Continuum limit of the lattice quantum graph Hamiltonian

Pavel Exner\textsuperscript{1,2} · Shu Nakamura\textsuperscript{3} · Yukihide Tadano\textsuperscript{4}

Received: 16 February 2022 / Revised: 4 August 2022 / Accepted: 4 August 2022 / Published online: 19 August 2022
© The Author(s), under exclusive licence to Springer Nature B.V. 2022

Abstract
We consider the quantum graph Hamiltonian on the square lattice in Euclidean space, and we show that the spectrum of the Hamiltonian converges to the corresponding Schrödinger operator on the Euclidean space in the continuum limit, and that the corresponding eigenfunctions and eigenprojections also converge in some sense. We employ the discrete Schrödinger operator as the intermediate operator, and we use a recent result by the second and third authors on the continuum limit of the discrete Schrödinger operator.

Keywords Lattice quantum graph · Vertex \(\delta\)-coupling · Schrödinger operator · Norm-resolvent convergence

Mathematics Subject Classification 81Q35 · 35J10 · 47B39

Pavel Exner
exner@ujf.cas.cz

Shu Nakamura
shu.nakamura@gakushuin.ac.jp

Yukihide Tadano
y.tadano@rs.tus.ac.jp

\textsuperscript{1} Doppler Institute for Mathematical Physics and Applied Mathematics, Czech Technical University, Břehová 7, 11519 Prague, Czech Republic

\textsuperscript{2} Department of Theoretical Physics, Nuclear Physics Institute, Czech Academy of Sciences, 25068 Řež, Czech Republic

\textsuperscript{3} Department of Mathematics, Faculty of Sciences, Gakushuin University, 1-5-1, Mejiro, Toshima, Tokyo 171-8588, Japan

\textsuperscript{4} Department of Mathematics, Faculty of Science Division I, Tokyo University of Science, 1-3, Kagurazaka, Shinjuku, Tokyo 162-8601, Japan
1 Introduction

In mathematics and physics, one often meets situations when we investigate a large structure being interested in its gross properties independent of the local structure. A classical example is the homogenization theory—see, e.g., [1–3] and references therein. Another example providing deep mathematical problems as well as a number of applications is represented by properties of large networks [4].

The present paper is devoted to a problem of this type appearing in the theory of quantum graphs, which is a short name for Schrödinger operators the configuration space of which is a metric graph [5]. To make such an operator self-adjoint, it is not enough to have the potential real-valued and sufficiently regular; one also has to define properly the conditions matching functions from the operator domain at the graph vertices [5, Thm. 1.4.4]. There are a large number of ways how to choose those conditions among which there is a smaller and distinguished subset, namely those preserving continuity at the vertices. In such a case, there is just one real parameter associated with each vertex; usually the term $\delta$-coupling is employed.

This paper is concerned with a family of such quantum graphs. It was observed in [6] that a square lattice graph with a varying $\delta$-coupling at the vertices and the vertex spacing tending to zero can approximate Schrödinger operator in $L^2(\mathbb{R}^\nu)$ provided the energy is rescaled by the dimension $\nu$; this approximation was illustrated on chaotic motion in billiards. What was left out there, however, was the existence of the limit and the type of the convergence. These are the questions addressed here. To get the answer, we combine two main elements. One is the recent result of two of the present authors [7] on the continuum limit of discrete Schrödinger operators, the other is the duality [8–10] between a Schrödinger operator on a metric graph and a suitable operator on the associated discrete graph.

One has to stress that while we use here the continuum limit of [7] as a tool to prove our main result, the purpose of the two approximations is different. The discretization of ‘continuous’ Schrödinger operators is a useful computational instrument, in the present case the aim is opposite, namely to describe global properties of large networks using PDE tools. In addition to examples mentioned in [6], this may be useful, for instance, to study effects in sheets of graphene and related systems often modeled by quantum graphs [11].

2 Problem statement and the main result

To begin with, we introduce in the standard way [5] the quantum graph Hamiltonian, that is, the Schrödinger operator on the metric graph. The latter will be in our case the $\nu$-dimensional square lattice graph of the lattice spacing $\ell > 0$,

\[ \Gamma = (\mathcal{V}, \mathcal{L}), \quad \mathcal{V} = \ell \mathbb{Z}^\nu, \quad \mathcal{L} = \{ \mathcal{L}_{jn} = [j, n] \mid j, n \in \mathcal{V}, |j - n| = \ell \} \]
where \([j, n]\) denotes the line segment connecting \(j\) and \(n \in \ell\mathbb{Z}^v\). The symbol \(V\) denotes the set of vertices, and \(\mathcal{L}\) is the set of edges in \(\Gamma\). We introduce the Hilbert space of functions on the graph by

\[
\mathcal{H}_1 = L^2(\Gamma) = \bigoplus_{\mathcal{L}_jn \in \mathcal{L}} L^2(\mathcal{L}_jn),
\]

with the inner product

\[
\langle \varphi, \psi \rangle_{\mathcal{H}_1} = \int_{\mathcal{L}_jn} \varphi_{jn}(t)\psi_{jn}(t) \, dt,
\]

where \(\varphi = (\varphi_{jn}), \psi = (\psi_{jn}) \in \mathcal{H}_1\).

We note the normalization factor \(\ell^{v-1}/v\) is introduced so that the restriction of a function \(\varphi\) on \(\mathbb{R}^v\) to \(\Gamma\) has norm asymptotically the same as the \(L^2\)-norm on \(\mathbb{R}^v\).

