NORMALIZED SOLUTIONS OF HIGHER-ORDER SCHRÖDINGER EQUATIONS

ALIANG XIA* AND JIANFU YANG

Department of Mathematics, Jiangxi Normal University
Nanchang, Jiangxi 330022, China

(Communicated by Congming Li)

Abstract. In this paper, we consider the existence of non-trivial solutions for the following equation

\[ H_{0,J} u = |u|^{p-2} u + \lambda u \quad \text{in} \quad \mathbb{R}^3, \]

where \( H_{0,J} \) is the higher-order Schrödinger operator with \( J \in \mathbb{N}, \quad 2 < p < \frac{4J+6}{J+2}, \)
and \( \lambda \in \mathbb{R} \) is a parameter. Let \( E(u) \) be the corresponding variational functional of problem (1).
We look for solutions of equation (1) by finding minimizers of the minimization problem

\[ E_\rho = \inf \{ E(u) | u \in H^J(\mathbb{R}^3) : \|u\|_{L^2(\mathbb{R}^3)} = \rho \}. \]

We show that problem (1) admits at least a solution provided that in the case \( J \) being odd, \( 2 < p < 3 \) and \( \rho > 0 \) small or \( 2 + J < p < \frac{4J+6}{J+2} \) and \( \rho > 0 \) large;
and for the case \( J \) being even, \( 3 < p < \frac{4J+6}{J+2} \) and \( \rho > 0 \) small.

1. Introduction. In this paper, we investigate the higher-order Schrödinger equation

\[ H_{0,J} u = |u|^{p-2} u + \lambda u \quad \text{in} \quad \mathbb{R}^3, \]

where \( H_{0,J} \) is the higher-order kinetic energy operator given by

\[ H_{0,J} = -\sum_{j=0}^{J} \frac{\alpha(j)\hbar^{2j}}{m^{2j-1}c^{2j-2}} \Delta^j, \]

with \( J \in \mathbb{N}. \) The constant \( \hbar = \frac{\hbar}{\sqrt{\pi}} \) denotes the reduced Planck constant, \( c \) is the speed of light in the vacuum, and

\[ \alpha(j) = \frac{(2j-2)!}{j!(j-1)!2^{2j-1}} \quad \text{for} \quad j \geq 1 \]

with \( \alpha(0) = -1. \) The operator \( H_{0,J} \) can be regarded as the finite approximation of the operator

\[ H_{0,\infty} = -\sum_{j=0}^{\infty} \frac{\alpha(j)\hbar^{2j}}{m^{2j-1}c^{2j-2}} \Delta^j, \]

2010 Mathematics Subject Classification. Primary: 35J30, 35J35.

Key words and phrases. Higher-order Schrödinger equations, normalized solutions, existence.

The first author is supported by the Foundation of Jiangxi Provincial Education Department, No: GJJ160335, the NNSF of China, No: 11701239 and the Program for Cultivating Youths of Outstanding Ability in Jiangxi Normal University. The second author is supported by the NNSF of China, Nos: 11671179 and 11771300.

* Corresponding author: Aliang Xia.
while the operator $\mathcal{H}_{0\infty}$ stems from the study of the pseudo-relativistic operator $\sqrt{-c^2\hbar^2\Delta + m^2c^4}$. Indeed, to describe the motion of a fast moving free particle, one has to take into account the effect of the speed of light, then from the Einstein’s mass-energy equivalence we obtain the kinetic energy

$$E = \sqrt{p^2c^2 + m^2c^4}$$

(3)

of a free particle of mass $m$ and momentum $p$. Applying the correspondence principle

$$E \leftrightarrow i\hbar \frac{\partial}{\partial t} \quad p \leftrightarrow -i\hbar \frac{\partial}{\partial x} = -i\hbar \nabla,$$

(4)

we obtain the pseudo-relativistic equation

$$i\hbar \frac{\partial \psi}{\partial t} = \sqrt{-c^2\hbar^2\Delta + m^2c^4} \psi$$

with the non-local pseudo-differential operator $\sqrt{-c^2\hbar^2\Delta + m^2c^4}$, which was studied in [17]. Further study of nonlinear problems with the operator $\sqrt{-c^2\hbar^2\Delta + m^2c^4}$ can be found in [5, 16, 21] etc.

Historically, in order to avoid the non-local operator, one squares both sides of (3) and obtains

$$E^2 = p^2c^2 + m^2c^4,$$

which yields the Klein-Gordon equation

$$\hbar \frac{\partial^2 \psi}{\partial t^2} = -c^2\hbar^2\Delta \psi + m^2c^4 \psi.$$  

(5)

Since (5) is of second order derivative in time, it cannot be interpreted as a Schrödinger type equation for a quantum state. Another way in this consideration is so called the Dirac equation, formulated by Dirac [7], that gave the fundamental relativistic equation of first order in time. On the other hand, in [4], using the Taylor series expansion of (3) in $p$, and by the correspondence principle (4), Carles and Moulay obtain the Schrödinger form equation

$$i\hbar \frac{\partial \psi}{\partial t} = mc^2 \psi - \sum_{j=0}^{\infty} \frac{a(j)\hbar^{2j}}{m^{2j-1}c^{2j-2}} \Delta^j \psi.$$  

(6)

The convergence of finite approximation $\mathcal{H}_{0J}$ to $\mathcal{H}_{0\infty}$ was considered in [4]. It is proved in [4] that equation (6) holds if $2v^2 < c^2$. Schrödinger type equations involving a higher-order Schrödinger operator and converging towards the semi-relativistic bound-state equation is known as the spinless Salpeter equation, it was studied in [11, 12], see also [6, 10, 13, 14] and references therein. The Cauchy problem of the higher-order Schrödinger equations without potential, i.e., for free particles, is considered in [4, 15]. It is treated in [4] the case of bounded potentials, e.g., particles in finite potential wells, and of linear potentials, that is, neutrons in free fall in the gravity field and electrons accelerated by an electric field. While in [3], it is studied the Hartree-Fock equations with harmonic-oscillator and Coulomb potentials. Moreover, the higher-order Schrödinger operator with quasi-periodic potentials in two dimensions is discussed in [14].

We remark that for $J = 1$, $\mathcal{H}_{0J}$ becomes the regular Schrödinger operator

$$\mathcal{H}_{01} = mc^2 - \frac{\hbar^2}{2m} \Delta.$$
If $J = 2$, we get
\[ \mathcal{H}_{02} = mc^2 - \frac{\hbar^2}{2m} \Delta - \frac{\hbar^4}{8m^3c^2} \Delta^2. \]

Such an operator was considered in [19, 20] for the Woods-Saxon problem. It is obvious that operators $\mathcal{H}_{01}$ and $\mathcal{H}_{02}$ enjoy different features.

In this paper, we restrict to the case $J > 1$. For notational simplicity, in the sequel we take without lose of generality that the constants equal to 1, that is,
\[ \mathcal{H}_{0J} = 1 - \sum_{j=1}^{J} \Delta^j. \]

Now, we consider the existence of solutions for equation (2) under the constraint
\[ \int_{\mathbb{R}^3} |u|^2 \, dx = 1 \tag{7} \]
with $2 < p < \frac{4J+6}{J}$. Since the operator $\mathcal{H}_{0J}$ emerges different properties by $J$ being even or odd, it is necessary to treat the operator $\mathcal{H}_{0J}$ separately.

