0))} \left[-4i z_0^{p/2} \left(h_0 + \frac{1}{h_0}\right) − 2i z_0 P \cdot m_2\right].$$

By the analogue for $\nu$ of relation (5.29), together with the fact that $m_2$ is $h_0$-independent, we conclude that

$$\{P, \nu\}(E) = 4i,$$

which leads directly to our claim.

\[\square\]

**Remark 5.6.** (a) In the Proposition above, we were unable to obtain the Poisson brackets between the $I_j$’s and $\nu$. Nevertheless, the functional independence of the integrals follows by combining Proposition 5.4 and Proposition 5.5.

(b) Proposition 5.4 shows that the Hamiltonian flow generated by $P$ does not give rise to nontrivial motion on the Jacobi variety of $C$, this is the reason for the introduction of the algebro-geometric variable $\nu$. (See also Proposition 6.2 below in this connection.) Note that in particular, this means that the periodic defocusing Ablowitz-Ladik equation is not an algebraically completely integrable system in the sense of Adler and van Moerbeke [AMV]. For other examples of integrable systems involving spectral curves which are not algebraically integrable, we refer the reader to [DLT] and [L2].
6. Solving the equations via factorization problems.

In this section, we will solve the Hamiltonian equations of motion generated by the commuting integrals in Section 5 via factorization problems.

We begin by writing down the Hamiltonian equations of motion by using Proposition 3.9, Theorem 3.6 and (5.4). To do so, for a map $F : gl(p, \mathbb{C}) \rightarrow \mathbb{C}$, we let $\nabla F(M) = (\partial F/\partial m_{ij})$ denote its gradient. In what follows, we will regard $\text{Re} I_j$, $\text{Im} I_j$ and $P$ as functions of $g^e$, $g^o$, where $\mathcal{E}(h) = g^e g^o(h)$, and $\tilde{\mathcal{E}}(h) = g^o(h)g^e$. We will use the notation of Proposition 3.9 (b). As an example, if $H = \text{Im} I_{p/2-j}$, then the corresponding central function $\varphi$ is given by $\varphi(X) = \text{Im} \oint_{|h|=1} E_j(X(h)) \frac{dh}{2\pi i n}$.

**Proposition 6.1.** The Hamiltonian equations of motion generated by $H = \text{Im} I_{p/2-j}$, $\text{Re} I_{p/2-j}$, and $P$ are given by

$$
\begin{align*}
\dot{g}^e &= (\Pi_{b_w}(D\varphi(\mathcal{E}(h)))) g^e - g^o(\Pi_{g^o_w}(D\varphi(\mathcal{E}(h)))), \\
\dot{g}^o &= (\Pi_{b_w}(D\varphi(\mathcal{E}(h)))) g^o(h) - g^o(h)(\Pi_{b_w}(D\varphi(\mathcal{E}(h)))),
\end{align*}
$$

(6.1)

where

$$
D\varphi(\mathcal{E}(h)) = \begin{cases} \\
\mathcal{E}(h)\nabla^T E_j(\mathcal{E}(h)), & \text{for } H = \text{Im} I_{p/2-j} \\
i\mathcal{E}(h)\nabla^T E_j(\mathcal{E}(h)), & \text{for } H = \text{Re} I_{p/2-j} \\
-ih\mathcal{E}(h)\nabla^T E_{p/2}(\mathcal{E}(h)), & \text{for } H = P,
\end{cases}
$$

(6.2)

and similarly for $D\varphi(\tilde{\mathcal{E}}(h))$.

In (6.1), the parameter $h$ is on the unit circle, however, we will remove this restriction later on. (See (6.29) below.) Before solving these equations, let us spell out $\nabla^T E_j(x)$ more explicitly. To do so, observe that $\nabla^T E_j(x)$ (see (5.3) above) obey the recursion relations

$$
\nabla^T E_{j+1}(x) = x \nabla^T E_j(x) - E_j(x) I, \quad 0 \leq j \leq p,
$$

(6.3)

where by convention $\nabla^T E_{p+1}(x) \equiv 0$. Since $\nabla^T E_0(x) = 0$, by solving the recursion relations backwards, we obtain

$$
\nabla^T E_j(x) = -\sum_{i=0}^{j-1} E_{j-1-i}(x)x^i, \quad j = 1, \cdots, p.
$$

(6.4)

The equations of motion generated by $P$ are the simplest to solve. Although we could easily write down the solutions of these equations without recourse to Proposition 6.1 above, however, we will do it by the proposition in order to achieve uniformity in our treatment.

**Proposition 6.2.** The Hamiltonian equations of motion generated by $P$ simplify to

$$
\begin{align*}
\dot{g}^e &= g^e \Lambda_1 - \Lambda_2 g^e \\
\dot{g}^o &= g^o(h) \Lambda_2 - \Lambda_1 g^o(h)
\end{align*}
$$

(6.5)
where
\[ \Lambda_1 = i \text{Re} \left( A_1 \left( \frac{P}{2} \right) \right)_0 = \text{diag}(0, iP, 0, iP, \ldots, 0, iP), \] (6.6)
and
\[ \Lambda_2 = i \text{Re} \left( A_{-1} \left( \frac{P}{2} \right) \right)_0 = \text{diag}(iP, 0, iP, 0, \ldots, iP, 0). \] (6.7)

The solutions of (6.5) are therefore given by
\[ g^e(t) = e^{-t\Lambda_2(0)} g^e(0) e^{t\Lambda_1(0)}, g^o(h, t) = e^{-t\Lambda_1(0)} g^o(h, 0) e^{t\Lambda_2(0)}. \] (6.8)

Hence
\[ \alpha_j(t) = \alpha_j(0)e^{i tP(0)}, \quad \rho_j(t) = \rho_j(0), \quad j = 0, \ldots, p - 1. \] (6.9)

Proof. According to (6.2) and (6.3),
\[ D\varphi(\mathcal{E}(h)) = i\hbar \sum_{j=0}^{\frac{p-1}{2}} E_j(\mathcal{E}(h)) \mathcal{E}(h)^\frac{p-j}{2}. \] (6.10)

