Incomplete information in scale-free networks

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Abstract

We investigate the effect of incomplete information on the growth process of scale-free networks - a situation that occurs frequently e.g. in real existing citation networks. Two models are proposed and solved analytically for the scaling behavior of the connectivity distribution. These models show a varying scaling exponent with respect to the model parameters but no break-down of scaling thus introducing the first models of scale-free networks in an environment of incomplete information. We compare to results from computer simulations which show a very good agreement.

Keywords: Random graphs, networks, Socio-economic networks, Stochastic processes, Growth processes

1 Introduction

Since the work on complex networks by Strogatz, Watts, Barabàsi and Albert (see [1, 2, 3, 4, 5, 6, 7, 8]) many researchers from such distinct fields as statistical mechanics [9, 10, 11, 12, 13], molecular biology [14, 15, 16, 17, 18, 19], ecology [20], physical chemistry [21, 22, 23], genetics e.g. [24, 25, 26, 27, 28] or social science [29, 30, 31] have studied the emerging complex structure and the behavior of networks in their respective field of research.

A special subset of scale-free non-equilibrium networks can emerge from a construction procedure in which at each time step $t$ one vertex is added.
and connected to $m$ existing vertices with preferential linking. This preference is proportional to the number of already existing connections of that particular vertex. By definition the average number of connections remains constant $\bar{k} = 2m$. The distribution of the degree of connections $P(k)$ is of particular interest as it provides for the possibility to distinguish different classes. One observes in scale-free-networks a behavior $P(k) \sim k^{-\gamma}$ that was first discussed by Simon [32] and in the context of citations networks by Price [33]. Extensions to a more complex linking procedure [34] or more general linkage properties [35] were recently discussed in detail.

To study the evolution of the distribution the continuum approximation is often used. At time $t$ the average number of connections $\bar{k}(s, t)$ of a vertex created at time $s$ is in an undirected network [8, 36]

$$\frac{\partial \bar{k}(s, t)}{\partial t} = m \cdot \frac{\bar{k}(s, t)}{\int_0^t du \bar{k}(u, t)}$$

(1)

Bianconi and Barabási [37] pointed to the effects of distributions of fitness of individuals to attract new connections. This can be already regarded as one prototypical example of incomplete information by interpreting the fitness in their model as an incomplete knowledge of all newer vertices about the individual properties or existence of the present vertices.

Mossa et al. [38] showed that the power-law behavior might be truncated due to information filtering. In their case a newly attached vertex is only aware of a certain subset of the existing vertices. This subset is however chosen randomly for each vertex individually. Therefore the incomplete information has no global properties but is instead a local property.

Here we want however to follow another route with two distinct models to deal with the more interesting case that the incomplete information is attached to the new vertices individually and still global with respect to the whole network. One model will mimic a generic and global effect that is present in all real citation networks while the other describes the influence of individual information unawareness.

## 2 Growing nets and latency

For a newly created vertex the only relevant information is a list of existing vertices to connect to and their respective degrees. Incomplete information
results in the ignorance of some of those vertices. This effect is here mediated via an 'awareness' function $\phi(s, t)$ that makes the newly connected vertex $t$ aware of the existing vertex $s$.

$$\phi(s, t) := \begin{cases} 
1 & s \text{ is known to } t \\
0 & \text{else} 
\end{cases}$$

Eq. (1) becomes then

$$\frac{\partial \bar{k}(s, t)}{\partial t} = m \cdot \frac{\bar{k}(s, t)\phi(s, t)}{\int_0^t \bar{k}(u, t)\phi(u, t)}$$

(2)

While there are many choices for $\phi(s, t)$ we will propose one particular structure to resemble actual effects in citation networks. We will further set $m = 1$ as this network property has no influence on the scaling exponent $\gamma$ in the models.

A newly created link might not be aware of the most recent created entities in the network. One encounters this situation actually very often: the author of a new WWW-page cannot be aware of other recently created pages that he would like to link to. Search engines do not provide for instantaneous listing of just created pages and therefore authors can find new pages just by chance.

This incomplete information about the vertices of the network results in a selection of 'older' vertices for linking. We model this by the setting

$$\phi_<(s, t) := \Theta (c \cdot t - s)$$

with some constant $0 < c \leq 1$ and $\Theta(x)$ the Heaviside-step-function. Therefore a newly created vertex is only aware of the oldest $c$ fraction of the existing vertices.

