Conservation Laws in a First Order Dynamical System of Vortices

N. S. Manton* and S. M. Nasir†

Department of Applied Mathematics and Theoretical Physics
University of Cambridge
Silver Street, Cambridge CB3 9EW, England

September, 1998

Abstract

Gauge invariant conservation laws for the linear and angular momenta are studied in a certain 2+1 dimensional first order dynamical model of vortices in superconductivity. In analogy with fluid vortices it is possible to express the linear and angular momenta as low moments of vorticity. The conservation laws are compared with those obtained in the moduli space approximation for vortex dynamics.

PACS: 11.27.+d, 11.30.-j, 74.20.De
1 Introduction

Recently, a 2+1 dimensional field theory was proposed to describe the non-dissipative dynamics of vortices in thin-film superconductors [8]. The model, which has $U(1)$ gauge invariance, has a Lagrangian of Schrödinger–Chern–Simons type containing terms linear in the first time derivatives of the fields, and for static fields, the Lagrangian reduces to the standard Ginzburg–Landau model. Interestingly, in this model two vortices can be shown to orbit around each other. It has been argued that such motion occurs in superconductors at very low temperature [1, 13].

For certain values of the coupling constants, the model has a large space of static solutions consisting of multi-vortices obeying Bogomol’nyi equations [4]. The space of such $N$-vortex solutions, whose parameters are the vortex positions, is known as the $N$-vortex moduli space. When the coupling constants have slightly different values, one can study vortex dynamics in the model by obtaining a reduced theory, where one projects the motion onto the moduli space of Bogomol’nyi vortices. This is the essence of the moduli space approximation to describe soliton dynamics [7]. The dynamical variables of the reduced theory are just the time-varying vortex positions. The moduli space approach cannot deal with the dynamics of interacting vortices and anti-vortices.

The moduli space approximation has been established rigorously by Stuart [16] in the context of slowly moving Bogomol’nyi vortices in the relativistic Abelian Higgs model. However, in the proposed Schrödinger–Chern–Simons type of model of vortices it is not yet certain that one can use the moduli space approximation faithfully in order to extract dynamics. Obtaining conservation laws provides an important consistency check.

The main conclusion in [8] was the Lagrangian of the reduced theory (eqn.(2.15) below). From this it is straightforward to obtain conserved quantities of the reduced dynamics, which can be interpreted as the linear and angular momenta. These have some interesting features. For example, the linear momentum turns out to be related to the mean of the vortex positions, which is not surprising, since in a first order dynamical system the linear momentum is often related to position. However, the conserved linear and angular momenta in the reduced theory were not directly related to the linear and angular momenta of the parent field theory in [8]. This omission is rectified here.

The conserved quantities of the parent field theory have to be derived with care. The naïve canonical linear and angular momenta are not gauge invariant; moreover, they are not conserved if the field configuration has non-trivial topology, because of currents at infinity. It is important to obtain gauge invariant conservation laws for the linear and angular momenta in the parent theory.

The relevant conserved quantities are, in fact, known. They have been obtained by Hassaine et al. [5] by identifying the model with the Jackiw–Pi model [6] in a background field. Here, we obtain the conservation laws from first principles using Noether’s theorem and we clarify
the issues of gauge invariance and currents at infinity. Following Papanicolau and Tomaras [9], who studied conservation laws in a not very different model, we also express the linear and angular momenta as moments of vorticity. This establishes an analogy between our model and models of fluid vortices [3]. Finally, by evaluating the linear and angular momenta for fields satisfying the Bogomol’nyi equations, we will show that they coincide with the linear and angular momenta in the reduced theory.

The outline of this paper is as follows. In Sect.2, we describe the model, and the reduced theory of vortex dynamics obtained using the moduli space approximation. In Sect.3, we obtain the conserved linear and angular momenta and express them in terms of vorticity. In Sect.4, we evaluate these expressions for Bogomol’nyi fields, and compare with the conserved quantities in the reduced theory. Sect.5 contains our conclusions.

2 The model

(i) The Schrödinger–Chern–Simons Lagrangian

Let φ be a complex (Higgs) scalar field representing the condensate of the superconducting electrons and let \( a_\alpha (\alpha = 0, 1, 2) \) be the U(1) gauge potential. We will use the subscript 0 to refer to time and the subscripts 1,2 to refer to the two directions in space. The Lagrangian of the model is

\[
L = T - V
\]

where the kinetic energy

\[
T = \int \left( \frac{\gamma}{2} (\bar{\phi} D_0 \phi - \phi D_0 \bar{\phi}) + \mu (B a_0 + E_2 a_1 - E_1 a_2) - \gamma a_0 \right) d^2 x
\]

and the potential energy

\[
V = \int \left( \frac{1}{2} B^2 + \frac{1}{2} D_i \phi D_i \bar{\phi} + \frac{\lambda}{8} (1 - |\phi|^2)^2 + a_i J_i^T \right) d^2 x.
\]

