K-THEORETIC QUASIMAP WALL-CROSSING

MING ZHANG AND YANG ZHOU

Abstract. In this paper, we prove a $K$-theoretic wall-crossing formula for $\epsilon$-stable quasimaps for all GIT targets in all genera. It recovers the genus-0 $K$-theoretic toric mirror theorem by Givental–Tonita [29] and Givental [20, 21], and the genus-0 mirror theorem for quantum $K$-theory with level structure by Ruan-Zhang [50]. The proofs are based on $K$-theoretic virtual localization on the master space introduced by the second author in [62].

Contents

1. Introduction 1
2. $K$-theoretic quasimap invariants 8
3. $K$-theoretic quasimap invariants with entangled tails 18
4. $K$-theoretic localization on the Master space 27
5. $K$-theoretic wall-crossing formula 35
Appendix A. $K$-theoretic pushforward formula for inflated projection bundles 60
References 65

1. Introduction

1.1. Overview. Quantum $K$-theory was introduced by Givental [17] and Y.P. Lee [42] as the $K$-theoretic version of Gromov-Witten theory. Let $X$ be a smooth Deligne–Mumford stack with projective coarse moduli over $\mathbb{C}$. The classical Gromov-Witten invariants of $X$ are defined as integrals of cohomology classes over the moduli space $\overline{M}_{g,n}(X,d)$ of stable maps. In quantum $K$-theory, the basic objects are vector bundles and coherent sheaves. Quantum $K$-invariants of $X$ are defined as holomorphic Euler characteristics of natural $K$-theory classes over $\overline{M}_{g,n}(X,d)$. Recently there has been increased interest in studying quantum $K$-invariants due to their connections to 3d gauge theories [36–38, 60] and representation theory [3, 4, 41, 46, 47, 51]. For example, the level structure defined in [50] is related to Chern-Simon levels in 3d $\mathcal{N} = 2$ gauge theory (c.f. [37, 38, 60]) and the space of conformal blocks (c.f. [51]).

The second author is supported by the Simons Collaboration Grant for Mathematicians and the Center of Mathematical Sciences and Applications, Harvard University.
As shown in [17, 42], the WDVV equation and most of Kontsevich-Manin axioms hold in quantum \( K \)-theory. In this sense, quantum \( K \)-theory has similar structures to Gromov-Witten theory. Moreover, it is shown in [29], [58] and [26] that quantum \( K \)-invariants and their permutation-equivariant versions are determined by cohomological Gromov-Witten invariants of the same manifold or orbifold. However, the precise relationship between \( K \)-theoretic and cohomological invariants is sophisticated. In general, Gromov-Witten invariants are rational numbers, while their \( K \)-theoretic counterparts are always integers. To relate them, one needs to apply the (virtual) Kawasaki-Riemann-Roch formula [56] to express holomorphic Euler characteristics of coherent sheaves on \( \overline{M}_{g,n}(X, d) \) in terms of intersection numbers over the corresponding inertia stack \( I\overline{M}_{g,n}(X, d) \). The complexity comes from the combinatorics of the strata of the inertia stack.

Unlike cohomological Gromov-Witten theory, the computations in quantum \( K \)-theory have been rather rare. One important way to compute genus-0 quantum \( K \)-invariants is by using \( K \)-theoretic mirror theorems, see e.g., [57], [36], and [37, Section 6.3]. To established the general mirror theorems in the \( K \)-theory setting, Givental introduced and studied a variant of quantum \( K \)-invariants in a series of papers [18–28]. These invariants are called permutation-equivariant quantum \( K \)-invariants. In the toric case, the mirror theorem [20, 21, 29] states that up to a change of variable (mirror map), a specific generating series (\( J \)-function) of genus-0 permutation-equivariant quantum \( K \)-invariants with descendants can be identified with an explicit \( q \)-hypergeometric series (\( I \)-function). Their proofs are based on the so-called adelelic characterization of the range of the \( J \)-function. This strategy has been generalized to quantum \( K \)-theory with level structure in [50].

In cohomological Gromov-Witten theory, the most general mirror theorem is obtained by using the wall-crossing strategy. Note that the moduli space \( \overline{M}_{g,n}(X, d) \) of stable maps is a compactification of the space of algebraic maps from smooth curves to \( X \). In many cases, there are other natural and simpler compactifications. For a large class of GIT quotients of affine varieties, a family of compactifications has been constructed in [6, 10], unifying and generalizing many previous constructions [8, 44, 45, 55]. These new theories are called \( \epsilon \)-stable quasimap theories, where \( \epsilon \in \mathbb{Q}_{>0} \) is the stability parameter. Examples of those GIT targets include complete intersections in toric Deligne–Mumford stacks, flag varieties of classical types, zero loci of sections of homogeneous bundles, and Nakajima quiver varieties.

The space of stability parameters have wall-and-chamber structure. Once the degree of quasimaps is fixed, there are only finitely many walls. There are two extreme chambers corresponding to \( \epsilon \) being sufficiently large or sufficiently close to 0. We denote them by \( \epsilon = \infty \) and \( \epsilon = 0+ \), respectively. For \( \epsilon = \infty \), the notion of \( \epsilon \)-stable quasimaps coincides with that of stable maps and therefore one obtains the Gromov–Witten theory. Roughly speaking, as \( \epsilon \) decreases, domain curves of \( \epsilon \)-stable quasimaps have fewer and fewer rational tails. Hence the moduli space \( Q^\epsilon_{g,n}(X, \beta) \) of \( \epsilon \)-stable quasimaps becomes simpler as \( \epsilon \) approaches 0. Similar to \( \overline{M}_{g,n}(X, d) \), all \( \epsilon \)-stable quasimap spaces have canonical perfect obstruction theories and hence are equipped with virtual fundamental cycles and virtual
structure sheaves. This allows us to define cohomological and $K$-theoretic quasimap invariants, which depend on the stability parameter $\epsilon$. In some cases, the ($\epsilon = 0^+$)-stable quasimap invariants are easier to compute. A wall-crossing formula is a relation between invariants from different stability chambers. It enables us to recover Gromov-Witten invariants from ($\epsilon = 0^+$)-stable quasimap invariants. This idea has been applied successfully in cohomological Gromov-Witten theory, see, for example, [33, 40, 43]. In particular, the genus-0 mirror theorem can be deduced from such wall-crossing formulas.

The cohomological quasimap wall-crossing formulas have been proved in various generalities in [6, 9, 11, 12] and the most general version is proved by the second author in [62]. The goal of this paper is to generalize the analysis in [62] to the $K$-theory setting and establish wall-crossing formulas in $K$-theoretic quasimap theory for all GIT targets in all genera. These formulas also work in the presence of twisting [27, 57] and level structure [50]. In genus zero, our formulas implies the $K$-theoretic toric mirror theorems proved in [20, 21, 29, 50] and the wall-crossing formula in [59].

1.2. Wall-crossing formulas. Let $W$ be an affine variety with a right action of a reductive group $G$. We assume that $W$ has at worst local complete intersection singularities. Let $\theta$ be a character of $G$ such that the $\theta$-stable locus $W^s(\theta)$ is smooth, nonempty, and coincides with the $\theta$-semistable locus $W^{ss}(\theta)$. In this paper, we consider the “stacky” GIT quotient $X = [W^{ss}(\theta)/G]$. Let $X$ denote the coarse moduli of $X$ and let $W/\emptyset G = \text{Spec} H^0(W, O_W)^G$ denote the affine quotient. The GIT set-up gives (see [62, §1.2] and [9, §3.1] for details) a morphism $[W/G] \to [\mathbb{C}^{N+1}/\mathbb{C}^*]$ for some $N \in \mathbb{Z}_{>0}$, inducing a closed embedding $X \to \mathbb{P}^N \times W/\emptyset G$. Hence $X$ is a smooth proper Deligne–Mumford stack over the affine quotient $W/\emptyset G$.

Let $\lambda$ denote the ground $\lambda$-algebra which contains the Novikov ring $\mathbb{Q}[\left\lbrack \mathbb{Q} \right\rbrack]$ (see Section 2.3 for the precise definition). Let $IX = \coprod I_r X$ be the cyclotomic inertia stack of $X$ and let $\tilde{IX} = \coprod \tilde{I}_r X$ be the rigidified cyclotomic inertia stack. There is a natural projection $\varpi : IX \to \tilde{IX}$, which exhibits $IX$ as the universal gerbe over $\tilde{IX}$ (cf. [1]). Let $ev_i : Q_{g,n}(X, \beta) \to IX$ be the evaluation map at the $i$-th marked point. We consider $ev_i : Q_{g,n}(X, \beta) \to IX$ and refer to it as the rigidified evaluation map at the $i$-th marked point. We denote by $K(IX)$ the Grothendieck group of topological complex vector bundles on $IX$ with rational coefficients. Let $q$ be a formal variable and let $t(q) = \sum_{j \in \mathbb{Z}} t_j q^j$ be a general Laurent polynomial in $q$ with coefficients $t_j \in K(IX) \otimes \lambda$. Let $L_i$ denote the $i$-th cotangent line bundle at the $i$-th marked point of coarse curves. When $W/\emptyset G$ is a point, the GIT quotient
\( X \) is proper. Consider the \( S_n \)-action on the quasimap moduli space \( Q_{g,n}^e(X, \beta) \) defined by permuting the \( n \) markings. Then there is a natural (virtual) \( S_n \)-module

\[
[t(L), \ldots, t(L)]^e_{g,n,\beta} := \sum_{m \geq 0} (-1)^m H^m(Q_{g,n}^e(X, \beta), \mathcal{O}_{Q_{g,n}^e(X, \beta)} \cdot \prod_{i=1}^n (\sum_j \text{ev}^*(t_j) L_j^i)),
\]

where \( \mathcal{O}_{Q_{g,n}^e(X, \beta)} \) is the virtual structure sheaf and the operation \( \cdot \) denotes the tensor product. For simplicity, we extend the definition of \( \text{ev}^* \) by linearity and denote \( \text{ev}^*(t(L_i)) := \sum_j \text{ev}^*(t_j) L_j^i \).

The permutation-equivariant \( K \)-theoretic \( e \)-stable quasimap invariant is defined as the complex dimension of its \( S_n \)-invariant submodule. Equivalently, we define

\[
\langle t(L), \ldots, t(L) \rangle_{S_n,\epsilon} := p_* \left( \mathcal{O}_{Q_{g,n}^e(X, \beta)} \cdot \prod_{i=1}^n \text{ev}^*(t(L_i)) \right),
\]

where \( p_* \) is the proper pushforward along the projection

\[
p : [Q_{g,n}^e(X, \beta)/S_n] \to \text{Spec } \mathbb{C}
\]

in \( K \)-theory. If \( W/\!\!/G \) is only quasiprojective but has a torus-action with good properties, we can define torus-equivariant \( K \)-theoretic quasimap invariants via the \( K \)-theoretic localization formula. For simplicity of notation, we assume that \( W/\!\!/0G \) is a point from now on. The theorems hold true in general and the proofs are verbatim.

In many applications, it is necessary to modify the virtual structure sheaf \( \mathcal{O}_{Q_{g,n}^e(X, \beta)} \) by tensoring it with a determinant line bundle or a \( K \)-family over \( Q_{g,n}^e(X, \beta) \). We discuss all such twistings in Section 2.4. Using the modified virtual structure sheaves, we can define twisted quasimap invariants with level structure. In this introduction, we will use the same notation for the corresponding invariants and generating series in the twisted theory.

The wall-crossing formula involves an important generating series of \( K \)-theoretic residues over \( g = 0 \), \( (\epsilon = 0+) \)-stable quasimap graph space. We denote it by \( I(Q, q) \) and call it the small \( I \)-function. We refer the reader to Definition 2.2 for the precise definition of the \( K \)-theoretic small \( I \)-function. These functions are rational functions in \( q \) modulo any power of Novikov’s variables, and explicitly computable in general (see Remark 2.3). Given a \( K \)-group valued rational function \( f(q) \), we denote by \( [f(q)]_+ \) the Laurent polynomial part in the partial fraction decomposition of \( f(q) \). For a curve class \( \beta \), we define the \( K \)-group valued Laurent polynomial

\[
\mu_\beta(q) \in K(I\mathcal{X}) \otimes \Lambda[q, q^{-1}]
\]

to be the coefficient of \( Q^3 \) in \( [(1-q)I(Q, q) - (1-q)]_+ \).

For a fixed curve class \( \beta \), the space \( Q_{g>0} \cup \{0+, \infty\} \) is divided into stability chambers by finitely many walls \( \{1/d \mid d \in Q_{>0}, d \leq \deg(\beta)\} \). Let \( \epsilon_0 = 1/d_0 \) be a wall, where \( d_0 \leq \deg(\beta) \). Let \( \epsilon_- < \epsilon_+ \) be the stability conditions in the two adjacent chambers separated by \( \epsilon_0 \). We also fix the genus \( g \) and the number of markings \( n \) such that \( 2g-2+n+\epsilon_0 \deg(\beta) > 0 \).
To compare the two virtual structure sheaves $\mathcal{O}^\text{vir}_{Q_{g,n}^+(X,\beta)}$ and $\mathcal{O}^\text{vir}_{Q_{g,n}^-(X,\beta)}$, we recall several natural morphisms between various moduli spaces. First, we have

$$
Q_{g,n}^+(X,\beta) \xrightarrow{c} Q_{g,n}^-(X,\beta)
$$

Here $d := \deg(\beta)$, the $\iota$'s are defined by composing the quasimap with (1) and forgetting the orbifold structure of the domain curves, and $c$ is defined by contracting all degree-$d_0$ rational tails to length-$d_0$ base points ([9, §3.2.2]). It is obvious that the morphisms $\iota$ and $c$ commute with the $S$-actions on the moduli spaces. Consider the $S_k$-action on $Q_{g,n+k}^-(\mathbb{P}^N, d - kd_0)$ defined by permuting the last $k$ markings. Let

$$
b_k : [Q_{g,n+k}^-(\mathbb{P}^N, d - kd_0)/S_k] \to Q_{g,n}^-(\mathbb{P}^N, d)
$$

be the map that replaces the last $k$ markings by base points of length $d_0$ ([9, §3.2.3]). For any $\epsilon$, let

$$
ev : Q_{g,n+k}^+(X, \beta) \to (IX)^n
$$

be the product of the rigidified evaluation maps at the first $n$ markings.

Let $K_c([Q_{g,n}^+(\mathbb{P}^N, d)/S_n])_Q$ denote the Grothendieck group of coherence sheaves on $[Q_{g,n}^+(\mathbb{P}^N, d)/S_n]$ with $\mathbb{Q}$ coefficients. It can be identified with the rational Grothendieck group $K^S_n(Q_{g,n}^+(\mathbb{P}^N, d))_Q$ of $S_n$-equivariant coherent sheaves on $Q_{g,n}^+(\mathbb{P}^N, d)$.

**Theorem 1.1** (Theorem 5.14). Assuming that $2g - 2 + n + \epsilon_0 \deg(\beta) > 0$, we have

$$
(\iota \times \ev)_* \mathcal{O}^\text{vir}_{Q_{g,n}^-(X,\beta)} - ((c \circ \iota) \times \ev)_* \mathcal{O}^\text{vir}_{Q_{g,n}^+(X,\beta)}
$$

$$
= \sum_{k \geq 1} \left( \sum_\beta (b_k \circ c \circ \iota) \times \ev)_* \left( \prod_{a=1}^k \ev_{\beta+a}^* \mu_{\beta_a}(L_{n+a}) : \mathcal{O}^\text{vir}_{Q_{g,n+k}^+(X,\beta')} \right) \right)
$$

in $K_0([Q_{g,n}^+(\mathbb{P}^N, d) \times (IX)^n)/S_n])_Q$, where $\beta$ runs through all ordered tuples

$$
\bar{\beta} = (\beta', \beta_1, \ldots, \beta_k)
$$

such that $\beta = \beta' + \beta_1 + \cdots + \beta_k$ and $\deg(\beta_i) = d_0$ for all $i = 1, \ldots, k$. The same formula also holds for the twisted virtual structure sheaves with level structure define in Section 2.4.

The above theorem implies a numerical wall-crossing formula. Before stating the result, let us introduce some notation so that the numerical wall-crossing terms are indexed by unordered tuples. Consider the permutation action of $S_k$ on the set of all ordered tuples $\bar{\beta} = (\beta', \beta_1, \ldots, \beta_k)$ in Theorem 1.1:

$$
\sigma(\bar{\beta}) := (\beta', \beta_{\sigma(1)}, \ldots, \beta_{\sigma(k)}), \quad \sigma \in S_k.
$$

We denote the orbit of $\bar{\beta}$ under the permutation action by

$$
\bar{\beta} = (\beta', \{\beta_1, \ldots, \beta_k\}).
$$
where \( \{\beta_1, \ldots, \beta\} \) is unordered and hence forms a multiset. Let \( S_\beta \subset S_k \) denote the stabilizer subgroup that fixes \( \beta \). We define the following \((S_n \times S_\beta)\)-permutation equivariant invariant

\[
\langle t(L), \ldots, t(L) \rangle_{S_n \times S_\beta, \epsilon^+}^{\epsilon_+} = \prod_{i=1}^n \langle t(L) \rangle_{\epsilon^+}^{\epsilon^+} \cdot \prod_{j=1}^{k} \langle t(L_{n+j}) \rangle_{\epsilon^+}^{\epsilon^+},
\]

where \( p_* \) is the proper pushforward along the projection \( p^\beta : [Q^+_{g,n+k}(X, \beta') \times S_n \times S_\beta] \rightarrow \text{Spec } \mathbb{C} \) in \( K \)-theory. Note that the above invariant only depends on the orbit \( \beta \) of \( \beta \).

All the maps involved in Theorem 1.1 induce isomorphisms of the relative cotangent spaces at the first \( n \) markings. By tensoring both sides of (2) with \( \prod_{i=1}^n t(L_i) \) and taking the proper pushforward along \( ([Q^+_{g,n}((\mathbb{P}^N, d) \times (IX)^n]) / S_n] \rightarrow \text{Spec } \mathbb{C} \), we obtain the following result:

**Theorem 1.2.** Assuming that \( 2g - 2 + n + \epsilon_0 \deg(\beta) > 0 \), we have

\[
\langle t(L), \ldots, t(L) \rangle_{S_n, \epsilon_-}^{\epsilon_-} - \langle t(L), \ldots, t(L) \rangle_{S_n, \epsilon_-}^{\epsilon_-} = \sum_{k=1}^m \sum_{\beta} \langle t(L), \ldots, t(L), \mu_{\beta_1}(L), \ldots, \mu_{\beta_k}(L) \rangle_{S_n \times S_\beta, \epsilon^+}^{\epsilon^+},
\]

where \( \beta \) runs through all

\[
\beta = (\beta', \{\beta_1, \ldots, \beta_k\}),
\]

with \( \{\beta_1, \ldots, \beta_k\} \) unordered, \( \beta = \beta' + \beta_1 + \cdots + \beta_k \) and \( \deg(\beta_i) = d_0 \) for all \( i = 1, \ldots, k \).

The same formula also holds for twisted permutation-equivariant \( \epsilon \)-quasimap invariants with level structure.

By definition, (2) is also an identity of \( S_n \)-equivariant \( K_0 \)-classes on \( Q^+_{g,n}((\mathbb{P}^N, d) \times (IX)^n) \). If we tensor (2) with \( \prod_{i=1}^n t(L_i) \) and take the proper pushforward along \( ([Q^+_{g,n}((\mathbb{P}^N, d) \times (IX)^n]) / S_n] \rightarrow \text{Spec } \mathbb{C} \), we obtain

**Theorem 1.3.** Assuming that \( 2g - 2 + n + \epsilon_0 \deg(\beta) > 0 \), we have

\[
\langle t(L), \ldots, t(L) \rangle_{S_n, \epsilon_-}^{\epsilon_-} - \langle t(L), \ldots, t(L) \rangle_{S_n, \epsilon_-}^{\epsilon_-} = \sum_{k=1}^m \sum_{\beta} \langle t(L), \ldots, t(L), \mu_{\beta_1}(L), \ldots, \mu_{\beta_k}(L) \rangle_{S_n \times S_\beta, \epsilon^+}^{\epsilon^+},
\]

as virtual \( S_n \)-modules, where \( \beta \) satisfies the same conditions as in Theorem 1.2 and

\[
\langle t(L), \ldots, t(L), \mu_{\beta_1}(L), \ldots, \mu_{\beta_k}(L) \rangle_{S_n \times S_\beta, \epsilon^+}^{\epsilon^+} = p^\beta_* \left( \langle g_{n+k, \beta'}, \epsilon^+ \rangle_{Q^+_{g,n+k}(X, \beta')} \prod_{i=1}^n \langle t(L_i), \epsilon^+ \rangle_{Q^+_{g,n+k}(X, \beta')} \prod_{j=1}^k \langle \mu_{\beta_j}(L_{n+j}), \epsilon^+ \rangle_{Q^+_{g,n+k}(X, \beta')} \right).
\]
with \( p^\beta : [Q_{g,n,k}^+ (X, \beta)/S_\beta] \to \text{Spec } \mathbb{C} \). The same formula also holds in the twisted permutation-equivariant \( \epsilon \)-stable quasimap \( K \)-theory with level structure.

Theorem 1.3 and Theorem 1.2 are in fact equivalent. On the one hand, it is clear that Theorem 1.3 implies Theorem 1.2. On the other hand, Theorem 1.2 produces an identity in an arbitrary \( \lambda \)-ring \( \Lambda \). In particular, we can choose \( \Lambda \) to contain the abstract algebra of symmetric functions \( Q[[N_1, N_2, \ldots ]] \) with the Adams operations \( \Psi^r(N_m) = N_{rm} \). As explained in Example 2 and Example 3 in [25], due to Schur–Weyl’s reciprocity, the permutation-equivariant invariant captures the entire information about the \( S_n \)-module \( [t(L), \ldots , t(L)]_{g,n,\beta}^\epsilon \). Hence Theorem 1.2 also implies Theorem 1.3.

We introduce the permutation-equivariant genus-\( g \) descendant potential of \( X \):

\[
F^\epsilon_g(t(q)) = \sum_{n=0}^{\infty} \sum_{\beta \geq 0} Q^\beta (t(L), \ldots , t(L))_{g,n,\epsilon}^{S_n, \epsilon}
\]

and define the following truncation of \((1 - q)I(Q, q) - (1 - q)\):

\[
\mu^\geq \epsilon(Q, q) = \sum_{\epsilon \leq 1/\deg(\beta) < \infty} \mu_\beta(q)Q^\beta.
\]

By repeatedly applying Theorem 1.2 to cross the walls in \([\epsilon, \infty)\), we obtain the following corollary, which holds for both the untwisted and twisted cases.

**Corollary 1.4.** For \( g \geq 1 \) and any \( \epsilon \), we have

\[
F^\epsilon_g(t(q)) = F^{\epsilon+\infty}_g(t(q) + \mu^\geq \epsilon(Q, q)).
\]

For \( g = 0 \), the same equality holds true modulo the constant and linear terms in \( t \).

In genus zero, Givental introduced an important generating series called the \( J \)-function. Its generalization to the \( K \)-theoretic \( \epsilon \)-stable quasimap theory is straightforward and we refer the reader to Definition 2.2 for the precise formula. The following theorem is proved in Section 5.5, which again holds for both the untwisted and twisted cases.

**Theorem 1.5** (Theorem 5.15). For any \( \epsilon \), we have

\[
J^\epsilon_{S_{\infty}}(t(q)) + \mu^\geq \epsilon(Q, q, Q) = J^\epsilon_{S_{\infty}}(t(q), Q).
\]

The small \( I \)-function is a specialization of the \( J \)-function in the chamber \( \epsilon = 0+ \). To be more precise, we have

\[
(1 - q)I(Q, q) = J^0_{S_{\infty}}(t(q), Q)|_{t(q) = 0}.
\]

Hence Theorem 1.5 generalizes the genus-zero toric mirror theorems in quantum \( K \)-theory [20, 21, 29] and quantum \( K \)-theory with level structure [50], and the \( K \)-theoretic wall-crossing formula in [59].
1.3. Comparison with the proof of the cohomological wall-crossing formulas in \[62\]. The main idea of this paper is the same as that of \[62\]. Namely, we deduce the wall-crossing formulas by applying the torus-localization formula to the master space constructed in \[62\]. However, \(K\)-theory is more sensitive to the stacky structure of the moduli spaces, comparing to cohomology theory. In particular, we mention some new features here. The first one is the emergence of the permutation-equivariant structure. When simplifying the localization contributions, we need to split a certain number of nodes simultaneously and the newly-created markings are unordered. It ‘forces’ us to define invariants over the quotient stack of a usual quasimap moduli space by some permutation group. In the cohomological setting, such modification will only change a invariant by multiplying a constant. However, in the \(K\)-theory setting, the effect of such modification is more complicated. In fact, this was first observed by Givental in \[18\] when analyzing the \(K\)-theoretic localization formula over the moduli space of stable maps. This phenomenon motivated him to introduce permutation-equivariant \(K\)-theoretic invariants.

To simplify the localization contributions, we need to refine a few key lemmas in \[62\] about the structures of certain morphisms. And the combinatorics used in the simplification is also more involved than that in \[62, \S7\].

1.4. Plan of the paper. This paper is organized as follows. In Section 2, we recall the basic notation in \(K\)-theory and define permutation-equivariant \(K\)-theoretic quasimap invariants and their generating functions. Twisted theories with level structure are discussed at the end of this section. In Section 3, we first recall the construction of moduli spaces of quasimaps with entangled tails in \[62\]. Then we define and study the virtual structure sheaves over these moduli spaces. In Section 4, we recall the master space introduced in \[62\]. It has a \(C^*\)-action. We compute the \(C^*\)-equivariant \(K\)-theoretic Euler classes of the virtual normal bundles of all fixed-point components in the master space. In Section 5, we first discuss some basic properties of the residue operation. Then we apply the \(K\)-theoretic localization formula to the master space and obtain the wall-crossing formula. Note that the combinatorics in simplifying the localization contributions is much more sophisticated than that in the cohomological setting \[62\].

1.5. Acknowledgments. The first author would like to thank David Anderson, Dan Edidin, Daniel Halpern-Leistner, Amalendu Krishna, Jeongseok Oh and Bhamidi Sreedhar for helpful discussions. The second author would like to thank Huai-liang Chang, Michail Savvas, Arnav Tripathy and Kai Xu for helpful discussions.

This project started when the second author was invited by Yongbin Ruan to visit the University of Michigan. Both authors would like to thank Yongbin Ruan for his encouragement and support.

2. \(K\)-theoretic quasimap invariants

In this section, we first introduce some basic notation in \(K\)-theory. Then we review the basics of (orbifold) quasimap theory and define permutation-equivariant \(K\)-theoretic invariants. Twisted theories will be in the end.
2.1. Basic notation in $K$-theory. For a Deligne–Mumford stack $X$, we denote by $K_o(X)$ the Grothendieck group of coherent sheaves on $X$ and by $K^0(X)$ the Grothendieck group of locally free sheaves on $X$. Suppose $X$ has a $C^*$-action. Let $K^C^*_o(X)$ and $K^C^*_o(X)$ denote the equivariant $K$-groups. We have canonical isomorphisms

$$K^C^*_o(X) \cong K_o([X/C^*]), \quad K^C^*_o(X) \cong K^0([X/C^*]).$$

Let $K(X)$ denote the Grothendieck group of topological complex vector bundles on $X$. Given a coherent sheaf (or vector bundle) $F$, we denote by $[F]$ or simply $F$ its associated $K$-theory class. Tensor product makes $K_o(X)$ a $K^0(X)$-module:

$$K^0(X) \otimes K_o(X) \to K_o(X), \quad ([E], [F]) \mapsto [E] \cdot [F] := [E \otimes_{O_X} F].$$

The unit in $K^0(X)$ is the class of the structure sheaf and we simply denote it by $1$. Throughout the paper, we consider Grothendieck groups with rational coefficients $K_o(X)_\mathbb{Q} := K_o(X) \otimes \mathbb{Q}$, $K^0(X)_\mathbb{Q} := K^0(X) \otimes \mathbb{Q}$ and $K(X)_\mathbb{Q} = K(X) \otimes \mathbb{Q}$.

For a flat morphism $f : X \to Y$, we have the flat pullback $f^* : K_o(Y) \to K_o(X)$. For a proper morphism $g : X \to Y$, we have the proper pushforward $f_* : K_o(X) \to K_o(Y)$ defined by

$$[F] \mapsto \sum_n (-1)^n [R^n f_* F].$$

In the case when $X$ is proper and $Y$ is a point, the proper pushforward can be identified with the holomorphic Euler characteristic

$$\chi(F) := \sum_{i\geq 0} \dim \mathbb{C} H^i(X, F).$$

A key tool in computing holomorphic Euler characteristics and proper pushforwards is the localization formula in equivariant $K$-theory. Such localization formula was first introduced by Atiyah-Segal [5] and developed by Thomason [53, 54] in the algebraic setting. A detailed discussion of the classical $K$-theoretic torus-localization formula can be found in [7, Chapter 5]. Virtual localization formulas in $C^*$-equivariant $K$-theory for stacks are proved in [34, 39]. We will use the version in [39] and recall the precise statement in Section 5.

We will use $q$ to denote the equivariant parameter (or weight), which corresponds to the standard representation $C_{std}$ of $C^*$. We have $K^C^*_o(pt) \cong \mathbb{Z}[q, q^{-1}]$.

Let $E$ be a vector bundle on $X$. We define the $K$-theoretic Euler class of $E$ by

$$\lambda_{-1}(E^\vee) := \sum_i (-1)^i \wedge^i E^\vee \in K^0(X),$$

where $\wedge^i E^\vee$ denote the $i$-th exterior power of $E^\vee$. Suppose $X$ has a $C^*$-action. For a $C^*$-equivariant vector bundle $E$, the same formula defines its $C^*$-equivariant $K$-theoretic Euler class $\lambda^C^*_{-1}(E^\vee) \in K^C^*_o(X)$. 
2.2. Stable quasimaps to GIT quotients. Let $W$ be an affine variety with a right action of a reductive group $G$. We assume that $W$ has at worst local complete intersection singularities. Let $\theta$ be a character of $G$. Denote by $W^s(\theta)$ the $\theta$-stable locus, and $W^{ss}(\theta)$ the $\theta$-semistable locus with respect to the linearization $L_\theta := W \times \mathbb{C}_\theta$. We assume that $W^s(\theta)$ is smooth, nonempty, and coincides with $W^{ss}(\theta)$. In this paper, we consider the “stacky” GIT quotient

$$X = [W^{ss}(\theta)/G].$$

Note that $X$ is a smooth proper Deligne–Mumford stack over the affine quotient $W/\!/_G = \text{Spec} H^0(W, \mathcal{O}_W)^G$.

Let $(C, x_1, \ldots, x_n)$ be an $n$-pointed, genus $g$ twisted curve with balanced nodes and trivialized gerbe markings (c.f. [1, §4]). A map $[u] : C \to X$ corresponds to a pair $(P, u)$ with

$$P \to C$$

a principal $G$-bundle on $C$ and

$$u : C \to P \times_G W$$

a section of the fiber bundle $P \times_G W \to C$. We call $[u]$ a quasimap to $X$ if $[u]$ is representable and $[u]^{-1}([W^{us}/G])$ is zero-dimensional. Here $W^{us}$ denotes the unstable locus. The locus $[u]^{-1}([W^{us}/G])$ is called the base locus of $[u]$ and points in the base locus are called base points.

