Evaluation of energy consumption for the object motion optimal control

A Bokhonsky¹,* and N Varminskaya²

¹ Department of Technical Mechanics and Mechanical Engineering, Sevastopol State University, Sevastopol, 299038, Russian Federation
² Department of Physics and General Technical Disciplines, Black Sea Higher Naval School named after P.S.Nakhimov, Sevastopol, 299028, Russian Federation

* bohon.alex@mail.ru

Abstract. The energy asymptotic reduction for the optimal control of the “acceleration-deceleration” type (with increasing of the degree of the polynomial that sets the type of optimal control) while realizing the motion purpose is revealed and investigated. For a fixed minimum possible motion time, the object is moved as an absolutely solid or deformable body from the initial state of quiescence to the final state of absolute quiescence. It is shown that the acceptable motion time of an elastic object is found from the moment ratios - the equality to zero of the relative displacement and the relative velocity at the end of the motion. Using the example of a system with one degree of freedom, it is shown that the moment relations are a system of transcendental equations with many mutual roots. The considered class of controls is applicable for the elastic objects optimal motion in modern technology.

1. Introduction

Optimal control methods for solving tasks of object motion usually involve the use of reasonable optimality criteria, the compilation of differential equations of object motion, taking into account constraints and, finally, the use of well-known methods of the classical optimal control theory.

Monographs by A G Butkovsky, N N Krasovsky, I A Karnovsky, Yu M Postman, V A Troitsky, F L Chernousko [1 – 5] and others are devoted to the optimal control of the motion and oscillations of elastic systems.

An important milestone in the progress of the optimal control theory was the discovery of the Pontryagin maximum principle and its subsequent effective application to solving of topical engineering problems [6].

A number of practical results were obtained in the design of optimal controls [7 – 11] based on the reversal principle of optimality (RPO), which implies the implementation of the following procedures: the formation of the purpose of motion and constraints; motion task (for example, displacement or acceleration functions in the polynomial form); determination of polynomial constants and subsequent factorization; according to the found motion function, obtaining the Euler-Poisson equations (as the first variation of the functional-criterion); designing of the functional criterion according to the relevant rules; verification of sufficient conditions for an extremum of a reversationally constructed functional (using the conditions of Jacobi, Weierstrass, Legendre), construction of a graphical image of the functional.
In fact, this way is a complete solution to the variational calculus problem (and therefore, in our opinion, it can be called the reversal calculus).

The purpose of the research is to evaluate the effect of the optimal control type of an object motion on energy consumption for its practical implementation.

Note that the reversionally constructed functionals-criterions [11] have an extremum (minimum). Below is an example of the use of sufficient conditions for estimating the functional extremum whose function (under the integral) is the square of the first derivative of the control.

2. Jacobi condition
An admissible curve must belong to the field of extremals and be a solution to the Euler equation, from which the Jacobi equation follows. For the functional \[ \int_{0}^{T} \left[ \dot{U}_e \right]^2 dt \] the Jacobi equation is

\[ \left( F_{U,U_e} - \frac{d}{dt} F_{U,U_e} \right) \tau(t) - \frac{d}{dt} \left[ F_{U,U_e} \cdot \dot{\tau}(t) \right] = 0, \quad F = \left[ \dot{U}_e \right]^2, \quad F_{U,U_e} = 0; \quad F_{U,U} = 0; \]

where

\[ F_{U,U} = 2, \quad \frac{d}{dt} \left[ F_{U,U_e} \cdot \dot{\tau}(t) \right] = 2\ddot{\tau}(t) = 0 \]

and therefore the Jacobi equation

\[ \frac{d^2 \tau(t)}{dt^2} = 0 \]

has a solution

\[ \tau(t) = C_1 t + C_2. \]

Since \( t = 0, \ \tau(0) = 0 \), then \( C_2 = 0, \ \tau(t) = C_1 t. \)

It may be noted that \( S_e(0) = 0, \ \tau(t) = \frac{dS_e(t,C)}{dC}, \) then at \( t = 0, \ \tau(0) = 0. \)

The function \( \tau(t) \) vanishes only at \( t = 0; \) the condition is satisfied, i.e. \( \tau(t) \) can be included in the central field of extremals with a center at \((0,0)\. So, the function \( \tau(t) = C_1 t \) belongs to the central field of extremals.

3. Weierstrass condition
For the studied functional, if the function \( E(t,\dot{U},p) \geq 0 \), then the functional has a minimum. So the Weierstrass function corresponds to the functional \( \int_{0}^{T} (t,\dot{U}_e^2, p) dt \) and

\[ E(t,\dot{U},p) = F(t,\dot{U}) - F(t,p) - (\dot{U} - p), \]

\[ F_p(t,p) = \dot{U}^2 - p^2 - 2p\dot{U} + p^2 = (\dot{U} - p)^2. \]

Therefore \( (\dot{U} - p)^2 \geq 0 \), i.e. the Weierstrass condition confirming the existence of a minimum of the functional-criterion is satisfied.

