Gaussian Graphical Regression Models with High Dimensional Responses and Covariates

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Abstract

Though Gaussian graphical models have been widely used in many scientific fields, limited progress has been made to link graph structures to external covariates because of substantial challenges in theory and computation. We propose a Gaussian graphical regression model, which regresses both the mean and the precision matrix of a Gaussian graphical model on covariates. In the context of co-expression quantitative trait locus (QTL) studies, our framework facilitates estimation of both population- and subject-level gene regulatory networks, and detection of how subject-level networks vary with genetic variants and clinical conditions. Our framework accommodates high dimensional responses and covariates, and encourages covariate effects on both the mean and the precision matrix to be sparse. In particular for the precision matrix, we stipulate simultaneous sparsity, i.e., group sparsity and element-wise sparsity, on effective covariates and their effects on network edges, respectively. We establish variable selection consistency first under the case with known mean parameters and then a more challenging case with unknown means depending on external covariates, and show in both cases that the convergence rate of the estimated precision parameters is faster than that obtained by lasso or group lasso, a desirable property for the sparse group lasso estimation. The utility and efficacy of our proposed method is demonstrated through simulation studies and an application to a co-expression QTL study with brain cancer patients.

Keywords: subject-specific Gaussian graphical model; high dimensional outcomes and covariates; non-asymptotic convergence rate; sparse group lasso; co-expression QTL.
1 Introduction

Gaussian graphical models, which shed light on the dependence structure among a set of response variables, have been applied to studies of, for example, gene regulatory networks from gene expression data (Peng et al., 2009; Li et al., 2012; Chen et al., 2016), brain connectivity networks from functional magnetic resonance imaging (fMRI) data (Qiu et al., 2016; Zhang et al., 2019), and firm-level financial networks from stock market data (Brunetti et al., 2015). Most existing models consider a homogeneous population obeying a common graphical model (Meinshausen and Bühlmann, 2006; Yuan and Lin, 2007; Friedman et al., 2008; Peng et al., 2009) or several stratified graphical models (Guo et al., 2011; Danaher et al., 2014).

In practice, graph structures may depend on individuals’ characteristics, leading to the notion of subject-specific graphical models. For example, in the context of gene expression networks, external covariates, such as genetic variants, clinical and environmental factors, may affect both the expression levels of individual genes and the co-expression relationships among genes. In biology, genetic variants that alter co-expression relationships are referred to as co-expression quantitative trait loci (QTLs), and identifying them is of keen scientific interest (Wang et al., 2013; van der Wijst et al., 2018a,b). Other factors such as cellular states and environmental conditions may also alter gene regulatory networks (Luscombe et al., 2004). With relevant external covariates, a fundamental interest, therefore, is to recover both the population-level and subject-level gene networks, and to monitor if and how the subject-level network varies as a function of external covariates. Characterizing individualized gene regulatory networks is key in developing gene therapies that target specific gene or pathway disruptions (van der Wijst et al., 2018b).

Though the literature on graphical models has been readily growing (see, for example, Meinshausen and Bühlmann, 2006; Yuan and Lin, 2007; Friedman et al., 2008; Peng et al., 2009), few systematic frameworks permit subject-specific graphical model estimation, especially when both response variables and external covariates are high-dimensional. Recently, Yin and Li (2011); Li et al. (2012); Cai et al. (2012); Chen et al. (2016) considered covariate dependent Gaussian graphical models, wherein the mean of the nodes depends on covariates, while the network structure is constant across all of the subjects. Guo et al. (2011); Danaher et al. (2014) jointly estimated several group-specific Gaussian graphical models, where the graph structure is allowed to vary with discrete external covariates; these approaches can only handle a small number of groups. Liu et al. (2010) proposed a graph-valued regression, which partitions the covariate space into several subspaces and fits separate Gaussian graphical models for each subspace using graphical lasso. This approach can only accommodate a small number of subspaces. Moreover, it is difficult to interpret the relationship between the covariates and the graphical models, as even the adjacent covariate subspaces may differ much. Cheng et al. (2014) considered a conditional Ising model for binary data where the
log-odds is modeled as a linear function of external covariates. Kolar et al. (2010) considered a nonparametric approach for conditional covariance estimation. However, their approach only permits one external covariate, which must be continuous. Ni et al. (2019) considered a conditional DAG model that allows the graph structure to vary with a finite number of discrete or continuous covariates. They assumed the ordering of the nodes is known a priori, which may not be always justifiable.

We propose a Gaussian graphical regression model that allows the network structure to vary with external covariates (discrete or continuous). Specifically, both the mean and the precision matrix are modeled as functions of covariates, enabling estimation of subject-specific graphical models; see Figure 1. To facilitate estimation, we show that our proposed model can be formulated as a sequence of linear regression models that include the interactions between the response variables (e.g., gene expressions) and the external covariates (e.g., genetic variants); Section 2.2. Our model accommodates the setting where both response variables and external covariates exceed the sample size, which is frequently encountered in genetic studies. The mean coefficients are assumed to be sparse, and are estimable (Yin and Li, 2011; Cai et al., 2012; Chen et al., 2016). To estimate coefficients in the covariate-dependent precision matrix, we impose a sparse group lasso penalty that encourages effective covariates to be sparse and their effects on edges to be sparse as well. The simultaneously sparse structure effectively leads to a parsimonious model, increasing model estimability and interpretability, though it brings considerable theoretical challenges, which are to be tackled as follows. We first conduct the theoretical investigation under a simpler setting where the mean coefficients are known, which allows us to focus on estimating the precision matrix coefficients that are simultaneously sparse. We then investigate a more challenging setting with unknown mean coefficients. In this case, estimating the precision matrix is more delicate with errors arising from the estimation of mean coefficients. For both cases, we derive the non-asymptotic rates of convergence, which improve the existing results, and establish selection consistency, which ensures that we correctly select edges in both the population- and subject-level networks with probability going to 1.

Our analysis may advance the related works on sparse group lasso (Simon et al., 2013) in linear regressions. Specifically, Chatterjee et al. (2012) considered regressions with hierarchical tree-structured norm regularizers. Their theoretical analysis requires the regularizer to be decomposable (Negahban et al., 2012), which is not satisfied for the sparse group lasso. Li et al. (2015) considered a multivariate group lasso regression and developed convergence rate for the estimator; Ahsen and Vidyasagar (2017) developed convergence rate for the sparse group lasso estimator with possibly overlapping groups. However, these two convergence rates do not show improvements over the lasso or the group lasso. Rao et al. (2015) studied classification with the sparse group lasso and developed bounds for sample complexity. Poignard (2020) studied the asymptotic properties of adaptive sparse group lasso, but in a low dimensional setting. Cai et al. (2019) studied the sample complexity and convergence
rate of the high-dimensional sparse group lasso estimator, and showed that the sparse group lasso estimator has provable improvements over the lasso and group lasso estimators. In contrast, though the subject-specific graphical model framework considered by us is more challenging than the usual linear regression setting as in Cai et al. (2019), we have obtained sharper upper bounds for estimation errors (see Remark 3).

To recap, our work contributes to both methodology and theory. As to methodology, we propose a flexible subject-specific graphical model which depends on a large number of external covariates. We employ a combined sparsity structure that encourages effective covariates and the effect of effective covariates on the network to be simultaneously sparse. With respect to theory, we carry out a thorough investigation of the simultaneously sparse estimator, including deriving tight non-asymptotic estimation error bounds and establishing variable selection consistency, when both the number of groups and the group sizes grow faster than the sample size. Our work addresses the theoretical challenges arising from regressing both the means and the precision matrices on external covariates. Moreover, as the simultaneously sparse regularizer is non-decomposable, the existing arguments using decomposable regularizers and null space properties (Negahban et al., 2012) are no longer applicable; see Section 4. Finally, we are able to derive a convergence rate that is faster than both the lasso and the group lasso (see Remark 1), a desirable property for the sparse group lasso estimation.

The rest of the article is organized as follows. Section 2 introduces the Gaussian graphical regression model and Section 3 discusses model estimation with known mean coefficients. Section 4 investigates theoretical properties of the estimator from Section 3. Section 5 presents a two-step estimation procedure and the related theoretical properties with unknown mean coefficients. Section 6 reports the simulation results, and Section 7 conducts a co-expression QTL analysis using a brain cancer genomic data set. Section 8 concludes the paper with a brief discussion. All of the technical proofs and lemmas are relegated to the Supplementary Materials.

2 Graphical Regression Models

2.1 Notation and Preamble

We start with some notation. Given a vector $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we use $\|\mathbf{x}\|_0 = \sum_{i=1}^{d} \mathbf{1}(x_i \neq 0)$, $\|\mathbf{x}\|_1 = \sum_{i=1}^{d} |x_i|$, $\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^{d} x_i^2}$ and $\|\mathbf{x}\|_{\infty} = \max_{i} |x_i|$ to denote the $\ell_0$, $\ell_1$, $\ell_2$ and $\ell_{\infty}$ norms, respectively. We use $\langle \mathbf{x}_1, \mathbf{x}_2 \rangle$ to denote the inner product of $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^d$. We write $[d] = \{1, 2, \ldots, d\}$. Given an index set $S \in [d]$, we use $\mathbf{x}_S \in \mathbb{R}^{|S|}$ to denote the sub-vector of $\mathbf{x}$ corresponding to index $S$. Given a matrix $\mathbf{X} \in \mathbb{R}^{d_1 \times d_2}$, we let $\|\mathbf{X}\|$ and $\|\mathbf{X}\|_{\infty} = \max_{ij} X_{ij}$ denote the spectral norm and element-wise max norm, respectively. Given $S \in [d_2]$, we use $\mathbf{X}_S \in \mathbb{R}^{d_1 \times |S_2|}$ to denote the sub-matrix with columns
We use \( \lambda_{\min}(\cdot) \) and \( \lambda_{\max}(\cdot) \) to denote the smallest and largest eigenvalues of a matrix, respectively. For two positive sequences \( a_n \) and \( b_n \), write \( a_n \preceq b_n \) or \( a_n = \mathcal{O}(b_n) \) if there exist \( c > 0 \) and \( N > 0 \) such that \( a_n < cb_n \) for all \( n > N \), and \( a_n = o(b_n) \) if \( a_n/b_n \to 0 \) as \( n \to \infty \); moreover, write \( a_n \asymp b_n \) if \( a_n \preceq b_n \) and \( b_n \preceq a_n \).

Suppose \( \mathbf{X} = (X_1, \ldots, X_p) \sim \mathcal{N}_p(0, \Sigma) \). Denote the precision matrix \( \Sigma^{-1} \) by \((\sigma_{ij})_{p \times p}\). Under a Gaussian distribution, \( \sigma_{ij} \neq 0 \) is equivalent to \( X_i \) and \( X_j \) being conditionally dependent given all of the other variables in \( \mathbf{X} \) (Lauritzen, 1996). Let \( \mathbf{X}_{-j} = \{X_k : 1 \leq k \neq j \leq p\} \). The results of Meinshausen and Bühlmann (2006) and Peng et al. (2009) relate \((\sigma_{ij})_{p \times p}\) to the coefficients in this linear regression model: for \( j \in [p] \),

\[
X_j = \sum_{k \neq j} \beta_{jk} X_k + \epsilon_j, \tag{1}
\]

where \( \epsilon_j \) is independent of \( \mathbf{X}_{-j} \) if and only if \( \beta_{jk} = -\sigma_{jk}/\sigma_{jj} \). Moreover, for such defined \( \beta_{jk} \), it holds that \( \text{Var}(\epsilon_j) = 1/\sigma_{jj} \). Consequently, estimating the conditional dependence structure (i.e., finding nonzero \( \sigma_{jk} \)'s) can be viewed as a model selection problem (i.e., finding nonzero \( \beta_{jk} \)'s) under the regression setting in (1).

Let \( \mathbf{U} = (U_1, \ldots, U_q) \in \mathbb{R}^q \) be a \( q \)-dimensional vector of covariates. One may consider a covariate-adjusted Gaussian graphical model (Yin and Li, 2011; Cai et al., 2012; Yin and Li, 2013; Chen et al., 2016):

\[
\mathbf{X} | \mathbf{U} = \mathbf{u} \sim \mathcal{N}_p(\mu(\mathbf{u}), \Sigma), \tag{2}
\]

where \( \mu(\mathbf{u}) = \mathbf{\Gamma} \mathbf{u} \) and \( \mathbf{\Gamma} \in \mathbb{R}^{p \times q} \). In expression QTL studies, the \( i \)th row of \( \mathbf{\Gamma} \) specifies the how the \( q \) genetic regulators affect the expression level of the \( i \)th gene, \( i \in [p] \). Estimating \( \mathbf{\Gamma} \) and \((\sigma_{ij})_{p \times p}\) can be formulated as a multivariate regression \( \mathbf{X} = \mathbf{\Gamma} \mathbf{u} + \mathbf{Z} \), where \( \mathbf{Z} \sim \mathcal{N}_p(0, \Sigma) \). Denote \( \mathbf{\Gamma} = (\gamma_1, \ldots, \gamma_p)^\top \). Similar to (1), we have that, for \( j \in [p] \),

\[
X_j = \mathbf{u}^\top \gamma_j + \sum_{k \neq j} \beta_{jk}(X_k - \mathbf{u}^\top \gamma_k) + \epsilon_j, \tag{3}
\]

such that \( \epsilon_j \) is independent with \( \mathbf{X}_{-j} \) if and only if \( \beta_{jk} = -\sigma_{jk}/\sigma_{jj} \). With such defined \( \beta_{jk} \), \( \text{Var}(\epsilon_j) = 1/\sigma_{jj} \).

### 2.2 Gaussian graphical regression with high dimensional responses and covariates

With a response vector \( \mathbf{X} = (X_1, \ldots, X_p)^\top \in \mathbb{R}^p \) and a covariate vector \( \mathbf{U} = (U_1, \ldots, U_q) \in \mathbb{R}^q \), we assume that

\[
\mathbf{X} | \mathbf{U} = \mathbf{u} \sim \mathcal{N}_p(\mu(\mathbf{u}), \Sigma(\mathbf{u})), \tag{4}
\]
where $\mu(u) = \Gamma u$ and $\Sigma(u)$ are the conditional mean vector and covariance matrix, respectively. Denote by $\Omega(u) = \Sigma^{-1}(u)$ and suppose that

$$
\Omega(u)_{jk} = \begin{cases} 
\sigma_{jj} & j = k, \\
-\omega_{jk}(u) & j \neq k,
\end{cases}
$$

where $\omega_{jk}(u) = \beta'_{jk0} + \sum_{h=1}^{q} \beta'_{jkh} u_h$, and $\beta'_{jkh} = \beta'_{kjh}$, $h \in \{0\} \cup [q]$, $j, k \in [p]$. Specifying $\Omega(u)_{jk}$’s to linearly depend on $u$, we allow both the sparsity patterns and the nonzero values (i.e., strengths of dependence) in $\Omega(u)$ to vary with covariates $u$; see Figure 1. Moreover, $\{\beta'_{jkh}\}_{j,k\in[p]}$ characterizes the population level regulatory network, and $\{\beta'_{jkh}\}_{j,k\in[p]}$ encodes the effect of $u_h$ on the regulatory network, where $h \in [q]$. See sufficient conditions on $\beta'_{jkh}$’s and $u$ in Section 8 for a positive definite $\Omega(u)$.

Model (4) is general: it includes the covariate-adjusted Gaussian graphical model (2) as a special case when $\beta'_{jkh} = 0$, that is, when all subjects share a common network characterized by $\{\beta'_{jkh}\}_{1 \leq j \leq p}$; if $\Gamma = 0$, then $X$ is independent of $U$ and (4) encompasses the regular Gaussian graphical model, i.e., $X \sim \mathcal{N}_p(0, \Sigma)$, as a special case.

Model (4) can be viewed as a multivariate regression problem $X = \Gamma u + Z$ with heteroskedastic errors $Z \sim \mathcal{N}_p(0, \Sigma(u))$. Define $\beta_{jkh} = \beta'_{jkh}/\sigma_{jj}$, for $h \in 0 \cup [p]$, $j, k \in [p]$. As in (1) and (3), the estimation of $\Gamma$ and $\Omega(u)$ relates to the regression:

$$
X_j = u^\top \gamma_j + \sum_{k \neq j}^{p} \sum_{h=1}^{q} \beta_{jkh} u_h (X_k - u^\top \gamma_k) + \epsilon_j,
$$

where $\text{Var}(\epsilon_j) = 1/\sigma_{jj}$ and $j \in [p]$. Thus, our proposed model extends (1) or (3) with an added interaction term between $X_{-j}$ and $u$, and the partial correlation between $X_j$ and $X_k$ conditional on all other variables in $X$ becomes a function of $u$, forming the basis of Gaussian graphical regression.
Given $U = u$, write $Z = X - \Gamma u = (Z_1, \ldots, Z_p)^\top$, and re-express (6) as

$$Z_j = \sum_{k \neq j}^p \beta_{jk0} Z_k + \sum_{k \neq j}^p \sum_{h=1}^q \beta_{jkh} u_h Z_k + \epsilon_j,$$

(7)

where $\text{Var}(\epsilon_j) = 1/\sigma_{jjj}$, $j \in [p]$. Denote by $\beta_j = (b_{j0}, b_{j1}, \ldots, b_{jq})^\top \in \mathbb{R}^{(p-1) \times (q+1)}$, where $b_{j0} = (\beta_{j10}, \ldots, \beta_{jp0}) \in \mathbb{R}^{p-1}$ encodes connections of node $j$ with $h \in [q]$ at the population level and $b_{jh} = (\beta_{j1h}, \ldots, \beta_{jph}) \in \mathbb{R}^{p-1}$ encodes the effects of the $h$th of $u$ on the edges of node $j$ with $h \in [q]$; see a more organizational and functional view of $\beta_j$ below:

$$
\begin{array}{llll}
\text{group 0} & \text{group 1} & \text{group q} \\
\beta_{j10}, \ldots, \beta_{jp0} & \beta_{j11}, \ldots, \beta_{jp1} & \beta_{j1q}, \ldots, \beta_{jpq} \\
\text{population level edges of node } j & \text{u}_1 \text{'s effect on edges of node } j & \text{u}_q \text{'s effect on edges of node } j
\end{array}
$$

(8)

In high dimensional settings (when both $p$ and $q$ are large) and to ensure the estimability of $\beta_j$ with $j \in [p]$, we impose on it simultaneous group sparsity and element-wise sparsity. Specifically, we assume $\beta_j$ is group sparse (with groups illustrated in (8)), which stipulates that effective covariates (e.g., genetic variants) are sparse. That is, only a few covariates may impact edges and those impactful covariates are termed effective covariates. We further assume $\beta_j$ is element-wise sparse with covariate effects on gene co-expressions being sparse. That is, effective covariates may influence only a few edges. These simultaneous sparsity assumptions are well supported by genetic studies (van der Wijst et al., 2018a). We exclude $b_{j0}$ from the group sparsity constraint (but not the element-wise sparsity constraint), as it determines the population level regulatory network.

The simultaneous sparsity assumptions bring substantial challenges to our theoretical development, as the corresponding regularizer is non-decomposable (Cai et al., 2019), and classic techniques that use decomposable regularizers and null space properties are non-applicable. By sharpening bound of the stochastic term in the estimation, we show in Section 4 that, even with $p, q$ far exceeding $n$, the proposed simultaneously sparse estimator enjoys a faster convergence rate than using each separate sparse structure (i.e., group sparse and element-wise sparse) alone.