The class $\beta$ of a quasimap is defined to be the group homomorphism

$$\beta : \text{Pic}([W/G]) \to \mathbb{Q}, \quad L \mapsto \deg([u]^*(L)).$$

We refer to the rational number $\deg(\beta) = \deg([u]^*(L_\theta))$ as the degree of the quasimap $[u]$. A group homomorphism $\beta : \text{Pic}([W/G]) \to \mathbb{Q}$ is called an effective curve class if it is the class of some quasimap $[u]$. We denote by $\text{Eff}(W, G, \theta)$ the semigroup of $L_\theta$-effective curve classes on $X$. For convenience, we write $\beta \geq 0$ if $\beta \in \text{Eff}(W, G, \theta)$ and $\beta > 0$ if the effective curve class is nonzero.

Fix a positive rational number $\epsilon$. A quasimap is called $\epsilon$-stable if the following three conditions hold:

1. The base points are disjoint from the trivialized gerbe markings and nodes of $(C, x_1, \ldots, x_n)$.
2. For every $y \in C$, we have $l(y) \leq 1/\epsilon$, where $l(y)$ is the length at $y$ of the subscheme $[u]^{-1}([W^{us}(\theta)/G])$.
3. The $\mathbb{Q}$-line bundle $(u^*L_\theta)^{\otimes \epsilon} \otimes \omega_{C,\log}$ is positive, where $\omega_{C,\log} := \omega_C(\sum_i x_i)$ is the log dualizing sheaf.

A quasimap is called $(0+)$-stable if it is $\epsilon$-stable for every sufficiently small positive rational number $\epsilon$, and $\infty$-stable if it is $\epsilon$-stable for every sufficiently large $\epsilon$.

It is straightforward to define families of quasimaps over any base scheme $S$. We denote by $Q^\epsilon_{g,n}(X, \beta)$ the moduli of genus-$g$ $\epsilon$-stable quasimaps to $X$ of curve class $\beta$ with $n$ markings. This is slightly different from the convention in [6], where the gerbe markings
are not trivialized. Let \( Q_{g,n}^e(X, \beta) \) be the moduli space defined in \([6, \S 2.3]\). Let \( \mathcal{G}_i \) be the \( i \)-th marking in the universal curve, which is a gerbe over \( Q_{g,n}(X, \beta) \). Then we have
\[
Q_{g,n}^e(X, \beta) = \mathcal{G}_1 \times Q_{g,n}^e(X, \beta) \times \cdots \times Q_{g,n}^e(X, \beta).
\]

Hence by \([6, \text{Theorem 2.7}]\), \( Q_{g,n}^e(X, \beta) \) is a Deligne–Mumford stack, proper over the affine quotient \( \overline{W/0G} \), with a perfect obstruction theory.

Let \( IX = \coprod_r I_r X \) be the cyclotomic inertia stack of \( X \) and let \( \bar{IX} = \coprod_r \bar{I}_r X \) be the rigidified cyclotomic inertia stack (c.f. \([1, \S 3.1]\)). There is a natural projection \( \varpi : IX \to \bar{IX} \), which exhibits \( IX \) as the universal gerbe over \( \bar{IX} \). For \( 1 \leq i \leq n \), there is an evaluation map \( \text{ev}_i : Q_{g,n}^e(X, \beta) \to IX \).

For later applications, we define the \textit{rigidified evaluation map}
\[
\text{ev}_i = \varpi \circ \text{ev}_i : Q_{g,n}^e(X, \beta) \to \bar{IX}.
\]

According to the general constructions in \([42, 48]\), the perfect obstruction theory induces a virtual structure sheaf
\[
\mathcal{O}_{Q_{g,n}^e(X, \beta)}^{\text{vir}} \in K_0(Q_{g,n}^e(X, \beta)).
\]
See Section 3.4 for the details of the construction of the virtual structure sheaf. The moduli stack of (0+)-stable (resp. \( \infty \)-stable) quasimaps is denoted by \( Q_{g,n}^{0+}(X, \beta) \) (resp. \( Q_{g,n}^{\infty}(X, \beta) \)).

To define the \( J \)-function and the \( I \)-function for the genus-0 theory, we will also need quasimap \textit{graph spaces}. Here we recall a special case of the definition in \([6]\). Given an effective curve class \( \beta \), choose \( \epsilon \in \mathbb{Q}_{>0} \) and \( A \in \mathbb{Z}_{>0} \) such that \( 1/A < \epsilon < 1/\deg(\beta) \). We view \( \mathbb{P}^1 \) as the GIT quotient \( \mathbb{C}^2/\mathbb{C}^* \) with the polarization \( \mathcal{O}_{\mathbb{P}^1}(A) \). Then the (0+)-stable quasimap graph space is defined by
\[
Q_{G_{0,1}^0}(X, \beta) := Q_{G_{0,1}}^0(X \times \mathbb{P}^1, \beta \times [\mathbb{P}^1]).
\]
The definition is independent of the choice of \( A \) and \( \epsilon \). This moduli space parametrizes quasimaps to \( X \times \mathbb{P}^1 \) with a unique rational component whose coarse moduli is mapped isomorphically onto \( \mathbb{P}^1 \).

We denote the first marking by \( x_* \). Consider the \( \mathbb{C}^* \)-action on \( \mathbb{P}^1 \) given by
\[
t[\zeta_0, \zeta_1] = [t\zeta_0, \zeta_1], \quad t \in \mathbb{C}^*.
\]
Set \(0 := [1 : 0]\) and \(\infty := [0 : 1]\). Then the tangent space of \(\mathbb{P}^1\) at \(\infty\) (resp. 0) is isomorphic to the standard representation (resp. the dual of the standard representation) of \(\mathbb{C}^*\). The \(\mathbb{C}^*\)-action (5) induces an action on \(QG_{0,1}^0(X, \beta)\). Let \(F_{*,0}^{0,\beta}\) be the distinguished fixed-point component consisting of \(\mathbb{C}^*\)-fixed quasimaps such that only the marking \(x_\star\) is over \(\infty\), while the other \(k\) markings and the entire class \(\beta\) are over 0 in \(\mathbb{P}^1\).

The restriction of the absolute perfect obstruction theory of \(QG_{0,1}^0(X, \beta)\) to \(F_{*,0}^{0,\beta}\) decomposes into moving and fixed parts. By [32], the fixed part of the obstruction theory defines a perfect obstruction theory on \(F_{*,0}^{0,\beta}\) and hence induces a virtual structure sheaf \(O_{vir}^{0,\beta} \in K_{\mathbb{C}^*}(F_{*,0}^{0,\beta})\). The moving part of the obstruction theory gives rise to a virtual normal bundle \(N_{vir}^{0,\beta}/QG_{0,1}^1(X, \beta) \in K_{\mathbb{C}^*}(F_{*,0}^{0,\beta})\).

### 2.3. K-theoretic quasimap invariants

We recall Givental–Tonita’s \(K\)-theoretic symplectic loop space formalism in the orbifold setting [23, 58]. To introduce permutation-equivariant \(K\)-theoretic invariants, we choose the ground coefficient ring to be a \(\lambda\)-algebra \(\Lambda\), i.e. an algebra over \(\mathbb{Q}\) equipped with abstract Adams operations \(\Psi^k, k = 1, 2, \ldots\). Here \(\Psi^k : \Lambda \to \Lambda\) are ring homomorphisms satisfying \(\Psi^r \Psi^s = \Psi^{rs}\) and \(\Psi^1 = \text{id}\). In this paper, we assume that \(\Lambda\) is over \(\mathbb{C}\) and includes the Novikov variables \(Q^\beta, \beta \in \text{Eff}(W, G, \theta)\) and the torus-equivariant \(K\)-ring of a point if we consider torus-actions on the target. We also assume that \(\Lambda\) is equipped with a maximal ideal \(\Lambda^+\) such that \(\Psi^i(\Lambda^+) \subset (\Lambda^+)^2\) for \(i > 1\). For example, one can choose

\[
\Lambda = \mathbb{Q}[[N_1, N_2, \ldots]][[Q]][[\lambda_1^\pm, \ldots, \lambda_N^\pm]],
\]

where \(N_i\) are the Newton polynomials (in infinitely or finitely many variables), \(Q\) denotes the Novikov variable(s), and \(\lambda_i\) denote the torus-equivariant parameters. The Adams operations \(\Psi^r\) act on \(N_m\) and \(Q\) by \(\Psi^r(N_m) = N_{rm}\) and \(\Psi^r(Q^\beta) = Q^{r^\beta}\), respectively, and they act trivially on the torus-equivariant parameters. One can take \(\Lambda^+\) to be the maximal ideal generated by \(N_i, \lambda_i\) and Novikov variables of positive degrees.

As mentioned in the introduction, for simplicity, we assume the affine quotient \(W/0\) is a point and hence \(X\) is proper. Let \(K(\bar{I}X)\) be the Grothendieck group of topological complex vector bundles on \(\bar{I}X\) with rational coefficients (see e.g. [2]). The *Mukai pairing* on \(K(\bar{I}X)\) is defined by

\[
(\alpha, \beta) = \chi(\bar{I}X, \alpha \cdot \iota^* \beta),
\]

where \(\iota\) is the involution on \(\bar{I}X\) reversing the banding.

We define the \(K\)-theoretic loop space by

\[
\mathcal{K} := [K(\bar{I}X) \otimes \mathbb{C}(q)] \hat{\otimes} \Lambda,
\]

where \(\mathbb{C}(q)\) is the field of complex rational functions in \(q\) and “\(\hat{\otimes}\)” means the completion in the \(\Lambda^+_\text{-adic}\) topology. In other words, *modulo any power of \(\Lambda^+_\), the elements of \(\mathcal{K}\) are rational functions of \(q\) with vector coefficients from \(K(\bar{I}X) \otimes \Lambda\). From now on, we simplify refer to them as rational functions.
By viewing elements in $\mathbb{C}(q) \otimes \Lambda$ as coefficients, we extend the Mukai pairing to $K$ via linearity. There is a natural $\hat{\Lambda}$-valued symplectic form $\Omega$ on $K$ defined by

$$\Omega(f, g) := [\text{Res}_{q=0} + \text{Res}_{q=\infty}] (f(q), g(q^{-1})) \frac{dq}{q},$$

where $f, q \in K$.

With respect to $\Omega$, there is a Lagrangian polarization $K = K_+ \oplus K_-$, where

$$K_+ = (K(\bar{I}X) \otimes \mathbb{C}[q, q^{-1}]) \otimes \Lambda$$

and

$$K_- = \{ f \in K | f(0) \neq \infty, f(\infty) = 0 \}.$$

In other words, $K_+$ is the space of $K(\bar{I}X) \otimes \Lambda$-valued Laurent polynomials in $q$ (in the $\Lambda_+$-adic sense) and $K_-$ consists of proper rational functions of $q$ regular at 0. For $f(q) \in K$, we write $f(q) = [f(q)]_+ + [f(q)]_-,$ where $[f(q)]_+ \in K_+$ and $[f(q)]_- \in K_-.$

In a series of work [18–28], Givental introduced and studied the permutation-equivariant quantum $K$-theory, which takes into account the $S_n$-action on the moduli spaces of stable maps by permuting the markings. We recall its natural generalization to the quasimap setting [59].

Let $\text{ev}^i : Q^{\epsilon}_{g,n}(X, \beta) \to \bar{I}X$ be the rigidified evaluation map (4) at the $i$-th marked point. We denote by $L_i$ the $i$-th cotangent line bundle at the $i$-th marked point of coarse curves. Consider the natural $S_n$-action on the quasimap moduli space $Q^{\epsilon}_{g,n}(X, \beta)$ by permuting the $n$ markings. Then for an arbitrary Laurent polynomial $t(q) = \sum_{j \in \mathbb{Z}} t_j q^j \in K_+$, there is a natural (virtual) $S_n$-module

$$[t(L), \ldots, t(L)]^{\epsilon}_{g,n,\beta} := \sum_{m \geq 0} (-1)^m H^m (Q^{\epsilon}_{g,n}(X, \beta), \mathcal{O}^{\text{vir}}_{Q^{\epsilon}_{g,n}(X, \beta)} \cdot n \prod_{i=1}^n \text{ev}^*_i (t(L_i))),$$

where $\text{ev}^*_i (t(L_i)) := \sum_j \text{ev}^*_j (t_j) L_i^j$. The permutation-equivariant invariant is defined as the dimension of its $S_n$-invariant submodule. Equivalently, we have the following

**Definition 2.1.** The correlator of the permutation-equivariant $K$-theoretic $\epsilon$-stable quasimap invariants is defined by

$$\langle t(L), \ldots, t(L) \rangle^{\epsilon}_{g,n,\beta} := p_* (\mathcal{O}^{\text{vir}}_{Q^{\epsilon}_{g,n}(X, \beta)} \cdot n \prod_{i=1}^n \text{ev}^*_i (t(L_i))),$$

where $p_*$ is the proper pushforward along the projection

$$p : [Q^{\epsilon}_{g,n}(X, \beta)/S_n] \to \text{Spec} \mathbb{C}.$$

By definition, $K$-theoretic invariants are integers (if $t(q)$ has integral coefficients). Definition 2.1 can be easily generalized to the case when there are different insertions. Suppose that we are given several Laurent polynomials $t^{(a)} = \sum_m t^{(a)}_m q^m$ with $a = 1, \ldots, s$. Let
(k_1, \ldots, k_s) be a partition of n. Then we generalize Definition 2.1 and define permutation-equivariant \( \epsilon \)-stable quasimap \( K \)-invariants with symmetry group \( S_{k_1} \times \cdots \times S_{k_s} \) by

\[
\langle t^{(1)}, \ldots, t^{(s)} \rangle_{g,n,\beta}^{S_{k_1} \times \cdots \times S_{k_s}, \epsilon} := \pi^*(O_{Y}^\vir)^{\langle \epsilon \rangle_{g,n,\beta}} \cdot \prod_{i=1}^{k_i} \langle \sum_{m} \ev^*_i(t^{(a)}_m)L^m_i \rangle_{g,n,\beta},
\]

where \( \pi^* \) is the proper pushforward along the projection

\[
\pi : [Q^\epsilon_{g,n}(X,\beta)/S_{k_1} \times \cdots \times S_{k_s}] \to \Spec \mathbb{C}.
\]

According to [25, Example 5], we have the permutation-equivariant binomial formula:

\[
\langle t + t', \ldots, t + t' \rangle_{g,n,\beta}^{S_{k_1} \times \cdots \times S_{k_s}, \epsilon} = \sum_{i+j=n} \langle t, \ldots, t, t', \ldots, t' \rangle_{g,n,\beta}^{S_{k_1} \times \cdots \times S_{k_s}, \epsilon}, \quad t, t' \in \mathcal{K}_+.
\]

It follows from the following equality of \( S_n \)-modules:

\[
\left[ t + t', \ldots, t + t' \right]_{g,n,\beta}^{S_{k_1} \times \cdots \times S_{k_s}, \epsilon} = \sum_{i+j=n} \text{Ind}_{S_i \times S_j}^{S_n} \left[ t, \ldots, t, t', \ldots, t' \right]_{g,n,\beta}^{S_{k_1} \times \cdots \times S_{k_s}, \epsilon}.
\]

Here \( \text{Ind}^G_H \) denotes the operation of inducing a \( G \)-module from an \( H \)-module. For any \( H \)-module \( V \), we have \((\text{Ind}^G_H V)^G = V^H \). By using the above binomial formula and induction on \( k_i \), one can prove the following permutation-equivariant multinomial formula:

\[
\langle \sum_{i=1}^{m} t_i, \ldots, \sum_{i=1}^{m} t_i \rangle_{g,n,\beta}^{S_{k_1} \times \cdots \times S_{k_m}, \epsilon} = \sum_{k_1 + k_2 + \cdots + k_m = n} \langle t_1, \ldots, t_1, t_2, \ldots, t_2, \ldots, t_m, \ldots, t_m \rangle_{g,n,\beta}^{S_{k_1} \times \cdots \times S_{k_m}, \epsilon},
\]

where \( t_1, \ldots, t_m \in \mathcal{K}_+ \). In general, suppose we have an \( S_n \)-equivariant proper morphism \( \tilde{\pi} : Q^\epsilon_{g,n}(X,\beta) \to Y \), where \( Y \) is a Deligne-Mumford stack on which \( S_n \) acts trivially. The permutation-equivariant multinomial formula still holds if we replace \( \langle \cdot \rangle_{g,n,\beta}^{S_{k_1} \times \cdots \times S_{k_m}, \epsilon} \) by the proper pushforward along the induced morphism

\[
\pi : [Q^\epsilon_{g,n}(X,\beta)/S_{k_1} \times \cdots \times S_{k_s}] \to Y.
\]

Define the genus-\( g \) descendant potential of \( X \):

\[
F^\epsilon_g(t(q)) = \sum_{n=0}^{\infty} \sum_{\beta \geq 0} Q^\beta \langle t(L), \ldots, t(L) \rangle_{g,n,\beta}^{S_{k_1} \times \cdots \times S_{k_s}, \epsilon}
\]

We now introduce two genus-0 generating series, the \( J \)-function and the \( I \)-function, using \( K \)-theoretic residues on the graph space \( QG^0_{g,1}(X,\beta) \). Let

\[
\tilde{\ev}_*: \times_QG^0_{g,1}(X,\beta) \to \tilde{I}X
\]

be the rigidified evaluation map at the unique marking \( x_* \) composed with the involution on \( \tilde{I}X \). Let \( r \) be the locally constant function on \( \tilde{I}X \) that takes value \( r \) on the component \( I_{r}X \) and set \( r_* = (\tilde{\ev}_*)^*(r) \).
Definition 2.2.  
(1) Define the permutation-equivariant \( K \)-theoretic big \( J \)-function by

\[
J_{S_{\infty}}^e(t(q), Q) := 1 - q + t(q) + (1 - q^{-1})(1 - q) \sum_{0 < \beta \leq 1/\epsilon} Q^\beta (\tilde{e}_\epsilon)_* \left( \frac{\mathcal{O}^\text{vir}_{F_{0,0}^e/\mathcal{O}^\text{vir}_{F_{0,0}^e}}}{\lambda^\text{vir}_{X,0}^e(N^\text{vir}_{F_{0,0}^e/\mathcal{O}^\text{vir}_{F_{0,0}^e}}(X, \beta))} \right)
\]

\[
+ \sum_{(k \geq 1, \beta \geq 0), (k, \beta) \neq (1, 0) \text{ or } k = 0, \deg(\beta) > 1/\epsilon} Q^\beta (\tilde{e}_1)_* \left( \frac{\mathcal{O}^\text{vir}_{Q_{0,1+k}^e(X, \beta)}}{1 - qL_1} \prod_{i=1}^k \text{ev}_i^*(t(L_i)) \right).
\]

where \( \tilde{L}_* \) is the line bundle formed by the relative orbifold cotangent space at \( x_* \).

(2) Define the \( K \)-theoretic (small) \( I \)-function by

\[
I(Q, q) := J_{S_{\infty}}^{e=0^+}(0, Q)/(1 - q) = 1 + (1 - q^{-1}) \cdot \sum_{\beta > 0} Q^\beta (\tilde{e}_\epsilon)_* \left( \frac{\mathcal{O}^\text{vir}_{F_{0,0}^e/\mathcal{O}^\text{vir}_{F_{0,0}^e}}}{\lambda^\text{vir}_{X,0}^e(N^\text{vir}_{F_{0,0}^e/\mathcal{O}^\text{vir}_{F_{0,0}^e}}(X, \beta))} \right).
\]

When \( \epsilon = \infty \), the definition of \( J \)-function coincides with that in quantum \( K \)-theory of orbifolds introduced in [58].

In general, by the \( K \)-theoretic localization theorem (Theorem 5.1), the action of the \( K \)-theoretic Euler class \( \lambda^\text{vir}_{X,0}^e(N^\text{vir}_{F_{0,0}^e/\mathcal{O}^\text{vir}_{F_{0,0}^e}}(X, \beta)) \) on \( K^e_0(F_{0,0}^e) \otimes \mathbb{Q}[q, q^{-1}] \mathbb{Q}(q) \) is invertible. Similarly, the action of \( 1 - qL_1 \) on \( K^e_0(Q_{0,1+k}^e) \otimes \mathbb{Q}[q, q^{-1}] \mathbb{Q}(q) \) is invertible. Indeed, this follows by applying [39, Proposition 5.13] to the embedding \( Q_{0,1+k}^e \hookrightarrow L_1 \) as the zero section. Hence by definition, the \( J \)-function and \( I \)-function are elements of the loop space \( \mathcal{K} \). We define

\[
\mu_{\beta}(q) \in K(\mathcal{I}X) \otimes \mathbb{C}[q, q^{-1}]
\]

to be the coefficient of \( Q^\beta \) in \([1 - q]I(Q, q) - (1 - q)]_+ \). For \( \epsilon \in \mathbb{Q}_{\geq 0} \cup \{0, \infty\} \), we define

\[
\mu^{\leq \epsilon}(Q, q) = \sum_{0 < \deg(\beta) \leq 1/\epsilon} \mu_{\beta}(q)Q^\beta.
\]

If we write

\[
I(Q, q) = 1 + \sum_{\beta > 0} I_{\beta}(q)Q^\beta,
\]

then \( \mu_{\beta}(q) = [(1 - q)I_{\beta}(q)]_+ \).

Remark 2.3. The \( K \)-theoretic small \( I \)-functions are of \( q \)-hypergeometric-type and explicitly computable in general. For example, the explicit formulas of \( I \)-functions for toric bundles and toric complete intersections can be found in [21]. See also [31, Example 5.3]. It is shown in [50] that \( K \)-theoretic \( I \)-functions with level structure recovers almost all Ramanujan’s mock theta functions. The formulas of \( I \)-functions for type A flag manifolds can be found in [52] and the \( I \)-function of the Grassmannian with nontrivial level structure is studied in [15] and [30]. For general nonabelian GIT quotients, one can use the \( K \)-theoretic abelian/nonabelian correspondence [61] to compute their \( K \)-theoretic \( I \)-functions.
As shown recently in [49], I-functions with non-trivial level structures play an important role in understanding 3d $\mathcal{N} = 2$ mirror symmetry.

2.4. Twisted theory and level structure. One can consider two types of twistings in $K$-theoretic quasimap theory. The first type was introduced in [27, 57] and includes Eulerian twistings as special cases. It can be used to compute $K$-theoretic invariants of total spaces of vector bundles or zero loci of sections of vector bundles. The second type was introduced in [50], which is related to Verlinde algebras. We will give a common generalization here.

For each non-zero integer $m$, fix an element $E(m)_{g,n,\beta} \in K^0_G(W)$. Let $\pi : C \to Q^e_{g,n}(X, \beta)$ be the universal curve, with universal principal $G$-bundle $P$ on it. Let $u : C \to \mathcal{P} \times_G W$ be the universal section. We define the following $K$-theory class

$$E^{(m)}_{g,n,\beta} := R^* \pi_* u^* (P \times_G E^{(m)})$$

on $Q^e_{g,n}(X, \beta)$. To avoid potential divergence issues of twisted virtual structure sheaves and invariants, we consider fiberwise scalar actions by $\mathbb{C}^*$ on the classes $E^{(m)}$. We choose two copies of $\mathbb{C}^*$ whose equivariant parameters are denoted by $t_+$ and $t_-$. Fix an integer $a$. Let $C_{t_+ a}$ denote the $a$-th tensor power of the standard representation of $\mathbb{C}^*$. For $m > 0$, we apply the construction (7) to the $G \times \mathbb{C}^*$-equivariant bundle $E^{(m)} \otimes C_{t_+ a}$ and obtain

$$E^{(m)}_{g,n,\beta}(t_+) \in K^0(Q^e_{g,n}(X, \beta)) \otimes \mathbb{Q}[t_+, t_-^1].$$

Similarly, for $m < 0$, we construct a class

$$E^{(m)}_{g,n,\beta}(t_-) \in K^0(Q^e_{g,n}(X, \beta)) \otimes \mathbb{Q}[t_-, t_-^1]$$

using $E^{(m)} \otimes C_{t_- b}$, for some fixed integer $b$.

To introduce the determinantal twisting or level structure (c.f. [50]), we fix another $G$-equivariant bundle $R$ on $W$ and an integer $l$. For example, we usually choose $R$ to be of the form $W \times \tilde{R}$, where $\tilde{R}$ is a finite-dimensional $G$-module. We define the level-$l$ determinant line bundle over $Q^e_{g,k}(X, \beta)$ by

$$D^{R,l} := (\det R \pi_*(R_{g,n,\beta}))^{-l}.$$

For $l \neq 0$, the level structure corresponds to a specific choice of non-trivial Chern–Simons level in 3d $\mathcal{N} = 2$ supersymmetric gauge theory [37, 38, 60]. It would be interesting to find the geometric construction of twists corresponding to generic Chern-Simons levels (c.f. [60, §4.1]).

The $K$-group $K^0(Q^e_{g,n}(X, \beta))$ has a natural $\lambda$-ring structure and we denote its Adams operations by $\Psi^m$, $m > 0$. For the sake of applications, we extend the Adams operations

---

1In general, the theory works for any rational number $l$ such that the determinant line bundle $D^{R,l}$ exists.
for negative values of $m$ by $\Psi^m(V) := \Psi^{-m}(V^\vee)$ for any $K$-theory class $V$. We also set $\Psi^m(t_\pm) = t_\pm^m$. By combining (7) and (8), we define the twisting class

$$T_{g,n,\beta}^{E, R,l} := \exp\left(\sum_{m<0} \Psi^m(E_{g,n,\beta}^{(m)}(t_-)) + \sum_{m>0} \Psi^m(E_{g,n,\beta}^{(m)}(t_+))\right) \cdot D^{R,l}$$

and the twisted virtual structure sheaf of level $l$

$$\mathcal{O}_{Q_{g,n}(X, \beta)}^{vir, E, R,l} := \mathcal{O}_{Q_{g,n}(X, \beta)}^{vir} \cdot T_{g,n,\beta}^{E, R,l}.$$ 

The twisting class $T_{g,n,\beta}^{E, R,l}$ is an element in $K^0(Q_{g,n}(X, \beta)) \otimes \mathbb{Q}[t_+, t_-^{-1}]$ where the completion depends on the signs of $a$ and $b$. For example, if we choose $a = b = 1$, then the twisting class lies in $K^0(Q_{g,n}(X, \beta)) \otimes \mathbb{Q}[t]\{t_+, t_-\}$, the ring of formal power series in $t_+$ and $t_-$ with coefficients in $K^0(Q_{g,n}(X, \beta)) \otimes \mathbb{Q}$.

The twisted invariants of level $l$ are defined by the same formula as in Definition 2.1, i.e., we define

$$(\mathbf{t}(L), \ldots, \mathbf{t}(L))_{g,n,\beta}^{S_n, E, R,l, \epsilon} := p_*(\mathcal{O}_{Q_{g,n}(X, \beta)}^{vir, E, R,l} \cdot \prod_{i=1}^n \text{ev}_i^{\epsilon}(\mathbf{t}(L_i))),$$

where $p$ denotes the unique morphism from $[Q_{g,n}(X, \beta)/S_n]$ to Spec $\mathbb{C}$. The pairing on the $K$-group is modified accordingly. To be more precise, let $E^{(m)}$ and $R$ denote the induced $K$-theory classes on $IX$. We define the twisting class

$$T := \exp\left(\sum_{m\neq 0} \Psi^m(E^{(m)})/m\right) \cdot (\det R)^{-l} \in K(IX).$$

Recall that $\pi : IX \to \tilde{IX}$ is the natural projection. We define the twisting class $T := \pi_*(T)$ in $K(\tilde{IX})$. The twisted $K$-theoretic pairing of level $l$ is defined by

$$(\alpha, \beta)^{R,l}_{E, \epsilon} := \chi\left(IX, \alpha \cdot t^\epsilon \beta \cdot T\right), \quad \alpha, \beta \in K(\tilde{IX})$$

We define the genus-$g$ twisted potential of level $l$ by (6) in which all invariants are replaced by their twisted counterparts. The definition of the twisted $J$-function of level $l$ is also similar to that in the untwisted theory. Note that the twisting class (9) can be defined similarly over genus-0 graph spaces. By abuse of notation, we still denote it by $T_{0,n,\beta}^{E, R,l}$. Using the notation introduced in Definition 2.2, we define the twisted small $I$-function of level $l$ by

$$I_{E, R,l}(Q, q) := 1 + (1 - q^{-1}) \cdot T^{-1} \cdot \sum_{\beta > 0} Q^\beta (\text{ev}_\epsilon)^\ast \left(\frac{\mathcal{O}_{Q_{g,n}(X, \beta)}^{vir, E, R,l}}{\chi_1(F_{g,n,0}^\epsilon/\mathbb{C}^* \cdot X_{g,n,0}^\epsilon, \gamma_{\epsilon,0})}\right).$$

Note that in the above formula, the restriction of the twisting class $T_{0,1+k,\beta}^{E, R,l} |_{F_{g,n,0}^\epsilon}$ can have non-trivial $\mathbb{C}^*$-weights. Let $I_{E, R,l}(q)$ be the coefficient of $Q^\beta$ in $I_{E, R,l}(Q, q)$. The
twisted $K$-theoretic big $J$-function of level $l$ is defined by

$$J_{\mathcal{S}_K}^* E^{(\bullet), R,l} (t(q), Q) = 1 - q + t(q) + (1 - q) \sum_{0 < \deg(\beta) \leq 1/\epsilon} t_\beta E^{(\bullet), R,l} (q) Q^\beta$$

$$+ T^{-1} \sum_{(k \geq 1, \beta \geq 0), (k, \beta) \neq (1,0) \text{ or } k=0, \deg(\beta) > 1/\epsilon} Q^\beta (e^\chi_1)_* \left( \frac{O^{\text{vir}}_{\mathcal{S}_K} E^{(\bullet), R,l}(X, \beta)}{1 - qL_1} \prod_{i=1}^k \text{ev}_i^* (t(L_i)) \right).$$

By definition, we have $J_{\mathcal{S}_K}^* E^{(\bullet), R,l} (Q, q) = J_{\mathcal{S}_K}^* E^{(\bullet), R,l}(0, Q)/(1 - q)$.

An important property of the twisting $K$-theory class is that it factorizes “nicely” over nodal strata. To be more precise, let $(C, P, u)$ be a quasimap node $p$ into two connected components $C'$ and $C''$. Consider the normalization exact sequence on $C$:

$$0 \to \mathcal{O}_C \to \mathcal{O}_{C'} \oplus \mathcal{O}_{C''} \to \mathcal{O}_p \to 0$$

coming from splitting $C$ at the node $p$. This gives rise to the following identity in $K$-theory:

$$(12) \quad H^*(C, P \times_G E) \oplus \mathcal{E}_u(p) = H^*(C', P|_{C'} \times_G E) \oplus H^*(C'', P|_{C''} \times_G E)$$

for any $E \in K^*_G(W)$. Since the two operations $\exp(\Psi^m(\cdot)/m)$ and $\det(\cdot)$ are additive-multiplicative, the twisting $K$-theory class in (10) factors. We refer the reader to [50, Proposition 2.9] for more details in the case of level structure.

3. $K$-THEORETIC QUASIMAP INVARIANTS WITH ENTANGLED TAILS

To construct the master space, the second author introduced weighted twisted curves and quasimaps with entangled tails in [62]. We review their definitions in this section and study the properties of (twisted) virtual structure sheaves on their moduli spaces.

