THE SCHRÖDINGER-WEIL REPRESENTATION AND JACOBI FORMS OF HALF-INTEGRAL WEIGHT

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Abstract. In this paper, we define the concept of Jacobi forms of half-integral weight using Takase’s automorphic factor of weight $1/2$ for a two-fold covering group of the symplectic group on the Siegel upper half plane and find covariant maps for the Schrödinger-Weil representation. Using these covariant maps, we construct Jacobi forms of half integral weight with respect to an arithmetic subgroup of the Jacobi group.

1. Introduction

For a given fixed positive integer $n$, we let

$$\mathbb{H}_n = \{ \Omega \in \mathbb{C}^{(n,n)} \mid \Omega = ^t\Omega, \quad \text{Im} \Omega > 0 \}$$

be the Siegel upper half plane of degree $n$ and let

$$\text{Sp}(n, \mathbb{R}) = \{ g \in \mathbb{R}^{(2n,2n)} \mid ^t g J_n g = J_n \}$$

be the symplectic group of degree $n$, where $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring $F$ for two positive integers $k$ and $l$, $^tM$ denotes the transposed matrix of a matrix $M$, $\text{Im} \Omega$ denotes the imaginary part of $\Omega$ and

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

We see that $\text{Sp}(n, \mathbb{R})$ acts on $\mathbb{H}_n$ transitively by

$$g \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1},$$

where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(n, \mathbb{R})$ and $\Omega \in \mathbb{H}_n$.

For two positive integers $n$ and $m$, we consider the Heisenberg group

$$H^{(n,m)}_{\mathbb{R}} = \{ (\lambda, \mu; \kappa) \mid \lambda, \mu \in \mathbb{R}^{(m,n)}, \kappa \in \mathbb{R}^{(m,m)}, \quad \kappa + \mu^t \lambda \text{ symmetric} \}$$

endowed with the following multiplication law

$$(\lambda, \mu; \kappa) \circ (\lambda', \mu'; \kappa') = (\lambda + \lambda', \mu + \mu'; \kappa + \kappa' + \lambda^t \mu' - \mu^t \lambda').$$

We let

$$G^J = \text{Sp}(n, \mathbb{R}) \ltimes H^{(n,m)}_{\mathbb{R}} \quad \text{(semi-direct product)}$$

be the Jacobi group endowed with the following multiplication law

$$(g, (\lambda, \mu; \kappa)) \cdot (g', (\lambda', \mu'; \kappa')) = (gg', (\tilde{\lambda} + \lambda', \tilde{\mu} + \mu'; \kappa + \kappa' + \tilde{\lambda}^t \mu' - \tilde{\mu}^t \lambda')).$$

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with \( g, g' \in \text{Sp}(n, \mathbb{R}), (\lambda, \mu; \kappa), (\lambda', \mu'; \kappa') \in H^{(n,m)}_\mathbb{R} \) and \((\tilde{\lambda}, \tilde{\mu}) = (\lambda, \mu)g'\). We let \( \Gamma_n = \text{Sp}(n, \mathbb{Z}) \) be the Siegel modular group of degree \( n \). We let

\[
\Gamma^J = \Gamma_n \ltimes H^{(n,m)}_\mathbb{Z}
\]

be the Jacobi modular group. Then we have the natural action of \( G^J \) on the Siegel-Jacobi space \( \mathbb{H}_{n,m} := \mathbb{H}_n \times \mathbb{C}^{(m,n)} \) defined by

\[
(g, (\lambda, \mu; \kappa)) \cdot (\Omega, Z) = \left( g \cdot \Omega, (Z + \lambda \Omega + \mu)(C\Omega + D)^{-1} \right),
\]

where \( g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(n, \mathbb{R}), (\lambda, \mu; \kappa) \in H^{(n,m)}_\mathbb{R} \) and \((\Omega, Z) \in \mathbb{H}_{n,m}\). We refer to \([27]-[34]\) for more details on materials related to the Siegel-Jacobi space.

The Weil representation for the symplectic group was first introduced by A. Weil in \([21]\) to reformulate Siegel’s analytic theory of quadratic forms (cf. \([16]\)) in terms of the group theoretical theory. It is well known that the Weil representation plays a central role in the study of the transformation behaviors of the theta series. Whenever we study the transformation formulas of theta series or Siegel modular forms of half integral weights, we are troubled by the ambiguity of the factor \( \det(C\Omega + D)^{1/2} \) in its signature. This means that we should consider the transformation formula on a non-trivial two-fold covering group of the symplectic group. In his paper \([19]\), Takase removed the ambiguity of \( \det(C\Omega + D)^{1/2} \) by constructing the right explicit automorphic factor \( J_{1/2} \) of weight \( 1/2 \) for \( \text{Sp}(n, \mathbb{R})_* \) on \( \mathbb{H}_n \):

\[
J_{1/2} : \text{Sp}(n, \mathbb{R})_* \times \mathbb{H}_n \longrightarrow \mathbb{C}^*.
\]

Here \( \text{Sp}(n, \mathbb{R})_* \) is the two-fold covering group of \( \text{Sp}(n, \mathbb{R}) \) in the sense of the real Lie group. See (4.14) for the precise definition. \( J_{1/2} \) is real analytic on \( \text{Sp}(n, \mathbb{R})_* \), holomorphic on \( \mathbb{H}_n \) and satisfies the relation

\[
J_{1/2}(g, \Omega)^2 = \det(C\Omega + D),
\]

where \( g = (g, c) \in \text{Sp}(n, \mathbb{R})_* \) with \( g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(n, \mathbb{R}) \). Using the automorphic factor \( J_{1/2} \) of weight \( 1/2 \), he expressed the transformation formula of theta series without ambiguity of \( \det(C\Omega + D)^{1/2} \). Moreover he decomposed the automorphic factor \( j(\gamma, \Omega) = \vartheta(\gamma \cdot \Omega)/\vartheta(\Omega) \) with a standard theta series \( \vartheta(\Omega) \) into a product of a character and the automorphic factor \( J_{1/2}(\gamma, \Omega) \). The automorphic factor \( J_{1/2} \) of Takase will play an important role in the further study of half-integral weight Siegel modular forms and half-integral weight Jacobi forms in the future.

This paper is organized as follows. In Section 2, we discuss the Schrödinger representation of the Heisenberg group \( H^{(n,m)}_\mathbb{R} \) associated with a symmetric nonzero real matrix of degree \( m \) which is formulated in \([22],[23]\). In Section 3, we define the Schrödinger-Weil representation \( \omega_\mathcal{M} \) of the Jacobi group \( G^J \) associated with a symmetric positive definite matrix \( \mathcal{M} \) and provide some of the actions of \( \omega_\mathcal{M} \) on the representation space \( L^2(\mathbb{R}^{(m,n)}) \) explicitly. In Section 4, we review Jacobi forms of integral weight, Siegel modular forms of half-integral weight, and define Jacobi forms of half-integral weight using the automorphic factor \( J_{1/2} \) of weight \( 1/2 \) for the metaplectic group \( \text{Sp}(n, \mathbb{R})_* \) on the Siegel upper half plane. In Section 5, we find the covariant maps for the Schrödinger-Weil representation \( \omega_\mathcal{M} \). In the final section
we construct Jacobi forms of half-integral weight with respect to an arithmetic subgroup of \( \Gamma' \) using the covariant maps obtained in Section 5.

**Notations:** We denote by \( \mathbb{Z} \) and \( \mathbb{C} \) the ring of integers, and the field of complex numbers respectively. \( \mathbb{C}^\times \) denotes the multiplicative group of nonzero complex numbers. \( T \) denotes the multiplicative group of complex numbers of modulus one. The symbol “:=” means that the expression on the right is the definition of that on the left. For two positive integers \( k \) and \( l \), \( F^{(k,l)} \) denotes the set of all \( k \times l \) matrices with entries in a commutative ring \( F \). For a square matrix \( A \in F^{(k,k)} \) of degree \( k \), \( \sigma(A) \) denotes the trace of \( A \). For any \( M \in F^{(k,l)} \), \( tM \) denotes the transposed matrix of \( M \). \( I_n \) denotes the identity matrix of degree \( n \). We put \( i = \sqrt{-1} \). For \( z \in \mathbb{C} \), we define \( z^{1/2} = \sqrt{z} \) so that \(-\pi/2 < \arg(z^{1/2}) \leq \pi/2 \). Further we put \( z^{n/2} = (z^{1/2})^\kappa \) for every \( \kappa \in \mathbb{Z} \).

## 2. The Schrödinger Representation of \( H^{(n,m)}_{\mathbb{R}} \)

First of all, we observe that \( H^{(n,m)}_{\mathbb{R}} \) is a 2-step nilpotent Lie group. The inverse of an element \( (\lambda, \mu; \kappa) \in H^{(n,m)}_{\mathbb{R}} \) is given by

\[
(\lambda, \mu; \kappa)^{-1} = (-\lambda, -\mu; -\kappa + \lambda^t \mu - \mu^t \lambda).
\]

Now we set

\[
[\lambda, \mu; \kappa] = (0, \mu; \kappa) \circ (\lambda, 0; 0) = (\lambda, \mu; \kappa - \mu^t \lambda).
\]

Then \( H^{(n,m)}_{\mathbb{R}} \) may be regarded as a group equipped with the following multiplication

\[
[\lambda, \mu; \kappa] \circ [\lambda_0, \mu_0; \kappa_0] = [\lambda + \lambda_0, \mu + \mu_0; \kappa + \kappa_0 + \lambda^t \mu_0 + \mu_0^t \lambda].
\]

The inverse of \( [\lambda, \mu; \kappa] \in H^{(n,m)}_{\mathbb{R}} \) is given by

\[
[\lambda, \mu; \kappa]^{-1} = [-\lambda, -\mu; \kappa + \lambda^t \mu + \mu^t \lambda].
\]

We set

\[
L = \left\{ [0, \mu; \kappa] \in H^{(n,m)}_{\mathbb{R}} \middle| \mu \in \mathbb{R}^{(m,n)}, \kappa = \kappa^t \in \mathbb{R}^{(m,m)} \right\}.
\]

Then \( L \) is a commutative normal subgroup of \( H^{(n,m)}_{\mathbb{R}} \). Let \( \hat{L} \) be the Pontrjagin dual of \( L \), i.e., the commutative group consisting of all unitary characters of \( L \). Then \( \hat{L} \) is isomorphic to the additive group \( \mathbb{R}^{(m,n)} \times \text{Symm}(m, \mathbb{R}) \) via

\[
\langle a, \hat{a} \rangle = e^{2\pi \imath \sigma(\mu^t \kappa + \bar{\kappa^t} \mu)}, \quad a = [\lambda, 0; \kappa] \in L, \quad \hat{\lambda} = (\hat{\mu}, \hat{\kappa}) \in \hat{L},
\]

where \( \text{Symm}(m, \mathbb{R}) \) denotes the space of all symmetric \( m \times m \) real matrices.

We put

\[
S = \left\{ [\lambda, 0; 0] \in H^{(n,m)}_{\mathbb{R}} \middle| \lambda \in \mathbb{R}^{(m,n)} \right\} \cong \mathbb{R}^{(m,n)}.
\]

Then \( S \) acts on \( L \) as follows:

\[
\alpha_\lambda([0, \mu; \kappa]) = [0, \mu; \kappa + \lambda^t \mu + \mu^t \lambda], \quad [\lambda, 0, 0] \in S.
\]

We see that the Heisenberg group \( \left( H^{(n,m)}_{\mathbb{R}}, \circ \right) \) is isomorphic to the semi-direct product \( S \ltimes L \) of \( S \) and \( L \) whose multiplication is given by

\[
(\lambda, a) \cdot (\lambda_0, a_0) = (\lambda + \lambda_0, a + \alpha_\lambda(a_0)), \quad \lambda, \lambda_0 \in S, \ a, a_0 \in L.
\]
On the other hand, $S$ acts on $\hat{L}$ by

$$\alpha^*_\lambda(\hat{a}) = (\hat{\mu} + 2\hat{\kappa}, \hat{\kappa}), \quad [\lambda, 0; 0] \in S, \quad a = (\hat{\mu}, \hat{\kappa}) \in \hat{L}.$$ 

Then, we have the relation $\langle \alpha_\lambda(a), \hat{a} \rangle = (a, \alpha^*_\lambda(\hat{a}))$ for all $a \in L$ and $\hat{a} \in \hat{L}$.