Model (7) can be viewed as an interaction model. However, our later development does not abide by the common hierarchical principle for the inclusion of interactions, that is, an interaction is allowed only if the main effects are present (Hao et al., 2018; She et al., 2018). This is because gene co-expressions may occur only for certain genetic variations (Wang et al., 2013; van der Wijst et al., 2018a), in which case, $\beta_{jk0}$ (i.e., effect of $u_k$ on edge $(j,k)$) can be nonzero while $\beta_{jk0}$ is zero (i.e., population level edge $(j,k)$). Section 8 discusses modifications of our proposal if hierarchy is to be enforced.

To ease the exposition of key ideas, we first assume a known $\Gamma$ in the ensuing development,
and focus on the estimation of \( \{\beta_{jkh}\}_{j,k\in[p],h\in[0]\cup[q]} \). In Section 5, we drop this assumption, develop an estimation procedure and derive theory when \( \Gamma \) is unknown.

3 Estimation

With \( n \) independent data \( \mathcal{D} = \{(u^{(i)},x^{(i)}), i \in [n]\} \), where \( u^{(i)} \in \mathbb{R}^q \), \( x^{(i)} \in \mathbb{R}^p \), let \( z^{(i)} = x^{(i)} - \Gamma u^{(i)}, i \in [n] \). Denote the samples of the \( j \)th \( z \) variable (response) by \( z_j = (z_j^{(1)}, \ldots, z_j^{(n)})^\top, j \in [p] \) and the samples of the \( h \)th \( u \) covariate by \( u_h = (u_h^{(1)}, \ldots, u_h^{(n)})^\top, h \in [q] \). The Gaussian graphical regression model on the \( j \)th response variable can be written as

\[
z_j = \sum_{k \neq j}^p \beta_{jk0}z_k + \sum_{k \neq j}^p \sum_{h=1}^q \beta_{jkh}u_h \odot z_k + \epsilon_j,  \tag{9}\]

where \( \odot \) denotes the Hadamard product (element-wise product of two equal-length vectors) and \( \text{Var}(\epsilon_j) = 1/\sigma_{jj}^2 \). We partition the index set \( \{1, \ldots, (p-1)(q+1)\} \) into \( q+1 \) groups, indexed as \( (0), (1), \ldots, (q) \subset \{1, \ldots, (p-1)(q+1)\} \), so that \( (\beta_j)_{(0)} = b_{j0} \) and \( (\beta_j)_{(h)} = b_{jh} \), \( h \in [q] \). For a group index subset \( G \in \{1, \ldots, q\} \), define \( (G) = \cup_{h \in G}(h) \) and \( (G^c) = \cup_{h \notin G}(h) \). As such, \( (\beta_j)_{(G)} \) represents the sub-vector of \( \beta_j \) in the union of groups \( h \in G \).

Denote the squared error loss function by

\[
\ell_j(\beta_j|\mathcal{D}) = \frac{1}{2n} \|z_j - W_{-j}\beta_j\|^2_2, 
\]

where \( W_{-j} = [z_1, z_1 \odot u_1, \ldots, z_1 \odot u_q, \ldots, z_{j-1} \odot u_q, z_{j+1}, z_{j+1} \odot u_1, \ldots, z_p \odot u_q] \in \mathbb{R}^{n \times (p-1)(q+1)} \).

To estimate \( \beta_j \), we consider a sparse group lasso penalized loss function:

\[
\ell_j(\beta_j|\mathcal{D}) + \lambda_1\|\beta_j\|_1 + \lambda_g\|\beta_{j,-0}\|_{1,2}, \tag{10}\]

where \( \|\beta_{j,-0}\|_{1,2} = \sum_{h=1}^q \|\beta_j(h)\|_2 \) and \( \lambda, \lambda_g \geq 0 \) are tuning parameters. The convex regularizing terms, \( \|\beta_j\|_1 \) and \( \|\beta_{j,-0}\|_{1,2} \), encourage element- and group-wise sparsity, respectively, though the group sparse penalty is not applied to \( (\beta_j)_{(0)} \). The combined sparsity penalty in (10) is termed the sparse group lasso penalty (Simon et al., 2013; Li et al., 2015).

We estimate each \( \beta_j \) separately using (10), which is computationally efficient, even when both \( p \) and \( q \) are large. As the estimates do not guarantee the symmetry of \( \Omega(u) \), we propose a post-processing step. Specifically, denote by \( \beta_{jkh} = \hat{\beta}_{jkh} \) and \( \hat{\beta}_{jkh} \) is estimated from (10) and \( \hat{\delta}^{ij} \) from (17), \( j, k \in [p], h \in \{0\} \cup [q] \). We later show that both \( \hat{\beta}_{jkh} \) and \( \hat{\beta}_{jkh} = \hat{\beta}_{jkh}1\{\hat{\beta}_{jkh} > |\hat{\beta}_{jkh}|\} + \hat{\beta}_{jkh}1\{|\hat{\beta}_{jkh}| < |\hat{\beta}_{jkh}|\} \).
Symmetrization can be also done via
\[
\hat{\beta}_{jkh}^t = \hat{\beta}_{jkh}^0 = \hat{\beta}_{jkh}^0 1 \{ |\hat{\beta}_{jkh}^0| < |\hat{\beta}_{jkh}^0| \} + \hat{\beta}_{jkh}^0 1 \{ |\hat{\beta}_{jkh}^0| > |\hat{\beta}_{jkh}^0| \}.
\]
However, it is conservative as \( \hat{\beta}_{jkh}^t \) is nonzero only when both \( \hat{\beta}_{jkh}^0 \) and \( \hat{\beta}_{jkh}^0 \) are nonzero, compared to (11) wherein \( \hat{\beta}_{jkh}^t \) is nonzero if either of \( \hat{\beta}_{jkh}^0 \) and \( \hat{\beta}_{jkh}^0 \) is nonzero. Though both are asymptotically equivalent (see Remark 4), (11) has a better finite sample performance (Meinshausen and Bühlmann, 2006; Peng et al., 2009), especially when \( p \) and \( q \) are large relative to \( n \).

We could estimate \( \beta_1, \ldots, \beta_p \) jointly by combining \( p \) loss functions, and minimizing \( \sum_{j=1}^p \ell_j(\beta_j | \mathcal{D}) + \lambda \sum_j \| \beta_j \|_1 + \lambda_g \sum_j \| \beta_{j,-0} \|_{1,2} \). It would have the benefit of preserving symmetry by restricting \( \beta_{jkh}^t = \beta_{jkh}^0 \), but it is much more computationally intensive than (10) by optimizing with respect to \( O(p^2 q) \) parameters simultaneously.

We select \( \lambda \) and \( \lambda_g \) in (10) via BIC (Yuan and Lin, 2007; Huang et al., 2010; Foygel and Drton, 2010). Specifically, among a set of working parameter values, we choose the combination of \( (\lambda, \lambda_g) \) that minimizes
\[
n \log \{ \ell_j(\hat{\beta}_j | \mathcal{D}) \} + s_j \times \log n,
\]
where \( \hat{\beta}_j \) is the estimate of \( \beta_j \) under the working tuning parameters, and \( s_j \) is the number of nonzero elements in \( \hat{\beta}_j \).

## 4 Theoretical Properties

We derive the non-asymptotic convergence rate of the sparse group lasso estimator from (10) and establish variable selection consistency. Our theoretical investigation is challenged by several unique aspects of the model. First, as the design matrix \( W_{-j} \in \mathbb{R}^{n \times (p-1)(q+1)} \) includes high-dimensional interaction terms, characterizing the joint distribution of each row in \( W_{-j} \) becomes difficult. Second, the rows in the design matrix \( W_{-j} \) are not identically distributed, with the \( i \)th row being
\[
(z_1^{(i)}, z_1^{(i)} \times u_1^{(i)}, \ldots, z_1^{(i)} \times u_q^{(i)}, \ldots, z_j^{(i)} \times u_1^{(i)}, z_j^{(i)} \times u_q^{(i)}, \ldots, z_{j+1}^{(i)} \times u_1^{(i)}, \ldots, z_p^{(i)} \times u_q^{(i)}) \text{ where } z^{(i)} \sim \mathcal{N}_p(0, \Sigma(\mu^{(i)})). \]
The complex joint distribution of \( W_{-j} \) requires a delicate treatment of it. Lastly, the combined penalty term \( \lambda \| \beta_j \|_1 + \lambda_g \| \beta_{j,-0} \|_{1,2} \) is not decomposable. Therefore, classic techniques for decomposable regularizers and null space properties are not applicable (Negahban et al., 2012). Standard treatments of the stochastic term (Bickel et al., 2009; Lounici et al., 2011; Negahban et al., 2012) such as \( \langle \epsilon, W_{-j} \Delta \rangle \leq \| W_{-j}^T \epsilon \|_\infty \| \Delta \|_1 \) or \( \langle \epsilon, W_{-j} \Delta \rangle \leq \| W_{-j}^T \epsilon \|_{\infty,2} \| \Delta \|_{1,2} \), where \( \Delta \in \mathbb{R}^{(p-1)(q+1)} \), would yield convergence rates that are comparable to those from the lasso or the group lasso. Utilizing the statistical properties and the computational optimality of the sparse group lasso estimator in (10), we derive
two interrelated bounds on the stochastic term. The first bound characterizes the cardinality measure of the covariate space, while the second one utilizes the Karush–Kuhn–Tucker (KKT) condition and properties of the combined regularizer. These bounds together give a sharp upper bound on the stochastic term, which is then used to determine the appropriate λ and λg; see Section S2.1 for details. In summary, by sharpening the bound of the stochastic term, we are able to demonstrate the improvement of our proposed estimator over both the lasso and the group lasso in the convergence rate.

Denote the true parameters by \( \beta_j, j \in [p] \), though in some contexts we also use them to denote the corresponding arguments in functions. Let \( S_j \) be the element-wise support set of \( \beta_j \), i.e., \( S_j = \{ l : (\beta_j)_l \neq 0, l \in [(p - 1)(q + 1)] \} \), and \( G_j \) be the group-wise support set of \( \beta_j \), i.e., \( G_j = \{ h : (\beta_j)_{(h)} \neq 0, h \in [q] \} \). Moreover, let \( s_j = |S_j|, s_{j,g} = |G_j| \), and assume \( s_j \geq 1, j \in [p] \). It follows that \( s_{j,g} \leq s_j, j \in [p] \). When there is no ambiguity, we write \( W \) without noting its dependence on \( j \) for notational ease. We denote by \( \sigma^2_{j,g} = 1/\sigma_j^2 \), \( j \in [p] \), and state a few regularity conditions.

**Assumption 1** Let \( u^{(i)} \) be non-random vectors. There exists a constant \( M_1 > 0 \) such that \( |\langle u^{(i)}, v \rangle| < M_1 \) for all \( \|v\| = 1 \) and \( v \in \mathbb{R}^q, i \in [n] \).

This assumption stipulates that, with \( \{u^{(i)}\}_{i \in [n]} \) being fixed, \( \{z^{(i)}\}_{i \in [n]} \) is the only source of randomness in (9). The boundedness condition on the inner product controls the moments of \( W_i^\top a \) for \( \|a\|_2 = 1 \) and \( a \in \mathbb{R}^{(p-1)(q+1)} \), where \( W_i \) is the \( i \)th row of \( W \), and is analogous to a bounded moment condition on the design matrix (Vershynin, 2010, 2012). Particularly by taking \( v = e_h \), a directional vector with the \( h \)th entry being 1 and 0 elsewhere, \( |\langle u^{(i)}, v \rangle| < M_1 \) implies that the \( |u^{(i)}_h| \leq M_1, h \in [q] \), or equivalently \( \|u^{(i)}\|_\infty \leq M_1 \).

**Assumption 2** Let \( \Sigma_W = \mathbb{E}(W^\top W/n) \). There exist positive constants \( \phi_1, \phi_2 \) such that \( \lambda_{\min}(\Sigma_W) \geq 1/\phi_1 > 0 \) and \( \lambda_{\min}(\Omega(u^{(i)})) \geq 1/\phi_2 > 0 \), \( i \in [n] \).

The boundedness of eigenvalues has been commonly considered in the high-dimensional regression literature (Peng et al., 2009; Chen et al., 2016; Cai et al., 2019).

**Theorem 1** Suppose Assumptions 1-2 hold, and \( n \geq A_1\{s_j \log(ep) + s_{j,g} \log(ep/s_{j,g})\} \) for some constant \( A_1 > 0 \). Then the sparse group lasso estimator \( \hat{\beta}_j, j \in [n] \), in (10) with

\[
\lambda = C \sigma_{j,g} \sqrt{\log(ep)/n + s_{j,g} \log(ep/s_{j,g})/(ns_j)}, \quad \lambda_g = \sqrt{s_j/s_{j,g}} \lambda,
\]

satisfies, with probability at least \( 1 - C_1 \exp[-C_2\{s_j \log(ep) + s_{j,g} \log(ep/s_{j,g})\}] \),

\[
\|\hat{\beta}_j - \beta_j\|^2 \lesssim \frac{\sigma^2_{j,g}}{n} \{s_j \log(ep) + s_{j,g} \log(ep/s_{j,g})\} + \frac{\sigma^2_{j,g}}{n}, \tag{14}
\]

where \( C, C_1, \) and \( C_2 \) are positive constants.
Remark 1 Given \( \beta_j \in \mathbb{R}^{(p-1)(q+1)} \) and \( s_{j,g} \leq s_j, j \in [p] \), applying the regular lasso regularizer \( \lambda \| \beta_j \|_1 \) alone would yield a convergence rate of \( (s_j/n) \log(pq) \) (Negahban et al., 2012), which is slower than that in (14), especially when \( q \) is large. Moreover, estimating with only the group lasso regularizer \( \lambda_g \| \beta_{j,-0} \|_{1,2} \) is not feasible when \( p > n + 1 \), as \( (\beta)_{(0)} \in \mathbb{R}^{(p-1)} \) is not included in the group lasso penalty term. If we assume \( (\beta)_{(0)} = 0 \) and consider a group lasso regularizer \( \lambda_g \| \beta_j \|_{1,2} \) (i.e., \( (\beta)_{(0)} \) is included), the group lasso estimator would have a convergence rate of \( (s_{j,g}/n) \log q + (s_{j,g}/n)p \) (Lounici et al., 2011), which is still larger than that in (14), particularly when \( p \) is large. Therefore, we conclude that the combined regularizer \( \lambda \| \beta_j \|_1 + \lambda_g \| \beta_{j,-0} \|_{1,2} \) improves upon both the regular lasso and group lasso regularizers, when the true underlying coefficients are both element-wise and group sparse.

Remark 2 Some group lasso literature (Yuan and Lin, 2006; Lounici et al., 2011) noted that the grouped \( \ell_1 \) penalty should compensate for the group size. It might be the case that \( \lambda_g \) be adjusted by \( \sqrt{p-1} \), as each group in \( \beta_j \) is of size \( p - 1 \). Indeed, with \( (\beta)_{(0)} = 0 \) and no element-wise sparsity within the nonzero groups (i.e., \( \| (\beta_j)_{(h)} \|_0 = p - 1 \) for \( h \in G_j \)), \( \sqrt{s_j/s_{j,g}} \) becomes \( \sqrt{q-1} \) in (13). Interestingly, our theoretical investigation reveals that, for the combined regularizer \( \lambda \| \beta_j \|_1 + \lambda_g \| \beta_{j,-0} \|_{1,2} \), \( \lambda_g = \sqrt{s_j/s_{j,g}} \lambda \) suffices to suppress the noise term; see (S7).

Remark 3 Cai et al. (2019) studied the convergence rate of the sparse group lasso estimator in a regular linear regression setting. With a different proof strategy, their theoretical analysis yields a convergence rate of \( \sigma^2 \{ s_j \log(es_{j,g}p) + s_{j,g} \log(eq/s_{j,g}) \} / n \), which is slower than (14), especially when \( s_{j,g} \) is large. Again with regular linear regression, Cai et al. (2019) showed that the minimax lower bound for the estimation error is \( \sigma^2 \{ s_j \log(es_{j,g}p/s) + s_{j,g} \log(eq/s_{j,g}) \} / n \), which is matched by the upper bound in (14) if \( s_j \approx s_{j,g} \).

We next show that our proposed sparse group lasso estimator achieves variable selection consistence under a mutual coherence condition.

Assumption 3 (Mutual coherence) Denote by \( \eta_j = 1 + \sqrt{s_j/s_{j,g}}, i \in [p] \). We assume that for some positive constant \( c_0 > 6\phi_1 \), the covariance matrix \( \Sigma_W \) satisfies that

\[
\max_{k \neq l} |\Sigma_W(k, l)| \leq \frac{1}{c_0(1 + 8\eta_j)s_j},
\]

where \( \Sigma_W(k, l) \) denotes the \((k, l)\)th element of \( \Sigma_W \).

Assumption 3 specifies that the correlation between columns in \( W \) cannot be excessive, which has been considered in establishing the \( \ell_\infty \) norm convergence of the lasso and the group lasso (Lounici, 2008; Lounici et al., 2011). Similar correlation conditions include the weak mutual coherence condition (Bunea et al., 2007), neighborhood stability condition (Meinshausen and
Bühlmann, 2006) and the irrepresentability condition (Zhao and Yu, 2006); see Van De Geer and Bühlmann (2009) for a discussion of these relationships.

**Theorem 2** Suppose Assumption 1-3 hold. If \( \log p \simeq \log q \) and \( n \geq A_1 \{ s_j \log(ep) + s_{j,g} \log(eq/s_{j,g}) \} \) for some constant \( A_1 > 0 \), then for \( j \in [p] \), the sparse group lasso estimator \( \hat{\beta}_j \) in (10) with \( \lambda \) and \( \lambda_g \) as in (13) satisfies

\[
\| \hat{\beta}_j - \beta_j \| \leq \left\{ 3\phi_1 \eta_j + \frac{18 \phi_1^2 (1 + 4 \eta_j)^2 \eta_j}{(c_0 - 2 \phi_1)(1 + 8 \eta_j)} \right\} \lambda,
\]

with probability \( 1 - C'_1 \exp(-C'_2 \log p) \), where \( C'_1, C'_2 \) are some positive constants. Define

\[
\hat{S}_j = \left\{ k : |(\hat{\beta}_j)_k| > \left\{ 3\phi_1 \eta_j + \frac{18 \phi_1^2 (1 + 4 \eta_j)^2 \eta_j}{(c_0 - 2 \phi_1)(1 + 8 \eta_j)} \right\} \lambda \right\}.
\]

In addition, if the minimum signal strength satisfies

\[
\min_{k \in \hat{S}_j} |(\hat{\beta}_j)_k| > \left\{ 3\phi_1 \eta_j + \frac{18 \phi_1^2 (1 + 4 \eta_j)^2 \eta_j}{(c_0 - 2 \phi_1)(1 + 8 \eta_j)} \right\} \lambda,
\]

we have that \( \mathbb{P}(\hat{S}_j = S_j) = 1 - C'_1 \exp(-C'_2 \log p), \ j \in [n] \).

