POINTS ON MODULAR CURVES OVER FINITE FIELDS

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Abstract. In this paper we propose a method of computing the number of points on the reduction of non-hyperelliptic modular curves of genus greater than or equal to 3 over finite fields.

1. Introduction

Let $N$ be a positive integer, and let

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid c \equiv 0 \mod N \right\}.$$ 

Let $X_0(N)$ denote the modular curve corresponding to $\Gamma_0(N)$ and $g_0(N)$ denote its genus. The modular curve $X_0(N)$ (with cusps removed) parameterizes isomorphism classes of pairs $(E, C)$, where $E$ is an elliptic curve and $C$ is a cyclic subgroup of $E$ of order $N$.

A curve $X$ defined over an algebraically closed field $k$ is called $d$-gonal if it admits a map $\phi : X \to \mathbb{P}^1$ over $k$ of degree $d$. The smallest possible $d$ is called the gonality of $X$ denoted by $\text{Gon}(X)$. If a curve $X$ is 2-gonal and its genus $g(X) \geq 2$, then $X$ is said to be hyperelliptic. If a curve $X$ is 3-gonal, then we call $X$ trigonal.

Ogg [4] determined all values of $N$ for which $X_0(N)$ is hyperelliptic, and Hasegawa and Shimura [2] determined all the trigonal curves $X_0(N)$. A crucial instrument used in their proofs is $\# \tilde{X}_0(N)(\mathbb{F}_p^2)$ which denote the number of points on the reduction of $X_0(N)$ over the finite fields $\mathbb{F}_p^2$ where $p$ is a prime with $p \nmid N$. Note that for a prime $p \nmid N$, the

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curve $X_0(N)$ has good reduction. Indeed Ogg [4] proposed a method to give a lower bound for $\#X_0(N)(\mathbb{F}_{p^2})$ by computing the pairs $(E, C)$ with supersingular elliptic curves $E$ and their cyclic subgroups $C$ of order $N$.

In this paper, we propose a method of computing the exact number of points on the reduction of non-hyperelliptic modular curves $X_0(N)$ of $g_0(N) \geq 3$ over any finite fields whose characteristic does not divide $N$. This method can be applied for another sort of modular curves defined over $\mathbb{Q}$.

Indeed, such a method is well-known for rational, elliptic or hyperelliptic modular curves.

2. Preliminaries

Suppose $X_0(N)$ is a non-hyperelliptic modular curve of $g := g_0(N) \geq 3$. In this section, we consider a method to find the canonical embedding of $X_0(N)$ which is described in [2, 3]. The canonical embedding of $X_0(N)$ is the embedding

$$X_0(N) \ni P \mapsto [\omega_1(P) : \cdots : \omega_g(P)] \in \mathbb{P}^{g-1}$$

determined by the canonical linear system. Its image is called a canonical curve.

The space $\Omega^1(X_0(N))$ of holomorphic differentials is isomorphic to the space of weight 2 cusp forms, $S_2(N)$, on $X_0(N)$. Indeed, let $\{f_1, \ldots, f_g\}$ be a basis for $S_2(N)$, then the set $\{f_i(\tau) d\tau\}$ forms a basis for $\Omega^1(X_0(N))$. Then the canonical embedding of $X_0(N)$ is given by

$$X_0(N) \ni P \mapsto [f_1(P) : \cdots : f_g(P)] \in \mathbb{P}^{g-1}.$$ 

This image is a curve of degree $2g - 2$ and it will be described by some set of projective equations of the form $F(f_1, \ldots, f_g) = 0$. We call these equations a canonical model of $X_0(N)$.

To construct a canonical model we take the $q$-expansions of a basis for the space $S_2(N)$ which can be computed by using a computer algebra system SAGE. Here $q = e^{2\pi i \tau}$ and $\tau$ is in the complex upper half plane. Then we compute a canonical model by finding combinations of powers of the $q$-expansions which yield identically zero series. We know that for almost all $N$ canonical models consist of polynomials of degree 2 from the following result.

**Theorem 2.1.** [1, 5] Let $X$ be a canonical curve of genus $\geq 4$ defined over an algebraically closed field. Then the ideal $I(X)$ of $X$ is generated by some quadratic polynomials, unless $X$ is trigonal or isomorphic to
a smooth plane quintic curve, in which cases it is generated by some quadratic and (at least one) cubic polynomials.

For the reader’s convenience, we make lists of \( N \) for which \( X_0(N) \) is rational, elliptic, hyperelliptic or of that \( \text{Gon}(X_0(N)) = 3 \).

**Theorem 2.2.** [2, 4] The following holds:

(a) \( X_0(N) \) is rational only for \( N: 1, 10, 12, 13, 16, 18, 25 \).
(b) \( X_0(N) \) is elliptic only for \( N: 11, 14, 15, 17, 19, 20, 21, 24, 27, 32, 36, 49 \).
(c) \( X_0(N) \) is hyperelliptic only for \( N: 22, 23, 26, 28, 29, 30, 31, 33, 35, 37, 39, 40, 41, 46, 47, 48, 50, 59, 71 \).
(d) \( \text{Gon}(X_0(N)) = 3 \) only for \( N: 34, 38, 43, 44, 45, 53, 54, 61, 64, 81 \).

