Holography on tessellations of hyperbolic space

Muhammad Asaduzzaman, Simon Catterall, Jay Hubisz, Roice Nelson and Judah Unmuth-Yockey

Department of Physics, Syracuse University, Syracuse NY 13244
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We compute boundary correlation functions for scalar fields on tessellations of two- and three-dimensional hyperbolic geometries. We present evidence that the continuum relation between the scalar bulk mass and the scaling dimension associated with boundary-to-boundary correlation functions survives the truncation of approximating the continuum hyperbolic space with a lattice.

I. INTRODUCTION

The holographic principle posits that the physical content of a gravitational system, with spacetime dimension \(d + 1\), can be understood entirely in terms of a dual quantum field theory living at the \(d\)-dimensional boundary of that space [1]. This conjecture is not proven, but it is supported by a great deal of evidence in the case of a gravitational theory in an asymptotically anti-de Sitter space. Furthermore, for a pure anti-de Sitter space, the dual quantum field theory is conformal. The posited duality can be expressed as an equality between the generating functional for a conformal field theory, and a restricted path integral over fields propagating in AdS:

\[
Z_{\text{CFT},d}[J(x)] = \int \mathcal{D}\phi \delta(\phi_0(x) - J(x)) e^{iS_{\text{AdS},d+1}}. \tag{1}
\]

The boundary values of the fields, \(\phi_0\), do not fluctuate, as they are equivalent to classical sources on the CFT side of the duality.

The earliest checks establishing the dictionary for this duality were performed by studying free, massive scalar fields, propagating on pure anti-de Sitter space [2, 3]. These established that the boundary-boundary two-point correlation function of such fields has a power law dependence on the boundary separation, where the magnitude of the scaling exponent, \(\Delta\), is related to the bulk scalar mass, \(m_0\), via the relation

\[
2\Delta = d \pm \sqrt{d^2 + 4m_0^2}, \tag{2}
\]

where \(m_0\) is expressed in units of the AdS curvature. The two choices for the scaling dimension are related to different treatments of the boundary action [4]. The “minus” branch of solutions (which can saturate the unitarity bound, \(\Delta = d/2 - 1\)) requires tuning to be accessible in the absence of supersymmetry.

In this paper we explore lattice scalar field theory on finite tessellations of negative-curvature spaces to determine which aspects of the AdS/CFT correspondence survive this truncation. Finite-volume and discreteness create both ultra-violet (UV) and infrared (IR) cutoffs, potentially creating both a gap in the spectrum and a limited penetration depth from the boundary into the bulk spacetime. Finite lattice spacing regulates the UV behavior of the correlators. Despite these artifacts, we show that such lattice theories do exhibit a sizable regime of scaling behavior, with this “conformal window” increasing with total lattice volume.

We specifically construct tessellations of both two- and three-dimensional hyperbolic spaces, construct scalar lattice actions, and compute the lattice Green’s functions to study the boundary-to-boundary correlators. We find general agreement with Eq. (2) in the large volume extrapolation.

Prior work has focused on the bulk behavior of spin models on fixed hyperbolic lattices and on using thermodynamic observables to map the phase diagram [5–8]. Here, the focus is on the structure of the boundary theory and, since free scalar fields are employed, the matter sector can be computed exactly including the boundary-boundary correlation function. This setup allows for a direct test of the continuum holographic behavior.

Another important aspect is the extension to three dimensions. Reference [9] performed a thorough investigation of the scalar field bulk and boundary propagators in two-dimensional Euclidean hyperbolic space using a triangulated manifold. Here we extend this discussion to boundary-boundary correlators in three dimensions.

The organization of this paper is as follows. In Section II we describe the class of tessellations we use in two dimensions and the construction of the discrete Laplacian operator needed to study the boundary correlation functions. In Section III we extend these calculations to three-dimensional hyperbolic geometries. Finally, we summarise our results in Section IV.

