A CLASSIFICATION FOR WAVE MODELS WITH TIME-DEPENDENT MASS AND SPEED OF PROPAGATION

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Abstract. In this paper, we study the long time behavior of energy solutions for a class of wave equation with time-dependent mass and speed of propagation. We introduce a classification of the potential term, which clarifies whether the solution behaves like the solution to the wave equation or Klein-Gordon equation. Moreover, \( L^q - L^2, q \in [1, 2] \) estimates for scale-invariant models are derived and applied to obtain global in time small data energy solutions for the semilinear Klein-Gordon equation in anti de Sitter spacetime.

1. Introduction

Let us consider the Cauchy problem for the wave equation with time-dependent mass and speed of propagation

\[
\begin{aligned}
    u_{tt} - a(t)^2 \Delta u + m(t)^2 u &= 0, & (t, x) \in (0, \infty) \times \mathbb{R}^n, \\
    (u(0, x), u_t(0, x)) &= (u_0(x), u_1(x)), & x \in \mathbb{R}^n.
\end{aligned}
\]

(1.1)

The Klein-Gordon type energy for the solution to (1.1) is given by

\[
E_{a,m}(t) = \frac{1}{2} \left( \|u_t(t, \cdot)\|_{L^2}^2 + a(t)^2 \|\nabla u(t, \cdot)\|_{L^2}^2 + m(t)^2 \|u(t, \cdot)\|_{L^2}^2 \right). \tag{1.2}
\]

One can observe many different effects for the behavior of \( E_{a,m}(t) \) as \( t \to \infty \) according to the properties of the speed of propagation \( a(t) \) and the coefficient \( m(t) \) in the mass term.

We first discuss properties of the energy in the case \( m(t) \equiv 0 \) in (1.1). If \( 0 < a_0 \leq a(t) \leq a_1 \) for any \( t \geq 0 \), then the energy \( E_{a,0}(t) \) is equivalent to

\[
E_{1,0}(t) = \frac{1}{2} \left( \|u_t(t, \cdot)\|_{L^2}^2 + \|\nabla u(t, \cdot)\|_{L^2}^2 \right).
\]

Although \( E_{1,0}(t) \) is a conserved quantity, oscillations of \( a(t) \) may have a very deteriorating influence on the energy behavior for the solution to (1.1) (see [5] and [18]). On the other hand, if \( a \in C^2 \) and

\[
|a^{(k)}(t)| \leq C_k (1 + t)^{-k} \quad \text{for} \quad k = 1, 2
\]

(so only very slow oscillations are allowed), then the so-called generalized energy conservation property holds [17]. This means, there exist positive constants \( C_0 \) and \( C_1 \) such that the inequalities

\[
C_0 E_{a,0}(0) \leq E_{a,0}(t) \leq C_1 E_{a,0}(0)
\]

(1.3)

are valid for all \( t \in (0, \infty) \), where the constants are independent of the data.

If \( a(t) \geq a_0 > 0 \) is an increasing function satisfying a suitable control on the oscillations, then one can prove the estimates [4]

\[
E_{a,0}(t) \leq C a(t) \left( E_{a,0}(0) + \|u_0\|_{L^2}^2 \right).
\]

(1.4)
We remark that an essential difference between (1.3) and (1.4) is that on the right-hand side of (1.4) it appears the $H^1$ norm of $u_0$, not only the $L^2$-norm of its gradient as in (1.3).

In the case $a(t) \equiv 1$, $E_{1,m}(t)$ is a conserved quantity for the classical Klein-Gordon equation, whereas it is known that the behavior of the potential energy $|u(t, \cdot)|_{L^2}$ change accordingly $\lim_{t \to \infty} tm(t) = 0$ or $\lim_{t \to \infty} tm(t) = 0$. To explain this effect, let us consider the energy

$$E_p(u)(t) = \frac{1}{2} \left( \|u_t(t, \cdot)\|^2_{L^2} + \|\nabla_x u(t, \cdot)\|^2_{L^2} + p(t)^2 \|u(t, \cdot)\|^2_{L^2} \right).$$

In the PhD thesis [2], the author studied decreasing coefficients $m = m(t)$ which satisfy among other things $\lim_{t \to \infty} tm(t) = 0$. In this case the potentials are called effective, i.e., the decays of the solution and its derivatives are related with the decays of the classical Klein-Gordon equation measured in the $L^q$ norm. Under some additional condition on $m$, the following energy estimate was derived

$$E_p(u)(t) \leq CE_p(u)(0),$$

with $p(t)^2 = m(t)$. For decreasing $m$ estimate (1.5) is better than an estimate like $E_{1,m}(t) \leq CE_{1,m}(0)$. In [2], the authors derived the energy estimate (1.5) for scale invariant models $m(t) = \frac{1}{1+q}, \mu > 0$, but now the constant $\mu$ has influence in the function $p(t)$.

In [10] the authors explained qualitative properties of solution to (1.1) in the case $\lim_{t \to \infty} tm(t) = 0$. If $(1 + t)m(t)^2 \in L^1(\mathbb{R}^+)$, it was proved a scattering result to free wave equation, whereas the potentials are called non-effective, i.e., the decays of the solution and its derivatives are related with the decays of the free wave equation measured in the $L^q$ norm. The energy estimate (1.5), with $p(t) = (1 + t)^{-1}u(t)$ and $\psi$ an increasing function, was obtained in the case of $(1 + t)m(t)^2 \notin L^1(\mathbb{R}^+)$.

In [10] the authors derived an explicit solution to the Cauchy problem for the well-known Klein-Gordon equation in anti de Sitter spacetime

$$u_{tt} - e^{2t}\Delta u + m^2 u = 0, \quad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x).$$

By using the obtained representation of solutions, in [11] it was proved some $L^q - L^{q'}$ estimates, with $q \in (1, 2]$ and $\frac{4}{q} + \frac{4}{q'} = 1$, but exception for the case of space dimension $n = 1$, we have some loss of regularity with respect to the initial data. Due to the lack of $L^q - L^{q'}$ estimates without loss of regularity, with $1 \leq q \leq r \leq \infty$, it is a challenging problem to derive the critical exponents for global (in time) small data energy solutions to the Cauchy problem for the semilinear Klein-Gordon equation in anti de Sitter spacetime

$$u_{tt} - e^{2t}\Delta u + m^2 u = |u|^p, \quad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x).$$

The same difficulties took place in the treatment of the classical Klein-Gordon semilinear equation with power nonlinearity $|u|^p$, being that only in the late nineties it was shown that, for space dimension $n \leq 3$, the critical exponent is the well known Fujita index $p_{Fuj} = 1 + \frac{2}{n}$, i.e., global (in time) existence of small data energy solutions holds for $p > p_{Fuj}$ (see [13]), whereas blow up results are established in [12] for $1 < p \leq p_{Fuj}$.

In this paper, our main goal is to introduce a classification (see Definition 2.2) for the potentials in (1.1) in terms of the time-dependent speed of propagation $a(t)$, consistently extended from the case of constant speed of propagation $a \equiv 1$. In the case of effective and non-effective potentials we derive sharp energy estimates and we show optimality of the results by the aid of scale-invariant models. Moreover, we explain some gaps from the thesis [2] and we derive $L^q - L^{2q}$ estimates, with $q \in [1, 2]$, for the model (1.6). As an application to our derived linear estimates,
we proved global existence (in time) of small data energy solutions for the Cauchy problem 1.7.

For the ease of reading, we summarize the scheme of the paper:

- in Section 2, we propose a classification for the potential term and we state our main results;
- in Section 3, we describe a diagonalization procedure to be used in sections 4 and 5;
- in Section 4, we prove the result for effective potential;
- in Section 5, we prove the result for non-effective potential;
- in Section 6, we prove the Scattering result;
- in Section 7, we discuss some scale invariant models and we prove Theorem 2.4;
- Section 8 completes the paper with concluding remarks and open problems.

2. Main Results

In this paper, we use the following notations.

Notation 1. Let $f, g : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be two strictly positive functions. We use the notation $f \approx g$ if there exist two constants $C_1, C_2 > 0$ such that $C_1 g(y) \leq f(y) \leq C_2 g(y)$ for all $y \in \Omega$. If the inequality is one-sided, namely, if $f(y) \leq C g(y)$ (resp. $f(y) \geq C g(y)$) for all $y \in \Omega$, then we write $f \lesssim g$ (resp. $f \gtrsim g$).

To state our results and assumptions on the coefficients of the equation in (1.1) we introduce some auxiliary functions.

Definition 2.1. Let $a \in C^2[0, \infty)$ be a strictly positive function. We define

$$A(t) = 1 + \int_0^t a(\tau) d\tau, \quad \eta(t) = a(t) A(t).$$

To study the interaction between $a$ and $m$ we assume the following conditions:

Hypothesis 1. We assume that $a \in C^2[0, \infty)$, $a(t) > 0$, with $a \not\in L^1$ and $a(0) = 1$, together with the estimates

$$\frac{|a^{(k)}(t)|}{a(t)} \lesssim \eta(t)^k \quad \text{for} \quad k = 1, 2.$$ (2.1)

Hypothesis 2. We assume that $m \in C^2[0, \infty)$ has the form

$$m(t) = \mu(t) \eta(t) > 0,$$

with $m(0) = 1$, may have an oscillating behavior. For this reason we suppose

$$|\mu^{(k)}(t)| \lesssim \mu(t) \eta(t)^k \quad \text{for} \quad k = 1, 2.$$ (2.2)

Definition 2.2. We propose the following classification of potential terms:

1. The potential term $m(t)^2 u$ generates scattering to the corresponding wave model if

$$\frac{A(t)}{a(t)} m(t)^2 \in L^1(\mathbb{R}^n).$$ (2.3)

2. The potential term $m(t)^2 u$ represents a non-effective potential if

$$\lim_{t \to \infty} \mu(t) = 0,$$ (2.4)

but $\frac{A(t)}{a(t)} m(t)^2 \not\in L^1[0, \infty)$.  

3. The potential term $m(t)^2 u$ generates an effective potential if

$$\lim_{t \to \infty} \mu(t) = \infty.$$ (2.5)
But there exists a grey zone. The models of the grey zone are in the boundary between effective and non-effective potentials, they can be described by the models
\[ u_{tt} - a(t)^2 \Delta u + \mu^2 \frac{a(t)^2}{A(t)^2} u = 0, \]
where \( \mu \geq 0 \) is a constant. For these models, the size of the constant \( \mu \) may have some influence in the long time behaviour, namely, for small \( \mu \) we are in the case of non-effective potentials and we are able to include it in Theorem 2.2 whereas for large \( \mu \) we are in the case of effective potential, but for simplicity of the proof we did not included it in Theorem 2.1. For this reason, we discuss some scale invariant models in sections 4.

