Abstract. The truncated variation, TV$_c$, is a fairly new concept introduced in [5]. Roughly speaking, given a càdlàg function $f$, its truncated variation is “the total variation which does not pay attention to small changes of $f$, below some threshold $c > 0$. The very basic consequence of such approach is that contrary to the total variation, TV$_c$ is always finite. This is appealing to the stochastic analysis where so-far large classes of processes, like semimartingales or diffusions, could not be studied with the total variation. Recently in [6], another characterization of TV$_c$ was found. Namely TV$_c$ is the smallest possible total variation of a function which approximates $f$ uniformly with accuracy $c/2$. Due to these properties we envisage that TV$_c$ might be a useful concept both in the theory and applications of stochastic processes.

For this reason we decided to determine some properties of TV$_c$ for some well-known processes. In course of our research we discover intimate connections with already known concepts of the stochastic processes theory.

Firstly, for semimartingales we proved that TV$_c$ is of order $c^{-1}$ and the normalized truncated variation converges almost surely to the quadratic variation of the semimartingale as $c \downarrow 0$. Secondy, we studied the rate of this convergence. As this task was much more demanding we narrowed to the class of diffusions (with some mild additional assumptions). We obtained the weak convergence to a so-called Ocone martingale. These results can be viewed as some kind of law of large numbers and the corresponding central limit theorem.

Finally, for a Brownian motion with a drift we proved the behavior of TV$_c$ on intervals going to infinity. Again, we obtained a LLN and CLT, though in this case they have a different interpretation and were easier to prove.

All the results above were obtained in a functional setting, viz. we worked with processes describing the growth of the truncated variation in time. Moreover, in the same respect we also treated two closely related quantities - the so-called upward truncated variation and downward truncated variation.

Key words and phrases. Stochastic processes, semimartingales, diffusions, truncated variation, total variation.

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1. Introduction and results

Recently, the following notion of the truncated variation has been introduced in [5]:

\[ \text{TV}^c(f, [a; b]) := \sup_n \sup_{a \leq t_1 < s_1 < t_2 < s_2 < \ldots < t_n < s_n \leq b} \sum_{i=1}^{n-1} \phi_c(|f(t_{i+1}) - f(t_i)|), \]

where \( \phi_c(x) = \max\{x - c, 0\} \), \( c \geq 0 \) and \( f : [a; b] \to \mathbb{R} \) is a càdlàg function. The trivial observation is that \( \text{TV}^0 \) is nothing else that the total variation (which will be also denoted by TV). The introduction of the truncation parameter \( c \) makes it possible to circumvent a classical problem of stochastic analysis; namely, that the total variation of the Brownian motion as well as of a ‘non-trivial’ diffusion process is almost surely infinite. This alone makes \( \text{TV}^c \) an interesting research object. Other properties of \( \text{TV}^c \) were found, amongst which the variational characterization of the truncated variation given by

\[ \text{TV}^c(f, [a; b]) = \inf \left\{ \text{TV}(g, [a; b]) : g \text{ such that } \|g - f\|_{\infty} \leq \frac{1}{2} c \right\}, \]

where \( \|g\|_{\infty} := \sup \{ |g(x)| : x \in [a; b] \} \). In other words, truncated variation is the lower bound for the total variation of functions approximating \( f \) with accuracy \( c \). It appears that the inf in the above expression is attained at some function \( g^c \). The properties just listed give hope that \( \text{TV}^c \) could be used in the stochastic analysis. This question is an active field of research, some promising results are contained in [7], like definition of a stochastic integral with respect to a semimartingale as a limit of the pathwise Riemann-Stieltjes stochastic integrals, and other are being investigated. A detailed description would be too vast for our introduction therefore we refer the reader to [7] and [6], and its debriefing in Section 2.

Having agreed that \( \text{TV}^c \) might be a useful tool, an important task is to describe the behavior of \( \text{TV}^c \) for a vast class of stochastic processes. This is the main aim of this paper. We will derive first order properties for continuous semimartingales and second order properties for continuous diffusions (under some mild technical assumptions) when \( c \searrow 0 \). Intuitively, these answer the question of how fast \( \text{TV}^c \) converges to the total variation, that is how fast it diverges to infinity. In the case of the Brownian motion with drift we will also study the behavior of \( \text{TV}^c \) on large time intervals.

Before presenting our results we define two concepts closely related to \( \text{TV}^c \). The upward truncated variation given by

\[ \text{UTV}^c(f, [a; b]) := \sup_n \sup_{a \leq t_1 < s_1 < t_2 < s_2 < \ldots < t_n < s_n \leq b} \sum_{i=1}^{n} \phi_c(f(s_i) - f(t_i)), \]

and the downward truncated variation given by

\[ \text{DTV}^c(f, [a; b]) := \sup_n \sup_{a \leq t_1 < s_1 < t_2 < s_2 < \ldots < t_n < s_n \leq b} \sum_{i=1}^{n} \phi_c(f(t_i) - f(s_i)). \]
The relation between $TV^c$, $UTV^c$, $DTV^c$ will become clear in Section 2.1. Given a càdlàg process $\{X_t\}_{t \geq 0}$ we define the following families of processes $\{TV^c(X, t)\}_{t \geq 0}$, $\{UTV^c(X, t)\}_{t \geq 0}$ and $\{DTV^c(X, t)\}_{t \geq 0}$ by

$$TV^c(X, t) := TV^c(X, [0; t]), \quad UTV^c(X, t) := UTV^c(X, [0; t]), \quad DTV^c(X, t) := DTV^c(X, [0; t]),$$

where all the above definitions are understood in a pathwise fashion. Obviously, all three processes are increasing. Moreover, for semimartingales and $c \downarrow 0$, under weak non-degeneracy conditions, their values diverge up to infinity. Thus a natural question arises what the growth rate of the (upward, downward) truncated variation is. Under a proper normalization we expect also some convergence to a non-trivial object. These questions are answered in the following section.

1.1. Behavior as $c \downarrow 0$. First order properties for continuous semimartingales. For a continuous semimartingale $\{X\}_{t \in [0; T]}$ we will denote its decomposition by

$$X_t := X_0 + M_t + A_t, \quad t \in [0; T],$$

where $M$ is a continuous local martingale such that $M_0 = 0$ and $A$ is a continuous finite variation process such that $A_0 = 0$. Given $T > 0$, by $C([0; T], \mathbb{R})$ we denote the usual space of continuous functions on $[0; T]$ endowed with the topology given by norm $\| \cdot \|_{\infty}$.

**Theorem 1.** Let $T > 0$ and let $\{X\}_{t \in [0; T]}$ be a continuous semimartingale as above. We have

$$\lim_{c \downarrow 0} c TV^c(X, t) \rightarrow \langle X \rangle_t, \quad a.s.$$  

$$\lim_{c \downarrow 0} c UTV^c(X, t) \rightarrow \langle X \rangle_t / 2, \quad a.s.$$  

$$\lim_{c \downarrow 0} c DTV^c(X, t) \rightarrow \langle X \rangle_t / 2, \quad a.s.$$  

In all cases the converge is understood in the $C([0; T], \mathbb{R})$ topology.

**Remark 2.** One can see that $TV^c$ is of order $c^{-1}$. Hence by the discussion above this is also the lower bound of the total variation of the approximation of $X$ in $\| \cdot \|_{\infty}$-ball of radius $c/2$. For diffusions we will find finer estimates in the next section.

Assumptions of Theorem 1 could be weakened slightly. Without additional effort we can prove the theorem for $A$ not being necessary continuous. This is however cumbersome from notational point of view, as we cannot work in $C([0; T], \mathbb{R})$ space. The problem of non-continuous semimartingales will be treated in full extent in future papers.

**Remark 3.** Theorem 1 could be considered as some kind of a law of large numbers. We will now provide a rough justification using the Wiener process $W$ as an example. One can imagine splitting an interval $[0; 1]$ into $c^{-2}$ parts. On each part $W$ performs a motion of order $c$. The contribution of the part to the total truncated variations is not negligible and is of order $c$. The contributions are random and “almost” independent for non-neighboring parts. Therefore there is no randomness in the limit.
Remark 4. The heuristics presented in the previous remark is nice at the intuitive level however a more precise description is required to perform the proof. In the case of a Wiener process with drift this will be a precise characterization of \( t_1, t_2, \ldots \) for which the sup in definition \( \text{(1.1)} \) is attained, which will lead to a natural renewal structure. In the case of a general semimartingale following the same path seems to be hopeless. To circumvent the problem we employed an abstract approach based on time change techniques in spirit of the Dambis, Dubins-Schwarz theorem [11, Chapt. V, Theorem 1.6].

Having explained “the law of large numbers nature” of the above result a natural question arises about the corresponding central limit theorem. This will be addressed in the next section for \( \{X_t\}_{t\geq0} \) being a diffusion satisfying some mild conditions.

1.2. Behavior as \( c \downarrow 0 \). Second order properties for diffusions. Let us now consider a general diffusion defined with equation

\[
dX_t = \sigma(X_t)dW_t + \mu(X_t)dt, \quad X_0 = 0,
\]

We will always assume that \( \sigma, \mu \) are Lipschitz functions and \( \sigma > 0 \). It is well known, [11, Sect. IX.2], that under these conditions the equation admits a unique strong solution. The main result of this section is

**Theorem 5.** Let \( T > 0 \) then

\[
\left( X, \text{UTV}^c(X,t) - \frac{1}{2} \left( \frac{\langle X \rangle_t}{c} + X_t \right), \text{DTV}^c(X,t) - \frac{1}{2} \left( \frac{\langle X \rangle_t}{c} - X_t \right), \text{TV}^c(X,t) - \frac{\langle X \rangle_t}{c} \right) \rightarrow^d (X, \tilde{M}_t, \tilde{M}_t, 2\tilde{M}_t), \text{ as } c \downarrow 0,
\]

where \( \tilde{M} \) is given by the change time formula:

\[
\tilde{M}_t := 12^{-1/2}B_{\langle X \rangle_t},
\]

where \( B \) is a standard Brownian motion such that \( B \) and \( X \) are independent. The convergence is understood as the weak convergence in \( C([0;T],\mathbb{R})^4 \) topology.

**Remark 6.** Let us notice that by [3, Proposition 5.33] from the joint convergence of \( X \) and three other processes related to UTV, DTV and TV one obtains their stable convergence as described in [3, Sect. VIII.5].

**Remark 7.** Let us now present an intuitive explanation of the result on the example of a Wiener process with drift, \( W \) and the truncated variation. Theorem \( \text{[4]} \) reads as

\[
c \text{TV}^c(W,t) \rightarrow t, \quad \text{a.s.}
\]

and by Theorem \( \text{[5]} \) and the fact that \( \langle W \rangle_t = t \) we obtain

\[
\text{TV}^c(X,t) - \frac{t}{c} \rightarrow^d 3^{-1/2}B_t.
\]
In this case the theorems are indeed an “almost classical” law of large numbers and central limit theorem. This stems from the fact that $TV^c$ in this case has a particularly nice, renewal structure.

