Partition Functions of $\mathcal{N} = (2, 2)$ Gauge Theories on $S^2$ and Vortices

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Abstract: We apply localization techniques to compute the partition function of a two-
dimensional $\mathcal{N} = (2, 2)$ R-symmetric theory of vector and chiral multiplets on $S^2$. The
path integral reduces to a sum over topological sectors of a matrix integral over the Cartan
subalgebra of the gauge group. For gauge theories which would be completely Higgsed
in the presence of a Fayet–Iliopoulos term in flat space, the path integral alternatively
reduces to the product of a vortex times an antivortex partition functions, weighted
by semiclassical factors and summed over isolated points on the Higgs branch. For
applications, we evaluate the partition function for some $U(N)$ gauge theories, showing
equality of the path integrals for theories conjectured to be dual by Hori and Tong and
deriving new expressions for vortex partition functions.

1. Introduction

Supersymmetric theories are attractive theoretical laboratories allowing the exact com-
putation of many observables irrespective of the strength of interactions. A recent line
of development exploits the power of localization techniques, initiated in [1,2], which
in many cases make it possible to exactly evaluate the path integral of a supersymmetric
quantum field theory placed on a compact manifold, possibly with the insertion of local
or non-local operators that respect some supersymmetry. For instance [3] computed the
path integral of an $\mathcal{N} = 2$ gauge theory on $S^4$; [4–6] of an $\mathcal{N} = 3$ and $\mathcal{N} = 2$ theory on
$S^3$, and [7] of an $\mathcal{N} = 1$ theory on $S^5$. We stress that in all these cases the gauge theories
are not topologically twisted. Other related works are [8–15] where different compact
manifolds are considered or various operators are inserted. Many of these papers build
on previous work [16,17] and especially on the $\Omega$-background [18,19] which in a sense
is another example of this philosophy: even though the manifold on which the field
theory is defined is non-compact, the $\Omega$-deformation effectively compactifies it.

We study the Euclidean path integral $Z_{S^2}$ of a two-dimensional $\mathcal{N} = (2, 2)$ theory of
vector multiplets and chiral multiplets, placed on the round sphere $S^2$. The $\mathcal{N} = (2, 2)$
supersymmetry algebra on $S^2$ is $\mathfrak{osp}^+(2|2) \cong \mathfrak{su}(2|1)$, whose bosonic subalgebra has an $\mathfrak{su}(2)$ factor of rotations of the sphere and a $u(1)$ R-symmetry factor reducing to the vector-like R-symmetry of the theory on $\mathbb{R}^2$ in the large radius limit. In particular the R-symmetry is part of the algebra, rather than being an outer automorphism of it. When placing the theory on $S^2$ one has the freedom of selecting the R-symmetry, among a family of choices differing by mixing with non-R Abelian symmetries. This choice determines some specific couplings in the Lagrangian. When the theory is actually conformal, the supersymmetry algebra enhances to the superconformal algebra which coincides with the one on flat space. Similar properties have been observed in three, four and five dimensions \[3,5,7,20\].

The localization technique is based on the observation \[1,2\] that in certain situations the path integral can be exactly equal to its semiclassical approximation. Given a supercharge $Q$ that squares to a bosonic symmetry of the theory, the path integral only gets contributions from classical configurations that are fixed points of $Q$ and from small quadratic fluctuations around them. The fixed point set is often discrete or of finite dimension, while the contribution from small fluctuations is in terms of a “one-loop determinant” which is easy to compute. Therefore, in these fortunate situations one is actually able to exactly compute the path integral. Local or non-local $Q$-invariant operators can be included in the path integral as well: in this case one is able to exactly compute their expectation value (VEV).

When localization is at work, the path integral is only sensitive to the cohomology of $Q$: $Q$-exact operators do not affect the integral. In the case of a two-dimensional $\mathcal{N} = (2, 2)$ theory of vector and chiral multiplets on $S^2$, this implies that the path integral does not depend on the gauge coupling $g$ nor on the parameters of the superpotential $W$ (it does depend on the constraints on R-charges), while it depends on the twisted superpotential $\tilde{W}$—including Fayet–Iliopoulos (FI) parameters and theta-angles—and on twisted masses.

We reduce the path integral $Z_{S^2}$ to a matrix integral (63) on the Cartan subalgebra of the gauge group, that we call “localization on the Coulomb branch”, very similar to the three-dimensional $\mathcal{N} = 2$ case \[4–6\] except for a sum over topological sectors which do not appear on 3-spheres.

Motivated by the work of \[21\] where it was observed in two three-dimensional examples that the path integral on the ellipsoid can be rewritten as a sum of products of a vortex and an antivortex partition function, weighted by semiclassical factors, we wonder whether a similar phenomenon takes place in our setup and look for a different way of performing localization. A physical way of understanding localization is to add to the Lagrangian a $Q$-exact term multiplied by a coupling $t$ that we take to infinity. In that limit the theory becomes semiclassical and the path integral is dominated by the saddle-points of the $Q$-exact term; independence of the result on $t$ completes the argument. We find that—for theories that in flat space are completely Higgsed when a FI term is turned on—an alternative choice of the deformation Lagrangian and of the path integration contour of auxiliary fields forces the path integral to localize on a finite sum of points on the Higgs branch. We call this “localization on the Higgs branch”.

For each isolated point on the Higgs branch, the path integral gets contributions from point-like vortices at the north pole and antivortices at the south pole of $S^2$. Close to the poles the supercharge $Q$ and the action take the same form as the theory on the so-called $\Omega$-background \[18,19\] studied in two dimensions by Shadchin \[22\]. Therefore $Z_{S^2}$ can be written as (95) in terms of the vortex partition function $Z_{\text{vortex}}$. Localization tells us that such expression must agree with the matrix integral discussed before. In fact the
expression on the Higgs branch is reminiscent of the $S^4$ case [3], where the instanton partition function $Z_{\text{inst}}$ of Nekrasov [18] appears, although in that case $Z_{\text{inst}}$ is in the integrand of a matrix integral, while in our case $Z_{\text{vortex}}$ is related to a matrix integral.

We compute $Z_{S^2}$ for a $U(N)$ gauge theory with $N_f$ chiral multiplets in the fundamental representation, $N_a$ antifundamentals and possibly one adjoint, and explicitly check that the result of the matrix integral coincides with the expression in terms of $Z_{\text{vortex}}$ (when the latter is known). This suggests that the method can be used to compute $Z_{\text{vortex}}$ in the presence of matter representations for which other means are not available. Already our most general expression has not appeared in the literature before.

Finally, motivated by a duality conjectured by Hori and Tong [23,24] between $SU(N)$ gauge theory with $N_f$ fundamentals (and $N_f > N$) and $SU(N_f - N)$ with the same number of fundamentals, we show that $Z_{S^2}$ coincide for the following pairs:

$$Z_{S^2}^{U(N), N_f}(\xi, m) = Z_{S^2}^{U(N_f-N), N_f}(\xi, -m) \quad \text{for } N_f > N$$

(1)

as functions of the FI parameter $\xi$ and the twisted masses $m$, and

$$Z_{S^2}^{SU(N), N_f}(b, m) = Z_{S^2}^{SU(N_f-N), N_f}(b, -m) \quad \text{for } N_f > N$$

(2)

as functions of the twisted mass $b$ associated to the baryon number and those $m$ associated to the flavor symmetry $SU(N_f)$. In the unitary case we extend the equality of partition functions to the case with antifundamentals.

The paper is organized as follows. In Sect. 2 we discuss supersymmetry on $S^2$ and the supersymmetric actions for vector and chiral multiplets that we will consider in the paper. In Sect. 3 we use localization to reduce the path integral to a matrix integral, summed over topological sectors, while in Sect. 4 we follow a different route and obtain a sum over Higgs branch points including point-like vortices and antivortices. In Sect. 5 we perform the matrix integral computation for $U(N)$ with $(N_f, N_a)$ flavors and possibly an adjoint chiral multiplet, reproducing the expression in terms of the vortex partition function. In Sect. 6 we prove the identity of $Z_{S^2}$ for some pairs of gauge theories. We conclude in Sect. 7 with some comments and open questions. Tedious computations are in the appendices.

A longer version of this paper is available online [25], where we present more examples, some consistency checks with mirror symmetry, and a comparison with the three-dimensional matrix integral for lens spaces $L(p, 1)$ [13,15] in the $p \to \infty$ limit, and with the three-dimensional integral for the index [8,10] in the limit where the radius of $S^1$ shrinks.

### 2. Supersymmetry on a Curved Manifold

We start by constructing an $\mathcal{N} = (2, 2)$ supersymmetric Euclidean gauge theory on a curved two-manifold. Our conventions for spinors follow [6]. We use complex two-dimensional anticommuting Dirac spinors, with multiplication given by

$$\epsilon \lambda \equiv \epsilon^\alpha C_{\alpha \beta} \lambda^\beta = \lambda \epsilon$$

(3)

where $C_{\alpha \beta} \equiv -i \epsilon_{\alpha \beta}$ is the antisymmetric charge conjugation matrix, and $\epsilon_{12} = -\epsilon_{21} = 1$ is the antisymmetric tensor. The gamma matrices $\gamma_{a=1,2}$ and the chirality matrix
$\gamma_3 \equiv -i\gamma_1\gamma_2$ are Pauli matrices—where we use $a, b = 1, 2$ for flat indices, $i, j = 1, 2, 3$ for flat indices including the chirality matrix, $\mu, \nu$ for curved indices—such that

$$\epsilon \gamma_i \lambda = -\lambda \gamma_i \epsilon, \quad \epsilon \gamma_{ij} \lambda = -\lambda \gamma_{ij} \epsilon$$

with $\gamma_{ij} = \gamma[i \gamma_j]$. They follow from the defining properties of the charge conjugation matrix:

$$C \gamma_i C = -\gamma_i^T, \quad C^2 = 1.$$ (5)

Gamma matrices also satisfy $\gamma_3 \gamma^a = i \epsilon^{ab} \gamma_b$. Later on we will consider commuting spinors as well, and we will need to contract spinors without the matrix $C$ implicit: in that case we will specify either a dagger or a transpose. For instance: $\epsilon \lambda = \epsilon^T C \lambda$. The covariant derivative is $D_\mu = \nabla_\mu - iA_\mu$.

The 2d $\mathcal{N} = (2, 2)$ supersymmetry is the dimensional reduction of the 4d $\mathcal{N} = 1$ or the 3d $\mathcal{N} = 2$ supersymmetry. In Euclidean signature the minimal spinor is a complex Weyl spinor, therefore $\mathcal{N} = (2, 2)$ supersymmetry has two Dirac spinor parameters $\epsilon, \bar{\epsilon}$. The algebra is acted upon by a $U(1)_R \times U(1)_A$ R-symmetry outer automorphism, where the action of $U(1)_R$ is vector-like on fermions while $U(1)_A$ is axial. Notice that all flavor symmetries commuting with the supercharges are necessarily vector-like.

The global conformal group 1 in two Lorentzian dimensions is $SO(2, 2) \cong SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$, and the $\mathcal{N} = (2, 2)$ global superconformal group is the complexification $OSp(2|2)$ whose bosonic subgroup is indeed $SL(2, \mathbb{R})^2 \times U(1)^2$ and it includes the R-symmetries. In Euclidean signature the global conformal group is $SO(3, 1) \cong SL(2, \mathbb{C})$, and the $\mathcal{N} = (2, 2)$ global superconformal group is the complexification $OSp(2|2, \mathbb{C})$ whose bosonic subgroup is $SL(2, \mathbb{C}) \times U(1)^2$. Let us explicitly construct vector multiplet and chiral multiplet representations of the $OSp(2|2, \mathbb{C})$ supergroup.

The gauge multiplet $V$ includes a vector $A_\mu$, two real scalars $\sigma$ and $\eta$ (that could be paired into a complex scalar $\hat{\sigma} = \sigma + i\eta$), a Dirac fermion $\lambda$ and a real auxiliary scalar $D$, all in the adjoint representation of the gauge group $G$. In Euclidean signature all fields get complexified and we will consider $\hat{\lambda}$ as a Dirac spinor independent of $\lambda$.

We consider the following supersymmetry variations:

$$\delta A_\mu = -i(\tilde{\epsilon} \gamma_\mu \lambda - \lambda \gamma_\mu \epsilon), \quad \delta \sigma = \frac{1}{2}(\tilde{\epsilon} \lambda - \bar{\lambda} \epsilon) \quad \delta \eta = -i(\tilde{\epsilon} \gamma_3 \lambda - \bar{\lambda} \gamma_3 \epsilon)$$

$$\delta \lambda = i\gamma_3 \epsilon F_{12} - D \epsilon + i\gamma^\mu \epsilon D_\mu \sigma - \gamma_3 \gamma^\mu \epsilon D_\mu \eta - \gamma_3 \epsilon [\sigma, \eta] + (i\sigma - \gamma_3 \eta)\gamma^\mu D_\mu \epsilon$$

$$\delta \bar{\lambda} = i\gamma_3 \bar{\epsilon} F_{12} + D \bar{\epsilon} - i\gamma^\mu \bar{\epsilon} D_\mu \sigma - \gamma_3 \gamma^\mu \bar{\epsilon} D_\mu \eta + \gamma_3 \bar{\epsilon} [\sigma, \eta] - (i\sigma + \gamma_3 \eta)\gamma^\mu D_\mu \bar{\epsilon}$$

$$\delta D = -i(\tilde{\epsilon} \gamma^\mu \lambda \bar{D} \mu \epsilon - i\frac{1}{2} \tilde{D}_\mu \lambda \gamma^\mu \epsilon + i\frac{1}{2} [\tilde{\epsilon} \lambda, \sigma] + i\frac{1}{2} [\bar{\lambda} \epsilon, \sigma] + i\frac{1}{2} [\tilde{\epsilon} \gamma_3 \lambda, \eta] + i\frac{1}{2} [\bar{\lambda} \gamma_3 \epsilon, \eta])$$

$$\delta D = -i(\tilde{D}_\mu \epsilon \gamma^\mu \lambda + \bar{\lambda} \gamma^\mu D_\mu \epsilon)$$ (6)

where $F_{12} = \frac{1}{4} F_{\mu \nu} \epsilon^{\mu \nu}$. For constant $\epsilon, \bar{\epsilon}$ they are the dimensional reduction of the 3d $\mathcal{N} = 2$ superalgebra with $\eta$ being the transverse component $A_3$.

Imposing the Killing spinor equation:

$$D_\mu \epsilon = \gamma_\mu \bar{\epsilon}, \quad D_\mu \bar{\epsilon} = \gamma_\mu \epsilon$$ (7)

1 To be more precise, the statements should be referred to the algebras. We follow the conventions of [26].
for some spinors $\tilde{\epsilon}, \bar{\epsilon}$, one can compute the following commutators:

\[
[\delta_{\epsilon}, \delta_{\bar{\epsilon}}] A_\mu = (\mathcal{L}^A_\xi A)_\mu + D_\mu A
\]

\[
[\delta_{\epsilon}, \delta_{\bar{\epsilon}}] \sigma = \mathcal{L}^A_\xi \sigma + i[A, \sigma] + \rho \sigma + 2\beta \eta
\]

\[
[\delta_{\epsilon}, \delta_{\bar{\epsilon}}] \eta = \mathcal{L}^A_\xi \eta + i[A, \eta] + \rho \eta - 2\beta \sigma
\]

\[
[\delta_{\epsilon}, \delta_{\bar{\epsilon}}] \lambda = \mathcal{L}^A_\xi \lambda + i[A, \lambda] + \frac{3}{2} \rho \lambda - i\alpha \lambda + i\beta \gamma_3 \lambda
\]

\[
[\delta_{\epsilon}, \delta_{\bar{\epsilon}}] \bar{\lambda} = \mathcal{L}^A_\xi \bar{\lambda} + i[A, \bar{\lambda}] + \frac{3}{2} \rho \bar{\lambda} + i\alpha \bar{\lambda} + i\beta \gamma_3 \bar{\lambda}
\]

\[
[\delta_{\epsilon}, \delta_{\bar{\epsilon}}] D = \mathcal{L}^A_\xi D + i[A, D] + 2\rho D
\]

\[
+ (\bar{\epsilon} D_\mu D^\mu \epsilon - D_\mu D^\mu \bar{\epsilon} \bar{\epsilon}) \sigma - i(\bar{\epsilon} \gamma_3 D_\mu \epsilon D^\mu \epsilon - D_\mu D^\mu \bar{\epsilon} \gamma_3 \bar{\epsilon}) \eta
\]

where we have split $\delta = \delta_{\epsilon} + \delta_{\bar{\epsilon}}$ and the parameters are defined as:

\[
\xi_\mu = i\bar{\epsilon} \gamma_\mu \epsilon \quad \Lambda = \bar{\epsilon} \epsilon \sigma - i\bar{\epsilon} \gamma_3 \epsilon \eta
\]

\[
\alpha = -\frac{1}{4} (D_\mu \bar{\epsilon} \gamma_\mu \epsilon - \bar{\epsilon} \gamma_\mu D_\mu \epsilon)
\]

\[
\rho = \frac{i}{2} (D_\mu \bar{\epsilon} \gamma_\mu \epsilon + \bar{\epsilon} \gamma_\mu D_\mu \epsilon) = \frac{1}{2} D_\mu \xi^\mu
\]

\[
\beta = \frac{1}{4} (D_\mu \bar{\epsilon} \gamma_3 \gamma_\mu \epsilon - \bar{\epsilon} \gamma_3 \gamma_\mu D_\mu \epsilon)
\]

The operator $\mathcal{L}^A_\xi$ is the gauge-covariant Lie derivative along the vector field $\xi^\mu$, acting on scalars, spinors and connection 1-forms as\(^2\)

\[
\mathcal{L}^A_\xi \sigma = \xi^\mu (\partial_\mu - iA_\mu) \sigma
\]

\[
\mathcal{L}^A_\xi \lambda = \xi^\mu (\nabla_\mu - iA_\mu) \lambda + \frac{1}{4} (\nabla_\mu \xi_\nu) \gamma^{\mu\nu} \lambda
\]

\[
\mathcal{L}^A_\xi A = \mathcal{L}^A_\xi A - d^A (\xi A) = \xi^\rho F_{\rho\mu} dx^\mu
\]

respectively. The algebra (8) implies that the commutator $[\delta_{\epsilon}, \delta_{\bar{\epsilon}}]$ is a translation by $\xi_\mu$, a gauge transformation by $\Lambda$, a dilation by $\rho$, a vector-like R-rotation by $\alpha$ and an axial R-rotation by $\beta$. The algebra closes when the second line in the variation of $D$ vanishes, which is achieved by imposing on the Killing spinors the extra conditions:

\[
D_\mu D^\mu \epsilon = h \epsilon, \quad D_\mu D^\mu \bar{\epsilon} = h \bar{\epsilon}
\]

with the same function $h$. We will see that on $S^2$ this is automatically satisfied by the Killing spinors. The remaining commutators are

\[
[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = [\delta_{\bar{\epsilon}_1}, \delta_{\bar{\epsilon}_2}] = 0
\]

The commutators (8), (12) describe the superconformal algebra $\mathfrak{osp}(2|2, \mathbb{C})$. In particular the dimensions of the fields $(A_\mu, \xi, \lambda, \bar{\lambda}, D)$ in the vector multiplet are $(1, 1, 3, 3, 2)$ respectively, the charges under the vector-like R-symmetry are $(0, 0, -1, 1, 0)$ and those under the axial R-symmetry are $(0, -1, 1, 1, 0)$.

\(^2\) The Lie derivative on spinors has been constructed in [27] and, although not manifestly, it is independent of the metric. $A_\mu$ acts in the correct gauge representation, and $d^A = d - iA$.
The chiral multiplet $\Phi$ includes a complex scalar $\phi$, a Dirac spinor $\psi$ and a complex auxiliary scalar $F$, all in some representation $R_\Phi$ of the gauge group $G$. The supersymmetry variations of a chiral multiplet of vector-like R-charge $q$ are:

$$
\delta \phi = \bar{\epsilon} \psi \quad \delta \psi = i \gamma^\mu \epsilon D_\mu \phi + i \epsilon \sigma \phi + \gamma_3 \epsilon \eta \phi + \frac{iq}{2} \gamma^\mu D_\mu \epsilon \phi + \bar{\epsilon} F
$$

$$
\delta \bar{\phi} = \bar{\psi} \epsilon \quad \delta \bar{\psi} = i \gamma^\mu \bar{\epsilon} D_\mu \bar{\phi} + i \bar{\epsilon} \bar{\phi} \sigma - \gamma_3 \bar{\epsilon} \bar{\phi} \eta + \frac{iq}{2} \gamma^\mu D_\mu \bar{\epsilon} \bar{\phi} + \epsilon \bar{F}
$$

$$
\delta F = \epsilon (i \gamma^\mu D_\mu \psi - i \sigma \psi + \gamma_3 \eta \psi - i \lambda \phi) + \frac{iq}{2} D_\mu \epsilon \gamma^\mu \psi
$$

$$
\delta \bar{F} = \bar{\epsilon} (i \gamma^\mu D_\mu \bar{\psi} - i \bar{\psi} \sigma - \gamma_3 \bar{\psi} \eta + i \bar{\phi} \lambda) + \frac{iq}{2} D_\mu \bar{\epsilon} \gamma^\mu \bar{\psi}.
$$

Imposing the Killing spinor equations one finds the commutators $[\delta_\epsilon, \delta_{\bar{\epsilon}}]$, from which one deduces that the dimensions of the fields ($\phi, \bar{\phi}, \psi, \bar{\psi}, F, \bar{F}$) are respectively $(q, q, q+\frac{1}{2}, q+\frac{1}{2}, q+\frac{1}{2}, q+\frac{1}{2})$, the charges under the vector-like R-symmetry are $(q, -q, q - 1, 1 - q, q - 2, 2 - q)$ and those under the axial R-symmetry are $(0, 0, 1, -1, 0, 0)$.

