All januarials constructed from Hecke groups

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Abstract

Professor Graham Higman defined januarial as a special instance of map constructed from embedding of a coset diagram for an action of \(\Delta(2, \ell, k)\), on finite sets yielding exactly two orbits of the product of the two generators, having equal sizes. In this paper we determine a condition for the existence of a januarial from \(\Delta(2, \ell, k)\), the quotients of Hecke groups \(H_\Lambda\), when acting on the projective lines over finite fields \(\text{PL}(F_q)\). We develop a method to find all the januarials from Hecke groups \(H_\Lambda\), when the triangle group \(\Delta(2, \ell, k)\) acts on \(\text{PL}(F_q)\). We evolve a formula for calculating genus of coset diagram depending on the fixed points. By using it, we determine genus of the januarials.

1. Introduction

It is well known that triangle group \(\Delta(m, \ell, k)\) has a presentation \(\langle s, t : s^m = t^\ell = (st)^k = 1 \rangle\) (e.g. see [2], [3] and [7]). For januarials, we need to fix \(m = 2\).

The widely studied Hecke groups \(H_\Lambda\) admit a presentation \(\langle a, b : a^2 = (ab)^\ell = 1 \rangle\) where \(a : z \to \frac{-1}{z}, b : z \to z + \Lambda\ell\) and \(\Lambda\ell = 2\cos(\pi/\ell)\). In another interpretation \(H_\Lambda \cong C_2 \otimes C_\ell\), that is, \(H_\Lambda = \Delta(2, \ell, \infty)\). Thus \(\Delta(2, \ell, k)\) are homomorphic images of \(H_\Lambda\). Throughout this paper we consider triangle groups as quotients of \(H_\Lambda\).

Let \(q\) be a prime-power \(p^r\). Then the projective line over Galois field \(F_q\), contains elements of \(F_q\) together with \(\infty\), and is denoted by \(\text{PL}(F_q)\).

A coset diagram is a graphical representation of an action of any group with finite presentation on a set. For instance, a coset diagram for the permutation action of Hecke group on projective line has \(\ell - \text{gon}\) for each cycle of \(y\) and an edge for each transposition of \(x\).

Genus of a connected orientable surface is the number of handles on it. Genus of a graph is the minimal integer \(m\) such that the graph can be drawn on the surface of genus \(m\) without crossings [10].

Let \(H_\Lambda \cong G = \langle x, y : x^2 = y^\ell = 1 \rangle\) where \(x, y\) are linear fractional transformations \(z \to \frac{az + b}{cz + d}\) where \(a, b, c\) and \(d \in \mathbb{Z}\) and \(ad - bc = 1\). Let \(\alpha\) be a non-degenerate homomorphism from \(G\) to \(\text{PGL}(2, q)\). The two non-degenerating homomorphisms \(\alpha\) and \(\beta\) are said to be conjugates if there exist an inner automorphism \(\rho\) such that \(\alpha \rho = \beta\). Let \(\bar{x} = aX, \bar{y} = aY\) and \(X, Y\) be the matrices representing \(x\) and \(y\) respectively. Then \(\theta = (trXY^2)/\det XY\) is invariant of the conjugacy classes of \(a\).
If \( S_1, S_2, S_3, \ldots, S_n \) are finite sets then inclusion-exclusion principle is symbolically expressed as

\[
|S_1 \cup S_2 \cup S_3 \cup \cdots \cup S_n| = \sum_{i=1}^{n} |S_i| - \sum_{1 \leq i < j \leq n} |S_i \cap S_j| + \sum_{1 \leq i < j < k \leq n} |S_i \cap S_j \cap S_k| - \cdots + (-1)^{n+1} |S_1 \cap S_2 \cap S_3 \cap \cdots \cap S_n|. \tag{1}
\]

As every integer can be uniquely written as the product of primes so let \( n = p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s} \). Using inclusion-exclusion principle the Euler’s phi function \( \phi(n) \) is expressible as (see [9])

\[
\phi(n) = n - \sum_{i=1}^{d} \frac{n}{p_i} + \sum_{1 \leq i < j \leq s} \frac{n}{p_i p_j} - \cdots + (-1)^{s-1} \frac{n}{p_1 p_2 \cdots p_s}. \tag{2}
\]

Professor Graham Higman FRS introduced januarials in his last lectures in 2001. He conceived the idea of januarials during his work on Hurwitz groups, that are non trivial quotients of \( \Delta(2,3,7) \) group, which he never published. After his death a brief account of his lectures concerning januarials was published [1].

