Random walks with long-range self-repulsion on proper time

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Abstract

We introduce a model of self-repelling random walks where the short-range interaction between two elements of the chain decreases as a power of the difference in proper time. Analytic results on the exponent $\nu$ are obtained. They are in good agreement with Monte Carlo simulations in two dimensions. A numerical study of the scaling functions and of the efficiency of the algorithm is also presented.
1 Introduction

Self-avoiding walks are a fascinating subject. Although they are much simpler than an Ising or Heisenberg spin system, they behave quite similarly in the critical region where the coherence length is much larger than the lattice spacing. There are deep reasons for this similarity. Indeed, using Symanzik representation [1, 2], the free energy of a spin system may be written as sum over partition functions of self-avoiding walks. Moreover the self-avoiding walks are exactly the limit of an $O(n)$-invariant spin system for $n$ going to zero [3, 4, 5, 6].

A long time ago Flory [7], using simple minded approximations, found that the critical exponent $\nu$ in dimension $D$ less than or equal to 4, was given by the simple expression

$$\nu = \frac{2}{3 + D}$$  \hspace{1cm} (1)

This result is quite puzzling; it is exact for $D = 1, 2$ and 4, but it is definitely wrong in $D = 3$ [3, 8, 9, 10, 11]. Moreover in $4 - \epsilon$ dimensions the exact results and Flory’s one differ at first order in $\epsilon$ [3]. The Flory approximation was strongly criticized by des Cloiseaux [12, 13], but he found the puzzling result that a more accurate variational approximation leads in three dimensions to the rather bad value of $\nu = 2/3$.

Quite recently it was found that des Cloiseaux’ method can be extended to the case of polymers with long-range repulsion [14]. Also in this case Flory’s and des Cloiseaux’ approaches give different results, however here it is possible to give very strong arguments suggesting that the variational approach of des Cloiseaux gives the exact result in the case where the space dimension becomes infinite.

The putative exact infinite-dimensional results can be extrapolated at finite dimensions using arguments based on the renormalization group and performing the approximation directly at finite dimension. One finally finds that for an interaction which decreases with the distance as a power law with exponent $\lambda$, $\nu$ is given by the generalization of des Cloiseaux’ formula [14]

$$\nu = \begin{cases} 
1 & \text{for } \lambda \leq 2 \\
2/\lambda & \text{for } 2 \leq \lambda \leq 4 \\
1/2 & \text{for } \lambda \geq 4 
\end{cases}$$ \hspace{1cm} (2)

with logarithmic corrections when $\lambda = 2, 4$. 

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Some of these predictions have been numerically tested \[15\].

A very different case concerns a potential which is short-range in space but in which the interaction among different elements of the polymer decreases as a power of the distance along the chain. An interesting theoretical analysis of this model can be done and the predictions can be tested using numerical methods due to the short-range nature of the Hamiltonian.

After this introduction we present the model in section 2. An approximate computation of the exponent $\nu$ is presented in section 3. The algorithm we use for numerical simulations is described in section 4, while the results we obtain are presented in section 5. Some conclusions are presented in section 6.

2 The model

Let $\mathcal{C}$ be the ensemble of random walks on the hypercubic lattice $\mathbb{Z}^D$ of $N$ steps, starting from the origin. Thus, if $\omega \in \mathcal{C}$

$$\omega = \{ \vec{\omega}_0, \vec{\omega}_1, \vec{\omega}_2, \ldots, \vec{\omega}_N \} \quad (3)$$

with $\vec{\omega}_0 = \vec{0}$ the origin of the lattice, $\vec{\omega}_i \in \mathbb{Z}^D$ the location at time $i$ of the walk and $|\vec{\omega}_i - \vec{\omega}_{i-1}| = 1$ for $i = 1, \ldots, N$.

We consider an interaction of the form

$$H_I[\omega] = \frac{1}{2} g N^\delta \sum_{i,j=0, i \neq j}^{N} \delta_{\vec{\omega}_i, \vec{\omega}_j} |i-j|^\lambda \quad (4)$$

This means that in the ensemble average $\langle \cdot \rangle$ in $\mathcal{C}$ we associate to each walk the statistical weight

$$m[\omega] = \frac{e^{-H_I[\omega]}}{Z_N[g]} \quad (5)$$

where

$$Z_N[g] = \sum_{\omega \in \mathcal{C}} e^{-H_I[\omega]} \quad (6)$$

To study the conformation of the walks in this ensemble we shall consider the square end-to-end distance

$$R_e^2[\omega] \equiv \vec{\omega}_N^2 \quad (7)$$
and the square radius of gyration

\[ R^2_g[\omega] \equiv \frac{1}{N + 1} \sum_{i=0}^{N} \left( \vec{\omega}_i - \frac{1}{N + 1} \sum_{j=0}^{N} \vec{\omega}_j \right)^2 \] (8)

Both of them are believed to have the asymptotic behaviour

\[ \langle R^2 \rangle \sim N^{2\nu} \] (9)

as \( N \to \infty \), with the same critical exponent \( \nu \).

We shall also consider the universal ratio

\[ A \equiv \frac{\langle R^2_g \rangle}{\langle R^2_e \rangle} \] (10)

For the ORW we have \( A = 1/6 \sim 0.16667 \), while for the SAW \( A \) is known only numerically and depends on the dimension of the embedding space. In two dimensions the most precise estimate is obtained by a Monte Carlo simulation [16] whose result is \( A = 0.14026 \pm 0.00011 \), where the error bar is 95% level of confidence.