The remaining commutators are $[\delta_{\epsilon_1}, \delta_{\bar{\epsilon}_1}] = [\delta_{\epsilon_2}, \delta_{\bar{\epsilon}_2}] = 0$, as before. The only exceptions are $[\delta_{\epsilon_1}, \delta_{\bar{\epsilon}_2}] F = 2q \epsilon_{[1} D^\mu D\mu \epsilon_{2]} \bar{\phi}$ and $[\delta_{\bar{\epsilon}_1}, \delta_{\epsilon_2}] \bar{F} = 2q \bar{\epsilon}_{[1} D^\mu D\mu \bar{\epsilon}_{2]} \bar{\phi}$: they vanish when the extra condition (11) is imposed on the Killing spinors.

It follows from (13) that if $F$ is the F-term of a neutral chiral multiplet of R-charge $q = 2$, then

$$
\delta F = D_\mu (i \epsilon \gamma^\mu \psi), \quad \delta \bar{F} = D_\mu (i \bar{\epsilon} \gamma^\mu \bar{\psi})
$$

(14)

are total derivatives. Such terms are superpotentials, and indeed their superconformal variations vanish up to total derivatives.

2.1. Supersymmetry on $S^2$. On the sphere $S^2$ the Euclidean global conformal group is $SO(3, 1)$ as on flat space, however it is realized differently than in the latter case (similar observations have been made in four [3] and three [5] dimensions): the maximal compact subgroup $SO(3)$ acts as rotations of $S^2$.

There are four complex Killing spinors on $S^2$. A basis can be chosen such that two of them, that we call “positive”, satisfy

$$
D_\mu \epsilon = \frac{i}{2r} \gamma_\mu \epsilon
$$

(15)

where $r$ is the radius of the sphere, while two that we call “negative” satisfy $D_\mu \epsilon = -\frac{i}{2r} \gamma_\mu \epsilon$. None of them is chiral. If we restrict the supersymmetry variations to positive Killing spinors $\epsilon$ and $\bar{\epsilon}$ (negative spinors would give the same result), one verifies that

$$
\rho = 0, \quad \alpha = \frac{i}{2r} \bar{\epsilon} \epsilon, \quad \beta = 0.
$$

(16)

In particular the algebra does not contain dilations nor axial R-rotations. In fact the superalgebra is $osp^*(2|2) \cong su(2|1)$, embedded as a real form into $osp(2|2, \mathbb{C})$. Its bosonic subalgebra is $su(2) \oplus u(1)_R$, where the first factor are isometries of $S^2$ and the second is the vector-like R-symmetry. We call such superalgebra the $\mathcal{N} = (2, 2)$
Euclidean supersymmetry on $S^2$. Notice that the R-symmetry is part of the algebra, rather than an outer automorphism of it.

It turns out that a two-dimensional theory—not necessarily conformal—that on flat space has $\mathcal{N} = (2, 2)$ supersymmetry with a $U(1)_R$ vector-like R-symmetry, can be placed on $S^2$ preserving the $OSp^*(2|2)$ supersymmetry (a systematic method to do so has been developed in [20]). In doing so one has the freedom to choose the R-charges $q$ in (13) (in a way which is compatible with superpotential interactions) and this results in different theories on $S^2$. If the theory on flat space is actually superconformal, then the R-charge of a chiral multiplet of dimension $\Delta$ is $q = 2\Delta$. For such choice of $q$ in (13), the supergroup $OSp^*(2|2)$ on $S^2$ enhances to the full superconformal group.

At this point we introduce coordinates $(\theta, \phi)$ on $S^2$ with metric:

$$ds^2 = r^2(d\theta^2 + \sin^2 \theta d\phi^2).$$

We choose a vielbein $e^1 = r d\theta, e^2 = r \sin \theta d\phi$.

2.2. Supersymmetric actions on $S^2$. Let us specialize the supersymmetry variations to positive Killing spinors $D_{\mu} \epsilon = \frac{i}{2} \gamma_{\mu} \epsilon, D_{\mu} \bar{\epsilon} = \frac{i}{2} \gamma_{\mu} \bar{\epsilon}$ on $S^2$:

$$\delta A_{\mu} = -\frac{i}{2} (\bar{\epsilon} \gamma_{\mu} \lambda - \lambda \gamma_{\mu} \epsilon) \delta \sigma = \frac{1}{2} (\bar{\epsilon} \lambda - \bar{\lambda} \epsilon) \delta \eta = -\frac{i}{2} (\bar{\epsilon} \gamma_{\lambda} \lambda - \bar{\lambda} \gamma_{\lambda} \epsilon)$$

$$\delta \lambda = i \gamma_{\lambda} \epsilon \left( F_{12} - \frac{\eta}{r} + i [\sigma, \eta] \right) - \epsilon \left( D + \frac{\sigma}{r} \right) + i \bar{\sigma} \epsilon \sigma - \gamma_{\lambda} \bar{\psi} \eta$$

$$\delta \bar{\lambda} = i \gamma_{\lambda} \bar{\epsilon} \left( F_{12} - \frac{\eta}{r} - i [\sigma, \eta] \right) + \bar{\epsilon} \left( D + \frac{\sigma}{r} \right) - i \bar{\sigma} \bar{\epsilon} \bar{\sigma} - \gamma_{\lambda} \bar{\psi} \bar{\eta}$$

$$\delta D = -\frac{i}{2} \bar{\epsilon} \gamma^{\mu} D_{\mu} \lambda - \frac{i}{2} \lambda \gamma^{\mu} \epsilon + \frac{i}{2} [\bar{\epsilon} \lambda + \bar{\lambda} \epsilon, \sigma] + \frac{1}{2} [\bar{\epsilon} \gamma_{\lambda} + \bar{\lambda} \gamma_{\epsilon}, \eta] - \frac{\bar{\epsilon} \lambda - \bar{\lambda} \epsilon}{2r}$$

and

$$\delta \phi = \bar{\epsilon} \psi \quad \delta \psi = \left( i \bar{\psi} \phi + i \sigma \phi + \gamma_{\lambda} \eta \phi - \frac{q}{2r} \phi \right) \epsilon + \bar{\epsilon} F$$

$$\delta \bar{\phi} = \bar{\psi} \epsilon \quad \delta \bar{\psi} = \left( i \bar{\psi} \bar{\phi} + i \bar{\phi} \sigma - \gamma_{\lambda} \bar{\phi} \eta - \frac{q}{2r} \bar{\phi} \right) \bar{\epsilon} + \epsilon \bar{F}$$

$$\delta F = \epsilon \left( i \bar{\psi} \psi - i \sigma \psi + \gamma_{\lambda} \eta \psi + \frac{q}{2r} \psi - i \lambda \phi \right)$$

$$\delta \bar{F} = \bar{\epsilon} \left( i \bar{\psi} \bar{\psi} - i \bar{\psi} \sigma - \gamma_{\lambda} \bar{\psi} \eta + \frac{q}{2r} \bar{\psi} + i \bar{\phi} \bar{\lambda} \right).$$

Supersymmetric Lagrangians on $S^2$ can be constructed by adding suitable $r^{-1}$ and $r^{-2}$ terms to the flat space Lagrangians, either working order by order (as in [3–6]) or by applying the method of [20].

The Yang–Mills (YM) action is $S_{YM} = \int d^2 \chi \frac{1}{8 \pi} L_{YM}$ with

$$L_{YM} = \text{Tr} \left\{ \frac{1}{2} \left( F_{12} - \frac{\eta}{r} \right)^2 + \frac{1}{2} \left( D + \frac{\sigma}{r} \right)^2 + \frac{1}{2} D_{\mu} \sigma D^{\mu} \sigma + \frac{1}{2} D_{\mu} \eta D^{\mu} \eta - \frac{1}{2} [\sigma, \eta]^2 + i \frac{\bar{\lambda} \gamma^{\mu} D_{\mu} \lambda + i \frac{\bar{\lambda} [\sigma, \lambda] + 1}{2} \bar{\lambda} \gamma_{\lambda} \eta, \lambda] . \right\}$$

(20)
Notice that the bosonic part is a sum of squares and therefore positive definite.\footnote{The Lorentzian YM Lagrangian has bosonic part:}

The usual flat space YM Lagrangian is recovered in the \( r \to \infty \) limit, and it is invariant under SUSY transformations with \( \partial_\mu \epsilon = \partial_\mu \bar{\epsilon} = 0 \). In fact the action (20) is even exact:

\[
\delta_\epsilon \delta \bar{\epsilon} \int d^2 x \ Tr \left( \frac{1}{2} \bar{\lambda} \lambda - 2 D \sigma - \frac{1}{r} \sigma^2 \right) = \bar{\epsilon} \epsilon \int d^2 x \ L_{YM}.
\]

Since the integrand on the left-hand-side is a neutral scalar invariant under rotations of \( S^2 \), \( \delta_\epsilon \delta \bar{\epsilon} \) can be commuted and the formula above is valid with \( \delta_\epsilon \delta \bar{\epsilon} \) as well.

Given an Abelian gauge multiplet, we can add a Fayet–Iliopoulos (FI) Lagrangian \( \mathcal{L}_{FI} \). Since \( \delta F_{12} = D_\mu (\epsilon^{\mu \nu} \delta A_\nu) \) and \( \delta D = - \frac{1}{r} D_\mu (\bar{\epsilon} \gamma^\mu \lambda + \bar{\lambda} \gamma^\mu \epsilon) \) are total derivatives, the FI Lagrangian

\[
\mathcal{L}_{FI} = -i \xi D + i \frac{\theta}{2 \pi} F_{12}
\]

gives a supersymmetric action on \( S^2 \) without further \( \frac{1}{r} \) corrections. The reason is that the FI action in two dimensions is superconformal, and the only corrections on \( S^2 \) for superconformal theories are the conformal couplings of scalars to curvature.

In fact the FI term is a special case of a \textit{twisted superpotential}. This is obtained by rewriting the vector multiplet \( V \) in terms of a twisted chiral multiplet \( \Sigma \), whose components are \( (\sigma + i \eta, \lambda, \bar{\lambda}, D - i F_{12}) \) transforming in the adjoint representation, and considering a holomorphic function \( \tilde{W}(\Sigma) \). A twisted superpotential interaction can be put supersymmetrically on \( S^2 \) for any gauge-invariant choice of \( \tilde{W} \). In the simplest case of a single Abelian vector multiplet, the Lagrangian is:

\[
\mathcal{L}_{\tilde{W}} = i \tilde{W}' \left( D - i F_{12} + \frac{\sigma + i \eta}{r} \right) - i \frac{1}{2} \tilde{W}'' \bar{\lambda} (1 + \gamma_3) \lambda - \frac{i}{r} \tilde{W},
\]

where \( \tilde{W}(\sigma + i \eta) \) is a holomorphic function of \( (\sigma + i \eta) \). Its twisted antichiral counterpart is:

\[
\overline{\mathcal{L}_{\tilde{W}}} = i \tilde{W}^{*'} \left( D + i F_{12} + \frac{\sigma - i \eta}{r} \right) - i \frac{1}{2} \tilde{W}^{*''} \bar{\lambda} (1 - \gamma_3) \lambda - \frac{i}{r} \tilde{W}^*,
\]

where \( \tilde{W}^*(\sigma - i \eta) \) is a holomorphic function of \( (\sigma - i \eta) \), which need not be the complex conjugate of \( \tilde{W} \). On the other hand, actions that would be real in Lorentzian signature are obtained by taking \( \tilde{W}^* \bar{z} \) as the complex conjugate of \( \tilde{W} z \).

For instance, the FI Lagrangian (22) follows from the twisted superpotential \( \tilde{W}(z) = \frac{1}{2} \left( - \xi + i \frac{\theta}{2 \pi} \right) z \). Another example is \( \tilde{W} = \frac{L}{4} z^2 \), which gives the Lagrangian:

\[
\mathcal{L}_{CS} = i L \ Tr \left( F_{12} \eta + D \sigma - \frac{1}{2} \bar{\lambda} \lambda + \frac{\sigma^2}{2r} - \frac{\eta^2}{2r} \right)
\]

where the length scale \( L \) has been included for dimensional reasons. The flat space limit of (25) can be obtained from the 3d Chern–Simons action on \( \mathbb{R}^2 \times S^1 \), by sending the \( z \) to \( i \).
radius of the circle $R \to 0$ and the CS level $k \to \infty$ while keeping $kR = L$ fixed [28].

A related Lagrangian is obtained with $\tilde{W} = i \frac{L}{4} \bar{z}^2$:

$$\mathcal{L} = i \tilde{L} \text{Tr} \left( F_{12} \sigma - D\eta - \frac{i}{2} \bar{\gamma} \eta \lambda - \frac{\sigma \eta}{r} \right)$$  \hspace{1cm} (26)$$

which does not follow from three dimensions.

The kinetic Lagrangian on $S^2$ for a matter chiral multiplet of R-charge $q$ is:

$$L_{\text{mat}} = D\mu \bar{\phi} D^\mu \phi + \bar{\phi} \sigma^2 \phi + \bar{\phi} \eta^2 \phi + i \bar{\phi} D\phi + \bar{F} F + \frac{i q}{r} \bar{\phi} \sigma \phi + \frac{q(2-q)}{4r^2} \bar{\phi} \phi - i \bar{\psi} \gamma^\mu D\mu \psi + i \bar{\psi} \sigma \psi - \bar{\psi} \bar{\gamma} \eta \psi + i \bar{\psi} \lambda \phi - i \bar{\phi} \bar{\lambda} \psi - \frac{q}{2r} \bar{\psi} \psi.$$  \hspace{1cm} (27)$$

Notice that for $q = 0$ there are no corrections to the flat space Lagrangian, which is classically superconformal. In fact the matter action is even exact:

$$\delta \bar{\epsilon} \delta \epsilon \int d^2 x \left( \bar{\psi} \psi - 2i \bar{\phi} \sigma \phi + \frac{q-1}{r} \bar{\phi} \phi \right) = \bar{\epsilon} \epsilon \int d^2 x \mathcal{L}_{\text{mat}}.$$  \hspace{1cm} (28)$$

The same formula is valid with $\delta \bar{\epsilon} \delta \epsilon$. The Lagrangian $L_{\text{mat}}$ depends holomorphically on the combination $\sigma + iq/2r$, once one fixes $D + \sigma/r = 0$ which is the supersymmetry condition in (41). Such on-shell constraint can be imposed only on background fields. This means that the effect on the path integral of shifting the R-charges proportionally to some flavor global charge can be obtained by giving an imaginary part to the twisted mass $\sigma_{\text{ext}}$ associated to that flavor global symmetry (compare with (66)).

We can add superpotential interactions:

$$L_W = F^{(q = 2, \text{neutral})}, \quad \overline{L_W} = \bar{F}^{(q = 2, \text{neutral})},$$  \hspace{1cm} (29)$$

where $F, \bar{F}$ are the F-terms of a neutral chiral multiplet of charge $q = 2$. Since we have $\delta F = D\mu (i \epsilon \gamma^\mu \psi)$ and $\delta \bar{F} = D\mu (i \bar{\epsilon} \gamma^\mu \bar{\psi})$, the action is supersymmetric. In more conventional terms the interaction is defined by the superpotential $W(\Phi_i)$, a holomorphic function of chiral multiplets, and the Lagrangian on $S^2$ is:

$$\mathcal{L}_W = \sum_j \frac{\partial W}{\partial \phi_j} F_j - \sum_{j,k} \frac{1}{2} \frac{\partial^2 W}{\partial \phi_j \partial \phi_k} \psi_j \psi_k.$$  \hspace{1cm} (30)$$

Its supersymmetry variation is a total derivative whenever $W(\Phi_i)$ is neutral and quasi-homogeneous of degree 2, assigning degree $q_i$ to $\Phi_i$.

Finally, if the theory has a global flavor symmetry $G_{\text{global}}$, we can introduce an external non-dynamical vector multiplet $V_{\text{ext}}$ coupled in a gauge-invariant way to the conserved current. Additional mass terms, called twisted masses, are obtained in flat space by giving VEV to $\sigma_{\text{ext}}, \eta_{\text{ext}}$ in the external vector multiplet. Supersymmetry on $S^2$ relates $D_{\text{ext}}, F_{12}^{\text{ext}}$ to $\sigma_{\text{ext}}, \eta_{\text{ext}}$ as in (41) and it requires all four terms to mutually commute. Therefore, up to flavor rotations, twisted masses take values in the complexified Cartan subalgebra of the global symmetry group $G_{\text{global}}$. Notice also that $F_{12}^{\text{ext}}$ should be quantized if we associate it to the background connection of a compact symmetry.

\hspace{1cm}

5 At linear order the vector multiplet couples to the current with minimal coupling, but in general gauge invariance might require the addition of seagull terms.
3. The $S^2$ Partition Function: Localization on the Coulomb Branch

We would like to compute the path integral:

$$Z_{S^2} = \int \mathcal{D}\varphi \ e^{-S[\varphi]}, \quad (31)$$

where $\varphi$ are all fields in the Lagrangian, for a theory of vector and chiral multiplets with Lagrangian made of the pieces we discussed: (20)–(30) plus twisted masses. We adopt the localization technique [1] along the lines of [3–5].

We choose a (commuting) positive Killing spinor $\epsilon_+$ on $S^2$ normalized as $\epsilon_+^\dagger \epsilon_+ = 1$. It defines a Killing vector $v_\mu$ (in particular $D^\mu v_\mu = 0$), its dual vector $w_\mu = \epsilon_+^\dagger \gamma_3 \gamma_\mu \epsilon_+$ and a scalar function $s = \epsilon_+^\dagger \gamma_3 \epsilon_+$:

$$v_\mu = \epsilon_+^\dagger \gamma_\mu \epsilon_+, \quad w_\mu = -i \epsilon_+^\dagger \gamma_3 \gamma_\mu \epsilon_+, \quad s = \epsilon_+^\dagger \gamma_3 \epsilon_+, \quad (32)$$

and we have $\epsilon_+^T C \epsilon_+ = 0$. The Killing vector $v_\mu$ vanishes at two antipodal points that we call north and south pole. In terms of a polar coordinate $\theta$ on $S^2$ such that $\theta = 0$ ($\theta = \pi$) at the north (south) pole, $s = \cos \theta$ and the vector fields in vielbein basis are $v^a = (0, \sin \theta)$ and $w^a = (\sin \theta, 0)$.

Then we construct the supercharges $Q, Q^\dagger$ corresponding to $\epsilon_+$. To do that, first we extract the supercharges$^6$ $Q_\alpha, Q^\dagger_\alpha$ from the commuting operator $\delta$:

$$\delta = \delta_\epsilon + \delta_\bar{\epsilon} = \epsilon^\alpha Q_\alpha + \bar{\epsilon}^\alpha Q^\dagger_\alpha \quad (33)$$

where $\epsilon, \bar{\epsilon}$ are anticommuting. Then we construct

$$Q \equiv \epsilon^\alpha Q_\alpha, \quad Q^\dagger \equiv \epsilon^\alpha Q^\dagger_\alpha \quad (34)$$

where now $\epsilon_+$ is commuting and $\epsilon^\alpha_+ = C \epsilon^\alpha_+$ is its charge conjugate. In the same way we can transform barred spinors to their charge conjugate: $\lambda^\dagger = C (\lambda^\dagger)^T$, recalling that $\lambda^\dagger$ is not the Hermitian conjugate of $\lambda$, etc... We will use the two notations interchangeably. Now $Q, Q^\dagger$ are anticommuting (their explicit expression is given in Appendix A) and in particular—up to a gauge transformation $\Lambda$—form the $su(1|1)$ superalgebra

$$\{Q, Q^\dagger\} = M + \frac{R}{2} + i \Lambda, \quad Q^2 = Q^\dagger 2 = 0 \quad (35)$$

where $M$ is the angular momentum that generates translations along $v_\mu$ and $R$ is the R-charge. Finally, we construct the supercharge $\tilde{Q} = Q + Q^\dagger$ so that:

$$\tilde{Q}^2 = M + \frac{R}{2} + i \Lambda, \quad \Lambda = -\sigma + is \eta, \quad (36)$$

and the explicit expression for the gauge transformation is computed in Appendix A.

We perform localization with respect to the supercharge $\tilde{Q}$. By the usual localization argument, as long as the action is $\tilde{Q}$-closed (which is the case if it is supersymmetric) the expectation value of $\tilde{Q}$-closed operators is unaffected by a deformation of the path integrand by $\tilde{Q}$-exact terms. Moreover the path integral is dominated by the fixed points of $\tilde{Q}$ and the small quadratic fluctuations around them (through a one-loop determinant).

$^6$ Recall that $Q^\dagger_\alpha$ is not the Hermitian conjugate of $Q_\alpha$. 

The second statement follows from the first one, as we now review. We can deform the action by the exact term $S_0 \rightarrow S_0 + t \delta S$, where $\delta S = \int (L_{YM} + L_{\psi})$ and

$$L_{YM} = \text{Tr} \frac{(Q \lambda) \lambda + \lambda^\dagger (Q \lambda^\dagger)}{4}, \quad L_{\psi} = \frac{(Q \psi) \psi + \psi^\dagger (Q \psi^\dagger)}{2}. \quad (37)$$

Here $\overline{\cdot}$ is a formal conjugation that acts as $\dagger$ on c-numbers and exchanges $\lambda, \phi, \psi, F$ with $\lambda^\dagger, \phi^\dagger, \psi^\dagger, F^\dagger$. As the name suggests, $L_{YM}$ is the same as in (20) up to total derivatives. The bosonic parts of these actions are

$$L_{YM} \big|_{\text{bos}} = \frac{1}{4} \text{Tr} (|Q \lambda|^2 + |Q \lambda^\dagger|^2), \quad L_{\psi} \big|_{\text{bos}} = \frac{1}{2} (|Q \psi|^2 + |Q \psi^\dagger|^2). \quad (38)$$

which are formally positive definite and vanish on the fixed points of $Q$. In the $t \rightarrow \infty$ limit only BPS configurations with $L_{YM} = L_{\psi} = 0$ contribute to the path integral: $Z$ equals a sum (or integral) over the BPS solutions, of the Gaussian path integral of the truncated quadratic action around them. The Gaussian path integral gives the on-shell action times a one-loop determinant. Since $\delta S$ vanishes on-shell, only $S_0$ contributes to the on-shell action; on the contrary when $t \rightarrow \infty$, $S_0$ is irrelevant with respect to $t \delta S$ in determining the spectrum of fluctuations around a saddle point. The result is independent of $t$, as it should.

