LOCALIZATION AND DIMENSION FREE ESTIMATES FOR MAXIMAL FUNCTIONS.

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ABSTRACT. In the recent paper [J. Funct. Anal. 259 (2010)], Naor and Tao introduce a new class of measures with a so-called micro-doubling property and present, via martingale theory and probability methods, a localization theorem for the associated maximal functions. As a consequence they obtain a weak type estimate in a general abstract setting for these maximal functions that is reminiscent of the ‘$n \log n$ result’ of Stein and Strömberg in Euclidean spaces.

The purpose of this work is twofold. First we introduce a new localization principle that localizes not only in the time-dilation parameter but also in space. The proof uses standard covering lemmas and selection processes. Second, we show that a uniform condition for micro-doubling in the Euclidean spaces provides indeed dimension free estimates for their maximal functions in all $L^p$ with $p > 1$. This is done introducing a new technique that allows to differentiate through dimensions.

1. INTRODUCTION

We say that $(X, d, \mu)$ is a metric measure space if $(X, d)$ is a separable metric space and $\mu$ a Radon measure on it. We denote by $B(x, r)$ the open ball centered at $x$ with radius $r$ with respect to the metric $d$, that is

$$B(x, r) = \{y \in X : d(x, y) < r\}.$$

We will assume that the measure $\mu$ is non-degenerate. This means that any ball with positive radius has non-zero measure. Given $T \subset (0, \infty)$, for a locally integrable function $f$ over $X$ we define the following centered maximal operator

$$M_T f(x) = \sup_{r \in T} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f(y)| \, d\mu(y).$$

When $T = (0, \infty)$, the operator $M = M_T$ represents the usual Hardy-Littlewood maximal operator. As in the case of evolution equations and semigroup theory we can think of $T$ as a set of times and then $M_T$ is the maximal operator on them.

We will also assume that the measure $\mu$ satisfies a doubling property. This simply says that the measure of a ball is comparable with the measure of certain dilation of it. The classical way of expressing the doubling property uses the dilation factor 2. That is, $\mu$ is doubling if there exists a constant $K > 0$ so that for each $x \in X$ and $R > 0$ one has $\mu(B(x, 2R)) \leq K \mu(B(x, R))$.

In these hypotheses one can reproduce the argument of Vitali’s covering lemma to show that $M_T$ is weakly bounded on $L^1(\mu)$. That is, there exist a constant $c_{\mu, 1} > 0$ so that

$$\mu(\{x \in X : M_T f(x) > \lambda\}) \leq \frac{c_{\mu, 1}}{\lambda} \int_X |f| \, d\mu,$$

for all $\lambda > 0$ and all locally integrable $f$ over $X$. Since $\|M_T f\|_{L^1(\mu)} \leq \|f\|_{L^\infty(\mu)}$, by interpolation one obtains the $L^p(\mu)$ bounds

$$\|M_T f\|_{L^p(\mu)} \leq C_{\mu, p} \|f\|_{L^p(\mu)}.$$

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for all \( p > 1 \). The order of magnitude of the constants \( c_{\mu,1} \) and \( C_{\mu,p} \) that we obtain is essentially that of the doubling constant \( K \) and \( K^{1/p}/(p-1) \) respectively.

In the case that \( X = \mathbb{R}^n \) and \( \mu = m_n \) is Lebesgue measure, the value of the doubling constant \( K \) is \( 2^n \). Hence the aforementioned constants \( c_{m_n,1} \) and \( C_{m_n,p} \) grow exponentially to infinity with the dimension. It has been a matter of interest to know if these constants can be bounded uniformly in dimension. E.M. Stein [31] (detailed proof in [32]) proved this for \( C_{m_n,p} \) with \( p > 1 \) when considering the Euclidean metric. J. Bourgain [7], [8], [9] and independently A. Carbery [12], showed that for the metric given by a norm, uniform bounds hold if \( p > 3/2 \). This was extended to all the range \( p > 1 \) by D. Müller in [27] for the maximal functions associated with the \( \ell^q \) metrics, with \( 1 \leq q < \infty \). Observe that this excludes the case \( q = \infty \), where the ‘balls’ are cubes with sides parallel to the coordinate axes. Recently J. Bourgain has solved this case by giving uniform bounds on \( L^p \) for all \( p > 1 \) (see [11]).

The case \( p = 1 \) is more complicated. J.M. Aldaz showed in [2] that for the \( \ell^\infty \) metric the constants \( c_{m_n,1} \) grow to infinity as \( n \to \infty \). As shown by Aubrun [6], in this case one has the estimate \( c_{m_n,1} \geq C_n (\log n)^{1-\varepsilon} \) for each \( \varepsilon > 0 \). It is still unknown if this also happens for other metrics, although there are partial results in the form of upper bounds for the possible growth of the constants. When considering the Euclidean metric a special argument allows to prove that \( c_{m_n,1} = O(n) \). For metrics given by general norms the best upper bound remains \( c_{m_n,1} = O(n \log n) \). Both results are due to E.M. Stein and J.O. Strömberg in [32]. In this last paper, instead of the usual doubling condition, they used the fact that in \( \mathbb{R}^n \) one has

\[
m_n(B(x, (1+1/n)R)) = (1+1/n)^n m_n(B(x, R)) \leq e \, m_n(B(x, R)).
\]

That is, the dilation by the factor \( (1+1/n) \) does not increase essentially the volume of a ball. This allows to perform a more efficient covering argument with less overlapping than in Vitali’s lemma.

A. Naor and T. Tao observed in [28] that in order to extend Stein and Strömberg \( n \log n \) bound to general metric measure spaces suffices to assume that dilations by the factor \( (1 + 1/n) \) preserve essentially the volumes of balls and that the measures of intersecting balls with the same radius are comparable. We next explain this conditions and state their result in detail.

Let \( (X, d, \mu) \) be a metric measure space. Given a positive number \( n \geq 1 \), we will say that \( \mu \) is \( n \)-micro-doubling if there exists a constant \( K_0 > 0 \) such that for each \( x \in X \) and \( R > 0 \) one has

\[
\mu(B((x, (1+1/n)R)) \leq K_0 \, \mu(B(x, R)).
\]

Of course \( n \) here is not necessarily related to any notion of dimension. We will refer to \( K_0 \) as the \( n \)-micro-doubling constant.

We will say that a measure \( \mu \) is weakly-doubling if the measure of two intersecting balls with the same radius is comparable. That is, there exists a constant \( K_1 > 0 \) such that for each \( x, y \in X \) and \( R > 0 \) such that \( B(x, R) \cap B(y, R) \neq \emptyset \) one has

\[
\mu(B(y, R)) \leq K_1 \, \mu(B(x, R)).
\]

Following the terminology of A. Naor and T. Tao in [28], we will say that a measure \( \mu \) is strong \( n \)-micro-doubling if it is both, \( n \)-micro-doubling and weakly-doubling. Equivalently, \( \mu \) is strong \( n \)-micro-doubling if there exist a constant \( K > 0 \) so that for each \( x \in X, R > 0 \), and \( y \in B(x, R) \) one has

\[
\mu(B(y, (1+1/n)R)) \leq K \, \mu(B(x, R)).
\]
It is easy to see that every Ahlfors-David $n$-regular metric measure space is strong $n$-micro-doubling. We recall that a metric measure space $(X, d, \mu)$ is said to be Ahlfors-David $n$-regular if there exists a constant $C \geq 1$ so that
\[
\frac{1}{C} R^n \leq \mu(B(x, R)) \leq C R^n,
\]
for all $x \in X$ and $R > 0$. In the Euclidean $n$-dimensional case, some examples of strong $n$-micro-doubling measures are given by the power densities $|x|^\alpha$, with $\alpha > -n$. See Section 3.1 for more details.

We will consider maximal operators associated with the following type of time sequences in $(0, \infty)$. A sequence $\{a_k\}_{k \in \mathbb{Z}} \subset (0, \infty)$ is lacunary if there exist $a > 1$ so that $a_{k+1} > aa_k$. If $a = n > 1$ in this setting of $n$-micro-doubling measures, we will say explicitly that the time sequence is $n$-lacunary. Now we are ready to state the maximal theorem presented by A. Naor and T. Tao in [28].

**Theorem 1.1.** Let $(X, d, \mu)$ be a measure metric space, with $\mu$ a strong $n$-micro-doubling measure with constant $K$. Then we have the following weak type estimates

i) if $T \subset (0, \infty)$ is an $n$-lacunary sequence, then
\[
\mu(\{x \in X : M_T f(x) > \lambda\}) \leq \frac{C_K}{\lambda} \|f\|_{L^1(\mu)},
\]

ii) if $T \subset (0, \infty)$ is a lacunary sequence with constant $a$, then
\[
\mu(\{x \in X : M_T f(x) > \lambda\}) \leq \frac{C_K \log n}{\lambda \log a} \|f\|_{L^1(\mu)},
\]

iii) if $T = (0, \infty)$ so that $M = M_T$ is the Hardy-Littlewood maximal operator, then
\[
\mu(\{x \in X : M f(x) > \lambda\}) \leq \frac{C_K n \log n}{\lambda} \|f\|_{L^1(\mu)},
\]

where the $C_K$ are constants only depending on $K$.

Note that part iii) is a generalization of the \textquoteleft$n \log n$\textquoteright bound by E.M. Stein and J.O. Strömberg (see [32]) in normed Euclidean spaces. In this setting, parts i) and ii) are due to M.T. Menárguez and the second author in [25], as well as the chain of implications i) \Rightarrow ii) \Rightarrow iii).

Moreover, A. Naor and T. Tao showed also in [28] that the $O(n \log n)$ bound is optimal even in the setting of Ahlfors-David $n$-regular spaces, by constructing a sequence of such spaces $(X_n, d_n, \mu_n)$ so that
\[
\|M_{\mu_n}\|_{L^1(\mu_n) \to L^{1, \infty}(\mu_n)} \geq C n \log n.
\]

Theorem 1.1 can be proved as in [25] or by an argument of Lindenstrauss present in [22] (see also [28]). However, the main contribution of A. Naor and T. Tao in [28] is the following nice localization principle for maximal operators, from which they obtained Theorem 1.1 as a corollary.

\footnote{Statement i) of the theorem does not appear explicitly in [28].}
Theorem 1.2. Let \((X, d, \mu)\) be a metric measure space and let \(\mu\) be \(n\)-micro-doubling with constant \(K_0\). If \(T \subset (0, \infty)\) and \(p \geq 1\), then we have the following localization property for the weak \(L^p\) norms of the associated maximal operator
\[
\|M_T\|_{L^p \to L^{p, \infty}} \leq C_{K_0} + C'_{K_0} \sup_{k \in \mathbb{Z}} \|M_k\|_{L^p \to L^{p, \infty}}.
\]
Here \(M_k\) denotes the restriction of \(M_T\) to times in \([n^k, n^{k+1})\), that is \(M_k = M_T \cap [n^k, n^{k+1})\), and \(C_{K_0}\) and \(C'_{K_0}\) are constants only depending on \(K_0\).²

The proof given by A. Naor and T. Tao of this result is probabilistic and relies on random martingales and Doob-type maximal inequalities. Here we present (see Theorem 2.1) a more geometrical proof based on covering lemmas and selection processes that is closer in spirit to the arguments in [32]. In addition, our result considers localization not only in time but also in space (see the statement of the theorem). The proof of this and the connection with Theorem 1.1 will be presented in Section 2.

