Mixed–frequency quantile regression with realized volatility to forecast Value-at-Risk*

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Abstract

The use of quantile regression to calculate risk measures has been widely recognized in the financial econometrics literature. When data are observed at mixed–frequency, the standard quantile regression models are no longer adequate. In this paper, we develop a model built on a mixed–frequency quantile regression to directly estimate the Value-at-Risk. In particular, the low–frequency component incorporates information coming from variables observed at, typically, monthly or lower frequencies, while the high–frequency component can include a variety of daily variables, like realized volatility measures or market indices. We derive the conditions for the weak stationarity of the daily return process suggested while the finite sample properties are investigated in an extensive Monte Carlo exercise. The validity of the proposed model is then explored through a real data application using the most important financial indexes. We show that our model outperforms other competing specifications, using backtesting and Model Confidence Set procedures.

Keywords: Value-at-Risk, Quantile Regression, Mixed-frequency variables, Model Confidence Set, Volatility.

JEL: C22; C52; C53; C58.

1 Introduction

During the last decades, an extensive financial econometrics literature was devoted to developing model–based approaches to calculate regulatory capital in line with the various requirements imposed by the Basel–II and Basel–III agreements for risk management (Kinateder, 2016). Among

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these methods, a special focus was given to forecasting methods for the Value–at–Risk (VaR) mea-
ure at a given confidence level $\tau$, $\text{VaR}(\tau)$, defined as the worst portfolio value movement (return)
to be expected at $1 - \tau$ probability over a specific horizon (Jorion, 1997), able to capture tail market
risk in a perspective fashion.

As the $\tau$–quantile of a portfolio return distribution, the $\text{VaR}(\tau)$ can be predicted as the pro-
duct of the portfolio volatility forecast times the quantile of the hypothesized distribution. For the
first component, volatility clustering, modeled by conditionally autoregressive models (such as the
ARCH/GARCH – Engle, 1982; Bollerslev, 1986), produces good forecasts capable of reproducing
well known stylized facts of financial time series, including skewed behavior and fat tails (Cont,
2001; Engle and Patton, 2001, among others). Further improvements were made possible by the
direct predictability of realized measures of financial volatility (Andersen et al., 2006b). While a
choice of a specific parametric distribution for the innovation term may be uninfluential for model
parameter estimation (Bollerslev and Wooldridge, 1992), unless a few extreme events (e.g. the
Flash Crash of May 2005) occur (Carnero et al., 2012), a wrong choice of distribution for the in-
novation term delivers an inadequate $\text{VaR}(\tau)$ forecasting: see for example Manganelli and Engle
(2001) and El Ghourabi et al. (2016).

As an alternative, the $\text{VaR}(\tau)$ can be directly derived through quantile regression methods
(Koenker and Bassett, 1978; Engle and Manganelli, 2004) where no distributional hypothesis is
required. A first suggestion in this direction comes from Koenker and Zhao (1996) who use quan-
tile regression for a particular class of ARCH models, i.e., the Linear ARCH models (Taylor, 1986),
chosen for its ease of tractability in deriving theoretical properties. Subsequent refinements are, for
instance, Xiao and Koenker (2009), Lee and Noh (2013), Zheng et al. (2018) for GARCH models,
Noh and Lee (2016) who consider asymmetry, Chen et al. (2012) who consider nonlinear regression
quantile approach with intra-day price, Bayer (2018) who combines VaR forecasts via penalized
quantile regressions, Taylor (2019) who considers a quantile regression approach to joint estimates
VaR and Expected Shortfall and the multivariate generalization of Merlo et al. (2021).

A relatively recent stream of literature investigates the value of information provided by data
available at both high– and low–frequency incorporated into the same model in assessing the dy-
namics of financial market activity: this is the case of the GARCH–MIDAS model proposed by
Engle et al. (2013) (building on the MI(xed)–DA(ta) Sampling approach by Ghysels et al., 2007),
the regime switching GARCH–MIDAS of Pan et al. (2017) and the recent paper by Xu et al. (2021)
who consider a MIDAS component in the Conditional Autoregressive Value–at–Risk (CA ViaR) of
Engle and Manganelli (2004).

In what follows, we propose a Mixed–Frequency Quantile ARCH model (MF–Q–ARCH, ex-
tending Koenker and Zhao, 1996): we show how the constant term in the quantile regression can
be written as a function of data sampled at lower frequencies (and hence become a low–frequency
component), while the high frequency component is regulated by the daily data. As a result, with

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the aim of capturing dependence on the business cycle, we benefit from the information contained in macroeconomic variables (cf. Mo et al., 2018; Conrad and Loch, 2015, among others), and we achieve a quite flexible representation of volatility dynamics. Since both components enter additively, our model can be seen as a quantile model version of the Component GARCH by Engle and Lee (1999).

In the proposed model, we also include a predetermined variable observed daily, typically a realized measure: this adds the “–X” component in the resulting MF–Q–ARCH–X model. This variable can capture extra information useful in modeling and forecasting future volatility and may improve the accuracy of VaR forecasts. Such a use in the quantile regression framework is not new in itself: the paper by Gerlach and Wang (2020) jointly forecast VaR and Expected Shortfall and Zhu et al. (2020) forecast VaR by adopting a GARCH–X model for the volatility term. Also the work of Žikeš and Baruník (2016) uses the realized measures in the context of quantile regressions to investigate the features of conditional quantiles of realized volatility and asset returns.

From a theoretical point of view, we provide the conditions for the weak stationarity of the daily return process suggested and we propose a sequential test aimed at finding the optimal number of daily return lags to be included in MF–Q–ARCH–X model. The finite sample properties are investigated through an extensive Monte Carlo exercise. The empirical application is carried out on the VaR predictive capability for the S&P 500, NASDAQ, and Dow Jones indices, showing that the proposed MF–Q–ARCH–X outperforms all the other competing models considered across all market indices when the VaR forecasts are evaluated by backtesting and constructing the Model Confidence Set (MCS, Hansen et al., 2011).

The rest of the paper is organized as follows. In Section 2 we introduce the notation and the basis for a dynamic model for the VaR(τ) and we provide details of the conditional quantile regression approach. Section 3 presents our MF–Q–ARCH–X model. Section 4 is devoted to the Monte Carlo experiment. Section 5 details the backtesting and MCS procedures. Section 6 illustrates the empirical application. Conclusions follow.

2 Approaches to VaR estimation

For the purposes of this paper we will adopt a double time index, i,t, where t = 1, . . . , T scans a low frequency time scale (i.e., monthly) and i = 1, . . . , Nf identifies the day of the month, with a varying number of days Nt in the month t, and an overall number N of daily observations N = \sum_{t=1}^{T} N_t. Let the daily returns r_{i,t} be, as customarily defined, the log–first differences of prices of an asset or a market index, and let the information available at time i,t be \mathcal{F}_{i,t}. In what follows, we are interested in the conditional distribution of returns, with the assumption:

\[ r_{i,t} = \sigma_{i,t} z_{i,t} \quad \text{with} \quad t = 1, \ldots, T, \ i = 1, \ldots, N_f, \] (1)
where $z_{i,t} \sim (0,1)$ having a cumulative distribution function denoted by $F(\cdot)$. The zero conditional mean assumption in Eq. (1) is not restrictive, in fact, when explicitly modeled, such a conditional mean is very close to zero, consistently with the market efficiency hypothesis.

Based on this setup, the conditional (one-step-ahead) VaR for day $i,t$ at $\tau$ level ($\text{VaR}_{i,t}(\tau)$) for $r_{i,t}$ is defined as

$$
\text{Pr}(r_{i,t} < \text{VaR}_{i,t}(\tau) | \mathcal{F}_{i-1,t}) = \tau,
$$
i.e., the $\tau$-th conditional quantile of the series $r_{i,t}$, given $\mathcal{F}_{i-1,t}$; consequently, we can write

$$
\text{VaR}_{i,t}(\tau) \equiv Q_{r_{i,t}}(\tau|\mathcal{F}_{i-1,t}) = \sigma_{i,t} F^{-1}(\tau),
$$

where $F^{-1}(\tau) = \inf \{z_{i,t} : F(z_{i,t}) \geq \tau\}$. For a given $\tau$, the traditional volatility–quantile approach to estimate the $\text{VaR}_{i,t}(\tau)$ is thus based on modeling $\sigma_{i,t}$ from a dynamic model of either the conditional variance of returns (following Engle, 1982; Bollerslev, 1986) or as a conditional expectation of a realized measure (Andersen et al., 2006a) and retrieving the constant $F^{-1}(\tau)$ either parametrically or nonparametrically. In either case, from an empirical point of view, it turns out that distribution tests mostly reject specific parametric choices, and that using the empirical distributions is prone to bias/variance problems and lack of stability through time.

