DYNAMICAL COMPLEXITY IN A DELAYED PLANKTON-FISH MODEL WITH ALTERNATIVE FOOD FOR PREDATORS

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Abstract. The present manuscript deals with a 3-D food chain ecological model incorporating three species phytoplankton, zooplankton, and fish. To make the model more realistic, we include predation delay in the fish population due to the vertical migration of zooplankton species. We have assumed that additional food is available for both the predator population, viz., zooplankton, and fish. The main motive of the present study is to analyze the impact of available additional food and predation delay on the plankton-fish dynamics. The positivity and boundedness (with and without delay) are proved to make the system biologically valid. The steady states are determined to discuss the stability behavior of non-delayed dynamics under certain conditions. Considering available additional food as a control parameter, we have estimated ranges of alternative food for maintaining the sustainability and stability of the plankton-fish ecosystem. The Hopf-bifurcation analysis is carried out by considering time delay as a bifurcation parameter. The predation delay includes complexity in the system dynamics as it passes through its critical value. The direction of Hopf-bifurcation and stability of bifurcating periodic orbits are also determined using the centre manifold theorem. Numerical simulation is executed to validate theoretical results.

1. Introduction. Theoretical ecology is a branch of science, which deeply studies ecological systems using mathematical models. The main work of ecologists is to unfold the novel and realistic insights about nature and make predictions about the diverse biological world. These predictions are helpful to understand various ecological phenomena, such as conservation of species, climate change, effects on...
the food chain, and the global carbon cycle. The marine ecosystem is a significant ecosystem that provides various biological, social, and economic services to humans. The phytoplankton and zooplankton species consist of small animals and the immature stages of larger animals in oceans. The plankton and fish biomass are the utmost essential components of marine life, as the extinction and survival of these species directly affect the ecological balance. Additional food or alternative food available for the zooplankton and fish population is a significant part of marine life due to its extensive utilization for the conservation and co-existence of plankton-fish species. The authors in [8] have explored the impact of available additional food on the prey-predator dynamics when the predation risk is low due to prey refuge. They have investigated that available alternative food can save the predator population from extinction in high prey refuge. Srinivasu et al. [27] have analyzed a mathematical model in which they have studied the global dynamics of the system when additional food is available for the predator, which is harvested at a constant rate. The biologists in [36, 37] have determined that the sufficient amount of available additional food to the predator population enhances their density and rate of predation, which reduces the density of target prey. But, [13] have investigated that the provision of alternative food to predators need not elevate the target predation. This conflict between empirical and recent theories led to a thorough study of the factor, viz., alternative food. Scientists in [35] have determined that the quality and quantity of additional food plays a vital role in the survival of plankton species. Extending the work of [35], Srinivasu et al. [33], have studied time-optimal control mechanisms to drive the state of the dynamics from a starting point to endpoint in minimum time using the quality of the alternative food as a control parameter. Scientists in [20,34] have studied the impacts of alternative food on their proposed system dynamics to understand the concepts of pest management and resource management. Researchers in [12, 26] have investigated that the natural supply of alternative food (non-prey) for predators is very significant for the survival of the predator when the preferred prey is infected. Many biologists, mathematician, and theoreticians [12,15,16,23,25,26,32] disclosed the consequences of providing alternate food to predators in a predator-prey dynamics and determined that almost all predators will divert to alternate prey when the preferred prey is infected or in limited numbers. Chakraborty et al. [2] have determined the role of available additional food for the co-existence of plankton species. They have observed that the extinction of the predator population reduces due to alternative food in the presence of toxicity.

In the real world, nothing is instantaneous as predator species not always successful in catching and killing prey. The delay-induced mathematical models have demonstrated more realistic but complex dynamics since a time delay can cause instability in the system by inducing various oscillations. and periodic solutions [4, 9, 18]. The impact of different types of time delays (maturation delay, predation delay, toxin liberation delay and, gestation delay, etc.) on various plankton dynamics has been studied by [3, 5, 7, 17, 21, 22, 24, 28, 31]. Mukhopadhyay and Bhattacharyya [19] have developed a stochastic extension of the plankton-fish dynamics to study the stability and bifurcation phenomena. The study [6] analyzed a plankton-fish model with digestion delay in the fish population to determine the threshold values of the bifurcation parameter to study the Hopf-bifurcation analysis. Sharma et al. [29] have demonstrated in their research that predation delay in fish species due to vertical migration of zooplankton species can induce excitability.
in the stable plankton-fish dynamics. Extensive work has been done to analyze the consequences of the availability of alternative food and predation delay on various prey-predator systems, separately. But, according to the best of our knowledge, the simultaneous effect of additional food and predation delay (in the fish population due to vertical migration of the zooplankton population) along with non-linear quadratic harvesting in the plankton-fish model is rarely seen. Motivated by [2,29], we present a delayed ecological model to study the significance of the availability of additional food sources to the zooplankton and fish population for the co-existence and survival of plankton-fish species. Here, we have proposed and analyzed plankton-fish dynamics consisting of the biomass of Phytoplankton $P(t)$, Zooplankton $Z(t)$, and Fish $F(t)$, respectively incorporating available additional food for both predator populations, viz., Zooplankton and Fish. We present our manuscript in the following sections.

We have proposed and analyzed the non-delayed dynamical system, its boundedness, positivity, stability analysis, and Hopf-bifurcation analysis in Section 2, Section 3, section 4, and Section 5, respectively. The non-delayed model, its positivity and boundedness, stability, Hopf-bifurcation, and direction of periodic solutions are discussed in Section 6 and Section 7. Numerical validation of analytical results is presented in Section 8 and the conclusion of the paper is given in Section 9.

| Parameter | Biological interpretation |
|-----------|--------------------------|
| $r_1$     | Growth rate of Phytoplankton. |
| $\alpha_1$ | Death rate phytoplankton. |
| $\beta_1$ | Maximum capture rate of phytoplankton by zooplankton. |
| $\beta_2$ | Maximum conversion rate of zooplankton. |
| $\gamma_1$ | Half saturation constant. |
| $\gamma_2$ | Half saturation constant. |
| $r_5$     | Additional food available for zooplankton. |
| $r_2$     | Death rate of zooplankton. |
| $a_1$     | Maximum capture rate of zooplankton by fish. |
| $a_2$     | Maximum conversion rate of fish. |
| $\theta$  | Rate of toxin produced by TPP. |
| $r_3$     | Additional food available for fish. |
| $\alpha_2$ | Rate of quadratic harvesting. |
| $r_4$     | Natural mortality rate of fish. |

2. The mathematical model. Let $P(t)$, $Z(t)$, and $F(t)$ be the population densities of the toxin-producing phytoplankton (TPP), specialist predator zooplankton, and Fish. It is assumed that phytoplankton is predated by zooplankton, which is a favorite food for the generalist predator fish. Most of the zooplankton species graze phytoplankton as their prey. Still, some zooplankton like krill, jellyfish, shrimps, and crabs prefer additional food (small zooplankton, the dead mass of sea organisms, larvae, bacteria, eggs, worms, organic particles, etc.) for their growth like Similarly, most of the ocean fish species consume zooplankton as their prey but some fish species, e.g., piscivorous fish, apex fish predators, marine mammals, and reptiles, etc. prefer planktivorous fish as additional food. Thus, alternate food is
available for both the zooplankton and the fish population. The phytoplankton and zooplankton species are both predated by Holling type-II response function. The toxin-producing phytoplankton species are harmful to zooplankton. Its effect is modeled by $P$. It is assumed that the death rate of fish species depends on two factors, namely, quadratic harvesting ($a_2$) and natural mortality ($r_4$) ([1, 30]).