2.1. Effective potential. To state our result in the case of effective potential we define
\[ \gamma(t) \doteq \max\{a(t), m(t)\} \] (2.6)
and the following energy
\[ E(u)(t) \doteq \frac{1}{2} \left( \|u_t(t, \cdot)\|_{L^2}^2 + a(t)^2 \|\nabla_x u(t, \cdot)\|_{L^2}^2 + m(t) \gamma(t) \|u(t, \cdot)\|_{L^2}^2 \right). \] (2.7)

**Theorem 2.1.** Let \((u_0, u_1) \in H^1 \times L^2\) and \(u\) be an energy solution of the Cauchy problem (1.1). We assume condition (2.6), Hypotheses 1 and 2. In addition, if \( \frac{\mu}{m} \in L^1[0, \infty) \), then we have the following estimate for the energy
\[ E(u)(t) \lesssim \gamma(t) E(u)(0), \quad \forall t \geq 0, \] (2.8)
where \( \gamma \) is given by (2.6).

**Remark 1.** If \( \frac{\mu}{m} \) is bounded, then
\[ \frac{\eta}{\mu} = \frac{a}{m} \frac{a}{A^2} \in L^1. \]
If \( \frac{\mu}{m} \notin L^1[0, \infty) \), we are near to models in the so-called grey zone.

**Remark 2.** From Theorem 2.1 we conclude that for increasing \( m \) the potential energy decay, i.e.,
\[ \|u(t, \cdot)\|_{L^2}^2 \lesssim \frac{1}{m(t)} E(u)(0), \quad \forall t \geq 0. \]
This estimate is better than the conjecture done in 2.

**Remark 3.** If \( \max\{a(t), m(t)\} = m(t) \), under additional regularity on the initial data and using the a priori estimates for \( \|u(t, \cdot)\|_{L^2} \), one may derive a better decay for the elastic energy, namely,
\[ a(t)^2 \|\nabla_x u(t, \cdot)\|_{L^2}^2 \lesssim a(t)(\|u_0\|_{H^2}^2 + \|u_1\|_{H^1}^2). \]

**Example 2.1.** Let \( a(t) = m(t) \), that is, take \( \mu(t) = A(t) \). Thanks to \( a \notin L^1 \), it is clear that \( \frac{\mu(t)}{m(t)} = \frac{a(t)}{A(t)} \in L^1[0, \infty) \) and the conclusion of Theorem 2.1 holds with \( \gamma(t) = a(t) \). In particular, for increasing \( a \) the potential energy decay
\[ \|u(t, \cdot)\|_{L^2}^2 \lesssim \frac{1}{m(t)} E(u)(0), \quad \forall t \geq 0. \]
In [19] the authors derived \( L^p - L^q \) estimates for the elastic and kinetic energy for this model.

**Example 2.2.** Let \( a(t) = (1 + t)^\ell \), with \( \ell > -1 \), and \( m(t) = \mu(1 + t)^{\ell - 1} \), with \( \epsilon > 0 \). Then \( \mu(t) \sim (1 + t)^\ell \) and the statement of Theorem 2.1 holds with \( \gamma(t) = (1 + t)^\ell \) for \( \epsilon \leq \ell + 1 \) and \( \gamma(t) = (1 + t)^{\ell - 1} \) for \( \epsilon > \ell + 1 \). (see Example 7.2 for the limit case \( \epsilon = 0 \)).
Example 2.3. Let \( a(t) = e^t \) and \( m(t) = \mu(1 + t) \), with \( \epsilon > 1 \). Then the statement of Theorem 2.1 holds with \( \gamma(t) = e^t \).

2.2. Non-effective potential.

To state our result in the case of non-effective potential we assume the following hypotheses:

**Hypothesis 3.** There exists a positive non-decreasing function \( \psi \in C^2(\mathbb{R}^+) \) with \( \psi(0) = 1 \) such that \( \frac{|\psi'(t)|}{\psi(t)} \) is increasing for large \( t \) and

\[
\frac{|\psi'(t)|}{\psi(t)} < c\eta(t), \quad c \in (0, 1).
\]

Besides (2.9) the following relation between \( m, \eta \) and \( \psi \) must be satisfied:

\[
\psi(t)\eta(t)\int_0^t \psi(t)^{-2}d\tau + \int_0^\infty \frac{1}{\eta(t)} \left| \frac{\psi'(t)}{\psi(t)} + m(t)^2 \right| d\tau \lesssim 1.
\]

Now we define

\[
E_p(u)(t) = \frac{1}{2} \left( \| u(t, \cdot) \|_{L^2}^2 + a(t) \| \nabla u(t, \cdot) \|_{L^2}^2 + p(t) \| u(t, \cdot) \|_{L^2}^2 \right),
\]

with

\[
p(t) \doteq \eta(t)\psi(t)\sqrt{q(t)}, \quad q(t) \doteq \max\{a(t), \psi(t)^{-2} \}.
\]

Then we have the following energy estimate:

**Theorem 2.2.** Let \((u_0, u_1) \in H^1 \times L^2\) and \( u \) be the solution of the Cauchy problem (1.1). We assume Hypotheses 1 to 3. Then we have the estimate

\[
E_p(u)(t) \lesssim q(t) E_p(u)(0), \quad \forall t \geq 0.
\]

**Remark 4.** As one may verified in the next examples, the condition (2.14) takes place in order to guarantee the existence of the function \( \psi \) in Hypothesis 3.

**Example 2.1.** (Scale invariant model)

Consider the Cauchy problem

\[
u_{tt} - (1 + t)2\Delta u + \frac{\mu^2}{(1 + t)^2} u = 0, \quad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x),
\]

where \( \ell > -1 \) and \( 0 < \mu^2 < \frac{1}{4} \). Let us take the function \( \psi \) from Hypothesis 3 as

\[
\psi(t) = (1 + t)^\sigma, \quad \text{with} \quad 2\sigma = 1 - \sqrt{1 - 4\mu^2}.
\]

It is clear that \( \psi'(t) + m(t)^2 = 0 \) and all the conditions from Hypotheses 2 to 3 are satisfied. If \((u_0, u_1) \in H^1 \times L^2\), then Theorem 2.2 implies the following estimates

\[
\| u(t, \cdot) \|_{L^2}^2 \lesssim (1 + t)^{1 + \sqrt{1 - 4\mu^2}}
\]

and

\[
\| u_t(t, \cdot) \|_{L^2}^2 + (1 + t)^2 \| \nabla u(t, \cdot) \|_{L^2}^2 \lesssim \begin{cases} (1 + t)^{\ell}, & \ell > \sqrt{1 - 4\mu^2}, \\ (1 + t)^{1 + \sqrt{1 - 4\mu^2}}, & \ell + 1 \leq \sqrt{1 - 4\mu^2}. \end{cases}
\]

**Remark 2.4.** The previous example can be treated in a different way, including also large parameter \( \mu \geq \frac{1}{4} \), see Example 7.2.

**Example 2.2.** Let \( a(t) = (1 + t)^\epsilon \) with \( \ell > -1 \). If \( m(t) = \frac{\mu(t)^2}{e^t} \), with \( \mu(t) \) satisfying conditions (2.2), (2.4) and \( \frac{\mu(t)^2}{e^t} \in L^1 \), then the conclusion of Theorem 2.2 holds with \( \psi(t) \) given by

\[
\psi(t) \doteq \exp \left( \int_0^t \frac{\mu(t)^2}{e^\tau} d\tau \right).
\]
Indeed,
\[
\psi''(\tau) + m(\tau)^2 = \frac{2\mu(t)\mu'(t)}{e + t} + \frac{\mu(t)^4}{(e + t)^2}
\]
Moreover, condition (2.4) implies that
\[
\int_0^t \psi(s)^{-2} ds \approx \frac{t}{\psi(t)^2},
\]
and $\frac{\psi''}{\psi}$ is increasing for large $t$. Indeed, integration by parts yields
\[
\int_0^t \psi(s)^{-2} ds = \frac{t}{\psi(t)^2} + 2 \int_0^t \frac{s}{\psi(s)^2 e + s} ds.
\]
On the one hand the right side is greater than $t\psi(t)^{-2}$. On the other hand,
$2t(e + t)^{-1}\mu(t)^2 < \epsilon$ for large $t$ and any $\epsilon > 0$. Then,
\[
\int_0^t \psi(s)^{-2} ds \leq \frac{1}{1 - \epsilon} \left( \frac{t}{\psi(t)^2} \right) \lesssim \frac{t}{\psi(t)^2}.
\]
Monotonicity is a consequence of
\[
\frac{d}{dt} \frac{t}{\psi(t)^2} = \left( 1 - \frac{2t\mu(t)^2}{(1 + t)} \right) \frac{1}{\psi(t)^2},
\]
which is positive for large $t$. Therefore, all the conditions from Hypotheses 1 to 3 are satisfied.
For instance, if $\mu(t)^2 = \frac{\mu^2}{\ln(e + t)} \ln^5(e + t)$ with $e^{[j+1]} = e^{5j}$ and $\ln^{[j+1]}(t) = \ln(\ln^{[j]}(t)), j = 1, 2, \cdots$, then $\psi(t) \approx \ln(\ln^{[j]}(e + t))^\gamma$.
If $\mu(t) = \frac{\mu}{\ln(e + t)}$, with $\mu > 0$ and $\gamma > \frac{1}{4}$, then $\psi(t)$ given by
\[
\psi(t) = \begin{cases}
\exp(\mu^2(\ln(e + t))^{1-2\gamma}), & \gamma \in \left( \frac{1}{4}, \frac{1}{2} \right), \\
\ln(e + t)^\gamma, & \gamma = \frac{1}{2}, \\
1, & \gamma > \frac{1}{2}.
\end{cases}
\]
Remark 2.5. In Example 2.2 if $\frac{\mu(t)^4}{e + t} \notin L^1$, one may still derive a result as in Theorem 2.2 (see [9]) by replacing $\psi$ by
\[
\psi(t) = \exp \left( \sum_{k=1}^{N} \gamma_k \int_0^t \frac{\mu(\tau)^{2k}}{(1 + \tau)^{d}} d\tau \right),
\]
for some integer $N > 1$. The sequence $\{\gamma_k\}_k$ in (2.14) is relate to the well-known Catalan numbers which can be found for instance in [13].

2.3. Scattering to wave equation.

In this section we will impose conditions for $a(t)$ and $m(t)$ such that the solutions $u = u(t, x)$ of Cauchy problem (1.1) behave asymptotically equal to the solution of corresponding wave equation with strictly increasing speed of propagation
\[
v_{tt} - a(t)^2 \Delta v = 0, \quad v(0, x) = v_0(x), \quad v_t(0, x) = v_1(x),
\]
with some suitable Cauchy data $(v_0, v_1)$.
Let us define the function space
\[
E = L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n).
\]
Before stating the result we define for any $\epsilon > 0$ the following closed subset of $E$:
\[
F_{\epsilon} := \{ U_0 \in E : U_0(\xi) = 0 \text{ for any } |\xi| \leq \epsilon \}.
\]
We remark that $\mathcal{L} = \cup_{\epsilon > 0} F_{\epsilon}$ is a dense subset of $E$. 