We have three types of $S$-orbits in $\hat{L}$.

**TYPE I.** Let $\hat{\kappa} \in \text{Symm}(m, \mathbb{R})$ be nondegenerate. The $S$-orbit of $\hat{a}(\hat{\kappa}) = (0, \hat{\kappa}) \in \hat{L}$ is given by

$$\hat{O}_{\hat{\kappa}} = \left\{ (2\hat{\kappa} \lambda, \hat{\kappa}) \in \hat{L} \mid \lambda \in \mathbb{R}^{(m,n)} \right\} \cong \mathbb{R}^{(m,n)}.$$

**TYPE II.** Let $(\hat{\mu}, \hat{\kappa}) \in \mathbb{R}^{(m,n)} \times \text{Symm}(m, \mathbb{R})$ with degenerate $\hat{\kappa} \neq 0$. Then

$$\hat{O}_{(\hat{\mu}, \hat{\kappa})} = \left\{ (\hat{\mu} + 2\hat{\kappa} \lambda, \hat{\kappa}) \mid \lambda \in \mathbb{R}^{(m,n)} \right\} \cong \mathbb{R}^{(m,n)} \times \{ \hat{\kappa} \}.$$

**TYPE III.** Let $\hat{y} \in \mathbb{R}^{(m,n)}$. The $S$-orbit $\hat{O}_{\hat{y}}$ of $\hat{a}(\hat{y}) = (\hat{y}, 0)$ is given by

$$\hat{O}_{\hat{y}} = \{ (\hat{y}, 0) \} = \hat{a}(\hat{y}).$$

We have

$$\hat{L} = \bigcup_{\hat{\kappa} \in \text{Symm}(m, \mathbb{R}) \backslash \kappa \text{ nondegenerate}} \hat{O}_{\hat{\kappa}} \bigcup_{\hat{y} \in \mathbb{R}^{(m,n)}} \left( \hat{O}_{\hat{y}} \bigcup_{(\hat{\mu}, \hat{\kappa}) \in \mathbb{R}^{(m,n)} \times \text{Symm}(m, \mathbb{R}) \backslash \hat{\kappa} \neq 0 \text{ degenerate}} \hat{O}_{(\hat{\mu}, \hat{\kappa})} \right)$$

as a set. The stabilizer $S_{\hat{\kappa}}$ of $S$ at $\hat{a}(\hat{\kappa}) = (0, \hat{\kappa})$ is given by

$$S_{\hat{\kappa}} = \{ 0 \}.$$

And the stabilizer $S_{\hat{y}}$ of $S$ at $\hat{a}(\hat{y}) = (\hat{y}, 0)$ is given by

$$S_{\hat{y}} = \left\{ [\lambda, 0; 0] \mid \lambda \in \mathbb{R}^{(m,n)} \right\} = S \cong \mathbb{R}^{(m,n)}.$$

In this section, for the present being we set $H = H_{(n,m)}^R$ for brevity. We see that $L$ is a closed, commutative normal subgroup of $H$. Since $(\lambda, \mu; \kappa) = (0, \mu; \kappa + \mu^t \lambda) \circ (\lambda, 0; 0)$ for $(\lambda, \mu; \kappa) \in H$, the homogeneous space $X = L \backslash H$ can be identified with $\mathbb{R}^{(m,n)}$ via

$$Lh = L \circ (\lambda, 0; 0) \mapsto \lambda, \quad h = (\lambda, \mu; \kappa) \in H.$$

We observe that $H$ acts on $X$ by

$$(Lh) \cdot h_0 = L(\lambda + \lambda_0, 0; 0) = (\lambda + \lambda_0),$$

where $h = (\lambda, \mu; \kappa) \in H$ and $h_0 = (\lambda_0, \mu_0; 0) \in H$.

If $h = (\lambda, \mu; \kappa) \in H$, we have

$$l_h = (0, \mu; \kappa + \mu^t \lambda), \quad s_h = (\lambda, 0; 0)$$

in the Mackey decomposition of $h = l_h \circ s_h$ (cf. [9]). Thus if $h_0 = (\lambda_0, \mu_0; 0) \in H$, then we have

$$s_h \circ h_0 = (\lambda_0, 0; 0) \circ (\lambda_0, \mu_0; 0) = (\lambda + \lambda_0, \mu_0; 0 + \lambda^t \mu_0)$$

and so

$$s_h \circ h_0 = (\lambda_0, 0; 0) \circ (\lambda_0, \mu_0; 0) = (\lambda + \lambda_0, \mu_0; \kappa_0 + \lambda^t \mu_0)$$

and

$$l_{s_h \circ h_0} = (0, \mu_0; \kappa + \mu_0 \lambda_0 + \lambda_0^t \mu_0 + \mu_0^t \lambda).$$
For a real symmetric matrix \( c = t^tc \in \text{Symm}(m, \mathbb{R}) \) with \( c \neq 0 \), we consider the unitary character \( \chi_c \) of \( L \) defined by
\[
\chi_c((0, \mu; \kappa)) = e^{\pi i \sigma(\kappa)} I, \quad (0, \mu; \kappa) \in L,
\]
where \( I \) denotes the identity mapping. Then the representation \( \mathcal{W}_c = \text{Ind}^H_{L} \chi_c \) of \( H \) induced from \( \chi_c \) is realized on the Hilbert space \( H(\chi_c) = L^2(X, d\hbar, \mathbb{C}) \cong L^2(\mathbb{R}^{m,n}), d\xi \) as follows. If \( h_0 = (\lambda_0, \mu_0; \kappa_0) \in H \) and \( x = Lh \in X \) with \( h = (\lambda, \mu; \kappa) \in H \), we have
\[
(\mathcal{W}_c(h_0)f)(x) = \chi_c(l_{shh_0}) (f(xh_0)), \quad f \in H(\chi_c).
\]
It follows from (2.1) that
\[
(\mathcal{W}_c(h_0)f)(\lambda) = e^{\pi i \sigma\{c(\kappa_0 + \mu_0)^2, \lambda_0 + 2\lambda^T \mu_0\}} f(\lambda + \lambda_0),
\]
where \( h_0 = (\lambda_0, \mu_0; \kappa_0) \in H \) and \( \lambda \in \mathbb{R}^{m,n} \). Here we identified \( x = Lh \) (resp. \( xh_0 = Lhh_0 \)) with \( \lambda \) (resp. \( \lambda + \lambda_0 \)). The induced representation \( \mathcal{W}_c \) is called the Schrödinger representation of \( H \) associated with \( \chi_c \). Thus \( \mathcal{W}_c \) is a monomial representation.

**Theorem 2.1.** Let \( c \) be a positive definite symmetric real matrix of degree \( m \). Then the Schrödinger representation \( \mathcal{W}_c \) of \( H \) is irreducible.

**Proof.** The proof can be found in [22], Theorem 3. \( \square \)

**Remark.** We refer to [22]-[26] for more representations of the Heisenberg group \( H_{\mathbb{R}}^{(n,m)} \) and their related topics.

### 3. The Schrödinger-Weil Representation

Throughout this section we assume that \( \mathcal{M} \) is a symmetric integral positive definite \( m \times m \) matrix. We consider the Schrödinger-Weil representation \( \mathcal{W}_M \) of the Heisenberg group \( H_{\mathbb{R}}^{(n,m)} \) with the central character \( \mathcal{W}_M((0, 0; \kappa)) = \chi_M((0, 0; \kappa)) = e^{\pi i \sigma(\mathcal{M} \kappa)}, \kappa \in \text{Symm}(m, \mathbb{R}) \) (cf. (2.2)). We note that the symplectic group \( \text{Sp}(n, \mathbb{R}) \) acts on \( H_{\mathbb{R}}^{(n,m)} \) by conjugation inside \( G^J \). For a fixed element \( g \in \text{Sp}(n, \mathbb{R}) \), the irreducible unitary representation \( \mathcal{W}_M^g \) of \( H_{\mathbb{R}}^{(n,m)} \) defined by
\[
(\mathcal{W}_M^g(h))(\lambda) = e^{\pi i \sigma(\mathcal{M} \kappa)} \text{Id}_{H(\chi_M)}, \quad \kappa \in \text{Symm}(m, \mathbb{R})
\]
has the property that
\[
(\mathcal{W}_M^g((0, 0; \kappa))) = \mathcal{W}_M((0, 0; \kappa)) = e^{\pi i \sigma(\mathcal{M} \kappa)} \text{Id}_{H(\chi_M)}, \quad \kappa \in \text{Symm}(m, \mathbb{R})
\]
Here \( \text{Id}_{H(\chi_M)} \) denotes the identity operator on the Hilbert space \( H(\chi_M) \). According to Stone-von Neumann theorem, there exists a unitary operator \( R_M(g) \) on \( H(\chi_M) \) such that \( R_M(gh) = R_M^g(h)R_M(g) \) for all \( h \in H_{\mathbb{R}}^{(n,m)} \). We observe that \( R_M(g) \) is determined uniquely up to a scalar of modulus one. From now on, for brevity, we put \( G = \text{Sp}(n, \mathbb{R}) \). According to Schur’s lemma, we have a map \( c_M : G \times G \rightarrow T \) satisfying the relation
\[
R_M(g_1 g_2) = c_M(g_1, g_2)R_M(g_1)R_M(g_2) \quad \text{for all } g_1, g_2 \in G.
\]
Therefore $R_M$ is a projective representation of $G$ on $H(\chi_M)$ and $c_M$ defines the cocycle class in $H^2(G, T)$. The cocycle $c_M$ yields the central extension $G_M$ of $G$ by $T$. The group $G_M$ is a set $G \times T$ equipped with the following multiplication

$$(g_1, t_1) \cdot (g_2, t_2) = (g_1 g_2, t_1 t_2 c_M(g_1, g_2)^{-1/m}), \quad g_1, g_2 \in G, \; t_1, t_2 \in T.$$ 

We see immediately that the map $\widetilde{R}_M : G_M \longrightarrow GL(H(\chi_M))$ defined by

$$(3.2) \quad \widetilde{R}_M(g, t) = t^m R_M(g) \quad \text{for all} \quad (g, t) \in G_M$$

is a true representation of $G_M$. As in Section 1.7 in [8], we can define the map $s_M : G \longrightarrow T$ satisfying the relation

$$c_M(g_1, g_2)^2 = s_M(g_1)^{-1} s_M(g_2)^{-1} s_M(g_1 g_2) \quad \text{for all} \quad g_1, g_2 \in G.$$

Thus we see that

$$(3.3) \quad G_{2,M} = \left\{ (g, t) \in G_M \mid t^2 = s_M(g)^{-1/m} \right\}$$

is the metaplectic group associated with $M$ that is a two-fold covering group of $G$. The restriction $R_{2,M}$ of $\widetilde{R}_M$ to $G_{2,M}$ is the Weil representation of $G$ associated with $M$. Now we define the projective representation $\pi_M$ of the Jacobi group $G^J$ by

$$(3.4) \quad \pi_M(hg) = \mathcal{W}_M(h) R_M(g), \quad h \in H^{(n,m)}_\mathbb{R}, \quad g \in G.$$

