Browder’s theorem with general parameter space

Eilon Solan and Omri N. Solan

Abstract. It follows from Browder (Summa Bras Math 4:183–191, 1960) that for every continuous function \( F : (X \times Y) \to Y \), where \( X \) is the unit interval and \( Y \) is a nonempty, convex, and compact subset of a locally convex linear vector space, the set of fixed points of \( F \), defined by \( C_F := \{ (x, y) \in X \times Y : F(x, y) = y \} \), has a connected component whose projection to the first coordinate is \( X \). We extend Browder’s result to the case that \( X \) is a connected and compact Hausdorff space.

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1. Introduction

Brouwer’s Fixed Point Theorem states that every continuous function from a nonempty, convex, and compact subset of a Euclidean space into itself has a fixed point. The following parametric extension of Brouwer’s Fixed Point Theorem is a special case of Theorem 2 in Browder [2].

Theorem 1.1. Let \( X = [0, 1] \), let \( Y \) be a nonempty, convex, and compact subset of a locally convex linear vector space, and let \( F : (X \times Y) \to Y \) be a continuous function. Define the set of fixed points of \( F \) by

\[
C_F := \{ (x, y) \in X \times Y : F(x, y) = y \}.
\] (1)

Then \( C_F \) has a connected component whose projection to the first coordinate is \( X \).

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Theorem 1.1 has been used in a variety of topics, like nonlinear complementarity theory (see, e.g., Eaves [5], or Allgower and Georg [1]), nonlinear elliptic boundary value problems (Shaw [15]), the study of global continua of solutions of nonlinear partial differential equations (see, e.g., Costa and Goncalves [4], or Massabo and Pejsachowitz [10]), theoretical economics (Citanna et al. [3]), and game theory (see, e.g., Herings and Peeters [7], or Solan and Solan [16]).

A natural question is whether Theorem 1.1 extends to the case that $X$ is not the unit interval, but rather any nonempty, connected, and compact Hausdorff space. Such an extension was in fact required by Munk and Solan [12] in their study in game theory. To this end, Munk and Solan [12] define a continuous function that assigns to each parameter a stochastic game (where the parameter set is a subset of a Euclidean space), and consider the set-valued function that assigns to each parameter the set of stationary discounted equilibria of the corresponding game. Since Nash equilibria are fixed points of a certain continuous function (Nash [14]), it follows by our extension to Theorem 1.1 (Theorem 2.1 below) that there is a connected component of discounted stationary equilibria whose projection to the set of parameters is the whole set of parameters. The limit of these connected components as the discount factor goes to 1 is then used to prove the desired result.

When $X$ admits a space-filling curve, then the extension of Theorem 1.1 follows\(^1\) from the case $X = [0, 1]$. By the Hahn–Mazurkiewicz Theorem (see, e.g., Willard [17], Theorem 31.5), a non-empty Hausdorff topological space admits a space-filling curve if and only if it is compact, connected, locally connected, and metrizable. One example of a compact, connected, and metrizable space that is neither locally connected nor does it admit a space-filling curve is

$$Z := \{(x, y) \in [0, 1] \times [-1, 1]: x = 0, \text{ or } x > 0 \text{ and } y = \sin(1/x)\}. \quad (2)$$

In this note we prove Theorem 1.1 when $X$ is a connected and compact Hausdorff space and $Y$ is a nonempty, convex, and compact subset of a locally convex linear vector space. In Sect. 3 we discuss extensions of Theorem 1.1. Another type of extension of Browder’s Theorem to the case where $F$ is an upper hemi-continuous set-valued function with contractible values (yet $X$ is still the unit interval) was provided by Mas-Colell [9].

In addition to extending Browder’s Theorem to a more general setup, our proof ensures that the topological properties of the connected component of $C_F$ whose projection to the first coordinate is $X$ are the same as those of $X$: both are connected and compact Hausdorff spaces. This allows to apply Browder’s Theorem to the connected component whose existence is guaranteed by the theorem. That is, suppose that $X$ is a compact, connected, locally connected, and metrizable Hausdorff space, and $Y$ and $F$ are as in Theorem 1.1. Let $D$ be a connected component of $C_F$ whose projection to

\(^1\)This extension follows since the image of a connected set is connected, and since every connected set is contained in a connected component.
the first coordinate is $X$, and the existence of which is guaranteed by Theorem 1.1. Suppose further that $G : D \times Y' \to Y'$ is a continuous function, where $Y'$ is a nonempty, convex, and compact subset of a locally convex linear vector space. Since $D$ is not necessarily locally connected, to apply Browder’s Theorem to $G$ we need a version of Theorem 1.1 that holds for parameter spaces that are not necessarily locally connected.

