Relatively-periodic solutions of planetary systems with satellites and systems with slow and fast variables

The partial case of the planar $N + 1$ body problem, $N \geq 2$, of the type of planetary system with satellites is studied. One of the bodies (the Sun) is assumed to be much heavier than the other bodies (“planets” and “satellites”), moreover the planets are much heavier than the satellites, and the “years” are much longer than the “months”. The existence of at least $2^{N-2}$ smooth 2-parameter families of symmetric periodic solutions in a rotating coordinate system is proved, such that the distances between each planet and its satellites are much shorter than the distances between the Sun and the planets. The existence of “gaps” in these families of solutions is proved, corresponding to $k : (k+1)$ resonances of angular frequencies of planets’ revolution around the Sun. Generating symmetric periodic solutions are described. Sufficient conditions for some periodic solutions to be orbitally stable in linear approximation are given. The results are extended to a class of Hamiltonian systems with slow and fast variables close to the systems of semidirect product type.

Key words: $n$-body problem, periodic solutions, orbital stability, averaging, slow and fast variables.

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§ 1. Introduction

We study the partial case of the planar $N + 1$ body problem, $N \geq 2$, that can be characterized as “the problem on the motion of a planetary system with satellites”.

An effective estimate for the number of smooth two-parameter families of symmetric periodic solutions of this problem in a rotating coordinate system is proved (theorems 2.1, 2.2(A) and corollary 2.1(Ξ) about “solutions of the first kind”). Sufficient conditions for orbital stability in linear approximation for some of these solutions are given (theorem 2.2(B)). Generating symmetric periodic solutions are described (theorem 2.1). The necessity of a nondegeneracy condition is proved (theorem 2.3 and corollary 2.1(♯)). The periodic solutions under our investigation are close to collections of independent “circular” solutions of the corresponding Kepler problems for each planet and each satellite. Via properties of periodic solutions of the Hill problem in the Lunar theory (Lemma 3.1 and Theorem 3.5), which were proved in [19] by means of the averaging method on a submanifold, the listed results are
Theorems 2.1 and 2.2 of the present work include as partial cases the result of F.R. Moulton [1] (that in turn generalizes results of G. Hill [2, 3] and E.W. Brown [4] on families of periodic solutions of the Hill problem and the restricted three-body problem, respectively, cf. [5; §17–19]) and a result by H. Poincaré [6] on the existence of periodic solutions and sufficient conditions of their orbital stability in linear approximation for the systems of the Sun–Earth–Moon and the Sun–two planets types (respectively). Theorem 2.1 implies the known results by G.A. Krassinsky [7] and E.A. Kudryavtseva [8, 9] on the number of periodic solutions of planetary systems without and with satellites (respectively), by V.N. Tkhai [10] on the number and the location of symmetric periodic solutions of the systems of the Sun–planets and Sun–planet–satellites types.

To be precise, the motions of planetary systems with satellites discovered in this paper are indeed relatively periodic (Definition 2.1) rather than periodic. This paper does not study orbital stability of these solutions, but it studies a weaker property of them, namely orbital stability in linear approximation (§4.3).

Our paper studies neither motions of such planetary systems as, e.g., Sun–Jupiter–asteroid, nor motions of such planetary systems with satellites as Sun–Saturn–Mimas–a particle of Saturn’s ring (recall, Mimas is Saturn’s satellite). The fact is that, in the former system, the asteroid plays the role of a “small planet” whose mass is much smaller than the mass of the “main planet”, the Jupiter, while in the latter system, the particle of Saturn’s ring plays the role of its “small satellite” whose mass is much smaller than the mass of the “main satellite”, Mimas. In contrast to these situations, our paper assumes (in order to reduce the number of small parameters) that the masses of all planets have the same order $\mu$ and masses of all satellites have the same order $\mu\nu$ (cf. (2.5)). That is, our paper does not consider systems with “small planets” (asteroids) or “small satellites” (like ring’s particles). Thus, results of our work are not applicable for explaining such phenomena in the Sun system as Kirkwood “hatches” in the asteroid bell or “gaps” in Saturn’s ring. Nevertheless, we justify the presence of similar “gaps” corresponding to resonances $(k+1) : k$ in our planetary systems.

Finally we remark that the question of interpretation of the discovered class of relatively periodic solutions of the $N+1$ body problem in terms of behaviour of planets and satellites of the real solar system is very interesting, needs an additional investigation and is not discussed in this work.
we construct a family of relatively periodic solutions of the unperturbed system, close to “generating” solutions. Along the way, we introduce coordinates in the phase space of the model (respectively, unperturbed or perturbed) problem, that bring the system to the form of the same name system in §3 (Lemma 5.1), besides we rediscover two families of periodic solutions of the Hill problem (discovered by G. Hill) via the method of averaging on a submanifold (§5.2). In §6, we start to derive Theorem 2.2(A) from Theorem 3.1. In §7, we prove (in Lemma 7.1) that the transformation (2.25) brings the $N + 1$ body problem to the perturbed system (2.22), and finish deriving Theorem 2.2(A) from Theorem 3.1. In §§, Theorems 2.1, 2.2(B), 2.3 are derived from Theorems 3.2, 3.3, 3.4.

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§ 2. Statement of the problem, main results
and methods of constructing solutions

Let us formulate the results of the paper more precisely.

2.1. Statement of the problem. The planar $N + 1$ body problem is described by the system of ODE’s

$$
\frac{dr_i}{dt} = \frac{\partial H(r, p)}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H(r, p)}{\partial r_i}, \quad 0 \leq i \leq N. \quad (2.1)
$$

Here we denote $\frac{\partial H(r, p)}{\partial p_i} := \left(\frac{\partial H}{\partial p_{i,1}}, \frac{\partial H}{\partial p_{i,2}}\right)$, $\frac{\partial H(r, p)}{\partial r_i} := \left(\frac{\partial H}{\partial q_{i,1}}, \frac{\partial H}{\partial q_{i,2}}\right)$,

$$
H = H(r, p) = H(r, p; g, \mu_0, \ldots, \mu_N) = \sum_{i=0}^{N} \frac{p_i^2}{2\mu_i} - \sum_{0 \leq i < j \leq N} \frac{g \mu_i \mu_j}{r_{ij}} =: G + U \quad (2.2)
$$

is the full energy of the system (equals the sum of kinetic and potential energies, $G$ and $U$), $r = (r_0, \ldots, r_N) \in Q \subset (\mathbb{R}^2)^{N+1}$ are “coordinates”, $p = (p_0, \ldots, p_N) \in (\mathbb{R}^2)^{N+1}$ are “momenta”, $r_i = (q_{i,1}, q_{i,2}) \in \mathbb{R}^2$ is the radius-vector of the $i$th body, $p_i = (p_{i,1}, p_{i,2}) \in \mathbb{R}^2$ is its momentum, $r_{ij} = |r_j - r_i|$, $0 \leq i, j \leq N$, are pairwise distances between bodies; $g > 0$ is the gravitation constant, $\mu_i > 0$ is the mass of the $i$th body. Here $Q \subset (\mathbb{R}^2)^{N+1}$ is the configuration manifold of the problem under consideration, consisting of all tuples of radius-vectors $r_i \in \mathbb{R}^2$, $0 \leq i \leq N$, such that $r_i \neq r_j$, $0 \leq i < j \leq N$. The system (2.1), (2.2) depends on $N + 2$ parameters $g, \mu_0, \ldots, \mu_N > 0$ and it is a Hamiltonian system with $2N + 2$ degrees of freedom, with the Hamilton function (2.2) and the symplectic structure

$$
\omega = dp \wedge dr = \sum_{i=0}^{N} dp_i \wedge dr_i := \sum_{i=0}^{N} \sum_{k=1}^{2} dp_{i,k} \wedge dq_{i,k}, \quad (2.3)
$$

defined on an open subset $T^* Q = Q \times (\mathbb{R}^2)^{N+1} \subset \mathbb{R}^{4(N+1)}$ with coordinates $r, p$.

In particular, the phase space of the problem has dimension $\dim(T^* Q) = 4N + 4$. Consider the submanifold $\hat{Q} := \{(r_0, \ldots, r_N) \in Q \mid \sum_{i=0}^{N} \mu_i r_i = 0\}$ in the
configuration manifold $Q$. It consists of all configurations of $N + 1$ particles in the Euclidean plane with masses $\mu_0, \ldots, \mu_N$ and the center of masses at the origin. There exists a natural symplectomorphism between $T^*Q$ and $4N$-dimensional symplectic submanifold $M^{4N} \subset T^*Q$:

$$M^{4N} := \left\{ (r_0, \ldots, r_N, p_0, \ldots, p_N) \in T^*Q \left| \sum_{i=0}^{N} \mu_i r_i = \sum_{i=0}^{N} p_i = 0 \right. \right\}. \quad (2.4)$$

The coordinates of the total momentum $p_0 + \ldots + p_N : T^*Q \to \mathbb{R}^2$ are first integrals of the system (2.1). It is enough to study the restriction of the system (2.1) to the $4N$-dimensional invariant submanifold $M^{4N} \approx T^*Q$.

**Remark 2.1.** Without loss of generality, we may choose the unities of mass, distance and time as it will be suitable. In fact, for any constants $a, b, c > 0$, the power transformation $\tilde{r}_i = b^2 c r_i$, $\tilde{p}_i = ab^{-1} c p_i$, $\tilde{t} = a^{-1} b^3 c t$, $\tilde{H} = a^2 b^{-2} c H$, $\tilde{\omega} = abc^2 \omega$, together with the transformation of parameters $\tilde{g} = a^2 g$, $\tilde{\mu}_i = c \mu_i$ brings the system (2.1), (2.2) to the Hamiltonian system with the Hamilton function $\tilde{H} = H(\tilde{r}, \tilde{p}; \tilde{\mu}_0, \ldots, \tilde{\mu}_N)$ and the symplectic structure $\tilde{\omega} = d\tilde{p} \wedge d\tilde{r}$. In particular, we do not need to assume that the gravitational constant $g$ is arbitrary, but we may assume its value to be a distinguished number that we will choose below. (This can be achieved via scaling the time, as we described above.)

**Definition 2.1.** A solution $(r(t), p(t))$ of the planar $N + 1$ body problem (2.1), (2.2) will be called *relatively periodic* (or *periodic*, by abuse of language) if the locations of all bodies (and, hence, momenta too) after the time-interval $T > 0$ can be obtained from their initial locations by rotating the plane by the same angle $\alpha$ around the centre of masses, for any initial time value, where $-\pi < \alpha \leq \pi$. The pair of real numbers $(T, \alpha)$ will be called the *relative period* of the solution, and the solution itself will be called $(T, \alpha)$-periodic.

Any solution obtained from a $(T, \alpha)$-periodic solution via shifting the time by a value $t$ and rotating by an angle $\varphi$ around the origin is a $(T, \alpha)$-periodic solution too. The union of the phase trajectories of all such solutions is a *two-dimensional torus* in the phase space, since it admits angular coordinates $t \mod T, \varphi \mod 2\pi$. All these solutions will be regarded as a single $(T, \alpha)$-periodic solution, and the union of their phase trajectories will be called the *phase orbit* of this solution.

Many relatively periodic solutions of the planar $N + 1$ body problem happen to be *symmetric* (Theorem 2.1). These solutions are characterized by the following property: at some time, all the bodies lie on the same line (i.e. a “parade” is observed) and their velocities are perpendicular to this line.

In the present work, the following partial case of the planar $N + 1$ body problem is considered, $N \geq 2$. We assume that the mass of one particle (the Sun) equals $\mu_0 = 1$ and is much greater than the masses of the other particles (the planets and satellites). Moreover the mass $\mu_i$ of the $i$th planet and the mass $\mu_{ij}$ of its $j$th satellite have the form

$$\mu_i = \mu m_i, \quad \mu_{ij} = \mu \nu m_{ij} \leq \mu_i, \quad 1 \leq i \leq n, \quad 1 \leq j \leq n_i, \quad (2.5)$$
where $0 < \mu, \nu \ll 1$ are small parameters and $m_i, m_{ij}$ are positive parameters far enough from zero (e.g. positive constants) with the properties
\[
\sum_{i=1}^{n} m_i = 1, \quad \min_{i=1}^{n} \sum_{j=1}^{n_i} \frac{m_{ij}}{m_i} = 1, \quad (2.6)
\]
moreover $m_{ij}$ are bounded for $n_i \geq 2$, where $n_i$ is the number of satellites of the $i$th planet and $1 + n + \sum_{i=1}^{n} n_i = N + 1$ is the number of all bodies. Thus, for each “double planet” ($n_i = 1$), the mass of the satellite equals $\mu m_i \theta_i/(1 - \theta_i)$ where the parameter $\theta_i := \nu m_{i1}/(m_i + \nu m_{i1}) \in (0, 1/2]$ is not necessarily small (since $m_i$ is not necessarily bounded).

We also assume that the distance $R_i$ between the Sun and the $i$th planet is of order $R \gg 1$, while each satellite is at the distance $r_{ij}$ of order 1 from its planet. Finally, “the years are much longer than the months”, i.e. the angular frequency $\varpi_i$ of the rotation of each planet around the Sun is of order $\varpi$, while the angular frequencies $\Omega_{ij}$ of the rotations of its satellites about it have order 1 where $0 < \omega \ll 1$. More precisely, let a set of non-vanishing real numbers
\[
\varpi_i = \varpi \Omega_{i0}, \quad \Omega_{ij}, \quad 1 \leq i \leq n, \quad 1 \leq j \leq n_i,
\]
called the set of angular frequencies satisfies the conditions
\[
c \leq |\Omega_{i0}| \leq |\Omega_{ij}| \leq 1, \quad 1 \leq i \leq n, \quad 1 \leq j \leq n_i, \quad (2.7)
\]
\[
||\Omega_{i0}| - |\Omega_{i'0}|| \geq c, \quad i < i', \quad ||\Omega_{ij}| - |\Omega_{ij'}|| \geq c, \quad 1 \leq j < j' \leq n_i. \quad (2.8)
\]
Here $c$ is a suitable real number in the interval $0 < c < 1$.

Suppose that the parameters $\omega, \mu, g, R$ satisfy the natural relations $\omega^2 R^3 = g$ and $1 = g \mu$ corresponding to Kepler’s second law for the planets ($\omega^2 R^3 = g \mu_0$) and the satellites ($\Omega_{ij}^2 r_{ij}^3 = g \mu_i$), for the chosen units of mass, distance and time. Thus, $g = \omega^2 R^3 = 1/\mu, \rho^3 = 1/R^3 = \omega^2 \mu \quad (2.10)$

and the problem has $N + 3$ independent parameters: $N$ parameters $m_i, m_{ij} > 0$ and three small parameters $\mu, \nu$ and $\varpi$ (here $\rho := 1/R$). We emphasize that the initial $N + 1$ body problem does not include the parameter $\varpi$ (and, hence, $\rho$ related with it), but it was introduced by us as an additional (“scaling”) small parameter. Namely, by means of this small parameter, we construct a $N + 3$-parameter family (with parameters $m_i, m_{ij}, \mu, \nu, \varpi > 0$) of “scaling” transformations of phase variables (2.23), (2.24), (2.25). These transformations bring solutions $(r(t), p(t))$, discovered by us, of the planetary system with satellites (with ratio of “months” to “years” $\varpi_i/\Omega_{ij}$ of order $\varpi \ll 1$) to motions $(\eta(t), \xi(t))$, that are close to “generating” (see below) circular motions $(Y^\beta(t), X^\beta(t))$ along circles whose radii have order 1. (The indicated “scaling” is necessary, since the radii of the corresponding circles for “non-scaled planets” and “satellites” equal $R_i, r_{ij}$ and have orders $R = \frac{1}{\rho} \gg 1$ and 1.) Therefore the only imposed restrictions to the parameters of the problem are as follows: the mass of the Sun is $\mu_0 = 1$, the distances from the satellites to their planets are of order 1, and the gravitational constant is $g = 1/\mu$. This does not cause any loss of generality due to remark 2.1.
2.2. Main results. Let us describe the “generating” relatively-periodic solutions of the $N+1$ body problem under consideration. These are the relatively-periodic solutions of the “model” problem (described below), whose phase trajectories form an $N$-dimensional invariant torus $\Lambda^0$ in a “model” phase space. In order to construct solutions of the initial $N+1$ body problem, that are close to “generating” ones, we will introduce three dynamical systems with $2N$ degrees of freedom, called model, unperturbed and perturbed problems, respectively. The first and the second systems (the model and unperturbed ones) depend on $N+1$ parameters $m_i, m_{ij} > 0$ and $\varpi \in \mathbb{R}$, and the second system is obtained from the first one by a perturbation of order $O(\varpi^2)$. The third system (the perturbed one) depends on $N+5$ parameters $m_i, m_{ij} > 0$ and $\varpi, \varepsilon, \mu, \nu, \rho \in \mathbb{R}, \varepsilon \neq 0$, and it is obtained from the second system by a regular 4-parameter perturbation with small parameters $\varepsilon, \mu, \nu, \rho$ (which are assumed to be 0 for the second system).

1) The model system is a Hamiltonian system with $2N$ degrees of freedom and $N+1$ parameters, with the following Hamilton function and the symplectic structure:

$$\varpi H_0(\mathbf{Y}_s, \mathbf{X}_s; m_*) + H_1(\mathbf{Y}_{**}, \mathbf{X}_{**}; m_*, m_{**}), \quad d\mathbf{X} \wedge d\mathbf{Y} = \omega_0 + \omega_1, \quad (2.11)$$

where the functions $H_0(\mathbf{Y}_s, \mathbf{X}_s) = H_0(\mathbf{Y}_s, \mathbf{X}_s; m_*)$, $H_1(\mathbf{Y}_{**}, \mathbf{X}_{**}) = H_1(\mathbf{Y}_{**}, \mathbf{X}_{**}; m_*, m_{**})$ and the closed 2-forms $\omega_0, \omega_1$ are defined by the formulae

$$H_0(\mathbf{Y}_s, \mathbf{X}_s) := \sum_{i=1}^{n} \left( \frac{X_i^2}{2m_i} - \frac{m_i}{|Y_i|} \right), \quad H_1(\mathbf{Y}_{**}, \mathbf{X}_{**}) := \sum_{i=1}^{n} \sum_{j=1}^{n_i} \left( \frac{X_{ij}^2}{2m_{ij}} - \frac{m_im_{ij}}{|Y_{ij}|} \right), \quad (2.12)$$

$$\omega_0 := d\mathbf{X} \wedge d\mathbf{Y} = \sum_{i=1}^{n} d\mathbf{X}_i \wedge d\mathbf{Y}_i, \quad \omega_1 := d\mathbf{X}_{**} \wedge d\mathbf{Y}_{**} = \sum_{i=1}^{n} \sum_{j=1}^{n_i} d\mathbf{X}_{ij} \wedge d\mathbf{Y}_{ij}. \quad (2.13)$$

Here $\mathbf{Y}_i, \mathbf{Y}_{ij} \in \mathbb{R}^{2} \setminus \{(0,0)\}$ are “coordinates”, $\mathbf{X}_i, \mathbf{X}_{ij} \in \mathbb{R}^{2}$ are “momenta”, $m_i, m_{ij} > 0$ and $\varpi \in \mathbb{R}$ are parameters. From now on, $\mathbf{Y}_* \in (\mathbb{R}^2)^n$ denotes the tuple of vectors $\mathbf{Y}_i$, and $\mathbf{Y}_{**} \in (\mathbb{R}^2)^{N-n}$ denotes the tuple of vectors $\mathbf{Y}_{ij}$ ($1 \leq i \leq n, 1 \leq j \leq n_i$). For $\varpi \neq 0$, the model problem $(2.11), (2.12), (2.13)$ splits into $N$ independent Kepler’s problems (for “scaled planets” and “satellites”).

Let us assume that $\varpi > 0$ and the collection $(2.7)$ is maximally relatively-resonant, i.e. has the form

$$\varpi_i = \varpi_1 + k_i \frac{2\pi}{T}, \quad \Omega_{ij} = \varpi_1 + K_{ij} \frac{2\pi}{T}, \quad 1 \leq i \leq n, \quad 1 \leq j \leq n_i, \quad (2.14)$$

where $k_i, K_{ij} \in Z, T > 0$. A solution $(\mathbf{Y}^0(t), \mathbf{X}^0(t)) = (\mathbf{Y}_s^0(t), \mathbf{Y}_{**}^0(t), \mathbf{X}_i^0(t), \mathbf{X}_{ij}^0(t))$ of the model problem $(2.11), (2.12), (2.13)$ will be called a main generating solution if it is a tuple of circular solutions

$$(\mathbf{Y}_\ell^0(t), \mathbf{X}_\ell^0(t)) := \frac{e^{i\varpi t}}{\Omega_{ij}^{2/3}}(1, i\Omega_{ij}m_{ij}), \quad (\mathbf{Y}_\ell(t), \mathbf{X}_\ell(t)) := \frac{e^{i\Omega_{ij} t}}{\Omega_{ij}^{2/3}}m_{ij}^{1/3}(1, i\Omega_{ij}m_{ij}) \quad (2.15)$$

of the corresponding Kepler problems (for the “scaled planets” around the Sun and the “satellites” around the planets) with angular frequencies $(2.7)$ of the form
(2.8), (2.9), (2.14), where \( \varpi_{\ell} = \varpi \Omega_{\ell 0} \) and the plane \( \mathbb{R}^2 \) is identified with \( \mathbb{C} \). Any solution \( (Y^\beta(t), X^\beta(t)) \), that is a tuple of circular solutions \( (Y_{\ell}^\beta(t), X_{\ell}^\beta(t)) := e^{i \beta t} (Y_{\ell}^\beta(0), X_{\ell}^\beta(0)) \), \( (Y_{\ell j}^\beta(t), X_{\ell j}^\beta(t)) := e^{i \beta t} (Y_{\ell j}^\beta(0), X_{\ell j}^\beta(0)) \), \( 1 \leq \ell \leq n, 1 \leq j \leq n_{\ell} \), with a given collection of frequencies, will be called a generating solution. Here \( \beta_{\ell}, \beta_{\ell j} \in \mathbb{R} / 2\pi \mathbb{Z} \) are arbitrary constants, \( \beta := (\beta_s, \beta_\ast) \in (\mathbb{R} / 2\pi \mathbb{Z})^N \). The union of phase trajectories of all generating solutions (with \( \varpi \neq 0 \)) is a \( N \)-dimensional torus \( \Lambda^\circ = \Lambda^\circ(\varpi, \omega_{\ast} ; m_s; m_\ast) \). Indeed: the polar angles of \( N \) radius vectors \( Y_{\ell}, Y_{\ell j} \) (drawn from the Sun to the planets and from the planets to their satellites) can be used as coordinates on it. Any generating solution can be obtained from the solution \( (Y^0(t), X^0(t)) \) by shifts along the flows of \( N \) pairwise commuting Hamiltonian vector fields with Hamiltonian functions \( I_i := [Y_i, X_i], I_{ij} := [Y_{ij}, X_{ij}] \). Due to the condition (2.14), generating solutions are \((T, \alpha)\)-periodic with relative period

\[
T > 0, \quad \alpha = \varpi T + 2\pi k \in (-\pi, \pi], \quad (\alpha \neq 0 \text{ при } N = n, \quad |\alpha| > \varpi^2 T \text{ при } N > n. \quad (2.17)
\]

The second part of the system is the non-autonomous Hamiltonian system with \( 2(N - n) \) degrees of freedom, which is given in the space with coordinates \( y_{ij} \in \mathbb{R}^2 \setminus \{(0, 0)\} \), \( x_i \in \mathbb{R}^2 \) \((1 \leq i \leq n, 1 \leq j \leq n_i) \), with the Hamilton function and the symplectic structure

\[
\varpi H_0(y_\ast, x_\ast; m_\ast), \quad \omega_0 = dx_\ast \wedge dy_\ast. \quad (2.18)
\]

cf. (2.12) and (2.13). Here \( y_{\ast}(t) = (y_1(t), \ldots, y_n(t)) \) denote the corresponding components of the solution \( (y_{\ast}(t), x_{\ast}(t)) = (y_1(t), \ldots, y_n(t), x_1(t), \ldots, x_n(t)) \) of the first part (2.18) of the system, and the functions \( \Phi(y_\ast, y_{\ast}) = \Phi(y_\ast, y_{\ast}; m_\ast) \) is defined by the formulae

\[
\Phi(y_\ast, y_{\ast}; m_\ast) := \sum_{i=1}^{n} \sum_{j=1}^{n_i} m_{ij} F(y_i, y_{ij}), \quad F(V, v) := \frac{V^2 v^2 - 3 \langle V, v \rangle^2}{2|V|^3}. \quad (2.20)
\]

Here \( y_i, y_{ij} \in \mathbb{R}^2 \setminus \{(0, 0)\} \) are “coordinates”, \( x_i, x_{ij} \in \mathbb{R}^2 \) are “momenta”, \( m_i, m_{ij} > 0 \) and \( \varpi \in \mathbb{R} \) are parameters of the unperturbed system. The first part (2.18) of the unperturbed system coincides with the corresponding part of the model system.
and for \( \varpi \neq 0 \) it is a collection of \( n \) independent Kepler’s problems for “scaled planets”. The second part (2.19) of the unperturbed system is \( O(\varpi^2) \)-close to the second part of the model system (2.11), and it is the collection of \( N - n \) “sidereal Hill problems” for the “satellites” (cf. §5.2), provided that \( \varpi \neq 0 \) and the “scaled planets” perform circular motions \( y_i(t) = Y_i^\beta(t), x_i(t) = X_i^\beta(t) \) by virtue of the first part (2.18) of the unperturbed problem.

For any \( |\varpi| > 0 \) small enough, denote by \((y^0(t), y^\alpha(t), x^0(t), x^\alpha(t)) = (y^0(t), x^0(t))\) such a \((T, \alpha)\)-periodic solution of the unperturbed problem (2.18), (2.19), that is \( O(\varpi) \)-close to the generating solution \((Y^0(t), X^0(t))\) and satisfies the following conditions: \( y^0(t) = Y^0_i(t), x^0(t) = X^0_i(t) \) and the vectors \( y^\alpha_i(0) \) are collinear with the abscissa axis. Let

\[
(y^\beta(t), x^\beta(t)) = (y^\alpha(t), y^\alpha(t), x^\alpha(t), x^\alpha(t)), \quad \beta = (\beta_*, \beta_{**}) \in (\mathbb{R}/2\pi\mathbb{Z})^N \tag{2.21}
\]

be a \((T, \alpha)\)-periodic solution of the unperturbed system, obtained from the indicated “main” solution \((y^0(t), x^0(t))\) by a composition of some shifts (in time periods \(\beta_i, \beta_{ij}\)) along the flows of \( N \) pairwise commuting Hamiltonian vector fields with the Hamilton functions \( I_i := \langle y_i, x_i \rangle, I_{ij} := \langle y_{ij}, x_{ij} \rangle \) and the symplectic structure \( d\mathbf{x} \wedge d\mathbf{y} \), where \( \beta_i, \beta_{ij} \in \mathbb{R}/2\pi\mathbb{Z} \) are arbitrary constants, \( \beta := (\beta_*, \beta_{**}) \). Let \( \Lambda = \Lambda(\varpi; \varpi, \Omega_{**}; m_*, m_{**}) \) be the union of phase trajectories of \((T, \alpha)\)-periodic solutions \((y^\beta(t), x^\beta(t))\), \( \beta \in (\mathbb{R}/2\pi\mathbb{Z})^N \). We prove (§5) by means of the averaging method that such a solution \((y^0(t), x^0(t))\) exists and it is \( O(\varpi^2) \)-close to the main generating solution \((Y^0(t), X^0(t))\) (hence any solution \((y^\beta(t), x^\beta(t))\) is \( O(\varpi^2) \)-close to the corresponding generating solution \((Y^\beta(t), X^\beta(t))\), and the \( N \)-dimensional torus \( \Lambda \) is \( O(\varpi^2) \)-close to the torus \( \Lambda^0 \) when \( 0 < |\varpi| \ll 1 \).

3) The perturbed problem is the Hamiltonian system with \( 2N \) degrees of freedom and \( N + 5 \) parameters, with the Hamilton functions and the symplectic structure

\[
\tilde{H} = \varpi\tilde{H}_0(\eta_*, \xi_*) + \varepsilon \left( \tilde{H}_1(\eta_{**}, \xi_{**}) + \varpi^2\tilde{\Phi}(\eta_*, \eta_{**}) \right), \quad \tilde{\omega} = d\eta_* \wedge d\eta_* + \varepsilon d\xi_* \wedge d\eta_{**},
\]

where the functions \( \tilde{H}_0(\eta_*, \xi_*) = \tilde{H}_0(\eta_*, \xi_*; \bar{m}_*, \mu), \tilde{H}_1(\eta_{**}, \xi_{**}) = \tilde{H}_1(\eta_{**}, \xi_{**}; \bar{m}_*, m_{**}, \nu), \tilde{\Phi} = \tilde{\Phi}(\eta_*, \eta_{**}; \bar{m}_*, m_{**}, \mu, \nu, \rho) \) are analytic, are in involution with the function

\[
\tilde{I} = \sum_{i=1}^n \left( [\eta_i, \xi_i] + \varepsilon \sum_{j=1}^{n_1} [\eta_{ij}, \xi_{ij}] \right)
\]

for any values of the parameters, and have the property \( \tilde{H}_0|_{\mu=0} = H_0, \tilde{H}_1|_{\nu=0} = H_1, \tilde{\Phi}|_{\mu=\nu=\rho=0} = \Phi \) (the functions \( \tilde{H}_0, \tilde{H}_1, \tilde{\Phi} \) and parameters \( \bar{m}_i \) are defined in more details in (7.18), (7.8) and (7.4), cf. (2.13). Here \( \eta = (\eta_*, \eta_{**}) \in \tilde{Q} = \tilde{Q}(m_*, m_{**}, \mu, \nu, \rho) \subset (\mathbb{R}^2)^N \) are “coordinates”, \( \xi = (\xi_*, \xi_{**}) \in (\mathbb{R}^2)^N \) are “momenta”, \( m_i, m_{ij} > 0 \) and \( \varpi, \varepsilon, \mu, \nu, \rho \in \mathbb{R} \) are parameters of the system, \( \varepsilon \neq 0 \). Here the domain

\[
\bigcup_{(m_*, m_{**}, \mu, \nu, \rho) \in \mathbb{R}^{N+3}} \tilde{Q}(m_*, m_{**}, \mu, \nu, \rho) \times (\mathbb{R}^2)^N \times \{(m_*, m_{**}, 0, 0, 0)\}
\]

contains the torus \( \Lambda^0(\varpi; \varpi, \Omega_{**}; m_*, m_{**}) \times \{(m_*, m_{**}, 0, 0, 0)\} \), coincides with the domain of (smooth and analytical) functions \( H_0, H_1, \tilde{H}_0, \tilde{H}_1, \tilde{\Phi} \) (as functions in
variables $\eta, \xi, m_*, m_{**}, \mu, \nu, \rho$, moreover the complement of this domain in $\mathbb{R}^{2N+3}$ is a real-algebraic subset. Here the functions $\hat{R}_0$ and $\hat{R}_1$ are defined by the conditions $\hat{H}_0 = H_0 + \mu \hat{R}_0$ and $\hat{H}_1 = H_1 + \nu \hat{R}_1$. It turns out (§6) that there exists a smooth family of tori $\hat{\Lambda} = \hat{\Lambda}(\omega, \varepsilon, \mu, \nu; \hat{\omega}, \hat{\Omega}_{**}; m_*, m_{**})$, containing the phase trajectories of all $(T, \alpha)$-periodic solutions near $\Lambda$, such that $\hat{\Lambda}_{|\varepsilon=0, \nu=0} = \Lambda$.

