Bethe vectors of quantum integrable models based on $U_q(\hat{\mathfrak{gl}}_N)$

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Abstract
We study quantum $U_q(\hat{\mathfrak{gl}}_N)$ integrable models solvable by the nested algebraic Bethe ansatz. Different formulas are given for the right and left universal off-shell nested Bethe vectors. It is shown that these formulas can be related by certain morphisms of the positive Borel subalgebra in $U_q(\hat{\mathfrak{gl}}_N)$ into analogous subalgebra in $U_{q^{-1}}(\hat{\mathfrak{gl}}_N)$.

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1. Introduction

The nested algebraic Bethe ansatz [1–3] in its original formulation allows one to find Bethe equations under the condition that the Bethe vectors (BVs) are eigenstates of the transfer matrix. Nevertheless, even when the Bethe parameters are free and do not satisfy any restriction, the structure of the BV (sometimes such BVs are called off-shell) is rather complicated. In the theory of solutions of the quantum Knizhnik–Zamolodchikov equation [4] the universal off-shell BVs were given by a certain trace over auxiliary spaces of the products of the monodromy matrices and $R$-matrices. This presentation allows one to investigate the structure of the nested off-shell BV. It also leads to explicit formulas for the nested BV when the quantum space of the integrable model becomes the tensor product of evaluation representations of the Yangian or the Borel subalgebra of the quantum affine algebra $U_q(\hat{\mathfrak{gl}}_N)$ [5].
The explicit formulas for the off-shell BV in terms of the monodromy matrix elements were obtained in [6, 7] within the framework of the so-called current approach. In this method off-shell BVs are identified with projections of the product of $U_q(\widehat{\mathfrak{gl}}_N)$ currents onto intersections of the standard and current Borel subalgebras in the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$. The theory of these projections was elaborated in the pioneering paper [8] and then fully developed in [9].

The main results of the paper [6] is development of the method initially discovered in [10] to calculate the projections of product of the currents in the case of quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$. It was shown in [7] that in order to obtain different presentations of the BV associated with two different embeddings $U_q(\widehat{\mathfrak{gl}}_{N-1})$ into $U_q(\widehat{\mathfrak{gl}}_N)$, one needs to explore two different current realizations of quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$. It becomes clear now that these two different current realizations are related by certain morphisms. In the present paper we show, in particular, how these maps allow us to obtain the dual (or left) off-shell BVs from the right ones.

The spectrum of the transfer matrix can be obtained without the use of explicit formulas for BV [1–3]. Explicit formulas, however, are very important for the calculation of BV scalar products, which, in turn, are the main tools for the analysis of correlation functions and form factors of local operators in the Bethe ansatz solvable models. To address this very complicated problem in relation to integrable models with $U_q(\widehat{\mathfrak{gl}}_N)$ symmetry, one has to get convenient formulas for the off-shell BV. Such types of presentations were obtained in [11] for the BV in the models with $\mathfrak{gl}_3$-invariant $R$-matrix. There the explicit formulas for off-shell BV were given in terms of sums over partitions of the sets of Bethe parameters. These expressions also include the Izergin determinant [12], which is the partition function of the six-vertex model with domain wall boundary conditions [13]. The properties of the Izergin determinant allow one to obtain compact formulas for the scalar products of BV in several important particular cases. Using these formulas and the solution of the inverse scattering problem [14, 15] we succeeded in calculating form factors of some local operators in the SU(3)-invariant XXX Heisenberg chain [16].

The main goal of this paper is to extend the results of [11] to include models with $U_q(\widehat{\mathfrak{gl}}_N)$ symmetry. Our starting point is the explicit formulas for the off-shell nested BV obtained in [6, 7] in terms of the summation over permutations of the whole set of Bethe parameters. First we observe that partial summations over permutations give the Izergin determinants. The remaining symmetrization then leads to explicit formulas for the right and left off-shell BV in terms of sums over partitions of the sets of Bethe parameters. We will obtain two different presentations for the same right BV corresponding to the different ways of embedding $U_q(\widehat{\mathfrak{gl}}_{N-1})$ into $U_q(\widehat{\mathfrak{gl}}_N)$.

Then using the special properties of the trigonometric $R$-matrix, we will define the special morphism of $U_q(\widehat{\mathfrak{gl}}_N)$ to $U_{q^{-1}}(\widehat{\mathfrak{gl}}_N)$ and prove that different presentations for the right off-shell BV can be related by this morphism. Further we will define one more antimorphism of $U_q(\widehat{\mathfrak{gl}}_N)$ to $U_{q^{-1}}(\widehat{\mathfrak{gl}}_N)$ which allows us to obtain the formulas in terms of sums over partitions for the left (or dual) off-shell BV.

1.1. Notations

In this paper we consider the quantum integrable models defined by the $N \times N$ monodromy matrix $T_{ij}(z)$ satisfying the commutation relation

$$R(u, v; q) \cdot (T(u) \otimes 1) \cdot (1 \otimes T(v)) = (1 \otimes T(v)) \cdot (T(u) \otimes 1) \cdot R(u, v; q), \quad (1.1)$$
where \( R(u, v; q) \in \text{End}(\mathbb{C}^N \otimes \mathbb{C}^N) \otimes \mathbb{C}[v/u] \), is a trigonometric R-matrix associated with the vector representation of \( U_q(\mathfrak{gl}_N) \). Here \( q \) is a complex parameter neither equal to zero nor a root of unity. The algebra \((1.1)\) also describes the commutation relation in the standard Borel subalgebra of the quantum affine algebra \( U_q(\hat{\mathfrak{gl}}_N) \). In what follows we will describe the morphisms of this subalgebra of \( U_q(\mathfrak{gl}_N) \) into the analogous subalgebra of \( U_q^{-1}(\mathfrak{gl}_N) \). This is a reason why we write the explicit dependence of the R-matrix on the parameter \( q \).

More explicitly this R-matrix can be written in the form

\[
R(u, v; q) = f_q(u, v) \sum_{1 \leq i < j \leq N} E_{ij} \otimes E_{ji} + \sum_{1 \leq i < j < k \leq N} (E_{ij} \otimes E_{jk} + E_{ij} \otimes E_{kj} + E_{jk} \otimes E_{ij} + E_{kj} \otimes E_{ji}),
\]

(1.2)

where \((E_{ij})_{ik} = \delta_{ij}\delta_{jk}, i, j, k = 1, \ldots, N\) are \(N \times N\) matrices with unit in the intersection of \(i\)th row and \(j\)th column and zero elsewhere, and the coefficient functions are defined as follows

\[
f_q(u, v) = \frac{q^u - q^{-1}v}{u - v}, \quad g_q(u, v) = \frac{(q - q^{-1})}{u - v},
\]

(1.3)

and

\[
g_q^{(l)}(u, v) = u g_q(u, v), \quad g_q^{(r)}(u, v) = v g_q(u, v).
\]

A concrete quantum integrable model is usually defined by a certain representation space \( V \) (or quantum space of the integrable models) where the entries of the monodromy matrix \( T_{i,j}(u) \) act. The space of states for this model is identified with a representation space of the Borel subalgebra of \( U_q(\mathfrak{gl}_N) \) generated by the vector \( |0\rangle \) satisfying the following conditions

\[
T_{i,j}(z)|0\rangle = 0, \quad j > i, \quad T_{i,i}(z)|0\rangle = \lambda_i(z)|0\rangle, \quad i = 1, \ldots, N.
\]

(1.5)

The functions \(\lambda_i(z)\) characterize the concrete integrable model and the vector \(|0\rangle\) is a common eigenvector of the diagonal entries of the monodromy matrix. Although the diagonal matrix elements do not commute, the commutators \([T_{i,j}(z), T_{j,i}(z')], \forall i, j\), annihilate the vacuum vector \(|0\rangle\), due to the commutation relations and (1.5).

