Local immobilization of particles in mass transfer described by the equation of the Jeffreys type

S. A. Rukolaine and A. M. Samsonov
The Ioffe Physical Technical Institute of the Russian Academy of Sciences,
26 Polytekhnicheskaya, St. Petersburg, 194021, Russia
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We consider the equation of the Jeffreys type as the basic one in three different models of mass transfer, namely, the Jeffreys type and two-phase models, and the $D_1$ approximation to the linear Boltzmann equation. We study two classic 1 + 1D problems in the framework of each model. The first problem is the transfer of a substance initially confined in a point. The second problem is the transfer of a substance from a stationary point source. We calculate the mean-square displacement (MSD) for the solutions of the first problem. The temporal behaviour of the MSD in the framework of the first and third models is found to be the same as that in the Brownian motion described by the standard Langevin equation. Besides, we find a remarkable phenomenon when a portion of the substance does not move.

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I. INTRODUCTION

The classic diffusion equation, based on Fick’s law, is widely used for approximate description of non-anomalous diffusion (dispersion of a substance or species) [1 2] and Brownian motion [3 5]. However, Fick’s law neglects the mass (inertia) of moving particles (molecules), and, therefore, the diffusion equation gives an appropriate and accurate model for diffusion phenomena only in weakly inhomogeneous media and/or for processes, which are slow in time, when relaxation time is short compared to a characteristic time scale. Otherwise the description of diffusion by the diffusion equation may fail [6]. Many biological media, e.g., cellular cytoplasm, are strongly inhomogeneous, therefore, diffusion in them is not Fickian and its description by the diffusion equation is questionable.

Note that the counterpart of Fick’s law is Fourier’s law in the theory of heat conduction [1 2]. The latter leads to the heat equation, similar to the diffusion equation. Fick’s law was postulated by analogy with Fourier’s law, which was proposed first [2].

The simplest modification of Fick’s law, taking into account the inertia of moving particles, is Cattaneo’s equation [8 9]. Strictly speaking, Cattaneo’s equation modifies Fourier’s law, since Cattaneo considered heat conduction, however, one can apply this to mass transfer as well. The modification leads to the telegraph equation, providing the finite speed of propagation [8 9 11 14]. The telegraph equation was proposed to be a substitution of the diffusion and heat equations. However, both two- and three-dimensional telegraph equations meet formal obstacles since solutions to initial value problem for them may become negative [15 16].

Long before Cattaneo H. Jeffreys proposed a relation for rheological description of Earth’s core [17 18], that can be considered as a combination of Fick’s (or Fourier’s) law and Cattaneo’s equation. We define this the law (relation) of the Jeffreys type. This law leads to a partial differential equation of the third order, called the equation of the Jeffreys type [8 9], also known as the simplest equation of the dual phase lag model of heat conduction [10 19 20]. We call this the Jeffreys type model. This model was used for description of viscoelastic fluids [8 21 22], Taylor dispersion [22 23] and heat transfer [8 10 11 14 20 21 28].

There is another two-temperature [10 14 19 20 24] or two-phase [20 22] model, which leads also to the equation of the Jeffreys type. This model is qualitatively different from the Jeffreys type one. Nevertheless, to the best of our knowledge, there is no clear distinction between cases when the equation of the Jeffreys type describes the Jeffreys type and the two-temperature (two-phase) models [10 14 20 24]. Moreover, in Ref. [24] the behaviour of the two-temperature model is illustrated by that of the Jeffreys type model. And in Ref. [33] the authors erroneously state that the two models are equivalent.

The diffusion equation is known to be the simplest approximation to the linear Boltzmann equation [34], and the latter can be considered as a mesoscopic model of the former. It is notable that the telegraph equation is the $P_1$ approximation (the next after the diffusion one) to the linear Boltzmann equation [34]. Recently, $D_N$ approximations to the linear Boltzmann equation were proposed [35]. They generalize the classic diffusion approximation, which corresponds to $N = 0$. We have found that the $D_{N=1}$ approximation (the next after the telegraph one) leads to the equation of the Jeffreys type, the model being qualitatively different from both the Jeffreys type and two-phase ones. Note that this model is similar to the one of Guyer and Krumhansl in the theory of second sound [10 13 56 37].

The primary motivation of this study was to investi-
gate the equation of the Jeffreys type as a substitution of
the diffusion equation instead of the telegraph one for
description of mass transfer. Eventually in this paper we
study and compare the three models (the Jeffreys type
and two-phase models, and the diffusion equation
approximation to the Boltzmann equation) as models of
mass transfer. We study two classic 1 + 1D problems, typical for
mass transfer, in the framework of each model, where, as
far as we know, the problems have not been studied. The
first problem is the transfer of a substance initially con-
fined in a point. The second problem is the transfer of a
substance from a stationary point source. We calculate the
mean-square displacement (MSD) for the solutions of
the problem, because the MSD is an integral quan-
tity whose temporal dependence characterizes diffusion
and Brownian motion. The temporal behaviour of the
MSD in the framework of the three models is found to be the same as that in the Brownian motion de-
scribed by the standard Langevin equation. We remind
here that the behaviour of the MSD in the framework of
the diffusion equation is wrong at small values of time,
where it must be ballistic. Besides, we find a remarkable
phenomenon when a portion of the substance
does not
move.

The rest of the paper is organized as follows. In Section II we briefly remind the phenomenological derivation
of the diffusion equation. In Section III we briefly recall
some facts about the telegraph equation. In Section IV we
describe the models of mass transfer related to the
equation of the Jeffreys type. In Section V we study the
diffusion of a substance initially confined in a point in
the framework of the three models. In Section VI we
calculate the mean-square displacement for the solutions
of the problems, considered in Section V. In Section VII
we study the diffusion of a substance from a stationary
point source also in the framework of the three models.
Section VIII contains some concluding remarks.

II. DIFFUSION EQUATION

The macroscopic law of mass balance for a substance
is expressed by the continuity equation [38]
\[ \frac{\partial u}{\partial t} + \text{div} J = f, \] (II.1)
where \( u \equiv u(x, t) \) is the concentration of the substance,
\( x = (x_1, x_2, x_3) \) is a point, \( t \) is time, \( J \equiv J(x, t) \) is
the flux of the substance, \( f \equiv f(x, t, u) \) is the net rate of
production or absorption (degradation) of the substance.

In the simplest approximation the flux is related to the
concentration by phenomenological Fick’s (first) law [4]
[38]
\[ J = -D \nabla u, \] (II.2)
where \( D \) is the diffusion coefficient.

The continuity equation [II.1] and Fick’s law lead to
the reaction diffusion equation
\[ \frac{\partial u}{\partial t} - D \Delta u = f. \] (II.3)

To determine a unique solution of the diffusion equation
one imposes the initial condition
\[ u|_{t=0} = u_0, \] (II.4)
where \( u_0 \equiv u_0(x) \) is the distribution of the concentration
at time \( t = 0 \).

Note that the diffusion equation is the simplest ap-
proximation to the linear Boltzmann equation [34], see
Eq. (A.1).

III. TELEGRAPH EQUATION

Fick’s law neglects the inertia of moving particles. Catt-
taneo’s equation [8–11]
\[ \tau \frac{\partial J}{\partial t} + J = -D \nabla u, \] (III.1)
where \( \tau \) is the relaxation time, modifies Fick’s law, taking
the inertia into account. Indeed, Cattaneo’s equation can
be written in the equivalent integral form
\[ J = -\frac{D}{\tau} \int_0^t e^{-t-t'}/\tau \nabla u(x, t') \, dt' + e^{-t/\tau} J_0, \] (III.2)
where \( J_0 \equiv J_0(x) \) is the distribution of flux at time \( t = 0 \).
Eq. (III.2) shows that Cattaneo’s equation takes into
account the prehistory of a process, since flux depends
on the gradient of the concentration at earlier time, the
dependence being exponentially decreasing with time. If
the relaxation time \( \tau \) in Cattaneo’s equation tends to
zero, one obtains in the limit Fick’s law.

The continuity equation [II.1] and Eq. (III.2) lead to
the integro-differential equation
\[ \frac{\partial u}{\partial t} - \frac{D}{\tau} \int_0^t e^{-(t-t')/\tau} \Delta u(x, t') \, dt' + e^{-t/\tau} \text{div} J_0 = f. \] (III.3)

This equation with the initial condition [II.4] is equi-



valent to the reaction telegraph (or damped wave) equation [39–40]
\[ \tau \frac{\partial^2 u}{\partial t^2} + \left(1 - \tau \frac{\partial f}{\partial u}\right) \frac{\partial u}{\partial t} - D \Delta u = f + \tau \frac{\partial f}{\partial t}, \] (III.3)
with the initial conditions
\[ u|_{t=0} = u_0, \quad \frac{\partial u}{\partial t}|_{t=0} = -\text{div} J_0 + f_0, \] (III.4)
where \( f_0 \equiv f_0(x, u_0) \) is the distribution of sources at time \( t = 0 \). If \( \tau = 0 \), the telegraph equa-
tion (III.3) becomes the diffusion equation (II.3).
The telegraph equation can also be obtained as the $P_1$ approximation to the linear Boltzmann equation [33], see Eq. (A.6).