We adopt the following hypothesis:

**Assumption A** \(V\) is a real-valued continuous function on \(\mathbb{R}^v\) and bounded from below. Furthermore, \((V(\cdot) + M)^{-1}\) is uniformly continuous for some \(M > 0\), and there is a \(c_1 > 0\) such that

\[
c_1^{-1}(V(x) + M) \leq V(y) + M \leq c_1(V(x) + M) \quad \text{if} \quad |x - y| \leq 1.
\]

We denote \(V_j = V(j)\) for \(j \in \mathcal{V}\) and set \(\alpha_j := \ell V_j\) for \(j \in \mathcal{V}\). The Sobolev space of order one on the graph \(\Gamma\) is then given by

\[
H^1(\Gamma) = \{(\varphi_{jn}) \in \mathcal{H}_1 \mid \varphi_{jn} \in H^1([j, n]), \varphi_{jn}(j) = \varphi_{jm}(j) \text{ for } j \in \mathcal{V} \text{ and } n, m \in \mathcal{V}(j)\},
\]

where \(\mathcal{V}(j) = \{n \mid |j - n| = \ell\}\) is the set of vertices adjacent to \(V_j\), in other words, the neighborhood of the point \(j\) in the discrete graph associated with \(\Gamma\).

On the domain \(\mathcal{Q}(H_1) = \{\varphi \in H^1(\Gamma) \mid \sum_{j \in \mathcal{V}} \alpha_j |\varphi_j|^2 < \infty\}\), we define the quadratic form \(q_\alpha\) by means of the formula

\[
q_\alpha(\varphi, \psi) = \langle \varphi', \psi' \rangle + \sum_{j \in \mathcal{V}} \alpha_j \overline{\varphi_j} \psi_j, \quad \varphi, \psi \in \mathcal{Q}(H_1),
\]

where \((\varphi')_{jn}(t) = \frac{d}{dt}\varphi_{jn}(t)\) on \(\mathcal{L}_jn\) and \(\varphi_j = \varphi_{jn}(j)\) for \(n \in \mathcal{V}(j)\). We denote the self-adjoint operator associated with \(q_\alpha\) by \(H_1\), that is, \(\langle \varphi, H_1 \psi \rangle = q_\alpha(\varphi, \psi)\) holds for \(\varphi, \psi \in \mathcal{D}(H_1)\). It is known [5, Sec. 1.4.3] that

\[
\mathcal{D}(H_1) = \left\{\psi = (\psi_{jn}) \in H^1(\Gamma) \cap \bigoplus_{\mathcal{L}_jn \in \mathcal{L}} H^2(\mathcal{L}_jn) \mid \sum_{n \in \mathcal{V}(j)} \psi'_jn(j) = \alpha_j \psi_j\right\}
\]

and \((H_1 \psi)_{jn}(t) = -\psi''_{jn}(t)\) on \(\mathcal{L}_jn\). We recall that \(\Gamma\) is regarded as a non-oriented graph and the derivatives entering the condition specifying \(\mathcal{D}(H_1)\) are all conventionally taken in the outward direction.
The second object to consider is the Schrödinger operator $H$ on $L^2(\mathbb{R}^\nu)$ given by

$$H\varphi(x) = -\Delta \varphi(x) + V(x)\varphi(x), \quad x \in \mathbb{R}^\nu$$

for $\varphi \in \mathcal{D}(H) = \{ \varphi \in H^2(\mathbb{R}^\nu) \mid V\varphi \in L^2(\mathbb{R}^\nu) \}$. We recall that under our assumption about the potential, $H$ is a self-adjoint operator on $L^2(\mathbb{R}^\nu)$.

Our main result claims that in the limit $\ell \to 0$ the operators $\nu H_1$ approximate $H$ in the sense of norm resolvent convergence; we recall that the approximation requires the energy to be rescaled by a dimensional factor as observed in [6].

**Theorem 2.1** Let $z \in \mathbb{C} \setminus \mathbb{R}$, and adopt Assumption A. Then, there is a bounded operator $\Psi : \mathcal{H}_1 \to L^2(\mathbb{R}^\nu)$ such that in the limit $\ell \to 0$ we have

$$\| (H - z)^{-1} - \Psi (\nu H_1 - z)^{-1} \Psi^* \|_{B(L^2)} \to 0,$$

$$\| (\nu H_1 - z)^{-1} - \Psi^* (H - z)^{-1} \Psi \|_{B(H_1)} \to 0.$$
We also have the convergence of the spectrum of $\nu H_1$ to that of $H$ with respect to the Hausdorff distance, which is defined by

$$d_H(X, Y) = \max \left\{ \sup_{x \in X} d(x, Y), \sup_{y \in Y} d(y, X) \right\},$$

where $X, Y \subset \mathbb{R}$ and $d(\cdot, \cdot)$ denotes the Euclidean distance.

**Corollary 2.4** Let $M > 0$ large enough to ensure that $-M < \inf \sigma(H)$. Then, for all sufficiently small $\ell > 0$ one has $-M \notin \sigma(\nu H_1)$ and

$$d_H(\sigma((H + M)^{-1}), \sigma((\nu H_1 + M)^{-1})) \to 0 \text{ as } \ell \to 0.$$

In particular, $\sigma(\nu H_1)$ converges to $\sigma(H)$ as $\ell \to 0$ locally in terms of the Hausdorff distance.

The proof is given in Appendix A.

### 3 Discrete Schrödinger operator and its convergence

To prove Theorem 2.1, we choose an appropriate discrete Schrödinger operator as the intermediate object and use the recent result by two of the present authors [7] on its continuum limit, cf. also [17, 18] for fresh related results. Let us first recall the basic notions. The Hilbert space of functions on the vertices,

$$\mathcal{H}_2 = \ell^2(\mathcal{V}) = \ell^2(\ell\mathbb{Z}^\nu),$$

is equipped with the norm

$$||u||^2_{\mathcal{H}_2} = \ell^\nu \sum_{j \in \mathcal{V}} |u_j|^2,$$

where $u = (u_j) \in \mathcal{H}_2$.

