The limit theorem for dependent Bernoulli tests

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Abstract. There is considered the generalized Bernoulli tests scheme in which it is assumed that the tests are connected unlike the usual Bernoulli tests scheme. There is defined the condition of weak dependence of tests. The integral limit theorem is proved for the distribution of the number of successes in carrying out a finite set of tests. There is investigated the convergence rate in the limit theorem. There is obtained the optimal remainder.

1. Formulation of problem. Definitions
In the study of various processes, for example, underground filtration of non-Newtonian fluids [1–7] nonlinear deformation of plates and shells [8–14] and multilayer structures [15–21], the interaction of low-temperature plasma with materials [22–28], etc., it becomes necessary to use relationships that link various characteristics of processes. These relationships are empirical, they are obtained as a result of processing the results of natural experiments. The Bernoulli scheme is often used for data processing, this work is devoted to the study of this scheme. This article belongs to the category of articles devoted to the investigation of the remainders in limit theorems. In articles [31-35] we study the convergence rate in various limit theorems for weakly dependent random terms.

Quite often it is necessary to meet with tasks consisting of a series of identical tests in which the same outcome appears repeatedly, for example, in the Monte Carlo simulation [36-40] which is used in modeling processes occurring in plasma, in the modeling of financial processes. Often in such problems it is required to be able to determine the probability of any given number of favorable outcomes in a series of tests. This question is well researched in the case of independent tests, and in the case of weakly dependent tests it is not completely resolved. Only special cases were subject to investigation. In this article there is proved the theorem of Moivre-Laplace on finding the limiting distribution of the number of favorable outcomes with the optimal convergence rate under the condition of weak dependence on the random factor.

The subject of our research is random testing. In this case a random test is the implementation of a certain set of conditions as a result of which some event may or may not occur.

Let consider the sequence of random tests \(E_1, \ldots, E_n, \ldots\). Each test \(E_k\) has two outcomes: “success” and “failure”, which we denote by \(e_1^{(k)}\) and \(e_2^{(k)}\) respectively. Thus, the space of elementary outcomes of the test \(E_k\) \(\Omega^{(k)} = e_1^{(k)} + e_2^{(k)}\). We assume that \(P(e_1^{(k)}) = p\), \(P(e_2^{(k)}) = 1 - p = q\), for any \(k=1,2,\ldots\).

1.1. Definition 1
The tests $E_1,\ldots,E_n$ are called independent if $P(e_1^{(i)} \ldots e_n^{(n)}) = P(e_1^{(i)}) \cdot P(e_n^{(n)})$ where $i_j = 1$ or $2$

1.2. Definition 2
We say that the tests $E_1,\ldots,E_n$ are connected in a simple Markov chain if under the condition that event $e_1^{(k)}$ appears in test $E_k$ the probability that the event $e_1^{(k+1)}$ will again appear in the test $E_{k+1}$ is $\alpha$; the probability of the event $e_1^{(k+1)}$ happening in the test $E_{k+1}$ if it is known that in the test $E_k$ $e_1^{(k)}$ happens is equal to $\beta$. Thus, the transition probabilities are given by matrix $\begin{pmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{pmatrix}$. Here it is assumed that $\alpha$ and $\beta$ are different from 0 and 1. It is obvious that such a tests scheme is a generalization of the independent tests scheme.

1.3. Definition 3
Let say that the tests $E_1,\ldots,E_n$ are weakly dependent and in this case they satisfy the strong mixing condition with the coefficient $\alpha(\tau)$ if

$$\sup_{A\in h'_1, B\in h_n'} \left| p(AB) - p(A)p(B) \right| = \alpha(\tau)$$

Here $h'_1$ is a $\sigma$-algebra of random events generated by random events $E_1,\ldots,E_n$, $h_n'$ is a $\sigma$-algebra of random events generated by random events $E_{i_2},\ldots,E_n$.

Let give one more definition of the dependence of the random tests $E_1,\ldots,E_n$.

Let $E$ be a random factor associated with two events: the $e_1$ - the factor acts, the $e_2$ - the factor does not act. Let $h$ be the $\sigma$-algebra generated by the random events $e_1$ and $e_2$, that is the random factor $E$. We introduce the notation $\beta^{(k)}_{ij} = P(A_i^{(k)}|A_j) - P(A_i^{(k)})$. Here $A_i^{(k)}$ is the elementary event of $\sigma$-algebra $h'_1$ generated by the test $E_k$ and $A_i^{(k)}$ is the elementary event of $\sigma$-algebra generated by the random factor $E$.

