Long-distance properties of frozen U(1) Higgs and axially U(1)-gauged four-Fermi models in $1 + 1$ dimensions

Hisashi Yamamoto

Uji Research Center
Yukawa Institute for Theoretical Physics
Kyoto University, Uji 611, Japan

Abstract

We study the long-distance relevance of vortices (instantons) in an $N$-component axially U(1)-gauged four-Fermi theory in $1 + 1$ dimensions, in which a naive use of $1/N$ expansion predicts the dynamical Higgs phenomenon. Its general effective lagrangian is found to be a frozen U(1) Higgs model with the gauge-field mass term proportional to an anomaly parameter ($b$). The dual-transformed versions of the effective theory are represented by sine-Gordon systems and recursion-relation analyses are performed. The results suggest that in the gauge-invariant scheme ($b = 0$) vortices are always relevant at long distances, while in non-invariant schemes ($b > 0$) there exists a critical $N$ above which the long-distance behavior is dominated by a free massless scalar field.

1 Introduction

In a previous paper [1] we have studied the finite-temperature phase structure of the quasi (2+1)-dimensional four-Fermi theory with a global chiral U(1) symmetry, in order to get insight into the infrared dynamics of quasi two-dimensional superconductors. Remarkably we found Kosterlitz-Thouless (KT) type and another Nambu-Goldstone (NG) type ordered phases. It is now a natural question to ask whether a dynamical Higgs mechanism operates or not if the model is coupled to an axial gauge field. This question may be well parallel to the analogous issue in superconductivity, namely, if a Meissner effect takes place or not when an external magnetic field is applied to quasi two-dimensional superconductors. It will not be too much to say that the qualitative long-distance properties at finite temperature are essentially similar to those at zero temperature, in which a local gauge symmetry is easier to treat.

On the other hand, four-Fermi interactions have recently attracted attentions in particle physics as a source of dynamical symmetry breaking in the Standard model [2]. In this context top quark constitutes a strong-coupling four-Fermi interaction and condensates to replace a fundamental Higgs boson. The main ingredient of these arguments is based on a Nambu-Jona-Lasinio mechanism [3] with a mean-field-type approximation. Although some detailed analyses of field-theoretical aspects have recently been given for nongauged models [4, 5, 6], those for gauged models including topological effects has not yet been addressed. In 3+1 dimensions it is not straightforward to treat analytically the many-body effect of topological excitations and one may rely on numerical simulations on a lattice. In such situations it would be instructive to study related low-dimensional models with an abelian gauge symmetry, where topological excitations are more tractable. Although results themselves may somewhat depend on a dimensionality and an abelian nature of the gauge symmetry and field-theoretical lessons which may prove useful for studying more realistic, higher-dimensional models with (non-abelian) gauge symmetries.

In view of these backgrounds, we wish to argue in the present article the non-perturbative long-distance properties...
Fermi theory coupled to an axial gauge field in $1 + 1$ dimensions, as a simple model to discuss the dynamical Higgs phenomenon. In standard notations the model is defined with the lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma \partial - \sigma - i\gamma_5\pi + A_5)\psi - (N/2g^2)(\sigma^2 + \pi^2) - (N/4e^2)F^2, \quad (1.1)$$

where $g$ and $e$ are respectively a dimensionless four-Fermi coupling and an axial-vector gauge coupling with a mass dimension, and a dot ($\cdot$) means to take a sum over fermion species $i = 1 \sim N$ (Lorentz indices are suppressed). $\mathcal{L}$ is invariant under the following axial gauge transformation:

$$\psi(x) \rightarrow e^{i\gamma_5\varphi(x)}\psi(x),$$

$$A(x) \rightarrow A(x) + \partial\varphi,$$

$$\sigma(x) \rightarrow e^{-2i\varphi(x)}(\sigma + i\pi)(x). \quad (1.2)$$

In their classic paper [7], Gross and Neveu studied the limit $N \rightarrow \infty$ of (\ref{1.1}) for fixed $g^2$ and $e^2$. For $A = 0$, $\mathcal{L}$ is invariant under scale and global chiral transformations. In the limit $N \rightarrow \infty$ the dynamical breaking of chiral symmetry develops for all non-zero $g^2$, and $\sigma$ acquires a vacuum expectation value (v.e.v.) $\langle \sigma \rangle = \sigma_c$, giving a mass to the fermions (dimensional transmutation) [7]. In $1 + 1$ dimensions, however, this symmetry-broken solution, which we call a NG vacuum, is unstable against higher-order corrections of $1/N$ expansion. Beyond the leading order, uncontrollable infrared divergences appear in the gap equations (cancellation conditions of $\sigma$ tadpoles: $\langle \sigma \rangle = 0$ where $\sigma \equiv \sigma - \sigma_c$) due to the fluctuations of a massless NG boson $\pi$ (fig.1) and render the leading-order solution meaningless for a general finite $N$ [3]. The absence of a stable NG vacuum is in accord with the general no-go theorem [3]. The meaningful solution for the nongauged model was later given by Witten [11], who argued that a stable solution is provided by assuming a v.e.v. only for a modulus field $\rho$ ($\langle \rho \rangle = \rho_c$) of the order-parameter field

$$\sigma + i\pi = \rho e^{i\chi}, \quad (1.3)$$

but not for a phase field $\chi$. At large $N$ the long-distance behavior is described mainly by a free massless phase field, i.e. with the effective lagrangian

$$\mathcal{L}_\chi = (N/8\pi)(\partial\chi)^2. \quad (1.4)$$

Other terms for $\chi$ (only) are all of higher distances. The chiral U(1) symmetry is broken to a KT phase [11, 12]. It is then expected that a KT boson (KT boson) in higher-order diagrams develops a singularity but only supplies finite renormalization to the effective lagrangian (\ref{1.4}). The absence of a stable NG particle generating an infrared singularity but only supplies finite renormalization to a KT boson.

Thus the vacuum specified by

$$\rho_c \neq 0,$$

which we call a KT vacuum, is expected for $N$ the long-distance behavior should develop in the low-temperature phase of the XY model. The power-law decay of a phase-wave correlation function

$$e^{-i\chi(x)}e^{i\chi(0)} \rightarrow \exp(-\rho_c|x|),$$

where $const.$ stand for some positive constant, is expected.

In ref. [11] Gross and Neveu have also studied the axial gauge coupling ($A \neq 0$). Based on a naive mean-field argument in the gauged model is accessible in $1 + 1$ dimensions and hence its picture is not to be criticized from the viewpoint of dilute-instanton-gas calculations [13]. The two-point effective action for $A$ shows that both of their propagators acquire poles at $p^2 = \rho_c^2$, indicating a power-law decay of a phase-wave correlation function

$$e^{-i\chi(x)}e^{i\chi(0)} \rightarrow \exp(-\rho_c|x|),$$

where $const.$ stand for some positive constant, is expected.

The naive mean-field argument if they are
Another point that should be considered carefully is a subtlety with respect to an axial-gauge invariance, which is also neglected in the old argument \[7\]. An anomaly could generally emerge if the fermions are integrated with a gauge non-invariant regularization. In 1 + 1 dimensions, however, this reflection manifest itself on the effective action just as a contact term $A^2$ of the axial gauge field \[13\] and does not bring out a theoretical difficulty; a massive abelian gauge (Proca) theory is quantum-mechanically consistent. Hence, following recent studies of anomalous gauge theories \[16, 17\], it is natural to consider that the quantum theory of such a type possesses a hidden one-parameter degree of freedom which cannot be uniquely determined from the theoretical consistency. Once such a mass term is allowed it should play an important role in the long-distance physics.

We thus believe it important to study whether the naive argument in favor of a dynamical Higgs mechanism could persist to hold or not even after incorporating above two points.

In (1 + 1)-dimensional models with an abelian symmetry, the relevant topological excitation is a vortex configuration in a euclidean space-time (instanton). As is suggested from our previous work \[1\], in order to incorporate vortex configurations in the present four-Fermi theory we should start from the KT vacuum \(1.3\) keeping a global chiral U(1) symmetry. This can conveniently be achieved with a radial parametrization of the order-parameter field \(1.3\) \[10, 1\]. In sect.2 we will show that a general large-\(N\) long-distance effective lagrangian on the KT vacuum consists of a U(1) Higgs model with a radial fluctuation frozen and of the possible gauge-field mass term, the coefficient \((\equiv b)\) of which is arbitrary. We will thus study the long-distance properties of the effective lagrangian in two cases separately, i.e. in a gauge-invariant scheme \((b = 0\), sect.3\) and in non-invariant (anomalous) schemes \((b > 0\), sect.4\).

The (normal) U(1) Higgs model itself is an interesting physical system, for example as effective theories of thin-film superconductors or a helium superfluid threatened by a magnetic field, which in fact contain vortex solutions and some early analyses have been reported \[18\]. In our knowledges, however, a proper quantum-mechanical treatment of the gauge-vortex dynamics has not yet appeared in references. With the help of a dual transformation, we will develop in sect.3 a renormalization-group (RG) analysis. We first formulate the partition function of the frozen U(1) Higgs model with an external scalar field. This scalar-vortex system, with a vortex chemical-potential controlled by a large chemical potential \((y \ll 1)\) can be treated as a coupled sine-Gordon (s-G) field theory. In this local field analysis the analysis is restricted to a small-fugacity \((y \ll 1)\) and in particular whether there could exist a phase where the mean-field treatment is justified at long distances.

Then in sect.4 we will study the general non-invariant scheme, i.e. with a non-zero gauge-field mass term. With the similar procedures to the above we will show vortex configurations interacting with both massive scalar and massive s-G fields and the recursion-relation analysis will be performed as well. Sect.5 is devoted to summary and concluding remarks. Appendix A includes some technical details for deriving momentum-shell recursion equations. In Appendix B is presented an approximate argument to support partly the conjecture for the original \(y = 1\) system in the \(b > 0\) scheme.

### 2 Effective lagrangian of the axially U(1)-gauged four-Fermi theory

In this section, based on the KT vacuum \(1.3\) of the axially gauged four-Fermi model \(1.1\) the large-\(N\) long-distance effective lagrangian corresponding to \(1.4\) for the non-gauged model will be derived. This will provide a basis on which we can later investigate vortex configurations.