3.1. The entanglement of weighted twisted curves. The space $\mathbb{Q}_{\geq 0} \cup \{0, \infty\}$ of stability conditions is divided into chambers by walls $\{1/d\}$. Let $\epsilon_0 = 1/d_0$ be a wall. We fix non-negative integers $g, n, d$ such that $2g - 2 + n + \epsilon_0 d > 0$. When $g = n = 0$, in additional we require that $\epsilon_0 d > 2$.

We consider $n$-pointed weighted twisted curves of genus $g$ and degree $d$. Here “weighted” means the assignment of a nonnegative integer to each irreducible component of twisted curves (c.f. [13, 35]). These integers will correspond to the degrees of quasimaps. Hence we will also refer to them as degrees. We suppress the degrees from the notation and denote an $n$-pointed weighted twisted curve by $(C, x_1, \ldots, x_n)$, or by $(C, x)$ for short. From now on, all curves are weighted twisted curves unless otherwise specified.

A rational tail (resp. bridge) of $(C, x)$ is a smooth rational irreducible component of $C$ whose normalization has one (resp. two) special points, i.e. the preimage of nodes or markings. Let $\mathcal{M}_{g,n,d}^{\text{wt}}(\epsilon)$ be the moduli stack of $n$-pointed weighted twisted curves of genus $g$ and degree $d$.

**Definition 3.1.** We define the open substack $\mathcal{M}_{g,n,d}^{\text{wt,ss}} \subset \mathcal{M}_{g,n,d}^{\text{wt}}$ of $\epsilon_0$-semistable weighted curves by the following conditions:
• the curve has no degree-0 rational bridge,
• the curve has no rational tail of degree strictly less that \( d_0 \).

Note that \( \mathcal{M}^{w,t,ss}_{g,n,d} \) is a smooth Artin stack of dimension \( 3g - 3 + n \).

We recall the blowup construction of the moduli stack of \( \epsilon_0 \)-semistable curves with entangled tails in [62]. Set

\[
m = \lfloor d/d_0 \rfloor, \quad \mathcal{U}_m = \mathcal{M}^{w,t,ss}_{g,n,d}.
\]

The integer \( m \) is the maximum of the number of degree-\( d_0 \) rational tails. Let \( \mathcal{Z}_i \subset \mathcal{U}_m \) be the reduced closed substack parametrizing curves with at least \( i \) rational tails of degree \( d_0 \). Note that \( \mathcal{Z}_1 \) is a normal crossing divisor and \( \mathcal{Z}_i \) is a codimension-\( i \) stratum in \( \mathcal{U}_m \).

We start with the deepest stratum \( \mathcal{Z}_m \) which is smooth. Let

\[
\mathcal{U}_{m-1} \to \mathcal{U}_m
\]

be the blowup along \( \mathcal{Z}_m \) and let \( \mathcal{E}_{m-1} \subset \mathcal{U}_{m-1} \) be the exceptional divisor. Inductively for \( i = m - 1, \ldots, 1 \), let

\[
\mathcal{Z}_i(i) \subset \mathcal{U}_i
\]

be the proper transform of \( \mathcal{Z}_i \) and let

\[
\mathcal{U}_{i-1} \to \mathcal{U}_i
\]

be the blowup along \( \mathcal{Z}_i(i) \) with exceptional divisor \( \mathcal{E}_{i-1} \subset \mathcal{U}_{i-1} \).

**Definition 3.2.** We call \( \widetilde{\mathcal{M}}_{g,n,d} := \mathcal{U}_0 \) the moduli stack of genus-\( g \) \( n \)-marked \( \epsilon_0 \)-semistable curves of degree \( d \) with entangled tails.

We refer the reader to [62, §2.3] for a concrete description of the closed points in \( \widetilde{\mathcal{M}}_{g,n,d} \).

Let

\[
\mathcal{M}^{w,t,ss}_{g,n+k,d-kd_0} \times' \left( \mathcal{M}^{w,t,ss}_{0,1,d_0} \right)^k
\]

denote the fiber product \( \mathcal{M}_{g,n+k,d-kd_0} \times_{\mathcal{M}_{0,1}}^k (\mathcal{M}_{0,1})^k \) formed by matching the sizes of the automorphism groups at the last \( k \)-markings. Since our markings are trivialized gerbes, we can define the gluing morphism

\[
\mathcal{G}_k : \mathcal{M}^{w,t,ss}_{g,n+k,d-kd_0} \times' \left( \mathcal{M}^{w,t,ss}_{0,1,d_0} \right)^k \to \mathcal{Z}_k \subset \mathcal{M}^{w,t,ss}_{g,n,d}
\]

which glues the \( k \) rational tails from \( \left( \mathcal{M}^{w,t,ss}_{0,1,d_0} \right)^k \) to the last \( k \) markings of the universal curve over \( \mathcal{M}^{w,t,ss}_{g,n+k,d-kd_0} \). Since \( \mathcal{M}^{w,t,ss}_{g,n+k,d-kd_0} \times' \left( \mathcal{M}^{w,t,ss}_{0,1,d_0} \right)^k \) is smooth, the gluing morphism factors through the normalization \( \mathcal{Z}_k^{\text{nor}} \) of \( \mathcal{Z}_k \).
For $1 \leq k \leq m$, we recall the structure of $\mathcal{Z}(k)$ from [62]. According to Lemma 2.2.2 and Lemma 2.2.3 in [62], there is a unique fibered diagram

$$
\begin{align*}
\widetilde{M}_{g,n+k,d-kd_0} \times' \left( \mathcal{M}_{0,1,d_0}^{\text{wt,ss}} \right)^k & \xrightarrow{\tilde{g}_k} \mathcal{Z}(k) \\
M_{g,n+k,d-kd_0}^{\text{wt,ss}} \times' \left( \mathcal{M}_{0,1,d_0}^{\text{wt,ss}} \right)^k & \xrightarrow{g_k^{\text{nor}}} \mathcal{Z}_{\text{nor}}(k)
\end{align*}
$$

(13)

where the horizontal arrows are étale of degree $k! / \prod_{i=n+1}^{n+k} r_i$. Here $r_i$ is the order of the automorphism group at the $(n+i)$-th marking of $\mathcal{M}_{g,n+k,d-kd_0}^{\text{wt,ss}}$.

3.2. The boundary divisors of $\widetilde{M}_{g,n,d}$ and inflated projective bundles. By the construction of $\widetilde{M}_{g,n,d}$ we have natural projections

$$\widetilde{M}_{g,n,d} \to U_i, \quad i = 0, \ldots, m.$$

**Definition 3.3.** Let $\xi$ be a geometric point of $\widetilde{M}_{g,n,d}$ and let $\{E_1, \ldots, E_k\}$ be a set of degree-$d_0$ rational tails of $\xi$. Then $E_1, \ldots, E_k$ are called entangled tails of $\xi$ if

1. the image of $\xi$ in $U_k$ lies in $\mathcal{Z}(k)$;
2. the tails $E_1, \ldots, E_k$ are those from $\left( \mathcal{M}_{0,1}^{\text{wt,ss}} \right)^k$ via the gluing morphism $\tilde{g}_k$;
3. the image of $\xi$ in $U_i$ does not lie in $\mathcal{Z}(i)$ for any $i < k$.

Let $D_{k-1} \subset \widetilde{M}_{g,n,d}$ be the proper transform of $\mathcal{E}_{k-1}$. According to [62, Lemma 2.5.4], the boundary divisor $D_{k-1}$ is the closure of the locally closed reduced locus where there are exactly $k$ entangled tails. By construction, we have morphisms $D_{k-1} \to \mathcal{Z}(k)$ and

$$\tilde{g}_k^* D_{k-1} \to \tilde{M}_{g,n+k,d-kd_0} \times' \left( \mathcal{M}_{0,1,d_0}^{\text{wt,ss}} \right)^k.$$

To describe the structure of the above morphism, the second author introduced the notion of inflated projective bundles in [62]. We recall the definition here.

Let $X$ be an algebraic stack and let $L_1, \ldots, L_k$ be line bundles on $X$. Let

$$P = \mathbb{P}(L_1 \oplus \cdots \oplus L_k) \to X$$

be the projective bundle. Consider the coordinate hyperplanes

$$H_i = \mathbb{P}(L_1 \oplus \cdots \oplus \{0\} \oplus \cdots \oplus L_k),$$

where the $\{0\}$ appears in the $i$-th place only. The construction of the inflated projective bundle is analogous to that of $\widetilde{M}_{g,n,d}$. More specifically, for $i = 1, \ldots, k-1$, let $Z_i \subset P$ be the union of the codimension-$i$ coordinate subspaces, i.e.

$$Z_i = \bigcup H_{j_1} \cap \cdots \cap H_{j_i},$$
where \( \{j_1, \ldots, j_i\} \) runs through all subsets of \( \{1, \ldots, k\} \) of size \( i \). First we set \( P_{k-1} = \mathbb{P}(L_1, \ldots, L_k) \). Inductively for \( i = k-1, \ldots, 1 \), let \( Z_{(i)} \subset P_i \) be the proper transform of \( Z_i \) and let
\[
P_{i-1} \to P_i
\]
be the blowup along \( Z_{(i)} \) with exceptional divisor \( E_{i-1} \subset P_{i-1} \).

**Definition 3.4.** We call \( \mathbb{P}(L_1, \ldots, L_k) := P_0 \to X \) the inflated projective bundle associated to \( L_1, \ldots, L_k \).

We denote by \( D_i \subset \mathbb{P}(L_1, \ldots, L_k) \) the proper transforms of \( E_i \), for \( i = 0, \ldots, k-2 \), and refer to it as the \( i \)-th tautological divisor of the inflated projective bundle. Note that \( \mathbb{P}(L_1, \ldots, L_k) \) is smooth over \( X \) of relative dimension \( k-1 \) and \( D_i \) are relative effective Cartier divisors. We denote by \( O_{\mathbb{P}}(-1) \) the pullback of the tautological bundle on \( P \).

For \( i = 1, \ldots, k \), let \( \Theta_i \) be the line bundles on \( \mathcal{M}_{g,n+k,d-kd_0} \times' (\mathcal{M}_{0,1,d_0})^k \) formed by the tensor product of two orbifold tangent lines to the curves, one at the \((n+i)\)-th marking of \( \mathcal{M}_{g,n+k,d-kd_0} \), and the other at the unique marking of the \( i \)-th copy of \( \mathcal{M}_{0,1} \).

**Lemma 3.5** ([62]). The diagram
\[
\begin{array}{ccc}
gl_k^* \mathcal{D}_{k-1} & \xrightarrow{\iota_p} & \mathcal{M}_{g,n,d} \\
\downarrow & & \downarrow \\
gl_k^* \mathcal{E}_{k-1} & \xrightarrow{\iota_e} & \Lambda_{k-1}
\end{array}
\]
is a fibered diagram. The morphism
\[
(14) \quad gl_k^* \mathcal{D}_{k-1} \to \mathcal{M}_{g,n+k,d-kd_0} \times' \left( \mathcal{M}_{0,1,d_0}^{wt,ss} \right)^k
\]
realizes \( gl_k^* \mathcal{D}_{k-1} \) as the inflated projective bundle
\[
\mathbb{P} := \mathbb{P}(\Theta_1, \ldots, \Theta_k) \to \mathcal{M}_{g,n+k,d-kd_0} \times' \left( \mathcal{M}_{0,1,d_0}^{wt,ss} \right)^k.
\]

Abusing the notation, we still denote the pullback of \( \Theta_i \) to \( gl_k^* \mathcal{D}_{k-1} \) by itself. The pullbacks of the boundary divisors \( \mathcal{D}_\ell \) to \( gl_k^* \mathcal{D}_{k-1} \) are described in Lemma 2.7.3 and Lemma 2.5.1 in [62]. We summarize them in the following lemma:

**Lemma 3.6.** Consider the morphism \( \iota_\mathcal{D} : gl_k^* \mathcal{D}_{k-1} \to \mathcal{M}_{g,n,d} \). Then we have

1. for \( 0 \leq \ell \leq k-2 \), the divisor pullback \( \iota_\mathcal{D}^* \mathcal{D}_\ell \) is equal to the \( \ell \)-th tautological divisor \( D_\ell \) of the inflated projective bundle (14);
2. for \( \ell \geq k \), the divisor pullback \( \iota_\mathcal{D}^* \mathcal{D}_\ell \) is equal to \( pr_1^*(\mathcal{D}'_{\ell-k}) \), where \( pr_1 \) is the composition of the projections
\[
pr_1 : gl_k^* \mathcal{D}_{k-1} \to \mathcal{M}_{g,n+k,d-kd_0} \times' \left( \mathcal{M}_{0,1,d_0}^{wt,ss} \right)^k \to \mathcal{M}_{g,n+k,d-kd_0};
\]
and \( \mathcal{D}'_{\ell-k} \) is the boundary divisor on \( \mathcal{M}_{g,n+k,d-kd_0} \).
(3) the pullback of the line bundle \( \mathcal{O}_{\mathfrak{M}_{g,n,d}}(D_{k-1}) \) is canonically isomorphic to
\[
\mathcal{O}_\mathbb{P}(-1) \otimes \iota^* \mathcal{O}_{\mathfrak{M}_{g,n,d}}(- \sum_{i=k}^{m-1} D_i).
\]

**Proof.** (1) and (2) are in Lemma 2.5.1 of [62]. (3) follows from the proof of the that lemma, where it is shown that \( \iota^* \mathcal{O}_{\mathfrak{M}_{g,n,d}}(D_{k-1}) \) is canonically isomorphic to the pullback of the relative \( \mathcal{O}(-1) \) of the projective bundle
\[
E_{k-1} \to Z^{k-1}.
\]
Note that in Lemma 2.5.1 of [62], we have
\[
\tilde{\mathfrak{g}}_k \mathfrak{e}_{k-1} \cong \mathbb{P}(\Theta_1 \oplus \cdots \oplus \Theta_k) \otimes \mathcal{O}_U(- \sum_{i=k}^{m-1} E_i).
\]
This is isomorphic to \( \mathbb{P}(\Theta_1 \oplus \cdots \oplus \Theta_k) \) but the relative tautological differs by \( \mathcal{O}_U(- \sum_{i=k}^{m-1} E_i) \).
Finally observe that the divisor pullback of \( E_i \) to \( \tilde{\mathfrak{M}}_{g,n,d} \) is equal to \( D_i \) by Lemma 2.7.2 of [62].

As a corollary, we have the following refinement of [62, Lemma 2.7.4].

**Lemma 3.7.** Along the map
\[
\iota_D : \tilde{\mathfrak{g}}_k D_{k-1} \to \tilde{\mathfrak{M}}_{g,n,d},
\]
the line bundle
\[
\mathcal{O}_{\mathfrak{M}_{g,n,d}}(D_0 + D_1 + \cdots + D_{m-1})
\]
pulls back to
\[
\mathcal{O}_{\tilde{\mathfrak{g}}_k D_{k-1}}(D_0 + \cdots + D_{k-2}) \otimes \mathcal{O}_\mathbb{P}(-1),
\]
where the divisors \( D_0, \ldots, D_{k-2} \) are the tautological divisors of the inflated projective bundle (14).

### 3.3. The calibration bundle and Master space

As before, we assume that \( 2g - 2 + n + \epsilon_0 d \geq 0 \), and \( \epsilon_0 d > 2 \) when \( g = 0 \).

**Definition 3.8.** When \( (g,n,d) \neq (0,1,d_0) \), the universal calibration bundle is defined to be the line bundle \( \mathcal{O}_{\mathfrak{M}_{g,n,d}^{\text{wt,ss}}}(-3) \); when \( (g,n,d) = (0,1,d_0) \), the universal calibration bundle is the relative cotangent bundle at the unique marking.

For an \( S \)-family of \( \epsilon_0 \)-semistable, genus-\( g \), degree-\( d \) weighted curves, its calibration bundle is the pullback of universal calibration bundle along the classifying morphism \( S \to \mathfrak{M}_{g,n,d}^{\text{wt,ss}} \).

Let us focus on the case \( (g,n,d) \neq (0,1,d_0) \). For a curve \( C \) with degree-\( d_0 \) rational tails \( E_1, \ldots, E_k \), its calibration bundle is naturally isomorphic to \( (\Theta_1 \otimes \cdots \otimes \Theta_k)^\vee \), where \( \Theta_i \) is the one dimensional vector space of infinitesimal smoothings of the node on \( E_i \). We refer the reader to [62, §2.8] for more details.
Definition 3.9. The moduli of $\epsilon_0$-semistable curves with calibrated tails is defined to be

$$M\tilde{m}_{g,n,d} := \mathbb{P}_{\tilde{M}_{g,n,d}}(\tilde{M}_{g,n,d} \oplus \mathcal{O}_{\tilde{M}_{g,n,d}}),$$

where $\tilde{M}_{g,n,d}$ is the calibration bundle of the universal family over $\tilde{M}_{g,n,d}$.

Following [62], we call $M\tilde{m}_{g,n,d}$ the Master space. Let $S$ be a scheme. An $S$-point of the Master space consists of

$$(\pi : C \to S, x, e, N, v_1, v_2)$$

where

- $(\pi : C \to S, x, e) \in \tilde{M}_{g,n,d}(S)$;
- $N$ is a line bundle on $S$;
- $v_1 \in \Gamma(S, M_S \otimes N), v_2 \in \Gamma(S, N)$ have no common zero, where $M_S$ is the calibration bundle for the family of curves $\pi : C \to S$.

For two families

$$(\pi : C \to S, x, e, N, v_1, v_2) \quad \text{and} \quad (\pi' : C' \to S', x', e', N', v'_1, v'_2),$$

an arrow between them consists of a triple

$$(f, t, \varphi),$$

where

- $f : S \to S'$ is a morphism;
- $t : (\pi : C \to S, x, e) \to f^*(\pi' : C' \to S', x', e')$ is a 2-morphism in $\tilde{M}_{g,n,d}(S)$;
- $\varphi : N \to f^*N'$ is an isomorphism of line bundles, such that the morphisms $1 \otimes \varphi : M_S \otimes N \to M_S \otimes f^*N' = f^*(M_{S'} \otimes N')$ and $\varphi$ sends $(v_1, v_2)$ to $(f^*v'_1, f^*v'_2)$.

3.4. The moduli and its virtual structure sheaf. Fix $g, n$ and a curve class $\beta$. Let $d = \deg(\beta)$. Recall that $L_{\theta}$ is the polarization on $[W/G]$. Let $\mathcal{Q}map_{g,n}(X, \beta)$ be the stack of genus-$g$, $n$-marked quasimaps to $X$ with curve class $\beta$. Consider the open substack $\mathcal{Q}map_{g,n}^{s}(X, \beta) \subset \mathcal{Q}map_{g,n}^{s}(X, \beta)$ parametrizing quasimaps with no rational tails of degree $< d_0$, no rational bridges of degree 0, or base point of length $> d_0$. There is a natural forgetful morphism

$$\mathcal{Q}map_{g,n}^{s}(X, \beta) \to M_{g,n,d}^{wts}$$

defined by taking the underlying curves weighted by the degrees of the quasimaps.

Definition 3.10. We define the stack of genus-$g$, $n$-pointed, $\epsilon_0$-semistable quasimaps with entangled tails to $X$ with curve class $\beta$ to be

$$\mathcal{Q}map_{g,n}^{\sim}(X, \beta) := \mathcal{Q}map_{g,n}^{s}(X, \beta) \times_{M_{g,n,d}^{wts}} \tilde{M}_{g,n,d}.$$

Definition 3.11. An $S$-family of $\epsilon_0$-semistable quasimaps with entangled tails is $\epsilon_+$-stable if the underlying family of quasimaps is $\epsilon_+$-stable. In other words, it is stable if there is no length-$d_0$ base point.
Let $\tilde{Q}_{g,n}^{ε+}(X, β)$ denote the moduli of genus-$g$, $n$-pointed $ε_+$-stable quasimaps to $X$ with entangled tails of curve class $β$. According to [62], there is a natural isomorphism

$$\tilde{Q}_{g,n}^{ε+}(X, β) \cong Q_{g,n}^{ε+}(X, β) \times \overline{\mathcal{M}}_{g,n,d}^{\text{wt,ss}}$$

and hence $\tilde{Q}_{g,n}^{ε+}(X, β)$ is a proper Deligne–Mumford stack.

Let $π : C \to Q_{g,n}^{ε+}(X, β)$ be the universal curve and let $[u] : C \to [W/G]$ be the universal map. According to [6, 10], the moduli stack $Q_{g,n}^{ε+}(X, β)$ has a relative perfect obstruction theory

$$(Rπ_*u^*\mathcal{T}_{[W/G]})^\vee \to \mathbb{L}Q_{g,n}^{ε+}(X, β)/\mathcal{M}_{g,n,d}^{\text{wt,ss}}$$

for the forgetful morphism $ν : Q_{g,n}^{ε+}(X, β) \to \mathcal{M}_{g,n,d}^{\text{wt,ss}}$. According to [48, Definition 2.2], using the above relative perfect obstruction theory, we can define a virtual pullback

$$ν' : K_0(\mathcal{M}_{g,n,d}^{\text{wt,ss}}) \to K_0(Q_{g,n}^{ε+}(X, β)),$$

and a virtual structure sheaf

$$O_{Q_{g,n}^{ε+}(X, β)}^{\text{vir}} := ν'_*O_{\mathcal{M}_{g,n,d}^{\text{wt,ss}}} \in K_0(Q_{g,n}^{ε+}(X, β)).$$

Note that we use $\mathcal{M}_{g,n,d}^{\text{wt,ss}}$ in place of $\mathcal{M}_{g,n}$. Let $ν' : Q_{g,n}^{ε+}(X, β) \to \mathcal{M}_{g,n}$ be the composition of $ν$ and the étale morphism $μ : \mathcal{M}_{g,n,d}^{\text{wt,ss}} \to \mathcal{M}_{g,n}$. We will get the same virtual structure sheaf if we take the virtual pullback of $O_{\mathcal{M}_{g,n}}$ along $ν'$. This follows from the functoriality of virtual pullbacks [48, Proposition 2.11] and the fact that $μ^*O_{\mathcal{M}_{g,n}} = O_{\mathcal{M}_{g,n,d}^{\text{wt,ss}}}$. Similarly, let $\tilde{π} : \tilde{C} \to \tilde{Q}_{g,n}^{ε+}(X, β)$ be the universal curve and let $[\tilde{u}] : \tilde{C} \to [W/G]$ be the universal quasimap. Let $\tilde{ν} : \tilde{Q}_{g,n}^{ε+}(X, β) \to \tilde{\mathcal{M}}_{g,n,d}$ denote the forgetful morphism. We have a relative perfect obstruction theory

$$(R\tilde{π}_*\tilde{u}^*\mathcal{T}_{[W/G]})^\vee \to \mathbb{L}\tilde{Q}_{g,n}^{ε+}(X, β)/\tilde{\mathcal{M}}_{g,n,d},$$

which in turn defines a virtual pullback $\tilde{ν}' : K_0(\tilde{\mathcal{M}}_{g,n,d}) \to K_0(\tilde{Q}_{g,n}^{ε+}(X, β))$. The virtual structure of $\tilde{Q}_{g,n}^{ε+}(X, β)$ is defined by

$$O_{\tilde{Q}_{g,n}^{ε+}(X, β)}^{\text{vir}} := \tilde{ν}'_*O_{\tilde{\mathcal{M}}_{g,n,d}} \in K_0(\tilde{Q}_{g,n}^{ε+}(X, β)).$$

Suppose we choose an integer $l$ and $K$-theory classes $R, E^{(m)} \in K_G(W)$, $m \neq 0$. As explained in Section 2.4, we can define twisted virtual structure sheaves $O_{\tilde{Q}_{g,n}^{ε+}(X, β)}^{\text{vir}, E^{(*)}, R,l}$ and $O_{Q_{g,n}^{ε+}(X, β)}^{\text{vir}, E^{(*)}, R,l}$ by (10).

Let $τ : \tilde{Q}_{g,n}^{ε+}(X, β) \to Q_{g,n}^{ε+}(X, β)$ be the forgetful morphism. The following lemma compares the two virtual structures and their twisted counterparts.

**Lemma 3.12.** We have

$$τ_*O_{\tilde{Q}_{g,n}^{ε+}(X, β)}^{\text{vir}} = O_{Q_{g,n}^{ε+}(X, β)}^{\text{vir}}.$$
and
\[
\tau_* \mathcal{O}^{\text{vir}}_{\tilde{Q}^+_{g,n}(X,\beta)}(E^{\bullet},R,l) = \mathcal{O}^{\text{vir}}_{Q^+_{g,n}(X,\beta)}(E^{\bullet},R,l).
\]

**Proof.** Consider the following fibered diagram.

\[
\begin{array}{ccc}
\tilde{Q}^+_{g,n}(X,\beta) & \xrightarrow{\tau} & Q^+_{g,n}(X,\beta) \\
\tilde{\nu} & & \downarrow \nu \\
\tilde{M}_{g,n,d} & \xrightarrow{\tau'} & M_{g,n,d}^{\text{wt,ss}}
\end{array}
\]

Since \(\tau'\) is a sequence of blowups along smooth centers, we have
\[
(15) \quad \tau'_* \mathcal{O}_{\tilde{M}_{g,n,d}} = \mathcal{O}_{M_{g,n,d}^{\text{wt,ss}}}.
\]

Note that the \(\tilde{\nu}\)-relative perfect obstruction theory is the pullback of \(\nu\)-relative perfect obstruction theory via \(\tau\). The first part of the lemma follows from (15) and the fact that virtual pullbacks commute with proper pushforwards (c.f. [48, Proposition 2.4]). Since the universal principal \(G\)-bundle over \(\tilde{Q}^+_{g,n}(X,\beta)\) is the pullback of that over \(Q^+_{g,n}(X,\beta)\), the same holds for twisting classes \(E^{(m)}_{g,n,\beta}(t_{\pm})\) and \(D^{R,l}\). Hence the second identity in the lemma follows from the first one and the projection formula.

\[\Box\]

### 3.5. Splitting off entangled tails

Recall that for \(k = 1, \ldots, m\), the boundary divisors \(D_{k-1} \subset M_{g,n,d}^{\text{wt,ss}}\) is the closure of the locally closed reduced locus where there are exactly \(k\) entangled tails. We describe the pullback of the virtual structure sheaf to \(\tilde{M}_{g,n,d}^{\text{wt,ss}}\) via the gluing morphism \(\iota_D : \tilde{\mathcal{D}}_{k-1} \to \tilde{M}_{g,n,d}^{\text{wt,ss}}\).

Define \(\tilde{Q}^+_{g,n}(X,\beta)\big|_{\tilde{\mathcal{D}}_{k-1}^*}\) by the following fibered diagram

\[
\begin{array}{ccc}
\tilde{Q}^+_{g,n}(X,\beta)\big|_{\tilde{\mathcal{D}}_{k-1}^*} & \xrightarrow{\iota_D} & \tilde{Q}^+_{g,n}(X,\beta) \\
\downarrow & & \downarrow \\
\tilde{\mathcal{D}}_{k-1}^* & \xrightarrow{\iota_D} & \tilde{M}_{g,n,d}^{\text{wt,ss}}
\end{array}
\]

and define
\[
\mathcal{O}^{\text{vir}}_{\tilde{Q}^+_{g,n}(X,\beta)}\big|_{\tilde{\mathcal{D}}_{k-1}^*} := \iota_D^!(\mathcal{O}^{\text{vir}}_{\tilde{Q}^+_{g,n}(X,\beta)}).
\]

Here \(\iota_D^!\) denotes the Gysin pullback. The twisted virtual structure sheaf \(\mathcal{O}^{\text{vir}}_{\tilde{Q}^+_{g,n}(X,\beta)}\big|_{\tilde{\mathcal{D}}_{k-1}^*}\) is defined similarly as the Gysin pullback of the twisted virtual structure sheaf on \(\tilde{Q}^+_{g,n}(X,\beta)\).
According to [62, §3.2], there is a fibered diagram
\[
\begin{array}{ccc}
\tilde{Q}^+_{g,n}(X,\beta)_{\overline{\mathcal{M}}_{k-1}} & \xrightarrow{p} & \prod_{\tilde{\beta}} \tilde{Q}^+_{g,n+k}(X,\beta') \times_{(IX)^k} \prod_{i=1}^k Q^+_{0,1}(X,\beta_i) \\
\tilde{g}^*_k \overline{\mathcal{M}}_{k-1} & \xrightarrow{\Phi} & \tilde{M}_{g,n+k,d-kd_0} \times' (\mathcal{M}^{vir})^k_{0,1,d_0}
\end{array}
\]
where \(\tilde{\beta} = (\beta', \beta_1, \ldots, \beta_k)\) runs through all the decompositions of effective curve classes \(\beta = \beta' + \beta_1 + \cdots + \beta_k\) such that \(\text{deg}(\beta_i) = d_0\) for \(i \geq 1\). By Lemma 3.5, the map \(p\) above is the inflated projective bundle \(\mathbb{P}(\Theta_1 \oplus \cdots \oplus \Theta_k)\), and in particular, it is flat. Consider the fibered diagram
\[
\begin{array}{ccc}
Q^+_{g,n+k}(X,\beta') \times_{(IX)^k} \prod_{i=1}^k Q^+_{0,1}(X,\beta_i) & \xrightarrow{\Phi} & Q^+_{g,n+k}(X,\beta') \times_{(IX)^k} \prod_{i=1}^k Q^+_{0,1}(X,\beta_i) \\
\xrightarrow{\text{ev}_k} & \Delta_{(IX)^k} & \xrightarrow{\text{ev}_k} (IX)^k \times (IX)^k,
\end{array}
\]
where the bottom line is the diagonal morphism.

**Lemma 3.13.** We have
\[
\mathcal{O}_{\tilde{Q}^+_{g,n}(X,\beta)}^{vir} = p^* \left( \sum_{\beta} \Delta^!_{(IX)^k} \mathcal{O}^{vir}_{\tilde{Q}^+_{g,n+k}(X,\beta')} \boxtimes \prod_{i=1}^k \mathcal{O}^{vir}_{Q^+_{0,1}(X,\beta_i)} \right)
\]
and
\[
\mathcal{O}_{\tilde{Q}^+_{g,n}(X,\beta)}^{vir, E^{(*)}, R,l} = p^* \left( \sum_{\beta} \Delta^!_{(IX)^k} \mathcal{O}^{vir, E^{(*)}, R,l}_{\tilde{Q}^+_{g,n+k}(X,\beta')} \boxtimes \prod_{i=1}^k \mathcal{O}^{vir, E^{(*)}, R,l}_{Q^+_{0,1}(X,\beta_i)} \right) \cdot p^*(\text{ev}_k)^*(T^{-1})^\otimes k,
\]
where \(T = \exp \left( \sum_{m \neq 0} \Psi^m(E^{(m)})) / m \right) \cdot (\det \tilde{R})^{-1}\) is the twisting class introduced in Section 2.4.