3 Heuristic analysis

In this Section we want to derive an estimate of the critical exponent \( \nu \) with a variational approach. Let us firstly discuss the case \( \delta = 0 \) [14].

Following Des Cloiseaux [12] one considers random rings instead of random walks, as this fact does not change the value of \( \nu \) and simplifies the calculations by permitting Fourier analysis along the chain. In addition it is simpler to work in continuum space rather than on a lattice. In this case one must somehow regularize the \( \delta \)-function appearing in (4). We will thus consider as equilibrium probability measure for the model

\[ dm(\omega) = \frac{1}{Z} \exp(-H) d^D\vec{\omega}_1 d^D\vec{\omega}_2 \ldots d^D\vec{\omega}_N \] (11)

where the Hamiltonian is given by

\[ H = \frac{1}{2} \sum_{i=0}^{N} (\vec{\omega}_i - \vec{\omega}_{i-1})^2 + \frac{1}{2} g \sum_{i,j \neq 0}^{N} \frac{V \left[ (\vec{\omega}_i - \vec{\omega}_j)^2 \right]}{|i-j|^{\lambda}} \] (12)
Here $V(x^2)$ can be any arbitrary short-range potential and for definiteness we will assume

$$V(x^2) = \exp \left( -\frac{x^2}{2a} \right)$$

(13)

The mean field approximation is based on a variational approach with a Gaussian trial measure [12, 14]

$$d m_0(\omega) = \frac{1}{Z_0} \exp(-H_0) d^D \bar{\omega}_1 d^D \bar{\omega}_2 \ldots d^D \bar{\omega}_N$$

(14)

with

$$H_0 = \frac{1}{2} \sum_{i,j=0}^N G_{ij}^{-1} \bar{\omega}_i \cdot \bar{\omega}_j$$

(15)

and

$$Z_0 = (2\pi)^{(N+1)D/2} (\det G)^{D/2}$$

(16)

The function $G_{ij}$ is determined by minimizing the functional

$$F[G] = \langle H - H_0 \rangle_0 - \log Z_0$$

(17)

where $\langle \rangle_0$ denotes the expectation value with respect to $d m_0$.

Because of invariance under translations along the chain we have $G_{ij} = G(i-j)$ and by symmetry $G(n) = G(-n)$. It is also convenient to introduce the Fourier transform

$$\tilde{G}(p) = \sum_{\tau=0}^{N-1} G(\tau)e^{ip\tau}$$

(18)

where $p = 0, 2\pi/N, \ldots, 2\pi(N-1)/N$, and $\tilde{G}(p)$ is real and positive due to the positive definiteness of $G_{ij}$.

The functional $F$ can be computed as

$$\frac{F[G]}{N} = D \int_{-\pi}^{\pi} \frac{dp}{2\pi} \tilde{G}(p) \left( 1 - \cos p \right) - \frac{D}{2} \left[ 1 + \log(2\pi) \right] - \frac{D}{2} \int_{-\pi}^{\pi} \frac{dp}{2\pi} \log \tilde{G}(p)$$

$$+ g \sum_{\tau=1}^{\infty} \frac{1}{\tau^\lambda} \left[ 1 + \frac{2}{a} \int_{-\pi}^{\pi} \frac{dp}{2\pi} \tilde{G}(p) \left( 1 - \cos p \tau \right) \right]^{-\frac{D}{2}}$$

(19)

where we have replaced sums over $p$ with the corresponding integrals as we are interested in the regime of very large $N$.
The minimization condition becomes
\[ \frac{1}{G(p)} = 2(1 - \cos p) \tag{20} \]
\[ \int \frac{dp}{2\pi} \tilde{G}(p) (1 - \cos p) \]\
\[ \sum_{\tau=1}^{\infty} \frac{1}{\tau^\lambda} (1 - \cos p\tau) \left[ 1 + \frac{2}{a} \int_{-\pi}^{\pi} \frac{dp'}{2\pi} \tilde{G}(p') (1 - \cos p'\tau) \right]^{-\frac{D+2}{2}} \]

The exponent \( \nu \) can be extracted, as easily seen, from the low-momentum behaviour of \( \tilde{G}(p) \) as
\[ \tilde{G}(p) \sim \frac{1}{|p|^{2\nu+1}} \tag{21} \]

The analysis of (20) is quite subtle and can be done following the original paper by Des Cloiseaux \[12\] and predicts \( \nu = 1/2 \) for \( D > 4 \) while for \( 2 < D < 4 \) gives the following result \[14\]

\[ \nu_{MF}(\lambda, D) = \begin{cases} 1/2 & \text{for } \lambda > \frac{1}{2}(4-D) \\ \frac{1}{D}(2-\lambda) & \text{for } \lambda < \frac{1}{2}(4-D) \end{cases} \tag{22} \]

For \( \lambda = 0, D = 2 \) additional logarithmic corrections appear as
\[ \langle R_g^2 \rangle \sim N^2 / \log N \tag{23} \]
and analogously for \( \lambda = \frac{1}{2}(4-D) \) we get
\[ \langle R_g^2 \rangle \sim N(\log N)^{2/D} \tag{24} \]

Thus we obtain an ORW for \( \lambda \) large enough. This has indeed to be expected, as we have the trivial rigorous bound
\[ H_I(\omega) < CN^{2-\lambda} \tag{25} \]
showing that at least for \( \lambda > 2 \) the model is an ORW.

