LIFTING GALOIS REPRESENTATIONS TO RAMIFIED COEFFICIENT FIELDS

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Abstract. Let \( p > 5 \) be a prime integer and \( K/\mathbb{Q}_p \) a finite ramified extension with ring of integers \( \mathcal{O} \) and uniformizer \( \pi \). Let \( n > 1 \) be a positive integer and \( \rho_n : G_{\mathbb{Q}} \to \text{GL}_2(\mathcal{O}/\pi^n) \) be a continuous Galois representation. In this article we prove that under some technical hypothesis the representation \( \rho_n \) can be lifted to a representation \( \rho : G_{\mathbb{Q}} \to \text{GL}_2(\mathcal{O}) \). Furthermore, we can pick the lift restriction to inertia at any finite set of primes (at the cost of allowing some extra ramification) and get a deformation problem whose universal ring is isomorphic to \( \mathbb{Z}_p[[X]] \). The lifts constructed are “nearly ordinary” (not necessarily Hodge-Tate) but we can prove the existence of ordinary modular points (up to twist).

1. Introduction

The present article is a continuation of the work done in [CP14], where we constructed, for a finite field \( \mathbb{F} \), lifts of representations \( \rho_n : G_{\mathbb{Q}} \to \text{GL}_2(W(\mathbb{F})/p^n) \) to \( \text{GL}_2(W(\mathbb{F})) \). Here we prove how to extend the results to finite ramified extensions \( K/\mathbb{Q}_p \) of ramification degree \( e \).

The method used in [CP14] followed the ideas of [Ram99] and [Ram02], adapted to the modulo \( p^n \) setting. As noticed in [CP14] (see the remark before Proposition 5.9) these methods do not generalize to representations \( \rho_n : G_{\mathbb{Q}} \to \text{GL}_2(\mathcal{O}/\pi^n) \) where \( \mathcal{O} \) is the ring of integers of \( K \). The obstacle is that the field extension given by the kernel of \( \rho_n \) (mod \( \pi^2 \)) (which we denote \( \rho_2 \)) as an element in \( H^1(G_{\mathbb{Q}}, Ad^0 \bar{\rho}) \) (where \( Ad^0 \bar{\rho} \) is the adjoint representation of the reduction mod \( \pi \) of \( \rho_n \)) is non-trivial, so no matters which finite set of auxiliary primes \( M \) we chose, the morphism of Theorem A of [CP14]

\[
H^1(G_M, Ad^0 \bar{\rho}) \to \bigoplus_{v \in M} H^1(G_v, Ad^0 \bar{\rho})/N_v
\]

will never be an isomorphism since both sides have the same dimension but the morphism has non trivial kernel (the cohomological element corresponding to \( \rho_2 \) is in the kernel).

The key innovation is to relax local conditions so that the morphism (1) is no longer an isomorphism, but a surjective map with one dimensional kernel. This will be enough for the lifting purpose, since it allows to adjust globally the local adjustments at each step. We cannot relax the hypothesis at primes \( v \neq p \), since the local deformation ring of \( \overline{\rho}\mid_{G_v} \) does not have a smooth quotient of dimension bigger than \( \dim H^1(G_v, Ad^0 \bar{\rho}) - \dim H^2(G_v, Ad^0 \bar{\rho}) \). Therefore, we need to impose a different local condition at the prime \( p \). The condition we impose is the same as in [CM09].

Definition. We say that a deformation is “nearly ordinary” if its restriction to the inertia subgroup is upper-triangular and its semisimplification is not scalar, i.e. if

\[
\rho|_{I_p} = \begin{pmatrix}
\psi_1 & * \\
0 & \psi_2
\end{pmatrix}
\]

with \( \psi_1 \neq \psi_2 \).

Using this local condition at \( p \), we are able to derive a slightly weaker version of the following theorem (see Theorem 4.1 for the precise statement), which is one of the main results of this work.
Theorem. Let \( \rho_n : G_\Q \to \GL_2(\O/p^n) \) be a continuous representation which is odd and nearly ordinary at \( p \). Assume that \( \text{Im}(\rho_n) \) contains \( \SL_2(\O/p) \) if \( n \geq e \) and that \( \rho \) is surjective otherwise. Let \( P \) be a set of primes of \( \Q \) containing the ramification set of \( \rho_n \). For each \( v \in P \setminus \{ p \} \) fix a local deformation \( \rho_v : G_v \to \GL_2(\O) \) that lifts \( \rho_n|_{G_v} \). Then there exists a continuous representation \( \rho : G_\Q \to \GL_2(\O) \) and a finite set of primes \( R \) such that:

- \( \rho \) lifts \( \rho_n \), i.e. \( \rho \equiv \rho_n \) (mod \( p^n \)).
- \( \rho \) is unramified outside \( P \cup R \).
- For every \( v \in P \), \( \rho|_{I_v} \simeq \rho_v|_{I_v} \) over \( \GL_2(\O) \).
- \( \rho \) is nearly ordinary at \( p \).
- All the primes of \( R \), except possibly one, are not congruent to 1 modulo \( p \).

In fact, the method provides us not only one lift of \( \rho_n \) but a lift to the coefficient ring \( \Z_p[[X]] \) (see Theorem \[13\]). The downside is that the family of representations we constructed is not ordinary but nearly ordinary, which implies that most points are not Hodge-Tate (in particular not modular). However, the freedom of the coefficient ring allows us to prove the existence of modular points, which is the second main result of the present article (see Theorem \[5,1\] for the precise statement).

Theorem. Let \( p \) be a prime, \( \O \) the ring of integers of a finite extension \( K/\Q_p \) with ramification degree \( e > 1 \) and \( \pi \) its local uniformizer. Let \( \rho_n : G_\Q \to \GL_2(\O/p^n) \) be a continuous representation satisfying

- \( \rho_n \) is odd.
- \( \text{Im}(\rho_n) \) contains \( \SL_2(\O/p) \) if \( n \geq e \) and \( \rho_n \) is surjective otherwise.
- \( \rho_n \) is ordinary at \( p \).

Let \( P \) be a set of primes containing the ramification set of \( \rho_n \), and for each \( v \in P \) pick a local deformation \( \rho_v : G_v \to \GL_2(\O) \) lifting \( \rho_n|_{G_v} \). Then there exists a finite set of primes \( Q \) and a continuous representation \( \rho : G_{P \cup Q} \to \GL_2(\O) \) such that

- \( \rho \) lifts \( \rho_n \), i.e. \( \rho \equiv \rho_n \) (mod \( p^n \)).
- \( \rho \) is modular.
- For every \( v \in P \), \( \rho|_{I_v} \simeq \rho_v|_{I_v} \) over \( \GL_2(\O) \).
- For every \( q \in Q \), \( \rho|_{I_q} \) is unipotent and \( q \not\equiv \pm 1 \) (mod \( p \)) for all but possibly one prime of \( Q \).
- \( \rho \) is ordinary at \( p \).

The strategy to prove both theorems is similar to the one in \[Ram02\] and \[CP14\]. We will construct, for each prime \( v \in P \), a set of deformations of \( \rho_n|_{G_v} \) to \( \O \) which contains \( \rho_v \) and a subspace \( N_v \) preserving its reductions in the sense of Proposition \[2.1\]. Also, for \( v = p \), we take \( C_p \) to be the set of nearly ordinary deformations, which will give a larger subspace \( N_p \) and therefore the image of the morphism \([11]\) has smaller dimension. Given this local setting, and after some manipulation of the groups appearing in \([11]\) (and its analogue for \( H^2(G_T, \Ad(\bar{\rho})) \)) we are able to make such map surjective (with a 1-dimensional kernel). This implies that the problem of lifting \( \bar{\rho} \) is unobstructed, which gives Theorem \[1.1\].

The tricky part here is that, differently from what happens in \[CP14\], in some cases the subspaces \( N_v \) preserve the reductions modulo \( \pi^n \) of the elements of \( C_v \), not for all \( n \) but for \( n \) bigger than certain integer \( \alpha \). To overcome this situation we will, following the ideas of \[KLR05\], lift by adding one set of auxiliary primes for each power of \( \pi \), until we reach the lift modulo \( \pi^n \) for the main method to work. In this way we will get Theorem \[5,1\].

Theorem \[5,1\] follows from studying the possible modular points appearing in the universal ring provided by Theorem \[4.13\]. Notice that we will obtain a modular lift of \( \rho_n \) each time we find a nearly ordinary lift \( \rho \) such that the characters appearing in the diagonal of \( \rho|_{I_p} \) can be written as an integral power of the cyclotomic character times a character of finite order.

The article is organized as follows: Section 2 concerns with the construction of the sets \( C_q \) and subspaces \( N_q \) for primes \( q \neq p \). In Section 3 we do the same for the nearly ordinary condition at \( p \). Section 4 treats the global argument for lifting and proves the first main theorem of this work. It is divided into two subsections, one for exponents at which we have the \( C_q \) and \( N_q \) for every
q ∈ P and one for the ones at which we do not. In Section 5 we prove the other main theorem which concerns modularity.