**Proof.** The first identity follows from the argument used in [62, Lemma 3.2.1] and the functoriality of virtual structure sheaves [42, Proposition 4]. In the twisted case, for various moduli spaces \(\mathcal{M}\) of quasimaps, we denote by \(T_{\mathcal{M}}\) the twisting class (9) over \(\mathcal{M}\). The family version of (12) implies that
\[
\Phi^* (T_{Q^+_{g,n+k}(X,\beta')} \boxtimes \prod_{i=1}^k T_{Q^+_{0,1}(X,\beta_i)}) = T_{Q^+_{g,n+k}(X,\beta')} \times_{(IX)^k} \prod_{i=1}^k Q^+_{0,1}(X,\beta_i) \cdot (\text{ev}_k)^* T^{\otimes k}.
\]
Since the universal families and universal principal \(G\)-bundles are preserved under the pullback of \(p\), we have
\[
p^* (T_{Q^+_{g,n+k}(X,\beta')} \times_{(IX)^k} \prod_{i=1}^k Q^+_{0,1}(X,\beta_i)) = Q^+_{g,n}(X,\beta)_{\overline{\mathcal{M}}_{k-1}}.
\]
This concludes the proof of the second identity. □

4. K-theoretic localization on the Master space

We recall in this section the definition of the master space and the description of its \(\mathbb{C}^*\)-fixed point loci studied in [62]. We will compute the \(\mathbb{C}^*\)-equivariant \(K\)-theoretic Euler classes of the virtual normal bundles of all fixed-point components.

4.1. The Master space and its \(\mathbb{C}^*\)-fixed loci. As before, we fix the numerical data \(g,n,d\). Let \(\epsilon_0 = 1/d_0\) be a wall. We assume that \(2g - 2 + n + \epsilon_0 d \geq 0\), and \(\epsilon_0 d > 2\) when \(g = n = 0\).

Recall from Definition 3.9 the moduli stack \(\widetilde{\mathcal{M}}_{g,n,d}\) of curves with calibrated tails. Let \(S\) be a scheme. An \(S\)-family of genus-\(g\), \(n\)-pointed \(\epsilon_0\)-semistable quasimaps with calibrated tails to \(X\) of curve class \(\beta\) is given by a tuple \((\pi : C \to S, x, e, u, N, v_1, v_2)\) where

- \((\pi : C \to S, x, e, u)\) is an \(\epsilon_0\)-semistable, genus-\(g\), \(n\)-pointed quasimaps to \(X\) with entangled tails of curve class \(\beta\), and
- \((\pi : C \to S, x, e, N, v_1, v_2)\) \(\in \widetilde{\mathcal{M}}_{g,n,d}(S)\).

Let \(\mathcal{Q}_{\epsilon_0}^{g,n}(X, \beta)\) denote the category parameterizing such families. According to [62, §4.1], \(\mathcal{Q}_{\epsilon_0}^{g,n}(X, \beta)\) is an Artin stack of finite type with finite-type separated diagonal.

A degree-\(d_0\) rational tail \(E \subset C\) is called a constant tail if \(E\) contains a base point of length \(d_0\).

Definition 4.1. An \(S\)-family of \(\epsilon_0\)-semistable quasimaps with calibrated tails

\[(\pi : C \to S, x, e, u, N, v_1, v_2)\]

is \(\epsilon_0\)-stable if over every geometric point \(s\) of \(S\),

1. any constant tail in \(C_s = \pi^{-1}(s)\) is an entangled tail;
2. if \(C_s\) has at least one rational tail of degree \(d_0\), then length-\(d_0\) base points only lie on degree-\(d_0\) rational tails of \(C_s\);
3. if \(v_1(s) = 0\), then \((\pi : C \to S, x, u)|_{s}\) is an \(\epsilon_+\)-stable quasimap;
4. if \(v_2(s) = 0\), then \((\pi : C \to S, x, u)|_{s}\) is an \(\epsilon_-\)-stable quasimap.

Let \(MQ_{\epsilon_0}^{g,n}(X, \beta)\) denote the category fibered in groupoids parameterizing genus-\(g\), \(n\)-pointed, \(\epsilon_0\)-stable quasimaps with calibrated tails to \(X\) of curve class \(\beta\). According to Proposition 5.0.1 in [62], \(MQ_{\epsilon_0}^{g,n}(X, \beta)\) is a Deligne–Mumford proper over \(\mathbb{C}\). The space \(MQ_{\epsilon_0}^{g,n}(X, \beta)\) is referred to as the Master space.

The construction of the virtual structure sheaf of \(MQ_{\epsilon_0}^{g,n}(X, \beta)\) is analogous to that of \(Q_{g,n}(X, \beta)\). Let \(\pi : C \to MQ_{\epsilon_0}^{g,n}(X, \beta)\) be the universal curve and let \(u : \mathcal{C} \to [W/G]\) be the universal map. Let \(\nu_M : MQ_{\epsilon_0}^{g,n}(X, \beta) \to \widetilde{\mathcal{M}}_{g,n,d}\) be the forgetful morphism. There is a natural relative perfect obstruction theory

\[(R\pi_* u^* T_{[W/G]})^\vee \to \mathbb{L}_{\nu_M}.\]
We can define the virtual structure sheaf via the virtual pullback:

\[ \mathcal{O}_{\overline{MQ}^{\epsilon_0}_{g,n}(X,\beta)}^{\text{vir}} := \nu^* \mathcal{O}_{\overline{M}^{\text{vir}}_{g,n,d} \in K_c(MQ^{\epsilon_0}_{g,n}(X,\beta))}. \]

The construction of the twisted virtual structure with level structure on \( MQ^{\epsilon_0}_{g,n}(X,\beta) \) is also parallel to that on \( Q^{\epsilon_0}_{g,n}(X,\beta) \).

Let \( d = \deg(\beta) \). We denote by \( I \) the least common multiple of \( 1, 2, \ldots, \lfloor d/d_0 \rfloor \). Consider the \( C^* \)-action on \( MQ^{\epsilon_0}_{g,n}(X,\beta) \) defined by scaling \( v_1 \):

\[ \lambda \cdot (\pi : C \to S, x, e, u, N, v_1, v_2) = (\pi : C \to S, x, e, u, N, \lambda^I v_1, v_2), \quad \lambda \in \mathbb{C}^*. \]

It is the \( I \)-th power of the \( C^* \)-action defined in [62, (6.1)]. The purpose of this modification is to trivialize the \( C^* \)-action on the fixed-point components and avoid fractional weights that usually show up in localization computations.

According to [62, §6], there are three types of fixed-point components.

4.1.1. \( \epsilon_+ \)-stable quasimaps with entangled tails. Let \( F_+ \subset MQ^{\epsilon_0}_{g,n}(X,\beta) \) be the Cartier divisor defined by \( v_1 = 0 \). It is a fixed-point component. We have an isomorphism

\[ F_+ \cong \tilde{Q}^{\epsilon_+}_{g,n}(X,\beta). \]

which identifies the universal principal \( G \)-bundles over the universal families and the perfect obstruction theories. Hence it also identifies their virtual structure sheaves

\[ \mathcal{O}_{F_+}^{\text{vir}} = \mathcal{O}_{\tilde{Q}^{\epsilon_+}_{g,n}(X,\beta)}^{\text{vir}}. \]

and twisted virtual structure sheaves with level structure. The virtual normal bundle is \( M_+ \), the calibration bundle of \( \tilde{Q}^{\epsilon_+}_{g,n}(X,\beta) \) in Definition 3.8, with a \( C^* \)-action of weight \( I \).

4.1.2. \( \epsilon_- \)-stable quasimaps. When \( g = 0, n = 1, \deg(\beta) = d_0 \), the moduli stack \( Q^{\epsilon_-}_{g,n}(X,\beta) \) is empty and \( v_2 \) is non-vanishing on \( MQ^{\epsilon_0}_{g,n}(X,\beta) \). Otherwise, the Cartier divisor \( F_- \subset MQ^{\epsilon_0}_{g,n}(X,\beta) \) defined by \( v_2 = 0 \) is a fixed component. We have an isomorphism

\[ F_- \cong Q^{\epsilon_-}_{g,n}(X,\beta). \]

Again, the above isomorphism identifies the universal principal \( G \)-bundles and the perfect obstruction theory, and hence virtual structure sheaves

\[ \mathcal{O}_{F_-}^{\text{vir}} = \mathcal{O}_{Q^{\epsilon_-}_{g,n}(X,\beta)}^{\text{vir}}. \]

and their twisted counterparts. The virtual normal bundle of \( F_- \) in the Master space is the line bundle \( M_\vee \), the dual of the calibration bundle \( M_- \) of \( Q^{\epsilon_-}_{g,n}(X,\beta) \). The \( C^* \)-action on \( M_\vee \) has weight \((-I)\).

The last type of fixed loci will be explained in the next subsection.
4.2. The correction terms. The other fixed-point components are closely related to the graph space $QG_{0,1}(X, \beta)$ where $\deg(\beta) = d_0$, and the $K$-theoretic localization contributions can be expressed in terms of the $I$-function. Recall that $F_{*,\beta} := F_{*,0}^0 \subset QG_{0,1}(X, \beta)$ denotes the fixed-point component where the unique marking $x_*$ is at $\infty$ and the quasimap $u$ has a base point of length $\deg(\beta) = d_0$ at $0$. According to [62, §6.3], the virtual normal bundle $N_{\text{vir}}^{\text{vir}}(F_{*,\beta})$ is isomorphic to

$$(R\pi_*(u^*\mathcal{T}_{[W/G]})(F_{*,\beta}))^{\text{mv}} \oplus T_\infty\mathbb{P}^1$$

in the $K$-theory, where $\pi : C \to QG_{0,1}(X, \beta)$ is the universal curve, $u : C \to [W/G]$ is the universal map, and the upper index “mv” denotes the moving part of the complex.

Define the following class in the localized $K$-group

$$I_\beta(q) := \frac{1}{\lambda_C^1(\mathbb{C}^*[\mathbb{T}_{[W/G]}](F_{*,\beta}))^{\text{mv}}(\mathbb{C})} \in K^C_*(F_{*,\beta}) \otimes \mathbb{Q}[q,q^{-1}] \mathbb{Q}(q).$$

Note that the tangent space $T_\infty\mathbb{P}^1$ has $\mathbb{C}^*$-weight 1 and hence $\lambda_C^1(T_\infty\mathbb{P}^1) = 1 - q^{-1}$. It follows that the $I$-function in Definition 2.2 can be rewritten as

$$I(Q, q) = \sum_{\beta \geq 0} Q^\beta(\xi_v)_* \left( I_\beta(q) \cdot \mathcal{O}_{F_{*,\beta}}^{\text{vir}} \right).$$

In the twisted case, we define

$$I^E_\beta(q) := \frac{1}{(\text{deg}(\beta))^{\text{mv}}(\mathbb{C})} \in K^C_*(F_{*,\beta}) \otimes \mathbb{Q}[q,q^{-1}] \mathbb{Q}(q).$$

Formula (17) still holds if we replace $I(Q, q), I_\beta(q)$ and $\mathcal{O}_{F_{*,\beta}}^{\text{vir}}$ by their twisted counterparts.

To describe the objects that the fixed-point components parametrize, we need the following definition. Consider an $e_0$-stable quasimap with calibrated tails

$$\xi = (C, x, e, u, N, v_1, v_2) \in MQ_{g,n}(X, \beta)(\mathbb{C}).$$

Definition 4.2. Let $E \subset C$ be a degree-$d_0$ rational tail and let $y \in E$ be a node (or marking if $g = 0, n = 1$). The tail $E$ is called a fixed tail if the automorphism group $\text{Aut}(E, y, u|_E)$ is infinite.

4.2.1. $g = 0, n = 1, \deg(\beta) = d_0$ case. In this case, the curve must be irreducible and $v_2$ is non-vanishing. Let $F_\beta$ be the fixed-point component defined by

$$F_\beta := \{ \xi \mid \text{the domain curve of } \xi \text{ is a single fixed tail, } v_1 \neq 0, v_2 \neq 0 \}.$$
Lemma 4.3. We have
\[ \tau_\ast \mathcal{O}_{F_\beta} = \theta^{r \ast} (\tilde{L}_\ast), \quad \tau_\ast \mathcal{O}^{\text{vir}}_{F_\ast, \beta} = \mathcal{O}^{\text{vir}}_{F_\beta}, \]
and
\[ \frac{1}{\lambda_1^\ast((N^{\text{vir}}_{F_\beta/MQ_{0,1}(X, \beta)})^\vee)} = (1 - q^{r \ast l}) \cdot \tau_\ast \mathcal{I}_\beta(q^{r \ast l}). \]

In the twisted case, we have
\[ \tau_\ast \mathcal{O}^{\text{vir}, \mathcal{E}(\ast), R, l}_{F_\ast, \beta} = \mathcal{O}^{\text{vir}, \mathcal{E}(\ast), R, l}_{F_\beta}. \]

Proof. Recall that \( \theta^{r \ast} (\tilde{L}_\ast) := \sum_{j=0}^{r - 1} \tilde{L}_j \). Thus the first identity is a standard fact for cyclic covers (c.f., for example, [16, §3.5]). Let \( \mathbb{E}_{MQ} \) denote the absolute perfect obstruction theory of \( MQ_{0,1}(X, \beta) \). The second and the third equalities follow from the analysis of the fixed and moving parts of the restriction \( \mathbb{E}_{MQ}|_{F_\beta} \) in [62, Lemma 6.4.2] and the functoriality of virtual structure sheaves (c.f. [42, Proposition 4]). The \( q^{r \ast l} \) comes from the fact that the \( \mathbb{C}^\ast \)-action (16) on the master space corresponds to a weight-(\( r \ast l \)) \( \mathbb{C}^\ast \)-action on the cotangent space \( T_{\beta}^\ast \mathbb{P}^1 \). The last equality between twisted virtual structure sheaves with level structure follows from the fact that the pullback of the universal curve over \( F_\beta \) along with its universal principal \( G \)-bundle via \( \tau \) are isomorphic to those over \( F_\ast, \beta \).

\( \square \)

Corollary 4.4. Write
\[ I(Q, q) = \sum_{\beta \geq 0} I_\beta(q) Q^\beta \quad \text{and} \quad I^{\mathcal{E}(\ast), R, l}(Q, q) = \sum_{\beta \geq 0} I^{\mathcal{E}(\ast), R, l}(q) Q^\beta \]

Then we have
\[ (\tilde{\mathbb{E}}_\ast)_\ast \tau_\ast \mathcal{O}^{\text{vir}}_{F_\beta} = \mathcal{O}^{\text{vir}, \mathcal{E}(\ast), R, l}_{F_\ast, \beta} = (1 - q^{r \ast l}) I_\beta(q^{r \ast l}), \quad \text{and} \]
\[ T^{-1} \cdot (\tilde{\mathbb{E}}_\ast)_\ast \tau_\ast \mathcal{O}^{\text{vir}, \mathcal{E}(\ast), R, l}_{F_\beta} = (1 - q^{r \ast l}) I^{\mathcal{E}(\ast), R, l}_{\beta}(q^{r \ast l}), \]

where \( r \) is the locally constant function on \( \overline{I}X \) that takes value \( r \) on \( \overline{I}_r X \).

4.2.2. 2g - 2 + n + \epsilon_0 d > 0 case.

Condition 4.5. Let \( \beta = (\beta', \{\beta_1, \ldots, \beta_k\}) \) be a tuple satisfying the following conditions

1. \( \{\beta_1, \ldots, \beta_k\} \) is a (unordered) multiset;
2. \( \beta = \beta' + \beta_1 + \cdots + \beta_k; \)
3. \( \deg(\beta_i) = d_0 \) for \( i = 1, \ldots, k; \)
For each $\beta = (\beta', \{\beta_1, \ldots, \beta_k\})$ satisfying Condition 4.5, we define a (possibly empty) substack

$$F_\beta = \{\xi | \xi \text{ has exactly } k \text{ entangled tails,}$$

which are all fixed tails, of degrees $\beta_1, \ldots, \beta_k\}$$

of $MQ_{g,n}^\circ(X, \beta)$. According to [62, §6.5], $F_\beta$ is closed (if nonempty), and $F_+, F_-$ and $F_\beta$ are all the fixed-point components of the $\mathbb{C}^*$-action on the Master space $MQ_{g,n}^\circ(X, \beta)$.

We now recall the structure of $F_\beta$ described in [62, §6.5]. Recall that $\mathcal{Z}_k \subset \mathcal{U}_k$ is the proper transform of the locus $\mathcal{Z}_k \subset \mathcal{M}_{g,n,d}$ where there are at least $k$ rational tails of degree $d_0$. Recall from Section 3.1 that

$$\tilde{\text{g}}_k^*: \tilde{\mathcal{M}}_{g,n+k,d-kd_0} \times' (\tilde{\mathcal{M}}_{0,1,d_0}^{\text{wt,ss}})^k \rightarrow \mathcal{Z}_k$$

is the morphism that is induced by gluing the universal curve of $(\mathcal{M}_{0,1,d_0}^{\text{wt,ss}})^k$ to the last $k$ markings of the universal curve $\tilde{\mathcal{M}}_{g,n+k,d-kd_0}$ as degree-$d_0$ rational tails. According to [62, Lemma 6.5.3], there is a forgetful morphism $F_\beta \rightarrow \mathcal{Z}_k$. We form the fibered diagram

$$\tilde{\text{g}}_k^* F_\beta \rightarrow \tilde{\mathcal{M}}_{g,n+k,d-kd_0} \times' (\tilde{\mathcal{M}}_{0,1,d_0}^{\text{wt,ss}})^k$$

$$\downarrow$$

$$F_\beta \rightarrow \mathcal{Z}_k$$

Note that the $k$ entangled tails are ordered in $\tilde{\text{g}}_k^* F_\beta$ and hence so are the curve classes $\beta_1, \ldots, \beta_k$. There is a natural $S_k$-action on $\tilde{\text{g}}_k^* F_\beta$ which permutes the $k$ entangled tails. It motivates the following condition:

**Condition 4.6.** Let $\bar{\beta} = (\beta', \beta_1, \ldots, \beta_k)$ be an ordered tuple satisfying the last two conditions in Condition 4.5.

We will refer to $\bar{\beta}$ as an ordered decomposition of the class $\beta$. Set $\bar{\beta}(i) = \beta_i, i = 1, \ldots, k$. We will use this notation when we want to emphasize that the class $\beta_i$ is the $i$-th component in the decomposition $\bar{\beta}$.

The stack $\tilde{\text{g}}_k^* F_\beta$ has the following decomposition

$$\tilde{\text{g}}_k^* F_\beta = \bigsqcup_{\bar{\beta} \prec \beta} \tilde{\text{g}}_k^* F_{\bar{\beta}},$$

where the disjoint union is over all ordered tuples satisfying Condition 4.6 and having $\beta$ as the underlying multiset, and the notation $\tilde{\text{g}}_k^* F_{\beta}$ denotes the substack parametrizing quasimaps whose curve classes of ordered entangled tails are given by $\bar{\beta}$. Note that for a
permutation $\sigma \in S_k$, it maps the component labeled by $\vec{\beta}$ to that labeled by $\sigma(\vec{\beta})$ where

$$\sigma(\vec{\beta}) := (\beta', \beta_{\sigma(1)}, \ldots, \beta_{\sigma(k)})$$

Let $C_{\tilde{g}_k F_{\vec{\beta}}}$ be the pullback to $\tilde{g}_k F_{\vec{\beta}}$ of the universal curve of $MQ^{c_0}_{g,n}(X, \beta)$. Let $C_{\beta'}$ be the pullback to $\tilde{g}_k F_{\vec{\beta}}$ of the universal curve of $\tilde{M}_{g,n+k,d-kd_0}$. Let $E_1, \ldots, E_k$ be the pullback to $\tilde{g}_k F_{\vec{\beta}}$ of the universal curve of $(\tilde{M}^{\text{wt,ss}}_{0,1,d_0})^k$. Note that $C_{\tilde{g}_k F_{\vec{\beta}}}$ is obtained by gluing $E_1, \ldots, E_k$ to the last $k$ markings of $C_{\beta'}$. We call $C_{\beta'}$ the main component of $C_{\tilde{g}_k F_{\vec{\beta}}}$. Let $T_{p_i} C_{\beta'}$ be the pullback to $\tilde{g}_k F_{\vec{\beta}}$ of the universal curve of $\tilde{M}_{g,n}^{+} + k, d - kd_0$. Let $E_1, \ldots, E_k$ be the pullback to $\tilde{g}_k F_{\vec{\beta}}$ of the universal curve of $(\tilde{M}^{\text{wt,ss}}_{0,1,d_0})^k$. Note that $C_{\tilde{g}_k F_{\vec{\beta}}}$ is obtained by gluing $E_1, \ldots, E_k$ to the last $k$ markings of $C_{\beta'}$. We call $C_{\beta'}$ the main component of $C_{\tilde{g}_k F_{\vec{\beta}}}$. Let $T_{p_i} C_{\beta'}$ be the orbifold tangent line bundles of the main component $C_{\beta'}$ at the orbifold node $p_i$. Denote $T_{p_1} C_{\beta'} \otimes T_{p_i} E_i$ by $\Theta$. According to Lemma [62, Lemma 2.5.5] and [62, §6.5], there are canonical isomorphisms

$$\Theta \cong T_{p_i} C_{\beta'} \otimes T_{p_i} E_i, \quad i = 2, \ldots, k,$$

and

$$(18) \quad \Theta \otimes k \cong M^{V}_{\beta'}$$

on $\tilde{g}_k F_{\vec{\beta}}$. Here, $M^{V}_{\beta'}$ is the (pullback of the) calibration bundle on $\tilde{M}_{g,n+k,d-kd_0}$. Consider the stack

$$Y \to \tilde{Q}^{\text{cal}}_{g,n+k}(X, \beta')$$

of $k$-th roots of the pullback to $\tilde{Q}^{\text{cal}}_{g,n+k}(X, \beta')$ of $M^{V}_{\beta'}$. Then (18) induces a morphism

$$(19) \quad \tilde{g}_k F_{\vec{\beta}} \to Y.$$

Now we focus on each component $\tilde{g}_k F_{\vec{\beta}}$ of $\tilde{g}_k F_{\vec{\beta}}$. As explained in [62, §6.5], the restrictions of quasimaps to $E_i$ give rise to morphisms

$$(20) \quad \tilde{g}_k F_{\vec{\beta}} \to F_{\star, \beta_i}, \quad i = 1, \ldots, k.$$ Let $ev_Y : Y \to (IX)^k$ denote the evaluation maps at the last $k$-markings and for each $i$, let

$e^{\star, \beta_i} : F_{\star, \beta_i} \to IX$

denote the evaluation map at the unique marking $x_\star$ composed with the involution on $IX$. Consider the fiber product of $ev_Y$ and $e^{\star, \beta_i}$ over $(IX)^k$:

$$Y \times (IX)^k \prod_{i=1}^{k} F_{\star, \beta_i}$$

By [62, Lemma 6.5.5], the morphism

$$\varphi : \tilde{g}_k F_{\vec{\beta}} \to Y \times (IX)^k \prod_{i=1}^{k} F_{\star, \beta_i},$$
induced by (19) and (20) is a representable, finite, and étale of degree $\prod_{i=1}^{k} r_i$. Here $r_i$ is the locally constant function whose value is the order of the automorphism group of the $i$-th node $p_i$. In fact, according to the proof of [62, Lemma 6.5.5], the morphism $\varphi$ is a fiber product of cyclic étale covers obtained by taking the $r_i$-th root of $(T_{p_i}E_i)^{\varphi_i}$.

Let $\mathcal{O}^\text{vir}_{\tilde{Y}} \in K_0(Y)$ be the flat pullback of $\mathcal{O}^\text{vir}_{\tilde{Q}_{g,n+k}(X,\beta')}$, and let $\mathcal{O}^\text{vir}_{\tilde{g}_k^k F_{\beta}}$ be the flat pullback of $\mathcal{O}^\text{vir}_{F_{\beta}}$, the virtual structure sheaf defined by the fixed part of the absolute perfect obstruction theory. We denote by $\tilde{L}(E_i)$ the orbifold cotangent line bundle of the rational tail $E_i$ at the orbifold node and by $\tilde{L}_{n+i}$ the orbifold cotangent line bundle at the $(n+i)$-th marking of $\tilde{Q}_{g,n+k}(X,\beta')$. Let $L(E_i)$ and $L_{n+i}$ denote the cotangent line bundles on the coarse curves. Recall from Section 3.2 that $\mathcal{D}_i \subset \tilde{\mathcal{M}}_{g,n,d}$ is the closure of the locally closed reduced locus where there are exactly $(i+1)$ entangled tails.

We want to express the correction terms in terms of Euler characteristic of sheaves over $Y \times_{(1)} \prod_{i=1}^{k} F_{\kappa_\beta}$, $\mathcal{O}^\text{vir}_{\tilde{g}_k^k F_{\beta}}$. One complication is that the pullback of the universal curve on $Y \times_{(1)} \prod_{i=1}^{k} F_{\kappa_\beta}$ is not isomorphic to the universal curve over $\tilde{g}_k^k F_{\beta}$. Nevertheless, we have the following comparison results between virtual structure sheaves and virtual normal bundles.

**Lemma 4.7.** We have

$$\varphi^*(\mathcal{O}^\text{vir}_{\tilde{g}_k^k F_{\beta}}) = \prod_{i=1}^{k} \theta^{\varphi_i}(\tilde{L}(E_i)),$$

$$\mathcal{O}^\text{vir}_{\tilde{g}_k^k F_{\beta}} = \varphi^*(\mathcal{O}^\text{vir}_{\tilde{Y}} \otimes \prod_{i=1}^{k} \mathcal{O}^\text{vir}_{F_{\kappa_\beta}}),$$

and

$$\frac{1}{\lambda^{-1}(\mathcal{N}_{F_{\beta}/MQ_{\tilde{M}}(X_{\beta})}^{\text{vir}}|_{\tilde{g}_k^k F_{\beta}})} = \varphi^*(\prod_{i=1}^{k} (1 - q^{\frac{r_i}{k}} L(E_i)^{\vee})),$$

$$\varphi^*(\tilde{L}(E_i)^{\vee}) \otimes \cdots \otimes \varphi^*(q^{\frac{r_i}{k}} L(E_k)^{\vee}),$$

where $\theta^{\varphi_i}(\tilde{L}(E_i))$ is the $r_i$-th Bott's cannibalistic class and $m := |d/d_0|$. 

**Proof.** The first statement follows from the fact that $\varphi$ is a fiber product of cyclic étale covers obtained by taking the $r_i$-th root of $(T_{p_i}E_i)^{\varphi_i}$ (c.f., for example, [16, §3.5]).

To prove the last two statements, we recall the comparison between the perfect obstruction theories of $g_k^k F_{\beta}$ and $Y \times_{(1)} \prod_{i=1}^{k} F_{\kappa_\beta}$ given in the proof of [62, Lemma 6.5.6].

Let $\pi_1: C_1 \to \tilde{g}_k^k F_{\beta}$ be the universal curve and let $u_1: C_1 \to [W/G]$ be the universal map. Recall that the complex $E_1 = (R\pi_1 u_1^\text{vir} [W/G])^\vee$ defines a perfect obstruction theory relative to $M\tilde{\mathcal{M}}_{g,n,d}$. Let $E_{MQ}$ denote the absolute perfect obstruction theory on $MQ_{g,n}(X,\beta)$. We have a distinguished triangle

$$E_{MQ} \rightarrow \mathcal{E} \rightarrow \mathcal{E}_1 \rightarrow 1.$$


Let \( \pi_2 : C_2 \to Y \times (IX)^k \) \( \prod_{i=1}^k F_{*,\beta_i} \) be the universal curve obtained from gluing the unique marking of \( F_{*,\beta_i} \) to the \((n+i)\)-th marking of \( Y \). Let \( u_2 : C_2 \to [W/G] \) be the universal map. Then the \( \mathbb{C}^* \)-fixed part of the complex \( \mathbb{E}_2 = (R\pi_{2*}u_{2*}T_{[W/G]})^Y \) defines a perfect obstruction theory of \( Y \times (IX)^k \) \( \prod_{i=1}^k F_{*,\beta_i} \) relative to \( \mathfrak{M}_{g,n+k,d-d_0k} \). By using a splitting-node argument similar to that of Lemma 3.13, one can prove that \( \mathcal{O}_{Y}^{vir} \otimes \prod_{i=1}^k \mathcal{O}_{F_{*,\beta_i}}^{vir} \) is equal to the virtual pullback of \( \mathcal{O}_{\mathfrak{M}_{g,n+k,d-d_0k}} \) defined by \( \mathbb{E}_2 \).

In [62], the second author constructed a principal \((\mathbb{C}^*)^k\)-bundle \( Y' \to Y \) such that the base change of the two universal families to \( Y' \) are isomorphic. More precisely, we have a fibered diagram

\[
\begin{array}{ccc}
Y' \times_Y \mathfrak{g}^k \mathbb{F}_{\beta} & \xrightarrow{\varphi'} & Y' \times (IX)^k \prod_{i=1}^k F_{*,\beta_i} \\
p_1 & \downarrow & \downarrow p_2 \\
\mathfrak{g}^k \mathbb{F}_{\beta} & \xrightarrow{\varphi} & Y \times (IX)^k \prod_{i=1}^k F_{*,\beta_i}
\end{array}
\]

which is equipped with an isomorphism of the universal curves

\[ \tilde{\varphi} : p_1^* \mathcal{C}_1 \to p_2^* \mathcal{C}_2 \]

that commutes with the universal maps to \([W/G]\). Hence it induces an isomorphism

\[ p_1^* \mathbb{E}_1 \cong p_2^* \varphi^* \mathbb{E}_2. \tag{21} \]

According to the proof of [62, Lemma 6.5.6], there is a natural isomorphism between the fixed parts

\[ \mathbb{E}_1^f \cong \varphi^* \mathbb{E}_2^f. \]

Since the flat pullback of \( \mathcal{O}_{\mathfrak{M}_{g,n+k,d-d_0k}} \) equals \( \mathcal{O}_{\mathfrak{M}_{\mathfrak{g},n+k,d-d_0k}} \), the above isomorphism implies the second statement of the lemma.

By the analysis in the proof of [62, Lemma 6.5.6], the pullback of the \( K \)-theoretic Euler class of the virtual normal bundle of \( F_{\beta} \) in \( MQ_{g,n}^\circ (X, \beta) \) comes from three contributions:

1. the moving part \( \mathbb{E}_1^{vir, \text{mv}} \);
2. \( \bigoplus_{i=1}^k l(E_i)[-1] \) where the \( i \)-th factor has \( \mathbb{C}^* \)-weight \(-r_i l/k\);
3. \( \Theta_1 \otimes \mathcal{O}(\sum_{i=0}^{\infty} \mathcal{D}_i) \) with \( \mathbb{C}^* \)-weight \( 1/k \).

