A FILTER ON A COLLECTION OF FINITE SETS AND EBERLEIN COMPACTA

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ABSTRACT. We introduce a σ-ideal on $\omega_1 \times \omega_1$ and a filter on the collection of graphs of strictly decreasing partial functions on $\omega_1$ taking values in $\omega_1$. We use them to prove that a certain space is a non-bisequential Eberlein compactum.

1. Introduction

For a subset $A \subseteq \omega_1 \times \omega_1$ of the set of pairs of countable ordinals, we will denote the vertical and horizontal sections of $A$ at $\alpha$ by

$$A_\alpha = \{\beta : (\alpha, \beta) \in A\}, \quad A^\beta = \{\alpha : (\alpha, \beta) \in A\},$$

and we will consider the σ-ideal

$$I = \{A \subseteq \omega_1 \times \omega_1 : \text{for all but countably many } \alpha, |A_\alpha| \leq \aleph_0 \text{ and } |A^\alpha| \leq \aleph_0\}.$$

We will prove the following

Lemma 1.1. For any $A$ in the σ-ideal $I$ and subsets $B_1, \ldots, B_n$ of $\omega_1 \times \omega_1$ not in $I$, there is a strictly decreasing function $f : S \rightarrow \omega_1$, $S \subseteq \omega_1$, whose graph omits $A$ and intersects each $B_i$, $i \leq n$.

We will denote by $[\omega_1 \times \omega_1]^{<\omega}$ the collection of all finite subsets of $\omega_1 \times \omega_1$, endowed with the pointwise topology, i.e., identifying a set with its characteristic function, we will consider $[\omega_1 \times \omega_1]^{<\omega}$ as the subspace of the Tychonoff product $\{0, 1\}^{\omega_1 \times \omega_1}$ consisting of functions with finite support.

Then the space

$$\mathcal{X} = \{A \in [\omega_1 \times \omega_1]^{<\omega} : A \text{ is the graph of a strictly decreasing function}\}$$

is an Eberlein compactum (the terminology is explained in the next section).

Peter Nyikos announced in [Nyi] that the compactum $\mathcal{X}$ is not bisequential. However, to our best knowledge, a proof of this result has not been published.

We will show that lemma 1.1 provides readily the following refinement of the Nyikos theorem.

Theorem 1.2. For any $B \subseteq \omega_1 \times \omega_1$ not in the σ-ideal $I$, the Eberlein compactum $\mathcal{X}_B = \{A \in \mathcal{X} : A \subseteq B\}$ is not bisequential.

Let us notice that the Eberlein compactum $\mathcal{X}$ was also considered by Leiderman and Sokolov [LS], but in a different context (cf. section 5).

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2. Some background

A sequence of non-empty subsets $A_0, A_1, \ldots$ of a topological space $X$ converges to a point $x \in X$ if for any neighbourhood $U$ of $x$ there exists an integer $n_0$ such that for any $n > n_0$ we have $A_n \subset U$. An ultrafilter $U \subset \mathcal{P}(X)$ is convergent to $x \in X$ if every neighbourhood $U$ of $x$ is an element of $U$.

We say that a topological space $X$ is bisequential if for any ultrafilter $U \subset \mathcal{P}(X)$ convergent to some element $x \in X$ there exists a sequence $U_0 \supset U_1 \supset U_2 \supset \ldots$ of elements of $U$ convergent to $x$. Equivalently, a topological space is bisequential if and only if it is a biquotient image of metrizable space. This is due to Michael [Mich, Theorem 3.D.2].

The notion of bisequentiality is closely related to Fréchet property. We say that a topological space $X$ is Fréchet, if for any subset $A \subset X$ and $x \in A$ there exists a sequence $x_0, x_1, \ldots$ of elements of $A$ converging to $x$. It is easy to see that every bisequential space is Fréchet.

Bisequential spaces enjoy many “good” properties: subspaces of bisequential spaces, countable products of bisequential spaces and continuous images of compact bisequential spaces are bisequential. Fréchet spaces behave much worse — there exist two compact Fréchet spaces whose product is not Fréchet.

A compact space $K$ is called an Eberlein compactum if it homeomorphically embeds into some Banach space $X$ with its weak topology. If we additionally assume that $X$ is a Hilbert space, we call $K$ an uniform Eberlein compactum.

It is not hard to see that the space $X$ described in the introduction is a closed subspace of $\{0, 1\}^{\omega_1 \times \omega_1}$. Using the following folklore theorem

Theorem 2.1. Let $X$ be a set and let $A$ be a family consisting of some finite subsets of $X$. If $K = \{\chi_A : A \in A\}$ is a closed subspace of $\{0, 1\}^X$ then $K$ is an Eberlein compactum.

we deduce that $X$ is an Eberlein compactum.

Note that $X$ serves as an example of non-bisequential Fréchet space — this is because Eberlein compacta are Fréchet (see e.g. [Arh]).

