A note on Almgren-Chriss optimal execution problem with geometric Brownian motion

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Abstract
We solve explicitly the Almgren-Chriss optimal liquidation problem where the stock price process follows a geometric Brownian motion. Our technique is to work in terms of cash and to use functional analysis tools. We show that this framework extends readily to the case of a stochastic drift for the price process and the liquidation of a portfolio.

Keywords: Almgren-Chriss, optimal liquidation, adjoint operator.

1 Introduction
Optimal liquidation is a problem faced by a trader when he needs to liquidate a large number of shares. The trader faces a tradeoff between fast execution, reducing a risk related to price changes and slow execution, allowing to avoid high trading costs. Since the seminal paper by Almgren and Chriss [2], various extensions of optimal liquidation problems have been studied, see for example [1, 3, 8]. The common framework to address this issue introduced in [2] assumes the following:

• the efficient price process follows an arithmetic Brownian motion (ABM),
• permanent market impact is linear,
• transaction costs are a linear function of the trading rate.

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The execution of a large order is then formulated in discrete time as a tradeoff between expected costs and risk of the trading strategy, with variance as a risk measure. Under this framework, there exists a unique optimal liquidation strategy, which is a deterministic function of time and the initial position of the trader.

Continuous versions of this problem have been considered, notably in [5, 6], where the author shows the ill-posedness of the mean-variance framework leading to time-inconsistent solutions. To overcome this issue, the authors suggest using alternative objective functions, particularly mean-quadratic variation. Under this choice, the authors solve a two-dimensional Hamilton-Jacobi-Bellman equation numerically. Moreover, in [8], the authors consider the optimal execution problem with CARA utility objective function. To the best of our knowledge, there is no closed-form solution to the continuous version of the Almgren-Chriss framework with quadratic variation as a risk measure and geometric Brownian motion (GBM) assumption for the efficient price process. In [7], the authors solve a modified version of the problem with GBM, accounting for the risk with a linear function of the trading rate.

In this paper, we solve the optimal liquidation problem under the Almgren-Chriss framework in continuous time, in the case where the efficient price process follows a GBM, and the risk measure is quadratic variation. Motivated by [4], we assume that trading costs are a quadratic function of the amount of cash, instead of shares. Thus, we reformulate the problem in terms of cash traded and derive in closed-form the optimal control of the trader liquidating his position. As the method is based on the resolution of a system of ODEs, it does not suffer from the curse of dimensionality. In particular, we show how to extend this framework to the case of the liquidation of a portfolio of $N$, possibly correlated, assets. It also enables us to treat the case where the return’s drift is a stochastic process without any BSDE methods with a singular condition.

The paper is organized as follows. In Section 2, we describe the Almgren Chriss framework in continuous time and reformulate the optimization problem in terms of cash. In Section 3, we obtain a closed-form solution of the Almgren Chriss framework with GBM for the efficient price process. Finally in Section 4, we present numerical applications under different market conditions.

2 The model

We define $(\Omega, \mathcal{F}_{t\in[0,T]}, \mathbb{P})$ a filtered probability space, on which all stochastic process are defined, and a trading horizon is $T > 0$.

2.1 Almgren-Chriss framework in continuous time

We rapidly recall the well-known Almgren-Chriss problem in continuous time. We consider the issue of the liquidation of $q_0 \in \mathbb{R}$ shares of a stock whose price at time $t$ is defined by $S_t$. The number of shares hold by the trader is defined by an absolutely continuous measurable process $q_t := q_0 - \int_0^t \hat{q}_s ds$ where $(\hat{q}_s)_{s\in[0,T]}$ is the trading rate, controlled by the trader. The transaction price is

$$\tilde{S}_t := S_t + \frac{\lambda}{2} \hat{q}_t + \gamma(q_t - q_0),$$

where $\lambda$, $\gamma$, and $\gamma$ are parameters.
where \( \lambda, \gamma \in \mathbb{R}^+ \) are constants related respectively to temporary and permanent price impact. Indeed, the term \( \frac{1}{2} \lambda q_t \) is the impact of trading \( q_t \) shares at time \( t \), whereas the term \( \gamma (q_t - q_0) \) is the impact generated by the flow of transactions up to time \( t \). In the original framework in discrete time, see \([2]\), and in most of the extensions in continuous time, see \([8]\) for example, the price process follows an ABM. The number of shares held by the trader satisfies the boundary condition \( q_T = 0 \). Therefore, the cost of this strategy during the trading period is

\[
C(\hat{q}) := \int_0^T \tilde{S}_t \hat{q}_t dt.
\]

Aiming at remedying the time inconsistency of the optimal strategies in the pre-commitment mean-variance framework, inspired by \([6]\), we replace the variance by the quadratic variation in the penalty. The optimal execution problem consists in the optimization of a mean-quadratic variation objective function over the strategies \((\hat{q}_t)_{t \in [0,T]} \in \mathcal{A}\) where

\[
\mathcal{A} := \left\{ (\hat{q}_t)_{t \in [0,T]}, \mathcal{F}_t \text{ - measurable such that } \int_0^T \hat{q}_s ds = q_0 \right\}.
\]

