Improved Hamilton-Jacobi Quantization for Nonholonomic Constrained System

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Abstract

The nonholonomic constrained system with second-class constraints is investigated using the Hamilton-Jacobi (HJ) quantization scheme to yield the complete equations of motion of the system. Although the integrability conditions in the HJ scheme are equivalent to the involutive relations for the first-class constrained system in the improved Dirac quantization method (DQM), one should elaborate the HJ scheme by using the improved DQM in order to obtain the first-class Hamiltonian and the corresponding effective Lagrangian having the BRST invariant nonholonomic constrained system.

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I. INTRODUCTION

Recently, several interesting constrained systems were investigated [1] in the framework of the improved Dirac quantization method (DQM) [2]. Based on the Carathéodory equivalent Lagrangians method [3], an alternative Hamilton-Jacobi (HJ) quantization scheme for the constrained systems was also proposed [4] and this HJ scheme was exploited to quantize singular systems with higher order Lagrangians such as the systems with elements of the Berezin algebra [5] and the Proca model [6]. One of the most interesting application of the HJ quantization scheme is the systems with second-class constraints [7,8], simply because the corresponding set of equations is not integrable [8]. If the singular system is transformed to become completely integrable, the Hamiltonian has the form suitable for application of the Hamilton-Jacobi equations.

In this paper we improve the HJ quantization scheme for the nonholonomic constrained system (NHCS) studied in the literature [9,10] to obtain the complete solutions of the HJ partial differential equations (PDEs) for the system and to compare them with those of the standard and improved DQMs. In section 2 we briefly recapitulate the HJ quantization scheme. In section 3, in this refined HJ scheme, the NHCS with nonholonomic primary constraint is reanalyzed to yield the complete solutions. In section 4 we treat the NHCS by using the standard DQM and in section 5 we construct the first-class Hamiltonian and the first-class effective Lagrangian corresponding to the integrable system in the improved DQM. Moreover, with the first-class effective Lagrangian, we have constructed the BRST invariant NHCS.

II. HAMILTON-JACOBI QUANTIZATION SCHEME

In this section, we briefly recapitulate the HJ quantization scheme [4,5]. We start with an unconstrained system with the Lagrangian $L$, for which we can obtain a completely equivalent Lagrangian described as
with \( i = 1, 2, \ldots, n \). These Lagrangians are equivalent to each other if there exists a function \( S(q_i, t) \) such that the Lagrangians \( L \) and \( L' \) have an extreme value of the action simultaneously. To guarantee this equivalence, one needs to find functions \( \alpha_i(q_j, t) \) and \( S(q_i, t) \) such that, for all neighborhood of \( \dot{q}_i = \alpha_i(q_j, t) \),

\[
L'(q_i, \dot{q}_i = \alpha_i(q_j, t), t) = 0, \tag{2.2}
\]

and \( L'(q_i, \dot{q}_i) \) is positive to yield at \( \dot{q}_i = \alpha_i(q_j, t) \),

\[
\frac{\partial L'}{\partial \dot{q}_i} = 0. \tag{2.3}
\]

Note that the Lagrangian \( L' \) has now a minimum at \( \dot{q}_i = \alpha_i(q_j, t) \) so that the solutions of the differential equations given by \( \dot{q}_i = \alpha_i(q_j, t) \) can yield the extremal action. Now exploiting Eqs. (2.1) and (2.2) one can obtain at \( \dot{q}_i = \alpha_i \)

\[
\frac{\partial S}{\partial t} = L - \frac{\partial S}{\partial q_i} \dot{q}_i. \tag{2.4}
\]

Similarly, combining Eq. (2.1) and (2.3) yields at \( \dot{q}_i = \alpha_i \) the HJ equation

\[
\frac{\partial S}{\partial q_i} = p_i, \tag{2.5}
\]

where \( p_i \) are the conjugate momenta. Inserting \( p_i \) into Eq. (2.4), one can obtain the HJ PDE in terms of the Hamiltonian \( H_0 \) as follows

\[
\frac{\partial S}{\partial t} = -H_0 = -p_i \dot{q}_i + L(q_i, \dot{q}_i). \tag{2.6}
\]

Next, we consider a constrained system in which canonical variables are not all independent. In the constrained system, the Lagrangian \( L \) is singular so that the determinant of the Hessian matrix \( H_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \) is zero and the accelerations of some variables \( \ddot{q}_i \) are not uniquely determined by the positions and the velocities at a given time.

