OPERATOR ALGEBRAS AND SUBPRODUCT SYSTEMS ARISING FROM STOCHASTIC MATRICES

ADAM DOR-ON AND DANIEL MARKIEWICZ*

Abstract. We study subproduct systems in the sense of Shalit and Solel arising from stochastic matrices on countable state spaces, and their associated operator algebras. We focus on the non-self-adjoint tensor algebra, and Viselter’s generalization of the Cuntz-Pimsner C*-algebra to the context of subproduct systems. Suppose that $X$ and $Y$ are Arveson-Stinespring subproduct systems associated to two stochastic matrices over a countable set $\Omega$, and let $T_+(X)$ and $T_+(Y)$ be their tensor algebras. We show that every algebraic isomorphism from $T_+(X)$ onto $T_+(Y)$ is automatically bounded. Furthermore, $T_+(X)$ and $T_+(Y)$ are isometrically isomorphic if and only if $X$ and $Y$ are unitarily isomorphic up to a $*$-automorphism of $\ell^\infty(\Omega)$. When $\Omega$ is finite, we prove that $T_+(X)$ and $T_+(Y)$ are algebraically isomorphic if and only if there exists a similarity between $X$ and $Y$ up to a $*$-automorphism of $\ell^\infty(\Omega)$. Moreover, we provide an explicit description of the Cuntz-Pimsner algebra $\mathcal{O}(X)$ in the case where $\Omega$ is finite and the stochastic matrix is essential.

1. Introduction

In this paper we study the structure of tensor and Cuntz-Pimsner algebras (in the sense of Viselter [Vis12]) associated to subproduct systems, and to what extent these algebras provide invariants for their subproduct systems. These algebras generalize the tensor and Cuntz-Pimsner operator algebras associated to C*-correspondences, which have been the focus of considerable interest by many researchers. Tensor algebras of a C*-correspondence, in particular, have been the subject of a deep study by Muhly and Solel [MS98, MS00, MS02], which has led into a far-reaching non-commutative generalization of function theory. We will focus on subproduct systems associated to stochastic matrices, and in this context we prove several results which have a close parallel in the work of Davidson, Ramsey and Shalit [DRS11] on the isomorphism problem of tensor algebras of subproduct systems over $\mathbb{C}$ with finite dimensional (Hilbert space) fibers.

A subproduct system over a W*-algebra $\mathcal{M}$ (and over the additive semigroup $\mathbb{N}$) is a family $\{X_n\}_{n \in \mathbb{N}}$ of W*-correspondences over $\mathcal{M}$ endowed with an isometric comultiplication $X_{n+m} \rightarrow X_n \otimes X_m$ which is an adjointable $\mathbb{N}$-bimodule map for every $n, m$. Subproduct systems were first defined and studied for their own sake by Shalit and Solel [SS09], and in the special case of $\mathcal{M} = \mathbb{C}$ they were also independently studied under the name of inclusion systems by Bhat and Mukherjee [BM10]. Subproduct systems had appeared implicitly earlier in the work of many researchers in the study of dilations of semigroups of completely positive maps (cp-semigroups for short) on von Neumann algebras and later C*-algebras (see for example [BS00, MS02, Mar03]). The study of cp-semigroups is closely related to the analysis of $E_\omega$-semigroups and product systems pioneered by Arveson and Powers (for a comprehensive introduction see [Arv03], and also [Ske03b] for product systems of Hilbert modules).

Given a correspondence $E$ over a C*-algebra $\mathcal{A}$, the Toeplitz C*-algebra $\mathcal{T}(E)$ and the Cuntz-Pimsner C*-algebra $\mathcal{O}(E)$ were introduced by Pimsner [Pim97], and modified by Katsura [Kat04] in the case of non-injective left action of $\mathcal{A}$. As is well-known, in general the Cuntz-Pimsner algebra does not provide a very strong invariant of the underlying correspondence. However, some information does remain. In the case of graph C*-algebras, for example, if a graph is row-finite, then its C*-algebra is simple if and only if the graph is cofinal and every cycle has an entry. And it

Date: January 29, 2014.

2000 Mathematics Subject Classification. Primary: 47L30, 46L55, 46L57. Secondary: 46L08, 60J10.

Key words and phrases. Non-self-adjoint operator algebras; tensor algebra; subproduct system; Cuntz-Pimsner algebra; cp-semigroup; stochastic matrix.

The first author was partially supported by GIF (German-Israeli Foundation) research grant No. 2297-2282.6/201, and the second author was partially supported by grant 2008295 from the U.S.-Israel Binational Science Foundation.

*corresponding author.

1
is easy to find two graphs with $d$ vertices and irreducible adjacency matrix whose C*-algebras are not isomorphic (see [Rae05]). In contrast, in section 5 we show that if $X$ is the Arveson-Stinespring subproduct system of a $d \times d$ irreducible stochastic matrix, then $\mathcal{O}(X) \cong C(\mathbb{T}) \otimes M_d(\mathbb{C})$. More generally, we also provide an explicit description for the Cuntz-Pimsner algebra of a subproduct system associated to essential finite stochastic matrices.

On the other hand, the non-self-adjoint tensor algebra $T_+(E)$ of a C*-correspondence $E$ over $A$ has often proven to be a strong invariant of the correspondence. Muhly and Solel [MS00] proved that if $E$ and $F$ are aperiodic C*-correspondences, then $T_+(E)$ is isometrically isomorphic to $T_+(F)$ if and only if $E$ and $F$ are isometrically isomorphic as $A$-bimodules. Similarly, Katsoulis and Kribs [KK04] and Solel [Sol04] proved that if $G$ and $G'$ are countable directed graphs, then the tensor algebras $T_+(G)$ and $T_+(G')$ are isomorphic as algebras if and only if $G$ and $G'$ are isomorphic as directed graphs. See also Davidson and Katsoulis [DK11] for another important example of this phenomenon of increased acuity of the normed (non-self-adjoint) algebras as opposed to C*-algebras, perhaps first recognized in Arveson [Arv67] and Arveson and Josephson [AJ69].

The tensor algebras of subproduct systems were first considered by Solel and Shalit [SS09] in the special case of $\mathcal{M} = C$, and they analyzed the problem of graded isomorphism of their tensor algebras. The general isomorphism problem for such subproduct systems was resolved by Davidson, Ramsey and Shalit [DRS11]. They proved that if $X, Y$ are subproduct systems of finite-dimensional Hilbert space fibers, then $T_+(X)$ and $T_+(Y)$ are isometrically isomorphic if and only if $X$ and $Y$ are (unitarily) isomorphic.

On the other hand, the recent work of Gurevich [Gur12] provides a useful contrast. Although in this paper we focus on subproduct systems over $\Omega$, it is possible to consider more general semigroups. Gurevich studied subproduct systems over the semigroup $\mathbb{N} \times \mathbb{N}$, with finite dimensional Hilbert space fibers. He proved in [Gur12] that subproduct systems of that type in a certain large class can be distinguished by their tensor algebras, however he also provided an example of two non-isomorphic subproduct systems over $\mathbb{N} \times \mathbb{N}$ of finite dimensional Hilbert space fibers whose tensor algebras are isometrically isomorphic.

The following are our main results. Suppose that $X$ and $Y$ are Arveson-Stinespring subproduct systems over $\mathcal{M} = \ell^\infty(\Omega)$, associated to two stochastic matrices over a countable set $\Omega$, and let $T_+(X)$ and $T_+(Y)$ be their tensor algebras. Then $T_+(X)$ and $T_+(Y)$ are isometrically isomorphic if and only if $X$ and $Y$ are unitarily isomorphic up to a *-automorphism of $\ell^\infty(\Omega)$. Every algebraic isomorphism from $T_+(X)$ onto $T_+(Y)$ is automatically bounded. Furthermore, when $\Omega$ is finite, $T_+(X)$ and $T_+(Y)$ are algebraically isomorphic if and only if there exists a similarity between $X$ and $Y$ up to a *-automorphism of $\ell^\infty(\Omega)$, in the appropriate sense.

We now describe the structure of this paper. In section 2 we review some preliminary material. In section 3 we describe the subproduct system of a stochastic matrix, and in Theorem 3.8 we provide an effective isomorphism theorem for such objects. In section 4 we review some basic facts about the Cuntz-Pimsner algebra of a subproduct system, and in section 5 we characterize the Cuntz-Pimsner C*-algebra of the subproduct system of essential stochastic matrices. Finally, in section 6 we begin the study of the tensor algebra of a general subproduct system, culminating in the main results for the case of stochastic matrices in section 7.

2. Preliminaries

Stochastic Matrices. We review the basic terminology and facts about stochastic matrices that are relevant to our study.

Definition 2.1. Let $\Omega$ be a countable set. A stochastic matrix is a function $P : \Omega \times \Omega \to \mathbb{R}_+$ such that for all $i \in \Omega$ we have $\sum_{j \in \Omega} P_{ij} = 1$. The set $\Omega$ is called the state set or space of the matrix $P$, and elements of $\Omega$ are called states of $P$.

Two stochastic matrices $P$ and $Q$ can be multiplied to obtain a new stochastic matrix $PQ$ defined by $(PQ)_{jk} = \sum_{i \in \Omega} P_{ij}Q_{jk}$. Henceforth, we will denote by $P^n$ the product of $P$ with itself $n$ times, and by $P_{ij}^{(n)} := (P^n)_{ij}$ the $(i, j)$-th entry of $P^n$. 


Definition 2.2. Let \( P \) be a stochastic matrix on a state space \( \Omega \). Denote by \( \text{Gr}(P) \) the matrix representing the directed graph of \( P \) which is defined to be
\[
\text{Gr}(P)_{ij} = \begin{cases} 
1, & P_{ij} > 0 \\
0, & P_{ij} = 0 
\end{cases}
\]

Definition 2.3. Let \( P \) and \( Q \) be stochastic matrices on a state space \( \Omega \), let \( \sigma : \Omega \to \Omega \) be a permutation with corresponding permutation matrix \( R_{\sigma} = [\delta_{i}(\sigma(j))] \).

We say that \( P \) is graph isomorphic to \( Q \) through \( \sigma \) and write \( P \sim_{\sigma} Q \) if \( R_{\sigma}^{-1} \text{Gr}(Q) R_{\sigma} = \text{Gr}(P) \).

We will also say that \( P \) is isomorphic to \( Q \) through \( \sigma \) and write \( P \cong_{\sigma} Q \) if \( R_{\sigma}^{-1} Q R_{\sigma} = P \).

Thus we have that \( P \sim_{\sigma} Q \) if the directed graphs of \( P \) and \( Q \) are isomorphic, while \( P \cong_{\sigma} Q \) if the weighted directed graphs of \( P \) and \( Q \) are isomorphic.

Definition 2.4. Let \( P \) be a stochastic matrix over a state set \( \Omega \). A path in \( P \) is a path in the directed graph of \( P \), that is a function \( \gamma : \{0,...,\ell\} \to \Omega \) such that \( P_{\gamma(k)\gamma(k+1)} > 0 \) for every \( 0 \leq k \leq \ell - 1 \). The path \( \gamma \) is said to be a cycle if \( \gamma(0) = \gamma(\ell) \). We will say that two states \( i,j \) communicate if and only if there exists a path from \( i \) to \( j \) and vice-versa.

It is clear that the communication relation is an equivalence relation. Note also that a path from \( i \) to \( j \) of length \( n \) exists if and only if \( P_{ij}^{n} > 0 \).

Definition 2.5. Let \( P \) be a stochastic matrix over a state set \( \Omega \), and let \( i \in \Omega \).
\begin{enumerate}
\item The period of \( i \) is \( r(i) = \gcd\{ n \mid P_{ii}^{(n)} > 0 \} \). If no such \( r(i) \) exists, or if \( r(i) = 1 \) we say that \( i \) is aperiodic.
\item \( i \in \Omega \) is said to be inessential in \( P \) if there is some \( j \in \Omega \) and \( n \in \mathbb{N} \) such that \( P_{ij}^{(n)} > 0 \) but \( P_{ji}^{m} = 0 \) for all \( m \in \mathbb{N} \). \( P \) is said to be essential if it has no inessential states.
\item \( P \) is said to be irreducible if any pair \( i,j \in \Omega \) communicates in \( P \).
\end{enumerate}

Clearly, irreducible stochastic matrices are automatically essential. Further note that for an essential state \( i \in \Omega \), the number \( r(i) \) is always well-defined.

Definition 2.6. Let \( P \) be a stochastic matrix over a state set \( \Omega \). A state \( i \in \Omega \) is said to be transient if the series \( \sum_{n \in \mathbb{N}} P_{ii}^{(n)} \) converges, and otherwise recurrent. If \( i \in \Omega \) is recurrent then it is null-recurrent if \( P_{ii}^{(n)} \to_{n \to \infty} 0 \), and positive-recurrent otherwise. We say that \( P \) is transient/recurrent if all states are transient/recurrent in \( P \), respectively.

In an irreducible stochastic matrix, all states have the same period and classification in terms of recurrence type. We also note that recurrent states are essential (see [Chu60, Part I, Section 4, Theorem 4]).

The next two theorems can be found in various forms in the literature, see [Chu60, Part I, Section 3], [Fel68, Chapter XV, Section 6]. We restate them in a form convenient for our purposes.

Theorem 2.7. (Irreducible decomposition for essential matrices)
Let \( P \) be an essential stochastic matrix over a state set \( \Omega \). Let \( (\Omega_{\alpha})_{\alpha \in A} \) be the partition of \( \Omega \) into equivalence classes of communicating states. With the appropriate enumeration of \( \Omega \), the matrix \( P \) decomposes into a block diagonal matrix whose diagonal blocks are irreducible stochastic matrices corresponding to the restriction of \( P \) to \( \Omega_{\alpha} \times \Omega_{\alpha} \), for \( \alpha \in A \).

As a corollary, we observe that complete reducibility is equivalent to essentiality.

Theorem 2.8. (Cyclic decomposition for periodic irreducible matrices)
Let \( P \) be an irreducible stochastic matrix over a state set \( \Omega \) with period \( r \), and let \( \omega \in \Omega \). For each \( \ell = 0,...,r-1 \), let \( \Omega_{\ell} = \{ j \in \Omega \mid P_{\omega j}^{(n)} > 0 \Rightarrow n \equiv \ell \mod r \} \). Then the family \( (\Omega_{\ell})_{\ell=0}^{r-1} \) is a partition of \( \Omega \). Furthermore if \( j \in \Omega_{\ell} \) then there exists \( N(j) \) such that for all \( n \geq N(j) \) we have \( P_{\omega j}^{(n+1)} > 0 \). In fact, with an appropriate enumeration of \( \Omega \), there exist stochastic matrices \( P_{0},...P_{r-1} \) such that \( P \) has the following cyclic block decomposition:
\[
\begin{bmatrix}
P_{0} & ... & 0 \\
... & ... & ... \\
0 & ... & P_{r-2} \\
P_{r-1} & ... & 0
\end{bmatrix}
\]

Where the rows of \( P_{\ell} \) in this matrix decomposition are indexed by \( \Omega_{\ell} \) for all \( 0 \leq \ell < r \).
Remark 2.9. We emphasize that the columns of $P_t$ in the above matrix decomposition are indexed by $\Omega_{(\ell)}$, where $s(\ell) = \ell + 1 \mod r$. Further, if $P$ is finite $d \times d$, then $P_0, \ldots, P_{r-1}$ are $\frac{d}{r} \times \frac{d}{r}$.

We note that the cyclic decomposition is independent of our initial choice of $\omega$ in the sense that for $P$ irreducible with period $r$ and cyclic decomposition $\Omega_0, \ldots, \Omega_{r-1}$ given by the previous theorem, if we were to pick a different $\omega' \in \Omega$ and a new partition $\Omega'_0, \ldots, \Omega'_{r-1}$ induced by it, a cyclic permutation of this partition would yield our original partition $\Omega_0, \ldots, \Omega_{r-1}$. Furthermore, if $i \in \Omega_{t_1}$ and $j \in \Omega_{t_2}$ are two states, let $0 \leq \ell < r$ be such that $\ell \equiv l_2 - l_1 \mod r$. Now if $P_{ij}^{(k)} > 0$ then one must have $k = nr + \ell$ for some $n \in \mathbb{N}$, and there exists some $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$ one has $P_{ij}^{(mr+\ell)} > 0$. (See [Chu60] Part I, Section 3, Theorems 3 & 4)

Recall that an irreducible stochastic $P$ is positive-recurrent if and only if it possesses a stationary distribution, i.e. a vector $\pi \in \ell^1(\Omega)$ such that $\pi_j \geq 0$ for all $j \in \Omega$, $\sum_i \pi_i = 1$ and $\pi_j = \sum_i \pi_i P_{ij}$ for all $j \in \Omega$. For the proof of the following theorem, see [Dur10] Theorem 6.7.2.

Theorem 2.10. (Convergence theorem for positive-recurrent irreducible matrices)
Let $P$ be an irreducible positive-recurrent stochastic matrix with stationary distribution $\pi$ and period $r \geq 1$. Let $\Omega_0, \ldots, \Omega_{r-1}$ be a cyclic decomposition of $\Omega$ with respect to $P$ as in Theorem 2.8. Given $i \in \Omega_{t_1}$ and $j \in \Omega_{t_2}$, let $0 \leq \ell < r$ be such that $\ell \equiv (l_2 - l_1) \mod r$. Then
\[
\lim_{m \to \infty} P_{ij}^{(mr+\ell)} = \pi_j r.
\]

Definition 2.11. Let $P$ be a stochastic matrix over $\Omega$. $i \in \Omega$ is said to be amenable in $P$ if $\limsup_{n \to \infty} \sqrt[n]{P_{ii}^{(n)}} = 1$. $P$ is said to be amenable if all states in $\Omega$ are amenable in $P$.

The proof of the following fact is well-known and will be omitted.

Lemma 2.12. Let $P$ be a recurrent irreducible stochastic matrix, then $P$ is amenable and for every $n \in \mathbb{N}$ we have that $P^n$ is recurrent.

Hilbert modules. We assume that the reader is familiar with the basic theory of Hilbert C*-modules, which can be found in [Pas73] [Lan95] [MT05]. We only give a quick summary of basic notions and terminology to clarify our conventions.

As usual inner product modules are right modules. The dual module of an inner product module $E$ over $\mathcal{A}$ is the set of all bounded $\mathcal{A}$-module maps from $E$ to $\mathcal{A}$, and it is denoted by $E^\prime$. A Hilbert C*-module is called self-dual if the canonical embedding of $E$ into $E^\prime$ is surjective.

Definition 2.13. Let $\mathcal{M}$ be a W*-algebra let $E$ be a Hilbert C*-module over $\mathcal{M}$. The $\sigma$-topology on $E$ is defined by the functionals $f(\cdot) = \sum_{n=1}^{\infty} w_n(\langle \xi_n, \cdot \rangle)$ where $\xi_n \in E$ and $w_n \in \mathcal{M}$, such that $\sum_{n=1}^{\infty} \|w_n\| \|\xi_n\| < \infty$.

Hilbert C*-modules over a W*-algebra will be called Hilbert W*-modules. If $E$ is a self-dual W*-module, then it is a dual Banach space (see [Pas73]), and the associated weak-* topology coincides with its $\sigma$-topology.

If $E$ is an inner product module over a W*-algebra $\mathcal{M}$, then the inner product module structure of $E$ can be naturally extended to $E^\prime$, which makes $E^\prime$ into a self-dual Hilbert W*-module over $\mathcal{M}$. In this case we will refer to $E^\prime$ as the self-dual extension of $E$. Furthermore, the canonical embedding of $E$ into $E^\prime$ maps onto a dense subset in the $\sigma$-topology of $E^\prime$, so that $E$ is self dual if and only if it is $\sigma$-topology closed in $E^\prime$.

For $E$ and $F$ Hilbert W*-modules, let $\mathcal{L}(E, F)$ denote the set of adjointable maps from $E$ to $F$. If $E$ and $F$ are self-dual W*-modules, then all bounded $\mathcal{M}$-module maps from $E$ to $F$ are adjointable. It also turns out that bounded module maps between any two inner product modules over a W*-algebra behave well with respect to self dual completions, as the following proposition states.

Proposition 2.14. Let $E$ and $F$ be inner-product modules over a W*-algebra $\mathcal{M}$, and let $T : E \to F$ be a bounded module map. Then $T$ has a unique extension to a bounded module map $T : E^\prime \to F^\prime$, and $\|T\| = \|T\|$. If $E = F$, then the map $T \to \overline{T}$, restricted to the algebra of adjointable operators on $E$, is a faithful $^*$-homomorphism from $\mathcal{L}(E)$ to $\mathcal{L}(E^\prime)$.
Definition 2.15. If \( E \) is a Hilbert \( C^* \)-module over \( B \), and \( A \) is another \( C^* \)-algebra, then \( E \) is called a \( C^* \)-correspondence from \( A \) to \( B \) if \( E \) is also a left \( A \)-module such that the left action is determined by a \( \ast \)-homomorphism \( \phi : A \to \mathcal{L}(E) \). In the case when \( A = B \), \( E \) is called a \( C^* \)-correspondence over \( A \). If \( N \) and \( M \) are \( W^* \)-algebras then \( E \) is called a Hilbert \( W^* \)-correspondence from \( N \) to \( M \) if in addition \( E \) is a self-dual module over \( M \) and the left action \( \phi \) of \( N \) is normal.

A key notion of \( C^* \) and \( W^* \)-correspondences is the internal tensor product. If \( E \) is a \( C^* \)-correspondence from \( A \) to \( B \) with left action \( \psi \), then on the algebraic tensor product \( E \otimes_{\text{alg}} F \) one defines a \( C^* \)-valued pre-inner product satisfying \( \langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle y_1, \psi(x_1, x_2) y_2 \rangle \) on simple tensors. The usual quotient and completion process yields the internal Hilbert \( C^* \)-module tensor product of \( E \) and \( F \), denoted by \( E \otimes F \) or \( E \otimes_{\psi} F \), which is a \( C^* \)-correspondence from \( A \) to \( C \). In the case where \( A, B \) and \( C \) are \( W^* \)-algebras, taking the self dual completion yields the self-dual tensor product denoted also by \( E \otimes_f F \) or \( E \otimes_{\psi} F \), which is a \( W^* \)-correspondence from \( A \) to \( C \).

The notion of self-dual direct sums of Hilbert \( C^* \)-modules over \( W^* \)-algebras was developed by Paschke.

Definition 2.16. Let \( \{E_i\}_{i \in I} \) be a family of self-dual \( W^* \)-modules over \( M \). The ultraweak direct sum of \( \{E_i\}_{i \in I} \) is the subset \( X \) of the Cartesian product of \( \{E_i\}_{i \in I} \) such that \( \{x_i\} \) is in \( X \) if the sum \( \sum_{i \in I} (x_i, x_i) \) converges ultraweakly. The inner product on \( X \) is defined to be \( \langle \{x_i\}, \{y_i\} \rangle = \sum_{i \in I} \langle x_i, y_i \rangle \), where the sum converges ultraweakly in \( M \). This direct sum is denoted by \( \bigoplus_{i \in I}^u E_i \) or by \( \oplus_{i \in I} E_i \) when the context is that of self-dual modules.

Base change. Suppose that \( E \) is a Hilbert \( C^* \)-module (correspondence) over \( A \) and \( \rho \) is a \( * \)-automorphism of \( A \). Then one can define a new \( C^* \)-module (correspondence) \( E^\rho \) over \( A \). As a set, \( E^\rho = E \), but its operations are defined as follows: \( \xi \cdot \rho = \xi \cdot \rho(a) \) (also \( a \cdot \rho \xi = \rho(a) \cdot \xi \) for correspondences) and \( \langle \xi, \eta \rangle^\rho = \rho^{-1}(\langle \xi, \eta \rangle) \) for all \( \xi, \eta \in E \) and \( a \in A \). In algebra, this operation on modules is sometimes called a change of rings or base change.

Definition 2.17. Let \( E \) and \( F \) be two \( C^* \)-modules over \( A \), and let \( \rho : A \to A \) be a \( * \)-automorphism. We will say that a bounded linear map \( V : E \to F \) is a \( \rho \)-module (\( \rho \)-correspondence) morphism if \( V : E \to F^\rho \) is an \( A \)-linear (\( A \)-correspondence) map, i.e. for all \( a \in A \) and \( \xi, \eta \in E \) one has \( V(\xi a) = V(\xi) \rho(a) \) (also \( V(a \xi) = \rho(a) V(\xi) \) for correspondences). We will say that \( V \) is \( \rho \)-adjointable if \( V \) is adjointable as a map \( E \to F^\rho \), and we will denote by \( V^{(\ast \rho)} \) its adjoint, i.e. \( V^{(\ast \rho)} : F \to E \) satisfies \( \langle V(\xi), \eta \rangle^\rho = \rho^{-1}(\langle \xi, V(\eta) \rangle) = \langle V^{(\ast \rho)}(\xi), \eta \rangle \) for all \( \xi \in F, \eta \in E \).

Note that a \( \rho \)-adjointable \( V \) must be a bounded \( \rho \)-module morphism and \( V^{(\ast \rho)} \) is a \( \rho^{-1} \)-adjointable map with \( (V^{(\ast \rho)})^{(\ast \rho^{-1})} = V \). Furthermore, if \( E, F \) and \( G \) are \( C^* \)-modules over \( A \) and we have \( V : E \to F \) a \( \rho \)-module/\( \rho \)-correspondence morphism and \( W : F \to G \) a \( \tau \)-module/\( \tau \)-correspondence morphism, then \( W \circ V : E \to G \) is a \( (\tau \circ \rho) \)-module/\( (\tau \circ \rho) \)-correspondence morphism respectively. Further, if \( V \) is \( \rho \)-adjointable and \( W \) is \( \tau \)-adjointable, then \( W \circ V \) is \( (\tau \circ \rho) \)-adjointable with \( (W \circ V)^{(\ast \tau \circ \rho)} = V^{(\ast \rho)} \circ W^{(\ast \tau)} \).

Furthermore, \( \sigma \)-topology continuity is automatic for \( \rho \)-adjointable maps. Indeed, suppose that \( V : E \to F \) is a \( \rho \)-adjointable map, and let \( \{\eta_n\}_n \) be a net in \( E \) converging in the \( \sigma \)-topology to \( \eta \). Let \( \xi_n \in F \) and \( w_n \in M \) be such that \( \sum ||w_n|| \cdot ||\xi_n|| < \infty \). Then we have that
\[
\sum_{n=1}^{\infty} w_n(\langle \xi_n, V(\eta_n - \eta) \rangle) = \sum_{n=1}^{\infty} (w_n \circ \rho)(\langle V^{(\ast \rho)}(\xi_n), \eta_n - \eta \rangle)
\]
And since \( \sum ||w_n \circ \rho|| \cdot ||V^{(\ast \rho)}(\xi_n)|| \leq ||V^{(\ast \rho)}|| \sum ||w_n|| \cdot ||\xi_n|| < \infty \), we have convergence of the net \( (V(\eta_n))_n \) to \( V(\eta) \) in the \( \sigma \)-topology in \( F \).

We also note that the identity map \( \iota_{E^\rho} : (E^\rho)' \to E^\rho \) is a \( \rho \)-module isometric isomorphism. It follows that if \( E \) is self-dual, then the same holds for \( E^\rho \). It also leads to the following fact.

Proposition 2.18. Let \( E \) and \( F \) be Hilbert \( W^* \)-modules over a \( W^* \)-algebra \( M \), and let \( \rho \) be a \( * \)-automorphism of \( M \). Suppose that \( V : E \to F \) is a \( \rho \)-module morphism. Then \( V \) has a unique extension to a bounded \( \rho \)-module morphism \( \overline{V} : E^\rho \to F^\rho \), and \( ||V|| = ||\overline{V}|| \).
Proof. Using Proposition 2.14 we obtain a unique W*-module map \( \bar{V} : E' \to (F^\rho)' \). By composing with \( \iota_\rho \) described in the paragraph preceding the proposition, we obtain that \( \bar{V} = \iota_\rho \cdot \bar{V} \) is a \( \rho \)-module morphism from \( E' \) to \( F' \) extending \( V \). Since \( \iota_\rho \) is a bijection, we also obtain uniqueness of the extension. The norm condition holds because \( \iota_\rho \) is isometric. \( \square \)

**Definition 2.19.** Let \( E \) and \( F \) be Hilbert W*-modules over \( \mathcal{M} \) and let \( \rho : \mathcal{M} \to \mathcal{M} \) be a *-automorphism. We say that a \( \rho \)-adjoointable map \( W : E \to F \) is a \( \rho \)-coisometry if \( W^{(*)}\rho \) is an isometry, and we will say that \( W \) is a \( \rho \)-unitary if it is an isometric surjective \( \rho \)-module morphism.

We observe that if \( U \) is a \( \rho \)-unitary then \( U^{-1} \) is a \( \rho^{-1} \)-module morphism and \( U^{(*)}\rho = U^{-1} \).

Further note that a \( \rho \)-module map \( U \) is a \( \rho \)-unitary if and only if it is a \( \rho \)-adjoointable module map satisfying \( UU^{(*)}\rho = \text{Id}_F \) and \( U^{(*)}\rho U = \text{Id}_E \). This follows from the analogous and well-known theorem for (Id-)unitaries, and the preceding discussion.