In terms of the radial parametrization of the order-parameter field \(1.3\) \[10, 1\],
four-Fermi system we will consider, is described by the partition function

\[ Z = \int \rho D\rho D\chi DAD\bar{\psi}D\psi \exp \left\{ i \int d^4x L' \right\}, \quad (2.1) \]

where a local gauge symmetry is not yet fixed. In (2.1), shifting a modulus field as \( \rho + \rho_c \), we have

\[ L' = \bar{\psi} \cdot (i \partial - \rho_c + A\gamma_5)\psi - (N/4e^2) F^2 - N \rho \rho - N p^2/2 \]

\[ -\rho_c \bar{\psi} \cdot (e^{i\chi_5} - 1) \psi - \rho \bar{\psi} \cdot e^{i\chi_5} \psi + \text{const.} \quad (2.2) \]

To the leading order of \( 1/N \) expansion, a gap equation (a cancellation condition of \( \rho \) tadpoles \( < \rho > 0 \), corresponding to the diagrams in fig.3) reads

\[ 0 = \frac{\rho_c}{g^2} - i \int \frac{d^4k}{(2\pi)^2} \text{Tr} \left( \frac{1}{k - \rho_c} \right) \quad (2.3) \]

With an ultraviolet cut-off scale \( \Lambda \), the lowest-lying-energy solution is given by

\[ \rho_c = \frac{\pi}{g^2} \Lambda = \frac{\pi}{g^2} s_A^2(\mu) \mu, \quad (2.4) \]

where a renormalized coupling \( g^2_R \) is defined by

\[ g^{-2} = g_R^{-2}(\mu) + (2\pi)^{-1} \ln(A^2/\mu^2) \quad (2.5) \]

with \( \mu \) a renormalization mass scale.

For all positive region of \( g^2 \) the gap \( \rho_c \) is positive and gives fermions a RG-invariant mass satisfying a homogeneous RG equation

\[ \left[ \mu (\partial/\partial \mu) + \beta(g^2_R)(\partial/\partial g^2_R) \right] \rho_c = 0, \quad (2.6) \]

with a \( \beta \) function

\[ \beta(g^2_R) = \mu (\partial/\partial \mu)g^2_R(\mu) = -(1/\pi)(g^2_R)^2. \quad (2.7) \]

Note that as is in the non-gauged model [10], even if fermions acquire a mass through a non-zero \( \rho_c \) the global chiral symmetry of full \( L' \) is kept unbroken. In other words, the present KT scheme of the four-Fermi dynamics drives a dimensional transmutation but not a spontaneous breaking of the global chiral symmetry.

Integrating over fermions we obtain \( 1/N \). The long-distance part is contained in \( S_2 \) including their mixing (fig.4). They read

\[ S_2 = N \int \frac{d^4p}{(2\pi)^2} \]

with

\[ \mathcal{L}_{\text{inv}}(p) = (2e^2)^{-1} A^\mu(-p) \]

\[ + (2\pi)^{-1} \rho_c U \]

\[ \mathcal{L}_{\text{anom}}(p) = (2\pi)^{-1} b A^\mu(-p) \]

where \( A^\mu(p) \) and \( \chi(p) \) represent Fourier components of \( A^\mu \) and \( \chi \).

Here \( U \) and \( V \) are functions of \( p^2 \) and \( \rho_c^2 \)

\[ U(p^2, \rho_c^2) = \frac{2}{\sqrt{p^2(\rho_c^2 - 4)}} \]

\[ V(p^2, \rho_c^2) = (\rho_c^2 U - 1)/\rho_c^2 \]

They have the following well-defined derivatives

\[ \rho_c^2 U(-\partial^2, \rho_c^2) = 1 \]

\[ V(-\partial^2, \rho_c^2) = 0 \]

accompanied by inverse powers of \( (\rho_c^2 - 4) \) which consists of two parts: one is a locally anomalous term, another is a non-invariant contact mass term. They arise in the evaluation of a local contribution to the chiral anomaly [13, 14, 15]. In (2.10) a parameter in the literatures of anomalous gauge theories, depending on a regularization procedure.

Since an ultraviolet regularization of the polarization diagram (fig.4(a)), it is such the lagrangian \( L' \) the following Pauli-Villars regulator

\[ \mathcal{L}_{\text{PV}} = \bar{\psi}_1^{\text{PV}} \cdot (i \partial + b_1 A + \Lambda) \psi_1^{\text{PV}} \]
with \( b_1^2 + b_2^2 = 1 \), the condition required for the cancellation of apparent logarithmic divergences. Of course, other regularization prescriptions such as a point-splitting method would also be available. In the present Pauli-Villars method, starting from \( \mathcal{L}' + \mathcal{L}_{PV} \), we in fact obtain (2.8) with \( b \) in (2.14) given explicitly by

\[
b = -1 + b_1^2 - b_2^2 = 2(b_1^2 - 1), \tag{2.14}
\]

which is an arbitrary constant.

\( \mathcal{L}_{inv} \) in (2.9) is (axially) gauge invariant and one should get only this part provided he imposes the gauge invariance on the gauge-field vacuum polarization by employing, for example, the above Pauli-Villars method with the special choice \( b_1 = 1 \) \( (b_2 = 0) \). Alternatively, for the practical purpose this gauge-invariant result is most easily achieved by use of the dimensional regularization, defined only in momentum integrals with all algebras of \( \gamma^\mu \) matrices unchanged from those of two-dimensional ones (a dimensional reduction) \([20]\), or that with an assumption that \( \gamma_5 \) anticommutes with all \( \gamma^\mu \) matrices in continuous \( d \) dimensions \([21]\). In this gauge-invariant system (2.9), if \( \chi \) is single valued, we can absorb it into a gauge-invariant vector field

\[
B^\mu(p) = A^\mu(p) + (ip^\mu/2)\chi(p), \tag{2.15}
\]

and a \( \chi \) integration is decoupled from the theory. Then, eq.(2.9) is reduced to a free part of the neutral massive vector theory with the propagator \( D_{\mu\nu} \)

\[
iD_{\mu\nu}(p) = \frac{\epsilon^2 (g_{\mu\nu} - (p_\mu p_\nu/p^2))(p^2 - \alpha)^{-1} - (\pi/\rho^2 U)(p_\mu p_\nu/p^2)}{(p_\mu p_\nu/p^2)}, \tag{2.16}
\]

which possesses a pole at \( p^2 = \alpha \). This mass \( \alpha^{1/2} \) is independent of the four-Fermi coupling \( g^2 \) and is identical with that of the Schwinger model \([13]\). Although our effective action (2.8) is gauge invariant the feature of the spectrum is the same as that predicted in the broken-symmetry \( (\sigma_c \neq 0) \) argument \([3]\). This may thus be called a gauge-invariant version of the dynamical Higgs mechanism in a \( (1+1) \)-dimensional chiral four-Fermi theory.

As is mentioned above, although \( b \) depends on distance, we are in principle free to choose it. Since the massive abelian gauge theory itself, any choice of \( b(\geq 0) \) would give a mass \( \rho_b \). In arbitrary, we therefore consider our effective action of theories, and will henceforth study this idea of the hidden-parameter generation of anomalous gauge theories \([16, 17]\).

To the leading order the two-point function of \( \rho_b \) via the use of the gap equation (2.3). The \( \rho \) field has a mass \( 2\rho_b \). In higher orders the integrals give finite corrections of \( O(N^0) \) to \( \mathcal{L}_{inv} \).

Even for the gauge-invariant system of the above Higgs picture if \( \chi \) is not single valued \( \mathcal{F} \) of \( B \) will induce singularities and the Higgs mechanism based on the replacement (2.15) appears to be too naive. This possibility can generally happen since \( ((1.2), (1.3)) \) and need not be single valued physically realized as topological excitations in the present abelian theory in \( 1+1 \) dimensions, it is useful to extract the long-distance gauge-invariant part \( \mathcal{L}_{inv} \). This effective gauge-invariant local operator \( \mathcal{F} \) and are with the following lagrangian:

\[
\mathcal{L}_{Hig} = -(\beta/\kappa) \mathcal{F}^2 \tag{2.17}
\]

where the gauge field has been rescaled

\[
\beta \equiv \left( N/4e_{ext}^2 \right) = \left( N/\rho_b \right),
\]

\[
\kappa \equiv \left( N/4\pi \right) \left( 1 + O(1/N) \right)
\]

\( \mathcal{L}_{Hig} \) is nothing but a \( U(1) \) Higgs model. The continuum version of the XY model can
same local gauge symmetry as the non-local action $L_{\text{inv}}$ and contains its lowest-derivative part for the gauge and Higgs dynamics. The higher-derivative terms which have been neglected include $(\partial F)^2$, $\partial^2 |D\Psi|^2$, etc. Rescaling $A^\mu \rightarrow e A^\mu$ we see both operators $F^2$ and $|D\Psi|^2$ possess canonical dimensions 2 and higher-derivative ones 4 and higher. Hence at long distances the latter is irrelevant in a RG point of view unless large anomalous dimensions are generated dynamically, the possibility of which cannot be excluded but shall not be considered here. Although we have so far not considered four and higher-point functions of $(A, \chi)$, it is straightforward to compute them. From the gauge invariance the results should anyhow consist of only higher powers of $F$ and $D\Psi$ like $(F^2)^2$ or $(|D\Psi|^2)^2$ than $L_{\text{Hig.}}$ and also of their higher derivatives. They are all irrelevant as seen from the naive dimensional counting. From the universality argument, $L_{\text{inv.}}$ should thus possess the same long-distance behavior as of $L_{\text{inv.}}$.

Our derivation of the effective lagrangian is based on $1/N$ expansion and hence, from its validity, parameters $\beta$ and $\kappa$ defined by (2.18), should formally satisfy

$$\beta e^2 \gg 1, \quad \kappa \gg 1.$$  

(2.19)

From general interests in $L_{\text{Hig.}}$ itself, however, we shall henceforth consider general regions of $\beta e^2 > 0$ and $\kappa > 0$. We expect that the qualitative properties at long distances are same between the full effective theory $L_{\text{inv.}}$ and its lowest-derivative part $L_{\text{Hig.}}$, for a general $N$ satisfying $1 \ll N \leq \infty$.