3. Proof of Lemma 1.1

Since $A \in \mathcal{I}$, there exists an ordinal $\alpha < \omega_1$ such that
\[
\forall \alpha < \beta < \omega_1 \quad |A_\beta| < \aleph_1 \land |A^{<\beta}| < \aleph_1.
\]

For any $i \in \{1, \ldots, n\}$, $B_i \in \mathcal{P}(\omega_1 \times \omega_1) \setminus \mathcal{I}$, therefore there exist uncountably many ordinals $\beta < \omega_1$ such that $|(B_i)_\beta| = \aleph_1$ or $|(B_i)^{<\beta}| = \aleph_1$.

Without loss of generality we can assume that $|(B_i)_\beta| = \aleph_1$ for $1 \leq i \leq k$ and $|(B_i)^{<\beta}| = \aleph_1$ for $k + 1 \leq i \leq n$. Therefore there exist ordinals $\beta_1, \beta_2, \ldots, \beta_k$ such that:

- $\alpha < \beta_1 < \beta_2 < \ldots < \beta_k$,
- the sets $(B_i)_{\beta_i}$ are uncountable for $1 \leq i \leq k$.

Analogously, there exist ordinals $\gamma_{k+1}, \gamma_{k+2}, \ldots, \gamma_n$ such that

- $\alpha < \gamma_n < \gamma_{n-1} < \ldots < \gamma_{k+1}$,
- the sets $(B_i)^{<\gamma_i}$ are uncountable for $k + 1 \leq i \leq n$.

Thus there exist ordinals $\gamma_1, \gamma_2, \ldots, \gamma_k$ such that:
Analogously, there exist ordinals $\beta_{k+1}, \beta_{k+2}, \ldots, \beta_n$ such that

- $\beta_k < \beta_{k+1} < \beta_{k+2} < \ldots < \beta_n$,
- $\beta_i \in (B_1)_{\beta_i}$ for each $1 \leq i \leq k$,
- $\gamma_i > \sup A_{\beta_i}$ for each $1 \leq i \leq k$.

It follows that $(\beta_1, \gamma_1) \in B_i$ and $(\beta_i, \gamma_i) \notin A$ for any $1 \leq i \leq n$. Moreover

$$\beta_1 < \beta_2 < \ldots < \beta_n \text{ and } \gamma_1 > \gamma_2 > \ldots > \gamma_n.$$ 

Therefore the function $f: \{\beta_1, \ldots, \beta_n\} \to \omega_1$ given by the formula $f(\beta_i) = \gamma_i$ has the desired properties.

4. Proof of theorem \[\text{[1, 2]}\]

Fix $B \in \mathcal{P}(\omega_1 \times \omega_1) \setminus \mathcal{I}$. It is clear that $\mathcal{X}_B$ is a closed subset of $\mathcal{X}$, and therefore is an Eberlein compactum.

For any $A, B_1, \ldots, B_n \subset B$ define

$$\mathcal{F}_A = \{C \in \mathcal{X}_B : A \cap C = \emptyset\}$$

and

$$\mathcal{F}_A(B_1, B_2, \ldots, B_n) = \{C \in \mathcal{X}_B : A \cap C = \emptyset \cap \forall i \leq n B_i \cap C \neq \emptyset\}.$$ 

Let $\mathcal{I}_B$ be the $\sigma$-ideal $\mathcal{I}_B = \{A \in \mathcal{I} : A \subset B\}$. Consider

$$\mathcal{F} = \{\mathcal{F}_A : A \in \mathcal{I}_B\} \cup \{\mathcal{F}_A(B_1, B_2, \ldots, B_n) : A \in \mathcal{I}_B \cap n \in \omega \cap \forall i \leq n B_i \in \mathcal{P}(B) \setminus \mathcal{I}_B\}.$$ 

Note that for any $A, A' \in \mathcal{I}_B$ and $B_1, \ldots, B_n, B'_1, \ldots, B'_n \in \mathcal{P}(B) \setminus \mathcal{I}_B$ we have $A \cup A' \in \mathcal{I}_B$ and

$$\mathcal{F}_A \cap \mathcal{F}_{A'} = \mathcal{F}_{A \cup A'}$$

$$\mathcal{F}_A(B_1, B_2, \ldots, B_n) \cap \mathcal{F}_{A'}(B'_1, B'_2, \ldots, B'_n) = \mathcal{F}_{A \cup A'}(B_1, B_2, \ldots, B_n, B'_1, B'_2, \ldots, B'_n).$$ 

It follows that intersection of any two elements of $\mathcal{F}$ is an element of $\mathcal{F}$. By lemma \[\text{[1, 1]}\] we infer that $\mathcal{F}$ consists of non-empty sets. Therefore $\mathcal{F}$ has the finite intersection property. Thus there exists an ultrafilter $\mathcal{U}$ on $\mathcal{X}_B$ extending $\mathcal{F}$. We will prove that $\mathcal{U}$ witnesses non-bisequentiality of $\mathcal{X}_B$.

Basic neighbourhoods of $\emptyset \in \mathcal{X}_B$ are of the form $\mathcal{F}_A$, where $A$ are finite subsets of $B$ and therefore are elements of $\mathcal{F}$. In particular, every neighbourhood of $\emptyset$ is an element of $\mathcal{U}$ which means that $\mathcal{U}$ converges to $\emptyset$.

