Multifractal Decompositions using Iterated Function Systems

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Abstract. We analyze two types of multifractal decompositions (MD) of fractals F generated by an Iterated Function System (IFS), they are the geometric and the statistical MD, the first is generated by an IFS and the second by an IFS with probability. In the first, F is decomposed in subsets M(ϕ) of points characterized by the same vector frequency ϕ, and we evaluate their Hausdorff dimension (HD). In the second, F is decomposed in subsets Jα of points with the same pointwise dimension α; however Jα is composed by an infinite subsets M(ϕ), therefore Jα is a multifractal, this implies that its HD is the maximum HD of its components M(ϕ), using a maximizing procedure we find ϕ* such that HD of M(ϕ*) is greater than any other M(ϕ) for a fixed α, this procedure gives in a natural form the auxiliary functions proposed by Cawlin and Mauldin. Thus we present a more simple description of the MD.

1. Introduction

The Iterated Function System (IFS) is a powerful tool for study the fractals [1], when it is conformed by contractive functions it has associated a unique fractal F which is invariant under the Hutchinson operator [2], furthermore this is the attractor of the IFS. When F is totally disconnected, each point of F could be put in a one-to-one correspondence with the elements of the code space [1], then F can be decomposed in an infinite number of fractals M(ϕ) characterized by its frequency, and given the Hausdorff dimension (HD) of them, we obtain the geometrical decomposition of F.

When a probability is associated to each function of the IFS, F is composed by multifractals instead of fractals, without taking into account this fact Cawley and Mouldin [3] made the statistical decomposition defining a convenient function (see (26)). In this work we show how is obtained that expression taking into consideration that the components of F are multifractals: In order to find the HD of these components we use a maximizing procedure, as a result of this procedure we obtain the function proposed by Cawlin and Mauldin.

2. The geometrical multifractal decomposition

Let the Iterated Function System (IFS) be the set of contractive functions \( \{X, w_0(x), w_1(x), \ldots, w_s(x)\} \) with contraction factors \( r_0, r_1, \ldots, r_s \), the Hutchinson’s operator (HO)
of the IFS is defined as $W(A) = \bigcup_{n=0}^{\infty} w_n(A)$, the HO has a fractal $F$ as its fixed set, i.e. $W(F) = F$, if the functions $w_j(x)$ satisfy that $w_j(F) \cap w_j(F) = \emptyset$, with $i \neq j$, then $F$ is a self-similar fractal with contraction factors $\{r_0, r_1, \ldots, r_{s+1}\}$ and its Hausdorff Dimension $D$ is given by the unique solution of
$$\sum_{j=0}^{s+1} r_j^D = 1.$$  

A subset of $F$ is obtained when $w_j$ is applied to the fractal $F$, it is called a fractal cylinder of first order (1-cylinder), and it is denoted by $C_j = w_j(F)$, its diameter is $\Delta(C_j) = \Delta(w_j(F)) = r_j \Delta(F)$, where $\Delta(F)$ is the diameter of $F$, thus applying $K$ times the HO to $F$ in a iterative way, the fractal is separated into $s$ cylinders of first order, $s^2$ cylinder of second order, $s^K$ cylinder of K-order, i.e.

$$F = W(F) = \bigcup_{z_i=0}^{s-1} C_{z_i};$$
$$F = W^2(F) = W \circ W(F) = \bigcup_{z_i=0}^{s-1} \bigcup_{z_j=0}^{s-1} C_{z_i}z_j;$$
$$F = W^K(F) = W \circ \cdots \circ W(F) = \bigcup_{z_i=0}^{s-1} \bigcup_{z_j=0}^{s-1} \cdots \bigcup_{z_{s-1}=0}^{s-1} C_{z_i}z_jz_{s-1};$$