In the rest of the paper we restrict ourselves to the Euclidean case. We study measures given by a radial density, so that they can be defined in all dimensions. It is known that if these measures are finite, and hence are neither weakly-doubling nor \(n\)-micro-doubling, the (weak) \(L^p\) operator norms of the associated maximal functions grow exponentially to infinity with the dimension at least for a range of values of \(p\) near 1 (see [1], [3] and [13]). There are cases in which this range of \(p\)'s may consist of the whole interval \([1, \infty)\). An example of this is given by the Gaussian measure (see [15]).

The situation changes if we consider measures that are strong \(n\)-micro-doubling in each \(\mathbb{R}^n\) with constants uniformly bounded in dimension. In this case Theorem 1.1 applies and the weak \(L^1\) operator norms of the maximal function grow at most like \(O(n \log n)\). In Section 3.1 we study maximal operators associated with uniformly weakly doubling measures. For such measures \(\mu\) the main result of this section asserts that \(M_\mu\) is uniformly bounded on \(L^p(\mu, \mathbb{R}^n)\) for all \(p > 1\). The same thing is true in weak \(L^1\) but restricting the action of \(M_\mu\) to radial functions.

The previous results are obtained via the following characterization: \(\mu\) is uniformly weakly doubling if and only if its density is essentially constant over dyadic annuli. This equivalence is proved using a new method of differentiation through dimensions presented in Section 3.2.

The case of decreasing densities is treated in Section 3.4. The situation here is that the uniform weakly doubling property can be obtained from surprisingly mild conditions.

Finally, Section 4 contains examples of doubling measures with associated maximal operators failing to have uniform bounds.

2. Localization Principles

In this section we will obtain Theorems 1.1 and 1.2 as a consequence of yet another localization theorem, whose proof avoids technical arguments from probability theory. The statement is the following:

²The constants \(C_{K_0}\) and \(C'_{K_0}\) obtained in [28] are of the order of \(O(K_0)\) and \(O(1 + \log \log K_0/(1 + \log n))\) respectively.
Theorem 2.1. Let \((X, d, \mu)\) be a metric measure space and let \(\mu\) be \(n\)-micro-doubling with constant \(K_0\). If \(T \subset (0, \infty)\) and \(p \geq 1\) are fixed, then for each locally integrable function \(f\) over \(X\) and each \(\lambda > 0\) one can find a disjoint collection of measurable sets \(\{A_k\}_{k \in \mathbb{Z}}\) such that
\[
\mu(\{x \in X : \lambda < M_T f(x) \leq 2\lambda\}) \\
\leq C_1 \left( \frac{1}{\lambda^p} \|f\|^p_{L^p(\mu)} + \sum_{k \in \mathbb{Z}} \mu(\{x \in X : M_k(f\chi_{A_k})(x) > C_2\lambda\}) \right),
\]
where \(C_1\) and \(C_2\) are constants that only depend on \(K_0\).

It is not difficult to show that Theorem 2.1 implies Theorem 1.2. To see this, given a locally integrable \(f\) and \(\lambda > 0\) we write
\[
E_\lambda = \{x \in X : M_T f(x) > \lambda\}, \\
F_\lambda = \{x \in X : \lambda < M_T f(x) \leq 2\lambda\}.
\]
Note that by the disjointness of the collection \(\{A_k\}_{k \in \mathbb{Z}}\) we have
\[
\sum_{k \in \mathbb{Z}} \mu(\{x \in X : M_k(f\chi_{A_k})(x) > C_2\lambda\}) \leq \frac{C^{-p}_2}{\lambda^p} \sup_{k \in \mathbb{Z}} \|M_k\|^p_{L^p(\mu)} \int_X |f|^p d\mu \\
\leq \frac{C^{-p}_2}{\lambda^p} \int_X |f|^p d\mu.
\]
Hence Theorem 2.1 gives the estimate
\[
\mu(F_\lambda) \leq \frac{C_{K_0}}{\lambda^p} \left( 1 + \sup_{k \in \mathbb{Z}} \|M_k\|^p_{L^p(\mu)} \right) \|f\|^p_{L^p(\mu)}.
\]
This implies
\[
\mu(E_\lambda) = \mu \left( \bigcup_{j=0}^{\infty} F_{2^j \lambda} \right) = \sum_{j=0}^{\infty} \mu(F_{2^j \lambda}) \\
\leq \sum_{j=0}^{\infty} \frac{C_{K_0}}{(2^j \lambda)^p} \left( 1 + \sup_{k \in \mathbb{Z}} \|M_k\|^p_{L^p(\mu)} \right) \|f\|^p_{L^p(\mu)} \\
= \frac{C_{K_0}}{\lambda^p} \left( 1 + \sup_{k \in \mathbb{Z}} \|M_k\|^p_{L^p(\mu)} \right) \|f\|^p_{L^p(\mu)},
\]
as wanted.

Also part \(i)\) of Theorem 1.1 follows from Theorem 1.2 using the following argument.

If \(T = \{a_j\}_{j \in \mathbb{Z}}\) is an \(n\)-lacunary sequence then each interval of the form \([n^k, n^{k+1}]\) contains at most one element of \(T\). Theorem 1.2 implies that
\[
\|M_T\|_{L^1 \rightarrow L^{1, \infty}} \leq C_K \sup_{j \in \mathbb{Z}} \|M_{\{a_j\}}\|_{L^1 \rightarrow L^{1, \infty}}.
\]
We only have to show now that \(M_{\{a_j\}}\) is bounded independently of \(j\). Since
\[
M_{\{a_j\}} f(x) = \frac{1}{\mu(B(x, a_j))} \int_{B(x, a_j)} |f(y)| \, d\mu(y),
\]
by the weak doubling property of $\mu$ and Fubini Theorem we have
\[
\|M_{\{a_j\}}f\|_{L^1} = \int_X \frac{1}{\mu(B(x, a_j))} \int_X \chi_{B(x, a_j)}(y) |f(y)| d\mu(y) d\mu(x) \\
\leq \int_X \int_X \frac{K_1}{\mu(B(y, a_j))} \chi_{B(y, a_j)}(x) |f(y)| d\mu(y) d\mu(x) \\
= \int_X \frac{K_1}{\mu(B(y, a_j))} \int_{B(y, a_j)} d\mu(x) |f(y)| d\mu(y) = K_1 \|f\|_{L^1}.
\]
This says, in particular, that each $M_{\{a_j\}}$ is bounded on $L^1$ with operator norm bounded by $K_1$.

The implications $i) \Rightarrow ii)$ and $ii) \Rightarrow iii)$ of Theorem 1.1 can be obtained through the same generic arguments given in [25]. The first one says that every maximal operator associated with a lacunary sequence is essentially majorized by $\log n$ operators associated with $n$-lacunary sequences. The second one says that the full maximal operator is majorized by $n$ operators, each associated with a lacunary sequence.

We now give a proof of Theorem 2.1

Proof of Theorem 2.1. Given a locally integrable $f$ and $\lambda > 0$ we write
\[
F_\lambda = \{x \in X : \lambda < M_T f(x) \leq 2\lambda\}.
\]
Once we determine what the sets $A_j$ are, we will have to prove that
\[
\mu(F_\lambda) \leq C_1 V_{\lambda,p}^f,
\]
where we have used the notation
\[
V_{\lambda,p}^f = \frac{1}{\lambda^p} \|f\|_{L^p(\mu)}^p + \sum_{k \in \mathbb{Z}} \mu(\{x \in X : M_k(f\chi_{A_k})(x) > C_2 \lambda\}).
\]

Note first that it is enough to prove the result for $p = 1$. To see this, given a function $f$ in $L^p(\mu)$ we consider $f = f^\lambda + f_\lambda$ where
\[
f^\lambda = f\chi_{\{|f| > \lambda\}} \in L^1(\mu), \\
f_\lambda = f\chi_{\{|f| \leq \lambda\}} \in L^\infty(\mu).
\]
Since
\[
M_T f(x) \leq M_T f^\lambda(x) + M_T f_\lambda(x) \leq M_T f^\lambda(x) + \lambda,
\]
for $M_T f(x) > 2\lambda$ to hold it is necessary that $M_T f^\lambda(x) > \lambda$. Hence,
\[
F_{2\lambda} \subset \{x \in X : \lambda < M_T f^\lambda(x) \leq 4\lambda\} = G_1 \cup G_2,
\]
with
\[
G_1 = \{x \in X : \lambda < M_T f^\lambda(x) \leq 2\lambda\}, \\
G_2 = \{x \in X : 2\lambda < M_T f^\lambda(x) \leq 4\lambda\}.
\]
Then since $f^\lambda \in L^1(\mu)$ we apply the result for $p = 1$ to $G_1$ and $G_2$ to obtain
\[
\mu(F_{2\lambda}) \leq 2 \max(\mu(G_1), \mu(G_2)) \\
\leq C \left( C_1 \int_X \frac{|f^\lambda|}{\lambda} d\mu + \sum_{k \in \mathbb{Z}} \mu(\{x \in X : M_k(f\chi_{A_k})(x) > C_2 \lambda\}) \right) \\
\leq C \left( C_1 \int_X \frac{|f|^p}{\lambda^p} d\mu + \sum_{k \in \mathbb{Z}} \mu(\{x \in X : M_k(f\chi_{A_k})(x) > C_2 \lambda\}) \right),
\]
for each \( p \geq 1 \). Since \( \lambda \) is arbitrary the result is proved for any \( p \geq 1 \).