On the other hand the mean-field result \( \nu_{MF} \) is definitely wrong for \( \lambda = 0, D \leq 4 \), which corresponds to the SAW limit \[17\] \[12\]. In two and three dimensions \( \nu_{SAW} \) is equal respectively to 3/4 and approximately 3/5 (see [4]) while \( \nu_{MF}(0) \) corresponds to 1 and 2/3. For \( D = 4 \) the mean-field approach correctly predicts logarithmic corrections to the random-walk behaviour, but
the power of the logarithm is higher than that obtained using the renormalization group which predicts [18]

$$\langle R^2_g \rangle \sim N(\log N)^{1/4}$$

(26)

Of course the variational approach will overestimate $\nu$ also for small $\lambda$. We do not have any theoretical control of this regime. However as we will discuss in Section 5, our numerical results are in reasonable agreement with the following conjecture for $\nu(\lambda, D)$

$$\nu(\lambda, D) = \min (\nu_{SAW}, \nu_{MF}(\lambda, D))$$

(27)

Let us finally notice that for large $N$, the mean-field equation (20) has a general scaling solution. Indeed in the limit of small $p$, which corresponds to large $N$ we can rewrite (20) as

$$\frac{1}{G(p)} = p^2$$

(28)

$$-g \frac{1}{2a} p^2 \int_1^\infty d\tau \tau^{2-\lambda} \left[ \frac{2}{a} \int_{-\pi}^\pi dp' \tilde{G}(p') \left( 1 - \cos p' \tau \right) \right]^{-\frac{D+2}{2}}$$

It is easy to check that the general solution has the form

$$\tilde{G}(p) = \frac{1}{p^2} \hat{f}_\lambda(g p^{\lambda-2+D/2})$$

(29)

which gives

$$\langle R^2_g \rangle = N \hat{f}_\lambda(g N^{2-\lambda-D/2})$$

(30)

Let us now discuss the case $\delta \neq 0$. For simplicity we shall set since now on $D = 2$, the generalization to arbitrary $D$ being trivial.

Let us firstly consider the case in which $\delta < 0$, namely the case in which we are weakening the coupling constant when approaching the asymptotic limit. One can think at the model in the intermediate region in a perturbative expansion around the ORW, that is around $g = 0$. To evaluate the dimension of the coupling constant, remark that $N^{2-\nu D}$ is the asymptotic behaviour for large $N$ of the average number of intersections of two walks of $N$ steps and Hausdorff dimension $1/\nu$ on a lattice of dimension $D$. Then

$$[g] = -[N](\delta - \lambda + 2 - \nu_{ORW} D) = 2 + 2 \delta - 2 \lambda$$

(31)
and

\[ [N] = -\frac{1}{\nu_{\text{ORW}}} = -2 \]  \hspace{1cm} (32)

Thus in the scaling region we expect that for \( gN^\delta \) small

\[ \langle R_g^2 \rangle = N F_\lambda(g, N) = N f_\lambda(g N^{1+\delta-\lambda}) \]  \hspace{1cm} (33)

Let us notice that for \( \delta = 0 \) this expression coincides with the scaling formula (30). For \( g = 0 \) the model is an ORW and thus we get \( f_\lambda(0) \neq 0 \). It follows that, when \( \delta < \lambda - 1 \) \( \langle R_g^2 \rangle \) scales as \( N \). On the other hand when \( \lambda - 1 \leq \delta < 0 \) the argument of the scaling function \( f_\lambda \) goes to infinity. Assuming in this limit \( f_\lambda(x) \sim x^\beta \) we get

\[ \langle R_g^2 \rangle \sim N^{1+\beta(1+\delta-\lambda)} \]  \hspace{1cm} (34)

The value of \( \beta \) is computed using the conjectured value for \( \nu \) when \( \delta = 0 \). In this way we obtain that \( f_\lambda(x) \) scales for large \( x \) as

\[ f_\lambda(x) \sim \begin{cases} x^{2\nu_{\text{SAW}}^{-1} - 1 - \frac{\delta}{2(1-\lambda)}} & \text{when } 0 \leq \lambda \leq 1/2 \\ x^{2(\nu_{\text{SAW}} - 1)} - \frac{\delta}{2(1-\lambda)} \sim x & \text{when } 1/2 \leq \lambda \leq 1 \end{cases} \]  \hspace{1cm} (35)

Then we obtain the prediction

\[ \langle R_g^2 \rangle \sim \begin{cases} N^{\frac{1}{2} + \frac{\delta}{2(1-\lambda)}} & \text{when } 0 \leq \lambda \leq 1/2, \delta \leq \lambda - 1 \\ N^{2+\delta-\lambda} & \text{when } 1/2 \leq \lambda \leq 1, \delta \leq \lambda - 1 \\ N & \text{when } \lambda \geq 1 \text{ and } \lambda < 1, \delta > \lambda - 1 \end{cases} \]  \hspace{1cm} (36)