For \( j = 1, \ldots, \frac{p}{2} - 1 \), it follows from (2.11) that
\[ \Pi_{\tilde{b}_w} \left( i\hbar E_j(\mathcal{E}(h)) \mathcal{E}(h)^\frac{p-j}{2} \right) = i\hbar E_j(\mathcal{E}(h)) \mathcal{E}(h)^\frac{p-j}{2}. \] (6.11)

Similarly, by using the fact that \( A_{-1} \left( \frac{P}{2} \right) \) is lower triangular, we find
\[ \Pi_{\tilde{b}_w} \left( i\hbar E_0(\mathcal{E}(h)) \mathcal{E}(h)^\frac{p}{2} \right) = i\hbar \mathcal{E}(h)^\frac{p}{2} - i\text{Im} \left( iA_{-1} \left( \frac{P}{2} \right) \right)_0 \] (6.12)
as \( E_0 \equiv 1 \). Therefore, on using (6.11) and (6.12), we obtain
\[ \Pi_{\tilde{b}_w} D\varphi(\mathcal{E}(h)) = D\varphi(\mathcal{E}(h)) - i\text{Re} \left( A_{-1} \left( \frac{P}{2} \right) \right)_0. \] (6.13)

By a similar calculation, we find
\[ \Pi_{\tilde{b}_w} D\varphi(\tilde{\mathcal{E}}(h)) = D\varphi(\tilde{\mathcal{E}}(h)) - i\text{Re} \left( A_1 \left( \frac{P}{2} \right) \right)_0. \] (6.14)

Hence, on substituting (6.13) and (6.14) into (6.1), and using the obvious facts that \( \mathcal{E}(h)^\frac{p-j}{2}g^e = g^e\tilde{\mathcal{E}}(h)^\frac{p-j}{2}, g^o(h)\mathcal{E}(h)^\frac{p-j}{2} = \tilde{\mathcal{E}}(h)^\frac{p-j}{2}g^o(h) \), we obtain the equations in (6.5). The formulas in (6.6) and (6.7) for \( \Lambda_1 \) and \( \Lambda_2 \) in terms of \( P \) then follow from Lemma 2.2 and (2.19). As \( P \) is a conserved quantity, it is easy to verify that the expressions in (6.8) give solutions to the equations in (6.5). Finally the solution formulas in (6.9) are obtained from (6.8) by multiplying out. \( \square \)

To solve the Hamiltonian equations generated by \( H = \text{Im} I_{p/2-j}, \text{Re} I_{p/2-j} \), we will make use of Proposition 3.9 (c), which means we have to solve explicitly for each \( t > 0 \) the following factorization problems
\[ e^{t D\varphi(\mathcal{E}(h, 0))} = k_1(h, t)b_1(h, t)^{-1}, \quad e^{t D\varphi(\tilde{\mathcal{E}}(h, 0))} = k_2(h, t)b_2(h, t)^{-1} \] (6.15)
for \( k_i(\cdot,t) \in \tilde{K}_w, b_i(\cdot,t) \in \tilde{B}_w, i = 1, 2 \). However, from the definition of \( \tilde{K}_w \), it is easy to show that it is enough to solve
\[
e^{-t D\varphi(\mathcal{E}(h,0))} e^{-t D\varphi(\mathcal{E}(h,0))^*} = b_1(h,t) b_1(h,t)^* ,
\]
\[
e^{-t D\varphi(\tilde{\mathcal{E}}(h,0))} e^{-t D\varphi(\tilde{\mathcal{E}}(h,0))^*} = b_2(h,t) b_2(h,t)^* \tag{6.16}
\]
for \( b_1(\cdot,t), b_2(\cdot,h) \in \tilde{B}_w \). Note that for \( i = 1, 2 \), \( b_i(\cdot,t) \) (resp. \( b_i(\cdot,t)^* \)) can be extended analytically in the interior (resp. exterior) of the unit circle \(|h| = 1\). So (6.16) is a Riemann-Hilbert problem. In order to solve this problem explicitly, we will first transform the product on the left hand side of (6.16) into a form which makes the problem more tractable. To this end, note that it follows from (6.4) that
\[
\nabla^T E_j(\mathcal{E}(h,0)) = - \sum_{i=0}^{j-1} E_{j-1-i}(\mathcal{E}(h,0)) \mathcal{E}(h,0)i, \quad j = 1, \cdots, p. \tag{6.17}
\]
As \( \mathcal{E}(h,0) \) commutes with \( \mathcal{E}(h,0)^* \) for \( h \in \partial\mathbb{D} \), it follows from (6.17) and (6.2) that \( D\varphi(\mathcal{E}(h,0)) \) commutes with \( D\varphi(\mathcal{E}(h,0))^* \) for \( h \in \partial\mathbb{D} \). In a similar way, we see that \( D\varphi(\tilde{\mathcal{E}}(h,0)) \) commutes with \( D\varphi(\tilde{\mathcal{E}}(h,0))^* \). Consequently, we can rewrite (6.16) as
\[
e^{-t(D\varphi(\mathcal{E}(h,0)) + D\varphi(\mathcal{E}(h,0))^*)} = b_1(h,t) b_1(h,t)^*, \tag{6.18}
\]
\[
e^{-t(D\varphi(\tilde{\mathcal{E}}(h,0)) + D\varphi(\tilde{\mathcal{E}}(h,0))^*)} = b_2(h,t) b_2(h,t)^*, \quad h \in \partial\mathbb{D}.
\]
Now if we compute \( D\varphi(\mathcal{E}(h,0))^* \) (resp. \( D\varphi(\tilde{\mathcal{E}}(h,0)) \)) more carefully, we find
\[
D\varphi(\mathcal{E}(h,0))^* = \pm D\varphi(\mathcal{E}(h,0)^{-1}),
\]
\[
D\varphi(\tilde{\mathcal{E}}(h,0))^* = \pm D\varphi(\tilde{\mathcal{E}}(h,0)^{-1}), \quad h \in \partial\mathbb{D}, \tag{6.19}
\]
where we pick the + sign for \( H = \text{Im} I_{p/2-j} \) and the − sign for \( H = \text{Re} I_{p/2-j} \). Hence (6.18) becomes
\[
e^{-t(D\varphi(\mathcal{E}(h,0)) \pm D\varphi(\mathcal{E}(h,0)^{-1}))} = b_1(h,t) b_1(h,t)^*, \tag{6.20}
\]
\[
e^{-t(D\varphi(\tilde{\mathcal{E}}(h,0)) \pm D\varphi(\tilde{\mathcal{E}}(h,0)^{-1}))} = b_2(h,t) b_2(h,t)^*, \quad h \in \partial\mathbb{D},
\]
where the choice of sign is described in the previous sentence.