On the intuitive level, by equation (1.2) one may say that for any path of $W$ on interval $[a; b]$, minimal "vertical" length of graph of any random function $f : [a; b] \to \mathbb{R}$, uniformly close to this path must be at least equal to

$$b - a + \sqrt{\frac{b - a}{3} R_c},$$

where $c = 2 \sup_{t \in [a; b]} |f(t) - W_t|$, and $R_c$ is a random variable such that it tends in distribution to a standard normal distribution $\mathcal{N}(0, 1)$ as $c \downarrow 0$. Note that for small $c$’s this lower bound is almost deterministic.

Remark 8. It is easy to check that $\langle \tilde{M} \rangle = \langle X \rangle$. Let $M$ be the local martingale in the semimartingale decomposition of $X$. It is natural to ask how the laws of $M$ and $\tilde{M}$ are related. The martingale of the form given by equation (1.6) were introduced in [9] and are called Ocone martingales. By results of [15] it follows that $M$ is an Ocone martingale only if $\sigma = \text{const}$ (i.e. $X$ is a Brownian motion with some stochastic drift).

Let us also notice that $\sigma = \text{const}$ is also the only case when $\langle X \rangle_t$ is a deterministic process.

Ocone martingales have particularly simple structure which sometimes makes it easy to draw conclusion about them. As an example we consider a situation when $\sigma \leq C$. Then

$$\mathbb{P} \left( \sup_{t \in [0; T]} \tilde{M}_t \geq a \right) \leq \mathbb{P} \left( \sup_{t \in [0; CT]} 12^{-1/2} B_t > a \right) = \mathbb{P} \left( \sup_{t \in [0; T]} B_t > (12/C)^{1/2} a \right),$$

hence $\tilde{M}$ has a Gaussian concentration. Further properties and references can be found in [15].

Remark 9. The assumption $\sigma > 0$ is equivalent to $\sigma \neq 0$. This follows by the fact that $\sigma$ is continuous so, under the assumption that $\sigma \neq 0$, either $\sigma > 0$ for any $x$ or $\sigma < 0$. In the latter case one can simply take $-\sigma$ instead of $\sigma$ and obtain a diffusion with the same law. The case when $\sigma$ may attain value 0 requires further studies. To see this let us consider "a very degenerate case" when $\sigma = 0$ on an interval $[x_0; x_1]$ for $x_0 < x_1$. For any $x \in (x_0; x_1)$ the diffusion degenerates locally to a deterministic process, a solution of an ordinary differential equation, with a bounded total variation. Hence the above formulation of the CLT does not make sense. While this case was relatively easy, the situation becomes more involved for border points $x_0, x_1$ or "isolated" 0’s. We suspect that in such cases a non-trivial correction term containing the local time may be required.

Remark 10. Similarly as in the case of the law of large numbers (see Remark 4) the proof splits into technically different parts.

The first one deals with the Wiener process with drift $X_t = W_t + \mu t$. We use here the fact that $TV^c(X, t)$ has a fairly simple renewal-like structure. Moreover, it is possible to derive explicit formulas for the Laplace transform of the increments of the truncated variation. Then a very simple argument allows to treat random drift, i.e. the case where $\mu$ is a random variable independent of $W$. 
The second step deals with diffusions with $\sigma = const$. Namely, on a small interval we have $X_{\Delta t + t} - X_t \approx \sigma(W_{\Delta t + t} - W_t) + \mu(X_t)\Delta t := Y_{\Delta t}$ which is essentially a Wiener process with a random drift as above. It turns out that we may control the quality of the approximation to conclude the proof using some metric-theoretic tricks and the Prohorov metric in this case.

As explained in Remark [26], this approach fails in the case of non-constant $\sigma$. Here we appeal to a time change technique and a Rényi mixing-like argument (see e.g. [13, p. 309]. A reader familiar with this kind of reasoning may recognize that this is why we get the independence in equation [1.6].

1.3. Large time results. For the Wiener process with drift it is possible to derive results for large time. In this section, we put $X := W_t + \mu t$.

Firstly, we present

**Fact 11.** Let $T > 0$ and $c > 0$. We have

$$\lim_{n \to +\infty} TV^c(X, nt) / n \to m^c_\mu t, \quad a.s.,$$

where the convergence is understood in $C([0; T], \mathbb{R})$ topology and

$$m^c_\mu = \begin{cases} \mu \coth(c\mu) & \text{if } \mu \neq 0, \\ c^{-1} & \text{if } \mu = 0. \end{cases}$$

Analogously we have

$$\lim_{n \to +\infty} UTV^c(X, nt) / n \to \frac{1}{2} n^c_\mu t, \quad a.s.,$$

and

$$\lim_{n \to +\infty} DTV^c(X, nt) / n \to \frac{1}{2} n^{-c}_\mu t, \quad a.s.$$

where again the convergence is understood in $C([0; T], \mathbb{R})$ topology and

$$n^c_\mu = \begin{cases} \mu \coth(c\mu) + \mu & \text{if } \mu \neq 0, \\ c^{-1} & \text{if } \mu = 0. \end{cases}$$

The quality of the above approximation is studied in

**Theorem 12.** Let $T > 0$ and $c > 0$. We have

$$TV^c(X, nt) - m^c_\mu nt \xrightarrow{\sigma^c_\mu \sqrt{n}} d B_t, \quad \text{as } n \to +\infty,$$

where $\xrightarrow{d}$ is understood as weak convergence in $C([0; T], \mathbb{R})$ topology; $m^c_\mu$ is given by [1.7] and

$$(\sigma^c_\mu)^2 = \begin{cases} \frac{2 - 2\mu \coth(c\mu)}{\sinh^2(c\mu)} + 1 & \text{if } \mu \neq 0, \\ 1/3 & \text{if } \mu = 0. \end{cases}$$

**Theorem 13.** Let $T > 0$ and $c > 0$. We have

$$UTV^c(X, nt) - \frac{1}{2} n^c_\mu nt \xrightarrow{\rho^c_\mu \sqrt{n}} d B_t, \quad \text{as } n \to +\infty,$$
and
\[
\frac{DTV^c(X, nt) - \frac{1}{2} n^{c-\mu} nt}{p^c_{\mu} \sqrt{n}} \to^d B_t, \quad as \ n \to +\infty,
\]
where \( \to^d \) is understood as weak convergence in \( C([0; T], \mathbb{R}) \) topology; \( n^{c-\mu} \) is given by (1.8) and
\[
(p^c_{\mu})^2 = \begin{cases} 
\frac{2 \exp(4c\mu)(\sinh(2c\mu)-2c\mu)}{(\exp(2c\mu)-1)^4} & \text{if } \mu \neq 0, \\
1/3 & \text{if } \mu = 0.
\end{cases}
\]

Remark 14. Fact [11] could be considered as a kind of law of large numbers. Indeed, \( TV^c \) builds up over time (cf. Subsection 2.1) and because of the homogeneity of \( X \) its truncated variation can be decomposed into a number of independent increments. These increments are also square integrable, therefore Theorem 12 and Theorem 13 hold.

The task of proving analogous facts for more general classes of processes seems to be elusive at the moment. Firstly, our methods failed in this case, but the reason seems to lie deeper than that. It is connected with the fact that the truncated variation depends on the paths in a rather complicated way, simplifying only when \( c \to 0 \). We suspect that it is possible to prove similar results for ergodic Markov processes. This however seem a little unsatisfactory as in this case the convergence stems merely from the fact that on distant intervals the process itself is nearly independent.

Remark 15. It is possible for the finite dimensional distributions of the normalized truncated variation processes appearing in Theorem [12] and Theorem [13] to obtain even stronger results, namely the Berry-Esséen-type estimates of the rate of convergence to normal distribution. The straightforward way to obtain such estimates is to use the already mentioned cumulative structure of the truncated variation processes of a Brownian motion with drift and [12, Theorem 8.2]. One can check that the appropriate moments exist (see formula equation (3.15) and observe that inter-renewal times in this case have the same distribution as the exit time of Brownian motion with drift from a strip. Thus we obtain that the difference between the cdf of the multidimensional projection of the limit distribution and the cdf of the finite dimensional distributions of the normalized truncated variation processes in Theorem [12] and Theorem [13] is of order \( \log(n)/\sqrt{n} \). We suspect that the results of Theorem [5] can be strengthened in a similar way. This will be a subject of further studies.

Let us now comment on the structure of the paper. In the next section we gather facts about the truncated variation and discuss potential application to the theory of stochastic processes. Section [3] is devoted to the proof of Theorem [5]. In section [4] we present the proof of Theorem [11]. Finally in Section [5] we sketch the proof of the large time results presented just above.

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2. Properties of the truncated variation

This section is based on results of [6]. For reader’s convenience we keep much of the notation introduced there.

Arguably the most interesting property of the \( \text{TV}^c \) was listed in equation (1.2). Another closely related property is given by

\[
(2.1) \quad \text{TV}^c(f, [a; b]) = \inf \{ \text{TV}(g, [a; b]) : g \text{ such that } \|g - f\|_{osc} \leq c, g(a) = f(a) \},
\]

where \( \|h\|_{osc} := \sup \{|h(x) - h(y)| : x, y \in [a; b]\} \). The infimum in equation (2.1) attained for some \( g^{0,c} : [a; b] \mapsto \mathbb{R} \), which is unique. Moreover, we also have the following explicit representation:

\[
(2.2) \quad g^{0,c}(s) = f(a) + \text{UTV}^c(f, [a; s]) - \text{DTV}^c(f, [a; s])
\]

and

\[
(2.3) \quad \|g^{0,c} - f\|_{\infty} \leq c.
\]

\( g^{0,c} \) is also closely related to the solution of the problem stated in equation (1.2). Let us put \( \alpha_0 := -\inf \{ g^{0,c}(s) - f(s) : s \in [a; b] \} - \frac{1}{2}\|g^{0,c} - f\|_{osc} \). The function \( g^c \) for which inf in equation (1.2) is attained is given by

\[
(2.4) \quad g^c(s) := \alpha_0 + g^{0,c}(s).
\]

The problem posed by equation (2.1) seems a little artificial at first. Its formulation has however a substantial advantage over the problem of equation (1.2) when considered in stochastic setting. Namely, when working with stochastic processes the solution given by equation (2.2) is adaptable to the same filtration as the process itself while the solution obtained in equation (2.4) requires some “knowledge of future”. We would like also to mention that condition \( \|g - f\|_{osc} \leq c \) in (2.1) implies that the increments of \( f \) are uniformly approximated by the increments of \( g^{0,c} \) with accuracy \( c \). This property might be useful for applications to numerical stochastic integration.

To give the reader some intuition about the functions introduced above we rephrase [6, Remark 2.4]: “\( g^c \) is the most lazy function possible, which changes its value only if it is necessary to stay in the tube defined by \( \|g^c - f\|_{\infty} \leq c/2 \)”. This can be seen on the following picture.
We hope that we convince the reader that the truncated variation is an interesting research object. Moreover, we hope that it will be useful both in the theory of stochastic processes and in their applications. The first step towards this goal were undertaken in [8] and [7] e.g. in [8] was calculated the Laplace transform of UTV\(_c\) and DTV\(_c\) for a Brownian motion with drift and in [7] are presented possible applications to the approximation of stochastic processes and stochastic integration.

