Rogue waves and lumps on the non-zero background in the $\mathcal{PT}$-symmetric nonlocal Maccari system

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Abstract. In this paper, the $\mathcal{PT}$-symmetric version of the Maccari system is introduced, which can be regarded as a two-dimensional generalization of the defocusing nonlocal nonlinear Schrödinger equation. Various exact solutions of the nonlocal Maccari system are obtained by means of the Hirota bilinear method, long-wave limit, and Kadomtsev-Petviashvili (KP) hierarchy method. Bilinear forms of the nonlocal Maccari system are derived for the first time. Simultaneously, a new nonlocal Davey-Stewartson-type equation is derived. Solutions for breathers and breathers on top of periodic line waves are obtained through the bilinear form of the nonlocal Maccari system. Hyperbolic line rogue-wave solutions and semi-rational ones, composed of hyperbolic line rogue wave and periodic line waves, are also derived in the long-wave limit. The semi-rational solutions exhibit a unique dynamical behavior. Additionally, general line soliton solutions on constant background are generated by restricting different tau-functions of the KP hierarchy, combined with the Hirota bilinear method. These solutions exhibit elastic collisions, some of which have never been reported before in nonlocal systems. Additionally, the semi-rational solutions, namely (i) fusion of line solitons and lumps into line solitons, and (ii) fission of line solitons into lumps and line solitons, are put forward in terms of the KP hierarchy. These novel semi-rational solutions reduce to $2N$-lump solutions of the nonlocal Maccari system with appropriate parameters. Finally, different characteristics of exact solutions for the nonlocal Maccari system are summarized. These new results enrich the structure of waves in nonlocal nonlinear systems, and help to understand new physical phenomena.

Keywords: $\mathcal{PT}$-symmetric nonlocal Maccari system · Rogue wave · Semi-rational solutions · Hirota bilinear method · KP hierarchy reduction method

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1. Introduction

Nonlinear evolution equations serve as models for various complex physical phenomena, therefore exact solutions of such equation have great significance\cite{1,2,3}. In particular, rogue waves (RWs), as a kind of exact solutions, have drawn much interest\cite{1,2,3,4,5,6,7,8,9,10}. RW is a special wave famous for its destructive force. Its amplitude can exceed the background value by a factor of up to 3\cite{15}. RW was first derived by Peregrine from the nonlinear Schrödinger equation\cite{15}. Subsequently, a series of other soliton equations have been shown to possess RW solutions\cite{16,17,18,19,20}. Manifestations of RWs have been identified in Bose-Einstein condensates\cite{21}, optical systems\cite{22,23}, superfluids\cite{24}, and finances\cite{25}. Semi-rational solutions consisting of RWs, lumps, solitons and periodic line waves also have attracted considerable attention\cite{26,27,28,29,30,31,32,33,34}. There are several methods to obtain the semi-rational solutions of nonlinear evolution equations, including the Darboux transform and the bilinear method.

Nonlinear evolution equations may be naturally divided in two vast classes, viz., local and nonlocal ones. Recently, Ablowitz and Musslimani have introduced a nonlocal reverse-space nonlinear

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Schrödinger (NLS) equation \[35\]

\[iu_t(x,t) - u_{xx}(x,t) - u^2(x,t)u_x(-x,t) = 0. \tag{1}\]

This equation generalizes the standard NLS equation by making it to satisfy the parity-time (PT) symmetry condition\[35\]. Ablowitz and Musslimani have obtained multi-soliton solutions of eq. (1) by means of the inverse scattering transform method\[35\]. Since then, the discrete versions\[36\]-[39] have been studied in depth. Subsequently, several extended one-dimensional and two-dimensional nonlocal nonlinear evolution equations have been introduced\[40\]-[45]. An issue of obvious interest is to generate a multidimensional integrable model with PT-symmetry and explore interaction dynamics in it, which is a motivation for the present work.

For nonlinear evolution equations with a dispersive linear part, wave modulation induced by weak nonlinearity is a topic of general interest too, with applications to plasma physics, nonlinear optics, hydrodynamics, etc. A limited number of model equations play a crucially important role for this purpose. They are produced by the reduction method applied to a family of nonlinear evolution equations under suitable approximations\[53\]-[56]. Of special interest are integrable model equations. To derive a (2+1)-dimensional integrable system with coordinates \((x,y)\) and temporal variable \(t\), Maccari\[57\] used a moving reference frame,

\[\xi = e^{p_1}(x - V_1 t), \quad \eta = e^{p_2}(y - V_2 t), \quad \tau = e^q t, \tag{2}\]

where \(p_1, p_2, q > 0\) and \(V(K) \equiv [V_1(K_1, K_2), V_2(K_1, K_2)]\) is the vector of the group velocity of the linearized equation, and \(\epsilon\) is the expansion parameter, supposed to be sufficiently small. In particular, starting from the celebrated Kadomtsev-Petviashvili (KP) equation

\[U_{xt}(x,y,t) - 3U_{xx}(x,y,t) + U_{xxxx}(x,y,t) + sU_{yy}(x,y,t) = 0, \tag{3}\]

\(s\) is an arbitrary real constant, its linear dispersive part can be described in terms of Fourier modes, with components of the group velocity

\[V_1(K_1, K_2) = -3K_1^2 - \frac{sK_2^2}{K_1^2}, \quad V_2(K_1, K_2) = \frac{2sK_2}{K_1}. \tag{4}\]

Then, the following asymptotic Fourier expansion is introduced,

\[U(x,y,t) = \sum_{n=-\infty}^{n=+\infty} \epsilon^n \psi_n(\xi, \eta, \tau; \epsilon)e^{i n (K_1 x + K_2 y - \omega t)}, \tag{5}\]

where \(\gamma_n = |n|\) and \(\gamma_0 = 1 + r\) are real constants, and \(\psi_{-n}(\xi, \eta, \tau; \epsilon) = \psi_n^*(\xi, \eta, \tau; \epsilon)\), because \(U(x,y,t)\) is real. It is assumed that the limit of \(\psi_n(\xi, \eta, \tau; \epsilon)\)'s, for \(\epsilon \to 0\), exists and is finite. The purpose of this assumption is to derive an evolution equation for the modulation amplitude, \(\psi_1(\xi, \eta, \tau; \epsilon \to 0) \equiv \Psi(\xi, \eta, \tau)\).

If \(p_1 \neq p_2\) in eq. (2) for the proper balance of terms, we get for \(n = 0\) (see eq. (5))

\[\Phi_1(\xi, \eta, \tau) + 2\sqrt{3}K_1 \Phi_\eta(\xi, \eta, \tau) - 6|\Psi(\xi, \eta, \tau)|^2_\xi = 0, \tag{6}\]

and for \(n = 1\)

\[i\Phi_1(\xi, \eta, \tau) - 6K_1 \Phi_\xi(\xi, \eta, \tau) + 6K_1 \Psi(\xi, \eta, \tau) \Phi(\xi, \eta, \tau) = 0, \tag{7}\]

with \(3K_1^4 = K_2^2, p_1 = \frac{3}{2}, p_2 = \frac{1}{2}, q = \frac{1}{2}, r = \frac{1}{4}\). By means of obvious rescaling, the following KP-type system is produced \[57\]-[58]

\[iu_t(x, y, t) + u_{xx}(x, y, t) + u(x, y, t)v(x, y, t) = 0, \quad \tag{8}\]

\[v_t(x, y, t) + v_y(x, y, t) + |u(x, y, t)|^2_x = 0. \]

Various exact solutions of local and nonlocal KP-type system\[8\] including solitons, breathers, rational and semi-rational solutions have been reported \[59\]-[64].

If \(p_1 = p_2\), for the proper balance of different terms, we get for \(n = 0\)

\[V_1 \Phi_\xi + V_2 \Phi_\eta - s \Phi_\eta + 6|\Psi|^2_\xi = 0, \tag{9}\]
Rogue waves and lumps on the non-zero background in the $\mathcal{PT}$-symmetric nonlocal Maccari system and for $n = 1$

\[
iK_1 \Psi_t - (6K_1 + V_1) \Psi_{\xi} - V_2 \Psi_{\eta} + s \Psi_{\eta\eta} + 6k_1^2 \Phi \Psi - 6|\Psi|^2 \Psi = 0,
\]

where $\Phi = \Phi(\xi, \eta, \tau)$, $\Psi = \Psi(\xi, \eta, \tau)$, $3K_1^4 = K_2^2$, $p_1 = p_2 = 1$, $q = 2$, $r = 1$, $\Phi(\xi, \eta, \tau) = \psi_0(\xi, \eta, \tau)$. After a rescaling, the following Maccari system is obtained [57]:

\[
iu_t(x,y,t) + L_1 u(x,y,t) + u(x,y,t)v(x,y,t) = 0,
L_2 v(x,y,t) - 2L_1 [u(x,y,t)u^*(x,y,t)] = 0,
\]

where $v(x,y,t)$ is a real function and linear differential operators $L_1$ and $L_2$ are defined as

\[
L_1 = \frac{1}{4}(1 - s\lambda^2) \frac{\partial^2}{\partial x^2} + s\lambda \frac{\partial}{\partial x} \frac{\partial}{\partial y} - s \frac{\partial^2}{\partial y^2},
L_2 = -\frac{1}{4}(1 + s\lambda^2) \frac{\partial^2}{\partial x^2} + s\lambda \frac{\partial}{\partial x} \frac{\partial}{\partial y} - s \frac{\partial^2}{\partial y^2},
\]

where $\lambda \equiv K_2/\sqrt{3K_1}$. This system belongs to a class of nonlinear evolution equations first studied by Shulman [65], which find applications to nonlinear optics, plasma physics and other physical fields. The integrability property and Lax pairs for system (11) has been demonstrated by Maccari [57]. Note that, for $\lambda = 0$, this system reduces to the Davey-Stewartson (DS) equation, whose integrability is a well-known fact [67]. Various exact solutions have been obtained for local and nonlocal DS equations [68]-[75]. Because the Maccari system (11) is quite complex in the general form, exact solutions have been reported for it in only in the special case of $s = -1$ [76].