In addition to Hypothesis 1, we assume \( a'(t) > 0 \) for all \( t \in [0, \infty) \), together with the estimate
\[
\sqrt{a(t)} \int_0^t \sqrt{a(\sigma)} d\sigma \lesssim A(t). \tag{2.16}
\]

**Theorem 2.3.** Let us assume Hypothesis 1, conditions (2.23), (2.16) and that \( a'(t) > 0 \) for all \( t \in [0, \infty) \). Then, for any initial data \((u_0, u_1) \in H^1 \times L^2\), there exists a linear, bounded operator \( W_+(D) : E \to E \) such that if the initial data of the Cauchy problems (1.1) and (2.15) are related by \((a(0)\nabla v_0, v_1) = W_+(D)(\langle 0 \rangle u_0, u_1)\), it follows that the asymptotic equivalence of solutions holds
\[
\lim_{t \to \infty} \frac{1}{\sqrt{a(t)}} \left\| (a(t)\nabla v(t, \cdot), v_t(t, \cdot)) - \left( \langle D(t) \rangle u(t, \cdot), u_0(t, \cdot) \right) \right\|_E = 0, \tag{2.17}
\]
where \( \langle D(t) \rangle \) denotes the pseudodifferential operator having the symbol
\[
h(t, \xi) = (|\xi|^2 a(t)^2 + N^2 \eta(t)^2)^{\frac{q}{2}}. \tag{2.18}
\]
Moreover, on the dense subset \( L \) we can state the decay rate as
\[
\frac{1}{\sqrt{a(t)}} \left\| (a(t)\nabla v(t, \cdot), v_t(t, \cdot)) - \left( \langle D(t) \rangle u(t, \cdot), u_0(t, \cdot) \right) \right\|_L \lesssim \int_t^\infty A(\tau) a(\tau) m^2(\tau) d\tau \tag{2.19}
\]
as \( t \) goes to infinity.

**Remark 2.6.** Condition (2.16) holds for a large class of examples like \( a(t) = (1 + t)^t \), \( t \geq 0 \), \( a(t) = e^t \) and \( a(t) = e^{\epsilon t} e^t \). More in general, it is true under the assumption \( a'(t) \leq C_1 a(t) \eta(t) \) for all \( 0 < C_1 < 2 \) in (2.1). Indeed, let us consider the function
\[
F(t) = \int_0^t \sqrt{a(\sigma)} d\sigma - C \frac{A(t)}{\sqrt{a(t)}}.
\]
Then
\[
F'(t) = \sqrt{a(t)} - C \sqrt{a(t)} + \frac{CA(t)}{2 \sqrt{a(t)}} a'(t) \leq \left( 1 - C + \frac{CC_1}{2} \right) \sqrt{a(t)} < 0,
\]
for \( C > 0 \) sufficiently large.

### 2.4. Semilinear Klein-Gordon equation in anti de Sitter spacetime.

Let us consider the Cauchy problem for the semilinear Klein-Gordon equation in anti de Sitter spacetime
\[
u^t - e^{2t} \Delta u + m^2 u = |u|^p, \quad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \tag{2.20}
\]
with \( m > 0 \) and \( p > 1 \). Let us define the function spaces
\[
D_q^p(\mathbb{R}^n) = (H^q \cap L^p) \times (H^q \cap L^p)
\]
with \( q \in [1, 2) \) and the norm \( \| (u, v) \|_{D_q^p} = \| u \|_{L^p} + \| u \|_{H^{q+1}} + \| v \|_{L^q} + \| v \|_{H^{\frac{n}{2}}}. \) In the following, we denote \( D_0^q = D_q. \)

**Theorem 2.4.** Let \( n \leq 4, \ m > 0 \) and
\[
2 \leq p \leq \frac{n}{n-2}.
\]
Then there exists a constant \( \varepsilon > 0 \) such that for all \((u_0, u_1) \in D_1(\mathbb{R}^n)\) with
\[
\| (u_0, u_1) \|_{D_1(\mathbb{R}^n)} \leq \varepsilon
\]
there exists a uniquely determined energy solution \( u \in C([0, \infty), H^1(\mathbb{R}^n)) \cap C^1([0, \infty), L^2(\mathbb{R}^n)) \) to (2.20). Moreover, the solution satisfies the following estimates
\[
\| u_t(t, \cdot) \|_{L^2} + e^t \| \nabla_x u(t, \cdot) \|_{L^2} \lesssim e^{t/2} (\| u_0 \|_{H^1} + \| u_1 \|_{L^2}), \quad \forall t \geq 0,
\]
where
\[ d(t) = \begin{cases} \frac{1}{t^{\frac{n}{2}}} & \text{for } n \geq 2, \\ 1 & \text{for } n = 1. \end{cases} \]

Remark 2.7. If \( n = 1 \), by using the embedding of \( \mathcal{H}^{1}(\mathbb{R}) \) into \( L^{\infty}(\mathbb{R}) \) and interpolation results, we no longer need to use Gagliardo-Nirenberg inequality in the proof of Theorem 2.4 and the conclusions are still true for all \( p > 1 \).

3. Diagonalization Procedure

We perform the Fourier transform of (1.1) with respect to \( x \) obtaining
\[
\begin{aligned}
\hat{u}_{tt} + \langle \xi \rangle_{a,m}(t)^2 \hat{u} &= 0, \\
\left( \hat{u}(0, \xi), \hat{u}_{t}(0, \xi) \right) &= \left( \hat{u}_{0}(\xi), \hat{u}_{1}(\xi) \right),
\end{aligned}
\]

(3.1)

where \( \langle \xi \rangle_{a,m}(t) = (|\xi|^2a(t)^2 + m(t)^2)^{1/2} \). We put
\[ U = (i \langle \xi \rangle_{a,m}(t) \hat{u}, \hat{u}_{t})^T, \]

so from (3.1) we derive the system
\[
\partial_{tt} U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} i \langle \xi \rangle_{a,m}(t) U + \frac{\partial_{tt} \langle \xi \rangle_{a,m}(t)}{\langle \xi \rangle_{a,m}(t)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} U.
\]

(3.2)

Let \( P \) be the (constant, unitary) diagonalizer of the principal part of (3.2), given by
\[ P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad P^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \]

that is, if we put \( V(t, \xi) = P^{-1} U(t, \xi) \), then we get
\[
\partial_{tt} V = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} i \langle \xi \rangle_{a,m}(t) V + R_{1}(t, \xi) V,
\]

(3.3)

where
\[ R_{1}(t, \xi) = \frac{1}{2} \frac{\partial_{tt} \langle \xi \rangle_{a,m}(t)}{\langle \xi \rangle_{a,m}(t)} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \]

We define the refined diagonalizer which depends on the not diagonal entries of \( R_{1}(t, \xi) \):
\[
K(t, \xi) = I + K_{1}(t, \xi), \quad K_{1}(t, \xi) = \frac{1}{4i} \frac{\partial_{tt} \langle \xi \rangle_{a,m}(t)}{\langle \xi \rangle_{a,m}(t)^{2}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
\]

(3.4)

By using Hypotheses 1 and 2 if \( \langle \xi \rangle_{a,m}(t) \eta(t)^{-1} \geq N \) we have
\[
\left| \frac{\partial_{tt} \langle \xi \rangle_{a,m}(t)}{\langle \xi \rangle_{a,m}(t)^{2}} \right| \leq \frac{1}{\langle \xi \rangle_{a,m}(t)^{2}} \times \frac{|a(t)\eta(t)|}{\langle \xi \rangle_{a,m}(t)^{3}} \leq C \eta(t),
\]

(3.5)

hence \( |\text{det } K| \geq 1 - C^{2}/N^{2} \). Therefore, \( K(t, \xi) \) is uniformly regular and bounded for a sufficiently large \( N \). We replace \( V(t, \xi) = K(t, \xi) W(t, \xi) \) and we get
\[
\partial_{tt} W = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} i \langle \xi \rangle_{a,m}(t) W + \frac{1}{2} \frac{\partial_{tt} \langle \xi \rangle_{a,m}(t)}{\langle \xi \rangle_{a,m}(t)} IW + R_{2}(t, \xi) W,
\]

(3.6)

where the matrix \( R_{2} \) is given by (see Lemma 5 in [4])
\[ R_{2}(t, \xi) = (\partial_{tt} K_{1} + K_{1} R_{1}) K^{-1}. \]
Thanks again to Hypotheses 1 and 2 the matrices $R_2(t, \xi)$ satisfies the following estimate
\[ \|R_2(t, \xi)\| \lesssim \frac{\eta(t)^2}{\langle \xi \rangle_{a,m}(t)}. \] (3.7)

Now let
\[ D(t, \xi) = \text{diag} \left( \exp \left( - \int_s^t i \langle \xi \rangle_{a,m}(\tau) d\tau \right), \quad \exp \left( \int_s^t i \langle \xi \rangle_{a,m}(\tau) d\tau \right) \right). \] (3.8)

We put $W(t, \xi) = \sqrt{\langle \xi \rangle_{a,m}(t)} D(t, \xi) Z(t, \xi)$ and we obtain
\[ \begin{cases} \partial_t Z = R_3(t, \xi) Z, \\ Z(s, \xi) = K^{-1}(s, \xi) P^{-1} U(s, \xi), \end{cases} \] (3.9)

where the matrix $R_3(t, \xi) = D^{-1}(t, \xi) R_2(t, \xi) D(t, \xi)$ satisfies again (3.8).

4. Effective Potential

Proof. (Theorem 2.1)

We claim that
\[ \mathcal{E}(t, \xi) \leq C \gamma(t) \mathcal{E}_0(\xi), \] (4.1)

uniformly with respect to $\xi \in \mathbb{R}^n$, where $\mathcal{E}(t, \xi)$ and $\mathcal{E}_0(\xi)$ are given by
\[ \mathcal{E}(t, \xi) = |\hat{\mu}_t(t, \xi)|^2 + (a(t)^2 |\xi|^2 + \gamma(t) m(t)) |\hat{\mu}_t(t, \xi)|^2, \] (4.2)
\[ \mathcal{E}_0(\xi) = |\hat{\mu}_0(\xi)|^2 + (1 + |\xi|^2) |\hat{\mu}_0(\xi)|^2. \] (4.3)

Indeed, by integrating this inequality with respect to $\xi$ and by Plancherel’s Theorem, estimate (2.8) will follow from (4.1). In order to prove (4.1), for some constant $N > 0$, we divide the extended phase space $[0, \infty) \times \mathbb{R}^n$ into the pseudodifferential zone $Z_{pd}(N)$ and into the hyperbolic zone $Z_{hyp}(N)$, defined by
\[ Z_{pd}(N) = \{ (t, \xi) \in [0, \infty) \times \mathbb{R}^n : \langle \xi \rangle_{a,m}(t) \eta(t)^{-1} \leq N \}, \]
\[ Z_{hyp}(N) = \{ (t, \xi) \in [0, \infty) \times \mathbb{R}^n : \langle \xi \rangle_{a,m}(t) \eta(t)^{-1} \geq N \}. \]

By using the definition of $m$ we derive that
\[ \langle \xi \rangle_{a,m}(t) \eta(t)^{-1} = (A(t)^2 |\xi|^2 + \mu(t)^2)^{1/2}. \]

Therefore, thanks to the effective condition (2.5), $Z_{pd}(N)$ is a compact subset of the extended phase space. Then there exists a constant $T > 0$ such that
\[ \mathcal{E}(t, \xi) \lesssim \mathcal{E}_0(\xi), \]
for all $0 \leq t \leq T$ and $(t, \xi) \in Z_{pd}(N)$.

