Semistability of graph products

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ABSTRACT
A graph product \( G \) on a graph \( \Gamma \) is a group defined as follows: For each vertex \( v \) of \( \Gamma \) there is a corresponding non-trivial group \( G_v \). The group \( G \) is the quotient of the free product of the \( G_v \) by the commutation relations \( [G_v, G_w] = 1 \) for all adjacent \( v \) and \( w \) in \( \Gamma \). Semistability at \( \infty \) is an asymptotic property of finitely presented groups that is needed in order to effectively define the fundamental group at \( \infty \) for a 1-ended group. It is an open problem whether or not all finitely presented groups have semistable fundamental group at \( \infty \). While many classes of groups are known to contain only semistable at \( \infty \) groups, there are only a few combination results for such groups. Our main theorem is such a result. It states that if \( G \) is a graph product on a finite graph \( \Gamma \) and each vertex group is finitely presented, then \( G \) has non-semistable fundamental group at \( \infty \) if and only if there is a vertex \( v \) of \( \Gamma \) such that \( G_v \) is not semistable, and the subgroup of \( G \) generated by the vertex groups of vertices adjacent to \( v \) is finite (equivalently \( \text{lk}(v) \) is a complete graph and each vertex group of \( \text{lk}(v) \) is finite). Hence if one knows which vertex groups of \( G \) are not semistable and which are finite, then an elementary inspection of \( \Gamma \) determines whether or not \( G \) has semistable fundamental group at \( \infty \).

1. Introduction
Given a graph \( \Gamma \) with vertex set \( V(\Gamma) \), and a group \( G_v \) for each \( v \in V(\Gamma) \), the graph product for \( (\Gamma, \{G_v\}_{v \in V(\Gamma)}) \) is the quotient of the free product of the \( G_v \) by the normal closure of the set of all commutators \( [g, h] \) where \( g \) and \( h \) are elements of adjacent vertex groups. Every right angled Coxeter group and every right angled Artin group is a graph product where vertex groups are copies of \( \mathbb{Z}_2 \) and \( \mathbb{Z} \) respectively. As an immediate corollary to our theorem we have that all right angled Artin and Coxeter groups have semistable fundamental group at \( \infty \) (a result first proved in [13]).

The question of whether or not all finitely presented groups have semistable fundamental group at \( \infty \) has been studied for over 40 years and is one of the premier questions in the asymptotic theory of finitely presented groups. If a finitely presented group \( G \) has semistable fundamental group at \( \infty \) then \( H^2(G; \mathbb{Z}G) \) is free abelian. The question of whether or not all finitely presented groups \( G \) are such that \( H^2(G; \mathbb{Z}G) \) is free abelian is attributed to H. Hopf and remains unanswered. Semistability is a quasi-isometry invariant of a group [4] or [7] and the class of groups with semistable fundamental group at \( \infty \) contains many classes of groups, including: Word hyperbolic groups (combining work of Bestvina-Mess [1], Bowditch [3], G. Levitt [9] and [21]), 1-relator groups [17], general Coxeter and Artin groups, most solvable groups [11] and various group extensions [14] and ascending HNN extensions [15].

There are two important combination results, both of which are used in the proof of our main theorem:
Theorem 1.1. (M. Mihalik, S. Tschantz [18]) If $G$ is the amalgamated product $A \ast_C B$ where $A$ and $B$ are finitely presented with semistable fundamental group at $\infty$, and $C$ is finitely generated, then $G$ has semistable fundamental group at $\infty$.

The next result implies the previous one if $A$ and $B$ are 1-ended and $C$ is infinite.

Theorem 1.2. (M. Mihalik [11]) Suppose $A$ and $B$ are 1-ended finitely generated subgroups of the finitely generated group $G$ such that $A$ and $B$ have semistable fundamental group at $\infty$, the set $A \cup B$ generates $G$ and the group $A \cap B$ contains a finitely generated infinite subgroup. Then $G$ has semistable fundamental group at $\infty$.

Our main theorem follows:

Theorem 1.3. Suppose $G$ is a graph product on the finite graph $\Gamma$ where each vertex group is finitely presented. Then $G$ does not have semistable fundamental group at $\infty$ if and only if there is a vertex $v$ of $\Gamma$ such that:

1. $G_v$ does not have semistable fundamental group at $\infty$ and
2. the link of $v$ is a complete graph with each vertex group finite.

If $G$ is a graph product on a finite graph $\Gamma$ where each vertex group is finitely presented, then our Theorem 3.3 implies that $G$ has semistable fundamental group at infinity if and only if each of the subgroups of $G$ on the connected components of $\Gamma$ have semistable fundamental group at $\infty$. From this point on, we only consider graph products on connected graphs. If $\Gamma_1$ is a subgraph of $\Gamma$ then let $\langle \Gamma_1 \rangle$ denote the subgroup of $G$ generated by the vertex groups of $\Gamma_1$. For a vertex of $\Gamma$, let $lk(v)$ (the link of $v$) be the full subgraph of $\Gamma$ on the vertices adjacent to $v$. Let $st(v)$ (the star of $v$) be $\{v\} \cup lk(v)$. Results in Section 2 show that when the graph product $G$ is not semistable (and $\Gamma$ is not complete), then $G$ “visually” splits as an amalgamated product $G = \langle st(v) \rangle \ast_{lk(v)} \langle \Gamma - \{v\} \rangle$, where $v$ is a vertex of $\Gamma$, $G_v$ is not semistable and $\langle lk(v) \rangle$ is a finite group. The theory of ends of finitely generated graph products is completely worked out by O. Varghese.

