Modified Jordan-Brans-Dicke theory with scalar current and the Eddington-Robertson $\gamma$-parameter

J. W. Moffat*† and V. T. Toth*

*Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2L 2Y5, Canada
†Department of Physics, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada

The Jordan-Brans-Dicke theory of gravitation, which promotes the gravitational constant to a dynamical scalar field, predicts a value for the Eddington-Robertson post-Newtonian parameter $\gamma$ that is significantly different from the general relativistic value of unity. This contradicts precision solar system measurements that tightly constrain $\gamma$ around 1. We consider a modification of the theory, in which the scalar field is sourced explicitly by matter. We find that this leads to a modified expression for the $\gamma$-parameter. In particular, a specific choice of the scalar current yields $\gamma = 1$, just as in general relativity, while the weak equivalence principle is also satisfied. This result has important implications for theories that mimic Jordan-Brans-Dicke theory in the post-Newtonian limit in the solar system, including our scalar-tensor-vector modified gravity theory (MOG).

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Jordan-Brans-Dicke theory [1, 2] is a theory of gravitation in which the gravitational constant $G$ is replaced with the inverse of a dynamical scalar field $\phi$. It can be demonstrated by straightforward derivation that this scalar field is effectively sourced by the curvature of space-time (see, e.g., [3]). There is, however, no scalar current: in the Lagrangian formulation, the variation of matter fields with respect to the scalar field is assumed to be zero.

Jordan-Brans-Dicke theory runs into severe observational constraints within the solar system. Notably, the theory predicts that the value of the post-Newtonian $\gamma$-parameter, first introduced by Eddington [4] and Robertson [5] and also Schiff [6], and effectively measuring the amount of spatial curvature produced by unit rest mass, will deviate from the standard general relativistic value of 1. Instead, its value will be $\gamma = (\omega + 1)/(\omega + 2)$ [2], where $\omega$ is the dimensionless coupling constant of the dynamical field. Constraints established by precision measurements of the Cassini spacecraft [7] require the uncomfortably large value of $|\omega| > 4 \times 10^4$.

Nonetheless, there is no a priori reason to exclude the possibility of a scalar current. A phenomenological matter Lagrangian could be constructed such that it depends explicitly on $G = \phi^{-1}$. The variation of such a Lagrangian with respect to $\phi$ would be non-zero, introducing a scalar current into the field equations. To demonstrate this, we write the scalar theory Lagrangian as follows:

$$\mathcal{L} = \frac{1}{16\pi} [(R - 2\Lambda)\phi + f(\phi, g^{\mu\nu}\partial_\mu\partial_\nu\phi)] \sqrt{-g} + \mathcal{L}_{O.F.},$$  \hspace{1cm} (1)

where $R$ is the Ricci scalar constructed from the metric $g_{\mu\nu}$, $g$ is the metric determinant, $\Lambda$ is the cosmological constant, $\phi$ is a scalar field, $f$ is an arbitrary function, and O.F. stands for terms that represent other fields, which, we assume, depend only on $\phi$, not on its derivatives. We set $c = 1$, use the $(+, -, -, -)$ metric signature, and define the Ricci tensor as $R_{\mu\nu} = \partial_\kappa\Gamma^\kappa_{\mu\nu} - \partial_\nu\Gamma^\kappa_{\mu\kappa} + \Gamma^\kappa_{\mu\lambda}\Gamma^\lambda_{\nu\kappa} - \Gamma^\kappa_{\nu\lambda}\Gamma^\lambda_{\mu\kappa}$, where the $\Gamma$ are the usual Christoffel-symbols.

The field equations of the theory are the Euler-Lagrange equations corresponding to (1):

$$\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - \frac{\partial}{\partial \phi}\frac{\partial \mathcal{L}}{\partial (g^{\mu\nu})_{\phi}^{\phi}} + \frac{\partial}{\partial \phi}\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}_{\phi}} = 0,$$  \hspace{1cm} (2)

$$\frac{\partial \mathcal{L}}{\partial \phi} - \nabla_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} = 0,$$  \hspace{1cm} (3)

