Research Article
Existence, Decay, and Blow-Up of Solutions for a Higher-Order Kirchhoff-Type Equation with Delay Term

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This article deals with the study of the higher-order Kirchhoff-type equation with delay term in a bounded domain with initial boundary conditions, where firstly, we prove the global existence result of the solution. Then, we discuss the decay of solutions by using Nakao’s technique and denote polynomially and exponentially. Furthermore, the blow-up result is established for negative initial energy under appropriate conditions.

1. Introduction

In this paper, we establish the higher-order Kirchhoff-type equation with delay term as follows:

\[
\begin{aligned}
&u_{tt} + \left( \int_0^L \left| A^{\infty} u \right|^2 \, dx \right) A^{\infty} u + \mu_1 |u_t(x,t)|^{-\gamma} u_t(x,t) \\
&+ \mu_2 |u_t(x,t-r)|^{-\gamma} u_t(x,t-r) = |u|^{\mu-1} u, \\
&u(x,0) = u_0(x), u_t(x,0) = u_1(x) \\
&u_t(x,t-r) = f_0(x,t-r) \\
&\frac{\partial^i u}{\partial t^i} = 0, i = 0, 1, \cdots, m-1
\end{aligned}
\]

\[x \in \Omega, t \in (0, T), \]

\[x \in \Omega, t \in (0, r), \]

\[x \in \partial \Omega.\]

(1)

where \( A = -\Delta, m \geq 1 \) is a natural number, \( q, \gamma \geq 0 \) are real numbers, \( p > 1 \) is a real number and is a bounded domain with smooth boundary \( \partial \Omega \) in \( \mathbb{R}^n, n = 1, 2, 3; \) \( \nu \) is the outer normal. \( r > 0 \) denotes time delay, and 1 and 2 are positive real numbers. The functions \((u_0, u_1, f_0)\) are the initial data belong to a suitable space.

The problem (1) is a general form of a model introduced by Kirchhoff [1]. To be more precise, Kirchhoff recommended a model denoted by the equation for \( f = g = 0, \)

\[
ph \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + g \left( \frac{\partial u}{\partial t} \right) = \left\{ \rho_0 + \frac{E h}{2l} \left( \frac{\partial u}{\partial x} \right)^2 \right\} \frac{\partial^2 u}{\partial x^2} + f(u).
\]

(2)

for \( 0 < x < L, t \geq 0, \) where \( u(x, t) \) is the lateral displacement, \( \rho \) is the mass density, \( h \) is the cross-section area, \( E \) is the Young modulus, \( L \) is the length, \( \rho_0 \) is the initial axial tension, and \( f, g \) are the external forces. Furthermore, (2) is called a degenerate equation when \( \rho_0 = 0 \) and nondegenerate one when \( \rho_0 > 0. \)

Time delays often appear in many various problems, such as thermal, economic phenomena, biological, chemical, and physical. Recently, the partial differential equations with time delay have become an active area, (see [2, 3] and references therein). Datko et al. [4] indicated that a small delay in a boundary control is a source of instability. An arbitrarily small delay may destabilize a system which is uniformly
asymptotically stable without delay unless additional conditions or control terms have been used in many cases [5]. Additional control terms will be necessary to stabilize hyperbolic systems including delay terms, (see [6–8] and references therein). In [6], Nicaise and Pignotti studied the equation as follows:

\[ u_{tt}(x, t) - \Delta u(x, t) + a_t u_t(x, t) + au_t(x, t - \tau) = 0, \]  

where \( a_0 \) and \( a \) are positive real parameters. The authors obtained that, under the condition \( 0 \leq a \leq a_0 \), the system is exponentially stable. In the case \( a \geq a_0 \), they obtained a sequence of delays that shows the solution is instable. In [8], Xu et al. obtained the same result similar to the [6] for sequence of delays that shows the solution is instable. In that work, the authors showed that an exponential stability result under the condition

\[ \alpha \leq \sqrt{1 - da_0}, \]  

where \( d \) is a constant such that

\[ \tau'(t) \leq d < 1, \forall t > 0. \]  

In recent years, some other authors investigate hyperbolic type equation with delay term (see [10–16]). Without delay term \( (\mu_t u_t(x, t) - \tau^{-1} u_t(x, t - \tau)), \) in 2004, Li [17] studied the higher-order Kirchhoff-type equation as follows:

\[ \begin{aligned} \mu_{tt} + (\int_{\Omega} |D^{m/2} u|^q dx)^{q/(q-1)} (-\Delta)^m u + u_t |u_t|^\gamma = |u|^p u, \end{aligned} \]  

where \( m > 1 \) is a positive integer, and \( q, p, r > 0 \) is a positive constant. The author obtained that the solution exists globally if \( p \leq r \), while if \( p > \max\{r, 2q\} \). He also established the blow-up result for \( E(0) < 0 \). Later, in 2007, Messaoudi and Houari [18] obtained the blow-up of solutions with \( E(0) > 0 \) of the equation (6). Then, Piskin and Polat [19] considered global existence and decay estimates utilizing Nakao’s inequality of the equation (6).

Without delay term, when \( m = 1 \) and \( q = 0 \), equation (1) takes the form of a semilinear hyperbolic equation as follows:

\[ u_{tt} - \Delta u + u_t |u_t|^{\gamma - 1} = |u|^{p-1} u. \]  

Goegele and Todorova [20] obtained the blow-up of solutions for \( E(0) < 0 \) if \( 1 < r < p \) \((1 < p < n/(n - 2))\) for \( n \geq 3 \), \( p > 1 \) for \( n < 3 \) of the equation (7). Under the condition of positive upper bounded initial energy, Vitillaro [21] proved the same results of equation (7). Also, Ohta [22, 23] studied related problems for the blow-up results of the equation (7).

