QUALITY CHOICE AND CAPACITY RATIONING IN ADVANCE SELLING

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ABSTRACT. This study considers a seller who sells a single product to strategic consumers sensitive to both price and quality over two periods: advance and spot. Customers’ valuations are uncertain in the first period and revealed over time. The seller’s decisions include whether to offer the product and, if so, the quality of the product, the prices in both periods, and whether to ration capacity in the advance period. The analysis is separated into two cases: unlimited capacity and limited capacity. The first case acts as a benchmark for the latter. It is found that in each case, the seller’s decisions on product offering and quality choice are fully determined by a single parameter, namely the cost coefficient of quality. The optimal rationing policy and its determinants, however, are distinct in these different settings. And the optimal rationing policy is contingent on whether the high- or low-quality product is offered. Further, our numerical studies show that the seller can benefit from capacity rationing and flexibility on quality choice. Specifically, the value of rationing is not evident, whereas the value of flexibility on quality choice is considerably significant.

1. Introduction. Advance selling refers to a marketing practice in which the seller offers opportunities for customers to make purchase commitments before the time of consumption [27]. The rapid development of information technology has led advance selling to become increasingly popular and to be applied by more and more sellers in many industries, see, e.g., traveling, consumer electronics, sporting events, etc. Generally, advance selling can benefit both the seller and customers. For instance, it can help the seller partially resolve uncertainty about future demand. Accordingly, the seller can make better operational decisions and thus increase profit. Moreover, customers are guaranteed availability and a possible discount if they buy the product in the advance period. However, they are often uncertain about their own valuation of the product since advance selling occurs before (even a long time before) the product becomes available.

Moreover, strategic consumer behavior is extensively found in both real-world and laboratory settings, as reported by [2], [8], [14], [16], [18], and [23]. Owing to

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the pervasiveness of strategic consumer behavior, treating strategic customers as myopic will lead sellers to suffer a considerable loss of profits. Indeed, [16] reports that the profit lost by ignoring strategic consumer behavior may be greater than 20%.

In the advance selling setting, a strategic consumer makes his or her purchasing decision by comparing the expected utility from advance buying with that from spot buying. When buying in the advance period, availability is guaranteed, whereas the valuation of the product is uncertain; conversely, when customers delay their purchase until the spot period, valuations are known but they may find the product out of stock. Hence, sellers must account for the extent to which the uncertainty about valuations as well as availability in the spot period affect customers’ purchasing behavior when determining whether to adopt advance selling and making other operational decisions.

In the current research, the authors consider a seller who offers a single product to strategic consumers in the spot period only or over two periods: advance and spot. The spot period is also the consumption period. In contrast to the literature on advance selling (see the detailed review in the next section), the seller is assumed to possess flexibility on quality choice and that consumers’ utilities (and accordingly the seller’s demand) are dependent on product quality. In this situation, the seller’s decision variables include the prices in the advance and spot periods, product quality, and capacity offered in the advance period when the seller’s capacity is constrained. It is apparent that the decisions on pricing and capacity rationing are interacted with the decision on quality choice. Given the quality and advance price announced and the capacity offered by the seller at the beginning of the advance period, customers who arrived early decide whether and when to buy the product by anticipating the price and availability in the spot period.

The context in this research is commonly seen in reality. For instance, a performance company must determine whether to invite top stars when organizing a concert since the size of the audience (or equivalently, the revenue of the company) is positively related to the popularity of the performers. Given the invited performers, the company needs to determine whether to sell tickets in advance to stimulate demand and if so, how many tickets to offer in the advance period. The prices in the advance and spot periods are the two remaining key decisions. Likewise, the organizer of an invitational tournament or academic conference may face similar questions. Indeed, other important questions must be considered by the organizer of an invitational tournament, such as the capabilities of the invited players/teams and the galvanizing impact of those players/teams on the host. However, questions on profitability may be equally important to the organizer. Further, the main difference in the focuses of the seller and organizer of an academic conference is that the former concentrates on profit maximization, while the latter needs to minimize total costs to meet the budget constraint.

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1In this research, the term “quality” refers to all of the related aspects of the product or service, such as functionality, performance, features, reliability, and popularity that affect the product/service’s perceived desirability.

2Although customers’ valuations may be affected by the detailed lineup and schedules, which are unavailable in the advance period, the major concern of potential audiences is who will appear in the concert. The stars invited by the company are the focus of the advertising campaigns and therefore are known to customers before the selling period. In other words, there is no information asymmetry about quality in this setting.

3Indeed, other important questions must be considered by the organizer of an invitational tournament, such as the capabilities of the invited players/teams and the galvanizing impact of those players/teams on the host. However, questions on profitability may be equally important to the organizer. Further, the main difference in the focuses of the seller and organizer of an academic conference is that the former concentrates on profit maximization, while the latter needs to minimize total costs to meet the budget constraint.
consumption and an individual consumer’s valuation at the time of consumption is affected by many variables such as mood, health, and scheduling conflicts.

The main research questions in this study are the following. The first is that whether or under what conditions the seller is better off offering advance selling. Further, when advance selling is optimal, how does the seller allocate the capacity between the advance and spot periods? In other words, does the seller satisfy all or only some proportion of demand in the advance period? How are prices set in the advance and spot periods? How does the seller’s decision on quality choice interact with that on capacity rationing? To answer these questions, two models are built in this research. The first model is based on the assumption that the seller’s capacity is assumed to be unconstrained and acts as the benchmark for the second model in which the assumption of unlimited capacity is relaxed to examine how the seller’s optimal decisions are affected by capacity.

The remainder of the paper is organized as follows. After a review of the literature in the next section, the model basics is presented in Section 3. The seller’s decisions in the unlimited and limited cases are discussed in Sections 4 and 5, respectively. Finally, the results are summarized in Section 6. All proofs are relegated to the appendix.

2. Related literature. This study belongs to the stream of research on advance selling. Alongside the huge development in information technology, more firms have adopted advance selling to stimulate customer demand and improve demand forecasting over the past two decades. Accordingly, more and more research on this issue has appeared in recent years.

First, one of the main questions in this study is the seller’s capacity decision in the advance period. In the literature, many studies focus on the seller’s production/procurement quantity decision in the context of advance selling, such as [5], [7], [11], [12], [13], and [20]. In these models, capacity is assumed to be endogenous, and the seller needs to determine the optimal order/production quantity to satisfy demand in the selling season. The optimal quantity consists of two parts: one is used to satisfy the deterministic demand that occurs in the advance period and the other is used to satisfy the stochastic demand in the spot period. That is, the quantity decision is made at the end of the advance period.

In contrast to these researches, this study assumes that the seller’s capacity is exogenously given and that the quantity decision relates to how many to offer to satisfy advance demand. From this viewpoint, the following three papers are close to ours. [27] assume that the seller’s capacity is exogenous and prove that supply in the advance period must be limited when capacity is small. Specifically, the seller should not offer advance selling when capacity is less than the number of customers with a favorable valuation. For the case of interdependent customer valuations, [29] show that the seller’s decisions on whether to offer advance selling, including full or limited advance selling, are dependent on both the seller’s capacity and the marginal cost. Moreover, [22] develop a model to simultaneously analyze the seller’s optimal inventory management, capacity rationing in the advance period, and pricing policies. In the above three studies, the product quality is assumed to be exogenous; whereas in the current research, our focus is on the interaction between the decisions on quality choice and capacity rationing. It is obvious that the research setting and the corresponding models in those studies are considerably different from ours.
Another issue in this study is the seller’s decision on quality choice. In the literature, few works consider the effect of product quality on the seller’s decisions in the context of advance selling. [28] propose a model in which the seller has private information about product quality, which is unknown to customers in the advance period and revealed only in the spot period. Their focus is on whether the seller has an incentive to signal quality in advance and how the seller can use capacity rationing in the advance period as a credible signal of product quality. Given the uncertainty over quality in the wine industry, [17] examine the role of advance selling as a form of operational flexibility in mitigating quality rating risk. The main difference of the current study from these studies is that the authors assume that product quality is endogenously determined and focus on how the decision on quality choice interplays with that on capacity rationing.

Further, there are some studies that address the quality issue in the presence of strategic consumers, but not in the context of advance selling. Among these, [19] considers a firm that sells two quality-differentiated products and can dynamically set prices over periods. The author’s focus is on whether product variety can be used to mitigate the effect of strategic consumer behavior. [10] consider dynamic pricing competition between two vertically differentiated firms and concentrate on the asymmetric effect of strategic consumer behavior on firms’ profits. Further, [30] investigate how consumer-generated quality information affects the dynamic pricing strategy of a firm that sells new experience goods to strategic consumers. These studies all assume that product quality is exogenously given, whereas in the current study, one the seller’s decisions is on product quality. Therefore, this research is considerably from the existing studies.

Finally, there are some researches focus on advance selling in supply chain settings are also related to the current study. [4] examine three selling strategies, i.e., advance selling, regular selling, and dynamic selling, of a manufacturer who produces and sells a seasonal product to a retailer when both the supply and demand are uncertain. They find that the manufacturer always prefers dynamic selling than both advance selling and regular selling. Differing from the traditional view that the retailer always benefits from advance selling, [31] show that in the supply chain setting, advance selling can hurt the retailer’s profits as well as the supply chain performance. Further, [6] focus on the retailer’s ordering and financing decision- under advancing and delayed payment in the supply chain setting and identify the conditions under which advance selling strategy or delayed payment strategy is preferable. Similar as in these studies, whether to offer advance selling is one of our main research questions. However, the issue of capacity rationing is not mentioned in these studies. Whereas in the current research, the focus is on whether and how the seller rations the capacity between the advance and spot periods.

3. The model. Consider a seller who offers a single product to strategic consumers over two periods: advance and spot. Consumption occurs only in the spot period, but purchase can occur in any period. In the following, the seller’s and customers’ decision problems and the game between the seller and customers are presented in a sequential way. All the players are risk-neutral.

3.1. Seller’s decisions. One of the seller’s decisions is to determine product quality, $q$. Suppose that the quality levels available to the seller are of two types: high
(H) and low (L) with $L < H$. As the quality level is determined, the seller needs to decide whether to offer advance selling and, if so, the advance price, $p_{1q}$, and capacity available in the advance period, $S_q$. Further, the seller needs to decide the spot price $p_{2q}$ regardless of whether offering advance selling.

The capacity of the seller, $T$, is assumed to be exogenously determined. Such an assumption is supported by many real-world examples. For instance, the number of seats in a venue used to hold a concert is fixed and cannot be adjusted in a short time. Further, it is natural to assume that the marginal cost of supplying one unit of a product with quality level $q \in \{H, L\}$ is dependent on $q$. Following [10], this study adopts a simple linear cost structure under which the seller incurs marginal cost $c q$ when offering one unit of a product with quality level $q$, where $c$ is the cost coefficient of quality.\footnote{The results of the current research can be easily extended to the situation in which the number of quality levels is more than two. However, the model and analysis become more complicated.}

3.2. Customers’ decisions. The market consists of a large population with a deterministic market size $N$ in which $N_1$ informed customers arrive in the advance period and $N_2$ ($N_2 = N - N_1$) uninformed customers arrive in the spot period. Here, $N_1$ is the number of customers aware of the product release in the advance period, whereas $N_2$ consumers hear about the product only in the spot period (or near the end of the advance period). In other words, $N_1$ customers may be interested in the product and show concern about its development and release. Although potential demand in the advance period is certain, actual demand and its allocation over the advance and spot periods are dependent on the seller’s decisions on the prices in these two periods, product quality, and capacity rationing in the advance period. Similarly, the number of consumers who arrive in the spot period and decide to purchase the product is determined by the seller’s decisions on the spot price as well as product quality.

Each customer is assumed to be infinitesimally small such that the interactions between them can be ignored. Upon arrival, $N_1$ customers decide whether and when to buy the product on the basis of product quality, the price and availability in the advance period, and their beliefs on the price and resulting availability in the spot period. Customers are assumed to have full information and accordingly form rational expectations about the seller’s decisions. That is, customers’ beliefs are consistent with the resulting equilibrium results. For those customers who arrive in the spot period, they only need to decide whether to buy since there is no other chance of obtaining the product. Note that given the seller’s capacity, $T$, and number of advance customers, $N_1$, the seller’s advance selling policy is one of the following three types: no advance selling where $S_q = 0$, limited advance selling where $S_q \in (0, \min\{T, N_1\})$, and full advance selling where $S_q = \min\{T, N_1\}$.

\footnote{In the literature, there are some studies, see, e.g., [1] and [15], assume a quadratic quality cost, i.e., $c q^2$. It can be expected that introducing such a cost function will not qualitatively change the results but will lead the analysis to be too complicated.}

\footnote{Where the calculation of $c$ is setting-contingent according to our definition of product/service quality. For instance, when the quality refers to the functionality of a product, the cost coefficient of quality may include the research and development cost, the setting cost, and the product cost. On the other hand, when the quality refers to the popularity of a performer or a player/team, $c$ is calculated by averaging the sum of the appearance fee, security fee, round trip air tickets and accommodation fee, etc.}
Consumers’ decisions on whether and when to buy are made based on their expected utilities from obtaining the product in different periods. Moreover, customers are assumed to be sensitive to both price and quality and heterogeneous in their marginal willingness to pay. In period $t$ ($t = 1$ and $2$ refer to the advance and spot periods, respectively), given quality level $q$ and unit price $p_{tq}$, a consumer with valuation $\theta$ obtains the following utility from consuming one unit of the product:

$$U(\theta, q, p_{tq}) = \theta + q - p_{tq}, \quad (1)$$

where the customer’s private valuation $\theta$ corresponds to the combined effect of all idiosyncratic factors, for instance, the individual preferences with regard to the flavor of a certain product, or the customer’s mood and/or health at the times of consumption. $\theta$ is assumed to be uniformly distributed on the interval $[0, 1]$. This uniform assumption allows us to derive analytical results, whereas a more general assumption of the distribution of customers’ valuations, for example, as in [28]-[29], renders the models analytically intractable. For this reason, the extension to a more general distribution of customers’ valuations are left to future research. Further, it is to be noted that the additive utility (1) can facilitate the analysis and is used by [28]. Another type of utility, namely multiplicative utility, can be expressed as $U(\theta, q, p_{tq}) = \theta q - p_{tq}$, and this is extensively used in the literature, as in [10], [19], [21], and [24]. All the results in the current study also hold for the case of multiplicative utility. Both the seller and the customers who arrive in the advance period are uncertain about these customers’ valuations but know the distribution. Consumers’ valuations are revealed in the spot period. Given the seller’s decisions, consumers decide whether and when to buy the product by choosing the options that maximize their expected utility. Each customer purchases at most one unit of the product.