A key (and apparently new) observation of this work is the following. Consider the planar $N+1$ body problem with the center of masses at the origin and with the parameters $m_i, m_{ij}, \mu, \nu > 0$, and the following canonical (by Lemma 7.1) coordinates in its $4N$-dimensional phase space $M^{4N} \approx T^* \hat{Q}$ (cf. (2.4)):

$$\hat{r}_i := c_i - r_0, \quad \hat{r}_{ij} := r_{ij} - r_i, \quad \text{rde} \quad c_i := \frac{m_i r_i + \nu \sum_{\ell=1}^{n_i} m_{i\ell} r_{i\ell}}{m_i + \nu \sum_{\ell=1}^{n_i} m_{i\ell}}, \quad (2.23)$$

$$\hat{p}_i := p_i + \sum_{\ell=1}^{n_i} p_{i\ell}, \quad \hat{p}_{ij} := p_{ij} - \frac{\nu m_{ij}}{m_i + \nu \sum_{\ell=1}^{n_i} m_{i\ell}} \left( p_i + \sum_{\ell=1}^{n_i} p_{i\ell} \right), \quad (2.24)$$

$1 \leq i \leq n, 1 \leq j \leq n_i$. It turns out (Lemma 7.1) that the power transformation

$$\eta_i := \rho \hat{r}_i, \quad \eta_{ij} := \hat{r}_{ij}, \quad \xi_i := \frac{\hat{p}_i}{\sqrt{\rho \mu}}, \quad \xi_{ij} := \frac{\hat{p}_{ij}}{\rho \mu \nu}, \quad \tilde{H} := \sqrt{\frac{\rho}{\mu} H}, \quad \tilde{\omega} := \sqrt{\frac{\rho}{\mu} \omega} \quad (2.25)$$

brings the $N+1$ body problem on $M^{4N} \approx T^* \hat{Q}$ under consideration to the $N+3$-parameter subfamily of the phase space $T^* \hat{Q}$ (2.22), in which the parameters $m_i, m_{ij}, \omega, \varepsilon, \mu, \nu, \rho$ are positive and related by fractional-power relations

$$\rho = \omega^{2/3} \mu^{1/3}, \quad \varepsilon = \omega^{1/3} \mu^{2/3} \nu. \quad (2.26)$$

When we talk about closeness of some solution (or of some submanifold of the phase space) of one system to a solution (resp. a submanifold) of another system, we assume that the phase spaces of the model, unperturbed and perturbed systems are identified with the corresponding open subsets of the vector space $(\mathbb{R}^2)^N \times (\mathbb{R}^2)^N$ with coordinates $(\eta, \xi)$. Besides, we identify the phase space $T^* \hat{Q} \approx M^{4N}$ (depending on the parameters $m_i, m_{ij}, \mu, \nu > 0$) of the $N+1$ body problem under consideration with the corresponding subset (with the corresponding values of $m_i, m_{ij}, \mu, \nu$ and any $\rho > 0$) of the phase space $\hat{Q} \times (\mathbb{R}^2)^N$ of the perturbed system (2.22) via the transformation (2.25).

**Symmetric** periodic solutions of the three-body problem were studied already by Poincaré [6]. Recall the definition of a symmetric solution of the planar $N+1$ body problem.

**Definition 2.2.** Consider a problem describing the motion of $N+1$ particles in a Euclidean plane. A solution of this problem will be called symmetric if there exists a line $l$ in the plane, called the axis of symmetry, and a time $t = t_0$ satisfying one of the following (equivalent) conditions called a “parade” of the particles:

1) at the time $t = t_0$, all points are placed on the line $l$ (i.e. a “parade” of the particles is observed) and their velocities are orthogonal to the line $l$;

2) the locations (and, hence, also the velocities) of all particles at any time $t \in \mathbb{R}$ can be obtained from their locations at the time $2t_0 - t$ by reflecting with respect to the axis $l$. 

All particles of the system are assumed to be numbered. Any solution of the $N + 1$ body problem obtained from a symmetric solution by shifting the time and by rotating the plane is also symmetric. Similarly to the case of $(T, \alpha)$-periodic solutions, we will not distinguish such solutions and will regard them as a single symmetric solution.

Exactly $2^{N-2}$ of the generating solutions $(Y^\beta(t), X^\beta(t))$ are symmetric. They are determined by the corresponding tuples of values $\beta_\ell, \beta_{ij} \in (0, \pi)$ with $\beta_1 = 0$. Here the tuples $\beta = (\beta_*, \beta_{**}) \in (0, \pi)^N$ and $(\beta_* + \pi k_*, \beta_{**} + \pi K_*)$ correspond to the same relatively-periodic solution (we assume without loss of generality that the tuple $(k_*, K_*) \in \mathbb{Z}^N$ has no common factors). Let

$$B \subset \{0, \pi\}^N \subset (\mathbb{R}/2\pi\mathbb{Z})^N$$

be a subset of cardinality $2^{N-2}$ containing exactly one element of each such a pair of tuples. The symmetric $(T, \alpha)$-periodic solutions are characterized by the condition that “parades” of the planets and satellites are observed, moreover they repeat each half of the period, $T/2$. This means that all the particles of the system are posed on a line, which turns by the angle $\alpha/2$ after the time-interval $T/2$.

**Theorem 2.1 (On the number of families of relatively-periodic solutions).** There exist constants $\omega_0, C > 0$ and continuous positive functions $\mu_0 = \mu_0(m_*, m_{**}, \omega, c), \nu_0 = \nu_0(m_*, m_{**}, \omega, c), 0 < \omega \leq \omega_0, 0 < c < 1$, such that, for any parameter values $\omega, c, \mu, \nu$ with the properties $0 < \omega \leq \omega_0, 0 < c < 1, 0 < \mu \leq \mu_0(m_*, m_{**}, \omega, c), 0 < \nu \leq \nu_0(m_*, m_{**}, \omega, c)$, for any tuple of angle frequencies (2.7) of the form (2.8), (2.9), (2.14) the following properties hold. Suppose that the parameters (2.16) satisfy either the nondegeneracy condition (2.17) or the following conditions (which are more delicate when satellites are present):

$$\alpha \neq 0, \quad \alpha - \frac{\omega_i^2}{4\Omega_{ij}} T \notin [-C\omega^3 T, C\omega^3 T] + 2\pi\mathbb{Z}$$

for $1 \leq i \leq n, 1 \leq j \leq n_i$. Then the $N + 1$ body problem of the type of planetary system with satellites, $N \geq 2$, has exactly $2^{N-2}$ symmetric $(T, \alpha)$-periodic solutions $(\eta^\beta(t), \xi^\beta(t)), \beta \in B \subset \{0, \pi\}^N$ (cm. (2.27)), that in coordinates (2.25) are $O(\omega^2)$-close to the symmetric generating solutions $(Y^\beta(t), X^\beta(t))$ (corresponding to independent circular rotations of the “scaled planets” around the Sun and the “satellites” around the planets) with the angular frequencies (2.7). Each of these $2^{N-2}$ solutions depends smoothly on the pair $(T, \alpha)$, provided that $\beta$, the parameters of the problem and the integers $k_i, K_{ij}$ in (2.14) are fixed. All such solutions (for all possible values of the parameters of the problem as above, and for fixed $\omega \in (0, \omega_0], k_i, K_{ij}$ and $\beta$) form a $2 + (N + 2)$-parameter subfamily of the $2 + (N + 4)$-parameter smooth family of solutions of the perturbed problem (2.22) (with fixed $\omega$) with parameters $T, \alpha$ and $m_i, m_{ij} > 0, |\mu| \leq \mu_0(m_*, m_{**}, \omega, c), |\nu| \leq \nu_0(m_*, m_{**}, \omega, c), |\rho| \leq \mu_0(m_*, m_{**}, \omega, c), |\epsilon| \leq \mu_0(m_*, m_{**}, \omega, c)$, where the parameters of the subfamily are related by the relations $\mu, \nu > 0$ and (2.26). For each of these solutions, parades are observed, which repeat each time-interval $\frac{T}{2}$: all of the particles of the system are posed on a line (which turns by the angle $\alpha/2$ after the time-interval $T/2$).
Let $\Lambda^0 \subset (\mathbb{R}^2)^{2N}$ be the $N$-dimensional torus formed by the phase trajectories of the generating solutions (see (2.16)). Let $\Sigma \subset (\mathbb{R}^2)^{2N}$ be a “transversal surface” (called a cross section) of codimension 2 in the phase space, that transversally intersects invariant two-dimensional tori lying on $\Lambda^0$ and corresponding to the $(T, \alpha)$-periodic solutions:

$$\Sigma := \left\{ \sum_{i=1}^{n} (\varphi_i + \sum_{j=1}^{n_i} \varphi_{ij}) = \frac{T_{\min}}{T} \sum_{i=1}^{n} (k_i \varphi_i + \sum_{j=1}^{n_i} K_{ij} \varphi_{ij}) = 0 \mod 2\pi \right\}. $$

Here $\varphi_i$ and $\varphi_{ij}$ are polar angles of the radius-vectors $\eta_i$ and $\eta_{ij}$, $k_i, K_{ij}$ are integers in (2.14), $T_{\min}$ is the minimal positive period, hence the integer $T/T_{\min}$ is the greater common divisor of the collection of integers $k_i, K_{ij}$. Let $\Lambda = \Lambda(\varpi, \varepsilon, \mu, \nu, \rho; \varpi, \Omega, \mu, m, m)$ be the $N$-dimensional torus of the indicated family of tori, containing the torus $\Lambda = \tilde{\Lambda}(\varpi, 0, 0, 0, 0; \varpi, \Omega, \mu, m)$. Let $\Psi$ be the generating function of the “succession map” $g_1^T_H(\varpi) - \varpi_1 I$ of the $N + 1$ body problem under consideration (see (4.1) and Definition 3.2). Consider the smooth function $\tilde{S} = \Psi|_{\Lambda \cap \Sigma}$ on the $(N - 2)$-dimensional torus $\Lambda \cap \Sigma$. Since the function $\tilde{S}$ is defined on a $(N - 2)$-dimensional torus, it has at least $N - 1$ critical points, moreover at least $2^{N-2}$ points counted with multiplicities [11]. We will prove (see Theorem 2.2) the same lower bound for the number of $(T, \alpha)$-periodic solutions of the problem under consideration. Observe that the function $\tilde{S}$ has at least one critical point, since it is defined on a closed manifold. We will also prove that each critical point of the function $\tilde{S}$ corresponds to a $(T, \alpha)$-periodic solution of the $N + 1$ body problem. Moreover, we will offer sufficient conditions that guarantee the orbital (structural) stability in linear approximation (see definition 4.1) of such a solution.

The following condition will be called the strong nondegeneracy condition:

$$0 < |\alpha| < \pi \text{ при } N = n, \quad \varpi^2 T < |\alpha| < \pi - \varpi^2 T \text{ при } N > n.$$  \hfill (2.29)

The following conditions will be called the property of having fixed sign:

1) all planets rotate “to the same side”, i.e. the angular frequencies of the rotations of the planets around the Sun have the same sign:

$$\varpi_i \varpi_{i'} > 0, \quad 1 \leq i < i' \leq n; \hfill (2.30)$$

2) the function $\tilde{S}$ on the $(N - 2)$-dimensional torus is either a Morse function or has at least one nondegenerate critical point of a local minimum (this condition is assumed to be always true if $N = 2$).

**Theorem 2.2 (On Stability of a Relatively-Periodic Solution).** Under the hypothesis of Theorem 2.1, there exists a smooth $N$-dimensional torus $\tilde{\Lambda}$ in the phase space of the problem with coordinates (2.25) that is $O(\varpi^2)$-close to the torus $\Lambda^0$, smoothly depends on the pair $(T, \alpha)$ and has the following properties.

(A) The phase orbits of all $(T, \alpha)$-periodic solutions of the $N + 1$ body problem that are $O(\varpi)$-close to the torus $\Lambda^0$ are contained in the torus $\tilde{\Lambda}$. Moreover their intersection points with the cross section $\Sigma$ coincide with critical points of the function $\tilde{S} := \Psi|_{\tilde{\Lambda} \cap \Sigma}$ defined on the $(N - 2)$-dimensional torus $\tilde{\Lambda} \cap \Sigma$. Here $\Psi$ is the generating function of the “succession map” $g_1^T_H(\varpi) - \varpi_1 I$ of the problem under
consideration (see (4.1) and Definition 3.2). The function $\Psi|_{\Lambda}$ is an even function in the collection $\varphi$ of angle variables $\varphi_i|_{\Lambda}$, $\varphi_{ij}|_{\Lambda}$. The phase orbits of the symmetric $(T, \alpha)$-periodic solutions contain the points $\varphi$ of the torus $\Lambda$ having the property $\varphi = -\varphi$.

(B) Suppose that the property of having fixed sign holds, moreover either the strong nondegeneracy condition (2.29) or the following conditions hold (which are more delicate when satellites are present):

$$\alpha \notin \{0, \pi\}, \quad \frac{\text{sgn } \omega_i}{2} + \frac{\text{sgn } \Omega_{ij}}{2} - \frac{\omega_i^2}{8|\Omega_{ij}|} T \notin \left[ -\frac{C}{2} \omega^3 T, \frac{C}{2} \omega^3 T \right] + \pi \mathbb{Z}, \quad (2.31)$$

$$\frac{\text{sgn } \Omega_{ij}}{2} + \frac{\text{sgn } \Omega_{ij}'}{2} - \frac{\omega_i^2}{|\Omega_{ij}|} + \frac{\omega_j^2}{|\Omega_{ij}'|} T \notin \left[ -C \omega^3 T, C \omega^3 T \right] + \pi \mathbb{Z} \quad (2.32)$$

for all $1 \leq i, i' \leq n$, $1 \leq j \leq n_i$ and $1 \leq j' \leq n_{i'}$. Then the $(T, \alpha)$-periodic solution corresponding to any nondegenerate critical point of a local minimum of the function $\tilde{S}$ is orbitally structurally stable in linear approximation.

Theorem 2.2(B) implies that, for $N = 3$, in the “generic case”, a half of the $(T, \alpha)$-periodic solutions that are close to the torus $\Lambda^0$ are orbitally stable in linear approximation (since the function $\tilde{S}$ is defined on a circle and, hence, has only critical points of local minima and maxima, which alternate on the circle).

The natural question arises: is the nondegeneracy condition (2.17) necessary for the validity of theorem 2.1? An answer happens to be affirmative in many cases.

In the following theorem, by “almost any” collection of masses $\mu_i > 0$, $1 \leq i \leq n$, we mean any collection belonging to the complement in $\mathbb{R}^n_{>0}$ to the union of a finite set of linear subspaces of $\mathbb{R}^n$. Moreover each of these subspaces depends on the collection of integers $k_i$ in (2.14), has codimension at least 2, and the number of these subspaces does not exceed $2^{n-2}$. The set $\mathcal{M}^\text{sym}$ of “almost all” collections of masses is described in more detail in §8.1. By the phase space of satellites, we regard the direct product of big balls in the phase spaces of the corresponding Kepler problems, except for a small neighbourhood of “the set of possible collisions”.

**Theorem 2.3** (on “gaps” in families of relatively-periodic solutions). Suppose that, under the hypothesis of theorem 2.1, the number $n$ of planets is at least 2 and there exist two planets whose angular frequencies satisfy the following resonance relation:

$$\frac{\omega_i}{\omega_{i'}} \in \left\{ \frac{k}{k+1} \left| k \in \mathbb{Z} \setminus \{-1, 0\} \right. \right\}. \quad (2.33)$$

In this case, $\alpha = 0$ automatically. Then there exist an open dense subset $\mathcal{M}^\text{sym} \subset \mathbb{R}^n_{>0}$ and a nonempty open subset $\mathcal{M} \subset \mathcal{M}^\text{sym}$ (see definition 8.1 and remark 8.1), both invariant under multiplication by any positive real number and having the following properties. For “almost any” collection of planets’ masses $\mu_i > 0$, namely for any collection of planets’ masses $\mu(m_1, \ldots, m_n) \in \mathcal{M}^\text{sym}$ (respectively for any collection of planets’ masses $\mu(m_1, \ldots, m_n) \in \mathcal{M}$, for example satisfying the inequality $|k_i| \in |c_{k_i} m_i| > \sum_{i \neq i'} |k_i| |c_{k_i} m_i|$, see (8.3), (8.4), (8.5)), there exist numbers $\mu_0, \nu_0 > 0$ and an open subset $U_0$ in the phase space of the planets containing the phase orbits of all symmetric “circular” solutions (respectively all circular solutions).
of the collection of the Kepler problems for planets with angular frequencies (2.7), such that the following condition holds. For any values $\mu, \nu, T, \tilde{\alpha} \in \mathbb{R}$ of the form

$$0 < \left(\frac{\nu}{\nu_0}\right)^3 \leq \mu \leq \mu_0, \quad |T - T| + |\tilde{\alpha}| \leq D\mu,$$

there exists no $(T, \tilde{\alpha})$-periodic solution of the $N+1$ body problem under consideration whose phase orbit has a nonempty intersection with the direct product $U$ of $U_0$ and the phase space of the satellites.

In particular, the region $U$ does not contain the phase orbit of any symmetric $T$-periodic solution (respectively $T$-periodic solution).

Consider the planetary system with two planets, a partial case of the three-body problem. In this case, the minimal positive period $T_{\min}$ equals $\frac{2\pi}{|\omega_2 - \omega_1|}$, thus the condition (2.33) means that the corresponding angle $\alpha_{\min} = \frac{2\pi \omega_1}{\omega_2 - \omega_1} - 2\pi k$ vanishes. We also observe that, in this case, the region $U$ in theorem 2.3 contains the whole two-dimensional torus $\Lambda^\circ$ and $\mathcal{M}_{\text{sym}} = \mathcal{M} = \mathbb{R}^2_{\geq 0}$ (i.e. “almost any” means “any”). Thus, theorems 2.1 and 2.3 for $N = n = 2$ imply the following.

**Corollary 2.1.** Consider the three-body problem of the type of planetary system with two planets. Fix angular frequencies of planets $\omega_1 \neq 0, \omega_2 \neq 0, |\omega_1| \neq |\omega_2|$, and consider the two-dimensional torus $\Lambda^\circ$ corresponding to the circular motions of planets with frequencies $\omega_1, \omega_2$. Put $T := \frac{2\pi}{|\omega_2 - \omega_1|}, \alpha = \frac{2\pi \omega_1}{\omega_2 - \omega_1} + 2\pi k \in (-\pi, \pi]$ for a suitable $k \in \mathbb{Z}$. In dependence on the ratio of these frequencies, one of the following statements holds.

(3) Suppose that the angular frequencies $\omega_1, \omega_2$ does not satisfy the special resonance condition (2.33). Then $\alpha \neq 0$ and there exists a number $\mu_0 = \mu_0(m_1, m_2, \omega_1, \omega_2) > 0$ such that, for any values $\mu, \tilde{\omega}_1, \tilde{\omega}_2, |\mu| + |\tilde{\omega}_1 - \omega_1| + |\tilde{\omega}_2 - \omega_2| \leq \mu_0$, there exists a two-dimensional torus $\tilde{\Lambda} = \tilde{\Lambda}(\mu, \tilde{\omega}_1, \tilde{\omega}_2)$ that smoothly depends on the triple $(\mu, \tilde{\omega}_1, \tilde{\omega}_2)$, coincides with the torus $\Lambda^\circ$ if $(\mu, \tilde{\omega}_1, \tilde{\omega}_2) = (0, \omega_1, \omega_2)$ and has the following property. If $0 < \mu \leq \mu_0$ then the torus $\tilde{\Lambda}(\mu, \tilde{\omega}_1, \tilde{\omega}_2)$ is the phase orbit of a symmetric $(\tilde{T}, \tilde{\alpha})$-periodic solution of the problem under consideration with parameters $\tilde{T} = \frac{2\pi}{|\omega_2 - \omega_1|}, \tilde{\alpha} = 2\pi \frac{\tilde{\omega}_2 - \omega_1}{\omega_2 - \omega_1} + 2\pi k$, for a suitable $k \in \mathbb{Z}$. 

(3) Suppose that the angular frequencies $\omega_1, \omega_2$ are in a special resonance (2.33). Then $\alpha = 0$ and, for any numbers $T, D > 0$ with $\frac{2\pi - \omega_1 T - \omega_2}{2\pi} \in \mathbb{Z}$, there exist a number $\mu_0 = \mu_0(m_1, m_2, \omega_1, \omega_2, T, D) > 0$ and a neighbourhood $U = U_{m_1, m_2, \omega_1, \omega_2, T, D}$ of the torus $\Lambda^\circ$ in the phase space such that, for any parameter value $\mu \in (0, \mu_0)$ of the three-body problem under consideration, $U$ does not contain any $(\tilde{T}, \tilde{\alpha})$-periodic orbit with parameters $\tilde{T}, \tilde{\alpha}$ of the form

$$|\tilde{T} - T| + |\tilde{\alpha}| \leq D\mu.$$
Fig. 1. A set of pairs \((\kappa = \frac{\omega_1}{\omega_2 - \alpha_1}, \mu)\) such that there exists (\(\exists\)) or does not exist (\(\nexists\)) a relatively-periodic solution of the 3-body problem

\((T, \alpha)\)-periodic motion. These \((T, \alpha)\)-periodic solutions were discovered already by H. Poincaré [6] who called them solutions of the first kind. In the degenerate case (2.33), Poincaré discovered periodic solutions corresponding to elliptic motions of the planets, solutions of the second kind.

2.3. Method of finding relatively-periodic solutions of the \(N + 1\) body problem. Let us describe our idea of constructing \((T, \alpha)\)-periodic solutions that appear in Theorems 2.1 and 2.2. Consider the model system (2.11), (2.12), (2.13) and the corresponding torus \(\Lambda^0 \subset (\mathbb{R}^2)^{2N}\) formed by trajectories of “generating” solutions with angular frequencies (2.7) of the form (2.8), (2.9). Consider also the unperturbed and perturbed systems, cf. (2.18), (2.19) and (2.22). We construct solutions in three stages (§5, 6, 7). On the first stage (§5), by using our construction (via the averaging method on a submanifold [12, 9]) of periodic solutions of the Hill problem [19], we construct an \(N\)-dimensional torus \(\Lambda \subset (\mathbb{R}^2)^{2N}\), which is \(O(\omega^2)\)-close to the torus \(\Lambda^0\), is formed by phase trajectories of the unperturbed system with angular frequencies (2.7) and depends smoothly on the small parameter \(\omega \in [-\omega_0, \omega_0]\) and the tuple of real numbers \(\Omega_{ij} (1 \leq i \leq n, 0 \leq j \leq n_i)\). On the second stage (§6 and §3.1), we fix a value \(\omega \in (0, \omega_0]\) and a “relatively resonance” tuple of frequencies of the form (2.14). Then, \(|\mu|, |\nu|, |\rho|, |\varepsilon| \ll 1\), we construct a torus \(\widetilde{\Lambda} \subset (\mathbb{R}^2)^{2N}\), which depends smoothly on the parameters \(T, \alpha, \mu, \nu, \rho, \varepsilon\) and contains phase orbits of all \((T, \alpha)\)-periodic solutions of the perturbed problem near the torus \(\Lambda\), moreover \(\widetilde{\Lambda} = \Lambda\) if \(\mu = \nu = \rho = \varepsilon = 0\). We prove that exactly \(2^{N-2}\) of these solutions are symmetric, depend smoothly on the indicated parameters, and if \(\mu = \nu = \rho = \varepsilon = 0\) then they coincide with the symmetric solutions of the unperturbed system whose phase orbits are contained in the torus \(\Lambda\). On the third stage (§7), we show (in Lemma 7.1) that the transformation (2.25) brings the \(N + 1\) body problem to a \(N + 3\)-parameter subfamily of the \(N + 5\)-parameter family of perturbed systems (2.22), where the parameters in the subfamily satisfy the relation (2.26).

It would be interesting to prove analogues of Theorems 2.1–2.3 for the \(N + 1\) body problems on the sphere and the Lobachevskiy plane. We believe that our methods should work in this case, due to periodicity of solutions of the Kepler problems on these surfaces (cf. the work [13] and references therein).
§ 3. The averaging method for a class of systems with slow and fast variables

In §§5–8, we will derive theorems 2.1–2.3 from the next theorems 3.1–3.4 on periodic solutions of dynamical systems having the following special form.

Let $p : M \to M_0$ be a surjective submersion (e.g., a locally trivial fibre bundle) of smooth manifolds. Suppose, we are given a symplectic structure $\omega_0$ on $M_0$ and a closed 2-form $\omega_1$ on $M$ such that

$$T_x M = (\text{Ker } dp_x) \oplus (\text{Ker } \omega_1_x)$$

at every point $x \in M$. Thus, the 2-form $\omega_1$ defines a symplectic structure on each fibre, moreover its field of kernels is transversal to the fibre at every point of $M$ and defines a symplectic flat connection of the fibre bundle $p$. The question of the existence of such a connection on a given symplectic fibre bundle (whose base is not necessarily a symplectic manifold) was studied in [14] (cf. also [15]). The pair $(p^* \omega_0, \omega_1)$ is called a splitted symplectic structure on the fibre bundle $(M, M_0, p)$.

Suppose that functions $\hat{H}_0 \in C^\infty(M_0), H_1 \in C^\infty(M)$ and a constant $\lambda \in \mathbb{R}$ are given. Put

$$\omega_0 := p^* \omega_0, \quad H_0 := \hat{H}_0 \circ p, \quad H := H_0 + \lambda H_1.$$ 

A vector field $v = v_{H,H_1}$ on $M$ will be called $\lambda$-Hamiltonian if

$$(\omega_0(\cdot, v) - dH) |_{\text{Ker } \omega_1} = 0, \quad (\omega_1(\cdot, v) - dH_1) |_{\text{Ker } dp} = 0.$$ 

In this case, the dynamical system $\dot{x}(t) = v(x(t))$ on $M$ will be called $\lambda$-Hamiltonian and denoted by

$$(M, M_0, p; \omega_0, \omega_1; H, H_1)^\lambda; \quad \text{(3.1)}$$

moreover the function $H$ will be called the Hamilton function, and $H_1$ will be called the $\lambda$-Hamilton function of the system (3.1). If $\lambda \neq 0$ then the system (3.1) is equivalent to the Hamiltonian system $(M, \omega := \omega_0 + \lambda \omega_1, H)$:

$$(M, M_0, p; \omega_0, \omega_1; H, H_1)^\lambda \cong (M, \omega := \omega_0 + \lambda \omega_1, H).$$

Here the symbol $\cong$ denotes the equivalence of $(\lambda)$-Hamiltonian systems, i.e. the coincidence of the corresponding $(\lambda)$-Hamiltonian vector fields. If $\lambda = 0$ then the system (3.1) is a semidirect product of Hamiltonian systems (such systems are studied by Yu. M. Vorobiev [16]; they include e.g. the restricted three-body problem, i.e. the three-body problem with $N = 2, n = 1, \nu = 0$).

Denote by $g^H_{H,H_1}$ the flow of the $\lambda$-Hamiltonian vector field $v_{H,H_1}$. Similarly to the case of Hamiltonian systems, the flow of the field $v_{H,H_1}$ always (even for $\lambda = 0$) preserves the 2-form $\omega$ and the Hamilton function $H$.

**Example 3.1.** Suppose that $M = M_0 \times M_1$ is a direct product, moreover (similarly to $\omega_0$ and $H_0$) a 2-form $\omega_1$ and a function $H_1$ are “lifted” from $M_1$ (i.e., they have the form $\omega_1 = p_1^* \omega_1$ and $H_1 = \hat{H}_1 \circ p_1$, for some 2-form $\omega_1$ on $M_1$ and a function $\hat{H}_1 \in C^\infty(M_1)$, where $p_1 : M \to M_1$ is the projection). Then the $\lambda$-Hamiltonian system (3.1) for any $\lambda \in \mathbb{R}$ is equivalent to the Hamiltonian system
The following mechanical problems can be transformed (by changing parameters) to systems with slow and fast variables. Consider a two-parameter family of $\varepsilon$-Hamiltonian systems on $M$ with a splitted symplectic structure $(\omega_0, \omega_1)$, the Hamiltonian $H = \varpi H_0 + \varepsilon H_1$ and $\varepsilon$-Hamiltonian $H_1$ where $|\varepsilon|, |\varpi| \ll 1$. Here the 2-forms $\omega_0, \omega_1$ and the functions $\hat{H}_0$ and $H_1$ depend, in general, on the small parameters $\varepsilon, \varpi$ and possibly on some other parameters of the system, moreover some relations between parameters may be posed. The local coordinates of a point $x_0 \in M_0$ are “slow variables”, while the local coordinates of a point “on a fibre” are “fast variables” of the system.

**Example 3.2.** The following mechanical problems can be transformed (by changing phase variables) to systems with slow and fast variables.

(i) The problem on slow motions of a charged particle in a magnetic field on a symplectic Riemannian manifold $(M_0, \hat{\omega}_0, g)$. Let a magnetic field be given by a closed 2-form $\hat{\omega}_0$ on $M_0$, and an electric field be given by a smooth function $U(q)$ on $M_0$ (called the electric potential). Motions of a charged particle in an electro-magnetic field are described by the Hamiltonian system

$$(M = T^*M_0, \omega = dp \wedge dq + \omega_0, H = T(q, p) + U(q))$$

on $M = T^*M_0$, where $\omega := p^*\hat{\omega}_0, p : T^*M \to M$ is the projection. Here $q = (q^i)_{i=1}^n$ are local coordinates on $M_0$, $p = (p_j)_{j=1}^n$ are conjugated momenta, $T(q, p) = \frac{1}{2}g^{ij}(q)dp_ip_j$ is the kinetic energy. Suppose that the magnetic field $\hat{\omega}_0$ is nondegenerate, and the local coordinates $q$ are canonical for it, i.e. $\hat{\omega}_0 = \sum_{j=1}^n dq^{2j-1} \wedge dq^{2j}$. Let us change the coordinates $q \to q + Jp := x_0$ (called the guiding-centre transformation, cf. [17; §3 and 6] or [18; §3], where $J$ denotes the matrix of $\omega$) and rescale the momenta $p \to p/\varpi =: x_1$. Then the symplectic structure and the Hamilton function take the form

$$\omega = \omega_0 - \varepsilon \omega_1, \quad H = U(x_0 - \varpi Jx_1) + \varepsilon T(x_0 - \varpi Jx_1, x_1)$$

where $\omega_1 = \sum_{j=1}^{n/2} dx^{2j-1}_1 \wedge dx^{2j}_1$ and the small parameters $\omega, \varepsilon$ are related by the condition $\varepsilon = \varpi^2$. If $U \equiv 0$, i.e. there is no electric field, then the obtained system is a system with slow and fast variables. Here the local coordinates $x_0$ on $M_0$ are “slow”, while the local coordinates $x_1$ on fibres are “fast”.

(ii) The planar $N + 1$ body problem of the type of a planetary system with satellites reduces (due to Lemma 7.1) to the “perturbed” system (2.22), which is a system with slow and fast variables. In this system, $M = M_0 \times M_1$ is the direct product of the phase spaces of “scaled planets” and “satellites”, while the small parameters are related by (2.26). See Example 3.3 for details.

From now on, we assume that $M = M_0 \times M_1$ is the direct product and $\omega_1 = p_1^*\hat{\omega}_1$.

Suppose that each symplectic manifold $(M_i, \hat{\omega}_i)$ is equipped with the Hamiltonian action of a circle $SO(2) = S^1 = \mathbb{R}/2\pi \mathbb{Z}$ with the Hamiltonian function $I_i$, $i = 0, 1$. The system (3.1) will be called $S^1$-symmetric (or $SO(2)$-symmetric) if the functions
$H, H_1$ are invariant under the diagonal action of the circle on $M$ (i.e. invariant under the flow of the $\lambda$-Hamiltonian field $v_{I_1,I_1}$ on $M$ where $I = I_0 + \lambda I_1$). All solutions of this system that differ by shifts along the commuting vector fields $v_{I_1,H_1}$ and $v_{I_1,I_1}$ will be regarded as a single solution, and the union of their phase trajectories will be called the phase orbit of this solution. Let $T, \alpha \in \mathbb{R}$, $T \neq 0$. A solution $\gamma(t)$ of a $S^1$-symmetric system will be called $(T, \alpha)$-periodic if it is defined on the whole time-axis, and $\gamma(t) = g_{I_1,H_1}^T H_1 g_{I_1,I_1}^{-\alpha}(\gamma(t))$ for some (and, hence, for any) $t \in \mathbb{R}$.