(1)

each $K$-cylinder is characterized by a sequence $\sigma_K = \{z_1, z_2, \ldots, z_K\}$ with $z_i \in \{0,1,\ldots,s-1\}$ we classify these cylinders by the frequency of the digits $\{0,1,\ldots,s-1\}$ in $\sigma_K$, denoting by $N_j(K)$ the times that the digit $j$ appears in $\sigma_K$, the frequency vector of a $K$-cylinder is given by:

$$f(\sigma_K) = (f_0(\sigma_K), f_1(\sigma_K), \ldots, f_{s-1}(\sigma_K)) \quad \text{with} \quad f_j(\sigma_K) = \frac{N_j(\sigma_K)}{K}$$

(2)

Let $M(\{f(K)\})$ the set of K-cylinder with the same frequency vector, i.e.

$$M(\{f(K)\}) = \{C_{\sigma_K} | f(\sigma_K) = f(K)\},$$

its cardinality is given by:

$$\text{Card } M(\{f(K)\}) = \frac{K!}{N_0(K)! \cdot N_1(K)! \cdot \cdots \cdot N_{s-1}(K)!} = \left[ f_0(K)^{N_0(\sigma_K)} f_1(K)^{N_1(\sigma_K)} \cdots f_{s-1}(K)^{N_{s-1}(\sigma_K)} \right]!$$

(3)

and the diameter of these cylinders are

$$\Delta(C_{\sigma_K}) = r_0^{N_0(\sigma_K)} r_1^{N_1(\sigma_K)} \cdots r_{s-1}^{N_{s-1}(\sigma_K)} \Delta(F)$$

(4)

therefore the box dimension of $M(f(K))$ is:

$$\text{Dim}_B(M(f(K))) = \frac{\sum_{j=0}^{s-1} f_j(K) \ln f_j(K)}{\sum_{j=0}^{s-1} f_j(K) \ln r_j}$$

(5)

Taking the limit when $K$ goes to infinity, we obtain the geometrical multifractal decomposition of $F$, because in this limit $\sigma_K$ gives the address of one point of $F$ and the $K$-cylinder goes to a one point of the fractal $F$, thus the set $M(f(K))$ goes to the set of points with the same vector frequency of its address, and the box counting dimension of $M(f(K))$ goes to the Hausdorff dimension of these points. Let $x$ be a point of $F$, with its address given by:

$$x = 0.z_1z_2\ldots z_K = \lim_{K \to \infty} w_{z_1} \circ w_{z_2} \circ \cdots \circ w_{z_K}(F)$$

(6)

then its point frequency vector is
\[ \tilde{q}(x) = \lim_{K \to \infty} \left( q_0(x), q_1(x), \ldots, q_{s-1}(x) \right) \] with \[ \varphi_j(x) = \lim_{K \to \infty} \frac{N_j(K)}{K} \] the set of points with the same frequency vector would be \[ M(\tilde{q}(x)) = \lim_{K \to \infty} M(f(x)) \] with a Hausdorff dimension:

\[ \text{Dim}_H \left( M(\tilde{q}(x)) \right) = \frac{\sum_{j=0}^{s-1} \varphi_j(x) \ln \varphi_j(x)}{\sum_{j=0}^{s-1} \ln \varphi_j(x)} \] where \( \bar{a} \cdot \ln \bar{b} = \sum_{j=0}^{s-1} a_j \ln b_j \). As \( F \) is the union of the sets \( M(\tilde{q}(x)) \), and each of them is a fractal, (8) gives the multifractal decomposition of \( F \) in terms of the frequency vector of the point address which belong to \( F \).

From (8) follows that the set \( M(\tilde{q}_N(x)) \) with \( \tilde{q}_N(x) = \{ q_j = r_j^N \} \) has its HD identical with the HD of \( F \), thus this set has the maximum HD of all fractals which conform \( F \),

\[ \text{Dim}_H(F) = \text{Dim}_H \left( M(\tilde{q}_N(x)) \right) \approx \text{Dim}_H \left( M(\tilde{q}(x)) \right) \]

For this property the components of \( \tilde{q}_N(x) \) are called the uniform weights [4].