We plan to report shortly on further findings.

2.1. **Joint structure of TV\(_c\), UTV\(_c\) and DTV\(_c\).** We will now describe the structure of TV\(_c\), UTV\(_c\) and DTV\(_c\). The construction is described in more details in [6, Section 2]. Let \(-\infty < a < b < +\infty\) and let \(f:\ [a; b] \to \mathbb{R}\) be a càdlàg function. For \(c > 0\) let us assume that (2.5)

\[
T^c_D f := \inf \left\{ s \geq a : \sup_{ t \in [a; s]} f( t ) - f( s ) \geq c \right\} \leq T^c_U f := \inf \left\{ s \geq a : f( s ) - \inf_{ t \in [a; s]} f( t ) \geq c \right\}.
\]

i.e. the first upward jump of function \(f\) of size \(c\) appears before the first downward jump of the same size \(c\) or both times are infinite, i.e. there is no upward or downward jump of size \(c\). Note that when this condition fails one may simply consider function \(-f\). Now we define sequences \(\left( T^c_{U,k} \right)_{k=0}^{\infty}, \left( T^c_{D,k} \right)_{k=-1}^{\infty}\), in the following way: \(T^c_{D,-1} = a, T^c_{U,0} = T^c_U f\) and for \(k \geq 0\):

\[
T^c_{D,k} := \inf \left\{ s \geq T^c_{U,k} : \sup_{ t \in [T^c_{U,k}; s]} f( t ) - f( s ) \geq c \right\},
\]

\[
T^c_{U,k+1} = \inf \left\{ s \geq T^c_{D,k} : f( s ) - \inf_{ t \in [T^c_{D,k}; s]} f( t ) \geq c \right\}.
\]
Next let us define two sequences of non-decreasing functions $m^c_k : [T^c_{D,k-1}; T^c_{U,k}] \to \mathbb{R}$ and $M^c_k : [T^c_{U,k}; T^c_{D,k}] \to \mathbb{R}$ for $k \geq 0$ such that $T^c_{D,k-1} < \infty$ and $T^c_{U,k} < \infty$ respectively, with the formulas

$$m^c_k(s) := \inf_{t \in [T^c_{D,k-1}; s]} f(t), \quad M^c_k(s) = \sup_{t \in [T^c_{U,k}; s]} f(t),$$

Similarly, let us define two finite sequences of real numbers $\{m^c_k\}$ and $\{M^c_k\}$, for such $k$'s that $T^c_{D,k-1} < \infty$ and $T^c_{U,k} < \infty$ by

$$m^c_k := m^c_k(T^c_{U,k}-) = \inf_{t \in [T^c_{D,k-1}; T^c_{U,k})} f(t),$$

$$M^c_k := M^c_k(T^c_{D,k}-) = \sup_{t \in [T^c_{U,k}; T^c_{D,k})} f(t).$$

The above definitions are simple however may be hard to read without pictures. We hope the following will be helpful. Note that we present the same function as in the previous example.

**Figure 2.2. Example of definition of $T^c_{U,k}$, $T^c_{D,k}$ and $M^c_k, m^c_k$.**

The main results of this section is (cf. [6, Theorem 2.3])

**Theorem 16.** For any càdlàg function $f : [a;b] \to \mathbb{R}$ such that $T^c_U f \leq T^c_D f$ we have

$$UTV^c(f, [a; s]) = DTV^c(f, [a; s]) = 0,$$

when $s \in [a; T^c_U)$ and

$$UTV^c(f, [a; s]) := \begin{cases}
\sum_{i=0}^{k-1} \{M^c_i - m^c_i - c\} + M^c_k(s) - m^c_k - c & \text{if } s \in [T^c_{U,k}; T^c_{D,k}], \\
\sum_{i=0}^{k} \{M^c_i - m^c_i - c\} & \text{if } s \in [T^c_{D,k}; T^c_{U,k+1}],
\end{cases}$$

$$DTV^c(f, [a; s]) := \begin{cases}
\sum_{i=0}^{k-1} \{M^c_i - m^c_{i+1} - c\} & \text{if } s \in [T^c_{U,k}; T^c_{D,k}], \\
\sum_{i=0}^{k-1} \{M^c_i - m^c_{i+1} - c\} + M^c_k - m^c_{k+1} - c & \text{if } s \in [T^c_{D,k}; T^c_{U,k+1}].
\end{cases}$$
Moreover, for any càdlàg function \( f : [a; b] \mapsto \mathbb{R} \) and any \( s \in [a; b] \) we have
\[
TV^c(f, [a; s]) = UTV^c(f, [a; s]) + DTV^c(f, [a; s]) .
\]

2.2. Basic properties of \( TV^c(f, [a; b]) \), \( UTV^c(f, [a; b]) \) and \( DTV^c(f, [a; b]) \). We will now list some properties, most of which is used in the paper. These are taken from \([6, \text{Section 2.4, Section 2.5}]\). Unless stated otherwise the functions considered below are càdlàg.

- For any strictly increasing and continuous function \( s : \mathbb{R} \to \mathbb{R} \)
\[
TV^c(f, [a; b]) = TV^c(f \circ s^{-1}, [s(a); s(b)]) ,
\]
the analogous equalities hold for \( UTV^c \) and \( DTV^c \).

- For any \( f : [a; b] \mapsto \mathbb{R} \) and any \( c > 0 \) we have
\[
DTV^c(f, [a; b]) = UTV^c(-f, [a; b]) .
\]

- For any \( s \in (a; b) \) we have
\[
TV^c(f, [a; b]) \geq TV^c(f, [a; s]) + TV^c(f, [s; t]) ,
\]
and the analogous inequalities hold for \( UTV^c \) and \( DTV^c \).

- On the other hand, for any \( s \in (a; b) \) we have
\[
TV^c(f, [a; b]) \leq TV^c(f, [a; s]) + TV^c(f, [s; t]) + c ,
\]
and the analogous inequalities hold for \( UTV^c \) and \( DTV^c \).

- For any \( f, g : [a; b] \to \mathbb{R} \) and \( c_1, c_2 \geq 0 \) we have
\[
TV^{c_1+c_2}(f + g, [a; b]) \leq TV^{c_1}(f, [a; b]) + TV^{c_2}(g, [a; b]) ,
\]
and the analogous inequalities hold for \( UTV^c \) and \( DTV^c \). Note that in above we admit some quantities to be infinite in case \( c_1 = 0 \) or \( c_2 = 0 \). In particular
\[
|TV^c(f + g, [a; b]) - TV^c(f, [a; b])| \leq TV(g, [a; b]) .
\]

These facts were not proved in \([6]\). We offer a proof in Fact \([17]\) below.

- For any \( f : [a; b] \mapsto \mathbb{R} \) mapping
\[
(0, +\infty) \ni c \mapsto TV^c(f, [a; b]) ,
\]
is convex and decreasing hence continuous. The same holds true for \( UTV^c \) and \( DTV^c \). Moreover, though not mentioned in \([6]\), it can be easily upgraded to functional setting.

E.g. we define functional \( T : (0; +\infty) \mapsto \mathcal{D} \) (Skorohod space of càdlàg functions) given by \( T(c)(t) := TV^c(f, [a; t]) \) is convex and decreasing in a point-wise sense.

- For any \( f : [a; b] \mapsto \mathbb{R} \) we have
\[
\lim_{c \searrow 0} TV^c(f, [a; b]) = TV(f, [a; b]) ,
\]
we recall that the right-hand side might be infinite.
Fact 17. For any \( f, g : [a; b] \to \mathbb{R} \) and \( c_1, c_2 \geq 0 \) we have
\[
TV^{c_1 + c_2}(f + g, [a; b]) \leq TV^{c_1}(f, [a; b]) + TV^{c_2}(g, [a; b]),
\]
and the analogous inequalities hold for UTV\(^c\) and DTV\(^c\).

Proof. The inequality for UTV\(^c\) holds by definition equation (1.3) and the inequality
\[
\max \{ f(s) + g(s) - f(t) - g(t) - c_1 - c_2, 0 \} = \max \{ f(s) - f(t) - c_1 + g(s) - g(t) - c_2, 0 \} \leq \max \{ f(s) - f(t) - c_1, 0 \} + \max \{ g(s) - g(t) - c_2, 0 \}.
\]
By equation (2.10) we have similar property for DTV\(^c\). Finally, to obtain equation (2.16) it is enough to utilize equation (2.8). □

3. Proof of Theorem 5

The proof structure reflects the outline contained in Remark 10. We start with

3.1. Proof for Wiener process with drift. In our proof we will use an Anscombe-like result. It is not much more than a reformulation of [13, Theorem 4.5.5] to our specific needs. From now on we will use “\( \lesssim \)" to denote the situation when an equality or inequality holds with some constant which is irrelevant for calculations. Our setting is as follows. Let us fix some \( T > 0 \) and
\[
(D_i(c), Z_i(c)), \quad i \geq 1,
\]
be sequences of i.i.d. random vectors indexed by certain parameter \( c \in (0, 1] \). We define
\[
M_c(t) := \min \left\{ i \geq 0 : \sum_{i=1}^{i+1} D_i(c) > t \right\},
\]
(3.1)
\[
P_c(t) := \left( \sum_{i=1}^{M_c(t)} Z_i(c) \right) - \frac{\mathbb{E}Z_1(c)}{\mathbb{E}D_1(c)} t, \quad t \in [0; T].
\]
(3.2)
Let us observe that such defined \( M_c, P_c \) are càdlàg processes. We will use the following assumptions

(A1): For any \( c > 0 \) we have \( D_1(c) > 0 \) a.s. and \( \mathbb{E}D_1(c) \to 0 \) as \( c \searrow 0 \).

(A2): We denote \( X_i(c) := Z_i(c) - (\mathbb{E}Z_1(c)/\mathbb{E}D_1(c))D_i(c) \). We have \( \mathbb{E}X_i(c) = 0 \). We assume that there exists \( \sigma > 0 \) such that
\[
\frac{\mathbb{E}X_1(c)^2}{\mathbb{E}D_1(c)} \to \sigma^2, \quad \text{as } \ c \searrow 0.
\]

(A3): There exists \( \delta \in (0, 2] \) such that
\[
\frac{\mathbb{E}|X_1(c)|^{2+\delta}}{\mathbb{E}D_1(c)} \to 0, \quad \text{as } \ c \searrow 0.
\]
(A4): There exists $\delta > 0, C > 0$ such that for any $c \in (0; 1]$ we have
$$\mathbb{E}|D_1(c)|^{1+\delta} \leq C(\mathbb{E}D_1(c))^{1+\delta}.$$ 

Before formulation of the fact we define
$$D := D([0; T], \mathbb{R}) := \{ f : [0; T] \mapsto \mathbb{R} : f \text{ is c.i.d. c.o.} \},$$
we equip this space with $\| \cdot \|_{\infty}$-norm. This may seem unusual, as the Skorohod metric (see [1, Chapter 3]) is a more natural choice for space $D$. Let us note however that in all cases we will obtain the convergence to a continuous limits. In such case both notions are equivalent (see [1, Section 18]).