The natural way to study the problem is to look for critical points of the functional
\[ E^{\text{odd}}(u) = \frac{1}{2} \left\{ \sum_{i=0}^{k-1} \int_{\mathbb{R}^3} |\nabla \Delta^i u|^2 \, dx - \sum_{i=1}^{k} \int_{\mathbb{R}^3} |\Delta^i u|^2 \, dx \right\} - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p \, dx \tag{8} \]
if $J = 2k+1$ for $k = 0, 1, 2, \ldots$; or
\[ E^{\text{even}}(u) = \frac{1}{2} \left\{ \sum_{i=0}^{k-1} \int_{\mathbb{R}^3} |\nabla \Delta^i u|^2 \, dx - \sum_{i=1}^{k} \int_{\mathbb{R}^3} |\Delta^i u|^2 \, dx \right\} - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p \, dx \tag{8} \]
if $J = 2k$ for $k = 1, 2, \ldots$ under the constraint (7). Denote by $H^J(\mathbb{R}^3)$ the usual Sobolev spaces. Then, any critical point of both $E(u) = E^{\text{odd}}(u)$ and $E^{\text{even}}(u)$ constrained to
\[ S_\rho := \{ u \in H^J(\mathbb{R}^3) : \|u\|_{L^2(\mathbb{R}^3)} = \rho \} \tag{9} \]
corresponds to a solution of (2). By a solution of (2) we mean a couple $(\lambda_\rho, u_\rho) \in \mathbb{R} \times H^J(\mathbb{R}^3)$, where $\lambda_\rho$ is the Lagrange multiplier associated with the critical point $u_\rho$ on $S_\rho$.

However, the functional $E^{\text{even}}(u)$ is not bounded from blow on $S_\rho$. Hence, we define $E^{\text{even}}(u) = -E^{\text{even}}(u)$, that is,
\[ E^{\text{even}}(u) = \frac{1}{2} \left\{ \sum_{i=1}^{k} \int_{\mathbb{R}^3} |\nabla \Delta^i u|^2 \, dx - \sum_{i=0}^{k-1} \int_{\mathbb{R}^3} |\nabla \Delta^i u|^2 \, dx \right\} + \frac{1}{p} \int_{\mathbb{R}^3} |u|^p \, dx, \tag{10} \]
and thus we set $E(u) = E^{\text{even}}(u)$ if $J$ is even.

Our main result is as follows.

**Theorem 1.1.** There exist $\rho_0 > 0$, $\rho_1 > 0$ and $\rho_2 > 0$ depending on $p$ such that all the minimizing sequences of
\[ E_\rho = \inf_{S_\rho} E(u) \]
are precompact in $H^J(\mathbb{R}^3)$ up to a translations provided that
(a) $0 < \rho < \rho_0$ if $J$ is odd and $2 < p < 3$;
(b) $\rho_1 < \rho < +\infty$ if $J$ is odd and $2 + J < p < \frac{4J+6}{3}$;
(c) $0 < \rho < \rho_2$ if $J$ is even and $3 < p < \frac{4J+6}{3}$.
In particular, there exists a couple \((\lambda_\rho, u_\rho) \in \mathbb{R} \times H^J(\mathbb{R}^3)\) solution of \(\text{(2)}\).

Since our problem \(\text{(2)}\) is setting in \(\mathbb{R}^3\), the main difficulty for variational problems is the lack of compactness. According to the concentration-compactness principle [18], the loss of the compactness occurs if vanishing and dichotomy happen. Correspondingly, for a minimizing sequence \(\{u_n\} \subset S_\rho\) of \(E(u)\), we have either (i) \(u_n \rightharpoonup 0\) or (ii) \(u_n \rightharpoonup \bar{u} \neq 0\) and \(0 < \|\bar{u}\|_{L^2(\mathbb{R}^3)} < \rho\). In order to show the relatively compactness of minimizing sequence \(\{u_n\} \subset S_\rho\) of \(E\), the necessary and sufficient condition is the following subadditivity inequality

\[
E_\rho < E_\mu + E\sqrt{\rho^2 - \mu^2} \tag{11}
\]

for all \(0 < \mu < \rho\). This inequality allows us to rule out vanishing and dichotomy. The classical approach to prove the subadditivity inequality (11) is to show the function \(s \to s^{-2}E_s\) is monotone decreasing. However, this assertion is not easy to prove. In [1, 2], Bellazzini and Siciliano find a way to recover the monotone decreasing condition, and applied it to the Schrödinger-Poisson equation.

We will deal with the problem \(E_\rho\) by the concentration-compactness principle. As the functional \(E(u)\) has different features for \(J\) being odd and even, we distinguish two cases to discuss. In the case that \(J\) is odd, we may show the subadditivity inequality for \(E_{\text{odd}}\rho\) if \(2 + \frac{J}{2} < p < \frac{4 + J}{4}\) by the standard scaling arguments. However, in the case \(J\) being even, the standard scaling arguments do not permit to show that the subadditivity inequality (11). Hence, we will use the techniques introduced in [2].

This paper is organized as follows. In Section 2, we present some useful inequalities and facts for future references.

2. Preliminaries. In this section, we present some useful inequalities and facts for future references.

Denote by \(H^m(\mathbb{R}^N)\) the Sobolev space endowed with the norm

\[
\|u\|_{H^m(\mathbb{R}^N)} := \sum_{\alpha_1 + \cdots + \alpha_N = m} \int_{\mathbb{R}^N} |D^\alpha u|^2 \, dx + \int_{\mathbb{R}^N} |u|^2 \, dx,
\]

where \(\alpha \in \mathbb{N}^N\) and \(D^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_N}^{\alpha_N}\). We recall the following useful inequalities:

(1) \((\text{Gagliardo-Nirenberg inequality})\) For \(u : \mathbb{R}^n \to \mathbb{R}\). Fix \(1 \leq q, r \leq +\infty\) and a natural number \(m\). Suppose also that a real number \(\alpha\) and a natural number \(j\) are such that

\[
\frac{1}{p} = \frac{j}{n} + \left(\frac{1}{r} - \frac{m}{n}\right)\alpha + \frac{1 - \alpha}{q}
\]

and

\[
\frac{j}{m} \leq \alpha \leq 1.
\]

Then every function \(u : \mathbb{R}^n \to \mathbb{R}\) that lies in \(L^q(\mathbb{R}^n)\) with \(m\)-th derivative in \(L^r(\mathbb{R}^n)\) also has \(j\)-th derivative in \(L^p(\mathbb{R}^n)\); Moreover, there exists a constant \(C\) depending only on \(m, n, j, q, r\) and \(\alpha\) such that

\[
\|D^j u\|_{L^p} \leq C\|D^m u\|_{L^r}\|u\|_{L^q}^{1 - \alpha}. \tag{12}
\]
(II) \textbf{(Interpolation inequality)} For \( j, p \geq 0 \), denote

\[
|u|_{j,p} = \left\{ \sum_{|\alpha|=j} \int_{\mathbb{R}^n} |D^\alpha u|^p \right\}^{\frac{1}{p}}.
\]

For \( \varepsilon > 0, m > j \geq 0, p \geq 0 \), we have

\[
|u|_{j,p} \leq \varepsilon |u|_{m,p} + \varepsilon^{-\frac{j}{m-j}} |u|_{0,p}.
\]

Now, we present an abstract result on a constrained minimization problem in Sobolev space \( H^J(\mathbb{R}^3) \). Let us consider the following problem

\[
E_\rho = \inf_{S_\rho} E(u),
\]

where

\[
E(u) = \frac{1}{2} \left\{ \sum_{i=0}^k \int_{\mathbb{R}^3} |\nabla \Delta^i u|^2 dx - \sum_{i=1}^k \int_{\mathbb{R}^3} |\Delta^i u|^2 dx \right\} + T(u),
\]

if \( J = 2k + 1 \) with \( k = 0, 1, 2, \ldots \); or

\[
E(u) = \frac{1}{2} \left\{ \sum_{i=1}^k \int_{\mathbb{R}^3} |\Delta^i u|^2 dx - \sum_{i=0}^{k-1} \int_{\mathbb{R}^3} |\nabla \Delta^i u|^2 dx \right\} - T(u),
\]

if \( J = 2k \) with \( k = 1, 2, \ldots \).

