STABILITY OF A FUNCTIONAL EQUATION DERIVING FROM QUARTIC AND ADDITIVE FUNCTIONS

MADJID ESHAGHI GORDJI

Abstract. In this paper, we obtain the general solution and the generalized Hyers-Ulam Rassias stability of the functional equation
\[ f(2x + y) + f(2x - y) = 4(f(x + y) + f(x - y)) - \frac{3}{2}(f(2y) - 2f(y)) + 2f(2x) - 8f(x). \]

1. Introduction

The stability problem of functional equations originated from a question of Ulam [28] in 1940, concerning the stability of group homomorphisms. Let \((G_1, \cdot)\) be a group and let \((G_2, \ast)\) be a metric group with the metric \(d(\cdot, \cdot)\). Given \(\epsilon > 0\), does there exist a \(\delta > 0\), such that if a mapping \(h : G_1 \to G_2\) satisfies the inequality \(d(h(x \cdot y), h(x) \ast h(y)) < \delta\) for all \(x, y \in G_1\), then there exists a homomorphism \(H : G_1 \to G_2\) with \(d(h(x), H(x)) < \epsilon\) for all \(x \in G_1\)? In other words, under what condition does there exist a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, D. H. Hyers [13] gave a first affirmative answer to the question of Ulam for Banach spaces. Let \(f : E \to E'\) be a mapping between Banach spaces such that
\[ \|f(x + y) - f(x) - f(y)\| \leq \delta \]
for all \(x, y \in E\), and for some \(\delta > 0\). Then there exists a unique additive mapping \(T : E \to E'\) such that
\[ \|f(x) - T(x)\| \leq \delta \]
for all \(x \in E\). Moreover if \(f(tx)\) is continuous in \(t\) for each fixed \(x \in E\), then \(T\) is linear. Finally in 1978, Th. M. Rassias [25] proved the following theorem.
Theorem 1.1. Let $f : E \to E'$ be a mapping from a norm vector space $E$ into a Banach space $E'$ subject to the inequality
\begin{equation}
\|f(x + y) - f(x) - f(y)\| \leq \epsilon (\|x\|^p + \|y\|^p)
\end{equation}
for all $x, y \in E$, where $\epsilon$ and $p$ are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E \to E'$ such that
\begin{equation}
\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2p} \|x\|^p
\end{equation}
for all $x \in E$. If $p < 0$, then inequality (1.1) holds for all $x, y \neq 0$, and (1.2) for $x \neq 0$. Also, if the function $t \mapsto f(tx)$ from $\mathbb{R}$ into $E'$ is continuous for each fixed $x \in E$, then $T$ is linear.

In 1991, Z. Gajda [9] answered the question for the case $p > 1$, which was raised by Rassias. This new concept is known as Hyers-Ulam-Rassias stability of functional equations (see [1, 3], [5-15], [22-24]).

In [19], W.-G. Park and J. H. Bae, considered the following functional equation:
\begin{equation}
f(2x + y) + f(2x - y) = 4(f(x + y) + f(x - y)) + 24f(x) - 6f(y).
\end{equation}
In fact they proved that a function $f$ between real vector spaces $X$ and $Y$ is a solution of (1.3) if and only if there exists a unique symmetric multi-additive function $B : X \times X \times X \times X \to Y$ such that $f(x) = B(x, x, x, x)$ for all $x$ (see [2, 4], [16-21], [26, 27]). It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (1.3), which is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic function.

We deal with the next functional equation deriving from quartic and additive functions:
\begin{equation}
f(2x + y) + f(2x - y) = 4(f(x + y) + f(x - y)) - \frac{3}{2}(f(2y) - 2f(y)) + 2f(2x) - 8f(x).
\end{equation}
It is easy to see that the function $f(x) = ax^4 + bx$ is a solution of the functional equation (1.4). In the present paper we investigate the general solution and the generalized Hyers-Ulam-Rassias stability of the functional equation (1.4).

2. General solution

In this section we establish the general solution of functional equation (1.4).

Theorem 2.1. Let $X,Y$ be vector spaces, and let $f : X \to Y$ be a function satisfies (1.4). Then the following assertions hold.

a) If $f$ is even function, then $f$ is quartic.

b) If $f$ is odd function, then $f$ is additive.

