Explicit continuation methods with L-BFGS updating formulas for linearly constrained optimization problems

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Abstract This paper considers an explicit continuation method with the trusty time-stepping scheme and the limited-memory BFGS (L-BFGS) updating formula (Eptctr) for the linearly constrained optimization problem. At every iteration, Eptctr only involves three pairs of the inner product of vector and one matrix-vector product, other than the traditional and representative optimization method such as the sequential quadratic programming (SQP) or the latest continuation method such as Ptctr [25], which needs to solve a quadratic programming subproblem (SQP) or a linear system of equations (Ptctr). Thus, Eptctr can save much more computational time than SQP or Ptctr. Numerical results also show that the consumed time of Eptctr is about one tenth of that of Ptctr or one fifteenth to 0.4 percent of that of SQP. Furthermore, Eptctr can save the storage space of an \((n + m) \times (n + m)\) large-scale matrix, in comparison to SQP. The required memory of Eptctr is about one fifth of that of SQP. Finally, we also give the global convergence analysis of the new method under the standard assumptions.

Keywords continuation method · trust-region method · SQP · structure-preserving algorithm · generalized projected gradient flow · large-scale optimization
1 Introduction

In this article, we consider the following linearly equality-constrained optimization problem

$$\min_{x \in \mathbb{R}^n} f(x)$$
subject to $Ax = b,$

where matrix $A \in \mathbb{R}^{m \times n}$ and vector $b \in \mathbb{R}^m$ may have random noise. This problem has many applications in engineering fields such as the visual-inertial navigation of an unmanned aerial vehicle maintaining the horizontal flight [8, 25], and there are many practical methods to solve it such as the sequential quadratic programming (SQP) method [3, 30] or the penalty function method [11].

For the constrained optimization problem (1), the continuation method [1, 9, 13, 19, 33, 39] is another method other than the traditional optimization method such as SQP or the penalty function method. The advantage of the continuation method over the SQP method is that the continuation method is capable of finding many local optimal points of the non-convex optimization problem by tracking its trajectory, and it is even possible to find the global optimal solution [4, 35, 43]. However, the computational efficiency of the continuation method may be higher than that of SQP. Recently, Luo, Lv and Sun [25] give a continuation method with the trusty time-stepping scheme and its consumed time is about one fifth of that of SQP for the linearly constrained optimization problem (1). Their method only needs to solve a linear system of equations with an $n \times n$ symmetric definite coefficient matrix at every iteration, which involves about $\frac{1}{2} n^3$ flops. SQP needs to solve a linear system of equations with an $(m+n) \times (m+n)$ coefficient matrix, which involves about $\frac{1}{2} (m+n)^3$ flops. In order to improve the computational efficiency further and save the storage of the continuation method [25] for the large-scale optimization problem, we consider a special limited-memory BFGS updating formula and the trusty time-stepping scheme in this article.

The rest of the paper is organized as follows. In section 2, we give a new continuation method with the trusty time-stepping scheme and the L-BFGS updating formula for the linearly equality-constrained optimization problem (1). In section 3, we analyze the global convergence of this new method. In section 4, we report some promising numerical results of the new method, in comparison to the traditional optimization method (SQP) and the latest continuation method (Ptctr) for some large-scale problems. Finally, we give some discussions and conclusions in section 5.
2 The explicit continuation method with L-BFGS updating formulas

In this section, we construct an explicit continuation method with the adaptive time-stepping scheme based on the trust-region updating strategy [44] for the linearly equality-constrained optimization problem (1). Firstly, we construct a generalized projected gradient flow based on the KKT conditions of linearly constrained optimization problem. Then, in order to efficiently follow the generalized gradient flow, we construct an explicit continuation method with an adaptive time-stepping scheme for this special ordinary differential equations (ODEs). Furthermore, we give a preprocessing method for the infeasible initial point.

2.1 The generalized projected gradient flow

For the linearly constrained optimization problem (1), it is well known that its optimal solution \( x^* \) needs to satisfy the Karush-Kuhn-Tucker conditions (p. 328, [30]) as follows:

\[
\nabla_x L(x, \lambda) = \nabla f(x) + A^T \lambda = 0, \quad (2)
\]

\[
Ax - b = 0, \quad (3)
\]

where the Lagrangian function \( L(x, \lambda) \) is defined by

\[
L(x, \lambda) = f(x) + \lambda^T (Ax - b). \quad (4)
\]

Similarly to the method of the negative gradient flow for the unconstrained optimization problem [23], from the first-order necessary conditions (2)-(3), we can construct a dynamical system of differential-algebraic equations for problem (1) [21, 22, 24, 36] as follows:

\[
\frac{dx}{dt} = -\nabla L(x, \lambda) = -\left( \nabla f(x) + A^T \lambda \right), \quad (5)
\]

\[
Ax - b = 0. \quad (6)
\]

By differentiating the algebraic constraint (6) with respect to \( t \) and replacing it into the differential equation (5), we obtain

\[
A \frac{dx}{dt} = -A \left( \nabla f(x) + A^T \lambda \right) = -A \nabla f(x) - AA^T \lambda = 0. \quad (7)
\]

If we assume that matrix \( A \) has full row rank further, from equation (7), we obtain

\[
\lambda = - (AA^T)^{-1} A \nabla f(x). \quad (8)
\]

By replacing \( \lambda \) of equation (8) into equation (5), we obtain the dynamical system of ordinary differential equations (ODEs) as follows:

\[
\frac{dx}{dt} = - \left( I - A^T (AA^T)^{-1} A \right) \nabla f(x). \quad (9)
\]
Thus, we also obtain the projected gradient flow for the constrained optimization problem \[39\].