3. Canonical models

In this section, we explain how to compute a canonical model of \( X_0(N) \). Consider \( X_0(42) \) of genus 5. In SAGE one can compute \( q \)-expansions of a basis for \( S_2(42) \) by using the following commands:

\[
M = \text{ModularForms}(\Gamma_0(42));
S = M.\text{cuspidal submodule}();
S.\text{q}\text{-expansion basis}(100);
\]

Then we have the following:

\[
\begin{align*}
    f_1 &= q + q^6 + q^7 - 2q^8 - 3q^9 - 2q^{10} - q^{12} - \cdots, \\
    f_2 &= q^2 - q^8 - q^9 - 2q^{10} - 2q^{11} + 2q^{13} - \cdots, \\
    f_3 &= q^3 - q^6 - 2q^9 + q^{12} + 2q^{18} + q^{21} - \cdots, \\
    f_4 &= q^4 - q^6 - q^9 - 2q^{11} + q^{12} + 2q^{13} + \cdots, \\
    f_5 &= q^5 + q^6 + q^7 - 2q^8 - 2q^9 - q^{10} - \cdots.
\end{align*}
\]

By Theorem 2.1, the defining ideal of the canonical curve in \( \mathbb{P}^4 \) of \( X_0(42) \) generated by quadratic polynomials, and hence it suffices to consider the relations of \( \frac{g(g+1)}{2} = 15 \) monomials \( \{f_if_j\} \) with \( 1 \leq i \leq j \leq 5 \) for getting a canonical model of \( X_0(42) \).

Put \( A = (a_{mn}) \) the \( 99 \times 15 \) matrix with \( a_{mn} \) being the coefficient of \( q^m \) in the \( q \)-expansion of the \( n \)-th element \( f_kf_l \) of \( \{f_if_j\} \).
Solving the linear equation $AX = 0$ with $X = \begin{pmatrix} x_1 \\ \vdots \\ x_{15} \end{pmatrix}$, we can find three relations between \{\(f_i \mid f_j\}\}, and they give a canonical model of $X_0(42)$ as follows:

\[
F_1 : \quad -y^2 + zx + vz,
\]
\[
F_2 : \quad -zy - z^2 + vx + vy + vz - v^2 + wz - wv,
\]
\[
F_3 : \quad z^2 - wx + wy - wz + w^2,
\]

where the variables $x, y, z, v, w$ are corresponding to $f_1, f_2, f_3, f_4, f_5$, respectively.

We omit an explanation for the canonical curves whose defining ideals contain a cubic polynomial for which one can refer [2, 3].

We list canonical models for $X_0(N)$ in Table 1 where $X_0(N)$ is a non-hyperelliptic curve of genus greater than or equal to 3 for $N \leq 50$. We note that the canonical models for $X_0(N)$ with $N = 34, 43, 45$ are directly from Table 1 in [3]. Indeed, such curves are of genus 3 and defined by plane quartic polynomials.

| $N$ | Canonical model of $X_0(N)$ |
|-----|--------------------------------|
| 34  | $x^4 + x^3z - 2x^2z^2 + 3xy^2z + xz^3 - y^4 + z^4$ |
| 38  | $-y^2 + zx - z^2 - wy - wz - w^2,$  
     | $y^2x - 3y^3 - zx^2 + 4zyx - 3zy^2 + z^2x - z^2y - z^3$  
     | $-wx^2 + wyx - 4wy^2 - wzx + w^3$ |
| 42  | $-y^2 + zx + vz,$  
     | $-zy - z^2 + vx + vy + vz - v^2 + wz - wv,$  
     | $z^2 - wx + wy - wz + w^2$ |
| 43  | $x^4 + 2x^3y + 2x^2y^2 + 2x^2yz + 4x^2z^2$  
     | $+xy^3 + 2xy^2z + 4xyz^2 + y^3z + 2y^2z^2 + 3yz^3 + 4z^4$ |
| 44  | $-x^2 - 4yx - 8y^2 - 4zx - 16zy - 16z^2 + w^2,$  
     | $-y^3 + z^2x + 4zyx + 4z^2x$ |
| 45  | $x^4 + 2x^3y + x^2y^2 + x^2yz - x^2z^2 - xy^2z + 3xyz^2$  
     | $-2xz^3 - y^3z + y^2z^2 + yz^3 + 4z^4$ |