Let the potential $V = V(x)$ be as before. We denote again $V_j = V(j)$ for $j \in \mathcal{V}$ and define a discrete Schrödinger operator $H_2$ on $\mathcal{H}_2$ by

$$(H_2\varphi)_j = -\Delta_d \varphi_j + V_j \varphi_j, \quad \Delta_d \varphi_j = \frac{1}{\ell^2} \sum_{n \in \mathcal{V}(j)} (\varphi_n - \varphi_j)$$

for $\varphi = (\varphi_j) \in \mathcal{H}_2$; it is easy to check that $H_2$ is a self-adjoint operator with its domain $\mathcal{D}(H_2) = \left\{ u = (u_j) \in \mathcal{H}_2 \left| (V_j u_j) \in \mathcal{H}_2 \right. \right\}$. The following result was proved in [7]:

**Theorem 3.1** (Nakamura–Tadano) Let $z \in \mathbb{C} \setminus \mathbb{R}$ and adopt Assumption A. Then, there is a bounded operator $\Phi : \mathcal{H}_2 \to L^2(\mathbb{R}^\nu)$ such that in the limit $\ell \to 0$ we have

$$\| (H - z)^{-1} - \Phi (H_2 - z)^{-1} \Phi^* \|_{\mathcal{B}(L^2)} \to 0,$$
\[ \|(H_2 - z)^{-1} - \Phi^*(H - z)^{-1}\Phi\|_{\mathcal{B}(\mathcal{H}_2)} \to 0. \]

For completeness, we sketch here the construction of \( \Phi \). Let \( \psi \in \mathcal{S}(\mathbb{R}^\nu) \) be such that
\[
\sum_{n \in \mathbb{Z}^\nu} |\hat{\psi}(\xi + n)|^2 = 1 \quad \text{for} \ \xi \in \mathbb{R}^\nu.
\]

For example, we may choose a function \( \hat{\psi} \in C^\infty_0(\mathbb{R}^\nu) \) satisfying the above property, and define \( \psi \) as its inverse Fourier transform. Then, \( \{\psi_{\ell,n} = \psi(\ell^{-1}(\cdot - j)) \mid j \in \mathcal{V}\} \) form an orthogonal system and the operator
\[
(\Phi \varphi)(x) = \sum_{j \in \mathcal{V}} \psi_{\ell,j}(x) \varphi_j dx, \quad x \in \mathbb{R}^\nu, \varphi \in \mathcal{H}_2,
\]
is an isometry from \( \mathcal{H}_2 \) into \( L^2(\mathbb{R}^\nu) \). We refer to [7] for details.

Thus, we know that \( H \) is approximated by the indicated discrete Schrödinger operator, and it will be sufficient to show that the latter is in turn approximated by the quantum graph Hamiltonian, and vice versa.

### 4 Approximation of the quantum graph Hamiltonian by the discrete Schrödinger operator

Our aim is to show that \( \nu H_1 \) and \( H_2 \) are close to each other with an appropriate identification map when the spacing \( \ell \) is small. Throughout this section, we suppose that \( V \) is bounded from below.

#### 4.1 Identification operators

Let \( I : \mathcal{H}_2 \to \mathcal{H}_1 \) be the embedding by linear interpolation, namely \( \varphi = (\varphi_j) \in \mathcal{H}_1 \mapsto I \varphi = (\varphi_{jn}) \in \mathcal{H}_2 \) defined by
\[
\varphi_{jn}(x(t)) = (1 - t)\varphi_j + t\varphi_n, \quad \text{where} \ x(t) = (1 - t)j + tn \in [j,n].
\]

We note \( I \) is bounded from \( \mathcal{H}_2 \) into \( \mathcal{H}_1 \).

Furthermore, we define the trace operator \( K : H^1(\Gamma) \to \mathcal{H}_2 \) by
\[
K : \varphi = (\varphi_{jn}) \in H^1(\Gamma) \mapsto (K \varphi)_j = \varphi_{jn}(j) \quad (\forall n \in \mathcal{V}(j)).
\]

#### 4.2 Preliminary estimates

**Lemma 4.1** For any \( \ell > 0 \), we have
\[
\|IK - 1\|_{\mathcal{B}(H^1(\Gamma), \mathcal{H}_1)} \leq \ell.
\]
Proof Given \( \varphi = (\varphi_{jn}) \in C^1(\Gamma) \), we write \( \varphi_j = \varphi_{jn}(j), \forall n \in \mathcal{V}(j) \). Let \( \tilde{\varphi} = IK\varphi \), in other words

\[
\tilde{\varphi}_{jn}(x(t)) = (1-t)\varphi_j + t\varphi_n, \quad \text{where } x(t) = (1-t)j + tn, \ 0 \leq t \leq 1.
\]

We identify \( \mathcal{L}_{jn} \cong [0, \ell] \) for the moment. Then, for \( t \in [0, \ell] \) we have

\[
\varphi_{jn}(t) = \int_0^t \varphi'_{jn}(s) \, ds - \frac{t}{\ell} \int_0^\ell \varphi'_{jn}(s) \, ds,
\]

since

\[
\tilde{\varphi}_{jn}(t) = \varphi_{jn}(j) + \frac{t}{\ell} \int_0^\ell \varphi'_{jn}(s) \, ds, \quad t \in [0, \ell].
\]