In the opposite situation (\( \delta > 0 \)) the coupling constant goes to infinity while approaching the asymptotic limit and we can expect that in this case the scaling behaviour is controlled by the SAW fixed point. Then

\[ [N] = -\frac{1}{\nu_{\text{SAW}}} = -\frac{4}{3} \]  \hspace{1cm} (37)

and

\[ [g] = -[N](\delta - \lambda + 2 - \nu_{\text{SAW}} D) \]  \hspace{1cm} (38)

so that we expect that for \( gN^\delta \) large

\[ \langle R_g^2 \rangle = N^{2\nu_{\text{SAW}}} \tilde{F}_\lambda(g, N) = N^{\frac{3}{2}} f_\lambda(g N^{\frac{1}{2}+\delta-\lambda}) \]  \hspace{1cm} (39)
Since for $g$ going to infinity at fixed $N$ we expect the model to describe a SAW, $\tilde{f}_\lambda(x)$ must converge to a constant for large $x$. It immediately follows that for $\delta > \lambda - \frac{1}{2}$ the behaviour is that of a SAW. On the other hand, when $\delta < \lambda - 1/2$ and $\lambda > 1/2$, the argument of the scaling function goes to zero when $N$ goes to infinity. As for $g = 0$ we have an ORW, we must have $\tilde{f}_\lambda(0) = 0$. Then, assuming for small $x$ the scaling form $\tilde{f}_\lambda(x) \sim x^\beta$ we get

$$\langle R_g^2 \rangle \sim N^{3/2 + \beta(1/2 + \delta - \lambda)}$$

This exponent $\beta$ is computed by requiring that for $\delta = 0$ this formula reproduces the result (27). We thus get that the scaling function $\tilde{f}_\lambda$ must scale for small argument as

$$\tilde{f}_\lambda(x) \sim \begin{cases} x^{\frac{4(\nu(\lambda) - \nu_{\text{SAW}})}{1 - 2\lambda}} & \text{when } 1/2 \leq \lambda \leq 1 \\ x^{\frac{4(\nu_{\text{ORW}} - \nu_{\text{SAW}})}{1 - 2\lambda}} & \text{when } \lambda \geq 1 \end{cases}$$

Then for positive $\delta$ we get

$$\langle R_g^2 \rangle \sim \begin{cases} N^{3/2} & \text{when } 0 \leq \lambda \leq 1/2 \text{ and } \lambda > 1/2, \delta > \lambda - 1/2 \\ N^{2 + \delta - \lambda} & \text{when } 1/2 \leq \lambda \leq 1, \delta \leq \lambda - 1/2 \\ N^{1 + \frac{\delta}{2\lambda - 1}} & \text{when } \lambda \geq 1 \delta \leq \lambda - 1/2 \end{cases}$$

Let us notice that all these scaling arguments do not take into account logarithmic corrections which we expect to be present for those values of $\delta$ where there is the transition to the purely SAW or ORW behaviour.

4 The algorithm

We have simulated the model by using the so-called pivot algorithm [19, 20, 21], which is known to be extremely efficient for the simulation of SAWs with fixed number of steps and free end-points as the computer time necessary to produce an independent walk is of the order of the number of steps in the walk, which is also the best possible behaviour because this is the order of time necessary simply to write down all the steps.

The algorithm is defined as follows [21]. Choose at random a point along the walk (the pivot), but not the first or the last one. Let it be the $k$-th point
\( \vec{\omega}_k \), with \( 0 < k < N \). Then choose at random an element \( g \) in the symmetry group of the lattice and propose a new walk \( \omega' \) defined by

\[
\vec{\omega}'_k = \begin{cases} 
\vec{\omega}_i & \text{for } 0 \leq i \leq k \\
\vec{\omega}_k + g(\vec{\omega}_i - \vec{\omega}_k) & \text{for } k + 1 \leq i \leq N
\end{cases}
\] (43)

The new walk is accepted according to a Metropolis test in order to generate the desired statistical ensemble.

For the limiting case of ORW and SAW it is known [21] that for a global observable \( A \) the integrated autocorrelation time \( \tau_{int,A} \) scales for large number of steps according to

\[
\tau_{int,A} \sim N^p
\] (44)

where (the exponent \( p \) should be the same for all global variables)

\[
p = \begin{cases} 
0 & \text{for ORW} \\
0.194 \pm 0.002 & \text{for SAW}
\end{cases}
\] (45)

For our models, as it can be seen from Table 1, where we report the integrated autocorrelation time for the end-to-end distance, we obtain similar results. When the values of the parameters are such that the walks are in the same universality class of ORW or SAW, the dynamical behaviour is compatible with (45). In the intermediate cases for which \( 1/2 \leq \nu \leq 3/4 \) the dynamic critical exponent \( p \) ranges correspondingly within the interval \([0, 0.19]\), and we find that it is in reasonable agreement with a linear interpolation of the form

\[
p_\nu = 0.39 (2 \nu - 1)
\] (46)

Let us now come to the computational complexity. In the practical implementation we used a hash-table with linear probing in order to check for the self-intersections of the walk. In the case of SAWs Madras and Sokal [21] showed that it is particularly convenient to insert the points in the hash-table starting from the pivot point and working outward (and of course stopping as soon as an intersection is detected). In this way the mean work per move turns out to be of order \( N^{1-p} \), thus smaller than the work done by inserting the points without a special order which is of order \( N \). We used in our case a similar trick. Indeed, chosen a random number \( r \) uniformly distributed in the unit interval, according to the Metropolis prescription, the proposed walk \( \omega' \) is accepted if

\[
H[\omega'] \leq S \equiv H[\omega] - \ln r
\] (47)
Our implementation works as follows: after the choice of the pivot point and of the transformation \( g \) we choose the random number \( r \) and compute the quantity \( S \). Then we begin to construct the new walk and we insert the points in the hash-table starting from the pivot point and working outward. Whenever we find an intersection we sum to the energy of the new walk the contribution from that intersection, but we stop if the accumulated value is already larger than \( S \), because the proposed walk has to be rejected.\(^1\)

In the limiting case of very strong repulsion, when the universality class is the same of \( \text{SAW} \), this implementation works exactly as the original one devised by Madras and Sokal. In the opposite limiting case, the \( \text{ORW} \)-one, practically all proposed walks will be accepted and thus no improvement can be expected. In the general situation we expect (although we didn’t check) that the computational work scales as \( N^{(1-p)} \) where \( p \) is the dynamic critical exponent for global observables and thus, in all cases, that the computer time necessary to produce a statistically independent measurement is of order \( N \).