Messaoudi [24] studied the following equation

\[ u_{tt} + \Delta^2 u + |u_t|^2 u_t = |u|^{p-2} u \]  

and obtained an existence result for the equation (8) and proved that the solution continues to exists globally if \( r \geq p \); however, if \( r < p \) and the initial energy is negative, the solution blows up in finite time. Chen [25] established that the solution of (8) blows up with \( E(0) > 0 \). In the presence of strong damping term \( (-\Delta u_t), \) Piskin and Polat [26] obtained the decay estimates by using Nakao’s inequality of equation (8).

When \( m = 1 \) and without delay term, equation (1) takes the form the following Kirchhoff-type equation:

\[ u_{tt} - M\left( \int_{\Omega} |D^m u|^2 dx \right)^{\gamma} \Delta u + u_t |u_t|^\gamma = |u|^p u. \]  

Many authors had studied existence and blow-up results at night time for equation (9) (see [27–30]). Ono [30] proved the blow-up results if \( p > \max\{r, 2q\} \) \((p > 2/(n - 4)\) for \( n \geq 5 \), \( p > 0 \) for \( n \leq 4 \)) and \( E(0) < 0 \) for equation (9). Later, Benaissa and Messaoudi [31] obtained the similar result for the generalized Kirchhoff-type equation as follows:

\[ u_{tt} - M\left( \int_{\Omega} e^{\phi(x)} |\nabla u|^2 dx \right) e^{\phi(x)} \text{div}(e^{\phi(x)} \nabla u) + a |u_t|^{r-2} u_t = b |u|^{p-2} u, \]  

where \( M : R^+ \rightarrow R^+ \) and \( \phi(x) \) are bounded functions. Then, Wu [32], verified the same result of the general Kirchhoff-type equation

\[ u_{tt} - M\left( |\nabla u|^2 \right) \Delta u + |u_t|^{r-2} u_t = |u|^{p-2} u, \]  

with the positive upper bounded initial energy. In 2013, Ye [33] considered the global existence results by constructing a stable set in \( H^2(\Omega) \) and showed the decay by using a lemma of Komornik for the nonlinear Kirchhoff-type equation (11) with dissipative term. Moreover, Ye [34] obtained the global existence results by constructing a stable set in \( H^n(\Omega) \) and showed the energy decay by using a lemma of V. Komornik for a nonlinear higher-order Kirchhoff-type equation with dissipative term is as follows:

\[ u_{tt} + \left| A^{1/2} u \right|^{2p} A u + a |u_t|^{r-2} u_t = b |u|^{p-2} u, \]  

where \( A = (-\Delta)^m, m > 1 \) is a positive integer.

Gao et al. [35] considered the Kirchhoff-type equation without delay term as follows:

\[ u_{tt} + M\left( |D^m u|^2 \right) (-\Delta)^m u + |u_t|^{r-2} u_t = |u|^{p-2} u. \]  

The authors obtained the blow-up of solutions for \( E(0) > 0 \) under appropriate conditions for equation (13).

In [36–40], some authors studied abstract evolution equations as follows:

\[ [P(u_t)]_{t} A(t, u) + Q(t, u_t) = F(u) \]  

on suitable Banach space, and they proved some global
nonexistence of solutions. Some other authors studied related problems (see [41–45]).

Motivated by the above works, we deal with the existence, decay, and blow-up results for the higher-order Kirchhoff type equation (1) with delay term and source term. There is no research, to our best knowledge, related to the higher-order Kirchhoff-type \((\int_D |A^{m/2}u|^q \, dx)^{q}\) equation (1) with delay \((u_{\tau}(x, t - \tau))^{q}\) terms; hence, our work is the generalization of the above studies.

This work consists of five sections in addition to the introduction: Firstly, in Sect. 2, we recall some lemmas and assumptions. Then, in Section 3, we get the global existence of solutions. Moreover, in Section 4, we establish the decay results by using Nakao’s technique. Finally, in Section 5, we obtain the blow-up of solutions for negative initial energy.

2. Preliminaries

In this part, we present some lemmas and assumptions for the proof of our result. Let \(H^m(\Omega)\) denote the Sobolev space with the norm

\[
\|u\|_{H^m(\Omega)} = \left( \sum_{|\alpha| \leq m} \|D^\alpha u\|^2_{L^2(\Omega)} \right)^{1/2},
\]

\(H^m_0(\Omega)\) denotes the closure in \(H^m(\Omega)\) of \(C^\infty_0(\Omega)\). For simplicity of notation, we denote by \(\|\cdot\|_p\) the Lebesgue space \(L^p(\Omega)\) norm, \(\|\cdot\|\) denotes \(L^2(\Omega)\) norm, and we write equivalent norm \(\|\cdot\|_{H^m_0(\Omega)}\) instead of \(H^m_0(\Omega)\). We denote by \(C_i(i = 1, 2, \ldots, n)\) various positive constants which may be different at different occurrences.