The telegraph equation is hyperbolic, providing the finite speed of signal propagation, and it was proposed to be a substitution of the parabolic diffusion and heat equations [8, 9, 12, 14]. However, two- and three-dimensional telegraph equations have a formal flaw since their solutions may take negative values even if the initial values are positive [13, 16]. Besides, the applicability of the telegraph equation to the description of heat transfer is doubtful [16, 41, 42].

IV. EQUATION OF THE JEFFREYS TYPE

A. Jeffreys type model

The relation, combining Fick’s law and Cattaneo’s equation, has the form [8, 9]

$$
\tau \frac{\partial J}{\partial t} + J = -\tau D_1 \frac{\partial \nabla u}{\partial t} - (D_1 + D_2) \nabla u \\
\equiv -(D_1 + D_2) \left[ \tau_2 \frac{\partial \nabla u}{\partial t} + \nabla u \right],
$$

where $D_1 > 0$, $D_1 + D_2 > 0$ and

$$
\tau_2 = \frac{\tau D_1}{D_1 + D_2}
$$

is another relaxation time, or, equivalently, the integro-differential form

$$
J = -D_1 \nabla u - \frac{D_2}{\tau} \int_0^t e^{-(t-t')/\tau} \nabla u(x, t') \, dt' \\
+ e^{-t/\tau} \left(D_1 \nabla u_0 + J_0\right),
$$

where $u_0 \equiv u_0(x)$ and $J_0 \equiv J_0(x)$ are the distributions of the concentration and flux, respectively, at time $t = 0$. We name the relations (IV.1) and (IV.2) the law of the Jeffreys type after H. Jeffreys who proposed similar relations for rheological description of the Earth core [17, 18].

Fick’s law and Cattaneo’s equation are particular cases of the law of the Jeffreys type. Indeed, if $\tau$ in Eqs. (IV.1) and (IV.2) tends to zero, one obtains in the limit Fick’s law with $D = D_1 + D_2$, while $D_1 = 0$ leads to Cattaneo’s equation.

The law of the Jeffreys type (IV.1) includes two different cases, $\tau > \tau_2$ and $\tau < \tau_2$, depending on whether the relaxation time $\tau$ is higher or lower than $\tau_2$. Both cases are considered in literature, see, e.g., [19–22]. The first inequality $\tau > \tau_2$ is equivalent to $D_2 > 0$. In this case the relation (IV.2) means that flux is determined by the concentration gradient both at the same moment and preceding time, the dependence on the past being exponentially decreased. The second inequality $\tau < \tau_2$ is equivalent to $D_2 < 0$. In this case the law of the Jeffreys type can be written in the equivalent form

$$
\nabla u = \frac{1}{D_1} \left[ -J + \frac{D_2}{\tau D_1} \int_0^t e^{-(t-t')/\tau_2} J(x, t') \, dt' \\
+ e^{-t/\tau_2} \left(D_1 \nabla u_0 + J_0\right) \right],
$$

which means that the gradient of the concentration is determined by flux both at the same moment and preceding time, the dependence on the past being exponentially decreased. Note that the relation (IV.3) can be used for setting boundary conditions for $u$ if mass transfer is considered in a finite domain.

The continuity equation (II.1) and the integro-differential law of the Jeffreys type (IV.2) lead to the integro-differential equation

$$
\frac{\partial u}{\partial t} - D_1 \Delta u - \frac{D_2}{\tau} \int_0^t e^{-(t-t')/\tau} \Delta u(x, t') \, dt' \\
+ e^{-t/\tau} \left(D_1 \Delta u_0 + \text{div} J_0\right) = f.
$$

This equation with the initial condition (III.4) is equivalent to the equation of the third order

$$
\tau \frac{\partial^2 u}{\partial t^2} + \left(1 - \tau \frac{\partial f}{\partial u}\right) \frac{\partial u}{\partial t} - \tau D_1 \frac{\partial \Delta u}{\partial t} - (D_1 + D_2) \Delta u \\
= f + \tau \frac{\partial f}{\partial t},
$$

with the initial conditions (III.4). We name Eq. (IV.4) the equation of the Jeffreys type [8]. The diffusion equation (II.3) and the telegraph equation (III.3) are the particular cases of the equation of the Jeffreys type for $\tau = 0$ and $D_1 = 0$, respectively.

Eq. (IV.1) can also be derived formally in the framework of the dual phase lag model [10, 19, 20]. The model applies heat transfer and Fourier’s law, however, one can extend this to mass transfer and Fick’s law as well. In this framework Fick’s law is replaced by the relation

$$
J(x, t + \tau) = -D \nabla u(x, t + \tau_2),
$$

where $\tau$ and $\tau_2$ are the time lags of the flux and the gradient of the concentration, respectively. Both sides of the relation are expanded with the use of Taylor’s formula. If only terms up to the first order are retained one obtains the relation

$$
\tau \frac{\partial J}{\partial t} + J = -D \left(\tau_2 \frac{\partial \nabla u}{\partial t} + \nabla u\right),
$$

which is nothing but Eq. (IV.1) with $D_1 + D_2 = D$. Note that if $\tau_2 = 0$ one obtains single phase lag model and Cattaneo’s equation.

However, Eq. (IV.5) (both for $\tau_2 > 0$ and $\tau_2 = 0$) together with the continuity equation yields delay equations leading to ill-posed initial value problems (with unstable solutions) [33, 43]. Therefore, the phase lag models cannot be considered as sensible physical ones. At the same time the formal “approximations” to the phase lag models lead to well-posed initial value problems.
B. Two-phase (two-temperature) model

In this model two phases of a substance (or species) are considered: free (mobile) and bound (immobile), see, e.g., [29-32]. The concentrations of these substances are denoted by \( u = u(x, t) \) and \( v = v(x, t) \), respectively, and satisfy the reaction diffusion system

\[
\begin{align*}
\frac{\partial u}{\partial t} - D_1 \Delta u + k_1 u - k_2 v &= f, \quad \text{(IV.6a)} \\
\frac{\partial v}{\partial t} + k_2 v - k_1 u &= 0, \quad \text{(IV.6b)}
\end{align*}
\]

with the initial conditions

\[
u|_{t=0} = u_0, \quad v|_{t=0} = v_0, \quad \text{(IV.7)}
\]

where \( D_1 \) is the diffusion coefficient of the free substance, \( k_1 \) and \( k_2 \) are the coefficients of interphase mass transfer, \( f = f(x, t, u) \) is the net rate of production and absorption (degradation) of the free substance, \( v_0 \equiv v_0(x) \) is the distribution of the concentration of the immobile substance at time \( t = 0 \). The coefficients \( k_1 \) and \( k_2 \) are evidently positive in this model.

The concentration \( v \) can be expressed through \( u \) from the equation (IV.6b) by

\[
v = k_1 \int_0^t e^{-k_2(t-t')} u(x, t') \, dt' + e^{-k_2t} v_0.
\]

Then the equation (IV.6a) leads to the equation

\[
\frac{\partial u}{\partial t} - D_1 \Delta u + k_1 u - \int_0^t e^{-k_2(t-t')} u(x, t') \, dt' - k_2 e^{-k_2t} v_0 = f.
\]

This equation with the first of the conditions (IV.7) is equivalent to the equation of the Jeffreys type

\[
\frac{\partial^2 u}{\partial t^2} + \left( k_1 + k_2 - \frac{\partial f}{\partial u} \right) \frac{\partial u}{\partial t} - D_1 \frac{\partial^2 u}{\partial t^2} - k_2 D_1 \Delta u = k_2 f + \frac{\partial f}{\partial t}, \quad \text{(IV.8)}
\]

with the initial conditions

\[
u|_{t=0} = u_0, \quad \left. \frac{\partial u}{\partial t} \right|_{t=0} = D_1 \Delta u_0 - k_1 u_0 + k_2 v_0 + f_0. \quad \text{(IV.9)}
\]

The equation for \( v \) is

\[
\frac{\partial^2 v}{\partial t^2} + (k_1 + k_2) \frac{\partial v}{\partial t} - D_1 \frac{\partial^2 v}{\partial t^2} - k_2 D_1 \Delta v = k_2 f \left( x, t, \frac{1}{k_1} \left( \frac{\partial u}{\partial t} + k_2 v \right) \right), \quad \text{(IV.10)}
\]

which is different from the equation (IV.8), if \( f \neq 0 \). The initial conditions for \( v \) are

\[
v|_{t=0} = v_0, \quad \left. \frac{\partial v}{\partial t} \right|_{t=0} = k_1 u_0 - k_2 v_0. \quad \text{(IV.11)}
\]

The counterpart of the two-phase model in the field of heat transfer is the two-temperature model [19, 24, 41, 43].

C. Relations between the coefficients of the two models

If sources and sinks (absorption) are absent, i.e., \( f = 0 \), then the equations of the Jeffreys type (IV.4) and (IV.8) are identical, and the coefficients are related by

\[
\tau = \frac{1}{k_1 + k_2} \quad \text{and} \quad D_2 = -\frac{k_1}{k_1 + k_2} D_1, \quad \text{(IV.12a)}
\]

or, vice versa,

\[
k_1 = -\frac{1}{\tau} D_2 \equiv \frac{1}{\tau} \frac{1}{\tau_2}, \quad \text{and} \quad k_2 = \frac{1}{\tau} \left( 1 + \frac{D_2}{D_1} \right) \equiv \frac{1}{\tau_2}, \quad \text{(IV.12b)}
\]

the diffusion coefficient \( D_1 \) being the same in the two models. At the same time, the initial conditions (III.4) and (IV.9) for the equations, concerning the time derivative, are different. Below, in Section V, it will be shown that this leads to qualitatively different behaviour of the solutions to the initial value problems for the equations of the Jeffreys type.