Alternatively, we can estimate directly $Q_{r_{i,t}}(\tau|\mathcal{F}_{i-1,t})$ using a quantile regression approach (Koenker and Bassett, 1978; Engle and Manganelli, 2004) which has become a widely used technique in many theoretical problems and empirical applications. While classical regression aims at estimating the mean of a variable of interest conditioned to regressors, quantile regression provides a way to model the conditional quantiles of a response variable with respect to a set of covariates in order to have a more robust and complete picture of the entire conditional distribution. This approach is quite suitable to be used in all the situations where specific features, like skewness, fat–tails, outliers, truncation, censoring and heteroskedasticity are present. The basic idea behind the quantile regression approach, as shown by Koenker and Bassett (1978), is that the $\tau$-th quantile of a variable of interest (in our case $r_{i,t}$), conditional on the information set $\mathcal{F}_{i-1,t}$, can be directly expressed as a linear combination of a $q + 1$ vector of variables $x_{i-1,t}$ (including a constant term), with parameters $\Theta_{\tau}$, that is:

$$
Q_{r_{i,t}}(\tau|\mathcal{F}_{i-1,t}) = x_{i-1,t}' \Theta_{\tau},
$$

An estimator for the $(q + 1)$ vector of coefficients $\Theta_{\tau}$ is obtained minimizing a suitable loss function (also known as check function):

$$
\hat{\Theta}_{\tau} = \arg\min_{\Theta} \sum \rho_{\tau}(r_{i,t} - x_{i-1,t}' \Theta_{\tau}),
$$

with $\rho_{\tau}(u) = u (\tau - \mathbb{I}(u < 0))$, where $\mathbb{I}(\cdot)$ denotes an indicator function. In our context, the advantage of such an approach is to avoid the need to specify the distribution of $z_{i,t}$ in Eq. (1), either
Following the approach by Koenker and Zhao (1996), we assume a Linear ARCH specification for \( r_{i,t} \), namely, a dependence of \( \sigma_{i,t} \) on past absolute values of returns:

\[
\sigma_{i,t} = \beta_0 + \beta_1 |r_{i-1,t}| + \ldots + \beta_q |r_{i-q,t}|, \quad \text{with } t = 1, \ldots, T, \ i = 1, \ldots, N_t, \quad (5)
\]

where we assume that \( 0 < \beta_0 < \infty, \beta_1, \ldots, \beta_q \geq 0 \). Thus, substituting the generic term \( x_{i-1,t} \) in Eq. (3) with the specific vector relative to the Linear ARCH model, we have

\[
\sigma_{i,t} = (1, |r_{i-1,t}|, \ldots, |r_{i-q,t}|)^\prime \begin{pmatrix} \beta_0, \beta_1, \ldots, \beta_q \end{pmatrix} = x_{i-1,t}^\prime \Theta. \quad (6)
\]

The Linear ARCH choice turns out to be convenient, since it allows for a direct comparability of the two setups to estimate the VaR(\( \tau \)) in Eq. (2):

\[
\text{VaR}_{i,t}(\tau) = \begin{cases} 
  x_{i-1,t}^\prime \Theta F^{-1}(\tau) & \text{volatility–quantile} \\
  x_{i-1,t}^\prime \Theta_{\tau} & \text{conditional quantile regression},
\end{cases}
\]

which establishes the equivalence \( \Theta F^{-1}(\tau) = \Theta_{\tau} \) which will prove useful later in our Monte Carlo simulations.

There is a second justification to adopt the Linear ARCH: the term \( \sigma_{i,t} \) defining the volatility of returns, can also be seen as the conditional expectation of absolute returns in the Multiplicative Error Model representation used by Engle and Gallo (2006):

\[
|r_{i,t}| = \sigma_{i,t} \eta_{i,t}. \quad (7)
\]

The term \( \eta_{i,t} \) is an i.i.d. innovation with non–negative support and unit expectation, and the Eq. (7) can be used to derive an estimate of the VaR. The Linear ARCH is therefore a simple and convenient nonlinear autoregressive model for \( |r_{i,t}| \) with multiplicative errors, which we hold as the maintained base specification to explore the merits of our proposal. Moreover, this lays the grounds for extending the approach, using other specifications for \( \sigma_{i,t} \) in Eq. (5) as functions of past volatility–related observable variables. Alternatives can be considered, for example:

\[
\sigma_{i,t} = \omega + \alpha_1 r_{v_{i-1,t}} + \ldots + \alpha_q r_{v_{i-q,t}}, \quad \text{with } t = 1, \ldots, T, \ i = 1, \ldots, N_t,
\]

with \( r_{v_{i,t}} \) the daily realized volatility. We will return on this point below.

In what follows we will introduce a MIDAS extension to the Linear ARCH model in a quantile regression framework, taking advantage of the well–known predictive power of low–frequency variables for the volatility observed at a daily frequency (e.g. Conrad and Kleen, 2020). We also augment the Linear ARCH to include a lagged realized measure of volatility (see also Gerlach
and Wang, 2020, within a CA ViAR context), in order to add the informational content of a more accurate measure to the volatility dynamics.

3 The MF–Q–ARCH–X model

3.1 Model specification and properties

In order to take advantage of the information coming from variable(s) observed at different frequency we introduce a low–frequency component in model (5). This low–frequency term represents a one–sided filter of \( K \) lagged realizations of a given variable \( MV_t \) (usually a macroeconomic variable), through a weighting function \( \delta(\omega) \), where \( \omega = (\omega_1, \omega_2) \). Our resulting Mixed–Frequency Quantile ARCH (MF–Q–ARCH) model becomes:

\[
r_{i,t} = \left[ (\beta_0 + \theta|WS_{t-1}|) + (\beta_1|r_{i-1,t}| + \ldots + \beta_q|r_{i-q,t}|) \right] z_{i,t} \quad (8)
\]

\[
\equiv \left[ (\beta_0 + \theta|WS_{t-1}|) + (\beta_1|r_{i-1,t}| + \ldots + \beta_q|r_{i-q,t}|) \right] z_{i,t}, \quad (9)
\]

where the parameter \( \theta \) represents the impact of the weighted summation of the \( K \) past realizations of \( MV_t \), observed at each period \( t \), that is, \( WS_{t-1} = \sum_{k=1}^{K} \delta_k(\omega)MV_{t-k} \). The importance of each lagged realization of \( MV_t \) depends on \( \delta(\omega) \), which can be assumed as a Beta or Exponential Almon lag function (see, for instance, Ghysels and Qian, 2019). Here we use the former function, that is:

\[
\delta_k(\omega) = \frac{(k/K)^{\omega_1-1}(1-k/K)^{\omega_2-1}}{\sum_{j=1}^{K} (j/K)^{\omega_1-1}(1-j/K)^{\omega_2-1}}. \quad (10)
\]

Eq. (10) is a rather flexible function able to accommodate various weighting schemes. Here we follow the literature and give a larger weight to the most recent observations, that is, we set \( \omega_1 = 1 \) and \( \omega_2 \geq 1 \). The resulting weights \( \delta_k(\omega) \) are at least zero and at most one, and their sum equals one, so that \( \sum_{k=1}^{K} \delta_k(\omega)MV_{t-k} \) is an affine combination of \( (MV_{t-1}, \ldots, MV_{t-K}) \).

In order to refine the VaR dynamics in our model, we include a predetermined variable \( X_{i,t} \), so that we can explore the empirical merits of such an extended specification, already present in the GARCH and MEM literature (Han and Kristensen, 2015; Engle and Gallo, 2006). Such a variable may be the realized volatility of the asset or a market volatility index (see the use of the VIX in Amendola et al., 2021, among others). The resulting eXtended Mixed–Frequency Quantile ARCH model, labelled MF–Q–ARCH–X, becomes:

\[
r_{i,t} = \left[ (\beta_0 + \theta|WS_{t-1}|) + (\beta_1|r_{i-1,t}| + \ldots + \beta_q|r_{i-q,t}| + \beta_X|X_{i-1,t}|) \right] z_{i,t}. \quad (11)
\]

In either Eq. (9) or (11), the first component (including the constant) depends only on the
low–frequency term (changing at every \( t \), according to the term \( WS_{t-1} \)), while the second comprises variables changing daily (i.e., every \( i,t \)) and include lagged returns and the high–frequency volatility term. In such a representation, the two components enter additively, in the spirit of the component model of Engle and Lee (1999):

\[ r_{i,t} = \left[ \sigma_{i,t}^{LF} + \sigma_{i,t}^{HF} \right] z_{i,t}, \tag{12} \]

which, for the MF–Q–ARCH–X, becomes

\[
\begin{align*}
    r_{i,t} &= \left[ \left( \beta_0 + \theta |WS_{t-1}| \right) + \left( \beta_1 |r_{i-1,t}| + \ldots + \beta_q |r_{i-q,t}| + \beta_X |X_{i-1,t}| \right) \right] z_{i,t}.
\end{align*}
\tag{13}
\]

In the following theorem we show that, under mild conditions, the process in (13) is weakly stationary:

**Theorem 1.** Let \( MV_t \) and \( X_{i,t} \) be weakly stationary processes. Assume that \( \beta_0 > 0, \beta_1, \ldots, \beta_q, \beta_X \geq 0 \) and \( \theta \geq 0 \). Let \( z^n \equiv (E|z_{i,t}|^p)^{1/p} < \infty \), for \( p = \{1,2\} \) and the polynomial

\[ \phi(\lambda) = z^n \left( \beta_1 \lambda^{q+1} + \beta_2 \lambda^q + \ldots + \beta_q \lambda^{q-2} \right) - \lambda^{q+2} \tag{14} \]

has all roots \( \lambda \) inside the unit circle. Then the process \( r_{i,t} \) in (13) is weakly stationary.