Theorem 1.4 (O. Varghese [22]). Suppose $G$ is a finitely generated graph product group on the graph $\Gamma$ (so that each $G_v$ is finitely generated and $\Gamma$ is finite) and $G$ has more than one end, then either:

1. $\Gamma$ is a complete graph such that one vertex group has more than one end and all others are finite, or
2. $G$ visually splits over a finite group. This means that there are non-empty full subgraphs $\Gamma_1$ and $\Gamma_2$ of $\Gamma$ (neither containing the other) such that $\Gamma = \Gamma_1 \cup \Gamma_2$, (so $\Gamma_1 \cap \Gamma_2$ separates $\Gamma$) and $\langle \Gamma_1 \cap \Gamma_2 \rangle$ is a finite group. In particular, $G$ visually decomposes as the (non-trivial) amalgamated product

$$G \cong \langle \Gamma_1 \rangle \ast_{\langle \Gamma_1 \cap \Gamma_2 \rangle} \langle \Gamma_2 \rangle.$$  

Condition (ii) is equivalent to the condition that there is complete separating subgraph $\Delta$ of $\Gamma$ such that each vertex group of $\Delta$ is finite. For graph products of groups, Theorem 1.4 is a visual version of J. Stallings splitting theorem [20] for general finitely generated groups. Stallings theorem states that a finitely generated group has more than 1-end if and only if it splits non-trivially over a finite group.

The following corollary is a combination result. Part (1) follows directly from Theorem 1.3 and Part (2) follows directly from Theorems 1.3 and 1.4.

Corollary 1.5. Suppose $G$ is a graph product on the finite connected graph $\Gamma$ and each vertex group of $\Gamma$ is finitely presented. If any of the following conditions are met, then $G$ has semistable fundamental group at $\infty$: 

...
(1) Each vertex group has semistable fundamental group at $\infty$.
(2) The group $G$ is 1-ended and $\Gamma$ is not complete.
(3) The group $G$ is 1-ended and at least two vertex groups are infinite.

The remainder of the paper is organized as follows: Section 2 contains basic results on graph products of groups. These results are all based on “standard presentations” of graph products of groups. Section 3 contains definitions and results on groups with semistable fundamental group at $\infty$. The results are all based on “standard presentations” of graph products of groups. Section 3 presented 1-ended group with semistable fundamental group at $\infty$.

A discussion of possible directions that would naturally follow the main combination result for the fundamental group of a graph of groups with finite edge groups. This result for $\mathbb{Z}$ contains definitions and results on groups with semistable fundamental group at $\infty$. If $G$ is a finitely presented 1-ended group with semistable fundamental group at $\infty$ we define the fundamental group at $\infty$ for $G$. The proof of the main theorem is produced in Section 4. Our Theorem 3.3 is a semistability combination result for the fundamental group of a graph of groups with finite edge groups. This result in proved in Section 5. Finally, a discussion of possible directions that would naturally follow the main theorem is given in Section 6.

2. Graph products of groups

The results of this section are directly or implicitly contained in §3 of [8]. In particular, Lemma 2.3 follows from Lemma 3.20 [8]. The proofs of the results here are elementary and we include them for completeness. Let $V(\Gamma)$ be the vertices and $E(\Gamma)$ the edges of a graph $\Gamma$ (so $E \subseteq V \times V$ and $e = (v, w) = (w, v)$). Suppose that for each vertex $v \in V(\Gamma)$, $G_v$ is a non-trivial group. The graph product for $(\Gamma, \{G_v\}_{v \in V(\Gamma)})$ is the quotient of the free product of the $G_v$ for $v \in V(\Gamma)$ by the normal closure of the set of all commutators $[a, b]$ where $a \in G_v$, $b \in G_w$ and $(v, w) \in E(\Gamma)$. For $v \in \Gamma$, let $P_v = \langle S_v : R_v \rangle$ be a presentation for $G_v$. The standard presentation for $G$ corresponding to $\{P_v\}_{v \in V(\Gamma)}$ is

$$\langle S : R \rangle$$

where $R = (\cup_{v \in V(\Gamma)} R_v) \cup \{(a, b) : a \in S_v, b \in S_w \text{ where } (v, w) \in E(\Gamma)\}$ and $S = \cup_{v \in V(\Gamma)} S_v$.

If $\Gamma_1$ is a full subgraph of $\Gamma$ we denote the subgroup of $G$ generated by the $G_v$ for $v \in \Gamma_1$ by $\langle \Gamma_1 \rangle_G$ (or simply $\langle \Gamma_1 \rangle$ when the over group $G$ is evident). If $G_1$ is the graph product for $(\Gamma_1, \{G_v\}_{v \in \Gamma_1})$ there is a natural homomorphism $m : G_1 \rightarrow G$ with image $\langle \Gamma_1 \rangle_G$ (induced by the inclusion of $\Gamma_1$ in $\Gamma$). We need to recall a few basic facts about graph products.

Lemma 2.1. Suppose $G$ is the graph product for $(\Gamma, \{G_v\}_{v \in V(\Gamma)})$. If $\Gamma_1$ is a full subgraph of $\Gamma$, then the subgroup $\langle \Gamma_1 \rangle_G$ is isomorphic to $G_1$, the graph product for $(\Gamma_1, \{G_v\}_{v \in \Gamma_1})$.