The localization argument immediately tells us that the partition function $Z_{S^2}$ will not depend on the YM coupling $g$, since $L_{YM}$ in (20) is exact, nor on the superpotential Lagrangian $L_W$, since

$$L_W = \int F^{(q = 2, \text{neutral})} = \int Q (\epsilon_+^T C \psi) \quad (39)$$

is exact (it does depend on the constraints that $W$ imposes on the R-charges); $Z_{S^2}$ will depend on twisted masses and the twisted superpotential $\tilde{W}$.

Let us study the BPS configurations. The form (38) assures that the saddle points of $\delta S$ coincide with its zeros, that is with the BPS configurations.\(^7\) For the gauge multiplet we have, up to total derivatives:

$$L_{YM} \big|_{\text{bos}} = \frac{1}{2} \text{Tr} \left\{ \left( F_{12} - \frac{\eta}{r} \right)^2 + \left( D + \frac{\sigma}{r} \right)^2 + (D_\mu \sigma)^2 + (D_\mu \eta)^2 - [\sigma, \eta]^2 \right\}. \quad (40)$$

The BPS equations $0 = Q \lambda = Q \lambda^\dagger$ are

$$0 = F_{12} - \frac{\eta}{r} = D + \frac{\sigma}{r} = D_\mu \sigma = D_\mu \eta = [\sigma, \eta]. \quad (41)$$

They imply that $\sigma$ and $\eta$ can be simultaneously diagonalized, then $F_{12}$ and $D$ are fixed in terms of them; due to diagonalization, $\sigma$ and $\eta$ are not charged under the background, therefore they are simply constant on $S^2$. The gauge flux

$$\frac{1}{2\pi} \int F = \frac{1}{2\pi} \int F_{12} e^1 \wedge e^2 = 2r^2 F_{12} \quad (42)$$

\(^7\) For a fermion $\lambda$, $Q \lambda = 0$ coincides with $|Q \lambda|^2 \equiv (Q \lambda)(Q \lambda) = 0$ only if we impose on fields the same reality conditions used to define $\overline{\cdot}$, like $D^\dagger = D, \sigma^\dagger = \sigma, etc...$ If we relaxed the reality conditions, the BPS equations would have many more solutions.
is quantized:
\[ F_{12} = \frac{m}{2r^2}, \quad \eta = \frac{m}{2r}, \quad (43) \]
where \( m \) belongs to a maximal torus of the gauge algebra and is GNO quantized \([29]\), meaning that for any representation \( R \) and weight \( \rho \in R, \rho(m) \in \mathbb{Z} \). The set of BPS configurations is therefore parametrized by a continuous variable \( \sigma \) and the discrete fluxes \( m \) of \( F_{12} \). Recall that at this stage \( \sigma \) and \( F_{12} \) represent dynamical as well as external fields.

For the chiral multiplet we find, up to total derivatives:
\[
\mathcal{L}_\psi \big|_{\text{bos}} = |D_\mu \phi|^2 + i \frac{1}{r} v^\mu \phi^\dagger D_\mu \phi + \phi^\dagger \left( \sigma^2 + \eta^2 + \frac{q^2}{4r^2} \right) \phi + |F|^2
\]
\[
+ s \left( \frac{2 - q}{r} \eta \phi \phi^\dagger + \phi^\dagger F_{12} \phi \right) + u^{\mu \nu} \phi^\dagger D_\mu \eta \phi. \quad (44)\]
In this case it is easier to study the BPS equations \( 0 = Q\psi = \mathcal{Q}\psi^\dagger \) directly:
\[
0 = -\sin \frac{\theta}{2} (2D_+ \phi + F) + \cos \frac{\theta}{2} (i \sigma + \eta - \frac{q}{2r}) \phi
\]
\[
0 = + \cos \frac{\theta}{2} (2D_- \phi + F) + \sin \frac{\theta}{2} (i \sigma - \eta - \frac{q}{2r}) \phi
\]
\[
0 = - \sin \frac{\theta}{2} (2D_- \phi^\dagger - F^\dagger) + \cos \frac{\theta}{2} \phi^\dagger (i \sigma + \eta - \frac{q}{2r})
\]
\[
0 = + \cos \frac{\theta}{2} (2D_+ \phi^\dagger - F^\dagger) + \sin \frac{\theta}{2} \phi^\dagger (i \sigma - \eta - \frac{q}{2r}) \quad (45)\]
where \( D_{\pm} = (D_1 \mp i D_2)/2 \). For generic \( q \) (we will study the case \( q = 0 \) in Sect. 4) and assuming that the bosonic fields in the vector multiplet are real, the solutions are:
\[
0 = \phi = \bar{\phi} = F = \bar{F}. \quad (46)\]
To be precise, solving the BPS equations (45) algebraically one finds another branch with \( 0 = F = \sigma \phi \) and \( 2 \cos \frac{\theta}{2} D_- \phi = \sin \frac{\theta}{2} (\eta + \frac{q}{2r}) \phi \), \( 2 \sin \frac{\theta}{2} D_+ \phi = \cos \frac{\theta}{2} (\eta - \frac{q}{2r}) \phi \). We combine them with the gauge BPS equations \( D_\mu \eta = 0 \) and \( F_{12} = \eta/r \), and choose the potential \( A_\psi = -r \cos \theta \eta \). Away from the poles at \( \theta = 0, \pi \) they imply \( \partial_\phi = -i \frac{q}{2} \phi \) as well as \( \partial_\phi = r (\sin \theta)^{-1} (\eta - \frac{q}{2r} \cos \theta) \phi \). They can be integrated: \( \phi = C \left( \tan \frac{\theta}{2} \right)^r \eta (\sin \theta)^{-q/2} e^{-i q \psi /2} \), however this solution is smooth only for \( C = 0 \).
Even the point-like vortex solutions that we will discuss in Sect. 4 are not present here due to \( D_\mu F_{12} = 0 \).

Next we need the one-loop determinant of quadratic fluctuations around the BPS configurations. Instead of using the localizing Lagrangian \( \mathcal{L}_\psi \) in (37), we prefer to use directly \( \mathcal{L}_{YM} \) (20) and \( \mathcal{L}_{\text{mat}} \) (27) which are \( Q \)-exact: they vanish and are extremized on the BPS configurations. They have the advantage of being invariant under \( SO(3) \) rotations of \( S^2 \), therefore the computation simplifies. We will recompute the one-loop determinants from \( \mathcal{L}_{YM} + \mathcal{L}_\psi \) using an index theorem in Sect. 3.4.

3.1. The vector multiplet. We compute the one-loop determinant of quadratic fluctuations of the YM action (20), to which we will soon have to add gauge-fixing terms, around the saddle points:
\[
0 = D_\mu \sigma = D_\mu \eta = [\sigma, \eta], \quad F_{12} = \frac{\eta}{r} = \frac{m}{2r^2}, \quad D = -\frac{\sigma}{r}. \quad (47)\]
Let us start with the bosonic part of $S = t \int_{YM}$. We separate all fields in a background part and a fluctuation part, as $\phi = \phi_0 + \frac{1}{\sqrt{t}} \tilde{\phi}$, and expand. Only terms quadratic in the fluctuations appear in the $t \to +\infty$ limit. We also need to fix the gauge. On a background the gauge-fixing action is:

$$L_{g-f} = -\bar{c}(D^\mu D_\mu c - i D^\mu [\tilde{A}_\mu, c]) - \frac{1}{2\xi} (D^\mu \tilde{A}_\mu)^2,$$

where $c, \bar{c}$ are anticommuting complex scalar ghost fields in the adjoint representation, $\xi$ is the gauge-fixing parameter\(^8\) in $R_\xi$-gauge which can be chosen at will, and all covariant derivatives are on the background $A_\mu^{(0)}$. Keeping only terms up to quadratic order in the fluctuations, we get:

$$L = \frac{1}{2} (\tilde{F}_{12} - \tilde{\eta})^2 + \frac{1}{2} (\tilde{D} + \tilde{\sigma})^2 + \frac{1}{2} (D_\mu \tilde{\sigma} - i [\tilde{A}_\mu, \sigma_0])^2 + \frac{1}{2} (D_\mu \tilde{\eta} - i [\tilde{A}_\mu, \eta_0])^2$$

$$- \frac{1}{2} ([\tilde{\sigma}, \eta_0] + [\sigma_0, \tilde{\eta}])^2 - \bar{c} D^\mu D_\mu c - \frac{1}{2\xi} (D^\mu \tilde{A}_\mu)^2.$$

We can immediately integrate over $\tilde{D}$, whose one-loop determinant is 1.

The determinant for the remaining fields is computed in Appendix B. It turns out to be a product over the roots $\alpha$ of the gauge group $G$. When $\alpha(m) \neq 0$ one finds, up to factors that are independent of the background $\sigma$, $m$ that we reabsorb in the normalization:

$$\frac{\text{Det} O_c}{\sqrt{\text{Det} O_{gauge}}} \bigg|_{\alpha} = \prod_{k=\lfloor \alpha(m) \rfloor}^{\infty} \frac{1}{(k + ir\alpha(\sigma))^{2k-1}(k + 1 + ir\alpha(\sigma))^{2k+3}},$$

where $O_c$ and $O_{gauge}$ are the kinetic operators for ghosts and for $(\tilde{A}_\mu, \tilde{\sigma}, \tilde{\eta})$ respectively. When $\alpha(m) = 0$ there are zero-modes (see Appendix B for details): after removing the zero eigenvalues one is left with

$$\frac{\text{Det'} O_c}{\sqrt{\text{Det'} O_{gauge}}} \bigg|_{\alpha} = \prod_{k=0}^{\infty} \frac{1}{|\alpha(\sigma)|} (k + ir\alpha(\sigma))^{2k-1}(k + 1 + ir\alpha(\sigma))^{2k+3}.$$ 

The zero-modes correspond to those of the background scalar $\sigma$, and span the subalgebra $\{\alpha(m) = 0\}$ which is left unbroken by the flux $m$. To integrate over them, we integrate over a Cartan subalgebra and use a Vandermonde determinant for the unbroken symmetry:

$$\int \text{zero-modes} = \frac{1}{|W_m|} \int \left( \prod_{n=1}^{\text{rank}(G)} d\sigma_n \right) \prod_{\alpha(m)=0} |\alpha(\sigma)|$$

where $|W_m|$ is the order of the Weyl group of the subalgebra $\{\alpha(m) = 0\}$. Notice that the Vandermonde measure cancels against the extra factor in the one-loop determinant (50).

---

\(^8\) The parameter $\xi$ used in this subsection and in Appendix B should not be confused with the FI parameter $\xi$ used in the rest of the paper.
Let us now consider the fermions $\lambda, \bar{\lambda}$. The quadratic expansion of the fermionic part of the YM action (20) around the background is:

$$L = i\frac{1}{2}\bar{\lambda} \gamma^\mu D_\mu \lambda + i\frac{1}{2}\bar{\lambda}[\sigma_0, \lambda] + \frac{1}{2}\bar{\lambda}\gamma_3[\eta_0, \lambda].$$

The one-loop determinant is computed in Appendix B. As before the final result is a product over the roots of $G$:

$$\text{Det} \mathcal{O}_\lambda = \prod_{\alpha \in G, k = \lfloor \alpha(m) \rfloor} \left( k + i r \alpha(\sigma) \right)^{2k} \left( k + 1 + i r \alpha(\sigma) \right)^{2k+2}. \quad (52)$$

The one-loop determinant for the gauge multiplet is the product of (49) and (52), neglecting the possible extra factor that cancels against the Vandermonde determinant. After many cancelations, the final result is:

$$Z_{\text{gauge}} = \prod_{\alpha \in G} \left( \frac{|\alpha(m)|}{2} + i r \alpha(\sigma) \right) = \prod_{\alpha > 0} \left( \frac{\alpha(m)^2}{4} + r^2 \alpha(\sigma)^2 \right). \quad (53)$$

3.2. The chiral multiplet. Now we consider the one-loop determinant of quadratic fluctuations of the matter Lagrangian $L_{\text{mat}} (27)$ around the saddle points: $0 = \phi = \bar{\phi} = F = \bar{F}$. The quadratic expansion of the bosonic part around the background is:

$$L = \bar{\psi} \left( -D^\mu D_\mu + \sigma^2 + \eta^2 + iD + \frac{iq\sigma}{r} + \frac{q(2 - q)}{4r^2} \right) \psi + \bar{F} F.$$

The auxiliary fields $F, \bar{F}$ give 1. The scalar $\phi$ in representation $R_\Phi$ is decomposed along the weights $\rho \in R_\Phi$, and in spin spherical harmonics $Y^{s}_{j, j_3}$ with effective spin $s = -\frac{1}{2}\rho(m)$. For each mode the eigenvalue is

$$\mathcal{O}_\phi = \frac{1}{r^2} \left( j + \frac{q}{2} - i r \rho(\sigma) \right) \left( j + 1 - \frac{q}{2} + i r \rho(\sigma) \right). \quad (54)$$

Taking the product over $\rho \in R_\Phi$, $j \geq |s|$ and $|j_3| \leq j$, the determinant is:

$$\text{Det} \mathcal{O}_\phi = \prod_{\rho \in R_\Phi} \prod_{j = \lfloor \rho(m) \rfloor} \left( j + \frac{q}{2} - i r \rho(\sigma) \right)^{2j+1} \left( j + 1 - \frac{q}{2} + i r \rho(\sigma) \right)^{2j+1}. \quad (55)$$

The fermionic part of the matter action (27) expanded at second order around the background is:

$$L = \bar{\psi} \left( -i\gamma^\mu D_\mu + i\sigma - \gamma_3\eta - \frac{q}{2r} \right) \psi.$$

The one-loop determinant is computed in Appendix B:

$$\text{Det} \mathcal{O}_\psi = \prod_{\rho \in R_\Phi} \prod_{k = \lfloor \rho(m) \rfloor} (-1)^{\frac{1}{2}} \left( k + \frac{q}{2} - i r \rho(\sigma) \right)^{2k} \left( k + 1 - \frac{q}{2} + i r \rho(\sigma) \right)^{2k+2}, \quad (56)$$
where we defined the function \(|x| = \frac{|x| + x}{2}\), which equals \(x\) for \(x \geq 0\) and vanishes for \(x \leq 0\). The one-loop determinant for the chiral multiplet is then given by the ratio

\[
Z_{\text{matter}} = \frac{\text{Det } O_\psi}{\text{Det } O_\phi} = \prod_{\rho \in R_\phi} (-1)^{\rho(m)} \prod_{k=0}^{\infty} \frac{k + \frac{|\rho(m)|}{2} + 1 - \frac{q}{2} + ir\rho(\sigma)}{k + \frac{|\rho(m)|}{2} + \frac{q}{2} - ir\rho(\sigma)}
\]

(57)
due to many cancelations. For \(\rho(m) \leq 0\) the last equality is trivial; for \(\rho(m) \geq 0\) the product in the second line on \(0 \leq k \leq \rho(m) - 1\) equals \((-1)^{\rho(m)}\), while the product on \(k \geq \rho(m)\) equals the product in the first line.

The expression in (57) is useful to exhibit zeros and poles of the matter one-loop determinant, but does not converge and requires regularization. We choose zeta-function regularization. Consider the Hurwitz zeta function \(\xi(z; q) = \sum_{n=0}^{\infty} (q + n)^{-z}\), which is absolutely convergent for \(\Re z > 1\) and \(\Re q > 0\). For each \(q\), it extends by analytic continuation to a meromorphic function of \(z \neq 1\), with a simple pole at \(z = 1\). In the region of absolute convergence we have \(\frac{\partial}{\partial z} \xi(z; q) = -\sum_{n=0}^{\infty} \frac{\log(q+n)}{(q+n)^2}\), therefore we can define the regularized sum:

\[
\left. \sum_{n=0}^{\infty} \log(q + n) \right| = -\frac{\partial}{\partial z} \xi(z; q) \bigg|_{z=0}.
\]

(58)

We can take the exponential and make use of \(\log \Gamma(q) = \frac{\partial}{\partial z} \xi(z; q) \bigg|_{z=0} + \log \sqrt{2\pi} [30]\) (a short proof is in [31]) to write:

\[
\left. \prod_{n=0}^{\infty} (q + n) \right| = \frac{\sqrt{2\pi}}{\Gamma(q)}.
\]

(59)

The regularized matter determinant is then:

\[
Z_{\text{matter}} = \prod_{\rho \in R_\phi} \frac{\Gamma\left(\frac{q}{2} - ir\rho(\sigma) - \frac{\rho(m)}{2}\right)}{\Gamma\left(1 - \frac{q}{2} + ir\rho(\sigma) - \frac{\rho(m)}{2}\right)}.
\]

(60)

3.3. Matrix integral for \(Z_{\mathcal{N}}\). The last step is to work out the on-shell value of the classical action on the BPS configurations. Exact terms do not contribute because they vanish there.

For each Abelian factor we can include a FI term. For instance, for a single \(U(1)\) factor:

\[
\mathcal{L}_{FI} = i \text{Tr} \left( -\xi D + \frac{\theta}{2\pi} F_{12} \right), \quad S_{\text{class}}^{\mathcal{L}_{FI}} = 4\pi i \xi \text{Tr}(r\sigma) + i\theta \text{Tr } m.
\]

(61)

Since \(\text{Tr } m \in \mathbb{Z}\), we see that \(\theta \in 2\pi\mathbb{Z}\) is equivalent to \(\theta = 0\). More generally, we can include a twisted superpotential \(\tilde{W}(z)\) as in (23) and (24). Its value on-shell is:

\[
S_{\tilde{W}}^{\text{class}} = -8\pi i r \Re \tilde{W} \left( \sigma + i\frac{m}{2r} \right).
\]

(62)

\[ \text{This function should not be confused with the integer part, sometimes denoted by the same symbol.} \]
Let us finally write down the $S^2$ partition function. We have a sum over fluxes $m$ and an integral over the gauge group Cartan weights $\sigma$. Since $r$ appears only in front of $\sigma$, we can rescale $r \sigma \rightarrow \sigma$ and discard $r$ (the dependence on $r$ is also independent of $m$).

If we sum over inequivalent fluxes $m$ (for instance for $U(N)$ that would mean over ordered integers), then the symmetry factor is $\frac{1}{|W_m|}$ in terms of the order $|W_m|$ of the Weyl group of the unbroken gauge group. Equivalently, we can sum over all fluxes $m$ (for $U(N)$ that means over all integer sequences) and divide by $|W|$, the order of the Weyl group of $G$.

If the theory has some non-R global symmetry $G_{\text{global}}$, let us consider its Cartan subalgebra. We can associate to the global symmetry a background vector multiplet, or equivalently its superfield strength, which is a twisted chiral multiplet. For each Cartan generator we can turn on the lowest component of the background twisted chiral multiplet $M_a$, which is a complex twisted mass $\tilde{m}_a$. Analogously to the lowest component $\frac{1}{i} (\sigma + \frac{m}{2r})$ of the dynamical superfield strength $\Sigma$, the twisted mass $\tilde{m}_a = \frac{1}{i} (\tau_a + i \frac{n_a}{2r})$ has a continuous real part $\tau_a$ and a quantized imaginary part, related to the integer flux $n_a$ for the global symmetry.$^{10}$

Let $f^a[\Phi]$ and $R[\Phi]$ be the non-R-charges and the R-charge of a chiral multiplet $\Phi$. Then the localized partition function is:

$$Z_{S^2} = \frac{1}{|W|} \sum_m \int \left( \prod_j \frac{d\sigma_j}{2\pi} \right) Z_{\text{class}}(\sigma, m) Z_{\text{gauge}}(\sigma, m) \prod_\Phi Z_\Phi(\sigma, m; \tau, n) \quad (63)$$

where the one-loop determinants are:

$$Z_{\text{gauge}} = \prod_{\alpha \in G} \left( \frac{|\alpha(m)|}{2} + i \alpha(\sigma) \right) = \prod_{\alpha > 0} \left( \frac{\alpha(m)^2}{4} + \alpha(\sigma)^2 \right)$$

$$Z_\Phi = \prod_{\rho \in R_\Phi} \Gamma \left( \frac{R[\Phi]}{2} - i \rho(\sigma) - if^a[\Phi] \tau_a - \frac{\rho(m) + f^a[\Phi]n_a}{2} \right) \Gamma \left( 1 - \frac{R[\Phi]}{2} + i \rho(\sigma) + if^a[\Phi] \tau_a - \frac{\rho(m) + f^a[\Phi]n_a}{2} \right) \quad (64)$$

and the classical piece, for a FI term and a more general twisted superpotential, is:

$$Z_{\text{class}} = e^{-4\pi i \xi \Tr \sigma - i \theta \Tr m} \exp \left\{ 8\pi i r \Re e \tilde{W} \left( \frac{\sigma}{r} + i \frac{m}{2r} \right) \right\} \quad (65)$$

Note that mixing the $R$-symmetry with global Abelian non-R-symmetries amounts to giving an imaginary part to $\tau_a$, like in [5]:

$$\tau_a \rightarrow \tau_a + i \frac{c_a}{2} \iff R \rightarrow R + \sum_a c_a F^a. \quad (66)$$

The factor of 1/2 in the mixing, compared to 1 in the $S^3$ case [5], has to do with the different relation between the superconformal $R$ and the dilation operator. Analyticity in the combination $\tau_a + i c_a/2$ was already observed in the matter Lagrangian $\mathcal{L}_{\text{mat}} (27)$.

The integrals in (63) are along $\mathbb{R}$ when all $R$-charges satisfy $R[\Phi] > 0$, in which case the one-loop determinants $Z_\Phi(\sigma)$ have no poles along the real axis. When some $R$-charges are zero or negative the integral is defined by analytic continuation, that is the contour has to be deformed to keep the poles on the same side as they were in the former case.

---

$^{10}$ $\sigma$ and $\tau_a$ have already been rescaled by $r$ to make them dimensionless.
3.4. One-loop determinants via index theorem. In Sect. 4 we will need the one-loop determinants around non-constant configurations, and in such case our previous computation is not valid. We then recompute the determinants—this time taking as localizing determinants around non-constant configurations, and in such case our previous computation.

Precisely, we use the Atiyah–Singer index theorem for transversally elliptic operators [32]. The method has been exploited in [3,12] to perform localization on $S^4$, and lots of details are contained therein.

