Abstract

We approximate \(d\)-variate periodic functions in weighted Korobov spaces with general weight parameters using \(n\) function values at lattice points. We do not limit \(n\) to be a prime number, as in currently available literature, but allow any number of points, including powers of 2, thus providing the fundamental theory for construction of embedded lattice sequences. Our results are constructive in that we provide a component-by-component algorithm which constructs a suitable generating vector for a given number of points or even a range of numbers of points. It does so without needing to construct the index set on which the functions will be represented. The resulting generating vector can then be used to approximate functions in the underlying weighted Korobov space. We analyse the approximation error in the worst-case setting under both the \(L_2\) and \(L_\infty\) norms. Our component-by-component construction under the \(L_2\) norm achieves the best possible rate of convergence for lattice-based algorithms, and the theory can be applied to lattice-based kernel methods and splines. Depending on the value of the smoothness parameter \(\alpha\), we propose two variants of the search criterion in the construction under the \(L_\infty\) norm, extending previous results which hold only for product-type weight parameters and prime \(n\). We also provide a theoretical upper bound showing that embedded lattice sequences are essentially as good as lattice rules with a fixed value of \(n\). Under some standard assumptions on the weight parameters, the worst-case error bound is independent of \(d\).

Keywords: Lattice rules, lattice algorithms, embedded lattice sequences, multivariate function approximation, component-by-component construction, composite number of points, non-prime number of points.

MSC Classification: 65D15, 65T40

1 Introduction

In this paper we provide a theoretical foundation for the component-by-component (CBC) construction of lattice algorithms and a practical CBC construction of embedded lattice sequences [4] for multivariate \(L_2\) and \(L_\infty\) approximation in the worst-case setting, for \(d\)-variate functions \(f\) in weighted Korobov spaces with smoothness parameter \(\alpha\) and general weights.
parameters $\gamma := \{\gamma_n\}_{n \in \mathbb{N}}$ (see Section 2 for details). The algorithm is based on function values at $n$ lattice points (see (1.1) below). Our error analysis for the CBC construction allows for any $n \geq 2$. Currently available literature on lattice-based algorithms provide error analysis restricted to prime $n$ and, in the case of $L_\infty$ approximation, further restricted to only “product”-type weight parameters.

Although our work here might look quite theoretical, it is motivated by strong practical needs. Composite values of $n$ enable practical applications of embedded lattice sequences, e.g., with $n$ being successive powers of 2, while non-product weight parameters are crucial for some uncertainty quantification problems involving PDEs with random coefficients (see POD and SPOD weights in [19] and further references below). Furthermore, our analysis also applies to lattice-based kernel methods, see [19].

More precisely, we consider one-periodic real-valued $L_2$ functions defined on $[0,1]^d$ with absolutely converging Fourier series

$$f(x) = \sum_{h \in \mathbb{Z}^d} \hat{f}(h) e^{2\pi i h \cdot x}, \quad \hat{f}(h) := \int_{[0,1]^d} f(x) e^{-2\pi i h \cdot x} \, dx,$$

where $\hat{f}(h)$ are the Fourier coefficients and $h \cdot x = h_1 x_1 + \cdots + h_d x_d$ denotes the usual dot product. The norm of our function space will be defined in terms of the Fourier coefficients (see (2.1) below), and when the smoothness parameter $\alpha$ of the weighted Korobov space is even then this has the interpretation that $f$ has square-integrable mixed partial derivatives of order $\alpha/2$.

The lattice algorithm $A_n(f)$ is defined as follows: we first truncate the Fourier expansion to a finite index set $A_d \subset \mathbb{Z}^d$ and then approximate the remaining Fourier coefficients by $n$ rank-1 lattice points, i.e.,

$$A_n(f)(x) := \sum_{h \in A_d} \hat{f}^n(h) e^{2\pi i h \cdot x}, \quad \hat{f}^n(h) := \frac{1}{n} \sum_{k=1}^{n} f(t_k) e^{-2\pi i h \cdot t_k},$$

(1.1)

where $z \in U_n^d$ is known as the generating vector which determines the quality of the approximation, with components from

$$U_n := \{ z \in \mathbb{Z} : 1 \leq z \leq n - 1 \text{ and } \gcd(z,n) = 1 \},$$

and the braces in (1.1) indicate that we take the fractional part of each component of a vector.

In this paper we follow [5, 6, 30, 34] to define the index set $A_d$ with some parameter $M > 0$ by

$$A_d(M) := \{ h \in \mathbb{Z}^d : r_{d,\alpha}(h) \leq M \},$$

(1.2)

where the quantity $r_{d,\alpha}(h)$ (see (2.2) below) moderates the decay of $|\hat{f}(h)|$, with a smaller value of $r_{d,\alpha}(h)$ indicating that the index $h$ is more significant. The set $A_d(M)$ therefore collects the most significant indices up to a threshold $M$.

Lattice rules were originally designed for multivariate integration, see, e.g., [7, 9, 36, 37, 38, 39, 44, 48], but the benefits of lattice-based algorithms for multivariate function approximation have also been recognized. A key development in recent years is the CBC construction of lattice generating vectors in high dimensions, with guaranteed good theoretical error bounds for integration and approximation in a variety of settings. Specifically for approximation, one approach is based on reconstruction lattices in which the generating vector $z$ is constructed to recover certain Fourier coefficients exactly so that $\hat{f}^n(h) = \hat{f}(h)$ for all $h \in A_d$ in (1.1), where $A_d$ can be any index set not restricted to the form (1.2),
Table 1: Comparison of convergence rates for approximation algorithms in the weighted Korobov space with smoothness parameter $\alpha > 1$; in this paper we extend the lattice algorithm results to composite $n$ and general weights

| $L_2$ | $\frac{\alpha}{2}$ | $\frac{\alpha}{2} + \frac{1}{1 + 1/\alpha}$ | $\frac{\alpha}{4}$ | $\frac{\alpha}{4}$ | $\frac{\alpha}{2} - \frac{1}{1 + 1/\alpha}$ | $\frac{1}{1 + 1/\alpha}$ | $\frac{1}{2} - \frac{1}{1 + 1/\alpha}$ | $\frac{1}{2}$ |
|-------|-------------------|---------------------------------|----------------|----------------|---------------------------------|----------------|---------------------------------|----------|
| $L_\infty$ | $\frac{\alpha - 1}{2}$ | $\frac{\alpha - 1}{2} + \frac{1}{1 + 1/\alpha}$ | $\frac{\alpha - 1}{2}$ | $\frac{\alpha - 1}{2}$ | $\frac{\alpha - 1}{2}$ | $\frac{\alpha - 1}{2}$ | $\frac{\alpha - 1}{2}$ | $\frac{\alpha - 1}{2}$ |

see, e.g., [3, 23, 47, 24, 27]. Combining multiple reconstruction lattices has also been shown to improve the error, see, e.g., [12, 20, 21, 22, 26]. Another approach constructs the generating vector $z$ to minimize some search criterion, with the aim to directly control the approximation error, see, e.g., [30, 34, 5, 6]. We follow the latter approach in this paper.

We measure the approximation error of the algorithm $A_n$ in the weighted Korobov space in the worst-case setting under the $L_2$ and $L_\infty$ norms:

$$e_{\text{wor-app}}(A_n; L_q) := \sup_{\|f\|_{d,\alpha,\gamma} \leq 1} \|f - A_n(f)\|_{L_q([0,1]^d)}, \quad q \in \{2, \infty\},$$

where $\|f\|_{d,\alpha,\gamma}$ denotes our Korobov space norm (see (2.1) below). Our goal is in constructing the lattice generating vector $z$ to have the largest possible rate of convergence $r$ in $e_{\text{wor-app}}(A_n; L_q) = O(n^{-r+\delta})$ for arbitrarily small $\delta > 0$, with the implied constant independent of $d$ under appropriate conditions on the weight parameters $\gamma$. This is the concept of strong tractability, see, e.g., [41, 42, 43].

Many papers study general multivariate approximation problems under different assumptions on the available information. The class $\Lambda_{\text{std}}$ of arbitrary linear information allows all linear functionals, while the class $\Lambda_{\text{all}}$ of standard information allows only function values. It is still an open problem whether algorithms in $\Lambda_{\text{std}}$ can achieve the same convergence rate as $\Lambda_{\text{all}}$. Our approximation (1.1) involves function values at lattice points, so it falls under the class $\Lambda_{\text{std}}$. Table 1 lists the optimal or best-known convergence rates in the weighted Korobov space, comparing general results from $\Lambda_{\text{all}}$ and $\Lambda_{\text{std}}$ to what can be achieved by lattice algorithms. (It needs to be noted that different papers use different conventions for the smoothness parameter $\alpha$ and therefore might use $2\alpha$ where we use $\alpha$.) We clearly see that lattice algorithms are not optimal. Indeed, it was proved in [3] that the best possible convergence rate for lattice-based algorithms is only $\frac{\alpha}{4}$. However, lattice algorithms are easy and efficient to construct and implement, compared to the general results from $\Lambda_{\text{std}}$ which are typically non-constructive, see, e.g., [25, 46].

The lattice approximation (1.1) with prime $n$ was already analyzed in the worst-case $L_2$ and $L_\infty$ settings in the weighted Korobov space with product weights, see [30] and [34], respectively, and the last column of Table 1. Motivated by the need from PDE applications [19], theoretical justification for the CBC construction with general weights was developed in the $L_2$ setting in [5], requiring a substantially more complicated analysis due to the difficulty of handling non-product weights in an inductive argument. Subsequently, fast CBC implementations were developed for special forms of weights (order-dependent, POD and SPOD weights) in [6], again requiring substantially more complex computational strategies, including the use of both fast Fourier and fast Hankel transforms. While the fast CBC implementations from [6] are applicable to composite $n$, the theoretical justification in [5] is restricted to prime $n$. The current paper removes this restriction by employing a different
where $t$ that use the same points. Thus with the same lattice generating vector

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does not have a circulant matrix, so they can be obtained efficiently using the fast Fourier transform.

Korobov space (see (2.4) below), and the coefficients

In [6] are already applicable to composite

t rates for

proof technique (see further below).

We use the same search criterion $S_{n,d,\alpha,\gamma}(z)$ from [5, 6] (see (3.3) below). Although

the approximation (1.1) depends on the index set $\mathcal{A}_d(M)$, the quantity $S_{n,d,\alpha,\gamma}(z)$ is independent of the index set and the value of $M$. This is advantageous for the computational cost since there is no need to create and maintain the index set during the CBC construction. It was shown, e.g., in [5] that the worst-case $L_2$ approximation error for our lattice approximation (1.1) satisfies

$$e^{\text{wor-app}}(A_n; L_2) \leq \left( \frac{1}{M} + M S_{n,d,\alpha,\gamma}(z) \right)^{1/2}. \quad (1.4)$$

We show in Section 5 that the worst-case $L_\infty$ approximation error satisfies

$$e^{\text{wor-app}}(A_n; L_\infty) \leq \left\{ \begin{array}{ll}
\left( \sum_{h \in \mathcal{A}_d(M)} \frac{1}{t_{d,\alpha,\gamma}(h)} + 3 M |\mathcal{A}_d(M)| S_{n,d,\alpha,\gamma}(z) \right)^{1/2} & \text{if } \alpha > 1,
\left( \sum_{h \in \mathcal{A}_d(M)} \frac{1}{t_{d,\alpha,\gamma}(h)} + 3 M [S_{n,d,\alpha,\gamma}(z)]^2 \right)^{1/2} & \text{if } \alpha > 2.
\end{array} \right.$$ 

These error bounds hold for general weights and for prime and composite $n$. Note that each bound involves a sum of two terms, representing the truncation error and the cubature error. We choose $M$ to balance the two terms.

We establish in Theorem 3.3 that the CBC construction with a general $n$ achieves essentially the same bound on the quantity $S_{n,d,\alpha,\gamma}(z)$ as for prime $n$ in [5], thus completing the theory for $L_2$ approximation. For $L_\infty$ approximation, the same CBC construction and bound on $S_{n,d,\alpha,\gamma}(z)$ can be applied when $\alpha > 1$, while for a higher smoothness $\alpha > 2$ we can alternatively carry out the CBC construction with $\alpha$ replaced by $\alpha/2$ and all weights $\gamma_a$ replaced by $\sqrt{n}/\gamma_a$, and then revise the bound accordingly. This yields the same convergence rates for $L_\infty$ approximation as in [34], thus extending the final entry in Table 1 to general weights and composite $n$. Recall that the fast implementations for special forms of weights in [6] are already applicable to composite $n$.