Finally, let us observe that the precise statements of both main theorems differ from the ones presented in this introduction and contain some technical assumptions which we expect to remove soon in a second version of the article (merging this version with some results of [CR14]).

1.1. Notation and conventions. In this article p will denote a rational prime, O the ring of integers of a ramified finite extension K/ℚp, π its local uniformizer and e the ramification degree. ℱ will denote the residual field ℓ/π. For a prime q we denote by Gq the local absolute Galois group Gal(ℚq/ℚ) and σ and τ stand for a Frobenius element and a generator of the tame inertia group of Gq. We denote

\[ e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \]

which form a basis for the space of 2 × 2 matrices with trace 0. The cyclotomic character will be denoted by χ. All our deformations will have fixed determinant. By ρn : Gq → GL2( ℓ/πn) we denote a continuous Galois representation, ˜ρ : Gq → GL2( ℱ) denotes its residual representation, and by ρ we denote a representation with image in GL2( ℓ). Finally v stands for the valuation in ℓ satisfying v(π) = 1.

For the definitions and main results of deformation theory, we refer to [Maz89].

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2. Local deformation theory at q ≠ p

Let q ≠ p a prime and let ρ : Gq → GL2( ℓ) be a continuous representation. We denote by ˜ρ its reduction mod π. We know that the elements of H1( Gq, Ad0 ˜ρ) act on the deformations with coefficients in ℓ/πr. Recall the notion of an element of H1( Gq, Ad0 ˜ρ) preserving a set of deformations.

Definition. Let Cq be a set of deformations \{ ρ : Gq → GL2( ℓ) \} and u ∈ H1( Gq, Ad0 ˜ρ). We say that u preserves Cq if for any ˜ρ : Gq → GL2( ℓ/πn) which is the reduction of a deformation of Cq we have that (1 + πm−1u) ˜ρ is also the reduction of some deformation of Cq.

We want to prove that for every q there exists a set Cq of deformations containing it that is preserved by a subspace Nq of certain dimension. We can prove such set exists for almost all possible ρ but some particular ones which are the following.

Definition. We say that a representation ρ : Gq → GL2( ℓ) is “bad” if

\[ \rho \simeq \begin{pmatrix} \psi_1 & \psi_1 - \psi_2 \\ 0 & \psi_2 \end{pmatrix} \]

over GL2( ℓ) and moreover the following holds:

- \( \tilde{\rho} \) is unramified and \( \tilde{\rho}^{\sigma}(\sigma) = 1d \).
- \( v(\psi_1(\sigma) - \psi_2(\sigma)) < v(\psi_1(\tau) - \psi_2(\tau)) \).

Proposition 2.1. Let ρ : Gq → GL2( ℓ) be a continuous representation. If ρ is not bad then there always exists a positive integer α, a set Cq of deformations of the reduction of ρ modulo πα to characteristic 0 that contains ρ (up to GL2( ℓ)-isomorphism), and a subset Nq ⊆ H1( Gq, Ad0 ˜ρ) such that:

- All the elements of Cq are isomorphic when restricted to inertia.
- \( N_q \subseteq H^1(G_q, Ad_0 \tilde{\rho}) \) has codimension equal to \( \dim H^2(G_q, Ad_0 \tilde{\rho}) \).
- Every u ∈ Nq preserves the mod \( \pi^n \) reductions of elements of Cq for \( n \geq \alpha \).

In other words, there exists a smooth deformation condition containing ρ of dimension equal to \( \dim H^1(G_q, Ad_0 \tilde{\rho}) - \dim H^2(G_q, Ad_0 \tilde{\rho}) \) if we start lifting from a big enough exponent.
Proposition 2.2. Let \( q \neq 2 \), be a prime number, with \( q \neq p \). Then every representation \( \rho : G_q \to \text{GL}_2(\mathbb{F}) \), up to twist by a character of finite order, belongs to one of the following three types:

- **Principal Series:** \( \rho \simeq \begin{pmatrix} \phi & 0 \\ 0 & 1 \end{pmatrix} \) or \( \rho \simeq \begin{pmatrix} 1 & \psi \\ 0 & 1 \end{pmatrix} \).
- **Steinberg:** \( \rho \simeq \begin{pmatrix} 0 & \mu \\ \mu & 1 \end{pmatrix} \), where \( \mu \in H^1(G_q, \mathbb{F}(\chi)) \) and \( \mu|_{I_q} \neq 0 \).
- **Induced:** \( \rho \simeq \text{Ind}^{G_q}_{M_q}(\xi) \), where \( M/Q_q \) is a quadratic extension and \( \xi : G_M \to \mathbb{F}^\times \) is a character not equal to its conjugate under the action of \( \text{Gal}(M/Q_q) \).

Here \( \phi : G_q \to \mathbb{F}^\times \) is a multiplicative character and \( \psi : G_q \to \mathbb{F} \) is an unramified additive character.

Proposition 2.3. Let \( \hat{\rho} : G_q \to \text{GL}_2(\mathbb{Z}_p) \) be a continuous representation. Then up to twist (by a finite order character times powers of the cyclotomic one) and \( \text{GL}_2(\mathbb{Z}_p) \) equivalence we have:

- **Principal Series:** \( \hat{\rho} \simeq \begin{pmatrix} \phi & n \pi^n(\phi^{-1}) \\ 0 & 1 \end{pmatrix} \), with \( n \in \mathbb{Z} \) satisfying \( \pi^n(\phi - 1) \in \mathbb{Z}_p \) or \( \hat{\rho} \simeq \begin{pmatrix} 1 & \psi \\ 0 & 1 \end{pmatrix} \).
- **Steinberg:** \( \hat{\rho} \simeq \begin{pmatrix} 0 & \pi^n \mu \\ \mu & 1 \end{pmatrix} \), with \( n \in \mathbb{Z} \geq 0 \).
- **Induced:** There exists a quadratic extension \( M/Q_q \) and a character \( \xi : G_M \to \mathbb{Z}_p^\times \) not equal to its conjugate under the action of \( \text{Gal}(M/Q_q) \) such that \( \hat{\rho} \simeq \langle v_1, v_2 \rangle \circ \hat{\sigma} \), where for \( \sigma \) a generator of \( \text{Gal}(M/Q_p) \) and \( \tau \in G_M \), the action is given by

\[
\tau(v_1) = \xi(\tau)v_1, \quad \tau(v_2) = \xi^\sigma(\tau)v_2, \quad \sigma(v_1) = v_2 \quad \text{and} \quad \sigma(v_2) = \xi(\sigma^2)v_1,
\]

or

\[
\hat{\rho}(\tau) = \begin{pmatrix} \xi(\tau) & \xi^\sigma(\tau) \\ 0 & \xi^2(\tau) \end{pmatrix} \quad \text{and} \quad \hat{\rho}(\sigma) = \begin{pmatrix} a & \xi^2(\tau) - a^2 \\ \pi^n & \pi \end{pmatrix}.
\]

where \( \xi^\sigma \) is the character of \( G_M \) defined by \( \xi^\sigma(a) = \xi(\sigma g \sigma^{-1}) \) and \( a \in O_L^\times \).

Proposition 2.4. Let \( \hat{\rho} \) be as above, then we have the following types of reduction:

- If \( \hat{\rho} \) is Principal Series, then \( \overline{\rho} \) is Principal Series or Steinberg, and the latter occurs only when \( q \equiv 1 \pmod p \).
- If \( \hat{\rho} \) is Steinberg, then \( \overline{\rho} \) is Steinberg or Principal Series, and the latter occurs only when \( \overline{\rho} \) is unramified.
- If \( \hat{\rho} \) is Induced, then \( \overline{\rho} \) is Induced, Steinberg or an unramified Principal Series. For the last two cases we must have \( q \equiv -1 \pmod p \).

To prove Proposition 2.1 we consider all the possible pairs of \( \text{GL}_2(\mathbb{F}) \) and \( \text{GL}_2(\mathbb{O}) \)-equivalence classes of representations that fits into our possible reduction types (indexing them first by the class of the mod \( \pi \) deformation), and in each case we define the deformation class. For the rest of this section \( \overline{\rho} \) is a deformation mod \( \pi \) and \( \rho \) a deformation to \( \mathbb{O} \) that reduces to \( \overline{\rho} \).

- **Case 1:** \( \overline{\rho} \) is ramified Principal Series. Proposition 2.1 implies that a mod \( \pi \) Principal Series can only come from a characteristic 0 principal series. The full study of this case is done in Case 1 of Section 4 of [CP14]. The work there is done for \( \mathbb{O} \) unramified but the same applies in our situation.

- **Case 2:** \( \overline{\rho} \) is Steinberg. When \( \overline{\rho} \) is Steinberg, it can be the reduction of any of the three characteristic 0 ramified types:

  - **Case 2.1:** \( \rho \) is Steinberg. The definition of \( C_q \) and \( N_q \) is essentially the same as [CP14], Case 2 of Section 4. Although in that work only the case where \( \mathbb{O}/\mathbb{Z}_p \) is unramified is treated, the ramification of \( \mathbb{O} \) does not affect the results.