Define by $\lambda_1(p_{tq}, S_q)$ the probability that a customer who wants to purchase the product with quality level $q$ in period $t$ actually obtains it, given the seller’s rationing policy $S_q$ and selling price $p_{tq}$. In other words, $\lambda_1(p_{tq}, S_q)$ can be seen as the capacity to the demand in period $t$. Obviously, there must be $\lambda_1(p_{tq}, S_q) \leq 1$. In the advance period, given the offered capacity $S_q$ and the selling price $p_{tq}$, customers buy the product if their utilities from consuming the product are non-negative, i.e., $U(\theta, q, p_{tq}) \geq 0$. This leads to that the demand of the product of quality $q$ in the advance period can be expressed as $N_1 \Pr\{U(\theta, q, p_{tq}) \geq 0\} = N_1 \Pr\{\theta \geq p_{tq} - q\} = N_1(1 - p_{tq} + q)$. Accordingly, $\lambda_1(p_{tq}, S_q)$ can be written as

$$\lambda_1(p_{tq}, S_q) = \min \left\{ \frac{S_q}{N_1(1 - p_{tq} + q)} \right\}.$$ 

In the spot period, the available capacity is $T - S_q$. And there are $N_1 - S_q + N_2 = N - S_q$ customers in the market. These customers buy the product only if their utilities from purchasing and consuming the product are nonnegative, i.e., $U(\theta, q, p_{2q}) \geq 0$. Accordingly, the demand in the spot period is $(N - S_q) \Pr\{U(\theta, q, p_{2q}) \geq 0\} = (N - S_q)(1 - p_{2q} + q)$. As a result, $\lambda_2(p_{2q}, S_q)$ can be expressed as

$$\lambda_2(p_{2q}, S_q) = \min \left\{ \frac{T - S_q}{(N - S_q)(1 - p_{2q} + q)} \right\}.$$ 

Given quality level $q$, for the customer with valuation $\theta$ wants to buy the product in the advance period, the expected utility is $E_\theta(\theta + q - p_{tq})$ if the product is available. When the product is out of stock, the customer has to wait until the spot period and tries to buy the product if the valuation exceeds to the spot price, which results
in the expected utility $E_\theta \max [\lambda_2(p_{2q}, S_q)(\theta + q - p_{2q}), 0]$. Since the probability of obtaining the product in the advance period is $\lambda_1(p_{1q}, S_q)$, the expected utility of the customer with valuation $\theta$ intends to buy in the advance period can be formulated as

$$u_{1q}(p_{1q}, S_q) = \lambda_1(p_{1q}, S_q) E_\theta (\theta + q - p_{1q})$$

$$+ [1 - \lambda_1(p_{1q}, S_q)] E_\theta \max [\lambda_2(p_{2q}, S_q)(\theta + q - p_{2q}), 0].$$

Further, if this customer only wants to buy the product in the spot period, the expected utility is

$$u_{2q}(p_{2q}, S_q) = E_\theta \max [\lambda_2(p_{2q}, S_q)(\theta + q - p_{2q}), 0].$$

Since $u_{2q}(p_{2q}, S_q) \geq 0$, it follows that the customer will purchase in the advance period if and only if $u_{1q}(p_{1q}, S_q) \geq u_{2q}(p_{2q}, S_q)$, which leads to

$$p_{1q} \leq q + E_\theta \{\theta - \max [\lambda_2(p_{2q}, S_q)(\theta + q - p_{2q}), 0] \}. \quad (2)$$

### 3.3. The game

Figure 1 illustrates the sequence of events. At the beginning of the advance period, a cohort of $N_1$ customers who are uncertain about their own valuations appear in the market. Upon observing the seller’s decisions on product quality, $q$, the advance price, $p_{1q}$, and capacity rationing, $S_q$, these customers decide whether to buy in advance or postpone their purchase decision until the spot period. In the spot period, another cohort of $N_2$ customers as well as the customers who arrive in the advance period and want to purchase the product in the spot period arrive. At this time, the product is available for consumption, and customers accordingly know their own valuations. Based on the spot price announced by the seller, those customers make decisions on whether to buy. At the end of the spot period, consumption takes place and there is no salvage value for any remaining capacity.

In what follows, we discuss the seller’s decisions in the advance and spot periods. The analysis is divided into two parts: one with unlimited capacity and the other with limited capacity. Such a separation enables us to see how the seller’s decisions on product offering, quality choice, and capacity rationing are affected by capacity.

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7Those customers who arrive in the advance period and want to purchase the product in the spot period can be divided into (i) those who intentionally postpone their purchase to the spot period to maximize their expected utility and (ii) those who want to purchase in the advance period but face a stockout because of the seller’s rationing decision, and thus have to postpone their purchase until the spot period.
Moreover, as one can see next, the results in the unlimited case act as the benchmark for the limited case.

4. **Unlimited capacity.** By using backward induction, the authors first find the optimal spot price that maximizes the seller’s profit in the spot period, and then solve the decision problem in the advance period. In this section, the superscript $U$ is used to denote the case with unlimited capacity. Moreover, there is $\min\{N_1, T\} = N_1$ as the seller’s capacity is unconstrained.

Under unlimited capacity, all customers arrive in the advance period are uncertain about their valuations and have the same valuation distribution. As a consequence, those customers’ responses to the seller’s decisions are uniform: all buy or none buy. Since $S_q \leq N_1$, it follows that the selling quantity in the advance period would be either $S_q$ or zero. The case in which no customers buy at the advance price can be replicated by $S_q = 0$. Therefore, the sales volume in the advance period equals the rationed capacity. In other words, the number of customers who stay in the market at the beginning of the spot period is $N - S_q$. In what follows, the seller’s decisions in the spot and advance periods are derived sequentially.

4.1. **Spot period.** In the spot period, information about both customers’ valuations and product quality is available to customers. According to (1), it is clear that customers with valuations $\theta \geq p_{2q} - q$ will buy the product. Given the seller’s rationing decision, $S_q$, the expected spot demand is $(N - S_q)(1 - p_{2q} + q)$. Because the seller’s capacity is unlimited, all demand is satisfied. Thus, actual demand is identical to expected demand. Accordingly, the seller’s expected profit in the spot period can be written as

$$\Pi_{2q}^U(p_{2q}|S_q) = (p_{2q} - cq)(N - S_q)(1 - p_{2q} + q).$$

The seller’s optimal spot price that maximizes $\Pi_{2q}^U(p_{2q}|S_q)$ is defined in the following lemma.

**Lemma 4.1.** Denote $c_1 = 1 - \frac{1}{q}$ and $c_2 = 1 + \frac{1}{q}$ for $q \in \{H, L\}$. The seller’s optimal spot price, denoted by $p_{2q}^U$, can be characterized as

$$\begin{align*}
(p_{2H}^U, p_{2L}^U) = \begin{cases} 
(H, L), & \text{if } c \leq c_1L \\
(H, p_{2L}^0), & \text{if } c_1L < c \leq c_1H \\
(p_{2H}^0, p_{2L}^0), & \text{if } c_1H < c \leq c_2H \\
(H + 1, p_{2L}^0), & \text{if } c_2H < c \leq c_2L \\
(H + 1, L + 1), & \text{if } c > c_2L,
\end{cases}
\end{align*}$$

where $p_{2q}^0 = \frac{1+(1+cq)}{2}$.

Lemma 4.1 indicates that the cost coefficient of quality $c$ plays a crucial role in determining the seller’s optimal spot price. The range of $c$ is divided into five intervals, each of which corresponds to a unique combination of $p_{2H}^U$ and $p_{2L}^U$. Equation (3) is fundamental to derive the seller’s decision on quality choice in the advance period.

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8No customers buy in the advance period occurs may be due to (i) the seller does not offer the product in the advance period or (ii) the seller price too high in the advance period such that no customers want to buy. Although the drivers of no customers buy are different, the consequence is the same, i.e., the selling quantity in the advance period equals zero.
period. Further, one can see from (3) that for a product with quality \( q \in \{ H, L \} \), three spot prices correspond to the different value intervals of \( c \), i.e.,

\[
p^U_{2q} = \begin{cases} 
q, & \text{if } c \leq c_{1q} \\
p^U_{2q}^0, & \text{if } c_{1q} < c \leq c_{2q} \\
q + 1, & \text{if } c > c_{2q}.
\end{cases}
\]

(4)

When \( c_{1q} < c \leq c_{2q} \), there is \( q < p^U_{2q}^0 \leq q + 1 \), where the equality holds only when \( c = c_{2q} \). It is indicated by (4) that \( p^U_{2q} \) increases in \( c \). This is natural since as the cost of quality increases (equivalently, the marginal cost increases), the seller has to charge a higher price to maintain a positive profit margin.

4.2. **Advance period.** In the advance period, the seller’s objective is to maximize the total profit in the advance and spot periods by determining the advance price, \( p_{1q} \), capacity rationing, \( S_q \), and whether to offer a high- or low-quality product given the optimal decision in the spot period. Since there are only two possible values of \( q \), the seller’s optimization problem can be solved sequentially: first the seller’s optimal decisions on prices and capacity rationing by assuming that the quality level is given and then the seller’s optimal decision on quality level by comparing the seller’s optimal total profit under different quality levels.

Given quality level \( q \), the seller’s expected profit can be written as

\[
\Pi^U_q(p_{1q}, S_q) = (p_{1q} - cq)S_q + (p^U_{2q} - cq)(N - S_q)(1 + q - p^U_{2q}).
\]

Because \( S_q \geq 0 \), \( \Pi^U_q(p_{1q}, S_q) \) increases in \( p_{1q} \). Further, the seller’s capacity is unlimited, i.e., \( T \to \infty \), which results in \( \lambda_2(p^U_{2q}, S_q) = 1 \) for any value of \( p^U_{2q} \). As a result, the optimal advance price can be expressed by using (2) as

\[
p_{1q} = E_{\theta} \min \{ p^U_{2q}, \theta + q \}.
\]

(5)

Hence, \( p_{1q} \leq p^U_{2q} \). In other words, the seller needs to offer discounted advance selling. [22] and [27] find that both premium and discounted advance selling are possible. For instance, [27] prove that premium (discounted) advance selling is optimal when the seller’s capacity is large (small). [22] show that a premium advance price is optimal when the seller’s capacity is high and the advance sales market is relatively small; otherwise, discounted advance price is optimal. In these two works, customers’ valuations are assumed to be Bernoulli distributed, which results in premium advance selling being optimal.\(^9\)

By substituting \( p^U_{1q} \) into \( \Pi^U_q(p_{1q}, S_q) \), the seller’s expected profit becomes a function of the single variable \( S_q \), that is,

\[
\Pi^U_q(S_q) = [(p^U_{1q} - cq) - (p^U_{2q} - cq)(1 + q - p^U_{2q})]S_q + N(p^U_{2q} - cq)(1 + q - p^U_{2q}).
\]

(6)

It is clear that \( \Pi^U_q(S_q) \) monotonically changes with regard to \( S_q \). Depending on whether the value of the term in the square brackets on the right-hand side of (6) is positive or negative, \( \Pi^U_q(S_q) \) may increase or decrease in \( S_q \). It can be inferred that it is optimal for the seller to offer full advance selling or sell the product only in the spot period, that is, \( S^U_q = N_1 \) or \( S^U_q = 0 \). The details of the seller’s optimal policy are presented in the following proposition.

\(^9\)Note that for large capacity, Bernoulli-distributed customer valuations are not the necessary condition that leads premium advance selling to be optimal. In a model in which customers’ valuations are generally distributed and customers who order in the advance period are offered a freebie by the seller, [5] show that the advance price may be higher than the spot price, since offering freebies raises the valuations of customers who purchase the product early.
Proposition 1. Let $c_{3q} = 1 + \frac{1}{3q}$ for $q \in \{H, L\}$ and $\alpha = \frac{1}{\tau} \in (0, 1)$. It is optimal for the seller to

(i) Offer a high-quality product when $c \leq c_{1H}$. Moreover, the amount offered in the advance period is any value between 0 and $N_1$;

(ii) Offer a high-quality product with full advance selling when $c_{1H} < c \leq c_{3H}$;

(iii) Offer a high-quality product with no advance selling when (a) $\alpha \leq \frac{1}{3}$ and $c_{3H} < c \leq c_{2H}$ or (b) $\alpha > \frac{1}{3}$ and $c_{3H} < c \leq c_{3L}$;

(iv) Offer a low-quality product with full advance selling when $\alpha \leq \frac{1}{3}$ and $c_{2H} < c \leq c_{3L}$;

(v) Offer a low-quality product with no advance selling when $c_{3L} < c \leq c_{2L}$;

(vi) Not offer any products when $c > c_{2L}$.

Proposition 1 indicates that the seller needs to take two steps to derive the optimal decisions on quality choice and capacity rationing: first check the difference on the quality levels between the two products and then check the cost coefficient of quality.