**Definition 3.1.** A $S^1$-symmetric system (3.1) will be called reversible (or $O(2)$-symmetric) if each $M_i$ is equipped with an anti-canonical involution $\tau_i : M_i \to M_i$ preserving the function $I_i$ (i.e. $\tau_i^* \omega_i = -\omega_i$ and $I_i \circ \tau_i = I_i$), $i = 0, 1$, moreover the functions $H, H_1$ are invariant under the (component-wise anti-canonical) involution $\tau := \tau_0 \times \tau_1 : M \to M$. A solution $\gamma(t)$ of the reversible system will be called symmetric if it is defined on a time-interval $(t_0 - \varepsilon, t_0 + \varepsilon) \subset \mathbb{R}$ and

$$\gamma(t_0) = g_{I_1,I_1}^{\varphi_0} \tau g_{I_1,I_1}^{-\varphi_0}(\gamma(t_0)) \quad \text{for some } \varphi_0 \in \mathbb{R} \mod 2\pi$$

(3.3)

(and, hence, $\gamma(2t_0 - t) = g_{I_1,I_1}^{\varphi_0} \tau g_{I_1,I_1}^{-\varphi_0}(\gamma(t))$ for any $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$).

Let us describe the model system on $M = M_0 \times M_1$:

$$(M, M_0, p; \omega_0, \omega_1; \varpi H_0, H_1)^0 \cong (M, \omega_0 + \omega_1, \varpi H_0 + H_1),$$

(3.4)

$\varpi \in \mathbb{R}$, where $H_1 = \hat{H}_1 \circ p_1$ for some function $\hat{H}_1 \in C^\infty(M_1)$, cf. (3.1) and (3.2). We will assume that each of the Hamiltonian systems $(M_0, \hat{\omega}_0, \hat{H}_0)$ and $(M_1, \hat{\omega}_1, \hat{H}_1)$ in the system (3.4) is the direct product of $S^1$-symmetric Hamiltonian systems:

$$(M_0, \hat{\omega}_0, \hat{H}_0) = \prod_{i=1}^n (M_{i0}, \hat{\omega}_{i0}, H_{i0}), \quad (M_1, \hat{\omega}_1, \hat{H}_1) = \prod_{i=1}^n \prod_{j=1}^{n_i} (M_{ij}, \hat{\omega}_{ij}, H_{ij}).$$

Moreover each factor $M_{ij} = S^1 \times (a_{ij}, b_{ij}) \times \mathbb{R}^2$ is equipped with coordinates $\varphi_{ij} \mod 2\pi, I_{ij}, q_{ij}, p_{ij}$ such that $\hat{\omega}_{ij} = dI_{ij} \wedge d\varphi_{ij} + dp_{ij} \wedge dq_{ij}$, the action of the circle on $(M_{ij}, \hat{\omega}_{ij})$ is given by the Hamiltonian $I_{ij}$, and the involution $\tau$ acts component-wise in the form $(\varphi_{ij}, I_{ij}, q_{ij}, p_{ij}) \mapsto (-\varphi_{ij}, I_{ij}, q_{ij}, -p_{ij})$. In particular,

$$\omega_0 = \sum_{i=1}^n \omega_{i0}, \quad \omega_1 = \sum_{i=1}^n \sum_{j=1}^{n_i} \omega_{ij},$$

$$H_0 = \sum_{i=1}^n H_{i0}, \quad H_1 = \sum_{i=1}^n \sum_{j=1}^{n_i} H_{ij}, \quad H_{ij} = H_{ij}(I_{ij}, q_{ij}, p_{ij})$$

where, from now on, we use the same notation for a function (or a 2-form) and its lift, by abuse of notation. We will assume that each coordinate cylinder $S^1 \times (a_{ij}, b_{ij}) \times \{0\} \times \{0\} \subset M_{ij}$ consists of relative equilibrium points, i.e. the co-vector $dH_{ij}$ is proportional to $dI_{ij}$ at any its point (with coefficient depending on the point), $1 \leq i \leq n, 0 \leq j \leq n_i$. Therefore the $2N$-dimensional symplectic submanifold

$$\prod_{i=1}^n \prod_{j=0}^{n_i} S^1 \times (a_{ij}, b_{ij}) \times \{0\} \times \{0\} \subset M_0 \times M_1$$

(3.5)
is invariant under the flow of the model system (3.4) and it is fibred by invariant 
$N$-dimensional tori $\prod_{i=1}^n \prod_{j=0}^{n_i} S^1 \times \{(I_{ij}, 0, 0)\}$ where $N := \sum_{i=1}^n (1 + n_i)$. Those 
solutions of the model system whose phase orbits are contained in the invariant 
submanifold (3.5) will be called the generating solutions. Consider one of these 
$N$-dimensional tori, $\Lambda^\circ$, and a $(4N-2)$-dimensional “cross section” $\Sigma$ in the $4N$-dimensional 
phase space $M_0 \times M_1$, which is transversal to the two-dimensional phase orbits of 
the generating solutions contained in the $N$-torus $\Lambda^\circ$.

Let us describe the unperturbed system. Suppose that a $S^1$-invariant function 
$F_{ij} = F_{ij}(I_{i0}, q_{i0}, p_{i0}, \varphi_{i1} - \varphi_{i0}, I_{ij}, q_{ij}, p_{ij})$ is given on each direct product $M_{i0} \times M_{ij}$, 
$1 \leq j \leq n_i$. Put

$$\Phi := \sum_{i=1}^n \sum_{j=1}^{n_i} F_{ij}.$$  \hspace{1cm} (3.6)

As the unperturbed system, we will regard the 0-Hamiltonian system

$$(M_0 \times M_1, M_0, p; \omega_0, \omega_1; \varpi H_0, H_1 + \varpi^2 \Phi)^0$$  \hspace{1cm} (3.7)

with parameter $0 < \varpi \ll 1$. Then the $S^1$-action is given via the Hamiltonian function $I_0 = \sum_{i=1}^n I_{i0}$ and the 0-Hamiltonian function $I_1 = \sum_{i=1}^n \sum_{j=1}^{n_i} I_{ij}$.

Both systems described above: the model one (3.4) and the unperturbed one 
(3.7), are systems with slow and fast variables, because of a small factor $\omega$ in their 
Hamiltonian function $\omega H_0$.

In the following theorems, we suppose that each Hamiltonian system $(M_{ij}, \omega_{ij}, H_{ij})$ 
possesses the following properties of periodicity and nondegeneracy:

1) all solutions of the Hamiltonian system $(M_{ij}, \omega_{ij}, H_{ij})$ are periodic with periods 
$T_{ij} \circ H_{ij}$, for some functions $T_{ij} = T_{ij}(h) \neq 0$;

2) the functions $H_{ij} = H_{ij}(I_{ij}, q_{ij}, p_{ij})$ satisfy the following nondegeneracy conditions 
at the points $(I_{ij}, 0, 0) \in (a_{ij}, b_{ij}) \times \{0\} \times \{0\}$:

$$\frac{\partial H_{ij}}{\partial I_{ij}} = \Omega_{ij}(I_{ij}), \quad \frac{\partial^2 H_{ij}}{\partial I_{ij}^2} \neq 0 = \frac{\partial H_{ij}}{\partial q_{ij}} = \frac{\partial H_{ij}}{\partial p_{ij}}, \quad \det \frac{\partial^2 H_{ij}}{\partial (q_{ij}, p_{ij})^2} = \Omega_{ij}(I_{ij}),$$  \hspace{1cm} (3.8)

where $\Omega_{ij}(I_{ij}) := 2\pi/T_{ij}(H_{ij}(I_{ij}, 0, 0))$, $1 \leq i \leq n$, $0 \leq j \leq n_i$ (in particular, the 
circles $\{I_{ij}\} \times S^1 \times \{0\} \times \{0\}$ are elliptic nondegenerate relative equilibria of this 
$S^1$-symmetric system).

Suppose also that the collection of “angular frequencies” $\Omega_{ij} = \Omega_{ij}(I_{ij})^0$ (for 
some frequencies $I_{ij}^0 \in [a_{ij}^0, b_{ij}^0]$) and the real number $\varpi > 0$ satisfy the following 
conditions:

3) the “relative resonance” condition

$$\varpi^{1-j} \Omega_{ij} = \omega_1 + k_{ij} \frac{2\pi}{T}, \quad 1 \leq i \leq n, \quad 0 \leq j \leq n_i,$$  \hspace{1cm} (3.9)

where the integers $k_{ij} \in \mathbb{Z}$ are not 0 simultaneously, $\bar{\ell} := \max\{0, \ell\}$, $\omega_1 := \varpi \Omega_{10}$ 
and $T > 0$;

4) the nondegeneracy condition (for $\alpha := \varpi T$ and $C_1 > |\Delta_{ij}|$, cf. (3.12) below)

$$\alpha \notin 2\pi \mathbb{Z} \text{ for } N = n, \quad \alpha \notin [-C_1 \varpi^2 T, C_1 \varpi^2 T] + 2\pi \mathbb{Z} \text{ for } N > n;$$  \hspace{1cm} (3.10)
where the function $\langle \hat{F}_{ij}^0 \rangle$ is obtained by averaging the function $F_{ij}|_{\{I_{ij}^0,0,0\} \times H_{ij}^{-1}(H_{ij}(I_{ij}^0,0))} =: \hat{F}_{ij}^0$ along $2\pi \over \Pi_{ij}$-periodic solutions of the system $(M_{ij}, \omega_{ij}, H_{ij})$.

**Theorem 3.1 (On the Number of Relatively-Periodic Solutions).** Suppose that each Hamiltonian system $(M_{ij}, \omega_{ij}, H_{ij})$ possesses the properties 1) and 2) of periodicity and nondegeneracy, and that the number of the systems is $N \geq 2$. Then, for any collection of segments $[a_{ij}, b_{ij}] \subset (a_{ij}, b_{ij})$, there exist real numbers $\omega_0, C_1, C_2 > 0$ such that the following conditions hold for any $\varpi \in (0, \omega_0]$. Suppose that, for some numbers $I_{ij}^0 \in [a_{ij}, b_{ij}]$, the real number $\varpi$ and the collection of “angular frequencies” $\Omega_{ij} = \Omega_{ij}(I_{ij}^0)$ satisfy the “relative resonance” condition 3), as well as either the nondegeneracy condition 4) or the more delicate nondegeneracy condition 4'). Then there exists $\varepsilon_0 > 0$ such that, for $0 < \varepsilon \leq \varepsilon_0$, any $S^1$-symmetric (“perturbed”) Hamiltonian system

$$\left( M, \omega_0 + \varepsilon \omega_1, \hat{H} \right) \cong (M, M_0, p; \omega_0, \omega_1; \hat{H}, \hat{H}_1 + \omega^2 \hat{\Phi})$$  \hspace{1cm} (3.13)$$
on M = M_0 \times M_1$ has at least $N - 1 (T, \alpha)$-periodic solutions close to generating solutions with angular frequencies $\varpi \Omega_{ij}, \alpha$, provided that $\hat{H} := \varpi \hat{H}_0 + \varepsilon \hat{H}_1 + \omega^2 \varepsilon \hat{\Phi}$, the function $\hat{H}_0$ “projects” to the factor $M_0$, and $\| \hat{H}_0 - H_0 \|_{C^2} + \| \hat{H}_1 - H_1 \|_{C^2} + \| \hat{\Phi} - \Phi \|_{C^2} \leq \varepsilon_0$. Moreover there exist at least $2N-2$ such solutions counted with multiplicities. The phase orbits of all such solutions are contained in some $N$-dimensional torus $\tilde{\Lambda}$ that is $O(\varpi)$-close (and even $O(\varpi^2)$-close in the case $d(\hat{F}_{ij}^0(0,0)) = 0$) to the torus $\Lambda^0 := \prod_{i=1}^{n} \prod_{j=0}^{N_{ij}} S^1 \times \{ (I_{ij}^0, 0, 0) \}$ with respect to a $C^1$-norm. The intersection points of these phase orbits with a cross section $\Sigma$ (i.e. with a transversal surface to the two-dimensional phase orbits of generating solutions on $\Lambda^0$) coincide with critical points of the function $\Psi_{\tilde{\Lambda}_{\Sigma}}$ where $\Psi$ is the generating function of the perturbed succession map $g_{H, \hat{H}_1 + \omega^2 \hat{\Phi}}^{T} : M \rightarrow M$, $\hat{I} := I_0 + \varepsilon I_1$ (see Definition 3.2 below). Suppose that the 0-Hamiltonian system $(3.7)$ is contained in a $r$-parameter family of $\varepsilon$-Hamiltonian systems of the form $(3.13)$ such that the functions $\hat{H}_0, \hat{H}_1, \hat{\Phi}$ depend smoothly (or analytically on $r \geq 2$ parameters $(\varepsilon, \varpi, \ldots)$, moreover $H_0 = H_0, \hat{H}_1 = H_1, \hat{\Phi} = \Phi$ for the zero values of parameters $(0, 0, \ldots)$. Then, for $\varepsilon = 0$, there exists a $(N+r-1)$-parameter family of $N$-dimensional tori $\Lambda$ that are invariant w.r.t. the corresponding 0-Hamiltonian systems and depend smoothly (respectively, analytically on the tuple $\Omega_{ij}$ of “angular frequencies” $\Omega_{ij} = \Omega_{ij}(I_{ij})$ and the
indicated tuple of parameters apart from $\varepsilon$ (in the domain $|(\varpi, \ldots)| < \varpi_0(\Omega_{**})$), moreover $\Lambda = \Lambda^0$ if $\varepsilon = \varpi = \ldots = 0$. Every torus $\Lambda$ of this family corresponding to a “relatively resonance” tuple of “angular frequencies” $\Omega_{ij}$ and a real number $\varpi$ (with $\varpi \neq 0$ if $n > 1$), is contained in a $(2 + r)$-parameter family of $N$-dimensional tori $\tilde{\Lambda}$, depending smoothly (respectively, analytically) on $T, \alpha$ and the indicated tuple of parameters (where $|\varepsilon| < \varepsilon_0(T, \alpha, k_{**}, \varpi, \ldots)$). If $\varepsilon = 0$ then the torus $\tilde{\Lambda}$ coincides with the torus $\Lambda$. If $\varepsilon, \varpi > 0$ then the torus $\tilde{\Lambda}$ possesses the above properties w.r.t. the $(T, \alpha)$-periodic solutions of the “perturbed system”.

Similarly to §4.4 (B1), one can show that, for any collection $(a^0, b^0)$ of real numbers $a_{ij}^0, b_{ij}^0$ under consideration and for small enough $0 < \varpi \ll 1$, the period $T$ can take an arbitrary value of the form $T \geq 2\pi a_0(a^0, b^0)/\omega$, hence the quantity $\varpi^2T$ can be arbitrarily small. Thus any of the nondegeneracy conditions (3.10) and (3.11) can always be fulfilled.

**Theorem 3.2 (On Symmetric Relatively-Periodic Solutions).** Suppose that, under the hypothesis of Theorem 3.1, each of three systems: the model system (3.4), the unperturbed system (3.7) and the perturbed system (3.13) is reversible (Definition 3.1). Then the perturbed system (3.13) admits exactly $2^{N-2}$ symmetric $(T, \alpha)$-periodic solutions that are $O(\varpi)$-close (and even $O(\varpi^2)$-close in the case $d(F_{ij}^0)(0,0) = 0$) to the generating symmetric solutions with the angular frequencies under consideration. Each of these $2^{N-2}$ solutions smoothly depends on the pair of parameters $(T, \alpha)$. Moreover the function $\Psi|_{\tilde{\Lambda}}$ is an even function in the collection $\varphi$ of angular frequencies $\varphi_{ij}|_{\tilde{\Lambda}}$, and the phase orbits of the symmetric $(T, \alpha)$-periodic solutions pass through the points $\varphi$ of the torus $\tilde{\Lambda}$ with the property $\varphi = -\varphi$.

**Theorem 3.3 (On Stability of a Relatively-Periodic Solution).** Under the hypothesis of Theorem 3.1, suppose that all the numbers $\frac{\partial^2 H_{ij}}{\partial i_j^2}(I_{ij}^0, 0, 0)$ have the same sign, e.g. negative (respectively positive). Suppose also that either the strong nondegeneracy condition holds:

$$\alpha \notin \pi \mathbb{Z} \quad \text{for} \quad N = n, \quad \alpha \notin [-C_1 \varpi^2 T, C_1 \varpi^2 T] + \pi \mathbb{Z} \quad \text{for} \quad N > n$$

and the following condition of having the same sign holds: all the signs

$$\eta_{ij} := \text{sgn} \left( \Omega_{ij} \text{Tr} \frac{\partial^2 H_{ij}(I_{ij}^0, 0, 0)}{\partial (q_{ij}, p_{ij})^2} \right), \quad 1 \leq i \leq n, \quad 0 \leq j \leq n_i,$$

are equal, or the following more delicate condition holds: for any set of real numbers $\alpha_{ij} \in \mathbb{R}, 1 \leq i \leq n, 0 \leq j \leq n_i$, such that

$$\alpha_{i0} = \eta_{i0} \alpha, \quad |\alpha_{ij} - \eta_{ij}(\alpha + \Delta_{ij} \varpi^2 T)| \leq C_2 \varpi^3 T, \quad 1 \leq i \leq n, \quad 1 \leq j \leq n_i,$$

the sum of any two, possibly coinciding, numbers of the set does not belong to the set $2\pi \mathbb{Z}$. Then the $(T, \alpha)$-periodic due to Theorem 3.1 phase orbit of the perturbed system (3.13) passing through any nondegenerate critical point of local minimum (respectively maximum) of the function $\Psi|_{\tilde{\Lambda} \cap \Sigma}$ is orbitally structurally stable in linear approximation (see Definition 4.1).
3.1. Proof of theorems 3.1 and 3.2 and a scheme of proof of Theorems 3.3 and 3.4. Let us describe two main stages of the proof of theorems 3.1–3.4.

Stage one is based on the averaging method on a submanifold \([21, 12, 9]\) (a similar assertion for systems with slow and fast variables see in \([22]\)). Using this method, we study the \((0\text{-Hamiltonian})\) unperturbed system \((3.7)\) taking into account that it is close to the “super-integrable” model system \((3.4)\). Namely, at first, we describe \((T, \alpha)\)-periodic solutions of the unperturbed system \((3.7)\) that are close to the generating solutions of the model system \((3.4)\). At second, we study the linearization of the “succession map” for these solutions (see Theorem 3.5).
Let us first consider the unperturbed system (3.7) in the simple partial case \((N, n) = (2, 1)\) and \(T = 2\pi/|\Omega_{11} - \varpi_1|\) (an analogue of the Hill problem from the Lunar theory). By assumption, the circle \(S^1 \times \{(I_{10}, 0, 0)\}\) is a relative equilibrium of the system \((M_{10}, \omega_{10}, \varpi H_{10})\), and the motion on it by virtue of this system is homogeneous with angular frequency \(\varpi \Omega_{10} = \varpi_1\) (cf. (3.9)). Hence, this circle consists of equilibria of the Hamiltonian system \((M_{10}, \omega_{10}, \varpi H_{10} - \varpi_1 I_{10})\) (which is an analogue of the “synodic Kepler problem” (5.13) for the “scaled planet”). Recall that the \(S^1\)-action is given by the Hamiltonian \(I_{10}\) and \(0\)-Hamiltonian \(I_{11}\). Let us study the 0-Hamiltonian system

\[
(M_{10} \times M_{11}, M_{10}, p; \omega_{10}, \omega_{11}; \varpi H_{10} - \varpi_1 I_{10}, H_{11} - \varpi_1 I_{11} + \varpi^2 F_{11})^0,
\]

which is obtained from a “linear combination” of two commuting 0-Hamiltonian systems: the unperturbed one and the system defining the \(S^1\)-action. Clearly, the problem on finding its \(T\)-periodic solutions of the form \((\varphi_{10}, I_{10}, 0, 0, \ldots)\) is equivalent to finding \((T, \alpha)\)-periodic solutions of the unperturbed system, where \(\alpha = 2\pi \varpi_1/|\Omega_{11} - \varpi_1|\). Put

\[
\hat{F}(\varphi, I, q, p; I_{10}) := F_{11}(I_{10}, 0, 0, \varphi, I, q, p)/\Omega_{10}^2(I_{10}),
\]

where \(\Omega_{10}(I_{10}) := 2\pi/T_{10}(H_{10}(I_{10}, 0, 0))\), cf. (3.6). Since \(\varpi_1 = \varpi \Omega_{10}\), we have

\[
\varpi_1^2 \hat{F} = \varpi^2 F_{11}(I_{10}, 0, 0, \varphi, I, q, p).
\]

Therefore, the problem of finding solutions of our system (3.15) of the form \((\varphi_{10}, I_{10}, 0, 0, \ldots)\) is equivalent to finding solutions of the Hamiltonian system

\[
(M_{11}, \omega_{11}, H_{11} - \varpi_1 I_{11} + \varpi^2 \hat{F})
\]

depending on two parameters \(\Omega_{10}, \varpi_1\). This system is called a \textit{generalized Hill problem} [19].

**Lemma 3.1 (On periodic solutions of the generalized Hill problem, cf. [19; Theorem 2])**. Consider a 1-parameter family of Hamiltonian systems with two degrees of freedom, with the Hamilton function and the symplectic structure

\[
H_{11}(I, q, p) = \varpi_1 I + \varpi^2 \hat{F}(\varphi, I, q, p), \quad \omega_{11} = dI \wedge d\varphi + dp \wedge dq
\]

on \(M_{11} = S^1 \times (a, b) \times \mathbb{R}^2\), with parameter \(\varpi_1 \in \mathbb{R}\). Suppose that some neighbourhood of the cylinder \(S^1 \times (a, b) \times \{0\} \times \{0\} \subset M_{11}\) is filled by periodic trajectories of the Hamiltonian system \((M_{11}, \omega_{11}, H_{11})\) with periods \(T_{11} \circ H_{11}\), for some function \(T_{11} = T_{11}(h) \neq 0\). Suppose that the function \(H_{11} = H_{11}(I, q, p)\) satisfies the nondegeneracy conditions (3.8) at the points \((I, 0, 0) \in (a, b) \times \{0\} \times \{0\}\), where \(\Omega_{11}(I) := 2\pi/T_{11}(H_{11}(I, 0, 0))\) (so, the system (3.16) is a generalization of the systems (5.9) and (5.12), which are equivalent to the Hill problem (5.11) if \(\varpi_1 \neq 0\), cf. §5.2). Then:

(A) there exist a continuous function \(\varpi_0 = \varpi_0(\Omega) > 0\) in \(\Omega \in \Omega_{11}((a, b))\) and a 2-parameter family of \(2\pi/|\Omega_{11} - \varpi_1|\)-periodic solutions \(\gamma_{\Omega, \varpi_1}(t) = (\varphi_{\Omega, \varpi_1}(t), I_{\varpi_1}(t), q_{\varpi_1}(t), p_{\varpi_1}(t))\) of the system (3.16) with parameters \(\Omega \in \Omega_{11}((a, b))\) and \(\varpi_1 \in (-\varpi_0(\Omega), \varpi_0(\Omega))\) such that \(\varphi_{\Omega, \varpi_1}(0) = 0\) and \(\Omega_{11}(I), 0(t) = (\Omega_{11}(I), 0, 0)\) for \(I \in (a, b)\). If the functions \(H_{11}, \hat{F}\) depend smoothly on the parameters \(\lambda_1, \ldots\), then the function \(\varpi_0 > 0\) depends smoothly on \(\Omega, \Lambda, \lambda_1, \ldots\), moreover the solution \(\gamma_{\Omega, \varpi_1}(t)\) depends smoothly on the parameters \(\varpi_1, \Omega, \lambda_1, \ldots\);
(B) if \( d(\hat{F}^\circ)(0,0) = 0 \) for some \( I^\circ \in (a,b) \), then \( \gamma_{\Omega^\circ,\varpi_1}(t) = ((\Omega^\circ - \varpi_1)t, I^\circ, 0, 0) + O(\varpi_1^2) \). Here \( \Omega^\circ := \Omega_{11}(I^\circ) \), and the function \( \langle \hat{F}^\circ \rangle \) is obtained by averaging the function \( \hat{F}^\circ := \hat{F}|_{H_{11}^{-1}(H_{11}(I^\circ,0,0))} \) along the \( 2\pi \) periodic solutions of the system \((M_{11},\omega_{11},H_{11})\):

(C) there exists a family of canonical frames \( e_{\Omega,\varpi_1,1}, e_{\Omega,\varpi_1,2}, e_{\Omega,\varpi_1,3}, e_{\Omega,\varpi_1,4} \) in the tangent spaces \( T_{x}M_{11} \), in which the linearization \( d(2\pi/|\Omega - \varpi_1|)_{H_{11}^{-1}}(H_{11}(I^\circ,0,0)) \) of the “succession map” at the point \( x := \gamma_{\Omega,\varpi_1}(0) \) is given by the matrix

\[
\begin{pmatrix}
1 & \frac{\partial^2 H_{11}}{\partial q^2}(\Omega_{11}^{-1}(\Omega),0,0)\frac{2\pi}{|\Omega - \varpi_1|} & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \cos \alpha_{11} & \sin \alpha_{11} \\
0 & 0 & -\sin \alpha_{11} & \cos \alpha_{11}
\end{pmatrix},
\]

for some smooth function \( \alpha_{11} = \alpha_{11}(\Omega, \varpi_1) \) whose Taylor expansion in the variable \( \varpi_1 \) at \( 0 \) has the form \( \alpha_{11} = \eta(\varpi_1 + \varpi_1^2 \Delta)\frac{2\pi}{|\Omega - \varpi_1|} + O(\varpi_1^3) \) for \( \Omega \in \Omega_{11}((a,b)) \), where the function \( \Delta = \Delta(\Omega) \) and the sign \( \eta \in \{1, -1\} \) are the same as in (3.12) and (3.14):

\[
\Delta(\Omega^\circ) := \frac{\Omega^\circ}{2} \text{Tr} \left( \left( \frac{\partial^2 H_{11}(I^\circ,0,0)}{\partial (q,p)^2} \right)^{-1} \frac{\partial^2 \langle \hat{F}^\circ \rangle(0,0)}{\partial (q,p)^2} \right) \quad \text{npu} \quad d(\hat{F}^\circ)(0,0) = 0,
\]

\[
\eta := \text{sgn} \left( \Omega^\circ \text{Tr} \frac{\partial^2 H_{11}(I^\circ,0,0)}{\partial (q,p)^2} \right).
\]

Moreover, the vectors \( e_{\Omega,\varpi_1,k} \) are bounded \((k = 1, 2, 3, 4)\), moreover the relations \( e_{\Omega,\varpi_1,1} = \partial/\partial \varphi, e_{\Omega,\varpi_1,2} = \partial/\partial I, e_{\Omega,\varpi_1,3} \in \mathbb{R}_{>0} \partial/\partial q \) hold for \( \hat{F} \equiv 0 \) and hold up to \( O(\varpi_1) \) in the general case. \( \Box \)

From Lemma 3.1, we easily obtain the following multidimensional generalization for any \( N \geq n \geq 1 \) and for any relatively-periodic solutions (including long periodic ones, i.e. those with arbitrary large period \( T \)).

**Theorem 3.5 (on Jordan-Kronecker blocks of linearization of the unperturbed succession map).** Suppose that, under the hypothesis of theorem 3.1, the number \( \varpi \in (0, \varpi_0] \) and the set of angular frequencies \( \Omega_{ij} = \Omega_{ij}(I_{ij}^\circ,0,0) \) satisfy the conditions 1), 2), 3) of periodicity, nondegeneracy and “relative resonance” (but not necessary the nondegeneracy condition 4)). Then there exists a \( N\)-dimensional torus \( \Lambda \) that is \( O(\varpi) \)-close (and even \( O(\varpi^2) \)-close in the case of \( d(\hat{F}^\circ)(0,0) = 0 \)) to the torus \( \Lambda^\circ \) with respect to a \( C^1 \)-norm and is formed by the phase orbits of \((T, \alpha)\)-periodic solutions of the unperturbed system (3.7). Moreover, \( p(\Lambda) = p(\Lambda^\circ) \) and, for any point \( x \in \Lambda \), there exist canonical frames \( e_{ij1}, e_{ij2}, e_{ij3}, e_{ij4} \) in the tangent spaces \( T_{x_{ij}}M_{ij} \) (where \( x_{ij} := pr_{ij}(x) \), \( pr_{ij} : M \rightarrow M_{ij} \) is the projection) such that the linear part \( d(g^T_{\varpi_0, H_{11} + \varpi^2 \Phi_0 a_{ii}})(x) \) of the unperturbed “succession map” at the point \( x \) with respect to this frame is given by a blockwise lower-triangular
matrix with the diagonal blocks

\[
\begin{pmatrix}
1 & \omega^{1-j}T \frac{\partial^2 H_{ij}}{\partial I_{ij}^2} (I_{ij}, 0, 0) & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \cos \alpha_{ij} & \sin \alpha_{ij} \\
0 & 0 & -\sin \alpha_{ij} & \cos \alpha_{ij}
\end{pmatrix},
\]

1 \leq i \leq n, 
0 \leq j \leq n_i.

Here \( \alpha_{ij} \) are some real numbers such that \( \alpha_{i0} = \eta_0 \alpha \), \( |\alpha_{ij} - \eta_{ij} (\alpha + \Delta_{ij} \varpi^2 T)| \leq C_2 \varpi^3 T \) for \( 1 \leq j \leq n_i \), where the numbers \( \Delta_{ij} \) and the signs \( \eta_{ij} \in \{1, -1\} \) are the same as in (3.12) and (3.14). Furthermore all non-diagonal blocks vanish, apart from those blocks whose column and row correspond to the factors \( M_{i0} \) and \( M_{ij} \) (respectively), \( 1 \leq j \leq n_i \), in the direct product \( M = \prod_{i=1}^n \prod_{j=0}^{n_i} M_{ij} \). The vectors \( e_{ijk} \) are bounded \( (k = 1, 2, 3, 4) \), and the relations \( e_{ij1} = \partial/\partial \varphi_{ij}, e_{ij2} = \partial/\partial I_{ij}, e_{ij3} \in \mathbb{R}_{>0} \partial/\partial q_{ij} \) hold either exactly if \( j = 0 \) or up to \( O(\varpi) \) if \( 1 \leq j \leq n_i \).

In particular, if the nondegeneracy condition 4) holds then \( \alpha_{ij} \neq 0 \) (mod 2\pi) for any \( i, j \); if the strong nondegeneracy condition holds then \( \alpha_{ij} \neq 0 \) (mod \( \pi \)).

Let us explain why the matrix in Theorem 3.5 is blockwise lower-triangular (instead of blockwise diagonal). Unlike the model system (3.4), which is a direct product, the unperturbed system (3.7) is only a semi-direct product. Namely: it “projects” to each factor \( M_{i0} \) and to each factor \( M_{ij} \times M_{ij} \), but (unlike to the model system) does not “projects” to the factors of the form \( M_{ij} \), \( 1 \leq i \leq n, 1 \leq j \leq n_i \). This observation has the following unexpected consequence: although each diagonal block of the indicated matrix is symplectic, the whole matrix is not symplectic. Indeed: the linear operator under consideration leaves invariant each subspace \( V_i := \bigoplus_{j=0}^{n_i} T_{x_{ij}} M_{ij}, 1 \leq i \leq n, \) as well as each its subspace \( T_{x_{ij}} M_{ij}, 1 \leq j \leq n_i \), moreover all the subspaces \( T_{x_{ij}} M_{ij}, 0 \leq j \leq n_i \), are symplectic and pairwise skew-orthogonal. This implies that, if the operator would be symplectic, then it should leave invariant the subspace \( T_{x_{0}} M_{i0} \) too (since this subspace coincides with the skew-orthogonal complement in \( V_i \) of the invariant subspace \( \bigoplus_{j=1}^{n_i} T_{x_{ij}} M_{ij} \)).

But the latter would mean that the matrix of the operator would be blockwise diagonal, which is false, since it is blockwise lower-triangular only, as explained above.

Notice that the diagonal blocks in Theorem 3.5 have the form \( \exp(\omega^{1-j}T V_{ij}) \) where

\[
V_{ij} = \begin{pmatrix}
0 & \frac{\partial^2 H_{ij}}{\partial I_{ij}^2} (I_{ij}, 0, 0) & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \hat{\Omega}_{ij} \\
0 & 0 & -\hat{\Omega}_{ij} & 0
\end{pmatrix}, \quad 1 \leq i \leq n, 
0 \leq j \leq n_i,
\]

\( \hat{\Omega}_{i0} = \eta_0 \Omega_{i0}, |\hat{\Omega}_{ij} - \eta_{ij} (\Omega_{ij} + \Delta_{ij} \varpi^2 T)| \leq C_2 \varpi^3 \) \( \alpha_{ij} = \hat{\Omega}_{ij} T \) (mod 2\pi), \( 1 \leq j \leq n_i \).