Under suitable assumptions on \( T \), we show as \([1, 2]\) that any minimizing sequence of problem (14) actually converges strongly.

\textbf{Lemma 2.1.} \textit{Let} \( T \) \textit{be a} \( C^1 \) \textit{functional on} \( H^J(\mathbb{R}^3) \) \textit{and} \( \{u_n\} \subset S_\rho \) \textit{be a minimizing sequence for} \( E_\rho \) \textit{such that} \( u_n \rightarrow \bar{u} \neq 0 \) \textit{and} \( \|\bar{u}\|_{L^2(\mathbb{R}^3)} := \mu \in (0, \rho) \).

\textit{Assume also that}

\[
T(u_n - \bar{u}) + T(\bar{u}) = T(u_n) + o(1),
\]

\[
T(\alpha_n (u_n - \bar{u})) - T(u_n - \bar{u}) = o(1),
\]

\textit{where} \( \alpha_n = \sqrt{\rho^2 - \mu^2} / \|u_n - \bar{u}\|_{L^2(\mathbb{R}^3)} \) \textit{and}

\[
E_\rho < E_\mu + E \sqrt{\rho^2 - \mu^2}
\]

\textit{for any} \( 0 < \mu < \rho \). \textit{Then} \( \bar{u} \in S_\rho \).

\textit{Moreover if, as} \( n, m \rightarrow \infty \),

\[
\langle T'(u_n) - T'(u_m), u_n - u_m \rangle = o(1),
\]

\[
\langle T'(u_n), u_n \rangle = O(1),
\]

\textit{then} \( \|u_n - \bar{u}\|_{H^J(\mathbb{R}^3)} \rightarrow 0 \).

\textit{Proof.} \textit{We only consider the case that} \( J \) \textit{being odd since the same argument can be carried out for the case} \( J \) \textit{being even}.

\textit{We argue by contradiction. Suppose on the contrary that} \( \mu < \rho \). \textit{Since} \( u_n - \bar{u} \rightarrow 0 \), \textit{we have}

\[
\|u_n - \bar{u}\|_{L^2(\mathbb{R}^3)}^2 + \|\bar{u}\|_{L^2(\mathbb{R}^3)}^2 = \|u_n\|_{L^2(\mathbb{R}^3)}^2 + o(1).
\]

\textit{Thus,}

\[
\alpha_n = \frac{\sqrt{\rho^2 - \mu^2}}{\|u_n - \bar{u}\|_{L^2(\mathbb{R}^3)}^2} \rightarrow 1.
\]
Since \( \{u_n\} \) is a minimizing sequence,
\[
\frac{1}{2} \left\{ \sum_{i=0}^{k} \int_{\mathbb{R}^3} |\nabla \Delta^i u_n|^2 dx - \sum_{i=1}^{k} \int_{\mathbb{R}^3} |\Delta^i u_n|^2 dx \right\} + T(u_n) = E_\rho + o(1)
\]
and by (15), we derive
\[
\frac{1}{2} \left\{ \sum_{i=0}^{k} \int_{\mathbb{R}^3} |\nabla \Delta^i (u_n - \bar{u})|^2 dx - \sum_{i=1}^{k} \int_{\mathbb{R}^3} |\Delta^i (u_n - \bar{u})|^2 dx \right\}
\]
\[
+ \frac{1}{2} \left\{ \sum_{i=0}^{k} \int_{\mathbb{R}^3} |\nabla \Delta^i \bar{u}|^2 dx - \sum_{i=1}^{k} \int_{\mathbb{R}^3} |\Delta^i \bar{u}|^2 dx \right\} + T(u_n - \bar{u}) + T(\bar{u})
\]
\[
= E_\rho + o(1).
\]
Using (16) and (20), we find
\[
\frac{1}{2} \left\{ \sum_{i=0}^{k} \int_{\mathbb{R}^3} |\nabla \Delta^i [\alpha_n (u_n - \bar{u})]|^2 dx - \sum_{i=1}^{k} \int_{\mathbb{R}^3} |\Delta^i [\alpha_n (u_n - \bar{u})]|^2 dx \right\}
\]
\[
+ \frac{1}{2} \left\{ \sum_{i=0}^{k} \int_{\mathbb{R}^3} |\nabla \Delta^i \bar{u}|^2 dx - \sum_{i=1}^{k} \int_{\mathbb{R}^3} |\Delta^i \bar{u}|^2 dx \right\} + T(\alpha_n (u_n - \bar{u})) + T(\bar{u})
\]
\[
= E_\rho + o(1).
\]
Observe that \( \|\alpha_n (u_n - \bar{u})\|_{L^2(\mathbb{R}^3)} = \sqrt{\rho^2 - \mu^2} \), then
\[
E \sqrt{\rho^2 - \mu^2} + E_\mu \leq E_\rho + o(1),
\]
which is in contradiction with (17). This implies that \( \|\bar{u}\|_{L^2(\mathbb{R}^3)} = \rho \).

In order to prove that \( u_n \to \bar{u} \) in \( H^J(\mathbb{R}^3) \), we may assume, by Ekeland variational principle [9], that \( \{u_n\} \) is a Palais-Smale sequence for functional \( E \). Since \( \bar{u} \in B_\rho \), then \( \|u_n - \bar{u}\|_{L^2(\mathbb{R}^3)} = o(1) \) and thus it remains to show that \( \|u_n - \bar{u}\|_{H^J(\mathbb{R}^3)} = o(1) \).

By assumptions, there exists a sequence \( \{\lambda_n\} \subset \mathbb{R} \) such that for functional \( E \),
\[
\langle E'(u_n) - \lambda_n u_n, v \rangle = o(1), \quad \forall v \in H^J(\mathbb{R}^3),
\]
where \( \langle \cdot, \cdot \rangle \) denotes the duality pairing. It follows that
\[
\langle E'(u_n) - \lambda_n u_n, u_n \rangle = o(1),
\]
since \( \|u_n\|_{H^J(\mathbb{R}^3)} \) is bounded. From this and assumption (19) we see that the sequence \( \{\lambda_n\} \) is bounded, and thus up to a subsequence there exits \( \lambda \in \mathbb{R} \) such that \( \lambda_n \to \lambda \).

We now have
\[
\langle E'(u_n) - E'(u_m), -\lambda_n u_n + \lambda_m u_m, u_n - u_m \rangle = o(1)
\]
as \( n, m \to \infty \). Therefore, using \( (\lambda_n - \lambda_m) \langle u_m, u_n - u_m \rangle = o(1) \), we have
\[
\sum_{i=0}^{k} \int_{\mathbb{R}^3} |\nabla \Delta^i (u_n - u_m)|^2 dx - \sum_{i=1}^{k} \int_{\mathbb{R}^3} |\Delta^i (u_n - u_m)|^2 dx
\]
\[
+ \langle T'(u_n) - T'(u_m), u_n - u_m \rangle - \lambda_n \|u_n - u_m\|_{L^2(\mathbb{R}^3)}^2
\]
\[
= o(1).
\]
Since \( \|u_n - u_m\|_{L^2(\mathbb{R}^3)}^2 = o(1), \lambda_n \to \lambda \), by (18) and interpolation inequality (13), we obtain that \( \{u_n\} \) is a Cauchy sequence in \( H^J(\mathbb{R}^3) \). Hence \( \|u_n - \bar{u}\|_{H^J(\mathbb{R}^3)} \to 0. \)
We will apply Lemma 2.1 to the functional $E(u)$ with

$$T(u) = -\int_{\mathbb{R}^3} F(u) \, dx = -\frac{1}{p} \int_{\mathbb{R}^3} |u|^p \, dx$$

in the proof of Theorem 1.1. The next lemma shows that the functional $E(u)$ is bounded from below on $S_\rho$.