The off-shell BV are constructed as special polynomials of the monodromy matrix elements \(T_{i,j}(z), i \leq j\), depending on sets of parameters (the Bethe parameters) and acting on \(|0\rangle\). These parameters are supposed to be generic complex numbers. If they satisfy the system of the nested Bethe ansatz equations, then the corresponding BV becomes an eigenvector of the transfer matrix \(\hat{R} \). We will also consider the dual (or left) off-shell BVs, which belong to the dual space \(V^*\). They are generated by the matrix elements \(T_{i,j}(z)\) acting on a vector \(|0\rangle\), which satisfies the conditions

\[
\langle 0|T_{i,j}(z) = 0, \quad j > i, \quad \langle 0|T_{i,i}(z) = \lambda_i(z)|0\rangle, \quad i = 1, \ldots, N.
\]

(1.6)

2. Formulas for BV from the current approach

The goal of this section is to recall one of the formulas for the right off-shell BVs formulated in theorem 2 of the paper [7] in the form of sums over all permutations of the Bethe parameters of the same sort.

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6 When there is no ambiguity, we will omit the subscript \( q \) in the rational functions (1.3) to simplify the formulas.

7 Sometimes such vectors are called on-shell BVs.
Let \( \bar{n} = \{ n_1, n_2, \ldots, n_{N-1} \} \) be a collection of positive integers and let \( \bar{t}_n \) be a set of variables

\[
\bar{t}_n = \{ t_{1}^{n}, \ldots, t_{n}^{n} : t_{1}^{n}, \ldots, t_{n}^{n} \} \quad (2.1)
\]

Here superscripts indicate the type of Bethe parameter and correspond to the simple roots of the algebra \( g_{N} \). There are \( N - 1 \) different sorts of Bethe parameters in the generic BV for \( U_{q}(g_{N}) \)-integrable model. The subscript counts the number of Bethe parameters of the same type. For a generic BV, we denote by \( n_i \) the total number of type \( i \) Bethe parameters.

Let us consider a direct product of the symmetric groups: \( S_{n} = S_{n_{1}} \times \cdots \times S_{n_{N-1}} \). For any function \( G(\bar{t}_n) \) we denote by

\[
\text{Sym}_{\bar{t}_n} G(\bar{t}_n) = \sum_{\sigma \in S_{n}} G(G(\bar{t}_n)), \quad \sigma = \{ \sigma_{1}, \sigma_{2}, \ldots, \sigma^{N-1} \}
\]

(2.2)
a symmetrization over groups of variables of same type \( k \).

\[
\bar{\sigma} \bar{t}_n = \{ t_{i}^{\sigma_{1}(n_1)} \ldots, t_{i}^{\sigma_{N-1}(n_{N-1})} : t_{i}^{\sigma_{1}(n_1)} \ldots, t_{i}^{\sigma_{N-1}(n_{N-1})} \} \quad (2.3)
\]

Let

\[
\beta(\bar{t}_n) = \prod_{k=1}^{N-1} \prod_{1 \leq \ell < \ell' \leq n_k} t_{\ell}^{j_k} t_{\ell'}^{j_k'}
\]

be a function of the formal variables \( t_{\ell}^{j_k} \), \( \ell = 1, \ldots, n_k, k = 1, \ldots, N - 1 \).

In order to describe formulas for the off-shell BV we need the following combinatorial data. Let \( \hat{m} = \{ m_{i}^{j} \} \) and \( \hat{s} = \{ s_{i}^{j} \} \) for \( 1 \leq i \leq j \leq N - 1 \) be two collections of the non-negative integers. We say that collections \( \hat{m} \) and \( \hat{s} \) are \( \bar{n} \) upper or lower (resp.) permissible, if they satisfy the following conditions

\[
m_{i}^{j} \geq m_{i+1}^{j} \geq \cdots \geq m_{N-1}^{j} \geq m_{N}^{j} = 0, \quad n_i = \sum_{j=1}^{i} m_{j}^{j}, \quad i = 1, \ldots, N - 1,
\]

(4.4)

and

\[0 = s_{0}^{j} \leq s_{1}^{j} \leq \cdots \leq s_{N-1}^{j} \leq s_{N}^{j}, \quad n_i = \sum_{j=1}^{N-1} s_{j}^{j}, \quad i = 1, \ldots, N - 1,
\]

(5.5)

respectively. We also follow the convention \( m_{N}^{j} = s_{0}^{j} = 0 \) for \( j = 1, \ldots, N - 1 \).

The collections of upper or lower permissible integers \( \hat{m} \) and \( \hat{s} \) can be visualized as upper or lower triangular matrices

\[
[\hat{m}] = \begin{pmatrix}
0 & m_{1}^{1} & m_{1}^{2} & \cdots & m_{1}^{N-1} \\
0 & m_{2}^{1} & m_{2}^{2} & \cdots & m_{2}^{N-1} \\
0 & m_{3}^{1} & m_{3}^{2} & \cdots & m_{3}^{N-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & m_{N}^{1} & m_{N}^{2} & \cdots & m_{N}^{N-1}
\end{pmatrix}
\]

(6.6)

and

\[
[\hat{s}] = \begin{pmatrix}
0 & s_{1}^{1} & s_{1}^{2} & \cdots & 0 \\
0 & s_{2}^{1} & s_{2}^{2} & \cdots & 0 \\
0 & s_{3}^{1} & s_{3}^{2} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & s_{N-1}^{1} & s_{N-1}^{2} & \cdots & s_{N-1}^{N-1}
\end{pmatrix}
\]

(7.7)
In words, $[\bar{m}]$ is an upper triangular matrix with integer entries, ordered on each line, and such that the sum of its $i$th column gives back $n_i$. In what follows, $[\bar{m}]$ (resp. $[\bar{s}]$) will denote upper (resp. lower) permissible collections of integers.

Let $\bar{m}^j$ and $\bar{s}^j$, $j = 1, \ldots, N - 1$ be the $j$th rows of the permissible matrices $[\bar{m}]$ and $[\bar{s}]$. Define a collection of vectors

$$
\bar{m}^j = \bar{m}^1 + \bar{m}^2 + \cdots + \bar{m}^{j-1} + \bar{m}^j, \quad j = 1, \ldots, N - 1,
$$

(2.8)

$$
\bar{s}^j = \bar{s}^1 + \bar{s}^2 + \cdots + \bar{s}^{N-2} + \bar{s}^{N-1}, \quad j = 1, \ldots, N - 1,
$$

(2.9)

with non-negative integer components. Set $\bar{m}^0 = 0$ and $\bar{s}^N = 0$. Denote by $\bar{m}_j^i$ and $\bar{s}_j^i$ the components of the vectors $\bar{m}^j$ and $\bar{s}^j$:

$$
\bar{m}^j = \{ n_1, n_2, \ldots, n_j, m_{j+1}^1, \ldots, m_{N-1}^j + \cdots + m_{N-1}^{j-1} \},
$$

$$
\bar{s}^j = \{ s_1^j + \cdots + s_{N-1}^j, \ldots, s_{j-1}^j + \cdots + s_{N-1}^j, n_j, \ldots, n_{N-2}, n_{N-1} \}.
$$

According to the conditions (2.5) and (2.4), $\bar{m}^{N-1} = \bar{s}^1 = \bar{n}$.

We introduce the following combinations of the $U_q(\mathfrak{gl}_N)$ monodromy matrix elements, that we call pre-BV:

$$
\mathcal{B}^\beta(\bar{t}_\beta) = \sum_{[\bar{m}]} \text{Sym} \bar{m} \left( \beta(\bar{t}_\beta) \prod_{1 \leq i < j \leq N} \left[ (m_{j-1}^i - m_{j+1}^i) ! \right]^{-1} \prod_{n_j - m_{j+1}^i < \ell < n_j - m_{j+1}^{i+1}} f(t_{j-1}^i, t_j^i)^{-1} \right)
\times \prod_{j=2}^{N-1} \left( \prod_{i=1}^{j-1} \prod_{\ell = 0}^{m_{j-1}^i-1} g^{(\ell)}(t_{j-1}^i - \ell, t_{j-1}^i) \prod_{\ell = m_{j}^i - \ell}^{n_j - m_{j+1}^i} f(t_{j-1}^i, t_j^i)^{-1} \right)
\times \prod_{1 \leq i < j \leq N-1} \left( \prod_{\ell = m_{j-1}^i - \ell}^{n_j - m_{j+1}^i} T_{j,i+1}(t_j^i) \prod_{j=1}^{N-1} \prod_{\ell = 1}^{n_j - m_{j}^i} T_{j,i}(t_j^i) \right),
$$

