Global Solutions and Stability Properties of the 5th Order Gardner Equation

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Abstract
In this work, we deal with the initial value problem of the 5th-order Gardner equation in \(\mathbb{R}\), presenting the local well-posedness result in \(H^2(\mathbb{R})\). As a consequence of the local result, in addition to \(H^2\)-energy conservation law, we are able to prove the global well-posedness result in \(H^2(\mathbb{R})\). Finally as a direct application, we prove that some globally defined functions, e.g. breather solutions of 5th order Gardner equation, are \(H^2(\mathbb{R})\) stable.

Keywords Higher order Gardner equation · Global well-posedness · Breather · Stability · Integrability

Mathematics Subject Classification Primary 37K15 · 35Q53 · Secondary 35Q51 · 37K10

1 Introduction

In this work, we are concerned with the focusing 5th order Gardner equation

\[
 u_t + u_{5x} + 10\mu^2 u_{3x} + \mathcal{K}_\mu(u) = 0, \quad \mu \in \mathbb{R}^+,
\]

\[
 \mathcal{K}_\mu(u) := 20\mu uu_{3x} + 10u^2u_{3x} + 120\mu^3 uu_{x} + 180\mu^2 u^2 u_{x} + 120\mu uu^3 u_{x} + 10u^3 + 40\mu u u_{x} u_{xx} + 40uu_{x} u_{xx} + 30u^4 u_{x}.
\]

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This higher order Gardner equation can be obtained from the corresponding 5th order focusing modified Korteweg-de Vries equation (shortly, 5th mKdV)

$$v_t + (v_4 + 10vv^2_x + 10v^2v_{xx} + 6v^5)_x = 0,$$

(1.2)

when one considers mKdV solutions of the form $v(t, x) = \mu + u(t, x)$, with $\mu \in \mathbb{R}^+$ and a suitable spatial translation$^1$. Moreover notice that when $\mu \ll 1$, equation (1.1) can be seen as a perturbed model of the 5th mKdV (1.2) and therefore describing small perturbations on the dynamics of the observed phenomena in (1.2), e.g. waves in shallow water and nonlinear patterns in elastic media. See [33,43,53,62] and references therein for further details on physical applications.

The 5th order Gardner Eq. (1.1), as well as the 5th mKdV equation, is a well-known completely integrable model [1,22,51], with infinitely many conservation laws and well-known (long-time) asymptotic behavior of its solutions obtained with the help of the inverse scattering transform [28]. As a physical model, the 5th Gardner (1.1) and the 5th mKdV (1.2) equations describe large-amplitude internal solitary waves, showing a dynamics which can look rather different from the KdV form. On the other hand, solutions of (1.1) are invariant under space and time translations. Indeed, for any $t_0, x_0 \in \mathbb{R}$, $u(t - t_0, x - x_0)$ is also a solution of both equations. Beside that, the scaling invariance is not respected by (1.1).

As seen in (1.1), the 5th order Gardner Eq. (1.1) contains mixed nonlinearities of the 5th KdV equation

$$v_t + (v_4 + 5v^2_x + 10vv_{xx} + 10v^3)_x = 0,$$

(1.3)

and 5th mKdV (1.2), and hence the well-posedness theory of (1.1) is highly relevant to the well-posedness of both equations. Ponce in [64] showed the local well-posedness of the 5th KdV in $H^s(\mathbb{R})$, $s \geq 4$ via the energy method in addition to the dispersive smoothing effect and a parabolic approximation method. Later, this local result has been improved by Kwon [49], now obtaining the local well-posedness in $H^s(\mathbb{R})$, $s > \frac{5}{2}$. Thereafter, Guo, Kwon and the second author [26] and Kenig and Pilod [35], independently, proved the local well-posedness in $H^s(\mathbb{R})$, $s \geq 2$. Both works were based on the short time Fourier restriction norm method [32], while an additional weight and the (frequency localized) modified energy were used to prove the crucial energy estimates, respectively. Thanks to the $H^2$-level energy conservation law, the local result extended to the global one.

On the other hand, the 5th mKdV (1.2) has been studied by Linares [52], proving the local well-posedness in $H^2(\mathbb{R})$ via the contraction mapping principle in addition to the dispersive smoothing effect [37,39]. Later, Kwon [50] improved the local result in $H^s(\mathbb{R})$, $s > \frac{5}{2}$, by using the standard Fourier restriction norm method [11] in addition to Tao’s $[k, Z]$-multiplier norm method [65].

We also refer to [29,34,37,40,63] for the local well-posedness of for higher order KdV and mKdV equations.

It is known that the Initial Value Problem (IVP) of 5th KdV (1.3) is a quasilinear problem in the sense that the solution map is (not uniformly) continuous, while the Cauchy problem of 5th mKdV is a semilinear problem in the sense that the flow map is Lipschitz continuous (via the Picard iteration method, and hence, is analytic). Thus, one may expect that the IVP of the 5th Gardner Eq. (1.1) is also a quasilinear problem due to the strong (high-low)

$^1$ Such a spatial translation is performed in order to provide a simpler expression of the N-soliton solution in Sect. 3. On the other hand, it is known that not only the first order linear term but also the third order term of the linear part in (1.1) are negligible in the study of the well-posedness theory compared to the fifth order term.
quadratic nonlinearity. Moreover, one expects to obtain the local well-posedness in $H^2(\mathbb{R})$ (and hence the global well-posedness in $H^2(\mathbb{R})$) from [26,35]. However, to prove the local well-posedness of the 5th Gardner equation is definitely non-trivial, thus one of aims in this work is to prove the local well-posedness for of the IVP for (1.1).

As related problems, we also refer to [12,36,47,48,66] for the well-posedness of 5th KdV, 5th mKdV and higher order equations in KdV hierarchy under the periodic boundary condition.

Concerning explicit solutions of higher order mKdV and Gardner models, Matsuno [54] proved the existence and built explicitly the N-soliton solution of the focusing mKdV hierarchy of equations by using inverse scattering technics and the bilinear Hirota decomposition. Recently, Gomes et al. [23] dealt with the defocusing mKdV with NVBC and the associated defocusing Gardner hierarchy, showing multisolitonic structures. Unfortunately, many of the solutions they obtained are singular solutions (up to the kink which is in $L^\infty$).

The 5th order Gardner Eq. (1.1), as a completely integrable system, has an infinite set of conserved quantities. Indeed some of the (first) standard conservation laws of the (1.1) are the mass

$$M[u](t) := \frac{1}{2} \int_{\mathbb{R}} u^2(t, x) dx = M[u](0),$$

the energy

$$E\mu[u](t) := \int_{\mathbb{R}} \left( \frac{1}{2} u_x^2 - 2\mu u^3 - \frac{1}{2} u^4 \right) (t, x) dx = E[u](0),$$

and the higher order energy, defined respectively in $H^2(\mathbb{R})$

$$E_{5\mu}[u](t) := \int_{\mathbb{R}} \left( \frac{1}{2} u_{xx}^2 - 10\mu uu_x^2 + 10\mu^2 u^4 - 5u^2 u_x^2 + 6\mu u^5 + u^6 \right) (t, x) dx = E_{5}[u](0).$$

### 1.1 Main Results

We are interested in the regularity properties of the 5th order Gardner Eq. (1.1) and long time behavior of $H^2$ global solutions to (1.1). In short, our main results in this work are based on the local and global existence of solutions for the 5th order Gardner Eq. (1.1), together with a characterization of its weak-ill-posedness properties. Finally we also show a direct application of the global existence theory for $H^2(\mathbb{R})$ solutions of (1.1), by presenting an orbital stability result for new solutions discovered for this model. In more details, in this work we are concerned with:

#### 1.1.1 Well-Posedness Theory

In comparison with the 5th mKdV (1.2), the nonlinearity of (1.1) consists of more terms which break the balance with the 5th order linear dispersive part of (1.1). Precisely, additional quadratic terms with three derivatives, pose technical problems, for instance, the failure of bilinear $X^{a,b}$ estimates, see Remark 1.1 below (also see Remark 2.3 in [26]). However, an analogous argument used in [26,35] enables us to attack the initial value problem of (1.1) in $H^2$.

The notion of the well-posedness, which is taken into account in this paper, is as follows:
**Definition 1.1** (Well-posedness) We say that the 5th Gardner Eq. (1.1) is local-in-time (or locally) well-posed in $H^s(\mathbb{R})$, if for any $R > 0$ and any $u_0 \in \{ f \in H^s(\mathbb{R}) : \| f \|_{H^s} \leq R \}$, there exist a local time $T = T(R) > 0$ and a unique solution $u$ to (1.1) in $C([0, T]; H^s(\mathbb{R})) \cap X_T$, for some auxiliary space $X_T$. Moreover, the solution map $u_0 \mapsto u(t)$ is continuous from $\{ f \in H^s(\mathbb{R}) : \| f \|_{H^s} \leq R \}$ to $C([0, T]; H^s(\mathbb{R}))$. The local result is extended to the global one, if $T > 0$ is independent of $R$.

Firstly, we are going to show that the 5th order Gardner Eq. (1.1) is locally well-posed in $H^2$ via the classical energy method in addition to the short time Fourier restriction norm method. We state the local well-posedness result as follows:

**Theorem 1.2** The 5th order Gardner Eq. (1.1) is locally well-posed in $H^s(\mathbb{R})$, $s \geq 2$.

For the proof of Theorem 1.2, we use the short-time Fourier restriction norm method in a frequency dependent time interval. This is introduced by Ionescu, Kenig and Tataru [32] in the context of KP-I equation in the Besov-type space setting, see also [20,44] for similar ideas in the other settings. The short-time Fourier restriction norm method has been further developed in, for instance, [24–27,30,35,47,48].

The main difficulty arising in (1.1) is the strong high-low bilinear interaction component of the following type

$$ (P_{\leq 0} u) \cdot (P_{\text{high}} u_{xxx}) $$

newly generated from the map $v(t, x) = \mu + u(t, x)$. The standard bilinear $X^{s,b}$-estimates ($\| u\|_{X^{s,b}} \| v\|_{X^{s,b}} \leq \| f\|_{X^{s,b}}$) fails in usual $X^{s,b}$ spaces for any $s \in \mathbb{R}$ (see Remark 1.1 below), where the $X^{s,b}$ norm is defined in (2.7), since the dispersive smoothing effect in a coherent case occurring in (1.7) is not enough to control the three derivatives in the high frequency mode. The following remark provides a counter-example to show the failure of the standard bilinear estimate:

**Remark 1.1** (Remark 2.3 in [26]) Similarly as the 5th KdV case, also as mentioned before, the standard $X^{s,b}$ bilinear estimate fails to hold:

$$ \| u \partial^3_x v \|_{X^{s,b}} \leq C \| u \|_{X^{s,b}} \| v \|_{X^{s,b}}, $$

due to the following high-low interactions causing the coherence, for instance,

$$ u(t, x) = F^{-1}[1_{\Omega}(\tau, \xi)](t, x) \quad \text{and} \quad v(t, x) = F^{-1}[1_{\Sigma}(\tau, \xi)], $$

where space-time frequency sets $\Omega$ and $\Sigma$ are given by

$$ \Omega = \{(\tau, \xi) \in \mathbb{R}^2 : |\tau-\xi^5| \leq 1, N \leq |\xi| \leq N+1\} \quad \text{and} \quad \Sigma = \{(\tau, \xi) \in \mathbb{R}^2 : |\tau-\xi^5| \leq 1, |\xi| \leq 1\}, $$

2 Here $P$ is a appropriate truncation operator in the Fourier space, thus $P_{\text{high}} u$ means the high frequency ($|\xi| \gg 1$) localized portion of $u$, while the frequency support of $P_{\leq 0} u$ is in $[-1, 1]$.

3 The $X^{s,b}$ spaces are equipped with the norm

$$ \| f \|_{X^{s,b}} = \langle \xi \rangle^s \langle \xi^5 \rangle^b \| \hat{f} \|_{L^2_{\xi}}, $$

where $\hat{f}$ is the space time Fourier coefficient (also denoted by $F(f)$) and $\langle \cdot \rangle = (1+|\cdot|^2)^{1/2}$. For more details, see Sect. 2.

4 It suffices to regard only $\partial^5_\xi$ as a linear part of (1.1), since $\partial^3_\xi$ is negligible in a sense of the dispersion effect.
for fixed large frequency $N \gg 1$. Indeed, a direct calculation gives LHS of \((1.8) = NN^s\), while RHS of \((1.8) = N^s\).

However, using $X^{s,b}$ structure in a short time interval ($\approx (\text{frequency})^{-2}$), one reduces the contribution of high frequency with low modulation, so that one handles high-low interaction component \((1.7)\) (see Remark 1.2 below and Proposition 2.7).

**Remark 1.2** [Remark 2.3 in [26]] The short time $X^{s,b}$ spaces ($F^s$ and $N^s$ to be introduced in Sect. 2.2) in the interval of the length ($\approx (\text{frequency})^{-2}$) resolves the low-high interaction counter-example presented in Remark 1.1. The corresponding sets in this setting are given by

$$\tilde{\Omega} = \{(\tau, \xi) \in \mathbb{R}^2 : |\tau - \xi^5| \leq N^2, N \leq |\xi| \leq N + N^{-2}\}$$

and

$$\tilde{\Sigma} = \{(\tau, \xi) \in \mathbb{R}^2 : |\tau - \xi^5| \leq 1, |\xi| \leq 1\},$$

and define $u$ and $v$ similarly as in Remark 1.1, but with respect to $\tilde{\Omega}$ and $\tilde{\Sigma}$, respectively. Then, one immediately obtains for any $s \in \mathbb{R}$ that

$$\left\| u \partial_x^3 v \right\|_{N^s} \sim N^2 N^3 N^{-1} N^{-2} N \sim N^s N \quad \text{and} \quad \left\| u \right\|_{F^s} \left\| v \right\|_{F^s} \sim N^s N.$$

A price to pay in this short-time argument is an energy-type estimate. However, the strong high-low interactions, where the low frequency component has the largest modulation, cause a trouble in the energy estimates when the Ionescu-Kenig-Tataru’s method is followed. A way to treat this interaction is to use a weight, which was suggested in [31] to handle the same interaction for the Benjamin-Ono equation (see also [30]). Note that the modified energy, initially introduced in [49] and further developed in [35,47,48,55,56], plays a similar role as an additional weight. See [26] and [35] for a comparison.

Note moreover that a scaling equivalence enables us to focus on small solutions to \((2.1)\) instead of \((1.1)\) (see Sect. 2). In order to close the energy method’s argument for \((2.1)\), we gather linear, nonlinear and energy estimates,

$$\begin{cases}
\left\| u \right\|_{F^s(T')} \lesssim \left\| u \right\|_{F^s(T')} + \left\| N_2(u) + N_3(u) + SN(u) \right\|_{N^s(T')}, \\
\left\| N_2(u) + N_3(u) + SN(u) \right\|_{N^s(T')} \lesssim \sum_{j=2}^5 \left\| u \right\|_{F^j(T'}), \\
\left\| u \right\|_{F^j(T')} \lesssim \left\| u_0 \right\|_{H^j} + \sum_{j=3}^6 \left\| u \right\|_{F^j(T)}.
\end{cases} \quad (1.9)$$

The continuity argument ensures a priori bound of solutions to \((2.1)\). Moreover, a similar estimate as in \((1.9)\) for the difference of two solutions completes the limiting argument (compactness argument). We note that the energy estimate for the difference of two solutions does not hold true in $F^s$ spaces due to the lack of the symmetry, but holds in the intersection of the weaker ($F^0$) and the stronger ($F^{2s}$) spaces, thus the Bona-Smith argument is essential to close the compactness method.

The global well-posedness follows immediately from the above local result and the conservation of the second order energy \((1.6)\).

**Theorem 1.3** The 5th order Gardner Eq. \((1.1)\) is globally well-posed in the energy space $H^2(\mathbb{R})$.

**Remark 1.3** It is well-known that local results can be extended to the global one in the energy space without the smallness assumption for defocusing equations (for simple models), while the smallness condition is necessary for the proof of the global well-posedness in the energy space $H^s(\mathbb{R})$, $s \geq 2$. 

$^5$ The persistence of regularities ensures the global well-posedness in $H^s(\mathbb{R})$, $s \geq 2$. 

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space for focusing equations. However, (1.1) admits the scaling equivalence, which is slightly different from the standard scaling symmetry (or invariance), but still plays an almost same role in the local (or perturbation) theory. Thus, one has Theorem 1.3 from Theorem 1.2 in addition to the (rescaled) conservation law (1.6). See Sect. 2, in particular Sect. 2.6, for more details.

On the other hand, an observation explained in Remark 1.1 above naturally poses an interesting question:

does the flow map from data to solutions fail to be (locally) uniformly continuous for all regularities?

As an immediate answer, we state the following (weak) ill-posedness result, which extends Alejo-Cardoso’s recent result [4] to all regularities:

**Theorem 1.4** The 5th order Gardner Eq. (1.1) is weakly ill-posed in $H^s(\mathbb{R})$, for $s > 0$ in the following sense: there exist $c, C > 0$, $0 < T \leq 1$, and two sequences $u_n$ and $v_n$ of solutions to (1.1) such that

$$ \sup_n \|u_n(t)\|_{H^s} + \sup_n \|v_n(t)\|_{H^s} \leq C, \quad t \in [0, T] $$

and initially

$$ \lim_{n \to \infty} \|u_n(0) - v_n(0)\|_{H^s} = 0, $$

but for every $t \in [0, T]$

$$ \liminf_{n \to \infty} \|u_n(t) - v_n(t)\|_{H^s} \geq c|\sin t| \sim c|t|. $$

Theorem 1.4 can be expected from the observation in the linear local smoothing effect [38,39]

$$ \left\| \partial_x^2 e^{-it\partial_x^5} u_0 \right\|_{L^\infty_t L^2_x} \lesssim \|u_0\|_{L^2} $$

compared to the three derivatives in the quadratic nonlinearity. In other words, the local smoothing effect, which recovers only two derivatives, is not enough to handle the nonlinear term $u \partial_x^3 u$, as already seen in Remark 1.1. Such a strong high-low interaction phenomena can be seen in other dispersive equations, for instance, the Benjamin-Ono equation (BO) and the Kadomtsev-Petviashvili I equation (KP-I). Early, constructing examples reflecting (1.7), the flow map has been shown not to be $C^2$ continuous [57,58], and uniformly continuous [45,46].

To prove Theorem 1.4, we invoke an argument introduced in [45] (but essentially it follows from [49]) in order to construct the approximate solutions, which indeed reveals the ill-posedness phenomenon. Using the local well-posedness theory, one shows that the approximate solutions are indeed "good" approximate solutions in $H^s$ sense, $s \geq 2$. Moreover, since the Eq. (1.1) is completely integrable (thus it admits infinitely many conservation laws), we are able to show the same conclusion in the regularity range not only for $s \geq 2$, but also for $0 < s < 2$ by using $L^2$ and $H^2$ conservation laws.

The strategy employed in [4] was to use breather solutions of the 5th order Gardner equation as a way to measure the regularity of the associated Cauchy problem in $H^s$. This allowed to find the sharp Sobolev index under which the local well-posedness of the problem is lost, meaning that the dependence of 5th order Gardner solutions upon initial data fails to be continuous. We refer to [19,42,50] for similar arguments.
Finally, together with the result in [4], we get the following

**Corollary 1.5** The 5th order Gardner Eq. (1.1) is (weakly) ill-posed in $H^s(\mathbb{R})$, for $s \in \mathbb{R}$, in the sense of the statement given in Theorem 1.4.

As already seen above, the 5th Gardner Eq. (1.1) contains the mixed nonlinearities of 5th order KdV and mKdV Eqs. (1.3)–(1.2), so that one can see both ill-posedness nature of semilinear and quasilinear equations. In the proof of Theorem 1.4, approximate solutions are constructed in the following way: the separation of the phase shift ($\mp t$) and the dispersion effect ($\Phi_N(t)$) in (A.1) inspired by the observation on the Burgers equation. However, in low regularity Sobolev space ($L^2$ or below), it is not clear to see such a phenomenon, see [45, 46, 49]. Nonetheless, the cubic nonlinearity (coming from 5th mKdV nonlinearities) reveals another ill-posedness phenomenon, breaking the uniform continuity of the flow map by the self-interaction of a single high frequency wave in low regularity spaces [4]. This nature can be seen in some semilinear equations, for instance [2, 13, 14, 19, 42, 50]. The mixed nonlinearities in (1.1), thus, ensure to claim the lack of the uniform continuity of the flow map of the 5th Gardner Eq. (1.1) in all regularity Sobolev spaces.

### 1.1.2 Orbital Stability for 5th Order Gardner Breather Solutions

Another interesting topic in the theory of PDEs is to study the global dynamics of a solution to a given equation, in particular, the stability theory for a special solution. In this paper, we shall focus on a special type of "wave packet" solution, defined as follows:

**Definition 1.6** (Breather solutions) We shall say that a nontrivial global strong solution $u = u(t, x)$ of (1.1) is a breather solution if $u$ is periodic in time (up to translation) and localized in space.

Important examples of breather solutions are the modified KdV [51, 67] and the Sine-Gordon [51] breathers. For recent works in the subject of stability, see, for instance, [3, 6–10, 59–61].