**Subproduct systems.**

**Definition 2.20.** Let \( \mathcal{M} \) be a W*-algebra, let \( X = \{X_n\}_{n \in \mathbb{N}} \) be a family of of Hilbert W*-correspondences over \( \mathcal{M} \) and let \( U = \{U_{n,m} : X_n \otimes X_m \to X_{n+m}\}_{n,m \in \mathbb{N}} \) be a family of maps. We will say that \((X,U)\) is a subproduct system over \( \mathcal{M} \) (and over the semigroup \( \mathbb{N} \)) if the following conditions are satisfied:

1. \( X_0 = \mathcal{M} \)
2. \( U_{n,m} \) is a coisometric mapping of W*-correspondences over \( \mathcal{M} \) for every \( n,m \in \mathbb{N} \)
3. The maps \( U_{n,0} \) and \( U_{0,n} \) are given by the left and right actions of \( \mathcal{M} \) on \( X_n \), respectively, and for every \( n,m \in \mathbb{N} \) we have the following identity:
   \[
   U_{n+m,p}(U_{n,m} \otimes I_{X_p}) = U_{n,m+p}(I_{X_n} \otimes U_{m,p})
   \]

In case the maps \( U_{n,m} \) are unitaries, we say that \( X \) is a product system.

When there is no ambiguity, we will suppress the reference to the family \( U \) of multiplication maps, and refer simply to a subproduct system \( X \).

**Example 2.21.** If \( E \) is a W*-correspondence over \( \mathcal{M} \) such that \( \mathcal{M} \cdot E = E \) (essential), the product system \( X^E \) over \( \mathbb{N} \) defined by \( X^E_n = E^\otimes n \) where \( U_{n,m} \) is the natural identification between \( E^\otimes n \otimes E^\otimes m \) and \( E^\otimes (n+m) \), is obviously a subproduct system. We call it the full product system.

**Definition 2.22.** Let \((X,U_X)\) and \((Y,U_Y)\) be subproduct systems over \( \mathcal{M} \) and \( \mathcal{N} \) respectively.

We define a subproduct system \( X \oplus Y \) over \( \mathcal{M} \oplus \mathcal{N} \), which we will call the direct sum of \( X \) and \( Y \).

1. The \( n \)-th fiber W*-correspondence is given by \( (X \oplus Y)_n := X_n \oplus Y_n \) with left and right multiplication of \( \mathcal{M} \oplus \mathcal{N} \) given by \((m \oplus n) \cdot (\xi \oplus \eta) \cdot (m' \oplus n') = m\xi m' \oplus n\eta n' \), and inner product given by \( (\xi \oplus \eta, \xi' \oplus \eta') = (\xi, \xi') \oplus (\eta, \eta') \).
2. The subproduct maps \( U_{X \oplus Y}^{n,m} \) are defined by
   \[
   U_{X \oplus Y}^{n,m}((\xi_n \oplus \eta_n) \otimes (\xi_m \oplus \eta_m)) = U_{X,n}^X(\xi_n \otimes \eta_n) \oplus U_{Y,m}^Y(\xi_m \otimes \eta_m)
   \]

**Definition 2.23.** Let \((X,U_X)\) and \((Y,U_Y)\) be two subproduct systems over a W*-algebra \( \mathcal{M} \). A family \( \mathcal{V} = \{V_n\} \) of maps \( V_n : X_n \to Y_n \) is called a morphism of subproduct systems if

1. \( \rho = V_0 : X_0 \to Y_0 \) is a *-automorphism,
2. For all \( n \neq 0 \) the map \( V_n \) is a \( \rho \)-coisometric \( \rho \)-correspondence morphism.
3. For all \( n,m \in \mathbb{N} \) the following identity holds:
   \[
   V_{n+m} \circ U_X^{n,m} = U_Y^{n,m} \circ (V_n \otimes V_m)
   \]

When the family \( \{V_n\} \) is a family of \( \rho \)-unitaries, we say that \( X \) and \( Y \) are (unitarily) isomorphic via \( \rho = V_0 \) and write \( X \cong_\rho Y \).

In \cite{SS09} the notion of isomorphism of subproduct systems was defined so that the map \( V_0 \) is the identity on \( \mathcal{M} \), yet the above variation will be more convenient for our purposes.

We now describe the general construction of Arveson-Stinespring subproduct systems.

Let \( \mathcal{M} \) be a W*-algebra, and let \( \theta \) be a completely positive contractive and normal map on \( \mathcal{M} \).

If \( \rho : \mathcal{M} \to B(H) \) is a normal representation, then we can define a new normal representation of
\(M\), which we will call a dilation of \(\theta\) via \(\rho\), as follows. We define \(M \otimes_\theta H\) to be the Hausdorff completion of the algebraic tensor product \(M \otimes H\) with respect to the sesquilinear form defined by
\[
\langle T_1 \otimes h_1, T_2 \otimes h_2 \rangle = \langle h_1, \rho(\theta(T_1^*T_2))h_2 \rangle, \quad \text{for all } T_1, T_2 \in M, h_1, h_2 \in H.
\]
Complete positivity of \(\theta\) ensures that this sesquilinear form is positive semidefinite. We define a normal representation \(\pi_\theta M\) of \(M \otimes_\theta H\) by \(\pi_\theta(S(T \otimes h)) = ST \otimes h\). Moreover, the map \(W_\theta : H \rightarrow M \otimes H\) given by \(W_\theta(h) = I \otimes h\) is a contraction which satisfies
\[
\rho(\theta(T)) = W_\theta^* \pi_\theta(T) W_\theta.
\]
One easily checks that \(M \otimes_\theta H\) is minimal in the sense that it is the smallest subspace of \(M \otimes_\theta H\) containing \(W_\theta H\) and reducing \(\pi_\theta\). It follows that if \((\pi, K, W)\) is a triple where \(\pi\) is a representation of \(M\) on \(K\) such that \(\rho(\pi(T)) = W^* \pi(T) W\) for all \(T \in M\) and \(K\) is minimal, then there is a unitary \(U : K \rightarrow M \otimes_\theta H\) such that \(U\) implements a unitary equivalence between \(\pi\) and \(\pi_\theta\) and \(UW = W_\theta U\).

We also define the associated intertwiner space
\[
\mathcal{L}_M(H, M \otimes_\theta H) := \{ X \in B(H, M \otimes_\theta H) \mid X \pi(T) = \pi_\theta(T) X \quad \forall T \in M \}.
\]

Suppose now that \(M\) is a von Neumann subalgebra of \(B(H)\) and \(\rho\) is the inclusion map. In this case the triple \((\pi_\theta, M \otimes_\theta H, W_\theta)\) will be called the minimal Stinespring dilation of \(\theta\). In [MS02], the intertwiner space \(\mathcal{L}_M(H, M \otimes_\theta H)\) was shown to be a \(W^*\)-correspondence over \(M'\), with left and right actions given by \(S \cdot X = (I \otimes S) \circ X\) and \(X \cdot S = X \circ S\) for \(S \in M'\) and \(X \in \mathcal{L}_M(H, M \otimes_\theta H)\), and the \(M'\)-valued inner product is given by \(\langle X, Y \rangle = X^*Y\), for \(X, Y \in \mathcal{L}_M(H, M \otimes_\theta H)\).

Now let \(\phi\) be another completely positive contractive normal map on \(M\). Then we obtain a representation \(\pi_{\theta,\phi}\) of \(M\) on \(M \otimes_\phi (M \otimes_\theta H)\) via dilation using \(\pi_\theta\), and in an analogous way we have that the \(M'\)-correspondence intertwiner space \(\mathcal{L}_M(H, M \otimes_\phi (M \otimes_\theta H))\), between the identity representation of \(M\) and \(\pi_{\theta,\phi}\).

**Definition 2.24.** Let \(\theta\) be a normal completely positive map on a W*-algebra \(M\). Let
\[
\mathcal{L}^\theta(n) = \mathcal{L}_M(H, M \otimes_{\theta^n} H) \quad \text{and} \quad \mathcal{L}^\theta(n, m) = \mathcal{L}_M(H, M \otimes_{\theta^n} M \otimes_{\theta^m} H), \quad n, m \in \mathbb{N}.
\]

The Arveson-Stinespring subproduct system of \(\theta\) is defined as follows:

1. The fibers are \((\mathcal{L}^\theta(n))_{n \in \mathbb{N}}\) with the aforementioned \(W^*\)-correspondence structure over \(M'\).
2. The subproduct maps \(U^\theta_{n,m} : \mathcal{L}^\theta(n) \otimes \mathcal{L}^\theta(m) \rightarrow \mathcal{L}^\theta(n+m)\) are defined by \(U^\theta_{n,m} = V^*_n \circ \Psi_{n,m}\) where,
   a. \(V_{n,m} : \mathcal{L}^\theta(n+m) \rightarrow \mathcal{L}^\theta(m, n)\) is defined by \(V_{n,m}(X) = \Gamma_{n,m} \circ X\) where \(\Gamma_{n,m} : M \otimes_{\theta^{n+m}} H \rightarrow M \otimes_{\theta^{n+m}} M \otimes_{\theta^{m+n}} H\) is defined to be \(\Gamma_{n,m}(S \otimes_{n+m} h) = S \otimes_{m+n} I \otimes_n h\).
   b. \(\Psi_{n,m} : \mathcal{L}^\theta(n) \otimes \mathcal{L}^\theta(m) \rightarrow \mathcal{L}^\theta(m, n)\) is defined by \(\Psi_{n,m}(X \otimes Y) = (I \otimes X)Y\).

We denote this subproduct system by \((\mathcal{L}^\theta, U^\theta)\).

It was proven in [SS09] that this is indeed a subproduct system. This fact relies on the work in [MS02], where it was shown that \(V_{n,m}\) is an isometric correspondence morphism, and that \(\Psi_{n,m}\) is a correspondence isomorphism.

A fundamental property of Arveson-Stinespring subproduct systems is that every subproduct system is (Id-)isomorphic to an Arveson-Stinespring subproduct system of a cp-semigroup (see [SS09 Corollary 2.10]).

### 3. Subproduct systems arising from stochastic matrices

Let \(\Omega\) be a countable set, and let \(L^\infty(\Omega)\) be the von Neumann algebra of bounded sequences indexed by \(\Omega\) acting on the Hilbert space \(L^2(\Omega)\). Let us denote by \(\{e_i\}_{i \in \Omega}\) the canonical orthogonal basis for \(L^2(\Omega)\), and by \(\{p_j\}_{j \in \Omega}\) the collection of rank one pairwise perpendicular projections in \(L^\infty(\Omega)\) defined by \(p_j(e_i) = \delta_{ij} e_i\).

In this section we will compute the Arveson-Stinespring subproduct system of the cp-semigroup generated by a single unital positive normal map on the von-Neumann algebra \(L^\infty(\Omega)\). It is easy to see that such a map is determined uniquely by a stochastic matrix on \(\Omega\). This simple observation will be used repeatedly, hence we record it here for emphasis. We omit the straightforward proof.
Proposition 3.1. There is a 1-1 correspondence between unital positive normal maps \( \theta : \ell^\infty(\Omega) \to \ell^\infty(\Omega) \) and stochastic matrices \( P \) over \( \Omega \), where the relationship is given by

\[
\langle e_i, \theta(p_j)e_i \rangle = P_{ij}
\]

The map just defined sends the composition of unital positive normal maps into the product of their respective stochastic matrices.

Of course representations of stochastic matrices on \( \Omega \) are dependent on its enumerations, hence permutations of \( \Omega \), or alternatively *-automorphisms of \( \ell^\infty(\Omega) \), will play a role in the continuation. Recall that the *-automorphisms of \( \ell^\infty(\Omega) \) are in 1-1 correspondence with the permutations of \( \Omega \), because minimal projections must be sent to minimal projections via *-automorphisms. We can associate to every permutation \( \sigma : \Omega \to \Omega \) the automorphism \( \rho_\sigma : \ell^\infty(\Omega) \to \ell^\infty(\Omega) \) given by \( \rho_\sigma(f) = f \circ \sigma^{-1} \). The inverse map is obtained as follows: if \( \rho \) is a *-automorphism, then for every \( j \in \Omega \) there exists a unique \( \sigma_\rho(j) \in \Omega \) such that \( \rho(p_j) = p_{\sigma_\rho(j)} \) and \( \sigma_\rho \) is a well-defined permutation.

Notation 3.2. We denote by * the Schur(entrywise) multiplication of matrices \( A = [a_{ij}] \) and \( B = [b_{lk}] \) given by \( A * B = [a_{ij}b_{lk}] \), and let Diag be the map on matrices given by \( \text{Diag}([a_{ij}]) = [\delta_{ij}a_{ij}] \).

Notation 3.3. Let \( P \) and \( Q \) be non-negative matrices indexed by \( \Omega \). Define the set \( E(P) := \{ (i, j) \mid P_{ij} > 0 \} \) which is the collection of edges in the weighted directed graph defined by \( P \), and the set \( E(P, Q) := \{ (i, j, k) \mid P_{ij}Q_{jk} > 0 \} \). Also denote by \( \sqrt{P} \) and \( P^\flat \) the matrices with \((i, k)\)-th entry given by

\[
(P^\flat)_{ik} := \begin{cases} (P_{ik})^{-1}, & \text{if } (i, k) \in E(P) \\ 0, & \text{else} \end{cases}
\]

Theorem 3.4. Let \( P \) be a stochastic matrix over a state space \( \Omega \), and let \( \theta \) be the unital positive normal map associated to \( P \) by the previous proposition. The Arveson-Stinespring subproduct system associated to \( \theta \) is naturally (Id-)isomorphic to the following subproduct system, which will be denoted by \( \text{Arv}(\theta) \) or \( \text{Arv}(P) \):

1. The \( n \)-th fiber is a \( W^* \)-correspondence over \( \ell^\infty(\Omega) \), given by

\[
\text{Arv}(P)_n = \{ [a_{ij}] \mid a_{ij} = 0 \ \forall \ (i, j) \notin E(P^n) , \ \sup_{j \in \Omega} \sum_{i \in \Omega} |a_{ij}|^2 < \infty \}
\]

where left and right actions are given by multiplication as diagonal matrices. Given \( A, B \in \text{Arv}(P)_n \), their \( W^* \)-correspondence inner-product is given by

\[
\langle A, B \rangle = \text{Diag}(A^*B).
\]

2. The subproduct maps are given by

\[
U_{n,m}(A \otimes B) = (\sqrt{P^{n+m}})^\flat \ast \left( (\sqrt{P^m} \ast A) \cdot (\sqrt{P^m} \ast B) \right)
\]

for \( n \neq 0 \) and \( m \neq 0 \), \( A \in \text{Arv}(P)_n \) and \( B \in \text{Arv}(P)_m \). The maps \( U_{0,n} \) and \( U_{m,0} \) are given by left and right multiplication by elements of \( \ell^\infty(\Omega) \) respectively, considered as diagonal matrices.

We call this presentation of \( \text{Arv}(P) \) the standard presentation of the Arveson-Stinespring subproduct system associated to the stochastic matrix \( P \).

Proof. For the computation of the \( n \)-th fibers, we fix an \( n \in \mathbb{N} \). We will follow the notation and construction described in Definition 2.22. For convenience we will write \( \mathcal{L}(n) \) instead of \( \mathcal{L}^\theta(n) \).

Let us denote \( H = \ell^2(\Omega) \), and consider the canonical inclusion of \( \ell^\infty(\Omega) \) into \( B(\ell^2(\Omega)) \). Notice that the set \( \{ p_j \otimes e_i \}_{(i,j) \in E(P^n)} \) constitutes an orthogonal set in \( \ell^\infty(\Omega) \otimes_{g_n} H \) since for \( i, j, k, \ell \in \mathbb{N} \),

\[
\langle p_k \otimes e_\ell, p_j \otimes e_i \rangle = \langle e_\ell, \theta^n(p_k^*p_j)e_i \rangle = \delta_{kj}\delta_{\ell i} \langle e_i, \theta^n(p_j)e_i \rangle = \delta_{kj}\delta_{\ell i} P_{ij}^{(n)}.
\]

Furthermore, it is straightforward to check that \( \{ p_j \otimes e_i \}_{(i,j) \in E(P^n)} \) is in fact an orthogonal basis for \( \ell^\infty(\Omega) \otimes_{g_n} H \).
We now show that $\mathcal{L}(n)$ and $Arv(P)_n$ are isomorphic as correspondences over $\mathcal{M}$. Let $X \in \mathcal{L}(n)$. Then there exist unique scalars $(a_{ijk})_{i,j,k \in \Omega}$ such that $a_{ijk} = 0$ for $(i, j) \notin E(P^n)$ and for all $k \in \Omega$,

$$X(e_k) = \sum_{(i,j) \in E(P^n)} a_{ijk}p_j \otimes e_i.$$  

Since $X \in \mathcal{L}(n)$, it is a continuous linear map satisfying $(T \otimes I)X = XT$ for all $T \in \ell^\infty(\Omega)$ (see definition $\mathcal{L}(n)$). On the other hand, if $T = (c_k)_{k \in \Omega} \in \ell^\infty(\Omega)$, then

$$XT(e_k) = \sum_{(i,j) \in E(P^n)} a_{ijk}c_kp_j \otimes e_i$$  

$$(T \otimes I)X(e_k) = \sum_{(i,j) \in E(P^n)} a_{ijk}c_jp_j \otimes e_i$$

Thus by uniqueness of representation, we must have for $j \neq k$ that $a_{ijk} = 0$. Hence, if we define $a_{ij} := a_{ijj}$, and denote $A = [a_{ij}]$, we obtain,

$$X(e_j) = \sum_{i : (i,j) \in E(P^n)} a_{ij}p_j \otimes e_i = p_j \otimes Ae_j$$

where $a_{ij} = 0$ for $(i, j) \notin E(P^n)$, and $Ae_j = (a_{ij})_{i \in \Omega} \in \ell^2(\Omega)$ for each $j \in \Omega$.

The boundedness condition on $X$ ensures that

$$\|X\| \geq \|X(e_j)\| = \|p_j \otimes Ae_j\|$$

Since $X(e_j) \perp X(e_{j'})$ for all $j \neq j'$, we also have by Pythagoras that for every $b \in \ell^2(\Omega)$:

$$\|X(b)\|^2 = \|\sum_{j \in \Omega} b_jX(e_j)\|^2 = \sum_{j \in \Omega} |b_j|^2 \|X(e_j)\|^2 \leq \|b\|_2^2 \sup_{j \in \Omega} \|p_j \otimes Ae_j\|^2$$

Thus $\|X\| = \sup_{j \in \Omega} \|p_j \otimes Ae_j\|$ and,

$$\|p_j \otimes Ae_j\|^2 = \langle p_j \otimes Ae_j, p_j \otimes Ae_j \rangle = \sum_{i \in \Omega} |a_{ij}|^2 \langle e_i, \theta^n(p_j)e_i \rangle = \sum_{i \in \Omega} |a_{ij}|^2 P_{ij}^{(n)}.$$  

In this way each $X \in \mathcal{L}(n)$ has a unique matrix $A = [a_{ij}]$ in the set $\mathcal{E}(n)$ of matrices indexed by $\Omega$ satisfying $a_{ij} = 0$ for $(i, j) \notin E(P^n)$ and $\sup_{i,j} \sum_i |a_{ij}|^2 P_{ij}^{(n)} < \infty$. Conversely, it is easy to see that any matrix in $\mathcal{E}(n)$ is obtained in this fashion, and this implements a bijection of $\mathcal{E}(n)$ with $\mathcal{L}(n)$. In the remainder, we will denote by $X_A = X_{[a_{ij}]} = X_A^{(n)} = X_{[a_{ij}]}^{(n)}$ the unique element associated to $A \in \mathcal{E}(P)_n$ determined by the identity

$$X_A(e_k) = p_k \otimes Ae_k, \quad k \in \mathbb{N}.$$  

A brief computation shows that the left and right actions are given by $T \cdot X_A = (I \otimes T) \circ X_A = X_{T \cdot A}$ and $X_A \cdot T = X_{A \cdot T}$, where $T \in \ell^\infty(\Omega)$ is thought of as a diagonal matrix when multiplied with the matrix $A$. Furthermore, given $A, B \in Arv(P)_n$, notice that $\langle X_A, X_B \rangle = X_A^{(n)}X_B^{(n)}$ is an element of $\ell^\infty(\Omega)$ hence a diagonal matrix. A direct computation shows that for every $k \in \Omega$

$$X_A^{(n)}X_B^{(n)}e_k = X_A^{(n)}(p_k \otimes Be_k) = \langle p_k \otimes Ae_k, p_k \otimes Be_k \rangle \cdot e_k$$

Hence,

$$\langle X_A, X_B \rangle_{jj} = \langle p_j \otimes Ae_j, p_j \otimes Be_j \rangle = \langle Ae_j, \theta^n(p_j)(Be_j) \rangle = \sum_{i \in \Omega} a_{ij}P_{ij}^{(n)} b_{ij} = \langle \text{Diag}(\sqrt{P^{(n)} \ast A})^* (\sqrt{P^{(n)} \ast B}) \rangle_{jj}$$

and it follows that $\langle X_A, X_B \rangle = \text{Diag}(\sqrt{P^{(n)} \ast A})^* (\sqrt{P^{(n)} \ast B})$. Establishing the fibrewise correspondence inner product for $\mathcal{E}(n)$ we denote the family $\mathcal{E} = \{\mathcal{E}(n)\}_{n \in \mathbb{N}}$.

Let us now focus on the subproduct maps. For that purpose, fix $n, m \in \mathbb{N}$. Let us observe that in analogy with the situation above, the set $\{p_k \otimes p_j \otimes e_i\}_{(i,j,k) \in E(P^n, P^m)}$ is an orthogonal basis
for $\ell^\infty(\Omega) \otimes_{\gamma_1} \ell^\infty(\Omega) \otimes_{\gamma_2} H$, since for all $i, j, k, i', j', k'$,

$$
\langle p_k \otimes p_j \otimes e_i, p_{k'} \otimes p_{j'} \otimes e_{i'} \rangle = \delta_{i'i'} \delta_{jj'} \delta_{kk'} \langle e_i, \theta^n(p_j \theta^n(p_k)) e_{i} \rangle = \delta_{i'i'} \delta_{jj'} \delta_{kk'} \langle e_i, \theta^n(p_j) e_{i} \rangle \mathbf{P}^{(m)}_{jk}
$$

Furthermore, given $Y \in \mathcal{L}(m,n)$, by a computation analogous to the one above involving the intertwiner condition, there exist scalars $c_{ijk}$ for $(i,j,k) \in E(P^n,P^m)$ such that for each $k \in \Omega$

$$
Y(e_k) = \sum_{(i,j,k) \in E(P^n,P^m)} c_{ijk} p_k \otimes p_j \otimes e_i
$$

We may also define $c_{ijk} = 0$ for $(i,j,k) \notin E(P^n,P^m)$. Furthermore, the norm of $Y$ given by $\|Y\| = \sup_{k \in \Omega} \|Y(e_k)\|$. We denote such $Y$ as $Y = Y_{c_{ijk}} = Y_{[c_{ijk}]}$ where $c_{ijk} = 0$ for $(i,j,k) \notin E(P^n,P^m)$.

One can similarly compute $Y^*_m$ and the inner product on $\mathcal{L}(m,n)$ which would make it into a $W^*$ correspondence along with the usual left and right actions.

As we shall see, these computations are unnecessary for the computation of our subproduct system.

Define $V_{n,m} : \mathcal{L}(n+m) \to \mathcal{L}(m,n)$ by the usual formula $V_{n,m}(X) = \Gamma_{n,m} \circ X$, where $\Gamma_{n,m} : \ell^\infty(\Omega) \otimes_{\gamma_1} H \to \ell^\infty(\Omega) \otimes_{\gamma_2} \ell^\infty(\Omega) \otimes_{\gamma_2} H$ is defined by $\Gamma_{n,m}(a \otimes h) = a \otimes i \otimes h$.

It is evident that $V_{n,m}^* \circ X = \Gamma_{n,m}^* \circ X$ so in order to compute $V_{n,m}^*$, all one needs to do is compute $\Gamma_{n,m}^*$. So indeed, we compute $\Gamma_{n,m}^*$ by computing the projection $Q = \Gamma_{n,m}^* \circ \Gamma_{n,m}$ onto the image of $\Gamma_{n,m}$ which is exactly $\ell^\infty(\Omega) \otimes_{\gamma_1} \mathcal{O} \otimes_{\gamma_2} H$. This has the form $\{p_k \otimes I \otimes e_i \}_{(i,k) \in E(P^n+m)}$ as an orthogonal basis. In fact, as Hilbert spaces with the corresponding bases, $\ell^\infty(\Omega) \otimes_{\gamma_1} \mathcal{O} \otimes_{\gamma_2} H \cong \ell^\infty(\Omega) \otimes_{\gamma_1} \mathcal{O} \otimes_{\gamma_2} H$ via $\Gamma_{n,m}^*$.

We run the aforementioned computation. Indeed,

$$
Q(p_k \otimes p_j \otimes (c_{ijk})_{i}) = \sum_{i : (i,k) \in E(P^n+m)} \frac{\langle p_k \otimes I \otimes e_i, p_k \otimes p_j \otimes (c_{ijk})_{i} \rangle^2}{\|p_k \otimes I \otimes e_i\|^2} p_k \otimes I \otimes e_i
$$

And since, $\langle p_k \otimes I \otimes e_i, p_k \otimes p_j \otimes (c_{ijk})_{i} \rangle = c_{ijk} P_{ij}^{(n)} P_{jk}^{(m)}$ and $\|p_k \otimes I \otimes e_i\|^2 = \|p_k \otimes e_i\|^2 = P_{ik}^{(n+m)}$

We obtain that,

$$
Q \left( \sum_{j \in \Omega} p_k \otimes p_j \otimes (c_{ijk})_{i} \right) = \sum_{i : (i,k) \in E(P^n+m)} \sum_{j \in \Omega} c_{ijk} P_{ij}^{(n)} P_{jk}^{(m)} p_k \otimes I \otimes e_i
$$

Now, since $\Gamma_{n,m}^* = \Gamma_{n,m}^* Q$, we have that

$$
\Gamma_{n,m}^* \left( \sum_{j \in \Omega} p_k \otimes p_j \otimes (c_{ijk})_{i} \right) = \sum_{i : (i,k) \in E(P^n+m)} \sum_{j \in \Omega} c_{ijk} P_{ij}^{(n)} P_{jk}^{(m)} p_k \otimes e_i
$$

We define the usual $\Psi : \mathcal{L}(n) \otimes \mathcal{L}(m) \to \mathcal{L}(m,n)$ by $\Psi_{n,m}(X_A^{(n)} \otimes X_B^{(m)}) = (I \otimes X_A^{(n)}) \circ X_B^{(m)}$ and obtain by a simple computation that for $A = [a_{ij}] \in \mathcal{E}(n)$ and $B = [b_{ik}] \in \mathcal{E}(m)$,

$$
\Psi_{n,m}(X_A^{(n)} \otimes X_B^{(m)}) = Y_{[a_{ij}, b_{ik}]}
$$

So the multiplication maps are $U_{n,m} : \mathcal{L}(n) \otimes \mathcal{L}(m) \to \mathcal{L}(n+m)$ given by $U_{n,m} = V_{n,m}^* \Psi_{n,m}$ and we obtain:

$$
U_{n,m}(X_A^{(n)} \otimes X_B^{(m)}) = (\Gamma_{n,m}^* \circ \Psi_{n,m}^*)_{[a_{ij}, b_{ik}]}(e_k) = \sum_{i : (i,k) \in E(P^n+m)} \frac{\sum_{j \in \Omega} a_{ij} P_{ij}^{(n)} b_{jk} P_{jk}^{(m)} p_k \otimes e_i}{P_{ik}^{(n+m)}}
$$

In other words,

$$
U_{n,m}(X_A^{(n)} \otimes X_B^{(m)}) = X_{(P^n+m)y \left[ (P^n+m, (P^n+m, B) \left[ (P^n+m, B) \right] \right]}
$$
Defining $\tilde{U}_{n,m} : \mathcal{E}(n) \otimes \mathcal{E}(m) \to \mathcal{E}(n+m)$ by the rule

$$\tilde{U}_{n,m}(A \otimes B) = (P^{n+m})^* \left[ (P^n * A) \cdot (P^m * B) \right]$$

Yields that $(\mathcal{E}, \tilde{U})$ is a subproduct system naturally isomorphic to $(\mathcal{L}, U)$ via the fibrewise map $A \mapsto X_A$ (for $A \in \mathcal{E}(n)$). Now a computation yields that the (Id-)isomorphism $V : \mathcal{E} \to \text{Arv}(P)$ defined fibrewise by $V_n(A) = \sqrt{P^n} * A$ imposes a structure of a subproduct system on $\text{Arv}(P)$, with the aforementioned subproduct maps in the theorem, induced from that of $(\mathcal{E}, \tilde{U})$.

Note that the structure of the W*-correspondences in a standard presentation depends only on the graph structure of the stochastic matrix. Information on the weighted graph is contained only in the subproduct maps.

Recall Definition 2.22 and Theorem 2.7.

**Proposition 3.5.** Let $P$ be a finite and essential stochastic matrix over $\Omega$. Assume that $P$ decomposes into block diagonal form with irreducible stochastic blocks $P(1), \ldots, P(\ell)$, and that $\Omega(1), \ldots, \Omega(\ell)$ are the state sets corresponding to the rows of $P(1), \ldots, P(\ell)$ respectively. Then $\text{Arv}(P(k))$ is a subproduct system over $\ell^\infty(\Omega_k)$ for every $1 \leq k \leq \ell$, and $\text{Arv}(P)$ is canonically Id-isomorphic to $\text{Arv}(P(1)) \oplus \ldots \oplus \text{Arv}(P(\ell))$, when identifying $\ell^\infty(\Omega(1)) \oplus \ldots \oplus \ell^\infty(\Omega(\ell))$ with $\ell^\infty(\Omega)$ in the natural way.

