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Research Article

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DOI: https://doi.org/10.21203/rs.3.rs-798652/v1

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Non-Field Analytical Method for Filtration of Slightly Compressible Fluid through Porous Media with Stagnation Areas

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ABSTRACT

In this study the method of Kulish has been used to derive a non-field solution of the equation, which models the process of unsteady filtration of a slightly compressible fluid within a domain consisting of both flow and stagnation areas under the influence of some pressure distribution at the boundary. The solution relates the local values of pressure and the corresponding pressure gradient and is valid everywhere within the domain including the boundary. The solution thus obtained is in the form of a series with respect to generalised differ-integral operators of fractional orders. The solution has been compared with the know solution of the filtration problem with no stagnation areas. Finally, an integral equation to estimate the pressure evolution at the boundary for a given filtration speed has been proposed.
Nomenclature

\( c_n \) coefficients in series expansion

\( D \) piezo-conductivity (diffusivity), m\(^2\)/s

\( f \) arbitrary function

\( k \) permeability, m\(^2\)

\( n, m \) summation indices

\( P \) pressure, Pa

\( t \) time, s

\( u \) flow speed, m/s

\( x \) spatial variable, m

\( z \) auxiliary variable

Indices

1 flow areas

2 stagnation areas

s surface

Special symbols

\( \mathcal{D}^\nu \) generalised differ-integral operator of order \( \nu \)

\( \mathfrak{s} \) generalised fractional differ-integral operator

\( \mathcal{L} \) generalised differ-integral operator

Greek symbols

\( \beta \) dimensionless volumetric fraction

\( \gamma \) mass transfer coefficient, s\(^{-1}\)

\( \delta \) relative truncation error

\( \zeta \) dummy integration variable

\( \xi \) dimensionless spatial variable

\( \kappa \) porosity per volume of flow areas, m

\( \mu \) dynamic viscosity, Pa\( \cdot \)s

\( \nu \) fractional order

\( \tau \) dimensionless time

\( \nu \) dummy integration variable

\( \phi \) volume fraction of flow areas
1. Introduction

Non-field methods are analytical methods that allow one to obtain closed-form solutions of partial differential equations. Unlike conventional analytical solutions, which provide information about the entire field within a domain of interest, non-field solutions relate local values of the unknown variable (e.g., temperature, mass concentration, velocity) and its gradient. Thus, non-field methods provide one with information about the evolution of the unknown variable in a location of interest without the need to know the entire field – hence the name of the method. One important characteristic of the non-field methods is that the solutions they yield are valid everywhere within the domain of interest including the domain boundaries. The latter feature is very useful in many practical applications, where it is sufficient to get information about the behaviour on either the domain boundary or in the locations of extrema (e.g., temperature and the corresponding heat flux, velocity and pressure/shear stress, etc.). It is also worth noting here that, in some cases, non-field methods allow obtaining analytical solutions, where the application of other methods is either impractical or does not yield desired results at all.

In the past two decades, non-field methods have been successfully used in a wide range of practical applications spanning from problems in thermo-fluid sciences to biomedical engineering and even econometrics. One of the non-field methods – nowadays known as the method of Kulish – has been used most extensively and was generalised recently. The method offers elegant non-field solutions in the form of series of fractional differ-integral operators.

In this paper, the method of Kulish is applied to the problem of unsteady filtration of a compressible fluid within a porous medium, in which both flow and stagnation areas are present. The latter problem is of great practical interest in design of water and oil filters, oil and gas industries, design and construction of hydro-systems (e.g., earth-fill dams), bioreactors (e.g., hemo-filtration) and many others. In spite of this, analytical solutions are not available in general, so that one has to resort to either approximate methods or laborious numerical simulations.

The global non-field solution of the unsteady filtration problem, presented in this study, relates the local values of pressure and its gradient – hence, flow speed – everywhere within the domain including the boundary between the flow and stagnation areas. The solution is general in the sense that it remains valid for any porosity value but not only some extreme cases.

For the sake of model validation and illustration, the solution thus obtained have been compared with the known solutions obtained for some particular cases by other methods.
2. Mathematical formulation

The process of unsteady filtration of a slightly compressible fluid under the influence of some pressure distribution at the boundary, \( P_s(t) \), can be modelled by

\[
(1 - \phi) \frac{\partial P_2}{\partial t} + \phi \frac{\partial P_1}{\partial t} = D \frac{\partial^2 P_1}{\partial x^2}, \quad 0 < x < \infty, \quad 0 < t < \infty \tag{1a}
\]

\[
\frac{\partial P_2}{\partial t} = \gamma (P_1 - P_2) \tag{1b}
\]

where \( \phi \) is the volume fraction of the flow areas; \( P_1 \) and \( P_2 \) are the pressure in the flow and stagnation areas, respectively; and \( \gamma \) denotes the constant coefficient of mass transfer between the flow and stagnation areas.