**Lemma 1** (see [46, 47] Sobolev-Poincaré inequality). If \(2 \leq p \leq (2n/(n - 2m))^+\) \((2 \leq p < \infty\) if \(n = 2m\)), then for some \(C_i\),

\[
\|u\|_p \leq C_i \|(-\Delta)^{m/2} u\|_{L^2(\Omega)} \quad \text{for } u \in H^m_0(\Omega),
\]

where we put \([a]^+ = \max \{0, a\}\), \(1/[a]^+ = \infty\) if \([a]^+ = 0\).

**Lemma 2** (see [48]). Let \(\phi(t)\) be nonincreasing and nonnegative function defined on \([0, T]\), \(T > 1\) and satisfies

\[
\phi^{1+\alpha}(t) \leq w_0(\phi(t) - \phi(t + 1)), \quad t \in [0, T]
\]

for \(w_0\) is a positive constant, and \(\alpha\) is a nonnegative constant. Then, we have for each \(t \in [0, T]\),

\[
\begin{cases}
\phi(t) \leq \phi(0) e^{-w_0 |t - \tau|^\alpha}, & \alpha = 0, \\
\phi(t) \leq (\phi(0) - w_0 \alpha |t - \tau|^\alpha)^{-1/\alpha}, & \alpha > 0,
\end{cases}
\]

where \(|t - \tau|^\alpha = \max \{t - \tau, 0\}\), and \(\tau(t) = \ln (w_0/w_0 - 1)\).

We make the assumptions on parameters \(r, p,\) and \(m\) as follows:

\[
\begin{aligned}
&(A1) \quad 1 < r < \infty, \quad n \leq 2m, \\
&(A2) \quad 1 < p < \frac{n}{n - 2m}, \quad n > 2m,
\end{aligned}
\]

3. Global Existence

In this part, we consider the global existence results of the problem (1). Firstly, we introduce the new function \(z\) similar to the [7],

\[
z(x, k, t) = u_t(x, t - \tau k), \quad x \in \Omega, \quad k \in (0, 1).
\]

Thus, we have

\[
\tau z_t(x, k, t) + z_k(x, k, t) = 0, \quad \text{in} \Omega \times (0, 1) \times (0, \infty).
\]

Hence, problem (1) can be transformed as follows:
We define the energy functional for any regular solution of (22) as follows:

\[
E(t) = \frac{1}{2} \| u_t \|^2 + \frac{1}{2(q + 1)} \| A^{m/2} u \|^{2(q+1)} - \frac{1}{p+1} \| u \|^{p+1}_{p+1} + \frac{c}{r+1} \int_\Omega \int_0^1 z^{r+1}(x,k,s) dk dx,
\]

such that

\[
\tau r |\mu_2| < c < \tau ((r+1)|\mu_1| - |\mu_2|).
\]

Also, have

\[
I(t) = I(u(t)) = \| A^{m/2} u \|^{2(q+1)} - \| u \|^{p+1}_{p+1} + \frac{c}{r+1} \int_\Omega \int_0^1 z^{r+1}(x,k,s) dk dx.
\]

We easily see that

\[
E(t) = J(t) + \frac{1}{2} \| u_t \|^2.
\]

Furthermore, we define

\[
\mathcal{W} = \{ u: u \in H^m_\Omega \cap H^{2m}(\Omega), I(u) > 0 \} \cup \{ 0 \}.
\]

Next, lemma gives that the energy functional \( E(t) \) is a nonincreasing.

**Lemma 3.** Assume that \((u,z)\) is the solution of (22), then for \( t \geq 0 \),

\[
E'(t) = -\left( \mu_1 - \frac{c}{\tau(r+1)} - \frac{\mu_2}{r+1} \right) \| u_t(t) \|^{r+1}_{r+1}
- \left( \frac{c}{\tau(r+1)} - \frac{\mu_2}{r+1} \right) \int_\Omega z^{r+1}(x,1,t) dx \leq 0.
\]

**Proof.** We multiply the first equation in (22) by \( u_t \), integrate over, and use integration by parts, and we obtain

\[
\frac{d}{dt} \left[ \frac{1}{2} \| u_t \|^2 + \frac{1}{2(q + 1)} \| A^{m/2} u \|^{2(q+1)} - \frac{1}{p+1} \| u \|^{p+1}_{p+1} \right] + \mu_1 \| u_t(t) \|^{r+1}_{r+1} \mu_2 \int_\Omega |z(x,1,t)|^{-1}z(x,1,t)u_t(x,t) dx = 0.
\]

Integrating (30) over \((0,t)\), we get

\[
\left[ \frac{1}{2} \| u_t(t) \|^2 + \frac{1}{2(q + 1)} \| A^{m/2} u \|^{2(q+1)} - \frac{1}{p+1} \| u \|^{p+1}_{p+1} \right] + \mu_1 \int_0^t \| u_t(s) \|^{r+1}_{r+1} ds + \mu_2 \int_0^t \int_\Omega |z(x,1,s)|^{-1}z(x,1,s)u_t(x,s) dx ds = \frac{1}{2} \| u_t \|^2.
\]

We multiply the second equation in (22) by \( \varsigma |z|^{-1}z \) and integrate the result over \( \Omega \times (0,1) \times (0,t) \), and we get

\[
\frac{c}{r+1} \frac{d}{dt} \int_0^t \int_\Omega |z(x,k,t)|^{-1}z(x,k,t)z_t(x,k,t) dx dt ds = -\frac{c}{\tau(r+1)} \int_0^t \int_\Omega \int_0^1 \frac{\partial}{\partial k} |z(x,k,t)|^{-1} dx dt ds
- \frac{c}{\tau(r+1)} \int_0^t \int_\Omega |z(x,1,t)|^{-1} - |z(x,0,t)|^{-1} dx dt ds
- \frac{c}{\tau(r+1)} \int_0^t \int_\Omega |z(x,1,t)|^{-1} dx dt
+ \frac{c}{\tau(r+1)} \int_0^t \| u_t(t) \|^{r+1}_{r+1} ds.
\]