We study stability properties of a breather solution to (1.1), which is presented in [4], by using the global existence result presented in Sect. 1.1.1. Specifically, this stability result is defined as follows

**Definition 1.7** (5th order Gardner breather) Let $\alpha, \beta \in \mathbb{R}\{0\}$, $\mu \in \mathbb{R}^+\{0\}$ such that $\Delta = \alpha^2 + \beta^2 - 4\mu^2 > 0$, and $x_1, x_2 \in \mathbb{R}$. The 5th order breather solution $B_{\mu} \equiv B_{\mu, 5}$ of the 5th order Gardner Eq. (1.1), is given explicitly by the formula

$$B_{\mu} = B_{\alpha, \beta, \mu, 5}(t, x; x_1, x_2) := 2\partial x\left[\arctan\left(\frac{G_{\mu}(t, x)}{F_{\mu}(t, x)}\right)\right],$$  \hspace{1cm} (1.10)

where

$$G_{\mu}(t, x) := \frac{\beta \sqrt{\alpha^2 + \beta^2}}{\alpha \sqrt{\Delta}} \sin(\alpha y_1) - \frac{2\mu \beta [\cosh(\beta y_2) + \sinh(\beta y_2)]}{\Delta},$$

$$F_{\mu}(t, x) := \cosh(\beta y_2) - \frac{2\mu \beta [\alpha \cos(\alpha y_1) - \beta \sin(\alpha y_1)]}{\alpha \sqrt{\alpha^2 + \beta^2 \sqrt{\Delta}}},$$

with $y_1$ and $y_2$

$$y_1 = x + \delta_5 t + x_1, \hspace{1cm} y_2 = x + \gamma_5 t + x_2.$$
and with velocities
\[
\begin{align*}
\delta_5 & := -\alpha^4 + 10\alpha^2\beta^2 - 5\beta^4 + 10(\alpha^2 - 3\beta^2)\mu^2, \\
\gamma_5 & := -\beta^4 + 10\alpha^2\beta^2 - 5\alpha^4 + 10(3\alpha^2 - \beta^2)\mu^2.
\end{align*}
\]

We present the following stability result for the new 5th order breather solutions (1.10):

**Theorem 1.8** Let \(\alpha, \beta \in \mathbb{R}\setminus\{0\}\) be given. Breather solutions (1.10) of the 5th order Gardner Eq. (1.1) are orbitally stable for \(H^2\) perturbations, whenever the parameter \(\mu \in (0, \sqrt{\alpha^2 + \beta^2})\).

Note that, as far as we know, no global existence results in \(H^2\) for higher-order equations in the hierarchy are known. Thus, the well-posedness results in \(H^2\) are needed in advance of the study on the stability theory.

The proof of this stability result is based on two main ingredients: the first one is a particular nonlinear identity, satisfied as far as we know, only by 5th order breather solutions (1.10) (see Lemma 3.2). On the other hand, the second ingredient is the universal fourth order ODE satisfied by (1.10) as it was previously shown in [5]. Using the nonlinear identity obtained from the first step, we prove that breather solutions (1.10) of the 5th order Gardner Eq. (1.1) satisfy a fourth order ODE (see [3, Theorem 3.5] for the same as the one satisfied by classical Gardner breather solutions).

To complete the proof of Theorem 1.8 (see Theorem 3.6), we introduce an \(H^2\) Lyapunov functional as the linear combination of the energy (1.5), the mass (1.4) and (1.6) in the following form:

\[
\mathcal{H}_\mu[u](t) := E_{\mu}[u](t) + 2(\beta^2 - \alpha^2)E_\mu[u](t) + (\alpha^2 + \beta^2)^2M[u](t).
\]

The rest of the proof uses perturbation of the functional \(\mathcal{H}_\mu[u](t)\) (see Lemma 3.5) and the spectral properties of the linearized operator \(L_\mu\) defined as (3.7) corresponding to the quadratic form arising in the small perturbation of the functional \(\mathcal{H}_\mu[u](t)\). The spectral properties, in particular the coercivity property of the quadratic form and existence of a unique negative eigenvalue for the linearized operator around the breather solution, ensure the \(H^2\)-stability of the 5th order Gardner breathers (1.10).

For more detailed statements and background about this stability property of 5th order Gardner breather solutions, see Sect. 3.

## 2 Well-Posedness Results

### 2.1 Setting

It is well-known that the integrability of equations (fixed coefficients of the nonlinearities) is no longer important for mathematical analysis in the local well-posedness theory.

**Remark 2.1** As mentioned in Sect. 1, (1.1) does not allow the scaling invariance. However, defining \(u_\lambda := \lambda u(\lambda^5 t, \lambda x), \lambda > 0\), ensures an equivalence between (1.1) and

\[
w_t + w_{5x} + 10\mu^2\lambda^2 w_{3x} + N_2(w) + N_3(w) + SN(w) = 0, \tag{2.1}
\]

where \(N_2(w)\) is the nonlinearity from the fifth order KdV given by

\[
N_2(w) = 20\mu\lambda w_x w_{xx} + 40\mu\lambda w w_{xxx} + 180\mu^2\lambda^2 w^2 w_x, \tag{2.2}
\]

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\( \mathcal{N}_3(w) \) is the nonlinearity from the fifth order mKdV given by
\[
\mathcal{N}_3(w) = 10w^2 w_{3x} + 10w^2 x + 40w w_x u_{xx} + 30w^4 w_x
\]
and \( SN(w) \) is the rest terms generated from the transformation \( u \mapsto \mu + u \), which is weaker compared to \( \mathcal{N}_2(w) \) and \( \mathcal{N}_3(w) \) in some sense, given by
\[
SN(w) = 120\mu^3 \lambda^3 w w_x + 120\mu \lambda w^3 w_x.
\]
That is, \( u_2, \lambda > 0 \) is a solution to (2.1), if and only if \( u \) is a solution to (1.1). See Sect. 2.6 for the details.

We use the notation \( \widehat{f} \) or \( \mathcal{F}(f) \) for the space-time Fourier transform of \( f \) defined by
\[
\widehat{f}(\tau, \xi) = \int_{\mathbb{R}^2} e^{-ix\xi} e^{-i\tau t} f(x, t) \, dx dt
\]
for any \( f \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}) \). Similarly, we use \( \mathcal{F}_x \) (or \( \widehat{\cdot} \)) and \( \mathcal{F}_t \) to denote the Fourier transform with respect to space and time variable respectively.

Let \( Z_+ \) denote the set of nonnegative integers. For \( k \in Z_+ \), let define dyadic intervals \( I_k \), \( k \in Z_+ \) as
\[
I_0 = [\xi : |\xi| \leq 2] \quad I_k = [\xi : |\xi| \in [2^k-1, 2^{k+1}]] \quad k \geq 1.
\]
Let \( \eta_0 : \mathbb{R} \to [0, 1] \) denote a smooth bump function supported in \([-2, 2]\) and equal to 1 in \([-1, 1]\) with the following property:
\[
\partial_t^j \eta_0(\xi) = O(\eta_0(\xi)/|\xi|^j), \quad j = 0, 1, 2,
\]
as \( \xi \) approaches end points of the support of \( \eta \). For \( k \in Z_+ \), let
\[
\chi_0(\xi) = \eta_0(\xi) \quad \text{and} \quad \chi_k(\xi) = \eta_0(\xi/2^j) - \eta_0(\xi/2^{j-1}), \quad k \geq 1,
\]
and
\[
\chi_{[k_1,k_2]} = \sum_{k=k_1}^{k_2} \chi_k \quad \text{for any} \ k_1 \leq k_2 \in Z_+.
\]
For the time-frequency decomposition, we use the cut-off function \( \eta_j \), but the same as \( \eta_j = \chi_j, \ j \in Z_+ \). For \( k \in Z \) let \( P_k \) denote the (smooth) truncation operators on \( L^2(\mathbb{R}) \) defined by \( P_k u(\xi) = \chi_k(\xi) \hat{u}(\xi) \). We also define the operators \( P_k \) on \( L^2(\mathbb{R} \times \mathbb{R}) \) by formulas \( \mathcal{F}(P_k u)(\xi, \tau) = \chi_k(\xi) \mathcal{F}(u)(\tau, \xi) \) for \( l \in Z \) let
\[
P_{\leq l} = \sum_{k \leq l} P_k, \quad P_{\geq l} = \sum_{k \geq l} P_k.
\]
For \( \xi \in \mathbb{R} \), \( w(\xi) = -\xi^5 \) is the dispersion relation associated to the Eq. (2.1).
\[\text{6 Originally, we have } w(\xi) = -\xi^5 + 10\mu^2 \xi^3 \text{ corresponding to the linear part of (1.1). However, for fixed } \mu \text{ and for large frequency } |\xi| \gg 1, \text{ the third order term are negligible compared to the fifth order term.}\]
2.2 Function Spaces

We introduce the $X^{s,b}$ spaces associated to (2.1), which is the completion of $\mathcal{S}'(\mathbb{R}^2)$ under the norm

$$\|f\|_{X^{s,b}} = \left\| \langle \tau - w(\xi) \rangle^b \langle \xi \rangle^s \tilde{f} \right\|_{L^2(\mathbb{R}^2)}, \tag{2.7}$$

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$. This Fourier restriction norm method was first implemented in its current form by Bourgain [11] and further developed by Kenig, Ponce and Vega [41] and Tao [65]. The Fourier restriction norm method turns out to be very useful in the study of low regularity theory for the dispersive equations. We denote the localized space by $X_T$ defined by standard localization to the interval $[-T,T]$.

As already mentioned in Sect. 1, the 5th Gardner Eq. (1.1) is a quasilinear equation where the flow map is not uniformly continuous. This fact can be seen from [49], which proves that the 5th order KdV Eq. (1.3) is weakly ill-posed, since this phenomenon occurs precisely in a strong interaction between low and high frequencies localized data of the form

$$(u_{\leq 0}) \cdot (\partial_x^3 u_{\geq 1})$$

which is also included in the nonlinearity of 5th order Gardner (1.1). For this reason, we must focus specifically on quadratic nonlinearities to prove the local well-posedness of the 5th Gardner Eq. (1.1). In what follows, we briefly introduce the functional spaces used in [26].

One of the purposes in this paper, as mentioned in Sect. 1, is to obtain $H^2$ global solutions to (1.1). Moreover, this regularity threshold is determined by the estimates of quadratic terms with three derivatives, which is already known from [26,35]. In what follows, we only focus on obtaining the estimates of cubic terms with three derivatives in $H^2$, since the cubic terms are another nontrivial and strong nonlinearities in (1.1). On the other hand, we expect that all estimates of these cubic terms can be obtained below $H^2$ compared to the quadratic nonlinearities, since the degree 3 of nonlinearities allows more smoothing effects in high-low interactions. However, here we do not explore such estimates below $H^2$ for our purpose.

We fix $k \in \mathbb{Z}_+$, and define the weighted Besov-type $(X^{0,\frac{1}{2}-1})$ space $X_k$ for frequency localized functions in $\tilde{I}_k$,

$$X_k = \left\{ f \in L^2(\mathbb{R}^2) : \text{supp } f \subset \mathbb{R} \times I_k, \| f \|_{X_k} < \infty \right\},$$

equipped with the norm

$$\| f \|_{X_k} := \sum_{j=0}^{\infty} 2^{j/2} \beta_{k,j} \| \eta_j(\tau - w(\xi)) f(\xi, \tau) \|_{L^2_{\xi,\tau}},$$

where

$$\beta_{k,j} = \begin{cases} 2^{j/2}, & k = 0, \\ 1 + 2^{(j-5k)/8}, & k \geq 1. \end{cases} \tag{2.8}$$

**Remark 2.2** The use of the weight $\beta_{k,j}$ is essential to control the localized energy for the quadratic terms in $\mathcal{N}_2(u)$, in particular, the high-low interaction components, where the low frequency component has the largest modulation. See Lemma 2.10. Moreover, it enables us

---

The basic method is similar to that used in [35], but it is chosen to avoid complicated calculations in the energy estimate.
to avoid the logarithmic divergence in $H^2$ appearing in the energy estimates for the cubic nonlinearities in $N_3(u)$, see Remark 2.11 and Propositions 2.12 and 2.13.

**Remark 2.3** An opposite effect of the use of the weight is to worsen the high × high → low interactions in the nonlinear estimates for the quadratic terms in $N_2(u)$

$$P_{\leq 0}(P_{\text{high}} u \cdot P_{\text{high}} v_{xxx}).$$

However, thanks to the representation of the quadratic nonlinearities as the compact, conservative form, i.e., $c_1 \partial_x u \partial_x^2 u + c_2 u \partial_x^3 u = c_1' \partial_x^2 u (\delta_x u \partial_x u) + c_2' (u \partial_x^2 u)$, one derivative is removed, and hence we are able to balance both purposes.

**Remark 2.4** Finally, the choice of a parameter $\frac{1}{8}$ in the weight for high frequency can be replaced by any parameter in $[1/8, 3/16]$. However, another choice of parameter is not able to improve the result, since the essential effect of the weight occurs in the high-low interactions, where the low frequency part has the largest modulation, as mentioned before.

At each frequency $2^k$, we define functional spaces based on $X_k$, uniformly on the $2^{-2^k}$ time scale.

$$F_k = \{ f \in L^2(\mathbb{R}^2) : \text{supp } f \subset \mathbb{R} \times I_k, \| f \|_{F_k} < \infty \},$$
equipped with the norm

$$\| f \|_{F_k} = \sup_{t_k \in \mathbb{R}} \| \mathcal{F}[f \cdot \eta_0(2^k(t-t_k))] \|_{X_k}$$

and

$$N_k = \{ f \in L^2(\mathbb{R}^2) : \text{supp } f \subset \mathbb{R} \times I_k, \| f \|_{N_k} < \infty \},$$
equipped with the norm

$$\| f \|_{N_k} = \sup_{u \in \mathbb{R}} \| (\tau - \omega(\xi) + i2^k)^{-1} \mathcal{F}[f \cdot \eta_0(2^k(t-t_k))] \|_{X_k}.$$ 

The standard way to construct localized spaces gives, for $T \in (0, 1]$, that

$$F_k(T) = \{ f \in C([-T,T] : L^2) : \| f \|_{F_k(T)} = \inf_{\tilde{f} = f} \| \tilde{f} \|_{F_k} \},$$

$$N_k(T) = \{ f \in C([-T,T] : L^2) : \| f \|_{N_k(T)} = \inf_{\tilde{f} = f} \| \tilde{f} \|_{N_k} \}.$$ 

We collect all pieces of spaces introduced above at dyadic frequency $2^k$ in the Littlewood-Paley way. For $s \geq 0$ and $T \in (0, 1]$, we define function spaces for solutions and nonlinear terms:

$$F^s(T) = \left\{ u : \| u \|_{F^s(T)}^2 = \sum_{k=0}^{\infty} 2^{2sk} \| P_k(u) \|_{F_k(T)}^2 < \infty \right\},$$

$$N^s(T) = \left\{ u : \| u \|_{N^s(T)}^2 = \sum_{k=0}^{\infty} 2^{2sk} \| P_k(u) \|_{N_k(T)}^2 < \infty \right\}.$$ 

In order to take the short time structure for IVP of (1.1), it is required to define the energy space as follows: for $s \geq 0$ and $u \in C([-T,T] : H^\infty)$

$$\| u \|_{E^s(T)}^2 = \| P_{\leq 0}(u(0)) \|_{L^2}^2 + \sum_{k \geq 1} \| P_k(u(t_k)) \|_{L^2}^2.$$
Remark 2.5  The short time Fourier restriction norm method used in this work was introduced by Ionescu, Kenig and Tataru [32], where the local well-posedness of KP-I equation in the energy space was proved, and further developed in [24–27,30,35,47,48] and references therein. We also refer to [20,44] for different formulas of short time analysis.

For the extension argument of functions in the spaces introduced above, we follow [32] to define the set $S_k$ of $k$-acceptable time multiplication factors for any $k \in \mathbb{Z}_+$:

$$S_k = \{ m_k : \mathbb{R} \to \mathbb{R} : \| m_k \|_{S_k} = \sum_{j=0}^{10} 2^{-2jk} \| \partial^j m_k \|_{L^\infty} < \infty \}. $$

Direct estimates using the definitions and (2.10) show that for any $s \geq 0$ and $T \in (0, 1]$

$$\left\{ \begin{array}{l}
\left\| \sum_{k \in \mathbb{Z}_+} m_k(t) \cdot P_k(u) \right\|_{F^s(T)} \lesssim (\sup_{k \in \mathbb{Z}_+} \| m_k \|_{S_k}) \cdot \| u \|_{F^s(T)}, \\
\left\| \sum_{k \in \mathbb{Z}_+} m_k(t) \cdot P_k(u) \right\|_{N^s(T)} \lesssim (\sup_{k \in \mathbb{Z}_+} \| m_k \|_{S_k}) \cdot \| u \|_{N^s(T)}, \\
\left\| \sum_{k \in \mathbb{Z}_+} m_k(t) \cdot P_k(u) \right\|_{E^s(T)} \lesssim (\sup_{k \in \mathbb{Z}_+} \| m_k \|_{S_k}) \cdot \| u \|_{E^s(T)}. 
\end{array} \right.$$ 

We finish this subsection with the following important lemma.

**Lemma 2.1**  [Properties of $X_k$]  Let $k, l \in \mathbb{Z}_+$ with $l \leq 5k$ and $f_k \in X_k$. Then

$$\sum_{j=l+1}^{\infty} 2^{j/2} \beta_{k,j} \left\| \eta_j(\tau - \omega(\xi)) \int_{\mathbb{R}} |f_k(\tau', \xi)| 2^{-l}(1 + 2^{-l}|\tau - \tau'|)^{-4} d\tau' \right\|_{L^2} + 2^{l/2} \left\| \eta_{l'}(\tau - \omega(\xi)) \int_{\mathbb{R}} |f_k(\tau', \xi)| 2^{-l}(1 + 2^{-l}|\tau - \tau'|)^{-4} d\tau' \right\|_{L^2} \lesssim \| f_k \|_{X_k}. \quad (2.9)$$

In particular, if $t_0 \in \mathbb{R}$ and $\gamma \in \mathcal{S}(\mathbb{R})$, then

$$\left\| \mathcal{F}[\gamma(2^l(t-t_0)) \cdot \mathcal{F}^{-1}(f_k)] \right\|_{X_k} \lesssim \| f_k \|_{X_k}. \quad (2.10)$$

**Proof**  See [26] for the proof. \hfill \Box

### 2.3 $L^2$-block Estimates

For $x, y \in \mathbb{R}_+$, $x \lesssim y$ means that there exists $C > 0$ such that $x \leq Cy$, and $x \sim y$ means $x \lesssim y$ and $y \lesssim x$. We also use $\lesssim_s$ and $\sim_s$ similarly, where the implicit constants depend on $s$. Let $a_1, a_2, a_3, a \in \mathbb{R}$. The quantities $a_{\text{max}} \geq a_{\text{sub}} \geq a_{\text{rh}} \geq a_{\text{min}}$ can be conveniently defined to be the maximum, sub-maximum, third-maximum and minimum values of $a_1, a_2, a_3, a$ respectively.

For $\xi_1, \xi_2 \in \mathbb{R}$, let denote the (quadratic) resonance function by

$$H = H(\xi_1, \xi_2) = w(\xi_1) + w(\xi_2) - w(\xi_1 + \xi_2) = \frac{5}{2} \xi_1 \xi_2 (\xi_1^2 + \xi_2^2 + (\xi_1 + \xi_2)^2).$$

Similarly, for $\xi_1, \xi_2, \xi_3 \in \mathbb{R}$, let

$$G(\xi_1, \xi_2, \xi_3) = w(\xi_1) + w(\xi_2) + w(\xi_3) - w(\xi_1 + \xi_2 + \xi_3) = \frac{5}{2} (\xi_1 + \xi_2)(\xi_2 + \xi_3)(\xi_3 + \xi_1)(\xi_1^2 + \xi_2^2 + \xi_3^2 + (\xi_1 + \xi_2 + \xi_3)^2). \quad (2.11)$$
be the (cubic) resonance function. Such resonance functions play an important role in the nonlinear $X^{s,b}$-type estimates.

**Remark 2.6** With the original dispersion relation $w(\xi) = -\xi^5 + 10\mu^2\xi^3$, we indeed obtain

$$H = H(\xi_1, \xi_2) = w(\xi_1) + w(\xi_2) - w(\xi_1 + \xi_2),$$

and

$$G(\xi_1, \xi_2, \xi_3) = w(\xi_1) + w(\xi_2) + w(\xi_3) - w(\xi_1 + \xi_2 + \xi_3)$$

$$= \frac{5}{2}(\xi_1 + \xi_2)(\xi_2 + \xi_3)(\xi_3 + \xi_1)(\xi_1^2 + \xi_2^2 + \xi_3^2 + (\xi_1 + \xi_2 + \xi_3)^2 - 12\mu^2).$$

Thus, for fixed $\mu$, once regarding the low frequency as $|\xi| \leq 12\mu^2$, all analyses below will not be affected by the choice of the dispersion relation between $w(\xi) = -\xi^5$ and $w(\xi) = -\xi^5 + 10\mu^2\xi^3$.