**Table 1. Canonical models for $X_0(N)$**
4. Points on modular curves over a finite field

Suppose $X_0(N)$ is a non-hyperelliptic modular curve of genus $g \geq 3$. Now we explain how to compute $\#\tilde{X}_0(N)(\F_q)$ where $q = p^k$ and $p \nmid N$. Suppose $\{F_1, F_2, \ldots, F_n\}$ is a canonical model of $X_0(N)$ with integer coefficients. Put $G_i := F_i \mod p$ for $i = 1, 2, \ldots, n$. Let $Y$ be the curve defined by $\{G_1, G_2, \ldots, G_n\}$ over $\F_p$. Our basic strategy is to compute the number of $\F_q$-rational points $\#Y(\F_q)$ on $Y$. However we don’t know whether it defines a non-singular curve. In fact, Galbraith [3] appointed that the canonical model of $X_0(38)$ he obtained first has bad reduction at the prime 3 even though 38 is not divisible by 3. By a proper change of coordinates, he could obtain a canonical model for $X_0(38)$ which has good reduction at 3. We note that the canonical model for $X_0(44)$ in Table 1 is not computed by the basis of $S_2(44)$ obtained from Singular but the basis $\{f(\tau), f(2\tau), f(4\tau), g(\tau)\}$ where $f(z)$ (resp. $g(\tau)$) is the normalized eigenform of Hecke operators in $S_2(11)$ (resp. $S_2(44)$). The canonical model for $X_0(44)$ obtained by using the basis of $S_2(44)$ from Singular has bad reduction at 3.

A computer algebra system Macaulay2 enables us to determine whether the reduction of a canonical model of $X_0(N)$ has good reduction over $\F_q$. First, we compute the arithmetic genus of $Y$ which should be equal to the (geometric) genus of $X_0(N)$. It can be computed by the following comments:

```math
R=ZZ/p[x_1,x_2,...,x_g]
I=ideal{G_1,j,G_2,...,G_n}
genus(I)
```

Second, we check $Y$ has no singularities over $\F_q$ by the following comments:

```math
R=GF(q)[x_1,x_2,...,x_g]
I=ideal{G_1,j,G_2,...,G_n}
sing=singularLocus(R/I)
codim(sing)
```

If it gives the co-dimension $g$ of singular locus, then we can conclude that $Y$ has no singularity over $\F_q$. However, we are not sure that $Y$ has no singularities over the algebraic closure $\bar{\F}_p$. Nevertheless it suffices to compute $\#Y(\F_q)$ for obtaining $\#\tilde{X}_0(N)(\F_q)$. 
Theorem 4.1. Suppose $X_0(N)$ is a non-hyperelliptic curve of genus $g \geq 3$, and its canonical model $\{F_1, F_2, \ldots, F_n\}$ consists of polynomials with integer coefficients. Let $Y$ be a curve defined by $\{G_1, G_2, \ldots, G_n\}$ over $\mathbb{F}_p$ where $G_i := F_i \mod p$ with $p \nmid N$. Suppose $Y$ has geometric genus $g$ and no singularities over $\mathbb{F}_q$ with $q = p^k$, then $\# Y(\mathbb{F}_q)$ is the same as $\# \tilde{X}_0(N)(\mathbb{F}_q)$.

Proof. If $Y$ is a non-singular curve, then the result is true. Suppose $Y$ has singular points $P_1, \ldots, P_m$ over a finite extension $\mathbb{F}_r$ of $\mathbb{F}_q$. For getting a smooth model for $Y$ we need to blow $Y$ up. Since the set $\{P_1, \ldots, P_m\}$ is Galois invariant, the blown up curve $Z$ will be defined over $\mathbb{F}_q$. And the blow-down map $\pi : Z \to Y$ is defined over $\mathbb{F}_q$ too. It follows that the fields of definition of points in $\pi^{-1}(P_i)$ must contain the field of definition of $P_i$, hence are not equal to $\mathbb{F}_q$. This proves that $\pi$ is a bijection on the $\mathbb{F}_q$-rational points, i.e. $\pi : Z(\mathbb{F}_q) \to Y(\mathbb{F}_q)$ is an isomorphism. By definition, $\# \tilde{X}_0(N)(\mathbb{F}_q)$ is $\# Z(\mathbb{F}_q)$, so the result is true.

Example 4.2. Let $Y$ denote the curve over $\mathbb{F}_5$ defined by the reduction $\{G_1, G_2, G_3\}$ modulo 5 of a canonical model of $X_0(42)$ described in (3.1). Using Macaulay2 we can check that $Y$ has arithmetic genus 5 and no singularities over $\mathbb{F}_{25}$. Plugging in all values of $x, y, z, v, w$ and counting those for which $G_1 \equiv G_2 \equiv G_3 \equiv 0 \pmod{5}$, we can get $\# \tilde{X}_0(42)(\mathbb{F}_5) = 12$ and $\# \tilde{X}_0(42)(\mathbb{F}_{25}) = 64$.

By using the method suggested in this paper, we compute $\# \tilde{X}_0(N)(\mathbb{F}_p)$ and $\# \tilde{X}_0(N)(\mathbb{F}_{p^2})$ in Table 2 where $X_0(N)$ is a non-hyperelliptic curve of $g_0(N) \geq 3$ for $N \leq 50$ and $p$ is the smallest prime $p \mid N$.
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