**Proof.** Considering $P$ with the mentioned decomposition, every $A \in \text{Arv}(P)_n$ decomposes uniquely to block diagonal form with blocks $A(1), \ldots, A(\ell)$ along the diagonal with $A(k) \in \text{Arv}(P(k))_n$ for all $1 \leq k \leq \ell$. Since the subproduct $U^{\text{Arv}(P)}$ is matrix multiplication (up to Schur products), the block diagonal form is preserved, and we must have that $\text{Arv}(P) \cong_\text{Id} \text{Arv}(P(1)) \oplus \ldots \oplus \text{Arv}(P(\ell))$ via the map sending $A$ to $A(1) \oplus \ldots \oplus A(\ell)$.

**Remark 3.6.** There is another construction of subproduct systems from completely positive normal maps called the GNS subproduct system mentioned in [SS09 Section 3]. The GNS subproduct system associated to a stochastic matrix was computed for finite stochastic matrices in [Vis12]. In our special case of the concrete von-Neumann algebra $\ell^\infty(\Omega) \subseteq B(\ell^2(\Omega))$, both the GNS and Arveson-Stinespring subproduct systems are over the same von-Neumann algebra, and although we do not include the proof here, it turns out that the GNS subproduct system for recurrent $P$ is naturally isomorphic to the Arveson-Stinespring subproduct system for the time reversal of $P$. In some sense, this is a duality phenomenon in the context of subproduct systems arising from stochastic matrices. We use the word “duality” in analogy with the well-known duality between GNS and Arveson-Stinespring product systems (see [SS09 Remark 3.4] and [MS05]) and product systems in general (see [Ske03a]). In any case, given this duality phenomenon, the choice of which framework to use in the analysis of the subproduct systems arising from stochastic matrices is a matter of convenience. We will proceed with the framework of Arveson-Stinespring subproduct systems.

The main point of the following is to tell exactly when there exists an isomorphism between two Arveson-Stinespring subproduct systems arising from a stochastic matrix, in terms of the matrices, and to recognize a certain class of stochastic matrices distinguishable by their Arveson-Stinespring subproduct systems. Recall the 1-1 correspondence between *-automorphisms of $\ell^\infty(\Omega)$ and permutations of $\Omega$ preceding Theorem 3.3.

**Notation 3.7.** Let $P$ and $Q$ be stochastic matrices. If for a permutation $\sigma$ we have $P \sim_\sigma Q$ and for all $(i, j, k) \in E(P^n, P^m)$ we have

$$P_{ij}^{(n)} \cdot P_{jk}^{(m)} = Q_{\sigma(i)\sigma(j)}^{(n)} \cdot Q_{\sigma(j)\sigma(k)}^{(m)}$$

We say that $P$ and $Q$ satisfy equation (3.1) via $\sigma$.

**Theorem 3.8.** Let $P$ and $Q$ be two stochastic matrices.

1. If $\text{Arv}(P) \cong_\rho \text{Arv}(Q)$ then $P$ and $Q$ satisfy equation (3.1) via $\rho$.
2. If $P$ and $Q$ satisfy equation (3.1) via $\sigma$, then $\text{Arv}(\theta) \cong_\rho \text{Arv}(\phi)$. 
Proof. When \( \sigma \) is a permutation of \( \Omega \), we denote for brevity, \( i' = \sigma(i) \).

(1): Assume \( V : \text{Arv}(P) \to \text{Arv}(Q) \) is a given (unitary) isomorphism of the subproduct systems.
Then \( \rho = V_0 : \ell^\infty(\Omega) \to \ell^\infty(\Omega) \) is induced by a permutation \( \sigma = \sigma_\rho \). Now for all \((i, j) \in E(P^n)\) denote \( E_{ij} \) to be the element in \( \text{Arv}(P)_n \) which is 1 at \((i, j)\) and zero otherwise, and define \( E_{i'j'} \) similarly in \( \text{Arv}(Q)_n \). Due to \( V_n \) being a \( \rho \)-correspondence morphism, we have,

\[
V_n(E_{ij}) = V_n(p_i E_{ij} p_j) = \rho(p_i V_n(E_{ij}) p_j) = p_i V_n(E_{ij}) p_j'.
\]

So we must have that \( V_n(E_{ij}) = b_{ij}^{(n)} \cdot E_{i'j'} \) for some \( b_{ij}^{(n)} \in \mathbb{C} \). Due to \( V_n \) being isometric we have that,

\[
1 = \|\langle E_{ij}, E_{ij} \rangle\| = \|E_{ij}\|^2 = \|b_{ij}^{(n)} \cdot E_{i'j'}\|^2 = \|b_{ij}^{(n)} \cdot E_{i'j'}, b_{ij}^{(n)} \cdot E_{i'j'}\|^2 = |b_{ij}^{(n)}|^2
\]

By the formula for the subproducts maps in \( \text{Arv}(P) \) and \( \text{Arv}(Q) \) we have that,

\[
U^{P}_{n,m}(E_{ij} \otimes E_{jk}) = \sqrt{\frac{\rho(n)^{P(m)}}{\rho(n+m)}} \cdot E_{ik}, \quad U^{Q}_{n,m}(E_{i'j'} \otimes E_{j'k'}) = \sqrt{\frac{\rho(n)^{Q(m)}}{\rho(n+m)}} \cdot E_{i'k'}
\]

But since \( V \) preserves subproducts, we obtain,

\[
b_{ik}^{(n+m)} \sqrt{\frac{\rho(n)^{P(m)}}{\rho(n+m)}} = b_{ij}^{(n)} b_{jk}^{(m)} \sqrt{\frac{\rho(n)^{Q(m)}}{\rho(n+m)}} b_{k'}^{(n)} \rho(n)^{Q(m)}
\]

So we obtain equation \(3.1\) by squaring the absolute value.

(2): Define \( V_0 : \ell^\infty(\Omega) \to \ell^\infty(\Omega) \) by \( V_0(p_i) = \rho(p_i) \). Define \( V_n : \text{Arv}(P)_n \to \text{Arv}(Q)_n \) by the change of variables \( V_n(A) = R_\sigma A R_\sigma^{-1} \). We need to show that \( V_n \) is a \( \rho \)-unitary between the two \( W^* \)-correspondences, and preserves the subproduct maps.

Indeed, it is immediate that \( V_n \) is a \( \rho \)-unitary since it preserves the inner product via \( \rho \) and the left and right actions via \( \rho \). Thus we use equation \(3.1\) to show that \( V_n \) preserves subproducts maps. Let \( A \in \text{Arv}(P)_n \) and \( B \in \text{Arv}(P)_m \). Then,

\[
U^{Q}_{n,m}(V_n(A) \otimes V_n(B)) = (\sqrt{Q^{n+m}})^* \left[ (\sqrt{Q^n} * R_\sigma A R_\sigma^{-1}) \cdot (\sqrt{Q^m} * R_\sigma B R_\sigma^{-1}) \right] =
R_\sigma \left[ (R_\sigma^{-1} (\sqrt{P^{n+m}})^* R_\sigma) \cdot \left\{ (\sqrt{P^n} R_\sigma) \cdot A \right\} \cdot (\sqrt{P^m} R_\sigma) \cdot B \right] R_\sigma^{-1} = V_{n+m} U^{P}_{n,m}(A \otimes B)
\]

Where the second last equality is due to equation \(3.1\) being satisfied with respect to \( \sigma \).

Example 3.9. Arveson-Stinespring subproduct systems are unable to distinguish all finite stochastic matrices. Define for \( r \in (0, 1) \) the stochastic matrix over \( \Omega = \{1, 2, 3\} \):

\[
P(r) = \begin{bmatrix} 0 & r & 1 - r \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

Note that for all \( r, q \in (0, 1) \) the matrices \( P(r) \) and \( P(q) \) are graph isomorphic via \( \sigma = \text{Id}_\Omega \) and that \( P(r) = P(r)^n \) for all \( n \geq 1 \). Thus, to check that equation \(3.1\) is satisfied with respect to \( \sigma = \text{Id}_\Omega \), we need only show that for \( (i, j, k) \in E(P(r), P(r)) = E(P(q), P(q)) \) we have that

\[
\frac{P(r)_{ij} \cdot P(r)_{jk}}{P(r)_{ik}} = \frac{P(q)_{ij} \cdot P(q)_{jk}}{P(q)_{ik}}
\]

Which is readily seen to be satisfied for any \( (i, j, k) \in E(P, P) \) and \( r, q \in (0, 1) \). So Arveson-Stinespring subproduct system can’t distinguish a continuous family of finite reducible stochastic matrices, since for \( r \neq q \) in \((0, \frac{1}{2})\) we have that \( P(r) \) and \( P(q) \) are not isomorphic (via any \( \sigma \)) due to the matrices having different sets of probabilities.

Note that in the last example, \( i = 1 \) is inessential.
Example 3.10. Arveson-Stinespring subproduct systems cannot distinguish all infinite irreducible stochastic matrices.

Take \( r \in (0, 1) \) and denote by \( P(r) \) the matrix over \( \Omega = \mathbb{Z} \) with entries \( P(r)_{i(i+1)} = r \), \( P(r)_{i(i-1)} = 1 - r \), and all other entries zero. Then one can check that any two paths of the same length \( \ell \geq 2 \) both starting at \( i \) and ending at \( k \) have the same probabilities. This means that, if \( \gamma \) and \( \gamma' \) are two such paths, then

\[
P^{\ell}_{m=1} P(r)_{i\gamma(m)} r^{(m+1)} = P^{\ell}_{m=1} P(r)_{i\gamma'(m)} r^{(m+1)}
\]

Thus, if we sum over all paths of length \( n + m \) starting at \( i \) and ending at \( k \) on the right, and on all paths of length \( n + m \) starting at \( i \), ending at \( k \) and passing through \( j \) on the left, such that \( (i,j,k) \in E(P^n,P^m) \), we obtain that

\[
d_{n,m} \cdot P(r)_{ij} P(r)_{jk} = P(r)_{ik}^{n+m}
\]

Where \( d_{n,m} \) is the number of \( j' \in \mathbb{N} \) such that \((i, j', k) \in E(P^n, P^m)\) (which is independent of \( r \)). Thus equation (3.1) holds for \( P(r) \) and \( P(q) \) for all \( r, q \in (0, 1) \), and we must have that \( Arv(P(r)) \cong_{id} Arv(P(q)) \). This means that Arveson-Stinespring subproduct systems cannot distinguish a continuous family of irreducible stochastic matrices, since for different \( r \in (0, \frac{1}{2}] \), the associated matrices \( P(r) \) are not isomorphic (via any \( \sigma \)), as they have different sets of probabilities. Further note that if you take \( r < \frac{1}{2} \) then \( P(r) \) is transient while \( P(\frac{1}{2}) \) is recurrent.

So a question arises, what known classes of stochastic matrices can Arveson-Stinespring subproduct systems distinguish up to isomorphism?

Theorem 3.11. Let \( P \) and \( Q \) be recurrent stochastic matrices. If \( Arv(P) \cong_{\sigma} Arv(Q) \) then \( P \cong_{\sigma} Q \).

Proof. Take \( \sigma = \sigma_{P} \) to be the permutation on \( \Omega \) which induces the graph isomorphism between \( P \) and \( Q \) satisfying equation (3.1). We denote for brevity, \( i' = \sigma(i) \). First assume that \( P \) and \( Q \) are both irreducible and assume also that \( P \) is \( r \)-periodic.

We now show that \( P^{(n)}_{ii} = Q^{(n)}_{i'i'} \) for every \( n \in \mathbb{N} \) and \( i \in \Omega \). Indeed, if \( P^{(n)}_{ii} = 0 \) then \( Q^{(n)}_{i'i'} = 0 \) due to the graph isomorphism. Now if \( P^{(n)}_{ii} > 0 \) then again \( Q^{(n)}_{i'i'} > 0 \) due to the graph isomorphism, and \( i \) and \( i' \) have the same period \( r \) in \( P \) and \( Q \) respectively, again due to the graph isomorphism. Note that one has that \( Arv(P^n) \cong_{\sigma} Arv(Q^n) \), that \( P^n \) and \( Q^n \) are aperiodic (since \( n = nr' \) for some \( n' \in \mathbb{N} \)), and that both \( P^n \) and \( Q^n \) are recurrent (due to Lemma 2.12). This reduces the problem to showing that \( P^{(n)}_{ii} = Q^{(n)}_{i'i'} \) where \( P \) is replaced by \( P^n \) and \( Q \) is replaced by \( Q^n \). Thus, for all \( i \in \Omega \) we have

\[
P^{(n)}_{ii} \cdot P^{(m)}_{ii} = \frac{Q^{(n)}_{i'i'} \cdot Q^{(m)}_{i'i'}}{Q^{(n+m)}_{i'i'}}
\]

Where these expressions are always well defined. It follows that,

\[
\left( P^{(M)}_{ii} \right)^{M} = \prod_{m=1}^{M-1} P^{(m)}_{ii} \frac{P^{(M)}_{ii}}{P^{(m+1)}_{ii}} = \prod_{m=1}^{M-1} Q^{(m)}_{i'i'} \frac{Q^{(m+1)}_{i'i'}}{Q^{(M)}_{i'i'}} = \left( Q^{(M)}_{i'i'} \right)^{M} \frac{P^{(M)}_{ii}}{Q^{(M)}_{i'i'}}
\]

Since \( P \) and \( Q \) are recurrent chains, they must be aperiodic. Thus by taking an \( M \)-th root and a limit in \( M \), we obtain that \( P^{(M)}_{ii} = Q^{(M)}_{i'i'} \).

Now by taking \( i = k \) in equation (3.1) and \( n = 1 \) we obtain for all \( i, j \in \Omega \)

\[
P^{(1)}_{ij} P^{(m)}_{ji} = Q^{(1)}_{i'j'} Q^{(m)}_{j'i'}
\]

By taking sums over \( m \) we obtain

\[
P_{ij} \sum_{m=1}^{M} P^{(m)}_{ji} = Q_{i'j'} \sum_{m=1}^{M} Q^{(m)}_{j'i'}
\]

Now since \( P^{(m)}_{ii} = Q^{(m)}_{i'i'} \) for all \( m \in \mathbb{N} \), we must have that,

\[
P_{ij} \cdot \frac{\sum_{m=1}^{M} P^{(m)}_{ji}}{\sum_{m=1}^{M} P^{(m)}_{ii}} = Q_{i'j'} \cdot \frac{\sum_{m=1}^{M} Q^{(m)}_{j'i'}}{\sum_{m=1}^{M} Q^{(m)}_{i'i'}}
\]
Now if $P$ is a recurrent irreducible stochastic matrix, by Doeblin’s ratio limit theorem (see Part I, Section 9, Theorem 5 in [Chu60]) we have for $P$ (and also for $Q$) that for any three states $i, j, k \in \Omega$,

$$\lim_{M \to \infty} \frac{\sum_{m=1}^{M} p^{(m)}_{ij}}{\sum_{m=1}^{M} p^{(m)}_{kj}} = 1$$

Thus, by taking $M \to \infty$ in equation (3.2) and using Doeblin’s theorem for both $P$ and $Q$, we must have that $P_{ij} = Q_{ij'}$.

Now, for general (reducible) recurrent stochastic matrices $P$ and $Q$, Theorem 2.7 enables us to decompose $P$ into $P(1), P(2), \ldots$ in block diagonal form, which induces a decomposition of $Q$ into $Q(1), Q(2), \ldots$ in block diagonal form via the graph isomorphism $\sigma$, such that for all $0 < k \in \mathbb{N}$ we have $Arv(P(k)) \cong_{\rho_k} Arv(Q(k))$ for some $*$-automorphism $\rho_k$, since equation (3.1) is satisfied via $\sigma_p$ (restricted to the appropriate set of indices in $\Omega$) for the pairs $P(k)$ and $Q(k)$ for every $0 < k \in \mathbb{N}$, and thus a reduction is made to the general case.

4. Cuntz-Pimsner algebras for subproduct systems

We describe the construction of Toeplitz and Cuntz-Pimsner algebras arising from subproduct systems (see [Vis11], [Vis12]). Let $X = (X_n)_{n \in \mathbb{N}}$ be a subproduct system. We use the following notations throughout this work. The $X$-Fock correspondence is the $\mathcal{W}^*$-correspondence (weak) direct sum of the fibers of the subproduct system:

$$\mathcal{F}_X := \bigoplus_{n \in \mathbb{N}} X_n$$

Denote by $Q_n \in \mathcal{L}(\mathcal{F}_X)$ the projection of $\mathcal{F}_X$ onto the $n$th fiber $X_n$. Define $Q_{[0,n]} = Q_0 + Q_1 + \ldots + Q_n$ and $Q_{[n,\infty]} = Id - Q_{[0,n-1]}$.

The $X$-shifts are the operators $S^{(n)}_\xi \in \mathcal{L}(\mathcal{F}_X)$ for $n \in \mathbb{N}$, $\xi \in X_n$ given by

$$S^{(n)}_\xi (\eta) := U_{n,m}(\xi \otimes \eta)$$

For $m \in \mathbb{N}$, $\eta \in X_m$. Since $\mathcal{F}_X$ is a $\mathcal{W}^*$-correspondence, $S^{(n)}_\xi$ are adjointable.

**Definition 4.1.** The Toeplitz algebra $\mathcal{T}(X)$ is the $C^*$-subalgebra of $\mathcal{L}(\mathcal{F}_X)$ generated by all $X$-shifts and $\mathcal{M}$,

$$\mathcal{T}(X) := C^*(\mathcal{M} \cup \{ S^{(n)}_\xi \mid \xi \in X_n, \ n \in \mathbb{N} \})$$

**Remark 4.2.** One can show that $S^{(n)}_\xi$ are adjointable even if one takes the $C^*$-correspondence direct sum for the Fock direct sum correspondence, and due to proposition 2.14, one obtains the same Toeplitz algebra as before.

The algebra $\mathcal{L}(\mathcal{F}_X)$ admits a natural action $\alpha$ of the unit circle $\mathbb{T} \subseteq \mathbb{C}$, called the gauge action, defined by $\alpha_\lambda(T) = W_\lambda TW_\lambda^*$ for all $\lambda \in \mathbb{T}$ where $W_\lambda : \mathcal{F}_X \to \mathcal{F}_X$ is the unitary defined by

$$W_\lambda(\oplus_{n \in \mathbb{N}} \xi_n) = \oplus_{n \in \mathbb{N}} \lambda^n \xi_n$$

Since $\alpha_\lambda(S^{(n)}_\xi) = S^{(n)}_{\lambda^n \xi}$, it follows that the Toeplitz algebra is an $\alpha$-invariant closed $C^*$-subalgebra, so the circle action can be restricted to a circle action on the Toeplitz algebra. One then shows that for every $S \in \mathcal{T}(X)$, the function $f(\lambda) = \alpha_\lambda(S)$ is norm continuous. This enables the definition of a conditional expectation $\Phi$ defined by

$$\Phi(S) = \int_{\mathbb{T}} \alpha_\lambda(S)d\lambda$$

Where $d\lambda$ is the normalized Haar measure on $\mathbb{T}$.

Let $\{k_n\}_{n=1}^\infty$ denote Fejér’s kernel function defined for $\lambda \in \mathbb{T}$ by

$$k_n(\lambda) = \sum_{j=-n}^{n} \left(1 - \frac{|j|}{n+1}\right)\lambda^j$$
Note that for $S \in \mathcal{T}(X)$, the existence of the canonical conditional expectation $\Phi$ permits the definition of Fourier coefficients for an element $S \in \mathcal{T}(X)$ by

$$\Phi_n(S) = \int T \alpha_\lambda(S) \lambda^{-n} d\lambda$$

Then define the Cesaro sums,

$$\sigma_n(S) := \sum_{j=-n}^{n} (1 - \frac{|j|}{n+1}) \Phi_j(S) = \int T \sum_{j=-n}^{n} (1 - \frac{|j|}{n+1}) \alpha_\lambda(S) \lambda^{-j} d\lambda = \int T \alpha_\lambda(S) k_n(\lambda) d\lambda$$

**Proposition 4.3.** For every $S \in \mathcal{T}(X)$, the Cesaro sums $\sigma_n(S)$ converge in norm to $S$.

The proof is an easy adaptation of the proof of Theorem VIII.2.2 in [Dav96].

**Definition 4.4.** For each $k \in \mathbb{Z}$ the $k$-th spectral subspace for $\alpha$ is defined by

$$\mathcal{T}(X)_k = \{ T \in \mathcal{T}(X) \mid \alpha_\lambda(T) = \lambda^k T \}$$

**Definition 4.5.** Let $X = (X_n)_{n \in \mathbb{N}}$ be a subproduct system. A monomial in $\mathcal{T}(X)$ is a composition of finitely many of the operators $S^{(n)}_\xi$ and their adjoints. Every such operator can be written as $\prod_{i=1}^{r} S^{(m_i)}_{\xi_i} S^{(n_i)}_{\zeta_i}$ for suitable $0 < i \in \mathbb{N}$, and $n_i, m_i \in \mathbb{N}$, $\xi_i \in X_{m_i}$, $\zeta_i \in X_{n_i}$. A monomial of this form is said to be of degree $\sum_{i=1}^{r} (m_i - n_i)$. For $k \in \mathbb{Z}$ define $\mathcal{T}_k(X)$ to be the closure of all homogeneous polynomials of degree $k$. Note that $\mathcal{T}_0(X)$ is a $C^*$-subalgebra of $\mathcal{T}(X)$.

The next corollary gives us a characterization of the graded structure of Toeplitz algebras, in terms of the natural circle action.

**Corollary 4.6.** Let $X$ be a subproduct system. Then, $\mathcal{T}_k(X) = \mathcal{T}(X)_k$.

**Proof.** It is trivial to show that $\mathcal{T}_k(X) \subseteq \mathcal{T}(X)_k$. To show the reverse inclusion, take $T \in \mathcal{T}(X)_k$, and let $T_n$ be a sequence of polynomials in $\mathcal{T}(X)$ converging to $T$ in norm. Since

$$\|T - \Phi_k(\sigma_n(T_n))\| \leq \|T - \sigma_n(T)\| + \|\Phi_k(\sigma_n(T)) - \Phi_k(\sigma_n(T_n))\| \leq \|T - \sigma_n(T)\| + \|T - T_n\|$$

We must have that $T$ is a norm limit of homogeneous polynomials of degree $k$, and we are done. \qed

A closed subspace $M \leq \mathcal{T}(X)$ is invariant under $\alpha$ or simply invariant if for all $\lambda \in \mathbb{T}$ one has $\alpha_\lambda(M) \subseteq M$, along with the previous proposition we obtain:

**Corollary 4.7.** Let $M \leq \mathcal{T}(X)$ be a closed invariant subspace. Then $M$ is the closed linear span of homogeneous polynomials in $M$. That is, $M = \overline{\text{sp}}(\bigcup (M \cap \mathcal{T}_k(X)))$.

**Proof.** Let $S \in M$. Then for every $k \in \mathbb{Z}$ denote $\Phi_k(S) := \int T \alpha_\lambda(S) \lambda^{-k} d\lambda$ which is in $M \cap \mathcal{T}_k(X)$. Since $\sigma_n(S)$ is a linear combination of $\{\Phi_k(S)\}_{k=-n}^{n}$, we are done. \qed

**Proposition 4.8.** Define a subset $\mathcal{J} \subseteq \mathcal{L}(\mathcal{F}_X)$ by

$$\mathcal{J} = \{ T \in \mathcal{L}(\mathcal{F}_X) \mid \lim_{n \to \infty} \|TQ_n\| = 0 \}$$

Then $\mathcal{J}$ is a closed left invariant ideal in $\mathcal{L}(\mathcal{F}_X)$.

**Proof.** $\mathcal{J}$ is obviously a left ideal, and is closed in norm. Indeed, take $S \in \mathcal{J}$, and $\epsilon > 0$, choose $T \in \mathcal{J}$ such that $|S - T| < \epsilon$. Thus we have that $\|SQ_n - TQ_n\| < \epsilon$ for every $n \in \mathbb{N}$, so we must have that also $\|SQ_n\| \to_{n \to \infty} 0$. Now, if $T \in \mathcal{J}$, we note that for each $\lambda \in \mathbb{T}$ and each $m \in \mathbb{N}$ $W_\lambda Q_m = Q_m W_\lambda$ and thus,

$$\|\alpha_\lambda(T)Q_m\| = \|W_\lambda TW_\lambda^* Q_m\| = \|W_\lambda TQ_m W_\lambda^*\| = \|TQ_m\| \to_{m \to \infty} 0$$

\qed

**Notation 4.9.** Let $X$ be a subproduct system, and let $\mathcal{A}$ be an invariant $C^*$-subalgebra of $\mathcal{L}(\mathcal{F}_X)$. Let $\mathcal{J}(\mathcal{A}) := \mathcal{A} \cap \mathcal{J}$, which is an invariant closed left ideal in $\mathcal{A}$.

**Proposition 4.10.** $\mathcal{J}(\mathcal{T}(X))$ is a two sided closed invariant ideal in $\mathcal{T}(X)$ and one has

$$\mathcal{J}(\mathcal{T}(X)) = \{ T \in \mathcal{T}(X) \mid \lim_{n \to \infty} \|TQ_{[n,\infty)}\| = 0 \}$$
Proof. The fact that \( \mathcal{J}(\mathcal{T}(X)) \) is a left ideal is shown in [Vis12] Theorem 2.5, and the equality of sets above follows from [Vis12] Corollary 2.7.

**Definition 4.11.** Let \( X = (X_n)_n \) be a subproduct system, and let \( \mathcal{T}(X) \) be its corresponding Toeplitz algebra. We define the Cuntz ideal of \( \mathcal{T}(X) \) to be \( \mathcal{J}(\mathcal{T}(X)) = \mathcal{J} \cap \mathcal{T}(X) \). The Cuntz-Pimsner algebra of \( X \) is then defined to be \( \mathcal{O}(X) := \mathcal{T}(X)/\mathcal{J}(\mathcal{T}(X)) \).

Going back to the full product system in example 2.21, we note that the Toeplitz and Cuntz-Pimsner algebras of \( X_F \) are the usual Toeplitz and Cuntz-Pimsner algebras of the \( W^* \)- correspondence \( E \), in the sense of Katsura [Kat04].

See [Vis12] Example 3.6 and Remark 3.7 for some motivation as to why Viselter defined the Cuntz-Pimsner algebra in this way for subproduct systems.

We denote by \( \overline{T} \) the image of \( T \in \mathcal{T}(X) \) under the canonical quotient from \( \mathcal{T}(X) \) onto \( \mathcal{O}(X) \).

**Proposition 4.12.** If \( Q_n \in \mathcal{T}(X) \) for all \( n \in \mathbb{N} \), then
\[
||\overline{T}||_{\mathcal{O}(X)} = \lim_{n \to \infty} ||TQ_{[n,\infty]}||_{\mathcal{T}(X)}
\]
Further, if \( T \in \mathcal{T}(X)_k \) then,
\[
||\overline{T}||_{\mathcal{O}(X)} = \limsup_{n \to \infty} ||TQ_n||_{\mathcal{T}(X)}
\]

**Proof.** Indeed, for every \( \epsilon > 0 \) there exists \( Q \in \mathcal{J}(\mathcal{T}(X)) \) such that
\[
||\overline{T}|| \geq ||T + Q|| - \epsilon \geq \limsup_{n \to \infty} ||(T + Q)Q_{[n,\infty]}|| - \epsilon \geq \lim_{n \to \infty} ||TQ_{[n,\infty]}|| - \epsilon = \lim_{n \to \infty} ||Q||_{\mathcal{O}(X)} ||Q_{[n,\infty]}|| - \epsilon
\]
Thus we have that \( ||\overline{T}|| \geq \limsup_{n \to \infty} ||TQ_{[n,\infty]}|| \). We also have \( ||\overline{T}|| = \inf_{Q \in \mathcal{J}(\mathcal{T}(X))} ||T + Q|| \), so by taking \( Q = -TQ_{[0,n]} \in \mathcal{J}(\mathcal{T}(X)) \) we obtain that \( ||\overline{T}|| \leq \inf_{n \to \infty} ||TQ_{[n,\infty]}|| \).

For the second equality, note that if \( T \in \mathcal{T}(X)_k \) for \( k \in \mathbb{Z} \) then for \( n \geq k \) we have \( T ||_{X_n} : X_n \to X_{n+k} \) and thus \( ||TQ_{[n,\infty]}||_{\mathcal{T}(X)} = \sup_{m \geq n} ||TQ_m||_{\mathcal{T}(X)} \).

**Remark 4.13.** Note that the circle action on \( \mathcal{T}(X) \) passes naturally to \( \mathcal{O}(X) \) due to \( \mathcal{J}(\mathcal{T}(X)) \) being invariant. This means that we still have \( \mathcal{O}_k(X) = \mathcal{O}(X)_k \) where these sets are defined to be the images of \( T_k(X) \) and \( \mathcal{T}(X)_k \) under the quotient map from \( \mathcal{T}(X) \) onto \( \mathcal{O}(X) \).

The following is a useful proposition used to establish \( * \)-isomorphisms between C*-algebras with a gauge action, it is a special case of Proposition 4.5.1 in [BODS] for when the compact group is the torus \( \mathbb{T} \).

**Proposition 4.14.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be two C*-algebras with gauge (torus) actions \( \alpha \) and \( \beta \) respectively. Assume further that \( \pi : \mathcal{A} \to \mathcal{B} \) is a \( * \)-homomorphism satisfying for every \( a \in \mathcal{A} \) and \( \lambda \in \mathbb{T} \) that \( \beta_{\lambda}(\pi(a)) = \pi(\alpha_{\lambda}(a)) \). Then \( \pi \) is injective if and only if it is injective on the fixed point algebra \( \mathcal{A}_0 = \{ a \in \mathcal{A} \mid \alpha_{\lambda}(a) = a \} \).