Now we consider the rank \( n - m \) of the Hessian where determinant of a sub-matrix of the Hessian is not zero and thus some velocities \( \dot{q}_a \, (a = 1, 2, \ldots, n - m) \) can be solved as a
function of coordinates \( q_i \) and momenta \( p_a \) to yield \( \dot{q}_a = \dot{q}_a(q_i, p_b) \). The remaining momenta \( p_\alpha (\alpha = n - m + 1, ..., n) \) are functions of \( q_i \) and \( p_a \) to yield

\[
p_\alpha = -H_\alpha(q_i, p_a), \tag{2.7}
\]

which are equivalent to the primary constraints \( p_\alpha + H_\alpha \) in the Dirac terminology \[7\]. The Hamiltonian \( H_0 \) then becomes

\[
H_0 = p_\alpha \dot{q}_\alpha + p_\alpha \dot{q}_\alpha - L(q_i, \dot{q}_a, \dot{q}_\alpha), \tag{2.8}
\]

which can be shown not to depend explicitly on the velocities \( \dot{q}_\alpha \).

With the redefinition: \( t_\alpha = (t_0, t_\alpha) = (t, q_\alpha), (\alpha = 0, n - m + 1, ..., n) \) and \( p_\beta = \frac{\partial S}{\partial t_\beta} \), Eqs. \(2.6\) and \(2.7\) yield the generalized HJ PDEs for \( \alpha = 0, n - m + 1, ..., n \),

\[
H'_\alpha \equiv p_\alpha + H_\alpha(t_\beta, q_a, p_a) = 0. \tag{2.9}
\]

Exploiting Eqs. \(2.8\) and \(2.9\), one can obtain

\[

dq_\alpha = \frac{\partial H'_\alpha}{\partial p_\alpha} dt_\alpha, \quad dp_\alpha = -\frac{\partial H'_\alpha}{\partial q_\alpha} dt_\alpha. \tag{2.10}
\]

Here note that we have used the extended index \( i \) \( (i = 0, 1, ..., n) \), instead of the index \( a \) \( (a = 1, 2, ..., n - m) \) used in the literature \[9\], to obtain the complete solutions to the system. Eq. \(2.10\) then yields

\[
dS = \left(-H'_\alpha + p_\alpha \frac{\partial H'_\alpha}{\partial p_\alpha}\right) dt_\alpha, \tag{2.11}
\]

from which we obtain the action of the form

\[
S = \int (-H_0 dt + p_i dq_i). \tag{2.12}
\]

Note that at the moment \( dq_i \) cannot be integrable to yield the desired effective Lagrangian, which will be realized by introducing auxiliary fields in the improved DQM in the next sections.

Now, in order to discuss the integrability conditions, one can introduce a linear operator \( X_\alpha (\alpha = 0, n - m + 1, ..., n) \) corresponding to Eq. \(2.10\) as
X_\alpha f = \{ f, H'_\alpha \} = \frac{\partial f}{\partial q_i} \frac{\partial H'_\alpha}{\partial p_i} - \frac{\partial H'_\alpha}{\partial q_i} \frac{\partial f}{\partial p_i}, \quad (2.13)

from which one can obtain the bracket relations among the linear operators X_\alpha

\[ [X_\beta, X_\alpha] f = \{ f, \{ H'_\alpha, H'_\beta \} \}. \quad (2.14) \]

Note that, if one can introduce operators X_\bar{\alpha} with an extended index \bar{\alpha} (\bar{\alpha} = 0, n - m + 1, ..., n, ...) such that these operators satisfy a closed Lie algebra

\[ [X_\beta, X_\bar{\alpha}] f = \{ f, \{ H'_\bar{\alpha}, H'_\beta \} \} = 0, \quad (2.15) \]

then the system of PDEs X_\beta f = 0 is complete and the total differential equations dq_i = \dot{f}_{i\beta} dt_\beta is called integrable. Since the total differential for any function F can be written as dF = \{ F, H'_\beta \} dt_\beta, the integrability conditions for \bar{\alpha} = 0, n - m + 1, ..., n, ..., are given as

\[ \dot{H}'_\bar{\alpha} = \{ H'_\bar{\alpha}, H'_0 \} + \{ H'_\bar{\alpha}, H'_\beta \} \dot{q}_\beta = 0. \quad (2.16) \]