The $L_2$ approximation result from this paper serves as an immediate and crucial upper bound for lattice-based kernel methods [19, 51, 52, 53], defined by

$$A_n^\text{ker}(f)(x) := \sum_{k=1}^{n} a_k K(x, t_k),$$

where $t_k$ are lattice points as in (1.1), $K(x, y)$ is the reproducing kernel of the weighted Korobov space (see (2.4) below), and the coefficients $a_k$ are such that $A_n^\text{ker}(f)$ interpolates $f$ at the $n$ lattice points $t_k$. These coefficients $a_k$ can be found by solving a linear system which has a circulant matrix, so they can be obtained efficiently using the fast Fourier transform. As explained in [19], kernel methods are optimal for $L_q$ approximation among all algorithms that use the same points. Thus with the same lattice generating vector $z$ we have

$$e^{\text{wor-app}}(A_n^\text{ker}; L_q) \leq e^{\text{wor-app}}(A_n; L_q), \quad 1 \leq q \leq \infty,$$

which is why our error bound and CBC construction can be applied directly. Recall that our approximation $A_n(f)$ in (1.1) depends on the index set $\mathcal{A}_d(M)$, whereas the kernel method $A_n^\text{ker}(f)$ involves no index set. Therefore, the kernel method can completely by-pass the index set, since our search criterion $S_{n,d,\alpha,\gamma}(z)$ is also independent of the index set.

It is important to note that the new results in this paper cannot be obtained by trivial generalisations of existing results for prime $n$. The analysis for a lattice generating vector $(z, z_a) \in \mathbb{Z}_n^d$ typically involves a certain sum over vectors $(\ell, \ell_a) \in \mathbb{Z}^d$ satisfying the congruence

$$\ell \cdot z + \ell_a z_a \equiv 0 \pmod{n}.$$
Existing averaging techniques for numerical integration with non-prime \( n \) work by first splitting the sum based on whether or not \( \ell_s \) is a multiple of \( n \) and then counting the repeated values of \( \ell \cdot z \mod n \), see, e.g., [9, proof of Theorem 5.8]. An attempt to use that technique for the approximation problem with non-prime \( n \) led to an undesirable error bound. Instead, in this paper we change the splitting to be based on the values of \( \gcd(\ell_s, n) \) together with \( \gcd(\ell \cdot z, n) \), see Lemma 4.1 and Lemma 4.2 ahead. This new proof technique can also be used in the context of integration, leading to the same result there but with a shorter proof.

Embedded or extensible lattice sequences for integration have been analysed in the past, see, e.g., [4, 10, 16, 17]. Here we follow a similar approach to [4] where we apply a mini-max strategy based on the ratios of a dimension-wise decomposition of \( S_{n,d,\gamma}(z) \) to construct the embedded lattice sequences for approximation (see Algorithm 6.1 below) for \( n \) being successive powers of a prime, i.e., \( n = p^m \) for \( m_1 \leq m \leq m_2 \). With \( N = p^m \), Theorem 6.2 proves that the embedded sequence is only a factor of \((\log N)^m\) worse than lattice rules with a fixed number of points. Numerical results in Section 7 confirms this in practice.

The paper is organised as follows. Section 2 introduces the weighted Korobov spaces. Section 3 provides a brief review of known results for \( L_2 \) approximation and states the new error bound for general \( n \). Section 4 is devoted to the technical proof of the main theorem. Section 5 derives an upper bound on the worst-case error in \( L_\infty \) norm and the corresponding search criterion for CBC construction. Section 6 constructs good generating vectors of embedded lattice sequences for a range of number of points. Section 7 presents numerical results which confirm our search criteria.

## 2 Weighted Korobov spaces

For \( \alpha > 1 \) and positive weight parameters \( \gamma = \{\gamma_u\}_{u \in \mathbb{N}} \), we consider the Hilbert space \( H_d \) of one-periodic \( L_2 \) functions defined on \([0, 1]^d\) with absolutely convergent Fourier series, with norm defined by

\[
\|f\|^2_{d,\alpha,\gamma} := \sum_{h \in \mathbb{Z}^d} |\hat{f}(h)|^2 r_{d,\alpha,\gamma}(h),
\]

(2.1)

where

\[
r_{d,\alpha,\gamma}(h) := \frac{1}{\gamma_{\text{supp}(h)}} \prod_{j \in \text{supp}(h)} |h_j|^\alpha,
\]

(2.2)

with \( \text{supp}(h) := \{1 \leq j \leq d : h_j \neq 0\} \). The parameter \( \alpha \) characterizes the rate of decay of the Fourier coefficients in the norm, so it is a smoothness parameter. We fix the scaling of the weights by setting \( \gamma_0 := 1 \), so that the norm of a constant function in \( H_d \) matches its \( L_2 \) norm and \( L_\infty \) norm. Various forms of weight parameters appear in the literature for multivariate integration, such as product weights [49, 50], order-dependent weights [11], POD weights [13, 14, 28, 29] and SPOD weights [8, 18]. In the simplest case of product weights, there is one weight \( \gamma_j \) for each coordinate index \( j \), and the weight for a set \( u \) of coordinate indices is given by \( \gamma_u = \prod_{j \in u} \gamma_j \). The other types of weights are more complicated and are motivated by applications, see the references above.

Some authors refer to this space as the weighted Korobov space, see [50] for product weights and [11] for general weights, while others call this a weighted variant of the periodic Sobolev space with dominating mixed smoothness [3]. We remark again that some references might use \( 2\alpha \) where we use \( \alpha \) and also sometimes the weight parameters might occur squared in the Hilbert norm.

When \( \alpha \geq 2 \) is an even integer, it can be shown that

\[
\|f\|^2_{d,\alpha,\gamma} = \sum_{u \subseteq (1:d)} \frac{1}{(2\pi)^{|u|}} \gamma_u \int_{[0,1]^d} \left( \int_{[0,1]^{d-|u|}} \left( \prod_{j \in u} \frac{\partial}{\partial x_j} \right)^{\alpha/2} f(x) \, dx \right)_u \, dx_u.
\]
So $f$ has square-integrable mixed partial derivatives of order $\alpha/2$ over all possible subsets of variables. Here and elsewhere in the paper, $\{1 : d\} = \{1, 2, \ldots, d\}$ and $x_u = (x_i)_{i \in u}$.

The inner product of $H_d$ is given by
\[
\langle f, g \rangle_{d,\alpha,\gamma} := \sum_{h \in \mathbb{Z}^d} \hat{f}(h) \overline{\hat{g}(h)} r_{d,\alpha,\gamma}(h),
\]
and the norm is $\|\cdot\|_{d,\alpha,\gamma} = \langle \cdot, \cdot \rangle_{d,\alpha,\gamma}^{1/2}$ which is consistent with (2.1). The reproducing kernel for $H_d$ is
\[
K_d(x, y) = \sum_{h \in \mathbb{Z}^d} e^{2\pi i h (x - y)} r_{d,\alpha,\gamma}(h),
\]
which satisfies (i) $K_d(x, y) = K_d(y, x)$ for all $x, y \in [0, 1]^d$; (ii) $K_d(\cdot, y) \in H_d$ for all $y \in [0, 1]^d$; (iii) $\langle f, K_d(\cdot, y) \rangle_d = f(y)$ for all $f \in H_d$ and all $y \in [0, 1]^d$. The last property is known as the reproducing property. Note that $r_{d,\alpha,\gamma}(h) = r_{d,\alpha,\gamma}(-h)$ and therefore $K_d(\cdot, \cdot)$ takes real values and can be written as a sum of cosine functions.

To simplify our notation, from this point on we write
\[
r(h) := r_{d,\alpha,\gamma}(h),
\]
except when we need to show the explicit dependence on $d$, $\alpha$ and $\gamma$.

3 Worst-case $L_2$ error with general $n$

The error for the lattice approximation in (1.1) clearly splits into two terms, i.e., the truncation error and the cubature error,
\[
(f - A_n(f))(x) = \sum_{h \in \mathcal{A}_d(M)} \hat{f}(h) e^{2\pi i h \cdot x} + \sum_{h \in \mathcal{A}_d(M)} \left( \hat{f}(h) - \hat{f}^n(h) \right) e^{2\pi i h \cdot x}. \tag{3.1}
\]
When measured in the $L_2$ norm, we have
\[
\|f - A_n(f)\|_{L_2([0, 1]^d)}^2 = \sum_{h \in \mathcal{A}_d(M)} |\hat{f}(h)|^2 + \sum_{h \in \mathcal{A}_d(M)} |\hat{f}(h) - \hat{f}^n(h)|^2.
\]
From [5] we have the following bound on the worst-case $L_2$ approximation error (1.3) where we now denote explicitly the dependence on the generating vector $z$ and the value of $M$ in the notation
\[
e_{n,d,M}^{\text{wor-approx}}(z; L_2) \leq \left( \frac{1}{M} + E_{n,d,\alpha,\gamma}(z) \right)^{1/2} \leq \left( \frac{1}{M} + M S_{n,d,\alpha,\gamma}(z) \right)^{1/2}, \tag{3.2}
\]
with
\[
E_{n,d,\alpha,\gamma}(z) := \sum_{h \in \mathcal{A}_d(M)} \sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} \frac{1}{r_{d,\alpha,\gamma}(h + \ell)},
\]
\[
S_{n,d,\alpha,\gamma}(z) := \sum_{h \in \mathbb{Z}^d} \sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} \frac{1}{r_{d,\alpha,\gamma}(h + \ell)}, \tag{3.3}
\]
where $\equiv_n$ means congruence modulo $n$.

The advantage of the search criterion $S_{n,d,\alpha,\gamma}(z)$ from [5] over $E_{n,d,\alpha,\gamma}(z)$ from [30] is that there is no dependence on the index set $\mathcal{A}_d(M)$, thus the error analysis is simpler and the construction cost is lower.
To handle general weights, instead of working directly with the error criterion \( S_{n,d,\alpha,\gamma}(z) \), the CBC construction in [5] works with a \textit{dimension-wise decomposition} of \( S_{n,d,\alpha,\gamma}(z) \), see Lemma 3.1 and Algorithm 3.2 below, both are valid for general \( n \). Lemma 3.1 can be shown by a recursive decomposition as in [5, Lemma 3.2] or it can be proved directly by induction.

**Lemma 3.1.** Given \( n \geq 2 \) (prime or composite), \( d \geq 1 \), \( \alpha > 1 \), and weights \( \gamma = \{\gamma_u\}_{u \in \mathbb{N}} \), we can write

\[
S_{n,d,\alpha,\gamma}(z) = \sum_{s=1}^{d} T_{n,d,s,\alpha,\gamma}(z_1, \ldots, z_s),
\]

where, for each \( s = 1, 2, \ldots, d \),

\[
T_{n,d,s,\alpha,\gamma}(z_1, \ldots, z_s) := \sum_{\omega \subseteq \{1: s+1:d\}} [2\zeta(2\alpha)]^{\omega} \theta_{n,s,\alpha}(z_1, \ldots, z_s; \{\gamma_u\}_{u \in \{1:s\}}),
\]

\[
\theta_{n,s,\alpha}(z_1, \ldots, z_s; \{\beta_u\}_{u \in \{1:s\}}) := \sum_{h \in \mathbb{Z}^d} \sum_{\ell \in \mathbb{Z}^d, \ell \neq 0} \frac{\beta_{\text{supp}(h)}}{\gamma_u,_{\ell}^{\text{supp}(h+\ell)}} r'(h)^{\beta_{\text{supp}(h+\ell)}},
\]

with \( r'(h) := \prod_{j \in \text{supp}(h)} |h_j|^n \).

We remark that the expression (3.5) takes as input argument a sequence of numbers \( \beta_u \) indexed by \( u \subseteq \{1:s\} \). The notation \( \{\gamma_u\}_{u \in \{1:s\}} \) in (3.4) indicates that we take different input arguments \( \beta_u = \gamma_{u,\text{spec}} \) by varying \( u \).

**Algorithm 3.2.** Given \( n \geq 2 \) (prime or composite), \( d \geq 1 \), \( \alpha > 1 \), and weights \( \gamma = \{\gamma_u\}_{u \in \mathbb{N}} \), construct the generating vector \( z^* = (z_1^*, \ldots, z_s^*) \) as follows: for each \( s = 1, \ldots, d \), with \( z_1^*, \ldots, z_{s-1}^* \) fixed, choose \( z_s \) from

\[
U_n = \{z \in \mathbb{Z} : 1 \leq z \leq n-1 \text{ and } \gcd(z,n) = 1\}
\]

to minimize \( T_{n,d,s,\alpha,\gamma}(z_1^*, \ldots, z_{s-1}^*, z_s) \) defined in (3.4).

Theorem 3.3 below is a non-trivial extension of [5, Theorem 3.5] to allow for composite \( n \), showing an upper bound on the search criterion \( S_{n,d,\alpha,\gamma}(z) \) guaranteed by the CBC construction. We devote Section 4 to its technical proof. Theorem 3.4 below is then a direct consequence of (3.2) and Theorem 3.3, by setting \( M \) to satisfy \( \frac{1}{\lambda} = M S_{n,d,\alpha,\gamma}(z) \), showing that the CBC construction for composite \( n \) also achieves the best possible convergence rate for \( L_2 \) approximation using lattice-based algorithms as concluded in [3]. Theorem 3.4 is therefore a non-trivial extension of [5, Theorem 3.6] to general \( n \).