Proof. a) Putting $x = y = 0$ in (1.4), we get $f(0) = 0$. Setting $x = 0$ in (1.4), by evenness of $f$, we obtain
\begin{equation}
f(2y) = 16f(y)
\end{equation}
for all \( y \in X \). Hence (1.4) can be written as

\[
(2.2) \quad f(2x + y) + f(2x - y) = 4(f(x + y) + f(x - y)) + 24f(x) - 6f(y)
\]

for all \( x, y \in X \). This means that \( f \) is a quartic function.

b) Setting \( x = y = 0 \) in (1.4) to obtain \( f(0) = 0 \). Putting \( x = 0 \) in (1.4), then by oddness of \( f \), we have

\[
(2.3) \quad f(2y) = 2f(y)
\]

for all \( y \in X \). We obtain from (1.4) and (2.3) that

\[
(2.4) \quad f(2x + y) + f(2x - y) = 4(f(x + y) + f(x - y)) - 4f(x)
\]

for all \( x, y \in X \). Replacing \( y \) by \(-2y\) in (2.4), it follows that

\[
(2.5) \quad f(2x - 2y) + f(2x + 2y) = 4(f(x - 2y) + f(x + 2y)) - 4f(x).
\]

Combining (2.3) and (2.5) to obtain

\[
(2.6) \quad f(x - y) + f(x + y) = 2(f(x - 2y) + f(x + 2y)) - 2f(x).
\]

Interchange \( x \) and \( y \) in (2.6) to get the relation

\[
(2.7) \quad f(x + y) + f(x - y) = 2(f(y - 2x) + f(y + 2x)) - 2f(y).
\]

Replacing \( y \) by \(-y\) in (2.7), and using the oddness of \( f \) to get

\[
(2.8) \quad f(x - y) - f(x + y) = 2(f(2x - y) - f(2x + y)) + 2f(y).
\]

From (2.4) and (2.8), we obtain

\[
(2.9) \quad 4f(2x + y) = 9f(x + y) + 7f(x - y) - 8f(x) + 2f(y).
\]

Replacing \( x + y \) by \( y \) in (2.9) it follows that

\[
(2.10) \quad 7f(2x - y) = 4f(x + y) + 2f(x - y) - 9f(y) + 8f(x).
\]

By using (2.9) and (2.10), we lead to

\[
(2.11) \quad f(2x + y) + f(2x - y) = \frac{79}{28}f(x + y) + \frac{57}{28}f(x - y) - \frac{6}{7}f(x) - \frac{11}{14}f(y).
\]

We get from (2.4) and (2.11) that

\[
(2.12) \quad 3f(x + y) + 5f(x - y) = 8f(x) - 28f(y).
\]

Replacing \( x \) by \( 2x \) in (2.4) it follows that

\[
(2.13) \quad f(4x + y) + f(4x - y) = 16(f(x + y) + f(x - y)) - 24f(x).
\]

Setting \( 2x + y \) instead of \( y \) in (2.4), we arrive at

\[
(2.14) \quad f(4x + y) - f(y) = 4(f(3x + y) + f(x - y)) - 4f(x).
\]

Replacing \( y \) by \(-y\) in (2.14), and using oddness of \( f \) to get

\[
(2.15) \quad f(4x - y) + f(y) = 4(f(3x + y) + f(x + y)) - 4f(x).
\]