When the difference between the two quality levels is large, i.e., $L < \frac{H}{\tau}$, the seller’s optimal policy can be characterized by five threshold values of $c$: $c_{1H}$, $c_{3H}$, $c_{2H}$, $c_{3L}$, and $c_{2L}$. Threshold $c_{2L}$ determines whether it is profitable for the seller to offer a product. When $c$ (accordingly, the marginal cost) is sufficiently large, the seller will not obtain a positive profit from the business and thus not offer any products. The second threshold, $c_{2H}$, determines whether the seller offers a high- or low-quality product. When $c$ is high, i.e., $c > c_{2H}$, offering a low-quality product is better than offering a high-quality product. This is due to that when $c > c_{2H}$, both the sale volume and the profit margins of the low-quality product are greater than that of the high-quality product.\(^{10}\)

Given that the seller is optimal to offer a high-quality product, two threshold values of $c$ are used to determine the seller’s policy on capacity rationing. When $c$

\(^{10}\)Since the price of low-quality product is less than that of the high-quality product, it is obvious that the sales volume would be greater when the seller offers the low-quality product than when the seller offers the high-quality product. The profit margins of the low-quality product are greater than that of the high-quality product can be interpreted as follows. From (5), there is $p_{1q}^U = p_{2q}^U - \frac{(p_{2q}^L - q)^2}{2}$ for $q \in \{H, L\}$. When $c_{3H} < c \leq c_{2L}$, the profit margins of the high-quality product in the advance and spot periods are $p_{1H}^U - cH = \frac{1}{2} + \tau H$ and $p_{2H}^U - cH = 1 + \tau H$, respectively, where $\tau = 1 - c$. According to the definition of $c_{2q}$, condition $c_{2H} < c \leq c_{2L}$ implies $-\frac{1}{\tau} < \tau < \frac{1}{\tau} < 0$. For the low-quality product, there are $p_{1L}^U - cL = \frac{1}{2} + \tau L - \frac{(1+\tau L)^2}{8}$ and $p_{2L}^U - cL = \frac{1+\tau L}{8}$. For the profit margins in the advance period, there is $(p_{1L}^U - cL) - (p_{1H}^U - cH) = -(p_{1H}^U - cH) - \frac{(1+\tau L)^2}{8} := f'(\tau)$. Differentiating $f'(\tau)$ with regard to $\tau$, one gets $f'(\tau) = -(H - L) - \frac{L(1+\tau L)}{4} < 0$, that is, $f'(\tau)$ strictly decreases in $\tau$. Further, the value of $f'(\tau)$ at the right endpoint of $\tau$ is $f'(\tau) = \frac{1}{2} - \frac{1}{2} = 1 - \alpha - \frac{(1-\alpha)^2}{3}$. Hence, $f'(\tau) = \frac{1}{2} - \frac{1}{2} = 1 - \alpha - \frac{(1-\alpha)^2}{3}$ can be seen as a function of $\alpha$. In the following, $f'(\tau) = \frac{1}{2} - \frac{1}{2} = 1 - \alpha - \frac{(1-\alpha)^2}{3}$ is replaced by $g'(\alpha)$ for the further analysis. The first derivative of $g(\alpha)$ with regard to $\alpha$ is given by $g'(\alpha) = \frac{3(1-\alpha)}{2} < 0$. Moreover, since the value of $\alpha$ at the right endpoint of $\alpha$ is greater than zero, i.e., $\alpha = \frac{1}{3} > 0$, it follows that $g(\alpha)$ (equivalently $f'(\tau) = \frac{1}{2} - \frac{1}{2}$) is positive over $\alpha \in (0, \frac{1}{3})$. Accordingly, there would be $f'(\tau) > 0$ for all $\tau \in \left(-\frac{1}{\tau}, \frac{1}{\tau}\right)$. This gives $p_{1H}^U - cH < p_{1L}^U - cL$. For the profit margins in the spot period, there is $(p_{1L}^U - cH) - (p_{2L}^U - cL) = (1 - 2\alpha)H + \frac{L(1+\tau L)}{8} < -\frac{1+\tau L}{8} < 0$, where the first inequality is due to $cH < -1$ and the second inequality is due to $cL > -1$. Moreover, when $c > c_{2L}$, there are $p_{1q}^U - cq = \frac{1}{2} + \tau q$ and $p_{2q}^U - cq = 1 + \tau q$ for $q \in \{H, L\}$. Since $c > c_{2L}$ implies $\tau < -\frac{1}{\tau} < 0$, it follows that both $p_{1H}^U - cq$ and $p_{2H}^U - cq$ decrease in $q$. In other words, there are $p_{1H}^U - cH < p_{1L}^U - cL$ and $p_{2H}^U - cH < p_{2L}^U - cL$.\]^{
is very small, i.e., \( c \leq c_{1H} \), there is \( p_{1q}^U - cq = (p_{2q}^U - cq)(1 + q - p_{2q}^U) \). Equation (6) suggests that \((p_{1q}^U - cq) - (p_{2q}^U - cq)(1 + q - p_{2q}^U)\) can be seen as the effective profit margin of selling in the advance period. Since such a profit margin equals zero (i.e., the seller’s total profit is independent of \( S_q \)), the amount offered in the advance period can be any value between zero and the number of customers who arrive in this period. As \( c \) increases, the effective profit margin no longer equals zero, but is positive for \( c_{1H} < c \leq c_{3H} \) and negative for \( c_{3H} < c \leq c_{2H} \). Thus, the seller initially offers full advance selling and then offers spot selling only. Further, when the seller determines to sell a low-quality product, threshold \( c_{3L} \) is used to decide whether to offer advance selling or spot selling only. Put simply, the effective profit margin of selling the low-quality product in the advance period is positive on the left side of \( c_{3L} \) and negative on the right. As a consequence, the seller offers full and no advance selling, respectively.

On the contrary, when the difference between the two quality levels is not so large, i.e., \( L > \frac{H}{3} \), the seller’s optimal policy is defined by four threshold values of \( c \): \( c_{1H}, c_{3H}, c_{3L}, \) and \( c_{2L} \). The unique difference between the current case and the case of \( L < \frac{H}{3} \) is the seller’s policy for \( c_{3H} < c \leq c_{3L} \). Note that there are \( c_{3L} > c_{2H} \) for \( \alpha < \frac{1}{4} \) and \( c_{3L} < c_{2H} \) for \( \alpha > \frac{1}{4} \). Recall from the above that the effective profit margin of selling the low-quality product in the advance period is always non-positive when \( c > c_{3L} \); therefore, offering a low-quality product with full advance selling does not hold in the current case.

5. **Limited capacity.** In this section, the focus is on the effect of constrained capacity on the seller’s optimal decisions on pricing and whether to offer advance selling and if so, how much to offer in the advance period. As in the unlimited case, the seller’s decisions in the spot and advance periods are discussed sequentially.

5.1. **Spot period.** Given the seller’s capacity rationing decision in the advance period, \( S_q \), and quality level, \( q \), the number of customers who want to purchase in the spot period is \((N - S_q)(1 - p_{2q} + q)\). In the unlimited case, this term is the seller’s actual demand in the spot period since then all demand can be satisfied. In the current case, the seller’s capacity is constrained and thus some potential demand may be lost. Given total capacity \( T \), the remaining capacity at the beginning of the spot period is \( T - S_q \). Therefore, actual demand in the spot period is \( \min\{T - S_q, (N - S_q)(1 - p_{2q} + q)\} \). Accordingly, the expected spot profit can be written as

\[
\Pi_{2q}^S(p_{2q}|S_q) = (p_{2q} - cq) \min\{T - S_q, (N - S_q)(1 - p_{2q} + q)\}.
\]

The seller’s optimal spot price under limited capacity is presented in the following lemma.

**Lemma 5.1.** Define

\[
p_{2q}^S(S_q) = q + \frac{N - T}{N - S_q}, \tag{7}
\]

for \( q \in \{H, L\} \). The seller’s optimal spot price can be characterized as

\[
p_{2q}^C = \begin{cases} p_{2q}^B(S_q), & \text{if } T - S_q \leq (N - S_q)(1 - p_{2q}^U + q) \\ p_{2q}^U, & \text{if } T - S_q \geq (N - S_q)(1 - p_{2q}^U + q) \end{cases}. \tag{8}
\]
By using (11), \( C \) Setting \( T \)

Hence, equation (8) indicates that the optimal spot price decreases in the seller’s remaining capacity in the spot period, \( T - S_q \). Since the salvage value is assumed to be zero, the seller would lower the spot price to run out of capacity when capacity is relatively large. Moreover, \( p_{2q}^C \geq p_{2q}^U \) indicates that the seller charges a higher spot price in the limited capacity case than in the unlimited case. Such a result is consistent with the reality that prices are always negatively correlated with capacity.

5.2. **Advance period.** Given quality level \( q \) and the optimal spot price \( p_{2q}^C \), one of the seller’s objectives in the advance period is to maximize

\[
\Pi_q^C(p_{1q}, S_q) = (p_{1q} - cq)S_q + (p_{2q}^C - cq) \min\{T - S_q, (N - S_q)(1 - p_{2q}^C + q)\}
\]

over \( p_{1q} \) and \( S_q \). Clearly, \( \Pi_q^C(p_{1q}, S_q) \) strictly increases in \( p_{1q} \). According to (2), the optimal advance price can be expressed as

\[
p_{1q}^C = q + E \theta \{ \theta - \max[\lambda_2(p_{2q}^C, S_q)(\theta + q - p_{2q}^C), 0] \}.
\]

To obtain the optimal \( p_{1q} \), it is necessary to derive the fill rate in the spot period. By substituting \( p_{2q}^C \) into \( \lambda_2(p_{2q}^C, S_q) \), there is

\[
\lambda_2(p_{2q}^C, S_q) = 1
\]

for both \( T - S_q \leq (N - S_q)(1 - p_{2q}^C + q) \) and \( T - S_q \geq (N - S_q)(1 - p_{2q}^C + q) \). This indicates that the fill rate in the spot period is independent of the seller’s remaining capacity. In other words, the seller determines the spot price such that any customer who wants to purchase and can afford the spot price will obtain the product.

Substituting (9) into \( p_{1q}^C \) gives rise to

\[
p_{1q}^C = E \theta \min\{p_{2q}^C, \theta + q\}.
\]

Hence, \( p_{1q}^C \leq p_{2q}^C \); that is, the seller can only offer discounted advance selling if capacity is constrained. Comparing (10) with (5) and using \( p_{2q}^C \geq p_{2q}^U \) lead to \( p_{1q}^C \geq p_{1q}^U \), suggesting that the optimal advance price is also higher in the limited case than in the unlimited case.

Based on (10), \( \Pi_q^C(p_{1q}, S_q) \) can be rewritten as a function of the single variable, \( S_q \), as follows:

\[
\Pi_q^C(S_q) = (p_{1q}^C - cq)S_q + (p_{2q}^C - cq) \min\{T - S_q, (N - S_q)(1 + q - p_{2q}^C)\}.
\]

Setting \( T - S_q = (N - S_q)(1 + q - p_{2q}^U) \) gives a \( S_q \), namely \( \mathcal{S}_q \), as follows:

\[
\mathcal{S}_q = N - \frac{N - T}{p_{2q}^U - q}.
\]

By using (11), \( \Pi_q^C(S_q) \) can be rewritten as

\[
\Pi_q^C(S_q) = \begin{cases} 
\Pi_q^B(S_q), & \text{if } S_q \geq \mathcal{S}_q \\
\Pi_q^U(S_q), & \text{if } S_q \leq \mathcal{S}_q
\end{cases}
\]
where
\[
\Pi^B_q(S_q) = \{E_q \min[p^B_{2q}(S_q), \theta + q] - cq\}S_q + [p^B_{2q}(S_q) - cq](T - S_q),
\]
\[
\Pi^U_q(S_q) = [E_q \min(p^U_{2q}, \theta + q) - cq]S_q + (p^U_{2q} - cq)(N - S_q)(1 + q - p^U_{2q})
\]
\[
= \frac{\Pi^U_q(N_1) - \Pi^U_q(0)}{N_1}S_q + (p^U_{2q} - cq)(1 + q - p^U_{2q})N.
\]

At the point \(S_q = \bar{S}_q\), there is \(\Pi^C_q(\bar{S}_q) = \Pi^B_q(\bar{S}_q) = \Pi^U_q(\bar{S}_q)\).

To develop the seller’s optimal rationing decision, it is necessary to derive how \(\Pi^C_q(S_q)\) varies with respect to \(S_q\). The property of \(\Pi^U_q(S_q)\) is discussed in the above section and that of \(\Pi^B_q(S_q)\) is presented in the following lemma.

Lemma 5.2. Denote \(\bar{S}_q = \frac{2N(2T-N)}{N+T}\), \(\bar{S}_q = \bar{N}_1 = \frac{N(3T-N)}{N+T}\), and \(T = \frac{N(3T-N)}{3N+T}\). \(\Pi^B_q(S_q)\) has the following properties:

(i) When \(T \leq \frac{N}{2}\), \(\Pi^B_q(S_q)\) is concave in \(S_q\);

(ii) When \(\frac{N}{2} < T \leq T\), \(\Pi^B_q(S_q)\) is convex and concave, respectively, in \(S_q\) on the left- and right-hand sides of \(\bar{S}_q\);

(iii) When \(T > T\), \(\Pi^B_q(S_q)\) is convex in \(S_q\).

Moreover, the maximizer of \(\Pi^B_q(S_q)\) is at
\[
S^*_B = \begin{cases} 
0, & \text{if } T \leq \frac{N}{2}; \\
\min\{T, N_1\} = N_1, & \text{if } \frac{N}{2} < T \leq T \text{ and } N_1 \leq \bar{N}_1; \\
\bar{S}_q < \min\{T, N_1\}, & \text{if } \frac{N}{2} < T \leq T \text{ and } N_1 > \bar{N}_1; \\
\min\{T, N_1\} = N_1, & \text{if } T > T.
\end{cases}
\]

Lemma 5.2 shows that the shape and optimizer of \(\Pi^B_q(S_q)\) are fundamentally dependent on the value of \(T\). Since \(\Pi^C_q(S_q)\) equals \(\Pi^B_q(S_q)\) when the seller’s remaining capacity in the spot period is not so large, it can be inferred that the seller’s optimal decisions on quality choice and capacity rationing are also dependent on \(T\), as shown in the following proposition.

Proposition 2. Denote \(c_{2L} = 1 + \frac{N + N_1 - 2T}{L(N + N(N_1))}\) and \(c_{4L} = 1 + \frac{\sqrt{2(3T-N)-2(2T-N)}}{4NL}\). The seller’s optimal decisions on product offering, quality choice, and capacity rationing can be defined as follows:

- \(T \leq \frac{N}{3}\). It is optimal for the seller to offer a high-quality product only in the spot period when \(c \leq 1\), offer a low-quality product only in the spot period when \(1 < c \leq c_{2L}\), and not offer any products when \(c > c_{2L}\).