**Remark 4** For the recovery of true signals in high-dimensional regression, minimum signal strength conditions such as (16) are necessary (Zhang, 2009; Wainwright, 2009). Following Remark 1, (16) implies that, when \( s_j \simeq s_{j,g} \), the sparse group lasso estimator can accommodate weaker signals compared to the regular lasso (Wainwright, 2009) and the group lasso (Lounici et al., 2011), a desirable property of the sparse group lasso estimator. The condition of \( \log p \simeq \log q \) allows \( p \) and \( q \) to grow at a polynomial rate relative to each other, and ensures a tighter bound on \( \| W^\top e_j \|_\infty \); see Chen et al. (2016). Moreover, the selection consistency result in Theorem 2 holds for both estimates in (11) and (12), as (15) characterizes the relationship between the fitted values and the true parameters.

**Remark 5** With \( \hat{\beta}_j \), a natural estimate of the variance \( \sigma^2_{\epsilon_j} = 1/\sigma_{\epsilon_j}^2 \) would be

\[
\hat{\sigma}^2_{\epsilon_j} = \frac{1}{n - \hat{s}_j} \| z_j - W \hat{\beta}_j \|_2^2 = \frac{1}{n - \hat{s}_j} z_j^\top (I_{n \times n} - P_{\hat{S}_j}) z_j,
\]

where \( P_{\hat{S}_j} \) is the projection matrix onto the column space of \( W_{\hat{S}_j} \). The estimator in (17) can alternatively be written as \( \hat{\sigma}^2_{\epsilon_j} = \frac{1}{n - \hat{s}_j} (1 - \gamma_n^2) \epsilon^\top \epsilon \), where \( \gamma_n^2 = \epsilon^\top P_{\hat{S}_j} \epsilon / \epsilon^\top \epsilon \) represents the fraction of bias in \( \hat{\sigma}^2_{\epsilon_j} \).

Under conditions in Theorem 2 and using a result in (S7), we get \( \gamma_n^2 \simeq \frac{\sigma^2_{\epsilon_j}}{n} \{ s_j \log(ep) + s_{j,g} \log(eq/s_{j,g}) \} \). Therefore, \( \hat{\sigma}^2_{\epsilon_j} \) is consistent, provided that \( \sigma^2_{\epsilon_j} \{ s_j \log(ep) + s_{j,g} \log(eq/s_{j,g}) \} / n \to 0 \).

## 5 Estimation with Unknown \( \Gamma \): a Two-Step Procedure

We now present a two-step estimation procedure when \( \Gamma \) is unknown, followed by its theoretical properties. Assuming a sparse \( \Gamma \) (Cai et al., 2012; Yin and Li, 2013; Chen et al.,
Step 1 estimates $\mathbf{\Gamma}$ using a $\ell_1$-penalized regression; in Step 2, we approximate each $z^{(i)}$ with $\hat{z}^{(i)} = x^{(i)} - \hat{\mathbf{\Gamma}}u^{(i)}$ where $\hat{\mathbf{\Gamma}}$ is estimated from the first step, and estimate $\beta_j$ based on $\{\hat{z}^{(i)}\}_{i \in [q]}$ by using the procedure described in Section 3. The two-step procedure is computationally feasible, particularly when both $p$ and $q$ are large, and has been considered in covariate-adjusted Gaussian graphical model estimation (Cai et al., 2012; Yin and Li, 2013; Chen et al., 2016).

**Step 1.** Denote by $\mathbf{H} = [\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(n)}]^\top$ and the sample of the $j$th variable by $\mathbf{x}_j = (x^{(1)}_j, \ldots, x^{(n)}_j)^\top$. We first estimate $\mathbf{\Gamma}$ with

$$\hat{\mathbf{\gamma}}_j = \arg \min_{\mathbf{\gamma} \in \mathbb{R}^q} \frac{1}{2n} \| \mathbf{x}_j - \mathbf{H} \mathbf{\gamma} \|_2^2 + \lambda_1 \| \mathbf{\gamma} \|_1,$$  \hspace{1cm} (18)

and denote the estimates by $\hat{\mathbf{\Gamma}} = (\hat{\mathbf{\gamma}}_1, \ldots, \hat{\mathbf{\gamma}}_p)^\top$.

**Step 2.** With $\hat{\mathbf{\Gamma}}$ obtained from Step 1, denote by $\hat{z}^{(i)} = x^{(i)} - \hat{\mathbf{\Gamma}}u^{(i)} = (\hat{z}_1^{(i)}, \ldots, \hat{z}_p^{(i)})^\top$, $i \in [n]$ and $\hat{z}_j = (\hat{z}_j^{(1)}, \ldots, \hat{z}_j^{(n)})^\top$, $j \in [p]$. We estimate $\beta_j$ with

$$\hat{\beta}_j = \arg \min_{\beta \in \mathbb{R}^{(p-1)(q+1)}} \frac{1}{2n} \| \hat{z}_j - \hat{\mathbf{W}}_{-j} \beta_j \|_2^2 + \lambda_1 \| \beta_j \|_1 + \lambda_2 \| \beta_{j-0} \|_{1,2},$$ \hspace{1cm} (19)

where $\hat{\mathbf{W}}_{-j} = [\hat{z}_1 \odot \mathbf{u}, \ldots, \hat{z}_1 \odot \mathbf{u}, \ldots, \hat{z}_{j-1} \odot \mathbf{u}, \hat{z}_{j+1} \odot \mathbf{u}, \ldots, \hat{z}_p \odot \mathbf{u}] \in \mathbb{R}^{n \times (p-1)(q+1)}$. To ease notation, we again write $\hat{\mathbf{W}}$ without emphasizing its dependence on $j$, when there is no ambiguity. As (18) and (19) are both convex, they can be optimized efficiently using existing algorithms (Tibshirani, 1996; Simon et al., 2013; Vincent and Hansen, 2014).

**Remark 6** Step 1 poses a regular lasso penalty on $\mathbf{\Gamma}$, as commonly considered in the covariate-adjusted Gaussian graphical model literature (Cai et al., 2012; Yin and Li, 2013; Chen et al., 2016). This is because each regression in Step 1 is only of dimension $q$, compared to $(p-1)(q+1)$ in Step 2, and, hence, a lasso penalty may suffice to reduce the sample complexity; see Rao et al. (2015); Cai et al. (2019) on sample complexity comparisons between (group) lasso and sparse group lasso. When $q$ is large, it may be necessary to consider a sparse group penalty (as in Step 2) that encourages $\mathbf{\Gamma}$ to be both element-wise and group sparse; see Section 8.

The theoretical development is challenging. Step 1 involves a regularized regression $\mathbf{x}_j = \mathbf{H} \mathbf{\gamma}_j + \mathbf{z}_j$ with heteroskedastic errors $z_j^{(i)} \sim \mathcal{N}(0, \Sigma(u^{(i)})_{jj})$, $i \in [n]$, which are considerably more difficult to analyze than the usual i.i.d normal errors. Moreover, in Step 2, both the response vector $\hat{z}_j$ and the design matrix $\hat{\mathbf{W}}$ inherit approximation errors from $\hat{\mathbf{\Gamma}} - \mathbf{\Gamma}$, which further complicates the analysis of the sparse group lasso estimator. Let $\mathcal{T}_j$ be the index set of nonzero entries in $\mathbf{\gamma}_j$, $j \in [p]$, and $t = \max |\mathcal{T}_j|$. We show that, using the same conditions as in Theorem 1, the convergence rate of $\hat{\beta}_j$ in (19) is slowed by a term of $\sqrt{t \log q}$, which reflects the estimation error from the first step. We further show that if correlations between
columns in the design matrix are small, $\hat{\beta}_j$ can achieve the same convergence rate as that in Theorem 1 (i.e., the noiseless case), and thus enjoys the oracle property.

**Assumption 4** There exists a constant $M_2 > 0$ such that $\|\beta_j\|_1 \leq \sigma_{\epsilon_j}M_2$, $j \in [p]$.

This condition controls the approximation errors in $\hat{z}_j$ when analyzing the second step of the estimation procedure. Similar conditions have been considered in other two-step procedures (Cai et al., 2012; Chen et al., 2016). The condition can be further relaxed to allow $M_2$ to diverge to infinity, in which case Theorems 3 and 4 hold after replacing $t$ with $tM_2^2$.

**Assumption 5 (Restricted Eigenvalue)** Let $C_j = \{v \in \mathbb{R}^q : \|v_{T_j}\|_1 \leq 2\|v_{T_j}\|_1\}$. We assume that, there exists $\kappa_j > 0$ such that, for $j \in [p]$, $\frac{\|Hv\|_2^2}{n} \geq \kappa_j\|v\|_2^2$, $\forall v \in C_j$.

This condition is needed for quantifying the error when estimating $\gamma_j$, $j \in [p]$. If the gram matrix $H^\top H/n$ satisfies this restricted eigenvalue condition, the $\ell_1$ regularized estimation leads to the desired prediction and estimation error rates (Bickel et al., 2009; Negahban et al., 2012).

**Theorem 3** Suppose that Assumptions 1-2 and 4-5 are satisfied, and $n \geq A_2(t\log q)\{s_j \log(ep) + s_{j,g} \log(eq/s_{j,g})\}$ for some constant $A_2 > 0$. Let $\lambda_1 = 14\sigma_n\sqrt{\tau_1 \log q}/n$ for any $\tau_1 > 0$. The minimizer $\hat{\gamma}_j$ in (18) satisfies

$$\|\hat{\gamma}_j - \gamma_j\|_2^2 \lesssim \frac{t\log q}{n},$$

$$\frac{1}{n}\|\hat{z}_j - z_j\|_2^2 \lesssim \frac{t\log q}{n},$$

with probability $1 - 3\exp(-\tau_1 \log q)$, $j \in [p]$. The minimizer $\hat{\beta}_j$ in (19) with

$$\lambda = C\sigma_{\epsilon_j} \sqrt{(t\log q)\{\log(ep)/n + s_{j,g} \log(eq/s_{j,g})/(ns_j)\}}, \quad \lambda_g = \sqrt{s_j/s_{j,g} \lambda},$$

satisfies

$$\|\hat{\beta}_j - \beta_j\|_2^2 \lesssim \frac{\sigma_{\epsilon_j}^2}{n} \{s_j \log(ep) + s_{j,g} \log(eq/s_{j,g})\} + \frac{\sigma_{\epsilon_j}^2}{n},$$

with probability $1 - C_3 \exp[C_4\{\log p - (\tau_1 - 1) \log q\}]$, for some positive constants $C_3$, $C_4$. Moreover, if Assumption 3 holds and $n \geq A_3\{s_j \log(ep) + s_{j,g} \log(eq/s_{j,g})\}$ for some constant $A_3 > 0$, then $\hat{\beta}_j$ in (19) with $\lambda_1$ and $\lambda_g$ in (13) satisfies

$$\|\hat{\beta}_j - \beta_j\|_2^2 \lesssim \frac{\sigma_{\epsilon_j}^2}{n} \{s_j \log(ep) + s_{j,g} \log(eq/s_{j,g})\} + \frac{\sigma_{\epsilon_j}^2}{n},$$
with probability $1 - C'_3 \exp[C'_4\{\log p - (\tau_1 - 1)\log q\}]$, for some positive constants $C'_3$ and $C'_4$.

**Remark 7** Compared to the tuning parameters in (13) with the noiseless case (i.e., when $\Gamma$ is known), the tuning parameters in (21) are multiplied by an additional term $t \log q$. When $\Gamma$ is unknown, as opposed to the oracle regression equation $z_j = W\beta_j + \epsilon_j$, we only have access to the noisy equation $\hat{z}_j = \hat{W}\beta_j + E_j$, where $E_j = \epsilon_j + (\hat{z}_j - z_j) + (W - \hat{W})\beta_j$. The additional term $\sqrt{t \log q}$ in $\lambda$ and $\lambda_g$ is needed to suppress the noise terms, including both $\langle \epsilon_j, W\Delta \rangle$ and $(\hat{z}_j - z_j) + (W - \hat{W})\beta_j$, where $\Delta = \hat{\beta}_j - \beta_j$. Consequently, the convergence rate in (22) is also multiplied by a term of $t \log q$.

**Remark 8** Under Assumption 3, the rate in (23) is the same as the oracle rate in (14). This is achievable as the stochastic term $\langle \epsilon_j, \hat{W}\Delta \rangle$ can be better bounded when the influence of the “noise” covariates (i.e., $\hat{W}_{S^c}$) on the effective covariates (i.e., $\hat{W}_S$) is controlled under Assumption 3. Conditions similar to Assumption 3 were also considered in Cai et al. (2012) and Yin and Li (2013) to establish the oracle inequality of the covariate-adjusted precision matrix estimator obtained from the two-step estimation procedure.

**Theorem 4** Suppose Assumptions 1-5 hold and $\lambda_1 = 14\sigma_n \sqrt{\tau_1 \log q / n}$ for any $\tau_1 > 0$. Assume $n \geq A_3\{s_j \log (ep) + s_{j,g} \log (eq/s_{j,g})\}$ for some constant $A_3 > 0$, $\log p \approx \log q$ and that $t = o(\sqrt{n/\log q})$. For $j \in [p]$, the sparse group lasso estimator $\hat{\beta}_j$ in (19) with $\lambda$ and $\lambda_g$ as in (13) satisfies,

$$
||\hat{\beta}_j - \beta_j||_{\infty} \leq \frac{9}{2} \left\{ \phi_1 \eta_j + \frac{12\phi_1(1 + 3\eta_j)^2}{(c_0 - 6\phi_1)(1 + 8\eta)} \right\} \lambda,
$$

with probability $1 - C_5 \exp[C_6\{\log p - (\tau_1 - 1)\log q\}]$, for some positive constants $C_5$, $C_6$.

Define $\mathcal{S}_j = \left\{ k : |\hat{\beta}_{j,k}| > \frac{9}{2} \left\{ \phi_1 \eta_j + \frac{12\phi_1(1 + 3\eta_j)^2}{(c_0 - 6\phi_1)(1 + 8\eta)} \right\} \lambda \right\}$. If, in addition, the minimum signal strength satisfies

$$
\min_{k \in S} |\beta_{j,k}| > \frac{9}{2} \left\{ \phi_1 \eta_j + \frac{12\phi_1(1 + 3\eta_j)^2}{(c_0 - 6\phi_1)(1 + 8\eta)} \right\} \lambda,
$$

then $\mathbb{P}(\mathcal{S}_j = S_j) = 1 - C_5 \exp[C_6\{\log p - (\tau_1 - 1)\log q\}]$, $j \in [n]$.

**Remark 9** The additional sparsity condition $t = o(\sqrt{n/\log q})$ is needed to establish the $\ell_\infty$ bound for $\hat{\beta}_j$ in the presence of the error in $\hat{\Gamma} - \Gamma$. Compared to the minimal signal strength condition (16) in the noiseless case, the condition in (25) is slightly stronger.

Using Remark 5 and (S33), it can be shown that $\hat{s}^2_{\epsilon_j} = \frac{1}{n-s_j}||\hat{z}_j - \hat{W}\hat{\beta}_j||_2^2$ is consistent, if $\sigma^2_{\epsilon_j} \{s_j \log (ep) + s_{j,g} \log (eq/s_{j,g})\} / n \to 0$.

### 6 Simulations

We investigate the finite sample performance of our proposed method by comparing it with some competing solutions. Specifically, we evaluate three competing methods. We first
consider our proposed Gaussian graphical model regression method defined in (19), referred to as \text{RegGMM} hereafter. We also consider a lasso estimator
\begin{equation}
\arg \min_{\beta \in \mathbb{R}^{(p-1)(q+1)}} \frac{1}{2n} \| \hat{z}_j - \hat{W}_{-j} \beta_j \|_2^2 + \lambda \| \beta_j \|_1,
\end{equation}
and a group lasso estimator
\begin{equation}
\arg \min_{\beta \in \mathbb{R}^{(p-1)(q+1)}} \frac{1}{2n} \| \hat{z}_j - \hat{W}_{-j} \beta_j \|_2^2 + \lambda_g (\| (\beta_j)_0 \|_1 + \sqrt{p-1} \| \beta_{j,0} \|_{1,2}),
\end{equation}
where the total number of groups is \((p - 1) + q\).

We simulate \(n\) samples \(\{ (u^{(i)}, x^{(i)}), i \in [n] \}\) from (4), where each sample has \(x^{(i)} \in \mathbb{R}^p\) (e.g., genes) and external covariate \(u^{(i)} \in \mathbb{R}^q\) (e.g., SNPs), including discrete and continuous covariates. Discrete covariates are generated from \(\{0, 1\}\) with equal probabilities, and continuous covariates are generated from \(\text{Uniform}[0,1]\). For \(\mathbf{\Gamma} \in \mathbb{R}^{p \times q}\), we randomly set a proportion of its entries to 0.5, and the rest to zero. Let \(s_{\mathbf{\Gamma}}\) denote this proportion of nonzero entries in \(\mathbf{\Gamma}\).

As to parameters \(\{ \beta_{jkh} \}_{j,k \in [p]}, h \in \{0\} \cup [q]\) and \(\sigma^{ii}, i \in [p]\), in (5), we first set \(\sigma^{ii} = 1, i \in [p]\). The population level network is assumed to follow a scale-free network model, with the degrees of nodes generated from a power-law distribution (Clauset et al., 2009) with parameter 2.5. We randomly select \(q_e\) out of \(q\) covariates to have nonzero effects, and the graphs for these \(q_e\) covariates follow an Erdos-Renyi model with edge probability \(v_e\); see the graph structures in Figure 2. The initial nonzero coefficients in \(\{ \beta_{jkh} \}_{j,k \in [p]}, h \in \{0\} \cup [q]\) are generated from \(\text{Uniform}([-0.5, -0.35] \cup [0.35, 0.5])\). For each \(j \in [p]\), we then rescale \(\{ \beta_{jkh} \}_{k \neq j \in [p], h \in [0] \cup [q]}\) by dividing each entry by \(\sum_{k \neq j \in [p], h \in [0] \cup [q]} |\beta_{jkh}|\). Finally, each rescaled \(\beta_{jkh}\) is averaged with \(\beta_{kjh}\) to ensure symmetry, \(h \in \{0\} \cup [q], k, j \in [p]\). This process results in diagonal dominance, which ensures the positive definiteness of the precision matrices. We set \(s_{\mathbf{\Gamma}} = 0.1, q_e = 5, v_e = 0.01, \) and consider \(n = 200, 400, p = 25, 50\) and \(q = 50, 100\).
Table 1: Estimation accuracy of $\Gamma$ in simulations with various sample sizes $n$, network sizes $p$ and covariate dimensions $q$.

For each simulation configuration, we generate 200 independent data sets, within each of which we randomly set half of the $q$ covariates to be discrete and the rest continuous. Given $u^{(i)}$, we are able to determine $\Omega(u^{(i)})$ and $\Sigma(u^{(i)})$; the $i$th sample $x^{(i)}$ is generated from $\mathcal{N}(\Gamma u^{(i)}, \Sigma(u^{(i)}))$, $i \in [n]$. When comparing the estimates of $\beta_j$’s obtained by the competing methods, we report the results after post-processing as in (11). For a fair comparison, tuning parameters in all of the methods are selected via BIC.

To evaluate the estimation accuracy, we report the estimation errors $\|\Gamma - \hat{\Gamma}\|_F$ (the Frobenius norm) and $\sum_{j=1}^{p} \|\hat{\beta}_j - \beta_j\|_2$, where $\hat{\beta}_j$’s, with a slight abuse of notation, denote the estimates of $\beta_j$’s obtained by various methods. For the selection accuracy, we report the true positive rate (TPR), false positive rate (FPR) and the $F_1$ score, calculated as $\frac{2TP}{2TP + FP + FN}$, where TP is the true positive count, FP is the false positive count, and FN is the false negative count. The highest possible value of $F_1$ is 1, indicating perfect selection.