Let $f, g, h \in L^2(\mathbb{R}^2)$ be compactly supported functions. We define a quantity by

$$J_2(f, g, h) = \int_{\mathbb{R}^4} f(\xi_1, \xi_1)g(\xi_2, \xi_2)h(\xi_1 + \xi_2 + H(\xi_1, \xi_2), \xi_1 + \xi_2)d\xi_1d\xi_2d\xi_1d\xi_2.$$

The change of variables in the integration yields

$$J_2(f, g, h) = J_2(g^\ast, h, f) = J_2(h, f^\ast, g),$$

where $f^\ast(\xi, \xi) = f(-\xi, -\xi)$. From the identities

$$\xi_1 + \xi_2 = \xi_3 \quad \text{and} \quad (\tau_1 - w(\xi_1)) + (\tau_2 - w(\xi_2)) = (\tau_3 - w(\xi_3)) + H(\xi_1, \xi_2),$$

on the support of $J_2(f^\ast, g^\ast, h^\ast)$, where $f^\ast(\tau, \xi) = f(\tau - w(\xi), \xi)$ with the property $\|f\|_{L^2} = \|f^\ast\|_{L^2}$, we see that $J(f^\ast, g^\ast, h^\ast)$ vanishes unless

$$2k_{\max} \sim 2k_{\med} \geq 1 \quad \text{and} \quad 2j_{\max} \sim \max(2j_{\med}, |H|).$$

For compactly supported functions $f_i \in L^2(\mathbb{R} \times \mathbb{R}), i = 1, 2, 3, 4$, we define

$$J_3(f_1, f_2, f_3, f_4) = \int_{\mathbb{R}^4} f_1(\xi_1, \xi_1)f_2(\xi_2, \xi_2)f_3(\xi_3, \xi_3)f_4(\xi_1 + \xi_2 + \xi_3)$$

$$+ G(\xi_1, \xi_2, \xi_3), \xi_1 + \xi_2 + \xi_3),$$

where the $f_i = \int_{\mathbb{R}^4} d\xi_1d\xi_2d\xi_3d\xi_1d\xi_2d\xi_3$. From the identities

$$\xi_1 + \xi_2 + \xi_3 = \xi_4 \quad \text{and} \quad (\tau_1 - w(\xi_1)) + (\tau_2 - w(\xi_2)) + (\tau_3 - w(\xi_3)) = (\tau_4 - w(\xi_4)) + G(\xi_1, \xi_2, \xi_3),$$

on the support of $J_3(f_1^\ast, f_2^\ast, f_3^\ast, f_4^\ast)$, we see that $J_3(f_1^\ast, f_2^\ast, f_3^\ast, f_4^\ast)$ vanishes unless

$$2k_{\max} \sim 2k_{\sub} \geq 1 \quad \text{and} \quad 2j_{\max} \sim \max(2j_{\sub}, |G|),$$

where $|\xi_i| \sim 2^i$ and $|\xi_i| \sim 2^i$, $i = 1, 2, 3, 4$. A direct calculation shows

$$|J_3(f_1, f_2, f_3, f_4)| = |J_3(f_2, f_1, f_3, f_4)| = |J_3(f_3, f_2, f_1, f_4)| = |J_3(f_1^\ast, f_2^\ast, f_4, f_3)|.$$
We give $L^2$-block estimates for the quadratic and cubic nonlinearities. The bi- and trilinear $L^2$-block estimates for the 5th order equations have already been introduced and used in several works, we refer to [15–18,26,35,47,48].

Lemma 2.2 Let $k_i \in \mathbb{Z}$, $j_i \in \mathbb{Z}_+$, $i = 1, 2, 3$. Let $f_{k_i,j_i} \in L^2(\mathbb{R} \times \mathbb{R})$ be nonnegative functions supported in $[2^{k_i-1}, 2^{k_i+1}] \times I_{j_i}$.

(a) For any $k_1, k_2, k_3 \in \mathbb{Z}$ with $|k_{\max} - k_{\min}| \leq 5$ and $j_1, j_2, j_3 \in \mathbb{Z}_+$, then we have

$$J_2(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3}) \lesssim 2^{j_{\min}/2} 2^{j_{\med}/2} 2^{-\frac{3}{2} k_{\max}} \prod_{i=1}^{3} \|f_{k_i,j_i}\|_{L^2}.$$  

(b) If $2^{k_{\min}} \ll 2^{k_{\med}} \approx 2^{k_{\max}}$, then for all $i = 1, 2, 3$ we have

$$J_2(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3}) \lesssim 2^{(j_1+j_2+j_3)/2} 2^{-3k_{\max}/2} 2^{-(k_i+j_i)/2} \prod_{i=1}^{3} \|f_{k_i,j_i}\|_{L^2}.$$  

(c) For any $k_1, k_2, k_3 \in \mathbb{Z}$ and $j_1, j_2, j_3 \in \mathbb{Z}_+$, then we have

$$J_2(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3}) \lesssim 2^{j_{\min}/2} 2^{k_{\min}/2} \prod_{i=1}^{3} \|f_{k_i,j_i}\|_{L^2}.$$  

Proof We refer to [17,18,35] for the proof. □

Corollary 2.3 Assume $k_i \in \mathbb{Z}$ and $j_i \in \mathbb{Z}_+$, $i = 1, 2, 3$ and $f_{k_i,j_i} \in L^2(\mathbb{R} \times \mathbb{R})$ be functions supported in $D_{k_i,j_i}$, $i = 1, 2$.

(a) For any $k_1, k_2, k_3 \in \mathbb{Z}$ with $|k_{\max} - k_{\min}| \leq 5$ and $j_1, j_2, j_3 \in \mathbb{Z}_+$, then we have

$$\|1_{D_{k_3,j_3}}(\tau, \xi) (f_{k_1,j_1} * f_{k_2,j_2})\|_{L^2} \lesssim 2^{j_{\min}/2} 2^{j_{\med}/2} 2^{-\frac{3}{2} k_{\max}} \prod_{i=1}^{2} \|f_{k_i,j_i}\|_{L^2}.$$  

(b) If $2^{k_{\min}} \ll 2^{k_{\med}} \approx 2^{k_{\max}}$, then for all $i = 1, 2, 3$ we have

$$\|1_{D_{k_3,j_3}}(\tau, \xi) (f_{k_1,j_1} * f_{k_2,j_2})\|_{L^2} \lesssim 2^{(j_1+j_2+j_3)/2} 2^{-3k_{\max}/2} 2^{-(k_i+j_i)/2} \prod_{i=1}^{2} \|f_{k_i,j_i}\|_{L^2}.$$  

(c) For any $k_1, k_2, k_3 \in \mathbb{Z}$ and $j_1, j_2, j_3 \in \mathbb{Z}_+$, then we have

$$\|1_{D_{k_3,j_3}}(\tau, \xi) (f_{k_1,j_1} * f_{k_2,j_2})\|_{L^2} \lesssim 2^{j_{\min}/2} 2^{k_{\min}/2} \prod_{i=1}^{2} \|f_{k_i,j_i}\|_{L^2}.$$  

Lemma 2.4 Let $k_i \in \mathbb{Z}$ and $j_i \in \mathbb{Z}_+$, $i = 1, 2, 3, 4$. Let $f_{k_i,j_i} \in L^2(\mathbb{R} \times \mathbb{R})$ be nonnegative functions supported in $I_{j_i} \times [2^{k_i-1}, 2^{k_i+1}]$.

(a) For any $k_i \in \mathbb{Z}$ and $j_i \in \mathbb{Z}_+$, $i = 1, 2, 3, 4$, we have

$$J_3(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3}, f_{k_4,j_4}) \lesssim 2^{(j_{\min}+j_{\med})/2} 2^{(k_{\min}+k_{\med})/2} \prod_{i=1}^{4} \|f_{k_i,j_i}\|_{L^2}.$$  

(b) Let $k_{\med} \leq k_{\max} - 10.$

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(b-1) If \((k_i, j_i) = (k_{thd}, j_{max})\) for \(i = 1, 2, 3, 4\), we have
\[
J_3(f_{k_1, j_1}, f_{k_2, j_2}, f_{k_3, j_3}, f_{k_4, j_4}) \\
\lesssim 2^{(j_1 + j_2 + j_3 + j_4)/2} 2^{-2k_{thd}/2} 2^{-j_{max}/2} \prod_{i=1}^{4} \| f_{k_i, j_i} \|_{L^2}.
\]

(b-2) If \((k_i, j_i) \neq (k_{thd}, j_{max})\) for \(i = 1, 2, 3, 4\), we have
\[
J_3(f_{k_1, j_1}, f_{k_2, j_2}, f_{k_3, j_3}, f_{k_4, j_4}) \\
\lesssim 2^{(j_1 + j_2 + j_3 + j_4)/2} 2^{-2k_{min}/2} 2^{-j_{max}/2} \prod_{i=1}^{4} \| f_{k_i, j_i} \|_{L^2}.
\]

**Proof** We refer to [35,47] for the proof. In [47], the second author established (cubic) \(L^2\)-block estimates for functions \(f_{k_i, j_i} \in L^2(\mathbb{R} \times \mathbb{Z})\), but the proof, here, is almost identical and easier, see [35].

**Corollary 2.5** Let \(k_i \in \mathbb{Z}\) and \(j_i \in \mathbb{Z}_+, i = 1, 2, 3, 4\). Let \(f_{k_i, j_i} \in L^2(\mathbb{R} \times \mathbb{R})\) be nonnegative functions supported in \(D_{k_i, j_i}\).

(a) For any \(k_i \in \mathbb{Z}\) and \(j_i \in \mathbb{Z}_+\), \(i = 1, 2, 3, 4\), we have
\[
\| \mathbf{1}_{D_{k_i,j_i}}(\tau, \xi)(f_{k_1, j_1} * f_{k_2, j_2} * f_{k_3, j_3}) \|_{L^2} \lesssim 2^{(j_{min} + j_{thd})/2} 2^{(k_{min} + k_{thd})/2} \prod_{i=1}^{4} \| f_{k_i, j_i} \|_{L^2}.
\]

(b) Let \(k_{thd} \leq k_{max} - 10\).

(b-1) If \((k_i, j_i) = (k_{thd}, j_{max})\) for \(i = 1, 2, 3, 4\), we have
\[
\| \mathbf{1}_{D_{k_i,j_i}}(\tau, \xi)(f_{k_1, j_1} * f_{k_2, j_2} * f_{k_3, j_3}) \|_{L^2} \\
\lesssim 2^{(j_1 + j_2 + j_3 + j_4)/2} 2^{-2k_{thd}/2} 2^{-j_{max}/2} \prod_{i=1}^{4} \| f_{k_i, j_i} \|_{L^2}.
\]

(b-2) If \((k_i, j_i) \neq (k_{thd}, j_{max})\) for \(i = 1, 2, 3, 4\), we have
\[
\| \mathbf{1}_{D_{k_i,j_i}}(\tau, \xi)(f_{k_1, j_1} * f_{k_2, j_2} * f_{k_3, j_3}) \|_{L^2} \\
\lesssim 2^{(j_1 + j_2 + j_3 + j_4)/2} 2^{-2k_{min}/2} 2^{-j_{max}/2} \prod_{i=1}^{4} \| f_{k_i, j_i} \|_{L^2}.
\]

We end this subsection introducing the Strichartz estimates for the family of the fifth-order operators \(\{e^{it\partial_x^5}\}_{t=-\infty}^{\infty}\).

**Lemma 2.6** (Strichartz estimates for \(e^{it\partial_x^5}\) operator [21]) Assume that \(-1 < \sigma \leq \frac{3}{2}\) and \(0 \leq \theta \leq 1\). Then there exists \(C > 0\) depending on \(\sigma\) and \(\theta\) such that
\[
\left\| D_{\sigma, \theta}^\theta e^{it\partial_x^5} \varphi \right\|_{L_t^p L^\theta_x} \leq C \| \varphi \|_{L^2}
\]
for \(\varphi \in L^2\), where \(p = \frac{2}{1-\theta}\) and \(q = \frac{10}{\theta(\sigma + 1)}\). In particular, we have
\[
\left\| e^{it\partial_x^5} P_k \varphi \right\|_{L_t^p L^\theta_x} \lesssim 2^{-k/2} \| P_k \varphi \|_{L^2}, \quad k \geq 1.
\]
2.4 Nonlinear Estimates

We first recall from [26] the bilinear estimates as follows:

**Proposition 2.7** (Nonlinear estimates for \( \mathcal{N}_2(u) \), [26])

(a) If \( s \geq 1 \), \( T \in (0, 1) \), and \( u, v \in F^s(T) \) then

\[
\| \mathcal{N}_2(u) \|_{N^s(T)} \lesssim \| u \|^2_{F^s(T)} + \| u \|^3_{F^s(T)},
\]

and

\[
\| \mathcal{N}_2(u) - \mathcal{N}_2(v) \|_{N^s(T)} \lesssim \left( \| u \|^2_{F^s(T)} + \| v \|_{F^s(T)} \right) \| u - v \|_{F^s(T)}
+ \left( \| u \|^2_{F^s(T)} + \| v \|^2_{F^s(T)} \right) \| u - v \|_{F^s(T)}.
\]

(b) If \( T \in (0, 1) \), \( u, v \in F^0(T) \cap F^2(T) \), then

\[
\| \mathcal{N}_2(u) - \mathcal{N}_2(v) \|_{N^0(T)} \lesssim \left( \| u \|^2_{F^2(T)} + \| v \|^2_{F^2(T)} \right) \| u - v \|_{F^0(T)}
+ \left( \| u \|^2_{F^2(T)} + \| v \|^2_{F^2(T)} \right) \| u - v \|_{F^0(T)}.
\]

**Proof** The proof for the quadratic term, follows as in [26]. On the other hand, one can easily control the cubic term in \( \mathcal{N}_2(u) \) rather than not only the quadratic term in \( \mathcal{N}_2(u) \), but also the cubic term in \( \mathcal{N}_3(u) \), since the cubic term in \( \mathcal{N}_2(u) \) contains only one (total) derivative. Hence we omit the details, but one can capture the estimates in the proof of Proposition 2.8. \( \square \)

**Proposition 2.8** (Nonlinear estimates for \( \mathcal{N}_3(u) \))

(a) If \( s \geq 2 \), \( T \in (0, 1) \), and \( u, v \in F^s(T) \) then

\[
\| \mathcal{N}_3(u) \|_{N^s(T)} \lesssim \| u \|^3_{F^s(T)} + \| u \|^5_{F^s(T)}. \tag{2.13}
\]

and

\[
\| \mathcal{N}_3(u) - \mathcal{N}_3(v) \|_{N^s(T)} \lesssim \left( \| u \|^2_{F^s(T)} + \| v \|_{F^s(T)} \right) \| u - v \|_{F^s(T)}
+ \left( \| u \|^4_{F^s(T)} + \| v \|^4_{F^s(T)} \right) \| u - v \|_{F^s(T)}.
\]

(b) If \( T \in (0, 1) \), \( u, v \in F^0(T) \cap F^2(T) \), then

\[
\| \mathcal{N}_3(u) - \mathcal{N}_3(v) \|_{N^0(T)} \lesssim \left( \| u \|^2_{F^2(T)} + \| v \|^2_{F^2(T)} \right) \| u - v \|_{F^0(T)}
+ \left( \| u \|^4_{F^2(T)} + \| v \|^4_{F^2(T)} \right) \| u - v \|_{F^0(T)}.
\]

**Proof** We first consider the cubic term in \( \mathcal{N}_3(u) \). From the support property (2.12) in addition to (2.11), we know that

\[
\max(|\tau_j - w(\xi_j)|; j = 1, 2, 3, 4)
\geq |(\xi_1 + \xi_2)(\xi_1 + \xi_3)(\xi_2 + \xi_3)(\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2)|. \tag{2.14}
\]

From the definition of \( N_k \) norm, the left-hand side of (cubic terms in) (2.13) is bounded by

\[
\sup_{t_k \in \mathbb{R}} \left\| (\tau_4 - w(\xi_4) + i2^{2k_4} - 2^{3k_4})1_{k_4}(\xi) \mathcal{F} \left[ \eta_0 \left( 2^{2k_4 - 2}(t - t_k) \right) P_{k_4} u \right] \mathcal{F} \eta_0 \left( 2^{2k_4 - 2}(t - t_k) \right) P_{k_4} u \right\|_{X_{k_4}} \tag{2.15}
\]
We set \( u_{k_i} = \mathcal{F} \left[ \eta_0 \left( 2^{2k_i/2} (t - t_{k_i}) \right) P_{k_i} u \right], i = 1, 2, 3 \). We decompose each \( u_{k_i} \) into \( u_{k_i} (\tau, \xi) = u_{k_i} (\tau, \xi) \eta_{j_i} (\tau - w(\xi)) \) with usual modification like \( f_{\leq j} (\tau) = f (\tau) \eta_{\leq j} (\tau - \mu(n)) \). Then, (2.15) is bounded by

\[
\sum_{j_4 \geq 0} \frac{2^{3k_4} 2^{j_4/2} \beta_{k_4, j_4}}{\max(2j_4, 2^{2k_4})} \sum_{j_1, j_2, j_3 \geq 2k_4} \left\| 1_{D_{k_4, j_4}} \cdot (u_{k_1, j_1} * u_{k_2, j_2} * u_{k_3, j_3}) \right\|_{L^2}.
\]

We, instead of Corollary 2.5, use the following observation to control

for particular case: Lemma 2.6 yields

\[
\| \mathcal{F}^{-1} [u_{k_i, j_i}] \|_{L^6} = \left\| \int e^{it \tau} e^{ix \xi} e^{i w(\xi)} u^\sharp_{k_i, j_i} (\tau, \xi) d\xi d\tau \right\|_{L^6} \lesssim \int \left\| \int e^{ix \xi} e^{i w(\xi)} u^\sharp_{k_i, j_i} (\tau, \xi) d\xi \right\|_{L^6} d\tau \lesssim 2^{-k_i/2} 2^{j_i/2} \| u^\sharp_{k_i, j_i} \|_{L^2},
\]

where \( u^\sharp_{k_i, j_i} (\tau, \xi) = u_{k_i, j_i} (\tau + w(\xi), \xi) \) with \( \| u^\sharp_{k_i, j_i} \|_{L^2} = \| u_{k_i, j_i} \|_{L^2} \). With this, Plancherel’s theorem and the Hölder inequality give

\[
\left\| 1_{D_{k_4, j_4}} \cdot (u_{k_1, j_1} * u_{k_2, j_2} * u_{k_3, j_3}) \right\|_{L^2} \lesssim 2^{-(k_1 + k_2 + k_3)/2} 2^{(j_1 + j_2 + j_3)/2} \sum_{i=1}^{3} \| u_{k_i, j_i} \|_{L^2}.
\]

(2.17)

**Case I.** (high-high-high \( \Rightarrow \) high). Let \( k_4 \geq 20 \) and \( |k_1 - k_4|, |k_2 - k_4|, |k_3 - k_4| \leq 5 \). Applying (2.17) to \( \left\| 1_{D_{k_4, j_4}} \cdot (u_{k_1, j_1} * u_{k_2, j_2} * u_{k_3, j_3}) \right\|_{L^2} \), one has

\[
(2.15) \lesssim \sum_{j_4 \geq 0} \frac{2^{3k_4} 2^{j_4/2} \beta_{k_4, j_4}}{\max(2j_4, 2^{2k_4})} \sum_{j_1, j_2, j_3 \geq 2k_4} 2^{-k_4/2} 2^{(j_1 + j_2 + j_3)/2} \sum_{i=1}^{3} \| u_{k_i, j_i} \|_{L^2}.
\]

Note that \( \beta_{k_4, j_4} \sim 1 \) when \( 0 \leq j_4 \leq 5k_4 \). Let denote the summand by \( \mathcal{M}_I \), i.e.,

\[
\mathcal{M}_I := \frac{2^{3k_4} 2^{j_4/2} \beta_{k_4, j_4}}{\max(2j_4, 2^{2k_4})} 2^{-3k_4/2} 2^{(j_1 + j_2 + j_3)/2}.
\]

(2.18)

Then, we know

\[
\mathcal{M}_I \lesssim 2^{j_i/2} 2^{-k_i/2} 2^{(j_1 + j_2 + j_3)/2}, \quad \text{when } 0 \leq j_4 \leq 2k_4,
\]

\[
\mathcal{M}_I \lesssim 2^{-j_i/2} 2^{3k_4/2} 2^{(j_i + j_2 + j_3)/2}, \quad \text{when } 2k_4 \leq j_4 \leq 5k_4
\]

and

\[
\mathcal{M}_I \lesssim 2^{-j_i/2} 2^{(j_i - 5k_4)/8} 2^{3k_4/2} 2^{(j_1 + j_2 + j_3)/2}, \quad \text{when } 5k_4 \leq j_4.
\]

Performing summations over \( j_i, i = 1, 2, 3, 4 \), we have

\[
(2.15) \lesssim 2^{k_i/2} \prod_{i=1}^{3} \| P_{k_i} u \|_{F_{k_i}}.
\]
Indeed, we have from the definition of \(X_k\)-norm and (2.4) in [25] that
\[
\sum_{j_1 \geq 2k_4} 2^{j_1/2} \|u_{k_1,j_1}\|_{L^2} \lesssim \sum_{j_1 > 2k_4} 2^{j_1/2} \beta_{k_1,j_1} \|\eta_{j_1}(\tau - \omega(\xi)) \int_{\mathbb{R}} |\tilde{u}_{k_1}(\xi, \tau')|\|^{2}\|_{L^2} \\
\lesssim 2^{-2k_4} (1 + 2^{-2k_4} |\tau - \tau'|)^{-4} \|\int_{\mathbb{R}} |\tilde{u}_{k_1}(\xi, \tau')|2^{-2k_4} (1 + 2^{-2k_4} |\tau - \tau'|)^{-4} d\tau'\|_{L^2} \\
\lesssim \|u_{k_1}\|_{X_{k_1}} \lesssim \|P_{k_1} u\|_{F_{k_1}} ,
\]
where \(\tilde{u}_{k_1} = \mathcal{F}[P_{k_1} u \cdot \eta_0(2^{2k_1}(t - t_{k_1}))]\).

**Remark 2.7** As seen in the proof of Case I (also for other cases except for the case when the resulting frequency (\(\xi_4\) is not the maximum frequency), the weight \(\beta_{k_4,j_4}\) does not play any role in the estimates, hence is negligible. See Case I and Case III (below) for comparison.

**Case II.** (high-high-low \(\Rightarrow\) high). Let \(k_4 \geq 20, |k_2 - k_4|, |k_3 - k_4| \leq 5\) and \(k_1 \leq k_4 - 10^8\). In this case, we have \(j_{\max} \geq 5k_4\) due to (2.14). The exactly same argument used in Case I (but use Corollary 2.5 (a) instead of (2.17)) to control \(\|1_{D_{k_4,j_4}} \cdot (u_{k_1,j_1} * u_{k_2,j_2} * u_{k_3,j_3})\|_{L^2}\) gives a better result as follows:
\[
(2.15) \lesssim 2^{k_1/2} \prod_{i=1}^{3} \|u_{k_i}\|_{X_{k_i}} \lesssim 2^{k_1/2} \prod_{i=1}^{3} \|P_{k_i} u\|_{F_{k_i}} .
\]
The last inequality holds true, thanks to (2.10), more precisely,
\[
\|u_{k_1}\|_{X_{k_1}} = \mathcal{F} \left[ \eta_0 \left( (2^{2k_1} - 2(t - k_{k_4})) \cdot P_{k_1} u \cdot \eta_0 \left( 2^{2k_1} (t - k_{k_4}) \right) \right) \right]_{X_{k_1}} \lesssim \|P_{k_1} u\|_{F_{k_1}} .
\]
We omit the details.