Apart from natural death, outbreaks of disease, mass starvation, harmful algal blooms, or extreme over-fishing, predation by other animals are the largest source of mortality of fishes in the oceans. This tri-trophic interaction system (with the biological interpretation of parameters given in table.1) is proposed by the following set of differential equations,

$$
\begin{align*}
\frac{dP}{dt} &= r_1 P - \alpha_1 P^2 - \beta_1 \frac{P}{\gamma_1 + P} Z, \\
\frac{dZ}{dt} &= r_3 Z + \beta_1 \beta_2 \frac{P}{\gamma_1 + P} Z - r_2 Z - a_1 \frac{Z}{\gamma_2 + Z} F - \theta \frac{P}{\gamma_1 + P} Z, \\
\frac{dF}{dt} &= r_3 F - \alpha_2 F^2 + a_1 a_2 \frac{Z}{\gamma_2 + Z} F - r_4 F.
\end{align*}
$$

(1)

3. Dynamical properties of the plankton system.

3.1. Positivity and boundedness.

**Theorem 3.1.** The dynamical system (1) has a unique and nonnegative solution with the initial values $(P(0), Z(0), F(0)) \in R^3_+$, where $R^3_+ = \{(x_1, x_2, x_3) : x_i \geq 0, i = 1, 2, 3\}$. Further, the set

$$
\Gamma = \{(P(t), Z(t), F(t)) \in R^3, U(t) \leq \frac{r_2^2}{\alpha_1} + \frac{\beta_1(r_3 - r_4)^2}{a_2 \alpha_2 (\beta_1 \beta_2 - \theta)} + \epsilon, \forall \epsilon > 0 \}
$$

and $\beta_1 \beta_2 > \theta$ is invariant for all the solutions in the interior of the positive octant.

**Proof.** The dynamical system (1) in the form of matrix is as follows, $\frac{dH}{dt} = H(x)$, where

$$
x = (x_1, x_2, x_3)^T = (P, Z, F)^T \in R^3, \quad H(x) = \begin{pmatrix} H_1(x) \\ H_2(x) \\ H_3(x) \end{pmatrix}
$$

and

$$
H(x) = \begin{pmatrix} r_1 P - \alpha_1 P^2 - \beta_1 \frac{P}{\gamma_1 + P} Z \\ r_3 Z + \beta_1 \beta_2 \frac{P}{\gamma_1 + P} Z - r_2 Z - a_1 \frac{Z}{\gamma_2 + Z} F - \theta \frac{P}{\gamma_1 + P} Z \\ r_3 F - \alpha_2 F^2 + a_1 a_2 \frac{Z}{\gamma_2 + Z} F - r_4 F \end{pmatrix}.
$$

Since, $H : R^3 \rightarrow R^3$ is locally Lipschitz continuous in $\Gamma$ along with $x(0) = x_0 \in R^3$, thus by fundamental theorem of ordinary differential equations, there must exist unique solution of (1). As, $H(x)|_{x_i(t)} = 0, x \in R^3 \geq 0$, then $[10, 11]$ ensures that $x(t) > 0$ for $t \geq 0$. From model system (1), $\frac{DP}{dt} F = 0, \frac{dF}{dt} z = 0$ and $\frac{dF}{dt} F = 0$. Therefore, the system (1) has unique +ve solution.

Next, we claim that all these solutions are uniformly bounded in the octant $\Gamma$.

Let

$$
U(t) = P(t) + \frac{\beta_1}{\beta_1 \beta_2 - \theta} Z(t) + \frac{\beta_1}{a_2 (\beta_1 \beta_2 - \theta)} F(t),
$$

(2)
The mathematical model (1) has the following steady states in the closed first octant around equilibrium states,

\[
\begin{align*}
\frac{dU}{dt} &\leq -\alpha_1(P(t) - \frac{r_1}{\alpha_1})^2 + \frac{r_1^2}{\alpha_1} - r_1 P(t) - \frac{r_2 \beta_1}{(\beta_1 \beta_2 - \theta)} Z(t) \\
&- \frac{a_2 \beta_1}{a_2 \beta_1} (F(t) - \frac{r_3 - r_4}{\alpha_2})^2 + \frac{\beta_1 (r_3 - r_4)^2}{a_2 \alpha_2 (\beta_1 \beta_2 - \theta)} - \frac{\beta_1 (r_3 - r_4)}{a_2 (\beta_1 \beta_2 - \theta)} F(t) \\
&\leq \frac{r_1^2}{\alpha_1} + \frac{\beta_1 (r_3 - r_4)^2}{a_2 \alpha_2 (\beta_1 \beta_2 - \theta)} - r_1 P(t) - \frac{r_2 \beta_1}{(\beta_1 \beta_2 - \theta)} Z(t) - \frac{\beta_1 (r_3 - r_4)}{a_2 (\beta_1 \beta_2 - \theta)} F(t),
\end{align*}
\]

where

\[
\eta = \min \{r_1, \frac{r_2 \beta_1}{(\beta_1 \beta_2 - \theta)} \}
\]

implies

\[
0 \leq U(t) \leq \frac{r_1^2}{\alpha_1} + \frac{\beta_1 (r_3 - r_4)^2}{a_2 \alpha_2 (\beta_1 \beta_2 - \theta)} + \frac{U(P(0), Z(0), F(0))}{e^{\eta t}}
\]

(using comparison theorem of ODE [11]).

As \( t \to \infty \), we have

\[
U(t) \leq \frac{r_1^2}{\alpha_1} + \frac{\beta_1 (r_3 - r_4)^2}{a_2 \alpha_2 (\beta_1 \beta_2 - \theta)},
\]

which implies that the solutions are bounded for

\[
0 \leq U(t) \leq \frac{r_1^2}{\alpha_1} + \frac{\beta_1 (r_3 - r_4)^2}{a_2 \alpha_2 (\beta_1 \beta_2 - \theta)}.
\]

Therefore, all the solutions of the given plankton system are lies in the octant,

\[
\Gamma = \{(P(t), Z(t), F(t)) \in R^{3+} : P \geq 0, Z \geq 0, F \geq 0\}
\]

\forall \epsilon > 0 and \( \beta_1 \beta_2 > \theta \). \hfill \Box

4. Stability analysis. In this section, we determine the stability of the dynamical system around equilibrium states.

The mathematical model (1) has the following steady states in the closed first octant \( R^{3+} = \{(P, Z, F) : P \geq 0, Z \geq 0, F \geq 0\} \).

- \( V_0(0, 0, 0) \), a trivial steady state always exist and unstable as \( r_1 \) is positive eigenvalue.

- \( V_1(\frac{r_4}{\alpha_1}, 0, 0) \), the zooplankton and fish free steady state exists and stable for \( r_4 > r_3 \) and \( r_2 (\gamma_1 \alpha_1 + r_1) + \theta_1 r_3 > \beta_1 \beta_2 \).

- \( V_2(\frac{r_4}{\alpha_1}, \frac{r_4 - r_3}{\alpha_1}, 0) \) exists if \( r_4 > r_3 \) means zooplankton free steady state exists if the quantity of available alternate food for fish is more than its natural death rate and stable for \( \beta_1 \beta_2 < \theta \).

- The fish free equilibrium \( V_3(P_3, Z_3, 0) \), where \( P_3 = \frac{(r_2 - r_5) \gamma_1}{\beta_1 \beta_2 - \theta + r_5 - r_2} \)

\[
Z_3 = r_1 \gamma_1 \beta_1 \beta_2 - \theta + r_5 - r_2)^2 + (r_2 - r_5) \gamma_1 (r_1 - \alpha_1 \gamma_1) (\beta_1 \beta_2 - \theta + r_5 - r_2) - \alpha_1 (r_2 - r_5)^2 \gamma_1^2 \beta_1 \beta_2 - \theta + r_5 - r_2)^2
\]
exists for \((r_1 - \alpha_1 \gamma_1)(\beta_1 \beta_2 - \theta + r_5 - r_2) > \alpha_1 (r_2 - r_5)\). The equilibrium state \(V_3\) is stable under the conditions \(r_1 < r_2, r_3 < r_4\) which conveys that the growth rate of phytoplankton should be less than mortality rate of zooplankton and availability of alternate food for fish should be less than its natural mortality rate and \((\beta_1 \gamma_1)(\gamma_2 + Z_3) > \alpha_1 a_2 (\gamma_1 + P_3)\).

- The positive interior equilibrium \(V_*(P_*, Z_*, F_*)\) exists, where
  \[
  Z_* = \frac{(r_1 - \alpha_1 P_*)(\gamma_1 + P_*)}{\beta_1}, \quad F_* = \frac{r_3 - r_4}{\alpha_2} + \frac{a_1 a_2 Z_*}{(\gamma_1 + Z_*) \alpha_2}
  \]
  and \(P_*\) is a positive zero of
  \[
  N(P_*) = N_1 P_*^5 + N_2 P_*^4 + N_3 P_*^3 + N_4 P_*^2 + N_5 P_* + N_6 = 0,
  \]
  where coefficients of \(N(P_*)\) are given in Appendix I.