The problem can be written as follows:

\[
\sup_{\hat{q} \in \mathcal{A}} \mathbb{E} \left[ -C(\hat{q}) - \frac{\kappa}{2} \langle C \rangle_T \right],
\]

with \( \kappa > 0 \) and

\[
\langle C \rangle_T := \int_0^T q_t^2 d\langle S \rangle_t.
\]

The use of quadratic variation leads to time-consistent strategies. Moreover, in contrast to the variance, quadratic variation takes into account the trajectory of liquidation. A direct integration by parts on \( C(q) \) gives

\[
C(q) = -q_0 S_0 - \int_0^T q_t dS_t + \frac{\lambda}{2} \int_0^T q_t^2 dt + \frac{\gamma}{2} q_0^2.
\]

Therefore, the problem writes as

\[
\sup_{q \in \mathcal{A}} \mathbb{E} \left[ \int_0^T q_t dS_t - \frac{\lambda}{2} \int_0^T q_t^2 dt - \frac{\kappa}{2} \int_0^T q_t^2 d\langle S \rangle_t \right]. \tag{2.1}
\]

When the price process follows an ABM, Problem (2.1) boils down to a simple calculus of variations problem, which has been solved, for example, in \([8]\). The case where the dynamics are given by a GBM is more intricate. In \([7]\), the authors consider it analytically intractable when a quadratic variation penalty is used. Moreover, in \([6]\), the authors derive a numerical solution of (2.1) by solving the corresponding Hamilton-Jacobi-Bellman equation. Note that strategies under ABM assumption are good proxies of the ones under GBM assumption in period of low volatility.
2.2 Reformulation in terms of cash

We now reformulate the optimal execution problem in terms of cash. We emphasize that we treat the very same problem as in (2.1), except that we modify the transaction costs such that the penalty for \( q_t \) becomes \( q_t S_t \).

We assume that the price process follows a GBM:

\[
\text{d}S_t = \sigma S_t \text{d}W_t.
\]

Multiplying above and below by \( S_t \), we obtain that

\[
\int_0^T q_t \text{d}S_t = \int_0^T \theta_t \text{d}y_t,
\]

where \( \text{d}y_t := \sigma \text{d}W_t \) is the return of the price process, and \( \theta_t := q_t S_t \) is the trader’s position expressed in dollars. Moreover, the quadratic variation penalty has the form

\[
\frac{\kappa \sigma^2}{2} \int_0^T \theta_t^2 \text{d}t.
\]

Applying Ito’s formula, we derive that the cash position \( \theta_t := \theta_t^u \) has the following dynamics:

\[
d\theta_t^u = u_t \text{d}t + \theta_t^u \text{d}y_t = u_t \text{d}t + \sigma \theta_t^u \text{d}W_t,
\]

where \( u_t = \dot{q}_t S_t \) is the trading’s rate in dollar at time \( t \).

Recall that in the classical Almgren-Chriss framework (2.1), trading costs are a quadratic function of the number of shares traded at time \( t \) defined by \( \dot{x}_t \) (the second term in (2.1)). The only modification we make here is to assume that instantaneous costs are a quadratic function of the amount of cash. According to [4], working with dollar holdings and returns is more consistent with common practice.

We define the set of admissible control processes \( (u_t)_{t\in[0,T]} \) as

\[
\mathcal{A} := \left\{ (u_t)_{t\in[0,T]} \text{ measurable, s.t } \int_0^T |u_t| \text{d}t < +\infty, \theta_T^u = 0 \right\}.
\]

where the last condition ensures the complete liquidation of the trader’s position at terminal time \( T \). Following the problem formulation in terms of cash instead of shares, we consider the following mean-quadratic variation optimization problem:

\[
\lim_{a \to +\infty} \sup_{u \in \mathcal{A}} \mathbb{E} \left[ \int_0^T \left( \frac{\lambda}{2} u_t^2 + \frac{\kappa \sigma^2}{2} (\theta_t^u)^2 \right) \text{d}t - \frac{a}{2} (\theta_T^u)^2 \right].
\]

(2.3)

The limit over \( a > 0 \) aims at representing the singular condition \( \theta_T^u = 0 \). Equation (2.3) can be seen as a classical linear-quadratic optimization problem, which is reduced to the resolution of a Riccati equation in dimension one. However such equations are not well suited for multidimensional extensions of this problem, that is to say the liquidation of a portfolio of \( N \) assets. Furthermore when adding a possibly non-Markovian drift \( (\alpha_t)_{t\in[0,T]} \) to the price process, one has to rely on BSDE methods to

\footnote{We write the superscript \( u \) since \( (u_t)_{t\in[0,T]} \) is the control process.}
compute the optimal control.

Our method, developed in the next section, has several advantages. First, it enables us to solve the original Almgren-Chriss problem explicitly, under the GBM assumption, only by assuming that instantaneous costs are a function of the amount of cash. In addition to this, it applies to the case of a stochastic drift \((\alpha_t)_{t \in [0,T]}\) without using the BSDE framework. Finally, an explicit solution can be obtained in the case of the liquidation of a portfolio of \(N\) possibly correlated assets.

We solve in the next section Problem (2.3) under the dynamics (2.2) for the trader's position. We treat the non-zero drift case in Section 5.1.

