Massive Neutrinos in Almost-Commutative Geometry

Christoph A. Stephan\textsuperscript{1, 2}

Abstract

In the noncommutative formulation of the standard model of particle physics by A. Connes and A. Chamseddine [1] one of the three generations of fermions has to possess a massless neutrino. This formulation is consistent with neutrino oscillation experiments and the known bounds of the Pontecorvo-Maki-Nakagawa-Sakata matrix (PMNS matrix). But future experiments which may be able to detect neutrino masses directly and high-precision measurements of the PMNS matrix might need massive neutrinos in all three generations.

In this publication we present an almost-commutative geometry which allows for a standard model with massive neutrinos in all three generations. This model does not follow in a straightforward way from Connes’ and Chamseddine’s version since it requires an internal algebra with four summands of matrix algebras, instead of three summands for the model with one massless neutrino.

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\textsuperscript{1} Unité Mixte de Recherche (UMR) 6207 du CNRS et des Universités Aix-Marseille 1 et 2
Sud Toulon-Var, Laboratoire affilié à la FRUMAM (FR 2291)
\textsuperscript{2} stephan@cpt.univ-mrs.fr
1 Introduction

Alain Connes’ noncommutative geometry [1–4] allows in an elegant way to unify gravity and the standard model of particle physics. A central role in this formalism is played by almost-commutative spectral triples \((A, \mathcal{H}, D)\) which decompose into an external and an internal, finite dimensional component. The external part encodes a compact 4-dimensional Euclidian spacetime and the internal one corresponds to a discrete 0-dimensional Kaluza-Klein space. The internal algebra consists of a sum of matrix algebras and determines together with the corresponding internal Hilbert space the particle content of the theory. Via the spectral action [1] one recovers the Einstein-Hilbert action combined with the bosonic action of a Yang-Mills-Higgs (YMH) theory. The set of YMH theories compatible with almost-commutative geometry is severely constrained. It could be shown that among the possible geometries those which produce the standard model of particle physics take a most prominent position [5]. The spectral triples were required to be irreducible and non-degenerate, in the sense that the Hilbert space was chosen to be as small as possible with non-degenerate fermion masses. For exact definitions see [5]. Heavy use was made of Krajewski’s diagrammatic method [6], which will be described briefly below.

The standard model with one generation of fermions turns out to be among the group of irreducible spectral triples with three and four summands in the internal algebra. The case of three algebras recovers the model of Connes and Chamseddine whereas the four algebra case resembles more the Connes-Lott model [7]. All of these minimal models do require a massless neutrino. It is possible to add massive neutrinos if one extends the particle content to three generations, but in the case of three summands at least one neutrino has to stay massless. In this paper we will present a model with massive neutrinos in all three generations based on the irreducible version of the standard model with four summands in the internal algebra.

2 Basic Definitions

In this section we will give the necessary basic definitions of almost commutative geometries from a particle physics point of view. For our calculations, only the finite part matters, so we restrict ourselves to real, \(S^0\)-real, finite spectral triples \((A, \mathcal{H}, D, J, \epsilon, \chi)\). The algebra \(A\) is a finite sum of matrix algebras \(A = \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{K}_i)\) with \(\mathbb{K}_i = \mathbb{R}, \mathbb{C}, \mathbb{H}\) where \(\mathbb{H}\) denotes the quaternions. A faithful representation \(\rho\) of \(A\) is given on the finite dimensional Hilbert space \(\mathcal{H}\). The Dirac operator \(D\) is a selfadjoint operator on \(\mathcal{H}\) and plays the role of the fermionic mass matrix. \(J\) is an antiunitary involution, \(J^2 = 1\), and is interpreted as the charge conjugation operator of particle physics. The \(S^0\)-real structure \(\epsilon\) is a unitary involution, \(\epsilon^2 = 1\). Its eigenstates with eigenvalue \(+1\) are the particle states, eigenvalue \(-1\) indicates antiparticle states. The chirality \(\chi\) as well is a unitary involution, \(\chi^2 = 1\), whose eigenstates with eigenvalue \(+1\) \((-1)\) are interpreted as right (left) particle states. These operators are required to fulfill Connes’ axioms for spectral triples:

- \([J, D] = [J, \chi] = [\epsilon, \chi] = [\epsilon, D] = 0\), \(\epsilon J = -J \epsilon\), \(D \chi = -\chi D\),
- \([\chi, \rho(a)] = [\epsilon, \rho(a)] = [\rho(a), J \rho(b) J^{-1}] = [[D, \rho(a)], J \rho(b) J^{-1}] = 0, \forall a, b \in A\).
• The chirality can be written as a finite sum $\chi = \sum_i \rho(a_i) J \rho(b_i) J^{-1}$. This condition is called orientability.

• The intersection form $\cap_{ij} := \text{tr}(\rho(p_i) J \rho(p_j) J^{-1})$ is non-degenerate, $\det \cap \neq 0$. The $p_i$ are minimal rank projections in $\mathcal{A}$. This condition is called Poincaré duality.

Now the Hilbert space $\mathcal{H}$ and the representation $\rho$ decompose with respect to the eigenvalues of $\epsilon$ and $\chi$ into left and right, particle and antiparticle spinors and representations:

$$\mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R \oplus \mathcal{H}_L^c \oplus \mathcal{H}_R^c \quad \rho = \rho_L \oplus \rho_R \oplus \rho_L^c \oplus \rho_R^c$$

In this representation the Dirac operator has the form

$$D = \begin{pmatrix} 0 & \mathcal{M} & 0 & 0 \\ \mathcal{M}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{M} \\ 0 & 0 & \mathcal{M}^* & 0 \end{pmatrix},$$

where $\mathcal{M}$ is the fermionic mass matrix connecting the left and the right handed fermions.

Connes’ axioms, the decomposition of the Hilbert space, the representation and the Dirac operator allow a diagrammatic depiction, known as Krajewski diagrams. As was shown in [6] and [5] this can be boiled down to simple arrows, which encode the intersection form and the fermionic mass matrix. From these informations all the ingredients of the spectral triple can be recovered. For our purpose a simple arrow and a double arrow are sufficient. The arrows allways point from right fermions (positive chirality) to left fermions (negative chirality). We may also restrict ourselves to the particle part, since the information of the antiparticle part is included by transposing the particle part. We will adopt the conventions of [5].

To complete our short survey, we will give a brief glimpse on how to construct the actual Yang-Mills-Higgs theory. We started out with the fixed (for convenience flat) Dirac operator of a 4-dimensional spacetime with a fixed fermionic mass matrix. To generate curvature we have to perform a general coordinate transformation and then fluctuate the Dirac operator. This can be achieved by lifting the automorphisms of the algebra to the Hilbert space, unitarily transforming the Dirac operator with the lifted automorphisms and then building linear combinations. Again we restrict ourselves to the finite case. Except for complex conjugation in $M_n(\mathbb{C})$ and permutations of identical summands in the algebra $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \ldots \oplus \mathcal{A}_N$, every algebra automorphism $\sigma$ is inner, $\sigma(a) = uau^{-1}$ for a unitary $u \in U(\mathcal{A})$. Therefore the connected component of the automorphism group is $\text{Aut}(\mathcal{A})^e = U(\mathcal{A})/(U(\mathcal{A}) \cap \text{Center}(\mathcal{A}))$. Its lift to the Hilbert space [3]

$$L(\sigma) = \rho(u) J \rho(u) J^{-1}$$

(2.1)

is multi-valued. To avoid the multi-valuedness in the fluctuations, we allow a central extension of the automorphism group. As we will see, central extensions will also allow to cancel anomalies.

The fluctuation $\mathcal{D}$ of the Dirac operator $D$ is given by a finite collection $f$ of real numbers $r_j$ and algebra automorphisms $\sigma_j \in \text{Aut}(\mathcal{A})^e$ such that

$$\mathcal{D} := \sum_j r_j L(\sigma_j) \mathcal{D} L(\sigma_j)^{-1}, \quad r_j \in \mathbb{R}, \quad \sigma_j \in \text{Aut}(\mathcal{A})^e.$$
We consider only fluctuations with real coefficients since \( \mathcal{D} \) must remain selfadjoint. The sub-matrix of the fluctuated Dirac operator \( \mathcal{D} \) which is equivalent to the mass matrix \( M \), is often denoted by \( \varphi \), the ‘Higgs scalar’, in physics literature.