We now relate the theory of finite direct sums of subproduct systems, to that of finite direct sums of Toeplitz and Cuntz algebras.

**Proposition 4.15.** Let \( (X,U^X) \) and \( (Y,U^Y) \) be subproduct systems over \( \mathcal{M} \) and \( \mathcal{N} \) respectively. Define the unitary element \( W_{X,Y} : \mathcal{F}_X \oplus \mathcal{F}_Y \to \mathcal{F}_X \oplus \mathcal{F}_Y \) by \( W_{X,Y}(\oplus_{n=0}^\infty (\xi_n \oplus \eta_n)) = (\oplus_{n=0}^\infty \xi_n) \oplus (\oplus_{n=0}^\infty \eta_n) \) for \( \xi_n \in X_n \) and \( \eta_n \in Y_n \). Further note that we have the natural inclusion \( \mathcal{T}(X) \oplus \mathcal{T}(Y) \subseteq \mathcal{L}(\mathcal{F}_X \oplus \mathcal{F}_Y) \).

We omit the straightforward proof of the following proposition.
5. Cuntz-Pimsner algebras of subproduct systems arising from finite essential stochastic matrices

In this section we compute the Cuntz-Pimsner algebras of subproduct systems arising from stochastic matrices, in the sense defined above, where \( \Omega \) is finite and \( P \) essential. We sometimes denote \( T(P) := T(Arv(P)) \) and \( O(P) := O(Arv(P)) \).

**Remark 5.1.** In the case of finite matrices, a matrix \( P \) is recurrent if and only if it is positive-recurrent if and only if it is essential. In particular, finite irreducible stochastic matrices, are positive-recurrent.

Note that the subproduct system associated to a finite stochastic matrix in standard presentation has no boundedness condition on elements of the fibers since \( P \) is finite. And we also have that \( \text{Diag} : M_d(\mathbb{C}) \to \ell^\infty(\Omega) \cong \mathbb{C}^d \) is a faithful conditional expectation.

We look at the shift operators of \( Arv(P) \). For every \( n \in \mathbb{N} \) and \( A \in Arv(P)_n \) we denote by \( S^{(n)}_A \) the operator defined for every \( B \in Arv(P)_m \) for \( m > 0 \) by

\[
S^{(n)}_A(B) = U_{n,m}(A \otimes B) = (\sqrt{P^{n+m}})^* \left[ (\sqrt{P^n} \ast A) \cdot (\sqrt{P^m} \ast B) \right]
\]

where for \( m = 0 \) it is simply the right multiplication of \( A \in Arv(P)_n \) by an element of \( \ell^\infty(\Omega) \).

The operator \( S^{(n)*}_A \) for \( A \in Arv(P)_n \) is given by the following formula

\[
S^{(n)*}_A(B) = \sqrt{P^n} \ast ( (\sqrt{P^n} \ast A)^* \cdot (\sqrt{P^{n+m}})^* \ast B)
\]

for \( B \in Arv(P)_{n+m} \) and \( m > 0 \). Further satisfying \( S^{(n)*}_A(B) = 0 \) for \( B \in Arv(P)_m \) where \( m < n \), and \( S^{(n)*}_A(B) = \text{Diag} [ A^* B ] \) for \( B \in Arv(P)_n \).

The Toeplitz algebra is defined to be

\[
T(P) := C^\ast \left( M \cup \{ S^{(n)}_A \mid n \in \mathbb{N}, \ A \in Arv(P)_n \} \right)
\]

We also recall that \( Q_n \in \mathcal{L}(\mathcal{F}_{Arv(P)}) \) is the projection onto the \( n \)-th summand of the Fock module direct sum.

**Proposition 5.2.** Let \( P \) be finite and stochastic. Then \( Q_n \in T(P) \) for every \( n \in \mathbb{N} \).

**Proof.** Let \( E_{ij} \) be the matrix with 1 in the \((i,j)\)-th coordinate and zeros everywhere else. Then we show directly that

\[
Q_{[n,\infty)} = \sum_{(i,j) \in E(P^n)} S^{(n)}_{E_{ij}} S^{(n)*}_{E_{ij}} \in T(P)
\]

Indeed, since \( S^{(n)*}_{E_{ij}}(A) = 0 \) for all \( m < n \) and \( A \in Arv(P)_m \), it would suffice to show that the right hand side in the above equation is the identity on \( E_{lk} \in Arv(P)_m \) for all \((l,k) \in E(P^m)\) and \( m \geq n \). For \( E_{lk} \in Arv(P)_m \) such that \( m > n \) we have

\[
S^{(n)*}_{E_{ij}}(E_{lk}) = \delta_{i,l} \cdot \sqrt{\frac{P^{(n)}_{ij} P^{(m)}_{jk}}{P^{(n+m)}_{lk}}} E_{jk}, \quad \text{and} \quad S^{(n)}_{E_{ij}}(E_{jk}) = \sqrt{\frac{P^{(n)}_{ij} P^{(m)}_{jk}}{P^{(n+m)}_{lk}}} E_{lk}
\]

So we have that

\[
S^{(n)}_{E_{ij}} S^{(n)*}_{E_{ij}}(E_{lk}) = \delta_{i,l} \cdot \sqrt{\frac{P^{(n)}_{ij} P^{(m)}_{jk}}{P^{(n+m)}_{lk}}} E_{lk}
\]

where \( \delta_{i,l} = 1 \) if \( i = l \) and zero otherwise. Now by taking sums over \( i,j \in \Omega \) and noting that a similar computation works for \( m = n \), we obtain our description of \( Q_{[n,\infty)} \) above.

Thus, one sees that \( Q_0 = I - Q_{[1,\infty)} \in T(P) \) and \( Q_n = Q_{[n,\infty)} - Q_{[n+1,\infty)} \in T(P) \). \( \square \)

**Definition 5.3.** We define for every \( A \in Arv(P)_n \) the operators

\[
T^{(n)}_A : Arv(P)_m \to Arv(P)_{n+m} \quad \text{by} \quad T^{(n)}_A(B) = S^{(n)}_{(\sqrt{P^n})^* A}(B)
\]

And

\[
W^{(n)}_A : Arv(P)_m \to Arv(P)_{n+m} \quad \text{by} \quad W^{(n)}_A(B) = A \cdot B
\]

for \( B \in Arv(P)_m \).
Proposition 5.4. Let $P$ be finite and stochastic. $T_A^{(n)}$ and $W_A^{(n)}$ are well-defined bounded operators in $\mathcal{L}(\mathcal{F}_{\text{Arv}(P)})$ for every $A \in \text{Arv}(P)_n$.

Proof. During this proof, summation is taken over all $i \in \Omega$ such that $(i,j) \in E(P^n)$, yet we abuse notation and simply write $i \in \Omega$ for brevity. Assume $A = [a_{ij}]$ and $B = [b_{jk}]$.

$$||T_A^{(n)}(B)|| = ||U_{n,m}(((\sqrt{P^n})^\delta * A) \otimes B)|| \leq ||(\sqrt{P^n})^\delta * A|| \cdot ||B||$$

Since one has,

$$||(\sqrt{P^n})^\delta * A||^2 = \sup_{j \in \Omega} \sum_{i \in \Omega} |a_{ij}|^2$$

The fact that $P$ is finite entails that $T_A^{(n)}$ is bounded and well-defined. We now show well-definedness and boundedness of $W_A^{(n)}$. Indeed, we have,

$$||W_A^{(n)}(B)||^2 = ||\left[\sum_{j \in \Omega} a_{ij}b_{jk}\right]|^2 = \sup_{k} \sum_{j \in \Omega} |a_{ij}|^2 \leq \sum_{i \in \Omega} \left(\sum_{j \in \Omega} |a_{ij}|^2\right)^\delta \left(\sum_{l \in \Omega} |b_{lk}|^2\right)^\delta ||B||^2$$

Where we’ve used the Cauchy Schwarz inequality in the inequality above. \hfill \square

The last proposition ensures that in the finite case, we have

$$\{T_A^{(n)} \mid A \in \text{Arv}(P)_n, \ n \in \mathbb{N}\} = \{S_A^{(n)} \mid A \in \text{Arv}(P)_n, \ n \in \mathbb{N}\}$$

Thus,

$$\mathcal{T}(P) = C^* \left(M \cup \{T_A^{(n)} \mid n \in \mathbb{N}, \ A \in \text{Arv}(P)_n\}\right)$$

We denote

$$\mathcal{T}^\infty(P) = C^* \left(M \cup \{W_A^{(n)} \mid n \in \mathbb{N}, \ A \in \text{Arv}(P)_n\}\right)$$

Note that by Theorem 3.4 the structure of $\mathcal{T}^\infty(P)$ depends only on the graph structure of $P$, and that $\mathcal{T}^\infty(P)$ is invariant under the circle action on $\mathcal{L}(\mathcal{F}_{\text{Arv}(P)})$ since $\alpha_\lambda(W_A^{(n)}) = W_{\lambda\alpha}^{(n)}$. Recall that $\mathcal{J} = \{T \in \mathcal{L}(\mathcal{F}_{\text{Arv}(P)}) \mid \lim_{n \to \infty} ||TQ_n|| = 0\}$ is an invariant closed left ideal of $\mathcal{L}(\mathcal{F}_{\text{Arv}(P)})$.

Proposition 5.5. Let $P$ be a finite irreducible stochastic matrix. For every $n \in \mathbb{N}$ and $A \in \text{Arv}(P)_n$ we have $T_A^{(n)} - W_A^{(n)} \in \mathcal{J}$.

Proof. Assume $P$ has period $r \geq 1$. Let $A = [a_{ij}] \in \text{Arv}(P)_n$ and $B = [b_{ij}] \in \text{Arv}(P)_m$. Evaluating

$$||[(T_A^{(n)} - W_A^{(n)})(B)||^2$$

using Cauchy-Schwarz inequality yields

$$||[(T_A^{(n)} - W_A^{(n)})(B)||^2 \leq \sup_k \left(\sum_{i,j} |a_{ij}|^2 c_m(i,j,k)\right) \cdot ||B||^2$$

Where

$$c_m(i,j,k) = \left\{\begin{array}{cl}
\left|\sqrt{\frac{P_{ik}^{(m)}}{P_{ij}^{(n+m)}}} - 1\right|^2 & : (i,j,k) \in E(P^n, P^m) \\
0 & : (i,j,k) \notin E(P^n, P^m)
\end{array}\right.$$}

So in order to show that $||(T_A^{(n)} - W_A^{(n)})Q_m|| \to 0$ as $m \to \infty$ one has $c_m(i,j,k) \to 0$. Decompose $P$ into stochastic matrices $P_0, ..., P_{r-1}$ as in Theorem 2.8 and let $\Omega_0, ..., \Omega_{r-1}$ be the state sets corresponding to the rows of $P_0, ..., P_{r-1}$ in the decomposition. Choose any $(i,j,k) \in \Omega^3$ and assume $i \in \Omega_t$, $j \in \Omega_{t+\ell_1}$, $k \in \Omega_{t+\ell_1+\ell_2}$. Assume by negation that we have a subsequence $\{c_{m_\alpha}(i,j,k)\}$ composed of non-zero entries only, such that $c_{m_\alpha}(i,j,k)$ does not tend to 0 as $\alpha \to \infty$. Then we must have that $(i,j,k) \in E(P^\alpha)$ for every $\alpha \in \mathbb{N}$. Due to $r$-periodicity of all states in
our matrix, we must have that \( n = n' r + \ell_1 \) and \( m_\alpha = m'_\alpha r + \ell_2 \). So we can find arbitrarily large \( m'_\alpha \in \mathbb{N} \) such that

\[
\sqrt{\frac{P_{jk}(n', r + \ell_2)}{P(n' + m'_\alpha r + \ell_1 + \ell_2)}} - 1 \geq \epsilon
\]

For some fixed \( \epsilon > 0 \). This contradicts the fact that \( P_{jk}(tr + \ell_2) \rightarrow_{t \to \infty} \pi_k r \) and \( P_{ik}(n' + \ell_1 + \ell_2) \rightarrow_{t \to \infty} \pi_k r \), which is due to Theorem 2.10.

Recall that if \( A \) is a gauge invariant \( C^* \)-subalgebra of \( \mathcal{L}(\mathcal{F}_{\mathcal{C}^*}(P)) \) then \( \mathcal{J}(A) := A \cap \mathcal{J} \) is an invariant closed left ideal in \( A \).

**Theorem 5.6.** Let \( P \) be an irreducible finite stochastic matrix. Let

\[
\mathcal{T}^c(P) := \mathcal{C}^*\left( \mathcal{C}^*(\Omega) \cup \left\{ T^{(n)}_A, W^{(n)}_A \mid n \in \mathbb{N}, \ A \in \mathcal{A}(P)_n \right\} \right)
\]

Then \( \mathcal{J}(\mathcal{T}^c(P)) \) is an invariant two sided closed ideal in \( \mathcal{T}^c(P) \) so that both \( \mathcal{J}(\mathcal{T}(P)) \) and \( \mathcal{J}(\mathcal{T}(\infty)(P)) \) are invariant two sided closed ideals in \( \mathcal{T}(P) \) and \( \mathcal{T}(\infty)(P) \) respectively, and if \( P \) is also irreducible then

\[
\mathcal{O}(P) := \mathcal{T}(P)/\mathcal{J}(\mathcal{T}(P)) \cong \mathcal{T}(\infty)(P)/\mathcal{J}(\mathcal{T}(\infty)(P))
\]

**Proof.** \( \mathcal{T}^c(P) \) is the closure of the linear span of all homogeneous polynomials in the variables \( T^{(n)}_A, W^{(n)}_A \) and their adjoints. Now \( \alpha_\lambda(W^{(n)}_A) = \lambda^n W^{(n)}_A \) and \( \alpha_\lambda(T^{(n)}_A) = \lambda^n T^{(n)}_A \), and thus one must have that \( \mathcal{T}^c(P) \) is an invariant \( C^* \)-subalgebra of \( \mathcal{L}(\mathcal{F}_{\mathcal{C}^*}(P)) \).

We show that \( \mathcal{J}(\mathcal{T}^c(P)) \) being a closed invariant left ideal, is also a right ideal in \( \mathcal{T}^c(P) \). Indeed, Take \( S \in \mathcal{J}(\mathcal{T}^c(P)) \) and \( T \in \mathcal{T}^c(P) \). Assume first that \( T \) is homogeneous of degree \( m \in \mathbb{Z} \). Then \( T : \mathcal{A}(P)_n \to \mathcal{A}(P)_{n+m} \) when \( n + m \geq 0 \). Thus,

\[
||STQ_n|| = ||SQ_{n+m}TQ_n|| \leq ||SQ_{n+m}|| \cdot ||T|| \to_{n \to \infty} 0
\]

Now since \( \mathcal{T}^c(P) \) is the closure of the linear span of homogeneous polynomials, one has this for general \( T \in \mathcal{T}^c(P) \). This shows that \( \mathcal{J}(\mathcal{T}(P)) \) and \( \mathcal{J}(\mathcal{T}(\infty)(P)) \) are closed two sided ideals in their respective algebras. By Proposition 5.5, we have that

\[
\mathcal{T}^c(P) = \mathcal{T}(P) + \mathcal{J}(\mathcal{T}^c(P)) = \mathcal{T}(\infty)(P) + \mathcal{J}(\mathcal{T}^c(P))
\]

And by Corollary I.5.6 in [Dav96] we have that

\[
\mathcal{T}(P)/\mathcal{J}(\mathcal{T}(P)) \cong \mathcal{T}(\infty)(P)/\mathcal{J}(\mathcal{T}(\infty)(P))
\]

as desired. \( \Box \)

Note that since for all \( B \in \mathcal{A}(P)_m \) we have \( W^{(n)}_A(B) = A \cdot B \), then a simple calculation shows that \( W^{(n)*}_A := (W^{(n)}_A)^*: \mathcal{A}(P)_{n+m} \to \mathcal{A}(P)_m \) is given by

\[
W^{(n)*}_A(B) = \text{Gr}(P^m(*)) \ast [A^* \cdot B]
\]

Usually \( A^*B \) may not be an element of \( \mathcal{A}(P)_m \) since it may have non zero entries outside the support of \( P^m \). Schur multiplication with \( \text{Gr}(P^m(*)) \) ensures that the output element is in \( \mathcal{A}(P)_m \).

**Proposition 5.7.** Let \( P \) be a finite and essential stochastic matrix over \( \Omega \). Assume that \( P \) decomposes into irreducible stochastic matrices \( P(1), P(2), ..., P(\ell) \) in block diagonal form, then

\[
\mathcal{O}(P) \cong \mathcal{O}(P(1)) \oplus ... \oplus \mathcal{O}(P(\ell))
\]

**Proof.** This follows immediately from Propositions 3.3 and the iteration of Proposition 1.15. \( \Box \)

The last proposition enables us to reduce the problem of computing the Cuntz-Pimsner algebra of \( \mathcal{A}(P) \) for finite essential \( P \), to that of computing the Cuntz-Pimsner algebra of \( \mathcal{A}(P) \) when \( P \) is finite and irreducible. Thus, we assume throughout the following discussion that \( P \) is irreducible, unless stated otherwise. Let \( P \) be a \( d \times d \) irreducible stochastic matrix with period \( r \geq 1 \). Denote \( q = \frac{d}{r} \).
Let $P$ be an irreducible $r$-periodic $d \times d$ matrix. Use Theorem 2.8 to write $P$ in the following form with appropriate enumeration:
\[
P = \begin{bmatrix}
0 & P_0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & P_{r-2} \\
P_{r-1} & \cdots & 0 & 0
\end{bmatrix}
\]

**Corollary 5.13.** Let hence both belong to

**Proof.** Let Schur-multiply it with
Which simply means that the matrix\n
**Remark 5.10.** Where $A_{0}, ..., A_{r-1} \in M_q(\mathbb{C})$, due to the matrix decomposition of $P$ assumed previously.

**Remark 5.9.** Note that the matrices $A_{0}, ..., A_{r-1}$ can have non-zero entries only in non-zero entries of the matrices comprising the cyclic decomposition of $P^n$ (For $n = 1$ these are just $P_{0}, ..., P_{r-1}$). Due to finiteness and positive-recurrence of $P$ (recall remark 5.7) along with Theorem 2.10, there exists $n_0$ such that for all $n \geq n_0$, the matrices comprising the cyclic decomposition of $P^n$ have all entries non-zero. Thus for $n \geq n_0$ any $A_{0}, ..., A_{r-1} \in M_q(\mathbb{C})$ can be taken in equation (5.1) so as to obtain an element in $Arv(P)_n$.

**Remark 5.10.** Due to the previous remark on equation (5.1), there exists $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$ and $B \in Arv(P)_m$ one actually has\n\[
W_B^{(m)} = A^* \cdot B
\]
Which simply means that the matrix $A^* \cdot B$ is already in $Arv(P)_m$, and one does not have to Schur-multiply it with $Gr(P^n)$, to obtain an element in $Arv(P)_m$.

**Definition 5.11.** Let $P$ be an irreducible $r$-periodic $d \times d$ stochastic matrix with cyclic decomposition as above. For $A_k \in M_q(\mathbb{C})$ with $0 \leq k \leq r-1$ and $n \in \mathbb{N}$ the operators $M_{A_0 \oplus \ldots \oplus A_{r-1}} : Arv(P)_m \rightarrow Arv(P)_m$ and $S_n : Arv(P)_m \rightarrow Arv(P)_{m+n}$ by
\[
M_{A_0 \oplus \ldots \oplus A_{r-1}}(B) = Gr(P^n) \ast [(A_0 \oplus \ldots \oplus A_{r-1}) \cdot B] \quad \text{and} \quad S_n(B) = Gr(P^{n+m}) \ast [F^n \cdot B]
\]
for $B \in Arv(P)_m$.

**Proposition 5.12.** Let $P$ be an irreducible $r$-periodic $d \times d$ stochastic matrix with cyclic decomposition as above. For every $A_k \in M_q(\mathbb{C})$ with $0 \leq k \leq r-1$ and $n \in \mathbb{N}$, we have that $M_{A_0 \oplus \ldots \oplus A_{r-1}} \circ S_n \in T^\infty(P)$.

**Proof.** Let $k \in \mathbb{N}$ be large enough so that,
\[
F^n \in Arv(P)_{kr+n} \quad \text{and} \quad I_d \circ A_0 \oplus \ldots \oplus A_{r-1} \in Arv(P)_{kr}
\]
Where $I_d$ is the identity matrix in $M_d(\mathbb{C})$. Then we have,
\[
M_{A_0 \oplus \ldots \oplus A_{r-1}} = W_{I_d}^{(kr)} \circ W_{A_0 \oplus \ldots \oplus A_{r-1}}^{(kr)} \quad \text{and} \quad S_n = W_{I_d}^{(kr)} \circ W_{F^n}^{(kr+n)}
\]
hence both belong to $T^\infty(P)$.

Denote for $A_0 \oplus \ldots \oplus A_{r-1} \in \oplus_0^{r-1} M_q(\mathbb{C})$ the element $\sigma(A_0 \oplus \ldots \oplus A_{r-1}) = A_1 \oplus \ldots \oplus A_{r-1} \oplus A_0 \in \oplus_0^{r-1} M_q(\mathbb{C})$ which is the backward cyclic shift.

**Corollary 5.13.** Let $P$ be an irreducible $r$-periodic $d \times d$ stochastic matrix with cyclic decomposition as above. For every $n \in \mathbb{N}$ and $A \in Arv(P)_n$ as in equation 5.1 for unique $A_k \in M_q(\mathbb{C})$ with $0 \leq k \leq r-1$, one has
\[
W_A^{(n)} - M_{A_0 \oplus \ldots \oplus A_{r-1}} \circ S_n \in J \quad \text{and} \quad S_1 \circ M_{A_0 \oplus \ldots \oplus A_{r-1}} \circ S_1^* - M_{\sigma(A_0 \oplus \ldots \oplus A_{r-1})} \in J$

Due to Proposition 4.12 we have that decomposition as above. Then, $O_0(P)$.

**Corollary 5.14.** Let $P$ be an $r$-periodic finite $d \times d$ irreducible stochastic matrix with cyclic decomposition as above. Then,

$$O_0(P) = \{ \overline{A_0 \oplus \cdots \oplus A_{r-1}} | A_k \in M_q(\mathbb{C}) \}$$

**Proposition 5.15.** Let $P$ be an $r$-periodic finite $d \times d$ irreducible stochastic matrix with cyclic decomposition as above. Then, $O_0(P) \cong \bigoplus_1^r M_q(\mathbb{C})$. Furthermore, if we let $\sigma : \mathbb{Z} \to Aut(O_0(P))$ be a $\mathbb{Z}$-action given by $\sigma_1(M_0 \oplus \cdots \oplus M_{r-1}) = M_{A_0 \oplus \cdots \oplus A_{r-1}}$. Then we have that

$$O(P) \cong \left( \bigoplus_1^r M_q(\mathbb{C}) \right) \rtimes_{\sigma} \mathbb{Z}$$

**Proof.** Define a map $\varphi : \bigoplus_0^{r-1} M_q(\mathbb{C}) \rightarrow O_0(P)$ by $\varphi(A_0 \oplus \cdots \oplus A_{r-1}) = M_{A_0 \oplus \cdots \oplus A_{r-1}}$. It follows that $\varphi$ is a surjective, bounded *-homomorphism, we show that it is 1-1. Assume $0 \neq C = A_0 \oplus \cdots \oplus A_{r-1} \in \bigoplus_0^{r-1} M_q(\mathbb{C})$. There exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ one has $I_d \in Arv(P)_{kr}$, and thus in $Arv(P)_{kr}$,

$$||MC(I_d)||^2 = ||C'||^2 = ||Diag[C^*C]||$$

Due to Proposition 4.12 we have that

$$||MC||^2 \geq \limsup_{k \rightarrow \infty} ||MC \circ Q_{kr}||^2 \geq ||Diag[C^*C]|| > 0$$

We must have that $\varphi$ is 1-1, and thus it is a *-isomorphism.

For the second part assume $O(P) \subseteq B(H)$ for some Hilbert space $H$. Due to Corollary 5.13 one has a faithful $\sigma$-covariant representation $(S_1, \varphi^{-1}, H)$ (in which the action of $\sigma$ is spatially implemented). Thus, by the universal property of crossed products, one attains a *-homomorphism $\Phi : \left( \bigoplus_1^r M_q(\mathbb{C}) \right) \rtimes_{\sigma} \mathbb{Z} \rightarrow B(H)$ satisfying $\Phi(\sum_{s \in \mathbb{Z}} A_s) = \sum_{s \in \mathbb{Z}} \varphi^{-1}(A_s)(S_1)^s$ the image of which is $O(Arv(P))$, due to Proposition the above. This assures us that this *-homomorphism respects the gauge actions and is injective on $\bigoplus_1^r M_q(\mathbb{C})$ (which is the fixed point algebra of the crossed product). So by Proposition 4.13 this *-homomorphism must be injective.

**Corollary 5.16.** Let $P$ be an irreducible $d \times d$ stochastic matrix. Then $O(P) \cong M_d(\mathbb{C}) \otimes C(T) \cong C(T; M_d(\mathbb{C}))$

**Proof.** By Proposition 5.15 we have that $O(P) \cong \left( \bigoplus_1^r M_q(\mathbb{C}) \right) \rtimes_{\sigma} \mathbb{Z}$, and we have that

$$\left( \bigoplus_1^r M_q(\mathbb{C}) \right) \rtimes_{\sigma} \mathbb{Z} \cong \left( C(\mathbb{Z}/r\mathbb{Z}) \otimes M_q(\mathbb{C}) \right) \rtimes_{\tilde{\delta}} \mathbb{Z}$$

where $\tilde{\delta}$ on $C(\mathbb{Z}/r\mathbb{Z}) \otimes M_q(\mathbb{C})$ is given by $\delta \otimes I$, and $\delta$ is the backward cyclic shift of the domain of functions in $C(\mathbb{Z}/r\mathbb{Z})$. Now let $G = \mathbb{Z}$, $H = r\mathbb{Z}$, $A = M_q(\mathbb{C})$ and let $\tau$ be the trivial action of $G$ on $A$. Then by [Cun80] [Corollary 2.8] we have that

$$\left( C(\mathbb{Z}/r\mathbb{Z}) \otimes M_q(\mathbb{C}) \right) \rtimes_{\tilde{\delta}} \mathbb{Z} \cong (C_0(G/H) \otimes A) \rtimes_{\tilde{\delta}} G \cong (A \rtimes_{\tau} H) \otimes K(L^2(G/H, \mu))$$

$$\cong (M_q(\mathbb{C}) \rtimes_{\tau} r\mathbb{Z}) \otimes K(L^2(\mathbb{Z}/r\mathbb{Z})) \cong C(T) \otimes (M_q(\mathbb{C}) \otimes M_r(\mathbb{C}))$$

Therefore $O(P) \cong C(T) \otimes M_q(\mathbb{C}) \otimes M_r(\mathbb{C}) \cong C(T) \otimes M_d(\mathbb{C}) \cong C(T; M_d(\mathbb{C}))$.

**Corollary 5.17** (Cuntz-Pimsner *-algebras in the finite essential case). Let $P$ be a finite and essential stochastic matrix over $\Omega$. Then

$$O(P) \cong C\left( T; \bigoplus_{k=1}^r M_{dk}(\mathbb{C}) \right)$$
Proof. Assume that $P$ decomposes into irreducible stochastic matrices $P(1), P(2), \ldots, P(\ell)$ in block diagonal form. Due to Proposition 5.7 and Corollary 5.10 we have

$$
\mathcal{O}(P) \cong \oplus_{k=1}^{\ell} \mathcal{O}(P(k)) \cong \oplus_{k=1}^{\ell} C(T; M_{d_k}(\mathbb{C})) \cong C(T; \oplus_{k=1}^{\ell} M_{d_k}(\mathbb{C}))
$$

Thus, we obtain the following result. We omit the proof, which is straightforward.

**Theorem 5.18** (Isomorphism between Cuntz-Pimsner C*-algebras in the finite essential case). Let $P$ and $Q$ be two finite essential stochastic matrices. Assume $P(1) \oplus P(2) \oplus \ldots \oplus P(\ell)$ and $Q(1) \oplus Q(2) \oplus \ldots \oplus Q(s)$ are irreducible decompositions for $P$ and $Q$ respectively, where $P(k) \in M_{d_k}(\mathbb{C})$ and $Q(k') \in M_{d_k}(\mathbb{C})$ are irreducible, and such that $d_1 \leq d_2 \leq \ldots \leq d_\ell$ and $t_1 \leq t_2 \leq \ldots \leq t_s$. Then $\mathcal{O}(P) \cong \mathcal{O}(Q)$ if and only if $\ell = s$ and $d_k = t_k$ for all $1 \leq k \leq \ell$.

6. **Tensor algebras and their graded structure**

In this section we begin the study of the non-self-adjoint norm-closed tensor algebra associated to a subproduct system over $\mathbb{N}$.