**Theorem 3.3.** Given \( n \geq 2 \) (prime or composite), \( d \geq 1 \), \( \alpha > 1 \), and weights \( \gamma = \{\gamma_u\}_{u \in \mathbb{N}} \), the generating vector \( z \) obtained from the CBC construction following Algorithm 3.2, satisfies for all \( \lambda \in \left(\frac{1}{n}, 1\right]\)

\[
S_{n,d,\alpha,\gamma}(z) \leq \left[ \frac{\kappa}{\varphi(n)} \left( \sum_{\emptyset \neq u \subseteq \{1:d\}} |u| \gamma_u^\lambda [2\zeta(\alpha]\lambda)]u \right) \left( \sum_{u \subseteq \{1:d\}} \gamma_u^\lambda [2\zeta(\alpha)]u \right) \right]^{1/\lambda},
\]

where \( \kappa := 2^{2n\lambda+1} + 1 \), \( \zeta(x) \) denotes the Riemann zeta function, i.e., \( \zeta(x) := \sum_{n=1}^{\infty} \frac{1}{n^x} \), \( \varphi(n) \) is the cardinality of \( U_n \), i.e., the Euler’s totient function.

**Theorem 4.4.** Given \( n \geq 2 \) (prime or composite), \( d \geq 1 \), \( \alpha > 1 \), and weights \( \gamma = \{\gamma_u\}_{u \in \mathbb{N}} \), the lattice approximation (1.1), with index set (1.2) and generating vector \( z \) obtained from the CBC construction following Algorithm 3.2, after taking \( M \) in (1.2) to be

\[
M = (S_{n,d,\alpha,\gamma}(z))^{-1/2},
\]

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satisfies for all \( \lambda \in (\frac{1}{n}, 1] \),
\[
e^{\text{wtr-dp}}_{n,d,m}(z; L_2) \leq \sqrt{2} |S_{\gamma(n)}|^{1/4} \leq \sqrt{2} \kappa^{1/(4\lambda)} \sum_{u \leq \{1:d\}} \max(|u|, 1) \gamma_u \lambda |2\zeta(\alpha \lambda)|^{[u]}^{1/\lambda} ,
\]
where \( \kappa, \zeta(\cdot) \) and \( \varphi(\cdot) \) are as in Theorem 3.3. The exponent of \( \varphi(n)^{-1} \) can be arbitrarily close to \( \frac{\alpha}{4} \).

The constant is independent of \( d \) if
\[
\sum_{u \subseteq \{1:n\}, \ |u| < \infty} \gamma_u^{1/[u]} \left(2e^{1/\alpha} \zeta\left(\frac{n}{n-\varphi}\right)\right)^{|u|} < \infty, \ \delta > 0.
\]

The last part of the theorem used \( \max(|u|, 1) \leq (e^{1/\alpha})^{[u]} = (1.4446 \ldots)^{[u]} \).

For \( n = p^m \) being a power of a prime \( p \), we have \( 1/\varphi(n) = [1 + 1/(p-1)]/n \leq 2/n \). For general \( n \), it can be verified that \( 1/\varphi(n) < 9/n \) for all \( n < 10^{10} \). Hence from the practical point of view, \( 1/\varphi(n) \) can be considered as a constant factor times \( 1/n \).

### 4 Proof of Theorem 3.3

Lemma 4.2 below provides the essential averaging argument required in the proof of Theorem 3.3. We begin by stating Lemma 4.1 which is essential to the proof of Lemma 4.2. The proof of the first part of Lemma 4.1 can be found in a general textbook on number theory, e.g., [1, Chapters 5]. Based on the first part, the second part of Lemma 4.1 can be easily proved by using the fact that the solutions are restricted to be coprime with \( n \), see, e.g., [2, Lemma 5.5] or [15, Theorem 2].

**Lemma 4.1.** For any positive integer \( n \), and given \( \ell_s \in \mathbb{Z} \) and \( c \in \mathbb{Z} \), define \( g = \gcd(\ell_s, n) \) and consider the linear congruence
\[
c + \ell_s z_s \equiv n 0. \tag{4.1}
\]

Then there are exactly \( g \) solutions for \( z_s \in \mathbb{Z}_n \) if \( g \mid c \) and no solution if \( g \nmid c \). Moreover, if we restrict the solutions to \( z_s \in \mathbb{U}_n \), i.e., \( z_s \in \mathbb{Z}_n \) and \( \gcd(z_s, n) = 1 \), then there are at most \( g \) solutions if also \( \gcd(c, n) = g \) holds, with the same definition of \( g = \gcd(\ell_s, n) \), and no solutions otherwise.

**Lemma 4.2.** Given \( n \geq 2 \) (prime or composite), for any \( s \geq 1 \), any input sequence \( \{\beta_s\}_{s \subseteq \{1:s\}} \), all \( (z_1, \ldots, z_{s-1}) \in \mathbb{U}_n^{s-1} \), and all \( \lambda \in (\frac{1}{n}, 1] \), we have
\[
\frac{1}{\varphi(n)} \sum_{s \subseteq \{1:s\}} |\theta_{n,s}(z_1, \ldots, z_s; \{\beta_s\}_{s \subseteq \{1:s\}})|^\lambda \leq \frac{\kappa}{\varphi(n)} \left( \sum_{s \subseteq \{1:s\}} \beta_s |2\zeta(\alpha \lambda)|^{[u]} \right) \left( \sum_{u \subseteq \{1:s\}} \beta_u |2\zeta(\alpha \lambda)|^{[u]} \right) ,
\]
where \( \kappa, \zeta(\cdot) \) and \( \varphi(\cdot) \) are as in Theorem 3.3.

**Proof.** For notational convenience we write in this proof \( z := (z_1, \ldots, z_{s-1}) \in \mathbb{U}_n^{s-1} \). From (3.5), with a slight change of notation for the indices \( (\ell, h) \in \mathbb{Z}^s \) and \( (\ell, h) \in \mathbb{Z}^s \), and the
Inequality \( \sum_k a_k \leq (\sum_k a_k^\lambda)^{1/\lambda} \) for all \( a_k \geq 0 \) and \( \lambda \in (1/\alpha, 1] \), we obtain

\[
\text{Avg} := \frac{1}{\varphi(n)} \sum_{z \in \mathbb{U}_n} \left[ \theta_{n, \alpha} (z_1, \ldots, z_s; \{ \beta_k \}_{k=1}^s) \right]^\lambda
\]

\[
\leq \frac{1}{\varphi(n)} \sum_{z \in \mathbb{U}_n} \sum_{(h, s) \in \mathbb{Z}^s} \sum_{(\ell, z, s) \neq 0} \frac{\beta^\lambda_{\supp(h, s)} \beta^\lambda_{\supp((h, s) + (\ell, z, s))}}{r^\prime((h, s), r^\prime((h, s) + (\ell, z, s)))^\lambda}
\]

\[
= \frac{1}{\varphi(n)} \sum_{g \in \{1\ldots n\}} \sum_{h \in \mathbb{Z}^s \setminus \{0\}} \sum_{\ell, z, s \in \mathbb{Z}^s \setminus \{0\}} \frac{\beta^\lambda_{\supp(h, s)} \beta^\lambda_{\supp((h, s) + (\ell, z, s))}}{r^\prime((h, s), r^\prime((h, s) + (\ell, z, s)))^\lambda}
\]

\[
= \frac{1}{\varphi(n)} \sum_{g \in \{1\ldots n\}} \sum_{h \in \mathbb{Z}^s \setminus \{0\}} \sum_{\ell, z, s \in \mathbb{Z}^s \setminus \{0\}} G^g(h, s, \ell, s)
\]

with

\[
G^g(h, s, \ell, s) := \sum_{h \in \mathbb{Z}^s \setminus \{0\}} \sum_{\ell, z, s \in \mathbb{Z}^s \setminus \{0\}} \beta^\lambda_{\supp(h, s)} \beta^\lambda_{\supp((h, s) + (\ell, z, s))} \frac{1}{r^\prime((h, s)) r^\prime((h, s) + (\ell, z, s)))^\lambda},
\]

where \( r^\prime(h) = \prod_j \min(1, |h_j|) \) as defined in Lemma 3.1, and the inequality holds because of Lemma 4.1, as we used that the congruence has at most \( g \) solutions for \( z \in \mathbb{U}_n \) if \( g = \gcd(z, n) = \gcd(h \cdot z, n) \) and no solution if this condition is not satisfied. This step on splitting according to the divisors \( g \) of \( n \) is the key difference compared to the previous proof technique.

We separate the above expression into three parts: (i) \( \ell_s = -h_s \); (ii) \( \ell_s \neq -h_s \) and \( g \mid h_s \); (iii) \( \ell_s \neq -h_s \) and \( g \not\mid h_s \), to obtain

\[
\text{Avg} \leq \frac{1}{\varphi(n)} \left( \sum_{g \in \{1\ldots n\}} \sum_{h \in \mathbb{Z}^s \setminus \{0\}} G^g(h, s, -h_s) \right)
\]

\[
= \frac{1}{\varphi(n)} \left( \sum_{g \in \{1\ldots n\}} \sum_{h \in \mathbb{Z}^s \setminus \{0\}} G^g(h, s, -h_s) \right)
\]

\[
= \frac{1}{\varphi(n)} \left( \sum_{g \in \{1\ldots n\}} \sum_{h \in \mathbb{Z}^s \setminus \{0\}} G^g(h, s, \ell, s) \right).
\]

In the following we will swap the order of the multiple sums. For \( \ell_s \neq 0 \), and with a relabeling of \( q = h + \ell \in \mathbb{Z}^s \), it is easy to show that

\[
\sum_{g \in \{1\ldots n\}} G^g(h, s, \ell, s) = \left( \sum_{h \in \mathbb{Z}^s \setminus \{0\}} \beta^\lambda_{\supp(h, s)} \frac{1}{r^\prime((h, s), r^\prime((h, s) + (\ell, z, s)))^\lambda} \right)
\]

\[
\left( \sum_{q \in \mathbb{Z}^s \setminus \{0\}} \beta^\lambda_{\supp(q, h_s + \ell_s)} \frac{1}{r^\prime((q, h_s + \ell_s), r^\prime((q, h_s + \ell_s) + (\ell, z, s)))^\lambda} \right)
\]

\[
= \begin{cases} 
\frac{1}{|\ell_s|}\varphi \Omega^p & \text{if } h_s = 0 \text{ and } \ell_s \neq 0, \\
\frac{1}{|h_s|}\varphi \Omega^p & \text{if } h_s \neq 0 \text{ and } \ell_s = -h_s, \\
\frac{1}{|h_s|}\varphi \Omega^p & \text{if } h_s \neq 0 \text{ and } \ell_s \neq -h_s, 
\end{cases}
\]
with the abbreviations
\[
P := \sum_{u \leq (1-s-1)} \beta_u^\lambda (2\zeta(\alpha\lambda))^{\vert u \vert} \quad \text{and} \quad Q := \sum_{u \leq (1-s-1)} \beta_u^\lambda (2\zeta(\alpha\lambda))^{\vert u \vert}.
\]

We now find an upper bound on \( F_1 \) as follows:
\[
F_1 := \sum_{g \in \{1,n\}} g \sum_{h_s \in \mathbb{Z}_n} \sum_{\tilde{h}_s \in \mathbb{Z}_n} G^g(h_s, -\tilde{h}_s) = \sum_{g \in \{1,n\}} g^{1-\alpha\lambda} \sum_{\tilde{h}_s \in \mathbb{Z}_n} G^g(\tilde{h}_s, -\tilde{h}_s) \leq \sum_{\tilde{h}_s \in \mathbb{Z}_n} \sum_{g \in \{1,n\}} G^g(\tilde{h}_s, -\tilde{h}_s) = \sum_{\tilde{h}_s \in \mathbb{Z}_n} \frac{1}{\vert \tilde{h}_s \vert^{\alpha\lambda}}Q^P = \vert 2\zeta(\alpha\lambda) \vert Q^P,
\]

where the inequality is obtained by dropping the condition \( \gcd(h_s, n/g) = 1 \) and using \( \alpha\lambda > 1 \). The second to last inequality follows from (4.2).