For convenience, we denote the projection matrix \( P \) as

\[
P = I - A^T (AA^T)^{-1} A.
\]

(10)

It is not difficult to verify \( P^2 = P \) and \( AP = 0 \). That is to say, \( P \) is a symmetric projection matrix and its eigenvalues are 0 or 1. From Theorem 2.3.1 in p. 73 of [15], we know that its matrix 2-norm is

\[
\|P\| = 1.
\]

(11)

We denote \( P^+ \) as the Moore-Penrose generalized inverse of \( P \) (p. 11, [38]). Since \( P \) is symmetric and its eigenvalues are 0 or 1, it is not difficult to verify

\[
P^+ = P.
\]

(12)

Thus, for a full rank matrix \( B \in \mathbb{R}^{n \times n} \), we obtain the generalized inverse \( (PB)^+ \) of \( PB \) as follows:

\[
(PB)^+ = B^+ P^+ = B^{-1} P.
\]

(13)

Similarly to the generalized gradient flow for an unconstrained optimization problem (p. 361, [17]), from the projected gradient flow (9), we can construct the generalized projected gradient flow for the constrained optimization problem (1) as follows:

\[
\frac{dx}{dt} = -\langle PH(x)P \rangle \nabla f(x), \quad x(0) = x_0,
\]

(14)

where \( H(x) \) is a symmetric positive definite matrix for any \( x \in \mathbb{R}^n \). Here, \( H(x) \) may be selected as the inverse of the Hessian matrix \( \nabla^2 f(x) \) of \( f(x) \) and \( PH(x)P \) can be regarded as a pre-conditioner of \( P \nabla f(x) \) to mitigate the stiffness of the ODEs (14). Consequently, we can adopt the explicit numerical method to compute the trajectory of the ODEs (14) efficiently [26,27].

Remark 1 If \( x(t) \) is the solution of the ODEs (14), it is not difficult to verify that \( x(t) \) satisfies \( A(dx/dt) = 0 \). That is to say, if the initial point \( x_0 \) satisfies \( Ax_0 = b \), the solution \( x(t) \) of the generalized projected gradient flow (14) also satisfies \( Ax(t) = b, \forall t \geq 0 \). This property is very useful when we construct a structure-preserving algorithm [16,40] to follow the trajectory of the ODEs (14) and obtain its equilibrium point \( x^* \).

Remark 2 If we assume that \( x(t) \) is the solution of the ODEs (14), from equations (10)-(11) and the positive definite property of \( H(x) \), we obtain

\[
\frac{df(x)}{dt} = (\nabla f(x))^T \frac{dx}{dt} = -((\nabla f(x))^T PH(x)P \nabla f(x))
\]

\[
= -((P \nabla f(x))^T H(x)(P \nabla f(x)) \leq 0.
\]
That is to say, \( f(x) \) is monotonically decreasing along the solution curve \( x(t) \) of the dynamical system (14). Furthermore, the solution \( x(t) \) converges to \( x^* \) when \( f(x) \) is lower bounded and \( t \) tends to infinity \([17,35,39]\), where \( x^* \) satisfies the first-order Karush-Kuhn-Tucker conditions (2)-(3). Thus, we can follow the trajectory \( x(t) \) of the ODEs (14) to obtain its equilibrium point \( x^* \), which is also one saddle point of the original optimization problem (1).

### 2.2 The explicit continuation method

The solution curve of the degenerate ordinary differential equations is not efficiently followed on an infinite interval by the traditional ODE method \([2,5,20]\), so one needs to construct the particular method for this problem (14). We apply the first-order implicit Euler method \([37]\) to the ODEs (14), then we obtain

\[
x_{k+1} = x_k - \Delta t_k (PH(x_k)(PV f(x_k)) ,
\]

where \( \Delta t_k \) is the time-stepping size.

Since the system of equations (15) is a nonlinear system which is not directly solved, we seek for its explicit approximation formula. We denote \( s_k = x_{k+1} - x_k \). By using the first-order Taylor expansion, we have the linear approximation \( \nabla f(x_k) + \nabla^2 f(x_k) s_k \) of \( \nabla f(x_{k+1}) \). By substituting it into equation (15) and using the zero-order approximation \( H(x_k) \) of \( H(x_{k+1}) \), we have

\[
s_k \approx -\Delta t_k (PH(x_k))(PV f(x_k) + \nabla^2 f(x_k) s_k) = -\Delta t_k (PH(x_k))(PV f(x_k)) - \Delta t_k P(H(x_k) P)(PV^2 f(x_k) s_k.
\]

From equation (15) and \( P^2 = P \), we have \( P s_k = s_k \). Let \( H(x_k) = (\nabla^2 f(x_k))^{-1} \). Then, we have \( H(x_k) P = (PV^2 f(x_k))^{-1} \). Thus, we regard

\[
P(H(x_k) P)(PV^2 f(x_k) Ps_k = P(PV^2 f(x_k))^{-1}(PV^2 f(x_k)) Ps_k \approx Ps_k = s_k.
\]

By substituting it into equation (16), we obtain the explicit continuation method as follows:

\[
s_k = -\frac{\Delta t_k}{1 + \Delta t_k} (PH_k)(Ps_k),
\]

\[
x_{k+1} = x_k + s_k,
\]

where \( g_k = \nabla f(x_k) \) and \( H_k = (\nabla^2 f(x_k))^{-1} \) or its quasi-Newton approximation in the projective space \( S_p^g = \{ x : x = x_k + Pd, d \in \mathbb{R}^n \} \).

If we let the projection matrix \( P = I \), the formula (18) is equivalent to the explicit continuation method given by Luo, Xiao and Lv \([26]\) for nonlinear equations. The explicit continuation method (18)-(19) is similar to the projected damped Newton method if we let \( \alpha_k = \Delta t_k / (1 + \Delta t_k) \) in equation (18). However, from the view of the ODE method, they are different. The projected damped Newton method is obtained.
by the explicit Euler scheme applied to the generalized projected gradient flow (14), and its time-stepping size \( \alpha_k \) is restricted by the numerical stability \([37]\). That is to say, the large time-stepping size \( \alpha_k \) can not be adopted in the steady-state phase.

The explicit continuation method (18)-(19) is obtained by the implicit Euler approximation method applied to the generalized projected gradient flow (14), and its time-stepping size \( \Delta t_k \) is not restricted by the numerical stability. Therefore, the large time-stepping size can be adopted in the steady-state phase for the explicit continuation method (18)-(19), and it mimics the Newton method near the equilibrium solution \( x^* \) such that it has the fast local convergence rate. The most of all, the new step size \( \alpha_k = \Delta t_k / (\Delta t_k + 1) \) is favourable to adopt the trust-region updating technique for adaptively adjusting the time-stepping size \( \Delta t_k \) such that the explicit continuation method (18)-(19) accurately tracks the trajectory of the generalized projected gradient flow in the transient-state phase and achieves the fast convergence rate near the equilibrium point \( x^* \).

**Remark 3** From equation (18) and the property \( AP = 0 \) of the projected matrix \( P \), it is not difficult to verify \( A s_k = 0. \) Thus, if the initial point \( x_0 \) satisfies the linear constraint \( Ax_0 = b \), the point \( x_k \) also satisfies the linear constraint \( Ax_k = b \). That is to say, the explicit continuation method (18)-(19) is a structure-preserving method.