Now we prove the result in the case that \( p = 1 \), that is
\[
\mu(F_\lambda) \leq C_1 V^f_{\lambda,1}
\]

We consider the following collections of balls
\[
B = \{ B(x, R) : x \in X, R \in T \}, \quad B_k = \{ B(x, R) : x \in X, R \in T \cap [n^k, n^{k+1}] \}.
\]
Given a ball \( B \in B \) we denote by \( z_B \) its center and by \( R_B \) its radius. We define the collection
\[
A = \left\{ B \in B : \lambda < \frac{1}{\mu(B)} \int_B |f| \, d\mu \leq 2\lambda \right\}.
\]
For each \( B \in A \) one can find a concentric ball \( \tilde{B} \) so that \( R_B/(1 + 1/n) \leq R_{\tilde{B}} < R_B \) and
\[
\frac{1}{\mu(B)} \int_{\tilde{B}} |f| \, d\mu > \lambda.
\]
We will write \( B^0 = B(z_B, R_B - R_{\tilde{B}}) \). Since
\[
F_\lambda \subset \bigcup_{B \in A} B^0,
\]
it suffices to prove that
\[
\mu \left( \bigcup_{B \in A} B^0 \right) \leq C_1 V^f_{\lambda,1}.
\]
It is not difficult to see (see the argument preceding (2.4) below) that \( \bigcup_{B \in A} B^0 \) is contained in a level set of \( M_T f \) and, as a consequence, has finite \( \mu \)-measure. Using that \( X \) is separable and the monotone convergence there exists \( A' \subset A \) with \( \# A' < \infty \) such that
\[
\mu \left( \bigcup_{B \in A'} B^0 \right) \geq \frac{1}{2} \mu \left( \bigcup_{B \in A} B^0 \right).
\]
Hence, it is enough to show that
\[
\mu \left( \bigcup_{B \in A'} B^0 \right) \leq C_1 V^f_{\lambda,1},
\]
with \( C \) independent of \( A' \). Writing
\[
A_k := \{ B \in A' : B \in B_k \},
\]
we have that \( A' = A^1 \cup A^2 \) with
\[
A^1 = \bigcup_{k \text{ odd}} A_k, \quad A^2 = \bigcup_{k \text{ even}} A_k.
\]
Assume that \( \mu \left( \bigcup_{B \in A^1} B^0 \right) \geq \mu \left( \bigcup_{B \in A^2} B^0 \right) \) (if not, interchange the names), then \( \mu \left( \bigcup_{B \in A^1} B^0 \right) \geq \frac{1}{2} \mu \left( \bigcup_{B \in A'} B^0 \right) \) and we just need to prove that
\[
\mu \left( \bigcup_{B \in A^1} B^0 \right) \leq C_1 V^f_{\lambda,1}.
\]
For the sake of simplicity in the notation we rename \( A = A^1 \).
Now we want to write the set $\bigcup_{B \in A} B^0$ as a union of disjoint sets. Observe that $A = \{B_j\}_{j=1}^J$ for certain $J$. We define $D_{B_1} = B_1^0$, and if $D_{B_1}, \ldots, D_{B_m}$ are already defined, we take

$$D_{B_{m+1}} = B_{m+1} \setminus \bigcup_{j=1}^m D_j^0.$$ 

Then, we have

$$\mu \left( \bigcup_{B \in A} B^0 \right) = \mu \left( \bigcup_{B \in A} D_B \right).$$

We also define the functions

$$g_B(x) = \frac{\mu(D_B)}{\mu(B)} \chi_B(x)$$

for $B \in A$ and

$$G_k(x) = \sum_{B \in A_k} g_B(x).$$

We start a selection process. Take $k_1$ as the largest $k \in \mathbb{Z}$ with $A_k \neq \emptyset$, then we take $\tilde{G}_{k_1} = G_{k_1}$ and $\tilde{A}_{k_1} = A_{k_1}$. Once $\tilde{G}_{k_1}, \ldots, \tilde{G}_{k_{m-1}}$ are determined for $m \geq 2$, we take $k_m$ as the largest $k < k_{m-1}$ so that $A_k \neq \emptyset$. We say that $B \in \tilde{A}_{k_m}$ if $B \in A_{k_m}$ and

$$\sum_{j=1}^{m-1} \tilde{G}_{k_j} \leq 1$$
on $B$. Then we define

$$\tilde{G}_{k_m}(x) = \sum_{B \in \tilde{A}_{k_m}} g_B(x).$$

Since $A$ is finite this process ends in a finite number of steps, and we have obtained $\tilde{G}_{k_1}, \ldots, \tilde{G}_{k_m}$ for certain $M$. We call

$$\tilde{A} = \bigcup_{m=1}^M \tilde{A}_{k_m},$$

and claim that

$$(2.1) \quad \mu \left( \bigcup_{B \in A} D_B \right) \leq (1 + K_0) \sum_{B \in \tilde{A}} \mu(D_B).$$

To prove this claim note that if $A \in A_{k_m} \setminus \tilde{A}_{k_m}$, then there exists $z \in A$ such that

$$\sum_{j=1}^{m-1} \tilde{G}_{k_j}(z) > 1.$$ 

Note that if $A \cap \tilde{B} \neq \emptyset$ for some $B \in \tilde{A}_{k_j}$ with $j < m$, then $A \subset B^* := B(x_B, (1 + 1/n)R_B)$. Thus for all $z \in A$ we have

$$\sum_{j=1}^{m-1} G_{k_j}^*(z) := \sum_{j=1}^{m-1} \sum_{B \in \tilde{A}_{k_j}} \frac{\mu(D_B)}{\mu(B)} \chi_B^*(z) > 1.$$ 

Then, by Tchebychev inequality
\[
\mu \left( \bigcup_{B \in \mathbb{A} \setminus \mathbb{A}_k} D_B \right) \leq \mu \left( \left\{ z \in X : \sum_{j=1}^M G_{kj}^* (z) > 1 \right\} \right) \\
\leq \sum_{j=1}^M \int_X G_{kj}^* (z) \, d\mu(z) = \sum_{B \in \mathbb{A}} \frac{\mu(D_B)}{\mu(B)} \mu(B^*) \\
\leq K_0 \sum_{B \in \mathbb{A}} \mu(D_B).
\]

Now that the claim is justified, we only need to prove
\[
(2.2) \quad \sum_{B \in \mathbb{A}} \mu(D_B) \leq C_1 V_{\lambda,1}^f.
\]

By the definition of \( \mathbb{A} \) we have
\[
(2.3) \quad \sum_{B \in \mathbb{A}} \mu(D_B) \leq \frac{1}{\lambda} \sum_{B \in \mathbb{A}} \frac{1}{\mu(B)} \int_B |f| \, d\mu(D_B) = \frac{1}{\lambda} \int_X |f| \left( \sum_{j=1}^M \tilde{G}_{kj} \right) \, d\mu.
\]

For each \( j = 1, \cdots, M \) we define \( \tilde{A}_j := \text{supp} \tilde{G}_{kj} \). We now take \( A_M = \tilde{A}_M \) and
\[
A_j = \tilde{A}_j \setminus \bigcup_{\ell = j+1}^{M} \tilde{A}_\ell.
\]

If \( z \in A_m \) we have \( \tilde{G}_{kj}(z) = 0 \) if \( j > m \), and then, by the way we selected \( \tilde{G}_{km} \)
\[
\sum_{j=1}^M \tilde{G}_{kj}(z) = \tilde{G}_{km}(z) + \sum_{j=1}^{m-1} \tilde{G}_{kj}(z) \leq \tilde{G}_{km}(z) + 1.
\]

Then we have that
\[
\sum_{j=1}^M \tilde{G}_{kj} \leq \sum_{j=1}^M \tilde{G}_{kj} \chi_{A_j} + 1,
\]
which combined with (2.1) and (2.3) yields
\[
\sum_{B \in \mathbb{A}} \mu(D_B) \leq \frac{1}{\lambda} \left( \int_X |f| \, d\mu + \sum_{j=1}^M \int_X |f| \chi_{A_j} \tilde{G}_{kj} \, d\mu \right).
\]

In order to bound the last sum note that
\[
\sum_{j=1}^M \int_X f \chi_{A_j} \tilde{G}_{kj} \, d\mu = \sum_{j=1}^M \sum_{B \in \mathbb{A}_k} \frac{1}{\mu(B)} \int_B |f| \chi_{A_j} \, d\mu \, \mu(D_B)
\]

Given \( B \in \mathbb{A}_k \), we say that \( B \in \mathbb{A}^*_{k_j} \) if
\[
\frac{1}{\mu(B)} \int_B |f| \chi_{A_j} \, d\mu \leq \frac{\lambda}{2}.
\]
Hence, we have
\[ \frac{1}{\lambda} \sum_{j=1}^{M} \sum_{B \in \tilde{A}^*_j} \frac{1}{\mu(B)} \int_B |f| \chi_{A_j} \, d\mu \mu(D_B) \leq \frac{1}{2} \sum_{B \in \tilde{A}} \mu(D_B) \]
and this term can be absorbed in the left hand side of (2.2).

We now consider those balls that do not belong to any of the classes $\tilde{A}^*_k$. We claim that if $z \in B^0$, then $\tilde{B} \subset B(z,R_B) \subset B^*$. The $n$-micro-doubling condition would imply then that $\mu(B) \geq \mu(B^*)/K_0 \geq \mu(B(z,R_B))/K_0$ and consequently if $B \in \tilde{A}_k \setminus \tilde{A}^*_k$
\[ \frac{\lambda}{2} < \frac{1}{\mu(B)} \int_B |f| \chi_{A_j} \, d\mu \leq \frac{K_0}{\mu(B(z,R_B))} \int_{B(z,R_B)} |f| \chi_{A_j} \, d\mu \]
\[ \leq K_0 M_k (f \chi_{A_j})(z). \]
Thus, for such $B$,
\[ B^0 \subset \left\{ x \in X : M_k (f \chi_{A_j})(x) > \frac{\lambda}{2K_0} \right\}. \] (2.4)
Therefore, using that for $B \in A$ one has
\[ \frac{1}{\mu(B)} \int_B |f| \chi_{A_j} \, d\mu \leq \frac{1}{\mu(B)} \int_B |f| \chi_{A_j} \, d\mu \leq 2\lambda, \]
we conclude that
\[ \frac{1}{\lambda} \sum_{j=1}^{M} \sum_{B \in \tilde{A}_j \setminus \tilde{A}^*_j} \frac{1}{\mu(B)} \int_B |f| \chi_{A_j} \, d\mu \mu(D_B) \]
\[ \leq \frac{1}{2} \sum_{j=1}^{M} \sum_{B \in \tilde{A}_j \setminus \tilde{A}^*_j} \mu(D_B) \]
\[ \leq \frac{1}{2} \sum_{j=1}^{M} \mu \left( \left\{ x \in X : M_k (f \chi_{A_j})(x) > \frac{\lambda}{2K_0} \right\} \right). \]
This finishes the proof, provided we justify the last claim.

In order to do so, suppose that $y \in \tilde{B}$, then
\[ |y - z| \leq |y - z_B| + |z_B - z| < R_{\tilde{B}} + (R_B - R_{\tilde{B}}) = R_B, \]
which means that $y \in B(z,R_B)$. Assume now that $y \in B(z,R_B)$
\[ |y - z_B| \leq |y - z| + |z - z_B| < R_B + R_B/n = (1 + 1/n)R_B, \]
hence $y \in B^*$. The claim is proved. \qed
3. Uniform bounds for maximal functions associated with rotation invariant measures in Euclidean spaces

In the sequel we will consider $X$ to be the Euclidean space $\mathbb{R}^n$, equipped with the Euclidean distance and a rotation invariant measure with a radial density, so that it can be defined in all dimensions. The goal of this section is to show that, in this setting, uniformly weakly doubling measures have associated maximal operators satisfying dimension free bounds on the spaces $L^p$.