Next we find an upper bound on \( F_2 \) as follows:
\[
F_2 := \sum_{g \in \{1,n\}} g \sum_{\tilde{h}_s \in \mathbb{Z}_n} \sum_{\ell_s \in \mathbb{Z}_n} G^g(h_s, \ell_s) = \sum_{g \in \{1,n\}} g \sum_{\tilde{h}_s \in \mathbb{Z}_n} \sum_{\ell_s \in \mathbb{Z}_n} G^g(0, \ell_s) + \sum_{g \in \{1,n\}} g \sum_{h_s \in \mathbb{Z}_n} \sum_{\ell_s \in \mathbb{Z}_n} G^g(h_s, \ell_s) = \sum_{g \in \{1,n\}} g^{1-\alpha\lambda} \sum_{\tilde{h}_s \in \mathbb{Z}_n} G^g(0, \tilde{h}_s) + \sum_{g \in \{1,n\}} g^{1-2\alpha\lambda} \sum_{h_s \in \mathbb{Z}_n} G^g(\tilde{h}_s, \ell_s) \leq \sum_{\tilde{h}_s \in \mathbb{Z}_n} \sum_{g \in \{1,n\}} G^g(0, \tilde{h}_s) + \sum_{h_s \in \mathbb{Z}_n} \sum_{\tilde{h}_s \in \mathbb{Z}_n} G^g(h_s, \ell_s) = \sum_{\tilde{h}_s \in \mathbb{Z}_n} \frac{1}{\vert \tilde{h}_s \vert^{\alpha\lambda}}Q^P + \sum_{h_s \in \mathbb{Z}_n} \frac{1}{\vert h_s \vert^{\alpha\lambda}} Q^2 \leq \vert 2\zeta(\alpha\lambda) \vert Q^P + \vert 2\zeta(\alpha\lambda) \vert^2 Q^2,
\]

where we separated the cases \( h_s = 0 \) and \( h_s \neq 0 \), and used again (4.2).

To find an upper bound on \( F_3 \), we write
\[
F_3 := \sum_{g \in \{1,n\}} g \sum_{h_s \in \mathbb{Z}_n} \sum_{\ell_s \in \mathbb{Z}_n} G^g(h_s, \ell_s) \leq \sum_{g \in \{1,n\}} g \sum_{h_s \in \mathbb{Z}_n} \sum_{\ell_s \in \mathbb{Z}_n} G^g(h_s, \ell_s) = \sum_{g \in \{1,n\}} g \sum_{h_s \in \mathbb{Z}_n} \sum_{\ell_s \in \mathbb{Z}_n} \frac{1}{\vert h_s \vert^{\alpha\lambda}} G^g,\]

where
\[
G^g := \sum_{h_s \in \mathbb{Z}_n} \sum_{\ell_s \in \mathbb{Z}_n} \frac{\beta_{\supp(h_s\cup(s))}^{\lambda}}{r(h)^{\lambda}} \frac{\beta_{\supp(h_s\cup\ell_s\cup(s))}^{\lambda}}{r(h + \ell)^{\lambda}}.
\]
Thus

Writing \( h_\ast = pg + q \) with \( q \) being congruent to the remainder modulo \( g \) and \( p \in \mathbb{Z} \) and writing \( \ell_\ast = kg \) with \( k \in \mathbb{Z} \setminus \{0\} \), we obtain

\[
F_3 \leq \sum_{g \in \{1, \ldots, n\}} g \sum_{q \in \mathbb{Z}_\ast} \sum_{p \in \mathbb{Z}} \frac{1}{|pg + q|^{\alpha \lambda}} \sum_{k \in \mathbb{Z}_\ast} \frac{1}{|(p + k)g + q|^{\alpha \lambda}} \tilde{G}^g

\]

\[
= \sum_{g \in \{1, \ldots, n\}} g \sum_{q \in \mathbb{Z}_\ast} \sum_{p \in \mathbb{Z}} \frac{1}{|pg + q|^{\alpha \lambda}} \left( \sum_{k \in \mathbb{Z}_\ast} \frac{1}{|k'g + q|^{\alpha \lambda}} - \frac{1}{|pg + q|^{\alpha \lambda}} \right) \tilde{G}^g
\]

\[
= \sum_{g \in \{1, \ldots, n\}} g \sum_{q \in \mathbb{Z}_\ast} \left[ \left( \sum_{p \in \mathbb{Z}} \frac{1}{|pg + q|^{\alpha \lambda}} \right)^2 - \left( \sum_{p \in \mathbb{Z}} \frac{1}{|pg + q|^{2\alpha \lambda}} \right) \right] \tilde{G}^g.
\]

To proceed further we need to obtain an upper bound on the inner sums over \( p \in \mathbb{Z} \). For any fixed \( g \in \{1, \ldots, n\} \) and \( q \) such that \( -\frac{n - 1}{2} \leq q \leq \frac{n - 1}{2} \) we have \( \frac{|q|}{g} \leq \frac{1}{2} \). Moreover, for \( p \in \mathbb{Z} \) and \( p \neq 0 \), by the triangle inequality we have

\[
\left| 1 + \frac{q}{pg} \right| \geq 1 - \frac{|q|}{|pg|} \geq 1 - \frac{|q|}{g} \geq \frac{1}{2}.
\]

Thus

\[
\sum_{p \in \mathbb{Z}} \frac{1}{|pg + q|^{\alpha \lambda}} = \sum_{p \in \mathbb{Z}_\ast} \frac{1}{|pg + q|^{\alpha \lambda}} + \frac{1}{|q|^{\alpha \lambda}} = \sum_{p \in \mathbb{Z}_\ast} \frac{1}{|pg|^{\alpha \lambda}} [1 + \frac{1}{|q|^{\alpha \lambda}}] + \frac{1}{|q|^{\alpha \lambda}} \leq \frac{2^{\alpha \lambda}}{g^{\alpha \lambda}} \sum_{p \in \mathbb{Z}_\ast} \frac{1}{|p|^{\alpha \lambda}} + \frac{1}{|q|^{\alpha \lambda}} = \frac{2^{\alpha \lambda}}{g^{\alpha \lambda}} (2\zeta(\alpha \lambda)) + \frac{1}{|q|^{\alpha \lambda}},
\]

and this leads to

\[
F_3 \leq \sum_{g \in \{1, \ldots, n\}} g \sum_{q \in \mathbb{Z}_\ast} \left[ \left( \frac{2^{\alpha \lambda}}{g^{\alpha \lambda}} [2\zeta(\alpha \lambda)] + \frac{1}{|q|^{\alpha \lambda}} \right)^2 - \frac{1}{|q|^{2\alpha \lambda}} \right] \tilde{G}^g
\]

\[
= \sum_{g \in \{1, \ldots, n\}} g \sum_{q \in \mathbb{Z}_\ast} \left[ \left( \frac{2^{\alpha \lambda}}{g^{\alpha \lambda}} [2\zeta(\alpha \lambda)]^2 + \frac{2^{\alpha \lambda + 1}}{g^{\alpha \lambda}} [2\zeta(\alpha \lambda)] - \frac{1}{|q|^{\alpha \lambda}} \right) \tilde{G}^g \leq \sum_{g \in \{1, \ldots, n\}} \left( \frac{2^{\alpha \lambda}}{g^{\alpha \lambda}} [2\zeta(\alpha \lambda)]^2 + \frac{2^{\alpha \lambda + 1}}{g^{\alpha \lambda}} [2\zeta(\alpha \lambda)]^2 \tilde{G}^g \leq \left( 2^{\alpha \lambda + 1} \right) [2\zeta(\alpha \lambda)]^2 \sum_{g \in \{1, \ldots, n\}} \tilde{G}^g \leq 2^{2\alpha \lambda + 1} [2\zeta(\alpha \lambda)]^2 \Omega^2,
\]

\[
11
\]
where with a relabeling of \( q = h + \ell \in \mathbb{Z}^{s-1} \), we used
\[
\sum_{\gamma \in \mathbb{Z}^{s-1}} \tilde{C}^\gamma = \sum_{h \in \mathbb{Z}^{s-1}} \frac{\beta_{\text{supp}(h)}^\lambda}{r'(h)^\lambda} \sum_{\gamma \in \mathbb{Z}^{s-1}} \frac{\beta_{\text{supp}(\gamma)}^\lambda}{r'(\gamma)^\lambda}
\]
\[
= \left( \sum_{u \subseteq \{1, s-1\}} \sum_{h \in \mathbb{Z}^{s-1}} \frac{\beta_{\text{supp}(h)}^\lambda}{\prod_{j \in u} |h_j|^\alpha \lambda} \right)^2
\]
\[
= \left( \sum_{u \subseteq \{1, s-1\}} \beta_{\text{supp}(u)}^\lambda \prod_{j \in u} \sum_{h \in \mathbb{Z}^{s-1}} \frac{1}{|h_j|^\alpha \lambda} \right)^2 = \Omega^2.
\]

Combining the upper bounds on \( F_1, F_2 \) and \( F_3 \), with \( \kappa := 2^{2n\alpha+1} + 1 \), we obtain an upper bound on \( \text{Avg} \) as follows:
\[
\text{Avg} \leq \frac{1}{\varphi(n)} (F_1 + F_2 + F_3)
\]
\[
\leq \frac{1}{\varphi(n)} \left( 2[2\zeta(\alpha \lambda)^2\varOmega + 2[2\zeta(\alpha \lambda)]^2\varOmega + 2[2\zeta(\alpha \lambda)]^2\varOmega^2 \right)
\]
\[
= \frac{1}{\varphi(n)} \left( 2[2\zeta(\alpha \lambda)]^2\varOmega + (2^{2n\alpha+1} + 1)[2\zeta(\alpha \lambda)]^2\varOmega^2 \right)
\]
\[
\leq \frac{\kappa}{\varphi(n)} \left( 2\zeta(\alpha \lambda) \varOmega \right)^{p + 2\zeta(\alpha \lambda) \varOmega}.
\]
Substituting the value of \( \varOmega \) and \( \varOmega \) into the above formula, we obtain
\[
\text{Avg} \leq \frac{\kappa}{\varphi(n)} \left( 2\zeta(\alpha \lambda) \sum_{u \subseteq \{1, s-1\}} \beta_{\text{supp}(u)}^\lambda [2\zeta(\alpha \lambda)]^{|u|} \right)
\]
\[
\times \left( \sum_{u \subseteq \{1, s-1\}} \beta_{\text{supp}(u)}^\lambda [2\zeta(\alpha \lambda)]^{|u|} + 2\zeta(\alpha \lambda) \sum_{u \subseteq \{1, s-1\}} \beta_{\text{supp}(u)}^\lambda [2\zeta(\alpha \lambda)]^{|u|} \right)
\]
\[
= \frac{\kappa}{\varphi(n)} \left( \sum_{u \subseteq \{1, s\}} \beta_{\text{supp}(u)}^\lambda [2\zeta(\alpha \lambda)]^{|u|} \right)
\]
\[
\times \left( \sum_{u \subseteq \{1, s-1\}} \beta_{\text{supp}(u)}^\lambda [2\zeta(\alpha \lambda)]^{|u|} + \sum_{u \subseteq \{1, s\}} \beta_{\text{supp}(u)}^\lambda [2\zeta(\alpha \lambda)]^{|u|} \right)
\]
\[
= \frac{\kappa}{\varphi(n)} \left( \sum_{u \subseteq \{1, s\}} \beta_{\text{supp}(u)}^\lambda [2\zeta(\alpha \lambda)]^{|u|} \right) \left( \sum_{u \subseteq \{1, s\}} \beta_{\text{supp}(u)}^\lambda [2\zeta(\alpha \lambda)]^{|u|} \right).
\]
This completes the proof. \( \square \)

With Lemma 3.1 and the new Lemma 4.2, we can complete the proof of Theorem 3.3 by following the argument in the proof of [5, Theorem 3.5]. (We have a slightly improved constant \( \kappa \) here, and we need to replace \( n - 1 \) by \( \varphi(n) \).)

5 Worst-case \( L_\infty \) error with general \( n \)

The following lemma gives an upper bound on the worst-case \( L_\infty \) error. It is a correction of [35, Equation (1.3)] (cf. [34, Lemma 1]) which mistakenly claimed (5.1) to be an equality.
Lemma 5.1. Given \( n \geq 2 \) (prime or composite), \( d \geq 1, \alpha > 1 \), weights \( \{ \gamma_n \}_{n \in \mathbb{N}} \), \( M > 0 \), let \( A_n \) be the lattice approximation defined by (1.1) with index set (1.2) and generating vector \( z \in \mathbb{Z}^d \). An upper bound on the worst-case \( L_\infty \) error is

\[
e_{n,d,M}^{\text{wor-app}}(z; L_\infty) \leq \left( \sum_{h \in A_d(M)} \frac{1}{r(h)} + 2 \sum_{h \in A_d(M)} \sum_{p \in A_d(M) (p-h) z = n 0} \frac{1}{r(p)} + \sum(T) \right)^{1/2},
\]

(5.1)

where

\[
\sum(T) := \sum_{h \in A_d(M)} \sum_{p \in A_d(M) (p-h) z = n 0} \sum_{\ell \in \mathbb{Z}^d \setminus (0, p-h)} \frac{1}{r(h + \ell)},
\]

(5.3)

and \( T \) being a matrix defined in the proof in (5.8).