4. Legendre condition
In this case, the condition has the form \( F_{U,U} = 2 \geq 0 \) and confirms the minimum of the functional.

In the well-known example of the optimal motion control using the variational method [8] for the
optimality criterion \[ J = \int_0^T U^2 \, dt, \]
where \[ \frac{d^2S}{dt^2} = U(t), \]
\( T \) – the total motion time, taking into account the purpose of the motion \( S(0) = 0, \dot{S}(0) = 0, S(T) = L, \dot{S}(T) = 0, \)
the following formulations for displacement, velocity and acceleration (control per unit mass) are obtained:
\[ S(t) = \frac{LT^2}{T^3} (3T - 2t); \quad V(t) = \frac{6Lt}{T^3} (T - t); \quad U(t) = \frac{6L}{T^3} (T - 2t) = a\left(\frac{T - 2t}{T}\right), \]
where \( a = \frac{6L}{T^2} \), \( L \) – the total displacement of the object in time \( T \).

Some possible generalization of the approach, leading to a qualitatively new result (significant savings in energy consumption for the control), is not excluded. As in the previous example, it is necessary to move the object from the initial state of quiescence to the new state of quiescence at a distance \( L \) for a fixed time \( T \).

Let’s present the control in the most general form:
\[ U(t) = \frac{a(T - 2t)^n}{T^n}, \]
where \( n \) is the integral odd number (\( n = 1,3,5,7 \ldots \)). Then, integrating expression (1) twice, we find the arbitrary constants from conditions \( S(0) = 0, \dot{S}(0) = 0, \)
and the acceleration from a condition \( S(T) = L, \)
we receive values of displacement, velocity and acceleration (control) for the any \( n \):
\[ S(t) = \frac{a(2Tnt - T^2 + T^n(T - 2t)^n + 4Tt)}{2(n^2 + 3n + 2)}, \quad V(t) = -\frac{a(T - 2t)^{n+1}}{n+1}, \]
\[ U(t) = \frac{a(T - 2t)^n}{T^n}, \quad a = \frac{4(n^2 + 3n + 2)LT^n}{2T^{n+1} \cdot T^n - T^{n+2} + (-T)^{n+2} + 4T^{n+1} \cdot T}. \]

The verification shows that for \( n = 1 \) the known case (3) follows, and for \( n = 3 \) and \( n = 5 \) follow
cases which are earlier received with use of the difficult reversion constructioning of controls (taking
into account the motion purpose, restrictions and properties of a skew symmetry of the controls [7 – 11]).

When \( n = 3 \) from (4) a particular case of solution follows that is constructed using RPO [11]:
\[ U = a\left(\frac{T - 2t}{T}\right)^3, \quad V = a \cdot \frac{t(T^3 - 3T^2t + 4T^3 + 2t^3)}{T^3}, \]
\[ S = \frac{LT^2}{T^3} (5T^3 - 10T^2t + 10T^3 + 4t^3), \]
where \( a = \frac{10L}{T^2} \).

If, for example, \( n = 5 \), then expressions for displacement, velocity, and acceleration are obtained,
which coincide with the result of reversal control designing
\[ S = \frac{LT^2}{3T^3} \cdot (21T^5 - 70T^4t + 140T^3t^2 - 168T^2t^3 + 112Tt^4 - 32t^5), \]
\[ V = \frac{14LT^2}{3T^3} \cdot (3T^3 - 15T^2t + 40T^3t^2 - 60T^2t^3 + 48Tt^4 - 16t^5), \]
\[ U = a(T - 2t)^5 / T^5, \]
where \( a = \frac{14L}{T^2} \).
The combined graphics of displacement, velocity and acceleration (control) for \( n = 1, 3, 5 \) are shown in Figure 1.

For separate cases \( (n = 1, 3, 5, 7) \) the motion generalized characteristics are calculated according to dependences:

- Energy \( A = 2 \int_0^T U V \, dt \),
- Influence principle (the Lagrange form) \( J = \int_0^T V^2 \, dt \),
- Multiplicative energy criterion \( EC = A J \),
- Control impulse \( P = 2 \int_0^T U(t) \, dt \).

![Figure 1](image)

The results are presented in table 1.

**Table 1. Generalized characteristics of the optimal controlled object motion**

| \( n \) | Energy \( A, J \) | Influence principle \( J, J \cdot s \) | Multiplicative energy criterion \( EC = A J, J^2 \cdot s \) | Control impulse \( P, \text{m}^2/\text{s} \) |
|---|---|---|---|---|
| 1 | 2.25 | 1.2 | 2.7 | 3 |
| 3 | 1.56 | 1.111 | 1.736 | 2.5 |
| 5 | 1.361 | 1.077 | 1.466 | 2.333 |
| 7 | 1.266 | 1.058 | 1.340 | 2.25 |
| 101 | 1.009 | 1.002 | 1.011 | 2.009 |

According to the results given in Table 1 it is observed the asymptotic regularity of exponential decrease of numerical values of the generalized characteristics while the polynomial degree increases.