Proof: see Appendix.

### 3.2 Inference on the MF–Q–ARCH–X Model

In order to make inference on the MF–Q–ARCH–X model we need to solve Eq. (4) where

\[
\begin{align*}
    x_{i-1,t} &= \left( 1, |WS_{t-1}|, |r_{i-1,t}|, \ldots, |r_{i-q,t}|, |X_{i-1,t}| \right) \tag{15}
    \Theta_\tau &= \left( \beta_0, \tau, \theta, \beta_1, \tau, \ldots, \beta_q, \tau, \beta_X, \tau \right). \tag{16}
\end{align*}
\]

The estimation of the vector \( \Theta_\tau \) is encumbered by the fact that the mixed–frequency term \( WS_{t-1} \) is not observable, as it depends on the unknown \( \omega_2 \) parameter of the weighting function \( \delta_k(\omega) \), also to be estimated. To make estimation feasible, we resort to the expedient of profiling out\(^1\) the parameter \( \omega_2 \), through a two-step procedure: we first fix \( \omega_2 \) at an initial arbitrary value, say \( \omega_2^{(b)} \), which turns the vector \( x_{i-1,t} \) into a completely observable counterpart, in short \( x_{i-1,t}^{(b)} \). This gives a

\(^1\)A profiling out strategy was used by Engle et al. (2013) for the parameter \( K \) in the GARCH–MIDAS model.
solution to the minimization of the loss function, which is dependent on $\omega_2^{(b)}$, that is,

$$\hat{\Theta}_\tau(\omega_2^{(b)}) \equiv \hat{\Theta}_\tau = \arg\min_{\Theta_\tau} \sum \rho_\tau \left(r_{i,t} - \left(x_{i-1,t}^{(b)}\right)' \Theta_\tau \right).$$

(17)

This procedure is repeated over a grid of $B$ values for $\omega_2$, so that we have $\{\hat{\Theta}_\tau^{(b)}\}_{b=1}^B$, and the chosen overall estimator is $\left(\hat{\omega}_2^*, \hat{\Theta}_\tau^{(*)}\right)$, corresponding to the smallest overall value of the loss function.

Accordingly, the MF–Q–ARCH–X estimator of the VaR is

$$\hat{Q}_{r,t,i}(\tau|\mathcal{F}_{i-1,t}) = \left(x_{i-1,t}^{(*)}\right)' \hat{\Theta}_\tau^{(*)}. \quad (18)$$

To obtain reliable VaR estimates in our model (18), an important issue is the choice of the optimal number of lags $q$ for the daily absolute returns in Eq. (5). To that end, we select the lag order suggested by a sequential likelihood ratio ($LR$) test on individual lagged coefficients (see also Koenker and Machado, 1999). In particular, for a given $\tau$, at each step $j$ of the testing sequence over a range of $J$ values, we compare the unrestricted model where the number of lags is set equal to $j$ (labelled U, with an associated loss function $V_{U,\tau}^{(j)}$), against a restricted model where the number of lags is $j - 1$ (labelled R, with an associated loss function $V_{R,\tau}^{(j-1)}$). In this setup, the null hypothesis of interest is

$$H_0 : \beta_j = 0, \quad (19)$$

i.e., the coefficient on the most remote lag is zero. The procedure starts contrasting a lag-1 model against a model with just a constant, then a lag-2 against a lag-1, and so on. For a given $\tau$, at each step $j$, we calculate the test statistic

$$LR_{\tau}^{(j)} = \frac{2 \left(V_{U,\tau}^{(j-1)} - V_{U,\tau}^{(j)}\right)}{\tau (1 - \tau) s(\tau)}, \quad (20)$$

where $s(\tau)$ is the so–called sparsity function estimated accordingly to Siddiqui (1960) and Koenker and Zhao (1996). Under the adopted configuration, $LR_{\tau}^{(j)}$ is asymptotically distributed as a $\chi_1^2$, so that we select $q$ to be the last value of $j$ in the sequence, for which we reject the null hypothesis.

### 4 Monte Carlo simulation

The finite sample properties of the sequential test and of the estimator of the MF–Q–ARCH model can be investigated by means of a Monte Carlo experiment. In what follows we consider $R = 5000$. 


replications of the data generating process (DGP):

\[ r_{i,t} = (\beta_0 + \theta |W S_{t-1}| + \beta_1 |r_{i-1,t}| + \beta_2 |r_{i-2,t}| + \beta_3 |r_{i-3,t}| + \beta_4 |r_{i-4,t}|) z_{i,t}, \]

where we assume a \( N(0,1) \) distribution for \( z_{i,t} \) and we set to zero the relevant initial values for \( r_{i,t} \). Moreover, the stationary variable \( MV_t \) entering the weighted sum \( WS_{t-1} \) is assumed to be drawn from an autoregressive AR(1) process \( MV_t = \varphi MV_{t-1} + e_t, \) with \( \varphi = 0.7 \) and the error term \( e_t \) following a Skewed \( t \)-distribution (Hansen, 1994), with degrees of freedom \( df = 7 \) and skewing parameter \( sp = -6 \). The frequency of \( MV_t \) is monthly and \( K = 24 \). The values of the parameters (collected in a vector \( \Theta \)) are detailed in the first column of the Tables 2–4. For the simulation exercise we consider \( N = 1250, N = 2500 \) and \( N = 5000 \) observations, to mimic realistic daily samples. Finally, three different levels of the VaR coverage level \( \tau \) are chosen: 0.01, 0.05, and 0.10.

In the Monte Carlo experiment, we start by evaluating the features of the LR test for the lag selection in Eq. (20). To that end, we test sequentially \( H_0 : \beta_j = 0 \) over \( J \) steps at a significance level \( \alpha \). Since the DGP is a fourth–order process, we expect to have a high rejection rate when the null involves a zero restriction on coefficients \( \beta_j, j = 1, \ldots, 4 \). In order to confirm the expected low rate of rejections, we extend the sequence of testing of further \( \beta_j \)'s, up to \( J = 6 \).

Looking at the Table 1, where we report the percentages of rejections for different VaR coverage levels \( \tau = 0.01, 0.05, 0.1 \) at the nominal significance level of \( \alpha = 5\% \) across replications, we validate the good behavior of the test. Overall, the sequential test procedure satisfactorily identifies the number of lags to be included in the MF–Q–ARCH model, with the performance improving with the number of observations, especially for \( H_0 : \beta_4 = 0; \) for the latter case, the percentage of rejections of the null increases considerably across coverage levels when \( N = 5000 \).

| Table 1: Percentage of rejection of the LR test for the null \( \beta_j = 0 \) |
|-----------------|-----------------|-----------------|
| \( N \) | 1250 | 2500 | 5000 |
| \( \tau = 0.01 \) | \( \beta_0 = 99.56 \) | 100.00 | 100.00 |
| | \( \beta_1 = 97.66 \) | 100.00 | 100.00 |
| | \( \beta_2 = 85.76 \) | 99.14 | 100.00 |
| | \( \beta_3 = 52.88 \) | 82.84 | 98.44 |
| | \( \beta_4 = 4.46 \) | 4.98 | 4.90 |
| | \( \beta_5 = 4.86 \) | 5.24 | 5.00 |
| \( \tau = 0.05 \) | \( \beta_0 = 100.00 \) | 100.00 | 100.00 |
| | \( \beta_1 = 99.94 \) | 99.94 | 100.00 |
| | \( \beta_2 = 93.16 \) | 93.16 | 99.70 |
| | \( \beta_3 = 6.30 \) | 6.30 | 6.56 |
| | \( \beta_4 = 5.14 \) | 5.14 | 5.56 |
| \( \tau = 0.1 \) | \( \beta_0 = 100.00 \) | 100.00 | 100.00 |
| | \( \beta_1 = 99.86 \) | 99.86 | 100.00 |
| | \( \beta_2 = 90.70 \) | 90.70 | 99.58 |
| | \( \beta_3 = 6.40 \) | 6.40 | 6.56 |
| | \( \beta_4 = 5.16 \) | 5.16 | 5.58 |

**Notes:** The table presents the percentage of rejection for the null in the first column, across all the Monte Carlo replicates, for three different configurations of \( N \) and \( \tau \).