It is necessary to emphasize here that the positive coefficient \( k_1 \) in the two-phase model (IV.6) corresponds to the negative(!) coefficient \( D_2 \) in the law of the Jeffreys type (IV.1), (IV.2), (IV.3). Conversely, the positive coefficient \( D_2 \) corresponds to the negative(!) coefficient \( k_1 \).

D. \( D_{N=1} \) approximation to the linear Boltzmann equation

We consider here an approximation to the linear Boltzmann equation (also referred to as the linear transport or radiative transfer equation) [34, 46, 47], which describes, e.g., neutron transport and radiative heat transfer (transport of thermal energy by photons), see Appendix A.3. We use the notation \( D_{N=1} \) instead of \( D_1 \), since the latter stands for the coefficient.

Consider the monoenergetic (one-speed) linear Boltzmann equation

\[
\frac{\partial \psi}{\partial t} + c \boldsymbol{\Omega} \cdot \nabla \psi + (\kappa + \sigma) \psi = \sigma \int_{S^2} K(\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}') \psi(x, \boldsymbol{\Omega}', t) \, d\Omega' + \frac{1}{4\pi} F, \quad \text{(IV.13)}
\]

where \( \psi \equiv \psi(x, \Omega, t) \) is the particle phase space density, i.e., the density of particles at the point \( x \) and at
time \( t \) moving along the direction \( \Omega \in S^2 \), \( S^2 \) is the unit sphere in \( \mathbb{R}^3 \), \( c \) is the velocity of particles, \( \kappa \) and \( \sigma \) are the absorption and scattering rates, respectively, \( K \) is the collision (or scattering) kernel, \( F \equiv F(x,t) \) is the source density (due to isotropic sources for simplicity).

Integration of the linear Boltzmann equation over the unit sphere, together with the normalization \( \int_{S^2} K(\Omega \cdot \Omega') \, d\Omega = 1 \), gives the continuity equation
\[
\frac{\partial u}{\partial t} + \text{div} \mathbf{J} + \kappa u = F, \tag{IV.14}
\]
where
\[
u(x,t) = \int_{S^2} \psi(x, \Omega, t) \, d\Omega \tag{IV.15}
\]
is the particle density, and
\[
\mathbf{J}(x,t) = c \int_{S^2} \mathbf{\Omega} \psi(x, \Omega, t) \, d\Omega \tag{IV.16}
\]
is flux. In the \( D_{N=1} \) approximation \[35\] the particle density and flux are related by the equation \[\Delta \mathbf{J} \], which can be written as
\[
\frac{\tau}{\partial t} \mathbf{\Omega} \mathbf{J} + \mathbf{J} = - (D_1 + D_2) \nabla \nu + \frac{\tau D_1}{4} (3\Delta \mathbf{J} + \nabla \text{div} \mathbf{J}), \tag{IV.17}
\]
where
\[
\tau = \frac{1}{\kappa + \sigma_1}, \quad D_1 = \frac{4c^2}{15(\kappa + \sigma_2)}, \quad D_2 = \left[ \frac{1}{\kappa + \sigma_1} - \frac{4}{5(\kappa + \sigma_2)} \right] \frac{c^2}{3} \quad \text{and} \quad \gamma = \kappa,
\]
see Appendix A (the parameter \( \gamma \) will be used elsewhere).

The continuity equation \[IV.14\] and relation \[IV.17\] imply that the particle density satisfies the equation of the Jeffreys type
\[
\frac{\tau}{\partial t} \partial^2 u + (1 + \tau \gamma) \frac{\partial u}{\partial t} - \tau D_1 \frac{\partial \Delta u}{\partial t} - [(1 + \tau \gamma) D_1 + D_2] \Delta u + \gamma u = F + \tau \frac{\partial F}{\partial t} - \tau D_1 \Delta F, \tag{IV.18}
\]
which is the same as the equation \[\Delta \mathbf{J} \]. Initial conditions for this equation are
\[
u_{|t=0} = u_0, \quad \frac{\partial u}{\partial t} \bigg|_{t=0} = - \gamma u_0 - \text{div} \mathbf{J} + F_0, \tag{IV.19}
\]
where \( F_0 = F_0(x) = \mathbf{F} \) is the distribution of sources at time \( t = 0 \).

In the absence of sources and absorption, i.e., if \( F = 0 \) and \( \kappa = \gamma = 0 \), the equation \[IV.18\] takes the form
\[
\frac{\tau}{\partial t} \partial^2 u + \frac{\partial u}{\partial t} - \tau D_1 \frac{\partial \Delta u}{\partial t} - (D_1 + D_2) \Delta u = 0, \tag{IV.20}
\]
which is the same as the equation \[IV.4\] with \( f = 0 \).

In a steady state the relation \[IV.17\] takes the form
\[
\mathbf{J} = -(D_1 + D_2) \nabla \nu + \frac{\tau D_1}{4} (3\Delta \mathbf{J} + \nabla \text{div} \mathbf{J}), \tag{IV.21}
\]
which differs qualitatively from Fick’s law.

It is necessary to emphasize that the relation \[IV.17\] is similar to but not the same as the equation of Guyer and Krumhansl \[B.4\]. The reason is that Guyer and Krumhansl considered the linearized Boltzmann equation rather than the linear one, the difference is explained in Ref. \[16\].

\section{Initial Value Problems for the Homogeneous Equation of the Jeffreys Type with Absorption}

In this section we study the classic one-dimensional transport problem for a substance initially confined in a point. We suppose that sources are absent, and the substance is absorbed (degraded), therefore, we set \( f = -\gamma \nu \), where \( \gamma \) is the absorption (degradation) rate. The study reveals a remarkable phenomenon when a finite portion of the substance does not move though this portion diminishes exponentially with time.

There is also qualitative difference between the two cases \( D_2 > 0 \) and \( D_2 < 0 \). In the first case the solution is wave-like because the characteristic values take complex values. In the second case the characteristic values are real, and, hence, the solution is not wave-like.

\subsection{The Jeffreys type model}

In the one-dimensional case the problem \[IV.4\], \[III.4\] with \( f = -\gamma \nu \) takes the form
\[
\tau \frac{\partial^2 \nu_{je}}{\partial t^2} + (1 + \tau \gamma) \frac{\partial \nu_{je}}{\partial t} - \tau D_1 \frac{\partial \Delta \nu_{je}}{\partial t} - (D_1 + D_2) \Delta \nu_{je} + \gamma \nu_{je} = 0, \quad x \in \mathbb{R}, \quad t > 0, \tag{V.1}
\]
\[
u_{je}|_{t=0} = u_0, \quad \frac{\partial \nu_{je}}{\partial t} \bigg|_{t=0} = -\gamma u_0 - \frac{\partial J_{je}}{\partial x}. \tag{V.2}
\]

The one-dimensional continuity equation \[II.1\] implies that if and only if \( \gamma = 0 \), i.e., absorption is absent, mass is conserved: \( \int_{-\infty}^{\infty} \nu_{je}(x,t) \, dx = \int_{-\infty}^{\infty} u_0(x) \, dx \equiv \text{const} \).

Consider the particular initial conditions
\[
u_{je}(x) = \delta(x) \quad \text{and} \quad J_{je}(x) = 0. \tag{V.3}
\]

The Fourier transform of the problem \[V.1\], \[V.2\], \[V.3\] yields
\[
\tau \frac{\partial^2 F_{je}}{\partial t^2} + \left[ 1 + \tau (D_1 \xi^2 + \gamma) \right] \frac{\partial F_{je}}{\partial t} + \left[ (D_1 + D_2) \xi^2 + \gamma \right] F_{je} = 0, \quad \xi \in \mathbb{R}, \quad t > 0, \tag{V.4}
\]
where
\[ \mathcal{F} u(x,t) = \int_{-\infty}^{\infty} u(x,\cdot) e^{-i\xi x} \, dx \]
defines the Fourier transform. The solution to the original problem is, therefore, given by
\[ u(x,\cdot) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F} u(\xi,\cdot) e^{-ix\xi} \, d\xi. \]
The characteristic values of the equation (V.4) are
\[ \lambda_{1,2}(\xi) = \frac{1}{2\tau} \left\{ -\left[ 1 + \tau \left( D_1 \xi^2 + \gamma \right) \right] \right. \\
\left. \pm \sqrt{\left[ 1 - \tau \left( D_1 \xi^2 + \gamma \right) \right]^2 - 4\tau D_2 \xi^2} \right\}, \] (V.6)
where the plus sign corresponds to \( \lambda_1 \). Note that, if \(-D_1 < D_2 \leq 0\), the characteristic values are real, otherwise, if \( D_2 > 0 \), there are two intervals on the real line, symmetric with respect to the origin, where the characteristic values are complex conjugates.

