ISOPERIMETRIC REGIONS IN $\mathbb{R}^n$ WITH DENSITY $r^p$

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ABSTRACT. We show that the unique isoperimetric hypersurfaces in $\mathbb{R}^n$ with density $r^p$ for $n \geq 3$ and $p > 0$ are spheres that pass through the origin.

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1. INTRODUCTION

Recently, there has been a surge of interest in manifolds with density, partly because of their role in Perelman’s proof of the Poincaré Conjecture. We consider the isoperimetric problem when volume and perimeter are weighted by the density function $r^p$ and prove the following theorem:

**Theorem 3.3** In $\mathbb{R}^n$ with density $r^p$, where $n \geq 3$ and $p > 0$, spheres that pass through the origin are uniquely isoperimetric with respect to weighted perimeter and volume.

The density $r^p$ is one of the simplest radial density functions, but it has some interesting properties. First, $r^p$ is homogeneous in degree $p$, which means that given an isoperimetric region of one volume, we can scale it to get an isoperimetric region of a different volume. Second, $r^p$ (or a constant multiple) is the only density for which spheres through the origin could be isoperimetric (see e.g. Rmk. 4.4). We can view our present problem as a venture either to prove a partial converse of this statement in the case that $p > 0$ or to extend the work of Dahlberg et al., who proved the result in $\mathbb{R}^2$ [DDNT, Thm. 3.16]. Díaz et al. [DHHT, Conj. 7.6] conjectured the generalization to $\mathbb{R}^n$ and reduced the problem to analyzing planar curves. Recently, Chambers [C, Thm. 1.1] proved that spheres centered at the origin are isoperimetric in $\mathbb{R}^n$ with any log-convex density.

We adapt Chambers’ proof to density $r^p$. Like Chambers, we first consider an isoperimetric region that is spherically symmetric (see Defn. 2.6), then prove the result in the general case. Given a spherically symmetric isoperimetric region, we prove that the generating curve for the boundary is a circle through the origin. The behavior of this curve is determined by a differential equation corresponding to the fact that isoperimetric hypersurfaces have constant generalized mean curvature [MP, Defn. 2.3]. By spherical symmetry and regularity, the rightmost point of the curve is on the $e_1$-axis, and the tangent vector at this point is vertical. Our Lemmas 4.5 and 4.7 show that if the osculating circle at the rightmost point of the curve, which we may assume to be $(1,0)$, goes through the origin, then the curve is a circle through the origin.

We suppose for contradiction that the initial osculating circle does not pass through the origin, then
take two cases according to whether its center is right or left of \((1/2, 0)\). In the case that the center is right of \((1/2, 0)\), the curve is like that in Chambers’ proof in that the curvature is greater at a point above the \(e_1\)-axis with tangent vector in the third quadrant than at the point of the same height with tangent vector in the second quadrant. As a result, the curve has a vertical tangent before it meets the \(e_1\)-axis again and then curves in to meet the axis at an angle (Fig. 1, right). In the left case, the opposite inequality regarding curvatures holds, and, as a result, the curve never returns to vertical before reaching the axis (Fig. 1, left).

The left case presents the new challenge of showing that there is only one point on the upper half of the curve where the tangent vector is horizontal (Prop. 7.21). Additionally, although the curve in the right case is similarly to that in Chambers, the proof is different in that we do not have the hypothesis that an isoperimetric hypersurface is mean convex, which is what Chambers used to prove that curvature was positive on the final segment of the curve. We achieve the same result by computations that depend on the fact that our curve ends right of the \(e_2\)-axis (Lemma 6.12), which is a property that may not hold for the generating curve in Chambers.

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2. Existence, Regularity, and Symmetry

Definition 2.1. A region \(E\) is a measurable subset of \(\mathbb{R}^n\). Its weighted volume is the integral of the density over \(E\). Its boundary is the topological boundary. Its weighted perimeter is the integral over the boundary of the density with respect to \((n-1)\)-dimensional Hausdorff measure. We say a region is isoperimetric if it minimizes weighted perimeter for fixed weighted volume.

The following theorem of Morgan and Pratelli guarantees the existence of isoperimetric regions of all volumes.

Theorem 2.2. [MP Thm. 3.3] Assume that \(f\) is a (lower-semicontinuous) radial density that diverges to infinity. Then there exist isoperimetric sets of all volumes.

Definition 2.3. (Regular Point) Let \(E\) be an isoperimetric region. We say that a point \(P \in \partial E\) is regular if there is an open set \(U\) containing \(P\) so that \(\partial E \cap U\) is a smooth embedded \((n-1)\)-dimensional manifold.

By [M Cor. 3.8, Rmk. 3.10], if \(E \subset \mathbb{R}^n\) is an isoperimetric region, then \(\partial E\) is regular except on a set of Hausdorff dimension at most \(n - 8\), after perhaps altering \(E\) by a negligible set of measure 0; henceforth we assume regions open. By the first variation formula, generalized mean curvature is constant on the set of regular points. The following proposition gives a sufficient condition for \(\partial E\) to be regular at a point.

Proposition 2.4. If \(P \in \partial E\) and \(E\) locally lies in a halfspace to one side of a hyperplane through \(P\), then \(\partial E\) is regular at \(P\), provided that the density function is positive at \(P\).
Proof. Since $E$ is an isoperimetric minimizer and the oriented tangent cone at $P$ lies in a halfspace, the oriented tangent cone is a hyperplane. The result follows by [M] Prop. 3.5, Rmk. 3.10. □

Corollary 2.5. All points in $\partial E$ of maximal distance from the origin are regular.

Definition 2.6. (Spherical Symmetrization) Given a region $E \subset \mathbb{R}^n$, let $A_E(r)$ denote the area of the intersection of $E$ with $S_r$, the sphere through the origin of radius $r$. We define the spherical symmetrization of $E$ to be the unique set $E^*$ such that for all $r \geq 0$, $A_E(r) = A_{E^*}(r)$ and $E^* \cap S_r$ is a closed spherical cap centered at $(r, 0, \ldots, 0)$.

Remark 2.7. Since the set of singularities on the boundary of an isoperimetric set $E \subset \mathbb{R}^n$ has dimension at most $(n - 8)$, it follows that if $E$ is spherically symmetric about the $e_1$-axis, then all points in $\partial E$ that are not on the $e_1$-axis are regular.

The following theorem demonstrates that for a radial density, spherical symmetrization preserves weighted volume but does not increase weighted perimeter. Moreover there are certain conditions under which the perimeter of a region remains the same after symmetrization only if the original region was spherically symmetric about some (oriented) line through the origin.

Theorem 2.8. [MP] Thm. 6.2] Let $f$ be a radial density on $\mathbb{R}^n$ and let $E$ be a set of finite perimeter. Then the spherical symmetrization $E^*$ satisfies

$$|E^*| = |E|$$

and

$$P(E^*) \leq P(E).$$

Suppose further that $E$ is an open set of finite perimeter, and let $\nu(x)$ denote the normal vector at any $x \in \partial E$. If $\mathcal{H}^{n-1} \left( x \in \partial E : \nu(x) = \frac{\pm e_1}{|\nu(x)|} \right) = 0$, and the set $I_E := \{ r > 0 : 0 < \mathcal{H}^{n-1}(E \cap S_r) < \mathcal{H}^{n-1}(S_r) \}$ is an interval, then $P(E^*) = P(E)$ if and only if $E = E^*$ up to rotation about the origin.

It is immediate that if $E$ is an isoperimetric region in Euclidean space with a radial density, then $E^*$ is also isoperimetric.

3. SPHERES THROUGH THE ORIGIN ARE UNIQUELY MINIMIZING

To prove our main result Theorem 3.3, we begin by showing that that any spherically symmetric isoperimetric region is a ball whose boundary is a sphere through the origin (Prop. 3.1). The proof of Proposition 3.1 comprises most of the paper, but we provide a sketch below. We apply this proposition to the symmetrized version of an arbitrary isoperimetric region to show that, in fact, any isoperimetric region is spherically symmetric about some oriented line through the origin (Prop. 3.2).

Proposition 3.1. Suppose that $E \subset \mathbb{R}^n$ is a spherically symmetric isoperimetric region in $\mathbb{R}^n$ with density $r^p$. Then $E$ is a ball whose boundary goes through the origin.

Proof. Assume without loss of generality that $E$ is spherically symmetric about the positive $e_1$-axis. Then $E$ can be generated by rotating a planar set $A$ about the $e_1$-axis. By regularity of $E$ (Defn. 2.3), we are assuming that $A$ is open and that its boundary is a curve (possibly having multiple connected components). Since $E$ is spherically symmetric about the positive $e_1$-axis, $A$ is also spherically symmetric about the positive $e_1$-axis. We define $\gamma \subset \partial A$ by beginning at the rightmost point on $\partial A$ and following the curve through this point in both directions until it intersects the $e_1$-axis again. This definition relies on regularity properties of $\partial E$; see the beginning of Section 4 for more details.

We assume that $\gamma : [\pm \beta, \beta]$ is an arclength parameterization so that $\gamma(0)$ is the rightmost point on $\partial A$ and $\gamma(\pm \beta)$ is the other intersection of $\gamma$ with the $e_1$-axis. Since $r^p$ is homogeneous, all isoperimetric regions are similar, and we can assume without loss of generality that $\gamma(0) = (1, 0)$. We will show that $\gamma$ is a circle through the origin. Given that $\gamma$ is a circle through the origin, $\gamma$ must comprise all of $\partial A$ by spherical symmetrization.

By Lemma 4.5 to prove that $\gamma$ is a circle through the origin, it suffices to prove that there exists an $s$ so that the associated canonical circle $C_s$ (see Defn. 4.2) has the same curvature as $\gamma$ at $\gamma(s)$ and $C_s$ goes through the origin. By Lemma 4.7 the canonical circle $C_0$ at the rightmost point has the same curvature
as \( \gamma \) at \( \gamma(0) \). Therefore, it suffices to prove that \( C_0 \) passes through the origin, which occurs if and only if the center of \( C_0 \) is \((1/2, 0)\).

Suppose that the center of \( C_0 \) is right of \((1/2, 0)\). By Proposition 4.8, \( \gamma(\beta) > 0 \) and \( \lim_{s \to \beta^-} \gamma'(s) > 0 \). As a result, there exists \( \epsilon > 0 \) so that \( \gamma \cdot \gamma' > 0 \) on \((\beta - \epsilon, \beta)\), contradicting Lemma 4.1, which is a consequence of spherical symmetry.

Now suppose that the center of \( C_0 \) is left of \((1/2, 0)\). By Proposition 4.9, \( \gamma(\beta) < 0 \) and \( \lim_{s \to \beta^-} \gamma'(s) < 0 \), which results in the same contradiction of spherical symmetry.

The only remaining possibility is that \( \gamma \) is a circle through the origin. Thus, \( \gamma = \partial A \) and, when rotated, \( \gamma \) generates a sphere through the origin. \( \square \)

Given Proposition 3.1, we can prove our claim that any isoperimetric region in \( \mathbb{R}^n \) with density \( r^p \) is spherically symmetric.

**Proposition 3.2.** If \( E \) is an isoperimetric region in \( \mathbb{R}^n \) with density \( r^p \), then \( E = E^* \), up to a rotation about the origin.

**Proof.** By regularity (Defn. 2.3), we are assuming \( E \) is open. By Theorem 2.8, it suffices to show that \( I_E \) is an interval and that
\[
\mathcal{H}^{n-1} \left( x \in \partial E : v(x) = \pm \frac{x}{|x|} \right) = 0.
\]
We call a point \( x \) with \( v(x) = \pm x/|x| \) tangential. Since symmetrization (Defn. 2.6) preserves weighted volume without increasing weighted perimeter, \( E^* \) is also isoperimetric. Applying Proposition 3.1, we conclude that \( E^* \) is a ball with boundary through the origin. It follows that \( I_E \) is an interval. Moreover, there exists no \( r \in I_E \) such that the spherical cap \( S_r \cap E \) is a full sphere. This will be important in our proof of (1). Suppose for contradiction that there exists a positive area subset of \( \partial E \) that is tangential. As in Morgan-Pratelli [MP, Pf. of Cor. 6.4], at any smooth point of density of this tangential subset of \( \partial E \), \( \partial E \) has the same generalized mean curvature as a sphere centered at the origin. It follows by uniqueness of solutions to elliptic partial differential equations that a component of \( \partial E \) is a sphere centered at the origin. \( E \) must contain an annular region centered at the origin with this spherical component as one of its bounding components. Thus, there exists an interval \((r_0, r_1)\) such that for any \( r \in (r_0, r_1) \), \( S_r \cap E \) is a full sphere, contradicting the fact that the boundary of \( E^* \) is a sphere through the origin. \( \square \)

Combining Propositions 3.1 and 3.2 along with Theorem 2.2, we have proved:

**Theorem 3.3.** In \( \mathbb{R}^n \) with density \( r^p \), where \( n \geq 3 \) and \( p > 0 \), spheres that pass through the origin are uniquely isoperimetric with respect to weighted perimeter and volume.