**Definition 6.1.** Let $X = (X_n)_{n \in \mathbb{N}}$ be a subproduct system over a $W^*$-algebra $\mathcal{M}$. The tensor algebra associated to $X$ is the norm-closed subalgebra of $\mathcal{L}(\mathcal{F}_X)$ given by

$$
T_+(X) := \overline{\text{Alg}\{ \mathcal{M} \cup \{ S_{\xi}^{(n)} \mid \xi \in X_n, \ n \in \mathbb{N} \}}
$$

The tensor algebra is clearly a gauge invariant closed subalgebra of $T(X)$, and the Fourier coefficient maps $\Phi_n$ restrict to idempotents. Therefore the tensor algebra is graded by the spaces

$$
T_+(X)_n = \Phi_n(T_+(X)) = \overline{\text{Span}\{ S_{\xi}^{(n)} \mid \xi \in X_n \}}.
$$

For a subproduct system $X$, we denote by $\Omega_X$ the vacuum vector of the Fock module of $X$, which is simply $1_{\mathcal{M}} \otimes 0 \in \mathcal{M} \oplus \bigoplus_{n=1}^{\infty} X_n$.

**Proposition 6.2.** Let $(X, U)$ be a subproduct system. For every $x \in X$ we have that $X_n$ is isometrically isomorphic as a Banach $\mathcal{M}$-bimodule to $T_+(X)_n$ via the map $\xi \mapsto S_{\xi}^{(n)}$.

Therefore, every element $T \in T_+(X)$ has a unique representation as an infinite series $T = \sum_{n=0}^{\infty} S_{\xi_{n}}^{(n)}$, where $\xi_n \in X_n$ satisfies $\Phi_n(T) = S_{\xi_n}^{(n)}$ (called its Fourier series representation for short), and the series converges Cesaro to $T$ in norm: if $\sigma(T) = \sum_{n=0}^{N} \left(\mathbb{1} - \frac{x_n}{N+1}\right) S_{\xi_n}^{(n)}$, then we have that $\lim_{N \to \infty} \|\sigma(T) - T\| = 0$. Furthermore, if $T, T' \in T_+(X)$ have Fourier series representations $T = \sum_{i=0}^{\infty} S_{\xi_{i}}^{(i)}$ and $T' = \sum_{i=0}^{\infty} S_{\xi_{i}}^{(i)}$, then

$$
TT' = \sum_{i=0}^{\infty} S_{\xi_{i}}^{(i)}, \quad \text{where} \quad \xi = \sum_{k=0}^{n} U_{k,n-k}(\xi_{k} \otimes \eta_{n-k}).
$$

**Proof.** Define $\Psi_n : X_n \to T_+(X)_n$ by $\Psi_n(\xi) = S_{\xi}^{(n)}$ for all $\xi \in X_n$. For every $\sum_{m=0}^{M} \eta_m$ with $\eta_m \in X_m$,

$$
|S_{\xi}^{(n)}(\sum_{m=0}^{M} \eta_m)|^2 = \sum_{m=0}^{M} |U_{n,m}(\xi \otimes \eta_m)|^2 = \sum_{m=0}^{M} |U_{n,m}(\xi \otimes \eta_m)|^2 \leq \sum_{m=0}^{M} |\xi \otimes \eta_m|^2 = |\xi \otimes (\sum_{m=0}^{M} \eta_m)|^2
$$

Thus, taking norms we obtain that

$$
\|S_{\xi}^{(n)}(\sum_{m=0}^{M} \eta_m)\| \leq \|\xi \otimes (\sum_{m=0}^{M} \eta_m)\| \leq \|\xi\| \cdot \|\sum_{m=0}^{M} \eta_m\|
$$

Thus, $\|S_{\xi}^{(n)}\| \leq \|\xi\|$. Moreover, $\|S_{\xi}^{(n)}(\Omega_X)\| = \|\xi\|$ hence $\Psi_n$ is an isometry. In addition, $\Psi_n$ preserves the left and right actions of $\mathcal{M}$: for $m \in \mathcal{M}$ and $\eta \in X_m$,

$$
(m \cdot \Psi_n(\xi))(\eta) = m \cdot U_{n,m}(\xi \otimes \eta) = U_{n,m}(m \cdot \xi \otimes \eta) = \Psi_n(m \cdot \xi)
$$

$$
(\Psi_n(\xi) \cdot m)(\eta) = \Psi_n(\xi)(m \cdot \eta) = U_{n,m}(\xi \otimes m \cdot \eta) = U_{n,m}(\xi \cdot m \otimes \eta) = (\Psi_n(\xi \cdot m))(\eta)
$$



Thus, \( \Psi_n \) is an \( \mathcal{M} \)-bimodule isometry with a dense image in \( T_+(X)_n \). Since the image of an isometry is closed, \( \Psi_n \) must be onto, and we have proven that \( \Psi_n \) has the desired properties.

Let \( T \in T_+(X) \). By the previous paragraph, there exist unique \( \xi_n \in X_n \) such that \( \Phi_n(T) = S_{\xi_n}^{(n)} \), and as discussed in Section 3, the series converges to \( T \) in the Cesaro sense. Thus the Fourier series decomposition is unique, in the sense that any two elements of \( T_+(X) \) with the same Fourier series must be identical.

Finally, let \( T, T' \in T_+(X) \) have Fourier series representations \( T = \sum_{i=0}^{\infty} S_{\xi_i}^{(i)} \) and \( T' = \sum_{i=0}^{\infty} S_{\eta_i}^{(i)} \).

\[
\begin{align*}
\sigma_N(T)\sigma_N(T') &= \left( \sum_{n=0}^{N} \left( 1 - \frac{n}{N+1} \right) S_{\xi_n}^{(n)} \right) \left( \sum_{j=0}^{N} \left( 1 - \frac{j}{N+1} \right) S_{\eta_j}^{(j)} \right) \\
&= \sum_{m=0}^{N} \sum_{k=0}^{m} \left( 1 - \frac{k}{N+1} \right) S_{\xi_k}^{(k)} \left( 1 - \frac{m-k}{N+1} \right) S_{\eta_{m-k}}^{(m-k)} \\
&= \sum_{m=0}^{N} \left( 1 - \frac{m}{N+1} \right) \sum_{k=0}^{m} \left( 1 - \frac{k}{N+1} \right) \left( 1 - \frac{m-k}{N+1} \right) S_{U_{k,m-k}(\xi_k \otimes \eta_{m-k})}^{(m)} \\
\end{align*}
\]

Thus,

\[
\Phi_n(TT') = \lim_{N \to \infty} \Phi_n(\sigma_N(T)\sigma_N(T')) = S_{\xi_n}^{(n)}, \quad \text{where} \quad \zeta = \sum_{k=0}^{n} U_{k,n-k}(\xi_k \otimes \eta_{n-k}).
\]

By uniqueness, the proposition is proven. \( \square \)

**Automatic continuity.** We are interested in the study of isomorphisms between tensor algebras. Under certain conditions, we will show that algebraic isomorphisms are automatically bounded. We will follow closely the ideas of Donsig, Hudson and Katsoulis [DHK01], Katsoulis and Kribs [KK04], and Davidson and Katsoulis [DK11].

Suppose that \( \mathcal{A} \) and \( \mathcal{B} \) are Banach algebras, and suppose that \( \varphi : \mathcal{A} \to \mathcal{B} \) is a surjective homomorphism. Let

\[
S(\varphi) = \{ b \in \mathcal{B} \mid \exists (a_n) \subseteq \mathcal{A} \text{ such that } a_n \to 0 \text{ and } \varphi(a_n) \to b \}.
\]

It is easy to see that the graph of \( \varphi \) is closed if and only if \( S(\varphi) = \{ 0 \} \), hence by the closed graph theorem we have that \( \varphi \) is continuous if and only if \( S(\varphi) = \{ 0 \} \). We will use the following adaptation of [Sin75, Lemma 2.1], which appeared first in Donsig, Hudson and Katsoulis [DHK01].

**Lemma 6.3 (Sinclair).** Suppose that \( \mathcal{A} \) and \( \mathcal{B} \) are Banach algebras and \( \varphi : \mathcal{A} \to \mathcal{B} \) is a surjective algebraic homomorphism. Let \( (b_n)_{n \in \mathbb{N}} \) be any sequence in \( \mathcal{B} \). Then there exists \( N \in \mathbb{N} \) such that for all \( n \geq N \),

\[
b_1 b_2 \ldots b_n S(\varphi) = b_1 b_2 \ldots b_{n+1} S(\varphi) \quad \text{and} \quad S(\varphi)b_n b_{n-1} \ldots b_1 = S(\varphi)b_{n+1} b_n \ldots b_1.
\]

**Theorem 6.4.** Let \( X \) be a subproduct system over a von Neumann algebra \( \mathcal{M} \) and let \( \mathcal{A} \) be a Banach algebra. Suppose that for all \( d \geq 0 \) and \( 0 \neq y \in T_+(X)_d \) there exists a sequence \( (b_n) \in \ker \Phi_0 \) such that \( y b_n b_{n-1} \ldots b_1 \neq 0 \) for all \( n \geq 1 \). Then any surjective algebraic homomorphism \( \varphi : \mathcal{A} \to T_+(X) \) is continuous.

**Proof.** Suppose towards a contradiction that there exists \( 0 \neq z \in S(\varphi) \). Let \( d \in \mathbb{N} \) be minimal such that \( y := \Phi_d(z) \neq 0 \), and let \( y = \Phi_d(z) \). By assumption, there exists a sequence \( (b_n) \subseteq \ker \Phi_0 \) such that for every \( n \), \( y b_n b_{n-1} \ldots b_1 \neq 0 \), and therefore \( z b_n b_{n-1} \ldots b_1 \neq 0 \). Notice that since \( b_n \in \ker \Phi_0 \) for all \( n \), we have that for all \( n \) and every \( k < n \), and every \( w \in S(\varphi) \),

\[
\Phi_k(w b_n b_{n-1} \ldots b_1) = 0
\]

It follows that

\[
S(\varphi)b_n b_{n-1} \ldots b_1 \subseteq \bigcap_{k<n} \ker \Phi_k
\]
However, by Lemma 6.6 there exists \( N \) such that for all \( n \geq N \),
\[
\mathcal{S}(\varphi) b_N b_{N-1} \cdots b_1 = \mathcal{S}(\varphi) b_n b_{n-1} \cdots b_1 \subseteq \bigcap_{k<n} \ker \Phi_k.
\]
Thus we obtain that
\[
\mathcal{S}(\varphi) b_N b_{N-1} \cdots b_1 \subseteq \bigcap_{k=1}^{\infty} \ker \Phi_k = \{0\},
\]
and so we reach a contradiction to the fact that \( z b_N b_{N-1} \cdots b_1 \neq 0 \). \( \square \)

This approach was used in [KK04, DK11], replacing a right acting sequence by a left acting one, in the special case when there exists \( b \in \ker \Phi_0 \) that satisfies \( \|by\| = \|y\| \) for all \( y \in T_+(X)_d \). That is analogous to the following corollary.

**Corollary 6.5.** Let \( X \) be a subproduct system over a von Neumann algebra \( M \) and let \( A \) be a Banach algebra. Suppose that for every \( 0 \neq y \in T_+(X) \) there exists \( b \in \ker \Phi_0 \) such that \( yb^n \neq 0 \) and for all \( n \geq 1 \). Then any surjective algebraic homomorphism \( \varphi : A \to T_+(X) \) is continuous.

**Admissible isomorphisms.** Our analysis of isomorphisms between tensor algebras is most effective for the following class of isomorphisms.

**Definition 6.6 (Admissibility).** Let \( X \) and \( Y \) be subproduct systems over a \( W^* \)-algebra \( M \). Let \( \varphi : T_+(X) \to T_+(Y) \) be an isomorphism. Denote for \( n, m \in \mathbb{N} \) the maps \( \varphi^{(n,m)} := \Phi_m \circ \varphi \circ T_+(X)_n \). \( \varphi \) is said to be admissible if the following conditions hold:

(A1) The maps \( \rho_\varphi := \Phi_0 \circ \varphi \mid_M \) and \( \rho_{\varphi^{-1}} := \Phi_0 \circ (\varphi^{-1}) \mid_M \) are *-automorphisms of \( M \).

(A2) For all \( n, m \in \mathbb{N} \) such that \( m < n \) we have that the maps \( \varphi^{(n,m)} \) and \( (\varphi^{-1})^{(m,n)} \) are continuous in the \( \sigma \)-topology induced by the identification of \( T_+(X)_k \) as a \( W^* \)-correspondence as in Proposition 6.6.

**Lemma 6.7.** Let \( X \) and \( Y \) be subproduct systems over a \( W^* \)-algebra \( M \), and let \( \varphi : T_+(X) \to T_+(Y) \) be an isomorphism. If \( \varphi \) is isometric then it is admissible.

**Proof.** If \( \varphi : T_+(X) \to T_+(Y) \) is an isometric isomorphism, since \( T_+(Y) \subseteq T(Y) \) we can regard \( \varphi \) as a map into the Toeplitz algebra. In our case, \( \varphi \mid_M : M \to T(Y) \) is a unitary contractive homomorphism, hence it is necessarily positive. Thus it preserves the involution from \( M \) to \( T(Y) \). Since we also have that \( \varphi(M) = \varphi(M)^* \subseteq T_+(Y)^* \) (considered inside \( T(Y) \)), we must have that \( \varphi(M) \subseteq T_+(Y) \cap T_+(Y)^* = M \). A similar argument then shows that \( \varphi^{-1}(M) \subseteq M \), so that \( \varphi(M) = M \) and \( \varphi \) satisfies (A1). To show (A2) note that since \( \varphi(M) = M \) and \( \varphi \) is multiplicative, we must have that \( \varphi^{(n,m)} \) is a \( \rho_\varphi \)-correspondence map for all \( n, m \in \mathbb{N} \). But we have seen after defining \( \rho \)-correspondence maps that they are necessarily \( \sigma \)-topology continuous. The identical argument then works for \( \varphi^{-1} \).

When an isomorphism \( \varphi : T_+(X) \to T_+(Y) \) fails to be isometric, then it may fail to satisfy condition (A1) even if we assume that \( \rho_\varphi \) and \( \rho_{\varphi^{-1}} \) are bijective, since \( \rho_\varphi \) might not be *-preserving, or equivalently contractive. Nevertheless, if in addition \( M \) is commutative, the following lemma provides automatic contractivity of \( \rho_\varphi \) and \( \rho_{\varphi^{-1}} \), assuming (A1) holds.

**Lemma 6.8.** Let \( A \) be a unital Banach algebra with \( \|1_A\| = 1 \), and let \( B \) be a unital Banach subalgebra of a commutative \( C^* \)-algebra \( B' \). If \( \varphi : A \to B \) is a unital algebraic homomorphism, then \( \varphi \) is contractive.

**Proof.** We may assume without loss of generality that \( B' \) is unital and \( 1_B' \in B \) since \( C^*(B) \subseteq B' \) is a unital commutative \( C^* \)-algebra. For all \( a \in A \), the element \( \varphi(a) \in B' \) must be normal, one has
\[
\|\varphi(a)\| = r_{B'}(\varphi(a)) = r_B(\varphi(a)) \leq r_A(a) \leq \|a\|,
\]
where the second equality follows from spectral permanence and the first inequality follows from unitarity of \( \varphi \).

We were not able to prove that composition of admissible isomorphisms is always admissible. In the cases when our analysis is most successful this is not an issue, because admissibility will be shown to be automatic. We already showed this fact for the isometric case, and it will also hold for tensor algebras arising from finite stochastic matrices (see Section 7).
Proposition 6.12. Let $\rho$ be a $\ast$-automorphism of a $W^*$-algebra $\mathcal{M}$. Then $\rho$ is an admissible graded bounded isomorphism if and only if $\rho_{\mathcal{M}} = \rho |_{\mathcal{M}}$. Ad

Definition 6.9. Let $(X, U^X)$ and $(Y, U^Y)$ be subproduct systems over the same $W^*$-algebra $\mathcal{M}$, and let $\varphi : (X, U^X) \to (Y, U^Y)$ be an isomorphism of subproduct systems. Then $\varphi$ is a graded graded bounded admissible isomorphism if $\varphi$ is an isomorphism of subproduct systems if $V_0 = \rho$, and for $0 \neq n \in \mathbb{N}$ the maps $V_n$ are bijective $\rho$-correspondence morphisms satisfying $\sup_{n\in\mathbb{N}} \{||V_n||, ||V_n^{-1}||\} < \infty$, such that

$$V_{n+m} U_{n,m}^X = U_{n,m}^Y (V_n \otimes V_m).$$

If $V : X \to Y$ is a $\rho$-similarity, we automatically have that $V^{-1} : Y \to X$ is a $\rho^{-1}$-similarity, since $V_0^{-1} = \rho^{-1}$ and $V_{n+m} U_{n,m}^Y = U_{n,m}^X (V_n^{-1} \otimes V_m^{-1})$ for all $n, m \in \mathbb{N}$.

Remark 6.10. Admissibility of a graded isomorphism $\varphi$ is reduced to showing that $\rho_{\mathcal{M}} = \varphi |_{\mathcal{M}}$ is contractive. Indeed, in that case, we have that $(A1)$ holds since then $\rho_{\mathcal{M}}$ is a $\ast$-automorphism (and as a consequence so is $\rho_{\mathcal{M}}^{-1} = \rho_{\mathcal{M}}^{-1}$). Furthermore, $(A2)$ is also satisfied since $\varphi^{(n,m)} = 0 = (\varphi^{-1})^{(n,m)}$ when $m < n$. See also Proposition 6.10.

Definition 6.11. Let $(X, U^X)$ and $(Y, U^Y)$ be subproduct systems over the same $W^*$-algebra $\mathcal{M}$, and let $\rho$ be a $\ast$-automorphism of $\mathcal{M}$. A family $V = \{V_n\}_{n\in\mathbb{N}}$ of maps $V_n : X_n \to Y_n$ is called a $\rho$-similarity $V : X \to Y$ of subproduct systems if $V_0 = \rho$, and for $0 \neq n \in \mathbb{N}$ the maps $V_n$ are bijective $\rho$-correspondence morphisms satisfying $\sup_{n\in\mathbb{N}} \{||V_n||, ||V_n^{-1}||\} < \infty$, such that

$$V_{n+m} U_{n,m}^X = U_{n,m}^Y (V_n \otimes V_m).$$

When we have a similarity $\rho$-isomorphism $V : X \to Y$, we can define a natural $\rho$-correspondence morphism from $\mathcal{F}_X$ to $\mathcal{F}_Y$ as follows. The map

$$W_V = \oplus_{n=0}^{\infty} V_n$$

is a well-defined map between the Fock $C^*$-direct sums, since $\sup_{n\in\mathbb{N}} \{||V_n||, ||V_n^{-1}||\} < \infty$. Moreover, it extends uniquely to a map which we also denote by $W_V$ in $\mathcal{L}(\mathcal{F}_X, \mathcal{F}_Y)$ between the $W^*$-direct sums, which is a $\rho$-correspondence morphism. Furthermore, we have that $W_V^{-1} = W_{V^{-1}}$ is a $\rho^{-1}$-correspondence morphism and

$$(6.1) \quad ||W_V|| \leq \sup_{n\in\mathbb{N}} ||V_n|| \quad \text{and} \quad ||W_V^{-1}|| \leq \sup_{n\in\mathbb{N}} ||V_n^{-1}||.$$

Proposition 6.12. Let $X$ and $Y$ be subproduct systems over a $W^*$-algebra $\mathcal{M}$.

1. Let $\rho$ be a $\ast$-automorphism of $\mathcal{M}$. If $V : X \to Y$ is a $\rho$-similarity, then $Ad \rho_V : \mathcal{T}_\rho(X) \to \mathcal{T}_\rho(Y)$ given by $Ad \rho_V(T) = W_V TW_V^{-1}$ for every $T \in \mathcal{T}_\rho(X)$ is a graded completely bounded admissible isomorphism satisfying $Ad \rho_V |_{\mathcal{M}} = \rho$ and

$$\max\{||Ad \rho_V|_{cb}, ||Ad \rho_V^{-1}|_{cb}\} \leq \sup_{n\in\mathbb{N}} ||V_n|| \cdot \sup_{n\in\mathbb{N}} ||V_n^{-1}||.$$

2. Let $\varphi : \mathcal{T}_\rho(X) \to \mathcal{T}_\rho(Y)$ be an admissible graded bounded isomorphism, and let $\rho_{\mathcal{M}} = \varphi |_{\mathcal{M}}$. Then the family $V_{\varphi} = (V_n^\varphi : X_n \to Y_n)$ uniquely determined by $S_{\varphi}^{(n)} = \varphi(S_{\xi}^{(n)})$ for $\xi \in X_n$ constitutes a $\rho_{\mathcal{M}}$-similarity satisfying

$$\sup_{n\in\mathbb{N}} ||V_n^\varphi|| \leq ||\varphi|| \quad \text{and} \quad \sup_{n\in\mathbb{N}} ||(V_n^\varphi)^{-1}|| \leq ||\varphi^{-1}||.$$

Thus, if $\varphi : \mathcal{T}_\rho(X) \to \mathcal{T}_\rho(Y)$ is an admissible graded bounded isomorphism, then $\varphi = Ad \rho_V$ and $\varphi$ is completely bounded with $||\varphi||_{cb} \leq ||\varphi|| ||\varphi^{-1}||$.

Proof. (1) Let us assume that $\{V_n\}$ is a family of maps constituting a $\rho$-similarity from $X$ to $Y$. The map $Ad \rho_V$ is multiplicative and bounded by $(6.1)$. Since $V^{-1}$ is a $\rho^{-1}$-similarity, the map $Ad V^{-1}$, is also a bounded graded homomorphism from $\mathcal{T}_\rho(Y)$ into $\mathcal{T}_\rho(X)$ and $Ad V^{-1} = (Ad \rho_V)^{-1}$, ensuring that $Ad \rho_V$ is a graded bounded admissible isomorphism between $\mathcal{T}_\rho(X)$ and $\mathcal{T}_\rho(Y)$. The norm inequalities follow trivially from equation $(6.1)$.

(2) Let us assume that $\varphi : \mathcal{T}_\rho(X) \to \mathcal{T}_\rho(Y)$ is a graded bounded admissible bijective homomorphism. Notice that $V_n^\varphi : X_n \to Y_n$ in the assertion can be written as $V_n^\varphi = \Psi_n^{-1} \varphi \Psi_n$, where $\Psi_n(\xi) = S_{\xi}^{(n)}$ is the map considered in Proposition 6.12. Notice that $V_n^\varphi = \rho_{\mathcal{M}}$ is a $\ast$-automorphism, and the proof of the identities $V_{n+m}^\varphi U_{n,m}^X = U_{n,m}^Y (V_n^\varphi \otimes V_m^\varphi)$ is straightforward. Moreover,

$$||V_n^\varphi(\xi)|| = ||(\Psi_n^{-1} \varphi \Psi_n)(\xi)|| = ||S_{\xi}^{(n)}(\Psi_n^{-1} \varphi \Psi_n)(\xi)|| = ||\varphi(S_{\xi}^{(n)})|| \leq ||\varphi|| \cdot ||S_{\xi}^{(n)}|| = ||\varphi|| \cdot ||\xi||.$$
Notice that $V_n^{-1} = (V_n^*)^{-1}$. Hence by a similar argument for $(V_n^*)^{-1}$ we obtain that $\|V_n^*\|$ and $\|(V_n^*)^{-1}\|$ are uniformly bounded by $\|\varphi\|$ and $\|\varphi^{-1}\|$ respectively.

For the last part of the proposition, using the norm estimates of item (2) and item (1) in tandem while noting that $\varphi = \text{Ad}_{V_n^*}$, we arrive at the completely bounded norm estimate for $\varphi$. \hfill $\square$

We obtain the following immediate corollary by noting that isometric isomorphisms are automatically admissible, or by Remark 6.10

**Corollary 6.13.** Let $X$ and $Y$ be subproduct systems over a $W^*$-algebra $\mathcal{M}$.

1. Let $\rho$ be a *-automorphism of $\mathcal{M}$. If $V : X \to Y$ is a unitary $\rho$-isomorphism, then $\text{Ad}_V : T_+(X) \to T_+(Y)$ given by $\text{Ad}_V(T) = W_TW_V^{-1} = W_TW_V^*$ for every $T \in T_+(X)$ is a completely isometric isomorphism satisfying $\text{Ad}_V |_{\mathcal{M}} = \rho$.

2. Let $\varphi : T_+(X) \to T_+(Y)$ be a graded isometric isomorphism, and let $\rho_{\varphi} = \varphi |_{\mathcal{M}}$. Then the family $V^\varphi = (V_n^\varphi : X_n \to Y_n)$ uniquely determined by $S^{(n)}_{V_n^\varphi}(\xi) = \varphi(S^{(n)}_{\xi})$ for $\xi \in X_n$ constitutes a unitary $\rho_{\varphi}$-isomorphism of subproduct systems satisfying $\varphi = \text{Ad}_{V^\varphi}$.

**Semi-graded bounded admissible isomorphisms.** We will consider a special class of bounded admissible isomorphisms which will provide a convenient platform for the analysis of the general case.

**Definition 6.14.** Let $X$ be a subproduct system. The minimal degree of an element $0 \neq T \in T_+(X)$, also denoted by $\text{mindeg}(T)$ is the smallest $n \in \mathbb{N}$ such that $\Phi_n(T) \neq 0$.

Let $Y$ be another subproduct system, and suppose that $\varphi : T_+(X) \to T_+(Y)$ is a bounded isomorphism. We say that $\varphi$ is semi-graded if for all $T \in T_+(X)$

$$\text{mindeg}(\varphi(T)) = \text{mindeg}(T),$$

or equivalently, for every $T \in T_+(X)$ and $S \in T_+(Y)$,

$$\text{mindeg}(\varphi(T)) \geq \text{mindeg}(T) \quad \text{and} \quad \text{mindeg}(\varphi^{-1}(S)) \geq \text{mindeg}(S).$$

**Proposition 6.15.** Let $X, Y$ be subproduct systems over a $W^*$-algebra $\mathcal{M}$ and suppose that $\varphi : T_+(X) \to T_+(Y)$ is a semi-graded bounded isomorphism. Then $\rho_{\varphi}^{-1} = \rho_{\varphi^{-1}}$ and if $\rho_{\varphi}$ is contractive then $\varphi$ is admissible. In particular, if $\mathcal{M}$ is commutative, then $\varphi$ is automatically admissible.

**Proof.** Note that since $\varphi$ is a semi-graded isomorphism, if $T \in T_+(X)$, then

$$\Phi_0^Y \varphi(T) = \Phi_0^Y \varphi \Phi_0^X(T).$$

Thus $\rho_{\varphi}$ is surjective and in fact, since the same argument works for $\varphi^{-1}$ we have that $\rho_{\varphi}^{-1} = \rho_{\varphi^{-1}}$.

Indeed, for $m \in \mathcal{M}$ we have $(\rho_{\varphi}^{-1} \circ \rho_{\varphi})(m) = \rho_{\varphi}^{-1}(\Phi_0 \varphi(m)) = \Phi_0 \varphi^{-1} \Phi_0 \varphi(m) = \Phi_0 \varphi^{-1} \varphi(m) = m$.

Thus if $\rho_{\varphi}$ is a contractive bijective map then both $\rho_{\varphi}$ and $\rho_{\varphi}^{-1}$ are *-automorphisms of $\mathcal{M}$ which establishes (A1). Since $\varphi$ is semi-graded, it follows that for all $n, m \in \mathbb{N}$ with $m < n$ we have $\varphi^{(n,m)} = 0 = (\varphi^{-1})^{(n,m)}$ and so (A2) is satisfied. Finally, notice that if $\mathcal{M}$ is commutative, then by Lemma 6.8 we have that $\rho_{\varphi}$ is contractive, hence $\varphi$ is admissible. \hfill $\square$

**Proposition 6.16 (Criteria for semi-gradedness).** Let $X, Y$ be subproduct systems and suppose that $\varphi : T_+(X) \to T_+(Y)$ is an admissible bounded isomorphism. Then the following are equivalent.

1. $\varphi$ is semi-graded
2. $\text{mindeg}(\varphi(T)) \geq \text{mindeg}(T)$ for all $T \in T_+(X)$
3. $\text{mindeg}(\varphi(S^{(1)}_{\xi})) \geq 1$ for every $\xi \in X_1$
4. $\varphi(\ker \Phi_0) = \ker \Phi_0$

**Proof.** We first prove that (2) $\iff$ (3). It is clear that (2) $\implies$ (3). For the converse, let us suppose that (3) holds. By continuity of $\varphi$, in order to prove (2) it suffices to show that $\text{mindeg}(\varphi(S^{(n)}_{\xi})) \geq n$ for every $n \geq 1$ and $\xi \in X_n$. Consider the set $S \subseteq T_+(X)_n$ given by

$$S = \text{Span} \left\{ S^{(1)}_{\xi_1} \cdots S^{(1)}_{\xi_n} \mid \xi_1, \ldots, \xi_n \in X \right\}.$$

By the natural identification given in Proposition 6.2, we get that $T_+(X)_n$ can be considered as a $W^*$-correspondence, for which the set $S$ is dense in the $\sigma$-topology. Next, since $\varphi$ is admissible,
for all $n,m \in \mathbb{N}$ such that $m < n$ the map $\Phi_n \circ \varphi \mid_{\mathcal{T}_+^{(X)}_n}: \mathcal{T}_+^{(X)}_n \to \mathcal{T}_+^{(Y)}_m$ is $\sigma$-topology to $\sigma$-topology continuous. Furthermore, when $m < n$ we have that for every $T \in \mathcal{S}$, $\Phi_n(\varphi(T)) = 0$, and hence $\Phi_m(\varphi(T)_n) \equiv 0$. We conclude that $\text{mindeg}(\varphi(S^{(n)}_\xi)) = n$.