In fact, if $m : G_1 \rightarrow G$ is the natural homomorphism with image $\langle \Gamma_1 \rangle_G$ and $q$ is the quotient homomorphism of $G$ by the normal closure of the union of the $G_v$ where $v \not\in V(\Gamma_1)$

$$G_1 \xrightarrow{m} G \xrightarrow{q} G/\langle \cup_{v \not\in V(\Gamma_1)} G_v \rangle$$

then the image of $q$ is isomorphic to $G_1$ and the composition $q \circ m$ is an isomorphism. In particular, $\langle \Gamma_1 \rangle_G \cong G_1$ is a retract of $G$.

Proof. For each $v \in V(\Gamma)$ let $P_v = \langle S_v : R_v \rangle$ be a presentation for $G_v$. Let $P = \langle S : R \rangle$ be the presentation for $G$ corresponding to the $P_v$. The quotient group of $G$ by the normal closure of the $G_v$ with $v \not\in V(\Gamma_1)$ has presentation $\langle S : R \cup (\cup_{v \not\in V(\Gamma_1)} S_v) \rangle$. Tietze moves that eliminate the generators of all $S_v$ for $v \not\in V(\Gamma_1)$ leave a presentation for $\langle \Gamma_1 \rangle$. This presentation is the presentation for $G_1$ corresponding to the $P_v$ for $v \in V(\Gamma_1)$ and $m \circ q$ is the identity on the generators of this presentation for $G_1$ and hence an isomorphism.

Lemma 2.1 applied to a standard presentation of a graph product implies:

Lemma 2.2. If $G$ is the graph product on $(\Gamma, \{G_v\}_{v \in V(\Gamma)})$ then $G$ is finitely generated (presented) if and only if $\Gamma$ is a finite graph and $G_v$ is finitely generated (presented) for each $v \in V(\Gamma)$.
Lemma 2.3. Suppose $\Delta$ is a subgraph of $\Gamma$ that separates the vertices of $\Gamma - \Delta$ into two non-empty disjoint sets $A$ and $B$ and no vertex of $A$ is connected to a vertex of $B$ by an edge. Let $\Gamma_A$ be the full subgraph of $\Gamma$ on the set of vertices in $A \cup \Delta$ and similarly for $\Gamma_B$. Let $G_A$, $G_B$ and $G_\Delta$ be the graph products on $\Gamma_A$, $\Gamma_B$ and $\Delta$ respectively, with vertex groups from $\Gamma$. Then $G$ "visually" decomposes as $G_A *_{G_\Delta} G_B$.

Proof. For each vertex $v \in \Gamma$, let $P_v$ be the presentation for $G_v$ with generators $S_v$ and relations $R_v$. A presentation for $G_A$ is $(S_A : R_A)$, where $R_A = \bigcup_{v \in (A \cup \Delta)} R_v$ $\cup \{(a, b) : a \in S_v, b \in S_w \text{ where } v, w \in A \cup \Delta \text{ and } (v, s) \in E(\Gamma)\}$.

Similarly one obtains a presentation for $G_B$ as $(S_B : R_B)$ and for $G_\Delta$ as $(S_\Delta : R_\Delta)$. The important thing to observe is there is no edge in $\Gamma$ from a vertex of $A$ to a vertex of $B$ and if $E$ is an edge of $\Gamma$ it is an edge of the full subgraph of $\Gamma$ on $A \cup \Delta$ or $B \cup \Delta$, or both - in which case $E$ is an edge of $\Delta$. Note that $S_A \cap S_B = S_\Delta$. Hence $(S_A \cup S_B : R_A \cup R_B)$ is the usual presentation for the amalgamated product $G_A *_{G_\Delta} G_B$, obtained from the presentations $(S_A : R_A)$ and $(S_B : R_B)$ by identifying generators of $G_\Delta$ in $S_A$ with their counterparts in $S_B$. This last presentation is also the presentation for $G$ corresponding to the presentations $P_v$ for $v \in V(\Gamma)$. $\square$

3. Semistability preliminaries

A continuous map between topological spaces $m : X \to Y$ is proper if for each compact set $C \subseteq Y$, $m^{-1}(C)$ is compact in $X$. Given proper maps $f, g : X \to Y$, we say $f$ is properly homotopic to $g$ if there is a proper map $H : X \times [0, 1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = h(x)$ for all $x \in X$.

The notion of semistable fundamental group at $\infty$ makes sense for a wide range of topological spaces. We are only interested in locally finite connected CW complexes in this article. R. Geoghegan’s book [7] is an excellent source for the basic theory of semistable spaces and groups. Suppose $X$ is a locally finite connected CW complex. Two proper rays $r, s : [0, \infty) \to X$ converge to the same end of $X$ if for any compact $C \subseteq X$, $r$ and $s$ eventually stay in the same component of $X - C$. We say $X$ has semistable fundamental group at $\infty$ if any two proper rays $r, s : [0, \infty) \to X$ that converge to the same end of $X$ are properly homotopic. Suppose $X$ has 1-end. For each $i \in \{0, 1, \ldots\}$, suppose $C_i$ and $D_i$ are compact sets such that $C_i$ (respectively $D_i$) is a subset of the interior of $C_{i+1}$ (respectively $D_{i+1}$). Say $r, s : [0, \infty) \to X$ are proper rays. If $X$ has semistable fundamental group at $\infty$ then the inverse systems $\pi_1(X - C_i, r)$ and $\pi_1(X - D_i, s)$ (with bonding maps induced by inclusion) are pro-isomorphic and the inverse limit of such a system is the fundamental group at $\infty$ of $X$.