The solution to the problem (V.4), (V.5) is
\[ \mathcal{F} u_{je}(\xi,t) = \frac{1}{\lambda_1(\xi) - \lambda_2(\xi)} \times \left\{ \left[ \lambda_1(\xi) e^{\lambda_2(\xi)t} - \lambda_2(\xi) e^{\lambda_1(\xi)t} \right] \right. \\
\left. - \gamma \left[ e^{\lambda_1(\xi)t} - e^{\lambda_2(\xi)t} \right] \right\}. \] (V.7)
The asymptotic behaviour of the characteristic values is described by
\[ \lambda_1(\xi) = -\frac{1}{\tau} \left( 1 + \frac{D_2}{D_1} \right) + O\left( \frac{1}{\xi^2} \right) \]
\[ \equiv -k_2 + O\left( \frac{1}{\xi^2} \right), \] (V.8a)
\[ \lambda_2(\xi) = -D_1 \xi^2 + \frac{D_2}{\tau D_1} - \gamma + O\left( \frac{1}{\xi^2} \right) \]
\[ \equiv -(D_1 \xi^2 + k_1 + \gamma) + O\left( \frac{1}{\xi^2} \right), \] (V.8b)
as \( \xi \to \pm \infty \), where we have used the relations (V.12) between the coefficients of the two models. Therefore, the asymptotic behaviour of the Fourier transform (V.7) with respect to \( \xi \) is
\[ \mathcal{F} u_{je}(\xi,t) = e^{-k_2 t} + O\left( \frac{1}{\xi^2} \right) \quad \text{as} \quad \xi \to \pm \infty. \] (V.9)
This means that the solution \( u_{je} \) has the form
\[ u_{je}(x,t) = u_{je}^s(x,t) + u_{je}^r(x,t), \] (V.10)
where
\[ u_{je}^s(x,t) = e^{-k_2 t} \delta(x) \] (V.11)
is the singular term, while the regular term \( u_{je}^r \) is a continuous function. The presence of the singular term means that in the Jeffreys type model a finite portion of the substance does not move, though this portion diminishes exponentially with time.

If \( \tau \gamma < 1 \) the asymptotic behaviour of the regular term with respect to \( t \) is
\[ e^{\gamma t} \mathcal{F} u_{je}^r \left( \frac{\xi}{\sqrt{t}}, t \right) \to e^{-D_{je}^r \xi^2} \quad \text{as} \quad t \to +\infty \]
with
\[ D_{je} = D_1 + \frac{D_2}{1 - \tau \gamma}, \]
which leads to the asymptotic behaviour
\[ e^{\gamma t} \sqrt{t} u_{je}^r \left( \sqrt{t} \xi, t \right) \to \frac{1}{2\sqrt{\pi D_{je}}} \exp\left( -\frac{x^2}{4D_{je}} \right) \]
as \( t \to +\infty \). This means that if \( \tau \gamma < 1 \) the solution \( u_{je} \) behaves asymptotically as \( t \to +\infty \) as the solution
\[ u_{DE}(x,t) = \frac{1}{2\sqrt{\pi D_{je}}} \exp\left( -\frac{x^2}{4D_{je} t} \right) \] (V.12)
of the diffusion equation
\[ \tau \frac{\partial u_{DE}}{\partial t} - D_{je} \frac{\partial^2 u_{DE}}{\partial x^2} + \gamma u_{DE} = 0, \quad x \in \mathbb{R}, \quad t > 0, \]
with the initial condition
\[ u_{DE}|_{t=0} = \delta(x). \] (V.13)

Figs. [2] and [3] show two solutions \( u_{je} \). In both cases absorption is absent, i.e., \( \gamma = 0 \), and, therefore, the mass of the substance is conserved. The solution, shown in Figs. [2] and [3] is obtained with the parameters \( \tau = 1 \), \( D_1 = 1 \) and \( D_2 = 4 \). The solution, shown in Fig. [3] is obtained with the parameters \( \tau = 1 \), \( D_1 = 1 \) and \( D_2 = -0.5 \) (this corresponds to \( k_1 = 0.5 \) and \( k_2 = 0.5 \)). The solution, shown in Fig. [2] is wave-like because the characteristic values (V.6) take complex values due to \( \lambda_2 > 0 \). In the second case \( \lambda_2 < 0 \), therefore, the characteristic values are real, and for this reason the solution, shown in Fig. [3] is not wave-like.

All the figures show also the diffusion asymptotics (V.12) with \( \gamma = 0 \), in this case \( D_{je} = D_1 + D_2 \). Fig. [1] shows also the solution of the telegraph equation
\[ \tau \frac{\partial^2 u_{TE}}{\partial t^2} + \frac{\partial u_{TE}}{\partial t} - (D_1 + D_2) \frac{\partial^2 u_{TE}}{\partial x^2} = 0, \]
\[ x \in \mathbb{R}, \quad t > 0, \]
FIG. 1. (Color online) The solution $u_{Je}$ to the problem (V.1), (V.2), (V.3) with $\tau = 1$, $D_1 = 1$, $D_2 = 4$ and $\gamma = 0$ in comparison with those of the diffusion and telegraph equations (see the text). The vertical lines stand for the singular terms $u_{sTE}$ and $u_{sJe}$. The portions of the regular term are $\int_{-\infty}^{\infty} u_{rJe}(x,0.02) \, dx \approx 0.10$, $\int_{-\infty}^{\infty} u_{rJe}(x,0.2) \, dx \approx 0.63$, $\int_{-\infty}^{\infty} u_{rJe}(x,0.6) \, dx \approx 0.95$ and $\int_{-\infty}^{\infty} u_{rJe}(x,2) \, dx \approx 1.000$.

FIG. 2. (Color online) The spatio-temporal images of solutions $u_{DE}$ and $u_{rJe}$ as in Fig. 1, $t \in [0.02, 4]$.

[this is Eq. (V.1)] with $\partial_x u_{Je} = 0$ and $\gamma = 0$] with the initial conditions

$$u_{TE}|_{t=0} = \delta(x), \quad \frac{\partial u_{TE}}{\partial t}|_{t=0} = 0.$$  

This solution is given by [11][12]

$$u_{TE}(x,t) = u_{sTE}(x,t) + u_{rTE}(x,t),$$

where

$$u_{sTE}(x,t) = \frac{e^{-t/2\tau}}{2\nu} \left[ \delta \left( \frac{x}{\nu} - t \right) + \delta \left( \frac{x}{\nu} + t \right) \right]$$

is the singular term, and

$$u_{rTE}(x,t) = \frac{e^{-t/2\tau}}{2\nu} \left[ \frac{1}{2\tau} \sqrt{t^2 - \frac{x^2}{\nu^2}} \right] I_0 \left( \frac{1}{2\tau} \sqrt{t^2 - \frac{x^2}{\nu^2}} \right) + t \left( \sqrt{t^2 - \frac{x^2}{\nu^2}} \right)^{-1} I_1 \left( \frac{1}{2\tau} \sqrt{t^2 - \frac{x^2}{\nu^2}} \right) H \left( t - \frac{|x|}{\nu} \right)$$

is the regular term, where $\nu = \sqrt{(D_1 + D_2)/\tau}$ is the velocity, $I_0$ and $I_1$ are the modified Bessel functions, $H(\cdot)$ is the Heaviside step function. The regular term is discontinuous at $x = \pm \nu t$.

The solution (V.7) and the one-dimensional continuity equation (II.1) with $f = 0$ imply that the Fourier
FIG. 3. (Color online) The solution \( u_{\text{Je}} \) to the problem \( \text{(V.1)} \), \( \text{(V.2)} \), \( \text{(V.3)} \) with \( \tau = 1, D_1 = 1, D_2 = -0.5 \) and \( \gamma = 0 \) in comparison with that of the diffusion equation (see the text). The vertical lines stand for the singular term \( u_{\text{Je}} \). The portions of the regular term are \( \int_{-\infty}^{\infty} u_{\text{Je}}^r(x, 0.1) \, dx \approx 0.05 \), \( \int_{-\infty}^{\infty} u_{\text{Je}}^r(x, 1) \, dx \approx 0.39 \) and \( \int_{-\infty}^{\infty} u_{\text{Je}}^r(x, 10) \, dx \approx 0.99 \).

The transform of flux is

\[
\mathcal{F}J(\xi, t) = \frac{i}{\xi} \lambda_1(\xi) \lambda_2(\xi) \left[ e^{\lambda_1(\xi)t} - e^{\lambda_2(\xi)t} \right].
\]

The asymptotic behaviour of the Fourier transform of flux is described by

\[
\left( \mathcal{F} \frac{\partial J}{\partial x} \right)(\xi, t) \equiv i \xi \mathcal{F}J(\xi, t) \to k_2 e^{-k_2 t}
\]
as \( \xi \to \pm \infty \), and means that flux \( J(x, t) \) has a finite discontinuity at \( x = 0 \), equal to \( k_2 e^{-k_2 t} \), which tends to zero as \( t \to +\infty \). Note that flux \( J(x, t) \) is an odd function with respect to \( x \).