(2.10)

and

$$
\mathcal{B}^\beta(\bar{t}_\beta) = \sum_{[\bar{m}]} \text{Sym} \bar{m} \left( \beta(\bar{t}_\beta) \prod_{1 \leq i < j \leq N} \left[ (s_{j}^i - s_{j+1}^i) ! \right]^{-1} \prod_{s_{j}^i - s_{j+1}^i < \ell < s_{j}^i - s_{j+1}^{i+1}} f(t_{j-1}^i, t_j^i)^{-1} \right)
\times \prod_{j=2}^{N-1} \left( \prod_{i=1}^{j-1} \prod_{\ell = 1}^{\beta(j)} g^{(\ell)}(t_{j}^{i+1} - \ell, t_{j}^{i+1}) \prod_{\ell = 1}^{n_j - s_{j+1}^{i+1} - \ell} f(t_{j}^i, t_{j}^{i+1})^{-1} \right)
\times \prod_{1 \leq i < j \leq N-1} \left( \prod_{\ell = 1}^{\beta(i)} T_{j,i+1}(t_j^i) \prod_{j=1}^{N-1} \prod_{\ell = \beta(j)} T_{j,i}(t_j^i) \right),
$$

(2.11)

where the ordered products of the non-commuting entries $A_i$ are defined as follows

$$
\prod_{i} A_i = A_{n}A_{n-1} \cdots A_{2}A_{1} \quad \text{and} \quad \prod_{i} A_i = A_{1}A_{2} \cdots A_{n-1}A_{n}.
$$

Theorem 2 in [7] states that the two pre-BVs produce two different presentations of the same off-shell BV:

$$
\mathcal{B}^\beta(\bar{t}_\beta) = \mathcal{B}^\beta(\bar{t}_\beta)|0\rangle = \mathcal{B}^\beta(\bar{t}_\beta)|0\rangle.
$$

(2.12)
The ordering in the products over \( \xi \) in the formulas (2.10) and (2.11) is not important, because of
the commutativity of the entries of \( T \)-operators with equal matrix indices. Moreover, the
order in the products of the non-commuting diagonal monodromy matrix elements in (2.10)
and (2.11) is also non-important, since these combinations will eventually act onto the highest
weight vector \(|0\rangle \) and produce products of scalar functions, due to (1.5).

The formulas (2.10) and (2.11) contain the summations over permutations of the
symmetric functions. We will calculate these summations in the following section and formulas
for the pre-BV will be presented as sums over different partitions of the set of Bethe parameters
(2.1).

3. Off-shell BV as sums over partitions

To save space and simplify formulas, we will use following convention for the products of the
commuting entries of the monodromy matrix \( T_{ij} \), vacuum eigenvalues \( \lambda_i \) and functions \( f(u, v) \).
Namely, whenever such an operator or a function depends on a set of variables, say \( \bar{\xi} \),
this means that we will deal with the product of these commuting operators or functions in respect
of the corresponding set:

\[
T_{ij}(\bar{\xi}) = \prod_{\xi \in \bar{\xi}} T_{ij}(\xi), \quad \lambda_i(\bar{\xi}) = \prod_{\xi \in \bar{\xi}} \lambda_i(\xi), \quad f(\bar{\xi}, \bar{\eta}) = \prod_{\xi, \eta \in \bar{\xi}} \prod_{\eta' \in \bar{\eta}} f(\xi, \eta).
\]  

(3.1)

In various formulas the Izergin determinant \( K_n(\bar{x}, \bar{y}) \) appears [12]. It is defined for two
sets \( \bar{x} \) and \( \bar{y} \) having the same cardinality \#\( \bar{x} = \#\bar{y} = n \):

\[
K_n(\bar{x}|\bar{y}) = \prod_{1 \leq i, j \leq k} (q x_i - q^{-1} y_j) \cdot \det \left[ \frac{q - q^{-1}}{x_i - y_j} \right].
\]  

(3.2)

Below we also use two modifications of the Izergin determinant

\[
K_n^{(l)}(\bar{x}|\bar{y}) = \prod_{i=1}^{n} x_i \cdot K_n(\bar{x}|\bar{y}), \quad K_n^{(r)}(\bar{x}|\bar{y}) = \prod_{i=1}^{n} y_i \cdot K_n(\bar{x}|\bar{y}),
\]  

(3.3)

which we call left and right Izergin determinants respectively.

3.1. Combinatorial description of the partitions

The main goal of this section is to transform the equations (2.10) and (2.11) for pre-BV into
new representations involving sums with respect to partitions of the Bethe parameters. Let us
first describe the general strategy of these transforms.

Consider for definiteness (2.10). This representation contains sums of different types.
One sum is taken over all possible permissible sets \([\bar{m}]\). Each individual term in this sum
includes summations over permutations of the same type of Bethe parameters. For example,
the extreme permissible set in (2.10) such that \( \bar{m}' = \delta/\bar{n} \) corresponds to the term

\[
\frac{1}{\bar{n}_1! \bar{n}_2! \cdots \bar{n}_{N-1}!} \text{Sym} \left[ \prod_{i=1}^{N-1} \prod_{\ell=1}^{\bar{n}_i} (T_{1,2}(\bar{\xi}) T_{2,3}(\bar{\xi}) \cdots T_{N-1,N}(\bar{\xi})) \right],
\]  

(3.4)

and the summation over permutations of the Bethe parameters can be easily performed due to
the commutativity of the entries of the monodromy matrix with the same matrix indices. It is
clear that this summation removes the combinatorial factors in (3.4).

We can treat other terms in the sum over permissible sets in (2.10) similarly. For any
permissible set \([\bar{m}]\) one can consider the sum over permutations of the Bethe parameters of
the same type, as the sum over special partitions of these parameters into subsets and further
permutations within every subset. The partitions of the Bethe parameters into subsets are
determined by the permissible set [\(\bar{m}\)]. They correspond to the Bethe parameters that enter the
products in (2.10) through the same monodromy matrix entry. Due to the commutativity of
such matrix elements the sums over permutations inside these subsets can be calculated via
the identities (3.13) and (3.14). They also remove the combinatorial factors. Details of this
calculation will be presented below in the proof of the proposition 3.1.

The presentation (2.11) can be transformed in the similar manner. As a result we will be
left with summations over all possible partitions of the sets of Bethe parameters dictated by
the permissible set [\(\bar{m}\) and [\(\bar{t}\)]. It is convenient to parameterize the subsets of every set \(\bar{r}^k\) by
two positive integers \(i\) and \(j\) satisfying the conditions

\[
1 \leq i \leq k \leq j \leq N - 1,
\]

and

\[
\#\bar{r}^k_{i,j} = m_j - m_j^{i+1} \quad \text{for} \quad i = 1, \ldots, k \quad \text{and} \quad j = k, \ldots, N - 1, \quad \forall k
\]

for the upper permissible matrix [\(\bar{m}\)]. For the lower permissible matrix [\(\bar{t}\)], we introduce
analogous partitions of the same sets (3.5) with similar conditions:

\[
\#\bar{r}^k_{i,j} = s_j - s_j^{i+1} \quad \text{for} \quad i = 1, \ldots, k \quad \text{and} \quad j = k, \ldots, N - 1, \quad \forall k.
\]

Note that for any given pair \((i, j)\), we have \(\#\bar{r}^k_{i,j} = \#\bar{r}^{k'}_{i,j}, \forall k, k'. \) Notice also that, because
of the conditions (2.4) and (2.5), we have indeed \(\#\bar{r}^k = \sum_{i=1}^{N-1} \sum_{j=k}^{N-1} \#\bar{r}_{i,j}^k\).

We introduce ordering rules ‘<’ and ‘≺’ of these pairs according to the following
conventions

\[
i, j < i', j' \quad \text{if} \quad i < i', \quad \forall j, j' \quad \text{or if} \quad i = i', j < j',
\]

and

\[
i, j < i', j' \quad \text{if} \quad j < j', \quad \forall i, i' \quad \text{or if} \quad j = j', i < i'.
\]

**Example 3.1.** To illustrate how combinatorial data encoded into permissible matrices (2.6)
and (2.7) is transferred into the description of partitions (3.5) we write explicitly the partitions
associated with this data in case of the \(U_q(\mathfrak{gl}_4)\) off-shell BV.