1.4. Definition 4
If for any $k$, in any row of the matrix $\{\beta^{(k)}_{ij}\}$ there exists a positive element, then we say that the tests $E_1,\ldots,E_n$ weakly depend on the random factor $E$. If, in addition, the elements of the matrix $\{\beta^{(k)}_{ij}\}$ do not depend on $k$, then we say that $E_1,\ldots,E_n$ weakly depend on $E$ equally.

Let consider the random tests $E_1,\ldots,E_n$ such that $P(e_i^{(k)}) = p$ for any $k=1,\ldots,n$. Let $\mu_n$ be the number of “successes” in the implementation of tests $E_1,\ldots,E_n$.

We are interested in the asymptotic demeanor of the distribution function of $\mu_n$ ($n \to \infty$) under various conditions of the $E_1,\ldots,E_n$ dependence.

Let assume that $E_1,\ldots,E_n$ are independent tests. It is known that expected value $M\mu_n = np$ and dispersion $D\mu_n = npq$. We denote $F_n(x) = P\left( \frac{\mu_n - np}{\sqrt{npq}} < x \right)$. Then the Muawr-Laplace theorem states that

$$\sup_{-\infty < x < \infty} |F_n(x) - \Phi(x)| \to 0 \quad (n \to \infty)$$

(1)
Here $\Phi(x)$ is the distribution function of the standard normal distribution. To the question of how rapidly the tendency to zero in (1) increases with increasing $n$ the Berry-Essen theorem [41] answers. From this theorem, as a particular case, follows

$$\sup_{-\infty < x < \infty} |F_n(x) - \Phi(x)| \leq \frac{p^2 + q^2}{\sqrt{npq}}$$

(2)

It should be noted that the order of estimate in (2) cannot be improved [41]. In the case of tests $E_1, \ldots, E_n$ connected in a simple Markov chain (Def. 2), the limiting demeanor of $F_n(x)$ is investigated by Gnedenko B.V. and a result similar to (1) is obtained [42], but in this case the convergence rate is not investigated.

If the tests $E_1, \ldots, E_n$ are weakly dependent and satisfy the strong mixing condition with the coefficient $\alpha(\tau) \leq c \tau^{-\omega}$ ($c$ is a constant, $\omega$ is an arbitrarily large positive number), then in [36] there is obtained an estimate for the convergence rate of the following form

$$\sup_{-\infty < x < \infty} |F_n(x) - \Phi(x)| = O\left(\frac{1}{\sqrt{n}}\right)$$

where $\varepsilon$ is an arbitrarily small positive number.

There is should be noticed that Definition 3 is more general than Definition 2.

We are interesting in the limiting demeanor of $F_n(x)$ ($n \to \infty$) the random tests $E_1, \ldots, E_n$ (Def. 3) satisfy the strong mixing condition and the condition of the same weak dependence on the random factor $E$ (Def. 4).

2. Statement of the theorem.

Let $E_1, \ldots, E_n, \ldots$ be a sequence of independent Bernoulli tests and let the limit relation

$$F_n(x) = \Phi(x) + O\left(\frac{p^2 + q^2}{\sqrt{npq}}\right)$$

($n \to \infty, \ -\infty < x < \infty$)

Then if $E_1, \ldots, E_n, \ldots$ satisfy the strong mixing condition (Def. 3) and the condition of the same weak dependence on the random factor $E$ then relation

$$F_n(x) = \Phi(x) + O\left(\frac{p^2 + q^2}{\sqrt{npq}}\right) + O\left(\frac{1}{\sqrt{n}}\right)$$

($n \to \infty, \ -\infty < x < \infty$)

3. Auxiliary assertion.

Let formulate an auxiliary assertion, necessary for the proof of the theorem.