Adding (2.14) to (2.15) to get the relation

\[
(2.16) \quad f(4x + y) + f(4x - y) = 4(f(3x + y) + f(3x - y)) - 4(f(x + y) + f(x - y)) - 8f(x).
\]
Replacing $y$ by $x + y$ in (2.4) to obtain
\begin{equation}
(2.17) \quad f(3x + y) + f(x - y) = 4(f(2x + y) - f(y)) - 4f(x).
\end{equation}
Replacing $y$ by $-y$ in (2.17), and using the oddness of $f$, we lead to
\begin{equation}
(2.18) \quad f(3x - y) + f(x + y) = 4(f(2x - y) + f(y)) - 4f(x).
\end{equation}
Combining (2.17) and (2.18) to obtain
\begin{equation}
(2.19) \quad f(3x + y) + f(3x - y) = 15(f(x + y) + f(x - y)) - 24f(x).
\end{equation}
Using (2.16) and (2.19) to get
\begin{equation}
(2.20) \quad f(4x + y) + f(4x - y) = 56(f(x + y) + f(x - y)) - 104f(x).
\end{equation}
Combining (2.13) and (2.20), we arrive at
\begin{equation}
(2.21) \quad f(x + y) + f(x - y) = 2f(x).
\end{equation}
Hence by using (2.12) and (2.21) it is easy to see that $f$ is additive. This completed the proof of theorem. $$\square$$

**Theorem 2.2.** Let $X, Y$ be vector spaces, and let $f : X \to Y$ be a function. Then $f$ satisfies (1.4) if and only if there exist a unique symmetric multi-additive function $B : X \times X \times X \times X \to Y$ and a unique additive function $A : X \to Y$ such that $f(x) = B(x,x,x,x) + A(x)$ for all $x \in X$.

**Proof.** Suppose $f$ satisfies (1.4). We decompose $f$ into the even part and odd part by setting
\begin{align*}
&f_e(x) = \frac{1}{2}(f(x) + f(-x)), \quad f_o(x) = \frac{1}{2}(f(x) - f(-x))
\end{align*}
for all $x \in X$. By (1.4), we have
\begin{align*}
f_e(2x + y) + f_e(2x - y)
&= \frac{1}{2}[f(2x + y) + f(-2x - y) + f(2x - y) + f(-2x + y)] \\
&= \frac{1}{2}[f(2x + y) + f(2x - y)] + \frac{1}{2}[f(-2x + (-y)) + f(-2x - (-y))] \\
&= \frac{1}{2}[4f(x + y) + f(x - y)] - \frac{3}{7}(f(2y) - 2f(y)) + 2f(2x) - 8f(x) \\
&\quad + \frac{1}{2}[4f(-x - y) + f(-x - (-y))] - \frac{3}{7}(f(-2y) - 2f(-y)) + 2f(-2x) - 8f(-x) \\
&= 4\left[\frac{1}{2}(f(x + y) + f(-x - y)) + \frac{1}{2}(f(-x + y) + f(x - y))\right] \\
&\quad - \frac{3}{7}\left[\frac{1}{2}(f(2y) + f(-2y)) - (f(y) - f(-y))\right] \\
&\quad + 2\left[\frac{1}{2}(f(2x) + f(-2x))\right] - 8\left[\frac{1}{2}(f(x) + f(-x))\right] \\
&= 4(f_e(x + y) + f_e(x - y)) - \frac{3}{7}(f_e(2y) - 2f_e(y)) + 2f_e(2x) - 8f_e(x)
\end{align*}
for all \( x, y \in X \). This means that \( f_e \) holds in (1.4). Similarly we can show that \( f_o \) satisfies (1.4). By above theorem, \( f_e \) and \( f_o \) are quartic and additive respectively. Thus there exists a unique symmetric multi-additive function \( B : X \times X \times X \times X \to Y \) such that \( f_e(x) = B(x, x, x, x) \) for all \( x \in X \). Put \( A(x) := f_o(x) \) for all \( x \in X \). It follows that \( f(x) = B(x) + A(x) \) for all \( x \in X \). The proof of the converse is trivially. \( \square \)

3. Stability

Throughout this section, \( X \) and \( Y \) will be a real normed space and a real Banach space, respectively. Let \( f : X \to Y \) be a function then we define \( D_f : X \times X \to Y \) by

\[
D_f(x, y) = 7[f(2x + y) + f(2x - y)] - 28[f(x + y) + f(x - y)] \\
+ 3[f(2y) - 2f(y)] - 14[f(2x) - 4f(x)]
\]

for all \( x, y \in X \).

