Indices of O-regular variation for weight functions and weight sequences

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Received: 28 March 2019 / Accepted: 15 July 2019 / Published online: 27 July 2019
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Abstract
A plethora of spaces in Functional Analysis (Braun–Meise–Taylor and Carleman ultradifferentiable and ultraholomorphic classes; Orlicz, Besov, Lipschitz, Lebesgue spaces, to cite the main ones) are defined by means of a weighted structure, obtained from a weight function or sequence subject to standard conditions entailing desirable properties (algebraic closure, stability under operators, interpolation, etc.) for the corresponding spaces. The aim of this paper is to stress or reveal the true nature of these diverse conditions imposed on weights, appearing in a scattered and disconnected way in the literature: they turn out to fall into the framework of O-regular variation, and many of them are equivalent formulations of one and the same feature. Moreover, we study several indices of regularity/growth for both functions and sequences, which allow for the rephrasing of qualitative properties in terms of quantitative statements.

Keywords Weight functions and weight sequences · O-regular variation · Matuszewska indices · Legendre conjugates

Mathematics Subject Classification Primary 26A12; Secondary 26A48 · 44A15 · 46E10 · 46E30

1 Introduction

The motivation of this paper arises from the study of ultraholomorphic and ultradifferentiable classes of functions, which consist of smooth or analytic functions defined in an appropriate
region $G$ of $\mathbb{R}$, $\mathbb{C}$ or the Riemann surface of the logarithm $\mathcal{R}$ whose derivatives satisfy one of the following estimates for some or all $A > 0$:

$$\sup_{z \in G, \, p \in \mathbb{N}_0} \frac{|f^{(p)}(z)|}{A^p M_p} < \infty, \quad \text{or} \quad \sup_{z \in G, \, p \in \mathbb{N}_0} \frac{|f^{(p)}(z)| \exp\left(-\frac{\varphi_\omega(A^p)}{A}\right)}{A^p M_p} < \infty, \quad (1)$$

where $\mathbb{N}_0 = \{0, 1, 2, \ldots\} = \mathbb{N} \cup \{0\}$, $M = (M_p)_{p \in \mathbb{N}_0}$ is a sequence of positive real numbers, $\varphi_\omega(x) := \sup\{xy - \omega(e^y) : y \geq 0\}$ and $\omega : [0, \infty) \to [0, \infty)$, see [7,23,30,32–35,39,40]. For instance, if $G = [a, b]$ is some compact interval of the real line and $\omega(t) = t$ or $M_p = p!$, then the class of smooth functions such that (1) holds for this choice coincides with the class of analytic functions in $G$.

One of the main topics regarding these classes is the characterization of the features of the ultradifferentiable or ultraholomorphic class in terms of properties of the corresponding sequence $M$ or function $\omega$. It is worthy to mention that some authors, specially when dealing with asymptotic expansions, have defined these classes using another estimate, see [8,17,18,20,24,37,41]. They assume that for some or all $A > 0$:

$$\sup_{z \in G, \, p \in \mathbb{N}_0} \frac{|f^{(p)}(z)|}{A^p p! M_p} < \infty. \quad (2)$$

In some sense it can be said that the sequence $\vec{M}$ measures the lack of analyticity in this situation. There is a link between the properties usually assumed for $M$ and the ones for $\vec{M} = (p!M_p)_{p \in \mathbb{N}_0}$. However these relations are not always straight and some considerations need to be made. Since our purpose is that this work can be applied to both situations, the results are stated in a general framework and suitable comments are provided showing how to use them in each case.

In this context, diverse conditions satisfied by $\vec{M}$ or by $\omega$ have been independently introduced adapted to the problem tackled in each work. Frequently, there are not further considerations about the nature of the different properties and their connections are not perfectly understood. In some recent works two indices associated to sequences have appeared: $\gamma(\vec{M})$, considered by Thilliez [41], and $\omega(\vec{M})$, defined by the second author in [37]. They have been shown to measure the limiting opening of the sectors in $\mathcal{R}$ such that for every sector of opening strictly smaller or, respectively bigger, the Borel map in the corresponding ultraholomorphic class is surjective or, respectively injective, see [20]. Similarly, the authors have introduced an analogous index $\gamma(\omega)$ for the function $\omega$ solving an extension problem for the pertinent classes, see [19,21].

This paper aims to untangle the true essence of these characteristics which have come out in the literature. The solution lies on the classical theory of regular variation, concretely on the notion of O-regular variation, see [4]. With this tool, the equivalence between the distinct conditions is provided and these qualitative properties are expressed in terms of some quantitative values, the growth orders and the Matuszewska indices, which turn out to coincide with the indices for functions and sequences mentioned above. Furthermore, the differences and similarities between the function approach and the sequence one are highlighted.

In the analysis of these conditions, we have found that, apart from the theory of ultraholomorphic and ultradifferentiable classes, they have repeatedly and independently appeared in their different forms in several areas of Functional Analysis, specially dealing with weighted structures. For instance, the N-functions defining the Orlicz spaces are usually assumed to satisfy conditions $\Delta_2$ and $\nabla_2$ which can be identified with the properties here studied, see [36]. The indices $\omega(\vec{M})$ and $\gamma(\vec{M})$ and some of the properties for sequences we will deal with.
have also been shown to be important for the Stieltjes moment problem in general Gelfand–Shilov spaces, see [9,25]. In the previous cases the mass of the function is concentrated at ∞, but there are also weighted spaces like weighted Bergman, Besov, Lebesgue, or Lipschitz spaces considered by Blasco and other authors, see [5] and the references therein, where the mass of the corresponding weight function is concentrated at 0, and similar properties appear, for example [5, (2.1) and (2.2)]. The same happens for the weighted Hölder classes studied by Dynkin in [12] where the modulus of continuity is regular if it satisfies certain conditions related to the ones treated in this paper. In these situations the results of this work could be applied after a suitable modification, see Remark 2.19 for further details.

Most of those weighted spaces are defined from the classical ones replacing the function $t \mapsto t^\alpha$ for some $\alpha > 0$ by a general function $t \mapsto \omega(t)$. The extension of the classical results to the weighted context highly depends on a power-like behavior of $\omega$ which leads to the theory of regular variation whose purpose is the systematic study of such type of behaviors. Hence the scope of this work might go beyond these examples and the results have been stated from a quite abstract point of view so they can be applied to diverse situations.

At this point we start describing the main achievements obtained in this paper and how they are organized. The second section starts by recalling the elementary facts about weight functions, regular and O-regular variation. The first important result, Lemma 2.10, establishes the equality between the upper Matuszewska index $\alpha(\sigma)$ and the inverse of the aforementioned index $\gamma(\sigma)$ under the basic assumption of the section: $\sigma : [0, \infty) \to [0, \infty)$ is nondecreasing with $\lim_{t \to \infty} \sigma(t) = \infty$. The main results of the section, Theorems 2.11 and 2.16, provide a list of equivalent conditions for some of the basic properties assumed for weight functions, for instance

$$(\omega_1) \quad \sigma(2t) = O(\sigma(t)),$$  

as $t \to \infty$ and

$$(\omega_{\text{snq}}) \quad \exists C \geq 1: \forall y > 0, \int_1^{\infty} \frac{\sigma(yt)}{t^2} dt \leq C \sigma(y) + C,$$

in the first case, see also Corollaries 2.13 and 2.14, and

$$(\omega_6) \quad \exists H \geq 1: \forall t \geq 0, 2\sigma(t) \leq \sigma(HT) + H,$$

in the second case, see also Corollary 2.14. Moreover, these results also connect these properties to the Matuszewska indices thanks to the almost monotonicity notions. Finally, the last subsection is devoted to the study of the relation between the index $\gamma$ of a function and the ones of its upper and lower Legendre conjugates.

In the third section, after summarizing the basic facts about weight sequences, we recall the main points of the theory of regularly and o-regularly varying sequences described in the works of Bojanić and Seneta [6] and Djurčić and Božin [10], respectively. The first task carried out in this section has been introducing the notion of growth order and Matuszewska indices, to the best of our knowledge new, and proving some elementary properties of those values, see from Proposition 3.5 to Remark 3.8. In Theorem 3.10, it is shown that $\gamma(\mathbb{M})$ equals the lower Matuszewska index of the sequence of quotients $\mathbf{m} := (m_p = M_{p+1}/M_p)_{p \in \mathbb{N}_0}$ and $\omega(\mathbb{M})$ coincides with the lower order of $\mathbf{m}$. The section concludes with Theorems 3.11 and 3.16 where the strong nonquasianalytic and the moderate growth conditions:

$$(\text{snq}) \quad \exists B \geq 0: \forall p \in \mathbb{N}_0, \sum_{q=p}^{\infty} \frac{M_q}{(q + 1)M_{q+1}} \leq B \frac{M_p}{M_{p+1}},$$

$$(\text{mg}) \quad \exists A \geq 0: \forall p, q \in \mathbb{N}_0, M_{p+q} \leq A^{p+q} M_p M_q,$$

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are characterized in terms of the Matuszewska indices of $m$ and compared with other conditions appearing in the literature. In the proof we have made use of Theorems 2.11 and 2.16 although it is possible to show them directly as it has been partially done by the first author in [16] with similar arguments.

The fourth section aims to compare the weight function approach with the weight sequence one through the counting function of the sequence of quotients $v_m(t) := \max\{ j \in \mathbb{N} : m_{j-1} \leq t \}$ and the associated function $\omega_M(t):= \sup_{p \in \mathbb{N}_0} \log \left( t^p / M_p \right)$ for all $t \geq 0$. In the first subsection, the duality relation between the orders and Matuszewska indices of these functions and the ones of the corresponding weight sequence is explained. In the second subsection, Corollary 4.11 shows under suitable assumptions that $\gamma(\omega_M) = \gamma(\omega_m) + 1$ by means of the Legendre conjugate. The strongly regular sequences, which appear in different issues, see the references in Sect. 4.3, are characterized in Corollary 4.12 in terms of the Matuszewska indices of $m$, $v_m$ and $\omega_M$. The section ends analyzing the connection between nonzero proximate orders and weight functions. Nonzero proximate orders have been used by Lastra, Malek and the second author in [24] to develop a summability theory for ultraholomorphic classes defined in terms of a weight sequence. Thanks to Corollary 4.16, we see that the information for ultraholomorphic classes defined in terms of a weight function is the same as the one from the weight sequence case.

The last section contains a counter-example of a weight sequence $M$ such that $\gamma(M)$ and $\gamma(\omega_M)$ do not coincide, so the corresponding properties associated with these indices are not equivalent. This fact clarifies the duality relations described in the previous section.

## 2 Weight functions and O-regular variation

### 2.1 Weight functions $\omega$ in the sense of Braun–Meise–Taylor

A function $\omega : [0, \infty) \rightarrow [0, \infty)$ is called a weight function if it is continuous, nondecreasing, $\omega(0) = 0$, and $\lim_{t \to \infty} \omega(t) = \infty$. If in addition $\omega(t) = 0$ for all $t \in [0, 1]$ we say that $\omega$ is a normalized weight function. Moreover, the following conditions are often considered when dealing with weighted spaces of functions:

- \((\omega_1)\) $\omega(2t) = O(\omega(t))$ as $t \to \infty$.
- \((\omega_2)\) $\omega(t) = O(t)$ as $t \to \infty$.
- \((\omega_3)\) $\log(t) = o(\omega(t))$ as $t \to \infty$.
- \((\omega_4)\) $\phi(\omega(t))$ is a convex function on $\mathbb{R}$.
- \((\omega_5)\) $\omega(t) = O(t)$ as $t \to \infty$.
- \((\omega_6)\) there exists $H \geq 1$ such that for all $t \geq 0$ $2\omega(t) \leq \omega(HT) + H$.
- \((\omega_7)\) there exists $H$, $C \geq 0$ such that for all $t \geq 0$ $\omega(t^2) \leq C \omega(HT) + C$.
- \((\omega_{\text{sq}})\) $\int_1^\infty \frac{\omega(t)}{t^2} \, dt < \infty$.
- \((\omega_{\text{sq}})\) there exists $C \geq 1$ such that for all $y > 0$, $\int_1^\infty \frac{\omega(yt)}{t^2} \, dt \leq C \omega(y) + C$.

The classical examples are the well-known Gevrey weights of index $s > 1$, $\omega(t) = t^{1/s}$, which define the Gevrey classes, they satisfy all listed properties except \((\omega_7)\). Another interesting example is $\omega(t) = \max\{0, \log(t)^s\}$, $s > 1$, which satisfies all listed properties except \((\omega_6)\). Note that, for a weight function, concavity implies subadditivity, i.e. $\omega(s + t) \leq \omega(s) + \omega(t)$, and this yields \((\omega_1)\).
Let \( I \) be an unbounded subinterval of \([0, \infty)\) and \( \sigma, \tau : I \to [0, \infty) \) be any pair of measurable functions, we call them equivalent and we write \( \sigma \sim \tau \) if there exists \( C \geq 1 \) such that for every \( t \in I \)

\[
C^{-1} \tau(t) - C \leq \sigma(t) \leq C \tau(t) + C.
\]

### 2.2 Regular variation and \( O \)-regular variation

The notion of regular variation was formally introduced in 1930 by Karamata [22] and has several applications in analytic number theory, complex analysis and, specially, in probability. The proofs of most of the results in this subsection are gathered in the book of Bingham et al. [4]. Until the end of this subsection, we assume that

\[
f : [A, \infty) \to (0, \infty), \quad \text{with } A > 0,
\]

is a measurable function.

We say that \( f \) is regularly varying if for every \( \lambda \in (0, \infty) \),

\[
\lim_{x \to \infty} \frac{f(\lambda x)}{f(x)} = g(\lambda) \in (0, \infty).
\]

(3)

There are three main results of this theory: the Uniform Convergence, the Representation and the Characterization Theorems [4, Th. 1.3.1, Th. 1.4.1 and Th. 1.5.2], the last one states that, if \( f \) is regularly varying, then there exists \( \rho \in \mathbb{R} \) such that the function \( g(\lambda) \) in (3) is equal to \( \lambda^\rho \). In this case, \( \rho \) is called the index of regular variation of \( f \), we write \( f \in R^\rho \) and \( RV := \bigcup_{\rho \in \mathbb{R}} R^\rho \). If \( \rho = 0 \), then \( f \) is said to be slowly varying.

Consequently, the behavior of a regularly varying function at \( \infty \) is in some sense similar to the behavior of a power-like function. If for \( n \in \mathbb{N} \), \( \log_n x \) denotes the \( n \)-th iteration of the logarithm, given \( n_i \in \mathbb{N} \) and \( \alpha_i \in \mathbb{R} \) for \( i = 0, 1, \ldots, k \), the classical example of a regularly varying function is

\[
f(x) = x^{\alpha_0} (\log_n x)^{\alpha_1} (\log_{n_2} x)^{\alpha_2} \cdots (\log_{n_k} x)^{\alpha_k}.
\]

For some of our purposes, the theory of regular variation is too restrictive and one may ask what remains valid if we replace \( \lim \) by \( \lim \sup \) and \( \lim \inf \) in (3). This extension of the class of regularly varying functions was defined by Karamata, Avakumović, considered by Matuszewska [29] and Feller [13] and systematically studied by Aljančić and Arandjelović [1] in 1977. We say that \( f \) is \( O \)-regularly varying [4, p. 61] if

\[
0 < f_{\text{low}}(\lambda) := \liminf_{x \to \infty} \frac{f(\lambda x)}{f(x)} \leq f_{\text{up}}(\lambda) := \limsup_{x \to \infty} \frac{f(\lambda x)}{f(x)} < \infty
\]

for every \( \lambda \geq 1 \), and we write \( f \in ORV \). This weaker notion preserves several desirable properties, in particular, the three main theorems of regular variation have their adapted version.

**Remark 2.1** We observe that \( f_{\text{low}}(\lambda) = 1 / f_{\text{up}}(1/\lambda) \) for every \( \lambda \geq 1 \). Consequently, if \( f \in ORV \), then (4) holds for every \( \lambda \in (0, \infty) \) and we deduce that \( RV \subseteq ORV \). Moreover, \( f \in ORV \) if and only if

\[
f_{\text{up}}(\lambda) = \limsup_{x \to \infty} \frac{f(\lambda x)}{f(x)} < \infty \quad \text{for every } \lambda \in (0, \infty).
\]
In this general context, the index $\rho$ of regular variation is split into two values, the Matuszewska indices [4, p. 68]. For any positive function, the upper Matuszewska index $\alpha(f)$ is defined by

$$\alpha(f) := \inf \left\{ \alpha \in \mathbb{R}; \exists C_\alpha > 0 \text{ s.t. } \forall \Lambda > 1, \limsup_{x \to \infty} \sup_{\lambda \in [1, \Lambda]} \frac{f(\lambda x)}{\lambda^\alpha f(x)} \leq C_\alpha \right\}$$

and the lower Matuszewska index $\beta(f)$ by

$$\beta(f) := \sup \left\{ \beta \in \mathbb{R}; \exists D_\beta > 0 \text{ s.t. } \forall \Lambda > 1, \liminf_{x \to \infty} \inf_{\lambda \in [1, \Lambda]} \frac{f(\lambda x)}{\lambda^\beta f(x)} \geq D_\beta \right\}.$$ 

**Remark 2.2** Since these sets are either empty or unbounded intervals from above for $\alpha$ and, respectively from below for $\beta$, we are allowed to use the classical conventions $\inf \emptyset = \sup \mathbb{R} = \infty$ and $\inf \mathbb{R} = \sup \emptyset = -\infty$.

Moreover, the inequality $\beta(f) \leq \alpha(f)$ always holds. The finiteness of these indices characterizes O-regular variation.

**Theorem 2.3** ([4], Th. 2.1.7) $f$ is O-regularly varying if and only if $\alpha(f) < \infty$ and $\beta(f) > -\infty$.

These indices admit a nicer and useful representation in terms of some almost monotonicity properties. We call a function $h : [a, +\infty) \to (0, +\infty)$, with $a > 0$, almost increasing, if there exists some $M > 0$ such that $h(x) \leq M h(y)$ for all $a \leq x \leq y < +\infty$. Analogously we call $h$ almost decreasing, if there exists some $m > 0$ such that $m h(y) \leq h(x)$ for all $a \leq x \leq y < +\infty$.

**Theorem 2.4** ([4] Th. 2.2.2) For $f$ as above,

$$\alpha(f) = \inf \{ \alpha \in \mathbb{R}; \exists C_\alpha > 0 \text{ s.t. } \forall \Lambda > 1, \limsup_{x \to \infty} \frac{f(x)}{x^\alpha} \text{ is almost decreasing} \},$$

$$\beta(f) = \sup \{ \beta \in \mathbb{R}; \exists D_\beta > 0 \text{ s.t. } \forall \Lambda > 1, \liminf_{x \to \infty} \frac{f(x)}{x^\beta} \text{ is almost increasing} \}.$$ 

**Remark 2.5** These indices are related to the classical lower order $\mu(f)$ and upper order $\rho(f)$ defined by

$$\mu(f) := \liminf_{x \to \infty} \frac{\log f(x)}{\log x}, \quad \rho(f) := \limsup_{x \to \infty} \frac{\log f(x)}{\log x},$$

in the following way: $\beta(f) \leq \mu(f) \leq \rho(f) \leq \alpha(f)$, see [4, Prop. 2.2.5]. If $f \in R_\rho$, then $\beta(f) = \mu(f) = \rho(f) = \alpha(f) = \rho$. However, in general, the inequalities are strict and the equality of indices and orders does not guarantee regular variation.

As a consequence of the almost monotone characterization the following properties are deduced.

**Remark 2.6** If $f, g : [A, \infty) \to (0, \infty)$, with $A > 0$, are measurable functions with

$$0 < \liminf_{x \to \infty} \frac{f(x)}{g(x)} \leq \limsup_{x \to \infty} \frac{f(x)}{g(x)} < \infty,$$

then $\beta(f) = \beta(g)$, $\mu(f) = \mu(g)$, $\rho(f) = \rho(g)$ and $\alpha(f) = \alpha(g)$.
For any $s > 0$ and $r \in \mathbb{R}$, we put $f_s(t) := f(t^s)$, $f^r(t) := (f(t))^r$ and $p_r(t) := t^r$, and we see that

$$
\beta(f_s) = s \beta(f), \quad \beta(f^r) = s \beta(f), \quad \beta(f \cdot p_r) = r + \beta(f).
$$

The same is valid if we replace the index $\beta$ by $\mu$, $\rho$ or $\alpha$. Moreover, if $s < 0$, then $\alpha(f_s) = s \beta(f)$, $\rho(f_s) = s \mu(f)$, $\mu(f_s) = s \rho(f)$ and $\beta(f^s) = s \alpha(f)$.