### 4. Structure of Proof

Sections 5, 6, and 7 are devoted to filling in the details of the proof of Proposition 3.1. Throughout these sections, we work within the following framework:

Let \( E \) be a spherically symmetric isoperimetric region. Then there is a set \( A \subset \mathbb{R}^2 \) such that \( E \) is the rotation of \( A \) about the \( e_1 \)-axis. We will analyze a certain curve on the boundary of \( A \). We begin at the point \( P \) on the \( e_1 \)-axis that is the rightmost point on \( \partial A \). By spherical symmetry, \( P \) is a point of \( E \) farthest from the origin, so \( \partial E \) is regular at \( P \) by Corollary 2.5. The tangent space to \( \partial A \) at \( P \) is spanned by \( e_2 \). We follow \( \partial A \), which has finite length, in both directions until it intersects the \( e_1 \)-axis at another point.

The result is a Jordan curve \( \gamma(s) : [-\beta, \beta] \to \mathbb{R}^2 \) such that \( \gamma(0) = P \) and \( \gamma(\pm \beta) \) is the other intersection of the curve with the \( e_1 \)-axis (Fig. 2). Since \( r^p \) is homogeneous, all isoperimetric regions are similar to each other. Therefore, we may assume without loss of generality that \( P = (1, 0) \). We assume that \( \gamma \) is a counterclockwise arclength parameterization. Let \( \gamma_1 \) and \( \gamma_2 \) denote the coordinates of \( \gamma \). Then \( \gamma_1(-s) = \gamma_1(s) \) and \( \gamma_2(-s) = -\gamma_2(s) \) for all \( s \).

By Corollary 2.5, \( \gamma \) is smooth at 0. By Remark 2.7, \( \gamma \) is smooth at all remaining points in \((-\beta, \beta)\). Since \( \gamma \) is smooth at 0 and 0 is the global maximum point of \( \gamma \), \( \gamma'(0) = (0, 1) \). We let \( \kappa(s) \) denote the curvature of \( \gamma \) at \( \gamma(s) \). Then \( \kappa(0) > 0 \); otherwise a contradiction to spherical symmetry would result. We will also consider the generalized mean curvature of the surface generated by \( \partial A \) at a point \( \gamma(s) \). As in
\[ H_f = H_0 + \frac{\partial \psi}{\partial \nu}, \]

where \( H_0 \) is the unaveraged Riemannian mean curvature and \( \nu \) is the outward unit normal vector. If \( \psi(x) = g(|x|) \) for some smooth function \( g \) and \( x \neq 0 \), we have

\[ H_f(x) = H_0(x) + g'(|x|) \frac{x}{|x|} \cdot \nu(x). \]

In \( \mathbb{R}^n \) with density \( r^p \), \( g(r) = \log(r^p) \). Henceforth, we will denote

\[ \frac{\partial \psi}{\partial \nu}(x) \]

by \( H_1(x) \). For concision, given a point \( \gamma(s) \), we refer to \( H_1(\gamma(s)) \) as \( H_1(s) \) with analogous notation for the values of \( H_0 \) and \( H_f \) at \( \gamma(s) \).

The following lemma of Chambers gives a trivial but useful result of spherical symmetrization.

**Lemma 4.1.** (Tangent Restriction) \([C, \text{Lemma 2.6}]\) For every \( s \in (0, \beta) \), \( \gamma(s) \cdot \gamma'(s) \leq 0 \).

At each point on \( \gamma \), we define a related circle that we call the canonical circle. We show in Proposition 5.1 that the curvature of the canonical circle is one of two terms in a formula for the mean curvature of the surface of revolution.

**Definition 4.2.** \([C, \text{Defns. 3.1, 3.2}]\) For \( s \in (0, \beta) \), let the canonical circle at \( s \), denoted \( C_s \), be the unique oriented circle centered on the \( e_1 \)-axis that passes through \( \gamma(s) \) and has unit tangent vector at \( \gamma(s) \) equal to \( \gamma'(s) \). If \( \gamma'(s) \) is a multiple of \( e_2 \), then \( C_s \) is an oriented vertical line. We define \( C_0 \) to be \( \lim_{s \to 0^+} C_s \). The regularity of the surface at \( \gamma(0) \) guarantees the existence of this limit. We let \( R(s) \) denote the radius of \( C_s \) and let \( \lambda(s) \) denote its signed curvature. Then \( \lambda(s) = 1/R(s) \) if \( C_s \) is counterclockwise oriented, and \( \lambda(s) = -1/R(s) \) if \( C_s \) is clockwise oriented. Finally, we let \( F(s) \) denote the abscissa of the center of \( C_s \).

The following lemma shows that spheres through the origin have constant generalized mean curvature. We apply this result to prove Lemmas \([4.5, 4.7]\) which imply that \( \gamma \) is a sphere through the origin, given that the curvature at the rightmost point is the same as the curvature of the circle through that point and the origin.

**Proposition 4.3.** In \( \mathbb{R}^n \) with density \( r^p \), hyperspheres through the origin have constant generalized mean curvature.
Proof. Let $S$ be a hypersphere through the origin and assume without loss of generality that $S$ can be obtained by rotating a circle $C$ in the plane about the $e_1$-axis. It suffices to prove that all points on $C$ have the same generalized mean curvature. $H_0$ is constant on $S$ since it is constant on $C$. It remains to prove that $H_1$ is constant on $C$.

Let the center of $C$ be $(a,0)$ with $a > 0$. Then the polar coordinates equation for $C$ is $r = 2a \cos \theta$. At a point $(r(\theta), \theta)$, the unit outward normal vector makes angle $2\theta$ to the positive $e_1$-axis, and the angle between the position vector and the unit outward normal vector is $\theta$. Supposing that $x$ has polar coordinates $(r, \theta)$, we have

$$g'(|x|) \frac{x}{|x|} \cdot v(x) = \frac{p}{r} \cos \theta = \frac{p}{r} \frac{r}{2a} = \frac{p}{2a}.$$ 

Therefore, $H_1$ is constant on $C$, as required.

Remark 4.4. These computations show that the only density on $\mathbb{R}^2 - \{0\}$ ($\mathbb{R}^n - \{0\}$) for which circles (spheres) through the origin are isoperimetrical is $r^\mu$, or a constant multiple thereof. On a circle $C$ through the origin, parameterized by $\alpha$, the quantity $\alpha(t)/|\alpha(t)| \cdot v(t)$ is a constant multiple of the magnitude of the position vector. Hence, for $H_1$ to be constant it must be the case that $g'(r)$ is inversely proportional to $r$. This occurs only if $g = \log(r^p) + c$ for some $p$ and some constant $c$.

Lemma 4.5. (cf. [C] Lemma 3.2) For any point $s \in [0, \beta]$, if $C_s$ passes through the origin and $\kappa(s) = \lambda(s)$, then $\gamma$ is a circle that passes through the origin.

Proof. Supposing that $C_s$ is arclength parameterized, to prove that $C_s$ agrees with $\gamma$ locally, it suffices by uniqueness theorems concerning solutions of ODEs to prove that both satisfy the differential equation $H_f = c$. This is clearly true since the tangent vectors of the two curves agree at $\gamma(s)$ and the generalized mean curvature of the surfaces generated by these curves is the same at $\gamma(s)$. To prove that $H_f = c$ at all points on $C_s$, it suffices to show that $H_f$ is constant on $C_s$. This follows from the computations in Lemma 4.3. Having proved that $\gamma$ and $C_s$ coincide locally, we claim that, in fact, $\gamma$ and $C_s$ must coincide everywhere.

Let $S = \{t \in [-\beta, \beta] : \gamma([s,t]) \subset C_s\}$. Since $\gamma$ and $C_s$ agree near $\gamma(s)$, $S$ is nonempty and therefore has a least upper bound $m$. Letting $\alpha$ be an arclength parameterization of $C_s$, it follows by smoothness of $\alpha$ and of $\gamma$ that $m \in S$, that $C_s$ is tangent to $\gamma$ at $\gamma(m)$, and that $\kappa(m) = \lambda(s)$. (To conclude smoothness of $\gamma$ at $m$, we are using our assumption that $m < \beta$.) By an identical argument to that in the first paragraph, there exists an open interval $I$ containing $m$ such that $\gamma(I) \subset C_s$, contradicting the fact that $m = \sup S$. We conclude that $m = \beta$. A similar argument shows that $\gamma$ coincides with $C_s$ on $[-\beta,s]$.

Remark 4.6. By radial symmetry, spheres centered at the origin also have constant generalized mean curvature. Thus, if $C_s$ is centered at the origin and $\kappa(s) = \lambda(s)$, then $\gamma$ is a circle that is centered at the origin. We use this result to obtain contradictions in several places.

Lemma 4.7. ([C] p. 12) We have $\kappa(0) = \lambda(0)$.

Proof. Showing that $\kappa(0) = \lambda(0)$ is equivalent to showing that $F(0) = 1 - 1/\kappa(0)$. If $\gamma(0) \neq 0$, then

$$F(s) = \frac{\gamma(s) \cdot \gamma'(s)}{\gamma'(s)}.$$ 

Since $\kappa(0) > 0$ and $\kappa$ is continuous at 0, there is a neighborhood of 0 on which $\gamma'(s) 
eq 0$ except when $s = 0$. By definition,

$$F(0) = \lim_{s \to 0} F(s) = \lim_{s \to 0} \frac{\gamma(s) \cdot \gamma'(s)}{\gamma'(s)} = 1 - \frac{1}{\kappa(0)}.$$ 

By Lemmas 4.5 and 4.7 if $C_0$ is a circle through the origin, then $\gamma$ is a circle through the origin. This means that if $F(0) = 1/2$, then $\gamma$ is a circle through the origin. We will argue by contradiction, taking cases according to whether $F(0) > 1/2$ or $F(0) < 1/2$. We call these cases the right case and the left case, respectively. In each case, we can obtain a result that contradicts spherical symmetry. We state
Meanwhile, by Meusnier’s formula, the second principal curvature is given by

$$\kappa_2 = \frac{\gamma'(s)}{\gamma(s)^3}$$

Proposition 4.8. (Right Tangent Lemma) If $F(0) > 1/2$, then $\gamma_1(\beta) < 0$, $\lim_{s \to \beta^-} \gamma'(s)$ is in the fourth quadrant, and $\lim_{s \to \beta^-} \gamma'(s) \neq (0, -1)$.

Proposition 4.9. (Left Tangent Lemma) If $F(0) < 1/2$, then $\gamma_1(\beta) < 0$, $\lim_{s \to \beta^-} \gamma'(s)$ is in the third quadrant, and $\lim_{s \to \beta^-} \gamma'(s) \neq (0, -1)$.

5. Preliminary Lemmas

This section contains results relevant to both cases. Proposition 5.1 and Corollary 5.2 give expressions for the mean curvature and generalized mean curvature at a point on the hypersurface generated by $\gamma$ in terms of the curvature of $\gamma$, the curvature of the canonical circle, and the normal derivative of the log of the density at that point. We then discuss computational techniques that we use to determine how these functions (and others) vary with arclength. Finally, Proposition 5.4 is used in both cases to compare the density at that point. We then discuss computational techniques that we use to determine how these functions (and others) vary with arclength. Finally, Proposition 5.4 is used in both cases to compare the density at that point. We then discuss computational techniques that we use to determine how these functions (and others) vary with arclength. Finally, Proposition 5.4 is used in both cases to compare the density at that point.

Proposition 5.1. [C] Prop. 3.1] Given a point $s \in [0, \beta)$, we have that

$$H_0(s) = \kappa(s) + (n - 2)\lambda(s).$$

Proof. We consider the principal curvatures of the surface at a point $P = \gamma(s)$. We treat the case that $y = \gamma_2(s) > 0$ and that $\gamma'(s) \neq (0, \pm 1)$. A similar argument shows that the equation holds if $\gamma_2(s) < 0$ and $\gamma'(s) \neq (0, \pm 1)$. We claim that there exists no interval on which $\gamma_2 = 0$ or $\gamma'$ is vertical; then it will follow by smoothness of $\gamma$ that the equation holds at the remaining points.

To prove the claim, recall that $\gamma$ is smooth at 0 and that, as a consequence of spherical symmetry, $\kappa(0) > 0$. Thus, $\gamma_2$ cannot be 0 on an interval including 0. On the other side of the curve, $\gamma(\beta)$ is defined to be the first point where the curve intersects the axis again, so even if a portion of the curve were a line segment along the $e_1$-axis, that segment would not be parameterized by the function $\gamma$. The curve cannot have vertical tangent vector on an interval either. If a portion of the curve were a vertical line segment, then this vertical line segment, when rotated, would generate a portion of a hyperplane, which would have zero mean curvature. However, $H_1$ (the normal derivative of the log of the density) would vary as one moved up or down along the line segment, contradicting the fact that the surface has constant generalized mean curvature.