**B. The two-phase model**

Here we study the behaviour of the net concentration \( w_{\text{TP}} = u_{\text{TP}} + v_{\text{TP}} \), where \( u_{\text{TP}} \) and \( v_{\text{TP}} \) are the concentrations of the free and bound phases, respectively. In the one-dimensional case the problems \( \text{(IV.3)}, \text{(IV.5)} \) and \( \text{(IV.10)}, \text{(IV.11)} \) with \( f = -\gamma u \) lead to the problem

\[
\begin{align*}
\frac{\partial^2 w_{\text{TP}}}{\partial t^2} + (k_1 + k_2 + \gamma) \frac{\partial w_{\text{TP}}}{\partial t} & - D_1 \frac{\partial^2 w_{\text{TP}}}{\partial x^2} \\
& - k_2 D_1 \frac{\partial^2 w_{\text{TP}}}{\partial x^2} + k_2 \gamma w_{\text{TP}} = 0, \quad x \in \mathbb{R}, \quad t > 0
\end{align*}
\]

\[\text{(V.14)}\]

\[
w_{\text{TP}}|_{t=0} = u_0 + v_0, \quad \left. \frac{\partial w_{\text{TP}}}{\partial t} \right|_{t=0} = D_1 \frac{\partial^2 u_0}{\partial x^2}. \quad \text{(V.15)}
\]

The total mass of the free and bound phases is conserved if and only if \( \gamma = 0 \), i.e., absorption is absent: 

\[
\int_{-\infty}^{\infty} w_{\text{TP}}(x, t) \, dx = \int_{-\infty}^{\infty} [u_0(x) + v_0(x)] \, dx \equiv \text{const.}
\]

Indeed, Eqs. \( \text{(V.6)} \) imply that the total mass obeys the equation \( \partial_t w_{\text{TP}} - D_1 \Delta w_{\text{TP}} = -\gamma u_{\text{TP}} \), or in the Fourier space \( \partial_t \mathcal{F}w_{\text{TP}} + \xi^2 D_1 \mathcal{F}w_{\text{TP}} = -\gamma \mathcal{F}u_{\text{TP}} \).

Consider the particular initial conditions

\[
u_0(x) = \alpha \delta(x) \quad \text{and} \quad v_0(x) = (1 - \alpha) \delta(x), \quad 0 \leq \alpha \leq 1. \quad \text{(V.16)}
\]

The Fourier transform of the problem \( \text{(V.14)}, \text{(V.15)}, \text{(V.16)} \) yields

\[
\frac{\partial^2 \mathcal{F}w_{\text{TP}}}{\partial t^2} + (k_1 + k_2 + \gamma + D_1 \xi^2) \frac{\partial \mathcal{F}w_{\text{TP}}}{\partial t} \\
+ k_2 \left( D_1 \xi^2 + \gamma \right) \mathcal{F}w_{\text{TP}} = 0, \quad \xi \in \mathbb{R}, \quad t > 0
\]

\[\text{(V.17)}\]

\[
\mathcal{F}w_{\text{TP}}|_{t=0} = 1, \quad \left. \frac{\partial \mathcal{F}w_{\text{TP}}}{\partial t} \right|_{t=0} = -\alpha D_1 \xi^2. \quad \text{(V.18)}
\]

The characteristic values of the equation \( \text{(V.17)} \) are

\[
\lambda_{1,2}(\xi) = \frac{1}{2} \left[ -\left(k_1 + k_2 + \gamma + D_1 \xi^2\right) \pm \sqrt{(k_1 - k_2 + \gamma + D_1 \xi^2)^2 + 4k_1k_2} \right].
\]

\[\text{(V.19)}\]

where the plus sign corresponds to \( \lambda_1 \). These values differ from the characteristic values \( \text{(V.6)} \) if \( \gamma \neq 0 \), however, their asymptotic behaviour is the same, see \( \text{(V.8)} \).

The solution to the problem \( \text{(V.17)}, \text{(V.18)} \) is

\[
\mathcal{F}w_{\text{TP}}(\xi, t) = \frac{1}{\lambda_1(\xi) - \lambda_2(\xi)} \times \left\{ \left[ \lambda_1(\xi) e^{\lambda_1(\xi)t} - \lambda_2(\xi) e^{\lambda_2(\xi)t} \right] - \alpha \left[ e^{\lambda_1(\xi)t} - e^{\lambda_2(\xi)t} \right] D_1 \xi^2 \right\}.
\]

\[\text{(V.20)}\]

Taking into account the asymptotic behaviour of the characteristic values, one can conclude that the asymptotic behaviour of the Fourier transform of the solution is

\[
\mathcal{F}w_{\text{TP}}(\xi, t) \to (1 - \alpha) e^{-k_2 t} + O \left( \frac{1}{\xi^2} \right) \quad \text{as} \quad \xi \to \pm \infty.
\]

\[\text{cf. with the asymptotics \text{(V.9)}}. \]

This means that the solution \( w_{\text{TP}} \) has the form

\[
w_{\text{TP}}(x, t) = w_{\text{TP}}^s(x, t) + w_{\text{TP}}^r(x, t), \quad \text{(V.21)}
\]

where

\[
w_{\text{TP}}^s(x, t) = (1 - \alpha) e^{-k_2 t} \delta(x). \quad \text{(V.22)}
\]

is the singular term, while the regular term \( w_{\text{TP}}^r \) is a continuous function \[\text{HS}. \]

The presence of the singular term
means that if \( \alpha < 1 \) in the two-phase model a finite portion of the substance does not move, though this portion diminishes exponentially with time.

The asymptotic behaviour of the Fourier transform (V.20) with respect to \( t \) is 
\[
e^{\gamma_{TP}t} \mathcal{F}u^\infty_{TP}\left(\frac{\xi}{\sqrt{t}}\right) = \frac{1}{2} \left( 1 + \frac{k_+}{s} \right) e^{-D_{TP} \xi^2}
\]
as \( t \to +\infty \), where
\[
D_{TP} = \frac{1}{2} \left( 1 - \frac{k_-}{s} \right) D_1, \quad \gamma_{TP} = \frac{k_+ - s}{2},
\]
\[
k_\pm = k_1 + \gamma \pm k_2 \quad \text{and} \quad s = \sqrt{k_1^2 + 4k_1k_2}.
\]

This means that as \( t \to +\infty \) the solution \( u_{TP} \) behaves asymptotically as 
\[
w^\infty_{TP} \equiv \frac{1}{2} \left( 1 + \frac{k_+}{s} \right) u_{DE}(x,t), \quad (V.23)
\]
where
\[
u_{DE}(x,t) = \frac{1}{2\sqrt{\pi D_{TP}t}} \exp\left( -\frac{x^2}{4D_{TP}t} - \gamma_{TP}t \right)
\]
is the solution of the diffusion equation
\[
\frac{\partial u_{DE}}{\partial t} - D_{TP} \frac{\partial^2 u_{DE}}{\partial x^2} + \gamma_{TP} u_{DE} = 0,
\]
\[
x \in \mathbb{R}, \quad t > 0,
\]
with the initial condition (V.13). If \( \gamma = 0 \) then \( k_+ = s \) and
\[
D_{TP} = \frac{k_2}{k_1 + k_2} D_1 = D_1 + D_2.
\]

Fig.4 shows the solution \( w_{TP} \) with \( \alpha = 1 \), i.e., \( u_0 = \delta(x) \) and \( v_0 = 0 \). The parameters are \( D_1 = 1, k_1 = 0.5, k_2 = 0.5 \) (this corresponds to \( \tau = 1 \) and \( D_2 = -0.5 \), cf. Fig.3) and \( \gamma = 0 \). The total mass of \( w_{TP} \) is conserved.

The figure also shows the diffusion asymptotics (V.23) with \( \gamma = 0 \), which is the same as the asymptotics (V.12) with \( \gamma = 0 \). For comparison the figure shows the solution \( u_{DE}^0 \) to the problem for the diffusion equation (IV.6a) with \( k_1 = k_2 = 0, f = -\gamma u \) and the initial condition \( u_0 = \delta(x) \), i.e., this would be the concentration of the free substance if intermediate mass transfer were absent.

Note that the solution \( w_{TP} \) with \( \alpha = 0 \), i.e., \( u_0 = 0 \) and \( v_0 = \delta(x) \), is the same as the solution \( u_{TC} \) to the problem (V.1), (V.2), (V.3) with the same parameters, see Fig.3.

C. \( D_{N=1} \) approximation to the linear Boltzmann equation

In the one-dimensional case the equation (IV.18) with \( F = 0 \) takes the form
\[
\tau \frac{\partial^2 u_{Bo}}{\partial t^2} + (1 + \tau \gamma) \frac{\partial u_{Bo}}{\partial t} - \tau D_1 \frac{\partial^3 u_{Bo}}{\partial x^2 \partial t} = - (D_1 + D_2) \frac{\partial^2 u_{Bo}}{\partial x^2} + \gamma u_{Bo} = 0, \quad x \in \mathbb{R}, \quad t > 0,
\]
with \( D'_2 = D_2 + \tau \gamma D_1 \). The initial conditions (IV.19) in this case are
\[
u_{Bo}|_{t=0} = u_0, \quad \frac{\partial u_{Bo}}{\partial t}\bigg|_{t=0} = -\gamma u_0 - \frac{\partial J_0}{\partial x}.
\]
This problem is similar to the problem (V.1), (V.2) in the framework of the Jeffreys type model.