**Theorem 3.1.** Let \( \psi : X \times X \to [0, \infty) \) be a function satisfies \( \sum_{i=0}^{\infty} \frac{\psi(0, 2^i x)}{16^i} < \infty \) for all \( x \in X \), and \( \lim_{n \to \infty} \frac{\psi(2^n x, 2^n y)}{16^n} = 0 \) for all \( x, y \in X \). If \( f : X \to Y \) is an even function such that \( f(0) = 0 \), and that

\[ ||D_f(x, y)|| \leq \psi(x, y) \]  

for all \( x, y \in X \), then there exists a unique quartic function \( Q : X \to Y \) satisfying (1.4) and

\[ ||f(x) - Q(x)|| \leq \frac{1}{48} \sum_{i=0}^{\infty} \frac{\psi(0, 2^i x)}{16^i} \]

for all \( x \in X \).

**Proof.** Putting \( x = 0 \) in (3.1), then we have

\[ ||3f(2y) - 48f(y)|| \leq \psi(0, y). \]

Replacing \( y \) by \( x \) in (3.3) and then dividing by 48 to obtain

\[ ||\frac{f(2x)}{16} - f(x)|| \leq \frac{1}{48} \psi(0, x) \]

for all \( x \in X \). Replacing \( x \) by \( 2x \) in (3.4) to get

\[ ||\frac{f(4x)}{16} - f(2x)|| \leq \frac{1}{48} \psi(0, 2x). \]

Combine (3.4) and (3.5) by use of the triangle inequality to get

\[ ||\frac{f(4x)}{16^2} - f(x)|| \leq \frac{1}{48} \left( \frac{\psi(0, 2x)}{16} + \psi(0, x) \right). \]
By induction on \( n \in \mathbb{N} \), we can show that

\[
\| \frac{f(2^nx)}{16^n} - f(x) \| \leq \frac{1}{48} \sum_{i=0}^{n-1} \psi(0, 2^i x).
\]

Dividing (3.7) by \( 16^m \) and replacing \( x \) by \( 2^m x \) to get

\[
\left\| \frac{f(2^{m+n}x)}{16^{m+n}} - \frac{f(2^mx)}{16^m} \right\| = \frac{1}{16^m} \| f(2^n 2^m x) - f(2^mx) \|
\]

\[
\leq \frac{1}{48 \times 16^m} \sum_{i=0}^{n-1} \psi(0, 2^i x)\]

\[
\leq \frac{1}{48} \sum_{i=0}^{\infty} \psi(0, 2^i 2^m x) \frac{1}{16^{n+i}}
\]

for all \( x \in X \). This shows that \( \{ \frac{f(2^nx)}{16^n} \} \) is a Cauchy sequence in \( Y \), by taking the \( \lim m \to \infty \). Since \( Y \) is a Banach space, then the sequence \( \{ \frac{f(2^nx)}{16^n} \} \) converges.

We define \( Q : X \to Y \) by \( Q(x) := \lim_{n \to \infty} \frac{f(2^nx)}{16^n} \) for all \( x \in X \). Since \( f \) is even function, then \( Q \) is even. On the other hand we have

\[
\| D_Q(x, y) \| = \lim_{n \to \infty} \frac{1}{16^n} \| D_f(2^n x, 2^ny) \|
\]

\[
\leq \lim_{n \to \infty} \frac{\psi(2^n x, 2^ny)}{16^n} = 0
\]

for all \( x, y \in X \). Hence by Theorem 2.1, \( Q \) is a quartic function. To shows that \( Q \) is unique, suppose that there exists another quartic function \( \hat{Q} : X \to Y \) which satisfies (1.4) and (3.2). We have \( Q(2^n x) = 16^n Q(x) \) and \( \hat{Q}(2^n x) = 16^n \hat{Q}(x) \) for all \( x \in X \). It follows that

\[
\| \hat{Q}(x) - Q(x) \| = \frac{1}{16^n} \| \hat{Q}(2^n x) - Q(2^n x) \|
\]

\[
\leq \frac{1}{16^n} \| \hat{Q}(2^n x) - f(2^nx) \| + \| f(2^n x) - Q(2^n x) \|
\]

\[
\leq \frac{1}{24} \sum_{i=0}^{\infty} \psi(0, 2^n 2^i x) \frac{1}{16^{n+i}}
\]

for all \( x \in X \). By taking \( n \to \infty \) in this inequality we have \( \hat{Q}(x) = Q(x) \). \( \square \)