With this technical point out of the way, we proceed in the case that $y = \gamma_2(s) > 0$ and that $\gamma'(s) \neq (0, \pm 1)$. One of the principal curvatures at $P$ is the of the curvature of $\gamma$ at this point. The cross section of the surface obtained by fixing the first coordinate is an $(n - 2)$-dimensional sphere of revolution. The remaining principal curvatures of the surface are the principal curvatures of the sphere, which are equal. Thus, to compute one of the principal curvatures of the sphere, it is sufficient to compute the second principal curvature of a 2-dimensional surface in the $n = 3$ case. This second principal curvature is the curvature of a circle of revolution $C$.

By assumption that $y = \gamma_2(s) > 0$, the curvature of the circle is $1/y$. We let $n$ denote the inward normal vector to the surface and $N$ denote the normal vector to the circle of revolution. Since $y > 0$, $C_\varepsilon$ is counterclockwise oriented if and only if $n$ is downward (i.e. $n$ has a negative $e_2$-component). Thus,

$$\lambda(s) = \begin{cases} \frac{1}{\gamma(s)}, & n \text{ downward} \\ \frac{1}{\gamma'(s)}, & n \text{ upward.} \end{cases}$$

Meanwhile, by Meusnier’s formula, the second principal curvature is given by

$$\kappa_2 = \frac{1}{y} \cos \phi,$$

where $\phi$ is the angle between $n$ and $N$. Again, since $y > 0$,

$$\cos \phi = \begin{cases} \frac{\gamma'(s)}{\gamma(s)}, & n \text{ downward} \\ \frac{\gamma'}{\gamma''}, & n \text{ upward}. \end{cases}$$
The first of these cases is depicted in Figure 3. In both cases, the second principal curvature is the curvature of the canonical circle.

Since one principal curvature of the surface at $\gamma(s)$ equals $\kappa(s)$ and all of the others equal $\lambda(s)$, the mean curvature of the surface at $\gamma(s)$ is given by

$$H_0(s) = \kappa(s) + (n-2)\lambda(s).$$

\[\square\]

**Corollary 5.2.** The generalized mean curvature of the surface at a point $\gamma(s)$ is given by

$$H_f(s) = \kappa(s) + (n-2)\lambda(s) + H_1(s).$$

In the left and right cases delineated on p. 7, for any $s \in [0, \beta)$ we can analyze how $\gamma$ and related functions are instantaneously changing at $\gamma(s)$ by computing the requisite derivatives on the osculating circle to $\gamma$ at $\gamma(s)$. A justification for this procedure follows.
For a given \( s \in [0, \beta) \), let \( A_s \) denote the unique oriented circle that is tangent to \( \gamma \) at \( \gamma(s) \) and has curvature \( \kappa(s) \). Note that if \( \kappa(s) = 0 \), then \( A_s \) is an oriented line with direction vector \( \gamma'(s) \). For a fixed \( s \), let \( \alpha \) be a arclength parameterization of \( A_s \) such that \( \alpha(\tilde{s}) = \gamma(s) \). Since \( A_s \) is tangent to \( \gamma \) at \( \gamma(s) \) and has curvature \( \kappa(s) \), we have \( \alpha'(\tilde{s}) = \gamma'(s) = \gamma''(s) \).

We let \( \kappa(t) \) denote the signed curvature of \( \alpha \) at \( t \) and \( \dot{H}_1(t) \) denote \( \partial g/\partial \nu \) at \( t \), where \( \nu \) is the unit outward normal to \( \alpha \) at \( t \). Both \( \kappa \) and \( \dot{H}_1 \) are smooth functions on their domains. Moreover, we consider the canonical circles to \( \alpha \) and related functions. For a fixed \( t \) we denote the canonical circle to \( A_s \) at \( \alpha(t) \) by \( \tilde{C}_t \). If \( \alpha_2(t) \neq 0 \), then we define \( \tilde{C}_t \) as before. If \( \alpha_2(t) = 0 \) and \( \alpha'(t) = (0, \pm 1) \), then we define \( \tilde{C}_t \) to be \( A_t \). If \( \alpha_2(t) = 0 \) and \( \alpha'(t) \neq (0, \pm 1) \), then \( \tilde{C}_t \) is undefined. For each \( t \) so that \( \tilde{C}_t \) is defined, the canonical circle is defined on a neighborhood of \( t \). Finally, we define the functions \( \tilde{\lambda}, \tilde{R}, \) and \( \tilde{F} \) by letting \( \tilde{\lambda}(t), \tilde{R}(t), \) and \( \tilde{F}(t) \) be the curvature of \( \tilde{C}_t \), the radius of \( \tilde{C}_t \), and the abscissa of the center of \( \tilde{C}_t \), respectively.

We use these functions to approximate their counterparts on \( \gamma \). Fix \( s \in [0, \beta) \), and let \( \alpha \) be a arclength parameterization of \( A_s \) such that \( \alpha(\tilde{s}) = \gamma(s) \). For a given \( t \), the canonical circle \( \tilde{C}_t \) depends only on \( \alpha(t) \) and on the inward normal at \( \alpha(t) \), which is a rotation of \( \alpha'(t) \) by \( \pi/2 \) (with the direction depending on the orientation of \( \alpha \)). It follows that \( \tilde{F}, \tilde{R}, \) and \( \tilde{\lambda} \) can be computed as in terms of \( \alpha \) and \( \alpha' \) and that their derivatives depend on \( \alpha \) and its first two derivatives. In particular, since \( \alpha'(\tilde{s}) = \gamma'(s) \) and \( \alpha''(\tilde{s}) = \gamma''(s) \), we have \( \tilde{F}'(\tilde{s}) = F'(s), \tilde{R}'(\tilde{s}) = R'(s), \) and \( \tilde{\lambda}'(\tilde{s}) = \lambda'(s) \).

As well as analyzing these functions, we also consider the angle the tangent vector makes to the horizontal.

**Definition 5.3.** We define the function \( \theta : S^1 \to (0, 2\pi) \) by letting \( \theta(v) \) be the angle in the specified interval that \( v \) makes to the positive \( e_1 \)-axis.

The next proposition of Chambers concerns two \( C^2 \) functions on an interval \( (a, b) \). Given \( h : (a, b) \to \mathbb{R} \geq 0 \), we let \( t_h(x) \) denote the unit tangent vector

\[
\frac{(1, h'(x))}{\|(1, h'(x))\|},
\]

and we let \( \kappa_0(x) \) denote the upward curvature of the graph of \( h \) at \( x \).

**Proposition 5.4.** [C Prop 3.8] Consider two \( C^2 \) functions \( f, g : (a, b) \to \mathbb{R} \geq 0 \) with \( b > a \) that satisfy the following:

1. \( \lim_{x \to b^-} t_f(x) \) and \( \lim_{x \to b^-} t_g(x) \) exist,
2. \( \lim_{x \to b^-} f(x) \) and \( \lim_{x \to b^-} g(x) \) exist,
3. \( f'(x) \geq 0 \) and \( g'(x) \geq 0 \) for all \( x \in (a, b) \),
4. \( \lim_{x \to b^-} f(x) \leq \lim_{x \to b^-} g(x) \) and \( \lim_{x \to b^-} \theta(t_f(x)) \geq \lim_{x \to b^-} \theta(t_g(x)) \),
5. \( \kappa_f(x) \leq \kappa_g(x) \) for all \( x \in (a, b) \).

Then for every \( x \in (a, b) \), \( f(x) \leq g(x) \) and \( \theta(t_f(x)) \geq \theta(t_g(x)) \). Furthermore, if there exists a point \( x_0 \in (a, b) \) such that \( \kappa_f(x_0) < \kappa_g(x_0) \), then there is some \( \phi > 0 \) such that \( \phi \leq \theta(t_f(x)) - \theta(t_g(x)) \) for all \( x \in (a, x_0) \).

**6. Proof of Right Tangent Lemma**

To prove Proposition 4.8, we assume that \( F(0) > 1/2 \), then investigate three portions of the curve, beginning at the rightmost point and traveling counterclockwise to its other intersection point with the \( e_1 \)-axis. One may think of the upper curve as the portion of \( [0, \beta) \) on which the tangent vector \( \gamma'(s) \) is in the second quadrant and the lower curve as the portion of \( [0, \beta) \) on which the tangent vector is in the third quadrant. We will prove that the lower curve ends in a vertical tangent at a point right of the \( e_2 \)-axis (Lemma 6.12) and that, past this point, curvature is positive and the tangent vector is strictly in the fourth quadrant (Lemma 6.14). The end behavior of the curve is similar to that of the generating curve in [C] except that our curve must terminate right of the \( y \)-axis, an additional feature which allows us to achieve a contradiction to spherical symmetry without an analogue of Chambers’ Second Tangent Lemma [C, Lemma 2.5]. As such, many intermediate results are also similar to results in [C] and are
satisfy the inequality

have discovered a set of sufficient conditions for the points

to be tangent vectors to the curve at two points, so

More generally, given points

where

quantities

cross-referenced.

Our analysis requires comparing curvatures at points of the same height on the upper and lower curves (technically, the portions of the curve parameterized by these intervals in \([0, \beta]\)). Specifically, we show that the curvature at the point on the left is strictly greater than the curvature at the corresponding point on the right (Proposition 6.1). By Corollary 5.2, it suffices to prove that \(\lambda\), the canonical circle curvature, is less at the point on the left and \(H_1\), the normal derivative of the log of the density, is strictly less at the point on the left. For any \(s \in [0, \beta]\), \(\gamma(s) \neq (0, 0)\), so the normal derivative of \(\log(p^\nu)\) at \(\gamma(s)\) is given by

\[
H_1(s) = \frac{p}{|\gamma(s)|} \frac{\gamma(s)}{|\gamma(s)|} \cdot v(s) = p \frac{\gamma(s)}{|\gamma(s)|^2} \cdot v(s). \tag{3}
\]

More generally, given points \((x_1, y_1), (x_2, y_2) \in \mathbb{R}^2 - \{0\}\), and unit vectors \(v_1\) and \(v_2\), one can compare the quantities

\[
\frac{(x_1, y_1)}{|(x_1, y_1)|^2} \cdot v_1^+ \quad \text{and} \quad \frac{(x_2, y_2)}{|(x_2, y_2)|^2} \cdot v_2^+,
\]

where \(v_1^+\) and \(v_2^+\) denote clockwise rotations of \(v_1\) and \(v_2\) by \(\pi/2\) radians. (In our context, \(v_1\) and \(v_2\) will be tangent vectors to the curve at two points, so \(v_1^+\) and \(v_2^+\) will be the unit outward normal vectors.) We have discovered a set of sufficient conditions for the points \((x_1, y_1)\) and \((x_2, y_2)\) and the vectors \(v_1\) and \(v_2\) to satisfy the inequality

\[
\frac{(x_1, y_1)}{|(x_1, y_1)|^2} \cdot v_1^+ \geq \frac{(x_2, y_2)}{|(x_2, y_2)|^2} \cdot v_2^+. \tag{4}
\]

In Definition 6.1, we define two unit vectors \(v_1\) and \(v_2\) to be admissible with respect to \((x_1, y_1)\) and \((x_2, y_2)\) if they satisfy these conditions.