Here, \( \gamma, \mu \) and \( \lambda \) are real constants with \( \lambda \) positive, \( D_\alpha \phi = (\partial_\alpha - i a_\alpha) \phi \) are the components of the covariant derivative of \( \phi \), \( B = \partial_1 a_2 - \partial_2 a_1 \) is the magnetic field, the electric field \( E_i = \partial_i a_0 - \partial_0 a_i \) and \( J_i^T \) is a constant transport current. We assume the summation convention in the spatial index \( i = 1, 2 \). The Schrödinger term (with coefficient \( \gamma \)) and the Chern–Simons term (with coefficient \( \mu \)) define the kinetic energy for the scalar field and gauge potential. The term \( \gamma a_0 \), introduced by Barashenkov and Harin [4], allows the possibility of a non-zero condensate in the ground state. The potential energy is the Ginzburg–Landau energy functional. Notice that the kinetic energy contains terms with only the first power of time derivatives. It was shown in [3] that \( L \) is Galilean invariant. Galilean invariance determines the following role for the transport current. Given any solution of the field equations in the absence of a transport
current, the effect of $J_i^T$ is simply to boost the solution with a velocity $v_i = \frac{1}{\gamma} J_i^T$. Having understood this role of the transport current, we will henceforth neglect it.

The field equations obtained by varying respectively $\bar{\phi}$, $a_i$ and $a_0$ are

$$i\gamma D_0 \phi = -\frac{1}{2} D_i D_i \phi - \frac{\lambda}{4} (1 - |\phi|^2) \phi$$  \hspace{1cm} (2.4)

$$\epsilon_{ij} \partial_j B = J_i + 2\mu \epsilon_{ij} E_j$$  \hspace{1cm} (2.5)

$$2\mu B = \gamma (1 - |\phi|^2)$$  \hspace{1cm} (2.6)

where $J_i$ is the supercurrent defined by

$$J_i = -\frac{i}{2} (\bar{\phi} D_i \phi - \phi D_i \bar{\phi})$$  \hspace{1cm} (2.7)

Eqn.(2.4) is the gauged non-linear Schrödinger equation, eqn.(2.5) is the two-dimensional version of Ampère’s law and eqn.(2.6) is a constraint. Such a constraint appears in other Chern–Simons type theories [6]. It is useful to note that this constraint is one of the Bogomol’nyi equations for vortices when $\gamma = \mu$ [4]. We shall assume that $1 - |\phi|^2$ and $D_i \phi$ decay rapidly as $|x| \to \infty$. Eqns.(2.4)–(2.7) imply that $D_0 \phi$, $B$ and $E_i$ decay similarly.

(ii) Vortices

The above model admits vortex solutions. Vortices appear whenever there is a non-trivial winding of the map between the boundary circle at spatial infinity and the manifold of ground states of the scalar field, the circle $|\phi| = 1$. The relation between the winding number $N$ and the magnetic flux is

$$\int B \, d^2 x = 2\pi N.$$  \hspace{1cm} (2.8)

$N$ can be interpreted as the vortex number. For a vortex $N = 1$ and for an anti-vortex $N = -1$. Henceforth, we suppose $N \geq 0$. Later, we will define a gauge invariant vorticity $\mathcal{V}$, whose integral is $2\pi N$. However, the vorticity is not simply $\mathcal{V} = B$.

Generally, a solution with $N$ vortices is not static, and we wish to understand how the vortices move. However, it is by now well-known that for special values of the couplings a large space of stable, static $N$-vortex solutions exists, for any $N > 0$ [17]. These solutions satisfy first order Bogomol’nyi equations, as well as the second order Ginzburg–Landau field equations [4]. For the model here, Bogomol’nyi vortices occur when $\lambda = 1$ and $\gamma = \mu$. The first order Bogomol’nyi equations are

$$(D_1 + iD_2) \phi = 0$$  \hspace{1cm} (2.9)

$$B = \frac{1}{2} (1 - |\phi|^2).$$  \hspace{1cm} (2.10)

Solutions of these equations also satisfy (2.4)–(2.7), with $D_0 \phi$ and $E_i$ vanishing. Bogomol’nyi vortices do not exert forces on each other and this is why a static configuration of $N$ vortices
can exist. The solutions of the Bogomol’nyi equations with winding number $N$ are uniquely specified by the unordered zeros of the scalar field, whose number, counted with multiplicity, is $N$. These zeros are the vortex positions and we denote them $\{x^s : 1 \leq s \leq N\}$. The space of solutions, called the $N$-vortex moduli space, is therefore topologically $\mathbb{C}^N/\Sigma_N$, where $\Sigma_N$ is the permutation group on $N$ objects and the two-dimensional real plane is identified with the complex plane $\mathbb{C}$. The $N$-vortex moduli space is a smooth manifold of dimension $2N$, despite the apparent singularities where vortex positions coalesce.

Hassaïne et al. have recently discovered stationary Bogomol’nyi-type vortex solutions in this model with $\gamma \neq \mu$. The fields satisfy (2.9) and (2.6), and in addition $a_0$ is proportional to $B$. One needs $\lambda = 2\gamma/\mu - \gamma^2/\mu^2$, and $\lambda$ must be positive. These vortices are sources for non-vanishing electric fields. However, we shall not consider these solutions here.