  - **Case 2.2:** \( \rho \) is Principal Series. Proposition 2.1 implies that \( q \equiv 1 \pmod p \). Let \( \rho_q = \begin{pmatrix} \psi & * \\ 0 & 1 \end{pmatrix} \). Without loss of generality, we can take \( * = 1 \). We have the following lemma:
Lemma 2.5. A deformation $\tilde{\rho} : G_q \to \text{GL}_2(\mathbb{O}/\pi^n)$ which has the form:

$$
\tilde{\rho}(\tau_q) = \begin{pmatrix} \psi(\tau) & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \tilde{\rho}(\sigma_q) = \begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix},
$$

has a unique lift to characteristic zero of the same form if and only if $\beta^2 + \gamma(\psi(\tau) - 1)\beta + \Psi \equiv 0 \pmod{\pi^n}$, where $\Psi$ is a fixed lift determinant.

Proof. This Lemma is part of a computation made in Proposition 3.4 of [Kha06]. There, it is done for deformations with coefficients in unramified coefficient field and its mod $p$ reductions, but the same proof works in general.

Let $j \in H^1(\mathbb{G}, Ad^0\tilde{\rho})$ be the element defined by:

$$
\jmath(\sigma_q) = e_2, \quad \jmath(\tau_q) = 0,
$$

and take $N_q$ to be the subspace it generates. Also let $C_q$ be the set of deformations to $\mathbb{O}$ which have the form given in Lemma 2.6. Observe that any mod $\pi^n$ reduction of an element of $C_q$ satisfies the equation $\beta^2 + \gamma(\psi(\tau) - 1)\beta + \Psi = 0$, and acting by $j$ on them does not change this (as $\pi$ divides $\psi(\tau) - 1$ so adding a multiple of $\pi^{n-1}$ to $\gamma$ does not change the equation modulo $\pi^n$). Then, Lemma 2.5 guarantees that $C_q$ and $N_q$ satisfy the property we are looking for.

- Case 2.3: $\rho$ is Induced. Proposition 2.4 tells us that necessarily $q \equiv -1 \pmod{\pi}$ and by results in Section 3 of [CP14] we have that $d_1 = d_2 = 1$ and therefore we can take $N_q = \{0\}$ and $C_q = \{\rho\}$.

- Case 3: $\overline{\rho}$ is Induced. By Proposition 2.4, when $\overline{\rho}$ is Induced, $\rho$ must be Induced as well. The way choice of $C_q$ and $N_q$ in this case is done in Case 3 of Section 4 of [CP14].

- Case 4: $\overline{\rho}$ is unramified. Being unramified, $\overline{\rho}$ allows lifts to any type of deformation. We must treat each of them separately as they have many subcases. The case of $\rho$ being Steinberg is dealt with in Case 4 of Section 4 of [CP14]. There are two other cases left to study. In each of them we will distinguish between three types of equivalence classes for $\overline{\rho}$, according to the image of Frobenius.

Case 4.1: $\rho$ is Principal Series. In this case we have $\rho = \begin{pmatrix} \phi & \pi^n (\phi - 1) \\ 0 & 1 \end{pmatrix}$ with $n \leq 0$. By Proposition 2.4 we necessarily have $q \equiv 1 \pmod{\pi}$.

- If $\overline{\rho}(\sigma_q) = \begin{pmatrix} \phi & 0 \\ 0 & 1 \end{pmatrix}$ with $\beta \neq 1$ we have $d_1 = 2$ and $d_2 = 1$ so we are looking for a one-dimensional subspace $N_q$. Observe that necessarily $\rho \simeq \begin{pmatrix} \phi & 0 \\ 0 & 1 \end{pmatrix}$ over $\mathbb{O}$ as $\psi(\sigma) \equiv \beta \neq 1 \pmod{\pi}$ so $(\psi - 1)/\pi$ is not an integer.

Let $u \in H^1(\mathbb{G}, Ad^0\rho)$ be the cocycle defined by $u(\sigma) = e_1$ and $u(\tau) = 0$. We can take $N_q = \langle u \rangle$ and $C_q$ the set of representations of the form $\rho \simeq \begin{pmatrix} \phi & 0 \\ 0 & \psi^{-1} \end{pmatrix}$ for $\psi : G_q \to \mathbb{O}^\times$ unramified.

- If $\overline{\rho}(\sigma_q) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ the corresponding dimensions are $d_1 = 2$ and $d_2 = 1$. In this case we have $\rho \simeq \begin{pmatrix} \phi & \pi^{n+1} \\ 0 & 1 \end{pmatrix}$, with $v_\pi(\phi - 1) = r$. Now take the cocycle defined by $u(\sigma) = 0$ and $u(\tau) = e_2$. Set $N_q = \langle u \rangle$ and $C_q$ the set of deformations of $\overline{\rho}$ such that $\rho \simeq \begin{pmatrix} \phi & \pi^{m+1} \\ 0 & 1 \end{pmatrix}$ for some $\beta \in \mathbb{O}^\times$.

Lemma 2.6. The set $C_q$ and subspace $N_q$ defined above satisfy that $N_q$ preserves $C_q \mod \pi^m$ for all $m$ such that $\phi$ is ramified mod $\pi^m$.

Proof. Assume that $\rho(\sigma) = \begin{pmatrix} a & -b \\ 0 & b \end{pmatrix}$ and $\rho(\tau) = \begin{pmatrix} x & -z \\ 0 & y \end{pmatrix}$. Our assumptions on $\rho$ not being bad tell us that $v(a - b) \geq v(x - y)$. If we have $v(a - b) > v(x - y)$ then we change $\sigma$ by $\tau \sigma$ (which is another Frobenius element). In this way, we can assume that $v(a - b) = v(x - y)$.

We want to prove that if we have a deformation $\tilde{\rho} : G_q \to \text{GL}_2(\mathbb{O}/\pi^n)$ that sends

$$
\tilde{\rho}(\sigma) = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \quad \text{and} \quad \tilde{\rho}(\tau) = \begin{pmatrix} x & z \\ 0 & y \end{pmatrix},
$$
with $x \neq y$, and is reduction of some element of $C_q$ (i.e. $c = \beta(a-b)$ and $z = \beta(x-y)$ for some $\beta, a, b, c, x, y, z \in \mathbb{O}$), then $(Id + \pi^{-1}u)\hat{\rho}$ is also the reduction of some element of $C_q$. Recall that

$$(Id + \pi^{-1}u)\hat{\rho}(\sigma) = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \quad \text{and} \quad (Id + \pi^{-1}u)\hat{\rho}(\tau) = \begin{pmatrix} x & z + \pi^{-1} \\ 0 & y \end{pmatrix}. $$

It is easily checked that the cocycle $v$ that sends $\sigma$ to $e_2$ and $\tau$ to $0$ is a coboundary. So $\hat{\rho}$ can also be tought as

$$(Id + \pi^{-1}u)\hat{\rho}(\sigma) = \begin{pmatrix} a & c + \lambda \pi^{-1} \\ 0 & b \end{pmatrix} \quad \text{and} \quad (Id + \pi^{-1}u)\hat{\rho}(\tau) = \begin{pmatrix} x & z + \pi^{-1} \\ 0 & y \end{pmatrix}$$

for any $\lambda \in \mathbb{F}$. Therefore, it is enough to find some $\lambda \in \mathbb{O}$ such that

$$\frac{x-y}{z+\pi^{-1}} = \frac{a-b}{c + \lambda \pi^{-1}}.$$

But this equation is equivalent to $\lambda(x-y) = (a-b)$, which has an integral solution given that $v(x-y) = v(a-b)$.

\[\square\]

**Remark.** In this case, it can be proved that $u$ preserves $C_q$ under the weaker condition $2v(a-b) \geq v(x-y)$. However, we do not know how to do this when this condition is not fulfilled.

If $\bar{\pi}(\sigma_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ we have $d_1 = 6$ and $d_2 = 3$ therefore we need to find a subspace $N_6$ of dimension 3. This case is a little more involved than the other two as there are non-trivial elements of $H^1(G_q, Ad^{\emptyset}\hat{\rho})$ that act trivially modulo $\pi^n$ for high powers of $\pi$. We follow the same ideas as in the study of the Steinberg-reducing-to-unramified case (the spirit of these ideas is taken from the approach to trivial primes followed in [RH08]).

Assume first that $\rho \simeq \begin{pmatrix} \phi & 0 \\ 0 & \phi_2 \end{pmatrix}$. We will take $C_q$ as the set of representations of the form

$$\rho' \simeq \begin{pmatrix} \gamma\phi & 0 \\ 0 & \gamma^{-1} \end{pmatrix}$$

with $\gamma : G_q \to \mathbb{O}^\times$ an unramified character, that lift the reduction mod $\pi^n$ of $\rho$, with $\alpha = v(\phi_1(\tau) - \phi_2(\tau)) + 2$. Clearly, the set $C_q$ is preserved by the cocycle $u_1$ that sends $\sigma \mapsto e_1$ and $\tau \mapsto 0$.