As the explicit solution of the factorization problem in (6.20) will involve constructing \( b_i(h,t), i = 1, 2 \), for values of \( h \) not on unit circle, it is necessary to introduce some Lie algebras and projection operators which complements those in Section 3. For this purpose, let \( \mathcal{A} \) be the ring of Laurent polynomials in the variable \( h \) and let \( gl_p(\mathcal{A}) \) be the Lie algebra of \( p \times p \) matrix functions with entries in \( \mathcal{A} \) equipped with the pointwise Lie bracket. We will consider the following Lie subalgebras of \( gl_p(\mathcal{A}) \):
\[
\bar{\mathfrak{b}} = \left\{ \sum_{j=0}^n X_j h^j \in gl_p(\mathbb{C}[h]) \mid X_0 \in \mathfrak{b} \right\}, \tag{6.21}
\]
\[
\bar{\mathfrak{f}} = \{ X \in gl_p(\mathcal{A}) \mid X(h) + X(h^{-1})^* = 0 \}. 
\]
Then analogous to (3.8), we have the splitting

\[ gl_p(A) = \tilde{\mathfrak{f}} \oplus \tilde{\mathfrak{b}}. \]  

(6.22)

For \( X \in gl_p(A) \), define

\[ (\Pi_+ X)(h) = \sum_{j>0} X_j h^j, \quad (\Pi_- X)(h) = \sum_{j<0} X_j h^j, \]  

(6.23)

then the projection operator onto \( \tilde{\mathfrak{b}} \) associated with the splitting in (6.22) is given by

\[ (\Pi_{\tilde{\mathfrak{b}}} X)(h) = (\Pi_+ X)(h) + \Pi_{\tilde{\mathfrak{b}}} X_0 + ((\Pi_- X)(\tilde{h}^{-1}))^*. \]  

(6.24)

**Theorem 6.3.** For \( H = Re I_{p/2−j}, Im I_{p/2−j} \), there exist unique holomorphic matrix-valued functions

\[ b_1(\cdot, t) : \mathbb{CP}^1 \setminus \{ \infty \} \rightarrow GL(p, \mathbb{R}), \]
\[ b_2(\cdot, t) : \mathbb{CP}^1 \setminus \{ \infty \} \rightarrow GL(p, \mathbb{R}) \]  

(6.25)

which are smooth in \( t \), solve the factorization problems

\[ e^{-t(D\varphi(\tilde{E}(h,0)) \pm D\varphi(\tilde{E}(h,0)^{-1}))} = b_1(h, t)b_1(\tilde{h}^{-1}, t)^*, \]
\[ e^{-t(D\varphi(\tilde{E}(h,0)) \pm D\varphi(\tilde{E}(h,0)^{-1}))} = b_2(h, t)b_2(\tilde{h}^{-1}, t)^*, \]  

(6.26)

\( h \in \mathbb{CP}^1 \setminus \{0, \infty\} \)

(\text{where the + sign corresponds to } H = Im I_{p/2−j} \text{ and the − sign corresponds to } H = Re I_{p/2−j} \text{)}

and satisfy

\[ b_i(0, t) \in B, b_i(h, t)^{-1} b_i(h, t) \in Im \Pi_{\tilde{\mathfrak{b}}} \quad i = 1, 2. \]  

(6.27)

Moreover, for \( h \in \mathbb{CP}^1 \setminus \{0, \infty\} \), the formulas

\[ g^e(t) = b_1(h, t)^{-1} g^e(0) b_2(h, t) = b_1(\tilde{h}^{-1}, t)^* g^e(0)(b_2(\tilde{h}^{-1}, t)^*)_1^{-1}, \]
\[ g^o(h, t) = b_2(h, t)^{-1} g^o(h, 0) b_1(h, t) = b_2(\tilde{h}^{-1}, t)^* g^o(h, 0)(b_1(\tilde{h}^{-1}, t)^*)_1^{-1} \]  

(6.28)

give solutions of the equations

\[ \dot{g}^e = (\Pi_{\tilde{\mathfrak{b}}} D\varphi(\tilde{E}(\cdot)))(h) g^e - g^e(\Pi_{\tilde{\mathfrak{b}}} D\varphi(\tilde{E}(\cdot)))(h), \]
\[ \dot{g}^o(h) = (\Pi_{\tilde{\mathfrak{b}}} D\varphi(\tilde{E}(\cdot)))(h) g^o(h) - g^o(h)(\Pi_{\tilde{\mathfrak{b}}} D\varphi(\tilde{E}(\cdot)))(h). \]  

(6.29)

Finally, for generic initial data \( g^e(0) \) and \( g^o(h, 0) \), \( b_1(h, t) \) and \( b_2(h, t) \) can be constructed by means of theta functions associated with the Riemann surface of the spectral curve \( \mathcal{C} = \{(h, z) \mid det(zI - g^e(0)g^o(h, 0)) = 0\} \) for values of \( t > 0 \) for which \( \alpha_j(t) \neq 0, j = 0, \cdots, p - 1. \)
Proof. We will prove the result for $H = \Im I_{p/2 - j}$. The argument for the other case is similar. We start with uniqueness of the factors $b_i(h, t), i = 1, 2$. To prove this, suppose $b_1^0(h, t), i = 1, 2$ is a second pair of solutions of the factorization problem. Then from $b_i(h, t)b_i(\bar{h}^{-1}, t)^* = b_i^0(h, t)b_i^0(\bar{h}^{-1}, t)^*$, we have

$$g_i(h, t) \equiv b_i^0(h, t)^{-1}b_i(h, t) = b_i^0(\bar{h}^{-1}, t)^*(b_i(\bar{h}^{-1}, t)^*)^{-1}, \ h \in \mathbb{C}^1 \setminus \{0, \infty\}. \quad (6.30)$$