Turning to the small sample properties of our estimator, the evaluation is done in terms of the original coefficients in the DGP, collected in the vector \( \Theta = (\beta_0, \theta, \beta_1, \ldots, \beta_q) \), using the relation-
ship with the quantile regression parameters $\Theta_\tau$, i.e., $\Theta = \Theta_\tau / F^{-1}(\tau)$. In Tables 2, 3 and 4 we report the Monte Carlo averages of the parameters ($\hat{\Theta}$) across replications for three levels of $\tau$, and the estimated Mean Squared Errors relative to the true values.

Overall, the proposed model presents good finite sample properties: independently of the $\tau$ level chosen, for small sample sizes, the estimates appear, in general, slightly biased, although, reassuringly, the MSE of the estimates relative to the true values always decreases as the sample period increases.

### Table 2: Monte Carlo estimates, $\tau = 0.01$

|       | $\Theta$ |       | $\Theta$ |       | $\Theta$ |
|-------|----------|-------|----------|-------|----------|
|      | $N = 1250$ | $N = 2500$ | $N = 5000$ | $N = 1250$ | $N = 2500$ | $N = 5000$ |
| $\beta_0$ | 0.050 | 0.079 | 0.040 | 0.064 | 0.019 | 0.058 | 0.009 |
| $\theta$ | 0.125 | 0.124 | 0.013 | 0.126 | 0.007 | 0.125 | 0.003 |
| $\beta_1$ | 0.300 | 0.286 | 0.009 | 0.292 | 0.005 | 0.296 | 0.002 |
| $\beta_2$ | 0.250 | 0.236 | 0.008 | 0.242 | 0.004 | 0.246 | 0.002 |
| $\beta_3$ | 0.200 | 0.187 | 0.008 | 0.194 | 0.004 | 0.196 | 0.002 |
| $\beta_4$ | 0.150 | 0.143 | 0.007 | 0.146 | 0.004 | 0.149 | 0.002 |
| $\omega_2$ | 2.000 | 1.993 | 0.010 | 1.991 | 0.010 | 1.984 | 0.010 |

**Notes:** The first column shows the true values of the $\Theta$ coefficients in the DGP. Simulations were replicated 5000 times, according to three different window lengths: $N = 1250$, $N = 2500$, and $N = 5000$. Columns $\hat{\Theta}$ report the averages of the estimated parameters across replications. Columns labeled $MSE$ refer to the Mean Square Error of the estimated coefficients relative to the true values.

### Table 3: Monte Carlo estimates, $\tau = 0.05$

|       | $\Theta$ |       | $\Theta$ |       | $\Theta$ |
|-------|----------|-------|----------|-------|----------|
|      | $N = 1250$ | $N = 2500$ | $N = 5000$ | $N = 1250$ | $N = 2500$ | $N = 5000$ |
| $\beta_0$ | 0.050 | 0.066 | 0.025 | 0.057 | 0.012 | 0.053 | 0.006 |
| $\theta$ | 0.125 | 0.123 | 0.008 | 0.125 | 0.004 | 0.125 | 0.002 |
| $\beta_1$ | 0.300 | 0.294 | 0.006 | 0.297 | 0.003 | 0.299 | 0.002 |
| $\beta_2$ | 0.250 | 0.242 | 0.006 | 0.246 | 0.003 | 0.248 | 0.001 |
| $\beta_3$ | 0.200 | 0.195 | 0.005 | 0.196 | 0.003 | 0.198 | 0.001 |
| $\beta_4$ | 0.150 | 0.146 | 0.005 | 0.148 | 0.002 | 0.149 | 0.001 |
| $\omega_2$ | 2.000 | 1.991 | 0.010 | 1.985 | 0.010 | 1.977 | 0.009 |

**Notes:** The first column shows the true values of the $\Theta$ coefficients in the DGP. Simulations were replicated 5000 times, according to three different window lengths: $N = 1250$, $N = 2500$, and $N = 5000$. Columns $\hat{\Theta}$ report the averages of the estimated parameters across replications. Columns labeled $MSE$ refer to the Mean Square Error of the estimated coefficients relative to the true values.

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2 As per the parameter $\omega_2$, the grid search is done over 100 values and the applied rescaling factor is equal to 1, as its value is unaffected by $\tau$. 

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Table 4: Monte Carlo estimates, $\tau = 0.1$

| True $\Theta$ | $\hat{\Theta}$ | MSE | $\hat{\Theta}$ | MSE | $\hat{\Theta}$ | MSE |
|---------------|--------------|-----|--------------|-----|--------------|-----|
| $\beta_0$     | 0.050        | 0.063 | 0.026        | 0.057 | 0.013        | 0.053 | 0.006 |
| $\theta$      | 0.125        | 0.124 | 0.009        | 0.124 | 0.004        | 0.125 | 0.002 |
| $\beta_1$     | 0.300        | 0.296 | 0.006        | 0.297 | 0.003        | 0.299 | 0.002 |
| $\beta_2$     | 0.250        | 0.244 | 0.006        | 0.246 | 0.003        | 0.248 | 0.002 |
| $\beta_3$     | 0.200        | 0.196 | 0.006        | 0.198 | 0.003        | 0.199 | 0.002 |
| $\beta_4$     | 0.150        | 0.145 | 0.005        | 0.148 | 0.003        | 0.149 | 0.001 |
| $\omega_2$    | 2.000        | 1.992 | 0.010        | 1.986 | 0.010        | 1.979 | 0.009 |

Notes: The first column shows the true values of the $\Theta$ coefficients in the DGP. Simulations were replicated 5000 times, according to three different window lengths: $N = 1250$, $N = 2500$, and $N = 5000$. Columns $\hat{\Theta}$ report the averages of the estimated parameters across replications. Columns labeled $MSE$ refer to the Mean Square Error of the estimated coefficients relative to the true values.

5 Model Evaluation

The bulk of the evaluation of our model lies in its capability to capture the coverage of VaR at a given $\tau$ level, especially in comparison to other models. In the next sections we illustrate two of the main procedures used in literature to evaluate the quality of the VaR estimates: the backtesting and the Model Confidence Set (MCS, Hansen et al., 2011) techniques (as recently done by Laporta et al., 2018, among others).

5.1 Backtesting

Building models able to predict VaRs efficiently is our main objective in this paper. For this reason, it is quite relevant to evaluate the quality of the VaR estimates by performing a set of targeted tests. Nowadays, in the context of risk management, backtesting is one of most recognized model’s evaluation procedure (see the reviews of Campbell, 2006; Nieto and Ruiz, 2016, among others). For this reason, in this study, we evaluate the VaR predictive ability of our model by means of four backtesting procedures: the Actual over Expected (AE) exceedance ratio, the Unconditional Coverage (UC, Kupiec, 1995), the Conditional Coverage (CC, Christoffersen, 1998), and the Dynamic Quantile (DQ, Engle and Manganelli, 2004) tests. The AE exceedance ratio is the number of times that the VaR measures have been violated over the expected VaR violations. The closer to one the ratio, the better is the model to forecast VaRs. The UC test is a $LR$–based test, where the null hypothesis assesses whether the actual frequency of VaR violations is equal to the chosen $\tau$ level. The UC test statistic is asymptotically $\chi^2$ distributed, with one degree of freedom. The UC test is complemented by the CC test, which verifies not only whether the observed frequency of violations is in line with the prefixed $\tau$, but also if these VaR violations are independently distributed over time. Also this CC test statistic is asymptotically $\chi^2$ distributed, but with two degrees
of freedom. The DQ test always verifies the independence of the VaR violations jointly with the correctness of the number of violations as the CC test, but it was shown (Berkowitz et al., 2011) to have more power over it. In particular, the DQ test consists of running a linear regression where the dependent variable is the sequence of VaR violations and the covariates are the past violations and possibly any other explanatory variables. Under the null hypothesis, the test statistic follows a $\chi^2_q$, with $q$ denoting the rank of the explanatory variable matrix.

The results of the backtesting procedures may be ambiguous as to the superiority of one model over the others: we therefore propose to adopt a MCS procedure in order to find the subset of superior models according to a suitable loss function.