The corresponding initial and boundary conditions are

\[
P_1 |_{t=0} = 0; \quad P_2 |_{t=0} = 0; \\
P_1 |_{x=0} = P_s(t); \quad P_1 |_{x=\infty} = 0; \quad P_2 |_{x=\infty} = 0,
\]

\[
\frac{\partial P_1}{\partial x} |_{x=0} = u
\]

respectively.

The task is to find the pressure gradient at the boundary, \( \frac{\partial P_1}{\partial x} |_{x=0} \). Provided the flow obeys the Darcy law, the pressure gradient defines the flow filtration speed as

\[
u = -\frac{k}{\mu \kappa} \frac{\partial P_1}{\partial x} \tag{2}
\]

where \( k \) is the permeability of the porous medium in question, \( \mu \) is the dynamic viscosity of the fluid, and \( \kappa \) denotes the porosity with respect to the volume of the flow areas.

It is worth noting here that the problem thus formulated is not reducible to the general problem considered in earlier works\(^2\). In view of this, the solution, presented in the following section, is of methodological interest for developing the general theory of non-field methods.

3. Solution procedure

In this section, a non-field solution for the problem modelled by Eqs. (1) is obtained by the method of Kulish, that is, in the form of a series with respect to generalised differ-integral operators of fractional orders\(^2\).
Expressing $P_2$ from Eq. (1b) with taking into account conditions (1c) and substituting into (1a) yields

$$\left[ \phi \frac{\partial}{\partial t} + (1 - \phi)\gamma - (1 - \phi)\gamma^2 \int_0^t e^{-\gamma(t - \zeta)}(\cdot) d\zeta - D \frac{\partial^2}{\partial x^2} \right] P_1 = 0$$

(3a)

with

$$P_1 |_{x=0} = P_s(t); \quad P_1 |_{x=\infty} = 0; \quad P_1 |_{t=0} = 0.$$  

(3b)

Equation (3a) can be written in the form

$$\left\{ e^{-\gamma t} \left[ \frac{\partial}{\partial t} + \gamma (\beta - 1) - \beta \gamma^2 \frac{\partial^{-1}}{\partial t^{-1}} \right] e^{\gamma t} - D \frac{\partial^2}{\phi \partial x^2} \right\} P_1 = 0,$$

(4)

where $\beta = (1 - \phi)/\phi$ is the dimensionless volumetric fraction.

Notice that other dimensionless variables are not introduced here in order later not to overcomplicate comparison with the known limiting cases $\phi = 1, \gamma = 0$ and $\gamma = \infty$.

In the latter equation, $\partial^{-1}/\partial t^{-1}$ denotes the differ-integral operator of order negative one, whereas the general definition of differ-integral operators – including those of fractional orders – is given through the Riemann-Liouville integral as

$$\mathfrak{D}^\nu f(t) = \frac{\partial^\nu f(t)}{\partial t^\nu} = \frac{1}{\Gamma(1 - \nu)} \frac{d}{dt} \int_0^t \frac{f(\zeta)d\zeta}{(t - \zeta)^\nu}, \quad -\infty < \text{Re}(\nu) < 1.$$

(5)

Following the general procedure of obtaining non-field solutions, Eq. (4) is now written in the form of the product of two operators, each of which contains the first order spatial derivatives only, that is,

$$\left( \mathfrak{D} - \sqrt{\frac{D}{\phi}} \frac{\partial}{\partial x} \right) \left( \mathfrak{D} + \sqrt{\frac{D}{\phi}} \frac{\partial}{\partial x} \right) P_1 = 0$$

(6)

From comparing Eqs. (4) and (6), it follows that the operator $\mathfrak{D}$ possesses the property
\[ Q^2 = e^{-\gamma t} \left[ \frac{\partial}{\partial t} + \gamma (\beta - 1) - \beta \gamma^2 \frac{\partial^{-1}}{\partial t^{-1}} \right] e^{\gamma t}. \] (7)