The projective representation $\pi_M$ of $G^J$ is naturally extended to the true representation $\omega_M$ of the group $G'_{2,M} = G_{2,M} \ltimes H^{(n,m)}_\mathbb{R}$. The representation $\omega_M$ is called the Schrödinger-Weil representation of $G'$. Indeed we have

$$(3.5) \quad \omega_M(h \cdot (g, t)) = t^m \mathcal{W}_M(h) R_M(g), \quad h \in H^{(n,m)}_\mathbb{R}, \quad (g, t) \in G_{2,M}.$$
generate the group $G_M \ltimes H^{(n,m)}_\mathbb{R}$. We can show that the representation $\tilde{R}_M$ is realized on the representation $H(\chi_M) = L^2(\mathbb{R}^{(m,n)})$ as follows: for each $f \in L^2(\mathbb{R}^{(m,n)})$ and $x \in \mathbb{R}^{(m,n)}$, the actions of $\tilde{R}_M$ on the generators are given by

\begin{equation}
(3.6) \quad \left( \tilde{R}_M(h_t(\lambda, \mu; \kappa)) f \right)(x) = t^m e^{\pi i \sigma(\lambda + 2x^t \mu)} f(x + \lambda),
\end{equation}

\begin{equation}
(3.7) \quad \left( \tilde{R}_M(t_M(b; t)) f \right)(x) = t^m e^{\pi i \sigma(\lambda b^t x)} f(x),
\end{equation}

\begin{equation}
(3.8) \quad \left( \tilde{R}_M(g_M(\alpha; t)) f \right)(x) = t^m (\det \alpha)^{\frac{m}{2}} f(x^t \alpha),
\end{equation}

\begin{equation}
(3.9) \quad \left( \tilde{R}_M(\sigma_n,M;t) f \right)(x) = t^m \left( \frac{1}{i} \right)^{\frac{nm}{2}} (\det M)^{\frac{n}{2}} \int_{\mathbb{R}^{(m,n)}} f(y) e^{-2 \pi i \sigma(M y^t x)} dy.
\end{equation}

We denote by $L^2_+ (\mathbb{R}^{(m,n)})$ (resp. $L^2_- (\mathbb{R}^{(m,n)})$) the subspace of $L^2(\mathbb{R}^{(m,n)})$ consisting of even (resp. odd) functions in $L^2(\mathbb{R}^{(m,n)})$. According to Formulas (3.7)-(3.9), $R_{2,M}$ is decomposed into representations of $R_{2,M}^\pm$

$$R_{2,M} = R_{2,M}^+ \oplus R_{2,M}^-,$$

where $R_{2,M}^+$ and $R_{2,M}^-$ are the even Weil representation and the odd Weil representation of $G$ that are realized on $L^2_+ (\mathbb{R}^{(m,n)})$ and $L^2_- (\mathbb{R}^{(m,n)})$ respectively. Obviously the center $Z_{2,M}^J$ of $G_{2,M}^J$ is given by

$$Z_{2,M}^J = \{ \{(I_{2n}, 1), (0, 0; \kappa) \} \in G_{2,M}^J \} \cong \text{Symm}(m, \mathbb{R}).$$

We note that the restriction of $\omega_M$ to $G_{2,M}$ coincides with $R_{2,M}$ and $\omega_M(h) = \omega_M(h)$ for all $h \in H^{(n,m)}_\mathbb{R}$.

**Remark 3.1.** In the case $n = m = 1$, $\omega_M$ is dealt in [1] and [10]. We refer to [3] and [7] for more details about the Weil representation $R_{2,M}$.

**Remark 3.2.** The Schrödinger-Weil representation is applied usefully to the theory of Jacobi’s theta sum [10] and the theory of Maass-Jacobi forms [12].

### 4. Jacobi Forms of Half-Integral Weight

Let $\rho$ be a rational representation of $GL(n, \mathbb{C})$ on a finite dimensional complex vector space $V_\rho$. Let $\mathcal{M} \in \mathbb{R}^{(m,m)}$ be a symmetric half-integral semi-positive definite matrix of degree $m$. Let $C^\infty(\mathbb{H}_{n,m}, V_\rho)$ be the algebra of all $C^\infty$ functions on $\mathbb{H}_{n,m}$ with values in $V_\rho$. 
For \( f \in C^\infty(\mathbb{H}_{n,m}, V_p) \), we define
\[
(f|_{\rho,\mathcal{M}}[(g, (\lambda, \mu; \kappa))])(\Omega, Z)
\]
(4.1) \[= e^{-2\pi i \sigma(\mathcal{M}(Z+\lambda\Omega+\mu))(C\Omega+D)^{-1}C(Z+\lambda\Omega+\mu)} \times e^{2\pi i \sigma(\mathcal{M}(\lambda \Omega \cdot 2 \lambda'Z + \kappa' \lambda))} \times \rho(C\Omega+D)^{-1}f(g\cdot \Omega, (Z + \lambda\Omega + \mu)(C\Omega+D)^{-1}),
\]
where \( g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(n, \mathbb{R}), \ (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)} \) and \((\Omega, Z) \in \mathbb{H}_{n,m} \).

**Definition 4.1.** Let \( \rho \) and \( \mathcal{M} \) be as above. Let
\[
H_{Z}^{(n,m)} = \{ (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)} \mid \lambda, \mu \in \mathbb{Z}^{(m,n)}, \ \kappa \in \mathbb{Z}^{(m,m)} \}.
\]
A Jacobi form of index \( \mathcal{M} \) with respect to \( \rho \) on a subgroup \( \Gamma \) of \( \Gamma_n \) of finite index is a holomorphic function \( f \in C^\infty(\mathbb{H}_{n,m}, V_p) \) satisfying the following conditions (A) and (B):

(A) \( f|_{\rho,\mathcal{M}}[\bar{\gamma}] = f \) for all \( \bar{\gamma} \in \Gamma \times H_{Z}^{(n,m)}. \)

(B) For each \( \mathcal{M} \in \Gamma_n \), \( f|_{\rho,\mathcal{M}}[\mathcal{M}] \) has a Fourier expansion of the following form :
\[
(f|_{\rho,\mathcal{M}}[\mathcal{M}])((\Omega, Z)) = \sum_{T = \frac{1}{2} \mathbb{Z} \cap \mathbb{R} \geq 0} \sum_{R \in \mathbb{Z}^{(n,m)}} c(T, R) \cdot e^{\frac{2\pi i}{\lambda_T} \sigma(T\Omega)} \cdot e^{2\pi i \sigma(RZ)}
\]
with a suitable \( \lambda_T \in \mathbb{Z} \) and \( c(T, R) \neq 0 \) only if \( \left( \frac{1}{\lambda_T} T \quad \frac{1}{2} R \right) \mathcal{M} \geq 0. \)

If \( n \geq 2 \), the condition (B) is superfluous by Koecher principle (cf. [36] Lemma 1.6). We denote by \( J_{\rho,\mathcal{M}}(\Gamma) \) the vector space of all Jacobi forms of index \( \mathcal{M} \) with respect to \( \rho \) on \( \Gamma \). Ziegler (cf. [36] Theorem 1.8 or [2] Theorem 1.1) proves that the vector space \( J_{\rho,\mathcal{M}}(\Gamma) \) is finite dimensional. In the special case \( \rho(A) = (\det(A))^k \) with \( A \in \text{GL}(n, \mathbb{C}) \) and a fixed \( k \in \mathbb{Z} \), we write \( J_{k,\mathcal{M}}(\Gamma) \) instead of \( J_{\rho,\mathcal{M}}(\Gamma) \) and call \( k \) the weight of the corresponding Jacobi forms. For more results on Jacobi forms with \( n > 1 \) and \( m > 1 \), we refer to [27]-[30] and [36].

**Definition 4.2.** A Jacobi form \( f \in J_{\rho,\mathcal{M}}(\Gamma) \) is said to be a cusp (or cuspidal) form if
\[
\left( \frac{1}{\lambda_T} T \quad \frac{1}{2} R \right) > 0 \text{ for any } T, R \text{ with } c(T, R) \neq 0.
\]
A Jacobi form \( f \in J_{\rho,\mathcal{M}}(\Gamma) \) is said to be singular if it admits a Fourier expansion such that a Fourier coefficient \( c(T, R) \) vanishes unless \( \det \left( \frac{1}{\lambda_T} T \quad \frac{1}{2} R \right) \mathcal{M} \) = 0.

Singular Jacobi forms were characterized by a certain differential operator and the weight by the author [29].

Without loss of generality we may assume that \( \rho \) is irreducible. Then we choose a hermitian inner product \( \langle , \rangle \) on \( V_p \) that is preserved under the unitary group \( U(n) \subset \text{GL}(n, \mathbb{C}) \). For two Jacobi forms \( f_1 \) and \( f_2 \) in \( J_{\rho,\mathcal{M}}(\Gamma) \), we define the Petersson inner product formally by
\[
\langle f_1, f_2 \rangle := \int_{\Gamma_n \backslash \mathbb{H}_{n,m}} \langle \rho(Y^{1/2})f_1(\Omega, Z), \rho(Y^{1/2})f_2(\Omega, Z) \rangle \kappa_{\mathcal{M}}(\Omega, Z) \ dv,
\]
(4.2)
For brevity, we set \( (\Omega) = \det(\Omega^{1/2} \Omega^{1/2}) \). A Jacobi form \( f \) in \( J_{\rho,M}(\Gamma) \) is said to be square integrable if \( \langle f, f \rangle < \infty \). We note that cusp Jacobi forms are square integrable and that \( \langle f_1, f_2 \rangle \) is finite if one of \( f_1 \) and \( f_2 \) is a cusp Jacobi form (cf. [36], p. 203).

For \( g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G \), we set
\[
J(g, \Omega) = C\Omega + D, \quad \Omega \in \mathbb{H}_n.
\]
Let \( M \) be an \( m \times m \) positive definite symmetric real matrix. We define the map \( J_M : G^J \times \mathbb{H}_{n,m} \rightarrow \mathbb{C}^* \) by
\[
J_M(g, (\Omega, Z)) = e^{2\pi i \sigma(M[\Omega + \mu](C\Omega + D)^{-1}C)} e^{-2\pi i \sigma(M(\lambda + 2\lambda^T Z + \kappa + \mu^T \lambda))},
\]
where \( g = (g, (\lambda, \mu; \kappa)) \in G^J \) with \( g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G \) and \((\lambda, \mu; \kappa) \in H(n,m)_R \). Here we use the Siegel’s notation \( S[X] := tXSX \) for two matrices \( S \) and \( X \).

We define the map \( J_{\rho,M} : G^J \times \mathbb{H}_{n,m} \rightarrow GL(V_\rho) \) by
\[
J_{\rho,M}(g, (\Omega, Z)) = J_M(g, (\Omega, Z)) \rho(J(g, \Omega)),
\]
where \( g = (g, h) \in G^J \) with \( g \in G \) and \( h \in H(n,m)_R \). For a function \( f \) on \( \mathbb{H}_n \) with values in \( V_\rho \), we can lift \( f \) to a function \( \Phi_f \) on \( G^J \):
\[
\Phi_f(\sigma) = (f|_{J_{\rho,M}[\sigma]})(iI_n, 0)
\]
\[
= J_{\rho,M}(\sigma, (iI_n, 0))^{-1} f(\sigma (iI_n, 0)), \quad \sigma \in G^J.
\]
A characterization of \( \Phi_f \) for a cusp Jacobi form \( f \) in \( J_{\rho,M}(\Gamma) \) was given by Takase [17] pp.162–164.

We allow a weight \( k \) to be half-integral.