Let us repeat the discussion in [3,12]. The supersymmetry transformations generated by $Q$ can be brought to a cohomological form by splitting the fields in four groups $\varphi_{e,o}, \hat{\varphi}_{e,o}$—where the subscript refers to even or odd statistics—and writing

$$Q \varphi_{e,o} = \hat{\varphi}_{o,e}, \quad Q \hat{\varphi}_{o,e} = \mathcal{R} \varphi_{e,o}$$

so that $(\varphi_e, \hat{\varphi}_o)$ and $(\varphi_o, \hat{\varphi}_e)$ form multiplets. Here $\mathcal{R}$ is the action of $U(1)^{M+\frac{n}{2}} \times G$, where $U(1)^{M+\frac{n}{2}}$ is a rotation of $S^2$ around the poles combined with an $R$-symmetry rotation and $G$ is the gauge group. Therefore $Q$ acts as a cohomological equivariant operator because $Q^2 \varphi_{e,o} = \mathcal{R} \varphi_{e,o}$ to be compared with (36), and $Q^2$ is nilpotent on $\mathcal{R}$-invariant field configurations. Since the $Q$-exact deformation term, that we can write as $\delta S = Q P$ for some $\mathcal{R}$-invariant $P$, is invariant under $Q$ and fields are paired as in (67), many cancelations take place and eventually the one-loop determinant reads [3]:

$$\frac{\det_{\ker D_{oe}} \mathcal{R}_o}{\det_{\ker D_{oe}} \mathcal{R}_e},$$

where $D_{oe}$ is the differential operator from the space of $\varphi_e$ to the space of $\varphi_o$ obtained by expanding $P$ at quadratic order around a saddle point and picking the term $\varphi_o D_{oe} \varphi_e$. Hence the one-loop determinant equals the product of weights of the group action of $\mathcal{R} = Q^2$. The weights can be extracted from the equivariant index:

$$\text{ind } D_{oe} = \text{tr}_{\ker D_{oe}} e^{\mathcal{R}} - \text{tr}_{\ker D_{oe}} e^{\mathcal{R}}.$$  \hspace{1cm} (68)

The index computes the Chern character, while the one-loop determinant is the Euler character. We can read off the weights $w_\alpha(\epsilon, \hat{\alpha})$ from $\text{ind } D_{oe}$, where $\epsilon, \hat{\alpha}$ are the equivariant parameters of the $U(1)^{M+\frac{n}{2}} \times G$ action, and apply the formula

$$\sum_\alpha c_\alpha e^{w_\alpha(\epsilon, \hat{\alpha})} \rightarrow \prod_\alpha w_\alpha(\epsilon, \hat{\alpha})^c_\alpha$$  \hspace{1cm} (69)

to relate the Chern to the Euler character.

The equivariant index $\text{ind } D_{oe}$ can be computed with the Atiyah–Singer index theorem, that we review. Consider: a manifold $M$, a pair of vector bundles $(E_0, E_1)$ over $M$, and the vector spaces $V_1 = \Gamma(E_1)$ of their sections; the action of a Lie group $G$, whose maximal torus is $T = U(1)^n$; an elliptic differential operator $D : V_0 \rightarrow V_1$ commuting with the $G$-action. We define the index $\text{ind } D(t)$ as a formal Laurent series in $t = (t_1, \ldots, t_n) \in T$ as:

$$\text{ind } D(t) = \text{tr}_{H^0} t - \text{tr}_{H^1} t$$  \hspace{1cm} (70)

where $H^0 = \ker D$, $H^1 = \coker D$ and the trace of $t$ is evaluated taking into account the action of $t$ on the subspaces. If $D$ is elliptic and $M$ is compact, then $H^{0,1}$ are finite dimensional vector spaces. In the more general case that $D$ is transversally elliptic, then $H^{0,1}$ can be infinite-dimensional but the eigensubspaces of the $T$-action are finite.
dimensional (in other terms, one can Laurent-expand in $t$ getting finite coefficients). The Atiyah–Singer index formula allows to compute the index from the fixed points of $G$ on $M$. Let $F$ be the set of fixed points of $G$ on $M$:

$$\text{ind } D(t) = \sum_{p \in F} \frac{\text{tr}_{E_0(p)} t - \text{tr}_{E_1(p)}}{\det TM(p)(1 - t)}.$$  \hspace{1cm} (71)

The differential operator $D_{oe}$ is obtained by picking the term $\varphi_0 D_{oe} \varphi_e$ in the second order expansion of $P$, when the localizing action is $\delta S = Q P$. We now consider $\delta S = \int (\mathcal{L}_{YM} + \mathcal{L}_\psi)$ that is

$$P \sim \sum_{\chi} (\overline{Q \chi}) \chi,$$  \hspace{1cm} (72)

therefore $D_{oe}$ is obtained from $Q$ by considering its action on $\chi$ and picking the terms with one derivative acting on $\{\varphi_e\}$. Moreover our manifold $M$ is $S^2$, and the only two fixed points of $U(1)_M$ on $S^2$ are the north pole (NP) and south pole (SP). We should then look at the complex of $Q$ at those points.

**Vector multiplet.** From the action of $Q$ on the vector multiplet in (135), the splitting of fields is:

$$\varphi_e = (A_\mu, \eta), \quad \varphi_0 = (\lambda^+ \epsilon - \epsilon^+ \lambda), \quad \hat{\varphi}_0 = (\lambda^+ \gamma^a \epsilon + \epsilon^+ \gamma^a \lambda), \quad \hat{\varphi}_e = (H)$$

where $H = D + \frac{1}{t} (\sigma + i \eta) - i s F_{12} + i \omega^\mu D_\mu \eta$ and $a = 1, 2, 3$. The scalar $\sigma - i \eta$, which is the parameter for gauge transformations in $\mathcal{R}$, is missing and it appears once ghosts fields are considered as well, as explained in [3]. Now $Q \lambda$ and $Q \lambda^\dagger$ contain the one-derivative terms $d A$ and $d \eta$, therefore we consider the de Rham complex

$$D_{dR} : \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2$$  \hspace{1cm} (73)

tensored by the gauge bundle in the adjoint representation $\mathrm{adj}(E)$. The equivariant index of the complex is defined as

$$\text{ind } D_{dR}(t) = \text{tr}_{\ker d_0} t - \text{tr}_{\text{coker } d_0} t - \text{tr}_{\ker d_1} t + \text{tr}_{\text{coker } d_1} t$$

where $d_\rho$ acts on $\Omega^\rho$, and can be folded as $d^* \oplus d : \Omega^1 \to \Omega^0 \oplus \Omega^2$, where $d^*$ is the adjoint of $d$, noticing that $\text{ind } D_{dR} = - \text{ind } (d^* \oplus d)$. The index formula (71) can be applied to the complexification of (73). We introduce a complex variable $z$ around the north pole, acted upon by $U(1)$ as $z \to iz$ (and $i$ is $i^{-1}$). Now $\Omega^0$ is generated by 1 and $\text{tr}_{\Omega^0} t = 1$; $\Omega^1$ is generated by $dz$, $d\bar{z}$ and $\text{tr}_{\Omega^1} t = t + t^{-1}$; $\Omega^2$ is generated by $dz \wedge d\bar{z}$ and $\text{tr}_{\Omega^2} t = 1$; $TM$ is generated by $\partial_z$, $\partial_{\bar{z}}$ and $\det TM(1 - t) = (1 - t)(1 - t^{-1})$. We get:

$$\text{ind } D_{dR}(t) = \frac{2 - t - t^{-1}}{(1 - t)(1 - t^{-1})} = 1.$$  \hspace{1cm} (74)

For the real complex the index is half of that. We also need to tensor by the adjoint gauge bundle, getting $\text{ind } D_{dR}(t, g) = \frac{1}{2} \chi_{\text{adj}}(g)$, where $g \in G$ and $\chi_{\text{adj}}$ is the character of the

11 The index of $D_{oe}$ depends on its symbol.
adjoint representation. Writing \( t = e^{\xi}, g = e^{\hat{a}} \), the \( U(1)_e \times G_\hat{a} \) equivariant index of the vector multiplet operator is

\[
\text{ind } D^\text{vm} = \frac{1}{2} \sum_{\rho \in \text{adj}(G)} e^{i \rho(\hat{a})}
\]

(75)
as a sum over the weights. Since vanishing weights give a contribution independent of everything, we can discard them and keep a sum over the roots \( \alpha \) of \( G \).

At the north pole we should expand (75) in powers of \( t \), extract weights and multiplicities and apply (69); at the south pole we should instead expand in powers of \( t^{-1} \). Their product is the Euler character

\[
Z_{\text{gauge}} = \prod_{\alpha \in G} \alpha(\hat{a}_{\text{NP}})^{1/2} \alpha(\hat{a}_{\text{SP}})^{1/2},
\]

(76)
where \( \hat{a}_{\text{NP}}, \hat{a}_{\text{SP}} \) are the values of the equivariant parameter at the two poles, which are extracted from the cohomology of \( Q \) (36): \( Q^2 = M + \frac{\theta}{2} + i \Lambda \), with \( i \Lambda = -i \sigma - \cos \theta \eta \). We get

\[
\epsilon = \frac{1}{r}, \quad \hat{a} = -i \sigma - \cos \theta \eta.
\]

(77)
Therefore up to an irrelevant phase:

\[
Z_{\text{gauge}} = \prod_{\alpha > 0} (r\alpha(\eta) + ir\alpha(\sigma))_{\text{NP}}(r\alpha(\eta) - ir\alpha(\sigma))_{\text{SP}}.
\]

(78)
In this computation we have neglected ghosts-for-ghosts needed to fix global gauge transformations on \( S^2 \): we have seen in Sect. 3.1 that their effect is to introduce a Vandermonde determinant for the unbroken gauge subalgebra in the denominator, which cancels an equal Vandermonde determinant in the numerator when integrating over the zero-modes.

For the constant background (47), the expression above agrees with (53). However the current expression is more generally valid on non-constant backgrounds as well.

**Chiral multiplet.** From the action of \( Q \) on the chiral multiplet in (136), the splitting of fields is:

\[
\varphi_e = (\phi), \quad \varphi_o = (\epsilon^T C \psi), \quad \hat{\varphi}_o = (-\epsilon^\dagger \psi), \quad \hat{\varphi}_e = (\tilde{H})
\]

where \( \tilde{H} = F + i \epsilon^T C \gamma^\mu \epsilon D_\mu \phi + \epsilon^T C \gamma_3 \epsilon \eta \phi \). Let us consider the north pole: \( Q_\psi \) contains \( D_- \phi = \tilde{\partial} \phi \) and the relevant complex is the Dolbeault complex

\[
\tilde{\partial}_q : \Omega^{(0,0)} \otimes K^{q/2} \rightarrow \Omega^{(0,1)} \otimes K^{q/2}
\]

(79)
with inverted grading and further tensored by the gauge bundle in the suitable representation \( R_\phi \). Here \( K \) is the canonical bundle which is present because \( \phi \) has R-charge \( q \) and this gives an extra contribution to \( \mathcal{R} \). Notice that for \( q = 1 \), in which case fermions are not charged under \( U(1)_e \), (79) is the same as the Dirac complex \( D_{\text{Dirac}} : S_+ \rightarrow S_- \) which appears in 4d [3]. Now \( \Omega^{(0,0)} \) is generated by 1, \( K \) by \( dz \) and \( \Omega^{(0,1)} \) by \( d\bar{z} \), therefore

\[
\text{ind } \tilde{\partial}_q = \frac{t^{q/2}}{1-t}.
\]

(80)
The full index for the chiral multiplet is then \( \text{ind } D_{\text{cm}} = \frac{n/2}{1-t} \sum \rho e^{i \rho(\hat{a})} \). At the north pole we expand in powers of \( t \) and extract the Euler character with (69):

\[
\sum_{k \geq 0} (-t^{q/2 + k}) \sum_{\rho} e^{i \rho(\hat{a})} \rightarrow \prod_{\rho} \prod_{k \geq 0} \left[ \left( \frac{q}{2} + k \right) \varepsilon + \rho(\hat{a}) \right]^{-1}.
\]

At the south pole instead we expand in powers of \( t^{-1} \), so that similarly:

\[
\sum_{k \geq 0} (t^{-1})^{1 - \frac{q}{2} + k} \sum_{\rho} e^{i \rho(\hat{a})} \rightarrow \prod_{\rho} \prod_{k \geq 0} \left[ (1 - \frac{q}{2} + k)(-\varepsilon) + \rho(\hat{a}) \right]^{-1}.
\]

Multiplying the two, substituting the equivariant parameters \( \varepsilon, \hat{a} \) as in (77) and regularizing the infinite products as in Sect. 3.2 we get, up to an irrelevant phase:

\[
Z_{\text{matter}} = \prod_{\rho \in K_{\Phi}} \frac{\Gamma\left(\frac{q}{2} - i \rho(\sigma) - r \rho(\eta)\right)_{NP}}{\Gamma\left(1 - \frac{q}{2} + i \rho(\sigma) - r \rho(\eta)\right)_{SP}}. \tag{81}
\]

For the constant background (47) this expression agrees with (60).

4. Localization on the Higgs Branch

It has been observed in [21] that the path integral on the three-dimensional ellipsoid \( S_{3b}^{3} \) of \( \mathcal{N} = 2 \) Maxwell–Chern–Simons theory with \( N_f \) chiral multiplets of charge +1 and \( N_a \) of charge −1 can be written as a sum of contributions each of which is the product of a classical action exponential, some one-loop determinants, a partition function for vortices and another for antivortices. This suggests that a similar factorization might take place in our two-dimensional setup. This is in fact the case, as we explicitly show in Sect. 5 for a \( U(N) \) gauge theory with \( N_f \) fundamentals, \( N_a \) antifundamentals and possibly one adjoint. It is natural to ask whether there is an a priori reason for such factorization, and our answer is positive.

In this section we show that when the gauge group admits FI terms that would lead to complete Higgsing in flat space, it is possible to perform an alternative localization in which the BPS configurations dominating the path integral are a finite number of points on the Higgs branch, supporting point-like vortices at the north pole and antivortices at the south pole. We call this “localization on the Higgs branch”. This has to be compared to the expression of \( Z_{S^2} \) obtained in Sect. 3, which comes from BPS configurations on the Coulomb branch.

To simplify the discussion, in this section we will mainly consider a \( U(N) \) gauge group with (anti)fundamentals; the construction can however be generalized to other gauge groups and matter representations (quivers, for instance). Moreover, we consider the case that all chiral multiplets have R-charge \( q = 0 \), which is possible unless there is a superpotential. We observed in Sect. 2.2 that the action is holomorphic in \( M + i \frac{q}{2} \) (where \( M \) is a twisted mass in units of \( r^{-1} \)), and so must be \( Z_{S^2} \). Therefore given the answer \( Z_{S^2} \) for \( q = 0 \), we can obtain \( Z_{S^2} \) for generic R-charges by analytic continuation in the twisted masses. Since a superpotential only affects \( Z_{S^2} \) by imposing constraints on the R-charges, we can similarly obtain \( Z_{S^2} \) for a theory with superpotential by neglecting the constraints first, computing \( Z_{S^2} \) at \( q = 0 \) and then performing the analytic continuation to reinstate the constraints.
At \( q = 0 \) a continuous Higgs branch opens up: besides (46), the BPS equations (45) with real bosonic vector multiplet fields and deformed by generic twisted masses \((\sigma \to \sigma + M)\) admit the solutions

\[
0 = F = D_\mu \phi = \eta \phi = (\sigma + M)\phi, \tag{82}
\]

where it is implied that \( \sigma \) and \( M \) act on \( \phi \) in the correct representation of \( G \times G_F \). If \( q > 0 \) the BPS equations still have Higgs branch solutions but for complex—not real—values of \( \sigma \), given by \( (\sigma + \frac{iq}{2r} + M)\phi = 0 \). Consequently such solutions are not saddle points of \( L_\psi \) with \( \sigma \) defined as after (37), see footnote 7: if we would like to have a Higgs branch at another value of \( q \) we should modify \( \sigma \), that is the path integration contour, accordingly.

We add to the deformation action \( \delta S = \int (L_{YM} + L_\psi) \) a new \( Q \)-exact and \( Q \)-closed term:

\[
L_H = Q \text{Tr} \left[ \frac{\epsilon_+ \lambda - \lambda_+ \epsilon_+}{2i} (\phi \phi^\dagger - \chi \mathbb{1}) \right], \tag{83}
\]

where \( \mathbb{1} \) has rank \( N \) as the gauge group, \( \phi \) transforms in a possibly reducible representation \( R_\Phi \) and includes all chiral multiplets, and \( \chi \) is a free parameter. We made a little abuse of notation since \( \lambda, \lambda^\dagger \) act on \( \phi \) in the representation \( R_\Phi \) and on \( \mathbb{1} \) in the adjoint. The parameter \( \chi \) is meaningful only if the gauge group has an Abelian factor, otherwise it disappears under the trace. The bosonic part of \( L_{YM} + L_\psi + L_H \) is, up to total derivatives,

\[
L_{\text{bos}} = \text{Tr} \left\{ \frac{1}{2} \left( F_{12} - \frac{\eta}{r} \right)^2 + \frac{1}{2} \left( D + \frac{\sigma}{r} \right)^2 + \frac{1}{2} (D_\mu \sigma)^2 + \frac{1}{2} (D_\mu \eta)^2 - \frac{1}{2} [\sigma, \eta]^2 \right. \\
\left. + \left[ i (D + \frac{\sigma}{r}) + s \left( F_{12} - \frac{\eta}{r} \right) - w^\mu D_\mu \eta \right] (\phi \phi^\dagger - \chi \mathbb{1}) \right\} + L_\psi \big|_{\text{bos}}, \tag{84}
\]

Notice that this action is not positive definite, however we can make it so with a trick. The path integral over \( D \) is Gaussian and can be performed exactly: it generates a term \( \frac{1}{2} \text{Tr} (\phi \phi^\dagger - \chi \mathbb{1})^2 \) and formally imposes the constraint

\[
D + \frac{\sigma}{r} + i (\phi \phi^\dagger - \chi \mathbb{1}) = 0. \tag{85}
\]

Now at the saddle point \( D + \frac{\sigma}{r} \) is imaginary, therefore effectively we have changed the path integration contour of the auxiliary field \( D \). The Lagrangian can then be brought to the form

\[
L_{\text{bos}} = \frac{1}{2} \text{Tr} \left\{ \left[ \sin \theta \left( F_{12} - \frac{\eta}{r} \right) + \cos \theta \ D_1 \eta \right]^2 + (D_2 \eta)^2 + (D_\mu \sigma)^2 - [\sigma, \eta]^2 \right. \\
\left. + \left[ \phi \phi^\dagger - \chi \mathbb{1} + \cos \theta \left( F_{12} - \frac{\eta}{r} \right) - \sin \theta \ D_1 \eta \right]^2 \right\} + L_\psi \big|_{\text{bos}}, \tag{86}
\]

which is a sum of squares.

The set of smooth configurations such that \( L_{\text{bos}} = 0 \), which coincide with its saddle points, consists of two branches. First we have a Higgs branch:

\[
0 = F_{12} - \frac{\eta}{r} = D_\mu \eta = D_\mu \sigma = [\sigma, \eta] = F = D_\mu \phi = \eta \phi = (\sigma + M)\phi = \phi \phi^\dagger - \chi \mathbb{1}. \tag{87}
\]
The last one is the usual D-term equation, which spelled out reads: \( \phi^T T^A_\phi \phi = \chi \delta^{A1} \) for \( A = 1, \ldots, \dim G \), where \( T^A_\phi \) are the gauge generators in representation \( R_\phi \) and \( T^1_\phi \) is the generator of \( U(1) \). The set of solutions strongly depends on the gauge group and matter content. For \( U(N) \) with \( N_f \) fundamentals \( \phi \) and \( N_a \) antifundamentals \( \tilde{\phi} \) and for generic twisted masses \( M \), up to gauge and flavor rotations the solution is: \( \phi_{a\beta} = \sqrt{\chi} \delta_{a\beta} \) (\( a \) is a gauge index, \( \beta \) a flavor index) for \( \chi > 0 \) and \( N_f \geq N; \tilde{\phi}_{ay} = -\sqrt{\chi} \delta_{ay} \) for \( \chi < 0 \) and \( N_a \geq N \); empty otherwise. This forces \( 12 = 0 = \eta = F_{12} \), and each eigenvalue of \( \sigma \) to equal a different eigenvalue of \(-M\) (of fundamentals or antifundamentals, depending on sign \( \chi \)):

\[
\sigma_a = -M_{ia}
\]

(88)

where vacua are parametrized by \( N \)-combinations \( I = (l_1, \ldots, l_N) \in C(N, N_f) \) of the first \( N_f \) (or \( N_a \)) integers. Summarizing, for generic twisted masses the set of Higgs branch solutions consists of some number of isolated vacua, in general different for \( \chi \geq 0 \). We will further study these solutions below.

Second we have a Coulomb branch:

\[
0 = D_A \sigma = [\sigma, m] = \phi, \quad \eta = r \chi \cos \theta + m/2r, \quad F_{12} = 2 \chi \cos \theta + m/2r^2
\]

(89)

for all \( \sigma \) and GNO quantized \( m \). The one-loop determinants have to be computed with the index theorem: the contribution of the gauge multiplet and of the chiral multiplet \( \phi \) according to (78)–(81) is \( 13 \)

\[
\prod_{\alpha > 0} \left( \frac{\alpha(m)^2}{4} + \alpha(\sigma)^2 \right) \times \prod_{\rho \in R_\phi} \frac{\Gamma(a_\rho - \rho(\chi))}{\Gamma(1 + a_\rho^* + \rho(\chi))}
\]

where \( a_\rho = -i\rho(\sigma) - \rho(m)/2 \), we set \( r = 1 \) and \( \alpha(\chi) = 0 \). For \( \chi \to \pm \infty \) we can use the limit:

\[
\frac{\Gamma(a - z)}{\Gamma(1 + a^* + z)} \xrightarrow{z\to\pm\infty} e^{-(2z+1)(\log |z| - 1)O(1) + O(\Im a)} \left( 1 + O\left(\frac{1}{z - 1}\right) \right).
\]

(90)

Let us set \( \rho_i(\chi) = q_i \chi \), so that \( q_i = \pm 1 \) are the charges under the \( U(1) \) factor. Then the leading contribution is \( \exp(-2(\sum_i q_i)\chi \log |\chi|) \); if \( \sum q_i = 0 \), the leading contribution is \( \exp(-\sum_i q_i \log |\chi|) \). In any case it is possible to send \( \chi \) to infinity with either one of the two signs so that the Coulomb branch contribution is suppressed and the path integral localizes on the Higgs branch.