(iii) The reduced theory

We consider the case where $\lambda$ is close to 1 and $\gamma = \mu$. We are interested in fields which remain close to $N$-vortex solutions of the Bogomol’nyi equations, but in which the vortex positions move slowly. In the moduli space approximation to vortex dynamics, one can obtain a reduced Lagrangian by simply inserting Bogomol’nyi solutions into (2.1) and taking the vortex positions dependent on time. Let us write $\phi$ as

$$\phi = e^{\frac{1}{2}h + i\chi}. \quad (2.11)$$

$h$ is gauge invariant, and tends to zero at spatial infinity, but is singular at the vortex positions. The first Bogomol’nyi equation (2.9) implies that

$$a_i = \frac{1}{2} \epsilon_{ij} \partial_j h + \partial_i \chi. \quad (2.12)$$

From the second Bogomol’nyi equation (2.10), one obtains

$$\partial_i \partial_i h = e^h - 1 + 4\pi \sum_{s=1}^{N} \delta^2(x - x^s). \quad (2.13)$$

We assume this equation holds, even if the vortex positions $x^s$ are slowly moving.

Let us now suppose that the vortex positions are distinct, which is the generic case. It is not difficult to allow for vortex coalescence. $h = \log |\phi|^2$ has the following expansion around the position of the $s$-th vortex

$$h = \log |x - x^s|^2 + a^s + b_1^s(x_1 - x_1^s) + b_2^s(x_2 - x_2^s) + \cdots \quad (2.14)$$

where $\{a^s, b_i^s\}$ are dependent on the positions of the other vortices relative to $x^s$. The usefulness of this expansion was discovered by Samols [11], developing work of Strachan [14]. $a^s$ plays no significant role in what follows, but $b_i^s$ does. $b_i^s$ is a measure of the lack of circular symmetry of $h$ around the vortex, and is exponentially small if the other vortices are far away. After
various integrations, and suppression of total time derivative terms, one obtains the manifestly
gauge invariant reduced Lagrangian \[8\]

\[
L_{\text{red}} = 2\pi\gamma \sum_{s=1}^{N} \left( (b_{2s}^s + \frac{1}{2}x_{2s}^s)\dot{x}_{1s}^s - (b_{1s}^s + \frac{1}{2}x_{1s}^s)\dot{x}_{2s}^s \right) - V_{\text{red}}
\]  

(2.15)

where an overdot denotes time-derivative. This leads directly to equations of motion for the
vortex positions. The potential (2.3) simplifies for solutions of the Bogomol’nyi equations to
the integral \[V_{\text{red}} = \frac{\lambda - \frac{1}{8}}{8} \int (1 - |\phi|^2)^2 d^2x\], plus a constant \(\pi N\), and this is a translationally
and rotationally invariant function of the vortex positions. Unfortunately, it appears that
\(V_{\text{red}}\) cannot be simplified to an explicit expression depending only on \(\{x^s, a^s, b_i^s\}\). The functions
\(b_i^s\) and \(V_{\text{red}}\) are not known explicitly as functions of the relative positions of
\(N\)-vortices, but they can be calculated numerically and this has been done for 2-vortices in [11, 12].

3 Conservation laws in the field theory

The linear and angular momenta for the field theory we are considering here were obtained in
[5]. Here, we give an independent derivation from first principles. Let \(\{\psi_c\} = \{\phi, \bar{\phi}, a_0, a_1, a_2\}\),
where \(c\) runs from 1 to 5. If under a variation of the fields \(\delta\psi_c\), the variation of the Lagrangian
density, \(L\), is \(\delta L = \partial_{\mu} \dot{X}^\mu\), then Noether’s theorem associates a conserved current with such a
variation. (Here and below, we suppress the infinitesimal quantity multiplying such variations.)
The Noether current, assuming the summation convention over the index \(c\), is

\[
\dot{j}^\mu = \frac{\partial L}{\partial(\partial_\mu \psi_c)} \delta\psi_c - \dot{X}^\mu.
\]  

(3.1)

By Noether’s theorem \(\partial_{\mu} \dot{j}^\mu = 0\), and it follows that the integral of the time component \(\dot{j}^0\) is
a conserved quantity provided that the spatial components of the current \(\dot{j}^1\) and \(\dot{j}^2\) fall off
sufficiently fast at spatial infinity.

(i) Energy

The simplest conserved quantity to consider is energy. This is related to invariance under
time translation. Naively, the variations of the fields are given by their time derivatives. However, one can supplement this by a gauge transformation with parameter \(-a_0\). The
variations of the fields are then

\[
\{\delta\psi_c\} = \{D_0\phi, \bar{D}_0\bar{\phi}, 0, -E_1, -E_2\}
\]  

(3.2)

and the change in \(L\) is \(\delta L = \partial_{\mu} X^\mu\), where

\[
X^0 = L + \gamma a_0 - \mu a_0 B, \quad X^1 = -\mu a_0 E_2, \quad X^2 = \mu a_0 E_1.
\]  

(3.3)
Using (3.1), the energy density is
\[ j^0 = \frac{1}{2} B^2 + \frac{1}{2} D_1 \phi D_1 \phi + \frac{\lambda}{8} (1 - |\phi|^2)^2. \] (3.4)

\( j^0 \) is gauge invariant. Moreover, its integral is conserved, because the spatial components of the currents
\[ j^1 = -\frac{1}{2} D_1 \phi D_0 \phi - \frac{1}{2} D_1 \phi D_0 \phi + B E_2 \] (3.5)
\[ j^2 = -\frac{1}{2} D_2 \phi D_0 \phi - \frac{1}{2} D_2 \phi D_0 \phi - B E_1 \] (3.6)
are gauge invariant, and hence decay rapidly at spatial infinity. Thus, the conserved energy is \( V \), as given in (2.3) (recall that the transport current is set to zero).