5 Numerical results

In order to test the ideas presented in Sect. 3 we have performed an extensive Monte Carlo simulation on a square lattice on walks of lengths ranging from 100 to 8000. The total CPU time for these runs was roughly 2000 hours of a VAX 6000-520.

In the numerical simulation the first problem one has to deal with is the initialization, that is, for each value of \( \lambda, \delta, g \) and \( N \) one has to generate a starting walk. This was done in two different ways according to the value of \( N \). When \( N \) was less than or equal to 2000 we generated a SAW using a dimerization routine \([22, 23, 24, 21]\). When instead \( N = 4000, 8000 \), as the dimerization routine is too costly (the computer time needed to generate a walk scales as \( \tau \sim N^{a \log_2 N + b} \), with \( a \approx 0.17 \) and \( b \approx 0.72 \) in two dimensions), we used the scanning method \([25]\) with scanning parameter equal to 3.\(^2\) None

1 This procedure works because the interaction is repulsive. For attractive interactions, as for instance for \( \text{SAWs} \) or trails at the \( \theta \)-point, the whole new walk must be defined and therefore the mean work per pivot-move will scale as the number of step \( N \).

2 The only reason why we used the dimerization routine for the low-\( N \) runs was its
of these two methods generates a random sample of walks with the correct equilibrium distribution, although the first is exact in the SAW-limit and the second in the ORW-one. It was thus necessary to run a certain number of thermalization iterations before measuring. As the convergence to equilibrium of a Markov chain is controlled by the exponential autocorrelation time $\tau_{\text{exp}}$ it is necessary to run a few $\tau_{\text{exp}}$ iterations to reach equilibrium. For the pivot algorithm $\tau_{\text{exp}}$ is proportional to $N$. For this reason we ran approximately $10N$ pivot iterations for thermalization before measuring. For each value of $\lambda$, $\delta$, $g$ and $N$ we have then performed $10^6$ iterations, except when $N = 100$ in which case the runs consist of $2 \times 10^6$ iterations. The integrated autocorrelation times for the squared end-to-end distance and for the square radius of gyration range from 3 to 20 and from 5 to 60 respectively. We have reported a few of them in Table 1.

In Table 3 and Table 4 we report the results of our runs for $\langle R_e^2 \rangle$ for different values of the parameters, respectively for negative and positive values availability at the time of the runs. In retrospect it would have probably been better to use the scanning program in all cases.
of $\delta$.

We have performed least-squares regressions on these data in order to
determine the critical exponent $\nu$. We fit $\langle R_e^2 \rangle$ to the Ansatz $aN^{2\nu}$ by per-
forming a weighted least-squares regression of its logarithm against $\log N$,
using the a priori error bars on the raw data points to determine both the
weights and the error bars. In order to control the systematic error due to
corrections to scaling we have done various fits in which the data points with
lowest $N$ were discarded. The results of these fits are reported in Table 4
and Table 5 for various $N_{\text{cut}}$, where $N_{\text{cut}}$ is the minimum $N$ included in the
fit. In many cases it is evident a systematic drift of the estimated exponent
with $N_{\text{cut}}$, an indication of strong corrections to the scaling. This effect is
however strongly dependent on the value of $g$. When the expected value of
$\nu$ is different from $1/2$ and $3/4$ one observes that for $g$ small the estimated
value of $\nu$ increases with $N_{\text{cut}}$, while for $g$ large the estimate decreases. For
the intermediate values of $g$ there is a flatness region where $\nu$ remains approx-
imately constant meaning that for these values the $g$-dependent corrections
are small compared to our statistical error. For every $\lambda$ and $\delta$ we have made
various runs for different values of $g$ in order to find the flat region were the
corrections are small enough, in order to obtain estimates of $\nu$ with a smaller
systematic error. Our final estimates are in reasonable agreement with the
proposed value of $\nu$, the discrepancy being less than a few per cent. The
worst cases are for $\lambda = 1$ and $\lambda = 0.5$. For instance for $\delta = 1/3$, $\lambda = 1$ our
data suggest $\nu \approx 0.68, 0.69$ while the expected value is $\nu = 2/3 \approx 0.667$.
Let us notice however that for $\lambda = 1$ and $\lambda = 0.5$ we expect the presence of
logarithmic corrections, i.e. a behaviour of the form

$$\langle R_e^2 \rangle_N = aN^{2\nu} \log^\beta N (1 + O(1/\log N, \log \log N/\log N))$$

The presence of logarithms makes the analysis very difficult. First of all
in this case the convergence to the asymptotic regime is extremely slow.
Moreover the presence of the term $\log^\beta N$ makes impossible an evaluation of
$\nu$. Indeed as we use data with $200 \leq N \leq 8000$, $\log^\beta N$ behaves, as far as the
fit is concerned, approximately as $N^{\approx 0.3\beta}$. Thus in a pure power-law fit one
really measures $2\nu + 0.3\beta$. Since we don’t have any theoretical knowledge of
$\beta$ it is thus impossible to draw any definite conclusion.