We already noticed that the Higgs branches at \( \chi \geq 0 \) are different: suppression of the Coulomb branch only works on (and thus selects) one of the two, depending on the matter content. We will see that this has a nice counterpart in the computation via localization on the Coulomb branch. In particular we choose \( \text{sign}(\chi) = \text{sign}(\sum q_i) \) when the latter is defined. For instance if \( q_i \) are all positive, the selected sign of \( \chi \) is the one compatible with the existence of Higgs branch solutions.

12 For simplicity in the text we assume that \( 0 = \eta^{\text{ext}} = F_{12}^{\text{ext}} \). If that is not the case, (87) is modified to \( (\eta + \eta^{\text{ext}})\phi = 0 \) and on solutions \( F_{12}^{\text{ext}} = \eta^{\text{ext}}/r \) will be turned on as well. The following discussion will be modified accordingly.

13 The expression is only valid for \( q = 0 \), which is the situation we are analyzing here.
On top of each of the smooth Higgs branch solutions there are singular point-like vortex solutions at the north pole $\theta = 0$ and antivortex solutions at the south pole $\theta = \pi$:

$$
\begin{align*}
\text{\theta = 0:} & \quad \{ D + \frac{\sigma}{r} = -i(\phi\phi^\dagger - \chi) \implies +iF_{12}, \quad D_\phi = 0 \} \\
\text{\theta = \pi:} & \quad \{ D + \frac{\sigma}{r} = -i(\phi\phi^\dagger - \chi) \implies -iF_{12}, \quad D_\phi = 0 \}.
\end{align*}
$$

(91)

In the vortex and antivortex solutions $\phi$ has a winding and it vanishes at the core; for $\chi > 0$ the vortex number $k = \frac{1}{2\pi}\int \text{Tr} F_{12}e^{12}$ is positive for vortices and negative for antivortices. When there is no continuous Higgs branch but only discrete points—which is the case for generic twisted masses—the size of vortices is not a modulus (as instead happens in 4d with instantons) but rather their size is of order $L \sim r/\sqrt{|\chi|}$, assuming that in (84) we adjust the dimensions with powers of $r$. Only in the limit $\chi \to \pm \infty$ we really have point-like vortices.

In fact we can take a deeper perspective on the localizing action $L_{YM} + L_{\psi} + L_H$: as we change the parameter $\chi$ from zero to infinity we interpolate between localization on the Coulomb branch and localization on the Higgs branch. It is easy to see that for $\chi = 0$ the bosonic action (84) localizes on the Coulomb branch and it is equivalent to $L_{YM} + L_{\psi}$ as discussed in Sect. 3: on the one hand the Coulomb branch contribution is no longer suppressed and it precisely coincides with that of Sect. 3; on the other hand the VEV $\phi^\dagger\phi$ of vortices vanishes and their size $L \sim r/\sqrt{|\chi|}$ blows up as $\chi \to 0$: vortices simply become the Coulomb branch configurations with constant flux. 14

4.1. The partition function. Let us now compute the path integral $Z_{S^2}$ by localization on the Higgs branch. We send $\chi \to \pm \infty$ according to the net charge under the $U(1)$ factor, in such a way that anything else than the Higgs branch contribution is suppressed. We also assume that the gauge group is completely Higgsed. In this case the classical smooth configurations are such that as many matter components as the rank $N$ of the gauge group get a VEV of order $\chi$, while all other VEVs are zero. To achieve that, we reach first a point (88) on the Coulomb branch where the flavors would be massless before turning on $\chi$: on that vacuum the classical action contribution to $Z_{S^2}$ is

$$
e^{-S_{\text{class}}} = e^{-4\pi i \xi \sum_{j=1}^{N} \sigma_j},$$

in case we do not have a more general twisted superpotential $\tilde{W}$. After turning on $\chi$ the $N$ components get a mass squared of order $\chi$.

The path integral is thus given by a sum over the solutions to the D-term equations (87). In each configuration, the one-loop determinants for the massive W-bosons and the matter fields with vanishing VEV are given by (53)–(60) (or equivalently (78)–(81)) evaluated at $m = 0$ (or $\eta = 0$) and at the values of $\sigma_j$ (88) in that vacuum. The latter combined with $M$ give the effective twisted masses of the matter fields, that we schematically call $m_{\text{eff}}$. The action $\int L_H$ in (83) linearized around the Higgsed vacuum solutions gives mass of order $\chi$ and mixes the broken Cartan gauge field and the chiral field acquiring VEV. The easiest way to compute their one-loop determinant in the $\chi \to \infty$ limit is to use the Coulomb branch computation for $U(1)$ with one flavor of charge 1 in Sect. 5.1: postponing the explanation to the end of this section, the result is

$$
Z_{\text{Higgsed}} = e^{-\frac{i}{2} \int F_{12}}.
$$

(92)

14 One might be worried that some vortices survive in the $\chi \to 0$ limit: that is the case when their size is a zero-mode. However this does not happen for generic twisted masses. See for instance [33].
This can be interpreted as an effective shift of the theta-angle by \( \pi \). When the chiral getting VEV has higher charge, the formula has to be modified accordingly.

On top of each smooth classical configuration, we have point-like vortices at the north pole and antivortices at the south pole. They differ by the sign of the magnetic flux, which is positive for vortices and negative for antivortices. The complex of \( Q \) close to the NP \( \{ \theta = 0 \} \) is at first order in \( \theta \) the same as the complex of the operator \( Q' \) used in \([22]\) to construct the \( \Omega \)-background \([18,19]\) in \( \mathbb{R}^2 \). One has to identify \( U(1)^{M+\frac{N}{2}} \) here with \( U(1)_e \) there in terms of the equivariant parameter \( \epsilon = \frac{1}{r} \), as well as \( i \Lambda = -i \sigma - \eta \) here with the equivariant parameter \( a \) there. Therefore the sum over the singular vortex configurations at the NP is computed by the vortex partition function \( Z_{\text{vortex}} \) in the \( \Omega \)-background. The one-loop determinants for W-bosons and vanishing matter fields do not depend on the flux at the pole, while \( Z_{\text{Higgsed}} \) above does. The classical weight of each vortex configuration is:

\[
e^{-S} = \exp \left[ - \int (-i \xi D + i \frac{\theta}{2\pi} F_{12}) \right] = e^{-2\pi \xi k - i \theta k} = z^k, \quad \text{with} \quad z = e^{-2\pi \xi - i \theta}
\]

where we used \( F_{12} = k/2r^2 \), \( D = i F_{12} \) at the NP. Taking into account (92) as well:

\[
Z_{\text{NP}} = Z_{\text{vortex}}((-1)^N z, \epsilon = \frac{1}{r}, a = -im_{\text{eff}}).
\] (93)

At the SP the equivariant rotational parameter is \( \epsilon = -\frac{1}{r} \). The flux \( F_{12} \) is negative: in fact antivortices are obtained from vortices by charge conjugation\(^{15} \) and this changes sign to the twisted masses. With \( F_{12} = -k/2r^2 \) and \( D = i F_{12} \) at the SP, we have weight \( e^{-S} = \bar{z}^k \) and therefore:

\[
Z_{\text{SP}} = Z_{\text{vortex}}((-1)^N \bar{z}, \epsilon = -\frac{1}{r}, a = im_{\text{eff}}).
\] (94)

The final expression for the path integral localized on the Higgs branch is:\(^{16} \)

\[
Z_{S^2} = \sum_{\text{D-term solutions}} e^{-4\pi i \xi} \sum_{j=1}^{N} \sigma_j Z'_{\text{1-loop}} Z_{\text{vortex}}((-1)^N z, \frac{1}{r}, -im_{\text{eff}})
\]

\[
\times Z_{\text{vortex}}((-1)^N \bar{z}, -\frac{1}{r}, im_{\text{eff}}),
\] (95)

where \( Z'_{\text{1-loop}} \) reminds us that the \( N \) non-vanishing chiral multiplets should not be included.

Finally let us justify (92). In Sect. 5.1 we compute \( Z_{S^2} \) for \( U(1) \) with one chiral of charge \( Q = 1 \) by localization on the Coulomb branch, obtaining \( Z_{S^2} = e^{-z+\bar{z}} \). On the other hand the vortex partition function is known to be \( Z_{\text{vortex}}(z, \epsilon) = e^{z/\epsilon} \) \([22,34–36]\), and the classical contribution is trivial. By comparing the two expressions we deduce (92).

The expression (95) is in some sense similar to the four-dimensional result of Pestun [3]. However while in that case one has an integral over the Coulomb branch of a partition function of non-perturbative configurations (there instantons), on \( S^2 \) we find either an

\(^{15} \) Given a vortex solution characterized by \( \int F = 2\pi k > 0 \) and \( D_\pm \Phi = 0 \), we can transform the gauge multiplet as \( V \to -V \) and the chiral multiplet as \( \Phi \to \Phi^\dagger \). Hence \( D_\pm \Phi \to (D_\pm \Phi)^\dagger \) and the vortex solution is mapped to an antivortex solution.

\(^{16} \) The formula has been written having \( U(N) \) with fundamentals, antifundamentals and adjoints in mind.
integral over the Coulomb branch of perturbative contributions, or a discrete sum of products of perturbative and non-perturbative contributions. We will show in Sect. 5 by direct evaluation that indeed for $U(N)$ with flavors and possibly one adjoint, the two expressions (63) and (95) coincide.

To conclude, we notice that (95) could be generalized to the more intricate case that a non-trivial twisted superpotential $\tilde{W}$ is present, but we leave that for future work.

5. Vortex Partition Functions

In this section, we explicitly compute the path integral $Z_{S^2}(\xi, \theta, \tau_j, \bar{\tau}_j, \tau_A)$ by localization on the Coulomb branch, for a $U(N)$ gauge theory with $N_f$ chirals in the fundamental representation, $N_a$ chirals in the antifundamental (we will succinctly refer to $(N_f, N_a)$ flavors), possibly one adjoint chiral, twisted masses $\tau_i, \bar{\tau}_j, \tau_A$ respectively, FI term $\xi$ and theta-angle $\theta$. We will see that it reproduces the result of localization on the Higgs branch, in terms of the vortex partition function. In the following we set to zero the magnetic fluxes for global symmetries, in order to simplify the computations.

5.1. $U(1)$ with $(N_f, N_a)$ flavors. Consider first a $U(1)$ gauge theory with $N_f$ chirals of charge $+1$ and masses $\tau_j$, and $N_a$ of charge $-1$ and masses $\bar{\tau}_j$. We take canonical R-charge $q = 0$: generic R-charges are eventually obtained by giving an imaginary part to $\tau_j, \bar{\tau}_j$. The matrix integral (63) is:

$$Z_{U(1)}^{(N_f, N_a)}(\xi, \theta; \tau, \bar{\tau}) = \sum_{\mathbb{Z}} e^{-i\theta m} \int \frac{d\sigma}{2\pi} e^{-4\pi i \xi \sigma} \prod_{j=1}^{N_f} \frac{\Gamma(-i\sigma - i\tau_j - \frac{m}{2})}{\Gamma(1 + i\sigma + i\tau_j - \frac{m}{2})} \prod_{f=1}^{N_a} \frac{\Gamma(i\sigma - i\bar{\tau}_f + \frac{m}{2})}{\Gamma(1 - i\sigma + i\bar{\tau}_f + \frac{m}{2})}.$$  (96)

Recall that the integration contour in $\sigma$ is defined along the real line when all R-charges are bigger than zero. The case when some R-charges are zero or negative is obtained by analytic continuation deforming the contour.

We evaluate the integral by closing the integration contour at infinity according to the asymptotic behavior of the integrand and summing the residues. The gamma functions have only poles, no zeros, so we can focus on the numerators. Assuming $N_f > N_a$, or $N_f = N_a$ and $\xi > 0$, we can close the contour in the lower half-plane, therefore only the chirals with positive charge give poles enclosed in the contour. The result for $N_f < N_a$, or $N_f = N_a$ and $\xi < 0$, can be obtained thanks to

$$Z_{U(1)}^{(N_f, N_a)}(\xi, \theta; \tau, \bar{\tau}) = Z_{U(1)}^{(N_f, N_a)}(-\xi, -\theta; \bar{\tau}, \tau),$$  (97)

which follows from a sign change in the dummy variables of (96), or charge conjugation. We show in Sect. 5.4 that a non-trivial identity implies analyticity in $\xi$ for $N_f = N_a$.

There are $N_f$ towers of poles, one for each fundamental $l = 1, \ldots, N_f$. The $l$-th tower of poles is given by $\sigma = -\tau_l - ik - \frac{l}{2} |m|, k \in \mathbb{N}$. The residue of the gamma

\[17\] The combination $\frac{1}{N_f} \sum_{i=1}^{N_f} \tau_i - \frac{1}{N_a} \sum_{j=1}^{N_a} \bar{\tau}_j$ is set to zero since the flavor symmetry is $S[U(N_f) \times U(N_a)]$. 

At a pole is $i(-1)^k/k!$ where we defined $k_1 = k + |m|$, while the value of $\Gamma$ in the denominator is $(k_1 - m)!$. The value of the remaining one-loop determinants at a pole is conveniently written in terms of
\[ M_j^l \equiv \tau_j - \tau_l, \quad \tilde{M}_j^l \equiv \tilde{\tau}_f + \tau_l, \quad (98) \]
They are the effective masses of the $j$-th fundamental and $f$-th antifundamental at the point of the Coulomb branch $\sigma = -\tau_l$ where the $l$-th fundamental is massless: $M_{1}^{l} = 0$. We can define $k_2 \equiv k_1 - m$ so that $\sum_{m \in \mathbb{Z}} \sum_{k \geq 0} = \sum_{k_1 \geq 0} \sum_{k_2 \geq 0}$ and apply the identity
\[ \Gamma(a + n) = (a)_n \Gamma(a), \]
in terms of the Pochhammer symbol $(a)_n = \prod_{j=0}^{n-1}(a + j)$. Finally we get the expression
\[ \sum_{l=1}^{N_f} e^{4\pi i \xi \tau_l} Z_{1\text{-loop}}^{(l)} Z_{\text{vortex}}^{(l)}(z) Z_{\text{av}}^{(l)}(\tilde{z}), \]
where the various pieces are:
\[ Z_{1\text{-loop}}^{(l)} = \left( \prod_{j \neq l}^{N_f} \frac{\Gamma(-iM_j^l)}{\Gamma(1 + iM_j^l)} \right) \left( \prod_{f}^{N_a} \frac{\Gamma(-i\tilde{M}_f^l)}{\Gamma(1 + i\tilde{M}_f^l)} \right) \]
\[ Z_{\text{vortex}}^{(l)}(z) = \sum_{k=0}^{\infty} \frac{\prod_{j \neq l}^{N_f} (-i\tilde{M}_j^l)_k}{\prod_{j \neq l}^{N_f} (1 + iM_j^l)_k} \frac{(-1)^N k^k z^k}{k!} \]
\[ = N_a F_{N_f - 1} \left( \left\{ -i\tilde{M}_l^l \right\}_{j \neq l}^{N_f} \left\{ 1 + iM_j^l \right\}_{j \neq l}^{N_f} \right) \left( -1 \right)^{N_f} z \]
\[ Z_{\text{av}}^{(l)}(\tilde{z}) = \sum_{k=0}^{\infty} \frac{\prod_{j \neq l}^{N_f} (-i\tilde{M}_j^l)_k}{\prod_{j \neq l}^{N_f} (1 + iM_j^l)_k} \frac{(-1)^N k^k \tilde{z}^k}{k!} \]
\[ = N_a F_{N_f - 1} \left( \left\{ -i\tilde{M}_l^l \right\}_{j \neq l}^{N_f} \left\{ 1 + iM_j^l \right\}_{j \neq l}^{N_f} \right) \left( -1 \right)^{N_f} \tilde{z} \]
and $z = e^{-2\pi i \xi - i\theta}$. Here $p_{F_l}$ is the generalized hypergeometric function. For $N_f > N_a$ the two series converge for any $z$; for $N_f = N_a$ they converge for $|z| < 1$ (which is the original domain of validity of our computation) and can be extended in the whole $z$-plane analytically. In fact they can be written as:
\[ Z_{\text{vortex}}^{(l)}(z) = Z_{\text{vortex}}(-z, 1, \left\{ -i\tilde{M}_j^l \right\}_{j \neq l}, \left\{ -i\tilde{M}_f^l \right\}_f) \]
\[ Z_{\text{av}}^{(l)}(\tilde{z}) = Z_{\text{vortex}}(-\tilde{z}, -1, \left\{ i\tilde{M}_j^l \right\}_{j \neq l}, \left\{ i\tilde{M}_f^l \right\}_f) \]
in terms of the vortex partition function for $U(1)$ with $(N_f, N_a)$ flavors in $\Omega$-background, that we can write as
\[ Z_{\text{vortex}}^{U(1)}(z, \varepsilon, m_j, \tilde{m}_f) = \sum_{k=0}^{\infty} \frac{z^k}{\varepsilon^{(N_f - N_a)k}} \frac{\prod_{f=1}^{N_f} \left( \frac{\tilde{m}_f}{\varepsilon} \right)_k}{k! \prod_{j=1}^{N_f - 1} \left( \frac{m_j}{\varepsilon} - k \right)_k}. \]
This expression agrees with [34], up to a redefinition of the masses. In [21] a similar structure has been found for the path integral $Z_{\Delta_3}^{3dN}$ of 3d $N = 2$ YM-CS $U(1)$ theory on the
three-dimensional ellipsoid. The expression in (99) agrees with our general expression (95).

5.2. $U(N)$ with $(N_f, N_a)$ flavors. Consider now a $U(N)$ gauge theory with flavors $(N_f, N_a)$. The matrix integral (63) is:

$$Z_{U(N)}^{(N_f, N_a)} = \frac{1}{N!} \sum_{m \in \mathbb{Z}^N} \int \prod_{r=1}^N d\sigma_r \, e^{-4\pi i \xi \sigma_r - i \theta m_r} \prod_{i < s}^N \left( \frac{(m_t - m_s)^2}{4} + (\sigma_t - \sigma_s)^2 \right)$$

$$\times \prod_{r=1}^N \prod_{j=1}^{N_f} \Gamma \left( -i \sigma_r - i \tau_j - \frac{m_r}{2} \right) \prod_{r=1}^{N_a} \Gamma \left( i \sigma_r - i \bar{\tau}_f + \frac{m_r}{2} \right). \quad (103)$$

We assume as before that $N_f > N_a$, or $N_f = N_a$ and $\xi > 0$, so we can close the integration contours in the lower half-planes and resum residues at the poles arising from fundamentals only.

In this section we compute (103) in terms of $Z_{U(1)}^{(N_f, N_a)}$ using a trick, while in the next section we will more conventionally sum over all poles. Note that (103) would be identical to the partition function of $N$ copies of $U(1)$ with $(N_f, N_a)$ flavors—though all with $\xi_r = \xi$ and $\theta_r = \theta$—were not for the one-loop determinant of the non-Abelian gauge fields, which however does not contribute poles to the integrand. We can thus introduce auxiliary distinct FI parameters $\xi' \equiv (\xi_1, \ldots, \xi_{N_f})$ and theta-angles $\theta' \equiv (\theta_1, \ldots, \theta_{N_a})$ for each Cartan $U(1)$ factor, eventually to be set to $\xi_r$ and $\theta_r$, and replace the non-Abelian determinant by $\sigma_r \leftrightarrow \frac{i}{4\pi} \frac{\partial}{\partial \xi_r}$, $m_r \leftrightarrow i \frac{\partial}{\partial \theta_r}$ acting on the integral. We can then write

$$Z_{U(N)}^{(N_f, N_a)}(\xi, \theta) = \frac{1}{N!} \Delta \prod_{r=1}^N Z_{U(1)}^{(N_f, N_a)}(\xi_r, \theta_r) \bigg|_{\xi_r = \xi, \theta_r = \theta}, \quad (104)$$

where $\Delta = \prod_{r<s} \left[ -\frac{1}{4} \left( \frac{\partial}{\partial \xi_r} - \frac{\partial}{\partial \xi_s} \right)^2 - \frac{1}{16\pi^2} \left( \frac{\partial}{\partial \theta_r} - \frac{\partial}{\partial \theta_s} \right)^2 \right]$ is a differential operator and $Z_{U(1)}$ has been computed before, with the dependence on twisted masses left implicit. We can further simplify the expression by introducing complex variables $w^l = -\log \zeta^l = 2\pi \xi^l + i \theta^l$. We get:

$$Z_{U(N)}^{(N_f, N_a)}(w, \bar{w}) = \frac{1}{N!} (-1)^{\frac{N(N-1)}{2}} (2\pi)^N \prod_{r=1}^N \left. Z_{U(1)}^{(N_f, N_a)}(w^r, \bar{w}^r) \right|_{w^r = w, \bar{w} = \bar{w}}, \quad (105)$$

where the differential operators are $\mathcal{D} = \prod_{r<s} (\partial_r - \partial_s)$ and $\overline{\mathcal{D}} = \prod_{r<s} (\overline{\partial}_r - \overline{\partial}_s)$,

$$Z_{U(1)}^{(N_f, N_a)}(w^r, \bar{w}^r) = \sum_{l=1}^{N_f} e^{i (w^r + \overline{w}^r) \tau_l} Z_{1\text{-loop}}^{(l)} Z_{v}^{(l)} (e^{-w^r}) Z_{av}^{(l)} (e^{-w^r}), \quad (106)$$

and the various pieces are as in (100). In (105) effectively we have a sum over $l = (l_1, \ldots, l_N)$ where each entry runs from 1 to $N_f$. Each term represents a point on the Coulomb branch where the flavors $(l_1, \ldots, l_N)$ have a massless component. Since both $\mathcal{D}$, $\overline{\mathcal{D}}$ are totally antisymmetric in $w^r$, $\overline{w}^r$ respectively, $Z_{U(1)}(w, \bar{w})$ factorizes into the product of a holomorphic and an antiholomorphic function of $w$, and eventually we set $w^r = w$, $\overline{w}^r = \overline{w}$, it follows that whenever two $l_r$’s are equal that term vanishes. In particular for $1 \leq N_f < N$ and generic twisted masses the localized partition function $Z_{g^2}$ vanishes. On the other hand each term is invariant under permutations of $(l_1, \ldots, l_N)$
(which is the action of the Weyl group) therefore we can restrict ourselves to sum over \(N\)-combinations \(C(N, N_f)\) of the first \(N_f\) integers, and remove \(N!\) from the denominator.