### 3. The statistical multifractal decomposition

The fractal \( F \) generated by the IFS is the support of a statistical measure \( \mu \) (SM) when each function \( w_j(x) \) has associated a probability \( p_j \), in this case we have an IFS with probabilities (IFSP) defined by:

\[ \left\{ X; w_0(x), w_1(x), \ldots, w_{s-1}(x); p_0, p_1, \ldots, p_{s-1} \right\} \] with \( \sum_{j=0}^{s-1} p_j = 1 \).

The SM of the first order cylinder is \( \mu(C_{x_{i_1}}) = p_{i_1} \), and the SM of a \( K \)-cylinder is:

\[ \mu(C_{x_{i_1} \ldots x_{i_k}}) = \mu(w_{i_1} \circ w_{i_2} \circ \ldots \circ w_{i_k}(F)) = p_{i_1} \cdot p_{i_2} \cdot \ldots \cdot p_{i_k} = p_{i_0} \cdot p_{i_1} \cdot \ldots \cdot p_{i_{s-1}} \]

this SM satisfies the power law \( \mu(C_{a_k}) = \left[ \Delta(C_{a_k}) \right]^\alpha_{a_k} \) with the Holder exponent:

\[ \alpha(\tilde{p}, \tilde{f}(K)) = \alpha_{a_k} = \frac{\ln \mu(C_{a_k})}{\ln \Delta(C_{a_k})} = \frac{\sum_{j=0}^{s-1} N_j(K) \ln p_j}{\sum_{j=0}^{s-1} \ln N_j(K) \ln r_j} \]

The SM of the set \( M(f(K)) \) is given by its cardinality

\[ \text{Card}(M(f(K))) = \left[ \Delta(C_{a_k}) \right]^{\text{Dim}_H(M(f(K)))} \]

times the SM of the \( K \)-cylinder, thus

\[ \mu(M(f(K))) = \left[ \Delta(C_{a_k}) \right]^{\alpha(\tilde{p}, \tilde{f}(K)) \cdot \text{Dim}_H(M(f(K)))} \]

As the SM has the property that \( 0 \leq \mu(A) \leq 1 \) for any set \( A \), then the exponent of the rhs of (14) satisfies that:
\[ \alpha(\bar{p}, \bar{r}(K)) \geq \text{Dim}_M(\bar{r}(K)) \]  

the equality is satisfied when the frequency vector \( f(K) \) is identical with the probability vector \( p \), this condition defines a special set where the SM is curdling [5], because when \( K \to \infty \) all the SM is concentrated on \( M(p) \) as follows from (14).

The statistical decomposition of \( F \) is on subsets with the same value of the Holder exponent (HE), from (12) follows that all the members of \( M(f(K)) \) have the same value of the HE, however the condition \( \alpha_{\alpha_0} = A \) is satisfied when the frequency vector has the property that \( \bar{r} \cdot \ln \bar{p} = A \bar{r} \cdot \ln \bar{r} \), it implies that the projection of \( f \) on the plane generated by the vectors \( \ln p \) and \( \ln r \) is on a line with slope \( A \), this condition is satisfied by a lot of frequency vectors, therefore there are several sets \( M(f(K)) \) with the same value of the HE, and the set \( J_\alpha \) is given by

\[ J_\alpha(K) = \left\{ C_{\alpha_0} \left| \alpha_{\alpha_0} = A \right. \right\} = \bigcup \left\{ M(\bar{r}(K)) \right\} \text{ such that } \bar{r}(K) \cdot \ln \bar{p} = A \bar{r}(K) \cdot \ln \bar{r} \]  

When \( K \) goes to infinity the HE goes to \( \alpha(x) \) the pointwise dimension of the point \( x \in F \), i.e.