In $Z_{hyp}(N)$ we use the calculations of Section 3. Thanks to $\frac{\mu(t)}{\mu(t)} \in L^1[0, \infty)$ we have
\[ \int_s^t \|R_3(\tau, \xi)\| d\tau \lesssim \int_s^\infty \frac{\eta(\tau)^2}{\langle \xi \rangle_{a,m}(\tau)} d\tau \lesssim \int_s^\infty \frac{\eta(\tau)^2}{m(\tau)} d\tau = \int_s^\infty \frac{\eta(\tau)^2}{\mu(\tau)} d\tau \leq C, \]

hence $|Z(t, \xi)| \leq C |Z(s, \xi)|$ and, by using Liouville’s formula, $|Z(t, \xi)| \geq C' |Z(s, \xi)|$. Therefore we have proved that in $Z_{hyp}(N)$ it holds
\[ C_1 \frac{\langle \xi \rangle_{a,m}(t)}{\langle \xi \rangle_{a,m}(s)} |U(s, \xi)|^2 \leq |U(t, \xi)|^2 \leq C_2 \frac{\langle \xi \rangle_{a,m}(t)}{\langle \xi \rangle_{a,m}(s)} |U(s, \xi)|^2. \] (4.4)

We remark that (4.4) is a two-sided estimate, that is, we have a precise description of the behavior of the energy in $Z_{hyp}(N)$.
Using again that $Z_{pd}(N)$ is a compact subset, we have that $s \in [0, T]$ and $\langle \xi, a, m(s) \rangle \geq C \langle \xi, a, m(0) \rangle$. Then, (1.3) implies

$$|\hat{u}(t, \xi)|^2 + a(t)^2 |\xi|^2 |\hat{u}(t, \xi)|^2 \lesssim \frac{\langle \xi, a, m(t) \rangle}{1 + |\xi|^2}|\xi_a(t)\xi_m(\xi)| \lesssim (a(t) + m(t)) \xi_a(\xi),$$

for all $t \geq 0$. Now, using again (1.3), for all $(t, \xi) \in Z_{hyp}(N)$ we have that

$$m(t)|\hat{u}(t, \xi)|^2 = m(t)\langle \xi, a, m(t) \rangle^2 \langle \xi, a, m(0) \rangle \xi_a(t)\xi_m(\xi)|^2 \lesssim \frac{m(t)}{\langle \xi, a, m(t) \rangle^2} \langle \xi, a, m(0) \rangle^2 |\hat{u}(0, \xi)|^2$$

$$\lesssim \frac{m(t)}{\langle \xi, a, m(t) \rangle^2} \langle \xi, a, m(0) \rangle^2 |\xi_a(0, \xi)|^2 + |\xi_a(0, \xi)|^2 \xi_a(t),$$

thanks to

$$\frac{m(t)}{\langle \xi, a, m(t) \rangle} \lesssim \left(1 + \frac{|\xi|^2 a(t)^2}{m(t)^2}\right)^{-1/2} \lesssim 1. \quad \square$$

5. Non-effective potential

In order to get some feeling for the behavior of solutions to (1.1) in the case of non-effective potential we can see (3) transform the time-dependent potential to a time-dependent damping and a new potential. If we introduce the change of variables given by $u(t, x) = \psi(t)v(t, x)$ the Cauchy problem (1.1) takes the form

$$\begin{align*}
v_{tt} - a(t)^2 \Delta v + 2 \frac{\psi'}{\psi}(t)v_t + \left(\frac{\psi''}{\psi}(t) + m(t)^2\right)v &= 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n, \\
(v(0, x), v_t(0, x)) &= (v_0(x), v_1(x)), \quad x \in \mathbb{R}^n.
\end{align*} \tag{5.1}$$

Aiming to exclude contributions to the energy coming from the time-dependent potential, thanks to the scattering assumption (2.8), a sufficient condition is

$$\frac{A(t)}{a(t)} \left(\frac{\psi''}{\psi}(t) + m(t)^2\right) \in L^1. \tag{5.2}$$

Under this assumption, we may use some ideas developed in (7) to derive asymptotic properties of solutions to wave equations with time-dependent non-effective dissipation (see also (21)).

Here we divide again the extended phase space $[0, \infty) \times \mathbb{R}^n$ into the pseudodifferential zone $Z_{pd}(N)$ and into the hyperbolic zone $Z_{hyp}(N)$ which are defined by

$$Z_{pd}(N) = \{(t, \xi) \in [0, \infty) \times \mathbb{R}^n : A(t)|\xi| \leq N\},$$

$$Z_{hyp}(N) = \{(t, \xi) \in [0, \infty) \times \mathbb{R}^n : A(t)|\xi| \geq N\}.$$

The separating curve is given by

$$\theta : (0, N] \to [0, \infty), \quad \theta_{N} = A^{-1}(N/|\xi|).$$

We put also $\theta_0 = \infty$, and $\theta_{N} = 0$ for any $|\xi| \geq N$. The pair $(t, \xi)$ from the extended phase space belongs to $Z_{pd}(N)$ (resp. to $Z_{hyp}(N)$) if and only if $t \leq \theta_{N}$ (resp. $t \geq \theta_{N}$).

In the $Z_{hyp}(N)$ we use the same energy and diagonalization procedure done in Section 3 to conclude

$$C_1 \langle \xi, a, m(s) \rangle^2 |U(s, \xi)|^2 \leq |U(t, \xi)|^2 \leq C_2 \langle \xi, a, m(t) \rangle^2 |U(s, \xi)|^2, \tag{5.3}$$
thanks to
\[
\int_s^t \| R_2(\tau, \xi) \| \, d\tau \lesssim \int_s^\infty \frac{\eta(\tau)^2}{|\xi| \lambda_m(\tau)} \, d\tau \lesssim \int_s^\infty \frac{a(\tau)}{|\xi| \lambda_m(\tau)^2} \, d\tau \lesssim \frac{1}{|\xi| \lambda_m(s)} \leq C.
\]
But differently from the effective potential, now \( Z_{pd}(N) \) is no longer a compact subset, so we have to apply a new strategy to derive estimates in this zone:

**Consideration in the pseudo-differential zone.** We will consider the following micro-energy in the pseudo-differential zone

\[
V = \left( \psi(t) \eta(t) \mathcal{A}, \psi(t) \mathcal{B} - \psi'(t) \mathcal{A} \right)^T, \quad V_0(\xi) = \left( \mathcal{A}_0(\xi), \mathcal{B}_0(\xi) - \mu(0)^2 \mathcal{A}_0(\xi) \right)^T.
\]

So we have

\[
\partial_t V(t, \xi) = A(t, \xi) V := \begin{pmatrix}
\frac{\psi'(t)}{\psi(t)} + 2 \frac{\psi''(t)}{\psi(t)} \langle \xi \rangle_{m, m}(t)^2 & \eta(t) \\
-\frac{1}{\eta(t)} \left( \frac{\psi''(t)}{\psi(t)} \langle \xi \rangle_{m, m}(t)^2 \right) & 0
\end{pmatrix} V. \tag{5.4}
\]

We want to prove that the fundamental solution \( E = E(t, s, \xi) \) to (5.4), that is, the solution to

\[
\partial_t E = A(t, \xi) E, \quad E(s, s, \xi) = I,
\]

is bounded for all \( t \in [0, \theta\xi] \). If we put \( E = (E_{ij})_{i,j=1,2} \), then we can write for \( j = 1, 2 \) the following system of coupled integral equations of Volterra type:

\[
E_{1j}(t, 0, \xi) = \eta(t) \psi(t)^2 \left( \delta_{1j} + \int_0^t \frac{1}{\psi(\tau)^2} E_{2j}(\tau, 0, \xi) \, d\tau \right), \tag{5.5}
\]

\[
E_{2j}(t, 0, \xi) = \delta_{2j} - \int_0^t \frac{1}{\eta(\tau)} \left( \frac{\psi''(\tau)}{\psi(\tau)} + \langle \xi \rangle_{m, m}(\tau)^2 \right) E_{1j}(\tau, 0, \xi) \, d\tau. \tag{5.6}
\]

By replacing (5.6) into (5.5) and after integration by parts we get

\[
E_{1j}(t, 0, \xi) = \eta(t) \psi(t)^2 \left( \delta_{1j} + \delta_{2j} \int_0^t \psi(\tau)^{-2} \, d\tau \right) - \eta(t) \psi(t)^2 \int_0^t \frac{1}{\eta(\tau)} \left( \frac{\psi''(\tau)}{\psi(\tau)} + \langle \xi \rangle_{m, m}(\tau)^2 \right) E_{1j}(\tau, 0, \xi) \, d\tau. \tag{5.7}
\]

By using (5.10) we conclude from (5.7) that

\[
|E_{1j}(t, 0, \xi)| \leq C + C \int_0^t \frac{1}{\eta(\tau)} \left| \frac{\psi''(\tau)}{\psi(\tau)} + \langle \xi \rangle_{m, m}(\tau)^2 \right| |E_{1j}(t, 0, \xi)| \, d\tau.
\]

Applying Gronwall’s type inequality we conclude

\[
|E_{1j}(t, 0, \xi)| \leq C \exp \left( C \int_0^t \frac{1}{\eta(\tau)} \left( \frac{\psi''(\tau)}{\psi(\tau)} + m(\tau)^2 + \langle \xi \rangle_{m, m}(\tau)^2 \right) \, d\tau \right).
\]

In \( Z_{pd}(N) \) we have \( A(t)|\xi| \leq N \). So, from the last estimate we get

\[
|E_{1j}(t, 0, \xi)| \leq C \exp \left( C \int_0^t \frac{1}{\eta(\tau)} \left( \frac{\psi''(\tau)}{\psi(\tau)} + m(\tau)^2 \right) \, d\tau \right).
\]

Finally, by using again (5.10) we get \( \|E_{1j}(t, 0, \xi)\| \leq C \). From the boundedness of \( \|E_{1j}(t, 0, \xi)\| \) we can estimate \( \|E_{2j}(t, 0, \xi)\| \leq C \). Therefore, we proved

\[
\|V(t, \xi)\| \leq C \|V_0(\xi)\| \text{ for all } t \in (0, \theta\xi]. \tag{5.8}
\]
Proof. (Theorem 2.2)
We claim that
\[ |\tilde{u}_t(t, \xi)|^2 + a(t)^2 |\xi|^2 |\tilde{u}(t, \xi)|^2 \lesssim a(t) \left( (1 + |\xi|^2) |\tilde{u}_0(\xi)|^2 + |\tilde{u}_1(\xi)|^2 \right) \]  (5.9)
and
\[ |\tilde{u}(t, \xi)|^2 \lesssim \frac{1}{\eta(t)^2 \psi(t)^2} \left( |\tilde{u}_0(\xi)|^2 + \frac{|\tilde{u}_1(\xi)|^2}{1 + |\xi|^2} \right), \]  (5.10)
uniformly with respect to $\xi \in \mathbb{R}^n$. By integrating these inequalities with respect to $\xi$ and by Plancherel's Theorem we have our desired estimate (2.12).