In [1] associates are used to find januarials from the modular group. But his method does not tell, how many times one should repeat the process of taking associates to ultimately get a januarial.

In this paper we prove when januarials exist and then use some pre-existing tools to get an appropriate method of finding januarials. Further we show all the distinct januarials constructible from the Hecke group \( H_{\Lambda} \).

2. A condition for the existence of januarials

In this section, we determine when januarials exist in the action of triangle groups that are the quotients of Hecke group \( H_{\Lambda} \) on \( PL(F_q) \).

**Theorem 1** A januarial exists in the action of quotients of \( H_{\Lambda} \), \( \Delta(2, \ell, k) \), on \( PL(F_q) \) if and only if \( k = (q + 1)/2 \), for all \( q > 3 \) where \( q = p^s \) and \( p \neq 2 \).

**Proof:** If \( k = (q + 1)/2 \) then \((xy)^{(q+1)/2} = 1\) and the action of \( xy \) on \( PL(F_q) \) gives the permutation \( \rho_{xy} \in S_{q+1} \) such that \((\rho_{xy})^{(q+1)/2} = 1\).

This means that the lengths \( r_i \) of cycles of \( \rho_{xy} \) are divisors of \((q + 1)/2\). Here two cases arise: when \( r_i \leq 2 \) for all \( i \) and when \( 2 < r_m < k \) for any cycle.

Case I: If \( r_i \leq 2 \) then \((xy)^2 \), which is linear fractional transformation, fixes all the points of \( PL(F_q) \) since the only linear fractional transformation which fixes more than two vertices is the identity transformation [4] so \((xy)^2 \) is trivial, that is \((xy)^2 = 1\). But order of \( xy \) which is \( k \) for \( q > 3 \) and \( k > 2 \), leads to a contradiction.

Case II: If \( 2 < r_m < k \) for any cycle then \((xy)^{r_m} \) fixes \( r_m > 2 \) points of the cycle since the only linear fractional transformation which fixes more than two vertices is the identity transformation [4] so \((xy)^{r_m} \) is trivial that is \((xy)^{r_m} = 1\). But order of \( xy \) which is \( k \) and \( k > r_m \), is a contradiction.

So the only possibility is that \( r_i = k \) for all \( i \). This implies that \( \rho_{xy} \) has exactly two cycles of length \( k \). That is, there are exactly two orbits of \( \langle xy \rangle \) of same size \( k = (q + 1)/2 \). Hence the result.
The converse follows from the definition of januarials.

Since the two equal sized orbits in januarials are the result of action of the cyclic group \( \langle xy \rangle \) on \( PL(F_p) \) so the question arises, does \( k = (q + 1)/2 \) assure the existence of two equal sized orbits under the action of every cyclic group of order \( k \) on \( PL(F_q) \)? We answer the question in the following theorem.

**Theorem 2** The action of the cyclic subgroup \( C_k \), of \( H_{\Lambda, \ell} \), on \( PL(\mathbb{F}_q) \) when \( k = (q + 1)/2 \) give exactly two orbits of the same size \( k \), for all \( q > 3 \) where \( q = p' \) and \( p \neq 2 \).