Note that the definition of the brackets (whose index \bar{\alpha} runs from 0 to n) in Eq. (2.16) slightly differs from that of usual Poisson brackets (whose index i runs from 1 to n). If H'_\beta does not possess time-dependence explicitly, the integrability conditions (2.16) are then equivalent to the consistency conditions in the DQM and the involution relations in the improved DQM, which will be discussed in the next sections.

III. NHCS IN HAMILTON-JACOBI SCHEME

In this section, we consider the nonholonomic constrained system (NHCS), where the primary constraint cannot be expressed in terms of the coordinates only, by introducing the

Lagrangian of the form [9,10]

\[ L_0 = \frac{1}{2} q_1^2 - \frac{1}{4}(\dot{q}_2 - \dot{q}_3)^2 + (q_1 + q_3)\dot{q}_2 - q_1 - q_2 - q_3^2, \quad (3.1) \]

with the canonical momenta

\[ p_1 = \dot{q}_1, \quad p_2 = \frac{1}{2}(\dot{q}_3 - \dot{q}_2) + q_1 + q_3, \quad p_3 = \frac{1}{2}(\dot{q}_2 - \dot{q}_3). \quad (3.2) \]
Since the rank of the Hessian matrix $H_{ij} \ (i, j = 1, 2, 3)$ is two, we have two independent relations of the momenta $p_1$ and $p_3$, and the dependent one $p_2$ given as

$$p_2 = -p_3 + q_1 + q_3 = -H_2,$$  \tag{3.3}

which is a nonholonomic primary constraint in the Dirac terminology \[7\].

On the other hand, the Hamiltonian given as

$$H_0 = \frac{1}{2}(p_1^2 - 2p_3^2) + q_1 + q_2 + q_3^2,$$  \tag{3.4}

and Eqs. (2.9) and (3.3) yield the generalized HJ PDEs for $H'_\alpha \ (\alpha = 0, 2)$

$$H'_0 = p_0 + H_0 = 0, \quad H'_2 = p_2 + p_3 - q_1 - q_3 = 0.$$  \tag{3.5}

Since the Hamilton equations are given by Eq. (2.10), the above $H'_\alpha \ (\alpha = 0, 2)$ functions generate the following set of equations of motion

$$dq_0 = dt, \quad dq_1 = p_1 dt, \quad dq_2 = dq_2, \quad dq_3 = -2p_3 dt + dq_2,$$

$$dp_0 = 0, \quad dp_1 = -dt + dq_2, \quad dp_2 = -dt, \quad dp_3 = -2q_3 dt + dq_2.$$  \tag{3.6}

Note that, since $dq_2$ is trivial, one could not obtain any information at this level.

Next, for the above $H'_0$ and $H'_2$, the integrability conditions (2.10) then imply

$$\dot{H}'_0 = \{H'_0, H'_2\}q_2 = -H'_3q_2 = 0, \quad \dot{H}'_2 = \{H'_2, H'_0\} = H'_3 = 0,$$  \tag{3.7}

with $H'_3$ being a nonholonomic secondary constraint of the form

$$H'_3 = 2p_3 - p_1 - 2q_3 - 1.$$  \tag{3.8}

This $H'_3$ then yields an additional integrability condition

$$\dot{H}'_3 = \{H'_3, H'_0\} + \{H'_3, H'_2\}q_2 = 4p_3 - 4q_3 + 1 - \dot{q}_2 = 0,$$  \tag{3.9}

to arrive at the desired information of $dq_2$ absent in Eq. (3.6),

$$\ddot{q}_2 - 2\dot{q}_2 + 2 = 0,$$  \tag{3.10}
so that we can now solve the equations of motion completely. Here one notes that using Eq. (3.2) the nonholonomic constraint (3.8) can be rewritten in terms of the $q_i$ and $\dot{q}_i$ as below