An almost commutative geometry is the tensor product of a finite noncommutative triple with an infinite, commutative spectral triple. By Connes’ reconstruction theorem [4] we know that the latter comes from a Riemannian spin manifold, which we will take to be any 4-dimensional, compact, flat manifold like the flat 4-torus. The spectral action of this almost commutative spectral triple reduced to the finite part is a functional on the vector space of all fluctuated, finite Dirac operators:

\[
V(\mathcal{D}) = \lambda \operatorname{tr}[(\mathcal{D})^4] - \frac{\mu^2}{2} \operatorname{tr}[(\mathcal{D})^2],
\]

where \( \lambda \) and \( \mu \) are positive constants [1]. The spectral action is invariant under lifted automorphisms and even under the unitary group \( U(A) \ni u \),

\[
V([\rho(u)J\rho(u)J^{-1}]\mathcal{D}[\rho(u)J\rho(u)J^{-1}]) = V(\mathcal{D}),
\]

and it is bounded from below. To obtain the physical content of a diagram and its associated spectral triple one has to find the minima \( \mathcal{D} \) of this action.

3 The Model

We will start with the Krajewski diagram of the standard model with four summands in the matrix algebra and only one fermion generation as it was shown in [5]:

\[
\begin{array}{cccc}
 a & b & c & d \\
 a & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
b & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
c & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
d & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
\end{array}
\]

Adding a massive neutrino, which also means adding a right-handed neutrino, can be achieved by drawing an extra arrow.

\[
\begin{array}{cccc}
 a & b & c & d \\
 a & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
b & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
c & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
d & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
\end{array}
\]

(3.1)
The determinant of the corresponding multiplicity matrix

\[
\mu = \begin{pmatrix}
0 & 0 & 2 & 1 \\
0 & 0 & -1 & -1 \\
2 & -1 & 0 & 0 \\
1 & -1 & 0 & 2
\end{pmatrix},
\]

(3.2)
is non-zero and so the axioms of noncommutative geometry remain fulfilled. Note that adding the right-handed neutrino by copying the form of the quark sector does not work,

\[
\begin{array}{cccc}
a & b & c & d \\
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ
\end{array}
\]
since the determinant of the multiplicity matrix

\[
\mu = \begin{pmatrix}
0 & 0 & 2 & 1 \\
0 & 0 & -1 & -1 \\
2 & -1 & 0 & 0 \\
2 & -1 & 0 & 0
\end{pmatrix},
\]

(3.3)
is zero. Due to this reason it is impossible to add a right-handed neutrino in the case of three summands in the internal algebra.

Now the internal algebra is chosen to be \( \mathcal{A} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \oplus \mathbb{C} \ni (a, b, c, d) \) From the Krajewski diagram 3.1 one reads off the representation of the algebra, ordered into left and right, particle and antiparticle part:

\[
\rho_L(a, b, c, d) = \begin{pmatrix}
b \otimes 1_3 & 0 \\
0 & b
\end{pmatrix}, \quad \rho_R(a, b, c, d) = \begin{pmatrix}
a_{13} & 0 & 0 & 0 \\
0 & \bar{a}_{13} & 0 & 0 \\
0 & 0 & \bar{a} & 0 \\
0 & 0 & 0 & \bar{d}
\end{pmatrix},
\]

\[
\rho_{L}^{c}(a, b, c, d) = \begin{pmatrix}
1_2 \otimes c & 0 \\
0 & d_{12}
\end{pmatrix}, \quad \rho_{R}^{c}(a, b, c, d) = \begin{pmatrix}
c & 0 & 0 & 0 \\
0 & c & 0 & 0 \\
0 & 0 & d & 0 \\
0 & 0 & 0 & d
\end{pmatrix},
\]