Inspired by the above considerations, we introduce an extension of system (11) satisfying the $\mathcal{PT}$ symmetry, which we refer to as the nonlocal Maccari system:

\[
iu_t(x,y,t) + L_1 u(x,y,t) + u(x,y,t)v(x,y,t) = 0,
L_2 v(x,y,t) - 2L_1 [u(x,y,t)u^*(-x,-y,t)] = 0,
\]

with the same operators $L_1$ and $L_2$ as defined by eq. (12). System (13) satisfies the condition of the $2D \mathcal{PT}$ symmetry. The concept of the $\mathcal{PT}$ symmetry was first proposed by Bender and Boettcher in quantum theory [77,78]. They had shown that the spectra of non-Hermitian Hamiltonian operators are all real as long as the $\mathcal{PT}$ symmetry is not broken. Inspired by that pioneering work, $\mathcal{PT}$ symmetry has made a series of important achievements in theoretical and experimental physics [79,80,81], and has also driven new development in other fields, including Lie algebras [82], the quantum field theory [83], complex crystals [84,85], optics and photonics. In particular, a new two-dimensional nonlinear model was recently introduced in Ref. [86], in which the $\mathcal{PT}$ symmetry remains unbroken for arbitrarily large values of the gain-loss coefficient. In the same work, an extended two-dimensional model with an imaginary part of the potential $\sim xy$, written in the Cartesian coordinates. The latter is not a $\mathcal{PT}$-symmetric model, but it also supports a continuous family of self-trapped states, which suggests an extension of the concept of the $\mathcal{PT}$ symmetry [86], which is another motive for addressing the nonlocal Maccari system [13].

In this paper, we present various exact solutions of the nonlocal Maccari system based on the Hirota bilinear method, long-wave limit, and the KP-hierarchy approach. In section 2 we derive the bilinear form of the nonlocal Maccari system, and present a new nonlocal DS-type equation. In section 3 and section 4 we obtain breather solutions on top of a periodic background and line RWs on the periodic background, by dint of the Hirota bilinear method and long-wave limit. In this context, semi-rational solutions feature unique dynamics. In section 5 and section 6 by restricting different tau functions of the KP hierarchy, general line solitons and general lump-soliton solutions are generated. In section 7 we discuss and summarize our results.
2. Bilinear forms of nonlocal Maccari system and its reduction

With the help of the independent variable transformation,
\[ u = \frac{g}{f}, \quad v = \left(\frac{1}{2}(1 - s\lambda^2)\partial_{xx} + 2s\lambda\partial_{xy} - 2s\partial_{yy}\right)(\ln f), \]
the nonlocal Maccari system \[13\] admits the following bilinear forms:
\[ (iD_t + \frac{1}{4}(1 - s\lambda^2)D_x^2 + s\lambda D_x D_y - sD_y^2)g(x, y, t) \cdot f(x, y, t) = 0, \]
\[ \left(\frac{-1}{4}(1 + s\lambda^2)D_x^2 + s\lambda D_x D_y - sD_y^2 + 2\right)f \cdot f = 2g(x, y, t) \cdot g^*(-x, -y, t), \]
where \( D \) is the Hirota’s bilinear differential operator\[2\] defined by
\[ D^n D^n f(x, t) \cdot g(x, t) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n f(x, t)g(x', t') \bigg|_{x' = x, t' = t}, \]
and the following condition must hold:
\[ f(x, y, t) = f^*(-x, -y, t). \]

- When \( \lambda = 0 \), the nonlocal Maccari system\[13\] reduces to a new nonlocal DS-type system:
  \[
  \begin{align*}
  iu_t + \frac{1}{4}u_{xx} - su_{yy} + uv &= 0, \\
  \frac{1}{4}v_{xx} + sv_{yy} &= -\frac{1}{2}|u|^2_{xx} + 2s|u|^2_{yy} = 0, \\
  |u|^2 &= u(x, y, t)u^*(-x, -y, t).
  \end{align*}
\]

- When \( s = 0 \), the nonlocal Maccari system\[13\] reduces to the defocusing nonlocal NLS equation:
\[ iu_t(x, t) + \frac{1}{4}u_{xx}(x, t) - \frac{1}{2}u^2(x, t)u^*(-x, t) = 0. \]

cf. its focusing counterpart \[1\].

Thus, the nonlocal Maccari system\[13\] may be considered as a two-dimensional generalization of the nonlocal NLS equation. Using eqs. \[14\] and \[15\], \( N \)-periodic wave solutions \( u \) and \( v \) of the nonlocal Maccari system\[13\] can be obtained by means of the Hirota method, in which \( f \) and \( g \) are written as follows:
\[ f = \sum_{\mu = 0, 1} \exp \left( \sum_{j < k}^{N} \mu_{jk}A_{jk} + \sum_{j=1}^{N} \mu_{j}\eta_{j} \right), \]
\[ g = \sum_{\mu = 0, 1} \exp \left( \sum_{j < k}^{N} \mu_{jk}A_{jk} + \sum_{j=1}^{N} \mu_{j}(\eta_{j} + i\Phi_{j}) \right). \]

Here
\[ \eta_{j} = iP_{j}x + iq_{j}y + \Omega_{j}t + \eta_{j}^{0}, \sin(\Phi_{j}) = -\frac{1}{8}\sqrt{[\lambda(P_{j}^{2} - 2q_{j})^{2} + P_{j}^{2}]} - 2[\lambda(P_{j}^{2} - 2q_{j})^{2} + P_{j}^{2}], \]
\[ \Omega_{j} = -\frac{2\sin(\Phi_{j})[s(2q_{j} - \lambda P_{j})^{2} - P_{j}^{2}]}{[s(2q_{j} - \lambda P_{j})^{2} + P_{j}^{2}]} - \lambda(P_{j} - P_{j}), \]
\[ \cos(\Phi_{j}) = \frac{1}{8}\sqrt{[s(2q_{j} - \lambda P_{j})^{2} + P_{j}^{2}]} + 1, \]
\[ \kappa_{1} = -[(\lambda(P_{j} - P_{j}) + 2(q_{k} - q_{j}))^{2} s + (P_{j} - P_{j})^{2}], \]
\[ \kappa_{2} = -[(\lambda(P_{j} + P_{j}) - 2(q_{k} + q_{j}))^{2} s - (P_{j} - P_{j})^{2}], \]
\[ e^{A_{jk}} = \frac{[\kappa_{1} + 4i(\Omega_{k} - \Omega_{j})]}{[\kappa_{2} + 4i(\Omega_{k} + \Omega_{j})]} e^{i\Phi_{j}}[\kappa_{1} + 4i(\Omega_{j} - \Omega_{j})] e^{i\Phi_{j}} \]
where condition \[\left[(\lambda P_{j}^{2} - 2q_{j})^{2} + P_{j}^{2}\right] - 2[\lambda(P_{j}^{2} - 2q_{j})^{2} + P_{j}^{2}] > 0\] must hold. Further, \( P_{j}, q_{j} \) are arbitrary real parameters, \( \eta_{j}^{0} \) is a complex constant, and \( j \) is an integer subscript. The notation \( \sum_{\mu = 0} \)
Rogue waves and lumps on the non-zero background in the $\mathcal{PT}$-symmetric nonlocal Maccari system indicates summation over all possible combinations of $\mu_1 = 0, 1, \mu_2 = 0, 1, \cdots, \mu_n = 0, 1$. The summation is running over all possible combinations of the $N$ elements.

3. General solutions for breathers on top of the periodic line-wave background

To demonstrate these periodic wave solutions, we first consider the case of $N = 1, P_1 = -1, q_1 = -1, \eta_1^0 = \frac{\pi}{3}, \lambda = 1, s = -2$, in eq. (20). The simplest periodic wave solutions $u$ and $v$ are generated as

$$u = \frac{(7 + i\sqrt{15})e^{-i(x+y) - \frac{3\sqrt{15}}{4}t + \frac{\pi}{2}} + 2(1 + e^{-i(x+y) - \frac{3\sqrt{15}}{4}t + \frac{\pi}{2}})^2}{8 + 8e^{-i(x+y) - \frac{3\sqrt{15}}{4}t + \frac{\pi}{2}}},$$

$$v = -\frac{3e^{-i(x+y) - \frac{3\sqrt{15}}{4}t + \frac{\pi}{2}}}{2(1 + e^{-i(x+y) - \frac{3\sqrt{15}}{4}t + \frac{\pi}{2}})^2}.$$  

(22)

It is a series of periodic line (effectively one-dimensional) waves. As can be seen in Fig. 1, a series of periodic line waves appear from the constant plane and annihilate rapidly, and its dynamic behaviors are similar to the line breather. However, the periodic line wave solution possesses one maximum amplitude and one minimum amplitude, which is different from the line breather with two minimum amplitudes and one maximum amplitude.