Stage two of the proof of Theorems 3.1 and 3.3 is based on generalizing the “method of generating function” (which was initially [6, 12, 23, 9, 24] introduced for Hamiltonian systems) to the case of an \( \varepsilon \)-Hamiltonian (“perturbed”) system that is \( C^2 \)-close to a 0-Hamiltonian (“unperturbed”) system. In this method, one
studies $T$-periodic trajectories of the perturbed system in a neighbourhood of a “nondegenerate” compact submanifold $\Lambda$ formed by the phase trajectories of $(T, \alpha)$-periodic solutions of the unperturbed system. Namely, one proves that the intersections points of $(T, \alpha)$-periodic trajectories of the perturbed system with the “cross section” $\Sigma$ (see the formulation of theorem 3.1) coincide with critical points of the function $\Psi|_{\Sigma \cap \tilde{\Lambda}}$. Since this function is defined on a $\mathbb{N} - 2$-dimensional torus $\Sigma \cap \tilde{\Lambda}$, the number of its critical points is at least $N - 1$, moreover it is at least $2^{N - 2}$ when counting with multiplicities [11]. This gives us the required estimate (formulated in Theorem 3.1) for the number of $(T, \alpha)$-periodic trajectories of the perturbed system. Here $\Psi$ is the generating function (cf. Definition 3.2 below) of the perturbed succession map $\tilde{\Lambda} = g_{R, \tilde{R}}^{T, \alpha} : M \to M$, moreover $\tilde{\Lambda} \subset M$ is a submanifold that is $C^1$-close to the submanifold $\Lambda$. A proof of Theorem 3.3 on orbital stability in linear approximation is based on some properties (and their “preservation” under perturbations) of the linearization of the unperturbed “succession map” $A$ at points of the torus $\Lambda$ (cf. [20; §3.3.2] for details).

**Definition 3.2 (Generating function).** Let $\varepsilon > 0$ and $A : M \to M$ be a symplectic self-map of a symplectic manifold $(M, \omega) = (M_0 \times M_1, \omega_0 + \varepsilon \omega_1)$. Denote $v_{ij1} := \varphi_{ij}, v_{ij2} := q_{ij}$ (“coordinates”), $u_{ij1} := I_{ij}, u_{ij2} := p_{ij}$ (“momenta”). Denote by $\alpha$ the differential 1-form of the type of $(u' - u)dv + (v - v')du'$ on $M$. More precisely, we define $\alpha$ by the formula

$$
\alpha(x) := \sum_{i=1}^{n} \sum_{k=1}^{2} \left( u'_{i0k} - u_{i0k} \right) dv_{i0k} + (v_{i0k} - v'_{i0k}) du'_{i0k} + \\
+ \varepsilon \sum_{j=1}^{n} \left( (u'_{ijk} - u_{ijk}) dv_{ijk} + (v_{ijk} - v'_{ijk}) du'_{ijk} \right)
$$

where $A : x = (v, u) \mapsto A(x) =: (v', u')$. In other words,

$$
\alpha(x)\xi := \omega(A(x) - x, dP(x)\xi), \quad \xi \in T_x M,
$$

(3.17)

where $P : x = (v, u) \mapsto P(x) := (v, u')$, and $\omega(\xi, \eta)$ denotes the value of the symplectic structure $\omega = \sum_{i=1}^{n} \sum_{k=1}^{2} (du_{i0k} \wedge dv_{i0k} + \varepsilon \sum_{j=1}^{n} du_{ijk} \wedge dv_{ijk})$ on the pair of vectors $\xi, \eta \in T_x M$. A function $\Psi = \Psi(x)$ will be called a generating function of the map $A$ if

$$
d\Psi(x) = \alpha(x), \quad x \in M.
$$

(3.18)

Let us show that such a function $\Psi$ exists, i.e. the form $\alpha$ is exact. One easily shows that the integral of the form $\alpha$ along any closed curve equals the integral of the symplectic structure $\omega$ along some two-dimensional torus. The latter integral vanishes, since the symplectic structure under consideration is exact (being the standard symplectic structure on $M = T^*Q$). Thus, the integral of the form $\alpha$ along any closed curve vanishes. This proves that the function $\Psi$ is well-defined up to an additive constant.

Let us derive, via the method of a generating function, Theorem 3.1 from the technical Theorem 3.5. For this, we will firsts (in Steps 1–3) construct a torus $\tilde{\Lambda} \subset M$
and show that the generating function $\Psi$ of the “perturbed succession map”

$$\tilde{A} := g_{H,H_1 + \omega^2 s}^T g_{I,I_1}^{-\alpha} : M \to M, \quad x = (\varphi, I, q, p) \mapsto (\varphi', \tilde{I}, \tilde{q}, \tilde{p}),$$

(3.19)

possesses the property

$$d\Psi(x) = \sum_{i=1}^{n} \left( (I_{i0} - I_{00}) d\varphi_{i0} + \varepsilon \sum_{j=1}^{n} (I_{ij} - I_{ij0}) d\varphi_{ij} \right), \quad x \in \tilde{\Lambda}. \quad (3.20)$$

After that (in Steps 4 and 5), we will justify the main idea of the method, namely: we will show that, for small enough perturbation and $\varepsilon \neq 0$, all critical points of the function $\Psi|_{\Sigma \setminus \tilde{\Lambda}}$ are fixed under the map $\tilde{A}$ (i.e., they are intersection points of $(T, \alpha)$-periodic trajectories of the perturbed system with the “cross section” $\Sigma$). Here, the standard transversality argument does not work (because the factor $\varepsilon$ in (3.20) is small, cf. Remark 3.1 below), however we manage to adapt it.

**Step 1.** Let us derive from technical Theorem 3.5 that the torus $\Lambda$ is nondegenerate in the following sense (3.21). Let us write the “unperturbed succession map”

$$A := g_{H_0,H_1 + \omega^2 s}^T g_{I_0,I_1}^{-\alpha} : M \to M$$

in the following form in the coordinates under consideration:

$$A = g_{H_0,H_1 + \omega^2 s}^T g_{I_0,I_1}^{-\alpha} : M \to M, \quad x = (\varphi, I, q, p) \mapsto (\varphi', I', q', p'),$$

where $(\varphi', I', q', p') = (\varphi'(\varphi, I, q, p), I'(\varphi, I, q, p), q'(\varphi, I, q, p), p'(\varphi, I, q, p))$. By nondegeneracy of the torus $\Lambda$, we mean that at every point $x = (\varphi, I, q, p) \in \Lambda$, the following determinant does not vanish:

$$\det \frac{\partial(\varphi' - \varphi, q' - q, p' - p)}{\partial(I, q, p)} \neq 0, \quad (\varphi, I, q, p) \in \Lambda, \quad (3.21)$$

i.e., the Jacobi matrix of the map $(I, q, p) \mapsto (\varphi' - \varphi, q' - q, p' - p)$ at the point $(I, q, p)$ is nondegenerate.

The decomposition $M = \bigoplus_{i=1}^{n} \bigoplus_{j=0}^{n_i} M_{ij}$ gives us an isomorphism $T_x M = \bigoplus_{i=1}^{n} \bigoplus_{j=0}^{n_i} T_{x_{ij}} M_{ij}$, where $p_{ij} : M \to M_{ij}$ is the projection, $x_{ij} := p_{ij}(x)$. Recall that the vectors

$$\frac{\partial}{\partial \varphi_{ij}}, \frac{\partial}{\partial q_{ij}}, \frac{\partial}{\partial q_{ij0}}, \frac{\partial}{\partial p_{ij}}$$

form a basis of the tangent space $T_{x_{ij}} M_{ij}$. Since the unperturbed system “projects” to each factor $M_{ij}$ and to each factor $M_{0i} \times M_{ij}$, we conclude that the matrix in (3.21) is blockwise lower-triangular with diagonal blocks

$$\frac{\partial(\varphi'_{ij} - \varphi_{ij}, q'_{ij} - q_{ij}, p'_{ij} - p_{ij})}{\partial(I_{ij}, q_{ij}, p_{ij})}.$$

Therefore, the nondegeneracy condition (3.21) is equivalent to nondegeneracy of each of these blocks.

Let $j = 0$. By the technical Theorem 3.5, the block

$$\frac{\partial(\varphi'_{ij} - \varphi_{ij}, q'_{ij} - q_{ij}, p'_{ij} - p_{ij})}{\partial(I_{i0}, q_{i0}, p_{i0})}$$

is conjugated to the corresponding minor of the corresponding block from the same theorem (since if $j = 0$ then the bases $e_{i01}, e_{i02}, e_{i03}, e_{i04}$ and $\frac{\partial}{\partial \varphi_{i0}}, \frac{\partial}{\partial q_{i0}}, \frac{\partial}{\partial q_{i0}}, \frac{\partial}{\partial p_{i0}}$ differ from each other just by changing the last two vectors by their linear combinations). From the explicit form of this block, we obtain that $\det \frac{\partial(\varphi'_{i0} - \varphi_{i0}, q'_{i0} - q_{i0}, p'_{i0} - p_{i0})}{\partial(I_{i0}, q_{i0}, p_{i0})} \neq 0$ (since $\alpha \notin 2\pi \mathbb{Z}$ due to the nondegeneracy condition (3.10) or to the more delicate condition (3.11)).

Let now $1 \leq j \leq n_i$. Suppose that $\det \frac{\partial(\varphi'_{ij} - \varphi_{ij}, q'_{ij} - q_{ij}, p'_{ij} - p_{ij})}{\partial(I_{ij}, q_{ij}, p_{ij})} = 0$. Then (because the subspace $T_{x_{ij}} M_{ij} \subset T_x M$ is invariant under the operator $dA(x)$)
for \( j > 0 \) there exists a nontrivial linear combination \( \xi_{ij} \in T_{x_{ij}}M_{ij} \) of vectors \( \frac{\partial}{\partial I_{ij}}, \frac{\partial}{\partial q_{ij}}, \frac{\partial}{\partial p_{ij}} \) such that the vector \( dA(x)\xi_{ij} - \xi_{ij} \) is proportional to the vector \( \frac{\partial}{\partial I_{ij}} \). On the other hand, it follows from the technical Theorem 3.5 that any vector of the form \( dA(x)\eta_{ij} - \eta_{ij} \) (with \( j > 0 \) and \( \eta_{ij} \in T_{x_{ij}}M_{ij} \)) is a linear combination of vectors \( e_{ij1}, e_{ij3}, e_{ij4} \). But (for small enough \( |\varepsilon| \)) such a linear combination can not be proportional to the vector \( \frac{\partial}{\partial I_{ij}} \), since, due to the same theorem, the vectors \( e_{ij1}, e_{ij3}, e_{ij4} \) are linearly independent (and form a basis of the space \( T_{x_{ij}}M_{ij} \)). Hence \( dA(x)\xi_{ij} - \xi_{ij} = 0, \) i.e. the vector \( \xi_{ij} \) is fixed under the map \( dA(x) \). Now, it follows from the same theorem that any fixed vector of the operator \( dA(x)|_{T_{x_{ij}}M_{ij}} \) is proportional to the vector \( e_{ij1} \) (since \( \alpha_{ij} \notin 2\pi\mathbb{Z} \) due to the nondegeneracy condition (3.10) or (3.11)). But (for small enough \( |\varepsilon| \)) such a vector can not be a linear combination of the vectors \( \frac{\partial}{\partial I_{ij}}, \frac{\partial}{\partial q_{ij}}, \frac{\partial}{\partial p_{ij}} \), since, by the same theorem, the vectors \( e_{ij1}, \frac{\partial}{\partial I_{ij}}, \frac{\partial}{\partial q_{ij}}, \frac{\partial}{\partial p_{ij}} \) are linearly independent (and form a basis of the subspace \( T_{x_{ij}}M_{ij} \)). Therefore \( \xi_{ij} = 0, \) a contradiction. This proves the inequality \( \det \left( \frac{\partial (\phi_{ij}' - \phi_{ij}, q_{ij}' - q_{ij}, p_{ij}' - p_{ij})}{\partial (I_{ij}, q_{ij}, p_{ij})} \right) \neq 0 \), and hence (3.21).

**Step 2.** Now let us describe a construction of the torus \( \tilde{\Lambda} \subset M \). Let the “perturbed succession map” \( \tilde{A} : M \to M \) has the form (3.19) in the coordinates under consideration, where \( (\tilde{\varphi}, \tilde{I}, \tilde{q}, \tilde{p}) = (\tilde{\varphi}(\varphi, I, q, p), \tilde{I}(\varphi, I, q, p), \tilde{q}(\varphi, I, q, p), \tilde{p}(\varphi, I, q, p)) \).

Define a set \( \tilde{\Lambda} \subset M \) as the following set of points near \( \Lambda \):

\[
\tilde{\Lambda} := \{ (\varphi, I, q, p) \mid (\tilde{\varphi}(\varphi, I, q, p), \tilde{q}(\varphi, I, q, p), \tilde{p}(\varphi, I, q, p)) = (\varphi, q, p) \}.
\]

It follows from the Implicit Functions Theorem and from the nondegeneracy condition (3.21) that, for a small enough size of perturbation \( \delta \geq |\varepsilon| \geq 0 \) (here \( \delta := |\varepsilon| + |\mu| + |\nu| + |\rho| \) in the case of the \( N + 1 \) body problem), the subset \( \tilde{\Lambda} \subset M \) has the form of a graph

\[
\tilde{\Lambda} = \{ (\varphi, I(\varphi), q(\varphi), p(\varphi)) \mid \varphi \in (\mathbb{R}/2\pi\mathbb{Z})^N \}
\]

for some smooth functions \( I(\varphi), q(\varphi), p(\varphi) \), i.e., it is a \( N \)-dimensional torus that depends smoothly on \( \varepsilon \) and other small parameters of the system (i.e., on \( \varepsilon, \mu, \nu, \rho \) in the case of the \( N + 1 \) body problem with fixed \( m_i, m_{ij} \)), as well as on the parameters \( \varepsilon, T > 0 \). The torus \( \tilde{\Lambda} \) is \( O(\delta) \)-close to the torus \( \Lambda \), due to the Implicit Functions Theorem and \( O(\delta) \)-closeness of the “perturbed” map \( \tilde{A} = g^T_{\tilde{H}, \tilde{H}_1 + \varepsilon^2 \tilde{g}_{\tilde{I}, \tilde{I}_1}^{-\alpha}} \) to the unperturbed one \( A = g^T_{H, H_1 + \varepsilon^2 g^{-\alpha}_{I, I_1}} \). From now on, we mean by \( O(1) \) real numbers whose absolute values do not exceed a positive value depending continuously on \( \varepsilon, T > 0 \) and the integers \( k_{ij} \in \mathbb{Z} \).

**IIIa. Let** \( \Psi \) **be a generating function of the map** \( \tilde{A} \), cf. (3.19), (??), (3.18). **It follows from the construction of the torus** \( \tilde{\Lambda} \) **(cf. Step 2) and the Implicite Functions Theorem that the map** \( \tilde{A} \) **shifts each point** \( x \in \Lambda \) **by a vector having the form**

\[
\tilde{A}(x) - x = (0, \tilde{I} - I, 0, 0) = O(\delta), \quad x = (\varphi, I, q, p) \in \tilde{\Lambda}.
\]

Here we wrote that the size of the shift has the same order \( O(\delta) \) as the order of perturbation, since if \( \delta = 0 \) then \( \tilde{\Lambda} = \Lambda \) and \( A(x) = x \) for any point \( x \in \Lambda \). Therefore \( d\Psi(x) \) has the form (3.20) at every point \( x \in \tilde{\Lambda} \).
Therefore, if $\varepsilon \neq 0$ then a point $x \in \tilde{\Lambda}$ is a critical point of the generating function $\Psi$ if and only if it is fixed under the map $\tilde{A}$ (i.e., this point is contained in the phase orbit of a $(T, \alpha)$-periodic solution of the perturbed system).

It remains to show that, if $\varepsilon \neq 0$ (and the perturbation is small), then any critical point of the function $\Psi|_{\Sigma \cap \tilde{\Lambda}}$ is also fixed under the map $\tilde{A}$.

**Step 4.** Let $\varepsilon \neq 0$ and a point $x \in \Sigma \cap \tilde{\Lambda}$ is a critical point of the function $\Psi|_{\Sigma \cap \tilde{\Lambda}}$. This means that the covector $(3.20)$ is a linear combination of the covectors $\sum_{i=1}^{n_i} \sum_{j=0}^{n_j} d\varphi_{ij}$ and $\sum_{i=1}^{n_i} \sum_{j=0}^{n_j} \omega^{1-j} \Omega_{ij} d\varphi_{ij} = \sum_{i=1}^{n_i} (\omega \Omega_{i0} d\varphi_{i0} + \sum_{j=1}^{n_j} \Omega_{ij} d\varphi_{ij})$ (since the plane $\Sigma$ is a common level set of the functions $\sum_{i=1}^{n_i} \sum_{j=0}^{n_j} \varphi_{ij}$ and $\sum_{i=1}^{n_i} \sum_{j=0}^{n_j} k_{ij} \varphi_{ij}$). By denoting the coefficients of this linear combination by $\lambda_1, \lambda_2 \in \mathbb{R}$, we obtain $\tilde{T}_i - T_i = \lambda_1 + \lambda_2 \omega \Omega_{i0}$, $\tilde{I}_{ij} - I_{ij} = \varepsilon^{-1}(\lambda_1 + \lambda_2 \Omega_{ij})$ for $1 \leq i \leq n$, $1 \leq j \leq n_i$.

Thus, the map $\tilde{A}$ shifts our point $x$ by the vector

$$\tilde{A}(x) - x = (0, \tilde{I} - I, 0, 0) = \lambda_1 V_1 + \lambda_2 V_2.$$  

(3.25)

Here we denoted

$$V_1 := \sum_{i=1}^{n_i} \left( \frac{\partial}{\partial I_{i0}} + \varepsilon^{-1} \sum_{j=1}^{n_j} \frac{\partial}{\partial \varphi_{ij}} \right), \quad V_2 := \sum_{i=1}^{n_i} \left( \omega \Omega_{i0} \frac{\partial}{\partial I_{i0}} + \varepsilon^{-1} \sum_{j=1}^{n_j} \Omega_{ij} \frac{\partial}{\partial \varphi_{ij}} \right).$$

We have to show that $\lambda_1 = \lambda_2 = 0$ in (3.25).

**Step 5.** Since the Hamilton functions $\tilde{H} = \omega \tilde{H}_0 + \varepsilon (\tilde{H}_1 + \omega^2 \tilde{\Phi})$ and $\tilde{I} = I_0 + \varepsilon I_1$ of our perturbed system and of the $S^1$-action are preserved by the map $\tilde{A}$, we have $\tilde{H}(\tilde{A}(x)) - H(x) = 0$. From this conservation laws, taking into account the inclusion $x \in \Lambda$, the relation (3.24) and $O(\varepsilon)$-closeness of the torus $\tilde{\Lambda}$ to $\Lambda$ (Step 2), we conclude that $\sum_{i=1}^{n_i} (\tilde{T}_i - T_i) + \varepsilon \sum_{i=1}^{n_i} (\tilde{I}_{ij} - I_{ij}) = 0$ and $\sum_{i=1}^{n_i} (\omega \Omega_{i0} + O(\varepsilon)) (\tilde{T}_i - T_i) + \varepsilon \sum_{i=1}^{n_i} (\Omega_{ij} + O(\varepsilon)) (\tilde{I}_{ij} - I_{ij}) = 0$. Due to (3.25), we conclude that the pair of real numbers $\lambda_1, \lambda_2 \in \mathbb{R}$ satisfies the following system of two linear relations that do not involve $\varepsilon$:

$$\lambda_1 N + \lambda_2 \sum_{i,j} \omega^{1-j} \Omega_{ij} = 0, \quad \lambda_1 \sum_{i,j} \omega^{1-j} \Omega_{ij} + \lambda_2 \sum_{i,j} (\omega^{1-j} \Omega_{ij})^2 = O((|\lambda_1| + |\lambda_2|) \delta),$$

where $\sum_{i,j} := \sum_{i=1}^{n_i} \sum_{j=0}^{n_j}$. We can rewrite this system in the form

$$\lambda_1 \langle V_1^o, V_1^o \rangle + \lambda_2 \langle V_2^o, V_2^o \rangle = 0, \quad \lambda_1 \langle V_1^o, V_2^o \rangle + \lambda_2 \langle V_2^o, V_2^o \rangle =: B = O((|\lambda_1| + |\lambda_2|) \delta).$$

Here $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^N$, and the vectors

$$V_1^o := \sum_{i=1}^{n_i} \left( \frac{\partial}{\partial \varphi_{i0}} + \sum_{j=1}^{n_j} \frac{\partial}{\partial \varphi_{ij}} \right), \quad V_2^o := \sum_{i=1}^{n_i} \left( \omega \Omega_{i0} \frac{\partial}{\partial \varphi_{i0}} + \sum_{j=1}^{n_j} \Omega_{ij} \frac{\partial}{\partial \varphi_{ij}} \right)$$

do not depend on $\varepsilon$. Since the vectors $V_1^o, V_2^o$ are noncollinear (due to (3.9) and $N \geq 2$), the matrix of the latter linear system is nondegenerate, and its solution is unique:

$$\lambda_1 = \frac{-\langle V_1^o, V_2^o \rangle B}{\langle V_1^o, V_1^o \rangle \langle V_2^o, V_2^o \rangle - \langle V_1^o, V_2^o \rangle^2}, \quad \lambda_2 = \frac{\langle V_2^o, V_1^o \rangle B}{\langle V_1^o, V_1^o \rangle \langle V_2^o, V_2^o \rangle - \langle V_1^o, V_2^o \rangle^2}.$$
Therefore $|\lambda_1| + |\lambda_2| = O(B) = O((|\lambda_1| + |\lambda_2|)\delta)$ when $\delta \to 0$. Hence, if the perturbation is small enough (i.e., $\delta \to 0$) then $\lambda_1 = \lambda_2 = 0$ in (3.25), i.e. $\tilde{A}(x) = x$ as required. Theorem 3.1 is completely proved. 

**Remark 3.1.** Let us explain the following: (a) a geometric meaning of the vectors $V_1^\circ, V_2^\circ$ and the equality (3.25); (b) how one could use these vectors for proving in a geometric manner (by means of a “transversality argument”) the key idea of the generating function method (cf. Steps 4–5 above), if a small factor $\varepsilon$ would not enter the “fast” components of the co-vector (3.20); (c) why is the standard “transversality argument” not working for systems with “slow” and “fast” variables.

(a) The vectors $V_1^\circ, V_2^\circ$ are nothing else than the values at the point $x^\circ \in \Lambda^\circ$ of the $0$-Hamiltonian vector fields corresponding to our $S^1$-action and model system. These vectors are non-collinear, since, by assumption of Theorem 3.1, one has $N \geq 2$ and the integers $k_{ij} \in \mathbb{Z}$ are not simultaneously 0 (due to (3.9)). By construction, the cross-section $\Sigma$ is the orthogonal complement to the plane span$(V_1^\circ, V_2^\circ)$ spanned by the vectors $V_1^\circ, V_2^\circ$. Therefore, the fact that a point $x \in \Sigma \cap \tilde{\Lambda}$ is a critical point of the function $\Psi|_{\Sigma \cap \tilde{\Lambda}}$ means that the gradient of the function $\Psi$ at the point $x$ belongs to the plane span$(V_1^\circ, V_2^\circ)$. But, due to (3.20), this gradient equals $\sum_{i=1}^{n}(I_{i0} - I_{00})\frac{\partial}{\partial \varphi_{i0}} + \varepsilon \sum_{i=1}^{n}(I_{ij} - I_{ij})\frac{\partial}{\partial \varphi_{ij}}$, i.e., it has the same coordinates as the vector of shifting (3.24), up to a factor $\varepsilon$ and a transposition of the basis vectors $\frac{\partial}{\partial \varphi_{ij}}$ and $\frac{\partial}{\partial I_{ij}}$. Hence, the shift vector belongs to the plane span$(V_1, V_2)$, i.e. (3.25) holds. On the other hand, due to the conservation laws (cf. Step 5 above), the endpoints of the shift vector (i.e., the points $x$ and $\tilde{A}(x)$) belong to the same common level set $\tilde{H}^{-1}(a) \cap \tilde{I}^{-1}(b)$ of the functions $\tilde{H}, \tilde{I}$. We want to manage to conclude that such a shift has to be trivial, i.e. $\tilde{A}(x) = x$.

(b) The standard transversality argument is as follows. Suppose that, firstly, $N > n$ (i.e., the “fast” variables and a small factor $\varepsilon$ are present), secondly, the gradients of the “unperturbed” functions $\tilde{H}_0, I_0$ at the point $x^\circ$ are non-collinear, thirdly, the “unperturbed” vectors $\lim_{\varepsilon \to 0}(\varepsilon V_1), \lim_{\varepsilon \to 0}(\varepsilon V_2)$ are also non-collinear, and fourthly, the “unperturbed” plane span$(\lim_{\varepsilon \to 0}(\varepsilon V_1), \lim_{\varepsilon \to 0}(\varepsilon V_2))$ and the common level set $H_0^{-1}(a^\circ) \cap I_0^{-1}(b^\circ)$ are transversal (in fact, the first three assumptions can be fulfilled simultaneously under the hypotheses of Theorem 3.1, but the first and fourth assumptions can not be fulfilled simultaneously, see below). Then, for small enough perturbation, the “perturbed” plane span$(V_1, V_2)$ and the common level set $\tilde{H}^{-1}(a) \cap \tilde{I}^{-1}(b)$ are also transversal. Hence, due to (a), the shift vector is zero, i.e. $\tilde{A}(x) = x$ as required. However our first and fourth assumptions can not be fulfilled simultaneously, since the “unperturbed” common level set contains the indicated “unperturbed” plane, and hence they can not be transversal.

(c) Let us summarize. The standard transversality argument can not be applied in our case (when “fast” variables are present). The reason is that the small factor $\varepsilon$ enters the co-vectors $d\tilde{H}, d\tilde{I}$ and the vectors $\varepsilon V_1, \varepsilon V_2$ in an inconsistent manner. Namely, it enters the “fast” components of the co-vectors $d\tilde{H}, d\tilde{I}$, but the “slow” components of the vectors $\varepsilon V_1, \varepsilon V_2$.

Stage two of the proof of Theorem 3.2 (on symmetric $(T, \alpha)$-periodic solutions) can be performed either by the method of Tkhay [10] (based on the Implicit
Functions Theorem and a simple refinement of Lemma 3.1 and Theorem 3.5 for reversible systems, without using a generating function), or by the following method by Krassinsky [7] (based on Theorem 3.1 and the explicit construction (3.22) of the torus $\tilde{\Lambda}$.

Let us prove first that the function $\Psi|_{\tilde{\Lambda}}$ is an even function in the variables $\varphi|_{\tilde{\Lambda}}$. Since the involution $\tau$ is anti-canonical, it maps phase trajectories to phase trajectories with reversing the time. In particular, it maps the segment of the phase trajectory from any point $x$ to the point $\tilde{A}(x)$ into the segment of a phase trajectory from the point $\tilde{x} := \tau(\tilde{A}(x))$ to the point $\tau(x)$ with reversing the time, therefore $\tau(x) = \tilde{A}(x)$. On the other hand, due to (3.22), any point $x = (\varphi, \mathbf{I}, \mathbf{q}, \mathbf{p}) \in \tilde{\Lambda}$ is sent by the map $\tilde{A}$ to a point of the form $\tilde{A}(x) = (\varphi, \tilde{\mathbf{I}}, \tilde{\mathbf{q}}, \tilde{\mathbf{p}})$. Hence, we conclude from Definition of the involution $\tilde{x} = \tau(\tilde{A}(x)) = (-\varphi, \tilde{\mathbf{I}}, \tilde{\mathbf{q}}, -\tilde{\mathbf{p}})$ and $\tilde{A}(x) = \tau(x) = (-\varphi, \mathbf{I}, \mathbf{q}, -\mathbf{p})$, therefore $\tilde{x} \in \tilde{\Lambda}$ (c.f. (3.22)). Now, we obtain from (3.20) the equality of co-vectors $d(\Psi|_{\tilde{\Lambda}}(x) = -d(\Psi|_{\tilde{\Lambda}}(\tilde{x})$ with respect to the variables $\varphi|_{\tilde{\Lambda}}$ on the torus $\tilde{\Lambda}$. Since (in these variables) the points $x$ and $\tilde{x}$ have coordinates $\varphi$ and $-\varphi$, the differential of the function $\Psi|_{\tilde{\Lambda}}$ has the form $d(\Psi|_{\tilde{\Lambda}}(\varphi) = -d(\Psi|_{\tilde{\Lambda}}(-\varphi) = d(\Psi|_{\tilde{\Lambda}}(-\varphi))$. Hence, the function $\Psi|_{\tilde{\Lambda}}(\varphi) - \Psi|_{\tilde{\Lambda}}(-\varphi)$ is constant on the torus $\tilde{\Lambda}$. Since this function equals 0 at the point $\varphi = 0$, it equals 0 everywhere, i.e. the function $\Psi|_{\tilde{\Lambda}}(\varphi)$ is even.

Since the function $\Psi|_{\tilde{\Lambda}}(\varphi)$ is even, we conclude that $d(\Psi|_{\tilde{\Lambda}})(0) = 0$. Hence, by Theorem 3.1, the solution of the “perturbed” system with initial condition of the form $x^0 = (0, \mathbf{I}^0, \mathbf{q}^0, \mathbf{p}^0) \in \tilde{\Lambda}$ is $(T, \alpha)$-periodic, i.e. $\tilde{A}(x^0) = x^0$. Let us show that this solution is symmetric. It suffices to show that $\tau(x^0) = x^0$ (c.f. §4.2). As we proved above, $\tau(x^0) = \tilde{A}(\tilde{A}(x^0)) = (0, \tilde{\mathbf{I}}^0, \tilde{\mathbf{q}}^0, -\tilde{\mathbf{p}}^0) \in \tilde{\Lambda}$. Thus, both points $\tau(x^0)$ and $x^0$ belong to $\tilde{\Lambda}$ and have the same coorinates $\varphi = 0$. Hence these points coincide (since $\tilde{\Lambda}$ has the form (3.23)), which proves that the solution under consideration is symmetric. In a similar way, one proves that any solution of the “perturbed” system with initial condition of the form $x^0 = (\beta, \mathbf{I}^0, \mathbf{q}^0, \mathbf{p}^0) \in \tilde{\Lambda}$ is also relatively-periodic and symmetric, where $\beta \in \{0, \pi\}^N$. Theorem 3.2 is completely proved.

Stage two of the proof of Theorem 3.4 (on “gaps” in the families of relatively-periodic solutions) is based on the averaging method on a submanifold [12; theorem 11.1], [9; theorems 2, 4]. We completely prove (by this method) Theorems 3.4 and 2.3 in the important special case: for planetary systems without satellites (cf. §8.1.1, proof of Corollary 2.1(‡), and §8.1.2).

§ 4. Relatively-periodic solutions, symmetricity, stability and nondegeneracy

Consider the problem about the motion of a system of $N + 1$ particles attracting by Newton’s law on a Euclidean plane $E^2$, $N \geq 2$. The equations of the motion have the form (2.1), (2.2).

4.1. Relatively-periodic solutions. Let us state the problem on finding $(T, \alpha)$-periodic solutions of the system, cf. Definition 2.1.
The motions in a rotating coordinate system with angular velocity \( \varpi_1 \) are described by the Hamiltonian system with the Hamilton function \( H = \varpi_1 I \), where

\[
I := \sum_{i=0}^N [r_i, p_i] \tag{4.1}
\]

is the “area integral”, also called the kinetic moment [25]. Here \([r_i, p_i]\) denotes the oriented area of the parallelogram spanned by the vectors \( r_i \) and \( p_i \). Therefore our problem is equivalent to finding \( T \)-periodic solutions of the Hamiltonian system with the Hamilton function \( H = \varpi_1 I \), where \( \varpi_1 = \frac{\alpha - 2\pi s}{T} \), \( s \) is any integer. Due to (2.14) and (2.16), we can define \( \varpi_1 \) as the angular frequency of any planet or satellite, e.g. the angular frequency of the first planet.