### 2.3 The L-BFGS quasi-Newton updating formula

For the large-scale problem, the numerical evaluation of the Hessian matrix \( \nabla^2 f(x_k) \) consumes much time and stores an \( n \times n \) matrix. In order to overcome these two shortcomings, we use the L-BFGS quasi-Newton formula ([6, 14] or pp. 222-230, [30]) to approximate the generalized inverse \( H(x_k)P \) of \( P \nabla^2 f(x_k) \). Recently, Ullah, Sabi and Shah [41] give an efficient L-BFGS updating formula for the system of monotone nonlinear equations. Here, in order to suit the generalized projected gradient flow (14), we revise their L-BFGS updating formula as

\[
H_{k+1} = \begin{cases} 
I - \frac{y_k y_k^T + s_k s_k^T}{y_k^T s_k} + 2 \frac{v_k^T y_k}{v_k^T s_k} s_k s_k^T, & \text{if } |s_k^T y_k| > \theta \|s_k\|^2, \\
I, & \text{otherwise},
\end{cases}
\]

where \( s_k = x_{k+1} - x_k, \ y_k = P \nabla f(x_{k+1}) - P \nabla f(x_k) \) and \( \theta \) is a small positive constant such as \( \theta = 10^{-6} \). The initial matrix \( H_0 \) can be simply selected by the identity matrix. When \( |s_k^T y_k| \geq \theta \|s_k\|^2 \), from equation (20), it is not difficult to verify

\[
H_{k+1} y_k = \frac{y_k y_k^T}{y_k^T s_k} s_k.
\]

That is to say, \( H_{k+1} \) satisfies the scaling quasi-Newton property. By using the Sherman-Morrison-Woodburg formula, from equation (20), when \( |s_k^T y_k| \geq \theta \|s_k\|^2 \), we have

\[
B_{k+1} = H_{k+1}^{-1} = I - \frac{s_k s_k^T}{s_k^T s_k} + \frac{y_k y_k^T}{y_k^T s_k}.
\]
The L-BFGS updating formula (20) has some nice properties such as the symmetric positive definite property and the positive lower bound of its eigenvalues.

**Lemma 1** Matrix $H_{k+1}$ defined by equation (20) is symmetric positive definite and its eigenvalues are greater than $1/2$.

**Proof.** (i) For any nonzero vector $z \in \mathbb{R}^n$, from equation (20), when $|s^T_k y_k| > \theta ||s_k||^2$, we have

$$
    z^T H_{k+1} z = ||z||^2 - 2(z^T y_k)(z^T s_k)/y^2_k s_k + 2(z^T s_k)^2 ||y_k||^2/(y^2_k s_k)
$$

$$
    = (||z|| - |z^T s_k/y^2_k s_k| ||y_k||)^2 + 2||z|| |z^T s_k/y^2_k s_k| ||y_k||
$$

$$
    - 2(z^T y_k)(z^T s_k)/y^2_k s_k + ||y_k||^2(z^T s_k/y^2_k s_k)^2 \geq 0.
$$

In the last inequality of equation (21), we use the Cauchy-Schwartz inequality $||z^T y|| \leq ||z|| ||y||$ and its equality holds if only if $z = ty$. When $z = ty$, from equation (21), we have $z^T H_{k+1} z = t^2 ||y||^2 = ||z||^2 > 0$. When $z^T s_k = 0$, from equation (21), we also have $z^T H_{k+1} z = ||z||^2 > 0$. Therefore, we conclude that $H_{k+1}$ is a symmetric positive definite matrix when $|s^T_k y_k| > \theta ||s_k||^2$. From equation (20), We apparently conclude that $H_{k+1}$ is a symmetric positive definite matrix since $H_{k+1} = I$ when $|s^T_k y_k| \leq \theta ||s_k||^2$.

(ii) It is not difficult to know that it exists at least $n - 2$ linearly independent vectors $z_1, z_2, \ldots, z_{n-2}$ such that $z^T_i s_k = 0, z^T_i y_k = 0 (i = 1 : (n - 2))$ hold. That is to say, matrix $H_{k+1}$ defined by equation (20) has at least $(n - 2)$ linearly independent eigenvectors whose corresponding eigenvalues are $1$. We denote the other two eigenvalues of $H_{k+1}$ as $\mu^{k+1}_i (i = 1 : 2)$ and their corresponding eigenvalues as $p_1$ and $p_2$, respectively. Then, from equation (20), we know that the eigenvectors $p_i (i = 1 : 2)$ can be represented as $p_i = y_k + \beta_i s_k$ when $\mu^{k+1}_i \neq 1 (i = 1 : 2)$. From equation (20) and $H_{k+1} p_i = \mu^{k+1}_i p_i (i = 1 : 2)$, we have

$$
    - \left(\mu^{k+1}_i + \beta_i s^T_k s_k/s^2_k y_k\right) y_k + \left(y^T_k y_k y^T_k s_k + 2\beta_i (y^T_k y_k)(s^T_k s_k)/(y^2_k s_k)^2 - \mu^{k+1}_i \beta_i\right) s_k = 0, \quad i = 1 : n.
$$

(22)

When $y_k = ts_k$, from equation (20), we have $H_{k+1} = I$. In this case, we conclude that the eigenvalues of $H_{k+1}$ are greater than $1/2$. When vectors $y_k$ and $s_k$ are linearly independent, from equation (22), we have

$$
    \mu^{k+1}_i + \beta_i s^T_k s_k/s^2_k y_k = 0,
$$

$$
    y^T_k y_k y^T_k s_k + 2\beta_i (y^T_k y_k)(s^T_k s_k)/(y^2_k s_k)^2 - \mu^{k+1}_i \beta_i = 0, \quad i = 1 : n.
$$

That is to say, $\mu^{k+1}_i (i = 1 : 2)$ are the two solutions of the following equation:

$$
    \mu^2 - 2\mu (y^T_k y_k)(s^T_k s_k)/(s^2_k y_k)^2 + (y^T_k y_k)(s^T_k s_k)/(s^2_k y_k)^2 = 0.
$$

(23)

Consequently, from equation (23), we obtain

$$
    \mu^{k+1}_1 + \mu^{k+1}_2 = 2(y^T_k y_k)(s^T_k s_k)/(s^2_k y_k)^2, \quad \mu^{k+1}_1 \mu^{k+1}_2 = (y^T_k y_k)(s^T_k s_k)/(s^2_k y_k)^2.
$$

