Zero-curvature point of minimal graphs

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Abstract
Motivated by a classical result of Finn and Osserman (J Anal Math 343(12):351–364, 1964), who proved that the Jenkins–Serrin graph over the square inscribed in the unit disk is extremal for the Gaussian curvature of the point $O$ (so-called centre) of the minimal graphs above the center $0$ of unit unit disk, provided the tangent plane is horizontal, we ask and answer to the question concerned the extremal of "second derivative" of the Gaussian curvature of such graphs provided that its curvature at $O$ is zero. We prove that the extremals are certain Jenkins–Serrin graph over the regular hexagon inscribed in the unit disk, provided that the Gaussian curvature vanishes and the tangent plane is horizontal at the centre.

Keywords Conformal minimal surface · Minimal graph · Curvature

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1 Introduction
Nonparametric minimal graph in $\mathbb{R}^3$ over a domain $D \subset \mathbb{C} \cong \mathbb{R}^2$ is given by

$$S = \{(u, v, f(u, v)) : (u, v) \in D\},$$

where $f$ is a solution of minimal surface equation:

$$(1 + f_u^2)f_{vv} - 2f_u f_v f_{uv} + (1 + f_v^2)f_{uu} = 0.$$
Let $M \subset \mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ be a minimal graph lying over the unit disc $\mathbb{D} \subset \mathbb{C}$. Let $x = (x_1, x_2, x_3) : \mathbb{D} \to M$ be a conformal harmonic parameterization of $M$ with $x(0) = 0$. Its projection $(x_1, x_2) : \mathbb{D} \to \mathbb{D}$ is a harmonic diffeomorphism of the disc which may be assumed to preserve the orientation. Let $z$ be the complex variable in $\mathbb{D}$, and write $x_1 + i x_2 = f$ in the complex notation. We denote by $f_z = \partial f / \partial z$ and $f_{\bar{z}} = \partial f / \partial \bar{z}$ the Wirtinger derivatives of $f$. The function $\omega$ defined by

$$f_{\bar{z}} = \omega f_z$$

is called the second Beltrami coefficient of $f$, and the above is the second Beltrami coefficient with $f$ as a solution.

Orientability of $f$ is equivalent to $J(f, z) = |f_z|^2 - |f_{\bar{z}}|^2 > 0$, hence to $|\omega| < 1$ on $\mathbb{D}$. Furthermore, the function $\omega$ is holomorphic whenever $f$ is harmonic and orientation preserving. (In general, it is meromorphic when $f$ is harmonic.) To see this, let

$$u + iv = f = h + \bar{g}$$

be the canonical decomposition of the harmonic map $f : \mathbb{D} \to \mathbb{D}$, where $h$ and $g$ are holomorphic functions on the disc. Then,

$$f_z = h', \quad f_{\bar{z}} = \bar{g}_z = \bar{g}', \quad \omega = f_{\bar{z}} / f_z = g' / h'.$$

In particular, the second Beltrami coefficient $\omega$ equals the meromorphic function $g'/h'$ on $\mathbb{D}$. In our case we have $|\omega| < 1$, so it is holomorphic map $\omega : \mathbb{D} \to \mathbb{D}$.

We now consider the Enneper–Weierstrass representation of the minimal graph $w = (u, v, t) : \mathbb{D} \to M \subset \mathbb{D} \times \mathbb{R}$ over $f$, following Duren [4, p. 183]. We have

$$u(z) = \Re f(z) = \Re \int_0^z \phi_1(\zeta) d\zeta$$
$$v(z) = \Im f(z) = \Re \int_0^z \phi_2(\zeta) d\zeta$$
$$t(z) = \Re \int_0^z \phi_3(\zeta) d\zeta$$

where

$$\phi_1 = 2(u)_z = 2(\Re f)_z = (h + \bar{g} + \bar{h} + g)_z = h' + g',$$
$$\phi_2 = 2(v)_z = 2(\Im f)_z = i(h + g - h - \bar{g})_z = i(g' - h'),$$
$$\phi_3 = 2(t)_z = \sqrt{-\phi_1^2 - \phi_2^2} = \pm 2i \sqrt{h'g'}.$$