By combining (31) and (32), we arrive at

\[
E(t) + \left( \mu_1 - \frac{c}{\tau(r+1)} - \frac{\mu_2}{r+1} \right) \int_0^t \| u_t(s) \|^{r+1}_{r+1} ds
+ \frac{c}{\tau(r+1)} \int_0^t \int_\Omega |z(x,1,s)|^{-1}z(x,1,s)u_t(x,s) dx ds = E(0).
\]

Utilizing the Young inequality on the fourth term of the left hand side of (33), we conclude that

\[
E(t) + \left( \mu_1 - \frac{c}{\tau(r+1)} - \frac{\mu_2}{r+1} \right) \int_0^t \| u_t(s) \|^{r+1}_{r+1} ds
+ \left( \frac{c}{\tau(r+1)} - \frac{\mu_2}{r+1} \right) \int_0^t \int_\Omega |z(x,1,s)|^{-1} dx ds = E(0).
\]

Deriving the (34), we have the desired result. Hence, the proof is completed.

**Remark 4.** From the condition (24), we obtain

\[
c_1 = \left( \mu_1 - \frac{c}{\tau(r+1)} - \frac{\mu_2}{r+1} \right) > 0, c_2 = \left( \frac{c}{\tau(r+1)} - \frac{\mu_2}{r+1} \right) > 0.
\]
Lemma 5. Assume that (19) and $p > 2q + 1$ hold. Let $u_0 \in \mathcal{W}$ and $u_1 \in H_0^1(\Omega)$, such that

$$\beta = C_* \left( \frac{2(q+1)(p+1)}{p-2q-1} E(0) \right)^{p-2q-1} < 1, \quad (36)$$

then $u \in \mathcal{W}$ for each $t \geq 0$.

**Proof.** It follows the continuity of $u(t)$, since $I(0) > 0$, such that

$$I(t) > 0, \quad (37)$$

for some interval near $t = 0$. Assume that $T_m > 0$ is a maximal time, when (26) holds on $[0, T_m]$.

By (25) and (26), we obtain

$$J(t) = \frac{1}{p+1} I(t) + \frac{p-2q-1}{2(q+1)(p+1)} \| A\beta u \|^2_{(q+1)} + \left( \frac{(p-r)}{(r+1)(p+1)} \right) \left( \int_{\Omega} \left| \nabla^{q+1} (x, k, t) \right| \right), \quad (38)$$

From (23), (38), and Lemma 3, we have

$$\| A\beta u \|^2_{(q+1)} \leq \frac{2(q+1)(p+1)}{p-2q-1} J(t) \leq \frac{2(q+1)(p+1)}{p-2q-1} E(t) \leq \frac{2(q+1)(p+1)}{p-2q-1} E(0). \quad (39)$$

Using Lemma 1 and (39), we get

$$\| u \|^{p+1}_{L_{p+1}} \leq C_* \| A\beta u \|^{p+1} = C_* \| A\beta u \|^{p-2q-1} \| A\beta u \|^2_{(q+1)}$$

$$\leq C_* \left( \frac{2(q+1)(p+1)}{p-2q-1} E(0) \right)^{p-2q-1} \| A\beta u \|^2_{(q+1)}$$

$$= \beta \| A\beta u \|^2_{(q+1)} \leq \| A\beta u \|^2_{(q+1)} \text{ on } t \in [0, T_m]. \quad (40)$$

Thus, from (26), we arrive at $I(t) > 0$ for all $t \in [0, T_m]$. $T_m$ is extended to $T$, by repeating the procedure. Hence, the proof is completed.

Lemma 6. Suppose that the assumptions of Lemma 5 hold. Then, there exists $\eta_1 = 1 - \beta$, such that

$$\| u \|^{p+1}_{L_{p+1}} \leq (1 - \eta_1) \| A\beta u \|^2_{(q+1)}. \quad (41)$$

**Proof.** By (40), we obtain

$$\| u \|^{p+1}_{L_{p+1}} \leq \beta \| A\beta u \|^2_{(q+1)}. \quad (42)$$

Let $\eta_1 = 1 - \beta$; therefore, we obtain the result.

Remark 7. By Lemma 6, we conclude that

$$\| A\beta u \|^2_{(q+1)} \leq \frac{1}{\eta_1} I(t). \quad (43)$$

Theorem 8. Assume that the assumptions (A2), $\mu_2 < \mu_1$, and $p > 2q + 1$ hold. Let $u_0 \in \mathcal{W}$ satisfying (36) and $f_0 \in L^2(\Omega \times (0, 1))$ be given. Then, the solution of problem (22) is global.