- \(\frac{N}{3} < T \leq T\).
  - \(N_1 \leq \frac{3T-N}{2}\). It is optimal for the seller to offer a high-quality product with full advance selling when \(c \leq 1\), offer a low-quality product with full advance selling when \(1 < c \leq c_{3L}\), offer a low-quality product only in the spot period when \(c_{3L} < c \leq c_{2L}\), and not offer any products when \(c > c_{2L}\).
  - \(\frac{3T-N}{3} < N_1 \leq \frac{N}{3}\). It is optimal for the seller to offer a high-quality product with full advance selling when \(c \leq 1\), offer a low-quality product with full advance selling when \(1 < c \leq c_{4L}\), offer a low-quality product only in the spot period when \(c_{4L} < c \leq c_{2L}\), and not offer any products when \(c > c_{2L}\).
  - \(N_1 > \frac{N}{3}\). It is optimal for the seller to offer a high-quality product with limited advance selling of \(S_H\) units when \(c \leq 1\), offer a low-quality product
with limited advance selling of $S_L$ units when $1 < c \leq c_{TL}$, offer a low-quality product only in the spot period when $c_{TL} < c \leq c_{2L}$, and not offer any products when $c > c_{2L}$.

- $T > T$. It is optimal for the seller to offer a high-quality product with full advance selling when $c \leq 1$, offer a low-quality product with full advance selling when $1 < c \leq c_{3L}$, offer a low-quality product only in the spot period when $c_{3L} < c \leq c_{2L}$, and not offer any products when $c > c_{2L}$.

Proposition 2 shows that the seller’s optimal decisions on quality choice and whether to offer a product are uniquely determined by the cost coefficient of quality. For any capacity level, the seller would prefer to offer a high-quality product for small $c$, offer a low-quality product for medium $c$, and not offer any products for sufficiently large $c$. These findings are the same as in the unlimited case and suggest that the seller needs to deliberate on the cost coefficient of quality when determining product quality and whether to offer a product.

Given that it is optimal for the seller to offer a high-quality product, the advance selling decision is determined by both the seller’s capacity and the number of customers who arrive in advance. When total capacity is very low such that at most a smaller part of all arrived customers can be satisfied (i.e., $T \leq \frac{N}{T}$), the product is offered only in the spot period such that all customers who arrived in the advance period have to postpone their purchase, as the seller only offers discounted advance selling in our context. To sell as much as possible the capacity at a higher price, the seller would choose to postpone all advance demand to the spot period. Such a result always holds irrespective of the number of advance customers. As the seller’s capacity is medium, i.e., $\frac{N}{T} < T \leq T$, the spot price may be too high to consume all capacity in the spot period. To increase actual demand and run out of capacity, now the seller would choose to satisfy advance demand. All advance demand will be satisfied if demand is relatively low. However, when the number of advance customers is large, the seller will ration capacity in the advance period to reserve sufficient capacity to sell at a higher price in the spot period. Finally, when capacity is sufficiently high, i.e., $T > T$, the seller has to offer full advance selling to sell as many units of the product as possible.

On the contrary, when offering a low-quality product is the seller’s best choice, the seller’s optimal decision on advance selling is simultaneously affected by the capacity level, cost coefficient of quality, and number of advance customers. This is different from the high-quality product case. When $c$ is relatively small, the advance selling policy is (nearly) the same as that for the high-quality product. However, when $c$ is relatively large, the marginal cost becomes very high such that the profit margin is positive only if the product is offered at the spot price. Thus, the seller only offers spot selling.

5.3. **Value of capacity rationing.** Proposition 2 shows that it is optimal for the seller to offer full advance selling or sell the product only in the spot period in most cases. Proposition 2 also indicates that for a moderate capacity level and a sufficiently large number of customers who arrive in the advance period, capacity rationing in the advance period always benefits the seller. An open question is the degree to which the seller benefits from capacity rationing in the advance period. Such an issue is addressed by means of numerical studies. Denote by $\Pi^*$ and $\Pi^*_N$, respectively the seller’s optimal profits when capacity rationing in the advance period is allowed and prohibited, where $\Pi^*$ and $\Pi^*_N$ can be easily obtained from the
proof of Proposition 2. Define \( \frac{H^* - H_N}{H_N} \times 100\% \) as the value of capacity rationing. The following parameter values are used throughout the study whenever numerical studies are conducted: \( H = 4, L = 2, N_1 = 50, \) and \( N_2 = 310. \)

Figure 2 illustrates the effects of the cost coefficient of quality, \( c, \) and seller’s capacity, \( T, \) on the value of capacity rationing. Although capacity rationing makes the seller strictly better off in all cases, the value of rationing is not evident. In those cases, the maximum of the value of rationing is only 0.804%, which occurs at \( T = 130 \) and \( c = 1.17. \) Further, the value of capacity rationing seemingly convexly increases in \( c. \) For small \( c, \) the seller benefits less from rationing. As \( c \) increases, rationing may generate a relatively large benefit for the seller. On the contrary, the value of capacity rationing is convex in \( T. \) When the value of \( T \) increases from 125 to 130, the value of rationing also increases. However, as \( T \) further increases, the value of rationing decreases. Specifically, when \( T = 140, \) the incremental profit that results from capacity rationing is less than 0.11% of the seller’s optimal profit when rationing is prohibited for any value of \( c. \) This finding is natural since when capacity is sufficiently large, the seller would have no reason to ration it.

5.4. Value of flexibility on quality choice. Proposition 2 demonstrates that it is optimal for the seller to offer a high-quality product when \( 1 < c \leq c_2L \) and a low-quality product when \( 1 < c \leq c_2L. \) Note that in proving Proposition 2, the optimal quality level is determined by comparing the seller’s optimal expected profits over different quality levels. This immediately gives rise to the following result.

Corollary 1. The seller is better off when the quality level is selectable (i.e., \( q \in \{H, L\} \)) than when the quality level is fixed (i.e., \( q = H \) or \( L \)) as long as selling any product is profitable.

Corollary 1 indicates that the seller always benefits from flexibility on quality choice, given that selling one unit of the product can generate a positive profit. Recall from above that capacity rationing can generate a higher profit than no rationing only under certain conditions. In other words, the applicability of possessing flexibility on quality choice is stronger than capacity rationing in the advance period. Moreover, it is shown above that the value of rationing is not evident in all
cases. An obvious issue that deserves to be addressed is thus whether the value of flexibility on quality choice is significant.

As in the discussion on the value of capacity rationing, the value of flexibility on quality choice is analyzed by using numerical studies. Denote by $\Pi_q^*$, $q = \{H, L\}$, the seller’s optimal profit when the quality level is fixed at $q$, where $\Pi_q^*$ can be obtained from the proof of Proposition 2. Further, define $\frac{\Pi^* - \Pi_q^*}{\Pi_q^*} \times 100\%$ as the values of flexibility on quality choice when the quality level is fixed at $q$.

Figure 3 illustrates the results of the numerical studies, showing that for the three levels of the seller’s capacity, the value of flexibility on quality choice convexly decreases in $c < 1$ and concavely increases in $c > 1$. This finding suggests that the seller can benefit more from possessing flexibility on quality choice when the cost coefficient of quality is small or large. On average, the values of flexibility on quality choice when $c \leq 1$ are 39.51%, 42.57%, and 46.13% for $T = 110$, $T = 150$, and $T = 190$, respectively. The values when $1 < c \leq c_{2L}$ are 1090.26%, 1121.30%, and 969.23% for $T = 110$, $T = 150$, and $T = 190$, respectively. In our studies, the maximum of $\frac{\Pi^* - \Pi_q^*}{\Pi_q^*} \times 100\%$ is 16800%, which occurs at $T = 150$ and $c = 1.24$.

Clearly, the value of flexibility on quality choice can significantly improve the seller’s profit, indicating that the seller should invest in flexibility on quality choice when the cost coefficient of quality is uncertain or varies over time.

Further, it can be expected from the proof of Proposition 2 that the following result would hold.

**Corollary 2.** Given $c \leq c_{2L}$, the higher flexibility on quality choice, the greater is the profitability of the seller.

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11Since it is optimal for the seller to offer a high-quality product when $c \leq 1$, the value of flexibility on quality choice equals zero if the seller’s quality level is high. Under this condition, the value of flexibility is discussed only when the quality level is low. In other words, the value of flexibility on quality choice is represented as $\frac{\Pi^* - \Pi_q^*}{\Pi_q^*} \times 100\%$ for $c \leq 1$. Similarly, for $1 < c \leq c_{2L}$, the value of flexibility on quality choice is represented as $\frac{\Pi^* - \Pi_q^*}{\Pi_q^*} \times 100\%$. 

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**Figure 3.** Value of flexibility on quality choice.
Finally, given the findings presented in Figure 3, another interesting question would be whether the presence or absence of rationing would make flexibility more or less valuable. Denote by $\Pi_{qN}^*$, $q = \{H, L\}$, the seller’s optimal profit when the quality level is fixed at $q$ and capacity rationing is prohibited, where $\Pi_{qN}^*$ can be obtained from the proof of Proposition 2. Further, define $\frac{\Pi_{qN}^* - \Pi_{qN}}{\Pi_{qN}^*} \times 100\%$ as the values of flexibility on quality choice when capacity rationing is prohibited and the quality level is fixed at $q$, the parameters are the same as in the above. Figure 4 illustrates the results of the numerical studies.

At first glance, the shape of the value of flexibility when capacity rationing is prohibited is almost same as that when capacity rationing is allowed. Specifically, the values of flexibility in the two cases are almost the same when the cost coefficient is relatively small, i.e., $c \leq 1$. When the product cost is large, i.e., $c > 1$, it can be seen from Figures 3 and 4 that the existence of capacity rationing will reduce the value of flexibility. However, the reduction is not so large, especially when the value of $c$ is not sufficiently large. In general, it can be concluded that the impact of the presence of capacity rationing on the value of flexibility on quality choice is not significant.

6. Conclusions. In this research, the authors consider a seller who offers a product to strategic consumers sensitive to both price and quality over two periods: advance and spot. Consumption occurs only in the spot period, whereas purchase can occur in any period. Customers’ valuations are uncertain in the advance period and are only revealed in the spot period. In our model, customers are divided into informed and uninformed. Informed customers arrive at the beginning of the advance period. Upon observing the product quality and capacity offered by the seller, they determine whether and when to buy based on their anticipation of the availability and price in the spot period. Uninformed customers appear in the market only in the spot period and buy the product only when it becomes available and their realized valuations are high. For the seller, the first decision is whether to offer a high- or low-quality product. This decision must be made before the selling period. Further, the seller must determine the prices in the advance and spot periods and whether
to offer advance selling. If selling in advance is optimal, he also needs to determine whether to meet all or some of advance demand.

The authors first consider the benchmark case in which the seller’s capacity is sufficiently large to satisfy all potential demand and then extend to the case in which the seller’s capacity is limited. It is found that the seller’s decisions on quality choice and product offering are fully determined by the cost coefficient of quality. In each case, there exist two thresholds of this coefficient: high and low. It is optimal for the seller to offer a high-quality product when the coefficient is below the low threshold, offer a low-quality product when the coefficient is in the interval between the two thresholds, and not offer any product when the coefficient is above the high threshold. Moreover, given that offering the product is optimal, the capacity rationing policy varies over the model settings. Under unlimited capacity, it is optimal for the seller to either offer full advance selling or sell the product only in the spot period. When the seller’s capacity is constrained, limited advance selling is optimal under certain conditions. Finally, the numerical studies indicate that the seller can benefit from capacity rationing and flexibility on quality choice. However, the value of capacity rationing is small, whereas the value of flexibility on quality choice is very high.

The findings indicate that the seller should carefully infer and calculate the cost coefficient of quality. As presented in the first two findings, this coefficient plays a critical role in determining whether to offer a product, the quality level of the product, and whether to ration capacity in the advance period. Given that the quality level is determined, the seller must deliberately design the capacity rationing policy since the rationing decision interacts with the decision on quality choice as well as on whether to allow returns.

The model and results of the current study can be extended in a number of directions. For example, it would be interesting to extend our model to the supply chain setting. For instance, a performance company may authorize one or more agents to sell concert tickets. In this situation, the company, agent(s), and potential audiences comprise a supply chain. Since the decisions on quality, prices, rationing, and returns are made by two or more independent firms, the model may thus be more complicated than ours. Although studies examine the issue of advance selling in the supply chain context, such as [4] and [31], the modeling framework and main research questions are considerably different from ours. Further, it is assumed in this research that customers’ utilities are affected by price and product quality. In reality, a customer who arrives in the advance period and decides to wait may regret not purchasing earlier if the realized valuation is high and the product is unavailable in the spot period. On the contrary, this customer may benefit from purchasing early if she buys the product in the advance period. In other words, customers’ utilities may be affected by the emotional consequences of their decisions. An extension of our model to the case with customer regret/rejoice may therefore be a direction worthy of future research. Additionally, capacity rationing may lead some consumers who want to buy during the advance period but do not obtain to be disappointed, which may result in the seller’s lost sales cost. A model with such a cost would be an interesting topic for future research. Finally, both the seller and customer are assumed to be risk-neutral in the current research. When the seller’s capacity is limited, the availabilities of the product in the advance and spot periods are uncertain. As mentioned in [3], [9], and [26], customers may behave risk aversely in uncertain situations. In the context of advance selling, the customers
who decide to buy in the advance period may suffer the mismatch risk between realization and expectation of the customer valuation. Therefore, the extension of the current study by introducing customers’ risk-aversion is an interesting issue that is worthy of further research.

Appendix.

Proof of Lemma 4.1. Since $\theta \in [0, 1]$, $p_{2q}$ must satisfy $p_{2q} \in [q, 1 + q]$. By taking first and second partial derivatives of $\Pi_U^i(p_{2q}|S_q)$ with respect to $p_{2q}$, there are

$$\frac{d\Pi_U^i(p_{2q}|S_q)}{dp_{2q}} = (N - S_q)(1 - 2p_{2q} + q + cq),$$

$$\frac{d^2\Pi_U^i(p_{2q}|S_q)}{dp_{2q}^2} = -2(N - S_q) < 0.$$

Given $S_q$, $\Pi_U^i(p_{2q}|S_q)$ is concave in $p_{2q}$. Denote by $p_{2q}^0$ the solution of $\frac{d\Pi_U^i(p_{2q}|S_q)}{dp_{2q}} = 0$. To seek for the optimal $p_{2q}$, namely $p_{2q}^U$, that maximizes $\Pi_U^i(p_{2q}|S_q)$, it is necessary to check the signs of $\frac{d\Pi_U^i(p_{2q}|S_q)}{dp_{2q}}$ at the endpoints of $p_{2q}$.

$$\frac{d\Pi_U^i(p_{2q}|S_q)}{dp_{2q}} \bigg|_{p_{2q}=q} = (N - S_q)(1 - q + cq),$$

$$\frac{d\Pi_U^i(p_{2q}|S_q)}{dp_{2q}} \bigg|_{p_{2q}=1+q} = (N - S_q)(cq - 1 - q),$$

where $c_1q = 1 - \frac{1}{q}$ and $c_2q = 1 + \frac{1}{q}$. For $q \in \{L, H\}$, there is $c_{1L} < c_{1H} < c_{2H} < c_{2L}$.