The last equation follows from the identity $\phi_1^2 + \phi_2^2 + \phi_3^2 = 0$ which is satisfied by the Enneper–Weierstrass datum $\phi = (\phi_1, \phi_2, \phi_3) = 2\partial w$ of any conformal minimal (equivalently, conformal harmonic) immersion $w : D \to \mathbb{R}^3$ from a conformal surface.
D. Let us introduce the notation \( p = f_z \). We have that
\[
p = f_z = (\Re f)_z + i(\Im f)_z = \frac{1}{2}(h' + g' + h' - g') = h'. \tag{1.4}
\]
By using also \( \omega = \frac{\overline{f_z}}{f_z} = g'/h' \) (see (1.3)), it follows that
\[
\begin{align*}
\phi_1 &= h' + g' = p(1 + \omega), \\
\phi_2 &= -i(h' - g') = -ip(1 - \omega), \\
\phi_3 &= \pm 2ip\sqrt{\omega}.
\end{align*}
\]
From the formula for \( \phi_3 \) we infer that \( \omega \) has a well-defined holomorphic square root:
\[
\omega = q^2, \quad q : \mathbb{D} \to \mathbb{D} \text{ holomorphic.} \tag{1.5}
\]
In terms of the Enepper–Weierstrass parameters \((p, q)\) given by (1.4) and (1.5) we obtain
\[
\phi_1 = p(1 + q^2), \quad \phi_2 = -ip(1 - q^2), \quad \phi_3 = -2ipq. \tag{1.6}
\]
(The choice of sign in \( \phi_3 \) is a matter of convenience; since we have two choices of sign for \( q \) in (1.5), this does no cause any loss of generality.) Hence,
\[
\mathbf{w}(z) = \left( \Re f(z), \Im f(z), \Im \int_0^z 2p(t)q(t)dt \right), \quad z \in \mathbb{D}.
\]
Let \( g : \mathbb{D} \to \mathbb{CP}^1 \) denote the Gauss map of the minimal graph \( x \). It is defined up to the choice of a stereographic projection of the unit sphere in \( \mathbb{R}^3 \) to \( \mathbb{R}^2 \times \{0\} \cong \mathbb{C} \). We choose the projection from \((0, 0, -1)\), which makes the plane \( \mathbb{R}^2 \times \{0\} \) with the upward orientation correspond to the origin \( 0 \in \mathbb{C} \subset \mathbb{CP}^1 \). This choice of \( g \) is given by the formula
\[
g = \frac{\phi_1 - i\phi_2}{\phi_3} = \frac{2pq^2}{-2ipq} = iq. \tag{1.7}
\]
(The formula used in [1, Eq. (2.79)], which corresponds to the stereographic projection from \((0, 0, 1)\), would give \( g = -i/\omega \).)

The curvature \( K \) of the minimal graph \( M \) is expressed in terms of \((h, g, \omega)\) (1.3), and in terms of the Enneper–Weierstrass parameters \((p, q)\), by
\[
K(f(z)) = -\frac{\lvert \omega' \rvert^2}{\lvert h'g'\rvert^2(1 + \lvert \omega \rvert)^4} = -\frac{4\lvert q' \rvert^2}{\lvert p \rvert^2(1 + \lvert q \rvert^2)^4}, \tag{1.8}
\]
where \( p = f_z \) and \( \omega = q^2 = \frac{\overline{f_z}}{f_z} \). (See Duren [4, p. 184].)

In order to motivate our problem let us recall Heinz–Hopf–Finn–Osserman problem
Problem 1.1 What is the supremum of $|K(O)|$ evaluated at the point $O$ (which we will call centre) above the center of the unit disk, over all minimal graphs lying over $\mathbb{D}$? Is

$$|K(O)| < \frac{\pi^2}{2}$$

(1.9)

the precise upper bound?

It was shown by Finn and Osserman [7] in 1964 that the upper bound in (1.9) is indeed sharp if $q(0) = 0$, which means that the tangent plane $T_0M = \mathbb{C} \times \{0\}$ being horizontal (and hence $f$ is conformal at 0). Although there is no minimal graph lying over the whole unit disc $\mathbb{D}$ whose centre curvature equals $\frac{\pi^2}{2}$, there is a sequence of minimal graphs whose centre curvatures converge to $\frac{\pi^2}{2}$, and the graphs converge to the Jenkins–Serrin graph lying over square inscribed into the unit disc. The associated Beltrami coefficient of the Jenkins–Serrin graph is $\omega(z) = z^2$, with $q(z) = z$. We refer to Duren [4, p. 185] for a survey of this subject.