By simple inspection,
\[ Q = e^{-\gamma t} e^{\gamma t}, \] (8)
where operator \( \mathfrak{S} \) is found by a formal expansion of the symbolic expression
\[ \mathfrak{S} = \sqrt{\mathfrak{D} + \gamma (\beta - 1) - \beta \gamma^2 \mathfrak{D}^{-1}} \] into a series with respect to \( \mathfrak{D}^{-1/2} \), that is,
\[ \mathfrak{S} = \sum_{n=0}^{\infty} c_n \mathfrak{D}^{1/2-n}, \] (9)
where \( c_0 = 1 \); \( c_1 = \gamma (\beta - 1)/2 \); \( c_2 = -(\beta \gamma^2 + c_1^2)/2 \); \ldots;
\[ c_n = -\frac{1}{2} \sum_{m=1}^{n-1} c_{n-m} c_m \quad n \geq 3. \]

Finally, the non-field solution of the problem, given by Eqs. (1), is obtained by writing the equation formed by the right multiplier in Eq. (6) for \( x = 0 \) and taking into account Eq. (9), that is,
\[ -\frac{\partial P_1}{\partial x} \bigg|_{x=0} \sqrt{D} \phi = e^{-\gamma t} \left( \sum_{n=0}^{\infty} c_n \mathfrak{D}^{1/2-n} \right) e^{\gamma t} P_s(t). \] (10)

For \( \phi = 1 \) \( (\beta = 0) \), Eq. (10) reduces to
\[ -\frac{\partial P_1}{\partial x} \bigg|_{x=0} \sqrt{D} = e^{-\gamma t} \sqrt{\mathfrak{D} - \gamma e^{\gamma t} P_s(t)} = e^{-\gamma t} (e^{\gamma t} \mathfrak{D}^{1/2} e^{-\gamma t}) e^{\gamma t} P_s(t) = \mathfrak{D}^{1/2} P_s(t), \] (11)
which coincides with the known solution of the filtration problem without stagnation areas\(^{14}\).

For \( \gamma = 0 \), Eq. (10) again reduces to Eq. (11) but with the effective piezo-conductivity \( D/\phi \).

Consider the case of a pressure jump at the boundary \( P_1(-0,t) = P_0 = \text{const} \) in more detail.
It has been shown\textsuperscript{11-14} that the pressure to the right from the boundary varies as

\[ P_s(t) = P_1(+0,t) = [1 - \exp(-\gamma t)]P_0 \]

(12)

It is possible to obtain the latter expression considering

\[ \left( \gamma \frac{\partial}{\partial t} + \phi \frac{\partial^2}{\partial t^2} - \gamma D \frac{\partial^2}{\partial x^2} - D \frac{\partial^2}{\partial x^2 \partial t} \right) P_1 = 0, \]

(13)

which is derivable from Eqs. (1) for the domain \(0 \leq t \leq T, -h \leq x \leq h\) as \(h \to 0\).

Indeed, Eq. (12) follows upon multiplying Eq. (13) by \(xt\) and integrating over the domain \(0 \leq t \leq T, -h \leq x \leq h\) as \(h \to 0\)\textsuperscript{12-14}. If \(\gamma \to \infty\), Eq. (12) reduces to \(P_1 = P_0\).

For the sake of illustration, consider the solution given by Eq. (10) in the case of \(\beta = 1\) (\(\phi = 1/2\)).

For the coefficients of the operator \(\mathcal{F}\), it follows

\[ c_{2n} = (-1)^n \left( \frac{1}{2} \right)^n \gamma^n; \quad c_{2n+1} = 0 \]

(14)

Then, from Eq. (10), taking into account\textsuperscript{1}

\[ \mathcal{F}^\mu t^\lambda = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + 1 - \nu)} t^{\lambda - \nu}; \quad \lambda > 1, \nu < 1 \]

(15)

and expanding \(\exp(\gamma t) - 1\) into power series, follows the expression for the pressure gradient at the boundary

\[ -\frac{\partial P_1}{\partial \xi} \bigg|_{\xi=0} = \frac{P_0 e^{-\sqrt{\tau}}}{{\sqrt{\tau}}} \left( 2 \frac{\tau}{1!!} + \frac{4}{3!!} \tau^2 + \frac{8}{5!!} \tau^3 + \frac{19}{7!!} \tau^4 + \frac{16}{9!!} \tau^5 + \frac{40}{11!!} \tau^6 + \frac{80}{13!!} \tau^7 + \frac{140}{15!!} \tau^8 + \frac{224}{17!!} \tau^9 + \frac{540}{19!!} \tau^{10} + \frac{1008}{21!!} \tau^{11} + \frac{1848}{23!!} \tau^{12} + \frac{3168}{25!!} \tau^{13} + \frac{6864}{27!!} \tau^{14} + \delta \right) \]

(16)
with the relative truncation error

$$0 < \delta < \frac{(2\tau)^{15}}{31!!} \left( 1 + \frac{\tau}{33} + \frac{\tau^2}{33 \cdot 35} + \frac{\tau^3}{33 \cdot 35 \cdot 37} + \ldots \right) < \frac{(2\tau)^{15}}{31!!} \frac{1}{1 - \tau/33}. $$

In Eq. (16), $\xi = x \sqrt{\phi \gamma / D}$ and $\tau = \gamma t$ denote the dimensionless spatial and temporal variables, respectively.