The invariant \( 4N \)-dimensional submanifold \( M^{4N} \approx T^* \hat{Q} \) (cf. (2.4)) of the \( N + 1 \) body problem is invariant under the phase flow of the system with the Hamilton function (4.1), and a relatively-periodic solution on \( M^{4N} \) is defined as above.

For the model system (2.11), (2.12), (2.13), a relatively-periodic solution is defined via the Hamiltonian action of the circle with the Hamiltonian \( I = \sum_{i=1}^n ([Y_i, X_i] + \sum_{j=1}^{n_i} [Y_{ij}, X_{ij}]) \) on the submanifold \((\mathbb{R}^2)^N \times (\mathbb{R}^2)^N\) with coordinates \((Y, X)\) and the symplectic structure \( dX \wedge dY \). For the unperturbed and perturbed systems (2.18), (2.19) and (2.22), a relatively-periodic solution is defined via the same \( S^1 \)-action on the phase spaces of these problems, by identifying the phase spaces of these problems with the corresponding domains in the space \((\mathbb{R}^2)^N \times (\mathbb{R}^2)^N\) via the coordinates \((y, x)\) and \((\eta, \xi)\) respectively.

### 4.2. Symmetric solutions

Let us show that the conditions 1 and 2 in Definition 2.2 of a symmetric solution are equivalent. For this we will use the invariance of the total energy \( H \) of the system under the following two involutions \( S_1 \) and \( S \) in the phase space \( T^* \hat{Q} \) (cf. (2.4)) defined as follows.

Let us fix a line \( l \) in the plane of motion passing through the centre of masses of the system of particles. Define the following three transformations in the phase space \( T^* \hat{Q} \) preserving the total energy \( H \) of the system:

1) the canonical involution \( S_1 : T^* \hat{Q} \to T^* \hat{Q} \) corresponding to the self-map (axial symmetry) of the configuration manifold \( \hat{Q} \) sending all particles of the system to their images under the symmetry with respect to the line \( l \);

2) the anti-canonical involution \( S \) (“reversion of time”) sending each pair \((q, p) \in T^* \hat{Q}\) to the pair \((q, -p)\) where \( q \) and \( p \) are the sets of “coordinates” and “momenta” of all particles of the system;

3) the anti-canonical involution \( \tau_1 := SS_1 = S_1 S \).

Each of these transformations is an involution, i.e. coincides with its inverse. The first involution is canonical, i.e. preserves the canonical symplectic structure \( dp \wedge dq \) on \( T^* \hat{Q} \). The second and the third involutions are anti-canonical, i.e. they affect the symplectic structure by changing its sign to the opposite. Thus all three involutions move trajectories of the system to trajectories, moreover the first involution preserves the time on trajectories, while the second and the third involutions “reverse the time”.

A solution satisfies the first (respectively the second) condition of symmetry if and only if the point of the phase space corresponding to the time \( t_0 \) (respectively
any time \( t \in \mathbb{R} \) of this solution is fixed (respectively is mapped to the point corresponding to the time \( 2t_0 - t \) of the same solution) under the involution \( \tau_1 = SS_l = S_lS \). This shows the equivalence of the conditions 1 and 2 in Definition 2.2 of a symmetric solution.

A solution \( \gamma(t) \) is symmetric and \((T, \alpha)\)-periodic if and only if its points \( \gamma(t_0) \) and \( \gamma(t_0 + T/2) \) are fixed under the involutions \( \tau_1 \) and \( \tau_{R_\alpha/2}(t) \) respectively where \( R_\varphi : \mathbb{R}^2 \to \mathbb{R}^2 \) denotes the rotation by the angle \( \varphi \).

4.3. A stable relatively-periodic solution. Suppose that a Hamiltonian system \((X, \omega, H)\) is \( S^1 \)-symmetric with respect to the Hamiltonian action of a circle \( S^1 \) on \( X \) via the Hamiltonian function \( I \). Then the function \( I \) is a first integral of the system. Consider the phase flow \( g^I_\tau H - \omega_1 I : X \to X, t \in \mathbb{R}, \) of the system with the Hamiltonian function \( H - \omega_1 I \). The map \( A := g^I_\tau H - \omega_1 I \) will be called the succession map, and its linear part \( dA(x) \) at a fixed point \( x \) will be called the monodromy operator at this point.

Let us define a “reduced” succession map for the two-dimensional torus \( \gamma \) corresponding to a \((T, \omega_1 T)\)-periodic solution. Let \( \Sigma \subset X \) be a small surface of codimension 2 that transversally intersects the torus \( \gamma \) at some point \( x_0 = \gamma \cap \Sigma \). Consider the restriction of the system to a regular common level set

\[ X_{H, I} := \{ H = \text{const}, \ I = \text{const} \} \cap \gamma \]

of the first integrals \( H \) and \( I \). Consider the small surface \( \sigma := \Sigma \cap X_{H, I} \) of codimension 2 in \( X_{H, I} \), which transversally intersects the torus \( \gamma \) at the point \( x_0 = \gamma \cap \sigma \). Consider the two-dimensional foliation on the manifold \( X_{H, I} \) whose fibres are invariant under the (commuting) flows of the systems with Hamilton functions \( H \) and \( I \); this condition uniquely determines fibres. Take the self-map \( \bar{A} \) of the surface \( \sigma \) sending any point \( x \in \sigma \) to the “next intersection point” of the fibre containing the point \( x \) with the surface \( \sigma \). In more detail, the map \( \bar{A} : \sigma' \to \sigma \) is defined in a sufficiently small neighbourhood \( \sigma' \subset \sigma \) of the point \( x_0 \in \sigma \), it is “close” to the map \( A|_{\sigma'} = g^I_{H - \omega_1 I}|_{\sigma'} \) and has the form \( \bar{A}(x) = g^I_{H - f_1(x) I} |_{\sigma'} \). Here \( f_1 \) and \( f_2 \) are some smooth functions on \( \sigma' \) defined by the conditions \( A(x) \in \sigma, \ x \in \sigma', \ f_1(x_0) = \omega_1, \ f_2(x_0) = T \). The map \( \bar{A} \) will be called the reduced succession map (or the Poincaré map), and its linear part \( \bar{A} = d\bar{A}(x_0) \) at the point \( x_0 \) will be called the reduced monodromy operator corresponding to the torus \( \gamma \). As is well-known [25], the transversal surface \( \sigma \) (called a cross section) is a symplectic submanifold and the self-map \( \bar{A} \) of this surface is also symplectic. In particular, the reduced monodromy operator \( \bar{A} \) is symplectic too.

Recall that a linear operator \( A \) is called \((\text{Liapunov}) \) stable if the norm of the operator \( A^n \) is bounded as \( n \to \infty \). A symplectic operator \( A \) is called structurally stable if it is stable and any symplectic operator that is close enough to \( A \) is stable too.

Definition 4.1. The two-dimensional torus \( \gamma \) and the corresponding relatively-periodic solution will be called orbitally structurally stable in linear approximation (OSSL) (respectively orbitally stable in linear approximation on a common level surface of the first integrals of energy and angular momentum (OSLI)) if the reduced monodromy operator \( A = d\bar{A}(x_0) \) corresponding to the torus \( \gamma \) is structurally
stable (respectively stable). The torus \( \gamma \) will be called \textit{isoenergetically nondegenerate} (IN) if 1 does not belong to the spectrum of the reduced monodromy operator \( A \), i.e. \( 1 \notin \text{spec } A \). The torus \( \gamma \) is called \textit{orbitally stable in linear approximation} (OSL) if the linear operator \( B = dB(x_0) \) is stable, where \( B : \Sigma' \rightarrow \Sigma \) is the map defined similarly to the Poincaré map \( \bar{A} : \sigma' \rightarrow \sigma \).

**Definition 4.2.** An eigenvalue \( \lambda \in \mathbb{C} \) of a symplectic operator \( A \) is called \textit{elliptic} [25] if it satisfies one of the following equivalent conditions:

1) the quadratic form \( Q(\xi) = \omega(A\xi, \xi) \) is (positive or negative) definite on the maximal invariant subspace where the operator \( A \) has no eigenvalues apart from \( \lambda \) and \( \bar{\lambda} \);

2) the Hermitian quadratic form \( \frac{1}{2i}\omega(\xi, \bar{\xi}) \) is (positive or negative) definite on the complex eigensubspace with eigenvalue \( \lambda \) of the complexified space.

The quadratic form \( Q \) is called the \textit{generating function} of the symplectic operator \( A \) (see also definition 3.2).

**Proposition 4.1** (см. [26]). (A) A symplectic operator is stable if and only if it is diagonalizable over \( \mathbb{C} \) and all its eigenvalues belong to the unit circle in \( \mathbb{C} \). (B) A symplectic operator is structurally stable if and only if all its complex eigenvalues are elliptic.

Let us mention some connections between the stability properties introduced above of an invariant two-dimensional torus \( \gamma \):

1) the following implications hold: IN \( \iff \) OSSL \( \Rightarrow \) OSL \( \Rightarrow \) OSLI. (The second implication is an important property of tori having the OSSL property; it follows from property 3 below. The first implication follows from proposition 4.1(B). The third implication is obvious. The inverse implications are in general false);

2) if all eigenvalues of a symplectic operator \( A \) are pairwise different and lie on the unit circle in \( \mathbb{C} \) then it is structurally stable, thus the torus \( \gamma \) is OSSL;

3) if the torus \( \gamma \) is isoenergetically nondegenerate (IN) then it is included into a smooth two-parameter family of isoenergetically nondegenerate two-dimensional tori \( \gamma_{H,I} \) where parameters of the family are values of the first integrals \( H \) and \( I \). If the torus \( \gamma \) is OSSL (and, hence, IN) then all the invariant tori of this family are also OSSL. Hence OSSL \( \Rightarrow \) OSL (and not only OSLI).

We stress that, if the torus \( \gamma \) is OSLI and IN, then the other tori of the family do not need to be OSLI, thus the torus \( \gamma \) does not need to be OSL.

Thus, among the stability properties introduced in Definition 4.1 for a relatively-periodic solution, the strongest one is the OSSL property, which is studied in the present paper (see theorems 2.2(B) and 3.3).

### 4.4. Nondegeneracy condition.

(A) Let us give \textit{sufficient} conditions for nondegeneracy (2.17) of a tuple of angular frequencies \( \varpi_i, \Omega_{ij} \). Due to (2.8), for small enough \( 0 < \varpi \ll 1 \), the nondegeneracy (2.17) implies the delicate nondegeneracy (2.28). In the presence of satellites \( (N > n) \), any of the nondegeneracy condition (2.17) and the delicate nondegeneracy condition (2.28) imply, due to the inequality \( |\alpha| \leq \pi \), that the relative period \( T \) of the solutions under investigation is \textit{“not too big”}:

\[
T < \frac{\pi}{\varpi^2} \quad \text{or} \quad T < \frac{\pi}{C\varpi^3} \quad \text{(for } N > n ),
\] (4.2)
respectively. In the case $n > 1$ (at least two planets) the relative period $T$ is always “big” for $0 < \omega \ll 1$. In more detail: $T = \frac{2\pi|k_2|}{|\omega_2 - \omega_1|} > \frac{\pi|k_2|}{\omega}$ due to (2.14), (2.8), (2.9). Therefore, if $N > n > 1$ then any of the relations (4.2) implies

$$\frac{\pi}{\omega} < T < \frac{\pi}{\omega^2} \quad \text{or} \quad \frac{\pi}{\omega} < T < \frac{\pi}{C\omega^3} \quad \text{(for } N > n > 1),$$

respectively. However, if $N > n = 1$ (only one planet, as in the case by V.N. Tkhay [10]), then, due to (2.8), the minimal positive relative period $T_{\text{min}} \in \left[\frac{2\pi \max_j |K_{ij}|}{1+c}, \frac{2\pi \min_j |K_{ij}|}{c-\omega}\right]$ is of order 1 when $\omega \ll 1$ and $c, K_{ij}$ are fixed.

(B) Let us give sufficient conditions for nondegeneracy (2.17), which we will call rough nondegeneracy conditions.

(B1) Let us show that the nondegeneracy condition (2.17) is always realizable for any $0 < \omega \ll 1$ and any relative period $T$ of the form

$$T = \frac{2\pi a}{\omega}, \quad a \geq a_0(k_2, \ldots, k_n) := \max \left\{7|k_i|, \sqrt{7(N-n+1)}, \frac{21}{2}\right\}, \quad (4.3)$$

for suitable collection of angular frequencies $\omega_i, \Omega_{ij}$ with integers $k_2, \ldots, k_n, K_{ij}$ of orders 1, 4. This will imply that the value $\omega^2 T$ can be made arbitrary small, hence the nondegeneracy condition (2.17) (and, hence, (2.28)) can be always fulfilled. Indeed: let us fix integers $k_2, \ldots, k_n \in \mathbb{Z} \setminus \{0\}$ having different absolute values, and real numbers $a \geq a_0(k_2, \ldots, k_n)$ and $c \in (0, c_0(a)]$, where $c_0(a) := \min\{\frac{1}{a}, \frac{1}{14(N-n+1)}\}$. For any $b \in \left[\frac{2}{7}, \frac{3}{7}\right]$ and $0 < \omega \leq \min\{\omega_0, \omega_0(a)\}$, where $\omega_0(a) := \frac{1}{4a}$, put

$$\omega_i := \omega b, \quad \Omega_{ij} := \omega_1 + K_{ij} \frac{\omega}{a} \quad (4.4)$$

for any integers $K_{ij} \in \mathbb{Z}$ satisfying the following conditions for any $i$ and $j \neq j'$:

$$|K_{ij}| \in \left[\frac{5a}{7\omega}, \frac{6a}{7\omega}\right], \quad ||K_{ij}|| - |K_{ij'}| \geq \ell, \quad \text{where} \ \ell \in \mathbb{N} \cap \left[\frac{ca}{\omega}, \frac{a}{7(N-n+1)\omega}\right]. \quad (4.5)$$

Such integers $\ell$ and $K_{ij}$ exist for $0 < \omega \leq \omega_0(a) = \frac{1}{4a}$, since the interval $\left[\frac{ca}{\omega}, \frac{a}{7(N-n+1)\omega}\right]$ (respectively $\left[\frac{5a}{7\omega}, \frac{6a}{7\omega}\right]$) is of the length $\frac{a}{7\omega} (\frac{1}{7(N-n+1)} - c) \geq \frac{2a^2}{7(N-n+1)} \geq 2$ (respectively $\frac{a}{7\omega} \geq (N-n+1)\ell$), and, hence, it always contains a positive integer (respectively $N-n$ different positive integers with pairwise distances $\geq \ell$). Since $\alpha = \omega_1 T = 2\pi ab (\mod 2\pi)$ and $\omega^2 T = 2\pi a \omega \leq \frac{\pi}{2}$, the required nondegeneracy condition (2.17) has the form $\{a(b - \omega), a(b + \omega)\} \cap \mathbb{Z} = \emptyset$. Hence it holds if $|ab - \frac{1}{4}, ab + \frac{1}{4}] \cap \mathbb{Z} = \emptyset$, i.e. $b \in \frac{1}{2a} + (\frac{1}{4a^2}, \frac{1}{4a}) + \frac{1}{a} \mathbb{Z}$. It follows from $\frac{1}{a} \geq \frac{1}{a}$ that, for any $a \geq a_0(k_2, \ldots, k_n)$, there exists $b \in \left[\frac{2}{7}, \frac{3}{7}\right] \cap \left((\frac{1}{4a}, \frac{3}{4a}) + \frac{1}{a} \mathbb{Z}\right)$. Moreover, for any

$$a \geq a_0(k_2, \ldots, k_n), \quad b \in \left[\frac{2}{7}, \frac{3}{7}\right] \cap \left((\frac{1}{4a}, \frac{3}{4a}) + \frac{1}{a} \mathbb{Z}\right) \quad (4.6)$$

the condition $b \in \left[\frac{2}{7}, \frac{3}{7}\right]$ and the nondegeneracy condition (2.17) hold automatically for the period (4.3) and the collection of angular frequencies of the form (4.4), (4.5). The constructed frequencies $\omega_i, \Omega_{ij}$ satisfy the inequalities $\frac{1}{a} \leq \frac{|\omega_i|}{\omega} \leq \frac{1}{a} < |\Omega_{ij}| < 1$, $||\omega_i| - |\omega_i'|| \geq \frac{1}{a} \geq c$, $||\Omega_{ij}|| - |\Omega_{ij'}| \geq \frac{\omega}{a} \ell \geq c$, and, hence, the inequalities (2.8)
and (2.9). This proves the realizability of any period \( T \) of the form (4.3). Thus, the system of relations (4.3)–(4.6), which we will call the rough nondegeneracy condition, gives a two-parameter family of nondegenerate tuples of frequencies \( \varpi_i, \Omega_{ij} \) with parameters \( T = \frac{\text{const}}{\varpi} \) and \( \varpi_1 \sim \varpi \) (for integers \( k_2, \ldots, k_n, K_{ij} \) of orders \( 1, \frac{1}{a} \)), where \( \varpi \ll 1 \). In particular, the rough nondegeneracy holds if \( a \geq a_0(k_2, \ldots, k_n) \), \( b \in \left( \frac{1}{a} \left[ \frac{3a}{T} + \frac{1}{4} \right] - \frac{3}{4a}, \frac{1}{a} \left[ \frac{3a}{T} + \frac{1}{4} \right] - \frac{1}{4a} \right) \), \( 0 < \varpi \leq \min\{\varpi_0, \varpi_0(a)\} \).

We notice that Theorems 2.1–2.3 do not assume that the relative period \( T > 0 \) in (2.16) is minimal. They assume only that it satisfies either the nondegeneracy condition (2.17), of the more delicate condition (2.28), or the more rough condition (4.3), (4.4), (4.5), (4.6), or the \( T_{\text{min}} \)-rough condition (cf. (B3)). For example, we can assume that \( T = \frac{\text{const}}{\varpi} \) (see above).

(B2) Let \( N = n > 1 \) (there are no satellites, as in the cases by G.M. Krasinskii [7] and, in particular, H. Poincaré [6]). As in the general case (B1), one constructs a two-parameter family of tuples of frequencies \( \varpi_i \) with parameters \( T = \frac{2\pi}{\varpi} \) and \( \varpi_1 \sim \varpi \) (for fixed integers \( k_2, \ldots, k_n \), real numbers \( a = \text{const} \geq a_0 \) and \( 0 < \varpi \leq \min\{\varpi_0, \varpi_0(a)\} \)), satisfying the rough nondegeneracy condition (4.3), (4.4), (4.6) with \( N = n \), where the numbers \( \Omega_{ij}, K_{ij} \) in (4.4) are ignored. As above, the rough nondegeneracy implies the nondegeneracy (2.17).

(B3) Let \( N > n = 1 \) (only one planet, as in the cases by V.N. Tkhay [10] and, in particular, G.W. Hill [2]), i.e. we study a system Sun–planet–satellites. If \( N = 2 \) (the system Sun–Earth–Moon, as in the case by G.W. Hill [2]), the minimal positive relative period is \( T_{\text{min}} = \frac{2\pi}{|H_{11} - \varpi_1|} \in \left[ \frac{2\pi}{1 + \varpi}, \frac{2\pi}{c - \varpi} \right] \) and the angle of turning is \( \alpha_{\text{min}} = \varpi_1 T_{\text{min}} \in (-\pi, \pi) \) (so, they have orders 1 and \( \varpi \), respectively) if \( 0 < \varpi \leq \min\{\varpi_0, \frac{\pi}{T} \} \). Therefore, the conditions of a relative resonance (2.14) and nondegeneracy (2.17) hold automatically for \( T = T_{\text{min}} \), for any \( \varpi_1, \Omega_{11} \) of the form (2.7), (2.8), (2.9) and \( 0 < \varpi \leq \min\{\varpi_0, \frac{\pi}{T} \} \). More generally: if \( N > n = 1 \) then, due to boundedness of \( T_{\text{min}} \) (cf. (A)), the angle of turning is \( \alpha_{\text{min}} = \varpi_1 T_{\text{min}} \in (-\pi, \pi) \), and the nondegeneracy condition (2.17) holds for \( T = T_{\text{min}} \) if the tuple of frequencies \( \varpi_1, \Omega_{1j} \) has the form (2.7), (2.8), (2.9), (2.14) and \( 0 < \varpi \leq \min\{\varpi_0, \frac{c}{1 + 2min_j |K_{1j}|} \} \). On the other hand, if \( n = 1 \) then the conditions (2.7), (2.8) and (2.9) hold automatically, provided that the tuple of frequencies has the form (2.14) for some relatively-prime tuple of nonvanishing integers \( K_{ij} \) with pairwise different absolute values, with the properties \( \max_j |K_{ij}| \leq \min_j |K_{ij}| =: r \in (c, 1] \) and \( \frac{\min_{c < c'} |K_{ij}| - |K_{ij'}|}{\max_j |K_{ij}|} =: r' \in (c, 1], \) and каких-либо чисел \( \varpi \in (0, \min\{\varpi_0, \frac{c}{1 + 2min_j K_{1j}} \}) \) (the latter segment has a nonempty interior, since \( 0 < \varpi < \min\{1 - \frac{1+c}{1+r}, 1 - \frac{2c}{2+r} \} \) and \( r, r' > c \)). The latter system of relations on \( K_{1j}, \varpi, \varpi_1, T \) gives a two-parameter family of nondegenerate (i.e. satisfying (2.17)) tuples of frequencies \( \varpi_1, \Omega_{1j} \) with parameters \( T = T_{\text{min}} \sim 1 \) and \( 0 < \varpi \ll 1 \) (and fixed \( K_{1j} \)) if \( n = 1 \). Such tuples of frequencies will be called \( T_{\text{min}} \)-roughly nondegenerate for \( n = 1 \).
§ 5. Constructing relatively-periodic solutions of the unperturbed problem (2.18), (2.19)

Let us fix real numbers $m_i, m_{ij}, \omega > 0$ and an arbitrary tuple of “angular frequencies” $\omega_i, \Omega_{ij} \in \mathbb{R} \setminus \{0\}$ of the form (2.7), (2.8). Let $(Y^0(t), X^0(t)) = (Y^0_\ell(t), Y^0_{m\ell}(t), X^0_{m\ell}(t))$ be the main generating solution (2.15) of the model system (2.11) corresponding to this tuple of frequencies.

Let us study the unperturbed system (2.18), (2.19) near the curve $(Y^0(t), X^0(t))$. We will describe a construction of its solution $(y^0(t), x^0(t)) = (y^0_\ell(t), y^0_{m\ell}(t), x^0_\ell(t), x^0_{m\ell}(t))$ that is $O(\omega)$–close to the solution $(Y^0(t), X^0(t))$ and satisfies that following conditions: $(y^0_\ell(t), y^0_{m\ell}(t)) = (Y^0_\ell(t), X^0_{m\ell}(t))$ and the vectors $y^0_\ell(0)$ are collinear with the abscissa axis (moreover, the solution is $(T, \alpha)$-periodic if (2.14), (2.16)).

If there are no satellites then the unperturbed system coincides with the model one, and the solution is already constructed: $(y^0(t), x^0(t)) = (Y^0(t), X^0(t))$.

Suppose that satellites are present. We have to find a solution $(y^0_{\ell j}(t), x^0_{\ell j}(t))$ of the non-autonomous Hamiltonian system (2.19), where $y_\ell(t) := Y^0_\ell(t)$. This system splits into the direct product of systems (corresponding to “satellites”) with the Hamilton functions and the symplectic structures

$$S_j^{(\ell)}(y_\ell, x_\ell; m_\ell, m_{\ell j}) + \omega^2 m_{\ell j} F(Y^0_\ell(t), y_\ell), \quad dx_\ell \wedge dy_\ell$$

(5.1)

corresponding to the $j$-th satellite of the $\ell$-th planet ($1 \leq \ell \leq n, 1 \leq j \leq n_\ell$),

$$S_j^{(\ell)}(y_\ell, x_\ell; m_\ell, m_{\ell j}) := \frac{x_{\ell j}^2}{2m_{\ell j}} - \frac{m_{\ell j} m_{\ell j}}{|y_\ell|}, \quad F(V, v) := \frac{V^2v^2 - 3(V, v)^2}{2|V|^5}.$$  

(5.2)

So, it suffices to construct, for each pair $(\ell, j)$, a solution $(y_\ell(t), x_\ell(t))$ of the system (5.1) that is $O(\omega)$–close to $(Y^0_\ell(t), X^0_{\ell j}(t))$ and satisfies the following conditions: it is relatively-periodic with the same relative period $(\frac{2\pi}{\Omega_{\ell j} - \omega_\ell}, \frac{2\pi\omega_\ell}{\Omega_{\ell j} - \omega_\ell})$ as for the solution $(Y^0_\ell(t), X^0_{\ell j}(t))$, and the radius-vector $y^0_{\ell j}(0)$ is collinear with the abscissa axis. In fact: if (2.14), (2.16) then such a solution will be automatically $(T, \alpha)$-periodic, since $(K_\ell - k_\ell)(\frac{2\pi}{\Omega_{\ell j} - \omega_\ell}, \frac{2\pi\omega_\ell}{\Omega_{\ell j} - \omega_\ell}) = (T, \alpha + 2\pi(k_\ell - k))$.

Let us apply the transformation $(y_\ell, x_\ell) := e^{i\omega_\ell t}(v, u)$, which is equivalent to passing to a synodic coordinate system (i.e. a coordinate system rotating with the “scaled planet” $Y^0_\ell(t) = e^{i\omega_\ell t}V^0$, where $V^0 := (\Omega_0^{-2/3}, 0)$). This transformation reduces the system (5.1) to the autonomous Hamiltonian system with the Hamilton function and the symplectic structure

$$S_j^{(\ell)}(v, u; m_\ell, m_{\ell j}) - \omega_\ell[v, u] + \omega_\ell^2 m_{\ell j} F(V, v)|_{v=(1,0)}, \quad du \wedge dv.$$ 

(5.3)

The system (5.3) with the parameter values $\omega_\ell = m_\ell = m_{\ell j} = 1$ describes the known Hill problem, and any system (5.3) with any nonzero parameter values reduces to it by some power transformation (cf. §5.2). The summand $F(V, v)|_{v=(1,0)} = -v_1^2 + v_2^2/2$ in the Hamilton function of the problem (5.3) (and, hence, of the Hill problem), as well as the function $F(V, v)$, will be called the Hill potential (or the “limiting potential of the action of the Sun to the satellite”).

Our purpose is the following. For small enough $|\omega_\ell|$, we have to construct a $\frac{2\pi}{|\Omega_{\ell j} - \omega_\ell|}$-periodic solution $(v^0(t), u^0(t))$ of the system (5.3) that is $O(\omega_\ell)$–close to
the \( \frac{2\pi}{|\ell_j - \omega t|} \) periodic solution \( e^{-i\omega t}(Y^0_{\ell_j}(t), X^0_{\ell_j}(t)) \) of the similar system with \( F \equiv 0 \) and has the following property: its initial point \( \mathbf{v}(0) \) belongs to the abscissa axis. For doing this, we will first check (in §5.1) that the system (5.3) satisfies the hypotheses of the technical Lemma 3.1(A), which means that the Hamilton function \( S_j(t)(\mathbf{v}, \mathbf{u}; \ell, m_{\ell_j}) \) of the planar Kepler problem satisfies the conditions (3.8) in some canonical coordinates \( \varphi_{\ell_j} \mod 2\pi, I_{\ell_j}, q_{\ell_j}, p_{\ell_j} \).

5.1. Normalizing the Kepler problem near circular orbits. Let us show that the Hamilton function of any planar Kepler’s problem has the form \( H = H(I, q, p) \) and satisfies the conditions (3.8) of Theorem 3.1, with respect to some canonical coordinates \( \varphi, I, q, p \) in some neighbourhood of the union of phase trajectories of circular solutions.

The planar Kepler problem is given by the Hamiltonian system

\[
\left( M = T^*(\mathbb{R}^2 \setminus \{0\}), \omega = dp \wedge dq, \quad H = \frac{p^2}{2m} - \frac{km}{|q|} = G + U \right). \tag{5.4}
\]

Here \( G = \frac{p^2}{2m} \) and \( U = -\frac{km}{|q|} \) are the kinetic and potential energies of the system, \( q \in \mathbb{R}^2 \setminus \{0\} \) and \( p \in \mathbb{R}^2 \) are the radius vector and the momentum of the particle, \( m, k > 0 \) are parameters. The solutions of the Kepler problem having negative energy levels \( H \) are periodic. Since the Kepler problem is invariant under all rotations of the plane, it has the first integral of angular momentum \( I = [q, p] \).

Note simple properties of circular motions in the Kepler problem:

1) For any \( r > 0 \), there is a unique (up to changing the direction of rotation) circular motion of the particle satisfying the system (5.4) along a circle of radius \( r \). The angular velocity of this motion equals \( \Omega = \pm \frac{\sqrt{k}}{r} \), while the energy \( H \) and the angular momentum \( I \) equal \( H = -\frac{km}{2r} \) and \( I = m\Omega r^2 = \pm m\sqrt{kr} \) respectively. In particular, the values \( H \) and \( I \) depend monotonically on the value \( \Omega \) (for \( \Omega > 0 \) or \( \Omega < 0 \)) and take all values in the domains \( H < 0 \) and \( I \neq 0 \) respectively. We will assume that the parameters \( r, \Omega, H, I \) of a circular motion are related by the formulae from above.

2) Circular motions correspond to the equilibrium (i.e. stationary) positions of the particle with respect to a rotating coordinate system with angular velocity \( \Omega \). Therefore, for any \( \alpha \neq 0 \), the solution of the Kepler problem corresponding to a circular motion is \((T, \alpha)\)-periodic with \( T = \frac{\pi}{\Omega} \). In other words, such a solution is \( \frac{\pi}{\Omega} \)-periodic with respect to a rotating coordinate system with angular velocity \( \Omega \) (as well as any angular velocity of the form \((1 + \frac{2\pi}{\alpha})\Omega\) where \( \ell \) is an integer).

The Kepler problem of our interest (5.1) with \( \varpi = 0 \) (or (5.3) with \( \varpi = 0 \)) has the form (5.4), where \( H = S_j(t), I = I_{\ell_j}, \)

\[
(r, p) = (v, u), \quad m = m_{\ell_j}, \quad k = m_{\ell}, \quad r = r_{\ell_j}, \quad \Omega = \Omega_{\ell_j}. \tag{5.5}
\]

Lemma 5.1 (Normalizing the Kepler Problem). Let \( H \) and \( I \) be the Hamilton function and the angular momentum first integral of the planar Kepler problem (5.4). In the domain \( \{r > 0, \varphi \neq 0\} \) of the phase space of the problem, consider the coordinates \( \varphi \mod 2\pi, I, q, p \) of the form

\[
\varphi = \psi - \frac{2r_\psi}{p_\psi}, \quad I = p_\psi, \quad q = \ln \frac{km^2 r}{p_\psi^2}, \quad p = rp_r. \tag{5.6}
\]
Here \( r, \psi \) are the polar coordinates in the plane of motion, \( p_r, p_\psi = I \) are the corresponding momenta. Then:

(a) the coordinates (5.6) are canonical, i.e. \( \omega = dI \wedge d\varphi + dp \wedge dq \);
(b) in the coordinates (5.6), the Hamilton function \( H \) of the Kepler problem does not depend on the angular coordinate \( \varphi \mod 2\pi \) and has the form

\[
\frac{H}{k^2 \mu_3} = \frac{I^2 + p_r^2}{2I^2e^{2\varphi}} - \frac{1}{I^2e^4} = -\frac{1}{2I^2} + \frac{q^2}{2I^2} + \frac{p_r^2}{2I^4} + o(q^2 + p_r^2)
\]
as \( q^2 + p_r^2 \to 0 \). Furthermore the involutions \( S_l, S \) in §4.2 with \( l = \mathbb{R} \times \{0\} \subset \mathbb{R}^2 \) have the form

\[
S_l(\varphi, I, q, p) = (-\varphi, -I, q, p), \quad S(\varphi, I, q, p) = (\varphi, -I, q, -p).
\]

One proves Lemma 5.1 in a direct way. \( \square \)

The canonical coordinates \( \varphi \mod 2\pi, I, q, p \) in Lemma 5.1, as follows from their construction, are quite “similar” to the canonical coordinates \( \psi \mod 2\pi, p_\psi = I, r, p_r \) corresponding to the polar coordinates \( \psi \mod 2\pi, r \) in the configuration space of the planar Kepler problem. For example, the involutions \( S_l, S \) have the form (5.7) in the coordinates \( \psi, p_\psi, r, p_r \) too:

\[
S_l(\psi, p_\psi, r, p_r) = (-\psi, -p_\psi, r, p_r), \quad S(\psi, p_\psi, r, p_r) = (\psi, -p_\psi, r, -p_r).
\]

For any \( \Omega \neq 0 \), denote by \( \gamma_\Omega \) the phase trajectory of the Kepler problem corresponding to the circular motion with angular velocity \( \Omega \). The invariant two-dimensional surface \( \bigcup_{\Omega \neq 0} \gamma_\Omega \) in the phase space formed by all these trajectories will be called the surface of circular motions. This surface is smooth and consists of two connected components, each of which is diffeomorphic to a punctured plane and bijectively projects onto the configuration manifold under the canonical projection.