To summarize this section we have constructed the long-distance effective lagrangian for the axially U(1)-gauged four-Fermi theory in $1+1$ dimensions, which reads

$$L_{\text{eff.}} = L_{\text{Hig.}} + L_{\text{anom.}}, \quad L_{\text{anom.}} = (b N/8\pi)A^2, \quad b \geq 0.$$  

(2.20)

3 Long-distance properties of the frozen U(1) Higgs model and a massive s-G system

In this section we consider the effective theory in the gauge-invariant scheme ($b = 0$), i.e. the frozen U(1) Higgs model defined with the lagrangian $L_{\text{Hig.}}$ in (2.17). As does so in $L_{\text{inv.}}$, without singular configurations for a phase order-parameter $\chi$ the Higgs mechanism operates in this model; the theory is reduced to that of a neutral massive vector boson with a natural U(1) charge is predicted to be screened. However, this naive picture could be changed once we allow singular configurations (vortices in a euclidean space-time) which cannot be absorbed into the vector field $A_e$.

The many-body dynamics of vortices in the model (2.17) has been argued in several different approaches were mainly taken to attack the problem. In one approach they relied upon a so called dilute-instanton-gas approximation based on the random varying of phase $\chi$: the gauge energy density of the system is proportional to $-\cos \theta$ and where the global gauge symmetry is not broken, hence, the random varying of phase $\chi$: the gauge-ansatz in addition to the massive pole at $p^2 = 0$, only if $Q$ is an integral multiple of $e$, the long-distance behavior in this semi-classical approximation is a dual transformation [29, 28]. In this approach also, although several possible phases were proposed in a wide range of parameters, the determination of the true long-distance behavior requires again a proper RG analysis which is not straightforward.

For these consequences to be justifiable it is important but unsolved problem to verify dynamically relevant and their distribution is sufficiently dilute. For this problem we must probe the long-distance picture in the semi-classical approximation of the Higgs mechanism.

Another typical approach [26, 27] is to consider the theory on a lattice and transform it to a system of a spin-wave (scalar field) plus vortices with the help of a dual transformation [29, 28]. In this approach, in addition to the massive pole at $p^2 = 0$, the main results are the energy density of the vortices in the (1+1)-dimensional U(1) Higgs model (2.17) with the method being complementary to early investigations.

In the rest of this section we will study the long-distance behavior of the frozen U(1) Higgs model (2.17) with the method based on the dual transformation. We keep the system (2.17) in the continuum.
our long-distance effective lagrangian is valid, and take into account explicitly an ensemble of single-vortex configurations in the partition function. Then we shall proceed along the similar line of the lattice approach \cite{21,27}. Namely, we first transform the model to a scalar-vortex system by the continuum version of a dual transformation \cite{30}, and next generalize it by adding a chemical potential of vortices, controlled by a fugacity parameter $y$. The small-$y$ regime ($0 < y \ll 1$) of this generalized system can be described by a local field theory, a massive scalar-G system (subsect.3.1). In this system we will be able to study the long-distance relevance of vortices in a systematic manner, i.e. by incorporating interaction effect among vortices through a momentum-shell recursion relation method (subsect.3.2).

3.1 Dual transformation in the cut-off continuum and a s-G system

We start from the following euclidean partition function of the frozen U(1) Higgs model \cite{2.17} with an ensemble of single-vortex configurations:

$$Z = \prod_x \int D\phi(x) \prod_{x_0} \sum_{n(x_0)} \prod_x \int D\chi(x) \delta [\chi(x) - n(x_0) \theta(x - x_0)] \exp \left\{ - \int d^2x \left[ \frac{\beta}{4} F^2 + \frac{\kappa}{2} (A - \partial \chi)^2 \right] \right\}, \quad (3.1)$$

where $x_0$ are the positions of vortex centers and $\theta(x - x_0)$ denotes the azimuthal angle of $x$ around $x_0$. In \cite{3.1} the case $n = 0$ corresponds to a usual unitary gauge ($\chi = 0$). Here we sum up over all vortex charges $n(x_0) \in Z$ defined at all possible centers $x_0$. The vortex centers are assumed to be separated from each other with the minimal length $a = \Lambda^{-1}$ where $\Lambda$ is the cut-off scale we are considering; in our effective theory of the four-Fermi model it is naturally given by the scale of order $\rho_c$. Performing the continuum version \cite{30} of a dual transformation \cite{23,28}

$$\exp \left\{ - \frac{\beta}{4} \int d^2x F^2 \right\} = \prod_x \int D\phi \exp \left\{ - \int d^2x \left[ \frac{1}{2\beta} \phi^2 - i\phi (\epsilon \cdot \partial A) \right] \right\}, \quad (3.2)$$

with $\epsilon \cdot \partial A \equiv \epsilon_{\mu\nu} \partial_\mu A_\nu$ ($\epsilon_{12} = -\epsilon_{21} = 1, \epsilon_{11} = \epsilon_{22} = 0$), and integrating over the gauge field $A$, we obtain

$$Z = \prod_x \int D\phi(x) \prod_{x_0} \sum_{n(x_0)} \prod_x \int D\chi(x) \delta [\chi(x) - n(x_0) \theta(x - x_0)] \exp \left\{ - \int d^2x \left[ \frac{1}{2\kappa} (\partial^2 - 2\kappa n) \phi^2 \right] \right\},$$

where

$$p(x) = (2\pi)^2 \delta^2 (x - x_0)$$

is a vortex source. The value of $p(x)$ is singular $\chi(x) = n(x_0) \theta(x - x_0)$ imposes $n(x_0) \delta^2 (x - x_0)$. Then, integrating over $\chi$,

$$Z = \prod_x \int D\phi(x) \prod_{x_0} \sum_{n(x_0) \in Z} \exp \left\{ - \frac{1}{2\kappa} \int d^2x \left[ (\partial \phi)^2 - 2\kappa \phi^2 \right] \right\},$$

which describes the system of many vortices with the presence of gauge interaction ($\beta < 2\kappa$) short-ranged

$$D(x - y; M) = \frac{1}{(2\pi)^2} \frac{1}{2\kappa} \left[ \delta^2 (x - y; M) \right],$$

in contrast to a long-range logarithmic interaction of the model. On the coincidence point $x = y$ the partition function $D(0; M) = (4\pi)^2$ is

$$D(0; M) = (4\pi)^2 \frac{1}{2\kappa} \left[ \delta^2 (x - y; M) \right],$$

with $Z \equiv M^2 \Lambda^{-2}$, which shows that a $M^2 > 0 (\beta < \infty)$. Thus, if we neglect the presence of gauge interaction ($\beta < 2\kappa$) short-ranged

$$Z_{\text{var}} = \prod_{x_0} \sum_{n(x_0)} \exp \left\{ - \frac{1}{2\kappa} \int d^2x \left[ (\partial \phi)^2 - 2\kappa \phi^2 \right] \right\},$$

from which the probability $P(x_0)$ to find

$$P = \frac{1}{Z_{\text{var}}} \exp \left\{ - \frac{1}{2\kappa} \int d^2x \left[ (\partial \phi)^2 - 2\kappa \phi^2 \right] \right\}.$$
with $X \equiv \pi \kappa - 2$, and does the chemical potential as

$$\frac{\pi \kappa}{2} \ln \left(1 + \frac{Z}{Z} \right). \tag{3.11}$$

Eq. (3.7) shows that the interaction length $r$ among vortices is roughly $r \sim M^{-1} = Z^{-1/2}a$ and so the area in which a vortex can affect is estimated by $(r/a)^2 \sim Z^{-1}$ in the unit of $a$. Hence, from (3.10) the number of other vortices lying closely enough to interact with a given vortex reads approximately as

$$n^* \equiv Z^{\frac{\kappa}{2}}(1 + Z)^{-(1 + \frac{\kappa}{2})}. \tag{3.12}$$

We expect this expression to be qualitatively valid also for a smooth momentum cut-off. Although this expression of $n^*$ is derived from a classical picture we can also define an effective one

$$n^*(\ell) = Z(\ell)^{\frac{\kappa}{2}}(1 + Z(\ell))^{-(1 + \frac{\kappa}{2})}, \tag{3.13}$$

for which quantum interaction effects are to be taken into account through a RG analysis. Here the dependences of $X(\ell)$ and $Z(\ell)$ on a scaling parameter $\ell$ are to be determined by RG. The quantity $n^*(\ell)$ thus changes dynamically as a function of $\ell$ and measures how vortices are effectively dilute (or dense) at long distances ($\ell \rightarrow \infty$). $P(\ell)$ is defined similarly as well.

Now, we generalize the system (3.5) by introducing an additional chemical potential for a single vortex, the prescription having been developed for the study of the XY model $^{29}$. This leads us to

$$Z(y) = \prod \int D\phi(x) \sum_{n(x_0) \in Z} \exp \left\{-\frac{1}{2\kappa} \int d^2x \left[(\partial \phi)^2 + M^2 \phi^2\right]ight. + (\ln y) \sum_{x_0} n(x_0)^2 + 2\pi i \sum_{x_0} n(x_0) \phi(x_0) \right\}, \tag{3.14}$$

where a chemical potential term $(\ln y) n^2(x_0)$ has been incorporated by hand to control fluctuations in $n(x_0)$. Although the original model corresponds to $y = 1$, we will henceforth consider this generalized system $Z(y)$, regarding $y$ as a free parameter (a fugacity) which controls the activation of single vortices. In the generalized system the probability $P$ of a single-vortex excitation and the quantity $n^*$ defined respectively in (3.11) and (3.12) are multiplied by $y$. Our interests then lie in how this parameter $y$ behaves especially in whether there exists the $y \rightarrow \infty$ long-distance limit ($\ell \rightarrow \infty$). If such a long-distance effect of vortices and the mean-field picture ($\ell \rightarrow 0$) of vortex activation $^{29}$.