Tables 1-2 report the average criteria, with standard errors in the parentheses, over 200 data replications. Table 1 shows that the first step of our estimation procedure achieves good performance, and the estimation error of $\Gamma$ decreases as $n$ increases, or as $p$ and $q$ decrease. Such observations conform to Theorem 3.

Table 2 shows that the proposed RegGMM outperforms the competing methods in both estimation accuracy and selection accuracy (i.e., the $F_1$ score), which holds true for different sample sizes $n$, network size $p$ and covariate dimensions $q$. This is consistent with our theoretical findings (see Remark 1). Moreover, the estimation errors of RegGMM decrease as $n$ increases, or as $p$ and $q$ decrease, confirming the theoretical results in Theorem 3. In contrast, under all the scenarios examined, the group lasso estimator cannot identify any effective covariate, possibly because the nonzero entries in the nonzero groups in $\beta_j$ are too sparse. As such, the group lasso estimator yields low false positive rates as well as low true positive rates. When $p$ and $q$ are both large (i.e., $p, q \geq 50$), RegGMM and the group lasso have comparable false positive rates, while RegGMM gives better true positive rates.
We apply our proposed method to the REMBRANDT study (GSE108476), which included a subcohort of \( n = 178 \) patients diagnosed with glioblastoma multiforme (GBM), the most aggressive subtype of brain cancer (Bleeker et al., 2012). GBM is highly complex and fatal, and existing therapies remain largely ineffective (Bleeker et al., 2012). It is imperative to explore new and effective treatment. All of these 178 GBM patients had undergone microarray and single nucleotide polymorphism (SNP) chip profiling, with both gene expression and SNP data available for analysis. Specifically, the extracted RNA from each tumor sample was processed using microarrays with 23,521 genes assayed on each array. Genomic DNA from each sample was hybridized to SNP chips, which covers 116,204 SNP loci with a mean intermarker distance of 23.6kb. The raw data files were pre-processed and normalized using standard pipelines; see Gusev et al. (2018) for more details.

We focus on a set of \( p = 73 \) genes (response variables) that belong to the human glioma
pathway in the Kyoto Encyclopedia of Genes and Genomes (KEGG) database (Kanehisa and Goto, 2000); see Figure S1. The covariates include local SNPs (i.e., SNPs that fall within 2kb upstream and 0.5kb downstream of the gene) residing near these 73 genes, resulting in a total of 118 SNPs. SNPs are coded with “0” indicating homozygous in the major allele and “1” otherwise. For each patient, age and gender are included in analysis. Consequently, there are \( q = 120 \) covariates, bringing a total of \( 73 \times 36 \times 121 = 317,988 \) parameters (including the intercepts) in the model. Our main objective is to recover both the population-level and subject-level gene networks, and to understand if and how the subject-level network varies with age, gender, and SNPs.

We apply the proposed two-step procedure in Section 5 to this data set. It is common in penalized regressions to standardize predictors to ensure they be on the same scale (Tibshirani, 1997). For example, the covariates in the model are standardized to have mean zero and variance one (Bien et al., 2013). The scheme does not alter interpretations of the model; see discussions in Section 8. Tuning parameters in both steps of the estimation procedure are selected via BIC, and post-processing, as in (11), generates the final estimates. Out of the 120 covariates considered, a total of 6 SNPs are estimated to have nonzero effects on the network.

We first examine the population level network. Figure 3 shows that most genes only have a few edges and a few hub genes have many edges, a commonly observed pattern in genetic pathways (Stuart et al., 2003). Most of the hubs genes are known to be associated with cancer. For example, PIK3CA is a protein coding gene and is one of the most highly mutated oncogenes identified in human cancers (Samuels and Velculescu, 2004); mutations in the PIK3CA gene are found in many types of cancer, including cancer of the brain, breast, ovary, lung, colon and stomach (Samuels and Velculescu, 2004). The PIK3CA gene is a part of the PI3K/AKT/MTOR signaling pathway, which is one of the core pathways in human GBM and other types of cancer (Network et al., 2008). TP53 is also a highly connected gene in the estimated network. This gene encodes a tumor suppressor protein containing transcriptional activation, and is the most frequently mutated gene in human cancer; the P53 signaling pathway is also one of the core pathways in human GBM and other types of cancer (Network et al., 2008). In Figure 3, we can identify several core pathways in human GBM including the PI3K/AKT/MTOR, Ras-Raf-MEK-ERK, calcium and p53 signaling pathways; see Table 3 for genes included in each pathway. These findings are in agreement to the existing literature on GBM genes and pathways (Network et al., 2008; Brennan et al., 2013; Maklad et al., 2019).

We next examine the covariate effects on the network. Identified are six co-expression QTLs, namely, rs6701524, rs1267622, rs10509346, rs9303511, rs503314 and rs759950. The network effects of rs6701524 are shown in Figure 4 (left panel). This SNP, residing in MTOR, is found to affect PIK3R2’s co-expression with PIK3R1, and also with SOS1, CALML5, GRB2 and CAMK2G. This is an interesting finding as PI3K/MTOR is a key
Figure 3: The graph corresponding to the population-level effect. The node sizes are proportional to mean expression levels. Edges that have positive (negative) effects on partial correlations are shown in red dashed (black solid) lines.

| name                        | genes                                                                 | references               |
|-----------------------------|----------------------------------------------------------------------|--------------------------|
| PI3K/ AKT/MTOR signaling pathway | PIK3CA, PIK3CB, PIK3CD, PIK3R3, PTEN, AKT1, AKT2, AKT3, MTOR, IGF1, PRKCA | Network et al. (2008)    |
| Ras-Raf-MEK-ERK signaling pathway | EGF, EGFR, GRB2, SOS1, SOS2, IGF1                                         | Brennan et al. (2013)    |
| calcium (Ca^{2+}) signaling pathway | CALM1, CALML3, CALML4, CALML5, CALML6, CAMK1, CAMK4, CAMK1D, CAMK1G, CAMK2A, CAMK2B, CAMK2D, CAMK2G, PRKCA | Maklad et al. (2019)    |
| p53 signaling pathway       | TP53, MDM2, DDB2, PTEN, IGF1 CDK4, CDK6, CDKN1A, CDKN2A                  | Network et al. (2008)    |

Table 3: Pathways and genes involves in each pathway.
pathway in GBM development and progression, and inhibition of PI3K/MTOR signaling was found effective in increasing survival with GBM tumor (Batsios et al., 2019). PIK3R2 is also found to interact directly with SOS1 in β-cell function (Barroso et al., 2003), GRB2 in growth factors (Bisson et al., 2011). This co-expression QTL can potentially regulate the co-expression of PIK3R1, PIK3R2 MTOR, and play an important role in activating the PI3K/MTOR pathway.

Shown in Figure 4 (right panel) are the network effects of rs1267622, a variant of BRAF that is commonly activated by somatic mutations in human cancer (Davies et al., 2002). The figure indicates that this SNP notably affects the co-expression of BAX with other genes. This agrees to the finding that RAF suppresses activation and translocation of BAX (Koziel et al., 2013). Moreover, based on our analysis, rs10509346 regulates the co-expressions of MAPK3, IGF1R and other genes; rs9303511 is associated with the co-expressions of GADD45B and other genes; rs503314 influences the co-expressions of CCND1 and PLCG2, CALML6; rs759950 may modify the co-expressions of GADD45G and CAMK2A. Graphs for these SNPs are shown in Figure S2. As co-expression QTL identification has sparked recent interest, these findings warrant more in-depth investigation.

8 Discussion

The precision matrix Ω(u) specified in (5) is valid (i.e., positive definite), if, for example, Ω(u) is diagonally dominant. That is, σkk ≥ ∑j≠k |Ω(u)jk|. From (5), it is seen that ∑j≠k |Ω(u)jk| ≤ ∥u∥∞∥β′k∥1, where β′k = (β′jk)j≠k∈[p],h∈Ω∪[q]. Thus, a sufficient condition for a positive definite Ω(u) is ∥u∥∞∥β′k∥1 ≤ σkk. With Assumption 1 stipulating ∥u∥∞ < M1,
this sufficient condition holds for $\beta_k'$ and $\sigma^{kk}$ such that $\|\beta_k'\|_1 \leq \sigma^{kk}/M_1$, $k \in [p]$, which leads to a positive definite $\Omega(u)$. This gives one class of feasible coefficients to ensure valid precision matrices; the explicit specification of all such classes warrants further research.

When estimating $\Gamma \in \mathbb{R}^{p \times q}$ in Section 5, we employ a regular lasso regularizer to encourage element-wise sparsity, as commonly done in the covariate-adjusted Gaussian graphical model literature (Yin and Li, 2011; Li et al., 2012; Cai et al., 2012; Chen et al., 2016). When $q$ is large (e.g., a large number of SNPs), it may be desirable to pose a sparse group lasso penalty on $\Gamma$ to encourage effective covariates and their effects on the means of nodes to be simultaneously sparse. In this case, $\Gamma$ can be estimated similarly as in Section 3, while the estimation of $\beta_j$’s remains intact. When establishing the properties of $\hat{\Gamma}$, we need to extend our current analysis of the sparse group lasso estimator to allow heteroskedastic Gaussian errors. We envision that it can be done by using a similar technique as in Kuchibhotla and Chakrabortty (2018).

Moreover, we can modify our method to accommodate the hierarchy between main effects and interaction terms by re-organizing $\beta_j$ as

$$\beta_j = (\beta_{j10}, \beta_{j11}, \ldots, \beta_{j1q}, \ldots, \beta_{jp0}, \beta_{jp1}, \ldots, \beta_{jpq}),$$

and imposing a modified sparse group lasso penalty

$$\lambda \|\beta_j^{-0}\|_0 + \lambda_g \|\beta_j\|_{1,2},$$

where $\beta_j^{-0}$ is $\beta_j$ after leaving out the main effects $\{\beta_{j10}, \ldots, \beta_{jp0}\}$ and groups in $\|\beta_j\|_{1,2}$ are as defined in (28). The penalty is designed in such a way that the element-wise sparsity is not imposed on the main effects, and interactions, if selected, will enter the model with non-zero main effects; a similar regularizer was adopted by (She et al., 2018) for penalized interaction models. With slight modifications, our established theoretical framework can still be used to study the theoretical properties of this modified regularizer.

Lastly, we briefly comment on the model interpretability after pre-processing (e.g. standardization) of the covariates (Bien et al., 2013). As such, each graphical regression (9) is in effect

$$\hat{z}_j = \sum_{k \neq j}^p \tilde{\beta}_{jk0} \frac{\hat{z}_k}{\text{sd}(z_k)} + \sum_{k \neq j}^p \sum_{h=1}^q \beta_{jkh} \frac{(u_{h} - \bar{u}_h)}{\text{sd}(u_h)} \odot \frac{\hat{z}_k}{\text{sd}(\hat{z}_k)} + \epsilon_j,$$

where $\bar{u}_h$ denotes the mean of $u_h$, $\text{sd}(\cdot)$ denotes the standard deviation, and $\hat{z}_j$, residual from the first estimation step, has mean zero. The above equation can be re-written as

$$\hat{z}_j = \sum_{k \neq j}^p \tilde{\beta}_{jk0} \hat{z}_k + \sum_{k \neq j}^p \sum_{h=1}^q \tilde{\beta}_{jkh} u_h \odot \hat{z}_k + \epsilon_j,$$

(29)
where $\tilde{\beta}_{jk0} = \frac{1}{\text{sd}(z_k)} \left\{ \beta_{jk0} - \sum_{h=1}^{q} \frac{u_h}{\text{sd}(u_h)} \beta_{jkh} \right\}$ and $\tilde{\beta}_{jkh} = \frac{\beta_{jkh}}{\text{sd}(u_h)\text{sd}(z_k)}$. Notice $\tilde{\beta}_{jkh}$ and $\beta_{jkh}$ only differ in scale by a positive scalar. Therefore, parameter estimates can be interpreted with non-standardized covariates after a re-calculation as in (29).

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Supplementary Materials

S1 Technical Lemmas

We state the technical lemmas that will be used in our proofs.

Lemma 1 (Lemma 1 in Bellec et al. (2018)) Let \( \text{pen} : \mathbb{R}^d \rightarrow \mathbb{R} \) be any convex function and \( \hat{\beta} \) be defined by

\[
\hat{\beta} \in \arg \min_{\beta \in \mathbb{R}^d} \left\{ \|y - W\beta\|_2^2 + \text{pen}(\beta) \right\},
\]

where \( W \in \mathbb{R}^{n \times d}, y \in \mathbb{R}^n \). Then for \( \beta \in \mathbb{R}^d \),

\[
\|y - W\hat{\beta}\|_2^2 + \text{pen}(\hat{\beta}) + \|W(\hat{\beta} - \beta)\|_2^2 \leq \|y - W\beta\|_2^2 + \text{pen}(\beta).
\]

Lemma 2 (Theorem F in Graybill and Marsaglia (1957)) Let \( \epsilon_j \sim \mathcal{N}_p(0, \sigma^2 I) \) and \( A \) be an \( p \times p \) idempotent matrix with rank equals to \( r \leq p \). Then, \( \epsilon_j^\top A \epsilon_j / \sigma^2 \) follows a \( \chi^2 \) distribution with \( r \) degrees of freedom.

Lemma 3 (Lemma 1 in Laurent and Massart (2000)) Suppose that \( U \) follows a \( \chi^2 \) distribution with \( r \) degrees of freedom. For any \( x > 0 \), it holds that

\[
P(U - r \geq 2\sqrt{rx} + 2x) \leq \exp(-x).
\]

Lemma 4 (Proposition 5.16 in Vershynin (2010)) Let \( X_1, \ldots, X_n \) be independent centered sub-exponential random variables. Let \( v_1 = \max_i \|X_i\|_{\psi_1}, \) where \( \|X_i\|_{\psi_1} = \sup_{d \geq 1} d^{-1} (E|X_i|^d)^{1/d} \) denotes the sub-exponential norm. There exists a constant \( c \) such that, for any \( t > 0 \),

\[
P \left( \left\| \sum_{i=1}^n X_i \right\| \geq t \right) \leq 2 \exp \left\{ -c \min \left( \frac{t^2}{v_1^2 n}, \frac{t}{z_1} \right) \right\}.
\]

Lemma 5 Consider independent vectors \( (y_1, x_1), \ldots, (y_n, x_n) \) in \( \mathbb{R} \times \mathbb{R}^p \) such that \( y_i = x_i^\top \beta + \epsilon_i, i \in [n] \). Let \( X = [x_1, \ldots, x_n]^\top \) be fixed and \( \epsilon_i \)'s be independent Gaussian errors with non-constant variances. Assume that \( \sup_{i \in [n]} \text{Var}(\epsilon_i) \) is bounded by a constant \( K_1 > 0 \). Suppose \( \|\beta\|_0 = k \) and there exists \( \kappa > 0 \) such that \( \|v^\top X^\top Xv\|_2/n \geq \kappa \|v\|^2_2 \) for any \( v \in \{v \in \mathbb{R}^q : \|v_T\|_1 \leq 2\|v_T\|_1 \} \), where \( T = \{j : \beta_j \neq 0\} \). Let

\[
\hat{\beta}_\lambda = \arg \min_{\theta \in \mathbb{R}^p} \frac{1}{2n} \sum_{i=1}^n (y_i - x_i^\top \theta)^2 + \lambda \|\theta\|_1.
\]
When \( \lambda = 14\sigma_n \sqrt{\tau_1 \log p/n + \tau_1 K(\log \log n)^{1/2}/n} \) for any \( \tau_1 > 0 \), the lasso estimate satisfies with probability at least \( 1 - 3p^{-\tau_1} \) that

\[
\| \hat{\beta}_\lambda - \beta \|_2 \lesssim \sigma_n \sqrt{\frac{k \log p}{n} + \frac{k^{1/2} \log p}{n}},
\]

where \( \sigma_n = \frac{1}{n} \sum_{i=1}^n \text{Var}(\epsilon_i) \).

The above result is adapted from Theorem 4.5 in Kuchibhotla and Chakrabortty (2018) by considering Gaussian errors and a fixed design matrix \( X \).

**Lemma 6 (Theorem 4.3 in Kuchibhotla and Chakrabortty (2018))** Let \( X_1, \ldots, X_n \) be independent random vectors in \( \mathbb{R}^p \). Assume each element of \( X_i \) is sub-Gaussian with \( \|X_{i,j}\|_{\psi_2} < K_2, \ i \in [n], \ j \in [p] \). Where \( \|X_{i,j}\|_{\psi_2} = \sup_{d \geq 1} d^{-1/2}(E|X_{i,j}|^d)^{1/d} \) denotes the sub-Gaussian norm. Let \( \hat{\Sigma}_X = X^\top X/n \) and \( \Sigma_X = E(X^\top X/n) \). Define

\[
\Upsilon_{n,k} = \sup_{\|v\|_0 \leq k, \|v\|_2 \leq 1} \frac{1}{n} \sum_{i=1}^n \text{Var}\{(X_i^\top v)^2\}.
\]

Then for any \( t > 0 \), with probability at least \( 1 - O(p^{-1}) \),

\[
\sup_{\|v\|_0 \leq k, \|v\|_2 \leq 1} \left| v^\top (\hat{\Sigma}_X - \Sigma_X) v \right| \lesssim \sqrt{\frac{\Upsilon_{n,k} k \log p}{n} + \frac{K_2^2 k \log(n \log p)}{n}}.
\]

**Lemma 7 (Proposition 2.1 in Vershynin (2012))** Consider independent sub-Gaussian random vectors \( X_1, \ldots, X_n \) in \( \mathbb{R}^{p_0} \), \( n \geq p_0 \), satisfying for some \( L > 0 \),

\[
\mathbb{P}(\|\langle X, x \rangle\| > t) \leq 2e^{-t^2/L^2} \text{ for } t > 0, \|x\|_2 = 1.
\]

Then with probability at \( 1 - d_1 \exp(-d_2 n) \), it holds that

\[
\left\| \frac{1}{n} \sum_{i=1}^n X_i \otimes X_i - E(X_i \otimes X_i) \right\| \lesssim (p_0/n)^{1/2},
\]

where \( d_1 \) and \( d_2 \) are positive constants.