We have four types of Bethe parameters \(\bar{r}^1, \ldots, \bar{r}^4\), and the permissible collections lead to
4 \(\times\) 4 triangular matrices of types (2.6) and (2.7). The rules (3.5) show that the sets of Bethe
parameters \(\bar{r}^1\) and \(\bar{r}^4\) (resp. \(\bar{r}^2\) and \(\bar{r}^3\)) are divided into four (resp. 6) subsets. We get the subsets

\[
\bar{r}^1 = \bar{r}^1_{1,1} \cup \bar{r}^1_{2,2} \cup \bar{r}^1_{3,3} \cup \bar{r}^1_{4,4}
\]

\[
\bar{r}^2 = \bar{r}^2_{2,2} \cup \bar{r}^2_{3,3} \cup \bar{r}^2_{4,4} \cup \bar{r}^2_{2,2} \cup \bar{r}^2_{3,3} \cup \bar{r}^2_{4,4}
\]

\[
\bar{r}^3 = \bar{r}^3_{3,3} \cup \bar{r}^3_{4,4} \cup \bar{r}^3_{2,2} \cup \bar{r}^3_{3,3} \cup \bar{r}^3_{4,4} \cup \bar{r}^3_{3,3}
\]

\[
\bar{r}^4 = \bar{r}^4_{4,4} \cup \bar{r}^4_{3,3} \cup \bar{r}^4_{4,4} \cup \bar{r}^4_{2,2} \cup \bar{r}^4_{3,3} \cup \bar{r}^4_{4,4}
\]

for (2.6) and the subsets

\[
\bar{r}^4 = \bar{r}^4_{4,4} \cup \bar{r}^4_{3,3} \cup \bar{r}^4_{4,4} \cup \bar{r}^4_{1,1}
\]

\[
\bar{r}^3 = \bar{r}^3_{4,4} \cup \bar{r}^3_{4,4} \cup \bar{r}^3_{4,4} \cup \bar{r}^3_{3,3} \cup \bar{r}^3_{4,4} \cup \bar{r}^3_{3,3}
\]

\[
\bar{r}^2 = \bar{r}^2_{4,4} \cup \bar{r}^2_{4,4} \cup \bar{r}^2_{4,4} \cup \bar{r}^2_{3,3} \cup \bar{r}^2_{4,4} \cup \bar{r}^2_{3,3}
\]

\[
\bar{r}^1 = \bar{r}^1_{4,4} \cup \bar{r}^1_{4,4} \cup \bar{r}^1_{4,4} \cup \bar{r}^1_{3,3} \cup \bar{r}^1_{4,4} \cup \bar{r}^1_{3,3}
\]

for (2.7). The subsets in the same column have the same cardinality, which is defined by the
formulas (3.6) and (3.7).
Now we can formulate the following

**Proposition 3.1.** The formulas \((2.10)\) and \((2.11)\) for the universal pre-BV can be written as sums over the partitions of the Bethe parameters \((3.5)-(3.7)\) as follows

\[
\begin{aligned}
B^\Phi(\vec{\ell}_n) &= \sum_{\text{part}} \prod_{k=1}^{N-1} \prod_{i,j \neq f',f} f(\vec{t}_{i,j}^k, \vec{t}_{i,j}^{k-1}) \prod_{k=2}^{N-1} \left( \prod_{i,j \neq f',f} t(\vec{t}_{i,j}^k, \vec{t}_{i,j}^{k-1}) \prod_{i<j} K^{(l)}(\vec{t}_{i,j}^k, \vec{t}_{i,j}^{k-1}) \right) \\
&\quad \times \prod_{1 \leq k \leq N-1} \left( \prod_{N \geq j > k} T_{j,k}(\vec{t}_{j,k}^k) \right) \prod_{k=2}^{N-1} \prod_{i,j = k,k} T_{k,k}(\vec{t}_{i,j}^k), \tag{3.10}
\end{aligned}
\]

and

\[
\begin{aligned}
\widetilde{B}^\Phi(\vec{\ell}_n) &= \sum_{\text{part}} \prod_{k=1}^{N-1} \prod_{i,j \neq f',f} f(\vec{t}_{i,j}^k, \vec{t}_{i,j}^{k-1}) \prod_{k=2}^{N-1} \left( \prod_{i,j \neq f',f} t(\vec{t}_{i,j}^k, \vec{t}_{i,j}^{k-1}) \prod_{i<j} K^{(l)}(\vec{t}_{i,j}^k, \vec{t}_{i,j}^{k-1}) \right) \\
&\quad \times \prod_{N-1 \geq k \geq 1} \left( \prod_{1 \leq j \leq k} T_{j,k+1}(\vec{t}_{j,k+1}^k) \right) \prod_{k=1}^{N-2} \prod_{i,k < i,j} T_{i,k+1}(\vec{t}_{i,k+1}^k). \tag{3.11}
\end{aligned}
\]

**Remark 3.1.** Note that, at fixed \(k\), the operators \(T_{i,j}\) in \((3.10)\) (resp. \(T_{i,j+1}\) in \((3.11)\)) depend on the same type of the Bethe parameters. The products of such operators can be reordered in a rather simple way using RTT-relations. We give an example of such reordering in section 5.1.

**Proof.** The basic idea is to replace the summations over permutations of the Bethe parameters and permissible matrices \([\vec{m}], [\vec{\imath}]\) in \((2.10)\) and \((2.11)\) by the summations over partitions of the sets of the Bethe parameters. Once this is done, the sum over permutations within any fixed subset can be calculated using certain identities of the rational functions \((3.13)-(3.14)\). To formulate these identities we introduce the following notations: for any sets \(\vec{x}\) and \(\vec{y}\) with equal cardinalities \#\(\vec{x}\) = \#\(\vec{y}\) = \(n\), we introduce rational functions

\[
B(\vec{y}) = \prod_{\ell < \ell'} f(y_{\ell}, y_{\ell'}), \quad G(\vec{y};\vec{x}) = \prod_{\ell=1}^n g(y_{\ell}, x_{\ell}), \quad F(\vec{y};\vec{x}) = \prod_{\ell < \ell'} f(y_{\ell}, x_{\ell}). \tag{3.12}
\]

Then the following identities of the rational functions are valid:

\[
\text{Sym}_2(B(\vec{y});G(\vec{y};\vec{x})F(\vec{y};\vec{x})) = K_n(\vec{y};\vec{x}), \tag{3.13}
\]

\[
\text{Sym}_2(B(\vec{x});G(\vec{y};\vec{x})F(\vec{y};\vec{x})) = K_n(\vec{x};\vec{y}). \tag{3.14}
\]

The proof of these identities, in the case of quantum integrable models associated with rational \(R\)-matrix, is given in appendix A of [19]. The proof in the trigonometric case is completely analogous.

According to the partitions defined by the permissible matrices \([\vec{m}]\) and \([\vec{\imath}]\) the rational function \(\beta(\vec{t}_n)\) can be rewritten as

\[
\beta(\vec{t}_n) = \prod_{k=1}^{N-1} \prod_{1 \leq \ell < \ell' \leq n_k} f(\vec{t}_{\ell,\ell'}^k, \vec{t}_{\ell,\ell'}^k) = \prod_{k=1}^{N-1} B(\vec{t}_k) = \prod_{i \leq k < j} B(\vec{t}_{i,j}^k), \tag{3.15}
\]

where \(\vec{t}_{i,j}^k\) are defined either by \((3.6)\) or by \((3.7)\).

We show the equivalence of \((2.10)\) and \((3.10)\) first, using partitions \((3.6)\). We will deconstruct the proof in a series of steps.
(1) The product of rational functions $\prod_{k=1}^{N-1} B(\hat{\mathcal{I}}_{k})$ entering the function $\beta(\hat{\mathcal{I}}_k)$ will be canceled out by the product of the rational functions in the first row of (2.10). This cancels some of the factorial factors. At this step the expression in (2.10) is symmetric with respect to the permutations of the set $\hat{\mathcal{I}}_{N-1}$ and the factorial factor $(m_{N-1}^{N-1})^{-1}$ disappear.