While this seems to be at a first glance to strong of an assumption, this is realized in exponentially growing systems like the WWW: suppose from the current real-time $\tau$ we can not know new pages younger than some period $\tau_c$. Then all existing pages that are capable of attracting a link are taken from the interval $[0; \tau - \tau_c]$. As the number of pages obeys however $\sim \exp (\alpha \cdot \tau)$ this translates in the $N$-notion to $[0; c \cdot N(\tau)]$. Here we have set $c := \exp (-\alpha \tau_c)$. Recall that the time $t$ of eq. (1) is actually the number of vertices so that we actually draw a vertex from $[0; c \cdot t]$.

Rewriting the definition of $\phi_<(s, t)$ we get $\phi_<(s, t) = \Theta (c - s/t)$ and see that this function scales with respect to $s/t$. We can then set $x = s/t$ and
rewrite equation \([2]\) to (with \(\bar{k}(s, t) = \kappa(x)\))

\[
-x \frac{d\kappa(x)}{dx} = \frac{\kappa(x)\phi_<(x)}{\int_0^x dx \kappa(x)\phi_<(x)} = \frac{\kappa(x)\phi_<(x)}{\int_0^c dx \kappa(x)} = \beta \cdot \kappa(x)\phi_<(x) \quad \text{with} \quad \kappa(1) = 1
\]

\(\beta^{-1}\) is the integral in the denominator \([36]\). The boundary condition \(\kappa(1) = 1\) reflects the fact, that upon creation a new node \(s = t\) has only one connection. The solution of this differential equation is

\[
\kappa(x) = \begin{cases} 
\bar{c} \cdot x^{-\beta} & 0 < t \leq c \\
\text{const} = 1 & t > c
\end{cases}
\]

As the function has to be continuous we must have \(\kappa(1) = \kappa(c)\). We can derive the value for \(\bar{c}\) and arrive at

\[
\kappa(x) = \begin{cases} 
\left(\frac{x}{c}\right)^{-\beta} & 0 < t \leq c \\
1 & t > c
\end{cases}
\]

The equation for \(\beta\) is then

\[
\frac{1}{\beta} = \int_0^c \kappa(x) \, dx = \frac{c}{1-\beta} \quad \iff \quad \beta = \frac{1}{1+c}
\]

By scaling arguments one can prove that in general the relation \(\gamma = 1 + 1/\beta\) holds \([36, 39, 40]\). Using this we conclude that here \(\gamma = 2 + c\). This is depicted in figure \([4]\) which compares this result with computer simulations.

We can further analyze the emerging networks by investigating the shortest-paths in these networks. Figure \([2]\) shows the increase of both the average \(l_{av}\) and the maximum \(l_{max}\) of the shortest path lengths \([41]\) in independently created networks with increasing \(c\) above \(c \approx 0.2\).

A smaller \(c \in [0.2; 1.0]\) reduces the number of available vertices in the growth process and the network gets more dense: the shortest path lengths get smaller and the probability of vertices with larger number of connections bigger - as can be seen from the previous result \(\gamma = 2 + c\).

For very small \(c < 0.2\) we see however an increase in \(l_{av}\) and \(l_{max}\). This is an artifact due to our initial setting of the first \(1/c\) vertices to form a chain. While the chain guarantees that none of the first vertices is preferred to another initial vertex, the path lengths are largely influenced now.
Figure 1: a) For $c = 0.4$ with $N=10^6$ and sampled over 100 replica we obtain a good fit to $N(k) = N \cdot p(k) \sim k^{-\gamma}$ with $\gamma_{\text{fit}} \approx 2.37$ in the interval $\log(k) \in [2.2; 5.5]$. The fitted curve was shifted for clarity. b) The exponent in the scaling law $P(k) \sim k^{-\gamma}$ as a function of the information awareness $c$ in simulated networks. The straight line is the function $\gamma = 2 + c$. For each point 600 independent networks of 10,000 vertices each were sampled and the best fit was used. The error bars stem from the fitting procedure of the cumulative distribution [36].
Figure 2: The average and the maximum of the minimal path lengths in the network grown with an information awareness of $c$. For every point we sampled over 100 independent networks. The first $1/c$ vertices were initialized to form a chain, so giving no preferences to any of them.
the smallest \( c = 0.025 \) used here we get an \( l_{\text{max}} = 40.99 \approx 1/c \). Here the chain length dominates the path lengths.

Suppose that WWW-pages are the vertices in this scenario and links are the edges of the network. This model then takes into account the time that e.g. search engines need to encounter new web-pages and make the general public aware of those sites. The anonymity of a vertex \( s \) is healed over time by sliding into the focus of new vertices as soon as \( s < c \cdot t \). Here the incomplete information refers to the knowledge of every new vertex uniformly.