1 Hyperbolic space is the Euclidean continuation of anti-de Sitter space.
II. TWO-DIMENSIONAL HYPERBOLIC GEOMETRY

Regular tessellations of the two-dimensional hyperbolic plane can be labeled by their Schlafli symbol, \{p,q\}, which denotes a tessellation by p-gons with the connectivity, q, being the number of p-gons meeting at a vertex. In order to generate a negative curvature space, the tessellation must satisfy \((p-2)(q-2)>4\). We create such a tessellation recursively by building out from a single elementary polygon. For the generation of the incidence matrix, it is quite straightforward to build the Laplacian matrix using the connectivity information of the tessellated disk. In this way, the lattice is stored solely in terms of its adjacency information. The lattice is then composed of flat equilateral triangles with straight edges, all of which are the same length throughout the lattice.

A typical example of the lattice (projected onto the unit disc) is shown in Fig. 1 where the tessellation has been mapped onto the Poincaré disk model and corresponds to the \{3,7\} combination. An image of the boundary connectivity can be seen in Fig. 2. We see the boundary has all manner of vertex connectivity, some even with seven-fold coordination.

In the continuum, the action for a massive scalar field in two Euclidean spacetime dimensions is given by

\[ S_{\text{con.}} = \frac{1}{2} \int d^2x \sqrt{g} (\partial_{\mu} \phi \partial^{\mu} \phi + m_0^2 \phi^2) \],

(3)

where \(m_0\) is the bare mass, and \(d^2x \sqrt{g}\) is the amount of volume associated with each point in spacetime. The corresponding discrete action on a lattice of p-gons is then

\[ S = \frac{1}{2} \sum_{\langle xy \rangle} p_{xy} V_e \left( \frac{\phi_x - \phi_y}{a^2} \right)^2 + \frac{1}{2} \sum_{x} n_x m_0^2 V_v \phi_x^2. \] (4)

Here \(V_e\) denotes the volume of the lattice associated with an edge, \(V_v\) denotes the volume associated with a vertex, \(a\) denotes the lattice spacing, \(p_{xy}\) denotes the number of p-gons which share an edge (in the case of an infinite lattice this is always two), and \(n_x\) is the number of p-gons around a vertex. For two-dimensional p-gons,

\[ V_v = V_e = \frac{a^2}{4} \cot \frac{\pi}{p}, \]

(5)

and an illustration of the volumes associated with links and edges are shown in Fig. 3. This definition of the mass and kinetic weights ensures that the sum of the weights gives the total volume of the lattice, i.e.

\[ \sum_{\langle xy \rangle} p_{xy} V_e = \sum_{x} n_x V_v = A_\triangle N_\triangle, \]

where \(A_\triangle\) is the area of an equilateral triangle, and \(N_\triangle\) is the number of triangles. This is equivalent to the usual weights determined by construction of the dual lattice. \(\sum_{\langle xy \rangle}\) denotes a sum over all nearest-neighbor vertices, and \(\sum_{x}\) is over all vertices. We can write the action from Eq. (4) as

\[ S = \sum_{x,y} \phi_x L_{xy} \phi_y \]

(6)

with \(L_{xy}\) given by

\[ L_{xy} = -\frac{p_{xy}}{2} \delta_{x,y+1} + \frac{1}{2} \left( \sum_{z} p_{xz} \delta_{z,x+1} + m_0^2 n_x \right) \delta_{x,y}. \]

(7)
Figure 3: The volume, $V_e$, associated with an edge is shown in yellow, and the volume, $V_v$, associated with a vertex is shown in blue. In two dimensions, since each $p$-gon has the same number of vertices as edges, these volumes are always the same (in this case they are both $1/3$ of the area of the total triangle).