We recall that the system describing the motion with respect to a rotating coordinate system with angular velocity \( \Omega \) is a Hamiltonian system with the Hamiltonian function \( H = -\Omega I \). Thus the circular motions correspond to the stationary points of such systems.

**Corollary 5.1.** The surface of circular motions of the planar Kepler problem is a region in the symplectic coordinate cylinder with respect to the coordinates \( \varphi \mod 2\pi, I, q, p \) from Lemma 5.1:

\[
\bigcup_{\Omega \in \mathbb{R} \setminus \{0\}} \gamma_\Omega = \{(\varphi \mod 2\pi, I, 0, 0) \mid I \neq 0\} = S^1 \times (\mathbb{R} \setminus \{0\}) \times \{(0, 0)\}.
\]

For any \( \Omega \in \mathbb{R} \setminus \{0\} \), at any point of the circle \( \gamma_\Omega \), the differential of the function \( H = -\Omega I \) equals 0, while the quadratic part (i.e. the quadratic form whose matrix is formed by the second partial derivatives) of this function has a diagonal form with respect to the coordinates (5.6):

\[
d^2(H - \Omega I)|_{\gamma_\Omega} = -\frac{3}{mr^2} \delta I^2 + \Omega \left( I\delta q^2 + \frac{\delta p_r^2}{I} \right).
\]

(5.8)
5.2. Constructing a family of periodic solutions of the Hill problem (the Moon theory). Рассмотрим частный случай плоской задачи трёх тел — систему типа Солнце–Земля–Луна ($N = 2, n = 1$). Изучим в этом случае невозмущённую систему и построим её относительно-периодические решения.

Пусть, как выше, “масштабированная планета” совершает круговое движение $(y_1(t), x_1(t)) = (Y_1(t), X_1(t)) = e^{i\omega_1 t}(\Omega_{10}^{-2/3}, i\Omega_{10}^{1/3} m_1)$ с угловой скоростью $\omega_1$. Тогда, согласно (5.3), невозмущённое движение “спутника” в синодической (т.е. вращающейся с угловой скоростью $\omega_1$) системе координат описывается гамильтоновой системой с функцией Гамильтона и симплектической структурой Гамильтона и симплектической структурой задачи Хилла (т.е. вращающейся с угловой скоростью $\omega_1$) в синодической системе координат описывается гамильтоновой системой с функцией Гамильтона и симплектической структурой

$$ H_1 = \frac{u^2}{2m} - \frac{km}{|\textbf{v}|} - \omega_1 [\textbf{v}, \textbf{u}] + \omega_1^2 m\hat{F}(\textbf{v}), \quad \omega = d\textbf{u} \wedge d\textbf{v}, \quad (5.9) $$

с параметрами $m, k > 0$ и $\omega_1 \neq 0$, где $m := m_{11}, k := m_1$ (как в (5.5)),

$$ \hat{F}(\textbf{v}) := -v_1^2 + v_2^2/2 = F(\textbf{v}, \textbf{v})|_{\textbf{v}=(1,0)} \quad (5.10) $$

— потенциал Хилла (“предельный потенциал действия Солнца на спутник”).

Задача (5.9) при единичных значениях параметров $m = k = \omega_1 = 1$ является задачей Хилла, с функцией Гамильтона и симплектической структурой

$$ \hat{H}_1 = \frac{\hat{u}^2}{2} - \frac{1}{|\hat{\textbf{v}}|} - [\hat{\textbf{v}}, \hat{\textbf{u}}] + \hat{\textbf{F}}(\hat{\textbf{v}}), \quad \hat{\omega} = d\hat{\textbf{u}} \wedge d\hat{\textbf{v}}. \quad (5.11) $$

Покажем, что задача (5.9) приводится к задаче Хилла (5.11) некоторой заменой переменных. Дробно-степенное преобразование $\hat{\textbf{v}} = k^{-1/3}\textbf{v}, \hat{\textbf{u}} = m^{-1}k^{-1/3}\textbf{u}, \quad \hat{H}_1 = m^{-1}k^{-2/3}H_1, \quad \hat{\omega} = m^{-1}k^{-2/3}\omega$ приводит гамильтонову систему (5.9) к системе с функцией Гамильтона и симплектической структурой

$$ \hat{\textbf{H}}_1 = \frac{\hat{\textbf{u}}^2}{2} - \frac{1}{|\hat{\textbf{v}}|} - [\hat{\textbf{v}}, \hat{\textbf{u}}] + \hat{\textbf{F}}(\hat{\textbf{v}}), \quad \hat{\omega} = d\hat{\textbf{u}} \wedge d\hat{\textbf{v}} \quad (5.12) $$

с параметром $\omega_1 > 0$. Преобразование $\tilde{\textbf{v}} = \omega_1^{2/3}\hat{\textbf{v}}, \tilde{\textbf{u}} = \omega_1^{-1/3}\hat{\textbf{u}}, \tilde{H}_1 = \omega_1^{-2/3}\hat{H}_1, \quad \tilde{\omega} = \omega_1^{-1/3}\hat{\omega}$ приводит систему (5.12) к требуемой задаче Хилла (5.11).

Рассмотрим “стандартную” (без параметров) “синодическую задачу Кеплера” с функцией Гамильтона и симплектической структурой

$$ \frac{\tilde{\textbf{p}}^2}{2} - \frac{1}{|\tilde{\textbf{r}}|} - [\tilde{\textbf{r}}, \tilde{\textbf{p}}], \quad d\tilde{\textbf{p}} \wedge d\tilde{\textbf{q}}. \quad (5.13) $$

Её функция Гамильтона получена из функции Гамильтона $\tilde{\textbf{H}}_1$ задачи Хилла (5.11) откидыванием последнего слагаемого (т.е. при $\tilde{F} \equiv 0$). Рассмотрим 1-параметрическое семейство “критовых” решений

$$ (\tilde{\textbf{q}}_{\tilde{\Omega}}(\tilde{t}), \tilde{\textbf{p}}_{\tilde{\Omega}}(\tilde{t})) = e^{i(\tilde{\Omega} - 1)\tilde{t}}(\tilde{\Omega}^{-2/3}, i\tilde{\Omega}^{1/3}) \quad (5.14) $$

задачи (5.13), где $\tilde{t}$ — время, $\tilde{\Omega} := \Omega/\omega_1$ — отношение угловой скорости сидерического кругового вращения “спутника” к угловой скорости кругового вращения “масштабированной планеты” (которая здесь считается равной 1).
EXAMPLE 5.1 (см. пример 1 в [19]). Многие семейства периодических решений задачи Хилла (5.11) хорошо известны и изучены [27], причем некоторые из этих семейств "достаточно близки" к семейству круговых решений (5.14) синоидической задачи Кеплера (5.13). Например, известны два 1-параметрических семейства \( f \) и \( g_+ \) периодических решений задачи Хилла, параметром на каждом из которых служит гамильтониан \( \tilde{H}_1 \in (-\infty, \infty) \), или минимальный положительный период \( \bar{T} \in (0, T_f) \) и \( \bar{T} \in (0, T_{g+}) \) соответственно, где \( T_f \approx 2\pi \) и \( T_{g+} \approx 4\pi \) [28; tables 3, 4], или отношение \( \tilde{\Omega} = 1 \pm \frac{2\pi}{\bar{T}} \neq 0 \) средней угловой частоты (сидерического) вращения "спутника" к угловой частоте вращения "масштабированной планеты". Оба семейства \( f \) и \( g_+ \) настаиваются (при \( \tilde{\Omega} \rightarrow -\infty \) или \( \tilde{\Omega} \rightarrow +\infty \)) квази-круговыми орбитами вокруг точки \( \tilde{\nu} = 0 \), причем направление движения на первом семействе — обратное (по часовой стрелке, \( \tilde{\Omega} < 0 \)), а на втором — прямое (против часовой стрелки, \( \tilde{\Omega} > 0 \)). Согласно численному результату М. Хена [28; tables 11, 12], семейства \( f \) и \( g_+ \) "достаточно близки" к семейству круговых решений (5.14) синоидической задачи Кеплера (5.13) на следующих начальных участках этих семейств: когда параметр \( \Gamma := -2\tilde{H}_1 \) семейства удовлетворяет оценке \( \Gamma \geq 1 \) и \( \Gamma \geq 5 \) соответственно (т.е. период решения удовлетворяет оценке \( \bar{T} \leq 1.51822 \) и \( \bar{T} \leq 0.92296 \) соответственно [28; tables 3, 4]). Наш подход дает построение начальных участков семейств \( f \) и \( g_+ \) (см. (5.17) ниже) и их "достаточную близость" к семейству круговых решений (5.14) при следующей оценке на период решения: \( \bar{T} < 2\pi \varpi_0 \), где \( \varpi_0 \in (0, 1) \) — некоторая (до足но малая) константа. Видимо, наша верхняя граница 2\( \pi \varpi_0 \) для периода меньше обоих значений 1.51822 и 0.92296, указанных Хеноном. Движение Луны в системе Солнце-Земля-Луна приближенно соответствует решению из семейства \( g_+ \) с периодом \( \bar{T} = 2\pi \cdot \frac{28}{92296} \approx \frac{2\pi}{13} \approx 0.48 \).

Рассмотрим задачу Хилла (5.11) (к которой, как мы показали, сводится задача (5.9), описывающая невозвышенное движение "спутника"). Мы хотим найти два 1-параметрических семейства её периодических решений (\( \tilde{\nu}_\Omega(\tilde{t}), \tilde{u}_\Omega(\tilde{t}) \)) при достаточно большом отношении средних частот |\( \tilde{\Omega} | \gg 1 \).

Наша основная идея состоит в следующем. Вместо задачи Хилла (5.11) мы рассмотрим (эквивалентную ей) "параметрическую задачу Хилла" (5.9) или (5.12) с малым параметром \( \varpi_1 \) и построим (в (5.19) и (5.18) ниже) её периодические решения методами теории возмущений (см. лемму 3.1(A)). Проведём построение искомого семейства \( \frac{2\pi}{|\Omega - \varpi_1|} \) периодических решений (см. (5.18) ниже) задачи (5.12) с помощью леммы 3.1(A).

Шаг 1. Опишем сначала "порождающие" круговые решения. При откidyвании последнего слагаемого (порядка \( \varpi_1^2 \)) из функции Гамильтона \( \tilde{H}_1 \) задачи (5.12), т.е. при \( \hat{F} \equiv 0 \), получаем "1-параметрическую синоидическую задачу Кеплера" (для "спутника") с гамильтонианом и симплектической структурой

\[
\frac{\hat{P}^2}{2} - \frac{1}{|\hat{r}|} \varpi_1[\hat{r}, \hat{p}], \quad d\hat{p} \wedge d\hat{q},
\]

c параметром \( \varpi_1 \in \mathbb{R} \). Рассмотрим семейство "круговых" решений этой задачи:

\[
(\hat{q}_\Omega, \varpi_1(t), \hat{p}_\Omega, \varpi_1(t)) = e^{i(\Omega - \varpi_1)t}(\Omega^{-2/3}, \hat{t}\Omega^{1/3}), \quad \Omega, \varpi_1 \in \mathbb{R}, \quad \Omega \neq 0,
\]  
(5.15)
действием $\varpi_1$ и $\Omega$ — угловые скорости сидерического кругового вращения “масштабированной планеты” и “спутника” соответственно. Круговое решение (5.15) является $\frac{2\pi}{|\varpi_1|}$-периодическим, его фазовая траектория $\gamma_0$ не зависит от $\varpi_1$.

проведем описанную выше процедуру для $\Omega := 1$. Изоэнергетическая поверхность $\Pi^\circ = \Pi^\circ_0 := \tilde{H}^{-1}(\tilde{\Omega}^{3/2}/2)$ задачи Кеплера заполнена $\frac{2\pi}{\Omega}$-периодическими траекториями задачи Кеплера. Рассмотрим окружность $\gamma_0 \subset \Pi^\circ$, т.е. фазовую траекторию задачи Кеплера, отвечающую круговому решению $(\tilde{q}_{\Omega,0}(t), \tilde{p}_{\Omega,0}(t))$ из (5.15) с угловой скоростью $\Omega$ (см. §5.1). Согласно лемме 5.1 и ее следствию 5.1, 1-параметрическая задача Хилла (5.12) удовлетворяет условиям (3.8). Поэтому, согласно лемме 3.1(A), существует столько малое число $\varepsilon_0 > 0$ и гладкое 1-параметрическое семейство $\frac{2\pi}{|\Omega|}$-периодических траекторий (“возмущенной”) системы (5.12) с параметром $\varpi_1 \in (-\varpi_0, 0)$, такое, что нулевому значению параметра $\varpi_1 = 0$ отвечает траектория кругового решения (т.е. окружность $\gamma_0$), а при $\varpi_1 \in (-\varpi_0, 0) \cup (0, \varpi_0)$ траектория семейства является единственной $\frac{2\pi}{|\Omega|}$-периодической траекторией системы (5.12), $O(\varpi_0)$-близкой к окружности $\gamma_0$. На каждой траектории этого семейства выберем параметризацию так, чтобы она задавала $\frac{2\pi}{|\Omega|}$-периодическое решение системы (5.12), положение которого в начальный момент времени принадлежит оси абсцисс. Полученное 1-параметрическое семейство $\frac{2\pi}{|\Omega|}$-периодических решений системы (5.12) (при фиксированном $\Omega \neq 0$) обозначим через

$$\left(\tilde{v}_{\Omega}(t), \tilde{u}_{\Omega}(t)\right), \quad \varpi_1 \in (-\varpi_0, \varpi_0). \quad (5.16)$$

Имеем $(\tilde{v}_0(t), \tilde{u}_0(t)) = (\tilde{q}_{\Omega,0}(t), \tilde{p}_{\Omega,0}(t))$.

проведем описанную выше процедуру для $\Omega := 1$. Из семейства (5.16) при $\Omega := 1$ мы получаем два (обнаруженных ещё Хиллом [2, 3]) 1-параметрических семейства $\frac{2\pi}{|\Omega - 1|}$-периодических решений

$$\left(\tilde{v}_{\tilde{\Omega}}(\tilde{t}), \tilde{u}_{\tilde{\Omega}}(\tilde{t})\right) := \left(\tilde{v}_{\tilde{\Omega}}(\tilde{t}), \tilde{u}_{\tilde{\Omega}}(\tilde{t})\right) := \left((\tilde{\Omega} - 1)^{-2/3} \tilde{v}_{(\tilde{\Omega} - 1)^{-1}}(\tilde{t}), (\tilde{\Omega} - 1)^{1/3} \tilde{u}_{(\tilde{\Omega} - 1)^{-1}}(\tilde{t})\right), \quad \tilde{\Omega} \in (-\infty, 1 - \varpi_0^{-1}) \cup (1 + \varpi_0^{-1}, \infty), \quad (5.17)$$

задачи Хилла (5.11). Параметр первого семейства отрицателен, а параметр второго — положителен. (Для получения формул (5.17) надо положить $\varpi_1 := (\tilde{\Omega} - 1)^{-1}$ в (5.16) и использовать преобразование $\tilde{v} := \varpi_1^{2/3} \tilde{v}, \tilde{u} := \varpi_1^{-1/3} \tilde{u}, \tilde{t} := \varpi_1 t$, приводящее задачу (5.12) к задаче Хилла.) Семейства решений (5.17) — это (обнаруженные Хиллом [2, 3]) наложенные участки хорошо известных и изученных семейств $f$ и $g_+$ периодических решений задачи Хилла [27]. В каждом из семейств $f$ и $g_+$ интервал изменения периода содержит указанный нами интервал $(0, 2\pi \varpi_0)$ (см. также пример 5.1). Оба эти семейства порождают семейства периодических решений круговой ограничённой задачи трёх тел (см.
[4, 1], [5; §17–19], [30, 31, 27]) и относительно-périодических решений задачи трёх тел (см. [1] и [5; §18–19]).

Из двух семейств (5.17) решений задачи Хилла получаем искомое 2-параметрическое семейство 
\[ \frac{2\pi}{|\Omega - \omega_1|} \text{-périодических решений системы (5.12):} \]

\[ \tilde{\gamma}_{\Omega, \omega_1}(t) = \left( \bar{v}_{\Omega, \omega_1}(t), \bar{u}_{\Omega, \omega_1}(t) \right) := \left( (\Omega - \omega_1)^{-2/3} \bar{v}_{\Omega, \omega_1}(\Omega - \omega_1 t), (\Omega - \omega_1)^{1/3} \bar{u}_{\Omega, \omega_1}(\Omega - \omega_1 t) \right), \quad (5.18) \]

\[ \Omega \neq 0, \quad \frac{\omega_1}{\Omega} \in \left( -\frac{\omega_0}{1 - \omega_0}, \frac{\omega_0}{1 + \omega_0} \right). \]

(Для этого надо положить \( \tilde{\Omega} := \Omega/\omega_1 \) в (5.17) и использовать обратное преобразование переменных.) Отметим, что решение (5.16) равно \( \tilde{\gamma}_{1 + \omega_1, \omega_1}(t) \).

**Remark 5.1.** Построенное (с помощью технической леммы 3.1(A)) 2-параметрическое семейство решений (5.18) "1-параметрической задачи Хилла" (5.12) гладко зависит от параметров \( \Omega, \omega_1 \) и при \( \omega_1 = 0 \) совпадает с семейством круговых решений (5.15). В частности, эти два семейства \( \frac{2\pi}{|\Omega - \omega_1|} \text{-périодических решений,} \) (5.18) и (5.15), являются \( O(\omega_1) \)–близкими. Показем, что из техническим леммы 3.1(B) следует, что эти семейства даже \( O(\omega_1^2) \)–близки. Рассмотрим функцию (\( F^0 \)) на изоэнергетической поверхности \( \Pi^0 \) задачи Кеплера (см. шаг 2), полученную усреднением функции \( F := F^0 |_{\Pi} \) по \( \frac{2\pi}{\Omega} \)-périодическим решением задачи Кеплера \( (M, \tilde{\omega}, \tilde{H}) \). Несложным вычислением, с учётом (5.10) и следствия 5.1, проверяется, что окружность \( \gamma = \Pi^0 \) является критическим множеством функции (\( F^0 \)). Поэтому из леммы 3.1(B) следует требуемая оценка

\[ \tilde{\gamma}_{\Omega, \omega_1}(t) = \left( \tilde{\varphi}_{\Omega, \omega_1}(t), \tilde{\psi}_{\Omega, \omega_1}(t) \right) + O(\omega_1^2). \]

Таким образом, мы построили следующее невозмущённое синодическое (т.е. описываемое задачей (5.9)) \( \frac{2\pi}{|\Omega - \omega_1|} \text{-périодическое движение “спутника”:} \)

\[ (v^0(t), u^0(t)) := k^{1/3}(\bar{v}_{\Omega, \omega_1}(t), m\bar{u}_{\Omega, \omega_1}(t)), \quad \frac{\omega_1}{\Omega} \in \left( -\frac{\omega_0}{1 - \omega_0}, \frac{\omega_0}{1 + \omega_0} \right), \quad (5.19) \]

см. (5.18), (5.16). Это решение отвечает круговому движению "масштабированной планеты" с угловой скоростью \( \omega_1 \) и сидерическому движению "спутника" со средней угловой частотой \( \Omega \). Поэтому искомое невозмущённое (сидерическое) движение “спутника” для задачи Солнце-Земля-Луна можно положить равным

\[ (y^0_{11}(t), x^0_{11}(t)) := e^{i\omega_1 t} m_1^{1/3}(\bar{v}_{\Omega, \omega_1, \omega_1}(t), m_{11}\bar{u}_{\Omega, \omega_1, \omega_1}(t)), \quad \frac{\omega_1}{\Omega_{11} - \omega_1} \in (\omega_0, 0). \]

5.3. Constructing families of relatively-periodic solutions of the unperturbed system. Рассмотрим невозмущённую систему (2.18), (2.19) для задачи \( N + 1 \) тел с любым числом планет и спутников, \( N \geq n \geq 1 \). Положим

\[ (y^0_{\ell_j}(t), x^0_{\ell_j}(t)) := e^{i\omega_\ell t} m_\ell^{1/3}(\bar{v}_{\Omega, \omega_\ell, \omega_\ell}(t), m_{\ell_j}\bar{u}_{\Omega, \omega_\ell, \omega_\ell}(t)), \quad \frac{\omega_\ell}{\Omega_{\ell_j} - \omega_\ell} \in (\omega_0, 0), \]

при \( 1 \leq j \leq n_\ell \), см. (5.18), (5.16). Построенное нами решение \( (y^0(t), x^0(t)) \) = \( (y^0_\ell(t), y^0_{\omega_\ell}(t), x^0_\ell(t), x^0_{\omega_\ell}(t)) \) является искомым решением невозмущённой системы (2.18), (2.19), \( O(\omega) \)–близким к порождающему решению \((Y^0(t), X^0(t))\). Более того, как выведено в замечании 5.1 из леммы 3.1(B), решение \((y^0(t), x^0(t))\) даже \( O(\omega^2) \)–близко к порождающему решению \((Y^0(t), X^0(t))\).
§ 6. Constructing relatively-periodic solutions of the perturbed system (2.22)

В этом параграфе мы начнём вывод теоремы 2.2(А) из леммы 3.1.

Как и в §5, фиксируем числа $m_i, m_{ij} > 0$, $\varpi \in (0, \varpi_0]$ и набор “угловых частот” $\varpi_i, \Omega_{ij} \in \mathbb{R} \setminus \{0\}$ вида (2.7), (2.8). Предположим, что набор частот является “относительно резонансным”, т.е. имеет вид (2.14), (2.16) (это условие автоматически выполнено при $N = 2$, т.е. в задаче 3 тел).

Пусть $\left( Y^0(t), X^0(t) \right) = \left( Y^0_0(t), Y^0_\ast(t), X^0_0(t), X^0_\ast(t) \right)$ — основное порождаю- щее решение (2.15) модельной системы (2.11), отвечающее этому набору частот.

Пусть $\left( y^0(t), x^0(t) \right) = \left( y^0_0(t), y^0_\ast(t), x^0_0(t), x^0_\ast(t) \right)$ — построенное в §5 $(T, \alpha)$-peri- одическое решение невозмущённой системы (2.18), (2.19), $O(\varpi)$-близкое к решению $(Y^0(t), X^0(t))$ и такое, что $(y^0_0(t), x^0_0(t)) = (Y^0_0(t), X^0_0(t))$ и радиус-векторы $y^0_{ij}(0)$ сонаправлены с осью абсцисс. Рассмотрим тор $\Lambda$, образованный фазовыми траекториями $(T, \alpha)$-периодических решений $(y^0(t), x^0(t))$ невозмущённой системы, $\beta = (\beta_*, \beta_*) \in \mathbb{R}/2\pi \mathbb{Z}$ (см. (2.21)). В §5 мы доказали выполнение всех условий леммы 3.1 для данной невозмущённой системы. Поэтому применима лемма 3.1, и из неё легко следует, что тор $\Lambda$ обладает свойствами, указанными в теореме 3.5.

Предположим, что набор частот удовлетворяет условию невырожденности (2.17) или более тонкому условию (2.28) из теоремы 2.1, где в более тонком условии (2.28) определим константу $C > 0$ равной константе $C_2 > 0$, отвечаю- щей тору $\Lambda$ согласно теореме 3.5. Если выполнено условие невырожденности (2.17), то при $0 < \varpi \leq 3c/(4C)$ выполнен (ввиду (2.8)) более тонкое условие (2.28). Поэтому можно считать, что выполнено более тонкое условие (2.28).

Изучим возмущённую систему (2.22) вблизи тора $\Lambda$. При фиксированных $m_i, m_{ij}, \varpi, \varpi_i, \Omega_{ij}$ система зависит от 4 параметров $\varepsilon, \mu, \nu, \rho \in \mathbb{R}$, а тор $\Lambda$ фиксирован. Мы хотим вывести существование (при фиксированных $m_i, m_{ij}, \varpi, \varpi_i, \Omega_{ij}$) числа $\mu_0 > 0$ и гладкого 4-периодического семейств торов $\hat{\Lambda}$ с параметрами $\varepsilon, \mu, \nu, \rho, |\varepsilon| + |\mu| + |\nu| + |\rho| \leq \mu_0$, обладающего нужными нам свойствами.

Проведём вывод аналогично выводу теоремы 3.1 из технической теоремы 3.5, методом производящей функции (см. §3.1, второй этап, шаги 1–5).

Шаг 1. Согласно методу производящей функции, нужно проверить, что тор $\hat{\Lambda}$ обладает свойством невырожденности (3.21). Это свойство мы вывели (в §3.1, шаг 1) из теоремы 3.5 и более тонкого условия (3.11) из теоремы 3.1.

Проверим условие (3.11). Его первая часть равносильна первой части (2.28). Для проверки второй части (3.11) найдём число $\Delta_{ij}$ в (3.12). В силу (7.20) имеем $F_{ij} = F_{ij}(\eta_i, \eta_{ij}) = m_{ij} F(\eta_i, \eta_{ij})$. По построению $\hat{F}_{ij} = (\hat{F}_{ij}(\eta_i^0, \eta_{ij}^0))|_{H_{ij}^{-1}(H_{ij}(t_0, \tilde{0}, \tilde{0}))}$, где $\eta_i^0 = \text{const}$, $|\eta_{ij}^0| = R_i/R = |\Omega_{0i}|^{-2/3}$ (см. теорему 3.1). Пусть $\langle \hat{F}_{ij} \rangle = \langle \hat{F}_{ij}(q_{ij}, p_{ij}) \rangle_{\Delta_{ij}}$ — функция, полученная усреднением потенциала Хилла $\hat{F}_{ij} = \hat{F}_{ij}(q_{ij}, p_{ij})$ по $\Delta_{ij}$-периодическим решениям задачи Кеплера ($M_{ij} = S^1 \times (\mathbb{R} \setminus \{0\}) \times \mathbb{R}^2$, $\omega_{ij} = dI_{ij} \wedge d\varphi_{ij} + dp_{ij} \wedge dq_{ij}, H_{ij} = S_j^{(i)}$) для “спутника”, где $\varphi_{ij}, I_{ij}, q_{ij}, p_{ij}$ — “нормализующие” канонические координаты для рассматриваемой задачи Кеплера (см. лемму 5.1). Согласно замечанию 5.1, $d\langle \hat{F}_{ij} \rangle(0, 0) = 0$. Несложным
вычислением находим также матрицу Гесса функции $\langle \tilde{F}_{ij} \rangle$ в нуле:

$\frac{\partial^2 \langle \tilde{F}_{ij} \rangle(0,0)}{\partial (q_{ij},p_{ij})^2} = \frac{\Omega_{ij}^2}{2} \left( \begin{array}{cc} -29/8 & 0 \\ 0 & 25/(8I_{ij}^2) \end{array} \right)$. 

Отсюда с учетом (5.8) имеем

$$
\Delta_{ij} = \frac{\Omega_{ij}^2}{2} \text{Tr} \left( \left( \frac{\partial^2 H_{ij}(I_{ij}^0,0,0)}{\partial (q_{ij},p_{ij})^2} \right)^{-1} \frac{\partial^2 \langle \tilde{F}_{ij} \rangle(0,0)}{\partial (q_{ij},p_{ij})^2} \right) = \\
= \frac{\Omega_{ij}^2}{2\Omega_{ij}} \text{Tr} \left( \left( \begin{array}{cc} 1 & 0 \\ 0 & I_{ij}^2 \end{array} \right) \left( \begin{array}{cc} -29/8 & 0 \\ 0 & 25/(8I_{ij}^2) \end{array} \right) \right) = -\frac{\Omega_{ij}^2}{4\Omega_{ij}}. 
$$

(6.1)

Потому требуемая вторая часть условия (3.11) имеет вид

$$
\alpha + \Delta_{ij} \varpi^2 T = \alpha - \frac{\varpi^2}{4\Omega_{ij}} T \not\in [-C_2 \varpi^3, C_2 \varpi^3] + 2\pi Z, 
$$

t.е. равносильна второй части условия (2.28) с константой $C := C_2$.

Итак, выполнено более тонкое условие невырожденности (3.11) в теореме 3.1. Отсюда следует (см. выше) свойство невырожденности (3.21) тора $\Lambda$.

§§ 2-5. Из явного построения функций $\tilde{H}_0, \tilde{H}_1, \tilde{F}$ (см. §7) видно, что они являются $S^1$-инвариантными. Возмущающий потенциал $\tilde{F}$ является аналитической функцией в окрестности любого тора $\Lambda^0$, на котором набор угловых частот $\Omega_{ij}$ удовлетворяет условиям (2.8) и (2.9) "отсутствия столкновений". Главная часть $\Phi := \tilde{F}|_{\mu=\nu=\rho=0}$ возмущающего потенциала имеет вид (3.6).

Согласно методу производящей функции (шаги 2–5), ввиду невырожденности (3.21) тора $\Lambda$, искомое семейство торов $\tilde{\Lambda}$ можно определить формулой (3.22). Поэтому из леммы 3.1 и результатов следующего §7 вытекает теорема 2.2(А) о существовании семейства торов $\tilde{\Lambda}$ с нужными свойствами. \ □

Попутно мы показали, что теорему 2.2(А) можно вывести также из технической теоремы 3.5 (или из теоремы 3.1) и результатов следующего §7.

Отметим, что в частном случае $N = 2, n = 1$ (система Солнце-Земля-Луна) наш результат показывает, что начальные участки (5.17) семейств $f$ и $g_+$ периодических решений задачи Хилла порождают начальные участки семейств периодических решений ограниченной задачи трёх тел (случай $\nu = 0$) и семейств относительно-периодических решений задачи трёх тел. Аналогичные участки семейств решений ограниченной задачи трёх тел (даже без ограничений на параметр $\tilde{\mu} = \frac{\mu m_1}{1+\mu m_1} \in (0,1)$) были найдены Брауном [4] методом Хилла [2, 3] разложения решений в ряд (см. также [1], [5]; §17–19, [30]). Аналогичные участки семейств решений задачи трёх тел (даже без ограничений на параметры $\tilde{\mu} = \frac{\mu(m_1+m_1\nu)}{1+\mu(m_1+m_1\nu)} \in [0,1]$) были найдены Мультоном [1, 5] методом малого параметра Пуанкаре (см. также [5]; §18–19).

§ 7. Reducing the $N + 1$ body problem to the perturbed system (2.22)

Рассмотрим на $(\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2$ гладкую функцию $F = F(V, v)$, определяемую формулой (2.20), см. (2.19). Напомним (см. (5.3)), что функцию $F(V, v)|_{v=(1,0)} =$
−v_1^2 + v_2^2/2, а также функцию \( F(V, v) \), мы называем потенциалом Хилла (или "пределным потенциалом действия Солнца на спутник").