Let us first prove (5.9). By using Cauchy-Schwarz inequality, (2.9) and the considerations in the pseudo-differential zone we conclude for all $t \leq \theta_{|\xi|}$ the estimates
\[ \frac{|V(t, \xi)|^2}{\psi(t)^2} \geq \eta(t)^2 |\tilde{u}(t, \xi)|^2 + |\tilde{u}_t(t, \xi)|^2 + \left| \frac{\psi'(t)}{\psi(t)} \right|^2 |\tilde{u}(t, \xi)|^2 - |\tilde{u}_t(t, \xi)| \left| \frac{2 \psi'(t)}{\psi(t)} \tilde{u}(t, \xi) \right| \]
\[ \geq \eta(t)^2 |\tilde{u}(t, \xi)|^2 + \frac{1}{2} |\tilde{u}_t(t, \xi)|^2 - \left| \frac{\psi'(t)}{\psi(t)} \right|^2 |\tilde{u}(t, \xi)|^2 \]
\[ \geq (1 - \epsilon^2) \eta(t)^2 |\tilde{u}_t(t, \xi)|^2 + \frac{1}{2} |\tilde{u}_t(t, \xi)|^2 \]
\[ \geq \frac{1 - \epsilon^2}{N^2} a(t)^2 |\xi|^2 |\tilde{u}_t(t, \xi)|^2 + \frac{1}{2} |\tilde{u}_t(t, \xi)|^2. \]
Therefore, by using (5.8) we have for all $t \leq \theta_{|\xi|}$
\[ |\tilde{u}_t(t, \xi)|^2 + a(t)^2 |\xi|^2 |\tilde{u}(t, \xi)|^2 \lesssim |V(t, \xi)|^2 \lesssim \frac{1}{\psi(t)^2} |V_0(\xi)|^2. \]  (5.11)

In the $Z_{hyp}(N) \cap \{|\xi| \geq N\}$, thanks to (5.9) we conclude
\[ |\tilde{u}_t(t, \xi)|^2 + a(t)^2 |\xi|^2 |\tilde{u}(t, \xi)|^2 \lesssim |U(t, \xi)|^2 \lesssim a(t) \left( (1 + |\xi|^2) |\tilde{u}_0(\xi)|^2 + |\tilde{u}_1(\xi)|^2 \right). \]

In the $Z_{hyp}(N) \cap \{|\xi| \leq N\}$, we have to glue the estimate (5.8) with (5.9). Putting $s = \theta_{|\xi|}$ and using (5.8) we have
\[ |\tilde{u}_t(t, \xi)|^2 + a(t)^2 |\xi|^2 |\tilde{u}(t, \xi)|^2 \lesssim \frac{\langle \xi \rangle_{a,m}(t)}{\langle \xi \rangle_{a,m}(s)} \left( \langle \xi \rangle_{a,m}(s)^2 |\tilde{u}(s, \xi)|^2 + |\tilde{u}_t(s, \xi)|^2 \right). \]
Due to $A(s)|\xi| = N$ and $m(s) \leq |\xi| a(s)$ we have that
\[ \frac{\langle \xi \rangle_{a,m}(t)}{\langle \xi \rangle_{a,m}(s)} \lesssim \eta(s)^{-1} \langle \xi \rangle_{a,m}(t) \lesssim a(t). \]
Therefore, using that $\psi$ is non-decreasing and (5.11) with $t = s$, we conclude
\[ |\tilde{u}_t(t, \xi)|^2 + a(t)^2 |\xi|^2 |\tilde{u}(t, \xi)|^2 \lesssim a(t) \left( |\tilde{u}_0(\xi)|^2 + |\tilde{u}_1(\xi)|^2 \right). \]

Now let us prove (5.10). For $t \leq \theta_{|\xi|}$ we have from (5.8) the estimate
\[ |\tilde{u}(t, \xi)|^2 \lesssim \frac{1}{\eta(t)^2 \psi(t)^2} |V_0(\xi)|^2. \]
In order to estimate $|\tilde{u}(t, \xi)|^2$ in the hyperbolic zone we split our considerations for $|\xi| \leq N$ and $|\xi| \geq N$. By definition, $\theta_{|\xi|} = 0$ for all $|\xi| \geq N$, and from (5.9) we have
\[ |\tilde{u}(t, \xi)|^2 \lesssim \frac{1}{\eta(t)^2 \psi(t)^2} \left( |\tilde{u}_0(\xi)|^2 + \frac{|\tilde{u}_1(\xi)|^2}{|\xi|^2} \right) \quad \text{for all } |\xi| \geq N. \]
On the other hand, for $|\xi| \leq N$, from (5.3) and (5.8) we conclude
\[ |\tilde{u}(t, \xi)|^2 \lesssim \frac{1}{\langle \xi \rangle_{a,m}(t) \langle \xi \rangle_{a,m}(s)} \left( \langle \xi \rangle_{a,m}(s)^2 |\tilde{u}(s, \xi)|^2 + |\tilde{u}_t(s, \xi)|^2 \right) \]
\[ \lesssim \frac{1}{\eta(t) \eta(s) \psi(s)^2} \left( |\tilde{u}_0(\xi)|^2 + |\tilde{u}_1(\xi)|^2 \right) \lesssim \frac{1}{\eta(t)^2 \psi(t)^2} \left( |\tilde{u}_0(\xi)|^2 + |\tilde{u}_1(\xi)|^2 \right). \]
thanks to $\frac{1}{\eta(t)\nu(t)^2}$ being increasing for $t$. Using that $a \not\in L^1$, the proof is completed.

\[\square\]

6. Scattering Theory

In order to define our scattering operator, first we have to prove some a priori estimates for the fundamental solution of a system associate to the Cauchy problem (1.1). To active this, we shall divide the extended phase space into two zones:

$$Z_{pd}(N) = \{(t, \xi) \in [0, \infty) \times \mathbb{R}^n : A(t)\|\xi\| \leq N\},$$
$$Z_{hyp}(N) = \{(t, \xi) \in [0, \infty) \times \mathbb{R}^n : A(t)\|\xi\| \geq N\}.$$

The separating curve is given by

$$\theta : (0, N] \to [0, \infty), \quad \theta_{\|\xi\|} = A^{-1}(N/\|\xi\|).$$

We put also $\theta_0 = \infty$, and $\theta_{\|\xi\|} = 0$ for any $\|\xi\| \geq N$. The pair $(t, \xi)$ from the extended phase space belongs to $Z_{pd}(N)$ (resp. to $Z_{hyp}(N)$) if and only if $t \leq \theta_{\|\xi\|}$ (resp. $t \geq \theta_{\|\xi\|}$).

Consideration in pseudo-differential zone.

We will consider the following micro-energy in the pseudo-differential zone:

$$U(t, \xi) = \frac{1}{\sqrt{a(t)}} (h(t, \xi) \tilde{u}, D_t \tilde{u})^T,$$

where $h$ is given by (6.2). Then

$$D_t U = \tilde{A}(t, \xi) U,$$

where

$$\tilde{A}(t, \xi) = \left( \frac{D_t h}{a(t)\|\xi\|^{2m(t)}}, -\frac{D_t a}{2n} \right).$$

We want to prove that the fundamental solution $E = E(t, s, \xi)$ to (6.2), that is, the solution to

$$D_t E = \tilde{A}(t, \xi) E \quad E(s, s, \xi) = I,$$

satisfies the estimate $\|E(t, s, \xi)\| \lesssim 1$. Indeed, in the pseudo-differential zone we shall only prove that the fundamental solution is bounded. If we put $E = (E_{ij})_{j=1,2}$, then we can write for $j = 1, 2$ the following system of coupled integral equations of Volterra type:

$$E_{11}(t, s, \xi) = \frac{h(t, \xi)}{a(t)} \left( \delta_{ij} \sqrt{a(s)} + i \int_s^t \sqrt{a(\tau)} E_{1j}(\tau, s, \xi) d\tau \right),$$

$$E_{2j}(t, s, \xi) = \frac{i}{\sqrt{a(t)}} \left( \delta_{2j} \sqrt{a(s)} + i \int_s^t \frac{a(\tau)\|\xi\|^2 + m(\tau)}{a(\tau)} \sqrt{a(\tau)} E_{1j}(\tau, s, \xi) d\tau \right).$$

We may assume that $a(0) = 1$. Replacing (6.4) in (6.3), with $s = 0$, and using that $a(t)$ is an increasing function we have

$$|E_{1j}(t, 0, \xi)| \leq \frac{h(t, \xi)}{\sqrt{a(t)}} (\delta_{ij} + t\delta_{2j})$$

$$+ \frac{h(t, \xi)}{\sqrt{a(t)}} \int_0^t \sqrt{a(\tau)} \int_0^\tau \frac{a(\sigma)\|\xi\|^2 + m(\sigma)^2}{a(\sigma)\xi} |E_{1j}(\sigma, 0, \xi)| d\sigma d\tau.$$  

\[1\] In the definition of the micro-energy we will use $D_t \tilde{a}$, where $D_t = \frac{1}{h} \partial_t$. 