In practice we can rescale the scalar field so that the kinetic term has unit weight. We set the edge length, $a$, to one, throughout. In the bulk, for a two-dimensional lattice, $p_{xy} = 2$ and $n_x = q$, but these values change for points on the boundary. We note that boundary terms must be added to appropriately approximate the infinite volume AdS/CFT correspondence, in which fields at the AdS boundary are not permitted to fluctuate. To simulate this, we include a large scalar mass, $M$, only on the boundary vertices and extrapolate fits as $M \to \infty$. The average boundary correlation function (propagator) is then computed from

$$C(r) = \frac{\sum_{x,y} L^{-1} \delta_{r,d(x,y)}}{\sum_{x,y} \delta_{r,d(x,y)}}, \quad (8)$$

where $d(x,y)$ is the distance measured between boundary sites $x$ and $y$. In practice we observe a power law, $C(r) \sim r^{-2\Delta}$, as can be seen in Fig. 4, which shows the correlator for five different masses on a $(3,7)$ tesselation with 10 layers containing a total of $N = 591$ vertices. We fit the correlator using the form,

$$\log C(r) = -2\Delta \log r + k, \quad (9)$$

with $k$ and $\Delta$ as fit parameters. The error bars are found from using the jackknife method over all boundary points. In addition to the average over boundary points, we also find a non-negligible systematic error from deciding the fit range. We calculate this error by repeating the analysis for all different possible reasonable fit ranges and re-sample from these results. We add the errors found from this method in quadrature to the jackknife error, and find that the systematic part is by far the largest contribution to the error.

We check to see if the power, $2\Delta$, obeys a similar relation to Eq. (2), and fit $2\Delta$ to the form,

$$2\Delta = A + \sqrt{A^2 + 4Bm_0^2}, \quad (10)$$

where $A$ and $B$ are fit parameters. The solid curve in Fig. 5 indicates the best fit (least squares minimum) to Eq. (10) for a fixed system size and boundary mass. We expect $A$ to correspond to a renormalized boundary dimension, $d$, and $B$ to a mass renormalization. In the
In the infinite-volume extrapolation, we identify the regime in which the fit parameters, $A$ and $B$, scale approximately linearly with the inverse boundary size, $N_{\text{bound}}$. In other words,

$$A = \frac{C}{N_{\text{bound}}} + A_\infty, \quad B = \frac{D}{N_{\text{bound}}} + B_\infty,$$

where $C$, $D$, $A_\infty$ and $B_\infty$ are fit parameters. An example of the large-volume data is shown in Fig. 6.

Once we have $A_\infty$ and $B_\infty$ for each boundary mass, we extrapolate those values to infinite boundary mass. Again we look for a window of masses in which the parameters scale linearly in the inverse squared boundary mass, such that

$$A_\infty = \frac{E}{M^2} + A_\infty(M_\infty) \quad (12)$$

$$B_\infty = \frac{F}{M^2} + B_\infty(M_\infty), \quad (13)$$

where $E$, $F$, $A_\infty(M_\infty)$, and $B_\infty(M_\infty)$ are fit parameters. Fig. 7 shows the large-boundary mass extrapolation for $B_\infty$ and a fit yielding $A_\infty(M_\infty) \approx 1.00(2)$ and $B_\infty(M_\infty) \approx 1.55(6)$.

III. THREE-DIMENSIONAL HYPERBOLIC GEOMETRY

We now transition to the case of three dimensions. First we describe the honeycomb used in this investigation, as well as its construction. Honeycombs are tilings of three-dimensional space, packings of polyhedra that fill the entire space with no gaps.

Similar to the two-dimensional case, in three dimensions, one can succinctly describe regular honeycombs with a Schläfli symbol, a recursive notation for regular tilings. \{p,q,r\} denotes a honeycomb of \{p,q\} cells, which are polyhedra (or tilings) of p-gons, where q of these surround each vertex \[10\]. Here we focus on the \{4,3,5\}, also known as the order-5 cubical honeycomb, because the \{4,3\} cubical cells pack 5 polyhedra around each edge. A projection of this lattice can be seen in Fig. 8. The excess of cubes around an edge gives a local
Each successive layer containing all cells one step further builds up the cubical honeycomb in layers of cells, with each cube. Reflections in the six faces of these cubes can generate regular honeycombs with any symmetry of the honeycomb. A fundamental tetrahedron can represent the space with tetrahedra. Each tetrahedron represents the elements of the tetrahedron in its four faces to fill out space. The deficit angle at an edge is \( \pi/2 \), and the remaining angles are \( \pi/p \), \( \pi/q \), and \( \pi/r \). Using sphere inversion, we recursively reflect the elements of the tetrahedron in its four faces to fill out the space with tetrahedra. Each tetrahedron represents a symmetry of the honeycomb. A fundamental tetrahedron can generate regular honeycombs with any \( \{p, q, r\} \) symmetry group with \( p, q, r \geq 3 \).