**Proof:** The proof is the same as that of Theorem 1.

3. Januarials from \( PGL(2, q) \)

Next we use the above condition to find januarials through a new method. We use the procedure described by F. Shaheen in [8]. Let \( X, Y \) denote the elements in \( GL(2, \mathbb{Z}) \) which corresponds to the elements \( x, y \) in \( G \). Then \( X, Y \) will satisfy the relations

\[
X^2 = Y^\ell = \lambda I
\]

for some scalar \( \lambda \). We choose \( X, Y \) to be the matrices

\[
\begin{bmatrix} a & ci \\ c & -a \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} e & fi \\ f & b-e \end{bmatrix}
\]

respectively, where \( i \neq 0 \), \( a, c, e, f, i \) belong to \( F_q \) and \( b \equiv s(mod p) \) for some \( s \) in \( F_q \).

Let \( \Delta \) be the determinant of \( X \). We assume that the determinant of \( Y \) is 1, so that we have

\[-(a^2 + ic^2) = \Delta \neq 0 \]

and

\[1 + if^2 + e^2 - eb = 0.\]

Let \( r \) be the trace of \( XY \), so that

\[r = a(2e - b) + 2icf.\]

Also

\[\det(XY) = \det X \det Y = \Delta.\]

We define a parameter \( \theta \) as \( r^2/\Delta \). For a pair \( \bar{x}, \bar{y} \) in \( PGL(2, q) \), satisfying the relations

\[\bar{x}^2 = \bar{y}^\ell = 1.\]

We denote by \( D(\theta, q, \ell) \), where \( \theta \) belongs to \( F_q \), the coset diagram corresponding to the action of \( G \) on \( PL(F_q) \).

Now by the condition in Theorem 1, we need to consider only those coset diagrams in which every vertex is fixed by \( (xy)^k \). We consider the case for \( (XY)^k = \lambda I \), for some scalar \( \lambda \).

By the following equation, given by Q. Mushtaq and F. Shaheen in [6],

\[
(XY)^k = \frac{1}{\Delta} \left( \begin{array}{cc} (k-1) & 1 \\ k-2 & 1 \end{array} \right) \Delta - \lambda \left( \begin{array}{cc} k-3 & 1 \\ 2 & 0 \end{array} \right) \Delta^2 - \ldots |XY
\]

we get \( (XY)^k = \lambda I \), for some scalar \( \lambda \), if and only if
\begin{equation}
\left(\frac{k - 1}{\phi}\right)k^2 - \left(\frac{k - 2}{\phi}\right)k + \left(\frac{k - 3}{\phi}\right) = 0.
\end{equation}

Substituting \( r^2 = \theta \Delta \) in the above equation and simplifying it we get an equation in \( \theta \) say \( g_k(\theta) \). Let \( \theta_{\ell,s} \) be the roots of \( g_k(\theta) \). Now using these \( \theta_{\ell,s} \) and by backward substitution we get \( X \) and \( Y \), and ultimately \( x, y \in \text{PGL}(2, \mathbb{Z}) \).

**Theorem 3** The action of quotients of \( H_\Delta, \Delta(2, \ell, k) \), on \( \text{PL}(F_p) \) for \( k = (p + 1)/2 \), yield januarials for all those values of \( \theta \) that are the roots of \( g_k(\theta) \) excluding those of \( g_{k/d}(\theta) \) for all \( d \mid k \).

**Proof:** The roots of \( g_{k/d}(\theta) \) which are also the roots of \( g_k(\theta) \) satisfy \((xy)^{k/d} = 1\). So there are 2d orbits of \( xy \) (by the arguments as in the proof of Theorem 1) and these values of \( \theta \) therefore do not yield januarials. All other values of \( \theta \) yield januilar because they give exactly two orbits of \( (xy) \) of the same size.

Now we find how many \( \ell - \text{januarials} \) exist and determine all the januarials that can be constructed from Hecke groups \( H_\Delta \).

4. All \( \ell \)-Januarials from \( \text{PGL}(2,q) \)

The following lemma is taken from [1]:

**Lemma 4** The number of conjugacy classes of elements in \( \text{PGL}(2,q) \) of order \((q + 1)/2\) is \( 1/2\phi((q + 1)/2) \). If \( z \) is any element of order \((q + 1)/2\) in \( \text{PGL}(2,q) \), then every one of these conjugacy classes intersects the subgroup generated by \( z \) in \( \{z^i \mid z \} \) for exactly one \( i \) coprime to \((q + 1)/2\).