To proceed further we split the action of \(\mathcal{D}\) on the exponential in (106) and on \(Z_v^{(l)}\), and similarly for \(\mathcal{D}\). The result can be written as:

\[
Z_{U(N)}(w, \overline{w}) = \sum_{l \in C(N, N_f)} e^{i(w + \overline{w})} \sum_{r=1}^{N} \tau_r \ Z_{1\text{-loop}}^{(l)} Z_v^{(l)}(e^{-w}) Z_{\text{av}}^{(l)}(e^{-\overline{w}}),
\]

(107)

where the one-loop factor now includes the massive W-bosons:

\[
Z_{1\text{-loop}}^{(l)} = \left[ \prod_{i < s}^{N} (M_{i_s}^{l})^2 \right] \left[ \prod_{r=1}^{N} Z_{1\text{-loop}}^{(l_r)} \right].
\]

(108)

The (anti)vortex contributions are \(Z_v^{(l)}(e^{-w}) = D^{(l)} \prod_{r=1}^{N} Z_v^{(l_r)}(e^{-w})|_{w' = w}\) and \(Z_{av}^{(l)}(e^{-\overline{w}}) = \overline{D}^{(l)} \prod_{r=1}^{N} Z_{av}^{(l_r)}(e^{-\overline{w}})|_{\overline{w}' = \overline{w}}\), and the differential operators are \(D^{(l)} = \prod_{l < s}^{N} [1 + i(\partial_l - \partial_s)/M_{i_s}^{l}]\) and \(\overline{D}^{(l)} = \prod_{l < s}^{N} [1 + i(\partial_s - \partial_l)/M_{i_s}^{l}]\). These expressions are only well-defined for \(l\) a combination without repetitions, as we are assuming; moreover each term is invariant under permutations of \(l\). Finally we can write the vortex contribution in a more explicit way, acting with the differentials. After getting rid of differentials, we can go back to \(z = e^{-w}\) and, using the identity \((a - l)_m (a - m)_l = (1 + \frac{m-l}{a})^{-1} (a + 1)_m (a + 1)_l\), we can recast the vortex contribution as:

\[
Z_v^{(l)}(z) = \sum_{k \in \mathbb{Z}_{\geq 0}^N} (-1)^{(N-1)|k|} z^{|k|} \prod_{r \in l}^{N} \frac{1}{(i M_{f}^{r})_{kr} k_r} \prod_{j \notin l}^{N} (-i M_{f}^{j} - k_r)_{kj} \prod_{j \notin l}^{N} (-i M_{f}^{j} - k_r)_{kj}.
\]

(109)

Let us explain the notation: \(r \in l\) is a fundamental flavor index taking one of the \(N\) values \((l_1, \ldots, l_N)\) picked up by \(l\), while \(j \notin l\) runs over the remaining \(N_f - N\) values; \(k\) is an \(N\)-tuple whose entries \(k_r\) we parameterize by \(r \in l\), and \(|k| = \sum_r k_r\). Similar manipulations can be done on \(Z_{av}^{(l)}(e^{-\overline{w}})\).

Eventually the two (anti)vortex contributions can be written as:

\[
Z_v^{(l)}(z) = Z_{\text{vortex}}(-1)^N z, 1, \{-i \tau_r\}_{r \in l}, \{-i \tau_j\}_{j \notin l}, \{-i \bar{\tau}_f\} \quad Z_{av}^{(l)}(z) = Z_{\text{vortex}}(-1)^N z, -1, \{i \tau_r\}_{r \in l}, \{i \tau_j\}_{j \notin l}, \{i \bar{\tau}_f\}
\]

(110)

in terms of the vortex partition function for \(U(N)\) with \((N_f, N_a)\) flavors:

\[
Z_{\text{vortex}}(z, \varepsilon, a_r, m_j, \tilde{m}_f) = \sum_{k \in \mathbb{Z}_{\geq 0}^N} \frac{z^{|k|}}{\varepsilon^{(N_f - N_a)|k|}} \prod_{f=1}^{N_a} \frac{\tilde{m}_f + a_r}{\varepsilon} \prod_{j \neq r} \frac{a_j - a_r}{\varepsilon - k_r} \prod_{j=N+1}^{N_f} \frac{m_j - a_r}{\varepsilon - k_r}.
\]

(111)
In this notation we have distinguished between the masses $a_r$ of the $N$ flavors with a component acquiring VEV, and the masses $m_j$ of the $N_f - N$ flavors with vanishing VEV, because they appear differently in the vortex partition function. This asymmetry is inherent to the brane inspired construction of vortex moduli spaces of [37], where the $k$-vortex theory for a 2d $U(N)$ gauge theory with $N_f$ fundamentals is a 0d $U(k)$ gauge theory with one adjoint, $N$ fundamentals and $N_f - N$ antifundamentals. The expression (111) agrees with (102) for $N = 1$, although the two are written in a slightly different notation.

The vortex partition function (111) also agrees with [22,35,36,38] for $N_f = N$, apart from a possible minus sign in front of $z$ which can be reabsorbed by a redefinition of the theta angle. Our final expression (107) has precisely the form of (95). Finally, the vortex partition function (111) can be written as a contour integral, as we spell out in Sect. 6.

5.3. $U(N)$ with flavors and one adjoint: $\mathcal{N} = (2, 2)^*$ SQCD. Consider a $U(N)$ gauge theory with $(N_f, N_a)$ flavors and one adjoint chiral multiplet. The matrix integral (63) is:

$$Z = \frac{1}{N!} \sum_{m \in \mathbb{Z}^N} \int \left[ \prod_{r=1}^{N_f} \frac{d\sigma_r}{2\pi} e^{-4\pi i \xi \sigma_r - i \theta m_r} \prod_{t,s=1}^{N} \frac{(m_t - m_s)^2}{4} + (\sigma_t - \sigma_s)^2 \right]$$

$$\times \prod_{r=1}^{N_f} \prod_{j=1}^{N_f} \frac{\Gamma(-i\sigma_r - i\tau_j - \frac{m_r}{2})}{\Gamma(1 + i\sigma_r + i\tau_j - \frac{m_r}{2})} \prod_{f=1}^{N_a} \frac{\Gamma(i\sigma_r - i\tilde{\tau}_f + \frac{m_r}{2})}{\Gamma(1 - i\sigma_r + i\tilde{\tau}_f + \frac{m_r}{2})}$$

$$\times \prod_{r,s=1}^{N} \frac{\Gamma(-i\sigma_r + i\sigma_s - i\tau_A - \frac{m_r - m_s}{2})}{\Gamma(1 + i\sigma_r - i\sigma_s + i\tau_A - \frac{m_r - m_s}{2})}.$$  (112)

As before $\tau_j, \tilde{\tau}_f$ are the twisted masses of the flavor symmetry $SU(N_f) \times SU(N_a)$ for fundamentals $Q$ and antifundamentals $\bar{Q}$, while $\tau_A$ is the twisted mass of the $U(1)$ flavor symmetry that rotates the adjoint $\Phi$. Moreover we assume $N_f > N_a$, or $N_f = N_a$ and $\xi > 0$, so to close the integration contours in the lower half-planes.

Let us start by listing poles and zeros of the integrand: fundamentals have poles at $\sigma_r = -\tau_j - i\left(\frac{|m_j|}{2} + k\right)$ and zeros at $\sigma_r = -\tau_j + i\left(\frac{|m_j|}{2} + k + 1\right)$, for some $j \in \{1, \ldots, N_f\}$ and $k \in \mathbb{N}$; antifundamentals have poles at $\sigma_r = \tilde{\tau}_f + i\left(\frac{|m_f|}{2} + k\right)$ and zeros at $\sigma_r = \tilde{\tau}_f - i\left(\frac{|m_f|}{2} + k + 1\right)$; the adjoint has poles at $\sigma_r - \sigma_s = -\tau_A - i\left(\frac{|m_r - m_s|}{2} + k\right)$ and zeros at $\sigma_r - \sigma_s = -\tau_A + i\left(\frac{|m_r - m_s|}{2} + k + 1\right)$. An $N$-pole\footnote{By $N$-pole we mean a point in $\mathbb{C}^N$ where the integrand has a simple pole in each of the $N$ variables $\sigma_r$.} in the lower half-planes is constructed by picking a number $n \leq N$ of poles from fundamentals to form a set, then adding to the set $N - n$ poles from the adjoint $\Phi_{rs}$: $\sigma_r = \sigma_s - \tau_A + i\left(\frac{|m_r - m_s|}{2} + k + 1\right)$ where $\sigma_s$ is already in the set, in such a way to form “tails”. In fact there is a one-to-one correspondence between poles of the integrand on one hand, and solutions to the D-term equation $\phi \phi^\dagger = \chi \mathbb{I}$ in (87), where we give VEV to the corresponding components, possibly dressed by (anti)vortices on the other hand.

The result of the integral could be computed straightforwardly, however for simplicity we specialize to the case of $\mathcal{N} = (2, 2)^*$ SQCD, i.e. a theory with $\mathcal{N} = (4, 4)$
supersymmetry broken to $\mathcal{N} = (2, 2)$ by a twisted mass term $\tau_A$ for the adjoint. In particular the matter spectrum consists of hypermultiplets, thus $N_f = N_a$, and we have the superpotential interaction

$$W = \hat{Q} \Phi$$

(113)

which reduces the flavor symmetry to $SU(N_f)_\text{diag} \times U(1)_A$ and poses constraints on masses and R-charges: $\tau_j + \tilde{\tau}_j + \tau_A = i \forall j$, implying $\tilde{M}_j^f = -M_j^f - \tau_A + i$. These relations imply that poles from the adjoint coincide with zeros from the antifundamentals, so that such poles are not really there. The fact that all $N$-poles involving the adjoint disappear is in correspondence with the fact that the F-term equation $\Phi Q = 0$ from $W$ in (113) forbids those Higgs branch solutions where $\Phi, Q$ get VEV simultaneously.

Eventually $N$-poles in the lower half-planes are parametrized by $N$-tuples $l = (l_1, \ldots, l_N)$ with $l_i = 1, \ldots, N_f$:

$$\{ \sigma_r = -\tau_{l_r} - i k_r - i \frac{i}{2} |m_r| \}_{r = 1, \ldots, N}.$$ 

After carrying out the computation one discovers, as before, that $N$-tuples $l$ with repetitions yield vanishing contribution (this is expected from the 1-1 correspondence with Higgs branch solutions, and could also be proven in a way similar to Sect. 5.2); since residues are invariant under permutations of $l$, we restrict ourselves to sum over $N$-combinations $l \in C(N, N_f)$ and remove $N!$ from the denominator. Then we proceed as in the $U(1)$ case of Sect. 5.1, and notice that if the adjoint was not there we would reproduce the result of Sect. 5.2.

The residues of the $\Gamma$’s developing a pole are $i (-1)^{k_{1r}!} / k_{1r}$, where $k_{1r} = k_r + |m_r|$, and the value of the corresponding $\Gamma$’s at the denominator is $(k_{1r} - m_r)!$. We define $k_{2r} = k_{1r} - m_r$ so that $\sum_{m \in \mathbb{Z}^N} \sum_{k \in \mathbb{Z}^N_{\geq 0}} \sum_{k_1 \in \mathbb{Z}^N_{\geq 0}} \sum_{k_2 \in \mathbb{Z}^N_{\geq 0}}$.

Evaluating all other terms we are left with an expression, that we call $Z_{U(1)}^{N_f, \text{adj}}$, as in (107) but with the following pieces. The classical term is the same. The one-loop term simplifies in the case of $\mathcal{N} = (2, 2)^*$ SQCD, as W-bosons cancel against fundamentals for $j \in l$ and the adjoint cancels against antifundamentals for $f \in \bar{l}$:

$$Z_{1}\text{-loop}^{(l)} = \prod_{r \in l} \prod_{j \notin l} \frac{\Gamma(-i M_j^f)}{\Gamma(1 + i M_j^f)} \prod_{f \notin l} \frac{\Gamma(1 + i M_f^r + i \tau_A)}{\Gamma(-i M_f^r - i \tau_A)}. \quad (114)$$

In particular for $\tau_A = 0$ we get $Z_{1}\text{-loop}^{(l)} = 1$. The vortex and antivortex contributions $Z_{v}^{(l)}(z)$ and $Z_{av}^{(l)}(\bar{z})$ can be written in the usual way (110) in terms of the vortex partition function:

19 Had we broken to $\mathcal{N} = (2, 2)$ by a superpotential mass term for the adjoint, the resulting partition function would have been the same as for $\mathcal{N} = (2, 2)$ $U(N)$ SQCD with $(N_f, \bar{N}_f)$ matter and a quartic superpotential, a subcase of what we considered in Sect. 5.2.

20 Concretely in the case of $U(2)$ with $N_f = 1$: $Q = (\sqrt{2} \chi, 0)$, $\Phi = (0, \sqrt{2} \chi)$ solves the D-term equation $QQ^\dagger + [\Phi, \Phi^\dagger] = \chi I$ with $\chi > 0$, and is in correspondence with a pole for $Q_1, \Phi_2$. However $\Phi Q = (0, \chi)$ so this solution disappears when imposing $\Phi Q = 0$. 
\[ Z_{\text{vortex}}^{(l)}(z, \varepsilon, a, m_A) = \sum_{k \in \mathbb{Z}_{\geq 0}^N} \left\{ \frac{z^k}{(-1)^{(N_f-1)|k|}} \prod_{f \in I} \left( \frac{a_{f r} + m_A}{\varepsilon} - k_r \right)_{k_r} \right\} \prod_{r \in I} \left[ \prod_{j \in I} \left( \frac{a_{j r}}{\varepsilon} - k_r \right)_{k_r} \prod_{j \not\in I} \left( \frac{a_{j r}}{\varepsilon} - k_r \right)_{k_r} \right] \]

(115)

where we defined \( a_j = -i \tau_j, m_A = -i \tau_A \) and \( a_{jr} = a_j - a_r = -i M^r_\ell \). This expression agrees with [35,36] for \( N_f = N \). Notice that here, compared with (111), the distinction between masses of flavors getting or not getting VEV is given by \( I \).

There are two notable limits to consider in (115). First, when \( m_A \to \infty \) the vortex partition function reduces to the one (111) for \( U(N) \) with \( N_f \) fundamentals, up to a renormalization of the FI term and a shift of the theta angle. This is expected because the superpotential (113) imposes \( \tilde{\tau}_j = -\tau_j - \tau_A + i \), therefore in the limit \( \tau_A \to \infty \) with \( \tau_j \) kept fixed, both the adjoint and the antifundamentals are integrated out. Second, when \( m_A = 0 \) we simply get:

\[ Z_{\text{vortex}}^{(l)} = \frac{1}{\left(1 + (-1)^{N_f / z}\right)^N}, \]

(116)

which does not depend on the twisted masses at all. Indeed for \( \tau_A = 0 \) the theory has enhanced \( \mathcal{N} = (4,4) \) supersymmetry which forces the effective twisted superpotential to be a linear function of the twisted masses (see for instance [39]), in particular \( Z_{\text{vortex}} \) cannot depend on them. We can thus compute \( Z_{\text{vortex}} \) in the limit \( \tau_j \to \infty \): the theory flows to \( N \) decoupled copies of the \( \mathcal{N} = (4,4) \) \( U(1) \) theory with one hyper, with renormalized theta angle \( z \to (-1)^{N_f-1}z \) from integrating out the other \( N_f - 1 \) hypers. The partition function of \( \mathcal{N} = (4,4) \) QED is (102) with \( \tilde{m} = \varepsilon \): \( Z_{\text{vortex}} = (1 - z)^{-1} \).

5.4. Analyticity in \( \xi \). In Sect. 5.1 we computed the \( S^2 \) partition function for a \( U(1) \) theory with \( (N_f, N_a) \) chiral. While for \( N_f \neq N_a \) the result was manifestly analytic in the FI parameter \( \xi \) as expected, this analyticity was not manifest for \( N_f = N_a \) because the closure of the integration contour depends on the sign of the FI parameter. In fact one can show that even in the latter case the partition function is analytic in \( \xi \), due to a nontrivial identity satisfied by the relevant generalized hypergeometric functions.

Concisely, one starts with the partition function for \( U(1) \) with \( (N_f, N_f) \) chiral computed for \( \xi > 0 \):

\[ Z_{U(1)}^{(N_f, N_f)}(w, \bar{w}) = \sum_{l=1}^{N_f} e^{i(w+\bar{w})\tau_l} Z_{1\text{-loop}}^{(l)} Z_{\text{vortex}}^{(l)}(e^{-w}) Z_{\text{av}}^{(l)}(e^{-\bar{w}}), \]

(117)

where the various pieces are in (100). The sum is over the \( N_f \) points of the Coulomb branch where a fundamental has vanishing effective twisted mass. Next one uses an identity for generalized hypergeometric functions \( _N F_{N-1} \) that allows to rewrite the function of \( x \) in terms of a sum of \( N \) hypergeometric functions of \( 1/x \) (see for instance [40]):

\[ ^{21} \text{We thank Sara Pasquetti for bringing this formula to our attention.} \]
\[
\frac{\prod_{k=1}^{N} \Gamma(a_k)}{\prod_{h=1}^{N-1} \Gamma(b_h)} N F_{N-1} \left( \begin{array}{c} \{a_k\}^N_{k=1} \\ \{b_h\}^N_{h=1} \end{array} \bigg| x \right) = \sum_{i=1}^{N} (-x)^{a_i} \frac{\Gamma(a_i) \prod_{j \neq i}^{N} \Gamma(a_j - a_i)}{\prod_{i=1}^{N-1} \Gamma(b_j - a_i)} N F_{N-1} \left( \begin{array}{c} a_i, \{1 + a_i - b_j\}^N_{j \neq 1} \\ \{1 + a_i - a_j\}^N_{j \neq 1} \end{array} \bigg| \frac{1}{x} \right). \tag{118}
\]

While the Taylor series in \( x \) for the LHS converges for \( |x| < 1 \), those for the RHS converge for \( |x| > 1 \), therefore the identity holds for the analytic continuations. Applying (118) to some of the factors in (117), one can rewrite the partition function in the form that would be obtained for \( \xi < 0 \) by closing the contour in the opposite half-plane.

6. Dualities

In [23,24] a duality—reminiscent of Seiberg duality [41] in four dimensions—was proposed between the SU(\( N \)) gauge theory with \( N_f > N \) fundamental flavors and SU(\( N_f - N \)) with the same number \( N_f \) of fundamentals. By gauging the U(1) baryon symmetry this implies a duality between U(\( N \)) and U(\( N_f - N \)).\(^{22}\) The duality for unitary groups is indeed straightforwardly suggested by the brane construction in [44]; it is also related by circle reduction to the 3d \( \mathcal{N} = 2 \) maximally chiral duality [45] between U(\( N \)) with \( N_f \) fundamentals and U(\( N_f - N \)) with \( N_f \) antifundamentals, up to charge conjugation in the latter. We show that indeed the \( S^2 \) partition functions of the two dual theories coincide, as functions of the FI parameter \( \xi \) and the twisted masses in the unitary case, and of the twisted masses only (including an extra mass for the baryonic symmetry) in the special unitary case.

Unitary group. We consider first the case of U(\( N \)) gauge theory with \( N_f \) fundamental flavors, because we can use the simpler expression of \( Z_{S^2} \) in terms of the vortex partition function. To avoid cluttering formulae, we use the notation

\[
-i \tau_j \equiv a_j \tag{119}
\]

for the twisted masses. The expression (107) is a sum over \( N \)-combinations \( I \in C(N, N_f) \). We show that for each combination \( I \) the corresponding term in \( Z_{U(N)} \) is equal to the term in \( Z_{U(N_f - N)} \) corresponding to the dual combination \( I^D \), where \( I^D \in C(N_f - N, N_f) \) is defined such that \( I \cap I^D = \emptyset \). In the duality the map of parameters is:

\[
a_j = -a_j^D, \quad \xi = \xi^D, \quad \theta = \theta^D + N_f \pi, \tag{120}
\]

where the superscript \( D \) labels quantities in the dual theory. For a fixed vacuum: \( I \leftrightarrow I^D \), \( a_r \in I = -a_r^D \) if \( r \notin I^D \) and \( a_s \notin I = -a_s^D \).

First, the exponential factor in (107) is clearly invariant since \( \sum_{j=1}^{N_f} a_j = 0 \). Second, the one-loop factor in (108) takes the following form:

\[
Z_{1\text{-loop}}^{(I)} = \left[ \prod_{t,s \in I} (a_t - a_s) \right] \left[ \prod_{r \in I} \prod_{j \neq r}^{N_f} \frac{\Gamma(a_j - a_r)}{\Gamma(1 - a_j + a_r)} \right]. \tag{121}
\]

\(^{22}\) Such duality has recently been considered in [42] in order to relate different UV non-Abelian GLSM descriptions of new 2d Calabi–Yau non-linear sigma models. A related duality in \( \mathcal{N} = (2, 2)^{\ast} \) SQCD has been investigated in [43].
The last product over \( j \) can be split into \((j \in \mathbb{L}, j \neq l_r)\) and \(j \notin \mathbb{L}\): the former exactly cancels against the determinant of W-bosons, leaving the latter:

\[
Z_{\text{1-loop}}^{(l)} = \prod_{r \in \mathbb{L}, j \notin \mathbb{L}} \frac{\Gamma(a_j - a_r)}{\Gamma(1 - a_j + a_r)}.
\]  

(122)

This is also invariant under the transformation after (120).