**Case III.** (high-high-high \(\Rightarrow\) low). Let \(k_3 \geq 20, |k_1 - k_3|, |k_2 - k_3| \leq 5\) and \(k_4 \leq k_3 - 10\).

**Remark 2.8** The trade-off of the use of the short time advantage (also, the use of the weight as in (2.8)) is to worsen some interactions for which the resulting frequency is lower than (at least) one of others, in particular, high-high-high \(\Rightarrow\) low and high-high-low \(\Rightarrow\) low interaction components. More precisely, in the case of the high-high-high \(\Rightarrow\) low, the time interval of length \(2^{-2k_4}\), on which the \(N_{k_4}\)-norm is taken, is longer than the interval of length \(2^{-2k_4}\), on which \(F_{k_4}\)-norm is taken, \(i = 1, 2, 3\). In order to cover whole intervals of length \(2^{-2k_4}\) in the estimates, one needs to divide the time interval of length \(2^{-2k_4}\) into \(2k_3^{-2k_4}\) intervals of length \(2^{-2k_3}\). Let choose \(\gamma : \mathbb{R} \rightarrow [0, 1]\) (a kind of the partition of unity), which is a smooth function supported in \([-1, 1]\) with \(\sum_{m \in \mathbb{Z}} \gamma^3(x - m) \equiv 1\). Then, the left-hand side of (cubic

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8 Thanks to the symmetry of frequencies, our assumption that \(\xi_1\) is the minimum frequency does not lose of the generality.
the summand by $M$

\[ \sum_{|m| \leq C 2^{k_3 - k_4}} \mathcal{F}[\eta_0(2^{2k_4}(t - t_k)) \gamma(2^{2k_3}(t - t_k) - m) P_k u] \]

(2.19)

The analogous procedure will be applied to the estimate of high-high-low ⇒ low interaction component below.

When $k_4 = 0$, (2.19) is bounded by

\[ \sum_{j_k \geq 0} 2^{5k_3} \sum_{j_1, j_2, j_3 \geq 2k_3} \| 1_{D_{0,j_4}} \cdot (u_{k_1,j_1} * u_{k_2,j_2} * u_{k_3,j_3}) \|_{L^2} \]

due to (2.8).

When $0 \leq j_4 \leq 5k_3 - 5$ or $\leq 5k_3 + 5 \leq j_4$, we apply Corollary 2.5 (a) to

\[ \| 1_{D_{k_4,j_4}} \cdot (u_{k_1,j_1} * u_{k_2,j_2} * u_{k_3,j_3}) \|_{L^2} \]

to obtain

\[ (2.19) \lesssim \left( \sum_{0 \leq j_4 \leq 5k_3 - 5} + \sum_{5k_3 + 5 \leq j_4} \right) 2^{5k_3} \sum_{j_1, j_2, j_3 \geq 2k_3} 2^{(j_{\min} + j_{\text{hd}})/2} 2^{k_3/2} \prod_{i=1}^3 \| u_{k_i,j_i} \|_{L^2} \]

We know $j_4 \neq j_{\max}$ in the former case, while $j_4 = j_{\max}$ and $2^{j_{\max}} \sim 2^{j_{\text{med}}} \gg |H|$ in the latter case.

On the other hand, when $5k_3 - 5 \leq j_4 \leq 5k_3 + 5 (2^{j_4} = 2^{j_{\max}} \sim |H| \gg 2^{j_{\text{med}}})$, we use (2.17). Then, similarly as the previous cases, we have (when $k_4 = 0$)\(^9\)

\[ (2.19) \lesssim 2^{7k_3} \prod_{i=1}^3 \| P_{k_i} u \|_{F_{k_i}} \]

When $k_4 \neq 0$, similarly as above, (2.19) is bounded by

\[ \sum_{j_4 \geq 0} \frac{2^{5k_3 - 2k_4} 2^{j_4/2} \beta_{k_4,j_4} 2^{(k_3+k_4)/2}}{\max(2^{j_4}, 2^{2k_4})} \sum_{j_1, j_2, j_3 \geq 2k_3} 2^{(j_{\min} + j_{\text{hd}})/2} \prod_{i=1}^3 \| u_{k_i,j_i} \|_{L^2} \]

thanks to Corollary 2.5 (a), except for the case when $5k_3 - 5 \leq j_4 \leq 5k_3 + 5$. Let denote the summand by $\mathcal{M}_{111}$, similarly as in (2.18) i.e.,

\[ \mathcal{M}_{111} := \frac{2^{5k_3 - 2k_4} 2^{j_4/2} \beta_{k_4,j_4} 2^{(k_3+k_4)/2}}{\max(2^{j_4}, 2^{2k_4})} 2^{(j_{\min} + j_{\text{hd}})/2} \]

\(^9\) One can see that the worst bound comes from the low frequency with high modulation case ($j_4 = j_{\max} > j_{\text{med}} + 5$).
If $2k_3 < 5k_4$, we know
\[
\mathcal{M}_{III} \lesssim 2^{j_4} 2^{k_3} 2^{-7k_4/2} (j_1 + j_2 + j_3)/2, \quad \text{when } 0 \leq j_4 \leq 2k_4,
\]
\[
\mathcal{M}_{III} \lesssim 2^{2k_3} 2^{-3k_4/2} (j_1 + j_2 + j_3)/2, \quad \text{when } 2k_4 \leq j_4 \leq 2k_3,
\]
\[
\mathcal{M}_{III} \lesssim 2^{-j_4/2} 2^{3k_3} 2^{-3k_4/2} (j_1 + j_2 + j_3)/2, \quad \text{when } 2k_3 \leq j_4 \leq 5k_4,
\]
\[
\mathcal{M}_{III} \lesssim 2^{-j_4/2} (j_1 - 5k_4)/8 2^{3k_3} 2^{-3k_4/2} (j_1 + j_2 + j_3)/2, \quad \text{when } 5k_4 \leq j_4 \leq 5k_3 - 4
\]
and
\[
\mathcal{M}_{III} \lesssim 2^{-j_4/2} (j_1 - 5k_4)/8 2^{3k_3} 2^{-3k_4/2} (j_1 + j_2 + j_3)/2, \quad \text{when } 5k_3 + 4 \leq j_4.
\]
Otherwise (when $5k_4 < 2k_3$), the estimates of $\mathcal{M}_{III}$ on $2k_4 \leq j_4 \leq 5k_3 - 4$ are replaced by
\[
\mathcal{M}_{III} \lesssim 2^{2k_3} 2^{-3k_4/2} (j_1 + j_2 + j_3)/2, \quad \text{when } 2k_4 \leq j_4 \leq 5k_4,
\]
\[
\mathcal{M}_{III} \lesssim 2^{(j_4 - 5k_4)/8} 2^{3k_3} 2^{-3k_4/2} (j_1 + j_2 + j_3)/2, \quad \text{when } 5k_4 \leq j_4 \leq 2k_3
\]
and
\[
\mathcal{M}_{III} \lesssim 2^{-j_4/2} (j_1 - 5k_4)/8 2^{3k_3} 2^{-3k_4/2} (j_1 + j_2 + j_3)/2, \quad \text{when } 2k_3 \leq j_4 \leq 5k_3 - 4
\]
On the other hand, when $5k_3 - 5 \leq j_4 \leq 5k_3 + 5$, we use (2.17) to obtain
\[
(2.20) \lesssim 2^{5k_3 - 2k_4} 2^{-5k_3/2} 2^{3/2} (k_3 - k_4) 2^{-3k_4/2} \sum_{j_1, j_2, j_3 \geq 2k_3} \prod_{i=1}^{3} 2^{j_i} \| u_{k_i j_i} \|_{L^2}. \tag{2.20}
\]
Summing over $j_i, i = 1, 2, 3, 4$, one has
\[
(2.19) \lesssim C_1(k_3, k_4) \prod_{i=1}^{3} \| P_{k_i} u \|_{F_{k_i}}. \tag{2.20}
\]
where
\[
C_1(k_3, k_4) = \begin{cases} 2^{7/2} k_3, & 2k_3 < 5k_4, \\ 2^{9/2} k_3^2 - 17/8 k_4, & 5k_4 \leq 2k_3. \end{cases}
\]

**Remark 2.9** A direct computation in the cubic term in $N_3(u)$, one has
\[
40uu_xu_{xx} + 10u^2 u_{xxx} + 10u_x^2 = 10(u^2 u_{xx})_x + 10(uu_x^2)_x.
\]
Then, one can reduce $2^{3k_3}$ in (2.19) by $2^{2k_3+k_4}$, and hence obtain a better result. However, our regularity threshold is $s = 2$, and hence we, here, do not explore the trilinear estimates in lower regularity.

**Case IV.** (high-low-low ⇒ high). Let $k_4 \geq 20, |k_3 - k_4| \leq 5$ and $k_1, k_2 \leq k_4 - 10$. Without loss of generality, we may assume that $k_1 \leq k_2$, thanks to the symmetry. Similarly as the **Case I**, it is enough to consider
\[
\sum_{j_4 \geq 0} \frac{2^{3k_4} 2^{j_4/2} \beta_{k_4, j_4}}{\max(2^{j_4}, 2^{2k_4})} \sum_{j_1, j_2, j_3 \geq 2k_4} \| 1_{D_{k_4, j_4}} \cdot (u_{k_1, j_1} * u_{k_2, j_2} * u_{k_3, j_3}) \|_{L^2}. \tag{2.21}
\]
Let denote the summand in (2.21) by $\mathcal{M}_{IV}$, i.e.,
\[
\mathcal{M}_{IV} := \frac{2^{3k_4} 2^{j_4/2} \beta_{k_4, j_4}}{\max(2^{j_4}, 2^{2k_4})} \| 1_{D_{k_4, j_4}} \cdot (u_{k_1, j_1} * u_{k_2, j_2} * u_{k_3, j_3}) \|_{L^2}.
\]
We further split this case into three cases: **Case IV-a** $k_2 = 0$, **Case IV-b** $k_1 = 0$ and $k_2 \neq 0$, and **Case IV-c** $k_1 \neq 0$.

**Case IV-a.** $k_2 = 0$. We do not distinguish Corollary 2.5 (b.1) and (b.2), since $2^{k_{\text{min}}} \leq 2^{k_{\text{had}}} \leq 1$. Then, from Corollary 2.5 (b), we have

$$
\mathcal{M}_{IV} \lesssim 2^{j_4} 2^{-2k_4} \prod_{i=1}^{3} 2^{j_i/2} \|u_{k_i,j_i}\|_{L^2}, \quad \text{when } 0 \leq j_4 \leq 2k_4
$$

and

$$
\mathcal{M}_{IV} \lesssim 2^{-j_4/2} \beta_{k_4,j_4} \prod_{i=1}^{3} 2^{j_i/2} \|u_{k_i,j_i}\|_{L^2}, \quad \text{when } 2k_4 \leq j_4.
$$

Summing over $j_i$, $i = 1, 2, 3, 4$, one has

$$
(2.15) \lesssim \prod_{i=1}^{3} \|u_{k_i}\|_{F_{k_i}}.
$$

**Case IV-b.** $k_1 = 0$ and $k_2 \neq 0$. Note that we have $j_{\text{max}} \geq 4k_4 + k_2$ due to (2.14). We use Corollary 2.5 (b.1) (the worst case occurring in $j_2 = j_{\text{max}}$) when $0 \leq j_4 \leq 4k_4 + k_2 - 5$, and (b.2) ($j_2 = j_{\text{max}}$ never happens) when $4k_4 + k_2 - 5 \leq j_4$ to control $\|1_{D_{k_4,j_4}} (u_{k_1,j_1} * u_{k_2,j_2} * u_{k_3,j_3})\|_{L^2}$ in $\mathcal{M}_{IV}$, then we have

$$
\mathcal{M}_{IV} \lesssim 2^{j_4} 2^{-3k_4} \prod_{i=1}^{3} 2^{j_i/2} \|u_{k_i,j_i}\|_{L^2}, \quad \text{when } 0 \leq j_4 \leq 2k_4,
$$

$$
\mathcal{M}_{IV} \lesssim 2^{-k_4} \prod_{i=1}^{3} 2^{j_i/2} \|u_{k_i,j_i}\|_{L^2}, \quad \text{when } 2k_4 \leq j_4 \leq 4k_4 + k_2 - 5
$$

and

$$
\mathcal{M}_{IV} \lesssim 2^{-j_4/2} \beta_{k_4,j_4} 2^{k_4} \prod_{i=1}^{3} 2^{j_i/2} \|u_{k_i,j_i}\|_{L^2}, \quad \text{when } 4k_4 + k_2 - 5 \leq j_4.
$$

Summing over $j_i$, $i = 1, 2, 3, 4$, one has

$$
(2.15) \lesssim k_4 2^{-k_4} \prod_{i=1}^{3} \|u_{k_i}\|_{F_{k_i}}.
$$

**Case IV-c.** $k_1 \neq 0$. Similarly as **Case IV-a** (if $|k_1 - k_2| < 5$ with Corollary 2.5 (b.2)) or **Case IV-b** (if $k_1 < k_2 - 5$), we have at most

$$
(2.15) \lesssim 2^{k_1/2} \prod_{i=1}^{3} \|u_{k_i}\|_{F_{k_i}}.
$$

**Case V.** (high-high-low $\Rightarrow$ low). Let $k_3 \geq 20$, $|k_2 - k_3| \leq 5$ and $k_1, k_4 \leq k_3 - 10$. We first divide this case into two cases: **Case V-a** $k_4 = 0$ and **Case V-b** $k_4 \neq 0$.

---

10 We use, here, $2^{j_{\text{max}}} \geq 2^{2k_4}$ to deal with a maximum modulation, since our purpose is to obtain the local well-posedness only in $H^s(\mathbb{R})$, $s \geq 2$. However, one may obtain the better result by performing a delicate calculation in addition to $2^{j_{\text{max}}} \geq |H|$, instead of $2^{j_{\text{max}}} \geq 2^{2k_4}$. For the same reason, so the high-high-low $\Rightarrow$ low case below as well.
Case V-a. $k_4 = 0$. From Remark 2.8, it suffices to consider

$$
\sum_{j_4 \geq 0} 2^{5k_3} \sum_{j_1, j_2, j_3 \geq 2k_3} \left\| I_{D_{0,j_4}} \cdot (u_{k_1,j_1} * u_{k_2,j_2} * u_{k_3,j_3}) \right\|_{L^2} \tag{2.22}
$$

due to (2.8). If $k_1 = 0$, by using Corollary 2.5 (a), we have

$$
(2.22) \lesssim 2^{4k_3} \prod_{i=1}^{3} \left\| u_{k_i} \right\|_{F_{k_i}}. \tag{2.23}
$$

More precisely, when $|\xi_2 + \xi_3| \ll \xi_3^{-2}$ (equivalently $|H| \ll 2^{2k_3}$) we know

$$
2^{(j_{\min} + j_{\text{hd}})/2} \leq 2^{(j_1 + j_2 + j_3 + j_4)/2} 2^{-2k_3}, \quad \text{when } 0 \leq j_4 \leq 2k_3
$$

and

$$
2^{(j_{\min} + j_{\text{hd}})/2} \leq 2^{(j_1 + j_2 + j_3)/2} 2^{-j_4/2}, \quad \text{when } 2k_3 \leq j_4.
$$

When $|\xi_2 + \xi_3| \gg \xi_3^{-2}$ (equivalently $|H| \gg 2^{2k_3}$), we know

$$
2^{(j_{\min} + j_{\text{hd}})/2} \leq 2^{(j_1 + j_2 + j_3 + j_4)/2} 2^{-2k_3}, \quad \text{when } 1 \leq j_4 \leq 2^{k_3}
$$

$$
2^{(j_{\min} + j_{\text{hd}})/2} \leq 2^{(j_1 + j_2 + j_3)/2} 2^{-j_4/2}, \quad \text{when } 2^{k_3} \leq j_4 \leq \frac{|H|}{2}
$$

and

$$
2^{(j_{\min} + j_{\text{hd}})/2} \leq 2^{(j_1 + j_2 + j_3)/2} 2^{-j_4/2}, \quad \text{when } 3|H|/2 \leq j_4.
$$

Note that the number of $j_i$ is finite ($\leq 10$) when $|H|/2 \leq j_4 \leq 3|H|/2$. Thus, the summation over $j_i, i = 1, 2, 3, 4$, yields (2.23).

Otherwise ($k_1 \neq 0$), similarly as Case IV-b, we have

$$
(2.22) \lesssim 2^{3k_3} 2^{k_1/2} \prod_{i=1}^{3} \left\| u_{k_i} \right\|_{F_{k_i}}. \tag{2.24}
$$

Case V-b. $k_4 \neq 0$. Similarly, it is enough to consider

$$
\sum_{j_4 \geq 0} 2^{5k_3 - 2k_4 - 2j_4/2} \beta_{k_4,j_4} \sum_{j_1, j_2, j_3 \geq 2k_3} \left\| I_{D_{k_4,j_4}} \cdot (u_{k_1,j_1} * u_{k_2,j_2} * u_{k_3,j_3}) \right\|_{L^2}. \tag{2.24}
$$

Let denote the summand in (2.24) by $M_V$, i.e.,

$$
M_V := \frac{2^{5k_3 - 2k_4 - 2j_4/2} \beta_{k_4,j_4}}{\max(2j_4, 2^{2k_3})} \left\| I_{D_{k_4,j_4}} \cdot (u_{k_1,j_1} * u_{k_2,j_2} * u_{k_3,j_3}) \right\|_{L^2}. \tag{2.24}
$$

If $k_1 = 0$, we use Corollary 2.5 (b) to control $\left\| I_{D_{k_4,j_4}} \cdot (u_{k_1,j_1} * u_{k_2,j_2} * u_{k_3,j_3}) \right\|_{L^2}$. Then, we know

$$
M_V \lesssim 2^{j_4} 2^{k_3} 2^{-9k_4/2} (j_1 + j_2 + j_3)/2, \quad \text{when } 0 \leq j_4 \leq 2k_4,
$$

$$
M_V \lesssim 2^{k_3} 2^{-5k_4/2} (j_1 + j_2 + j_3)/2, \quad \text{when } 2k_4 \leq j_4 \leq 5k_4,
$$

$$
M_V \lesssim 2^{3k_3/2} 2^{-3k_4/2} (j_1 + j_2 + j_3)/2, \quad \text{when } 5k_4 \leq j_4 \leq 4k_3 + k_4 - 5
$$

and

$$
M_V \lesssim 2^{-j_4/2} 2^{(j_4 - 5k_4)/8} 2^{3k_3/2} 2^{-3k_4/2} (j_1 + j_2 + j_3)/2, \quad \text{when } 4k_3 + k_4 - 5 \leq j_4.
$$
Summing over $j_i, i = 1, 2, 3, 4$, one has
\[
(2.22) \lesssim \max \left( k_3 2^{3k_3/2} 2^{-3k_4}, 2^{3k_3/2} 2^{-2k_4} \right) \prod_{i=1}^{3} \| u_{k_i} \|_{F_{k_i}}.
\]

Otherwise $(k_1 \neq 0)$, analogous arguments as Case V-b (for $|k_1 - k_4| \geq 5$ case) and Case V-a, in particular $k_1 = 0$, case, (for $|k_1 - k_4| \leq 5$ case) can be applied, and hence we have (but, omit the details)
\[
(2.22) \lesssim C_2(k_1, k_3, k_4) \prod_{i=1}^{3} \| u_{k_i} \|_{F_{k_i}},
\]
where
\[
C_2(k_1, k_3, k_4) = \begin{cases} 
2^{3(k_3 - k_4)/2} \max \left( k_3 2^{-k_4}, 1, 2^{3k_3/2} 2^{-k_4/2} k_1/2, 2^{-k_4/2} k_1/2 \right), & k_1 \leq k_4 - 5, \\
2^{3k_3/2} 2^{-k_4/2} k_1, & k_4 \leq k_1 - 5, \\
2^{4k_3} 2 k_1, & |k_1 - k_4| \leq 5.
\end{cases}
\]

The estimate of $\text{low-low-low} \Rightarrow \text{low}$ interaction component can be easily obtained, and hence we omit the details. On the other hand, the estimate of quintic term in $\mathcal{N}_3(u)$ will be taken into account in the estimate of $\mathcal{S}\mathcal{N}(u)$ below. Thus, by collecting all, we complete the proof.

\[\square\]

**Proposition 2.9** (Nonlinear estimates for $\mathcal{S}\mathcal{N}(u)$) (a) If $s \geq 2, T \in (0, 1]$, and $u, v \in F^s(T)$ then
\[
\| \mathcal{S}\mathcal{N}(u) \|_{N^s(T)} \lesssim \| u \|^2_{F^s(T)} + \| u \|^4_{F^s(T)}.
\]

and
\[
\| \mathcal{S}\mathcal{N}(u) - \mathcal{S}\mathcal{N}(v) \|_{N^s(T)} \lesssim \left( \| u \|^2_{F^s(T)} + \| v \|^2_{F^s(T)} + \| u \|^3_{F^s(T)} + \| v \|^3_{F^s(T)} \right) \| u - v \|_{F^s(T)}.
\]

(b) If $T \in (0, 1], u, v \in F^0(T) \cap F^2(T)$, then
\[
\| \mathcal{S}\mathcal{N}(u) - \mathcal{S}\mathcal{N}(v) \|_{N^0(T)} \lesssim \left( \| u \|^2_{F^2(T)} + \| v \|^2_{F^2(T)} + \| u \|^3_{F^2(T)} + \| v \|^3_{F^2(T)} \right) \| u - v \|_{F^0(T)}.
\]

**Proof** The quadratic term in $\mathcal{S}\mathcal{N}(u)$ can be easily treated compared to one in $\mathcal{N}_2(u)$, due to a less number of derivatives, similarly as the cubic term in $\mathcal{N}_3(u)$ compared to one in $\mathcal{N}_3(u)$. The remaining of the proof (also for the quadratic and cubic terms in $\mathcal{S}\mathcal{N}(u)$ and $\mathcal{N}_3(u)$, respectively) is based on the following direct computation
\[
\| \mathbf{1}_{D_{k_1, j_1}} \cdot (u_{k_1, j_1} * \cdots * u_{k_{i-1}, j_{i-1}}) \|_{L^2} \lesssim 2^{-(k_{\text{max}} + k_{\text{med}})/2} 2^{-(j_{\text{max}} + j_{\text{med}})/2} k_1/2 2 k_{i-1}/2 \prod_{i=1}^{\ell-1} 2^{j_i/2} 2^{k_i/2} \| u_{k_i, j_i} \|_{L^2},
\]
which can be obtained by Cauchy-Schwarz inequality. Due to a less number of derivatives (indeed, one (total) derivative) in $\mathcal{S}\mathcal{N}(u)$, the analogous (but much simpler) argument used in the proof of Proposition 2.8 immediately yields Proposition 2.9. In particular, the total derivative form enables us to drop one derivative taken in a high frequency mode, see Remark 2.9. We omit the details.

\[\square\]
2.5 Energy Estimates

Assume that \( u, G \in C([-T, T]; L^2) \) satisfy

\[
\begin{align*}
& \begin{cases}
  u_t + u_{5x} = G, \quad (x, t) \in \mathbb{R} \times (-T, T) \\
  u(0, x) = u_0(x)
\end{cases}
\end{align*}
\]

A direct calculation gives

\[
\sup_{|t_k| \leq T} \|u(t_k)\|_{L^2}^2 \leq \|u_0\|_{L^2}^2 + \sup_{|t_k| \leq T} \left| \int_{\mathbb{R} \times [0, T]} u \cdot G \, dx \, dt \right|.
\] (2.25)

To control the second term (for \( N_2(u), N_3(u) \) and \( S\mathcal{N}(u) \)) of the right-hand side of (2.25), we need the following lemmas.