**Lemma 2.2.** If $2 < p < \frac{4J+6}{3}$, then for every $\rho > 0$ the functional $E(u)$ is bounded from below and coercive on $S_\rho$.

**Proof.** By the Gagliardo-Nirenberg inequality, we have

$$\|u\|_{L^p(\mathbb{R}^3)} \leq C \|D^J u\|^\alpha_{L^2(\mathbb{R}^3)} \|u\|^{1-\alpha}_{L^2(\mathbb{R}^3)},$$

where

$$\alpha = \left(\frac{1}{2} - \frac{1}{p}\right) \frac{3}{J} \quad (21)$$

satisfying $0 \leq \alpha \leq 1$ if $2 \leq p \leq \frac{4J+6}{3}$. The interpolation inequality then implies

$$E^{\text{odd}}(u) \geq \frac{1}{2} \sum_{i=0}^{k-1} \int_{\mathbb{R}^3} |\nabla \Delta^i u|^2 \, dx + \frac{1}{4} \int_{\mathbb{R}^3} |\nabla \Delta^k u|^2 \, dx - C_\rho - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p \, dx$$

$$\geq \frac{1}{2} \sum_{i=0}^{k-1} \int_{\mathbb{R}^3} |\nabla \Delta^i u|^2 \, dx + \frac{1}{4} \int_{\mathbb{R}^3} |\nabla \Delta^k u|^2 \, dx$$

$$- \tilde{C}_\rho \left( \int_{\mathbb{R}^3} |\nabla \Delta^k u|^2 \, dx \right)^{\frac{ap}{2p}} - C_\rho, \quad (22)$$

where $C_\rho$ and $\tilde{C}_\rho$ are positive constants which depending on $\rho$. Since $p < \frac{4J+6}{3}$, it results $\alpha p < 2$, which implies $E^{\text{odd}}(u)$ is bounded from below and coercive on $S_\rho$.

On the other hand, by the interpolation inequality, we have

$$E^{\text{even}}(u) \geq \frac{1}{2} \sum_{i=1}^{k-1} \int_{\mathbb{R}^3} |\Delta^i u|^2 \, dx + \frac{1}{4} \int_{\mathbb{R}^3} |\Delta^k u|^2 \, dx$$

$$- \tilde{C}_\rho \left( \int_{\mathbb{R}^3} |\Delta^k u|^2 \, dx \right)^{\frac{ap}{2p}} - C_\rho. \quad (23)$$

It follows that $E^{\text{even}}(u)$ is bounded from below and coercive on $S_\rho$. The proof is complete. \(
\square
\)

3. **The existence for $J$ being odd.** This section is devoted to prove the existence result when $J$ is odd, that is, Theorem 1.1 (a) and (b). As the functional $E^{\text{odd}}_\rho$ behaves differently for $2 < p < 2 + J$ and $2 + J < p < \frac{4J+6}{3}$, we distinguish two cases (i) $2 < p < 2 + J$ and (ii) $2 + J < p < \frac{4J+6}{3}$ to discuss.

We commence with establishing the subadditivity inequality.

**Lemma 3.1.** Assume that $J = 2k + 1$, $k = 0, 1, 2, \ldots$, then

(i) there exists $\rho_0 > 0$ such that $E^{\text{odd}}_\mu < 0$ for all $\mu \in (0, \rho_0)$ provides $2 < p < 3$;

(ii) there exists $\rho_1 > 0$ such that $E^{\text{odd}}_\mu < 0$ for all $\mu \in (\rho_1, +\infty)$ provides $2 + J < p < \frac{4J+6}{3}$.


(iii) moreover, we have
\[ E^\text{odd}_\rho < E^\text{odd}_\mu + E^\text{odd}_{\sqrt{\rho^2 - \mu^2}} \]  
(24)
for \(0 < \mu < \rho\).

Proof. Define
\[ u_\theta(x) = \theta^{1-\frac{2}{p}} u\left(\frac{x}{\theta^{\beta}}\right), \]
we have the following scaling laws:
\[
A(u_\theta) := \frac{1}{2} \left\{ \sum_{i=0}^{k} \int_{\mathbb{R}^3} |\nabla \Delta^i u_\theta|^2 dx - \sum_{i=1}^{k} \int_{\mathbb{R}^3} |\Delta^i u_\theta|^2 dx \right\} \\
= \frac{k}{2} \sum_{i=0}^{k} \theta^{2-2(2i+1)\beta} \int_{\mathbb{R}^3} |\nabla \Delta^i u|^2 dx - \frac{k}{2} \sum_{i=1}^{k} \theta^{2-4i\beta} \int_{\mathbb{R}^3} |\Delta^i u|^2 dx \\
T(u_\theta) = -\frac{1}{p} \int_{\mathbb{R}^3} |u_\theta|^p dx = -\theta^{(1-\frac{2}{p})\mu + 3\beta} \int_{\mathbb{R}^3} |u|^p dx.
\]
Therefore,
\[ E^\text{odd}(u_\theta) = \theta^2 \left( E^\text{odd}(u) + f(\theta,u) \right), \]
where
\[ f(\theta,u) = \frac{1}{2} \sum_{i=0}^{k} \left( \theta^{-2(2i+1)\beta} - 1 \right) \int_{\mathbb{R}^3} |\nabla \Delta^i u|^2 dx \]
\[ -\frac{1}{2} \sum_{i=1}^{k} \left( \theta^{-4i\beta} - 1 \right) \int_{\mathbb{R}^3} |\Delta^i u|^2 dx - \frac{1}{p} \left( \theta^{(1-\frac{2}{p})\mu + 3\beta} - 1 \right) \int_{\mathbb{R}^3} |u|^p dx. \]
Observing that for \( \beta = -2 \), we get
\[ E^\text{odd}(u_\theta) = \sum_{i=0}^{k} \frac{\theta^{2+4(2i+1)}}{2} \int_{\mathbb{R}^3} |\nabla \Delta^i u|^2 dx - \sum_{i=1}^{k} \frac{\theta^{2+8i}}{2} \int_{\mathbb{R}^3} |\Delta^i u|^2 dx \\
- \frac{\theta^{4p-6}}{p} \int_{\mathbb{R}^3} |u|^p dx. \]  
(25)

Apparently, \( 4p - 6 > 2 + 4J \) if \( p > 2 + J \). Hence, for \( \theta \) sufficiently large, we have \( E(u_\theta) < 0 \) which proves the second claim. Similarly, we have \( E(u_\theta) < 0 \) for \( \theta \) sufficiently small if \( 4p - 6 < 6 \), that is, \( p < 3 \), which proves the first claim.