Finally let us consider the vortex partition function in (111) for \(U(N)f\) with \((N_f, 0)\) flavors: it is a Taylor series in \(z\) in which the \(z^k\) term is a sum over \(N\)-tuples \(k\) of positive integers (that we will call partitions) with fixed \(|k| \equiv \sum_r k_r = k\). Writing

\[
Z_{\text{vortex}} = \sum_{k=0}^{\infty} z^k Z_k:
\]

\[
Z_k = \sum_{k \in \mathbb{Z}_N^k} \varepsilon^{-N_f k} \prod_{r \in \mathbb{L}} k_r! \left[ \prod_{j \neq r} \left( \frac{\alpha_j - \alpha_r}{\varepsilon} - k_r \right) \right]^{a_j - \alpha_r} \left[ \prod_{j \notin \mathbb{L}} \left( \frac{\alpha_j - \alpha_r}{\varepsilon} - k_r \right) \right]^{a_j - \alpha_r + k_r}.
\]  

(123)

Notice that here, as opposed to (111), the distinction between masses of flavors getting or not getting VEV is given by \(l\). It turns out that \(Z_k\) can be written as a contour integral:

\[
Z_k = \frac{1}{\varepsilon^k k!} \int \left[ \prod_{j=1}^k \frac{d\varphi_j}{2\pi i} \right] \left[ \prod_{i<j} \frac{(\varphi_i - \varphi_j)^2}{(\varphi_i - \varphi_j)^2 - \varepsilon^2} \right] \times \prod_{j=1}^k \frac{1}{\prod_{r \in \mathbb{L}} (\varphi_j - \alpha_r) \prod_{s \notin \mathbb{L}} (\alpha_s - \varphi_j - \varepsilon)}.
\]  

(124)

The contours are taken along the real lines \(\Re^k\) and closed in the upper half-planes (for \(N_f = 1\) there would be an ambiguity because of a pole at infinity). To begin with, here we have assumed that the twisted masses and \(\varepsilon\) are real; we will generalize to complex values below. The indentations of the integration contours are defined by assigning to the parameter \(\varepsilon\) and the masses \(\alpha_r \notin \mathbb{L}\) and \(\alpha_s \notin \mathbb{L}\) small positive imaginary parts, with the imaginary part of \(\varepsilon\) larger than that of the masses.

It is easy to see that \(k\)-poles inside the contours are classified by partitions \(k\) of \(k\) and each configuration is given by:

\[
\{\varphi_j\} = \begin{cases} 
\vdots \\
a_r, \ a_r + \varepsilon, \ \ldots, \ a_r + (k_r - 1) \varepsilon \quad r \in \mathbb{L}, \\
\vdots 
\end{cases}
\]  

(125)

that is for each \(r \in \mathbb{L}\) we have a “tail” of \(k_r\) values starting from \(a_r\). For each of such configurations we actually have \(k!\) \(k\)-poles obtained by permutations. The definition of the contours above is valid when \(\varepsilon, a_r\) are real: for generic complex parameters the contours are defined to include all and only the poles in (125). One can also show that the residue of a \(k\)-pole of type \(k\) equals the corresponding term in the sum in (123).

Given the integral representation of \(Z_k\) in (124), we can now close the contours in the lower half-planes. The integral is unaffected as long as there are no poles at infinity, which happens for \(N_f > 1\). To bring the contour integral back to a form that we can
compare with (124), we first define “dual” integration variables $\varphi^D_j = -\varphi_j - \varepsilon$, so that the integration contour of $d\varphi^D_j$ closes in the upper half-plane, and $a^D_j = -a_j$; then we change the direction of the contours to the original orientation; we move the contours back to $\mathbb{R}^k$ and give a small positive imaginary part to $a^D_j$ which can be done without crossings. Eventually we reach:

$$Z_k = \frac{1}{\varepsilon^k k!} \int \left[ \prod_{j=1}^k \frac{d\varphi^D_j}{2\pi i} \right] \left[ \prod_{1<j}^k \frac{(\varphi^D_i - \varphi^D_j)^2}{(\varphi^D_i - \varphi^D_j)^2 - \varepsilon^2} \right] \times \prod_{j=1}^k \frac{1}{\varphi^D_j - a^D_j} \prod_{r \in I} (a^D_r - \varphi^D_j - \varepsilon)$$

with the contours defined as after (124). This indeed produces the vortex partition function of $U(N_f - N)$ with $N_f$ flavors, with twisted masses obtained from the map after (120).

We have thus proved that, at fixed $I$ and $I^D$, the two vortex partition functions of $U(N)$ and $U(N_f - N)$ with $N_f$ flavors coincide under the map after (120). Because of the extra sign in (110), the vortex and antivortex contributions in $Z_\mathbb{S}^2$ (107) agree if $z = (-1)^{N_f} z^D$, which amounts to a shift of the theta angle. Since all factors in (107) match, the two path integrals $Z_\mathbb{S}^2$ are equal:

$$Z_{U(N)}^{(N_f,0)}(\xi, \theta, a_j) = Z_{U(N_f-N)}^{(N_f,0)}(\xi, \theta + N_f \pi, -a_j).$$

(126)

**Special unitary group.** Given the partition function of the $U(N)$ gauge theory with $N_f$ fundamentals, expressed by the matrix integral in (63), we can deduce the partition function of the $SU(N)$ gauge theory with $N_f$ fundamentals by performing an integral over the theta angle and a Fourier transform over the FI parameter:

$$Z_{SU(N)}^{(N_f,0)}(b; a_j) = \int_0^{2\pi} \frac{d\theta}{2\pi} \int_{-\infty}^{+\infty} \frac{d\xi}{4\pi i \varepsilon b} e^{\xi b} Z_{U(N)}^{(N_f,0)}(\xi, \theta; a_j).$$

(127)

Eventually $b$ will be the twisted mass for the baryonic symmetry $U(1)_B$ of the $SU(N)$ theory, which gives charge 1 to baryons. Indeed the integral over $\theta$ gives a Kronecker delta $\delta_0$, $\sum_{r=1}^N m_r$ restricting the magnetic flux $m$ to the Cartan subalgebra of $SU(N)$, whereas the integral over the FI parameter gives a Dirac delta $\delta(\sum_{r=1}^N \sigma_r - b)$ imposing that the trace of the matrix $\sigma$ is $b$.

We then change basis from the “natural” Cartan of $U(N)$ to the “natural” Cartan of $SU(N)$ and $U(1)$ in $U(N) \cong [SU(N) \times U(1)]/\mathbb{Z}_N$, setting:

$$\sigma_1 = y + x_1, \quad \sigma_j = y - x_{j-1} + x_j \quad \text{for } j = 2, \ldots, N - 1, \quad \sigma_N = y - x_{N-1},$$

where $y$ is the center of mass of the fields $\sigma_i$ and $x_1, \ldots, x_{N-1}$ are the components with respect to the Cartan of $SU(N)$. Taking into account the Jacobian, the integration measures are related as $\prod_{r=1}^N d\sigma_r = N dy \prod_{i=1}^{N-1} dx_i$. In terms of the new variables $y$ and $x_i$, the Dirac delta function becomes $\delta(Ny - b)$. After evaluating the integral $\int dy$ thanks to the delta function, we remain with an integral over the Cartan of $SU(N)$, with
the replacement above and $y = b/N$ inside the integrand. This is precisely the partition function $Z_{SU(N)}$ of an $SU(N)$ gauge theory with $N_f$ flavors and with a $U(1)_B$ baryonic symmetry normalized to give charge 1 to baryons, to which we associate a twisted mass parameter $b$, expectation value of the corresponding background field $\sigma_B$.

In particular, given the identity of partition functions $Z_{SU(N)}^{(N_f,0)} = Z_{SU(N_f-N)}^{(N_f,0)}$ proven above, we get the identity:

$$Z_{SU(N)}^{(N_f,0)}(b, a_j) = Z_{SU(N_f-N)}^{(N_f,0)}(b, -a_j) \quad (128)$$

for the duality in the $SU$ case by applying (127) to both sides of (126). Baryons $B$ in the electric theory and $b$ in the magnetic theory are defined as

$$B^{i_1...i_N} = \epsilon^{a_1...a_N} Q_{a_1}^{i_1} \ldots Q_{a_N}^{i_N}, \quad b_{j_1...j_N} = \epsilon^{\beta_1...\beta_N} q_{j_1}\beta_1 \ldots b_{j_N}\beta_N \quad (129)$$

where $\tilde{N} = N_f - n$, $\alpha_t$ and $\beta_s$ are electric and magnetic color indices, and $i_k, j_k$ are flavor indices. The duality map between baryons is

$$B^{i_1...i_N} = \epsilon^{i_1...i_N j_1...j_N} b_{j_1...j_N}. \quad (130)$$

The quantum numbers of baryonic operators are consistent with the duality map: they have baryon charge 1, canonical R-charge 0 and transform in the same representation of $SU(N_f)$.

### 6.1. Duality for $U(N)$ with $(N_f, N_a)$ flavors and $N_f > N_a + 1$

Brane constructions [44] suggest that a duality could hold between $U$-type gauge theories in the presence of both $N_f$ fundamentals and $N_a$ antifundamentals. If we assume for definiteness that $N_f \geq N_a$, the duality suggested by the brane creation effect [46] is between a $U(N)$ gauge theory with $(N_f, N_a)$ flavors and a $U(N_f - N)$ theory with $(N_a, N_f)$ flavors, similarly to the maximally chiral duality of 3d theories studied in [45]. For the sake of homogeneity with the rest of the section, we relabel the magnetic gauge representations by charge conjugation, so that the magnetic side becomes a $U(N_f - N)$ theory with $(N_f, N_a)$ flavors. We call $Q_{\alpha}^{i}, \tilde{Q}_{\bar{i}}^{\bar{\alpha}}$ the quark and antiquark superfields of the electric theory and $q_{i\beta}, \tilde{q}_{\bar{i}\bar{\beta}}$ those of the magnetic theory. In order for mesonic branches of the putative dual theories to match, the magnetic theory must contain singlets $M_{i}^{j}$ dual to the electric mesons $\tilde{Q}_{\bar{i}}^{\bar{\alpha}} Q_{\alpha}^{i}$, coupled to the dual (anti)quark superfields by the usual superpotential $W_{\text{dual}} = \tilde{q}_{\bar{i}\bar{\beta}} M_{i}^{j} q_{i\beta}$.

Postponing a careful study of the field theories and of a possible duality between them to future work, we show here agreement between the partition functions of a $U(N)$ theory with $(N_f, N_a)$ flavors and $N_f > N_f$ and a $U(N_f - N)$ theory with $(N_a, N_f)$ flavors, gauge singlets $M_{i}^{j}$ and superpotential $W_{\text{dual}} = \text{Tr} \tilde{q} M q$, up to a FI term for the Abelian global symmetries on whose possible origin we will comment below. We will also require $N_f > N_a + 1$. In such cases, even in the absence of twisted masses and bare FI parameter, the classical Coulomb branch of the flat space theory is lifted, up to a set of isolated points, by one-loop logarithmic corrections to the effective FI parameter. For both theories the flavor symmetry is $SU(N_f) \times SU(N_a) \times U(1)_A$ and the map of parameters is:

$$\xi = \xi^D, \quad \theta = \theta^D + (N_f - N_a)\pi, \quad \tau_j = -\tau_j^{D} + \frac{i}{2}, \quad \tilde{\tau}_j = -\tilde{\tau}_j^{D} + \frac{i}{2} \quad (131)$$
We also impose the constraint $\frac{1}{N_f} \sum_j \tau_j = \frac{1}{N_a} \sum_f \bar{\tau}_f$ (and similarly in the magnetic theory) because we parametrize the twisted masses as $\tau_j = m_A + \delta \tau_j$, $\bar{\tau}_f = m_A + \delta \bar{\tau}_f$ and similarly $\tau^D_j = m^D_A + \delta \tau^D_j$, $\bar{\tau}^D_f = m^D_A + \delta \bar{\tau}^D_f$ in terms of the Cartan of $SU(N_f) \times SU(N_a) \times U(1)_A$, with $0 = \sum_j \delta \tau_j = \sum_f \delta \bar{\tau}_f = \sum_j \delta \tau^D_j = \sum_f \delta \bar{\tau}^D_f$. The relations among masses are then:

$$\delta \tau_j = -\delta \tau^D_j, \quad \delta \bar{\tau}_f = -\delta \bar{\tau}^D_f, \quad m_A = -m^D_A + \frac{i}{2}. \quad (132)$$

Note that the effective mass of a meson $M^I_i = \tilde{Q}_i Q^j$ in the electric theory is $\tau_j + \bar{\tau}_i$, hence the effective mass parameter associated to $W_{\text{dual}}$ is $i$ which correctly corresponds to R-charge 2 and vanishing flavor charges.

As in the previous section, we compare the contributions to the partition functions from the vacua $l$ and $l^D$. First, the equality

$$\exp \left[ 4\pi i \xi \sum_{r \in l} \tau_r \right] = e^{4\pi i \xi^D \left( -N_f m_A^D + \frac{i}{2} N \right)} \exp \left[ 4\pi i \xi^D \sum_{r \in l^D} \tau^D_r \right] \quad (133)$$

shows that the classical actions on the two sides agree, up to an extra factor on the magnetic side that could be ascribed to a FI term for the Abelian global symmetries $U(1)_A \times U(1)_R$. If there were a duality between the charge-conjugation-invariant theories with $N_f = N_a$, where such a term could not be generated, one could hope to derive global FI terms in the other cases by integrating out massive matter fields, as done in the 3d case in [45] starting from Aharony duality [47].

The one-loop determinants of $W$-bosons and fundamentals combine, as in (122), into

$$\prod_{r \in l, j \not\in l} \frac{\Gamma(-i \tau_j + i \tau_r)}{\Gamma(1 + i \tau_j - i \tau_r)}$$

which is manifestly invariant under the duality. On the other hand the equality

$$\prod_{r \in l, i = 1}^{N_a} \prod_{i = 1}^{N_f} \frac{\Gamma(-i \bar{\tau}_i - i \tau_r)}{\Gamma(1 + i \bar{\tau}_i + i \tau_r)} = \prod_{r \in l^D, i = 1}^{N_a} \prod_{i = 1}^{N_f} \frac{\Gamma(-i \bar{\tau}^D_i - i \tau_r)}{\Gamma(1 + i \bar{\tau}^D_i + i \tau_r)} \cdot \prod_{r \in l, i = 1}^{N_a} \prod_{i = 1}^{N_f} \frac{\Gamma(-i \bar{\tau}_i - i \tau^D_r)}{\Gamma(1 + i \bar{\tau}_i + i \tau^D_r)}$$

shows that the one-loop of antifundamentals in the electric theory (LHS) equals the one-loop of gauge singlets $M^r_i$ (first term on the RHS) times the one-loop of antifundamentals (second term) in the magnetic theory.

Finally the partition function $Z_k$ for $k$ vortices in the $U(N)$ theory with $(N_f, N_a)$ flavors is the term in (111) multiplying $\tilde{z}^k$:

$$Z_k = \sum_{k \in \mathbb{Z}^N_{\geq 0}} \varepsilon^{(N_a - N_f) k} \prod_{r \in l} k_r \prod_{s \not\in l} \left( \frac{a_s - a_r}{\varepsilon} - k_r \right)_{k_s} \prod_{j \not\in l} \left( \frac{a_j - a_r}{\varepsilon} - k_r \right)_{k_r} \prod_{f = 1}^{N_a} \left( \frac{\tilde{m}_f + a_r}{\varepsilon} \right)_{k_r}$$

where as usual $a_r = -i \tau_r$ and $\tilde{m}_f = -i \bar{\tau}_f$. Such expression is reproduced by the contour integral:
\[ Z_k = \frac{1}{\epsilon^k k!} \int \left[ \prod_{j=1}^{k} \frac{d\varphi_j}{2\pi i} \right] \left[ \prod_{i<j}^{k} (\varphi_i - \varphi_j)^2 \right] \times \prod_{j=1}^{k} \frac{N_a^{\mu_j}}{\prod_{r \in I} (\varphi_j - a_r) \prod_{s \not\in I} (a_s - \varphi_j - \epsilon)} . \]  

(134)

The contours are defined in the same way as after (124). Repeating the manipulations of the previous section and moving the contours in the lower half-planes, it is straightforward to see that the vortex partition functions in the two dual theories match. For this to work it is important that \( N_f > N_a + 1 \), otherwise there is a pole at infinity that prevents the contours to be harmlessly deformed.

When \( N_f = N_a \) or \( N_f = N_a + 1 \) one can check by direct evaluation that the partition functions \( Z_k \) do not match under the map (131), therefore if a duality exists at all, the map has to be modified somehow.

7. Discussion

We conclude with some open questions and directions for future work. The question that motivated the present work was to understand from a more fundamental point of view the factorization, observed in [21] in two examples, of the \( S^3_b \) partition function [11] into products of K-theoretic vortex and antivortex partition functions. Although we have succeeded in explaining a similar equality in two dimensions, the three-dimensional case remains to be addressed. A natural guess would be to consider exactly the same \( Q \)-exact deformation \( \mathcal{L}_H \) (83) but in three dimensions.

We can gain some intuition by considering how the north and south pole of \( S^2 \) uplift to \( S^3 \) or the Lens space \( L(p, 1) \). The preimages of the poles under the projection \( S^3 \rightarrow S^2 \) are the Hopf fibers over them, forming two linked great circles in \( S^3 \). We expect the point-like (anti)vortex configurations at the poles to lift to similar singular configurations on \( S^3 \) or \( L(p, 1) \), possibly with KK momentum along the extra circles. The 2d (anti)vortex partition functions would then lift to K-theoretic (anti)vortex partition functions. When the round \( S^3 \) is squashed into the ellipsoid \( S^3_b \) defined by the equation \( b^2 |z_1|^2 + b^{-2} |z_2|^2 = 1 \) in \( \mathbb{C}^2 \), the radii of the two circles become \( b \) and \( 1/b \) respectively (in units of the radius \( R \) of \( S^3 \)): they match the sizes of the extra circles in the K-theoretic vortex and antivortex partition functions of [21]; we expect a similar factorization in terms of K-theoretic (anti)vortex partition functions to occur for \( L(p, 1) \), and similarly for \( S^2 \times S^1 \). It would be interesting to make these considerations concrete.

Our result that the matrix integral equals the vortex partition function times the antivortex partition function (weighted by semiclassical factors and summed over isolated points on the Higgs branch) suggests that the matrix integral can be used to determine the vortex partition function for theories where other means are not available. In particular, the computations in the literature of \( Z_{\text{vortex}} \) are based either on an ADHM-like construction of the moduli space [37] derived from string theory, which however has applicability limited to some gauge groups and matter representations, or the moduli matrix approach [36,48]. The matrix integral seems a simple and flexible alternative method.

A useful development of our work would be to include twisted chiral multiplets, possibly with periodic imaginary part, besides chiral multiplets. That would allow a more thorough exploration of mirror symmetry. We hope to report progress in that
direction in the near future. It would also be interesting to include local or non-local operators in our path integral computation, for instance loops or extra vortex operators.