So far the best inequality for graphs whose tangent planes are not horizontal has been given by R. Hall, who proved [8] (1982)

$$|K(O)| < \frac{16\pi^2}{27}$$

(1.10)

He also in [9] showed that the estimate (1.10) is not sharp by a very small numerical improvement.

In this paper we consider points of minimal surfaces that have zero Gaussian curvature. In that case, the gradient of the Gaussian curvature vanishes at that point, and we call those points stationary points of the Gaussian curvature. We will estimate the "second derivative" of Gaussian curvature at stationary points.

1.1 Zero Gaussian curvature points

For a non-parametric surface $\zeta = f(u, v)$, the Gaussian curvature of the surface at the point $w = (u, v, \zeta)$ is given by

$$K(w) = \frac{f_{uu}f_{vv} - f_{uv}^2}{(1 + f_u^2 + f_v^2)^2}.$$ 

The aim of this paper is to study the behaviors of the surface near the the point $(u, v)$ of zero Gaussian curvature, i.e. near the point in which we have

$$f_{uu}f_{vv} - f_{uv}^2 = 0.$$ 

Notice that, it follows from (1.8) that the zero-points of a minimal surface are isolated. We will show in Lemma 2.1 below that if $K(0) = 0$ then the following unrestricted limit exists
\[ \mathcal{K}''(O) := \lim_{w \to 0} \frac{\mathcal{K}(w)}{2(|w|^2 + \langle \nabla f(w), w \rangle)^2}. \]

Now the following problem is natural

**Problem 1.2** What is the supremum of \( |\mathcal{K}''(O)| \) over all minimal graphs lying over \( \mathbb{D} \), provided that the Gaussian curvature is equal to zero in 0.

We first prove the following general result

**Theorem 1.3** Let \( S : \zeta = f(u, v) \) be a minimal surface over the unit disk \( \mathbb{D} \). Assume also at the point \( O = (0, 0, 0) \in S \) the Gaussian curvature of \( S \) vanishes. Then

\[
|\mathcal{K}''(O)| < \frac{256\pi^4}{729}.
\]

Then we prove the following

**Theorem 1.4** (The main result) Let \( S = \{(u, v, f(u, v)) : (u, v) \in \mathbb{D}\} \) be a non-parametric minimal surface above the unit disk, with vanishing Gaussian curvature \( \mathcal{K} \) and with a horizontal tangent plane at the centre \( O \). Then it exists

\[
-\mathcal{K}''(O) := \lim_{|w| \to 0} \frac{|\mathcal{K}(w)|}{|w|^2}
\]

which is equal to \( 2(f_{uu}^2 + f_{vv}^2) \) and there hold the sharp inequality

\[
2(f_{uu}^2 + f_{vv}^2) < \left( \frac{2\pi}{3}\right)^4.
\] (1.11)

In other words, for every constant \( C < \left( \frac{2\pi}{3}\right)^4 \), there is a minimal surface \( S_1 = \{(u, v, f_1(u, v)) : (u, v) \in \mathbb{D}\} \) over the unit disk with a zero curvature at the point above the center and horizontal tangent plane so that \(-\mathcal{K}''_1(O) \geq C\). The equality in (1.11) is never attained.

## 2 Proof of results

We start the proofs by the following lemma

**Lemma 2.1** Let \( \zeta = f(u, v) \) be a solution of minimal surface equation. Assume also at the point \( O = (0, 0, 0) \),

\[
\mathcal{K}(w) = \frac{f_{uu}f_{vv} - f_{uv}^2}{(1 + f_{u}^2 + f_{v}^2)^2} = 0.
\]