**Definition 6.1.** Consider a pair of points \((x_1, y_1)\) and \((x_2, y_2)\) with \(y > 0\), and a pair of unit vectors, \(v_1\) and \(v_2\), which lie strictly in the second and third quadrants, respectively. Let \(v_1^f\) denote the reflection of \(v_1\) over the \(e_1\)-axis. Let \(C_i\) denote the canonical circle with respect to \(v_i\) at \((x_i, y_i)\), with center \((a_i, 0)\) and radius \(R_i\). As depicted in Figure 4, \(v_1\) and \(v_2\) are admissible with respect to \((x_1, y_1)\) and \((x_2, y_2)\) if the following occur:

1. \(a_1 > R_1\),
2. \(\theta(v_2) \geq \theta(v_1^f)\),
3. \(x_1 - a_1 \geq a_1 - x_2\).
Proposition 6.2. Consider a pair of points \((x_1, y)\) and \((x_2, y)\) in the upper half plane with \(x_1 \geq x_2\). Let \(v_1\) and \(v_2\) be two unit vectors, and \(v_1^+\) and \(v_2^+\) denote the clockwise rotations of these respective vectors through \(\pi/2\) radians. If \(v_1\) and \(v_2\) are admissible with respect to \((x_1, y)\) and \((x_2, y)\), then

\[
\frac{(x_1, y)}{|(x_1, y)|^2} \cdot v_1^+ > \frac{(x_2, y)}{|(x_2, y)|^2} \cdot v_2^+
\]

Proof. Let \((x_1', y)\) be the reflection of \((x_1, y)\) over the vertical line \(x = a_1\). It follows that \(x_1' = a_1 - (x_1 - a_1)\). By symmetry, \(C_1\) is also the canonical circle with respect to \(v_1\) at \((x_1', y)\). We will show that

\[
\frac{(x_1, y)}{|(x_1, y)|^2} \cdot v_1^+ > \frac{(x_1', y)}{|(x_1', y)|^2} \cdot v_1^+
\]

and that

\[
\frac{(x_1', y)}{|(x_1', y)|^2} \cdot v_1^+ \geq \frac{(x_2, y)}{|(x_2, y)|^2} \cdot v_2^+. \tag{5}
\]

To prove (5), we parameterize \(C_1\) by \(\alpha(t) = (a_1 + R_1 \cos t, R_1 \sin t)\) for \(t\) in \([0, 2\pi)\). Taking \(t_1 \in (0, \pi/2)\) so that \(\alpha(t_1) = (x_1, y)\), we have by symmetry that \((x_1', t) = \alpha(\pi - t_1)\). Using this parametrization to simplify the quantities in (5), we have

\[
\frac{(x_1, y)}{|(x_1, y)|^2} \cdot v_1^+ = \frac{(a_1 + R_1 \cos t_1, R_1 \sin t_1)}{|\alpha(t_1)|^2} \cdot (\cos t_1, \sin t_1) = \frac{a_1 \cos t_1 + R_1}{|\alpha(t_1)|^2}
\]

and

\[
\frac{(x_1', y)}{|(x_1', y)|^2} \cdot v_1^+ = \frac{(a_1 - R_1 \cos t_1, R_1 \sin t_1)}{|\alpha(\pi - t_1)|^2} \cdot (- \cos t_1, \sin t_1) = \frac{-a_1 \cos t_1 + R_1}{|\alpha(\pi - t_1)|^2}.
\]

whence

\[
\frac{(x_1, y)}{|(x_1, y)|^2} \cdot v_1^+ - \frac{(x_1', y)}{|(x_1', y)|^2} \cdot v_1^+ = \frac{(a_1 \cos t_1 + R_1)|\alpha(\pi - t_1)|^2 - (-a_1 \cos t_1 + R_1)|\alpha(t_1)|^2}{|\alpha(t_1)|^2|\alpha(\pi - t_1)|^2}.
\]

The denominator is positive, so we need only show that the numerator is positive to conclude that (5) holds. A short computation reveals that

\[
(a_1 \cos t_1 + R_1)|\alpha(\pi - t_1)|^2 - (-a_1 \cos t_1 + R_1)|\alpha(t_1)|^2 = 2a_1(a_1^2 - R_1^2) \cos t_1 > 0.
\]

To prove (6), we first note that since \(v_1\) and \(v_2\) are admissible with respect to \((x_1, y)\) and \((x_2, y)\), we have that \(x_2 \geq a_1 - (x_1 - a_1) = x_1'\). Moreover, \(x_1'\) must be positive, as \(a_1 - (x_1 - a_1) > a_1 - R_1 > 0\). It follows that

\[
\frac{1}{|(x_1', y)|} \geq \frac{1}{|(x_2, y)|}.
\]

Therefore, to prove (6), it suffices to show that

\[
\frac{(x_1', y)}{|(x_1', y)|} \cdot v_1^+ \geq \frac{(x_2, y)}{|(x_2, y)|} \cdot v_2^+.
\]

We note that the left-hand side of (7) is \(\cos(\theta(v_1^+) - \theta((x_1', y)))\) and the right-hand side is equal to \(\cos(\theta(v_2^+) - \theta((x_2, y)))\). Since \(v_1\) is strictly in the second quadrant, \(v_2\) is strictly in the third, and \(x_2 \geq x_1' > 0\), it follows that

\[
0 < \theta(v_1^+) - \theta((x_1', y)), \theta(v_2^+) - \theta((x_2, y)) < \pi.
\]

As cosine is decreasing on \((0, \pi)\), it suffices to show that

\[
\theta(v_2^+) - \theta((x_2, y)) \geq \theta(v_1^+) - \theta((x_1', y)). \tag{8}
\]

As noted above, \(x_2 \geq x_1'\), so \(\theta((x_2, y)) \leq \theta((x_1', y))\). By the admissibility of \(v_1\) and \(v_2\), we have that \(\theta(v_2^+) \geq \theta(v_1^+)\). Combining these inequalities establishes (8), completing our proof of (6).

Having proved Proposition 6.2, we define the upper and lower curves and prove various properties that hold on these intervals. Our definition of the upper curve is motivated by the following observation.
Lemma 6.3. (cf. [C] Lemma 3.5) Given that \( F(0) > 1/2 \), we have \( \kappa''(0) > 0 \).

Proof. By a similar argument to that in [C] Lemma 3.5, it suffices to prove that \( \hat{H}_{1}''(0) < 0 \). Let \( a = F(0) \) and \( r = R(0) \). Note that our assumption that
\[
F(0) > \frac{1}{2} = \frac{\gamma_1(0)}{2}
\]
is equivalent to the inequality \( a > r \). Parameterizing \( A_0 \) by
\[
\alpha(t) = \left( a + r \cos \left( \frac{t}{r} \right), r \sin \left( \frac{t}{r} \right) \right),
\]
we can then compute that
\[
\hat{H}_{1}''(0) = \frac{p}{|\alpha(0)|^4} \frac{a}{r^2} (r^2 - a^2) < 0.
\]
\[\square\]

Definition 6.4. (cf. [C] Defn. 3.4) Let the upper curve \( K \) be defined as the set of all \( t \in [0, \beta) \) such that for all \( s \) in \([0, \beta] \) the following properties are satisfied:
1. \( \gamma'(s) \) lies in the second quadrant,
2. \( \kappa(s) \geq \lambda(s) > 0 \).

By similar arguments to those in [C] Lemma 3.11, there exist \( \rho_1, \rho_2 > 0 \) such that \( \gamma'(s) \) lies in the second quadrant for all \( s \in [0, \rho_1] \) and \( \kappa(s) \geq \lambda(s) > 0 \) for all \( s \in [0, \rho_2] \). Taking \( \rho = \min(\rho_1, \rho_2) \), it follows that \([0, \rho] \subseteq K \). Thus, \( K \) is nonempty and \( \sup K > 0 \). We let \( \delta = \sup K \). The following lemma extends our assumption that \( F(0) > R(0) \) and allows us to check the first condition of admissibility.

Lemma 6.5. If \( s \in K \), then \( F(s) > R(s) \).

Proof. By hypothesis, \( F(0) > R(0) \). By a similar argument to that in [C] Lemma 5.3, \( F' \) and \( \lambda' \) are nonnegative on \( K \). Since \( R(s) = 1/\lambda(s) \), \( R(s) \leq R(0) \), and we have \( F(s) \geq F(0) > R(0) \geq R(s) \), as desired.
\[\square\]

We will soon prove several properties of \( \delta \), but first we require one more lemma.

Lemma 6.6. Let \( s \in (0, \delta) \). If \( \kappa(s) = \lambda(s) > 0 \), then \( \lambda'(s) = 0 \), but \( \kappa'(s) > 0 \).

Proof. Differentiating Equation (3) gives \( \kappa'(s) + (n-2)\lambda'(s) + H'_1(s) = 0 \). By the hypothesis that \( \kappa(s) = \lambda(s) \), we have that \( A_s = C_s \). It follows that the canonical circle to \( A_s \) at each point is \( A_s \), so \( \lambda \) is constant. In particular, \( \lambda'(s) = \tilde{\lambda}'(\tilde{s}) = 0 \).

Given this result, to prove that \( \kappa'(s) > 0 \), it suffices to prove that \( H'_1(s) < 0 \). Parameterizing \( A_s \) by
\[
\alpha(t) = \left( a + r \cos \left( \frac{t}{r} \right), b + r \sin \left( \frac{t}{r} \right) \right),
\]
we can compute that
\[
H'_1(s) = \frac{-p(a^2 + b^2 - r^2)(-b \cos \left( \frac{t}{r} \right) + a \sin \left( \frac{t}{r} \right))}{r|\alpha(t)|^4}.
\]
Since \( A_s = C_s \), we have that \( b = 0, a = F(s), \) and \( r = R(s) \). By Lemma 6.5, \( a > r > 0 \). Thus, we have
\[
H'_1(s) = H'_1(\tilde{s}) = \frac{-p(a^2 - r^2)(a \sin \left( \frac{\tilde{s}}{r} \right))}{r|\alpha(\tilde{s})|^4} < 0.
\]
\[\square\]

Proposition 6.7. (cf. [C] Prop. 3.12) The following properties of \( \delta \) hold:
1. \( \delta < \beta \),
2. \( \delta \in K \),
3. \( \gamma(\delta) \geq F(s) \) for any \( s \in [0, \delta] \),
4. \( \gamma(\delta) > 0 \),
5. \( \gamma'(\delta) = (-1, 0) \).
Proof. The proofs of (1)-(3) are identical to their counterparts in [C] Prop. 3.12. Setting \( s = 0 \) in the inequality \( \gamma_1(\delta) \geq F(s) \) we have

\[
\gamma_1(\delta) \geq F(0) > \frac{\gamma_1(0)}{2} > 0.
\]

The proof that \( \gamma'(\delta) = (-1, 0) \) is similar to that in [C] Prop. 3.12; however, we do not require a second sub-case, and we apply our Lemma 6.6 in place of [C] Lemma 3.4. \( \square \)

**Lemma 6.8.** We have that \( \kappa'(\delta) > 0 \).

**Proof.** Differentiating the ODE \( H_f = c \), we obtain \( \kappa'(\delta) + (n-2)\lambda'(\delta) + H_f'(\delta) = 0 \). Let \((a, b)\) be the center of \( A_\delta \) and \( r \) be its radius. Since \( \gamma'(\delta) = (-1, 0) \), we have \( a = F(\delta) > 0 \). Moreover, since \( \kappa(\delta) \geq \lambda(\delta) \), it follows that \( b \geq 0 \). Parameterizing \( A_\delta \) as in (9), we see that \( \gamma'(\delta) = \alpha(\pi r/2) \), where \( r \) is the radius of \( A_\delta \). Thus, \( \lambda'(\delta) = \lambda'(\pi r/2) \), which equals 0 by the computations in [C] Lemma 5.3.

Since \( H_f'(\delta) = \tilde{H}_f'(\pi r/2) \), it suffices to prove that \( \tilde{H}_f'(\pi r/2) < 0 \).

Looking to (10), we claim that \( a^2 + b^2 > r^2 \). To prove so, let \( R = R(\delta) \) be the radius of \( C_\delta \). Since \( \gamma'(\delta) = (-1, 0) \), we have that \( R = r + b \) and \( a = F(\delta) \). We apply Lemma 6.6 to give \( a > R = r + b \). Since both sides of \( a - b > r \) are positive, we may square to give \( (a - b)^2 > r^2 \). Since \( b \geq 0 \), this implies that \( a^2 + b^2 > r^2 \). Therefore, we have that

\[
\tilde{H}_f'(\frac{\pi r}{2}) = -\frac{pa(a^2 + b^2 - r^2)}{2r(\alpha(\frac{\pi r}{2}))^3} < 0.
\]

\( \square \)

**Definition 6.9.** (cf. [C] Defn. 3.5) Let the lower curve \( L \) be defined as the set of all \( t \) in \([\delta, \beta]\) such that for all \( s \in [\delta, \beta] \) the following hold:

1. \( \gamma'(s) \) is in the third quadrant with \( \gamma'(s) \neq (-1, 0) \) if \( s > \delta \).
2. If \( s \) is the unique point in \( K \) with \( \gamma_2(s) = \gamma_2(s) \), then \( \kappa(s) \leq \kappa(s) \).

Since \( \gamma'(\delta) = (-1, 0) \), \( \kappa'(\delta) > 0 \), and \( \kappa'(\delta) > 0 \), these conditions hold on an interval \([\delta, \delta + \varepsilon]\). Thus, \( L \) is nonempty and has a supremum, which we denote by \( \eta \). We define \( h : (\gamma_2(\eta), \gamma_2(\delta)) \to (0, \delta) \) by letting \( h(y) = \) the unique \( t \in (0, \delta) \) such that \( \gamma_2(t) = y \). Similarly, we define \( k : (\gamma_2(\eta), \gamma_2(\delta)) \to (\delta, \eta) \) by letting \( k(y) = \) the unique \( t \in (\delta, \eta) \) such that \( \gamma_2(t) = y \). Now, Consider the functions \( f \) and \( g \) on \((\gamma_2(\eta), \gamma_2(\delta))\) defined by

\[
f(y) = 2\gamma_1(\delta) - \gamma_1(h(y))
\]
and

\[
g(y) = \gamma_1(k(y)).
\]

The function \( g \) gives the \( e_1 \)-coordinate of a point in the lower curve with a given \( e_2 \)-coordinate. If we begin with the point in the upper curve with a given \( e_2 \)-coordinate, then \( f \) gives the \( e_1 \)-coordinate of the reflection of this point over the line \( x = \delta \). We can use these functions to prove two properties of the lower curve.