\[ \alpha(x) = \lim_{K \to \infty} \frac{\ln \mu(C_{\alpha_0})}{\ln \Delta(C_{\alpha_0})} = \frac{\bar{q}(x) \cdot \ln p}{\bar{q}(x) \cdot \ln r} \]  

and \( F \) is decomposed in sets \( J_\alpha \) conformed by points with the same value of the pointwise dimension:

\[ J_{\alpha_0} = \left\{ x \mid \alpha(x, \bar{p}) = \alpha \right\} \], this is a multifractal set due to the fact that it is the union of the infinite sets \( M(q(x)) \), in symbols:

\[ J_{\alpha_0} = \bigcup \left\{ M(\bar{q}(x)) \right\} \text{ such that } \bar{q}(x) \cdot \ln \bar{p} = \alpha \bar{q}(x) \cdot \ln \bar{r} \]  

As \( J_\alpha \) is a multifractal its Hausdorff dimension is given by the maximum HD of its fractals component sets [6]:

\[ \text{Dim}_H(J_\alpha) = D(\alpha) = \text{Max} \text{ Dim}_H(M(\bar{q}(x))) \]  

4. Maximizing procedure

The HD of \( J_\alpha \) is determined by the frequency vector \( q^*(x) \) which maximize (8) with the following restrictions:

\[ \frac{\bar{q} \cdot \ln \bar{q}}{\bar{q} \cdot \ln \bar{r}} = \alpha \quad \text{and} \quad \sum_{j=0}^{n-1} q^*_j(x) = 1 \]  

This frequency vector can be found maximizing the Lagrange function:

\[ I(\bar{q}) = \frac{\bar{q} \cdot \ln \bar{q}}{\bar{q} \cdot \ln \bar{r}} - \frac{\bar{q} \cdot \ln \bar{p}}{\bar{q} \cdot \ln \bar{r}} - \lambda \sum_{j=0}^{n-1} q^*_j \]  

where \( q \) and \( \lambda \) are the indeterminated Lagrange multipliers. As the derivative of (21) respect to the frequency vector is given by

\[ \frac{\partial I(\bar{q})}{\partial q_k} = \frac{\ln e q^*_k p_k^{q_k}}{\bar{q} \cdot \ln \bar{r}} - \frac{\ln r_k \sum_{j=0}^{n-1} q^*_j \ln q^*_j p_j^{q_j}}{\left(\bar{q} \cdot \ln \bar{r}\right)^2} - \lambda \]  

the components of vector \( q^*(x) \) satisfy that

\[ \lambda q^*_j \cdot \ln \bar{r} = \ln e q^*_k p_k^{q_k} = \left\{ \text{Dim}_H(M(q^*)) - q \alpha(q^*) \right\} \ln r_k \]  

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where we use (8) and (17). As \( q^α(x) \) depends of the undetermined function \( q \), we have that 
\[
α\bigl(q^α\bigr) = α\bigl(q\bigr)
\]
and \( \text{Dim}_M(q^α) = D(α) = D(q) \), and (23) can be rewritten in the form:
\[
λq^α \cdot \ln \bar{r} = \ln e q^α p_k^{-q} - \ln r_k^{p(α)-q(α)} = \ln \frac{e q^α_k}{p_k^{p(α)-q(α)}}
\]
(24)
the lhs of (24) depends of all the components of \( q^α(x) \) and the rhs is a function of only the \( k \)-component, it implies that the rhs is a constant, which is selected as the unity, then we have that
\[
q^α_k = p_k r_k^{p(α)} \quad \text{with} \quad β\bigl(q\bigr) = D\bigl(q\bigr) - qα\bigl(q\bigr)
\]
(25)
and by (20b) it satisfies:
\[
\sum_{k=0}^{l-1} p_k r_k^{β(α)} = 1
\]
(26)
This is the auxiliary function defined by Cawley and Mouldin [3]. Taking the implicit differentiation of this equation is obtained that
\[
\frac{dβ(α)}{dq} = \frac{-\bar{q}^α \cdot \ln \bar{p}}{q^α \cdot \ln \bar{r}} = -α\bigl(q\bigr)
\]
(27)
and
\[
\frac{d^2β(α)}{dq^2} = -\frac{1}{q^α \cdot \ln \bar{r}} \sum_{k=0}^{l-1} q^α_k \left( \ln p_k - α\bigl(q\bigr) \ln r_k \right)^2
\]
(28)
on the other hand from the definition of \( β(α) \) this derivative is given by
\[
\frac{dβ(α)}{dq} = \left( \frac{dD}{dα} - q \right) \frac{dα}{dq} - α
\]
(29)
Combined equations (27) and (29) we have that:
\[
\frac{dD}{dα} = q
\]
(30)