**Lemma 8 (Lemma 12 in Loh and Wainwright (2011))** For any symmetric matrix \( \Sigma \in \mathbb{R}^{p \times p} \) and if \( |v^\top \Sigma v| \leq \delta_1 \) for any \( v \in \{v : \|v\|_0 \leq 2s \text{ and } \|v\|_2 = 1\} \), then

\[
|v^\top \Sigma v| \leq 27\delta_1 (\|v\|_2^2 + \frac{1}{s} \|v\|_1^2), \text{ for any } v \in \mathbb{R}^p.
\]
S2 Proofs of Main Results

S2.1 Proof of Theorem 1

As \( \hat{\beta}_j \) is a minimizer of the objective function \((10)\) and the sparse group penalty function in \((10)\) is convex, Lemma 1 implies that

\[
\frac{1}{2n} \| z_j - W \hat{\beta}_j \|_2^2 + \lambda \| \hat{\beta}_j \|_1 + \lambda_g \| \hat{\beta}_{j,0} \|_{1,2} + \frac{1}{2n} \| W (\hat{\beta}_j - \beta_j) \|_2^2 \leq \frac{1}{2n} \| z_j - W \beta_j \|_2^2 + \lambda \| \beta_j \|_1 + \lambda_g \| \beta_{j,0} \|_{1,2}.
\]

Writing \( \Delta = \hat{\beta}_j - \beta_j \) and reorganizing terms in the above inequality gives

\[
\frac{1}{n} \| W \Delta \|_2^2 + \lambda \| \hat{\beta}_j \|_1 + \lambda_g \| \hat{\beta}_{j,0} \|_{1,2} \leq \frac{1}{n} \langle \epsilon_j, W \Delta \rangle + \lambda \| \beta_j \|_1 + \lambda_g \| \beta_{j,0} \|_{1,2}.
\]

Using the fact that \( \| \hat{\beta}_j \|_1 = \| (\hat{\beta}_j)_{S_j} \|_1 + \| (\hat{\beta}_j)_{S_j^c} \|_1 \), \( \| \beta_j \|_1 = \| (\beta_j)_{S_j} \|_1 \), \( \| \hat{\beta}_{j,0} \|_{1,2} = \| (\hat{\beta}_j)_{(S_j^c)} \|_{1,2} + \| (\hat{\beta}_j)_{(S_j^c)} \|_{1,2} \), \( \| \beta_{j,0} \|_{1,2} = \| (\beta_j)_{(S_j^c)} \|_{1,2} \) and applying the triangle inequalities of \( \| \cdot \|_1 \) and \( \| \cdot \|_{1,2} \), we arrive at

\[
\frac{1}{n} \| W \Delta \|_2^2 + \lambda \| \Delta_{S_j} \|_1 + \lambda_g \| \Delta_{(S_j^c)} \|_{1,2} \leq \frac{1}{n} \langle \epsilon_j, W \Delta \rangle + \lambda \| \Delta_{S_j} \|_1 + \lambda_g \| \Delta_{(S_j^c)} \|_{1,2}. \tag{S1}
\]

Defining \( \hat{S}_j = \{ l : (\hat{\beta}_j)_l \neq 0, l \in [(p - 1)(q + 1)] \} \) and letting \( \tilde{S}_j = S_j \cup \hat{S}_j \), we obtain

\[
\langle \epsilon_j, W \Delta \rangle = \langle \epsilon_j, P_{\tilde{S}_j} W \tilde{S}_j \Delta_{\tilde{S}_j} \rangle = \langle P_{\tilde{S}_j} \epsilon_j, W \Delta \rangle \leq \frac{1}{2a_1} \| W \Delta \|_2^2 + \frac{a_1}{2} \| P_{\tilde{S}_j} \epsilon_j \|_2^2,
\]

where the last inequality comes from that \( 2ab \leq ta^2 + b^2/t \) for any \( t > 0 \). Here, \( P_{\tilde{S}_j} \) is the orthogonal projection matrix onto the column space of \( W_{\tilde{S}_j} \). As opposed to the classic techniques that bound \( \langle \epsilon, W \Delta \rangle \) with \( \| W^T \epsilon \|_\infty \| \Delta \|_1 \) or \( \| W^T \epsilon \|_\infty,2 \| \Delta \|_{1,2} \) (Bickel et al., 2009; Lounici et al., 2011; Negahban et al., 2012), we bound this term in \((S2)\) with \( \| W \Delta \|_2^2/(2a_1) + a_1 \| P_{\tilde{S}_j} \epsilon_j \|_2^2/2 \), which is useful in our proof to more sharply bound the term \( \| P_{\tilde{S}_j} \epsilon_j \|_2^2 \). This indeed is a challenging step as the group lasso penalty term in \((10)\) is not decomposable with respect to \( S_j \) and, hence, the existing techniques based on decomposable regularizers and null space properties are non-applicable. We provide a new proof, which is divided into three steps.

**Step 1:** We first bound \( \| P_{S_j} \epsilon_j \|_2^2 \) with some cardinality measures. Given any \( J \subset [(p - 1)(q + 1)] \) and \( \gamma \in \{0, 1\}^{(p-1)(q+1)} \) satisfying \( \gamma_J = 1 \) and \( \gamma_{J^c} = 0 \), we write \( G(J) = \{ h : (\gamma)(h) \neq 0, h \in [q] \} \). In this step, we aim to show that, given \( 0 \leq s_{j,g} \leq q + 1 \) and
0 \leq s_j \leq (p - 1)(q + 1)$, the following holds

\[
\mathbb{P} \left[ \sup_{|\mathcal{J}| = s_j, |\mathcal{G}(\mathcal{J})| = s_{j,g}} \left\| \mathcal{P}_{\mathcal{J}} \mathbf{e}_j \right\|_2^2 \geq 6 \sigma^2_{\mathcal{J}} \left\{ s_j \log(ep) + s_{j,g} \log(eq/s_{j,g}) + t \sigma^2_{\mathcal{J}} \right\} \right] \leq c_1 \exp(-c_2 t),
\]

where $c_1$, $c_2$ are positive constants.

As the projection matrix $\mathcal{P}_{\mathcal{J}}$ is idempotent, Lemma 2 implies that

\[
\left\| \mathcal{P}_{\mathcal{J}} \mathbf{e}_j \right\|_2^2 / \sigma^2_{\mathcal{J}} \sim \chi^2_d, \quad d < s_j.
\]

Next, we find the size of $\{ \mathcal{J} \subset [(p - 1)(q + 1)], |\mathcal{J}| = s_j, |\mathcal{G}(\mathcal{J})| = s_{j,g} \}$ by considering two cases (i) $s_{j,g} = s_j$ and (ii) $s_{j,g} < s_j$. These are the only two possible cases since $S_j$ includes all nonzero elements in $(\beta_j)_0$ and $(\beta_j)_1, \ldots, (\beta_j)_q$.

**case (i):** $s_{j,g} = s_j$. Here, $\{ \mathcal{J} \subset [(p - 1)(q + 1)], |\mathcal{J}| = s_j, |\mathcal{G}(\mathcal{J})| = s_{j,g} \}$ contains $(s_j^q)$ $(p - 1)^{s_j}$ elements. It follows from Stirling’s approximation that $\log(s_j^q) \leq s_{j,g} \log(eq/s_{j,g})$. Therefore, $\log \left\{ \binom{s_j^q}{s_{j,g}} (p - 1)^{s_j} \right\} \leq s_j \log p + s_{j,g} \log(eq/s_{j,g})$.

**case (ii):** $s_{j,g} < s_j$. The number of elements in $\{ \mathcal{J} \subset [(p - 1)(q + 1)], |\mathcal{J}| = s_j, |\mathcal{G}(\mathcal{J})| = s_{j,g} \}$ is bounded above by $\binom{s_j^q}{s_{j,g}} (p - 1)^{s_j}$. By Stirling’s approximation, we have $\log \left\{ \binom{s_j^q}{s_{j,g}} (p - 1)^{s_j} \right\} \leq s_j \log (e(p - 1)(s_{j,g} + 1)/s_j) \leq s_j \log(ep)$. Therefore, $\log \left\{ \binom{s_j^q}{s_{j,g}} (p - 1)^{s_j} \right\} \leq s_j \log(ep) + s_{j,g} \log(eq/s_{j,g})$.

Combining these two cases, we conclude that $\{ \mathcal{J} \subset [(p - 1)(q + 1)], |\mathcal{J}| = s_j, |\mathcal{G}(\mathcal{J})| = s_{j,g} \}$ is bounded above by $s_j \log(ep) + s_{j,g} \log(eq/s_{j,g})$. Applying Lemma 3 and applying the union bound leads to the desired result in Step 1.

**Step 2:** Using the result from Step 1, we find an upper bound for $\left\| \mathcal{P}_{\mathcal{J}_j} \mathbf{e}_j \right\|_2^2$. First, we define

\[
r_{s,s_g} = \sup_{|\mathcal{J}| = s, |\mathcal{G}(\mathcal{J})| = s_g} \left\| \mathcal{P}_{\mathcal{J}} \mathbf{e}_j \right\|_2^2 - M \sigma^2_{\mathcal{J}} \left\{ s \log(ep) + s_g \log(eq/s_g) \right\} + r_{s,s_g},
\]

and $r = \sup_{1 \leq s \leq (p - 1)(q + 1), 0 \leq s_g \leq q} r_{s,s_g}$, where $M > 0$ is a constant to be specified later. Then,

\[
\left\| \mathcal{P}_{\mathcal{J}_j} \mathbf{e}_j \right\|_2^2 \leq M \sigma^2_{\mathcal{J}_j} \left\{ s_j \log(ep) + s_{j,g} \log(eq/s_{j,g}) \right\} + r_{s,s_g}.
\]

(S3)
With $M = 9$, the result from Step 1 gives
\[
\mathbb{P}\{r \geq t\sigma_{s_i}^2\} \leq \sum_{s=1}^{(p-1)(q+1)} \sum_{s_g=0}^{q} \mathbb{P}\{r_{s,s_g} \geq t\sigma_{s_i}^2\} \leq \sum_{s=1}^{(p-1)(q+1)} \sum_{s_g=0}^{q} c_1 \exp[-c_2 t - 3c_2 \{s \log(ep) + s_g \log(eq/s_g)\}].
\]

**Step 3:** This step derives an inequality for $\|\mathcal{P}_{\hat{S}_j} \epsilon_j\|_2^2$ by utilizing the computational optimality of $\hat{\beta}_j$. Since the objective function is convex, $\hat{\beta}_j$ is a stationary point of

\[
\frac{1}{2n} \|z_j - W\beta_j\|^2 + \lambda \|\beta_j\|_1 + \lambda_g \|\beta_j - 0\|_{1,2}.
\]

By the KKT condition, for any $l \in \hat{S}_j \cap (0)$, $h \in [q]$, $\hat{\beta}_j$ must satisfy that

\[
\lambda \text{sign}\{((\hat{\beta}_j)_l)\} = \frac{1}{n} \langle w_i, z_j - W\hat{\beta}_j \rangle.
\]

(S4)

Similarly, for any $l \in \hat{S}_j \cap (h)$, it must satisfy that

\[
\lambda \text{sign}\{((\hat{\beta}_j)_l)\} + \lambda_g \frac{(\hat{\beta}_j)_l}{\|((\hat{\beta}_j)_{(h)})\|_2^2} = \frac{1}{n} \langle w_i, z_j - W\hat{\beta}_j \rangle.
\]

(S5)

Squaring both sides of (S4) and (S5) and summing over all $l \in \hat{S}_j$ gives

\[
\lambda^2 \hat{\sigma}_j + \lambda_g^2 \hat{\sigma}_{j,g} \leq \frac{1}{n^2} \|W^T(z_j - W\hat{\beta}_j)\|_2^2,
\]
due to $\text{sign}\{((\hat{\beta}_j)_l)\}(\hat{\beta}_j)_l \geq 0$.

Next, we have $W^T \mathbf{v} = \sum_{i=1}^{p} z_i \mathbf{v}^\top \mathbf{u}^\top$, where $\mathbf{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_p)$, $\mathbf{v}_l \in \mathbb{R}^{q+1}$ and $\|\mathbf{v}\|_2 = 1$. Since $z^{(i)} \sim \mathcal{N}(\mathbf{0}, \Sigma_{(\mathbf{u}^{(i)})})$, it is true $W^T \mathbf{v} \sim \mathcal{N}(0, \mathbf{b}^\top_i \Sigma \mathbf{b}_i)$, where $\mathbf{b}_i = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p)$. It follows from Assumption 1 and the bounded eigenvalue condition in Assumption 2 that $\mathbf{b}^\top_i \Sigma \mathbf{b}_i = O(1)$. Therefore, $W^T \mathbf{v}$ is sub-Gaussian with a bounded sub-Gaussian norm. Using the result in Lemma 7 and the bounded eigenvalue condition of $\Sigma \mathbf{W}$ in Assumption 2, we obtain $|\|W_{\hat{S}_j}\|/\sqrt{n} \leq M_1$ for some $M_1 > 0$, with probability $1 - d_1 \exp(-d_2 n)$. Since $\hat{S}_j \in \hat{S}$, it holds that

\[
\lambda^2 \hat{\sigma}_j + \lambda_g^2 \hat{\sigma}_{j,g} \leq \frac{2M_1}{n}\|W\Delta\|_2^2 + \frac{2M_1}{n}\|\mathcal{P}_{\hat{S}_j} \epsilon_j\|_2^2.
\]

(S6)
Combining (S3) and (S6) and letting
\[ \lambda = C \sigma_{\epsilon_j} \sqrt{\log(ep)/n + s_{j,g} \log(eq/s_{j,g})/(ns_j)}, \quad \lambda_g = \sqrt{s_j/s_{j,g} \lambda}, \]
where \( C = 3(M_1a_2)^{1/2} \) for some \( a_2 > 0 \), we arrive at
\[
(1 - \frac{2}{a_2}) \| \mathbf{P}_j \mathbf{e}_j \|_2^2 \leq 9\sigma_{\epsilon_j}^2 \{ s_j \log(ep) + s_{j,g} \log(eq/s_{j,g}) \} + \frac{2}{a_2} \| \mathbf{W} \Delta \|_2^2 + r. \tag{S7}
\]
This, together with (S1) and (S2), implies
\[
\frac{\| \mathbf{W} \Delta \|_2^2}{n} + \lambda \| \Delta_{S_j} \|_1 + \lambda_g \| \Delta_{(g_j)} \|_{1,2} \leq \frac{9a_1a_2}{2(a_2 - 2)} \frac{\sigma_{\epsilon_j}^2 \{ s_j \log(ep) + s_{j,g} \log(eq/s_{j,g}) \}}{n} + \frac{a_1a_2}{a_2 - 2} \frac{\| \mathbf{W} \Delta \|_2^2}{n} + \frac{a_1a_2}{2(a_2 - 2)n} r + \lambda \| \Delta_{S_j} \|_1 + \lambda_g \| \Delta_{(g_j)} \|_{1,2}. \tag{S8}
\]
Next, we have that
\[
\frac{\| \Delta_{S_j} \|_1}{\sqrt{s_j}} + \frac{\| \Delta_{(g_j)} \|_{1,2}}{\sqrt{s_{j,g}}} \leq \frac{\| \Delta_{S_j} \|_2 + \| \Delta_{(g_j)} \|_2}{\frac{\sqrt{s_j}}{\sqrt{s_{j,g}}}} \leq 2\phi_1 \| \Sigma_W^{1/2} \Delta \|_2,
\]
where the first inequality is due to that \( \| \Delta_{S_j} \|_1 \leq \sqrt{s_j} \| \Delta_{S_j} \|_2, \| \Delta_{(g_j)} \|_{1,2} \leq \sqrt{s_{j,g}} \| \Delta_{(g_j)} \|_2 \) and the second inequality holds because \( \| \Delta_{S_j} \|_2 + \| \Delta_{(g_j)} \|_2 < 2\| \Delta \|_2 \) trivially with \( \Delta_{S_j} \) and \( \Delta_{(g_j)} \) being the sub-vectors of \( \Delta \), and \( \lambda_{\min}(\Sigma_W) \geq 1/\phi_1 > 0 \) in Assumption 2. Consequently,
\[
\lambda \| \Delta_{S_j} \|_1 + \lambda_g \| \Delta_{(g_j)} \|_{1,2} \leq 2C\phi_1 \sqrt{\epsilon_j} \| \Sigma_W^{1/2} \Delta \|_2 \leq a_3 C \phi_1 \epsilon_j + \frac{1}{a_3} \| \Sigma_W^{1/2} \Delta \|_2^2,
\]
where the last inequality comes from that \( 2ab \leq ta^2 + b^2/t \) for any \( t > 0 \). Plugging this into (S8), we obtain
\[
\left\{ \frac{1}{2a_1} - \frac{a_1}{a_2 - 2} \right\} \frac{\| \mathbf{W} \Delta \|_2^2}{n} \leq \left\{ \frac{9a_1a_2}{2(a_2 - 2)} + C a_3 \phi_1 \right\} \frac{\sigma_{\epsilon_j}^2 \{ s_j \log(ep) + s_{j,g} \log(eq/s_{j,g}) \}}{n} + \frac{1}{a_3} \| \Sigma_W^{1/2} \Delta \|_2^2 + \frac{a_1a_2}{2(a_2 - 2)n} r. \tag{S9}
\]
It remains to bound the distance between \( \| \mathbf{W} \Delta \|_2^2/n \) and \( \| \Sigma_W^{1/2} \Delta \|_2^2 \). To proceed, we first
show with probability $1 - C' \exp\{-(\log p + \log q)\}$,

$$\sup_{\mathbf{v} \in \mathbb{K}_0(2C_{\beta_j}s_j)} \left| \mathbf{v}^\top \left( \frac{\mathbf{W}^\top \mathbf{W}}{n} - \Sigma_\mathbf{W} \right) \mathbf{v} \right| \leq \frac{1}{L}, \quad \text{(S10)}$$

where $L$ is an arbitrarily large constant and $\mathbb{K}_0(2C_{\beta_j}s_j) = \{ \mathbf{v} : \|\mathbf{v}\|_0 \leq 2C_{\beta_j}s_j \text{ and } \|\mathbf{v}\|_2 = 1 \}$ for some positive constant $C_{\beta_j}$.