The notation that we will use is the following. By $\mu$ we will always denote a rotation invariant measure with a radial density $w$. That is $d\mu(x) = w(x) \, dx$ and $w(x) = w_0(|x|)$ with $w_0 : [0, \infty) \to [0, \infty)$. Properly speaking, $w$ and $d\mu$ are objects that change with the dimension. In particular, their integrability properties may be different from one Euclidean space to another. By a slight abuse of notation, we keep, however the same symbols to denote them in $\mathbb{R}^n$ for all $n$.

As usual, for a measurable set $E \subset \mathbb{R}^n$, $|E|$ will denote its Lebesgue measure, $d\sigma_{n-1}$ will be the measure on $S^{n-1}$ induced by Lebesgue measure on $\mathbb{R}^n$ and $\omega_{n-1} = \sigma_{n-1}(S^{n-1})$. Also, for each $x \in \mathbb{R}^n$ and $R > 0$, we will denote by $B(x, R)$ the Euclidean ball centered at $x$ with radius $R$.

For a ball whose center is the origin we will use the notation $B_R$.

3.1. Weakly-doubling measures in Euclidean spaces and uniform bounds. As a first and simple example that uniformity in the weak doubling condition implies uniform bounds we present

**Theorem 3.1.** Let $\mu$ be a rotation invariant measure over $\mathbb{R}^n$ that is absolutely continuous with respect to Lebesgue measure. If $\mu$ is weakly doubling with constant $K_1$ and $f$ is a radial function we have

$$\mu(\{x \in \mathbb{R}^n : M_\mu f(x) > \lambda\}) \leq \frac{2(K_1 + 1)}{\lambda} \|f\|_{L^1(\mathbb{R}^n, d\mu)}.$$

Observe that if $K_1$ is uniformly bounded in $n$, the above inequality provides a dimension-free bound for $M_\mu$. This is a generalization of a result presented by M.T. Menárguez and the second author in [25]. There, it was shown that for all radial $f$ over $\mathbb{R}^n$ and $\lambda > 0$ one has

$$|\{x \in \mathbb{R}^n : M_\mu f(x) > \lambda\}| \leq \frac{4}{\lambda} \|f\|_{L^1(\mathbb{R}^n, dx)}.$$  

(3.1)

A previous generalization was due to A. Infante, who pointed out (see [20], [21]) that the same is true for $M_\mu$, if $\mu$ is an ‘increasing’ measure. By this we mean that $\mu(B(x, R)) \geq \mu(B(y, R))$ whenever $y = ax$ with $a \geq 1$. In [4] there is an independent approach with results similar to our Theorems 3.1 and 4.3. These two results already appeared in [14].

The main result of the section asserts that the uniform weakly-doubling property of a measure $\mu$ ensures uniform bounds in the dimension not only for radial functions but, in fact, over the whole class of functions $L^p(\mathbb{R}^n, d\mu)$, if $p > 1$.

**Theorem 3.2.** If there is a positive integer $N$ so that $\mu$ is uniformly weakly-doubling in $\mathbb{R}^n$ for $n \geq N$, then for each $p > 1$ there exists $C_p$ only depending on $\mu$ and $p$ so that

$$\|M_\mu f\|_{L^p(\mathbb{R}^n, \mu)} \leq C_p \|f\|_{L^p(\mathbb{R}^n, \mu)},$$

for all $f \in L^p(\mathbb{R}^n, \mu)$. 

In order to prove this result we first obtain the following characterization of radial measures that are uniformly weakly-doubling.

**Theorem 3.3.** There exists $N \in \mathbb{N}$ so that $d\mu = w(x)dx$, with $w(x) = w_0(|x|)$, is uniformly weakly-doubling in $\mathbb{R}^n$ for each $n \geq N$ if and only if $w$ is essentially constant on dyadic annuli, i.e. there exists $\beta \geq 1$ so that for all $R > 0$ one has

$$\begin{align*}
\text{ess sup}_{R \leq |x| \leq 2R} w(x) \leq \beta \text{ ess inf}_{R \leq |x| \leq 2R} w(x).
\end{align*}$$

(3.2)

The proof of this employs a new method of differentiation through dimensions that is presented in Section 3.2. With this characterization at hand, instead of proving Theorem 3.2 we will prove the equivalent:

**Theorem 3.4.** With the above notation, let $w$ be essentially constant over dyadic annuli with constant $\beta$. Then, there exists $N$ so that $\mu$ is locally finite in $\mathbb{R}^n$ for $n \geq N$ and, for each $p > 1$, there exists a constant $C_p$ only depending on $p$ and $\beta$ so that for $n \geq N$ one has

$$\begin{align*}
\|M_\mu f\|_{L^p(\mathbb{R}^n, \mu)} \leq C_p \|f\|_{L^p(\mathbb{R}^n, \mu)},
\end{align*}$$

for all $f \in L^p(\mathbb{R}^n, \mu)$.

The densities $|x|^\alpha$, $\alpha \in \mathbb{R}$, provide examples of measures with this property. It is easy to see that in each case they are essentially constant over dyadic annuli with constant $2^{|\alpha|}$.

At this point, let us make the simple observation that if $w$ is essentially constant over dyadic annuli, then $w$ can be compared pointwise with a continuous function in $\mathbb{R}^n \setminus \{0\}$. In the sequel, we will assume, as we may, that under this hypothesis, $w$ is indeed continuous with the possible exception at $x = 0$.

Theorem 3.4 will be obtained as a consequence of the following results.

**Lemma 3.5.** Let $\mu$ be locally finite in $\mathbb{R}^n$ and assume that its density $w$ is essentially constant over dyadic intervals with constant $\beta$. Then we have the following pointwise inequality

$$\begin{align*}
M_\mu f(x) \leq C \left( M f(x) + \frac{1}{w(x)} M(fw)(x) + \mathcal{H}_\mu f(x) \right),
\end{align*}$$

where $C$ depends only on $\beta$, $M$ is the usual Hardy-Littlewood maximal operator and $\mathcal{H}_\mu$ is defined as

$$\begin{align*}
\mathcal{H}_\mu f(x) = \sup_{R \geq |x|} \frac{1}{\mu(B_R)} \int_{B_R} |f(y)|d\mu(y).
\end{align*}$$

This transforms the problem of finding $L^p(\mu)$ bounds for $M_\mu$ to the one of finding them for $\mathcal{H}_\mu$ and for $M$.

**Theorem 3.6.** In the same hypotheses of Lemma 3.5 assume that in addition we have the weighted inequalities

$$\begin{align*}
\|Mf\|_{L^p(w)} &\leq W_1 \|f\|_{L^p(w)}, \\
\|Mf\|_{L^p(w^{1-p})} &\leq W_2 \|f\|_{L^p(w^{1-p})}.
\end{align*}$$

(3.3)

Then one has

$$\begin{align*}
\|M_\mu f\|_{L^p(\mu)} \leq C \|f\|_{L^p(\mu)},
\end{align*}$$

where the constant $C$ only depends on $\beta$, $W_1$ and $W_2$. 
Note that the required weighted inequalities are equivalent to \( w \in A_p \cap A_{p'} = A_{\min\{p,p'\}} \), where \( A_p = A_p(\mathbb{R}^n) \) denotes the usual Muckenhoupt class of weights. The existence of \( W_1 \) and \( W_2 \) in (3.3) is guaranteed due to the following

**Lemma 3.7.** Let \( w \) be essentially constant over dyadic intervals with constant \( \beta \). Then for each \( p > 1 \) there exist \( N \in \mathbb{N} \) depending only on \( p \) and \( \beta \) so that \( w \in A_p(\mathbb{R}^n) \) for all \( n \geq N \).

The constants \( W_1 \) and \( W_2 \) can be taken, in fact, independent of the dimension, as the following result by J. Duoandikoetxea and L. Vega, appeared in [16], shows.

**Theorem 3.8.** Let \( w_0 \) be a nonnegative function on \((0, \infty)\), so that \( w = w_0(|\cdot|) \in A_p(\mathbb{R}^N) \). Then for all \( n \geq N \) one has \( w \in A_p(\mathbb{R}^n) \). Moreover,
\[
\| M f \|_{L^p(\mathbb{R}^n, w)} \leq C \| f \|_{L^p(\mathbb{R}^n, w)},
\]
with a constant \( C \) that might depend on \( p \) and \( w \) but not on \( n \).

To complete the proof of Theorem 3.4, we have to show that if \( w \) is essentially constant over dyadic annuli, then there exists \( N \in \mathbb{N} \) so that \( w \) is locally integrable in \( \mathbb{R}^n \) for \( n > N \). This is immediate once we have the following auxiliary lemma, which will be used in different proofs throughout the paper.

**Lemma 3.9.** Let \( w \) be essentially constant over dyadic intervals with constant \( \beta \). Then there exists \( N \in \mathbb{N} \) only depending on \( \beta \), so that for \( n \geq N \) one has in \( \mathbb{R}^n \) the control
\[
\mu(B_R) = \int_{B_R} w(x) \, dx \leq 2 \beta w_0(R)|B_R|.
\]

In the remaining part of the section we present the proofs of Theorem 3.6, Lemmas 3.5, 3.7 and 3.9 and Theorem 3.1 in this given order.

We start showing how Lemma 3.5 implies Theorem 3.6.

**Proof of Theorem 3.6.** By Lemma 3.5 one has
\[
\| M_\mu f \|_{L^p(\mu)} \leq C \left( \| M f \|_{L^p(\mu)} + \left\| \frac{M(fw)}{w} \right\|_{L^p(\mu)} + \| \mathcal{H}_\mu f \|_{L^p(\mu)} \right),
\]
where the constant \( C \) depends only on \( \beta \).

By assumption we have
\[
\| M f \|_{L^p(\mu)} \leq W_1 \| f \|_{L^p(\mu)}.
\]

We also would like to have
\[
\left\| \frac{1}{w} M(fw) \right\|_{L^p(\mu)} \leq C \| f \|_{L^p(\mu)}.
\]
Taking \( g = fw \) this is equivalent to
\[
\| M_g \|_{L^p(w^{1-p})} \leq C \| g \|_{L^p(w^{1-p})},
\]
which we know true by assumption with \( C = W_2 \).
For $\mathcal{H}_\mu$ we use the standard argument for Hardy type operators. It is obvious that $\mathcal{H}_\mu$ is bounded on $L^\infty(\mu)$ with constant $1$. We will show that it is also weakly bounded on $L^1(\mu)$ with operator norm $1$. Then by real interpolation it is bounded on $L^p(\mu)$ with operator norm controlled by an absolute constant.

To see the weak type inequality take $\lambda > 0$ and consider $E_\lambda = \{ x \in \mathbb{R}^n : \mathcal{H}_\mu f(x) \geq \lambda \}$. If $x \in E_\lambda$ there exists $R_x > |x|$ so that

$$\frac{1}{\mu(B_{R_x})} \int_{B_{R_x}} |f(y)| d\mu(y) \geq \lambda.$$  

Note that then $B_{R_x} \subset E_\lambda$, and that

$$E_\lambda = \bigcup_{x \in E_\lambda} B_{R_x}.$$  

Then $E_\lambda = B_R$ for certain $R > 0$ and monotonicity gives

$$\mu(E_\lambda) = \mu(B_R) = \sup_{x \in E_\lambda} \mu(B_{R_x}) \leq \sup_{x \in E_\lambda} \frac{1}{\lambda} \int_{B_{R_x}} |f(y)| d\mu(y) \leq \frac{1}{\lambda} \int_{\mathbb{R}^n} |f(y)| d\mu(y).$$

\[ \square \]

**Proof of Lemma 3.3** We will bound the mean value over an arbitrary ball $B(x, R)$. Fixing $x$ and $R$, we consider different cases.