In the region $y \ll 1$, the sum over $^n \exp \left\{(\ln y) \sum_{n(x_0)}\right\}

= \ln \left\{1 + 2 \sum_{n(x_0) = 1}^{\infty} y_m \cos [2 \pi n(x_0)] \right\} \tag{3.15}

with

$$y_1 = 2y \left[1 + \frac{\kappa}{2} + \frac{\kappa}{2} \left(1 + \frac{\kappa}{2}\right)\right] + \ln y + \frac{\kappa}{2} \ln (\ln y),$$

$$y_2 = -y^2 \left[1 + \frac{\kappa}{2} + \left(1 + \frac{\kappa}{2}\right)\right] + \frac{\kappa}{2} \ln (\ln y),$$

$$y_3 = \left(\frac{\kappa}{2}\right)^2 \ln (\ln y),$$

$$y_4 = y^4 + \ldots$$

and $y_m (m \geq 5) \sim O(y^p)$ in (3.16) only contributes to the most dominant terms in $y_4$ and higher-order harmonics, a depressing the leading contributions. Rescaling $\phi$ in the continuum expression we get

$$Z_{msG} = \prod \int D\phi(x) \exp \left\{-\frac{A^2}{2} y_m \sum_{m \in N} y_m \right\}, \tag{3.16}$$

with $y_m$ given by (3.16). As in the last section, massive s-G system starting from the
Before proceeding to the actual RG analyses let us give some remarks.

In ref. \[27\] Jones et al. have commented on the RG behavior of the massive s-G theory derived from the lattice frozen U(1) Higgs model. They argued in favor of the KT phase transition and predicted an existence of the phase \((\kappa < \kappa_c)\) where the Higgs mechanism operates. However, their argument is based on a hypothesis that in each of the steps of RG the super-renormalizable mass term is irrelevant and can be ignored. This hypothesis may be plausible for the short-distance behavior but not so for the long-distance behavior which determines the phase structure of the model. Classically a massive term is infrared relevant. However, as is seen in the massless s-G theory (pure XY model) \[22\], a cosine operator which has the classical mass dimension zero, could acquire large \((=2)\) anomalous dimensions from quantum corrections so that there can appear the (low-temperature) KT phase where vortices are irrelevant at long distances \((y_m(\ell \to \infty) \to 0)\). Therefore, it is a non-trivial dynamical problem how the massive s-G system \((3.17)\) behaves at long distances. To draw definite conclusions we must carefully investigate the mixed renormalization-effects among parameters \(\kappa, M^2\) and \(y_m\).

For this purpose it is important that RG equations should be constructed in terms of a recursion-relation method, as has been done for the massless s-G theory in a real space-time \[32\] and in a momentum space \[33, 34\]. Since from the beginning we are working in the continuum with an ultraviolet cut-off, it is natural to adopt the latter, i.e. a momentum-shell method. It should be stressed that other conventional field-theoretical schemes such as those based on ultraviolet divergences \[33\] are not suited for our purpose; in such schemes the long-distance effect of super-renormalizable terms such as a mass term could erroneously be missed. This fact is actually exemplified in the standard O(\(N\)) vector non-linear \(\sigma\) model with a finite external magnetic field \(h\), defined with the lagrangian

\[
\mathcal{L} = \frac{1}{2t} \left[ (\partial \vec{\varpi})^2 + (\vec{\varpi} \cdot \partial \vec{\varpi})^2 \right] + \frac{h}{t} (1 - \varpi^2)^{1/2}, \tag{3.18}
\]

where \(t\) is a coupling constant (temperature). In the conventional scheme of dimensional renormalizations \[22\] the last term is regarded only as an infrared cut-off and does not affect the ultraviolet divergences, so that the result there appears no dependence on \(h\) for the non-zero \(h\) in the critical region near the ultraviolet cut-off. In contrast, in the momentum-shell recursion-relation method \[34\] behavior revealing a crossover phenomenon occurs between these schemes also occurring in the \(\sigma\) model with no external fields \[37\].

We think that these failures are commonly happen in those utilization of eqations. Thus it seems essentially important for our purpose to study the long-distance behavior of the system \((3.17)\) with the recursion-relation method, along the original idea of Wilson \[31\]. This approach of the momentum-shell recursion has been applied to a massless s-G theory \[33, 34\] and it is not difficult to employ it for the massless s-G theory \((3.17)\).

### 3.2 Construction of recursion relations

As has been argued above we shall construct recursion equations for the s-G system \((3.17)\). The applications of the method to the massless s-G theory \[33, 34\] have already appeared in refs. \[33, 34\], and we mainly follow the notation of ref. \[34\]. The RG of the system \((3.17)\) can be treated by a perturbation theory: whereas higher-order harmonics possess the coefficients of higher powers of \(y\), other conventional schemes such as those based on ultraviolet divergences \[33\] are not suited for our purpose; in such schemes the long-distance effect of super-renormalizable terms such as a mass term could erroneously be missed. This fact is actually exemplified in the standard O(\(N\)) vector non-linear \(\sigma\) model with a finite external magnetic field \(h\), defined with the lagrangian

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We think that these failures are commonly happen in those utilization of eqations. Thus it seems essentially important for our purpose to study the long-distance behavior of the system \((3.17)\) with the recursion-relation method, along the original idea of Wilson \[31\]. This approach of the momentum-shell recursion has been applied to a massless s-G theory \[33, 34\] and it is not difficult to employ it for the massless s-G theory \((3.17)\).
perturbation theory of the original U(1) Higgs model \([\ref{2.17}]\). Our main interest then lies in whether there could exist the parameter region in which such a Higgs picture is valid. This question can legitimately be examined by a small-fugacity \((|y_m| \ll 1)\) perturbation theory.

The action of the massive s-G system consists of a free part \(S_0\) of the massive scalar field and of its harmonic term \(S_I\) originated from the vortex configurations in the U(1) Higgs model, i.e. \(S_{\text{ms-G}} = S_0 + S_I\) where

\[
S_0 = \frac{1}{2} \int d^2x \left[ (\partial \phi)^2 + M^2 \phi^2 \right], \quad (3.19)
\]

\[
S_I = -\Lambda^2 \sum_{m=1} y_m \int d^2x \cos m \varphi_x \quad (3.20)
\]

with \(\varphi_x \equiv 2\pi \sqrt{\kappa} \phi_x \equiv 2\pi \sqrt{\kappa} \phi(x)\). In the momentum-shell renormalization, the field \(\phi\) is decomposed into a low \(\phi'\) \((|p| < \Lambda' = \Lambda - d\Lambda)\) and a high \(h\) \((\Lambda' < |p| < \Lambda)\) momentum parts

\[
\phi = \phi' + h, \quad (3.21)
\]

in which is assumed a smooth momentum-slicing procedure \([31], [34]\). Although this procedure is not explicitly specified here, it is in fact argued in ref. \([31]\) that such a slicing procedure exists in principle. As is shown in refs. \([31], [34]\) the present treatment applied to the massless s-G system in fact gives the same recursion equations as those obtained by other methods including those based on local singularities, at least to leading order \([32], [35]\). For the present massive s-G system defined at large distances we rather believe that the momentum-shell recursion relation method shall be more appropriate than others, by the reason we have remarked in subsect.3.1.

Owing to the above field decomposition the partition function of the system \([\ref{3.17}]\) reads

\[
Z_{\text{ms-G}} = \int D\phi' \exp \{ -S_0(\phi') \} \ Z' = \text{const.} \int D\phi' \exp \{ -F(\phi') \},
\]

\[
Z' = \int Dh \exp \{ -S_h - S_I(\phi' + h) \}. \quad (3.22)
\]

Here \(S_h \equiv S_0(h)\) is a free massive scalar-field action for the high-momentum part \(h\) and \(F(\phi')\) represents a free energy of the low-momentum part \(\phi'\). Our task is to calculate \(F(\phi')\) by a cumulant expansion with respect to \(S_h\). Explicitly,

\[
F(\phi') = S_0(\phi') - \ln(Z'Z_h^{-1})
\]
\[
\frac{dY_3}{d\ell} = 2Y_3 - \delta \Delta \left( 9Y_3 - 2\delta \Delta Y_1Y_2 + \frac{1}{12}\delta^2 \Delta^2 Y_1^3 \right), \quad (3.30)
\]
\[
\frac{dZ}{d\ell} = \left( 2 - \frac{1}{8}\delta^2 \Delta Y_1^2 \right)Z, \quad (3.31)
\]
where we have rescaled \(4\pi e^{\nu/2}y_m = Y_m\).

### 3.3 Solutions of recursion equations

Having constructed the recursion RG equations we now solve them numerically and investigate the behavior of RG flows. For simplicity we set \(c = 1\) in the numerical calculations. (The choice of \(c\) does not alter the qualitative properties of RG flows.) In the followings we fix the initial conditions for \(Y_m\) by \(Y_1(0) = 0.1\), which means from (3.16) to take \(y \simeq 4.0 \times 10^{-3}\). Then, respecting other relations in (3.16) provides \(y_2 \sim -y^2 \sim -1.6 \times 10^{-5}\) and \(y_3 \sim (2/3)y^3 \sim 4.2 \times 10^{-8}\), and the initial conditions for \(Y_2\) and \(Y_3\) are fixed as

\[
Y_2(0) (= 4\pi y_2) = -2 \times 10^{-4}, \quad Y_3(0) (= 4\pi y_3) = 5 \times 10^{-7}. \quad (3.32)
\]

Let us first see the solutions for the massless \((Z = 0)\) s-G system to compare with those for the massive \((Z > 0)\) system. Physically this system is related to the pure (non-gauged) XY model or a two-dimensional Coulomb-gas system, and the lowest-order solution of RG with only a single harmonics \((Y_1)\) is already known [32, 34]. Here we will show the RG flows including the effects of both higher-order corrections and higher harmonics. The RG flows projected on the \((X,Y_1)\) plane are depicted in fig.5. As is seen in this figure, their qualitative behaviors are the same as those obtained by the lowest-order \((O(y_1^2))\) analyses [32, 34]. The flow diagram exhibits the well known two-phase structure distinguished by the KT phase boundary. As is read from (3.27)~(3.30) the flow equations are approximated by

\[
\frac{dX}{d\ell} \simeq -Y_1^2, \quad \frac{dY_1}{d\ell} \simeq -XY_1, \quad \frac{dY_2}{d\ell} \simeq \frac{dY_3}{d\ell} \simeq 0, \quad (3.33)
\]
in the neighborhood of \(X \simeq Y_m \simeq 0\) where the phase boundary in the \((X,Y_1)\) plane is described by \(X \simeq Y_1\). Figs.6(a) and (b) show the behaviors of \(Y_1(\ell), Y_2(\ell)\) and \(Y_3(\ell)\) as functions of a RG step \(\ell\), starting from the representative points in the small \((X(0) = 0.075)\) and the large \((X(0) = 0.125)\) \(X\) regimes. It is observed that the higher harmonics harmonic \((m = 1)\), except in the first (fig.6(a)), \(Y_2(\ell)\) immediately \((\ell \ll 1)\) decreases at \(\ell \sim 0.62\). Also \(Y_3(\ell)\) initially at \(\ell \sim 0.74\). All \(Y_m(=1\cdots3)(\ell)\) continues \(\ell = 13.25 \sim 13.75\) where they turn to increasing (fig.6(b)) all flows finally converge to the KT although \(Y_2(\ell)\) and \(Y_3(\ell)\) behave again.