We will construct two more cocycles $u_2$ and $u_3$ that act trivially on reductions modulo $\pi^n$ of deformations on $C_q$ for $n \geq \alpha$. Let $\hat{\rho} : G_q \to GL_2(\mathbb{O}/\pi^n)$ be the mod $\pi^n$ reduction of an element in $C_q$. Let $\hat{\rho}(\sigma) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and $\hat{\rho}(\tau) = \begin{pmatrix} 0 & 0 \\ 0 & \tau \end{pmatrix}$. Let $u \in H^1(G_q, Ad^{\emptyset}\hat{\rho})$. To prove that $u$ acts trivially on $\hat{\rho}$ we need to find a matrix $C \in GL_2(\mathbb{O}/\pi^n)$ such that $C \equiv Id \mod \pi^n$ and $C\hat{\rho}C^{-1} = (Id + \pi^{-1}u)\hat{\rho}$. One can find such matrix by taking $C = \begin{pmatrix} 1 + \pi^n & \pi^n \bar{\rho} \\ \pi^n & 1 + \pi^n \bar{\rho} \end{pmatrix}$ and explicitly computing $C\hat{\rho} = (Id + \pi^{-1}u)\hat{\rho}$ at $\sigma$ and $\tau$. In this way, one finds out that if $v(\phi_1(\tau) - \phi_2(\tau)) > v(\phi_1(\sigma) - \phi_2(\sigma))$ then the cocycles $u_2$ and $u_3$ sending $\sigma$ to $e_2$ and $e_3$ respectively and $\tau$ to $0$ act trivially on the reductions of elements of $C_q$. The corresponding base change matrices $C_i$ that conjugate $(Id + \pi^{-1}u_i)\hat{\rho}$ into $\hat{\rho}$ are

$$C_2 = \begin{pmatrix} 1 & -\pi^{-2} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad C_3 = \begin{pmatrix} 1 & 0 \\ \pi^{-2} & 1 \end{pmatrix}.$$

It remains to check what happens when $v(\phi_1(\tau) - \phi_2(\tau)) \leq v(\phi_1(\sigma) - \phi_2(\sigma))$. Observe that we can always assume that $v(\phi_1(\tau) - \phi_2(\tau)) = v(\phi_1(\tau) - \phi_2(\tau))$ in this case, by simply changing $\sigma$ for $\tau\sigma$. In this case we can take $u_2$ sending $\sigma$ to $e_2$ and $\tau$ to $\lambda e_2$ and $u_3$ sending $\sigma$ to $e_3$ and $\tau$ to $\lambda e_3$ for $\lambda = (x-y)/(a-b) \in \mathbb{F}$ (notice that this does not depend on $\hat{\rho}$). Again, the action of both cocycles will be trivial and the base change matrices will be the same as before.

It remains to consider the case where $\rho \simeq \begin{pmatrix} \phi & 0 \\ 0 & (\phi-1)/\pi^n \end{pmatrix}$ for $r > 0$. Let $\alpha = v((\phi-1)/\pi^n) + 2$ and let $C_q$ be the set of deformations of the mod $\pi^n$ reduction of $\rho$ such that $\rho \simeq \begin{pmatrix} \phi & 0 \\ 0 & (\phi-1)/\pi^n \end{pmatrix}$ for some $\beta \in \mathbb{O}^\times$. By doing the exact same calculation as in Lemma 2.6 it can be proved that the cocycle $u_1$ that sends $\sigma$ to $0$ and $\tau$ to $e_2$ preserves the reductions of elements in $C_q$. We still need two
more elements preserving \( C_q \). As in the previous case, we have two cocycles that act trivially on mod \( \pi^n \) reductions of elements of \( C_q \) for \( n \geq \alpha \). Let \( \tilde{\rho} \) be a mod \( \pi^n \) reduction of some element in \( C_q \) given by \( \tilde{\rho}(\sigma) = (\begin{smallmatrix} a & c \\ 0 & d \end{smallmatrix}) \) and \( \tilde{\rho}(\tau) = (\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}) \). As in Lemma \( \text{2.6} \) we can assume that \( v(a - b) = v(x - y) \).

Let \( \lambda = \frac{x - y}{a - b} = \frac{z}{c} \).

Let \( u_2 \) be the cocycle that sends \( \sigma \) to \( \lambda e_1 \) and \( \tau \) to \( e_1 \) and \( u_3 \) the one that sends \( \sigma \) to \( \lambda e_2 \) and \( \tau \) to \( e_2 \). We claim that these act trivially on \( \tilde{\rho} \) if \( n \geq v(z) + 2 \).

It can be checked that the base change matrices given by

\[
C_2 = \begin{pmatrix} 1 & 0 \\ -\pi^{n-2} & 1 \end{pmatrix} \quad \text{and} \quad C_3 = \begin{pmatrix} 1 + \pi^{n-2} & 0 \\ 0 & 1 \end{pmatrix}
\]

serve to prove the trivialness of the action of \( u_2 \) and \( u_3 \) respectively which concludes this case.

**Case 4.2: \( \rho \) is Induced.** Proposition \( \text{2.4} \) says that whenever \( \rho \) is induced and \( \overline{\rho} \) is unramified, \( \overline{\rho}(\sigma) \) has eigenvalues 1 and \( -1 \) (up to twist) and \( q \equiv -1 \) (mod \( p \)). In this case we have that \( d_1 = 3 \) and \( d_2 = 2 \). We want to find a set \( C_q \) and a subspace \( N_q \) of dimension 1 preserving it. Let \( C_q = \{ \rho \} \). As in the Principal Series case, we will be able to find non trivial cocycles that act trivially on mod \( \pi^n \) reductions of \( \rho \) for \( n \) big enough. We split into the two possible families of \( \text{GL}_2(\mathbb{Z}_p) \)-equivalence classes of induced representations given by Proposition \( \text{2.6} \).

\( \cdot \) If \( \rho(\sigma) = (\begin{smallmatrix} 0 & c \\ 1 & a \end{smallmatrix}) \) and \( \rho(\tau) = (\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}) \), it can be checked that the cocycle \( u \) sending \( \tau \) to 0 and \( \sigma \) to \( e_2 \) is non trivial. If \( r = v(x - y) \) then \( u \) acts trivially for all \( n \geq r + 2 \) and the base change matrix is given by

\[
C = \begin{pmatrix} 1 & \pi^{r-1}u_1 \\ \pi^{n-r-1}u_2 & 1 \end{pmatrix}
\]

where \( u_1, u_2 \in \mathbb{O} \) are such that \( u_1 - tu_2 \equiv \pi^{r-1} \) (mod \( \pi^r \)).

\( \cdot \) If \( \rho(\sigma) = (\begin{smallmatrix} -a & c \\ b & d \end{smallmatrix}) \) and \( \rho(\tau) = (\begin{smallmatrix} x & z \\ y & w \end{smallmatrix}) \), as in the previous case, it can be checked that the cocycle \( u \) that sends \( \sigma \) to 0 and \( \tau \) to \( e_2 \) is non trivial. Again, the action of this cocycle in the reduction modulo \( \pi^n \) of \( \rho \) is trivial for \( n \geq v(z) + 2 \). In this case, the base change matrix is given by

\[
C = \begin{pmatrix} 1 + \frac{\pi^{n-2}}{b\pi^{n-2}} & -\frac{\pi^{n-2}}{2a} \\ \frac{b\pi^{n-2}}{2a} & 1 \end{pmatrix}
\]

3. **Local deformation theory at \( p \)**

At the prime \( p \) we will impose the deformation condition of being “nearly ordinary” (as in [CM09]). This section is mainly about gathering previously done calculations, and all the deformations appearing are deformations of the local Galois group \( G_p \).

**Definition.** We say that a deformation of \( G_p \) is “nearly ordinary” if its restriction to the inertia subgroup is upper-triangular and its semisimplification is not scalar, i.e. if

\[
\rho|_{I_p} = \begin{pmatrix} \psi_1 & * \\ 0 & \psi_2 \end{pmatrix}
\]

with \( \psi_1 \neq \psi_2 \).

We will prove the following theorem.

**Theorem 3.1.** Let \( \rho_n : G_p \rightarrow \mathbb{O}/\pi^n \) be a nearly ordinary deformation and \( \overline{\rho} \) its mod \( \pi \) reduction. If the quotient of the characters in the diagonal of \( \overline{\rho} \) is not the cyclotomic character then there is a family of nearly ordinary deformations \( C_p \) to characteristic 0 such that \( \rho_n \) is the reduction of a member of \( C_p \) and a subspace \( N_p \subseteq H^1(G_p, Ad^0\rho) \) of codimension equal to \( \dim H^1(G_p, Ad^0\rho) + 1 \) preserving \( C_p \) in the sense of Proposition \( \text{2.7} \). If \( \psi_1/\psi_2 = \chi \) and \( \overline{\rho} \) is indecomposable and non-flat the set \( C_p \) and subspace \( N_p \) also exists.
Proof. Let

$$\overline{\rho} \simeq \begin{pmatrix} \psi_1 & \ast \\ 0 & \psi_2 \end{pmatrix}. $$

By twisting $\overline{\rho}$ by $\psi_2^{-1}$ we can assume that $\psi_2 = 1$. Let $U$ be the set of upper-triangular $2 \times 2$ matrices of trace 0. To prove the theorem we will construct for each possible $\overline{\rho}$, the corresponding set and subspace and verify that its dimensions satisfy the statement of the theorem. In most of the cases $C_p$ will consist on all nearly ordinary deformations of $\overline{\rho}$ and $N_p$ will be the image of $H^1(G_p, U)$ in $H^1(G_p, Ad^0 \overline{\rho})$. Tables 3 and 4 of [Ram02] show that this works for $\overline{\rho}$ indecomposable, as we are excluding the case of $\overline{\rho}$ flat.