Let \( \mathcal{S} = \{ S \in \mathbb{C}^{(n,n)} | S = tS, \ Re(S) > 0 \} \) be a connected simply connected complex manifold. Then there is a uniquely determined holomorphic function \( \det^{1/2} \) on \( \mathcal{S} \) such that
\[
(\det^{1/2} S)^2 = \det S \quad \text{for all } S \in \mathcal{S}.
\]
\[
(\det^{1/2} S)^2 = (\det S)^{1/2} \quad \text{for all } S \in \mathcal{S} \cap \mathbb{R}^{(n,n)}.
\]

For each integer \( k \in \mathbb{Z} \) and \( S \in \mathcal{S} \), we put
\[
\det^{k/2} S = (\det^{1/2} S)^k.
\]
For brevity, we set \( G = Sp(n, \mathbb{R}) \). For any \( g \in G \) and \( \Omega, \Omega' \in \mathbb{H}_n \), we put
\[
\varepsilon(g; \Omega', \Omega) = \det^{-\frac{k}{2}} \left( \frac{g \cdot \Omega' - g \cdot \Omega}{2i} \right) \det^{\frac{k}{2}} \left( \frac{\Omega' - \Omega}{2i} \right)
\]
\[
\times |\det J(g, \Omega')|^{-1/2} |\det J(g, \Omega)|^{-1/2}.
\]
We refer to [13, 14, 17] for more detail about \( \varepsilon(g; \Omega, \Omega) \).

For each \( \Omega \in \mathbb{H}_n \), we define the function \( \beta_\Omega : G \times G \to T \) by
\[
(4.10) \quad \beta_\Omega(g_1, g_2) = \varepsilon(g_1; \Omega, g_2(\Omega)), \quad g_1, g_2 \in G.
\]
Then \( \beta_\Omega \) satisfies the cocycle condition and yields the cohomology class of \( \beta_\Omega \) of order two;
\[
(4.11) \quad \beta_\Omega(g_1, g_2)^2 = \alpha_\Omega(g_2) \alpha_\Omega(g_1g_2)^{-1} \alpha_\Omega(g_1),
\]
where
\[
(4.12) \quad \alpha_\Omega(g) = \frac{\det J(g, \Omega)}{\det J(g, \Omega)}, \quad g \in G, \ \Omega \in \mathbb{H}_n.
\]
For any \( \Omega \in \mathbb{H}_n \), we let
\[
G_\Omega = \{(g, \varepsilon) \in G \times T \mid \varepsilon^2 = \alpha_\Omega(g)^{-1}\}
\]
be the two-fold covering group with multiplication law
\[
(g_1, \varepsilon_1)(g_2, \varepsilon_2) = (g_1g_2, \varepsilon_1 \varepsilon_2 \beta_\Omega(g_1, g_2)).
\]
The covering group \( G_\Omega \) depends on the choice of \( \Omega \in \mathbb{H}_n \), i.e., the choice of a maximal compact subgroup of \( G \). However for any two elements \( \Omega_1, \Omega_2 \in \mathbb{H}_n \), \( G_{\Omega_1} \) is isomorphic to \( G_{\Omega_2} \) (cf. [19]).

We put
\[
(4.13) \quad G_* := G_{iI_n}.
\]
Takase [19, p. 131] defined the automorphic factor \( J_{1/2} : G_* \times \mathbb{H}_n \to \mathbb{C}^* \) of weight 1/2 by
\[
(4.14) \quad J_{1/2}(g_*, \Omega) := e^{-1} \varepsilon(g; \Omega, iI_n) |\det J(g, \Omega)|^{1/2},
\]
where \( g_* = (g, \varepsilon) \in G_* \) and \( \Omega \in \mathbb{H}_n \). It is easily checked that
\[
(4.15) \quad J_{1/2}(g_*h_*, \Omega) = J_{1/2}(g_*, h \cdot \Omega)J_{1/2}(h_*, \Omega)
\]
for all \( g_* = (g, \varepsilon), h_* = (h, \eta) \in G_* \) and \( \Omega \in \mathbb{H}_n \). We also see easily that
\[
(4.16) \quad J_{1/2}(g_*, \Omega)^2 = \det(C\Omega + D)
\]
for all \( g_* = (g, \varepsilon) \in G \) with \( g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G \).

Let \( \pi_* : G_* \to G \) be the projection defined by \( \pi_*(g, \varepsilon) = g \). Let \( \Gamma \) be an arithmetic subgroup of the Siegel modular group \( \Gamma_n \) of finite index. Let \( \Gamma_* = \pi_*^{-1}(\Gamma) \subset G_* \). Let \( \chi \) be a finite order unitary character of \( \Gamma_* \). Let \( k \in \mathbb{Z} \) be an odd integer. We say that a holomorphic function \( \phi : \mathbb{H}_n \to \mathbb{C}^* \) is a Siegel modular form of a half-integral weight \( k/2 \) with level \( \Gamma \) if it satisfies the condition
\[
(4.17) \quad \phi(\gamma_* \cdot \Omega) = \chi(\gamma_*) J_{1/2}(\gamma_*, \Omega)^k \phi(\Omega)
\]
for all \( \gamma_* \in \Gamma_* \) and \( \Omega \in \mathbb{H}_n \). Here \( \Gamma_* \) acts on \( \mathbb{H}_n \) through the projection \( \pi_* \). We denote by \( M_{k/2}(\Gamma, \chi) \) the vector space of all Siegel modular forms of weight \( k/2 \) with level \( \Gamma \). Let \( S_{k/2}(\Gamma, \chi) \) be the subspace of \( M_{k/2}(\Gamma, \chi) \) consisting of \( \phi \in M_{k/2}(\Gamma, \chi) \) such that
\[
|\phi(\Omega)| \det(\Im \Omega)^{k/4} \text{ is bounded on } \mathbb{H}_n.
\]
An element of $S_{k/2}(\Gamma, \chi)$ is called a Siegel cusp form of weight $k/2$. The Petersson norm on $S_{k/2}(\Gamma, \chi)$ is defined by
\[
||\phi||^2 := \int_{\Gamma \backslash \mathbb{H}_n} |\phi(\Omega)|^2 \det(\Im \Omega)^{k/2} \, dv_\Omega,
\]
where $dv_\Omega = (\det Y)^{-(n+1)}dX \, dY$ ($\Omega = X + iY \in \mathbb{H}_n$) is a $G_*$-invariant volume element on $\mathbb{H}_n$.

**Remark 4.1.** Using the Schrödinger-Weil representation, Takase [20] established a bijective correspondence between the space of cuspidal Jacobi forms and the space of Siegel cusp forms of half-integral weight which is compatible with the action of Hecke operators. For example, if $m$ is a positive integer, the classical result (cf. [2] and [9])
\[(4.18) \quad J_{m,1}^\text{cusp}(\Gamma_n) \cong S_{m-1/2}(\Gamma_0(4))
\]
can be obtained by the method of the representation theory. Here $\Gamma_n$ is the Siegel modular group of degree $n$, $\Gamma_0(4)$ is the Hecke subgroup of $\Gamma_n$ and $J_{m,1}^\text{cusp}(\Gamma_n)$ denotes the vector space of cuspidal Jacobi forms of weight $m$ and index 1.

We now define the notion of Jacobi forms of half-integral weight as follows.

**Definition 4.3.** Let $\Gamma \subset \Gamma_n$ be a discrete subgroup of finite index. We put $\Gamma_* = \pi_*^{-1}(\Gamma)$ and
\[
\Gamma_*^f = \Gamma_* \times H_Z^{(m,m)}.
\]
A holomorphic function $f : \mathbb{H}_{n,m} \to \mathbb{C}$ is said to be a Jacobi form of a weight $k/2$ in $\frac{1}{2} \mathbb{Z}$ (for odd $k$) with respect to $\Gamma_*^f$ and index $\mathcal{M}$ for the character $\chi$ of $\Gamma_*^f$ if it satisfies the following transformation formula
\[(4.19)\quad f(\gamma_* \cdot (\Omega, Z)) = \chi(\gamma_*) J_{k,\mathcal{M}}(\gamma_* \cdot (\Omega, Z)) f(\Omega, Z) \quad \text{for all } \gamma_* \in \Gamma_*,
\]
where $J_{k,\mathcal{M}} : \Gamma_* \times \mathbb{H}_{n,m} \to \mathbb{C}$ is an automorphic factor defined by
\[(4.20) \quad J_{k,\mathcal{M}}(\gamma_* \cdot (\Omega, Z)) := e^{2\pi i \sigma(\mathcal{M}(Z+\lambda \Omega+\mu)(\xi \Omega+D)^{-1}C^t(Z+\lambda \Omega+\mu))} \times e^{-2\pi i \sigma(\mathcal{M}(\Omega^t\lambda^t+2\lambda^t Z + \kappa + \mu^t \lambda))} J_{1/2}(\gamma_* \cdot \Omega)^k,
\]
where $\gamma_* = (\gamma, (\lambda, \mu; \kappa)) \in \Gamma_*$ with $\gamma_* = (\gamma, e)$, $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$, $\lambda, \mu, \kappa \in H_Z^{(m,m)}$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$.

5. Covariant Maps for the Schrödinger-Weil representation

As before we let $\mathcal{M}$ be a symmetric positive definite $m \times m$ real matrix. We keep the notations in Section 4.

We define the mapping $\mathcal{F}(\mathcal{M}) : \mathbb{H}_{n,m} \to L^2(\mathbb{R}^{(m,n)})$ by
\[(5.1) \quad \mathcal{F}(\mathcal{M})(\Omega, Z)(x) = e^{\pi i \sigma(\mathcal{M}(x \Omega^t x + 2 x^t Z))}, \quad (\Omega, Z) \in \mathbb{H}_{n,m}, \ x \in \mathbb{R}^{(m,n)}.
\]
For brevity we put $\mathcal{F}^{(M)}_{\Omega,Z} := \mathcal{F}^{(M)}(\Omega,Z)$ for $(\Omega,Z) \in \mathbb{H}_{n,m}$. Takase [19] proved that $G_{2,M}$ is isomorphic to $G_{2,I_2}$ (cf. (3.3) and (4.13)). Therefore we will use $G_* := G_{2,I_2}$ instead of $G_{2,M}$.

We set
\[ G_*^J := G_* \rtimes H_{Z}^{(n,m)}. \]
We note that $G_*^J$ acts on $\mathbb{H}_{n,m}$ via the natural projection of $G_*^J$ onto $G_*^J$.

Now we assume that $m$ is odd. We define the automorphic factor $J_*^{M} : G_*^J \times \mathbb{H}_{n,m} \rightarrow \mathbb{C}^\times$ for $G_*^J$ on $\mathbb{H}_{n,m}$ by
\[ J_*^{M}(g_*,(\Omega,Z)) = e^{\pi i \sigma(M(Z+\lambda \Omega + \mu)(C\Omega + D)^{-1}C^{-1}(Z+\lambda \Omega + \mu))} \times e^{-\pi i \sigma(M(\lambda \Omega + 2\lambda' Z + \kappa + \mu \epsilon))} J_{/2}(g, \epsilon), (\Omega,Z)^m, \]
where $g_* = ((g, \epsilon), (\lambda, \mu; \kappa)) \in G_*^J$ with $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_*$, $(\lambda, \mu; \kappa) \in H_{R}^{(n,m)}$ and $(\Omega,Z) \in \mathbb{H}_{n,m}$.

**Theorem 5.1.** Let $m$ be an odd positive integer. The map $\mathcal{F}^{(M)} : \mathbb{H}_{n,m} \rightarrow L^2(\mathbb{R}^{(m,n)})$ defined by (5.1) is a covariant map for the Schrödinger-Weil representation $\omega_M$ of $G_*^J$ and the automorphic factor $J_*^{M}$ for $G_*^J$ on $\mathbb{H}_{n,m}$ defined by Formula (5.3). In other words, $\mathcal{F}^{(M)}$ satisfies the following covariance relation
\[ \omega_M(g_*) \mathcal{F}^{(M)}_{\Omega,Z} = J_*^{M}(g_*, (\Omega,Z))^{-1} \mathcal{F}^{(M)}_{g_*^*, (\Omega,Z)} \]
for all $g_* \in G_*^J$ and $(\Omega,Z) \in \mathbb{H}_{n,m}$.

