FAILURE OF THE RYLL-NARDZEWSKI THEOREM
ON THE CAR ALGEBRA

VITONOFRIO CRISMALE AND STEFANO ROSSI

Abstract. Spreadability of a sequence of random variables is a
distributional symmetry that is implemented by suitable actions
of $J_Z$, the unital semigroup of strictly increasing maps on $Z$ with
cofinite range. We show that $J_Z$ is left amenable but not right
amenable, although it does admit a right Følner sequence. This
enables us to prove that on the CAR algebra $\text{CAR}(Z)$ there exist
spreadable states that fail to be exchangeable. Moreover, we also
show that on $\text{CAR}(Z)$ there exist stationary states that fail to be
spreadable.

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invariant states, de Finetti's theorem

1. Introduction

In classical probability exchangeable sequences of random variables
are completely understood. First, by virtue of a general version of de
Finetti’s theorem exchangeability is equivalent to conditional indepen-
dence and identical distribution with respect to the tail algebra of the
sequence itself, see e.g. [17]. Second, exchangeability is the same as
spreadability, which is the content of a well-known result due to Ryll-
Nardzewski, [23]. Putting these two statements together, one obtains
what is known as the extended de Finetti theorem, which represents
an accomplished characterization of exchangeability.

The equivalences established in the extended de Finetti theorem, though,
will in general cease to hold in the wider context of non-commutative
probability, where a variety of novel phenomena may occur. For in-
stance, the $W^*$-formalism adopted by Köstler in [19] yields examples
of quantum stochastic processes which are spreadable while not be-
ing exchangeable. However, concrete models from non-commutative
probability in the $C^*$-formalism do exist where the extended de Finetti
theorem continues to hold. Boolean, monotone, and $q$-deformed (with
$|q| < 1$) processes are all a case in point, [6, 10, 14, 8, 11]. In the
class of the concrete settings alluded to above, the case of the CAR
algebra certainly stands out for its relevance to quantum physics. As
for exchangeability, virtually everything is known. Indeed, exchange-
able (or symmetric) states on the CAR algebra make up a Choquet
simplex whose extreme points are product states of a single even state
on 2 by 2 matrices, see [5]. Moreover, a state on the CAR algebra is
symmetric if and only if the corresponding stochastic process is condi-
tionally independent and identically distributed with respect to its tail
algebra, see [6]. Even so, spreadability for states on the CAR algebra
has not been understood fully insofar as a description of all spreadable
states is still missing. The present paper aims in part to bridge this
gap. In particular, in Theorem 4.2 we prove that on the CAR algebra
there exist spreadable states that are not exchangeable, and stationary
states that are not spreadable. Now, spreadable states are the invariant
states under the action of the unital semigroup \(J_\mathbb{Z}\) of strictly increasing
maps of \(\mathbb{Z}\) to itself whose range is a cofinite set. In order to prove
existence of spreadable states with prescribed values on suitable ele-
ments of the CAR algebra, it comes in useful to delve further into the
properties of the semigroup \(J_\mathbb{Z}\) in terms of amenability. In particular,
in Theorem 3.3 we prove that \(J_\mathbb{Z}\) is left amenable despite having expo-
nential growth, which is shown in Proposition 3.4. In addition, \(J_\mathbb{Z}\) is
not right amenable, although it has a right Følner sequence, as proved
in Proposition 3.1. It is ultimately this circumstance that allows us to
obtain a good supply of spreadable states with prescribed properties
which prevent them from being exchangeable. In more detail, states
of this type can be obtained by averaging on the right Følner sequence
a carefully chosen quasi-free state associated with a positive Toeplitz
operator, Proposition 4.1.

Going back to \(J_\mathbb{Z}\), we would like to stress that its left amenability is a
result which has an interest in its own. For instance, the semigroup \(J_\mathbb{Z}\)
is loosely related to the Thompson monoid \(F^+\), which has very recently
been associated with a new distributional invariance principle in [20].
More precisely, \(J_\mathbb{Z}\) contains a semigroup \(D_\mathbb{Z}\) such that \(J_\mathbb{Z} \cong \mathbb{Z} \ltimes D_\mathbb{Z}\) (the
semidirect product is with respect to a suitable action \(\eta\) of \(\mathbb{Z}\)) and \(D_\mathbb{Z}\)
is isomorphic with a quotient of \(F^+\), whose amenability is not known.
A few words on the organization of the paper are in order. After setting
the notation and recalling the necessary definitions from \(C^*\)-dynamical
systems and quantum stochastic processes in Section 2, we directly
move on to deal with \(J_\mathbb{Z}\) in Section 3. In Section 4 the focus is then
on spreadable and stationary states on the CAR algebra and its subal-
gebra \(\mathfrak{C}\) generated by the so-called position operators. The techniques
we develop to treat the CAR algebra work for $\mathcal{C}$ as well. In particular, on $\mathcal{C}$, too, there exist spreadable states that are not exchangeable, Corollary 4.4. There is however a big difference between the two $C^*$-algebras. In stark contrast with $\text{CAR}(\mathbb{Z})$, which has a great many exchangeable states, its subalgebra $\mathcal{C}$ has in fact only the vacuum as such state, Proposition 4.3.

2. Preliminaries

If $\mathfrak{A}$ is a unital $C^*$-algebra, we denote by $\text{End}(\mathfrak{A})$ the set of all unital *-endomorphisms of $\mathfrak{A}$. This is a unital semigroup with respect to the map composition. A $C^*$-dynamical system is a triplet $(\mathfrak{A}, S, \Gamma)$, where $\mathfrak{A}$ is a unital $C^*$-algebra, $S$ a unital semigroup, and $\Gamma : S \to \text{End}(\mathfrak{A})$ a unital homomorphism, namely $\Gamma_{gh} = \Gamma_g \circ \Gamma_h$ for all $g, h \in S$. If $S = G$ is a group, for any $C^*$-dynamical system $(\mathfrak{A}, G, \alpha)$, the endomorphism $\alpha_g$ is a *-automorphism of $\mathfrak{A}$ for all $g \in G$.

As is commonly done in the literature, $\mathcal{S}(\mathfrak{A})$ denotes the weakly-* compact convex set of all states (normalized, positive, linear functionals) on $\mathfrak{A}$. For any given $(\mathfrak{A}, S, \Gamma)$, we can define the convex subset $\mathcal{S}_{S}(\mathfrak{A}) \subset \mathcal{S}(\mathfrak{A})$ of those states of $\mathfrak{A}$ which are invariant under the action $\Gamma$ of $S$ as

$$\mathcal{S}_{S}(\mathfrak{A}) := \left\{ \varphi \in \mathcal{S}(\mathfrak{A}) \mid \varphi \circ \Gamma_g = \varphi, \ g \in S \right\}.$$ 

This is a weakly-* compact convex set as it is closed in $\mathcal{S}(\mathfrak{A})$.