**Lemma 6.10.** (cf. [C] Lemma 3.13) For each \( s \in [\delta, \eta] \), we have the following:

\[
\gamma_1(\delta) - \gamma_1(\delta) \geq \gamma_1(\delta) - \gamma_1(s).
\]

\[
\theta(\gamma'(s)) \geq 2\pi - \theta(\gamma'(s)).
\]

**Proof.** Both inequalities are trivially true if \( s = \delta \). Now let \( s \in (\delta, \eta) \) be fixed, and let \( y = \gamma_2(s) \). By the definition of \( L \) (Def. 6.9), \( f \) and \( g \) satisfy the hypotheses of Proposition 5.4. From the inequality \( f \leq g \) in Proposition 5.4 above is immediate. To arrive at (14), let \( t_f(y) \) and \( t_g(y) \) denote the unit tangent vectors to the graph of \( f \) and \( g \) at \( y \). Note that \( \gamma'(\delta, \eta) \) is the set \( \{ g(y), y \in (\gamma_2(\eta), \gamma_2(\delta)) \} \), and the reflection of \( \gamma'(0, \delta) \) across the line \( x = \delta \) is the set \( \{ f(y), y \in (\gamma_2(\eta), \gamma_2(\delta)) \} \). Let \( y = \gamma_2(s) \). Then we obtain the tangent vector \( t_g(y) \) from \( \gamma'(s) \) by rotating \( \theta'(s) \) clockwise through \( \pi \) radians and reflecting the resulting vector in the first quadrant over the line \( y = x \). Therefore, we have

\[
\theta(t_g(y)) = \frac{\pi}{2} - (\theta(\gamma'(s)) - \pi) = \frac{3\pi}{2} - \theta(\gamma'(s)).
\]
Similarly, we obtain $t_f(y)$ from $\gamma'(\bar{x})$ by reflecting over the line $x = \delta$ and reflecting over the line $y = x$. Thus,

$$\theta(t_f(y)) = \frac{\pi}{2} - (\pi - \theta(\gamma'(\bar{x}))) = \theta(\gamma'(\bar{x})) - \frac{\pi}{2}.$$  

Substituting these results into the second inequality in (5.4) completes the proof. 

\begin{proposition}
Let $s \in (\delta, \eta)$, and suppose that $\gamma'(s) \neq (0, -1)$. If $s$ is the unique point in $K$ so that $\gamma_2(s) = \gamma_2'(s)$ then $\kappa(s) < \kappa'(s)$.
\end{proposition}

\begin{proof}
Since $H_f$ is constant,

$$\kappa(s) + (n-2)\lambda(s) + H_1(s) = \kappa(\bar{x}) + (n-2)\lambda(\bar{x}) + H_1(\bar{x}).$$

It can be shown using right triangle trigonometry and (14) from Lemma 6.10 that $\lambda(s) \leq \lambda(\bar{x})$. Thus, to prove that $\kappa(s) > \kappa(\bar{x})$, it suffices to prove that $H_1(s) < H_1(\bar{x})$. We show that $\gamma'(s)$ and $\gamma'(\bar{x})$ are admissible with respect to $\gamma(\bar{x})$ and $\gamma(s)$ and then appeal to Proposition 6.2. Since $\gamma'(s)$ is not equal to $(0, \pm 1)$, $\gamma'(s)$ lies strictly in the third quadrant. By Lemma 6.5 $F(\bar{x}) > R(\bar{x})$. Thus the first condition in the definition of admissibility is met.

By Lemma 6.10 $\theta(\gamma'(s)) \geq 2\pi - \theta(\gamma'(\bar{x}))$, satisfying the second condition of admissibility. Furthermore, by the same lemma we have $\gamma(\bar{x}) - \gamma(\delta) \geq \gamma(\delta) - \gamma(s)$. By Proposition 6.7 $\gamma(\delta) \geq F(\bar{x})$, so $\gamma(\bar{x}) - F(\bar{x}) \geq F(\bar{x}) - \gamma(s)$, and the final condition for admissibility is satisfied.

Because $\gamma'(s)$ and $\gamma'(\bar{x})$ are admissible with respect to $\gamma(\bar{x})$ and $\gamma(s)$, we conclude by Proposition 6.2 that

$$\frac{\gamma(\bar{x})}{|\gamma(\bar{x})|^2} \cdot \gamma'(s) \cdot \gamma'(\bar{x}) \geq \frac{\gamma(s)}{|\gamma(s)|^2} \cdot \gamma'(s) \cdot \gamma'(\bar{x}).$$

By (5), it follows that $H_1(s) < H_1(\bar{x})$, as required.

By a similar argument to that in Lemma 3.14 along with Proposition 6.11 $\eta < \beta$, $\eta \in L$, and $\gamma'(\eta) = (0, -1)$. In addition to these properties of $\eta$, we can also show using the curvature comparison that $\gamma_1(\eta) > 0$. Then proving that $\eta(\beta) > 0$ is a matter of showing that $\gamma_1$ is increasing on $(\eta, \beta)$. To establish the second claim of the Right Tangent Lemma, we consider the functions $\kappa$ and $\gamma'$ on $(\eta, \beta)$.

\begin{lemma}
We have that $\gamma_1(\eta) > 0$.
\end{lemma}

\begin{proof}
By Lemma 6.10 $\gamma_1(\bar{x}) - \gamma_1(\delta) \geq \gamma(\delta) - \gamma(\eta)$. Furthermore, $\gamma_1(\delta) = F(\delta)$, since $\gamma'(\delta) = (0, -1)$. Since $F$ is non-decreasing on $K$, $F(\delta) \geq F(\bar{x})$. Therefore, $\gamma_1(\bar{x}) - F(\bar{x}) \geq \gamma_1(\bar{x}) - \gamma_1(\eta)$. Finally, $\gamma_1(\bar{x}) - F(\bar{x}) < R(\bar{x})$, because $R(\bar{x})$ is the distance from $(F(\bar{x}), 0)$ to $\gamma(\eta)$. It follows that $\gamma_1(\eta) \geq F(\bar{x}) - (\gamma_1(\bar{x}) - \gamma_1(\eta)) > F(\bar{x}) - R(\bar{x})$. By Lemma 6.5 this final expression is positive.

\begin{lemma}
Let $s \in (0, \beta)$. If $\gamma_1(s) \geq 0$ and $\gamma'(s)$ is in the fourth quadrant, then $\kappa(s) > 0$.
\end{lemma}

\begin{proof}
Since $\gamma'(s)$ is in the fourth quadrant and $\gamma_2(s) > 0$, $\lambda(s) \leq 0$. Since $\gamma(s)$ is in the first quadrant and $\nu(s)$ is in the third, $\gamma(s) \cdot \nu(s) \leq 0$, which implies that $H_1(s) \leq 0$. Meanwhile, we have that $H_f(0) = H_1(0)$, because $H_1(0) > 0$, $\kappa(0) > 0$ by spherical symmetry, and $\kappa(0) = \lambda(0)$ (Proposition 4.7). Hence, it must be the case that $\kappa(s) > 0$.

\begin{lemma}
For $s \in (\eta, \beta)$, $\gamma'(s)$ lies strictly in the fourth quadrant and $\kappa(s) > 0$.
\end{lemma}

\begin{proof}
Define $A \subset (\eta, \beta)$ so that $s \in A$ if and only if for all $t \in (\eta, s)$, $\gamma'(t)$ lies strictly in the fourth quadrant and $\kappa(t) > 0$. Note that $A$ is nonempty because $\kappa(\eta) > 0$, $\gamma'(\eta) = (0, -1)$, and $\kappa$ is continuous at $\eta$. Thus, $A$ has a supremum $\omega$. To prove the lemma we show that $\omega = \beta$.

Suppose for contradiction that $\omega < \beta$. Then $\gamma$ is smooth at $\omega$; in particular, $\gamma'(\omega)$ and $\kappa'(\omega)$ are defined. Since $\gamma'(t)$ lies in the fourth quadrant for all $t \in (\eta, \omega)$, $\gamma'(\omega)$ is in the fourth quadrant. Since $\kappa > 0$ on $(\eta, \omega)$, $\gamma'(\omega)$ is not equal to $(0, -1)$. Furthermore, $\gamma_1(\omega) > 0$, as $\gamma_1(\eta) > 0$ (Lemma 6.12).
\( \gamma' \) lies in the fourth quadrant on \((\eta, \omega)\). If \( \gamma'(\omega) \) were equal to \((1, 0)\), then we would have \( \gamma'(\omega) \cdot \gamma(\omega) = \gamma(\omega) > 0 \), contradicting the Tangent Restriction Lemma (Lemma 4.1). Thus \( \gamma'(\omega) \) lies strictly in the fourth quadrant. By Lemma 6.13, \( \kappa(\omega) > 0 \). Thus, by continuity of \( \gamma' \) and \( \kappa \) on \([0, \beta]\), \( A \) could be extended past \( \omega \), contradicting the definition of \( \omega \).

**Proof of the Right Tangent Lemma (Lemma 4.8).** It follows from Lemma 6.14 that \( \eta(\beta) > 0 \), as \( \gamma'(s) \) lies strictly in the fourth quadrant for all \( s \in (\eta, \beta) \), and \( \eta(\eta) > 0 \). As \( \kappa > 0 \) and \( \gamma' \) is in the fourth quadrant on \((\eta, \beta)\), the angle \( \theta(s) \) that \( \gamma'(s) \) makes with the \( e_1 \)-axis, measured counterclockwise in radians, must be a strictly increasing function on \((\eta, \beta)\) that is bounded above by \( 2\pi \). Therefore, \( \lim_{s \to \beta} \theta(s) \) exists and is in \((\theta(\eta), 2\pi]\). It follows that \( \lim_{s \to \beta} \gamma'(s) \) exists, lies in the fourth quadrant, and is not \((0, -1)\).

7. **Proof of Left Tangent Lemma**

In the previous section, the key to proving the Right Tangent Lemma was to show that the curvature was greater at a point on the lower curve than at its corresponding point on the upper curve, allowing us to find \( \eta < \beta \) where \( \gamma'(\eta) = (0, -1) \). Now, in the left case (Prop. 4.9), we will prove the opposite inequality concerning curvatures at corresponding points, with the aim of showing that \( \lim_{s \to \beta} \gamma'(s) \) is strictly in the third quadrant. This case, however, presents new obstacles. One difficulty we eliminate is the possibility that there are multiple points on the portion of \( \gamma \) parameterized by \([0, \beta]\) where the tangent vector is \((-1, 0)\). In the right case, the lower curve naturally terminated at a point where the tangent vector was \((0, -1)\). However, the goal in the left case will be to show that the lower curve does not terminate before \( \beta \), allowing us to apply the curvature comparison all the way up to \( \beta \). We begin with a new definition of admissibility for the left case and an analogue of Proposition 6.2.

**Definition 7.1.** Consider two points \((x_1, y_1)\) and \((x_2, y_2)\) and two unit vectors \(v_1\) and \(v_2\), strictly in the second and third quadrants, respectively. Let \(C, a, R, x_1',\) and \(v_1'\) be as in Definition 6.1. Finally, let \((x^*, 0)\) be the unique point on the \(e_1\)-axis so that \(v_2\) is tangent at \((x^*, y)\) to the circle centered at the origin that passes through \((x^*, y)\). We say that \(v_1\) and \(v_2\) are *admissible* with respect to \((x_1, y)\) and \((x_2, y)\) if the following hold:

1. \(0 < a_1 < R_1\)
2. \(\theta(v_2) \leq \theta(v_1')\)
3. \(R_2 \leq R_1\)
4. \(x_2 \in [x^*, x_1']\).

Figures 5 and 6 depict vectors \(v_1\) and \(v_2\) that are admissible with respect to \((x_1, y)\) and \((x_2, y)\) when \(x_2 \geq 0\) and when \(x_2 < 0\).

**Proposition 7.2.** If \(v_1\) and \(v_2\) are admissible with respect to \((x_1, y)\) and \((x_2, y)\), then \(H_1\) is larger at \((x_2, y)\) with respect to \(v_2\) than at \((x_1, y)\) with respect to \(v_1\).

**Proof.** We take cases according to whether \(x_2 \geq 0\) or \(x_2 < 0\). In the case that \(x_2 \geq 0\), \(|(x_2, y)| \geq |(x_1', y)|\), and the result follows by a similar argument to that in Proposition 6.2. In the case case that \(x_2 < 0\), we will prove two inequalities:

\[
\frac{(x_1, y)}{|(x_1, y)|^2} \cdot v_1^* < \frac{(x^*, y)}{|(x^*, y)|^2} \cdot v_2^*.
\]

\[
\frac{(x^*, y)}{|(x^*, y)|^2} \cdot v_2^* \leq \frac{(x_2, y)}{|(x_2, y)|^2} \cdot v_2^*.
\]

Beginning with (15), note that since \(a_1 > 0\), we must have that \(|(x_1, y)| > R_1\). Additionally, \(|(x^*, y)| = R_2\). Combining these observations with the inequality \(R_1 \geq R_2\), we have \(|(x_1, y)| > R_1 \geq R_2 = |(x^*, y)|\). It follows that

\[
\frac{1}{|(x_1, y)|} < \frac{1}{|(x^*, y)|},
\]

so proving (15) has been reduced to showing that
Figure 5. The vectors $v_1$ and $v_2$ are admissible with respect to $(x_1, y)$ and $(x_2, y)$ with $x_2 \geq 0$.