To better understand the validity of our conjecture (22) we have made
two runs with higher statistics at $\lambda = 0.75$ and $\lambda = 0.25$ with $\delta = 0$. Each
data point corresponds here to $9 \times 10^6$ iterations and in order to avoid any initialization bias we have discarded the first 100 $N$ iterations. The results are reported in Table 2. A good agreement is seen although a systematic trend is visible in both cases.

We have then checked if $\langle R^2_e \rangle$ obeys the scaling laws (30) and (39). In Fig. 1 we plot our estimates of $\langle R^2_e \rangle/N$ versus $N^{1+\delta-\lambda}$ for $\delta < 0$ and $\lambda = 0.00, 0.25, 0.50, 0.90$ with $N > 200$. The agreement seems quite good. However a more detailed examination of the scaling plot shows that the data points do not belong to a unique curve within error bars. Indeed one sees that different runs with the same values of $g$ and $\delta$ belong to distinct curves which approach each other only when $N$ becomes large. This fact has to be expected. Indeed, as the analysis of the exponent $\nu$ shows, for the values of $N$ which we are considering the corrections to scaling are still large and thus we expect analogously large violations to the scaling behaviour given by (39). Analogously in Fig. 2 we plot our estimates $\langle R^2_e \rangle/N^{3/2}$ versus $N^{1/2+\delta-\lambda}$ for $\delta < 0$ and $\lambda = 0.75, 0.90$ and $N > 200$. Here again the agreement is only approximate. The situation is even worse for $\lambda = 1.00$ and $\delta > 0$: in this case the points are scattered and no scaling can be observed. This can be explained by the presence of logarithmic terms, which break the scaling laws and make the approach to the asymptotic regime extremely slow. As a final check we have studied the behaviour of the universal ratio $A$, which, using

|   | $R^2_e$ | $\nu$   | $R^2_e$ | $\nu$   |
|---|---------|---------|---------|---------|
|   | $\lambda = 0.25$ |         | $\lambda = 0.75$ |         |
| 500 | 4862(10) | 0.7433(11) | 1874(4) | 0.6238(6) |
| 1000 | 13554(31) | 0.7447(17) | 4449(10) | 0.6239(8) |
| 2000 | 37927(91) | 0.7460(27) | 10556(25) | 0.6242(13) |
| 4000 | 106704(277) | 0.7500 | 25073(60) |         |
| 8000 | 300005(874) | 0.6250 | 59567(155) |         |

Table 2: Results for the runs with higher statistics and comparison with the expected value for the exponent $\nu$. In both cases $\delta = 0$ and $g = 1$. 


Figure 1: Scaling function for the end to end distance in the case $\delta < 0$. Small pentagons, rectangles, triangles and diamonds correspond respectively to the values $\lambda = 0.90$, 0.50, 0.25 and 0.
Figure 2: Scaling function for the end to end distance in the case \( \delta > 0 \). Small pentagons and triangles correspond respectively to the values \( \lambda = 0.90 \) and 0.75.
the scaling laws (39) and (39) must have the form

$$A = h_\lambda(gN^{1+\delta-\lambda})$$  \hspace{1cm} (49)

for $\delta < 0$ and

$$A = \bar{h}_\lambda(gN^{1/2+\delta-\lambda})$$  \hspace{1cm} (50)

for $\delta > 0$. Moreover for $x \to 0$ both functions must converge to the ORW-value $1/6$. In Fig. 3 and 4 we present the scaling plots for $A$ in the two cases. The agreement is reasonable, although a closer inspection shows again the presence of systematic deviations.

6 Conclusions

In conclusion the results of our simulations are in good agreement with the theoretical arguments of Sect. 3, suggesting that our conjectured value for $\nu$, if not exact, is certainly a very good approximation in the range of parameters we have examined.

A Flory argument would predict for the exponent $\nu$

$$\nu_F = \frac{3 - \lambda + \delta}{2 + D}$$  \hspace{1cm} (51)

It is interesting to remark that such a value is a good approximation, according to our analysis, only for $\lambda = \delta = 0$. The deep reason for such an agreement escapes us.