(24)
From equation (24), it is not difficult to obtain

\[ 1/\mu_1^{k+1} + 1/\mu_2^{k+1} = 2, \mu_i^{k+1} > 0, \ i = 1:2. \] (25)

Therefore, from equation (25), we conclude that \( \mu_i^{k+1} > \frac{1}{2} (i = 1:2) \). Consequently, the eigenvalues of \( H_{k+1} \) are greater than 1/2. □

If \( s_{k-1} \) is obtained from the explicit continuation method (18), we have \( P_{s_{k-1}} = s_{k-1} \) since \( P^2 = P \). By combining it with the L-BFGS updating formula (20), the explicit continuation method (18)-(19) can be simplified by

\[
d_k = \begin{cases} 
-p_{g_k}, & \text{if } |s_k^T y_{k-1}| \leq \theta \|y_{k-1}\|^2, \\
p_{g_k} + \gamma_{k-1}(s_k^T p_{g_k} + y_{k-1}^T s_k^T p_{g_k}) - 2 \|y_{k-1}\|^2 (s_k^T p_{g_k}) \gamma_{k-1}^{-1} & \text{otherwise,} 
\end{cases}
\]

\[ s_k = \frac{\Delta t_k}{1 + \Delta t_k} d_k, \quad x_{k+1} = x_k + s_k, \] (26)

where \( p_{g_k} = P g_k = P \nabla f(x_k) \) and \( y_{k-1} = P \nabla f(x_k) - P \nabla f(x_{k-1}) \). Thus, it does not need to store the matrix \( H_k \) in practical computation. Furthermore, it only requires three pairs of the inner product of vector and one matrix-vector product \( P_{s_k} = P g_k \) to obtain the trial step \( s_k \) and involves in \( O((n-m)n) \) flops when we use the QR decomposition or the singular value decomposition to obtain the projection matrix \( P \) in subsection 2.4.

2.4 The treatments of infeasible initial points and projection matrices

We need to compute the projected gradient \( P g_k \) at every iteration in the updating formula (26). In order to reduce the computational complexity, we use the QR factorization (pp.276-278, [15]) to factor \( \mathbf{A}^T \) into a product of an orthogonal matrix \( Q \in \mathbb{R}^{n \times n} \) and an upper triangular matrix \( R \in \mathbb{R}^{n \times m} \):

\[ \mathbf{A}^T = QR = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R_1 \\ 0 \end{bmatrix}. \] (28)

where \( Q_1 = Q(1 : n, 1 : m), Q_2 = Q(1 : n, m + 1 : n), R_1 = R(1 : r, 1 : m) \) is upper triangular and nonsingular. Then, from equations (10), (28), we simplify the projection matrix \( P \) as

\[ P = I - Q_1 Q_1^T = Q_2 Q_2^T. \] (29)

In practical computation, we adopt the different formulas of the projection \( P \) according to \( m \leq n/2 \) or \( m \geq n/2 \). Thus, we give the computational formula of the projected gradient \( P g_k \) as follows:

\[ P g_k = \begin{cases} 
g_k - Q_1 (Q_1^T g_k), & \text{if } m \leq \frac{1}{2} n, \\
Q_2 (Q_2^T g_k), & \text{otherwise.} 
\end{cases} \] (30)
For a real-world optimization problem (1), we probably meet the infeasible initial point $x_0$. That is to say, the initial point can not satisfy the constraint $Ax = b$. We handle this problem by solving the following projection problem:

$$\min_{x \in \mathbb{R}^m} \|x - x_0\|^2 \text{ subject to } Q^T_k x = b,$$

where $b_r = (R_1 R_1^T)^{-1} R_1 b$. By using the Lagrangian multiplier method and the QR factorization (28) of matrix $A^T_k$ to solve problem (31), we obtain the initial feasible point $x_0^F$ of problem (1) as follows:

$$x_0^F = x_0 - Q_1 (Q_1^T Q_1)^{-1} (Q_1^T x_0 - b_r) = x_0 - Q_1 (Q_1^T x_0 - b_r).$$

For convenience, we set $x_0 = x_0^F$ in line 4, Algorithm 1.

2.5 The trusty time-stepping scheme

Another issue is how to adaptively adjust the time-stepping size $\Delta t_k$ at every iteration. We borrow the adjustment method of the trust-region radius from the trust-region method due to its robust convergence and fast local convergence [10]. After the preprocessing of the initial point $x_0$, it is feasible. According to the structure-preserving property of the explicit continuation method (18)-(20), $x_{k+1}$ will preserve the feasibility. That is to say, $x_{k+1}$ satisfies $Ax_{k+1} = b$. Therefore, we use the objective function $f(x)$ instead of the nonsmooth penalty function $f(x) + \sigma \|Ax - b\|$ as the cost function.

When we use the trust-region updating strategy to adaptively adjust time-stepping size $\Delta t_k$ [18], we need to construct a local approximation model of the objective $f(x)$ around $x_k$. Here, we adopt the following quadratic function as its approximation model:

$$q_k(x_k + s) = f(x_k) + s^T g_k + \frac{1}{2} s^T H_k^{-1} s.$$  \hfill (33)

In practical computation, we do not store the matrix $H_k$. Thus, we use the explicit continuation method (18)-(20) and regard $(H_k P) (H_k P)^+ \approx I$ to simplify the quadratic model $q_k(x_k + s_k) - q_k(x_k)$ as follows:

$$m_k(s_k) = g_k^T s_k - \frac{0.5 \Delta t_k}{1 + \Delta t_k} g_k^T s_k = \frac{1 + 0.5 \Delta t_k}{1 + \Delta t_k} g_k^T s_k \approx q_k(x_k + s_k) - q_k(x_k).$$ \hfill (34)

where $g_k = \nabla f(x_k)$. We enlarge or reduce the time-stepping size $\Delta t_k$ at every iteration according to the following ratio:

$$\rho_k = \frac{f(x_k) - f(x_{k+1})}{m_k(0) - m_k(s_k)}.$$ \hfill (35)

A particular adjustment strategy is given as follows:

$$
\Delta t_{k+1} = \begin{cases} 
\eta_1 \Delta t_k, & \text{if } 0 \leq |1 - \rho_k| \leq \eta_1, \\
\Delta t_k, & \text{if } \eta_1 < |1 - \rho_k| < \eta_2, \\
\gamma_2 \Delta t_k, & \text{if } |1 - \rho_k| \geq \eta_2,
\end{cases}
$$ \hfill (36)
where the constants are selected as $\eta_1 = 0.25$, $\gamma_1 = 2$, $\eta_2 = 0.75$, $\gamma_2 = 0.5$ according to numerical experiments. When $\rho_k \geq \eta_t$, we accept the trial step $s_k$ and let $x_{k+1} = x_k + s_k$, where $\eta_t$ is a small positive number such as $\eta_t = 1.0 \times 10^{-6}$. Otherwise, we discard it and let $x_{k+1} = x_k$.