If $|x| \geq 2R$ and $y \in B(x, R)$ then

$$\frac{1}{2} |x| - R \leq |y| \leq |x| + R \leq \frac{3}{2} |x|.$$  

Since $w$ is essentially constant over dyadic annuli, we have $\beta^{-1}w(x) \leq w(y) \leq \beta w(x)$. Hence

$$\frac{1}{\mu(B(x, R))} \int_{B(x, R)} |f(y)| w(y) dy \leq \frac{\beta^2}{w(x) |B(x, R)|} \int_{B(x, R)} |f(y)| w(x) dy \leq \beta^2 M f(x).$$

In the case that $R/2 \leq |x| \leq 2R$ one has that if $y \in B(x, R) \setminus B_{R/2}$, then $|x|/4 \leq |y| \leq 3|x|$, which implies $\beta^{-2} w(|x|) \leq w(y) \leq \beta^2 w(|x|)$. Hence,

$$\mu(B(x, R)) \geq \mu(B(x, R) \setminus B_{R/2}) \geq \frac{1}{\beta^2} w(x) |B(x, R) \setminus B_{R/2}| \geq \frac{1}{2\beta^2} w(x) |B(x, R)|.$$  

Therefore we have

$$\frac{1}{\mu(B(x, R))} \int_{B(x, R)} |f(y)| w(y) dy \leq \frac{2\beta^2}{w(x) |B(x, R)|} \int_{B(x, R)} |f(y)| w(y) dy \leq \frac{2\beta^2}{w(x)} M(fw)(x).$$
Last, we consider that $|x| \leq R/2$. We split the ball $B(x, R)$ into two disjoint pieces $B_{R/2}$ and $B(x, R) \setminus B_{R/2}$ and integrate over them separately. For the first one

$$\frac{1}{\mu(B(x, R))} \int_{B_{R/2}} |f(y)| \, d\mu(y) \leq \frac{1}{\mu(B_{R/2})} \int_{B_{R/2}} |f(y)| \, d\mu(y) \leq \mathcal{H}_\mu f(x).$$

For the second one note that if $y \in B(x, R) \setminus B_{R/2}$ then $R/2 \leq |y| \leq 3R$ and then $\beta^{-2}w_0(R) \leq w(y) \leq \beta^2w_0(R)$. This implies that

$$\mu(B(x, R)) \geq \mu(B(x, R) \setminus B_{R/2}) \geq \frac{1}{\beta^2} w_0(R) |B(x, R) \setminus B_{R/2}|$$

and then we get

$$\frac{1}{\mu(B(x, R))} \int_{B(x, R) \setminus B_{R/2}} |f(y)| \, d\mu(y) \leq \frac{2\beta^4}{w_0(R) |B(x, R)|} \int_{B(x, R) \setminus B_{R/2}} |f(y)| \, w_0(R) \, dy \leq 2\beta^4 Mf(x).$$

\[ \square \]

**Remark.** Both, Theorem 3.6 and Lemma 3.5 do not require $\mu$ to be radial. For the applications, however, this requirement is the most natural in order to define $\mu$ and $M_\mu$ simultaneously in all dimensions and to study whether or not there are uniform bounds as $n \to \infty$.

**Proof of Lemma 3.7** We have to prove that $w \in A_p(\mathbb{R}^N)$ for some $N \in \mathbb{N}$. That is, there exists a constant $C > 0$ so that for all $x \in \mathbb{R}^N$ and $R > 0$ one has

$$\left( \int_{B(x, R)} w(y) \, dy \right) \left( \int_{B(x, R)} w(y)^{1/(1-p)} \, dy \right)^{p-1} \leq C.$$

Observe that $w^{1/(1-p)}$ is also constant over dyadic annuli. This is easy because

$$\sup_{R \leq |x| \leq 2R} w(x)^{1/(1-p)} = \left( \inf_{R \leq |x| \leq 2R} w(x) \right)^{1/(1-p)} \leq \left( \beta^{-1} \sup_{R \leq |x| \leq 2R} w(x) \right)^{1/(1-p)} = \beta^{1/(p-1)} \inf_{R \leq |x| \leq 2R} w(x)^{1/(1-p)}.$$
Then we can choose $N$ so that \( (3.4) \) holds for $w$ and $w^{1/(1-p)}$ in $\mathbb{R}^N$. If $|x| \leq 2R$, one has $B(x, R) \subset B_{3R}$ and then

$$\int_{B(x, R)} w(x) \, dx \left( \int_{B(x, R)} w(x)^{1/(1-p)} \, dx \right)^{p-1} \leq 3^p \int_{B_{3R}} w(x) \, dx \left( \int_{B_{3R}} w(x)^{1/(1-p)} \, dx \right)^{p-1} \leq 3^p 2\beta w(0)(3R) \left( 2\beta/(p-1) \right) \left( w(0)(3R)^{1/(1-p)} \right)^{p-1} \leq (22^p)^p \beta^2.$$ 

Assume conversely that $|x| > 2R$. If $y \in B(x, R)$, then $|x|/2 \leq |y| \leq 3|x|/2$ and consequently $\beta^{-1}w(x) \leq w(y) \leq \beta w(x)$.

Hence,

$$\int_{B(x, R)} w(y) \, dy \left( \int_{B(x, R)} w(y)^{1/(1-p)} \, dy \right)^{p-1} \leq \beta w(x) \left( \left( \beta^{-1}w(x) \right)^{1/(1-p)} \right)^{p-1} \leq \beta^2.$$ 

\[ \square \]

**Proof of Lemma 3.9** Assuming that we are in $\mathbb{R}^N$ with $2^N > \beta$ we have

$$\mu(B_R) = \omega_{N-1} \int_0^R w_0(t) t^{N-1} \, dt = \omega_{N-1} \sum_{j=0}^{\infty} \int_{2^{j-1}R}^{2^j R} w(t) t^{N-1} \, dt$$

$$\leq \omega_{N-1} \sum_{j=0}^{\infty} \int_{2^{j-1}R}^{2^j R} \beta^{j+1} w_0(R) t^{N-1} \, dt$$

$$\leq \beta w_0(R) \omega_{N-1} R^N \sum_{j=0}^{\infty} \left( \frac{\beta}{2N} \right)^j = \frac{\beta}{1 - \beta/2N} w_0(R) |B_R|.$$ 

Taking $N$ big enough one has that $1/(1 - \beta/2N) \leq 2$, as stated. \[ \square \]

For the proof of Theorem 3.1 we will follow [25]. First, associated with a weight $v$ on $\mathbb{R}$ we define the non-centered maximal function

\[ (3.5) \quad \hat{M}_v F(x) = \sup_{a \leq x \leq b} \frac{1}{v([a, b])} \int_{[a, b]} |F(t)| v(t) \, dt, \]

for each $F$ locally integrable with respect to $v$. Then, using a simple covering argument for intervals in $\mathbb{R}$ one has the estimate

\[ (3.6) \quad v(\{ r \geq 0 : \hat{M}_v F(r) > \lambda \}) \leq \frac{2}{\lambda} \int_{\mathbb{R}} |F(r)| v(r) \, dr, \]

(see [26], [18] or [23]).

We make now the following definition. Given a measurable set $E \subset \mathbb{R}^n$ we define its projection onto the sphere $S^{n-1}$ by

$$\Sigma_E = \{ \theta \in S^{n-1} : r\theta \in E \text{ for some } r > 0 \}.$$
The following geometrical result can be found in [25].

**Lemma 3.10.** For each ball $B(x, R) \in \mathbb{R}^n$ there exists a set $D$ such that

(a) $B(x, R) \subset B_n$,  
(b) $\Sigma_D = \Sigma_{B(x, R)}$,  
(c) for each $\theta \in \Sigma_D$ there exist $0 \leq a_\theta \leq b_\theta$ such that $r\theta \in D$ if and only if $a_\theta \leq r \leq b_\theta$,  
(d) $|x|\theta \in D$ for each $\theta \in \sigma_D$ (this means $a_\theta \leq |x| \leq b_\theta$),  
(e) $D$ is contained in the union of $B(x, R)$ and another ball $B(z, R)$ with $z \in B(x, R)$.

Using this, we prove Theorem 3.1.  

**Proof of Theorem 3.1.** By conditions (a) and (c) and the hypothesis that $\mu$ is weakly doubling, we have

$$
\frac{1}{\mu(B(x, R))} \int_{B(x, R)} |f(y)| \, d\mu(y) \leq \frac{\mu(D)}{\mu(B(x, R))} \frac{1}{\mu(D)} \int_D |f(y)| \, d\mu(y) \leq \frac{1 + K_1}{\mu(D)} \int_D |f(y)| \, d\mu(y).
$$

Assume that $f(x) = f_0(|x|)$ and set $v(t) = w_0(t)t^{n-1}$. Now we integrate along each ray coming from the origin and use conditions (b)–(d)

\[
\int_D |f(y)| \, d\mu(y) = \int_{\Sigma_D} \int_{a_\theta}^{b_\theta} |f_0(t)|v(t) \, dt \, d\sigma_n(\theta) = \int_{\Sigma_D} v([a_\theta, b_\theta]) \int_{a_\theta}^{b_\theta} |f_0(t)|v(t) \, dt \, d\sigma_n(\theta) \leq \int_{\Sigma_D} v([a_\theta, b_\theta]) \tilde{M}_v f_0(|x|) \, d\sigma_n(\theta).
\]

Note that

\[
\int_{\Sigma_D} v([a_\theta, b_\theta]) \, d\sigma_n(\theta) = \int_{\Sigma_D} \int_{a_\theta}^{b_\theta} v(t) \, dt \, d\sigma_n(\theta) = \mu(D),
\]

Hence we have proved that

\[
\frac{1}{\mu(B(x, R))} \int_{B(x, R)} |f(y)| \, d\mu(y) \leq (1 + K_1) \tilde{M}_v f_0(|x|),
\]

and, since $R$ is arbitrary, $M_\mu f(x) \leq (1 + K_1)\tilde{M}_v f_0(|x|)$. Integrating in polar coordinates we have

\[
\mu \left\{ x \in \mathbb{R}^n : Mf(x) > \lambda \right\} \leq \omega_{n-1} v \left( \left\{ r \geq 0 : \tilde{M}_v f_0(r) > \frac{\lambda}{1 + K_1} \right\} \right).
\]

By (3.6) the latter term is bounded by

\[
\frac{2(1 + K_1)}{\lambda} \omega_{n-1} \int_0^\infty |f_0(t)|v(t) \, dt = \frac{2(K_1 + 1)}{\lambda} \int_{\mathbb{R}^n} |f(x)| \, d\mu(x).
\]

□
3.2. Differentiation through dimensions. In this section we introduce a new technique that will be useful in settling several questions mentioned in this work, among them, the proof of Theorem 3.3. It starts with the following observation: take, in each Euclidean space $\mathbb{R}^n$, a ball $B^n$ of a fixed radius $R$ and with center at a fixed distance $s$ from the origin. If $w_0 \in L^1_{\text{loc}}([0, \infty), t^{N-1} dt)$, for some $N \geq 1$, and $m_n$ denotes Lebesgue measure in $\mathbb{R}^n$, then the limits
\[
\lim_{n \to \infty} \int_{B^n} w_0(|x|) \ dm_n(x)
\]
exist a.e. depending on $R$ and $s$. Observe that our integrability condition on $w_0$ ensures that the function $w(x) = w_0(|x|)$ is locally integrable in each $\mathbb{R}^n$ whenever $n \geq N$. Let us remark at this point that for balls $B^n$ of the same radius one always has $\lim_{n \to \infty} m_n(B^n) = 0$. This is not however the reason for this phenomenon that we will refer to as “differentiation through dimensions”. The precise statement of this type of differentiation is contained in the following result.