Next, we show the subadditivity inequality (24).
Let \( u_n \) be a minimizing sequence in \( S_\mu \) of \( E^\text{odd}_\mu \). For \( \beta = 0 \), we have
\[ f(\theta,u_n) = -\frac{1}{p} (\theta^{p-2} - 1) \int_{\mathbb{R}^3} |u_n|^p dx. \]
Hence,
\[ \lim_{n \to +\infty} f(\theta,u_n) < 0 \quad \text{for } \theta > 1 \]

since \( \lim_{n \to +\infty} \|u_n\|_{L^p(\mathbb{R}^3)} > 0 \) and \( p > 2 \). Let \( u_{n,\theta}(x) = \theta^{1-\frac{2}{p}} u_n \left(\frac{x}{\theta^{\beta}}\right) \). Observing that
\[ \|u_{n,\theta}\|_{L^2(\mathbb{R}^3)} = \theta^2 \|u_n\|_{L^2(\mathbb{R}^3)}, \]

we have
\[ E_{\theta \mu}^{\text{odd}} \leq \lim_{n \to +\infty} E_{\theta \mu}^{\text{odd}}(u_n) \]
\[ = \lim_{n \to +\infty} \theta^2 \left( E_{\theta}^{\text{odd}}(u_n) + f(\theta, u_n) \right) \]
\[ < \theta^2 \lim_{n \to +\infty} E_{\theta \mu}^{\text{odd}}(u_n) \]
\[ = \theta^2 E_{\mu}^{\text{odd}}, \]
that is,
\[ E_{\theta \mu}^{\text{odd}} < \theta^2 E_{\mu}^{\text{odd}}, \quad \text{for } \theta > 1. \] 
(26)

Since \( \mu < \rho \), we distinguish the following cases:
1. \( \mu < \sqrt{\rho^2 - \mu^2} \);
2. \( \mu = \sqrt{\rho^2 - \mu^2} \), that is, \( \rho = \sqrt{2} \mu \);
3. \( \mu > \sqrt{\rho^2 - \mu^2} \).

For the first case, by (26), we have
\[ E_{\rho}^{\text{odd}} = E_{\rho}^{\text{odd}} \sqrt{\rho^2 - \mu^2} < \frac{\rho^2}{\rho^2 - \mu^2} E_{\rho}^{\text{odd}} \sqrt{\rho^2 - \mu^2} \]
\[ = \frac{\rho^2 - \mu^2 + \mu^2}{\rho^2 - \mu^2} E_{\rho}^{\text{odd}} \sqrt{\rho^2 - \mu^2} \]
\[ = \frac{\mu^2}{\rho^2 - \mu^2} E_{\rho}^{\text{odd}} \sqrt{\rho^2 - \mu^2} + \frac{E_{\rho}^{\text{odd}}}{\sqrt{\rho^2 - \mu^2}} \]
\[ < E_{\rho}^{\text{odd}} \sqrt{\rho^2 - \mu^2} + E_{\mu}^{\text{odd}}. \]

For the second case, we choose \( \theta = \sqrt{2} \) in (26) and the assertion follows. For the third case, by (26), we have
\[ E_{\rho}^{\text{odd}} = E_{\rho}^{\text{odd}} \sqrt{\rho^2 - \mu^2} < \frac{\rho^2}{\mu^2} E_{\rho}^{\text{odd}} \]
\[ = \frac{\rho^2 - \mu^2 + \mu^2}{\mu^2} E_{\rho}^{\text{odd}} \]
\[ = \frac{\rho^2 - \mu^2}{\mu^2} E_{\rho}^{\text{odd}} \sqrt{\rho^2 - \mu^2} + \frac{E_{\rho}^{\text{odd}}}{\sqrt{\rho^2 - \mu^2}} \]
\[ < E_{\rho}^{\text{odd}} \sqrt{\rho^2 - \mu^2} + E_{\mu}^{\text{odd}}. \]

Now we are in position to prove Theorem 1.1 (a) and (b).

Proof of Theorem 1.1 (a) and (b). Let \( \{u_n\} \subset S_\rho \) be a minimizing sequence of \( E_{\rho}^{\text{odd}} \). Notice that for any sequence \( y_n \in \mathbb{R}^3 \), \( u_n(\cdot + y_n) \) is still a minimizing sequence for \( E_{\rho}^{\text{odd}} \). By Lemma 2.1, it suffices to show that there is a sequence \( y_n \in \mathbb{R}^3 \) such that the weak limit of \( u_n(\cdot + y_n) \) belongs to \( S_\rho \), and then the convergence becomes strongly in \( H^J(\mathbb{R}^3) \). Thus, the assertion follows. To this purpose, we now rule out the vanishing.

Suppose the vanishing happens, that is,
\[ \lim_{n \to \infty} \left( \sup_{y \in \mathbb{R}^3} \int_{B(y, 1)} |u_n|^2 dx \right) = 0, \]
where \( B(a,r) = \{ x \in \mathbb{R}^3 : |x-a| < r \} \). Then, \( u_n \to 0 \) in \( L^p(\mathbb{R}^3) \) for \( 2 < p < \frac{4j+6}{3} \).

This yields a contradiction to the fact \( E^\text{odd}_\rho < 0 \), which given by Lemma 3.1. So we necessarily have that

\[
\sup_{y \in \mathbb{R}^3} \int_{B(y,1)} |u_n|^2 dx \geq 2\delta > 0
\]

for some \( \delta > 0 \). Therefore, there exists \( y_n \in \mathbb{R}^3 \) such that

\[
\int_{B(0,1)} |u_n(\cdot + y_n)|^2 dx \geq \delta > 0
\]

and, due to the compactness of embedding \( H^j(B(0,1)) \subset L^2(B(0,1)) \), see [8], we deduce that the weak limit \( \bar{u} \) of sequence \( u_n(\cdot + y_n) \) is non-trivial. By Lemma 3.1, the subadditivity condition holds and it was verified in Proposition 3.1 in [1] that

\[ T(u) = -\frac{1}{p} \int_{\mathbb{R}^3} |u|^p dx \]

satisfies (15)-(16) and (18)-(19). Hence, the conclusion follows by Lemma 2.1.

\[ 4. \text{ The existence for } J \text{ being even.} \]

This section is devoted to prove the existence result when \( J \) is even, that is, Theorem 1.1 (c). In this case, the standard scaling arguments do not permit to show that the subadditivity inequality

\[
\tilde{E}^\even_\mu < \tilde{E}^\even_\mu + \frac{\tilde{E}^\even_\mu}{\sqrt{\rho^2 - \mu^2}}
\]  

(27)

holds for \( 0 < \mu < \rho \). Hence, the possibility of dichotomy for an arbitrary minimizing sequence cannot be excluded. By techniques introduced in [2], we are able to prove (27) holds for \( J \) being even at least for small value of \( \rho \). Hence, the compactness of minimizing sequences of \( \tilde{E}^\even_\mu \) retains up to translations.

The main idea is to show that the function \( \mu \mapsto \frac{\tilde{E}^\even_\mu}{\mu^2} \) is monotone decreasing, which yields (27). Indeed, if the function \( \mu \mapsto \frac{\tilde{E}^\even_\mu}{\mu^2} \) is monotone decreasing for \( \mu \in (0, \rho) \), we have

\[
\frac{\mu^2}{\rho^2} \tilde{E}^\even_\rho < \tilde{E}^\even_\mu \quad \text{and} \quad \frac{\rho^2 - \mu^2}{\rho^2} \tilde{E}^\even_\mu < \tilde{E}^\even_\mu + \tilde{E}^\even_\mu - \frac{\tilde{E}^\even_\mu}{\sqrt{\rho^2 - \mu^2}},
\]

which implies

\[
\tilde{E}^\even_\rho = \frac{\mu^2}{\rho^2} \tilde{E}^\even_\rho + \frac{\rho^2 - \mu^2}{\rho^2} \tilde{E}^\even_\mu < \tilde{E}^\even_\mu + \tilde{E}^\even_\mu - \frac{\tilde{E}^\even_\mu}{\sqrt{\rho^2 - \mu^2}}
\]

for any \( \mu \in (0, \rho) \).