**Proof.** For an element $g_* = ((g, \epsilon), (\lambda, \mu; \kappa)) \in G_*^J$ with $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$, we put $(\Omega_*,Z_*) = g_* \cdot (\Omega,Z)$ for $(\Omega,Z) \in \mathbb{H}_{n,m}$. Then we have
\[ \Omega_* = g \cdot \Omega = (A \Omega + B)(C \Omega + D)^{-1}, \]
\[ Z_* = (Z + \lambda \Omega + \mu)(C \Omega + D)^{-1}. \]
In this section we use the notations $t(b)$, $g(\alpha)$ and $\sigma_n$ in Section 3. Since the following elements $h(\lambda, \mu; \kappa)_\epsilon$, $t(b; \epsilon)$, $g(\alpha; \epsilon)_\epsilon$ and $\sigma_{n, \epsilon}$ of $G_*^J$ defined by
\[ h(\lambda, \mu; \kappa)_\epsilon = ((I_{2n}, \epsilon), (\lambda, \mu; \kappa)) \quad \text{with} \quad \epsilon = \pm 1, \quad \lambda, \mu \in \mathbb{R}^{(m,n)}, \quad \kappa \in \mathbb{R}^{(m,m)}, \]
\[ t(b; \epsilon) = ((t_0(b), \epsilon), (0,0;0)) \quad \text{with} \quad \epsilon = \pm 1, \quad b = t \in \mathbb{R}^{(m,m)}, \]
\[ g(\alpha; \epsilon) = ((g_0(\alpha), \epsilon), (0,0;0)) \quad \text{with} \quad \epsilon = \pm 1, \quad \alpha \in GL(n,\mathbb{R}), \]
\[ \sigma_{n, \epsilon} = ((\sigma_n, \epsilon), (0,0;0)) \quad \text{with} \quad \epsilon^2 = (-i)^n. \]
generate the group $G_*^J$, it suffices to prove the covariance relation (5.4) for the above generators.

**Case I.** $g_* = h(\lambda, \mu; \kappa)_\epsilon$ with $\epsilon = \pm 1$ and $\lambda, \mu \in \mathbb{R}^{(m,n)}$, $\kappa \in \mathbb{R}^{(m,m)}$. 
In this case, we have
\[
\Omega_* = \Omega, \quad Z_* = Z + \lambda \Omega + \mu.
\]
We put
\[
h(\lambda, \mu; \kappa)_+ := ((I_{2n}, 1), (\lambda, \mu; \kappa))
\]
and
\[
h(\lambda, \mu; \kappa)_- := ((I_{2n}, -1), (\lambda, \mu; \kappa)).
\]
It is easily seen that
\[
J_{1/2}(I_{2n}, 1, \Omega) = 1 \quad \text{and} \quad J_{1/2}(I_{2n}, -1, \Omega) = -1.
\]
Therefore we get
\[
J_M^*(h(\lambda, \mu; \kappa)_+, (\Omega, Z)) = e^{-\pi i \sigma \{\mathcal{M}(\lambda \Omega t + 2 \lambda^tZ + \kappa + \mu^t\lambda)}
\]
and
\[
J_M^*(h(\lambda, \mu; \kappa)_-, (\Omega, Z)) = -e^{-\pi i \sigma \{\mathcal{M}(\lambda \Omega t + 2 \lambda^tZ + \kappa + \mu^t\lambda)}.
\]
According to Formulas (3.5) and (3.6), for \( x \in \mathbb{R}^{(m,n)}, \)
\[
\left( \omega_{\mathcal{M}}(h(\lambda, \mu; \kappa)_+) \mathcal{F}_{\Omega,Z}^{(\mathcal{M})} \right)(x) = e^{-\pi i \sigma \{\mathcal{M}(\kappa t + \lambda x + 2 x^t\mu)} \mathcal{F}_{\Omega,Z}^{(\mathcal{M})} \right)(x + \lambda)
\]
On the other hand, according to Formula (5.3), for \( x \in \mathbb{R}^{(m,n)}, \)
\[
J_M^*(h(\lambda, \mu; \kappa)_+, (\Omega, Z))^{-1} \mathcal{F}_{\Omega,Z}^{(\mathcal{M})} \mathcal{F}_{\Omega,Z}^{(\mathcal{M})}(x) = e^{-\pi i \sigma \{\mathcal{M}(\kappa t + \lambda x + 2 x^t\mu)} \mathcal{F}_{\Omega,Z}^{(\mathcal{M})} \mathcal{F}_{\Omega,Z}^{(\mathcal{M})}(x + \lambda)
\]
Therefore we prove the covariance relation (5.4) in the case \( \tilde{g}_* = h(\lambda, \mu; \kappa)_+ \) with \( \lambda, \mu, \kappa \) real.

Similarly we can prove the covariance relation (5.4) in the case \( \tilde{g}_* = h(\lambda, \mu; \kappa)_- \) with \( \lambda, \mu, \kappa \) real. In fact,
\[
\left( \omega_{\mathcal{M}}(h(\lambda, \mu; \kappa)_-) \mathcal{F}_{\Omega,Z}^{(\mathcal{M})} \right)(x) = e^{-\pi i \sigma \{\mathcal{M}(\kappa t + \lambda x + 2 x^t\mu)} \mathcal{F}_{\Omega,Z}^{(\mathcal{M})} \mathcal{F}_{\Omega,Z}^{(\mathcal{M})}(x + \lambda)
\]
\[\text{Case II.} \quad \tilde{g}_* = t(b; \epsilon) \text{ with } \epsilon = \pm 1 \text{ and } b = \epsilon b \in \mathbb{R}^{(n,n)}.
\]
In this case, we have
\[
\Omega_* = \Omega + b, \quad Z_* = Z \quad \text{and} \quad J_M(\tilde{g}_*, (\Omega, Z)) = 1.
\]
We put
\[
t(b)_+ = (((b), 1), (0, 0; 0))
\]
and
\[ t(b) = ((t(b), -1), (0, 0; 0)). \]

It is easily seen that
\[ J_{1/2}((t(b), 1), \Omega) = 1 \quad \text{and} \quad J_{1/2}((t(b), -1), \Omega) = -1. \]

Therefore we get
\[ J^*_M(t(b)_+,(\Omega,Z)) = 1 \quad \text{and} \quad J^*_M(t(b)_-,(\Omega,Z)) = -1. \]

According to Formula (3.7) together with Formula (3.5), we obtain
\[
\left( \omega_M(t(b)_+).\mathcal{F}^{(M)}_{\Omega,Z} \right) (x) = e^{\pi i \sigma(M x b^t x)} \mathcal{F}^{(M)}_{\Omega,Z} (x), \quad x \in \mathbb{R}^{(m,n)}.
\]

On the other hand, according to Formula (5.3), for \( x \in \mathbb{R}^{(m,n)} \), we obtain
\[
J^*_M(t(b)_+,(\Omega,Z))^{-1} \mathcal{F}^{(M)}_{t(b)_+,(\Omega,Z)} (x)
= \mathcal{F}^{(M)}_{\Omega+b,Z} (x)
= e^{\pi i \sigma(M x (\Omega+b)^t x + 2 x^t Z)}
= e^{\pi i \sigma(M x b^t x)} \mathcal{F}^{(M)}_{\Omega,Z} (x).
\]

Therefore we prove the covariance relation (5.4) in the case \( \tilde{g}_s = t(b)_+ \) with \( b = t b \in \mathbb{R}^{(n,n)} \).

Similarly we can prove the covariance relation (5.4) in the case \( \tilde{g}_s = t(b)_- \) with \( \lambda, \mu, \kappa \) real. In fact,
\[
\left( \omega_M(t(b)_-).\mathcal{F}^{(M)}_{\Omega,Z} \right) (x) = J^*_M(t(b)_-,(\Omega,Z))^{-1} \mathcal{F}^{(M)}_{t(b)_-,(\Omega,Z)} (x)
= -e^{\pi i \sigma(M x b^t x)} \mathcal{F}^{(M)}_{\Omega,Z} (x).
\]

**Case III.** \( \tilde{g}_s = ((g(\alpha), \epsilon), (0, 0; 0)) \) with \( \epsilon = \pm 1, \pm i \) and \( \alpha \in GL(n, \mathbb{R}) \).

In this case, we have
\[ \Omega_\epsilon = t^\alpha \Omega \alpha \quad \text{and} \quad Z_\epsilon = Z \alpha. \]

We put
\[
\begin{align*}
g(\alpha)_+ & : = ((g(\alpha), 1), (0, 0; 0)), \\
g(\alpha)_- & : = ((g(\alpha), -1), (0, 0; 0)), \\
g(\alpha)^+ & : = ((g(\alpha), i), (0, 0; 0)), \\
g(\alpha)^- & : = ((g(\alpha), -i), (0, 0; 0)).
\end{align*}
\]

Then we can see easily that
\[
\begin{align*}
J_{1/2}((g(\alpha), 1), \Omega) & = (\det \alpha)^{-1/2}, \\
J_{1/2}((g(\alpha), -1), \Omega) & = -(\det \alpha)^{-1/2}, \\
J_{1/2}((g(\alpha), i), \Omega) & = i (\det \alpha)^{-1/2}, \\
J_{1/2}((g(\alpha), -i), \Omega) & = -i (\det \alpha)^{-1/2}.
\end{align*}
\]
Using Formulas (3.5), (3.8) and (5.3), we can show the covariance relation (5.4) in the case \( \tilde{g}_s = ((g(\alpha), e), (0, 0; 0)) \) with \( \epsilon = \pm 1, \pm i \) and \( \alpha \in GL(n, \mathbb{R}) \).

**Case IV.** \( \tilde{g}_s = ((\sigma_n, \epsilon), (0, 0; 0)) \) with \( \sigma_n = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \) and \( \epsilon^2 = (-i)^n \).

In this case, we have

\[
\Omega_s = -\Omega^{-1} \quad \text{and} \quad Z_s = Z \Omega^{-1}.
\]

In order to prove the covariance relation (5.4), we need the following useful lemma.

**Lemma 5.1.** For a fixed element \( \Omega \in \mathbb{H}_n \) and a fixed element \( Z \in \mathbb{C}^{m,n} \), we obtain the following property

\[
\int_{\mathbb{R}^{(m,n)}} e^{\pi \sigma(x \Omega^t x + 2 x^t Z)} dx_{11} \cdots dx_{mn} = \left( \det \frac{\Omega}{i} \right)^{-\frac{m}{2}} e^{-\pi \sigma(Z \Omega^{-1} \Omega)} \tag{5.5}
\]

where \( x = (x_{ij}) \in \mathbb{R}^{m,n} \).