(2) At this step, for $k = 1, \ldots, N-2$, we can select the following product of rational functions in the rhs of (2.10)
\[
B(\hat{\mathcal{I}}_{N-1}^{N-1}) G(\hat{\mathcal{I}}_{k, N-1}^{N-2}) F(\hat{\mathcal{I}}_{k, N-1}^{N-1}).
\]
Since all the other factors in the rhs of (2.10) are symmetric with respect to the permutations in the set $\hat{\mathcal{I}}_{k, N-1}$, we can perform the symmetrization over these sets and apply the identity (3.13) to obtain the product over $k$ of the Izergin determinants $K(\hat{\mathcal{I}}_{k, N-1}^{N-1})$ which are symmetric with respect to permutations in the sets $\hat{\mathcal{I}}_{k, N-1}^{N-2}$. Moreover the factorial factors $(m_{N-1}^{N-2} - m_{N-2}^{N-2})^{-1}$ will disappear since the expression under summation is symmetric with respect to permutations in the sets $\hat{\mathcal{I}}_{N-2}$ and $\hat{\mathcal{I}}_{N-2}$.

(3) Next we perform an analogous procedure to obtain the product of the Izergin determinants over $k = 1, \ldots, N-3$
\[
K(\hat{\mathcal{I}}_{k, N-1}^{N-3}) K(\hat{\mathcal{I}}_{k, N-1}^{N-3}),
\]
which is symmetric with respect to the permutations in the sets $\hat{\mathcal{I}}_{k, N-1}^{N-3}$ and $\hat{\mathcal{I}}_{k, N-1}^{N-3}$. Now the combinatorial factors $(m_{N-3}^{N-3} - m_{N-3}^{N-3})^{-1}$ will disappear due to the symmetry of the summand with respect to permutations in the sets $\hat{\mathcal{I}}_{N-3}$ and $\hat{\mathcal{I}}_{N-3}$.

(4) We iterate these symmetrizations over the sets up to the final step, which corresponds to the symmetrizations over the sets $\hat{\mathcal{I}}_{1, k}$, $k = 2, \ldots, N-1$. We obtain in this way the product of Izergin determinants $\prod_{k=1}^{N-2} K(\hat{\mathcal{I}}_{1, k})$ and the last combinatorial factor $\prod_{k=1}^{N-3} (m_k^{N-1} - m_{k+1})^{-1}$ disappears, due to symmetry arguments.

Besides the product of Izergin determinants described above, we are also left with the product of rational functions, which can be written as in the first line of (3.10) using the ordering of the sets introduced by (3.8).

The proof of equivalence between (2.11) and (3.11) is analogous. The only difference is that we have to start the whole procedure with the partitions (3.7), and begin with the symmetrizations over sets $\hat{\mathcal{I}}_{1,1}$, $\hat{\mathcal{I}}_{1,2}$ and so on. Then, we use the identity (3.14) to produce the product of the Izergin determinants.

4. Morphisms

The aim of this section is to describe certain morphisms which relate the algebras $U_q(\hat{\mathfrak{g}}_k)$ and $U_{q^{-1}}(\hat{\mathfrak{g}}_k)$ [20]. It will be shown that one of these morphisms relates pre-BV (3.10) and (3.11). Using the second morphism one can obtain dual or left pre-BV in the form of sums over partitions of the Bethe parameters from the formulas for the right pre-BV (3.10) and (3.11).

4.1. Properties of the R-matrix

Note that the functions (1.3) introduced above, and the modified Izergin determinants (3.3) obey the following relations
\[
\begin{align*}
\mathbb{g}_q^{(i)}(v, u) &= \mathbb{g}_q^{(i)}(u, v), & \mathbb{g}_q^{(i)}(u^{-1}, v^{-1}) &= \mathbb{g}_q^{(i)}(u, v), \\
\mathbb{f}_q^{(i)}(v, u) &= \mathbb{f}_q(u, v), & \mathbb{f}_q^{(i)}(u^{-1}, v^{-1}) &= \mathbb{f}_q(u, v), \\
\mathbb{K}_q^{(i)}(\vec{v} | \vec{u}) &= \mathbb{K}_q^{(i)}(\vec{u} | \vec{v}), & \mathbb{K}_q^{(i)}(\vec{u}^{-1} | \vec{v}^{-1}) &= \mathbb{K}_q^{(i)}(\vec{u} | \vec{v}).
\end{align*}
\]
We also have the following properties, that can be proved by direct calculation:

\[ R_{12}(u, v) R_{21}(v, u) = I_q(u, v) I_q(v, u) \otimes 1, \quad R_{12}(u, v)^{t; 2} = R_{21}(u, v), \]

\[ U_1 U_2 R_{12}(u, v) U_1^{-1} U_2^{-1} = R_{21}(u, v), \quad U = \sum_{i=1}^{N} E_{i, N+1-i}, \]

\[ R_{21}(v, u; q) = R_{21}(u^{-1}, v^{-1}; q) = R_{12}(u, v; q^{-1}), \quad R_{12}(v^{-1}, u^{-1}) = R_{12}(u, v), \] (4.2)

where \( ^t \) denotes the transposition (\( E_{ij} \))^t = E_{ji}, \( i, j \). We have used auxiliary space notations, where the indices on an operator indicate in which space(s) it acts non-trivially, and \( R_{21}(u, v) = P_{12} R_{12}(u, v) P_{12} \) with \( P_{12} \) the permutation of the two spaces 1 and 2.

### 4.2. Isomorphism and antimorphisms

As one can easily deduce from the properties of the \( R \)-matrix (4.2), there exist the following morphisms from \( U_q(\hat{\mathfrak{gl}}_N) \) to \( U_{q^{-1}}(\hat{\mathfrak{gl}}_N) \).

**Proposition 4.1.**

(i) The map \( \psi \) defined by

\[ \psi(T(u)) = UT^t(u)U^{-1}, \] (4.3)

where \( U \) is given in (4.2), defines an isomorphism from \( U_q(\hat{\mathfrak{gl}}_N) \) to \( U_{q^{-1}}(\hat{\mathfrak{gl}}_N) \).

(ii) The map \( \psi \) given by

\[ \psi(T(u)) = T^t(u^{-1}), \] (4.4)

defines an anti-isomorphism from \( U_q(\hat{\mathfrak{gl}}_N) \) to \( U_{q^{-1}}(\hat{\mathfrak{gl}}_N) \). In (4.3) and (4.4), \( T(u) \in U_q(\hat{\mathfrak{gl}}_N) \) while \( \psi(T(u)) \) and \( \psi(T(u)) \) belong to \( U_{q^{-1}}(\hat{\mathfrak{gl}}_N) \).

**Proof.** We start with the defining relation of \( U_q(\hat{\mathfrak{gl}}_N) \)

\[ R_{12}(u, v; q) \cdot T_1(u) \cdot T_2(v) = T_2(v) \cdot T_1(u) \cdot R_{12}(u, v; q), \] (4.5)

apply the transpositions in space 1 and then in space 2, and conjugation by \( U_1 U_2 \) to get

\[ U_1 T_1^t(u) U_1^{-1} \cdot U_2 T_2^t(v) U_2^{-1} \cdot R_{12}(u, v; q) = R_{12}(u, v; q) \cdot U_2 T_2^t(v) U_2^{-1} \cdot U_1 T_1^t(u) U_1^{-1}, \] (4.6)

where we have used (4.2). Using again (4.2), we can rewrite (4.6) as

\[ \psi(T_1(u)) \cdot \psi(T_2(v)) \cdot R_{21}(v, u; q^{-1}) = R_{21}(v, u; q^{-1}) \cdot \psi(T_2(v)) \cdot \psi(T_1(u)). \] (4.7)

After relabeling 1 ↔ 2 and \( u \leftrightarrow v \), one recognizes in (4.7) the defining relations for \( U_{q^{-1}}(\hat{\mathfrak{gl}}_N) \). This proves (i).