Proof: Consider \( f \in H_d \) and the lattice approximation (1.1), and recall that the error is given by (3.1). Using the reproducing property of the kernel (2.4), and making use of the definition of the corresponding inner product (2.3), we follow the argument in the proof of [34, Lemma 1] to write

\[
(f - A_n(f))(x) = \sum_{h \in \mathbb{Z}^d} \langle f, \tau_h \rangle_{d, \alpha, \gamma} e^{2\pi i h \cdot x} = \left( \sum_{h \in \mathbb{Z}^d} \tau_h e^{-2\pi i h \cdot x} \right)_{d, \alpha, \gamma},
\]

(5.4)

where

\[
\tau_h(\ell) = \begin{cases} 
- \frac{1}{r(\ell)} & \text{if } h \in A_d(M), \ell \neq h \text{ and } h \cdot z \equiv n \ell \cdot z, \\
\frac{1}{r(\ell)} & \text{if } h \notin A_d(M) \text{ and } \ell = h, \\
0 & \text{otherwise}.
\end{cases}
\]

We can directly read off the Fourier coefficients of the functions \( \tau_h \): for \( \ell \in \mathbb{Z}^d \),

\[
\overline{\tau_h}(\ell) = \begin{cases} 
\frac{1}{r(\ell)} & \text{if } h \in A_d(M), \ell \neq h \text{ and } h \cdot z \equiv n \ell \cdot z, \\
0 & \text{otherwise}.
\end{cases}
\]

Based on these we obtain

\[
\langle \tau_h, \tau_p \rangle_{d, \alpha, \gamma} = \begin{cases} 
\frac{1}{r(h + \ell)} & \text{if } h, p \in A_d(M) \text{ and } p \cdot z \equiv n h \cdot z, \\
- \frac{1}{r(p)} & \text{if } h \in A_d(M), p \notin A_d(M) \text{ and } p \cdot z \equiv n h \cdot z, \\
- \frac{1}{r(h)} & \text{if } h \notin A_d(M), p \in A_d(M) \text{ and } p \cdot z \equiv n h \cdot z, \\
\frac{1}{r(h)} & \text{if } h = p \notin A_d(M), \\
0 & \text{otherwise}.
\end{cases}
\]

(5.5)
Here we point out that the second and third cases in (5.5) are negative, which corrects both [34, Lemma 1] and [35, Equation (1.2)].

Applying the Cauchy–Schwarz inequality to (5.4), we obtain

\[ \| (f - A_\alpha(f))(x) \| \leq \| f \|_{d, \alpha, \gamma} \left\| \sum_{h \in \mathbb{Z}^d} \tau_h e^{-2\pi i h \cdot x} \right\|_{d, \alpha, \gamma}. \]

Note that the second norm is with respect to the functions \( \tau_h \) and the right-hand side is thus still a function of \( x \). Equality is attained when \( f(t) \) and \( \sum_{h \in \mathbb{Z}^d} \tau_h(t) e^{-2\pi i h \cdot x} \) are linearly dependent and hence the upper bound is attainable. An upper bound on the worst-case \( L_\infty \) error can hence be obtained as follows

\[ \varepsilon_{n,d,M}^{\text{wor-app}}(x; L_\infty) = \sup_{x \in [0,1]^d} \left\| \sum_{h \in \mathbb{Z}^d} \tau_h e^{-2\pi i h \cdot x} \right\|_{d, \alpha, \gamma}. \]

Define the matrix \( T \) by

\[ T := \left[ \left\{ \langle \tau_h, \tau_p \rangle_{d, \alpha, \gamma} \right\}_{h, \tau \in A_d(M)} \right]. \] (5.8)

Let \( \text{trace}(T) \) denote the sum of its diagonal elements, and \( \sum(T) \) the sum of all its elements, as in (5.3). Using (5.5) and (5.7), we then obtain the first claimed bound (5.1). To proof the second bound we follow [35] to bound the middle term in (5.1) as

\[ \sum_{h \in A_d(M)} \sum_{p \in A_d(M)} \frac{1}{r(p)} \leq \sum_{h \in A_d(M)} \sum_{p \in \mathbb{Z}^d \setminus \{h\}} \frac{1}{r(p)} \]

\[ = \sum_{h \in A_d(M)} \sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} \frac{1}{r(h + \ell)} = \text{trace}(T) \leq \sum(T). \]

This completes the proof.

There are two parts in the upper bound (5.2) and we will evaluate them separately. We will need bounds on the cardinality of \( A_d(M) \) for general weights.

**Lemma 5.2.** For all \( d \geq 1, \alpha > 1 \), weights \( \{ \tau_h \}_{u \in \mathbb{N}} \), and \( M > 0 \), the cardinality of the index set (1.2) satisfies the upper bound

\[ |A_d(M)| \leq M^d C_{1,d,q,\alpha,\gamma}, \quad \text{for all} \quad q > \frac{1}{\alpha}, \]

\[ 14 \]
Proof. Denote by $\gamma_0$. Therefore, with also

Moreover, if $M \geq 1$ then we have the lower bound

$$|\mathcal{A}_d(M)| \geq (\gamma_1(M))^{1/\alpha}. \tag{5.9}$$

Proof. The upper bound is proved in [5, Lemma 3.1]. We proceed to prove a simple lower bound. Suppose $M \geq 1$. Then since $\gamma_0 = 1$ we have $0 \in \mathcal{A}_d(M)$ and

Restricting the sum to a single subset $u = \{1\}$ gives $|\mathcal{A}_d(M)| \geq 1 + 2[(\gamma_1(M))^{1/\alpha}]$. If $\gamma_1(M) \geq 1$, then $1 + 2[(\gamma_1(M))^{1/\alpha}] \geq 1 + [(\gamma_1(M))^{1/\alpha}] \geq (\gamma_1(M))^{1/\alpha}$. If $\gamma_1(M) < 1$, then $1 + 2[(\gamma_1(M))^{1/\alpha}] = 1 > (\gamma_1(M))^{1/\alpha}$. Hence in all cases we have the lower bound (5.9). \qed

The first part in (5.2) is the truncation error and we will bound it in the following lemma, which extends [31, Lemma 6] from product weights to general weights. Note that the stronger condition $M \geq 1$ is required.

**Lemma 5.3.** For all $d \geq 1$, $\alpha > 1$, weights $\{\gamma_u\}_{u \in \mathbb{N}}$, $M \geq 1$, and index set (1.2), we have

$$\sum_{h \not\in \mathcal{A}_d(M)} \frac{1}{r(h)} \leq C_{2,d,\tau,\alpha,\gamma} M^{-\frac{1}{\beta \tau}} \quad \text{for all } \tau \in \left(\frac{1}{\beta}, 1\right),$$

where

$$C_{2,d,\tau,\alpha,\gamma} \coloneqq (\gamma_1(M))^{\frac{\tau}{1-\tau}} \left(\sum_{u \in \{1:d\}} \gamma_u^\tau |2\zeta(\alpha\tau)|^{|u|}\right)^{1/\tau}.$$

**Proof.** Denote by $h^{(i)}$ for $i = 1, 2, \ldots$ an ordering of $h \in \mathbb{Z}^d$ such that $\frac{1}{r(h^{(i)})} \geq \frac{1}{r(h^{(i+1)})} \geq \cdots$. For all $i$ and $\tau > 1/\alpha$, we then have

$$\frac{1}{r(h^{(i)})^\tau} \leq \frac{1}{i} \sum_{j=1}^i \frac{1}{r(h^{(j)})^\tau} \leq \frac{1}{i} \sum_{h \in \mathbb{Z}^d} \frac{1}{r(h)^\tau} = \frac{1}{i} \sum_{u \in \{1:d\}} \gamma_u^\tau |2\zeta(\alpha\tau)|^{|u|}.$$

Therefore, with also $\tau < 1$, we have

$$\sum_{h \not\in \mathcal{A}_d(M)} \frac{1}{r(h)} = \sum_{i > |\mathcal{A}_d(M)|} \frac{1}{r(h^{(i)})} \leq \sum_{i > |\mathcal{A}_d(M)|} \left(\frac{1}{i} \sum_{u \in \{1:d\}} \gamma_u^\tau |2\zeta(\alpha\tau)|^{|u|}\right)^{1/\tau}$$

$$\leq \left(\sum_{u \in \{1:d\}} \gamma_u^\tau |2\zeta(\alpha\tau)|^{|u|}\right)^{1/\tau} \int_{|\mathcal{A}_d(M)|}^\infty x^{-1/\tau} \, dx$$

$$= \left(\sum_{u \in \{1:d\}} \gamma_u^\tau |2\zeta(\alpha\tau)|^{|u|}\right)^{1/\tau} \frac{1}{|\mathcal{A}_d(M)|^{(1-\tau)/\tau}} \frac{\tau}{1-\tau}. \tag{5.10}$$

Combining (5.9) and (5.10) completes the proof. \qed

The paper [22] studied worst-case $L_\infty$ approximation by a combination of multiple rank-1 lattice rules. We remark that our Lemmas 5.2 and 5.3 can be used to extend the results in [22] from product weights to general weights.

The second part in our worst-case $L_\infty$ error bound (5.2) is $3\sum(T)$, which bounds the cubature error. We look at two ways of bounding $\sum(T)$, both making use of the quantity $S_{n,d,\alpha,\gamma}(z)$ defined in (3.3).
Lemma 5.4. Given \( n \geq 2 \) (prime or composite), \( d \geq 1 \), \( \alpha > 1 \), weights \( \{\gamma_u\}_{u \in \mathbb{N}} \), and generating vector \( \mathbf{z} \in \mathbb{Z}^d \), the quantity \( \text{sum}(T) \) defined by (5.3) satisfies

\[
\text{sum}(T) \leq M \left| A_d(M) \right| S_{n,d,\alpha,\gamma}(\mathbf{z}).
\]  

(5.11)

Moreover, if \( \alpha > 2 \) then

\[
\text{sum}(T) \leq M \left[ S_{n,d,\alpha,\gamma}(\mathbf{z}) \right]^2,
\]

(5.12)

where \( S_{n,d,\alpha,\gamma}(\mathbf{z}) \) is defined by the expression (3.3) with \( \alpha \) replaced by \( \alpha/2 \) and each weight \( \gamma_u \) replaced by \( \sqrt{\gamma_u} \).

Proof. The first bound (5.11) is obtained (as shown in [34]) by bounding the sum over \( \mathbf{p} \) in (5.3) by the cardinality of \( A_d(M) \) and then using \( 1 \leq \frac{M}{|\mathbf{p}|} \) for \( \mathbf{h} \in A_d(M) \).

To derive the second bound (5.12), we write (5.3) as (also shown in [34])

\[
\text{sum}(T) = \sum_{h \in A_d(M)} \sum_{\mathbf{p} \in A_d(M)} \sum_{q \in \mathbb{Z}^d} \frac{1}{r(q)} = \sum_{h \in A_d(M)} \left( \sum_{q \in \mathbb{Z}^d} \frac{1}{r(q)} \right)^2.
\]

Now we use that \( \sum_k a_k \leq (\sum_k a_k^2)^{1/2} \) for all \( a_k \geq 0 \) and that \( 1 \leq \frac{M^{1/2}}{|r(h)|} \) for \( \mathbf{h} \in A_d(M) \) to obtain, for \( \alpha > 2 \),

\[
\text{sum}(T) \leq \left( \sum_{q \in \mathbb{Z}^d} \frac{1}{r(q)} \right)^{1/2} \left( \sum_{h \in A_d(M) \setminus \{q\}} 1 \right)^2 \\
\leq M \left( \sum_{q \in \mathbb{Z}^d} \frac{1}{r(q)} \right)^{1/2} \left( \sum_{h \in A_d(M) \setminus \{q\}} \frac{1}{r(h)} \right)^{1/2},
\]

from which the second bound (5.12) follows by making use of the definition (3.3) since \( [r_{d,\alpha,\gamma}(\mathbf{h})]^{1/2} = r_{d,\alpha,\gamma}(\mathbf{h}) \). The condition \( \alpha > 2 \) is needed so that the corresponding error bound (3.6) for \( S_{n,d,\alpha,\gamma}(\mathbf{z}) \) is valid.

We remark that (5.11) was analyzed in [34] so \( S_{n,d,\alpha,\gamma}(\mathbf{z}) \) was proposed as one of the CBC search criteria for \( L_\infty \) approximation for product weights and a prime number of points. Another quantity \( X_d(\mathbf{z}) \) which depends on the index set \( A_d(M) \) was introduced in [34] to obtain a better convergence rate when \( \alpha > 2 \). We show that our second bound (5.12) leads to the same better rate when \( \alpha > 2 \), and has the advantage that it does not involve any index set.

Theorem 5.5. Given \( n \geq 2 \) (prime or composite), \( d \geq 1 \), \( \alpha > 1 \), and weights \( \gamma = \{\gamma_u\}_{u \in \mathbb{N}} \), consider the lattice approximation (1.1) with index set (1.2).