Then \( \nabla f(0) = 0 \), and the following (unrestricted) limit exists

\[
\mathcal{K}''(O) := \lim_{w \to 0} \frac{\mathcal{K}(w)}{2(|w|^2 + \langle \nabla f(w), w \rangle)^2}.
\]
and is equal to the function

\[
(1 + f_u^2)^3 f_{uuu}^2 + 2 f_v f_{uuu} f_u \left( 3 (f_u^2 + f_v^2) - f_u^2 f_v^2 \right) f_{uuu} + (1 + f_v^2)^3 f_{vvv}^2 \\
\left( (1 + f_u^2 + f_v^2)^2 (1 + f_u^2 + f_v^2 (1 - 3 f_u^2)) \right)
\]

evaluated in \( w = (0, 0) \). Here \( w = (w, f(w)) \). Further \( f_{uu} = 0, f_{vv} = 0 \) and \( f_{uv} = 0 \) and \( f_{uuu}, f_{uuu}, f_{uvv}, f_{vvv} \) are uniquely determined by \( K''(O) \), \( f_u \) and \( f_v \) up to their sign at that point \( w \).

In particular if at \( w, f_u = 0 \) and \( f_v = 0 \), then

\[
K''(O) = -2 (f_{uuu}(0) + f_{vvv}(0)). \quad (2.1)
\]

**Proof** If \( K(O) = 0 \) then at \( w = (0, 0) \) we have

\[
f_{uu} f_{vv} - f_{uv}^2 = 0,
\]

and

\[
f_{uv} = \sqrt{f_{uu} f_{vv}}.
\]

Moreover \( f_{uu} \) and \( f_{vv} \) have the same sign. Assume, without loosing of generality that both are non-negative. Then we use the equation

\[
f_{uu}(1 + f_u^2) - 2 f_u f_v f_{uv} + f_{vv}(1 + f_v^2) = 0,
\]

which becomes

\[
f_{uu} + f_{vv} + (\sqrt{f_{uu} f_v} - \sqrt{f_{vv} f_u})^2 = 0.
\]

Therefore

\[
f_{uu}(0) = f_{vv}(0) = f_{uv}(0) = 0. \quad (2.2)
\]

Now for \( w = (u, v, f(u, v)) = (w, f(w)) \) we have

\[
K(w) = K(O) + \langle \nabla K(O), w \rangle + \frac{1}{2} \langle H(O) w, w \rangle + o(|w|^2).
\]

Here

\[
H = \begin{pmatrix}
K_{uu} & K_{uv} \\
K_{uv} & K_{vv}
\end{pmatrix}
\]
is the Hessian matrix of $K$. So

$$
\mathcal{H} = \begin{pmatrix}
\frac{-2f_{uuu}^2 + 2f_{uvu} f_{uuu}}{(1 + f_v^2 + f_u^2)^2} & \frac{-f_{uuv} f_{uuu} + f_{uvu} f_{uuu}}{(1 + f_v^2 + f_u^2)^2} \\
\frac{-f_{uuv} f_{uuu} + f_{uvu} f_{uuu}}{(1 + f_v^2 + f_u^2)^2} & \frac{-2f_{uuu}^2 + 2f_{uvu} f_{uuu}}{(1 + f_v^2 + f_u^2)^2}
\end{pmatrix}
$$

Because of (2.2), we obtain that $\nabla K(O) = 0$. In order to prove (2.1), for $w = re^{it} = rk$ we have

$$
Q(t) = -\lim_{r \to 0} \frac{K(w)}{|w|^2} = \left\{ \mathcal{H}[K](O) e^{it}, e^{it} \right\}.
$$

We need to show that

$$
R(t) = \frac{Q(t)}{1 + \langle \nabla f(0), e^{it} \rangle^2}
$$

do not depend on $t$.

Thus, by (2.2),

$$
Q(t) = \frac{A + B}{(1 + f_v^2 + f_u^2)^2}
$$

(2.3)

where

$$
A = -2 \sin^2 t f_{uuv}^2 + 2 \sin t (\sin tf_{uvu} - \cos tf_{uuu}) f_{vvu}
$$

and

$$
B = -2 \cos^2 t f_{vvu}^2 + 2 \cos t (\sin tf_{vvu} + \cos tf_{uuu}) f_{uuu}.
$$