Now, starting again from (4.5), and applying transpositions in space 1 and then in space 2, and the transformation \( (u, v) \rightarrow (u^{-1}, v^{-1}) \), we get

\[ T_1^t(u^{-1}) \cdot T_2^t(v^{-1}) \cdot R_{21}(u^{-1}, v^{-1}; q) = R_{21}(u^{-1}, v^{-1}; q) \cdot T_2^t(v^{-1}) \cdot T_1^t(u^{-1}). \] (4.8)

Using once more (4.2), it can be rewritten as

\[ \psi(T_1(u)) \cdot \psi(T_2(v)) \cdot R_{12}(u, v; q^{-1}) = R_{12}(u, v; q^{-1}) \cdot \psi(T_2(v)) \cdot \psi(T_1(u)). \] (4.9)

One recognizes in (4.9) the image of defining relations for \( U_{q^{-1}}(\hat{\mathfrak{gl}}_N) \) under an antimorphism, i.e. \( \psi(T_1(u) \cdot T_2(v)) = \psi(T_2(v)) \cdot \psi(T_1(u)) \). This proves (ii). \( \square \)
5. Closed formulas for dual off-shell BV

Define the following combinations of the monodromy matrix elements

\[
C_\circ(t_\circ) = \sum_{\text{part}} \prod_{k=1}^{N-1} f(t_{i,k}^2, t_{i,k}^2) \prod_{k=2}^{N-1} \left( \prod_{i<j} f(t_{i,k}^2, t_{i,k}^{2-1}) \prod_{i<j} K^{(e)}(t_{i,k}^2)^{2-1} \right) \times \prod_{k=2}^{N-1} \prod_{i,j=k,k} T_{k,k} t_{i,j} \prod_{k=1}^{N-1} \left( \prod_{k<j} T_{k,k} t_{i,j} \right),
\]

(5.1)

\[
\tilde{C}_\circ(t_\circ) = \sum_{\text{part}} \prod_{k=1}^{N-1} f(t_{i,k}^2, t_{i,k}^2) \prod_{k=2}^{N-1} \left( \prod_{i<j} f(t_{i,k}^2, t_{i,k}^{2-1}) \prod_{i<j} K^{(e)}(t_{i,k}^2)^{2-1} \right) \times \prod_{k=2}^{N-1} \prod_{i,j=k,k} T_{k,k} t_{i,j} \prod_{k=1}^{N-1} \left( \prod_{k<i,j} T_{k,k} t_{i,j} \right).
\]

(5.2)

We have the following.

**Proposition 5.1.**

- The morphism \( \phi \) relates the universal off-shell pre-BV

\[
\phi(B_q^\circ(t_\circ)) = B_q^{\circ_{\omega}}(t_\circ),
\]

(5.3)

where \( \omega \) maps the sets of the Bethe parameters into the fully permuted sets:

\[
\omega : t_\circ \rightarrow \omega t_\circ = \{ t_1^{n-1}, \ldots, t_{m-1}^{n-1}, t_1^{n-2}, \ldots, t_{m-2}^{n-2}, \ldots; t_1^2, \ldots, t_{m-1}^2; t_1^1, \ldots, t_{m-1}^1 \},
\]

and accordingly for the sets of cardinalities:

\[
\omega : n \rightarrow \omega n = \{ n_{N-1}, n_{N-2}, \ldots, n_2, n_1 \}.
\]

- The combinations (5.1) and (5.2) are related to the pre-BV (3.10) and (3.11) by the antismorphism \( \psi \)

\[
\psi(B_q^\circ(t_\circ)) = C_q^{\circ_{\omega}}(t_\circ^{-1}) \quad \text{and} \quad \psi(\tilde{B}_q^\circ(t_\circ)) = \tilde{C}_q^{\circ_{\omega}}(t_\circ^{-1}),
\]

(5.4)

where

\[
\tilde{t}_\circ^{-1} = \{ (t_1^{-1}), \ldots, (t_1^{-1}); (t_2^{-1}), \ldots, (t_1^{-1}); \ldots; (t_1^{n-2}); \ldots; (t_1^{n-1})^{-1} \}
\]

is the set of the inverses of the Bethe parameters.

**Proof.** We start with the first item. We write expression for pre-BV (3.10) for the algebra \( U_q^{-1}(\mathfrak{gl}_n) \) and using the sets \( \tilde{n} \leftrightarrow \omega \tilde{n}, \tilde{t}_\circ \leftrightarrow \omega \tilde{t}_\circ \):

\[
B_q^{-\omega}(\tilde{t}_\circ) = \sum_{\text{part}} \prod_{k=1}^{N-1} f(t_{i,k}^{-1}, t_{i,k}^{-1}) \prod_{k=2}^{N-1} \left( \prod_{i<j} f(t_{i,k}^{-1}, t_{i,k}^{2-1}) \prod_{i<j} K^{(e)}(t_{i,k}^{-1})^{2-1} \right) \times \prod_{k=2}^{N-1} \prod_{i,j=k,k} \tilde{T}_{k,k} t_{i,j} \prod_{k=1}^{N-1} \left( \prod_{k<i,j} \tilde{T}_{k,k} t_{i,j}^{-1} \right) \prod_{k=2}^{N-1} \prod_{i,j=k,k} \tilde{T}_{k,k} t_{i,j}^{-1}. \]

(5.5)

Here, \( \tilde{T}_{i,j}(t) \) are the matrix elements of the \( U_q^{-1}(\mathfrak{gl}_n) \) monodromy matrix.
The next step is to use the formulas (4.1) to change the rational functions $f_q \rightarrow f_q$ and $K_{q,r}^{(l)} \rightarrow K_q^{(l)}$, and apply the morphism $\varphi^{-1}$ to obtain, starting from (5.5):

\[
\varphi^{-1}(B_{q,r}^{\bar{i}_{t_0}}(i_{t_0})) = \sum_{\text{part}} \prod_{k=1}^{N-1} \prod_{i,j<i',j'} f_q(\hat{\tau}^{N-k}_{i,j}, \hat{\tau}^{N-k}_{i',j'}) \prod_{k=2}^{N-1} \sum_{i,j=1}^{N-k} t_q(\hat{\tau}^{N-k+1}_{i,j}, \hat{\tau}^{N-k}_{i,j}) \times \prod_{k=2}^{N-1} \prod_{i,j=k,k} T_{N+1-k,N+1-k}(\hat{\tau}^{N-k}_{i,j}).
\]

Let us consider the products in each line of (5.6) separately.

- In the product of the first line we renumber the partition (3.5) of the set $i_{t_0}$. We introduce new pairs of integers $j = N - i$ and $i = N - j$ which can be also used to numerate the partition (3.5) since it is clear that for any given $k$ which defines the pair $i, j$ the new integers $i, j$ will also satisfy the condition $1 \leq i \leq N - k \leq j \leq N - 1$. One can verify that the condition $i, j < i', j'$ will be reformulated as $i, j < i', j$. Then, after renaming the integers: $k \rightarrow N - k, i \rightarrow j, i \rightarrow i'$ and $j \rightarrow j'$, one can see that the first product in (5.6) coincides literally with the first product in (3.11)

\[
\prod_{k=1}^{N-1} \prod_{i,j<i',j'} f(\hat{\tau}^{N-k}_{i,j}, \hat{\tau}^{N-k}_{i',j'}). \]

- Using the same arguments, the same change of dumb integers and renaming $k \rightarrow N - k + 1$, $i \rightarrow j$ and $j \rightarrow j'$, we can prove that the second product in (5.6) is equal to

\[
\prod_{k=2}^{N-1} \prod_{i,j=k,k} f(\hat{\tau}^{N-k-1}_{i,j}, \hat{\tau}^{N-k}_{i',j'}). \]

- In the third product of (5.6) we change numeration of the partition of the set $\hat{\tau}^{N-k+1}$ in such a way to obtain the product

\[
\prod_{k=2}^{N-1} \prod_{i,j<i',j'} K^{(l)}(\hat{\tau}^{N-k}_{i,j}, \hat{\tau}^{N-k}_{i',j'}). \]

- Renaming the partitions $[\hat{\tau}^{N-k}_{i,j}] \rightarrow [\hat{\tau}^{N-k}_{N-j,N-k}]$ and changing the dumb integers $k \rightarrow N - k$ and $j \rightarrow N - j + 1$, we obtain for the fourth product in (5.6)

\[
\prod_{N-1 \geq k \geq 1} \left( \prod_{1 \leq j \leq k} T_{k+1-1}(\hat{\tau}^{N-k}_{i,j}) \right). \]

- Finally in the fifth product of (5.6), we renumber the partition $[\hat{\tau}^{N-k}_{i,j}] \rightarrow [\hat{\tau}^{N-k}_{N-j,N-i}]$ and rename the dumb variables $k \rightarrow N - k, N - j \rightarrow i$ and $N - i \rightarrow j$. It leads to the product

\[
\prod_{k=1}^{N-2} \prod_{k,k<i,j} T_{k+1,k+1}(\hat{\tau}^{N-k}_{i,j}). \]