From here, we infer that

\[
\left| \varphi_{jn}(t) - \tilde{\varphi}_{jn}(t) \right| \leq \int_0^\ell |\varphi'_{jn}(s)| \, ds \leq \ell \left( \int_0^\ell |\varphi'_{jn}(s)|^2 \, ds \right)^{1/2}
\]

using the Schwarz inequality, and consequently, we have

\[
\int_{\mathcal{L}_{jn}} \left| \varphi_{jn}(t) - \tilde{\varphi}_{jn}(t) \right|^2 \, dt \leq \ell^2 \int_{\mathcal{L}_{jn}} \left| \varphi'_{jn}(t) \right|^2 \, dt
\]

Summing up this over the edges \( \mathcal{L}_{jn} \), we find

\[
\| \varphi - \tilde{\varphi} \|_{H^1} \leq \ell \| \varphi' \|_{H^1} \leq \ell \| \varphi \|_{H^1(\Gamma)},
\]

and by the density argument, we get the estimate for any \( \varphi \in H^1(\Gamma) \).

Lemma 4.2 For any \( \ell > 0 \), we have

\[
\| I^* - K \|_{\mathcal{B}(H^1(\Gamma), H_2)} \leq \frac{\ell}{\sqrt{5}}.
\]

Proof A simple computation yields for \( \varphi \in C^1(\Gamma) \) the relation

\[
(I^*\varphi)_j = \frac{1}{v} \sum_{n \in \mathcal{V}(j)} \int_0^\ell \left( 1 - \frac{t}{\ell} \right) \varphi_{jn}(t) \, dt
\]

\[
= \frac{1}{2v} \sum_{n \in \mathcal{V}(j)} \left\{ \left[ -\left( 1 - \frac{t}{\ell} \right)^2 \varphi_{jn}(t) \right]_0^\ell + \int_0^\ell \left( 1 - \frac{t}{\ell} \right)^2 \varphi'_{jn}(t) \, dt \right\}
\]

\[
= \varphi_j + \frac{1}{2v} \sum_{n \in \mathcal{V}(j)} \int_0^\ell \left( 1 - \frac{t}{\ell} \right)^2 \varphi'_{jn}(t) \, dt.
\]
This further implies

\[ \| I^* \varphi - K \varphi \|_{H^2}^2 = \frac{\ell^\nu}{4 \nu^2} \sum_{j \in \mathcal{V}} \left| \sum_{n \in \mathcal{V}(j)} \int_0^\ell \left( 1 - \frac{t}{\ell} \right)^2 \varphi'_{jn}(t) \, dt \right|^2 \]

\[ \leq \frac{\ell^\nu}{2 \nu} \sum_{j \in \mathcal{V}} \sum_{n \in \mathcal{V}(j)} \int_0^\ell \left( 1 - \frac{t}{\ell} \right)^4 \, dt \int_0^\ell |\varphi'_{jn}(t)|^2 \, dt \]

\[ = \frac{\ell^2}{5} \| \varphi' \|_{H^1}^2 \leq \frac{\ell^2}{5} \| \varphi \|_{H^1(\Gamma)}^2, \]

which also holds for any \( \varphi \in H^1(\Gamma) \) proving thus the claim. \( \square \)

Since \( I \) is bounded, Lemmata 4.1 and 4.2 in combination with the triangle inequality give the following result:

**Corollary 4.3** There is a \( C > 0 \) such that for all \( \ell > 0 \) we have

\[ \| I I^* - 1 \|_{B(H^1(\Gamma), H^1)} \leq C \ell. \]

### 4.3 Explicit formula for \( K(\nu H_1 - z)^{-1} I \)

Now we are going to derive an explicit expression for the sandwiched resolvent of the operator \( \nu H_1 \) which will play the key role in the proof of our main theorem. Following the standard convention, we write points of the resolvent set as \( z = k^2 \).

**Lemma 4.4** Let \( \psi = (\psi_{jn}) \in \left( \bigoplus_{jn} H^2(L_{jn}) \right) \cap H^1(\Gamma), \varphi \in H_2, \) and \( k^2 \notin \mathbb{R} \). Then, there are operators \( M_1, M_2 \in B(H_2) \) such that

\[ (\nu H_1 - k^2) \psi = I \varphi \quad (4.1) \]

holds if and only if

\[ -\nu \psi''_{jn} - k^2 \psi_{jn} = (I \varphi)_{jn} \quad \text{on} \ L_{jn}, \quad (4.2) \]

and

\[ (H_2 - k^2 + M_1) K \psi = (1 + M_2) \varphi. \quad (4.3) \]

Moreover, \( M_1 \) and \( M_2 \) satisfy \( \|(H_2 - k^2)^{-1} M_1\| = \mathcal{O}(\ell) \) and \( \|(H_2 - k^2)^{-1} M_2\| = \mathcal{O}(\ell) \) as \( \ell \to 0 \).