For example, at \( L = 1 \text{ m}, T = 1 \text{ s} \) with increasing \( n (n \to \infty) \) characteristics of \( A, J, EC \) aspire to 1, and \( P \) aspires to 2.
The trend of energy consumption decreasing of optimum operated control attracts attention. With use of input data of Table 1 function \( A = A(n) \) is approximated with analytical one \( A = a + be^{-cn} \).

After determining the constants \( a, b, c \) (with the least squares method) the dependence \( A(n) = 1.0 + 1.2e^{-0.19n} \) was obtained, the graphic is shown in Figure 2.

![Figure 2. Control energy reduction depending on polynomial degree](image)

The relative motion (oscillations) of an elastic system with one degree of freedom with optimal portable motion is investigated. Skew-symmetric controls of this type are applicable for optimal portable motion (for example, acceleration and deceleration) of elastic systems.

The differential equation of the relative motion of an elastic system with one degree of freedom without resistance is written in the form:

\[
\frac{d^2 x_r}{dt^2} + k^2 x_r = -\frac{14L(T-2t)^5}{T^7},
\]

where \( k = \frac{2\pi}{T_1} \) is the natural frequency, \( T_1 \) – the period of free oscillations.

The motion starts from the state of quiescence \( (x_r(0) = 0, \dot{x}_r(0) = 0) \). After integration it was received:

\[
x_r(t) = \frac{140L \cdot \left(-48k^2T^2 + 384 + T^4k^4\right) \cdot \sin(kt) + \frac{14L}{T^6k^6} \cdot \left(-80k^2T^2 + 1920 + T^4k^4\right) \cdot \cos(kt) - \frac{14L(1920 + (T-2t)^4k^4) - 80(T-2t)^2k^2) (T-2t)}{T^7k^6}}{\pi^2 k^2}.
\]

The moment relations \( x_r(T) = 0, \dot{x}_r(T) = 0 \) are written accounting \( T = nT_1, \ k = 2\pi/T_1 \).

For the maximum displacement and total motion time were taken \( L = 1 \text{ m}, \ T = 1 \text{ s} \).

Now the moment ratios depend only on the parameter \( n \) and represent a system of transcendental equations:

\[
x_r(T) = 0, \quad \dot{x}_r(T) = 0.
\] (5)

The mutual roots of system (5) on the interval \( 4 \geq n \geq 1 \) are the intersection points of the functions \( x_r(n) \) and \( \dot{x}_r(n) \) on the x-axis. After transformation the moment relations (as a system of transcendental equations) have the form:

\[
\left(n^4 \pi^4 - 12n^2 \pi^2 + 24\right) \sin 2\pi n + \left(20n^3 \pi^3 - 120n\pi - n^4 \pi^4\right) \cos 2\pi n + 20n^3 \pi^3 - 120n\pi - n^5 \pi^5,
\]

\[
\left(n^4 \pi^4 - 12n^2 \pi^2 + 24\right) \cos 2\pi n + \left(20n^3 \pi^3 + 120n\pi + n^4 \pi^4\right) \sin 2\pi n + 60n^2 \pi^2 - 120 - 5n^4 \pi^4 = 0.
\]

Graphic interpretation of moment ratios is represented in Figure 3.

\( x_r(n), \dot{x}_r(n) \)
Figure 3. Dependencies of moment ratios on the polynomial degree (three mutual roots in a given interval)

One of the mutual roots is specified graphically \( n = 2.2804 \) and, therefore, the total motion time \( T = nT_1 = 2.2804 \). Graphics of the relative motion \( x_r(t) \) and \( x_r'(t) \) are represented in Figure 5, and graphics of the portable motion \( S_e(t) \), \( V_e(t) \), \( U_e(t) \) respectively in Figure 6.

\[ x_r(n), \ x_r'(n) \]

\[ S_e(t), \ V_e(t), \ U_e(t) \]

Figure 4. Graphic root specification

Figure 5. Relative object motion (oscillations)

Figure 6. Portable object motion: \( S_e(t) \) – displacement, \( V_e(t) \) – velocity, \( U_e(t) \) – acceleration

As one would expect, at the moment of time \( t = T \), the portable velocity \( V_e(T) = 0 \), the relative displacement and velocity are zero: \( x_r(T) = 0 \), \( x_r'(T) = 0 \).

In this case, the equality \( |U_e(T)| = |a_r(T)| \) is true, which indicates that with control deactivation there is the absolute quiescence state. The obtained decision confirms the achievement of the motion purpose and energy savings on its practical implementation.

5. Conclusions

1. Increasing the odd degree of the polynomial, which describes the optimal control (acceleration-deceleration type) of the object motion, the control energy asymptotically approaches a certain limit
value, at which the motion purpose is achieved for a given fixed time.

2. The skew-symmetric controls of the considered type are applicable for optimal portable motion of elastic systems with the choice of an acceptable minimum motion time from the set of mutual roots of a system of transcendental equations into which the moment ratios of relative movement turn (equality to zero of relative displacement and velocity at the end of motion).

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