$$H'_3 = -\dot{q}_1 + \dot{q}_2 - \dot{q}_3 - 2q_3 - 1. \quad (3.11)$$

With the aid of Eq. (3.10), we can now completely find the solutions for the equations of motion (3.6) as

$$q_1(t) = Ae^{2t} - t + C_1, \quad p_1(t) = 2Ae^{2t} - 1,$$
$$q_2(t) = 2Ae^{2t} + t + C_2, \quad p_2(t) = -t + C_1,$$
$$q_3(t) = \frac{A}{2} e^{2t} + Be^{-2t} + \frac{1}{2}, \quad p_3(t) = \frac{3}{2} Ae^{2t} + Be^{-2t} + \frac{1}{2}, \quad (3.12)$$

where $A$, $B$, $C_1$ and $C_2$ are arbitrary constants of integration. Note that the results (3.12) are exactly same as those of Ref. [9,10] except the existence of $C_1$ in our solutions.

**IV. NHCS IN STANDARD DIRAC QUANTIZATION METHOD**

In this section, we analyze the Hamiltonian structure of the Lagrangian (3.1) in the standard DQM to compare with that in the HJ scheme. With the definition of the canonical momenta (3.2) one can obtain the nonholonomic primary constraint of the form, which is the same as $H'_2$ defined in Eq. (3.5),

$$\Omega_1 = p_2 + p_3 - q_1 - q_3 \approx 0. \quad (4.1)$$

Now we define the Hamiltonian $H$ with a Lagrangian multiplier $v$,

$$H = H_0 + v\Omega_1, \quad (4.2)$$

from which, requiring the time stability of the primary constraint (4.1), one can easily find secondary constraint, which is equal to $H'_3$ in Eq. (3.8),

$$\dot{\Omega}_1 = \{\Omega_1, H\} = \Omega_2 = 2p_3 - p_1 - 2q_3 - 1 \approx 0. \quad (4.3)$$

The time stability of $\Omega_2$ yields
\( \dot{\Omega}_2 = \{ \Omega_2, H \} = 4p_3 - 4q_3 + 1 - v = 0, \) (4.4)

to fix the \( v \) as \( v = 4p_3 - 4q_3 + 1 \). These constraints in Eqs. (1.1) and (1.3) make the system second-class with \( \Delta_{ab} = \{ \Omega_a, \Omega_b \} = \epsilon_{ab} \) and \( \epsilon_{12} = 1 \).

Next, to obtain the equations of motion for \( q_i \) \((i = 1, 2, 3)\), we can proceed to construct the Poisson brackets \( \{ q_i, H \} \) and \( \{ p_i, H \} \) for the physical variables \( (q_i, p_i) \) to yield

\[
\begin{align*}
\dot{q}_1 &= q_1, \\
\dot{q}_2 &= 4p_3 - 4q_3 + 1, \\
\dot{q}_3 &= 4p_2 + 6p_3 - 4q_1 - 8q_3 + 1, \\
\dot{p}_1 &= 4p_3 - 4q_3, \\
\dot{p}_2 &= -1, \\
\dot{p}_3 &= 4p_2 + 8p_3 - 4q_1 - 10q_3 + 1, \\
\end{align*}
\] (4.5)

which, together with the constraints (1.1) and (1.3), reproduce the solutions (3.12). Note that the equation of motion for \( \dot{q}_2 \) in Eq. (4.5) is the same as the above fixed value of \( v \).

Note that the Poisson brackets in the HJ scheme are the same as those in the DQM since \( \Omega_a \) do not depend on time explicitly. Moreover, if the integrability conditions (3.7) and (3.9) are rewritten in terms of \( \Omega_1 (= H'_2) \) and \( \Omega_2 (= H'_3) \) and \( v (= \dot{q}_2) \), one can easily reproduce Eqs. (1.3) and (1.4) to explicitly show that the integrability conditions in HJ scheme is equivalent to the consistency conditions in DQM.