These indices might be difficult to handle. Fortunately, if some additional property holds, then a combination of [4, Th. 2.1.5, Coro. 2.1.6, Th. 2.1.7] leads to this suitable representation.

**Theorem 2.7** If $\alpha(f) < \infty$ or $\beta(f) > -\infty$, then

$$
\alpha(f) = \lim_{\lambda \to \infty} \frac{\log(f_{\text{up}}(\lambda))}{\log \lambda} = \inf_{\lambda > 1} \frac{\log(f_{\text{up}}(\lambda))}{\log \lambda},
$$

$$
\beta(f) = \lim_{\lambda \to \infty} \frac{\log(f_{\text{low}}(\lambda))}{\log \lambda} = \sup_{\lambda > 1} \frac{\log(f_{\text{low}}(\lambda))}{\log \lambda}.
$$

Hence, this theorem holds for O-regularly varying functions but also under weaker conditions. In particular, the next lemma, which is a consequence of Theorem 2.4, ensures that Theorem 2.7 is available for monotone (not necessarily ORV) functions. This situation occurs when dealing with weight functions or sequences.

**Lemma 2.8** (i) If $f$ is nondecreasing, then $\beta(f) \geq 0$. Hence $f \in \text{ORV}$ if and only if $\alpha(f) < \infty$.

(ii) If $\beta(f) > 0$, then there exists an nondecreasing function $g$ satisfying (5).

(iii) If $f$ is nonincreasing, then $\alpha(f) \leq 0$. Hence $f \in \text{ORV}$ if and only if $\beta(f) > -\infty$.

(iv) If $\alpha(f) < 0$, then there exists a nonincreasing function $g$ satisfying (5).

### 2.3 Index $\gamma$ for nondecreasing functions and its relation to $\alpha$

In [19,21], the index $\gamma(\omega)$ was introduced in order to measure the limit opening which the Borel map defined in the corresponding ultraholomorphic class in a sector of the Riemann surface of the logarithm is surjective for. The definition is based on [30, Prop. 1.3] and has the same spirit as Thilliez’s index $\gamma(M)$ for sequences considered in [41].

This index was originally defined for a weight function $\omega$, but due to the general approach of this paper we will work in a more general framework. Until the end of the section we deal with:

$$
\sigma : [0, \infty) \to [0, \infty), \quad \text{nondecreasing with} \lim_{t \to \infty} \sigma(t) = \infty.
$$

Although even weaker assumptions may be considered, this approach is enough to cover at the same time the weight function and the weight sequence case. We can also treat other weighted structures introduced and used in different fields of Functional Analysis as it has been explained in the introduction, see also Remark 2.19. Let $\sigma$ and $\gamma > 0$ be given, we say that $(P_{\sigma,\gamma})$ holds if there exists $K > 1$ such that

$$
\limsup_{t \to \infty} \frac{\sigma(K^t \gamma t)}{\sigma(t)} < K.
$$

**Note:** If $(P_{\sigma,\gamma})$ holds for some $K > 1$, then also $(P_{\sigma,\gamma'})$ is satisfied for all $\gamma' \leq \gamma$ with the same $K$ and since $\sigma$ is nondecreasing we might restrict ourselves to $\gamma > 0$. 

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Finally, we put
\[ \gamma(\sigma) := \sup \{ \gamma > 0 : (P_{\sigma,\gamma}) \text{ is satisfied} \} \] (6)

If none condition \((P_{\sigma,\gamma})\) holds true, then we put \( \gamma(\sigma) := 0 \).

Remark 2.9 We want to compare \( \gamma(\sigma) \) with the Matuszewska indices introduced above. In the classical literature, \( \alpha(\sigma) \) and \( \beta(\sigma) \) are defined only for positive functions. For \( \sigma \) as above, by \( \alpha(\sigma) \) and \( \beta(\sigma) \) we mean the corresponding indices of the restriction of \( \sigma \) to some interval \([A, \infty)\), with \( A > 0 \), which is possible because \( \sigma \) is nondecreasing with \( \lim_{t \to \infty} \sigma(t) = \infty \) and, by Remark 2.6, the indices do not depend on the restriction considered.

Lemma 2.10 The relation
\[ \alpha(\sigma) = \frac{1}{\gamma(\sigma)} \]
holds. Since \( \gamma(\sigma), \alpha(\sigma) \in [0, \infty) \), for the extreme values this means that \( \gamma(\sigma) = \infty \) if and only if \( \alpha(\sigma) = 0 \) and \( \gamma(\sigma) = 0 \) if and only if \( \alpha(\sigma) = \infty \).

Proof By Lemma 2.8.(i), \( \beta(\sigma) \geq 0 \), so Theorem 2.7 can be applied and write
\[ \alpha(\sigma) = \inf_{\lambda > 1} \frac{\log(\sigma^{\uparrow}(\lambda))}{\log \lambda}. \]
We rewrite the definition of the index \( \gamma(\sigma) \) defined in (6) to obtain:
\[ \gamma(\sigma) = \sup \left\{ \gamma > 0 : \exists \Lambda > 1 : \limsup_{t \to \infty} \frac{\sigma(\Lambda t)}{\sigma(t)} < \Lambda^{1/\gamma} \right\} \]
\[ = (\inf\{ \tau > 0 : \exists \Lambda > 1 : \sigma^{\uparrow}(\Lambda) < \Lambda^{\tau} \})^{-1}. \]
Let now \( \alpha > \alpha(\sigma) \), then there exists some \( \lambda > 1 \) such that \( (\log(\sigma^{\uparrow}(\lambda))/\log \lambda) < \alpha \) and consequently \( (\gamma(\sigma))^{-1} \leq \alpha \). Conversely, let \( \tau > (\gamma(\sigma))^{-1} \) be given, then there exists some \( \Lambda > 1 \) such that \( (\log(\sigma^{\uparrow}(\Lambda))/\log \Lambda) < \tau \) and so \( \alpha(\sigma) < \tau \) follows.

From this connection and according to Remark 2.6 we deduce the following properties, most of them have been obtained in [21] directly from the definition:
(i) For any \( s > 0 \), \( \gamma(\sigma^s) = \gamma(\sigma)/s \) and \( \gamma(\sigma_s) = \gamma(\sigma)/s \) where \( \sigma_s(t) = \sigma(t^s) \) and \( \sigma^s(t) = (\sigma(t))^s \).
(ii) Let \( \omega, \sigma : [0, \infty) \to [0, \infty) \), nondecreasing with \( \lim_{t \to \infty} \omega(t) = \lim_{t \to \infty} \sigma(t) = \infty \) and \( \sigma \sim \omega \) be given. Then \( \sigma \) and \( \omega \) satisfy (5) so \( \gamma(\omega) = \gamma(\sigma) \).

2.4 Main theorems

Aljančić and Arandjelović [1] give several equivalent representations of the indices \( \alpha(f) \) and \( \beta(f) \) for O-regularly varying functions. In Theorems 2.11 and 2.16, this information is extended for monotone, but not necessarily O-regularly varying, functions. In Corollaries 2.13, 2.14 and 2.17 we deduce the relation between these indices and some of the classical conditions usually assumed for weight functions. The proof of the main results is based on Bari, Stechkin [2, Lemmas 2 and 3] and Meise, Taylor [30, Prop. 1.3].

We will denote by \( \mathbb{N} \) the set \( \{1, 2, 3, \ldots\} \), write \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \) and \( [x] := \min\{k \in \mathbb{Z} : x \leq k\} \) for \( x \in \mathbb{R} \).
Theorem 2.11 Let $\sigma : [0, \infty) \to [0, \infty)$, nondecreasing with $\lim_{t \to \infty} \sigma(t) = \infty$ and $\alpha > 0$ be given. We take $a \geq 0$ such that $\sigma(x) > 0$ for $x \geq a$. Then the following are equivalent:

(i) there exists $C > 0$ such that $\int_1^\infty \frac{\sigma(yt)}{t^{1+\alpha}} dt \leq C \sigma(y) + C$ for all $y > 0$,

(ii) there exists a nondecreasing function $\kappa : [0, +\infty) \to [0, +\infty)$ such that $\sigma \sim \kappa$, $\kappa(0) = \frac{\sigma(0)}{a^2}$, $\kappa$ satisfies (i) and $\kappa(t^{1/\alpha})$ is concave,

(iii) $\lim_{t \to 0} \limsup_{t \to +\infty} \frac{\varepsilon^\alpha \sigma(t)}{\sigma(\varepsilon t)} = 0$,

(iv) there exists $K > 1$ such that $\limsup_{t \to +\infty} \frac{\sigma(Kt)}{\sigma(t)} < K^\alpha$,

(v) $\gamma(\sigma) > 1/\alpha$,

(vi) $\alpha(\sigma) < \alpha$ (with the convention in Remark 2.9),

(vii) there exists $\gamma \in (0, \alpha)$ such that $t \mapsto \frac{\sigma(t)}{t^{\gamma}}$ is almost decreasing in $[a, \infty)$ if $a > 0$, and in $[\varepsilon, \infty)$ for any $\varepsilon > 0$ if $a = 0$,

(viii) there exists $C > 0$ such that $\int_a^\infty \frac{t^\alpha}{\sigma(t)} dt \leq C \frac{y^\alpha}{\sigma(y)}$ for all $y \geq a$,

(ix) there exists $C > 0$ such that $\sum_{k=\lfloor a \rfloor + 1}^{\infty} \frac{k^{\alpha-1}}{\sigma(k)} \leq C \frac{p^\alpha}{\sigma(p)}$ for all $p \in \mathbb{N}$ with $p \geq \lceil a \rceil + 1$,

(x) for every $\theta \in (0, 1)$ there exists $k \in \mathbb{N}$, $k \geq 2$, such that $\sigma(kp) \leq \theta k^\alpha \sigma(p)$ for every $p \in \mathbb{N}$ with $p \geq \lceil a \rceil + 1$,

(xi) there exists $k \in \mathbb{N}$, $k \geq 2$ such that $\limsup_{p \to \infty} \frac{\sigma(kp)}{\sigma(p)} < k^\alpha$,

(xii) there exists $C > 0$ such that $\sum_{k=p}^{\infty} \frac{\sigma(k)}{k^{1+\alpha}} \leq C \frac{\sigma(p)}{p^\alpha}$ for every $p \in \mathbb{N}$ with $p \geq \lceil a \rceil$.

Proof (i) $\Rightarrow$ (ii) For all $y > 0$ we define

$$\kappa_0(y) = \xi_y(y) := \int_1^\infty \frac{\sigma(yt)}{t^{1+\alpha}} dt = y^\alpha \int_y^\infty \frac{\sigma(s)}{s^{1+\alpha}} ds.$$ 

Using that $\sigma$ is nondecreasing, we observe that $\kappa_0$ is continuous nondecreasing in $[0, +\infty)$, $\kappa_0(0) = \sigma(0)/\alpha$ and $\kappa_0(y) \geq \sigma(y)/\alpha$ for all $y \geq 0$. By (i), we see that $\kappa_0(y) \leq C \sigma(y) + C$ for all $y > 0$, so $\sigma \sim \kappa_0$, then one can check that $\kappa_0$ also satisfies (i) for some constant $C'$.

Iterating the procedure, we construct $\kappa(y) := \xi_{\kappa_0}(y) \in C^1(0, \infty)$, $\kappa$ is nondecreasing, $\kappa(0) = \sigma(0)/\alpha^2$, $\kappa \sim \kappa_0 \sim \sigma$, $\kappa$ satisfies (i). Finally, for all $y > 0$ we see that

$$(\kappa(y^{1/\alpha}))' = \left(y \int_{y^{1/\alpha}}^\infty \frac{\kappa_0(s)}{s^{1+\alpha}} ds\right)' = \int_{y^{1/\alpha}}^\infty \frac{\kappa_0(s)}{s^{1+\alpha}} ds - \frac{\kappa_0(y^{1/\alpha})}{\alpha y} = \int_{y^{1/\alpha}}^\infty \frac{\kappa_0(s) - \kappa_0(y^{1/\alpha})}{s^{1+\alpha}} ds.$$ 

Since $\kappa_0(y^{1/\alpha})$ is nondecreasing we conclude that $(\kappa(y^{1/\alpha}))'$ is nonincreasing, so $\kappa(y^{1/\alpha})$ is concave.

(ii) $\Rightarrow$ (iii) We will show that $\kappa$ satisfies (iii), and we conclude using that $\kappa \sim \sigma$. Since $\kappa(t^{1/\alpha})$ is concave and $\kappa(0) \geq 0$, $\kappa(2t^{1/\alpha}) \leq \kappa((2t)^{1/\alpha}) + \kappa(0) \leq 2\kappa(t^{1/\alpha})$ for all $t \geq 0$, so we put $A := 2^{1/\alpha} > 1$ and we see that $\kappa(Ay) \leq A^\alpha \kappa(y)$ for all $t \geq 0$, or more generally, $A^{-n\alpha} \kappa(A^n y) \leq A^{-j\alpha} \kappa(A^j y)$ for all $y \geq 0$ and every $j \in \mathbb{N}$ with $j \leq n$. Using that $\kappa$...
satisfies (i) and that κ and $t^\alpha$ are nondecreasing, we see that

$$n \frac{\kappa(A^n y)}{A^n \alpha} \leq \sum_{j=1}^{n} \frac{\kappa(A/j y)}{A/j \alpha} = \frac{A^{\alpha}}{\log A} \sum_{j=1}^{n} \frac{\kappa(A/j y)}{A(j+1)^{\alpha}} \int_{A/j}^{A(j+1)/t} \frac{dt}{t} \leq \frac{A^{\alpha}}{\log A} \sum_{j=1}^{n} \int_{A/j}^{A(j+1)/t} \frac{\kappa(y t)}{t^{1+\alpha}} dt \leq \frac{A^{\alpha}}{\log A} \int_{1}^{\infty} \frac{\kappa(y t)}{t^{1+\alpha}} dt \leq B \kappa(y),$$

for $y \geq y_0$ large enough for some suitable constant $B > 0.$ Hence for every $\varepsilon \in (0, 1/A]$ there exists $n$ such that $\varepsilon \in (1/A^{n+1}, 1/A^n),$ so for $s \geq A^{n+1} y_0$ we observe that

$$\frac{\varepsilon^{\alpha} \kappa(s)}{\kappa(s)} \leq \frac{A^{-n \alpha} \kappa(s)}{\kappa(A^{-(n+1)} s)} \leq \frac{A^{\alpha} B}{n+1},$$

which proves that $\kappa$ satisfies (iii).

(iii) $\Rightarrow$ (iv) Immediate.

(iv) $\Rightarrow$ (v) Immediate.

(v) $\Rightarrow$ (vi) Deduced from Lemma 2.10.

(vi) $\Rightarrow$ (vii) By Theorem 2.4 taking $\alpha / \sigma(\alpha) < \gamma < \alpha.$

(vii) $\Rightarrow$ (viii) By (vii) and suitably enlarging the constant when $a = 0,$ we see that $t \mapsto t^\gamma / \sigma(t)$ is almost increasing in $[a, \infty).$ Hence for all $y \geq a$

$$y^{-\alpha} \int_a^y \frac{t^{\alpha}}{\sigma(t)} dt = y^{-\alpha} \int_a^y \frac{t^{\alpha-\gamma} t^\gamma}{\sigma(t)} dt \leq M \frac{y^{\gamma-\alpha}}{\sigma(y)} \int_0^y \frac{t^{\alpha-\gamma}}{t} dt = \frac{M}{(\alpha - \gamma) \sigma(y)}.$$

(viii) $\Rightarrow$ (ix) Since $\sigma$ is nondecreasing, for every $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ with $k \geq a$ we see that

$$\int_k^{k+1} \frac{t^{\alpha}}{\sigma(t)} dt \geq \frac{1}{\sigma(k+1)} \frac{(k+1)^{\alpha-k} - k^{\alpha}}{\alpha} = \frac{(k+1)^{\alpha-1}}{\alpha \sigma(k+1)} (1 - \left(\frac{k}{k+1}\right)^{\alpha}).$$

We write $a_k = (k+1)(1 - (k/k+1)^{\alpha})$ for all $k \in \mathbb{N}_0$ and we see that $(a_k)_{k \in \mathbb{N}_0}$ is a sequence of positive real numbers with $\lim_{k \to \infty} a_k = \alpha,$ so we fix $0 < D < \min_{k \in \mathbb{N}_0} (a_k).$ Hence by (viii) for $y = p \geq \lceil a \rceil + 1,$ we observe that

$$C \frac{p^{\alpha}}{\sigma(p)} \geq \int_a^p \frac{t^{\alpha}}{\sigma(t)} dt \geq \int_{\lceil a \rceil}^{\lceil a \rceil+1} \frac{t^{\alpha}}{\sigma(t)} dt \geq \frac{D}{\alpha} \sum_{k=\lceil a \rceil}^{p} \frac{(k+1)^{\alpha-1}}{\alpha \sigma(k+1)}.$$

(ix) $\Rightarrow$ (x) For simplicity we write $k_0 := \lceil a \rceil + 1.$ First we show that the sequence $(p^{\alpha} / \sigma(p))^\infty_{p=k_0}$ is almost increasing. By (ix) and the monotonicity of $\sigma,$ for any $q \geq p \geq k_0 + 3$ we observe that

$$C \frac{q^{\alpha}}{\sigma(q)} \geq \sum_{k=k_0}^{q} \frac{k^{\alpha-1}}{\sigma(k)} \geq \sum_{k=k_0}^{p} \frac{k^{\alpha-1}}{\sigma(k)} \geq \frac{1}{\sigma(p)} \int_{k_0+1}^{p} \frac{t^{\alpha}}{t} dt \geq \frac{p^{\alpha}}{p \alpha} \left(\frac{p-1}{p}\right)^{\alpha} \left(1 - \left(\frac{k_0+1}{k_0+2}\right)^{\alpha}\right).$$

Since $(p-1)/p \geq 1/2,$ we deduce that $(p^{\alpha} / \sigma(p))^\infty_{p=k_0+3}$ is almost increasing. We conclude that the same holds for $(p^{\alpha} / \sigma(p))^\infty_{p=k_0}$ by suitably enlarging the almost monotonicity constant. Now, for every $m \geq k_0,$ and every $k \in \mathbb{N}, k \geq 2,$ we apply (ix) for $p = km$ and the almost monotonicity of $(p^{\alpha} / \sigma(p))^\infty_{p=k_0}$ and we see that

$$C \frac{(km)^{\alpha}}{\sigma(km)} \geq \sum_{j=k_0}^{km} \frac{j^{\alpha-1}}{\sigma(j)} \geq \sum_{j=m}^{km} \frac{j^{\alpha-1}}{\sigma(j)} \geq B \frac{m^{\alpha}}{\sigma(m)} \int_{m}^{km} \frac{dt}{t} = B \frac{m^{\alpha}}{\sigma(m)} \log(k).$$
Then, (i) holds for $y$.

By suitably enlarging the constant we see that (xii) holds for all $y$.

Consequently, for all $p \geq p_0$

$$\sum_{\ell=p+1}^{\infty} \frac{\sigma(\ell)}{\ell^{1+\alpha}} = \sum_{s=0}^{\infty} \frac{\sigma(s)}{s^{1+\alpha}} \leq \sum_{s=0}^{\infty} \frac{\sigma(k^{s+1}p)}{k^{s+1}p} \frac{dt}{t^{1+\alpha}}.$$

Then

$$\sum_{\ell=p+1}^{\infty} \frac{\sigma(\ell)}{\ell^{1+\alpha}} \leq \sigma(p) \sum_{s=0}^{\infty} \frac{\theta^{s+1}k^{(s+1)\alpha}}{(k^s p)^{\alpha}} \int_{k^s p}^{k^{s+1}p} \frac{dt}{t} \leq \theta \log(k)k^\alpha \frac{\sigma(p)}{p^\alpha} \sum_{s=0}^{\infty} \theta^s.$$ 

Consequently, for all $p \geq p_0$

$$\sum_{\ell=p}^{\infty} \frac{\sigma(\ell)}{\ell^{1+\alpha}} \leq \left( \frac{\theta \log(k)k^\alpha}{1 - \theta} + 1 \right) \frac{\sigma(p)}{p^\alpha}. $$

By suitably enlarging the constant we see that (xii) holds for all $p \in \mathbb{N}$ with $p \geq \lceil a \rceil$.