Integration by parts yields
\[
|E_{ij}(t,0,\xi)| \leq \frac{h(t,\xi)}{\sqrt{a(t)}} (\delta_{ij} + t \delta_{2j}) \\
+ \frac{h(t,\xi)}{\sqrt{a(t)}} \int_0^t \frac{a(\tau)^2 |\xi|^2 + m(\tau)^2}{h(\tau,\xi)} |E_{ij}(\tau,0,\xi)| \left( \int_\tau^t \sqrt{a(\sigma)} d\sigma \right) d\tau.
\]
From the definition of the pseudo-differential zone we can get
\[
A(t) \frac{h(t,\xi)}{\sqrt{a(t)}} \sim 1,
\]
and thanks to hypothesis (2.16) we have that
\[
\frac{h(t,\xi)}{\sqrt{a(t)}} \left( 1 + t + \int_0^t \sqrt{a(\sigma)} d\sigma \right) \leq C.
\]
Hence, applying Gronwall’s inequality, condition (2.3) and from the definition of the pseudo-differential zone we can get
\[
|E_{1j}(t,0,\xi)| \leq \exp \left( \int_0^t a(\tau)^2 |\xi|^2 + m(\tau)^2 \sqrt{a(\tau)} d\tau \right) \leq C.
\]
Now, using again (2.3) and that \(|E_{1j}(t,0,\xi)|\) is bounded in (6.4), we also conclude that \(|E_{2j}(t,0,\xi)|\) \(\leq C\).

**Consideration in hyperbolic zone.**

Define the micro-energy
\[
U_W(t,\xi) = (a(t)|\xi| \mathbf{\hat{u}}, D_t \mathbf{\hat{u}})^T.
\]
Then \(U_W\) satisfies
\[
D_t U_W = A(t,\xi) U_W,
\]
with
\[
A(t,\xi) = \begin{pmatrix}
\frac{D_a(t)}{a(t)} & a(t)|\xi| \\
0 & \frac{m(t)^2}{a(t)|\xi|^2} + \frac{a(t)|\xi|^2}{a(t)}
\end{pmatrix}
\]
Let
\[
M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad M^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.
\]
If we define \(U^{(0)} = M^{-1} U_W\), then
\[
D_t U^{(0)} = (D(t,\xi) + R_a(t) + R_{a,m}(t)) U^{(0)},
\]
with
\[
D(t,\xi) = \begin{pmatrix} a(t)|\xi| & 0 \\ 0 & -a(t)|\xi| \end{pmatrix}, \quad R_a(t) = \frac{1}{2} \frac{D_a(t)}{a(t)} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}
\]
and
\[
R_{a,m}(t,\xi) = \frac{1}{2} \frac{m(t)^2}{a(t)|\xi|^2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.
\]
Let \(E_a = E_a(t,s,\xi)\) be the fundamental solution of the operator \(D_t - D(t,\xi) - R_a(t)\), that is, \(E_a\) satisfies the Cauchy problem
\[
(D_t - D(t,\xi) - R_a(t)) E_a(t,s,\xi) = 0, \quad E(s,s,\xi) = I.
\]
It is well known (see [4]) that
\[
\|E_a(t,s,\xi)\|_{L^\infty(\mathbb{R}^n)} \lesssim \frac{\sqrt{a(t)}}{\sqrt{a(s)}}.
\]
Moreover, using Liouville’s formula, we arrive at
\[ \det E_a(t, s, \xi) = \exp \left( i \int_s^t \text{tr}(D(\tau, \xi) + R_a(\tau))d\tau \right) = \frac{a(t)}{a(s)}. \]

Hence,
\[ \| E_a^{-1}(t, s, \xi) \|_{L^\infty(\mathbb{R}^n_+)} \lesssim \frac{\sqrt{a(s)}}{a(t)}. \]

Now the goal is to construct the fundamental solution to the operator \( D_1 - D(t, \xi) - R_a(t) - R_{a,m}(t) \). For this purpose let us introduce
\[ P(t, s, \xi) = E_a(t, s, \xi)^{-1} R_{a,m}(t, \xi) E_a(t, s, \xi). \]

Applying Peano-Backer formula we have that
\[ Q_{a,m}(t, s, \xi) = I + \sum_{k=1}^\infty \int_s^t \cdots \int_s^{t_{k-1}} P(t_1, s, \xi) P(t_2, s, \xi) \cdots P(t_k, s, \xi) dt_k \cdots dt_2 dt_1 \]
is the solution to the Cauchy problem
\[ D_1 Q_{a,m}(t, s, \xi) = P(t, s, \xi) Q_{a,m}(t, s, \xi), \quad Q_{a,m}(s, s, \xi) = I. \]

Therefore, if \( E_{a,m}(t, s, \xi) = E_a(t, s, \xi) Q_{a,m}(t, s, \xi) \) follows that
\[ D_1 E_{a,m}(t, s, \xi) = (D(t, \xi) + R_a(t) + R_{a,m}(t, \xi)) E_{a,m}(t, s, \xi), \quad E_{a,m}(s, s, \xi) = I. \]

From the definition of \( P(t, s, \xi) \) we can derive that
\[ \| P(t, s, \xi) \|_{L^\infty} \leq \| R_{a,m}(t, \xi) \|. \]

So, using the definition of the hyperbolic zone and the hypothesis (2.3) we arrive at
\[ \| Q_{a,m}(t, s, \xi) \|_{L^\infty} \leq \exp \left( \int_s^t \| P(\tau, s, \xi) \|_{L^\infty} d\tau \right) \leq \exp \left( \int_s^t \frac{A(\tau)m(\tau)^2}{a(\tau)} d\tau \right) \leq C. \]

Consequently,
\[ \| E_{a,m}(t, s, \xi) \|_{L^\infty} \lesssim \| E_a(t, s, \xi) \|_{L^\infty} \| Q_{a,m}(t, s, \xi) \|_{L^\infty} \lesssim \frac{\sqrt{a(t)}}{\sqrt{a(s)}}. \]

Now, let us introduce
\[ H(t, \xi) := \begin{pmatrix} \frac{h(t, \xi)}{\sqrt{a(t)}} & 0 \\ 0 & 1 \end{pmatrix}. \]

It is clear that in the hyperbolic zone we have \( \frac{h(t, \xi)}{\sqrt{a(t)}} \approx C \). Then the inverse matrix \( H^{-1} \) exists and \( \| H(t, \xi) \|, \| H^{-1}(t, \xi) \| \approx C \) for all \( t \geq \theta(\xi) \).

Thanks to
\[ U(t, \xi) = \frac{1}{\sqrt{a(t)}} H U_W = \frac{1}{\sqrt{a(t)}} (h(t, \xi) \tilde{u}, D_1 \tilde{u})^T, \]
where \( U \) is defined in (2.4), we conclude from the previous calculation in the hyperbolic zone that
\[ |U(t, \xi)| \leq \frac{\| U \|}{\sqrt{a(t)}} H(t, \xi) M E_a(t, s, \xi) Q_{a,m}(t, s, \xi) M^{-1} H^{-1}(s, \xi) U(s, \xi) | \leq C |U(s, \xi)|. \]
Proof. (Theorem 2.3)
With \( s = \theta|\xi| \) and the notation introduced in the pseudo-differential zone and in the hyperbolic zone we define

\[
\mathcal{E}(t, s, \xi) = \begin{cases} 
E(t, 0, \xi), & 0 \leq t \leq \theta|\xi|, \\
\frac{\sqrt{a(s)}}{\sqrt{a(t)}} H(t, \xi) ME_a(t, s, \xi)Q_{a,m}(t, s, \xi)M^{-1}H^{-1}(s, \xi)E(s, 0, \xi), & t \geq \theta|\xi|.
\end{cases}
\]

We have proved that \( \|\mathcal{E}(t, s, \xi)\| \leq C \) for all \( t, \xi \).

Note that the matrix functions \( \mathcal{E}(t, s, \xi) \) and \( \frac{\sqrt{a(s)}}{\sqrt{a(t)}} ME_a(t, s, \xi)M^{-1} \) generates a Fourier multiplier to the operators

\[
S(t, s, D) : \frac{1}{\sqrt{a(s)}}((D(s))u(s), D_t u(s))^T \mapsto \frac{1}{\sqrt{a(t)}}((D(t))u(t), D_t u(t))^T,
\]

\[
S_1(t, s, D) : \frac{1}{\sqrt{a(s)}}(a(s)|D|v(s), D_t v(s))^T \mapsto \frac{1}{\sqrt{a(t)}}(a(t)|D|v(t), D_t v(t))^T,
\]

for the solutions \( u \) and \( v \) to the Cauchy problems (1.1) and (2.15), respectively. Therefore we shall prove that the limit

\[
W_+(D) = \lim_{t \to \infty} S_1^{-1}(t, 0, D)S(t, 0, D)
\]

exists in \( E \). To study the operator \( S_1^{-1}(t, 0, D)S(t, 0, D) \) it is sufficient to study in the phase space the bounded multiplier

\[
ME_a^{-1}(t, 0, \xi)M^{-1}H(t, \xi)ME_a(t, s, \xi)Q_{a,m}(t, s, \xi)M^{-1}H^{-1}(s, \xi)E(s, 0, \xi).
\]

Let us prove that the limit \( W_+(\xi) \) exists for all \( |\xi| \geq c > 0 \). Thanks to \( E_a^{-1}(t, 0, \xi) = E_a(0, t, \xi), E_a(0, t, \xi)E_a(t, s, \xi) = E_a(0, s, \xi) \) and \( \lim_{t \to \infty} H(t, \xi) = I \), we get that

\[
\lim_{t \to \infty} E_a^{-1}(t, 0, \xi)M^{-1}H(t, \xi)ME_a(t, s, \xi) = E_a(0, s, \xi).
\]

Thus we shall investigate the limit

\[
\lim_{t \to \infty} Q_{a,m}(t, \theta|\xi|, \xi). 
\]

We obtain from the Peano-Backer formula that

\[
\sum_{k=1}^{\infty} t^k \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} P(t, 0, \xi) dt_k \ldots dt_1 = Q_{a,m}(t, 0, \xi) - Q_{a,m}(s, 0, \xi).
\]

Therefore,

\[
\|Q_{a,m}(t, \theta|\xi|, \xi) - Q_{a,m}(s, \theta|\xi|, \xi)\|_{L^\infty} \leq \sum_{k=1}^{\infty} \int_0^t \|P(t_1, \theta|\xi|, \xi)\|_{L^\infty} dt_1 \leq \int_0^t \|P(t_1, \theta|\xi|, \xi)\|_{L^\infty} dt_1 \leq \int_0^t \|R_{a,m}(t_1, \xi)\|_{L^\infty} dt_1,
\]

Our assumption (2.3) implies that in the hyperbolic zone \( R_{a,m}(\cdot, \xi) \in L^1 \). Thus,

\[
\|Q_{a,m}(t, \theta|\xi|, \xi) - Q_{a,m}(s, \theta|\xi|, \xi)\|_{L^\infty}
\]

becomes arbitrarily small for sufficiently large times \( s, t \). So the limit (6.7) exists uniformly in \( \xi \) for \( |\xi| \geq c > 0 \) and thanks to the Banach-Steinhaus’s theorem we can define the operator \( W_+ \).
The conclusion of the theorem follows thanks to
\[
\frac{1}{\sqrt{a(t)}} \left( (a(t) \nabla v(t, \cdot), v_t(t, \cdot)) - ((D(t))u(t, \cdot), u_t(t, \cdot)) \right)
\]
\[= S_1(t, 0, D) \left( S_1^{-1}(t, 0, D)S(t, 0, D) - W_+(D) \right)((D(0))u_0, u_1),
\]
and for the decay rate we only use that
\[
\alpha
\]
some constant
\[\text{the Cauchy problem (7.1) takes the form}
\]
\[A
\]
Assume that
\[m
\]
with
\[\delta.
\]
The conclusion of the theorem follows thanks to
\[
\text{Applying a result from [3] we get that the solution to (7.1) satisfies}
\]
\[\|Q_{a,m}(t, 0, \xi) - Q_{a,m}(\infty, 0, \xi)\|_{L^\infty} \lesssim \int_1^\infty \|R_{a,m}(\tau)\| \exp \left( \int_0^\tau \|R_{a,m}(\tau)\| d\tau \right) dt_1
\]
\[\lesssim \int_1^\infty A(\tau) \frac{A(\tau)}{\tau^2} m(\tau)^2 d\tau.
\]
The proof is completed. 