### 5.2 Model Confidence Set

By the MCS methodology we construct a “superior set of models” (SSM) among several competing ones. Initially proposed by Hansen et al. (2011), it permits to discriminate within an initial set of models $\mathcal{M}^0$, labelled as $l = 1, \ldots, M$, the SSM with dimension $m^* \leq M$, denoted by $\hat{\mathcal{M}}^{*}_{1-\delta}$. In other words, the SSM $\hat{\mathcal{M}}^{*}_{1-\delta}$ is a subset of the full class of models containing the best specifications with a confidence level $1 - \delta$. In our VaR framework, such a procedure relies on loss functions based on quantiles mapping the distance between the observed returns, $r_{i,t}$, and the estimated VaR. Let $\hat{VaR}_{l,i,t}(\tau)$ be the one-step-ahead VaR forecasts for day $i$ and month $t$, obtained from model $l$ at level $\tau$. Formally, the loss function associated to the model $l$ is:

$$
\ell_{l,i,t} = \ell(\tau, r_{i,t}, \hat{VaR}_{l,i,t}(\tau)).
$$

(21)

In line with González-Rivera et al. (2004) and Laporta et al. (2018), we use an asymmetric loss function\footnote{Other loss functions in this field are discussed in Amendola and Candila (2016).} penalizing more heavily the negative returns falling below the VaR, that is:

$$
\ell(r_{i,t}, \tau) = \left( \tau - 1_{r_{i,t} < \hat{VaR}_{l,i,t}(\tau)} \right) \left( r_{i,t} - \hat{VaR}_{l,i,t}(\tau) \right),
$$

(22)

where $1_{(\cdot)}$ is an indicator function equals to one if the argument is true.

The relative performance of two models, say $l$ and $k$, is evaluated by means of the loss differential $d_{l,k,i,t}$, that is:

$$
d_{l,k,i,t} = \ell_{l,i,t} - \ell_{k,i,t}, \quad \forall l, k \in \mathcal{M}^0.
$$

(23)

Let $\mu_{lk} \equiv E(d_{l,k,i,t})$ be the expectation of the loss differential. Clearly, when $\mu_{lk} < 0$, model $l$ is preferred to model $k$. The MCS procedure consists of a sequence of equal predictive ability tests, with the null hypothesis given by:

$$
H_{0,l} : \mu_{lk} = 0, \forall l, k \in \mathcal{M}.
$$

(24)
with $\mathcal{M} \subset \mathcal{M}^0$. As in Cipollini et al. (2021), the test used in the MCS procedure is based on the semi–quadratic (SQ) test statistic (Hansen et al. (2011) and Clements et al. (2009)) given by:

$$T_{SQ} = \sum_{l \neq k \in \mathcal{M}} \frac{\overline{d}^2_{lk}}{\hat{\var}(d_{lk})},$$

(25)

where $\overline{d}_{lk}$ is the sample loss differential between model $l$ and model $k$ and $\hat{\var}(\overline{d}_{lk})$ is the estimate variance of $\overline{d}_{lk}$. Intuitively, large values for $T_{SQ}$ provide evidence against model $l$ with respect the others in $\mathcal{M}$. Consequently, model $l$ should be eliminated from $\mathcal{M}$ according to a defined elimination rule. For details on the proper elimination rule, we refer to Hansen et al. (2011). Therefore, the steps of the MCS procedure can be synthesized as follows:

1. Configure the set of models to evaluate with $\mathcal{M} = \mathcal{M}^0$.

2. Test the null hypothesis in (24), with a significance level $\delta$, using the loss function in (22) to calculate the loss differential.

3. If the null hypothesis is not rejected at the $\delta$ significance level, then $\hat{\mathcal{M}}^*_1 - \delta = \mathcal{M}$. Alternatively, eliminate a model from $\mathcal{M}$ according to the defined elimination rule and repeat Step 2.

6 Empirical Analysis

In this section we apply the MF–Q–ARCH–X model to estimate VaRs at different $\tau$ levels for three main daily financial indices: the S&P 500, the NASDAQ and the Dow Jones. We compare the estimate VaRs with several well–known competitive models: the Linear ARCH estimated in the quantile context (Q–ARCH), the Symmetric Absolute Value (SAV), Asymmetric Slope (AS) and Indirect GARCH (IG) specifications of the CAViaR (Engle and Manganelli, 2004) model, the GARCH, the GARCH with Student’s $t$ error term (GARCH–t), the GJR–GARCH (Glosten et al., 1993), the GJR–GARCH with Student’s $t$ error term (GJR–t), the RiskMetrics (Riskmetrics, 1996) model and the GARCH–MIDAS, with the same low–frequency variables used in the proposed MF–Q–ARCH–X model. All the functional forms of these models are reported in Table 5.

The period of investigation covers almost 20 years, from January 2001 to February 2020 on a daily basis. The data have been collected from the realised library of the Oxford–Man Institute (Heber et al., 2009), while the monthly variables come from the Federal Reserve Economic Data (FRED) database: the U.S. Industrial Production ($IP_t$), the Housing Starts ($HS_t$), taken as month–to–month percentage change, and the National Activity Index ($NAI_t$) (as done by Conrad and Loch, 2015, for instance). Summary statistics for the daily log–returns, for the 5-minute realized volatilities (which is included in the MF–Q–ARCH–X model as “–X” variable) of the three indices and
| Model               | Functional form                                                                                                                                                                                                 | Err. Distr.   |
|---------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------|
| MF–Q–ARCH–X         | $\sigma_{i,t} = (\beta_0 + \theta |W S_{t-1}| + \beta_1|r_{i-1,t}^2| + \ldots + \beta_q|r_{i-q,t}^2| + \beta_x|X_{i-1,t}|)$<br>${W S_{t-1}} = \sum_{k=1}^{K} S_k(\omega)MV_{t-k}$ | $z_{i,t} \overset{i.i.d}{\sim} (0,1)$ |
| MF–Q–ARCH           | $\sigma_{i,t} = (\beta_0 + \theta |W S_{t-1}| + \beta_1|r_{i-1,t}^2| + \ldots + \beta_q|r_{i-q,t}^2|)$<br>${W S_{t-1}} = \sum_{k=1}^{K} S_k(\omega)MV_{t-k}$ | $z_{i,t} \overset{i.i.d}{\sim} (0,1)$ |
| Q–ARCH              | $\sigma_{i,t} = (\beta_0 + \theta |W S_{t-1}| + \beta_1|r_{i-1,t}^2| + \ldots + \beta_q|r_{i-q,t}^2|)$<br>${W S_{t-1}} = \sum_{k=1}^{K} S_k(\omega)MV_{t-k}$ | $z_{i,t} \overset{i.i.d}{\sim} (0,1)$ |
| SAV                 | $Va R_{i,t}(\tau) = \beta_0 + \beta_1 Va R_{i-1,t}(\tau) + \beta_2|r_{i-1,t}|$                                                                                                                             |              |
| AS                  | $Va R_{i,t}(\tau) = \beta_0 + \beta_1 Va R_{i-1,t}(\tau) + (\beta_2 1_{(r_{i-1,t} > 0)} + \beta_3 1_{(r_{i-1,t} < 0)})|r_{i-1,t}|$                                               |              |
| IG                  | $Va R_{i,t}(\tau) = -\sqrt{\beta_0 + \beta_1 Va R_{i-1,t}^2(\tau) + \beta_2 r_{i-1,t}^2}$                                                                                                     |              |
| GARCH               | $r_{i,t} |F_{i-1} = \sqrt{h_{i,t}} \eta_{i,t}$<br>$h_{i,t} = \omega_0 + \alpha_1 r_{i-1,t}^2 + \beta_1 h_{i-1,t}$                                                                                   | $\eta_{i,t} \overset{i.i.d}{\sim} N(0,1)$ |
| GARCH-t             | $r_{i,t} |F_{i-1} = \sqrt{h_{i,t}} \eta_{i,t}$<br>$h_{i,t} = \omega_0 + \alpha_1 r_{i-1,t}^2 + \beta_1 h_{i-1,t}$                                                                                   | $\eta_{i,t} \overset{i.i.d}{\sim} t_v$ |
| GJR                 | $h_{i,t} = \omega_0 + (\alpha_1 + \gamma 1_{(r_{i-1,t} < 0)}) r_{i-1,t}^2 + \beta_1 h_{i-1,t}$                                                                                               | $\eta_{i,t} \overset{i.i.d}{\sim} N(0,1)$ |
| GJR-t               | $h_{i,t} = \omega_0 + (\alpha_1 + \gamma 1_{(r_{i-1,t} < 0)}) r_{i-1,t}^2 + \beta_1 h_{i-1,t}$                                                                                               | $\eta_{i,t} \overset{i.i.d}{\sim} t_v$ |
| GARCH–MIDAS         | $\xi_{i,t} = (1 - \alpha_1 - \beta_1 - \gamma / 2) + (\alpha_1 + \gamma 1_{(r_{i-1,t} < 0)}) r_{i-1,t}^2 + \beta_1 \xi_{i-1,t}$<br>$\gamma = \exp \left[ m + \sum_{j=1}^{K} S_j(\omega)MV_{t-j} \right]$ | $\eta_{i,t} \overset{i.i.d}{\sim} N(0,1)$ |

**Notes:** The table reports the functional forms for the MF–Q–ARCH–X, MF–Q–ARCH, Q–ARCH, Symmetric Absolute Value (SAV), Asymmetric Slope (AS), Indirect GARCH (IG), GARCH, GARCH with Student’s t distribution for the errors (GARCH-t), GJR, GJR with Student’s t distribution for the errors, and GARCH–MIDAS specifications.
In particular, we perform an in-sample analysis where running the first differences of IP and pandemic.