In our case, a set of 48 symmetry tetrahedra form each cube. Reflections in the six faces of these cubes build up the cubical honeycomb in layers of cells, with each successive layer containing all cells one step further in the cell adjacency graph of the honeycomb. The number of cubes in each layer are \( 1, 6, 30, 126, 498, \ldots \), with the total number of cubes up to each level the sum of the entries in this sequence. This can be seen in Fig. 9. We store all the cubes, faces, edges, and vertices that we see during the reflections, taking care to avoid duplicates. In the infinite-volume limit, we would fill the whole of hyperbolic space with cubes.

So far we have described the geometrical construction. We use this information to derive incidence information for all of the elements of the honeycomb, i.e. to determine which vertices, edges, facets, and cubes connect to each other. This we encode as a list of flags. A flag is a sequence of elements, each contained in the next, with exactly one element from each dimension. All possible flags encode the full incidence information of our partially-built honeycombs. The incidence encoding is agnostic to geometrical distances.

We generate lists of flags out to various distances in the cell adjacency graph. The further one recurses, the less edge-effects appear in the incidence information. For example, after adding six layers of cubes, we get enough cells to completely surround all eight vertices of the central cube.

Using the lattice described above, we work with the same model as in two dimensions, and take a naive discretization of the scalar field action (this time in three dimensions) given by,

\[
S_{\text{lat}} = \frac{1}{2} \sum_{(xy)} p_{xy} V_e (\phi_x - \phi_y)^2 + \frac{1}{2} \sum_x n_x V_e m_0^2 \phi_x^2.
\]

Here \( \sum_{(xy)} \) is over nearest neighbors and \( p_{xy} \) is the number of cubes around an edge. In the infinite lattice case, \( p_{xy} \) is always five, but we leave it as a variable to allow for consideration of the case when the lattice is finite and has a boundary. \( n_x \) denotes the number of cubes which share a vertex. Again, in the infinite case this is always 20, but we leave it as a variable to allow for consideration of the case when the lattice is finite and has a boundary. \( n_x \) denotes the number of cubes which share a vertex. Again, in the infinite case this is always 20, but we leave it as a variable to allow for consideration of the case when the lattice is finite and has a boundary.

The weights again are chosen such that \( \sum_{(xy)} p_{xy} V_e = \sum_x n_x V_e = V_\square N_\square \), with \( V_\square \) being the volume of a cube, and \( N_\square \) the number of cubes. Above, \( a \) is reinserted for clarity but we assume the lattice edge length is one, as before.

We rewrite the lattice action to clearly identify the inverse lattice propagator, even in the presence of a boundary. To do this, we start by expanding and col-

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[Figure 8: An in-space view of the order-5 cubic honeycomb.]
Let’s express the action in terms of the inverse lattice vertex $x$. Using the fact that $V_e = (2/3)V_v$, we simplify further to get

$$S_{\text{lat}} = - \frac{1}{3} \sum_{(xy)} \langle xy \rangle p_{xy} V_e (\phi_x \phi_y + \phi_y \phi_x)
+ \sum_x \left( \sum_y \left( \frac{p_{xy}}{3} + \frac{m_0^2}{2} n_x \right) \phi_x^2 \right).$$

The cube edge lengths appear to vary in length in the Poincaré ball model; however, the lattice here has a fixed edge length, $a$.