The partition function of an $\mathcal{N} = 2$ gauge theory on $S^4$ has proven very useful to uncover surprising aspects of the dynamics of multiple M5-branes—whose reduction on a Riemann surface $\Sigma$ can give either an $\mathcal{N} = 2$ [49] or an $\mathcal{N} = 1$ [50–52] four-dimensional theory depending on the twist, some aspects being discussed in [53–55]—most notably the AGT correspondence [56]. A similar role is being played by the $\delta$ commuting operator $A$. Localizing Supercharges

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### A. Localizing Supercharges

The commuting operator $\delta = \delta_\epsilon + \delta_\overline{\epsilon} = \epsilon^\alpha Q_\alpha + \overline{\epsilon}^\alpha \overline{Q}_\alpha$, where $\epsilon$, $\overline{\epsilon}$ are anticommuting positive Killing spinors, defines the supercharges $Q_\alpha, \overline{Q}_\alpha$. Multiplying them back by commuting Killing spinors, we obtain anticommuting operators. We choose $\epsilon_+^\dagger$ such that $\epsilon_+^\dagger \epsilon_+ = 1$ and $\epsilon_+^\dagger C \epsilon_+ = 0$, and form $Q = \epsilon_+^\dagger Q_\alpha, Q^\dagger = \epsilon_+^\dagger \overline{Q}_\alpha = -(\epsilon_+^\dagger C) \overline{Q}_\alpha$. For convenience, we also rewrite barred spinors as $\overline{\lambda} = C(\lambda^\dagger)^\dagger$, recalling that $\lambda^\dagger$ is independent from $\lambda$, and drop the suffix from $\epsilon_+$ recalling that it is here commuting. We get for the gauge multiplet:

\[
\begin{align*}
QA_\mu &= \frac{i}{2} \lambda^\dagger \gamma_\mu \epsilon \\
Q\sigma &= \frac{i}{2} \lambda^\dagger \epsilon \\
Q\eta &= \frac{i}{2} \lambda^\dagger \gamma_3 \epsilon \\
Q \lambda &= i \gamma_3 \epsilon \left( F_{12} - \frac{\eta}{r} + i [\sigma, \eta] \right) - \epsilon \left( D + \frac{\sigma}{r} \right) + i \gamma^\mu \epsilon D_\mu \sigma - \gamma_3 \gamma^\mu \epsilon D_\mu \eta \\
Q^\dagger \lambda &= -i \epsilon^\dagger \gamma_3 \left( F_{12} - \frac{\eta}{r} - i [\sigma, \eta] \right) + \epsilon^\dagger \left( D + \frac{\sigma}{r} \right) + i \epsilon^\dagger \gamma^\mu D_\mu \sigma + \epsilon^\dagger \gamma_3 \gamma^\mu D_\mu \eta \\
QD &= -i \epsilon^\dagger \gamma_3 \lambda + \frac{i}{2} \gamma_\mu \epsilon + \frac{1}{2} [\lambda^\dagger, \sigma] + \frac{1}{2} [\lambda^\dagger, \gamma_3 \epsilon, \eta] + \frac{1}{2r} \lambda^\dagger \epsilon \\
Q^\dagger \overline{D} &= \frac{i}{2} \epsilon^\dagger \gamma^\mu D_\mu \lambda + \frac{i}{2} [\sigma, \epsilon^\dagger \lambda] + \frac{1}{2} [\eta, \epsilon^\dagger \gamma_3 \lambda] + \frac{1}{2r} \epsilon^\dagger \lambda
\end{align*}
\]

\[\text{135}\]

23 Alternatively one could simply consider $\mathcal{N} = 4$ SYM on $\Sigma$ [60], which however gives a 2d non-linear sigma model and does not fall within our framework.
and for the chiral multiplet:

\[
\begin{align*}
Q\phi &= 0 \\
Q\phi^\dagger &= \psi^\dagger \epsilon \\
Q\psi &= \left( i\gamma^\mu D_\mu \phi + i\sigma \phi + \gamma_3 \eta \phi - \frac{q}{2r} \phi \right) \epsilon \\
Q^\dagger\psi^\dagger &= \epsilon^\dagger \left( -i\gamma^\mu D_\mu \phi^\dagger + i\phi^\dagger \sigma + \gamma_3 \phi^\dagger \eta - \frac{q}{2r} \phi^\dagger \right) \\
QF &= \epsilon^\dagger C \left( i\gamma^\mu D_\mu \psi - i\sigma \psi + \gamma_3 \eta \psi + \frac{q}{2r} \psi - i\lambda \phi \right) \\
Q^\dagger F^\dagger &= \left( -iD_\mu \psi^\dagger \gamma^\mu - i\psi^\dagger \sigma + \psi^\dagger \gamma_3 \eta + \frac{q}{2r} \psi^\dagger + i\phi^\dagger \lambda^\dagger \right) C \epsilon^* \\
Q^\dagger F &= 0.
\end{align*}
\]

Then \(Q, Q^\dagger\) satisfy the algebra \(\{Q, Q^\dagger\} = M + \frac{R}{2}, \quad Q^2 = Q^\dagger 2 = 0\), up to a gauge transformation \(\Lambda\), where \(M\) is the angular momentum that generates rotations along the Killing vector field \(v_\mu = \epsilon^\dagger \gamma_\mu \epsilon\) and \(R\) is R-charge. For instance:

\[
\begin{align*}
\{Q, Q^\dagger\} A_\mu &= -i \nu^\mu F_{\nu\mu} + D_\mu \Lambda \\
\{Q, Q^\dagger\} \phi &= -i \nu^\mu D_\mu \phi + i\Lambda \phi + \frac{q}{2r} \phi \\
\{Q, Q^\dagger\} \lambda &= -i \nu^\mu D_\mu \lambda + \frac{s}{2r} \gamma_3 \lambda + i\Lambda \lambda - \frac{1}{2r} \lambda \\
\{Q, Q^\dagger\} \psi &= -i \nu^\mu D_\mu \psi + \frac{s}{2r} \gamma_3 \psi + i\Lambda \psi + \frac{q - 1}{2r} \psi \\
\{Q, Q^\dagger\} F &= -i \nu^\mu D_\mu F + i\Lambda F + \frac{q - 2}{2r} F.
\end{align*}
\]

In particular \(\{Q, Q^\dagger\} = M + \frac{R}{2} + i\Lambda\), with \(\Lambda = -\sigma + i\epsilon^\dagger \gamma_3 \epsilon \eta\). Finally \(Q = Q + Q^\dagger\).

### B. One-Loop Determinants

We compute here the one-loop determinant for various kinetic operators appearing in the main text. Before doing that, let us recall a few facts about spin spherical harmonics on a background.

#### B.1. Spin spherical harmonics

On a sphere \(S^2\) or radius \(r\), the standard scalar spherical harmonics \(Y^0_{j, j_3}\) are the eigenfunctions of the Laplacian

\[
\nabla^2 = \frac{1}{r^2} \left( \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta) + \frac{1}{\sin^2 \theta} \partial^2_\psi \right).
\]

They are parametrized by \(j \in \mathbb{N}\) and \(|j_3| \leq j, j_3 \in \mathbb{Z}\) with eigenvalues \(-j(j+1)/r^2\).

Now consider a particle with spin \(s_z\) (representations of the spin group \(SO(2)\) are one-dimensional) moving in a background magnetic flux \(F\) with \(\frac{1}{2\pi} \int F = m\), the particle transforming as the weight \(\rho\) of some representation. Since \(A = -\frac{m}{2} \omega^1\), the covariant derivative is

\[
D_\mu = \partial_\mu + is_z \omega^1_\mu - i \frac{\rho(m)}{2} \omega^1_\mu,
\]

(139)
therefore the particle has effective spin \( s = s_c - \frac{\rho(m)}{2} \). Let us consider the
(anti)holomorphic derivatives
\[
D_\pm = \frac{D_1 \mp i D_2}{2} = \frac{1}{2r} \left( \partial_\theta \mp \frac{i}{\sin \theta} \partial_\varphi \right) \mp \frac{s}{2r} \cos \theta.
\]

(140)

\( D_+ \) maps a spin \( s \) to a spin \( s + 1 \) harmonic, while \( D_- \) reduces \( s \) by one unit. In fact
\( D^\mu D_\mu = 2[D_+, D_-] \) and \([D_+, D_-] = -\frac{s}{2}\). To be very explicit about the spin of the
operators, \( D_s^2 = 2(D_s^{(s-1)} - D_s^{(s+1)}) \).

The eigenfunctions of \( D_s^2 \) are the spin spherical harmonics \( Y_{s,j_3} \), with \(|s|, |j_3| \leq j \)
and \( j - s \in \mathbb{N}, j - j_3 \in \mathbb{N} \), and the eigenvalues are:
\[
r^2 D_s^2 Y_{s,j_3} = - \left[ j(j+1) - s^2 \right] Y_{s,j_3}.
\]

(141)

For integer \( s \) they can be constructed from the scalar harmonics:
\( Y_{s,j_3} \propto (D_s)^s Y_0^{s,j_3} \) for \( 0 \leq s \leq j \) and \( Y_{s,j_3} \propto (D_+)^s Y_0^{s,j_3} \) for \(-j \leq s \leq 0 \), while we get zero if \(|s| > j\). If
\( s \in \mathbb{N} + \frac{1}{2} \) we can start from \( Y_{1/2,1/2} = e^{i\psi/2} \cos \theta / 2 \). It will be useful to consider also
the operators:
\[
D_+ D_- Y_{s,j_3} = \frac{s^2 - s - j^2 - j}{4r^2} Y_{s,j_3}, \quad D_- D_+ Y_{s,j_3} = \frac{s^2 + s - j^2 - j}{4r^2} Y_{s,j_3}.
\]

(142)

B.2. Gauge one-loop determinant. Consider the action
\[
\mathcal{L} = \frac{1}{2} \left( \vec{F}_{12} - \frac{\vec{\eta}}{r} \right)^2 + \frac{1}{2} \left( \vec{D} + \frac{\vec{\sigma}}{r} \right)^2 + \frac{1}{2} (D_\mu \bar{\sigma} - i[A_\mu, \sigma_0]) - \bar{c} D^\mu D_\mu c
\]
\[+ \frac{1}{2} (D_\mu \bar{\eta} - i[A_\mu, \eta_0]) - \frac{1}{2} (\bar{\sigma}, \eta_0) + [\sigma_0, \bar{\eta}])^2 - \frac{1}{2s} (D^\mu \bar{A}_\mu)^2
\]

(143)

quadratic in the fluctuations \((\bar{A}_\mu, \bar{\sigma}, \bar{\eta}, \bar{D}, \bar{c}, c)\). First, the term \( \frac{1}{2} (\vec{D} + \frac{\vec{\sigma}}{r})^2 \) integrated
over \( \vec{D} \) gives determinant 1, and we discard it. Then let us look at \((\bar{A}_\mu, \bar{\sigma}, \bar{\eta})\). We write
the vectors in holomorphic coordinates, using
\[
g_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \ \\ \frac{1}{2} & 0 \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} 1 & 0 \ \\ 0 & 1 \end{pmatrix}, \quad \varepsilon_{\mu\nu} = \begin{pmatrix} 0 & \frac{i}{2} \ \\ -\frac{i}{2} & 0 \end{pmatrix}, \quad \varepsilon^{\mu\nu} = \begin{pmatrix} 0 & -2i \ \\ 2i & 0 \end{pmatrix}
\]

and \( V_\pm = (V_1 \mp iV_2) / 2 \). Once we go to holomorphic coordinates, \( \bar{A}_\mu^\pm = \vec{A}_\mu \). We
decompose all modes along the weights \( \rho \) of the adjoint representation. The Cartan
generators are decoupled from everything else at quadratic order: since their one-loop
determinant will not depend on the background \( A_\mu^{(0)}, \sigma_0, \eta_0 \), we discard them, leaving
behind with a decomposition along the roots \( \alpha \) of \( G \).

Let us choose the gauge \( \xi = -1 \). Then the Lagrangian can be written in matrix
notation as \((\bar{A}_\mu^+, \bar{\sigma}^+, \bar{\eta})^T \mathcal{M} (\bar{A}_\mu, \bar{\sigma}, \bar{\eta})\) where the matrix \( \mathcal{M} \) is:
\[
\mathcal{M} = \begin{pmatrix}
-4D_+D_- + \sigma_0^2 + \eta_0^2 & -i \eta_0 \sigma_0 + \frac{1}{2} & i \eta_0 \sigma_0 + \frac{1}{2} \\
-i \eta_0 \sigma_0 + \frac{1}{2} & -4D_-D_+ + \sigma_0^2 + \eta_0^2 & -i \eta_0 \sigma_0 + \frac{1}{2} \\
-i \sigma_0 D_+ & -i \sigma_0 D_- & -4D_+D_- + \sigma_0^2 + \eta_0^2
\end{pmatrix}
\]

(144)
This matrix is Hermitian. To compute the determinant we need a basis of eigenfunctions. We expand \( \tilde{\eta}, \tilde{\sigma} \) on \( Y_{s,j_3}^s, \tilde{A}_+ \) on \( Y_{s,j_3}^{s+1} \) and \( \tilde{A}_- \) on \( Y_{s,j_3}^{s-1} \), with \( s = -\frac{\alpha(m)}{2} \). We also introduce the notation \( s_\pm = \sqrt{j(j+1) - s(s \pm 1)} \). If we normalize the harmonics as \( |Y_{s,j_3}^s|^2 = 1 \), then the commutation relations fix the relations among them via \( D_{\pm} \), up to a phase that we choose as follows:

\[
D_+ Y_{s,j_3}^s = \frac{s_+}{2r} Y_{s+1,j_3}^s, \quad D_- Y_{s,j_3}^s = -\frac{s_-}{2r} Y_{s-1,j_3}^s. \tag{145}
\]

We can now substitute in the matrix, and consider all possible values of \( j \). For \( j \geq \frac{\alpha(m)}{2} + 1 \) all three harmonics \( Y_{s\pm 1,j_3}^s, Y_{s,j_3}^s \) exist; the determinant of (144) is

\[
\det \mathcal{M} = \frac{1}{64r^8} (j^2 + 2r^2 \alpha(\sigma)^2)(j + 1)^2 + r^2 \alpha(\sigma)^2) \left( j(j + 1) - \frac{\alpha(m)^2}{4} \right)^2. \tag{146}
\]

The coefficient of \( r \) in front does not depend on the background and can be reabsorbed in the normalization. These eigenvalues have multiplicity \( 2j + 1 \).

For \( j = \frac{\alpha(m)}{2} + \frac{1}{2} \) only two harmonics exist. For instance, when \( \alpha(m) \geq 1 \) then \( Y_{s+1,j_3}^s \) and \( Y_{s,j_3}^s \) exist while \( D_- Y_{s,j_3}^s = 0 \). Removing the second row/column in (144), the determinant is \( \frac{1}{4r^6} (\alpha(m)^2 + 1)^2 + r^2 \alpha(\sigma)^2) \) with multiplicity \( |\alpha(m)| + 1 \). When \( \alpha(m) \leq -1 \) we reach the same conclusion. The case \( j = \alpha(m) = 0 \) has to be treated separately. Only the harmonic \( Y_{s=0,j_3=0}^s \) exists. Removing the first and second row/column in (144), the two eigenvalues are 0 and \( \frac{1}{r^2} (1 + r^2 \alpha(\sigma)^2) \). The zero eigenvalue corresponds to the zero-mode of the background \( \sigma_0 \); we have to remove the eigenvalue from the determinant, and integrate over the zero-mode.

For \( j = \frac{\alpha(m)}{2} - 1 \geq 0 \) only one harmonic exists. For instance, when \( \alpha(m) \geq 2 \) then \( Y_{s+1,j_3}^s \) exists while \( D_- Y_{s+1,j_3}^s = 0 \). Removing the second, third and fourth row/column in (144), we are left with \( \frac{1}{r^4} (\alpha(m)^2 + r^2 \alpha(\sigma)^2) \) with multiplicity \( |\alpha(m)| - 1 \). If \( |\alpha(m)| = 1 \) this case does not arise (but we can still formally use the formula if we include the multiplicity); if \( \alpha(m) = 0 \) this case does not arise as well, but we have to treat it separately.

The ghost term is \( -\bar{c} \hat{D}^2 c \equiv \bar{c} \mathcal{O}_c : c \) decomposing in a basis of spin spherical harmonics the determinant is:

\[
\text{Det} \mathcal{O}_c = \prod_{\alpha \in G} \prod_{j \geq \frac{\alpha(m)}{2}} \left( j(j + 1) - \frac{\alpha(m)^2}{4} \right)^{2j+1}. \tag{147}
\]

Let us put all pieces together, discarding numerical factors of \( r \). In the end we have to take the product over all roots \( \alpha \in G \). Since the adjoint is a real representation, for each root \( \alpha \) we have also \(-\alpha\). If we consider the product of a root and its opposite:

\[
\prod_{[\alpha,-\alpha]} (k^2 + r^2 \alpha(\sigma)^2) = \prod_{[\alpha,\alpha]} (k + ir\alpha(\sigma))^2 = \prod_{[\alpha,\alpha]} (k - ir\alpha(\sigma))^2.
\]

We will use this relation often below. When \( \alpha(m) \neq 0 \) we find:

\[
\frac{\text{Det} \mathcal{O}_c}{\sqrt{\text{Det} \mathcal{O}_{\text{gauge}}} \alpha} = \prod_{k = \frac{|\alpha(m)|}{2}}^{\infty} \frac{1}{(k + ir\alpha(\sigma))^{2k-1}(k + 1 + ir\alpha(\sigma))^{2k+3}}. \tag{148}
\]
When \( \alpha(m) = 0 \) we compute \( \text{Det}' \) because there is a zero eigenvalue, corresponding to a zero-mode of the background \( \sigma \) along the root \( \alpha \), that we remove. Also \( \mathcal{O}_c \) has a zero eigenvalue. The determinants give:

\[
\frac{\text{Det}' \mathcal{O}_c}{\sqrt{\text{Det}' \mathcal{O}_{\text{gauge}}}}\bigg|_\alpha = \frac{1}{|\alpha(\sigma)|} \prod_{k = |\alpha(m)|/2}^{\infty} \frac{1}{(k + i r \alpha(\sigma))^{2k - 1} (k + 1 + i r \alpha(\sigma))^{2k + 3}}
\]

(B.3) Gaugino one-loop determinant. The fermionic part of the YM action (20) expanded at quadratic order around the background is:

\[
\mathcal{L} = \frac{i}{2} \bar{x} \gamma^\mu D_\mu \lambda + \frac{i}{2} \bar{x} [\sigma_0, \lambda] + \frac{1}{2} \bar{x} \gamma_5 [\eta_0, \lambda].
\]

As before, modes along the Cartan generators give a background-independent constant that we discard. We are left with a decomposition along the roots \( \alpha \) of \( G \).

The fermions \( \bar{x}, \lambda \) have two components of spin \( \pm \frac{1}{2} \), so we decompose them into spin spherical harmonics. For \( j \geq \frac{1}{2} + \frac{|\alpha(m)|}{2} \) the effective spins are \( s = \pm \frac{1}{2} - \frac{\alpha(m)}{2} \).

The matrix acting on the subspace \( (Y_{j, j_3}^{1/2 - |\alpha(m)|/2}, Y_{j, j_3}^{-1/2 + |\alpha(m)|/2}) \) is

\[
\mathcal{M} = \frac{1}{2r} \left( \begin{array}{cc} ir \alpha(\sigma) + \frac{\alpha(m)}{2} & 2 i r D_+ \\ 2 i r D_- & -ir \alpha(\sigma) - \frac{\alpha(m)}{2} \end{array} \right).
\]

However the product of spinors is \( \bar{x} \lambda = \bar{x}^a e_{a\beta} \lambda^\beta = -\bar{x}^2 \lambda^1 + \bar{x}^1 \lambda^2 \). Therefore we should really compute the determinant of \( \mathcal{O}_\lambda \equiv \left( \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right) \mathcal{M} \). The determinant is:

\[
\det \mathcal{O}_\lambda = \frac{1}{4 r^2} \left( j(j + 1) + r^2 \alpha(\sigma)^2 + \frac{1}{4} \right) = \frac{1}{4 r^2} \left( j(j + 1) - ir \alpha(\sigma) \right) (j + 1 + i r \alpha(\sigma))
\]

with multiplicity \( 2j + 1 \).

For \( j = \frac{|\alpha(m)|}{2} - \frac{1}{2} \geq 0 \) only one harmonic exists. For instance, when \( \alpha(m) \geq 1 \) then \( \lambda = (Y_{j, j_3}^{1/2 - |\alpha(m)|/2}, 0) \) and the eigenvalue is \( -\frac{1}{2r} (\frac{|\alpha(m)|}{2} + i r \alpha(\sigma)) \) with multiplicity \( |\alpha(m)| \).

When \( \alpha(m) \leq -1 \) then \( \lambda = (0, Y_{j, j_3}^{-1/2 + |\alpha(m)|/2}) \) and the eigenvalue is the same as above but with opposite sign. For \( \alpha(m) = 0 \) this case does not exist, but we can still formally use the formula as long as we include the multiplicity.

The one-loop determinant, discarding numerical factors of \( r \), is then:

\[
\text{Det} \mathcal{O}_\lambda = \prod_{\alpha \in G} \prod_{k = |\alpha(m)|/2}^{\infty} (k + i r \alpha(\sigma))^{2k} (k + 1 + i r \alpha(\sigma))^{2k + 2},
\]

To write this expression we changed sign to \( \alpha(\sigma) \) since we take the product over \( \alpha \) and the adjoint is a real representation. We we also used that \( \prod_{\alpha \in G} (-1)^{|\alpha(m)|} = (-1)^{\sum_{\alpha \geq 0} |\alpha(m)|} = (-1)^{2|\delta(m)|} = 1 \), where \( |x| = \frac{|x + x^\dagger|}{2} \) and \( \delta \) is the Weyl vector.
B.4. Fermion one-loop determinant. The fermionic part of the matter action (27) expanded at second order around the background is:

\[ \mathcal{L} = \bar{\psi} \left( -i \gamma^\mu D_\mu + i \sigma - \gamma_3 \eta - \frac{q}{2r} \right) \psi. \]

There are two components in \( \psi \) of spin \( \pm \frac{1}{2} \), therefore the effective spins are \( s = \pm \frac{1}{2} - \frac{\rho(m)}{2} \). As in the previous section, we compute the determinant of the operator multiplied by \( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \).

For \( j \geq \frac{1}{2} + \frac{\rho(m)}{2} \), both harmonics \( Y_{j, j+1}^{\frac{1}{2} - \frac{\rho(m)}{2}} \) and \( Y_{j, j-1}^{\frac{1}{2} - \frac{\rho(m)}{2}} \) exist, then the operator is:

\[ O_\psi = \frac{1}{r} \left( ir \rho(\sigma) - \frac{q}{2} - \frac{\rho(m)}{2} \right) \left( 2ir D_+ - 2i \rho(\sigma) + \frac{q}{2} - \frac{\rho(m)}{2} \right). \]  

The determinant of the matrix is:

\[ \det O_\psi = \frac{1}{r^2} \left( j + \frac{q + 1}{2} + i \rho(\sigma) \right) \left( j + 1 - \frac{q + 1}{2} + i \rho(\sigma) \right) \]  

and each mode has degeneracy \( 2j + 1 \).

For \( j = \frac{\rho(m)}{2} - \frac{1}{2} \geq 0 \), one of the two harmonics with \( s = \frac{1}{2} - \frac{\rho(m)}{2} \) does not exist because \( |s| \) exceeds \( j \). For instance, when \( \rho(m) \geq 1 \) then \( Y_{j, j+1}^{\frac{1}{2} - \frac{\rho(m)}{2}} \) exists while \( D_- Y_{j, j-1}^{\frac{1}{2} - \frac{\rho(m)}{2}} = 0 \). Then we take \( \psi = (Y_{j, j+1}^{\frac{1}{2} - \frac{\rho(m)}{2}}, 0) \), only the upper-left corner in (152) is non-trivial and the eigenvalue is \( -\frac{1}{r} \left( \frac{\rho(m)}{2} + \frac{q}{2} - i \rho(\sigma) \right) \) with degeneracy \( |\rho(m)| \). When \( \rho(m) \leq -1 \) only \( Y_{j, j-1}^{\frac{1}{2} - \frac{\rho(m)}{2}} \) exists. Taking \( \psi = (0, Y_{j, j+1}^{\frac{1}{2} - \frac{\rho(m)}{2}}) \) we pick the lower-right corner in (152) getting the same eigenvalue as before but with an extra minus sign, and same degeneracy \( |\rho(m)| \). For \( \rho(m) = 0 \) this case does not exist, but we can still formally use the formula as long as we include the multiplicity.

Multiplying all eigenvalues, up to multiplicative factors of \( r \) that we disregard, we get the one-loop determinant

\[ \text{Det} O_\psi = \prod_{\rho \in R_\Phi} (-1)^{|\rho(m)|} \prod_{k=\frac{|\rho(m)|}{2}}^{\infty} \left( k + \frac{q}{2} - i \rho(\sigma) \right)^{2k} \left( k + 1 - \frac{q}{2} + i \rho(\sigma) \right)^{2k+2}. \]  

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