Lemma 3.11. Take $w_0 \in L^1_{\text{loc}}([0, \infty), t^{N-1} dt)$ for some $N \geq 1$. Then, for almost every $T > 0$ and for all $s \geq 0$ and $R > 0$ so that $s^2 + R^2 = T^2$, if we take points $z^n \in \mathbb{R}^n$ with $|z^n| = s$ and we denote $B(z^n, R) = \{y \in \mathbb{R}^n : |z^n - y| < R\}$, the following holds
\[
\lim_{n \to \infty} \int_{B(z^n, R)} w_0(|x|) \ dm_n(x) = w_0(T).
\]

Proof. The idea is to exploit the fact that in high dimensions the measure of a ball concentrates around ‘maximal circles’. We will assume that $w_0$ is positive.

Fix $T > 0$, and take positive $s$, $z^n \in \mathbb{R}^n$ with $|z^n| = s$ and $R$ with $s^2 + R^2 = T^2$. Observe that
\[
\int_{B(z^n, R)} w_0(|x|) \ dm_n(x) = \frac{n}{\omega_{n-1} R^n} \int_{(s-R)_+}^{s+R} w_0(t) A_n(t) t^{n-1} dt.
\]
where $A_n(t) = A_n^{(s,R)}(t) = |\{\theta \in \mathbb{S}^{n-1} : t \in B(z^n, R)\}|_{n-1}$. Define
\[
\varphi_n(t) = \varphi_n^{(s,R)}(t) = \frac{n}{\omega_{n-1} R^n} A_n(t) t^{n-1} \chi_{[(s-R)_+,s+R]}(t).
\]
With this notation we have
\[
\int_{B(z^n, R)} w_0(|x|) \ dm_n(x) = \int_{-\infty}^{\infty} w_0(t) \varphi_n^{(s,R)}(t) dt =: \Phi_n^{(s,R)} w_0.
\]
For a continuous $w_0$ the proof follows if we show that $\varphi_n \rightharpoonup \delta_T$ in the sense of distributions. For this it is enough to check that $\varphi_n$ is an approximation of the identity at the point $t = T$, that is
i) $\varphi_n \geq 0$ for all $n \in \mathbb{N}$,
ii) $\int_{-\infty}^{\infty} \varphi_n(t) dt = 1$ for all $n \in \mathbb{N}$,
iii) $\forall \varepsilon > 0$ one has $I_\varepsilon(n) = \int_{|t-T| > \varepsilon} \varphi_n(t) dt \xrightarrow{n \to \infty} 0$.

Note that [iii] is trivial and [ii] is immediate from the observation that
\[
\omega_{n-1} \int_{(s-R)_+}^{s+R} A_n(t) t^{n-1} dt = |B(z^n, R)|_n.
\]
To see [iii] observe that $I_\varepsilon(n) = I_{1\varepsilon}(n) + I_{2\varepsilon}(n)$ where
\[
I_{1\varepsilon}(n) = \int_{-\infty}^{T-\varepsilon} \varphi_n(t) dt = \frac{|B(z^n, R) \cap B_{T-\varepsilon}|}{|B(z^n, R)|_n} \leq \left(\frac{R_1}{R}\right)^n \xrightarrow{n \to \infty} 0,
\]
\[
I_{2\varepsilon}(n) = \int_{T+\varepsilon}^{\infty} \varphi_n(t) dt = \frac{|B(z^n, R) \setminus B_{T+\varepsilon}|}{|B(z^n, R)|_n} \leq \left(\frac{R_2}{R}\right)^n \xrightarrow{n \to \infty} 0,
\]
and $R_1$ and $R_2$ are the radii of the minimal balls that contain $B(z^n, R) \cap \partial B_{T- \varepsilon}$ and $B(z^n, R \setminus \partial B_{T+ \varepsilon}$ respectively. It is obvious that $R_1, R_2 < R$. This proves Lemma 3.11 for $T > 0$, if $w_0$ is continuous in $[0, \infty)$.

For a general $w_0$ we have to show that, if $P_T = \{(s, R) : s \geq 0, R > 0, s^2 + R^2 = T^2\}$, the set $E_0(w_0)$ defined as

$$
E_0(w_0) = \left\{ T \in [\varepsilon, T_0] : \exists (s, R) \in P_T : \limsup_{n \to \infty} \Phi_n(s, R)w_0 - \liminf_{n \to \infty} \Phi_n(s, R)w_0 > 0 \right\},
$$

has measure 0 for every $\varepsilon, T_0$ with $0 < \varepsilon < T_0$.

To that end, we fix $T \in [\varepsilon, T_0]$. For $t \in [(s - R)_+, s + R]$ we denote by $y(t)$ the diameter of the set $\partial B_t \cap B(z^n, R)$. Observe that $y(t) \leq R$ and that $y$ increases with $t$ up to the point $t = T$, where it attains its maximum, and then decreases in $(T, s + R)$. Also, for each $t \in [(s - R), s + R]$ we call $\alpha(t)$ the angle between the segment connecting the origin with $z^n$ and the one joining the origin with any point in $\partial B_t \cap \partial B(z^n, R)$. Clearly $y(t) = t \sin \alpha(t)$.

Observe first that the function $\varphi_n^{(s, R)}$ is decreasing in the interval $[T, s + R]$. To see this note that

$$
A_n(t) = t^{n-1} \frac{A_n(t) t^{n-1}}{\omega_{n-2}} = t^{n-1} \int_0^{\alpha(t)} (\sin \beta)^{n-2} d\beta = \int_0^{\alpha(t)} u^{n-2} \frac{du}{\sqrt{1 - (u/t)^2}},
$$

and that, both, the integrand and $y(t)$ decrease with $t$ in this interval. Hence,

$$
\int_T^{s+R} \varphi_n^{(s, R)}(t)w_0(t) dt \leq M_0 \overline{w}_0(T),
$$

where $M_0$ denotes the one-sided maximal operator

$$
M_0f(t) = \sup_{h>0} \frac{1}{h} \int_t^{t+h} |f(s)| ds,
$$

and $\overline{w}_0$ is the restriction of $w_0$ to the interval $[\varepsilon, 2T_0]$.

Let us assume that $s > 0$. If $\frac{\pi}{2} < \alpha(t) \leq \pi$, we will use that

$$
t^{1-N} \varphi_n^{(s, R)}(t) \leq \frac{n t^{n-N}}{R^n}.
$$

This case may happen only if $R > s$ and, then, $t \leq \sqrt{R^2 - s^2} < R$. For $\alpha(t) < \frac{\pi}{2}$ we have the estimate

$$
t^{1-N} \varphi_n^{(s, R)}(t) \leq \frac{n t^{n-N}}{\omega_{n-1} R^n} \omega_{n-2} \alpha(t) \sin^{n-2} \alpha(t) \leq \frac{n \omega_{n-2}}{\omega_{n-1} R^n} \frac{\pi}{2} y(t)^{n-N}.
$$

Combining (3.8) and (3.9) we see that for all $t_0 < T$ we have

$$
\lim_{n \to \infty} \left[ \sup_{t < t_0} t^{1-N} \varphi_n^{(s, R)}(t) \right] = 0.
$$

As a consequence, if $t_0 < T$

$$
\limsup_{n \to \infty} \int_0^{t_0} \varphi_n^{(s, R)}(t) w_0(t) dt \leq \lim_{n \to \infty} \left[ \sup_{t < t_0} t^{1-N} \varphi_n^{(s, R)}(t) \right] \int_0^{t_0} w_0(t) t^{N-1} dt = 0.
$$

Still for $\alpha(t) < \frac{\pi}{2}$, we make the observation that

$$
A_n(t) = \omega_{n-2} \int_0^{\alpha(t)} (\sin \beta)^{n-2} \cos \beta \cos^{-1} \beta d\beta,
$$

which is decreasing for $\alpha(t) < \frac{\pi}{2}$.
The final observation is that there exists $\tilde{\varphi}_n^{(s,R)}(t) = \frac{n}{n - 1} \frac{\omega_{n-2} y(t)^{n-1}}{R^n}$, then we have the double estimate

$$
\tilde{\varphi}_n^{(s,R)}(t) \leq \varphi_n^{(s,R)}(t) \leq \tilde{\varphi}_n^{(s,R)}(t) \cdot \cos^{-1} \alpha(t).
$$

It is important to point out that $\tilde{\varphi}_n^{(s,R)}(t)$ is increasing in $[0, T]$ and that

$$
\int_{[0,T]} \tilde{\varphi}_n^{(s,R)}(t) \, dt \leq \int_{[0,T]} \varphi_n^{(s,R)}(t) \, dt \leq 1.
$$

The final observation is that there exists $t_1 = t_1(s, R) < T$, so that $\alpha(t)$ decreases in $[t_1, T]$. Hence, for $t_0 \in [\max\{t_1, \varepsilon\}, T)$ we have

$$
\int_{t_0}^{T} \varphi_n^{(s,R)}(t)w_0(t) \, dt \leq \int_{t_0}^{T} \tilde{\varphi}_n^{(s,R)}(t)w_0(t) \cos^{-1} \alpha(t) \, dt
$$

where $M_{-}$ is the maximal operator

$$
M_{-} f(t) = \frac{1}{h} \int_{t-h}^{t} |f(s)| \, ds.
$$

For this $t_0$,

$$
\limsup_{n \to \infty} \Phi_n^{(s,R)} w_0 = \limsup_{n \to \infty} \left( \int_{t_0}^{T} \varphi_n^{(s,R)}(t)w_0(t) \, dt + \int_{t_0}^{T} \tilde{\varphi}_n^{(s,R)}(t)w_0(t) \, dt \right)
$$

$$
\leq \cos^{-1} \alpha(t_0) M_{-} \varpi_0(T).
$$

Taking the limit as $t_0 \to T^-$, we have

$$
\limsup_{n \to \infty} \Phi_n^{(s,R)} w_0 \leq \cos^{-1} \alpha(T) M_{-} \varpi_0(T) = \frac{T}{s} M_{-} \varpi_0(T)
$$

$$
\leq \left( 1 + \frac{R}{s} \right) M_{-} \varpi_0(T).
$$

Therefore, if $R/s \leq k$ for $k \in \mathbb{N}$ we have

$$
\limsup_{n \to \infty} \Phi_n^{(s,R)} w_0 \leq M_{-} \varpi_0(T) + (1 + k) M_{-} \varpi_0(T) \leq 3k \tilde{M} \varpi_0(T),
$$

with $\tilde{M}$ denoting, as in (3.5), the non centered one dimensional maximal operator

$$
\tilde{M} f(t) = \sup_{a \leq t \leq b} \frac{1}{b-a} \int_{a}^{b} |f(s)| \, ds.
$$