By differentiating w.r.t. $u$ and $v$ the minimal surface equation we get

$$
\begin{align*}
-2f_u f_{uuu}^2 + (1 + f_u^2) f_{uuv} &+ 2f_v f_u f_{uuu} - 2f_u^2 f_{uuu} + (1 + f_v^2) f_{uuu} = 0, \\
f_{vvv} (1 + f_u^2) - 2f_v f_{uuu}^2 &- 2f_v f_u f_{uuu} + 2f_v f_{vvu} f_{uuu} + (1 + f_v^2) f_{uuu} = 0.
\end{align*}
$$

Therefore

$$
\begin{align*}
f_{uuv} &= \frac{2f_v f_{uvu} (f_u + f_v^3) + (1 + f_v^2)^2 f_{uuu}}{-1 - f_u^2 + f_v^2 (-1 + 3f_u^2)} \\
(2.4)
\end{align*}
$$

and

$$
\begin{align*}
f_{uuu} &= \frac{f_{vvv} (1 + f_u^2)^2 + 2 (f_v + f_u^3) f_u f_{uuu}}{-1 - f_u^2 + f_v^2 (-1 + 3f_u^2)}.
\end{align*}
\quad(2.5)
$$
Then inserting to (2.3) we get

\[ Q(t) = Y(t) \left( f_{uuu}^2 (1 + f_u^2)^3 + 2 f_u f_{uuu} f_v f_{uvv} \left( f_v^2 (3 - f_u^2) + 3 (1 + f_u^2) f_{uuu} + (1 + f_v^2)^3 f_{uuu}^2 \right) \right) \left( (1 + f_v^2 + f_u^2)^2 (1 + f_v^2 + f_u^2 (1 - 3 f_u^2))^2 \right), \]

where

\[ Y(t) = -2 \left( 1 + \sin^2 t f_v^2 + \sin(2t) f_u f_v + \cos^2 t f_u^2 \right) = -2 \left( 1 + \left( \nabla f(0), e^{i t} \right)^2 \right). \]

Since

\[ Y'(t) = -4 \cos(2t) f_v f_u + 2 \sin(2t) \left( -f_v^2 + f_u^2 \right) \]

we get that \( Y'(t) = 0 \) if and only if \( f_v = f_u = 0 \). In this particular case we get

\[ -Q(0) = 2 (f_{uuu}^2 + f_{uvv}^2). \]

\[ \square \]

### 2.1 Proof of Theorem 1.3

The proof is based on Lemma 2.1 and R. Hall sharp result best lower bound of the gradient of a harmonic diffeomorphism of the unit disk onto itself [8].

**Proof of Theorem 1.3** Assume that \( w = f(z) = u(z) + iv(z) \) is the harmonic diffeomorphisms that produces the Enneper-Weisstrass parameterization so that \( f(0) = 0 \).

The unit normal at \( w = (u, v, t) \in S \), in view of [4, p. 169] is given by

\[ n_w = -\frac{1}{1 + |q(z)|^2} (2 \Re q(z), 2 \Im q(z), -1 + |q(z)|^2). \]

It is also given by the formula

\[ n_w = \frac{1}{\sqrt{1 + f_u^2 + f_v^2}} (-f_u, -f_v, 1). \]

Then we have the relations

\[ f_v(u(x, y), v(x, y)) = \frac{2a(x, y)}{-1 + a(x, y)^2 + b(x, y)^2} \quad (2.6) \]

\[ f_u(u(x, y), v(x, y)) = \frac{2b(x, y)}{-1 + a(x, y)^2 + b(x, y)^2}. \]
where $a = \Re q, b = \Im q$. So

$$\nabla f(u(x, y), v(x, y)) = \frac{-2i\bar{q}(z)}{1 - |q(z)|^2}$$

(2.8)

and

$$\mathcal{K}''(O) = \lim_{r \to 0} \frac{\mathcal{K}(kr)}{-2r^2(1 + \langle \nabla f(rk), k \rangle)^2}.$$  