**Proof.** It is sufficient to show that $\| u_r \|^2 + \| A\beta u \|^2_{(q+1)}$ is bounded independently of $t$. To obtain this, by using (23) and (26), we have

$$E(0) \geq E(t) = \frac{1}{2} \| u_r \|^2 + \frac{1}{2(q+1)} \| A\beta u \|^2_{(q+1)} - \frac{1}{p+1} \| u \|^{p+1}_{L_{p+1}}$$

$$+ \frac{c}{r+1} \int_{\Omega} \int_{0}^{q+1} \frac{z}{(x, k, t)} dk dx = \frac{1}{2} \| u_r \|^2$$

$$+ \frac{p-2q-1}{2(q+1)(p+1)} \| A\beta u \|^2_{(q+1)}$$

$$+ \frac{(p-r)}{(r+1)(p+1)} \left( \int_{\Omega} \int_{0}^{q+1} \frac{z}{(x, k, t)} dk dx \right)$$

$$+ \frac{1}{p+1} I(t) \geq \frac{1}{2} \| u_r \|^2 + \frac{p-2q-1}{2(q+1)(p+1)} \| A\beta u \|^2_{(q+1)}, \quad (44)$$

since $I(t) \geq 0$. Thus,

$$\| u_r \|^2 + \| A\beta u \|^2_{(q+1)} \leq CE(0), \quad (45)$$

where $C = \max \{ 2, (2(q+1)(p+1)/p-2q-1) \}$. Therefore, we obtain the global existence of solutions. Therefore, we completed the proof.

4. Decay of Solution

In this part, we obtain the decay of solutions of the problem (22) by using Nakao’s technique.

Theorem 9. Assume that the assumption (A2) and (36) hold. Let $u_0 \in \mathcal{W}$, $f_0 \in L^2(\Omega \times (0, 1))$, be given. Hence, we have following decay estimates:

$$E(t) \leq \begin{cases} E(0) e^{-|t|^{-\beta}}, & \text{if } r = 1, \\ (E(0)^{-\beta} + C_7 \alpha (|t-1|^{-1})^{-1/\alpha}, & \text{if } r > 1, \end{cases} \quad (46)$$

where $\omega_1$, $\alpha$, and $C_7$ are positive constants which will be defined later.

**Proof.** We integrate (29) over $[t, t+1], t > 0$, to get

$$E(t) - E(t+1) = [D(t)]^{r+1}, \quad (47)$$
where

\[ |D(t)|^{r+1} = c_1 \int_1^{+\infty} \|u(t)\|_{r+1}^{r+1} ds + c_2 \int_1^{+\infty} z^{r+1}(x, 1, s) dx ds. \]  

(48)

From (48) and Hölder inequality, we see that

\[ \int_1^{+\infty} |u(t)|^2 dx dt + \int_1^{+\infty} |z(x, 1, s)|^2 dx ds \leq \omega(\Omega)|D(t)|^2, \]  

(49)

where \( \omega(\Omega) = \text{vol}(\Omega) \). Therefore, by (49), there exists \( t_i \in [0, t+1/4] \) and \( t_j \in [t+1/4, t+1] \), so that

\[ \|u(t_i(t))\|^2 + \|z(x, 1, t_i(t))\|^2 \leq \omega(\Omega)|D(t)|^2, i = 1, 2. \]  

(50)

We multiply the first equation in (22) by \( u \) and integrate over \( \Omega \times [t_1, t_2] \). Use integration by parts, Hölder’s inequality, adding, and subtracting the term \( \int_{t_1}^{t_2} \int_{\Omega} \partial_t z^{r+1}(x, k, t) \) with \( dx dt \), we have

\[ \int_{t_1}^{t_2} \int_{\Omega} |u(t)|^2 dx dt \leq \|u(t_1)\|_2 \|u(t_2)\|_2 + \int_{t_1}^{t_2} |u(t)|^2 dx dt + \int_{t_1}^{t_2} \int_{\Omega} z^{r+1}(x, k, t) \]  

\[ - \mu_1 \int_{t_1}^{t_2} \int_{\Omega} |u(t)|^2 dx dt \]  

\[ - \mu_2 \int_{t_1}^{t_2} \int_{\Omega} |z(x, 1, t)|^{r-1} z(x, 1, t) dx dt. \]  

(51)

Now, we estimate the right hand side for (51). From (39), (50), and Lemma 1, we obtain

\[ \|u(t_i(t))\|_2 \|u(t_i(t))\|_2 \leq C_1 D(t) \sup_{t_1 \leq t \leq t_2} E^{1/2}(s), \]  

(52)

where \( C_1 = 2C_1((2q+1)/(p+1)\omega(\Omega)|D(t)|^2)^{1/2(\eta+1)}. \)

By using (32) that

\[ \int_{t_1}^{t_2} \int_{\Omega} z^{r+1}(x, k, t) \]  

\[ \leq \left( \int_{t_1}^{t_2} \int_{\Omega} |u(t)|^2 ds dv \right)^{1/2} \left( \int_{t_1}^{t_2} \int_{\Omega} |u(t)|^{r+1} ds dv \right)^{1/2} \leq (t_2 - t_1)|D(t)|^{r+1}. \]  

(53)

Utilizing Hölder inequality, we get

\[ \int_{t_1}^{t_2} \int_{\Omega} |u(t)|^{r-1} u dx dt \leq \left( \int_{t_1}^{t_2} \int_{\Omega} |u(t)|^{r+1} dx dt \right)^{1/2} \left( \int_{t_1}^{t_2} |u(t)| dx dt \right)^{1/2}. \]  

(54)

Utilizing the Sobolev-Poincare inequality and (39), we have

\[ \int_{t_1}^{t_2} \int_{\Omega} |u(t)|^2 dx dt \leq C_1 \int_{t_1}^{t_2} \int_{\Omega} |u(t)|^2 dx dt + C_3 \int_{t_1}^{t_2} I(t) dt. \]  