The characteristic equation of above system at \(V_*\) is given by
\[
\lambda^3 + A_1 \lambda^2 + (A_2 + B_2) \lambda + (B_1 + A_3) = 0, \tag{2}
\]
where
\[
A_1 = -(a_{100} + b_{010} + c_{001}), \quad A_2 = a_{100} b_{010} - a_{010} b_{100} + a_{100} c_{001} + c_{001} b_{010},
\]
\[
A_3 = -a_{100} b_{010} c_{001} + a_{010} b_{010} c_{001}, \quad B_1 = a_{100} c_{010} b_{001}, \quad B_2 = -B_001 c_{010},
\]
\[
A_3 = -a_{100} b_{100} c_{001} + a_{100} b_{100} c_{001}, \quad B_1 = a_{100} c_{010} b_{001}, \quad B_2 = -B_001 c_{010},
\]
\[
a_{100} = r_1 - 2a_1 P_* - \frac{\beta_1 \gamma_1 Z_*}{(\gamma_1 + P_*)^2}, \quad a_{010} = -\frac{\beta_1 P_*}{(\gamma_1 + P_*)}, \quad a_{001} = 0,
\]
\[
b_{100} = \frac{(\beta_1 \beta_2 - \theta) \gamma_1 Z_*}{(\gamma_1 + P_*)^2}, \quad b_{010} = r_5 - r_2 + \frac{(\beta_1 \beta_2 - \theta) P_*}{(\gamma_1 + P_*)} - \frac{a_1 \gamma_1 F_*}{(\gamma_2 + Z_*)}.
\]
\[
B_{001} = -\frac{a_1 Z_*}{(\gamma_2 + Z_*)}, \quad c_{010} = \frac{a_1 a_2 \gamma_2 F_*}{(\gamma_2 + Z_*)^2}, \quad c_{001} = r_3 - 2a_2 F_* - r_4 + \frac{a_1 a_2 Z_*}{\gamma_2 + Z_*}.
\]

The equation (2) can be written as
\[
\lambda^3 + H_1 \lambda^2 + H_2 \lambda + H_3 = 0, \tag{3}
\]
where \(H_1 = A_1, H_2 = A_2 + B_2\) and \(H_3 = B_1 + A_3\). The Routh-Hurwitz criterion certifies that the zeros of (3) have \(-ve\) real parts, i.e. the \(+ve\) interior equilibrium \(V_*\) is LAS (Locally asymptotically stable) under the following condition, \((T_1) : H_i > 0, i = 1, 3\) and \(H_1 H_2 - H_3 > 0\).

5. Hopf-bifurcation Analysis.

**Theorem 5.1.** The system (1) enters into Hopf-bifurcation around the interior point \(V_*\) as \(r_5\) passes through its critical value \(r_5^*\) under the following conditions,
1. \(H_i(r_5^*) > 0, i=1,3; H_1(r_5^*) H_2(r_5^*) - H_3(r_5^*) = 0,\)
2. \((H_1(r_5^*) H_2(r_5^*))' \neq (H_3(r_5^*))'.\)

**Proof.** We consider \(r_5\) as a bifurcation parameter, the given plankton system shows excitability if there exists a critical value \(r_5^*\) of \(r_5\) such that \(H_1(r_5^*) H_2(r_5^*) - H_3(r_5^*) = 0\). Thus, the characteristic equation
\[
\lambda^3 + H_1 \lambda^2 + H_2 \lambda + H_3 = 0, \tag{4}
\]
must have of the following form at \(r_5 = r_5^*\)
\[
(\lambda^2 (r_5^*) + H_2(r_5^*) (\lambda (r_5^*) + H_1 (r_5^*)) = 0. \tag{5}
\]
which clearly have roots $-H_1(r_5^*)$ and $\pm i\sqrt{H_2(r_5^*)}$. But, in general, $\lambda_1(r_5) = u(r_5) + \nu v(r_5)$, $\lambda_2(r_5) = u(r_5) - \nu v(r_5)$, and $\lambda_3(r_5) = H_1(r_5)$. Substituting values of $\lambda_i, i = 1, 2$ in (4) and calculating the derivatives, we get
\[
\begin{cases}
M_1(r_5)u'(r_5) - M_2(r_5)v'(r_5) + M_3(r_5) = 0, \\
M_1(r_5)u'(r_5) + M_2(r_5)v'(r_5) + M_4(r_5) = 0,
\end{cases}
\]
where
\[
M_1(r_5) = 3u^2(r_5) + 2H_1(r_5)u(r_5) + H_2(r_5) - 3u^2(r_5),
M_2(r_5) = 6u(r_5)v(r_5) + 2H_1(r_5)v(r_5),
M_3(r_5) = u^2(r_5)H'_1(r_5) + H'_2(r_5)u(r_5) + H'_1(r_5) - H'_1'(r_5)v^2(r_5),
M_4(r_5) = 2u(r_5)v(r_5)H'_1(r_5) + H'_2(r_5)v(r_5).
\]
Taking $u(r_5^*) = 0$ and $v(r_5^*) = \sqrt{H_2(r_5^*)}$, we obtain
\[
M_1(r_5^*) = -2H_2(r_5^*), \quad M_2(r_5^*) = 2H_1(r_5^*)\sqrt{H_2(r_5^*)},
M_3(r_5^*) = H'_3(r_5^*) - H'_1(r_5^*)H_2(r_5^*), \quad M_4(r_5^*) = H'_2(r_5^*)\sqrt{H_2(r_5^*)},
\]
Solving (6) for $u'(r_5)$, we get
\[
(u'(r_5))_{r_5 = r_5^*} = \frac{-M_2(r_5^*)M_4(r_5^*) + M_1(r_5^*)M_3(r_5^*)}{M_1^2(r_5^*) + M_2^2(r_5^*)} = \frac{-(H_1(r_5^*)H_2(r_5^*))' - H'_1(r_5^*)}{2(H_1(r_5^*)^2 + H_2(r_5^*))} \neq 0
\]
(using given hypothesis).}

6. Delayed model system. In this section, we include predation delay in fish population and obtained the following set of differential equations,
\[
\begin{align*}
\frac{dP}{dt} &= r_1P - \alpha_1P^2 - \beta_1\frac{P}{\gamma_1 + P}Z, \\
\frac{dZ}{dt} &= r_5Z + \beta_1\beta_2\frac{P}{\gamma_1 + P}Z - r_2Z - a_1\frac{Z}{\gamma_2 + Z}F(t - \tau) - \theta\frac{P}{\gamma_1 + P}Z, \\
\frac{dF}{dt} &= -\alpha_2F^2 + a_1a_2\frac{Z}{\gamma_2 + Z}F - r_4F.
\end{align*}
\]

6.1. Positivity and boundedness.

**Theorem 6.1.** The positive interior equilibrium $V_*(P^*, Z^*, F^*)$ of the dynamical system (7) is invariant in $+ve$ quadrant.

**Proof.** We want to show that $\forall \, 0 \leq t < T^*$, $(P^*) > 0$, $P(t) > 0$, $Z(t) > 0$ and $F(t) > 0$ with the initial conditions $P(0) > 0$, $Z(0) > 0$ and $F(0) > 0$, otherwise, it can be assumed that $K$ where $0 < K < T^*$ such that $\forall \, t \in [0, K)$, $P(t) > 0$, $Z(t) > 0$ and $F(t) > 0$ and one of $P(K)$, $Z(K)$ and $F(K)$ is zero for any $t \in [-\tau, K)$.