Observe that if $s = 0$, then $\varphi_n^{(0,T)}(t)$ is an increasing function in the interval $[0, T]$. Arguing as before to remove the integration on $[0, \varepsilon]$, we have

$$
\limsup_{n \to \infty} \Phi_n^{(0,T)} w_0 \leq M_{-} \varpi_0(T) \leq \tilde{M} \varpi_0(T).
$$

To finish the proof, for $T > 0$ and $k \in \mathbb{N}$ we denote $P_T^{k} = \{(s, R) : s, R > 0, R/s \leq k, s^2 + R^2 = T^2\} \cup \{(0, T)\}$ and we define for $\lambda > 0$ the set $E_{\lambda}(w_0)$ as

$$
\left\{ T \in [\varepsilon, T_0] : \exists (s, R) \in P_T^{k} : \limsup_{n \to \infty} \Phi_n^{(s,R)} w_0 - \liminf_{n \to \infty} \Phi_n^{(s,R)} w_0 > \lambda \right\}.
$$
Note that if \( g \) is continuous, the previous considerations and (3.6) give
\[
\left| E^k_\lambda(w_0) \right| = \left| E^k_0(w_0 - g) \right| \leq \left| \left\{ T > 0 : 6k\mathcal{M}(\overline{w_0 - g})(T) > \lambda \right\} \right|
\leq \frac{12k}{\lambda} \| \overline{w_0 - g} \|_{L^1},
\]
This last term can be made as small as needed with an appropriate choice of \( g \). Therefore \( \left| E^k_\lambda(w_0) \right| = 0 \). As a consequence, \( \left| E^k_0(w_0) \right| = \left| \bigcup_{\lambda > 0} E^k_\lambda(w_0) \right| = 0 \) and, hence, \( \left| E_0(w_0) \right| = \left| \bigcup_{k \in \mathbb{N}} E^k_0(w_0) \right| = 0 \), as wanted.

\[\square\]

3.3. Some applications of the differentiation through dimensions. As a consequence of this new technique of differentiation, we prove here Theorem 3.3 and show the equivalence between the two properties of weakly doubling and strong \( n \)-micro-doubling when they hold uniformly with the dimension.

**Proof of Theorem 3.3** First suppose that \( \mu \) is uniformly weakly doubling with constant \( K_1 \) in each \( \mathbb{R}^n \) with \( n \geq N \) for some \( N \in \mathbb{N} \). We have to prove that \( w \) is essentially constant over dyadic annuli.

With \( P_T \) as in the previous proof, let
\[
\mathcal{T} = \left\{ T > 0 : \lim_{n \to \infty} \frac{\mu(B(z^n, R))}{|B(z^n, R)|} = w_0(T), \quad \forall (|z^n|, R) \in P_T \right\}.
\]
We know from Lemma 3.11 that \( \mathcal{T} \) has full measure in \((0, \infty)\).

Now, for \( R \in \mathcal{T} \) and \( T \in \mathcal{T} \cap [R, 2R] \) we take \( z^n \in \mathbb{R}^n \) with \( |z^n|^2 + R^2 = T^2 \). Observe that
\[
0 \leq |z^n| \leq \sqrt{(2R)^2 - R^2} = \sqrt{3}R,
\]
so that \( B_R \cap B(z^n, R) \neq \emptyset \). By the hypothesis on \( \mu \)
\[
\frac{1}{K_1} \frac{\mu(B_R)}{|B_R|} \leq \frac{\mu(B(z^n, R))}{|B_R|} \leq K_1 \frac{\mu(B_R)}{|B_R|}.
\]
Taking limits when \( n \to \infty \), as we may since \( R, T \in \mathcal{T} \) we get
\[
(3.10) \quad \frac{1}{K_1} w_0(R) \leq w_0(T) \leq K_1 w_0(R).
\]
This shows that \( w_0 \) is essentially constant on dyadic intervals with constant \( \beta = K_1 \).

The reverse implication is straightforward and will be omitted. \[\square\]

We finish this section with a proof of the following important equivalence.

**Theorem 3.12.** A rotation invariant measure \( \mu \) is uniformly strong \( n \)-micro-doubling in each \( \mathbb{R}^n \) for \( n \geq N \) if and only if \( \mu \) is uniformly weakly doubling.

**Proof.** In view of Theorem 3.3 we only need to prove that a measure given by a radial density that is essentially constant over dyadic annuli, is also uniformly \( n \)-micro-doubling.

Assume that \( w \) is essentially constant over dyadic annuli with constant \( \beta \) and take \( n \geq N \) for the \( N \in \mathbb{N} \) obtained in Lemma 3.9. Given \( x \in \mathbb{R}^n \) and \( R > 0 \), we will write \( B^*(x, R) = B(x, (1 + 1/n)R) \). We first consider the case \( |x| \geq 3R \), which implies that \( (1 + 1/n)R \leq 2|x|/3 \). If \( y \in \)
\(B^*(x, R)\), then \(|x|/3 \leq |y| \leq 5|x|/3\) which means that \(\beta^{-2} w(x) \leq w(y) \leq \beta^2 w(x)\). From this we get
\[
\mu(B^*(x, R)) = \int_{B^*(x, R)} w(y) \, dy \leq \beta^2 \, w(x) \, |B^*(x, R)| \\
\leq e\beta^2 \, w(x) \, |B(x, R)| \leq e\beta^4 \, \mu(B(x, R)).
\]

If \(|x| \leq 3R\) we split \(B^*(x, R)\) into two disjoint pieces. For the one intersecting \(B_{R/2}\) we use Lemma 3.9 as follows
\[
\mu(B^*(x, R) \cap B_{R/2}) \leq \mu(B_{R/2}) \leq 2\beta \, w_0(R/2) \, |B_{R/2}| \leq \beta^2 \, w_0(R) \, |B_R|.
\]
In the complementary piece \(w\) is essentially constant: if \(y \in B^*(x, R) \setminus B_{R/2}\), then \(R/2 \leq |y| \leq 5R\), which means that \(\beta^{-3} w_0(R) \leq w(y) \leq \beta^3 \, w_0(R)\). Hence
\[
\mu(B^*(x, R) \setminus B_{R/2}) \leq \beta^3 \, w_0(R) \, |B^*(x, R)| \leq e\beta^3 \, w_0(R) \, |B_R|.
\]
On the other hand note that if \(y \in B(x, R) \setminus B_{R/2}\) then \(R/2 \leq |y| \leq 4R\) and consequently \(\beta^{-2} w_0(R) \leq w(y) \leq \beta^2 \, w_0(R)\). This yields
\[
\mu(B(x, R)) \geq \mu(B(x, R) \setminus B_{R/2}) \geq \frac{1}{\beta^2} \, w_0(R) \, |B(x, R) \setminus B_{R/2}| \\
\geq \frac{1}{2\beta^2} \, w_0(R) \, |B_R|.
\]

**Remark.** Observe that the only thing that we have really proved here is that a uniform weakly doubling property implies a uniform micro-doubling property. The reciprocal is not true. In the last Section, we show an example of a density for which the associated measure is uniformly \(n\)-microdoubling but is not uniformly weakly doubling.

### 3.4. Decreasing densities.
Radial measures with decreasing densities have some interesting and sometimes surprising properties. For instance, to ensure that a measure with such a density is uniformly weakly bounded, it is enough to check this condition just in one concrete Euclidean space.

**Proposition 3.13.** Let \(w_0\) be a decreasing function over \([0, \infty)\). If \(\mu\) is weakly doubling in \(\mathbb{R}^N\) for some \(N \in \mathbb{N}\), then \(\mu\) is uniformly weakly doubling in \(\mathbb{R}^n\) for all \(n \geq N\).

**Proof.** Assume that \(\mu\) is weakly doubling with constant \(K_1\) in \(\mathbb{R}^N\) for some \(N \in \mathbb{N}\). By Theorem 3.3 we only need to show that \(w\) is essentially constant over dyadic annuli. Given \(R > 0\), if \(R \leq |x| \leq 2R\), since \(w_0\) is decreasing, one has \(w_0(2R) \leq w(x) \leq w_0(R)\). Thus, we only need to check that \(w(R) \leq C w(2R)\) with \(C\) independent of \(R\). To do so, consider a point \(z \in \mathbb{R}^N\) so that \(|z| = R/2\). Take \(z' = 3z\) and \(z'' = 5z\) and consider the balls \(B = B(z, R/2)\), \(B' = B(z', R/2)\) and \(B'' = B(z'', R/2)\).
Letting \( w \in \mathcal{F}_{\text{rad}} \), we have \( w \) uniformly bounded on \( \mathbb{R}^n \) for all \( n \geq N \).\( \square \)

Also, if the density is decreasing, the reciprocal implication in Theorem 3.1 is true.

**Proposition 3.14.** Let \( w_0 \) be a decreasing function over \([0, \infty)\). Assume that there exists \( C > 0 \) so that for some \( N > 0 \) one has \( \|M_\mu\|_{L^1(\mathbb{R}^n, d\mu)} \leq C \) for all \( n \geq N \). Then \( \mu \) is uniformly weakly doubling in \( \mathbb{R}^n \) for all \( n \geq N \).

Observe that by Theorem 3.2 the hypothesis in the previous proposition also implies that \( M_\mu \) is uniformly bounded on \( L^p(\mathbb{R}^n, d\mu) \) for all \( n \geq N \). It is remarkable that a boundedness condition over radial functions implies one for general functions. The proof of this Proposition is based on differentiation through dimensions.