The finitely presented group $G$ has semistable fundamental group at $\infty$ if for some (equivalently any) finite connected CW complex $X$, with $\pi_1(X) = G$, the universal cover of $X$ has semistable fundamental group at $\infty$. For simplicity we say $G$ is semistable. If $G$ is 1-ended and semistable, then the fundamental group at $\infty$ of $G$ is the fundamental group at $\infty$ of the universal cover of $X$. Again, this is independent of finite complex $X$ as long as $\pi_1(X) = G$.

Over the last 40 years a substantial theory has been built to study the semistability of finitely presented groups. It is as yet unknown if all finitely presented groups are semistable. Using M. Dunwoody’s accessibility theorem for finitely presented groups, the problem is reduced to the same problem for 1-ended groups [12]. For our purposes, we only need a few of the results in the literature. In [4] S. Brick showed that semistability is a quasi-isometry invariant of finitely presented groups. This immediately implies:

Theorem 3.1. If $A$ is a finitely presented group and $A$ has finite index in the group $B$, then $A$ has semistable fundamental group at $\infty$ if and only if $B$ has semistable fundamental group at $\infty$. 
Theorem 3.2 (Theorem 2.2, [10]). Suppose X and Y are locally finite connected and infinite CW-complexes. Then X × Y is 1-ended with semistable fundamental group at ∞. In particular, if G and H are finitely presented infinite groups, then G × H is 1-ended with semistable fundamental group at ∞.

The next result is proved in Section 5. A version of this theorem was proved around the same time as this result (see [5], Proposition 3.1):

Theorem 3.3. Suppose the group G has a graph of groups decomposition G where each edge group is finite and each vertex group is finitely presented. The group G has semistable fundamental group at ∞ if and only if each vertex group of G has semistable fundamental group at infinity.

4. Proof of the main theorem

Main Theorem Suppose G is a graph product on the finite connected graph Γ where each vertex group is finitely presented. Then G does not have semistable fundamental group at ∞ if and only if there is a vertex v of Γ such that:

1. Gv does not have semistable fundamental group at ∞ and
2. the link of v is a complete graph with each vertex group finite.

Proof. If v is a vertex of Γ then condition (2) of the theorem holds for v, if and only if ⟨lk(v)⟩ is a finite group. Now suppose conditions (1) and (2) of the theorem hold for a vertex v of Γ. If Γ = st(v) then G = Gv × ⟨lk(v)⟩ and G does not have semistable fundamental group at ∞ by Theorem 3.1. If Γ ≠ st(v), then Lemma 2.3 implies that G visually decomposes as (Γ − {v}) ∗ ⟨lk(v)⟩⟨st(v)⟩ where ⟨st(v)⟩ = Gv × ⟨lk(v)⟩. Combining this with Theorem 3.3, we immediately see that G does not have semistable fundamental group at ∞. The “if” portion of the Theorem is proved.

If G is a graph product on the (finite) graph Γ and no vertex v of Γ is such that conditions (1) and (2) of the theorem hold for v then we must show that G has semistable fundamental group at ∞. In order to do this, it is enough to arrive at a contradiction from the assumption: The group G is a non-semistable graph product of finitely presented groups on a (finite) graph Γ such that no vertex of Γ satisfies conditions (1) and (2) of the Theorem, and the graph Γ has as few vertices as possible. We make this assumption throughout the remainder of this section.

Lemma 4.1. The graph Γ and the group G satisfy the following:

(A) The graph Γ is not equal to st(w) for any vertex w.
(B) The group G is 1-ended. In particular for each vertex w of Γ, ⟨lk(w)⟩ is infinite.
(C) There is at least one vertex w in Γ such that Gw is not semistable.
(D) If v is a vertex of Γ and Gv is infinite, then ⟨st(v)⟩ is 1-ended and semistable.

Proof. (Of Part (A)) Assume Γ = st(w). Then G = Gw × ⟨lk(w)⟩. Since G is not semistable, Theorem 3.2 implies either Gw or ⟨lk(w)⟩ is finite (and so the other has finite index in G). If ⟨lk(w)⟩ is finite, then Theorem 3.1 implies Gw is not semistable. But then w satisfies conditions (1) and (2) of the Theorem—contrary to our assumption.

If Gw is finite, then ⟨lk(w)⟩ has finite index in G and so ⟨lk(w)⟩ is not semistable. For v a vertex in lk(w), let lk1(v) denote the link of v in lk(w). The minimality of Γ implies there is a vertex v ∈ lk(w) such that Gv is not semistable and ⟨lk1(v)⟩ is finite. But, then each vertex group of the complete graph lk(v) = lk1(v) ∪ {w} is finite and v satisfies conditions (1) and (2)—contrary to our assumption. Instead, part (A) is verified.

(Part (B)) Suppose G has more than 1-end. Part (A) implies Γ is not complete. Theorem 1.4 implies there are non-empty full subgraphs Γ1 and Γ2 of Γ such that Γ1 ∪ Γ2 = Γ, Γ1 ∩ Γ2 separates Γ and
Lemma 4.2. If \( v \) is a vertex of \( \Gamma \) then \( v \) is adjacent to a vertex \( w \) such that \( G_w \) is not semistable.