![Figure 1](image1.png)

Figure 1. The evolution of the periodic line-wave solution $|u|$ of the nonlocal Maccari system.

In order to obtain breather solutions, we impose the following restrictions in eq. (20):

$$N = 2n, P_j = P_{n+j}, q_j = -q_{n+j}, \eta_j^0 = \eta_{n+j}^0.$$  

(23)

For instance, a particular case of eq. (23), with

$$N = 2, P_1 = P_2, q_1 = -q_2, \eta_1^0 = \eta_2^0,$$  

(24)

is considered. Three kinds of one-breather solutions are generated by changing the free parameter $s$, as shown seen in Fig. 2.

Furthermore, for the case of $N = 2n + 1$, a family of breathers on top of the periodic line-wave background are presented, taking parameters

$$N = 2n + 1, P_j = P_{n+j}, q_j = -q_{n+j}, \eta_j^0 = \eta_{n+j}^0, P_{2n+1}q_{2n+1} \neq 0,$$  

(25)

in eq. (20). For example, these may be

$$N = 3, P_1 = P_2, q_1 = -q_2, P_3 = q_3, \eta_1^0 = \eta_2^0, P_3q_3 \neq 0.$$  

(26)

The hybrid solutions, consisting of one breather and periodic line waves for the nonlocal Maccari system, can be thus generated, see Fig. 3.
Figure 2. Three kinds of one-breather solutions of the nonlocal Maccari system with parameters $P_1 = i$, $P_2 = i$, $q_1 = \frac{1}{2}$, $q_2 = -\frac{i}{2}$, $\eta_{01} = \eta_{02} = 0$, $\lambda = 1$, displayed at $t = 0$: (a): $s = -2$; (b): $s = -\frac{2}{3}$; (c): $s = -\frac{1}{5}$.

Figure 3. Three kinds of one-breather solutions on top of a periodic line-wave background, obtained for the nonlocal Maccari system with parameters $P_1 = i$, $P_2 = i$, $P_3 = -1$, $q_1 = \frac{1}{2}$, $q_2 = -\frac{i}{2}$, $q_3 = -1$, $\eta_{01} = \eta_{02} = 0$, $\eta_{03} = -\frac{\pi}{2}$, $\lambda = 1$, displayed at $t = 0$: (a): $s = -2$; (b): $s = -\frac{1}{2}$; (c): $s = -\frac{1}{3}$.

4. General line rogue waves on top of the periodic line-wave background

In this section, we focus on RWs and semi-rational solutions of the nonlocal Maccari system. To generate RWs, the long-wave limit may be used. Taking parameters in eq. (20) as

$$N = 2, q_1 = \lambda_1 P_1, q_2 = \lambda_2 P_2, \lambda_1 = \lambda_2 = \kappa \neq 0,$$

and considering the limit of $P_j \to 0$ ($j = 1, 2$), the first-order rational solutions $u$ and $v$ are obtained, in which $f$ and $g$ can be expressed as

$$f = -16[(\lambda - 2\kappa)^2 s + 1](x + \kappa y)^2 + 16(\lambda - 2\kappa)^2 s) - 1]^2 t^2 + [1 + (\lambda - 2\kappa)^2 s]^2,$$

$$g = -16[(\lambda - 2\kappa)^2 s + 1](x + \kappa y)^2 + 2[(\lambda - 2\kappa)^2 s - 1]^2 (t + \frac{-i - i(\lambda - 2\kappa)^2 s}{-2 + 2(\lambda - 2\kappa)^2 s})^2$$

$$+ [1 + (\lambda - 2\kappa)^2 s]^2.$$ (28)

As can be seen from the above expressions, to ensure that the rational solutions $u$ and $v$ are smooth, condition $(\lambda - 2\kappa)^2 s + 1 < 0$ must hold. The first-order rational solution $|u|$ is a line rogue wave in the $(x, y)$-plane, its dynamical behavior being similar to that of the first-order RW solution introduced in Refs. [48, 50, 70, 90]. Additionally, higher-order RWs composed of more line RWs can also be generated for larger $N$. Taking parameters in eq. (20) as

$$N = 4, q_j = \lambda_j P_j (j = 1, 2, 3, 4), \lambda_1 = \lambda_2 \neq 0, \lambda_3 = \lambda_4 \neq 0,$$

(29)
and imposing the limit of $P_j \to 0 \ (j = 1, 2)$, the second-order rational solutions $u$ and $v$ are obtained, in which

$$f = x^4 + 81 y^4 + \left(\frac{321}{4} t^2 - 18 y^2 - \frac{279}{2}\right)x^2 + \left(\frac{2889}{4} t^2 + \frac{2673}{2}\right)y^2 + \left(\frac{287}{2} t^2 + 9\right)xy$$

$$+ \frac{105625}{72} t^2 - \frac{40299}{8} t + \frac{9801}{2},$$

$$g = x^4 + 81 y^4 - \left(18 y^2 + \frac{315}{2}\right)x^2 + \frac{2349}{2} y^2 - 27 xy + \frac{105625}{72} t^4 - \frac{5525}{2} it^3 + \left(\frac{321}{4} x^2ight)$$

$$+ \frac{287}{2} xy + \frac{2889}{4} y^2 - \frac{55903}{8} t^2 - (76 x^2 + 144 xy + 684 y^2 - 5355)it + \frac{12177}{2}.$$

The corresponding solution $|u|$ is shown in Fig.4. It follows from the above expressions that the solutions are subject to the boundary condition $\lim_{x \to \pm \infty} |u| = 1$, i.e., $|u|$ approaches a constant background in the $(x,y)$-plane. A cross-shaped RW appears at $t \approx -3/2$, describing the interaction between two line RWs. With the increase of the interaction strength, the central region of the cross-shaped RW creates two sharp peaks at $t \approx -3/4$. The cross-shaped RW is separated into two hyperbolic line RWs at $t \approx 0$, whose behavior is similar to that of the rational-soliton in the $(1 + 1)$-dimensional system \cite{88, 89}; then they merge back into the constant background, the strong interaction resulting in the change of the waveform. We stress that the dynamical behavior of the higher-order line RWs of nonlocal Maccari system \cite{13} is different from that of the corresponding higher-order line RWs in other nonlocal systems, such as the nonlocal two-dimensional NLS equation \cite{48, 90} and nonlocal DS equation \cite{31, 70}.

Figure 4. The temporal evolution in the $(x, y)$-plane of second-order rogue-wave solutions of the nonlocal Maccari system with parameters $\lambda_1 = \lambda_2 = 3$, $\lambda_3 = \lambda_4 = -3$, $\lambda = 1$ and $s = -1$.

Semi-rational solutions of nonlocal Maccari system \cite{13} can also be obtained by taking the long-wave limit of a part of exponential functions in the expressions for $f$ and $g$ given by eq. (20), which describe the interplay of RWs with the periodic waves. We first consider the simplest semi-rational solution by setting

$$N = 3, q_1 = \lambda_1 P_1, q_2 = \lambda_2 P_2, \eta_i^0 = \eta_2^0 = i\pi,$$

$$\lambda_1 = \lambda_2 = -3, \ P_3 = 4, q_3 = 4, \eta_4^0 = -\frac{\pi}{2}, \lambda = 1, s = -2,$$

and taking the limit of $P_1, P_2 \to 0$ in eq. (20). Under parameter constraints

$$\lambda_1 = \lambda_2 = -3, \ P_3 = 4, q_3 = 4, \eta_4^0 = -\frac{\pi}{2}, \lambda = 1, s = -2,$$
functions \(f\) and \(g\) become

\[
f = \left(3142800(y - \frac{1}{3}x - \frac{97i}{90})^2 + 35283600t^2 + 2117025\right) e^{(-4ix-4iy-\frac{1}{2}\pi)}
+ 3142800(y - \frac{1}{3}x)^2 + 35283600t^2 + 2117025, \\
g = \left(-3142800(y - \frac{1}{3}x - \frac{97i}{90})^2 - 35283600(t - \frac{97i}{198})^2 + 2117025\right) e^{(-4ix-4iy-\frac{1}{2}\pi)}
+ 349200(x - 3y)^2 + 35283600(t - \frac{97i}{198})^2 + 2117025. \tag{33}
\]

The corresponding semi-rational solution \(|u|\) is displayed in Fig. 5. Four panels display the emergence and annihilation of a line RW on top of the background of periodic line waves. As seen in Fig. 5, a line RW arises from the periodic line-wave background at \(t \approx -\frac{1}{2}\), and the interaction produces a series of sharp peaks along the line RW at \(t \approx 0\). Then, all wave peaks on the line RW \((x - 3y = 0)\) quickly merge into the periodic line waves. This behavior is obviously different from that featured by the corresponding solutions of the nonlocal DS-I equation with the flat background, cf. Ref. [31].

![Figure 5](image_url)

**Figure 5.** The evolution of the line rogue waves on top of a background of periodic line waves, as solutions of the nonlocal Maccari system with parameters \(\lambda_1 = \lambda_2 = -3, P_3 = q_3 = 4, \lambda = 1\) and \(s = -2\).