(ii) Momentum

Let us now find the linear momentum components, \( P_i \), associated with translation in the \( x_i \)-direction. First, consider translation in the \( x_1 \)-direction. The naïve variations of the fields are given by the spatial derivatives in the \( x_1 \)-direction. One supplements this by a gauge transformation with parameter \(-a_1\). The variations of the fields are then
\[ \{ \delta \psi \} = \{ D_1 \phi, D_1 \phi, E_1, 0, B \} \] (3.7)
and the change in \( \mathcal{L} \) is \( \delta \mathcal{L} = \partial_\mu \tilde{X}^\mu \), where
\[ X^\mu = -\mu a_1 B + \gamma a_1, \quad X^1 = \mathcal{L} - \mu a_1 E_2, \quad X^2 = \mu a_1 E_1. \] (3.8)
The density of the linear momentum in the \( x_1 \)-direction, calculated using (3.1), is
\[ j^{01} = -\gamma (J_1 + a_1). \] (3.9)
Notice that \( j^{01} \) is not gauge invariant. Moreover, the spatial components of the currents are
\[ j^{11} = -\gamma \frac{i}{2} (\phi D_0 \phi - \phi D_0 \phi) - \frac{1}{2} B^2 - \frac{1}{2} |D_1 \phi|^2 + \frac{1}{2} |D_2 \phi|^2 + \frac{\lambda}{8} (1 - |\phi|^2)^2 + \gamma a_0 \] (3.10)
and
\[ j^{12} = -\frac{1}{2} (D_2 \phi D_1 \phi + D_2 \phi D_1 \phi). \] (3.11)
\( j^{11} \) is not gauge invariant, and hence does not fall off sufficiently fast at infinity for the integral of \( j^{01} \) to be conserved. The remedy for both problems is to note that \( X^\mu \) is not uniquely defined, but can be altered by adding total derivative terms. One chooses an improved \( \tilde{X}^\mu \), with \( \partial_\mu X^\mu = \partial_\mu \tilde{X}^\mu \), in such a way that the resulting current is gauge invariant. \( \tilde{X}^\mu \) can be taken as
\[ \tilde{X}^0 = X^0 + \gamma \partial_1 (x_2 a_2) - \gamma \partial_2 (x_2 a_1) \] (3.12)
\[ \tilde{X}^{11} = X^{11} + \gamma \partial_2 (x_2 a_0) - \gamma \partial_0 (x_2 a_2) \] (3.13)
\[ \hat{X}^\prime_2 = X^\prime_2 + \gamma \partial_0 (x_2 a_1) - \gamma \partial_1 (x_2 a_0). \]  

(3.14)

Using \( \hat{X}^\mu \), the improved density of the linear momentum in the \( x_1 \)-direction is

\[ \tilde{j}^0 = -\gamma (J_1 + x_2 B) \]  

(3.15)

and the spatial components of the current are

\[ \tilde{j}^i = -\gamma \frac{i}{2} (\partial^i \phi - \phi^i \partial \phi) - \frac{1}{2} B^2 - \frac{1}{2} |D_1 \phi|^2 + \frac{1}{2} |D_2 \phi|^2 + \frac{\lambda}{8} (1 - |\phi|^2)^2 - \gamma x_2 E_2 \]  

(3.16)

\[ \tilde{j}^2 = -\frac{1}{2} (\overline{D_2 \phi} D_1 \phi + D_2 \phi \overline{D_1 \phi}) + \gamma x_2 E_1. \]  

(3.17)

Clearly, \( \tilde{j}^1 \) and \( \tilde{j}^2 \) are now gauge invariant and fall off sufficiently fast at spatial infinity. Similarly, one can consider translations in the \( x_2 \)-direction. One concludes that the conserved linear momentum is

\[ P_i = -\int \tilde{j}^0 d^2 x = \gamma \int (J_i + \epsilon_{ij} x_j B) \ d^2 x. \]  

(3.18)

(The choice of minus sign, here and in \( (3.26) \) below, is deliberate, and ensures agreement between the conservation laws in the field theory and those in the reduced theory. It is made because the field variations are due to passive coordinate variations, whereas later we will be actively varying the vortex positions in the reduced theory.)

(iii) Angular momentum

Now, let us obtain the conservation law for angular momentum, \( M \), by considering the generator of rotations, \( \epsilon_{ij} x_i \partial_j \), combined with a gauge transformation with parameter \( -\epsilon_{ij} x_i a_j \).

Here, care is needed to evaluate the Lie derivatives correctly on the scalar field and gauge potential. The variations of the fields are

\[ \{ \delta \psi_c \} = \{ \epsilon_{ij} x_i D_j \phi, \epsilon_{ij} x_i \overline{D_j \phi}, \epsilon_{ij} x_i E_j, -\partial_1 B, -\partial_2 B \}. \]  

(3.19)

The change in \( \mathcal{L} \) is \( \delta \mathcal{L} = \partial_\mu X^{\mu} \), where

\[ X^{\mu_0} = -\epsilon_{ij} x_i a_j (\mu B - \gamma), \quad X^{\mu_1} = -x_2 \mathcal{L} - \mu \epsilon_{ij} x_i a_j E_2, \quad X^{\mu_2} = x_1 \mathcal{L} + \mu \epsilon_{ij} x_i a_j E_1. \]  

(3.20)

The angular momentum density obtained using (3.1) is

\[ j^{\mu_0} = -\gamma \epsilon_{ij} x_i (J_j + a_j). \]  