**Theorem 3.2.** Let \( \psi : X \times X \to [0, \infty) \) be a function satisfies

\[
\sum_{i=0}^{\infty} 16^i \psi(0, 2^{-i-1} x) < \infty
\]

for all \( x \in X \), and \( \lim 16^n \psi(2^{-n} x, 2^{-n} y) = 0 \) for all \( x, y \in X \). Suppose that an even function \( f : X \to Y \) satisfies \( f(0) = 0 \), and (3.1). Then the limit
Q(x) := \lim_n 16^n f(2^{-n}x) exists for all x ∈ X and Q : X → Y is a unique quartic function satisfies (1.4) and

(3.8) \quad \|f(x) - Q(x)\| \leq \frac{1}{3} \sum_{i=0}^{\infty} 16^i \psi(0, 2^{-i-1}x)

for all x ∈ X.

Proof. By putting x = 0 in (3.1), we get

(3.9) \quad \|3f(2y) - 4f(y)\| \leq \psi(0, y).

Replacing y by \frac{1}{2} in (3.9) and result dividing by 3 to get

(3.10) \quad \|16f(2^{-1}x) - f(x)\| \leq \frac{1}{3} \psi(0, 2^{-1}x)

for all x ∈ X. Replacing x by \frac{x}{2} in (3.10) it follows that

(3.11) \quad \|16f(4^{-1}x) - f(2^{-1}x)\| \leq \frac{1}{3} \psi(0, 2^{-2}x).

Combining (3.10) and (3.11) by use of the triangle inequality to obtain

(3.12) \quad \|16^2 f(4^{-1}x) - f(x)\| \leq \frac{1}{3} \left( \frac{\psi(0, 2^{-2}x)}{16} + \psi(0, 2^{-1}x) \right).

By induction on n ∈ N, we have

(3.13) \quad \|16^n f(2^{-n}x) - f(x)\| \leq \frac{1}{3} \sum_{i=0}^{n-1} 16^i \psi(0, 2^{-i-1}x).

Multiplying (3.13) by 16^n and replacing x by 2^{-m}x to obtain

\quad \|16^{m+n} f(2^{-m-n}x) - 16^m f(2^{-m}x)\| = 16^m \|f(2^{-n}2^{-m}x) - f(2^{-m}x)\|

\leq \frac{16^m}{3} \sum_{i=0}^{n-1} 16^i \psi(0, 2^{-i-1}x)

\leq \frac{1}{3} \sum_{i=0}^{\infty} 16^{m+i} \psi(0, 2^{-i-1}2^{-m}x)

for all x ∈ X. By taking the \lim_{m→\infty}, it follows that \{16^n f(2^{-n}x)\} is a Cauchy sequence in Y. Since Y is a Banach space, then the sequence \{16^n f(2^{-n}x)\} converges. Now we define Q : X → Y by

Q(x) := \lim_n 16^n f(2^{-n}x)

for all x ∈ X. The rest of proof is similar to the proof of Theorem 3.1. □

Theorem 3.3. Let \psi : X × X → [0, \infty) be a function such that

(3.14) \quad \sum \frac{\psi(0, 2^i x)}{2^i} < \infty
and
\[ \lim_{n \to \infty} \frac{\psi(2^n x, 2^n y)}{2^n} = 0 \]
for all \( x, y \in X \). If \( f : X \to Y \) is an odd function such that
\[ \|Df(x,y)\| \leq \psi(x,y) \]
for all \( x, y \in X \). Then there exists a unique additive function \( A : X \to Y \) satisfies (1.4) and
\[ \|f(x) - A(x)\| \leq \frac{1}{2} \sum_{i=0}^{\infty} \psi(0,2^i x) \]
for all \( x \in X \).