The contributions (2) and (3) give rise to the equality

\[ \frac{1}{\lambda^{-1}((L_{\pi_{2*}u_{2*}T_{[W/G]}})^Y)} = \frac{\prod_{i=1}^k (1 - q^{r_i l} L(E_i)^Y)}{1 - q^{-1} r l E_1 \cdot L_{n+1} \cdot \mathcal{O}(\sum_{i=0}^{m-1} \mathcal{D}_i)}. \tag{22} \]

To compute the contribution (1), we consider group actions on \( p_1^* \mathcal{C}_1 \), \( i = 1, 2 \). There is a \( \mathbb{C}^* \times (\mathbb{C}^*)^k \) action on \( p_1^* \mathcal{C}_1 \) where the first factor acts by (16) and the second factor acts on \( Y' \). There is a \( (\mathbb{C}^*)^k \times (\mathbb{C}^*)^k \) action on \( p_2^* \mathcal{C}_2 \) where the first factor acts by the product of the \( \mathbb{C}^* \)-actions (5) on the graph spaces \( QG_{0,1}(X, \beta_i) \) and the second factor acts on \( Y' \). Let

\[ (q_1, \ldots, q_k, \lambda_1, \ldots, \lambda_k) \]
be the equivariant parameters of \((\mathbb{C}^*)^k \times (\mathbb{C}^*)^k\) and let \(q\) be the equivariant parameter of \(\mathbb{C}^*\) as before. By definition, we have
\[
\frac{1}{\lambda_{\mathbb{C}^* \times (\mathbb{C}^*)^k}(p_2^*E_{\text{mv}})} = \mathcal{I}_{\beta_1}(q_1) \boxtimes \cdots \boxtimes \mathcal{I}_{\beta_k}(q_k).
\]
Then it follows from [62, Lemma 6.5.4] and (21) that
\[
\frac{1}{\lambda_{\mathbb{C}^* \times (\mathbb{C}^*)^k}(p_1^*E_{\text{mv}})} = \mathcal{I}_{\beta_1}(q_1^{r_1/k}) \boxtimes \cdots \boxtimes \mathcal{I}_{\beta_k}(q_k^{r_k/k}) \lambda^{-1}.
\]
By the construction of \(Y'\) (c.f. [62, Section 6.5]), under the natural equivalence between \(K^\mathbb{C}_* \times (\mathbb{C}^*)^k(Y' \times Y \tilde{g}^*_kF_{\tilde{\beta}})\) and \(K^\mathbb{C}_* \tilde{g}^*_kF_{\tilde{\beta}}\), the \((\mathbb{C}^*)^k\)-weights \(\lambda_i\) over the principal bundle \(Y' \times Y \tilde{g}^*_kF_{\tilde{\beta}} \rightarrow \tilde{g}^*_kF_{\tilde{\beta}}\) correspond to the pullbacks of the coarse cotangent line bundles \(L(\mathcal{E}_i)\). Therefore the above equality descends to
\[
\frac{1}{\lambda_{\mathbb{C}^*}(E_1^\text{mv})} = \mathcal{I}_{\beta_1}(q_1^{r_1/k}L(\mathcal{E}_1))^\vee \boxtimes \cdots \boxtimes \mathcal{I}_{\beta_k}(q_k^{r_k/k}L(\mathcal{E}_k)^\vee).
\]
Combining with (22), we conclude the proof of the third statement.

**Remark 4.8.** For the various moduli spaces \(\mathcal{M}\) in the previous lemma, we denote by \(T_\mathcal{M}\) the twisting class (9) over \(\mathcal{M}\). Let \(\text{ev}^k\) denote the evaluation map from \(Y' \times (IX)^k \prod_{i=1}^k F_{*,\beta_i}\) (or \(Y \times (IX)^k \prod_{i=1}^k F_{*,\beta_i}\)) to \((IX)^k\). It follows from the proof of Lemma 3.13 that
\[
T_{Y' \times (IX)^k \prod_{i=1}^k F_{*,\beta_i}} = (T_{Y'} \boxtimes \prod_{i=1}^k T_{F_{*,\beta_i}}(q_i)) \cdot (\text{ev}^k)^*(T^{-1})^{\otimes k}.
\]
By using similar arguments as in the proof of Lemma 4.7, we can show that
\[
T_{\tilde{g}^*_kF_{\tilde{\beta}}} = \varphi^*(T_{Y'} \boxtimes \prod_{i=1}^k T_{F_{*,\beta_i}}(q_i^{r_i/k}L(\mathcal{E}_i)^\vee)) \cdot (\text{ev}^k)^*(T^{-1})^{\otimes k}.
\]

## 5. \textit{K}-THEORETIC WALL-CROSSING FORMULA

In this section, we first recall the virtual \(\text{K}\)-theoretic localization formula in [39] and review some elementary properties of the residue operation. Then we prove the \(\text{K}\)-theoretic wall-crossing formulas by applying the localization formula to the master space.

### 5.1. **Virtual \(\text{K}\)-theoretic localization formula.** Let \(\mathcal{X}\) be an admissible Deligne-Mumford stack of finite type with a \(\mathbb{C}^*\)-action and a \(\mathbb{C}^*\)-equivariant almost perfect obstruction theory in the sense of Definition 2.10 and Definition 5.11 in [39]. Denote the \(\mathbb{C}^*\)-equivariant weight by \(q\). We have the following virtual \(\text{K}\)-theoretic localization formula.
Theorem 5.1 ([39]). Let $F$ denote the $\mathbb{C}^*$-fixed locus of $\mathcal{X}$ and let $\iota : F \to \mathcal{X}$ be the embedding. Under the assumption that the virtual normal bundle $N^{\text{vir}}$ of $F$ in $\mathcal{X}$ has a global resolution $[N_0 \to N_1]$ of locally free sheaves on $F$, we have

$$\mathcal{O}_\mathcal{X}^{\text{vir}} = \mathcal{O}_F^{\text{vir}} \cdot E \overset{\mathcal{X}_0^{\mathbb{C}^*} - 1}{=\sim} \mathcal{O}_F^{\text{vir}} \otimes_{\mathbb{Q}[q,q^{-1}]} \mathbb{Q}.$$ 

According to [39, Remark 5.2], a $\mathbb{C}^*$-equivariant perfect obstruction theory is a special case of a $\mathbb{C}^*$-equivariant almost perfect obstruction theory. By Proposition 5.13 and Remark 5.14 in [39], a Deligne-Mumford stack with a $\mathbb{C}^*$-action is always admissible after possibly reparameterizing the action of $\mathbb{C}^*$. It is clear from Lemma 4.7 and the discussions in Section 4.1.1 and Section 4.1.2 that the virtual normal bundle of each $\mathbb{C}^*$-fixed-point component in $MQ_{g,n}^a(X, \beta)$ has a global two-term locally-free resolution. Hence the virtual $K$-theoretic localization formula (23) is valid for the $\mathbb{C}^*$-action on the Master space.

5.2. Residue operation and its properties. Let $\mathcal{X}$ be a Deligne-Mumford stack. Consider the vector space $K_0(\mathcal{X}) \otimes \mathbb{Q}(q)$ of rational function in $q$ with coefficients in the $K$-group $K_0(\mathcal{X})_\mathbb{Q} := K_0(\mathcal{X}) \otimes \mathbb{Q}$. We define a residue operation

$$\text{Res}(-) : K_0(\mathcal{X}) \otimes \mathbb{Q}(q) \to K_0(\mathcal{X})_\mathbb{Q}$$

by

$$\text{Res}(f(q)) := \left[\text{Res}_{q=0} + \text{Res}_{q=\infty}\right](f(q)) \frac{dq}{q}.$$ 

Here $\text{Res}_{q=0}(-)dq/q$ and $\text{Res}_{q=\infty}(-)dq/q$ are the obvious extensions of the corresponding residue operations on $\mathbb{Q}(q)$. From now on, we simply refer to $\text{Res}(f(q))$ as the residue of $f(q)$.

We first summarize two elementary properties of the residue operation in the following lemma.

Lemma 5.2. 

(1) For any Laurent polynomial $g(q) \in K_0(\mathcal{X}) \otimes \mathbb{Q}[q,q^{-1}]$, we have

$$\text{Res}(g(q)) = 0.$$ 

(2) Let $r$ be an integer and $L$ be a line bundle on $\mathcal{X}$. For any $f(q) \in K_0(\mathcal{X}) \otimes \mathbb{Q}(q)$, the change of variable $q \mapsto q^rL$ does not change the residue, i.e., we have

$$\text{Res}(f(q)) = \text{Res}(f(q^rL)).$$ 

Proof. By linearity it suffices to assume $g(q) \in \mathbb{Q}[q,q^{-1}]$. Thus the results are standard. □

Corollary 5.3. In the situation of Theorem 5.1, for any proper morphism $q : \mathcal{X} \to \mathcal{Y}$ between Deligne-Mumford stacks such that $q$ is $\mathbb{C}^*$-equivariant with respect to the trivial $\mathbb{C}^*$-action on $\mathcal{Y}$, we have

$$\text{Res}\left(q_*(\mathcal{F}, \frac{[\mathcal{O}_F^{\text{vir}} : \mathcal{E}]}{\lambda_1^{\mathbb{C}^*}((N^{\text{vir}})^\vee)})\right) = 0,$$

for any $\mathcal{E} \in K_0^*(\mathcal{X})_\mathbb{Q}$. 

Proof. Consider the trivial \(\mathbb{C}^*\)-action on \(\mathcal{Y}\). Then \(q\) is \(\mathbb{C}^*\)-equivariant. We apply \(q_*\) to both sides of (23). Then the corollary follows from the observation that \(q_*(\mathcal{X}, [\mathcal{O}_{\mathcal{X}}]|_\cdot \mathcal{E}) \in K^\mathbb{C}^*_s(\mathcal{Y}) \otimes \mathbb{Q}[q, q^{-1}]\).

An important example for us is the residue of the rational function of the form

\[
f(q) = \frac{g(q)}{1 - q^{-1}L},
\]

where \(g(q) \in K_0(\mathcal{X}) \otimes \mathbb{Q}(q)\) and \(L\) is a line bundle on \(\mathcal{X}\). A convenient way to compute the residue

\[
\text{Res}
\left(\frac{g(q)}{1 - q^{-1}L}\right)
\]

is by using the Laurent expansions of \(g(q)/(1 - q^{-1}L)\) at 0 and \(\infty\). More precisely, we have

\[
\frac{g(q)}{1 - q^{-1}L} = \frac{g(q)qL^{-1}}{1 - qL^{-1}}
\]

in \(K_0(\mathcal{X}) \otimes \mathbb{Q}(q)\). The right hand side of the above equation has the following formal Laurent series expansion around 0:

\[
(24) \quad - \frac{g(q)}{1} \sum_{i=1}^{\infty} q^i L^{-i} \in K_0(\mathcal{X})((q)).
\]

It follows that

\[
\text{Res}_{q=0} \left( \frac{g(q)}{1 - q^{-1}L} \right) \frac{dq}{q} = - \sum_{i=1}^{\infty} [g(q)]_{-i} L^{-i},
\]

where \([g(q)]_{-i}\) denotes the coefficient of \(q^{-i}\) in the formal Laurent series expansion of \(g(q)\) at \(q = 0\). Note that \([g(q)]_{-i} \neq 0\) only for finitely many \(i > 0\). To compute the residue at \(\infty\), we make the change of variable \(w = 1/q\). By using similar arguments, we can show that

\[
\text{Res}_{q=\infty} \left( \frac{g(q)}{1 - q^{-1}L} \right) \frac{dq}{q} = - \text{Res}_{w=0} \left( \frac{g(1/w)}{1 - w L} \right) \frac{dw}{w} = - \sum_{i=0}^{\infty} [g(1/q)]_{-i} L^i
\]

We summarize the above discussion in the following lemma.

**Lemma 5.4.** Let \(g(q) \in K_0(\mathcal{X}) \otimes \mathbb{Q}(q)\) and let \(L\) be a line bundle on \(\mathcal{X}\). Then

\[
\text{Res} \left( \frac{g(q)}{1 - q^{-1}L} \right) = - \sum_{i=1}^{\infty} [g(q)]_{-i} L^{-i} - \sum_{i=0}^{\infty} [g(1/q)]_{-i} L^i.
\]

**Corollary 5.5.** Let \(g \in K_0(\mathcal{X})_{\mathbb{Q}}\) and let \(L\) be a line bundle on \(\mathcal{X}\). We have

\[
\text{Res} \left( \frac{g}{1 - q^{-1}L} \right) = -g
\]

and

\[
\text{Res} \left( \frac{g}{1 - qL} \right) = g.
\]
For later computations, we need one more elementary identity of the residue operation. Consider a rational function \( h(q) \in K_\circ(X) \otimes \mathbb{Q}(q) \). We write \( h(q) \) as a unique sum of a Laurent polynomial \([h(q)]_+\) and a proper rational function \([h(q)]_-\). The notation is consistent with those introduced in Section 2.3.

**Lemma 5.6.** For any \( h(q) \in K_\circ(X) \otimes \mathbb{Q}(q) \), we have
\[
\sum_{i=0}^\infty q^i \text{Res}(q^{-i}h(q)) = [h(q)]_-.
\]

**Proof.** By the linearity of the residue operation, we have
\[
\sum_{i=0}^\infty q^i \text{Res}(q^{-i}h(q)) = \sum_{i=0}^\infty q^i \text{Res}(q^{-i}[h(q)]_+) + \sum_{i=0}^\infty q^i \text{Res}(q^{-i}[h(q)]_-).
\]
The first term of the right hand side is zero because \( q^{-i}[h(q)]_+ \) is a Laurent polynomial. Therefore, we only need to prove the second term on the right hand side is equal to \([h(q)]_-\).

We first analyze the residues at \( \infty \). Note that \( q^{-i}[h(q)]_- \) vanishes at \( \infty \) for \( i \geq 0 \). Therefore we have
\[
\text{Res}_{q=\infty}(q^{-i}[h(q)]_-) \frac{dq}{q} = 0.
\]

Now we compute the residues at 0. By definition, \([h(q)]_-\) can be expanded into a formal power series around 0
\[
[h(q)]_- = \sum_{j=0}^\infty a_j q^j, \quad \text{where } a_j \in K_\circ(X)_\mathbb{Q}.
\]
The lemma follows from the following identity:
\[
\text{Res}_{q=0}(q^{-i}(\sum_{j=0}^\infty a_j q^j)) \frac{dq}{q} = a_i, \quad i \geq 0.
\]
\[
\square
\]

5.3. **The** \( g = 0, n = 1 \) **and** \( d = d_0 \) **case.** The statements of the results and their proofs in the rest of the section apply to both the untwisted and the twisted theories. To simplify the exposition, we will drop \( E^{(\bullet)}, R \) and \( l \) from the notation of the twisted theory.

Recall that \( r \) denotes the locally constant function on \( \bar{IX} \) that takes value \( r \) on the component \( \bar{I}_rX \). Set \( r_1 = \mathcal{E}_{x_1}(r) \). Recall that \( \tilde{L}_1 \) is the orbifold cotangent line bundle at the unique marking and \( \tilde{E}_{x_1} : Q^\bullet_{0,1}(X, \beta) \to \bar{IX} \) is the rigidified evaluation map at the unique marking \( x_1 \) composed with the involution on \( \bar{IX} \).

**Lemma 5.7.**
\[
(\tilde{E}_{x_1})_*(\mathcal{O}^\text{vir}_{Q^\bullet_{0,1}(X, \beta)} \cdot \tilde{L}_1^\ell) = \text{Res}(q^{-\ell}(1 - q)I_\beta(q)), \quad \ell \in \mathbb{Z}.
\]
The same identity holds if we replace \( \mathcal{O}^\text{vir}_{Q^\bullet_{0,1}(X, \beta)} \) and \( I_\beta(q) \) by their twisted counterparts.
Proof. Let \( \tilde{\text{ev}}_1 : MQ^0_{0,1}(X, \beta) \to \bar{I}X \) be the rigidified evaluation map at the unique marking \( x_1 \) composed with the involution on \( \bar{I}X \). We apply Corollary 5.3 to the master space \( MQ^0_{0,1}(X, \beta) \) with the \( \mathbb{C}^* \)-action (16) and \( \tilde{\text{ev}}_1 \). The fixed locus \( F \) is the disjoint union of \( F_+ \) and \( F_\beta \), which are studied in Section 4.1.1 and Corollary 4.4, respectively. Thus we obtain

\[
\text{Res} \left( (\tilde{\text{ev}}_1)_* \left( \frac{\mathcal{O}^\text{vir}_{Q^+_{0,1}(X, \beta)}}{1 - q^{-1}M^\vee_+} \cdot L^1_1 \right) \right) + \text{Res} \left( q^{-\text{tr}}(1 - q^r)I_\beta(q^r) \right) = 0.
\]

We conclude the proof by applying Corollary 5.5 to the first term and Lemma 5.2, part (2) to the second term. \( \square \)

Let \( L_1 \) denote the coarse cotangent line bundle at the unique marking of \( Q^+_{0,1}(X, \beta) \). We have the relation \( L_1 = \tilde{L}^r_1 \).

**Corollary 5.8.** We have

\[
(\tilde{\text{ev}}_1)_* \left( Q^+_{0,1}(X, \beta), \frac{\mathcal{O}^\text{vir}_{Q^+_{0,1}(X, \beta)}}{1 - qL_1} \right) = [(1 - q)I_\beta(q)]_-
\]

The same identity holds if we replace \( \mathcal{O}^\text{vir}_{Q^+_{0,1}(X, \beta)} \) and \( I_\beta(q) \) by their twisted counterparts.

**Proof.** Consider the formal geometric series expansion:

\[
\frac{1}{1 - qL_1} = \sum_{i=0}^{\infty} q^i L^i_1.
\]

By Lemma 5.7, Lemma 5.2, and Lemma 5.6, we compute

\[
(\tilde{\text{ev}}_1)_* \left( Q^+_{0,1}(X, \beta), \frac{\mathcal{O}^\text{vir}_{Q^+_{0,1}(X, \beta)}}{1 - qL_1} \right) = \sum_{i=0}^{\infty} q^i (\tilde{\text{ev}}_1)_* \left( Q^+_{0,1}(X, \beta), \frac{\mathcal{O}^\text{vir}_{Q^+_{0,1}(X, \beta)}}{1 - qL_1} \cdot L^1_1 \right)
\]

\[
= \sum_{i=0}^{\infty} q^i \cdot \text{Res}(q^{-i}(1 - q)I_\beta(q))
\]

\[
= [(1 - q)I_\beta(q)]_-. \]

\( \square \)

4. The main case. We study the case \( 2g - 2 + n + d \epsilon_0 > 0 \). As in the previous subsection, we use \( \mathcal{O}^\text{vir} \) to denote both the untwisted and twisted virtual structure sheaves. We will mention the needed modifications and facts for the twisted case in the proofs.
By the $K$-theoretic virtual localization formula (23), we have

$$
\mathcal{O}_{\nu Q_{g,n}^0(X,\beta)}^{\nu} = (t_{F_+})_* \left( \frac{\mathcal{O}^{\nu}_{\nu Q_{g,n}^0(X,\beta)}}{\lambda_{1}^{\nu}\left( N^{\nu}_{F_+/MQ_{g,n}^0(X,\beta)} \right)} \right) + (t_{F_-})_* \left( \frac{\mathcal{O}^{\nu}_{\nu Q_{g,n}^0(X,\beta)}}{\lambda_{1}^{\nu}\left( N^{\nu}_{F_-/MQ_{g,n}^0(X,\beta)} \right)} \right)
$$

(25)

$$
+ \sum_{i=1}^{m} \sum_{\beta} (t_{F_{\beta}})_* \left( \frac{\mathcal{O}_{F_{\beta}}^{\nu}}{\lambda_{1}^{\nu}\left( N^{\nu}_{F_{\beta}/MQ_{g,n}^0(X,\beta)} \right)} \right),
$$

where $m := [d/d_0]$, the sum is over all $\beta$ satisfying Condition 4.5, and $t_{F_+}$, $t_{F_-}$ and $t_{F_{\beta}}$ are the embeddings of the corresponding fixed-point components into $MQ_{g,n}^0(X,\beta)$.

We define the morphism

$$
\tau_0 : MQ_{g,n}^0(X,\beta) \to Q_{g,n}^{\nu}(\mathbb{P}^N, d)
$$

by

- composing the quasimaps with (1),
- taking the coarse moduli of the domain curves,
- taking the $\epsilon_-$-stabilization of the obtained quasimaps to $\mathbb{P}^N$.

Let $ev : MQ_{g,n}^0(X,\beta) \to (IX)^n$ be the product of the rigidified evaluation maps at the $n$ markings. Consider the trivial $\mathbb{C}^*$-action on $Q_{g,n}^{\nu}(\mathbb{P}^N, d) \times (IX)^n$. Then $\tau_0 \times ev$ is a $\mathbb{C}^*$-equivariant proper morphism. Define the map

$$
\tau : \tilde{Q}_{g,n}^{\nu}(X,\beta) \to Q_{g,n}^{\nu}(\mathbb{P}^N, d)
$$

(26)

as the restriction of $\tau_0$ to $F_+ \cong \tilde{Q}_{g,n}^{\nu}(X,\beta)$. Note that the restriction of $\tau_0$ to $F_- \cong Q_{g,n}^{\nu}(X,\beta)$ is the map $\iota$ defined in the introduction. In general, suppose $M$ is a quasimap space with a $\mathbb{C}^*$-equivariant proper morphism $\tau : M \to Q_{g,n}^{\nu}(\mathbb{P}^N, d)$ and a product of evaluation morphisms at certain markings $ev : M \to (IX)^n$ that are both obvious from the context. To simplify the notation, we write

$$
\chi(M, E \cdot F) := (\tau \times ev)_*(E \cdot F), \text{ for } E \in K_{C^*}^Q(M)_Q, F \in K_{C^*}^Q(M)_Q.
$$

As mentioned in Section 4.1.1 and Section 4.1.2, the virtual normal bundles of $F_+$ and $F_-$ in the Master space are given by $\mathbb{M}_+$ with weight $1$ and $\mathbb{M}_-$ with weight $-1$, respectively. Hence, we have

$$
\lambda_{1}^{\nu}\left( N^{\nu}_{F_+/MQ_{g,n}^0(X,\beta)} \right) = 1 - q^{-1}\mathbb{M}_+^{\nu} \text{ and } \lambda_{1}^{\nu}\left( N^{\nu}_{F_-/MQ_{g,n}^0(X,\beta)} \right) = 1 - q^{\mathbb{M}_-}.
$$

Consider the proper pushforward of the relation (25) along $\tilde{\tau}$. By Corollary 5.3, we have

$$
0 = \text{Res} \left( \chi(\tilde{Q}_{g,n}^{\nu}(X,\beta), \frac{\mathcal{O}_{\nu Q_{g,n}^0(X,\beta)}}{1 - q^{-1}\mathbb{M}_+^{\nu}}) \right) + \text{Res} \left( \chi(Q_{g,n}^{\nu}(X,\beta), \frac{\mathcal{O}_{\nu Q_{g,n}^0(X,\beta)}}{1 - q^{\mathbb{M}_-}}) \right)
$$

$$
+ \sum_{i=1}^{m} \sum_{\beta} \text{Res} \left( \chi(F_{\beta}, \frac{\mathcal{O}_{F_{\beta}}^{\nu}}{\lambda_{1}^{\nu}\left( N^{\nu}_{F_{\beta}/MQ_{g,n}^0(X,\beta)} \right)} \right).$$
Note that the residue operation commutes with proper pushforward. Using Corollary 5.5 and Corollary 5.2 (2), we obtain

\begin{equation}
0 = -\chi \left( \tilde{Q}^+_{g,n}(X, \beta), \mathcal{O}^{\text{vir}}_{Q^+_{g,n}(X, \beta)} \right) + \chi \left( Q^-_{g,n}(X, \beta), \mathcal{O}^{\text{vir}}_{Q^-_{g,n}(X, \beta)} \right) \\
+ \sum_{\sigma=1}^{m} \sum_{\mu} \chi \left( F_{\sigma}, \text{Res} \left( \frac{\mathcal{O}^{\text{vir}}_{F_{\sigma}}}{\lambda_{\sigma-1}^{-1}(\mathcal{N}^{\text{vir,v}}_{F_{\sigma}/MQ_{g,n}^0(X, \beta)})} \right) \right)
\end{equation}

By Lemma 3.12, the first term on the right side is equal to \(-\chi(Q^+_{g,n}(X, \beta), \mathcal{O}^{\text{vir}}_{Q^+_{g,n}(X, \beta)})\).

Now let us compute the residues over \(F_{\sigma}\). By the construction of the gluing morphism (13), it descends to an étale morphism of degree \(1/\prod_{i=n+1}^{n+k} r_i\)

\[ \left( \tilde{\mathcal{M}}_{g,n+k,d-kd_0} \times' \left( \mathcal{M}^{\text{wt,ss}}_{0,1,d_0} \right)^k / S_k \right) \rightarrow \tilde{3}(k), \]

where the symmetric group \(S_k\) permutes the last \(k\) factors. In fact, this morphism is the fiber product of the universal gerbes at the \(k\) nodes of the universal curve of \(\tilde{3}(k)\). Recall from Section 4.2.2 that we have the following fibered diagram

\[
\begin{array}{ccc}
\tilde{\mathcal{M}}_{g,n+k,d-kd_0} \times' \left( \mathcal{M}^{\text{wt,ss}}_{0,1,d_0} \right)^k / S_k & \rightarrow & \tilde{3}(k) \\
\downarrow \tilde{g}_{\beta} & & \\
F_{\beta} & \rightarrow & \tilde{3}(k)
\end{array}
\]

where \(S_k\) acts on \(\tilde{g}_{\beta} F_{\beta} / S_k\) by permuting the \(k\) ordered entangled tails. It follows that the morphism \(\tilde{g}_{\beta}\) is the fiber products of the universal gerbes at the \(k\) nodes of the universal curve of \(\tilde{3}(k)\). In particular, we have

\[ (\tilde{g}_{\beta})_* \mathcal{O}_{[\tilde{g}_{k} F_{\beta} / S_k]} = \mathcal{O}_{F_{\beta}} \]

and

\[ (\tilde{g}_{\beta})^* \mathcal{O}^{\text{vir}}_{F_{\beta}} = \mathcal{O}^{\text{vir}}_{[\tilde{g}_{k} F_{\beta} / S_k]}. \]

The same pullback relation holds for twisted virtual structure sheaves with level structure. Hence by the projection formula, we obtain

\begin{equation}
\chi \left( F_{\beta}, \frac{\mathcal{O}^{\text{vir}}_{F_{\beta}}}{\lambda_{\sigma-1}^{-1}(\mathcal{N}^{\text{vir,v}}_{F_{\beta}/MQ_{g,n}^0(X, \beta)})} \right) = \chi \left( \tilde{g}_{k} F_{\beta} / S_k, \frac{\mathcal{O}^{\text{vir}}_{[\tilde{g}_{k} F_{\beta} / S_k]}}{\lambda_{\sigma-1}^{-1}(\mathcal{N}^{\text{vir,v}}_{F_{\beta}/MQ_{g,n}^0(X, \beta)})} \right).
\end{equation}

Let \(\tilde{\beta}\) be an ordered tuple whose underlying multiset is \(\tilde{3}(k)\). Note that

\[ [\tilde{g}_{k} F_{\beta} / S_k] = [\tilde{g}_{k} F_{\tilde{\beta}} / S_{\tilde{\beta}}]. \]
where $S_{\vec{\beta}}$ is the stabilizer subgroup of $S_k$ that fixes $\vec{\beta}$ under the action (3). By Lemma 4.7, the residue of the right hand side of (28) equals

\[
\chi\left(\left[\prod_{i=1}^{k} F_{*,\beta_i}/S_{\vec{\beta}}\right], \varphi^* \text{Res} \left( \frac{\prod_{i=1}^{k} (1 - q^{r_i} L(\mathcal{E}_i)^{\nu})}{1 - q^{r_i} L(\mathcal{E}_i) \cdot L_{n+1} \cdot \mathcal{O}(\sum_{i=0}^{m-k-1} \mathcal{D}_i')} \right) \cdot \mathcal{O}_{Y_{\nu,\beta_i}}^{\text{vir}} \right)
\]

Recall that we have a morphism

\[
\varphi : \left[\prod_{i=1}^{k} r_i \right] \rightarrow \left[ Y \times (IX)^k \prod_{i=1}^{k} F_{*,\beta_i}/S_{\vec{\beta}} \right]
\]

which is étale of degree $\prod_{i} r_i$. Using the projection formula and Lemma 4.7 again, we rewrite (29) as

\[
\chi\left(\left[\prod_{i=1}^{k} F_{*,\beta_i}/S_{\vec{\beta}}\right], \text{Res} \left( \frac{\prod_{i=1}^{k} \theta^{r_i}(L(\mathcal{E}_i)) \cdot (1 - q^{r_i} L(\mathcal{E}_i)^{\nu})}{1 - q^{r_i} L(\mathcal{E}_i) \cdot L_{n+1} \cdot \mathcal{O}(\sum_{i=0}^{m-k-1} \mathcal{D}_i')} \right) \cdot \left( \mathcal{O}_{Y_{\nu,\beta_i}}^{\text{vir}} \otimes \prod_{i=1}^{k} \mathcal{O}_{F_{*,\beta_i}}^{\text{vir}} \right) \right).
\]

Here $\chi$ denotes the proper pushforward along the morphism $(b_k \circ \tau \circ p) \times \text{ev}$, where

- $\tau : [\tilde{Q}_{g,n+k}^+(X, \beta')]/S_{\vec{\beta}} \rightarrow [Q_{g,n+k}^-(\mathbb{P}^N, d - kd_0)/S_k]$ is defined similarly to (26),
- $b_k : [Q_{g,n+k}^-(\mathbb{P}^N, d - kd_0)/S_k] \rightarrow Q_{g,n+k}^+(\mathbb{P}^N, d)$ is the map that replaces the last $k$ markings by base points of length $d_0$,
- $\text{ev} : [(Y \times (IX)^k \prod_{i=1}^{k} F_{*,\beta_i})/S_{\vec{\beta}}] \rightarrow (IX)^n$ is the product of the rigidified evaluation maps at the first $n$ markings.

By Remark 4.8, the formula (30) is also valid in the twisted case if we replace $\mathcal{I}_{\beta_i}$, $\mathcal{O}_{Y_{\nu,\beta}}^{\text{vir}}$ and $\mathcal{O}_{F_{*,\beta_i}}^{\text{vir}}$ by their twisted counterparts.