Since a single-charged \((n(x_0) = \pm 1)\) contribution to the second and the third harmonics behave similarly at long distances provided with those for the massive \((Z > 0)\) system. Physically this system is related to the pure (non-gauged) XY model or a two-dimensional Coulomb-gas system, and the lowest-order solution of RG with only a single harmonics \((Y_1)\) is already known [32, 34]. The non-gauged chiral four-Fermi model exists in this regime persists to hold in our extended system. The small-\(X\) (high-temperature) regime single vortices are activated with a small chemical potential \((\Delta y \ll 1)\). The power-law decay \((\Delta y \ll 1)\) regime constitutes the KT phase which is characterized by a power-law decay of the correlation function. The disordered (vortex) phase is realized in the small-\(X\) (low-temperature) regime the disordered (vortex) phase is realized.

Fig.7 exhibits the behaviors of \(Y_1(\ell)\), \(Y_2(\ell)\), and \(Y_3(\ell)\) as functions of a RG step \(\ell\), starting from the representative points in the small \((X(0) = 0.075)\) and the large \((X(0) = 0.125)\) \(X\) regimes. It is observed that the higher harmonics harmonic \((m = 1)\), except in the first (fig.6(a)), \(Y_2(\ell)\) immediately \((\ell \ll 1)\) decreases at \(\ell \sim 0.62\). Also \(Y_3(\ell)\) initially at \(\ell \sim 0.74\). All \(Y_m(=1\cdots3)(\ell)\) continues \(\ell = 13.25 \sim 13.75\) where they turn to increasing (fig.6(b)) all flows finally converge to the KT although \(Y_2(\ell)\) and \(Y_3(\ell)\) behave again.

Since a single-charged \((n(x_0) = \pm 1)\) contribution to the second and the third harmonics behave similarly at long distances provided with those for the massive \((Z > 0)\) system. Physically this system is related to the pure (non-gauged) XY model or a two-dimensional Coulomb-gas system, and the lowest-order solution of RG with only a single harmonics \((Y_1)\) is already known [32, 34]. The non-gauged chiral four-Fermi model exists in this regime persists to hold in our extended system. The small-\(X\) (high-temperature) regime single vortices are activated with a small chemical potential \((\Delta y \ll 1)\). The power-law decay \((\Delta y \ll 1)\) regime constitutes the KT phase which is characterized by a power-law decay of the correlation function. The disordered (vortex) phase is realized in the small-\(X\) (low-temperature) regime the disordered (vortex) phase is realized.
Y₁(ℓ) stops at a certain ℓ(= ℓ₁) the value of which depends on an initial condition and that Y₁(ℓ) turns to an exponential increasing at large ℓ(> ℓ₁). Fig.8(a) shows the behaviors of Y₁(ℓ) and Z(ℓ) as functions of a step ℓ, where Z(0) is chosen to be 0.01. Z(ℓ) remains small within small steps (ℓ < ℓ₁ ≈ 2), where the second term (anomalous dimensions of y₁Λ²) in the square bracket on the right-hand side (r.h.s.) of the recursion equation (3.28) is larger than or comparable with the first term (= 2, canonical dimensions of y₁Λ²) and hence Y₁(ℓ) decreases obeying the quantum scaling as that for Z = 0. However, after some steps (ℓ ∼ ℓ₁) where Z(ℓ) reaches to be O(1), Z(ℓ) grows large and reduces the anomalous dimensions of y₁Λ² significantly, Δ(ℓ ≫ ℓ₁) ≈ 0. As a result Y₁(ℓ) turns to an exponential increasing according to a classical scaling law (see also fig.8(b) plotted by a logarithmic scale). Fig.8(b) also indicates that this crossover and the scaling behaviors in the massive system are common in three harmonics Yₘ(ℓ)(m=1~3), although in this figure is comprised large-fugacity (Yₘ(ℓ) ≫ 1) behaviors extrapolated from the perturbative results. We have checked numerically that these qualitative properties, more or less, hold for all initial conditions in a large X regime. There are no flows converging in the long-distance limit to a Yₘ = 0(m=1~3) fixed line. In any case including a small X regime flows eventually (ℓ → ∞) go to the region where all Yₘ are large.

For the massive s-G system, we thus conclude that there is no phase transition nor a Higgs phase defined by Yₘ(ℓ → ∞) → 0. Instead we have observed crossover phenomena from the classical to the quantum scaling regimes in some intermediate scales (ℓ₁) which themselves depend on initial conditions. These crossover phenomena are driven by an increasing of the effective mass parameter Z(ℓ)(≈ gauge coupling). Fig.8(a) also tells us that the effective probability of a vortex in the generalized system defined by yP(ℓ) ≈ (1/2)y₁(ℓ)P(ℓ) with P in (3.10) increases monotonically and that the quantity yₙ*(ℓ) ≈ (1/2)y₁(ℓ)n*(ℓ) which roughly measures the effective vortex density in the generalized system remains almost constant (increases slightly), starting from the very dilute region 3.92 × 10⁻⁴. Therefore it is not obvious whether the diluteness of the vortex distribution can actually be realized or not in the long-distance limit (ℓ → ∞) even if it is chosen so initially (ℓ = 0) at some finite scales.

4 Long-distance properties of the frozen U(1) Higgs model

In this section we turn our attention to (anomalous) schemes (b ≠ 0). The effective gauge-field mass term proportional to mass term proportional to

4.1 Double dual transformation

As in the previous case, performing the average over the gauge field we obtain the

Lₕig. + Lₐnom.:

\[ Z = \prod_x \int Dφ_1(x) \prod_x \int Dχ(x) \exp \left\{ -\int d^2x \left[ \frac{1}{2κ_+} (\partial φ_1)^2 - \frac{κ}{2} \right] \right\} \]

with

\[ κ_+ \equiv κ + \frac{b N}{4π} \]

This system may be regarded as the field of \( \frac{1}{2} \) coupled to a massive scalar, \( (2π)^{-1} ε_{μν} £_μ £_ν χ(x) \).

The XY model lagrangian can be introduce a fictitious gauge field \( A' \) and performing the dual transformation [8]

\[ \exp \left\{ -\frac{κ_+}{2} \int d^2x (\partial χ)^2 \right\} \]

\[ = \prod_x \int DA'(x) \exp \left\{ -\int d^2x \int \frac{1}{2} \phi'^2 - \frac{1}{2} \phi'^2 - i\phi' \right\} \]

\[ = \prod_x \int DA'(x) \exp \left\{ -\int d^2x \int \frac{1}{2} φ'^2 - i\phi' \right\} \]

\[ = \prod_x \int DA'(x) \exp \left\{ -\int d^2x \int \frac{1}{2} φ'^2 - i\phi' \right\} \]
where $\beta' \equiv (e')^{-2}$ and we have neglected overall constant factors. In the second equality of (4.3), we have rescaled $e' A' \to A'$ and introduced a new scalar field $\phi_0(x)$ via the dual transformation. After the double dual transformation (4.1) and (4.3), the $\chi$ dependence in the effective action emerges only through the vortex source operator $p(x)$. Hence the $\chi$ configurations which can affect the partition function, are only vortex configurations $\chi(x) = n(x_0) \theta(x-x_0)$ with arbitrary integer charges $n(x_0)$ and centers $x_0$, as has been specified in (4.3) by the delta functional as the gauge-fixing procedure for the gauge-invariant system.

Substituting (4.3) into (4.1) and integrating over $\chi$ with the above consideration, we get the following total partition function:

$$Z = \prod_x \int D\phi_0(x) D\phi_1(x) \exp \left\{ - \int d^2x \left[ \frac{1}{2\kappa_0} (\partial \phi_0)^2 + \frac{1}{2\kappa_+} (\partial \phi_1)^2 + M_i^2 \phi_1^2 \right] + 2\pi i \sum_{x_0} n(x_0) \left[ \phi_0 + \left( \frac{\kappa}{\kappa_+} \right) \phi_1 \right] (x_0) \right\}$$

(4.4)

with $M_i^2 \equiv \kappa_+ / \beta$. We thus have seen that our effective theory $L_{\text{Hig}} + L_{\text{anom}}$ is equivalent to a system of vortices interacting with both massless ($\phi_0$) and massive ($\phi_1$) scalar fields.