**Fact 18.** Let $T > 0$ and assume that (A1)-(A4) hold. Then
$$P_c \to^d \sigma B, \quad \text{as } c \searrow 0,$$
where $\sigma^2$ is the same as in (A2), $id(x) = x$, and the convergence is understood as weak convergence in $D([0; T], \mathbb{R})$.

**Proof.** We define
$$S_c(n) := \sum_{i=1}^{n} Z_i(c), \quad V_c(n) := \sum_{i=1}^{n} D_i(c), \quad n \in \mathbb{N}.$$ 
Moreover, let us denote $f(c) := \frac{\mathbb{E}Z_1(c)}{\mathbb{E}D_1(c)}$ and we recall that $X_i(c) := Z_i(c) - f(c)D_i(c)$. We define a family of auxiliary processes
$$P_c^1(t) := H_c(|g(c)\epsilon|), \quad t \geq 0,$$
where $H_c(n) := S_c(n) - f(c)V_c(n)$ and $g(c) := (\mathbb{E}D_1(c))^{-1}$. By (A1) $g(c) \to +\infty$ as $c \searrow 0$.

Now the proof follows by [13, Theorem 4.5.5, p. 290]. The assumptions of [13, Theorem 4.5.5] consist of seven conditions denoted by $T_1, S_4, S_5, S_7, S_8, S_9$ and $J_20$. These conditions read as:

- $(T_4)$: $(\kappa_{\epsilon, k}, \xi_{\epsilon, k}), k = 1, 2, ..., is a sequence of i.i.d. random vectors that take values in $[0; +\infty) \times \mathbb{R}$;
- $(S_4)$: $n_8 \mathbb{P}(\kappa_{\epsilon, k} > u) \to \pi_1(u)$ as $\epsilon \to 0$ for all $u > 0$, which are points of continuity of the limiting function $\pi_1(u)$;
- $(S_5)$: $n_8 \mathbb{E}K_{\epsilon, k}1_{\{\kappa_{\epsilon, k} \leq u\}} \to c(u)$ as $\epsilon \to 0$ for some $u > 0$, which is a point of continuity of $\pi_1(u)$;
- $(S_7)$: $n_8 \mathbb{P}(\xi_{\epsilon, k} > u) \to 0$ as $\epsilon \to 0$ for every $u > 0$;
- $(S_8)$: $n_8 \mathbb{E}|\xi_{\epsilon, k}|1_{\{\kappa_{\epsilon, k} \leq u\}} \to a$ as $\epsilon \to 0$ for some $u > 0$;
- $(S_9)$: $n_8 \mathbb{E}^2|\xi_{\epsilon, k}|1_{\{\kappa_{\epsilon, k} \leq u\}} \to b^2$ as $\epsilon \to 0$ for some $u > 0$;
- $(J_20)$: $c = c(u) - \int_0^u sd\pi_1(s) > 0$, where $\pi_1(s)$ and $c(u)$ are obtained in $(S_4)$ and $(S_5)$ respectively.
Before verifying assumptions we list how our notation translates to the one of \cite{13}. \(c\) is \(e\), \([g(c)]\) is \(n_t\), \(D_i(c)\) is \(\kappa_{e,i}\) and \(X_i(t)\) is \(\xi_{e,i}\). Condition \(T_4\) (p. 287) is obviously fulfilled. Conditions \(S_4\) and \(S_5\) (p. 283) hold with \(\pi(u) = 0\) and \(c(u) = 1\) respectively. Indeed, let us fix \(u > 0\). \(S_4\) writes as

\[
[g(c)]P(D_1(c) > u) \leq [g(c)]u^{-1-\delta}E[D_1(c)]^{1+\delta} \lesssim [g(c)]g(c)^{-(1+\delta)} \rightarrow 0, \quad \text{as } c \searrow 0,
\]

where we used assumptions (A1), (A4) and the Chebyshev inequality. We will use a few times an obvious inequality

\[
(3.7) \quad \quad \quad |x|^{\delta_1+\delta_2} \geq |u|^{\delta_1}|x|^{\delta_2},
\]

valid for any \(\delta_1, \delta_2 > 0\) and \(|x| \geq |u|\). We check that

\[
(3.8) \quad \quad \quad [g(c)]E[D_1(c)1_{\{D_1(c)>u\}}] \leq [g(c)]u^{-\delta}E[D_1(c)]^{1+\delta} \rightarrow 0, \quad \text{as } c \searrow 0,
\]

again by (A1), (A4) and equation (3.7). The expression in condition \(S_5\) writes in our notation as \([g(c)]E[D_1(c)1_{\{D_1(c)\leq u\}}].\) By equation (3.8) its limit does not depend on \(u\) and is the same as the one of

\[
[g(c)]E[D_1(c) \rightarrow 1, \quad \text{as } c \searrow 0,
\]

which follows by (A1) and the definition of \(g(c)\).

We will now verify conditions \(S_7, S_8, S_9\) (p. 287-288) with \(a = 0\) and \(b^2 = \sigma^2\). Let \(u > 0\), the condition \(S_7\) writes as

\[
[g(c)]P(|X_1(c)| \geq u) \leq [g(c)]u^{-(2+\delta)}E[X_1(c)]^{2+\delta} \rightarrow 0, \quad \text{as } c \searrow 0,
\]

where we used assumption (A3) and the Chebyshev inequality. Further we have

\[
[g(c)]E(|X_1(c)|1_{\{|X_1(c)|>u\}}) \leq [g(c)]u^{-(1+\delta)}E[X_1(c)]^{2+\delta} \rightarrow 0, \quad \text{as } c \searrow 0,
\]

where we used assumption (A3) and equation (3.7). Now \(S_8\) follows directly from above and the equality

\[
E(X_1(c)1_{\{|X_1(c)|>u\}}) = -E(X_1(c)1_{\{|X_1(c)|\leq u\}})
\]

which is a consequence of the fact that \(E X_i(c) = 0\). Let us now observe that

\[
[g(c)]E(X_1(c)^21_{\{|X_1(c)|>u\}}) \leq [g(c)]u^{-\delta}E[X_1(c)]^{2+\delta} \rightarrow 0, \quad \text{as } c \searrow 0,
\]

where we again used assumption (A3) and equation (3.7). By the above considerations we have that \(\lim_{c \searrow 0}[g(c)]\text{Var}(X_1(c)1_{\{|X_1(c)|\leq u\}})\) is the same as \(\lim_{c \searrow 0}[g(c)]\text{Var}(X_1(c))\). Now \(S_9\) follows directly from (A2). Finally, \(J_{20}\) (p.285) holds with \(c = 1\) see also \[13\] (4.5.2).

Now, it is straightforward to identify the limit using the description in \[13\] p. 284 and p. 288. Indeed, the process \(\kappa_0\) (p. 284) is simply given by \(\kappa_0(t) = t\) (notice that on the right hand side of formula (4.5.1) in \[13\] p. 284 one should replace \(z\) by \(y\)) so its inverse \(\nu_0\) is also \(\nu_0(t) = t\) (which proves equation (3.1)). The process \(\xi_0\) is the same as in \(A_{65}\) (p. 288). Let us note that Silvětrov’s \(\rightarrow U\) is the same convergence we need, see \[13\] Definition 2.4.2.

\[
\square
\]

Let \(W\) be a standard Wiener process and \(\mu \in \mathbb{R}\). We denote a Wiener process with drift \(\mu\) by

\[
(3.9) \quad \quad \quad X_t := W_t + \mu t, \quad t \geq 0.
\]
Our first result is the following

**Lemma 19.** Let $T > 0$ and $X$ be a Wiener process with drift given by (3.3). We have

\[
(3.10) \quad \left( X_t - \mu t, TV^c(X, t) - \frac{t}{c} \right) \to^d \left( W_t, 3^{-1/2} B_t \right), \quad \text{as } c \searrow 0,
\]

where $(W, B)$ are independent standard Wiener processes. The convergence is understood as weak convergence in $\mathcal{C}([0; T], \mathbb{R})^2$ topology.

**Proof.** We fix $a, b \in \mathbb{R}$ and define $A^c_t := aTV^c(X, t) + bX_t - \left( \frac{a}{c} + b \mu \right) t$. Assume that we proved that

\[
(3.11) \quad A^c \to^d \left( a^2/3 + b^2 \right) \tilde{B}, \quad \text{as } c \searrow 0,
\]

weakly in topology of $\mathcal{C}([0; T], \mathbb{R})$, where $\tilde{B}$ is some standard Brownian motion. The convergence for $(a, b) = (1, 0)$ yields that $\{TV^c(X, t) - \frac{t}{c}\}_{c>0}$ is tight, hence also is the sequence of vectors on the left side of equation (3.10). Now, applying the Cramèr-Wold device [1] Theorem 7.7] we easily justify that the convergence of finite-dimensional distributions, hence (3.10) indeed holds.

Now we are to prove (3.11). We transparently transfer all quantities of Section 2 to the stochastic setting by applying them in a pathwise fashion, i.e. $f(t) = X_t$. We denote

\[G_i(c) := (M_i^c - m_i^c - c) + (M_i^c - m_{i+1}^c - c), \quad H_i(c) := (M_i^c - m_i^c - c) - (M_i^c - m_{i+1}^c - c).\]

By Theorem [16] and continuity of $X$ we have $TV^c(X, T_{U,k}) = \sum_{i=0}^{k-1} Y_i(c)$ (in fact this holds under additional assumption (2.5) but this is irrelevant in the limit). By (2.3) and again by Theorem [16] we have

\[
(3.12) \quad \|X_{T_{U,k}} - \sum_{i=0}^{k-1} H_i(c)\|_{\infty} \leq c, \quad \text{a.s.}
\]

(Note that $X_0 = 0$.) We fix some $a, b \in \mathbb{R}$ and for any $i \geq 0$ write

\[
(3.13) \quad Z_i(c) := aG_i(c) + bH_i(c).
\]

We denote also

\[
(3.14) \quad D_i(c) := T_{U,i} - T_{U,i-1}, \quad i \geq 1, \text{ and } D_0(c) := T_{U,0}.
\]

The following simple observation will be crucial for the further proof. Let us notice that by the strong Markov property of $X$ and its space homogeneity we have that $\{Z_i(c)\}_{i \geq 1}$ and $\{D_i(c)\}_{i \geq 1}$ are i.i.d. sequences. For $i = 0$ the distributions are different because of “starting conditions”. The first part, i.e. the values for $i = 0$ disappear in the limit. For notational simplicity from now on, we will implicitly assume that $i \geq 1$.