**Remark 7.1.** (A) The Hill potential \( F(x, y) \) is in fact the third coefficient of the power series of the function \( \frac{1}{|x + \rho y|} \) in the variable \( \rho \) at zero:

\[
\frac{1}{|x + \rho y|} = \frac{1}{\sqrt{x^2 + 2\rho(x, y) + \rho^2y^2}} =: \frac{1}{|x|} - \rho \frac{\langle x, y \rangle}{|x|^3} - \rho^2 F_{0, \rho}(x, y), \quad (7.1)
\]

\[
F_{0, \rho}(x, y) = F(x, y) - \rho \langle x, y \rangle \frac{3x^2y^2 - 5\langle x, y \rangle^2}{2|x|^7} - \rho^2 \frac{3x^4y^4 - 30x^2(y, x)^2y^2 + 35\langle x, y \rangle^4}{8|x|^9} + \ldots, \quad \rho \to 0. \quad (7.2)
\]

A more general analytic potential

\[
F_{\theta, \rho}(x, y) = \theta F_{0, -\theta \rho}(x, y) + (1 - \theta) F_{0, (1-\theta)\rho}(x, y) \quad (7.3)
\]

appears in the three-body problem (with \( 0 < \theta < 1, \mu, \omega, \rho > 0 \) and (2.10)) and in the restricted three-body problem (with \( \theta = 0, \mu, \omega, \rho > 0 \) and (2.10)), see §7.7 and (7.4). We remark that \( \frac{\partial F}{\partial y}(x, y) = \frac{x^2y - 3\langle x, y \rangle x}{|x|^6} \).

(B) The unperturbed system (7.4) shows that the variables \( x_i \) and \( y_{ij} \) are automatically slow and fast variables respectively, provided that \( \omega \) is small.

Let \( \eta_i, \eta_{ij}, \xi_i, \xi_{ij} \) be the coordinates (2.25) in the phase space \( T^* \tilde{Q} \), \( 1 \leq i \leq n, \quad 1 \leq j \leq n_i \). Denote

\[
\bar{m}_i = m_i + \nu \sum_{j=1}^{n_i} m_{ij}, \quad \bar{m}_i = \frac{m_i}{1 + \mu m_i}, \quad \bar{m}_{ij} = \frac{m_{ij}}{m_i}, \quad \bar{m}_{ij} = \frac{m_{ij}m_i}{m_i + \nu m_{ij}} \quad (7.4)
\]

the total mass of the \( i \)th satellite system, and the "reduced" masses of planets and satellites. Introduce the following functions on \( T^* \tilde{Q} \) the Hamilton functions

\[
\tilde{K}_i = \frac{\xi_i^2}{2\bar{m}_i} - \frac{\bar{m}_i}{|x_i|}, \quad \tilde{S}_{ij}^{(i)} = \frac{\eta_{ij}^2}{2\bar{m}_{ij}} - \frac{m_i m_{ij}}{|y_{ij}|} \quad (7.5)
\]

of the Kepler problems, the angular momenta

\[
I_{i0} = [x_i, \xi_i], \quad I_{ij} = [y_{ij}, \eta_{ij}], \quad 1 \leq i \leq n, \quad 1 \leq j \leq n_i \quad (7.6)
\]

of "scaled planets" and "satellites", and the "perturbation functions"

\[
K_{ii'} = \langle \xi_i, \xi_{i'} \rangle - \frac{\bar{m}_i \bar{m}_{i'}}{|x_i - x_{i'}|}, \quad S_{ijj'}^{(i)} = \frac{\langle \eta_{ij}, \eta_{ij'} \rangle}{m_i} - \frac{m_i m_{ij}}{|y_{ij} - y_{ij'}|}, \quad 1 \leq i < i' \leq n, \quad 1 \leq j < j' \leq n_i, \quad (7.7)
\]

1 \leq i < i' \leq n, \quad 1 \leq j < j' \leq n_i, \quad (7.7)

of the planetary system and the satellite systems, respectively (corresponding to pair-wise interactions of planets, respectively satellites of the same planet).

As a "perturbation potential", let us consider the function

\[
\hat{\Phi} = \hat{\Phi}(\eta_*, \eta_*, \bar{m}_*, \bar{m}_*, \mu, \nu, \rho) := \sum_{i=1}^{n} \bar{m}_i \Phi_i + \mu \sum_{1 \leq i < i' \leq n} \bar{m}_i \bar{m}_{i'} \Phi_{ii'} \quad (7.8)
\]
in the configuration variables \( \eta_i, \eta_{ij} \in \mathbb{R}^2 \) and the parameters \( \bar{m}_i, \bar{m}_{ij}, \mu, \nu, \rho \in \mathbb{R} \).

Here the functions \( \Phi_i = \Phi_i(\eta_i, \eta_{is}, \bar{m}_{is}, \nu, \rho) \) and \( \Phi_{ii'} = \Phi_{ii'}(\eta_i - \eta_{i'}, \eta_{is}, \eta_{is'}, \bar{m}_{is}, \bar{m}_{is'}, \nu, \rho) \) are defined by the formulae

\[
\Phi_i := \frac{1}{\nu \rho^2} \left( \frac{1}{|\eta_i|} - \frac{m_i/\bar{m}_i}{|\eta_i - \nu \rho \delta_i|} - \nu \sum_{j=1}^{n_i} \frac{m_{ij}/\bar{m}_i}{|\eta_i + \rho \eta_{ij} - \rho \nu \delta_i|} \right),
\]

\[
\Phi_{ii'} := \frac{1}{\nu \rho^2 \bar{m}_i \bar{m}_{i'}} \left( \frac{\bar{m}_i \bar{m}_{i'}}{|\eta_i - \eta_{i'}|} - \nu \sum_{j=1}^{n_i} \frac{\nu m_{ij} m_{ij'}}{|\eta_i - \eta_{i'} - \nu \rho (\delta_i - \delta_{i'})|} - \nu \sum_{j'=1}^{n_{i'}} \frac{\nu m_{i'j'} m_{i'j'}}{|\eta_i - \eta_{i'} + \nu (\eta_{ij} - \eta_{i'j'}) - \nu \rho (\delta_i - \delta_{i'})|} \right) - \frac{m_i m_{i'}}{|\eta_i - \eta_{i'} - \nu \rho (\delta_i - \delta_{i'})|} - \nu \sum_{j'=1}^{n_{i'}} \frac{\nu m_{i'j'} m_{i'j'}}{|\eta_i - \eta_{i'} + \nu (\eta_{ij} - \eta_{i'j'}) - \nu \rho (\delta_i - \delta_{i'})|}.
\]

Here

\[
\delta_i := \sum_{j=1}^{n_i} \frac{\bar{m}_{ij} \eta_{ij}}{m_i} = \sum_{j=1}^{n_i} \bar{m}_{ij} \eta_{ij} / (1 + \nu \sum_{j=1}^{n_i} \bar{m}_{ij})
\]

is the radius vector drawn from a planet to the centre of masses of the system of its satellites, multiplied by \( \sum_{j=1}^{n_i} \bar{m}_{ij} / \bar{m}_i \). One easily shows (see (7.1)) that the function \( \Phi_i \) is analytic in all its variables in the region

\[
\left\{ 1 + \nu \sum_{j'=1}^{n_i} \bar{m}_{ij'} \neq 0, \quad |\rho| \left( 1 + |\nu| \sum_{j'=1}^{n_i} |\bar{m}_{ij'}| \right) |\eta_{ij'}| < |\eta_i| \right\}_{j=1}^{n_i},
\]

while the function \( \Phi_{ii'} \) is analytic in all its variables in the domain

\[
\left\{ 1 + \nu \sum_{j'=1}^{n_i} \bar{m}_{ij'} \neq 0, \quad |\rho| \left( 1 + |\nu| \sum_{j'=1}^{n_i} |\bar{m}_{ij'}| \right) |\eta_{ij'}| < |\eta_i| \right\}_{j=1}^{n_i},
\]

while the function \( \Phi_{ii'} \) is analytic in all its variables in the domain

\[
\left\{ 1 + \nu \sum_{j=1}^{n_i} \bar{m}_{ij} \neq 0, \quad 1 + \nu \sum_{j'=1}^{n_{i'}} \bar{m}_{i'j'} \neq 0, \quad |\rho| \left( 1 + |\nu| \sum_{j'=1}^{n_{i'}} |\bar{m}_{i'j'}| \right)|\eta_{ij'}| < \frac{|\eta_i|}{2}, \quad 1 \leq j \leq n_i, \quad |\rho| \left( 1 + |\nu| \sum_{j'=1}^{n_{i'}} |\bar{m}_{i'j'}| \right)|\eta_{ij'}| < \frac{|\eta_{i'}|}{2}, \quad 1 \leq j' \leq n_{i'} \right\}.
\]

The functions \( \Phi_{ii'} \) are expressed in terms of \( \Phi_1, \ldots, \Phi_n \) as follows:

\[
\Phi_{ii'} = \frac{m_{i'j'}}{m_{i'}} \Phi_i(\eta_i - \eta_{i'}, \eta_{is}, \bar{m}_{is}, \nu, \rho) + \Phi_{i'}(\eta_{i'} - \eta_i, \eta_{is'}, \bar{m}_{is'}, \nu, \rho) + \nu \sum_{j'=1}^{n_{i'}} \frac{m_{i'j'}}{m_{i'}} \Phi_i(\eta_i - \eta_{i'}, \rho \eta_{i'j'} + \nu \rho \delta_{i'}, \eta_{is}, \nu, \rho).
\]
We set $\Phi_i := 0$ if $n_i = 0$ (i.e. the $i$th planet has no satellites), and we set $\Phi_{ii'} := 0$ if $n_i = n_{i'} = 0$. If $n_i = 1$ (i.e. the $i$th planet is a double planet, $\theta_i := \nu m_{i1}/(m_i + \nu m_{i1})$) then we have $\Phi_i = \frac{\theta_i(1-\theta_i)}{\nu m_{i'}} F_{\theta_i-p}(\eta_i, \eta_{i1})$ and

$$\Phi_{ii'} = \Phi_i(\eta_{i'}, \eta_i, \eta_{i'i'}, \tilde{m}_{i'i'}, \nu, \rho) + \frac{\theta_i(1-\theta_i)}{\nu m_{i'}} F_{\theta_i-p}(\eta_i - \eta_{i'}, \eta_i + \nu \rho \delta_{i'}, \eta_{i1}) +$$

$$+ \theta_i(1-\theta_i) \sum_{j'i' = 1}^{n_i} m_{i'i'} F_{\theta_i-p}(\eta_i - \eta_{i'} + \nu \rho \delta_{i'}, \eta_{i1}).$$

**Lemma 7.1 (Equivalence of the $N+1$ body problem to an $\varepsilon$-Hamiltonian system).** Let $\hat{Q}$ be the $2N$-dimensional vector space formed by all configurations of $N+1$ particles with masses (2.5) and the centre of masses at the origin in a Euclidean plane. Define linear coordinates on $\hat{Q}$ to be the collection of radius vectors $x_i, y_{ij}$ : $\hat{Q} \to \mathbb{R}^2$, $1 \leq i \leq n$, $1 \leq j \leq n_i$ (see (7.5), (7.6)). There exists a collection of linear functions $\xi_i, \eta_{ij} : \hat{Q}^* \to \mathbb{R}^2$, $1 \leq i \leq n$, $1 \leq j \leq n_i$ (momenta) having the following properties. In the coordinates $x_i, y_{ij}, \xi_i, \eta_{ij}$ on $T^* \hat{Q} \cong \hat{Q} \times \hat{Q}^*$, the canonical symplectic structure $\omega = dp \wedge dq$, the Hamiltonian function $H$ of the $N+1$ body problem and the first integral $I$ of angular momentum (see (7.3) and (4.1)) have the form

$$\omega = \frac{\rho}{\omega} \tilde{w}, \quad H = \frac{\rho}{\omega} \tilde{H}, \quad I = \frac{\rho}{\omega} \tilde{I}$$

(7.16)

where

$$\tilde{w} = w_0 + \varepsilon w_1, \quad \tilde{H} = w \tilde{H}_0 + \varepsilon \tilde{H}_1 + \omega^2 \tilde{\Phi}, \quad \tilde{I} = I_0 + \varepsilon I_1,$$

(7.17)

$$\omega_0 = d\xi \wedge dx = \sum_{i=1}^{n} d\xi_i \wedge dx_i, \quad \omega_1 = dn \wedge dy = \sum_{i=1}^{n_i} \sum_{j=1}^{n} d\eta_{ij} \wedge dy_{ij},$$

$$\tilde{H}_0 = \sum_{i=1}^{n} \tilde{K}_i + \mu \sum_{1 \leq i < i' \leq n} \tilde{K}_{i'i'}, \quad \tilde{H}_1 = \sum_{i=1}^{n} \left( \sum_{j=1}^{n_i} \tilde{S}_{ij}^{(i)} + \nu \sum_{1 \leq j < j' \leq n_i} \tilde{S}_{jj'}^{(i)} \right),$$

(7.18)

$$I_0 = [x, \xi] = \sum_{i=1}^{n} I_{0i}, \quad I_1 = [y, \eta] = \sum_{i=1}^{n_i} \sum_{j=1}^{n} I_{ij},$$

(7.19)

see (7.5), (7.6), (7.7). Here the small parameters $0 < \omega, \varepsilon, \mu, \nu, \rho \ll 1$ are related by the conditions $\rho = \omega^{2/3} \mu^{1/3}$ and $\varepsilon = \omega^{1/3} \mu^{2/3} \nu = \nu \rho^2 / \omega$, $\rho = 1/\tilde{R}$. The “perturbation potential” $\tilde{\Phi} = \tilde{\Phi}(x_i, \nu, \mu, \nu, \rho)$ has the form (7.8), is an analytic function on the direct product of the regions

$$\left\{ \begin{array}{l}
1 + \nu \sum_{i'=1}^{n_i} |\tilde{m}_{i'i'}| \neq 0, \\
|\rho| \left(1 + |\nu| \sum_{i'=1}^{n_i} |\tilde{m}_{i'i'}| \right) |y_{ij}| < \min \left( |x_i|, \frac{1}{2} \min_{i' \neq i} |x_i - x_{i'}| \right)
\end{array} \right\}^{n_i}_{j=1},$$

1 \leq i \leq n, and satisfies the condition

$$\tilde{\Phi}_{\nu+\rho=0} = \sum_{i=1}^{n_i} \sum_{j=1}^{n} m_{ij} \left( F(x_i, y_{ij}) + \mu \sum_{i'=1}^{n} \tilde{m}_{i'i'} F(x_i - x_{i'}, y_{ij}) \right),$$

(7.20)
see (2.20). In particular, \( \tilde{H}_1 = \tilde{\Phi} = 0 \) in the case \( n_i = 0 \) of a planetary system without satellites.

**Remark 7.2.** Lemma 7.1 implies equivalences of the following \((\varepsilon-)\)Hamiltonian systems for \( \omega, \varepsilon, \mu, \nu, \rho > 0 \):

\[
(T^*Q, \omega, H) \cong (T^*Q, \omega, H) \cong (T^*Q, T^*Q_0, p; \omega_0, \omega_1; \tilde{H}, \tilde{H}_1 + \omega^2\tilde{\Phi})^\varepsilon
\]

where \( Q_0 \) is the configuration space of planets, and \( p : T^*Q \to T^*Q_0 \) is the projection. The third of these systems, called the unperturbed system, is not only Hamiltonian, but also \( \varepsilon \)-Hamiltonian (see (3.1)). Hence it naturally extends to any nonnegative values \( \omega, \varepsilon, \mu, \nu, \rho \geq 0 \) of small parameters (despite the fact that the symplectic structure degenerates if one of the parameters vanishes). For the limiting values of the parameters \( \omega > 0 \) and \( \mu = \nu = 0 \) (and, hence, \( \varepsilon = \rho = 0 \)), the third system becomes a 0-Hamiltonian system \((T^*Q, T^*Q_0, p; \omega_0, \omega_1; \omega H_0, H_1 + \omega^2\Phi)^0\) called unperturbed where \( H_0 := H_0|_{\mu=0}, H_1 := H_1|_{\nu=0}, \Phi := \Phi|_{\mu=0,\nu=0} \). The system \((T^*Q, T^*Q_0, p; \omega_0, \omega_1; \omega H_0, H_1)^0\), called the model system, is \( \omega^2 \)-close to the unperturbed system. It follows from lemma 7.1 that the unperturbed system indeed has the form \((??)\) in the configuration space \( \tilde{Q} \).

Due to (7.1) and (7.15), the functions \( \Phi_i, \Phi_{ii'} \) in (7.9), (7.10) have the following form for \( |\nu| \leq \nu_0, |\rho| \leq \rho_0 \):

\[
\Phi_i = \nu \sum_{j=1}^{n_i} \frac{m_{ij}}{m_i} F_{0,\rho}(x_i, \delta_i) + \sum_{j=1}^{n_i} \frac{m_{ij}}{m_i} F_{0,\rho}(x_0, y_{ij} - \nu \delta_i) = \\
= \sum_{j=1}^{n_i} \frac{m_{ij}}{m_i} F_{0,\rho}(x_i, y_{ij}) - \nu F(x_i, \delta_i) + O(\nu \rho),
\]

\[
\Phi_{ii'} = \sum_{j=1}^{n_i} \frac{m_{ij}}{m_i} F_{0,\rho}(x_i - x_{i'}, y_{ij}) + \sum_{j'=1}^{n_i'} \frac{m_{ij'}}{m_{i'}} F_{0,\rho}(x_{i'} - x_i, y_{ij'}) - \\
- \nu F(x_i - x_{i'}, \delta_i) - \nu F(x_{i'} - x_i, \delta_{i'}) + O(\nu \rho).
\]

Hence the “perturbation potential” \( \tilde{\Phi} \) in (7.8) satisfies the condition

\[
\tilde{\Phi} = \sum_{i=1}^{n} \sum_{j=1}^{n_i} m_{ij} \left( F_{0,\rho}(x_i, y_{ij}) + \mu \sum_{i'=1}^{n} \tilde{m}_{i'} F_{0,\rho}(x_i - x_{i'}, y_{ij}) \right) - \\
- \nu \sum_{i=1}^{n} \left( \tilde{m}_i F(x_i, \delta_i) + \mu \sum_{i'=1}^{n} \tilde{m}_{i'} F(x_i - x_{i'}, \delta_i) \right) + O(\nu \rho), \quad \nu, \rho \to 0,
\]

which implies (7.20), see (7.2).

Let us explain a geometric meaning of the Hill potential \( F(x, y) \) when the \( i \)th planet has \( n_i > 1 \) satellites \((1 \leq i \leq n)\). Consider the function \( \Phi_i = \Phi_i(x_i, y_{i*}, \tilde{m}_{i*}, \nu, \rho) \) defined by the formula (7.9) and called the “potential of interaction of all satellites of the \( i \)th planet with the Sun”. Due to (7.21), the function \( \Phi_i|_{\rho=0} \) equals a linear combination of the functions \( F(x_i, y_{ij}), 1 \leq j \leq n_i, \) and \( F(x_i, \delta_i) \).
7.1. The Poincaré transformation in the $n+1$ body problem. In order to prove lemma 7.1, we will explicitly construct the variables of momenta $\xi_i, n_{ij}$ and will show that the function $H$ in (7.16), (7.17), (7.18) equals the total energy of the system. One easily shows that those summands in $H$ that do not depend on the momenta give the potential energy $U$.

Let us compute the kinetic energy $G$. We will explore the fact that the transition from the coordinates $(r_0, r_1, \ldots, r_N)$ in the configuration space to the coordinates $x_i, y_{ij}$ (see §7?) can be done by applying twice the following transformation called the Poincaré transformation.

Let us consider the configuration manifold $Q$ of a planetary system (i.e. the system of $n+1$ particles in a Euclidean plane). It consists of all ordered collections of radius vectors $r_0, r_1, \ldots, r_n$ with associated masses $c_0 = 1, c_1 = \lambda m_1, \ldots, c_n = \lambda m_n$ where $0 < \lambda < 1$. Thus the manifold $Q$ is naturally identified with the vector space $\mathbb{R}^{2(n+1)}$ with coordinates $r_0, r_1, \ldots, r_n$. Consider the linear transformation $L = L_{c_0, c_1, \ldots, c_n}$ in $\mathbb{R}^{2(n+1)}$ that corresponds to introducing the new linear coordinates on the space $Q$ corresponding to the following collection of radius vectors:

$$\tilde{r}_0 = \frac{r_0 + c_1 r_1 + \ldots + c_n r_n}{1 + c_1 + \ldots + c_n}, \quad \tilde{r}_1 = r_1 - r_0, \quad \ldots, \quad \tilde{r}_n = r_n - r_0$$

where $\tilde{r}_0 = c := \frac{r_0 + c_1 r_1 + \ldots + c_n r_n}{1 + c_1 + \ldots + c_n}$ is the radius vector of the centre of masses of the system.

**Definition 7.1.** The transformation $L = L_{c_0, c_1, \ldots, c_n}$ is called the Poincaré transformation on the configuration manifold of the planetary system.

Actually one could consider another transformation, namely the Jacobi transformation $\tilde{r}_0 = c, \tilde{r}_1 = r_1 - c, \ldots, \tilde{r}_n = r_n - c$. But this transformation would lead to more awkward formulae. Moreover it would bring us to a desired result only in the case of a usual planetary system, i.e. having no satellites.

Consider the dual space $Q^*$, i.e. the space of all linear functions on the space $Q$ (or, equivalently, the cotangent space to $Q$ at its any point). This space consists of all collections $p_0, p_1, \ldots, p_n$ whose each item $p_i$ is a linear function on the plane, i.e. a co-vector. In fact, we can define the value of the linear function corresponding to such a collection on the configuration $(r_0, r_1, \ldots, r_n) \in Q$ to be $\sum_{i=0}^n \langle p_i, r_i \rangle$. It is clear that the nondegenerate transformation $L$ on $\mathbb{R}^{2(n+1)} \cong Q$ induces a linear transformation $L^*$ on $(\mathbb{R}^{2(n+1)})^* \cong Q^*$. Denote the image of the collection $p_0, p_1, \ldots, p_n$ under the transformation $L^*$ by $\tilde{p}_0, \tilde{p}_1, \ldots, \tilde{p}_n$.

Consider the real valued function $G = \sum_{i=0}^n \frac{p_i^2}{2c_i}$ of kinetic energy on the space $Q^*$. Besides we consider the function $I = \sum_{i=0}^n \langle r_i, p_i \rangle$ of angular momentum on the space $T^*Q$. Finally consider the function of the total momentum $P = p_0 + p_1 + \ldots + p_n$ on $Q^*$ whose values belong to the space of co-vectors, i.e. of linear functions on the plane.

**Lemma 7.2.** Under the Poincaré transformation $L = L_{c_0, =1, c_1, \ldots, c_n}$ on the configuration space $Q$ of the $n+1$ body problem, the functions $G$ and $P$ on $Q^*$ transform as follows:

(A) The kinetic energy $G = \sum_{i=0}^n \frac{\tilde{p}_i^2}{2c_i}$ has the form

$$G = \frac{\tilde{p}_0^2}{2c_0} + \sum_{i=1}^n \frac{\tilde{p}_i^2}{2c_i} + \frac{1}{2} \langle \tilde{p}_1 + \ldots + \tilde{p}_n \rangle^2$$

(7.23)
where \( \tilde{c}_0 = 1 + c_1 + \ldots + c_n = 1 + \lambda(m_1 + \ldots + m_n) \). The expression (7.23) can be rewritten as follows:

\[
G = \frac{\tilde{p}_0^2}{2c_0} + \sum_{i=0}^{n} \frac{\tilde{p}_i^2}{2c_i} + \sum_{1 \leq i < i' \leq n} \langle \tilde{p}_i, \tilde{p}_{i'} \rangle
\] (7.24)

where \( \tilde{c}_i := c_0c_i / (c_0 + c_i) = c_i / (1 + c_i), \ 1 \leq i \leq n \).

(B) The total momentum \( \mathbf{P} = \mathbf{p}_0 + \mathbf{p}_1 + \ldots + \mathbf{p}_n \) transforms to the momentum of the “heaviest” particle:

\[
\mathbf{P} = \tilde{\mathbf{p}}_0.
\] (7.25)

The function of angular momentum \( I = \sum_{i=0}^{n} [\mathbf{r}_i, \mathbf{p}_i] \) on the phase space \( X = T^*\mathcal{Q} \) is \( L \)-invariant: \( I = \sum_{i=0}^{n} [\tilde{\mathbf{r}}_i, \tilde{\mathbf{p}}_i] \).

Proof. Items A and B directly follow by substituting into the functions \( G \) and \( \mathbf{P} \) the following explicit formulae for the transformation \( L^* \) of momenta: \( \mathbf{p}_0 = \frac{1}{1+c_1+\ldots+c_n} \tilde{\mathbf{p}}_0 - \ldots - \tilde{\mathbf{p}}_n, \mathbf{p}_i = \tilde{\mathbf{p}}_i + \frac{c_i}{1+c_1+\ldots+c_n} \tilde{\mathbf{p}}_0, 1 \leq i \leq n \).

The invariance of the angular momentum follows from its invariance under the transformation on the space \( X = T^*\mathcal{Q} \) induced by any linear transformation \( \tilde{\mathbf{r}}_i = \sum_j a_{ij} \mathbf{r}_j \) on the space \( \mathcal{Q} \) with a nondegenerate matrix \( \|a_{ij}\| \). The latter holds, since the transformation of momenta has the form \( \tilde{\mathbf{p}}_i = \sum_k b_{ik} \mathbf{p}_k \) where \( \sum_k b_{ik}a_{ij} = \delta_{jk} \), hence

\[
\sum_{i=0}^{n} [\tilde{\mathbf{r}}_i, \tilde{\mathbf{p}}_i] = \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{k=0}^{n} [\mathbf{r}_j, \mathbf{p}_k] a_{ij} b_{ik} = \sum_{j=0}^{n} [\mathbf{r}_j, \mathbf{p}_j] = I.
\]

\( \square \)

7.2. Proof of the main Lemma 7.1. Let us consider the case of a planetary system without satellites. Observe that the transition from the coordinates \( \mathbf{r} \) to the coordinates \( \mathbf{x} \) is a composition of the Poincaré transformation \( L \) and the homothety \( \tilde{\mathbf{r}}_i = R\mathbf{x}_i, 1 \leq i \leq N \). By setting \( \tilde{\mathbf{p}}_i = \sqrt{\frac{\mu}{R}} \xi_i, 1 \leq i \leq N, \tilde{\mathbf{p}}_0 = 0 \), one obtains from (7.24) the desired expression for the kinetic energy \( G \). In fact, \( GR \) equals

\[
\sum_{i=1}^{N} \frac{\xi_i^2}{2m_i} + \mu \sum_{1 \leq i < j \leq N} \langle \xi_i, \xi_j \rangle.
\]

Hence, in the partial case of a planetary system without satellites, the function \( H \) in (7.16), (7.17), (7.18) indeed equals the total energy \( G + U \) of the system. The symplectic structure \( d\tilde{\mathbf{p}} \wedge d\tilde{\mathbf{r}} = \sqrt{\mu R} d\xi \wedge d\mathbf{x} \) also has the desired form, since \( \sqrt{\mu R} = \frac{1}{\omega R} \).

In the general case of a planetary system with satellites, we observe that a transition from the radius vectors \( \{\mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_N\} \) to the coordinates \( \mathbf{x}_i, \mathbf{y}_{ij} \) can be obtained via performing the following three transformations. At first, one should perform the Poincaré transformation \( L \) to the whole system (see above). At second, one performs the transformation \( L \) to each satellite system \( \tilde{\mathbf{r}}_{ij}, 0 \leq j \leq n_i \). Finally one performs the “scaling” homothety \( \tilde{\mathbf{r}}_{i0} = \mathbf{c}_i = R\mathbf{x}_i, 1 \leq i \leq n, \tilde{\mathbf{r}}_{ij} = \mathbf{y}_{ij}, 1 \leq j \leq n_i \). For the sake of simplicity, we will assume that all planets have satellites, i.e. all \( n_i \) are positive.

Step 1. The first performing of the Poincaré transformation \( L \) to the initial configuration variables gives, due to (7.23),

\[
G = \frac{\tilde{p}_0^2}{2c_0} + \sum_{i=1}^{N} \frac{\tilde{p}_i^2}{2c_i} + \frac{1}{2} (\tilde{\mathbf{p}}_1 + \ldots + \tilde{\mathbf{p}}_N)^2 = \frac{\tilde{p}_0^2}{2c_0} + \sum_{i=1}^{n} \tilde{G}_i + \frac{1}{2} (\tilde{\mathbf{p}}_1 + \ldots + \tilde{\mathbf{p}}_n)^2.
\]
Here \( \tilde{G}_i = \frac{\tilde{p}_i^2}{2\tilde{m}_i} + \sum_{j=1}^{n_i} \frac{\tilde{p}_{ij}^2}{2\mu m_{ij}} + \sum_{j=1}^{n_i} \frac{\tilde{p}_{ij}^2}{2\mu m_{ij}} \) is the kinetic energy of the \( i \)th satellite system, and \( \tilde{P}_i = \tilde{p}_i + \sum_{j=1}^{n_i} \tilde{p}_{ij} \) is its total momentum.

Step 2. Put \( \tilde{P}_0 = 0 \) and perform separately the Poincaré transformation \( L \) to each satellite system. As a result, we have from the formula (7.24)

\[
\tilde{G}_i = \frac{\tilde{p}_i^2}{2\tilde{m}_i} + \sum_{j=1}^{n_i} \frac{\tilde{p}_{ij}^2}{2\mu m_{ij}} + \frac{1}{\mu m_i} \sum_{1 \leq j < j' \leq n_i} \langle \tilde{p}_{ij}, \tilde{p}_{ij'} \rangle
\]

where \( \tilde{m}_i \), \( \tilde{m}_{ij} \) are as in (7.4). By the formula (7.25), we have \( \tilde{P}_i = \tilde{p}_i \).

By using the previous step, we obtain

\[
G = \sum_{i=1}^{n} \left( \frac{\tilde{p}_i^2}{2\tilde{m}_i} + \sum_{j=1}^{n_i} \frac{\tilde{p}_{ij}^2}{2\mu m_{ij}} + \sum_{1 \leq j < j' \leq n_i} \frac{\tilde{p}_{ij} \tilde{p}_{ij'}}{\mu m_i} \right) + \frac{1}{2} \left( \sum_{i=1}^{n} \tilde{p}_i \right)^2
\]

\[
= \sum_{i=1}^{n} \left( \frac{\tilde{p}_i^2}{2\tilde{m}_i} + \sum_{j=1}^{n_i} \frac{\tilde{p}_{ij}^2}{2\mu m_{ij}} + \sum_{1 \leq j < j' \leq n_i} \frac{\tilde{p}_{ij} \tilde{p}_{ij'}}{\mu m_i} \right) + \sum_{1 \leq i < i' \leq n} \langle \tilde{p}_i, \tilde{p}_{i'} \rangle
\]

where \( \tilde{m}_i = \frac{m_i}{1 + \mu m_{ij}}, 1 \leq i \leq n \).

Step 3. Now perform the following “scaling” of coordinates and momenta:

\[
\tilde{r}_i = R\xi_i, \quad \tilde{r}_{ij} = y_{ij}, \quad \tilde{p}_i = \sqrt{\frac{\mu}{R}}\xi_i, \quad \tilde{p}_{ij} = \mu\nu\eta_{ij}
\]

(7.26)

for \( 1 \leq i \leq n, 1 \leq j \leq n_i \). This gives:

\[
G = \sum_{i=1}^{n} \left( \frac{\xi_i^2}{2\tilde{m}_i R} + \frac{\nu^2}{2\tilde{m}_i} + \frac{\nu^2}{m_i} \sum_{1 \leq j < j' \leq n_i} \langle \eta_{ij}, \eta_{ij'} \rangle \right) + \frac{\mu}{R} \sum_{1 \leq i < i' \leq n} \langle \xi_i, \xi_{i'} \rangle.
\]

By taking into account that \( \mu\nu R = \varepsilon \), we have the desired formula:

\[
\omega R = \sum_{i=1}^{n} \left( \omega \frac{\xi_i^2}{2\tilde{m}_i} + \varepsilon \sum_{j=1}^{n_i} \frac{\eta_{ij}^2}{2\tilde{m}_{ij}} + \varepsilon \nu \sum_{1 \leq j < j' \leq n_i} \langle \eta_{ij}, \eta_{ij'} \rangle \right) + \omega \mu \sum_{1 \leq i < i' \leq n} \langle \xi_i, \xi_{i'} \rangle.
\]

Due to (7.26), the symplectic structure has the form

\[
\omega = dp \wedge dr = d\tilde{p} \wedge d\tilde{r} = \sqrt{\frac{\mu}{R}} \sum_{i=1}^{n} \left( d\xi_i \wedge dx_i + \varepsilon \sum_{j=1}^{n_i} d\eta_{ij} \wedge dy_{ij} \right),
\]

since, recall, \( \varepsilon = \nu \sqrt{\frac{1}{R}} \). This symplectic structure has the desired form (7.16), (7.17), (7.18), since \( \sqrt{\mu R} = \frac{1}{\omega R} \). In a similar way, one proves the formulae for the first integral of angular momentum.