Higher-order semi-rational solutions composed of two line RWs and periodic line waves can be obtained in a similar way for \(N = 5\) in eq. (20). The parameters are chosen as

\[
N = 5, Q_k = \lambda_k P_k, \eta_k^0 = i\pi \ (k = 1, 2, 3, 4), \tag{34}
\]

with \(P_k \to 0\) in eq. (20). Under the constraints

\[
\lambda_1 = \lambda_2 = -3, \lambda_3 = \lambda_4 = 3, P_5 = \sqrt{2}, q_5 = -\sqrt{2}, \eta_5^0 = -\frac{\pi}{2}, \tag{35}
\]
Rogue waves and lumps on the non-zero background in the $\mathcal{PT}$-symmetric nonlocal Maccari system functions $f$ and $g$ become

$$f = \left\{ x^4 + \frac{105625}{72} t^4 + 81y^4 + \frac{321}{4} t^2 - 18y^2 - \frac{3168}{25} x^2 + \frac{2889}{4} t^2 + \frac{23328}{25} y^2 - \frac{559623}{100} t^2 ight.$$

$$+ \left( \frac{287}{2} t^2 - \frac{828}{25} \right) x y + i\sqrt{2} \left( \frac{9}{5} x^3 + \frac{117}{5} x^2 \frac{y}{2} - \left( \frac{421}{20} t^2 + \frac{81}{5} y^2 + \frac{2961}{25} \right) x - \left( \frac{17487}{20} t^2 + \frac{1053}{5} y^2 \right) \right)^{1/2} y + \frac{97119}{25} \right\} e^{(i\sqrt{2} x - i\sqrt{2} y - \frac{1}{2}\pi + x^4 + 81y^4 + \frac{105625}{72} t^4 + \left( \frac{321}{4} t^2 - 18y^2 - \frac{279}{2} \right) x^2}$$

$$+ \left( \frac{2889}{4} t^2 + \frac{2673}{2} y^2 - \frac{40299}{8} t^2 + \left( \frac{287}{2} t^2 + 9 \right) x y + \frac{9801}{2} \right) \right\}.$$ 

$$g = \left\{ -x^4 - 81y^4 - \frac{105625}{72} t^4 + \frac{5525}{2} t^3 - \left( \frac{321}{4} t^2 + \frac{287}{2} x y + \frac{2889}{4} t^2 - \frac{754673}{100} \right) t^2 - \frac{19278}{25} y^2 \right.$$

$$+ \left( 18y^2 + \frac{3618}{25} \right) x^2 + \sqrt{2} \left( \frac{421}{20} i x + \frac{17487}{20} i y \right) t^2 + \left( \frac{126}{5} x + \frac{4122}{5} y \right) t - \frac{9}{5} x^3 - \frac{117}{5} i x^2 y$$

$$+ \left( \frac{81}{2} y^2 + \frac{2781}{25} \right) x^2 + \frac{1053}{25} y^2 + \frac{34263}{25} i y + \left( 76x^2 + 144xy + 684y^2 - \frac{146988}{25} \right) i t + \frac{1728}{25} x y$$

$$- \frac{129897}{25} \right\} e^{(i\sqrt{2} x - i\sqrt{2} y - \frac{1}{2}\pi + x^4 + 81y^4 + \frac{105625}{72} t^4 - \frac{5525}{2} t^3 + \frac{2349}{2} y^2 + \left( \frac{321}{4} x^2 + \frac{287}{2} x y \right) \right\}.$$ 

$$+ \left( \frac{2889}{4} t^2 - \frac{55903}{8} \right) t^2 - \left( 18y^2 + \frac{315}{2} \right) x^2 - \left( 76x^2 + 144xy + 684y^2 - \frac{5355}{2} \right) i t - 27xy + \frac{12177}{8}.$$ 

producing semi-rational solutions $|U|$, which consist of two line RWs and periodic line waves. As can be seen in Fig. 6, two hyperbolic line RWs appear on the periodic line-wave background at $t \approx -1/2$, and produce a sharp peak at the top of curved RW, which seems more evident at $t \approx 0$. Finally, the hyperbolic RW rapidly disappears, merging into the background of the periodic line waves. In comparison to the results reported in Refs. [31, 70, 90], the corresponding solution always describes the evolution of the cross-shaped RW on top of the background of periodic line waves.

**Figure 6.** The evolution of the second-order line rogue waves on top of the background of periodic line waves, obtained as a solution of the nonlocal Maccari system with parameters $\lambda_1 = \lambda_2 = 3$, $\lambda_3 = \lambda_4 = -3$, $P_3 = - q_3 = \sqrt{2}$, $\lambda_1 = 1$, and $s = -2$. 
5. General line solitons on top of a constant background

In this section, we consider general soliton solutions to the nonlocal Maccari system, built on top of a constant background. It is difficult to get multi-soliton solutions of the nonlocal system by means of perturbation expansion combined with the long-wave limit. Recently, general soliton solutions for a nonlocal NLS equation were produced using a combination of the Hirota’s bilinear method and KP hierarchy reduction [91]. This finding has triggered rapid progress in studies of solitons in nonlocal systems [99, 100, 102, 103]. Inspired by that work, we consider tau-functions of the nonlocal Maccari system.

**Lemma 1.** Referring to the Sato theory [74, 94, 95, 96, 97], the bilinear equations in the KP hierarchy

\[
(D_{x_1}^2 - D_{x_2})\tau_{n+1} \cdot \tau_n = 0,
\]

\[
(D_{x_1}^2 + D_{x_2})\tau_{n+1} \cdot \tau_n = 0,
\]

\[
(D_{x_1}D_{x_2} - 2)\tau_n \cdot \tau_n = -2\tau_{n+1} \cdot \tau_{n-1},
\]

(37)

give rise to the following tau-functions,

\[
\tau_n = \det_{1 \leq j,k \leq N} (m^{(n)}_{j,k}),
\]

(38)

with matrix elements \( m^{(n)}_{i,j} \) satisfying the following difference relations,

\[
\partial_{x_1} m^{(n)}_{j,k} = \varphi^{(n)}_j \psi^{(n)}_k, \quad m^{(n+1)}_{j,k} = m^{(n)}_{j,k} + \varphi^{(n)}_j \psi^{(n+1)}_k,
\]

\[
\partial_{x_2} m^{(n)}_{j,k} = \varphi^{(n+1)}_j \psi^{(n)}_k + \varphi^{(n)}_j \psi^{(n-1)}_k, \quad \partial_{x_1} m^{(n)}_{j,k} = -\varphi^{(n-1)}_j \psi^{(n+1)}_k,
\]

\[
\partial_{x_2} m^{(n)}_{j,k} = -\varphi^{(n-2)}_j \psi^{(n+1)}_k - \varphi^{(n-1)}_j \psi^{(n)}_k,
\]

\[
\partial_{x_1} \varphi^{(n)}_j = \varphi^{(n+v)}_j, \quad \partial_{x_1} \varphi^{(n)}_j = -\psi^{(n-v)}_s (v = -2, -1, 1, 2).
\]

(39)

Here \( m^{(n)}_{i,j} \), \( \varphi^{(n)}_i \), and \( \psi^{(n)}_j \) are functions of variables \( x_1, x_2, x_1 \) and \( x_2 \).

Furthermore, by changing the independent variables,

\[
x_2 = -\frac{i}{2} t, x_1 = x + \frac{i \sqrt{s-1}}{2i} y, x_1 = x + (\lambda - \frac{i \sqrt{s-1}}{2i}) y, x_2 = \frac{i}{2} t,
\]

(40)

\[
x_2 = -\frac{i}{2} t, x_1 = x + \frac{i \sqrt{s-1}}{2i} y, x_1 = x + (\lambda - \frac{i \sqrt{s+1}}{2i}) y, x_2 = \frac{i}{2} t,
\]

(41)

\[
x_2 = -\frac{i}{2} t, x_1 = x + \frac{i \sqrt{s+1}}{2i} y, x_1 = x + (\lambda - \frac{i \sqrt{s-1}}{2i}) y, x_2 = \frac{i}{2} t,
\]

(42)

or

\[
x_2 = -\frac{i}{2} t, x_1 = x + \frac{i \sqrt{s+1}}{2i} y, x_1 = x + (\lambda - \frac{i \sqrt{s+1}}{2i}) y, x_2 = \frac{i}{2} t,
\]

(43)

the bilinear equations in the KP hierarchy (37) are reduced to the bilinear equation (15) of the nonlocal Maccari system for

\[
f = \tau_0, g = \tau_1, g^* = \tau_{-1}.
\]

(44)

Thus the nonlocal Maccari system [13] gives rise to the following tau-functions:

\[
u = \frac{\tau_1}{\tau_0}, \quad v = \left[ \frac{1}{2}(1 - s \lambda^2) \partial_{xx} + 2s \lambda \partial_{xy} - 2s \partial_{yy} \right] (\ln \tau_0),
\]

(45)

where tau-functions \( \tau_0 \) and \( \tau_1 \) satisfy relations

\[
\tau_0^*(-x, -y, t) = C \tau_0(x, y, t), \quad \tau_1^*(-x, -y, t) = C \tau_{-1}(x, y, t).
\]

(46)

To derive soliton solutions under nonzero boundary condition, we choose functions \( m^{(n)}_{j,k} \), \( \varphi^{(n)}_s \) and \( \psi^{(n)}_s \) in eq. (39) as follows,

\[
m^{(n)}_{j,k} = c_j \delta_{jk} + \frac{1}{p_j + q_k} \varphi^{(n)}_j \psi^{(n)}_k, \quad \varphi^{(n)}_j \psi^{(n)}_k = p_j^n e^{x_j} e^{x_k}, \quad \varphi^{(n)}_j \psi^{(n)}_k = (-q_k)^{-n} e^{x_k},
\]