1. The generating vector \( \mathbf{z} \) obtained from Algorithm 3.2 based on the search criterion \( S_{n,d,\alpha,\gamma}(\mathbf{z}) \) in (3.3) satisfies

\[
\epsilon_{n,d,M}(\mathbf{z}; L_\infty) = O\left( [S_{n,d,\alpha,\gamma}(\mathbf{z})]^{\tau_1} \right) = O\left( \varphi(n)^{-\tau_1} \right) \quad \text{for all } \tau \in \left( \frac{\alpha}{2}, 1 \right),
\]

where \( M \) is given by (5.14) (with \( n \) sufficiently large so that \( M \geq 1 \)) and

\[
\tau_1 := \frac{1 - \tau}{2\tau(1 - \tau + \alpha^2 + \alpha \tau^2)}, \quad \text{which can be arbitrarily close to } \frac{\alpha - 1}{4}.
\]
2. For $\alpha > 2$, the generating vector $z$ obtained from Algorithm 3.2 based on the search criterion $S_{n,d,\sqrt[3]{\gamma n}}(z)$, i.e. (3.3) with $\alpha$ replaced by $\alpha/2$ and weights $\gamma_n$ replaced by $\sqrt[3]{\gamma_n}$, satisfies

$$e_{n,d,M}^{\text{wor-app}}(z; L_\infty) = O\left(\left[\frac{S_{n,d,\alpha/2,\sqrt[3]{\gamma n}}(z)}{\sqrt[3]{\gamma_n}}\right]^{2\tau}\right) = O\left(\varphi(n)^{\gamma_{\Sigma}}\right) \text{ for all } \tau \in \left(\frac{1}{\alpha}, \frac{1}{2}\right),$$

where $M$ is given by (5.15) (with $n$ sufficiently large so that $M \geq 1$) and

$$r_2 := \frac{1 - \tau}{2\tau(1 - \tau + \alpha\tau)}, \quad \text{which can be arbitrarily close to } \frac{\alpha - 1}{2} - \frac{1}{\alpha}.$$  

The implied constants are independent of $d$ if (3.7) holds.

**Proof.** For both claims we start from the error bound (5.2) from Lemma 5.1 and combine this with the bounds from Lemmas 5.2, 5.3, and 5.4.

For the first claim, we use the first bound (5.11) from Lemma 5.4 to obtain

$$e_{n,d,M}^{\text{wor-app}}(z; L_\infty) \leq \left(C_{d,\tau,\alpha,\gamma_\theta} M^{-\frac{1}{\alpha\gamma}} + 3 M^{\gamma+1} C_{d,\gamma,\alpha,\gamma} S_{n,d,\alpha,\gamma}(z)\right)^{1/2}, \quad (5.13)$$

for all $q \in \left(\frac{1}{\alpha}, \infty\right)$ and $\tau \in \left(\frac{1}{\alpha}, 1\right)$, where $C_{d,\tau,\alpha,\gamma_\theta}$ is as defined in Lemma 5.3 and $C_{d,\gamma,\alpha,\gamma}$ is as defined in Lemma 5.2. We take $q = \tau$ and choose $M$ to equate the two terms inside the brackets in (5.13) to arrive at

$$M = \left(C_{d,\tau,\alpha,\gamma_\theta} S_{n,d,\alpha,\gamma}(z)\right)^{-1} \left[\frac{\alpha\gamma - 1}{\alpha + \alpha\gamma - 1}\right]^{1/2}. \quad (5.14)$$

Provided that $n$ is sufficiently large, (5.14) will satisfy $M \geq 1$ and this leads to

$$e_{n,d,M}^{\text{wor-app}}(z; L_\infty) \leq \sqrt{2} \left(3 C_{d,\tau,\alpha,\gamma_\theta} C_{d,\gamma,\alpha,\gamma} S_{n,d,\alpha,\gamma}(z)\right)^{1/2}. \quad (5.15)$$

Using Theorem 3.3 and taking $\lambda = \tau$, we conclude that

$$e_{n,d,M}^{\text{wor-app}}(z; L_\infty) \leq C_{d,\tau,\alpha,\gamma} \varphi(n)^{\frac{1}{\alpha + \alpha\gamma - 1}},$$

where

$$C_{d,\tau,\alpha,\gamma} := \sqrt{2} \left[3 \gamma_{\Sigma}^{-1} \left(2^{2\alpha\gamma + 1} + 1\right)^{1/\alpha\gamma} \left(1 - \frac{\alpha\gamma - 1}{\alpha + \alpha\gamma - 1}\right)^{1/\alpha + \alpha\gamma - 1} \right],$$

which can be bounded independently of $d$ if (3.7) holds.

For the second claim, we assume that $\alpha > 2$. Then from the second bound (5.12) from Lemma 5.4 we obtain

$$e_{n,d,M}^{\text{wor-app}}(z; L_\infty) \leq \left(C_{d,\tau,\alpha,\gamma} M^{-\frac{1}{\alpha\gamma}} + 3 M S_{n,d,\alpha,\gamma}(z)^{1/2}\right)^{1/2},$$

for all $\tau \in \left(\frac{1}{\alpha}, 1\right)$. We again equate the two terms inside the brackets to obtain

$$M = \left(C_{d,\tau,\alpha,\gamma} S_{n,d,\alpha,\gamma}(z)^{-1/2}\right)^{1/2}. \quad (5.15)$$
Provided that \( n \) is sufficiently large, (5.15) will satisfy \( M \geq 1 \) and this now leads to
\[
\varepsilon_{n,d,M}^{\text{wor-app}}(z; L_{\infty}) \leq \sqrt{2} \left( 3 C_{2,d,r,\alpha,\gamma}^{\max} \left( S_{n,d,\frac{1}{\alpha},\gamma}(z) \right)^2 \right)^{1-\tau/\alpha}. 
\]
We can now apply Theorem 3.3, but with \( \alpha \) replaced by \( \tilde{\alpha} := \alpha/2 \) and \( \gamma \) replaced by \( \tilde{\gamma} := \sqrt{\gamma} \), to obtain for all \( \lambda \in (1/\tilde{\alpha}, 1) = (2/\alpha, 1) \),
\[
S_{n,d,\tilde{\alpha},\tilde{\gamma}}(z) \leq \left( \frac{(2^{\alpha}+1)+1}{\varphi(\lambda)} \right)^{1/\lambda} \left( \sum_{u \leq (1-d)} \max(|u|, 1) \gamma_z^d \beta \left[ 2\zeta(\alpha\lambda)|u| \right]^{1/\lambda} \right)^{2/\lambda}.
\]
Taking \( \lambda = 2\tau \) and restricting \( \tau < 1/2 \), we can thus obtain
\[
\varepsilon_{n,d,M}^{\text{wor-app}}(z; L_{\infty}) \leq C_{4,d,r,\alpha,\gamma}(n)^{-\frac{1-\tau}{\alpha}}\tau A, 
\]
where
\[
C_{4,d,r,\alpha,\gamma} := \sqrt{2} \left( 3 \gamma_z^{-1} (2^{2\alpha}+1) \right)^{1-\tau/\alpha} \left( \frac{\tau}{1-\tau} \right)^{\frac{1}{1+\frac{1-\tau}{\alpha}}} \times \left( \sum_{u \leq (1-d)} \max(|u|, 1) \gamma_z^d \beta \left[ 2\zeta(\alpha\tau)|u| \right]^{1/\tau} \right)^{1+\frac{1-\tau}{\alpha}},
\]
which can be bounded independently of \( d \) if (3.7) holds.

\[\Box\]

### 6 Embedded lattice rules

In this section, we apply techniques from [4] to construct good generating vectors of embedded lattice rules for a range of number of points. Recall from Algorithm 3.2 that the search criterion for a fixed \( n \) is \( T_{n,d,s,\alpha,\gamma}(z_1, \ldots, z_s) \) which contributes to the dimension-wise decomposition of \( S_{n,d,\alpha,\gamma}(z) \) as in Lemma 3.1. In Algorithm 6.1 below, we will construct embedded lattice rules by a mini-max strategy based on the ratios of \( T_{n,d,s,\alpha,\gamma}(z_1, \ldots, z_s) \) against the “best” choices of generating vectors for a range of values of \( n \).

**Algorithm 6.1.** Given \( m_2 > m_1 \geq 1 \), prime \( p \), \( d \geq 1 \), \( \alpha > 1 \), and weights \( \gamma = \{\gamma_u\}_{u \in \mathbb{N}} \), for each \( m = m_1, \ldots, m_2 \), we obtain the generating vector \( \mathbf{z}(m) = (z_1(m), \ldots, z_s(m)) \) using Algorithm 3.2 with \( n = p^m \) and store the corresponding values of \( T_{p^m,d,s,\alpha,\gamma}(z_1(m), \ldots, z_s(m)) \) for \( s = 1, \ldots, d \).

Then we construct the generating vector \( \mathbf{z}^{\text{emb}} = (z_1^{\text{emb}}, \ldots, z_s^{\text{emb}}) \) as follows: for each \( s = 1, \ldots, d \), with \( z_1^{\text{emb}}, \ldots, z_s^{\text{emb}} \) fixed, choose \( z_s \) from
\[
U_{p^m} = \{ z \in \mathbb{Z} : 1 \leq z \leq p^m-1 \text{ and } \gcd(z, p) = 1 \},
\]
to minimize
\[
X_{p,m_1,m_2,d,s,\alpha,\gamma}(z_1^{\text{emb}}, \ldots, z_s^{\text{emb}}) := \max_{m_1 \leq m \leq m_2} \frac{T_{p^m,d,s,\alpha,\gamma}(z_1^{\text{emb}}, \ldots, z_s^{\text{emb}}, z_s)}{T_{p^m,d,s,\alpha,\gamma}(z_1(m), \ldots, z_s(m))}.
\]

From the definition (6.1) it follows that the generating vector \( \mathbf{z}^{\text{emb}} \) obtained by Algorithm 6.1 satisfies, for each \( m \) between \( m_1 \) and \( m_2 \),
\[
S_{p^m,d,s,\alpha,\gamma}(\mathbf{z}^{\text{emb}}) = \sum_{s=1}^d T_{p^m,d,s,\alpha,\gamma}(z_1^{\text{emb}}, \ldots, z_s^{\text{emb}})
\]
\[
\leq \sum_{s=1}^d X_{p,m_1,m_2,d,s,\alpha,\gamma}(z_1^{\text{emb}}, \ldots, z_s^{\text{emb}}) T_{p^m,d,s,\alpha,\gamma}(z_1(m), \ldots, z_s(m))
\]
\[
\leq \left( \max_{s \in (1:d)} X_{p,m_1,m_2,d,s,\alpha,\gamma}(z_1^{\text{emb}}, \ldots, z_s^{\text{emb}}) \right) S_{p^m,d,s,\alpha,\gamma}(\mathbf{z}(m)).
\]
The quantity \( \max_{z \in \{1, \ldots, d\}} X_{p, m_1, m_2, d, s, \alpha, \gamma}(z_1^{\text{emb}}, \ldots, z_s^{\text{emb}}) \) therefore indicates how much worse \( z^{\text{emb}} \) obtained from Algorithm 6.1 is compared with \( z^{(m)} \) obtained from Algorithm 3.2 for each number of points \( p^m, m_1 \leq m \leq m_2, \) due to the reason that the upper bounds on the worst-case error measured under both \( L_2 \) and \( L_\infty \) are in terms of \( S_{n,d,\alpha}(z) \) or its variants \( S_{n,d,\alpha/2}(z) \), see Theorem 3.4 and 5.5.

Theorem 6.2 shows a theoretical upper bound on the ratio (6.1).