This proves the relation (5.3) of the first item of the proposition.
For the proof of the second item we consider the first relation in (5.4). The second one can be proved analogously. We rewrite again the expression of the pre-BV (3.10) for $U_{q^{-1}}(\hat{\mathfrak{gl}}_N)$ and $\tilde{t}_n \leftrightarrow \tilde{t}_n^{-1}$

$$B^\mu_{q^{-1}}((\tilde{t}_n)^{-1}) = \sum_{\text{part}} \prod_{k=1}^{N-1} \prod_{j \neq l, f} f_{q^{-1}}((\tilde{t}_{i,j}^k)^{-1}, (\tilde{t}_{i,j}^k)^{-1}) \prod_{k=2}^{N-1} \prod_{i \neq l, f} f_{q^{-1}}((\tilde{t}_{i,j}^k)^{-1}, (\tilde{t}_{i,j}^k)^{-1}) \prod_{k=2}^{N-1} \prod_{i \neq l, f} \tilde{T}_{k,i}((\tilde{t}_{i,j}^k)^{-1}) \prod_{1 \leq k \leq N-1} \left( \prod_{N \geq j \geq k} \tilde{T}_{k,j}((\tilde{t}_{i,j}^k)^{-1}) \right). \tag{5.7}$$

Here again $\tilde{T}_{i,j}(t)$ are the matrix elements of the $U_{q^{-1}}(\hat{\mathfrak{gl}}_N)$ monodromy matrix. Now we apply the antimorphism $\psi$ to (5.7), using once more the relation (4.1) to obtain

$$\psi(B^\mu_{q^{-1}}((\tilde{t}_n)^{-1})) = \sum_{\text{part}} \prod_{k=1}^{N-1} \prod_{j \neq l, f} f_{q^{-1}}((\tilde{t}_{i,j}^k)^{-1}, (\tilde{t}_{i,j}^k)^{-1}) \prod_{k=2}^{N-1} \prod_{i \neq l, f} f_{q^{-1}}((\tilde{t}_{i,j}^k)^{-1}, (\tilde{t}_{i,j}^k)^{-1}) \prod_{k=2}^{N-1} \prod_{i \neq l, f} \tilde{T}_{k,i}((\tilde{t}_{i,j}^k)^{-1}) \prod_{1 \leq k \leq N-1} \left( \prod_{N \geq j \geq k} \tilde{T}_{k,j}((\tilde{t}_{i,j}^k)^{-1}) \right). \tag{5.8}$$

In (5.8) the property of the antimorphism $\psi$ is used to reverse the order of the product of the non-commuting operators. The rhs of (5.8) coincides with the combination (5.1). Thus, the second item of the Proposition is proved. \hfill \Box

Repeating arguments used for the proof of the proposition 5.1, we can deduce from properties of $\mathfrak{B}^\mu(\tilde{t}_n)$, similar properties for $\mathfrak{B}^\nu(\tilde{t}_n)$ and $\mathfrak{C}^\mu(\tilde{t}_n)$. As an illustration, we prove the following.

**Proposition 5.2.** The combinations of the monodromy matrix elements (5.1) and (5.2) are the dual pre-BV because their left action onto vacuum vector defined by (1.6) produces the dual off-shell BV

$$\mathfrak{C}^\mu(\tilde{t}_n) = \langle 0 | \mathfrak{C}^\mu(\tilde{t}_n) = \langle 0 | \mathfrak{C}^\mu(\tilde{t}_n), \tag{5.9}$$

which are eigenvectors under the left action of the transfer matrix $T(t) = \sum_{i=1}^{N} T_{t,i}(t)$:

$$\mathfrak{C}^\mu(\tilde{t}_n) \tau(t; \tilde{t}_n) = \mathfrak{C}^\mu(\tilde{t}_n) T(t), \tag{5.10}$$

with eigenvalue

$$\tau(t; \tilde{t}_n) = \sum_{i=1}^{N} \lambda_i^+ (t) \prod_{j=1}^{n_i} f_{t,j}^{-1} \prod_{j=1}^{n_i} f_{t,j}, \tag{5.11}$$

provided the Bethe equations

$$\frac{\lambda_i^+(t)}{\lambda_{i+1}^+(t)} = (-1)^{n_i-1} \prod_{m=1}^{n_i} f_{t,i_m} \prod_{m=1}^{n_i} f_{t,i_m}^{-1} \prod_{m=1}^{n_i} f_{t,i_m+1} \tag{5.12}$$

are satisfied.

**Proof.** We start with the relation (2.12), written in the algebra $U_{q^{-1}}(\hat{\mathfrak{gl}}_N)$ and for the set of parameters $\tilde{t}_n^{-1}$. Then, we apply $\psi^{-1}$ to get (5.9).
For the remaining part of the proposition, we use a result proved in the paper [17]. It states that in $U_q(\mathfrak{gl}_N)$, we have
\[ T(t; \vec{\tau}) = \tau(t; \vec{\tau}) T(\vec{\tau}), \]
if the Bethe equations (5.12) are satisfied. We reformulate this result for $U_q(\hat{\mathfrak{gl}}_N)$, in the following form:
\[ T_{q^{-1}}(t) \mathcal{B}_{q^{-1}}(\vec{\tau}) = \tau_{q^{-1}}(t; \vec{\tau}) T_{q^{-1}}(\vec{\tau}) + \sum_{j=1}^{N-1} \sum_{i=1}^{N} \mathcal{O}_{ij} \left\{ \prod_{m=1}^{n_i-1} t_{q^{-1}}(t'_m, t'_j) \prod_{m=1}^{n_{j-1}} t_{q^{-1}}(t'_{m+1}, t'_j) \tilde{T}_{i,j}(t'_j) \right\} |0\rangle, \] (5.14)
where $T_{q^{-1}}(t) = \sum_{i=1}^{N} \tilde{T}_i(t)$ is the transfer matrix in $U_{q^{-1}}(\hat{\mathfrak{gl}}_N)$, and $\mathcal{O}_{ij}$ are some operators, whose explicit form is not needed for our proof.

A direct calculation, using properties (4.1), shows that $\tau_{q^{-1}}(t^{-1}; \vec{\tau}) = \tau_q(t; \vec{\tau})$. It is also clear that $\psi^{-1}(T_{q^{-1}}(t^{-1})) = T_q(t)$. Then, applying $\psi^{-1}$ to (5.14) written using a spectral parameter $t^{-1}$ and the sets $\vec{\tau}^{-1}$, we get in $U_q(\mathfrak{gl}_N)$
\[ \mathcal{C}_q(\vec{\tau}) T(t) = \mathcal{C}_q(\vec{\tau}) \tau(t; \vec{\tau}) + \sum_{i=1}^{N-1} \sum_{j=1}^{N} \sum_{l=0}^{n_i-1} \left\{ \prod_{m=1}^{n_i-1} t_q(t'_m, t'_j) \prod_{m=1}^{n_{j-1}} t_q(t'_{m+1}, t'_j) T_{i,j}(t'_j) \right\} \psi^{-1}(\mathcal{O}_{ij}), \] (5.15)
where we have used once more the relations (4.1). This ends the proofs. \[ \square \]

### 5.1. BV for $U_q(\hat{\mathfrak{gl}}_N)$-symmetric integrable models

In the paper [18] the action of the monodromy matrix elements onto right off-shell BVs was calculated in the framework of the current approach and the formulas for the dual off-shell BV were announced. Let us verify that the formulas used in [18] can be obtained from the generic formulas proved in the present paper.