(47)
Rogue waves and lumps on the non-zero background in the $\mathcal{P}\mathcal{T}$-symmetric nonlocal Maccari system

where

$$\xi_j = p_j^{-2} x_{-2} + p_j^{-1} x_{-1} + p_j x_1 + p_j^2 x_2 + \xi_j \delta_{jk},$$

$$\eta_k = -q_k^{-2} x_{-2} + q_k^{-1} x_{-1} + q_k x_1 - q_k^2 x_2 + \eta_k \delta_{jk},$$

$p_j, q_j, \xi_0$ and $\eta_0$ are arbitrary complex constants, and $\delta_{jk}$ is the Kronecker's symbol. Using the transformation of variable given by eq. [40], the above tau-function $\tau_n$ can be rewritten as,

$$\tau_n = \prod_{k=1}^{N} e^{\eta_k + \xi_k} \left| c_j \delta_{jk} e^{-(\xi_j + \eta_k)} + \left( -\frac{p_j}{q_k} \right)^n \frac{1}{p_j + q_k} \right|_{N \times N}, \quad (48)$$

with

$$\xi_j = (p_j + \frac{1}{p_j}) x + \left( \frac{\lambda(p_j^2 + 1) \sqrt{-s} - p_j^2 + 1}{2 \sqrt{-sp_j}} \right) y + \frac{i}{2} \left( p_j^2 - \frac{1}{p_j^2} \right) t + \xi_j \delta_{jk},$$

$$\eta_j = (q_j + \frac{1}{q_j}) x + \left( \frac{\lambda(q_j^2 + 1) \sqrt{-s} - q_j^2 + 1}{2 \sqrt{-sq_j}} \right) y - \frac{i}{2} \left( q_j^2 - \frac{1}{q_j^2} \right) t + \eta_j \delta_{jk}. \quad (49)$$

In order to construct the line solitons on top of a constant background, the following lemma is presented.

**Lemma 2.** Setting $N = 2M, q_j = p_j^s, c_{M+j} = -c_j^s, p_{M+j} = -p_j, \xi_{m+j0} = \xi_j \delta_{jk}, \eta_{m+j0} = \eta_0 \delta_{jk}$ in eq. [48], where $p_j$ and $c_j$ are complex parameters, $\xi_0$ and $\eta_0$ are real, one has

$$\tau_n^*(x, y, t) = \tau_n(x, y, t),$$

which means $\tau(0, y, t) = f(x, y, t)$ and $\tau_1(x, y, t) = g(x, y, t)$. A detailed proof of lemma 2 is given in Appendix A. The above results can be summarized as the following theorems.

**Theorem 1** The nonlocal Maccari system admits general soliton solutions

$$u = \frac{g}{f}, \quad v = \left( \frac{1}{2} \left( 1 - s \lambda^2 \right) \partial_{xx} + 2s \lambda \partial_{xy} - 2s \partial_{yy} \right) (\ln f), \quad (50)$$

where

$$\tau_n = \left| c_j \delta_{jk} e^{\xi_j + \eta_j} + \left( -\frac{p_j}{p_k} \right)^n \frac{1}{p_j + p_k} \right|_{2M \times 2M}, \quad (51)$$

$$\xi_j = (p_j + \frac{1}{p_j}) x + \left( \frac{\lambda(p_j^2 + 1) \sqrt{-s} - p_j^2 + 1}{2 \sqrt{-sp_j}} \right) y + \frac{i}{2} \left( p_j^2 - \frac{1}{p_j^2} \right) t + \xi_j \delta_{jk},$$

$q_j = p_j^s, c_{M+j} = -c_j^s, p_{M+j} = -p_j, \xi_{m+j0} = \xi_j \delta_{jk}, \eta_{m+j0} = \eta_0 \delta_{jk}$, where $p_j$ and $c_j$ are complex parameters, while $\xi_0$ and $\eta_0$ are real.

In what follows, to produce soliton solutions, we first present an explicit form of the two-soliton solutions, by setting $M = 1$ in Theorem 1. The solutions $u$ and $v$ are then expressed as

$$u = \frac{g}{f}, \quad v = \left( \frac{1}{2} \left( 1 - s \lambda^2 \right) \partial_{xx} + 2s \lambda \partial_{xy} - 2s \partial_{yy} \right) (\ln f), \quad (52)$$
where

\[ f_1 = \left| c_1 e^{-(\xi_1 + \xi_1^*)} + \frac{1}{p_1 + p_1^*} \right| - c_1^* e^{-(\xi_2 + \xi_2^*)} + \frac{1}{p_2 + p_2^*} \]

\[ = -|c_1|^2 e^{i(p_1^2 + p_1^2) - \frac{1}{p_1^2}} - \frac{1}{p_1 + p_1^*} \]

\[ g_1 = \left| c_1 e^{-(\xi_1 + \xi_1^*)} + \frac{1}{p_1 + p_1^*} \right| - c_1^* e^{-(\xi_2 + \xi_2^*)} + \frac{1}{p_2 + p_2^*} \]

\[ = -|c_1|^2 e^{i(p_1^2 + p_1^2) - \frac{1}{p_1^2}} - \frac{1}{p_1 + p_1^*} \]

and

\[ \xi_1 + \eta_1^* = (p_1 + p_1^* + \frac{1}{p_1 + p_1^*}) x + \left( \frac{\lambda(p_1^2 + 1)\sqrt{-s - p_1^2} + 1}{2\sqrt{-sp_1}} + \frac{\lambda(p_1^2 + 1)\sqrt{s} - p_1^2 + 1}{2\sqrt{-sp_1^*}} \right) y + \frac{i}{2}(p_1^2 - \frac{1}{p_1^2}) t + \xi_{10} + \eta_{10} \]

\[ \xi_2 + \eta_2^* = - (\xi_1 + \eta_1^*) + i(p_1^2 - \frac{1}{p_1^2} + p_1^2 - \frac{1}{p_1^2}) t. \]

It can be seen from the above expressions that the two-soliton solution is controlled by two free parameters, \( p_1 \) and \( c_1 \). There are three patterns of two-soliton solutions for a given parameter \( p_1 \). When \( c_1 = -1 + \frac{1}{3} i \), the corresponding solutions are two anti-dark solitons; when \( c_1 = \frac{1}{3} + 2i \), the solutions are dark-anti-dark solitons; when \( c_1 = 1 + \frac{1}{3} i \), the solutions are two dark solitons, see
Rogue waves and lumps on the non-zero background in the $\mathcal{PT}$-symmetric nonlocal Maccari system. Fig. 7(a,b,c). We stress that the two solitons form a parallel pair, and their dynamical behavior is different from that of the cross solitons in nonlocal systems considered in Refs. [49, 50, 92]. As can be seen in Fig. 7(d,e,f), the two solitons pass through each other without changes in their velocity and waveforms, which suggests that there the interaction between the two solitons is strictly elastic, as in other integrable systems.

Furthermore, solutions for collisions of $2M$ line solitons can be found for larger $M$. The patterns of $2M$ line solitons are controlled by parameters $p_k$ and $c_k$ ($k = 1, 2, \cdots, M$). For example, taking $M = 2$, three patterns of four-soliton interactions are presented for given parameters $p_1$ and $p_2$ in Fig. 8.

6. General lumps on solitons background

In this section, we focus on semi-rational solutions produced by dint of the KP-hierarchy method. To obtain semi-rational solutions of the nonlocal Maccari system (13), we introduce the following differential operators [74, 97]:

$$\Xi^s = \sum_{k=0}^{n_0} c_{sk} (p_s \partial_{p_s})^{n_0-k}, \quad \Upsilon^j = \sum_{l=0}^{n_0} d_{jl} (q_j \partial_{q_j})^{n_0-l},$$

and choose the following functions,

$$\phi^{(n)}_s = \Xi^s p_s^n e^{\xi^s},$$
$$\psi^{(n)}_j = \Upsilon^j (q_j)^{-n} e^{\eta^j},$$
$$M^{(n)}_{s,j} = \Xi^s \Upsilon^j \frac{1}{p_s + q_j} [\delta_{sj} + (-\frac{p_s}{q_j})^n e^{\xi^s + \eta^j}].$$

Figure 8. Three types of four-soliton solutions of the nonlocal Maccari system displayed at $t = 5$: (a) four antidark solitons with parameters $p_1 = 1 + i$, $p_2 = 2 - i$, $\lambda = 1$, $s = -2$, $c_1 = -\frac{1}{3} - \frac{1}{5} i$, and $c_2 = -\frac{1}{3} - \frac{1}{5} i$; (b) a two-dark-two-antidark-solitons complex with parameters $p_1 = 1 + i$, $p_2 = 2 - i$, $\lambda = 1$, $s = -2$, $c_1 = -\frac{1}{3} - 2i$, and $c_2 = -\frac{1}{3} - 2i$; (c) four dark solitons with parameters $p_1 = 1 + i$, $p_2 = 2 - i$, $\lambda = 1$, $s = -2$, $c_1 = 1 + \frac{1}{3} i$, and $c_2 = 1 - \frac{1}{3} i$. 