where $\nabla_\mu$ is the covariant derivative with respect to $x^\mu$. These equations can be recast in the form,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda + \frac{1}{\sqrt{-g}} \frac{1}{\phi} \frac{\partial f \sqrt{-g}}{\partial g^{\mu\nu}} - \frac{\partial R}{\partial g^{\mu\nu}} \frac{\partial \phi}{\phi} + \frac{2}{\sqrt{-g}} \frac{\partial}{\partial \phi} \left( \sqrt{-g} g^{\mu\nu}_{\kappa\lambda} \frac{\partial R}{\partial g^{\mu\nu}_{\kappa\lambda}} \frac{\partial \phi}{\phi} \right) = \frac{8\pi}{\phi} T_{\mu\nu},$$  \hspace{1cm} (4)

$$R - 2\Lambda + \frac{\partial f}{\partial \phi} - \nabla_\mu \left( \frac{\partial f}{\partial (\partial_\mu \phi)} \right) = 16\pi J,$$  \hspace{1cm} (5)

1 The other Eddington-Robertson parameter, $\beta$, is identically 1 in Jordan-Brans-Dicke theory, just as in general relativity.
where $T_{\mu\nu} = -(2/\sqrt{-g})\partial L_{O.F.}/\partial g^{\mu\nu}$ and $J = -(1/\sqrt{-g})\partial L_{O.F.}/\partial \phi$. The existence of a non-zero variation of matter fields with respect to $\phi$ represents a significant generalization of the archetypal scalar field theory of Jordan, Brans and Dicke.

Equation (4) can be rewritten using covariant derivatives, yielding

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda + \frac{1}{\sqrt{-g}} \frac{1}{\phi} \left( \frac{\partial f}{\partial \phi} \right) \frac{\nabla \Lambda \phi}{\phi} = \frac{8\pi}{\phi} T_{\mu\nu},$$

which, apart from the presence of $T_{\mu\nu}$, are the equations of Jordan-Brans-Dicke theory in the standard form. To the first order, terms quadratic in derivatives vanish; the second derivative in (12) can, in turn, be eliminated by a suitable gauge choice (for a thorough derivation, see Appendix A of [8]). In the post-Newtonian metric [9], the $\gamma$-parameter can be read off as the ratio of the $ii$ and 00 components of (12). In the absence of a cosmological term, $\Lambda = 0$, we get

$$\gamma = \frac{(\omega + 1)T - \phi J}{(\omega + 2)T + \phi J}.$$  

If the scalar current vanishes ($J = 0$), we get back the usual post-Newtonian result for Jordan-Brans-Dicke theory:

$$\gamma = \frac{\omega + 1}{\omega + 2}.$$  

This result is frequently cited as a reason for rejecting Jordan-Brans-Dicke theory within the solar system, as precision measurements by the Cassini spacecraft yielding $\gamma = 1 = (2.1 \pm 2.3) \times 10^{-5}$, for instance, are consistent with the theory only if $|\omega| \gtrsim 4 \times 10^{15}$ [11]. However, if a scalar current is present, the situation changes. Specifically, we can choose a scalar current in the form

$$\phi J = -\frac{1}{2} T,$$

which is equivalent to

$$-\phi \frac{1}{\sqrt{-g}} \frac{\partial L_{O.F.}}{\partial \phi} = \frac{1}{\sqrt{-g}} \frac{\partial L_{O.F.}}{\partial g^{\mu\nu}} g^{\mu\nu}.$$
This choice can be made, in part, because \( J \) is not a conserved quantity, just as \( T \) is not conserved. In this case, equations (12) and (13) read

\[
R_{\mu\nu} = \frac{8\pi}{\phi} \left( T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} + \frac{1}{4\pi} \frac{\omega + 1}{2\omega + 3} \phi \Lambda g_{\mu\nu} \right) + \frac{\omega}{\phi^2} \frac{\partial \phi}{\partial \phi} + \frac{\nabla_\mu \nabla_\nu \phi}{\phi},
\]

(18)

\[
\nabla_\mu \nabla^\mu \phi = -\frac{2\phi \Lambda}{2\omega + 3}.
\]

(19)

Considering the trace of the bracketed term in Eq. (18), if

\[
|\Lambda| \ll \left| \frac{2\omega + 3}{\omega + 1} \phi^{-1} T \right|,
\]

(20)

the general relativistic result that is also consistent with solar system data,

\[
\gamma \approx 1,
\]

(21)

is easily satisfied.