Clearly the function $g_i(\cdot, t)$ defined in (6.30) above can be extended to an analytic function everywhere, hence by Liouville’s theorem, $g_i(h, t) = c_i(t)$. To determine $c_i(t)$, note that $g_i(0, t) = b_i^0(0, t)^{-1}b_i(0, t) \in \mathfrak{b}$. On the other hand, $g_i(\infty, t) = (g_i(0, t)^{-1})^*$ is upper triangular with positive diagonal entries on the diagonal. Hence $c_i(t) \equiv I$.

To establish the existence of $b_i(h, t), i = 1, 2$, note that $\mathcal{E}(h, t)$ (resp. $\bar{\mathcal{E}}(h, t)$) exists for $h \in \mathbb{C}P \setminus \{0, \infty\}$ since it exists for $h \in \partial \mathbb{D}$. Hence we can obtain $b_i(h, t), i = 1, 2$ as solutions of the equations

$$\dot{b}_1(h, t) = -b_1(h, t)(\Pi_\delta D\varphi(\mathcal{E}(\cdot, t)))(h), \quad (6.31)$$

and

$$\dot{b}_2(h, t) = -b_2(h, t)(\Pi_\delta D\varphi(\bar{\mathcal{E}}(h, t)))(h). \quad (6.32)$$

Clearly, the analyticity properties and (6.27) are satisfied by definition. We next consider the product $b_1(h, t)b_1(\bar{h}^{-1}, t)^*$. By differentiating, we have

$$\frac{d}{dt}(b_1(h, t)b_1(\bar{h}^{-1}, t)^*)$$

$$= -b_1(h, t)\left((\Pi_\delta D\varphi(\mathcal{E}(\cdot, t)))(h) + ((\Pi_\delta D\varphi(\mathcal{E}(\cdot, t)))(\bar{h}^{-1}))^*\right)b_1(\bar{h}^{-1}, t)^*. \quad (6.33)$$

Now, from the definition of $\Pi_\delta$, it is straightforward to check that

$$(\Pi_\delta X)(h) + ((\Pi_\delta X)(\bar{h}^{-1}))^* = X(h) + (X(\bar{h}^{-1}))^*. \quad (6.34)$$

Consequently, we obtain

$$((\Pi_\delta D\varphi(\mathcal{E}(\cdot, t)))(h) + ((\Pi_\delta D\varphi(\mathcal{E}(\cdot, t)))(\bar{h}^{-1}))^*)$$

$$= D\varphi(\mathcal{E}(h, t)) + (D\varphi(\mathcal{E}(\bar{h}^{-1}, t)))^* \quad (6.35)$$

Substitution of (6.35) into (6.33) therefore gives the relation

$$\frac{d}{dt}(b_1(h, t)b_1(\bar{h}^{-1}, t)^*)$$

$$= -b_1(h, t)(D\varphi(\mathcal{E}(h, t)) + D\varphi(\mathcal{E}(h, t)^{-1}))b_1(\bar{h}^{-1}, t)^*$$

$$= -b_1(h, t)b_1(\bar{h}^{-1}, t)^*(D\varphi(\mathcal{F}(h, t)) + D\varphi(\mathcal{F}(h, t)^{-1})) \quad (6.36)$$
where we have used the fact that \( \varphi \) is a central function, and where \( \mathcal{F}(h, t) = (b_1(\bar h^{-1}, t)^*)^{-1} \mathcal{E}(h, t)b_1(\bar h^{-1}, t)^* \). Now, by direct differentiation, using (6.31), the equation

\[
\dot{\mathcal{E}}(h, t) = (\Pi \nu D\varphi(\mathcal{E}(\cdot, t)))(h)\mathcal{E}(h, t) - \mathcal{E}(h, t)(\Pi \nu D\varphi(\mathcal{E}(\cdot, t)))(h),
\]

and (6.35), we find that

\[
\frac{d}{dt} \mathcal{F}(h, t) = (b_1(\bar h^{-1}, t)^*)^{-1}(D\varphi(\mathcal{E}(h, t)) + D\varphi(\mathcal{E}(h, t)^{-1}))\mathcal{E}(h, t)b_1(\bar h^{-1}, t)^*
\]

\[
- (b_1(\bar h^{-1}, t)^*)^{-1}\mathcal{E}(h, t)(D\varphi(\mathcal{E}(h, t)) + D\varphi(\mathcal{E}(h, t)^{-1}))b_1(\bar h^{-1}, t)^*
\]

\[= 0.\]

Therefore, \( \mathcal{F}(h, t) = \mathcal{E}(h, 0) \) and so (6.36) becomes

\[
\frac{d}{dt}(b_1(h, t)b_1(\bar h^{-1}, t)^*)
\]

\[= - b_1(h, t)b_1(\bar h^{-1}, t)^*(D\varphi(\mathcal{E}(h, 0)) + D\varphi(\mathcal{E}(h, 0)^{-1})).\]

This shows \( b_1(h, t) \) satisfies the first relation in (6.26). In a similar fashion, we can show that \( b_2(h, t) \) satisfies the second relation in (6.26).

Finally, we will show that \( g^e(t) \) and \( g^o(h, t) \) as defined in (6.28) satisfy (6.29).