Integrating the given model system (7), we have
\[
\begin{align*}
P(K) &= P(0)e^{\int_0^K (r_1 - \alpha_1P - \beta_1\frac{P}{\gamma_1 + P})ds} \\
Z(K) &= Z(0)e^{\int_0^K (r_5 + \beta_1\beta_2\frac{P}{\gamma_1 + P} - r_2 - a_1\frac{Z}{\gamma_2 + Z}F(t - \tau) - \theta\frac{P}{\gamma_1 + P})ds} \\
F(K) &= F(0)e^{\int_0^K (r_3 - \alpha_2F + a_1a_2\frac{Z}{\gamma_2 + Z}F - r_4F)ds}.
\end{align*}
\]
Since $P(t)$, $Z(t)$ and $F(t)$ are all continuous functions in $[-\tau, K)$, there exist $S > 0$ such that $\forall t \in [-\tau, K)$

\[
P(K) = P(0)e^{\int_0^t (r_1 - \alpha_1 P - \beta_1 \frac{P}{1+P}) ds} > P(0)e^{-KS},
\]

\[
Z(K) = Z(0)e^{\int_0^t (r_5 + \beta_1 \frac{P}{1+P} - r_2 - a_1 \frac{1}{1+Z} + F(t-\tau) - \theta P) ds} > Z(0)e^{-KS},
\]

\[
F(K) = F(0)e^{\int_0^t (r_3 - \alpha_2 F + a_1 a_2 \frac{Z}{1+Z} - r_4) ds} > F(0)e^{-KS}.
\]

Taking $t \to K$, we get $P(K) > 0, Z(K) > 0$ and $F(K) > 0$, a contradiction. Thus $P(t) > 0, Z(t) > 0$ and $F(t) > 0$ for any $0 \leq t < T^*$.

6.2. Delayed stability analysis. Now, we are interested to observe the impact of time lag $\tau$ on the given dynamical system (7). So, initially we linearize the
The characteristic equation (15) can be written in an exponential polynomial in $\lambda$ as,

$$L_1(\lambda, \tau) + L_2(\lambda, \tau)e^{-\lambda\tau} = 0,$$

where $L_1(\lambda, \tau) = \lambda^3 + A_1\lambda^2 + A_2\lambda + A_3$, $L_2(\lambda, \tau) = B_1 + B_2\lambda$.

Next, to apply the rules given in [18] we verify the following five properties.

(i) $L_1(0, \tau) + L_2(0, \tau) = A_3 + B_1 \neq 0$.

(ii) $L_1(\omega_1, \tau) + L_2(\omega_1, \tau) = -\omega_1^3 - A_1\omega_1^2 + \omega_2\omega_1 + A_3 + (B_1 + \omega B_2)e^{-\omega_1\tau} \neq 0$.

(iii) $\lim_{|\lambda| \to \infty} |L_1(\lambda, \tau)| = \lim_{|\lambda| \to \infty} \left| \frac{(B_1 + B_2\lambda)}{\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3} \right| = 0 < 1,

(iv) $G_2(\omega) = |L_1(\omega, \tau)|^2 - |L_2(\omega, \tau)|^2$ is a polynomial of degree 6. Thus it has finite number of zeros.

(v) Every positive zero $\omega_1(\tau)$ of $G_2(\omega_1(\tau)) = 0$ is differentiable and continuous in $\tau$ whenever it exists (using implicit function theorem).

Therefore, we can check that for some $\tau > 0$, $\lambda = \omega_1(\omega_1 > 0)$ is a root of the characteristic equation (15) and by substituting $\lambda = \omega_1(\omega_1 > 0)$ in equation (15), we can get the equation given below,

$$-\omega_1^3 - A_1\omega_1^2 + A_2\omega_1 + A_3 = -(B_1 + \omega B_2)e^{-\omega_1\tau}.$$
Separating the real and imaginary parts, we get

\[ \lambda \]

\[ \rho \]

\[ \omega \]

\[ \tau \]

\[ \omega_1^0 + (A_1^2 - 2A_2)\omega^4 + (A_2^2 - 2A_1A_3 - B_2^2)\omega_1^2 + (A_3^2 - B_1^2) = 0. \]

Put \( \omega = \rho \) in (19), we get a cubic equation.

\[ Y(\rho) = \rho^3 + M_1 \rho^2 + M_2 \rho + M_3 = 0, \]

where

\[ M_1 = (A_1^2 - 2A_2), \quad M_2 = (A_2^2 - 2A_1A_3 - B_2^2), \quad M_3 = A_3^2 - B_1^2. \]

Let us assume \( M_3 < 0 \) then this implies \( Y(0) < 0, \ Y(\infty) = \infty \) and equation (19) has at least single positive root \( \omega_0 \).

Now, the following theorem shows the existence of Hopf-bifurcation for \( \tau > 0 \) as bifurcation parameter.

**Theorem 6.2.** Suppose \( M_3 < 0 \) then there exists \( \tau_k \) such that \( V_\ast \) is LAS for \( \tau \in (0, \tau_k) \) and unstable when \( \tau > \tau_k \). Furthermore, system (1) undergoes a Hopf-bifurcation with the occurrence of periodic oscillation for \( \tau = \tau_k \) and the critical value of time delay (\( \tau \)) is given by,

\[ \tau_k = \frac{1}{\omega_1^0} \arctan \frac{B_2\omega_1^0(A_1\omega_1^2 - A_3) + B_1(A_2\omega_1^0 - \omega_1^3)}{(A_1\omega_1^2 - A_3) - B_2\omega_1^0(A_2\omega_1^0 - \omega_1^3)} + \frac{2k\pi}{\omega_1^0} \]

for \( k = 0, 1, 2, ... \) provided \( B_1((A_2 - 3\omega_1^2)\sin \omega_1^0 \tau + 2\omega_1^0 A_1 \cos \omega_1^0 \tau) - \omega_1^3 B_2(\omega_1^0 A_1 A_2 - 3\omega_1^2) \cos \omega_1^0 \tau - 2\omega_1^0 A_1 \sin \omega_1^0 \tau + B_2) \neq 0 \).

**Proof.** Let the condition \( M_3 < 0 \) holds true then (19) has at least unique +ve zero. Thus equation (15) has pair of imaginary zeros \( \pm i\omega_1^0 \) (say).

After solving (17) and (18), we get,

\[ \tau_k = \frac{1}{\omega_1^0} \arctan \frac{B_2\omega_1^0(A_1\omega_1^2 - A_3) + B_1(A_2\omega_1^0 - \omega_1^3)}{(A_1\omega_1^2 - A_3) - B_2\omega_1^0(A_2\omega_1^0 - \omega_1^3)} + \frac{2k\pi}{\omega_1^0} \]

for \( k = 0, 1, 2, ... \). Taking \( \lambda(\tau) = \sigma(\tau) + i\omega_1^0(\tau) \) in (15), we can observe that conjugate pair of imaginary zeros \( \lambda_{\pm}(\tau_0) = \pm i\omega_1^0(\tau_0) \) of (6.2) exists at \( \tau = \tau_0 \), which passes through imaginary axis from left to right if the transversality condition that is \( \sigma(\tau_0) > 0 \) or right to left if \( \sigma(\tau_0) < 0 \), where \( \sigma(\tau_0) = \text{sign}(\frac{d\Re(\lambda)}{d\tau})_{\lambda=i\omega_1^0} \), which can be obtained as

\[ \left(\left(\frac{d\Re(\lambda)}{d\tau}\right)^{-1}\right)_{\sigma=0,\tau=\tau_0} = \frac{L}{\omega_1^0 (B_1^2 + \omega_1^0 B_2^2)} \neq 0, \]

where \( L = B_1((A_2 - 3\omega_1^2)\sin \omega_1^0 \tau + 2\omega_1^0 A_1 \cos \omega_1^0 \tau) - \omega_1^3 B_2((A_2 - 3\omega_1^2) \cos \omega_1^0 \tau - 2\omega_1^0 A_1 \sin \omega_1^0 \tau + B_2). \)
7. Direction of periodic solutions. After deriving the conditions for occurrence of Hopf-bifurcation. Now, we shall derive the direction of bifurcating equilibrium state $V^*$, (on the lines of Hassard et al. [14]).
For proof, See Appendix.2.

**Theorem 7.1.** (Using Hassard et al. [14]) The sign of $\mu_2$ confirms about the direction of the Hopf bifurcation: if $\mu_2 > (\mu_2 < 0)$, then the Hopf bifurcation is supercritical (subcritical) and the stability and instability of bifurcating periodic solutions can be verified through $\beta_2 < 0 (\beta_2 > 0)$, the sign of $T_2$ tells us about the increase or decrease of bifurcating periodic solution by $T_2 > 0 (T_2 < 0)$.