4. Results and discussion

It is worth noting that an alternative form of the non-field solution for the pressure gradient at the boundary can be found by the Laplace transform technique in the form

$$-\frac{\partial P_1}{\partial \xi} \bigg|_{\xi=0} = \frac{1}{\sqrt{\pi}} \frac{d}{d\tau} \int_0^\tau \frac{1}{\sqrt{\tau - \xi}} \left( \frac{d}{d\xi} \int_0^\xi \exp \left[ - \left( 1 + \frac{\beta}{2} \right) \left( \xi - z \right) \right] I_0 \left[ \frac{\beta}{2} \left( \xi - z \right) \right] + \right.
$$

$$+ \left. (1 + \beta) \int_0^{\xi - z} \exp \left[ - \left( 1 + \frac{\beta}{2} \right) \nu \right] I_0 \left( \frac{\beta}{2} \nu \right) d\nu \right) P_s(z) dz d\xi$$

(17)

where again $\xi = x \sqrt{\phi \gamma / D}$, $\tau = \gamma t$, $\beta = (1 - \phi)/\phi$, and $I_0$ denotes the modified Bessel function.$^{15}$

The latter expression, however, is not convenient for practical calculations, because it contains two differential operators and three indefinite integrals.

In contrast, Eq. (16) is very useful when numerical evaluation is required. The relative error does not exceed

$$\delta = \frac{2}{2 \cdot 31!!} \frac{(2\tau)^{15}}{1 - \tau/33}$$

(18)

For instance, $\delta/2 \approx 3 \cdot 10^{-3}$ for $\tau = 5$.

Figure 1 shows comparison between the boundary pressure gradient non-dimensionalised with respect to the corresponding pressure jump $-\frac{\sqrt{\pi}}{P_0} \frac{\partial P_1}{\partial \xi} \bigg|_{\xi=0}$ versus the dimensionless time $\tau$ for $\beta = 0$, $\phi = 1$ (red curve) and $\beta = 1$, $\phi = 1/2$ (blue curve), respectively.
Fig. 1. Transient behaviour of the boundary pressure gradient non-dimensionalised with respect to the corresponding pressure jump \(-\frac{\sqrt{\pi}}{P_0} \frac{\partial P_1}{\partial \xi} \bigg|_{\xi=0}\) for $\beta = 0$, $\phi = 1$ (red) and $\beta = 1$, $\phi = 1/2$ (blue).

As can be seen from the figure and as expected, for smaller values of porosity, larger values of the pressure gradients are needed to achieve the same filtration speed for a given pressure jump. In addition, it can be observed that the memory of the process about the initial state increases with an increase of $\beta$. Thus, in case of $\beta = 1$, it requires about a trice longer span of time to dissipate the initial pressure pulse than in case of $\beta = 0$. This can be understood by recalling that, in case of $\beta = 1$, only one half of the domain volume is available to the flow and, hence, the same amount of momentum is to be dissipated in a smaller volume – obviously, this requires a longer time.

To conclude, it is worth revisiting Eq. (11).

Upon applying $\mathfrak{D}^{-1/2}$ on both sides of Eq. (11) by using Eq. (5) and rearranging the terms, the solution becomes

$$P_s(t) = \phi(0) - \frac{1}{\sqrt{\pi D}} \int_0^t \frac{\partial P_1}{\partial x} \frac{d\zeta}{\sqrt{t-\zeta}}.$$  

(19)
The latter is identical to the non-field solutions of classical problems of heat and mass transfer\textsuperscript{3-4} and the Stokes problem in fluid dynamics\textsuperscript{5}.

For the case, considered in this study, the latter result, upon using Eq. (2), can be generalised to provide the first estimate of the pressure evolution at the boundary for a given filtration speed, that is,

\[ P_s(t) = P_s(0) - \frac{k}{\mu \kappa \sqrt{\pi D}} \int_0^t u(\zeta) d\zeta. \]

(20)

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**Acknowledgments**

The authors would like to thank the Department of Thermodynamics & Fluid Mechanics of the Faculty of Mechanical Engineering at the Czech Technical University in Prague for the financial support.

**Author Contributions**

Both the authors equally contributed into this study.

**Additional Information**

**Competing financial interests:** The authors declare no competing financial interests.