Lemma 2.10 ([26]) Let \( T \in (0, 1] \) and \( k_1, k_2, k_3 \in \mathbb{Z}_+ \).

(a) Assume \( k_1 \leq k_2 \leq k_3 \) and \( |k_3 - k_1| \leq 5, u_i \in F_{k_i}(T), i = 1, 2, 3 \). Then

\[
\left| \int_{\mathbb{R} \times [0, T]} u_1 u_2 u_3 \, dx \, dt \right| \leq 2^{-\frac{1}{2}k_3} \prod_{i=1}^{3} \|u_i\|_{F_{k_i}(T)}.
\]

(b) Assume \( k_1 \leq k_2 \leq k_3 \) and \( k_3 \geq 10, 2^{k_1} \ll 2^{k_2} \sim 2^{k_3} \) and \( u_i \in F_{k_i}(T), i = 1, 2, 3 \).

If \( k_1 \geq 1 \), then

\[
\left| \int_{\mathbb{R} \times [0, T]} u_1 u_2 u_3 \, dx \, dt \right| \leq 2^{-2k_3 - \frac{1}{2}k_1} \prod_{i=1}^{3} \|u_i\|_{F_{k_i}(T)}.
\]

If \( k_1 = 0 \), then

\[
\left| \int_{\mathbb{R} \times [0, T]} (\partial_x u_1) u_2 u_3 \, dx \, dt \right| \leq 2^{-2k_3} \prod_{i=1}^{3} \|u_i\|_{F_{k_i}(T)}.
\]

(c) Assume \( k_1 \leq k - 10 \). Then

\[
\left| \int_{\mathbb{R} \times [0, T]} P_k(u) P_k(\partial_x^3 u \cdot P_k v) \, dx \, dt \right| \leq 2^{\frac{1}{2}k_1} \|P_k v\|_{F_k^2(T)} \sum_{|k' - k| \leq 10} \|P_{k'} u\|_{F_{k'}(T)}^2.
\]

(d) Under the same condition as in (c), we have

\[
\left| \int_{\mathbb{R} \times [0, T]} P_k(u) P_k(\partial_x^3 u \cdot P_k \partial_x v) \, dx \, dt \right| \leq 2^{\frac{1}{2}k_1} \|P_k v\|_{F_k^2(T)} \sum_{|k' - k| \leq 10} \|P_{k'} u\|_{F_{k'}(T)}^2.
\]

Lemma 2.11 Let \( T \in (0, 1], k_1, k_2, k_3, k_4 \in \mathbb{Z}_+, \) and \( u_i \in F_{k_i}(T), i = 1, 2, 3, 4 \). We further assume \( k_1 \leq k_2 \leq k_3 \leq k_4 \) with \( k_4 \geq 10 \). Then

(a) For \( |k_1 - k_4| \leq 5 \), we have

\[
\left| \int_{[0, T] \times \mathbb{R}} u_1 u_2 u_3 u_4 \, dx \, dt \right| \leq 2^{-k_4/2} \prod_{i=1}^{4} \|u_i\|_{F_{k_i}(T)}.
\] (2.26)
(b) For \(|k_2 - k_4| \leq 5\) and \(k_1 \leq k_4 - 10\), we have
\[
\left| \int_{[0,T] \times \mathbb{R}} u_1 u_2 u_3 u_4 \, dxdt \right| \lesssim 2^{-k_4} 2^{k_1/2} \prod_{i=1}^{4} \|u_i\|_{F_{k_i}(T)}.
\]
(2.27)

(c) Let \(|k_3 - k_4| \leq 5\) and \(k_2 \leq k_4 - 10\).
In general, we have
\[
\left| \int_{[0,T] \times \mathbb{R}} u_1 u_2 u_3 u_4 \, dxdt \right| \lesssim 2^{-k_4} 2^{-k_2/2} \prod_{i=1}^{4} \|u_i\|_{F_{k_i}(T)}.
\]
(2.28)

In particular, if \(k_1 = 0\) and \(k_2 \geq 1\), we have
\[
\left| \int_{[0,T] \times \mathbb{R}} u_1 u_2 u_3 u_4 \, dxdt \right| \lesssim 2^{-2k_4} 2^{-k_2/2} \prod_{i=1}^{4} \|u_i\|_{F_{k_i}(T)}.
\]
(2.29)

If \(0 < k' \leq k_4 - 10\), we have
\[
\left| \int_{[0,T] \times \mathbb{R}} P_{k'}(u_1 u_2) u_3 u_4 \, dxdt \right| \lesssim 2^{-2k_4} 2^{-k_2/2} \prod_{i=1}^{4} \|u_i\|_{F_{k_i}(T)}.
\]
(2.30)

If \(k' = 0\), we have
\[
\left| \int_{[0,T] \times \mathbb{R}} P_0(\partial_x(u_1 u_2)) u_3 u_4 \, dxdt \right| \lesssim 2^{-2k_4} 2^{k_1/2} \prod_{i=1}^{4} \|u_i\|_{F_{k_i}(T)}.
\]
(2.31)

**Remark 2.10** In [35], a weaker estimate (2.28) is enough to control the cubic term with one derivative, while, in this paper, (2.29)–(2.31) are necessary to control the cubic terms with three derivatives. On the other hand, under the periodic boundary condition, (2.28) is optimal, due to the lack of smoothing effect. We refer to [35] and [47] for a part of proof and the periodic case (also for the comparison), respectively.

**Proof of Lemma 2.11** We only prove (a) and (c). The proof of (b) is analogous to the proof of (c), thus we omit it. For part (b), see [35,47].

(a) We apply a similar argument as in Remark 2.8 to the interval \([0,T]\). Let choose \(\rho : \mathbb{R} \to [0,1]\) to make a partition of unity, that is, \(\sum_{m \in \mathbb{Z}} \rho^4(x - m) \equiv 1\), for all \(x \in \mathbb{R}\). It follows that
\[
\left| \int_{[0,T] \times \mathbb{R}} u_1 u_2 u_3 u_4 \, dxdt \right| \lesssim \sum_{|m| \leq 2^{2k_4}} \left| \int_{[0,T]} \prod_{i=1}^{4} (\rho(2^{2k_4}t - m) I_{[0,T]}(t) u_i) \, dt \right|.
\]

Set
\[
A := \left\{ m : \rho(2^{2k_4}t - m) I_{[0,T]}(t) \text{ non-zero and } \neq \rho(2^{2k_4}t - m) \right\}.
\]

Note that \(|A| \leq 4\). We split
\[
\sum_{|m| \leq 2^{2k_4}} = \sum_{m \in A} + \sum_{m \in A^c}.
\]
It suffices to show (a) on the second summation, since otherwise, the same argument in addition to
\[ \sup_{j \in \mathbb{Z}_+} 2^{j/2} \left\| \eta_j(\tau - w(\xi)) : F[1_{[0,1]}(t) \rho(2^{2k} t - m)u] \right\|_{L^2} \lesssim \left\| \rho(2^{2k} t - m) \tilde{u} \right\|_{X_k} \]
gives better result, thanks to the absence of $2^{2k_4}$ (see [25,35] for the details).

On the second summation ($\sum_{m \in A^c}$), we can ignore $1_{[0,T]}(t)$. Similarly as in Sect. 2.4, let $u_{k_i} = F[\rho(2^{2k_4} t - m)\tilde{u}(\xi_i)]$ and $u_{k_i,j_i} = \eta_{j_i}(\tau_i - w(\xi_i))u_{k_i}$, $i = 1, 2, 3, 4$. Parseval’s identity and (2.9) yield
\[
\sum_{m \in A^c} \left| \int_{\mathbb{R}^2} \prod_{i=1}^{4} (\rho(2^{2k_4} t - m)1_{[0,T]}(t)u_i) \; dx \; dt \right| \lesssim \sup_{m \in A^c} 2^{2k_4} \sum_{j_1,j_2,j_3,j_4 \geq 2k_4} \left| \int_{\mathbb{R}^2} \prod_{i=1}^{4} F^{-1}[u_{k_i,j_i}](t, x) \; dx \; dt \right|. \tag{2.32}
\]
Hölder inequality and (2.16) ensure
\[
\left| \int_{\mathbb{R}^2} \prod_{i=1}^{4} F^{-1}[u_{k_i,j_i}](t, x) \; dx \; dt \right| \lesssim \|u_{k_1,j_1}\|_{L^2} \prod_{i=2}^{4} \left\| F^{-1}[u_{k_i,j_i}] \right\|_{L^6}
\lesssim 2^{-3k_4/2} 2^{-j_1/2} \prod_{i=1}^{4} 2^{j_i/2} \|u_{k_i,j_i}\|_{L^2},
\]
together with (2.32), one concludes (2.26) and we complete the proof.

(c) The proof of (2.28) can be found in [35]. The proof of (2.29) follows the proof of (2.28) with a modification $j_{\max} \geq 4k_4 + k_2 - 10$ instead of $j_{\max} \geq 2k_4$. Thus we omit the detail.

Note that
\[ \begin{cases} 2^{k'} \sim 2^{k_2}, & \text{if } k_1 \leq k_2 - 4, \\ 2^{k'} \ll 2^{k_2}, & \text{if } |k_1 - k_2| \leq 4, \end{cases} \tag{2.33} \]
since $2^{k'} \sim |\xi_1 + \xi_2| \lesssim 2^{k_2}$\(^\text{11}\).

We first show (2.30). Similarly as above, it suffices to estimate on $\sum_{m \in A^c}$. Using $2^{j_{\max}} \gtrsim 2^{4k_4} 2^{k'}$ in addition to (2.33), one immediately obtains from Lemma 2.4 (b) that
\[
\text{LHS of (2.30)} \lesssim 2^{-2k_4} 2^{k_1/2} \sum_{j_1,j_2,j_3,j_4 \geq 2k_4} \prod_{i=1}^{4} 2^{j_i/2} \|u_{k_i,j_i}\|_{L^2} \lesssim 2^{-2k_4} 2^{k_1/2} \prod_{i=1}^{4} \|u_i\|_{F_{k_i}(T)}. 
\]

The proof of (2.31) is analogous to the proof of Lemma 4.1 (b) (in particular (4.6)) in [26]. The left-hand side of (2.31) can be replaced by
\[
\sum_{k' \leq 0} 2^{k'} \left| \int_{[0,T] \times \mathbb{R}} P_{k'}(u_1u_2)u_3u_4 \; dx \; dt \right|.
\]

\(^\text{11}\) The case $|\xi_1 + \xi_2| \sim 2^{k_2}$, when $|k_1 - k_2| \leq 4$, exists, if both $\xi_1$ and $\xi_2$ have same sign. However, under this condition, one has the same conclusion as (2.29).
If $k_2 = 0$, similarly as the proof of (2.30), we have
\[ \text{LHS of (2.31)} \lesssim \left( \sum_{k' \leq 0} 2^{k'/2} \right) 2^{-2k_3} \prod_{i=1}^{4} \|u_i\|_{F_{k_i}(T)} \lesssim 2^{-2k_3} \prod_{i=1}^{4} \|u_i\|_{F_{k_i}(T)} \cdot \]

Otherwise ($k_1 \geq 1$), the same argument in the proof of (2.30) yields
\[ \text{LHS of (2.31)} \lesssim \left( \sum_{k' \leq 0} 2^{k'/2} \right) 2^{-2k_3} 2^{k_1/2} \prod_{i=1}^{4} \|u_i\|_{F_{k_i}(T)} \lesssim 2^{-2k_3} 2^{k_1/2} \prod_{i=1}^{4} \|u_i\|_{F_{k_i}(T)} \cdot \]

Thus, we complete the proof. \hfill \Box

**Remark 2.11** Using the weight, one can have at least $2^{k_3/4}$ more derivative gain, while $2^{k_1}$ derivative loss occurs. Indeed, a direct computation gives
\[ \sum_{j_1 \geq 2k_4} 2^{j_1/2} \|u_{k_1, j_1}\|_{L^2} \lesssim 2^{2k_3-8k_4} \|u_1\|_{F_{k_1}(T)} \cdot \]

Such derivative gain may be helpful to avoid the occurrence of the logarithmic divergence in $H^2$-energy estimates (see [47]). Moreover, the derivative loss in low frequencies is not big in $H^2$, so be handled in $H^2$. This approach may be applied to LWP of the fifth-order mKdV $H^2(\mathbb{T})$ (improvement of [47], in authors’ forthcoming project).

**Proposition 2.12** Let $s \geq 2$ and $T \in (0, 1]$. Then, for the solution $u \in C([-T, T]; H^\infty(\mathbb{T}))$ to (2.1), we have
\[ \|u\|_{E^s(T)}^2 \lesssim \|u_0\|_{H^s}^2 + \sum_{j=3}^{6} \|u\|_{F^s(T)}^j \cdot \tag{2.34} \]

**Proof of Proposition 2.12** The definition of the $E^s(T)$ norm says
\[ \|u\|_{E^s(T)}^2 - \|P_{\leq 0}(u_0)\|_{L^2}^2 = \sum_{k \geq 1} \sup_{t_k \in [-T, T]} 2^{2s k} \|P_k(u(t_k))\|_{L^2}^2 \cdot \]

Then, we immediately have
\[ 2^{2s k} \|P_k(u(t_k))\|_{L^2}^2 - 2^{2s k} \|P_k(u_0)\|_{L^2}^2 \leq 2^{2s k} \left| \int_{\mathbb{R} \times [0, t_k]} P_k(u) P_k(N_2(u)) \, dx \, dt \right| \]
\[ + 2^{2s k} \left| \int_{\mathbb{R} \times [0, t_k]} P_k(u) P_k(N_3(u)) \, dx \, dt \right| \]
\[ + 2^{2s k} \left| \int_{\mathbb{R} \times [0, t_k]} P_k(u) P_k(SN(u)) \, dx \, dt \right| \]
\[ =: I_1(k) + I_2(k) + I_3(k) , \]

thanks to (2.25). Proposition 4.2 (in addition to Remark 4.3) in [26] yields
\[ \sum_{k \geq 1} I_1(k) \lesssim \|u\|_{F^s(T)}^3 + \|u\|_{F^s(T)}^4 , \]

for $s \geq \frac{5}{4}$.

---

12 The case $|\xi_1 + \xi_2| \leq 1$ cannot happen when $k_1 = 0$ and $k_2 \geq 1$.
We now focus on $I_2(k)$. Note that a direct calculation gives
\[40uu_xu_{xx} + 10u^2u_{xxx} + 10u_x^3 = 10(u^2)_xu_{xx} + 10(u^2u_{xx})_x + 10u_x^3.\]

We split $I_2(k)$ (in particular cubic part in $N_3(u)$) into $I_{2,1} + I_{2,2} + I_{2,3}$, where
\[I_{2,1} := 2^{2sk} \left| \int_{\mathbb{R} \times [0,t_k]} P_k(u) P_k((u^2)_x u_{xx}) \, dx \, dt \right|,\]
\[I_{2,2} := 2^{2sk} \left| \int_{\mathbb{R} \times [0,t_k]} P_k(u) P_k((u^2u_{xx})_x) \, dx \, dt \right|\]
and
\[I_{2,3} := 2^{2sk} \left| \int_{\mathbb{R} \times [0,t_k]} P_k(u) P_k(u_x^3) \, dx \, dt \right|.

We first estimate $I_{2,1}$. We further decompose $I_{2,1}$ as follows:
\[I_{2,1} \lesssim 2^{2sk} \sum_{k' \leq k \leq 10} \left| \int_{\mathbb{R} \times [0,t_k]} P_k(u) P_k(P_{k'}((u^2)_x u_{xx})) \, dx \, dt \right|\]
\[+ 2^{2sk} \sum_{k' > k \geq 10, k_3 \geq 0} \left| \int_{\mathbb{R} \times [0,t_k]} P_k^2(u) P_k^3((u^2)_x) P_k(u_{xx}) \, dx \, dt \right|\]
\[=: I_{2,1,1} + I_{2,1,2}.

Note that $k' \leq k_2 + 10$ in $I_{2,1,1}$. Lemma 2.11 yields
\[I_{2,1,1} \lesssim 2^{2sk} \sum_{k_1 \leq k_2 \leq k \leq 10} 2^{k_1/2} 2^{k_2} \left\| P_{k_1} u \right\|_{F_{k_1}(T)} \left\| P_{k_2} u \right\|_{F_{k_2}(T)} \sum_{|k_0 - k| \leq 10} \left\| P_{k_0} u \right\|^2_{F_{k_0}(T)}\]
\[\lesssim 2^{2sk} \sum_{|k_0 - k| \leq 10} 2^{k_0} \left\| P_{k_0} u \right\|^4_{F_{k_0}(T)}\]
\[\lesssim 2^{2sk} \sum_{k_1 \geq k_2 \geq 10} 2^{k_1/2} \left\| P_{k_1} u \right\|_{F_{k_1}(T)}^2 \sum_{|k_0 - k| \leq 10} \left\| P_{k_0} u \right\|^2_{F_{k_0}(T)}\]
\[\lesssim 2^{2sk} \left\| P_k u \right\|^2_{F_k(T)} \left\| u \right\|^2_{F_{\frac{k_0}{2}}(T)}\]

We divide the summation over $k_3$ in $I_{2,1,2}$ by $\sum_{k_3 \leq k \leq 10} + \sum_{k_3 - k \leq 10} + \sum_{k_3 \geq k + 10}$. Then, by the support property, we know that the integral vanishes unless $|k' - k| \leq 10$ on the first and second summations and $k' \geq k + 10$ on the last summation. Note on the last summation that $|k' - k_3| \leq 10$.

On the first summation, the following cases of $k_1$ and $k_2$ (assuming $k_1 \leq k_2$ by the symmetry) are possible:

1. $k_1 \leq k - 10$ and $|k_2 - k| \leq 10$
2. $|k_1 - k_2| \leq 10$ and $k_2 \geq k + 10$
3. $|k_1 - k_2| \leq 10$ and $|k_2 - k| \leq 10$

It suffices to assume in the first case that $k_1 \leq k_3$, since two derivatives are taken in the $k_3$-frequency mode. We use (2.29) and (2.28) with the use of the weight (see Remark 2.11).
when \( k_1 = 0 \) and \( k_1 \geq 1 \), respectively, to estimate \( I_{2,1,2} \), precisely, \( I_{2,1,2} \) is bounded by

\[
2^{2sk} \| P_0 u \|_{F_0(T)} \sum_{k_3 \leq k - 10} 2^{k_3} \| P_{k_3} u \|_{F_{k_3}(T)} \sum_{|k_2 - k| \leq 10} \| P_{k_2} u \|^2_{F_{k_2}(T)} \\
+ 2^{2sk} \sum_{1 \leq k_1 \leq k_3 \leq 10} 2^{3k_1} 2^{k_2 - 1} \| P_{k_1} u \|_{F_{k_1}(T)} \| P_{k_3} u \|_{F_{k_3}(T)} \sum_{|k_2 - k| \leq 10} \| P_{k_2} u \|^2_{F_{k_2}(T)} \\
\lesssim 2^{2sk} \| P_k u \|^2_{F_k(T)} \| u \|^2_{F^2(T)}.
\]

Under the second case, by (2.30), \( I_{2,1,2} \) is bounded by

\[
2^{2sk} \sum_{k_3 \leq k - 10} 2^{2sk_3} 2^{k_2 - 2k_3} \prod_{j=1}^3 \| P_{k_j} u \|_{F_{k_j}(T)} \| P_k u \|_{F_k(T)} \\
\lesssim \max(2^{2k}, 2^{(s-1)k}) \| P_k u \|_{F_k(T)} \| u \|_{F^2(T)}^3,
\]

for \( s \geq 0 \).