8. Numerical simulation. In this section, we will observe the numerical validation of all the analytical findings with and without delay.

8.1. Simulation without Delay. In this subsection, we study the dynamics of a given plankton system (1) around multiple steady states. Firstly we consider a set of parameters $[K_1]$:
$$r_1 = 1.4, \alpha_1 = 0.05, \beta_1 = 2, \gamma_1 = 10, r_5 = 0.02, r_2 = 1, \beta_2 = 1, a_1 = 1.45, \gamma_2 = 20, \theta = 0.0126, r_4 = 0.6, r_3 = 0.1, \alpha_2 = 0.009, \text{and} \ a_2 = 0.689.$$

The trivial steady state $V_0(0,0,0)$ always exists and unstable as $-0.9400$, $-0.0400$, and $1.4000$ are eigen values of the corresponding jacobian matrix. If we take $r_3 = 0.06$, $\beta_1 = 0.5$, $\beta_2 = 0.8$, and $\theta = 1$ in set $[K_1]$, we get the zooplankton and fish free
Figure 7. Bifurcation diagrams for $1 \leq \tau \leq 3$

Figure 8. Co-existence of all species according to table 2 with $r_3$ on x-axis and $r_5$, $P(t)$, $Z(t)$ and $F(t)$ on y-axis.

Table 2. Impact of additional food ($r_3$ and $r_5$) on the co-existence of species.

| $r_3$ | $r_5$ | $P(t)$  | $Z(t)$  | $F(t)$  |
|-------|-------|---------|---------|---------|
| 0.01  | 0.01  | 23.8297 | 3.5257  | 6.6514  |
| 0.05  | 0.01  | 25.1787 | 4.806   | 6.7049  |
| 0.08  | 0.01  | 26.0261 | 1.7771  | 6.8449  |
| 0.1   | 0.01  | 26.6036 | 1.2770  | 6.9687  |
| 0.2   | 0.01  | 27.9978 | 0.0000  | 11.1111 |
| 0.2   | 0.4   | 27.8282 | 0.1586  | 11.9851 |
| 0.2   | 0.9   | 26.0543 | 1.7483  | 20.0424 |
| 0.2   | 1     | 25.6022 | 2.1321  | 21.8148 |
| 0.2   | 1.1   | 25.0996 | 2.5444  | 23.6513 |

equilibrium point $V_1(27.9991, 0, 0)$ (Fig.1), which is LAS, as -1.3999, -0.6546, and -0.0400 are the eigen values of the corresponding jacobian matrix around $V_1$ and the stability conditions for $V_1$ given in Section 4 i.e. $r_4 > r_3(0.1 > 0.06)$ and $r_2(\gamma_1a_1 + r_1 + \theta r_1 > \beta_1\beta_2(1.91764 > 0.04)$ are satisfied. Taking $r_3 = 0.06$, $r_4 = 0.1$, and $\theta = 2.5$ in $[K_1]$, we obtain the zooplankton free equilibrium $V_2(28, 0, 55.5556)$
ues and the transversality condition, \( (\pm i\omega_1, 0) \) with \( \omega_1 = 0.4200 \) of (20). The corresponding eigen values are -0.7727 and \( \pm 0.4200i \), the existence of negative or purely imaginary values and the transversality condition, \( \left( \frac{d\Re(\lambda)}{d\tau} \right)_{\tau = \tau_0} = 0.19664 \neq 0 \) confirm the existence of Hopf bifurcation with the existence of periodic solutions (Fig.6). The critical value of time delay for which stability exchanges takes place is \( \tau_0 = 1.5 \) such
that $V_*$ remains stable in $[0,1.5)$ (Fig.5) and bifurcation occurs for $\tau_0 \geq 1.5$ (Fig.7). It implies that the interior equilibrium converges in the range, $0 \leq \tau < 1.5$, become unstable for $\tau \geq 1.5$, and limit cycles exists at $\tau = 1.6, 2, 3...$ (Fig.6). Thus, we can observe that the system loses its stability as $\tau$ crosses its critical value $\tau_0 = 1.5$. The stability determining quantities for direction of Hopf-bifurcation at $\tau = \tau_0$ are given by $c_1(0) = -0.087219057897901 - 0.129589891502112i$, $\mu_2 = 0.046641207432032$, $\beta_2 = -0.174438115795801$, and $T_2 = 0.152901502427691$. Thus, we have observed that the Hopf-bifurcation is supercritical, bifurcating periodic orbits are stable, and increases as $\tau$ passes through $\tau_0$.

9. Conclusion. In this study, we have discussed the role of available additional food and time delay for the co-existence of all plankton fish species. The predation delay in the fish population due to the vertical migration of zooplankton is of great interest due to its possible detrimental impact on fisheries. The significance of the available additional food lies in the fact that it enhances the re-survival of extinct species, as shown in Table 2. It is also notable from Table 2 that if $r_3$ increases, the phytoplankton and fish population increases, and zooplankton decline. But as $r_5$ increases, phytoplankton and fish species decrease, and zooplankton grows. To capture the oscillatory co-existence, we have performed the Hopf-bifurcation analysis taking additional food ($r_5$) as a bifurcation parameter. It is determined that the dynamical system (1) remains stable for $0.1 \leq r_5 < 0.8$ and becomes unstable beyond it. We have determined that the model system (7) remains LAS for $\tau < \tau_0$ around $V_*$ (Fig.5) and Hopf-bifurcation occurs when $\tau$ crosses its critical value $\tau_0 = 1.5$ and limit cycles occur at $\tau = 1.6, 2.3$ (Fig.6). It is observed that the Hopf-bifurcation is supercritical, the bifurcating periodic solutions are stable with increasing periods.

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We shall numerically prove that there exist a positive root of the polynomial $N(P_1)$, namely, $P_*$ and using this positive root we can easily find $Z_*$ and $F_*$. 

**Appendix.2.** Let $y_1 = P - P_*$, $y_2 = Z - Z_*$, $y_3 = F - F_*$ and using the Taylor theorem to expand given system (7) about $V_c(P_*, Z_*, F_*)$ we get,

\[
\begin{align*}
\frac{dy_1}{dt} & = a_{100}y_1(t) + a_{010}y_2(t) + a_{001}y_3(t) + \sum_{i+j+k \geq 2} a_{ijk}y_1^i(t)y_2^j(t)y_3^k(t) \\
& = F_1(y_1, y_2, y_3) \\
\frac{dy_2}{dt} & = b_{100}y_1 + b_{010}y_2(t) + b_{001}y_3(t - \tau) + \sum_{i+j+k \geq 2} b_{ijk}y_1^i(t)y_2^j(t)y_3^k(t) \\
& = F_2(y_1, y_2, y_3) \\
\frac{dy_3}{dt} & = c_{100}y_1 + c_{010}y_2(t) + c_{001}y_3(t) + \sum_{i+j+k \geq 2} c_{ijk}y_1^i(t)y_2^j(t)y_3^k(t - \tau) \\
& = F_3(y_1, y_2, y_3)
\end{align*}
\]
where
\[ a_{ijk} = \frac{1}{\prod_{j} k_{j}} \frac{\partial^{i+j+k} F_1}{\partial P^i \partial Z^j \partial F^k}, \quad b_{ijk} = \frac{1}{\prod_{j} k_{j}} \frac{\partial^{i+j+k} F_2}{\partial P^i \partial Z^j \partial F^k}, \]
\[ c_{ijk} = \frac{1}{\prod_{j} k_{j}} \frac{\partial^{i+j+k} F_3}{\partial P^i \partial Z^j \partial F^k}. \]