As in the previous case, in order to systematically study whether vortex excitations are actually relevant or not at long distances, we add a chemical potential term $(\ln y) \sum_{x_0} n^2(x_0)$ to reduce the vortex activation $(y \ll 1)$. Then taking the sum over vortex charges and identifying the sum over vortex centers $\sum_{x_0}$ with the continuum integral $\int^\infty_0 dx$, we obtain the following coupled system of massless and massive s-G fields:

$$Z = \prod_x \int D\phi_0(x) D\phi_1(x) \exp \left\{ - \int d^2x \left[ \frac{1}{2} (\partial \phi_0)^2 + \frac{1}{2} (\partial \phi_1)^2 + M_i^2 \phi_1^2 \right] - \Lambda^2 \sum_{m \in N} y_m \cos \left( 2\pi \frac{1}{\kappa_0} \sum_{j=0}^1 \sqrt{\kappa_j} \phi_j \right) \right\},$$

(4.5)

where

$$\kappa_1 \equiv \kappa^2 / \kappa_+ = \kappa - \kappa_0,$$

(4.6)

and the fields have been rescaled as $\phi_0 \to \sqrt{\kappa_0} \phi_0$ and $\phi_1 \to \sqrt{\kappa_+} \phi_1$. Here $y_m = y_m(y)$ is of order $O(y^m)$. Eqs. (4.1), (4.3) and (4.5) indicate that for the regularization scheme $b < 0$ ($\kappa_0 < 0$) the theory is non-unitary and we consider only the case $\kappa_0 > 0$ which is realized by the $b > 0$ schemes.

### 4.2 Recursion equations for the coupled s-G system

Using again the recursion-relation method, we have rescaled $e' A' \to A'$ and introduced a new scalar field $\phi_0(x)$ via the dual transformation. After the double dual transformation (4.1) and (4.3), the $\chi$ dependence in the effective action emerges only through the vortex source operator $p(x)$. Hence the $\chi$ configurations which can affect the partition function, are only vortex configurations $\chi(x) = n(x_0) \theta(x-x_0)$ with arbitrary integer charges $n(x_0)$ and centers $x_0$, as has been specified in (4.3) by the delta functional as the gauge-fixing procedure for the gauge-invariant system.

Substituting (4.3) into (4.1) and integrating over $\chi$ with the above consideration, we get the following total partition function:

$$Z = \prod_x \int D\phi_0(x) D\phi_1(x) \exp \left\{ - \int d^2x \left[ \frac{1}{2\kappa_0} (\partial \phi_0)^2 + \frac{1}{2\kappa_+} (\partial \phi_1)^2 + M_i^2 \phi_1^2 \right] + 2\pi i \sum_{x_0} n(x_0) \left[ \phi_0 + \left( \frac{\kappa}{\kappa_+} \right) \phi_1 \right] (x_0) \right\}$$

(4.4)

with $M_i^2 \equiv \kappa_+ / \beta$. We thus have seen that our effective theory $L_{\text{Hig}} + L_{\text{anom}}$ is equivalent to a system of vortices interacting with both massless ($\phi_0$) and massive ($\phi_1$) scalar fields.

As in the previous case, in order to systematically study whether vortex excitations are actually relevant or not at long distances, we add a chemical potential term $(\ln y) \sum_{x_0} n^2(x_0)$ to reduce the vortex activation $(y \ll 1)$. Then taking the sum over vortex charges and identifying the sum over vortex centers $\sum_{x_0}$ with the continuum integral $\int^\infty_0 dx$, we obtain the following coupled system of massless and massive s-G fields:

$$Z = \prod_x \int D\phi_0(x) D\phi_1(x) \exp \left\{ - \int d^2x \left[ \frac{1}{2} (\partial \phi_0)^2 + \frac{1}{2} (\partial \phi_1)^2 + M_i^2 \phi_1^2 \right] - \Lambda^2 \sum_{m \in N} y_m \cos \left( 2\pi \frac{1}{\kappa_0} \sum_{j=0}^1 \sqrt{\kappa_j} \phi_j \right) \right\},$$

(4.5)

where

$$\kappa_1 \equiv \kappa^2 / \kappa_+ = \kappa - \kappa_0,$$

(4.6)

and the fields have been rescaled as $\phi_0 \to \sqrt{\kappa_0} \phi_0$ and $\phi_1 \to \sqrt{\kappa_+} \phi_1$. Here $y_m = y_m(y)$ is of order $O(y^m)$. Eqs. (4.1), (4.3) and (4.5) indicate that for the regularization scheme $b < 0$ ($\kappa_0 < 0$) the theory is non-unitary and we consider only the case $\kappa_0 > 0$ which is realized by the $b > 0$ schemes.
The effect of the first-order cumulant is merely to renormalize the fugacity parameters $y_m$ of the original harmonics,

$$< S_I > = -\Lambda^2 \sum_{m=1}^{2} y_m \left( 1 - m^2 \delta d\ell \right) \int d^2 x \cos m \varphi' + O((d\ell)^2)$$  \hspace{1cm} (4.9)

with $\delta \equiv \sum_{j=0}^{\infty} \Delta_j \delta_j$, $\Delta_j \equiv (1 + Z_j)^{-1}$, $Z_j \equiv \delta_{ij} M_{ij}^2 \Lambda^{-2}$, $\delta_j \equiv X_j + 2 \equiv \pi \kappa_j$ and $\varphi' \equiv 2\pi \sum_{j=0}^{\infty} \sqrt{\kappa_j} \phi_j \equiv \sum_{j=0}^{\infty} \varphi_j$. However, the second-order cumulant generates new operators as well as the wave-function renormalizations. Up to irrelevant higher-derivative operators, it reads

$$(-1/2) \left( < S_I^2 > - < S_I >^2 \right) \begin{align*} &\int d^2 x \left\{ c\pi^2 \delta y_i^2 \delta d\ell \sum_{j=0}^{1} \delta_j (\partial \phi_j^2)^2 - \pi \gamma_1 \delta^2 \Lambda^2 \delta d\ell \cos 2\varphi' \\
&- (c\pi^2/8) \gamma_1^2 \delta \sum_{j=0}^{1} \partial^2 \varphi_j \sin 2\varphi' \\
&+ 2c\pi^2 \delta \delta_0 \delta y_i^2 \delta d\ell \partial \phi_0 \partial \phi_i' + O((d\ell)^2, \partial^4) \right\}. \hspace{1cm} (4.10) \end{align*}$$

Therefore, in order to complete the $O(y^2, \partial^2)$ RG, we must start from the generalized action $S = S_0 + S_I$ with the bare interaction

$$S_I = \int d^2 x \left[ -\Lambda^2 \sum_{m=1}^{2} y_m \cos m \varphi - \frac{1}{2} \sum_{j=0}^{1} w_j \partial^2 \varphi_j \sin 2\varphi' + v \partial \phi_0 \partial \phi_1 \right], \hspace{1cm} (4.11)$$

where $v$ and $w_j$ are new coupling constants. They are initially $(\ell = 0)$ zeroes but are generated at $\ell > 0$ by the $O(y^2)$ perturbation, as is manifested in (4.10). The second term in (4.11) produces new contributions to the first-order cumulant

$$< S_I > = \int d^2 x \left\{ -y_1 \Lambda^2 \left( 1 - \delta d\ell \right) \cos \varphi' \\
-\Lambda^2 \left[ y_2 - \left( 4\gamma_2 \delta + \pi^{-1} W \cdot \delta \right) d\ell \right] \cos 2\varphi' \\
- \frac{1}{2} \left( 1 - 4\delta d\ell \right) \sum_{j=0}^{1} w_j \partial^2 \varphi_j \sin 2\varphi' + v \partial \phi_0 \partial \phi_1' + O((d\ell)^2) \right\} \hspace{1cm} (4.12)$$

with $W \cdot \delta \equiv \sum_{j=0}^{1} W_j \delta_j \Delta_j$ and $W_j \equiv 4\pi w_j$. In the $O(y^2)$ approximation, the second-order cumulant for the new action (4.11) is precisely the one in (4.10).
and for the special choice $b = 1$

$$\tag{4.20} (X \equiv) X_0 = X_1 = \frac{N}{8} - 2,$$

on which initial conditions for $X_0$ and $X_1$ are identical and the equality is not altered by the RG $\left(\text{4.14}\right)$, that is $X_0(\ell) = X_1(\ell)$ for any step $\ell$.

First, in the strong gauge-coupling limit $Z_1 \to \infty$ recursion equations in $\left(\text{4.14}\right)$ are equal to those for the pure massless $s$-G system in the second-order approximation $\left(\text{3.27}\right)$ and $\left(\text{3.28}\right)$ with $Y = 0$ and $Y_1^3 \to 0$). They describe the KT transition with the phase boundary $X_0 = Y_1$ (fig.5).

Next, if $Z_1 = 0$ the system is a two-component massless $s$-G system in which recursion equations in $\left(\text{4.14}\right)$ are reduced to

$$\frac{dX_j}{d\ell} = -\frac{1}{8} X_j^2 (X_0 + X_1 + 4) Y^2, \quad \frac{dY_1}{d\ell} = -(X_0 + X_1 + 2) Y_1. \tag{4.21}$$

The critical line exists at $Y_1 = 0$, $X_0 + X_1 + 2 = 0$ which corresponds to $N = 24$ in the large-$N$ approximation $\left(\text{4.19}\right)$ with an arbitrary $b > 0$. Fig.9 exhibits the RG flows ($b = 1$) projected on a $(X, Y_1)$ plane. The phase boundary around the critical point $(X = -1, Y_1 = 0)$ is written by

$$x^2 - \frac{y^2}{8} = 0 \tag{4.22}$$

with $x, y$ denoting small deviations from the critical point, i.e. $X = -1 + x$, $Y_1 = y$.

Roughly speaking, if the mass parameter $Z_1(\ell)$ of $\phi_1$ would scale up at long distances ($\ell \to \infty$) as for the pure massive $s$-G system (sect.3), the RG flows, starting from a very small but a non-zero $Z_1(0)$, should exhibit a crossover from the two-component massless $s$-G ($Z_1 = 0$) to the pure massless $s$-G ($Z_1 \to \infty$) systems. The critical behavior in the long-distance limit shall thus be described by the latter.