Figure 6. The vectors $v_1$ and $v_2$ are admissible with respect to $(x_1, y)$ and $(x_2, y)$ with $x_2 \leq 0$.

\[
\frac{(x_1, y)}{|(x_1, y)|} \cdot v_1^+ \leq \frac{(x^*, y)}{|(x^*, y)|} \cdot v_2^+ \neq 0.
\]

This inequality is immediate when we recognize that

\[
\frac{(x^*, y)}{|(x^*, y)|} \cdot v_2^+ = \cos(\theta(v_2^+)) = 1.
\]

To prove (16), we will rewrite the right side of the inequality using the subtraction identity for cosine. As noted above, $\theta(v_2^+) = \theta((x^*, y))$, so

\[
\cos(\theta(v_2^+)) = \frac{x^*}{|(x^*, y)|} \quad \text{and} \quad \sin(\theta(v_2^+)) = \frac{y}{|(x^*, y)|}.
\]
Hence, we have
\[
\frac{(x_2, y)}{|(x_2, y)|^2} \cdot v_2^* = \frac{1}{|(x_2, y)|} \cos(\theta(v_2^*)) - \theta((x_2, y))
\]
\[
= \frac{1}{|(x_2, y)|} \left( \cos(\theta(v_2^*)) \cos(\theta((x_2, y))) + \sin(v_2^*) \sin(\theta((x_2, y))) \right)
\]
\[
= \frac{1}{|(x_2, y)|} \left( \frac{x^* x_2}{|(x^*, y)|} \frac{2}{|(x_2, y)|} + \frac{y}{|(x^*, y)|} \frac{y}{|(x_2, y)|} \right)
\]
\[
= \frac{1}{|(x^*, y)|} \left( \frac{x^* x_2}{|(x^*, y)|^2} + \frac{y^2}{|(x_2, y)|^2} \right).
\]
By (4) in Definition 7.1 and the assumption that \(x_2 < 0\), we have \(x^* \leq x_2 < 0\). We multiply through by \(x_2\) to obtain \(x^* x_2 \geq x_2^2 > 0\). Substituting this into the above equation, we have
\[
\frac{(x_2, y)}{|(x_2, y)|^2} \cdot v_2^* \geq \frac{1}{|(x^*, y)|} \left( \frac{x_2^2 + y^2}{|(x_2, y)|^2} \right) = \frac{1}{|(x^*, y)|^2} = \frac{(x^*, y)}{|(x^*, y)|^2},
\]
completing the second case. \(\square\)

Before we define the upper and lower curves, we require several lemmas. Proposition 7.7 and Proposition 7.8, which concern points where the unit tangent vector is in the second quadrant, are later used to check the conditions for admissibility. Meanwhile, we determine some properties that hold at points on the curve with positive first coordinates.

**Lemma 7.3.** Suppose that \(s \in (0, \beta)\) and that \(\gamma_1(s) > 0\). Then \(\gamma_1'(s) < 0\).

**Proof.** Suppose for contradiction that \(\gamma_1'(s) \geq 0\). If \(\gamma'(s)\) were in the first quadrant and not equal to \((1, 0)\), this would violate the Tangent Restriction Lemma (Lemma 4.1). If \(\gamma'(s) = (1, 0)\), then, by Lemma 6.14 \(\kappa(s) > 0\), which implies by continuity of \(\gamma'\) that there exists \(t > s\) so that \(\gamma_1(t) > 0\), \(\gamma_2(s) > 0\), and \(\gamma'(t)\) is strictly in the first quadrant, producing the same contradiction to the Tangent Restriction Lemma. Thus, if \(\gamma_1'(s) \geq 0\), then \(\gamma'(s)\) must be in the fourth quadrant and not equal to \((1, 0)\). However, this also yields a contradiction, because, replacing \(\eta\) with \(t\), we could then apply Lemmas 6.14 and 6.13 to achieve the same contradiction as in the right case. These lemmas would apply because \(\gamma_1(s) \geq 0\). \(\square\)

**Lemma 7.4.** Suppose that \(s \in (0, \beta)\) and that \(\gamma_1(s) \geq 0\). Then \(\gamma_1(t) > 0\) for all \(t \in [0, s)\).

**Proof.** It is clear that \(\gamma_1(0) > 0\). To prove the result on \((0, s)\), consider the set \(C = \{t \in [0, s) : \gamma_1(t) > 0\}\) for all \(u \in [s, t)\). By Lemma 7.3 \(\gamma_1'(s) < 0\), so there exists \(\varepsilon > 0\) such that \((s - \varepsilon, s) \subset C\). Since \(C\) is nonempty and bounded below, it has a greatest lower bound. Let \(c = \inf C\). It suffices to prove that \(c = 0\). Suppose for contradiction that \(c > 0\). By continuity of \(\gamma_1\), \(\gamma_1(c) = 0\); if \(\gamma_1(c)\) were positive then we could extend \(C\) farther back, whereas if it were negative, then \(c\) would not be the greatest lower bound. By Lemma 7.3 \(\gamma_1'(c) < 0\). It follows that there exists \(\varepsilon' > 0\) such that \(\gamma_1 < 0\) on \((c, c + \varepsilon')\), contradicting the fact that \(c\) is the greatest lower bound of \(C\). \(\square\)

Now we consider the initial canonical circle \(C_0\). By spherical symmetry, \(F(0) \geq 0\). It must actually be the case that \(F(0) > 0\); otherwise, by Remark 4.6 \(\gamma\) would be a circle centered at the origin, contradicting the fact that balls through the origin are not stable \([RCBM\text{, Thm. 3.10}]\). Given this strict inequality, it follows by the computations in the proof of Lemma 6.3 that \(\kappa''(0) < 0\).

A natural next step would be to extend the inequality \(F(0) < R(0)\). We will eventually prove that \(F(s) < R(s)\) for all \(s\) with \(\gamma_1(s) \geq 0\) (Proposition 7.9). Since \(R'\) may alternate signs, this is slightly more complicated than merely reversing the inequalities in the proof of Lemma 6.5. To show that the sign of \(R'\) does not matter, we define an auxiliary function \(G: (-\beta, \beta) \to \mathbb{R}\) by letting \(G(s) = \gamma_1(s) \gamma_1(s) - R(s)\). For a given \(s\), we can compute the derivatives of \(\bar{F}\) and \(\bar{G}\), and \(\bar{G}\) on the approximating circle \(A_t\) to prove the following lemma (cf. [C, Lemma 5.3]).

**Lemma 7.5.** Let \(s \in [0, \beta)\). If \(\gamma_1(s) \geq 0\) and \(\kappa(s) \leq \lambda(s)\), then \(F'(s) \leq 0\) and \(G'(s) \leq 0\).
Proof. We take three cases according to whether \( \kappa(s) > 0 \), \( \kappa(s) = 0 \), or \( \kappa(s) < 0 \). If \( \kappa(s) = 0 \), then \( A_s \) is the oriented line through \( \gamma(s) \) that has direction vector \( \gamma'(s) \). By Lemma 7.3, \( \gamma_1'(s) < 0 \). We parameterize \( A_s \) by \( \alpha(t) = \gamma(s) + t\gamma'(s) \). Let \( \tilde{F}(t) \) and denote the \( e_1 \)-coordinate of the canonical circle to \( A_s \) at \( \alpha(t) \), let \( \tilde{R}(t) \) denote its radius, and let \( \tilde{G} = \tilde{F} - \tilde{R} \). Then we can compute that
\[
\tilde{F}'(t) = \frac{1}{\gamma_1'(s)} < 0 \quad \text{and} \quad \tilde{G}'(t) = \frac{1 + \gamma_1'(s)}{\gamma_1'(s)}.
\]
Since \( \gamma \) is an arclength parameterization, the numerator of \( \tilde{G}' \) is necessarily nonnegative. Thus, \( \tilde{G}' \leq 0 \).

If \( \kappa \neq 0 \), let \((a,b)\) be the center of \( A_s \), and let \( r \) be the radius. If \( \kappa > 0 \), then \( b < 0 \), whereas if \( \kappa < 0 \), then \( b > 0 \). We parameterize \( A_s \) in \([0, 2\pi r] \) so that \( \alpha(s) = \gamma(s) \). Then, as in [C] Lemma 5.3, we obtain expressions for \( \tilde{F} \) and \( \tilde{R} \) and differentiate. In each case, the signs of \( \tilde{F}'(s) \) and \( \tilde{G}'(s) \) are determined by the sign of \( b \).

Although we used both hypotheses of Proposition 7.5 in the proof, it is actually the case that the first hypothesis implies the second, as we prove below.

Lemma 7.6. Let \( s \in [0, \beta) \). If \( \gamma_1(s) \geq 0 \), then \( \kappa(s) \leq \lambda(s) \).

Proof. Due to Lemma 4.7 and the fact that \( \kappa'(0) = 0 \), \( \lambda \) and \( \kappa \) are equal up to order two at 0. However, \( \lambda''(0) = 0 \), whereas \( \kappa''(0) < 0 \). Hence, there exists \( t > 0 \) so that \( \kappa \leq \lambda \) on \([0, t] \). Let \( S = \{ t \in [0, \beta) : \kappa \leq \lambda \text{ and } \gamma_1 \geq 0 \text{ on } [0, t) \} \), and let \( u = \sup S \). Since the inequalities that define \( S \) are not strict, it follows by smoothness of \( \gamma \) that \( u \in S \). If \( \gamma_1(u) = 0 \), then, by Lemma 7.4, \( \gamma_1(s) \geq 0 \) only if \( s \in [0, u] \). Thus, to prove that \( \kappa(s) \leq \lambda(s) \) for all \( s \) with \( \gamma(s) \geq 0 \), it suffices to prove that \( \gamma_1(u) = 0 \).

Suppose for contradiction that \( \gamma_1(u) > 0 \). We will show that \( u \) is not an upper bound for \( S \), but, instead, that there exists \( \varepsilon > 0 \) so that \([u, u + \varepsilon) \subset S \). We can obviously find \( \varepsilon_1 > 0 \) so that \( \gamma_1 \geq 0 \) on \([u, u + \varepsilon_1) \). It remains to show that there exists \( \varepsilon_2 > 0 \) so that \( \kappa \leq \lambda \) on \([u, u + \varepsilon_2) \). The proof will be similar to that of Lemma 3.4 in [C].

First, we can prove by contradiction that \( \kappa(u) = \lambda(u) \). Given this assumption, we have that \( C_u = A_u \), so \( \lambda'(u) = \lambda'(u) = 0 \). Thus, to guarantee the existence of a \( \varepsilon_2 > 0 \) so that \( \kappa \leq \lambda \) on \([u, u + \varepsilon_2) \), it suffices to show that \( \kappa'(u) < 0 \). Since \( \kappa \leq \lambda \) and \( \gamma_1 \geq 0 \) on \([0, u] \), it follows from Proposition 7.5 that \( \tilde{G}' \leq 0 \) on \([0, u] \). Therefore, \( G(u) \leq G(0) < 0 \) by assumption that \( F(0) < R(0) \), and it follows by a similar argument to that in the proof of Lemma 6.6 that \( \kappa'(u) < 0 \). While the inequality \( a > 0 \) was immediate in the case that \( a > r \), here it is more subtle. The fact that \( a \geq 0 \) follows from a similar argument to that in [C] Lemma 3.3. To prove strict inequality, note that if \( a = 0 \), then \( \gamma \) is a circle centered at the origin, which contradicts the fact that balls centered at the origin are not stable ([RCBM] Thm. 3.10).

We use Lemmas 7.3 and 7.6 to prove two propositions used in checking the conditions for admissibility (Props. 7.8 and 7.9), but first we require one additional lemma.

Lemma 7.7. Suppose that \( s \in (0, \beta) \) and that \( \gamma'(s) \) is in the second quadrant. Then \( \gamma_1(s) > 0 \).

Proof. By Lemma 4.1, \( \gamma'(s) \neq (0, 1) \). Thus, \( \gamma_2'(s) \geq 0 \) and \( \gamma_1'(s) < 0 \). If \( \gamma_1(s) < 0 \) or \( \gamma_1(s) = 0 \) and \( \gamma_2'(s) > 0 \), then we can obtain a contradiction to Lemma 4.1. It remains to cover the case in which \( \gamma_1(s) = 0 \) and \( \gamma'(s) = (-1, 0) \). By Proposition 7.6, \( \kappa(s) \geq \lambda(s) \). If \( \kappa(s) = \lambda(s) \), then \( \gamma \) is a circle centered at the origin, contradicting the fact that centered balls are not stable ([RCBM] Thm. 3.10). Now, suppose that \( \kappa(s) < \lambda(s) \). Since \( |\gamma(t)| \) is a non-increasing function of \( t \) and \( C_s \) is centered at the origin, \( \gamma(t) \) must be contained in \( C_s \) for \( t \geq s \). However, since \( \kappa(s) < \lambda(s) \), the curve locally leaves the disk bounded by \( C_s \).