(3.21)

Neither this density nor the spatial components of the current are gauge invariant, nor do they fall off sufficiently fast at spatial infinity. Again, the remedy is to find an improved \( \hat{X}^\mu \), with \( \partial_\mu X^{\mu} = \partial_\mu \hat{X}^{\mu} \). One may take

\[ \hat{X}^{\mu_0} = X^{\mu_0} - \gamma \partial_1 (a_2 r^2/2) + \gamma \partial_2 (a_1 r^2/2) \]  

(3.22)

\[ \hat{X}^{\mu_1} = X^{\mu_1} - \gamma \partial_2 (a_0 r^2/2) + \gamma \partial_0 (a_2 r^2/2) \]  

(3.23)
\[
\dot{X}^{\mu 2} = X^{\mu 2} - \gamma \partial_0 (a_1 r^2/2) + \gamma \partial_1 (a_0 r^2/2).
\]

Then, the improved angular momentum density is
\[
\dot{j}^{\mu 0} = -\gamma (\epsilon_{ij} x_i J_j - \frac{1}{2} r^2 B)
\]
which is clearly gauge invariant. Likewise, \( \dot{j}^{\mu 1} \) and \( \dot{j}^{\mu 2} \) are gauge invariant and do fall off sufficiently fast. The conserved angular momentum is therefore
\[
M = -\int \dot{j}^{\mu 0} d^2 x = \gamma \int (\epsilon_{ij} x_i J_j - \frac{1}{2} r^2 B) \, d^2 x.
\]

(iv) Vorticity

Let us now define the vorticity
\[
\mathcal{V} = \epsilon_{ij} \partial_i J_j + B.
\]
Substituting for \( J_i \), using (2.7), the vorticity can be written as
\[
\mathcal{V} = -i \epsilon_{ij} D_i \phi D_j \phi + B(1 - |\phi|^2),
\]
which is the definition in [9], and is a gauge invariant generalisation of the notion of vorticity discussed in [10]. In the sector with vortex number \( N \),
\[
\int \mathcal{V} \, d^2 x = 2\pi N,
\]
using (3.27) and Stokes’ theorem. Integrating by parts, one may express the linear and angular momenta as the following moments of the vorticity
\[
P_i = \gamma \epsilon_{ij} \int x_j \mathcal{V} \, d^2 x
\]
and
\[
M = -\frac{\gamma}{2} \int r^2 \mathcal{V} \, d^2 x.
\]
We noted in the introduction that in a first order dynamical system the linear momentum can often be taken as a measure of position. The formula (3.30) shows that this idea applies here. The components of momentum are proportional to the components of the centre of vorticity. Eqns. (3.30) and (3.29) imply that \( R_i = -\frac{1}{2\pi N \gamma} \epsilon_{ij} P_j \) is the centre of vorticity [9], and it does not move.

The conservation of the angular momentum (3.31) shows that no net vorticity can escape to infinity, assuming that singularities do not form, and therefore a vortex cannot escape to infinity, unless accompanied by an anti-vortex.

The vorticity has the following interesting property for Bogomol’nyi vortices. Substituting (2.10) in (2.3), the energy density for Bogomol’nyi vortices is seen to be
\[
\mathcal{E}^{\text{Bog}} = \frac{1}{2} |D_i \phi|^2 + B^2.
\]
On the other hand, using both Bogomol’nyi equations, the vorticity (3.28) can be rewritten as
\[ \mathcal{V} = |D_i \phi|^2 + 2B^2. \] (3.33)
Thus, for Bogomol’nyi vortices \( \mathcal{V} = 2\xi^{\text{Bog}} \).

4 Conservation laws in the reduced dynamics

(i) The conserved quantities

Conservation laws of the reduced dynamics can be obtained directly from \( L^\text{red} \), eqn. (2.15). (Note that the discussion of conservation laws at the end of ref. [8] is slightly wrong.) In general, a variation \( \delta x_s^i = \xi_s^i \) is a symmetry if \( \delta L^\text{red} = \frac{d}{dt} X \) for some \( X \). Noether’s theorem states that
\[ \sum_{s=1}^{N} \frac{\partial L^\text{red}}{\partial \dot{x}_s^i} \xi_s^i - X \] (4.1)
is then conserved.

A translation of all the vortex positions in the \( x_1 \)-direction is a symmetry. Here, for all \( s \),
\[ \delta x_1^s = 1, \quad \delta x_2^s = 0, \] (4.2)
and \( b_s^i \) and \( V^\text{red} \) are invariant. One finds that \( X = -\pi \gamma \sum_{s=1}^{N} x_2^s \), and the conserved component of momentum is
\[ P_1^\text{red} = 2\pi \gamma \sum_{s=1}^{N} (b_2^s + x_2^s). \] (4.3)
Similarly, translation in the \( x_2 \)-direction is a symmetry, and
\[ P_2^\text{red} = -2\pi \gamma \sum_{s=1}^{N} (b_1^s + x_1^s) \] (4.4)
is conserved. It was shown by Samols that \( \sum_{s=1}^{N} b_2^s = 0 \). The linear momentum in the reduced dynamics is therefore simply
\[ P_i^\text{red} = 2\pi \gamma \epsilon_{ij} \sum_{s=1}^{N} x_j^s, \] (4.5)
and directly related to the mean of the vortex positions. Conservation of momentum implies that the vortices circulate about their fixed mean position.