Recall that in the Enneper–Weierstrass parameters the curvature can be expressed as

$$-\mathcal{K}(w(z)) = \frac{4|q'|^2}{|p|^2(1 + |q|^2)^2},$$

where $w(z) = (f(z), t(z)) \in S$. So

$$\mathcal{K}''(O) = \lim_{r \to 0} \frac{-\mathcal{K}(rk)}{|rk|^2(1 + \langle \nabla f(rk), k \rangle)^2}
= \lim_{r \to 0} \frac{-\mathcal{K}(rk)}{|f^{-1}(rk)|^2 |rk|(1 + \langle \nabla f(rk), k \rangle)^2}
= \frac{4|q''(0)|^2}{|p(0)|^2(1 + |q(0)|^2)^2} \lim_{r \to 0} \frac{M^2(kr)}{(1 + \langle \nabla f(rk), k \rangle)^2},$$

where

$$M(kr) = \frac{1}{J(f, z)} \overline{g^7k - h'k} = \frac{1}{|g'(z)|^2(1 - |q(z)|^4)} |g'(z)| \cdot |1 - q(z)^2k^2|.$$

Here $z = f^{-1}(rk)$ and $J(f, z)$ is the Jacobian. Thus

$$\lim_{r \to 0} M(kr)^2 = \frac{|1 - q(0)^2k^2|^2}{|p(0)|^2(1 - |q(0)|^4)^2}.$$

Further

$$\frac{|1 - q(0)^2k^2|^2}{(1 + \langle \nabla f(0), k \rangle)^2} = \frac{|1 - q(0)^2k^2|^2}{1 + \left(\frac{-2i\bar{q}(0)}{1 - |q(0)|^2}, k\right)^2} = (1 - |q(0)|^2)^2.$$

Thus

$$\mathcal{K}''(O) = \frac{4|q''(0)|^2(1 - |q(0)|^2)^2}{|p(0)|^4(1 + |q(0)|^2)^2(1 - |q(0)|^4)^2} = \frac{4|q''(0)|^2}{|p(0)|^4(1 + |q(0)|^2)^5}. \quad (2.9)$$
since \(|q(z)| \leq 1\), and \(q'(0) = 0\), we get
\[
\left| \frac{q(z) - q(0)}{1 - q(z)q(0)} \right| \leq |z|^2
\]
which implies that \(|q''(0)| \leq 2(1 - |q(0)|^2)\). Observe that \(p(0) = f_z(0)\), and so by Hall’s result [8]
\[
|p(0)|^2 \geq \frac{27}{4\pi^2} \frac{1}{1 + |q(0)|^2}.
\]
Therefore
\[
\mathcal{K}''(O) \leq \frac{16(1 - |q(0)|^2)^2}{|p(0)|^4(1 + |q(0)|^2)^6} \leq \frac{16 \cdot (4\pi^2)^2}{27^2}.
\]
Thus
\[
\mathcal{K}''(O) \leq \frac{256\pi^4}{729},
\]
what we wanted to prove.

The following example produces a Jenkins–Serrin graph. We will show that it is extremal for the second derivative of Gaussian curvature (see the proof of Theorem 1.4).

**Example 2.2** Let

\[
g(z) = 3zF\left[\frac{1}{6}, 1, \frac{7}{6}, -z^6\right]_\pi
\]

and

\[
h(z) = \frac{3z^5F\left[\frac{5}{6}, 1, \frac{11}{6}, -z^6\right]}{5\pi}
\]

where \(F\) is the Gauss Hypergeometric function. Then

\[
f(z) = g(z) + \overline{h(z)}
\]

is a harmonic diffeomorphism of the unit disk onto the regular hexagon \(G\) inscribed in the unit disk. Then

\[
p(z) = g'(z) = \frac{3}{\pi + \pi z^6}
\]
Fig. 1 A Jenkins–Serrin graph over the hexagon

and

\[ q(z) = \sqrt{\frac{h'(z)}{g'(z)}} = z^2. \]

Moreover \( f(z) = g(z) + h(z) \) maps the unit disk onto the regular hexagon that defines a 6—sides Jenkins–Serrin graph. See Fig. 1.

The third coordinate of conformal harmonic parametrization is given by

\[ t(z) = \frac{\log \left[ \frac{1 + r^6 - 2r^3 \sin(3s)}{1 + r^6 + 2r^3 \sin(3s)} \right]}{2\pi}, \quad z = re^{is}. \]

See also the paper of Duren and Thygerson [5] for additional details for those Jenkins–Serrin graphs.