(59)
Hence, from (57) and (59), we obtain

$$\int_{t_2}^{t_1} E(t) dt \leq \frac{1}{2} C[D(t)]^2 + C_4 C_2 \left[ \sup_{t \in [s, t_2]} E^{\frac{1}{2}}(s) D(t) + [D(t)]^2 \right]$$

$$+ (t_2 - t_1)[D(t)]^{r+1} + 2C,$$

$$\cdot \left( \frac{(2q + 1)(p + 1)}{p - 2q - 1} \frac{E(0)}{r + 1} \right)^{1/2} \left[ \sup_{t \in [s, t_2]} E^{\frac{1}{2}}(s) [D(t)]^{\frac{1}{r+1}} \right].$$

(60)

Integrating \((d/dt)E(t)\) over \([t, t_2]\), we conclude that

$$E(t) = E(t_2) + \int_{t}^{t_2} \left( \mu_1 - \frac{C}{\tau(r + 1)} - \frac{H_2}{r + 1} \right) \|u_1(s)\|^{r+1} ds$$

$$+ \int_{t}^{t_2} \left( \frac{C}{\tau(r + 1)} - \frac{H_2 r}{r + 1} \right) \int_\Omega |z(x, s)|^{r+1} dx ds.$$

(61)

Thus, since \(t_2 - t_1 \geq 1/2\), we arrive at

$$\int_{t_2}^{t_1} E(t) dt \geq (t_2 - t_1) E(t_2) \geq \frac{1}{2} E(t_2).$$

(62)

Hence,

$$E(t_2) \leq \int_{t_1}^{t_2} E(t) dt.$$  

(63)

As a result, from (47), (60), (61), (63), and since \(t_1, t_2 \in [t, t + 1]\), we get

$$E(t) \leq 2 \int_{t_1}^{t_2} E(t) dt + \int_{t}^{t_2} \left( \mu_1 - \frac{C}{\tau(r + 1)} - \frac{H_2}{r + 1} \right) \|u_1(s)\|^{r+1} ds$$

$$+ \int_{t}^{t_2} \left( \frac{C}{\tau(r + 1)} - \frac{H_2 r}{r + 1} \right) \int_\Omega |z(x, s)|^{r+1} dx ds$$

$$= 2 \int_{t_1}^{t_2} E(t) dt + [D(t)]^{r+1}.$$  

(64)

Then, from (60), we have

$$E(t) \leq \left( \frac{1}{2} C + C_3 C_2 \right) [D(t)]^2 + C_4 C_2 [D(t)]^{r+1}$$

$$+ C_3 [D(t)] + [D(t)]^{r+1} E^{\frac{1}{2}}(t).$$

(65)

Thus, utilizing Young inequality, we have

$$E(t) \leq C_3 [D(t)]^2 + [D(t)]^{r+1} + [d(t)]^{2r}.$$  

(66)

□

Hence, we have the decay estimates as follows:

Case 1. If \(r = 1\), by (66), we obtain

$$E(t) \leq 3C_3 [D(t)]^2 = 3C_3 [E(t) - E(t - 1)].$$  

(67)

Utilizing Lemma 2, we have

$$E(t) \leq E(0) e^{-\omega_1 |t - 1|},$$  

(68)

where \(\omega_1 = \ln(3C_3C_3C_3 - 1)\).

Case 2. If \(r > 1\), by (66), we have

$$E(t) \leq C_3 [D(t)]^2 (1 + [D(t)]^{r-1} + [D(t)]^{2r-2}).$$  

(69)

Then, by (47), since \(E(t) \leq E(0), \forall t \geq 0\), we see that

$$E(t) \leq C_3 \left( 1 + E^{r-1/r+1}(0) + E^{2(r-1)/r+1}(0) \right) [D(t)]^2$$

$$\leq C_3 [D(t)]^2, t \geq 0.$$  

Therefore, by (71) and Lemma 2, we obtain

$$E(t) \leq \left( E(0) - a \right) C_4^{-1} [t - 1]^{1/r}.$$  

(72)

Thus, we completed the proof of Theorem 9.

5. Blow-Up of Solution

In this part, we get the blow-up of solutions for negative initial energy, in the case \(r > 1\).

Theorem 10. Let \((u_0, u_1) \in (H^{2m} \cap H^{2m}_0(\Omega)) \times H^{2m}_0(\Omega)\) and \(f_\alpha \in L^2(\Omega \times (0, 1))\) be given. Assume that \(p > \max \{2, r, 2q + 1\}\) and the assumptions (A1)-(A2) hold. Then, the solution of (22) blows up in a finite time with \(E(0) < 0\).

Proof. Setting

$$H(t) = -E(t),$$  

(73)

from Lemma 3, we obtain

$$H'(t) \geq - \left( \mu_1 - \frac{C}{\tau(r + 1)} - \frac{H_2}{r + 1} \right) \|u_1(t)\|^{r+1}$$

$$- \left( \frac{C}{\tau(r + 1)} - \frac{H_2 r}{r + 1} \right) \int_\Omega |z^{r+1}(x, 1, t)| dx.$$  

(74)

Thus,

$$0 < H(0) \leq H(t) \leq \frac{1}{p + 1} \|u\|^{p+1}, t > 0.$$  

(75)
Let
\[ M(t) = \|u\|_{2}^{2}. \] (76)

Differentiating (76) twice, we get
\[ M'(t) = 2 \int_{\Omega} u_{t}udx, \quad \text{and} \quad M''(t) = 2\|u\|^{2} + 2 \int_{\Omega} u_{tt}udx. \] (77)