(xii) $\Rightarrow$ (i) For all $y \geq 1$ by the monotonicity of $\sigma$

$$\int_{y}^{\infty} \frac{\sigma(t)}{t^{1+\alpha}} dt \leq \sum_{k=\lceil y \rceil}^{\infty} \int_{k}^{k+1} \frac{\sigma(t)}{t^{1+\alpha}} dt \leq \sum_{k=\lceil y \rceil}^{\infty} \frac{\sigma(k+1)}{(k+1)^{1+\alpha}} \left( \frac{k+1}{k} \right)^{1+\alpha} \leq 2^{1+\alpha} \sum_{k=\lceil y \rceil}^{\infty} \frac{\sigma(k)}{k^{1+\alpha}}.$$ 

Consequently, for all $y \geq y_0 := \max(1, \lceil a \rceil)$ applying (xii) and the monotonicity of $\sigma$

$$\int_{y}^{\infty} \frac{\sigma(t)}{t^{1+\alpha}} dt \leq 2^{1+\alpha} C \frac{\sigma(\lceil y \rceil)}{\lceil y \rceil^\alpha} \leq 4^{1+\alpha} C \frac{\sigma(y)}{y^\alpha}.$$ 

Then, (i) holds for $y \geq y_0$. By suitably modifying the constant it also holds for $y \in (0, y_0)$, because

$$y^\alpha \int_{y}^{\infty} \frac{\sigma(t)}{t^{1+\alpha}} dt = y^\alpha \left( \int_{y}^{y_0} \frac{\sigma(t)}{t^{1+\alpha}} dt + \int_{y_0}^{\infty} \frac{\sigma(t)}{t^{1+\alpha}} dt \right) \leq \frac{\sigma(y_0)}{\alpha} + 4^{1+\alpha} C \sigma(y_0).$$

$\square$

**Remark 2.12** The value of $\alpha$ is stable for $\sim$ because the equivalence implies (5) for nondecreasing functions tending to infinity at infinity, so all the conditions above are stable under equivalence for nondecreasing functions tending to infinity.

The list of equivalent definitions of $\alpha(\sigma)$ given in [1] can be increased, we write

$$\alpha(\sigma) = \inf \{ \alpha > 0; \text{ any of the conditions in Theorem 2.11 holds for } \sigma \},$$

if the previous set is empty we write $\alpha(\sigma) = 0$. It is worthy to notice that if any of the statements in the previous theorem holds for $\alpha$ it is also valid for $\alpha - \varepsilon$ for some small enough $\varepsilon > 0$. Hence the set of values satisfying any of these conditions is left-open.

Since for $\alpha = 1$ condition (i) in the previous theorem is ($\omega_{snq}$) condition, we obtain the following corollary.
Corollary 2.13 Let \( \sigma \) be as above. The following are equivalent:

(i) \( \sigma \) satisfies \((\omega_{snq})\),
(ii) \( \sigma \) satisfies every/some of the equivalent conditions \((ii)-(xii)\) in Theorem 2.11 for \( \alpha = 1 \).

In particular, \( \sigma \) satisfies \((\omega_{snq})\) if and only if \( \gamma(\sigma) > 1 \) if and only if \( \alpha(\sigma) < 1 \).

Similarly, we can translate condition \((\omega_1)\) in terms of the index \( \alpha \). The following corollary can be seen as a restatement of a well-known result of W. Feller (e.g. see [4, Coro. 2.0.6]).

Corollary 2.14 Let \( \sigma \) be as above. The following are equivalent:

(i) \( \sigma \) satisfies \((\omega_1)\),
(ii) There exists \( \alpha > 0 \) such that \( \sigma \) satisfies every/some of the equivalent conditions \((i)-(xii)\) in Theorem 2.11.

In particular, \( \sigma \) satisfies \((\omega_1)\) if and only if \( \gamma(\sigma) > 0 \) if and only if \( \alpha(\sigma) < \infty \).

Proof It is immediate to check that, if \( \sigma \) satisfies \((\omega_1)\), then it exists \( \alpha > 0 \) such that Theorem 2.11.(iv) holds for \( K = 2 \). If Theorem 2.11.(iv) holds for some \( K \geq 2 \), then \( \sigma \) satisfies \((\omega_1)\) by monotonicity. If Theorem 2.11.(iv) holds for some \( 1 < K < 2 \), we fix \( n \in \mathbb{N} \) such that \( 2 \leq K^n \), by monotonicity, we observe that

\[
\limsup_{t \to \infty} \frac{\sigma(2t)}{\sigma(t)} \leq \limsup_{t \to \infty} \frac{\sigma(K^n t)}{\sigma(t)} \leq \limsup_{t \to \infty} \frac{\sigma(K^n t)}{\sigma(K^{n-1} t)} \cdots \frac{\sigma(K t)}{\sigma(t)} < K^{n \alpha}.
\]

Hence \( \sigma \) satisfies \((\omega_1)\). \( \square \)

Since \( \sigma \) is nondecreasing \( \beta(\sigma) \geq 0 \) and so, according to Theorem 2.3, \( \sigma \in ORV \) if and only if \( \alpha(\sigma) < \infty \) if and only if \( \gamma(\sigma) > 0 \) if and only if \( \sigma \) satisfies \((\omega_1)\).

Remark 2.15 Conditions \((\omega_2)\), \((\omega_5)\) and \((\omega_{nq})\) are instead connected to the order \( \rho(\sigma) \). For \( \sigma \) as above, thanks to the relation between \( \alpha(\sigma) \) and \( \rho(\sigma) \) (see Remark 2.5) we see that each assertion implies the following:

(i) \( \alpha(\sigma) < 1 \),
(ii) \( \rho(\sigma) < 1 \),
(iii) there exists \( \alpha \in (0, 1) \) such that \( \omega(t) = O(t^\alpha) \) as \( t \to \infty \),
(iv) \( \sigma \) satisfies \((\omega_{nq})\),
(v) \( \sigma \) satisfies \((\omega_5)\),
(vi) \( \sigma \) satisfies \((\omega_2)\),
(vii) \( \rho(\sigma) \leq 1 \),

and only the implication (ii) \( \Rightarrow \) (iii) can be reversed. Hence if \( \sigma \) satisfies \((\omega_{snq})\), applying Corollary 2.13, we see that \( \sigma \) satisfies the conditions (i)–(vii) and, by Corollary 2.14, \( \sigma \) has also \((\omega_1)\). Part of this information was well-known but dispersed, see [30, Coro. 1.4] and the main novelty is its connection to \( O \)-regular variation.

Finally, the growth condition \((\omega_6)\) is associated with the lower Matuszewska index \( \beta(\sigma) \) and it is possible to establish a result analogous to Theorem 2.11.

Theorem 2.16 Let \( \sigma \) be as above and \( \beta \geq 0 \). We take \( a \geq 0 \) such that \( \sigma(x) > 0 \) for all \( x \geq a \) and \( \sigma(x) = 0 \) for every \( x \leq [a] - 1 \). The following are equivalent:

\[ \square \] Springer
(i) there exists $C > 0$ such that $\int_1^y \frac{\sigma(t)}{t^{\beta}} \, dt \leq C \sigma(y) \frac{\sigma(y)}{y^{\beta}}$ for all $y \geq 1$,
(ii) $\lim_{k \to \infty} \liminf_{t, \to \infty} \frac{\sigma(k t)}{k^{\beta} \sigma(t)} = \infty$,
(iii) there exists $K > 1$ such that $\lim_{t \to \infty} \frac{\sigma(K t)}{\sigma(t)} > K^\beta$,
(iv) $\beta(\sigma) > \beta$ (with the convention in Remark 2.9),
(v) there exists $y > \beta$ such that $\frac{\sigma(t)}{t^{\beta}}$ is almost increasing in $[a, \infty)$ if $a > 0$ and in $[e, \infty)$ for all $\varepsilon > 0$ if $a = 0$,
(vi) there exists $C > 0$ such that $\frac{1}{y^{\beta-1}} \int_{y}^{\infty} \frac{\sigma(t)}{\sigma(y)} \, dt \leq \frac{C}{\sigma(y)}$ for all $y \geq a$ if $a > 0$ and for all $y \geq \varepsilon$ if $a = 0$ where $\varepsilon > 0$ is arbitrary but fixed and $C$ depends on $\varepsilon$,
(vii) there exists $C > 0$ such that $\sum_{k=p}^{\infty} \frac{k^{\beta-1}}{\sigma(k)} \leq C \sigma(p)$ for every $p \in \mathbb{N}$ with $p \geq a$,
(viii) for every $\theta \in (0, 1)$ there exists $k \in \mathbb{N}$, $k \geq 2$, such that $\sigma(p) \leq \theta k^{-\beta} \sigma(kp)$ for every $p \in \mathbb{N}$,
(ix) there exists $k \in \mathbb{N}$, $k \geq 2$, such that $\liminf_{p \to \infty} \frac{\sigma(k p)}{\sigma(p)} > k^\beta$,
(x) there exists $C > 0$ such that $\sum_{k=1}^{p} \frac{\sigma(k)}{k^{1+\beta}} \leq C \frac{\sigma(p)}{p^{\beta}}$ for every $p \in \mathbb{N}$.

**Proof** (i) $\Rightarrow$ (ii) First, we assume that $\beta = 0$. By (i) and the monotonicity of $\sigma$ for all $y \geq 1$ and every $k > 1$ we see that
\[
C \sigma(ky) \geq \int_1^{ky} \frac{\sigma(t)}{t^{\beta}} \, dt \geq \int_y^{ky} \frac{\sigma(t)}{t} \, dt \geq \sigma(y) \log(k),
\]
Then (ii) holds for $\beta = 0$. Secondly, if $\beta > 0$, applying (ii) twice and using the monotonicity of $\sigma$ for all $y \geq 1$ and every $k > 1$ we observe that
\[
C^2 \frac{\sigma(ky)}{(ky)^{\beta}} \geq \int_y^{ky} C \frac{\sigma(t)}{t^{\beta}} \, dt \geq \int_y^{ky} \int_y^t \frac{\sigma(u)}{u^{\beta}} \, du \frac{dt}{u} \geq \sigma(y) \int_y^{ky} \left[ \frac{1}{\beta y^{\beta}} - \frac{1}{\beta t^{\beta}} \right] \, dt \geq \sigma(y) \frac{\log(k) - 1 + k^{-\beta}}{\beta y^{\beta}}.
\]
Since $\lim_{k \to \infty} \left[ \beta \log(k) - 1 + k^{-\beta} \right] = \infty$, (ii) holds.
(ii) $\Rightarrow$ (iii) Immediate.
(iii) $\Rightarrow$ (iv) There exists $K > 1$ such that $\sigma_{low}(K) > K^\beta$. Since $\sigma$ is nondecreasing one may apply Theorem 2.7 and deduce that (iv) holds.
(iv) $\Rightarrow$ (v) Immediate by Theorem 2.4.
(v) $\Rightarrow$ (vi) Analogous to (vii) $\Rightarrow$ (viii) in Theorem 2.11.
(vii) $\Rightarrow$ (vii) If $\beta > 0$, it is analogous to (viii) $\Rightarrow$ (ix) in Theorem 2.11. If $\beta = 0$, we use that for every $k \geq a$
\[
\int_k^{k+1} \frac{1}{\sigma(t)} \, dt \geq \frac{1}{(k+1)\sigma(k+1)},
\]
and we conclude as for $\beta > 0$.
(vii) $\Rightarrow$ (viii) Applying condition (vii) twice and using the monotonicity of $\sigma$, for all $p \in \mathbb{N}$ with $p \geq a$ and every $m \geq 2$ we see that
\[
C^2 \frac{p^{\beta}}{\sigma(p)} \geq \sum_{k=p}^{\infty} \frac{\sigma(k)^{\beta-1}}{\sigma(k)} \geq \sum_{k=p}^{\infty} \frac{1}{k} \sum_{\ell=k}^{\infty} \frac{\ell^{\beta-1}}{\ell^{\beta}} \geq \sum_{k=p}^{\infty} \frac{1}{k} \sum_{\ell=k}^{mp} \frac{\ell^{\beta-1}}{\ell^{\beta}} \geq \sum_{k=p}^{\infty} \frac{1}{k} \sum_{\ell=k}^{mp} \frac{1}{k^{\beta}(mp)^{\beta}},
\]
and
\[
\sum_{k=p}^{\infty} \frac{1}{k} \sum_{\ell=k}^{mp} \frac{1}{k^{\beta}(mp)^{\beta}} \geq \frac{1}{mp^{\beta}(mp)^{\beta}} \sum_{k=p}^{mp} \frac{1}{k^{\beta+1}}.
\]
Then, given $\theta \in (0, 1)$, we take $m$ large enough such that

\[
\frac{\sigma (m)p}{m^\theta \sigma (p)} \geq \frac{1}{C^2 (\beta + 1)^2} \left[ \log (m^{\beta + 1}) - 1 + \frac{1}{m^{\beta + 1}} \right] \geq \frac{1}{\theta},
\]

for all $p \in \mathbb{N}$ with $p \geq a$. For $1 \leq p < a$, $\sigma (p) = 0$ and (viii) trivially holds.

(vii) $\Rightarrow$ (ix) Immediate.

(ix) $\Rightarrow$ (x) First, we prove that there exists $\epsilon \in (0, 1)$ such that the sequence $(\sigma (p)/p^{\beta + \epsilon})_{p=1}^\infty$ is almost increasing. By (ix), there exists $\epsilon \in (0, 1)$ and $p_0 \in \mathbb{N}$ such that for every $p \geq p_0$, $\sigma (k p) > \sigma (p) k^{\beta + \epsilon}$. Using the monotonicity of $\sigma$, we see that for every $p, q \in \mathbb{N}$ with $q \geq p \geq p_0$ there exists $s \in \mathbb{N}_0$ with $k^s p < q < k^{s+1} p$ and we observe that

\[
\frac{\sigma (q)}{q^{\beta + \epsilon}} \geq \frac{\sigma (k^s p)}{(k^{s+1} p)^{\beta + \epsilon}} = \frac{(k^{\beta + \epsilon})^s \sigma (p)}{(k^{s+1} p)^{\beta + \epsilon}} \geq \frac{1}{k^{\beta + \epsilon}} \sigma (p).
\]

By the monotonicity of $\sigma$, we conclude that $(\sigma (p)/p^{\beta + \epsilon})_{p=1}^\infty$ is almost increasing. We use this property to show that for all $p \in \mathbb{N}$

\[
\sum_{k=1}^{p} \frac{\sigma (k)}{k^{\beta + 1}} \leq C \frac{\sigma (p)}{p^{\beta + \epsilon}} \sum_{k=1}^{p} \frac{1}{k^{1-\epsilon}} \leq C \frac{\sigma (p)}{p^{\beta + \epsilon}} \int_0^p \frac{dt}{t^{1-\epsilon}} = \frac{C \sigma (p)}{\epsilon} p^{\beta - \epsilon}.
\]

(x) $\Rightarrow$ (i) By the monotonicity of $\sigma$, for $y \geq 1$ we see that

\[
\int_1^y \frac{\sigma (t)}{t^{1+\beta}} dt \leq \sum_{k=1}^{\lfloor y \rfloor - 1} \sigma (k+1) \int_k^{k+1} \frac{dt}{t^{1+\beta}} + \sigma (y) \int_{\lfloor y \rfloor}^y \frac{dt}{t^{1+\beta}} \leq \sum_{k=1}^{\lfloor y \rfloor - 1} \frac{\sigma (k+1)}{k^{1+\beta}} + \frac{\sigma (y)}{(\lfloor y \rfloor)^{1+\beta}},
\]

where the sum does not appear if $\lfloor y \rfloor = 1$. Using that $(k+1)/k \leq 2$ and $y/\lfloor y \rfloor \leq 2$ for all $y, k \geq 1$ and (x) we observe that

\[
\int_1^y \frac{\sigma (t)}{t^{1+\beta}} dt \leq 2^{1+\beta} \sum_{k=1}^{\lfloor y \rfloor - 1} \frac{\sigma (k+1)}{(k+1)^{1+\beta}} + 2^{\beta} \frac{\sigma (y)}{y^\beta} \leq 2^{1+\beta} C \frac{\sigma (\lfloor y \rfloor)}{(\lfloor y \rfloor)^{1+\beta}} + 2^{\beta} \frac{\sigma (y)}{y^\beta} \leq (2^{1+2\beta} C + 2^\beta) \frac{\sigma (y)}{y^\beta}.
\]

The index $\gamma (\sigma)$ naturally appears in the study of the surjectivity of the Borel map in ultraholomorphic classes, see [21]. According to the last result, one might analogously define an index

\[
\overline{\gamma} (\sigma) := \inf \{ \gamma > 0 : \exists A > 1 : \lim_{t \to \infty} \frac{\sigma (A^{\gamma} t)}{\sigma (t)} > A \}.
\]

(7)

We have refrained from providing its detailed study, similar to that of $\gamma (\sigma)$, but we can mention that $1/\overline{\gamma} (\sigma) = \beta (\sigma)$, and that $\sigma$ has $(\omega_k)$ if and only if $\overline{\gamma} (\sigma)$ is finite. If $\beta > 0$, $\sigma$ is as above and in addition $\alpha (\sigma) < \infty$, then one can show that $\beta (\sigma) > \beta > 0$ if and only if there exists a nondecreasing function $\kappa : [1, +\infty) \to [0, +\infty)$ such that $\sigma \sim_k \kappa$. 

\begin{align*}
\overline{\gamma} (\sigma) &= \inf \{ \gamma > 0 : \exists A > 1 : \lim_{t \to \infty} \frac{\sigma (A^{\gamma} t)}{\sigma (t)} > A \}. \tag{7}
\end{align*}
satisfies Theorem 2.16.(i) and $\kappa(t^{1/\beta})$ is convex, recovering the missing equivalent condition which is available for $\alpha(\sigma)$ but is not for $\beta(\sigma)$ in general. Finally, the considerations made in Remark 2.12 for $\alpha(\sigma)$ are also valid for $\beta(\sigma)$.

Using that $\lim_{t \to \infty} \sigma(t) = \infty$, with a proof similar to the one of Corollary 2.14, we see that Theorem 2.16.(iii) is satisfied for $\beta = 0$ if and only if $\sigma$ satisfies $(\omega_6)$ and we obtain the desired characterization of this growth property.

**Corollary 2.17** Let $\sigma$ be as above. The following are equivalent:

(i) $\sigma$ satisfies $(\omega_6)$,

(ii) $\sigma$ satisfies every/some of the equivalent conditions (i)–(x) in Theorem 2.16 for $\beta = 0$.

In particular, $\sigma$ satisfies $(\omega_6)$ if and only if $\beta(\sigma) > 0$.

**Remark 2.18** According to Remark 2.5, it is worthy to notice that, if $\sigma$ is of regular variation of index $\omega$, which happens for most of the examples appearing in the applications, then all the information provided by the previous results is concentrated in one value since $\beta(\sigma) = \mu(\sigma) = \rho(\sigma) = \omega \in [0, \infty)$. In this case, $\sigma$ always satisfies $(\omega_1)$, and $\omega < 1$ if and only if $\sigma$ satisfies $(\omega_{snq})$. If $\omega = 0$, $\sigma$ is said to be of slow variation and we see that $\sigma$, regularly varying, does not satisfy $(\omega_6)$ if and only if it is of slow variation.