On the other hand, in order to consistently quantize the NHCS, one should obtain the Dirac brackets

\[
\begin{align*}
\{ q_i, p_j \}_D &= \delta_{ij} - \delta_{i1}(\delta_{j1} + \delta_{j3}) - 2\delta_{i2}\delta_{j3} + 2\delta_{i3}\delta_{j1}, \\
\{ q_i, q_j \}_D &= -2(\delta_{i2}\delta_{j3} - \delta_{i3}\delta_{j2}) - (\delta_{i1}\delta_{j2} - \delta_{i2}\delta_{j1}) + (\delta_{i3}\delta_{j1} - \delta_{i1}\delta_{j3}), \\
\{ p_i, p_j \}_D &= -2(\delta_{i1}\delta_{j3} - \delta_{i3}\delta_{j1}),
\end{align*}
\] (4.6)

where the Dirac brackets for any functions \( A(q, p) \), \( B(q, p) \) are defined as \( \{ A, B \}_D = \{ A, B \} - \{ A, \Omega_a \} \Delta^{ab} \{ \Omega_b, B \} \), with \( \Delta^{ab} \) being the inverse of \( \Delta_{ab} \).

V. NHCS IN IMPROVED DIRAC QUANTIZATION METHOD

A. Gauge invariant Lagrangian for NHCS

Now, according to the improved DQM [34], we embed the second-class constrained system into first-class one via systematic first-class prescription, where one introduces an
auxiliary canonical pairs $(\theta, \pi_{\theta})$ satisfying $\{\theta, \pi_{\theta}\} = 1$ to yield modified first-class constraints
\(\tilde{\Omega}_a\) satisfying a closed Lie algebra $\{\tilde{\Omega}_a, \tilde{\Omega}_b\} = 0$. Following the improved DQM \[\text{[1,2]}\] we can find effective first-class constraints as
\[
\tilde{\Omega}_1 = \Omega_1 + \theta, \quad \tilde{\Omega}_2 = \Omega_2 - \pi_{\theta}, \quad (5.1)
\]
whose Poisson brackets strongly vanish in the extended phase space due to the introduction of the auxiliary canonical pairs of $(\theta, \pi_{\theta})$. Note that, in the limit $(\theta, \pi_{\theta}) \to 0$, $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ reduce into $H_2'$ and $H_3'$ in Eqs. (3.5) and (3.8) of the HJ scheme, respectively. On the other hand, we can also obtain first-class physical variable in this first-class embedded phase space as

\[
\begin{align*}
\tilde{q}_1 &= q_1 - \theta, \quad \tilde{q}_2 = q_2 + \pi_{\theta}, \quad \tilde{q}_3 = q_3 + \pi_{\theta} + 2\theta, \\
\tilde{p}_1 &= p_1 + \pi_{\theta}, \quad \tilde{p}_2 = p_2, \quad \tilde{p}_3 = p_3 + \pi_{\theta} + 2\theta.
\end{align*}
\]
(5.2)

Note that the Dirac algebra (4.6) in the original phase space is mapped into the Poisson algebra of the first-class physical variables in the extended phase space. Moreover, since the second-class nature of the NHCS is sometimes suffering from unfavorable problems such as ordering upon quantization, it is preferred to convert the second-class system into the first-class one so that one can perform consistent quantization. Using the first-class physical variables (5.2), we can now construct the first-class Hamiltonian as
\[
\tilde{H} = H_0 + (-4p_3 + 4q_3 - 1)\theta + (p_1 - 2p_3 + 2q_3 + 1)\pi_{\theta} + \frac{1}{2}\pi_{\theta}^2, \quad (5.3)
\]
where $H_0$ is the original Hamiltonian (3.4). Moreover, one can also construct the equivalent first-class Hamiltonian of the form
\[
\tilde{H}' = \tilde{H} + \pi_{\theta}\tilde{\Omega}_2, \quad (5.4)
\]

\[\text{to satisfy the Gauss law constraints } \{\tilde{\Omega}_1, \tilde{H}'\} = \tilde{\Omega}_2 \text{ and } \{\tilde{\Omega}_2, \tilde{H}'\} = 0. \text{ Note that these constraint Lie algebra are the same as those in the HJ scheme in the } (\theta, \pi_{\theta}) \to 0 \text{ limit to show that the involution relations are equivalent to the integrability conditions in the HJ scheme in this limit, as in the standard DQM.} \]
Now, exploiting the gauge invariant first-class effective Hamiltonian, we perform the Legendre transformation to integrate out all the involved momenta, via the partition function

\[ Z = \int \prod_{i=1}^{3} dq_i dp_i d\theta d\pi_\theta \prod_{a,b=1}^{2} \delta(\tilde{\Omega}_a)\delta(\Gamma_b) \det | \{ \tilde{\Omega}_a, \Gamma_b \} | e^{i \int dt L}, \]  