We denote by $\mathcal{P}_\mathbb{Z}$ the group of finite permutations of the set $\mathbb{Z}$. Its elements are bijective maps of $\mathbb{Z}$ which only moves finitely many integers. The group operation is given by the map composition.

We denote by $\mathbb{L}_\mathbb{Z}$ the unital semigroup of all strictly increasing maps of $\mathbb{Z}$ to itself. For any fixed $h \in \mathbb{Z}$, the $h$-right hand-side partial shift is the element $\theta_h$ of $\mathbb{L}_\mathbb{Z}$ given by

$$\theta_h(k) := \begin{cases} k & \text{if } k < h, \\ k + 1 & \text{if } k \geq h. \end{cases}$$

Analogously, the $h$-left hand-side partial shift is the element $\psi_h$ of $\mathbb{L}_\mathbb{Z}$ given by

$$\psi_h(k) := \begin{cases} k & \text{if } k > h, \\ k - 1 & \text{if } k \leq h. \end{cases}$$

The unital semigroup generated by all (left and right) partial shifts is denoted by $\mathbb{I}_\mathbb{Z}$. Furthermore, we denote by $\mathbb{D}_\mathbb{Z} \subset \mathbb{I}_\mathbb{Z}$ and by $\mathbb{E}_\mathbb{Z} \subset \mathbb{I}_\mathbb{Z}$ the submonoids generated by all right and left partial shifts, respectively.

Finally, let $\mathbb{J}_\mathbb{Z} \subset \mathbb{I}_\mathbb{Z}$ be the unital semigroup of all strictly increasing
Definition 2.1. A stochastic process \((\mathcal{A}, \mathcal{H}, \{\iota_j\}_{j \in \mathbb{Z}}, \xi)\) is said to be

- **stationary** if
  \[
  \langle \iota_{j_1}(a_1) \cdots \iota_{j_n}(a_n) \xi, \xi \rangle = \langle \iota_{j_1+1}(a_1) \cdots \iota_{j_n+1}(a_n) \xi, \xi \rangle;
  \]

- **exchangeable** if for any \(\sigma \in \mathbb{P}_{\mathbb{Z}},\)
  \[
  \langle \iota_{j_1}(a_1) \cdots \iota_{j_n}(a_n) \xi, \xi \rangle = \langle \iota_{\sigma(j_1)}(a_1) \cdots \iota_{\sigma(j_n)}(a_n) \xi, \xi \rangle;
  \]

- **spreadable** if for any \(g \in \mathbb{L}_{\mathbb{Z}},\)
  \[
  \langle \iota_{j_1}(a_1) \cdots \iota_{j_n}(a_n) \xi, \xi \rangle = \langle \iota_{g(j_1)}(a_1) \cdots \iota_{g(j_n)}(a_n) \xi, \xi \rangle.
  \]

where the equalities hold true for all \(n \in \mathbb{N}, j_1, j_2, \ldots, j_n \in \mathbb{Z},\) and \(a_1, a_2, \ldots, a_n \in \mathcal{A}.\)

A stochastic process \((\mathcal{A}, \mathcal{H}, \{\iota_j\}_{j \in \mathbb{Z}}, \xi)\) can equivalently be assigned through a state \(\varphi\) on the free product \(C^*\)-algebra \(*_{\mathbb{Z}}\mathcal{A}.\) We recall that \(*_{\mathbb{Z}}\mathcal{A}\) is the unital \(C^*\)-algebra uniquely determined up to isomorphism by the following universal property: there are unital monomorphisms \(i_j : \mathcal{A} \to *_{\mathbb{Z}}\mathcal{A}\) such that for any unital \(C^*\)-algebra \(\mathcal{B}\) and unital morphisms \(\Phi_j : \mathcal{A} \to \mathcal{B}, j \in \mathbb{Z},\) there exists a unique unital homomorphism \(\Phi : *_{\mathbb{Z}}\mathcal{A} \to \mathcal{B}\) such that \(\Phi \circ i_j = \Phi_j\) for all \(j \in \mathbb{Z}.\) For an extensive account of the theory of free products we refer the reader to [2].

On the one hand, with any stochastic process \((\mathcal{A}, \mathcal{H}, \{\iota_j\}_{j \in \mathbb{Z}}, \xi)\) it is possible to associate a state \(\varphi\) on the free product \(*_{\mathbb{Z}}\mathcal{A}\) by setting
\[
\varphi(i_{j_1}(a_1)i_{j_2}(a_2) \cdots i_{j_n}(a_n)) := \langle \iota_{j_1}(a_1)\iota_{j_2}(a_2) \cdots \iota_{j_n}(a_n) \xi, \xi \rangle,
\]
for every $n \in \mathbb{N}$, and integers $j_1 \neq j_2 \neq \ldots \neq j_n$ and $a_1, a_2, \ldots, a_n \in \mathcal{A}$.

On the other hand, all states on the free product $*_{\mathbb{Z}} \mathcal{A}$ arise in this way, see [6, 7]. Indeed, starting with a state $\varphi \in S(*_{\mathbb{Z}} \mathcal{A})$, the corresponding stochastic process is recovered through the GNS representation $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ of $\varphi$ by defining, for every $j \in \mathbb{Z}$, $\iota_j(a) := \pi_\varphi(i_j(a)), a \in \mathcal{A}$.

Note that corresponding to any map $g : \mathbb{Z} \rightarrow \mathbb{Z}$ by universality there is a $*$-endomorphism $\alpha_g$ of $*_{\mathbb{Z}} \mathcal{A}$ uniquely determined by $\alpha_g(i_j(a)) = i_{g(j)}(a)$, for all $j \in \mathbb{Z}, a \in \mathcal{A}$. One has that $\alpha_{fg} = \alpha_f \circ \alpha_g$ for all $f, g$ maps of $\mathbb{Z}$ to itself. This means in particular that $\mathbb{P}_\mathbb{Z}$ and $\mathbb{L}_\mathbb{Z}$ act on $*_{\mathbb{Z}} \mathcal{A}$. Finally, $\mathbb{Z}$ naturally acts on $*_{\mathbb{Z}} \mathcal{A}$ as well through the $*$-automorphism $\alpha_\tau$ corresponding to the map $\tau : \mathbb{Z} \rightarrow \mathbb{Z}$ we defined above; the corresponding invariant states are denoted by $S_\mathbb{Z}(*_{\mathbb{Z}} \mathcal{A})$.