We will proceed now in the direction of utilizing Fact [18]. To do this, we need to calculate moments, fortunately enough [14] provides us with sufficient tools. Using the notation from [14] we may write

\[
(T_{D,i}^c - T_{U,i}^c, M_i^c - m_i^c - c) =^d (T_c, X(T_c) + c),
\]
where \( T_c, X \) are defined in \cite{14} Introduction. Hence the formula \cite{14} (1.1) reads as

\[
\mathbb{E}\exp(\alpha (M_i^c - m_i^c - c) - \beta (T_{D,i}^c - T_{U,i}^c)) = \frac{\delta \exp(-\alpha \mu c) \exp(\alpha c)}{\delta \cosh(\delta c) - (\alpha + \mu) \sinh(\delta c)},
\]

where \( \delta = \sqrt{\mu^2 + 2\beta} \). This formula is valid if \( \alpha < \delta \coth(\delta c) - \mu \) and \( \beta > 0 \). If \( \mu \neq 0 \) we may also put \( \beta = 0 \). One may check that the pair

\[
(T_{U,i+1}^c - T_{D,i}^c, M_i^c - m_i^c - c)
\]

is independent of \((T_{D,i}^c - T_{U,i}^c, M_i^c - m_i^c - c)\). It becomes obvious when one recalls definitions of Section 2 ((2.6) and (2.7) in particular) and apply the strong Markov property of \( X \). Moreover, we notice that the law of \((T_{U,i+1}^c - T_{D,i}^c, M_i^c - m_i^c - c)\) is the same as the one of \((T_{D,i} - T_{U,i}^c, M_i^c - m_i^c - c)\) if we change the drift coefficient to \(-\mu\). Therefore, by \cite{14} (1.1) we get

\[
\mathbb{E}\exp(\alpha (M_i^c - m_i^c + 1 - c) - \beta (T_{U,i+1}^c - T_{D,i}^c)) = \frac{\delta \exp(-\alpha \mu c) \exp(\alpha c)}{\delta \cosh(\delta c) - (\alpha + \mu) \sinh(\delta c)},
\]

where \( \delta = \sqrt{\mu^2 + 2\beta} \) (with the same restrictions as before). These are enough information to check the moment conditions required in Fact 18. Calculations are easy and straightforward however lengthy. We decided not to include all of them in the paper. Instead, we list crucial steps and provide the reader with the Mathematica notebook with all details. Combining the above equations and putting \( \alpha = 0 \) (note that this is always possible for \( c \)'s small enough) we get

\[
\mathbb{E}\exp(-\beta D_i(c)) = \frac{2\beta + \mu^2}{\beta + \mu^2 + \beta \cosh\left(2c\sqrt{2\beta + \mu^2}\right)}.
\]

Differentiation yields

\[
\mathbb{E}D_i(c) = \frac{2 \sinh(c\mu)^2}{\mu^2} = 2c^2 + O(c^4).
\]

One can check that the formula above is valid for \( \mu = 0 \) when we take the limit. This applies also to the subsequent moments formulae. Moreover

\[
\mathbb{E}D_i(c)^2 = \frac{16}{3} c^4 + O(c^6), \quad \mathbb{E}D_i(c)^4 = \frac{7936}{105} c^8 + O(c^{10}).
\]

This is enough to check conditions (A1) of Fact 18 as well as (A4) with \( \delta = 3 \). Analogously, by putting \( \beta = 0 \) we calculate that

\[
\mathbb{E}\exp(\alpha Z_i(c)) = \frac{4\mu^2}{((a - b)(1 - e^{-2c\mu}) - \alpha - 2\mu)((a + b)(1 - e^{2c\mu}) + \alpha + 2\mu)}.
\]

Again, by differentiation one gets

\[
\mathbb{E}Z_i(c) = \frac{2 \sinh(c\mu)(a \cosh(c\mu) + b \sinh(c\mu))}{\mu}.
\]

\[http://www.mimuw.edu.pl/~pmilos/moments.nb\]. The file can be viewed with a free application available on \[http://www.wolfram.com/products/player/\].
And therefore
\[(3.20) \quad \frac{\mathbb{E}Z_1(c)}{\mathbb{E}D_1(c)} = \mu(b + a \coth(c\mu)) = \frac{a}{c} + b\mu + O(c).\]

Now we have
\[\mathbb{E}\exp(\alpha Z_1(c) - \beta D_1(c)) = \frac{2(2\beta + \mu^2)}{-a^2\alpha^2 + b^2\alpha^2 + 2ab\alpha + 2(\beta + \mu^2) + (a^2\alpha^2 + 2\beta - bab(\alpha + 2\mu)) \cosh(2\alpha\sqrt{2\beta + \mu^2}) - 2a\sqrt{2\beta + \mu^2} \sinh(2\alpha\sqrt{2\beta + \mu^2})}.\]

Following axiom (A2) we denote \(X_i(c) := Z_i(c) - (\mathbb{E}Z_1(c)/\mathbb{E}D_1(c))D_1(c)\).

Using this one may check that
\[\mathbb{E}X_i(c)^2 = \frac{3a^2 - b^2 - 4abc\mu + (a^2 + b^2) \cosh(2\mu) - 4a^2\mu \coth(c\mu) + 2ab \sinh(2\mu)}{\mu^2}.\]

Now it is straightforward to check (A2) of Fact 18, viz.
\[\frac{\mathbb{E}X_i(c)^2}{\mathbb{E}D_1} = \frac{1}{2} \cosh(\mu)^2 \left(3a^2 - b^2 - 4abc\mu + (a^2 + b^2) \cosh(2\mu) - 4a^2\mu \coth(c\mu) + 2ab \sinh(2\mu)\right).\]

Finally, one can check that \(\mathbb{E}X_i(c)^4 \leq c^4\) and hence (A3) is verified with \(\delta = 2\). Having checked all conditions we conclude that for \(P_c(t)\) defined by (3.2) and (3.13), (3.14) we have
\[P_c(t) - \left(\frac{a}{c} + b\mu\right) t \to^d \left(\frac{a^2}{3} + b^2\right)^{1/2} \tilde{B}, \quad \text{as } c \searrow 0.\]

Therefore in order to prove (3.11) it is enough to show that \(P_c(t) - a\text{TV}^c(X, t) - bX_t \to^d 0\). By the property equation (3.12) and the continuity of \(X\) it follows easily that it suffices to concentrate on the case \((a, b) = (1, 0)\), that is \(A_t = \text{TV}^c(X, t)\). Since \(D_0(c)\) has different distribution than \(D_i(c)\) for \(i \geq 1\) we introduce two auxiliary objects
\[\tilde{M}_e(t) := \min\left\{n \geq 0 : \sum_{i=0}^n D_i(c) > t\right\}, \quad \tilde{S}_c(n) = \sum_{i=0}^n Z_i(c),\]

and
\[\tilde{P}_c(t) := \tilde{S}_c(\tilde{M}_e(t)).\]

This differs slightly from \(P_c\), however, one easily checks that \(\tilde{P}_c - P_c \to^d 0\). By Theorem 16 we see that the processes \(\text{TV}^c(X, t)\) and \(\tilde{S}_c(\tilde{M}_e(t))\) coincide at random times \(T^c_{i, j}, i \geq 1\) moreover, both are increasing, hence, for any \(T \geq 0\) and \(\varepsilon > 0\)
\[\mathbb{P}\left(\sup_{t \in [0, T]} |\text{TV}^c(X, t) - \tilde{S}_c(\tilde{M}_e(t))| > \varepsilon\right) \leq \mathbb{P}\left(\sup_{t \in [0, T]} Z_{\tilde{M}_e(t)}(c) > \varepsilon\right).\]
Using this we estimate
\[
\mathbb{P}\left( \sup_{t \in [0, T]} \left| TV^\mu_c (t) - \hat{S}_c \left( \tilde{M}_c (t) \right) \right| > \varepsilon \right) \\
\leq \mathbb{P}\left( \max_{k \leq 2T / E D_1 (c) + 1} Z_k (c) \geq \varepsilon \right) + \mathbb{P}\left( \tilde{M}_c (T) \geq \frac{2T}{E D_1 (c)} + 1 \right).
\]
The first term could be estimated by the Chebyshev inequality and the estimates of \( E Z_1 (c)^4 \) and \( E D_1 (c) \)
\[
\mathbb{P}\left( \max_{k \leq 2T / E D_1 (c) + 1} |Z_i (c)| > \varepsilon \right) \leq \left( \frac{2T}{E D_1 (c)} + 1 \right) \frac{E Z_1 (c)^4}{\varepsilon^4} \to 0, \quad \text{as } c \to 0.
\]
The convergence of the second term to 0 could be established by Fact [18].

3.2. **Proof for diffusions with \( \sigma = \text{const} \).** We start with a yet simpler case. Namely, let \( W \) be a standard Brownian motion and \( X \) be a random variable. Let us define process \( Z \) by
\[
Z_t := W_t + X_t, \quad t \geq 0.
\]
**Lemma 20.** Let \( T > 0 \). Let us assume that that \( W \) and \( X \) are independent then
\[
\left( X, W, TV^c(Z, t) - \frac{t}{c}, Z_t \right) \rightarrow^d \left( X, W, 3^{-1/2} B \right), \quad \text{as } c \searrow 0.
\]
where \( B \) is a standard Brownian motion and \( X, W, B \) are independent. The convergence is understood in weak sense in the product topology of \( \mathbb{R} \times C([0, T], \mathbb{R})^2 \).

**Proof.** We will proceed by the very definition of the weak convergence. Let \( f : \mathbb{R} \times C([0, T], \mathbb{R})^2 \to \mathbb{R} \) be a bounded continuous function. We have
\[
\lim_{c \searrow 0} \mathbb{E} f \left( X, W, TV^c(Z, t) - \frac{t}{c}, Z_t \right) = \lim_{c \searrow 0} \mathbb{E} \mathbb{E} \left( f \left( x, W, TV^c(Z, t) - \frac{t}{c} \right) \bigg| X = x \right)
= \mathbb{E} \mathbb{E} \left( f \left( x, W, TV^c(Z, t) - \frac{t}{c} \right) \bigg| X = x \right)
= \mathbb{E} \left( f \left( x, W, 3^{-1/2} B \right) \bigg| X = x \right) = \mathbb{E} f \left( X, W, 3^{-1/2} B \right).
\]
where we used Lemma [19] and the Lebesgue dominated convergence theorem.

We will deal now with diffusion given by an equation
\[
(3.21) \quad dX_t = dW_t + \mu(X_t)dt, \quad X_0 = 0,
\]
i.e. we set \( \sigma \equiv 1 \) in equation (1.4). We assume also that \( \mu \) is bounded and Lipschitz. This process is essentially a Brownian motion with “a variable drift”. We denote
\[
(3.22) \quad \mu^* = \sup_{x \in \mathbb{R}} |\mu(x)| < +\infty.
\]
We will us the discretion technique. To this end we need to be able to control the increments of \( X \). The following simple lemma is the first, most crude step of our analysis.
Lemma 21. Let \( t \geq 0 \) and \( \delta > 0 \) then for any \( b > 0 \) we have

\[
P \left( \sup_{s \in [t, t+\delta]} |X_s - X_t| \geq (\mu^* + b)\delta \right) \leq 2 \exp \left( -b^2\delta/2 \right).
\]

Proof. We know that

\[ X_t = X_0 + W_t + \int_0^t \mu(X_s)ds. \]

Hence, we have \( X_s - X_t \in (W_s - W_t - \mu^*(s-t), W_s - W_t + \mu^*(s-t)) \). Now the lemma follows by [11, Proposition II.1.8]. \( \square \)

Let us fix \( T > 0 \), \( n = 1, 2, \ldots \) and denote \( t^n_i := i\frac{T}{n}, i \in \{0, 1, \ldots, n\} \). We define the “approximated” truncated variation process by

\[
ATV^{n,c}(t) := \sum_{i=0}^{[nt]-1} TV^c(X, [t^n_i; t^n_{i+1}]) + TV^c(X, [t^n_{[nt]}; t]),
\]

Its name is justified by

Lemma 22. We have

\[
ATV^{n,c}(t) - TV^c(X, t) \to 0, \text{ a.s. when } c \searrow 0,
\]

and the convergence is understood in \( C([0; T], \mathbb{R}) \) topology.