In Figure 1 we present the patterns of the daily log–returns (in percentage scale) of the three indices considered. Some commonalities can be found: first, the increased variability during the Dot–Com bubble (March 2001 to November 2001) and the Great Recession period (December 2007 to June 2009), identified by shared areas according to the U.S. recession periods (NBER dating); second, a tendency to a generalized downturn in the indices, due to the beginning of the Covid–19 pandemic.

The plots of the monthly variables are reported in Figure 2. The top and central panels show the first differences of IP and HS, while the bottom panel shows the NAI pattern. It can be noted that, during the Great Recession, IP (which is a proxy of the aggregate demand) and NAI present some negative peaks, which in turn may correspond to an increased volatility in the stock markets.

In what follows, we illustrate the results of the implementation of the MF–Q–ARCH–X model. In particular, we perform an in-sample analysis where running the LR test implemented in Section 3.2, we define the optimal number of daily lags in absolute log–returns to be included.
Finally, in the out-of-sample analysis, we illustrate model performances in a comparative fashion using both backtesting and MCS procedures.

**In-sample analysis**

Tables 7 reports the p-values of LR test using $\tau = 0.05$, with an in-sample dataset from April 2016 to January 2001, i.e., approximately the 80% of the full sample period, for all the indices under investigation. The application of the LR test suggests the inclusion up to eight lagged daily log–returns in the models for all the three indices.

| Index       | $\beta_1 = 0$ | $\beta_2 = 0$ | $\beta_3 = 0$ | $\beta_4 = 0$ | $\beta_5 = 0$ | $\beta_6 = 0$ | $\beta_7 = 0$ | $\beta_8 = 0$ | $\beta_9 = 0$ |
|-------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| S&P 500     | 0.000         | 0.000         | 0.000         | 0.000         | 0.000         | 0.000         | 0.002         | 0.001         | 0.785         |
| NASDAQ      | 0.000         | 0.000         | 0.000         | 0.000         | 0.000         | 0.004         | 0.005         | 0.000         | 0.265         |
| Dow Jones   | 0.000         | 0.000         | 0.000         | 0.000         | 0.000         | 0.000         | 0.000         | 0.000         | 0.618         |

*Notes:* The table reports the p-values of LR test according to the procedure highlighted in Section 3.2, for the null in column. Sample period: from January 2001 to April 2016.

As regards the number of lagged realizations entering the low–frequency component, we choose $K = 12$. The patterns of the resulting weighting schemes selected by the profiling procedure illustrated in Section 3.2 for the MF–Q–ARCH and MF–Q–ARCH–X models, and for all the monthly
variables used in this work, are depicted in Figure 3. Interestingly, all the patterns of the resulting weighting schemes are very reasonable: further observations have weights close to zero. Looking at the weighting schemes for MF–Q–ARCH–X and MF–Q–ARCH using $\Delta H_S$ in the low–frequency component (Panels 3c and 3d, respectively), the profiling procedure signals the same importance (approximately) to the macro–variable lagged realizations for the Dow Jones and S&P 500. For the other panels, there is no evidence of huge differences among the indices under investigation, in terms of the weights associated to the past macro–variable observations.

**Out-of-sample evaluation**

The empirical analysis is completed by the out-of-sample analysis, where we compare the performances of the MF–Q–ARCH and the MF–Q–ARCH–X models in VaR prediction with those mentioned at the beginning of this section. In line with Lazar and Xue (2020), the one-step-ahead VaR forecasts are obtained with parameters estimated every ten days, using a rolling window of size 1500 observations. The out-of-sample period represents approximately 20% of the full sample, and covers the period from May 2016 to February 2020. As an example, the plots of the VaR estimates with the daily log–returns for the proposed MF–Q–ARCH–X, with $\Delta I_P$ as MIDAS variable

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**Notes:** Plots of the U.S. Industrial Production (top panel, expressed in month-to-month percentage change), U.S. Housing Starts (central panel, expressed in month-to-month percentage change) and U.S. National Activity Index (bottom panel). Monthly frequency, 230 observations. Shaded areas represent U.S. recession periods (NBER dating).
Figure 3: Weighting schemes for the MF–Q–ARCH–X and MF–Q–ARCH models

(a) MF–Q–ARCH–X; LF term: $\Delta IP_t$

(b) MF–Q–ARCH; LF term: $\Delta IP_t$

(c) MF–Q–ARCH–X; LF term: $\Delta HS_t$

(d) MF–Q–ARCH; LF term: $\Delta HS_t$

(e) MF–Q–ARCH–X; LF term: $\text{NAI}_t$

(f) MF–Q–ARCH; LF term: $\text{NAI}_t$

Notes: Plots of weighting schemes, for the S&P 500 (green lines), NASDAQ (red lines), and Dow Jones (blue lines), according to the profiling procedure described in the text, for the in-sample period.
and the realized volatility as “–X” term, are reported in Figure 4.

Figure 4: MF–Q–ARCH–X VaR estimates, with $\Delta IP_t$ as low–frequency variable

Notes: Plot of the S&P 500, NASDAQ and Dow Jones daily log-returns (black lines) and of the VaR estimates obtained from the MF–Q–ARCH–X model, with $\Delta IP_t$ as MIDAS variable (blue lines). Percentage scale. Sample period: from May 2016 to February 2020.

Backtesting results

The results of the backtesting procedures are synthesized, for each index under investigation, in Tables from 8 to 10. In terms of the AE ratio, it can be noted that the MF–Q–ARCH models, with and without the “–X” component, generally perform better than the competing models, and more importantly, better than the main competitor (the Q–ARCH model). Sometimes, like for the S&P 500 index, the MF–Q–ARCH model with $NAI_t$ as low-frequency term has an AE ratio of one, which means that the expected and actual number of VaR violations coincide. Considering all the indices used in this work, the proposed models have the AE ratios ranging from 0.793 to 1.189. Conversely, the AR ratios of the GARCH, GARCH-t, GJR, GJR-t and GARCH–MIDAS models swing between 0.688 and 1.210, which represents a wider interval with respect to that of the MF–Q–ARCH–X models. Even the performance of the CAViaR specifications appears inferior than that of the Q–ARCH–based models with low–frequency and “–X” components. In fact, the AR ratios of the SAV, AS and IG range between 0.708 and 1.522 (which is indeed a severe underestimation of the VaR). These results witness the benefits of including the macro–variables within the Q–ARCH framework. With reference to the other three backtesting procedures, it is worth
noting that the proposed MF–Q–ARCH–X and MF–Q–ARCH models always pass the tests used, at the 5% significance level (signalled by shades of gray in the tables), independently of the index considered. On the other hand, the standard GARCH–based models seem to produce inferior VaR forecasts, given that these models do not always pass the backtesting procedures. Interestingly, the specifications belonging to the CAViaR have an inconclusive performance. The SAV model overcomes the backtesting procedure only for the Dow Jones, the AS for S&P 500 and Dow Jones, and the IG only for the NASDAQ. To conclude, if the proposed models appear to brilliantly pass the backtesting procedures, irrespectively of the index considered, the other models do not have very good performances.