Using the first equation in (22), to have
\[ M''(t) = 2\|u\|^{2} - 2\|\mathcal{A}u\|_{2}^{2(q+1)} - 2\mu_{1} \int_{\Omega} |u_{t}(x, t)|^{-1}u_{t}u_{t}(x, t)dx \]
\[ - 2\mu_{2} \int_{\Omega} |z(x, 1, t)|^{-1}uz(x, 1, t)dx + 2\|u\|_{p+1}^{r}, \] (78)

we add and subtract the term \(2(p+1)H(t)\), and then (78) becomes the form
\[ M''(t) \geq (p + 3)\|u\|^{2} + 2(p + 1)H(t) \]
\[ + \left( \frac{p + 1}{q + 1} - 2 \right)\|\mathcal{A}u\|_{2}^{2(q+1)} \]
\[ - 2\mu_{1} \int_{\Omega} |u_{t}(x, t)|^{-1}u_{t}u_{t}(x, t)dx \]
\[ - 2\mu_{2} \int_{\Omega} |z(x, 1, t)|^{-1}uz(x, 1, t)dx + \frac{2c(p + 1)}{r + 1} \int_{\Omega} \int_{0}^{1} z^{r+1}(x, k, s)dkdx. \] (79)

Now, we define
\[ L(t) = H(t)^{-\kappa} + 2\varepsilon M'(t). \] (80)

Differentiating (80), we obtain
\[ L'(t) = (1 - \kappa)H(t)^{-\kappa}H'(t) + 2\varepsilon M''(t). \] (81)

Replacing (79) in (81), we arrive at
\[ L'(t) \geq (1 - \kappa)H(t)^{-\kappa}H'(t)2\varepsilon(p + 3)\|u_{t}\|^{2} \]
\[ + 4\varepsilon(p + 1)H(t) + 2\varepsilon \left( \frac{p + 1}{q + 1} - 2 \right)\|\mathcal{A}u\|_{2}^{2(q+1)} \]
\[ - 4\mu_{1} \int_{\Omega} |u_{t}(x, t)|^{-1}u_{t}u_{t}(x, t)dx \]
\[ - 4\mu_{2} \int_{\Omega} |z(x, 1, t)|^{-1}uz(x, 1, t)dx \]
\[ + \frac{4c(p + 1)}{r + 1} \int_{\Omega} \int_{0}^{1} z^{r+1}(x, k, s)dkdx. \] (82)

From (75) and utilizing Hölder inequality, we get
\[ \left[ \int_{\Omega} |u_{t}(x, t)|^{-1}u_{t}u_{t}(x, t)dx \right] \leq \|u_{t}\|_{r+1}^{r+1}d_{r+1}^{r+1} \]
\[ \leq c_{1} \|u\|_{r+1}^{r+1}H(t)^{-\kappa} + \frac{\rho^{-r'}H(t)^{-\kappa}H'(t)}{C_{1}^{r+1}}, \] (83)

From Young’s inequality and (74), we have
\[ \left[ \int_{\Omega} |u_{t}(x, t)|^{-1}u_{t}u_{t}(x, t)dx \right] \]
\[ \leq c_{3} \left( \rho^{1/2}\|\mathcal{M}u\|_{r+1}^{r+1}H(t)^{-\kappa} + \rho^{-r'}H(t)^{-\kappa}H'(t) \right). \] (84)

where \( \bar{k} = 1/p + 1 + 1/(p + 1)^{2} > 0, \rho > 0, r' = r + 1/r, \) letting \( 0 < k < \bar{k} \). In a similar way, we obtain
\[ \left[ \int_{\Omega} u|z(x, 1, t)|^{-1}z(x, 1, t)dx \right] \]
\[ \leq c_{3} \left( \rho^{1/2}\|\mathcal{M}u\|_{r+1}^{r+1}H(t)^{-\kappa} + \rho^{-r'}H(t)^{-\kappa}H'(t) \right). \] (85)

By using (82), (84), and (85), to have
\[ L'(t) \geq \left[ (1 - \kappa) - 4\varepsilon(\mu_{1} + \mu_{2})H^{k-k'}(0)\rho^{-r'} \right]H(t)^{-\kappa}H'(t) \]
\[ - 4\varepsilon(\mu_{1} + \mu_{2})H^{k-k'}(0)\rho^{1/2}\|\mathcal{M}u\|_{r+1}^{r+1} + 2\varepsilon(p + 3)\|u\|^{2} \]
\[ + 4\varepsilon(p + 1)H(t) + 2\varepsilon \left( \frac{p + 1}{q + 1} - 2 \right)\|\mathcal{A}u\|_{2}^{2(q+1)} \]
\[ + \frac{4c(p + 1)}{r + 1} \int_{\Omega} \int_{0}^{1} z^{r+1}(x, k, s)dkdx, \] (86)

for \( \varepsilon \) sufficiently small, we obtain
\[ \left[ (1 - \kappa) - 4\varepsilon(\mu_{1} + \mu_{2})H^{k-k'}(0)\rho^{-r'} \right] \geq 0. \] (87)