3 Solving explicitly the Almgren-Chriss problem with GBM

Throughout this section, we work on the following functional space:

\[
H^2 := \left\{ (v_t)_{t \in [0,T]} : \mathbb{E}\left[ \int_0^T v_t^2 dt \right] < +\infty \right\},
\]

with its associated inner product and norm

\[
\langle u, v \rangle_t = \mathbb{E}\left[ \int_0^t u_s v_s ds \right], \quad ||u|| = \mathbb{E}\left[ \int_0^t u_s^2 ds \right].
\]

We also define for all \(t \in [0,T]\) the exponential martingale \(M_t := \exp\left( \sigma W_t - \frac{\sigma^2}{2} t \right)\) and the associated change of measure \(\frac{dQ}{dP}|_{\mathcal{F}_T} = M_T\). We begin with a lemma characterizing the trader's position.

**Lemma 3.1.** The unique solution of (2.2) is given by

\[
\int_0^t M_s M_s^{-1} u_s ds.
\]

For all \(v \in H^2\), we define the operator

\[(Kv)_t := \int_0^t M_s M_s^{-1} v_s ds.\]

The adjoint process \((K^*v)\) is equal for all \(s \in [0,T]\) to

\[(K^*v)_s := \int_s^T \mathbb{E}^Q[v_t | \mathcal{F}_s] dt.\]

The proof is given in Appendix A.1 and relies on a straightforward application of Ito’s formula. Therefore the optimization problem (2.3) can be rewritten, with a fixed \(a > 0\), as

\[
\sup_{u \in A} -\frac{\lambda}{2} ||u||^2 - \frac{k\sigma^2}{2} ||Ku||^2 - \frac{a}{2} (Ku)^2_T.
\] (3.1)
The problem is a supremum over a concave function of \( u \), which is Gateaux-differentiable on \( H^2 \). Thus first order condition gives:

\[
\frac{\kappa \sigma^2}{\lambda} K^* K u + u + \frac{a}{\lambda} (K u)_T = 0,
\]

or equivalently

\[
\frac{\kappa \sigma^2}{\lambda} \int_s^T \int_0^t \mathbb{E}^Q[M_t M_{-r}^{-1} u_r | \mathcal{F}_s] d\tau dt + u_s + \frac{a}{\lambda} M_T \int_0^T M_r^{-1} u_r d\tau = 0.
\]

For all \( (s, s_0) \in [0, T]^2 \) such that \( s \geq s_0 \), we apply \( \mathbb{E}^Q[\cdot | \mathcal{F}_{s_0}] \) on both sides of (3.3). This leads to the following technical lemma.

**Lemma 3.2.** We define \( v(s) := \mathbb{E}^Q[u_s | \mathcal{F}_{s_0}] \) such that \( v(s_0) = u_{s_0} \), and assume that it is differentiable with respect to \( s \).

We also set

\[
z(t) = e^{\sigma^2 (t-s_0)} \theta_{s_0} + \int_{s_0}^t e^{\sigma^2 (t-\tau)} v(\tau) d\tau,
\]

where we recall that \( \theta_{s_0} := (K u)_{s_0} \).

i) Equation (3.2) can be rewritten

\[
v(s) + \frac{\kappa \sigma^2}{\lambda} \int_s^T z(t) dt + \frac{a}{\lambda} z(T) = 0.
\]

ii) The couple \((v, z)\) satisfies the following system of differential equations

\[
\begin{cases}
  v'(s) = \frac{\kappa \sigma^2}{\lambda} z(s) \\
  z'(s) = \sigma^2 z(s) + v(s),
\end{cases}
\]

with boundary conditions

\[
\begin{cases}
  v(T) = -\frac{a}{\lambda} z(T) \\
  z(s_0) = \theta_{s_0}.
\end{cases}
\]

Thus, the control problem (2.3) is reduced to the resolution of a linear system of ODEs with constant coefficients. We can now state our main theorem.

**Theorem 3.3.** Consider the problem (2.3), the optimal control is given explicitly for all time \( t \in [0, T] \) by

\[
u^*_t = \theta^*_t \Gamma(t),
\]

where \( \Gamma(\cdot) \) is a deterministic function of time defined in (A.1) and the optimal trader’s position satisfies

\[
\theta^*_t = \theta^*_0 \exp \left( \int_0^t (\Gamma(s) - \frac{\sigma^2}{2}) ds + \sigma W_t \right).
\]

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2See Appendix A.2 for well-definedness of the first order condition.

3It will be shown ex-post, by a direct verification argument, that \( v(\cdot) \) is differentiable.
The proof is included in the one of Theorem 5.1, where we prove a similar result in a more general framework by allowing a stochastic drift in the dynamics of the price process, and is reported in Appendix A.4. The theorem shows that the optimal control is a linear function of the trader’s position. Therefore, we find an aggressive in-the-money selling strategy, similar to [7], in the sense that the trader liquidates faster when the stock price increases and conversely. This is illustrated in the following section. Moreover, the trader’s position is a geometric Brownian motion, so that it always stays positive, in contrast to [7]. As the function $\Gamma(t) \rightarrow -\infty$ superlinearly, we have $\theta_t^u \rightarrow 0$.