**Theorem 6.2.** Given \( m_2 > m_1 \geq 1, \) prime \( p, d \geq 1, \) and \( \alpha > 1, \) and weights \( \gamma = \{\gamma_u\}_{u \subseteq \mathbb{N}}, \) let \( z^{\text{emb}} = (z_1^{\text{emb}}, \ldots, z_s^{\text{emb}}) \) be obtained from the CBC construction following Algorithm 6.1. Then the maximum in (6.2) is of order arbitrarily close to

\[
(m_2 - m_1 + 1)^\alpha.
\]

In other words, we are penalised by a factor of only \( (\log N)^\alpha \) with \( N = p^{m_2}. \)

**Proof.** Following Algorithm 6.1, for \( s = 1, \ldots, d, \) with \( z_1^{\text{emb}}, \ldots, z_{s-1}^{\text{emb}} \) chosen and fixed, the next choice \( z_s^{\text{emb}} \) by the algorithm satisfies for any \( \lambda \in \left(\frac{1}{d}, 1\right], \)

\[
[X_{p, m_1, m_2, d, s, \alpha, \gamma}(z_1^{\text{emb}}, \ldots, z_{s-1}^{\text{emb}}, z_s^{\text{emb}})]^\lambda
\]

\[
\leq \frac{1}{\varphi(p^m_s)} \sum_{z_s \in \mathbb{U}_{p^m_s}[1:1:d]} [X_{p, m_1, m_2, d, s, \alpha, \gamma}(z_1^{\text{emb}}, \ldots, z_{s-1}^{\text{emb}}, z_s)]^\lambda
\]

\[
\leq \frac{1}{\varphi(p^m)} \sum_{z_s \in \mathbb{U}_{p^m}[1:1:d]} \left[ \sum_{m_1 = 1}^{m_2} T_{p^m, d, s, \alpha, \gamma}(z_1^{(m)}, \ldots, z_{s-1}^{(m)}, z_s) \right]^\lambda
\]

\[
= \sum_{m_1 = 1}^{m_2} \frac{1}{\varphi(p^m)} \sum_{z_s \in \mathbb{U}_{p^m}[1:1:d]} \left[ T_{p^m, d, s, \alpha, \gamma}(z_1^{(m)}, \ldots, z_{s-1}^{(m)}, z_s) \right]^\lambda,
\]

(6.3)

where in the first inequality we replaced the minimum over \( z_s \) by the average and in the second inequality we replaced the maximum over \( m \) by the sum, while in the third inequality we used \( \sum_{k=1}^{s} \lambda_{a_k} \leq \sum_{k=1}^{s} \lambda_{a_k} \) for \( a_k \geq 0 \) and then swapped the order of the sum over \( m \) and the average over \( z_s. \)

The numerator in (6.3) is an average of \( [T_{p^m, d, s, \alpha, \gamma}(z_1^{(m)}, \ldots, z_{s-1}^{(m)}, z_s)]^\lambda \) over \( z_s \in \mathbb{U}_{p^m}[1:1:d], \) which is exactly the same as the average over \( z_s \in \mathbb{U}_{p^m}, \) since from (3.4)–(3.5) we see that the expression only depends on \( z_s \) through the value of \( (z_s \mod p^m). \) Writing \( n = p^m, \) we have from Lemma 4.2 that, for any \( z_1, \ldots, z_{s-1} \in \mathbb{U}, \)

\[
\frac{1}{\varphi(n)} \sum_{z_s \in \mathbb{U}} [T_{n, d, s, \alpha, \gamma}(z_1, \ldots, z_{s-1}, z_s)]^\lambda
\]

\[
\leq \sum_{w \subseteq \mathbb{Z}(1:1:d)} \left[ 2\zeta(2\alpha) \right]^{\lambda_{|w|}} \frac{1}{\varphi(n)} \sum_{z_s \in \mathbb{U}} \left[ \gamma_{w, a}(z_1, \ldots, z_{s-1}, z_s; \{\gamma_{w, a}\}_{a \subseteq \mathbb{Z}(1:1)}) \right]^\lambda
\]

\[
\leq \frac{K}{\varphi(n)} \sum_{w \subseteq \mathbb{Z}(1:1:d)} \left[ 2\zeta(2\alpha) \right]^{\lambda_{|w|}} \left( \sum_{x \in \mathbb{Z}(1:1)} \gamma_{w, a}(x) \right) \left( \sum_{x \in \mathbb{Z}(1:1)} \gamma_{w, a}(x) \right) \left( \sum_{x \in \mathbb{Z}(1:1)} \gamma_{w, a}(x) \right).
\]

(6.4)

For the denominator in (6.3), we can get a rough lower bound by restricting (3.5) to the terms with \( h = 0, \ell = (0, \ldots, 0, \ell_s), \ell_s \neq 0 \) and \( \ell_s \equiv 0, \) where \( n = p^m. \) We obtain for any \( z_1, \ldots, z_s \in \mathbb{U}, \)

\[
T_{n, d, s, \alpha, \gamma}(z_1, \ldots, z_s) \geq 2^{2\zeta(2\alpha)} \gamma_{z_s} / n^m.
\]

(6.5)

Substituting (6.4) and (6.5) into (6.3), we conclude that the upper bound to

\[
X_{p, m_1, m_2, d, s, \alpha, \gamma}(z_1^{\text{emb}}, \ldots, z_{s-1}^{\text{emb}}, z_s^{\text{emb}})
\]

is of the order

\[
\left( \sum_{m_1 = 1}^{m_2} \frac{1}{\varphi(p^m)} \right)^{1/\lambda} = \left( \frac{p}{p-1} \sum_{m_1 = 1}^{m_2} (p^{\lambda-1})^m \right)^{1/\lambda},
\]

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where the order can be arbitrarily close to \((m_2 - m_1 + 1)^\alpha\) as \(\lambda\) approaches 1/\(\alpha\).

# 7 Numerical results

For different \(d\), \(\alpha\) and \(\gamma\), we will compare values of \(S_{n,d,\alpha,\gamma}(z)\) for vectors \(z\) obtained by Algorithm 3.2 for numbers of points \(n = 2^m\) with those for prime \(n\) in Subsection 7.1, as well as compare values of \(S_{n,d,\alpha,\gamma}(z)\) for embedded lattice rules obtained by Algorithm 6.1 with those for vectors \(z\) obtained by Algorithm 3.2 against numbers of points \(n = 2^m\) in Subsection 7.2.

We consider some special forms of weights which are motivated by applications in uncertainty quantification, see, e.g., [19, 13, 14, 28, 29, 8, 18].

1. For product weights [49, 50], there is a positive weight parameter \(\gamma_j\) associated with each coordinate variable \(x_j\), and

\[
\gamma_u = \prod_{j \in u} \gamma_j \quad \text{and} \quad \gamma_\emptyset = 1.
\]

2. For product and order dependent (POD) weights [13, 14, 28, 29], there are two sequences \(\{\gamma_j\}_{j \geq 1}\) and \(\{\Gamma_\ell\}_{\ell \geq 0}\) such that

\[
\gamma_u = \Gamma_{|u|} \prod_{j \in u} \gamma_j \quad \text{and} \quad \gamma_\emptyset = \Gamma_0 = 1.
\]

3. For smoothness-driven product and order dependent (SPOD) weights [8, 18], there is a smoothness degree \(\sigma \in \mathbb{N}\) and sequences \(\{\Gamma_\ell\}_{\ell \geq 0}\) and \(\{\gamma_j,\nu\}_{j \geq 1, 1 \leq \nu \leq \sigma}\) such that

\[
\gamma_u = \sum_{\nu_u \in \{1: \sigma\}^{|u|}} \Gamma_{|u|} \prod_{j \in u} \gamma_{j,\nu} \quad \text{and} \quad \gamma_\emptyset = \Gamma_0 = 1.
\]

In this section, we consider two different smoothness parameters \(\alpha \in \{2, 4\}\) and three different choices of weights:

(a) Product weights: \(\gamma_j = j^{-1.5\alpha}\);
(b) POD weights: \(\Gamma_\ell = \ell! / a^{\ell}, \quad \gamma_j = a j^{-1.5\alpha}\);
(c) SPOD weights: \(\sigma = \alpha / 2, \Gamma_\ell = \ell! / a^{\ell}, \quad \gamma_{j,\nu} = a (2j^{-1.5\alpha})^\nu;\)

with the re-scaling parameter \(a = (d!)^{1/d}\) for numerical stability. These weights were chosen in [6] to ensure that the implied constant in the error bound is independent of \(d\) and such that they are distinguishable in the plot.

## 7.1 Comparison between lattice rules constructed with prime numbers and composite numbers

In this subsection, we want to see whether the empirical values of the convergence rates of \(S_{n,d,\alpha,\gamma}(z)\) with \(z\) constructed for composite \(n\) will differ from those constructed for prime \(n\) in [6].

We use the fast CBC constructions developed in [6] for approximation based on \(S_{n,d,\alpha,\gamma}(z)\) together with the techniques from [45] for composite \(n\) to implement Algorithm 3.2. In Figure 1, we plot the values of \(S_{n,d,\alpha,\gamma}(z)\) against the number of points \(n = 2^m\) for \(m = 9, 10, \ldots, 17\), as well as the values of \(S_{n,d,\alpha,\gamma}(z)\) against the prime numbers of points for \(n \in \{503, 1009, 2003, 4001, 8009, 16007, 32003, 64007, 128021\}\).
According to Theorem 3.3, the theoretical rate of convergence of \( S_{n,d,\alpha,\gamma}(z) \) is \( O(\varphi(n)^{-\alpha+\delta}) \), \( \delta > 0 \). Table 2 lists the empirical rates of convergence \( O(n^{-r}) \) for the twelve groups of lines in Figure 1, where all entries are the values of \( r \).

Our key observation is that Figure 1, which is plotted against \( n \) being powers of 2, effectively coincides with the figure in [6] which was plotted against primes. This is consistent with Theorem 3.3. The empirical rates in Table 2 exhibit the expected trend between \( \alpha = 2 \) and \( \alpha = 4 \).

As in [6] we note that different values of \( d \) do not affect the empirical rates of convergence, which is consistent with Theorem 3.4. We remark that since the initial approximation error \( \max_{u \in \{1,d\}} \gamma^1_n \) may be different, the empirical values of \( S_{n,d,\alpha,\gamma}(z) \) vary with the dimension \( d \) and weight parameters, so the relative heights of the lines are irrelevant.

In summary, we have demonstrated that both the theory and construction of lattice algorithms for function approximation extend well from prime \( n \) to composite \( n \), and from product weights to more complicated forms of weights that arise from practical applications.
7.2 Comparison between embedded lattice rules and near-optimal lattice rules

In Figure 2 and 3, we plot the values of $S_{n,d,\alpha,\gamma}(z)$, of which $z_{\text{emb}}$ is constructed by Algorithm 6.1 for the range $m = 9, \ldots, 17$, as well as the values of $S_{n,d,\alpha,\gamma}(z^{(m)})$ with $z^{(m)}$ constructed by Algorithm 3.2 for comparison. Table 3 lists the empirical rates of convergence $O(n^{-r})$ for the twelve groups in Figure 2 and 3, where all entries are the values of $r$.

Given $d = 100$, Figure 4 shows the values of $X_{2,9,17,d,s,\alpha,\gamma}(z_{\text{emb}}^{1}, \ldots, z_{\text{emb}}^{s})$ against $s = 1, 2, \ldots, d$ for product weights, POD weights and SPOD weights with $\alpha = 2$ and $\alpha = 4$. The maxima of $X_{2,9,17,d,s,\alpha,\gamma}(z_{\text{emb}}^{1}, \ldots, z_{\text{emb}}^{s})$ over $1 \leq s \leq 100$ are shown in Table 4.

We observe from both Figure 2 and 3 that each group of lines of $S_{n,d,\alpha,\gamma}(z)$ for embedded lattice rules lie above $S_{n,d,\alpha,\gamma}(z^{(m)})$ for near-optimal lattice rules constructed by Algorithm 3.2. The groups for $\alpha = 2$ are much closer than those for $\alpha = 4$. The empirical rates in Table 3 of embedded lattice rules are close to those of near-optimal lattice rules and also exhibit the expected trend between $\alpha = 2$ and $\alpha = 4$.

As in [4], we see from Figure 4 that, given the quantity $d = 100$, $X_{2,9,17,d,s,\alpha,\gamma}(z_{\text{emb}}^{1}, \ldots, z_{\text{emb}}^{s})$ increase initially when $s$ increases, and then decrease from some dimensions and wiggle around some values onward. In our experiments, Table 4 shows that, in the case of $\alpha = 2$, $\max_{1 \leq s \leq 100} X_{2,9,17,d,s,\alpha,\gamma}(z_{\text{emb}}^{1}, \ldots, z_{\text{emb}}^{s})$ is at most 2.08, while in the case of $\alpha = 4$, $\max_{1 \leq s \leq 100} X_{2,9,17,d,s,\alpha,\gamma}(z_{\text{emb}}^{1}, \ldots, z_{\text{emb}}^{s})$ is at most 25.72.
Figure 3: The values of $S_{n,d,\alpha,\gamma}(\mathbf{z})$ for embedded lattice rules and near-optimal lattice rules against $n = 2^m$ for different weights with $\alpha = 4$. Each group includes 5 lines representing $d \in \{5, 10, 20, 50, 100\}$.

Table 3: Empirical convergence rates for the twelve groups in Figure 2 and 3

| Product weights | POD weights | SPOD weights |
|-----------------|-------------|--------------|
| $\alpha = 2$    | $\alpha = 4$| $\alpha = 4$|
| embedded        | 1.5         | 1.3          | 1.2         |
| near-optimal    | 1.5         | 1.3          | 1.2         |

Considering the worst-case $L_2$ error bound (1.4), Theorem 3.4 and (6.2) show that,

$$
e_{\text{wor-app}}^{p_m,d,M}(\mathbf{z}^{\text{emb}}; L_2) \leq \sqrt{2} \left[ S_{p_m,d,\alpha,\gamma}(\mathbf{z}^{\text{emb}}) \right]^{1/4}
\leq \left( \max_{s \in \{1, \ldots, d\}} X_{p,m_1,m_2,d,s,\alpha,\gamma}(z_1^{\text{emb}}, \ldots, z_s^{\text{emb}}) \right)^{1/4} \sqrt{2} \left[ S_{p_m,d,\alpha,\gamma}(\mathbf{z}^{(m)}) \right]^{1/4}
\lesssim \left( \frac{\max_{s \in \{1, \ldots, d\}} X_{p,m_1,m_2,d,s,\alpha,\gamma}(z_1^{\text{emb}}, \ldots, z_s^{\text{emb}})}{p_m} \right)^{1/4} \lambda^{-1/\alpha}$$

for all $\lambda \in (1/\alpha, 1]$.

where we used that $\varphi(p^m) = p^m(1 - \frac{1}{m})$. As $\lambda$ can be arbitrarily close to $1/\alpha$, $
\left( \max_{s \in \{1, \ldots, d\}} X_{p,m_1,m_2,d,s,\alpha,\gamma}(z_1^{\text{emb}}, \ldots, z_s^{\text{emb}}) \right)^{1/\alpha}$ indicates how many times more points are needed for $\mathbf{z}^{\text{emb}}$ obtained from Algorithm 6.1 to achieve the same error bound as $\mathbf{z}^{(m)}$ obtained from Algorithm 3.2 for each $n = p^m, m = m_1, \ldots, m_2$. In the case of $\alpha = 2$, the embedded lattice rules are at most 1.45 times worse than the near-optimal lattice rules obtained from Algorithm 3.2 for specific range of number of points; in the case of $\alpha = 4$, this factor increases to 2.26.