In the following, $p_{2q}^U$ is derived according to the value of $c$.

- $c \leq c_{1L}$. In this case, both $\frac{d\Pi_U^i(p_{2q}|S_q)}{dp_{2q}} \bigg|_{p_{2q}=L} > 0$ and $\frac{d\Pi_U^i(p_{2q}|S_q)}{dp_{2q}} \bigg|_{p_{2q}=L} < 0$ are non-positive, that is, $\Pi_U^i(p_{2q}|S_q)$ is strictly decreasing in $p_{2q} \in [L, 1 + L]$. Thus, the optimal spot price given quality level $L$ is $p_{2L}^U = L$. In a similar way, there is $p_{2H}^U = H$.

- $c_{1L} < c \leq c_{1H}$. For product $L$, now there are $\frac{d\Pi_U^i(p_{2q}|S_q)}{dp_{2q}} \bigg|_{p_{2q}=L} > 0$ and $\frac{d\Pi_U^i(p_{2q}|S_q)}{dp_{2q}} \bigg|_{p_{2q}=L+1} < 0$. Then $p_{2L}^U = p_{2L}^0 = 1 + (1+c)L$. For product with quality $H$, the analysis is the same as in the above bullet and there is $p_{2H}^U = H$.

- $c_{1H} < c \leq c_{2H}$. Since $\frac{d\Pi_U^i(p_{2q}|S_q)}{dp_{2q}} \bigg|_{p_{2q}=q} > 0$ and $\frac{d\Pi_U^i(p_{2q}|S_q)}{dp_{2q}} \bigg|_{p_{2q}=q+1} < 0$ for $q \in \{H, L\}$, it follows that $p_{2q}^U = p_{2q}^0$.

- $c_{2H} < c \leq c_{2L}$. The analysis for product $L$ is same as in the above two bullets. For product $H$, $\Pi_{2H}(p_{2H}|S_H)$ strictly increases in $p_{2H}$ since both $\frac{d\Pi_U^i(p_{2q}|S_H)}{dp_{2q}} \bigg|_{p_{2q}=H} = 0$ and $\frac{d\Pi_U^i(p_{2q}|S_H)}{dp_{2q}} \bigg|_{p_{2q}=H+1} < 0$ are positive. This leads to $p_{2H}^U = H + 1$.

- $c > c_{2H}$. In this case, both $\frac{d\Pi_U^i(p_{2q}|S_q)}{dp_{2q}} \bigg|_{p_{2q}=q} < 0$ and $\frac{d\Pi_U^i(p_{2q}|S_q)}{dp_{2q}} \bigg|_{p_{2q}=q+1} < 0$ are positive for $q \in \{H, L\}$. Thus, $p_{2q}^U = q + 1$.

This completes the proof. \square
Proof of Proposition 1. Since $\Pi^U_q(S_q) (= \Pi^U_q(p^U_{1q}, S_q))$ monotonically changes with regard to $S_q$, only the values of $\Pi^U_q(S_q)$ at the endpoints of $S_q$ needs to be discussed. The results are proved based on the intervals in which $c$ located.

(i) $c \leq c_{1L}$: In this case, $p^U_{2q} = q$. Accordingly, there would be $\Pi^U_q(N_1)|_{p^U_{2q}=q} = \Pi^U_q(0)|_{p^U_{2q}=q} = N\tau q$, where $\tau = 1 - c$. Since $c \leq c_{1L} < 1$, it follows that $N\tau q$ strictly increases in $q$. As a result, the seller will not provide the product with low quality level. To show the seller is optimal to sell product $H$, it is necessary to identify $N\tau H > 0$. According to the prerequisite condition $c \leq c_{1L}$, there are $\tau L > 1$ and thus $N\tau H > N\tau L > N > 0$. Moreover, there is $p^U_{1H} - cH = (p^U_{2H} - cH)(1 + H - p^U_{2H})$. That is, $\Pi^U_q(S_H)$ is independent of $S_H$. Therefore, the amount offered in the advance period can be any value between zero and $N_1$.

(ii) $c_{1L} < c \leq c_{1H}$: According to the seller’s optimal spot prices: $p^U_{2L} = p^U_{2H} = p^U_{0L} = p^U_{0H} = N\tau q$, where the inequality is due to the prerequisite conditions: $f_1 > 0$, where the inequality is due to the prerequisite conditions: $c > c_{1L} \Rightarrow 1 - \tau L > 0$ and $c \leq c_{1H} \Rightarrow c \leq 1 \Rightarrow 1 + 3\tau L > 0$. Therefore, the seller will provide full advance selling if it is optimal to sell product $L$.

In the following, whether selling product $H$ or selling $L$ is optimal is discussed. This is proved by comparing $\Pi^U_q(N_1)|_{p^U_{2H}=H}$ with $\Pi^U_q(N_1)|_{p^U_{2L}=p^U_{2H}}$.

$$\Pi^U_q(N_1)|_{p^U_{2L}=p^U_{2H}} - \Pi^U_q(N_1)|_{p^U_{2H}=H} = \frac{N_1(1 + 3\tau L)(1 - \tau L)}{8} - \frac{N(1 + \tau L)^2}{4} - N\tau H,$$

where $\tau \in [\frac{1}{L}, \frac{1}{H}]$. The right-hand side of (A1) can be written as a function of $\tau$:

$$f(\tau) = \frac{N_1(1 + 3\tau L)(1 - \tau L)}{8} + \frac{N(1 + \tau L)^2}{4} - N\tau H.$$  

The first and second derivatives of $f(\tau)$ are given by $f'(\tau) = \frac{N_1L - 3N_1L^2\tau}{4} + \frac{NL(1 + \tau L)}{4} - NH$ and $f''(\tau) = \frac{L^2(2N - 3N_1)}{4}$, respectively. Obviously, there are $f''(\tau) \geq 0$ for $N \geq \frac{3N_1}{2}$ and $f''(\tau) \leq 0$ for $N \leq \frac{3N_1}{2}$. Moreover, the values of $f'(\tau)$ at the two endpoints of $\tau$ can be expressed as

$$f'(\tau \uparrow \frac{1}{L}) = -\frac{N_1L}{2} - N(H - L) < 0$$

and

$$f'(\tau \downarrow \frac{1}{H}) = \frac{N_1HL - 3N_1L^2 + 2NHL + 2NL^2 - 4NH^2}{5H},$$

respectively. Define $g(\alpha) = (2N - 3N_1)\alpha^2 + (2N + N_1)\alpha - 4N$, where $\alpha = \frac{L}{H} \in (0, 1)$. Accordingly, $f'(\tau = \frac{1}{H})$ can be rewritten as $f'(\tau = \frac{1}{H}) = \frac{Hg(\alpha)}{4}$. It is apparent that $f'(\tau = \frac{1}{H})$ and $g(\alpha)$ share the same sign. Taking first and second derivatives of $g(\alpha)$ with regard to $\alpha$ lead to $g'(\alpha) = 2(2N - 3N_1)\alpha - 2N + N_1$ and $g''(\alpha) = 2(2N - 3N_1)$. When $N \geq \frac{3N_1}{2}$, $g(\alpha)$ is convex in $\alpha$. Because $g(\alpha \downarrow 0) = -4N < 0$ and $g(\alpha \uparrow 1) = -2N_1 < 0$, it is known that $g(\alpha)$ is negative over $\alpha \in (0, 1)$. On the other side, when $N < \frac{3N_1}{2}$, there is $g''(\alpha) < 0$. That is, $g'(\alpha)$ strictly decreases in $\alpha$. Since the value of $g'(\alpha)$ at
the left endpoint of $\alpha$ is negative, i.e., $g'(\alpha \downarrow 0) = -2N + N_1 < 0$, it follows that $g'(\alpha)$ is negative for any value of $\alpha$. By combining this with $g(\alpha \downarrow 0) < 0$, it can be inferred that $g(\alpha)$ is always negative for $N < \frac{3N_1}{2}$. In summary, there is $f'(\overline{\tau}) < 0$.

Recall from above (i.e., $f''(\overline{\tau}) \geq 0$ for $N \geq \frac{3N_1}{2}$ and $f''(\overline{\tau}) \leq 0$ for $N \leq \frac{3N_1}{2}$) that $f'(\overline{\tau})$ always monotonically change with respect to $\overline{\tau}$. Since both $f'(\overline{\tau} \uparrow \frac{1}{H})$ and $f'(\overline{\tau} = \frac{1}{H})$ are negative, it follows $f'(\overline{\tau}) < 0$ for all $\overline{\tau} \in \left[\frac{1}{p}, \frac{1}{H}\right]$, i.e., $f'(\overline{\tau})$ strictly decreases in $\overline{\tau}$. The value of $f(\overline{\tau})$ at the left endpoint of $\overline{\tau}$ is

$$f\left(\overline{\tau} = \frac{1}{H}\right) = \frac{N_1(1 + 3\alpha)(1 - \alpha)}{8} + \frac{N(1 + \alpha)^2}{4} - N := l(\alpha).$$

The first and second derivatives of $l(\alpha)$ with regard to $\alpha$ are $l'(\alpha) = \frac{N_1(1-3\alpha)}{1} + \frac{N(1+\alpha)}{2}$ and $l''(\alpha) = \frac{2N_1 - 3N_1}{2}$. There are $l''(\alpha) \geq 0$ for $N \geq \frac{3N_1}{2}$, and $l''(\alpha) \leq 0$ for $N \leq \frac{3N_1}{2}$, that is, $l'(\alpha)$ always monotonically changes with respect to $\alpha$. Due to $l'(\alpha \downarrow 0) = \frac{2N_1 + N_1}{2} > 0$ and $l'(\alpha \uparrow 1) = \frac{2N_1 - N_1}{2} > 0$, there would be $l'(\alpha) > 0$ for all $\alpha \in (0, 1)$. Together with the fact of $l(\alpha \uparrow 1) = 0$, it can be inferred that $l(\alpha)$ is non-positive for any value of $\alpha$. Such a conclusion further results in $f(\overline{\tau}) < 0$ for $\overline{\tau} \in \left[\frac{1}{p}, \frac{1}{H}\right]$. As a consequence, the seller is optimal to sell the high-quality product. Furthermore, there is $p_{L_H}^{U} - cH = (p_{L_H}^{U} - cH)(1 + H - p_{L_H}^{U})$. Thus, the amount offered in the advance period can be any value between zero and $N_1$.

(iii) $c_{L_H} < c \leq c_{2H}$. In this case, the seller’s optimal spot price is $p_{2q}^{U} = p_{2q}^{0}$ for $q \in \{H, L\}$. Accordingly, there must be

$$
\Pi_q^{U}(N_1)|_{p_{2q}^{U}=p_{2q}^{0}} - \Pi_q^{U}(0)|_{p_{2q}^{U}=p_{2q}^{0}} = \frac{N_1(1 + 3\overline{\tau}q)(1 - \overline{\tau}q)}{8}.
$$

(A2)

Using the prerequisite condition $c > c_{1q}$ gives $1 - \overline{\tau}q > 0$. Therefore, the sign of (A2) is dependent on that of $1 + 3\overline{\tau}q$. Denote $c_{3q} = 1 + \frac{1}{3\overline{\tau}}$, there are

$$
\begin{cases} 
\Pi_q^{U}(N_1)|_{p_{2q}^{U}=p_{2q}^{0}} \geq \Pi_q^{U}(0)|_{p_{2q}^{U}=p_{2q}^{0}}, & \text{if } c \leq c_{3q} \\
\Pi_q^{U}(N_1)|_{p_{2q}^{U}=p_{2q}^{0}} < \Pi_q^{U}(0)|_{p_{2q}^{U}=p_{2q}^{0}}, & \text{if } c > c_{3q}.
\end{cases}
$$

(A3)

According to the definition of $c_{3q}$, there is $c_{3H} < c_{3L}$ and $c_{3H} < c_{2H}$. To derive the results, it is also necessary to compare $c_{3L}$ with $c_{2H}$. When $\alpha \leq \frac{1}{3}$, there is $c_{2H} \leq c_{3L}$. Under this situation, there is always $\Pi_q^{U}(N_1)|_{p_{2L}^{U}=p_{2L}^{0}} \geq \Pi_q^{U}(0)|_{p_{2L}^{U}=p_{2L}^{0}}$ due to the prerequisite condition $c \leq c_{2H}$. The analysis can be separated into two cases: $c \leq c_{3H}$ and $c > c_{3H}$. On the other hand, when $\alpha > \frac{1}{3}$, there is $c_{3L} < c_{2H}$ and the analysis must be divided into three cases: $c \leq c_{3H}$, $c_{3H} < c \leq c_{3L}$, and $c > c_{3L}$. In what follows, the seller’s optimal policies are derived case by case.

- $\alpha \leq \frac{1}{3}$ and $c_{L_H} < c \leq c_{3L}$. In this case, the results can be obtained by comparing $\Pi_q^{U}(N_1)|_{p_{2H}^{U}=p_{2H}}$ with $\Pi_q^{U}(N_1)|_{p_{2L}^{U}=p_{2L}}$.

$$
\begin{align*}
\Pi_q^{U}(N_1)|_{p_{2H}^{U}=p_{2H}} - \Pi_q^{U}(N_1)|_{p_{2L}^{U}=p_{2L}} &= \frac{N_1[(1 + 3\overline{\tau}H)(1 - \overline{\tau}H) - (1 + 3\overline{\tau}L)(1 - \overline{\tau}L)]}{8} \\
+ \frac{N[(1 + \overline{\tau}H)^2 - (1 + \overline{\tau}L)^2]}{4}.
\end{align*}
$$
Define \( l(q) = N_1(1 + 3\bar{\tau}q)(1 - \bar{\tau}q) + 2N(1 + \bar{\tau}q)^2 \). Equation (A3) can rewritten as

\[
\Pi_H^U(N_1)\bigg|_{p_{2H}^U=p_{2L}^U} - \Pi_L^U(N_1)\bigg|_{p_{2L}^U=p_{2L}^U} = \frac{l(H) - l(L)}{8}.
\]

Differentiating \( l(q) \) with regard to \( q \) gives \( l'(q) = 2N_1\bar{\tau}(1 - \bar{\tau}q) + 4N_2\bar{\tau}^2q + 4N\bar{\tau} \). According to the prerequisite condition \( c > c_{1q} \), the term \((1 - \bar{\tau}q)\) is positive. Thus, \( l(q) \) is strictly increasing in \( q \), this implies \( l(H) > l(L) \) and \( \Pi_H^U(N_1)\bigg|_{p_{2H}^U=p_{2H}^U} > \Pi_L^U(N_1)\bigg|_{p_{2L}^U=p_{2L}^U} \). That is to say, the seller is optimal to offer the high-quality product with full advance selling.