VI. THE MEAN SQUARE DISPLACEMENT

The mean-square displacement (MSD) is an integral quantity whose temporal dependence characterizes diffusion and Brownian motion. Of interest is to calculate the MSD in the framework of each model for comparison with that in diffusion and Brownian motion. This comparison is of particular interest for small \( t \) since the asymptotics of the above solutions for large \( t \) is diffusive.

In this section we calculate the MSD, defined by \( \langle x^2(t) \rangle = \int_{-\infty}^{\infty} x^2 u(x,t) \, dx \), for the solutions of the problem, considered in the previous section, with the initial condition \( u|_{t=0} = \delta(x) \). The solution \( u(\cdot,t) \) is a probability distribution function for any \( t \geq 0 \), i.e., the necessary condition is \( \int_{-\infty}^{\infty} u(x,t) \, dx = 1 \). Therefore, absorption is necessarily absent, i.e., \( \gamma = 0 \).
Concerning the diffusion equation it is well known that the MSD for the solution to the problem
\[ \frac{\partial u_{DE}}{\partial t} - D \frac{\partial^2 u_{DE}}{\partial x^2} = 0, \quad x \in \mathbb{R}, \quad t > 0, \]
\[ u_{DE}|_{t=0} = \delta(x), \]
linearly depends on time and is equal to
\[ \langle x_{DE}^2(t) \rangle = 2Dt, \quad t \geq 0. \]
However, this temporal behaviour of the MSD is wrong at small values of time, where it must be ballistic.

The MSD in the framework of the Jeffreys type model is defined through the solution \( u_{Je} \) to the problem \( (V.1), (V.2), (V.3) \) with \( \gamma = 0 \). Therefore, the MSD is the solution to the problem
\[ \tau \frac{d^2}{dt^2} \langle x_{Je}^2 \rangle + \frac{d}{dt} \langle x_{Je}^2 \rangle = 2(D_1 + D_2), \quad t > 0, \]
\[ \langle x_{Je}^2 \rangle|_{t=0} = 0, \quad \frac{d}{dt} \langle x_{Je}^2 \rangle|_{t=0} = 0. \]
Hence, the MSD is equal to
\[ \langle x_{Je}^2(t) \rangle = 2(D_1 + D_2) \left[ t - \tau \left( 1 - e^{-t/\tau} \right) \right] \]
\[ \sim \begin{cases} D_1 + D_2 & \text{as } t \to 0, \\ 2(D_1 + D_2) \tau & \text{as } t \to \infty. \end{cases} \quad (VI.1) \]

The Fourier transform of this problem yields
\[ \frac{\tau}{\partial^2} \mathcal{F} u_{Je} + \left[ 1 + \tau \left( D_1 \xi^2 + \gamma \right) \right] \mathcal{F} \frac{\partial u_{Je}}{\partial t}
+ \left[ (D_1 + D_2) \xi^2 + \gamma \right] \mathcal{F} u_{Je} = 1, \quad \xi \in \mathbb{R}, \quad t > 0, \]
\[ \mathcal{F} u_{Je}|_{t=0} = 0, \quad \frac{\partial \mathcal{F} u_{Je}}{\partial t}|_{t=0} = 1. \]

The solution to the latter problem is
\[ \mathcal{F} u_{Je}(\xi, t) = \frac{1}{\lambda_1(\xi) - \lambda_2(\xi)} \times \]
\[ \left\{ \frac{1}{\tau} \left[ \frac{e^{\lambda_1(\xi)t} - 1}{\lambda_1(\xi)} - \frac{e^{\lambda_2(\xi)t} - 1}{\lambda_2(\xi)} \right] + \left[ e^{\lambda_1(\xi)t} - e^{\lambda_2(\xi)t} \right] \right\}, \]
where \( \lambda_{1,2} \) are the characteristic values, given by Eq. \( (V.6) \).

The asymptotic behaviour \( (V.8) \) of the characteristic values leads to the asymptotic behaviour
\[ \mathcal{F} u_{Je}(\xi, t) = O \left( \frac{1}{\xi^2} \right) \quad \text{as} \quad \xi \to \pm \infty. \]
Hence the solution \( u_{JE} \) is a continuous function of \( x \).

The mass of the substance is equal at any time to that in the similar problems for the diffusion and telegraph equations:

\[
\int_{-\infty}^{\infty} u_{DE}(x, t) \, dx = \int_{-\infty}^{\infty} u_{TE}(x, t) \, dx
\]

\[
= \int_{-\infty}^{\infty} u_{JE}(x, t) \, dx = \frac{1 - e^{-\gamma t}}{\gamma}, \quad t \geq 0, \quad (\text{VII.3})
\]

where \( u_{DE} \) is the solution of the diffusion equation

\[
\frac{\partial u_{DE}}{\partial t} - (D_1 + D_2) \frac{\partial^2 u_{DE}}{\partial x^2} + \gamma u_{DE} = \delta(x),
\]

with the initial condition

\[
u_{DE}\big|_{t=0} = 0,
\]

and \( u_{TE} \) is the solution of the telegraph equation

\[
\tau \frac{\partial^2 u_{TE}}{\partial t^2} + (1 + \tau \gamma) \frac{\partial u_{TE}}{\partial t} - (D_1 + D_2) \frac{\partial^2 u_{TE}}{\partial x^2} + \gamma u_{TE} = \delta(x), \quad x \in \mathbb{R}, \quad t > 0,
\]

with the initial conditions

\[
u_{TE}\big|_{t=0} = 0, \quad \frac{\partial u_{TE}}{\partial t} \bigg|_{t=0} = \delta(x).
\]

Note that the mass does not depend on \( \tau \).

Fig. 5 shows the solution \( u_{JE} \), obtained with the parameters \( \tau = 1, D_1 = 1, D_2 = 4 \) and \( \gamma = 1 \). The figure shows also the steady state solution \( u_{\infty} \) of the equation (VII.1).

For comparison the figure shows also the solutions \( u_{DE} \) and \( u_{TE} \). One can see the vertical front of the solution \( u_{TE} \). The solution \( u_{JE} \) is intermediate between the solutions of the diffusion and telegraph equations.

### B. Two-phase model

Here we study the behaviour of the net concentration \( w_{TP} = u_{TP} + v_{TP} \), where \( u_{TP} \) and \( v_{TP} \) are the concentrations of the free and bound phases, respectively. In the one-dimensional case the problems (IV.8), (IV.9) and (IV.10), (IV.11) with \( f = -\gamma u + \delta(x) \), \( u_0 = 0 \) and \( v_0 = 0 \) lead to the problem

\[
\frac{\partial^2 w_{TP}}{\partial t^2} + (k_1 + k_2 + \gamma) \frac{\partial w_{TP}}{\partial t} - D_1 \frac{\partial^2 w_{TP}}{\partial x^2} \partial t
\]

\[
- k_2 D_1 \frac{\partial^2 w_{TP}}{\partial x^2} + k_2 \gamma w_{TP} = (k_1 + k_2) \delta(x),
\]

\[
x \in \mathbb{R}, \quad t > 0, \quad (\text{VII.4})
\]

\[
w_{TP}\big|_{t=0} = 0, \quad \frac{\partial w_{TP}}{\partial t} \bigg|_{t=0} = \delta(x). \quad (\text{VII.5})
\]

Note that the equation \( (\text{VII.4}) \), expressed through the parameters \( \tau, D_1 \) and \( D_2 \), takes the form

\[
\tau \frac{\partial^2 w_{TP}}{\partial t^2} + (1 + \tau \gamma) \frac{\partial w_{TP}}{\partial t} - D_1 \frac{\partial^2 w_{TP}}{\partial x^2} \partial t
\]

\[
- (D_1 + D_2) \frac{\partial^2 w_{TP}}{\partial x^2} + \left(1 + \frac{D_2}{D_1}\right) \gamma w_{TP} = \delta(x),
\]

cf. with the equation (VII.1), the difference is in the last term of the left-hand side.

The Fourier transform of the problem (VII.4), (VII.5) yields

\[
\frac{\partial^2 F w_{TP}}{\partial t^2} + (k_1 + k_2 + \gamma + D_1 \xi^2) \frac{\partial F w_{TP}}{\partial t}
\]

\[
+ k_2 (D_1 \xi^2 + \gamma) F w_{TP} = k_1 + k_2, \quad \xi \in \mathbb{R}, \quad t > 0,
\]

\[
F w_{TP}\big|_{t=0} = 0, \quad \frac{\partial F w_{TP}}{\partial t} \bigg|_{t=0} = 1.
\]

The solution to this problem is

\[
F w_{TP}(\xi, t) = \frac{1}{\lambda_1(\xi) - \lambda_2(\xi)} \times \left\{ (k_1 + k_2) \left[ e^{\lambda_1(\xi) t} - 1 - e^{\lambda_2(\xi) t} - 1 \right] + \left[ e^{\lambda_1(\xi) t} - e^{\lambda_2(\xi) t} \right] \right\},
\]

where \( \lambda_{1,2} \) are the characteristic values, given by Eq. (V.19).

The asymptotic behaviour \( (\text{VIII.8}) \) of the characteristic values leads to the asymptotic behaviour

\[
F w_{TP}(\xi, t) \sim O \left( \frac{1}{\xi^2} \right) \quad \text{as} \quad \xi \to \pm \infty.
\]

This means that the solution \( w_{TP} \) is a continuous function of \( x \).