The submonoids $I_\mathbb{Z}, J_\mathbb{Z} \subset \mathbb{L}_\mathbb{Z}$ act on $*_{\mathbb{Z}} \mathcal{A}$ as well by restriction. For the purposes of the present paper, it is important to recall that the set equalities $S_{*_{\mathbb{Z}} \mathcal{A}} = S_{\mathbb{P}_\mathbb{Z}}(\mathbb{Z}) = S_{\mathbb{L}_\mathbb{Z}}(\mathbb{Z})$ hold, see [9, Remark 4]. In addition, in general one has $S_{*_{\mathbb{Z}} \mathcal{A}}(\mathbb{P}_\mathbb{Z}) \subseteq S_{*_{\mathbb{Z}} \mathcal{A}}(\mathbb{L}_\mathbb{Z}) \subseteq S_{*_{\mathbb{Z}} \mathcal{A}}(\mathbb{Z})$, see [8, Formula (2.5)].

In light of the one-to-one correspondence between stochastic processes $(\mathcal{A}, \mathcal{H}, \{\iota_j\}_{j \in \mathbb{Z}}, \xi)$ and states on the free product $C^*$-algebra $*_{\mathbb{Z}} \mathcal{A}$, we have that a process:

(i) is spreadable if and only if the corresponding state belongs to $S_{*_{\mathbb{Z}} \mathcal{A}}(\mathbb{P}_\mathbb{Z})$, or, which is the same, to $S_{*_{\mathbb{Z}} \mathcal{A}}(\mathbb{L}_\mathbb{Z})$, and the state itself is then said to be spreadable;

(ii) is exchangeable if and only if the corresponding state belongs to $S_{*_{\mathbb{Z}} \mathcal{A}}(\mathbb{P}_\mathbb{Z})$, and the state itself is then said to be symmetric;

(iii) is stationary or shift-invariant if and only if the corresponding state belongs to $S_{*_{\mathbb{Z}} \mathcal{A}}(\mathbb{Z})$, and the state itself is then said to be stationary.

3. ON THE AMENABLEITY OF THE MONOID $J_\mathbb{Z}$

This section is devoted to a thorough study of the properties of amenability of the monoid $J_\mathbb{Z}$. In an effort to keep the exposition as self-contained as possible, we start by recalling a couple of definitions to do with amenable semigroups.

A discrete semigroup $S$ is said to be left (or right) amenable if there exists a state $\varphi$ on $\ell^\infty(S)$ such that $\varphi(l_s f) = \varphi(f)$ (or $\varphi(r_s f) = \varphi(f)$), for every $s \in S$ and $f \in \ell^\infty(S)$, where $l_s f(t) := f(st)$ (or $r_s f(t) := f(ts)$), for any $t \in S$. For convenience we recall that the weakly-$*$ compact convex set of all states of $\ell^\infty(S)$ can easily be identified with the set of all normalized positive finitely additive measures on $(S, \mathcal{P}(S))$,
where $\mathcal{P}(S)$ is the $\sigma$-algebra of all subsets of $S$, see e.g. [22]. Unlike the case of groups, left amenability and right amenability are not the same notion.

A left (right) Følner sequence of a countable discrete semigroup $S$ is a sequence $\{F_n : n \in \mathbb{N}\}$ of finite subsets of $S$ such that for any $h \in S$ one has

$$
\lim_{n \to \infty} \frac{|F_n \Delta h F_n|}{|F_n|} = 0 \quad \left( \lim_{n \to \infty} \frac{|F_n \Delta h F_n|}{|F_n|} = 0 \right)
$$

where $\Delta$ denotes the symmetric difference between sets, i.e. $A \Delta B := (A \cup B) \setminus (A \cap B)$.

We start by showing that $\mathbb{J}_\mathbb{Z}$ has a right Følner sequence.

**Proposition 3.1.** The semigroup $\mathbb{J}_\mathbb{Z}$ has a right Følner sequence.

**Proof.** As we recalled in Section 2, the semigroup isomorphism $\mathbb{J}_\mathbb{Z} \cong \mathbb{Z}_{\eta} \ltimes \mathbb{D}_\mathbb{Z}$ holds, where

$$
\eta_l(\theta_m) = \tau^l \theta_m \tau^{-l} = \theta_{m+l}
$$

for every $l, m \in \mathbb{Z}$. Thus, any $f \in \mathbb{J}_\mathbb{Z}$ uniquely determines $s \in \mathbb{Z}$ and $h \in \mathbb{D}_\mathbb{Z}$ such that $f = h \tau^s$. In addition, using the relations $\theta_k \theta_l = \theta_l \theta_{k-1}$ when $l < k$, any $h \in \mathbb{D}_\mathbb{Z} \setminus \{\text{id}_\mathbb{Z}\}$ can be put in the following form:

$$
h = \theta_{h_1}^{p_1} \theta_{h_2}^{p_2} \cdots \theta_{h_r}^{p_r},
$$

for $r \in \mathbb{N}$, $h_1 < h_2 < \cdots < h_r \in \mathbb{Z}$, and $p_1, p_2, \ldots, p_r \in \mathbb{N}$.

The proof is constructive. Indeed, we will show that the sequence $\{F_n : n \in \mathbb{N}\}$ with

$$
F_n := \left\{ \theta_{-n}^{h_n} \theta_{-n+1}^{h_{n-1}} \cdots \theta_0^{h_0} \cdots \theta_{n-1}^{h_{n-1}} \theta_n^{h_n} \tau^l : \sum_{i=-n}^n h_i \leq n^2, -n \leq l \leq n \right\}
$$

will do. To this aim, we start by computing the cardinality of each $F_n$.