Theorem 2. The nonlocal Maccari system (13) can be summarized in the following Theorem. The proof of lemma 3 is given in Appendix B. According to Lemma 3, the semi-rational solutions of

\[ M_{s,j}^{(n)} = (-\frac{p_s}{q_j})^n e^{\xi_j + n \eta_j} \sum_{k=0}^{n_0} a_{sk}(p_s \partial_{p_s} + \xi_s + n)^{n_0-k} \]

\[ \times \sum_{l=0}^{n_0} d_{jl}(q_j \partial_{q_j} + \eta_j - n)^{n_0-l} \frac{1}{p_s + q_j} + \tilde{c}_{sj}, \]

where

\[ \xi_j = (p_j + \frac{1}{p_j}) x + \left( \frac{\lambda(p_j^2 + 1)\sqrt{-s - p_j^2} + 1}{2\sqrt{-sp_j}} \right) y + \frac{i}{2}(p_j^2 - \frac{1}{p_j^2}) t + p_j \tilde{\xi}_j, \]

\[ \eta_j = (q_j + \frac{1}{q_j}) x + \left( \frac{\lambda(q_j^2 + 1)\sqrt{s - q_j^2} + 1}{2\sqrt{-sq_j}} \right) y - \frac{i}{2}(q_j^2 - \frac{1}{q_j^2}) t + q_j \tilde{\eta}_j, \]

\[ \xi_j' = (p_j - \frac{1}{p_j}) x + \left( \frac{\lambda(p_j^2 - 1)\sqrt{-s - p_j^2} - 1}{2\sqrt{-sp_j}} \right) y + i(p_j^2 + \frac{1}{p_j^2}) t + p_j \tilde{\xi}_j, \]

\[ \eta_j' = (q_j - \frac{1}{q_j}) x + \left( \frac{\lambda(q_j^2 - 1)\sqrt{s - q_j^2} - 1}{2\sqrt{-sq_j}} \right) y - i(q_j^2 + \frac{1}{q_j^2}) t + q_j \tilde{\eta}_j, \]

Here \( p_i \) and \( a_{sk} \) are arbitrary complex constants, \( \delta_{ij} = 0, 1 \), \( n_i \) are arbitrary positive integers, and \( \tilde{c}_{sj} = \delta_{sj} c_{s,n_s} c_{j,n_j} \).

Lemma 3 Consider the \( 2N \times 2N \) matrix for the tau function \( M_{s,j}^{(n)} \) defined in (56). Taking

\[ p_{N+j} = -p_j, c_{N+s,N+j} = -\tilde{c}_{j,s}, \]

\[ b_{s,N+j} = b_{j,N+s}, c_{s,N+j} = c_{s,j}, \]

\[ b_{s,N+j} = b_{j,N+s}, c_{s,N+j} = c_{s,j}, \]

\[ \tilde{\xi}_j = \tilde{\xi}_s, \tilde{\eta}_j = \tilde{\xi}_s, \]

\[ \tilde{\eta}_j = \tilde{\xi}_s, \]

\[ \tilde{\xi}_s, \tilde{\eta}_s = \tilde{\xi}_s, \tilde{\eta}_s, \]

\[ \tilde{\eta}_s = \tilde{\xi}_s, \]

\[ \tilde{\eta}_s = \tilde{\xi}_s, \]

\[ \tilde{\xi}_s, \tilde{\eta}_s = \tilde{\xi}_s, \]

for \( s, j = 1, 2, \cdots, N \), then we have

\[ \tau_n^s(-x, -y, t) = \tau_{-n}(x, y, t). \]

The proof of lemma 3 is given in Appendix B. According to Lemma 3, the semi-rational solutions of the nonlocal Maccari system [13] can be summarized in the following Theorem.

Theorem 2. The nonlocal Maccari system [13] has semi-rational solutions

\[ u = \frac{\tau_1}{\tau_0}, \quad v = \left( \frac{1}{2} (1 - s \lambda^2) \partial_{xx} + 2s \lambda \partial_{xy} - 2s \partial_{yy} \right) (\ln \tau_0), \]

where

\[ \tau_n(x, y, t) = \left| \begin{array}{cc} M_{j,s}^{(n)} & M_{j,s}^{(n)} \\ M_{j,N+s}^{(n)} & M_{j,N+s}^{(n)} \end{array} \right|_{0<s,j \leq N}, \]

and the matrix elements are given by

\[ M_{j,s}^{(n)} = (-\frac{p_s}{p_j})^n e^{\xi_j + n \eta_j} \sum_{k=0}^{n_0} c_{jk}(p_j \partial_{p_j} + \xi_j + n)^{n_0-k} \]

\[ \times \sum_{l=0}^{n_0} c_{sk}(p_s \partial_{p_s} + \xi_s + n)^{n_0-l} \frac{1}{p_j + p_s} + \tilde{c}_{sj}, \]

with

\[ \xi_j = (p_j + \frac{1}{p_j}) x + \left( \frac{\lambda(p_j^2 + 1)\sqrt{-s - p_j^2} + 1}{2\sqrt{-sp_j}} \right) y + \frac{i}{2}(p_j^2 - \frac{1}{p_j^2}) t + p_j \tilde{\xi}_j, \]

\[ \xi_j' = (p_j - \frac{1}{p_j}) x + \left( \frac{\lambda(p_j^2 - 1)\sqrt{-s - p_j^2} - 1}{2\sqrt{-sp_j}} \right) y + i(p_j^2 + \frac{1}{p_j^2}) t + p_j \tilde{\xi}_j, \]
Remark 1: Setting $\tilde{c}_{s,j} = 0$ and $b_{s+N,j} = b_{s,j+N} = 0$ ($1 \leq s, j \leq N$) reduces the semi-rational solutions [59] to rational solutions of the nonlocal Maccari system [13], which are $2N$-lump-type solutions.

Remark 2: With $\tilde{c}_{s,j} \neq 0$, these semi-rational solutions [59] describe the interaction between $2N$-lumps and $2N$-solitons of the nonlocal Maccari system [13].

6.1. General lumps solutions

$$p_{N+j} = -p_j, \tilde{c}_{N+s,j} = -\tilde{c}_{s,j}, \tilde{c}_{s,j} = \tilde{c}_{s,j}, \tilde{c}_{N+j} = i b_{N+j}, \tilde{c}_{s,N+j} = i b_s, N+j,$$

$$b_{s,N+j} = b_{j,N+s}, c_{N+s,j} = c_{s,j}, c_s,N+j = c_{s,j}, \tilde{\eta}_j = -\tilde{\eta}_j, \tilde{c}_{s,N+j} = \tilde{c}_{s,j}, q_j = p_j, c_{sk} = d_{sk}. \tag{63}$$

To illustrate the $2N$-lump solutions of the nonlocal Maccari equation, we first take $N = 1, n_0 = 1, \tilde{c}_{11} = 0$ and $b_{12} = 0$ in eq. [59], we obtain

$$u = g, \quad v = \left[ \frac{1}{2} (1 - s \lambda^2) \partial_{xx} + 2s \lambda \partial_{xy} - 2s \partial_{yy} \right] (\ln f), \tag{64}$$

where

$$f = \begin{vmatrix} \zeta_1 \zeta_1' + \frac{p_1 p_1^*}{(p_1 + p_2^*)^2} & \zeta_1 \zeta_2' + \frac{p_1 p_2^*}{(p_1 + p_2^*)^2} \\ \z_2 \z_1' + \frac{p_2 p_1^*}{(p_2 + p_1^*)^2} & \z_2 \z_2' + \frac{p_2 p_2^*}{(p_2 + p_2^*)^2} \end{vmatrix},$$

$$g = \begin{vmatrix} \zeta_1(1) (\zeta_1' - 1) + \frac{p_1 p_1^*}{(p_1 + p_2^*)^2} & \zeta_1(1) (\zeta_2' - 1) + \frac{p_1 p_2^*}{(p_1 + p_2^*)^2} \\ \z_2(1) (\z_1' - 1) + \frac{p_2 p_1^*}{(p_2 + p_1^*)^2} & \z_2(1) (\z_2' - 1) + \frac{p_2 p_2^*}{(p_2 + p_2^*)^2} \end{vmatrix}, \tag{65}$$

with

$$\zeta_s = \zeta_s - \frac{p_s}{p_s + p_j^*} + c_{11}, \quad \zeta_s' = \zeta_s - \frac{p_j^*}{p_j^* + p_s} + s_{11}, \tag{66}$$

where $\zeta$ and $\eta$ ($s, j = 1, 2$) are defined in eq. [62]. $p_2 = -p_1, \tilde{\xi}_2 = -\tilde{\xi}_1, p_1, c_{11}, \tilde{\xi}_1$ being complex parameters. The corresponding rational solutions are two lumps in the $(x, y)$-plane, whose shape is controlled by parameters $p_1$ and $\tilde{\xi}_1$. When $\tilde{\xi}_1 = 0$, as discussed in Ref. [83], the shape of each lump is controlled by parameters $\frac{p_1 R_{12}}{P_{12m}}$: a dark lump for $0 < \frac{p_1 R_{12}}{P_{12m}} \leq \frac{1}{3}$; a four-petal lump for $\frac{1}{3} < \frac{p_1 R_{12}}{P_{12m}} \leq 3$.

Figure 9. Three species of two-lump solutions of the nonlocal Maccari equation are displayed at $t = 2$: (a) a bright-bright-lump solution for parameters $p_1 = 1 + \frac{i}{2}, \lambda = 1,$ and $s = -1$; (b) a four-petal-four-petal-lump solution for parameters $p_1 = 1 + i, \lambda = 1,$ and $s = -1$; (c) a dark-dark-lump solution for parameters $p_1 = 1 + 2i, \lambda = 1,$ and $s = -1$.
and a bright lump for $\frac{p_2^2}{p_1^{Re}} > 3$. Here $p_1^{Re}$ and $p_1^{Im}$ are the real part and the imaginary part of $p_1$.