(5.5)

where \( \Gamma_a (a = 1, 2) \) is a gauge-fixing function and the effective Lagrangian is given as

\[ L = p_i \dot{q}_i + \pi_\theta \dot{\theta} - \tilde{H}'. \]  

(5.6)

Integrating out the momenta \( \pi_\theta \) and \( p_i \) and exploiting the corresponding equations of motion and the constraints \( \tilde{\Omega}_a \), we can obtain the desired first-class Lagrangian

\[ L = L_0 + L_{WZ}, \]

\[ L_{WZ} = (2\dot{q}_1 - \dot{q}_2 - 2\dot{\theta} + 3)\theta - \frac{1}{2}\dot{\theta}^2, \]  

(5.7)

where \( L_0 \) is given by Eq. (3.1). This Lagrangian (5.7) is invariant under the transformation

\[ \delta q_1 = -\epsilon_2, \quad \delta q_2 = \epsilon_1, \quad \delta q_3 = \epsilon_1 + 2\epsilon_2, \quad \delta \theta = -\epsilon_2, \]  

(5.8)

with \( \epsilon^1 = \epsilon_1 + 4\epsilon_2 \) and \( \epsilon^2 = -\epsilon_1 - 2\epsilon_2 \), which is obtained from the definition of \( \delta c \equiv \{ c, Q \} \) with the symmetry generator \( Q = \epsilon_a \tilde{\Omega}_a \). On the other hand, one can easily see that the canonical momenta (3.2) are modified in this first-class system as

\[ p_1 = \dot{q}_1 + 2\theta, \quad p_2 = \frac{1}{2}(\dot{q}_3 - \dot{q}_2) + q_1 + q_3 - \theta, \]

\[ p_3 = \frac{1}{2}(\dot{q}_2 - \dot{q}_3), \quad \pi_\theta = -\dot{\theta} - 2\theta, \]  

(5.9)

to yield the modified nonholonomic constraint \( \tilde{\Omega}_2 \) in terms of the \( q_i, \dot{q}_i \) and \( \dot{\theta} \),

\[ \tilde{\Omega}_2 = -\dot{q}_1 + \dot{q}_2 - \dot{q}_3 - 2q_3 - 1 + \dot{\theta}, \]  

(5.10)

which reduces into Eq. (3.11) in the vanishing limit of the auxiliary field \( \theta \).

### B. BRST invariant NHCS

In this section we will obtain the BRST invariant Lagrangian in the framework of the Batalin-Fradkin-Vilkovisky formalism [11] which is applicable to theories with the first-class constraints by introducing two canonical sets of ghosts and anti-ghosts together with
auxiliary fields \((C^a, \bar{P}_a), (P^a, \bar{C}_a), (N^a, B_a), (a = 1, 2)\) which satisfy the super-Poisson algebra
\[
\{C^a, \bar{P}_b\} = \{P^a, \bar{C}_b\} = \{N^a, B_b\} = \delta_b^a \delta_a^b.
\]

In the NHCS, the nilpotent BRST charge \(Q_B\), the fermionic gauge fixing function \(\Psi\) and the BRST invariant minimal Hamiltonian \(H_m\) are given by
\[
Q_B = C^a \bar{\Omega}_a + P^a B_a, \quad \Psi = \bar{C}_a \chi^a + \bar{P}_a N^a, \quad H_m = \bar{H}' - C^1 \bar{P}_2
\]  
(5.11)

which satisfy the relations \(\{Q_B, H_m\} = 0, Q_B^2 = \{Q_B, Q_B\} = 0, \{\Psi, Q_B\}, Q_B = 0\). The effective quantum Lagrangian is then given with \(H_{tot} = H_m - \{Q_B, \Psi\}\) as follows
\[
L_{eff} = p_i \dot{q}_i + \pi_\theta \dot{\theta} + B_a \dot{N}^a + \bar{P}_a \dot{C}^a + \bar{C}_a \dot{\bar{P}}^a - H_{tot}.
\]  
(5.12)