**Proof** We note (4.1) implies (4.2) by the definition of \( H_1 \). We denote \( \varphi_{jn} = (I \varphi)_{jn} \) and recall that

\[ \varphi_{jn}(x) = \left( 1 - \frac{x}{\ell} \right) \varphi_j + \frac{x}{\ell} \varphi_n = \varphi_j + \frac{x}{\ell} (\varphi_n - \varphi_j), \quad x \in [0, \ell] \supseteq L_{jn}. \]

\( \square \)
Given the boundary values $\psi_j = \psi_j n(0)$ and $\psi_n = \psi_j n(\ell)$, we can solve the equation (4.2) explicitly using the standard ODE method, obtaining thus the expression

$$\psi_{jn}(x) = \frac{\sin(k' x)}{\sin(k' \ell)} \psi_n + \frac{\sin(k' (\ell - x))}{\sin(k' \ell)} \psi_j$$

$$+ \frac{1}{k'^2} \left( \frac{\sin(k' x)}{\sin(k' \ell)} - \frac{x}{\ell} \right) \varphi_n + \frac{1}{k'^2} \left( \frac{\sin(k' (\ell - x))}{\sin(k' \ell)} - 1 + \frac{x}{\ell} \right) \varphi_j,$$

where $k' = k/\sqrt{\nu}$. In particular, this yields

$$\psi'_{jn}(j) = \psi'_{jn}(0) = \frac{k'}{\sin(k' \ell)} (\psi_n - \psi_j) + \frac{k'(1 - \cos(k' \ell))}{\sin(k' \ell)} \psi_j$$

$$+ \frac{1}{k'^2} \left( \frac{k'}{\sin(k' \ell)} - \frac{1}{\ell} \right) \varphi_n - \frac{\varphi_j}{\nu} + \frac{1 - \cos(k' \ell)}{k' \sin(k' \ell)} \frac{\varphi_j}{\nu}.$$

Substituting this into the boundary condition in the definition of $H_1$,

$$\sum_{n \in \mathcal{V}_j} \psi'_{jn}(j) = \alpha_j \psi_j,$$

we get the relation

$$\frac{k'}{\sin(k' \ell)} \sum_{n \in \mathcal{V}_j} (\psi_n - \psi_j) + \frac{k'(1 - \cos(k' \ell))}{\sin(k' \ell)} |\mathcal{V}_j| \psi_j$$

$$+ \frac{1}{k'^2} \left( \frac{k'}{\sin(k' \ell)} - \frac{1}{\ell} \right) \sum_{n \in \mathcal{V}_j} \varphi_n - \frac{\varphi_j}{\nu} + \frac{1 - \cos(k' \ell)}{k' \sin(k' \ell)} |\mathcal{V}_j| \frac{\varphi_j}{\nu} = \alpha_j \psi_j$$

for each $j \in \mathcal{V}$. Recalling that $\alpha_j = \ell V_j$ and $|\mathcal{V}_j| = 2v$, we can rewrite it as

$$- \frac{1}{\ell^2} \sum_{n \in \mathcal{V}_j} (\psi_n - \psi_j) + \left( \frac{\sin(k' \ell)}{k' \ell} \right) V_j \psi_j - k^2 \left( \frac{1 - \cos(k' \ell)}{(k' \ell)^2 / 2} \right) \psi_j$$

$$= - \left( \frac{\sin(k' \ell) - k' \ell}{(k' \ell)^3} \right) \sum_{n \in \mathcal{V}_j} \varphi_n - \frac{\varphi_j}{\nu} + \left( \frac{1 - \cos(k' \ell)}{(k' \ell)^2 / 2} \right) \varphi_j.$$

Next we note that by the Taylor series expansion we have

$$\frac{\sin(k' \ell)}{k' \ell} = 1 + O(\ell^2), \quad \frac{1 - \cos(k' \ell)}{(k' \ell)^2 / 2} = 1 + O(\ell^2), \quad \frac{\sin(k' \ell) - k' \ell}{(k' \ell)^3} = O(1)$$

as $\ell \to 0$, hence setting

$$(M_1 \psi)_j := \left( \frac{\sin(k' \ell)}{k' \ell} - 1 \right) V_j \psi_j - k^2 \left( \frac{1 - \cos(k' \ell)}{(k' \ell)^2 / 2} - 1 \right) \psi_j.$$
\[(M_2\varphi)_j := -\frac{1}{\nu} \left( \frac{\sin(k'\ell) - k'\ell}{(k'\ell)^3} \right) \sum_{n \in V_j} (\varphi_n - \varphi_j) + \left( \frac{1 - \cos(k'\ell)}{(k'\ell)^2/2} - 1 \right) \varphi_j, \]

we can rewrite the above relation in the form (4.3),

\[(H_2 - k^2 + M_1)K\psi = (1 + M_2)\varphi.\]

Finally, we use the following claim to conclude the proof.

**Lemma 4.5** Suppose \( V \) is bounded from below. Then, for each \( z \in \mathbb{C} \setminus \mathbb{R} \), there is a \( C > 0 \) such that

\[ \|\triangle_d (H_2 - z)^{-1}\|_{\mathcal{B}(H_2)} \leq C\ell^{-1}, \quad \|V(H_2 - z)^{-1}\|_{\mathcal{B}(H_2)} \leq C\ell^{-1} \quad \text{for } 0 < \ell \leq 1. \]

Using this result in combination with the above explicit expressions of \( M_1 \) and \( M_2 \), we get the estimates

\[ \|V(H_2 - z)^{-1}\| = O(\ell) \quad \text{and} \quad \|(H_2 - z)^{-1} M_2\| = O(\ell) \quad \text{as } \ell \to 0 \text{ for any } z \in \mathbb{C} \setminus \mathbb{R}. \]

**Proof of Lemma 4.5** Since \( V \) is by assumption bounded from below, \( \triangle_d \) and \( V \) are relatively form bounded with respect to \( H_2 = -\triangle_d + V \). On the other hand, we note that \( \|\triangle_d\|_{\mathcal{B}(H_2)} = 2\nu\ell^{-2} \), and therefore,

\[ \|\triangle_d (H_2 - z)^{-1}\| \leq \|\triangle_d\|^{1/2} \cdot \|\triangle_d\|^{1/2} (H_2 - z)^{-1} \| \leq C\ell^{-1}. \]

Then, we also have

\[ \|V(H_2 - z)^{-1}\| = \|(H_2 + \triangle_d)(H_2 - z)^{-1}\| \leq \|H_2(H_2 - z)^{-1}\| + C\ell^{-1}, \]

and this completes the proof.