By [15, p. 161], we have

$$
|F_n| = (2n + 1) \sum_{k=0}^{n^2} \binom{2n + 1 + k - 1}{k} = (2n + 1) \binom{n^2 + 2n + 1}{n^2}
$$

Let now $f$ be a fixed element in $\mathbb{J}_\mathbb{Z}$. If $f = \theta_{i_1}^{k_1} \theta_{i_2}^{k_2} \cdots \theta_{i_r}^{k_r} \tau^s$, with $i_1 < i_2 < \cdots < i_r$ and $k_j \in \mathbb{N}$ for $j = 1, 2, \ldots, r$, let us denote by $M$ the maximum of the finite set $\{|i_1|, |i_2|, \ldots, |i_r|\}$. Thanks to (3.1), the product of a generic element of $F_n$ with $f$ on the right takes the form:

$$
\theta_{-n}^{h_n} \theta_{-n+1}^{h_{n-1}} \cdots \theta_{0}^{h_0} \cdots \theta_{n-1}^{h_{n-1}} \theta_n^{h_n} \tau^l \theta_{i_1}^{k_1} \theta_{i_2}^{k_2} \cdots \theta_{i_r}^{k_r} \tau^s
$$

$$
= \theta_{-n}^{h_n} \theta_{-n+1}^{h_{n-1}} \cdots \theta_{0}^{h_0} \cdots \theta_{n-1}^{h_{n-1}} \theta_n^{h_n} \theta_{i_1+l}^{k_1} \theta_{i_2+l}^{k_2} \cdots \theta_{i_r+l}^{k_r} \tau^{l+s}.
$$
Now if \( |l| \leq n - N \) (for \( n \) big enough) where \( N := \max \{|s|, M\} \) (\( N \) does not depend on \( n \)), the above word is seen to be still in \( F_n \) provided that \( q + u \leq n^2 \), where \( q = \sum_{i=-n}^{n} h_i \) and \( u := \sum_{j=1}^{r} k_j \). In more detail, if we define \( n_0 := \max_{|j| \leq n} \{j : h_j \neq 0\} \), the right-hand side of the above equality can be rewritten as

\[
\theta^{h-n} \theta^{h-n+1} \cdots \theta^{h_{n_0}} \theta^{k_1} \theta^{k_2} \cdots \theta^{k_r} r^{l+s}.
\]

Now if \( i_1 + l \geq n_0 \), the word is seen at once to sit in \( F_n \) thanks to the conditions imposed on the indices \( l \) and \( q \). If \( i_1 + l < n_0 \), then by virtue of the commutation rules \( \theta_k \theta_l = \theta_l \theta_k \), for all \( l < k \), there exists \( j \in \{1, 2, \ldots, r\} \) such that our word rewrites as

\[
\theta^{h-n} \theta^{h-n+1} \cdots \theta^{k_1} \theta^{k_{i_1+l}} \cdots \theta^{k_{i_j+l}} \theta^{h_{n_0}} \theta^{k_{i_j+l+1}} \cdots \theta^{k_r} r^{l+s}
\]

where \( i_1 + l < \cdots < i_j + l \leq n_0 - j \leq i_{j+1} + l < \cdots < i_r + l \). By iterating this procedure as many times as necessary, one ends up with an ordered word which lies in \( F_n \) thanks to the constraints on \( l \) and \( q \). But then we have the inequality

\[
|F_n \cap F_n f| \geq (2n - 2N + 1) \sum_{k=0}^{n^2-u} \binom{2n+k}{k}.
\]

Therefore, by (3.2) we have:

\[
\frac{|F_n \cap F_n f|}{|F_n|} \geq \frac{2n - 2N + 1}{2n + 1} \sum_{k=0}^{n^2-u} \binom{2n+k}{k} \left( \frac{n^2+2n+1}{n^2} \right).
\]

Now, the limit of \( \frac{2n-2N+1}{2n+1} \) for \( n \to \infty \) is clearly 1 as is the limit of \( \sum_{k=0}^{n^2-u} \binom{2n+k}{k} \left( \frac{n^2+2n+1}{n^2} \right) \). This can be seen by showing that

\[
\lim_{n \to \infty} \frac{\sum_{k=n^2-u+1}^{n^2} \binom{2n+k}{k} \left( \frac{n^2+2n+1}{n^2} \right)}{n^2} = 0.
\]

Since the function mapping \( k \) to \( \binom{2n+k}{k} \) is increasing, the expression above can be bounded in the following way:

\[
\frac{\sum_{k=n^2-u+1}^{n^2} \binom{2n+k}{k} \left( \frac{n^2+2n+1}{n^2} \right)}{n^2} \leq u \left( \frac{2n + n^2}{n^2} \right) \frac{(n^2)!}{(2n + 1)!} = u \frac{2n + 1}{(n + 1)^2},
\]

and the thesis follows. \( \square \)
Remark 3.2. The semigroup $\mathbb{J}_\mathbb{Z}$ also has a left Følner sequence. More precisely, if we set
\[
G_n := \left\{ \tau^l \theta_{-n}^l \theta_{-n+1}^l \ldots \theta_0^l \ldots \theta_{n-1}^l \theta_n^l : \sum_{i=-n}^{n} h_i \leq n^2, -n \leq l \leq n \right\},
\]
it is not too hard to verify that for any $f = \tau^s \theta_{i_1}^{k_{i_1}} \theta_{i_2}^{k_{i_2}} \ldots \theta_{i_r}^{k_{i_r}} \in \mathbb{J}_\mathbb{Z}$ one has
\[
\lim_{n} \frac{|fG_n \cap G_n|}{|G_n|} = 1.
\]
This can be seen much in the same way as in the proof above, replacing the $n_0$ appearing in that proof with $\min_{|j| \leq n} \{ j : h_j \neq 0 \}$.

In order to prove the left amenability of $\mathbb{J}_\mathbb{Z}$, we first need to recall some facts. First, any left-cancellative semigroup $S$ (i.e. given $s, t, t' \in S$ such that $st = st'$, then $t = t'$) which admits a left Følner sequence is left amenable, as proved by Namioka, see [21, Corollary 4.3]. We will also make use of a notion from semigroup theory which amounts to a weak form of left cancellativity. This is the so-called Klawe condition, [18]: a semigroup $S$ satisfies the Klawe condition if for any $f, g, s \in S$ the equality $sf = sg$ implies that there exists $t \in S$ such that $ft = gt$.

Theorem 3.3. The monoid $\mathbb{J}_\mathbb{Z}$ is left amenable but not right amenable.

Proof. Left amenability follows from Remark 3.2 and left cancellativity thanks to the result of Namioka we recalled above.