Note that the set $C_F$ may be connected but not locally connected; this happens, for example, when $X = [0, 1]$, $Y = [-1, 1]$, and

$$F(t, x) := \begin{cases} y, & x = 0, \\ (1 - x)y + x\sin(\frac{1}{x}), & x \neq 0, \end{cases}$$

where $C_F$ is the set $Z$ of Eq. (2).

2. The main result

The main result of this note is the following extension of Theorem 1.1.

**Theorem 2.1.** Let $X$ be a connected and compact Hausdorff space, let $Y$ be nonempty, compact, and convex set in a locally convex linear vector space $V$, and let $F : X \times Y \to Y$ be a continuous function. Then $C_F,$ the set of fixed point of $F$ that is defined in Eq. (1), has a connected component whose projection over the first coordinate is $X$.

Extensions of Theorem 2.1 are discussed in Sect. 3. For the proof we will need the concept of the fixed-point index, which we describe in Sect. 2.1. The proof of Theorem 2.1 appears in Sect. 2.2. Though Browder’s [2] original proof uses the fixed-point index as well, it relies on the fact that the unit interval has two extreme points, and it is not clear whether it can be easily extended to any connected and compact Hausdorff space.

2.1. The fixed point index

Roughly, the fixed-point index is a mapping that counts with signs the number of fixed points of a continuous function in a given set. Given a locally convex linear vector space $V$, the fixed-point index is a function, denoted fpi, that assigns an integer to every pair $(Z, f)$, where $Z \subseteq V$ is compact and $f : Z \to V$ is a continuous function that has no fixed points on $\partial Z$, the boundary of $Z$. The function fpi satisfies the following properties:

- Normalization: If $f$ is a constant function whose value is an element of the interior of $Z$, then $\text{fpi}(Z, f) = 1$.
- Additivity: For every compact sets $Z_1, \ldots, Z_K$ with disjoint interiors and every continuous function $f : \bigcup_{k=1}^K Z_k \to V$ such that $f$ has no fixed points on $\bigcup_{k=1}^K \partial Z_k$, we have

$$\text{fpi} \left( \bigcup_{k=1}^K Z_k, f \right) = \sum_{k=1}^K \text{fpi}(Z_k, f).$$
Continuity: For every compact set \( Z \), the function \( f \mapsto \text{fpi}(Z, f) \) is continuous over the set of continuous functions \( f : Z \to V \) that do not have fixed points in \( \partial Z \).

It follows from the above properties that if \( f \) has no fixed points in \( Z \), then \( \text{fpi}(Z, f) = 0 \).

For an exposition of the fixed-point index, as well as proofs of the existence and uniqueness of the fixed-point index, see, e.g., Nagumo [13], Granas and Dugundji [6, Chapter 12], or McLennan [11, Chapter 13].

We will present a reformulation of the fixed-point index that is adapted for our purposes. For every topological space \( C \) denote by \( \mathcal{E}(C) \) the collection of all clopen subsets of \( C \), namely, all subsets that are both closed and open. For every subset \( Y \) of a finite-dimensional real vector space and every function \( f : Y \to Y \), denote the set of fixed points of \( f \) by

\[
C_f := \{ y \in Y : f(y) = y \} \subseteq Y.
\]

For every two sets \( X \) and \( Y \), every function \( F : (X \times Y) \to Y \), and every \( x \in X \), denote by \( F_x : Y \to Y \) the function defined by

\[
F_x(y) := F(x, y), \forall y \in Y.
\]

**Theorem 2.2.** Let \( Y \) be nonempty, compact, and convex set in a locally convex linear vector space \( V \), and let \( \mathcal{F} \) be the class of all continuous functions \( F : (X \times Y) \to Y \), for some nonempty, connected, and compact Hausdorff space \( X \). There exists a function \( \text{FPI} \) that assigns an integer to each pair \((D, F)\) in \( \mathcal{F} \), such that the following properties hold for every continuous function \( F : (X \times Y) \to Y \) in \( \mathcal{F} \):

(I1) **Normalization:** \( \text{FPI}(C_F, F) = 1 \).

(I2) **Additivity:** for every disjoint sets \( (D_k)_{k=1}^K \) in \( \mathcal{E}(C_F) \),

\[
\text{FPI} \left( \bigcup_{k=1}^K D_k, F \right) = \sum_{k=1}^K \text{FPI}(D_k, F).
\]

(I3) **Compatibility:** if \( X = \{ x \} \) is a singleton, then for every compact set \( Z \) of \( Y \) such that \( \partial Z \cap C_{F_x} = \emptyset \)

\[
\text{FPI}(Z \cap C_F, F) = \text{fpi}(Z, F_x).
\]

(I4) **Functoriality:** For every nonempty, connected, and compact Hausdorff space \( X' \) and every continuous function \( g : X' \to X \) define \( \widetilde{g} = (g, \text{Id}) : (X' \times Y) \to (X \times Y) \), \( F' = F \circ \widetilde{g} \), and \( D' = \widetilde{g}^{-1}(D) \). Then

\[
\text{FPI}(D, F) = \text{FPI}(D', F').
\]

Condition (I4) implies that the index can be computed using every single fiber. That is, let \( x \in X \), set \( X' = \{ x \} \), and let \( g : X' \to X \) be the inclusion map. Then \( D' \) and \( F' \) are the restrictions of \( D \) and \( F \) respectively to \( \{ x \} \times Y \).