Table 8: S&P 500 backtesting results

| Model       | LF var. | Mean   | SD    | AE    | LR_{uc} | LR_{cc} | DQ   |
|-------------|---------|--------|-------|-------|---------|---------|------|
| MF–Q–ARCH–X | ΔP_t   | −1.055 | 0.562 | 0.938 | 0.654   | 0.901   | 0.698|
| MF–Q–ARCH–X | ΔH_{S_t} | −1.083 | 0.570 | 0.812 | 0.169   | 0.368   | 0.830|
| MF–Q–ARCH–X | NAI_t  | −1.071 | 0.583 | 0.938 | 0.654   | 0.901   | 0.989|
| MF–Q–ARCH   | ΔP_t   | −1.085 | 0.433 | 0.917 | 0.548   | 0.663   | 0.735|
| MF–Q–ARCH   | ΔH_{S_t} | −1.121 | 0.434 | 0.958 | 0.766   | 0.493   | 0.473|
| MF–Q–ARCH–X | NAI_t  | −1.131 | 0.417 | 1.000 | 1.000   | 0.295   | 0.122|
| Q–ARCH      | −1.104 | 0.426 | 0.938 | 0.654 | 0.426   | 0.738   |      |
| SAV         | −0.948 | 0.456 | 1.125 | 0.383 | 0.187   | 0.016   |      |
| AS          | −0.941 | 0.470 | 1.167 | 0.248 | 0.470   | 0.415   |      |
| IG          | −1.211 | 0.357 | 0.708 | 0.029 | 0.033   | 0.118   |      |
| GARCH       | −1.167 | 0.360 | 0.729 | 0.044 | 0.051   | 0.023   |      |
| GARCH–t     | −1.129 | 0.360 | 0.875 | 0.364 | 0.228   | 0.009   |      |
| GJR         | −1.188 | 0.434 | 0.729 | 0.044 | 0.051   | 0.195   |      |
| GJR–t       | −1.149 | 0.456 | 0.833 | 0.223 | 0.296   | 0.149   |      |
| RiskMetrics | −1.186 | 0.414 | 0.750 | 0.063 | 0.078   | 0.108   |      |
| GARCH–MIDAS | ΔP_t   | −1.238 | 0.543 | 0.688 | 0.019   | 0.047   | 0.046|
| GARCH–MIDAS | ΔH_{S_t} | −1.242 | 0.480 | 0.729 | 0.044   | 0.051   | 0.171|
| GARCH–MIDAS | NAI_t  | −1.197 | 0.518 | 0.812 | 0.169   | 0.223   | 0.043|

Notes: The table reports the backtesting results for the models in the first column, for the out-of-sample period, which starts on May 2, 2016 and ends on February 28, 2020 (960 days). The models with the “–X” label include the realized volatility (see Eq. (13)). Column “LF var.” reports the low-frequency variables. Columns Mean and SD are the sample averages and standard deviations for the VaR measures. Column AE denotes the Actual over Expected exceedance ratio. Columns LR_{uc} and LR_{cc} report the p-values of the Unconditional and Conditional coverage tests. Finally, column DQ represents the p-value of the Dynamic Quantile test. Shades of gray denote that the model passes the LR_{uc}, LR_{cc} and DQ tests at significance level α = 0.05.

MCS results

The inclusion in the SSM is detailed in Table 11, according to the MCS procedure and the loss function in (22), whose averages are reported in column $\bar{\ell}$. The table includes the SSM for all the indices and with two significance levels: 0.25 and 0.10. In line with what found earlier, also the MCS signals the superiority of the proposed MF–Q–ARCH–X model, independently of the index and significance level employed. Looking at the NASDAQ index, the only models entering the SSM
are the three MF–Q–ARCH–X, based on the macro–variables variables considered. Interestingly, neither the GARCH–based models or the GARCH–MIDAS specifications enter the SSM. Instead, some models taking into account the leverage effect through some asymmetric terms (like the GJR-t and the AS) are sometimes included in the SSM.

### 7 Concluding Remarks

This paper explored the possibility of including mixed–frequency components in a quantile regression approach to VaR estimation, specifically introducing a low–frequency and a high–frequency component within a dynamic model of volatility. As an illustration, here we used a Linear ARCH–X which includes lagged realized volatility as an extended contribution to the short–run volatility dynamics. For the MF–Q–ARCH–X model, the accuracy of Value–at–Risk is increased by assuming that the underlying slow evolution of the volatility of returns is influenced by some macroeconomic variables (indeed observed at lower frequencies) and some additional predetermined daily variables. In estimating the MF–Q–ARCH–X by quantile regression, no distribution for the returns is necessary and robustness to outliers in the data is guaranteed.
by Bernardi et al. (2017), Petrella and Raponi (2019), and Du and Pei (2020), among others. An-

tions identified in the Model Confidence Sets. MIDAS. In addition, only our MF–Q–ARCH–X models consistently enter the set of superior mod-

competing models belonging to the parametric framework, like the GARCH and the GARCH–

specification brilliantly pass the backtesting procedures; the same does not hold for some popular

S&P 500, NASDAQ, and Dow Jones indices. All models based on the proposed MF–Q–ARCH–X

evaluated, both in- and out-of-sample, against several popular alternatives, with applications to the

component. In the empirical application, the proposed MF–Q–ARCH–X model was extensively

robust to some misspecification in the weighting parameter which may affect the low–frequency

Monte Carlo exercise. Overall, the estimates appear satisfactory and the resulting VaR forecasts are

include in the high–frequency component of the MF–Q–ARCH–X model.

a sequential test with the aim of finding the optimal number of daily lagged absolute returns to

the conditions for the weak stationarity of our MF–Q–ARCH–X process. Moreover, we proposed

Further research may focus on the multivariate extension of the proposed specification, as done

From a statistical point of view, we suggested an estimation procedure, having first established

the conditions for the weak stationarity of our MF–Q–ARCH–X process. Moreover, we proposed

sequential test with the aim of finding the optimal number of daily lagged absolute returns to

to some misspecification in the weighting parameter which may affect the low–frequency

component. In the empirical application, the proposed MF–Q–ARCH–X model was extensively

evaluated, both in- and out-of-sample, against several popular alternatives, with applications to the

S&P 500, NASDAQ, and Dow Jones indices. All models based on the proposed MF–Q–ARCH–X

specification brilliantly pass the backtesting procedures; the same does not hold for some popular

competing models belonging to the parametric framework, like the GARCH and the GARCH–

MIDAS. In addition, only our MF–Q–ARCH–X models consistently enter the set of superior mod-

els identified in the Model Confidence Sets.

Further research may focus on the multivariate extension of the proposed specification, as done

by Bernardi et al. (2017), Petrella and Raponi (2019), and Du and Pei (2020), among others. An-

Table 10: Dow Jones backtesting results

| Model          | LF var. | Mean  | SD    | AE    | LR uc | LR cc | DQ  |
|----------------|---------|-------|-------|-------|-------|-------|-----|
| MF–Q–ARCH–X    | ΔIP     | −1.071| 0.589 | 0.898 | 0.460 | 0.575 | 0.724|
| MF–Q–ARCH–X    | ΔHS f   | −1.105| 0.565 | 0.835 | 0.229 | 0.469 | 0.697|
| MF–Q–ARCH–X    | NAI f   | −1.097| 0.583 | 0.856 | 0.295 | 0.568 | 0.801|
| MF–Q–ARCH     | ΔIP     | −1.065| 0.481 | 0.981 | 0.894 | 0.259 | 0.528|
| MF–Q–ARCH     | ΔHS f   | −1.113| 0.462 | 0.793 | 0.128 | 0.167 | 0.623|
| MF–Q–ARCH–X   | NAI f   | −1.089| 0.476 | 0.919 | 0.558 | 0.144 | 0.545|
| Q–ARCH        |         | −1.081| 0.475 | 0.877 | 0.372 | 0.478 | 0.644|
| SAV           |         | −0.980| 0.456 | 1.044 | 0.757 | 0.356 | 0.835|
| IG            |         | −1.215| 0.416 | 0.793 | 0.128 | 0.018 | 0.003|
| GARCH         |         | −1.185| 0.407 | 0.814 | 0.173 | 0.028 | 0.006|
| GARCH–t       |         | −1.152| 0.416 | 0.877 | 0.372 | 0.023 | 0.010|
| GJR           |         | −1.210| 0.472 | 0.772 | 0.093 | 0.118 | 0.392|
| GJR–t         |         | −1.181| 0.500 | 0.835 | 0.229 | 0.302 | 0.598|
| RiskMetrics   |         | −1.206| 0.465 | 0.772 | 0.093 | 0.011 | 0.003|
| GARCH–MIDAS   | ΔIP     | −1.232| 0.509 | 0.772 | 0.093 | 0.011 | 0.001|
| GARCH–MIDAS   | ΔHS f   | −1.219| 0.471 | 0.772 | 0.093 | 0.118 | 0.049|
| GARCH–MIDAS   | NAI f   | −1.220| 0.502 | 0.856 | 0.295 | 0.003 | 0.000|

Notes: The table reports the backtesting results for the models in the first column, for the out-of-

sample period, which starts on May 2, 2016 and ends on February 28, 2020 (958 days). The models

with the “–X” label include the realized volatility (see Eq. (13)). Column “LF var.” reports the low–

frequency variables. Columns Mean and SD are the sample averages and standard deviations for

the VaR measures. Column AE denotes the Actual over Expected exceedance ratio. Columns LR

and LR cc, report the p-values of the Unconditional and Conditional coverage tests. Finally, column