The next result was first proved by Q. Mushtaq in [3] for \( y \) of order 3 and then generalized for \( y \) of order \( \ell \) by F. Shaheen in [8] as:

**Theorem 5** Any element \( g \) whose order is not equal to 1, 2 or 6, of \( \text{PGL}(2,q) \) is the image of \( xy \) under some non-degenerate homomorphism from \( G \) to \( \text{PGL}(2,q) \).

From the above two results we can easily see that:

**Corollary 6** For any prime \( p > 3 \), the number of distinct \( \ell \)-januarials constructible from \( \text{PGL}(2,p) \) is \( 1/2\phi((p + 1)/2) \).

We also show that the number \( \ell \)-januarials constructible from \( \text{PGL}(2,p) \) using the method described in Section 2 is \( 1/2\phi(k) \).

**Theorem 7** The number of roots, \( N \), of \( g_k(\theta) \) excluding those of \( g_{k/d}(\theta) \), for all \( d \in \mathbb{Z} \) such that \( d \mid k \), is given by

\[ N = \frac{1}{2}\phi(k). \] (3)

**Proof:** Let \( N_k \) denote the set of roots of \( g_k(\theta) \). Since every positive integer can be uniquely written in the form of product of primes so let

\[ k = p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}, \]

where \( p_{i,r} \) are primes such that \( p_1^{r_1} < p_2^{r_2} < \cdots < p_s^{r_s} \) and \( r_{i,s} \) are integers.

\( N_{k/p_i} \cup N_{k/p_2} \cup \cdots \cup N_{k/p_s} \) contain the roots of \( g_{k/d}(\theta) \), for all \( d \in \mathbb{Z} \) such that \( d \mid k \), as \( N_{k/d} \subset N_{k/p_i} \), for at least one \( i \in \{1, 2, \ldots, s\} \). So

\[ N = |N_k| - \left| N_{k/p_1} \cup N_{k/p_2} \cup \cdots \cup N_{k/p_s} \right|. \]
Now by equation (1)

\[ |N_{k/p_1} \cup N_{k/p_2} \cup \cdots \cup N_{k/p_s}| = \sum_{i=1}^{s} |N_{k/p_i}| - \sum_{1 \leq i < j \leq s} |N_{k/p_i} \cap N_{k/p_j}| + \sum_{1 \leq i < j \leq s} |N_{k/p_i} \cap N_{k/p_j} \cap \cdots \cap N_{k/p_s}| \]

so

\[ N = |N_k| - \sum_{i=1}^{s} |N_{k/p_i}| + \sum_{1 \leq i < j \leq s} |N_{k/p_i} \cap N_{k/p_j}| - \sum_{1 \leq i < j \leq s} |N_{k/p_i} \cap N_{k/p_j} \cap \cdots \cap N_{k/p_s}|. \]

Now

\[ N_{k/p_i} \cap N_{k/p_j} = N_{k/p_i} \cap N_{k/p_j}. \]

Similarly

\[ N_{k/p_i} \cap N_{k/p_j} \cap \cdots \cap N_{k/p_s} = N_{k/p_{i_2} \cdots p_s}. \]

and equation (5) becomes

\[ N = |N_k| - \sum_{i=1}^{s} |N_{k/p_i}| + \sum_{1 \leq i < j \leq s} |N_{k/p_i} \cap N_{k/p_j}| - \sum_{1 \leq i < j \leq s} |N_{k/p_i, p_j, p_s}| + \cdots + (-1)^s |N_{k/p_{i_2} \cdots p_s}|. \]

The degree of \( g_k (\theta) \) and hence \( |N_k| \) is \((k - 1)/2\) when \( k \) is odd and \( k/2 - 1 \) when \( k \) is even. So there are two cases.