Proof. By (2.11) one easily verifies that \( ATV^{n,c}(t) \leq TV^c(X, t) \). On the other hand, by (2.12),

\[
TV^c(X, t) - ATV^{n,c}(t) \leq nc, \text{ for any } t \in [0; T].
\]

We will take now a detour of the main flow of the proof in order to collect weak convergence facts used below. First we recall the Prokhorov metric. Let \( (S, d) \) be a metric space and \( \mathcal{P}(S) \) be the space of Borel probability measures on \( S \). We topologise \( \mathcal{P}(S) \) with the Prokhorov metric

\[
d_P(P, Q) := \inf \{ \epsilon > 0 : P(F) \leq Q(F^\epsilon) + \epsilon, \text{ for all closed } F \subset S \},
\]

in the above expression \( F^\epsilon := \{ x \in S : \inf_{y \in F} d(x, y) < \epsilon \} \). It is well-known that when \( (S, d) \) is separable then convergence with respect to \( d_P(\cdot, \cdot) \) is equivalent to weak convergence. We refer the reader to [2, Chapter 3] and [2, Theorem 3.3.1] in particular. Given two random variables \( X, Y \) with values in the same space we will write

\[
d_P(X, Y) := d_P(\mathcal{L}(X), \mathcal{L}(Y)),
\]

where \( \mathcal{L}(X) \) denotes the law of \( X \).

In some parts of our analysis we will need the space of càdlàg functions \( D([0; T], \mathbb{R}) \) introduced by [3,3]. We will also use the following product space

\[
C \times D := C([0; T], \mathbb{R}) \times D([0; T], \mathbb{R}),
\]

always with the norm given by \( \|(f, g)\| := \|f\|_\infty + \|g\|_\infty \).
Lemma 23. Let \((X, Y)\) be random variables with values in \(C \times D\), moreover let \(A\) be an event. Then
\[
d_P ((X, Y), (X_{1A}, Y)) \leq 2(1 - \mathbb{P}(A)).
\]
Proof. It is enough to apply [2 Theorem 3.1.2] with \(\mu = \mathcal{L}((X, Y), (X_{1A}, Y))\).

Lemma 24. Let \(X := (X_1, X_2)\) and \(Y := (Y_1, Y_2)\) be a random variable with values in \(C \times D\) such that
\[
\mathbb{P} (\|X_1 - Y_1\|_\infty \geq \epsilon/2) \leq \epsilon/2 \quad \text{and} \quad \mathbb{P} (\|X_2 - Y_2\|_\infty \geq \epsilon/2) \leq \epsilon/2,
\]
then
\[
d_P (X, Y) \leq \epsilon.
\]
Proof. We calculate
\[
\mathbb{P} (\|(X_1 - Y_1, X_2 - Y_2)\| \geq \epsilon) \leq \mathbb{P} (\|X_1 - Y_1\| \geq \epsilon/2) + \mathbb{P} (\|X_2 - Y_2\| \geq \epsilon) \leq \epsilon,
\]
now the proof follows directly by application of [2 Theorem 3.1.2].

We are ready to prove the main result of this part of the proof which is an upgrade of Lemma 19 to “simplified diffusions” given by equation (3.21).

Fact 25. Let \(T > 0\). We have
\[
(3.26) \quad \left(t, TV_c(X, t) - \frac{t}{c}\right) \xrightarrow{d} (X, 3^{-1/2}B), \quad \text{as } c \searrow 0,
\]
where the convergence is understood as weak convergence in \(C([0; T], \mathbb{R}^d)\) topology and \(B\) is a Brownian motion independent of \(X\).

Proof. We recall that \(t^n_i := \frac{t}{n}T\), fix some \(A \geq \mu^* + 1\) and define random sets
\[
A^n_i := [X^n_t - A/n^{1/4}; X^n_t + A/n^{1/4}].
\]
We also define random variables
\[
\mu^n_i := \mu(X^n_i).
\]
and events
\[
E_i^n := \{X_s \in A^n_i, \text{ for } s \in [t^n_i; t^n_{i+1}]\}, \quad E^n := \bigcap_{i \in \{0, 1, \ldots, n-1\}} E_i^n.
\]
Using Lemma 21 we check that for \(n\) large enough we have \(\mathbb{P}(E^n_i) \geq 1 - 2n \exp(-n^{1/2}/2)\). Consequently, \(\mathbb{P}(E^n) \to 1\) as \(n \to +\infty\). For \(n \in \mathbb{N}\) we define c.i.i.d. processes \(\{X^n_t\}_{t \in [0; T]}\) which approximate our diffusion:
\[
X^n_t := X^n_{t^n_i} + \mu^n_i(t - t^n_i) + W_t - W_{t^n_i}, \quad \text{whenever } t \in [t^n_i; t^n_{i+1}).
\]
One easily checks that \(X^n \to X\) a.s. with respect to \(\|\cdot\|_\infty\). Let us recall (3.23), we define its counterpart for \(X^n\), viz.,
\[
H^{n,c}(t) := \sum_{i=0}^{[nt]-1} TV^c\left(X^n, [t^n_i; t^n_{i+1}]\right) + TV^c\left(X^n, [t^n_{[nt]}; t]\right).
\]
One checks (using the same method as in the proof of Lemma 22) that
\[
H^{n,c}(t) - TV^c(X^n, t) \to 0, \quad \text{a.s. when } c \searrow 0,
\]
norm $\| \cdot \|_\infty$. On each interval $t \in [t_i^n, t_{i+1}^n)$ we have

$$X_i^n - X_t = \int_{t_i^n}^t (\mu_i^n - \mu(X_s)) ds.$$  

(3.27)

We observe that conditionally on $E_i^n$ this expression defines a function of $t$ which is Lipschitz with constant $w_n \leq L n^{-1/4}$ for some $L > 0$. This follows by the fact that $\mu$ is a Lipschitz function itself. By (2.13) applied with $c_1 = c$ and $c_2 = 0$, conditionally on $E_i^n$, we have that

$$TV^c(X^n, [t_i^n, t]) - w_n(t - t_i^n) \leq TV^c(X, [t_i^n, t]) \leq TV^c(X^n, [t_i^n, t]) + w_n(t - t_i^n),$$

for any $t \in [t_i^n, t_{i+1}^n]$. Further,

$$1_{E_i^n} H^{n, c}(t) - w_n T \leq 1_{E_i^n} ATV^{n, c}(t) \leq 1_{E_i^n} H^{n, c}(t) + w_n T,$$

(3.28)

for any $t \in [0; T]$. In other words: $\|1_{E_i^n} ATV^{n, c}(t) - 1_{E_i^n} H^{n, c}(t)\|_\infty \leq 2 w_n T$. Lemma 24 implies that

$$d_P ((X^n, 1_{E_i^n} ATV^{n, c}), (X^n, 1_{E_i^n} H^{n, c})) \leq 4 w_n T.$$  

(3.29)

It will be crucial that this estimate is uniform in $c$. Let us denote $L^{n, c} := (H^{n, c}(t) - c/t)$. Lemma 20 applied term by term to $H^{n, c}$ yields the functional convergence

$$L^{n, c}, X^n \to^d (3^{-1/2} B, X^n), \quad \text{as } c \searrow 0,$$

(3.30)

where $B$ and $X^n$ are independent. In the above, we understand the convergence as the functional one in $C \times D$ (see also (3.25)).

The rest of the proof will follow by a metric-theoretic considerations. Let us denote

$$X_1(c) := (TV^c(X, t - t/c, X), \quad X_2(c, n) := (TV^c(X, t - t/c, X^n),$$

$$X_3(c, n) := (ATV^{n, c}(t - t/c, X^n), \quad X_4(c, n) := (1_{E_i^n} ATV^{n, c}(t - t/c), X^n),$$

$$X_5(c, n) := (1_{E_i^n} H^{n, c}(t - t/c), X^n), \quad X_6(c, n) := (H^{n, c}(t - t/c, X^n),$$

$$X_7(n) := \left(3^{-1/2} B, X^n \right), \quad X_8 := \left(3^{-1/2} B, X \right).$$

Let us fix some $\epsilon > 0$. We find $n_{1,2}$ such that for any $n \geq n_{1,2}$ we have $d_P (X_1(c), X_2(c, n)) \leq \epsilon$ which is possible by Lemma 24 and convergence $X^n \to X$. We find $n_{3,4}$ such that for any $n \geq n_{3,4}$ we have $d_P (X_3(c, n), X_4(c, n)) \leq \epsilon$ which is possible by Lemma 23 and estimation of the probability of $E_i^n$. Further we find $n_{4,5}$ such that for any $n \geq n_{4,5}$ we have $d_P (X_4(c, n), X_5(c, n)) \leq \epsilon$ which is given by equation (3.29). Next, we check that for any $n \geq n_{3,4}$ we have $d_P (X_5(c, n), X_6(c, n)) \leq \epsilon$ as well. Finally, we choose $n_{7,8}$ such that for any $n \geq n_{7,8}$ we have $d_P (X_7(n), X_8) \leq \epsilon$ which holds by Lemma 24. We denote $N = \max(n_{1,2}, n_{3,4}, n_{4,5}, n_{7,8})$, obviously for this $N$ all the above inequalities hold simultaneously for any $c > 0$.

Now we choose $c_0$ such that for any $c \leq c_0$ we have $d_P (X_2(c, N), X_3(c, N)) \leq \epsilon$ and $d_P (X_6(c, N), X_7(c, N)) \leq \epsilon$. The first one is possible by Lemma 22 and Lemma 24 and the second one by equation (3.30) and again Lemma 24. Using the triangle inequality multiple times one obtains

$$d_P (X_1(c), X_8) \leq 8 \epsilon, \quad \text{for any } c \leq c_0,$$

This yields convergence equation (3.26) since $\epsilon$ was arbitrary. □
Remark 26. We strongly believe that it is not possible to improve the above proof to general diffusions. The main reason is that without $\sigma = const$ assumption equation equation (3.27) is not longer true. Consequently, the estimate in equation (3.27) does not depend only on $w_n$ but also on $c$. Even worse, one can check that the estimate diverges to infinity as $c \searrow 0$. We could change $n$ and $c$ simultaneously in a smart way so that the estimate is still useful. However a new problem emerges then, namely estimate in Lemma 22 also depend on $n$ and $c$. It appears that it is not possible to change $n$ and $c$ is such way that both estimates converge to 0 when $c \searrow 0$.