Note that while the fixed-point index is defined for pairs \((\hat{D}, f)\) such that \( f \) has no fixed point on \( \partial \hat{D} \), the function \( \text{FPI} \) is defined for pairs \((D, F)\) where all points in \( D \) are fixed points of \( F \).
Proof of Theorem 2.2. We first argue that we can assume w.l.o.g. that $F$ has no fixed points on $\partial Y$ for each $F \in \mathcal{F}$. Indeed, let $\hat{Y} \subset V$ be a convex and compact set whose interior contains $Y$. Given a function $F : (X \times Y) \rightarrow Y$, define a function $\widehat{F} : (X \times \hat{Y}) \rightarrow Y$ by

$$\widehat{F}(x, y) := F(x, \pi_Y(y)), \quad \forall y \in Y,$$

where $\pi_Y : \hat{Y} \rightarrow Y$ is the projection. For every continuous function $F : Y \rightarrow Y$, the function $\widehat{F}$ is continuous and has no fixed points in $X \times (\hat{Y} \setminus Y)$. We can then study the fixed-point index when the underlying space is $\hat{Y}$ rather than $Y$.

We next define $\text{FPI}(D, F)$ when $X = \{x\}$ is a singleton. We abuse notations and identify $X \times Y$ with $Y$. Let $F : (X \times Y) \rightarrow Y$ in $\mathcal{F}$. For every $D \in \mathcal{E}(C_F)$ let $Z \subseteq Y$ be a compact set that satisfies (a) $Z \supseteq D$, (b) $\partial Z \cap D = \emptyset$, and (c) $Z \cap C_F = D$. Such a set exists since $D$ is compact, and $F$ has no fixed points on $\partial Y$. It follows that $\text{fpi}(Z, F)$ is well defined. Set

$$\text{FPI}(D, F) := \text{fpi}(Z, F).$$

It is standard to show that the definition is independent of the choice of $Z$.

We now define $\text{FPI}(D, F)$ when $X$ is a general nonempty, connected, and compact Hausdorff space. For every $x \in X$ define $D_x := \{y \in Y : (x, y) \in D\} \subseteq Y$. Standard arguments show that the function $x \mapsto \text{FPI}(D_x, F_x)$ is locally constant. Since $X$ is connected and since $\text{FPI}(D_x, F_x)$ is an integer for every $x \in X$, this function is constant. Define $\text{FPI}(D, F)$ to be this constant.

It is not difficult to show that the properties of the fixed-point index imply that Conditions (I1)–(I4) hold.

One consequence of the definition is that if $E \in \mathcal{E}(C_F)$, and if $E \cap (\{x\} \times Y) = \emptyset$, then $\text{FPI}(E, F) = 0$. Indeed, by (I4) we have $\text{FPI}(E, F) = \text{FPI}(\emptyset, F_x)$, and by (I2) the latter is equal to 0.

2.2. The proof of Theorem 2.1

Theorem 2.1 follows from the following two lemmas.

Lemma 2.3. There exists a connected component $D \subseteq C_F$ that satisfies the following property: for every $E \in \mathcal{E}(C_F)$ that contains $D$ there exists $E' \in \mathcal{E}(C_F)$ that satisfies $D \subseteq E' \subseteq E$ and $\text{FPI}(E', F) \neq 0$.

Lemma 2.4. Let $D \subseteq C_F$ be a connected component that satisfies the condition of Lemma 2.3. The projection of $D$ to the first coordinate is $X$.

We note that since $X$ and $Y$ are compact, and since $F$ is continuous, the set $C_F$ is nonempty and compact.