(3.4)
where \( 1_3 \) is the unit matrix and the complex conjugates where chosen in order to reproduce the standard model. This representation acts on the Hilbert space \( \mathcal{H} = \mathcal{H}^{PL} \oplus \mathcal{H}^{PR} \oplus \mathcal{H}^{AL} \oplus \mathcal{H}^{AR} \) with the left- and right-handed particle subspaces

\[
\mathcal{H}^{PL} = \begin{pmatrix}
u^{L}_e \\
\nu^{L}_e \\
\nu^{L}_e \\
\nu^{L}_e
\end{pmatrix}, \quad \mathcal{H}^{PR} = \begin{pmatrix}
u^{R}_e \\
\nu^{R}_e \\
\nu^{R}_e \\
\nu^{R}_e
\end{pmatrix},
\]

(3.5)
and antiparticle subspaces

\[ H^{AL} = \begin{pmatrix} u^c \end{pmatrix}^L, \quad H^{AR} = \begin{pmatrix} u^c R \end{pmatrix} \]

Note the fermions automatically appear as left-handed doublets and right-handed singlets.

The particle part \( \Delta \) of the Dirac operator is

\[ \Delta = \begin{pmatrix} 0 & 0 & M_1 & M_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & M_3 & M_4 \\ M_1^* & 0 & 0 & 0 & 0 & 0 \\ M_2^* & 0 & 0 & 0 & 0 & 0 \\ 0 & M_3^* & 0 & 0 & 0 & 0 \\ 0 & M_4^* & 0 & 0 & 0 & 0 \end{pmatrix} \]

We choose the four mass matrices \( M_1, M_2, M_3 \) and \( M_4 \) of the initial Dirac operator as

\[ M_1 = \begin{pmatrix} m_1 \\ 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 \\ m_2 \end{pmatrix}, \quad M_3 = \begin{pmatrix} m_3 \\ 0 \end{pmatrix}, \quad \text{and} \quad M_4 = \begin{pmatrix} 0 \\ m_4 \end{pmatrix}, \]

with \( m_1, m_2, m_3, m_4 \in \mathbb{C} \) arbitrary.

In the next step the lift \( L \) has to be worked out. The unitaries of \( \mathcal{A} \) close to the identity are: \( U^e(\mathbb{C}) = U(1), U^e(\mathbb{H}) = SU(2) \) and \( U^e(M_3(\mathbb{C})) = U(3) \). With definition 2.1 and the algebra representation one finds for the particle part of the lift

\[ L^P((\det w)^p, u, (\det w)^q w, (\det w)^r) = \rho_L^P(\cdots) \rho_L^A(\cdots) \oplus \rho_R^P(\cdots) \rho_R^A(\cdots) = \text{diag} \left[ ((\det w)^q u \otimes w, (\det w)^r u, (\det w)^{q-p} w, (\det w)^{q-p} w, (\det w)^0, (\det w)^{p+r} \right] \]

where \( u \in SU(2), w \in U(3) \) and \( p, q, r \in \mathbb{Q} \). The exponents \( p, q \) and \( r \) are fixed to recover the standard model charges up to normalisation

\[ q = \frac{p-1}{3} \quad \text{and} \quad r = -p. \]

Next the Higgs potential \( \varphi \) can be calculated. It follows for the particle part of the Higgs potential

\[ \varphi^P = \sum_i r_i L^P((\det w_i)^p, u_i, (\det w_i)^q \bar{w}_i, (\det w_i)^r) \times \Delta \times \left[ L^P((\det w_i)^p, u_i, (\det w_i)^q \bar{w}_i, (\det w_i)^r) \right]^{-1} \]

\[ = \begin{pmatrix} 0 & 0 & \varphi_1 & \varphi_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varphi_3 & \varphi_4 \\ \varphi_1^* & 0 & 0 & 0 & 0 & 0 \\ \varphi_2^* & 0 & 0 & 0 & 0 & 0 \\ 0 & \varphi_3^* & 0 & 0 & 0 & 0 \\ 0 & \varphi_4^* & 0 & 0 & 0 & 0 \end{pmatrix} \]
with $\tilde{w} = (\det w)^{-1/3}w \in SU(3)$ and

$$
\begin{align*}
\varphi_1 &= \sum_i r_i u_i M_1 (\det w_i)^{-p}, \\
\varphi_2 &= \sum_i r_i u_i M_1 (\det w_i)^p, \\
\varphi_3 &= \sum_i r_i u_i M_3 (\det w_i)^{-p}, \\
\varphi_4 &= \sum_i r_i u_i M_4 (\det w_i)^p.
\end{align*}
$$