Now, we recall some definitions in [2].

Let \( u \in H^j(\mathbb{R}^3), u \not\equiv 0 \). A continuous path \( g_u : \theta \in \mathbb{R}^+ \mapsto g_u(\theta) \in H^j(\mathbb{R}^3) \) such that \( g_u(1) = u \) is said to be a scaling path of \( u \) if

\[
\Theta_{g_u}(\theta) := \|g_u(\theta)\|_{L^2(\mathbb{R}^N)}\|u\|_{L^2(\mathbb{R}^N)}^{-1}
\]

is differentiable and \( \Theta_{g_u}'(1) \neq 0 \). Denote by \( \mathcal{G}_u \) the set of the scaling paths of \( u \).

In the application, it is relevant to consider the family of scaling paths of \( u \) parameterized with \( \beta \in \mathbb{R} \) given by

\[
\mathcal{G}^\beta_u = \left\{ g_u(\theta) = \theta^{1-\frac{N}{2j}}u(x/\theta^\beta) \right\} \subset \mathcal{G}_u.
\]

Observe that all paths of this family have the associated function \( \Theta_{g_u}(\theta) = \theta^2 \).
Furthermore, fixed \( u \neq 0 \), we define the following real valued function

\[
h_{g_u}(\theta) := \tilde{E}^{\text{even}}(g_u(\theta)) - \Theta_{g_u}(\theta) \tilde{E}^{\text{even}}(u), \quad \theta \geq 0.
\]

Finally, let \( u \neq 0 \) be fixed and \( g_u \in G_u \). We say that the scaling path \( g_u \) is admissible for the functional \( \tilde{E}^{\text{even}} \) if \( g_u \) is a differentiable function.

The following result allows us to find a minimizer of \( \tilde{E}^{\text{even}} \).

**Proposition 4.1.** Assume that for every \( \rho > 0 \), all the minimizing sequences \( \{u_n\} \) for

\[
\tilde{E}_\rho^{\text{even}} = \inf_{S_\rho} \tilde{E}^{\text{even}}(u)
\]

has a weak limit, up to translations, different from zero. Assume

\[
\tilde{E}_\rho^{\text{even}} \leq \tilde{E}_\mu^{\text{even}} + \tilde{E}^{\text{even}} \frac{\sqrt{\rho^2 - \mu^2}}{\rho^2} \quad \text{for all } 0 < \mu < \rho
\]

and the following conditions

\[
-\infty < \tilde{E}_s^{\text{even}} < 0 \quad \text{for all } s > 0 \quad (\tilde{E}^{\text{even}}(0) = 0),
\]

\[
s \mapsto \tilde{E}_s^{\text{even}} \quad \text{is continuous},
\]

\[
\lim_{s \to 0} \frac{\tilde{E}_s^{\text{even}}}{s^2} = 0.
\]

Then for every \( \rho > 0 \) the set

\[
M(\rho) = \bigcup_{\mu \in (0, \rho]} \{ u \in S_\mu : \tilde{E}^{\text{even}}(u) = \tilde{E}_\mu^{\text{even}} \}
\]

is nonempty. If in addition

\[
\forall u \in M(\rho) \exists g_u \in G_u \text{ admissible such that } \frac{d}{d\theta} h_{g_u}(\theta) \big|_{\theta=1} \neq 0,
\]

then (27) holds. Moreover, if \( \{u_n\} \) is a minimizing sequence weakly convergent to a certain \( \tilde{u} \), then \( ||u_n - \tilde{u}||_{H^1(\mathbb{R}^3)} \to 0 \) and \( \tilde{E}^{\text{even}}(\tilde{u}) = \tilde{E}_\rho^{\text{even}} \).

**Proof.** The proof is similar to that of Theorem 2.1 in [2], so we omit it here. \(\square\)

Next, we need the following estimate.

**Lemma 4.2.** Assume that \( 3 < p < \frac{4J+16}{3} \). If \( J = 2k, \ k = 1, 2, \cdots \), then there exists \( \rho_2 > 0 \) such that \( \tilde{E}_\mu^{\text{even}} < 0 \) for all \( \mu \in (0, \rho_2) \).

**Proof.** We define

\[
u(\theta) = \theta^{1-\frac{2}{p}} u \left( \frac{x}{\theta^{\beta}} \right).
\]

Then,

\[
\tilde{E}^{\text{even}}(\nu) := \frac{1}{2} \left\{ \sum_{i=1}^{k} \theta^{2-4i}\beta \int_{\mathbb{R}^3} |\Delta^i u|^2 dx - \sum_{i=0}^{k-1} \theta^{2-2(2i+1)} \beta \int_{\mathbb{R}^3} |\nabla \Delta^i u|^2 dx \right\} + \frac{\theta^{(1-\frac{2}{p})\beta + 3\beta}}{p} \int_{\mathbb{R}^3} |u|^p dx.
\]

For \( \beta = -2 \), we have

\[
\tilde{E}^{\text{even}}(\nu) = \sum_{i=1}^{k} \frac{\theta^{2+8i}}{2} \int_{\mathbb{R}^3} |\Delta^i u|^2 dx - \sum_{i=0}^{k-1} \frac{\theta^{2+4(2i+1)}}{2} \int_{\mathbb{R}^3} |\nabla \Delta^i u|^2 dx
\]
\[ + \frac{\theta^{4p-6}}{p} \int_{\mathbb{R}^3} |u|^p dx. \]

The fact \( 4p - 6 > 6 \) if \( p > 3 \) implies \( \bar{E}^{\text{even}}(u_\theta) \to 0^- \) whenever \( \theta \to 0 \). The assertion follows. \( \square \)

Now we are in position to prove Theorem 1.1 (c).

**Proof of Theorem 1.1 (c).** We will verify that the hypotheses of Proposition 4.1 are fulfilled, the conclusion then follows from Proposition 4.1.

A direct consequence of Lemma III.1 and Lemma I.1 in [18] gives the following weak subadditivity inequality
\[
\bar{E}_\rho^{\text{even}} \leq \bar{E}_\mu^{\text{even}} + \bar{E}_{\sqrt{s^2-\rho^2}}^{\text{even}} \quad \text{for all } 0 < \mu < \rho.
\]

Therefore, (28) is valid.

Now, we verify condition (29).

If \( 3 < p < \frac{4J+6}{3} \), by Lemma 2.2, we know that \( \bar{E}_s^{\text{even}} > -\infty \) for all \( s > 0 \). So we need to show that \( \bar{E}_s^{\text{even}} < 0 \) for every \( s > 0 \).

By Lemma 4.2, we know there exists a \( \bar{\rho} > 0 \) such that \( \bar{E}_s^{\text{even}} < 0 \) for \( s \) in a certain interval \((0, \bar{\rho})\). For every \( s \in (\bar{\rho}, \rho] \) we have
\[
\bar{E}_s^{\text{even}} \leq \bar{E}_{\bar{\rho}}^{\text{even}} + \bar{E}_{\sqrt{s^2-\bar{\rho}^2}}^{\text{even}} < 0
\]

since \( s^2 - \bar{\rho}^2 < \bar{\rho}^2 \). This shows that \( \bar{E}_s^{\text{even}} < 0 \) for \( s \) in a large interval \((0, \rho]\).