Write $\mathbf{W}_i^\top \mathbf{v} = \sum_{l \neq i} \gamma_l^{(i)} \mathbf{v}_l^\top \mathbf{u}^{(i)}$, where $\mathbf{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_p) \in \mathbb{K}_0(2C_{\beta_j}s_j)$, $\mathbf{v}_l \in \mathbb{R}^{q+1}$. Since $\mathbf{W}_i^\top \mathbf{v} \sim \mathcal{N}(0, \mathbf{b}_i^\top \Sigma(\mathbf{u}^{(i)}) \mathbf{b}_i)$, where $\mathbf{b}_i = (\mathbf{v}_1^\top \mathbf{u}^{(i)}, \ldots, \mathbf{v}_p^\top \mathbf{u}^{(i)})$, with Assumption 1 and the bounded eigenvalue condition in Assumption 2, we deduce

$$\frac{1}{n} \sum_{i=1}^n \mathbb{E}\{ (\mathbf{W}_i^\top \mathbf{v})^4 \} \leq 3 \left( \frac{1}{n} \sum_{i=1}^n \mathbf{b}_i^\top \Sigma(\mathbf{u}^{(i)}) \mathbf{b}_i \right)^2 = O(1).$$

It then follows from Lemma 5 that with probability $1 - C' \exp\{-(\log p + \log q)\}$,

$$\left| \mathbf{v}^\top \left( \frac{\mathbf{W}^\top \mathbf{W}}{n} - \Sigma_\mathbf{W} \right) \mathbf{v} \right| = o(1).$$

Thus, we have shown (S10) holds with probability at least $1 - C' \exp\{-(\log p + \log q)\}$. Combining this with the result in Lemma 8, we have, with probability at least $1 - C' \exp\{-(\log p + \log q)\}$,

$$\left| \Delta^\top \left( \frac{\mathbf{W}^\top \mathbf{W}}{n} - \Sigma_\mathbf{W} \right) \Delta \right| \leq \frac{1}{L'} \left( \|\Delta\|_2^2 + \frac{1}{s_j} \|\Delta\|_1^2 \right), \quad \text{(S11)}$$

where $L'$ is an arbitrarily large positive constant. Plugging (S11) into (S9) and choosing proper constants $a_1$, $a_2$ and $a_3$ (e.g., $a_1 = 2$, $a_2 = 6$, $a_3 = 6$), we have

$$\frac{1}{2} \|\Sigma_\mathbf{W}^{1/2}{\Delta}^{1/2}\|^2_2 \gtrsim \frac{\sigma_j^2 \{sj\log(ep) + sj,g\log(eq/sj,g)\}}{n} + \frac{1}{L'} \left( \|\Delta\|_2^2 + \frac{1}{s_j} \|\Delta\|_1^2 \right) + \frac{\sigma_j^2}{n}, \quad \text{(S12)}$$

with probability $1 - c_1 \exp\{-c_j^2 \{sj\log(ep) + sj,g\log(eq/sj,g)\}\}$, due to that

$$\mathbb{P} \left[ r \geq M_0 \sigma_j^2 \{sj\log(ep) + sj,g\log(eq/sj,g)\} \right] \leq c_1 \exp\{-c_j^2 \{sj\log(ep) + sj,g\log(eq/sj,g)\}\},$$

for a large positive constant $M_0$. Next, taking $a_1 = 2 - \sqrt{2}$ and $a_2 = 6$ in (S8) and using the expressions for $\lambda$, $\lambda_y$, we have with probability at least $1 - c_1 \exp\{-c_j^2 \{sj\log(ep) + sj,g\log(eq/sj,g)\}\}$ that

$$\frac{\|\Delta_{S_j}\|_1}{\sqrt{s_j}} + \frac{\|\Delta_{(g_j)}\|_{1,2}}{\sqrt{s_{j,g}}} \leq \sigma_j \sqrt{\frac{sj\log(ep) + sj,g\log(eq/sj,g)}{n}} + \frac{\|\Delta_{S_j}\|_1}{\sqrt{s_j}} + \frac{\|\Delta_{(g_j)}\|_{1,2}}{\sqrt{s_{j,g}}}. \quad \text{(S13)}$$
Adding $\|\Delta S_j\|_1/\sqrt{s_j}$ to both sides of (S13), we get

$$\frac{\|\Delta\|_1}{\sqrt{s_j}} \leq \sqrt{c_j} + 3\|\Delta\|_2.$$  (S14)

Plugging (S14) into (S12) and with $\lambda_{\min}(\Sigma W) \geq 1/\phi_1 > 0$ in Assumption 2, we have

$$\|\Delta\|_2^2 \lesssim \sigma^2_j \{s_j \log(ep) + s_{j,g} \log(eq/s_{j,g})\} + \frac{\sigma^2_j}{n}.$$  

with probability at least $1 - C_1 \exp[-C_2\{s_j \log(ep) + s_{j,g} \log(eq/s_{j,g})\}]$, for some positive constants $C_1$, $C_2$.

$$\square$$

### S2.2 Proof of Theorem 2

We first establish the element-wise error bound of $\hat{\beta}_j$ in (15), which is accomplished in three steps.

**Step 1:** With $\Psi = W^T W / n$, this step shows that with probability at least $1 - 2 \exp\{-c_3(\log p + \log q)\}$,

$$\|\Psi \Delta\|_\infty \leq \frac{3\eta_j \lambda}{2}$$  (S15)

for a constant $c_3 > 0$. We also show with probability at least $1 - 2 \exp\{-c'_3 \log p\}$ that

$$\|\Delta S_j\|_1 \leq 4\eta_j \|\Delta S_j\|_1.$$  (S16)

The KKT condition states that $\theta$, an optimizer of (10), satisfies

$$\begin{cases}
(W^T(z_j - W \theta)/n)_l = \text{sign}(\theta_l) \lambda & \text{if } \theta_l \neq 0, l \in (0) \\
(W^T(z_j - W \theta)/n)_l = \text{sign}(\theta_l) \lambda + \lambda_g \frac{\theta_l}{\|\theta(l)\|_2} & \text{if } \theta_l \neq 0, l \in (h) \\
|\langle W^T(z_j - W \theta)/n \rangle_l| < \eta_j \lambda & \text{if } \theta_l = 0.
\end{cases}$$

Thus, any solution $\beta_j$ must satisfy that

$$\left\| \frac{1}{n} W^T (z_j - W \beta_j) \right\|_\infty \leq \eta_j \lambda.$$  

If we can show that with high probability

$$\frac{1}{n} \|W^T \epsilon_j\|_\infty \leq \frac{\eta_j \lambda}{2},$$  (S17)
we conclude that with high probability
\[ \| \Psi \Delta \|_\infty \leq \frac{3\eta_j \lambda}{2}. \]

Now we prove (S17). Define \( V_l = w_l^\top \epsilon_j / n \), \( l \in [(p - 1)(q + 1)] \), where \( w_l \) is the \( l \)th column of \( W \). As \( V_l \) is a sum of independent exponential random variables and \( \text{Var}(w_l(i)) \leq M_1^2 \phi_2 \), applying Lemma 4 gives that
\[
\mathbb{P} \left( |V_l| > \frac{\eta_j \lambda}{2} \right) \leq 2 \exp \left( -\frac{c \eta_j^2 \lambda^2 n}{4M_1^2 \phi_2 \sigma^2_{e_j}} \right) \leq 2 \exp \{ -C'_0 (\log p + \log q) \},
\]
where \( C'_0 = cC^2/(4M_1^2 \phi_2) \), and \( C \) as defined in (13), and the last inequality is due to
\[ \eta_j \lambda \geq C \sigma_{e_j} \sqrt{\frac{\log p}{n}} + C \sigma_{e_j} \sqrt{\frac{\log q}{n}}. \]
With constant \( C \) in (13) chosen sufficiently large, we have \( C'_0 > 1 \). Applying the union bound inequality gives
\[
\mathbb{P} \left( \frac{1}{n} \| W^\top \epsilon_j \|_\infty \geq \frac{\eta_j \lambda}{2} \right) \leq \mathbb{P} \left( \max_l |V_l| \geq \frac{\eta_j \lambda}{2} \right) \leq 2 \exp \{ -(C'_0 - 1)(\log p + \log q) \}.\]

Next, we show that \( \| \Delta s_j \|_1 \leq 3\eta_j \| \Delta s_j \|_1 \) with probability greater than \( 1 - \exp \{ -c'_3 (\log p) \} \). The definition of \( \hat{\beta}_j \) implies that
\[
\frac{1}{2n} \| z_j - W \hat{\beta}_j \|_2^2 + \lambda \| \hat{\beta}_j \|_1 + \lambda_g \| \hat{\beta}_{j-0} \|_{1,2} \leq \frac{1}{2n} \| \epsilon_j \|_2^2 + \lambda \| \beta_j \|_1 + \lambda_g \| \beta_{j-0} \|_{1,2}.
\]
Developing the left hand side of the above inequality gives
\[
\lambda \| \hat{\beta}_j \|_1 + \lambda_g \| \hat{\beta}_{j-0} \|_{1,2} \leq \lambda \| \beta_j \|_1 + \lambda_g \| \beta_{j-0} \|_{1,2} + \frac{1}{n} \Delta^\top W^\top \epsilon_j. \quad (S19)
\]
Recall that \( V_l = w_l^\top \epsilon_j / n \), \( l \in [(p - 1)(q + 1)] \), where \( w_l \) is the \( l \)th column of \( W \). Using similar arguments as in (S17) and again applying Lemma 4 gives that
\[
\mathbb{P} \left( |V_l| > \frac{\lambda}{2} \right) \leq 2 \exp \left( -\frac{c \lambda^2 n}{4M_1^2 \phi_2 \sigma^2_{e_j}} \right) \leq 2 \exp \{ -C''_0 (\log p) \},
\]
where \( C''_0 = cC^2/(4M_1^2 \phi_2) \), \( C \) as defined in (13), and the last inequality is due to \( \lambda^2 n \geq \)
\[ C^2 \sigma_j^2 \log p. \] Applying the union bound inequality then gives
\[ \mathbb{P} \left( \frac{1}{n} \| W^T e_j \|_\infty \geq \frac{\lambda}{2} \right) \leq \mathbb{P} \left( \max_i | V_i | \geq \frac{\lambda}{2} \right) \leq 2 \exp \{ \log q - (C''_0 - 1) (\log p) \}. \]

As \( \log p \approx \log q \), when \( C \) is chosen sufficiently large, we have, for some \( c'_3 > 0 \), \( \exp \{ \log q - (C''_0 - 1) (\log p) \} \leq \exp \{ -c'_3 \log p \} \). Writing event \( A \) as
\[ A = \left\{ \frac{1}{n} \| W^T e_j \|_\infty \leq \frac{\lambda}{2} \right\}, \quad \text{(S21)} \]
we have \( \mathbb{P}(A^c) \leq 2 \exp \{ -c'_3 \log p \} \). Given \( A \), (S19) leads to
\[ 2 \| \hat{\beta}_j \|_1 + 2 \sqrt{\frac{s_j}{s_{j,g}}} \| \hat{\beta}_{j,-0} \|_{1,2} \leq 2 \| \beta_j \|_1 + 2 \sqrt{\frac{s_j}{s_{j,g}}} \| \beta_{j,-0} \|_{1,2} + \| \Delta \|_1, \]
as \( \lambda_g = \sqrt{s_j / s_{j,g}} \) specified in (13).

Adding \( \| \hat{\beta}_j - \beta_j \|_1 \) and \( \sqrt{\frac{s_j}{s_{j,g}}} \| \hat{\beta}_{j,-0} - \beta_{j,-0} \|_{1,2} \) to both sides, and by noting \( \| \Delta \|_1 = \| \hat{\beta}_j - \beta_j \|_1 \), we get
\[ \| \hat{\beta}_j - \beta_j \|_1 + 2 \| \hat{\beta}_j \|_1 + 2 \sqrt{\frac{s_j}{s_{j,g}}} \| \hat{\beta}_{j,-0} \|_{1,2} + \sqrt{\frac{s_j}{s_{j,g}}} \| \beta_{j,-0} \|_{1,2} \leq 2 \| \hat{\beta}_j - \beta_j \|_1 + 2 \| \beta_j \|_1 + 2 \sqrt{\frac{s_j}{s_{j,g}}} \| \beta_{j,-0} \|_{1,2} + 2 \sqrt{\frac{s_j}{s_{j,g}}} \| \beta_{j,-0} - \beta_{j,-0} \|_{1,2}, \]
which leads to
\[ \| \hat{\beta}_j - \beta_j \|_1 + \sqrt{\frac{s_j}{s_{j,g}}} \| \hat{\beta}_{j,-0} - \beta_{j,-0} \|_{1,2} \leq 2(\| \hat{\beta}_j - \beta_j \|_1 + \| \beta_j \|_1 - \| \hat{\beta}_j \|_1) + 2 \sqrt{\frac{s_j}{s_{j,g}}} (\| \beta_{j,-0} - \beta_{j,-0} \|_{1,2} + \| \beta_{j,-0} \|_{1,2} - \| \hat{\beta}_{j,-0} \|_{1,2}). \]

Since \( \| (\hat{\beta}_j)_l - (\beta_j)_l \|_1 + \| (\beta_j)_l \|_1 - \| (\hat{\beta}_j)_l \|_1 = 0 \) for \( l \in S_j^c \) and \( \| (\hat{\beta}_j)_h - (\beta_j)_h \|_{1,2} + \| (\beta_j)_h \|_{1,2} - \| (\hat{\beta}_j)_h \|_{1,2} = 0 \) for \( h \in G_j^c \), using the triangular inequality, we have that
\[ \| \Delta_s \|_1 \leq \| \Delta \|_1 \leq 4 \| \Delta_s \|_1 + 4 \sqrt{\frac{s_j}{s_{j,g}}} \| \Delta_{s_j} \|_{1,2}. \]

Therefore, conditional on event \( A \) in (S21), we have that
\[ \| \Delta_s \|_1 + \sqrt{\frac{s_j}{s_{j,g}}} \| \Delta_{s_j} \|_{1,2} \leq 4 \| \Delta_s \|_1 + 4 \sqrt{\frac{s_j}{s_{j,g}}} \| \Delta_{s_j} \|_{1,2}, \]
which further implies $\|\Delta S_j\|_1 \leq 4\eta_j \|\Delta S_j\|_1$, as $\|\Delta S_j\|_{1,2} \leq \|\Delta S_j\|_1$.

**Step 2:** The step bounds the diagonal and off-diagonal elements of $\Psi$. We first bound the diagonal elements, i.e., $\Psi_{ll} = \|w_l\|_2^2/n$, with $w_l^{(i)} \sim N(0, \sigma_{il}^2)$. Under Assumptions 1 and 2, $\sigma_{il}^2 \leq M^2_1 \phi_2$ and $\Sigma_W(l, l) = \sum_{i=1}^{n} \sigma_{il}^2 / n \geq 1/\phi_1$. Using the concentration inequality for sub-exponential random variables in Lemma 4, we have

$$P(|\Psi_{ll} - \Sigma_W(l, l)| > \Sigma_W(l, l)/2) \leq 2 \exp(-c_4 n),$$

for some positive constant $c_4$. Immediately,

$$P(\Psi_{ll} \notin [1/(2\phi_1), 2M_1^2\phi_2]) \leq 2 \exp(-c_4 n) \tag{S22}$$

because

$$P(\Psi_{ll} \notin [1/(2\phi_1), 2M_1^2\phi_2]) \leq P(|\Psi_{ll} - \Sigma_W(l, l)| > \Sigma_W(l, l)/2).$$

Similarly, for the off diagonal elements, i.e., $\Psi_{kl} = w_k^\top w_l / n$, by noting that $\|\Sigma(u^{(i)})\|_{\infty} \leq \phi_2$, we have

$$P\left\{\Psi_{kl} \notin \left[ -\frac{1}{c_0(1 + 8\eta_j)s_j}, \frac{3}{c_0(1 + 8\eta_j)s_j} \right]\right\}$$

$$\leq P(|\Psi_{kl} - \Sigma_W(k, l)| \geq 2\Sigma_W(k, l)) \leq 2 \exp(-c_5 n), \tag{S23}$$

for a positive constant $c_5$.

**Step 3:** This step establishes that, conditional on event $\mathcal{A}$ and that

$$\Psi_{ll} \in [1/(2\phi_1), 2M_1^2\phi_2], \quad \Psi_{kl} \in \left[ -\frac{1}{c_0(1 + 8\eta_j)s_j}, \frac{3}{c_0(1 + 8\eta_j)s_j} \right], \tag{S24}$$

we have

$$\min_{\|v_{S_j}\|_1 \leq 3\eta_j \|v_{S_j}\|_2} \frac{\|Wv\|_2}{\sqrt{n\|v_{S_j}\|_2}} \geq \sqrt{\frac{1}{2\phi_1} - \frac{1}{c_0}} > 0.$$

First, we have that

$$\frac{\|Wv_{S_j}\|_2^2}{n\|v_{S_j}\|_2^2} = \frac{v_{S_j}^\top \text{diag}(\Psi)v_{S_j}}{\|v_{S_j}\|_2^2} + \frac{v_{S_j}^\top (\Psi - \text{diag}(\Psi))v_{S_j}}{\|v_{S_j}\|_2^2} \geq \frac{1}{2\phi_1} - \frac{1}{c_0(1 + 8\eta_j)s_j} \|v_{S_j}\|_1^2 \|v_{S_j}\|_2^2,$$

where $\text{diag}(\Psi)$ is a diagonal matrix with the diagonal elements being identical to those of.
\( \Psi \), and the last inequality follows from (S24). Furthermore,

\[
\frac{\|Wv\|^2}{n\|v_s\|^2} \geq \frac{\|Wv_s\|^2}{n\|v_s\|^2} + 2\frac{v_s^\top \Psi v_s}{n\|v_s\|^2} \\
\geq \frac{1}{2\phi_1} - \frac{1}{c_0(1 + 8\eta_j)s_j} \frac{\|v_s\|^2}{2} - \frac{2}{c_0(1 + 8\eta_j)s_j} \frac{\|v_s\|_1}{\|v_s\|^2} \\
\geq \frac{1}{2\phi_1} - \frac{1 + 8\eta_j}{c_0(1 + 8\eta_j)s_j} \frac{\|v_s\|^2}{2} \geq \frac{1}{2\phi_1} - \frac{1}{c_0} > 0,
\]

where we have used the results that \( \|\Delta s_j\|_1 \leq 4\eta_j\|\Delta s_j\|_1 \) on event \( \mathcal{A} \) and the fact that \( \|v_s\|_1 \leq \sqrt{s_j}\|v_s\|_2 \). Thus, we have finished Step 3.

Lastly, based on the results from Steps 1-3, we find the \( \ell_\infty \) bound of the error of \( \hat{\beta}_j \). For \( l \in [(p - 1)(q + 1)] \), it is true that

\[
\left( \Psi(\hat{\beta}_j - \beta_j) \right)_l = \Psi_{ll}(\hat{\beta}_j - \beta_j)_l + \sum_{k \neq l} \Psi_{kl}(\hat{\beta}_j - \beta_j)_k.
\]

Given (S24), we have

\[
\left| \left( \Psi(\hat{\beta}_j - \beta_j) \right)_l - \Psi_{ll}(\hat{\beta}_j - \beta_j)_l \right| \leq \frac{3}{c_0(1 + 8\eta_j)s_j} \sum_{k \neq l} |(\hat{\beta}_j - \beta_j)_k|,
\]

and also

\[
\|\hat{\beta}_j - \beta_j\|_\infty \leq 2\phi_1 \|\Psi \Delta\|_\infty + \frac{6\phi_1}{c_0(1 + 8\eta_j)s_j} \|\Delta\|_1. \quad (S25)
\]

With \( \Delta = \hat{\beta}_j - \beta_j \), and conditioning on \( \|\Psi \Delta\|_\infty \leq \frac{3\eta_j\lambda}{2} \) and \( \|\Delta s_j\|_1 \leq 4\eta_j\|\Delta s_j\|_1 \) from Step 1, we have that

\[
\frac{\|W\Delta\|^2}{n} \leq \|\Psi \Delta\|_\infty \|\Delta\|_1 \leq \frac{3\eta_j\lambda}{2} (1 + 4\eta_j) \sqrt{s_j} \|\Delta s_j\|_2, \quad (S26)
\]

while Step 3 also gives that \( \|W\Delta\|^2/n \geq \{1/(2\phi_1) - 1/c_0\} \|\Delta s_j\|_2^2 \). Combining the above two inequalities yields that \( \{1/(2\phi_1) - 1/c_0\} \|\Delta s_j\|_2^2 \leq \frac{3\eta_j\lambda}{2} (1 + 4\eta_j) \sqrt{s_j} \|\Delta s_j\|_2^2 \), and, therefore,

\[
\|\Delta s_j\|_2 \leq 3\eta_j\lambda(1 + 4\eta_j) \frac{c_0\phi_1}{c_0 - 2\phi_1} \sqrt{s_j}.
\]

With \( \|\Delta s_j\|_1 \leq 4\eta_j\|\Delta s_j\|_1 \), it follows that \( \|\Delta\|_1 \leq (1 + 4\eta_j)\|\Delta s_j\|_1 \leq (1 + 4\eta_j)\sqrt{s_j} \|\Delta s_j\|_2 \), and, therefore,

\[
\|\Delta\|_1 \leq 3\eta_j\lambda(1 + 4\eta_j)^2 \frac{c_0\phi_1}{c_0 - 2\phi_1} s_j.
\]

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Plugging this into (S25), we get the desired result that with probability at least 1 − \(C_1' \exp\{-C_2' \log p\}\),

\[
\|\hat{\beta}_j - \beta_j\|_\infty \leq \left(3\phi_1 \eta_j + \frac{18\phi_1^2 (1 + 4\eta_j)^2 \eta_j}{(c_0 - 2\phi_1)(1 + 8\eta_j)}\right) \lambda,
\]

for some positive constants \(C_1', C_2'\).