For $\overline{\rho}$ decomposable, the same set $C_p$ and subspace $N_p$ work. Table 2 of [Ram02] computes the dimensions of cohomology groups of $Ad^0 \overline{\rho}$ and the dimension of $H^1(G_p, U)$ can be easily obtained by explicitly computing $H^0(G_p, U)$ and $H^1(G_p, U^*)$ and then using local Tate duality and the local Euler-Poincare characteristic. In all these cases $\dim H^1(G_p, U)$ agrees with $\dim H^1(G_p, Ad^0 \overline{\rho}) - \dim H^2(G_p, Ad^0 \overline{\rho}) + 1$ and the morphism $H^1(G_p, U) \to H^1(G_p, Ad^0 \overline{\rho})$ is injective. Also $H^2(G_p, U) = 0$ which guarantees that $N_p$ preserves $C_p$. □

To ease the statements of the main theorems we make the following definitions.

Definition. A representation $\rho$ of $G_Q$ is nicely nearly ordinary if $\rho|_{G_p}$ is nearly ordinary and moreover one of the following two conditions holds:

- The quotient of the characters appearing in the diagonal of $\overline{\rho}|_{G_p}$ is not the cyclotomic character.
- The quotient of the characters appearing in the diagonal of $\overline{\rho}|_{G_p}$ is the cyclotomic character and $\overline{\rho}|_{G_p}$ is indecomposable and non-flat.

4. Global deformation theory

In this section we prove the one of the main results of this article.

Theorem 4.1. Let $n \geq 2$ be an integer and $\rho_n : G_Q \to GL_2(\mathbb{O}/\pi^n)$ be a continuous representation which is odd and nicely nearly ordinary (see Definition 3) at $p$. Assume that $\text{Im}(\rho_n)$ contains $SL_2(\mathbb{O}/p)$ if $n \geq e$ and that $\rho$ is surjective otherwise. Let $P$ be a set of primes of $\mathbb{Q}$ containing the ramification set of $\rho_n$. For each $v \in P \setminus \{p\}$ fix a local deformation $\rho_v : G_v \to GL_2(\mathbb{O})$ that lifts $\rho_n|_{G_v}$ and is not bad (see Definition 3). Then there exists a continuous representation $\rho : G_Q \to GL_2(\mathbb{O})$ and a finite set of primes $R$ such that:

- $\rho$ lifts $\rho_n$, i.e. $\rho \equiv \rho_n \pmod{\pi^n}$.
- $\rho$ is unramified outside $P \cup R$.
- For every $v \in P$, $\rho|_{I_v} \simeq \rho_v|_{I_v}$ over $GL_2(\mathbb{O})$.
- $\rho$ is nearly ordinary at $p$.
- All the primes of $R$, except possibly one, are congruent to 1 modulo $p$.

The proof of this theorem essentially consists on finding a way to lift $\rho_n$ to characteristic 0 one power of $\pi$ at a time. We will split the proof into two sections, essentially because the local results we have so far are split in two different cases depending on whether the exponent $m$ in each step is big enough or not. Let $\alpha$ be the integer obtained in the following way: for each $v \in P$, Proposition 2.1 gives an integer $\alpha_v$ such that there is a set $C_v$ and a subspace $N_v$ preserving its reductions mod $\pi^{\alpha_v}$ for $n \geq \alpha_v$. Let $\alpha$ be the maximum of the $\alpha_v$’s for $v \in P$. When lifting from $\pi^n$ to $\pi^{m+1}$ for $m \geq \alpha$ we are in a global setting similar to the one in [Ram02]. The existence of the sets $C_q$ and subspaces $N_q$ let us mimic the argument given there. When working modulo $m$ for $m < \alpha$, we do not count with these sets and subspaces for all the primes of $P$ and therefore are unable to overcome the local obstructions. In this case, we will follow the ideas from [KLR05] and will lift the first steps to reach the exponent $\alpha$ by adding a finite number of auxiliary primes at each step.

As $\mathbb{O}$ is the ring of integers of a ramified extension we have that $\mathbb{O}/\pi^2$ is isomorphic to the dual numbers and therefore the projection mod $\pi^2$ of $\rho_n$ defines an element in $H^1(G_Q, Ad^0 \overline{\rho})$ which we will call $f$. Observe that our hypotheses imply that the image of the projection of $\rho_n$ to $\mathbb{O}/\pi^2$ contains $SL_2(\mathbb{O}/\pi^2)$ and thus $f \neq 0$ as an element in $H^1(G_Q, Ad^0 \overline{\rho})$. 

4.1. Getting to mod $\pi^n$. Assume that the exponent $n$ we start with is strictly smaller than the natural number $\alpha$ from Proposition 2.1 (if this is not the case we are done). The idea is to adjust the main argument of [KLR05] to our situation. Recall the following definition.

**Definition.** A prime $q$ is nice for $\overline{p}$ if it satisfies the following properties

- The prime $q$ is not congruent to $\pm 1$ mod $p$.
- The representation $\rho_n$ is unramified at $q$.
- The eigenvalues of $\overline{p}(\sigma_q)$ have ratio $q$.

We say that $q$ is nice for $\rho_n$ if furthermore

- The eigenvalues of $\rho_n(\sigma_q)$ have ratio $q$.

At a nice prime $q$ we consider the set

$$C_q = \{ \text{deformations } \rho \text{ of } \overline{p}|_{G_q} : \rho(\sigma_q) = \left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right) \},$$

and the subspace $N_q \subseteq H^1(G_q, Ad^0 \overline{p})$ generated by the cocycle $u$ sending $\sigma_q$ to $0$ and $\tau_q$ to $\left( \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} \right)$. It is easy to check that $u$ preserves the reductions of elements of $C_q$.

The work on [KLR05] is based on the existence of nice primes that are either null or non null at certain elements of both $H^1(U, Ad^0 \overline{p})$ and $H^1(U, (Ad^0 \overline{p})^*)$ for different sets of primes $U$. We claim that the same arguments work in this settings except for the cases when the element $f \in H^1(G_P, Ad^0 \overline{p})$ attached to $\rho_n$ is involved. We will sort this obstacle by adding the following primes.

**Definition.** We say that a prime $q$ is a special prime for $f$ if

- The representation $\rho_n$ is unramified at $q$.
- $\rho_n(\sigma_q) = \left( \begin{smallmatrix} 1 & \pi \\ 0 & 1 \end{smallmatrix} \right)$.
- The prime $q \equiv 1 \pmod {\pi^n}$.

Note that for such primes $q$ we have that $f|_{G_q} \neq 0$ as an element in $H^1(G_q, Ad^0 \overline{p})$. We need some partial results to state and prove the main result of this case (Theorem 4.7).

**Lemma 4.2.** Let $f, f_1, \ldots, f_r \in H^1(G_Q, Ad^0 \overline{p})$ and $\phi_1, \ldots, \phi_s \in H^1(G_Q, (Ad^0 \overline{p})^*)$ be linearly independent sets.

a) Let $I \subseteq \{1, \ldots, r\}$ and $J \subseteq \{1, \ldots, s\}$. There is a Chebotarev set of primes $v$ such that

- $v$ is nice for $\rho_n$.
- $f_i|_{G_v} \neq 0$ if $i \in I$ and $f_i|_{G_v} = 0$ if $i \notin I$.
- $\phi_j|_{G_v} \neq 0$ if $j \in J$ and $\phi_j|_{G_v} = 0$ if $j \notin J$.

b) Also, there is a Chebotarev set of primes $w$ such that

- $w$ is a special prime for $f$; henceforth $f|_{G_w} \neq 0$.
- $f_i|_{G_w} = 0$ for all $1 \leq i \leq r$.

**Remark.** For special primes we can also define a set $C_v$ of deformations to characteristic $0$ and a subspace $N_v \subseteq H^1(G_v, Ad^0 \overline{p})$ of codimension $\text{dim } H^2(G_v, Ad^0 \overline{p})$ preserving it. This is explicitly done in Lemma 4.1 of [CPL14]. The procedure is the same as the one employed in cases 4.1 and 4.2 of Section 2. We will not reproduce the results here in an effort to preserve elegance.