**Proof.** Suppose \( v \) is a vertex of \( \Gamma \) and every vertex \( w \in \langle v \rangle \) is such that \( G_w \) is semistable. Let \( \Gamma_1 \) be the full subgraph of \( \Gamma \) on the vertices of \( \Gamma \setminus \{ v \} \). If \( u \) is a vertex of \( \Gamma_1 \) such that \( G_u \) is not semistable, then \( u \in \Gamma_1 \setminus \langle v \rangle \), and the link of \( u \) in \( \Gamma_1 \) agrees with the link of \( u \) in \( \Gamma \). The minimality of \( \Gamma \) implies that the group \( \langle \Gamma_1 \rangle \) is semistable.

If \( G_v \) is semistable, and \( u \) is a vertex of \( \langle v \rangle \) then \( G_u \) is semistable. The minimality of \( \Gamma \) and Lemma 4.1(A) imply that \( \langle \langle v \rangle \rangle \) is semistable. If \( G_v \) is not semistable, then Lemma 4.1(D) implies that \( \langle \langle v \rangle \rangle \) is semistable. We have that in any case, both \( \langle \Gamma_1 \rangle \) and \( \langle \langle v \rangle \rangle \) are semistable. As \( G = \langle \Gamma_1 \rangle * \langle \langle v \rangle \rangle \), Theorem 1.1 implies that \( G \) is semistable - contrary to our assumption.

**Lemma 4.3.** Suppose \( v \) is a vertex of \( \Gamma \) such that \( G_v \) is not semistable. Let \( \Gamma_1 \) be the full subgraph of \( \Gamma \) on the vertices of \( \Gamma \setminus \{ v \} \). The group \( G_{\Gamma_1} \) is not semistable.

**Proof.** Suppose \( G_{\Gamma_1} \) is semistable. Note that \( G = G_{\Gamma_1} * \langle \langle v \rangle \rangle \). Since \( \langle \langle v \rangle \rangle \) is semistable (Lemma 4.1(D)), Theorem 1.1 implies \( G \) is semistable, the desired contradiction.

Let \( v \) be a vertex of \( \Gamma \) such that \( G_v \) is not semistable. Let \( \Gamma_1 \) be the full subgraph of \( \Gamma \) on the vertices of \( \Gamma \setminus \{ v \} \) and \( G_1 = \langle \Gamma_1 \rangle \). For \( w \) a vertex of \( \Gamma_1 \), let \( \langle k_1 \rangle \) be the link of \( w \) in \( \Gamma_1 \).

\[ \langle k_1 \rangle \]

If \( w \) is a vertex of \( \Gamma_1 \) and \( w \not\in \langle v \rangle \), then \( \langle k_1 \rangle \subseteq \langle v \rangle \). Lemma 4.3 implies \( G_1 \) is not semistable. Since \( \Gamma_1 \) has fewer vertices than \( \Gamma \), there is a vertex \( \Gamma_2 \) such that \( G_{\Gamma_2} \) is not semistable and \( \langle k_2 \rangle \) is finite. Since \( \langle k_2 \rangle \) is not finite, \( \langle v \rangle \subseteq \langle k_2 \rangle \) and so

\[ \langle k_2 \rangle \]

Let \( \Gamma_2 \) be the full subgraph of \( \Gamma \) on the vertices of \( \Gamma \setminus \{ w \} \) and \( G_2 = \langle \Gamma_2 \rangle \). For \( u \) a vertex of \( \Gamma_2 \), let \( \langle k_2 \rangle \) be the link of \( u \) in \( \Gamma_2 \). As argued above (for \( v \)), there is a vertex \( z \in \langle k_2 \rangle \) such that \( G_z \) is not semistable and \( \langle k_2 \rangle \) is finite. The only vertex of \( \langle k_2 \rangle \) such that \( \langle k_2 \rangle \) with non-semistable group is \( v \); so \( z = v \), \( \langle k_2 \rangle \) is finite and

\[ k_2 \]

Combining, we have

\[ \langle k_2 \rangle \subseteq \langle v \rangle \cup \langle k_1 \rangle \cup \langle k_2 \rangle \]

**Lemma 4.1(D) implies the groups \( \langle \langle v \rangle \rangle \) and \( \langle \langle v \rangle \rangle \) are 1-ended and semistable. Since \( v, w \in \langle \langle v \rangle \rangle \cap \langle \langle v \rangle \rangle \), Theorem 1.2 implies the group \( S = \langle \langle v \rangle \rangle \cup \langle \langle w \rangle \rangle \) is 1-ended and semistable. In particular, \( G \neq S \). Let \( \Gamma_3 \) be the full subgraph of \( \Gamma \) on the vertices of \( \Gamma \setminus \{ v, w \} \). For \( z \) a vertex of \( \Gamma_3 \), let \( k_3 \) be the link
of $z$ in $\Gamma$. Let $G_3 = \langle \Gamma \rangle$. If $z$ is a vertex of $\Gamma$ such that $G_z$ is not semistable, then $z \not\in \text{lk}_1(w) \cup \text{lk}_2(v)$ (this set only contains vertices whose groups are finite). In particular, $z \not\in \text{lk}(w) \cup \text{lk}(v)$. That means $\text{lk}_3(z) = \text{lk}(z)$ and hence $(\text{lk}_3(z))$ is not finite. The minimality of $\Gamma$ implies that $G_3$ is semistable. Since $(\text{lk}(w) \cup \text{lk}(v)) - \{v, w\}$ separates $w$ and $v$ from the vertices of $\Gamma - (\text{st}(v) \cup \text{st}(w))$ (see Fig. 1), Lemma 2.3 implies

$$G = G_3 * ((\text{lk}(w) \cup \text{lk}(v)) - \{v, w\}) (\text{st}(v) \cup \text{st}(w)).$$