According to (2) of Lemma 5.2, after the change of variable

\[
q \mapsto q^{k/\ell} L(\mathcal{E}_i)^k \otimes L_{n+1}^k = \cdots = q^{k/\ell} L(\mathcal{E}_i)^k \otimes L_{n+k}^k
\]

(30) becomes

\[
\chi\left(\left[\prod_{i=1}^{k} F_{*,\beta_i}/S_{\vec{\beta}}\right], \text{Res} \left( \frac{\prod_{i=1}^{k} \theta^{r_i}(L(\mathcal{E}_i)) \cdot (1 - q^{r_i} L_{n+i})}{1 - q^{-1} \mathcal{O}(\sum_{i=0}^{m-k-1} \mathcal{D}_i')} \right) \cdot \left( \mathcal{O}_{Y_{\nu,\beta_i}}^{\text{vir}} \otimes \prod_{i=1}^{k} \mathcal{O}_{F_{*,\beta_i}}^{\text{vir}} \right) \right).
\]

Consider the natural morphism

\[
p_k : \left[\prod_{i=1}^{k} F_{*,\beta_i}/S_{\vec{\beta}}\right] \rightarrow \left[ Y \times (IX)^k \prod_{i=1}^{k} F_{*,\beta_i}/S_{\vec{\beta}} \right],
\]
where the target is the fiber product of \( \text{ev}_Y \) and \( \tilde{\text{ev}}_{\tau, \beta_i} \) over \((\bar{I}X)^k\). Note that \( p_k \) is a gerbe whose elements in the automorphism groups act on \( L(\mathcal{E}_i) \) as \( \mathbf{r}_i \)-th roots of unity. It follows that \((p_k)_*(\prod_{i=1}^k \tilde{L}^{n_i}(\mathcal{E}_i)) = 0\) unless \( a_i \) is divisible by \( \mathbf{r}_i \) for all \( i \). Hence

\[
(p_k)_*\left(\prod_{i=1}^k \theta^{r_i}(\tilde{L}(\mathcal{E}_i))\right) = 1.
\]

By applying the projection formula to \( p_k \), we rewrite (31) as

\[
\chi\left(\left[\left(Y \times (\bar{I}X)^k \prod_{i=1}^k F_{*, \beta_i}\right)/S_\beta\right]\right), \text{Res}\left(\prod_{i=1}^k \frac{(1 - q^{r_i}L_{n+i})I_{\beta_i}(q^{r_i}L_{n+i})}{1 - q^{-1}O(\sum_{i=0}^{m-k-1} \mathcal{D}_i)}\right), \left(O_{Y,S_\beta} \boxtimes \prod_{i=1}^k O_{F_{*, \beta_i}}\right).
\]

Recall that we have morphisms

\[
\left[Y \times (\bar{I}X)^k \prod_{i=1}^k F_{*, \beta_i}\right]/S_\beta \xrightarrow{p_Y} [Y/S_\beta]_\beta \xrightarrow{\Sigma_{g,n+k}(Y, \beta')} [\bar{Q}_{g,n+k}^\tau(X, \beta')]/S_\beta.
\]

By taking the pushforward along \( p_Y \), we obtain

\[
\chi\left([Y/S_\beta]\right), \text{Res}\left(\prod_{i=1}^k \frac{\text{ev}_{n+i}^*(1 - q^{r_i}L_{n+i})I_{\beta_i}(q^{r_i}L_{n+i})}{1 - q^{-1}O(\sum_{i=0}^{m-k-1} \mathcal{D}_i)}\right), \left(O_{[Y/S_\beta]}\right).
\]

Here we have extended the definition of \( \text{ev}_{n+i}^* \) by linearity: we first compute \( (1 - q^{r_i}L_{n+i})I_{\beta_i}(q^{r_i}L_{n+i}) \) by viewing \( L_{n+i} \) as a formal variable, and then we agree that

\[
\text{ev}_{n+i}(q^a L_{n+i}^b) := q^a L_{n+i}^b \text{ev}_{n+i}(a), \quad a, b \in \mathbb{Z}, \alpha \in K(\bar{I}X) \otimes \Lambda.
\]

The above equality follows from the elementary fact that given two finite groups \( H \subset G \) and a \( H \)-module \( V \), we have \( \text{Ind}_H^G V = V^H \). Here \( \text{Ind}_H^G V \) denotes the induced \( G \)-module.

Note that \( Y \rightarrow \bar{Q}_{g,n+k}^\tau(X, \beta') \) is a gerbe banded by \( \mu_k \). Hence we have

\[
p_* O_{[Y/S_\beta]} = O_{[\bar{Q}_{g,n+k}^\tau(X, \beta')/S_\beta]}.
\]

By applying the projection formula again, we conclude that the correction terms in the wall-crossing formula (27) equal

(32)

\[
\sum_{k=1}^m \sum_{\beta} \chi\left(\left[\bar{Q}_{g,n+k}^\tau(X, \beta')/S_\beta\right]\right), \text{Res}\left(\prod_{i=1}^k \frac{\text{ev}_{n+i}^*(1 - q^{r_i}L_{n+i})I_{\beta_i}(q^{r_i}L_{n+i})}{1 - q^{-1}O(\sum_{i=0}^{m-k-1} \mathcal{D}_i)}\right), \left(O_{[\bar{Q}_{g,n+k}^\tau(X, \beta')/S_\beta]}\right).
\]

Here we abuse the notation and write the summation sign \( \sum_{\beta} \) outside of the proper push-forward. By our convention, \( \chi \) denote the proper pushforward along the map \((b_k \circ \tau) \times \text{ev} \). The formula (32) is also valid in the twisted case if we replace \( I_{\beta_i} \) and \( O_{[\bar{Q}_{g,n+k}^\tau(X, \beta')/S_\beta]} \) by their twisted counterparts.
In the rest of the section, we will simplify (32) and prove the following:

**Proposition 5.9.** The wall-crossing contribution (32) is equal to

$$\sum_{k=1}^{m} \chi \left( [Q_{g,n+k}^+(X,\beta')/S_k], \prod_{i=1}^{k} \ev_{n+i}^*(L_{n+i}) \cdot O^\vir_{[Q_{g,n+k}^+(X,\beta')/S_k]} \right)$$

in $K_0((Q_{g,n+k}^+(\mathbb{P}^d, d) \times (IX)^n)/S_n)Q$.

To simplify the exposition, we introduce some notation here. For any rational function $f(q)$ in $q$ with coefficients in some $K$-group, we denote by $(f)_0$ the formal Laurent series expansion of $f(q)$ at $q = 0$ and by $(f)_\infty$ the formal Laurent series expansion of $f(1/q)$ at $q = 0$. Recall that the notation $[g(q)]_s$ denotes the coefficient of $q^s$ of a formal Laurent series $g(q)$ at $q = 0$. Consider the permutation action of $S_k$ of $(IX)^k$. Let $S$ be a subgroup of $S_k$. For $G \subset K_0((Q_{g,n+k}^+(X,\beta')/S)])$, we denote

$$\langle G \rangle^S := \chi \left( [Q_{g,n+k}^+(X,\beta')/S], G \cdot O^\vir_{[Q_{g,n+k}^+(X,\beta')/S]} \right).$$

When using the bracket notation, we omit the moduli space and the pullback via evaluation maps. Recall that for a partition $\vec{\beta} = (\beta', \beta_1, \ldots, \beta_k)$, we also denote $\beta_i$ by $\vec{\beta}_{(i)}$. This notation is introduced to avoid confusion when evaluation maps are omitted in the bracket notation. In the discussion below, a typical example of $F$ is of the form $\sum_{\vec{\beta}} \prod_i C_{\vec{\beta}_{(i)}}$, where $C_{\vec{\beta}_{(i)}}$ are certain classes depending on $I_{\vec{\beta}_{(i)}}$. We denote

$$\prod_{i=1}^{k} f_{\vec{\beta}_{(i)}} := \left( \prod_{i=1}^{k} \ev_{n+i}^* \right) \left( 1 - q^rL_{n+i} \right) I_{\vec{\beta}_{(i)}} \left( q^rL_{n+i} \right).$$

As the first step in simplifying (32), we evaluate the residues within and prove the following:

**Proposition 5.10.** We have

$$\sum_{k=1}^{m} \sum_{\vec{\beta}} \left\langle \left( \prod_{i=1}^{k} f_{\vec{\beta}_{(i)}} \right) \frac{\prod_{i=1}^{k} f_{\vec{\beta}_{(i)}}}{1 - q^{-1}O(\sum_{i=0}^{m-k-1} D_i')} \right\rangle_k^{S_k}$$

$$= - \sum_{k=1}^{m} \sum_{\vec{\beta}} \sum_{s \geq 1} \sum_{i=1}^{k} \left( \prod_{i=1}^{k} (f_{\vec{\beta}_{(i)}})_0 \right) \left( \prod_{i=1}^{k} (f_{\vec{\beta}_{(i)}})_\infty \right) \left( \prod_{i=1}^{k} (f_{\vec{\beta}_{(i)}})_0 \right) \left( \prod_{i=1}^{k} (f_{\vec{\beta}_{(i)}})_\infty \right)$$

$$\sum_{k=2}^{m} \sum_{r=1}^{k-1} \sum_{\vec{\beta}_{(i)}} \sum_{s \geq 1} \sum_{t \geq 0} \sum_{j_1 + \cdots + j_r = t} \sum_{j_u \geq 0} \sum_{i=1}^{k-r} \left( \prod_{i=1}^{k-r} (f_{\vec{\beta}_{(i)}})_0 \right) \left( \prod_{i=1}^{k-r} (f_{\vec{\beta}_{(i)}})_\infty \right) \left( \prod_{i=1}^{k-r} (f_{\vec{\beta}_{(i)}})_0 \right) \left( \prod_{i=1}^{k-r} (f_{\vec{\beta}_{(i)}})_\infty \right)$$

$$+ \sum_{k=2}^{m} \sum_{r=1}^{k-1} \sum_{\vec{\beta}_{(i)}} \sum_{s \geq 1} \sum_{t \geq 0} \sum_{j_1 + \cdots + j_r = t+1} \sum_{j_u > 0} \sum_{i=1}^{k-r} \left( \prod_{i=1}^{k-r} (f_{\vec{\beta}_{(i)}})_0 \right) \left( \prod_{i=1}^{k-r} (f_{\vec{\beta}_{(i)}})_\infty \right) \left( \prod_{i=1}^{k-r} (f_{\vec{\beta}_{(i)}})_0 \right) \left( \prod_{i=1}^{k-r} (f_{\vec{\beta}_{(i)}})_\infty \right).$$
Suppose $\tilde{\beta}' = (\beta''', \beta_1', \ldots, \beta_r')$ is an ordered decomposition of the class $\beta'$ satisfying Condition 4.6. Set $d' := \deg(\beta')$ and $\tilde{\beta}'_{(i)} := \beta'_i$ for $i = 1, \ldots, r$. Consider the diagram

$$
\begin{array}{c}
\tilde{Q}^+_{g,n+k}(X, \beta') \\
\downarrow \tau \times ev \\
\tilde{Q}^+_{g,n+k+r}(X, \beta'') \xrightarrow{(b \circ \tau) \times ev} Q^+_{g,n+k}(\mathbb{P}^N, d') \times (IX)^{n+k}.
\end{array}
$$

As before, $\tau$ is defined similarly to (26), $b$ replaces the last $r$ markings by base points of length $d_0$, and $ev$ is the product of the rigidified evaluation maps at the first $n+k$ markings. To prove Proposition 5.10, we need the following lemma.

**Lemma 5.11.** For $a = 1, 2, \ldots, m - k$ and $s \geq 1$, we have the following identities between $S_{n+k}$-equivariant $K_o$-classes on $Q^+_{g,n+k}(\mathbb{P}^N, d') \times (IX)^{n+k}$:

$$
(\tau \times ev)_*\left(\mathcal{O}\left(-\sum_{i=0}^{a-1} \mathcal{D}'_i\right) \cdot \mathcal{O}^{vir}_{Q^+_{g,n+k}(X, \beta')}\right)
= \sum_{r=0}^{a} \left[ \sum_{\beta'''} \left( (b \circ \tau) \times ev \right)_* \left( \prod_{u=1}^{r} \mathcal{O}^{vir}_{Q^+_{g,n+k+r}(X, \beta''')} \right) \right]_{S_r},
$$

(34)

$$
(\tau \times ev)_*\left(\mathcal{O}\left(-s \sum_{i=0}^{m-k-1} \mathcal{D}'_i\right) \cdot \mathcal{O}^{vir}_{Q^+_{g,n+k}(X, \beta')}\right) - (\tau \times ev)_*\left(\mathcal{O}\left(-s \sum_{i=0}^{m-k-1} \mathcal{D}'_i\right) \cdot \mathcal{O}^{vir}_{Q^+_{g,n+k+r}(X, \beta''')}\right)
= \sum_{r=1}^{m-k} \left[ \sum_{j_1 + \cdots + j_s = r} \left( (b \circ \tau) \times ev \right)_* \left( \prod_{u=1}^{r} \mathcal{O}^{vir}_{Q^+_{g,n+k+r}(X, \beta''')} \right) \right]_{S_r},
$$

(35)

and

$$
(\tau \times ev)_*\left(\mathcal{O}\left(s \sum_{i=0}^{m-k-1} \mathcal{D}'_i\right) \cdot \mathcal{O}^{vir}_{Q^+_{g,n+k+r}(X, \beta''')}\right) - (\tau \times ev)_*\left(\mathcal{O}\left((s - 1) \sum_{i=0}^{m-k-1} \mathcal{D}'_i\right) \cdot \mathcal{O}^{vir}_{Q^+_{g,n+k+r}(X, \beta''')}\right)
= \sum_{r=1}^{m-k} \left[ \sum_{j_1 + \cdots + j_s = r} \left( (b \circ \tau) \times ev \right)_* \left( \prod_{u=1}^{r} \mathcal{O}^{vir}_{Q^+_{g,n+k+r}(X, \beta''')} \right) \right]_{S_r},
$$

(36)

where $[.]_{S_r}$ denotes taking the $S_r$-invariant part. The same formulas hold in the twisted setting if we replace the virtual structure sheaves and $f_{\tilde{\beta}'}_{(u)}$ by their twisted counterparts.

**Proof.** For $r = 0, 1, 2, \ldots$, we define the operator

$$
(\cdot)|_{\mathcal{D}'_{r-1}} : K_o(Q^+_{g,n+k}(X, \beta')) \to K_o(Q^+_{g,n+k}(X, \beta'))
$$

...
as
\[(37)\quad \mathcal{F}|_{\mathcal{D}'_{r-1}} := (\mathcal{g}_{\mathcal{D}'_{r-1}})^* i_{\mathcal{D}'_{r-1}}^* \mathcal{F},\]
where \(\mathcal{g}_{\mathcal{D}'_{r-1}}\) and \(i_{\mathcal{D}'_{r-1}}\) are the obvious arrows in the following fibered diagram

\[
\begin{array}{c}
\tilde{Q}^{e+}_{g,n+k}(X,\beta')|_{\mathcal{D}'_{r-1}} \\
\downarrow \\
\mathcal{D}'_{r-1} \\
\downarrow \\
\tilde{Q}^{e+}_{g,n+k}(X,\beta')
\end{array}
\]

Note that for any \(\alpha \in K^0(\tilde{Q}^{e+}_{g,n+k}(X,\beta'))\), we have
\[(\alpha \cdot \mathcal{F})|_{\mathcal{D}'_{r-1}} = \alpha \cdot (\mathcal{F}|_{\mathcal{D}'_{r-1}}).
\]
Indeed, we have
\[(\alpha \cdot \mathcal{F})|_{\mathcal{D}'_{r-1}} = (\mathcal{g}_{\mathcal{D}'_{r-1}})^* i_{\mathcal{D}'_{r-1}}^* (\alpha \cdot \mathcal{F}) = (\mathcal{g}_{\mathcal{D}'_{r-1}})^* i_{\mathcal{D}'_{r-1}}^* \alpha \cdot i_{\mathcal{D}'_{r-1}}^* \mathcal{F} = \alpha \cdot (\mathcal{F}|_{\mathcal{D}'_{r-1}}).
\]

For any \(\mathcal{F} \in K_0(\tilde{Q}^{e+}_{g,n+k}(X,\beta'))\), we have
\[\mathcal{O}(-\mathcal{D}'_i) \cdot \mathcal{F} = \mathcal{F} - \mathcal{F}|_{\mathcal{D}'_i}.
\]
This follows from the definition of the refined Gysin map (see, for example, [42, §2.1]). By repeatedly using this relation, we have for any \(\mathcal{F} \in K_0(\tilde{Q}^{e+}_{g,n+k}(X,\beta'))\),
\[(38)\quad \mathcal{O}(-\sum_{i=0}^{a-1} \mathcal{D}'_i) \cdot \mathcal{F} = \mathcal{F} - \sum_{r=1}^{a} \mathcal{O}(-\sum_{i=r}^{a-1} \mathcal{D}'_i) \cdot \mathcal{F}|_{\mathcal{D}'_{r-1}}.
\]

We further consider the fibered diagram
\[
\begin{array}{c}
[\tilde{Q}^{e+}_{g,n+k}(X,\beta')|_{\mathcal{g}_{\mathcal{D}'_{r-1}}^* \mathcal{D}'_{r-1}/\mathcal{S}_r}] \\
\downarrow \\
[\tilde{g}_{\mathcal{D}_{r-1}}^* \mathcal{D}_{r-1}/\mathcal{S}_r] \\
\downarrow \\
[\mathcal{D}'_{r-1}]
\end{array}
\]

Note that the horizontal arrows are étale gerbes. Hence replacing \(\mathcal{D}'_r\) by \([\tilde{g}_{\mathcal{D}_{r-1}}^* \mathcal{D}_{r-1}/\mathcal{S}_r]\) in (37) does not change the definition of \(\mathcal{F}|_{\mathcal{D}'_{r-1}}\). In particular \(\mathcal{O}^{\text{vir}}_{\tilde{Q}^{e+}_{g,n+k}(X,\beta')|_{\mathcal{D}'_{r-1}}}\) is equal to the pushforward of \(\mathcal{O}^{\text{vir}}_{\tilde{Q}^{e+}_{g,n+k}(X,\beta')|_{\mathcal{g}_{\mathcal{D}'_{r-1}}^* \mathcal{D}'_{r-1}/\mathcal{S}_r}}\) Recall that
\[p : \tilde{Q}^{e+}_{g,n+k}(X,\beta')|_{\mathcal{g}_{\mathcal{D}'_{r-1}}^* \mathcal{D}'_{r-1}} \to \coprod_{\beta''} \tilde{Q}^{e+}_{g,n+k+r}(X,\beta'') \times_{\mathcal{S}_{r-1}} \prod_{u=1}^{r} Q^{e+}_{0,1}(X,\beta'_u)
\]
is an inflated projective bundle. By Lemma 3.6 (2), the restriction of \(\mathcal{O}(-\sum_{i=0}^{a-1} \mathcal{D}'_i)\) to \(\tilde{Q}^{e+}_{g,n+k}(X,\beta')|_{\mathcal{g}_{\mathcal{D}'_{r-1}}^* \mathcal{D}'_{r-1}}\) is equal to the pullback of \(\mathcal{O}(-\sum_{i=0}^{a-1} \mathcal{D}'_i)\) as \(S_{n+k} \times S_r\)-equivariant
sheaves, where $D_i''$ are the boundary divisors on $\mathcal{M}_{g,n+k+r,d'}$. Note that we have a commutative diagram

$$
\begin{array}{ccc}
\tilde{Q}_{g,n+k}^-(X,\beta')|_{\bar{\mathcal{D}}_{r-1}} & \xrightarrow{\tau \times \text{ev}} & \tilde{Q}_{g,n+k}^-([\mathbb{P}^{N-1}, d']) \times (\tilde{I}X)^{n+k} \\
\downarrow p & & \uparrow (\text{boc} \times \text{ev}) \\
\prod_{\beta'} \tilde{Q}_{g,n+k+r}^-(X,\beta'') \times (\tilde{I}X)^{r} & \xrightarrow{\text{pr}_1} & \prod_{\beta'} \tilde{Q}_{g,n+k+r}^+(X,\beta'')
\end{array}
$$

where the maps are $S_{n+k}$-equivariant and $S_r$-invariant. Hence as $S_{n+k} \times S_r$-equivariant $K_0$-classes on $Q_{g,n+k}^-([\mathbb{P}^{N-1}, d']) \times (\tilde{I}X)^{n+k}$, where $S_r$ acts trivially, we have

$$(\tau \times \text{ev})_* (\mathcal{O}(- \sum_{i=r}^{a-1} D_i') \cdot \mathcal{O}_{\tilde{Q}_{g,n+k}^-([\mathbb{P}^{N-1}, d'])^{n+k}}) = \left(\left( (b \circ \tau) \times \text{ev} \right)_* \left( (\text{pr}_1)_* p_* \left( \mathcal{O}_{\tilde{Q}_{g,n+k}^+(X,\beta')|_{\bar{\mathcal{D}}_{r-1}}} \right) \cdot \mathcal{O}(- \sum_{i=0}^{a-1-r} D_i'') \right) \right)_{S_r}.$$

We denote by $\mathcal{O}_{\tilde{Q}_{g,n+k+r}^-([\mathbb{P}^{N-1}, d'])^{n+k}}$ the structure sheaf of the fiber product. By Lemma 3.13,

$$(b \circ \tau) \times \text{ev}_* (\text{pr}_1)_* p_* \left( \mathcal{O}_{\tilde{Q}_{g,n+k}^+(X,\beta')|_{\bar{\mathcal{D}}_{r-1}}} \right) = (b \circ \tau) \times \text{ev}_* \sum_{\beta'} (\text{pr}_1)_* p_* \left( \mathcal{O}_{\tilde{Q}_{g,n+k}^+(X,\beta')|_{\bar{\mathcal{D}}_{r-1}}} \right) \cdot \mathcal{O}_{\tilde{Q}_{g,n+k+r}^+(X,\beta'')} \otimes (\tilde{I}X)^{k} \prod_{u=1}^{r} \mathcal{O}_{\tilde{Q}_{0,1}^+(X,\beta_u)}$$

Appendix

$$(b \circ \tau) \times \text{ev}_* \sum_{\beta'} (\text{pr}_1)_* \left( \mathcal{O}_{\tilde{Q}_{g,n+k+r}^+(X,\beta'')} \otimes (\tilde{I}X)^{k} \prod_{u=1}^{r} \mathcal{O}_{\tilde{Q}_{0,1}^+(X,\beta_u)} \right)$$

$$(b \circ \tau) \times \text{ev}_* \sum_{\beta'} \left( \prod_{u=1}^{r} \text{ev}_{*n+k+u} (\text{ev}_{1})_* (\mathcal{O}_{\tilde{Q}_{0,1}^+(X,\beta_u)}) \right) \cdot \mathcal{O}_{\tilde{Q}_{g,n+k+r}^+(X,\beta'')}.$$

Here $\text{pr}_1$ denotes the projection of $\prod_{\beta'} \tilde{Q}_{g,n+k+r}^+(X,\beta'') \times (\tilde{I}X)^{r} \prod_{u=1}^{r} \tilde{Q}_{0,1}^+(X,\beta_u)$ onto its first factors.

By Lemma 5.7, we have

$$(\text{ev}_{1})_* (\mathcal{O}_{\tilde{Q}_{0,1}^+(X,\beta_u)}) = \text{Res} \left( (1 + q^r) \bar{I}_{\beta_u} (q^{r}) \right)$$
According to Lemma 5.2, part (2), the above class also equals \( \text{Res}(f_{\tilde{\beta}'(u)}) \) after the change of variable \( q \mapsto qL_{n+k+u} \). Summarizing, we have

\[
(\tau \times \text{ev})_* \left( \mathcal{O}(\sum_{i=0}^{a-1} \mathcal{D}_i^r) \cdot \mathcal{O}_{\tilde{Q}^+_{g,n+k}(X,\beta')}^{\text{vir}} | \mathcal{D}_r^{a-1} \right)
= \left[ \prod_{r=1}^{a} ((b \circ \tau) \times \text{ev})_* \left( \prod_{u=1}^{r} \text{ev}^*_{n+k+u} \text{Res}(f_{\tilde{\beta}'(u)}) \cdot \mathcal{O}_{\tilde{Q}^+_{g,n+k+r}(X,\beta')}^{\text{vir}} \cdot \mathcal{O}(\sum_{i=0}^{a-1-r} \mathcal{D}_i^r) \right) \right] S_r.
\]

Substituting these into (38), we have

\[
(\tau \times \text{ev})_* \left( \mathcal{O}(\sum_{i=0}^{a-1} \mathcal{D}_i^r) \cdot \mathcal{O}_{\tilde{Q}^+_{g,n+k}(X,\beta')}^{\text{vir}} \right)
= (\tau \times \text{ev})_* \left( \mathcal{O}_{\tilde{Q}^+_{g,n+k}(X,\beta')}^{\text{vir}} \right)
- \sum_{r=1}^{a} \left[ \left( (b \circ \tau) \times \text{ev})_* \left( \prod_{u=1}^{r} \text{ev}^*_{n+k+u} \text{Res}(f_{\tilde{\beta}'(u)}) \cdot \mathcal{O}_{\tilde{Q}^+_{g,n+k+r}(X,\beta')}^{\text{vir}} \cdot \mathcal{O}(\sum_{i=0}^{a-1-r} \mathcal{D}_i^r) \right) \right] S_r.
\]

When \( a = 1 \), this is already the desired formula. In general we use induction. For \( a > 1 \), suppose that (34) is true with \( a \) replaced by \( a - r \), \( \beta' \) replaced by \( \beta'' \) and \( n + k \) replaced by \( n + k + r \), with \( r \geq 1 \), then we have

\[
(\tau \times \text{ev})_* \left( \mathcal{O}_{\tilde{Q}^+_{g,n+k+r}(X,\beta'')}^{\text{vir}} \cdot \mathcal{O}(\sum_{i=0}^{a-1-r} \mathcal{D}_i^r) \right)
= \sum_{i=0}^{a-r} \left[ \left( (b \circ \tau) \times \text{ev})_* \left( \prod_{u=1}^{i} \text{ev}^*_{n+k+r+u} (-\text{Res}(f_{\tilde{\beta}''(u)})) \cdot \mathcal{O}_{\tilde{Q}^+_{g,n+k+r+u}(X,\beta'')}^{\text{vir}} \right) \right] S_i
\]

as \( S_{n+k+r} \)-equivariant \( K_0 \)-classes on \( \mathcal{Q}^{r}_{g,n+k+r}(\mathbb{P}^{N-1}, d'') \times (IX)^{n+k+r} \). In particular, the equation holds as \( S_{n+k} \times S_r \)-equivariant \( K_0 \)-classes. Note that the inner sum is over all ordered decompositions \( \beta'' = (\beta''_1, \beta''_2, \ldots, \beta''_i) \) with \( \tilde{\beta}''_i := \beta''_i \) and the maps have new meaning: e.g. ev is the evaluation at the first \( n + k + r \) markings.

Tensoring the relation with \( \prod_{u=1}^{r} \text{ev}^*_{n+k+u} \text{Res}(f_{\tilde{\beta}'(u)}) \) and taking the pushforward along

\[
\mathcal{Q}^{r}_{g,n+k+r}(\mathbb{P}^{N-1}, d'') \times (IX)^{n+k+r} \to \mathcal{Q}^{r}_{g,n+k}(\mathbb{P}^{N-1}, d'') \times (IX)^{n+k},
\]
which replaces the last $r$-markings by length-$d_0$ base points and forgets the last $r$ copies of $IX$, we obtain

\[
\sum_{\bar{\beta'}}((b \circ \tau) \times ev)_s\left(\prod_{u=1}^{r} ev^*_u \kappa_{u+k} \cdot \text{Res}(f_{\bar{\beta'}}^{(u)}) \cdot \mathcal{O}_{Q^{g,n+k+r+1}}(X,\beta') \cdot \mathcal{O}(-\sum_{i=0}^{a-1-r} \mathcal{D}'_i)\right)
\]

This is the second term in (39). Hence we have

\[
(\tau \times ev)_s\left(\mathcal{O}(-\sum_{i=0}^{a-1} \mathcal{D}'_i) \cdot \mathcal{O}_{Q^{g,n+k}}(X,\beta')\right)
\]

\[
= (\tau \times ev)_s\left(\mathcal{O}_{Q^{g,n+k}}(X,\beta')\right)
\]

\[
- \sum_{i=0}^{a} \sum_{\bar{\beta'},\bar{\beta''}}\left[\sum_{i=0}^{r} ((b \circ \tau) \times ev)_s\left(\prod_{u=1}^{r} ev^*_u \kappa_{u+k+r+v} \cdot \text{Res}(f_{\bar{\beta'}}^{(u)}) \cdot \mathcal{O}_{Q^{g,n+k+r+1}}(X,\beta') \cdot \mathcal{O}(-\sum_{i=0}^{a-1-r} \mathcal{D}'_i)\right)\right]
\]

This finishes the proof of (34).