First, note that by using the relation \( \bar{\mathcal{E}}(h, 0) = g^e(0)^{-1}\mathcal{E}(h, 0)g^o(0) \) and (6.26), we have

\[
b_1(h, t)^{-1}g^e(0)b_2(h, t) = b_1(\bar h^{-1}, t)^*g^e(0)(b_2(\bar h^{-1}, t)^*)^{-1},
\]

\[
b_2(h, t)^{-1}g^o(0)b_1(h, t) = b_2(\bar h^{-1}, t)^*g^o(0)(b_1(\bar h^{-1}, t)^*)^{-1}.
\]

Differentiate \( g^e(t) = b_1(h, t)^{-1}g^e(0)b_2(h, t) \) and \( g^o(h, t) = b_2(h, t)^{-1}g^o(0)b_1(h, t) \) with respect to \( t \), we find

\[
\dot{g}^e(t) = -b_1(h, t)^{-1}b_1(h, t)g^e(t) + g^e(t)b_2(h, t)^{-1}b_2(h, t),
\]

\[
\dot{g}^o(h, t) = -b_2(h, t)^{-1}b_2(h, t)g^o(h, t) + g^o(h, t)b_1(h, t)^{-1}b_1(h, t).
\]

On the other hand, by differentiating the first relation in (6.26) with respect to \( t \), and multiply the resulting expression on the left by \( b_1(h, t)^{-1} \) and on the right by \( (b_1(\bar h^{-1}, t)^*)^{-1} \), we obtain

\[
- b_1(\bar h^{-1}, t)^*(D\varphi(\mathcal{E}(h, 0)) + D\varphi(\mathcal{E}(h, 0)^{-1}))(b_1(\bar h^{-1}, t)^*)^{-1}
\]

\[= b_1(h, t)^{-1}b_1(h, t) + b_1(\bar h^{-1}, t)^*(b_1(\bar h^{-1}, t)^*)^{-1}.
\]

But from the fact that \( \varphi \) is a central function and (6.40), we see that

\[
b_1(\bar h^{-1}, t)^*(D\varphi(\mathcal{E}(h, 0)) + D\varphi(\mathcal{E}(h, 0)^{-1}))(b_1(\bar h^{-1}, t)^*)^{-1}
\]

\[= D\varphi(\mathcal{E}(h, t)) + D\varphi(\mathcal{E}(h, t)^{-1}).\]
where $\mathcal{E}(h, t) = g^e(t)g^o(h, t)$. Thus (6.42) becomes
\[-(D\varphi(\mathcal{E}(h, t)) + D\varphi(\mathcal{E}(h, t)^{-1})) = b_1(h, t)^{-1}b_1(h, t) + (b_1(\tilde{h}^{-1}, t)^{-1}b_1(\tilde{h}^{-1}, t))^*.
\]
(6.44)

Now let
\[X(h, t) = b_1(h, t)^{-1}b_1(h, t) - (b_1(\tilde{h}^{-1}, t)^{-1}b_1(\tilde{h}^{-1}, t))^*,
\]
\[Y(h, t) = D\varphi(\mathcal{E}(h, t)) - D\varphi(\mathcal{E}(\tilde{h}^{-1}, t)^{-1}).
\]
(6.45)

Clearly, $X(\cdot, t) \in \tilde{k}$, thus $\Pi_\xi X(\cdot, t) = 0$. On the other hand, as $D\varphi(\mathcal{E}(h, t)^{-1}) = D\varphi(\mathcal{E}(\tilde{h}^{-1}, t)^*)$, we also have $Y(\cdot, t) \in \tilde{k}$ and $\Pi_\xi Y(\cdot, t) = 0$. Consequently, when we apply $\Pi_\xi$ to both sides of (6.44), we obtain
\[b_1(h, t)^{-1}b_1(h, t) = -(\Pi_\xi D\varphi(\mathcal{E}(\cdot, t)))(h).
\]
(6.46)

Similarly, from the second relation in (6.26), we can show that
\[b_2(h, t)^{-1}b_2(h, t) = -(\Pi_\xi D\varphi(\tilde{\mathcal{E}}(\cdot, t)))(h).
\]
(6.47)

Finally, substituting (6.46) and (6.47) into (6.41) gives (6.29). This completes the proof of the theorem modulo the assertion on the construction of $b_1(h, t)$ and $b_2(h, t)$ via Riemann theta functions.

We now turn to the construction of $b_1(h, t)$ and $b_2(h, t)$ via theta functions. Again, we will give details for $H = \text{Im } I_{p/2-j}$, leaving the other case to the interested reader. The following proposition shows we can construct $b_2(h, t)$ from $b_1(h, t)$ and the solution of a finite dimensional factorization problem.

**Proposition 6.4.** Let $l(t) \in B$, $u(t) \in K$ be the solution of the factorization problem
\[b_1(0, t)^{-1}g^e(0) = u(t)l(t)^{-1}.
\]
(6.48)

Then
\[b_2(h, t) = g^e(0)*b_1(h, t)u(t).
\]
(6.49)

**Proof.** Since $\tilde{\mathcal{E}}(h, 0) = g^e(0)^{-1}\mathcal{E}(h, 0)g^e(0)$, the factorization problem for $b_2(h, t)$ in (6.26) can be rewritten as
\[e^{-t(D\varphi(\mathcal{E}(h, 0)) + D\varphi(\mathcal{E}(h, 0)^{-1}))} = g^e(0)b_2(h, t)b_2(\tilde{h}^{-1}, t)^*(g^e(0))^{-1}.
\]
(6.50)

Therefore, when we compare this with the first relation in (6.26), we obtain
\[b_2(h, t)b_2(\tilde{h}^{-1}, t)^* = g^e(0)*b_1(h, t)(g^e(0)*b_1(\tilde{h}^{-1}, t))^*.
\]
(6.51)

Now let $l(t) \in B$, $u(t) \in K$ be the solution of the factorization problem in (6.48). Then
\[g^e(0)*b_1(h, t) = (l(t) + O(h))u(t)^*
\]
(6.52)
and
\[ g^e(0)^*b_1(h^{-1}, t) = (l(t) + O(h^{-1}))u(t)^*. \] (6.53)