5. Results
The results given by (27) and (30) imply that \( D(α) \) and \( β(q) \) are a couple of Legendre transforms, therefore both functions contains the same information, thus for find the dimension spectrum \( D(α) \); first \( β(q) \) is founded solving (26); second \( α(q) \) is evaluated using (27); third \( D(q) \) is obtained from (25); fourth \( D(α) \) is obtained when the parameter \( q \) is eliminating for \( D(q) \) and \( α(q) \).

There is a special case when
\[
p_k = r_k^{C} \quad \text{for all} \quad k, \quad \text{as} \quad \sum_{k=0}^{l-1} p_k = 1
\]
(31)
then \( C \) corresponds with \( D \) the Hausdorff dimension of \( F \).

Using this result in (17) the value of pointwise dimension is \( α=D \), therefore all the points \( x \in F \) have the same pointwise dimension, this implies that \( β\bigl(q\bigr) = D - qD \) introducing into (25), we have
\[
q^α_k = r_k^{D} = p_k
\]
(32)
this is the frequency vector \( \bar{q}_N\bigl(x\bigr) \).

The behavior of the function \( β(q) \) is the following: from (27) we conclude that \( β\bigl(q\bigr) < 0 \), so that \( β(q) \) is strictly decreasing, from (28) we have that \( β\bigl(q\bigr) > 0 \) so that \( β(q) \) is a convex function.

On the other hand, we obtain the behavior of the spectrum dimension \( D(q) \) from equations (25), (27) and (28):
\[ D'(q) = q\alpha'(q) = -q\beta'(q) \] (33)

therefore \( D(q) \) is strictly increasing from \(-\infty\) to 0 and strictly decreasing from 0 to \(\infty\).

6. Conclusions
We give a theoretical support of the relation (23) proposed by Cawlin and Mauldin, it determines implicitly the function \( \beta(q) \) which describes the multifractal decomposition. We show that if a probability vector is associated with an IFS, then the decomposition is given in terms of multifractals \( J_\alpha \), thus find their corresponding Hausdorff dimension requires the introduction of an extremal principle which implies the equation (23).

The multifractal decomposition is used in geophysics, turbulence, heartbeat dynamics [7], and recently it has been a very useful approach to study problems related with genomics [8], as the study of human genome [9].

References
[1] Barnsley M F 1986 Fractals Everywhere (Boston: Academic Press)
[2] Hutchinson J 1981 Fractals and self-similarity Ind. J. Math 30 713-747
[3] Cawley R and Mauldin R D 1992 Multifractal decompositions on moran fractals Adv. Math. 92 196-236
[4] Edgar G A 1998 Integral, Probability, and Fractal Measures (New York: Springer Verlag)
[5] Mandelbrot B 1982 The Fractal Geometry of Nature (New York: W. H. Freeman)
[6] Billingsley P 1965 Ergodic Theory and Information (New York: John Wiley)
[7] Plamen C I, Golberger A L and Stanley H E 2002 Fractal and Multifractal Approaches in Physiology in The Science of Disasters eds Bunde A, Kropp J and Schellnhuber (Berlin: Springer) pp 219-257
[8] Gutiérrez J M, Rodríguez M A and Abramson G 2001 Multifractal analysis of DNA sequences using a novel chaos-game representation Physica A 300 271–284
[9] Moreno et al 2011 The human genome: a multifractal analysis BMC Genomics 12 506