**Remark 2.19** The results presented in this subsection are prepared to be applied to weight functions whose mass is concentrated at $\infty$. However, in the literature of weighted spaces it is common to find a function $h : (0, \infty) \to (0, \infty)$ nonincreasing with $\lim_{t \to 0} h(t) = \infty$ or a function $H : (0, \infty) \to (0, \infty)$ nondecreasing with $\lim_{t \to 0} H(t) = 0$ as weight functions for the corresponding structure. Since in that context similar conditions appear for such functions, see [5, (2.1) and (2.2)] or the definition of regular modulus of continuity in [12], one might be tempted to obtain an analogous version of Theorems 2.11 and 2.16 which could be done by defining $\sigma(t) = h(1/t)$ in the first case and $\sigma(t) = 1/H(1/t)$ in the second one.

In the following examples we can compute the indices and deduce the corresponding properties for $\sigma$ according to the previous corollaries:

(i) The Gevrey weights $\omega(t) = t^s, 0 < s \leq 1$, are regularly varying of index $\rho = \alpha(\omega) = \rho(\omega) = \mu(\omega) = \beta(\omega) = s$ and $\gamma(\omega) = 1/s$.

(ii) The weights $\omega(t) = t/(\log(e + t))^\alpha, \alpha \in \mathbb{R}$, are also regularly varying of index $\rho = \alpha(\omega) = \rho(\omega) = \mu(\omega) = \beta(\omega) = \gamma(\omega) = 1$.

(iii) The weights $\omega(t) = \max(0, \log(t)^s), s > 1$, are regularly varying, in fact they are slowly varying so $\rho = \alpha(\omega) = \rho(\omega) = \mu(\omega) = \beta(\omega) = 0$ and $\gamma(\omega) = \infty$.

Moreover, for any $\sigma$ as above and any $s > 0$, by Remark 2.6, we observe that $\alpha(\sigma) < 1/s$ (or resp. $\beta(\sigma) > 1/s$) if and only if some/every of the conditions in Theorem 2.11 (or resp. some/every of the conditions in Theorem 2.16) holds for $\sigma(t) = \sigma(t^s)$ and $\alpha = 1$ (or resp. $\beta = 1$). Hence the list of equivalent definitions of the indices $\alpha$ and $\beta$ might be increased. In particular, we see that $\sigma$ satisfies $(\omega_1)$ (or respectively $(\omega_6)$) if and only if $\sigma_s$ satisfies $(\omega_1)$ (or resp. $(\omega_6)$) and we also see that $\alpha(\sigma) < 1/s$ if and only if $\sigma_s$ satisfies $(\omega_{snq})$. The same is valid if $\sigma_s$ is replaced by $\sigma^s(t) = (\sigma(t))^s$.

It is worthy to notice that conditions $(\omega_3)$ and $(\omega_4)$ are related to the Legendre-Fenchel-Young-conjugate $\varphi^*_\beta(x) := \sup\{xy - \sigma(e^x) : y \geq 0\}$, defined for $x \geq 0$ and they seem not to be connected to O-regular variation.

Finally, for condition $(\omega_7)$, introduced and described in [38, Lemmas 3.6.1, 5.4.1], [34, Lemma 5.9 (5.12)] and [19, Appendix A] and for $\sigma$ as above satisfying this condition, we
can show that $\alpha(\sigma) = 0$ as follows: if $\sigma$ satisfies $(\omega_7)$ there exist $H, C, t_0 \geq 0$ such that for all $t \geq t_0$, $\sigma(t^2) \leq C \sigma(Ht)$, so for every $\lambda \in (1, \infty)$ there exists $t_\lambda \geq Ht_0$ such that for all $t \geq t_\lambda$, $\sigma(\lambda t) \leq \sigma(t^2/H^2) \leq C \sigma(t)$. Hence $\sigma^{up}(\lambda) \leq C$ for every $\lambda \in (1, \infty)$. This implies the desired property by Theorem 2.7, so $\beta(\sigma) = \mu(\sigma) = \rho(\sigma) = \alpha(\sigma) = 0$ and we recover the well-known incompatibility between $(\omega_7)$ and $(\omega_6)$. Note that $(\omega_7)$ is just a sufficient condition for having $\gamma(\sigma) = +\infty$.

2.5 Legendre conjugates and the index $\gamma$

As it was pointed out in Remark 2.6, for any $\sigma$ we observe that

$$\beta(\sigma(t)t) = \beta(\sigma(t)) + 1, \quad \beta(\sigma(t)) = \beta(\sigma(t)/t) + 1,$$

and the same holds for $\alpha$. Motivated by the relation between weight sequences and weight functions described in Sect. 4 and by its necessity for the applications, see [21], we want to obtain some similar relation for the index $\gamma$, that is, we look for functions $\kappa_1, \kappa_2$ such that

$$\gamma(\sigma) = \gamma(\kappa_1) + 1, \quad \gamma(\kappa_2) = \gamma(\sigma) + 1.$$  \hspace{1cm} (8)

For this purpose we consider the Legendre conjugates. Let $\sigma : [0, \infty) \rightarrow [0, \infty)$ be nondecreasing with $\lim_{t \rightarrow \infty} \sigma(t) = \infty$, then for any $s \geq 0$ we define the so-called upper Legendre conjugate (or upper Legendre envelope) of $\sigma$ by

$$\sigma^*(s) := \sup_{t \geq 0} \{\sigma(t) - st\}.$$

We summarize some basic properties, see [33, Remark 1.5] and [3, (8), p. 156]. By definition, $\sigma^*(0) = \infty$. If $\sigma$ has in addition $(\omega_5)$, then $\sigma^*(s) < \infty$ for all $s > 0$. In this case, the function $\sigma^* : (0, \infty) \rightarrow [0, \infty)$ is nonincreasing, convex and continuous with $\lim_{s \rightarrow 0} \sigma^*(s) = \infty$ and $\lim_{s \rightarrow \infty} \sigma^*(s) = \lim_{t \rightarrow 0} \sigma(t)$.

On the other hand for any $h : (0, \infty) \rightarrow [0, \infty)$ which is nonincreasing and $\lim_{t \rightarrow 0} h(t) = \infty$ for $t \geq 0$ we can define the so-called lower Legendre conjugate (or envelope) by

$$h_* (t) := \inf_{s > 0} \{h(s) + ts\}.$$

We observe that $h_*$ is nondecreasing, concave and continuous with $\lim_{t \rightarrow \infty} h_*(t) = \infty$ and $\lim_{t \rightarrow 0} h_*(t) = \lim_{s \rightarrow \infty} h(s)$.

**Remark 2.20** Let $\sigma : [0, \infty) \rightarrow [0, \infty)$ be nondecreasing with $\lim_{t \rightarrow \infty} \sigma(t) = \infty$ satisfying $(\omega_5)$ and $h : (0, \infty) \rightarrow [0, \infty)$ be nonincreasing with $\lim_{t \rightarrow 0} h(t) = \infty$. We observe that

(i) The function $(\sigma^*)_* : (0, \infty) \rightarrow (0, \infty)$ is nondecreasing, concave and continuous with $\lim_{t \rightarrow \infty} (\sigma^*)_*(t) = \infty$ and it is indeed the least concave majorant of $\sigma$ (in the sense that, if $\varepsilon : [0, +\infty) \rightarrow [0, +\infty)$ is concave and $\sigma \leq \varepsilon$, then $(\sigma^*)_* \leq \varepsilon$).

(ii) The function $(h_*)^* : (0, \infty) \rightarrow (0, \infty)$ is well-defined since a direct computation leads to $(h_*)^*(u) \leq h(u)$ for all $u > 0$, so $h_*$ does not need to satisfy $(\omega_5)$. Moreover, $(h_*)^*$ is nonincreasing, convex and continuous with $\lim_{t \rightarrow 0} h_*(t) = \infty$, it is indeed the largest convex minorant of $h$ (in the sense that, if $k : [0, +\infty) \rightarrow [0, +\infty)$ is convex and $h \geq k$, then $(h_*)^* \geq k$).

When we consider the upper and lower Legendre conjugates the information is transferred from 0 to $\infty$ and vice versa, so for any positive function $f$ defined in an interval $I \subseteq (0, \infty)$
it is helpful to introduce the function \( f^i(t) := f(1/t) \) in the corresponding subinterval of \((0, \infty)\). The first result compares the indices of \( \sigma \) and \((\sigma^*)^i\).

**Proposition 2.21** Let \( \sigma : [0, \infty) \rightarrow [0, \infty) \) be nondecreasing with \( \lim_{t \to \infty} \sigma(t) = \infty \). Assume that \( \sigma \) satisfies \((\omega_5)\) and that \( \sigma \) is equivalent to its least concave majorant, i.e., \( \sigma \sim (\sigma^*)_* \). Then

\[
\gamma(\sigma) = \gamma(((\sigma^*)^i)^i) + 1. \tag{9}
\]

**Proof** We fix \( \gamma < \gamma(\sigma) \), so there exists \( K, H > 1 \) and \( \epsilon \in (0, 1) \) such that for all \( t \geq 0 \)

\[
\sigma(K^\gamma t) \leq K^{1-\epsilon} \sigma(t) + H.
\]

Using that \((\sigma^*)^i\) is nondecreasing, for all \( s > 0 \) we see that

\[
((\sigma^*)^i)(K^\gamma^{-1} s) = \sup_{t \geq 0} \left\{ \sigma(t) - \frac{t}{K^\gamma^{-1} s} \right\} = \sup_{u \geq 0} \left\{ \sigma(K^\gamma u) - \frac{K u}{s} \right\}
\]

\[
\leq \sup_{u \geq 0} \left\{ K^{1-\epsilon} \sigma(u) - \frac{K u}{s} \right\} + H
\]

\[
\leq K^{1-\epsilon}(\sigma^*)^i(K^{-\epsilon} s) + H \leq K^{1-\epsilon}(\sigma^*)^i(s) + H.
\]

Since \( \lim_{s \to \infty} (\sigma^*)^i(s) = \infty \), we deduce that \( \gamma - 1 < \gamma(((\sigma^*)^i)^i) \).

Conversely, we fix \( \gamma < \gamma(((\sigma^*)^i)^i) \), so there exists \( K > 1 \) and \( \epsilon_0 \in (0, 1) \) such that for all \( \epsilon \in (0, \epsilon_0) \) there exists \( H_\epsilon > 0 \) such that for all \( t > 0 \)

\[
((\sigma^*)^i)(K^\gamma t) \leq K^{1-\epsilon}(\sigma^*)^i(t) + H_\epsilon.
\]

Hence, for all \( t > 0 \) we observe that

\[
((\sigma^*)^i)_*(K^{\gamma+1-\epsilon} t) \leq K^{1-\epsilon}(\sigma^*)_*(t) + H_\epsilon.
\]

Then \( \gamma + 1 - \epsilon < \gamma(((\sigma^*)^i)_*) \) and, since this holds for all \( \epsilon \in (0, \epsilon_0) \), we deduce that \( \gamma + 1 \leq \gamma(((\sigma^*)^i)_*) \). Since the value of the index \( \gamma \) is stable for \( \sim \) we conclude that \( \gamma + 1 \leq \gamma(\sigma) \). \( \square \)

Hence we have found a candidate for \( \kappa_1 \) in (8), we will show that under suitable assumptions \( \kappa_2 \) can be chosen as \((\sigma^*)_\ast\).

**Proposition 2.22** Let \( \sigma \) be as above. Assume that \( \sigma^i \) is equivalent to its largest convex minorant, i.e., \( \sigma^i \sim ((\sigma^i)_\ast)_\ast \). Then

\[
\gamma(\sigma) + 1 = \gamma(((\sigma^i)_\ast)_\ast). \tag{10}
\]

**Proof** We observe that \((\sigma^i)_\ast : [0, \infty) \rightarrow [0, \infty) \) is nondecreasing with \( \lim_{t \to \infty} (\sigma^i)_\ast(t) = \infty \), concave and continuous. Then \(((\sigma^i)_\ast)_\ast \ast \) is well-defined and has the properties described in Remark 2.20. We can follow the proof of Proposition 2.21 and show that

\[
\gamma(((\sigma^i)_\ast)_\ast) = \gamma(((\sigma^i)_\ast)_\ast^i) + 1.
\]

Since \( \sigma^i \sim ((\sigma^i)_\ast)_\ast \), then \( \sigma \sim (((\sigma^i)_\ast)_\ast^i) \) and, by the stability of the index \( \gamma \) for \( \sim \), we conclude that (10) is valid. \( \square \)

If \( \sigma \sim \kappa \) with \( \kappa \) concave or if \( \sigma^i \sim \tau \) with \( \tau \) convex, then \( \sigma \) is equivalent to its least concave majorant or \( \sigma^i \) is equivalent to its largest convex minorant, respectively, and Propositions 2.21
and 2.22 are also valid. Nevertheless, it will helpful to see if these properties are satisfied under some standard assumption. For \( f : (0, \infty) \to (0, \infty) \) Peetre [31] shows that

\[
f(s) \leq C \max \left(1, \frac{s}{t} \right) f(t)
\]

(11)
is sufficient for \( f \) to satisfy \((2C)^{-1} F(x) \leq f(x) \leq F(x)\) for all \( x > 0 \) where \( F \) is its least concave majorant. Note that condition (11) holds if and only if \( f \) is almost increasing in \((0, \infty)\) and \( f(t)/t \) is almost decreasing in \((0, \infty)\). Since we want to allow the function \( f \) to take the value 0, a suitable modification of (11) is needed which entails the appearance of an additional summand which does not destroy the equivalence relation.

**Proposition 2.23** Let \( f : [0, \infty) \to [0, \infty) \) be such that there exists \( C \geq 1 \) with

\[
f(s) \leq Cf(t) \quad \text{and} \quad f(t) \frac{s}{t} \leq C(f(s) + 1),
\]

(12)
for all \( t > s \geq 0 \). Then there exists \( A \geq 1 \) such that for every \( x \in (0, \infty) \)

\[
AF(x) - A \leq f(x) \leq F(x),
\]

(13)
where \( F \) is the least concave majorant of \( f \).

**Proof** The least concave majorant of \( f \) can be represented by

\[
F(x) = \sup \{ \lambda_1 f(x_1) + \lambda_2 f(x_2) ; \lambda_1 + \lambda_2 = 1, \; \lambda_1 x_1 + \lambda_2 x_2 = x, \; \lambda_i \geq 0 \}
\]

and \( f(x) \leq F(x) \) for all \( x \in [0, \infty) \). Since \( F(0) = f(0) \), we can assume, without loss of generality, that \( 0 \leq x_1 < x \) and that \( x_2 > x \). In this situation, using (12), we see that

\[
\lambda_1 f(x_1) + \lambda_2 f(x_2) \leq C \lambda_1 f(x) + C \lambda_2 \frac{x_2}{x} (f(x) + 1) \leq 2 Cf(x) + C,
\]

because \( \lambda_1 \leq 1 \) and \( \lambda_2 x_2 \leq x \). Hence (13) holds.

In the case of the largest convex minorant, which was not considered by Peetre, a similar result can be obtained inspired by the previous one with a slightly different proof.

**Proposition 2.24** Let \( h : (0, \infty) \to (0, \infty) \) be such that there exists \( C \geq 1 \) and \( \beta > 0 \) such that

\[
h(s) + C \geq \frac{1}{C} \min \left(1, \frac{t^\beta}{s^\beta} \right) h(t),
\]

(14)
for all \( t, s \in (0, \infty) \). Then there exists \( A \geq 1 \) such that for every \( x \in (0, \infty) \)

\[
H(x) \leq h(x) \leq AH(x) + A,
\]

(15)
where \( H \) is the largest convex minorant of \( h \).

**Proof** The largest convex minorant of \( h \) can be represented by

\[
H(x) = \inf \{ \lambda_1 h(x_1) + \lambda_2 h(x_2) ; \lambda_1 + \lambda_2 = 1, \; \lambda_1 x_1 + \lambda_2 x_2 = x, \; \lambda_i \geq 0 \}
\]

and \( H(x) \leq h(x) \) for all \( x \in (0, \infty) \). We fix \( x \in (0, \infty) \), for every \( x_1, x_2 \in (0, \infty) \), \( \lambda_1, \lambda_2 \in [0, \infty) \) with \( \lambda_1 + \lambda_2 = 1 \) and \( \lambda_1 x_1 + \lambda_2 x_2 = x \) by (14) we see that

\[
\lambda_1 h(x_1) + \lambda_2 h(x_2) \geq \frac{\lambda_1}{C} \min \left(1, \frac{x^\beta}{x_1^\beta} \right) h(x) + \frac{\lambda_2}{C} \min \left(1, \frac{x^\beta}{x_2^\beta} \right) h(x) - C.
\]
If \( x_1 = x_2 = x \), then \( \lambda_1 h(x_1) + \lambda_2 h(x_2) = h(x) \). Otherwise, we can assume, without loss of generality, that \( x_1 < x \) and that \( x_2 > x \). In this situation,

\[
\lambda_1 h(x_1) + \lambda_2 h(x_2) \geq \frac{h(x)}{C} \left( \lambda_1 + \lambda_2 \frac{x^\beta}{x_2^\beta} \right) - C = \frac{h(x)}{C} \left( 1 - \lambda_2 \left( 1 - \frac{x^\beta}{x_2^\beta} \right) \right) - C.
\]

Using that \( \lambda_2 x_2 \leq x \) and that \( x_2 > x \), we observe that

\[
\lambda_2 \left( 1 - \frac{x^\beta}{x_2^\beta} \right) \leq \frac{x}{x_2} \left( 1 - \frac{x^\beta}{x_2^\beta} \right) \leq \sup_{u \in [0,1]} u(1 - u^\beta) = \frac{1}{(1 + \beta)(\beta + 1)/\beta} = K_\beta < 1.
\]

Therefore \( \lambda_1 h(x_1) + \lambda_2 h(x_2) \geq h(x)(1 - K_\beta)C^{-1} - C \) and we conclude that (15) holds. \( \square \)

Thanks to the connection between the \( \gamma \) index, the upper Matuszewska index and the almost monotonicity properties, it is possible to ensure that (9) and (10) are valid under some standard assumptions.

**Corollary 2.25** Let \( \sigma \) be as above. If \( \gamma(\sigma) > 1 \), then (9) holds.

**Proof** If \( \gamma(\sigma) > 1 \), then \( \alpha(\sigma) < 1 \) so \( \sigma(t)/t \) is almost decreasing in \([a, \infty)\) with \( a > 0 \) such that \( \sigma(t) > 0 \) in \([a, \infty)\). Since \( \sigma(t) \leq \alpha(a) \) for all \( t \in [0, a] \) and, by the monotonicity of \( \sigma \), we see that (12) holds. Hence, by Proposition 2.23, \( \sigma \) is equivalent to its least concave majorant and by Remark 2.15 it satisfies \((\omega_5)\), then we conclude applying Proposition 2.21. \( \square \)

**Corollary 2.26** Let \( \sigma \) be as above. If \( \gamma(\sigma) > 0 \), then (10) holds.

**Proof** By Lemma 2.10, \( \alpha(\sigma) < \infty \), so by Theorem 2.4 there exists \( a > 0 \) and \( \alpha > 0 \) such that \( \sigma(t)^r \sim a \) is almost decreasing in \([a, \infty)\). Consequently, \( \sigma^t(t)^a \) is almost increasing in \((0, 1/a]\). Since \( \sigma^t \) is bounded in \([1/a, \infty) \) by \( \sigma^t(1/a) \) and \( \sigma^t \) is nonincreasing we deduce that (14) is valid and we conclude by Propositions 2.24 and 2.22. \( \square \)

**Remark 2.27** Regarding the index \( \overline{\gamma}(\sigma) \), see (7), and following similar ideas as those in Propositions 2.21 and 2.22, we can see that:

(i) Let \( \sigma \) be as above. If \( \sigma \) satisfies \((\omega_5)\) and it is equivalent to its least concave majorant, then

\[
\overline{\gamma}(\sigma) = \overline{\gamma}(\sigma^t)^t + 1.
\]

Consequently \( \sigma \) satisfies \((\omega_6)\), i.e. \( \overline{\gamma}(\sigma) < +\infty \), if and only if \( (\sigma^t)^t \) has \((\omega_6)\).