**Proof.** This is a slight modification of Fact 5 of [KLR05]. The main problem with nice primes in ramified extensions is that if $v$ is a nice prime then $f|_{G_v} = 0$. The use of special primes for $f$ solves this problem, since almost by definition if $v$ is a special prime, $f|_{G_v} \neq 0$. In order to check that Chebotarev conditions at the different $f_i$’s and $\phi_j$’s are disjoint from the condition of being nice for $\rho_n$, and that this last condition only overlaps with extension corresponding to the element $f$, we need to understand the Galois structure of the corresponding extensions. We will prove a ramified version of Lemma 5.8 of [CPL14]. Following the notation of that article (which is the original notation of [Ram99]), let $K = \mathbb{Q}(Ad^0 \overline{p})\mathbb{Q}(\mu_n)$. For each $f_i$ let $L_i$ the extension of $K$ given by its kernel and for each $\phi_j$ let $M_j$ be the corresponding one. Finally let $K' = K \cdot \mathbb{Q}(Ad^0 \rho_n)$ and $L_f$ be the extension of $K$ given by $f$. Let $L = L_f \prod L_i$ and $M = \prod M_i$. We claim that $K' \cap LM = L_f$. 


For this, let $\mathcal{H} = \text{Gal}(K'/K) \subseteq \text{PGL}_2(\mathbb{O}/\pi^n)$ and $\pi_1 : \text{PGL}_2(\mathbb{O}/\pi^n) \to \text{PGL}_2(\mathbb{F})$. Observe that $\mathcal{H}$ consists on the classes of matrices in $\text{Im}(\rho_n)$ which are trivial in $\text{PGL}_2(\mathbb{F})$, i.e. $\mathcal{H} = \text{Im}(\text{Ad}^0\rho_n) \cap \text{Ker}(\pi_1)$. Recall that our hypotheses imply $\text{PSL}_2(\mathbb{O}/\pi^n) \subseteq \text{Im}(\text{Ad}^0\rho_n) \subseteq \text{PGL}_2(\mathbb{O}/\pi^n)$, and therefore $\text{PSL}_2(\mathbb{O}/\pi^n) \cap \text{Ker}(\pi_1) \subseteq \mathcal{H} \subseteq \text{Ker}(\pi_1)$. As $[\text{PSL}_2(\mathbb{O}/\pi^n) : \text{PGL}_2(\mathbb{O}/\pi^n)] = 2$ and $\text{Ker}(\pi_1)$ is a $p$ group we have that $\mathcal{H} = \text{Ker}(\pi_1)$.

Recall that $\text{Gal}(F/K) \cong (\text{Ad}^0\rho)^* \times (\text{Ad}^0\rho^*)^*$ as $\mathbb{Z}[G_Q]$-module and by Lemma 7 of [Ram99], this is its decomposition as $\mathbb{Z}[G_Q]$ simple modules. This implies that $K'/LM$ is the direct sum of the quotients of $\text{Gal}(K'/K) \cong \mathcal{H}$ isomorphic to $\text{Ad}^0\rho$ or $(\text{Ad}^0\rho)^*$. To prove that the only such quotient is $\text{Gal}(L_f/K)$ observe that any surjective morphism $\mathcal{H} \to \text{Ad}^0\rho$ or $(\text{Ad}^0\rho)^*$ must contain $[\mathcal{H} : \mathcal{H}]$ inside its kernel. We will prove in Lemma 4.3 below that such commutator is equal to the subgroup of $\mathcal{H}$ formed by the matrices congruent to the identity modulo $\pi^2$. This finishes the proof as implies that such quotient necessarily factors through $\text{Gal}(L_f/K)$.

\[\square\]

**Lemma 4.3.** If $H \subseteq \text{SL}_2(\mathbb{O}/\pi^n)$ is the subgroup consisting of matrices congruent to the identity modulo $p$ then its commutator subgroup $[H : H]$ is the subgroup $H'$ of $\text{SL}_2(\mathbb{O}/\pi^n)$ formed by the matrices congruent to the identity modulo $\pi^2$.

**Proof.** It is easy to check that $H/H'$ is abelian, implying that $[H : H] \subseteq H'$. For the other inclusion, observe that $H'$ is generated by the set of elements of the form

\[
\begin{pmatrix}
1 & \pi^2 x \\
0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
\pi^2 y & 1
\end{pmatrix}
\text{ and }
\begin{pmatrix}
1 + \pi^2 z & 0 \\
0 & (1 + \pi^2 z)^{-1}
\end{pmatrix}
\]

for $x, y, z \in \mathbb{O}$. It is easy to verify the following identities for any $a, b \in \mathbb{O}$:

- \[
\begin{pmatrix}
1 + \pi & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 + \pi & 0 \\
0 & (1 + \pi)^{-1}
\end{pmatrix} = \begin{pmatrix}
1 - \pi^2 a (1 + \pi) & 0 \\
0 & 1
\end{pmatrix} \in [H : H].
\]

- \[
\begin{pmatrix}
1 & 0 \\
\pi & 1
\end{pmatrix}
\begin{pmatrix}
1 + \pi & 0 \\
0 & (1 + \pi)^{-1}
\end{pmatrix} = \begin{pmatrix}
1 + \pi^2 & 0 \\
0 & (1 + \pi)^{-1}
\end{pmatrix} \in [H : H].
\]

This shows that the first two families of generators of $H'$ lie inside $[H : H]$. In order to prove that $[H : H] = H'$ it only remains to check that $\begin{pmatrix}
1 + \pi^2 z & 0 \\
0 & (1 + \pi^2 z)^{-1}
\end{pmatrix} \in [H : H]$ for any $z \in \mathbb{O}$.

But $\begin{pmatrix}
1 & 0 \\
\pi & 1
\end{pmatrix} = \begin{pmatrix}
1 - \pi^2 z & 0 \\
0 & (1 + \pi^2 z)^{-1}
\end{pmatrix} \in [H : H]$. Multiplying this element by matrices of the form $\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}$ we get $\begin{pmatrix}
1 - \pi^2 z & 0 \\
0 & (1 + \pi^2 z)^{-1}
\end{pmatrix} \in [H : H]$ (as we can raise the power of $\pi$ appearing in the place $(2,1)$ eventually making it $0$ modulo $\pi^n$). The same argument applies to matrices of the form $\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}$ we get $\begin{pmatrix}
1 - \pi^2 z & 0 \\
0 & (1 + \pi^2 z)^{-1}
\end{pmatrix} \in [H : H]$.

\[\square\]

Lemma 4.3. will let us prove the existence of auxiliary primes that kill global obstructions. We introduced special primes because otherwise we would have not been able to modify the behaviour of $f$.

**Lemma 4.4.** Let $\rho_n$ and $P$ as before. Then there exists a finite set $P'$ consisting of nice primes for $\rho_n$ and eventually one special prime for $f$ such that $\Pi^n_1_{P_1} (\text{Ad}^0\bar{\rho})$ and $\Pi^n_2_{P_2} (\text{Ad}^0\bar{\rho})$ are both trivial.

**Proof.** If $f \notin \Pi^n_1 (\text{Ad}^0\bar{\rho})$ this follows from taking basis $\{f_1, \ldots, f_r\}$ and $\{\phi_1, \ldots, \phi_s\}$ of $\Pi^n_1 (\text{Ad}^0\bar{\rho})$ and $\Pi^n_2 (\text{Ad}^0\bar{\rho})$ respectively and choosing, by applying Lemma 4.2 sets of nice primes $q_1, \ldots, q_r$ and $q'_1, \ldots, q'_s$ such that

- $f_i|G_{q_j} = 0$ if $i \neq j$ and $f_i|G_{q_j} \neq 0$.
- $\phi_i|G_{q'_j} = 0$ if $i \neq j$ and $\phi_i|G_{q'_j} \neq 0$.

If, otherwise, $f \in \Pi^n_2 (\text{Ad}^0\bar{\rho})$, we do the same but taking a special prime for $f$ instead of a nice prime.

\[\square\]

From the previous lemmas, we can assume that $\Pi^n_1 (\text{Ad}^0\bar{\rho})$ and $\Pi^n_2 (\text{Ad}^0\bar{\rho})$ are both trivial by enlarging $P$ if necessary. This imply, as in [KLR05], the following two key propositions.
Proposition 4.5. Let $S$ be a finite set of primes and $\rho_m : G_S \to \text{GL}_2(\mathbb{O}/\pi^n)$ a continuous representation such that $\text{III}^1_S(Ad^0\hat{\rho}) = \text{III}^2_S(Ad^0\hat{\rho}) = 0$. Then there exists a set $Q$ of nice primes for $\rho_m$ such that the map

$$H^1(G_{S\cup Q}, Ad^0\hat{\rho}) \to \bigoplus_{v \in S} H^1(G_v, Ad^0\hat{\rho})$$

is an isomorphism.

Proof. Given the existence of auxiliary primes, this is just Lemma 8 of [KLR05].

Proposition 4.6. Let $\rho_m$, $S$ and $Q$ as in the Proposition 4.5. For each $q_i \in Q$ pick an element $h_i \in H^1(G_{q_i}, Ad^0\hat{\rho})$. Then there is a finite set $T$ of nice primes for $\rho_m$ and an element $g \in H^1(G_{S\cup Q\cup T}, Ad^0\hat{\rho})$ satisfying

- $g|_{G_{q_i}} = 0$ for $v \in S$.
- $g|_{\sigma_{q_i}} = h_i|_{\sigma_{q_i}}$ for $q_i \in Q$.
- $g|_{\sigma_v} = 0$ for $v \in T$.

Proof. This is Corollary 11 of [KLR05], which follows from Lemmas 8 and 9 and Proposition 10 from the same work. It can be easily checked that the proofs given in that paper adapt well to our setting.