Fig. 6 shows the solution \( w_{TP} \), obtained with the parameters \( D_1 = 1, k_1 = 0.5, k_2 = 0.5 \) (this corresponds to \( \tau = 1 \) and \( D_2 = -0.5 \)) and \( \gamma = 1 \). The figure shows also the steady state solution \( w_{\infty} \), of the equation (VII.4).

For comparison the figure shows also the solution \( u_{JE} \) for the problem for the equation (IV.10) with \( k_1 = k_2 = 0, f = -\gamma u + \delta(x) \) and the homogeneous initial condition, i.e., this would be the concentration of the free substance if the interphase mass transfer were absent.

### C. \( D_{N=1} \) approximation to the linear Boltzmann equation

In the one-dimensional case the problem (IV.18), (IV.19) with \( F = \delta(x) \), \( u_0 = 0 \) and \( J_0 = 0 \) takes the
The solution \( u_{Je} \) to the problem (VII.1), (VII.2) with \( \tau = 1, D_1 = 1, D_2 = 4 \) and \( \gamma = 1 \) in comparison with those of the diffusion and telegraph equations (see the text).

The solution \( w_{TP} \) to the problem (VII.4), (VII.5) with \( D_1 = 1, k_1 = 0.5, k_2 = 0.5 \) and \( \gamma = 1 \) in comparison with that of the diffusion equation (see the text).

Form

\[
\tau \frac{\partial^2 u_{Bo}}{\partial t^2} + (1 + \tau \gamma) \frac{\partial u_{Bo}}{\partial t} - \tau D_1 \frac{\partial^3 u_{Bo}}{\partial x^2 \partial t} - (D_1 + D'_2) \frac{\partial^2 u_{Bo}}{\partial x^2} + \gamma u_{Bo} = \delta(x) - \tau D_1 \frac{\partial^2 \delta(x)}{\partial x^2},
\]

\( x \in \mathbb{R}, \ t > 0 \), (VII.6)

\[
u_{Bo}|_{t=0} = 0, \quad \left. \frac{\partial u_{Bo}}{\partial t} \right|_{t=0} = \delta(x), \quad (VII.7)
\]
with \( D'_2 = D_2 + \tau \gamma D_1 \). The Fourier transform of this problem yields

\[
\tau \frac{\partial^2 \mathcal{F}u_{Bo}}{\partial t^2} + \left[ 1 + \tau (D_1 \xi^2 + \gamma) \right] \frac{\partial \mathcal{F}u_{Bo}}{\partial t} + \left[ (D_1 + D'_2) \xi^2 + \gamma \right] \mathcal{F}u_{Bo} = 1 + \tau D_1 \xi^2, \quad \xi \in \mathbb{R}, \quad t > 0,
\]

\[ \mathcal{F}u_{Bo}|_{t=0} = 0, \quad \frac{\partial \mathcal{F}u_{Bo}}{\partial t} |_{t=0} = 1, \]

and leads to the following solution:

\[
\mathcal{F}u_{Bo}(\xi, t) = \frac{1}{\lambda_1(\xi) - \lambda_2(\xi)} \times \left\{ \begin{array}{l}
\left( \frac{1}{\tau} + D_1 \xi^2 \right) \left[ e^{\lambda_1(\xi)t} - 1 \right] - e^{\lambda_2(\xi)t} \xi^2
+ \left[ e^{\lambda_1(\xi)t} e^{\lambda_2(\xi)t} \right] \left[ e^{\lambda_1(\xi)t} - e^{\lambda_2(\xi)t} \right] \end{array} \right\},
\]

where \( \lambda_1, \lambda_2 \) are the characteristic values, given by Eq. (V.6) with \( D'_2 \) instead of \( D_2 \).

The asymptotic behaviour (V.8) of the characteristic values leads to the asymptotic behaviour

\[
\mathcal{F}u_{Bo}(\xi, t) \approx \frac{1 - e^{-k_2' t}}{k_2'} + O\left( \frac{1}{\xi^2} \right) \quad \text{as} \quad \xi \to \pm \infty,
\]

where \( k_2' \) is given by the relation (V.12a) for \( k_2 \) with \( D'_2 \) instead of \( D_2 \), i.e.,

\[
k_2' = \frac{1}{\tau} \left( 1 + \frac{D'_2}{D_1} \right).
\]

Therefore, the solution \( u_{Bo} \) has the form

\[
u_{Bo}(x, t) = u_{Bo}^s(x, t) + u_{Bo}^i(x, t),
\]

where

\[
u_{Bo}^s(x, t) = \frac{1 - e^{-k_2' t}}{k_2'} \delta(x),
\]

is the singular term, while the regular term \( u_{Bo}^i \) is a continuous function. The presence of the singular term means that in this model a finite portion of the substance does not move, and this portion increases with time up to the value \( 1/k_2' \) as \( t \to \infty \).

The steady state solution of the equation (VII.6) satisfies the equation

\[ - (D_1 + D'_2) \frac{\partial^2 u_{\infty}^{Bo}}{\partial x^2} + \gamma u_{\infty}^{Bo} = \delta(x) - \tau D_1 \frac{\partial^2 \delta(x)}{\partial x^2}, \]

\( x \in \mathbb{R} \). The Fourier transform of the steady state solution is

\[
\mathcal{F}u_{\infty}^{Bo}(\xi) = \frac{1 + \tau D_1 \xi^2}{(D_1 + D'_2) \xi^2 + \gamma} \equiv \frac{1}{D_1 + D'_2} \left[ \tau D_1 + \frac{D_1 + D'_2}{(D_1 + D'_2) \xi^2 + \gamma} \right].
\]

Therefore, the steady state solution is

\[
u_{\infty}^{Bo}(x) = \delta_{\infty}^{Bo}(x) + \nu_{\infty}^{Bo}(x),
\]

where

\[
u_{\infty}^{Bo}(x) = \frac{\tau D_1}{D_1 + D'_2} \delta(x) \equiv \frac{1}{k_2' \gamma} \delta(x),
\]

is the singular term, and

\[
u_{\infty}^{Bo}(x) = \frac{D_1 + D'_2}{2(D_1 + D'_2) \gamma} \exp\left( -\sqrt{\frac{\gamma}{D_1 + D'_2}} |x| \right)
\]

is the regular (continuous) term.

Fig. 7 shows the solution \( u_{Bo}^i \), obtained with the parameters \( c = \sqrt{15}/4, \sigma = 0.5, \tau_1 = 0.5, \tau_2 = 0.5 \), and \( F = \delta(x) \) (this corresponds to \( \tau = 1, D_1 = 1, D'_2 = 0.25, D'_2 = 0.75 \) and \( \gamma = 0.5 \)). All the figures show also the steady state solution \( u_{\infty}^{Bo} \) given by Eqs. (VII.8)–(VII.10).

VIII. CONCLUDING REMARKS

We have considered three models of non-anomalous mass transfer, leading to the equation of the Jeffreys type. In the framework of the Jeffreys type model this equation combines the diffusion and telegraph ones through the law of the Jeffreys type, which combines, in its turn, Fick’s law and Cattaneo’s equation. In the framework of the two-phase model the equation of the Jeffreys type describes the concentrations of the free (mobile) and bound (immobile) phases of a substance as well as the net concentration. The equation of the Jeffreys type in the
FIG. 7. (Color online) The solution $u_{B_0}$ to the problem (VII.6), (VII.7) with $\tau = 1$, $D_1 = 1$, $D_2 = 0.25$ and $\gamma = 0.5$ in comparison with those of the diffusion and telegraph equations (see the text). The vertical lines stand for the singular term $u_{sB_0}$.

form of the $D_{N=1}$ approximation ranks after the diffusion and telegraph equations in the hierarchy of the spherical harmonics approximations to the linear Boltzmann equation.

Solutions of the equation of the Jeffrey type show qualitatively different behaviour in all these models. The two-phase model shows nothing unusual while the Jeffrey type model and the $D_{N=1}$ approximation to the linear Boltzmann equation exhibit distinctive features.

The first problem we have studied is the transfer of a substance initially confined in a point. In this case the Jeffrey type model and $D_{N=1}$ approximation coincide. The study has revealed that in these models a finite portion of the substance does not move, though this portion diminishes exponentially with time. Besides, we have calculated the mean square displacement (MSD) for the solutions of the first problem. The temporal behaviour of the MSD in the Jeffrey type model and in the $D_{N=1}$ approximation is found to be the same as that in the Brownian motion described by the standard Langevin equation, i.e., it is ballistic as $t \to 0$ and diffusive as $t \to \infty$.

The second problem we have studied is the transfer of a substance from a stationary point source. The study has revealed that in the $D_{N=1}$ approximation a finite portion of the substance does not move, and this portion increases with time up to a value as $t \to \infty$.

Tentative interpretation of the local immobilization phenomena is that in a dense crowd inner particles have no space to move, but when the crowd is scattered the particles become mobile.

The two problems we have studied are one-dimensional. An important question requires further consideration: are the solutions of the three-dimensional problems for the equation of the Jeffrey type left non-negative?