4 Numerical results

We simulate one Brownian motion trajectory, and plot the corresponding stock price process, as well as trading strategy $(u_t^*)_{t \in [0,T]}$ and trader’s cash position $(\theta_t^*)_{t \in [0,T]}$ and in shares $(\theta_t^*/S_t)_{t \in [0,T]}$ for different values of $\sigma$. We take a stock with initial price $S_0 = 100$ following a GBM without drift (whose trajectories for different values of $\sigma$ are in Figure 5), a portfolio of $10^3$ shares to liquidate over $T = 20$ days, with $\lambda = \kappa = 0.2$. In Figure 1, we see an increase of the cash position at the beginning, which can be misleading but is only due to the initial increase of the stock price process. This is also represented in the trading strategy of Figure 2, where we see that the trader liquidates his position faster when the stock process has a higher volatility. Figures 3 and 4 show the position and the trading strategy in terms of shares. We also compare in Figure 6 our trading strategy in shares to the one in [7], which is defined as

$$q_t^* := \left(\frac{T-t}{T}\right) \left(q_0 - \kappa T \int_0^t S_u \, du\right), \quad (4.1)$$

where $q_0 = \frac{\theta_0}{S_0}$ is the initial number of shares hold by the trader, and $dS_t = \sigma S_t \, dW_t$. The trader still liquidates faster with a high volatility but his trading strategy, in this rather extreme regime, can go negative.

Figure 1: Evolution of the cash position with respect to time.

Figure 2: Trading strategy in cash with respect to time.
We now fix $\sigma = 0.1$ and $\kappa = 0.2$. The various cases of the impact of the transaction costs $\lambda$ on the trader’s behavior are represented in Figures 7, 8, 9 and 10. Obviously, the price process is insensitive to a variation of $\lambda$. Moreover, the trading strategies in Figures 8 and 10 are decreasing functions of $\lambda$ meaning that the trader liquidates his position using smaller sell orders when transactions costs are higher. This is also shown in the trader’s position in Figures 7 and 9.
Finally, we set $\sigma = 0.1, \lambda = 0.2$ and study the influence of the risk aversion parameter $\kappa$. In Figures 12 and 14, we see that a highly risk averse trader will liquidate faster than a low risk averse trader. This is shown in terms of his position in Figures 11 and 13.
We now show how to extend our framework to the case of a stochastic drift for the price process and the liquidation of a portfolio of $N$ assets.

5 Extensions of the model

5.1 Stochastic drift

We now consider the case of a stochastic drift, that is we solve

$$
\lim_{a \to +\infty} \sup_{u \in A} \mathbb{E} \left[ \int_0^T \alpha_t \theta_t - \left( \frac{\lambda}{2} u_t^2 + \frac{\kappa \sigma^2}{2} (\theta_t^u)^2 \right) dt - \frac{a}{2} (\theta_T^u)^2 \right],
$$

where $(\alpha_t)_{t \in [0,T]}$ is a stochastic drift of the price process. We consider a slight modification of the problem where we neglect the part $\alpha_t S_t$ of the price’s drift.\footnote{It can be shown that, if there exists $\eta > 0$ such that $\sup_{t} |\alpha_t| < \eta$, then the trading strategy derived in this section is arbitrary closed (as a function of $\eta, T, \sigma$) to the optimal strategy without simplification of the drift.} Therefore, we simplify the dynamics of the price process, and assume

$$
d\theta_t^u = u_t dt + \sigma \theta_t^u dW_t. \tag{5.1}
$$

The first-order condition associated to this optimization problem writes as

$$
\frac{\kappa \sigma^2}{\lambda} \int_s^T \int_0^t \mathbb{E}^Q[M_t M_{\tau}^{-1} u_{\tau} | \mathcal{F}_s] d\tau dt + u_s + \frac{a}{\lambda} M_T \int_0^T M_{\tau}^{-1} u_{\tau} d\tau = \frac{1}{\lambda} \int_s^T \mathbb{E}^Q[\alpha_t | \mathcal{F}_s] dt.
$$

Then, the analogous of Equation (3.2) can be rewritten

$$
v(s) + \frac{\kappa \sigma^2}{\lambda} \int_s^T z(t) dt + \frac{a}{\lambda} z(T) = \frac{1}{\lambda} \int_s^T \mathbb{E}^Q[\alpha_t | \mathcal{F}_{s_0}] dt.
$$

where the couple $(v, z)$ satisfies the following system of differential equations

$$
\begin{cases}
  v'(s) = \frac{\kappa \sigma^2}{\lambda} z(s) - \frac{1}{\lambda} \mathbb{E}^Q[\alpha_s | \mathcal{F}_{s_0}]
  \\
  z'(s) = \sigma^2 z(s) + v(s),
\end{cases}
$$

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{figure13}
\caption{Evolution of the share’s position with respect to time.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{figure14}
\caption{Trading strategy in shares with respect to time.}
\end{figure}
with conditions
\[
\begin{aligned}
  v(T) &= -\frac{a}{N} z(T) \\
  z(s_0) &= \theta_{s_0}.
\end{aligned}
\]

We finally obtain the following theorem:

**Theorem 5.1.** The optimal control at any time \( t \in [0, T] \) is given by
\[
u_t^* = \theta_t^u \Gamma(t) + \nu(t),
\]
where the optimal trader’s position is defined as
\[
\theta_t^u = H_t \int_0^t H_s^{-1} \nu(s) ds.
\]
with \( dH_t = \Gamma(t) H_t dt + \sigma H_t dW_t \), and \( \Gamma(\cdot), \nu(\cdot) \) are deterministic functions defined in (A.1).