(3.12)

For the Dirac operator with mass matrices \(3.8\) it follows that \(\varphi_1\) and \(\varphi_2\) as well as \(\varphi_3\) and \(\varphi_4\) allow the following operations: One can combine \(\varphi_1\) and \(\varphi_2\) \((\varphi_3\) and \(\varphi_4\)) into a single matrix

$$
\varphi_{m/n} = \sum_i r_i u_i (M_m, M_n) \begin{pmatrix} (\det w_i)^{-p} & 0 \\ 0 & (\det w_i)^p \end{pmatrix}
$$

$$
= \sum_i r_i u_i \begin{pmatrix} (\det w_i)^{-p} & 0 \\ 0 & (\det w_i)^p \end{pmatrix} (M_m, M_n).
$$

(3.13)

This is possible since \((M_m, M_n)\) are diagonal matrices. One sees further that

$$
\begin{pmatrix} (\det w_i)^{-p} & 0 \\ 0 & (\det w_i)^p \end{pmatrix} \in SU(2)
$$

(3.14)

and thus

$$
u_i \begin{pmatrix} (\det w_i)^{-p} & 0 \\ 0 & (\det w_i)^p \end{pmatrix} \in SU(2).
$$

(3.15)

With the observation that any quaternion \(h \in \mathbb{H}\) can be written as \(h = ru\) with \(r \in \mathbb{R}\) and \(u \in SU(2)\) it follows that \(\varphi_1/2\) and \(\varphi_3/4\) take the simple form

$$
\varphi_{m/n} = h \begin{pmatrix} m_m & 0 \\ 0 & m_n \end{pmatrix} \text{ with } h = \begin{pmatrix} x & y \\ -\bar{y} & \bar{x} \end{pmatrix} \in \mathbb{H}.
$$

(3.16)

It is now easy to minimise the Higgs potential

$$V(\varphi) = \lambda \text{tr}(\varphi^* \varphi)^2 - \frac{1}{2} \mu^2 \text{tr}(\varphi^* \varphi)
$$

(3.17)

with respect to the variables \(x, y \in \mathbb{C}\). To break up the calculation into smaller steps one starts with

$$
\varphi^* \varphi = \begin{pmatrix}
\varphi_{1/2} \varphi_{1/2}^* \otimes 1_3 & 0 & 0 & 0 \\
0 & \varphi_{3/4} \varphi_{3/4}^* & 0 & 0 \\
0 & 0 & \varphi_{1/2} \varphi_{1/2}^* \otimes 1_3 & 0 \\
0 & 0 & 0 & \varphi_{3/4} \varphi_{3/4}^* \\
\end{pmatrix}.
$$

(3.18)

It follows that

$$
\text{tr}(\varphi^* \varphi) = 6 \text{tr}(\varphi_{1/2}^* \varphi_{1/2}) + 2 \text{tr}(\varphi_{3/4}^* \varphi_{3/4}^*)
$$

(3.19)
and
\[ \text{tr}(\varphi \varphi^*)^2 = 6 \text{tr}(\varphi_{1/2}^* \varphi_{1/2})^2 + 2 \text{tr}(\varphi_{3/4}^* \varphi_{3/4})^2. \] (3.20)