Proposition 7.8. Let \( s \in [0, \beta) \). If \( \gamma'(s) \) is in the second quadrant, then \( F(s) > 0 \).

Proof. We know that \( \gamma \) must eventually curve down and arrive at the \( e_1 \)-axis. Thus, there are points where \( \gamma' \) is in the third or fourth quadrant, and, by the Intermediate Value Theorem, combined with the fact that \( \gamma'(s) \neq (0, 1) \) (Lemma 4.1), there is a point \( t > s \) such that \( \gamma'(t) = (-1, 0) \). By Lemma 7.7, \( \gamma_1(t) > 0 \); moreover, by Lemma 7.4, \( \gamma_1(t) > 0 \) on the interval \([s, t] \). Therefore, \( F(t) \leq 0 \) on \([s, t] \), from which it follows that \( F(s) \geq F(t) = \gamma_1(t) > 0 \).
Proposition 7.9. If \( \gamma'(s) \) is in the second quadrant, then \( F(s) < R(s) \).

Proof. Since \( \gamma'(s) \) is in the second quadrant, \( \gamma_1(s) > 0 \). In fact, for all \( t \in [0, s] \), \( \gamma_1(t) > 0 \), so \( \kappa \leq \lambda \) on \([0, s]\). Consequently, by Lemma 7.5 \( G' \leq 0 \) on \([0, s]\). By hypothesis that \( F(0) < \gamma_1(0)/2, G(0) < 0 \). Therefore, \( G(s) \leq G(0) < 0 \).

Having proved the propositions necessary for checking the definitions of admissibility, we define the upper and lower curves and prove that the curvature at a point on the lower curve is less than the curvature at its counterpart on the upper curve (Prop. 7.15).

Definition 7.10. A point \( s \) is in the upper curve \( K \subset (0, \beta) \) if and only if for all \( t \in (0, s) \), \( \gamma'(t) \) is strictly in the second quadrant.

Note that \( K \) is nonempty because \( \gamma'(0) = (0, 1) \) and \( \kappa(0) > 0 \) (both consequences of spherical symmetry) and because \( \kappa \) is continuous at 0. Thus, \( K \) has a least upper bound \( \delta \). Since \( \gamma' \) is strictly in the second quadrant on \([0, \delta) \), \( \gamma_2(\delta) > 0 \), so \( \delta < \beta \), from which it follows that \( \gamma \) is smooth at \( \delta \). In particular, \( \gamma' \) is continuous at \( \delta \), from which it can be proved that \( \gamma'(\delta) = (-1, 0) \).

Definition 7.11. We define the lower curve \( L \subset [\delta, \beta) \) as follows: \( s \in L \) if and only if for all \( t \in [\delta, s] \), the following hold:

1. \( \gamma'(t) \) is in the third quadrant, with \( \gamma'(t) \neq (-1, 0) \) if \( t > \delta \).
2. If \( \gamma \) is the unique point in \( K \) so that \( \gamma_2(t) = \gamma_2(t) \), then \( \gamma(t) < \kappa(t) \).

Since \( \delta \in L \), \( L \) is nonempty and therefore has a supremum, which we denote by \( \eta \).

By Proposition 7.21 \( \delta \) is the only point in \([0, \beta]\) at which the tangent vector is \((-1, 0)\). We can use this fact to prove that \( \eta > \delta \). In addition to Proposition 7.21 our proof that \( \eta > \delta \) utilizes the following lemma, which shows that at any point on \( \gamma \) where the tangent vector is \((-1, 0)\) and the curvature is 0, the curvature has a negative derivative.

Proposition 7.12. Let \( s \in (0, \beta) \), and suppose that \( \gamma'(s) = (-1, 0) \). If \( \kappa(s) \geq 0 \), then \( \kappa'(s) < 0 \).

Proof. In the case that \( \kappa(s) > 0 \), the result follows by a similar argument to the proof of Lemma 6.8. Now, suppose that \( \kappa(s) = 0 \). The osculating circle to \( \gamma \) at \( \gamma(s) \) is an oriented horizontal line which we parameterize by \( \alpha(t) = \gamma(s) + t(-1, 0) \). For each \( t \), \( \tilde{R}(t) = \gamma_2(s) \), so \( \tilde{\lambda}(t) \) is constant. Meanwhile, for all \( t \),

\[
\tilde{H}_1(t) = \frac{p}{|\alpha(t)|^2} \gamma_2(s)
\]

By Lemma 7.7 \( \gamma_1(s) > 0 \). Differentiating (17), we have

\[
\tilde{H}_1'(s) = \tilde{H}_1(0) = -2\frac{p \gamma_2(s)}{|\alpha(0)|^4} (-\gamma_1(s)) > 0.
\]

Thus, \( \kappa'(s) < 0 \).

Lemma 7.13. Given that \( \gamma([0, \beta]) \) has tangent vector \((-1, 0)\) only at \( \delta \), we have \( \eta > \delta \).

Proof. It suffices to prove that there exist \( \varepsilon_1, \varepsilon_2 > 0 \) so that for all \( s \in (\delta, \delta + \varepsilon_1) \), \( \gamma'(s) \) is in the third quadrant with \( \gamma'(s) \neq (-1, 0) \), and for all \( s \in [\delta, \delta + \varepsilon_2) \), \( \kappa(s) \leq \kappa(\delta) \). For the existence of such an \( \varepsilon_2 \), we observe that since \( \gamma' \) is in the second quadrant for all \( s \in [0, \delta] \), \( \kappa(\delta) \geq 0 \). Therefore, by Proposition 7.12, \( \kappa'(\delta) < 0 \). To prove that there is a \( \varepsilon_1 > 0 \) as described, it suffices to prove the strict inequality \( \kappa(\delta) > 0 \). By Proposition 7.12, if \( \kappa(\delta) = 0 \), then \( \kappa'(\delta) < 0 \). Hence, there exists \( q \in (\delta, \beta) \) with \( \gamma_2(q) > \gamma_2(\delta) \). By the Intermediate Value Theorem, applied to \( \gamma_2 \) on \([0, \beta]\), there is a later point at the same height as \( \gamma(\delta) \). Since \( \gamma' \neq (0, 1) \) on \([0, \beta]\) this implies the existence of \( q' \geq q \) with \( \gamma'(q') = (-1, 0) \), a contradiction.

Proposition 7.14. Given that \( \gamma([0, \beta]) \) has tangent vector \((-1, 0)\) only at \( \delta \), let \( s \in L \) with \( s > \delta \), and let \( \bar{s} \) be the unique point in \( K \) so that \( \gamma_2(\bar{s}) = \gamma_2(s) \). Then the following two inequalities hold:

\[
\gamma(\bar{s}) - \gamma(s) \leq \gamma(\delta) - \gamma(s), \tag{18}
\]

\[
\theta(\gamma'(s)) \leq 2\pi - \theta(\gamma'(\bar{s})). \tag{19}
\]
Proof. We define $k, h$ as in the proof of Proposition 5.4, but now, to apply Proposition 7.15, we define $f, g : (\gamma_2(\eta), \gamma_2(\delta)) \to \mathbb{R}$ by $f(y) = \gamma_1(k(y))$ and $g(y) = 2\gamma_1(\delta) - \gamma_1(h(y))$. □

**Proposition 7.15.** Given that $\gamma([0, \beta))$ has tangent vector $(-1, 0)$ only at $\delta$, let $s \in L$ with $s > \delta$ and $\gamma'(s) = (0, -1)$. Letting $\hat{s}$ be the unique point in $K$ so that $\gamma_2(s) = \gamma_2(\hat{s})$, we have $\kappa(s) < \kappa(\hat{s})$.

Proof. Since generalized mean curvature is constant on $\gamma$, we have $\kappa(s) + (n-2)\lambda(\hat{s}) + H_1(s) = \kappa(\hat{s}) + (n-2)\lambda(\hat{s}) + H_1(\hat{s})$. By Proposition 7.2, we can prove that $\gamma'(s)$ and $\gamma'(\hat{s})$ are admissible with respect to $\gamma(\hat{s})$ and $\gamma(s)$. Therefore, the above inequality is sufficient by Proposition 7.15. To show that $\gamma'(s)$ is tangent to the circle through the origin that passes through $(x', y)$, we have $0 = (x', y) \cdot y' = x' \gamma_1(y') + \gamma_2(y')$. Meanwhile, by Lemma 4.10, $0 \geq \gamma(s) \cdot y' = \gamma(s) \gamma_1(y') + \gamma_2(y')$. Since $y = \gamma_2(s)$, it follows that $x' \gamma_1(y') \geq \gamma_1(y')$. Dividing through by $\gamma_1(y')$ gives $x' \leq \gamma_1(\hat{s})$. □

**Proposition 7.16.** If there exists no $s \neq \delta$ so that $\gamma'(s) = (0, -1)$, then $\eta = \beta$.

Proof. Suppose for contradiction that $\eta < \beta$. We will show that $\eta$ is not an upper bound for $L$, but instead, that $L$ can be extended. Recall that by defining a local inverse function $h : (\gamma_2(\eta), \gamma_2(\delta)) \to (0, \delta)$ as on p. 13, we can explicitly write $\pi = h(\gamma_2(s))$. By continuity of $\gamma'$, of $\kappa$, and of $\kappa \circ h \circ \gamma_2$, $\gamma'(\eta)$ is in the third quadrant, and $\kappa(\eta) < \kappa(\pi)$. To show that $\eta \in L$, we need only show that $\gamma'(\eta) \neq (-1, 0)$. If $\gamma'(\eta) = (-1, 0)$, this would contradict the fact that $\gamma$ does not have multiple horizontal tangents. Meanwhile, if $\gamma'(\eta) = (0, -1)$, this would contradict (19), which holds at $\eta$ by continuity of $\gamma$ on $(0, \beta)$ and by our assumption that $\eta < \beta$. Thus, $\gamma'(\eta)$ is strictly in the third quadrant. Finally, by an identical argument to that in Proposition 7.15, $\kappa(\eta) < \kappa(\pi)$. □

Having shown that $\eta = \beta$, we are near to proving Lemma 4.9 with the assumption that $\delta$ is the only point at which $\gamma'$ equals $(-1, 0)$. First, we show that $\gamma_1(\beta) < 0$. In order to discuss $\lim_{t \to \beta^-} \gamma'(s)$, we must first show that the limit exists. For this, we prove in Proposition 7.19 that $\kappa$ is eventually negative. The proof requires the result of Proposition 7.17 as well as a lemma giving a bound on $\gamma'$ (Lemma 7.18).

**Proposition 7.17.** Given that $F(0) < 1/2$ and that $\gamma([0, \beta))$ has tangent vector $(-1, 0)$ only at $\delta$, we have that $\gamma_1(\beta) < 0$.

Proof. To prove that $\gamma_1(\beta) < 0$, we begin with Proposition 7.14, which states that if $s \in L$, and $\pi$ is the corresponding point in $K$ such that $\gamma_2(s) = \gamma_2(\delta)$, then $\gamma_1(\pi) - \gamma_1(\delta) \leq \gamma_1(\delta) - \gamma_1(s)$. Since $\beta = sup L$ and $\gamma$ is continuous at $\beta$, this inequality also holds for $s = \beta$. Noting that $\beta = 0$, we have $\gamma_1(0) - \gamma_1(\delta) \leq \gamma_1(\delta) - \gamma_1(\beta)$. Since $\gamma'(\delta) = (-1, 0)$, $\gamma(\delta) = F(\delta)$. In turn, since $F$ is non-increasing on the upper curve, $F(\delta) \leq F(0)$. Consequently, $\gamma_1(\delta) - F(0) \leq \gamma_1(\delta) - F(0) \leq \gamma_1(\delta) - F(0) - \gamma_1\beta$. Rearranging gives $\gamma_1(\beta) \leq 2\gamma(0) = \gamma_1(\beta) < 0$. □

**Lemma 7.18.** Given that $\gamma([0, \beta))$ has tangent vector $(-1, 0)$ only at $\delta$, there exists $\xi > 0$ so that $\gamma_1(s) \leq -\xi$ for all $s \in L$.