Further

\[ |K''(O)| = \frac{|q''(0)|^2}{|\rho(0)|^4} = \frac{16\pi^4}{81}. \quad (2.12) \]

Observe that \( \frac{16\pi^4}{81} < \frac{256\pi^4}{729} \) (see (2.10)).

2.2 The proof of Theorem 1.4

We need the following lemma.

Lemma 2.3 [7, Lemma 1] Let \( \phi_1, \phi_2 \) be any two distinct solutions of the minimal surface equation such that \( \phi_1 \) and \( \phi_2 \), together with all their derivatives of order up to and including \( n \), coincide at some point \( a \). Then a neighborhood \( D = D_a \) of that
point a may be mapped homeomorphically onto a neighborhood of 0 in the complex \(\zeta\)-plane in such a way that the function \(\phi = \phi_1 - \phi_2\), will be of the form

\[
\phi(z) = \Re \left[ \zeta(z)^N \right]
\]

for some \(N \geq n + 1\), \(z \in D_a\).

**Proof of Theorem 1.4** Assume that \(S : \zeta = f(w)\) is a minimal surface above the unit disk with a zero-Gaussian curvature and horizontal tangent plane at the point above the center and let \(S^\circ : \zeta = f^\circ(u, v)\) be the Jenkins–Serrin minimal graph above the regular hexagon constructed in Example 2.2. We assert that \(|K''(O)| < |K''_\circ(O)|\). Assume the converse \(|K''(O)| \geq |K''_\circ(O)|\) and argue by a contradiction. Then as in [7], by using the dilatation \(L(\zeta) = \lambda \zeta\) for some \(\lambda \geq 1\) we get the surface

\[
S_\lambda = L(S) = \left\{ (u, v, \lambda f \left( \frac{u}{\lambda}, \frac{v}{\lambda} \right) : |u + iv| < \lambda \right\},
\]

whose Gaussian curvature

\[
K_\lambda(w) = \frac{1}{\lambda^2} \frac{(f_{uu}(u/\lambda, v/\lambda)f_{vv}(u/\lambda, v/\lambda) - f_{uv}(u/\lambda, v/\lambda)^2)}{(1 + f_u(u/\lambda, v/\lambda)^2 + f_v(u/\lambda, v/\lambda)^2)^2}.
\]

(2.13)

Here and in the sequel \(w = (w, f(w))\). Observe that such transformation does not change the gradient \(\nabla f(0)\). Moreover \(K_\lambda(0) = 0\). Since

\[
|K''(O)| = \lim_{w \to 0} \frac{-K(w)}{|w|^2} > 0,
\]

because of (2.13), it exists \(\lambda_* \geq 1\) so that

\[
K''_{\lambda_*}(O) = K''_\circ(O).
\]

(2.14)

Let

\[
f^*(u, v) = \lambda_* f \left( \frac{u}{\lambda_*}, \frac{v}{\lambda_*} \right).
\]

Since \(K_*(O) = K_\circ(O) = 0\) and \(\nabla f^*(0) = 0 = \nabla f^\circ(0)\), it follows that all derivatives up to the order 2 of \(f_*\) and \(f_\circ\) vanishes at zero.

From (2.14) we obtain that

\[
(f^*_{uuu})^2 + (f^*_{vvv})^2 = (f^\circ_{uuu})^2 + (f^\circ_{vvv})^2.
\]

(2.15)

We can also w.l.g. assume that \(f^*_{uuu}\) and \(f^\circ_{uuu}\) as well as \(f^*_{vvv}\) and \(f^\circ_{vvv}\) have the same sign. If not, then we choose \(\lambda_* \leq -1\) and repeat the previous procedure with

\[
S_1 = L(S) = \left\{ (u, v, \lambda f \left( \frac{u}{\lambda}, \frac{v}{\lambda} \right) : |u + iv| < |\lambda| \right\}.
\]
From (2.1), (2.4), (2.5) and (2.15) we obtain that $f^\omega_{uuu} = f^\omega_{uuu}$, $f^\omega_{uvv} = f^\omega_{uvv}$, $f^\omega_{uuu} = f^\omega_{uvv}$, $f^\omega_{uuu} = f^\omega_{uvv}$.

Thus the function $F(u, v) = f^\omega(u, v) - f^\omega_0(u, v)$ has all derivatives up to the order 3 equal to zero in the point $w = 0$.