Now we choose the unitary gauge \(\chi^1 = \Omega_1, \chi^2 = \Omega_2\) and perform the path integration over the fields \(B_1, N^1, \bar{C}_1, p^1, \bar{P}_1\) and \(C^1\) to yield
\[
L_{eff} = p_i \dot{q}_i + \pi_\theta \dot{\theta} + 2B_2 \dot{N}^2 + \bar{P}_2 \dot{C}^2 + \bar{C}_2 \dot{\bar{P}}^2 + \bar{P}_2 \dot{P}^2 - \pi_\theta \dot{N}^2
\]
\[-\frac{1}{2}(p_1^2 - 2p_3^2) - q_1 - q_2 - q_3^2 - (-4p_3 + 4q_3 - 1)\theta + \frac{1}{2}\pi_\theta^2
\]
\[+(2p_3 - p_1 - 2q_3 - 1)(N^2 + B_2).
\]  
(5.13)

Next, using the variations with respect to \(p_i, \pi_\theta, P\) and \(\bar{P}\), one obtain the relations
\[
p_1 = \dot{q}_1 - N^2 - B_2, \quad p_2 = \frac{1}{2}(\dot{q}_3 - \dot{q}_2) + q_1 + q_3 + \theta + N^2 + B_2,
\]
\[p_3 = \frac{1}{2}(\dot{q}_2 - \dot{q}_3) - 2\theta - N^2 - B_2, \quad \pi_\theta = -\dot{\theta} - N^2,
\]
\[P^2 = -\dot{C}^2, \quad \bar{P}_2 = \dot{\bar{C}}_2,
\]  
(5.14)

to, with the choice of \(N^2 = -B_2 + 2\theta\), yield the effective Lagrangian
\[
L_{eff} = L_0 + L_{WZ} + L_{gh},
\]
\[L_{gh} = -\frac{1}{2}(B_2)^2 - (2\theta + \dot{\theta})B_2 + \dot{\bar{C}}_2 \dot{C}^2.
\]  
(5.15)

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1Here the super-Poisson bracket is defined as \(\{A, B\} = \frac{\delta A}{\delta q}|_r \frac{\delta B}{\delta p}|_l - (1)^{\eta_A \eta_B} \frac{\delta B}{\delta q}|_r \frac{\delta A}{\delta p}|_l\) where \(\eta_A\) denotes the ghost number in \(A\) and the subscript \(r\) and \(l\) the right and left derivatives.
which is invariant under the BRST transformation

\[ \delta_B q_1 = -\lambda C^2, \quad \delta_B q_2 = \lambda C^1, \quad \delta_B q_3 = \lambda(C^1 + 2C^2), \]

\[ \delta_B \theta = -\lambda C^2, \quad \delta_B \bar{C}_a = -\lambda B_a, \quad \delta_B C_a = \delta_B B_a = 0, \]

(5.16)

with \( \hat{C}^1 = C^1 + 4C^2 \) and \( \hat{C}^2 = -C^1 - 2C^2 \), which are generalized transformation rules of Eq. (5.8), including the ghost fields. Here one notes that the first-class nonholonomic constraint \( \tilde{\Omega}_2 \) in Eq. (5.10) is now generalized to include the ghost term contributions as follows

\[ \tilde{\Omega}_2 = -\dot{q}_1 + \dot{q}_2 - \dot{q}_3 - 2q_3 - 1 + \dot{\theta} + B_2, \]

(5.17)

and \( \tilde{\Omega}_1 \) in Eq. (5.1) is trivially satisfied via using the relations (5.14).

\section*{VI. CONCLUSION}

In conclusion, using the Hamilton-Jacobi (HJ) quantization scheme, we have investigated the nonholonomic constrained system, which possesses the structure of second-class constraints, to compare with the standard and improved Dirac quantization methods (DQMs). We have shown that the integrability conditions in the HJ scheme are equivalent to the involutive relations for the first-class constrained system in the improved DQM by constructing the first-class Hamiltonian and the corresponding effective Lagrangian in the framework of the improved DQM. Furthermore, with this effective Lagrangian, we have also constructed the BRST invariant nonholonomic constrained system. Through further investigation it is interesting to apply the improved HJ scheme to the constrained field systems as well as constrained point particle ones.

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