According to the above discussions, we give the detailed implementation of the explicit continuation method with the trusty time-stepping scheme for the linearly equality-constrained optimization problem (1) in Algorithm 1.

**Algorithm 1** The explicit continuation method with the trusty time-stepping scheme for linearly constrained optimization (the Eptctr method)

**Input:**
- the objective function $f(x)$, the linear constraint $Ax = b$, the initial point $x_0$ (optional), the terminated parameter $\varepsilon$ (optional).

**Output:**
- the optimal approximation solution $x^*$.

1. Set $x_0 = \text{ones}(n, 1)$ and $\varepsilon = 10^{-6}$ as the default values.
2. Initialize the parameters: $\eta_0 = 10^{-6}$, $\gamma_1 = 0.25$, $\gamma_2 = 0.75$, $\gamma_3 = 0.5$, $\theta = 10^{-6}$.
3. Factorize matrix $A^T$ with the QR factorization (28).
4. Compute $x_0 \leftarrow x_0 - Q_1 (Q_1^T x_0 - b_r)$, such that $x_0$ satisfies the linear system of constraints $Ax = b$.
5. Set $k = 0$. Evaluate $f_0 = f(x_0)$ and $g_0 = \nabla f(x_0)$.
6. Compute the projected gradient $p_{k0} = P g_0$ according to the formula (30). Set $y_{-1} = 0$ and $s_{-1} = 0$.
7. Set $\Delta_0 = 10^{-2}$.
8. while $\|p_{k0}\| > \varepsilon$ do
9. if $|\eta_{1}^{k-1} y_{k-1}| > \theta |\eta_{2}^{k-1} y_{k-1}|$ then
10. $s_k = -\frac{\Delta_k}{1 + 3 \gamma_2} \left( p_{k0} - \frac{y_{k-1} (\nu_k^2 p_{k0} + y_{k-1} (\nu_k^2 p_{k0})^T)}{\gamma_{k-1}^2 y_{k-1}} \right)$.
11. else
12. $s_k = -\frac{\Delta_k}{1 + 3 \gamma_2} P g_0$.
13. end if
14. Compute $x_{k+1} = x_k + s_k$.
15. Evaluate $f_{k+1} = f(x_{k+1})$ and compute the ratio $\rho_k$ from equations (34)-(35).
16. if $\rho_k \leq \eta_1$ then
17. Set $x_{k+1} = x_k$, $f_{k+1} = f_k$, $p_{k+1} = P g_k$, $g_{k+1} = g_k$, $y_k = y_{k-1}$.
18. else
19. Evaluate $g_{k+1} = \nabla f(x_{k+1})$.
20. Compute $p_{k+1} = P g_{k+1}$ according to the formula (30). Set $y_k = P_{k+1} - P g_k$ and $s_k = x_{k+1} - x_k$.
21. end if
22. Adjust the time-stepping size $\Delta_{k+1}$ based on the trust-region updating scheme (36).
23. Set $k \leftarrow k + 1$.
24. end while

### 3 Algorithm Analysis

In this section, we analyze the global convergence of the explicit continuation method (18)-(19) with the trusty time-stepping scheme and the L-BFGS updating formula (20) for the linearly equality-constrained optimization problem (i.e. Algorithm 1).
Firstly, we give a lower-bounded estimate of \( m_k(0) - m_k(s_k) \) \((k = 1, 2, \ldots)\). This result is similar to that of the trust-region method for the unconstrained optimization problem [34]. For simplicity, we assume that the rank of matrix \( A \) is full.

**Lemma 2** Assume that the quadratic model \( q_k(x) \) is defined by equation (34) and \( s_k \) is computed by the explicit continuation method (18)-(20). Then, we have

\[
m_k(0) - m_k(s_k) \geq \frac{\Delta t_k}{4(1 + \Delta t_k)} \|P s_k\|^2,
\]

where \( p_{g_k} = P g_k = \nabla f(s_k) \) and the projection matrix \( P \) is defined by equation (10).

**Proof.** From equation (20) and Lemma 1, we know that \( H_k \) is symmetric positive definite and its eigenvalues are greater than 1/2. According to the eigenvalue decomposition of \( H_k \), we know that it exists an orthogonal matrix \( Q_k \) such that \( H_k = Q_k^T S_k Q_k \) holds, where \( S_k = \text{diag}(\mu_1^k, \ldots, \mu_n^k) \) and \( \mu_i^k \) \((i = 1 : n)\) are the eigenvalues of \( H_k \). We denote the smallest eigenvalue of \( H_k \) is \( \mu_{\text{min}}^k \). From the explicit continuation method (18) and \( P^T = P \), we know that \( s_k = P s_k \). By combining it with the explicit continuation method (18) and the quadratic model (34), we have

\[
m_k(0) - m_k(s_k) \geq -\frac{1}{2} g_k^T s_k = -\frac{1}{2} (P g_k)^T s_k = \frac{\Delta t_k}{2(1 + \Delta t_k)} (P g_k)^T H_k (P g_k)
\]

\[
\geq \mu_{\text{min}}^k \frac{\Delta t_k}{2(1 + \Delta t_k)} \|P g_k\|^2 \geq \frac{\Delta t_k}{4(1 + \Delta t_k)} \|P g_k\|^2 = \frac{\Delta t_k}{4(1 + \Delta t_k)} \|P g_k\|^2.
\]

In the first inequality in equation (38), we use the property \((1 + 0.5\Delta t_k)/(1 + \Delta t_k) \geq 0.5\) when \( \Delta t_k \geq 0 \). Consequently, we prove the result (37). \( \square \)

In order to prove that \( p_{g_k} \) converges to zero when \( k \) tends to infinity, we need to estimate the lower bound of time-stepping sizes \( \Delta t_k \) \((k = 1, 2, \ldots)\). We denote the constrained level set \( S_f \) as

\[
S_f = \{x: f(x) \leq f(x_0), A x = b\}.
\]