**Proof.** Setting \( x = 0 \) in (3.16) to get
\[ \|f(2y) - 2f(y)\| \leq \psi(o, y) \]
Replacing \( y \) by \( x \) in (3.17) and result dividing by 2, then we have
\[ \left\| \frac{f(2x)}{2} - f(x) \right\| \leq \frac{1}{2} \psi(0, x) \]
Replacing \( x \) by \( 2x \) in (3.18) to obtain
\[ \left\| \frac{f(4x)}{2} - f(2x) \right\| \leq \frac{1}{2} \psi(0, 2x) \]
Combine (3.18) and (3.19) by use of the triangle inequality to get
\[ \left\| \frac{f(4x)}{4} - f(x) \right\| \leq \frac{1}{2} (\psi(0, x) + \frac{1}{2} \psi(0, 2x)) \]
Now we use iterative methods and induction on \( n \) to prove our next relation.
\[ \left\| \frac{f(2^n x)}{2^n} - f(x) \right\| \leq \frac{1}{2} \sum_{i=0}^{n-1} \psi(0,2^i x) \]
Dividing (3.21) by \( 2^m \) and then substituting \( x \) by \( 2^m x \), we get
\[ \left\| \frac{f(2^{m+n} x)}{2^{m+n}} - \frac{f(2^m x)}{2^m} \right\| = \frac{1}{2^m} \left\| \frac{f(2^{n+1} x)}{2^{n+1}} - f(2^m x) \right\| \]
\[ \leq \frac{1}{2^{m+1}} \sum_{i=0}^{n-1} \psi(0,2^i 2^m x) \]
\[ \leq \frac{1}{2} \sum_{i=0}^{\infty} \psi(0,2^{i+m} x) \]
(3.22)
Taking $m \to \infty$ in (3.22), then the right hand side of the inequality tends to zero. Since $Y$ is a Banach space, then $A(x) = \lim_n \frac{f(2^n x)}{2^n}$ exits for all $x \in X$. The oddness of $f$ implies that $A$ is odd. On the other hand by (3.15) we have

$$D_A(x, y) = \lim_n \frac{1}{2^n} \|D_f(2^n x, 2^n y)\| \leq \lim_n \frac{\psi(2^n x, 2^n y)}{2^n} = 0.$$ 

Hence by Theorem 1.2, $A$ is additive function. The rest of the proof is similar to the proof of Theorem 3.1.

**Theorem 3.4.** Let $\psi : X \times X \to [0, \infty)$ be a function satisfies

$$\sum_{i=0}^{\infty} 2^i \psi(0, 2^{-i-1} x) < \infty$$

for all $x \in X$ and $\lim 2^n \psi(2^{-n} x, 2^{-n} y) = 0$ for all $x, y \in X$. Suppose that an odd function $f : X \to Y$ satisfies (3.1). Then the limit $A(x) := \lim_n 2^n f(2^n x)$ exists for all $x \in X$ and $A : X \to Y$ is a unique additive function satisfying (1.4), and

$$\|f(x) - A(x)\| \leq \sum_{i=0}^{\infty} 2^i \psi(0, 2^{-i-1} x)$$

for all $x \in X$.

**Proof.** It is similar to the proof of Theorem 3.3. □

**Theorem 3.5.** Let $\psi : X \times X \to Y$ be a function such that

$$\sum_{i=0}^{\infty} \frac{\psi(0, 2^i x)}{2^i} \leq \infty \quad \text{and} \quad \lim_n \frac{\psi(2^n x, 2^n y)}{2^n} = 0$$

for all $x \in X$. Suppose that a function $f : X \to Y$ satisfies the inequality

$$\|D_f(x, y)\| \leq \psi(x, y)$$

for all $x, y \in X$, and $f(0) = 0$. Then there exist a unique quartic function $Q : X \to Y$ and a unique additive function $A : X \to Y$ satisfying (1.4) and

$$\|f(x) - Q(x) - A(x)\| \leq \frac{1}{48} \left[ \sum_{i=0}^{\infty} \left( \psi(0, 2^i x) + \psi(0, -2^i x) \right) \right]$$

(3.23)

for all $x, y \in X$.