Fig.10 reveals the RG flows numerically solved for the coupled $s$-G system with $b = 1$. They are projected on the $(X, Y_1)$ plane and the initial value of $Z_i$ is taken to be $Z_i(0) = 0.1$. We have checked that $Z_i(\ell)$ in fact grows large at long distances. Accordingly, the crossover behavior can be seen as expected above. Since $Z_i(0)$

\footnote{Do not confuse this case with the non-gauged ($e = 0$) limit of the original four-Fermi model, where there is no anomaly from the beginning. The non-gauged model ($\approx XY$ model) is related to the one-component massless $s$-G system (sect.3).}

is small the flows are initially obeyed to $\left(\text{3.29}\right)$ and $\left(\text{3.30}\right)$ with $Y = 0$ and $Y_1^3 \to 0$). Beyond some steps where $Z_1$ starts to be frozen and the flows approach a massless $s$-G system $\left(Z_1 \to \infty\right)$. As $b \to 0$, the critical point $X_0 = Y_1 = 0$, and we are in a KT-like regime. In the zero-th order approximation $\left(b \pi \kappa N \sim 1\right)$ which reduces in the large $N$ limit to

$$N > N_c \approx \frac{b \pi \kappa}{2} \tag{4.23}$$

Note that this critical point $X_0 = 0$ does not change by choice of a slicing procedure, as is the case in the usual massless $s$-G system. As $b \to 0$, the phase boundary hits the $Y_1$ axis than that for a pure massless $s$-G system (due to non-zero $Y_1$) to the above estimations for those for the latter.

In the KT phase, vortices are irrelevant at long distances ($\ell \to \infty$) as seen from (4.17) and (4.19). This consequence is consistent with that obtained for the pure massive $s$-G system (sect.3) deduced from the $\left(\text{2.27}\right)$ and $\left(\text{2.28}\right)$ with $y = 0$ and $Y_1^3 \to 0$). The similar critical behaviors are also seen for other coupling constants $\left(\phi_2, \phi_3 \right)$ as functions of $\ell$. Figures we have chosen $b = 1$ so that $V(0)$, $V(1)$, and $10^2 W$ as functions of $\ell$, in the vortex phase $\left(\text{fig.11(a)}\right)$ all the coupling constants $\phi_j$ are irrelevant in the KT phase $\left(\text{fig.11(b)}\right)$.

\footnote{Although we plotted in fig.11(a) only the data to $\ell = 30$, we checked that the flows continue to converge to zero values at sufficiently long distances ($\ell \to \infty$).}
5 Summary and Concluding remarks

Previous studies of the so called dynamical symmetry breaking in the (gauged) four-Fermi models has mostly relied upon the mean-field type treatments. As a simple and typical case, the naive application of $1/N$ expansion to the axially U(1)-gauged four-Fermi theory in $1 + 1$ dimensions predicts the dynamical Higgs phenomenon [7] which, in contrast to the non-gauged model [8], is stable at least qualitatively against higher-order corrections. In this article we have then addressed the problem whether such a picture is really valid or not, and have investigated the possibility that topological excitations (vortices) of the model could be dynamically relevant to the long-distance properties. The latter issue is important for the former general question because topological excitations are non-mean-field-like objects and, if relevant, usually work toward disordering the system and restoring the symmetry broken by the mean-field ansatz, as is known in the dilute-instanton-gas arguments. This problem is of general importance, with no special regard to the space-time dimensionality.

To simplify the problem we have first derived the large-$N$ long-distance effective lagrangian along with the Witten’s prescription in a non-gauged model [9]. We have kept a global chiral-U(1) symmetry without assuming any v.e.v. for the angle part $\chi$ of the axial-U(1) order-parameter $\sigma + i \pi$, and have put a v.e.v. for the radial part the value of which is determined by the large-$N$ gap equation. The derived two-point effective action consists of the non-local part which is (axially) U(1) gauge-invariant, and of the contact mass term for the gauge field the coefficient ($b$) of which is arbitrary and depends on a fermion-loop regularization scheme.

If the (axially) gauge-invariant scheme is chosen, one only obtains the non-local part. This part displays the dynamical Higgs mechanism in a gauge-invariant fashion provided $\chi$ is restricted to be single-valued. The lowest-derivative part of this non-local action is a frozen U(1) Higgs model with two parameters. For this effective theory we have studied the long-distance relevance of multi-valued (vortex) configurations of $\chi$. With the help of the dual transformation, the partition function of the frozen U(1) Higgs model with an ensemble of single-vortex configurations has been shown to be equivalent with the system of many vortices interacting with a massive scalar field. This system has been further generalized to incorporate an additional chemical potential $y$, which controls the activity of vortices. By hand, we have examined the dynamical response to the compulsive reduction of the vortex activation. The small $y$ regime of the system with small fugacities ($y_m \sim O(1)$) does not reach in the long distance limit, hence vortices are always relevant at the crossover scale ($\ell = \ell_c$), we have observed that quantum fluctuations are frozen due to a rapid increasing of the effective fugacities $y_m(\ell)$ obeys a classical scaling.

Therefore it is strongly suggested that the effective theory of the axially U(1)-gauged scheme, corresponding to the $y = 1$ limit, does not reach in the long distance limit, and hence is completely irrelevant and the Higgs mechanism is due to a general disordering (symmetry-restoring) nature of non-paired vortices, which controls the activity of vortices. Reducing the inter-vortex distance beyond a critical value, we have examined the dynamical response to the compulsive reduction of the vortex activation. The small $y$ regime of the system is described by the massive s-G system, corresponding to the $y = 1$ limit. Utilizing the double dual transformation we have shown that the model is equivalent to the system of vortices interacting with both massless and massive scalar fields. This system has been mapped in the similar way as above to the coupled system of massless and massive s-G fields. The recursion-relation analysis has been performed with the result that a KT-type transition occurs at some critical value of the Higgs coupling which is given to the zero-th order approximation by (4.23) in the small $y$ regime (fig.12(b)). Thus the results given in Appendix B suggest that in the...
present four-Fermi theory undergoes a KT-type phase transition at
\[ N = N_c \approx \frac{8(1 + b)}{b}, \]  
(5.1)
in the number \( N \) of fermion species (fig.12(b)). The large-\( N \) \( (> N_c) \) regime of the system constitutes a KT-like phase where topological excitations are irrelevant and the long-distance properties are characterized by a free massless scalar field. This phase disappears \( (N_c \to \infty) \) in the gauge-invariant limit \( (b \to 0) \) at which \( \kappa_0 \) is zero and the massless field is frozen. In this limit the theory is thus consistently continued to the gauge-invariant scheme \( (b = 0) \), and the combined results in both gauge-invariant and non-invariant schemes provides a unified view for the phase structure of the four-Fermi model in general schemes \( b \geq 0 \).

There are some points left to be clarified such as to determine the macroscopic order parameter characterizing the phase structure. To make the phase structure more precise and concrete it would be necessary to consider the v.e.v. of a Wilson loop operator \[ \text{(27)} \] without relying upon a dilute-instanton-gas approximation. One unanswered technical problem in our recursion-relation analysis is that we have assumed a smooth momentum slicing procedure but have not specified it explicitly. Although the general possibility of taking such a slicing is argued in literatures \[ \text{(31)} \] it would be desirable to present it in the explicit form.

In refs. \[ \text{(38)} \], Banks et al. and Halpern studied the bosonization of the non-gauged SU(\( N \)) Thirring model. It may be an interesting further direction to apply the method to the present gauged model and to compare the results with those obtained here.

Anyhow the analysis developed in this article presents the prototype that the long-distance picture of the dynamical symmetry breaking based on a naive mean-field treatment should be modified even qualitatively due to the topological effects. Note that their existence itself is not special to 1+1 dimensions nor to the abelian nature of the gauge group. Then, it may be suspected that similar modifications of the long-distance picture could occur as well in higher-dimensional four-Fermi and Higgs models including those with non-abelian gauge symmetries. A semi-classical argument supporting this conjecture already exists in a certain model \[ \text{(39, 28)} \]. In the RG point of view one important difference between four and lower than four dimensions may be that the gauge coupling constant is dimensionless in the former while has positive mass dimensions in the gauge non-invariant scheme, thus the long-ranged logarithmic interaction determines the long-distance properties of the system. The procedures developed here will provide a useful tool for the long-distance quantum dynamics in higher dimensions.

Our results obtained for the gauge-invariant scheme indicate that some low-dimensional systems in solid-state physics, for example, the Meissner effect would not occur at finite temperature. It is to be treated quantum mechanically. Although the KT-type phase transition observed in the gauge non-invariant scheme is very interesting, we are not certain whether such an explicit breaking mass term of the gauge field could be relevant or not to real condensed-matter systems.

With the different motivation from ours, Ichinose and Mukaida \[ \text{(40)} \] recently performed a recursion-relation analysis of the massive s-G model with a single harmonic \( (m = 1) \) and drew the similar conclusion to ours for the (lattice) frozen U(1) Higgs model.

The author would like to thank I. Ichinose, S. Iida, M. Kato, M. Ninomiya and K. Shizuya for useful discussions and comments. He is also grateful to S. Iida and members of YITP (Uji) for their helps in his computer jobs.