(ii) Let \( \sigma \) be as above. If \( \sigma^t \) is equivalent to its largest convex minorant, then

\[
\overline{\gamma}(\sigma) + 1 = \overline{\gamma}(\sigma^t)^t.
\]

Hence, \( \sigma \) satisfies \((\omega_6)\) if and only if \((\sigma^t)^t \), does so.

**Remark 2.28** The relations (9) and (10) for the index \( \gamma \) can be rewritten in a different form that recalls the classical relation for conjugate indices appearing in the study of the convex conjugates. For this purpose, one needs to consider the notion of \( O \)-regular variation at 0. We say that a measurable function \( h : (0, a) \to (0, \infty) \) with \( a > 0 \) is \( O \)-regularly varying at 0, if \( h^t : (1/a, \infty) \to (0, \infty) \) is \( O \)-regularly varying. This notion has already appeared in different
works dealing with O-regular variation and Orlicz spaces, see [27]. We are interested in the following quantities:

\[\alpha_0(h) := \inf \{ \alpha \in \mathbb{R} : x \mapsto h(x)^{\frac{1}{x^\alpha}} \text{ is almost decreasing}\},\]

\[\beta_0(h) := \sup \{ \beta \in \mathbb{R} : x \mapsto h(x)^{\frac{1}{x^{\beta}}} \text{ is almost increasing}\}.\]

By Theorem 2.4, one might check that \(\alpha_0(h) = -\beta(h')\) and \(\beta_0(h) = -\alpha(h')\). We put \(\alpha^\infty(\sigma) = \alpha(\sigma)\) and \(\beta^\infty(\sigma) = \beta(\sigma)\). Then (9) can be expressed as

\[\frac{1}{\alpha^\infty(\sigma)} + \frac{1}{\beta^0(\sigma^*)} = 1,\]

and (10) can also be written in terms of \(\alpha^\infty((\sigma^i)_*)\) and \(\beta^0(\sigma^i)\). Moreover, under suitable assumptions, similar relations can be obtained for \(\alpha_0(\sigma^*)\) and \(\beta^\infty(\sigma)\).

3 Weight sequences and O-regular variation

3.1 Weight sequences and growth indices

In what follows, \(\mathbb{M} = (M_p)_{p \in \mathbb{N}_0}\) always stands for a sequence of positive real numbers, and we always impose that \(M_0 = 1\), where \(\mathbb{N}_0 = \{0, 1, 2, \ldots\} = \mathbb{N} \cup \{0\}\). The names of the conditions given by Thilliez and, for the convenience of the reader, the corresponding descriptive acronyms employed by the third author [39] have been used. We say that:

(i) \(\mathbb{M}\) is logarithmically convex (for short, (lc)) if for all \(p \in \mathbb{N}\),

\[M_{2p} \leq M_{p-1} M_{p+1} - M_p.\]

(ii) \(\mathbb{M}\) is of or has moderate growth (briefly, (mg)) whenever there exists \(A > 0\) such that

\[M_{p+q} \leq A^{p+q} M_p M_q, \quad p, q \in \mathbb{N}_0.\]

(iii) \(\mathbb{M}\) satisfies the strong nonquasianalyticity condition (for short, (snq)) if there exists \(B > 0\) such that

\[\sum_{q=p}^{\infty} \frac{M_q}{(q+1) M_{q+1}} \leq B \frac{M_p}{M_{p+1}}, \quad p \in \mathbb{N}_0.\]

If \(\mathbb{M}\) is (lc) and \(\lim_{p \to \infty} (M_p)^{1/p} = \infty\) we say that \(\mathbb{M}\) is a weight sequence. According to Thilliez [41], if \(\mathbb{M}\) is (lc), has (mg) and satisfies (snq), we say that \(\mathbb{M}\) is a strongly regular sequence.

For a sequence \(\mathbb{M}\) we define the sequence of quotients \(\mathbf{m} = (m_p)_{p \in \mathbb{N}_0}\) by

\[m_p := \frac{M_{p+1}}{M_p}, \quad p \in \mathbb{N}_0.\]

We observe that for every \(p \in \mathbb{N}\) one has \(M_p = m_{p-1} m_{p-2} \cdots m_1 m_0\) and \(M_0 = 1\). Hence the knowledge of one of the sequences amounts to that of the other. Consequently one may work directly with \(\mathbf{m}\) in the next subsections and the sequences of quotients of sequences \(\mathbb{M}, \mathbb{L}, \ldots\) will be denoted by lowercase letters \(\mathbf{m}, \mathbf{l}\) and so on without misunderstanding. Moreover, (snq) can be easily stated in terms of the sequence of quotients and, as the next lemma shows, the same holds for (lc), (mg), and the notion of weight sequence. Statements (i) and (ii) are classical and deduced from a direct computation and (iii) was given by Petzsche and Vogt [33, Lemma 5.3].
Lemma 3.1  For every sequence $\mathbb{M}$ the following holds:

(i) $\mathbb{M}$ is logarithmically convex if and only if $m$ is nondecreasing.

(ii) If $\mathbb{M}$ is logarithmically convex, $\lim_{p \to \infty} (M_p)^{1/p} = \infty$ if and only if $\lim_{p \to \infty} m_p = \infty$. Hence $\mathbb{M}$ is a weight sequence if and only if $\mathbb{M}$ is nondecreasing and $\lim_{p \to \infty} m_p = \infty$.

(iii) If $\mathbb{M}$ is logarithmically convex, then the following are equivalent:

(iii.a) $\mathbb{M}$ has (mg),

(iii.b) $\sup_{p \geq 1} m_p / M_p^{1/p} < \infty$,

(iii.c) $\sup_{p \geq 1} m_{2p} / m_p < \infty$.

We say that $\mathbb{M}$ and $\mathbb{L}$ are equivalent and we write $\mathbb{M} \approx \mathbb{L}$ if there exists $C \geq 1$ such that for all $p \in \mathbb{N}_0$, $C^{-p} L_p \leq M_p \leq C^p L_p$. There is also an equivalence relation at the level of the sequence of quotients, we write $\mathbf{m} \subseteq \mathbf{e}$ if there exists $c \geq 1$ such that for all $p \in \mathbb{N}_0$, $c^{-1} \epsilon_p \leq m_p \leq c \epsilon_p$. If $\mathbf{m} \approx \mathbf{e}$, then $\mathbb{M} \approx \mathbb{L}$ and the converse fails in general. However, if $\mathbb{M}$ and $\mathbb{L}$ are (lc) and one of them has (mg), then $\mathbb{M} \approx \mathbb{L}$ implies $\mathbf{m} \approx \mathbf{e}$.

The growth index $\gamma(\mathbb{M})$ was defined and considered by Thilliez [41, Sect. 1.3] in the study of ultraholomorphic classes of functions. The original definition was given for strongly regular sequences and $\gamma > 0$, but one can consider it for any sequence $\mathbb{M}$ and $\gamma \in \mathbb{R}$. We say $\mathbb{M}$ satisfies property $(P_\gamma)$ if there exists a sequence of real numbers $\ell = (\ell_p)_{p \in \mathbb{N}_0}$ such that:

(i) $\mathbf{m} \approx \ell$, that is, there is a constant $a \geq 1$ such that $a^{-1} m_p \leq \ell_p \leq a m_p$, for all $p \in \mathbb{N}_0$,

(ii) $(m + 1)^{-\gamma} \ell_p$ is nondecreasing.

If $(P_\gamma)$ is satisfied, then $(P_{\gamma'})$ is satisfied for $\gamma' \leq \gamma$. It is natural to consider its growth index $\gamma(\mathbb{M})$ defined by

$$\gamma(\mathbb{M}) := \sup\{\gamma \in \mathbb{R} : (P_\gamma) \text{ is fulfilled}\}$$

with the conventions in Remark 2.2.

For the study of the injectivity of the asymptotic Borel map, see [37], the second author has defined the growth index $\omega(\mathbb{M})$ for any sequence $\mathbb{M}$ by

$$\omega(\mathbb{M}) := \liminf_{p \to \infty} \frac{\log m_p}{\log p}.$$  

By definition, the value of $\gamma(\mathbb{M})$ and of $\omega(\mathbb{M})$ is stable for $\approx$. For weight sequences with (mg), in particular for strongly regular sequences, these values are also stable for $\approx$, thanks to the equivalence between $\approx$ and $\simeq$. In Corollary 3.14 and Remark 4.7, we will eventually show the stability under $\approx$ for arbitrary weight sequences.

We mention some interesting examples. In particular, those in (i) and (iii) appear in the applications of summability theory to the study of formal power series solutions for different kinds of equations.

(i) The sequences $\mathbb{M}_{\alpha, \beta} := (p^\alpha \prod_{m=0}^p \log^\beta (e + m))_{p \in \mathbb{N}_0}$, where $\alpha > 0$ and $\beta \in \mathbb{R}$, are strongly regular (in case $\beta < 0$, the first terms of the sequence have to be suitably modified in order to ensure (lc)). In case $\beta = 0$, we have the best known example of a strongly regular sequence, $G_{\alpha} := (p^\alpha)_{p \in \mathbb{N}_0}$, called the Gevrey sequence of order $\alpha$. In this case, $\gamma(\mathbb{M}_{\alpha, 0}) = \omega(\mathbb{M}_{\alpha, 0}) = \alpha$.

(ii) The sequence $\mathbb{M}_{0, \beta} := (\prod_{m=0}^p \log^\beta (e + m))_{p \in \mathbb{N}_0}$, with $\beta > 0$, is (lc), (mg) and $\mathbf{m}$ tends to infinity, but (snq) is not satisfied and $\gamma(\mathbb{M}_{0, \beta}) = \omega(\mathbb{M}_{0, \beta}) = 0$.

(iii) For $q > 1$, $\mathbb{M}_q := (q^{p^2})_{p \in \mathbb{N}_0}$ is (lc) and (snq), but not (mg) and $\gamma(\mathbb{M}_q) = \omega(\mathbb{M}_q) = \infty$.  

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Most of the classical examples of strongly regular sequences satisfy that $\omega(M) = \gamma(M)$. Moreover, in the next section we will show that the values of the indices coincide for a large class of sequences, the ones whose sequence of quotients is regularly varying. However, it is possible to construct a strongly regular sequence for which the values are different, arbitrarily chosen, positive real numbers (see Remark 4.13).

### 3.2 O-regularly varying sequences

In 1973, Bojanić and Seneta [6] show that, under a suitable adaptation, one may consider regularly varying sequences satisfying similar properties to the ones of regularly varying functions. Even if all the results in this subsection, except the last one, were shown by Bojanić and Seneta, we refer to [4] for the proofs, as in the previous sections. This notion may be too restrictive, we need to consider O-regular variation for sequences, stated by Aljančić in 1981 and detailed by Djurčić and Božin [10] in 1997. Until the end of the section:

Let $a = (a_p)_{p \in \mathbb{N}}$ and $b = (b_p)_{p \in \mathbb{N}}$ are sequences of positive real numbers.

The sequence $a$ is said to be regularly varying if $\lim_{p \to \infty} a_{\lfloor \lambda p \rfloor} / a_p \in (0, \infty)$, for every $\lambda \in (0, \infty)$ and O-regularly varying if $\limsup_{p \to \infty} a_{\lfloor \lambda p \rfloor} / a_p < \infty$, for every $\lambda \in (0, \infty)$.

Regularly and O-regularly varying sequences are embeddable as regularly and, respectively, O-regularly varying step functions.

**Theorem 3.2** ([4], Th. 1.9.5, [10], Th. 1) Let $a$ be as above and $f_a(x) := a_{\lfloor x \rfloor}$ for $x \geq 1$. Then

(i) $a$ is regularly varying if and only if the function $f_a$ is regularly varying.

(ii) $a$ is O-regularly varying if and only if the function $f_a$ is O-regularly varying.

Hence, if $a$ is regularly varying, then it is O-regularly varying.

Regarding O-regular variation, from this result, we obtain the Uniform Convergence Theorem, see [10, Th. 2] and the Representation Theorem, specially interesting in the construction of pathological examples, as in Remark 4.13 and Sect. 5.

**Theorem 3.3** ([10], Th. 3, Representation Theorem for O-regularly varying sequences) Let $a$ be an O-regularly varying sequence. Then there exist bounded sequences of real numbers $(d_p)_{p \in \mathbb{N}}$ and $(\xi_p)_{p \in \mathbb{N}}$ such that

$$a_p = \exp \left( d_p + \sum_{j=1}^{p} \frac{\xi_j}{j} \right), \quad p \in \mathbb{N}.$$ 

Conversely, such a representation for a sequence $(a_p)_{p \in \mathbb{N}}$ implies that it is O-regularly varying.

To the best of our knowledge, the notions of orders and Matuszewska indices for sequences have not been considered. Only in the paper by Djurčić and Božin [10] one can find relevant information from which some of our statements concerning this topic can be inferred. In this subsection, a possible formalization of these concepts, based on Theorem 3.2, is proposed, providing a simple description, analyzing their behavior under elementary sequence transformations and showing some stability properties.

For $a$ as above, we define its upper Matuszewska index $\alpha(a)$, its lower Matuszewska index $\beta(a)$, its upper order $\rho(a)$ and its lower order $\mu(a)$ by
\[
\begin{align*}
\alpha(a) &:= \alpha(f_a), \quad \beta(a) := \beta(f_a), \quad \rho(a) := \rho(f_a), \quad \mu(a) := \mu(f_a), \\
\end{align*}
\]
where \( f_a(x) = a_{[x]} \) for all \( x \geq 1 \).

**Remark 3.4** By Remark 2.5, it immediately follows that \( \beta(a) \leq \mu(a) \leq \rho(a) \leq \alpha(a) \) and, using Theorems 2.3 and 3.2, we see that \( a \) is O-regularly varying if and only if \( \beta(a) > -\infty \) and \( \alpha(a) < \infty \).

If the sequence \( a \) is regularly varying of index \( \omega \in \mathbb{R} \), by Theorem 3.2 and Remark 2.5, we deduce that \( \beta(a) = \mu(a) = \rho(a) = \alpha(a) = \omega \). However, the equality of the indices does not imply regular variation.

A sequence \( a \) is said to be almost increasing, if there exists some \( M > 0 \) such that \( a_p \leq Ma_q \) for all \( p, q \in \mathbb{N} \) with \( p < q \). Analogously, \( a \) is almost decreasing if there exists some \( m > 0 \) such that \( ma_q \leq a_p \) for all \( p, q \in \mathbb{N} \) with \( p < q \). Thanks to these definitions and using that \( [x] \leq x \leq 2[x] \) for all \( x \geq 1 \), it is possible to skip the step function \( f_a \) and give a simple characterization of the indices and orders only in terms of the sequence \( a \).

**Proposition 3.5** Let \( a \) be as above. We have that

\[
\begin{align*}
\alpha(a) &= \inf \{ \alpha \in \mathbb{R} ; (a_p/p^\alpha)_{p \in \mathbb{N}} \text{ is almost decreasing} \}, \\
\rho(a) &= \limsup_{p \to \infty} \frac{\log(a_p)}{\log(p)}, \\
\beta(a) &= \sup \{ \beta \in \mathbb{R} ; (a_p/p^\beta)_{p \in \mathbb{N}} \text{ is almost increasing} \}, \\
\mu(a) &= \liminf_{p \to \infty} \frac{\log(a_p)}{\log(p)}.
\end{align*}
\]

When applying ramification arguments in the classes of functions defined in terms of a given weight sequence \( \mathbb{M} \), transformations of this sequence will appear. Using this last characterization result, the indices for the transforms and for the original sequence can be compared as indicated below.

**Proposition 3.6** Let \( a \) be a sequence of positive numbers. For any \( r \in \mathbb{R} \) and \( s > 0 \), we have that

\[
\begin{align*}
\alpha(a^s) &= s \alpha(a), \\
\beta(a^s) &= s \beta(a), \\
\rho(a^s) &= s \rho(a), \\
\mu(a^s) &= s \mu(a),
\end{align*}
\]
where \( a^s := (a_p^s)_{p \in \mathbb{N}} \), and we also obtain that

\[
\begin{align*}
\alpha(g_r \cdot a) &= r + \alpha(a), \\
\beta(g_r \cdot a) &= r + \beta(a), \\
\rho(g_r \cdot a) &= r + \rho(a), \\
\mu(g_r \cdot a) &= r + \mu(a),
\end{align*}
\]
where \( g_r := (p^r)_{p \in \mathbb{N}} \) and \( g_r \cdot a = (p^r a_p)_{p \in \mathbb{N}} \).

One may notice the stability of the orders, the Matuszewska indices and the notion of O-regular variation for sequences under \( \simeq \).

**Lemma 3.7** Let \( a = (a_p)_{p \in \mathbb{N}} \) and \( b = (b_p)_{p \in \mathbb{N}} \) be as above with \( a \simeq b \). Then, we see that

\[
\begin{align*}
\alpha(a) &= \alpha(b), \\
\beta(a) &= \beta(b), \\
\rho(a) &= \rho(b), \\
\mu(a) &= \mu(b).
\end{align*}
\]
Hence \( a \) is O-regularly varying if and only if \( b \) also is.

In the next subsection, these results will be applied for the sequence of quotients \( m = (m_{p-1})_{p \in \mathbb{N}} \) of \( \mathbb{M} \), then the stability of those indices under \( \simeq \) is a first approach. However, in the context of ultraholomorphic and ultradifferentiable classes, it is always possible to switch \( \mathbb{M} \) for an equivalent sequence under \( \approx \) and so, the appropriate question is the stability for this weaker relation. A partial but sufficient solution is given at the end of the current section (see Corollary 3.14 and Remark 4.7).
Remark 3.8 Since there is not a uniform definition of the sequence of quotients, one may alternatively consider the shifted sequence \( \mathbf{s}_m = (m_p)_{p \in \mathbb{N}} \), as some authors \([7,23,32,33,40]\) have. We see below that both approaches are equivalent in this context and one can switch from \( \mathbf{m} \) to \( \mathbf{s}_m \) when needed.

For \( \alpha \geq 0 \) we observe that \( p^\alpha \leq (p + 1)^\alpha \leq 2^\alpha p^\alpha \), so for \( \gamma \in \mathbb{R} \) we deduce that \( (a_p p^\gamma)_{p \in \mathbb{N}} \) is almost increasing (resp. almost decreasing) if and only if \( (a_{p+1} p^\gamma)_{p \in \mathbb{N}} \) is almost increasing (resp. almost decreasing). Hence, even if \( \mathbf{a} = (a_p)_{p \in \mathbb{N}} \) and the corresponding shifted sequence \( \mathbf{s}_a := (a_{p+1})_{p \in \mathbb{N}} \) are not equivalent for \( \simeq \), we see that

\[
\alpha(a) = \alpha(s_a), \quad \beta(a) = \beta(s_a), \quad \rho(a) = \rho(s_a), \quad \mu(a) = \mu(s_a).
\]

Consequently, by Remark 3.4, \( \mathbf{a} \) is O-regularly varying if and only if \( \mathbf{s}_a \) also is.

### 3.3 Orders, Matuszewska indices and growth indices

In this subsection, we will see that the indices \( \gamma(\mathbb{M}) \) and \( \omega(\mathbb{M}) \) can be expressed in terms of the lower Matuszewska index \( \beta(\mathbf{m}) \) and the lower order \( \mu(\mathbf{m}) \), respectively. First, we obtain the central connection between logarithmic convexity and O-regular variation which is the analogous version of Lemma 2.8.

Lemma 3.9 Let \( \mathbb{M} \) be a sequence of positive real numbers with sequence of quotients \( \mathbf{m} = (m_{p-1})_{p \in \mathbb{N}} \). For any \( \gamma \in \mathbb{R} \), we have that

(i) If there exists \( t = (t_p)_{p \in \mathbb{N}_0} \) nondecreasing such that \( ((p + 1)^{-\gamma} m_p)_{p \in \mathbb{N}_0} \simeq t \), then \( \beta(\mathbf{m}) \geq \gamma \). Hence, if \( \mathbb{M} \) is (lc), then \( 0 \leq \beta(\mathbf{m}) \leq \mu(\mathbf{m}) \leq \rho(\mathbf{m}) \leq \alpha(\mathbf{m}) \).