Iterating this procedure we have the \( \bar{E}_s^{\text{even}} < 0 \) for every \( s > 0 \).

For every minimizing sequence \( \{u_n\} \subset S_\rho \) of \( \bar{E}_\rho^{\text{even}} \) with \( \rho > 0 \), we may prove as the proof of Theorem 1.1 (a) or (b) that \( u_n \rightharpoonup \bar{u} \neq 0 \) up to translations. Moreover the weak limit \( \bar{u} \) belongs to \( M(\rho) \). Indeed, if \( \|\bar{u}\|_{L^2(\mathbb{R}^3)} = \rho \), it is trivial. If \( \|\bar{u}\|_{L^2(\mathbb{R}^3)} = \mu \) and \( \mu \in (0, \rho) \), we can show \( \bar{E}^{\text{even}}(\bar{u}) = \bar{E}_\mu^{\text{even}} \) as Proposition 3.1 in [2]. Thus, \( \bar{u} \in M(\rho) \).

Next, we prove that the function \( s \mapsto \bar{E}_s^{\text{even}} \) satisfies (30) and (31).

We claim that
\[
\lim_{n \to \infty} \bar{E}_{\rho_n}^{\text{even}} = \bar{E}_\rho^{\text{even}}
\]
if \( \rho_n \to \rho \). In fact, for every \( n \in \mathbb{N} \), let \( w_n \in S_{\rho_n} \) be such that
\[
\bar{E}^{\text{even}}(w_n) < \bar{E}_{\rho_n}^{\text{even}} + \frac{1}{n} < \frac{1}{n}.
\]

By Lemma 2.2, \( \{w_n\} \) is bounded in \( H^1(\mathbb{R}^3) \) since \( \{\rho_n\} \) is bounded. So we may find
\[
\bar{E}_{\rho}^{\text{even}} \leq \bar{E}_\rho^{\text{even}} \left( \frac{\rho}{\rho_n} w_n \right)
\]
\[
= \frac{1}{2} \left( \frac{\rho}{\rho_n} \right)^2 \left\{ \sum_{i=1}^k \int_{\mathbb{R}^3} |\Delta^i w_n|^2 dx - \sum_{i=0}^{k-1} \int_{\mathbb{R}^3} |\nabla \Delta^i w_n|^2 dx \right\}
\]
\[
+ \frac{1}{p} \left( \frac{\rho}{\rho_n} \right)^p \int_{\mathbb{R}^3} |w_n|^p dx
\]
\[
= \bar{E}_\rho^{\text{even}}(w_n) + o(1)
\]
\[
< \bar{E}_{\rho_n}^{\text{even}} + o(1).
\]
On the other hand, given a minimizing sequence \( \{v_n\} \subset S_\rho \) for \( E_{\rho}^{\text{even}} \), we have
\[
E_{\rho n}^{\text{even}} \leq E_{\rho}^{\text{even}} \left( \frac{\rho_n}{\rho} v_n \right) = E_{\rho}^{\text{even}}(v_n) + o(1) = E_{\rho}^{\text{even}} + o(1)
\]
which, together with (33), yields \( \lim_{n \to \infty} E_{\rho n}^{\text{even}} = E_{\rho}^{\text{even}} \). The claim follows.

Now, we show that
\[
\lim_{\rho \to 0} \frac{E_{\rho}^{\text{even}}}{\rho^2} = 0.
\]
We argue indirectly. Suppose on the contrary that \( \lim_{\rho \to 0} \frac{E_{\rho}^{\text{even}}}{\rho^2} = -2c < 0 \) for some \( c \in (0, 1/2) \). Therefore, \( E_{\rho}^{\text{even}} < -\frac{9}{2} c \rho^2 \) if \( \rho \) is small. Let \( \{\rho_n^\rho\} \subset S_\rho \) be a minimizing sequence for \( E_{\rho}^{\text{odd}} \) such that
\[
E_{\rho}^{\text{even}}(\rho_n^\rho) \leq E_{\rho}^{\text{even}} + \frac{\rho^2}{n}.
\]
Then, \( u_n^\rho \) satisfies
\[
\frac{\lambda_n^\rho}{2} \leq \sum_{j=1}^{J} \Delta^j u_n^\rho + |u_n^\rho|^{p-2} u_n^\rho - \lambda_n^\rho = \sigma_n^\rho \to 0
\]
as \( n \to \infty \). By (36) and Gagliardo-Nirenberg inequality, we have
\[
\lambda_n^\rho \leq -c,
\]
since \( (1-\alpha)p > 2 \). By (36) and Sobolev embedding,
\[
\frac{k}{2} \int_{\mathbb{R}^3} |\Delta^i u_n^\rho|^2 dx - \frac{k-1}{2} \int_{\mathbb{R}^3} \nabla |\Delta^i u_n^\rho|^2 dx + \int_{\mathbb{R}^3} |u_n^\rho|^p dx \leq \frac{k}{2} \int_{\mathbb{R}^3} |\Delta^i u_n^\rho|^2 dx - \frac{k-1}{2} \int_{\mathbb{R}^3} \nabla |\Delta^i u_n^\rho|^2 dx - \chi_n^\rho \int_{\mathbb{R}^3} |u_n^\rho|^2 dx
\]
which, together with (37), yields \( \lim_{n \to \infty} E_{\rho n}^{\text{even}} = E_{\rho}^{\text{even}} \). The claim follows.
where $C$ is a positive constant and $o(1) \to 0$ as $n \to \infty$. On the other hand, by interpolation inequality (13), we can show that

$$
\sum_{i=1}^{k} \int_{\mathbb{R}^3} |\Delta^i u_n|^2 \, dx - \sum_{i=0}^{k-1} \int_{\mathbb{R}^3} |\nabla \Delta^i u_n|^2 \, dx + c \int_{\mathbb{R}^3} |u_n|^2 \, dx \geq c \|u_n\|^2_{H^1(\mathbb{R}^3)},
$$

(38)

since $\|u_n\|_{L^2(\mathbb{R}^3)} = \rho$ is small. Therefore, we deduce from (37) and (38) that there exists $\bar{c} > 0$ such that

$$
\|u_n\|^2_{H^1(\mathbb{R}^3)} \geq \bar{c} > 0
$$

for $n$ large enough. However, by (23), we have

$$
\tilde{E}^{even}(u_n) \geq C \int_{\mathbb{R}^3} |\Delta^k u_n|^2 \, dx \geq C\bar{c} > 0,
$$

if $\rho$ is small, which contradicts to (35). Hence, (34) holds true.

Finally, we show the strict subadditivity inequality holds true, which implies the strong convergence of minimizing sequences of $\tilde{E}^{even}$. By Proposition 4.1, it is enough to verify that the functional $\tilde{E}^{even}$ satisfies (32) for small $\rho > 0$.