In order to prove that $\mathbb{J}_\mathbb{Z}$ fails to be right amenable, we will argue by contradiction. If $\mathbb{J}_\mathbb{Z}$ were right amenable, then its opposite semigroup $\mathbb{J}_\mathbb{Z}^\text{op}$ would be left amenable (we recall that, as a set, $\mathbb{J}_\mathbb{Z}^\text{op}$ is just $\mathbb{J}_\mathbb{Z}$ with the new product $f \cdot \text{op} g := gf$, for any $f, g \in \mathbb{J}_\mathbb{Z}^\text{op}$). Now $\mathbb{J}_\mathbb{Z}^\text{op}$ has a left Følner sequence by Proposition 3.1. By applying [16, Proposition 2.5], we would find that $\mathbb{J}_\mathbb{Z}^\text{op}$ would satisfy the Klawe condition, which in this case reads as follows. For any $f, g, s \in \mathbb{J}_\mathbb{Z}$ the equality $fs = gs$ implies that there exists $t \in \mathbb{J}_\mathbb{Z}$ such that $tf = tg$, that is $f = g$ by left cancellativity of $\mathbb{J}_\mathbb{Z}$. But this does not hold true, as is seen by taking $f = \theta_j$ and $g = s = \theta_{j-1}$, $j \in \mathbb{Z}$, and recalling that $\theta_k \theta_l = \theta_l \theta_{k-1}$ when $l < k$. \hfill $\Box$

The next step to accomplish our analysis of $\mathbb{J}_\mathbb{Z}$ is to show it has exponential growth. This is a result worth stressing because amenability is very often inferred from subexponential growth, which is of course only a sufficient condition. More precisely, Theorem 4.4 in [16] shows that left amenability follows from assuming subexponential growth and the Klawe condition.
Going back to $\mathbb{J}_Z$, we first need to point out that owing to the relations $\theta_l = \tau^l \theta_0 \tau^{-l}, \ l \in \mathbb{Z}$, the monoid $\mathbb{J}_Z$ is actually finitely generated, with $\tau, \tau^{-1}, \theta_0$ being its generators. We recall that the growth function $f : \mathbb{N} \to \mathbb{N}$ of a finitely generated (semi)group at $n$ is just the number of different words of length $n$ in the given generators. Thus, in our case we have $f(n) \leq 3^n$, $n \in \mathbb{N}$. We next aim to show that, however, $f(n) \geq C a^n$, $n \in \mathbb{N}$, for some real constants $C$ and $a$ with $a > 1$.

Proposition 3.4. The monoid $\mathbb{J}_Z$ has exponential growth.

Proof. We start by defining the sequence of sets

$$A_n := \left\{ \theta_{h-n} \theta_{h-n+1} \cdots \theta_0 \theta_{h_n} \theta_{h_{n-1}} \theta_{h_n} : h_i \geq 1, \ \sum_{i=-n}^{n} h_i = 3n + 1 \right\}.$$ 

If we set $k_i := h_i - 1$, $-n \leq i \leq n$, we have $k_i \geq 0$ and $\sum_{i=-n}^{n} k_i = n$. Therefore, the cardinality of each $A_n$ is given by

$$|A_n| = \binom{2n+1+n-1}{n} = \binom{3n}{n} = \frac{(3n)!}{n!(2n)!}.$$ 

Using Stirling’s approximation of the factorial, $|A_n|$ is then seen to satisfy the asymptotic relation $|A_n| = O((\frac{27}{4})^n \sqrt{n})$.

If we rewrite the elements of $A_n$ in terms of the generators $\tau, \tau^{-1}, \theta_0$, after the due simplifications we obtain words of the form

$$\tau^{-n} \theta_0^{h_{-n}} \tau \theta_0^{h_{-n+1}} \tau \cdots \tau \theta_0^{h_{-1}} \tau \theta_0^{h_0} \tau \theta_0^{h_2} \cdots \tau \theta_0^{h_n} \tau^{-n}.$$ 

Now the length of such a word is $7n$. Phrased differently, the set of all words of length equal to $7n$ in the generators contains the set $A_n$. Therefore, when $n$ is big enough, we must have

$$f(7n) \geq |A_n| = O\left(\left(\frac{27}{4}\right)^n \frac{1}{\sqrt{n}}\right) \geq C 6^n$$ 

for some constant $C > 0$. In particular, we find the inequality $f(n) \geq C(\sqrt{6})^n$ for $n$ sufficiently large. \qed

4. Stationary and spreadable states on the CAR algebra

The Canonical Anticommutation Relations (CAR for short) algebra over $\mathbb{Z}$ is the universal unital $C^*$-algebra CAR($\mathbb{Z}$), with unit $I$, generated by the set $\{a_j, a_j^\dagger : j \in \mathbb{Z}\}$ (i.e. the Fermi annihilators and creators respectively), satistying the relations

(4.1) $(a_j)^* = a_j^\dagger$, $\{a_j^\dagger, a_k\} = \delta_{j,k} I$, $\{a_j, a_k\} = \{a_j^\dagger, a_k^\dagger\} = 0$, $j, k \in \mathbb{Z}$. 

where \{\cdot,\cdot\} is the anticommutator and \(\delta_{i,k}\) is the Kronecker symbol.

Note that by definition

\[
\text{CAR}(Z) = \bigcup \{\text{CAR}(F) : F \subset Z \text{ finite}\}
\]

is the (dense) subalgebra of the localized elements, and \(\text{CAR}(F)\) is the \(C^*\)-subalgebra generated by the finite set \(\{a_j, a_j^\dagger : j \in F\}\).

\(\text{CAR}(Z)\) is a \(\mathbb{Z}_2\)-graded algebra. The grading is induced by the parity automorphism \(\Theta\) acting on the generators as

\[
\Theta(a_j) = -a_j, \quad \Theta(a_j^\dagger) = -a_j^\dagger, \quad j \in \mathbb{Z}.
\]

Consequently, the \(\text{CAR}\) algebra decomposes as \(\text{CAR}(Z) = \text{CAR}(Z)_+ \oplus \text{CAR}(Z)_-\), where

\[
\text{CAR}(Z)_+ := \{a \in \text{CAR}(Z) \mid \Theta(a) = a\}, \quad \text{CAR}(Z)_- := \{a \in \text{CAR}(Z) \mid \Theta(a) = -a\}.
\]

Elements in \(\text{CAR}(Z)_+\) and in \(\text{CAR}(Z)_-\) are called even and odd, respectively.

A state \(\varphi\) on \(\text{CAR}(Z)\) is said to be even if \(\varphi \circ \Theta = \varphi\), which is the same as \(\varphi|_{\text{CAR}(Z)_-} = 0\).

The \(C^*\)-algebra \(\text{CAR}(Z)\) has a distinguished (faithful) irreducible representation on the Fermi Fock space \(\mathcal{F}_-(\ell^2(\mathbb{Z}))\). In this representation, for every \(j \in \mathbb{Z}\), the operator \(a_j^\dagger\) (or \(a_j\)) acts as the Fermi creator (or annihilator) of a particle in the state \(e_j\), where \(\{e_j : j \in \mathbb{Z}\}\) is the canonical orthonormal basis of \(\ell^2(\mathbb{Z})\). For an exhaustive account of the Fermi Fock space the reader is referred to Chapter 5.2 of [4]. The vector state associated with the Fock vacuum vector \(\Omega \in \mathcal{F}_-(\ell^2(Z))\) (i.e. the one corresponding to the state with no particles at all) is called the vacuum state.