**Proof.** Assume that \( M_\mu \) satisfies the weak \( L^1(\mu) \) inequality
\[
\mu \left( \{ x \in \mathbb{R}^n : M_\mu f(x) > \lambda \} \right) \leq \frac{C_*}{\lambda} \int_{\mathbb{R}^n} |f(x)| \, d\mu(x),
\]
for all \( f \in L^1_{\text{rad}}(\mathbb{R}^n) \) and \( n \geq N \) for certain \( N \in \mathbb{N} \), with \( C_* \) independent of \( f \) and \( n \). As before, it is enough to show that \( w \) is essentially constant over dyadic annuli. Since \( w_0 \) is decreasing, it suffices to prove that \( w_0(R) \leq Cw_0(2R) \) with \( C \) independent of \( R \). Fix \( R > 0 \) and for each \( n \geq N \), take \( z^n \in \mathbb{R}^n \) so that \(|z^n| = R\). Set \( f = \chi_{B_{r}} \) with \( 0 < \varepsilon < R \) and
\[
\lambda = \frac{\mu(B(z^n, R + \varepsilon))}{\mu(B_{r})}.
\]
Clearly we have \( B_{r} \subset \{ x \in \mathbb{R}^n : M_\mu f(x) > \lambda \} \), which together with our hypothesis gives
\[
\mu(B_{r}) \leq \frac{C_*}{\lambda} \int_{\mathbb{R}^n} |f(x)| \, d\mu(x) = C_* \, \mu(B(z^n, R + \varepsilon)).
\]
Letting \( \varepsilon \to 0^+ \), by monotonicity we obtain \( \mu(B_{r}) \leq C_* \mu(B(z^n, R)) \). In particular we have
\[
\frac{\mu(B_{r})}{|B_{r}|} \leq C_* \frac{\mu(B(z^n, R))}{|B(z^n, R)|},
\]
and now we differentiate through dimensions. Letting \( n \to \infty \) in the last inequality, Lemma 3.11 gives \( w_0(R) \leq C_*w_0(\sqrt{2}R) \) from which we deduce that \( w_0(R) \leq Cw_0(2R) \) with \( C = C_*^2 \). \( \square \)
We now make a connection between the previous hypotheses and the Muckenhoupt $A_1$ condition. We recall that for a positive $w \in L^1_{\text{loc}}(\mathbb{R}^n)$ the $A_1$-constant we call $[w]_{A_1(\mathbb{R}^n)}$ is defined as the smallest constant $C$ so that
\[ \int_{B(x,R)} w(y) \, dy \leq C \, \text{ess inf}_{y \in B(x,R)} w(y), \]
for all $x \in \mathbb{R}^n$ and $R > 0$. Under the hypothesis that $w_0$ is decreasing in $[0, \infty)$ and $d\mu(x) = w_0(|x|) \, dx$ is uniformly weakly doubling we will show that $w(x) = w_0(|x|) \in A_1(\mathbb{R}^n)$ for all sufficiently large $n$. Moreover, the $A_1$-constants can be bounded independently of $n$. The precise statement is the following.

**Proposition 3.15.** Let $w_0$ be a decreasing function over $[0, \infty)$ and essentially constant over dyadic intervals with constant $\beta$. Then there exist $N > 0$ and $C > 0$ only depending on $\beta$ so that $[w]_{A_1(\mathbb{R}^n)} \leq C$ for all $n \geq N$.

**Proof.** Assume that $w$ is essentially constant over dyadic intervals with constant $\beta$ and take $n \geq N$ for the $N$ given in Lemma 3.9. We have to find a constant $C > 0$ so that for all $x \in \mathbb{R}^n$ and $R > 0$ one has
\[ \frac{\mu(B(x,R))}{|B(x,R)|} \leq C \inf_{y \in B(x,R)} w(y). \]
If $|x| \leq 2R$, since $w$ is decreasing and using Lemma 3.9 and that $w$ is essentially constant over dyadic annuli one has
\[ \frac{\mu(B(x,R))}{|B(x,R)|} \leq \frac{\mu(B_R)}{|B_R|} \leq 2\beta w_0(R) \leq 2\beta^3 w_0(4R). \]
This finishes the proof in this case because now using again that $w_0$ is decreasing we obtain
\[ w_0(4R) \leq \inf_{y \in B_{4R}} w(y) \leq \inf_{y \in B(x,R)} w(y). \]
If $|x| \geq 2R$ then $w$ is essentially constant over $B(x,R)$. Indeed if $y \in B(x,R)$ then $|x|/2 \leq |y| \leq 3|x|/2$, which implies that $\beta^{-1} w(x) \leq w(y) \leq \beta w(x)$. Hence,
\[ \frac{\mu(B(x,R))}{|B(x,R)|} \leq \beta w(x) \leq \beta^2 \inf_{y \in B(x,R)} w(y). \]

The amazing thing here is that the reciprocal statement of this last proposition is also true.

**Proposition 3.16.** Let $w_0$ be a non-negative function over $[0, \infty)$ and set $w(x) = w_0(|x|)$ so that $w \in A_1(\mathbb{R}^n)$ for $n \geq N > 0$. If there exists $C_* > 0$ so that $[w]_{A_1(\mathbb{R}^n)} \leq C_*$ for all $n \geq N$, then $w_0$ is essentially constant over dyadic intervals and comparable with a decreasing function.

**Proof of Proposition 3.16.** It is enough to prove that $w$ is comparable with a decreasing function $\bar{w}$ because then $\bar{w} \in A_1$ and, therefore, $\bar{w}$ is (weakly) doubling. By Proposition 3.13 and Theorem 3.3 we would have that $\bar{w}$ is essentially constant over dyadic intervals, and hence so is $w$.

It suffices to check that there is a constant $q > 0$ so that $w_0(t) \leq q w_0(s)$ whenever $0 \leq s \leq t$. For $0 < s < t$ and $n \geq N$, take $x_n, y_n \in \mathbb{R}^n$ so that $x_n$ is a multiple of $y_n$ and $|x_n| = s$, $|y_n| = t$. Consider the ball $B(x_n, R)$ with $R^2 = t^2 - s^2$. 
By hypothesis we have, in the almost everywhere sense,
\[ \mu(B(x_n, R)) \leq C_* w(x_n) = C_* w_0(s). \]
Letting \( n \to \infty \), by the differentiation through dimensions of Lemma 3.11 we get that
\[ w_0(t) = w_0(\sqrt{s^2 + R^2}) \leq C_* w_0(s), \]
for almost every \( s \leq t \). \( \square \)

4. Further results and remarks

4.1. Examples of measures that are not uniformly weakly-doubling. Each measure \( \mu \) with density \( w(x) = |1 - |x||^{-\alpha} \) for \( 0 < \alpha < 1 \) is doubling on \( \mathbb{R}^n \) for each \( n \in \mathbb{N} \), but is not uniformly weakly doubling. To see this observe that in each \( \mathbb{R}^n \) one has
\[ \mu(B(1)) \geq C_1 n^\alpha |B_1|. \]
Using differentiation through dimensions, if \( |z| = 1 \), then \( \mu(B(z, 1)) \to \infty \) \( w(\sqrt{2}) = (\sqrt{2} - 1)^{-\alpha} \). This says that the measures of the intersecting balls \( B_1 \) and \( B(z, 1) \) are not comparable with constants independent of the dimension.

Using this idea we can also prove that there are not uniform weak \( L^1(\mu) \) bounds for the associated maximal operator.

**Theorem 4.1.** Let \( \mu \) be the measure defined above and let \( C_{1,n} \) be the smallest constant satisfying
\[ \mu(\{x \in \mathbb{R}^n : M_\mu f(x) > \lambda\}) \leq C_{1,n} \frac{1}{\lambda} \|f\|_{L^1(\mathbb{R}^n, \mu)}, \]
for all \( f \in L^1(\mathbb{R}^n, \mu) \). Then, one has \( C_{1,n} \geq cn^\alpha \).

**Proof.** Using discretization (see [19] and [24]), or the argument in the proof of Proposition 3.14 we may consider \( M_\mu \) acting over finite sums of Dirac deltas instead of integrable functions. For \( \delta_0 = \lim_{\varepsilon \to 0} \chi_{B_\varepsilon}(x)/\mu(B_\varepsilon) \), in the sense of distributions, the weak \( L^1(\mu) \) inequality reads
\[ \mu(\{x \in \mathbb{R}^n : M_\mu \delta_0(x) > \lambda\}) \leq C_{1,n} \frac{1}{\lambda}. \]
Note that \( M_\mu \delta_0(x) = 1/\mu(B(x, |x|)) \) for each \( x \neq 0 \), which makes \( M_\mu \delta_0 \) a radially decreasing function. Then taking \( \lambda = M_\mu \delta_0(z) \) with \( |z| = 1 \) we have \( \{x \in \mathbb{R}^n : M_\mu \delta_0(x) > M_\mu \delta_0(z)\} \subset B_1 \) and from (4.1) we obtain
\[ C_{1,n} \geq \frac{\mu(B_1)}{\mu(B(z, 1))}. \]
This, together with the previous observation that \( \mu(B_1) \geq Cn^\alpha|B_1| \) and that \( \mu(B(z,1))/|B_1| \to w(\sqrt{2}) \) as \( n \to \infty \), proves the result.

\[ \Box \]

Remark. One can prove with some extra work that \( \mu \) is uniformly \( n \)-micro-doubling. It is also weakly doubling but in this case with a constant \( K_1 \sim n^\alpha \) in each \( \mathbb{R}^n \). In particular, Theorems 1.1 and 4.1 imply that \( C_{1,n} \leq c n^{1+\alpha} \log n \).

4.2. Families of measures changing with the dimension. As we have seen in the preceding sections, maximal operators associated with measures given by power densities have \( L^p \) operator norms bounded with respect to the dimension. An interesting observation by J.M. Aldaz and J. Pérez Lázaro in [3] showed that given an exponent \( p \) (as large as wanted) there exist families of power weights such that the \( L^p \) bounds of the associated maximal operators grow to infinity as \( n \to \infty \). The twist here is that the powers change from one dimension to another. To be more precise they considered measures \( \nu_{\alpha,n} \) given by the densities \(|x|^{-\alpha n}\) over \( \mathbb{R}^n \) with \( 0 < \alpha < 1 \). Their result is the following (see Theorem 3.12 in [3]).

**Theorem 4.2.** Given \( p_0 \in [1, \infty) \), there exist \( \alpha_0 \in (0, 1) \) and \( a > 1 \) such that for all \( p \in [1, p_0] \) and all \( \alpha \in [\alpha_0, 1) \) one has
\[
c_{\nu_{\alpha,n},p} \geq \frac{a^{(1-\alpha)n}}{6}.
\]

It is implicit in the proof given in [3] that \( \alpha_0 \to 1 \) as \( p_0 \to \infty \). This leads to the question of whether, fixing \( \alpha, M_{\nu_{\alpha,n}} \) may satisfy a uniform \( L^p \) bound for large \( p \). We can apply the method used in Theorem 15 for the Gaussian measure to show that this is not the case when \( \alpha > 1/2 \).

**Theorem 4.3.** For each \( \alpha \in (1/2, 1) \) there exists a constant \( a > 1 \) such that for all \( p \in [1, \infty) \)
\[
c_{\nu_{\alpha,n},p} \geq ca^{n/p},
\]
even if the action is restricted to radially decreasing functions.

The proof of this result can be found in [14] and in [4].

A consequence of this result is that for these families of measures the constants of the \( n \)-micro-doubling and the weak doubling conditions grow to infinity with the dimension (see Theorems 1.1, 3.1 and 3.6).

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