**Proof of Lemma 5.1.** By a simple computation, we see that

\[
e^{\pi \sigma(x \Omega^t x + 2 x^t Z)} = e^{-\pi \sigma(Z \Omega^{-1} \Omega)} \cdot e^{\pi \sigma((x + Z \Omega^{-1}) \Omega^t (x + Z \Omega^{-1}))}.
\]

Since the real Jacobi group \( Sp(n, \mathbb{R}) \times H_{\mathbb{R}}^{(n,m)} \) acts on \( \mathbb{H}_{n,m} \) holomorphically, we may put

\[
\Omega = i A^t A, \quad Z = i V, \quad A \in \mathbb{R}^{(n,n)}, \quad V = (v_{ij}) \in \mathbb{R}^{(m,n)}.
\]

Then we obtain

\[
\int_{\mathbb{R}^{(m,n)}} e^{\pi \sigma(x \Omega^t x + 2 x^t Z)} dx_{11} \cdots dx_{mn}
\]

\[
= e^{-\pi \sigma(Z \Omega^{-1} \Omega)} \int_{\mathbb{R}^{(m,n)}} e^{\pi \sigma((x + i V (i A^t A)^{-1})(i A^t A)^{(i A^t A)^{-1}})} dx_{11} \cdots dx_{mn}
\]

\[
= e^{-\pi \sigma(Z \Omega^{-1} \Omega)} \int_{\mathbb{R}^{(m,n)}} e^{\pi \sigma((x + V (A^t A)^{-1}) A^t A (x + V (A^t A)^{-1}))} dx_{11} \cdots dx_{mn}
\]

\[
= e^{-\pi \sigma(Z \Omega^{-1} \Omega)} \int_{\mathbb{R}^{(m,n)}} e^{-\pi \sigma((u A)^{(u A)})} du_{11} \cdots du_{mn} \quad \text{(Put} \quad u = x + V (A^t A)^{-1} = (u_{ij}) \text{)}
\]

\[
= e^{-\pi \sigma(Z \Omega^{-1} \Omega)} \int_{\mathbb{R}^{(m,n)}} e^{-\pi \sigma((w^t w)} (\det A)^{-m} dw_{11} \cdots dw_{mn} \quad \text{(Put} \quad w = u A = (w_{ij}) \text{)}
\]

\[
= e^{-\pi \sigma(Z \Omega^{-1} \Omega)} (\det A)^{-m} \cdot \prod_{i=1}^{m} \prod_{j=1}^{n} \int_{\mathbb{R}} e^{-\pi w_{ij}^2} dw_{ij}
\]

\[
= e^{-\pi \sigma(Z \Omega^{-1} \Omega)} (\det A)^{-m} \quad \text{(because} \quad \int_{\mathbb{R}} e^{-\pi w_{ij}^2} dw_{ij} = 1 \quad \text{for all} \quad i, j \text{)}
\]

\[
= e^{-\pi \sigma(Z \Omega^{-1} \Omega)} (\det (A^t A))^{-m} \cdot \frac{m}{2}
\]

\[
= e^{-\pi \sigma(Z \Omega^{-1} \Omega)} \left( \frac{\Omega}{i} \right)^{-\frac{m}{2}}
\]

This completes the proof of Lemma 5.1. \( \square \)

According to Formulas (3.5) and (3.9), for \( x \in \mathbb{R}^{(m,n)} \), we obtain
On the other hand, it is possible to substitute $\mathcal{M}^{1/2} y$, then $\frac{du}{\sqrt{(\det \mathcal{M})^n}}$. Therefore, according to Lemma 5.1, we obtain

$$
\left(\omega_{\mathcal{M}}(\mathcal{M})_{(n)}^{(m)} \mathbb{F}_{\Omega, Z}^{(n)} \right)(x)
$$

$$
eq e^{m} \left(\frac{1}{i} \right) \left( \det \mathcal{M} \right)^{\frac{n}{2}} \int_{\mathbb{R}^{(m,n)}} e^{\pi i \sigma(u \Omega^{1/2} u^t (Z-x))} \left( \det \mathcal{M} \right)^{-\frac{n}{2}} du
$$

$$
eq e^{m} \left(\frac{1}{i} \right) \left( \det \mathcal{M} \right)^{\frac{n}{2}} \int_{\mathbb{R}^{(m,n)}} e^{\pi i \sigma(u \Omega^{1/2} u^t (Z-x))} \left( \det \mathcal{M} \right)^{-\frac{n}{2}} du
$$

$$
eq e^{m} \left(\frac{1}{i} \right) \left( \det \frac{\Omega}{i} \right)^{-\frac{m}{2}} e^{-\pi i \sigma\left\{ \mathcal{M}^{1/2}(Z-x) \Omega^{-1} (Z-x) \mathcal{M}^{1/2} \right\}} \quad \text{(by Lemma 5.1)}
$$

$$
eq e^{m} \left( \det \Omega \right)^{-\frac{m}{2}} e^{-\pi i \sigma\left\{ \mathcal{M}(Z-x) \Omega^{-1} (Z-x) \right\}}
$$

On the other hand,

$$J_{1/2}(\sigma_n, \epsilon, \Omega)
$$

$$= e^{-1} \left( \det \mathcal{M} \right)^{-\frac{1}{2}} \left( \frac{\sigma_n \cdot \Omega \cdot \Omega^{-1} \cdot (i \mathbb{I}_n)}{2i} \right) \det^{1/2} \left( \frac{\Omega - (i \mathbb{I}_n)}{2i} \right) \left| J(\sigma_n, i \mathbb{I}_n) \right|^{-\frac{1}{2}}
$$

$$= e^{-1} \left( \det \mathcal{M} \right)^{-\frac{1}{2}} \left( -i \Omega^{-1} (\Omega - i \mathbb{I}_n) \right) \det^{1/2} \left( \frac{\Omega - (i \mathbb{I}_n)}{2i} \right) \left( i^n \right)^{-\frac{1}{2}}
$$

$$= e^{-1} \left( \det \mathcal{M} \right)^{-\frac{1}{2}} \left( i^{-n/2} \right)
$$

$$= e^{-1} \left( \det \mathcal{M} \right)^{-\frac{1}{2}} \left( \det \Omega \right)^{1/2} i^{-n/2}
$$

$$= e^{-1} \left( \det \Omega \right)^{1/2}.
$$

Thus, according to Formula (5.3), for $x \in \mathbb{R}^{(m,n)}$, we obtain
\[
J_s^*(\tilde{g}_s, (\Omega, Z))^{-1} \mathcal{F}_{\tilde{g}_s, (\Omega, Z)}(x)
= e^{-\pi i \sigma(MZ\Omega^{-1}x)} J_{1/2}((\sigma_n, \epsilon, \Omega))^{-m} \mathcal{F}_{\Omega^{-1}, Z\Omega^{-1}}(x)
= e^m (\det \Omega)^{-\frac{\sigma}{2}} e^{-\pi i \sigma(MZ\Omega^{-1}x)} e^{\pi i \sigma(M(x(-\Omega^{-1})x + 2x^t(\Omega^{-1})))}
= e^m (\det \Omega)^{-\frac{\sigma}{2}} e^{-\pi i \sigma(MZ\Omega^{-1}x + x\Omega^{-1}x - 2Z\Omega^{-1}x)}.
\]

Therefore we have proved the covariance relation (5.4) in the case \(\tilde{g}_s = ((\sigma_n, \epsilon), (0, 0; 0))\) with \(\epsilon^2 = (-i)^n\). Since \(J_s^*\) is an automorphic factor for \(G_s^J\) on \(\mathbb{H}_{n,m}\), we see that if the covariance relation (5.4) holds for two elements \(\tilde{g}_s, \tilde{h}_s\) in \(G_s^J\), then it holds for \(\tilde{g}_s, \tilde{h}_s\). Finally we complete the proof.

\[\square\]

It is natural to raise the following question:

**Problem**: Find all the covariant maps for the Schrödinger-Weil representation \(\omega_M\) on \(\mathbb{H}_{n,m}\).

### 6. Construction of Jacobi Forms of Half-Integral Weight

Let \((\pi, V_\pi)\) be a unitary representation of \(G_s^J\) on the representation space \(V_\pi\). Let \(\Gamma\) be an arithmetic subgroup of the Siegel modular group \(\Gamma_n\). We set \(\Gamma_s = \pi_s^{-1}(\Gamma)\) and
\[
\Gamma_s^J = \Gamma_s \ltimes H_Z^{(n,m)}.
\]

We assume that \((\pi, V_\pi)\) satisfies the following conditions (A) and (B):

(A) There exists a vector valued map
\[
\mathcal{F} : \mathbb{H}_{n,m} \longrightarrow V_\pi, \quad (\Omega, Z) \mapsto \mathcal{F}_{\Omega, Z} := \mathcal{F}(\Omega, Z)
\]
satisfying the following covariance relation
\[
(6.1) \quad \pi(\tilde{g}_s) \mathcal{F}_{\Omega, Z} = \psi(\tilde{g}_s) J_s(\tilde{g}_s, (\Omega, Z))^{-1} \mathcal{F}_{\tilde{g}_s, (\Omega, Z)} \quad \text{for all} \quad \tilde{g}_s \in G_s^J, \quad (\Omega, Z) \in \mathbb{H}_{n,m},
\]
where \(\psi\) is a character of \(G_s^J\) and \(J_s : G_s^J \times \mathbb{H}_{n,m} \longrightarrow GL(1, \mathbb{C})\) is a certain automorphic factor for \(G_s^J\) on \(\mathbb{H}_{n,m}\).

(B) There exists a linear functional \(\theta : V_\pi \longrightarrow \mathbb{C}\) which is semi-invariant under the action of \(\Gamma_s^J\), in other words, for all \(\tilde{\gamma}_s \in \Gamma_s^J\) and \((\Omega, Z) \in \mathbb{H}_{n,m},
\[
(6.2) \quad \langle \pi^*(\tilde{\gamma}_s) \theta, \mathcal{F}_{\Omega, Z} \rangle = \langle \theta, \pi(\tilde{\gamma}_s)^{-1} \mathcal{F}_{\Omega, Z} \rangle = \chi(\tilde{\gamma}_s) \langle \theta, \mathcal{F}_{\Omega, Z} \rangle,
\]
where \(\pi^*\) is the contragredient of \(\pi\) and \(\chi : \Gamma_s^J \longrightarrow T\) is a unitary character of \(\Gamma_s^J\).

Under the assumptions (A) and (B) on a unitary representation \((\pi, V_\pi)\), we define the function \(\Theta\) on \(\mathbb{H}_{n,m}\) by
\[
(6.3) \quad \Theta(\Omega, Z) := \langle \theta, \mathcal{F}_{\Omega, Z} \rangle = \theta(\mathcal{F}_{\Omega, Z}), \quad (\Omega, Z) \in \mathbb{H}_{n,m}.
\]

We now shall see that \(\Theta\) is an automorphic form on \(\mathbb{H}_{n,m}\) with respect to \(\Gamma_s^J\) for the automorphic factor \(J_s\).
Lemma 6.1. Let \((\pi, V_{\pi})\) be a unitary representation of \(G_*^J\) satisfying the above assumptions (A) and (B). Then the function \(\Theta\) on \(\mathbb{H}_{n,m}\) defined by (6.3) satisfies the following modular transformation behavior

\[
\Theta(\tilde{\gamma}_s \cdot (\Omega, Z)) = \psi(\tilde{\gamma}_s)^{-1} \chi(\tilde{\gamma}_s)^{-1} J_s(\tilde{\gamma}_s, (\Omega, Z)) \Theta(\Omega, Z)
\]

for all \(\tilde{\gamma}_s \in \Gamma_*^J\) and \((\Omega, Z) \in \mathbb{H}_{n,m}\).