The term \( \nu(\cdot) \) is a linear function of both \( \alpha_t \) and \( \mathbb{E}^Q[\alpha_T | \mathcal{F}_t] \), representing the influence of the drift on the optimal strategy. It is an increasing function of the drift \( \alpha_t \) meaning that we aim at liquidating faster our position when the stock price increases. Moreover, it is a decreasing function of \( \mathbb{E}^Q[\alpha_T | \mathcal{F}_t] \): when the expected drift at the terminal time is high, the trader prefers to liquidate slower, waiting for a future stock price increase. As in the zero-drift case, we observe an aggressive in-the-money selling strategy.

### 5.2 Multi-dimensional case

This model extends directly to the problem of optimal execution of a portfolio of \( N \) assets. We define the return of the \( i \)-th asset as
\[
dy_t^i = \sigma_t dW_t^i,
\]
where \((W^1, \ldots, W^N)\) are Brownian motions with non-singular covariance matrix \( \Sigma = (\sigma_i \sigma_j \rho_{i,j})_{1 \leq i,j \leq N} \), \( \sigma_i > 0 \) is the volatility of the \( i \)-th asset and \( \rho_{i,j} \) is the correlation between the \( i \)-th and the \( j \)-th Brownian motion. The cash position of the trader with respect to the \( i \)-th asset is defined by
\[
d\theta_t^{u,i} = u_t^i dt + \theta_t^{u,i} dy_t^i = u_t^i dt + \sigma_t \theta_t^{u,i} dW_t^i,
\]
where \( (u_t^i)_{t \in [0,T]} \) is the trading rate on the \( i \)-th asset. Therefore the optimization problem (2.3) rewrites as
\[
\lim_{a \to +\infty} \sup_{u \in \mathcal{A}} \mathbb{E} \left[ \int_0^T \sum_{i=1}^N -\frac{\lambda}{2} (u_t^i)^2 - \frac{\kappa}{2} \left( \sum_{i=1}^N \sigma_i^2 (\theta_t^{u,i})^2 dt + \sum_{i,j=1}^N \rho_{i,j}^2 \sigma_i \sigma_j \theta_t^{u,i} \theta_t^{u,j} dt \right) - \frac{a}{2} \sum_{i=1}^N (\theta_t^{u,i})^2 \right],
\]
where
\[
\mathcal{A} := \left\{ (u_t^i)_{t \in [0,T], i \in \{1, \ldots, N\}} \text{ measurable, s.t for all } i \in \{1, \ldots, N\} \int_0^T |u_t^i| dt < +\infty, \theta_T^{u,i} = 0 \right\}.
\]
We define \((K^i u_t^i)_{t \in [0,T], i \in \{1, \ldots, N\}}\) as the solution of the SDE (5.2):

\[
(K^i u_t^i) = M_t^i \int_0^t (M_s^i)^{-1} u_t^i ds,
\]

where \(\frac{dQ^i}{dt} \big|_{\mathcal{F}_t} = M_T^i := \exp \left( \sigma_i W_T^i - \frac{(\sigma_i)^2 T}{2} \right)\). The adjoint operator is defined as

\[
(K^{*i} u_t^i) = \int_s^T \mathbb{E}^Q^i [u_t^i | \mathcal{F}_s] dt.
\]

For a fixed \(a > 0\), the optimization problem rewrites

\[
\sup_{u \in A} \frac{\lambda}{2} \|u\|^2 - \frac{K}{2} \langle Ku, \Sigma Ku \rangle - \frac{a}{2} \sum_{i=1}^N (K^i u_t^i)^2 T.
\]

The first order condition gives the following system

\[
u_t^i + \frac{K}{2\lambda} \left( 2\sigma_t^2 K^{*i} K^i u_t^i + \sum_{j \neq i} \rho_t^{i,j} \sigma_t^{i,j} K^{*i} K^j u_t^j \right) + \frac{a}{\lambda} (K^i u_t^i)_T = 0, \quad i = 1, \ldots, N, \quad (5.3)
\]

or equivalently for all \(i = 1, \ldots, N,\)

\[
u_t^i + \frac{K}{2\lambda} \left( 2\sigma_t^2 \int_s^T \left( \mathbb{E}^Q^i [M_t^i (M_s^i)^{-1} u_t^i | \mathcal{F}_s] d\tau dt + \sum_{j \neq i} \rho_t^{i,j} \sigma_t^{i,j} \mathbb{E}^Q^i [M_t^i (M_s^i)^{-1} u_t^i | \mathcal{F}_s] d\tau dt \right) \right) + \frac{a}{\lambda} (K^i u_t^i)_T = 0.
\]