7. Scale invariant models

Let us consider the Cauchy problem for a class of scale invariant models with
time-dependent mass and speed of propagation
\[
u_{tt} - a(t)^2 \Delta u + m(t)^2 u = 0, \quad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x),
\]
with
\[m(t) = \mu \frac{a(t)}{A(t)}
\]
and the function \(a \not\in L^1\) is given by
\[a(t) = A(0)^{-\alpha} A(t)^\alpha
\]
for some constant \(\alpha \in \mathbb{R}\), i.e.,
\[
a'(t) = \alpha \frac{a(t)}{A(t)}, \quad A(t) = A(0) + \int_0^t a(\tau) d\tau.
\]
Assume that \(A(0) \in (0, 1]\). Applying the change of variable
\[v(\tau, x) = u(t, x), \quad \tau + 1 = A(t), \quad \tau_0 = A(0) - 1 \in (-1, 0]
\]
the Cauchy problem (7.1) takes the form
\[
v_{\tau\tau} - \Delta v - \alpha \frac{v}{1 + \tau} v + \frac{\mu^2}{(1 + \tau)^2} v = 0, \quad v(\tau_0, x) = u_0(x), \quad v_t(\tau_0, x) = u_1(x).
\]
Now, the number \(\delta \equiv (\alpha - 1)^2 - 4\mu^2\) plays a fundamental role.

- If \(\delta < 0\), applying the change of variable \(v(\tau, x) = (1 + \tau)^{-\alpha/2} w(\tau, x)\) (see [15]) we get the equation
\[
w_{\tau\tau} - \Delta w + \frac{\alpha}{(1 + \tau)^2} w = 0, \quad w(\tau_0, x) = w_0(x), \quad w_t(\tau_0, x) = w_1(x),
\]
where \(\sigma = \frac{\alpha}{2} - \frac{\alpha^2}{4} + \mu^2 = \frac{1 - \delta}{4} \geq 1/4\) and
\[w_0(x) \doteq (1 + \tau_0)^{\alpha/2} u_0(x), \quad w_1(x) \doteq \frac{\alpha}{2} (1 + \tau_0)^{-1} u_0(x) + (1 + \tau_0)^{\alpha/2} u_1(x).
\]
Applying a result from [3] we get that the solution to (7.1) satisfies
\[
E(u(t)) \lesssim a(t) E(u(0)), \quad \forall t \geq 0,
\]
where
\[
E(u(t)) \doteq \frac{1}{2} \left( \|u_t(t, \cdot)\|_{L^2}^2 + a(t)^2 \|\nabla u(t, \cdot)\|_{L^2}^2 + m(t) a(t) \|u(t, \cdot)\|_{L^2}^2 \right).
\]
In particular we get
\[
\|u(t, \cdot)\|_{L^2} \lesssim \frac{1}{\sqrt{m(t)}} E(u(0)) \sim \frac{\sqrt{A(t)}}{\sqrt{a(t)}} E(u(0)), \quad \forall t \geq 0.
\]

\[\]]
• If \( \delta \geq 0 \), applying the change of variable (see [15])

\[
v(\tau, x) = (1 + \tau)^\sigma w(\tau, x), \quad \sigma = \frac{1 - \alpha}{2} + \frac{\sqrt{\delta}}{2},
\]

we get the equation

\[
w_{\tau\tau} - \Delta w + \frac{1 + \sqrt{\delta}}{(1 + \tau)} w_\tau = 0, \quad w(\tau_0, x) = w_0(x), \quad w_\tau(\tau_0, x) = w_1(x),
\]

where

\[
w_0(x) = (1 + \tau_0)^{-\sigma} u_0(x), \quad w_1(x) = (1 + \tau_0)^{-\sigma} u_1(x) - \sigma(1 + \tau_0)^{-1} a(\tau_0)^{-1} u_0(x).
\]

If \((u_0, u_1) \in H^1 \times L^2\), applying results from [20] we get

\[
\|w(\tau, \cdot)\|_{L^2} \lesssim \begin{cases} 1, & \delta > 0, \\ \ln(\epsilon + \tau), & \delta = 0, \end{cases}
\]

and

\[
\|w_\tau(\tau, \cdot)\|_{L^2} + \|\nabla w(\tau, \cdot)\|_{L^2} \lesssim \begin{cases} (1 + \tau)^{-\frac{(1 + \alpha)\sqrt{\delta}}{2}}, & \delta \in [0, 1), \\ (1 + \tau)^{-1}, & \delta \geq 1. \end{cases}
\]

Therefore, using that the solution to (41) satisfies \( u(t, x) = (1 + \tau)^\sigma w(\tau, x) \) we conclude

\[
\|u(t, \cdot)\|_{L^2} \lesssim \begin{cases} a(t)^{\frac{1 - \alpha - \sqrt{\delta}}{2\alpha}}, & \delta > 0, \\ a(t)^{\frac{1}{2\alpha}} \ln a(t), & \delta = 0, \end{cases} \quad (7.6)
\]

and

\[
a(t)\|\nabla u(t, \cdot)\|_{L^2} \lesssim \begin{cases} a(t)^{\frac{2 + \alpha - 1 - \sqrt{\delta}}{2\alpha}}, & \delta \in [0, 1), \\ a(t)^{-1}, & \delta \geq 1. \end{cases}
\]

We may derive that same estimate for \( \|u_{\tau\tau}(t, \cdot)\|_{L^2} \) and thanks to \( \frac{2(\sigma + \alpha) - 1 - \sqrt{\delta}}{\alpha} \leq 1 \) we conclude

\[
\|u_\tau(t, \cdot)\|_{L^2} + a(t)\|\nabla u(t, \cdot)\|_{L^2} \lesssim \begin{cases} \sqrt{a(t)}, & \delta \in [0, 1), \\ a(t)^{-\frac{1 - \sqrt{\delta}}{2}}, & \delta \geq 1. \end{cases} \quad (7.7)
\]

**Example 7.1.** (Exponential speed of propagation)
Consider the Cauchy problem

\[
u_{\tau\tau} - e^{2t} \Delta u + \mu^2 u = 0, \quad u(0, x) = u_0(x), \quad u_\tau(0, x) = u_1(x),
\]

i.e., model (7.1) with \( \mu > 0 \) and \( \alpha = 1 \). Then \( \delta = -4\mu^2 < 0 \) and thanks to (64) we conclude that

\[
E(u)(t) := \frac{1}{2} \left( \|u(t, \cdot)\|_{L^2}^2 + e^{2t} \|\nabla u(t, \cdot)\|_{L^2}^2 + \mu e^t \|u(t, \cdot)\|_{L^2}^2 \right) \lesssim e^t E(u)(0).
\]

**Example 7.2.** (Polynomial speed of propagation)
Consider the Cauchy problem

\[
u_{\tau\tau} - (1 + t)^{2\ell} \Delta u + \frac{\mu^2}{(1 + t)^2} u = 0, \quad u(0, x) = u_0(x), \quad u_\tau(0, x) = u_1(x),
\]

where \( \ell > -1 \) and \( \tilde{\mu} > 0 \). This model can be written in the form (7.1) with \( \alpha = \frac{\ell}{\ell + 1} \) and \( \tilde{\mu} = \mu(\ell + 1) \). In this case

\[
\delta = \frac{1 - 4\tilde{\mu}^2}{(\ell + 1)^2}.
\]
If \((u_0, u_1) \in H^1 \times L^2\), thanks to (7.4), (7.6) and (7.7) we have the following estimates
\[
\|u(t, \cdot)\|_{L^2}^2 \lesssim \begin{cases}
(1 + t), & \delta < 0, \\
(1 + t)(\ln(\epsilon + t))^2, & \delta = 0, \\
(1 + t)^{1+\sqrt{1-4\mu^2}}, & \delta > 0,
\end{cases}
\]
and
\[
\|u_t(t, \cdot)\|_{L^2}^2 + (1 + t)^{2\delta}\|\nabla u(t, \cdot)\|_{L^2}^2 \lesssim \begin{cases}
(1 + t)^{\delta}, & \delta < 1, \\
(1 + t)^{-1+\sqrt{1-4\mu^2}}, & \delta \geq 1.
\end{cases}
\]

Remark 5. In Example 7.2 the case \(\delta \geq 1\) correspond to \(-1 < \ell < 0\) satisfying \(\ell + 1 \leq \sqrt{1-4\mu^2}\). In particular, this condition is equivalent to say that for negative \(\ell\), the last decay for the kinetic and elastic energies is worst than the first one in the case \(\delta < 1\).

7.1. \(L^q - L^2\) estimates, \(q \in [1, 2]\).
In this section we show that additional regularity \(L^q\), with \(q \in [1, 2]\), may improve the estimates for the solution and its derivatives. As discussed in the Section 4 applying the change of variable (7.2) we arrive in the Cauchy problem (7.3). Depending on the signal of \(\delta = (a - 1)^2 - 4\mu^2\) we have two situations:

(1) If \(\delta \leq 0\), \((u_0, u_1) \in D_q\), then applying Theorem 4.3 from [15] the solution \(u\) for the Cauchy problem (7.1) satisfies the following estimates:
\[
\|u(t, \cdot)\|_{L^2} \lesssim \frac{1}{\sqrt{a(t)}}(1 + \ln(a(t)^{\frac{1}{2}}))d(t)(\|u_0\|_{H^1 \cap L^2} + \|u_1\|_{L^2 \cap L^2}),
\]
for all \(t \geq 0\), where \(\gamma = 1\) if \(\delta = 0\), \(\gamma = 0\) if \(\delta < 0\) and
\[
d(t) = \begin{cases}
1, & \text{for } n > 2 - \frac{q}{2}, \\
(\ln(a(t)^{\frac{1}{2}})^{2\frac{q}{\gamma q}}, & \text{for } n = 2 - \frac{q}{2},
\end{cases}
\]
We notice that additional regularity in the initial data improves the estimates for the potential energy, see (7.2) for the case \(q = 2\).