**Proof.** For any \(\tilde{\gamma}_s \in \Gamma_*^J\) and \((\Omega, Z) \in \mathbb{H}_{n,m}\), according to the assumptions (6.1) and (6.2), we obtain

\[
\Theta(\tilde{\gamma}_s \cdot (\Omega, Z)) = \langle \theta, \mathcal{F}_{\tilde{\gamma}_s} \cdot (\Omega, Z) \rangle \\
= \langle \theta, \psi(\tilde{\gamma}_s)^{-1} J_s(\tilde{\gamma}_s, (\Omega, Z)) \pi(\tilde{\gamma}_s) \mathcal{F}_{\Omega, Z} \rangle \\
= \psi(\tilde{\gamma}_s)^{-1} J_s(\tilde{\gamma}_s, (\Omega, Z)) \langle \theta, \mathcal{F}_{\Omega, Z} \rangle \\
= \psi(\tilde{\gamma}_s)^{-1} \chi(\tilde{\gamma}_s)^{-1} J_s(\tilde{\gamma}_s, (\Omega, Z)) \langle \theta, \mathcal{F}_{\Omega, Z} \rangle \\
= \psi(\tilde{\gamma}_s)^{-1} \chi(\tilde{\gamma}_s)^{-1} J_s(\tilde{\gamma}_s, (\Omega, Z)) \Theta(\Omega, Z).
\]

\(\square\)

Now for a positive definite integral symmetric matrix \(\mathcal{M}\) of degree \(m\), we define the holomorphic function \(\Theta_{\mathcal{M}} : \mathbb{H}_{n,m} \rightarrow \mathbb{C}\) by

\[
\Theta_{\mathcal{M}}(\Omega, Z) := \sum_{\xi \in \mathbb{Z}^{(m,n)}} e^{\pi i \sigma(\mathcal{M}([\xi^T \xi + 2\xi^\top Z])}, \quad (\Omega, Z) \in \mathbb{H}_{n,m}.
\]

Theorem 6.1. Let \(m\) be an odd positive integer. Let \(\mathcal{M}\) be a symmetric positive definite integral matrix of degree \(m\) such that \(\det(\mathcal{M}) = 1\). Let \(\Gamma\) be an arithmetic subgroup of \(\Gamma_n\) generated by all the following elements

\[
t(b) = \begin{pmatrix} I_n & b \\ 0 & I_n \end{pmatrix}, \quad g(\alpha) = \begin{pmatrix} \alpha^t & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \quad \sigma_n = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},
\]

where \(b = b^t \in \mathbb{Z}^{(n,n)}\) with even diagonal and \(\alpha \in \mathbb{Z}^{(n,n)}\). Then for any \(\tilde{\gamma}_s \in \Gamma_*^J\), the function \(\Theta_{\mathcal{M}}\) satisfies the functional equation

\[
\Theta_{\mathcal{M}}(\tilde{\gamma}_s \cdot (\Omega, Z)) = \rho_{\mathcal{M}}(\tilde{\gamma}_s) J^*_{\mathcal{M}}(\tilde{\gamma}_s, (\Omega, Z)) \Theta_{\mathcal{M}}(\Omega, Z), \quad (\Omega, Z) \in \mathbb{H}_{n,m},
\]

where \(\rho_{\mathcal{M}}\) is a uniquely determined character of \(\Gamma_*^J\) and \(J^*_{\mathcal{M}} : G_*^J \times \mathbb{H}_{n,m} \rightarrow \mathbb{C}^*\) is the automorphic factor for \(G_*^J\) on \(\mathbb{H}_{n,m}\) defined by the formula (5.3).

**Proof.** For an element \(\tilde{\gamma}_s = ((\gamma, \epsilon), (\lambda, \mu; \kappa)) \in \Gamma_*^J\) with \(\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma\) and \((\lambda, \mu; \kappa) \in H_2^{(n,m)}\), we put \((\Omega, Z_s) = \tilde{\gamma}_s \cdot (\Omega, Z)\) for \((\Omega, Z) \in \mathbb{H}_{n,m}\). Then we have

\[
\Omega_s = \gamma \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}, \\
Z_s = (Z + \lambda \Omega + \mu)(C\Omega + D)^{-1}.
\]

We define the linear functional \(\vartheta\) on \(L^2(\mathbb{R}^{(m,n)})\) by
\[ \vartheta(f) = \langle \vartheta, f \rangle := \sum_{\xi \in \mathbb{Z}^{(m,n)}} f(\xi), \quad f \in L^2(\mathbb{R}^{(m,n)}) \]

We note that \( \Theta_M(\Omega, Z) = \vartheta(\mathcal{F}^{(M)}) \). Since \( \mathcal{F}^{(M)} \) is a covariant map for the Schrödinger-Weil representation \( \omega_M \) by Theorem 5.1, according to Lemma 6.1, it suffices to prove that \( \vartheta \) is semi-invariant for \( \omega_M \) under the action of \( \Gamma' \), in other words, \( \vartheta \) satisfies the following semi-invariance relation

\[ (6.7) \quad \left\langle \vartheta, \omega_M(\tilde{\gamma}_s) \mathcal{F}^{(M)}_{\Omega,Z} \right\rangle = \rho_M(\tilde{\gamma}_s)^{-1} \left\langle \vartheta, \mathcal{F}^{(M)}_{\Omega,Z} \right\rangle \]

for all \( \tilde{\gamma}_s \in \Gamma'_s \) and \( (\Omega, Z) \in \mathbb{H}_{n,m} \).

In this section we use the notations \( t(b), g(\alpha) \) and \( \sigma_n \) in Section 3. Since the following elements \( h(\lambda, \mu; \kappa)_\epsilon, \ t(b; \epsilon), \ g(\alpha; \epsilon)_\alpha \) and \( \sigma_{n, \epsilon} \) of \( G^J_s \) defined by

\[
\begin{align*}
  h(\lambda, \mu; \kappa)_\epsilon &= ((I_{2n}, \epsilon), (\lambda, \mu; \kappa)) \quad \text{with } \epsilon = \pm 1, \ \lambda, \mu \in \mathbb{Z}^{(m,n)}, \ \kappa \in \mathbb{Z}^{(m,m)}, \\
  t(b; \epsilon) &= ((t(b), \epsilon), (0, 0; 0)) \quad \text{with } \epsilon = \pm 1, \ b = t^{(n,m)} \text{ even diagonal}, \\
  g(\alpha; \epsilon) &= ((g(\alpha), \epsilon), (0, 0; 0)) \quad \text{with } \epsilon = \pm 1, \ \pm i, \ \alpha \in GL(n, \mathbb{Z}), \\
  \sigma_{n, \epsilon} &= ((\sigma_n, \epsilon), (0, 0; 0)) \quad \text{with } \epsilon^2 = (-i)^n.
\end{align*}
\]

generate the group \( \Gamma'_s \). Therefore it suffices to prove the semi-invariance relation (6.7) for the above generators of \( \Gamma' \).

**Case I.** \( \tilde{\gamma}_s = h(\lambda, \mu; \kappa)_\epsilon \) with \( \epsilon = \pm 1, \ \lambda, \mu \in \mathbb{Z}^{(m,n)}, \ \kappa \in \mathbb{Z}^{(m,m)} \).

In this case, we have

\[ \Omega_* = \Omega \quad \text{and} \quad Z_* = Z + \lambda \Omega + \mu. \]

We put

\[ h(\lambda, \mu; \kappa)_+ := ((I_{2n}, 1), (\lambda, \mu; \kappa)) \]

and

\[ h(\lambda, \mu; \kappa)_- := ((I_{2n}, -1), (\lambda, \mu; \kappa)). \]

It is easily seen that

\[ J_{1/2}((I_{2n}, 1), \Omega) = 1 \quad \text{and} \quad J_{1/2}((I_{2n}, -1), \Omega) = -1. \]

Therefore we get

\[ J^*_M(h(\lambda, \mu; \kappa)_+, (\Omega, Z)) = e^{-\pi i \sigma(M(\lambda \Omega^t \lambda + 2 \lambda^t Z + \kappa + \mu \Omega))} \]

and

\[ J^*_M(h(\lambda, \mu; \kappa)_-, (\Omega, Z)) = -e^{-\pi i \sigma(M(\lambda \Omega^t \lambda + 2 \lambda^t Z + \kappa + \mu \Omega))}. \]
According to the covariance relation (5.4),
\[
\langle \vartheta, \omega_M(h(\lambda, \mu; \kappa)_+) \mathcal{F}_{\Omega, Z}^{(M)} \rangle = \langle \vartheta, J_M^*(h(\lambda, \mu; \kappa)_+, (\Omega, Z))^{-1} \mathcal{F}_{h(\lambda, \mu; \kappa)_+, (\Omega, Z)}^{(M)} \rangle
\]
\[
= J_M^*(h(\lambda, \mu; \kappa)_+, (\Omega, Z))^{-1} \langle \vartheta, \mathcal{F}_{\Omega, Z}^{(M)} + \lambda \Omega + \mu \rangle
\]
\[
= J_M^*(h(\lambda, \mu; \kappa)_+, (\Omega, Z))^{-1} \sum_{A \in Z^{(m, n)}} e^{\pi i \sigma} \{ M(A \Omega + 2 A \Omega + \lambda \Omega + \mu) \}
\]
\[
= J_M^*(h(\lambda, \mu; \kappa)_+, (\Omega, Z))^{-1} \cdot e^{-\pi i \sigma} (M(\lambda \Omega + 2 \lambda \Omega))
\]
\[
\times \sum_{A \in Z^{(m, n)}} e^{2 \pi i \sigma (M A \mu)} e^{\pi i \sigma} \{ M((A + \lambda) \Omega + (A + \lambda) + 2 (A + \lambda) \Omega) \}
\]
\[
= e^{\pi i \sigma} (M(\kappa + \mu')) \langle \vartheta, \mathcal{F}_{\Omega, Z}^{(M)} \rangle.
\]
Here we used the fact that \(\sigma(M A \mu)\) is an integer. Similarly we obtain
\[
\langle \vartheta, \omega_M(h(\lambda, \mu; \kappa)_-) \mathcal{F}_{\Omega, Z}^{(M)} \rangle = - e^{\pi i \sigma} (M(\kappa + \mu')) \langle \vartheta, \mathcal{F}_{\Omega, Z}^{(M)} \rangle.
\]
We put
\[
\rho_M(h(\lambda, \mu; \kappa)_+) = e^{-\pi i \sigma} (M(\kappa + \mu'))
\]
and
\[
\rho_M(h(\lambda, \mu; \kappa)_-) = - e^{-\pi i \sigma} (M(\kappa + \mu')).
\]
Therefore \(\vartheta\) satisfies the semi-invariance relation (6.7) in the case \(\gamma_\epsilon = h(\lambda, \mu; \kappa)_\epsilon\) with \(\epsilon = \pm 1, \lambda, \mu \in Z^{(m, n)}, \kappa \in Z^{(m, m)}\).

**Case II.** \(t(b; \epsilon) = ((t_0(b), \epsilon), (0, 0; 0))\) with \(\epsilon = \pm 1, b = t b \in Z^{(m, n)}\) even diagonal.

In this case, we have
\[
\Omega_* = \Omega + b, \quad Z_* = Z \quad \text{and} \quad J_M(\bar{g}, (\Omega, Z)) = 1.
\]
We put
\[
t(b)_+ = ((t_0(b), 1), (0, 0; 0))
\]
and
\[
t(b)_- = ((t_0(b), -1), (0, 0; 0)).
\]
It is easily seen that
\[
J_{1/2}((t_0(b), 1), \Omega) = 1 \quad \text{and} \quad J_{1/2}((t_0(b), -1), \Omega) = -1.
\]
Therefore we get
\[
J_M^*(t(b)_+, (\Omega, Z)) = 1 \quad \text{and} \quad J_M^*(t(b)_-, (\Omega, Z)) = -1.
\]
According to the covariance relation (5.4), we obtain

\[
\langle \vartheta, \omega_M(t(b)_+ \mathcal{F}^{(M)}_{\Omega, \mathcal{Z}}) \rangle = \langle \vartheta, J^*_M(t(b)_+, (\Omega, \mathcal{Z}))^{-1} \mathcal{F}^{(M)}_{t(b)_+ (\Omega, \mathcal{Z})} \rangle = \sum_{A \in \mathbb{Z}^{(m,n)}} e^{\pi i \sigma \mathcal{M}(A(\Omega+b)A + 2A^tZ)} \mathcal{F}^{(M)}_{\Omega, \mathcal{Z}}.
\]

Here we used the fact that \(\sigma(MAb)\) is an even integer. Similarly we obtain

\[
\langle \vartheta, \omega_M(t(b)_- \mathcal{F}^{(M)}_{\Omega, \mathcal{Z}}) \rangle = -\langle \vartheta, \mathcal{F}^{(M)}_{\Omega, \mathcal{Z}} \rangle.
\]

We put

\[
\rho_M(t(b)_+) = 1 \quad \text{and} \quad \rho_M(t(b)_-) = -1.
\]

Therefore \(\vartheta\) satisfies the semi-invariance relation (6.7) in the case \(t(b; \epsilon) = ((t_0(b), \epsilon), (0, 0; 0))\) with \(\epsilon = \pm 1, \quad b = t^tb \in \mathbb{Z}^{(n,m)}\).