Substitute (6.52) and (6.53) into (6.51), we obtain
\[ b_2(h, t)b_2(h^{-1}, t) = (l(t) + O(h))(l(t)^* + O(h^{-1})). \] (6.54)

Therefore, from the uniqueness of solution of the factorization problem (Theorem 6.3), we conclude that
\[ b_2(h, t) = l(t) + O(h) = g^e(0)^*b_1(h, t) u(t). \]

To construct \( b_1(h, t) \), we invoke the formula in (6.2) and (6.17), according to which we have
\[
(D\varphi(\mathcal{E}(h(P), 0)) + D\varphi(\mathcal{E}(h(P), 0)^{-1})) \hat{v}(P) = \mu_j(P) \hat{v}(P) \] (6.55)

where \( \mu_j \) is meromorphic on the hyperelliptic Riemann surface \( C \). From the first relation in (6.26),
\[
e^{-\mu_j(P)}b_1(h(P), t)^{-1}\hat{v}(P)
= b_1(h(P), t)^{-1}e^{-t(D\varphi(\mathcal{E}(h(P), 0)) + D\varphi(\mathcal{E}(h(P), 0)^{-1}))}\hat{v}(P)
= b_1(h(P)^{-1}, t)^*\hat{v}(P) \] (6.56)

for \( h(P) \in \mathbb{CP}^1 \setminus \{0, \infty\} \). Since \( \mathcal{E}(h, t) = b_1(h, t)^{-1}\mathcal{E}(h, 0)b_1(h, t) \), we have
\[
\mathcal{E}(h(P), t)b_1(h(P), t)^{-1}\hat{v}(P) = z(P)b_1(h(P), t)^{-1}\hat{v}(P). \] (6.57)

On the other hand, as
\[
\mathcal{E}(h(P), t) = (\overline{\mathcal{E}(h(P)^{-1}, t)^*})^{-1} = b_1(h(P), t)^*\mathcal{E}(h(P), 0)(\overline{b_1(h(P)^{-1}, t)^*})^{-1}, \] (6.58)

we also have
\[
\mathcal{E}(h(P), t)b_1(h(P)^{-1}, t)^*\hat{v}(P) = z(P)b_1(h(P)^{-1}, t)^*\hat{v}(P). \] (6.59)

Thus if we let
\[
v_\pm^t(P) = b_1(h(P), t)^{-1}\hat{v}(P),
\] (6.60)

then (6.56), (6.57) and (6.59) give
\[
e^{-t\mu_j(P)}v_+^t(P) = v_-^t(P), \quad h(P) \in \mathbb{CP}^1 \setminus \{0, \infty\}
\]
\[
\mathcal{E}(h(P), t)v_\pm^t(P) = z(P)v_\pm^t(P). \] (6.61)
In this way, we are led to scalar factorization problems. Note that because $b_1(0,t)$ is lower triangular, it follows from (4.50) and (6.60) that
\[
\left((v'_+)_{2j}\right) \geq -D(0) + \left(\frac{p}{2} - j - 1\right) P_+ + \left(\frac{p}{2} - j\right) Q_-
\]
\[
\left((v'_+)_{2j+1}\right) \geq -D(0) + \left(\frac{p}{2} - j - 1\right) P_+ + \left(\frac{p}{2} - j - 1\right) Q_-
\]
(6.62)
on $C \setminus \{h(P) = \infty\}$. Similarly, because $b_1(0,t)^*\ is upper triangular, we find that
\[
\left((v'_-)_{2j}\right) \geq -D(0) - \left(\frac{p}{2} - j - 1\right) P_+ - \left(\frac{p}{2} - j\right) Q_+
\]
\[
\left((v'_-)_{2j+1}\right) \geq -D(0) - \left(\frac{p}{2} - j - 1\right) P_+ - \left(\frac{p}{2} - j - 1\right) Q_+
\]
(6.63)
on $C \setminus \{h(P) = 0\}$.

We will first solve the following scalar factorization problem (cf. [RSTS], [DL])
\[
e^{-\frac{2}{\iota} (\omega_+^j(P))} = \omega_+^j(P), \quad P \in C \setminus \{h(P) = 0, \infty\},
\]
(6.64)
\[
(\omega_+^j) \geq -D(0), \quad \text{on } C \setminus \{h(P) = \infty\},
\]
(6.65)
\[
(\omega_+^j) \geq -D(0), \quad \text{on } C \setminus \{h(P) = 0\}.
\]

To do so, we fix a canonical homology basis $\{a_j, b_k\}_{1 \leq j, k \leq g}$ of the Riemann surface associated with the spectral curve, and let $\{\omega_i\}_{1 \leq i \leq g}$ be a cohomology basis dual to $\{a_j, b_k\}$, i.e.,
\[
\int_{a_j} \omega_i = \delta_{ij}, \quad \int_{b_j} \omega_i = \Omega_{ij},
\]
(6.66)
where $(\Omega_{ij})$ is the Riemann matrix. Let
\[
\theta(z) = \theta(z, \Omega) = \sum_{m \in \mathbb{Z}^g} \exp\{2\pi i(m, z) + \pi i(m, \Omega m)\}
\]
be the Riemann theta function associated with the matrix $\Omega$. Let $\omega = (\omega_1, \cdots, \omega_g)$. Choose a nonsingular $e \in \mathbb{C}^g$ in the theta divisor, i.e. $\theta(e) = 0$, the prime form $E_e(x, y) \equiv \theta(e + \int_x^y \omega) \neq 0$ with the additional property that $E_e(P_\pm, P)$ are not identically zero in $P$. (See Lemma 3.3 of [M].) Let $P_0$ be a fixed point on the finite part of the Riemann surface, then by Corollary 3.6 of [M], there exists an effective divisor $D_{g-1}$ of degree $g - 1$ such that $e = \Delta - \int_{(g-1)P_0} \omega$, where $\Delta$ is the vector of Riemann constants. Also note that
\[
\theta \left(e + \int_{D(0)}^{D_{g-1} + P} \omega\right) = \theta \left(\Delta + \int_{D(0)}^{(g-1)P_0 + P} \omega\right) = 0 \iff P \in D(0).
\]
(6.67)