**Proof.** We have

$$\|D_f(x, y)\| \leq \frac{1}{2} [\psi(x, y) + \psi(-x, -y)]$$
for all \( x, y \in X \). Since \( f_e(0) = 0 \) and \( f_e \) is an even function, then by Theorem 3.1, there exists a unique quartic function \( Q : x \to Y \) satisfying

\[
\| f_e(x) - Q(x) \| \leq \frac{1}{48} \sum_{i=0}^{\infty} \frac{\psi(0, 2^i x) + \psi(0, -2^i x)}{2 \times 16^i}
\]

for all \( x \in X \). On the other hand \( f_0 \) is odd function and

\[
\| D f_0(x, y) \| \leq \frac{1}{2} \left[ \psi(x, y) + \psi(-x, -y) \right]
\]

for all \( x, y \in X \). Then by Theorem 3.3, there exists a unique additive function \( A : X \to Y \) such that

\[
\| f_0(x) - A(x) \| \leq \frac{1}{2} \sum_{i=0}^{\infty} \frac{\psi(0, 2^i x) + \psi(0, -2^i x)}{2 \times 2^i}
\]

for all \( x \in X \). Combining (3.24) and (3.25) to obtain (3.23). This completes the proof of theorem. \( \square \)

By Theorem 3.5, we are going to investigate the Hyers-Ulam-Rassias stability problem for functional equation (1.4).

**Corollary 3.6.** Let \( \theta \geq 0, P < 1 \). Suppose \( f : X \to Y \) satisfies the inequality

\[
\| D f(x, y) \| \leq \theta (\| x \|^p + \| y \|^p)
\]

for all \( x, y \in X \) and \( f(0) = 0 \). Then there exists a unique quartic function \( Q : X \to Y \) and a unique additive function \( A : X \to Y \) satisfying (1.4), and

\[
\| f(x) - Q(x) - A(x) \| \leq \frac{\theta}{48} \| x \|^p \left( \frac{16}{16 - 2^p} + \frac{96}{1 - 2^{p-1}} \right)
\]

for all \( x \in X \).

By Corollary 3.6, we solve the following Hyers-Ulam stability problem for functional equation (1.4).

**Corollary 3.7.** Let \( \epsilon \) be a positive real number, and let \( f : X \to Y \) be a function satisfies

\[
\| D f(x, y) \| \leq \epsilon
\]

for all \( x, y \in X \). Then there exist a unique quartic function \( Q : X \to Y \) and a unique additive function \( A : X \to Y \) satisfying (1.4), and

\[
\| f(x) - Q(x) - A(x) \| \leq \frac{362}{45} \epsilon
\]

for all \( x \in X \).

By applying Theorems 3.2 and 3.4, we have the following theorem.
Theorem 3.8. Let $\psi : X \times X \to Y$ be a function such that
\[ \sum_{i=0}^{\infty} 16^i \psi(0, 2^{-i-1}x) \leq \infty \quad \text{and} \quad \lim_{n \to \infty} 16^n \psi(2^n x, 2^n x) = 0 \]
for all $x \in X$. Suppose that a function $f : X \to Y$ satisfies the inequality
\[ \| Df(x, y) \| \leq \psi(x, y) \]
for all $x, y \in X$ and $f(0) = 0$. Then there exist a unique quartic function $Q : X \to Y$ and a unique additive function $A : X \to Y$ satisfying (1.4), and
\[ \| f(x) - Q(x) - A(x) \| \leq \sum_{i=0}^{\infty} \left( \frac{16^i}{3} + 2^i \right) \left( \frac{\psi(0, 2^{-i-1}x) + \psi(0, -2^{-i-1}x)}{2} \right) \]
for all $x, y \in X$.

Corollary 3.9. Let $\theta \geq 0$, $P > 4$. Suppose $f : X \to Y$ satisfies the inequality
\[ \| Df(x, y) \| \leq \theta (\| x \|^p + \| y \|^p) \]
for all $x, y \in X$, and $f(0) = 0$. Then there exist a unique quartic function $Q : X \to Y$ and a unique additive function $A : X \to Y$ satisfying (1.4), and
\[ \| f(x) - Q(x) - A(x) \| \leq \frac{\theta}{3 \times 2^p} \| x \|^p \left( \frac{1}{1 - 2^{4-p}} + \frac{1}{1 - 2^{1-p}} \right) \]
for all $x \in X$.

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