Proof of Lemma 2.3. Assume to the contrary that the claim does not hold. Then for every connected component $D$ of $C_F$ there exists $E_D \in \mathcal{E}(C_F)$ such that $D \subseteq E_D$ and for every $E' \in \mathcal{E}(C_F)$ that satisfies $D \subseteq E' \subseteq E_D$ we have $\text{FPI}(E', F) = 0$. By taking $E' = E_D$ we obtain that $\text{FPI}(E_D, F) = 0$, and hence $\text{FPI}(\hat{E}, F) = 0$ for every clopen subset $\hat{E}$ of $E_D$, whether or not $\hat{E}$
contains \( D \). Indeed, if \( \tilde{E} \) does not contain \( D \), then since \( D \) is connected and \( \tilde{E} \) is clopen, \( \tilde{E} \) is disjoint from \( D \), hence by (I2) we have

\[
0 = \text{FPI}(E_D \setminus \tilde{E}, F) = \text{FPI}(E_D, F) - \text{FPI}(\tilde{E}, F) = -\text{FPI}(\tilde{E}, F).
\]

Since \( C_F \) is compact and \((E_D)_D\) is an open cover of \( C_F \), there is a finite collection of its connected components, denoted \((D_k)_{k=1}^K\), such that \((E_{D_k})_{k=1}^K\) covers \( C_F \).

The set \( C_F \) can be presented as a disjoint union

\[
C_F = \bigcup_{k=1}^K \left( E_{D_k} \setminus \left( \bigcup_{j<k} E_{D_j} \right) \right).
\]

Since the sets \((E_{D_k})_{k=1}^K\) are clopen, so is the set \( E_{D_k} \setminus \left( \bigcup_{j<k} E_{D_j} \right) \), for each \( k, 1 \leq k \leq K \). Since \( E_{D_k} \setminus \left( \bigcup_{j<k} E_{D_j} \right) \subseteq E_{D_k} \), it follows from the definition of \( E_{D_k} \) that \( \text{fp} \circ (E_{D_k} \setminus \left( \bigcup_{j<k} E_{D_j} \right), F) = 0 \). By (I1) and (I2) we have

\[
1 = \text{FPI}(C_F, F) = \sum_{k=1}^K \text{FPI} \left( E_{D_k} \setminus \left( \bigcup_{j<k} E_{D_j} \right), F \right) = 0,
\]

a contradiction. \( \square \)

**Proof of Lemma 2.4.** Assume to the contrary that the claim does not hold. Then there exists \( x \in X \) that is not in the projection of \( D \) to the first coordinate. It follows that \( D \cap (\{x\} \times Y) = \emptyset \).

It is known (e.g., Kuratowski [8], Theorem 2 on Page 169) that in a compact Hausdorff space, a connected component is equal to the intersection of all clopen sets containing it. Denoting by \((E_{\alpha})_{\alpha}\) all the clopen sets in \( \mathcal{E}(C_F) \) that contain \( D \), we have

\[
(\{x\} \times Y) \cap \left( \bigcap_{\alpha} E_{\alpha} \right) = \emptyset.
\]

Since all the sets in this intersection are compact, a finite intersection of these sets is already empty: there are \((\alpha_k)_{k=1}^K\) such that

\[
(\{x\} \times Y) \cap \left( \bigcap_{k=1}^K E_{\alpha_k} \right) = \emptyset. \tag{3}
\]

Set \( E^* := \bigcap_{k=1}^K E_{\alpha_k} \). As the intersection of finitely many clopen sets, \( E^* \in \mathcal{E}(C_F) \). As mentioned at the end of Sect. 2.1, Eq. (3) implies that \( \text{FPI}(E', F) = 0 \), for every \( E' \subseteq E^* \) such that \( E' \in \mathcal{E}(C_F) \). This contradicts the choice of \( D \). Indeed, we selected \( D \) to satisfy Lemma 2.3, yet the set \( E = E^* \) does not satisfy the conclusion of the lemma. \( \square \)
3. Extensions

Our proof of Theorem 2.1 holds whenever (a) $V$ satisfies the fixed point property, (b) the fixed point index is well defined on $V$, and (c) $Y$ is a neighborhood retract of $V$. As noted by Andrew McLennan, one such case is when $Y$ is a compact absolute retract. Recall that a space is an absolute retract if and only if it is a contractible absolute neighborhood retract (Theorem 11.2.1 in Granas and Dugundji [6] or Theorem 8.2 in McLennan [11]).

Theorem 2 in Browder [2], which is a more general version of Theorem 1.1, is the following.

**Theorem 3.1.** (Browder [2]) Let $X = [0,1]$, let $Y$ be a nonempty, compact, and convex set in a locally convex linear space $V$, let $U \subseteq X \times V$ be open, and let $F : \overline{U} \to Y$ be a continuous function. Suppose that $C_F \cap \partial U = \emptyset$ and that $\text{fpi}(U_0, F_0) \neq 0$. Then $C_F$ has a connected component whose projection over the first coordinate is $X$.

With minor adaptations, our proof shows that Theorem 3.1 is valid whenever $X$ is a connected and compact Hausdorff space and $Y$ is as in the previous paragraph.

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Eilon Solan and Omri N. Solan
The School of Mathematical Sciences
Tel Aviv University
Tel Aviv 6997800
Israel
e-mail: eilons@post.tau.ac.il;
omrisola@mail.tau.ac.il

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