The coefficients of the linear and non-linear terms are
\[ a_{100} = r_1 - 2\alpha_1 \beta_1 \gamma_1 - \frac{\beta_1 \gamma_1}{(\gamma_1 + p_s)^2}, \quad a_{010} = \frac{\beta_1 \gamma_1}{(\gamma_1 + p_s)^3}, \quad a_{001} = 0, \quad a_{200} = -2\alpha_1, \]
\[ a_{110} = -\frac{\beta_1 \gamma_1}{(\gamma_1 + p_s)^2}, \quad a_{300} = 0, \quad a_{210} = \frac{2\beta_1 \gamma_1}{(\gamma_1 + p_s)^3}, \quad b_{100} = \frac{(\beta_1 \beta_2 - \theta) \gamma_1}{(\gamma_1 + p_s)^2}, \]
\[ b_{010} = r_5 - r_2 + \frac{(\beta_1 \beta_2 - \theta) \gamma_1}{(\gamma_1 + p_s)^2}, \quad b_{210} = -\frac{2(\beta_1 \beta_2 - \theta) \gamma_1}{(\gamma_1 + p_s)^3}, \]
\[ B_{001} = -\frac{a_1 z_s}{(\gamma_2 + z_s)}, \quad b_{200} = -\frac{2(\beta_1 \beta_2 - \theta) \gamma_1}{(\gamma_1 + p_s)^3}, \quad b_{020} = -\frac{2a_1 \gamma_2 F_z}{(\gamma_2 + z_s)^3}, \]
\[ b_{030} = \frac{6a_1 \gamma_2 F_z}{(\gamma_2 + z_s)^4}, \quad b_{110} = \frac{(\beta_1 \beta_2 - \theta) \gamma_1}{(\gamma_1 + p_s)^2}, \quad B_{011} = -\frac{a_1 \gamma_2}{(\gamma_2 + z_s)^3}, \]
\[ b_{001} = \frac{6(\beta_1 \beta_2 - \theta) \gamma_1 Z_s}{(\gamma_1 + p_s)^4}, \quad B_{021} = \frac{2a_1 \gamma_2}{(\gamma_2 + z_s)^3}, \quad c_{010} = \frac{a_1 a_2 \gamma_2 F_z}{(\gamma_2 + z_s)^2}, \]
\[ c_{001} = r_3 - 2\alpha_2 F_z - r_4 + \frac{a_1 a_2 Z_s}{\gamma_2 + Z_s}, \quad c_{020} = -2\alpha_2, \quad c_{011} = \frac{a_1 a_2 \gamma_2}{(\gamma_2 + Z_s)^2}, \]
\[ c_{021} = -\frac{2a_1 a_2 \gamma_2}{(\gamma_2 + Z_s)^3}. \]

Let \( \tau = \tau_k + \nu \), \( \bar{v}(t) = v(t) - v(t + \delta) \) for \( \delta \in [-1, 0] \) and after some simplification, system (24) becomes a functional differential equation in \( C = C([-1, 0], \mathbb{R}^3) \) as
\[ \dot{v}(t) = L_\nu(v_t) + f(\nu, v_t) \] (25)
where \( v(t) = (v_1(t), v_2(t), v_3(t))^T \in \mathbb{R}^3 \) and \( L_\nu : C \rightarrow \mathbb{R}^3 \), \( f : \mathbb{R} \times C \rightarrow \mathbb{R}^3 \) are given, respectively, by
\[
L_\nu(\varsigma) = (\tau_k + \nu)[A_{11} \varsigma(0) + A_{22} \varsigma(-1)]
\]
(26)
\[
A_{11} = \begin{bmatrix}
a_{100} & a_{010} & 0 \\
b_{100} & b_{010} & 0 \\
0 & c_{100} & c_{001}
\end{bmatrix}, \quad A_{22} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & B_{001} \\
0 & 0 & 0
\end{bmatrix}
\]
and
\[
f(\nu, \varsigma) = (\tau_k + \nu) \begin{bmatrix}
b_{100} \varsigma_2(0) + b_{110} \varsigma_1(0) + b_{200} \varsigma_2(0) + b_{010} \varsigma_2(0) + b_{000} \varsigma_3(0) - a_{110} \varsigma_1(0) + a_{220} \varsigma_2(0) \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
(27)

By Riesz representation theorem, \( \exists \) a function \( \varsigma(\theta, \nu) \) of bounded variation for \( \theta \in [-1, 0] \) such that
\[ L_\nu(\varsigma) = \int_{-1}^{0} d\varphi(\theta, \nu) \varsigma(\theta) \quad for \quad \varsigma \in C \] (28)
In fact, we can take
\[ \varphi(\delta, \nu) = (\tau_k + \nu)[A_{11} \delta(\delta) - A_{22} \delta(\delta + 1)] \] (29)
where \( \delta \) denote the Dirac delta function.
For $\zeta \in C([-1, 0], \mathbb{R}^3)$, define

$$A(\nu)\zeta(\delta) = \begin{cases} \frac{d\zeta(\delta)}{d\delta} & \delta \in [-1, 0), \\
\int_{-1}^{\delta} d\varphi(s, \nu)\zeta(s) & \delta = 0 \end{cases}$$

and

$$R(\nu)\zeta(\delta) = \begin{cases} 0 & \delta \in [-1, 0), \\
\varphi(\nu) & \delta = 0 \end{cases}$$

Then system (25) is equivalent to

$$\dot{v}(t) = A(\nu)v + R(\nu)v,$$

For $\psi \in C^1([0, 1], (\mathbb{R}^3)^*)$, define

$$A^*\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds} & s \in (0, 1], \\
\int_{-1}^{s} d\varphi(t, 0)\psi(-t) & s = 0 \end{cases}$$

and a bilinear inner product

$$<\psi(s), \zeta(\delta)> = \bar{\psi}(0)\zeta(0) - \int_{-1}^{0} \int_{\delta}^{0} \bar{\psi}(s)\zeta(s)ds
\int_{-1}^{\delta} \bar{\psi}(s)\zeta(s)ds$$

where $\varphi(\delta) = \varphi(\delta, 0)$. Then $A(0)$ and $A^*$ are adjoint operators. From the results of last section, we know that $\pm i\omega_{10}\tau_k$ are the eigenvalues of $A(0)$ and $\mp i\omega_{10}\tau_k$ are the eigenvalues of $A^*$.

**Theorem 7.2** Let $Q(\delta) = (1, q_1, q_2)^T e^{i\omega_{10}\tau_k\delta}$ be the eigenvector of $A(0)$ corresponding to the eigenvalue $i\omega_{10}\tau_k$ and $Q^*(s) = D(1, q_1^*, q_2^*) e^{i\omega_{10}\tau_k s}$ be the eigenvector of $A^*$ corresponding to the eigenvalue $-i\omega_{10}\tau_k$.

Then

$$<Q^*, Q> = 1 \quad \text{and} \quad <Q^*, Q> = 0,$$

where

$$q_1 = \frac{-(a_{010} + \omega_{10})}{a_{010}} \quad q_2 = \frac{c_{010}(\omega_{10} - a_{010})}{(\omega_{10} - c_{001})a_{001}} \quad q_1^* = \frac{-a_{010} - \omega_{10}}{b_{100}}, \quad q_2^* = \frac{B_{001} e^{i\omega_{10}\tau_k} (a_{100} + \omega_{10})}{(c_{001} + \omega_{10})100} \quad D = \frac{1}{1 + q_1^* q_1 + q_2^* q_2 + B_{001} \tau_k e^{i\omega_{10}\tau_k} q_1^* q_1}.$$

Further, we shall calculate the coordinates to observe the center manifold $C_0$ at $\nu = 0$. Let us assumed that $v_1$ be the solution of (30) when $\nu = 0$.