Collecting terms to get

$$S_{\text{lat}} = \frac{1}{2} \sum_{(xy)} p_{xy} V_e (\phi_x - \phi_y)^2
+ \frac{1}{2} \sum_x n_x V_v m_0^2 \phi_x^2
= - \frac{1}{2} \sum_{(xy)} p_{xy} V_e (\phi_x \phi_y + \phi_y \phi_x)
+ \frac{1}{2} \sum_x \left( \sum_y \left( p_{xy} V_e + m_0^2 n_x V_v \right) \phi_x^2 \right),$$

where $\sum_y$ in the second term is over points neighboring vertex $x$. For a lattice with uniform edge length $a = 1$, these correspond to $1/12$ and $1/8$ respectively.
In the case of an infinite lattice, this simplifies to

\[ L_{xy} = -\delta_{x,y+1} + 12 \left( 1 + \frac{m_0^2}{2} \right) \delta_{x,y}, \]

which is expected for a lattice with 12-fold coordination. Using Eqs. (18) and (19), we construct an inverse lattice propagator for the hyperbolic lattice considered here, and use it in numerical computations.

Again, the boundary correlator is given by inverting the matrix corresponding to the discrete scalar inverse propagator. A typical set of correlators are shown in Fig. 11, corresponding to four bulk masses and squared propagator. A typical set of correlators are shown in the matrix corresponding to the discrete scalar inverse and use it in numerical computations.

Using Eqs. (18) and (19), we construct an inverse lattice propagator and get

\[ S_{\text{lat}} = S_{\text{kinetic}} + S_{\text{mass}} = \sum_{x,y} \phi_x L_{xy} \phi_y \]

with

\[ S_{\text{kinetic}} = -\sum_{x,y} \phi_x \frac{p_{xy} \cdot \delta_{x,y+1}}{3} \phi_y \]

and

\[ S_{\text{mass}} = \sum_{x,y} \phi_x \left( \left( \sum_z \frac{p_{xz} \cdot \delta_{z,x+1}}{3} \right) + \frac{m_0^2}{2} n_x \right) \delta_{x,y} \phi_y. \]

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By repeating this analysis with multiple volumes, we consider the extrapolation to infinite volume. We fit a power law to this window for a series of fixed, squared bulk mass. By far the largest source of error in this analysis is the systematic error in choosing those sources on the boundary. The distance shown in Fig. 11 is the distance on the boundary. This is computed by starting at the source vertex, taking a step to all neighboring boundary vertices, then taking a step from those vertices to their neighboring boundary vertices, skipping vertices that have already been visited, and so on, until all vertices have been visited. The error bars are produced using the jackknife method on the sources.

Clearly, a distance window exists in which the correlator follows a power law. This power-law behavior is observed for all masses explored in this study, and seems to solely be a consequence of the lattice geometry. We fit a power law to this window for a series of fixed, squared bulk mass. By far the largest source of error in this analysis is the systematic error in choosing a fit range. Because of the drastic slope of the correlator in Fig. 11, varying the fit range has a relatively large effect. The jackknife error is added in quadrature with this systematic error to produce the final errors on each power-law fit. From this fit, we obtain the power v.s. the squared mass. In the continuum, in the case of anti-de Sitter space, the boundary-boundary correlator is expected to show the behavior from Eq. (2) with boundary dimension \( d = 2 \). We attempt a fit using Eq. (10). An example of the fits can be seen in Fig. 12. We note that the power, \( 2\Delta \), is well-defined even in the regime of negative squared mass, indicating the operator \( -\nabla^2 + m_0^2 \) is positive in this regime. Based on these numerical results, the behavior of \( \Delta \) here matches well with the expected behavior of \( \Delta_+ \) expressed in Ref. [4]. However, we do not expect it to match exactly, since we do not approach the continuum limit on this lattice.

By repeating this analysis with multiple volumes, we consider the extrapolation to infinite volume. Here we consider three different volumes, corresponding to five,
Figure 13: The finite-size scaling of the fit parameter, $A$, from Eq. (10). The volumes have been rescaled by 1000 for readability. All three volumes use the same boundary mass of $M^2 = 10$.