**Lemma 3** Assume that \( f: \mathbb{R}^n \rightarrow \mathbb{R} \) is continuously differentiable and its gradient \( g(x) \) satisfies the following Lipschitz continuity:

\[
\|g(x) - g(y)\| \leq L_c \|x - y\|, \forall x, y \in S_f.
\]

where \( L_c \) is the Lipschitz constant. We suppose that the sequence \( \{x_k\} \) is generated by Algorithm 1. Then, there exists a positive constant \( \delta_{\text{thr}} \), such that

\[
\Delta t_k \geq \gamma_2 \delta_{\text{thr}}
\]

holds for all \( k = 1, 2, \ldots \), where \( \Delta t_k \) is adaptively adjusted by the trust-region updating scheme (34)-(36).
Proof. From Lemma 1, we know that the eigenvalues of $H_k$ are greater than 1/2 and it has at least $n - 2$ eigenvalues which equal 1. When $|s_{k-1}^T y_{k-1}| \geq \theta \|y_{k-1}\|^2$, we denote the other two eigenvalues of $H_k$ as $\mu_1^k$ and $\mu_2^k$. By substituting it into equation (24), we obtain

$$\mu_1^k \mu_2^k = \frac{\|y_{k-1}\|^2 \|s_{k-1}\|^2}{(s_{k-1}^T y_{k-1})^2} \leq \frac{\|y_{k-1}\|^2 \|s_{k-1}\|^2}{\theta^2 \|y_{k-1}\|^4} = \frac{1}{\theta^2} \|y_{k-1}\|^2. \quad (42)$$

From Lemma 2 and Algorithm 1, we know $f(x_k) \leq f(x_0)$ ($k = 1, 2, \ldots$). From the explicit continuation method (18)-(20) and Remark 3, we know that $A x_k = A x_0 = b$ ($k = 1, 2, \ldots$). Thus, from the Lipschitz continuity (40) of $g(x)$, we have

$$\|y_{k-1}\| \leq \|P\| \|g(x_k) - g(x_{k-1})\| \leq L_C \|x_k - x_{k-1}\| = L_C \|s_{k-1}\|. \quad (43)$$

By substituting it into equation (42) and using $\mu_k^k > \frac{1}{2} (i = 1, 2)$, we obtain

$$\frac{1}{2} < \mu_k^k < \frac{2L_C^2}{\theta^2}, \quad i = 1, 2. \quad (44)$$

That is to say, the eigenvalues of $H_k$ are less than or equal to $M_H$, where $M_H = \max\{1, 2L_C^2/\theta^2\}$. According to the eigenvalue decomposition theorem, we know that there exists an orthogonal matrix $Q_k$ such that $H_k = Q_k^T S_k Q_k$ holds, where $S_k = \text{diag}(\mu_1^k, \ldots, \mu_k)$ and $\mu_i^k$ ($i = 1 : n$) are the eigenvalues of $H_k$. Consequently, we have

$$\|H_k(P_{g_k})\| = \|((Q_k^T S_k Q_k)P_{g_k})\| = \|S_k(Q_k P_{g_k})\| \leq M_H \|P_{g_k}\|. \quad (45)$$

From the first-order Taylor expansion, we have

$$f(x_k + s_k) = f(x_k) + \int_0^1 s_k^T g(x_k + ts_k) dt. \quad (46)$$

Thus, from equations (34)-(37), (46) and the Lipschitz continuity (40) of $g(x)$, we have

$$|p_k - 1| = \left| \frac{(f(x_k) - f(x_k + s_k)) - (m_k(0) - m_k(s_k))}{m_k(0) - m_k(s_k)} \right| \leq 1 + \frac{1}{1 + 0.5 \Delta t_k} \left| \int_0^1 s_k^T (g(x_k + ts_k) - g(x_k)) dt \right| + \frac{0.5 \Delta t_k}{1 + 0.5 \Delta t_k} \leq \frac{2L_C(1 + \Delta t_k)}{\Delta t_k} \|s_k\|^2 + \frac{0.5 \Delta t_k}{1 + 0.5 \Delta t_k}. \quad (47)$$

By substituting equation (18) and equation (45) into equation (47), we have

$$|p_k - 1| \leq \frac{2L_C \Delta t_k \|P_{g_k}(P_{g_k})\|^2}{1 + 0.5 \Delta t_k} + \frac{0.5 \Delta t_k}{1 + 0.5 \Delta t_k} \leq \frac{2L_C \Delta t_k \|P_{g_k}(P_{g_k})\|^2}{1 + 0.5 \Delta t_k} + \frac{0.5 \Delta t_k}{1 + 0.5 \Delta t_k} \leq \frac{(2L_C M_H^2 + 0.5) \Delta t_k}{1 + 0.5 \Delta t_k}. \quad (48)$$
In the last inequality of equation (48), we use the property \( \|P\| = 1 \). We denote
\[
\delta_M \triangleq \frac{\eta_1}{2L\mathcal{C}M^2 + 0.5}.
\] (49)
Then, from equation (48)-(49), when \( \Delta t_k \leq \delta_M \), it is not difficult to verify
\[
|\rho_k - 1| \leq (2L\mathcal{C}M^2 + 0.5)\Delta t_k \leq \eta_1.
\] (50)

We assume that \( K \) is the first index such that \( \Delta t_K \leq \delta_M \) where \( \delta_M \) is defined by equation (49). Then, from equations (49)-(50), we know that \( |\rho_K - 1| \leq \eta_1 \).

By using the results of Lemma 2 and Lemma 3, we prove the global convergence of Algorithm 1 for the linearly constrained optimization problem (1) as follows.

**Theorem 1** Assume that \( f : \mathbb{R}^n \to \mathbb{R} \) is continuously differentiable and its gradient \( \nabla f(x) \) satisfies the Lipschitz continuity (40). Furthermore, we suppose that \( f(x) \) is lower bounded when \( x \in S_f \), where the constrained level set \( S_f \) is defined by equation (39). The sequence \( \{x_k\} \) is generated by Algorithm 1. Then, we have
\[
\lim_{k \to \infty} \inf \|Pg_k\| = 0,
\] (51)
where \( g_k = \nabla f(x_k) \) and matrix \( P \) is defined by equation (10).