For any \(s \geq s_0,\) apply \(\mathbb{E}^Q^i \left[ \cdot | \mathcal{F}_{s_0} \right]\) on both sides of the equations. Simple but tedious computations lead to

\[
\mathbb{E}^Q^i \left[ u_t^i | \mathcal{F}_{s_0} \right] + \frac{K \sigma_t^2}{\lambda} \left( \int_s^T e^{\sigma_t^2 (t-s_0)} (K^i u_t^i)_{s_0} dt + \int_s^T \int_s^t e^{\sigma_t^2 (t-s_0)} \mathbb{E}^Q^i [u_t^i | \mathcal{F}_{s_0}] d\tau dt \right) + \frac{K}{\lambda} \sum_{j \neq i} \rho_t^{i,j} \sigma_t^{i,j} \left( \int_s^T e^{\sigma_t^2 (t-s_0)} (K^j u_t^j)_{s_0} dt + \int_s^T \int_s^t e^{\sigma_t^2 (t-s_0)} \mathbb{E}^Q^j [u_t^j | \mathcal{F}_{s_0}] d\tau dt \right) + \frac{a}{\lambda} (K^i u_t^i)_{s_0} = 0.
\]

By denoting for all \(i = 1, \ldots, N,\) \(v_t^i (s) = \mathbb{E}^Q^i \left[ u_t^i | \mathcal{F}_{s_0} \right],\) and \(\theta_t^i = (K^i u_t^i)_{s_0}\) the system becomes

\[
v_t^i (s) + \frac{K \sigma_t^2}{\lambda} \left( \int_s^T e^{\sigma_t^2 (t-s_0)} \theta_t^i d\tau dt + \int_s^T \int_s^t e^{\sigma_t^2 (t-s_0)} v_t^i (\tau) d\tau d\tau \right) + \frac{K}{\lambda} \sum_{j \neq i} \rho_t^{i,j} \sigma_t^{i,j} \left( \int_s^T e^{\sigma_t^2 (t-s_0)} \theta_t^j d\tau + \int_s^T \int_s^t e^{\sigma_t^2 (t-s_0)} v_t^j (\tau) d\tau dt \right) + \frac{a}{\lambda} (K^i u_t^i)_{s_0} \left( \int_s^T e^{\sigma_t^2 (t-s_0)} \theta_t^i d\tau + \int_s^T \int_s^t e^{\sigma_t^2 (t-s_0)} v_t^i (\tau) d\tau d\tau \right) = 0.
\]

We define

\[
z_t^i := e^{\sigma_t^2 (t-s_0)} \theta_t^i + \int_s^T e^{\sigma_t^2 (t-s_0)} v_t^i (\tau) d\tau,
\]

\[
z_t^{i,j} := e^{\sigma_t^2 (t-s_0)} \theta_t^j + \int_s^T e^{\sigma_t^2 (t-s_0)} v_t^j (\tau) d\tau,
\]

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and obtain for all $i = 1, \ldots, N$:

\[
v^i(s) + \frac{\kappa \sigma_i^2}{\lambda} \left( \int_s^T z^i(t) dt \right) + \frac{\kappa}{\lambda} \sum_{j \neq i}^N \rho^{ij} \sigma_i \sigma_j \int_s^T z^{ij}(t) dt + \frac{a}{\lambda} z^i(T) = 0.
\]

Therefore first-order condition (5.3) is equivalent to the system of differential equations

\[
\begin{align*}
v^i(s) - \frac{\kappa \sigma_i^2}{\lambda} z^i(s) - \frac{\kappa}{\lambda} \sum_{j \neq i}^N \rho^{ij} \sigma_i \sigma_j z^{ij}(s) &= 0, \\
z^i(s) &= \sigma_i^2 z^i(s) + v^i(s), \\
z^{ij}(s) &= \sigma_i \sigma_j \rho^{ij} z^{ij}(s) + v^j(s),
\end{align*}
\]

with initial conditions

\[
\begin{align*}
v^i(T) &= -\frac{a}{\lambda} z^i(T) \\
z^i(s_0) &= \theta_{s_0}^i, \\
z^{ij}(s_0) &= \theta_{s_0}^{ij}.
\end{align*}
\]

We obtain a system of linear differential equations with constant coefficients. Thus, by noting that for all $i = 1, \ldots, N$ and $s_0 \in [0, T]$, $v^i(s_0) = u^i_{s_0}$, we obtain the controls $u^i_t$ for all $t \in [0, T]$ and $i = 1, \ldots, N$ by solving this system of ODEs.

### 6 Conclusion

In this article, we present a way to solve the traditional Almgren-Chriss liquidation problem when the underlying asset is driven by a GBM. By working in terms of cash and using functional analysis tools, we can provide the optimal control of the problem explicitly. We provide an extension to the case of a GBM with stochastic drift and the liquidation of a portfolio of correlated assets. In particular, our method does not suffer from the curse of dimensionality.

### A Appendix

#### A.1 Proof of Lemma 3.1

An application of Ito’s formula gives

\[
d(Ku)_t = u_t dt + \sigma(Ku)_t dW_t,
\]

hence solving (2.2). The adjoint of $K$ is the operator $K^*$ such that for all $(u, v) \in \mathcal{A}$,

\[
\langle Ku, v \rangle = \langle u, K^*v \rangle.
\]
Using Bayes formula, we have
\[
\langle Ku, v \rangle = \mathbb{E} \left[ \int_0^T (Ku)_t v_t dt \right] 
= \mathbb{E} \left[ \int_0^T \int_0^t M_t M^{-1}_s u_s v_t ds dt \right] 
= \mathbb{E} \left[ \int_0^T \int_s^T M_t M^{-1}_s u_s v_t ds ds \right] 
= \mathbb{E} \left[ \int_0^T u_s (K^* v)_s ds \right] 
= \langle u, K^* v \rangle,
\]
where \((K^* v)_s = \int_s^T M_t M^{-1}_s v_t dt = \int_s^T \mathbb{E}[v_t | \mathcal{F}_s] dt\).