3.1. Lemma

If $E_1, \ldots, E_n, \ldots$ satisfy the conditions of Definition 3 and Definition 4, then for any events $A \in \mathcal{H}_n, \ldots, A \in \mathcal{H}_n, B \in \mathcal{H}_n, \ldots, B \in \mathcal{H}_n$

$$|P(A_1 \ldots A_r B_1 \ldots B_t) - P(A_1 \ldots A_r) P(B_1 \ldots B_t)| \leq C_1 \frac{\sqrt{\alpha(d)}}{l + s}$$

Where $d = t_1 - r_1$ is the distance between sets $\{r_1, \ldots, r_r\}$ and $\{t_1, \ldots, t_t\}$ ($\{r_1, \ldots, r_r\}$ and $\{t_1, \ldots, t_t\}$ are subsets of $\{1, 2, 3, \ldots\}$, $C_1$ is a constant.
Further we denote some constants by $C_i$.

3.2. Proof of the lemma
We denote the conditional probability of the event $A_i^{(k)}$ under the condition that some events $A_i^{(m)},...,A_m^{(m)}$ have occurred by $\hat{P}(A_i^{(k)} | A_j), \ m=1,...,k-1$

From the condition of Definition 4 it follows that $\hat{P}(A_i^{(k)} | A_j) - P(A_i^{(k)}) > 0$ that is the matrix $\{ p_{ij}^k \}$ contains a positive element in each line. It is obviously that $\mu > 0$. The quantity $\mu$ does not depend on $k$ because the tests $E_k$ weakly depend on the factor $E$ equally.

Further

$$\hat{P}(A_i^{(k)}) = \frac{\hat{P}(A_i^{(k)} | A_j)}{1 + (\hat{P}(A_i^{(k)} | A_j) - P(A_i^{(k)}))/\hat{P}(A_i^{(k)})} \leq \frac{1}{1 + \mu}$$

By the formula of multiplication of probabilities

$$P(A_1...A_i) \leq \left( \frac{1}{1 + \mu} \right)^i$$

$$P(B_1...B_i) \leq \left( \frac{1}{1 + \mu} \right)^i$$

(3)

$$P(A_1...A_iB_1...B_i) \leq \left( \frac{1}{1 + \mu} \right)^{i+1}$$

From the strong mixing condition it follows that

$$|P(A_1...A_iB_1...B_i) - P(A_1...A_i)P(B_1...B_i)| \leq \alpha(d)$$

(4)

Using the estimates (3) and (4) we obtain

$$|P(A_1...A_iB_1...B_i) - P(A_1...A_i)P(B_1...B_i)| = \sqrt{P(A_1...A_iB_1...B_i) - P(A_1...A_i)P(B_1...B_i)} \leq$$

$$\leq \sqrt{\alpha(d)P(A_1...A_iB_1...B_i) + P(A_1...A_i)P(B_1...B_i)} \leq C_2 \sqrt{\alpha(d) \left( \frac{1}{1 + \mu} \right)^{i+s}} \leq C_3 \sqrt{\alpha(d) \frac{1}{l+s}}$$

Lemma is proved.

4. Proof of the theorem.
We proceed directly to the proof of the theorem.

Let $\Pi^{pl+p_0}$ be the set of all natural numbers of the form $pl + p_0, l=0,1,2,..., p$ and $p_0$ are natural numbers such that $p \geq 1$ and $p_0 \leq p - 1$.

We denote $\delta(A,B) = P(AB) - P(A)P(B); \ A^l (i=1,...,n)$ are events from $\sigma$-algebra $h^{(i)}$ generated by random tests $E_i, N=\{1,...,n\}; \bigcap_{i=1}^{pl+p_0} \cap_N A^{pl+p_0}$ is an elementary volume.

Let carry out the transformation:
\[ P(A_1\ldots A^n) = P(A_1\ldots A^n) - P(A_{2l})P(A_{2l+1}) + P(A_{2l})P(A_{2l+1}) = \\
= \delta(A_{2l}, A_{2l+1}) + P(A_{2l})P(A_{2l+1}) = \delta(A_{2l}, A_{2l+1}) + \left[ \delta(A_{4l}, A_{4l+2}) + P(A_{4l})P(A_{4l+2}) \right] * \\
* \left[ \delta(A_{4l+1}, A_{4l+3}) + P(A_{4l+1})P(A_{4l+3}) \right] = \delta(A_{2l}, A_{2l+1}) + \delta(A_{4l}, A_{4l+2}) * \delta(A_{4l+1}, A_{4l+3}) + \\
+ P(A_{4l})P(A_{4l+2})P(A_{4l+1})P(A_{4l+3}) = \text{etc} \\
\]