DQ represents the p-value of the Dynamic Quantile test. Shades of gray denote that the model

passes the LR uc, LR cc and DQ tests at significance level α = 0.05.
Table 11: Superior set of models provided by the MCS procedure for each index

| Model | Index     | LF var. | \(\bar{\ell}\) SS | \(\bar{\ell}\) SSM | \(\bar{\ell}\) SS | \(\bar{\ell}\) SSM |
|-------|-----------|---------|---------------------|---------------------|---------------------|---------------------|
| MF–Q–ARCH–X | \(\Delta IP_t\) | 7.985 | 25, 10 | 9.479 | 25, 10 | 7.942 | 25, 10 |
| MF–Q–ARCH–X | \(\Delta HS_t\) | 7.907 | 25, 10 | 9.592 | 25, 10 | 7.967 | 25, 10 |
| MF–Q–ARCH–X | \(NAl_t\) | 7.997 | 25, 10 | 9.679 | 25, 10 | 8.023 | 25, 10 |
| MF–Q–ARCH | \(\Delta IP_t\) | 8.412 | 10.294 | 8.594 | 10.294 |
| MF–Q–ARCH | \(\Delta HS_t\) | 8.578 | 10.442 | 8.603 | 10.442 |
| MF–Q–ARCH | \(NAl_t\) | 8.626 | 10.419 | 8.673 | 10.419 |
| Q–ARCH | | 8.425 | 10.408 | 8.583 | 10.408 |
| SAV | | 8.167 | 25, 10 | 10.488 | 25, 10 | 8.279 | 25, 10 |
| AS | | 8.124 | 25, 10 | 10.399 | 25, 10 | 8.248 | 25, 10 |
| IG | | 8.640 | 10.505 | 8.872 | 10.505 |
| GARCH | | 8.527 | 10.376 | 8.841 | 10.376 |
| GARCH–t | | 8.494 | 10.420 | 8.824 | 10.420 |
| GJR | | 8.535 | 10.348 | 8.785 | 10.348 |
| GJR–t | | 8.459 | 10.386 | 8.736 | 10.386 |
| RiskMetrics | | 8.464 | 10.348 | 8.786 | 10.348 |
| GARCH–MIDAS | \(\Delta IP_t\) | 8.769 | 10.438 | 9.099 | 10.438 |
| GARCH–MIDAS | \(\Delta HS_t\) | 8.757 | 10.341 | 8.924 | 10.341 |
| GARCH–MIDAS | \(NAl_t\) | 8.658 | 10.461 | 9.081 | 10.461 |

Notes: The table reports the inclusion in the SSM according to the MCS procedure. Column “LF var.” reports the low–frequency variables. Column \(\bar{\ell}\) represents the averages of the loss function in (21), used in the MCS procedure. The out-of-sample period, which starts on May 2, 2016 and ends on February 28, 2020 (960, 959 e 958 days, respectively, for S&P 500, NASDAQ and Dow Jones). The models with the “–X” label include the realized volatility (see Eq. (13)). Symbols \(\bar{\ell}\) and \(\bar{\ell}\) denote inclusion in the SSM at significance level \(\alpha = 0.25\) and \(\alpha = 0.10\), respectively.

Another interesting point would be the inclusion of an asymmetric term in the proposed model, both for what concerns the daily returns and the low–frequency component, as done by Amendola et al. (2019), for instance. Finally, the current approach involving the low–frequency and “–X” terms could be enriched by jointly estimating both the VaR and the Expected Shortfall measures, as recently developed by Fissler et al. (2016), Taylor (2019), and Pan et al. (2021).
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**Appendix**

**Proof of Theorem 1**

*Proof.* Let \( \| x \|_p = (E|x|^p)^{1/p} \), and recall that \( MV_t \) and \( X_{i,t} \) are assumed to be weakly stationary processes. Let \( s \) be the compact time notation in lieu of \( i, t \), that is,

\[
s \equiv \sum_{j=1}^{i-1} N_j + i.
\]

Moreover, let \( \sigma_s = (\beta_0 + \beta_1|r_{s-1}| + \cdots + \beta_q|r_{s-q}| + \theta|W_{S_{s-1}}| + \beta_X|X_{s-1}|) \). Note that \( WS_s \), obtained as an affine combination of \( (MV_{t-1}, \cdots, MV_{t-K}) \), is weakly stationary.
From the model in (13), we can write:

$$\|r_s\|_p = \|\sigma_s z_s\|_p = \|\sigma_s\|_p \cdot \|z_s\|_p,$$

(A.1)

given the independence between $\sigma_s$ and $z_s$. For $p = 1$, the right hand side (RHS) of (A.1) is zero, because $z_s \sim (0, 1)$.

Let us now focus on $p = 2$; let us replace the second term of the RHS of (A.1), having assumed that $\|z_s\|_2 = z^* < \infty$:

$$\|r_s\|_r = z^* \left( E(\beta_0 + \beta_1 |r_{s-1}| + \cdots + \beta_q |r_{s-q}| + \theta |W_{S_{s-1}}| + \beta_X |X_{s-1}|)^2 \right)^{1/2}$$

(A.2)

$$\leq z^* (\beta_0 + \beta_1 \|r_{s-1}\|_2 + \cdots + \beta_q \|r_{s-q}\|_2 + \theta \|W_{S_{s-1}}\|_2 + \beta_X \|X_{s-1}\|_2).$$

(A.3)

Let us now translate this expression in matrix notation. Therefore, let us collect terms in a vector indexed by $s$, that is,

$$\xi_s = \left( \|r_s\|_2, \cdots, \|r_{s-q+1}\|_2, \|W_{S_s}\|_2, \|X_s\|_2 \right)'$$

and let the $(q + 2) \times (q + 2)$ dimensional companion matrix $A$, the vectors $b$ and $c$

$$A = \begin{bmatrix}
  z^* \beta_1 & z^* \beta_2 & \cdots & z^* \beta_{q-1} & z^* \beta_q & z^* \theta & z^* \beta_x \\
  1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
  0 & 1 & \cdots & 0 & 0 & 0 & 0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  0 & 0 & \cdots & 1 & 0 & 0 & 0 \\
  0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
  0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
  0 & 0 & \cdots & 0 & 0 & 0 & 0
\end{bmatrix}, \quad b = \begin{bmatrix}
  z^* \beta_0 \\
  0 \\
  0 \\
  \vdots \\
  0 \\
  0 \\
  0 \\
  0
\end{bmatrix} \quad \text{and} \quad c = \begin{bmatrix}
  0 \\
  0 \\
  \|W_{S_s}\|_2 \\
  \|X_s\|_2
\end{bmatrix},$$

where we have made us of the fact that, because of the stationarity of $W_{S_s}$ and $X_s$, the vector $c$ does not depend on time. Thus, we have:

$$\xi_s \leq A \xi_{s-1} + b + c.$$
Substituting recursively $\xi_{s-1}$ backwards, and letting $I_{q+2}$ be the identity matrix of size $(q + 2)$,

\[
\xi_s \leq A (A \xi_{s-2} + b + c) + b + c \\
\leq A^2 \xi_{s-2} + Ab + b + Ac + c \\
\leq A^2 \xi_{s-2} + (I_{q+2} + A) b + (I_{q+2} + A) c \\
\leq A^3 \xi_{s-3} + (I_{q+2} + A + A^2) b + (I_{q+2} + A + A^2) c \\
\vdots \\
\leq A^m \xi_{s-m} + (I_{q+2} + A + A^2 + \cdots + A^{m-1}) b + (I_{q+2} + A + A^2 + \cdots + A^{m-1}) c. 
\] (A.8)

Recall the characteristic polynomial of $A$ is $\phi(\lambda)$, defined by Eq. (14), namely,

\[
\phi(\lambda) = z^* \left( \beta_1 \lambda^{q+1} + \beta_2 \lambda^q + \cdots + \beta_q \lambda^{q-2} \right) - \lambda^{q+2}, 
\] (A.9)

which has all eigenvalues $\lambda$ lie inside the unit circle. When $m \to \infty$, for the eigen–decomposition theorem, this implies that

\[
\lim_{m \to \infty} A^m = 0, 
\] (A.10)

and that

\[
\lim_{m \to \infty} (I_{q+2} + A + A^2 + \cdots + A^{m-1}) = (I_{q+2} - A)^{-1}. 
\] (A.11)

Putting terms together, therefore, as $m \to \infty$ we can say that

\[
\xi_s \leq (I_{q+2} - A)^{-1} b + (I_{q+2} - A)^{-1} c < \infty, 
\] (A.12)

that is the RHS converges to a finite expression not depending on time, establishing the result. \qed