Proof. If $\gamma'$ is non-increasing, then $\gamma'(s) < 3\pi/2 - \tau$ for all $s \in L$. Since $\gamma'(\delta) = (-1, 0)$ and $\gamma'$ is continuous on $[0, \beta)$, there exists $s_0 > \delta$ such that $\theta(\gamma'(s)) < 3\pi/4$ on $[\delta, s_0]$. By Proposition 7.15, $\kappa(s_0) < \kappa(s)$. Letting $y_0 = \gamma_2(s_0)$, we have that the upward curvatures of the graphs of the functions $f$ and $g$ defined in Proposition 7.14 satisfy $\kappa_f(y_0) < \kappa_g(y_0)$. Therefore, by Proposition 5.4, there exists $\phi > 0$ such that

$$\theta(t_f(y)) \geq \theta(t_g(y)) + \phi$$

(20)
for all $y \in (0, y_0)$. Take $\tau = \min(\phi, \pi/4)$. If $s \in [\delta, s_0]$, then $\theta(y'(s)) < 5\pi/4 = 3\pi/2 - \pi/4 \leq 3\pi/2 - \tau$. If $s \in (s_0, \beta)$, let $y = y_2(s)$. Then $\theta(t_y(y)) = 3\pi/2 - \theta(y'(s))$, and $\theta(t_y(y)) = \theta(y'(s)) - \pi/2$. Substituting into (20), we obtain $\theta(y'(s)) \leq 2\pi - \theta(y'(s)) - \phi$. Finally, since $y > 0$, it follows by Lemma 7.3 that $\theta(y'(s)) > \pi/2$. Therefore, $\theta(y'(s)) < 3\pi/2 - \phi \leq 3\pi/2 - \tau$. □

**Proposition 7.19.** Given that $\gamma([0, \beta])$ has tangent vector $(-1, 0)$ only at $\delta$, there exists $\epsilon > 0$ such that $\kappa < 0$ on $(\beta - \epsilon, \beta)$.

**Proof.** We show that for $s$ close to $\beta$, we can make $(n - 2)\lambda(s) + H_1(s)$ larger than $c$, the constant of the differential equation $H_f = c$. First, we show that by taking $s$ sufficiently close to $\beta$, we can make $\lambda(s)$ large. The radius of the canonical circle at $\gamma$ satisfies

$$R(s)^2 = (\gamma(s) - F(s))^2 + \gamma_2(s)^2 = \left(\gamma_1(s) - \frac{\gamma(s) \cdot \gamma'(s)}{\gamma'(s)}\right)^2 + \gamma_2(s)^2 = \frac{\gamma_2(s)^2}{\gamma'(s)^2}.$$ 

Since $\gamma'(s) < 0$, we have $R(s) = \gamma_2(s) - \gamma_1'(s)$ and $\lambda(s) = -\gamma_1'(s)/\gamma_2(s)$.

By Lemma 7.18, $\lambda(s) \geq \xi/\gamma_2(s)$. Since $\gamma_2(\beta) = 0$ and $\gamma$ is continuous, there exists $\epsilon_1 > 0$ so that if $s \in (\beta - \epsilon_1, \beta)$, then $\gamma_2(s) < \xi/c$. Consequently, for all $s \in (\beta - \epsilon_1, \beta)$, $\lambda(s) > c$.

Now we will show that for $s$ sufficiently large, $H_1$ is positive. By Proposition 7.17 and continuity of $\gamma$, there exists $\epsilon_2 > 0$ such that $\gamma_1 < 0$ on $(\beta - \epsilon_2, \beta)$. For any $s$ in this interval $\gamma(s)$ and $v(s)$ are both strictly in the second quadrant, so $H_1(s) > 0$.

Set $\epsilon = \min(\epsilon_1, \epsilon_2)$, and suppose that $s \in (\beta - \epsilon, \beta)$. By our observations above and our assumption that $n \geq 3$, we have $(n - 2)\lambda(s) + H_1(s) > (n - 2)\lambda(s) \geq \lambda(s) > c$. Therefore, $\kappa(s)$ must be less than 0 to compensate.

**Proof of the Left Tangent Lemma (4.9).** By Proposition 7.17, $\gamma_1(\beta) < 0$. By 7.19, there exists $\epsilon > 0$ such that $\kappa < 0$ on $(\beta - \epsilon, \beta)$. On this interval, $\theta \circ \gamma'$ is a decreasing function of $s$. Since $\theta \circ \gamma'$ is decreasing and bounded below by $\pi$, $\lim_{s \to \beta^-} \gamma'(s)$ exists. Moreover, since $\gamma'$ is strictly in the third quadrant on $L$ and $\theta \circ \gamma'$ is decreasing on $(\beta - \epsilon, \beta)$, $\lim_{s \to \beta^-} \gamma'(s)$ is in the third quadrant but not equal to $0$.

**7.1. Proof That There Is Only One Horizontal Tangent.** Finally, we supply a proof of the result used from Proposition 7.13 onward that $\delta$ is the only point in $[0, \beta]$ with tangent vector $(-1, 0)$. It is expedient to consider the sets $T_0 = \{s \in [0, \beta] : y'(s) = (0, 0) \}$ and $U = \{s \in [0, \beta] : y'(s) = (0, 0) \}$. Consider the supremum $\delta$ of the upper curve $K$ (Def'n 7.10). By assuming that $\delta$ was the only point in $[0, \beta]$ where the tangent vector was $(-1, 0)$ (the fact that we are about to prove), we could show that $\kappa(\delta) > 0$. However, without this assumption, it must be the case that $\kappa(\delta) \geq 0$, because $\gamma'$ is strictly in the second quadrant on $(0, \delta)$ (cf. proof of Lemma 6.8). Thus, $\delta \in T$. Since $T$ is nonempty, it has a least upper bound $\nu$.

**Lemma 7.20.** The supremum of $T$ satisfies the following:

1. $\nu < \beta$
2. $\nu$ is the maximum element of $U$.
3. $\kappa(\nu) > 0$.

**Proof.** To prove that $\nu < \beta$, it suffices to show that there there exists $\epsilon_0 > 0$ so that if $s \in (\beta - \epsilon_0, \beta)$ and $y'(s) = (0, 0)$, then $\kappa(s) < 0$. To achieve this result, we consider the ODE $H_f = c$. We know that the constant $c$ is positive, because $H_1(0) = p$, $\kappa(0) > 0$ by spherical symmetry, and $\lambda(0) = \kappa(0)$ by Proposition 4.7.

Since $y_2(\beta) = 0$ and the curve is continuous at $\beta$, there exists $\epsilon_0 > 0$ so that for any $s$ in $(\beta - \epsilon_0, \beta)$, we have $y_2(s) < 1/c$. Let $s \in (\beta - \epsilon_0, \beta)$ and suppose that $y'(s) = (0, 0)$. Then we have $\lambda(s) = 1/y_2(s) > c$. Meanwhile the unit outward normal at $s$ is $v(s) = (0, 1)$, so

$$H_1(s) = \frac{p}{|y'(s)|^2} (\gamma_1(s), y_2(s)) \cdot (0, 1) = \frac{p}{|y'(s)|^2} y_2(s) > 0.$$ 

Given that $n \geq 3$, we have that $(n - 2)\lambda(s) + H_1(s) \geq \lambda(s) + H_1(s) > c$, which means that $\kappa(s)$ must be negative to compensate.
Given that \( v < \beta \), it can be shown by continuity of \( \gamma' \) and \( \kappa \) on \((0, \beta)\) that \( \gamma'(v) = (-1, 0) \) and that \( \kappa(v) \geq 0 \). Since \( \gamma'(v) = (-1, 0), v \in U = \{ s \in [0, \beta) : \gamma'(s) = (-1, 0) \} \). We claim that \( v \) is the largest point in \( U \). By definition of \( T \), there exists \( s > v \) so that \( \gamma'(s) = (-1, 0) \) and \( \kappa(s) \geq 0 \). Meanwhile, if there were an \( s > v \) so that \( \gamma'(s) = (-1, 0) \) and \( \kappa(s) < 0 \), then \( s \) would be a local minimum point of \( \gamma \). Since \( \gamma_2(\beta) = 0 \), there would exist \( t > s \) so that \( t \) was a local maximum point of \( \gamma_2 \). Since \( \gamma'(v) \neq (0,1) \) on \((0, \beta)\) (Lemma 4.1), \( \gamma'(t) \neq (1,0) \). Thus, it must be the case that \( \gamma'(t) = (-1,0) \), contradicting the fact that \( v = \sup T \). We conclude that \( v \) is the maximum element of \( U \). Again, since \( \gamma'(v) \neq (0,1) \) on \((0, \beta)\), this means that \( \gamma_2' < 0 \) on \((v, \beta)\).

Finally, to prove that \( \kappa(v) > 0 \), suppose for contradiction that \( \kappa(v) = 0 \). By Lemma 7.12, there exists \( \epsilon > 0 \) so that \( \kappa < 0 \) on \((v, v + \epsilon) \). Since \( \gamma'(v) = (-1, 0) \), this would imply the existence of an interval following \( v \) on which the tangent vector was strictly in the second quadrant, contradicting the fact that \( \gamma_2' < 0 \) on \((v, \beta)\) (cf. proof of Lemma 7.13). Thus, \( \kappa(v) > 0 \).

**Proposition 7.21.** There is only one point \( \delta \in [0, \beta) \) so that \( \gamma'(\delta) = (-1, 0) \).

**Proof.** Suppose for contradiction that \( U - \{v\} \) is nonempty. Since \( \gamma'(v) = (-1, 0) \) and \( \kappa(v) > 0 \), there exists \( \epsilon > 0 \) so that \( \gamma' \) is strictly in the second quadrant on \((v - \epsilon, v) \) and \( \gamma' \) is strictly in the third quadrant on \((v, v + \epsilon) \). Since \( \gamma' \) is strictly in the second quadrant on \((v - \epsilon, v) \), \( U - \{v\} = \{ s \in [0, v - \epsilon) : \gamma'(s) = (-1, 0) \} \); that is, \( U - \{v\} \) is a closed subset of \([0, v - \epsilon) \). As such, \( U - \{v\} \) is closed in \([0, v - \epsilon) \), which means that \( U - \{v\} \) is a compact subset of \( \mathbb{R} \) and has a maximum element \( u \).

We claim that \( \gamma'(s) \) is strictly in the second quadrant for all \( s \in (u, v) \). To prove so, suppose for contradiction that there exists \( s \in (u, v) \) so that \( \gamma'(s) \) is not strictly in the second quadrant. By Lemma 7.4, \( \gamma(s) > 0 \). Hence, we apply Lemma 7.3 to conclude that \( \gamma'(s) \) is in the third quadrant. Since there is an \( \epsilon > 0 \) so that \( \gamma' \) is strictly in the second quadrant on \((v - \epsilon, v) \) and \( \gamma' < 0 \) on \((0, \delta) \) (Lemma 7.3), there exists \( t \in (s, v) \) so that \( \gamma'(t) = (-1, 0) \), contradicting maximality of \( u \) in \( U - \{v\} \).

We define \( w \) to be the unique point in \((v, \beta) \) so that \( \gamma_2(w) = \gamma_2(u) \). We will ultimately achieve contradiction by showing that \( \gamma'(w) = (-1, 0) \). In turn, we will accomplish this by curvature comparison. Let \( s \in (v, w) \), and let \( \tilde{s} \) be the unique point in \((u, v) \) so that \( \gamma_2(\tilde{s}) = \gamma_2(s) \). We claim that \( \kappa(s) \leq \kappa(\tilde{s}) \). Since \( \kappa'(v) < 0 \) (Lemma 7.12), we already know that this inequality holds for all \( s \) sufficiently close to \( v \). Additionally, recall that there exists \( \epsilon > 0 \) so that \( \gamma'(s) \) is strictly in the third quadrant for all \( s \in (v, v + \epsilon) \). We will prove \( \gamma'(s) \) is strictly in the third quadrant for all \( s \in (v, w) \) as well.

Let \( W = \{ s \in (v, w) : \gamma'(t) \) is strictly in the third quadrant and \( \gamma'(t) < \kappa(t) \) for all \( t \in (v, s) \} \). Since \( W \) is nonempty and bounded above, \( W \) has a supremum, which we shall denote by \( z \). By analogous computations to those in the proof of Proposition 7.14, the following inequalities hold for all \( s \in (v, z) \):

\[
\gamma(\tilde{s}) - \gamma(v) \leq \gamma(v) - \gamma(s), \tag{21}
\]

\[
\theta(\gamma'(s)) \leq 2\pi - \theta(\gamma'(\tilde{s})). \tag{22}
\]

It can also be proved that \( \lambda(s) \geq \lambda(\tilde{s}) \) for all \( s \in (v, z) \). By continuity of all relevant quantities on \((0, \beta)\), it follows that these inequalities hold at \( z \) as well.

Finally, since \( \gamma' \) is strictly in the second quadrant on \((u, v) \) and strictly in the third quadrant on \( W \), it can be proved by a similar argument to that in Proposition 7.15 that \( \gamma' = \pi \). It follows that the inequalities \( \tag{21} \) and \( \tag{22} \) hold for all \( s \in (v, w) \). By \( \tag{22}, \theta(\gamma'(w)) \leq 2\pi - \theta(\gamma'(\tilde{w})) = 2\pi - \theta(\gamma'(u)) = \pi \). Since \( \theta \circ \gamma' \in (\pi, 3\pi/2) \) on \((v, w) \), it must be the case that \( \theta(\gamma'(w)) = \pi \). That is, \( \gamma'(w) = (-1, 0) \), contradicting the fact that there exists no \( s > v \) with \( \gamma'(s) = (-1, 0) \). \( \square \)

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