The \(\text{CAR}\) algebra \(\text{CAR}(Z)\) is isomorphic to the \(C^*\)-infinite tensor product of \(M_2(\mathbb{C})\) with itself:

\[
\text{CAR}(Z) \cong \bigotimes_{\mathbb{Z}} M_2(\mathbb{C})^{C^*},
\]

via a Jordan–Klein–Wigner transformation (see [24], Exercise XIV). Moreover, in Example 3.2 of [12] the \(\text{CAR}\) algebra is also shown to be isomorphic with the infinite graded tensor product of \((M_2(\mathbb{C}), \text{ad}(U))\) with itself, where \(M_2(\mathbb{C})\) is understood as a \(\mathbb{Z}_2\)-graded \(C^*\)-algebra with
grading being induced by the adjoint action of the unitary (Pauli) matrix
\( U := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \).

It is worth recalling that the vacuum state can also be obtained as an
infinite product in the sense of Araki-Moriya, see [1], of a particular
even state on \( \mathbb{M}_2(\mathbb{C}) \). More precisely, by Theorem 5.3 in [5] any extreme
symmetric state on CAR(\( \mathbb{Z} \)) is of the form \( \times_\mathbb{Z} \rho_\lambda \) for some \( 0 \leq \lambda \leq 1 \),
where \( \rho_\lambda \) is the state on \( \mathbb{M}_2(\mathbb{C}) \) given by \( \rho_\lambda(T) := \text{Tr}(TD_\lambda), \ T \in \mathbb{M}_2(\mathbb{C}) \)
and \( D_\lambda := \begin{pmatrix} \lambda & 0 \\ 0 & 1 - \lambda \end{pmatrix} \). The vacuum state is the one corresponding
to \( \lambda = 1 \).

Finally, the CAR algebra can also be seen as a quotient of the free pro-
duct \( \ast_{\mathbb{Z}} \mathbb{M}_2(\mathbb{C}) \). Indeed, if we define \( A := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \), it is easy to see that
the quotient of \( \ast_{\mathbb{Z}} \mathbb{M}_2(\mathbb{C}) \) modulo the relations \( \{ i_j(A^*), i_k(A) \} = \delta_{j,k}I \)
and \( \{ i_j(A), i_k(A) \} = \{ i_j(A^*), i_k(A^*) \} = 0 \), for all \( j, k \in \mathbb{Z} \), is isomor-
phic with the CAR algebra by (4.1).

Note that \( \mathbb{Z}, \mathbb{P}_\mathbb{Z}, \mathbb{J}_\mathbb{Z} \) act naturally on \( \text{CAR}(\mathbb{Z}) \) by displacing the in-
dices of the generators according to the given map of \( \mathbb{Z} \). These actions
obviously come from the action at the level of free product we in-
troduced towards the end of Section 2. Therefore, studying invariant
Fermi stochastic processes is the same as analyzing the invariant states
of \( \text{CAR}(\mathbb{Z}) \) under the corresponding action. Also note that exchange-
able states are automatically spreadable and spreadable states are of
course stationary.

As we recalled in the introduction, de Finetti’s theorem provides
quite a satisfactory description of exchangeable random variables as
those which are conditionally independent and identically distributed w.r.t. the tail algebra. This characterization continues to hold true
for the CAR algebra, as shown in [6, Theorem 5.4]. However, the Ryll-
Nardzewski theorem [23] that exchangeable sequences are the same as
spreadable sequences is no longer true in the CAR algebra.

Our next goal is to show that there exist shift-invariant states on
\( \text{CAR}(\mathbb{Z}) \) that are not spreadable, and spreadable states that are not
symmetric. To this end, we start by singling out a class of (quasi-free)
shift-invariant states, which we do in the next proposition.
In the following \( i \) will denote the imaginary unit of \( \mathbb{C} \).
Proposition 4.1. On CAR($\mathbb{Z}$) there exists a stationary state $\omega$ such that

$$\omega(a_m^\dagger a_n) = \frac{i}{\pi^2(m-n)^2} \cdot \frac{3C}{\pi^2(m-n)^2},$$

for all $m, n \in \mathbb{Z}$ with $m > n$ and some positive constant $C$.

Proof. As recalled in [13], it is possible to obtain (gauge invariant quasi-free) stationary states on the CAR algebra by setting for every $m, n \in \mathbb{N}$ and $i_1, \ldots, i_m, j_1, \ldots, j_n \in \mathbb{Z}$

$$\varphi(a_{i_1}^\dagger \cdots a_{i_m}^\dagger a_{j_1} \cdots a_{j_n}) = \delta_{m,n} \det [Q_{i_k,j_l}]_{k,l=1}^n,$$

where $0 \leq Q \leq I$ is a Toeplitz operator on $l^2(\mathbb{Z})$, and $Q_{m,n} = \langle Q e_m, e_n \rangle$ are its matrix elements in the canonical basis of $l^2(\mathbb{Z})$. Therefore, we need to show that a suitable choice of $Q$ yields a state with the desired properties.

To this end, we start by recalling that a Toeplitz operator is represented by a bi-infinite matrix $[Q_{m,n}]_{m,n \in \mathbb{Z}}$ where the entries $Q_{m,n}$ depend only on $(m-n) =: k$. In other words, the entries of such a matrix are constant along all $k$-th diagonals, $k \in \mathbb{Z}$. Notice that $k = 0$ corresponds to the leading diagonal. For every $k \in \mathbb{Z}$, we denote by $d_k$ the value taken by the entries of our matrix on the $k$-th diagonal.

We now recall that the operator corresponding to such a matrix will be bounded if and only if the Fourier series $\sum_{k \in \mathbb{Z}} d_k z^k$ is a function in $L^\infty(\mathbb{T})$, see e.g. [3]. We next verify that the choice

$$d_k := \begin{cases} 
1 & \text{if } k = 0 \\
-i \frac{3}{\pi^2 k^2} & \text{if } k > 0 \\
i \frac{3}{\pi^2 k^2} & \text{if } k < 0
\end{cases}$$

produces a positive Toeplitz operator. First, note that the Fourier series $\sum_{k \in \mathbb{Z}} d_k z^k$ certainly defines an essentially bounded function since its sum is even continuous on $\mathbb{T}$ by total convergence. Second, note that

$$d_0 \geq \sum_{k \neq 0} |d_k|$$

as follows from $\sum_{k>0} \frac{1}{k^2} = \frac{\pi^2}{6}$. We next show that this inequality implies that the corresponding operator $Q$ is positive.