We prove that (3) $\implies$ (1). Since (2) $\iff$ (3), it suffices to show that $\text{mindeg}(\varphi^{-1}(S^{(1)}_\eta)) \geq 1$ for $\eta \in Y_1$. Represent uniquely $\varphi^{-1}(S^{(1)}_\eta) = m + T$ for unique $m \in \mathcal{M}$ and $\text{mindeg}(T) \geq 1$. Since $\varphi$ does not decrease minimal degree, we obtain that $\text{mindeg}(\varphi(T)) \geq 1$, and thus from $S^{(1)}_\eta = \varphi(m) + \varphi(T)$, due to degree considerations we get $\rho_{\varphi}(m) = \Phi_0(\varphi(m)) = 0$. By injectivity of $\rho_{\varphi}$ we get that $m = 0$.

Finally, we note that the implications (1) $\implies$ (3), (1) $\implies$ (4) and (4) $\implies$ (3) are trivial. □

The concept of semi-gradedness in the sense of condition (4) in the previous Proposition, appeared in the work by Muhly and Solel [MS00, section 5] in their study of tensor algebras of correspondences.

**Proposition 6.17.** Let $X$ and $Y$ be subproduct systems over the same $W^*$-algebra, and let $\varphi: \mathcal{T}_+^{(X)} \to \mathcal{T}_+^{(Y)}$ be a semi-graded bounded admissible isomorphism. Let $\tilde{\varphi}: \mathcal{T}_+^{(X)} \to \mathcal{T}_+^{(Y)}$ be the unique bounded map satisfying

$$\tilde{\varphi}(S^{(n)}_\xi) = \Phi_n(\varphi(S^{(n)}_\xi)), \quad \forall \xi \in X_n.$$

Then $\tilde{\varphi}$ is a graded completely bounded admissible isomorphism such that $\tilde{\varphi}^{-1} = \tilde{\varphi}^{-1}$, and $\|\tilde{\varphi}\|_{cb} \leq \|\varphi\|_{\text{mindeg}}$.  

**Proof.** We first make a few observations. Notice that since $\varphi$ is a semi-graded isomorphism, if $T \in \mathcal{T}_+^{(X)}$ and $n = \text{mindeg}(T)$, then

$$\Phi_n^{(Y)}(\varphi(T)) = \Phi_n^{(Y)}(\varphi(T)).$$

It follows that for all $n$,  

$$(6.2) \quad T = \Phi_n(T) = \Phi_n(\varphi^{-1}(\varphi(T))) = \Phi_n(\varphi^{-1}(\varphi(T))), \quad \forall T \in \mathcal{T}_+^{(X)}_n.$$

Recall $\rho_\varphi = \Phi_0 \circ \varphi \mid_{\mathcal{M}}$. Since $\varphi$ is admissible, $\rho_\varphi$ is a *-automorphism. For each $n \in \mathbb{N}$, let us define $V_n: X_n \to Y_n$ by the map $V_n(\xi) = (\Psi_n)^{-1}\Phi_n(\varphi(\Psi_n(\xi)))$, where $\Psi_n(\xi) = S^{(n)}(\xi)$ (inserted in the appropriate Toeplitz algebra; the superscripts $X$ and $Y$ are clear from the context and hence omitted). Let also $V'_n: Y_n \to X_n$ be the map given by $V'_n(\eta) = (\Psi_n)^{-1}\Phi_n(\varphi^{-1}(\xi_n))$.

Notice that $V_n$ is clearly well-defined and $\|V_n\| \leq \|\varphi\|$. It follows from Proposition 6.2 that $V_n$ is a $\rho_\varphi$-correspondence morphism for every $n$. The same reasoning applies to $V'_n$ to show that it is a $\rho_\varphi^{-1}$-correspondence morphism, and in particular $\|V'_n\| \leq \|\varphi^{-1}\|$. Furthermore, by equation (6.2), we have that $V'_n^{-1} = V_n$. It follows that $\text{sup}_{n \in \mathbb{N}}\|V_n\| \leq \|\varphi\|$ and $\text{sup}_{n \in \mathbb{N}}\|V'_n\| \leq \|\varphi^{-1}\|$.

In order to obtain that $V = (V_n)$ is a $\rho_\varphi$-similarity, it remains to show that $V_{n+m}^{(X)} = U_{n,m}^{(X)}(V_n \otimes V_m)$ for all $n,m$. So let $\xi \in X_n$, $\eta \in X_m$, and let $\xi' = V_n(\xi)$ and $\eta' = V_m(\eta)$, so that $S^{(n)}_{\xi'} = \Phi_n(\varphi(S^{(n)}_\xi))$. Notice that

$$S^{(n+m)}_{\eta'} = \Phi_n(\varphi(S^{(n)}_{\eta})), \quad \Phi_n(\varphi(S^{(n)}_{\xi} \cdot S^{(m)}_{\eta})) = \Phi_n(\varphi(S^{(n)}_{\xi} \cdot S^{(m)}_{\eta})).$$

and because $\varphi$ is semi-graded we also have that

$$\Phi_n(\varphi(S^{(n)}_{\xi} \cdot S^{(m)}_{\eta})) = \Phi_n(\varphi(S^{(n)}_{\xi} \cdot S^{(m)}_{\eta})).$$

Therefore we obtain the desired identity as follows:

$$S^{(n+m)}_{U_{n,m}(\xi' \otimes \eta')} = \Phi_n(U_{n,m}(\xi \otimes \eta)).$$

By proposition [6.12] we have that $\tilde{\varphi} = \text{Ad}_\varphi$ is a graded completely bounded admissible isomorphism from $\mathcal{T}_+^{(X)}$ to $\mathcal{T}_+^{(Y)}$. It is clear that it satisfies the required property, and its uniqueness is also clear. Finally, the norm inequality follows from Proposition 6.12. □
Proposition 6.18. Let $X, Y$ be subproduct systems and let $\varphi : \mathcal{T}_+(X) \to \mathcal{T}_+(Y)$ be a semi-graded isometric isomorphism. Let $\tilde{\varphi} : \mathcal{T}_+(X) \to \mathcal{T}_+(Y)$ be the unique map satisfying

$$\tilde{\varphi}(S_{\xi}^{(n)}) = \Phi_n(\varphi(S_{\xi}^{(n)})], \quad \forall \xi \in X_n.$$ 

Then $\tilde{\varphi}$ is a graded completely isometric isomorphism such that $\tilde{\varphi}^{-1} = \tilde{\varphi}^{-1}$.

Proof. First notice that by Lemma 6.17 the map $\varphi$ is admissible. Therefore by the previous proposition, we have that the map $\tilde{\varphi}$ is a completely bounded graded isomorphism satisfying $\|\tilde{\varphi}\|_{cb} \leq \|\varphi\|_{cb}$ and $\tilde{\varphi}^{-1} = \tilde{\varphi}^{-1}$. Since $\varphi$ is isometric, we have that $\|\tilde{\varphi}\|_{cb} \leq 1$ and also $\|\tilde{\varphi}^{-1}\|_{cb} \leq 1$ similarly. It follows that $\tilde{\varphi}$ is completely isometric. \(\square\)

We will show in Section 7 that given a bounded/isometric admissible isomorphism between tensor algebras arising from stochastic matrices, then there exists a semi-graded bounded/isometric isomorphism between them. The two previous propositions will then allow us to reach the desired goal of the existence of a graded isometric/bounded isomorphism between the algebras.

Reducing projections. We now define the notion of reducing projection, which will be useful in converting bounded isomorphisms into semi-graded isomorphisms, in the context of subproduct systems arising from stochastic matrices.

Definition 6.19. Let $(X, U)$ be a subproduct system over the $W^*$-algebra $\mathcal{M}$. A projection $p \in \mathcal{M}$ is said to be reducing for $X$ if

$$U_{n,m}^*(pX_{n+m}p) \subseteq pX_np \otimes pX_mp.$$ 

Proposition 6.20. Let $X$ be a subproduct system over a $W^*$-algebra $\mathcal{M}$, and suppose that $p \in \mathcal{M}$ is a projection. Then $p$ is reducing for $X$ if and only if $pXp = \{pX_np\}_{n \in \mathbb{N}}$ with subproduct maps $\{U_{n,m} | pX_np \mapsto pX_mp\}$ is a subproduct system over $pM\mathcal{P}$. Moreover, if $\mathcal{M}$ is commutative, then $p$ is reducing if and only if for every $\xi \in X_n$ and $\eta \in X_m$,

$$(6.3) \quad U_{n,m}(p\xi(1 - p) \otimes (1 - p)\eta)p) = 0,$$

or alternatively

$$U_{n,m}(p\xi \otimes \eta)p) = U_{n,m}(p\xi_\eta \otimes \eta)p).$$

Proof. $\Rightarrow$: Suppose that $p \in \mathcal{M}$ is reducing for $X$. Then $pXp$ is readily shown to be a family of $W^*$-correspondences over $pM\mathcal{P}$ with respect to the restricted actions and inner product. Regarding self-duality, note that if $E$ is a right Hilbert $W^*$-module over $\mathcal{M}$, then $\mathcal{M}$ is faithfully represented in a Hilbert space $H$, and $E$ can be regarded as a von Neumann module inside $B(H, E \otimes H)$, in the terminology of [Ske00]. It is easy to see that if $p \in \mathcal{M}$ is a projection, then $Ep$ is identified with a von Neumann module over $pM\mathcal{P}$ in $B(pHp, Ep\otimes pHp)$, which, hence, is self-dual.

The thing only left to check is that the restriction of $U_{n,m}$ to $pX_np \otimes pX_mp$ is a coisometric $W^*$-correspondence map onto $pX_{n+m}$, which is an immediate consequence of the fact that $p$ is reducing.

$\Leftarrow$: Notice that since $U_{n,m}$ is a coisometry onto $pX_{n+m}$ when restricted to $pX_np \otimes pX_mp$, we obtain immediately that $p$ is reducing.

For the remaining statement of the proposition, suppose that $\mathcal{M}$ is commutative. Note that in that case we always have the orthogonal direct sum decomposition $pX_np \otimes X_mp = (pX_np \otimes pX_mp) \oplus (pX_n(1 - p) \otimes (1 - p)X_mp)$. If (6.3) is satisfied, then $(pX_n(1 - p) \otimes (1 - p)X_mp) \subseteq \ker U_{n,m}$. It follows by orthogonality and the fact that $U^*_{n,m}$ is a correspondence map that $U^*_{n,m}(pX_{n+m}) \subseteq pX_np \otimes pX_mp$. Hence $p$ is reducing. The proof of the converse is straightforward. \(\square\)

Proposition 6.21. Let $X$ be a subproduct system over the $W^*$-algebra $\mathcal{M}$, and $p \in \mathcal{M}$ reducing for $X$. Then one has a canonical graded contractive isomorphism given by the restriction map $Res_p : p\mathcal{T}_+(X)p \to \mathcal{T}_+(pXp)$ defined by $Res_p(pTp) = pTp \mid _{pXp}$. 

Proof. $Res_p$ is easily shown to be a contractive surjective homomorphism. We show it is injective. Indeed, if we represent $T = \sum_{n=0}^{\infty}S_{\xi}^{(n)}$ for unique $\xi_n \in X_n$, then $pTp = \sum_{n=0}^{\infty}S_{p\xi_n}^{(n)}$. If $Res_p(pTp) = 0$, then for every $n \in \mathbb{N}$ we have $\Phi_n(Res_p(pTp)) = S_{p\xi_n}^{(n)} \mid _{pXp} = 0$. But since $\|S_{p\xi_n}^{(n)} \mid _{pXp} \| \geq \|S_{p\xi_n}^{(n)} \mid _{pXp} \| = \|p\xi_n\|$, we must have $S_{p\xi_n}^{(n)} = 0$ for all $n \in \mathbb{N}$, ensuring that $pTp = 0$. \(\square\)
We now focus our attention on the case of subproduct systems over a commutative W*-algebra. Note that in this case, by Proposition 6.13 all bounded semi-graded isomorphisms of tensor algebras of subproduct systems over a commutative W*-algebra $\mathcal{M}$ are automatically admissible.

**Lemma 6.22.** Let $X$ and $Y$ be subproduct systems over a commutative W*-algebra $\mathcal{M}$, and let $\varphi : T_+(X) \to T_+(Y)$ be an (algebraic) isomorphism. Then for every pair of projections $p, q \in \mathcal{M}$ with $pq = 0$ and any $T \in T_+(X)$ such that $\mindeg(T) \geq 1$ we have that

$$\mindeg(\varphi(pTq)) \geq 1.$$ 

*Proof.* Let $T \in T_+(X)$ be given such that $\mindeg(T) \geq 1$ and let $p' = \Phi_0(\varphi(p))$ and $q' = \Phi_0(\varphi(q))$. Note that since $pq = 0$ and both $\varphi$ and $\Phi_0$ are homomorphisms we have that $p'q' = 0$. Thus,

$$\Phi_0(\varphi(pTq)) = p'\Phi_0(\varphi(T))q' = p'q'\Phi_0(\varphi(T)) = 0$$

where we used commutativity of $\mathcal{M}$ to interchange the order of the elements in the product. \hfill $\square$

**Definition 6.23.** Let $X$ and $Y$ be subproduct systems over a commutative W*-algebra $\mathcal{M}$, and let $\varphi : T_+(X) \to T_+(Y)$ be an (algebraic) isomorphism. We say that a projection $p \in \mathcal{M}$ is $\varphi$-singular if for some $\xi \in X_1$ we have

$$\mindeg(\varphi(pS_\xi^{(1)})) = 0$$

Otherwise, we say that $p$ is $\varphi$-regular.

The following proposition will be useful in reducing the problem of finding a semi-graded bounded isomorphism even further, in the admissible case.

**Proposition 6.24.** Let $X$ and $Y$ be subproduct systems over a commutative W*-algebra $\mathcal{M}$, and let $\varphi : T_+(X) \to T_+(Y)$ be an admissible bounded isomorphism. If there exists a family of pairwise-perpendicular $\varphi$-regular projections $\{p_i\}_{i \in I}$ such that $\sum_{i \in I} p_i = 1_{\mathcal{M}}$, then $\varphi$ is semi-graded.

*Proof.* By Proposition 6.16 it suffices to show that for every $\xi \in X_1$ we have that $\mindeg(\varphi(S_\xi^{(1)})) \geq 1$. Since $\sum_{i \in I} p_i = 1$ by assumption, due to Lemma 6.22 and normality of $\varphi : \mathcal{M} \to \mathcal{M}$ (it being a *-automorphism) we have the following chain of equalities taking place in $\mathcal{M}$, where the sums are convergent in the weak* topology:

$$\Phi_0(\varphi(S_\xi^{(1)})) = \rho_\varphi(\sum_{i \in I} p_i) \cdot \Phi_0(\varphi(S_\xi^{(1)})) \cdot \rho_\varphi(\sum_{j \in I} p_j) = \sum_{i \in I} \sum_{j \in I} \rho_\varphi(p_i)\Phi_0(\varphi(S_\xi^{(1)}))\rho_\varphi(p_j)$$

$$= \sum_{i \in I} \sum_{j \in I} \Phi_0(\varphi(p_is_\xi^{(1)}p_j)) = \sum_{i \in I} \Phi_0(\varphi(p_is_\xi^{(1)}p_i)) = 0.$$

$\square$

We will need the following facts about reducing projections in the next section, for dealing with subproduct systems arising from stochastic matrices.

**Proposition 6.25.** Let $X$ and $Y$ be subproduct systems over a commutative W*-algebra $\mathcal{M}$, and let $\varphi : T_+(X) \to T_+(Y)$ be a bounded isomorphism, and $p \in \mathcal{M}$ a nonzero projection. Let $q = \Phi_0(\varphi(p))$.

1. $q$ is a nonzero self-adjoint projection in $\mathcal{M}$.
2. If $p$ is reducing for $X$ then $q$ is reducing for $Y$, and $q\varphi(p)q = q$.
3. Suppose that $p$ is reducing and let $\varphi_{p,q} : pT_+(X)p \to qT_+(Y)q$ be given by $\varphi_{p,q}(x) = q\varphi(x)q$.
   Then $\varphi_{p,q}$ is a bounded isomorphism.

*Proof.* (1) Since $p$ is an idempotent we have that $\varphi(p)^2 = \varphi(p)$, and thus $q^2 = \Phi_0(\varphi(p))^2 = \Phi_0(\varphi(p)^2) = \Phi_0(\varphi(p)) = q$, and since $\mathcal{M}$ is commutative, we must have that $q$ is a self-adjoint projection. Furthermore, it must be nonzero. Indeed, $\varphi(p) \neq 0$ is an idempotent, and we must have $\mindeg(\varphi(p)) = \mindeg(\varphi(p)^2) = 2\mindeg(\varphi(p))$. Hence $\mindeg(\varphi(p)) = 0$.

   (2) We show now that if $p$ is reducing for $X$, then $q$ must be reducing for $Y$. Let $\xi \in Y_n$ and $\eta \in Y_m$, and let us consider the following Fourier series representations:

$$\varphi^{-1}(\xi) = \sum_{k=0}^\infty S_{\xi k}^{(k)} \quad \text{and} \quad \varphi^{-1}(\eta) = \sum_{k=0}^\infty S_{\eta k}^{(k)}.$$
By applying $\Phi$ a bounded linear map. We will show that it is multiplicative and bijective.

Now let $T = \varphi(p) - q$, and note that it is an operator of minimal degree strictly greater than 0. By applying $\varphi$ on both sides of the previous equation we obtain that

$$(q + T)S_{\xi}^{(n)}S_{\eta}^{(m)}(q + T) = (q + T)S_{\xi}^{(n)}(q + T)S_{\eta}^{(m)}.$$  

By applying $\Phi_{n+m}$ on both sides of the equation, we have that

$$S_{U,m(q\xi \otimes \eta \eta)}^{(n+m)} = qS_{\xi}^{(n)}S_{\eta}^{(m)} = qS_{\xi}^{(n)}qS_{\eta}^{(m)} = S_{U,m(q\xi \otimes \eta \eta)}^{(n+m)}.$$  

This establishes that $q$ is reducing for $Y$.

Let us show that $q\varphi(p)q = q$. Write uniquely the Fourier series representation $\varphi(p) = q + \sum_{m = 1}^{\infty} S_{\xi}^{(m)}$. Since $\varphi(p)^2 = \varphi(p)$ we have that

$$\zeta_1 = q\zeta_1 + \zeta_1 q$$

$$\zeta_m = q\zeta_m + \left( \sum_{k=1}^{m-1} U_{k,m-k}(\zeta_k \otimes \zeta_{m-k}) \right) + \zeta_m q, \quad m \geq 2$$

By multiplying the first equation by $q$ on both sides, we obtain $q\zeta_1 q = 2q\zeta_1 q$ and so $q\zeta_1 q = 0$. Similarly, multiplying the second equation by $q$ from both sides, and using induction and the reducibility of $q$, we obtain

$$q\zeta_m q = q\zeta_m q + \left( \sum_{k=1}^{m-1} U_{k,m-k}(q\zeta_k q \otimes q\zeta_{m-k} q) \right) + q\zeta_m q = 2q\zeta_m q$$

and thus, $q\zeta_m q = 0$ for all $m$. Therefore, $q\varphi(p)q = q$.

(3) Let us suppose that $p$ is reducing, and $\varphi$ is a bounded isomorphism. It is clear that $\varphi_{p,q}$ is a bounded linear map. We will show that it is multiplicative and bijective.

Let $\xi \in X_n$ and $\eta \in X_m$. Write uniquely the Fourier series representations

$$\varphi(S_{\xi}^{(n)}) = \sum_{k=0}^{\infty} S_{\xi}^{(k)}$$

and

$$\varphi(S_{\eta}^{(m)}) = \sum_{k=0}^{\infty} S_{\eta}^{(k)}$$

By an argument similar to the one earlier, using Proposition 6.2 and the reducibility of $q$ we obtain

$$q\varphi(S_{\xi}^{(n)}) \varphi(S_{\eta}^{(m)}) q = q \sum_{k=0}^{\infty} S_{\xi}^{(k)} \sum_{\ell=0}^{\infty} U_{\xi,\ell}(q\xi \otimes \eta \eta) q = q \sum_{k=0}^{\infty} S_{\xi}^{(k)} \sum_{\ell=0}^{\infty} U_{\xi,\ell}(q\xi \otimes \eta \eta) q = q \varphi(S_{\xi}^{(n)}) q \varphi(S_{\eta}^{(m)}) q$$

Thus by boundedness of $\varphi$, we obtain that $\varphi_{p,q}$ is multiplicative.

Let us show that $\varphi_{p,q}$ is injective. Suppose that $0 \neq pTp \in pT_+(X)p$ satisfies $q\varphi(pTp)q = 0$. Notice that for all $n \geq 0$, $q\Phi_n(q\varphi(pTp)q) = \Phi_n(q\varphi(pTp)q) = 0$.

On the other hand, if $d = \text{mindeg}(\varphi(pTp))$ then

$$\Phi_d(\varphi(pTp)) = \Phi_d(\varphi(p)p\varphi(pTp)p\varphi(p)) = q\Phi_d(\varphi(pTp))q = 0$$

We conclude that $\Phi_n(\varphi(pTp)) = 0$ for all $n$, and therefore $\varphi(pTp) = 0$. By injectivity of $\varphi$, it follows that $pTp = 0$, which is a contradiction.

We now show surjectivity. Let $qTq \in qT_+(Y)q$ be given. By surjectivity of $\varphi$ there exists $S \in T_+(X)$ such that $\varphi(S) = qTq$. Using the fact that $q\varphi(p)q = q$ we obtain

$$\varphi_{p,q}(pSp) = q\varphi(pSp)q = q\varphi(p)\varphi(S)\varphi(p)q = q\varphi(p)Tq\varphi(p)q = qTq.$$
7. Tensor algebras of subproduct systems arising from stochastic matrices

We now head towards handling the specific case of our interest: the characterization of isomorphisms between tensor algebras arising from subproduct systems of stochastic matrices over a countable state space $\Omega$. In this case $\mathcal{M} = \ell^\infty(\Omega)$, and we have a natural family of pairwise orthogonal projections summing up to 1, namely $\{p_i\}_{i \in \Omega}$. Thus one can try to apply Proposition 6.24 in the admissible case and Proposition 6.25.

**General Properties.** Firstly, we characterize several general properties and concepts in the particular circumstances arising from stochastic matrices: $\rho$-similarities, admissibility and automatic continuity of algebraic isomorphisms.

When $P$ is a stochastic matrix, we will write $T_+(P) := T_+(Arv(P))$.

**Theorem 7.1.** Let $P$ and $Q$ be stochastic matrices.

1. If there exists a $\rho$-similarity of $Arv(P)$ and $Arv(Q)$ then $P \sim_\rho Q$.
2. If $P$ and $Q$ are finite and essential such that $P \sim_\rho Q$, then there exists a $\rho_\sigma$-similarity of $Arv(P)$ and $Arv(Q)$.

**Proof.** When $\sigma$ is a permutation of $\Omega$, we denote for brevity, $i' = \sigma(i)$.

1: Assume $V : Arv(P) \rightarrow Arv(Q)$ is a given $\rho$-similarity of the subproduct systems. Then $\rho = V_0 : \ell^\infty(\Omega) \rightarrow \ell^\infty(\Omega)$ is induced by a permutation $\sigma = \sigma_\rho$. Now for all $(i,j) \in E(P)$ denote $E_{ij}$ to be the element in $Arv(P)_1$ which is 1 at $(i,j)$ and zero otherwise. Due to $V_1$ being a $\rho$-correspondence morphism, we have,

$$V_1(E_{ij}) = V_1(p_iE_{ij}p_j) = \rho(p_i)V_1(E_{ij})\rho(p_j) = p_iV_1(E_{ij})p_j$$

So we must have that $0 \neq V_1(E_{ij}) = b_{ij} : E_{i'j'}$ for some $0 \neq b_{ij} \in \mathbb{C}$. Hence $(i',j') \in E(Q)$, and $P \sim_\sigma Q$.

2: Assume $P \sim_\sigma Q$. Define $V_0 : \ell^\infty(\Omega) \rightarrow \ell^\infty(\Omega)$ by $V_0 = \rho_\sigma = \rho$, that is, $V_0(p_i) = p_{\sigma(i)}$.

Define $V_n : Arv(P)_n \rightarrow Arv(Q)_n$ by $V_n(A) = (\sqrt{Q^m})^\ast [R_\sigma(\sqrt{P^m} \ast A)R_\sigma^{-1}]$. We need to show that $V_n$ is a bijective, subproduct preserving, $\rho$-morphorphism with sup$_{n \in \mathbb{N}} \{\|V_n\|,\|V_n^{-1}\|\} < \infty$.

It is immediate that $V_n$ is bijective with $\rho^{-1}$-morphism inverse $V_n^{-1}(A) = (\sqrt{P^m})^\ast [R_{\sigma^{-1}}(\sqrt{Q^m} \ast A)R_{\sigma^{-1}}^{-1}]$. We show that sup$_{n \in \mathbb{N}} \{\|V_n\|\} < \infty$ and the proof for $V_n^{-1}$ is analogous. Indeed, for $A = [a_{ij}] \in Arv(P)_n$ we have

$$\|V_n(A)\|^2 = \sum_i \sup_{j} \|P^{(n)}_{ij}Q_{i'j'}^{(n)}\| |a_{ij}|^2 \leq \left( \sup_{(i,j) \in E(P^m)} \|P^{(n)}_{ij}Q_{i'j'}^{(m)}\| \right) \|A\|^2$$

Denote

$$c_n(i,j) = \begin{cases} \frac{P^{(n)}_{ij}Q_{i'j'}^{(m)}}{\sqrt{Q_{i'j'}^{(m)}}} : (i,j) \in E(P^m) \\ 0 : (i,j) \notin E(P^m) \end{cases}$$

Due to finiteness of $P$, in order to show sup$_{n \in \mathbb{N}} \{\|V_n\|\} < \infty$, it would suffice to show that for all $(i,j) \in \Omega^2$ one has lim$_{n \rightarrow \infty} c_n(i,j) < \infty$. Since $P$ and $Q$ are essential, Theorem 2.7 allows us to assume WLOG that $P$ and $Q$ are irreducible. Indeed, if $i$ and $j$ do not belong to the same irreducible component in a decomposition of $P$ into irreducibles, we would just have $c_n(i,j) = 0$ for all $n \in \mathbb{N}$. Thus, assuming $P$ (as well as $Q$) is irreducible and $r$-periodic, with $i \in \Omega_t$ and $j \in \Omega_{t_2}$ where $\Omega_0, \ldots, \Omega_{t-1}$ is a cyclic decomposition for $P$ given by Theorem 2.8 noting that the only instance for which $c_n(i,j) > 0$ is when $n$ is of the form $n = n'r + \ell$, where $\ell = t_2 - t_1$ (mod $r$), we may apply Theorem 2.11 to obtain that lim$_{n \rightarrow \infty} c_n(i,j) < \infty$. Thus we establish that $V_n$ is a bijective $\rho$-morphism with sup$_{n \in \mathbb{N}} \{\|V_n\|\} < \infty$.

We now show that $V_n$ preserves subproducts. Indeed, for $A \in Arv(P)_n$ and $B \in Arv(P)_m$ we have

$$U_{n,m}^{Q}(V_n(A) \otimes V_m(B)) = (\sqrt{Q^{n+m}})^\ast [R_\sigma(\sqrt{P^n} \ast A)R_\sigma^{-1}] \cdot (\sqrt{Q^{m+n}})^\ast [R_\sigma(\sqrt{P^m} \ast B)R_\sigma^{-1}] = \sqrt{V_nA \otimes V_mB}$$
Thus the family $V_n : Arv(P) \to Arv(Q)$ is a $\rho_n$-similarity, as required.

Thus, for finite essential stochastic matrices, only their graph structure determines whether or not there exists a $\rho$-similarity of the subproduct systems.

**Theorem 7.2** (Automated continuity). Let $P$ and $Q$ be a stochastic matrices. If $\varphi : T_+(Q) \to T_+(P)$ is an algebraic isomorphism, then $\varphi$ is bounded.

**Proof.** By Corollary 6.5, it suffices to show that for every $0 \neq T \in T_+(P)_d$ there exists $Z \in \ker \Phi_0$ such that $TZ^n \neq 0$ for all $n \geq 1$. For the purpose of the proof, we say that $i \in \Omega$ is returning for $P$ if $P_{ii}^{(\ell)} > 0$ for some (and hence infinitely many) $1 \leq \ell \in \mathbb{N}$, and we say for $i, j \in \Omega$ that $i$ leads to $j$ if $P_{ij}^{(\ell)} > 0$ for some $1 \leq \ell \in \mathbb{N}$.

Let $T = S_A^{(d)} \neq 0$ and denote $A = [a_{ij}]$. There exist $i, j \in \Omega$ such that $p_i T p_j \neq 0$, or equivalently $p_i A p_j \neq 0$, since $p_i T p_j = a_{ij} S_A^{(d)}$. Note that for $d = 0$ we must have $i = j$. We split the proof into three cases.

1. There is a pair $i, j \in \Omega$ such that $p_i A p_j \neq 0$ and $j \in \Omega$ is returning for $P$. Then let $Z = S_{E_{ij}}^{(\ell)}$, where $1 \leq \ell \in \mathbb{N}$ is such that $P_{jj}^{(\ell)} > 0$. Note that for each $n \geq 1$ there exists $c_n > 0$ so that $(S_{E_{ij}}^{(\ell)})^n = c_n S_{E_{ij}}^{(n\ell)}$. Therefore $TZ^n = S_A^{(d)} (S_{E_{ij}}^{(\ell)})^n = c_n S_A^{(d)} S_{E_{ij}}^{(n\ell)} = c'_n S_B^{(d+n\ell)}$ where $c'_n$ is a nonzero scalar and $B = (\sqrt{P_{dd}^{(\ell)}})^b * \sqrt{P_{dd}^{(\ell)}} * (A p_j)$, which is non-zero since $A p_j \neq 0$, and the matrices $P^{(d)}$ and $P^{d+n\ell}$ have non-zero entries where $A p_j$ does.