Now let $d\phi_+$ be the unique meromorphic differential of the second kind with vanishing $a$-periods with poles only at $\{h(P) = \infty\}$ such that $(d\phi_+ - d\mu_j)(P)$ is regular in $C \setminus \{h(P) = 0\}$. Set
\[
\omega_+^j(P) = \exp(t\phi_+(P)) \frac{\theta \left(e + \int_{D(0)}^{D_{g-1} + P} \omega + t\Phi_+\right)}{\theta \left(e + \int_{D(0)}^{D_{g-1} + P} \omega\right)},
\]
(6.68)
where \( \phi_+(P) = \int_{P_0}^P d\phi_+ \) and \( \Phi_+ \) is the vector of \( b \)-periods of \( d\phi_+ \), i.e.,

\[
(\Phi_+)_j = \frac{1}{2\pi i} \int_{b_j} d\phi_+, \quad j = 1, \cdots, g.
\] (6.69)

By using (6.69), the basic property

\[
\theta(z + \gamma' + \Omega\gamma) = \theta(z) \exp\left(2\pi i(-\gamma, z) - \frac{1}{2}(\gamma, \Omega\gamma)\right),
\]

and (6.67), it is straightforward to check that \( \omega_+^t \) is single-valued and meromorphic in \( C \setminus \{h(P) = \infty\} \) with \( (\omega_+^t)_j \geq -D(0) \). Moreover, \( e^{-t\mu_j(P)}\omega_+^t(P) \) is meromorphic in \( C \setminus \{h(P) = 0\} \) with \( (e^{-t\mu_j}\omega_+^t) \geq -D(0) \) because \( d\phi_+(P) - d\mu_j(P) \) is regular in \( C \setminus \{h(P) = 0\} \). Thus \( \omega_+^t \) and \( \omega_-^t \equiv e^{-t\mu_j}\omega_+^t \) solves the scalar factorization problem (6.64).

Next, we compare (6.64) and (6.61), this gives

\[
\omega_+^t(P)^{-1}v_+^t(P) = \omega_-^t(P)^{-1}v_-^t(P), \quad C \setminus \{h(P) = 0, \infty\}. \] (6.70)

Therefore,

\[
\tilde{v}^t(P) \equiv \begin{cases} 
\omega_+^t(P)^{-1}v_+^t(P), & P \in C \setminus \{h(P) = \infty\} \\
\omega_-^t(P)^{-1}v_-^t(P), & P \in C \setminus \{h(P) = 0\}
\end{cases}
\] (6.71)

is meromorphic on \( C \). Moreover, it follows from (6.62), (6.63) and the expressions for \( \omega_+^t \) that

\[
(\tilde{v}^t_2)_j \geq -\tilde{D}(t) - \left(\frac{p}{2} - j - 1\right)P_+ + \left(\frac{p}{2} - j - 1\right)P_- \\
- \left(\frac{p}{2} - j\right)Q_+ + \left(\frac{p}{2} - j\right)Q_-,
\] (6.72)

\[
(\tilde{v}^t_{2j+1}) \geq -\tilde{D}(t) - \left(\frac{p}{2} - j - 1\right)P_+ + \left(\frac{p}{2} - j - 1\right)P_- \\
- \left(\frac{p}{2} - j - 1\right)Q_+ + \left(\frac{p}{2} - j - 1\right)Q_-,
\]

where \( \tilde{D}(t) \) is the divisor of zeros of the function

\[
\theta \left(e + \int_{D(0)}^{D_{g-1}+P} \omega + t\Phi_+\right). \] (6.73)

Since \( \mathcal{E}(h(P), t)v_+^t(P) = z(P)v_+^t(P) \), it follows from the definition of \( \tilde{v}^t(P) \) that

\[
\mathcal{E}(h(P), t)\tilde{v}^t(P) = z(P)\tilde{v}^t(P). \] (6.74)

But on the other hand,

\[
\mathcal{E}(h(P), t)\tilde{v}(t, P) = z(P)\tilde{v}(t, P), \] (6.75)
where the last component of $\tilde{v}(t, P)$ is equal to 1 and

$$
(\tilde{v}_{2j}(t, \cdot)) \geq -D(t) - \left( \frac{p}{2} - j - 1 \right) P_+ + \left( \frac{p}{2} - j - 1 \right) P_-
- \left( \frac{p}{2} - j \right) Q_+ + \left( \frac{p}{2} - j \right) Q_-,
$$

$$(\tilde{v}_{2j+1}(t, \cdot)) \geq -D(t) - \left( \frac{p}{2} - j - 1 \right) P_+ + \left( \frac{p}{2} - j - 1 \right) P_-
- \left( \frac{p}{2} - j - 1 \right) Q_+ + \left( \frac{p}{2} - j - 1 \right) Q_-.$$  \tag{6.76}

Clearly, $\tilde{v}^t(P) = \tilde{v}_{p-1}^t(P)\tilde{v}(t, P)$. Let $\tilde{D}_0(t)$ be the divisor of zeros of $\tilde{v}_{p-1}^t(P)$ so that

$$(\tilde{v}_{p-1}^t) = \tilde{D}_0(t) - \tilde{D}(t).$$  \tag{6.77}

Then it follows from the relation connecting $\tilde{v}^t(P)$ and $\tilde{v}(t, P)$ above, (6.72) and (6.77) that

$$(\tilde{v}_{2j}(t, \cdot)) = (\tilde{v}_{2j}^t) - (\tilde{v}_{p-1}^t)
\geq -\tilde{D}_0(t) - \left( \frac{p}{2} - j - 1 \right) P_+ + \left( \frac{p}{2} - j - 1 \right) P_-
- \left( \frac{p}{2} - j \right) Q_+ + \left( \frac{p}{2} - j \right) Q_-.$$  \tag{6.78}