Define

$$z(t) = <Q^*, v_1>, \quad W(t, \delta) = u_1(\delta) - 2Re\{z(t)Q(\delta)\}$$

for the center manifold $C_0$, we get

$$W(t, \delta) = W(z(t), \bar{z}(t), \delta)$$

where

$$W(z(t), \bar{z}(t), \delta) = W_{20}(\delta)\frac{\dot{z}^2}{2} + W_{11}(\delta)z\bar{z} + W_{02}(\delta)\frac{\bar{z}^2}{2} + \ldots$$

$z$ and $\bar{z}$ are local coordinates for center manifold $C_0$ in the direction of $Q$ and $Q^*$ respectively. Here we consider only real solutions as $W$ is real if $v_2$ is real. For the solution $v_1 \in C_0$ of (30), since $\nu = 0$, we have

$$\dot{z}(t) = i\omega_{10}\tau_k z + Q^*(0)f(0, W(z, \bar{z}, 0) + 2Re(zq(\delta))) = i\omega_{10}\tau_k z + Q^*(0)f_0(z, \bar{z}).$$
This equation can be rewritten as

$$\dot{z}(t) = \omega_1 \tau_k z + g(z, \bar{z})$$

where

$$g(z, \bar{z}) = \tilde{Q}^*(0) f_0(z, \bar{z})$$

$$= g_{20}(\theta) \frac{z^2}{2} + g_{11}(\theta) z \bar{z} + g_{02}(\theta) \frac{\bar{z}^2}{2} + g_{21}(\theta) \frac{z^2 \bar{z}}{2} + \ldots$$ \hspace{1cm} (34)

From (32) and (33), we have

$$v_t(\theta) = W(t, \delta) + 2 \text{Re}\{z(t)q(\delta)\}$$

$$= W_{20}(\delta) \frac{z^2}{2} + W_{11}(\delta) z \bar{z} + W_{02}(\delta) \frac{\bar{z}^2}{2} + \frac{(1, q_1, q_2)^T e^{i \omega_1 t} \tau_k \delta z}{2}$$

$$+ \frac{(1, q_1, q_2)^T e^{-i \omega_1 t} \tau_k \delta \bar{z} + \ldots}{2}$$ \hspace{1cm} (35)

Now, from (27) and (34), it follows that

$$g(z, \bar{z}) = \tau_\delta \tilde{D}(1, q_1^*, q_2^*) \left[ \begin{array}{c} a_{110} v_{11}(0) v_{22}(0) + a_{200} v_{12}(0) \\ b_{200} v_{12}^2(0) + b_{110} v_{11}(0) v_{22}(0) + b_{020} v_{22}^2(0) + B_{011} v_{22}(0) v_{12}(1) - 1 \\ c_{002} v_{22}(0) + c_{011} v_{12}(0) v_{22}(0) \end{array} \right]$$

or

$$g(z, \bar{z}) = \tilde{D}[g_{20} Z^2 + g_{11} Z \bar{Z} + g_{02} \bar{Z}^2 + g_{21} Z^2 \bar{Z}]$$ \hspace{1cm} (36)

Comparing its coefficients with (34), we find

$$g_{11} = D_{\tau_k} \left\{ (a_{110} + q_1^* b_{110}) 2 \text{Re} q_1 + 2 a_{200} + q_1^* ((2 b_{200} + 2 b_{020}) q_1 \bar{q}_2) + B_{011} (q_1 \bar{q}_2 e^{-i \omega_1} + q_2 \bar{q}_1 e^{-i \omega_1}) + q_2^* (2 q_2 \bar{q}_2 c_{002} + c_{011} 2 \text{Re} q_1 \bar{q}_2) \right\}$$

$$g_{20} = 2 \tilde{D}_{\tau_k} \left\{ q_1 a_{110} + a_{200} + q_1^* (b_{110} q_1 + b_{200} + b_{020} q_2^2 + B_{011} q_1 q_2 e^{-i \omega_1}) + q_2^* (c_{002} q_2^2 + c_{011} q_1 q_2) \right\}$$

$$g_{02} = 2 \tilde{D}_{\tau_k} \left\{ q_1 \bar{q}_2 a_{110} + a_{200} + q_1^* (b_{110} \bar{q}_1 + b_{200} + b_{020} \bar{q}_2 + B_{011} \bar{q}_1 \bar{q}_2 e^{i \omega_1}) + q_2^* (c_{002} q_2^2 + c_{011} q_1 q_2) \right\}$$

$$g_{21} = 2 \tilde{D}_{\tau_k} \left\{ (a_{110} + q_1(2W_{11}(3))) (2) q_1 + 2 q_1(2W_{20}(3)) + (W_{11}(2) q_2) + a_{200} ((2W_{11}(1)) + (W_{20}(1) q_1 + q_1^* (b_{110} W_{11}(3)) q_1 + q_1(2W_{20}(3))) + q_2(2W_{11}(3)) \right\}$$

$$+ (q_2(2W_{20}(3)) + B_{011} (q_1 W_{11}(3)) (1) + q_1^* (b_{110} W_{11}(3)) q_1 + q_1(2W_{20}(3)) + \frac{\tilde{D}(\omega_1 T) e^{i \omega_1 t} \tau_k \delta z}{2} + \frac{\tilde{D}(\omega_1 T) e^{-i \omega_1 t} \tau_k \delta \bar{z} + \ldots}{2} \right\}$$ \hspace{1cm} (37)

Since, components of $W_{20}$ and $W_{11}$ are in $g_{21}$, we still to compute them.

Now, from (30) and (32), we have

$$W = \hat{u}(t) - \dot{z} Q - \bar{z} \bar{Q}$$

$$\left\{ \begin{array}{l} A(0) W - 2 \text{Re} \{ \tilde{Q}^*(0) f_0 \} = \tilde{Q}(0) \delta \in [-1, 0] \\ A(0) W - 2 \text{Re} \{ \tilde{Q}^*(0) f_0 \} = 0 \end{array} \right. \hspace{1cm} (38)$$

where

$$H(z, \bar{z}, \delta) = H_{20}(\delta) \frac{z^2}{2} + H_{11}(\delta) z \bar{z} + H_{02}(\delta) \frac{\bar{z}^2}{2} + \ldots$$ \hspace{1cm} (39)

Substituting (39) into (38) and comparing the coefficients, we get

$$A(0) W - 2 \omega_1 \tau_k I W_{20}(\delta) = -H_{20}(\delta),$$

$$A(0) W_{11}(\delta) = -H_{11}(\delta)$$ \hspace{1cm} (40)
From (38) and for $\delta \in [-1, 0)$

$$H(z, \bar{z}, \delta) = -\bar{Q}^*(0)f_0Q(\delta) - Q^*(0)f_0\bar{Q}(\delta)$$
$$= -g(z, \bar{z})Q(\delta) - \bar{g}(z, \bar{z})\bar{Q}(\delta)$$

(41)

Using (34) in (41) and comparing coefficients with (39), we can obtain

$$H_{20}(\delta) = -g_{20}Q(\delta) - \bar{g}_{02}\bar{Q}(\delta)$$

(42)

and

$$H_{11}(\delta) = -g_{11}q(\delta) - \bar{g}_{11}\bar{q}(\delta)$$

(43)

From the definition of $A(0)$, (40) and (42), we obtain

$$\dot{W}_{20}(\delta) = 2\omega_{10}\tau_kW_{20}(\delta) + g_{20}Q(\delta) + \bar{g}_{02}\bar{Q}(\delta)$$

Solving it and for $Q(\delta) = (1, q_1, q_2)^T e^{i\omega_{10}\tau_k \delta}$, we have

$$W_{20}(\delta) = \frac{i g_{20}}{\omega_{10}\tau_k}Q(0)e^{i\omega_{10}\tau_k \delta} + \frac{i g_{02}}{3\omega_{10}\tau_k}\bar{Q}(0)e^{-i\omega_{10}\tau_k \delta} + E_1 e^{2i\omega_{10}\tau_k \delta}$$

(44)

Similarly, from (40) and (43) it follows that,

$$W_{11}(\delta) = -\frac{i g_{11}}{\omega_{10}\tau_k}Q(0)e^{i\omega_{10}\tau_k \delta} + \frac{i g_{11}}{\omega_{10}\tau_k}\bar{Q}(0)e^{-i\omega_{10}\tau_k \delta} + E_2$$

(45)

where $E_1 = (E_1^{(1)}, E_1^{(2)}, E_1^{(3)})^T$ and $E_2 = (E_2^{(1)}, E_2^{(2)}, E_2^{(3)})^T$ are three dimensional constant vectors, and can be determined by setting $\theta = 0$ in $H(z, \bar{z}, \theta)$.

Again, from the definition of $A(0)$ and (40), we have

$$\int_{-1}^{0} d\varphi(\delta) W_{20}(\delta) = 2i\omega_{10}\tau_kW_{20}(0) - H_{20}(0)$$

(46)

and

$$\int_{-1}^{0} d\varphi(\delta) W_{11}(\delta) = -H_{11}(0)$$

(47)

where $\varphi(\delta) = \varphi(0, \delta)$.