**Proof.** According to Lemma 3 and Algorithm 1, we know that there exists an infinite subsequence \( \{x_k^i\} \) such that trial steps \( s_k^i \) are accepted, i.e., \( \rho_k^i \geq \eta_a, i = 1, 2, \ldots \). Otherwise, all steps are rejected after a given iteration index, then the time-stepping size will keep decreasing, which contradicts (41). Therefore, from equations (35) and (37), we have
\[
f(x_0) - \lim_{k \to \infty} f(x_k) = \sum_{k=0}^{\infty} (f(x_k) - f(x_{k+1})) \geq \eta_a \sum_{i=0}^{\infty} (m_k(0) - m_k(s_k)) \geq \eta_a \sum_{i=0}^{\infty} \frac{\Delta t_k}{4(\Delta t_k + 1)} \|Pg_k\|.
\] (52)

From the result (41) of Lemma 3, we know that \( \Delta t_k \geq \gamma_2 \delta_M \) \( (k = 1, 2, \ldots) \). By substituting it into equation (52), we have
\[
f(x_0) - \lim_{k \to \infty} f(x_k) \geq \eta_a \sum_{i=0}^{\infty} \frac{\gamma_2 \delta_M}{4(\gamma_2 \delta_M + 1)} \|Pg_k\|.
\] (53)
Since \( f(x) \) is lower bounded when \( x \in S_f \) and the sequence \( \{f(x_k)\} \) is monotonically decreasing, we have \( \lim_{k \to \infty} f(x_k) = f^* \). By substituting it into equation (53), we obtain the result (51). \( \square \)
4 Numerical Experiments

In this section, some numerical experiments are executed to test the performance of Algorithm 1 (the Eptctr method). The codes are executed by a Dell G3 notebook with the Intel quad-core CPU and 20Gb memory. We compare Eptctr with SQP (the built-in subroutine fmincon.m of the MATLAB2018a environment) \cite{12,14,29,30,42} Ptctr \cite{25}) for some large-scale linearly constrained-equality optimization problems which are listed in Appendix A. SQP is the traditional and representative optimization for the constrained optimization problem. Ptctr is significantly better than SQP for linearly constrained optimization problems according to the numerical results in \cite{25}. Therefore, we select these two typical methods as the basis for comparison.

The termination conditions of the three compared methods are all set by

\begin{align}
\|\nabla_x L(x_k, \lambda_k)\|_\infty & \leq 1.0 \times 10^{-6}, \\
\|Ax_k - b\|_\infty & \leq 1.0 \times 10^{-6}, \quad k = 1, 2, \ldots,
\end{align}

where the Lagrange function \(L(x, \lambda)\) is defined by equation (4) and \(\lambda\) is defined by equation (8).

We test those ten problems with \(n \approx 5000\). The numerical results are arranged in Table 1 and illustrated in Figure 1. From Table 1, we find that three methods can correctly solve those ten test problems and the consumed time of Eptctr is significantly less than those of the other two methods for every test problem, respectively. The consumed time of Eptctr is about one tenth of that Ptctr or one fifteenth to 0.4 percent of that of SQP for the test problem.

From those test data, we find that Eptctr works significantly better than the other two methods, respectively. One of the reasons is that Eptctr only involves three pairs of the inner product of two vectors and one matrix-vector product (\(p g_k = Pg_k\)) to obtain the trial step \(s_k\) and involves about \((n - m)n\) flops at every iteration. However, Ptctr needs to solve a linear system of equations with an \(n \times n\) symmetric definite coefficient matrix and involves about \(\frac{1}{3}n^3\) flops (p. 169, \cite{15}) at every iteration. SQP needs to solve a linear system of equations with dimension \((m + n)\) when it solves a quadratic programming subproblem at every iteration (pp. 531-532, \cite{30}) and involves about \(\frac{2}{3}(m + n)^3\) flops (p. 116, \cite{15}). Furthermore, Eptctr can save the storage space of an \((n + m) \times (n + m)\) large-scale matrix, in comparison to SQP. The required memory of Eptctr is about one fifth of that of SQP.

5 Conclusion and Future Work

In this paper, we give an explicit continuation method with the trusty time-stepping scheme and the L-BFGS updating formula (Eptctr) for linearly equality-constrained optimization problems. This method only involves three pairs of the inner product of vector and one matrix-vector product (\(p g_k = Pg_k\)) at every iteration, other than the
Table 1: Numerical results of test problems with $n \approx 5000$.

| Problems          | Pctr steps (time) | $f(x^*)$ | Mem /Gb | Eptctr steps (time) | $f(x^*)$ | Mem /Gb | SQP steps (time) | $f(x^*)$ | Mem /Gb |
|-------------------|-------------------|----------|---------|--------------------|----------|---------|------------------|----------|---------|
| Exam. 1 (n = 5000, m = n/2) | 11 (15.46) | 3.64E+04 | 3.41    | 11 (1.94)          | 3.64E+04 | 0.51    | 2 (34.93)       | 3.64E+04 | 2.43    |
| Exam. 2 (n = 4800, m = n/2)   | 17 (16.06) | 5.78E+03 | 4.09    | 15 (1.58)          | 5.78E+03 | 0.31    | 14 (116.16)      | 5.78E+03 | 1.53    |
| Exam. 3 (n = 4800, m = 2/3n) | 12 (23.51) | 2.86E+03 | 3.40    | 12 (2.11)          | 2.86E+03 | 0.54    | 3 (56.06)       | 2.86E+03 | 3.08    |
| Exam. 4 (n = 5000, m = n/2)   | 11 (17.07) | 493.79  | 3.41    | 11 (2.02)          | 493.79  | 0.51    | 8 (123.04)      | 493.79  | 2.43    |
| Exam. 5 (n = 5000, m = n/2)   | 14 (16.79) | 432.15  | 3.97    | 13 (2.04)          | 432.15  | 0.51    | 11 (178.41)     | 432.15  | 2.43    |
| Exam. 6 (n = 4800, m = 2/3n) | 13 (23.58) | 2.06E+03 | 3.57    | 13 (2.17)          | 2.06E+03 | 0.54    | 11 (211.35)     | 2.06E+03 | 3.10    |
| Exam. 7 (n = 5000, m = n/2)   | 10 (15.02) | 5.94E+04 | 3.22    | 13 (2.07)          | 5.94E+04 | 0.51    | 20 (544.37)     | 5.94E+04 | 2.43    |
| Exam. 8 (n = 4800, m = n/3)   | 38 (38.29) | 776.88  | 7.36    | 133 (3.21)         | -1.21E+04 | 0.31 | 28 (243.09)     | 784.94  | 1.53    |
| Exam. 9 (n = 5000, m = n/2)   | 12 (22.59) | 2.21E+05 | 3.59    | 8 (1.94)           | 2.21E+05 | 0.51    | 29 (490.36)     | 2.21E+05 | 2.43    |
| Exam. 10 (n = 4800, m = n/3)  | 16 (13.51) | 2.00    | 3.92    | 20 (1.39)          | 2.00    | 0.31    | 14 (109.59)     | 2.00    | 1.53    |