We are now able to state and prove the main theorem of this section. Recall from for $v$ a nice prime there is a set $C_v$ of deformations to characteristic 0 and a subspace $N_v \subseteq H^1(G_v, Ad^0\hat{\rho})$ preserving its reductions.

Theorem 4.7. Let $\rho_n$ and $P$ as in Theorem 4.4 and let $\alpha$ be an integer greater or equal than $n$. Pick, for each $v \in P$ a lift

$$\rho_{v,\alpha} : G_v \to \text{GL}_2(\mathbb{O}/\pi^n)$$

of $\rho_n|_{G_v}$. Then, there is a finite set of nice primes $P'$ for $\rho_n$ and a lift

$$\rho_n : G_{P\cup P'} \to \text{GL}_2(\mathbb{O}/\pi^n)$$

of $\rho_n$ such that $\rho_n|_{G_v} \simeq \rho_{v,\alpha}$ for every $v \in P$ and $\rho_n|_{G_v}$ is a reduction of some member of $C_v$ for every $v \in P'$.

Proof. We will prove the theorem by induction on $\alpha$. If $\alpha = n$ the statement is trivial. Assume the theorem is true for $\alpha = m$, and apply it with the collection of local deformations given by the reductions mod $\pi^m$ of the local representations $\rho_{v,m+1}$. Then, there is a lift $\rho_m : G_{P\cup P'} \to \text{GL}_2(\mathbb{O}/\pi^m)$ such that $\rho_m|_{G_v} = \rho_{v,m+1}$ (mod $\pi^m$), where $P'$ consists of nice primes for $\rho_n$. We will add two sets of nice primes in order to first get a lift of $\rho_m$ to $\mathbb{O}/\pi^{m+1}$ and then locally adjust this lift. Since $\text{III}_P^1(Ad^0\hat{\rho}) = 0$, $\rho_m$ has no global obstructions. Also observe that $\rho_m$ is unobstructed at the primes of $P$, as $\rho_m|_{G_v}$ lifts to $\rho_{v,m+1}$ and at the primes of $P'$ too, as $\rho_m|_{G_v}$ is the reduction of some member of $C_v$. Therefore $\rho_m$ is both globally and locally unobstructed implying that it lifts to some

$$\tilde{\rho}_{m+1} : G_{P\cup P'} \to \text{GL}_2(\mathbb{O}/\pi^{m+1}).$$

To complete the proof, we need to fix the local behaviour of $\tilde{\rho}_{m+1}$. We will do this in two steps. First of all, pick for each $v \in P \cup P'$ a class $u_v \in H^1(G_v, Ad^0\hat{\rho})$ such that

- $(1 + \pi^m u_v)\tilde{\rho}_{m+1}|_{G_v} \simeq \rho_{v,m+1}$ for $v \in P$.
- $(1 + \pi^m u_v)\tilde{\rho}_{m+1}|_{G_v}$ is a reduction of a member of $C_v$ for $v \in P'$.

Now, let $Q$ be the set of nice primes produced by applying Proposition 4.5 to $S = P \cup P'$. As the map

$$H^1(G_{P\cup P'\cup Q}, Ad^0\hat{\rho}) \to \bigoplus_{v \in P\cup P'} H^1(G_v, Ad^0\hat{\rho})$$
is an isomorphism, there is a class \( g_{1} \in H^{1}(G_{P \cup P' \cup Q}, Ad^{0} \bar{\rho}) \) such that \( g_{1}|_{G_{v}} = u_{v} \) for all \( v \in P \cup P' \). Acting by this element on \( \bar{\rho}_{m+1} \) fixes its local shape at the places of \( P \cup P' \) but may ruin it at the newly added primes of \( Q \). We will solve this issue by adding a second set of auxiliary primes.

We pick, for each \( q_{i} \in Q \) a class \( h_{i} \in H^{1}(G_{q_{i}}, Ad^{0} \bar{\rho}) \) such that
\[
(1 + \pi^{m}(h_{i} + g_{1}))\bar{\rho}_{m+1}(\sigma_{q_{i}}) = \begin{pmatrix} q_{i} & 0 \\ 0 & 1 \end{pmatrix}.
\]

Let \( T \) and \( g_{2} \) be respectively the set of nice primes and the element of \( H^{1}(G_{P \cup P' \cup Q \cup T}, Ad^{0} \bar{\rho}) \) obtained from applying Proposition 4.7 with \( S = P \cup P' \) and \( Q \) and \( h_{i} \) as above. It is easy to check that
- \( (1 + \pi^{m}g)\bar{\rho}_{m+1}|_{G_{v}} \simeq \rho_{v,m+1} \) for \( v \in P \).
- \( (1 + \pi^{m}g)\bar{\rho}_{m+1}|_{C_{v}} \) for \( v \in P' \cup Q \cup T \).

It follows that \( \rho_{m+1} = (1 + \pi^{m}g)\bar{\rho}_{m+1} \) satisfies what we need, completing the proof. \( \square \)

4.2. Exponent \( \alpha \) and above. Assume we have a representation \( \rho_{\alpha} \) as in Theorem 4.1 with \( n \geq \alpha \) (since otherwise we apply Theorem 4.7). To ease the notation, let \( P \) denote the set \( P \cup P' \) if we applied Theorem 4.7.

Recall from Sections 2 and 3 that for exponents bigger than \( \alpha \) we have defined for each \( v \in P \) a set of deformations \( C_{v} \) of \( \rho_{\alpha} \) to characteristic 0 and a subspace \( N_{v} \subseteq H^{1}(G_{v}, Ad^{0} \bar{\rho}) \) such that \( N_{v} \) preserves the reductions of the elements in \( C_{v} \), in the sense of Proposition 2.1. They also satisfy that
\[
\dim N_{v} = \dim H^{1}(G_{v}, Ad^{0} \bar{\rho}) - \dim H^{2}(G_{v}, Ad^{0} \bar{\rho}) \text{ for } v \in P \setminus \{p\} \text{ and } \dim N_{p} = \dim H^{1}(G_{P}, Ad^{0} \bar{\rho}) - \dim H^{2}(G_{P}, Ad^{0} \bar{\rho}) + 1.
\]

In this setting, we can mimic the arguments of [Ram02] in our situation with some minor modifications. We start by collecting a series of results that will prove useful.

Lemma 4.8. Let \( r = \dim \Pi^{+}_{v}(Ad^{0} \bar{\rho})^{*} \) and \( s = \sum_{v \in P} \dim H^{2}(G_{v}, Ad^{0} \bar{\rho}) \). Then
\[
\dim H^{1}(G_{P}, Ad^{0} \bar{\rho}) = r + s + 2.
\]

Proposition 4.9. Let \( \{f, f_{1}, \ldots, f_{r+s+1}\} \) be a basis of \( H^{1}(G_{P}, Ad^{0} \bar{\rho}) \), where \( f \) is the element attached to \( \rho_{\alpha} \) mod \( \pi^{2} \). There is a set \( Q = \{q_{1}, \ldots, q_{r}\} \) of nice primes for \( \rho_{\alpha} \) not in \( P \) such that:
- \( \Pi^{+}_{v \cup Q}(Ad^{0} \bar{\rho})^{*} = 0 \) and \( \Pi^{+}_{v \cup Q}(Ad^{0} \bar{\rho}) = 0 \).
- \( f|_{G_{q_{j}}} = 0 \) for \( i \neq j \) and \( f|_{G_{q_{j}}} = 0 \) for all \( j \).
- \( f|_{G_{q_{j}}} \notin N_{q_{j}} \).
- The inflation map \( H^{1}(G_{P}, Ad^{0} \bar{\rho}) \to H^{1}(G_{P \cup Q}, Ad^{0} \bar{\rho}) \) is an isomorphism.

Proof. Lemma 4.8 is just the Lemma before Lemma 10 in [Ram02]. The proof of Proposition 4.9 mimics that of Fact 16 of [Ram02]. Observe that, in the spirit of what is remarked in the proof of Proposition 4.2, the condition for a prime \( q \) being nice for \( \rho_{\alpha} \) is not compatible with \( f|_{G_{q}} \notin N_{q} \) as every subspace \( N_{v} \) is chosen such that \( f|_{G_{v}} \in N_{v} \). In particular, \( f \) is always in the kernel of the restriction map
\[
H^{1}(G_{P \cup Q}, Ad^{0} \bar{\rho}) \to \bigoplus_{v \in P \cup Q} H^{1}(G_{v}, Ad^{0} \bar{\rho})/N_{v}.
\]
As we have checked in Proposition 4.2, conditions for the rest of the \( f_{i} \)'s are independent from being \( \rho_{\alpha} \)-nice, so the same proof as in [Ram02] works. \( \square \)

Lemma 4.10. Let \( \langle f, f_{1}, \ldots, f_{d} \rangle \) be the kernel of the map
\[
H^{1}(G_{P \cup Q}, Ad^{0} \bar{\rho}) \to \bigoplus_{v \in P} H^{1}(G_{v}, Ad^{0} \bar{\rho})/N_{v}.
\]
Then \( r \geq d \).