**Appendix A. Evaluation of cumulants in the recursion-relation analysis**

In this appendix we present the calculation of cumulants appearing in the recursion-relation analysis of the massive s-G system \[ \text{(3.17)} \] in subsect.3.2. Besides the original kinetic term \( (\partial \phi)^2 \) we consider here only non-derivative harmonic operators \( \cos m \phi \) within \( O(y^3) \) approximation. There is no difficulty in extending the method to that applicable for the coupled s-G system treated in subsect.4.2 where instead we take into account all \( O(y^2, \partial^2) \) terms consistently.
The first-order cumulant \( <S_I> \) is calculated easily with the result
\[
<S_I> = -\lambda^2 \sum_{m=1} y_m A_m^2(0) \int d^2 x \cos m\varphi_x,
\]
where \( A(x) = e^{-Gx} \) is defined with the Green function \( G_x \) for \( h \)
\[
(2\pi^2 \kappa)^{-1} G_x = <h(x)h(0)> = \int_{N<\varphi<\Lambda} d\varphi \frac{e^{i\varphi}}{p^2 + M^2}.
\]
To \( O(\Lambda^{-1} d\Lambda) \) we evaluate
\[
A_m^2(0) = 1 - m^2 \pi \alpha (1 + Z)^{-1} d\lambda.
\]

In the second-order cumulant we consider terms of \( O(y_1^2) \sim O(y^2) \) and of \( O(y_1 y_2) \sim O(y^3) \)
\[
(-1/2)(<S_I^2> - <S_I>^2)
= (-1/4)y_1^2 A_0^2(0) \Lambda^4 \int d^2 x \int d^2 y
[(A_{xy}^2 - 1) \cos \varphi_x + \varphi_y] + (A_{xy}^2 - 1) \cos (\varphi_x - \varphi_y)]

+ (1/2)y_1 y_2 A_0^2(0) \Lambda^4 \int d^2 x \int d^2 y
[(A_{xy}^2 - 1) \cos (\varphi_x' + 2\varphi_y') + (A_{xy}^2 - 1) \cos (\varphi_x' - 2\varphi_y')] 
\simeq (1/4)y_1^2 A_0^2(0) \Lambda^4 \int d^2 \xi \int d^2 \eta \int d^2 z (\partial \varphi_x')^2

+ (1/2)y_1 y_2 A_0^2(0) \Lambda^4 \int d^2 \xi \int d^2 z[C_x(2 + C_1) \cos \varphi_x' + B_\xi(2 + B_\eta) \cos 3\varphi_x']

- (1/4)y_1^2 A_0^2(0) \Lambda^4 \int d^2 \xi \int d^2 \eta \int d^2 z \cos 2\varphi_x',
\]
where \( \xi \equiv x - y, \eta \equiv (x + y)/2, A_{xy} \equiv A(x - y), B_\xi \equiv A^2(\xi) - 1, C_\xi \equiv A^{-2}(\xi) - 1. \)

In the present approximation taken in subsect.3.2 we have neglected in the last semi-equality higher-derivative terms such as \( (\partial \varphi_x')^2 \eta_n, (\partial \varphi_x')^2 m \varphi_x' \) and \( (\partial \varphi_x')^2 \sin m \varphi_x'. \) To \( O(\Lambda) \) we evaluate
\[
\int d^2 \xi \ B_\xi = \int d^2 \xi \ C_\xi = \frac{1}{2} \int d^2 \xi \ B_\xi^2 = \frac{1}{2} \int d^2 \xi \ C_\xi^2,
\]
\[
= 4\pi^2 \kappa^2 (1 + Z)^{-2} \Lambda^{-3} d\Lambda,
\]
\[
\int d^2 \xi \xi \^2 C_\xi = 4 c\pi^2 \kappa (1 + Z)^{-1} \Lambda^{-5} d\Lambda,
\]
where \( c \equiv \int d\lambda \lambda^3 \tilde{j}_0(\lambda) \) is a dimensionless constant depending on a momentum-slicing procedure [34, 35]. Here \( \tilde{j}_0(\lambda) \) denotes a Bessel function modified so that

\( \lambda \) integral converges [34] according to a smooth momentum-slicing which we have not specified here. Inserting (A.3) and
\[
(-1/2)(<S_I^2> - <S_I>^2) \rightarrow c \pi^2 \kappa^2 y_1^2(1 + Z)^{-1}

+ 8\pi^2 \kappa^3 y_1 y_2(1 + Z)^{-2}

- 3\pi^2 \kappa^2 y_1^2(1 + Z)^{-1},
\]
Similarly we evaluate the third-order
\[
(1/6)(<S_I^3> - 3 <S_I^2>) = \frac{1}{24} \int d^2 \xi \int d^2 \eta \int d^2 z \cos 2\varphi_x'

[(B_1 B_2 + B_3 B_\xi + B_\xi B_\eta + B_\xi B_\eta)

+ (cyclic permutations)]
\simeq \frac{1}{24} \int d^2 \xi \int d^2 \eta \int d^2 z \cos 2\varphi_x',
\]
To \( O(\Lambda) \), a careful evaluation of \( \xi, \eta \) in
\[
(1/6)(<S_I^3> - 3 <S_I^2>) \rightarrow - \frac{1}{3} \int d^2 \xi \int d^2 \eta \int d^2 z \cos 2\varphi_x',
\]

Appendix B. An approximate solution for \( b > 0 \) system

We have seen in sect.4 that the condition \( x_0 \gtrsim 0 \) is satisfied at least in the region \( x_0 > 0 \) near \( Y_b \) and \( b > 0 \) only implies that our effective lagrangian is valid in \( b \) since the original system (4.4) corresponds to a perturbation is not valid. Since the
coupled s-G system exist for any non-zero value of \( X_0 \) and since we have considered operators being most infrared-important in a RG sense, it is plausible that our original model with a sufficiently large \( N \) is indeed in the KT phase. Although it is difficult to give the proof, we will develop here an approximate argument supporting partly this conjecture. The idea is as follows. Starting from the system (4.4), we integrate over all massive \( (\phi) \) scalar modes \( (0 < |p| < \Lambda) \). In a certain region of parameters \( e \) and \( N \), the short-ranged interaction among vortices due to \( \phi_1 \) can be neglected. The system is then well approximated by that only of a massless scalar fugacity \( y \) generated by the \( \phi \) where \( c \) coupled s-G system exist for any parameters. As in the pure XY model, an infrared divergence in \( D \) generated by \( \phi \), the terms in the first line of (B.2) represents the chemical potential of each vortex interaction due to them. As has been reviewed in subsect.3.1, the interaction \( (D(x; M_1)) \) due to \( \phi_1 \) is short-ranged and decays exponentially outside the range of \( O(Z_1^{-1}) \). On the other hand, from (B.2) the probability to find a vortex on a given position reads

\[
P = \exp(-2c_0^2/\Lambda)
\]

Hence, the average number of other vortices around a given vortex is evaluated to

\[
n_1^* \approx \exp(-2c_0^2/\Lambda)
\]

For example, in the limit: (i) \( Z_1 \to 0 \) both \( P \) and \( n_1^* \) vanish and the interaction is in the large-\( N \) and the very weak but

\[
Z_1 = \hat{\eta}^{N
\]

with \( \hat{\eta} \equiv e/\Lambda \) being a dimensionless gauge potential

\[
Z = \prod_x \left\{ \int D\phi_0(x) \sum_n \exp \left\{ -\frac{1}{2} n_0^2(x) \right\} \right\}
\]

This is equivalent to the massless s-G model where the perturbative RG analysis of the transition to occur at the critical point.

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Figure Captions

Fig.1: A $\sigma$ (a dashed line) tadpole diagram appearing in the second order gap equation of $1/N$ expansion, which includes infrared divergence. The solid lines are fermion propagators and the wavy line represents the propagation of a massless NG ($\pi$) boson.

Fig.2: $\rho$ (dashed lines) tadpole diagrams appearing in the second order gap equation of $1/N$ expansion, which contribute to infrared divergences. The solid lines are fermion propagators and the dotted line represents the propagation of a massless KT ($\chi$) boson. Infrared divergences contained in both diagrams are cancelled with each other.

Fig.3: $\rho$ (dashed lines) tadpole diagram. The solid line is a fermion propagator.

Fig.4: Feynman diagrams which contribute to the $(A,\chi)$ two-point effective action in the leading order of $1/N$ expansion. The wavy and dotted lines represent the gauge ($A$) and the KT ($\chi$) bosons.

Fig.5: The RG flows for the massless ($Z = 0$) s-G system, projected on a $(X,Y)$ plane. The dotted line is a phase boundary.

Fig.6(a): The RG behaviors of $Y_1(\ell)$, $Z(\ell)$, $Y_2(\ell)$ and $Y_3(\ell)$ as functions of a scale parameter $\ell$, in the small-$X$ ($X(0) = 0.075$) regime of the massless s-G system.

Fig.6(b): The RG behaviors of $Y_1(\ell)$, $Z(\ell)$, $Y_2(\ell)$ and $Y_3(\ell)$ as functions of a scale parameter $\ell$, in the large-$X$ ($X(0) = 1.125$) regime of the massless s-G system.

Fig.7: The RG behaviors of $Y_1(\ell)$, $Z(\ell)$, $Y_2(\ell)$ and $Y_3(\ell)$ as functions of a scale parameter $\ell$, in the large-$X$ ($X(0) = 1.0$) regime of the massive ($Z(0) = 0.01, 0.1$) s-G system.

Fig.8(a): The RG behaviors of $Y_1(\ell)$, $Z(\ell)$, $Y_2(\ell)$ and $Y_3(\ell)$ as functions of a scale parameter $\ell$, in the large-$X$ ($X(0) = 1.0$) regime of the massive ($Z(0) = 0.01, 0.1$) s-G system.
massive \((Z(0) = 0.01)\) s-G system.

**Fig.8(b):** The RG scaling behaviors of \(Y_1(\ell), Y_2(\ell), Y_3(\ell)\) and \(Z(\ell)\) as functions of a scale parameter \(\ell\), plotted by a logarithmic scale in the vertical axis. Initial conditions are the same as those for fig.8(a).

**Fig.9:** The RG flows for the two-component massless s-G system \((b = 1)\), projected on a \((X, Y_1)\) plane. The dotted line is a phase boundary.

**Fig.10:** The RG flows for the coupled s-G system \((b = 1)\) with \(Z_1(0) = 0.1\), projected on a \((X, Y_1)\) plane. The dotted line is a phase boundary.

**Fig.11(a):** The RG behaviors of \(Y_1(\ell), 10^2Y_2(\ell)\) and \(10^2W(\ell)\) as functions of a scale parameter \(\ell\), in the small-\(X\) \((X(0) = -0.02)\) regime of the coupled s-G system \((b = 1)\) with \(Z_1(0) = 0.1\).

**Fig.11(b):** The RG behaviors of \(Y_1(\ell), 10^3Y_2(\ell)\) and \(10^2W(\ell)\) as functions of a scale parameter \(\ell\), in the large-\(X\) \((X(0) = 0.05)\) regime of the coupled s-G system \((b = 1)\) with \(Z_1(0) = 0.1\).

**Fig.12(a):** The schematic behavior of the expected RG flows in the generalized scalar-vortex system with \(b = 0\). The vertical line represents \(y\) and the horizontal line stands for other parameters \((\kappa, \beta, ...\) collectively.

**Fig.12(a):** The schematic behavior of the expected RG flows in the generalized scalar-vortex system with \(b > 0\). The vertical line represents \(y\) and the horizontal line stands for other parameters \((\kappa_j, \beta, ...\) collectively.