traditional optimization method such as SQP or the latest continuation method such as Ptctr [25], which needs to solve a quadratic programming subproblem (SQP) or a linear system of equations (Ptctr). Thus, Eptctr involves about $(n-m)n$ flops, Ptctr involves about $\frac{1}{3}n^3$ flops, and SQP involves about $\frac{2}{3}(m+n)^3$ flops at every iteration. This means that Eptctr can save much more computational time than SQP or Ptctr. Numerical results also show that the consumed time of Eptctr is about one tenth of that of Ptctr or one fifteenth to 0.4 percent of that of SQP for the test problem with $n \approx 5000$. Furthermore, Eptctr can save the storage space of an $(n+m) \times (n+m)$ large-scale matrix, in comparison to SQP. The required memory of Eptctr is about one fifth of that of SQP. Therefore, Eptctr is worth exploring further, and we will extend it to the general nonlinear optimization problem in the future.

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Fig. 1: The consumed CPU time (s) of Ptcrt, Eptcr and SQP for test problems with \( n \approx 5000 \).

A Test Problems

Example 1.

\[ m = \frac{n}{2} \]

\[ \min_{x \in \mathbb{R}^n} f(x) = \sum_{k=1}^{n/2} \left( x_{2k-1}^2 + 10x_{2k}^2 \right), \text{ subject to } x_{2i-1} + x_{2i} = 4, i = 1, 2, \ldots, m. \]

This problem is extended from the problem of [31]. We assume that the feasible initial point is \((2, 2, \ldots, 2, 2)\).

Example 2.

\[ m = \frac{n}{3} \]

\[ \min_{x \in \mathbb{R}^n} f(x) = \sum_{k=1}^{n/3} \left( (x_{3k-1} - 2)^2 + 2(x_{3k} - 1)^3 \right) - 5, \text{ subject to } x_{3i-2} + 4x_{3i-1} + 2x_{3i} = 3, i = 1, 2, \ldots, n/3. \]

We assume that the infeasible initial point is \((-0.5, 1.5, 1, 0, \ldots, 0, 0)\).

Example 3.

\[ m = (2/3)n \]

\[ \min_{x \in \mathbb{R}^n} f(x) = \sum_{i=1}^{n/3} x_i^2, \text{ subject to } x_{3i-2} + 2x_{3i-1} + x_{3i} = 3, 2x_{3i-2} - x_{3i-1} - 3x_{3i} = 4, i = 1, 2, \ldots, n/3. \]
This problem is extended from the problem of [32]. The infeasible initial point is \((1, 0.5, -1, \ldots, 1, 0.5, -1)\).

**Example 4.**

\[ m = n/2 \]

\[ \min_{x \in \mathbb{R}^n} f(x) = \sum_{k=1}^{n/2} \left( x_k^2 - 1 \right) - 1, \text{ subject to } x_{2i-1} + x_{2i} = 1, i = 1, 2, \ldots, n/2. \]

This problem is modified from the problem of [28]. We assume that the infeasible initial point is \((1, 1, \ldots, 1)\).

**Example 5.**

\[ m = n/2 \]

\[ \min_{x \in \mathbb{R}^n} f(x) = \sum_{k=1}^{n/2} \left( (x_{2k-1} - 2)^4 + 2(x_{2k} - 1)^6 \right) - 5, \text{ subject to } x_{2i-1} + 4x_{2i} = 3, i = 1, 2, \ldots, m. \]

We assume that the feasible initial point is \((-1, 1, 1, \ldots, 1, -1, 1)\).

**Example 6.**

\[ m = (2/3)n \]

\[ \min_{x \in \mathbb{R}^n} f(x) = \sum_{k=1}^{n/3} \left( x_k^3 - 1 \right) - 1, \text{ subject to } x_{2i-1} + x_{2i} = 4, i = 1, 2, \ldots, n/2. \]

This problem is extended from the problem of [32]. We assume that the infeasible initial point is \((2, 0, \ldots, 0)\).

**Example 7.**

\[ m = n/2 \]

\[ \min_{x \in \mathbb{R}^n} f(x) = \sum_{k=1}^{n/2} \left( x_k^2 + 3x_k^2 \right) - 1, \text{ subject to } x_{2i-1} + x_{2i} = 4, i = 1, 2, \ldots, n/2. \]

This problem is extended from the problem of [7]. We assume that the infeasible initial point is \((2, 2, 0, \ldots, 0)\).

**Example 8.**

\[ m = n/3 \]

\[ \min_{x \in \mathbb{R}^n} f(x) = \sum_{k=1}^{n/3} \left( x_k^3 - 1 \right) - 1, \text{ subject to } 2x_{2i-2} + 2x_{2i-1} + x_{2i} = 1, 2x_{2i-2} - 2x_{2i-1} - 3x_{2i} = 4, i = 1, 2, \ldots, m/2. \]

We assume that the infeasible initial point is \((1.5, 0, 0, \ldots, 0)\).

**Example 9.**

\[ m = n/2 \]

\[ \min_{x \in \mathbb{R}^n} f(x) = \sum_{k=1}^{n/2} \left( x_k^3 - 1 \right) - 1, \text{ subject to } x_{2i-1} + x_{2i} = 4, i = 1, 2, \ldots, m. \]

This problem is extended from the problem of [31]. We assume that the infeasible initial point is \((2, 2, 2, \ldots, 2)\).

**Example 10.**

\[ m = n/3 \]

\[ \min_{x \in \mathbb{R}^n} f(x) = \sum_{k=1}^{n/3} \left( x_k^3 - 1 \right) - 1, \text{ subject to } x_{2i-2} + 2x_{2i-1} + 2x_{2i} = 1, i = 1, 2, \ldots, m. \]

This problem is modified from the problem of [43]. The feasible initial point is \((1, 0, 0, \ldots, 1, 0, 0)\).
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