Theorem 1.1 implies $G$ is semistable - the desired contradiction. \qed

5. Proper relative simplicial approximation

In this section we review two results from [16] that will help prove Theorem 3.3. All spaces are simplicial complexes and subcomplexes are full subcomplexes of the over complex. If $X$ is a simplicial complex, then we say a subcomplex $Z$ separates a vertex $v \in X - Z$ from a subcomplex $Y$ of $X$ if any edge path in $X$ from $v$ to a vertex of $Y$ contains a vertex of $Z$. Our model for the following theorem is when the space $X$ is the Cayley 2-complex for a group split non-trivially as $G = A *_C B$ where $A$ and $B$ are finitely presented and $C$ is finitely generated.

We are interested in proper homotopies $M : [0, \infty) \times [0, 1] \rightarrow X$ of proper edge path rays $r$ and $s$ into a connected locally finite simplicial 2-complex $X$, where $r$ and $s$ have image in a subcomplex $Y$ of $X$. Simplicial approximation allows us to assume that $M$ is simplicial (see Lemma 5.3). If $G$ is the amalgamated product $A *_C B$, then the space $X$ is the Cayley 2-complex for $G$, and the space $Y$ is the Cayley subcomplex of $X$ for $A$. Say $\{Z_i\}_{i=1}^\infty$ is a collect of connected subcomplexes of $Y$ such that only finitely many $Z_i$ intersect any compact subset of $X$, and such that each $Z_i$ separates vertices of $Y$ from vertices of $X - Y$. In the $A *_C B$ setting, the $Z_i$ are the $aC$ cosets for each $a \in A$.

The next result describes how to simplicially excise the parts of $[0, \infty) \times [0, 1]$ that are not mapped into $Y$ (and perhaps a bit more is removed). What is removed from $[0, \infty) \times [0, 1]$ is a disjoint union of open sets $E_j$ for $j \in J$, each homeomorphic to $\mathbb{R}^2$ and $M([0, \infty) \times [0, 1] - (\bigcup_{j \in J} E_j)) \subset Y$. Also, $M$ maps the topological boundary of each $E_j$ into some $Z_i$. When $E_j$ is bounded in $[0, \infty) \times [0, 1]$ (contained in a compact set), there is an embedded edge path loop $\alpha_j$ in $[0, \infty) \times [0, 1]$ that bounds $E_j$, and $M(\alpha_j)$ has image in one of the $Z_i$. In this case the $E_j$ and $\alpha_j$ form a finite subcomplex of $[0, \infty) \times [0, 1]$ homeomorphic a closed ball which contains a certain equivalence class of triangles that are mapped into $X - Y$. When $E_j$ is unbounded, $\alpha_j$ is an embedded proper edge path line that bounds $E_j$. Again $M(\alpha_j)$ has image in one of the $Z_i$ and $E_j$ and $\alpha_j$ form a subcomplex of $[0, \infty) \times [0, 1]$ homeomorphic to the closed upper half plane. Again, in this case $E_j$ will contain a certain equivalence class of triangles, each of which is mapped into $X - Y$. 

![Figure 1. Separating $\Gamma$.](image-url)
Definition 1. We call the pair \((E, \beta)\) a disk pair in the simplicial complex \([0, \infty) \times [0, 1]\) if \(E\) is an open subset of \([0, \infty) \times [0, 1]\) homeomorphic to \(\mathbb{R}^2\), \(E\) is a union of cells, \(\alpha\) is an embedded edge path bounding \(E\) and \(E\) union \(\alpha\) is a closed subspace of \([0, \infty) \times [0, 1]\) homeomorphic to a closed ball or a closed half space in \([0, \infty) \times [0, 1]\). When \(\alpha\) is finite, we say the disk pair is finite, otherwise we say it is unbounded. Note that if \((E, \beta)\) is a disk pair, then \(\beta\) is collared in \(E\).

When there is enough information about the \(Z_i\), we intend to apply the theorem to alter the homotopy \(M\) to a proper homotopy in \(Y\).

Theorem 5.1 (Theorem 6.1, [16]). (excise) Suppose \(M : [0, \infty) \times [0, 1] \to X\) is a proper simplicial homotopy rel\([\ast]\) of proper edge path rays \(r\) and \(s\) into a connected locally finite simplicial 2-complex \(X\), where \(r\) and \(s\) have image in a subcomplex \(Y\) of \(X\). Say \(Z = \{Z_i\}_{i=1}^\infty\) is a collection of connected subcomplexes of \(Y\) such that only finitely many \(Z_i\) intersect any compact subset of \(X\). Assume that each vertex of \(X - Y\) is separated from \(Y\) by exactly one \(Z_i\).