(ii) If \( \beta(\mathbf{m}) > \gamma \), then there exists \( t = (t_p)_{p \in \mathbb{N}_0} \) nondecreasing such that \( ((p + 1)^{-\gamma} m_p)_{p \in \mathbb{N}_0} \simeq t \).

According to Remark 3.8 and Proposition 3.5, the lower order \( \mu(\mathbf{m}) \) and the growth index \( \omega(\mathbb{M}) \) coincide for any sequence \( \mathbb{M} \). The relation between \( \gamma(\mathbb{M}) \) and Matuszewska indices can be deduced from the previous result. A weaker version of it, for strongly regular sequences and \( \gamma > 0 \) is contained in \([17, \text{Prop. 4.15}]\) where the connection with O-regular variation was unknown.

Theorem 3.10 Let \( \mathbb{M} \) be a sequence of positive real numbers with sequence of quotients \( \mathbf{m} = (m_{p-1})_{p \in \mathbb{N}} \). Then

\[
\gamma(\mathbb{M}) = \beta(\mathbf{m}), \quad \omega(\mathbb{M}) = \mu(\mathbf{m}).
\]

From this main connection, we see that the properties satisfied by \( \beta \) and \( \mu \) appearing in Remarks 3.4 and 3.8, Proposition 3.6 and Lemma 3.9 also hold for \( \gamma(\mathbb{M}), \omega(\mathbb{M}) \). Some of these properties were already proved by V. Thilliez and the second author after introducing the corresponding indices. In particular, for \( \mathbb{M} \) (lc) this means that \( 0 \leq \gamma(\mathbb{M}) \leq \omega(\mathbb{M}) \leq \infty \) and we deduce that for (lc) sequences the original definition of \( \gamma(\mathbb{M}) \) given by V. Thilliez, where the supremum is taken only for \( \gamma > 0 \), coincides with the general one considered in this paper.

### 3.4 Main theorems

Applying Theorems 2.11 and 2.16 to suitable step functions, we obtain similar results for the indices \( \beta(\mathbf{m}) \) and \( \alpha(\mathbf{m}) \) where \( \mathbf{m} = (m_{p-1})_{p \in \mathbb{N}} \) is the sequence of quotients of a weight sequence \( \mathbb{M} \), so \( \mathbf{m} \) is nondecreasing and tends to infinity, and the same holds for \( f_\mathbf{m} \) and \( f_\mathbf{s}_m \).
Theorem 3.11 Let \( M \) be a weight sequence and \( \beta \geq 0 \). The following are equivalent:

(i) there exists \( C > 0 \) such that \( \sum_{k=p}^{\infty} \frac{(k+1)^{\beta-1}}{m_k} \leq C \frac{(p+1)^\beta}{m_p} \) for all \( p \in \mathbb{N}_0 \),

(ii) there exists \( \varepsilon > 0 \) such that \( \left( m_p / (p+1)^{\beta+\varepsilon} \right)_{p \in \mathbb{N}} \) is almost increasing,

(iii) there exists a sequence \( h \simeq m \) such that \( ((p+1)^{-\beta} h_p)_{p \in \mathbb{N}_0} \) is nondecreasing and \( \inf_{p \geq 1} \frac{h_p}{h_p} > 2^\beta \),

(iv) \( \lim_{k \to \infty} \inf_{p \to \infty} \frac{m_{kp}}{k^\beta m_p} = \infty \),

(v) there exists \( k \in \mathbb{N}, k \geq 2 \) such that \( \lim_{p \to \infty} \frac{m_{kp}}{m_p} > k^\beta \),

(vi) for every \( \theta \in (0, 1) \) there exists \( k \in \mathbb{N}, k \geq 2 \), such that \( m_p \leq \theta k^{-\beta} m_{kp} \) for every \( p \in \mathbb{N} \),

(vii) \( \beta(m) > \beta \),

(viii) \( \gamma(M) > \beta \).

(ix) there exists \( C > 0 \) such that \( \sum_{k=0}^{p} \frac{m_k}{(k+1)^{\beta+\varepsilon}} \leq C \frac{m_p}{(p+1)^\beta} \) for all \( p \in \mathbb{N}_0 \).

Proof

(i)\( \Rightarrow \) (ii) By Theorem 2.16[(vii) \( \Rightarrow \) (v)] for \( \sigma_1(t) = m_{\lfloor t \rfloor} - 1 \) for \( t \geq 1 \) and \( \sigma_1(t) = m_0 \) for \( t \in [0, 1) \), there exists \( \varepsilon > 0 \) such that \( (m_p/(p+1)^{\beta+\varepsilon})_{p \in \mathbb{N}_0} \) is almost increasing. Hence, by Remark 3.8, (ii) holds.

(ii)\( \Rightarrow \) (iii) For every \( p \in \mathbb{N} \) we define the sequence \( h_p := p^{\beta+\varepsilon} \inf_{q \geq p} (q^{-\beta+\varepsilon}) m_q \) and \( h_0 := h_1/2^\beta \). With a direct calculation, we observe that \( ((p+1)^{-\beta} h_p)_{p \in \mathbb{N}_0} \) is nondecreasing and, by (ii), we see that there exists \( C > 1 \) such that \( m_p/C \leq h_p \leq m_p \) for all \( p \in \mathbb{N} \), so \( h \simeq m \). Finally, for all \( p \in \mathbb{N} \) we compute

\[
\frac{h_{2p}}{h_p} = 2^{\beta+\varepsilon} \frac{\inf_{q \geq 2p} (q^{-\beta+\varepsilon}) m_q}{\inf_{q \geq p} (q^{-\beta+\varepsilon}) m_q} \geq 2^{\beta+\varepsilon}.
\]

Hence \( \inf_{p \geq 1} \frac{h_{2p}}{h_p} > 2^\beta \).

(iii)\( \Rightarrow \) (iv) By (iii), there exists \( \varepsilon > 0 \) such that \( \inf_{p \geq 1} \frac{h_{2p}}{h_p} > 2^\beta+\varepsilon \). Then for all \( k \in \mathbb{N} \), \( k \geq 2 \), there exists \( n \in \mathbb{N} \) such that \( 2^n \leq k < 2^{n+1} \) and for every \( p \in \mathbb{N} \) we observe that

\[
\frac{h_{kp}}{k^\beta h_p} \geq \frac{h_{2^n p}}{(2^{n+1})^\beta h_p} \geq 2^{n\varepsilon - \beta}.
\]

Therefore (iv) is valid for \( h \), then it also holds for \( m \) because \( h \simeq m \).

(iv)\( \Rightarrow \) (v) Immediate.

(v)\( \Rightarrow \) (vi) It follows from Theorem 2.16[(ix) \( \Rightarrow \) (viii)] for \( \sigma_2(t) = m_{\lfloor t \rfloor} \) for all \( t \geq 0 \).

(vi)\( \Rightarrow \) (vii) For \( \sigma_2(t) = f_m(t) = m_{\lfloor t \rfloor} \) for all \( t \geq 0 \), Theorem 2.16[(viii) \( \Rightarrow \) (iv)] implies \( \beta(\sigma) = \beta(f_m) = \beta(s_m) > \beta \) and, by Remark 3.8, \( \beta(m) > \beta \).

(vii)\( \Leftrightarrow \) (viii) By Theorem 3.10.

(vii)\( \Leftrightarrow \) (ix)\( \Leftrightarrow \) (i) By Theorem 2.16 for \( \sigma_1(t) = f_m(t) = m_{\lfloor t \rfloor} - 1 \) for \( t \geq 1 \) and \( \sigma_1(t) = m_0 \) for \( t \in [0, 1) \).

The importance of Theorem 3.11 lies on the relation between the listed conditions and some of the classical properties for weight sequences appearing in the literature. For \( \beta = 0 \), (i) is precisely (snq) for \( M \) and for \( \beta = 1 \) it is condition \((\gamma_1)\) of Petzsche [32] for \( m \). The authors have employed, when studying the surjectivity of the Borel map [20], an extension of \((\gamma_1)\) introduced by Schmets and Valdivia [40] for \( r \in \mathbb{N} \), that can be defined for \( r > 0 \) as follows: we say that \( m \) satisfies \((\gamma_r)\) if there exists \( C > 0 \) such that

\[
(\gamma_r) \sum_{k=p}^{\infty} \frac{1}{(m_k)^{1/r}} \leq C \frac{(p+1)}{(m_p)^{1/r}}, \quad p \in \mathbb{N}_0.
\]
In particular, the next corollary was needed in the cited work.

**Corollary 3.12** Let $\mathbb{M}$ be a weight sequence and $\beta > 0$. Then

(i) $\gamma(\mathbb{M}) > 0$ if and only if $\mathbb{M}$ satisfies (snq).
(ii) $\gamma(\mathbb{M}) > 1$ if and only if $m$ satisfies ($\gamma_1$).
(iii) $\gamma(\mathbb{M}) > \beta$ if and only if $m$ satisfies ($\gamma_\beta$).

**Proof** (i) and (ii) follow directly from Theorem 3.11. (iii) First, we note that $M^{1/\beta}$ is also a weight sequence, so by Proposition 3.6 and (ii), $\gamma(\mathbb{M}) > 1$ if and only if $m^{1/\beta}$ satisfies ($\gamma_1$) if and only if $m$ satisfies ($\gamma_\beta$). \(\square\)

The reader might have noticed that condition (v) in Theorem 3.11 appears frequently in the context of weighted spaces in the case $\beta = 0$ as condition ($\beta_3$) in [38] or as condition 12.(2) in [7] and in the case $\beta = 1$ as ($\beta_1$) in [32,38]. It is worthy to mention that, due to the previously commented index shift nuisance, sometimes these conditions are asked to be satisfied by $m$ and others by $s_m$, but thanks to Remark 3.8, we know that both approaches are equivalent.

In the literature, see Komatsu [23], Petzsche [32], Bonet et al. [7], the Carleman ultradifferentiable classes are usually defined by imposing control of the derivatives as in (1). However, in some cases, and specially when dealing with ultraholomorphic classes (see, for example, [8,20,41]), the control as in (2) is preferred, so highlighting the defect of analyticity of the functions belonging to the class. Accordingly, the properties of the class are deduced from conditions on the sequence $\mathbb{M}: = G_1 \mathbb{M} = (p!M_p)_{p \in \mathbb{N}_0}$ or on the sequence $\mathbb{M}$, respectively. Even if most of the conditions can be translated from one approach to the other in the first case $\mathbb{M}$ is often supposed to be a weight sequence whereas in the second situation the basic hypothesis is satisfied by $M$. This difference might be troublesome since if $M$ is a weight sequence then $\mathbb{M}$ also is, but the opposite is not true in general. Hence one might think if Theorem 3.11 is valid under weaker assumptions. In this sense, we observe the following:

**Corollary 3.13** Let $\beta \geq 0$ and $\mathbb{M}$ be a sequence such that for some $r \geq 0$ the sequence $G_r \mathbb{M}: = (p^r M_p)_{p \in \mathbb{N}_0}$ is a weight sequence. Then the equivalences for $\mathbb{M}$ and $\beta$ in Theorem 3.11 hold.

**Proof** Using Proposition 3.6 and Remark 3.8, we can check that each condition in Theorem 3.11 holds for $m$ and $\beta$ if and only if this same condition is valid for $(p^r m_{p-1})_{p \in \mathbb{N}}$ and $\beta + r$. Hence we deduce the desired equivalence for the conditions applying Theorem 3.11 to $G_r \mathbb{M}$. \(\square\)

By Lemma 3.7, we know that the value of the index $\beta(m) = \gamma(\mathbb{M})$ is stable for $\simeq$. Consequently, all the conditions listed in Theorem 3.11 are stable for $\simeq$ for weight sequences. However, the natural requirement for weight sequences is the stability for the weaker relation $\approx$. In [32, Th. 3.4], the stability of ($\gamma_1$) condition for $\approx$ is indirectly deduced and to the best of our knowledge there is no direct proof of this fact which is used to obtain the desired stability as follows.

**Corollary 3.14** Let $\mathbb{M}$ and $\mathbb{L}$ be sequences with $\mathbb{M} \approx \mathbb{L}$. Assume that there exists $r \geq 0$ such that $G_r \mathbb{M}$ and $G_r \mathbb{L}$ are weight sequences. Then $\gamma(\mathbb{M}) = \gamma(\mathbb{L})$ holds true. Consequently, all the conditions in Theorem 3.11 are stable for $\approx$ for such sequences.

**Proof** We might assume without loss of generality that $\mathbb{M}$ and $\mathbb{L}$ are weight sequences, otherwise, since $G_r \mathbb{M} \approx G_r \mathbb{L}$ also holds, one will show that $\gamma(G_r \mathbb{M}) = \gamma(G_r \mathbb{L})$ and we conclude using Proposition 3.6 and Corollary 3.13.
Let \(0 < s < \gamma(\mathbb{M})\), then \(\mathbb{M}^{1/s}\) and \(\mathbb{L}^{1/s}\) are also a weight sequence with \(\mathbb{M}^{1/s} \approx \mathbb{L}^{1/s}\) and, by Proposition 3.6, we obtain that \(\gamma(\mathbb{M}^{1/s}) > 1\). By Corollary 3.12(ii), applied to \(\mathbb{M}^{1/s}\), we see that \(\mathbb{M}^{1/s}\) satisfies \((\gamma_1)\). By [32, Th. 3.4], we deduce that \(\ell^{1/s}\) has \((\gamma_1)\). Then, reversing the arguments above for \(\mathbb{L}\) instead of \(\mathbb{M}\), we get \(s < \gamma(L)\) and which proves \(\gamma(\mathbb{M}) \leq \gamma(\mathbb{L})\).

The converse inequality follows analogously. □

**Remark 3.15** One can go one step further and study what remains true if \(\mathbb{M}\) is an arbitrary sequence, not necessarily a weight sequence, such that \(\mathbb{M} \simeq \ell\) with \(\mathbb{L}\) a weight sequence. In this case, since the value of \(\gamma\) index and condition \((\gamma_1)\) are stable for \(\simeq\) for any pair of sequences, applying Theorem 3.11 for \(\mathbb{L}\) and \(\beta = 1\), we see that \(\gamma(\mathbb{M}) > 1\) if and only if \(\mathbb{M}\) has \((\gamma_1)\).

Moreover, since \(\gamma(\mathbb{M}) > 1\) implies that there exists \(\varepsilon > 0\) such that \((m_p/p^{1+\varepsilon})_{p \in \mathbb{N}}\) is almost increasing and tends to infinity, as in Theorem 3.11 [(ii) \(\Rightarrow\) (iii)] one might construct a weight sequence \(\mathbb{L} = (H_p/p!)_{p \in \mathbb{N}_0}\) with \(\mathbb{M} \simeq ((p+1)^{\ell_p})_{p \in \mathbb{N}}\). Hence, for an arbitrary sequence \(\mathbb{M}\) we see that the following are equivalent:

(i) \(\gamma(\mathbb{M}) > 1\),
(ii) there exists a weight sequence \(\mathbb{L}\) such that \(\mathbb{M} \simeq ((p+1)^{\ell_p})_{p \in \mathbb{N}_0}\) and \(\hat{\ell}\) satisfies \((\gamma_1)\),
(iii) there exists a weight sequence \(\mathbb{H}\) such that \(\mathbb{M} \simeq \mathbb{h}\) and \(\mathbb{h}\) satisfies \((\gamma_1)\),

Subsequently, we obtain an analogous result for the index \(\alpha(\mathbb{M})\) which, as we will show, is related to moderate growth condition.

**Theorem 3.16** Let \(\mathbb{M}\) be a weight sequence and \(\alpha > 0\). The following are equivalent:

(i) there exists \(C > 0\) such that \(\sum_{k=0}^{p} \frac{(k+1)^{\alpha-1}}{m_k} \leq C \frac{p+1}{m_p} \) for all \(p \in \mathbb{N}_0\),
(ii) there exists \(\varepsilon \in (0, \alpha)\) such that \((m_p/p^{\alpha-\varepsilon})_{p \in \mathbb{N}}\) is almost decreasing,
(iii) there exists a sequence \(\mathbb{h} \simeq \mathbb{m}\) such that \((p+1)^{-\alpha}h_p)_{p \in \mathbb{N}_0}\) is nonincreasing and \(\sup_{p \geq 1} \frac{h_p}{h_p} < 2^\alpha\),
(iv) \(\lim_{k \to \infty} \lim_{p \to \infty} \frac{mk_p}{k^\alpha m_p} = 0\),
(v) there exists \(k \in \mathbb{N}, k \geq 2\), such that \(\lim_{p \to \infty} \frac{mk_p}{m_p} < k^\alpha\),
(vi) for every \(\theta \in (0, 1)\) there exists \(k \in \mathbb{N}, k \geq 2\), such that \(mk_p \leq \theta k^\alpha m_p\) for every \(p \in \mathbb{N}\),
(vii) \(\alpha(\mathbb{M}) < \alpha\),
(viii) there exists \(C > 0\) such that \(\sum_{k=p+1}^{\infty} \frac{m_k}{(k+1)^{1+\alpha}} \leq \frac{C m_p}{(p+1)^p}\) for every \(p \in \mathbb{N}_0\).

**Proof** Analogous to the proof of Theorem 3.11 using Theorem 2.11 instead of Theorem 2.16. We only note that the sequence \(\mathbb{h}\) constructed in the proof of (ii) \(\Rightarrow\) (iii) might be defined for \(p \in \mathbb{N}\) by \(h_p := (p+1)^{\alpha-\varepsilon} \sup_{q \geq p} m_q q^{\alpha-\varepsilon}\) and \(h_0 := h_1/2^\alpha\). □

Using that \(\mathbb{m}\) is nondecreasing we see that \(\sup_{p \geq 1} m_{2p}/m_p < \infty\) if and only if there exists \(\alpha > 0\) such that Theorem 3.16(v) holds. Hence, applying Lemma 3.1, we obtain the following corollary.

**Corollary 3.17** Let \(\mathbb{M}\) be a weight sequence. The following are equivalent:

(i) \(\mathbb{M}\) satisfies \((mg)\),
(ii) there exists \(\alpha > 0\) such that \(\mathbb{M}\) satisfies every/some of the equivalent conditions (i)–(viii) in Theorem 3.16.

In particular, \(\mathbb{M}\) satisfies \((mg)\) if and only if \(\alpha(\mathbb{M}) < \infty\) if and only if \(\mathbb{M}\) is \(O\)-regularly varying.
Remark 3.18 As it was pointed out for functions in Remark 2.12, the set
\[ B_m := \{ \beta > 0 ; \text{any of the conditions in Theorem 3.11 holds for } m \text{ and } \beta \} \]
is either empty or an open subinterval of \((0, \infty)\) and, correspondingly, \(\beta(m) = 0\) or \(\beta(m) = \sup B_m\). Similarly, the set
\[ A_m := \{ \alpha > 0 ; \text{any of the conditions in Theorem 3.16 holds for } m \text{ and } \alpha \} \]
is either empty or an open subinterval of \((0, \infty)\) and, correspondingly, \(\alpha(m) = \infty\) or \(\alpha(m) = \inf A_m\).

Remark 3.19 We observe that Corollary 3.13 is also valid if we replace Theorem 3.11 by
Theorem 3.16. In this situation, the stability for \(\approx\) is directly obtained using that \(\equiv\) and \(\approx\) are equivalent if \(M\) has moderate growth, and that \(M\) has \((mg)\) if and only if for some \(r \geq 0\) the sequence \(G_r M\) has \((mg)\).

Finally, using the definition of the exponent of convergence of a nondecreasing sequence, see [15, p. 65], as in [37, Th. 3.4], we obtain the following information for the index \(\omega(M) = \mu(m)\) of a weight sequence \(M\) which is related to the nonquasianalyticity condition. We recall that \(M\) is said to be nonquasianalytic \((nq)\), if \(\sum_{p=0}^{\infty} M_p / (p + 1)M_{p+1} < +\infty\). Here the terminology may differ depending on the definition of the space, that is, if it is defined in terms of \(\hat{M}\), as in (1), or of \(M\), as in (2).