### A.2 Gateaux differentiability in (3.1)

We define the map \(\Xi : \mathbb{H}^2 \to \mathbb{R}\) by
\[
\Xi(u) = -\frac{\lambda}{2} \|u\|^2 - \frac{\kappa \sigma^2}{2} \|Ku\|^2 - \frac{a}{2} (Ku)^2.
\]

As \(K\) is a linear operator of \(u \in \mathbb{H}^2\), and \(\lambda, \kappa > 0\), we deduce that \(\Xi\) is continuous, strictly concave and Gateaux differentiable; with Gateaux derivative given, for any \(h \in \mathbb{H}^2\), by
\[
\Xi(u)[h] = -\lambda \langle u, h \rangle - \kappa \sigma^2 \langle Ku, Kh \rangle - a(Kh).
\]

By setting \(\Xi(u)[h] = 0\), and using the definition of an adjoint operator, we obtain Equation (3.2).

### A.3 Proof of Lemma 3.2

By the fact that
\[
\mathbb{E}^Q \left[ \int_s^T \int_0^t \mathbb{E}^Q[M_t M^{-1}_t u_{\tau} | \mathcal{F}_{s_0}] d\tau d\tau | \mathcal{F}_{s_0} \right] = \int_s^T \int_0^t \mathbb{E}^Q[M_t M^{-1}_t u_{\tau} | \mathcal{F}_{s_0}] d\tau d\tau 
= \int_s^T \left( \int_0^{s_0} M_{s_0}^{-1} u_{\tau} \mathbb{E}^Q[M_t | \mathcal{F}_{s_0}] d\tau + \int_{s_0}^t \mathbb{E}^Q[M_t M^{-1}_t u_{\tau} | \mathcal{F}_{s_0}] d\tau \right) d\tau 
= \int_s^T \left( e^{\sigma^2(t-s_0)} (Ku)_{s_0} dt + \int_{s_0}^t \mathbb{E}^Q[M_t M^{-1}_t u_{\tau} | \mathcal{F}_{s_0}] d\tau \right) dt 
= \int_s^T \left( e^{\sigma^2(t-s_0)} (Ku)_{s_0} dt + \int_{s_0}^t e^{\sigma^2(\tau-s_0)} \mathbb{E}^Q[u_{\tau} | \mathcal{F}_{s_0}] d\tau \right) dt.
\]

Condition (3.3) can be rewritten
\[
\frac{\kappa \sigma^2}{\lambda} \int_s^T e^{\sigma^2(t-s_0)} (Ku)_{s_0} dt + \frac{\kappa \sigma^2}{\lambda} \int_s^T \int_{s_0}^t e^{\sigma^2(\tau-s_0)} \mathbb{E}^Q[u_{\tau} | \mathcal{F}_{s_0}] d\tau d\tau + \frac{a}{\lambda} e^{\sigma^2(T-s_0)} \theta_{s_0} 
+ \frac{a}{\lambda} \int_{s_0}^T e^{\sigma^2(T-\tau)} \mathbb{E}^Q[u_{\tau} | \mathcal{F}_{s_0}] + \mathbb{E}^Q[u_{s} | \mathcal{F}_{s_0}] = 0,
\]
which proves the first statement of the theorem. We obtain the second point by a straightforward derivation of the functions \(z\) and \(v\).
A.4 Proof of Theorem 5.1

The solution of the ODE for $z$ is given by

$$z(s) = \theta_{s_0} e^{\sigma^2(s-s_0)} + \int_{s_0}^{s} e^{\sigma^2(s-u)} v(u) du,$$

and we can rewrite

$$v'(s) = \frac{k\sigma^2}{\lambda} (\theta_{s_0} e^{\sigma^2(s-s_0)} + \int_{s_0}^{s} e^{\sigma^2(s-u)} v(u) du) - \frac{1}{\lambda} \mathbb{E}^Q[\alpha_s | \mathcal{F}_{s_0}].$$

Multiplying by $e^{-\sigma^2 s}$ on both sides we have

$$e^{-\sigma^2 s} v'(s) = \frac{k\sigma^2}{\lambda} (e^{-\sigma^2 s_0} \theta_{s_0} + \int_{s_0}^{s} e^{-\sigma^2 u} v(u) du) - \frac{e^{-\sigma^2 s}}{\lambda} \mathbb{E}^Q[\alpha_s | \mathcal{F}_{s_0}],$$

and defining $w(s) = e^{-\sigma^2 s} v(s)$ we obtain

$$w'(s) = \frac{k\sigma^2}{\lambda} (e^{-\sigma^2 s_0} \theta_{s_0} + \int_{s_0}^{s} w(u) du) - \sigma^2 w(s) - \frac{e^{-\sigma^2 s}}{\lambda} \mathbb{E}^Q[\alpha_s | \mathcal{F}_{s_0}].$$