\(- \alpha \leq \frac{1}{3} \) and \( c_{3H} < c \leq c_{2H} \). Conditions \( \alpha \leq \frac{1}{3} \) and \( c > c_{3H} \) imply \( \Pi_L^U(N_1)\bigg|_{p_{2L}^U=p_{2L}^U} \geq \Pi_L^U(0)\bigg|_{p_{2L}^U=p_{2L}^U} \) and \( \Pi_H^U(N_1)\bigg|_{p_{2H}^U=p_{2H}^U} < \Pi_H^U(0)\bigg|_{p_{2H}^U=p_{2H}^U} \), respectively. Further, condition \( c > c_{3H} \) leads to \( c > c_{1H} > c_{1L} \). Under this situation, it is known from the above case that \( \Pi_H^U(N_1)\bigg|_{p_{2H}^U=p_{2H}^U} > \Pi_L^U(N_1)\bigg|_{p_{2L}^U=p_{2L}^U} \). This further leads to \( \Pi_H^U(0)\bigg|_{p_{2H}^U=p_{2H}^U} > \Pi_L^U(0)\bigg|_{p_{2L}^U=p_{2L}^U} \), that is, the seller is optimal to offer the high-quality product only in the spot period.

\(- \alpha > \frac{1}{3} \) and \( c_{1H} < c \leq c_{3H} \). Condition \( c \leq c_{3H} \) implies \( \Pi_H^U(N_1)\bigg|_{p_{2H}^U=p_{2H}^U} > \Pi_H^U(0)\bigg|_{p_{2H}^U=p_{2H}^U} \) for \( q \in \{H, L\} \). Further, condition \( c > c_{1H} \) leads to (see the case of \( \alpha \leq \frac{1}{3} \) and \( c_{1H} < c \leq c_{3H} \) \( \Pi_H^U(N_1)\bigg|_{p_{2H}^U=p_{2H}^U} > \Pi_L^U(N_1)\bigg|_{p_{2L}^U=p_{2L}^U} \). As a consequence, the seller is optimal to offer the high-quality product with full advance selling.

\(- \alpha > \frac{1}{3} \) and \( c_{3H} < c \leq c_{3L} \). Conditions \( c > c_{3H} \) and \( c \leq c_{3L} \) imply \( \Pi_H^U(N_1)\bigg|_{p_{2H}^U=p_{2H}^U} > \Pi_H^U(0)\bigg|_{p_{2H}^U=p_{2H}^U} \) and \( \Pi_L^U(N_1)\bigg|_{p_{2L}^U=p_{2L}^U} \geq \Pi_L^U(0)\bigg|_{p_{2L}^U=p_{2L}^U} \), respectively. Moreover, condition \( c > c_{3H} > c_{1H} \) gives \( \Pi_H^U(N_1)\bigg|_{p_{2H}^U=p_{2H}^U} > \Pi_L^U(N_1)\bigg|_{p_{2L}^U=p_{2L}^U} \) in summary, the seller is optimal to offer the high-quality product only in the spot period.

\(- \alpha > \frac{1}{3} \) and \( c_{3L} < c \leq c_{2H} \). Since \( c > c_{3L} \) leads to \( \Pi_H^U(N_1)\bigg|_{p_{2H}^U=p_{2H}^U} > \Pi_H^U(0)\bigg|_{p_{2H}^U=p_{2H}^U} \) for \( q \in \{H, L\} \), the results can be obtained by comparing \( \Pi_H^U(0)\bigg|_{p_{2H}^U=p_{2H}^U} \) with \( \Pi_L^U(0)\bigg|_{p_{2L}^U=p_{2L}^U} \). The difference between them can be expressed as

\[
\Pi_H^U(0)\bigg|_{p_{2H}^U=p_{2H}^U} - \Pi_L^U(0)\bigg|_{p_{2L}^U=p_{2L}^U} = \frac{N}{4}\bar{\tau}(H - L)[2 + \bar{\tau}(H + L)].
\]

Condition \( c > c_{3L} \) implies \( \bar{\tau} < 0 \) and condition \( c < c_{2H} \) implies both \( 1 + \bar{\tau}H \) and \( 1 + \bar{\tau}L \) are positive. Therefore, the left-hand side of (A4) is negative. This indicates that the seller is optimal to offer the low-quality product only in the spot period.

(iv) \( c_{2H} < c \leq c_{2L} \). In this case, \( p_{2H}^U = H + 1 \) and \( p_{2L}^U = p_{2L}^0 \). For the product with high quality level, there is \( \Pi_H^U(N_1)\bigg|_{p_{2H}^U=H+1} = N_1\{E_0[\min(0, \theta - 1)] + (H + 1 - cH)\} \) and \( \Pi_H^U(0)\bigg|_{p_{2H}^U=H+1} = 0 \). Condition \( c > c_{2H} \) means \( 1 + H - cH < 0 \). Accordingly, there is \( \Pi_H^U(N_1)\bigg|_{p_{2H}^U=H+1} < 0 \). For the low-quality product, there is

\[
\Pi_L^U(N_1)\bigg|_{p_{2L}^U=p_{2L}^0} - \Pi_L^U(0)\bigg|_{p_{2L}^U=p_{2L}^0} = \frac{N_1(1 + 3\bar{\tau}L)(1 - \bar{\tau}L)}{8}.
\]
α. In what follows, the analysis is separated into three cases based on the value of α and the value interval of c.

- α ≤ \( \frac{1}{3} \) and \( c_{2L} < c \leq c_{3L} \). Condition \( c \leq c_{3L} \) indicates \( \Pi_L(U(N_1))|_{p_{2q}} \geq \Pi_L(U(0))|_{p_{2q}} \). Since \( \Pi_L(U(0))|_{p_{2q}} \geq 0 = \Pi_H(U(H+1))|_{p_{2q}} = \Pi_H(U(H+1))|_{p_{2q}} = \Pi_H(U(0))|_{p_{2q}} \), it can be inferred that the seller is optimal to offer the low-quality product with full advance selling.

- α ≤ \( \frac{1}{3} \) and \( c_{3L} < c \leq c_{2L} \). Condition \( c > c_{3L} \) implies \( \Pi_L(U(N_1))|_{p_{2q}} < \Pi_L(U(0))|_{p_{2q}} \). Since \( \Pi_L(U(0))|_{p_{2q}} \geq 0 = \Pi_H(U(H+1))|_{p_{2q}} = \Pi_H(U(H+1))|_{p_{2q}} = \Pi_H(U(0))|_{p_{2q}} \), it follows that the seller is optimal to offer the low-quality product only in the spot period.

- α > \( \frac{1}{3} \). Because \( c_{2L} < c > c_{3L} \), the analysis is similar as in the above case.

(v) \( c > c_{2L} \). Under this situation, there are \( \Pi_L(U(0))|_{p_{2q}} = 0 > \Pi_L(U(N_1))|_{p_{2q}} \) for \( q \in \{H, L\} \). Since any policies cannot generate positive profits, the seller had better not offer any products.

Proposition 1 is obtained by classifying the above discussions.

Proof of Lemma 5.1. Setting \( T = S_q = (N - S_q)(1 - p_{2q} + q) \) gives \( p_{2q}^B(S_q) = q + \frac{N - T}{N - S_q} \).

In what follows, the analysis is divided into two cases based on the value of \( p_{2q} \).

- \( p_{2q} \leq p_{2q}^B(S_q) \). Given \( S_q \), \( \Pi_{2q}(p_{2q}|S_q) = (p_{2q} - cq)(T - S_q) \) strictly increases in \( p_{2q} \). Therefore, the optimal \( p_{2q} \) that maximizes \( \Pi_{2q}(p_{2q}|S_q) \) is \( p_{2q}^B(S_q) \).

- \( p_{2q} > p_{2q}^B(S_q) \). In this case, \( \Pi_{2q}(p_{2q}|S_q) = (p_{2q} - cq)(N - S_q)(1 - p_{2q} + q) = \Pi_{2q}^U(p_{2q}|S_q) \). It is known from the proof of Lemma 4.1 that given \( S_q \), \( \Pi_{2q}(p_{2q}|S_q) \) is strictly concave in \( p_{2q} \). According to the range of \( p_{2q} \), the optimal \( p_{2q} \) can be defined as: \( p_{2q}^C(S_q) = p_{2q}^U \) if \( p_{2q} > p_{2q}^B(S_q) \) and \( p_{2q}^C(S_q) = p_{2q}^B(S_q) \) if \( p_{2q} \leq p_{2q}^B(S_q) \).

By summarizing the above discussions, there is \( p_{2q}^C = \max\{p_{2q}^B, p_{2q}^C(S_q)\} \). Further, by using (7), \( p_{2q}^B \leq \frac{2}{S_q} p_{2q}^B(S_q) \) can be rewritten as \( T - S_q \leq \frac{2}{S_q} (N - S_q)(1 - p_{2q}^B + q) \).

Proof of Lemma 5.2. By taking first and second derivatives of \( \Pi_{2q}^B(S_q) \) with respect to \( S_q \) and using (7), there are

\[
\frac{d\Pi_{2q}^B(S_q)}{dS_q} = \frac{(N - T)[N(3T - N) - S_q(N + T)]}{2(N - S_q)^3} \quad \text{and} \quad \frac{d^2\Pi_{2q}^B(S_q)}{dS_q^2} = \frac{(N - T)[2N(2T - N) - S_q(N + T)]}{(N - S_q)^4}.
\]

It is evident that when \( 2T - N \leq 0 \), there must be \( \frac{d^2\Pi_{2q}^B(S_q)}{dS_q^2} \leq 0 \) for any value of \( S_q \in [0, \min\{T, N_1\}] \). When \( 2T > N \), there exists an \( S_q \), namely \( \tilde{S}_q \), such that \( \frac{d^2\Pi_{2q}^B(S_q)}{dS_q^2} > 0 \) for \( S_q < \tilde{S}_q \) and \( \frac{d^2\Pi_{2q}^B(S_q)}{dS_q^2} < 0 \) for \( S_q > \tilde{S}_q \), where \( \tilde{S}_q = \frac{2N(2T - N)}{N + T} \) is obtained by setting \( \frac{d^2\Pi_{2q}^B(S_q)}{dS_q^2} = 0 \). This may lead to two different cases. First, if \( \tilde{S}_q \) is beyond the range of \( S_q \), i.e., \( \tilde{S}_q > \min\{T, N_1\} \), there is always \( S_q \leq \tilde{S}_q \). In this case, \( \Pi_{2q}(S_q) \) is convex in \( S_q \in [0, \min\{T, N_1\}] \). On the other side, when \( \tilde{S}_q \leq \min\{T, N_1\} \), \( \Pi_{2q}(S_q) \) is initially convex and then concave in \( S_q \). Therefore, it is necessary to compare \( \tilde{S}_q \) with \( \min\{T, N_1\} \) for the case of \( T > \frac{N}{2} \).
\begin{itemize}
  \item $N_1 > \frac{N}{2}$ and $T \leq N_1$. In this case, $\min\{T, N_1\} = T$. Subtracting $T$ from $\tilde{S}_q$ gives
  \[
  \tilde{S}_q - T = \frac{(T - 2N)(N - T)}{N + T} < 0.
  \]
  In other words, $\tilde{S}_q$ is less than $\min\{T, N_1\}$ when $T \leq N_1$.
  \item $N_1 > \frac{N}{2}$ and $T > N_1$. Now there is $\min\{T, N_1\} = N_1$. The difference between $\tilde{S}_q$ and $\min\{T, N_1\}$ is given by
  \[
  \tilde{S}_q - N_1 = \frac{(4N - N_1)T - N(2N + N_1)}{N + T} \begin{cases} 
    \leq 0, & \text{if } T \leq T' \\
    > 0, & \text{if } T > T',
  \end{cases}
  \]
  where $T' = \frac{N(2N + N_1)}{4N - N_1}$. That is, $\tilde{S}_q \leq \min\{T, N_1\}$ when $T \leq T'$ and $\tilde{S}_q > \min\{T, N_1\}$ otherwise.
  \item $N_1 \leq \frac{N}{2}$. As $T > \frac{N}{2}$, there is always $\min\{T, N_1\} = N_1$. For this case, the analysis is same as in the second bullet.
\end{itemize}

By classifying the above discussions, (i)-(iii) follow.

Now we turn to derive the optimal $S_q$ that maximizes $\Pi^B_q(S_q)$. For the case of $T \leq \frac{N}{2}$, the value of $\frac{d\Pi^B_q(S_q)}{dS_q}$ at the left endpoint of $S_q$ can be represented as
\[
\left. \frac{d\Pi^B_q(S_q)}{dS_q} \right|_{S_q=0} = \left( \frac{T - N(3T - N)}{2N^2} \right) \begin{cases} 
  \geq 0, & \text{if } T \leq \frac{N}{4} \\
  < 0, & \text{if } \frac{N}{4} < T \leq \frac{N}{2},
  \end{cases}
\]

Since $\Pi^B_q(S_q)$ is concave in $S_q$, it follows that when $T \leq \frac{N}{4}$, the first derivative of $\Pi^B_q(S_q)$ with respect to $S_q$ would be negative for all $S_q \in [0, \min\{T, N_1\}]$. In other words, $\Pi^B_q(S_q)$ decreases in $S_q$. As a result, the optimal $S_q$ that maximizes $\Pi^B_q(S_q)$ is at $S_q = 0$. On the other side, when $\frac{N}{4} < T \leq \frac{N}{2}$, it is necessary to check the sign of $\frac{d\Pi^B_q(S_q)}{dS_q}$ at the right endpoint of $S_q$. According to the relationship between $T$ and $N_1$, the discussion is divided into two cases.