**Case III.** \(\tilde{\gamma}_* = g(\alpha; \epsilon) = ((g(\alpha), \epsilon), (0, 0; 0))\) with \(\epsilon = \pm 1, \quad \pm i, \quad \alpha \in GL(n, \mathbb{Z})\).

In this case, we have

\[
\Omega_* = ^t\alpha \Omega \alpha \quad \text{and} \quad Z_* = Z\alpha.
\]

We put

\[
\begin{align*}
  g(\alpha)_+ &= (g(\alpha), 1, (0, 0; 0)), \\
  g(\alpha)_- &= (g(\alpha), -1, (0, 0; 0)), \\
  g(\alpha)^+ &= (g(\alpha), i, (0, 0; 0)), \\
  g(\alpha)^- &= (g(\alpha), -i, (0, 0; 0)).
\end{align*}
\]

Then we can see easily that

\[
\begin{align*}
  J_{1/2}((g(\alpha), 1, \Omega)) &= (\det \alpha)^{-1/2}, \\
  J_{1/2}((g(\alpha), -1, \Omega)) &= -(\det \alpha)^{-1/2}, \\
  J_{1/2}((g(\alpha), i, \Omega)) &= i(\det \alpha)^{-1/2}, \\
  J_{1/2}((g(\alpha), -i, \Omega)) &= -i(\det \alpha)^{-1/2}.
\end{align*}
\]
According to the covariance relation (5.4), we obtain
\[
\langle \partial_x, \omega_M(g(\alpha_{+})), F^{(M)}_{\Omega(Z)} \rangle = \langle \partial_x, J^*(M)(g(\alpha_{+}), (\Omega, Z))^{-1} F^{(M)}_{g(\alpha_{+})}, (\Omega, Z) \rangle = (\det \alpha)^{\frac{m}{2}} \langle \partial_x, F^{(M)}_{\Omega, Z} \rangle
\]
\[
= (\det \alpha)^{\frac{m}{2}} \langle \partial_x, F^{(M)}_{Z_{\alpha}} \rangle = (\det \alpha)^{\frac{m}{2}} \sum_{A \in \mathbb{Z}^{(m,n)}} F^{(M)}_{\Omega, Z} \langle A \rangle
\]
\[
= (\det \alpha)^{\frac{m}{2}} \sum_{A \in \mathbb{Z}^{(m,n)}} e^{\pi i \sigma(M(A^t \alpha Z_{\alpha}^t(A^t \alpha^t Z_{\alpha})+2 A^t \alpha^t Z)}} \langle \partial_x, F^{(M)}_{\Omega, Z} \rangle.
\]

Similarly we obtain
\[
\langle \partial_x, \omega_M(g(\alpha_{-})), F^{(M)}_{\Omega(Z)} \rangle = (-1)^m (\det \alpha)^{\frac{m}{2}} \langle \partial_x, F^{(M)}_{\Omega, Z} \rangle,
\]
\[
\langle \partial_x, \omega_M(g(\alpha_{+})), F^{(M)}_{\Omega(Z)} \rangle = i^m (\det \alpha)^{\frac{m}{2}} \langle \partial_x, F^{(M)}_{\Omega, Z} \rangle,
\]
\[
\langle \partial_x, \omega_M(g(\alpha_{-})), F^{(M)}_{\Omega(Z)} \rangle = (-i)^m (\det \alpha)^{\frac{m}{2}} \langle \partial_x, F^{(M)}_{\Omega, Z} \rangle.
\]

Now we put
\[
\rho_M(g(\alpha_{+})) = (\det \alpha)^{\frac{m}{2}},
\]
\[
\rho_M(g(\alpha_{-})) = (-1)^m (\det \alpha)^{\frac{m}{2}},
\]
\[
\rho_M(g(\alpha_{+})) = i^m (\det \alpha)^{\frac{m}{2}},
\]
\[
\rho_M(g(\alpha_{-})) = (-i)^m (\det \alpha)^{\frac{m}{2}}.
\]

Therefore \( \partial_x \) satisfies the semi-invariance relation (6.7) in the case \( \tilde{\gamma}_s = ((g(\alpha), \epsilon), (0, 0; 0)) \) with \( \epsilon = \pm 1, \pm i, \alpha \in GL(n, \mathbb{Z}) \).

**Case IV.** \( \tilde{\gamma}_s = ((\sigma_n, \epsilon), (0, 0; 0)) \) with \( \sigma_n = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \) and \( \epsilon^2 = (-i)^n \).

In this case, we have
\[
\Omega_s = -\Omega^{-1} \quad \text{and} \quad Z_s = Z \Omega^{-1}.
\]

In the process of the proof of Theorem 5.1, using Lemma 5.1, we already showed that
\[
(6.8) \quad \int_{\mathbb{R}^{(m,n)}} e^{\pi i \sigma(M(y^t \Omega^t y+2 y^t Z))} dy = (\det \mathcal{M})^{\frac{n}{2}} \left( \det \frac{\Omega}{i} \right)^{-\frac{m}{2}} e^{-\pi i \sigma(MZ \Omega^{-1} Z)}.
\]

By (6.8), we see that
\[
(6.9) \quad \hat{F}^{(M)}_{\Omega, Z}(\mathcal{M} \cdot x) = (\det \mathcal{M})^{\frac{n}{2}} \left( \det \frac{\Omega}{i} \right)^{-\frac{m}{2}} e^{-\pi i \sigma(M(Z-x)(Z-x))}.
\]

where \( \hat{f} \) is the Fourier transform of \( f \) defined by
\[
\hat{f}(x) = \int_{\mathbb{R}^{(m,n)}} f(y) e^{-2 \pi i \sigma(y^t x)} dy, \quad x \in \mathbb{R}^{(m,n)}.
\]
On the other hand, in the process of the proof of Case IV in Theorem 5.1, we showed that
\[ J_M^* \left( \bar{g}_s, (\Omega, Z) \right) = e^{-m} \left( \det \Omega \right)^{\frac{m}{2}} e^{\pi i \sigma (MZ \Omega^{-1} Z)}. \]

According to the covariance relation (5.4), Formula (6.9) and Poisson summation formula, we obtain

\[
\left\langle \vartheta, \omega_M(\bar{\gamma}_s), \mathcal{F}^{(M)}_{\Omega, Z} \right\rangle \\
= \left\langle \vartheta, J_M^*(\bar{\gamma}_s, (\Omega, Z))^{-1} \mathcal{F}^{(M)}_{\bar{\gamma}_s}((\Omega, Z)) \right\rangle \\
= J_M^*(\bar{\gamma}_s, (\Omega, Z))^{-1} \left\langle \vartheta, \mathcal{F}^{(M)}_{-\Omega^{-1} Z \Omega^{-1}} \right\rangle \\
= e^m \left( \det \Omega \right)^{-\frac{m}{2}} e^{-\pi i \sigma (MZ \Omega^{-1} Z + A \Omega^{-1} A - 2 A \Omega^{-1} Z)} \\
= e^m \left( \det \Omega \right)^{-\frac{m}{2}} \sum_{A \in \mathbb{Z}^{(m,n)}} e^{-\pi i \sigma (M(Z-A) \Omega^{-1} (Z-A))} \\
= e^m \left( \det \Omega \right)^{-\frac{m}{2}} \sum_{A \in \mathbb{Z}^{(m,n)}} \mathcal{F}^{(M)}_{\Omega, Z}(M \ A) \quad \text{(by Formula (6.9))} \\
= e^m \left( \det I_n \right)^{\frac{m}{2}} \sum_{A \in \mathbb{Z}^{(m,n)}} \mathcal{F}^{(M)}_{\Omega, Z}(A) \quad \text{(because } \det(M) = 1 \text{)} \\
= e^m \left( \det I_n \right)^{\frac{m}{2}} \sum_{A \in \mathbb{Z}^{(m,n)}} \mathcal{F}^{(M)}_{\Omega, Z}(A) \quad \text{(by Poisson summation formula)} \\
= e^m (-i)^{\frac{mn}{2}} \left\langle \vartheta, \mathcal{F}^{(M)}_{\Omega, Z} \right\rangle
\]

We put
\[ \rho_M(\bar{\gamma}_s) = e^m (-i)^{\frac{mn}{2}}. \]

Therefore \( \vartheta \) satisfies the semi-invariance relation (6.7) in the case \( \bar{\gamma}_s = ((\sigma_n, \epsilon), (0, 0; 0)) \) with \( \sigma_n = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \) and \( \epsilon^2 = (-i)^n \). The proof of Case IV is completed.

Since \( J_M^* \) is an automorphic factor for \( G^J_n \) on \( \mathbb{H}_{n,m} \), we see that if the formula (6.6) holds for two elements \( \bar{\gamma}_1, \bar{\gamma}_2 \) in \( \Gamma_s^* \), then it holds for \( \bar{\gamma}_1 \bar{\gamma}_2 \). Finally we complete the proof of Theorem 6.1.

\[ \square \]

**Corollary 6.1.** Let \( \Gamma_s^J \) and \( \rho_M \) be as before in Theorem 6.1. If \( m \) is odd, \( \Theta_M(\Omega, Z) \) is a Jacobi form of a half-integral weight \( \frac{m}{2} \) and index \( \frac{M}{2} \) with respect to an arithmetic subgroup \( \Gamma_s^J \) for a character \( \rho_M \) of \( \Gamma_s^J \).
Remark 6.1. Let \( a = (a_1, a_2) \in \mathbb{Z}^n \times \mathbb{Z}^n \) with \( a_1, a_2 \in \mathbb{Z}^n \). Takase \cite{19} considered the following theta series defined by

\[
\vartheta_a(\Omega, Z) := \sum_{\ell \in \mathbb{Z}^n} e^{\pi i \left( (\ell + a_1) \Omega^t (\ell + a_2) + 2 (\ell + a_1)^t (Z + a_2) \right)},
\]

where \( \Omega \in \mathbb{H}_n \) and \( Z \in \mathbb{C}^n \). We put

\[
\vartheta_a^*(\Omega, Z) := e^{-a_1^t a_2} \cdot \vartheta_a(\Omega, Z).
\]

We let \( \Gamma_0 \) be a discrete subgroup of \( \Gamma_n \) consisting of \( \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) such that

1. \( L \gamma = L \), where \( L = \mathbb{Z}^n \times \mathbb{Z}^n \).
2. \((xA + yC)^t (xB + yD) \equiv x^t y \pmod{2\mathbb{Z}}\) for all \((x, y) \in L\).

We put \( \Gamma_{*,0} = \pi_{*}^{-1}(\Gamma_0) \) and

\[
\Gamma_{*,0}^J = \Gamma_{*,0} \rtimes H_{Z}^{(n,1)}.
\]

He proved that for any \( \tilde{\gamma}_s = ((\gamma, \epsilon), (\lambda, \mu; t)) \in \Gamma_{*,0}^J \) with \( \gamma \in \Gamma_0 \), the following transformation formula

\[
\vartheta_{a(\gamma \tilde{\gamma}_s)}(\Omega, Z) = \rho((\gamma, \epsilon)) \chi_a((\lambda, \mu; t)) J_1(\tilde{\gamma}_s, (\Omega, Z)) \vartheta_{a}^*(\Omega, Z)
\]

holds, where \( \chi_a \) denotes the unitary character of \( L \times \mathbb{R} \) defined by

\[
\chi_a((\lambda, \mu; t)) = e^{2\pi i \left( 1 + \frac{1}{2} \lambda^t \mu - \lambda^t a_2 + a_1^t \mu \right)}, \quad (\lambda, \mu; t) \in L \times \mathbb{R}
\]

and \( 1 = (1) \) denotes the \( 1 \times 1 \) matrix. Here \( \rho : \Gamma_{*,0} \longrightarrow T \) is a certain unitary character that is given explicitly in \cite{19} Theorem 5.3, p. 134.

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