Setting \( s = r + 1 + p + 1 \) such that
\[ \|\mathcal{M}u\|_{r+1}^{s+1} \leq c_{1} \left( \|\mathcal{A}u\|_{2}^{2(q+1)} + \|\mathcal{M}u\|_{r+1}^{r+1} \right), \] (88)

where \( c = 4(\mu_{1} + \mu_{2})H^{k-k'}(0)\rho^{1/2}\varepsilon c_{1} \) and taking \( (p + 1/q + 1 - 2) > c \). Thus, we have
\[ L'(t) \geq 2\varepsilon \left( \frac{p + 1}{q + 1} - 2 - c \right)\|\mathcal{A}u\|_{2}^{2(q+1)} \]
\[ - ec\|\mathcal{M}u\|_{r+1}^{r+1} + 4\varepsilon(p + 1)H(t) + 2\varepsilon(p + 3)\|u\|^{2} \]
\[ + \frac{4c(p + 1)}{r + 1} \int_{\Omega} \int_{0}^{1} z^{r+1}(x, k, s)dkdx. \] (89)
By using the notations \( a_1 = 2(p + 1/q + 1 - c) \), \( a_2 = c \), \( a_3 = 4(p + 1) \), and \( a_4 = 2(p + 3) \), (89) takes the form

\[
L'(t) \geq a_1 \epsilon \|A^{m/2}H\|_2^{2(q+1)} - \epsilon a_2 \|\mu\|_{p+1}^p + \epsilon a_3 H(t) + \epsilon a_4 \|\mu_t\|^2.
\]

(90)

Similarly to the approach of Messaoudi [49], we assume that \( p = 2a_1 + (p - 2a_5) \), where \( a_5 < \min \{a_1, a_2, a_3, a_4\} \), and then (90) becomes the form

\[
L'(t) \geq (a_1 - a_5) \epsilon \|A^{m/2}H\|_2^{2(q+1)} + \epsilon (a_5 - a_2) \|\mu\|_{p+1}^p + \epsilon (a_3 - a_5) H(t) + \epsilon (a_4 - a_5) \|\mu_t\|^2.
\]

(91)

Then,

\[
L'(t) \geq \delta \epsilon \left[ \|A^2u\|_2^{2(q+1)} + \|\mu_t\|_{p+1}^p + H(t) + \|u_t\|^2 \right].
\]

(92)

We conclude that

\[
L'(t) \geq \delta \epsilon \left[ \|u\|_{p+1}^p + H(t) + \|u_t\|^2 \right],
\]

(93)

where \( \delta > 0 \) is the minimum of the coefficients of \( \|u\|_{p+1}^p \), \( H(t) \), \( \|u_t\|^2 \). Pick out \( \epsilon \) such that

\[
L(0) = H^{1-k}(0) + 2\epsilon \int_{\Omega} u_t u_0 dx > 0.
\]

(94)

As a result, we get setting \( \omega = 1/1 - k \), and since \( k < 1 \), we see that \( 1 < \omega < 1/1 - k \). Set

\[
L(t) = H(t)^{1-k} + 2\epsilon \int_{\Omega} u_t u dx.
\]

(95)

Then,

\[
L(t) = H(t)^{1/\omega} + 2\epsilon \int_{\Omega} u_t u dx \leq H(t)^{1/\omega} + 2\epsilon \int_{\Omega} u_t u dx + 2E_1 \left( \|u\|_{p+1}^p \right)^{p+1/\omega}.
\]

(96)

Utilizing Young, Hölder’s inequalities, and (96), we conclude that

\[
L(t)^{\omega} \leq \left[ H(t)^{1/\omega} + 2\epsilon \int_{\Omega} u_t u dx + 2E_1 \left( \|u\|_{p+1}^p \right)^{p+1/\omega} \right]^{\omega} \leq 2^{\omega-1} \left[ H(t) + 2\epsilon \int_{\Omega} u_t u dx + 2E_1 \left( \|u\|_{p+1}^p \right)^{p+1/\omega} \right] \leq c_4 H_0 \left[ H(t) + \|u\|_{p+1}^p \right],
\]

(97)

where \( c_4 = \max \{2^{\omega-1}, \beta^q\} \). Furthermore, for \( p > 1 \), utilizing Hölder and Young inequalities, we obtain

\[
\|u\|_{p+1}^p \leq c_6 \|u\|_1^2 \left( 1 - \beta^{p+1} \right) \leq c_6 \|u\|_1^2, \quad c_6 = c_1 \beta^q.
\]

(98)

Then, (97) becomes the form

\[
L(t)^{\omega} \leq c_6 \left( H(t) + \|u\|_1^2 + \|u\|_{p+1}^p \right),
\]

(99)

By combining (93) and (99), we conclude that

\[
L'(t) \leq c_7 L(t)^{\omega}, \quad c_7 > 0, \quad \omega > 1.
\]

(100)

Therefore, a simple integration over \((0, t)\), we have the desired result. Hence, we completed the proof. \( \square \)

6. Conclusions

Time delays often appear in many various problems, such as, thermal, economic phenomena, biological, chemical, and physical. Recently, the partial differential equations with time delay have become an active area (see [2, 3] and references therein). In recent years, there has been published much work concerning the wave equation with constant delay or time-varying delay. However, to the best of our knowledge, there were no global existence, decay, and blow-up results for the higher-order Kirchhoff-type equation with delay term. Firstly, we have been obtained the global existence result. Later, we have been established the decay results by using Nakao’s technique. Finally, we have proved the blow-up of solutions with negative initial energy for the problem (1) under the sufficient conditions in a bounded domain. In the next work, we will extend our current study to more general case of the problem (1).
Data Availability
No data were used to support the study.

Conflicts of Interest
The authors declare that they do not have any conflicts of interest.

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