As $\frac{p_2^2}{p_1^{Re}} = \frac{p_2^2}{p_1^{Im}}$, two lumps have the same structure. Three species of two-lumps patterns, namely, two bright lumps, two four-petal lumps, and two dark lumps, are shown in Fig. 9.

Similarly, four-lump solutions of the nonlocal Maccari system (13) are generated by setting

$$N = 2, n_0 = 1, \bar{c}_{s,j} = 0, b_{s,j} = 0, c_{s,0} = 1, (s \neq j, s, j = 1, 2, 3, 4),$$

in eq. (59), the corresponding solutions $u$ and $v$ being

$$u = \frac{g}{f}, \quad v = \left[ \frac{1}{2} (1 - s\lambda^2) \partial_{xx} + 2s\lambda \partial_{xy} - 2s \partial_{yy} \right] (\ln f), \quad (67)$$

with

$$f(x, y, t) = \begin{vmatrix} M_{11}^{(0)} & M_{12}^{(0)} & M_{13}^{(0)} & M_{14}^{(0)} \\ M_{21}^{(0)} & M_{22}^{(0)} & M_{23}^{(0)} & M_{24}^{(0)} \\ M_{31}^{(0)} & M_{32}^{(0)} & M_{33}^{(0)} & M_{34}^{(0)} \\ M_{41}^{(0)} & M_{42}^{(0)} & M_{43}^{(0)} & M_{44}^{(0)} \end{vmatrix},$$

$$g(x, y, t) = \begin{vmatrix} M_{11}^{(1)} & M_{12}^{(1)} & M_{13}^{(1)} & M_{14}^{(1)} \\ M_{21}^{(1)} & M_{22}^{(1)} & M_{23}^{(1)} & M_{24}^{(1)} \\ M_{31}^{(1)} & M_{32}^{(1)} & M_{33}^{(1)} & M_{34}^{(1)} \\ M_{41}^{(1)} & M_{42}^{(1)} & M_{43}^{(1)} & M_{44}^{(1)} \end{vmatrix},$$

(68)

where $M_{s,j}^{(n)} (s, j = 1, 2, 3, 4, n = 0, 1)$ are defined in eq. (61). Three species of four-lump solutions, namely, four bright lumps, four-lump-four-petal lumps, and four dark lumps, are displayed in Fig. 10.

**Figure 10.** Three species of four-lump solutions of the nonlocal Maccari system, displayed at $t = 0$: (a) a four-bright-lump solution for parameters $p_1 = 1 + \frac{1}{4}i$, $p_2 = -1 + \frac{1}{4}i$, $\lambda = 1$, and $s = -1$; (b) a four–lump-four-petal solution for parameters $p_1 = 1 + i$, $p_2 = -1 + \frac{3}{4}i$, $\lambda = 1$, and $s = -1$; (c) a four-dark-lump solution for parameters $p_1 = \frac{1}{2} + i$, $p_2 = -\frac{1}{2} + \frac{3}{8}i$, $\lambda = 1$, and $s = -1$. 
6.2. General lump-soliton solutions

To illustrated the dynamical behavior of semi-rational solutions composed of 2N-lumps and 2N-solitons, we first consider the case of $N = 1, n_0 = 1$ in eq. (59), with

$$f = \left(\frac{e^{\xi_1 + \eta_1}}{p_1 + p_2} \left(\xi_1 \xi_1' + \frac{p_1 p_2}{(p_1 + p_2)^2} \right) + c_{11} |c_{11}|^2 \right) \left(\frac{e^{\xi_2 + \eta_2}}{p_1 + p_2} \left(\xi_2 \xi_2' + \frac{p_1 p_2}{(p_1 + p_2)^2} \right) + i b_{21} |c_{11}|^2 \right) \left(\frac{e^{\xi_1 + \eta_1}}{p_1 + p_2} \left(\xi_1 \xi_1' + \frac{p_1 p_2}{(p_1 + p_2)^2} \right) + c_{12} |c_{12}|^2 \right) \left(\frac{e^{\xi_2 + \eta_2}}{p_1 + p_2} \left(\xi_2 \xi_2' + \frac{p_1 p_2}{(p_1 + p_2)^2} \right) + i b_{21} |c_{12}|^2 \right),$$

$$g = \left(\frac{e^{\xi_1 + \eta_1}}{p_1 + p_2} \left(\xi_1 \xi_1' + \frac{p_1 p_2}{(p_1 + p_2)^2} \right) + c_{11} |c_{11}|^2 \right) \left(\frac{e^{\xi_2 + \eta_2}}{p_1 + p_2} \left(\xi_2 \xi_2' + \frac{p_1 p_2}{(p_1 + p_2)^2} \right) + i b_{21} |c_{11}|^2 \right) \left(\frac{e^{\xi_1 + \eta_1}}{p_1 + p_2} \left(\xi_1 \xi_1' + \frac{p_1 p_2}{(p_1 + p_2)^2} \right) + c_{12} |c_{12}|^2 \right) \left(\frac{e^{\xi_2 + \eta_2}}{p_1 + p_2} \left(\xi_2 \xi_2' + \frac{p_1 p_2}{(p_1 + p_2)^2} \right) + i b_{21} |c_{12}|^2 \right),$$

where $\xi_s, \eta_j, \xi_s', \eta_j'$ ($s, j = 1, 2$) are defined in eqs. (62) and (66), where $p_2 = -p_1, \tilde{c}_{22} = -c_{11}, b_{12} = b_{21}, \tilde{\xi}_2 = -\xi_1$, $p_1, c_{11}, \xi_1$ being complex parameters, and $b_{12}$ is a real one. Further, setting

$$p_1 = 1 + \frac{i}{2}, p_2 = -1 - \frac{i}{2}, \tilde{c}_{11} = 1 + i, \tilde{c}_{22} = -1 + i, b_{1,2} = b_{2,1} = 0, c_{11} = 1, \lambda = 1, s = -2,$$

in eq. (69), it is seen in Fig. (11) that the corresponding semi-rational solution $|u|$ describes fusion of two dark solitons and two bright lumps into two dark solitons. At the initial stage of the evolution, two bright lumps are observed on top of the background of two dark solitons. Interaction between lumps and solitons begins in the course of the evolution. Eventually, two -bright lumps merge into two dark solitons at $t \to +\infty$. This dynamical phenomenology has not been previously reported in works dealing with other nonlocal systems. When taking

![Figure 11](imageURL)

**FIGURE 11.** The evolution of a semi-rational solution $|u|$ with parameters given by eq. (70), exhibiting fusion of two lumps and two dark line solitons into two dark line ones.

$$p_1 = 1 + \frac{i}{2}, p_2 = -1 - \frac{i}{2}, \tilde{c}_{11} = 1 + i, \tilde{c}_{22} = -1 + i, b_{1,2} = b_{2,1} = 0, c_{11} = 1, \lambda = 1, s = -2,$$

in eq. (69), it is seen in Fig. (12) that the corresponding semi-rational solution $|u|$ describes fusion of two anti-dark solitons into two bright lumps and two anti-dark solitons. At the initial stage, the solution features two anti-dark solitons on top of the constant background. In the course of the evolution, two bright lumps arise from the two anti-dark solitons. Eventually, two lumps and two anti-dark solitons are completely separated.

7. Discussion and conclusion

In this paper, the nonlocal Maccari system (13) is introduced, featuring the specific $\mathcal{PT}$ symmetry. When $\lambda = 0$, a new nonlocal DS-type equation (18) is derived. For $s = 0$, the nonlocal Maccari system (13) reduces to the defocusing nonlocal NLS equation (19), which may be regarded as a two-dimensional generalization of the nonlocal NLS equation (1). Various exact solutions are presented for the nonlocal Maccari system, produced by means of the long-wave limit and KP-hierarchy approach, combined with the Hirota bilinear method. We have derived solutions for the breather, and breather on top of the periodic background solutions, applying the bilinear form to the nonlocal Maccari system.
We have also derived hyperbolic line-RW (rogue-wave) solutions, and hyperbolic line-RW ones on top of the background of periodic line waves, considering the long-wave limit. Nontrivial dynamics exhibited by these solutions is displayed in Figs. 2-6.

Furthermore, by constructing different tau-functions of the KP hierarch, general line-soliton solutions with elastic collisions between the solitons are derived, as shown in Figs. 7 and 8. Additionally, the semi-rational solutions, which consist of $2N$ lumps and $2N$ solitons, are presented in terms of the KP-hierarchy method. Their novel dynamic behavior is shown in Figs. 11 and 12. These lump-soliton solutions reduce to $2N$-lump solutions of the nonlocal Maccari system, for appropriate parameters.

Main differences between the exact solutions of the nonlocal Maccari system and other nonlocal equations, such as the nonlocal DS (Davey-Stewartson) equation, nonlocal KP-type system, and nonlocal $(2 + 1)$-dimensional NLS equation, are summarized as follows:

• Periodic solutions. The nonlocal Maccari system \(13\) produces breathers, and breathers on top of the background of periodic line waves. The nonlocal DS-I equation gives rise to periodic solitons \(73\) and line breather solutions \(31, 70\). The nonlocal $(2+1)$-dimensional NLS equation produces line breather solution\(90\).