Under the last case, by (2.27), \( I_{2,1,2} \) is bounded by

\[
2^{2sk} \sum_{k_3 \leq k - 10} 2^{5k_3} \prod_{j=1}^3 \| P_{k_j} u \|_{F_{k_j}(T)} \| P_k u \|_{F_k(T)} \\
\lesssim 2^{2sk} \| P_k u \|^2_{F_k(T)} \| u \|_{F^2(T)}^{5/2}.
\]

On the second summation, the possible cases of \( k_1 \) and \( k_2 \) are same as before. Using (2.27), (2.30) and (2.26), one concludes that \( I_{2,1,2} \) is bounded by

\[
2^{2sk} \sum_{k_1 \leq k - 10} 2^{2k_1/2} \prod_{j=1}^3 \| P_{k_j} u \|_{F_{k_j}(T)} \| P_k u \|_{F_k(T)} \\
+ 2^{2sk} \sum_{|k_1 - k| \leq 10 \atop |k_2 - k| \leq 10} 2^{2sk_2} \prod_{j=1}^3 \| P_{k_j} u \|_{F_{k_j}(T)} \| P_k u \|_{F_k(T)} \\
+ 2^{2sk} \sum_{|k_1 - k| \leq 10 \atop |k_2 - k| \leq 10} 2^{2sk} \prod_{j=1}^3 \| P_{k_j} u \|_{F_{k_j}(T)} \| P_k u \|_{F_k(T)} \\
\lesssim 2^{2sk} \| P_k u \|^2_{F_k(T)} \| u \|^2_{F^2(T)}.
\]

On the last summation, the following cases of \( k_1 \) and \( k_2 \) (assuming \( k_1 \leq k_2 \) by the symmetry) are possible:

1. \( k_1 \leq k_2 - 10 \) and \( |k_2 - k_3| \leq 10 \)
2. \( |k_1 - k_2| \leq 10 \) and \( k_2 \geq k_3 + 10 \)
3. \( |k_1 - k_2| \leq 10 \) and \( |k_2 - k_3| \leq 10 \)
Since $k$-frequency is the lowest frequency, hence one similarly or easily has

\[
2^{2s}k \sum_{k_1 \leq k_2 - 10 \atop \left| k_2 - k_3 \right| \leq 10} 2^{2k_3} 2^{2k_1} 2^{k_3/2} \sum_{j=1}^{3} \left\| P_{k_j} u \right\|_{F_k(T)} \left\| P_k u \right\|_{F_k(T)}
\]

\[
+ 2^{2s}k \sum_{k_2 \geq k_2 - 10 \atop \left| k_1 - k_2 \right| \leq 10} 2^{2k_3} 2^{k_2} 2^{k_3/2} \sum_{j=1}^{3} \left\| P_{k_j} u \right\|_{F_k(T)} \left\| P_k u \right\|_{F_k(T)}
\]

\[
+ 2^{2s}k \sum_{\left| k_1 - k_2 \right| \leq 10 \atop \left| k_2 - k_3 \right| \leq 10 \atop k_3 \geq k + 10} 2^{2k_3} 2^{k_2} 2^{k_3/2} \sum_{j=1}^{3} \left\| P_{k_j} u \right\|_{F_k(T)} \left\| P_k u \right\|_{F_k(T)}
\]

\[
\lesssim \max \left( 2^{15 \frac{s}{2}}, 2 \left( 2^{\frac{27}{2} - s} \right)^k \right) \left\| P_k u \right\|_{F_k(T)} \left\| u \right\|_{F^s(T)}^3
\]

for $s \geq \frac{9}{8}$, thanks to Lemma 2.11 (c) and (b).

The estimate of $I_{2,2}$ is very similar as before. In view of the estimate of $I_{2,1}$, one knows that the worst case appears when the frequency support of $u_{xx}$ is $I_k$. However, a direct calculation (integration by parts) gives

\[
\left\| \int_{\mathbb{R} \times [0,t_k]} P_k(u)(P_k'(u^2)P_k(u_{xx}))_x \, dx \, dt \right\| = \left\| \int_{\mathbb{R} \times [0,t_k]} (P_k(u))_x (P_k'(u^2)P_k(u_{xx})) \, dx \, dt \right\| = \frac{1}{2} \left\| \int_{\mathbb{R} \times [0,t_k]} ((P_k(u))_x)^2 (P_k'(u^2))_x \, dx \, dt \right\|
\]

which is exactly same as $I_{2,1}$ (in particular, $I_{2,1,1}$ and $I_{2,1,2}$ under $|k_3 - k| \leq 10$). The rigorous justification of this observation can be seen in the commutator estimates, see the proof of Lemma 4.1 (c) in [26] for the details or see the proof of Proposition 2.13 below. Moreover, one can see that the derivatives are fairly distributed in $I_{2,3}$, and hence it can be easily or similarly controlled as the estimate of $I_{2,1}$. We omit the details.

On the other hand, the remaining part,

\[
2^{2s}k \left\| \int_{\mathbb{R} \times [0,t_k]} P_k(u) P_k(F(u)) \, dx \, dt \right\|
\]

where $F(u) = (u^p)_x$, $p = 2, 4, 5$, can be immediately handled by using

\[
\left\| \int_{\mathbb{R} \times [0,t_k]} \prod_{j=1}^{p+1} u_j \, dx \, dt \right\| \lesssim 2^{(k_1 + \cdots + k_{p+1})/2} \prod_{j=1}^{p+1} \left\| u_j \right\|_{F_k(T)}
\]

where $u_j = P_{k_j} u \in F_{k_j}(T)$, $j = 1, \cdots, p + 1$ and assuming that $k_1 \leq \cdots \leq k_{p+1}$, for $p = 2, 4, 5$.

Collecting all, we have

\[
\sum_{k \geq 1} (I_2(k) + I_3(k)) \lesssim \left\| u \right\|_{F^s(T)}^2 + \left\| u \right\|_{F^s(T)}^4 + \left\| u \right\|_{F^s(T)}^5 + \left\| u \right\|_{F^s(T)}^6
\]

for $s \geq 2$, thus we complete the proof of (2.34).
Let \( u_1 \) and \( u_2 \) be solutions to (2.1). Define \( v = u_1 - u_2 \), then \( v \) solves
\[
v_t + v_{5x} + \mathcal{N}_2(u_1, u_2) + \mathcal{N}_3(u_1, u_2) + \mathcal{S}\mathcal{N}(u_1, u_2) = 0,
\]
\[
v(0, x) = u_1(0, x) - u_2(0, x),
\]
where
\[
\mathcal{N}_2(u_1, u_2) = \mathcal{N}_2(u_1) - \mathcal{N}_2(u_2), \quad \mathcal{N}_3(u_1, u_2) = \mathcal{N}_3(u_1) - \mathcal{N}_3(u_2) \quad \text{and}
\]
\[
\mathcal{S}\mathcal{N}(u_1, u_2) = \mathcal{S}\mathcal{N}(u_1) - \mathcal{S}\mathcal{N}(u_2).
\]

**Proposition 2.13** Let \( s \geq 2 \) and \( T \in (0, 1] \). Then, for solutions \( v \in C([-T, T]; H^s(\mathbb{T})) \) to (2.35) and \( u_1, u_2 \in C([-T, T]; H^s(\mathbb{T})) \) to (2.1), we have
\[
\|v\|^2_{E^0(T)} \lesssim \|v_0\|^2_{L^2} + \left( \sum_{j=1}^{2} \left( \|u_j\|_{F^s(T)}^2 + \|u_j\|_{F^s(T)}^2 + \|u_j\|_{F^s(T)}^2 + \|u_j\|_{F^s(T)}^2 \right) \right)
\]
and
\[
\|v\|^2_{F^s(T)} \lesssim \|v_0\|^2_{H^s} + \left( \sum_{j=1}^{2} \left( \|u_j\|_{F^s(T)}^2 + \|u_j\|_{F^s(T)}^2 + \|u_j\|_{F^s(T)}^2 + \|u_j\|_{F^s(T)}^2 \right) \right)
\]
\[
+ \left( \|u_1\|_{F^{2s}(T)} + \|u_2\|_{F^{2s}(T)} \right) \|v\|_{F^0(T)} \|v\|_{F^s(T)}.
\]

**Remark 2.12** One can see that the cubic terms with three derivatives are not harmful even in \( F^s \), while the same terms are the main enemy under the periodic boundary condition. The principal reason is due to the lack of the smoothing effect under the periodic condition. Compare Lemma 2.11 (c) and Lemma 6.4 (c) and (d) in [47].

**Proof** We first concentrate on the estimate on \( \|v\|^2_{E^0(T)} \). From the definition of \( \|v\|^2_{E^0(T)} \) and (2.25), it suffices to control
\[
\begin{align*}
2^{2sk} \|P_k(v(t_k))\|^2_{L^2} - 2^{2sk} \|P_k(v_0)\|^2_{L^2} &\lesssim 2^{2sk} \int_{[0,t_k]} P_k(v) P_k(\mathcal{N}_2(u_1, u_2)) \, dxdt \\
&\quad + 2^{2sk} \int_{[0,t_k]} P_k(v) P_k(\mathcal{N}_3(u_1, u_2)) \, dxdt \\
&\quad + 2^{2sk} \int_{[0,t_k]} P_k(v) P_k(\mathcal{S}\mathcal{N}(u_1, u_2)) \, dxdt \\
&=: 2^{2sk}\tilde{I}_1(k) + 2^{2sk}\tilde{I}_2(k) + 2^{2sk}\tilde{I}_3(k).
\end{align*}
\]
Proposition 4.4 in [26] yields
\[
\sum_{k \geq 1} \tilde{I}(k) \lesssim \left( \sum_{j=1}^{2} \left( \|u_j\|_{F^s(T)}^2 + \|u_j\|_{F^s(T)}^2 \right) \right) \|v\|^2_{F^0(T)}.
\]
and
\[
\sum_{k \geq 1} 2^{2sk} \tilde{I}(k) \lesssim \left( \sum_{j=1}^{2} \left( \| u_j \|_{F^1(T)}^2 + \| u_j \|_{F^3(T)}^2 \right) \right) \| v \|_{F^3(T)}^2 \\
+ \left( \| u_1 \|_{F^{3s}(T)}^2 + \| u_2 \|_{F^{3s}(T)}^2 \right) \| v \|_{F^0} \| v \|_{F^s(T)}.
\]
for \( s \geq 2 \). Moreover, since the quintic term in \( \mathcal{N}_3(u_1, u_2) \) and \( \mathcal{S}\mathcal{N}(u_1, u_2) \) contains only one derivative, one can easily handle them compared to the cubic term in \( \mathcal{N}_3(u_1, u_2) \). Thus, in what follows, we only focus on \( \tilde{I}_2(k) \) (in particular, the cubic terms), similarly as the proof of Proposition 2.12.

We write \( \tilde{I}_2(k) = \tilde{I}_{2,1} - \tilde{I}_{2,2} + \tilde{I}_{2,3} \), where
\[
\tilde{I}_{2,1} := \frac{1}{2} \left| \int_{\mathbb{R} \times [0, t_k]} P_k(v) P_k((u_1^2 + u_2^2)_x v_{xx} + (v(u_1 + u_2))_x (u_1 + u_2)_xx) \, dxdt \right|,
\]
\[
\tilde{I}_{2,2} := \frac{1}{2} \left| \int_{\mathbb{R} \times [0, t_k]} P_k(v) P_k((u_1^2 + u_2^2)v_{xx} + v(u_1 + u_2)(u_1 + u_2)_xx) \, dxdt \right|
\]
\[
(:= \tilde{I}_{2,2,1} + \tilde{I}_{2,2,2})
\]
and
\[
\tilde{I}_{2,3} := \left| \int_{\mathbb{R} \times [0, t_k]} P_k(v) P_k(v_x(u_1^2 + u_1, x u_2, x + u_2^2, x)) \, dxdt \right|.
\]

Here, we only consider \( \tilde{I}_{2,2} \) in order to provide a rigorous proof of the estimate of \( I_{2,2} \) in the proof of Proposition 2.12. Moreover, it is easier to handle \( \tilde{I}_{2,1,2} \) than \( \tilde{I}_{2,1,1} \) (or similar), since less derivatives are taken in \( v \), hence it is enough to estimate only \( \tilde{I}_{2,1,1} \). We reduce \( \tilde{I}_{2,1,1} \) as
\[
\left| \int_{\mathbb{R} \times [0, t_k]} P_k(v_x) P_k(u^2 v_{xx}) \, dxdt \right|.
\]
A direct calculation gives
\[
\left| \int_{\mathbb{R} \times [0, t_k]} P_k(v_x) P_k(u^2 v_{xx}) \, dxdt \right|
\leq \sum_{k' \leq k-10} \left| \int_{\mathbb{R} \times [0, t_k]} P_k(v_x) P_k(P_k'(u^2)v_{xx}) \, dxdt \right|
\]
\[
+ \sum_{k' \geq k-9, k_3 \geq 0} \left| \int_{\mathbb{R} \times [0, t_k]} P_k^2(v_x) P_k(u^2) P_k(v_{xx}) \, dxdt \right|.
\]
Since
\[
P_k(v_x) P_k(P_k'(u^2)v_{xx}) = P_k(v_x) P_k(v_{xx}) P_k'(u^2) + P_k(v_x)[P_k, P_k'(u^2)]v_{xx}
= \frac{1}{2} ((P_k(v_x))^2)_x P_k'(u^2) + P_k(v_x)[P_k, P_k'(u^2)]v_{xx},
\]
\( \square \) Springer
where \([A, B] = AB - BA\), the integration by parts yields
\[
\sum_{k' \leq k - 10} \int_{\mathbb{R} \times [0, t_k]} \left((P_k(v_x))^2\right) P_{k'}(u^2) \, dx \, dt \leq \sum_{k' \leq k - 10} \int_{\mathbb{R} \times [0, t_k]} \left((P_k(v_x))^2\right) P_{k'}((u^2)_x) \, dx \, dt,
\]
which is already considered in the proof of Proposition 2.12 (in particular, \(I_{2,1,1}\)). Thus, we have
\[
\sum_{k \geq 1} \sum_{k' \leq k - 10} \int_{\mathbb{R} \times [0, t_k]} \left((P_k(v_x))^2\right) P_{k'}((u^2)_x) \, dx \, dt \lesssim \|u\|^2_{F^s(T)} \|v\|^2_{F^0(T)}
\]
and
\[
\sum_{k \geq 1} 2^{2sk} \sum_{k' \leq k - 10} \int_{\mathbb{R} \times [0, t_k]} \left((P_k(v_x))^2\right) P_{k'}((u^2)_x) \, dx \, dt \lesssim \|u\|^2_{F^s(T)} \|v\|^2_{F^s(T)},
\]
for \(s \geq 2\).

On the other hand, a direct computation, in addition to the mean value theorem, (2.5) and (2.4), gives
\[
\mathcal{F} \left( [\tilde{P}_k, \tilde{P}_{k'}(u^2)](v_{xx}) \right)(\tau, \xi) = C \int_{\mathbb{R}^2} \mathcal{F}(\tilde{P}_{k'}((u^2)_x))(\tau', \xi') \cdot \mathcal{F}(v_x)(\tau - \tau', \xi - \xi') \cdot m(\xi, \xi') \, d\xi' d\tau',
\]
where,
\[
|m(\xi, \xi')| = \left| \frac{(\xi - \xi')(\chi_k(\xi) - \chi_k(\xi - \xi'))}{\xi'} \right| \lesssim |(\xi - \xi_1)\chi_k'(\xi - \theta \xi_1)| \lesssim \sum_{|k - k'| \leq 4} \chi_{k'}(\xi - \xi_1),
\]
for \(0 \leq \theta \leq 1\). Thus, an analogous argument yields
\[
\sum_{k \geq 1} \sum_{k' \leq k - 10} \int_{\mathbb{R} \times [0, t_k]} P_k(v_x)[P_k, P_{k'}(u^2)]v_{xx} \, dx \, dt \lesssim \|u\|^2_{F^s(T)} \|v\|^2_{F^0(T)}
\]
and
\[
\sum_{k \geq 1} 2^{2sk} \sum_{k' \leq k - 10} \int_{\mathbb{R} \times [0, t_k]} P_k(v_x)[P_k, P_{k'}(u^2)]v_{xx} \, dx \, dt \lesssim \|u\|^2_{F^s(T)} \|v\|^2_{F^s(T)},
\]
for \(s \geq 2\).

The rest of the proof, which is the estimate of
\[
\sum_{k' \geq k - 9, k_3 \geq 0} \int_{\mathbb{R} \times [0, t_k]} P_k^2(v_x) P_{k'}(u^2) P_{k_3}(v_{xx}) \, dx \, dt,
\]
is almost identical to the proof of the estimate of \(I_{2,1,2}\) in the proof of Proposition 2.12. Thus, we have
\[
\sum_{k \geq 1} \sum_{(2.36)} \lesssim \|u\|^2_{F^s(T)} \|v\|^2_{F^0(T)}
\]
and
\[ \sum_{k \geq 1} 2^{2sk} (2.36) \lesssim \|u\|_{F^s(T)}^2 \|v\|_{F^s(T)}^2, \]
for \( s \geq 2 \). Thus, we complete the proof. \( \square \)

### 2.6 Local and Global Well-Posedness

The local-well-posedness argument (the classical energy method) is now standard. We refer the readers to [26,30,32,35,47] and references therein, for more details, and we, here, give a sketch of proof.

We first state fundamental properties of \( X^s,b \)-type norms.

**Proposition 2.14** Let \( s \geq 0 \), \( T \in (0,1) \), and \( u \in F^s(T) \), then
\[ \sup_{t \in [-T,T]} \|u(t)\|_{H^s} \lesssim \|u\|_{F^s(T)} \]

**Proposition 2.15** Let \( T \in (0,1) \), \( u, v \in C([-T,T]:H^\infty) \) and
\[ \partial_t u + \partial_x^5 u = v \text{ on } \mathbb{R} \times (-T,T) \]
Then we have
\[ \|u\|_{F^s(T)} \lesssim \|u\|_{E^s(T)} + \|v\|_{N_s(T)}, \]
for any \( s \geq 0 \).

See Appendix in [26] for the proofs. We also refer to [30,32,35].

One can observe that (1.1) admits the scaling equivalence with (2.1): For \( \lambda > 0 \), if \( u \) is a solution to (1.1), then \( u_\lambda \), defined by
\[ u_\lambda(t,x) := \lambda u(\lambda^5 t, \lambda x), \]
is a solution to (2.1). Moreover, a direct calculation yields
\[ \|u_{0,\lambda}\|_{\tilde{H}^{s_c}} = \lambda^{s_c + \frac{1}{2}} \|u_0\|_{\tilde{H}^{s_c}}, \]
which says the scaling exponent \( s_c = -\frac{1}{2} \). Thus, a small data local well-posedness of (2.1) ensures the local-in-time well-posedness of (1.1) for an arbitrary data.

From Duhamel’s principle, we know that the solution to (2.1) is of the following integral form:
\[ u(t) = W(t)u_0 + \int_0^t W(t-s)(N_2(u)(s) + N_3(u)(s) + SN(u)(s)) \, ds, \]
where \( W(t) \) is defined as in (2.6). We assume
\[ \|u_0\|_{\tilde{H}^{s_c}} \leq \epsilon \ll 1. \] \hspace{1cm} (2.37)
Remark that for any fixed \( \mu \in \mathbb{R^+} \), we can choose \( 0 < \lambda_0 \ll 1 \) sufficiently small such that the initial data satisfy (2.37) and \( \mu \lambda \leq 1 \) for all \( \lambda \leq \lambda_0 \). The second condition ensures that our local well-posedness argument does not depend on \( \mu \).

We fix \( s \geq 2 \). Proposition 2.15, Propositions 2.7, 2.8 and 2.9, and Proposition 2.12 ensures
\[
\begin{align*}
\|u\|_{F^s(T)} & \lesssim \|u_0\|_{\tilde{H}^{s_c}} \quad \text{and} \quad \|u\|_{E^s(T)} \quad \text{for} \quad s \geq 2, \\
\|N_2(u) + N_3(u) + SN(u)\|_{N_s(T)} & \lesssim \sum_{j=2}^5 \|u\|_{F^s(T)}^j, \\
\|u\|_{E^s(T)}^2 & \lesssim \|u_0\|_{\tilde{H}^{s_c}}^2 \quad \text{and} \quad \sum_{j=3}^6 \|u\|_{F^s(T)}^j, \\
\end{align*}
\]
for any $T' \in [0, T]$, which, in addition to the smallness condition (2.37) and continuity argument (see Lemma 6.3 in [35] for the details), implies a priori bound:

$$\sup_{t \in [-T, T]} \|u(t)\|_{H^s} \lesssim \|u_0\|_{H^s}.$$  \hfill (2.38)

To complete the proof, we need

**Proposition 2.16** Assume $s \geq 2$. Let $u_1, u_2 \in F^s(T)$ be solutions to (2.1) with small initial data $u_{1,0}, v_{2,0} \in H^\infty$. Let $v = u_1 - u_2$ and $v_0 = u_{1,0} - u_{2,0}$. Then we have

$$\|v\|_{F^0(T)} \lesssim \|v_0\|_{L^2}$$

and

$$\|v\|_{F^s(T)} \lesssim \|v_0\|_{H^s} + \|u_{1,0}\|_{H^2s} \|v_0\|_{L^2}.$$  

It immediately follows from Proposition 2.15, Propositions 2.7, 2.8 and 2.9, and Proposition 2.13 under (2.38).

For fixed $u_0 \in H^s$, a density argument enables us to choose a sequence $(u_{0,n})_{n=1}^\infty \subset H^\infty$ such that $u_{0,n} \to u_0$ in $H^s$ as $n \to \infty$. Let $u_n(t) \in H^\infty$ is a solution to (2.1) with initial data $u_{0,n}$. Using a similar argument as above and Proposition 2.16, one shows $\{u_n\}$ is a Cauchy sequence. Indeed, for $K \in \mathbb{Z}_+$, let $u^K_{0,n} = P_{\leq K}(u_{0,n})$. Then, $u^K_n = P_{\leq K}u_n$ satisfies the frequency localized equation $(P_{\leq K}(2.1))$ with the initial data $u^K_{0,n}$.