Lemma 3.20 Let \(M\) be a sequence such that \(G_r M = (p!^r M_p)_{p \in \mathbb{N}_0}\) is a weight sequence for some \(r \geq 0\). Then
\[ \omega(M) = \mu(M) = \sup \{ \mu > -r ; \sum_{k=0}^{\infty} \frac{1}{(k + 1)^r m_k}^{1/(\mu + r)} < \infty \}. \]
In particular, if the hypothesis hold for some \(r \in [0, 1]\), then it is valid for \(r = 1\), that is, \(M\) is a weight sequence and we see that if \(\omega(M) > 0\), then \(M\) is \((nq)\) and if \(M\) is \((nq)\), then \(\omega(M) \geq 0\).

Then the classical relations between the properties of \(M\) can be re-obtained in terms of these indices. As it happens for functions, see Remark 2.18, if \(m\) is regularly varying, as it is in many of the examples in the applications, then all the information is tied to only one value, the index \(\rho = \beta(m) = \mu(m) = \rho(m) = \omega(M)\) of regular variation of \(m\).

Remark 3.21 The conditions listed in the results of this section appear related to diverse problems of analysis with several different names. Due to its relevance in the context of ultradifferentiable classes it is worthy to mention that in the classical work of H. Komatsu [23], whose notation has been employed by many authors, \((lc)\) is \((M.1)\), \((mg)\) is \((M.2)\) and \((\gamma_1)\) is \((M.3)\).

4 Associated weight functions and O-regular variation

In this context, there is a canonical form to go from weight sequences to weight functions. Therefore, it is natural to study how the information obtained from O-regular variation affects this procedure.
4.1 Duality between $m$, $v_m$ and $\omega_M$

Let $M$ be a sequence of positive real numbers with $M_0 = 1$, then the associated function $\omega_M : [0, \infty) \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\omega_M(t) := \sup_{p \in \mathbb{N}_0} \log \left( \frac{t^p}{M_p} \right) \quad \text{for all } t > 0, \quad \omega_M(0) := 0.$$  

For an abstract introduction of the associated function we refer to [28, Chap. I], see also [23, Def. 3.1]. If $\lim_{p \to +\infty} (M_p)^{1/p} = \infty$, then $\omega_M(t) < \infty$ for any finite $t$, $\omega_M(t) = 0$ for all $t$ small enough, $\omega_M$ is continuous, nondecreasing, $\lim_{t \to \infty} \omega_M(t) = \infty$. Moreover, if $M$ is a weight sequence then $\omega_M$ satisfies ($\omega_3$) and ($\omega_4$).

Given a weight sequence $M$, we may also define the counting function for the sequence of quotients $m$, $v_m : (0, \infty) \to \mathbb{N}_0$ given by

$$v_m(t) := \# \{ j \in \mathbb{N}_0 : m_j \leq t \} = \max \{ j \in \mathbb{N} : m_{j-1} \leq t \}.$$  

This function is nondecreasing and $\lim_{t \to \infty} v_m(t) = \infty$. Since $\omega_M$ and $v_m$ are measurable and positive, we may consider their Matuszewska indices and their upper and lower orders $\beta(\omega_M)$, $\alpha(\omega_M)$, $\alpha(v_m)$, $\mu(\omega_M)$, $\mu(v_m)$, $\rho(\omega_M)$, $\rho(v_m)$, which belong to $[0, \infty]$. Hence the results in Sect. 2 connecting these indices with the classical properties are available.

For a weight sequence $M$, we recover the classical relation, between $v_m$ and the associated function $\omega_M$. One has that

$$\omega_M(t) = \int_0^t \frac{v_m(r)}{r} \, dr, \quad t > 0. \quad (16)$$

The classical correspondence between the properties of $m$, $v_m$ and $\omega_M$ (see [7, Lemma 12] and [35, Lemma 2.2]) suggests that one may also find relations for their Matuszewska indices. The counting function of $m$ can be seen as the generalized inverse of the associated function $f_m$. In a recent work of Djeurčić et al. [11], the connection of O-regular variation with the generalized inverse of a positive nondecreasing unbounded function $f : [X, \infty) \to (0, \infty)$, given by

$$f^{-1}(x) := \inf \{ y \geq X ; f(y) > x \} = \sup \{ y \geq X ; f(y) \leq x \}$$

for all $x \geq f(X)$, has been partially studied. Even if some information can be inferred from their proofs, there is not an explicit correspondence between the indices of $f$ and $f^{-1}$. In our situation, we will show the correspondence between the ones of the sequence of quotients and its counting function. We will start by the Matuszewska indices $\alpha$ and $\beta$, see Remark 4.7 for the information about $\rho$ and $\mu$.

**Proposition 4.1** Let $M$ be a weight sequence. Then

$$\beta(m) = \frac{1}{\alpha(v_m)}, \quad \alpha(m) = \frac{1}{\beta(v_m)}$$

(with the typical conventions for 0 and $\infty$).

**Proof** First we assume that $0 < \gamma < \beta(m)$, so, by Proposition 3.5, $\gamma^{-1} m_p / p \in \mathbb{N}$ is almost increasing, then $(\gamma^{-1} m_p / p \in \mathbb{N}$ is almost increasing with constant $D \geq 1$. For every $t \geq s \geq m_0$, there exist $p, q \in \mathbb{N}_0$ such that $q \geq p, s \in [m_p, m_{p+1})$ and $t \in [m_q, m_{q+1})$. If $q = p$, we see that

$$\frac{v_m(s)}{s^{1/\gamma}} = \frac{v_m(t)}{s^{1/\gamma}} \geq \frac{v_m(t)}{t^{1/\gamma}},$$

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and if \( q \geq p + 1 \geq 1 \), we get
\[
\frac{\nu_m(s)}{s^{1/\gamma}} \geq \frac{p + 1}{(m_{p+1})^{1/\gamma}} \geq \frac{q}{D(m_q)^{1/\gamma}} \geq \frac{q + 1}{q \ D(t)^{1/\gamma}} \geq v_m(t) \frac{2D^{1/\gamma}}{2},
\]
that is, \( v_m(t)/t^{1/\gamma} \) is almost decreasing, then \( 1/\gamma \geq \alpha(v_m) \) and \( 1/\beta(m) \geq \alpha(v_m) \). Similarly, if \( \gamma > \alpha(m) \), then \( 1/\gamma \leq \beta(v_m) \) and \( 1/\alpha(m) \leq \beta(v_m) \).

Reciprocally, if \( \gamma > \alpha(v_m) \), there exists \( \varepsilon > 0 \) such that \( \gamma - \varepsilon > \alpha(v_m) \), so \( \gamma - \varepsilon > 0 \), since \( \alpha(v_m) \geq 0 \). Hence, \( v_m(t)/t^{\gamma-\varepsilon} \) is almost decreasing which implies that there exists \( d \in (0, 1) \) such that for every \( \lambda \geq 1 \) and all \( t \geq m_0 \) we get
\[
v_m(t) \geq d(v_m(\lambda t))/\lambda^{\gamma-\varepsilon}.
\]
We fix \( Q \in \mathbb{N} \), large enough, such that \( Q^{(e/2)/(\gamma-\varepsilon/2)} \geq 1 \) and taking \( \lambda = Q^{1/(\gamma-\varepsilon/2)} \) we see that
\[
v_m(t)Q \geq v_m(Q^{1/(\gamma-\varepsilon/2)}t), \quad t \geq m_0.
\](17)

Using (17), for \( p \in \mathbb{N} \), we observe that
\[
m_p = \sup\{t \geq m_0; v_m(t) \leq p\} \leq \sup\{t \geq m_0; v_m(Q^{1/(\gamma-\varepsilon/2)}t) \leq Qp\} = Q^{-1/(\gamma-\varepsilon/2)} \sup\{s \geq Q^{1/(\gamma-\varepsilon/2)}m_0; v_m(s) \leq Qp\} \leq \frac{m_p \ Q_p}{Q^{1/(\gamma-\varepsilon/2)}}.
\]

Hence we have shown that there exist \( Q \in \mathbb{N} \), \( Q \geq 2 \) and \( \delta > 0 \) such that
\[
\lim_{p \to \infty} \frac{m_p \ Q_p}{Q^{1/\gamma} m_p} \geq Q^\delta > 1.
\]

By Theorem 3.11, we obtain that \( 1/\gamma \leq \beta(m) \) and \( 1/\alpha(m) \leq \beta(m) \). Analogously, if \( 0 < \gamma < \beta(v_m) \), using Theorem 3.11, we get that \( 1/\gamma \geq \alpha(m) \) and \( 1/\beta(v_m) \geq \alpha(m) \). \( \square \)

Applying Corollaries 2.14, 2.17, 3.12 and 3.17, we recover the classical equivalence for the growth conditions of \( M \) and \( v_m \):

**Corollary 4.2** Let \( M \) be a weight sequence. Then

(i) \( M \) has \( (mg) \) if and only if \( v_m \) satisfies \( (o_6) \).

(ii) \( M \) satisfies \( (snq) \) if and only if \( v_m \) satisfies \( (o_1) \).

Now, we want to connect the O-regular variation character of \( v_m \) and of \( o_{[0]} \), which can be done using (16) and the next result that compares the O-regular variation of a function and its derivative.

**Theorem 4.3** ([4], Th. 2.6.1, Coro. 2.6.2) Let \( f : [X, \infty) \to (0, \infty) \) be a locally integrable function. We define \( F(x) := \int_x^\infty f(t)/dt \). Then,

(i) If \( \alpha(f) < \infty \), then \( \lim \sup_{x \to \infty} f(x)/F(x) < \infty \).

(ii) If \( \beta(f) > 0 \), then \( \lim \inf_{x \to \infty} f(x)/F(x) > 0 \).

(iii) We have that \( \alpha(F) \leq \lim \sup_{x \to \infty} f(x)/F(x) \).

(iv) We have that \( \beta(F) \geq \lim \inf_{x \to \infty} f(x)/F(x) \).

Moreover, we get that
\[
0 < \lim \inf_{x \to \infty} \frac{f(x)}{F(x)} \leq \lim \sup_{x \to \infty} \frac{f(x)}{F(x)} < \infty
\]
if and only if \( \alpha(f) < \infty \) and \( \beta(f) > 0 \). In this case, \( \alpha(F) = \alpha(f) \) and \( \beta(F) = \beta(f) \).
Since \( v_m : [m_0, \infty) \to (0, \infty) \) is a locally integrable function, an easy consequence of (16) and Theorem 4.3 is the following:

**Theorem 4.4** Let \( \mathbb{M} \) be a weight sequence. Then

(i) \( \alpha(v_m) \geq \alpha(\omega_\mathbb{M}) \).

(ii) \( \beta(v_m) = \beta(\omega_\mathbb{M}) \).

**Proof** (i) If \( \alpha(v_m) = \infty \), the result is trivial. If \( \alpha(v_m) < \infty \), we fix \( \alpha > \alpha(v_m) \), then \( t \mapsto v_m(t)/t^{\alpha} \) is almost decreasing. By (16), for every \( y \geq x \geq m_0 \) we see that

\[
\frac{\omega_\mathbb{M}(y)}{y^{\alpha}} \leq \frac{1}{y^{\alpha}} \left( \int_{m_0}^{x} v_m(u) \frac{du}{u} + \int_{x}^{y} v_m(u) \frac{du}{u} \right) \leq \frac{\omega_\mathbb{M}(x)}{x^{\alpha}} + \frac{C v_m(x)}{\alpha x^{\alpha}}.
\]

By Theorem 4.3.(i), there exists \( D > 0 \) such that \( v_m(t) \leq D \omega_\mathbb{M}(t) \) for \( t \) large enough, so \( \alpha \geq \alpha(\omega_\mathbb{M}) \).

(ii) First we will show that \( \beta(v_m) \leq \beta(\omega_\mathbb{M}) \). If \( \beta(v_m) = 0 \), the inequality holds. Assume that \( \beta(v_m) > 0 \), by Theorem 4.3.(ii) there exists \( D, t_0 > 0 \) such that \( v_m(t) \geq D \omega_\mathbb{M}(t) \) for \( t \geq t_0 \) and, by Theorem 2.16, \( \liminf_{t \to \infty} v_m(\lambda t) \lambda^{-\beta} \omega_\mathbb{M}(t) > e^\beta / D \) for all \( \lambda \geq \lambda_0 \). From (16) and the monotonicity of \( v_m \) for every \( t > 0 \) we get

\[
\omega_\mathbb{M}(et) = \int_{0}^{et} v_m(u) \frac{du}{u} \geq \int_{t}^{et} v_m(u) \frac{du}{u} \geq v_m(t).
\]

Hence

\[
\liminf_{t \to \infty} \omega_\mathbb{M}(\lambda t, et) \geq \liminf_{t \to \infty} \frac{Dv_m(\lambda t)}{e^\beta \lambda v_m(\lambda t)} > 1,
\]

and, by Theorem 2.16, we conclude that \( \beta < \beta(\omega_\mathbb{M}) \) so \( \beta(v_m) \leq \beta(\omega_\mathbb{M}) \).

Secondly, we will see that \( \beta(v_m) \geq \beta(\omega_\mathbb{M}) \). If \( \beta(\omega_\mathbb{M}) = 0 \), the result is trivial. If \( \beta(\omega_\mathbb{M}) > 0 \), by Corollary 2.17, \( \omega_\mathbb{M} \) has \( (\omega) \), and by [23, Prop. 3.6] this implies that \( \mathbb{M} \) has \( (mg) \). Then, by Lemma 3.1, this implies that there exists \( A > 1 \) such that \( \sup_{p \in \mathbb{N}} m_p / M_p^{1/p} \leq A < \infty \). Hence, for every \( t \geq m_0 \), there exists \( p \in \mathbb{N} \) such that \( t \in [m_p, m_{p+1}) \) and we have that

\[
\omega_\mathbb{M}(t) \leq \omega_\mathbb{M}(m_{p+1}) = (p + 1) \log \left( \frac{m_{p+1}}{M_{p+1}^{1/p+1}} \right) \leq (p + 1) \log A = v_m(t) \log A.
\]

Assume that \( \beta(\omega_\mathbb{M}) > \beta > 0 \), by Theorem 2.16, \( \liminf_{t \to \infty} \omega_\mathbb{M}(\lambda et) \lambda^{-\beta} \omega_\mathbb{M}(et) > e^\beta \log A \) for all \( \lambda \geq \lambda_0 \), so

\[
\liminf_{t \to \infty} \frac{v_m(\lambda et)}{(\lambda et)^{\beta} v_m(t)} \geq \liminf_{t \to \infty} \frac{1}{e^\beta \log A \lambda^\beta \omega_\mathbb{M}(et)} > 1,
\]

and, by Theorem 2.16, we conclude that \( \beta < \beta(v_m) \) then \( \beta(v_m) \geq \beta(\omega_\mathbb{M}) \). \( \square \)

**Corollary 4.5** Let \( \mathbb{M} \) be a weight sequence. Then the following are equivalent:

(i) \( 0 \leq \liminf_{t \to \infty} \frac{v_m(t)}{\omega_\mathbb{M}(t)} \leq \limsup_{t \to \infty} \frac{v_m(t)}{\omega_\mathbb{M}(t)} < \infty \),

(ii) \( \beta(v_m) > 0 \) and \( \alpha(v_m) < \infty \),

(iii) \( \beta(\omega_\mathbb{M}) > 0 \) and \( \alpha(\omega_\mathbb{M}) < \infty \).

In this case, \( \beta(\omega_\mathbb{M}) = \beta(v_m) \) and \( \alpha(\omega_\mathbb{M}) = \alpha(v_m) \).
Proof (i) ⇔ (ii) Immediate from Theorem 4.3.
(i) ⇒ (iii) Again by Theorem 4.3, we have that \( \beta(\omega_M) = \beta(\nu_m) > 0 \) and \( \alpha(\omega_M) = \alpha(\nu_m) < \infty \).
(iii) ⇒ (i) By Theorem 4.4, we see that \( \beta(\nu_m) = \beta(\omega_M) > 0 \) then, by Theorem 4.3(ii), we see that \( \lim_{t \to \infty} \frac{\nu_m(t)}{\omega_M(t)} > 0 \). From (16) and the monotonicity of \( \nu_m \) we obtain (18).
Since \( \alpha(\omega_M) < \infty \), by Theorem 2.11, we get
\[
\limsup_{t \to \infty} \frac{\nu_m(t)}{\omega_M(t)} \leq \limsup_{t \to \infty} \frac{\omega_M(et)}{\omega_M(t)} < \infty.
\]
\( \square \)
Finally, we want to show the connection between the indices of \( \omega_M \) and the ones of \( M \).

Corollary 4.6 Let \( M \) be a weight sequence. Then
(i) \( \gamma(M) = \beta(m) = 1/\alpha(v_m) \leq 1/\alpha(\omega_M) = \gamma(\omega_M) \).
(ii) \( \alpha(m) = 1/\beta(v_m) = 1/\beta(\omega_M) \).
(iii) If \( M \) has in addition \( (mg) \), then \( \gamma(M) = \gamma(\omega_M) \).

Proof (i) and (ii) follow from Lemma 2.10, Proposition 4.1, Theorems 3.10 and 4.4.
(iii) If \( \alpha(\omega_M) = \infty \), then by Theorem 4.4 \( \alpha(\nu_m) = \infty \) and \( \gamma(M) = \gamma(v_m) = \gamma(\omega_M) = 0 \). Assume that \( \alpha(\omega_M) < \infty \). Since \( M \) has \( (mg) \), by Corollary 3.17, \( \alpha(m) < \infty \) so, by (ii) we also have that \( \beta(\omega_M) > 0 \). Hence we can apply Corollary 4.5 and deduce that \( \alpha(\omega_M) = \alpha(v_m) = 1/\beta(m) \), as desired.
\( \square \)

We close this section with several observations.

Remark 4.7 In the book of Goldberg and Ostrovskii [14, Ch. 2, Th. 1.1], it is shown that \( \omega_M \) and \( v_m \) belong to the same growth category and from their computations it can be deduced that \( \rho(\omega_M) = \rho(v_m) \) and \( \mu(\omega_M) = \mu(v_m) \).

In [18, Th. 3.5], the authors have shown that for any weight sequence \( M \) the following equality holds
\[
\rho(\omega_M) = \rho(v_m) = \frac{1}{\mu(m)} = \frac{1}{\omega(M)}.
\]
With a more elaborated argument, it is also possible to show that \( \mu(\omega_M) = \mu(v_m) = 1/\rho(m) \).

As a consequence, we observe that if \( M \) and \( L \) are weight sequences with \( M \approx L \) then there exists \( A \geq 1 \) such that
\[
\omega_L(A^{-1}t) \leq \omega_M(t) \leq \omega_L(A t), \quad t > 0,
\]
and we can show that \( \rho(\omega_M) = \rho(\omega_L) \) so \( \omega(M) = \omega(L) \). Together with Corollary 3.14, this means that one can extend Lemma 3.7, that is, we have stability for \( \approx \), for the two relevant indices \( \gamma(M) \) and \( \omega(M) \) in the study of the surjectivity and injectivity of the Borel map in ultraholomorphic classes in sectors, see [20].

Remark 4.8 Corollary 4.6 allows us to recover the following well-known relations between the conditions of \( M \) and \( \omega_M \) and explains why there is not a complete symmetry. For any weight sequence \( M \) we get
(i) \( M \) has \( (mg) \) if and only if \( \omega_M \) has \( (\omega_R) \).
(ii) If \( m \) satisfies \( (\gamma_1) \), then \( \omega_M \) has \( (\omega_{snq}) \).
(iii) If \( M \) satisfies \( (snq) \), then \( \omega_M \) has \( (\omega_1) \).
Moreover, if $\mathbb{M}$ has \((mg)\), then the implications in (ii) and (iii) can be reversed.

**Remark 4.9** The example constructed in Sect. 5 provides a weight sequence $\mathbb{M}$ such that $\gamma(\mathbb{M}) = 0$ and $\gamma(\omega_{\mathbb{M}}) = \infty$, which shows that the assumption \((\omega_{\text{snq}})\) for $\omega_{\mathbb{M}}$ is really weaker than condition \((\gamma_1)\) for $\mathbb{M}$ underlining the importance of the difference between $\gamma(\mathbb{M})$ and $\gamma(\omega_{\mathbb{M}})$ in general.