From (38), we know when $\delta = 0$,

$$H(z, \bar{z}, 0) = -2Re(\bar{Q}^*(0)f_0Q(0)) + f_0(z, \bar{z})$$
$$= -\bar{Q}^*(0)f_0Q(0) - Q^*(0)f_0\bar{Q}(0) + f_0(z, \bar{z})$$

That is,

$$H_{20}(\delta)\frac{\bar{z}^2}{2} + H_{11}(\delta)z\bar{z} + H_{02}(\delta)\frac{\bar{z}^2}{2} + ......$$
$$= -Q(0)\{g_{20}\frac{\bar{z}^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + .....\}$$
$$-\bar{Q}(0)\{g_{20}\frac{\bar{z}^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + .....\} + f_0(z, \bar{z})$$

(48)

By (27), we have

$$f_0 = \tau_k \left[ \begin{array}{l} a_{110}v_{11}(0)v_{2t}(0) + a_{200}v_{11}^2(0) \\ b_{200}v_{11}^2(0) + b_{110}v_{11}(0)v_{2t}(0) + b_{020}v_{2t}^2(0) + B_{001}v_{2t}(0)v_{3z}(-1) \\ c_{002}v_{3z}^2(0) + c_{011}v_{2t}(0)v_{3z}(0) \end{array} \right]$$
From (32), we have
\[ v_1(\delta) = W(t, \delta) + 2\text{Re}\{z(t)Q(\delta)\} = W(t, \delta) + z(t)Q(\delta) + \bar{z}(t)\bar{Q}(t) \]
\[ = W_{20}(\delta)\frac{z^2}{2} + W_{11}(\delta)z\bar{z} + W_{02}(\delta)\frac{\bar{z}^2}{2} + \ldots. \]

Thus, we can obtain,
\[ f_0 = 2\tau_k \left[ \begin{array}{c} \xi_{11} \\ \xi_{21} \\ \xi_{31} \end{array} \right] \frac{z^2}{2} + \tau_k \left[ \begin{array}{c} \xi_{12} \\ \xi_{22} \\ \xi_{32} \end{array} \right] z\bar{z} + \ldots \quad (49) \]

where
\[ \xi_{11} = a_{110}q_1 + a_{200}, \quad \xi_{21} = b_{110}q_1 + b_{200} + b_{020}q_1^2 + B_{011}q_1q_2e^{-i\omega_1}, \]
\[ \xi_{31} = c_{002}q_2^2 + c_{011}q_1q_2, \quad \xi_{12} = 2\text{Re}(q_1)a_{110} + 2a_{200}, \]
\[ \xi_{22} = 2\text{Re}(q_1)b_{110} + 2b_{200} + 2b_{020}q_1q_2 + B_{011}(q_1q_1e^{i\omega_1} + q_1q_2e^{-i\omega_1}), \]
\[ \xi_{32} = 2\text{Re}(q_1)b_{110} + 2c_{200}q_2q_2. \]

Comparing the coefficients in (48) and using (49), we get
\[ H_{20}(0) = -g_{20}Q(0) - \bar{g}_{02}\bar{Q}(0) + 2\tau_k \left[ \begin{array}{c} \xi_{11} \\ \xi_{21} \\ \xi_{31} \end{array} \right] \]
\[ H_{11}(0) = -g_{11}Q(0) - \bar{g}_{11}\bar{Q}(0) + \tau_k \left[ \begin{array}{c} \xi_{12} \\ \xi_{22} \\ \xi_{32} \end{array} \right] \quad (50, 51) \]

Since \( \omega_{10}\tau_k \) is the eigenvalue of \( A(0) \) corresponding to eigenvector \( Q(0) \), then
\[ \{i\omega_{10}\tau_k I - \int_{-1}^{0} e^{i\omega_{10}\tau_\delta} d\varphi(\delta)\}Q(0) = 0 \quad \text{and} \]
\[ \{-i\omega_{10}\tau_k I - \int_{-1}^{0} e^{-i\omega_{10}\tau_\delta} d\varphi(\delta)\}Q(0) = 0. \]

Substituting (44) and (50) into (46), we find
\[ \{2i\omega_{10}\tau_k I - \int_{-1}^{0} e^{2i\omega_{10}\tau_\delta} d\varphi(\delta)\}E_1 = 2\tau_k \left[ \begin{array}{c} \xi_{11} \\ \xi_{21} \\ \xi_{31} \end{array} \right] \]

or
\[ \left[ \begin{array}{ccc} 2\omega_{10} - a_{100} & -a_{010} & 0 \\ -b_{100} & 2\omega_{10} - b_{010} & -B_{001} \\ 0 & -c_{100} & 2i\omega_{10} - c_{001} \end{array} \right] E_1 = \left[ \begin{array}{c} 2\xi_{11} \\ 2\xi_{21} \\ 2\xi_{31} \end{array} \right]. \]

Simplification gives,
\[ E_1^{(1)} = \frac{2}{M}[[\xi_{11}(2\omega_{10} - b_{010})(2i\omega_{10} - c_{001}) - c_{010}B_{001}] \]
\[ + a_{010}(\xi_{21}(2i\omega_{10} - c_{001}) + c_{010}M), \]
\[ E_1^{(2)} = \frac{2}{M}[[2\omega_{10} - a_{100}][\xi_{21}(2\omega_{10} - c_{001}) + c_{010}B_{001}] \]
\[ + \xi_{11}[2\omega_{10} - c_{001}]b_{100} - a_{010}[\xi_{31}(2\omega_{10} - c_{001}) + \xi_{21}c_{100}], \]
\[ E_1^{(3)} = \frac{2}{M}[[2\omega_{10} - a_{100}][\xi_{31}(2\omega_{10} - b_{010}) + \xi_{21}c_{100}] + a_{010}[\xi_{31}(2\omega_{10} - c_{001}) + \xi_{21}c_{100}]. \]
and 

\[
M = [(2\omega_{10} - a_{100})(2\omega_{10} - A_{010}c_{001}] - a_{010}[b_{100}(2\omega_{10} - c_{001})]
\]

Similarly, substituting (45) and (51) into (47), we obtain

\[
\begin{pmatrix}
a_{100} & a_{010} & a_{001} 
b_{100} & b_{010} & b_{001} 
c_{100} & c_{010} & c_{001}
\end{pmatrix}
\begin{pmatrix}
E_2
\end{pmatrix} =
\begin{pmatrix}
\xi_{21} 
\xi_{22} 
\xi_{23}
\end{pmatrix}
\]

On solving, we can obtain

\[
\begin{align*}
E_2^{(1)} &= \frac{1}{N}[\xi_{12}(b_{010}c_{001} - c_{010}B_{001}) - a_{010}(\xi_{22}c_{001} - B_{001}\xi_{32})] \\
E_2^{(2)} &= \frac{1}{N}[a_{100}(b_{001}\xi_{20} - \xi_{32}B_{001}) - \xi_{12}(b_{100}c_{001})] \\
E_2^{(3)} &= \frac{1}{N}[a_{100}(b_{010}\xi_{32} - c_{010}\xi_{22}) - a_{010}((b_{100})\xi_{32} + c_{100}b_{100}\xi_{12})]
\end{align*}
\]

and

\[
N = a_{100}[b_{010}c_{001} - b_{001}c_{010}] - a_{010}[(b_{100} + B_{100})c_{001} - c_{100}b_{001}] + a_{001}[(b_{100} + B_{100})c_{010} - c_{100}b_{101}]
\]

Thus, we can determine \(W_{20}(\theta), W_{11}(\theta)\) from (44), (45) and \(g_{21}\) can be computed from (37).

Finally, we can compute the following quantities:

\[
c_1(0) = \frac{t\{g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\}}{2\omega_1 r_k} + \frac{g_{21}}{2} \mu_2 = -\frac{Re\{c_1(0)\}}{Re\{\frac{dx(r_k)}{dr}\}}, \quad \beta_2 = 2Re\{c_1(0)\},
\]

\[
T_2 = -\frac{\Im\{c_1(0)\} + \mu_2 \Im\{\frac{dx(r_k)}{dr}\}}{\omega_1 r_k}, \quad k=0,1,2,\ldots, \text{ where } g_{ij}\text{ are given by (37)}.
\]