3.3. Proof for general diffusion. Now we proceed to the general case. Before proving Theorem 5 we present some measure-theoretic considerations. In the reasoning below by $\mathcal{W}$ we denote the Wiener measure on $C([0; T], \mathbb{R})$, see e.g. [11, Proposition I.3.3], and by $\mathcal{H}$ we denote the Cameron-Martin space, see [11, Definition VIII.2.1]. Moreover by $\mathcal{H}$ we denote algebra (i.e. class closed under finite sums and finite intersections) generated by open balls with centers in $\mathcal{H}$. We have

Lemma 27. Let $h : C([0; T], \mathbb{R}) \mapsto \mathbb{R}_+$ be a measurable mapping such that $\int h(f)\mathcal{W}(df) = 1$. Then for any $\epsilon > 0$ there exists $m \in \mathbb{N}$, sets $A_1, A_2, \ldots , A_m \in \mathcal{H}$ and $h_1, h_2, \ldots , h_m \in \mathbb{R}_+$ such that

$$\int_{\mathcal{C}} |h_\epsilon(f) - h(f)|\mathcal{W}(df) \leq \epsilon, \quad (3.31)$$

where

$$h_\epsilon(f) := \sum_{i=1}^{m} h_i 1_{A_i}(f).$$

Moreover, one may choose such $A_1, A_2, \ldots , A_m$ that for all $i \leq m$, $\mathcal{W}(\partial A_i) = 0$.

Proof. In the proof we will write $\mathcal{C}$ instead of $C([0; T], \mathbb{R})$ and $B(f, r)$ will denote an open ball with convention $B(f, 0) = \emptyset$. Let us notice that without loss of generality we can assume that $h$ is bounded by some $l > 0$ and has compact support, say contained in ball $B(0, R)$. Indeed for any function $h$ and any $\epsilon > 0$ we can choose $l, R$ such that $\int_{\mathcal{C}} |h(f)1_{\{h \leq l\}}1_{\{f \in B(0, R)\}} - f(f)|\mathcal{W}(df) \leq \epsilon/2$. Now it is enough to approximate $h(f)1_{\{h \leq l\}}1_{\{f \in B(0, R)\}}$ with accuracy $\epsilon/2$. Therefore from now on we will work implicitly with the assumptions listed above.

Let us denote

$$S_k := \{f \in \mathcal{C} : h(f) \in (k\epsilon/2, (k+1)\epsilon/2]\},$$

for $k \in \{0, 1, \ldots , 2l/\epsilon\}$. We note that by our assumption sets $S_k$ are bounded. We put $\delta := \epsilon^2/(4l^2)$. By the regularity of $\mathcal{W}$ (see [11, Theorem 1.1.1]) we can find open sets $O_k$ such that

$$S_k \subset O_k \quad \text{and} \quad \mathcal{W}(O_k \setminus S_k) \leq \delta/2. \quad (3.32)$$

It is well known that $\mathcal{C}$ is a separable space and $\mathcal{H}$ is its dense subspace so one can easily find a countable subset $\{f_1, f_2, \ldots \} \subset \mathcal{H}$ which is dense in $\mathcal{C}$. For each $f_i$ we define $r_i^k := \sup \{r : B(f_i, r) \subset O_k\}/2$ (by convention we put $r_i^k := 0$ if the set is empty). One promptly
proves that $O_k = \bigcup_i B(f_i, r^k_i)$. By the continuity of measure there exists $i_k \in \mathbb{N}$ such that
\[ \mathbb{W}(O_k) - \mathbb{W}(\bigcup_{i \leq i_k} B(f_i, r^k_i)) \leq \delta/2. \]

Let us denote $A_k := \bigcup_{i \leq i_k} B(f_i, r^k_i)$. We now define
\[ h(\delta) := \sum_{k} \frac{k^2}{\delta} 1_{A_k}(f). \]

We will now show that $h(\delta)$ is a good approximating function. We recall that by the construction $A_k \subset O_k$ and $\mathbb{W}(O_k \setminus A_k) \leq \delta/2$. This together with equation (3.32) yields that $\mathbb{W}(S_k \Delta A_k) \leq \delta$.

To check $\mathbb{W}(\partial A_k) = 0$ is is enough to prove that for any $f \in H$ and any $r > 0$ we have $\mathbb{W}(\partial B(f, r)) = 0$. By [11, Theorem VIII.2.2] it is enough to show that $\mathbb{W}(\partial B(0, r)) = 0$. This holds by the fact that the sup of the Wiener process has a continuous density (see [11, Section III.3]).

Finally we present

**Proof.** (of Theorem 5). We first will show that in order to prove equation (1.5) it is enough to prove
\[ (X, TV^c(X, t) - \frac{\langle X_t \rangle}{c}) \rightarrow_d (X, 2M), \text{ as } c \searrow 0, \]

where $M$ is the same as in Theorem 5. Since $X_0 = 0$ by equation (2.3) there exists process \{$R_c(t)$\}$_{t \in [0; T]}$ such that $\|R_c\|_\infty \leq c$ almost surely and
\[ X_t = UTV^c(X, t) - DTV^c(X, t) + R_c(t). \]

This together with equation (2.8) yields that
\[ UTV^c(X, t) = \frac{1}{2} (TV^c(X, t) + X_t - R_c(t)), \]

Therefore
\[ UTV^c(X, t) - \frac{1}{2} \left( \frac{\langle X_t \rangle}{c} + X_t \right) = \frac{1}{2} \left( TV^c(X, t) - \frac{\langle X_t \rangle}{c} \right) - \frac{1}{2} R_c(t). \]

Now the convergence follows simply by fact that $TV^c(X, t) - \langle X_t \rangle/c$ is a continuous transformation of equation (3.33) and by [1 Corollary 2, p.31], [11 Theorem 4.1].
argument proves the convergence of $\mathrm{DTV}^c(X, t) - \frac{1}{2} \langle (X_t)/c - X_t \rangle$. The joint convergence in equation (1.5) can be established in the same way.

It will be more convenient to work with an additional assumption that

$$C_1 \geq \sigma \geq C_2 > 0,$$

for some constants $C_1, C_2 > 0$. At the end of the proof we will remove this assumption.

Diffusion equation (1.4) writes in the integral form as

$$X_t = \int_0^t \sigma(X_s) dW_s + \int_0^t \mu(X_s) ds.$$

Let us define $\beta_t := \int_0^t \sigma(X_s)^2 ds = \langle X_t \rangle$, its inverse $\alpha_t := \inf \{ s \geq 0 : \beta_s > t \}$ and $\tilde{X}_t := X_{\alpha_t}$, $t \in [0; T_0]$, where $T_0 := C_2^2 T$.

By the time-change formula [10, Theorem 8.5.7] we obtain that $\tilde{X}$ is also a diffusion fulfilling equation

$$\tilde{X}_t = \tilde{W}_t + \int_0^t \frac{\mu(\tilde{X}_s)}{\sigma^2(\tilde{X}_s)} ds,$$

for some Brownian motion $\tilde{W}$. We chose such $T_0$ that the definition is valid (i.e. $\alpha_{2T} \leq T$). We note also that $x \mapsto \frac{\mu(x)}{\sigma^2(x)}$ is a Lipschitz function. Let us now denote the natural filtration of $\tilde{X}$ (and $\tilde{W}$) by $\mathcal{F}$. Making the reverse change of time we get $X_t = \tilde{X}_{\beta_t}$. We denote also $G_t := \mathcal{F}_{\beta_t}$. Now we can apply Fact 25. We know that

$$(3.37) \quad \left( \mathrm{CTV}^c(\tilde{X}, t), \tilde{X} \right) \rightarrow^d (B, \tilde{X}),$$

where $\mathrm{CTV}^c(X, t) := \mathrm{TV}^c(X, t) - \frac{t}{2}$ and $B$ and $\tilde{X}$ are independent. Let us also note that $\mathrm{CTV}^c$ can be regarded as a measurable mapping $\mathrm{CTV}^c : \mathcal{C}([0; T], \mathbb{R}) \mapsto \mathcal{C}([0; T], \mathbb{R})$.

Now, let $K \in \mathcal{H}$ be non-empty set. We check that the measure $\mathbb{P} \left( \tilde{X} \in \cdot | X \in K \right)$ is absolutely continuous with respect to $\mathbb{P} \left( \tilde{X} \in \cdot \right)$. Indeed one needs only to check that $\mathbb{P} (X \in K) > 0$. By the Radon-Nikodym theorem [4, Theorem A.1.3] there exists a measurable function $h$ such that

$$(3.38) \quad \mathbb{P} \left( \tilde{X} \in d\tilde{f} | X \in K \right) = h(f) \mathbb{P} \left( \tilde{X} \in d\tilde{f} \right).$$

Using this fact we can leverage equation (3.37). Let us first note that by the portmanteau theorem [11, Theorem I.2.1] and [11, Theorem I.2.2] and standard topological considerations we know that equation (3.37) is equivalent to

$$(3.39) \quad \mathbb{P} \left( \left\{ \mathrm{CTV}^c(\tilde{X}) \in K_1 \right\} \cap \left\{ \tilde{X} \in K_2 \right\} \right) \rightarrow \mathbb{P} (B \in K_1) \mathbb{P} (\tilde{X} \in K_2), \quad \forall K_1, K_2 \in \mathcal{H}.$$
we obtain

where

One easily checks that for any

\[ (3.40) \]

Using \[1, \text{Theorem I.2.2}\] in the same spirit as in the case of equation (3.39) we get

Using the standards arguments we obtain that for any \(K\)

We define

where \(P\) is the same as the Wiener measure. We now approximate \(h\) with accuracy \(\epsilon = 1/n\) with simple function \(h^n\) satisfying conditions of Lemma, (27) (we use additional superscript \(n\) to denote the case we are referring to). Hence we have

\[ (3.40) \]

We define

\[ a^n_c := P(X \in K) \int_c h^n(f)1_{CTV^c(f) \in K_1}1_{f \in K_2} P(\tilde{X} \in df). \]

One easily checks that for any \(c > 0\) there is \(|a_c - a^n_c| \leq 1/n\). Applying equation (3.39) we obtain

\[ a^n_c = P(X \in K) \sum_{i=1}^{m^n} h^n_i P \left( \left\{ CTV^c(\tilde{X}) \in K_1 \right\} \cap \left\{ \tilde{X} \in A^n_i \cap K_2 \right\} \right) \]

\[ \to c \searrow 0 P(B \in K_1) P(X \in K) \sum_{i=1}^{m^n} h^n_i P(\tilde{X} \in A^n_i \cap K_2) \]

\[ = P(B \in K_1) P(X \in K) \int_{K_2} h^n(f) P(\tilde{X} \in df) =: a^n. \]

It is easy to check that \(|a^n - a| \leq 1/n\), where

\[ a := P(B \in K_1) P(X \in K) \int_{K_2} h(f) P(\tilde{X} \in df) = P(B \in K_1) P \left( \left\{ \tilde{X} \in K_2 \right\} \cap \left\{ X \in K \right\} \right). \]

Using the standards arguments we obtain that for any \(K_1, K_2, K \in \mathcal{H}\)

\[ P \left( \left\{ CTV^c(\tilde{X}) \in K_1 \right\} \cap \left\{ \tilde{X} \in K_2 \right\} \cap \left\{ X \in K \right\} \right) \]

\[ \to c \searrow 0 P(B \in K_1) P(X \in K) \int_{K_2} h(f) P(\tilde{X} \in df) = P(B \in K_1) P \left( \left\{ \tilde{X} \in K_2 \right\} \cap \left\{ X \in K \right\} \right). \]

Using \(1, \text{Theorem I.2.2}\) in the same spirit as in the case of equation (3.39) we get

\(CTV^c(\tilde{X}), \tilde{X}, X) \to^d (B, \tilde{X}, X), \) as \(c \searrow 0\).

where \(B\) is independent of \((\tilde{X}, X)\) hence also of \(\beta\). Changing the time according to this process we obtain

\[ TV^c(\tilde{X}, \beta_t) = \frac{\beta_t}{c} \to^d B_{\beta_t}, \]
This equation is well-defined as long as \( t \leq T_{00} = T_0/C_1^2 = TC_2^2/C_1^2 \). Our final step is to use (2.9) in order to get
\[
TV^c(X, t) - \langle X \rangle_t \xrightarrow{c} d B_t.
\]
So far we have obtained convergence in the space \( C([0; T_{00}], \mathbb{R}) \). Taking the initial value of \( T \) larger (which is possible as our diffusion is well defined on the whole line) we can obtain the convergence in \( C([0; T], \mathbb{R}) \).