For the sake of contradiction assume that $\mathcal{X}_B$ is bisequential. Then there exists a decreasing sequence $\mathcal{U}_0, \mathcal{U}_1, \ldots$ of elements of $\mathcal{U}$ such that for any basic neighbourhood $\mathcal{F}_A$ of $\emptyset$ there exists a positive integer $i$ such that $\mathcal{U}_i \subset \mathcal{F}_A$.

Define for any $i \in \omega$

$$A_i = \{(\alpha, \beta) \in B : \mathcal{U}_i \subset \mathcal{F}_{\{\alpha, \beta\}}\}.$$ 

Then $\bigcup_{i \in \omega} A_i = B$. Moreover

$$\mathcal{U}_i \subset \bigcap_{(\alpha, \beta) \in A_i} \mathcal{F}_{\{\alpha, \beta\}} = \mathcal{F}_{A_i}.$$ 

Thus $\mathcal{F}_{A_i} \in \mathcal{U}$.
Suppose that $A_i \notin \mathcal{I}_B$ for some $i \in \omega$. Then
\[ \mathcal{X} \setminus \mathcal{F}_{A_i} = \mathcal{F}_\emptyset(A_i) \in \mathcal{F} \subset \mathcal{U}, \]
which is a contradiction as $\mathcal{U}$ is an ultrafilter and $\mathcal{F}_{A_i} \in \mathcal{U}$. Therefore $A_i \in \mathcal{I}_B$ for any $i \in \omega$.

Note that
\[ \bigcup_{i \in \omega} A_i \in \mathcal{I}_B \]
because $\mathcal{I}_B$ is a $\sigma$-ideal. On the other hand
\[ \bigcup_{i \in \omega} A_i = \omega_1 \times \omega_1, \]
which is a contradiction. Thus $\mathcal{X}_B$ is non-bisequential.

5. Uniform Eberlein compacta

We already mentioned that the space $\mathcal{X}$ was also studied by Leiderman and Sokolov. They proved that $\mathcal{X}$ is an Eberlein compactum which isn’t a uniform Eberlein compactum. In this section we present another proof of this fact using non-bisequentiality of $\mathcal{X}$.

We will need a few other results.

Benyamini, Rudin and Wage gave the following characterization of uniform Eberlein compacta [BRW, Lemma 1.2]:

**Theorem 5.1.** Every uniform Eberlein compactum $K$ is a continuous image of a closed subspace of $\alpha(\lambda)\omega$ for some cardinal $\lambda$, where $\alpha(\lambda)$ is the one-point compactification of a discrete space of cardinality $\lambda$.

Moreover, one can take as $\lambda$ the weight of $K$.

The following theorem is a well-known fact proved by Nyikos.

**Theorem 5.2.** If $K$ is a uniform Eberlein compactum whose weight is smaller than the first measurable cardinal, then $K$ is bisequential.

*Proof.* Let $\lambda$ be the weight of $K$. It follows from Theorem 5.1 that there exists a closed subset $F \subset \alpha(\lambda)^\omega$ and a continuous surjection $f : F \rightarrow K$.

By assumption, $\lambda$ is smaller than the first measurable cardinal, therefore $\alpha(\lambda)$ is bisequential (see [Mich, Example 10.15]). Thus $\alpha(\lambda)^\omega$ is bisequential as well, because it is a countable product of bisequential spaces. Thus $F$ is bisequential as a subspace of bisequential space.

Since $F$ is a closed subset of a compact space $\alpha(\lambda)^\omega$, it follows that $F$ is compact. Therefore $K = f(F)$ is a continuous image of a compact bisequential space. Thus $K$ is bisequential.

**Corollary 5.3.** $\mathcal{X}$ is not a uniform Eberlein compactum.

*Proof.* $\mathcal{X}$ has weight $\aleph_1$ and is not bisequential. It follows from the previous theorem that $\mathcal{X}$ is not a uniform Eberlein compactum.
6. Comments

One can replace \( \omega_1 \) by an ordinal number \( \alpha \) in the description of the space \( X \). If \( \alpha < \omega_1 \) then the resulting space is countable, which of course is bisequential. If \( \alpha > \omega_1 \), then we get a non-bisequential space, because \( X \) embeds homeomorically into it. Therefore, in a sense, \( X \) is the smallest example of a non-bisequential Eberlein compactum of this type.

We get another modification by replacing \( \omega_1 \) by a partially ordered set \( T \). As a result we get a space which we will denote by \( X_T \). If \( T \) is a tree then \( X_T \) is an Eberlein compactum.

Note that if \( T \) has a branch of length at least \( \omega_1 \), then \( X \) can be homeomorphically embedded into \( X_T \) which implies that \( X_T \) isn’t bisequential. Of course trees of height greater than \( \omega_1 \) have such a branch. Two interesting questions arise:

**Question 6.1.** For which trees \( T \) of height less or equal than \( \omega_1 \) is \( X_T \) bisequential?

**Question 6.2.** Let \( T \) be an Aronszajn tree. Is \( X_T \) bisequential?

The author wasn’t able to answer these questions.

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