Similarly,

$$(\tilde{v}_{2j+1}(t, \cdot)) \geq -\tilde{D}_0(t) - \left( \frac{p}{2} - j - 1 \right) P_+ + \left( \frac{p}{2} - j - 1 \right) P_-
- \left( \frac{p}{2} - j - 1 \right) Q_+ + \left( \frac{p}{2} - j - 1 \right) Q_-.$$  \tag{6.79}

Clearly, we must have $\tilde{D}_0(t) \geq D(t)$. Since $\deg D(t) = \deg \tilde{D}_0(t) = g$, we must have $\tilde{D}_0(t) = D(t)$. But $D(t)$ is a general divisor by Proposition 4.12, so from $(\tilde{v}_{p-1}^t) = \tilde{D}_0(t) - \tilde{D}(t) = D(t) - D(t)$, we must have $\tilde{v}_{p-1}^t$ = constant and this implies $\tilde{D}(t) = D(t)$. Thus we can solve for $\tilde{v}^t(P)$ and hence $v_{p-1}^t(P)$ up to multiples because of Proposition 4.12. Indeed, by making use of the prime form, we can write down the explicit expression

$$
\tilde{v}_{2j}^t(P) = c_{2j}(t) \left( \frac{E_e(P_-, P)}{E_e(P_+, P)} \right)^{\frac{p}{2} - j - 1} \cdot \left( \frac{E_e(Q_-, P)}{E_e(Q_+, P)} \right)^{\frac{p}{2} - j}
\times \frac{\theta \left( e + \int_{D(t)}^{D_{p-1} + P} \omega \right)}{\theta \left( e + \int_{D(t)}^{D_{p-1} + P} \omega \right)}
$$

where $c_{2j}(t)$ has to be determined. Similarly, we have

$$
\tilde{v}_{2j+1}^t(P) = c_{2j+1}(t) \left( \frac{E_e(P_-, P)}{E_e(P_+, P)} \right)^{\frac{p}{2} - j - 1} \cdot \left( \frac{E_e(Q_-, P)}{E_e(Q_+, P)} \right)^{\frac{p}{2} - j - 1}
\times \frac{\theta \left( e + \int_{D(t)}^{D_{p-1} + P} \omega \right)}{\theta \left( e + \int_{D(t)}^{D_{p-1} + P} \omega \right)}
$$

\tag{6.80}

\tag{6.81}
where \( c_{2j+1}(t) \) is also as yet undetermined.

We are now ready to construct \( b_1(h, t) \) and in the process, we will also determine \( c_j(t) \). For given \( h \in \mathbb{CP}^1 \) which is not a branch point of the coordinate function \( h(P) \), there exist \( p \) points \( P_0(h), \ldots, P_{p-1}(h) \) of the Riemann surface \( C \) lying over \( h \). Therefore we can define the matrices

\[
V_\pm(h, t) = (v_\pm^1(P_0(h)) \cdots v_\pm^t(P_{p-1}(h)))
\]

\[
\hat{V}^\theta(h) = (\hat{\nu}(P_0(h)) \cdots \hat{\nu}(P_{p-1}(h)))
\]

where \( \hat{\nu}(P) \) can be obtained from the formula for \( \hat{\nu}'(P) \) by setting \( t = 0 \) (since \( \omega_\pm^{t=0}(P) \equiv 1 \) and \( b_1(h, t = 0) = I \) and so can be computed in terms of theta functions. Of course, \( v_\pm^t(P) \) are also in terms of theta functions. With these matrices, it follows that

\[
b_1(h, t) = \hat{V}^\theta(h)V_+(h, t)^{-1}.
\]

Of course, \( v_\pm^t(P) = \omega_\pm^t(P)\hat{\nu}'(P) \) are determined only up to the \( c_j(t) \)’s. Write

\[
v_\pm^t(P) = c(t)v_\pm^t(t, P)
\]

where \( v_\pm^t(t, P) \) are known and \( c(t) = \text{diag}(c_0(t), \ldots, c_{p-1}(t)) \) is to be determined.

Then

\[
V_\pm(h, t) = c(t)V_\pm^\theta(h, t)
\]

where \( V_\pm^\theta(h, t) = (v_\pm^1(t, P_0(h)) \cdots v_\pm^t(t, P_{p-1}(h))) \). With these definitions,

\[
b_1(h, t) = \hat{V}^\theta(h)V_+^\theta(h, t)^{-1}c(t)^{-1}.
\]

As \( b_1(h, 0) = I \), the above relation determines \( c(0) \) via the formula

\[
c(0) = \hat{V}^\theta(h)V_+^\theta(h, 0)^{-1}.
\]

To determine \( c(t) \) for \( t > 0 \), we bring in

\[
k_1(h, t) = e^{tD\varphi(E(h, 0))}b_1(h, t)
\]

\[
e^{tD\varphi(E(h, 0))}\hat{V}^\theta(h)V_+^\theta(h, t)^{-1}c(t)^{-1}
\]

which is required to be unitary for \( h \in \partial\mathbb{D} \). Therefore, if we equate the expression for \( k_1(1, t) \) from (6.88) with the corresponding one for \( (k_1(1, t)^*)^{-1} \), we obtain \( |c(t)|^2 = c(t)^*c(t) \). Explicitly,

\[
|c(t)|^2 = (\hat{V}^\theta(1)V_+^\theta(1, t)^{-1})^*e^{t(D\varphi(E(1, 0)) + D\varphi(E(1, 0)^{-1}))}\hat{V}^\theta V_+^\theta(1, t)^{-1}c(t)^{-1}.
\]

Write \( c(t) = |c(t)|e^{i\eta(t)} \), where \( e^{i\eta(t)} = \text{diag}(e^{i\eta_0(t)}, \ldots, e^{i\eta_{p-1}(t)}) \). It remains to determine \( e^{i\eta(t)} \). However, this is fixed by the condition that

\[
b_1(0, t) = \hat{V}^\theta(0)V_+^\theta(0, t)^{-1}|c(t)|^{-1}e^{-i\eta(t)} \in B
\]

as the diagonal entries of the elements in \( B \) are positive.

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