Figure 14: The finite-size scaling of the fit parameter, $B$, from Eq. (10). The volumes have been rescaled by 1000 for readability. All three volumes use the same boundary mass of $M^2 = 10$.

six and seven-layers of cubes. These correspond to 2643, 10497, and 41511 cubes, respectively. Using the fit parameters from multiple volumes allows us to extrapolate to infinite cubes. In Figs. 13 and 14 we see the finite-size scaling of the fit parameters, $A$ and $B$, respectively, from Eq. (10). The fit is of the form of Eq. (11), with $N_{\text{bound}}$ being the size of the two-dimensional boundary of the three-dimensional hyperbolic lattice. We find $A_\infty \simeq 2.12(6)$, and $B_\infty \simeq 1.53(9)$ in the infinite volume limit, with the squared boundary mass $M^2 = 10$.

IV. CONCLUSIONS

In this paper we have studied the behavior of boundary correlations of massive scalar fields propagating on discrete tessellations of hyperbolic space. Both two and three dimensions are examined and good quantitative agreement with the continuum formula relating the power of the boundary correlator to bulk mass is obtained. Specifically, the functional form for the dependence of the boundary scaling dimension on bulk mass is reproduced accurately, including the inferred dimension of the boundary theory. A single parameter, $B$, controls the renormalization of the lattice mass relative to its continuum cousin.

In the future we plan to extend these calculations to four dimensions and to investigate the effects of allowing for dynamical fluctuations in the discrete geometries in order to simulate the effect of gravitational fluctuations. In such scenarios, the effects of the back reaction of matter fields on the geometries can be explored. This should allow us to probe holography in regimes that are difficult to explore using analytical approaches.

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[1] Juan Martin Maldacena. The Large N limit of superconformal field theories and supergravity. Int. J. Theor. Phys., 38:1113–1133, 1999.
[2] S. S. Gubser, Igor R. Klebanov, and Alexander M. Polyakov. Gauge theory correlators from noncritical string theory. Phys. Lett., B428:105–114, 1998.
[3] Edward Witten. Anti-de Sitter space and holography. Adv. Theor. Math. Phys., 2:253–291, 1998.
[4] Igor R. Klebanov and Edward Witten. Ads/cft correspondence and symmetry breaking. Nuclear Physics B, 556(1):89 – 114, 1999.
[5] R Krcmar, A Gendiar, K Ueda, and T Nishino. Ising model on a hyperbolic lattice studied by the corner transfer matrix renormalization group method. Journal of Physics A: Mathematical and Theoretical, 41(12):125001, mar 2008.
[6] Seung Ki Baek, Harri Mäkelä, Petter Minnhagen, and Beom Jun Kim. Ising model on a hyperbolic plane with a boundary. Phys. Rev. E, 84:032103, Sep 2011.
[7] Dario Benedetti. Critical behavior in spherical and hyperbolic spaces. *Journal of Statistical Mechanics: Theory and Experiment*, 2015(1):P01002, Jan 2015.

[8] Nikolas P. Breuckmann, Benedikt Placke, and Ananda Roy. Critical properties of the Ising model in hyperbolic space. *Phys. Rev. E*, 101:022124, Feb 2020.

[9] Richard C Brower, Cameron V Cogburn, A Liam Fitzpatrick, Dean Howarth, and Chung-I Tan. Lattice setup for quantum field theory in $\text{ads} \_2$. *arXiv preprint arXiv:1912.07606*, 2019.

[10] Harold Stephen Macdonald Coxeter. Regular honeycombs in hyperbolic space. In *Proceedings of the International Congress of Mathematicians* of 1954.

[11] Tristan Needham. *Visual Complex Analysis*. Oxford University Press, USA, 1999.

[12] Roice Nelson and Henry Segerman. Visualizing hyperbolic honeycombs. *Journal of Mathematics and the Arts*, 11(1):4–39, 2017.