(i) $T \leq N_1$. Now $\min\{T, N_1\} = T$ and there is
\[
\left. \frac{d\Pi^B_q(S_q)}{dS_q} \right|_{S_q=T} = \frac{1}{2} < 0.
\]

(ii) $T > N_1$. In this case, $\min\{T, N_1\} = N_1$. Accordingly, there is
\[
\left. \frac{d\Pi^B_q(S_q)}{dS_q} \right|_{S_q=N_1} = \left( \frac{T - N(3T - N) - N_1(N + T)}{2(N - N_1)^3} \right) \begin{cases} 
  \geq 0, & \text{if } N_1 \leq \overline{N}_1 \\
  < 0, & \text{if } N_1 > \overline{N}_1,
  \end{cases}
\]

where $\overline{N}_1 = \frac{N(3T - N)}{N + T}$. Since $\overline{N}_1 < T$, the results in the above two cases can be summarized as
\[
\left. \frac{d\Pi^B_q(S_q)}{dS_q} \right|_{S_q=\min\{T, N_1\}} \begin{cases} 
  \geq 0, & \text{if } N_1 \leq \overline{N}_1 \\
  < 0, & \text{if } N_1 > \overline{N}_1.
  \end{cases}
\]

It can be inferred that when $N_1 \leq \overline{N}_1$, $\Pi^B_q(S_q)$ strictly increases in $S_q$ and is maximized at $S_q = \min\{T, N_1\} = N_1$; otherwise, $\Pi^B_q(S_q)$ initially increases and then decreases in $S_q$ and is maximized at $S_q = \tilde{S}_q = \frac{N(3T - N)}{N + T}$, where $\tilde{S}_q$ is obtained by
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Figure A1. Sketch of $\Pi_q^C(S_q)$ for $T \leq \frac{N}{2}$.

setting $\frac{d\Pi_B^B(S_q)}{dS_q} = 0$. Moreover, there is $\hat{S}_q < \min\{T, N_1\}$ due to $\frac{d\Pi_B^B(S_q)}{dS_q} |_{S_q=0} > 0$
and $\frac{d\Pi_B^B(S_q)}{dS_q} |_{S_q=\min\{T,N_1\}} < 0$.

When $\frac{N}{2} < T \leq T$, the values of $\frac{d\Pi_B^B(S_q)}{dS_q}$ at points $S_q = 0$ and $S_q = \bar{S}_q$ are

\[
\left. \frac{d\Pi_B^B(S_q)}{dS_q} \right|_{S_q=0} = \frac{(N-T)(3T-N)}{2N^2} > 0 \quad \text{and}
\]
\[
\left. \frac{d\Pi_B^B(S_q)}{dS_q} \right|_{S_q=\bar{S}_q} = \frac{(N+T)^3}{54N^2(N-T)} > 0,
\]
respectively. Since $\Pi_B^B(S_q)$ is initially convex and then concave in $S_q$, it can be inferred from the signs of $\frac{d\Pi_B^B(S_q)}{dS_q} |_{S_q=0}$, $\frac{d\Pi_B^B(S_q)}{dS_q} |_{S_q=\bar{S}_q}$, and $\frac{d\Pi_B^B(S_q)}{dS_q} |_{S_q=\min\{T,N_1\}}$ that $\Pi_B^B(S_q)$ is increasing in $S_q \in [0, \bar{S}_q]$ and increasing or initially increasing and then decreasing in $S_q \in (\bar{S}_q, \min\{T, N_1\}]$. Therefore, the optimal $S_q$ that maximizes $\Pi_B^B(S_q)$ is at $S_q = \min\{T, N_1\} = N_1$ when $N_1 \leq \bar{N}_1$ and at $S_q = \bar{S}_q$ when $N_1 > \bar{N}_1$.

Finally, when $T > T > \frac{N}{2}$, there is $\frac{d\Pi_B^B(S_q)}{dS_q} |_{S_q=0} > 0$. By combining the result that $\Pi_B^B(S_q)$ is convex in $S_q$, it is known that $\Pi_B^B(S_q)$ increases in $S_q \in [0, \min\{T, N_1\}]$ and is maximized at $S_q = \min\{T, N_1\}$. In this case, there is $T > \bar{T} > N_1$ and thus $\min\{T, N_1\} = N_1$.

The following propositions are used to prove Proposition 2.

Proposition A1. Suppose $T \leq \frac{N}{2}$ and denote $c_{s_q} = 1 + \frac{N-2T}{qN}$ for $q \in \{H, L\}$. 

• For $c_1q < c \leq c_2q$, $\overline{S}_q \leq 0$ if $c \leq c_5q$ and $\overline{S}_q > 0$ otherwise.

• $\Pi^C_q(S_q)$ can be illustrated by Figure A1.

Proof. Based on Lemma 4.1, there would be

$$\overline{S}_q = \begin{cases} -\infty, & \text{if } c \leq c_1q \\ N - \frac{2(N-T)}{1-q_0}, & \text{if } c_1q < c \leq c_2q \\ T, & \text{if } c > c_2q \end{cases}$$

Given $c_1q < c \leq c_2q$, there is

$$\overline{S}_q = N - \frac{2(N-T)}{1-q_0} \begin{cases} \leq 0, & \text{if } \overline{\tau} \geq \frac{2T-N}{qN} \\ > 0, & \text{if } \overline{\tau} < \frac{2T-N}{qN} \end{cases}$$

According to the definitions of $c_1q$ and $c_2q$, there is $c_1q < c_5q < c_2q$. Further, $\overline{\tau} \leq \frac{2T-N}{qN}$ can be expressed as $c \leq 1 + \frac{N-2T}{qN} = c_5q$. As a consequence, the results in the first bullet hold. Notice that there is $c_5q \geq c_3q$ for $T \leq \frac{N}{3}$, where the equality holds only when $T = \frac{N}{3}$.

For the second bullet, it is known from (12) that $\Pi^C_q(S_q) = \Pi^U_q(S_q)$ for $S_q \leq \overline{S}_q$ and $\Pi^C_q(S_q) = \Pi^B_q(S_q)$ for $S_q \geq \overline{S}_q$. The properties of function $\Pi^B_q(S_q)$ are presented in Lemma 5.2. It is shown that in different value intervals of $T$, $\Pi^B_q(S_q)$ may be concave, initially convex and then concave, or convex in $S_q$. Moreover, it can be known from $\frac{d\Pi^U_q}{dS_q}(S_q) = \frac{\Pi^U_q(N_1)-\Pi^U_q(0)}{N_1}$ that $\Pi^U_q(S_q)$ monotonically changes with regard to $S_q$ in some cases and is independent of $S_q$ in other cases. From the proof of Proposition 2, there is

$$\Pi^U_q(N_1) - \Pi^U_q(0) \begin{cases} = 0, & \text{if } c \leq c_1q \\ > 0, & \text{if } c_1q < c \leq c_3q \\ < 0, & \text{if } c > c_3q \end{cases}$$

Therefore, $\Pi^U_q(S_q)$ is independent of, increases in, and decreases in $S_q$ for $c \leq c_1q$, $c_1q < c \leq c_3q$, and $c > c_3q$, respectively.

It is shown in the first bullet that $\overline{S}_q$ is non-positive when $c \leq c_5q$. Since $S_q \geq 0$, it follows that $\Pi^C_q(S_q)$ equals $\Pi^B_q(S_q)$ over $S_q \in [0, \min\{T, N_1\}]$. Moreover, it is demonstrated in Lemma 5.2 that when $T \leq \frac{N}{3}$, $\Pi^B_q(S_q)$ is concave and is maximized at $S_q = 0$. In other words, $\Pi^B_q(S_q)$ is decreasing and concave in $S_q > 0$. As a consequence, $\Pi^C_q(S_q)$ for $T \leq \frac{N}{3}$ and $c \leq c_5q$ can be illustrated by Figure A1(a). Further, when $c_5q < c \leq c_2q$, $\overline{S}_q$ is positive. At the right-hand side of $\overline{S}_q$, $\Pi^C_q(S_q)$ equals $\Pi^B_q(S_q)$ and is concavely decreasing in $S_q$. Over the range $[0, \overline{S}_q]$, $\Pi^C_q(S_q)$ equals $\Pi^U_q(S_q)$ and is linear in $S_q$. Since $c_5q > c_3q$, there would be $\Pi^U_q(N_1) - \Pi^U_q(0) < 0$ for $c > c_5q$. And it can be inferred that $\Pi^C_q(S_q)$ for $c_5q < c \leq c_2q$ can be illustrated by Figures A1(b). Additionally, when $c > c_2q$, there is $\overline{S}_q = T$. Because $S_q$ must satisfy $S_q \leq \min\{T, N_1\}$, there would be $\Pi^U_q(S_q) = \Pi^U_q(S_q)$ for $S_q \leq T$. Therefore, $\Pi^C_q(S_q)$ in this case can be illustrated by Figures A1(c).

\[\Box\]

**Proposition A.2.** Suppose $\frac{N}{3} < T \leq \frac{N}{2}$ and denote $c_{4q} = 1 + \frac{N+\min\{T,N_1\}-2T}{q(N-N_1)}$ and $c_{6q} = 1 + \frac{T}{qN}$ for $q \in \{H, L\}$.
Figure A2. Sketch of $\Pi^C_q(S_q)$ for $\frac{N}{3} < T \leq \frac{N}{2}$ and $N_1 \leq \frac{3T-N}{2}$.

Figure A3. Sketch of $\Pi^C_q(S_q)$ for $\frac{N}{3} < T \leq \frac{N}{2}$ and $\frac{3T-N}{2} < N_1 \leq N_1$. 
Figure A4. Sketch of $\Pi_q^c(S_q)$ for $\frac{N}{q} < T \leq \frac{N}{2}$ and $N_1 > N_1$.

- For $c_{1q} < c \leq c_{2q}$, there are
  \[
  \begin{cases}
  \hat{S}_q \geq \overline{S}_q, & \text{if } c \leq c_{6q} \quad \text{and} \quad \overline{S}_q \leq N_1, & \text{if } c \leq c_{4q} \\
  \hat{S}_q < \overline{S}_q, & \text{if } c > c_{6q} \quad \text{and} \quad \overline{S}_q > N_1, & \text{if } c > c_{4q}.
  \end{cases}
  \]

- $\Pi_q^C(S_q)$s for $N_1 \leq \frac{3T-N}{2}, \frac{3T-N}{2} < N_1 \leq N_1$, and $N_1 > N_1$ can be illustrated by Figures A2-A4, respectively.

Proof. By the definitions of $\hat{S}_q$ and $\overline{S}_q$, there are
\[
\hat{S}_q - \overline{S}_q = 2(N - T) \left( \frac{1}{1 - \bar{c}_q} - \frac{N}{N + T} \right) \begin{cases}
\leq 0, & \text{if } \bar{c} \leq -\frac{T}{qN} \\
> 0, & \text{if } \bar{c} > -\frac{T}{qN}.
\end{cases}
\]
\[
\overline{S}_q - N_1 = N - N_1 - \frac{2(N - T)}{1 - \bar{c}_q} \begin{cases}
\geq 0, & \text{if } \bar{c} \leq -\frac{2T(N-N_1)}{q(N-N_1)} \\
< 0, & \text{if } \bar{c} > -\frac{2T(N-N_1)}{q(N-N_1)}.
\end{cases}
\]
for $c_{1q} < c \leq c_{2q}$. Since $\bar{c} \leq -\frac{T}{qN}$ and $\bar{c} \leq \frac{2T - N - N_1}{q(N - N_1)}$, can be rewritten as $c < \frac{1}{1 + \frac{T}{qN}} = c_{6q}$ and $c \geq 1 + \frac{T}{qN} = c_{4q}$, respectively, the first bullet holds.

For the second bullet, given $T > \frac{N}{3}$, there is $c_{5q} < c_{3q}$. For $c \leq c_{5q}$, there is $\Pi^C_q(S_q) = \Pi^B_q(S_q)$. When $c > c_{5q}$, it is shown in (12) that $\Pi^C_q(S_q)$ equals $\Pi^U_q(S_q)$ and $\Pi^B_q(S_q)$ at the left- and right-hand sides of $\bar{q}$, respectively. Because $S_q$ must satisfy $S_q \leq \min\{T, N_1\}$, it is necessary to check whether $S_q$ is greater than the minimum of $T$ and $N_1$ or not. By the definition of $\bar{q}$, it is known that $\bar{q}$ is always less than $T$. This leads to that when $c > c_{2q}$, $\Pi^C_q(S_q)$ always equals $\Pi^U_q(S_q)$. Moreover, the relationship between $\bar{q}$ and $N_1$ is shown in the first bullet. If $T \leq N_1$, it is obvious $\bar{q} < \min\{T, N_1\}$; otherwise, it is necessary to check whether $c$ is greater than $c_{4q}$ or not to determine whether $\bar{q}$ is greater than $N_1$ or not. According to the definition of $c_{4q}$, there is $c_{5q} < c_{4q} < c_{2q}$. The difference between $c_{4q}$ and $c_{3q}$ is given by

$$c_{4q} - c_{3q} = \frac{4N_1 - (6T - 2N)}{3q(N - N_1)} \begin{cases} \leq 0, & \text{if } N_1 \leq \frac{3T - N}{2} \\ > 0, & \text{if } N_1 > \frac{3T - N}{2}. \end{cases}$$

Under $N_1 \leq \frac{3T - N}{2}$, there is $c_{4q} < c_{5q} < c_{4q} < c_{3q} < c_{2q}$. It is known from the proof of Lemma 5.2 that $\Pi^B_q(S_q)$ is decreasing and concave in $S_q$ as long as $T \leq \frac{N}{2}$. Since $\Pi^C_q(S_q) = \Pi^B_q(S_q)$ for $c \leq c_{5q}$, it is evident that now $\Pi^C_q(S_q)$ can be illustrated by Figure A2(a). When $c_{5q} < c \leq c_{4q}$, there are $0 < \bar{q} < \min\{T, N_1\} = N_1$ and $\Pi^U_q(S_q)$ is shown in Figure A2(b). Further, when $c_{4q} < c \leq c_{3q}$, $\Pi^C_q(S_q)$ must equal $\Pi^U_q(S_q)$ because $\bar{q} < \min\{T, N_1\} = N_1$. Since $\Pi^U_q(N_1) - \Pi^U_q(0) > 0$, and $\Pi^C_q(S_q)$ strictly increases in $S_q$ and $\Pi^C_q(S_q)$ can be illustrated by Figure A2(c). Additionally, when $c > c_{3q}$, there are $\bar{q} > \min\{T, N_1\} = N_1$ and $\Pi^U_q(N_1) - \Pi^U_q(0) < 0$. Hence $\Pi^C_q(S_q)$ is shown by Figure A2(d).