There is also the opposite scenario in which some vertices are aware of the full information (that is the number of connections all the existing vertices possess) and others are just ignorant and connect with equal probability to any of the existing ones. Here the incompleteness of information is restricted to a subset of individuals:

3 Growing nets and partial ignorance

Suppose that a newly added vertex is with some probability \( p \) aware of all the connectivities of the other vertices. In this case it is attached with preferential linking described above. With probability \( 1 - p \) it is connected without preference. We want to deduce the effect on the connectivity distribution from the master equation for the average number of connections of degree \( k \) at time \( t \)

\[
N(k, t + 1) = p \cdot \left[ N(k, t) + \frac{k - 1}{\bar{k}} N(k - 1, t) - \frac{k}{\bar{k}} N(k, t) \right] \\
+ (1 - p) \cdot \left[ N(k, t) + \frac{1}{t} N(k - 1, t) - \frac{1}{t} N(k, t) \right] + \delta_{k,1} \tag{4}
\]

Here \( \bar{k} = 2 \) is the average degree of each vertex. The first term describes the preferential linking with its in- and outflow while the second term provides for the additional connections or loss thereof with equal probability. Notice that there are currently \( t \) vertices in the network, so \( 1/t \) is the probability of hitting any one of those. The third term is finally responsible for the newly added 'guy'. By changing to continuous time we get from eq. (4)

\[
\frac{\partial}{\partial t}(t \cdot P(k, t)) = \left[ p \frac{k - 1}{k} + 1 - p \right] P(k - 1, t) - \left[ p \frac{k}{k} + 1 - p \right] P(k - 1, t) + \delta_{k,1}
\]
where \( P(k, t) = N(k, t)/t \) is the density of vertices with degree \( k \) at time \( t \).

We can now solve for the stationary distribution for \( t \to \infty \). We arrive at

\[
P(k) = \frac{p k - p + \bar{k} - p \bar{k}}{p k + 2 \bar{k} - p k} P(k - 1) + \frac{\bar{k}}{p k + 2 \bar{k} - p k} \delta_{k,1}
\]

This is further written as

\[
P(k) = 2 \left( \frac{\Gamma \left( \frac{4}{p} \right) \Gamma \left( \frac{2}{p} - 2 + k \right)}{\Gamma \left( \frac{2}{p} - 1 \right) \Gamma \left( \frac{4}{p} - 1 + k \right)} \right)
\]

(5)

Using the relationship

\[
\frac{\Gamma (k + a)}{\Gamma (k + b)} = k^{a-b} \left[ 1 + O \left( k^{-1} \right) \right]
\]

for large \( k \) we conclude that \( \gamma = 1 + \frac{2}{p} \) for large \( k \) with a divergence for the exponent when approaching \( p = 0 \) as in this case we have no preferential linking at all. In this case the starting master equation (4) leads correctly to the Poisson-distribution \( P_{p=0}(k) = 2^{-k} \). The diverging behavior of the exponent was for instance also found by Krapivsky and Redner [43] in their treatment of growing networks with redirection. Figure 4 shows the results from computer experiments for this model. The smaller the \( p \) the more difficult it is to see any indication of the power-law.

## 4 Conclusion

In this paper we developed two distinct models to describe the effect of 1) global incomplete information caused by penetration rates while constructing a citation network and 2) local incomplete information of individual vertices that are attached with a probability of 'non-knowledge'. We derived the scaling behavior of the degree distribution for large degrees in both cases and compared this to computer experiments. Both models approach the analytic value [4] of \( \gamma = 3 \) when reaching full information. The incomplete information in the two models does not destroy the scale-free-behavior of the systems while Mossa et al. [38] found a cross-over from scale-free-behavior to an exponential in another model which takes information into account. By
Figure 3: The cumulative number of connections $N_{\text{cum}}(k) = \sum_{k' = k}^{\infty} N(k')$ of degree $k$ in networks of size $L$ averaged over 500 independent runs. The data was shifted for a better overview. The straight line through the data of $L = 10,000$ is the derived result of eq. (5) with $p = 0.4$ while the broken line indicates an asymptotic power-law for the cumulative number of a distribution with $\gamma = 1 + 2/p = 6$. 
comparison one can see the influence incomplete information may have on
the global structure of growing networks.

We will work out particulars on real-world-networks and the influence of
incomplete information in a forthcoming study.

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