The result (15) has been used as an argument against theories that, within the solar system, yield the same solution as Jordan-Brans-Dicke theory to the first post-Newtonian order. We mention in particular our scalar-tensor-vector (STVG) modified gravity theory (MOG) \([10, 11]\), which, according to an extensive analysis by Deng, et al. \([8]\), shows the same behavior in the solar system as Jordan-Brans-Dicke theory. This problem is avoided by a suitable choice of \( J \) yielding (21), as demonstrated above.

Nonetheless, we note that in the case of \( J \neq 0 \), the theory is no longer a metric theory: material particles carry a scalar charge and no longer move along geodesics. To determine the equations of motion for a test particle, we use a test particle Lagrangian in the form

\[
L_{TP} = -m \sqrt{g_{\mu\nu} u^\mu u^\nu} - q\phi,
\]

(22)

where \( q \) is the scalar charge associated with a particle of mass \( m \), moving with four-velocity \( u^\mu = dx^\mu/d\tau \) and \( \tau \) is the proper time along the particle’s world line. Integration of (16) over a three-volume encompassing a test particle gives

\[
q = -\frac{1}{2} \phi^{-1} m,
\]

(23)

and \( \frac{1}{2} \phi^{-1} m \simeq \frac{1}{2} G_N m \) at the present epoch (\( G_N \) is Newton’s constant of gravitation.) The equation of motion obtained by varying (22) contains an extra term when compared to the standard geodesic equation of motion:

\[
m \left( \frac{d^2 x^\kappa}{d\tau^2} + \Gamma^\kappa_{\mu\nu} u^\mu u^\nu \right) - q g^\kappa\lambda \frac{\partial \phi}{\partial x^\lambda} = 0.
\]

(24)

Given (23), we obtain

\[
m \left( \frac{d^2 x^\kappa}{d\tau^2} + \Gamma^\kappa_{\mu\nu} u^\mu u^\nu \right) + mg^\kappa\lambda \frac{1}{2\phi} \frac{\partial \phi}{\partial x^\lambda} = 0.
\]

(25)

We observe that \( m \) cancels out in the equation of motion, hence the theory satisfies the weak equivalence principle.

Finally, we note that equation (19) can be rewritten in the familiar form

\[
(\Box + \mu^2)\phi = 0,
\]

(26)

with \( \Box = \nabla_\mu \nabla^\mu \) and \( \mu \) given by

\[
\mu^2 = \frac{2\Lambda}{2\omega + 3}.
\]

(27)

This last term can be interpreted as the mass \( \mu \) of the scalar field \( \phi \). Using \( \Lambda \simeq 1.2 \times 10^{-52} \text{ m}^{-2} \), we obtain the mass of an ultralight scalar field, \( \mu \simeq 3.9\sqrt{2/(2\omega + 3)} \times 10^{-69} \text{ kg} \).

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[1] C. Brans and R. H. Dicke. Mach’s Principle and a Relativistic Theory of Gravitation. *Phys. Rev.*, 124(3):925–935, 1962.
[2] P. Jordan. *Schwerkraft und Weltall, Grundlagen der Theoretische Kosmologie*. Vieweg und Sohn, Braunschweig, 1955.
[3] S. Weinberg. *Gravitation and Cosmology*. John Wiley & Sons, 1972.
[4] A. S. Eddington. *The Mathematical Theory of Relativity*. Cambridge University Press, 1957.
[5] H. P. Robertson. Note on the preceding paper: The two body problem in general relativity. *Ann. Math.*, 39(1):101–104, January 1938.
[6] L. I. Schiff. On experimental tests of the general theory of relativity. *Amer. J. Phys.*, 28(4):340–343, 1960.
[7] B. Bertotti, L. Iess, and P. Tortora. A test of general relativity using radio links with the Cassini spacecraft. *Nature (London)*, 425:374–376, September 2003.
[8] X.-M. Deng, Y. Xie, and T.-Y. Huang. Modified scalar-tensor-vector gravity theory and the constraint on its parameters. *Phys. Rev. D*, 79(4):044014–+, February 2009.
[9] C. M. Will. *Theory and Experiment in Gravitational Physics*. Cambridge University Press, 1993.
[10] J. W. Moffat. Scalar-tensor-vector gravity theory. *Journal of Cosmology and Astroparticle Physics*, 2006(03):004, 2006.
[11] J. W. Moffat and V. T. Toth. Fundamental parameter-free solutions in Modified Gravity. *Class. Quant. Grav.*, 26:085002, 2009.