2. There is no pair $i, j \in \Omega$ such that $p_i A p_j \neq 0$ and $j \in \Omega$ is returning for $P$, however there is a pair $i, j \in \Omega$ such that $p_i A p_j \neq 0$ and $j$ leads to a returning state $k \in \Omega$. In that case there exist $m, \ell \geq 1$ such that $S_{E_{jk}}^{(m)} \neq 0$ and $S_{E_{kk}}^{(\ell)} \neq 0$. Let us take $Z = S_{E_{jk}}^{(m)} + S_{E_{kk}}^{(\ell)}$. Note that since $S_{E_{jk}}^{(m)} S_{E_{kk}}^{(\ell)} = 0$, we have $Z^n = S_{E_{jk}}^{(m)} (S_{E_{kk}}^{(\ell)})^{n-1} + (S_{E_{kk}}^{(\ell)})^n$. Thus, since $A p_k = 0$ we must have that $TZ^n = S_A^{(d)} (S_{E_{jk}}^{(m)} (S_{E_{kk}}^{(\ell)})^{n-1} + (S_{E_{kk}}^{(\ell)})^n) = S_A^{(d)} S_{E_{jk}}^{(m)} (S_{E_{kk}}^{(\ell)})^{n-1}$

Let $T' = S_A^{(m)} S_{E_{jk}}^{(\ell)}$. Notice that $p_i T' p_k \neq 0$, and in this case $k$ is returning for $T'$. Therefore, by the argument of case (1) we see that $T' (S_{E_{kk}}^{(\ell)})^{n-1} \neq 0$, and hence $TZ^n \neq 0$, for all $n \geq 1$.

3. There is no pair $i, j \in \Omega$ such that $p_i A p_j \neq 0$ and $j \in \Omega$ is returning for $P$ or $j$ leads to a returning state $k \in \Omega$. Let $i, j \in \Omega$ be such that $p_i A p_j \neq 0$. Then there exists a sequence of pairwiseperpendicular non-zero projections, which constitutes a basis for $\ell^\infty(\Omega)$, gets sent to a family of pairwise perpendicular non-zero projections $\{\rho_\varphi(p_i)\}_{i \in \Omega}$, due to Proposition 6.23 item (1) and $\rho_\varphi$ being a homomorphism. This implies, in particular, that this new family is a linearly independent set. For

Henceforth, there is no ambiguity when referring to isomorphisms $\varphi : T_+(P) \to T_+(Q)$: they are always bounded, and there is no need to mention boundedness explicitly.

**Proposition 7.3** (Automatic admissibility for finite matrices). Let $P$ and $Q$ be stochastic matrices over finite $\Omega$. Then every (algebraic) isomorphism $\varphi : T_+(P) \to T_+(Q)$ is admissible.

**Proof.** We show that $\rho_\varphi = \Phi_0 \circ \varphi |_{\ell^\infty(\Omega)}$ is injective, since then, finite dimensionality would imply bijectiveness, and Lemma 6.8 will show that $\rho_\varphi$ is a $\ast$-automorphism, thus satisfying (A1) in Definition 6.6. Indeed, to show injectivity, note that the finite family $\{p_i\}_{i \in \Omega}$ of pairwise perpendicular non-zero projections, which constitutes a basis for $\ell^\infty(\Omega)$, gets sent to a family of pairwise perpendicular non-zero projections $\{\rho_\varphi(p_i)\}_{i \in \Omega}$, due to Proposition 6.23 item (1) and $\rho_\varphi$ being a homomorphism.
(A2) in Definition 6.6 note that \( \varphi^{(n,m)} \) as maps between finite dimensional W*-correspondences are automatically \( \sigma \)-topology continuous.

\( \square \)

Reducing and Singular projections. We proceed to characterize reducing projections in \( \text{Arv}(P) \). Recall the terminology of paths from Definition 2.4. We will sometimes mention paths and cycles without mentioning the stochastic matrix \( P \) when it is determined unambiguously by the context.

**Proposition 7.4.** Let \( P \) be a stochastic matrix and let \( p \) be a projection in \( \ell^\infty(\Omega) \). Let \( C_p \subset \Omega \) be such that \( p = \sum_{i\in C_p} p_i \). Then \( p \) is reducing for \( \text{Arv}(P) \) if and only if for any path \( \gamma : \{0, ..., \ell \} \to \Omega \) in the directed graph of \( P \) with \( \gamma(0), \gamma(\ell) \in C_p \), we have that \( \gamma(k) \in C_p \) for all \( 0 \leq k \leq \ell \).

**Proof.** (\( \Rightarrow \)) : We prove the contrapositive. Suppose that there exists a path \( \gamma \) of length \( n + m \) such that \( i := \gamma(0), k := \gamma(n + m) \in C_p \), while \( j := \gamma(n) \notin C_p \). Decompose \( \gamma \) into two paths, \( \gamma_n \) from \( i \) to \( j \) and \( \gamma_m \) from \( j \) to \( k \), which assures us that \( P_{ij}^{(n)}, P_{jk}^{(m)} > 0 \). Thus,

\[
U_{n,m}(pE_{ij}^{(n)}(1 - p) \otimes (1 - p)E_{jk}^{(m)}p) = U_{n,m}(E_{ij}^{(n)} \otimes E_{jk}^{(m)}) = \sqrt{\frac{P_{ij}^{(n)}P_{jk}^{(m)}}{P_{ik}^{(n+m)}}}E_{ik}^{(n+m)} \neq 0
\]

Which shows that \( p \) is not reducing.

(\( \Leftarrow \)) : Again we prove the contrapositive. If \( p \) is not reducing, then

\[
0 \neq U_{n,m}(pA(1 - p) \otimes (1 - p)Bp)
\]

for some \( A \in \text{Arv}(P)_n \) and \( B \in \text{Arv}(P)_m \). Thus there exists some \( i, k \in C_p \) and \( j \notin C_p \) such that

\[
0 \neq U_{n,m}(p_iAp_j \otimes p_jBp_k)
\]

In particular, \( p_iAp_j \) and \( p_jBp_k \) are nonzero in the subproduct system, so that by considering their supports we obtain \( P_{ij}^{(n)}, P_{jk}^{(m)} > 0 \). Hence there exists a path \( \gamma \) with \( \gamma(0), \gamma(n + m) \in C_p \) while \( \gamma(n) \notin C_p \). \( \square \)

**Definition 7.5.** Let \( P \) be a stochastic matrix over a state set \( \Omega \). A path \( \gamma \) of length \( \ell \) is said to be streamlined if for every \( 0 \leq k \leq \ell - 1 \) one has \( \gamma(k) \neq \gamma(k + 1) \).

Given two different states \( i, j \in \Omega \), we note that if there exists a path from \( i \) to \( j \), then a simple culling procedure yields a streamlined path from \( i \) to \( j \). The situation for cycles is slightly different: in general, a cycle \( \gamma \) of length \( \ell \) from \( i \) to \( i \) only yields a streamlined cycle by culling when there exists \( 1 \leq n \leq \ell - 1 \) such that \( \gamma(n) \neq i \).

**Corollary 7.6.** Let \( P \) be a stochastic matrix over a state set \( \Omega \). A projection \( p_i \) for \( i \in \Omega \) is reducing for \( \text{Arv}(P) \) if and only if there are no streamlined cycles through \( i \) in the directed graph of \( P \).

**Remark 7.7.** If \( P \) and \( Q \) are stochastic over \( \Omega \) and \( i \in \Omega \) is such that \( p_i \) is \( \varphi \)-singular for some \( \varphi : T_+(P) \to T_+(Q) \) then it must be the case that \( P_{ii} > 0 \). Indeed, for some \( \xi \in \text{Arv}(P)_1 \), we have that \( \Phi_0(\varphi(p_iS_{\xi}^{(1)}p_i)) \neq 0 \) so that \( p_iS_{\xi}^{(1)}p_i \neq 0 \). Therefore, since \( p_iS_{\xi}^{(1)}p_i \) is supported in \( (i,i) \) we have that \( S_{E_{ii}}^{(1)} \neq 0 \), and therefore \( P_{ii} > 0 \).

**Remark 7.8.** If \( i \in \Omega \) is such that \( p_i \) is reducing for \( \text{Arv}(P) \), then \( p_i\text{Arv}(P)p_i \) is a subproduct system over \( \mathbb{C} \), with all fibers being dimension 0 (except for the 0-th fiber) or all of dimension 1. Since \( U_{n,m} |_{p_i\text{Arv}(P)_n p_i \otimes p_i\text{Arv}(P)_m p_i} \) remains coisometric for all \( n, m \in \mathbb{N} \), it must in fact be isometric due to dimension considerations. Thus, \( p_i\text{Arv}(P)p_i \) is isomorphic to a product system as in example 2.27 with \( E = 0 \) or \( E = \mathbb{C} \). The tensor algebras are then given up to a canonical isomorphism by \( \mathbb{C} \) or \( \mathbb{A}[\mathbb{D}] \) respectively, where the latter is the disc algebra.

**Proposition 7.9.** Let \( i \in \Omega \). If for some stochastic \( Q \) and isomorphism \( \varphi : T_+(P) \to T_+(Q) \) we have that \( p_i \) is \( \varphi \)-singular then \( p_i \) must be reducing for \( \text{Arv}(P) \).

**Proof.** We prove the contrapositive. Suppose that \( p_i \) is not reducing. By Corollary 7.6 there exists a streamlined cycle \( \gamma \) of length \( \ell \geq 2 \) such that \( \gamma(0) = \gamma(\ell) = i \). Hence, for some \( c_\gamma > 0 \) we have

\[
S_{E_{\gamma(1)}}^{(1)} \cdots S_{E_{\gamma(\ell-1)}}^{(1)} = c_\gamma S_{E_{ii}}^{(\ell)}.
\]
By Remark 7.8, we must have \( S_{E_{\eta}}^{(1)} \neq 0 \). We note that for some \( d_{i,\ell} > 0 \) we have

\[
(S_{E_{\eta}}^{(1)})^\ell = d_{i,\ell} \cdot S_{E_{\eta}}^{(\ell)}.
\]

Since \( \gamma \) is streamlined, by Lemma 0.22, it follows that we must have \( \mindeg(\varphi(S_{E_{\eta}}^{(1)})) \geq 1 \) for all \( 0 \leq k \leq \ell - 1 \). So, by equation (7.1), we must have \( \mindeg(\varphi(S_{E_{\eta}}^{(1)})) \geq \ell \). Therefore, since \( \ell^\infty(\Omega) \) (being commutative) does not contain any nilpotent elements, it follows from equation (7.2) that we must have that \( \mindeg(\varphi(S_{E_{\eta}}^{(1)})) \geq 1 \), contradicting \( \varphi \)-singularity of \( p_1 \).

Thus we see that, in the context of subproduct systems arising from stochastic matrices, singular minimal projections (with respect to some isomorphism) can only arise from reducing minimal projections. In particular, the following special case of our main goal is resolved.

**Corollary 7.10.** Let \( P \) and \( Q \) be two essential stochastic matrices over \( \Omega \) with no \( 1 \times 1 \) irreducible block. If \( \varphi : T_+(P) \to T_+(Q) \) is an admissible isomorphism, then it is semi-graded. Therefore, there exists a completely bounded graded (admissible) isomorphism from \( T_+(P) \to T_+(Q) \), hence there exists a \( \rho \)-similarity between \( Arv(P) \) and \( Arv(Q) \) and \( P \sim_{\sigma_\rho} Q \).

**Proof.** Using Proposition 2.24, we can decompose \( P \) into irreducible blocks, which will be at least \( 2 \times 2 \). For every \( i \in \Omega \) we must have that \( p_i \) is non-reducing for \( Arv(P) \) (and \( Arv(Q) \)) since there exists a streamlined cycle through \( i \) in the directed graph of \( P \). This makes \( p_i \) a \( \varphi \)-regular projection for every \( i \in \Omega \) due to Proposition 7.9. Since \( \varphi \) is admissible, by Proposition 0.24, we obtain that \( \varphi \) is semi-graded. Using Proposition 0.17 yields a the graded completely bounded admissible isomorphism, and by Proposition 6.12, we obtain the desired \( \rho \)-similarity. It follows from Theorem 7.1 that \( P \sim_{\sigma_\rho} Q \).

We will develop further machinery to deal with the case of algebraic isomorphisms between tensor algebras of general (non-essential) stochastic matrices.

**Proposition 7.11.** Let \( P \) and \( Q \) be stochastic matrices over \( \Omega \). Let \( \varphi : T_+(P) \to T_+(Q) \) be an isomorphism and \( i \in \Omega \) be such that \( p_i \) is reducing. Then,

1. There exists \( j \in \Omega \) such that \( p_j = \Phi_0(\varphi(p_i)) \) and \( p_j \) is reducing. Furthermore, the map \( \hat{\varphi}_{p_i,p_j} := \text{Res}_{p_j} \circ \varphi_{p_i,p_j} \circ \text{Res}_{p_j}^{-1} : T_+(p_i Arv(P)p_i) \to T_+(p_j Arv(Q)p_j) \) is an isometric isomorphism.
2. If \( R \) is a stochastic matrix over \( \Omega \) and \( \psi : T_+(Q) \to T_+(R) \) is an isomorphism, let \( k \in \Omega \) be such that \( p_k = \Phi_0(\psi(p_j)) \), then \( p_k = \Phi_0(\psi \circ \varphi(p_i)) \) and \( \psi_{p_j,p_k} \circ \varphi_{p_i,p_j} = (\psi \circ \varphi)_{p_i,p_k} \).

In particular, \( (\varphi_{p_i,p_j})^{-1} = (\varphi^{-1}_{p_j,p_i})^{\gamma} = (\varphi^{-1}_{p_j,p_i})^{p_j,p_i} \) and \( (\hat{\varphi}_{p_i,p_j})^{-1} = (\varphi^{-1}_{p_j,p_i})^{p_j,p_i} \).

**Proof.** (1): Let \( q = \Phi_0(\varphi(p_i)) \). We show that \( q \) is minimal. If not then write \( q = q_1 + q_2 \) both non-zero in \( q^\infty(\Omega) \). Since \( (\varphi_{p_i,q})^{-1} \) is an idempotent in \( p_i T_+(P)p_i \) and \( \varphi_{p_i,q} \) is an isomorphism by item (3) of Proposition 6.25, we must have, by similar arguments to those in item (1) of Proposition 0.24, that \( \Phi_0(\varphi_{p_i,q}(q_1)) \) is a non-zero projection in \( p_i T_+(P)p_i \). Similarly, \( \Phi_0(\varphi_{p_i,q}(q_2)) \) is also a non-zero projection \( p_1 T_+(P)p_i \), which is perpendicular to \( \Phi_0(\varphi_{p_i,q}(q_1)) \). This contradicts minimality of \( p_i \). Now, since \( \hat{\varphi}_{p_i,p_j} \) is a map from \( T_+(p_i Arv(P)p_i) \) into \( T_+(p_j Arv(Q)p_j) \), each being isomorphic to either \( \mathbb{D} \) or \( \mathbb{C} \), both embed into a commutative \( C^* \) algebra, so by Lemma 6.8, our map \( \hat{\varphi}_{p_i,p_j} \) is isometric.

2. Using the previous item, let \( j, k, r \in \Omega \) be such that \( p_j = \Phi_0(\varphi(p_i)) \), \( p_k = \Phi_0(\psi(p_j)) \) and \( p_r = \Phi_0(\psi(\varphi(p_i))) \). By Proposition 6.25, we have that \( p_j = p_j \varphi(p_i) p_j \) and \( p_k = p_k \psi(p_j) p_k \). Therefore,

\[
p_k = p_k \psi(p_j) p_k = p_k \psi(p_j \varphi(p_i) p_j) p_k = p_k \psi(p_j) (\varphi(p_i)) \psi(p_j) p_k.
\]

Applying \( \Phi_0 \) to both sides and using the fact that \( \Phi_0 \) is a homomorphism we obtain that

\[
p_k = \Phi_0(p_k) = \Phi_0(p_k \psi(p_j \varphi(p_i) p_j)) = \Phi_0(\psi(\varphi(p_i))) \Phi_0(\psi(p_j)) \Phi_0(p_k) = p_k \psi(p_j) p_k.
\]

Therefore \( p_k \leq p_r \). By minimality of the projections, we obtain \( p_k = p_r \) as desired.

For the remainder of the statement, given \( p_i T_+(P)p_i \), a computation yields

\[
(\psi_{p_j,p_k} \circ \varphi_{p_i,p_j})(p_i T_+(P)p_i) = p_k \psi(p_j) (\psi \circ \varphi)(p_i T_+(P)p_i) = p_k \psi(p_j) p_k (\psi \circ \varphi)(p_i T_+(P)p_i) = p_k (\psi \circ \varphi)(p_i T_+(P)p_i) = (\psi \circ \varphi)(p_i T_+(P)p_i).
\]
Where the second equality follows along the lines of the proof of multiplicativity of $\psi_{p_1,p_k}$ in item three of Proposition 6.20.

Proposition 7.12. Let $P$ and $Q$ be stochastic, let $i \in \Omega$ and let $\varphi : T_+(P) \to T_+(Q)$ be an isomorphism. Let $i' \in \Omega$ be such that that $p_{i'} = \Phi_0(\varphi(p_i))$. Then $p_i$ is $\varphi$-singular if and only if $p_{i'}$ is $\varphi^{-1}$-singular.

Proof. If $p_i$ is $\varphi$-regular then $p_i$ is $\hat{\varphi}_{p_{i},p_{i'}}$-regular. Thus $\hat{\varphi}_{p_{i},p_{i'}}$ is semi-graded by Proposition 6.24. Since $\varphi^{-1}_{p_{i},p_{i'}} = (\hat{\varphi}_{p_{i},p_{i'}})^{-1}$, we have that $p_{i'}$ is $\varphi^{-1}_{p_{i},p_{i'}}$-regular and so $p_{i'}$ is $\varphi^{-1}$-regular. By Proposition 7.11 item (2) we get that $p_i = \Phi_0(\varphi(p_{i'}))$ so that by applying the above we get that if $p_{i'}$ is $\varphi^{-1}$-regular, then $p_i$ is $\varphi$-regular.

An equivalence between singular projections. For every stochastic matrix $P$ over a state set $\Omega$ we define the set of singular states of $P$

$$I_P := \{ j \in \Omega \mid j \text{ is } \varphi \text{-singular for some isomorphism } \varphi : T_+(P) \to T_+(Q) \}$$

Now, if $\varphi : T_+(P) \to T_+(Q)$ is any algebraic isomorphism, and $j \in I_P$ then there is a projection $p_{j'}$ such that $p_{j'} = \Phi_0(\varphi(p_j))$ and $j' \in I_Q$ and $\hat{\varphi}_{p_{i},p_{i'}} : T_+(p_j\text{Arv}(Q)p_j) \to T_+(p_{j'}\text{Arv}(P)p_{j'})$ is an isometric isomorphism of Banach algebras, by Proposition 7.11. Since $P_{j,j} > 0$, and $Q_{j',j'} > 0$, we must have that $T_+(p_j\text{Arv}(P)p_j)$ and $T_+(p_{j'}\text{Arv}(Q)p_{j'})$ are both isomorphic to the disc algebra $A(D)$ so that the spectra of these commutative Banach algebras can be naturally identified with $\mathbb{D}$ where the vacuum state is identified with the point 0. Composition by $\hat{\varphi}_{p_{i},p_{i'}}$ induces a homeomorphism $f_{j'}^\varphi = (\hat{\varphi}_{p_{i},p_{i'}})^* : \mathbb{D} \to \mathbb{D}$ so that when it is restricted to $\mathbb{D}$, it becomes a biholomorphic automorphism of $\mathbb{D}$ (See [DRS11] Lemma 4.4) for this fact in much greater generality), and $j$ is $\varphi$-regular if and only if $(\hat{\varphi}_{p_{i},p_{i'}})^*$ preserves the vacuum state, that is, if and only if $f_{j'}^\varphi(0) = 0$. Biholomorphisms of the disk have a well-known description as Moebius transformations. Thus, we have proven the following statement.

Proposition 7.13. Let $P,Q$ be stochastic matrices, let $\varphi : T_+(P) \to T_+(Q)$ be an algebraic isomorphism, and let $j \in I_P$. Let $p_{j'} = \Phi_0(\varphi(p_j))$. Then there exist $w_j \in \mathbb{D}$ and $\theta_j \in [0, 2\pi)$ such that $f_{j'}^\varphi = (\hat{\varphi}_{p_{i},p_{i'}})^* : \mathbb{D} \to \mathbb{D}$ is given by

$$f_{j'}^\varphi(z) = e^{i\theta_j} \cdot \frac{z-w_j}{1-w_jz}$$

Furthermore, $p_j$ is $\varphi$-regular if and only if $f_{j'}^\varphi(0) = 0$.

The preceeding discussion and Proposition 7.12 motivate the following definition

Definition 7.14. Let $P$ and $Q$ be stochastic matrices over $\Omega$. An isomorphism $\varphi : T_+(P) \to T_+(Q)$ is said to be regular if for all $i \in \Omega$, the projection $p_i$ is $\varphi$-regular.

Due to Proposition 7.12 we have that $\varphi$ is regular if and only if $\varphi^{-1}$ is regular and, by Proposition 6.24 $\varphi$ is regular and regularizable and if and only if $\varphi$ is semi-graded.

Definition 7.15. Let $P$ be a stochastic matrix over $\Omega$. Define $\sim_P$ to be the equivalence relation given by pairs $(i,k) \in I_P^2$ such that there exists a finite chain $(i = j_0, j_1, ..., j_s = k)$ with the property that any two subsequent pairs $(j_1,j_2)$ there exist $j',k' \in \Omega$ with two streamlined paths from $j'$ to $k'$ in the graph of $P$, one through $j_1$ and the other through $j_2$. Let $\mathcal{R}_P$ be a distinct set of representatives for $\sim_P$, henceforth taken to be fixed.

Remark 7.16. Let $P$ and $Q$ be stochastic matrices. We shall show that $\sim_P$ satisfies the following three properties:

$(\Diamond_1)$ If $\varphi : T_+(P) \to T_+(Q)$ is an isomorphism and $p_j$ is $\varphi$-regular for all $j \in \mathcal{R}_P$, then $p_j$ is $\varphi$-regular for all $j \in \Omega$.

$(\Diamond_2)$ For every $\Lambda = \{\Lambda_j\} \in \mathcal{T}^{\mathcal{R}_P}$ there exists an isometric isomorphism $\alpha_\Lambda : T_+(P) \to T_+(P)$ such that for every $j \in I_P$ we have $f_{\Lambda_j}^\mu(z) = \lambda_\mu z$ for the unique $\mu \in \mathcal{R}_P$ such that $j \sim_P \mu$.

$(\Diamond_3)$ Every isomorphism $\varphi : T_+(P) \to T_+(Q)$ induces an equivalence preserving bijection $\Upsilon_\varphi : I_P \to I_Q$, and hence a bijection $\Upsilon_j : \mathcal{R}_P \to \mathcal{R}_Q$. Moreover, $\Upsilon_{\varphi^{-1}} = (\Upsilon_\varphi)^{-1}$. 

These three properties of $\sim_P$ will suffice to show that every isomorphism can be “deformed” into a regular isomorphism.

First we prove a few auxiliary results about the relation $\sim_P$.

**Proposition 7.17.** Let $P$ and $Q$ be stochastic matrices, and let $\varphi : T_+ (P) \to T_+ (Q)$ be an isomorphism. Assume that $j \in \Omega$ is $\varphi$-singular. Assume further that for $i, k$ there exists a streamlined path $\gamma$ from $i$ to $k$ through $j$ of length $\ell \geq 2$. Then for any path $\gamma'$ from $i$ to $k$ and through $j'$, we have that $p_{j'}$ is reducing for $Arv(P)$. If in addition $P^s_{ij'} > 0$ for some $n \geq 1$, then $p_{j'}$ must be $\varphi$-singular.

**Proof.** If $p_{j'}$ is non-reducing for $Arv(P)$, there exists a streamlined cycle through $j'$ of length $r \geq 2$ and thus $\mindeg(\varphi(S^{(nr)}_{E_{ij'}})) \geq n \cdot \mindeg(\varphi(S^{(r)}_{E_{ij'}})) \geq nr$ for all $n \geq 1$ due to Lemma [6.22]. Now, if $\gamma$ is of length $\ell$ with $\gamma(m) = j$, and $\gamma'$ is of length $\ell'$ with $\gamma'(m') = j'$ both assumed to be streamlined, then up to some positive scalar, which we denote by $\approx$, we have

$$S_{E_{ik}}^{(s)} \approx S_{E_{ij}(1)}^{(1)} \cdot \ldots \cdot S_{E_{ij}(m-1)_{j'}}^{(1)} \cdot S_{E_{ij}}^{(s)} \cdot \ldots \cdot S_{E_{ij}(t-1)_{j'}}^{(1)}$$

Now if $\mindeg(\varphi(S^{(s)}_{E_{ij}})) = t$, due to the above and the fact $\Phi_0(\varphi(S^{(s)}_{E_{ij}}))$ is a non-zero scalar multiple of $p_{j'}$, we still have the equality $\mindeg(\varphi(S^{(s)}_{E_{ij}})) = t$ for all $s \geq 0$.

On the other hand for $n \geq 1$,

$$S_{E_{ik}}^{(s+n)} \approx S_{E_{ij}(1)}^{(1)} \cdot \ldots \cdot S_{E_{ij}(m-1)_{j'}}^{(1)} \cdot S_{E_{ij}}^{(s)} \cdot \ldots \cdot S_{E_{ij}(t-1)_{j'}}^{(1)}$$

Due to Proposition [7.9] we have that $p_{j'}$ is $\varphi$-regular, so we get that $\mindeg(S^{(nr)}_{E_{ij'}}) \geq nr$, concluding that $\mindeg(\varphi(S^{(nr)}_{E_{ij}})) \geq \ell' + nr$ for all $n \geq 1$, obtaining a contradiction. It follows that $p_{j'}$ must be reducing for $Arv(P)$.

For the second part, if by negation $p_{j'}$ is $\varphi$-regular, and $1 \leq r \in \mathbb{N}$ is minimal such that $P^r_{ij'} > 0$, with the same assumptions on $\gamma$ and $\gamma'$ as before, we have the following equality for $n \geq 1$,

$$S_{E_{ik}}^{(s+nr)} \approx S_{E_{ij}(1)}^{(1)} \cdot \ldots \cdot S_{E_{ij}(m-1)_{j'}}^{(1)} \cdot (S_{E_{ij}}^{(r)})^n \cdot \ldots \cdot S_{E_{ij}(t-1)_{j'}}^{(1)}$$

So we will have $\mindeg(\varphi(S^{(s+nr)}_{E_{ij}})) \geq \ell' + nr$ for all $n \geq 1$, obtaining a contradiction again. \(\square\)

**Proposition 7.18.** Let $P$ be stochastic and $i \neq j$ both in $\Omega$ such that $(i, j) \in E(P)$, with $p_i, p_j$ both reducing for $Arv(P)$. Assume $\varphi : T_+ (P) \to T_+ (Q)$ is an isomorphism. Then

$$\mindeg(\varphi(S^{(1)}_{E_{ij}})) = 1$$

**Proof.** Surely it is the case that $\mindeg(\varphi(S^{(1)}_{E_{ij}})) \geq 1$ due to Lemma [6.22]. Since $p_i, p_j$ are reducing for $Arv(P)$, there exist $i', j' \in \Omega$ such that $p_{i'}, p_{j'}$ are reducing for $Arv(Q)$, and $\varphi p_i, \varphi p_j$ and $\varphi p_{i'}, \varphi p_{j'}$ are bounded isomorphisms. Assume by negation that $k \geq 2$ is minimal such that $p_{i'} \varphi(S^{(1)}_{E_{ij}}) p_{j'} = \sum_{n=k}^{\infty} a_n S^{(n)}_{E_{ij}}$ with $a_k \neq 0$ and $S^{(k)}_{E_{ij}} \neq 0$ (which also shows that a path of length at most $k$ from $i'$ to $j'$ exists in $Q$). Thus $\mindeg(\varphi(S^{(1)}_{E_{ij}})) \geq 2$, and since for all $n > 1$ we have that $S^{(n)}_{E_{ij}}$ is either a multiple of at least two operators of minimal degree 1, or a multiple of $S^{(1)}_{E_{ij}}$, it must be that $\mindeg(\varphi(S^{(n)}_{E_{ij}})) \geq 2$ for all $n \geq 1$. Write $p_{i'} \varphi^{-1}(S^{(1)}_{E_{ij}}) p_{j'} = \sum_{n=m}^{\infty} b_n S^{(n)}_{E_{ij}}$ for $b_m \neq 0$ and compute

$$S^{(1)}_{E_{ij}} = \Phi_1(p_{i'} \varphi^{-1}(S^{(1)}_{E_{ij}}) p_{j'}) = \Phi_1(\varphi^{-1}(S^{(1)}_{E_{ij}}) p_{j'}) = \Phi_1(\varphi(\sum_{n=m}^{\infty} b_n S^{(n)}_{E_{ij}}))$$

$$= \sum_{n=m}^{\infty} b_n \Phi_1(\varphi(S^{(n)}_{E_{ij}})) = 0$$

This shows that $S^{(1)}_{E_{ij}} = 0$ and in particular that $(i', j') \notin E(Q)$. Let $\ell \geq 2$ be the minimal length of a path from $i'$ to $j'$. Since for all $n \geq \ell$ the operator $S^{(n)}_{E_{ij}}$ can be written as a multiple of at
least 2 operators of minimal degree 1, we have that \( \text{mindeg}(\varphi^{-1}(S_{E_{ij}}^{(n)})) \geq 2 \) for all \( n \geq \ell \). By a similar computation to the one above we obtain

\[
S_{E_{ij}}^{(1)} = \Phi_1(p_i\varphi^{-1}(\varphi(S_{E_{ij}}^{(1)}))p_j) = \Phi_1(\varphi^{-1}(p_i\varphi(S_{E_{ij}}^{(1)}))p_j)) = \Phi_1(\varphi^{-1}(\sum_{n=k}^{\infty} a_nS_{E_{ij}}^{(n)})) = \\
= \sum_{n=k}^{\infty} a_n\Phi_1(\varphi^{-1}(S_{E_{ij}}^{(n)})) = 0
\]

Which contradicts \((i,j) \in E(P)\). \(\square\)

**Proposition 7.19.** Let \( P \) and \( Q \) be stochastic matrices, and let \( \varphi : T_+(P) \to T_+(Q) \) be an isomorphism. Assume that \( j \in \Omega \) is \( \varphi \)-singular. Assume further that for \( i, k \in \Omega \) there exists a streamlined path from \( i \) to \( k \) through \( j \) of length \( \ell \) in the directed graph of \( P \). Then any streamlined path from \( i \) to \( k \) in the directed graph of \( P \) is of length not greater than \( \ell \).