When $N_1 > \frac{3T - N}{2}$, the analysis can be separated into cases: $\frac{3T - N}{2} < N_1 \leq \bar{N}_1$ and $N_1 > \bar{N}_1$. This is due to that the shape and the maximum of $\Pi^B_q(S_q)$ is different when $N_1 \leq \frac{N}{2}$ from when $N_1 > \frac{N}{2}$. For the case of $\frac{3T - N}{2} < N_1 \leq \bar{N}_1$, the analysis is similar as that for $N_1 \leq \frac{3T - N}{2}$ since that the shapes of $\Pi^B_q(S_q)$ in the two cases are the same. In what follows, the focus is on the case of $N_1 > \bar{N}_1$. When $c \leq c_{5q}$, $\Pi^C_q(S_q)$ equals $\Pi^B_q(S_q)$. It is identified in the proof of Lemma 5.2 that $\Pi^B_q(S_q)$ is concave in $S_q$ and is maximized at $S_q = \bar{S}_q$. Therefore, $\Pi^C_q(S_q)$ can be shown by Figure A4(a). Further, when $c > c_{5q}$, the shape of $\Pi^C_q(S_q)$ may be dependent on the relationship between $S_q$ and $\bar{S}_q$. In the first bullet, it is demonstrated that there are $\bar{S}_q < \bar{S}_q$ for $c \leq c_{6q}$ and $\bar{S}_q > \bar{S}_q$ for $c > c_{6q}$. According to the definition of $c_{6q}$, there is $c_{3q} < c_{6q} < c_{2q}$. To derive the shape of $\Pi^B_q(S_q)$, it is also necessary to determine the relationship between $c_{6q}$ and $c_{4q}$. Differentiating $c_{6q}$ with regard to $N_1$ gives

$$\frac{dc_{6q}}{dN_1} = \frac{2(N - T)}{q(N - N_1)^2} > 0.$$

Since $c_{4q} |_{N_1 = 1} = c_{6q}$, it can be inferred that $c_{4q}$ is greater than $c_{6q}$ for all $N_1 > \bar{N}_1$. As a consequence, there must be $c_{5q} < c_{6q} < c_{6q} < c_{4q} < c_{2q}$. For $c_{5q} < c \leq c_{3q}$, there are $0 < \bar{q} < \bar{S}_q$ and $\Pi^U_q(N_1) - \Pi^U_q(0) > 0$.

\textsuperscript{a1} Because $\frac{3T - N}{2} < T$, it follows $N_1 < T$. 
This indicates that $\Pi_q^C(S_q)$ equals $\Pi_q^U(S_q)$ and strictly increases in $S_q \leq \overline{S}_q$ and equals $\Pi_q^B(S_q)$ and strictly concave in $S_q > \overline{S}_q$. Evidently, $\Pi_q^C(S_q)$ is maximized at $\Pi_q^B(\overline{S}_q)$. In other words, $\Pi_q^C(S_q)$ can be illustrated by Figure A4(b). The shape of $\Pi_q^C(S_q)$ for $c_{3q} < c \leq c_{6q}$ is similar as that for $c_{3q} < c \leq c_{3q}$ and the unique difference between them lies in that in the former case, $\Pi_q^C(S_q)$ strictly decreases in $S_q \in [0, \overline{S}_q]$. Therefore, $\Pi_q^C(S_q)$ can be shown by Figure A4(c). Further, for $c_{6q} < c \leq c_{4q}$, $\overline{S}_q$ is greater than $\overline{S}_q$. The shape of $\Pi_q^C(S_q)$ over $S_q \in [0, \overline{S}_q]$ is same as that for $c_{3q} < c \leq c_{6q}$. At the right-hand side of $\overline{S}_q$, $\Pi_q^C(S_q)$ equals $\Pi_q^B(S_q)$ and decreases in $S_q$. Thus, $\Pi_q^C(S_q)$ can be shown by Figure A4(d). Finally, for $c > c_{4q}$, there is $\overline{S}_q > N_1$ that results in that $\Pi_q^C(S_q)$ equals $\Pi_q^U(S_q)$ and strictly decreases in $S_q \in [0, N_1]$. In this case, $\Pi_q^C(S_q)$ can be shown by Figure A4(e). □

**Proposition A3.** Suppose $\frac{N}{r} < T \leq \overline{T}$, $\Pi_q^C(S_q)$ for $N_1 \leq \frac{3T-N}{2r}$, $\frac{3T-N}{2r} < c \leq \overline{N}_1$, and $N > \overline{N}_1$ can be illustrated by Figures A5-A7, respectively.

*Proof.* The proof is similar as that of Proposition A2. □

**Proposition A4.** Suppose $T > \overline{T}$, $\Pi_q^C(S_q)$ can be illustrated by Figure A8.

*Proof.* The proof is similar as that of Proposition A1. □

**Proposition A5.** Given $T \leq \frac{N}{r}$, the seller is optimal to sell the high-quality product only in the spot period when $c \leq 1$, sell the low-quality product only in the spot period when $1 < c \leq c_{2L}$, and not sell any products when $c > c_{2L}$. 
- Case 1: \( c_{3q} < c < c_{4q} \). Since \( c > c_{3q} \) and \( \Pi_q^H(N_1) - \Pi_q^L(0) < 0 \), \( \Pi_q^C(S_H) \) can be illustrated by Figure A1(b). Apparently, \( \Pi_q^C(S_H) \) is maximized at \( S_H = 0 \) and the maximum of \( \Pi_q^C(S_H) \) can be expressed as \( \Pi_q^C(0) \). On the other side, \( c < c_{3q} \) implies that \( \Pi_q^C(S_L) \) is shown by Figure A1(a) and the maximum of \( \Pi_q^C(S_L) \) is \( \Pi_q^B(0) \). Thus, the seller’s optimal decisions on quality choice and advance selling can be derived by comparing \( \Pi_q^C(0) \) and \( \Pi_q^B(0) \).
with $\Pi_L^F(0)$. According to Lemma 4.1 and (6), $\Pi_H^U(0)$ can be written as

$$\Pi_H^U(0) = \frac{N(1 + \tau H)^2}{4}.$$  

(A6)

Using (A6) and subtracting $\Pi_L^F(0)$ from $\Pi_H^U(0)$ lead to

$$\Pi_H^U(0) - \Pi_L^F(0) = \frac{N(1 + \tau H)^2}{4} - \left(\frac{N - T}{N} + \tau L\right)T := f(\tau),$$  

(A7)

where $\tau \in \left[-\frac{1}{H}, -\frac{N - 2T}{NH}\right)$. The second derivative of $f(\tau)$ is $f''(\tau) = \frac{NH^2}{4} > 0$, that is, $f(\tau)$ is convex in $\tau$. Since $f(\tau = -\frac{1}{H}) = (\alpha - \frac{N - T}{N})T < 0^{a2}$ and $f(\tau \uparrow -\frac{N - 2T}{NH}) = \frac{T(2T - N)(H - L)}{NH} < 0$, it follows that $f(\tau) < 0$ for

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\(^{a2}\)In the current case, there is $c_2 H < c_5 L$. In other words, $\alpha$ satisfies $\alpha < \frac{N - 2T}{N}$. Therefore, there is $f(\tau = -\frac{1}{H}) < 0$.  

Figure A7. Sketch of $\Pi_H^U(S_q)$ for $\frac{N}{2} < T \leq T$ and $N_1 > \overline{N}_1$.  

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all $c \in \left[ \frac{1}{H}, \frac{N-2T}{NH} \right)$. Together with $\Pi^P_H(0) > 0$, it is evident that the seller is optimal to offer the low-quality product and sell only in the spot period.

- $c_{2H} < c \leq c_{5L}$. In this case, $\Pi^U_H(N_1) - \Pi^U_H(0)$ is negative and $\Pi^C_H(S_H)$ is shown by Figure A1(a). It can be found that $\Pi^C_H(S_H)$ equals $\Pi^U_H(S_H)$ and is maximized at $S_H = 0$. On the other hand, for the low-quality product, $\Pi^C_L(S_L)$ is shown by Figure A1(a) and is maximized at $S_L = 0$. Now the seller’s optimal policy is dependent on which one of $\Pi^U_H(0)$ and $\Pi^P_L(0)$ is greater than the other. Given $c > c_{2H}$, we know $p^U_H = H + 1$ from Proposition 1. Accordingly, there is $\Pi^U_H(0) = 0$. For the low-quality product, there is $\Pi^P_L(0) > 0$, where the inequality is due to $c \leq c_{5L}$. As a consequence, the seller is optimal to offer the low-quality product and sell only in the spot period.

- $c_{5L} < c \leq c_{2L}$. In this case, the analysis for the high-quality product is same as in the case of $c_{2H} < c \leq c_{5L}$. For the low-quality product, conditions $c_{5L} < c \leq c_{2L}$ and $\Pi^U_L(N_1) - \Pi^U_L(0) < 0$ indicate that $\Pi^C_L(S_L)$ is illustrated by Figure A1(b). The maximum of $\Pi^C_L(S_L)$ is $\Pi^C_L(0) = \frac{N(1+qL)}{4} \geq 0$, where the equality holds only when $c = c_{2L}$. Thus, the seller is optimal to offer the low-quality product and sell only in the spot period.

- $c > c_{2L}$. For $q \in \{H, L\}$, $\Pi^C_q(N_1) - \Pi^C_q(0) < 0$ and $\Pi^C_q(S_q)$ is shown by Figure A1(c). Hence the seller’s optimal decisions can be determined by
comparing $\Pi'_H(0)$ with $\Pi'_L(0)$. Given $p_q^U = q + 1$, there is $\Pi'_q(0) = 0$. It can be concluded that the seller is optimal to not offer any products.

- Case 2: $c_5H < c_5L < c_{2H} < c_{2L}$.
  - $c < c_5H$. The analysis is same as in Case 1.
  - $c_5H < c < c_5L$. Now $\Pi'_H(S_H)$ and $\Pi'_L(S_L)$ are shown by Figures A1(b) and A1(a), respectively. It is obvious that there is $\max_S\Pi'_H(S_H) = \Pi'_H(0)$ and $\max_S\Pi'_L(S_L) = \Pi'_L(0)$. The remainder proof is similar as when $c_5H < c < c_2L$ in Case 1. In the current case, $c_5H < c < c_5L$ indicates $\tau \in \left[ -\frac{N-2T}{NH}, -\frac{N-2T}{NL} \right]$. By using (A7), there is $f(\tau) = \frac{1}{\alpha}g(\alpha)$, where $g(\alpha) = (2T + N)\alpha + 2T - N$. Evidently, $g(\alpha)$ strictly increases in $\alpha$. Further, $c_5L < c_2H$ implies $\alpha > \frac{N-2T}{N}$. Since $g(\alpha = \frac{N-2T}{N}) = \frac{2T(N-2T)}{N} > 0$, it follows $g(\alpha) > 0$ for all $\alpha > \frac{N-2T}{N}$. Therefore, there is $f(\tau) > 0$ and $f(\tau) < 0$. Due to $f''(\tau) > 0$ and $f(\tau) < 0$, it can be inferred that $f(\tau) < 0$ for $\tau \in \left[ -\frac{N-2T}{NH}, -\frac{N-2T}{NL} \right]$. Thus, the seller is optimal to offer the low-quality product only in the spot period.
  - $c_5L < c < c_{2H}$. In this case, both $\Pi'_H(S_H)$ and $\Pi'_L(S_L)$ can be illustrated by Figure A1(b). Given $c_5L < c < c_{2H}$, there would be $\max_S\Pi'_H(S_H) = \Pi'_H(0) = \frac{N}{4}(1+\pi H)^2 < \frac{N}{4}(1+\pi L)^2 = \Pi'_L(0) = \max_S\Pi'_L(S_L)$. As a result, the seller is optimal to offer the low-quality product only in the spot period.
  - $c_2H < c < c_{2L}$. The analysis is nearly same as when $c_5L < c < c_{2L}$ in Case 1.
  - $c > c_{2L}$. The analysis is same as in Case 1.

Proposition A5 is obtained by classifying the above discussions.

**Proposition A6.** Denote $c_{TL} = 1 + \sqrt{\frac{2(3T-N)-2(2T-N)}{2NL}}$. Given $\frac{N}{3} < T \leq \overline{T}$, the seller’s optimal decisions are defined according to the value of $N_1$ as follows.

- $N_1 \leq \frac{3T-N}{2}$. The seller is optimal to offer the high-quality product with full advance selling when $c \leq 1$, offer the low-quality product with full advance selling when $1 < c \leq c_3L$, offer the low-quality product only in the spot period when $c_3L < c \leq c_{2L}$, and not offer any products when $c > c_{2L}$.

- $\frac{3T-N}{2} < N_1 < \overline{N}_1$. The seller is optimal to offer the high-quality product with full advance selling when $c \leq 1$, offer the low-quality product with full advance selling when $1 < c \leq c_4L$, offer the low-quality product only in the spot period when $c_4L < c \leq c_{2L}$, and not offer any products when $c > c_{2L}$.

- $N_1 > \overline{N}_1$. The seller is optimal to offer the high-quality product with limited advance selling of $\overline{S}_H$ units when $c \leq 1$, offer the low-quality product with limited advance selling of $\overline{S}_L$ units when $1 < c \leq c_{TL}$, offer the low-quality product only in the spot period when $c_{TL} < c < c_{2L}$, and not offer any products when $c > c_{2L}$.

**Proof.** The proof is similar as that of Proposition A5.

**Proposition A7.** Given $T > \overline{T}$, the seller is optimal to offer the high-quality product with full advance selling when $c \leq 1$, offer the low-quality product with full advance selling when $1 < c \leq c_3L$, offer the low-quality product only in the spot period when $c_3L < c \leq c_{2L}$, and not offer any products when $c > c_{2L}$.

**Proof.** The proof is similar as that of Proposition A5.
Proof of Proposition 2. Proposition 2 is obtained by summarizing Propositions A5-A7.

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