**4.4 Approximation theorem**

Now we are in position to compare the resolvents of the operators \( \nu H_1 \) and \( H_2 \) using the identification map \( I \).

**Theorem 4.6** Let \( z \in \mathbb{C} \setminus \mathbb{R} \), then there is a \( C > 0 \) such that

\[ \|(H_2 - z)^{-1} - I^*(\nu H_1 - z)^{-1} I\|_{\mathcal{B}(H_2)} \leq C\ell, \]

\[ \|(\nu H_1 - z)^{-1} - I (H_2 - z)^{-1} I^*\|_{\mathcal{B}(H_1)} \leq C\ell. \]

**Proof** We recall that \( z = k^2 \in \mathbb{C} \setminus \mathbb{R} \). By Lemma 4.4, we have

\[ (H_2 - z + M_1)K(\nu H_1 - z)^{-1} I = 1 + M_2 \] (4.4)
on \( \mathcal{H}_2 \). We use the identity

\[
H_2 - z + M_1 = (H_2 - z)(1 + (H_2 - z)^{-1} M_1)
\]

which implies

\[
(H_2 - z + M_1)^{-1} = (1 + (H_2 - z)^{-1} M_1)^{-1}(H_2 - z)^{-1}
\]

as long as \( \ell \) is sufficiently small so that the first factor on the right-hand side makes sense. Combining this with (4.4), we get

\[
K(vH_1 - z)^{-1} I = (1 + (H_2 - z)^{-1} M_1)^{-1}(H_2 - z)^{-1}(1 + M_2),
\]

and therefore,

\[
K(vH_1 - z)^{-1} I - (H_2 - z)^{-1}
\]

\[
= -(1 + (H_2 - z)^{-1} M_1)^{-1}((H_2 - z)^{-1} M_1(H_2 - z)^{-1} - (H_2 - z)^{-1} M_2).
\]

This implies, again by virtue of Lemma 4.4,

\[
\|(H_2 - z)^{-1} - K(vH_1 - z)^{-1} I\|_{\mathcal{B}(\mathcal{H}_2)} = O(\ell) \quad \text{as} \quad \ell \to 0.
\]

Now we use Lemma 4.2 and the triangle inequality to conclude that

\[
\|(H_2 - z)^{-1} - I^*(vH_1 - z)^{-1} I\|_{\mathcal{B}(\mathcal{H}_2)} = O(\ell) \quad \text{as} \quad \ell \to 0,
\]

since \((vH_1 - z)^{-1}\) is bounded as a map from \( \mathcal{H}_1 \) to \( H^1(\Gamma) \). In a similar way, we use Lemma 4.1 and Corollary 4.3 to get the other estimate,

\[
\|I(H_2 - z)^{-1} I^* - (vH_1 - z)^{-1} I\|_{\mathcal{B}(\mathcal{H}_1)} = O(\ell)
\]

as \( \ell \to 0 \). This completes the proof. \( \square \)

5 Proof of Theorem 2.1

To finish the task, it is now sufficient to combine Theorem 4.6 with Theorem 3.1. We define the identification operator by

\[
\Psi := \Phi I^*.
\]

Using the fact that \( \Phi \) is bounded, in fact an isometry, we then have

\[
\|(H - z)^{-1} - \Psi(vH_1 - z)^{-1} \Psi^*\|
\]

\[
\leq \|(H - z)^{-1} - \Phi(H_2 - z)^{-1} \Phi^*\| + \|\Phi(H_2 - z)^{-1} \Phi^* - \Psi(vH_1 - z)^{-1} \Psi^*\|
\]

\[
= \|(H - z)^{-1} - \Phi(H_2 - z)^{-1} \Phi^*\| + \|\Phi((H_2 - z)^{-1} - I^*(vH_1 - z)^{-1} I)\Phi^*\|
\]

\( \square \) Springer
\begin{align*}
&\leq \norm{(H - z)^{-1} - \Phi(H_2 - z)^{-1} \Phi^*} + \norm{(H_2 - z)^{-1} - I^*(\nu H_1 - z)^{-1} I} \\
&\to 0 \text{ as } \ell \to 0.
\end{align*}

The proof of the other estimate is almost identical, so we omit the computation; by that the proof of Theorem 2.1 is finished.

\textbf{Acknowledgements} P.E. was supported by the Czech Science Foundation within the project 21-07129S and by the EU project CZ.02.1.01/0.0/0.0/16_019/0000778. S.N. was partially supported by JSPS Grant Numbers 15H03622 (2015–2019) and 21K03276 (2021–2024). Y.T. was partially supported by JSPS Grant Numbers 20J00247 (2020–2021) and 21K20337 (2021–2023). This work was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University. The authors appreciate useful comments made by the referees. In particular, Appendix A is added following one of their suggestions.

\textbf{Data availability} There are no data associated with this manuscript.

\section*{Declarations}

\textbf{Conflict of interest} The authors have no conflict of interest.