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Appendix A: Approximations to the linear Boltzmann equation in the framework of the spherical harmonics method

One of the methods to obtain approximate solutions of the linear Boltzmann equation (IV.13) is the spherical harmonics method. In this method the particle phase space density is expanded into the generalized
Fourier series

\[ \psi(x, \Omega, t) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \psi_n^m(x, t) Y_n^m(\Omega), \quad (A.1) \]

where \( Y_n^m \) are the spherical harmonics \([49]\), the coefficients are expressed by

\[ \psi_n^m(x, t) = \int_{S^2} \psi(x, \Omega, t) Y_n^m(\Omega) d\Omega, \]

where the overline means the complex conjugate. Note that the expansion (A.1) can be expressed through the particle density \( u \) (IV.15) and flux \( J \) (IV.16). Indeed, note that

\[ \psi_0^0(x, t) Y_0^0 \equiv \frac{1}{4\pi} u(x, t) \]

and

\[ \sum_{m=-1}^{1} \psi_1^m(x, t) Y_1^m(\Omega) = \frac{3}{4\pi c^2} J(x, t) \cdot \Omega. \]

Therefore, the expansion (A.1) takes the form

\[ \psi(x, \Omega, t) = \frac{1}{4\pi} u(x, t) + \frac{3}{4\pi c^2} J(x, t) \cdot \Omega \]

\[ + \sum_{n=2}^{\infty} \sum_{m=-n}^{n} \psi_n^m(x, t) Y_n^m(\Omega). \]

The collision kernel is also expanded into the spherical harmonics:

\[ K(\Omega \cdot \Omega') = \sum_{n=0}^{\infty} K_n \sum_{m=-n}^{n} Y_n^m(\Omega) Y_n^m(\Omega'), \quad (A.2) \]

where

\[ K_n = 2\pi \int_{-1}^{1} K(\mu) P_n(\mu) d\mu, \]

\( P_n \) are the Legendre polynomials, with \( K_0 = 1 \) due to the normalization \( \int_{S^2} K(\Omega \cdot \Omega') d\Omega = 1 \), which is equivalent to \( \int_{-1}^{1} K(\mu) d\mu = (2\pi)^{-1} \). The expansions (A.1) and (A.2) are substituted into the linear Boltzmann equation. Due to orthogonality of the spherical harmonics this leads to an infinite system of coupled partial differential equations for the functions \( \psi_n^m \).

The first equation of the infinite system is the continuity equation (IV.14). The second (vector) equation can be obtained with the help of integrating the linear Boltzmann equation, multiplied by \( \Omega \), over the unit sphere. The second equation relates the gradient of the particle density \( \nabla u \), flux \( J \), its time derivative \( \partial J/\partial t \) and coefficients \( \psi_2 \).

1. **Diffusion approximation**

The classic diffusion approximation is obtained if the coefficients \( \psi_n^m, n > 1 \), in the expansion (A.1) are negligible and the coefficients \( \psi_1^1 \) are quasi-stationary. The latter condition is equivalent to quasi-stationarity of flux \( J \), i.e., \( \partial J/\partial t \approx 0 \). In this case the particle density and flux are related by the (truncated second) equation

\[ (\kappa + \sigma_1) J + \frac{c^2}{3} \nabla u = 0, \quad (A.3) \]

where

\[ \sigma_n = \sigma (1 - K_n), \quad n = 1, 2, \ldots, \]

clearly, \( \sigma_n > 0 \). Note that the equation (A.3) is identical to Fick’s law (II.2). The continuity equation (IV.14) and equation (A.3) imply that the particle density satisfies the diffusion equation

\[ \frac{\partial u}{\partial t} - \frac{c^2}{3 (\kappa + \sigma_1)} \Delta u + \kappa u = F. \quad (A.4) \]

2. **\( P_N \) approximations**

The classic \( P_N \) approximations are obtained if the coefficients \( \psi_n^m, n > N \geq 1 \), in the expansion (A.1) are negligible. Particularly, in the \( P_1 \) approximation the particle density and flux are related by the (truncated second) equation

\[ \frac{\partial J}{\partial t} + (\kappa + \sigma_1) J + \frac{c^2}{3} \nabla u = 0, \quad (A.5) \]

which is the extension of Eq. (A.3). Note that Eq. (A.5) is similar to Cattaneo’s equation (III.1). The continuity equation (IV.14) and equation (A.5) imply that the particle density satisfies the telegraph equation

\[ \frac{\partial^2 u}{\partial t^2} + (2\kappa + \sigma_1) \frac{\partial u}{\partial t} - \frac{c^2}{3} \Delta u + \kappa (\kappa + \sigma_1) u \\
= (\kappa + \sigma_1) F + \frac{\partial F}{\partial t}. \quad (A.6) \]

3. **\( D_N \) approximations**

Recently, \( D_N \) approximations were proposed \([35]\). They generalize the diffusion approximation, which can be considered as the \( D_0 \) approximation. The \( D_N \) approximations are obtained if the coefficients \( \psi_n^m, n > N + 1 \), in the expansion (A.1) are negligible and the coefficients \( \psi_N^N \) are quasi-stationary. The coefficients \( \psi_N^N \) can be expressed through \( \psi_n^m \), and the \( D_N \) approximation is described by \( \psi_n^m, n = 0, \ldots, N \). In the case \( N = 1 \) the coefficients \( \psi_2 \) can be expressed through flux \( J \). As a
result, in the $D_{N=1}$ approximation the particle density and flux are related by the equation
\[
\frac{\partial J}{\partial t} + (\kappa + \sigma_1) J + \frac{c^2}{3} \nabla u = \frac{c^2}{15 (\kappa + \sigma_2)} (3\Delta J + \nabla \text{div} J), \quad (A.7)
\]
which is the generalization of Eq. [A.5]. (We used the notation $D_{N=1}$ instead of $D_1$, since the latter stands for the coefficient.) The continuity equation [IV.14] and (A.7) imply that the particle density satisfies the equation of the Jeffreys type
\[
\frac{\partial^2 u}{\partial t^2} + (2\kappa + \sigma_1) \frac{\partial u}{\partial t} - \frac{4c^2}{15 (\kappa + \sigma_2)} \frac{\partial \Delta u}{\partial t}
- \left[ \frac{c^2}{3} + \frac{4c^2}{15 (\kappa + \sigma_2)} \kappa \right] \Delta u + (\kappa + \sigma_1) u 
= (\kappa + \sigma_1) F + \frac{\partial F}{\partial t} - \frac{4c^2}{15 (\kappa + \sigma_2)} \Delta F. \quad (A.8)
\]

**Appendix B: Model of Guyer and Krumhansl**

In this section we consider heat transfer. The energy equation without sources and sinks has the form
\[
C \frac{\partial T}{\partial t} + \text{div} q = 0, \quad (B.1)
\]
where $T \equiv T(x,t)$ is temperature, $q \equiv q(x,t)$ is heat flux, $C$ is the volumetric heat capacity.

The equation of Guyer and Krumhansl relating heat flux and temperature, is \[B.10\] \[B.30\]
\[
\frac{\partial q}{\partial t} + \frac{1}{\tau_R} q = -\frac{c^2 C}{3} \nabla T + \frac{\tau_N c^2}{5} (\Delta q + 2 \nabla \text{div} q), \quad (B.2)
\]
where $c$ is the average speed of phonons, $\tau_N$ is a relaxation time for momentum-conserving collisions (normal process) and $\tau_R$ is a relaxation time for momentum-nonconserving collisions (“unklapp” process) in the phonon gas. An equivalent equation was obtained in the framework of extended irreversible thermodynamics [14].

The energy equation [B.1] and the equation of Guyer and Krumhansl imply that temperature satisfies the homogeneous equation of the Jeffreys type
\[
\frac{\partial^2 T}{\partial t^2} + \frac{1}{\tau_R} \frac{\partial T}{\partial t} - \frac{3\tau_N c^2}{5} \frac{\partial \Delta T}{\partial t} - \frac{c^2}{3} \Delta T = 0. \quad (B.3)
\]
This equation is related to the equation [IV.20] by $\tau = \tau_R$, $D_1 = 3\tau_N c^2/5$ and $D_2 = (\tau_R/3 - 3\tau_N/5)c^2$, besides, the inequalities $D_2 \geq 0$ are equivalent to $5\tau_R \geq 2\tau_N$. Initial conditions for the equation (B.3) are
\[
T\big|_{t=0} = T_0, \quad \frac{\partial T}{\partial t}\big|_{t=0} = -\frac{1}{C} \text{div} q_0,
\]
where $T_0 \equiv T_0(x)$ and $q_0 \equiv q_0(x)$ are initial temperature and heat flux, respectively. These are the initial conditions [IV.19] with $F = 0$ and $\gamma = 0$.

The equation of Guyer and Krumhansl [B.2], written through $\tau$, $D_1$ and $D_2$, has the form
\[
\tau \frac{\partial q}{\partial t} + q = -(D_1 + D_2) C \nabla T + \frac{\tau D_1}{3} (\Delta q + 2 \nabla \text{div} q), \quad (B.4)
\]
which differs of the similar equation [IV.17] in the framework of the $D_{N=1}$ approximation to the linear Boltzmann equation. In a steady state the equation (B.4) takes the form
\[
q = -(D_1 + D_2) C \nabla T + \frac{\tau D_1}{3} (\Delta q + 2 \nabla \text{div} q),
\]
which differs qualitatively from Fourier’s law.

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