**S2.3 Proof of Theorem 3**

We start the proof by noting that the two events in (20) hold with the specified probability by applying Lemma 5, which is applicable because

\[
\sup_{i \in [n]} \text{Var}(z^{(i)}_j) = O(1), \quad \frac{1}{n} \sum_{i=1}^n \text{Var}(z^{(i)}_j) = O(1),
\]

due to \(\{\Sigma(u^{(i)})\}_{jj} \leq \phi_2, j \in [p]\) with \(\lambda_{\text{min}}(\Omega(u^{(i)})) \geq 1/\phi_2 > 0\) in Assumption 2.

In what follows, we show (22) and (23), conditional on that the two events, as specified in (20), hold.

When \(\Gamma\) is unknown, compared to the oracle regression equation \(z_j = W\beta_j + \epsilon_j\), we only have access to the noisy equation

\[
\hat{z}_j = \hat{W}\beta_j + E_j,
\]

where \(E_j = \epsilon_j + (\hat{z}_j - z_j) + (W - \hat{W})\beta_j\). As \(\hat{\beta}_j\) is a minimizer of the convex objective function (19), Lemma 1 implies

\[
\frac{1}{2n} \|\hat{z}_j - \hat{W}\beta_j\|_2^2 + \lambda \|\hat{\beta}_j\|_1 + \lambda_0 \|\hat{\beta}_{j,-0}\|_{1,2} + \frac{1}{2n} \|\hat{W}(\hat{\beta}_j - \beta_j)\|_2^2 \leq \frac{1}{2n} \|\hat{z}_j - \hat{W}\beta_j\|_2^2 + \lambda \|\beta_j\|_1 + \lambda_0 \|\beta_{j,-0}\|_{1,2}.
\]

With \(\Delta = \hat{\beta}_j - \beta_j\), reorganizing terms in the above inequality leads to

\[
\frac{1}{n} \|\hat{W}\Delta\|_2^2 + \lambda \|\hat{\beta}_j\|_1 + \lambda_0 \|\hat{\beta}_{j,-0}\|_{1,2} \leq \frac{1}{n} \langle E_j, \hat{W}\Delta \rangle + \lambda \|\beta_j\|_1 + \lambda_0 \|\beta_{j,-0}\|_{1,2}.
\]

Next, with \(\Delta_{E_j} = (\hat{z}_j - z_j) + (W - \hat{W})\beta_j\), we have that

\[
\frac{1}{n} \langle E_j, \hat{W}\Delta \rangle = \frac{1}{n} \langle \epsilon_j, \hat{W}\Delta \rangle + \frac{1}{n} \langle \Delta_{E_j}, \hat{W}\Delta \rangle \leq \frac{1}{n} \langle \epsilon_j, \hat{W}\Delta \rangle + \frac{1}{2n} \|\Delta_{E_j}\|_2^2 + \frac{1}{2n} \|\hat{W}\Delta\|_2^2.
\]
Using similar arguments as in (S1), we obtain

\[
\frac{1}{2n} \|\hat{\mathbf{W}} \Delta\|_2^2 + \lambda \|\Delta_{S_j}\|_1 + \lambda_g \|\Delta_{(g_j)}\|_{1,2} \leq \frac{1}{n} \langle \mathbf{e}_j, \hat{\mathbf{W}} \Delta \rangle + \frac{1}{2n} \|\Delta_{E_j}\|_2^2 + \lambda \|\Delta_{S_j}\|_1 + \lambda_g \|\Delta_{(g_j)}\|_{1,2}.
\] (S27)

Consider the two stochastic terms \(\langle \mathbf{e}_j, \hat{\mathbf{W}} \Delta \rangle\) and \(\|\Delta_{E_j}\|_2^2\). For the latter, recall that our proof is conditional on the events of (20). Then, with probability \(1 - \exp(\log p + \log q - \tau_1 \log q)\),

\[
\frac{1}{n} \|\Delta_{E_j}\|_2^2 \leq \frac{1}{n} \|\mathbf{z}_j - \hat{\mathbf{z}}_j\|_2^2 + \frac{\max_j \|\mathbf{W}_j - \hat{\mathbf{W}}_j\|_2^2}{n} \cdot \|\mathbf{\beta}_j\|_1^2 \preceq \sigma_{\epsilon_j}^2 \frac{t \log q}{n},
\] (S28)

where the last inequality is true due to Assumption 2 and (20).

Next, we bound \(\langle \mathbf{e}_j, \hat{\mathbf{W}} \Delta \rangle\). Defining \(\hat{\mathbf{S}}_j = \{l : (\hat{\mathbf{\beta}}_j)_l \neq 0, l \in [(p - 1)(q + 1)]\}\) and letting \(\hat{\mathbf{S}}_j = \mathbf{S}_j \cup \hat{\mathbf{S}}_j\), we have that

\[
\langle \mathbf{e}_j, \hat{\mathbf{W}} \Delta \rangle = \langle \mathbf{e}_j, \hat{\mathbf{P}}_{\hat{\mathbf{S}}_j} \hat{\mathbf{W}}_{\hat{\mathbf{S}}_j} \Delta_{\hat{\mathbf{S}}_j} \rangle = \langle \hat{\mathbf{P}}_{\hat{\mathbf{S}}_j} \mathbf{e}_j, \hat{\mathbf{W}}_j \Delta_{\hat{\mathbf{S}}_j} \rangle \leq \frac{1}{2a_1} \|\hat{\mathbf{W}} \Delta\|_2^2 + \frac{a_1}{2} \|\hat{\mathbf{P}}_{\hat{\mathbf{S}}_j} \mathbf{e}_j\|_2^2,
\] (S29)

where \(\hat{\mathbf{P}}_{\hat{\mathbf{S}}_j}\) is the orthogonal projection matrix onto the column space of \(\hat{\mathbf{W}}_{\hat{\mathbf{S}}_j}\). Using the same argument as in (S3), we have

\[
\|\hat{\mathbf{P}}_{\hat{\mathbf{S}}_j} \mathbf{e}_j\|_2^2 < M \sigma_{\epsilon_j}^2 \{(s_j + \hat{s}_j) \log(ep) + (s_{g_j} + \hat{s}_{g_j}) \log(eq/s_{g_j})\} + \hat{\tau},
\] (S30)

where

\[
\hat{\tau} = \sup_{1 \leq s \leq [(p - 1)(q+1)]} \left( \sup_{0 \leq s_g \leq q} \left( \|\hat{\mathbf{P}}_{\hat{\mathbf{S}}_j} \mathbf{e}_j\|_2^2 \left\{ s \log(ep) + s_g \log(eq/s_g) \right\} - M \sigma_{\epsilon_j}^2 \{ s \log(ep) + s_g \log(eq/s_g) \} \right) \right).
\]

Setting \(M = 9\), by Step 1 in Section S2.1, we have

\[
\mathbb{P}\{\hat{\tau} \geq t \sigma_{\epsilon_j}^2\} < \sum_{s=1}^{(p-1)(q+1)} \sum_{s_g=0}^{q} c_1 \exp(-c_2 t) \exp[-3c_2 \{ s \log(ep) + s_g \log(eq/s_g) \}].
\]

We move to bound \(\|\hat{\mathbf{P}}_{\hat{\mathbf{S}}_j} \mathbf{e}_j\|_2^2\) by using the computational optimality of \(\hat{\mathbf{\beta}}_j\). As in (S4) and (S5), it follows that

\[
\lambda^2 \hat{s}_j + \lambda_g^2 \hat{s}_{g,j} \leq \frac{1}{n^2} \|\hat{\mathbf{W}}_{\hat{\mathbf{S}}_j}(\hat{\mathbf{z}}_j - \hat{\mathbf{W}} \hat{\mathbf{\beta}}_j)\|_2^2.
\] (S31)
We also have that $\|\hat{W}_{S_j} - W_{S_j}\|/n \leq \hat{s}_j t \log q/n$ conditional on the events of (20). The bound on the spectral norm of $\hat{W}_{S_j} - W_{S_j}$ may not be further improved since we only have the column-wise $\ell_2$ bound of $W_{S_j} - W_{\hat{S}_j}$, i.e., $\sqrt{t \log q/n}$ from (20). Together with the result in Lemma 7, we have that $\|\hat{W}_{S_j}\| \leq M_3 \log q/n$ for some $M_3 > 0$. It then follows from the Cauchy-Schwarz inequality that

$$\lambda^2 \hat{s}_j + \lambda_g^2 \hat{s}_{j,g} \leq \frac{3M_3 \log q/n}{n} \|\hat{W}\Delta\|_2^2$$

(S32)

$$+ \frac{3M_3 \log q/n}{n} \|\hat{P}_{S_j} \epsilon_j\|_2^2 + \frac{3M_3 \log q/n}{n} \|\Delta_{E_j}\|_2^2.$$ 

Set $\lambda = C\sigma_{e_j} \sqrt{(t \log q/n)\{\log(ep)/n + s_{j,g} \log(eq/s_{j,g})/(ns_j)\}}$ and $\lambda_g = \sqrt{s_j/s_{j,g}} \lambda$, where $C = 3(a_2 M_3)^{1/2}$ for some $a_2 > 0$. Combining (S30) and (S32), we have that

$$(1 - \frac{3}{a_2} ) \|\hat{P}_{S_j} \epsilon_j\|_2^2 \leq 9\sigma_{e_j}^2 \{s_j \log(ep) + s_{j,g} \log(eq/s_{j,g})\}$$

(S33)

$\quad + \frac{3}{a_2} \|\hat{W}\Delta\|_2^2 + \frac{3}{a_2} \|\Delta_{E_j}\|_2^2 + \hat{r}.$

Plugging the above inequality and (S29) into (S27), we have

$$\frac{\|\hat{W}\Delta\|_2^2}{2n} + \lambda \|\Delta_{S_j}\|_1 + \lambda_g \|\Delta_{(g_j)}\|_{1,2}$$

(S34)

$$\leq \frac{1}{2a_1} \|\hat{W}\Delta\|_2^2 + \frac{9a_1 a_2}{2(a_2 - 3)} \sigma_{e_j}^2 \{s_j \log(ep) + s_{j,g} \log(eq/s_{j,g})\}$$

$$+ \frac{3a_1}{2(a_2 - 3)} \|\hat{W}\Delta\|_2^2 + \frac{a_1 a_2}{2(a_2 - 3)n} \hat{r} + \frac{1}{2n} \|\Delta_{E_j}\|_2^2$$

$$+ \frac{a_1 a_2}{2(a_2 - 3)n} \|\Delta_{E_j}\|_2^2 + \lambda \|\Delta_{S_j}\|_1 + \lambda_g \|\Delta_{(g_j)}\|_{1,2}.$$ 

As in Section S2.1, we have that

$$\frac{\|\Delta_{S_j}\|_1}{\sqrt{s_j}} + \frac{\|\Delta_{(g_j)}\|_{1,2}}{\sqrt{s_{j,g}}} \leq \|\Delta_{S_j}\|_2 + \|\Delta_{(g_j)}\|_2 \leq 2\phi_1 \|\Sigma_{1/2}\Delta\|_2.$$ 

Consequently,

$$\lambda \|\Delta_{S_j}\|_1 + \lambda_g \|\Delta_{(g_j)}\|_{1,2} \leq 2C\phi_1 \sqrt{e_j'} \|\Sigma_{1/2}\Delta\|_2.$$ (S35)
where $e_j = \sigma_j^2(t \log q)\{s_j \log(ep) + s_{j,g} \log(eq/s_{j,g})\}/n$. Combining (S34) and (S35), we have

$$
\left\{ \frac{1}{2} - \frac{1}{2a_1} - \frac{3a_1}{2(a_2 - 3)} \right\} \left\| \dot{W} \Delta \right\|^2_n 
\leq \left\{ \frac{9a_1a_2}{2(a_2 - 3)} + C a_3 \phi_1(t \log q) \right\} \frac{\sigma_j^2\{s_j \log(ep) + s_{j,g} \log(eq/s_{j,g})\}}{n} 
\quad + \frac{1}{a_3} \left\| \Sigma^{1/2}_W \Delta \right\|^2 n \quad + \frac{a_1a_2}{2(a_2 - 2)n} \hat{r} 
\quad + C_n \left\{ \frac{1}{2n} + \frac{a_1a_2}{2(a_2 - 3)n} \right\} t \log q.
$$

Therefore, by choosing proper constants $a_1$, $a_2$ and $a_3$ (e.g., $a_1 = 4$, $a_2 = 51$, $a_3 = 4$), we have, with probability at least $1 - c_1\exp[-c_2\{s_j \log(ep) + s_{j,g} \log(eq/s_{j,g})\}]$,

$$
\frac{\left\| \dot{W} \Delta \right\|^2_n - \left\| \Sigma^{1/2}_W \Delta \right\|^2 n}{2n} \lesssim \frac{\sigma_j^2(t \log q)\{s_j \log(ep) + s_{j,g} \log(eq/s_{j,g})\}}{n},
$$

where we have used the fact that

$$
P \left[ \hat{r} \geq M_0 \sigma_j^2\{s_j \log(ep) + s_{j,g} \log(eq/s_{j,g})\} \right] 
\leq c_1 \exp[-c_2\{s_j \log(ep) + s_{j,g} \log(eq/s_{j,g})\}],
$$

for a large constant $M_0$.

We then bound the difference between $\left\| \dot{W} \Delta \right\|^2_n/n$ and $\left\| \Sigma^{1/2}_W \Delta \right\|^2 n$. To do this, we first show that, with probability $1 - c_0 \exp[c_7\{\log p - (\tau_1 - 1)\log q\}]$,

$$
\sup_{v \in \mathbb{K}_0(2C_{\beta},s_j)} \left\| v^\top \left( \frac{\dot{W} \dot{W}}{n} - \Sigma_W \right) v \right\| \leq 1/L,
$$

where $L$ is a large constant and $\mathbb{K}_0(2C_{\beta},s_j) = \{ v : \|v\|_0 \leq 2C_{\beta} s_j \text{ and } \|v\|_2 = 1 \}$ for some positive constant $C_{\beta}$. Notice that

$$
\left\| v^\top \left( \frac{\dot{W} \dot{W}}{n} - \frac{W^\top W}{n} \right) v \right\| \leq 2\left| v^\top \dot{W}^\top(\dot{W} - W)v \right| + \frac{\| (\dot{W} - W)v \|_2^2}{n},
$$

where we have used that $\|v\|_1 \leq \sqrt{2C_{\beta} s_j}\|v\|_2$. For the second term on the right hand-side, we have with probability $1 - 3\exp\{\log p - (\tau_1 - 1)\log q\}$,

$$
\frac{\| (\dot{W} - W)v \|_2}{n} \leq \max_j \| W_{-j} - \dot{W}_{-j} \|_2 \cdot \|v\|_1 \lesssim \sqrt{\frac{2C_{\beta} s_j \log q}{n}} = o(1),
$$

and, moreover,

$$
\frac{|v^\top \dot{W}^\top(\dot{W} - W)v|}{n} \leq \frac{\| (\dot{W} - W)v \|_2 \cdot \|Wv\|_2}{n}.
$$

(S38)
Next, we have \( W_i^T v = \sum_{i \neq j} z_i^{(i)} v_i^T u^{(i)} \), where \( v = (v_1, \ldots, v_p) \), \( v_i \in \mathbb{R}^{q+1} \). Since \( z_i^{(i)} \) is from \( \mathcal{N}(0, \Sigma(u^{(i)})) \), we have that \( W_i^T v \) follows \( \mathcal{N}(0, b_i^T \Sigma(u^{(i)}) b_i) \), where \( b_i = (v_1^T u^{(i)}, \ldots, v_p^T u^{(i)}) \). It follows from Assumption 1 and the bounded eigenvalue condition in Assumption 2 that

\[
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\{(W_i^T v)^4\} \leq 3 \left( \frac{1}{n} \sum_{i=1}^{n} b_i^T \Sigma(u^{(i)}) b_i \right)^2 = \mathcal{O}(1).
\]

As \( W_i \) is coordinate-wise sub-Gaussian with the bounded sub-Gaussian norm, \( i \in [n], l \in [(p-1)(q+1)] \) (diagonal elements of \( \Sigma_i \) are bounded due to Assumption 2), it follows from Lemma 6 that with probability \( 1 - \mathcal{O}\{(pq)^{-1}\} \)

\[
\left| v^T \left( \frac{W^T W}{n} - \Sigma_W \right) v \right| = o(1).
\]

Since \( \|\Sigma_W\| \) is bounded by \( \phi_2 \) in Assumption 2, we have that \( \|Wv\|_2 = \mathcal{O}(1) \) with probability \( 1 - \mathcal{O}\{(pq)^{-1}\} \). Putting this together with (S38), we have shown (S37).

Next, conditioning on (S37) and using the result in Lemma 8, we have

\[
\left| \Delta^T \left( \frac{W^T W}{n} - \Sigma_W \right) \Delta \right| \leq \frac{1}{L'} \left( \|\Delta\|^2_2 + \frac{1}{s_j} \|\Delta\|_1^2 \right)^{\frac{1}{2}}.
\]

Plugging this into (S36), we have

\[
\frac{\|W\Delta\|^2_2}{2n} \lesssim \frac{\sigma_j^2 (t \log q) \{s_j \log(ep) + s_{j,g} \log(eq/s_{j,g})\}}{n} + \frac{1}{L'} \left( \|\Delta\|^2_2 + \frac{1}{s_j} \|\Delta\|_1^2 \right) + \frac{\sigma_j^2}{n}.
\]

By choosing an appropriate \( a_1 \) given \( a_2 \) in (S34), we have with probability \( 1 - c_1 \exp\{-c'_2 \{s_j \log(ep) + s_{j,g} \log(eq/s_{j,g})\}\} \),

\[
\frac{\|\Delta_{S_j}\|_1}{\sqrt{s_j}} + \frac{\|\Delta_{(G_j)}\|_{1,2}}{\sqrt{s_{j,g}^2}} \leq \sqrt{e'_j} + \frac{\|\Delta_{S_j}\|_1}{\sqrt{s_j}} + \frac{\|\Delta_{(G_j)}\|_{1,2}}{\sqrt{s_{j,g}}}. \quad \text{(S41)}
\]

Adding \( \|\Delta_{S_j}\|_1/\sqrt{s_j} \) to both sides of (S41), we get

\[
\frac{\|\Delta\|_1}{\sqrt{s_j}} \leq \sqrt{e'_j} + 3\|\Delta\|_2. \quad \text{(S42)}
\]

Plugging (S42) into (S40) and by Assumption 5, we have

\[
\|\Delta\|_2 \lesssim \frac{\sigma_j^2}{n} (t \log q) \{s_j \log(ep) + s_{j,g} \log(eq/s_{j,g})\} + \frac{\sigma_j^2}{n},
\]

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with probability at least $1 - C_3 \exp[C_4\{\log p - (\tau_1 - 1)\log q\}]$, for some positive constants $C_3$, $C_4$.