Then there is an index set \(J\) and for each \(j \in J\), there is a disk pair \((E_j, \alpha_j)\) in \([0, \infty) \times [0, 1]\) where the \(E_j\) are disjoint, \(M\) maps \(\alpha_j\) to \(Z_{i(j)}\) (for some \(i(j) \in \{1, 2, \ldots\}\)) and \(M([0, \infty) \times [0, 1] - \bigcup_{j \in J} E_j) \subseteq Y\).

Remark 5.2. If \(Z_i \in Z\) is a finite subcomplex of \(X\) and \(M\) maps \(\alpha_j\) to \(Z_i\) then since \(M\) is proper, \(\alpha_j\) is a circle (and not a line), so that \(E_j\) is bounded in \([0, \infty) \times [0, 1]\). Also since \(M\) is proper, \(M^{-1}(Z_i)\) is compact, and so \(M\) maps only finitely many \(\alpha_j\) to \(Z_i\).

Claim 6.1.1 of [16] implies that under the hypotheses of Theorem 5.1: If \(K\) is a finite subcomplex of \([0, \infty) \times [0, 1]\), there are only finitely many \(j \in J\) such that \(\alpha_j\) has an edge in \(K\).

The Simplicial Approximation Theorem (see [Theorem 3.4.9, [19]]) applies to finite simplicial complexes. The next result is an elementary extension of that result.

Lemma 5.3 (Lemma 6.5, [16]). Suppose \(M : [0, \infty) \times [0, 1] \to X\) is a proper map to a simplicial complex \(X\) where \(M|[0,\infty)\times[0,1]\) and \(M|[0,\infty)\times[1]\) are (proper) edge paths and \(M(0,t) = M(0,0)\) for all \(t \in [0, 1]\). Then there is a proper simplicial approximation \(M'\) of \(M\) that agrees with \(M\) on \(([0, \infty) \times [0, 1]) \cup ([0] \times [0, 1])\).

Proof. (Of Theorem 3.3) We first consider the case when the groups \(A, B\) and \(G = A \ast_C B\) are finitely presented and \(C\) is finite. Suppose that \(G\) has semistable fundamental group at \(\infty\). Assume (for the sake of a contradiction) that \(A\) does not have semistable fundamental group at \(\infty\). Choose finite presentations \(P_1 = \langle A \cup C : R_1 \rangle\) and \(P_2 = \langle B \cup C' : R_2 \rangle\) for \(A\) and \(B\) respectively, where \(C = \{c_1, \ldots, c_n\}\) and \(C' = \{c_1', \ldots, c_n'\}\) generate \(C\) in \(A\) and \(B\) respectively. Also assume that in \(G\), \(c_i = c_i'\) for all \(i \in \{1, \ldots, n\}\) and \(R_1\) contains relations \(Q_1\) such that \(P_3 = \langle C : Q_1\rangle\) finitely presents \(C\).

Let \(S = A \cup B \cup C\) and \(R = R_1 \cup R_2\). Then a presentation for \(G\) is \(P = \langle S : R \rangle\) where each \(c_i'\) in a relation of \(R_2\) is replaced by \(c_i\). Let \(X\) be the standard 2-complex for \(P\) (one vertex \(\ast\), a labeled loop at \(\ast\) for each element of \(S\), and a 2-cell attached at \(\ast\) according to each relation \(r \in R\)).

Let \(\tilde{X}\) be the universal cover of \(X\). The space \(\tilde{X}\) is the Cayley 2-complex for \((G, P)\) with 1-skeleton the (labeled) Cayley graph for \((G, S)\). Each vertex of \(\tilde{X}\) is mapped by the covering map \(p : \tilde{X} \to X\) to \(\ast\), each edge of \(\tilde{X}\) is mapped to a loop with the same label and each 2-cell of \(\tilde{X}\) is mapped to a 2-cell with the same boundary labels.

Let \(\ast\) be the identity vertex of \(\tilde{X}\), let \(Y_\ast\) be the maximal (full) subcomplex of \(\tilde{X}\) containing \(\ast\) and such that each edge of \(Y_\ast\) is labeled by a letter in \(A \cup C\). The space \(Y_\ast\) is the (simply connected) Cayley 2-complex for \((A, P_1)\).

If \(v \in A\), let \(Z_v\) be the maximal full subspace of \(Y_\ast\) containing \(v\) such that each edge of \(Z_v\) is labeled by an element of \(C\), then \(Z_v \subset Y_\ast\), and \(Z_v\) is a copy of the Cayley 2-complex of \((C, P_3)\). Since \(C\) is a finite group, \(Z_v\) is a finite (simply connected) complex.

We begin with a triangulation of \(\tilde{X}\) that refines our cellular structure. We may assume there are no relations in \(R\) of length 1. Add a vertex to the interior of every edge so the edge is exchanged for two
edges). Now, every 2-cell is bounded by at least 4 edges. Triangulate each 2-cell, by selecting a point in the interior of each 2-cell to be a vertex, and defining edges from the new vertex to each of the other vertices of that cell. This defines a triangulation of \( \tilde{X} \).

There are proper edge path rays \( r \) and \( s \) in \( Y \), such that \( r \) and \( s \) are based at \( * \), converge to the same end of \( Y \) and are not properly homotopic in \( Y \) (since \( A \) is not semistable). As \( G \) is semistable, the proper rays \( r \) and \( s \) (which converge to the same end of \( \tilde{X} \)) are properly homotopic rel\( \{v\} \) in \( \tilde{X} \). Suppose \( H : [0, \infty) \times [0, 1] \) is a proper homotopy rel\( \{v\} \) of \( r \) to \( s \) in \( \tilde{X} \).