• Soliton solutions. The nonlocal Maccari system \(13\) generates three kinds of two-soliton solutions, viz., dark-dark solitons, dark anti-dark solitons, and anti-dark anti-dark solitons. In these complexes, two bound solitons stay parallel to each other. This features is not found in the nonlocal DS equation and $(2+1)$-dimensional NLS equation. The nonlocal DS-I equation produces a dark cross soliton \(71\) and a double-peak dromion \(73\).

• Rational solutions. The second-order rational solution of nonlocal Maccari system\(13\) is a hyperbolic line RW and 2-lump solutions. However, the second-order rational solution of the nonlocal DS-II equation features a cross-shape line RW \(72\) and dark-anti-dark rational travelling waves \(99\).

• Semi-rational solutions. We have presented: (a) hyperbolic line RWs on top of the background of periodic line waves; (b) fusion of two line solitons and lumps into line solitons; (c) fission of two line solitons into lumps and line solitons of the nonlocal Maccari system \(13\). The semi-rational solutions of the nonlocal 2D NLS equation are composed of a cross-shaped line RW and periodic line waves \(48\). The semi-rational solutions of the nonlocal KP-type system are composed of a soliton and two lumps \(61\).

These new results enrich the structure of waves in nonlocal systems, which suggest that the nonlocal Maccari system \(13\) is quite a relevant nonlocal extension of the nonlocal NLS equation \(1\). The system also provides a useful model for understanding new features of nonlinear dynamics of $\mathcal{PT}$-symmetric systems.
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Appendix A.

In this appendix, we give the proof of Lemma 2.

Proof As

\[(\xi_j + \eta_j)(x, y, t) = (p_j + p_j^* + \frac{1}{p_j} + \frac{1}{p_j^*})x + \left(\frac{\lambda(p_j^2 + 1)\sqrt{-s} - p_j^2 + 1}{2\sqrt{-sp_j}} + \frac{\lambda(p_j^*)^2 + 1)\sqrt{-s} - p_j^*2 + 1}{2\sqrt{-sp_j^*}}\right)y \]
\[+ \frac{i}{2}(p_j^2 - \frac{1}{p_j^*} + p_j^* - \frac{1}{p_j})t + \xi_{j0} + \eta_{j0},\]

and

\[(\xi_{M+j} + \eta_{M+j})(x, y, t) = (p_{M+j} + p_{M+j}^* + \frac{1}{p_{M+j}} + \frac{1}{p_{M+j}^*})x + \left(\frac{\lambda(p_{M+j}^2 + 1)\sqrt{-s} - p_{M+j}^2 + 1}{2\sqrt{-sp_{M+j}}} + \frac{\lambda(p_{M+j}^*)^2 + 1)\sqrt{-s} - p_{M+j}^*2 + 1}{2\sqrt{-sp_{M+j}^*}}\right)y \]
\[+ \frac{i}{2}(p_{M+j}^2 - \frac{1}{p_{M+j}^*} + p_{M+j}^* - \frac{1}{p_{M+j}})t + \xi_{M+j0} + \eta_{M+j0},\]

so we can obtain

\[(\xi_{M+j} + \eta_{M+j})^*(-x, -y, t) = (\xi_j + \eta_j)(x, y, t),\]
\[(\xi_j + \eta_j)^*(-x, -y, t) = (\xi_{M+j} + \eta_{M+j})(x, y, t).\]
Besides, the tau function can be rewritten as

\[
\tau_n(x,y,t) = \prod_{k=1}^{2M} e^{\xi_k + \xi_k^*} \left[ c_j \delta_{jk} e^{-(\xi_j + \xi^*_j)} + \left( -\frac{p_j}{p_{M+k}} \right)^n \frac{1}{p_j + p_{M+k}} \right] \left( -\frac{p_j}{p_{M+j+k}} \right)^n \frac{1}{p_j + p_{M+j+k}} e^{-(\xi_{M+j} + \xi^*_{M+j})} (75)
\]

On the other hand

\[
\tau_n(x,y,t) = \prod_{k=1}^{2M} e^{\xi_k + \xi_k^*} \left[ c_j \delta_{jk} e^{-(\xi_j + \xi^*_j)} + \left( -\frac{p_j}{p_{M+k}} \right)^n \frac{1}{p_j + p_{M+k}} \right] \left( -\frac{p_j}{p_{M+j}} \right)^n \frac{1}{p_j + p_{M+j}} e^{-(\xi_{M+j} + \xi^*_{M+j})} (75)
\]

thus

\[
\tau^*_n = \prod_{k=1}^{2M} e^{\xi_k + \xi_k^*} \left[ c_j^* \delta_{jk} e^{-(\xi_j + \xi^*_j)}(-x,-y,t) + \left( -\frac{p_j}{p_{M+k}} \right)^n \frac{1}{p_j + p_{M+k}} \right] \left( -\frac{p_j}{p_{M+j+k}} \right)^n \frac{1}{p_j + p_{M+j+k}} e^{-(\xi_{M+j} + \xi^*_{M+j})} (75)
\]

\[
\tau^*_n(-x,-y,t) = \tau_n(x,y,t).
\]

Appendix B.

In this appendix, we give the proof of Lemma 3.

Proof Since \( p_{N+j} = -p_j, q_j = p_j^*, \tilde{\eta}_j = -\xi_j^*, \tilde{\xi}_{N+s} = \tilde{\xi}_s \), we have

\[
(\xi_{N+s} + \eta_{N+j})^*(-x,-y,t) = (\xi_j + \eta_j)(x,y,t),
(\xi_s + \eta_{N+j})^*(-x,-y,t) = (\xi_{N+j} + \eta_j)(x,y,t),
\]

\[
\xi_{N+s}^*(-x,-y,t) = \xi_j^*(x,y,t),
\eta_{N+s}^*(-x,-y,t) = \xi_j(x,y,t).
\]

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Furthermore

\[
M^{(n)*}_{N+s,N+j}(-x, -y, t) = \left(-\frac{p_{\xi_j}^*}{q_{\xi_j}^*}\right)^{n} e^{(\xi_{N+s} + \eta_{N+s})^*} (-x, -y, t) \sum_{k=0}^{n_0} c^{*}_{N+s,k} (p_{N+s}^* \partial_{p_{N+s}}^* + \xi_{N+s}^* (-x, -y, t) + n)^{n_0-k} \\
\times \sum_{l=0}^{n_0} d_{N+j,l}^* (q_{N+j}^* \partial_{q_{N+j}}^* + \eta_{N+j}^* (-x, -y, t) - n)^{n_0-l} \frac{1}{p_{N+s}^* + q_{N+j}^*} + \tilde{c}_{N+s,N+j}^*
\]

\[
= -\left(-\frac{p_{\xi_j}}{q_{\xi_j}}\right)^{-n} e^{\xi_{i} + \eta_{N+s}} \sum_{k=0}^{n_0} d_{N+s,k} (q_{N+s} \partial_{q_{N+s}} + \eta_{N+s}^* + n)^{n_0-k} \\
\times \sum_{l=0}^{n_0} c_j (p_j \partial_{p_j} + \xi_j^* - n)^{n_0-l} \frac{1}{p_j + q_{N+s}} - \tilde{c}_{j,N+s}
\]

\[
= -M^{(-n)}_{j,N+s}(x, y, t),
\]  

(77)

On the other hand

\[
M^{(n)*}_{s,N+j}(-x, -y, t) = -M^{(-n)}_{N+j,N+s}(x, y, t).
\]  

(78)

\[
M^{(n)*}_{s,N+j}(-x, -y, t) = \left(-\frac{p_{\xi_j}^*}{q_{\xi_j}^*}\right)^{n} e^{(\xi_{s} + \eta_{N+j})^*} (-x, -y, t) \sum_{k=0}^{n_0} c_{s,k} (p_{s}^* \partial_{p_{s}}^* + \xi_{s}^* (-x, -y, t) + n)^{n_0-k} \\
\times \sum_{l=0}^{n_0} d_{N+j,l}^* (q_{N+j}^* \partial_{q_{N+j}}^* + \eta_{N+j}^* (-x, -y, t) - n)^{n_0-l} \frac{1}{p_{s}^* + q_{N+j}^*} + \tilde{c}_{s,N+j}^*
\]

\[
= -\left(-\frac{p_{\xi_j}}{q_{\xi_j}}\right)^{-n} e^{\xi_{i} + \eta_{N+s}} \sum_{k=0}^{n_0} d_{N+s,k} (q_{N+s} \partial_{q_{N+s}} + \eta_{N+s}^* + n)^{n_0-k} \\
\times \sum_{l=0}^{n_0} c_j (p_j \partial_{p_j} + \xi_j^* - n)^{n_0-l} \frac{1}{p_j + q_{N+s}} - \tilde{c}_{j,N+s}
\]

\[
= -M^{(-n)}_{j,N+s}(x, y, t),
\]  

(79)

These results imply

\[
\tau^*_{n}(-x, -y, t) = \begin{vmatrix} 
M^{(n)*}_{s,j}(-x, -y, t) & M^{(n)*}_{s,N+j}(-x, -y, t) \\
M^{(n)*}_{N+s,j}(-x, -y, t) & M^{(n)*}_{N+s,N+j}(-x, -y, t) 
\end{vmatrix}
\]

\[
= \begin{vmatrix} 
-M^{(-n)}_{N+s,N+j} & -M^{(-n)}_{j,N+s} \\
-M^{(-n)}_{j,N+s} & -M^{(-n)}_{j,s} 
\end{vmatrix}
\]  

(81)

this completes the proof.

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