To prove (35), we make similar simplifications as follows:

\[
(\tau \times ev)_s\left(\mathcal{O}(-s) \sum_{i=0}^{m-1} \mathcal{D}'_i \cdot \mathcal{O}_{Q^{g,n+k}}(X,\beta')\right) - (\tau \times ev)_s\left(\mathcal{O}(-s) \sum_{i=0}^{m-1} \mathcal{D}'_i \cdot \mathcal{O}_{Q^{g,n+k}}(X,\beta')\right)
\]

\[
= (\tau \times ev)_s\left(\mathcal{O}(-s) \sum_{i=0}^{m-1} \mathcal{D}'_i \cdot \left(\mathcal{O}(-\sum_{i=0}^{m-1} \mathcal{D}'_i) - \mathcal{O}_{Q^{g,n+k}}(X,\beta')\right)\right)
\]

\[
= - \sum_{r=1}^{a} (\tau \times ev)_s\left(\mathcal{O}(-s) \sum_{i=0}^{m-1} \mathcal{D}'_i \cdot \mathcal{O}_{Q^{g,n+k}}(X,\beta')\right)\big|_{\mathcal{D}'_{r-1}}
\]
By Lemma 3.7, we have
\[
(\tau \times \text{ev})_* \left( (\mathcal{O} \circ (s \sum_{i=0}^{m-k-1} \mathcal{D}'_i) \cdot \mathcal{O} \circ (s \sum_{i=r}^{m-k-1} \mathcal{D}'_i) \cdot \mathcal{O}^\text{vir}_{g, n+k}(X, \beta')) \right) |_{\mathbf{D}'_{r-1}}
\]
\[
= \left[ ((b \circ \tau) \times \text{ev})_* \left( (\text{pr}_1)_* \mathcal{P}_s \left( \mathcal{O}^\text{vir}_{g, n+k}(X, \beta') \right) \cdot \mathcal{O}^\text{vir}_{\text{gl}, \mathbf{D}'_{r-1}} \cdot (s \sum_{j=0}^{r-2} \mathbf{D}_j) \cdot \mathcal{O}(- \sum_{i=0}^{a-1-r} \mathcal{D}'_i) \right) \right]_{S_r},
\]
where the divisors $\mathbf{D}_j, j = 0, \ldots, r - 2$ are the tautological divisors of the inflated projective bundle $p$. It follows from Lemma 3.13, Lemma A.1 and Lemma 5.7 that
\[
(\text{pr}_1)_* \mathcal{P}_s \left( \mathcal{O}^\text{vir}_{g, n+k}(X, \beta') \right) \cdot \mathcal{O}^\text{vir}_{\text{gl}, \mathbf{D}'_{r-1}} \cdot (s \sum_{j=0}^{r-2} \mathbf{D}_j) \cdot \mathcal{O}(- \sum_{i=0}^{a-1-r} \mathcal{D}'_i)
\]
\[
= - \sum_{\beta'} \sum_{j_1 + \cdots + j_r = s} \prod_{j_u > 0} \mathcal{P}_s \left( \mathcal{O}^\text{vir}_{g, n+k+r}(X, \beta') \right) \cdot \mathcal{O}(- \sum_{i=0}^{a-1-r} \mathcal{D}'_i)
\]
To summarize, we have
\[
(\tau \times \text{ev})_* \left( (\mathcal{O} \circ (s \sum_{i=0}^{m-k-1} \mathcal{D}'_i) \cdot \mathcal{O} \circ (s \sum_{i=r}^{m-k-1} \mathcal{D}'_i) \cdot \mathcal{O}^\text{vir}_{g, n+k}(X, \beta')) \right) - (\tau \times \text{ev})_* \left( (\mathcal{O} \circ (s \sum_{i=0}^{m-k-1} \mathcal{D}'_i) \cdot \mathcal{O} \circ (s \sum_{i=r}^{m-k-1} \mathcal{D}'_i) \cdot \mathcal{O}^\text{vir}_{g, n+k}(X, \beta')) \right)
\]
\[
= \sum_{r=1}^{m-k} \sum_{\beta'} \sum_{j_1 + \cdots + j_r = s} ((b \circ \tau) \times \text{ev})_* \left( \prod_{u=1}^{r} \mathcal{P}_s \left( \mathcal{O}^\text{vir}_{g, n+k+r}(X, \beta') \right) \cdot \mathcal{O}(- \sum_{i=0}^{a-1-r} \mathcal{D}'_i) \right) \right]_{S_r \times S_i}
\]
\[
= \sum_{r=1}^{m-k} \sum_{\beta'} \sum_{j_1 + \cdots + j_r = s} ((b \circ \tau) \times \text{ev})_* \left( \prod_{u=1}^{r} \mathcal{P}_s \left( \mathcal{O}^\text{vir}_{g, n+k+r}(X, \beta') \right) \cdot \mathcal{O}(- \sum_{i=0}^{a-1-r} \mathcal{D}'_i) \right) \right]_{S_r \times S_i}
\]
The identity (36) can be proved similarly by using the relation
\[
\mathcal{O}(s \sum_{i=0}^{m-k-1} \mathcal{D}'_i) \cdot \mathcal{F} = \mathcal{F} + \sum_{r=0}^{m-k-1} \mathcal{O}(s \sum_{i=r}^{m-k-1} \mathcal{D}'_i) \cdot \mathcal{F}|_{\mathbf{D}'_r}
\]
for any \( F \in K_\ast(\tilde{Q}^+_{g,n+k}(X, \beta')) \). We omit the details. \( \Box \)

**Proof of Proposition 5.10.** We first evaluate the residues at 0. Consider the following formal expansion at \( q = 0 \):

\[
\frac{1}{1 - q^{-1}O(\sum_{i=0}^{m-k-1} \mathcal{D}_i')} = -qO\left(- \sum_{i=0}^{m-k-1} \mathcal{D}_i'\right) - \sum_{s \geq 1} q^{s+1}O\left(- (s + 1) \sum_{i=0}^{m-k-1} \mathcal{D}_i'\right).
\]

Then by using (34) and (35), we have

\[
\sum_{k=1}^{m} \sum_{\vec{\beta}} \left\langle \text{Res}_{q=0} \left( \frac{\prod_{i=1}^{k} f_{\vec{\beta}(i)}}{1 - q^{-1}O(\sum_{i=0}^{m-k-1} \mathcal{D}_i')} \right) \frac{dq}{q} \right\rangle_{k} S_{k} = - \sum_{k=1}^{m} \sum_{\vec{\beta}} \left\langle \left( \prod_{i=1}^{k} (f_{\vec{\beta}(i)})_0 \right)_{-1} \cdot O\left(- \sum_{i=0}^{m-k-1} \mathcal{D}_i'\right) \right\rangle_{k} S_{k}.
\]

\[
= - \sum_{k=1}^{m} \sum_{\vec{\beta}} \left\langle \left( \prod_{i=1}^{k} (f_{\vec{\beta}(i)})_0 \right)_{-s-1} \cdot O\left(- (s + 1) \sum_{i=0}^{m-k-1} \mathcal{D}_i'\right) \right\rangle_{k} S_{k}.
\]

\[
= - \sum_{k=1}^{m} \sum_{\vec{\beta}} \left\langle \left( \prod_{i=1}^{k} (f_{\vec{\beta}(i)})_0 \right)_{-1} \right\rangle_{k} S_{k}.
\]

\[
- \sum_{k=2}^{m} \sum_{\vec{\beta}, \vec{\beta'}} \sum_{t \geq 2} \sum_{r=1}^{k-1} \sum_{j_1, \ldots, j_r = t} \left\langle \left( \prod_{i=1}^{k-r} (f_{\vec{\beta}(i)})_0 \right)_{-t-1} \cdot \prod_{u=1}^{r} \left( - \text{Res}(q^{-j_u f_{\vec{\beta}(u)}}) \right) \right\rangle_{k} S_{k-r} \times S_{r}.
\]

\[
- \sum_{k=1}^{m} \sum_{\vec{\beta}} \sum_{s \geq 1} \left\langle \left( \prod_{i=1}^{k} (f_{\vec{\beta}(i)})_0 \right)_{-s-1} \cdot O\left(- s \sum_{i=0}^{m-k-1} \mathcal{D}_i'\right) \right\rangle_{k} S_{k}.
\]
Hence we obtain a recursive formula:

\[
- \sum_{k=1}^{m} \sum_{\beta} \sum_{s \geq 1} \left( \prod_{i=1}^{k} (f_{\tilde{\beta}(i)})_{0} \right) \cdot O(-s \sum_{i=0}^{m-k-1} D_{i}) S_{k}
\]

\[
- \left( - \sum_{k=1}^{m} \sum_{\beta} \sum_{s \geq 1} \left( \prod_{i=1}^{k} (f_{\tilde{\beta}(i)})_{0} \right) \cdot O(-s \sum_{i=0}^{m-k-1} D_{i}) S_{k} \right)
\]

\[
= - \sum_{k=1}^{m} \sum_{\beta} \left( \prod_{i=1}^{k} (f_{\tilde{\beta}(i)})_{0} \right) S_{k}
\]

\[
- \sum_{k=1}^{m} \sum_{\beta} \sum_{s \geq 1} \sum_{t \geq 0} \sum_{r \geq 1} \left( \prod_{i=1}^{k-r} (f_{\tilde{\beta}(i)})_{0} \right) \prod_{u=1}^{r} \left( - \text{Res}(q^{-u} f_{\tilde{\beta}(u)}) \right) \right) S_{k} \times S_{r}
\]

By induction, we have

\[
\sum_{k=1}^{m} \sum_{\beta} \left( \prod_{i=1}^{k} (f_{\tilde{\beta}(i)})_{0} \right) \cdot O(-s \sum_{i=0}^{m-k-1} D_{i}) S_{k}
\]

\[
= - \sum_{k=1}^{m} \sum_{\beta} \sum_{s \geq 1} \left( \prod_{i=1}^{k} (f_{\tilde{\beta}(i)})_{0} \right) \cdot O(-s \sum_{i=0}^{m-k-1} D_{i}) S_{k}
\]

\[
= - \sum_{k=1}^{m} \sum_{\beta} \left( \prod_{i=1}^{k} (f_{\tilde{\beta}(i)})_{0} \right) S_{k}
\]

\[
- \sum_{k=2}^{m} \sum_{\beta, \beta'} \sum_{s \geq 1} \sum_{t \geq 0} \sum_{r \geq 1} \sum_{j_{1}+\cdots+j_{r}=t} \left( \prod_{i=1}^{k-r} (f_{\tilde{\beta}(i)})_{0} \right) \prod_{u=1}^{r} \left( - \text{Res}(q^{-u} f_{\tilde{\beta}(u)}) \right) \right) S_{k} \times S_{r}
\]

Now we compute the residue at \( q = \infty \). Set \( w = 1/q \) and consider the following formal expansion at \( w = 0 \):

\[
\frac{1}{1 - w O(\sum_{i=0}^{m-k-1} D_{i})} = 1 + \sum_{t \geq 0} w^{t+1} O(t \sum_{i=0}^{m-k-1} D_{i}^{'} \cdot O(\sum_{i=0}^{m-k-1} D_{i}')).
\]
Then by using (36), we have

\[
\sum_{k=1}^{m} \sum_{\beta} \left\langle \text{Res}_{q=\infty} \left( \frac{\prod_{i=1}^{k} f_{\bar{\beta}(i)}}{1 - q^{-1} \mathcal{O}(\sum_{i=0}^{m-k-1} \mathcal{D}'_i)} \right) \frac{dq}{q} \right\rangle_{S_k}^{\beta} = - \sum_{k=1}^{m} \sum_{\beta} \left\langle \text{Res}_{w=0} \left( \frac{\prod_{i=1}^{k} f_{\bar{\beta}(i)}}{1 - q^{-1} \mathcal{O}(\sum_{i=0}^{m-k-1} \mathcal{D}'_i)} \right) \frac{dw}{w} \right\rangle_{S_k}^{\beta}
\]

\[
= - \sum_{k=1}^{m} \sum_{\beta} \left\langle \left[ \prod_{i=1}^{k} (f_{\bar{\beta}(i)})_\infty \right]_0 \right\rangle_{S_k}^{\beta}
\]

\[
= - \sum_{k=1}^{m} \sum_{\beta} \sum_{t \geq 0} \left\langle \left[ \prod_{i=1}^{k} (f_{\bar{\beta}(i)})_\infty \right]^{-t-1} \cdot \mathcal{O}(\sum_{i=0}^{m-k-1} \mathcal{D}'_i) \cdot \mathcal{O}(t \sum_{i=0}^{m-k-1} \mathcal{D}'_i) \right\rangle_{S_k}^{\beta}
\]

\[
= - \sum_{k=1}^{m} \sum_{\beta} \left\langle \left[ \prod_{i=1}^{k} (f_{\bar{\beta}(i)})_\infty \right]^{-t-1} \cdot \mathcal{O}(t \sum_{i=0}^{m-k-1} \mathcal{D}'_i) \right\rangle_{S_k}^{\beta}
\]

Similar to the previous case, we obtain a recursive formula:

\[
- \sum_{k=1}^{m} \sum_{\beta} \sum_{t \geq 0} \left\langle \left[ \prod_{i=1}^{k} (f_{\bar{\beta}(i)})_\infty \right]^{-t} \cdot \mathcal{O}(t \sum_{i=0}^{m-k-1} \mathcal{D}'_i) \right\rangle_{S_k}^{\beta}
\]

\[
- \left( - \sum_{k=1}^{m} \sum_{\beta} \sum_{t \geq 0} \left\langle \left[ \prod_{i=1}^{k} (f_{\bar{\beta}(i)})_\infty \right]^{-t-1} \cdot \mathcal{O}(t \sum_{i=0}^{m-k-1} \mathcal{D}'_i) \right\rangle_{S_k}^{\beta} \right)
\]

\[
= - \sum_{k=1}^{m} \sum_{\beta} \left\langle \left[ \prod_{i=1}^{k} (f_{\bar{\beta}(i)})_\infty \right]_0 \right\rangle_{S_k}^{\beta}
\]

\[
- \sum_{k=1}^{m} \sum_{\beta} \sum_{t \geq 0} \sum_{r=1}^{k-1} \sum_{j_1, \ldots, j_r = t+1} \sum_{j_u > 0} \left\langle \left[ \prod_{i=1}^{k} (f_{\bar{\beta}(i)})_\infty \right]^{-t-1} \cdot \mathcal{O}(t \sum_{i=0}^{m-k-1} \mathcal{D}'_i) \right\rangle_{S_k}^{\beta} \cdot \mathcal{O}(t \sum_{i=0}^{m-k-1} \mathcal{D}'_i) \right\rangle_{S_k}^{\beta}
\]

\[
- \sum_{k=1}^{m} \sum_{\beta} \sum_{t \geq 0} \sum_{r=1}^{k-1} \sum_{j_1, \ldots, j_r = t+1} \sum_{j_u > 0} \left\langle \left[ \prod_{i=1}^{k} (f_{\bar{\beta}(i)})_\infty \right]^{-t-1} \cdot \mathcal{O}(t \sum_{i=0}^{m-k-1} \mathcal{D}'_i) \right\rangle_{S_k}^{\beta} \cdot \mathcal{O}(t \sum_{i=0}^{m-k-1} \mathcal{D}'_i) \right\rangle_{S_k}^{\beta}
\]

\[
- \sum_{k=1}^{m} \sum_{\beta} \sum_{t \geq 0} \sum_{r=1}^{k-1} \sum_{j_1, \ldots, j_r = t+1} \sum_{j_u > 0} \left\langle \left[ \prod_{i=1}^{k} (f_{\bar{\beta}(i)})_\infty \right]^{-t-1} \cdot \mathcal{O}(t \sum_{i=0}^{m-k-1} \mathcal{D}'_i) \right\rangle_{S_k}^{\beta} \cdot \mathcal{O}(t \sum_{i=0}^{m-k-1} \mathcal{D}'_i) \right\rangle_{S_k}^{\beta}
\]
By induction, we have
\[
\sum_{k=1}^{m} \sum_{\beta} \left\langle \text{Res}_{q=\infty} \left( \frac{\prod_{i=1}^{k} f_{\beta_{(i)}}}{1 - q^{-1} \mathcal{O}(\sum_{i=0}^{m-k-1} D_i')} \right) dq \right\rangle S_k
\]
\[
= -\sum_{k=1}^{m} \sum_{\beta} \sum_{t \geq 0} \left\langle \left( \prod_{i=1}^{k} (f_{\beta_{(i)}}(\infty))^{-t} \cdot \mathcal{O}(t \sum_{i=0}^{m-k-1} D_i') \right) dq \right\rangle S_k
\]
\[
= -\sum_{k=1}^{m} \sum_{r} \sum_{u=1}^{r} \sum_{t \geq 0} \sum_{s \geq 1} \left\langle \left( \prod_{i=1}^{k} (f_{\beta_{(i)}}(\infty))^{-t} \right) \sum_{j_1 + \cdots + j_t = t+1} \prod_{j_u > 0} \left( \sum_{i=1}^{k-r} \text{Res}_{q_n} \left( \sum_{u=1}^{r} \text{Res}_{q_n} (q^n f_{\beta_{(u)}}) \right) \right) \right\rangle S_{k-r} \times S_r
\]

This concludes the proof of Proposition 5.10 \(\square\)

Next, we want to simplify the right side of (33) and show that it equals
\[
-\sum_{k=1}^{m} \sum_{\beta} \chi \left( \left\langle \left( \prod_{i=1}^{k} f_{\beta_{(i)}}(X, \beta')/S_k \right) \prod_{i=1}^{k} \text{ev}_{n+i} \left( -\text{Res} \left( \frac{f_{\beta_{(i)}}}{1 - q^{-1}} \right) \right) \cdot \mathcal{O}_{\mathcal{G}^{\infty}}(X, \beta')/S_k \right) \right)
\]
\[
= -\sum_{k=1}^{m} \sum_{\beta} \left\langle \prod_{i=1}^{k} \text{Lau}(f_{\beta_{(i)}}) \right\rangle S_k
\]

where
\[
\text{Lau}(f_{\beta_{(i)}}) := -\text{Res}(f_{\beta_{(i)}}/(1 - q^{-1})) = \sum_{s \geq 1} [(f_{\beta_{(i)}})_0]_{-s} + \sum_{t \geq 0} [(f_{\beta_{(i)}})_{\infty}]_{-t}
\]
is the \(K\)-theory class obtained by evaluating the Laurent polynomial part of \(f_{\beta_{(i)}}\) at \(q = 1\), i.e., \(\text{Lau}(f_{\beta_{(i)}}) = [f_{\beta_{(i)}}]_{+|q=1} \).

For the fixed curve class \(\beta\), we denote by \(D_{\beta}\) the set of all degree-\(d_0\) curve classes that appears in the decompositions of \(\beta\) satisfying Condition 4.5. Recall that \(f_{\beta_{(i)}} = (1 - q^{t} L_{n+i}) I_{\beta_{(i)}}(q^{t} L_{n+i})\). We observe that the dependence of both (33) and (40) on the \(I\)-function is through the coefficients of the formal Laurent series expansions of \((1 - q) I_{\gamma}(q), \gamma \in D_{\beta}\) at 0 and \(\infty\). Hence it suffices to prove the identity by viewing those expansions as independent series which can be defined for any tuple \((g_0, g_{\infty})\), where \(g_0 := (g_0(q))_{\gamma \in D_{\beta}}\) and \(g_{\infty} := (g_{\infty}(q))_{\gamma \in D_{\beta}}\) are any Laurent series indexed by classes in \(D_{\gamma}\). More precisely, given such a tuple, we set \((f_{\beta_{(i)}})_0 = g_{\beta_{(i)}}^0 \left( q^i L_{n+i} \right) \) and \((f_{\beta_{(i)}})_{\infty} = g_{\beta_{(i)}}^\infty \left( q^i L_{n+i} \right) \) and define the right sides of (33) and (40). Denote the universal expressions of the right side of (33) and (40) by \(\text{Loc}(g_0, g_{\infty})\) and \(\text{Cor}(g_0, g_{\infty})\), respectively. We prove the following
Lemma 5.12. We have

\[ \text{Loc}(g_0, g_\infty) = \text{Cor}(g_0, g_\infty) \]

for any tuples \((g_0, g_\infty)\) of formal Laurent series indexed by \(D_\beta\).

Proof. For any \(\gamma \in D_\beta\), we choose an integer \(l_\gamma \geq 0\) and a class \(\delta^0_\gamma \in K(\bar{I}X) \otimes \Lambda\). Consider the formal directional derivative

\[ \nabla \sum_{\gamma \in D_\beta} \delta^0_\gamma q^l_\gamma \text{ of the difference } \text{Loc}(g_0, g_\infty) - \text{Cor}(g_0, g_\infty) \]

in the direction

\[ (41) \quad g_0 = (g_\beta^0) \mapsto g_0 + \sum_{\gamma \in D_\beta} \delta^0_\gamma q^l_\gamma := (g_\beta^0 + \delta^0_\gamma q^l_\gamma). \]

More precisely, for any function \(F\) depending on \(g_0\) and \(g_\infty\), we define

\[ \nabla \sum_{\gamma \in D_\beta} \delta^0_\gamma q^l_\gamma F(g_0, g_\infty) = \lim_{h \to 0} \frac{F(g_0 + h \sum_{\gamma \in D_\beta} \delta^0_\gamma q^l_\gamma, g_\infty) - F(g_0, g_\infty)}{h}. \]

By definition, for a decomposition \(\vec{\beta}\), we have \((f_{\vec{\beta}(i)})_0 = g^0_{\vec{\beta}(i)} (q^r L_{n+i})\) and \((f_{\vec{\beta}(i)})_\infty = g^\infty_{\vec{\beta}(i)} (q^r L_{n+i})\). Hence the change (41) induces

\[ (f_{\vec{\beta}(i)})_0 \mapsto (f_{\vec{\beta}(i)})_0 + \delta_{\vec{\beta}(i)} f d^0 L_{n+i} q^l_{\vec{\beta}(i)} \]

and keeps \((f_{\vec{\beta}(i)})_\infty\) the same. Under the assumption that \(l_\gamma \geq 0\), it is easy to see that

\[ \nabla \sum_{\gamma} \delta^0_\gamma q^l_\gamma \text{Lau}(f_{\vec{\beta}(i)}) = 0 \text{ and } \nabla \sum_{\gamma} \delta^0_\gamma q^l_\gamma \text{Res}(q^j(f_{\vec{\beta}(i)})) = \delta^0_{\vec{\beta}(i)} L_{n+i}^j \text{ for } j = -l_{\vec{\beta}(i)} r \text{ and } 0 \]

otherwise. Given a class \(\gamma \in D_\beta\), we set \(\beta^\gamma := \beta - \gamma\) and denote by \(\vec{\beta}^\gamma\) an ordered decomposition of \(\beta^\gamma\) satisfying Condition 4.6. We will use similar notation for \(\beta'\). By the Lebnitz rule stated in Lemma 5.13, we obtain

\[ \nabla \sum_{\gamma} \delta^0_\gamma q^l_\gamma \text{Cor}(g_0, g_\infty) = 0. \]
When we differentiate the right side of (33), only the first and third summations contribute non-zero terms. Using the Leibnitz rule in Lemma 5.13, we have
\[
\nabla \sum_{\gamma} \delta_\gamma^0 q^r \text{Loc}(g_0, g_\infty)
\]
\[
= - \sum_{k=2}^{m} \sum_{\gamma \in D_\beta, \tilde{\gamma}} \sum_{s \geq 1} \left\langle e^{\psi^*_1}(\delta_\gamma^0) L_{n+1}^\gamma \cdot \left[ \prod_{i=1}^{k-1} (f_{\tilde{\beta}^{(i)}}) \right]_{-s-t_r} \right\rangle_k
\]
\[
+ \sum_{k=2}^{m} \sum_{\gamma \in D_\beta, \tilde{\gamma}} \sum_{s \geq 1} \left\langle e^{\psi^*_1}(\delta_\gamma^0) L_{n+1}^\gamma \cdot \left[ \prod_{i=1}^{k-1} (f_{\tilde{\beta}^{(i)}}) \right]_{-s-t_r} \right\rangle_k
\]
\[
- \sum_{k=3}^{m} \sum_{r=1}^{k-2} \sum_{\gamma \in D_\beta, \tilde{\gamma}, \check{\gamma}} \sum_{s \geq 1} \sum_{j_1+\cdots+j_r = t} \left\langle e^{\psi^*_1}(\delta_\gamma^0) L_{n+1}^\gamma \cdot \left[ \prod_{i=1}^{k-r-1} (f_{\tilde{\beta}^{(i)}}) \right]_{-t-s-t_r} \right\rangle_k
\]
\[
\cdot \prod_{u=1}^{r-1} \left( - \text{Res}(q^{-j_u} f_{\tilde{\beta}^{(u)}}) \right) \rangle_k
\]
\[
+ \sum_{k=3}^{m} \sum_{r=2}^{k-1} \sum_{\gamma \in D_\beta, \tilde{\gamma}, \check{\gamma}} \sum_{s \geq 1} \sum_{j_1+\cdots+j_r-1 = t-t_r} \left\langle e^{\psi^*_1}(\delta_\gamma^0) L_{n+1}^\gamma \cdot \left[ \prod_{i=1}^{k-r} (f_{\tilde{\beta}^{(i)}}) \right]_{-t-s} \right\rangle_k
\]
\[
\cdot \prod_{u=1}^{r-1} \left( - \text{Res}(q^{-j_u} f_{\tilde{\beta}^{(u)}}) \right) \rangle_k.
\]

The above class is zero because the first line of (42) cancels with the second line and the third line cancels with the fourth line after replacing \( r \) by \( r-1 \) and \( t \) by \( t - t_r \). Therefore, we have
\[
\nabla \sum_{\gamma} \delta_\gamma^0 q^r \text{Loc}(g_0, g_\infty) = 0.
\]

Let \( \nabla \sum_{\gamma \in D_\beta} \delta_\gamma^0 q^r \) denote the formal directional derivative in the direction
\[
g_\infty = (g_\gamma^0) \mapsto g_\infty + \sum_{\gamma \in D_\beta} \delta_\gamma^0 q^r := (g_\gamma^0 + \delta_\gamma^0 q^r),
\]
where \( q^r \) and \( \delta_\gamma^0 \) are non-negative powers of \( q \) in \( g_\gamma^0 \) and \( g_\infty \). Using similar computations as in the previous case, we can show that
\[
\nabla \sum_{\gamma} \delta_\gamma^0 q^r \text{Loc}(g_0, g_\infty) = 0.
\]

We omit the proof.

The vanishing of formal directional derivatives implies that the difference \( \text{Loc}(g_0, g_\infty) - \text{Cor}(g_0, g_\infty) \) does not depend on the coefficients of the nonnegative powers of \( q \) in \( g_\gamma^0 \) and the positive powers of \( q \) in \( g_\infty \). Hence we can set
\[
[g_\gamma^0]_l = 0, \ l \geq 0 \quad \text{and} \quad [g_\infty]_l = 0, \ l > 0.
\]
Under this additional assumption, we have

$$-\text{Res}(q^{-j} f_{\beta^{(u)}}) = [(f_{\beta^{(u)}})_{\infty}]_{-j}, \quad \text{Res}(q^{j'} f_{\beta^{(u)}}) = [(f_{\beta^{(u)}})_{0}]_{j'}$$

for any $j \geq 0, j' > 0$. Hence

$$\sum_{j_1 + \ldots + j_r = t} \prod_{u=1}^r \left( -\text{Res}(q^{-j_u} f_{\beta^{(u)}}) \right) = \prod_{u=1}^r (f_{\beta^{(u)}})_{\infty}^{-t},$$

$$\sum_{j_1 + \ldots + j_{r+1} = t+1} \prod_{u=1}^r \text{Res}(q^{j_u} f_{\beta^{(u)}}) = \prod_{u=1}^r (f_{\beta^{(u)}})_{0}^{t+1}.$$ # \text{Using the above identities, we can simplify the difference} \text{Loc}(f_0, f_\infty) - \text{Cor}(f_0, f_\infty) \text{ as follows:} #

\begin{align*}
\text{Loc}(g_0, g_\infty) - \text{Cor}(g_0, g_\infty) & = -\sum_{k=1}^m \sum_{\beta} \sum_{s \geq 1} \left\langle \prod_{i=1}^k (f_{\beta^{(i)}})_{0} - s \right\rangle_k - \sum_{k=1}^m \sum_{\beta} \sum_{t \geq 0} \left\langle \prod_{i=1}^k (f_{\beta^{(i)}})_{\infty}^{-t} \right\rangle_k \\
& \quad - \sum_{k=2}^m \sum_{r=1}^{k-1} \sum_{\beta, \bar{\beta}} \left( \sum_{l \geq 0} \sum_{s \geq 1} \left\langle \prod_{i=1}^{k-r} (f_{\bar{\beta}^{(i)}})_{0} - s \cdot \prod_{u=1}^r (f_{\beta^{(u)}})_{\infty}^{-t} \right\rangle_k S_{k-r \times S_r} \\
& \quad + \sum_{t \geq 0} \sum_{s \geq 1} \left\langle \prod_{i=1}^{k-r} (f_{\bar{\beta}^{(i)}})_{\infty} - s \cdot \prod_{u=1}^r (f_{\beta^{(u)}})_{0}^{t+1} \right\rangle_k S_{k-r \times S_r} \right) \\
& \quad + \sum_{k=1}^m \sum_{\beta} \left\langle \prod_{i=1}^k (\sum_{s \geq 1} [(f_{\beta^{(i)}})_{0} - s \cdot \sum_{t \geq 0} (f_{\beta^{(i)}})_{\infty}^{-t}] \right\rangle_k S_k \\
& = -\sum_{k=1}^m \sum_{\beta} \sum_{s \geq 1} \left\langle \prod_{i=1}^k (f_{\beta^{(i)}})_{0} - s \right\rangle_k - \sum_{k=1}^m \sum_{\beta} \sum_{t \geq 0} \left\langle \prod_{i=1}^k (f_{\beta^{(i)}})_{\infty}^{-t} \right\rangle_k \\
& \quad - \sum_{k=2}^m \sum_{r=1}^{k-1} \sum_{\beta, \bar{\beta}} \sum_{l \geq 0} \sum_{s \geq 1} \left\langle \prod_{i=1}^{k-r} (f_{\bar{\beta}^{(i)}})_{0} - s \cdot \prod_{u=1}^r (f_{\beta^{(u)}})_{\infty}^{-t} \right\rangle_k S_{k-r \times S_r} \\
& \quad + \sum_{k=1}^m \sum_{\beta} \left\langle \prod_{i=1}^k (\sum_{s \geq 1} [(f_{\beta^{(i)}})_{0} - s \cdot \sum_{t \geq 0} (f_{\beta^{(i)}})_{\infty}^{-t}] \right\rangle_k S_k \\
& = 0
\end{align*}

The last equality follows from the permutation-equivariant multinomial formula.
Lemma 5.13 (Permutation-equivariant Leibniz rule). Let $T_i, 1 \leq i \leq m$ be linear maps from the ring of formal Laurent series $[K((X)) \otimes \Lambda]]((q))$ to its coefficient ring $K((X)) \otimes \Lambda$. Let $(k_1, \ldots, k_m)$ be a partition of a positive integer $n$ and let $f_1, \ldots, f_m$ be formal Laurent series. For $l_i \in \mathbb{Z}, \delta_i \in K((X)) \otimes \Lambda$, consider the directional derivative $\nabla \sum l_i q_{i}$ along the direction $(f_1, \ldots, f_m) \mapsto (f_1 + \delta_1 q_{i}, \ldots, f_m + \delta_m q_{i})$.

Then we have

\[
\nabla \sum l_i q_{i} \left< T_1(f_1)^{\otimes k_1} \cdot T_2(f_2)^{\otimes k_2} \cdots T_m(f_m)^{\otimes k_m} \right>_n^{S_{k_1} \times \cdots \times S_{k_m}}
\]

\[
= \sum_{a=1}^{m} \left< \nabla l_a q_{a} T_a(f_a) \cdot T_1(f_1)^{\otimes k_1} \cdots T_a(f_a)^{\otimes k_a-1} \cdots T_m(f_m)^{\otimes k_m} \right>_n^{S_{k_1} \times \cdots \times S_{k_a-1} \times \cdots \times S_{k_m}}.
\]

Here we adopt the bracket notation from which the pullbacks $ev_{\tau}$ are suppressed. In particular, the equality still holds if we make the change of variable $q \mapsto q^r L$, where $r \in \mathbb{Z}$ and $L$ denotes the coarse cotangent line bundles at the corresponding markings, and extend the definition of $T_i$ via linearity: $T_i(q^r L^\alpha) := L^r T_i(q^\alpha)$ with $a, b \in \mathbb{Z}, \alpha \in K((X)) \otimes \Lambda$.

**Proof.** Let $h \in \mathbb{C}$. By the permutation-equivariant binomial formula, we have

\[
\left< (T_1(f_1) + h \delta_1 q_{1})^{\otimes k_1} \cdots T_m(f_m) + h \delta_m q_{m})^{\otimes k_m} \right>_n^{S_{k_1} \times \cdots \times S_{k_m}}
\]

\[
= \left< T_1(f_1)^{\otimes k_1} \cdots T_m(f_m)^{\otimes k_m} + h \sum_{a=1}^m T_a(f_a)^{\otimes (k_a-1)} \cdots T_m(f_m)^{\otimes k_m} \right>_n^{S_{k_1} \times \cdots \times S_{k_a-1} \times \cdots \times S_{k_m}} + O(h^2).
\]

Hence the lemma follows from the definition of the directional derivative $\nabla \sum l_i q_{i}$ and the equality $\nabla l_a q_{a} T_a(f_a) = T_a(\delta_a q_{a})$.

**Proof of Proposition 5.9.** Recall that $[(1 - q) I_{\beta}(q)]_+ = \mu_{\beta}(q)$ for $\beta > 0$. Hence we have

\[
\text{Lau}_\tau ((1 - q^r L) I_{\beta}(q^r L)) = [(1 - q^e L) I_{\beta}(q^e L)]_+ |_{q^e = 1} = \mu_{\beta}(L).
\]

We conclude the proof by combining Proposition 5.10, Lemma 5.12, Lemma 3.12, and the above equality.

By substituting the expression in Proposition 5.9 into the $K$-theoretic localization formula (27) over the master space, we obtain the main theorem of this paper:

**Theorem 5.14.** Assuming that $2g - 2 + n + \epsilon_0 \deg(\beta) > 0$, we have

\[
\left< \iota \times ev \otimes Q_{g, n}^{vir}(X, \beta) - (\iota \circ \iota) \times ev \otimes Q_{g, n}^{vir}(X, \beta) \right>
\]

\[
= \sum_{k \geq 1} \left< \sum_{\beta}(b_{\beta} \circ \iota) \times ev \otimes Q_{g, n+k}^{vir}(X, \beta) \right>
\]

\[
\left< \prod_{a=1}^{k} ev_{n+a}^{*} \mu_{\beta_{a}}(L_{n+a}) \cdot O_{Q_{g, n+k}^{vir}(X, \beta)} \right>
\]
in $K_0\left([Q_{g,n}^\epsilon(\mathbb{P}^N, d) \times (\bar{TX})^n]/S_n\right)_Q$, where $\beta$ runs through all ordered decompositions of $\beta$ satisfying Condition 4.6. The same formula also holds for the twisted virtual structure sheaves with level structure.

5.5. The genus-0 case. In genus zero, the following theorem generalizes the genus-zero toric mirror theorems in quantum $K$-theory [20, 21, 29] and quantum $K$-theory with level structure [50], and the $K$-theoretic wall-crossing formula in [59].

**Theorem 5.15.** For any $\epsilon$, we have

$$J_{S_\infty}^\infty(\mathbf{t}(q) + \mu_{\geq \epsilon}(Q, L), Q) = J_{S_\infty}^{\epsilon}(\mathbf{t}(q), Q).$$

The same formula holds for twisted permutation-equivariant $J$-functions with level structure.