Lemma 5.3 implies there is a simplicial structure on \([0, \infty) \times [0, 1]\) and a proper simplicial homotopy \( M : [0, \infty) \times [0, 1] \to \tilde{X} \) of \( r \) to \( s \). We will apply Theorem 5.1 to \( M \) and then obtain a proper homotopy of \( r \) to \( s \) with image in \( Y \) (the desired contradiction).

In the hypothesis of Theorem 5.1, let \( \tilde{X} \) play the role of \( X \), let \( Y_{\ast} \) play the role of \( Y \). Since \( C \) is finite, \( C \) has infinite index in \( A \) (otherwise \( A \) is finite and finite groups are all semistable). Let \( \{v_1 C, v_2 C, \ldots\} \) be the \( C \)-cosets of \( A \). Let \( Z_i = Z_{v_i} \) and \( \{Z_i\}_{i=1}^{\infty} \) play the role of \( Z \) in Theorem 5.1. By Theorem 5.1 there are open subsets \( E_j \) for \( j \in J \), each homeomorphic to \( \mathbb{R}^2 \) and edge paths \( \alpha_j \) bounding \( E_j \) such that \( (E_j, \alpha_j) \) is a disk pair. Since each \( Z_i \) is finite, Remark 5.2 guarantees that each edge path \( \alpha_j \) is finite. By Theorem 5.1, \( M \) restricted to \([0, \infty) \times [0, 1] - \bigcup_{i=1}^{\infty} E_j \) has image in \( Y \) and for each \( j \), \( M(\alpha_j) \) has image in say \( Z_{i(j)} \). Let \( M_1 \) be the homotopy that agrees with \( M \) on \([0, \infty) \times [0, 1] - \bigcup_{i=1}^{\infty} E_j \) and on \( E_j \) is a simplicial homotopy that kills \( \alpha_j \) in \( Z_{i(j)} \). To complete the proof of Theorem 3.3 we need to show that \( M_1 \) is proper.

If \( K \) is a finite subcomplex in \( Y \) and \( M_1^{-1}(K) \) is not a finite complex, then there is an unbounded sequence of vertices in \([0, \infty) \times [0, 1]\) that are mapped by \( M_1 \) into \( K \). Since \( M \) is proper, there is an unbounded sequence of vertices in the union of the disks \( E_j \) that \( M_1 \) maps into \( K \). Since each \( E_j \) contains only finitely many vertices, there are infinitely many \( E_j \) each of which has a vertex that \( M_1 \) maps to \( K \). But Remark 5.2 guarantees that only finitely many disks are mapped by \( M_1 \) to a given \( aZ \) for \( a \in A \). This means that infinitely many distinct \( aZ \) intersect \( K \), a contradiction. Basically the same argument shows that if \( G \) is an HNN-extension \( A_f : G_1 \to G_2 \) with finite associated subgroups \( G_1 \) and \( G_2 \), and \( G \) has semistable fundamental group at \( \infty \) then \( A \) has semistable fundamental group at \( \infty \). Extending to graphs of groups decomposition with finite edge groups is now straightforward.

6. Potential new directions

Right angled Artin (Coxeter) groups are graph products with infinite cyclic (resp. \( \mathbb{Z}_2 \)) vertex groups. As noted earlier all of these groups were previously known to be semistable at \( \infty \). A simplicial complex is flag if every complete subgraph of its 1-skeleton bounds a simplex. If \( L \) is a finite simplicial complex, then the right angled Artin group \( A_L \) has finite presentation with generating set equal to the set of vertices of \( L \) and relation \( sts^{-1}t^{-1} = 1 \) if \( s \) and \( t \) are adjacent vertices in \( L \). Theorem B of N. Brady and J. Meier’s article [2] classifies which right angled Artin groups are \( m \)-connected (\( m \)-acyclic) at infinity. In particular, they prove:

**Theorem 6.1** (Corollary 5.2, [2]). Let \( L \) be a finite flag complex which is different from the 0- or 1-simplex. Then the right angled Artin group \( A_L \) is simply connected at infinity if and only if \( L \) is simply connected and contains no cut vertex.

The main theorem of M. Davis and J. Meier [6] gives geometric conditions on a finite simplicial complex that determine precisely when a corresponding (general) Coxeter group is simply connected at infinity. Generalizing these results to graph products of groups would be interesting.

We extended the definition of semistability at \( \infty \) to finitely generated groups in [11]. Nearly all of the major semistability results for finitely presented groups generalize to finitely generated groups. The main advantage of considering finitely generated groups with semistable fundamental group at \( \infty \) is that they can be used to show certain finitely presented groups have semistable fundamental group at \( \infty \) (for instance see [11] where solvable groups are shown to have semistable fundamental group at \( \infty \)).
In order to prove our Main Theorem in the setting of finitely generated groups we would simply need the corresponding results for Theorems 1.1, 1.2, 3.1, 3.2, and 3.3 where the finitely presented hypothesis is relaxed to finitely generated. With the exception of Theorem 1.1, it is an easy exercise to check that the proofs of the other results extend to the finitely generated setting. The proof of Theorem 1.1 is a formidable technical argument and not so simple to analyze.

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