By doing \( k = C_i \log_2 n \) we get

\[ P(A_1\ldots A^n) = \prod_{i=1}^n P(A_i') + \sum_{k=1}^{C_i \log_2 n} C_k \max \delta(A^{2i+0}, A^{2i+0'}) \tag{5} \]

Let \( E_1, \ldots, E_n \ldots \) be the set of independent Bernoulli tests with probability of “success” \( p, \mu_n \) is the number of “successes” in the implementation of \( E_1, \ldots, E_n \). Further let \( E_1^*, E_2^*, \ldots, E_n^* \) be the set of Bernoulli tests satisfying the strong mixing condition (Def. 3) and the condition of the same weak dependence on the random factor \( E \) (Def. 4) with probability of “success” equal to \( p \) in each test; \( \mu_n^* \) is the number of “successes” in the implementation of \( E_1^*, E_2^*, \ldots, E_n^* \). Then

\[ P \left( \frac{\mu_n - np}{\sqrt{npq}} < x \right) = \sum_{\mu_n - np} P(A_1\ldots A^n) \]

\[ P \left( \frac{\mu_n^* - np}{\sqrt{npq}} < x \right) = \sum_{\mu_n - np} P(A_1\ldots A^n) \]

From this and from (5) it follows that

\[ \left| P \left( \frac{\mu_n^* - np}{\sqrt{npq}} < x \right) - P \left( \frac{\mu_n - np}{\sqrt{npq}} < x \right) \right| = \\
= \left| \sum (P(A_1\ldots A^n) - P(A_1)\ldots P(A^n)) \right| = \\
= \sum_{\mu_n - np} \sum_{i=1}^{C_i \log_2 n} C_i (\max \delta(A^{2i+0}, A^{2i+0'}) \leq \\
\leq \sum_{i=1}^{C_i \log_2 n} \left| \sum_{\mu_n^* - np} \delta(A^{2i+0}, A^{2i+0'}) \right| \]

where \( \hat{p}_0, \hat{p}_0' \) are the values at which the maximum of the sum on the right in (6).

Let \( \Omega \) be the set of all elementary volumes of \( A_1, \ldots, A_n \). It is obvious that the inequality

\[ \left| \sum_{\mu_n^* - np} \delta(A^{2i+0}, A^{2i+0'}) \right| \leq \max \sum_{G} \delta(A^{2i+0}, A^{2i+0'}) \]

where \( G \) is an arbitrary subset of \( \Omega \).
Further it can be shown that there exists a constant $C_8$ such that the inequality

$$\max_G \sum_{A_i \times B_j} \delta \left( A_i^{q_{ij} \hat{p}_0}, A_j^{q_{ij} \hat{p}_0} \right) \leq C_8 \max_{A_i \times B_j} \left| \sum_{A_i \times B_j} \delta \left( A_i^{q_{ij} \hat{p}_0}, A_j^{q_{ij} \hat{p}_0} \right) \right|$$

(7)

where $A_i^{q_{ij} \hat{p}_0} \in A_i$, $A_j^{q_{ij} \hat{p}_0} \in B_j$, $A_i \times B_j$ is a direct multiplication of sets $A_i$ and $B_j$ from $\Omega$.

Using (6) and (7) we obtain

$$\Delta = \left| P \left( \frac{\mu_n^* - np}{\sqrt{npq}} < x \right) - P \left( \frac{\mu_n - np}{\sqrt{npq}} < x \right) \right| \leq \sum_{i=1}^{C_{l1} \log_2 n} C_i \left| \delta \left( A_i^{q_{ij} \hat{p}_0}, A_j^{q_{ij} \hat{p}_0} \right) \right|$$

Here $A_i, B_j$ are sets on which a maximum is reached. Taking into account the additivity of $\delta(o, o)$ and the assertion of the lemma in which we set the mixing coefficient $\phi(d) = \frac{C_{10}}{d}$ we have

$$\Delta \leq \sum_{i=1}^{C_{l1} \log_2 n} C_{l1} \frac{1}{n} \leq C_{l2} \frac{1}{\sqrt{n}}$$

(8)

The assertion of the theorem follows from (8).

5. Conclusion.
In the article there is considered the Bernoulli tests scheme with dependent tests. The condition for weak dependence of the tests is defined. For this condition it is established that the convergence rate in the integral limit theorem for the number of occurrence of an event in the realization of a finite number of dependent tests coincides with the convergence rate in the case of independent tests.

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