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Figure 3: Scaling function for the universal ratio $A$ in the case $\delta < 0$. Small pentagons, rectangles, triangles and diamonds correspond respectively to the values $\lambda = 0.90, 0.50, 0.25$ and 0.
Figure 4: Scaling function for the universal ratio $A$ in the case $\delta > 0$. Small pentagons and triangles correspond respectively to the values $\lambda = 0.90$ and $0.75$. 
Table 3: Values of $R^2_e$ for $\delta \leq 0$. All error bars are two standard deviations.
Table 4: Values of $R^2_e$ for $\delta \geq 0$. All error bars are two standard deviations.
| \( \delta \) | g | 100  | 200  | 500  | 1000 | 2000 | 4000 | \( \nu_{\text{th}} \) |
|-----|---|------|------|------|------|------|------|----------------|
| -1  | 10 | 0.5003(8) | 0.5000(12) | 0.4985(19) | 0.4999(30) | 0.4953(59) | 0.5000 |
| -3/4| 2.0| 0.5505(8) | 0.5532(12) | 0.5534(18) | 0.5538(29) | 0.5556(58) | 0.5625 |
| -3/4| 5.0| 0.5588(8) | 0.5585(12) | 0.5584(18) | 0.5578(29) | 0.5575(56) | 0.5625 |
| -3/4| 8.0| 0.5617(8) | 0.5616(12) | 0.5621(18) | 0.5638(28) | 0.5637(57) | 0.6250 |
| -1/2| 1.0| 0.6141(8) | 0.6178(12) | 0.6199(18) | 0.6179(28) | 0.6210(57) | 0.6250 |
| -1/2| 3.0| 0.6246(8) | 0.6255(12) | 0.6253(19) | 0.6269(30) | 0.6270(59) | 0.6250 |
| -1/2| 5.0| 0.6290(8) | 0.6284(12) | 0.6272(19) | 0.6232(30) | 0.6174(62) | 0.6250 |
| -1/4| 1.0| 0.6869(7) | 0.6884(10) | 0.6895(14) | 0.6894(20) | 0.6886(32) | 0.6854(67) | 0.6825 |
| -1/4| 2.0| 0.6902(9) | 0.6901(13) | 0.6894(20) | 0.6915(33) | 0.6942(68) | 0.6825 |
| -1/4| 4.0| 0.6969(9) | 0.6952(14) | 0.6945(22) | 0.6946(35) | 0.6948(72) | 0.6825 |
| \( \lambda = 0.25 \) | | | | | | | |
| -3/4| 10 | 0.5068(8) | 0.5062(12) | 0.5046(19) | 0.5022(30) | 0.5002(60) | 0.5000 |
| -1/2| 5.0 | 0.5807(8) | 0.5810(12) | 0.5797(19) | 0.5787(30) | 0.5844(60) | 0.5833 |
| -1/2| 10 | 0.5897(8) | 0.5872(13) | 0.5869(19) | 0.5873(31) | 0.5866(62) | 0.5833 |
| -1/2| 34 | 0.6329(9) | 0.6177(14) | 0.6051(22) | 0.5958(35) | 0.5919(70) | 0.5919 |
| -3/8| 5.0 | 0.6263(8) | 0.6246(13) | 0.6243(20) | 0.6233(31) | 0.6226(64) | 0.6250 |
| -3/8| 13 | 0.6513(9) | 0.6453(14) | 0.6408(21) | 0.6354(35) | 0.6335(70) | 0.6666 |
| -1/4| 0.7 | 0.6400(8) | 0.6439(12) | 0.6476(19) | 0.6457(30) | 0.6520(60) | 0.6666 |
| -1/4| 4.7 | 0.6732(9) | 0.6723(14) | 0.6703(21) | 0.6667(34) | 0.6626(71) | 0.6666 |
| 0  | 1.0 | 0.7378(9) | 0.7404(14) | 0.7423(22) | 0.7410(36) | 0.7443(75) | 0.7500 |
| \( \lambda = 0.5 \) | | | | | | | |
| -1/2| 22 | 0.5382(8) | 0.5277(12) | 0.5212(20) | 0.5158(32) | 0.5147(53) | 0.5000 |
| -1/4| 1.0 | 0.5794(9) | 0.5805(13) | 0.5808(20) | 0.5801(31) | 0.5855(63) | 0.6250 |
| -1/4| 3.3 | 0.6076(7) | 0.6061(10) | 0.6047(15) | 0.6026(20) | 0.6030(26) | 0.5956(66) |
| -1/4| 6.0 | 0.6215(7) | 0.6187(11) | 0.6153(15) | 0.6124(22) | 0.6112(35) | 0.6085(70) |
| 0  | 0.5 | 0.6774(8) | 0.6835(12) | 0.6893(19) | 0.6942(33) | 0.6950(54) | 0.7500 |
| 0  | 1.0 | 0.6980(9) | 0.7012(14) | 0.7039(21) | 0.7051(34) | 0.7042(70) | 0.7500 |
| 0  | 2.0 | 0.7139(6) | 0.7153(8) | 0.7171(12) | 0.7183(17) | 0.7174(29) | 0.7218(59) |
| 0  | 10.0 | 0.7410(8) | 0.7417(12) | 0.7415(18) | 0.7422(26) | 0.7416(43) | 0.7384(89) |
| \( \lambda = 0.9 \) | | | | | | | |
| -1/10| 0.2 | 0.5142(8) | 0.5135(12) | 0.5138(16) | 0.5149(23) | 0.5124(38) | 0.5137(72) |
| -1/10| 0.5 | 0.5258(8) | 0.5236(11) | 0.5216(16) | 0.5206(22) | 0.5235(35) | 0.5186(70) |
| -1/10| 1.0 | 0.5381(8) | 0.5352(11) | 0.5334(16) | 0.5326(22) | 0.5334(35) | 0.5311(72) |