There is also symmetry under a rotation, where, for all \( s \)
\[ \delta x_1^s = -x_2^s, \quad \delta x_2^s = x_1^s. \] (4.6)
\( V^\text{red} \) is invariant, but the rotation leads to the variations
\[ \delta b_1^s = -b_2^s, \quad \delta b_2^s = b_1^s \] (4.7)
It follows that \( L^{\text{red}} \) is strictly invariant, with \( X = 0 \), so one has the following conserved angular momentum in the reduced dynamics

\[
M^{\text{red}} = -2\pi \gamma \sum_{s=1}^{N} \left( \frac{1}{2} x_i^s x_i^s + b_i^s x_i^s \right).
\]

This conservation law probably implies that no vortex can escape to infinity, but we do not have enough information on the functions \( b_i^s \) to prove this. If the \( s \)-th vortex went to infinity, \( x_i^s x_i^s \) would diverge while \( b_i^s x_i^s \) would tend to zero. This could be compensated by singularities in the values of \( b_i^{s'} \) (\( s' \neq s \)). Now such singularities do occur as vortices coalesce \([11]\), but they appear (at least for two vortices) to cancel in \( \sum_{s=1}^{N} b_i^s x_i^s \), leaving a finite result.

(ii) Comparison with the field theory

We compare the conserved quantities in our field theory with the corresponding conserved quantities obtained directly from the reduced Lagrangian, \( L^{\text{red}} \), by assuming the fields satisfy the Bogomol’nyi equations at all times, possibly with time-varying vortex positions. This is sensible if \( \lambda \) is close to 1 and \( \mu = \gamma \).

First of all, the conserved energy is \( E = \pi N + V^{\text{red}} \). This is consistent with the reduced dynamics, where the Hamiltonian is simply \( V^{\text{red}} \) (the constant \( \pi N \) is dropped), and \( V^{\text{red}} \) is conserved.

The main task is to evaluate the expressions (3.30) and (3.31) for linear and angular momentum. Using (2.11) and (2.12), it can be shown that for solutions of the Bogomol’nyi equations

\[
J_i = -\frac{1}{2} \epsilon_{ij} \partial_j h e^h, \quad B = -\frac{1}{2} \partial_i \partial_i h.
\]

From (3.27), the vorticity \( \mathcal{V} \) becomes

\[
\mathcal{V} = \frac{1}{2} \partial_i \partial_i (e^h - h).
\]

Another expression for \( \mathcal{V} \) is

\[
\mathcal{V} = \frac{1}{2} \partial_i (\partial_i h(e^h - 1)) = \frac{1}{2} \partial_i (\partial_i h \partial_j \partial_j h),
\]

where use has been made of (2.13) and temporarily we ignore the logarithmic singularities of \( h \). Note that \( \mathcal{V} \) is a smooth function despite the singularities of \( h \).

In what follows we will again suppose that \( \phi \) has \( N \) simple zeros. In order to carry out the integrals involving moments of \( \mathcal{V} \) let us remove, from \( \mathbb{R}^2 \), \( N \) discs of small radius \( \epsilon \) centred at the vortex positions, and call the resulting region \( \mathbb{R}^2_0 \). As \( \mathcal{V} \) is a smooth function, integrations over the discs will be \( O(\epsilon^2) \) or smaller, and hence can be neglected in the limit \( \epsilon \to 0 \). Thus, in the following, the effective region of integration is \( \mathbb{R}^2_0 \), and the singularities of \( h \) may be ignored in the formulae (4.10) and (4.11) for \( \mathcal{V} \).
Let $C^s$, where $s$ runs from 1 to $N$, denote the boundary of the disc around the $s$-th vortex position $x^s$ and let $C^0$ denote the boundary circle at spatial infinity. Further, let $\theta^s$ be the polar angle relative to $x^s$ with $\theta^s = 0$ in the positive $x_1$-direction. Then, the outward unit normal along $C^s$ is $n^s = (\cos \theta^s, \sin \theta^s)$ and the coordinates of points on $C^s$ can be written as $x_i = x_i^s + \epsilon n_i^s$. The differential line element on $C^s$ is $dl = \epsilon d\theta^s$.

Now, using (4.14), and remembering the discussion above, the linear momentum (3.30) can be written with $O(\epsilon^2)$ error as

$$P_i = \frac{\gamma}{2} \epsilon_{ij} \int_{R^3} (x_j \partial_k \partial_h (e^h - h - 1)) d^2 x. \quad (4.12)$$

The replacement of $(e^h - h)$ by $(e^h - h - 1)$ is convenient at this stage. Using Green’s lemma in two dimensions,

$$P_i = -\frac{\gamma}{2} \epsilon_{ij} \sum_{s=1}^{N} \int_{C^s} (x_j \partial_k (e^h - h - 1) - (e^h - h - 1) \partial_k x_j) n_k^s dl + \frac{\gamma}{2} \epsilon_{ij} \sum_{s=1}^{N} \int_{C^s} (x_j \partial_k h (e^h - 1) n_k^s - (e^h - h - 1) n_j^s) dl. \quad (4.13)$$