(2) If \(\delta > 0\), \((u_0, u_1) \in D_q^{\kappa - 1}\) and \(\kappa \in [0, 1]\) then applying Theorem 4.7 from [15] the solution \(u\) for the Cauchy problem (7.1) satisfies the following estimates:
\[
\|u(t, \cdot)\|_{H^\kappa} \lesssim \|(u_0, u_1)\|_{D_q^{\kappa - 1}} \begin{cases}
\left(a(t)^{-\frac{1}{2}}(1 + \ln(a(t)^{\frac{1}{2}}))^{\frac{2 - \frac{q}{\gamma q}}{2 - \frac{q}{2}}}, & \text{for } n > 2 - \frac{q}{2}, \\
\frac{1 + \sqrt{\delta}}{\sqrt{a(t)}}, & \text{for } n = 2 - \frac{q}{2},
\end{cases}
\]
Moreover, \(a(t)^{-1}\|\partial_t u(t, \cdot)\|_{L^2[\mathbb{R}^n]}\) satisfies the same decay estimates as \(\|\nabla u(t, \cdot)\|_{L^2[\mathbb{R}^n]}\) which are obtained from (7.6) after taking \(\kappa = 1\).

In this case we notice that additional regularity on the initial data only improves the estimates for the potential energy if \(a(t)\) is increasing, see (7.6) for the case \(q = 2\).

7.2. Proof of Theorem 7.4
According to Duhamel’s principle, a solution to (2.20) satisfies the non-linear integral equation
\[
u(t, x) = K_0(t, 0, x) *_{(x)} u_0(x) + K_1(t, 0, x) *_{(x)} u_1(x) + \int_0^t K_1(t, s, x) *_{(x)} u(s, x)|^p ds,
\]
where \(K_j(t, 0, x) *_{(x)} u_j(x), j = 0, 1\), are the solutions to the corresponding linear Cauchy problem
\[
\begin{align*}
u_{tt} - c^2 a\Delta u + m^2 u &= 0, \quad u(0, x) = \delta_{0j} u_0(x), \quad u_t(0, x) = \delta_{1j} u_1(x),
\end{align*}
\]
with \( \delta_{kj} = 1 \) for \( k = j \), and zero otherwise. The term \( K_1(t, s, x) * (s) f(s, x) \) is the solution of the parameter-dependent Cauchy problem

\[
 u_{tt} - e^{2t} \Delta u + m^2 v = 0, \quad u(s, x) = 0, \quad u_t(s, x) = f(s, x).
\]  

(7.9)

In order to derive semilinear results for (7.9), it is not sufficient to use the linear estimates from Section 7.1, but in addition one has to derive \( L^3 - L^2 \) estimates for the parameter dependent Cauchy problem (7.9).

The equation in (7.9) may be written as model (7.1) with \( \alpha = 1 \) and \( \delta = -4m^2 < 0 \). Applying the change of variable

\[
 v(\tau, x) = u(t, x), \quad \tau + 1 = e^\tau, \quad \tau_s + 1 = e^s
\]

the Cauchy problem (7.9) takes the form

\[
 v_{\tau\tau} - \Delta v + \frac{1}{1 + \tau} v_{\tau} + \frac{m^2}{(1 + \tau)^2} v = 0, \quad v(\tau_s, x) = 0, \quad v(\tau_s, x) = (1 + \tau_s)^{-1} f(s, x).
\]  

(7.10)

By using the representation of solutions given by (15), one may derive the following:

**Proposition 7.1.** If \( m > 0 \) and \( f(s, \cdot) \in D_q, q \in [1, 2] \), then the solution to (7.9) satisfies the following estimates:

\[
 \| \partial_t K_1(t, s, \cdot) * f(s, \cdot) \|_{L^2} + e^t \| \nabla_s K_1(t, s, \cdot) * f(s, \cdot) \|_{L^2} \lesssim e^{\frac{t-s}{2}} \| f(s, \cdot) \|_{L^2}
\]  

(7.11)

\[
 \| K_1(t, s, \cdot) * f(s, \cdot) \|_{L^2} \lesssim e^{\frac{t-s}{2}} (d(t, s) \| f(s, \cdot) \|_{L^2} + e^{-t} \| u(t, \cdot) \|_{L^2}),
\]  

(7.12)

for all \( t \geq s \geq 0 \), where

\[
 d(t, s) = \begin{cases} 
 1 & \text{for } n > \frac{2}{2-q}, \\
 \frac{(t-s)^{2-n}}{2-n} & \text{for } n = \frac{2}{2-q}.
\end{cases}
\]

For the consideration of the semilinear models we shall use \( q = 1 \).

**Proof.** (Theorem 2.4) We define

\[
 X = \{ u \in C([0, \infty), H^1(\mathbb{R}^n)) \cap C^1([0, \infty), L^2(\mathbb{R}^n)) : \| u \|_X < \infty \},
\]

with the norm

\[
 \| u \|_X = \sup_{t \geq 0} \left( e^{t/2} (d(t)^{-1} \| u(t, \cdot) \|_{L^2} + \| \nabla_x u(t, \cdot) \|_{L^2} + e^{-t} \| u(t, \cdot) \|_{L^2}) \right).
\]

For any \( u \in X \) we define

\[
 Pu(t, x) := u^{lin}(t, x) + Nu(t, x),
\]

where \( u^{lin}(t, x) := K_0(t, 0, x) * (u_0(x) + K_1(t, 0, x) * u_1(x)) \) and

\[
 Nu(t, x) = \int_0^t E_1(t, s, x) * |u(s, x)|^p ds.
\]

Thanks to the derived estimates in Example 7.1 and Section 7.1, the linear part \( u^{lin} \) of the solution is in \( X \), and

\[
 \| u^{lin} \|_X \lesssim (\| u_0, u_1 \|_{D_1(\mathbb{R}^n)}).
\]

For any \( u, v \in X \), the nonlinear part of the solution satisfy

\[
 \| Nu \|_X \lesssim \| u \|_X^p \]  

(7.13)

\[
 \| Nu - Nv \|_X \lesssim \| u - v \|_X (\| u \|_X^{p-1} + \| v \|_X^{p-1})
\]  

(7.14)

Hence the operator \( P \) maps \( X \) into itself and the existence of a unique global solution \( u \) follows by contraction (7.14) and continuation argument for small data.
By using the derived linear estimates (7.11) and (7.12), we prove (7.13), but we omit the proof of (7.14), since it is analogous to the proof of (7.13). Indeed, for $\ell + j = 0, 1$ it holds

$$
\|\partial_t^{(\ell + j)} \nabla_x^j u(t, \cdot)\|_{L^2} \lesssim \int_0^t e^{\frac{(2\ell - 11)}{2} d(t)(1-j-\ell)} e^{\frac{j}{4}(j+\ell)} \|u(\cdot, \cdot)\|_p \|u(\cdot, \cdot)\|_{L^2 \cap L^1} ds.
$$

Using

$$
\|u(\cdot, \cdot)\|_{L^2 \cap L^1} = \|u(\cdot, \cdot)\|_p + \|u(\cdot, \cdot)\|_{L^1} = \|u(\cdot, \cdot)\|_p + \|u(\cdot, \cdot)\|_{L^2}
$$

and applying Gagliardo-Nirenberg inequality, for all $p \geq 2$ and $k = 1, 2$ we get

$$
\|u(\cdot, \cdot)\|_{L^p} \lesssim \|u(\cdot, \cdot)\|_{L^2}^{1-\theta(kp)} \|\nabla_x u(\cdot, \cdot)\|_{L^2}^{\theta(kp)} \lesssim e^{-\frac{2k}{p}d(s)(1-\theta(kp))} \|u\|_{X}^p,
$$

provided that $p \leq \frac{n}{n-2}$ if $n \geq 3$, i.e., $\theta(kp) \in [0, 1]$. Therefore

$$
\|\partial_t^{(\ell + j)} \nabla_x^j u(t, \cdot)\|_{L^2} \lesssim e^{\frac{(2\ell - 11)}{2} d(t)(1-j-\ell)} \int_0^t e^{\frac{j}{4}(j+\ell)} ds \lesssim e^{\frac{(2\ell - 11)}{2} d(t)(1-j-\ell)} \|u\|_{X}^p,
$$

for all $p > 1$. This concludes the proof. \qed

8. Concluding remarks and Open problems

(1) The range of admissible $p$ in Theorem 2.3 came from the use of Gagliardo-Nirenberg inequality, so it’s quite likely a technical restriction relate to the choice of function spaces we take for the data and solutions.

(2) Through this paper we assume $a \notin L^1$. In [8] the authors studied models for the damped wave models with integrable in time speed of propagation. In a forthcoming paper we shall study the Cauchy problem (1.1) in the case $a \in L^1$, which includes the Klein-Gordon equation in de Sitter spacetime, an important model that appear in Mathematical Cosmology.

(3) Consider the semilinear problem for the scale invariant model

$$
\begin{align*}
\left\{ 
\begin{array}{l}
u_{tt} - (1 + t)^{2\ell} \Delta u + \frac{\tilde{\mu}^2}{(1 + t)^{\ell}} u = |u|^p \\
(u(0, x), u_t(0, x)) = (u_0(x), u_1(x)),
\end{array}
\right.
\end{align*}
$$

where $\ell > 0$ and $\tilde{\mu} = \mu(\ell + 1) > 0$. Applying the change of variable

$$
v(\tau, x) = u(t, x), \quad \tau + 1 = \frac{(1 + t)^{\ell + 1}}{\ell + 1},
$$

this model can be transformed into the following Cauchy problem

$$
\begin{align*}
\left\{ 
\begin{array}{l}
v_{\tau\tau} - \Delta v + \frac{\alpha}{(1 + \tau)^{\ell}} v_{\tau} + \frac{\beta^2}{(1 + \tau)^{2\ell}} v = \beta^{-2}(1 + \tau)^{-2\alpha} |v|^p \\
v(\tau_0, x), v_{\tau}(\tau_0, x) = (u_0(x), (1 + \tau_0)^{-\ell} u_1(x)),
\end{array}
\right.
\end{align*}
$$

with $\alpha = \frac{\ell}{\ell + 1}, \beta = (\ell + 1)^{\frac{\ell}{\ell + 1}}$ and $\tau_0 = -\frac{\ell}{\ell + 1}$. By using data in $L^1 \cap L^2$, in Section 7.3 one may observe some improvement in the decay rate for the energy solutions. This hints to the possibility to derive global (in time) existence of small data energy solutions to this model. However, due to the fact that the dissipation is non-effective, the approach used in Theorem 2.3 does not bring any sharp result for the critical exponent. It is really a challenging problem to derive sharp global existence results to this model.
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