We are yet to remove assumption (3.36). For any \( N > 0 \) we put
\[
\sigma^N(x) := \begin{cases} 
\sigma(x), & \text{if } |x| \leq N, \\
\sigma(N), & \text{if } x > N, \\
\sigma(-N), & \text{if } x < -N,
\end{cases}
\]
and
\[
\mu^N(x) := \begin{cases} 
\mu(x), & \text{if } |x| \leq N, \\
\mu(N), & \text{if } x > N, \\
\mu(-N), & \text{if } x < -N.
\end{cases}
\]
We define a family of diffusions by
\[
dX^N_t := \sigma^N(X^N_t)dW_t + \mu(X^N_t)dt, \quad X^N_0 = 0.
\]
We assume that this diffusion is driven by the same \( W \) as in (1.4) and that \( X, X^N \) are coupled in such a way that \( X^N_t = X_t \) and \( \langle X^N_t \rangle = \langle X_t \rangle \) whenever \( t \leq \tau^N := \inf \{ t \geq 0 : |X^N_t| > N \} = \inf \{ t \geq 0 : |X_t| > N \} \). The solution of (1.4) is a continuous process and exists on the whole line, therefore for any \( T > 0 \) we have
\[
1_{\{\tau^N \leq T\}} \rightarrow_N \rightarrow_{+\infty} 0, \quad \text{a.s.}
\]
We notice now that \( X^N \) fulfills (3.36), hence the thesis of Theorem 5 is already proved for it. The quantities studied in the proof are equal for \( X^N \) and \( X \) on the set \( \{\tau^N \leq T\} \). Using the metric-theoretic arguments as in the proof of Fact 25 one easily concludes the proof. \( \Box \)

4. PROOF OF THEOREM 1

As indicated in Introduction the proof splits into two parts. In the first one we will prove Theorem 1 in the case when \( X \) is a Wiener process with a drift. This will serve as a key step for the second part of the proof in which, using time change techniques we will elevate the result to a general class of semimartingales.

4.1. Proof for Wiener process with drift. This is much simpler compared to the proof of Lemma 19, therefore we provide only a sketch leaving details to the reader. Let \( X \) be a Wiener process with drift, i.e.
\[
X_t := W_t + \mu t,
\]
for a standard Wiener process \( W \) and \( \mu \in \mathbb{R} \). We have

Lemma 28. Let \( T > 0 \) and let \( \{X\}_{t \in [0; T]} \) be a Wiener process with drift. Then
\[
\lim_{c \searrow 0} c TV^c(X, t) \rightarrow \langle X \rangle_t, \quad \text{a.s.}
\]
The converge is understood in the \( C([0; T], \mathbb{R}) \) topology.

Proof. Firstly, we recall \( S_c(n) \) defined in equation (3.5) and \( Z_t(c) \) given by equation (3.13). We want to show that process \( X_t(c) := cS_c([g(c)t]) \) converges to a linear function. Let us
consider
\[ M_n(c) := c \sum_{i=1}^{n} (Z_i(c) - \mathbb{E}Z_i(c)). \]

It is a centered martingale. Differentiation of equation (3.18) yields that (we have \(a = 1, b = 0\) in this case)
\[ \mathbb{E}(Z_i(c) - \mathbb{E}Z_i(c))^2 = \frac{2 \cosh(2c\mu) \sinh(c\mu)^2}{\mu^2} = 2c^2 + O(c^3). \]

Therefore, by the Doob inequality and equation (3.17) we have
\[ \mathbb{E} \left( \sup_{t \in [0; T]} [cS_c([g(c)t]) - c[g(c)t]\mathbb{E}Z_i(c)]^2 \right) \leq LT^2c^2, \]
for some constant \(L\). Using equation (3.19) and equation (3.20) one obtains
\[ X_t(c) \to id, \quad \text{a.s.,} \]
where \(id(t) = t\) and the converge holds in \(C([0; T], \mathbb{R})\) topology. Now one proves an analogous convergence for process \(V_c([g(c)t])\), which is roughly speaking the inverse of \(M_c\). To finish the proof one needs to argue similarly as in the second part of the proof of Lemma 19.

4.2. **Proof for semimartingales.** Now we assume that \(X_t = X_0 + M_t + A_t\), where \(M\) is a continuous local martingale and \(A\) is a process with bounded variation.

To avoid notational inconveniences we assume that \(M, A\) are defined on \([0; +\infty)\) (one can simply put a constant process after \(T\)). Let us now introduce an additional standard Brownian motion \(\beta\) independent of \(X\) and denote
\[ X^\epsilon := X + \epsilon \beta, \quad \epsilon > 0. \]
This is a simple trick to avoid the case when \(\langle X \rangle\) is not strictly increasing. Indeed we have \(\langle X_t^\epsilon \rangle = \langle X_t \rangle + \epsilon t\). Obviously this is a strictly increasing function. Moreover its inverse, denoted by \(\alpha\), is almost surely Lipschitz with constant smaller then \(\epsilon^{-1}\). Let us denote also \(M^\epsilon := M + \epsilon \beta\). The DDS theorem [11, Theorem V.1.6] ensure that there exists a Brownian motion \(B\) such that
\[ M_t^\epsilon := B_{\langle X_t^\epsilon \rangle}, \quad t \in [0; T]. \]

Using (2.9) we have
\[ c \text{TV}^c(X^\epsilon, t) = c \text{TV}^c(B_t + A_{\alpha t}, \langle X^\epsilon \rangle_t). \]

Let us fix \(N > 0\). Applying (2.14) to the paths of \(B_t + A_{\alpha t}\) we get
\[ |\text{TV}^c(B_t + A_{\alpha t}, \langle X_t^\epsilon \rangle \wedge N) - \text{TV}^c(B_t, \langle X_t \rangle \wedge N) | \leq \text{TV}(A_{\alpha t}, \langle X^\epsilon \rangle_t \wedge N). \]

Using Lemma 28 and the above estimate one gets that
\[ c \text{TV}^c(B_t + A_{\alpha t}, \langle X_t^\epsilon \rangle \wedge N) \to_{c \downarrow 0} \langle X_t^\epsilon \rangle \wedge N \quad \text{a.s.} \]
where convergence is understood in $C([0; T], \mathbb{R})$ topology. Moreover, the limit agrees with the limit of equation (4.1) on the set $\{\langle X^\epsilon \rangle \leq N\}$. Hence we obtain
\[ c \text{TV}^c(X^\epsilon, t) \to \langle X^\epsilon \rangle, \quad \text{a.s.} \]

Our aim now is to get rid of $\epsilon$. We fix some $\alpha \in (0, 1)$ and notice that by (2.13) we have
\[ c \text{TV}^c(X, t) \leq c \text{TV}^{\alpha c}(X^\epsilon, t) + c \text{TV}^{(1-\alpha)c}(\epsilon B, t). \]

Therefore we have
\[
\limsup_{c \to 0} c \text{TV}^c(X, t) \leq \alpha^{-1}\langle X^\epsilon_t \rangle + (1 - \alpha)^{-1}\epsilon^2 t = \alpha^{-1}\langle X_t \rangle + ((1 - \alpha)^{-1} + \alpha^{-1})\epsilon^2 t.
\]

By converging $\epsilon \to 0$ and $\alpha \to 1$ one can obtain
\[
\limsup_{c \to 0} c \text{TV}^c(X, t) \leq \langle X_t \rangle, \quad \text{a.s.}
\]

Analogously one obtains a lower-bound for $\liminf$. Therefore we proved that for any $t > 0$ we have
\[
\lim_{c \to 0} c \text{TV}^c(X, t) = \langle X_t \rangle, \quad \text{a.s.}
\]

This is a one dimensional convergence but one easily extends it to the finite dimensional one. Moreover, since the trajectories are almost surely increasing the finite dimensional convergence can be upgraded to the functional one. This follows by the simple fact that if $f_n \in C([0; T], \mathbb{R})$ is a sequence of continuous increasing functions converging point-wise to a continuous function then the convergence is in fact uniform.

In order to prove the convergence for UT$^c$ it suffices to use equation (3.33). DT$^c$ follows similarly.

### 5. Proof of Large Times Results

In this section we will only prove Theorem 12. It follows by a similar argument as in the proof of Lemma 19. This time $c$ is fixed and $n$ will go to infinity. The analogues of equation (3.13) and equation (3.14) are given by
\[
Z_i(n) := n^{-1/2}Y_i(c).
\]

and
\[
D_i(n) := n^{-1}(T_{\tilde{U}_{i-1}} - T_{\tilde{U}_{i-1}} - 1), \quad i \geq 1, \quad \text{and} \quad D_0(n) := n^{-1}T_{\tilde{U},0}.
\]

By equation (3.20) (with $a = n^{1/2}, b = 0$) we have
\[
\frac{\mathbb{E}Z_i(n)}{\mathbb{E}D_i(n)} = n^{-1/2}\mu \coth(c\mu).
\]
We define $X_i(n) := Z_i(n) - \frac{\mathbb{E}Z_i(n)}{\mathbb{E}D_i(n)} D_i(n)$. Repeating calculations as in the proof of Lemma 19 one obtains

$$ \text{Var}(X_i(n)) = \frac{3 + \cosh(2c\mu) - 4c\mu \coth(c\mu)}{n\mu^2}. $$

One checks that $\text{Var}(X_i(n))/\mathbb{E}D_i(n) = (\sigma_c^2)^2$ as in Theorem 12. Now in order to obtain this theorem it is enough to apply Fact 18. This is an easy task. Above we already checked (A1), (A2) and (A3) are trivial. (A4) holds with any $\delta > 0$.

We skip the proof of 11 which is a simpler version of the proof in Section 4.1. Proofs of Theorem 13 follows similarly to the one above with an exception that $Z_i(n) = M_c^n - m_c^n - c$ in the case of UTV$^c$ and $Z_i(n) = M_c^n - m_c^{n+1} - c$ in the case of DTV$^c$.

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