Furthermore one has
\[ \varphi_{m/n}^* = \begin{pmatrix} \bar{m}_m & 0 \\ 0 & \bar{m}_n \end{pmatrix} \begin{pmatrix} \bar{x} & -y \\ y & \bar{x} \end{pmatrix} \] (3.21)
and
\[ \varphi_{m/n}^* \varphi_{m/n} = \begin{pmatrix} \bar{m}_m & 0 \\ 0 & \bar{m}_n \end{pmatrix} \begin{pmatrix} |x|^2 + |y|^2 & 0 \\ 0 & |x|^2 + |y|^2 \end{pmatrix} \begin{pmatrix} m_m & 0 \\ 0 & m_n \end{pmatrix} \]
\[ = r \begin{pmatrix} |m_m|^2 & 0 \\ 0 & |m_n|^2 \end{pmatrix}, \] (3.22)
with \( r := |x|^2 + |y|^2 \) as the new variable. Putting everything into \( V(\varphi) \) gives
\[ V(\varphi) = \lambda (6 |m_1|^4 + 6 |m_2|^4 + 2 |m_3|^4 + 2 |m_4|^4) r^2 \]
\[ - \frac{\mu^2}{2} (6 |m_1|^2 + 6 |m_2|^2 + 2 |m_3|^2 + 2 |m_4|^2) r \]
\[ =: \lambda \alpha r^2 - \frac{\mu^2}{2} \beta r. \] (3.23)

The minimum in \( r \) is found by differentiating with respect to \( r \) and is
\[ \varphi = \frac{\alpha \mu^2}{\beta 4 \lambda}. \] (3.24)

Now one can construct a minimum of the Higgs potential as
\[ \varphi = \sqrt{\varphi} \Delta. \] (3.25)

The minimum is given by the initial mass matrices \( M_1, M_2, M_3 \) and \( M_4 \) up to a fixed numerical factor \( \sqrt{\varphi} \in \mathbb{R}^+ \). The masses \( m_1, m_2, m_3 \) and \( m_4 \) are identified as the quark masses, the neutrino mass and electron mass.

In the last step the little group and the charges of the fermions have to be found. \( U(1) \times SU(2) \times SU(3) \) is the unbroken gauge group. The little group \( G_\ell \) is defined by \( \rho_L^\ell(g_\ell) \varphi - \varphi = 0 \) for all \( g_\ell \in G_\ell \). This is only possible if the representation is diagonal. It follows that \( U(1) \times SU(2) \to U(1) \subset SU(2) \ni v \) so that \( G_\ell = U(1) \times SU(3) \).
The charges of the fermions are then given by the lift of $G_ℓ$:

$$L^{PL} = \begin{pmatrix}
(d \w)\frac{p}{2} \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix} \otimes \bar{w} & 0 \\
0 & (d \w)^p \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix} \\
0 & 0 & (d \w)^p \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix}
\end{pmatrix}$$

$$L^{PR} = \begin{pmatrix}
(d \w)^{p+\frac{p}{2}} \bar{w} & 0 & 0 & 0 \\
0 & (d \w)^{-p+\frac{p}{2}} \bar{w} & 0 & 0 \\
0 & 0 & (d \w)^0 & 0 \\
0 & 0 & 0 & (d \w)^{-2p}
\end{pmatrix}, \quad (3.26)$$

the electric charges being the exponents of the $U(1)$ elements.

Finally $p$ and $v$ have to be fixed in view of experimental data. The neutrino has to be neutral under the little group and so $(d \w)^pv = 1$ and it follows that $v = (d \w)^{-p}$. With $p = -1/2$ the electric charge of the electron is $Q_ℓ = -1$ and the quark charges are $Q_d = -1/3$ and $Q_u = +2/3$. Note that the charges for the left-handed and right-handed fermions are equal and that the colour and the electric charge couple vectorially, as desired. The right-handed neutrino is completely neutral. We also remark at this point that the gauge and mixed gravitational anomalies can be canceled only by the use of a particular central extension.

4 Conclusion

We presented an almost-commutative geometry with four summands in the internal algebra which allows massive neutrinos in all generations of the standard model. This model is reducible in the sense of [5], i.e. the right-handed neutrino can be erased without violating any axiom of noncommutative geometry. One may speculate that this reducibility is connected to the relatively small mass of the neutrino compared with the other fermions.

It should be pointed out that from the experimental point of view a massless neutrino in one of the fermion generations is not excluded. For a complete classification of all mass matrices and corresponding PMNS mixing matrices compatible with the experiment we refer to [8].

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