There is no contribution from $C^0$, the circle at infinity, as $e^h - h - 1$ vanishes there. In calculating the integrals along $C^s$ we will ignore terms which are $O(\epsilon)$ or smaller. On $C^s$, one finds from (2.14) that

$$\partial_k h = \frac{2n_k^s}{\epsilon} + b_k^s + \ldots, \quad (4.15)$$

and $e^h = O(\epsilon^2)$. Therefore

$$P_i = -\frac{\gamma}{2} \epsilon_{ij} \sum_{s=1}^{N} \int_{C^s} (-x_j^s + \epsilon n_j^s)(\frac{2n_k^s}{\epsilon} + b_k^s)n_k^s + (\log \epsilon^2 + a_s + 1)n_j^s) dl. \quad (4.16)$$

Noting that $\int_{C^s} n_j^s dl = 0$ and $\int_{C^s} n_j^s n_k^s dl = \pi \epsilon \delta_{jk}$, one concludes that

$$P_i = 2\pi \gamma \epsilon_{ij} \sum_{s=1}^{N} x_j^s, \quad (4.17)$$

which is the same as the expression (4.3), derived in the reduced dynamics.

Before proceeding to compute the angular momentum $M$ we note the following useful identity

$$r^2 \mathcal{V} = \frac{1}{2} r^2 \partial_i (\partial_j h \partial_j \partial_i h) = \frac{1}{2} \partial_i q_i, \quad (4.18)$$

where

$$q_i = r^2 \partial_i h \partial_j \partial_j h - 2x_j \partial_j h \partial_i h + x_i \partial_j h \partial_j h. \quad (4.19)$$

As $r^2 \mathcal{V}$ is a smooth function, we follow the same procedure as in evaluating $P_i$, namely, remove $N$ small discs centred at the positions of vortices. With $O(\epsilon^2)$ error,

$$M = -\frac{\gamma}{2} \int (r^2 \mathcal{V}) d^2 x = -\frac{\gamma}{4} \int_{R^3} \partial_i q_i d^2 x = \frac{\gamma}{4} \sum_{s=1}^{N} \int_{C^s} q_i n_i^s dl. \quad (4.20)$$
Again there is no contribution coming from $C^0$, the circle at infinity, as $\partial_i h$ vanishes there. We rewrite $M$ as

$$M = \frac{\gamma}{4} \sum_{s=1}^{N} (I_1^s - 2I_2^s + I_3^s), \quad (4.21)$$

where

$$I_1^s = \int_{C^s} (r^2 \partial_i h \partial_j h) n_i^s dl, \quad (4.22)$$

$$I_2^s = \int_{C^s} (x_j \partial_j h \partial_i h) n_i^s dl, \quad (4.23)$$

and

$$I_3^s = \int_{C^s} (x_i \partial_j h \partial_i h) n_i^s dl. \quad (4.24)$$

Noting from (2.14) that on $C^s$, $\partial_i \partial_i h = -1 + O(\epsilon^2)$, we obtain

$$I_1^s = \int_{C^s} (x_k^s x_k^s + 2\epsilon n_k^s x_k^s)(\frac{2}{\epsilon} + b^s_i n_i^s)(-1) dl = -4\pi x_k^s x_k^s, \quad (4.25)$$

where, as usual, terms of $O(\epsilon)$ or smaller have been neglected. Similarly,

$$I_2^s = \int_{C^s} (\frac{2}{\epsilon} x_j^s n_j^s + 2 + x_j^s b_j^s)(\frac{2}{\epsilon} + b^s_i n_i^s) dl = 8\pi + 6\pi b_j^s x_j^s, \quad (4.26)$$

and

$$I_3^s = \int_{C^s} (x_i^s n_i^s + \epsilon)(\frac{4}{\epsilon^2} + 4 b_j^s b_j^s) dl = 8\pi + 4\pi b_j^s x_j^s. \quad (4.27)$$

Thus, putting all the above integrals together,

$$M = -2\pi \gamma \sum_{s=1}^{N} \left( \frac{1}{2} x_i^s x_i^s + b_i^s x_i^s + 1 \right), \quad (4.28)$$

which apart from a constant additive term agrees with (4.8).

The constant term which appears in (4.28) has the following meaning. A single vortex situated at the origin has no linear momentum. However, its total angular momentum is $-2\pi\gamma$. Thus, associated with each vortex there is a net spin.

(iii) Contribution of the supercurrent to the momenta

It is of some interest to separate the contributions of the supercurrent and the magnetic field to the linear and angular momenta, for fields satisfying the Bogomol’nyi equations. We note that the vorticity $\mathcal{V}$ can be written as

$$\mathcal{V} = \mathcal{V}_J + B \quad (4.29)$$

where the contribution due to the supercurrent is

$$\mathcal{V}_J = \epsilon_{ij} \partial_i J_j. \quad (4.30)$$
Note that the integral of $\mathcal{V}_J$ over the plane is zero, so the total vorticity, or vortex number, is entirely due to the magnetic field. From (4.10),

$$\mathcal{V}_J = \frac{1}{2} \partial_i \partial_j (e^b) = \frac{1}{2} \partial_i \partial_j (\partial_j \partial_i h).$$ (4.31)

It is not difficult to show that for $k = 1, 2$

$$\int x_k \mathcal{V}_J \ d^2 x = 0.$$ (4.32)

The supercurrent therefore makes no contribution to the linear momentum, and

$$P_i = \gamma \varepsilon_{ij} \int x_j B \ d^2 x.$$ (4.33)

Further, it can be shown that

$$\int (x_k)^2 \mathcal{V}_J \ d^2 x = -4\pi N.$$ (4.34)

Hence, the supercurrent contribution to the angular momentum $M$ is

$$M_J = -\frac{\gamma}{2} \int r^2 \mathcal{V}_J \ d^2 x = 4\pi \gamma N,$$ (4.35)

and the contribution due to $B$ is therefore

$$M_B = -\frac{\gamma}{2} \int r^2 B \ d^2 x = -2\pi \gamma \sum_{s=1}^{N} \left( \frac{1}{2} x_i^s x_i^s + b_i^s x_i^s + 3 \right).$$ (4.36)

Thus, in the reduced dynamics, $\mathcal{V}_J$ contributes just a constant to the angular momentum.