By Lemma 2.1, for $F(w) = f^\omega(w) - f^\omega_0(w)$ we are in situation of Lemma 2.3, with $n = 3$. We conclude that there exists a homeomorphism of a neighborhood of the origin onto a neighborhood of the origin in the $\zeta$ plane such that $F(z) = \Re\{\zeta(z)^N\}$ for $N \geq 4$. In particular, the level locus $F = 0$ in this neighborhood consists of $N$ arcs intersecting only at the origin which divide the neighborhood into $2N$ sectors in which $F$ is alternately positive and negative.

Let us now examine any component $R$ of the open set of points in the hexagon $G$ at which $F \neq 0$. At a boundary point of $R$ which is interior to $G$ we must have $F = 0$.

Thus the boundary of $R$ cannot consist entirely of interior points of $G$, since for the difference of two solutions the maximum principle holds [3, p. 322] and hence we would have $F = 0$ in $R$. Thus the boundary of $R$ must contain at least one point of the boundary of $G$. If it contains an inner point of a side of $G$ then it must contain all points on that side, since on each side $f^\omega \to \pm \infty$ but $f^\omega$ is bounded. Thus the set $F = 0$ has at most eight components whose boundaries contain inner points of a side of $G$. On the other hand, the set $F \neq 0$ has at least 6 components, since this is true at the origin and otherwise we could find an arc lying in a single component $R$ which joined two different sectors at the origin. Then one of the other sectors at the origin would lie in a component whose boundary was entirely interior to $G$ which, as we have seen, is impossible. We therefore conclude that some component $R$ of $F \neq 0$ has as boundary points only interior points of $G$ and one or more of the vertices of $G$. However, the Finn’s maximum principle applies also in this case (see [6]), and we would again find $F = 0$ in $R$. Thus the assumption that the surface $S^{\omega}$ has the same second derivative of the Gaussian curvature at the origin as Jenkins–Serrin graph $S^\omega$ and can be extended across the sides of $G$ leads to a contradiction, and in view of (2.12), inequality (1.11) is proved.

Prove the sharpness part. A similar statement for the extremal Gaussian curvature at the centre for the case that the tangent plane is horizontal has been proved in [7, Proposition 3]. However that proof does not work in this case. Assume that $\omega = \zeta^4$.

Then $f$ defined in (2.11) is a solution of Beltrami equation $\overline{f}_z = \omega f_z$ satisfying the initial conditions $f_z(0) > 0$ and $f(0) = q(0) = q'(0) = 0$. Further $f$ maps the unit disk onto the regular hexagon $G$.

Furthermore, for $0 < k < 1$ assume that $\omega_k = k^2 \omega$. Then solve the second Beltrami equation $\overline{f}_z = \omega_k f_z$ that map the unit disk $\mathbb{D}$ onto itself satisfying the initial condition $f(0) = 0$ and $f_z(0) > 0$ [10]. This mapping exists and is unique [4, p. 134]. Then this mapping produces a minimal surface $S_k$ over the unit disk. Moreover for $k = n/(n + 1)$, the sequence $f_n$ converges (up to some subsequence) in compacts of the unit disk, to a mapping $f^\omega$ that maps the unit disk into the unit disk. By using again the uniqueness theorems [2, Theorem B& Theorem 1], because $f^\omega(0) = f(0)$ and $f_z(0) > 0$, it follows that $f^\omega \equiv f$. 

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Let $w_n$ be the point above 0 of minimal surface $S'' = S_{n/(n+1)}$. Then $f_n(0) = 0$. Moreover the second derivative of the Gaussian curvatures $K_n(w_n)$ of $S''$, in view of the formula (1.8), is equal to

$$K''_n(w_n) = \frac{4|q''_n(0)|^2}{|p_n(0)|^4(1 + |q_n(0)|^2)^6}$$

and converges to the second derivative of the Gaussian curvature $K''(w)$. To prove the last fact, observe that $f_n$ and also $f$ are quasiconformal in a disk around 0 and the family is normal. This is why $q_n$ and $p_n$ and $q''_n$ converges in compacts to the corresponding $q$, $p$ and $q''$. This implies that (1.11) cannot be improved.

\[\square\]

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