It remains to show (23). Notice that the mutual coherence condition implies that $\hat{S} \subset S$ when $\lambda = C_\sigma \sqrt{\log(ep)/n + s_{j,g} \log(eq/s_{j,g})/(ns_j)}$ and $\lambda_g = \sqrt{s_{j,g}/s_{j}} \lambda$. This can be shown using the same arguments as in Lemma 6.1-6.2 of Van De Geer and Bühlmann (2009). When $\hat{S} \subset S$, we have $\hat{s}_j \leq s_j$ and (S31) can be written as

$$\lambda^2 \hat{s}_j + \lambda_g^2 \hat{s}_{j,g} \leq \frac{3M_3}{n} \| \hat{W} \Delta \|_2^2 + \frac{3M_3}{n} \| \hat{P}_{\hat{S}} \epsilon_j \|_2^2 + \frac{3M_3}{n} \| \Delta E_j \|_2^2.$$ 

Correspondingly, (S36) is improved to

$$\| \hat{W} \Delta \|_2^2 - \frac{\| \Sigma \hat{W}^T \Delta \|_2^2}{2n} \gtrsim \frac{\sigma_j^2 \{s_j \log(ep) + s_{j,g} \log(eq/s_{j,g})\}}{n}. \quad (S43)$$

The rest of the proof follows similarly to the proof of (22), and we arrive at the desired convergence rate of $\frac{\sigma_j^2}{n} \{s_j \log(ep) + s_{j,g} \log(eq/s_{j,g})\} + \frac{\sigma_j^2}{n}$.

**S2.4 Proof of Theorem 4**

We first establish the $\ell_\infty$ norm bound of $\hat{\beta}_j$ in (24) with three steps.

**Step 1:** In this step, we show that, with probability $1 - c_8 \exp[c_9\{\log p - (\tau_1 - 1)\log q\}]$ for some $c_8, c_9 > 0$,

$$\| \hat{\Psi} \Delta \|_\infty \leq \frac{3\eta_j \lambda}{2}. \quad (S44)$$

And it also holds with probability $1 - c'_8 \exp[c'_9\{\log p - (\tau_1 - 1)\log q\}]$ that, for some $c'_8, c'_9 > 0$,

$$\| \Delta_{S_j} \|_1 \leq 4\eta_j \| \Delta_{S_j} \|_1. \quad (S45)$$

Under the KKT condition, if $\theta$ is an optimum of (19), then

$$\left\{ \begin{array}{ll}
(\hat{W}^T (\hat{z}_j - \hat{W} \theta)/n)_l = \text{sign}(\theta_l) \lambda & \text{if } \theta_l \neq 0, l \in (0) \\
(\hat{W}^T (\hat{z}_j - \hat{W} \theta)/n)_l = \text{sign}(\theta_l) \lambda + \lambda_g \theta_l/\| \theta \|_2 & \text{if } \theta_l \neq 0, l \in (h) \\
(\hat{W}^T (\hat{z}_j - \hat{W} \theta)/n)_l < \eta_j \lambda & \text{if } \theta_l = 0.
\end{array} \right.$$ 

As such, any solution $\hat{\beta}_j$ satisfies that

$$\| \frac{1}{n} \hat{W}^T (\hat{z}_j - \hat{W} \hat{\beta}_j) \|_\infty \leq \eta_j \lambda.$$
If we can show that with high probability
\[ \frac{1}{n} \left\| \hat{W}^\top E_j \right\|_\infty \leq \frac{\eta_j \lambda}{2}, \quad (S46) \]
then we can reach the desired conclusion that with high probability
\[ \left\| \hat{\Psi} \Delta \right\|_\infty \leq \frac{3\eta_j \lambda}{2}. \]

Now we consider the inequality in (S46), and note that
\[ \frac{1}{n} \left\| \hat{W}^\top E_j \right\|_\infty \leq \frac{1}{n} \left\| W^\top \epsilon_j \right\|_\infty + \frac{1}{n} \left\| (\hat{W} - W)^\top \epsilon_j \right\|_\infty + \frac{1}{n} \left\| \hat{W}^\top \Delta E_j \right\|_\infty. \]

Consider term (I). Define \( V_l = w_l^\top \epsilon_j / n, j \in [(p - 1)(q + 1)] \). Using a similar argument as in (S18), we have
\[ \mathbb{P} \left( |V_l| > \frac{\eta_j \lambda}{2} \right) \leq 2 \exp \left( - \frac{c\eta_j^2 \lambda^2 n}{4M_1^2 \phi_2 \sigma_{\epsilon_j}^2} \right) \leq 2 \exp \{-C'_0(\log p + \log q)\}, \]
where \( C'_0 = cC/(4M_1^2 \phi_2) > 1 \). Using the union bound inequality, we have
\[ \mathbb{P} \left( \frac{1}{n} \left\| W^\top \epsilon_j \right\|_\infty \geq \frac{\eta_j \lambda}{2} \right) \leq \mathbb{P} \left( \max_l |V_l| \geq \frac{\eta_j \lambda}{2} \right) \leq 2 \exp\{- (C'_0 - 1)(\log p + \log q)\}. \]

For term (II), it is true that
\[ \frac{1}{n} \left\| (\hat{W} - W)^\top \epsilon_j \right\|_\infty = \frac{1}{n} \max_l |\langle \hat{w}_l - w_l, \epsilon_j \rangle| \leq \frac{\max_l \left\| \hat{w}_l - w_l \right\|_2 \left\| \epsilon_j \right\|_2}{\sqrt{n}}. \]

Using Lemma 4, we have that for any large constant \( M_4 > 0 \),
\[ \mathbb{P} \left( \left\| \epsilon_j \right\|_2^2 > M_4^{1/2} \sigma_{\epsilon_j}^2 \right) \leq \mathbb{P} \left( \left\| \epsilon_j \right\|_2^2 - \sigma_{\epsilon_j}^2 \geq (M_4 - 1)\sigma_{\epsilon_j}^2 \right) \leq 2 \exp(-b_0 \log q). \]

We further have \( \mathbb{P}(\| \hat{w}_l - w_l \|_2 / \sqrt{n}) \lesssim t^{1/2} \lambda_1 \) with probability \( 1 - 3 \exp(-\tau_1 \log q) \) and \( t^{1/2} \lambda_1 = o(\eta_j \lambda) \), which is true as \( t = o(\sqrt{n}/\log q) \). Applying the union bound, we have
\[ \mathbb{P} \left( \frac{1}{n} \left\| (\hat{W} - W)^\top \epsilon_j \right\|_\infty \geq \frac{\eta_j \lambda}{2} \right) \leq b'_0 \exp\{\log p + \log q - \tau_1 \log q\}. \quad (S47) \]
Moving to term (III), notice that there exists a large constant $M'_4 > 0$ such that

$$\frac{1}{n} \| \hat{W}^\top \Delta E_j \|_\infty = \| (W + \hat{W} - W)^\top \Delta E_j / n \|_\infty \leq \frac{1}{n} \| W^\top \Delta E_j \|_\infty + \frac{\max_l \| w_l - w_l \|_2}{\sqrt{n}}, \frac{\| \Delta E_j \|_2}{\sqrt{n}} \leq \frac{1}{n} \| W^\top \Delta E_j \|_\infty + M'_4 t\lambda^2,$$

with probability $b''_0 \exp\{ \log p + \log q - \tau_1 \log q \}$, where the last inequality follows from (20) and (S28). Since $t^{1/2} \lambda = o(\eta_j \lambda)$, it suffices to bound the term of $\| W^\top \Delta E_j \| / n$. To this end, we have

$$\frac{1}{n} \| W^\top \Delta E_j \|_\infty < \max_l \| w_l \|_2 \sqrt{n} \cdot \| \Delta E_j \|_2 \sqrt{n}.$$

As $w_l$ has independent Gaussian entries with a bounded sub-Gaussian norm, using a similar argument as for term (II) we have

$$\mathbb{P} \left( \frac{1}{n} \| W^\top \Delta E_j \|_\infty \geq \frac{\eta_j \lambda}{2} \right) \leq b''_0 \exp\{ \log p + \log q - \tau_1 \log q \}. \quad (S48)$$

Combining terms (I)-(III), we have finished showing (S44) of Step 1.

Next, we prove that $\| \Delta \tilde{S} \|_1 \leq 4 \eta_j \| \Delta S \|_1$. By the definition of $\hat{\beta}_j$, we have

$$\frac{1}{2n} \| \hat{z}_j - \hat{W} \hat{\beta}_j \|_2^2 + \lambda \| \hat{\beta}_j \|_1 + \lambda_g \| \hat{\beta}_{j,-0} \|_{1,2} \leq \frac{1}{2n} \| E_j \|_2^2 + \lambda \| \beta_j \|_1 + \lambda_g \| \beta_{j,-0} \|_{1,2}.$$

Developing the left hand side of the above inequality, we have

$$\lambda \| \hat{\beta}_j \|_1 + \lambda_g \| \hat{\beta}_{j,-0} \|_{1,2} \leq \lambda \| \beta_j \|_1 + \lambda_g \| \beta_{j,-0} \|_{1,2} + \frac{1}{n} \Delta^\top \hat{W}^\top E_j.$$

Define an event

$$\mathcal{A}_1 = \left\{ \frac{1}{n} \| W^\top \Delta E_j \|_\infty \leq \frac{\lambda}{2} \right\}. \quad (S49)$$

We again write

$$\frac{1}{n} \| \hat{W}^\top E_j \|_\infty \leq \frac{1}{n} \| W^\top \epsilon_j \|_\infty + \frac{1}{n} \| (\hat{W} - W)^\top \epsilon_j \|_\infty + \frac{1}{n} \| W^\top \Delta E_j \|_\infty.$$

Consider term (I). Define $V_l = w_l^\top \epsilon_j / n, j \in [(p - 1)(q + 1)]$. Using a similar argument as in (S20), we have

$$\mathbb{P} \left( \frac{1}{n} \| W^\top \epsilon_j \|_\infty \geq \frac{\lambda}{2} \right) \leq \mathbb{P} \left( \max_l | V_l | \geq \frac{\lambda}{2} \right) \leq \exp\{ - c'_3 \log p \}.$$
For term (II), using the same argument as in (S47) and by noting \( t^{1/2} \lambda_1 = o(\lambda) \), which is true as \( t = o(\sqrt{n/\log q}) \), we have
\[
P \left( \frac{1}{n} \| (\hat{W} - W)^\top \epsilon_j \|_\infty \geq \frac{\lambda}{2} \right) \leq b'_0 \exp \{ \log p + \log q - \tau_1 \log q \}.\]

For term (III), using the same argument as in (S48) and again noting \( t^{1/2} \lambda_1 = o(\lambda) \), we have
\[
P \left( \frac{1}{n} \| W^\top \Delta_{E_j} \|_\infty \geq \frac{\lambda}{2} \right) \leq b''_0 \exp \{ \log p + \log q - \tau_1 \log q \}.\]

Combining terms (I)-(III), we have \( P(\mathcal{A}_1^n) \leq c'_8 \exp \{ c'_9 \{ \log p - (\tau_1 - 1) \log q \} \} \). Conditioning on \( \mathcal{A}_1 \) in (S49), we have that
\[
2\| \hat{\beta}_j \|_1 + 2\sqrt{\frac{s_j}{\| \hat{\beta}_j,0 \|_{1,2}}} \leq 2\| \beta_j \|_1 + 2\sqrt{\frac{s_j}{\| \beta_j,0 \|_{1,2}}} + \| \Delta \|_1.
\]

The rest of the arguments is similar to Step 1 in the proof of Theorem 2, which leads us to the desired result of \( \| \Delta S_j^\top \|_1 \leq 4\eta_j \| \Delta S_j \|_1 \) given \( \mathcal{A}_1 \).

**Step 2:** In this step, we bound the diagonal and off-diagonal elements of \( \hat{\Psi} \). First, we consider the diagonal elements of \( \hat{\Psi} \), i.e. \( \hat{\Psi}_{ii} \)'s. Note that
\[
\frac{\| \hat{\omega}_i \|_2 - \| \hat{\omega}_i - \omega_i \|_2}{\sqrt{n}} \leq \frac{\| \hat{\omega}_i \|_2}{\sqrt{n}} \leq \frac{\| \hat{\omega}_i \|_2}{\sqrt{n}} + \frac{\| \hat{\omega}_i - \omega_i \|_2}{\sqrt{n}}.
\]

Since \( \| \hat{\omega}_i - \omega_i \|_2 / \sqrt{n} = O(t^{1/2} \lambda_1) = o(1) \) and \( P(\hat{\Psi}_{ii} \notin [1/(2\phi_1), 2M_1^2 \phi_2]) \leq 2 \exp(-c_4 n) \) from (S22), we have that \( 1/(3\phi_1) \leq \hat{\Psi}_{ii} \leq 3M_1^2 \phi_2 \) with probability \( 1 - 2 \exp(-c_4 n) \). Next, we consider the off-diagonal elements. It holds for \( k \neq l \) that
\[
\| \hat{\omega}_k^\top \omega_l - \hat{\omega}_k^\top \hat{\omega}_l \|_2 \leq \| \hat{\omega}_k^\top (\omega_l - \hat{\omega}_l) \|_2 + \| \hat{\omega}_l^\top (\hat{\omega}_k - \omega_k) \|_2.
\]

Since \( \| \hat{\omega}_l - \omega_l \|_2 / \sqrt{n} = O(t^{1/2} \lambda_1) = o(1) \) and (S23) shows
\[
P \left\{ \hat{\Psi}_{kl} \notin \left[ -\frac{1}{c_0(1 + 8\eta_j)s_j}, \frac{3}{c_0(1 + 8\eta_j)s_j} \right] \right\} \leq 2 \exp(-c_5 n),
\]
the following holds with probability \( 1 - \exp(-c_4^* n) \),
\[
\max_{k \neq l} \hat{\Psi}_{kl} \in \left[ -\frac{2}{c_0(1 + 8\eta_j)s_j}, \frac{4}{c_0(1 + 8\eta_j)s_j} \right].
\]
Step 3: We show that conditional on $A_1$ and that
\[
\max_l \hat{\Psi}_l \in [1/(3\phi_1), 3M_1^2\phi_2], \quad \max_{k\neq l} \hat{\Psi}_{kl} \in [-\frac{2}{c_0(1 + 8\eta_j)s_j}; \frac{4}{c_0(1 + 8\eta_j)s_j}], \tag{S50}
\]
\[
\text{it holds that } 
\min_{\|v_{S_j}\|_1 \leq 3\eta_j\|v_{S_j}\|_1} \frac{\|\hat{W}v\|_2}{\sqrt{n}\|v_{S_j}\|_2} \geq \sqrt{\frac{1}{3\phi_1} - \frac{2}{c_0}} > 0.
\]

First, given (S50), we have that
\[
\|\hat{W}v_{S_j}\|_2^2 \geq \frac{\|\hat{W}v_{S_j}\|_2^2}{n\|v_{S_j}\|_2^2} = \frac{\|v_{S_j}\|_2^2}{\|v_{S_j}\|_2^2} + \frac{\|v_{S_j}\|_2^2}{\|v_{S_j}\|_2^2} 
\geq \frac{1}{3\phi_1} - \frac{2}{c_0(1 + 8\eta_j)s_j}\|v_{S_j}\|_2^2.
\]

Furthermore, given (S50) and $A_1$, we have that
\[
\|\hat{W}v\|_2^2 \geq \frac{\|\hat{W}v_{S_j}\|_2^2}{n\|v_{S_j}\|_2^2} + \frac{2}{n\|v_{S_j}\|_2^2} \|v^\top_{S_j} \hat{\Psi} v_{S_j}\|_2^2 
\geq \frac{1}{3\phi_1} - \frac{2}{c_0(1 + 8\eta_j)s_j}\|v_{S_j}\|_2^2 - \frac{4}{c_0(1 + 8\eta_j)s_j}\|v_{S_j}\|_2^2 
\geq \frac{1}{3\phi_1} - \frac{2}{c_0(1 + 8\eta_j)s_j}\|v_{S_j}\|_2^2 > \frac{1}{3\phi_1} - \frac{2}{c_0} > 0,
\]

where we have used the results that $\|\Delta_{S_j}\|_1 \leq 4\eta_j\|\Delta_{S_j}\|_1$ and the fact that $\|v_{S_j}\|_1 \leq \sqrt{s_j}\|v_{S_j}\|_2$.

Lastly, with results from Steps 1-3, we find the $\ell_\infty$ bound of $\beta_j$. For $l \in [(p - 1)(q + 1)]$, it is true that
\[
\left(\hat{\Psi}(\hat{\beta}_j - \beta_j)\right)_l = \hat{\Psi}_l(\hat{\beta}_j - \beta_j)_l + \sum_{k\neq l} \hat{\Psi}_{kl}(\hat{\beta}_j - \beta_j)_k
\]
Given (S50) from Step 2, we have
\[
\left|\left(\hat{\Psi}(\hat{\beta}_j - \beta_j)\right)_l - \hat{\Psi}_l(\hat{\beta}_j - \beta_j)_l\right| \leq \frac{4}{c_0(1 + 8\eta_j)s_j} \sum_{k\neq l} |(\hat{\beta}_j - \beta_j)_k|,
\]
and also
\[
\|\hat{\beta}_j - \beta_j\|_\infty \leq 3\phi_1 \|\hat{\Psi}\Delta\|_\infty + \frac{12\phi_1}{c_0(1 + 8\eta_j)s_j}\|\Delta\|_1. \tag{S51}
\]

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With $\Delta = \hat{\beta}_j - \beta_j$ and given $\|\hat{\Psi} \Delta\|_\infty \leq \frac{3\eta_j \lambda}{2}$ and $\|\Delta S_j\|_1 \leq 4\eta_j \|\Delta S_j\|_1$ from Step 1, we have

$$\frac{\|W \Delta\|^2}{n} \leq \|\hat{\Psi} \Delta\|_\infty \|\Delta\|_1 \leq \frac{3\eta_j \lambda}{2} (4\eta_j + 1)\sqrt{s_j} \|\Delta S_j\|_1.$$  

We also have from Step 3 that $\|W \Delta\|^2/n \geq \{1/(3\phi_1) - 2/c_0\} \|\Delta S_j\|^2_2$ given (S50) and $A_1$. Combining these two inequalities and by noting $\|\Delta\|_1 \leq (1 + 4\eta_j)\sqrt{s_j} \|\Delta S_j\|_2$, we have that

$$\|\Delta\|_1 \leq \frac{3\eta_j \lambda}{2} (1 + 4\eta_j) \frac{3c_0 \phi_1}{c_0 - 6\phi_1} s_j.$$  

Plugging this into (S51), we obtain that

$$\|\hat{\beta}_j - \beta_j\|_\infty \leq \frac{9}{2} \left\{ \phi_1 \eta_j + \frac{12\phi_1^2 (1 + 4\eta_j)^2}{(c_0 - 6\phi_1)(1 + 8\eta_j)} \right\} \lambda.$$  

with probability $1 - C_5 \exp\{C_6\{\log p - (\tau_1 - 1)\log q\}\}$, for some positive constants $C_5$ and $C_6$.

## S3 Additional results of data analysis

### Additional references

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Figure S1: The KEGG human glioma pathway. This figure is downloaded from https://www.genome.jp/kegg/ (Kanehisa and Goto, 2000).
Figure S2: Graphs depending on each covariate (i.e., different SNPs). Edges that have positive (negative) effects on partial correlations are shown in red dashed (black solid) lines.