To this end, for every $n \in \mathbb{N}$ define a bounded operator $Q^{(n)}$ whose entries $[Q^{(n)}_{i,j}]$ are the same as those of $Q$ for $|i|, |j| \leq n$ and 0 otherwise. Each $Q^{(n)}$ is a positive operator in that it is represented by a $(2n+1)$-squared Toeplitz matrix which is Hermitian, diagonally dominant, and with positive diagonal entries. The conclusion will then be reached if we ascertain that $Q$ is the limit of the sequence $\{Q^{(n)} : n \in \mathbb{N}\}$ in the weak operator topology. Because the sequence is bounded in norm, with $\|Q^{(n)}\| \leq \|Q\|$ for every $n \in \mathbb{N}$, it is enough to check for any
fixed \(i, j \in \mathbb{Z}\) one has \(\lim_n Q_{i,j}^{(n)} = Q_{i,j}\). But this is certainly true since \(Q_{i,j}^{(n)} = Q_{i,j}\) as soon as \(n \geq \max\{|i|, |j|\}\).

Finally, in order to satisfy the condition \(Q \leq I\), it is enough to replace \(Q\) with \(\frac{Q}{\|Q\|}\), hence the thesis holds with \(C := \frac{1}{\|Q\|}\). \(\square\)

**Theorem 4.2.** There holds the chain of strict inclusions

\[
S_{\mathbb{P}_\mathbb{Z}}(\text{CAR}(\mathbb{Z})) \subsetneq S_{\mathbb{J}_\mathbb{Z}}(\text{CAR}(\mathbb{Z})) \subsetneq S_\mathbb{Z}(\text{CAR}(\mathbb{Z}))
\]

**Proof.** We start by observing that any state as in Proposition 4.1 provides an example of a stationary state which by construction fails to be spreadable.

Exhibiting a spreadable state that is not exchangeable requires far more work to do. To this aim, pick a state \(\omega\) as in Proposition 4.1, then define a sequence \(\{\omega_n : n \in \mathbb{Z}\}\) of states by setting

\[
\omega_n := \frac{1}{|F_n|} \sum_{h \in F_n} \omega \circ \alpha_h,
\]

where \(\{F_n\}_{n \in \mathbb{N}}\) is the right Følner sequence of \(\mathbb{J}_\mathbb{Z}\) exhibited in Proposition 3.1, and \(\mathbb{J}_\mathbb{Z} \ni h \mapsto \alpha_h \in \text{End}(\text{CAR}(\mathbb{Z}))\) is its natural action on the CAR algebra. By weak-* compactness of \(S(\text{CAR}(\mathbb{Z}))\), the sequence above weakly-* converges (up to taking a subsequence) to some state \(\tilde{\omega}\).

First, we prove that \(\tilde{\omega}\) is spreadable, that is \(\tilde{\omega} \circ \alpha_k = \tilde{\omega}\) for any \(k \in \mathbb{J}_\mathbb{Z}\).

This is seen by means of a standard \(\frac{\varepsilon}{3}\)-argument, which we nevertheless include in full below. We have:

\[
|\tilde{\omega}(a) - \tilde{\omega}(\alpha_k(a))| \leq |\tilde{\omega}(a) - \omega_n(a)| + |\omega_n(a) - \omega_n(\alpha_k(a))| + |\omega_n(\alpha_k(a)) - \tilde{\omega}(\alpha_k(a))|.
\]

Obviously, it is only the second term of the above sum that needs to be taken care of. For any fixed \(\varepsilon > 0\) this can be done as follows:

\[
|\omega_n(a) - \omega_n(\alpha_k(a))| = \frac{1}{|F_n|} \left| \sum_{h \in F_n} \omega(\alpha_h(a)) - \sum_{h \in F_n} \omega(\alpha_h(\alpha_k(a))) \right| = \frac{1}{|F_n|} \left| \sum_{h \in F_n} \omega(\alpha_h(a)) - \sum_{h \in F_n} \omega(\alpha_h(\alpha_k(a))) \right| \\
\leq \frac{1}{|F_n|} \left| \sum_{h \in F_n \Delta F_n k} \omega(\alpha_h(\alpha_k(a))) \right| \leq \frac{|F_n \Delta F_n k|}{|F_n|} \|a\| \leq \frac{\varepsilon}{3}.
\]
as soon as \( n \) is big enough.

We claim that \( \tilde{\omega}(a_1 a_2^\dagger) = -\frac{3iC}{\pi^2} \) and \( \tilde{\omega}(a_2 a_1^\dagger) = \frac{3iC}{\pi^2} \). From this it easily follows that \( \tilde{\omega} \) cannot be exchangeable, for the equality \( \tilde{\omega}(a_1 a_2^\dagger) = \tilde{\omega}(a_2 a_1^\dagger) \) does not hold.

We now move on to prove the claim. We only focus on the first equality as the second can be got to in the same way. For every fixed \( n \in \mathbb{N} \), we define the subset \( S_n \subset F_n \) as the set of all maps \( h : \mathbb{Z} \to \mathbb{Z} \) in \( F_n \) such that \( h(2) - h(1) > 1 \). We next bound the cardinality of each \( S_n \) from above. Recall that a generic element of \( F_n \) has the form

\[
h = \theta_{-n}^h \theta_{-n+1}^h \cdots \theta_0^h \theta_{n-1}^h \tau^l
\]

with \( \sum_{i=-n}^{n} h_i \leq n^2 \), \( -n \leq l \leq n \). A moment's reflection shows that for any \( l \) with \( |l| \leq n \), such an \( h \) will sit in \( S_n \) if and only if \( h_2 + l \neq 0 \) (with \( 2 + l \) still between \(-n \) and \( n \)). This implies that

\[
\begin{align*}
|S_n| &\leq (2n + 1) \sum_{k=0}^{n^2-1} \left( \frac{2n - 1 + k}{k} \right) = (2n + 1) \left( \frac{n^2 + 2n - 1}{n^2 - 1} \right).
\end{align*}
\]