Table 5: Values of \( \nu_{\text{eff}} \) for \( \delta \leq 0 \), for various values of \( N_{\text{cut}} \).
| $\delta$ | g | 100 | 200 | 500 | 1000 | 2000 | 4000 | $\nu_{th}$ |
|-------|---|-----|-----|-----|------|------|------|----------|
|       |    |     |     |     |      |      |      |          |
|       | 0  | 1   | 0.6698(9) | 0.6733(14) | 0.6745(22) | 0.6761(35) | 0.6763(70) | 0.7000 |
| 0.60  | 0  | 2   | 0.6886(9)  | 0.6898(15)  | 0.6903(23)  | 0.6899(37)  | 0.6845(77)  |        |
| 0.60  | 0  | 4   | 0.7072(10) | 0.7063(16)  | 0.7041(25)  | 0.7020(41)  | 0.6945(85)  |        |
| 0.60  | 0  | 5   | 0.7147(10) | 0.7134(16)  | 0.7128(25)  | 0.7128(41)  | 0.7135(85)  |        |
|       |    |     |     |     |      |      |      |          |
|       | 0  | 0.5 | 0.5996(9)  | 0.6023(14)  | 0.6045(21)  | 0.6086(34)  | 0.6061(68)  | 0.6250 |
| 0.75  | 0  | 1.0 | 0.6292(9)  | 0.6234(14)  | 0.6239(22)  | 0.6204(35)  | 0.6186(70)  |        |
| 0.75  | 0  | 5.0 | 0.6745(10)| 0.6718(16)  | 0.6687(26)  | 0.6628(43)  | 0.6653(89)  |        |
|       | 1/10| 0.5 | 0.6537(9)  | 0.6582(14)  | 0.6612(22)  | 0.6614(35)  | 0.6713(72)  | 0.6750 |
|       | 1/10| 1.0 | 0.6754(10)| 0.6766(15)  | 0.6783(24)  | 0.6790(38)  | 0.6803(77)  |        |
|       | 1/10| 2.0 | 0.6927(10)| 0.6918(16)  | 0.6908(25)  | 0.6907(41)  | 0.6844(85)  |        |
|       | 1/4 | 1.0 | 0.7385(10)| 0.7397(17)  | 0.7416(27)  | 0.7413(44)  | 0.7380(91)  | 0.7500 |
|       | 1/4 | 2.0 | 0.7375(10)| 0.7386(12)  | 0.7387(26)  | 0.7364(42)  | 0.7372(89)  |        |
|       |    |     |     |     |      |      |      |          |
|       | 0  | 0.3 | 0.5470(8)  | 0.5476(11)  | 0.5481(15)  | 0.5487(22)  | 0.5476(35)  | 0.5475(70)| 0.5500 |
| 0.90  | 0  | 0.5 | 0.5608(8)  | 0.5608(11)  | 0.5609(16)  | 0.5598(23)  | 0.5624(36)  | 0.5598(71)|        |
| 0.90  | 0  | 1.0 | 0.5798(8)  | 0.5788(11)  | 0.5770(16)  | 0.5762(23)  | 0.5763(36)  | 0.5722(72)|        |
|       | 2/10| 0.2 | 0.6271(8)  | 0.6336(11)  | 0.6382(16)  | 0.6411(23)  | 0.6428(37)  | 0.6408(74)| 0.6500 |
|       | 2/10| 0.5 | 0.6631(8)  | 0.6656(12)  | 0.6679(17)  | 0.6687(25)  | 0.6689(41)  | 0.6737(83)|        |
|       | 2/10| 1.0 | 0.6803(9)  | 0.6793(13)  | 0.6787(19)  | 0.6778(28)  | 0.6733(46)  | 0.6687(94)|        |
|       |    |     |     |     |      |      |      |          |
|       | 0  | 0.1 | 0.5141(8)  | 0.5136(12)  | 0.5139(16)  | 0.5131(23)  | 0.5109(36)  | 0.5106(72)| 0.5000 |
| 1.00  | 0  | 0.5 | 0.5411(8)  | 0.5394(12)  | 0.5393(16)  | 0.5392(23)  | 0.5386(36)  | 0.5353(72)|        |
| 1.00  | 0  | 1.0 | 0.5579(8)  | 0.5546(11)  | 0.5529(16)  | 0.5520(23)  | 0.5511(37)  | 0.5521(73)|        |
|       | 1/3 | 0.20 | 0.6741(8) | 0.6811(13)  | 0.6859(18)  | 0.6876(26)  | 0.6888(42)  | 0.6902(86)| 0.6666 |
|       | 1/3 | 0.50 | 0.6972(9) | 0.6969(14)  | 0.6949(20)  | 0.6921(29)  | 0.6918(48)  | 0.6844(99)|        |
|       | 1/3 | 1.00 | 0.7014(10)| 0.6987(14)  | 0.6979(21)  | 0.6969(31)  | 0.6992(51)  | 0.6937(105)|        |
|       | 1/3 | 2.15 | 0.7099(11)| 0.708(18)   | 0.7083(28)  | 0.7049(46)  | 0.7058(97)  |        |        |
|       | 1/2 | 1.41 | 0.7348(13)| 0.7341(16)  | 0.7326(22)  | 0.7320(32)  | 0.7418(48)  | 0.7466(98)| 0.7500 |
|       | 1/2 | 1.79 | 0.7332(13)| 0.7315(17)  | 0.7302(21)  | 0.7300(31)  | 0.7428(47)  | 0.7467(98)|        |
|       | 1/2 | 2.83 | 0.7514(29)| 0.7518(46)  | 0.7547(92)  |        |        |        |        |
|       | 2/3 | 1.00 | 0.7402(27)| 0.7410(35)  | 0.7391(111) |        |        |        | 0.7500 |

Table 6: Values of $\nu_{eff}$ for $\delta \geq 0$, for various values of $N_{cut}$.  

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