The calculations above make it interesting to compute the third and the fourth moments of $\mathcal{V}_J$, for fields satisfying the Bogomol’nyi equations. One can show, using (4.31), that

$$\int (x_k)^3 \mathcal{V}_J \ d^2 x = -12\pi \sum_{s=1}^{N} x_k^s.$$ (4.37)

To obtain the fourth moment it is necessary to compute the integral of $h$ over the plane. This integral is finite despite the logarithmic singularities of $h$. Using (2.13), the vorticity (1.10) can be written as

$$\mathcal{V} = \frac{1}{2} \partial_i \partial_i \partial_j h - \frac{1}{2} \partial_i \partial_i h.$$ (4.38)

Then,

$$M = -\frac{\gamma}{2} \int r^2 \mathcal{V} \ d^2 x = \gamma \int h \ d^2 x - \pi \gamma \sum_{s=1}^{N} x_i^s x_i^s + 4\pi \gamma N.$$ (4.39)

Equating this with (4.28), we find the noteworthy result

$$\int h \ d^2 x = -2\pi \sum_{s=1}^{N} (b_i^s x_i^s + 3).$$ (4.40)

Using (4.31), and Green’s lemma, we find

$$\int \mathcal{V}_J \ d^2 x = 0$$

where there is no summation over the index $k$. Integrating again, and using (4.10), we conclude that

$$\int \mathcal{V}_J \ d^2 x = -24\pi \sum_{s=1}^{N} ((x_k^s)^2 + b_i^s x_i^s + 3).$$ (4.42)
5 Conclusions

In this paper, we have succeeded in obtaining conservation laws from first principles for the linear and angular momenta in the Schrödinger–Chern–Simons dynamical model of vortices as proposed in [8]. Quite similar to fluid vortices, the linear and angular momenta can be expressed as low moments of a suitably defined vorticity. Our expressions agree with those in [8] in the absence of any transport current. The conserved quantities in the presence of a transport current are those that follow using the Galilean invariance of the dynamics.

For a range of values of the couplings, vortex dynamics in the model reduces, approximately, to motion in the moduli space of Bogomol’nyi vortices. The expressions for the linear and angular momenta in the reduced theory have been shown to agree with those obtained by evaluating the linear and angular momenta in the parent field theory, assuming the fields satisfy the Bogomol’nyi equations. This agreement was not inevitable, and supports the use of the moduli space approximation. Various integrals involving the vorticity have been evaluated explicitly to make the comparison possible. One consequence of the calculations is that each vortex has a constant net spin. Our results can probably be extended somewhat, to a larger range of couplings, by exploiting the electrically excited vortex solutions of Hassaïne et al. [5].

The conservation of linear momentum implies that the centre of vorticity, which becomes the mean of the vortex positions in the reduced theory, does not move. Conservation of angular momentum probably implies that net vorticity cannot escape to infinity, and hence no vortex can escape to infinity in the reduced dynamics, but this result is not proved. Recently, there have been numerical studies of vortices in the model considered here [13]. The conservation laws should be a useful guide to the accuracy of such numerical studies.

Acknowledgements

NSM is grateful to N. Papanicolaou and P. Horváthy, and their collaborators, for very helpful discussions and hospitality.
References

[1] Aitchison I J R, Ao P, Thouless D L and Zhu X -M 1995 Phys. Rev. B 51 6531

[2] Barashenkov I V and Harin A O 1994 Phys. Rev. Lett. 72 1575

[3] Batchelor G K 1967 An introduction to fluid dynamics §7.3 (Cambridge: Cambridge University Press)

[4] Bogomol’nyi E B 1976 Sov. J. Nucl. Phys. 24 449

[5] Hassaïne M, Horváthy P A and Yera J -C 1998 Ann. Phys. 263 276

[6] Jackiw R and Pi S -Y 1990 Phys. Rev. D 42 3500

[7] Manton N S 1982 Phys. Lett. B 110 54

[8] Manton N S 1997 Ann. Phys. 256 114

[9] Papanicolaou N and Tomaras T N 1993 Phys. Lett. A 179 33

[10] Papanicolaou N and Tomaras T N 1991 Nucl. Phys. B 360 425; Komineas S and Papanicolaou N 1998 Nonlinearity 11 265

[11] Samols T M 1992 Commun. Math. Phys. 145 149

[12] Shah P A 1994 Nucl. Phys. B 429 259

[13] Stone M 1995 Int. J. Mod. Phys. B 9 1359

[14] Strachan I A B 1992 J. Math. Phys. 33 102

[15] Stratopoulos G 1998 unpublished

[16] Stuart D 1994 Commun. Math. Phys. 159 51

[17] Taubes C H 1980 Commun. Math. Phys. 72 277