Case I: When \( k \) is odd.

\[ |N_k| = (k - 1)/2, \]
\[ |N_{k/p_i}| = (k/p_i - 1)/2, \]
\[ |N_{k/p_i, p_j, p_s}| = (k/p_i p_j - 1)/2 \]

and

\[ |N_{k/p_i} \cap N_{k/p_j} \cap \cdots \cap N_{k/p_s}| = (k/p_1 p_2 \cdots p_s - 1)/2. \]

Then equation (5) becomes

\[ N = \frac{(k - 1)/2 - \sum_{i=1}^{s} (k/p_i - 1)/2 + \sum_{1 \leq i < j \leq s} (k/p_i p_j - 1)/2 - \sum_{1 \leq i < j \leq s} (k/p_i p_j p_i - 1)/2}{2} + \cdots + (-1)^s(k/p_1 p_2 \cdots p_s - 1)/2 \]

implying that

\[ N = \frac{1}{2} \left[ (k - 1) - \sum_{i=1}^{s} (k/p_i - 1) + \sum_{1 \leq i < j \leq s} (k/p_i p_j - 1) - \sum_{1 \leq i < j \leq s} (k/p_i p_j p_i - 1) + \cdots + (-1)^s(k/p_1 p_2 \cdots p_s - 1) \right]. \]

Combining the degree it can be easily seen that

\[ |[k/p_i : 1 \leq i \leq s]| = \binom{s}{1}. \]
Putting these values in equation (6) we get

\[
\frac{\phi(k)}{k} = \frac{k}{p_1} + \frac{k}{p_2} + \cdots + (-1)^{s-1} \frac{k}{p_s}
\]

and

\[
\frac{\phi(k)}{k} = \frac{k}{p_1} + \frac{k}{p_2} + \cdots + (-1)^{s-1} \frac{k}{p_s}
\]

Putting these values in equation (6) we get

\[
N = \frac{1}{2} \left[ k - \binom{s}{0} - \sum_{i=1}^{s} \binom{s}{i} + \sum_{i<j \leq s} (k/p_i) - \binom{s}{j} \right]
\]

or

\[
N = \frac{1}{2} \left[ k - \sum_{i=1}^{s} k/p_i + \sum_{i<j \leq s} (k/p_i) - \sum_{i<j \leq s} \sum_{i<j \leq s} (k/p_i p_j) \right.
\]

\[
\left. + \cdots + (-1)^{s-1} (k/p_1 p_2 \cdots p_s) \right]
\]

Since

\[
\binom{s}{0} - \binom{s}{1} + \binom{s}{2} - \binom{s}{3} + \cdots + (-1)^{s-1} \binom{s}{s} = 0
\]

therefore using (8) in equation (7) we get

\[
N = \frac{1}{2} \left[ k - \sum_{i=1}^{s} k/p_i + \sum_{i<j \leq s} (k/p_i) - \sum_{i<j \leq s} \sum_{i<j \leq s} (k/p_i p_j) \right]
\]

But by equation (2)

\[
\phi(k) = k - \sum_{i=1}^{s} k/p_i + \sum_{i<j \leq s} k/p_i p_j - \cdots + (-1)^{s-1} k/p_1 p_2 \cdots p_s
\]

imply that

\[
N = \frac{1}{2} \phi(k).
\]

Case II: When \( k \) is even, we have \( p_1 = 2 \). We have further two cases;

(a) When \( r_1 > 1 \) then all the \( k/p_i, k/p_i p_j, \cdots k/p_1 p_2 \cdots p_s \) are even so we have

\[
|N_1| = (k/2 - 1),
\]

\[
|N_{k/p_1}| = (k/2 p_1 - 1),
\]

\[
|N_{k/p_p}| = (k/2 p_2 p_3 - 1)
\]

and

\[
|N_{k/p_1 \cap N_{k/p_2} \cap \cdots N_{k/p_s}}| = (k/2 p_1 p_2 \cdots p_s - 1)
\]