We note $y(s) = \int_{s_0}^{s} w(u) du$, satisfying the following differential equation

$$y''(s) = \frac{k\sigma^2}{\lambda} y(s) - \sigma^2 y'(s) + \frac{k\sigma^2}{\lambda} (e^{-\sigma^2 s_0} \theta_{s_0} - \frac{e^{-\sigma^2 s}}{\lambda} \mathbb{E}^Q[\alpha_s | \mathcal{F}_{s_0}]).$$

Solving this ODE without second member, we have

$$y(s) = C_1 e^{\gamma_1 s} + C_2 e^{\gamma_2 s},$$

where $C_1, C_2 \in \mathbb{R}$, $\gamma_1 = -\frac{\sigma^2 - \sqrt{\Delta}}{2}, \gamma_2 = -\frac{\sigma^2 + \sqrt{\Delta}}{2}, \Delta = \sigma^2(\sigma^2 + 4\frac{\gamma_1}{\lambda}) > 0$. A particular solution is given by the function $y(s) = -\theta_{s_0} e^{-\sigma^2 s_0} + \frac{e^{-\sigma^2 s}}{k\sigma^2} \mathbb{E}^Q[\alpha_s | \mathcal{F}_{s_0}]$. The general solution is therefore given by:

$$y(s) = C_1 e^{\gamma_1 s} + C_2 e^{\gamma_2 s} - \theta_{s_0} e^{-\sigma^2 s_0} + \frac{e^{-\sigma^2 s}}{k\sigma^2} \mathbb{E}^Q[\alpha_s | \mathcal{F}_{s_0}].$$

To find $C_1, C_2$ we use the fact that $y(s_0) = 0$ and $y'(T) = w(T) = e^{-\sigma^2 T} v(T) = -\frac{e^{-\sigma^2 T}}{\lambda} a(T)$. Substituting the previous expression of $y$, and making $a \to +\infty$ to ensure liquidation at terminal time, we obtain

$$C_1^\infty(s_0) = \beta^\infty(s_0) \left( e^{\gamma_1 T - \sigma^2 s_0} (\theta_{s_0} - \frac{\alpha_{s_0}}{k\sigma^2}) + e^{\gamma_2 s_0 - \sigma^2 T} \frac{\mathbb{E}^Q[\alpha_T | \mathcal{F}_{s_0}]}{k\sigma^2} \right),$$

$$C_2^\infty(s_0) = \beta^\infty(s_0) \left( - e^{\gamma_1 T - \sigma^2 s_0} (\theta_{s_0} - \frac{\alpha_{s_0}}{k\sigma^2}) - e^{\gamma_2 s_0 - \sigma^2 T} \frac{\mathbb{E}^Q[\alpha_T | \mathcal{F}_{s_0}]}{k\sigma^2} \right),$$

where $\beta^\infty(s_0) = \frac{1}{e^{\gamma_1 s_0 + \frac{\alpha_{s_0}}{k\sigma^2}} - e^{\gamma_1 T + \gamma_2 s_0}} > 0$. Note that $v(s_0) = u_{s_0} = e^{\sigma^2 s_0} y'(s_0)$, which gives

$$u_{s_0} = \theta_{s_0} \Gamma(s_0) + v(s_0),$$
where
\[
\begin{align*}
\Gamma(s_0) &= \beta^\infty(s_0)(e^{\gamma_1 s_0 + \gamma_2 T\gamma_1} - e^{\gamma_1 T + \gamma_2 s_0 \gamma_2}), \\
\nu(s_0) &= \beta^\infty(s_0) \left( \gamma_1 e^{(\gamma_1 + \sigma^2)s_0} \left(-\frac{\alpha_{s_0}}{\kappa \sigma^2} e^{\gamma_2 T - \sigma^2 s_0} + e^{\gamma_2 s_0 - \sigma^2 T} \frac{\mathbb{E}^Q[\alpha_T | F_{s_0}]}{\kappa \sigma^2} \right) + \gamma_2 e^{(\gamma_2 + \sigma^2)s_0} \left(\frac{\alpha_{s_0}}{\kappa \sigma^2} e^{\gamma_1 T - \sigma^2 s_0} - e^{\gamma_1 s_0 - \sigma^2 T} \frac{\mathbb{E}^Q[\alpha_T | F_{s_0}]}{\kappa \sigma^2}\right) \right) - \frac{\alpha_{s_0}}{\kappa}.
\end{align*}
\]

Substituting this expression in (5.1), the trader’s position becomes
\[
d\theta_u^t = (\nu(t) + \Gamma(t)\theta_u^t)dt + \sigma\theta_u^t dW_t.
\]
Therefore, we have the optimal position defined by
\[
\theta_u^{\ast t} = H_t \int_0^t H_s^{-1} \nu(s)ds,
\]
where \(dH_t = \Gamma(t)H_t dt + \sigma H_t dW_t\). The optimal control is finally given explicitly at any time \(t \in [0, T]\) by
\[
u^\ast_t = \theta_u^{\ast t}\Gamma(t) + \nu(t).
\]

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