**Proof.** Before we begin, we note that all states considered here are reducing due to Proposition 7.14 thus making the use of Proposition 7.18 possible. Assume by negation that while \( \gamma \) is a streamlined path of length \( \ell \) from \( i \) to \( k \) with \( \gamma(m) = j \), we have a streamlined path \( \gamma' \) of length \( r > \ell \) from \( i \) to \( k \). Thus, up to a positive scalar multiple, which we denote by \( \approx \), we have the following chain of equalities

\[
S_{E_{\gamma'(1)}}^{(1)} \cdots S_{E_{\gamma'(\ell-1)k}}^{(1)} \approx S_{E_{\gamma(1)}}^{(r)} \approx S_{E_{\gamma(m)}}^{(1)} \cdots S_{E_{\gamma(m-1)j}}^{(1)} \cdot (S_{E_{\gamma(j)}}^{(1)})^{r-\ell} \cdot S_{E_{\gamma(\ell)k}}^{(1)} \cdots S_{E_{\gamma(\ell-1)k}}^{(1)}
\]

But since \( j \) is \( \varphi \)-singular, after applying \( \varphi \) to both sides, the right hand side of the equation would be of minimal degree equal to \( \ell \) while the left hand side of the equation would be of minimal degree \( r > \ell \), obtaining a contradiction. \(\square\)

**Corollary 7.20.** Let \( P \) and \( Q \) be stochastic matrices, and let \( \varphi : T_+(P) \to T_+(Q) \) be an isomorphism. Assume that \( j_1, j_2 \in \Omega \) are \( \varphi \)-singular, such that there exist \( i, k \in \Omega \) and two streamlined paths from \( i \) to \( k \), one through \( j_1 \) of length \( \ell_1 \) and the other through \( j_2 \) of length \( \ell_2 \). Then \( \ell_1 = \ell_2 \).

**Proposition 7.21** (Property \( Q_1 \)). Let \( P \) and \( Q \) be stochastic matrices and \( \varphi : T_+(P) \to T_+(Q) \) an isomorphism. Assume that for each \( j \in \mathcal{R}_P \) we have that \( p_j \) is \( \varphi \)-regular. Then for all \( j \in \Omega \) we have that \( p_j \) is \( \varphi \)-regular and thus \( \varphi \) is regular.

**Proof.** Suppose towards a contradiction that there there exists \( j \in I_P \) and \( \varphi \)-singular, and let \( j' \in \mathcal{R}_P \) be such that \( j \sim_P j' \). Then there exists a finite chain \((j = s_0, s_1, \ldots, s_\ell, s_\ell = j')\) of elements in \( I_P \), with \((s_k, s_{k+1})\) such that there exist \( i, k \in \Omega \) with two streamlined paths from \( i \) to \( k \), one through \( s_k \) and the other through \( s_{k+1} \) in the graph of \( P \). By Remark 7.7 every \( s \in I_P \) must satisfy that \( P_{ss} > 0 \), so due to the second part of Proposition 7.17 we must have that \( s_k \) and \( s_{k+1} \) are either both \( \varphi \)-regular or both \( \varphi \)-singular. But \( j' \in \mathcal{R}_P \) is \( \varphi \)-regular by assumption while \( j \) is \( \varphi \)-singular, which leads to a contradiction. \(\square\)

Let \( P \) be a stochastic matrix. We say that a triple \((i, k, n) \in \Omega \times \Omega \times \mathbb{N}\) is \( P \)-proper (or just \( \mu \)-proper when \( P \) is determined by the context) if \((i, k) \in E(P^n)\) and there exists a streamlined path \( \gamma \) from \( i \) to \( k \) in the graph of \( P \) with length \( \ell < n \) and there exists \( 0 \leq s \leq \ell \) such that \( \gamma(s) \in I_P \). Note that by definition of \( \sim_P \), if \((i, k, n)\) is proper, it determines a unique element \( \mu \in \mathcal{R}_P \) corresponding to the class of the element \( \gamma(s) \in I_P \). In this case we will say that \( \mu \in \mathcal{R}_P \) is the singular class associated to \((i, k, n)\).

**Theorem 7.22** (Property \( Q_2 \)). Let \( P \) be a stochastic matrix, and let \( \Lambda = (\lambda_\mu)_{\mu \in \mathcal{R}_P} \in T^{\mathcal{R}_P} \) be given. There exists a unique isometric \((1\text{-}d)\)-automorphism \( V^\Lambda = \{ V^\Lambda_n \} \) of the subproduct system \( Arv(P) \) satisfying the following condition: for every \( n \geq 1 \) and for each pair \((i, k) \in E(P^n)\),

\[
V^\Lambda_n(E_{ik}) = \begin{cases} 
\lambda_\mu^{-\ell}E_{ik}, & \text{if } (i, k, n) \text{ is proper, with associated singular class } \mu \in \mathcal{R}_P \\
E_{ik}, & \text{otherwise}
\end{cases}
\]

Moreover, the \((1\text{-}d)\)-similarity \( \alpha_\Lambda : T_+(P) \to T_+(P) \) given by \( \alpha_\Lambda = \text{Ad}V^\Lambda \) is an isometric automorphism satisfying \( f^\Lambda_{\alpha_\Lambda}(z) = \lambda_\mu z \) for all \( j \in I_P \), where \( j \sim_P^{\mu} \in \mathcal{R}_P \).
Proof. By Corollary 7.20 we see that for each $n \in \mathbb{N}$ the map $V_n^A$ on $\{E_{ik} \mid (i, k) \in E(P^n)\}$ is well-defined and clearly extends uniquely to a unitary correspondence morphisms on $\text{Span}(E_{ik} \mid (i, k) \in E(P^n))$. Thus we obtain a unique unitary $W^*$-correspondence morphisms $V_n^A$ on $\text{Arr}(P)_n$.

We need to show that $U_{n,m}^P(V_n^A \otimes V_m^A) = V_{n+m}^A(U_{n,m}^P)$ for every $n, m \in \mathbb{N}$. It suffices to show that for all $(i, j, k) \in E(P^n, P^m)$ we have

\begin{equation}
U_{n,m}^P(V_n^A(E_{ij}) \otimes V_m^A(E_{jk})) = V_{n+m}^A(U_{n,m}^P(E_{ij} \otimes E_{jk}))
\end{equation}

Let a triple $(i, j, k) \in E(P^n, P^m)$ be given for $n, m \in \mathbb{N}$. We split the proof into two cases.

Case 1: Suppose that $(i, k, n + m)$ is proper, with associated singular class $\mu \in \mathcal{R}_P$. Let $\gamma$ be a streamlined path of length $\ell < n + m$ from $i$ to $k$ such that $\gamma(s) \sim_\mu \mu$ for some $0 \leq s \leq \ell$. By definition $V_n^A(E_{ik}) = \lambda_{i\mu}^{n-m-\ell}E_{ik}$. We first show that at least one of $(i, j, n)$ and $(j, k, m)$ must be proper with associated singular class $\mu$. Clearly there exist two paths $\gamma^{(n)}$ from $i$ to $j$ of length $n$ and $\gamma^{(m)}$ from $j$ to $k$ of length $m$. Notice that the concatenation $\gamma'$ of $\gamma^{(n)}$ and $\gamma^{(m)}$ provides a path from $i$ to $k$ which by Proposition 7.19 cannot be streamlined, since $\ell < n + m$. Therefore, there exists some repeated index $j'$ in the path $\gamma'$, hence $P^{(r)}_{j'j'} > 0$ for some $r \geq 1$. By Proposition 7.14 we must have that $j' \in I_P$, and by definition of the equivalence relation we have therefore $j' \sim_\mu \mu$. Let us suppose without loss of generality that $(i, j, n)$ is proper (necessarily with associated singular class $\mu$), we have $V_n^A(E_{ij}) = \lambda_{i\mu}^{n-\ell}E_{ij}$. Let $\ell_2$ be the length of the streamlined path from $j$ to $k$ obtained from the culling procedure applied to $\gamma^{(n)}$. We have $\ell = \ell_1 + \ell_2$ by Corollary 7.20. If $(j, k, m)$ is also proper (necessarily with associated singular class $\mu$), we have $V_{n+m}^A(E_{ij}) = \lambda_{j\mu}^{n-m-\ell}E_{jk}$. On the other hand, notice that by the same argument we used above, if $\gamma^{(m)}$ is not streamlined, we must have that $(j, k, m)$ is proper. Therefore, if $(j, k, m)$ is improper we must have $\ell_2 = m$, i.e. $\gamma^{(m)}$ already streamlined, and we also have for this case the formula $V_n^A(E_{jk}) = \lambda_{j\mu}^{n-m-\ell}E_{jk}$. Since $n + m - \ell = (n - \ell_1) + (m - \ell_2)$, we obtain (7.3).

Case 2: Suppose that $(i, k, n + m)$ is improper, so that $V_{n+m}^A(E_{ik}) = E_{ik}$. Then we clearly must have that $(i, j, n)$ and $(j, k, m)$ are also improper. It follows that $V_{n+m}^A(E_{ij}) = E_{ij}$ and $V_{n+m}^A(E_{jk}) = E_{jk}$, and therefore (7.3) holds.

Finally, the map $\alpha_A = \text{Ad}_{V_n^A}$, given by Corollary 6.13 clearly satisfies the stated properties since it leaves the vacuum state invariant and if $j \in I_P$, we have $(\alpha_A)p_{j}\frac{S_{E_{ij}}}{\mu} = \lambda_{\mu}S_{E_{ij}}$ where $\mu \in \mathcal{R}_P$ is the unique element with $j \sim_\mu \mu$. This is because any streamlined path from $j$ to $k$ must be of length $\ell = 0$.

Proposition 7.23 (Property $\cong_3$). Let $P$ and $Q$ be stochastic matrices and let $\varphi : \mathcal{T}_+(P) \rightarrow \mathcal{T}_+(Q)$ be an isomorphism. Then $\varphi$ induces an equivalence preserving bijection $\Upsilon_\varphi : I_P \rightarrow I_Q$ uniquely determined by the identity $p_{\Upsilon_\varphi(j)} = \Phi_0(\varphi(p_j))$ for every $j \in I_P$. Furthermore, we have $\Upsilon_{\varphi^{-1}} = \Upsilon_\varphi^{-1}$.

Proof. By Proposition 7.22 we have that $p_j$ is a reducing projection for $\text{Arr}(P)$ for every $j \in I_P$. By Propositions 7.11 and 7.12 we see that for all $j \in I_P$ we must have that there is a unique $j' \in I_Q$ such that $p_{j'} = \Phi_0(\varphi(p_j))$, hence $\Upsilon_\varphi$ is a well-defined injection. Furthermore, by symmetry and Proposition 7.11, we have that it is onto and $\Upsilon_{\varphi^{-1}} = \Upsilon_\varphi^{-1}$. To show that $\Upsilon_\varphi$ preserves equivalence, let $j_1, j_2$ be two elements in $I_P$ such that there exist two streamlined paths (which must be of the same length due to Corollary 7.20) $\gamma$ from $i$ to $k$ of length $\ell$ with $\gamma(t) = j_1$, and $\gamma'$ from $i$ to $k$ of length $\ell$ with $\gamma'(r) = j_2$. Thus, we have that up to a positive scalar, which we denote by $\approx$,

\begin{align*}
S_1^{(1)} & \cdots S_1^{(1)} \quad S_1^{(1)} \cdots S_1^{(1)} \approx S_1^{(1)} \cdots S_1^{(1)}
\end{align*}

Denote $p_j' = \Phi_0(\varphi(p_j))$, $p_j'' = \Phi_0(\varphi(p_j''))$, $p_{\ell'} = \Phi_0(\varphi(p_{\ell'}))$ and $p_{\ell''} = \Phi_0(\varphi(p_{\ell''}))$. By Proposition 7.18 we have the equality $\varphi(S_{E_{\ell''}}(\ell'')) = \ell''$. Thus, write uniquely $\varphi(S_{E_{\ell''}}(\ell'')) = \sum_{n=\ell''}^{\ell''} \varphi(S_{E_{\ell''}}(\ell''))$. For $\xi_n \in \text{Arr}(Q)_n$ with $\xi_n = \pi_{i}(\xi_n) \pi_{k} \neq 0$. By applying $\Phi_\ell$ to the equation above, we have that up to a positive scalar,

\begin{align*}
p_{\ell'}\Phi_\ell(\varphi(S_{E_{\ell''}}(\ell''))) & \approx p_{\ell''} \Phi_\ell(\varphi(S_{E_{\ell''}}(\ell'')))\pi_{\ell'} \Phi_\ell^{-1}(\varphi(S_{E_{\ell''}}(\ell'')))
\end{align*}

Finally, apply $\Phi_\ell$ to the equation above, we have that up to a positive scalar,

\begin{align*}
p_{\ell'}\Phi_\ell(\varphi(S_{E_{\ell''}}(\ell''))) & \approx p_{\ell''} \Phi_\ell(\varphi(S_{E_{\ell''}}(\ell'')))\pi_{\ell'} \Phi_\ell^{-1}(\varphi(S_{E_{\ell''}}(\ell'')))\pi_{\ell''} \Phi_\ell^{-1}(\varphi(S_{E_{\ell''}}(\ell''))),
\end{align*}

which is the desired result.
Thus we obtain two streamlined paths from $i'$ to $k'$, one through $j_1'$ and the other through $j_2'$ and we are done. □

**Main results.** We now use the three properties of $\sim_P$ to establish the existence of a regular isomorphism, from that of a general isomorphism. We were inspired by [DRS11] Proposition 4.7.

**Theorem 7.24.** Let $P$ and $Q$ be stochastic matrices over $\Omega$. If there exists an algebraic isomorphism $\varphi : T_+(P) \to T_+(Q)$ then there exists a regular isomorphism from $T_+(P)$ to $T_+(Q)$, which can be taken to be isometric if $\varphi$ is isometric.

**Proof.** Let $\varphi : T_+(Q) \to T_+(P)$ be an algebraic isomorphism. For each $(\Lambda, \Theta) \in \mathbb{T}^{RP} \times \mathbb{T}^{RQ}$, let $\alpha_\Lambda$ and $\alpha_\Theta$ be the automorphisms of $T_+(P)$ and $T_+(Q)$, respectively, provided by property $\Diamond_2$, and consider the isomorphism from $T_+(Q)$ to $T_+(P)$ given by

$$\Psi_{\Lambda, \Theta} = \varphi \circ \alpha_\Lambda \circ \varphi^{-1} \circ \alpha_\Theta \circ \varphi$$

We will prove that there exists a pair $(\Lambda_0, \Theta_0)$ such that $\psi = \Psi_{\Lambda_0, \Theta_0}$ has the property that $f_j(0) = 0$ for every $j \in \mathcal{R}_P$. By Proposition 7.13 and property $\Diamond_1$, such an isomorphism $\psi$ is regular. Furthermore, if $\varphi$ is isometric, then $\psi$ is isometric by property $\Diamond_2$.

For simplicity, for every $j \in \mathcal{R}_P$ and for each pair $(\Lambda, \Theta) \in \mathbb{T}^{RP} \times \mathbb{T}^{RQ}$, let us denote $f^j_{\Lambda, \Theta} = f^j_{\Psi_{\Lambda, \Theta}}$. Let $j' = \overline{\Lambda}_{\varphi}(j)$, and notice that $j = \overline{\Lambda}_{\varphi}^{-1}(j') = (\overline{\Lambda}_{\varphi})^{-1}(j')$ by property $\Diamond_3$. Therefore, by contravariance, we have that

$$f^j_{\Lambda, \Theta}(0) = f^j_{\varphi^{-1}} \circ f^j_{\alpha_\Theta} \circ f^j_{\alpha_\Lambda} \circ f^j_{\varphi}$$

Let us denote by $T_j = f^j_{\varphi}$ and $\theta_j' = \varphi_j'$. By Proposition 7.11 and property $\Diamond_3$, we have that $f^j_{\varphi^{-1}} = T_j^{-1}$. Thus we have that for every $j \in \mathcal{R}_P$,

$$f^j_{\Lambda, \Theta}(0) = f^j_{\varphi} \circ f^j_{\alpha_\Theta} \circ f^j_{\alpha_\Lambda} \circ f^j_{\varphi} = T_j(\theta_j' T_j^{-1}(\lambda_j(T_j(0))))$$

Since $T_j$ is a Moebius transformation, it is an elementary fact that there exist $\theta_j', \lambda_j \in \mathbb{T}$ such that $f^j_{\Lambda, \Theta}(0) = 0$. Indeed, if $T_j(0) = 0$ this is trivial. If $T_j(0) \neq 0$, then $C_j = \mathbb{T} \cdot T_j(0)$ is a circle centered at origin (and not containing it). On the other hand, since $T_j$ is a Moebius transformation, $T_j^{-1}(C_j)$ is a circle and it clearly contains the origin, since $T_j(0) \in C_j$. Therefore, the larger circle $C_j' = \mathbb{T} \cdot T_j^{-1}(C_j)$ obtained by its rotation contains the interior of $T_j^{-1}(C_j)$. It follows that $T_j(C_j')$ is a disk that will contain the interior of $C_j$, and therefore it contains the origin. □

**Corollary 7.25.** Let $P$ and $Q$ be stochastic matrices over $\Omega$. If there exists an isometric isomorphism $\varphi : T_+(P) \to T_+(Q)$ then there exists a graded completely isometric isomorphism $\varphi : T_+(P) \to T_+(Q)$.

**Proof.** If there exists an isometric isomorphism $\varphi : T_+(P) \to T_+(Q)$ then there exists a regular isometric isomorphism and since it must be admissible by Lemma 6.7, Proposition 6.24 implies that it is semi-graded. Thus, by Proposition 6.18 there exists a graded completely isometric isomorphism. □

Further, if one takes $\Omega$ to be a finite set in Theorem 7.24, then one obtains

**Corollary 7.26.** Let $P$ and $Q$ be stochastic matrices over finite $\Omega$. If there exists an isomorphism $\varphi : T_+(P) \to T_+(Q)$ then there exists a graded completely bounded isomorphism $\varphi : T_+(P) \to T_+(Q)$.

**Proof.** Since $\Omega$ is finite, the regular $\varphi$ assured by Theorem 7.24 must be admissible by Proposition 7.3 and by an appeal to Proposition 6.24 we see that $\varphi$ is semi-graded. Finally, by Proposition 6.17 we arrive at the desired isomorphism $\varphi$. □

We now rephrase our results in a unified manner. Recall Theorems 3.11 and 7.1

**Theorem 7.27 (Isometric tensor algebra isomorphism).** Let $P$ and $Q$ be stochastic matrices over a set $\Omega$. Then the following are equivalent
(1) There exists a \( \rho \)-unitary isomorphism from \( \text{Arv}(P) \) to \( \text{Arv}(Q) \) for some \( * \)-automorphism \( \rho \) of \( \mathcal{E}^\infty(\Omega) \).

(2) There exists a graded completely isometric isomorphism \( \varphi : \mathcal{T}_+(P) \to \mathcal{T}_+(Q) \)

(3) There exists an isometric isomorphism \( \varphi : \mathcal{T}_+(P) \to \mathcal{T}_+(Q) \)

If moreover \( P \) and \( Q \) are recurrent, then the above conditions are equivalent to \( P \cong_* Q \) for some permutation \( \sigma \) of \( \Omega \).

**Theorem 7.28** (Algebraic tensor algebra isomorphism for finite matrices). Let \( P \) and \( Q \) be finite stochastic matrices over a set \( \Omega \). Then the following are equivalent

1. There exists a \( \rho \)-similarity isomorphism from \( \text{Arv}(P) \) to \( \text{Arv}(Q) \) for some \( * \)-automorphism \( \rho \) of \( \mathcal{E}^\infty(\Omega) \).
2. There exists a graded completely bounded isomorphism \( \varphi : \mathcal{T}_+(P) \to \mathcal{T}_+(Q) \)
3. There exists an algebraic isomorphism \( \varphi : \mathcal{T}_+(P) \to \mathcal{T}_+(Q) \)

If moreover \( P \) and \( Q \) are essential, then the above conditions are equivalent to \( P \sim_* Q \) for some permutation \( \sigma \) of \( \Omega \).

**Example 7.29.** For every \( 0 < r < 1/2 \) let \( P_r = \begin{bmatrix} r & (1-r) \\ r & (1-r) \end{bmatrix} \). Then it follows from the previous theorem that \( \mathcal{T}_r(P_r) \) is isomorphic to \( \mathcal{T}_s(P_s) \) for every \( r \neq s \in (0, 1/2) \), however the two algebras are isometrically isomorphic only when \( r = s \).

We note that when \( P \) and \( Q \) are essential (and \( \Omega \) is possibly infinite), it is possible to prove more directly, and without recourse to Theorem 7.24, that the existence of an admissible algebraic (resp. isometric) isomorphism from \( \mathcal{T}_+(P) \) to \( \mathcal{T}_+(Q) \) implies the existence of a graded algebraic (resp. isometric) isomorphism from \( \mathcal{T}_+(P) \) to \( \mathcal{T}_+(Q) \). For instance, under those conditions one can proceed as in the proof of Corollary 7.10 after fixing the \( 1 \times 1 \) irreducible blocks by using Theorem 7.1. The main point of Theorem 7.24 was to deal with the general, non-essential case.

**Acknowledgments**

We would like to thank Orr Shalit for his many helpful remarks which substantially refined this work. We also thank Kenneth Davidson, Ilan Hirshberg, N. Christopher Phillips and Baruch Solel for many useful comments. The first author would also like to thank Ariel Yadin, for many discussions and insights on stochastic matrices.

**References**

[AJ69] William B. Arveson and Keith B. Josephson, *Operator algebras and measure preserving automorphisms II*, J. Functional Analysis 4 (1969), 100–134.

[Arv67] William B. Arveson, *Operator algebras and measure preserving automorphisms*, Acta Math. 118 (1967), 95–109.

[Arv03] -, *Noncommutative dynamics and E-semigroups*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2003.

[BM10] B. V. Rajarama Bhat and Mithun Mukherjee, *Inclusion systems and amalgamated products of product systems*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 13 (2010), no. 1, 1–26.

[BO08] Nathanial P. Brown and Narutaka Ozawa, *C\(^*\)-algebras and finite-dimensional approximations*, Graduate Studies in Mathematics, vol. 88, American Mathematical Society, Providence, RI, 2008.

[BS00] B. V. Rajarama Bhat and Michael Skeide, *Tensor product systems of Hilbert modules and dilations of completely positive semigroups*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 3 (2000), no. 4, 519–575.

[Chu60] Kai Lai Chung, *Markov chains with stationary transition probabilities*, Die Grundlehren der mathematischen Wissenschaften, Bd. 104, Springer-Verlag, Berlin, 1960.

[Dav96] Kenneth R. Davidson, *C\(^*\)-algebras by example*, Fields Institute Monographs, vol. 6, American Mathematical Society, Providence, RI, 1996.

[DHK01] A. P. Donsig, T. D. Hudson, and E. G. Katsoulis, *Algebraic isomorphisms of limit algebras*, Trans. Amer. Math. Soc. 353 (2001), no. 3, 1169–1182 (electronic).

[DK11] Kenneth R. Davidson and Elias G. Katsoulis, *Operator algebras for multivariable dynamics*, Mem. Amer. Math. Soc. 209 (2011), no. 982, viii+53.

[DRS11] Kenneth R. Davidson, Christopher Ramsey, and Orr Moshe Shalit, *The isomorphism problem for some universal operator algebras*, Adv. Math. 228 (2011), no. 1, 167–218.
[Dur10] Rick Durrett, *Probability: theory and examples*, fourth ed., Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge University Press, Cambridge, 2010.

[Fel68] William Feller, *An introduction to probability theory and its applications. Vol. I*, Third edition, John Wiley & Sons Inc., New York, 1968.

[Gre80] Philip Green, *The structure of imprimitivity algebras*, J. Funct. Anal. 36 (1980), no. 1, 88–104.

[Gur12] Maxim Gurevich, *Subproduct systems over \( \mathbb{N} \times \mathbb{N} \)*, J. Funct. Anal. 262 (2012), no. 10, 4270–4301.

[Kat04] Takeshi Katsura, *On \( C^* \)-algebras associated with \( C^* \)-correspondences*, J. Funct. Anal. 217 (2004), no. 2, 366–401.

[KK04] Elias Katsoulis and David W. Kribs, *Isomorphisms of algebras associated with directed graphs*, Math. Ann. 330 (2004), no. 4, 709–728.

[Lan95] E. C. Lance, *Hilbert \( C^* \)-modules*, London Mathematical Society Lecture Note Series, vol. 210, Cambridge University Press, Cambridge, 1995, A toolkit for operator algberasts.

[Mar03] Daniel Markiewicz, *On the product system of a completely positive semigroup*, J. Funct. Anal. 200 (2003), no. 1, 237–280.

[MS98] Paul S. Muhly and Baruch Solel, *Tensor algebras over \( C^* \)-correspondences: representations, dilations, and \( C^* \)-envelopes*, J. Funct. Anal. 158 (1998), no. 2, 389–457.

[MS00], *On the Morita equivalence of tensor algebras*, Proc. London Math. Soc. (3) 81 (2000), no. 1, 113–168.

[MS02], *Quantum Markov processes (correspondences and dilations)*, Internat. J. Math. 13 (2002), no. 8, 863–906.

[MS05], *Duality of \( W^* \)-correspondences and applications*, Quantum probability and infinite dimensional analysis, QP–PQ: Quantum Probab. White Noise Anal., vol. 18, World Sci. Publ., Hackensack, NJ, 2005, pp. 396–414.

[MT05] V. M. Manuilov and E. V. Troitsky, *Hilbert \( C^* \)-modules*, Translations of Mathematical Monographs, vol. 226, American Mathematical Society, Providence, RI, 2005, Translated from the 2001 Russian original by the authors.

[Pas73] William L. Paschke, *Inner product modules over \( B^* \)-algebras*, Trans. Amer. Math. Soc. 182 (1973), 443–468.

[Pim97] Michael V. Pimsner, *A class of \( C^* \)-algebras generalizing both Cuntz-Krieger algebras and crossed products by \( \mathbb{Z} \)*, Free probability theory (Waterloo, ON, 1995), Fields Inst. Commun., vol. 12, Amer. Math. Soc., Providence, RI, 1997, pp. 189–212.

[Rae05] Iain Raeburn, *Graph algebras*, CBMS Regional Conference Series in Mathematics, vol. 103, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2005.

[Sin75] Allan M. Sinclair, *Homomorphisms from \( C_0(R) \)*, J. London Math. Soc. (2) 11 (1975), no. 2, 165–174.

[Ske00] Michael Skeide, *Generalised matrix \( C^* \)-algebras and representations of Hilbert modules*, Math. Proc. R. Ir. Acad. 100A (2000), no. 1, 11–38.

[Ske03a] , *Commutants of von Neumann modules, representations of \( B^*(E) \) and other topics related to product systems of Hilbert modules*, Advances in quantum dynamics (South Hadley, MA, 2002) (Providence, RI), Contemp. Math., vol. 335, Amer. Math. Soc., Providence, RI, 2003, pp. 253–262.

[Ske03b] , *Dilation theory and continuous tensor product systems of Hilbert modules*, Quantum probability and infinite dimensional analysis (Burg, 2001), QP–PQ: Quantum Probab. White Noise Anal., vol. 15, World Sci. Publ., River Edge, NJ, 2003, pp. 215–242.

[Sol04] Baruch Solel, *You can see the arrows in a quiver operator algebra*, J. Aust. Math. Soc. 77 (2004), no. 1, 111–122.

[SS09] Orr Moshe Shalit and Baruch Solel, *Subproduct systems*, Doc. Math. 14 (2009), 801–868.

[Vis11] Ami Viselter, *Covariant representations of subproduct systems*, Proc. Lond. Math. Soc. (3) 102 (2011), no. 4, 767–800.

[Vis12] , *Cuntz-Pimsner algebras for subproduct systems*, Internat. J. Math. 23 (2012), no. 8, 1250081 (32 pp.).

Adam Dor-On and Daniel Markiewicz, Department of Mathematics, Ben-Gurion University of the Negev, P.O.B. 653, Beersheva 84105, Israel.

E-mail address: adamd@math.bgu.ac.il, danielm@math.bgu.ac.il