We have from the triangle inequality that

$$\sup_{t \in [-T, T]} \|u_m - u_n\|_{H^s} \lesssim \sup_{t \in [-T, T]} \|u_m - u^K_m\|_{H^s} + \sup_{t \in [-T, T]} \|u^K_m - u^K_n\|_{H^s}$$

$$+ \sup_{t \in [-T, T]} \|u^K_n - u_n\|_{H^s}.$$  

The first and last terms are bounded by $\epsilon$, thanks to a priori bound, and the second term is bounded by $\epsilon$, thanks to Proposition 2.13, precisely,

$$\sup_{t \in [-T, T]} \|v^K_m - v^K_n\|_{H^s(T)} \lesssim \|v^K_{0,m} - v^K_{0,n}\|_{H^s} + K^s \|v^K_{0,m} - v^K_{0,n}\|_{L^2} \lesssim \epsilon.$$  

Hence the $3\epsilon$-argument completes the proof and we obtain a solution as the limit. The uniqueness and the continuity of dependence follow from an analogous argument.

**Remark 2.13** In view of all analyses above, we do not use the integrability of the Gardner Eq. (2.1) to prove the local well-posedness, and thus we can apply our argument to prove the local result of (2.1) with arbitrary coefficients.

Small solutions $u$ to (2.1) satisfies the (rescaled) conservation laws (1.4)-(1.5)-(1.6), namely

$$M[u](t) := \frac{1}{2} \int_{\mathbb{R}} u^2(t, x)dx = M[u](0),$$

$$E_\mu[u](t) := \int_{\mathbb{R}} \left( \frac{1}{2} u_x^2 - 2\mu \lambda u^3 - \frac{1}{2} u^4 \right) (t, x)dx = E_\mu[u](0),$$
and

$$E_{5\mu}[u](t) := \int_{\mathbb{R}} \left( \frac{1}{2} u_{xx}^2 - 10\mu \lambda u_x^2 + 10\mu^2 \lambda^2 u^4 - 5u_x^2 u^2 + 6\mu \lambda u^5 + u^6 \right)(t, x)dx = E_{5\mu}[u](0).$$  \hspace{1cm} (2.39)

Using above conserved quantities and the Sobolev embedding in addition to the smallness condition, one proves Theorem 1.3.

3 Stability of Breathers of the 5th Order Gardner Equation

Once we have shown the existence of global solutions of the Cauchy problem for the 5th order Gardner Eq. (1.1), we study stability properties of a special solution. This is usually called as breather solution, and it was presented in Definition 1.7. Moreover, it satisfies the following nonlinear identities:

**Lemma 3.1** Let $B_{\mu} \equiv B_{\mu,5}$ be the breather solution (1.7) of the 5th order Gardner Eq. (1.1). Then

1. $B_{\mu} = \tilde{B}_{\mu,x}$, with $\tilde{B}_{\mu} = \tilde{B}_{\alpha,\beta,\mu}$ given by the smooth $L^\infty$-function

$$\tilde{B}_{\mu}(t, x) := 2 \arctan \left( \frac{G_{\mu}}{F_{\mu}} \right).$$

2. For any fixed $t \in \mathbb{R}$, we have $(\tilde{B}_{\mu})_t$ well-defined in the Schwartz class, satisfying

$$B_{\mu,4x} + \tilde{B}_{\mu,t} + 10(\mu + B_{\mu})^2 B_{\mu,xx} + 10(\mu + B_{\mu}) B_{\mu,x}^2 + 6(10\mu^3 B_{\mu}^2 + 10\mu^2 B_{\mu}^3 + 5\mu B_{\mu}^4 + B_{\mu}^5) = 0.$$ \hspace{1cm} (3.1)

**Proof** The first item above is a direct consequence of the definition of $B_{\mu}$ in (1.7). On the other hand, (3.1) is a consequence of (1.1) and integration in space (from $-\infty$ to $x$) of (1.1). \hfill \square

Finally, we show that breather solutions (1.7) of the 5th order Gardner Eq. (1.1) satisfy the following non trivial identity, which as far as we know, it is not known for higher order breathers (7th and higher order):

**Lemma 3.2** Let $B_{\mu} \equiv B_{\mu,5}$ be the breather solution (1.7) of the 5th order Gardner Eq. (1.1). Then, for all $t \in \mathbb{R}$,

$$\tilde{B}_{\mu,t} = (\alpha^2 + \beta^2)^2 B_{\mu} + 2(\alpha^2 - \beta^2 - 5\mu^2) \left( B_{\mu,xx} + 2B_{\mu}^3 + 6\mu B_{\mu}^2 \right).$$ \hspace{1cm} (3.2)

**Proof** See appendix B for a detailed proof of this nonlinear identity. \hfill \square

Now using identity (3.2), we prove that breather solutions (1.7) of the 5th order Gardner Eq. (1.1) satisfy a fourth order ODE, which indeed is the same as the one satisfied by classical Gardner breather solutions (see [3, Theorem 3.5] for further details)

**Theorem 3.3** Let $B_{\mu} \equiv B_{\mu,5}$ be the breather solution (1.7) of the 5th order Gardner Eq. (1.1). Then, for any fixed $t \in \mathbb{R}$, $B_{\mu}$ satisfies the nonlinear stationary equation

$$\mathcal{W}(B_{\mu}) := B_{\mu,4x} - 2(\beta^2 - \alpha^2)(B_{\mu,xx} + 6\mu B_{\mu}^2 + 2B_{\mu}^3) + (\alpha^2 + \beta^2)^2 B_{\mu} + 10B_{\mu} B_{\mu,x} + 10B_{\mu} B_{\mu,xx} + 6B_{\mu}^5 + 10\mu B_{\mu}^2 B_{\mu,xx} + 40\mu^2 B_{\mu}^3 + 30\mu B_{\mu}^4 = 0.$$ \hspace{1cm} (3.3)
Therefore, \( H_{\mu} \) and \( B_{\mu} \), we can introduce a \( H_{\mu,xx} \) where in the last line we have used the identity (3.2).

**Proof** We use the identity (3.1) to substitute the \( B_{\mu,4x} \) term in the left-hand side of (3.3), simplifying it as:

\[
\mathcal{W}(B_{\mu}) = -\left( \bar{B}_{\mu,t} + 10(\mu + B_{\mu})^2 B_{\mu,xx} + 10(\mu + B_{\mu}) B_{\mu,xx}^2 \right.
+ 6(10\mu B_{\mu}^2 + 10\mu^2 B_{\mu}^3 + 5\mu B_{\mu}^4 + B_{\mu}^5) \\
- 2(\beta^2 - \alpha^2)(B_{\mu,xx} + 6\mu B_{\mu}^2 + 2B_{\mu}^3) + (\alpha^2 + \beta^2)^2 B_{\mu} + 10B_{\mu} B_{\mu,xx} \\
+ 6B_{\mu}^5 + 10B_{\mu} B_{\mu,xx} + 20B_{\mu} B_{\mu,xx} + 40\mu^2 B_{\mu}^3 + 30\mu B_{\mu}^4 \\
\left. = - \bar{B}_{t} + (\alpha^2 + \beta^2)^2 B_{\mu} + 2(\alpha^2 - \beta^2 - 5\mu^2)(B_{\mu,xx} + 2B_{\mu}^3 + 6\mu B_{\mu}^2) = 0, \right. \\
\]

where in the last line we have used the identity (3.2).

Note that being the shift parameters \( x_1, x_2 \) in (1.7) selected as independent of time, a simple argument guarantees that the previous Theorem 3.3 still holds under time dependent, translation parameters \( x_1(t) \) and \( x_2(t) \).

**Corollary 3.4** Let \( B_{\mu}^0 \equiv B_{\alpha,\beta,\mu}^0(t, x; 0, 0) \) be any Gardner breather as in (1.7), and \( x_1(t), x_2(t) \in \mathbb{R} \) two continuous functions, defined for all \( t \) in a given interval. Consider the modified breather

\[
B_{\mu}(t, x) := B_{\alpha,\beta,\mu}^0(t, x; x_1(t), x_2(t)), \quad \text{(cf. (1.7)).}
\]

Then \( B_{\mu} \) satisfies (3.3), for all \( t \) in the given interval.

**Proof** From the invariance of the Eq. (3.3) under spatial translations, we conclude. \( \square \)

Even more, we can characterize variationally these breather solutions of the 5th order Gardner equation. Explicitly, considering the \( H^2(\mathbb{R}) \) conserved quantity (1.6)

\[
E_{5\mu}[u](t) := \int_{\mathbb{R}} \left( \frac{1}{2} u_{xx}^2 - 10\mu uu_x^2 + 10\mu^2 u^4 - 5u^2 u_x^2 + 6\mu u^5 + u^6 \right) dx, \quad (3.4)
\]

we can introduce a \( H^2 \) functional, associated to the breather solution. Namely, we define this functional as a linear combination of the energy (1.5), the mass (1.4) and (3.4) in the following way

\[
\mathcal{H}_{\mu}[u](t) := E_{5\mu}[u](t) + 2(\beta^2 - \alpha^2) E_{\mu}[u](t) + (\alpha^2 + \beta^2)^2 M[u](t). \quad (3.5)
\]

Therefore, \( \mathcal{H}_{\mu}[u] \) is a conserved quantity, well-defined for \( H^2 \)-solutions of (1.1). Additionally, we have that

**Lemma 3.5** Breather solutions \( B_{\mu} \) (1.7) of the 5th order Gardner Eq. (1.1) are critical points of the Lyapunov functional \( \mathcal{H}_{\mu} \) (3.5). In fact, for any \( z \in H^2(\mathbb{R}) \) with sufficiently small \( H^2 \)-norm, and \( B_{\mu} = B_{\alpha,\beta,\mu} \) any 5th Gardner breather solution, one has, for all \( t \in \mathbb{R} \), that

\[
\mathcal{H}_{\mu}[B_{\mu} + z] - \mathcal{H}_{\mu}[B_{\mu}] = \frac{1}{2} Q_{\mu}[z] + N_{\mu}[z],
\]

with \( Q_{\mu} \) being the quadratic form defined in (3.6) below, and \( N_{\mu}[z] \) satisfying \( |N_{\mu}[z]| \leq K \|z\|^3_{H^2(\mathbb{R})} \).
Then there exist $x_1$, $x_2$ associated to the linearized operator $L$ for a constant $K > 0$. Then, there exist positive parameters $\eta$ with $\eta > 0$ such that the solution $u(t)$ of the Cauchy problem for the 5th order Gardner Eq. (1.1) with initial data $u_0$ satisfies

$$\sup_{t \in \mathbb{R}} \| u(t) - B_\mu(t; x_1(t), x_2(t)) \|_{H^2(\mathbb{R})} \leq A_0 \eta,$$

with

$$\sup_{t \in \mathbb{R}} |x_1'(t)| + |x_2'(t)| \leq K A_0 \eta,$$

for a constant $K > 0$.  

\[\Box\] Springer
Proof We take \( u = u(t) \in H^2(\mathbb{R}) \) as the corresponding local in time solution of the Cauchy problem associated to (1.1), with initial condition \( u(0) = u_0 \in H^2(\mathbb{R}) \). Therefore, once we guaranteed for the case of the breather solution of the 5th order Gardner equation, that it satisfies the same 4th order ODE (3.3) as the classical Gardner breather, that a suitable coercivity property holds for the bilinear form \( Q_\mu \) associated to the breather solution of (1.1), and the existence of a unique negative eigenvalue of the linearized operator \( L_\mu \) given in (3.7), the stability proof follows the same steps as the \( H^2 \)-stability of classical Gardner breathers [3, Theorem 6.1] (see also [7]). Namely, we proceed assuming that the maximal time of stability \( T \) is finite and we arrive to a contradiction. \( \square \)

Appendix A. Proof of Theorem 1.4

The aim of this section is to prove the weak ill-posedness of (1.1) for \( s > 0 \), which, in addition to the first author’s recent work [4], completely justifies that the 5th Gardner Eq. (1.1) is a quasilinear equation in the sense that the flow map from data to solutions is not (locally) uniformly continuous for all regularities, see Corollary 1.5. Since the weak-ill-posedness phenomenon occurs due to the strong high-low interaction in the quadratic nonlinearity with three derivatives, Theorem 1.2 in [49] seems to guarantee the lack of uniform continuity of the flow map associated to (1.1) for \( s > 0 \). This section contributes to prove that the Eq. (1.1) is indeed weakly ill-posed for \( s > 0 \).

The proof basically follows the argument used in [49], initially introduced by Koch-Tzvetkov [45]. Since the (weak) ill-posedness phenomenon arises from the strong high-low quadratic nonlinearity (high frequency waves with low frequency perturbations), the main part of the proof is identical to the argument in [49]. Thus, we, here, provide an additional estimate to be needed for the other nonlinearities.

In view of the argument presented in Sect. 2.6, it suffices to show the ill-posedness of (2.1) with small initial data.

A.1 Setting

We first define the approximate solution, which is an ansatz to cause the (weak) ill-posedness phenomenon. Let \( \phi, \tilde{\phi} \in C^\infty_0(\mathbb{R}) \) be smooth bump functions satisfying

\[
\phi \equiv 1, \quad |x| < 1, \quad \text{and} \quad \phi \equiv 0, \quad |x| > 2
\]

and

\[
\tilde{\phi} \equiv 1, \quad x \in \text{supp}(\phi) \quad \text{and} \quad \tilde{\phi} \phi \equiv \phi,
\]

respectively. For \( N \geq 1 \) and \( 0 < \delta < 1 \), set

\[
\phi_N(x) := \phi\left(\frac{x}{N^4+\delta}\right), \quad \tilde{\phi}_N(x) := \tilde{\phi}\left(\frac{x}{N^4+\delta}\right).
\]

Let \( \epsilon > 0 \) be a sufficiently small for the initial data to satisfy (2.37). Let

\[
u_{0,t}^\pm(x) := \pm \epsilon N^{-3} \phi_N(x)
\]

and \( u_0^\pm(t, x) \) be the solution to (2.1) with the initial data \( u_{0,t}^\pm(x) \). Let \( \Phi_N(t) := (N^5 - 10\mu^2\lambda^2 N^3) t \) and

\[
\nu_{h}^\pm(t, x) := N^{-\frac{4+\delta}{2}} \phi_N(x) \cos(Nx - \Phi_N(t) \mp t)
\]

(A.1)
be a high frequency part of the approximate solution, and thus define the approximate solution as
\[ u_{ap}^\pm(t, x) := u_l^\pm(t, x) + u_r^\pm(t, x). \]

Then the main task is to prove the following proposition:

**Proposition A.1** *(Proposition 6.2 in [49]) Let \( \max(0, 2 - 2s) < \delta < 1 \). Let \( u_N^\pm \) be the unique solution to (2.1) with initial data
\[ u_N^\pm(0, x) = \pm \epsilon N^{-3} \phi_N(x) + N^{-4+\delta} \phi_N(x) \cos(Nx). \]

Then, we have
\[ \| u_N^\pm - u_{ap}^\pm \|_{H^s} = o(1), \] (A.2)
for \( s > 0 \) and \( |t| < 1 \), as \( N \to \infty \).

Once (A.2) holds true, one conclude that
\[ \| u_N^+ - u_N^- \|_{H^s} = N^{-4+\delta} \]
\[ \| \phi_N(x) \cos(Nx - \Phi_N(t) - t) - \cos(Nx - \Phi_N(t) + t) \|_{H^s} + o(1) \]
\[ = 2N^{-4+\delta} \| \phi_N(x) \sin(Nx - \Phi_N(t)) \|_{H^s} | \sin t | + o(1), \]
which, in addition to Lemma A.2 below, implies
\[ \lim_{N \to \infty} N^{-4+\delta} \| \phi_N(x) \sin(Nx + \gamma) \|_{H^s} = c_0 \| \phi \|_{L^2}, \]
for some \( c_0 > 0 \).

We recall from [45,49] the following useful lemmas to prove Proposition A.1.

**Lemma A.2** *(Lemma 2.3 in [45]) Let \( s \geq 0 \), \( \delta > 0 \) and \( \gamma \in \mathbb{R} \). Then,
\[ \lim_{N \to \infty} N^{-4+\delta} \| \phi_N(x) \sin(Nx + \gamma) \|_{H^s} = c_0 \| \phi \|_{L^2}, \]
for some \( c_0 > 0 \).

**Lemma A.3** *(Lemma 6.3 in [49]) Let \( K \) be a positive integer and \( K - 2 - s \geq k \geq 0 \). Then, we have
\[ \| \partial_x^k u_{l_0^\pm}(t, \cdot) \|_{L^2} \lesssim K N^{-2-s-k(4+\delta)} \] (A.3)
\[ \| \partial_x^k u_{l_0^\pm}(t, \cdot) \|_{L^\infty} \lesssim K N^{-3-k(4+\delta)} \] (A.4)
\[ \| u_{l_0^\pm}(t, \cdot) - u_{l_0^\pm}(\cdot) \|_{L^2} \lesssim K N^{-15-3\delta} \] (A.5)

**Proof** The proof of (A.3) and (A.4) follows from a direct computation and Theorem 1.2, in particular, a priori bound (2.38). Moreover, the proof of (A.5) follows from a direct calculation in (2.1) and (A.3)–(A.4). The proof is almost identical to the proof of Lemma 6.3 in [49], thus we omit the details.
Lemma A.4 Let
\[ P^\pm(t, x) := u^\pm_{ap,t} + u^\pm_{ap,3x} + 10\mu^2\lambda^2u^\pm_{ap,3x} + N_2(u^\pm_{ap}) + N_3(u^\pm_{ap}) + SN(u^\pm_{ap}), \quad (A.6) \]
where \( N_2(\cdot), N_3(\cdot) \) and \( SN(\cdot) \) are defined as in (2.2)-(2.3), respectively. Let \( s > 0 \), \( 0 < \delta < 2 \) and \( |t| \leq 1 \). Then, we have
\[ \| P^\pm(t, \cdot) \|_{L^2_x} \lesssim N^{-s-\delta} + N^{\frac{2\delta}{2} - 2s} + N^{-1 - \delta - 3s} + N^{1 - \frac{3(4+\delta)}{2} - 4\delta - 2s} + N^{1 - 2(4+\delta) - 5s} \tag{A.7} \]
Moreover, if \( \sigma > 0 \), we have
\[ \| P^\pm(t, \cdot) \|_{H^s_x} \lesssim N^{-s-\delta+\sigma} + N^{\frac{2\delta}{2} - 2s+\sigma} + N^{-1 - \delta - 3s+\sigma} + N^{1 - \frac{3(4+\delta)}{2} - 4\delta+\sigma} + N^{1 - 2(4+\delta) - 5s+\sigma}. \tag{A.8} \]

Proof It suffices to consider \( P^+ \), since an identical argument holds true for \( P^- \). We drop the super-index +. We decompose \( P \) into \( P_1 + P_2 \), where \( P_2 = N_3(u_{ap}) - N_3(u_l) + SN(u_{ap}) - SN(u_l) \) and \( P_1 = P - P_2 \). Lemma 6.4 in [49] exactly shows (A.7) and (A.8) for \( P_1 \). Our setting of \( \phi, \tilde{\phi} \) and \( u_l \) is essential to deal with
\[ \Lambda := \epsilon N^{-\frac{4+\delta}{2} - s} \phi_N(x) (\partial_t + \partial_x^5 + 10\mu^2\lambda^2\partial_x^3 + \epsilon^{-1} u_l^3 \partial_x^3 \cos (N x - \Phi_N(t) - t) \]
contained in \( P_1 \) (compared to \( F_4 \) in the proof of Lemma 6.4 in [49]). Indeed, a direct calculation in addition to \( u_{0,l}(x) := \epsilon N^{-3}\tilde{\phi}_N(x) \) and \( \phi \tilde{\phi} = \phi \) gives
\[ \Lambda = N^{-\frac{4+\delta}{2} - s} \phi_N(x) (u_l N^3 - \epsilon) \sin (N x - \Phi_N(t) - t) \]
\[ = N^{-\frac{4+\delta}{2} - s} N^3 \phi_N(x) (u_l - u_{0,l}) \sin (N x - \Phi_N(t) - t), \]
which is handled by using (A.5). Thus, it suffices to show (A.7) and (A.8) for \( P_2 \). Putting first \( u_{ap} = u_l + u_h \) into \( 10u^2_{ap}u_{ap,3x} - 10u^2_l u_l,x \) in \( P_2 \), one has
\[ 10u^2_l u_{h,xxx} + 20u_l u_h u_{l,xxx} + 20u_l u_h u_{h,xxx} + 10u^2_l u_{l,xxx} + 10u^2_h u_{h,xxx}. \tag{A.9} \]
Note that
\[ u_{h,x} = N^{-\frac{4+\delta}{2} - s} (\partial_x \phi_N(x) \cos (N x - \Phi_N(t) - t) + \phi_N(x) \partial_x \cos (N x - \Phi_N(t) - t)) \]
\[ = N^{-\frac{4+\delta}{2} - s} \left( N^{-\frac{4+\delta}{2} - s} \phi_{N,x}(x) \cos (N x - \Phi_N(t) - t) \right) \]
\[ - N \phi_N(x) \sin (N x - \Phi_N(t) - t). \]
Thus, one can see that the worst term arises from the case when the derivative acts on \( \cos (N x - \Phi_N(t) - t) \). Using Lemmas A.2 and A.3, one estimates
\[ \|(A.9)\|_{L^2_x} \lesssim N^{-3-s} + N^{-\frac{4+\delta}{2} - 2s} + N^{-1 - \delta - 3s}. \]