(b) When \( r_1 = 1 \) we have \( k/p_1, k/p_1 p_j, \cdots k/p_1 p_2 \cdots p_s \) odd whereas all other are even. Simplifying equation (4) for both cases II (a) and (b) we get

\[
N = \frac{1}{2} \phi(k).
\]

So the result is true for all the cases.

\[\square\]

In Corollary \[8\] and Theorem \[7\] it can be seen that all the distinct \( \ell \)-januarials can be constructed from \( PGL(2, p) \) using the method described in Section 2. Hence this method of obtaining januarial is precise as compared to the method of obtaining januarial using associates which is described in \[1\].
5. Genus of the coset diagram for the action of $H_{\Lambda_\ell}$

It is pertinent to find a formula for calculating genus (of the coset diagram for the action of $H_{\Lambda_\ell}$ on $PL(F_p)$) using the fixed points in a coset diagram instead of edges and vertices, counting which is time consuming for higher primes and increase the possibility of errors.

Graham Higman’s formula for the genus $g$ of a coset diagram is

$$g = \frac{1}{2} \left( 2 - (v - e + f) \right)$$

where $g$ denotes the genus of the coset diagram, $v$ is the number of all the orbits and cycles of $y$, $e$ is the number of nontrivial orbits or cycles of $x$ and $f$ is the number of all the orbits or cycles of $xy$.

**Theorem 8** Genus $g$ of the coset diagram for the action of $H_{\Lambda_\ell} = \langle x, y : x^2 = y^\ell = 1 \rangle$ on $PL(F_p)$ is

$$g = 1 - \frac{1}{4k\ell} \left[ 2(\ell - k) - \ell(p + 1) + k\ell(2(\eta_x + \eta_{xy}) + \eta_y) - 2(k\eta_y + \ell\eta_{xy}) \right]$$

where $k$ is the order of $xy$ in the coset diagram, $\eta_x, \eta_y$ and $\eta_{xy}$ are the number of fixed points of $x, y$ and $xy$ respectively.

**Proof:** The genus $g$ of a coset diagram can be calculated by using Graham Higman’s formula, that is

$$g = \frac{1}{2} \left( 2 - (v - e + f) \right)$$

where $v$ is the number of orbits of $y$, $e$ is the number of non-trivial orbits of $x$, $f$ is the number of all orbits of $xy$.

Let

$$v = \mu_y + \eta_y$$

where $\mu_y$ is number of non trivial cycles of $y$ and $\eta_y$ be the number of trivial cycles or fixed points of $y$. As $y^\ell = 1$ and $y$ makes a cycle of length $\ell$ so that values of $\mu_y$ and $\eta_y$ would depend on the quotient and the remainder respectively after dividing $p + 1$ by $\ell$, the remainder would be $\eta_y$. Then

$$\ell\mu_y = p + 1 - \eta_y$$

that is

$$v = \left(\frac{p + 1 - \eta_y}{\ell}\right) + \eta_y$$

or

$$v = \left(\frac{p + 1 + (\ell - 1)\eta_y}{\ell}\right)$$

Similarly, $f$ is the number of all cycles of $xy$. We split $f$ as

$$f = \mu_{xy} + \eta_{xy}$$

where $\mu_{xy}$ is the number of non-trivial cycles of $xy$ and $\eta_{xy}$ is the number of trivial cycles. As $(xy)^k = 1$ and $xy$ makes a cycle of length $k$.