Proof. Follows from the formulas \( \dim \oplus_{v \in P} H^{1}(G_{v}, Ad^{0} \bar{\rho})/N_{v} = s + 1 \) and \( \dim H^{1}(G_{P \cup Q}, Ad^{0} \bar{\rho}) = r + s + 2 \). \( \square \)

Lemma 4.11. There is a finite set of nice primes \( \{t_{r+1}, \ldots, t_{d}\} \) for \( \rho_{\alpha} \) such that
local deformation $\rho$ of primes $Q$

Theorem 5.1. Lift with a previously fixed weight seems out of the scope of this work. We will prove the following.

Proposition 4.12 gives a family of lifts of $\rho$ gives a family of lifts of $\rho$ .\vspace{3mm}

□

Proof. This is Lemma 14 of [Ram02], which relies on Proposition 10. It is easy to check that the proof given there adapts to the ramified setting, using the results we have available so far, as it does not involve picking nice primes that satisfy properties at $f$.

The set $Q \cup T$ will serve as the auxiliary set for Theorem 4.1. So far we have the following properties:

- For $1 \leq i \leq r$: $f_i|_{G_v} = 0$ for all $v \in Q \cup T$ except $q_i \in Q$ for which $f_i|_{G_{q_i}} \notin N_{q_i}$.
- For $r + 1 \leq i \leq d$: $f_i|_{G_v} = 0$ for all $v \in Q \cup T$ except $t_i \in T$ for which $f_i|_{G_{t_i}} \notin N_{t_i}$.
- The restriction $(f_{d+1}, \ldots, f_{d+s+1}) \to \bigoplus_{v \in P} H^1(G_v, Ad^0\bar{\rho})/N_v$ is an isomorphism.
- $f|_{G_v} \in N_v$ for every $v \in P \cup Q \cup T$.

It easily follows from these properties that

Proposition 4.12. The map

$$H^1(G_{P;Q\cup T}, Ad^0\bar{\rho}) \to \bigoplus_{v \in P\cup Q\cup T} H^1(G_v, Ad^0\bar{\rho})/N_v$$

is surjective and has one dimensional kernel generated by $f$.

From this, we can easily deduce Theorem 4.1 in the same way as Theorem 1 is proved in [Ram02]. Moreover we have the following result

Theorem 4.13. Let $\rho_n : G_Q \to GL_2(\mathcal{O}/\mathfrak{p}^n)$ and $\rho_v : G_v \to GL_2(\mathcal{O})$ as in Theorem 4.1. Consider the collection $\mathcal{L}$ of deformation conditions given by the pairs $(C_v, N_v)$ for $v \in P \cup Q \cup T$. Then the deformation problem with fixed determinant and local conditions $\mathcal{L}$ has universal deformation ring $R_n \simeq Z[\mathcal{O}[X]]$.

Proof. Proposition 4.12 tells us that $H^1_{\mathcal{L}}(G_{P;Q\cup T}, Ad^0\bar{\rho}) = (f)$. As $\Pi^2_{P;Q\cup T}(Ad^0\bar{\rho}) = 0$, we also know that the problem is unobstructed. This proves the theorem. □

5. Modularity

So far we have constructed, for a mod $\mathfrak{p}^n$ representation $\rho_n$, which is nearly ordinary at $p$, a global deformation $\rho_0 : G_0 \to GL_2(Z_p[[X]])$ that lifts $\rho_0$ and is also nearly ordinary at $p$. This gives a family of lifts of $\rho_n$ to rings of dimension and characteristic 0. The purpose of this section is to prove that when $\rho_n$ is ordinary at $p$, at least one of these lifts is modular. However, getting a lift with a previously fixed weight seems out of the scope of this work. We will prove the following.

Theorem 5.1. Let $p$ be a prime, $\mathcal{O}$ the ring of integers of a finite extension $K/\mathbb{Q}_p$, with ramification degree $e > 1$ and $\pi$ its local uniformizer. Let $\rho_n : G_Q \to GL_2(\mathcal{O}/\mathfrak{p}^n)$ be a continuous representation satisfying

- $\rho_n$ is odd.
- $\text{Im}(\rho_n)$ contains $SL_2(\mathcal{O}/\mathfrak{p})$ if $n > e$ and $\rho_n$ is surjective otherwise.
- $\rho_n$ is nicely ordinary (both nicely nearly ordinary and ordinary) at $p$.

Let $P$ be a set of primes containing the set of ramification of $\rho_n$, and for each $v \in P$ pick a local deformation $\rho_v : G_v \to GL_2(\mathcal{O})$ lifting $\rho_n|_{G_v}$, which is not bad. Then there exists a finite set of primes $Q$ and a continuous representation $\rho : G_{P;Q} \to GL_2(\mathcal{O})$ such that

- $\rho$ lifts $\rho_n$, i.e. $\rho \equiv \rho_n \pmod{\mathfrak{p}^n}$.
- $\rho$ is modular.
- For every $v \in P$, $\rho|_{I_v} \simeq \rho_v|_{I_v}$ over $GL_2(\mathcal{O})$.
- For every $q \in Q$, $\rho|_{I_q}$ is unipotent and $q \not\equiv 1 \pmod{p}$ for all but possibly one $q \in Q$.
- $\rho$ is ordinary at $p$. 

Proof. Theorem 4.13 implies the existence of a universal ring $R_u \simeq \mathbb{Z}_p[[X]]$ that parametrizes nearly ordinary deformations with fixed determinant $\omega \chi$ and satisfying certain local conditions at the primes of $P \cup Q$.

Observe that for each morphism of local $\mathbb{Z}_p$-algebras $\gamma : R_u \to \mathcal{O}$ we have a nearly ordinary deformation $\rho_\gamma$ with coefficients in $\mathcal{O}$. If we look at the local structure at $p$ of this deformation we find that

$$\rho_\gamma|_p \simeq \begin{pmatrix} \omega_1 \psi_1 \chi^b & 0 \\ \omega_2 \psi_2 \chi^{k-b} & \ast \end{pmatrix}$$

for $\omega_1$, $\omega_2$ unramified characters, $\psi_1$ and $\psi_2$ of finite order and $b \in \mathbb{Z}_p$ (using that any $p$-adic character can be written as a power of the cyclotomic character times a character of finite order times an unramified character). Twisting $\rho_\gamma$ by $\psi_2^{-1} \chi^{b-k}$ we get, for each $\gamma \in \text{Hom}_{\text{loc}}-\mathbb{Z}_p(R_u, \mathcal{O})$ an ordinary deformation of $\overline{\rho}$. Denote the representation $\psi_2^{-1} \chi^{b-k} \rho_\gamma$ by $\hat{\rho}_\gamma$.

As we are asking $\mathcal{O}$ to be modular, the work of [DT94] ensures that $R_u$ contains a twist of characteristic zero modular points. This is, there is a morphism $\gamma_k : R_u \to \mathbb{Z}_p$ such that the twisted deformation $\hat{\rho}_k$ is ordinary of weight $k$ (and therefore modular). Specifically, [DT94] guarantees that $\mathcal{O}$ has an ordinary lift of classical weight and arbitrary determinant. After twisting this lift by the corresponding power of the cyclotomic character, it lies in our family of deformations with fixed determinant. This implies that the family of representations that we have constructed is part of the Hida family of $\hat{\rho}_k$. Let $\mathcal{H}$ be this Hida family, then there is a morphism $\Omega : \text{Hom}_{\text{loc}}-\mathbb{Z}_p(R_u, \mathcal{O}) \to \mathcal{H}$. As $\rho_\gamma$ admits different lifts to $\mathcal{O}$, this morphism is not constant. Recall this Hida family is equipped with his corresponding weight map $w : H \to W$.

On the other hand, we have a morphism $\theta : \mathbb{Z}_p[[X]] \to \mathcal{O}/\pi^n$ which induces $\rho_\gamma$. We know that any morphism $\hat{\theta} : \mathbb{Z}_p[[X]] \to \mathcal{O}$ that lifts $\theta$ induces a lift $\rho_\hat{\theta}$ of $\rho_\gamma$ to $\mathcal{O}$. But $\text{Hom}_{\text{loc}}-\mathbb{Z}_p(\mathbb{Z}_p[[X]], \mathcal{O}) \simeq \mathcal{M}_\mathcal{O}$ (as defining such a morphism amounts to choosing an element of the maximal ideal of $\mathcal{O}$ where to send $X$) and the set of morphisms $\theta$ that lift $\theta$ correspond to an open set $\mathcal{U} \subseteq \mathcal{M}_\mathcal{O}$ under this identification.

Overall, if we restrict the previously constructed morphism we have $\Omega|_\mathcal{U} : \mathcal{U} \to \mathcal{H}$. The image of this morphism must be an open set (as $\Omega$ is analytic), and if we compose it with the weight map we get an open set $w \circ \Omega(\mathcal{U}) \subseteq W$. This open set necessarily contains a classical point, and any pre-image of this classical point gives raise to an ordinary lift of $\rho_\gamma$ of integer weight which we call $\rho$. Finally, by the main theorem of [SW01], $\rho$ is modular. \qed

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