\section*{Appendix A. Proof of Corollary 2.4}

\textbf{Proof of Corollary 2.4} Since we already have
\[
d_H(\sigma((H + M)^{-1}), \sigma((H_2 + M)^{-1})) \to 0, \quad \text{as } \ell \to 0,
\]
(see [7]), it suffices to show
\[
d_H(\sigma((H_2 + M)^{-1}), \sigma((\nu H_1 + M)^{-1})) \to 0, \quad \text{as } \ell \to 0.
\]
\begin{equation}
(A.1)
\end{equation}

We note that Corollary 4.3 implies
\begin{equation}
\left| \norm{I^* \varphi}_{\mathcal{H}_2} - \norm{\varphi}_{\mathcal{H}_1} \right| = \langle \varphi, (II^* - 1) \varphi \rangle_{\mathcal{H}_1} \leq \norm{I^* I - 1}_{B(H^1(\Gamma))} \norm{\varphi}_{H^1(\Gamma)} \norm{\varphi}_{\mathcal{H}_1}
\begin{equation}
(A.2)
\end{equation}

In fact, by Corollary 4.3, we immediately have
\[
\left| \norm{I^* \varphi}_{\mathcal{H}_2} - \norm{\varphi}_{\mathcal{H}_1} \right| = \langle \varphi, (II^* - 1) \varphi \rangle_{\mathcal{H}_1} \leq \norm{I^* I - 1}_{B(H^1(\Gamma))} \norm{\varphi}_{H^1(\Gamma)} \norm{\varphi}_{\mathcal{H}_1}
\begin{equation}
(A.2)
\end{equation}

Hence, we have
\[
\left| \norm{I^* \varphi}_{\mathcal{H}_2} - \norm{\varphi}_{\mathcal{H}_1} \right| = \frac{\norm{I^* \varphi}_{\mathcal{H}_2} - \norm{\varphi}_{\mathcal{H}_1}}{\norm{I^* \varphi}_{\mathcal{H}_2} + \norm{\varphi}_{\mathcal{H}_1}} \leq \frac{\norm{I^* \varphi}_{\mathcal{H}_2}^2 - \norm{\varphi}_{\mathcal{H}_1}^2}{\norm{\varphi}_{\mathcal{H}_1}}
\begin{equation}
\leq C \ell \norm{\varphi}_{H^1(\Gamma)}.
\end{equation}

\textcopyright Springer
Let $z \in \mathbb{C} \setminus \mathbb{R}$. We note that $(\nu H_1 - z)^{-1}$ is bounded from $\mathcal{H}_1$ to $H^1(\Gamma)$, uniformly in $\ell$. By (A.2), we have
\[
\left| \| I^*(\nu H_1 - z)^{-1} I \varphi \|_{\mathcal{H}_2} - \| (\nu H_1 - z)^{-1} I \varphi \|_{\mathcal{H}_1} \right| 
\leq C \ell \| (\nu H_1 - z)^{-1} I \varphi \|_{H^1(\Gamma)} \leq C' \ell \| \varphi \|_{\mathcal{H}_2}.
\]
This implies
\[
\left| \| I^*(\nu H_1 - z)^{-1} I \|_{B(\mathcal{H}_2)} - \| (\nu H_1 - z)^{-1} I \|_{B(\mathcal{H}_1)} \right| \leq C \ell.
\]
Then, we consider
\[
\left| \| (\nu H_1 - z)^{-1} I \|_{B(\mathcal{H}_2, \mathcal{H}_1)} - \| (\nu H_1 - z)^{-1} I \|_{B(\mathcal{H}_1)} \right|
= \left| \| I^*(\nu H_1 - z)^{-1} I \|_{B(\mathcal{H}_1, \mathcal{H}_2)} - \| (\nu H_1 - z)^{-1} I \|_{B(\mathcal{H}_1)} \right|
\]
but the right-hand side is bounded by $C \ell$ as well as the above argument, simply by replacing $z$ by $\bar{z}$, and $I \varphi$ by $\varphi$. Thus, we have
\[
\left| \| I^*(\nu H_1 - z)^{-1} I \|_{B(\mathcal{H}_2)} - \| (\nu H_1 - z)^{-1} I \|_{B(\mathcal{H}_1)} \right| \leq C \ell.
\]
Combining this with Theorem 4.6, we have
\[
\left| \| (H_2 - z)^{-1} I \|_{B(\mathcal{H}_2)} - \| (\nu H_1 - z)^{-1} I \|_{B(\mathcal{H}_1)} \right| \leq C \ell \quad \text{(A.3)}
\]
where $0 < \ell \leq 1$.

Let $R > 0$, fixed. We consider the resolvents at $z = \mu + i$, $\mu \in [-R, R]$. Here we show the estimate (A.3) holds uniformly in such $z$. By the resolvent equation, we know
\[
\| (A - (\mu + i))^{-1} - (A - (\mu' + i))^{-1} \| \leq |\mu - \mu'|, \quad \mu, \mu' \in \mathbb{R},
\]
in general, where $A$ is a self-adjoint operator. Let $\varepsilon > 0$, and let $\{\mu_j\}_{j=1}^N = \{n\varepsilon/3 \in [-R, R] \mid n \in \mathbb{Z}\}$. We choose $\ell_0 > 0$ so small that
\[
\left| \| (H_2 - (\mu_j + i))^{-1} I \|_{B(\mathcal{H}_2)} - \| (\nu H_1 - (\mu_j + i))^{-1} I \|_{B(\mathcal{H}_1)} \right| < \varepsilon/3
\]
for $0 < \ell \leq \ell_0$ and $j = 1, \ldots, N$. Then, by the $\varepsilon/3$-argument, we learn
\[
\left| \| (H_2 - (\mu + i))^{-1} I \|_{B(\mathcal{H}_2)} - \| (\nu H_1 - (\mu + i))^{-1} I \|_{B(\mathcal{H}_1)} \right| < \varepsilon
\]
for all $\mu \in [-R, R]$, $0 < \ell \leq \ell_0$, and this proves the uniform bound.