For $u \in M(\rho)$, we define $v(\theta, u) = \theta^{3/2} u(\theta^{-1} x)$. Then,

$$
\|v(\theta, u)\|_{L^2(\mathbb{R}^3)} = \|u\|_{L^2(\mathbb{R}^3)}
$$

and

$$
\tilde{E}^{even}(v(\theta, u)) = \frac{1}{2} \left\{ \sum_{i=1}^{k} \theta^{-4i} \int_{\mathbb{R}^3} |\Delta^i u|^2 \, dx - \sum_{i=0}^{k-1} \theta^{-2(2i+1)} \int_{\mathbb{R}^3} |\nabla \Delta^i u|^2 \, dx \right\} + \frac{\theta^{3-2p}}{p} \int_{\mathbb{R}^3} |u|^p \, dx.
$$

Since the map $\theta \mapsto \tilde{E}^{even}(v(\theta, u))$ is differential and $u$ achieves the minimum on $S_\mu$ for $\mu \in (0, \rho]$, we get

$$
\frac{d}{d\theta} \tilde{E}^{even}(v(\theta, u)) \bigg|_{\theta=1} = 0,
$$

that is,

$$
\sum_{i=1}^{k} 2i \int_{\mathbb{R}^3} |\Delta^i u|^2 \, dx - \sum_{i=0}^{k-1} (2i + 1) \int_{\mathbb{R}^3} |\nabla \Delta^i u|^2 \, dx - \frac{6 - 3p}{2p} \int_{\mathbb{R}^3} |u|^p \, dx = 0.
$$

(39)

Now, for $u \neq 0$, choosing the family of scaling paths of $u$ parameterized with $\beta \in \mathbb{R}$ given by

$$
\mathcal{G}_u^\beta = \left\{ g_u(\theta) = \theta^{1-\frac{4}{3} \beta} u \left( \frac{x}{\theta^\beta} \right) \right\} \subset \mathcal{G}_u
$$

and the associated function $\Theta_{g_u}(\theta) = \theta^2$, we deduce

$$
h_{g_u}(\theta) = \tilde{E}^{even}(g_u(\theta)) - \Theta_{g_u}(\theta) \tilde{E}^{even}(u)
$$

$$
= \frac{1}{2} \left\{ \sum_{i=1}^{k} (\theta^{2-4i\beta} - \theta^2) \int_{\mathbb{R}^3} |\Delta^i u|^2 \, dx - \sum_{i=0}^{k-1} (\theta^{2-2(2i+1)\beta} - \theta^2) \int_{\mathbb{R}^3} |\nabla \Delta^i u|^2 \, dx \right\} + \frac{1}{p} \left( \theta^{1-\frac{2}{3} \beta} \right)^p - \theta^2 \int_{\mathbb{R}^3} |u|^p \, dx,
$$
which shows that the paths in $G_u^\beta$ are admissible, that is, $h_{g_u}(\theta)$ is differential for every $g_u \in G_u^\beta$. Moreover, for $g_u \in G_u^\beta$, we have

$$\frac{d}{d\theta} h_{g_u}(\theta) \bigg|_{\theta=1} = -\sum_{i=1}^{k} 2i\beta \int_{\mathbb{R}^3} |\Delta^i u|^2 dx + \sum_{i=0}^{k} (2i + 1)\beta \int_{\mathbb{R}^3} |\nabla \Delta^i u|^2 dx$$

$$+ \frac{(1 - \frac{3}{2} \beta) p + 3 \beta - 2}{p} \int_{\mathbb{R}^3} |u|^p dx.$$ 

We claim that there exists an admissible scaling path $g_u$ in $G_u^\beta$ such that

$$\frac{d}{d\theta} h_{g_u}(\theta) \bigg|_{\theta=1} \neq 0.$$ 

We argue by contradiction. Were it not the case, there would exist a sequence $\{u_n\} \subset M(\rho)$ with $\rho \geq \|u_n\|_{L^2(\mathbb{R}^3)} = \rho_n \to 0$ such that for all $\beta \in \mathbb{R}$

$$-\sum_{i=1}^{k} 2i\beta \int_{\mathbb{R}^3} |\Delta^i u_n|^2 dx + \sum_{i=0}^{k} (2i + 1)\beta \int_{\mathbb{R}^3} |\nabla \Delta^i u_n|^2 dx$$

$$+ \frac{(1 - \frac{3}{2} \beta) p + 3 \beta - 2}{p} \int_{\mathbb{R}^3} |u_n|^p dx = 0.$$ 

Using (39), we have

$$\left(1 - \frac{2}{p}\right) \int_{\mathbb{R}^3} |u_n|^p dx = 0.$$ 

Since $p > 2$, $\|u_n\|_{L^p(\mathbb{R}^3)} = 0$, which yields a contradiction. The claim is true and the proof is complete.

REFERENCES

[1] J. Bellazzini and G. Siciliano, Stable standing waves for a class of nonlinear Schrödinger-Poisson equations, Z. Angew. Math. Phys., 62 (2011), 267-280.

[2] J. Bellazzini and G. Siciliano, Scaling properties of functionals and existence of constrained minimizers, J. Funct. Anal., 261 (2011), 2486-2507.

[3] R. Carles, W. Lucha and E. Moulay, Higher-order Schrödinger and Hartree-Fock equations, J. Math. Phys., 56 (2015), 122301, 17 pp.

[4] R. Carles and E. Moulay, Higher order Schrödinger equations, J. Phys. A, 45 (2012), 395304, 11 pp.

[5] X. Chen and J. Yang, Regularity and symmetry of solutions of an integral equation, Acta Math. Sci., 32 (2012), 1759-1780.

[6] H. L. Cycon, R. G. Froese, W. Kirsch and B. Simon, Schrödinger Operators with Application to Quantum Mechanics and Global Geometry, Study ed. Texts and Monographs in Physics, Springer-Verlag, Berlin, 1987.

[7] P. A. M. Dirac, The quantum theory of the electron, Proc. R. Soc. A, 117 (1928), 610-624.

[8] Y. Ebihara and T. Schonbek, On the (non)compactness of the radial Sobolev spaces, Hiroshima Math. J., 16 (1986), 665-669.

[9] I. Ekeland, On the variational principle, J. Math. Anal. Appl., 47 (1974) 324-353.

[10] A. N. Gorban and I. V. Karlin, Schrödinger operator in an overfull set, Europhys. Lett., 42 (2007), 113-118.

[11] R. L. Hall and W. Lucha, Schrödinger upper bounds to semirelativistic eigenvalues, J. Phys. A, 38 (2005), 7997-8002.

[12] R. L. Hall and W. Lucha, Schrödinger secant lower bounds to semirelativistic eigenvalues, Int. J. Mod. Phys. A, 22 (2007), 1899-1904.

[13] B. Helffer, Semi-Classical Analysis for the Schrödinger Operator and Applications, Lecture Notes in Mathematics Vol. 1336, Springer-Verlag, Berlin, 1988.
Y. Karpeshina and R. Shterenberg, Extended states for polyharmonic operators with quasi-periodic potentials in dimension two, *J. Math. Phys.*, **53** (2012), 103512, 8pp.

J. M. Kim, A. Arnold and X. Yao, Global estimates of fundamental solutions for higher-order Schrödinger equations, *Monatsh. Math.*, **168** (2012), 253–266.

E. Lenzmann, Well-posedness for semi-relativistic Hartree equations of critical type, *Math. Phys. Anal. Geom.*, **10** (2007), 43–64.

E. H. Lieb and M. Loss, *Analysis*, Graduate Studies in Mathematics 14, AMS, 2001.

P. L. Lions, The concentration-compactness principle in the Calculus of Variations. The locally compact case. I, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **1** (1984), 109–145.

W. Lucha and F. Schöberl, Semirelativistic Bound-State Equations: Trivial Considerations, *EPJ Web of Conferences*, **80** (2014), 00049.

W. Lucha and F. Schöberl, The spinless relativistic Woods Saxon problem, *International Journal of Modern Physics A*, **29** (2014), 1450057, 15pp.

J. Tan, Y. Wang and J. Yang, Nonlinear Fractional field equations, *Nonlinear Anal. TMA*, **75** (2012), 2098–2110.

Received March 2018; revised July 2018.

E-mail address: xiaaliang@126.com

E-mail address: jfyang.2000@yahoo.com