$$k\mu_{xy} = p + 1 - \eta_{xy}$$

or

$$f = \left(\frac{p + 1 - \eta_{xy}}{k}\right) + \eta_{xy}$$

or

$$f = \left(\frac{p + 1 + (k - 1)\eta_{xy}}{k}\right)$$
Let $\eta_x$ denote the number of fixed points of $x$. Then

$$p + 1 = 2e + \eta_x$$

or

$$e = \frac{p + 1 - \eta_x}{2}$$

putting values of $v, f,$ and $e$ in Graham Higman’s formula to obtain the required result. That is,

$$g = \frac{1}{2} \left\{ 2 - \frac{p + 1 + (\ell - 1)\eta_x}{\ell} - \frac{p + 1 - \eta_x}{2} + \frac{p + 1 + (k - 1)\eta_y}{k} \right\}$$

or

$$g = 1 - \frac{1}{4k\ell} \left[ (2(k + \ell) - k\ell)(p + 1) + k\ell(2(\eta_x + \eta_y) + \eta_x) - 2(k\eta_y + \ell\eta_y) \right].$$

For januarials we have $k = \frac{p + 1}{2}$ and $\eta_{xy} = 0$. Hence the following corollary is in order.

**Corollary 9**  *Genus of a januarial is*

$$g = -\frac{p + 1 - \eta_y}{2\ell} + \frac{1}{4}(p + 1 - 2\eta_y - \eta_x)$$

where $\eta_x$ and $\eta_y$ are the fixed points of $x$ and $y$ respectively.

6. Illustration

![Figure 1: Coset diagram $D(7, 31, 4)$.](image)
In the following we illustrate the above mentioned method for its better understanding. If we take $p = 31$, then by Theorem 1 we get $k = 16$, and

$$g_{16}(\theta) = 1x^7 - 14x^6 + 78x^5 - 220x^4 + 330x^3 - 252x^2 + 84x - 8.$$  

The zeros of $g_{16}(\theta)$ excluding those of $g_8(\theta)$, $g_4(\theta)$ and $g_2(\theta)$ are 7, 16, 19 and 28.

Now for a 4- januarial if we consider $\theta = 7$, we get

$$x : z \rightarrow \frac{3z + 30}{10z - 3}$$

$$y : z \rightarrow \frac{42}{14z + 8}$$

The action of $x$ and $y$ on $PL(F_{31})$ give permutations:

$$x^r = (0, 21)(1, 18)(2, 24)(3, 29)(4, 7)(5, 28)(6, 9)(8, 16)(10, 15)$$

$$y^r = (0, 13, 26, \infty)(1, 16, 9, 29)(2, 27, 3, 12)(4, 21, 18, 19)$$

$$xy^r = (0, 18, 16, 5, 17, 24, 27, 23, 26, 2, 14, 3, 1, 19, 28, 22)$$

Hence the coset diagram $D(7, 31, 4)$ and the two orbits of $xy$ highlighted with bold lines are depicted in the Figures [1] and [2].

According to Corollary 2, genus of this januarial is 4.

**References**

[1] M. Conder and T. Riley, Graham Higman’s lectures on januarials, Quart. J. Math., 65(1), 2014, 113-131.
[2] M. R. Darafsheh and G. R. Razaezadeh, Factorization of groups involving symmetric and alternating groups, Int. J. Math. and Math. Science, 27(3), 2000, 161-167.
[3] L. Levai, G. Rosenberger and B. Soumignier, All finite generalized triangle groups, Trans. AMS, 347(9), 1995, 3625-3627.
[4] Q. Mushtaq, Coset diagrams for the modular group, D. Phil. Thesis, Oxford University, 1983.
[5] Q. Mushtaq, Parametrization of all Homomorphisms from PGL(2, Z) into PGL(2, q), Comm. in Algebra, 20(4), 1992, 1023-1040.
[6] Q. Mushtaq and F. Shaheen, On the subgroups of polyhedral groups, Math. Japon., 35, 1990, 849-853.
[7] Q. Mushtaq and F. Shaheen, Finite presentation of alternating groups, Acta Math. Sinica, 11(2), 1995, 221-224.
[8] F. Shaheen, Coset diagrams for a two generator group, Ph.D. Thesis, Quaid-i-Azam University, Islamabad, 1991.
[9] J. H van Lint and R. M Wilson, A course in combinatorics, Cambridge University Press, 1992.
[10] A. T. White, Graphs, groups and surfaces, North Holland Pub. Co., Amsterdam, 1973.