**Proof.** It follows from Corollary 1.4 that the above equality holds modulo the constant terms in $\mathbf{t}$. Let $\epsilon_- < \epsilon_0 = \frac{1}{d_0} < \epsilon_+$ and let

$$\mu^\epsilon(Q, q) = \sum_{\text{deg}(\beta) = 1/\epsilon_0} \mu_\beta(q)Q^\beta.$$

It suffices to prove that

(43) $$J_{S_\infty}^{\epsilon_+}(\mu^\epsilon(Q, q), Q) = J_{S_\infty}^{\epsilon_-}(0, Q).$$

The right hand side of (43) is equal to

$$1 - q + (1 - q) \sum_{0 \leq \text{deg}(\beta) \leq 1/\epsilon_0} I_\beta(q)Q^\beta + \sum_{\text{deg}(\beta) > 1/\epsilon_0} Q^\beta(\mathbf{ev}_1)_*\left(\frac{\mathcal{O}^\text{vir}_{Q^\epsilon_{0,1}(X, \beta)}}{1 - qL_1}\right).$$

According to Theorem 5.14, we have

$$\sum_{\text{deg}(\beta) > 1/\epsilon_0} Q^\beta(\mathbf{ev}_1)_*\left(\frac{\mathcal{O}^\text{vir}_{Q^\epsilon_{0,1}(X, \beta)}}{1 - qL_1}\right) = \sum_{(k=0, \text{deg}(\beta) > 1/\epsilon_0) \text{ or } (k \geq 1, \beta \geq 0, (k, \beta) \neq (1, 0))} Q^\beta(\mathbf{ev}_1)_*\left(\frac{\mathcal{O}^\text{vir}_{Q^\epsilon_{0,1+k}(X, \beta)/S_k}}{1 - qL_1}\right) \cdot \prod_{i=1}^k [\mathbf{ev}_i]_*(\mu_{\epsilon_0}(Q, L_{i+1}))$$

By comparing to the left hand side of (43), we see that (43) is equivalent to

$$(1 - q) \sum_{\text{deg}(\beta) = 1/\epsilon_0} I_\beta(q)Q^\beta = \sum_{\text{deg}(\beta) = 1/\epsilon_0} Q^\beta(\mathbf{ev}_1)_*\left(\frac{\mathcal{O}^\text{vir}_{Q^\epsilon_{0,1}(X, \beta)}}{1 - qL_1}\right) + \mu^\epsilon(Q, q).$$

This follows immediately from Corollary 5.8. $\square$
Appendix A. $K$-theoretic pushforward formula for inflated projection bundles

Lemma A.1. Let $X$ be any Deligne–Mumford stack with an $S_k$-action, and $\Theta_1, \ldots, \Theta_k$ be an $S_k$-equivariant tuple of line bundles on $X$. Let $p : \tilde{\mathbb{P}} \to X$ be the inflated projective bundle associated to $\Theta_1, \ldots, \Theta_k$. For $i = 1, \ldots, k - 1$, let $D_i$ be the $i$-tautological divisor. Then for any $a \in K_0(X/S_k)$ and any integer $t \geq 0$, we have

\[
p_* \left( [\mathcal{O}(t(D_0 + \cdots + D_{k-2}))] \cdot [\mathcal{O}_{\tilde{\mathbb{P}}}(-t)] \cdot p^* a \right) = [\text{Sym}^t(\Theta_1 \oplus \cdots \Theta_k)] \cdot a
\]

and for any integer $t < 0$, we have

\[
p_* \left( [\mathcal{O}(t(D_0 + \cdots + D_{k-2}))] \cdot [\mathcal{O}_{\tilde{\mathbb{P}}}(-t)] \cdot p^* a \right) = -\left( \sum_{j_1 + \cdots + j_k = t} [(-\Theta_1^{-j_1}) \otimes \cdots \otimes (-\Theta_k^{-j_k})] \right) \cdot a
\]

We will prove this lemma at the end of this section. First, we need some preparation.

Set

\[L = \mathcal{O}(D_0 + \cdots + D_{k-2}) \otimes \mathcal{O}(-1).\]

Note that we have canonical isomorphism

\[\mathcal{O}(D_0 + 2D_1 + \cdots + (k-1)D_{k-2}) \cong \mathcal{O}(k) \otimes \Theta_1 \otimes \cdots \Theta_k,
\]

whose proof is similar to that in the original version of Lemma 2.6.3 of [Zhou]. Thus we have

\[L \cong \mathcal{O}(k-1) \otimes \mathcal{O}(-(D_1 + \cdots + (k-2)D_{k-2})) \otimes \Theta_1 \otimes \cdots \otimes \Theta_k.
\]

Lemma A.2. In $K_0(\tilde{\mathbb{P}}/S_k)$ we have

\[(L + (-\Theta_1)) \cdots (L + (-\Theta_k)) = 0.
\]

Proof of Lemma A.2. Let

\[x_i : \mathcal{O}(-1) \to \Theta_i, \quad i = 1, \ldots, k
\]

be the tautological morphisms over $\mathbb{P}(\Theta_1 \oplus \cdots \Theta_k)$. Then we have

\[x_1 \cdots x_i \cdots x_k : \mathcal{O}(-(k-1)) \to \Theta_1 \cdots \hat{\Theta}_i \cdots \Theta_k.
\]

This is equivalent to

\[\varphi_i : \Theta_i \to \mathcal{O}(k-1) \otimes \Theta_1 \cdots \Theta_k.
\]

Let

\[H_i = \mathbb{P}(\bigoplus_{j \neq i} \Theta_i), \quad i = 1, \ldots, k
\]

be the coordinate hyperplanes.
Let \( b : \bar{\mathbb{P}} \to \mathbb{P}(\Theta_1 \oplus \cdots \oplus \Theta_k) \) be the composition of the sequence of blowups. The vanishing locus of \( b^* \varphi_i \) defines a Cartier divisor supported on \( D_0 \cup \cdots \cup D_{k-2} \). Along any irreducible component \( D \) of some \( D_j \), the vanishing order of \( b^* \varphi_i \) is equal
\[
\# \{ a \mid b(D) \subset H_a, 1 \leq a \leq k, a \neq i \}.
\]
Note that for each irreducible component \( D \) of \( D_j \), there are precisely \( (j+1) \) of \( H_1, \ldots, H_k \) that contain \( p(D) \). Hence,

1. for \( i = 1, \ldots, k \), \( b^* \varphi_i \) factors through \( d_i : \Theta_i \to L \).
2. at each point \( p \) of \( \bar{\mathbb{P}} \), there exists \( i \) such that \( d_i \) is an isomorphism near \( p \).

This implies that we have a \( S_k \)-equivariant Koszul resolution
\[
0 \to \bigwedge^k ((\Theta_1 \oplus \cdots \oplus \Theta_k) \otimes L^{-1}) \to \cdots \to 1 \bigwedge^1 ((\Theta_1 \oplus \cdots \oplus \Theta_k) \otimes L^{-1}) \to \mathcal{O} \to 0.
\]
The relation (44) is obtained by taking the \( K \)-theory class of the above long exact sequence.

Lemma A.3. For \( i \geq 1, t \geq 0 \), \( R^i p_* L^{\otimes t} = 0 \).

Proof. It suffices to prove the Lemma in the case when \( X \) is a point. We will describe \( \bar{\mathbb{P}} \) as a toric variety and show that \( L \) is nef.

Let \( N = \mathbb{Z}^k / \Delta \), where \( \Delta \) is the small diagonal. Then \( \mathbb{P}^{k-1} \) is the toric variety associated to the fan \( \Sigma_{k-1} \) whose rays are spanned by the image in \( N \) of standard basis \( e_i \) of \( \mathbb{Z}^k \). We also denote them by \( e_i \). Thus a typical maximal dimensional cone \( \sigma_{k-1} \in \Sigma_{k-1} \) is spanned by
\[
e_1, \ldots, e_{k-1}.
\]
When we say some object is typical, we mean that the other objects are obtained via \( S_k \)-symmetry. Whenever we subdivide some typical cone, we do the same thing to other cone using \( S_k \)-symmetry.

We claim that \( \bar{\mathbb{P}} \) is the toric variety associated to the fan \( \Sigma_1 \), a typical maximal dimensional cone \( \sigma_1 \) of which is spanned by
\[
(1^a, 0^{k-a}), \quad a = 1, \ldots, k - 1.
\]
Here and below we use \( 1^a \) or \( 0^a \) to respectively denote 1 or 0 repeated \( a \) times.

Indeed, blowing up along intersection of the proper transform of \( H_1, \ldots, H_\ell \) amounts to star-subdividing the fan by adding the ray
\[
(1^\ell, 0^{k-\ell}).
\]
The first blowup \( U_{k-2} \to U_{k-1} \) corresponds to the fan \( \Sigma_{k-2} \) obtained by star-subdivide the \( \Sigma_{k-1} \) by adding the ray spanned by
\[
(1^{k-1}, 0)
\]
and also the rays in it $S_k$-orbits. Thus a typical maximal dimensional cone $\sigma_{k-2} \in \Sigma_{k-2}$ is spanned by $$e_1, \ldots, e_{k-2}, (1^{k-1}, 0).$$

Rays in the $S_k$-orbits of $(1^{k-1}, 0)$ correspond to the exceptional divisor. Inductively, we define $U_\ell = X_{\Sigma_\ell}$, a typical maximal dimensional cone of which is spanned by $$e_1, \ldots, e_\ell, (1^{\ell+1}, 0^{k-\ell-1}), \ldots, (1^{k-1}, 0).$$

The blowup $U_{\ell-1} \to U_\ell$ replaces $e_\ell$ by $(1^{\ell}, 0^{k-\ell})$. This proves the claim.

We now compute the divisors. For a ray $\rho = (\rho_1, \ldots, \rho_k)$ in $\Sigma_1$, we write $|\rho| = \sum_{j=1}^k \rho_j$.

Then $$D_i = \sum_{|\rho| = i+1} D_\rho.$$ And $L$ is linearly equivariant to the $Q$-Cartier divisor

$$k - 2 \sum_{i=0}^{k-2} \sum_{|\rho| = i+1} \frac{k-i-1}{k} D_i = \frac{k-|\rho|}{k} \sum_{\rho} D_\rho.$$ We choose the (unique) Cartier data

$$m = \left( \frac{1-k}{k}, \frac{1}{k}, \ldots, \frac{1}{k} \right) \in M = N^\vee$$

in $\sigma_1 \in \Sigma_1$. The Cartier data in other cones are determined by $S_k$-symmetry. Note that all maximal cones that contain $e_1$ share the same $m$. There are $(k-1)!$ of them. Let $\varphi$ be the support function determined by $S_k$-symmetry and the formula

$$\varphi(v) = \langle m, v \rangle, \quad v \in \sigma_1.$$ It is easy to see that $\varphi$ is non-positive and $\{v \mid \varphi(v) \geq -1/k\}$ is the $k-1$-simplex whose vertices are

$$-e_1, \ldots, -e_k \in N = \mathbb{Z}^k/\Delta.$$ Hence the support function is convex.\(^2\) Hence $L$ is nef. By the Demazure vanishing theorem, $L^\otimes t$ has no higher cohomology for $t \geq 0$. \(\square\)

Recall that we have constructed

$$d_i : \Theta_i \to L.$$ Note that here $\Theta_i$ actually means $p^* \Theta_i$. We have $p_* p^* \Theta_i = \Theta_i$. Using adjunction we have

$$d : \bigoplus_{i=1}^k \Theta_i \to p_*(L).$$

**Lemma A.4.** For $t \geq 1$, the composition

$$d^t : \text{Sym}^t \bigoplus_{i=1}^k \Theta_i \xrightarrow{\text{Sym}^t(d)} \text{Sym}^t(p_*(L)) \to p_*(L^\otimes t)$$

are isomorphisms for $t \geq 1$.\(^2\)

\(^2\)Note that the definition of convexity in [14] is different from the usual convention.
Proof. Again it suffices to prove statements when $X$ is a point. Consider the graded ideal

$$a \subset \mathbb{C}[x_1, \ldots, x_k]$$

generated by monomials

$$x_1 \cdots x_i \cdots x_k, \quad i = 1, \ldots, k.$$  

Thus the subscheme

$$V_{\mathbb{P}^{k-1}}(a) \subset \mathbb{P}^{k-1}$$

is the codimension-2 toric strata.

Let $[a^t]_m$ denote the degree $m$ part of $a^t$, for any $t, m \geq 1$. It is easy to see that $a^t$ is generated by monomials

$$x^a := x_1^{a_1} \cdots x_k^{a_k}, \quad a = (a_1, \ldots, a_k)$$

such that $\sum_i a_i = (k - 1)t$ and $0 \leq a_i \leq t$ for $i = 1, \ldots, k$. We rewrite it as

$$\frac{x^t}{x^b},$$

where $t = (t, \ldots, t)$ and $b = (b_1, \ldots, b_k)$ with $\sum_i b_i = t$ and $b_i \geq 0, i = 1, \ldots, t$. Thus we have

$$[a^t]_{t(k-1)} = \text{Sym}^t[a]_{k-1}.$$  

Then the map $d^t$ is identified with the $d^t$ in

$$[a^t]_{t(k-1)} \xrightarrow{d^t} H^0(\mathbb{P}, L^\otimes t)$$

Again it suffices to prove statements when $X$ is a point. Consider the graded ideal

$$a \subset \mathbb{C}[x_1, \ldots, x_k]$$

generated by monomials

$$x_1 \cdots x_i \cdots x_k, \quad i = 1, \ldots, k.$$  

Thus the subscheme

$$V_{\mathbb{P}^{k-1}}(a) \subset \mathbb{P}^{k-1}$$

is the codimension-2 toric strata.

Let $[a^t]_m$ denote the degree $m$ part of $a^t$, for any $t, m \geq 1$. It is easy to see that $a^t$ is generated by monomials

$$x^a := x_1^{a_1} \cdots x_k^{a_k}, \quad a = (a_1, \ldots, a_k)$$

such that $\sum_i a_i = (k - 1)t$ and $0 \leq a_i \leq t$ for $i = 1, \ldots, k$. We rewrite it as

$$\frac{x^t}{x^b},$$

where $t = (t, \ldots, t)$ and $b = (b_1, \ldots, b_k)$ with $\sum_i b_i = t$ and $b_i \geq 0, i = 1, \ldots, t$. Thus we have

$$[a^t]_{t(k-1)} = \text{Sym}^t[a]_{k-1}.$$  

Then the map $d^t$ is identified with the $d^t$ in

$$[a^t]_{t(k-1)} \xrightarrow{d^t} H^0(\mathbb{P}, L^\otimes t)$$

Let $\bar{a} \subset \mathcal{O}_{\mathbb{P}^{k-1}}$ be the ideal sheaf associated to $a$. It suffices to show that images of both vertical arrows are $H^0(\mathbb{P}^{k-1}, \bar{a}^t \otimes \mathcal{O}_{\mathbb{P}^{k-1}}(t(k - 1)))$. For the left vertical arrow, it amounts to check that for any homogeneous $f \in \mathbb{C}[x_1, \ldots, x_k]$ of degree $t(k - 1)$, any integer $N > 0$ such that

$$x_i^N f \in a^t, \quad i = 1, \ldots, k,$$

we have $f \in a^t$. Since $a^t$ is generated by monomials, it suffices to prove this for $f = x_1^{a_1} \cdots x_k^{a_k}, \sum_i a_i = t(k - 1)$. Then $x_i^N f \in a^t$ reads

$$a_1^{a_1} \cdots a_i^{a_i+N} \cdots a_k^{a_k} = (x_1^{a_1} \cdots x_i^{a_i+1} \cdots x_k^{a_k}) \cdot (x_1^{c_1} \cdots x_i^{c_i} \cdots x_k^{c_k})$$

for some $\sum_j a_j = t(k - 1), 0 \leq a_j \leq t$, $c_j \geq 0$. Since $a_i \leq t$, $\sum_j a_i' \geq t(k - 2)$. Hence $\sum_j a_j \geq t(k - 2)$. Hence $a_i \leq t$. Now we have shown that $a_i \leq t$ for all $i = 1, \ldots, k$. We conclude that $f \in a^t$.

We now consider the right vertical arrow. Suppose $f \in H^0(\mathbb{P}^{k-1}, \bar{a}^t \otimes \mathcal{O}_{\mathbb{P}^{k-1}}(t(k - 1)))$, then by the previous paragraph it comes from $[a^t]_{t(k-1)}$. Hence $f$ lies in the image of the right vertical arrow by the commutativity of the diagram. Now suppose $f \in H^0(\mathbb{P}^{k-1}, \mathcal{O}_{\mathbb{P}^{k-1}}(t(k - 1)))$ whose image in $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}^{k-1}}(t(k - 1)))$ vanishes along $t(D_1 + \cdots + (k - 2)D_{k-2})$. We want to show that $f$ is in $[a^t]_{t(k-1)}$. Since the arrow is torus equivariant,
without loss of generality we may assume that \( f = x_1^{a_1} \cdots x_k^{a_k}, \sum_i a_i = t(k - 1). \) Since the pullback of \( f \) vanishes along \( t(k - 2)D_{k-2}, \) we must have
\[
a_1 + \cdots + \hat{a}_i + \cdots + a_k \geq t(k - 2)
\]
for each \( i = 1, \ldots, k. \) Thus we must have \( a_i \leq t. \) Hence \( f \in [a^t]_{t(k-1)}. \) This completes the proof.

**Proof of Lemma A.1.** The \( t \geq 0 \) case follows immediately from Lemma A.3 and Lemma A.4. We focus on the \( t < 0 \) case. Recall that by Lemma A.2, we have
\[
(L + (-\Theta_1)) \cdots (L + (-\Theta_k)) = 0.
\]
From the above equality, we can deduce
\[
(1 - \lambda^{-1}L)(\sum_{i=0}^{k-1} \lambda^{-i} \alpha_i) = (1 + \lambda^{-1}(-\Theta_1)) \cdots (1 + \lambda^{-1}(-\Theta_k)),
\]
where \( \lambda \) is a formal variable and
\[
\alpha_i = \sum_{j=0}^{i} L^\otimes j \otimes \text{Sym}^{i-j}((-\Theta_1) \oplus \cdots \oplus (-\Theta_k)).
\]
Thus we have
\[
\frac{1}{1 - \lambda^{-1}L} = \frac{\sum_{i=0}^{k-1} \lambda^{-i} \alpha_i}{(1 + \lambda^{-1}(-\Theta_1)) \cdots (1 + \lambda^{-1}(-\Theta_k))}.
\]
The formal expansions of both sides of the above equation around \( \lambda = 0 \) gives that the following identity
\[
\sum_{j \geq 1} \lambda^j (L^{-1})^\otimes j = -(\sum_{i=0}^{k-1} \lambda^{-i} \alpha_i) \cdot \left( \sum_{j \geq 1} \lambda^j \sum_{j_1 + \cdots + j_k = j} (-\Theta_1^{-j_1}) \otimes \cdots \otimes (-\Theta_k^{-j_k}) \right)
\]
in \( K^0(X[[\lambda]]). \) By Lemma A.3 and Lemma A.4, we have
\[
p_* (L^\otimes j) = \text{Sym}^j (\Theta_1 \oplus \cdots \oplus \Theta_k)
\]
for \( j \geq 0. \) Hence \( p_*(\alpha_i) = 0 \) for \( 1 \leq i \leq k - 1 \) and we have
\[
\sum_{j \geq 1} \lambda^j p_* (L^{-1})^\otimes j = -\sum_{j \geq 1} \lambda^j \sum_{j_1 + \cdots + j_k = j} (-\Theta_1^{-j_1}) \otimes \cdots \otimes (-\Theta_k^{-j_k}).
\]
We conclude the proof of the lemma by comparing the coefficients of \( \lambda \) in the above equation.
References

[1] Dan Abramovich, Tom Graber, and Angelo Vistoli, *Gromov-Witten theory of Deligne-Mumford stacks*, Amer. J. Math. **130** (2008), no. 5, 1337–1398. MR 2450211

[2] Alejandro Adem and Yongbin Ruan, *Twisted orbifold K-theory*, Comm. Math. Phys. **237** (2003), no. 3, 533–556. MR 1993337

[3] Mina Aganagic and Andrei Okounkov, *Elliptic stable envelopes*, arXiv e-prints (2016), arXiv:1604.00423.

[4] Mina Aganagic and Andrei Okounkov, *Quasimap counts and Bethe eigenfunctions*, Mosc. Math. J. **17** (2017), no. 4, 565–600. MR 3734654

[5] M. F. Atiyah and G. B. Segal, *The index of elliptic operators. II*, Ann. of Math. (2) **87** (1968), 531–545. MR 236951

[6] Daewoong Cheong, Ionuț Ciocan-Fontanine, and Bumsig Kim, *Orbifold quasimap theory*, Math. Ann. **363** (2015), no. 3-4, 777–816. MR 3412343

[7] Neil Chriss and Victor Ginzburg, *Representation theory and complex geometry*, Modern Birkhäuser Classics, Birkhäuser Boston, Ltd., Boston, MA, 2010, Reprint of the 1997 edition. MR 2838836

[8] Ionuț Ciocan-Fontanine and Bumsig Kim, *Moduli stacks of stable toric quasimaps*, Adv. Math. **225** (2010), no. 6, 3022–3051. MR 2729000

[9] ________, *Wall-crossing in genus zero quasimap theory and mirror maps*, Algebr. Geom. **1** (2014), no. 4, 400–448. MR 3272909

[10] Ionuț Ciocan-Fontanine, Bumsig Kim, and Davesh Maulik, *Stable quasimaps to GIT quotients*, J. Geom. Phys. **75** (2014), 17–47. MR 3126932

[11] Emily Clader, Felix Janda, and Yongbin Ruan, *Higher-genus quasimap wall-crossing via localization*, arXiv e-prints (2017), arXiv:1702.03427.

[12] ________, *Higher-genus wall-crossing in the gauged linear sigma model*, arXiv e-prints (2017), arXiv:1706.05038.

[13] Kevin Costello, *Higher genus Gromov-Witten invariants as genus zero invariants of symmetric products*, Ann. of Math. (2) **164** (2006), no. 2, 561–601. MR 2247968

[14] David A. Cox, John B. Little, and Henry K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011. MR 2810322

[15] Hai Dong and Yaoxiong Wen, *Level correspondence of K-theoretic I-function in Grassmann duality*, arXiv e-prints (2020), arXiv:2004.10661.

[16] Hélène Esnault and Eckart Viehweg, *Lectures on vanishing theorems*, DMV Seminar, vol. 20, Birkhäuser Verlag, Basel, 1992. MR 1193913

[17] Alexander Givental, *On the WDVV equation in quantum K-theory*, vol. 48, 2000, Dedicated to William Fulton on the occasion of his 60th birthday, pp. 295–304. MR 1786492

[18] Alexander Givental, *Permutation-equivariant quantum K-theory II. Fixed point localization*, preprint (2015), arXiv:1508.04374.

[19] ________, *Permutation-equivariant quantum K-theory III. Lefschetz’ formula on \(\overline{M}_{0,n}/S_n\) and adelic characterization*, preprint (2015), arXiv:1508.06697.
[20] ______, Permutation-equivariant quantum K-theory IV. $\mathcal{D}_q$-modules, preprint (2015), arXiv:1509.00830.
[21] ______, Permutation-equivariant quantum K-theory V. Toric $q$-hypergeometric functions, preprint (2015), arXiv:1509.03903.
[22] ______, Permutation-equivariant quantum K-theory VI. Mirrors, preprint (2015), arXiv:1509.07852.
[23] ______, Permutation-equivariant quantum K-theory VII. General theory, preprint (2015), arXiv:1510.03076.
[24] ______, Permutation-equivariant quantum K-theory VIII. Explicit reconstruction, preprint (2015), arXiv:1510.06116.
[25] ______, Permutation-equivariant quantum K-theory I. Definitions. Elementary K-theory of $\mathcal{M}_{0,n}/S_n$, Mosc. Math. J. 17 (2017), no. 4, 691–698. MR 3734658
[26] ______, Permutation-equivariant quantum K-theory IX. Quantum Hirzebruch-Riemann-Roch in all genera, preprint (2017), arXiv:1709.03180.
[27] ______, Permutation-equivariant quantum K-theory XI. Quantum Adams-Riemann-Roch, preprint (2017), arXiv:1711.04201.
[28] Alexander Givental, Permutation-equivariant quantum K-theory X. Quantum Hirzebruch-Riemann-Roch in genus 0, SIGMA Symmetry Integrability Geom. Methods Appl. 16 (2020), Paper No. 031, 16. MR 4089511
[29] Alexander Givental and Valentin Tonita, The Hirzebruch-Riemann-Roch theorem in true genus-0 quantum K-theory, Symplectic, Poisson, and noncommutative geometry, Math. Sci. Res. Inst. Publ., vol. 62, Cambridge Univ. Press, New York, 2014, pp. 43–91. MR 3380674
[30] Alexander Givental and Xiaohan Yan, Quantum K-theory of Grassmannians and non-abelian localization, arXiv e-prints (2020), arXiv:2008.08182.
[31] Eduardo González and Chris Woodward, Quantum Kirwan for quantum K-theory, arXiv e-prints (2019), arXiv:1911.03520.
[32] T. Graber and R. Pandharipande, Localization of virtual classes, Invent. Math. 135 (1999), no. 2, 487–518. MR 1666787
[33] Shuai Guo, Felix Janda, and Yongbin Ruan, A mirror theorem for genus two Gromov-Witten invariants of quintic threefolds, arXiv e-prints (2017), arXiv:1709.07392.
[34] Daniel Halpern-Leistner, $\Theta$-stratifications, $\Theta$-reductive stacks, and applications, Algebraic geometry: Salt Lake City 2015, Proc. Sympos. Pure Math., vol. 97, Amer. Math. Soc., Providence, RI, 2018, pp. 349–379. MR 3821155
[35] Yi Hu and Jun Li, Genus-one stable maps, local equations, and Vakil-Zinger’s desingularization, Math. Ann. 348 (2010), no. 4, 929–963. MR 2721647
[36] Hans Jockers and Peter Mayr, Quantum K-theory of Calabi-Yau manifolds, J. High Energy Phys. (2019), no. 11, 011, 20. MR 4069552
[37] ______, A 3d gauge theory/quantum K-theory correspondence, Adv. Theor. Math. Phys. 24 (2020), no. 2, 327–458. MR 4125364
[38] Hans Jockers, Peter Mayr, Urmi Ninad, and Alexander Tabler, Wilson loop algebras and quantum K-theory for Grassmannians, arXiv e-prints (2019), arXiv:1911.13286.
[39] Young-Hoon Kiem and Michail Savvas, \textit{K-Theoretic Generalized Donaldson-Thomas Invariants}, arXiv e-prints (2019), arXiv:1912.04966.
[40] Bumsig Kim and Hyenho Lho, \textit{Mirror theorem for elliptic quasimap invariants}, Geom. Topol. 22 (2018), no. 3, 1459–1481. MR 3780438
[41] Peter Koroteev, Petr P. Pushkar, Andrey Smirnov, and Anton M. Zeitlin, \textit{Quantum K-theory of Quiver Varieties and Many-Body Systems}, arXiv e-prints (2017), arXiv:1705.10419.
[42] Y.-P. Lee, \textit{Quantum K-theory. I. Foundations}, Duke Math. J. 121 (2004), no. 3, 389–424. MR 2040281
[43] Hyenho Lho and Rahul Pandharipande, \textit{Stable quotients and the holomorphic anomaly equation}, Adv. Math. 332 (2018), 349–402. MR 3810256
[44] Alina Marian, Dragos Oprea, and Rahul Pandharipande, \textit{The moduli space of stable quotients}, Geom. Topol. 15 (2011), no. 3, 1651–1706. MR 2851074
[45] Andrei Mustață and Magdalena Anca Mustață, \textit{Intermediate moduli spaces of stable maps}, Invent. Math. 167 (2007), no. 1, 47–90. MR 2264804
[46] Andrei Okounkov, \textit{Lectures on K-theoretic computations in enumerative geometry}, Geometry of moduli spaces and representation theory, IAS/Park City Math. Ser., vol. 24, Amer. Math. Soc., Providence, RI, 2017, pp. 251–380. MR 3752463
[47] Petr P. Pushkar, Andrey V. Smirnov, and Anton M. Zeitlin, \textit{Baxter Q-operator from quantum K-theory}, Adv. Math. 360 (2020), 106919, 63. MR 4035952
[48] Feng Qu, \textit{Virtual pullbacks in K-theory}, Ann. Inst. Fourier (Grenoble) 68 (2018), no. 4, 1609–1641. MR 3887429
[49] Yongbin Ruan, Yaoxiong Wen, and Zijun Zhou, \textit{Quantum K-theory of toric varieties, level structures, and 3d mirror symmetry}, arXiv e-prints (2020), arXiv:2011.07519.
[50] Yongbin Ruan and Ming Zhang, \textit{The level structure in quantum K-theory and mock theta functions}, arXiv e-prints (2018), arXiv:1804.06552.
[51] , \textit{Verlinde/Grassmannian Correspondence and Rank 2 \( \delta \)-wall-crossing}, arXiv e-prints (2018), arXiv:1811.01377.
[52] K. Taipale, \textit{K-theoretic J-functions of type A flag varieties}, Int. Math. Res. Not. IMRN (2013), no. 16, 3647–3677. MR 3090705
[53] R. W. Thomason, \textit{Lefschetz-Riemann-Roch theorem and coherent trace formula}, Invent. Math. 85 (1986), no. 3, 515–543. MR 848684
[54] , \textit{Algebraic K-theory of group scheme actions}, Algebraic topology and algebraic K-theory (Princeton, N.J., 1983), Ann. of Math. Stud., vol. 113, Princeton Univ. Press, Princeton, NJ, 1987, pp. 539–563. MR 921490
[55] Yukinobu Toda, \textit{Moduli spaces of stable quotients and wall-crossing phenomena}, Compos. Math. 147 (2011), no. 5, 1479–1518. MR 2834730
[56] Valentin Tonita, \textit{A virtual Kawasaki-Riemann-Roch formula}, Pacific J. Math. 268 (2014), no. 1, 249–255. MR 3207609
[57] , \textit{Twisted K-theoretic Gromov-Witten invariants}, Math. Ann. 372 (2018), no. 1-2, 489–526. MR 3856819
[58] Valentin Tonita and Hsian-Hua Tseng, \textit{Quantum orbifold Hirzebruch-Riemann-Roch theorem in genus zero}, arXiv e-prints (2013), arXiv:1307.0262.
[59] Hsian-Hua Tseng and Fenglong You, *K-theoretic quasimap invariants and their wall-crossing*, arXiv e-prints (2016), arXiv:1602.06494.

[60] Kazushi Ueda and Yutaka Yoshida, *3d N=2 Chern-Simons-matter theory, Bethe ansatz, and quantum K-theory of Grassmannians*, arXiv e-prints (2019), arXiv:1912.03792.

[61] Yaoxiong Wen, *K-Theoretic I-function of V//ϕG and Application*, arXiv e-prints (2019), arXiv:1906.00775.

[62] Yang Zhou, *Quasimap wall-crossing for GIT quotients*, arXiv e-prints (2019), arXiv:1911.02745.

**Department of Mathematics, the University of British Columbia**

*Email address: zhangming@math.ubc.ca*

**Center of Mathematical Sciences and Applications, Harvard University**

*Email address: yangzhou@cmsa.fas.harvard.edu*