A small difference between \( P_1 \) and \( F \) in Lemma 6.4 in [49] does not make any trouble. Indeed, our setting of \( u_h \) corresponds to (2.1), so that one can immediately apply the argument in the proof of Lemma 6.4 in [49] to our case. Moreover, the cubic term with one derivative in \( N_2(u_{ap}) \) can be dealt with similarly as \( SN(u_{ap}) \).
An analogous argument yield
\[
\|u^{3}_{ap,x} - u^{3}_{l,x}\|_{L^2} \lesssim N^{-5-2(4+\delta)-s} + N^{-1-\frac{3(4+\delta)}{2}-2s} + N^{-1-\delta-3s},
\]
\[
\|u_{ap}u_{ap,x}u_{ap,xx} - u_{l}u_{l,x}u_{l,xx}\|_{L^2} \lesssim N^{-8-\delta-s} + N^{-\frac{4+\delta}{2}-2s} + N^{-1-\delta-3s},
\]
\[
\|u^{4}_{ap}u_{ap,x} - u^{4}_{l}u_{l,x}\|_{L^2} \lesssim N^{-11-s} + N^{-8-\frac{4+\delta}{2}-2s} + N^{-5-(4+\delta)-3s} + N^{-2-\frac{3(4+\delta)}{2}-4s} + N^{1-2(4+\delta)-5s},
\]
\[
\|u_{ap}u_{ap,x} - u_{l}u_{l,x}\|_{L^2} \lesssim N^{-2-s} + N^{-\frac{4+\delta}{2}-2s}
\]
and
\[
\lim_{N \to \infty} \|w^+\|_{H^s} = o(1) \quad \text{as} \quad N \to \infty \quad \text{and} \quad \text{drop the super-index}+.
\]
For \( s \geq 2 \), the local well-posedness theory is available. A direct calculation gives
\[
\Gamma w + N_2(u_N) - N_2(u_{ap}) + N_3(u_N) - N_3(u_{ap}) + SN(u_N) - SN(u_{ap}) + \mathcal{P} = 0,
\]
where \( \Gamma := \partial_t + \partial^5 + 10\mu^2x^2\partial_x^3 \) and \( \mathcal{P} \) is as in (A.6). For \( 2 \leq \sigma \), the local well-posedness, in particular (2.38), ensures
\[
\|u_N\|_{C_T H^\sigma} + \|u_N\|_{F^\sigma(T)} \lesssim \|u_{N}(0)\|_{H^\sigma} \lesssim N^{\sigma-s}. \tag{A.10}
\]
Moreover, a direct calculation and the local theory (for \( u_{l} \)) gives
\[
\|u_{ap}\|_{C_T H^\sigma} + \|u_{ap}\|_{F^\sigma(T)} \lesssim N^{\frac{2\delta}{2}+\frac{2\delta}{2}} + N^{\sigma-s}. \tag{A.11}
\]
Using Propositions 2.15, Propositions 2.7, 2.8, 2.9 and 2.13, and (A.7) under (A.10) and (A.11), one concludes
\[
\|w\|_{F^0(T)} \lesssim \|\mathcal{P}\|_{L^1_T L^2_x} = O(N^{-s-\beta}),
\]
for \( \beta = \min(\delta, -\frac{2-\delta}{2} + s) > 0 \), which, in addition to Proposition 2.14, implies
\[
\|w\|_{L^\infty_T L^2_x} = O(N^{-s-\beta}). \tag{A.12}
\]
Furthermore, an analogous argument (but using (A.8) instead of (A.7)) in addition to
\[
\|u_{ap}\|_{F^2s(T)} \|w\|_{F^0(T)} = O(N^{s}N^{-s-\beta}) = O(N^{-\beta}),
\]
ensures \( \|w\|_{F^1(T)} = O(N^{-\beta}) \), which concludes (A.2) as \( N \to \infty \) for \( s \geq 2 \).

To fill the regularity range \( 0 < s < 2 \), we use the conservation law and the interpolation theorem. \( H^2 \) conservation law (2.39) and a direct calculation yield
\[
\|u_{N}\|_{H^2} \lesssim N^{2-s} \quad \text{and} \quad \|u_{ap}\|_{H^2} \lesssim N^{2-s},
\]
respectively, which concludes
\[ \|w\|_{H^s} \lesssim N^{2-s}. \]  
\[(A.13)\]

The interpolation between (A.12) and (A.13) ensures
\[ \|w\|_{H^s} \lesssim \|w\|^\frac{1-s}{2}_{L^2} \|w\|^\frac{s}{H^3} \lesssim N^{-\frac{\beta(2-s)}{s}}, \]
which proves (A.2) as \( N \to \infty \) for \( 0 < s < 2 \).

## 4 Appendix B. Proof of Lemma 3.2.

We are going to prove the identity (3.2)
\[ \tilde{B}_{\mu,t} = (\alpha^2 + \beta^2) B_{\mu} + 2(\alpha^2 - \beta^2 - 5\mu^2)(B_{\mu,xx} + 2B_{\mu}^3 + 6\mu B_{\mu}^3). \]

Firstly and for the sake of simplicity, we will use the following notation:
\[
\begin{align*}
A_1 &:= (\alpha^2 + \beta^2)^2, & A_2 &:= 2(\alpha^2 - \beta^2 - 5\mu^2), \\
\Delta &:= \alpha^2 + \beta^2 - 4\mu^2, & e^z &= \cosh(z) + \sinh(z), \\
D &:= f^2 + g^2, & \text{where } f, g \text{ and its derivatives are given by:}
\end{align*}
\]

\[
\begin{align*}
f &= \cosh(\beta y_2) - \frac{2\beta\mu}{\alpha \sqrt{\alpha^2 + \beta^2}} (\alpha \cos(\alpha y_1) - \beta \sin(\alpha y_1)), \\
f_1 &= f_x = \beta \sinh(\beta y_2) + \frac{2\beta\mu}{\alpha^2 + \beta^2} (\beta \cos(\alpha y_1) + \alpha \sin(\alpha y_1)), \\
f_2 &= f_t = \beta y_5 \sinh(\beta y_2) + \frac{2\beta\mu}{\alpha^2 + \beta^2} (\beta \cos(\alpha y_1) + \alpha \sin(\alpha y_1)), \\
f_3 &= f_{xx} = \beta^2 \cosh(\beta y_2) + \frac{2\alpha\beta\mu}{\alpha^2 + \beta^2} (-\alpha \cos(\alpha y_1) + \beta \sin(\alpha y_1)), \\
f_4 &= f_{xxx} = \beta^3 \sinh(\beta y_2) - \frac{2\alpha^2\beta\mu}{\alpha^2 + \beta^2} (\beta \cos(\alpha y_1) + \alpha \sin(\alpha y_1))
\end{align*}
\]

and
\[
\begin{align*}
g &= \frac{\beta\sqrt{\alpha^2 + \beta^2}}{\alpha \sqrt{\Delta}} \sin(\alpha y_1) - \frac{2\beta\mu e^{\beta y_2}}{\Delta}, \\
g_1 &= g_x = \frac{\beta\sqrt{\alpha^2 + \beta^2}}{\sqrt{\Delta}} \cos(\alpha y_1) - \frac{2\beta^2 \mu e^{\beta y_2}}{\Delta}, \\
g_2 &= g_t = \frac{\beta y_5 \sqrt{\alpha^2 + \beta^2}}{\sqrt{\Delta}} \cos(\alpha y_1) - \frac{2\beta^2 \gamma_5 e^{\beta y_2}}{\Delta}, \\
g_3 &= g_{xx} = -\frac{\alpha\beta \sqrt{\alpha^2 + \beta^2}}{\sqrt{\Delta}} \sin(\alpha y_1) - \frac{2\beta^3 \mu e^{\beta y_2}}{\Delta}, \\
g_4 &= g_{xxx} = -\frac{\alpha^2 \beta \sqrt{\alpha^2 + \beta^2}}{\sqrt{\Delta}} \cos(\alpha y_1) - \frac{2\beta^4 \mu e^{\beta y_2}}{\Delta},
\end{align*}
\]
where velocities \((\gamma_5, \delta_5)\) are given in (1.7). From the explicit expression of the breather solution (1.7) but now written in terms of the above derivatives (B.1)–(B.2), we obtain that:

\[
B_{\mu} = 2 \frac{g_1 f - f_1 g}{D} \quad \text{and} \quad \tilde{B}_{\mu,t} = 2 \frac{g_2 f - f_2 g}{D}.
\] (B.3)

Moreover we get

\[
B_{\mu}^2 = 4 \left( \frac{g_1 f - f_1 g}{D} \right)^2 \quad \text{and} \quad \tilde{B}_{\mu}^3 = 8 \left( \frac{g_1 f - f_1 g}{D} \right)^3.
\] (B.4)

Now, we compute \(B_{\mu,xx}\). First we get

\[
B_{\mu,x} = -\frac{2}{D^2} \left( f^3 g_3 - f^2 (2 f_1 g_1 + f_3 g) + f g (2 f_1^2 + g g_3 - 2 g_1^2) + g^2 (2 f_1 g_1 - f_3 g) \right),
\]

and then

\[
B_{\mu,xx} = \frac{2 M_1}{D^3},
\] (B.5)

where

\[
M_1 := \left( f^5 g_4 - f^4 (3 f_1 g_3 + 3 f_3 g_1 + f_4 g) + 2 f^3 (3 f_1^2 g_1 + 3 f_1 f_3 g + g^2 g_4 - 3 g g_1 g_3 - g_1^3) - 2 f^2 g (3 f_1^3 - 9 f_1 g_1^2 + f_4 g)^2 ight. + f g^2 \left( -18 f_1^2 g_1 + 6 f_1 f_3 g + g^2 g_4 - 6 g g_1 g_3 + 6 g_1^3 \right) + g^3 (2 f_1^3 + f_1 (3 g g_3 - 6 g_1^2) + g (3 f_3 g_1 - f_4 g)) \right).
\] (B.6)

and therefore from (B.3), (B.4), (B.5) and (B.6), we get

\[
A_1 B_{\mu} + A_2 (B_{\mu,xx} + 2 B_{\mu}^3 + 6 \mu B_{\mu}^2) = \frac{M_2}{D^3},
\] (B.7)

where

\[
M_2 := 2 \left( A_1 D^2 (f g_1 - f_1 g) + A_2 (8 (f g_1 - f_1 g)^3 + 12 \mu D (f g_1 - f_1 g)^2 + M_1) \right).
\] (B.8)

Now, we verify that, after expanding \(f'\)’s and \(g'\)’s terms (B.1)–(B.2) and lengthy rearrangements, the above term (B.8) simplifies as follows:

\[
M_2 = 2 D^2 (g_2 f - g f_2).
\]

Finally, remembering (B.7), we have that

\[
A_1 B_{\mu} + A_2 (B_{\mu,xx} + 2 B_{\mu}^3 + 6 \mu B_{\mu}^2) = \frac{M_2}{D^3} = \frac{2 D^2 (g_2 f - g f_2)}{D^3} = \tilde{B}_{\mu,t},
\]

and we conclude.
References

1. Ablowitz, M., Clarkson, P.: Solitons, Nonlinear Evolution Equations and Inverse Scattering, London Mathematical Society Lecture Note Series, vol. 149. Cambridge University Press, Cambridge (1991)
2. Alejo, M.A.: On the ill-posedness of the Gardner equation. J. Math. Anal. Appl. 396(1), 256–260 (2012)
3. Alejo, M.A.: Nonlinear stability of Gardner breathers. J. Differ. Equ. 264(2), 1192–1230 (2018)
4. Alejo, M.A., Cardoso, E.: On the ill-posedness of the 5th-order Gardner equation. São Paulo J. Math. Sci. (2019). https://doi.org/10.1007/s40863-019-00150-7
5. Alejo, M.A., Cardoso, E.: Dynamics of Breathers in the Gardner hierarchy: Universality of the variational characterization, preprint arXiv:1901.10409v1
6. Alejo, M.A., Cortez, M. F., Kwak, C., Muñoz, C.: On the dynamics of zero-speed solutions for Camassa-Holm type equations. Int. Math. Res. Not. IMRN 2021(9), 6543–6585 (2021)
7. Alejo, M.A., Muñoz, C.: Nonlinear stability of mKdV breathers. Commun. Math. Phys. 37, 2050–2080 (2013)
8. Alejo, M.A., Muñoz, C.: Dynamics of complex-valued modified KdV solitons with applications to the stability of breathers. Anal. PDE 8(3), 629–674 (2015)
9. Alejo, M.A., Muñoz, C., Palacios, J.M.: On the variational structure of breather solutions I: Sine-Gordon equation. J. Math. Anal. Appl. 453(2), 1111–1138 (2017)
10. Alejo, M.A., Muñoz, C., Palacios, J. M.: On the variational structure of breather solutions II: Periodic mKdV equation. Electron. J. Differ. Equ., Paper No. 56, p 26 (2017)
11. Bourgain, J.: Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. Parts I, II. Geom. Funct. Anal. 3 (1993) 107–156, 209–262
12. Bourgain, J.: On the Cauchy problem for periodic KdV-type equations, Proceedings of the Conference in Honor of Jean-Pierre Kahane (Orsay, 1993). J. Fourier Anal. Appl. Special Issue, 17–86 (1995)
13. Burq, N., Gérard, P., Tzvetkov, N.: An instability property of the nonlinear Schrödinger equation on $S^d$. Math. Res. Lett. 9(2–3), 323–335 (2002)
14. Burq, N., Gérard, P., Tzvetkov, N.: Two singular dynamics of the nonlinear Schrödinger equation on a plane domain. Geom. Funct. Anal. 13(1), 1–19 (2003)
15. Cavalcante, M., Kwak, C.: The initial-boundary value problem for the Kawahara equation on the half-line. Nonlinear Differ. Equ. Appl. 27, 45 (2020). https://doi.org/10.1007/s00030-020-00648-6
16. Cavalcante, M., Kwak, C.: Local well-posedness of the fifth-order KdV-type equations on the half-line. Commun. Pure Appl. Anal. 18(5), 2607–2661 (2019)
17. Chen, W., Guo, Z.: Global well-posedness and I-method for the fifth-order Korteweg-de Vries equation. J. Math. Anal. 114, 121–156 (2011)
18. Chen, W., Li, J., Miao, C., Wu, J.: Low regularity solutions of two fifth-order KdV type equations. J. Anal. Math. 107, 221–238 (2009)
19. Christ, M., Colliander, J., Tao, T.: Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations. Amer. J. Math. 125(6), 1235–1293 (2003)
20. Christ, M., Colliander, J., Tao, T.: A priori bounds and weak solutions for the nonlinear Schrödinger equation in Sobolev space of negative order. J. Funct. Anal. 254, 368–395 (2008)
21. Cui, S., Tao, S.: Strichartz estimates for dispersive equations and solvability of the Kawahara equation. J. Math. Anal. Appl. 304, 683–702 (2005)
22. Gardner, C.S., Kruskal, M.D., Miura, R.: Korteweg-de Vries equation and generalizations. II. Existence of conservation laws and constants of motion. J. Math. Phys. 9(8), 1204–1209 (1968)
23. Gomes, J.F., França, G.S., Zimerman, A.H.: Nonvanishing boundary condition for the mKdV hierarchy and the Gardner equation. J. Phys. A: Math. Theor. 45, 015207 (2012)
24. Guo, Z.: Local well-posedness and a priori bounds for the modified Benjamin-Ono equation. Adv. Differ. Equ. 16(11–12), 1087–1137 (2011)
25. Guo, Z.: Local well-posedness for dispersion generalized Benjamin-Ono equations in Sobolev spaces. J. Differ. Equ. 252, 2053–2084 (2012)
26. Guo, Z., Kwak, C., Kwon, S.: Rough solutions of the fifth-order KdV equations. J. Funct. Anal. 265, 2791–2829 (2013)
27. Guo, Z., Oh, T.: Non-existence of solutions for the periodic cubic NLS below $L^2$. Int. Math. Res. Not. IMRN 2018(6), 1656–1729 (2018)
28. Grimshaw, R., Slunyaev, A., Pelinovsky, E.: Generation of solitons and breathers in the extended Korteweg-de Vries equation with positive cubic nonlinearity. Chaos 20(1), 01310201–01310210 (2010)
29. Grünrock, A.: On the hierarchies of higher order mKdV and KdV equations. Cent. Eur. J. Math. 8(3), 500–536 (2010)
30. Guo, Z., Peng, L., Wang, B., Wang, Y.: Uniform well-posedness and inviscid limit for the Benjamin-Ono-Burgers equation. Adv. Math. 228, 647–677 (2011)
31. Ionescu, A., Kenig, C.: Global well-posedness of the Benjamin-Ono equation in low-regularity spaces. J. Amer. Math. Soc. 20(3), 753–798 (2007)
32. Ionescu, A., Kenig, C., Tataru, D.: Global well-posedness of the KP-I initial-value problem in the energy space. Invent. Math. 173(2), 265–304 (2008)
33. Kakutani, T.: Weakly nonlinear Hydromagnetic waves in a cold collision free plasma. J. Phys. Soc. Japan 26, 5 (1969)
34. Kato, T.: Well-posedness for the fifth order KdV equation. Funkcialaj Ekvacioj 55(1), 17–53 (2012)
35. Kenig, C., Pilod, D.: Well-posedness for the fifth-order KdV equation in the energy space. Trans. Amer. Math. Soc. 367, 2551–2612 (2015)
36. Kenig, C., Pilod, D.: Local well-posedness for the KdV hierarchy at high regularity. Adv. Diff. Eq. 21, 801–836 (2016)
37. Kenig, C.E., Ponce, G., Vega, L.: On the hierarchy of the generalized KdV equations, Singular limits of dispersive waves (Lyon: NATO Adv. Sci. Inst. Ser. B Phys., vol. 320. Plenum, New York 1994, 347–356 (1991)
38. Kenig, C.E., Ponce, G., Vega, L.: Well-posedness of the initial value problem for the Korteweg-de Vries equation. J. Amer. Math. Soc. 4(2), 323–347 (1991)
39. Kenig, C.E., Ponce, G., Vega, L.: Oscillatory integrals and regularity of dispersive equations. Indiana U. Math. J 40, 33–69 (1991)
40. Kenig, C.E., Ponce, G., Vega, L.: Higher-order nonlinear dispersive equations. Proc. Amer. Math. Soc. 122(1), 157–166 (1994). https://doi.org/10.2307/2160855
41. Kenig, C.E., Ponce, G., Vega, L.: On the ill-posedness of some canonical dispersive equations. Duke Math. J. 106(3), 617–633 (2001)
42. Kichenassamy, S., Olver, P.J.: Existence and non-existence of solitary wave solutions to higher order model evolution equations. SIAM J. Math. Anal. 3, 1141–1166 (1992)
43. Koch, H., Tzvetkov, N.: A priori bounds for the 1D cubic NLS in negative Sobolev spaces, Int. Math. Res. Not. IMRN 16 (2007), Art. ID rnm053, 36. DOI https://doi.org/10.1093/imrn/rnm053. MR2353092 (2010d:35307)
44. Koch, H., Tataru, D.: On finite energy solutions of the KP-I equation. Int. Math. Res. Not. IMRN 16 (2007), Art. ID rnm053, 36. DOI https://doi.org/10.1093/imrn/rnm053. MR2353092 (2010d:35307)
45. Lamb, G.L.: Elements of Soliton Theory. Pure Appl. Math, Wiley, New York (1980)
46. Linares, F.: A higher order modified Korteweg-de Vries equation. Comput. Appl. Math. 14(3), 253–267 (1995)
47. Marchant, T.R., Smyth, N.F.: The extended Korteweg-de Vries equation and the resonant flow of a fluid over topography. J. Fluid Mech. 221, 263–288 (1990)
48. Matsuno, Y.: Bilinearization of nonlinear evolution equations: Higher Order mKdV. J. Phys. Soc. Jpn, 49, no. 2 (1980)
49. Molinet, L., Pilod, D., Vento, S.: Unconditional uniqueness for the modified Korteweg-de Vries equation on the line, to appear in Rev. Mat. Iber (2018)
50. Molinet, L., Pilod, D., Vento, S.: On unconditional well-posedness for the periodic modified Korteweg-de Vries equation, to appear in J. Math. Soc. Japan (2018)
51. Muñoz, C., Saut, J.C., Tzvetkov, N.: Ill-posedness issues for the Benjamin-Ono and related equations. SIAM J. Math. Anal. 33, 982–988 (2001)
52. Muñoz, C., Saut, J.C., Tzvetkov, N.: Well-posed and ill-posed results for the Kadomtsev-Petviashvili-I equation. Duke Math. J. 115(2), 353–384 (2002)
53. Muñoz, C.: Instability in nonlinear Schrödinger breathers. Proyecciones 36(4), 653–683 (2017)
54. Muñoz, C., Palacios, J. M.: Nonlinear stability of 2-solitons of the Sine-Gordon equation in the energy space. Ann. IHP Analyse Nonlinéaire 36(4), 977–1034 (2019)
55. Muñoz, C., Ponce, G.: Breathers and the dynamics of solutions to the KdV type equations. Commun. Math. Phys. 367(2), 581–598 (2019)
62. Olver, P.J.: Hamiltonian perturbation theory and water waves. Contemp. Math. 28, 231–249 (1984)
63. Pilod, D.: On the Cauchy problem for higher-order nonlinear dispersive equations. J. Differ. Equ. 245(8), 2055–2077 (2008). https://doi.org/10.1016/j.jde.2008.07.017
64. Ponce, G.: Lax pairs and higher order models for water waves. J. Differ. Equ. 102(2), 360–381 (1993)
65. Tao, T.: Multilinear weighted convolution of $L^2$ functions and applications to nonlinear dispersive equations. Amer. J. Math. 123(5), 839–908 (2001)
66. Tsugawa, K.: *Parabolic smoothing effect and local well-posedness of fifth-order semilinear dispersive equations on torus*, Harmonic analysis and nonlinear partial differential equations, 177-193, RIMS, Kyoto, 2016. arXiv:1707.09550 [math.AP]
67. Wadati, M.: The modified Korteweg-de vries equation. J. Phys. Soc. Jpn. 34(5), 1289–1296 (1973)

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