This finishes the proof of the main lemma 7.1. \( \square \)
§ 8. Deriving theorems 2.1–2.3 from theorems 3.1–3.4

Due to lemma 7.1 and remark 7.2, the $N + 1$ body problem of the type of planetary system with satellites is equivalent to the $\varepsilon$-Hamiltonian system (3.13) with small parameters $0 < \omega, \varepsilon, \mu, \nu, \rho < 1$ related by the conditions $\varepsilon = \omega^{1/3} \mu^{2/3} \nu$ and $\rho = \omega^{2/3} \mu^{1/3}$. Moreover, the functions $\tilde{H}_0, \tilde{H}_1, \Phi$ are $S^1$-invariant, the function $\tilde{H}_0 = H_0 + \mu R_0$ “projects” to $M_0 := T^*Q_0$, the function $\tilde{H}_1 = H_1 + \nu R_1$ “projects” to $M_1 := T^*Q_1$, and their “principal parts” equal the sums $H_0 = \sum_{i=1}^n H_{i0}$ and $H_1 = \sum_{i=1}^n \sum_{j=1}^n H_{ij}$. Furthermore, due to lemma 5.1, each summand has the form $H_{ij} = H_{ij}(I_{ij}, q_{ij}, p_{ij})$ and satisfies the conditions (3.8). The perturbation potential $\tilde{\Phi}$ is an analytic function in a neighbourhood of any torus $\Lambda^\circ$, provided that the collection of angular frequencies $\Omega_{ij}$ satisfies the conditions (2.8) and (2.9) of “lack of collisions”. The principal part $\Phi := \tilde{\Phi}_{|\mu=\nu=\rho=0}$ of the perturbation potential has the form (3.6).

So, the $N + 1$ body problem of the type of planetary system with satellites considered in theorem 2.1 is equivalent to an $\varepsilon$-Hamiltonian system belonging to the class of “perturbed” systems in theorem 3.1.

Proof of theorems 2.1 and 2.2. Step 1. In theorems 2.1 and 3.1, the “relative resonance” conditions (2.14) and (3.9) on the collection of frequencies are equivalent. The nondegeneracy condition (2.17) from theorem 2.1 is equivalent to the nondegeneracy condition (3.10) from theorem 3.1.

Let us suppose that the more delicate nondegeneracy condition (2.28) from theorem 2.1 holds. Let us prove the nondegeneracy conditions (3.11) from theorem 3.1. The first condition in (3.11) is equivalent to the first condition in (2.28). In order to prove the second condition in (3.11), let us evaluate the number $\Delta_{ij}$ in (3.12). Due to (7.20), we have $F_{ij} = F_{ij}(x_i, y_{ij}) = m_{ij}F(x_i, y_{ij})$. By construction, $F^\circ_{ij} = (F_{ij}(x_i, \cdot)|_{H^{-1}_i)}(I_{ij}, (r_{ij}^0, 0, 0))$, where $x_i = \text{const}$, $|x_i| = R_i/R = |\Omega_{i0}|^{-2/3}$ (see theorem 3.1). Let $(F^\circ_{ij}) = \langle F^\circ_{ij}\rangle(q_{ij}, p_{ij})$ be the function obtained by averaging the Hill potential $F^\circ_{ij} = F_{ij}(q_{ij}, p_{ij})$ along the $2\pi$-periodic solutions of the Kepler problem $(M_{ij}, \omega_{ij}, H_{ij})$ for the satellite. By an easy calculation, taking into account (2.20) and corollary 5.1, we find the differential and the Hesse matrix of the function $\langle F^\circ_{ij}\rangle$ at the point $(0, 0)$:

$$d\langle F^\circ_{ij}\rangle(0, 0) = 0, \quad \frac{\partial^2 \langle F^\circ_{ij}\rangle(0, 0)}{\partial (q_{ij}, p_{ij})^2} = \Omega_{ij}^2 I_{ij} \left( \begin{array}{cc} -29/8 & 0 \\ 0 & 25/(8I_{ij}^2) \end{array} \right).$$

This and (5.8) imply that

$$\Delta_{ij} = \frac{\Omega_{ij}}{2} \text{Tr} \left( \left( \frac{\partial^2 H_{ij}(I_{ij}, 0, 0)}{\partial (q_{ij}, p_{ij})^2} \right)^{-1} \frac{\partial^2 \langle F^\circ_{ij}\rangle(0, 0)}{\partial (q_{ij}, p_{ij})^2} \right) =$$

$$= \frac{\Omega_{ij}^2}{2\OM_{ij}} \text{Tr} \left( \begin{array}{cc} 1 & 0 \\ 0 & I_{ij}^2 \end{array} \right) \left( \begin{array}{cc} -29/8 & 0 \\ 0 & 25/(8I_{ij}^2) \end{array} \right) = -\frac{\Omega_{ij}^2}{4\OM_{ij}}.$$

Therefore the second desired condition in (3.11) has the form

$$\alpha + \Delta_{ij}\omega^2T = \alpha - \frac{\nu^2}{4\OM_{ij}} T \not\in [-C_2\omega^3T, C_2\omega^3T] + 2\pi\Z,$$
i.e. it is equivalent to the second condition in (2.28) with the constant $C := C_2$.

Thus, all the conditions of theorem 3.1 are fulfilled. Hence this theorem implies theorem 2.2(A).

Step 2. Let us check that, for the $N + 1$ body problem under consideration, the model problem, the unperturbed and the perturbed problems are reversible. The construction of the functions $\hat{H}, \hat{H}_0, \hat{H}_1$ shows that they (and, hence, also $\tilde{\Phi}$) are invariant under each of the involutions $S_l$ and $S$ from §4.2. Hence they are invariant under the composition $\tau = S_lS = SS_l$. Due to (5.7), the involution $\tau = S_lS = SS_l$ acts component-wise in the form $(\varphi_{ij}, I_{ij}, q_{ij}, p_{ij}) \mapsto (-\varphi_{ij}, I_{ij}, q_{ij}, -p_{ij})$. From here, by taking into account the $\tau$-invariance of the functions $\hat{H}_0, \hat{H}_1, \tilde{\Phi}$, we obtain the reversibility of the model system, of the unperturbed and the perturbed systems. Hence theorem 3.2 implies theorem 2.1.

Step 3. Let us derive theorem 2.2(B) from theorem 3.3. By lemma 5.1 or (5.8), all numbers $\frac{\partial^2 H_{ij}}{\partial I_{ij}^2}(I_{ij}^c, 0, 0)$ are negative. The sign $\eta_{ij}$ from (3.14) equals

$$\eta_{ij} = \text{sgn} \left( \Omega_{ij} \text{Tr} \frac{\partial^2 H_{ij}(I_{ij}^c, 0, 0)}{\partial (q_{ij}, p_{ij})^2} \right) = \text{sgn} \Omega_{ij}. $$

Hence, by the first property of having fixed sign in theorem 2.2, all the signs $\eta_{i0} = \text{sgn} \Omega_{i0}$ are the same, moreover $\eta_{ij} \Delta_{ij} < 0$ for $1 \leq j \leq n_i$. Suppose that the conditions (2.31) and (2.32) hold for $C := C_2$. Then $\alpha \not\in \pi \mathbb{Z}$ and

$$\frac{\eta_{i0} + \eta_{ij}}{2} \alpha + \frac{\eta_{ij} \Delta_{ij}}{2} \omega^2 T \not\in \left[ -\frac{C_2}{2} \omega^3 T, \frac{C_2}{2} \omega^3 T \right] + \pi \mathbb{Z},$$

(8.1)

$$\frac{\eta_{ij} + \eta_{ij'}}{2} \alpha + \frac{\eta_{ij} \Delta_{ij} + \eta_{ij'} \Delta_{ij'}}{2} \omega^2 T \not\in \left[ -C_2 \omega^3 T, C_2 \omega^3 T \right] + \pi \mathbb{Z}$$

(8.2)

for $1 \leq j \leq n_i$ and $1 \leq j' \leq n_{i'}$. Consider any collection of real numbers $\alpha_{ij}$, $1 \leq i \leq n$, $0 \leq j \leq n_i$, such that

$$\alpha_{i0} = \eta_{i0} \alpha, \quad |\alpha_{ij} - \eta_{ij} (\alpha + \Delta_{ij} \omega^2 T)| \leq C_2 \omega^3 T, \quad 1 \leq i \leq n, \quad 1 \leq j \leq n_i.$$ 

Then:

1) the sum $\alpha_{i0} + \alpha_{i'0} = 2\eta_{i0} \alpha$ does not belong to $2\pi \mathbb{Z}$, since $\alpha \not\in \pi \mathbb{Z}$;

2) for $1 \leq j \leq n_i$, the sum $\alpha_{i'0} + \alpha_{ij} \in (\eta_{i0} + \eta_{ij}) \alpha + \eta_{ij} \Delta_{ij} \omega^2 T + [-C_2 \omega^3 T, C_2 \omega^3 T]$ does not belong to $2\pi \mathbb{Z}$ because of (8.1);

3) for $1 \leq j \leq n_i$ and $1 \leq j' \leq n_{i'}$, the sum $\alpha_{ij} + \alpha_{ij'} \in (\eta_{ij} + \eta_{ij'}) \alpha + (\eta_{ij} \Delta_{ij} + \eta_{ij'} \Delta_{ij'}) \omega^2 T + [-2C_2 \omega^3 T, 2C_2 \omega^3 T]$ does not belong to $2\pi \mathbb{Z}$ because of (8.2).

Thus, the sum of any two, possibly coinciding, numbers of the set $\alpha_{ij}$ does not belong to the set $2\pi \mathbb{Z}$. Hence, the hypothesis of theorem 3.3 holds, and therefore this theorem implies theorem 2.2(B).

8.1. Existence of “gaps” in the families of relatively-periodic solutions. In this section, we give a more exact definition of the notion “almost any” in theorem 2.3 (meaning “any” for $N \geq n = 2$, see corollary 2.1(‡)) and of the subsets $\mathcal{M} \subset \mathcal{M}^{\text{sym}} \subset \mathbb{R}_>^n$. Besides we derive theorem 2.3 and corollary 2.1(‡) from theorem 3.4.
Consider the sequence of positive real numbers
\[
c_k := \sqrt{r_c^2} \left( \frac{1}{2} - \frac{1}{4\kappa} \right) A_\kappa + \left( \frac{5}{4} r_c + \frac{1}{4 r_c} - \frac{3}{2} \right) \frac{B_\kappa}{\kappa}, \quad \text{где} \quad r_c := \left( \frac{\kappa + 1}{\kappa} \right)^{2/3},
\]
(8.3)

\[A_\kappa := \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\cos(\kappa t) dt}{\sqrt{r_c^2 + \frac{1}{r_c} - 2 \cos t}}, \quad B_\kappa := \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\cos(\kappa t) dt}{(r_c^2 + \frac{1}{r_c} - 2 \cos t)^{3/2}},\]

\(\kappa \in \mathbb{Z} \setminus \{-1, 0\}\). It will interest us only up to a nonzero multiplicative factor. The following properties of the sequence \(c_\kappa\) and its entries can be used for approximate computations:

\[c_\kappa = \frac{1}{4\pi} A + \frac{1}{3\pi} B + o(1), \quad A_\kappa = \frac{1}{2\pi} A + o(1), \quad B_\kappa = \frac{\kappa^2}{2\pi} B + o(\kappa^2), \quad |\kappa| \to \infty,\]

and \(\frac{5}{4} r_c + \frac{1}{4 r_c} - \frac{3}{2} = \frac{2}{3\kappa} - \frac{5}{81\kappa^3} + O(\frac{1}{\kappa^3})\), where

\[A := \int_{-\infty}^{\infty} \frac{\cos u du}{\sqrt{\frac{1}{3} + u^2}}, \quad B := \int_{-\infty}^{\infty} \frac{\cos u du}{(\frac{1}{3} + u^2)^{3/2}} > 0.\]

Put
\[\kappa_{ii'} := \frac{\omega_i}{\omega_{i'} - \omega_i}, \quad i \neq i', \quad 1 \leq i, i' \leq n.\]

(8.4)

Then \(\frac{\omega_{i'}}{\omega_i} = \frac{\kappa_{ii'} + 1}{\kappa_{ii'} - 1}\), \(\kappa_{ii'} = -\kappa_{ii'} - 1\), thus the numbers \(\kappa_{ii'}\) and \(\kappa_{ii'}\) are either both integer or both non-integer, moreover \(\kappa := \kappa_{ii'} \in \mathbb{R} \setminus \{0, -1\}\). The number \(\frac{\kappa + 1}{\kappa}\) equals the ratio \(\frac{\omega_{i'}}{\omega_i}\) of the angular frequencies of two planets along the circular orbits. Hence, due to Kepler’s second law, the number \(r_c = (\frac{\kappa + 1}{\kappa})^{2/3}\) in (8.3) equals the ratio of the radii of these orbits. Let us define the number \(c_\kappa \in \mathbb{R}\) for any \(\kappa \in \mathbb{R}\) as follows: either by the formula (8.3) if \(\kappa \in \mathbb{Z} \setminus \{-1, 0\}\), or by the formula

\[c_\kappa := 0 \quad \text{if} \quad \kappa \in \mathbb{R} \setminus (\mathbb{Z} \setminus \{-1, 0\}).\]

(8.5)

Consider the following collection of complex-valued functions on the \(n\)-dimensional torus \((S^1)^n\) with angular coordinates \(\varphi = (\varphi_1, \ldots, \varphi_n)\):

\[f_{ll'}(\varphi) := \kappa_{ll'} c_{ll'} e^{i\kappa_{ll'}(\varphi_{l'} - \varphi_i)}, \quad l \neq l', \quad 1 \leq l, l' \leq n,\]

(8.6)

where \(i = \sqrt{-1} \in \mathbb{C}\) is the imaginary unit and \(c_\kappa \geq 0\) is defined in (8.3) and (8.5).

The functions (8.6) are \(2\pi\)-periodic in each argument, moreover they are equivariant:

\[f_{ll'}(\varphi_1 + t, \ldots, \varphi_n + t) = f_{ll'}(\varphi_1, \ldots, \varphi_n), \quad t \in \mathbb{R},\]

\[f_{ll'}(\varphi_1 + \omega_1 t, \ldots, \varphi_n + \omega_n t) = e^{i\omega_l t} f_{ll'}(\varphi_1, \ldots, \varphi_n), \quad t \in \mathbb{R}.\]

**Definition 8.1.** Let us fix the angular frequencies \(\omega_i\) of the planets satisfying the properties (2.14), (2.16), (2.8), (2.9). A collection of planets’ masses \(\mu(m_1, \ldots, m_n) \in\)
$\mathbb{R}_{\geq 0}^n$ will be called unclosing for a phase point $\varphi = (\varphi_1, \ldots, \varphi_n) \in (S^1)^n$, or simply $\varphi$-unclosing\(^1\), if at least one of the following numbers does not vanish:

$$f_l(\varphi; m_1, \ldots, m_n) := \sum_{l' = 1}^{n} m_{l'} f_{l'}(\varphi) \in \mathbb{C}, \quad 1 \leq l \leq n, \quad (8.7)$$

see (8.3)–(8.6). A phase point $\varphi \in (S^1)^n$ will be called symmetric if it is fixed under the involution $(S^1)^n \to (S^1)^n$, $\varphi \mapsto -\varphi$, of the $n$-dimensional torus, i.e. its coordinates have the form $\varphi_l \in \{0, \pi\} \mod 2\pi$, $1 \leq l \leq n$. Denote by $\mathcal{M}$ (respectively $\mathcal{M}^\text{sym}$) the set of all collections of planets’ masses $\mu(m_1, \ldots, m_n) \in \mathbb{R}_{\geq 0}^n$ that are $\varphi$-unclosing for any (respectively for any symmetric) phase point $\varphi \in (S^1)^n$.

**Remark 8.1.** The number of symmetric phase points equals $2^n$. For any phase point $\varphi \in (S^1)^n$, the set of $\varphi$-closing collections of planets’ masses is the intersection of a linear subspace of $\mathbb{R}^n$ with $\mathbb{R}_{\geq 0}^n$. Hence $\mathcal{M}^\text{sym}$ is open in $\mathbb{R}_{\geq 0}^n$. Moreover it is dense in $\mathbb{R}_{\geq 0}^n$ whenever it is nonempty. The subset $\mathcal{M} \subset \mathbb{R}_{\geq 0}^n$ is open in $\mathbb{R}_{\geq 0}^n$, since the torus $(S^1)^n$ is compact and the functions $f_l = f_l(\varphi; m_1, \ldots, m_n)$ are continuous, see (8.7). Suppose that $\kappa_{i,i'} \in \mathbb{Z}$ for some $i \neq i'$, $1 \leq i, i' \leq n$. Then the set of $\varphi$-closing collections of planets’ masses is contained in a plane of codimension $\geq 2$ (since the system of functions (8.7) is linear in $(m_1, \ldots, m_n)$ and its rank is at least 2). Moreover any collection of masses with $|\kappa_{i,i'}|c_{\kappa_{i,i'}}m_{i'} > \sum_{l \neq i,i'} |\kappa_{i,l}|c_{\kappa_{i,l}}m_l$ is $\varphi$-unclosing for any phase point $\varphi \in (S^1)^n$ (i.e. belongs to $\mathcal{M}$). Hence the open sets $\mathcal{M} \subset \mathcal{M}^\text{sym}$ are nonempty, thus “almost any” collection of planets’ masses (see the paragraph before theorem 2.3) belongs to $\mathcal{M}^\text{sym}$. 

Consider the natural angular coordinates on the torus $\Lambda^\circ$:

$$x \mapsto \{\varphi_l, \varphi_j, 1 \leq l \leq n, 1 \leq j \leq n_l\}, \quad x \in \Lambda^\circ.$$  

The following statement generalizes theorem 2.3.

**Proposition 8.1.** Consider the $N + 1$ body problem of the type of planetary system with (or without) satellites, $N \geq n \geq 2$. Under the hypothesis of theorem 2.1, fix the angular frequencies $\omega_1, \ldots, \omega_n$ of planets having the form (2.14), (2.16), (2.8), (2.9). Suppose that there exists at least one pair of planets with indices $i \neq i'$, whose frequencies are in a special resonance (2.33). In this case, one automatically has $\alpha = 0$, $\kappa_{i,i'} \in \mathbb{Z} \setminus \{0, -1\}$ and $c_{\kappa_{i,i'}} > 0$. Fix the two-dimensional torus $\gamma \subset \Lambda^\circ$ corresponding to a $T$-periodic solution of the model system. Let us suppose that the collection of planets’ masses $\mu(m_1, \ldots, m_n) \in \mathbb{R}_{\geq 0}^n$ is $\varphi$-unclosing for some (and, hence, any) point $x = \{\varphi_l, \varphi_j\} \in \gamma$, see (8.7).

Then, for any real number $D > 0$, there exist numbers $\mu_0, \nu_0 > 0$ and a neighbourhood $U_0$ of the projection of the two-dimensional torus $\gamma$ to the phase space of planets such that the following property holds. For any values $\mu, \nu$ such that $0 < \left(\frac{\nu}{\nu_0}\right)^3 <$

\(^1\)A collection of masses $\mu(m_1, \ldots, m_n) \in \mathbb{R}_{\geq 0}^n$ is $\varphi$-closing if and only if, for any index $l = 1, \ldots, n$, the planar polygonal line $A_{l1}(\varphi) \ldots A_{ln}(\varphi) \subset \mathbb{C}$ is closed, provided that the segments of this polygonal line have the form $A_{l,l'+1}(\varphi) = m_{l'} f_{l'}(\varphi)$ for $1 \leq l' < l - 1$ and the form $A_{l,l'-1}(\varphi) = m_{l'} f_{l'}(\varphi)$ for $l + 1 \leq l' < n$. 


E. A. KUDRYAYTSEVA

56

\[ \mu \leq \mu_0 \text{ (respectively } 0 < \mu \leq \mu_0 \text{ if there are no satellites), the direct product } \]

\[ U \text{ of the neighbourhood } U_0 \text{ and the phase space of satellites does not contain any } \]

\[(\tilde{T}, \tilde{\alpha})\text{-periodic trajectory of the } N+1 \text{ body problem under consideration, provided } \]

\[ \text{that the parameters } \tilde{T}, \tilde{\alpha} \text{ have the form} \]

\[ |\tilde{T} - T| + |\tilde{\alpha}| \leq D\mu. \]

In particular, the assertions of theorem 2.3 hold.

8.1.1. Система Солнце–две планеты. Let us show that theorem 3.4 (or the averaging method on a submanifold [12; theorem 11.1, 9; theorem 4]) implies corollary 2.1(\#) on non-existence of periodic solutions of a planetary system with two planets \( (N = n = 2) \). The Hamilton function and the symplectic structure of the perturbed problem are

\[ \varpi \tilde{H}_0 = \varpi \left( \tilde{K}_1 + \tilde{K}_2 + \mu K_{12} \right), \quad \varpi_0 = d\xi_1 \wedge dx_1 + d\xi_2 \wedge dx_2. \]

Here \( \tilde{K}_i = \frac{\xi_i^2}{2\tilde{m}_i} - \frac{\tilde{m}_i}{|x_i|}, \) is the Hamiltonian function of the Kepler problem corresponding to the \( i \)th planet, \( \tilde{m}_i = \frac{m_i}{1+\mu m_i}, \) \( i = 1, 2, \) \( K_{12} = (\xi_1, \xi_2) - \frac{m_1 m_2}{|x_1 - x_2|}. \)

Suppose that the ratio of frequencies \( \varpi_1, \varpi_2 \) is rational, i.e. they have the form

\[ \varpi_1 = \frac{2\pi k_1}{T}, \quad \varpi_2 = \frac{2\pi k_2}{T} \]

where \( k_1, k_2 \) are nonzero integers, \( k_1 \neq \pm k_2, \) \( T > 0. \) Then the solutions of the unperturbed problem \( (\mu = 0) \) corresponding to the independent circular motions of planets with angular frequencies \( \varpi_1, \varpi_2 \) are \( T \)-periodic. Conversely, if the solution is \( T \)-periodic then the pair of angular frequencies \( (2.7) \) is proportional to a pair of integers, with coefficient \( \frac{2\pi}{T}. \)

**Proof of corollary 2.1(\#).** Step 1. Recall that the unperturbed problem (corresponding to \( \mu = 0 \)) splits into two independent planar Kepler’s problem. Hence its \( T \)-periodic phase trajectories form the six-dimensional submanifold

\[ \Theta = \{ K_1 = \text{const}, \ K_2 = \text{const} \} \]

in the eight-dimensional phase space. In fact, due to periodicity of solutions of the Kepler problem with negative energy levels, the period of any its closed trajectory is a smooth (and strictly monotone, see §5) function in the value of energy.

Let us find the averaged perturbation \( \langle R_0^2 \rangle \), i.e. the function obtained by averaging the perturbation function \( R_0^2 = R_0|_{\Theta} = (K_{12} + \frac{1}{2}\xi^2)|_{\Theta} \) along the periodic solutions of the unperturbed problem. In more detail, let us find the differential of the function \( \langle R_0^2 \rangle \) at any point of the torus \( \Lambda^{\circ} \subset \Theta. \)

Step 2. With respect to polar coordinates \( \psi, r \) on the plane of motion, we have

\[ \tilde{K}_i = \frac{p_r^2 + p_{\psi_i}^2}{2\tilde{m}_i} - \frac{\tilde{m}_i}{r_i}, \quad i = 1, 2, \]

\[ K_{12} = \left( p_{r_1} p_{r_2} + \frac{p_{\psi_1} p_{\psi_2}}{r_1 r_2} \right) \cos(\psi_1 - \psi_2) + \]
\[ + \left( p_{r_1} \frac{p_{\psi_2}}{r_2} - p_{\psi_1} \frac{p_{r_2}}{r_1} \right) \sin(\psi_1 - \psi_2) - \frac{\bar{m}_1 \bar{m}_2}{\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\psi_1 - \psi_2)}}. \]

**Step 3.** Let us transfer to the coordinates \( \varphi_i, I_i, q_i, p_i, i = 1, 2 \), from lemma 5.1. By this lemma,\( \Lambda^0 = \{ p_1 = p_2 = q_1 = q_2 = 0, \ I_1 = \text{const}, \ I_2 = \text{const} \}, \)

moreover the restriction of the linearized unperturbed system to the tangent bundle \( T_{\Lambda^0} \Theta = \{ \xi | dI_1(\xi) = dI_2(\xi) = 0 \} \) to \( \Theta \) has the following form (with respect to the coordinates \( \varphi_i, d\varphi_i, dq_i, dp_i, i = 1, 2 \), on this bundle):

\[
\frac{d\varphi_i}{dt} = \omega_i, \quad \frac{d(d\varphi_i)}{dt} = 0, \quad \frac{d(dq_i)}{dt} = \omega_i \frac{dp_i}{I_i}, \quad \frac{1}{I_i} \frac{d(dp_i)}{dt} = -\omega_i dq_i, \quad (8.8)
\]

\( i = 1, 2 \). Hence, at each point \( (\varphi_1, \varphi_2) \) of the torus \( \Lambda^0 \), the differential of the function \( K_{12}|_{\Theta} \) has the following form:

\[
d(K_{12}|_{\Theta})(\varphi_1, \varphi_2) = \bar{m}_1 \bar{m}_2 r_1 r_2 \frac{r_3}{r_{12}^3} \left( \sin \varphi_{12} \left( d\varphi_{12} + \frac{2 dp_1}{I_1} - \frac{2 dp_2}{I_2} \right) \right) + \\
+ \left( \frac{r_1}{r_2} - \cos \varphi_{12} \right) dq_1 + \left( \frac{r_2}{r_1} - \cos \varphi_{12} \right) dq_2 - \\
- \frac{I_1 I_2}{r_1 r_2} \left( \cos \varphi_{12}(dq_1 + dq_2) + \sin \varphi_{12} \left( d\varphi_{12} + \frac{dp_1}{I_1} - \frac{dp_2}{I_2} \right) \right) \quad (8.9)
\]

where \( \varphi_{12} := \varphi_1 - \varphi_2, \ r_{12} := \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos \varphi_{12}} \). The perturbation function has the form \( R_0^0 = R_0|_{\Theta} = (K_{12} + \frac{1}{2} \xi^2)|_{\Theta} \). One easily shows that the contribution of the summand \( \frac{1}{2} \xi^2|_{\Theta} = \frac{1}{2} \sum_{i=1}^2 (p_{r_i}^2 + p_{\psi_i}^2/r_i^2) \) to the averaged perturbation \( \langle R_0^0 \rangle \) has a trivial differential at any point \( (\varphi_1, \varphi_2) \in \Lambda^0 \), i.e. \( d\langle R_0^0 \rangle(\varphi_1, \varphi_2) = d(K_{12}|_{\Theta})(\varphi_1, \varphi_2) \).

**Step 4.** Consider the rational number \( \kappa = \kappa_{12} = \frac{k_1}{k_2}, c = \frac{\omega_1}{\omega_2 - \omega_1}, \) see (8.4). If \( \kappa \in \mathbb{Z} \) then define the number \( c_\kappa \) by the formula (8.3).

By integrating the values of the co-vector \( d(K_{12}|_{\Theta}) \) (see (8.9)) on the solutions of the linearized system (8.8), one obtains the following:

1) if \( \kappa \in \mathbb{Q} \setminus \mathbb{Z} \) then the differential of the function \( \langle R_0^0 \rangle \) vanishes at any point of the two-dimensional torus \( \Lambda^0 \);

2) if \( \kappa \in \mathbb{Z} \) then this differential has the following form at any point \( (\varphi_1, \varphi_2) \in \Lambda^0 \):

\[
d\langle R_0^0 \rangle(\varphi_1, \varphi_2) = \bar{m}_1 \bar{m}_2 \left( \frac{\kappa c_\kappa}{r_1} \left( \cos(\kappa \varphi_{12}) dq_1 + \sin(\kappa \varphi_{12}) \frac{dp_1}{I_1} \right) \right) - \\
- \frac{(\kappa + 1)c_{-\kappa - 1}}{r_2} \left( \cos((\kappa + 1) \varphi_{12}) dq_2 - \sin((\kappa + 1) \varphi_{12}) \frac{dp_2}{I_2} \right) \quad (8.10)
\]

and, hence, it does not vanish (since \( c_\kappa \neq 0, c_{-\kappa - 1} \neq 0 \), see (8.3)).

Since any point of the torus \( \Lambda^0 \) is noncritical for the function \( \langle R_0^0 \rangle \), Theorem 3.4 (of the averaging method on a submanifold [12; theorem 11.1], [9; theorem 4]) implies Corollary 2.1(3). \( \square \)
8.1.2. The general case of a planetary system with satellites. Let us derive Proposition 8.1 from Theorem 3.4. By lemma 7.1, the summands $\tilde{H}_0, \omega_0$ in the decompositions (7.17) of the Hamiltonian function and the symplectic structure of the $N+1$ body problem have the form

$$\tilde{H}_0 = \sum_{i=1}^{n} \tilde{K}_i + \mu \sum_{1 \leq i < i' \leq n} K_{ii'}, \quad \omega_0 = \sum_{i=1}^{n} d\xi_i \wedge dx_i.$$ 

Here $\tilde{K}_i$ is the Hamiltonian function of the Kepler problem of the $i$th planet that is similar to the Hamiltonian function $K_1$, and $K_{ii'}$ is the function similar to $K_{12}$. Put $\kappa_{ii'} = \frac{\varpi_i - \varpi_{i'}}{\omega_i - \omega_{i'}}$, $1 \leq i, i' \leq n$, $i \neq i'$, see (8.4).

From the equality (8.10) in the case $N = n = 2$, we immediately obtain the analogous formula in general case $N \geq n \geq 2$:

$$d\langle R_0^2 |_{\Lambda^0} \rangle = d\langle R_0^2 \rangle(\varphi_1, \ldots, \varphi_n) = \sum_{1 \leq i < i' \leq n} \xi_{i,i'}^* (\varphi_i, \varphi_{i'}).$$

Here $\xi_{i,i'}^* (\varphi_i, \varphi_{i'})$ is the co-vector analogous to the co-vector (8.10), which is denoted by $\xi_{12}^*(\varphi_1, \varphi_2)$. In more detail, we have

$$d\langle R_0^2 |_{\Lambda^0} \rangle = \sum_{i=1}^{n} \sum_{i' \neq i}^{n} \tilde{m}_i \kappa_{ii'} c_{\kappa_{ii'}} \left( \cos(\kappa_{ii'} \varphi_{ii'}) dq_i - \sin(\kappa_{ii'} \varphi_{ii'}) dp_i / I_i \right),$$

where $\varphi_{ii'} := \varphi_i - \varphi_{i'}$. This co-vector vanishes at those points of the torus $\Lambda^0$ where the functions $f_l$, $1 \leq l \leq n$, simultaneously vanish, see (8.7). Hence theorem 3.4 implies the absence of $(\bar{T}, \bar{\alpha})$-periodic solutions in some neighbourhood of the torus $\gamma$, provided that the parameters $\omega, \mu, \varepsilon > 0$ are small enough and are related by the inequalities $\omega \varepsilon / \mu_0 \leq \mu \leq \mu_0$. Due to the relation $\varepsilon = \omega^{1/3} \mu^{2/3} \nu$, these inequalities have the form $\omega^{4/3} \mu^{2/3} \nu / \mu_0 \leq \mu \leq \mu_0$, i.e. the form $\omega^4 (\nu / \mu_0)^3 \leq \mu \leq \mu_0$. Therefore theorem 3.4 indeed implies proposition 8.1.

Theorem 2.3 obviously follows from Proposition 8.1, Definition 8.1 of the subsets $\mathcal{M} \subset \mathcal{M}^{\text{sym}} \subset \mathbb{R}^n_{>0}$, and Remark 8.1.

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**E. A. Kudryavtseva**
Moscow State University
*E-mail: eakudr@mech.math.msu.su*