### 4.2 A formula relating the indices of $\omega_{\mathbb{M}}^*$ and $\omega_{\hat{\mathbb{M}}}$

As commented before Corollary 3.13, sometimes the ultradifferentiable classes are defined replacing the sequence $\mathbb{M}$ by $\hat{\mathbb{M}} := (p!M_p)_{p\in\mathbb{N}_0}$, see also (1) and (2), and the requirement of $\mathbb{M}$ being a weight sequence is substituted by the weaker condition of $\hat{\mathbb{M}}$ being a weight sequence. The relation between the indices of both sequences is stated in Proposition 3.6. However, the connection of the indices of $\omega_{\mathbb{M}}$ and $\omega_{\hat{\mathbb{M}}}$, which is motivated by the study of the surjectivity of the Borel map, see [19,21], does not follow directly from the theory of $O$-regular variation. The connection of the index $\gamma$ of both functions is based on the study of the Legendre conjugates carried out in Sect. 2.5. For this purpose, we recall the next property of $\omega^*$, for its proof we refer to [21, Lemma 3.1.(ii) and (3.5)].

**Lemma 4.10** Let $\mathbb{M} = (M_p)_{p\in\mathbb{N}_0}$ be any sequence of positive real numbers with $M_0 = 1$ and $\lim_{p\to\infty} (M_p)^{1/p} = \infty$. Then for all $s > 0$

$$\omega_{\hat{\mathbb{M}}}^*(s)^i \leq \omega_{\mathbb{M}}(s) \leq (\omega_{\hat{\mathbb{M}}}^*(se))^i.$$  

Using the previous Lemma we can show:

**Corollary 4.11** Let $\hat{\mathbb{M}} = (M_p)_{p\in\mathbb{N}_0}$ be any sequence of positive real numbers with $M_0 = 1$ such that $\hat{\mathbb{M}} = (p!M_p)_{p\in\mathbb{N}_0}$ is a weight sequence and $\lim_{p\to\infty} (M_p)^{1/p} = \infty$. Assume that $\gamma(\omega_{\hat{\mathbb{M}}}) > 1$. Then we obtain

$$\gamma(\omega_{\hat{\mathbb{M}}}) = \gamma(\omega_{\mathbb{M}}) + 1.$$  

**Proof** By Corollary 2.25 applied to $\sigma = \omega_{\hat{\mathbb{M}}}$, we see that $\gamma(\omega_{\hat{\mathbb{M}}}) = \gamma((\omega_{\hat{\mathbb{M}}})^*) + 1$. This means that $\gamma((\omega_{\hat{\mathbb{M}}})^*) > 0$, so $(\omega_{\hat{\mathbb{M}}})^*$ satisfies $(\omega_1)$.

By Lemma 4.10 this implies that $(\omega_{\hat{\mathbb{M}}})^* \sim \omega_{\mathbb{M}}$ and, consequently, since both functions are nondecreasing and tend to $\infty$ at $\infty$, $\gamma(\omega_{\mathbb{M}}) = \gamma((\omega_{\hat{\mathbb{M}}})^*)$. \qed

### 4.3 Strongly regular sequences

The ultradifferentiable spaces defined in terms of a strongly regular sequence are known to be well-behaved with respect to extension and division properties, see [41] and the references therein. Moreover the classical result of Bonet et al. [7, Th. 14], stated in a more general framework in [39, Sect. 6] by the third author, can be translated into the following form: the ultradifferentiable space defined, as in (1), by a weight sequence $\hat{\mathbb{M}}$ satisfying $\beta(\hat{\mathbb{M}}) > 0$ can be defined in terms of the associated function $\omega_{\hat{\mathbb{M}}}$ if and only if $\alpha(\hat{\mathbb{M}}) < \infty$. Hence the spaces coincide whenever $\hat{\mathbb{M}}$ is strongly regular.

With the information in Remark 3.4, Corollaries 3.12, 3.17 and 4.5 we obtain the next characterization of strongly regular sequences which helps us to understand their nature.

**Corollary 4.12** Let $\mathbb{M}$ be a weight sequence. The following are equivalent:

- \(\omega_{\mathbb{M}}\) is strongly regular.
- \(\omega_{\mathbb{M}}\) is strongly regular.
- \(\omega_{\mathbb{M}}\) is strongly regular.
- \(\omega_{\mathbb{M}}\) is strongly regular.
(i) \( M \) is strongly regular,
(ii) There exists \( k \in \mathbb{N}, k \geq 2 \), such that
\[
1 < \liminf_{p \to \infty} \frac{m_{kp}}{m_p} \leq \limsup_{p \to \infty} \frac{m_{kp}}{m_p} < \infty,
\]
(iii) \( \alpha(m) < \infty \) and \( \beta(m) > 0 \),
(iv) \( m \) is \( O \)-regularly varying and \( \beta(m) > 0 \),
(v) \( 0 < \liminf_{t \to \infty} \frac{\nu_m(t)}{\omega_M(t)} \leq \limsup_{t \to \infty} \frac{\nu_m(t)}{\omega_M(t)} < \infty \),
(vi) \( \alpha(\omega_m) < \infty \) and \( \beta(\omega_m) > 0 \),
(vii) \( \alpha(\omega_M) < \infty \) and \( \beta(\omega_M) > 0 \).

In this case, we also get that \( \alpha(\omega_M) = \alpha(\nu_m) = 1/\beta(m) \), \( \beta(\omega_M) = \beta(\nu_m) = 1/\alpha(m) \).

**Remark 4.13** From the last Corollary we see that, if \( M \) is strongly regular, then \( \gamma(M), \omega(M) \in (0, \infty) \). Most of the classical examples of strongly regular sequences satisfy that \( \omega(M) = \gamma(M) \). However, in [18, Example 4.18] we have constructed a strongly regular sequence with \( \gamma(M) = 2 \) and \( \omega(M) = 5/2 \). This construction can be extended: given four mutually distinct positive values \( 0 < \beta < \mu < \rho < \alpha < \infty \), for all \( a > b > 1 \) and \( n \in \mathbb{N}_0 \) we define
\[
\xi(t) := \begin{cases} 
\alpha, & \text{for } t \in [2^{an}, 2^{an+1}), \\
\beta, & \text{for } t \in [2^{bn}, 2^{bn+1}), 
\end{cases}
\]
and \( \xi(t) = 1 \) for all \( t \in [1, 2) \), see [16, Sect. 2.2.5] where other pathological examples have been constructed. We can consider the function
\[
\omega(x) = \exp \left( \int_1^x \xi(u) \frac{du}{u} \right), \quad t > 1,
\]
that is a nondecreasing function. A straightforward but tedious computation leads to
\[
\mu(\omega) = \frac{(b-1)\alpha + (a-b)\beta}{a-1}, \quad \rho(\omega) = \frac{a(b-1)\alpha + (a-b)\beta}{b(a-1)},
\]
then taking
\[
b := \frac{\alpha - \mu}{\alpha - \rho}, \quad a := b \frac{\rho - \beta}{\mu - \beta} = \frac{\alpha - \mu}{\alpha - \rho} \frac{\rho - \beta}{\mu - \beta},
\]
we obtain
\[
\beta(\omega) = \beta, \quad \mu(\omega) = \mu, \quad \rho(\omega) = \rho, \quad \alpha(\omega) = \alpha.
\]
Finally, the sequence \( M \) defined in terms of its sequence of quotients by \( m_p := \omega(p) \) for \( p \) large enough, is strongly regular and
\[
\beta(m) = \beta, \quad \mu(m) = \mu, \quad \rho(m) = \rho, \quad \alpha(m) = \alpha.
\]

### 4.4 Proximate orders and weight functions

We recall now the notion of so-called proximate orders, see [14,42] resp. [37, Definitions 4.1, 4.2] and [18, Definition 2.1] and the references therein.

**Definition 4.14** A function \( \varphi : (c, +\infty) \to [0, +\infty) \) for some \( c \geq 0 \) is called a **proximate order** if the following conditions are satisfied:
(A) \( \varrho \) is continuous and piecewise continuously differentiable in \((c, +\infty)\),
(B) \( \varrho(t) \geq 0 \) for all \( t > c \),
(C) \( \lim_{t \to \infty} \varrho(t) = \rho < +\infty \),
(D) \( \lim_{t \to \infty} t \varrho'(t) \log(t) = 0 \).

If the value \( \rho \) in (C) is positive, then \( \varrho \) is called a **nonzero proximate order**, if \( \rho = 0 \), then \( \varrho \) is called a **zero proximate order**.

Assuming that the function \( d_M(t) = \log(\omega_M(t))/\log t \) is a nonzero proximate order, or if it is close to one in the sense of [18, Def. 4.1], the summability theory developed by Lastra, Malek and the second author, see [24], is available. One may naturally try to check if this theory is also available for spaces of functions defined in terms of a weight function and if this approach provides any new insight. We point out the classical connection between proximate orders and regular variation.

**Lemma 4.15** ([26], Sect. I.12, p. 32) Let \( \varrho \) be a proximate order with \( \lim_{t \to \infty} \varrho(t) = \rho \). Then, the function \( V(t) = t^{\varrho(t)} \in R_\rho \).

If there exists a proximate order \( \varrho \) and positive constants \( A, B \) such that

\[
A \leq \frac{\sigma(t)}{t^{\varrho(t)}} \leq B \quad \text{for } t \text{ large enough},
\]

we say that \( \sigma \) **admits \( \varrho \) as a proximate order.** As a consequence of Remark 2.6 and Lemma 4.15 we obtain the following corollary.

**Corollary 4.16** Let \( \sigma : [0, \infty) \to [0, \infty) \), nondecreasing with \( \lim_{t \to \infty} \sigma(t) = \infty \) admitting \( \varrho \) as a proximate order with \( \lim_{t \to \infty} \varrho(t) = \rho \). Then \( \beta(\sigma) = \mu(\sigma) = \rho(\sigma) = \alpha(\sigma) = \rho \).

Consequently, if \( \sigma \) admits a nonzero proximate order, it satisfies (\( \omega_1 \)) and (\( \omega_6 \)).

In [34,38], with each normalized weight function \( \omega \) satisfying (\( \omega_3 \)) a weight matrix \( \Omega := \{ \mathbb{W}^l = (W^l_j)_{j \in \mathbb{N}_0} : l > 0 \} \) has been associated, defined by

\[
W^l_j := \exp \left( \frac{1}{l} \varphi^*_\omega(\ell j) \right),
\]

where \( \varphi^*_\omega(x) := \sup \{ xy - \omega(e^z) : y \geq 0 \} \). We summarize some facts which are shown in [34, Section 5]: for all \( l > 0 \) we have \( \mathbb{W}^l \) is a weight sequence and \( \omega \) has (\( \omega_6 \)) if and only if some/each \( W^l \) satisfies (mg). In case (\( \omega_6 \)) is satisfied, \( \Omega \) is constant, i.e. \( \mathbb{W}^x \approx \mathbb{W}^y \) for all \( x, y \) > 0 by recalling [34, Lemma 5.9], so the associated ultradifferentiable resp. ultraholomorphic class defined by the weight \( \omega \) can already be represented by a single sequence \( \mathbb{W}^x \).

In the same way as Carleman ultradifferentiable or ultraholomorphic classes may be defined by imposing control of the derivatives by a sequence \( \hat{M} = (p!M_p)_{p \in \mathbb{N}_0} \), as in (1), or by a sequence \( M \), as in (2), and so the properties of the class are deduced from conditions on the sequence \( \hat{M} \), respectively \( M \), one could be tempted to take two corresponding different approaches for introducing ultradifferentiable or ultraholomorphic classes with respect to a Braun-Meise-Taylor weight function or with respect to the associated weight matrix: to work with a given weight function \( \omega \) playing the role of \( \hat{M} \), or consider instead the weight (\( \omega^* \)'s), playing (as explained above, see Proposition 2.21 or Corollary 4.11) the previous role of \( M \). However, as far as the interest of proximate orders in this respect is concerned, no difference appears in both settings since, if any of both weights admits a nonzero proximate order it will have (\( \omega_6 \)) by Corollary 4.16, the same will be true for the other weight function according.
to Remark 2.27, and the corresponding associated weight matrices will be constant, so that the classes defined by any of these procedures can be defined by a single weight sequence and no new, richer structure is obtained in any case.

## 5 A (counter)-example comparing $\gamma(\bar{M})$ and $\gamma(\bar{\omega}_M)$

The objective of the example constructed in this section is to show that the inequality in Corollary 4.6.(i) can be strict. The example is suitable for both approaches, commented before Corollary 3.13 and at the beginning of Sect. 4.2, that is, for classes defined in terms of $\bar{M}$ or of $\bar{\omega}_M = (p!^M_p)_{p \in \mathbb{N}_0}$. According to Corollary 4.6.(iii), the sequences $\bar{M}$ and $\bar{\omega}_M$ can not satisfy (mg), but even if (mg) is violated $\gamma(\bar{M})$ and $\gamma(\bar{\omega}_M)$ might be equal.

We construct a (counter)-example of a weight sequence $\bar{M}$, satisfying the following properties:

(i) $\gamma(\bar{M}) = 0$. Consequently, by Proposition 3.6 and Corollary 3.12 $\gamma(\hat{\omega}) = 1$, so $\bar{M}$ does not satisfy (snq) and $\hat{\omega}$ does not satisfy ($\gamma_1$).

(ii) $\omega(\bar{M}) = \infty$. Hence $\rho(\bar{M}) = \alpha(\bar{M}) = \infty$ and $\omega(\hat{\omega}) = \rho(\hat{\omega}) = \alpha(\hat{\omega}) = \infty$.

(iii) $\beta(\omega_{\bar{M}}) = \mu(\omega_{\bar{M}}) = \rho(\omega_{\bar{M}}) = \alpha(\omega_{\bar{M}}) = 0$, then $\gamma(\omega_{\bar{M}}) = \infty$ and $\omega_{\bar{M}}$ has $(\omega_{\text{snq}})$ (see Corollary 2.13),

(iv) $\beta(\omega_{\hat{\omega}}) = \mu(\omega_{\hat{\omega}}) = \rho(\omega_{\hat{\omega}}) = \alpha(\omega_{\hat{\omega}}) = 0$, then $\gamma(\omega_{\hat{\omega}}) = \infty$ and $\omega_{\hat{\omega}}$ has also $(\omega_{\text{snq}})$.

**Note:** For any weight sequence $\bar{L} \sim \bar{M}$, due to the stability of the indices under equivalence, see Corollary 3.14 and Remark 4.7, conditions (i) and (ii) also hold for $\bar{L}$. Moreover, since $\omega_{\bar{M}}$ has ($\gamma_1$) we deduce that $\omega_{\bar{M}} \sim \omega_{\bar{L}}$ and also conditions (iii) and (iv) hold true for $\omega_{\bar{L}}$.

### 5.1 Construction of $\bar{M}$

We give now the explicit construction of $\bar{M}$, inspired by the Representation Theorem 3.3, and prove all the desired properties. For all $p \geq 1$, we put

$$m_p := \exp \left( \sum_{k=1}^{p} \delta_k \right), \quad m_0 := 1,$$

for some sequence $(\delta_k)_{k=1}^{\infty}$ of nonnegative real numbers. By definition such sequence of quotients is nondecreasing, so $\bar{M}$ is (lc).

The sequence $(\delta_k)_k$ is introduced as follows: First consider two sequences of natural numbers $(a_j)_{j \geq 1}$ and $(b_j)_{j \geq 1}$, which are defined recursively by $a_1 := 1$ and for every $j \in \mathbb{N}$, $b_j := 2a_j$ and $a_{j+1} := b_j^2$. Then for all $j \in \mathbb{N}$ we get

$$a_j = 2^{2^{(2j-1) - 1}}, \quad b_j = 2^{2j - 1}.$$

The sequence $(\delta_k)_{k \geq 1}$ is defined now by

$$\delta_k := c_j = 2^{2^{j+1}} \text{ if } a_j + 1 \leq k \leq b_j; \quad \delta_p := 0 \text{ if } b_j + 1 \leq k \leq a_{j+1}, \quad j \geq 1.$$  

**First immediate consequence:** We observe that $m_{a_j} \geq \exp(\delta_{a_j}) = \exp(c_j)$ then $\lim_{p \to \infty} m_p = \infty$ and $\bar{M}$ is a weight sequence.

**Second immediate consequence:** $\gamma(\bar{M}) = 0$, i.e., (i) holds. For all $k \in \mathbb{N}$, $kb_j < a_{j+1}$ for all $j \geq 1$ large enough (depending on given $k$). For all $k \in \mathbb{N}$ and all $j \geq 1$ large enough $m_{kb_j}/m_{b_j} = 1$ and, by Theorem 3.11, we conclude that $\gamma(\bar{M}) = 0$.  

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For convenience we put $L_p := \log(m_p)/p$, for all $p \geq 1$, and we will show that

$$
\lim_{p \to \infty} L_p = \lim_{p \to \infty} \frac{\log(m_p)}{p} = \lim_{p \to \infty} \frac{1}{p} \sum_{j=1}^{p} \delta_j = \infty. \quad (19)
$$

For all $j \geq 1$ we get

$$
L_{b_j} = \frac{1}{b_j} \sum_{i=1}^{j} (b_j - a_i)c_i = \frac{1}{b_j} \sum_{i=1}^{j} a_i c_i, \quad L_{a_{j+1}} = \frac{1}{a_{j+1}} \sum_{i=1}^{j} (b_j - a_i)c_i = \frac{1}{a_{j+1}} \sum_{i=1}^{j} a_i c_i.
$$

By definition $p \mapsto L_p$ is nonincreasing on each $[b_j, a_{j+1}]$, $j \geq 1$. With a direct computation, we can show that the choice of $c_j$ is sufficient for $p \mapsto L_p$ being nondecreasing on $[a_j, b_j]$, for all $j \geq 2$. An easy calculation leads to $2L_{b_j} \geq c_j \geq L_{a_j} \geq c_{j-2}$ for all $j \geq 3$. Hence (19) is valid.

**Third immediate consequence:** $\omega(\mathbb{M}) = \infty$ because $\lim_{p \to \infty} \log(m_p)/\log p = \infty$ by (19). By Remark 3.4 and Proposition 3.6, condition (ii) is valid and we conclude that neither $\mathbb{M}$ nor $\hat{\mathbb{M}}$ have (mg).

### 5.2 Proving $\gamma(\omega_{\mathbb{M}}) = \infty$

The most arduous part of the example is the proof of $\gamma(\omega_{\mathbb{M}}) = \infty$. The goal is to show that $\omega_{\mathbb{M}}$ has $(P_{\omega_{\mathbb{M}},1/s})$ for all $s > 0$ small enough. Using (16) and that $\gamma(\omega_{\mathbb{M}}) > 1/s$ if and only if $\gamma^{'}(\omega_{\mathbb{M}})_{1/s} > 1$, where $(\omega_{\mathbb{M}})_{1/s}(t) = \omega_{\mathbb{M}}(t^{1/s})$, we can show that for $\omega_{\mathbb{M}}$ to satisfy $(P_{\omega_{\mathbb{M}},1/s})$, it suffices to prove that there exists $C \geq 1$ such that for all $p \in \mathbb{N}$

$$
1 + \frac{\left(\hat{m}_{p+1}\right)^{s}}{p} \sum_{j=p}^{\infty} \frac{1}{\left(\hat{m}_{j+1}\right)^{s}} \leq (C - s) \log\left(\frac{\hat{m}_{p}}{(M_{p})^{1/p}}\right) + C. \quad (20)
$$

With a careful computation one can show that (20) holds for all $s \in (0, 1)$. Then $\gamma(\omega_{\mathbb{M}}) = \infty$.

**Final consequence:** $\gamma(\omega_{\mathbb{M}}) = \infty$ by Corollary 4.11. Hence (iii) and (iv) hold by Remark 2.5, Lemma 2.10 and Corollary 2.13.

**Acknowledgements** The first two authors are partially supported by the Spanish Ministry of Economy, Industry and Competitiveness under the project MTM2016-77642-C2-1-P. The first author is partially supported by the University of Valladolid through a Predoctoral Fellowship (2013 call) co-sponsored by the Banco de Santander. The third author is supported by Austrian Science Fund FWF-Project J 3948-N35, as a part of which he is an external researcher at the Universidad de Valladolid (Spain) for the period October 2016–September 2018.

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