But then we have

\[
\begin{align*}
\tilde{\omega}(a_1 a_2^\dagger) &\equiv \lim_{n} \frac{1}{|F_n|} \sum_{h \in F_n} \omega(a_h(a_1 a_2^\dagger)) = \lim_{n} \frac{1}{|F_n|} \sum_{h \in F_n} \omega(a_{h(1)}a_{h(2)}^\dagger) \\
&= \lim_{n} \frac{1}{|F_n|} \left( \sum_{h \in S_n} \omega(a_{h(1)}a_{h(2)}^\dagger) + \sum_{h \in S_n^c} \omega(a_{h(1)}a_{h(2)}^\dagger) \right) \\
&= \lim_{n} \frac{1}{|F_n|} \sum_{h \in S_n} \omega(a_{h(1)}a_{h(2)}^\dagger) + \lim_{n} \frac{|S_n^c|}{|F_n|} \left( -\frac{3iC}{\pi^2} \right) \\
&= -\frac{3iC}{\pi^2},
\end{align*}
\]

where we have used that \( |\omega(a_{h(1)}a_{h(2)}^\dagger)| \leq 1 \) for every \( h \in S_n \) and \( \frac{|S_n|}{|F_n|} \) tends to 0, which we need to verify. From (3.2) and (4.4) we find

\[
\frac{|S_n|}{|F_n|} \leq \left( \frac{n^2 + 2n - 1}{n^2 - 1} \right) = \frac{n^2(2n+1)}{(n^2+2n+1)(n^2+2n)} = O\left( \frac{1}{n} \right).
\]

We finally turn our attention to the so-called self-adjoint part of \( \text{CAR}(\mathbb{Z}) \). This is by definition the unital \( C^* \)-algebra generated by the position operators, say \( \mathcal{C} := C^*(x_j : j \in \mathbb{Z}) \), where \( x_j := a_j + a_j^\dagger \).
for every \( j \in \mathbb{Z} \). As a consequence of (4.1), one has that the \( x_j \)'s anticommute with one another and their square is the identity, that is
\[
(4.5) \quad x_j x_k + x_k x_j = 0, \quad \text{for all } j \neq k \text{ and } x_j^2 = I, \quad \text{for all } j \in \mathbb{Z}.
\]

We are going to prove that there is a marked difference between \( \text{CAR}(\mathbb{Z}) \) and its subalgebra \( \mathcal{C} \) in that the latter has only one symmetric state, the vacuum state.

**Proposition 4.3.** The vacuum state is the only symmetric state on \( \mathcal{C} \), the self-adjoint part of \( \text{CAR}(\mathbb{Z}) \).

**Proof.** By (4.5) one gets that the linear span of words of type \( I, x_i \) for \( i \in \mathbb{Z} \), and finally \( x_{j_1} \cdots x_{j_l} \), with \( l \geq 2 \) and \( j_1, \ldots, j_l \in \mathbb{Z} \) different from one another, is dense in \( \mathcal{C} \).

Let \( \omega \) be a state on \( \mathcal{C} \) invariant under permutation. We first show that \( \omega(x_j) = 0 \) for any \( j \in \mathbb{Z} \). Clearly, it is enough to prove that \( \omega(x_1) = 0 \) since \( \omega(x_j) = \omega(x_1) \) for any \( j \in \mathbb{Z} \). Therefore, we have the equality \( \omega(x_1) = \omega(\frac{1}{n} \sum_{j=1}^{n} x_j) \) for any natural \( n \). The conclusion will follow if we show that \( \|\frac{1}{n} \sum_{j=1}^{n} x_j\| \) converges to 0. This is a matter of easy computations. Indeed, by (4.5) we have
\[
\|x_1 + \ldots + x_n\|^2 = \|(x_1 + \ldots + x_n)^2\|
\]
\[
= \left\| nI + \sum_{i<j} (x_ix_j + x_jx_i) \right\| = n,
\]
hence \( \|\frac{1}{n} \sum_{j=1}^{n} x_j\| = \frac{1}{\sqrt{n}} \to 0 \) for \( n \to \infty \). Longer words can be handled more easily. Indeed, for any length \( l \geq 2 \) and any set \( \{j_1, j_2, \ldots, j_l\} \subset \mathbb{Z} \) of indices different from one another, we have \( \omega(x_{j_1} x_{j_2} \cdots x_{j_l}) = \omega(x_{j_2} x_{j_1} \cdots x_{j_l}) = -\omega(x_{j_1} x_{j_2} \cdots x_{j_l}) \), and thus \( \omega(x_{j_1} x_{j_2} \cdots x_{j_l}) = 0 \). The conclusion then follows by density as the restriction of the vacuum state to \( \mathcal{C} \) assumes the same values on the above words. \( \square \)

We would like to end our discussion by pointing out that on \( \mathcal{C} \) as well there exist stationary states that are not spreadable, and spreadable states that are not the vacuum state.

**Corollary 4.4.** There holds the chain of strict inclusions
\[
S_{P_2}(\mathcal{C}) \subsetneq S_{I_2}(\mathcal{C}) \subsetneq S_{\mathbb{Z}}(\mathcal{C})
\]

**Proof.** A stationary state that is not spreadable is obtained by restricting to \( \mathcal{C} \) the state \( \omega \) in Proposition 4.1. Indeed, by (4.2) and (4.3) one easily sees that \( \omega(x_1 x_2) = \omega(a_1 a_2^\dagger) + \omega(a_1^\dagger a_2) = -\frac{6 \hbar c}{\pi^2} \). On the other hand, we have \( \omega(x_1 x_3) = \omega(a_1 a_3^\dagger) + \omega(a_1^\dagger a_3) = -\frac{3 \hbar c}{2 \pi^2} \), which means the restriction of \( \omega \) to \( \mathcal{C} \) is not spreadable.
A spreadable state that is not the vacuum state is obtained by restricting to $\mathfrak{c}$ the state $\tilde{\omega}$ in the proof of Theorem 4.2. Indeed, $\tilde{\omega}$ does not vanish on $x_1x_2$. More precisely, we have

$$\tilde{\omega}(x_1x_2) = \tilde{\omega}(a_1^\dagger a_2) + \tilde{\omega}(a_1^\dagger a_2) = -\frac{6\iota C}{\pi^2},$$

since $\tilde{\omega}((a_1a_2)\sharp) = \lim_{n \to \infty} \frac{1}{|F_n|} \sum_{h \in F_n} \omega((a_{h(1)}a_{h(2)})\sharp) = 0$ by (4.2), where $\sharp$ is either 1 or $\dagger$.

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VITONOFRIO CRISMALE, Dipartimento di Matematica, Università degli studi di Bari, Via E. Orabona, 4, 70125 Bari, Italy
Email address: vitonofrio.crismale@uniba.it

Stefano Rossi, Dipartimento di Matematica, Università degli studi di Bari, Via E. Orabona, 4, 70125 Bari, Italy
Email address: stefano.rossi@uniba.it