We can consider similarly for the worst-case error measured under $L_\infty$ norm, see Theorem 5.5. In the case of $\alpha = 2$, the embedded lattice rules are at most 1.45 times worse than the near-optimal lattice rules obtained from Algorithm 3.2 for the specific range of number
Figure 4: Given \( d = 100 \), the values of \( X_{2,9,17,d,s,\alpha,\gamma}(z_{1}^{\text{emb}}, \ldots, z_{s}^{\text{emb}}) \) against \( s = 1, 2, \ldots, d \) for different weights with \( \alpha = 2 \) (bottom three groups) and \( \alpha = 4 \) (top three groups).

| \( \alpha \) | product | POD | SPOD |
|-----------|---------|-----|------|
| 2         | 2.08    | 1.91| 1.85 |
| 4         | 23.88   | 25.72| 23.16|

Table 4: With \( d = 100 \), the maximum of \( X_{2,9,17,d,s,\alpha,\gamma}(z_{1}^{\text{emb}}, \ldots, z_{s}^{\text{emb}}) \) over \( s = 1, \ldots, d \) for different weights and \( \alpha \) in Figure 4.

of points; the same factor holds in the case of \( \alpha = 4 \) (in this case, we need to adjust the smoothness parameter to \( \tilde{\alpha} = \alpha/2 \) and weight parameters to \( \tilde{\gamma} = \sqrt{\gamma} \)).

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References

[1] T. M. Apostol, *Introduction to analytic number theory*, Springer-Verlag, New York, 1976.

[2] B. Alomair, A. Clark, R. Poovendran, *The power of primes: Security of authentication based on a universal hash-function family*, J. Math. Cryptol., 4 (2010), 121–148.

[3] G. Byrenheid, L. Kämmerer, T. Ullrich, T. Volkmer, *Tight error bounds for rank-1 lattice sampling in spaces of hybrid mixed smoothness*, Numer. Math., 136 (2017), 993–1034.

[4] R. Cools, F. Y. Kuo, D. Nuyens, *Constructing embedded lattice rules for multivariate integration*, SIAM J. Sci. Comput., 28 (2006), 2162–2188.
[5] R. Cools, F. Y. Kuo, D. Nuyens, I. H. Sloan, *Lattice algorithms for multivariate approximation in periodic spaces with general weights*, Contemp. Math., 754 (2020), 93–113.

[6] R. Cools, F. Y. Kuo, D. Nuyens, I. H. Sloan, *Fast CBC construction of lattice algorithms for multivariate approximation with POD and SPOD weights*, Math. Comput., 90 (2021), 787–812.

[7] R. Cools, D. Nuyens, *A Belgian view on lattice rules*, in: Monte Carlo and Quasi-Monte Carlo Methods 2006 (A. Keller, S. Heinrich, and H. Niederreiter, eds.), Springer, 2008, pp. 3–21.

[8] J. Dick, F. Y. Kuo, Q. T. Le Gia, Ch. Schwab, *Multilevel higher order QMC Petrov-Galerkin discretisation for affine parametric operator equations*, SIAM J. Numer. Anal., 54 (2016), 2541–2568.

[9] J. Dick, F. Y. Kuo, I. H. Sloan, *High-dimensional integration: the Quasi-Monte Carlo way*, Acta Numer., 22 (2013), 133–288.

[10] J. Dick, F. Pillichshammer, B. J. Waterhouse, *The construction of good extensible rank-1 lattices*, Math. Comput., 77 (2008), 2345–2373.

[11] J. Dick, I. H. Sloan, X. Wang, H. Woźniakowski, *Good lattice rules in weighted Korobov spaces with general weights*, Numer. Math., 103 (2006), 63–97.

[12] C. Gross, M. A. Iwen, L. Kämmerer, T. Volkmer, *A deterministic algorithm for constructing multiple rank-1 lattices of near-optimal size*, Adv. Comput. Math., 47 (2021), 86.

[13] I. G. Graham, F. Y. Kuo, J. A. Nichols, R. Scheichl, Ch. Schwab, I. H. Sloan, *Quasi-Monte Carlo finite element methods for elliptic PDEs with lognormal random coefficients*, Numer. Math., 131 (2015), 329–368.

[14] I. G. Graham, F. Y. Kuo, D. Nuyens, R. Scheichl, and I. H. Sloan, *Circulant embedding with QMC: analysis for elliptic PDE with lognormal coefficients*, Numer. Math., 140 (2018), 479–511.

[15] O. Grošek, Š. Purubský, *Coprime solutions to ax ≡ b (mod n)*, J. Math. Cryptol., 7 (2013), 217–224.

[16] F. J. Hickernell, H. S. Hong, P. L’Ecuyer and C. Lemieux, *Extensible lattice sequences for quasi-Monte Carlo quadrature*, SIAM J. Sci. Comput., 22 (2000), 1117–1138.

[17] F. J. Hickernell, H. Niederreiter, *The existence of good extensible rank-1 lattices*, J. Complexity, 19 (2003), 286–300.

[18] V. Kaarnioja, F. Y. Kuo, I. H. Sloan, *Uncertainty quantification using periodic random variables*, SIAM J. Numer. Anal., 58 (2020), 1068–1091.

[19] V. Kaarnioja, Y. Kazashi, F. Y. Kuo, F. Nobile, I. H. Sloan, *Fast approximation by periodic kernel-based lattice-point interpolation with application in uncertainty quantification*, Numer. Math., 150 (2022), 33–77.

[20] L. Kämmerer, *Multiple rank-1 lattices as sampling schemes for multivariate trigonometric polynomials*, J. Fourier. Anal., 24 (2018), 17–44.

[21] L. Kämmerer, *Constructing spatial discretizations for sparse multivariate trigonometric polynomials that allow for a fast discrete Fourier transform*, Appl. Comput. Harmon. Anal., 47 (2019), 702–729.

[22] L. Kämmerer, *Multiple lattice rules for multivariate $L_{\infty}$ approximation in the worst-case setting*, arXiv:1909.02290.

[23] L. Kämmerer, *Reconstructing hyperbolic cross trigonometric polynomials from sampling along rank-1 lattices*, SIAM J. Numer. Anal., 51 (2013), 2773–2796.
[24] L. Kämmerer, D. Potts, T. Volkmer, Approximation of multivariate periodic functions by trigonometric polynomials based on rank-1 lattice sampling, J. Complexity, 31 (2015), 543–576.
[25] D. Krieg, M. Ullrich, Function values are enough for $L_2$-approximation: Part II, J. Complexity, 66 (2021), 101569.
[26] L. Kämmerer, T. Volkmer, Approximation of multivariate periodic functions based on sampling along multiple rank-1 lattices, J. Approx. Theory, 246 (2019), 1–27.
[27] F. Y. Kuo, G. Migliorati, F. Nobile, D. Nuyens, Function integration, reconstruction and approximation using rank-1 lattices, Math. Comp., 90 (2021), 1861–1897.
[28] F. Y. Kuo, D. Nuyens, Application of quasi-Monte Carlo methods to elliptic PDEs with random diffusion coefficients – a survey of analysis and implementation, Found. Comput. Math., 16 (2016), 1631–1696.
[29] F. Y. Kuo, Ch. Schwab, I. H. Sloan, Quasi-Monte Carlo finite element methods for a class of elliptic partial differential equations with random coefficient, SIAM J. Numer. Anal., 50 (2012), 3351–3374.
[30] F. Y. Kuo, I. H. Sloan, H. Woźniakowski, Lattice rules for multivariate approximation in the worst case setting, in: Monte Carlo and Quasi-Monte Carlo Methods 2004 (H. Niederreiter and D. Talay, eds.), Springer, 2006, pp. 289–330.
[31] F. Y. Kuo, I. H. Sloan, H. Woźniakowski, Lattice rule algorithms for multivariate approximation in the average case setting, J. Complexity, 24 (2008), 283–323.
[32] F. Y. Kuo, G. W. Wasilkowski, H. Woźniakowski, Multivariate $L_\infty$ approximation in the worst case setting over reproducing kernel Hilbert spaces, J. Approx. Theory, 152 (2008), 135–160.
[33] F. Y. Kuo, G. W. Wasilkowski, H. Woźniakowski, On the power of standard information for multivariate approximation in the worst case setting, J. Approx. Theory, 158 (2009), 97–125.
[34] F. Y. Kuo, G. W. Wasilkowski, H. Woźniakowski, Lattice Algorithms for Multivariate $L_\infty$ Approximation in the Worst-Case Setting, Constr. Approx., 30 (2009), 475–493.
[35] F. Y. Kuo, G. W. Wasilkowski, H. Woźniakowski, Correction to: Lattice Algorithms for Multivariate $L_\infty$ Approximation in the Worst-Case Setting, Constr. Approx., 52 (2020), 177–179.
[36] P. L’Ecuyer, D. Munger, On figures of merit for randomly shifted lattice rules, in: Monte Carlo and Quasi-Monte Carlo Methods 2010 (L. Plaskota and H. Woźniakowski, eds.), Springer, 2012, pp. 133–159.
[37] C. Lemieux, Monte Carlo and Quasi-Monte Carlo Sampling, Springer, New York, 2009.
[38] G. Leobacher, F. Pillichshammer, Introduction to Quasi-Monte Carlo Integration and Applications, Springer, 2014.
[39] H. Niederreiter, Random Number Generation and Quasi-Monte Carlo Methods, SIAM, 1992.
[40] E. Novak, I. H. Sloan, H. Woźniakowski, Tractability of approximation for weighted Korobov spaces on classical and quantum computers, Found. Comput. Math. 4 (2004), 121–156.
[41] E. Novak, H. Woźniakowski, Tractability of Multivariate Problems, Volume I: Linear Information, EMS, Zürich, 2008.
[42] E. Novak, H. Woźniakowski, Tractability of Multivariate Problems, Volume II: Standard Information for Functionals, EMS, Zürich, 2010.
[43] E. Novak, H. Woźniakowski, Tractability of Multivariate Problems, Volume III: Standard Information for Operators, EMS, Zürich, 2012.
[44] D. Nuyens, *The construction of good lattice rules and polynomial lattice rules*, in: Uniform Distribution and Quasi-Monte Carlo Methods (P. Kritzer, H. Niederreiter, F. Pillichshammer, A. Winterhof, eds.), Radon Series on Computational and Applied Mathematics Vol. 15, De Gruyter, 2014, pp. 223–256.

[45] D. Nuyens, R. Cools, *Fast component-by-component construction of rank-1 lattice rules with a non-prime number of points*, J. Complexity, 22 (2006), 4–28.

[46] N. Nagel, M. Schäfer, T. Ullrich, *A new upper bound for sampling numbers*, Found. Comput. Math., (2021), 1–24.

[47] D. Potts, T. Volkmer, *Sparse high-dimensional FFT based on rank-1 lattice sampling*, Appl. Comput. Harmon. Anal., 41 (2016), 713–748.

[48] I. H. Sloan, S. Joe, *Lattice Methods for Multiple Integration*, Oxford University Press, Oxford, 1994.

[49] I. H. Sloan, H. Woźniakowski, *When are quasi-Monte Carlo algorithms efficient for high-dimensional integrals?*, J. Complexity, 14 (1998), 1–33.

[50] I. H. Sloan, H. Woźniakowski, *Tractability of multivariate integration for weighted Korobov classes*, J. Complexity, 17 (2001), 697–721.

[51] G. Wahba, *Spline Models for Observational Data*, SIAM, Philadelphia, 1990.

[52] X. Y. Zeng, K. T. Leung, F. J. Hickernell, *Error analysis of splines for periodic problems using lattice designs*, in: Monte Carlo and Quasi-Monte Carlo Methods 2004 (H. Niederreiter and D. Talay, eds), Springer, 2006, pp. 501–514.

[53] X. Y. Zeng, P. Kritzer, F. J. Hickernell, *Spline methods using integration lattices and digital nets*, Constr. Approx., 30 (2009), 529–555.