We now prove the local convergence of the spectrum with respect to the Hausdorff distance. We fix $R > 0$, and consider $\sigma(\nu H_1)$ and $\sigma(H_2)$ in $[-R, R]$. Let $\varepsilon > 0$ and we set
\[
\rho_\varepsilon = [-R, R] \cap \{ \mu \in \mathbb{R} \mid \text{dist}(\mu, \sigma(\nu H_1)) \geq \varepsilon \}.
\]
We note, for each $\mu \in \rho_\varepsilon$, 
\[
\|(vH_1 - (\mu + i))^{-1}\|_{B(H_1)} \leq \frac{1}{\sqrt{1 + \varepsilon^2}}.
\]

We choose $\ell_0 > 0$ so small that 
\[
\|(H_2 - (\mu + i))^{-1}\|_{B(H_2)} - \|(vH_1 - (\mu + i))^{-1}\|_{B(H_1)} \leq \frac{1}{\sqrt{1 + \varepsilon^2/4}} - \frac{1}{\sqrt{1 + \varepsilon^2}}
\]
for $\mu \in [-R, R]$ and $0 < \ell \leq \ell_0$. Then, this implies 
\[
\|(H_2 - (\mu + i))^{-1}\|_{B(H_2)} \leq \frac{1}{\sqrt{1 + \varepsilon^2/4}}
\]
if $\mu \in \rho_\varepsilon$ and $0 < \ell \leq \ell_0$, and hence, $(\mu - \varepsilon/2, \mu + \varepsilon/2) \subset \rho(H_2)$. In particular, $\mu \in \rho(H_2)$ if $\mu \in \rho_\varepsilon$, i.e., $\rho_\varepsilon \subset \rho(H_2)$, provided $0 < \ell \leq \ell_0$. This, in turn, implies $\sigma(H_2) \cap [-R, R]$ is included in the $\varepsilon$-neighborhood of $\sigma(vH_1)$.

Replacing $vH_1$ and $H_2$, we also learn $\sigma(vH_1) \cap [-R, R]$ is included in the $\varepsilon$-neighborhood of $\sigma(H_2)$ if $\ell$ is sufficiently small. In other words, we now have 
\[
\sup\limits_{\mu \in \sigma(vH_1) \cap [-R, R]} d(\mu, \sigma(H_2)) < \varepsilon, \quad \sup\limits_{\mu \in \sigma(H_2) \cap [-R, R]} d(\mu, \sigma(vH_1)) < \varepsilon,
\]
if $\ell > 0$ is sufficiently small. This local convergence implies (A.1) by the spectrum mapping theorem. This completes the proof.

References

1. Allaire, G., Piatnitski, A.: Homogenization of the Schrödinger equation and effective mass theorems. Commun. Math. Phys. 258(1), 1–22 (2005)
2. Bakhvalov, N., Panasenko, G.: Homogenisation: Averaging Process in Periodic Media Mathematical Problems in Mechanics of Composite Materials. Mathematics and its Applications (Soviet Series), vol. 36. Kluwer, Dordrecht (1989)
3. Birman, Sh., Suslina, T.A.: Homogenization with corrector for periodic differential operators. Approximation of solutions in the Sobolev class $H^1(\mathbb{R}^d)$. Algebra i Analiz 18(6), 1–130 (2006)
4. Lovász, L.: Large Networks and Graph Limits, Colloquium Publication, vol. 60. American Mathematical Society, Providence, R.I. (2012)
5. Berkolaiko, G., Kuchment, P.: Introduction to Quantum Graphs, Mathematical Surveys and Monographs, vol. 186. American Mathematical Society, Providence, R.I. (2013)
6. Exner, P., Hejčík, P., Šeba, P.: Approximation by graphs and emergence of global structures. Rep. Math. Phys. 57(3), 445–455 (2006)
7. Nakamura, S., Tadano, Y.: On a continuum limit of discrete Schrödinger operators on square lattice. J. Spectr. Theory 11(1), 355–367 (2021)
8. Cattaneo, C.: The spectrum of the continuous Laplacian on a graph. Monatsh. Math. 124(3), 215–235 (1997)
9. Exner, P.: A duality between Schrödinger operators on graphs and certain Jacobi matrices. Ann. Inst. H. Poincaré, Phys. Théor. 66(4), 359–371 (1997)
10. Pankrashkin, K.: Unitary dimension reduction for a class of self-adjoint extensions with applications to graph-like structures. J. Math. Anal. Appl. 396(2), 640–655 (2012)
11. Kuchment, P., Post, O.: On the spectra of carbon nano-structures. Commun. Math. Phys. 275(3), 805–826 (2007)
12. Post, O.: Spectral Analysis on Graph-Like Spaces. Lecture Notes in Mathematics, vol. 2039. Springer, Berlin (2011)
13. Exner, P., Post, O.: A general approximation of quantum graph vertex couplings by scaled Schrödinger operators on thin branched manifolds. Commun. Math. Phys. 322(1), 207–227 (2013)
14. Post, O., Simmer, J.: Graph-like spaces approximated by discrete graphs and applications. Math. Nachr. 294(11), 2237–2278 (2021)
15. Demuth, M., Krishna, M.: Determining Spectra in Quantum Theory. Birkhäuser, Boston (2005)
16. Simon, B.: Functional Integration in Quantum Physics, 2nd edn. AMS Chelsea, Providence, R.I. (2005)
17. Cornean, H., Garde, H., Jensen, A.: Norm resolvent convergence of discretized Fourier multipliers. J. Fourier Anal. Appl. 27(4), 71 (2021)
18. Isozaki, H., Jensen, A.: Continuum limit for lattice Schrödinger operators. Rev. Math. Phys. 34, 2250001 (2022)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.