In order to rewrite the $U_q(\mathfrak{gl}_N)$ off-shell BV in the form used in our previous papers we rename the sets of Bethe parameters
\[ \vec{i}, \vec{\tau} \rightarrow \vec{u}, \vec{v}. \]
We rename as follows the subsets of these sets
\[ \vec{i}^1 = \vec{i}^{1}_{1,1} \cup \vec{i}^{1}_{1,2} \rightarrow \vec{u}_1 \cup \vec{u}_1 = \vec{u}, \]
\[ \vec{i}^2 = \vec{i}^{2}_{1,2} \cup \vec{i}^{2}_{2,2} \rightarrow \vec{v}_1 \cup \vec{v}_2 = \vec{v}. \] (5.16)

With this renaming of the parameters and partitions, the formulas (3.10) and (3.11) yield the following expressions for the off-shell BV:
\[ \mathcal{B}(\vec{u}, \vec{v}) = \sum_{\text{part}} \mathcal{K}(\vec{v}_1|\vec{u}_1) f(\vec{u}_1, \vec{u}_1) f(\vec{v}_1, \vec{v}_1) T_{13}(\vec{u}_1) T_{12}(\vec{u}_1) T_{23}(\vec{v}_1) \lambda_{12}(\vec{v}_1)|0\rangle, \]
\[ \mathcal{B}(\vec{u}, \vec{v}) = \sum_{\text{part}} \mathcal{K}(\vec{v}_1|\vec{u}_1) f(\vec{u}_1, \vec{u}_1) f(\vec{v}_1, \vec{v}_1) T_{13}(\vec{u}_1) T_{23}(\vec{v}_1) T_{12}(\vec{u}_1) \lambda_{12}(\vec{u}_1)|0\rangle, \] (5.17)
respectively. The second formula in (5.17) coincides literally with the formula (5.1) in [18] after changing the overall normalization of the BV

$$\mathcal{B}(\bar{u}, \bar{v}) \rightarrow \frac{\mathcal{B}(\bar{u}, \bar{v})}{f(\bar{u}, \bar{u})_{\lambda_2(\bar{u})\lambda_2(\bar{v})}}.$$  

(5.18)

This change of normalization of the BV is useful for the calculation of the action of the monodromy matrix elements onto them because it removes certain poles in the BV and makes formulas for this action quite effective (see details in [18]).

Analogously, formulas (5.1) and (5.2) yield the following expressions for the dual or left off-shell BV

$$\mathcal{C}(\bar{u}, \bar{v}) = \sum_{\text{part}} K^{(\ell)}(\bar{v}_1|\bar{u}_1)f(\bar{u}_1, \bar{u}_2)\hat{T}_2(\bar{v}_1)\hat{T}_2(\bar{u}_2)|0\rangle T_{12}(\bar{u}_1)T_{13}(\bar{u}_2)|0\rangle,$$

$$\mathcal{C}(\bar{u}, \bar{v}) = \sum_{\text{part}} K^{(\ell)}(\bar{v}_1|\bar{u}_1)f(\bar{u}_1, \bar{u}_2)\hat{T}_2(\bar{v}_1)\hat{T}_2(\bar{u}_2)|0\rangle T_{12}(\bar{u}_1)T_{13}(\bar{u}_2)|0\rangle,$$  

(5.19)

and after the same change of normalization as in (5.18), the second formula in (5.19) coincides literally with formula (5.2) in the paper [18].

As we have mentioned above in remark 3.1, in (5.17) the product of operators depending on the same type of Bethe parameters can be reordered. Then, these formulas can be rewritten in a different form:

$$\mathcal{B}(\bar{u}, \bar{v}) = \sum_{\text{part}} K^{(\ell)}(\bar{v}_1|\bar{u}_1)f(\bar{u}_1, \bar{u}_2)\hat{T}_2(\bar{v}_1)\hat{T}_2(\bar{u}_2)|0\rangle T_{12}(\bar{u}_1)T_{13}(\bar{u}_2)|0\rangle,$$

$$\mathcal{B}(\bar{u}, \bar{v}) = \sum_{\text{part}} K^{(\ell)}(\bar{v}_1|\bar{u}_1)f(\bar{u}_1, \bar{u}_2)\hat{T}_2(\bar{v}_1)\hat{T}_2(\bar{u}_2)|0\rangle T_{12}(\bar{u}_1)T_{13}(\bar{u}_2)|0\rangle.$$  

(5.20)

The equivalence between (5.17) and (5.20) can be proved in the context of rational $R$-matrices using the methods presented in [11]. Applying morphisms $\psi$ to the formulas (5.20) we obtain an alternative presentations for the dual BVs

$$\mathcal{C}(\bar{u}, \bar{v}) = \sum_{\text{part}} K^{(\ell)}(\bar{v}_1|\bar{u}_1)f(\bar{u}_1, \bar{u}_2)\hat{T}_2(\bar{v}_1)\hat{T}_2(\bar{u}_2)|0\rangle T_{12}(\bar{u}_1)T_{13}(\bar{u}_2)|0\rangle,$$

$$\mathcal{C}(\bar{u}, \bar{v}) = \sum_{\text{part}} K^{(\ell)}(\bar{v}_1|\bar{u}_1)f(\bar{v}_1, \bar{u}_2)\hat{T}_2(\bar{v}_1)\hat{T}_2(\bar{u}_2)|0\rangle T_{12}(\bar{u}_1)T_{13}(\bar{u}_2)|0\rangle.$$  

(5.21)

Conclusion

In this paper we have obtained explicit formulas for the right and left (dual) off-shell BV in the form of sums over partitions of the sets of Bethe parameters using morphisms of the algebra $U_q(\hat{gl}_N)$. Our starting formulas were presentations of the off-shell BV in terms of sums over permutations in the sets of Bethe parameters obtained previously in [6, 7] by the current approach. Formulas for left or dual off-shell Bethe vectors are necessary to address the problem of calculation of the scalar products of the BV.

In a previous paper [18], we computed the action of the monodromy matrix elements onto nested off-shell BV in integrable models with $GL(3)$ trigonometric $R$-matrix. The morphisms (4.3) and antismorphism (4.4) introduced in this paper allows one to easily relate the different formulas of these actions. For example, one can obtain using antismorphism $\psi$ the left action of the monodromy matrix elements onto dual off-shell BV from the corresponding formulas of the right action onto right off-shell BV and many other useful relations.
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References

[1] Kulish P P and Reshetikhin N Yu 1983 Diagonalization of GL(N) invariant transfer matrices and quantum N-wave system (Lee model) *J. Phys. A: Math. Gen.* 16 L591–6
[2] Kulish P P and Reshetikhin N Yu 1981 Generalized Heisenberg ferromagnet and the Gross–Neveu model *Zh. Eksp. Theor. Fiz.* 80 214–28
[3] Kulish P P and Reshetikhin N Yu 1982 GL(3)-invariant solutions of the Yang–Baxter equation and associated quantum systems *Zap. Nauchn. Sem. POMI.* 120 92–121
[4] Tarasov V and Varchenko A 1994 Jackson integral representations of solutions of the quantized Knizhnik–Zamolodchikov equation *Algebra Anal.* 6 90–137
[5] Tarasov V and Varchenko A 1995 *St. Petersburg Math. J.* 6 275–313 (Engl. transl.)
[6] Khoroshkin S and Pakuliak S 2008 A computation of an universal weight function for the quantum affine algebra $U_q(\hat{gl}_N)$ *J. Math. Kyoto Univ.* 48 277–321
[7] Oskin A, Pakuliak S and Silantyev A 2009 On the universal weight function for the quantum affine algebra $U_q(\hat{sl}_N)$ *Algebra Anal.* 21 196–240
[8] Enriquez B and Rubtsov V 1999 Quasi-Hopf algebras associated with $sl_2$ and complex curves *Isr. J. Math.* 112 61–108
[9] Enriquez B, Khoroshkin S and Pakuliak S 2007 Weight functions and Drinfeld currents *Commun. Math. Phys.* 276 691–725
[10] Khoroshkin S and Pakuliak S 2005 Weight function for $U_q(\hat{sl}_N)$ *Theor. Math. Phys.* 145 1373–99
[11] Belliard S, Pakuliak S, Ragoucy E and Slavnov N A 2013 Bethe vectors of GL(3)-invariant integrable models *J. Stat. Mech.* P02020
[12] Izergin A G 1987 Partition function of the six-vertex model in a finite volume *Dokl. Akad. Nauk SSSR* 297 331–3
[13] Korepin V E 1982 Calculation of norms of Bethe wave functions *Commun. Math. Phys.* 86 391–418
[14] Maillet J M and Terras V 2000 On the quantum inverse scattering problem *Nucl. Phys. B* 575 627–44
[15] Kitanine N, Maillet J M and Terras V 1999 Form factors of the XXZ Heisenberg spin-1/2 finite chain *Nucl. Phys. B* 554 647–78
[16] Belliard S, Pakuliak S, Ragoucy E and Slavnov N A 2013 Form factors in SU(3)-invariant integrable models *J. Stat. Mech.* P04033
[17] Frappat L, Khoroshkin S, Pakuliak S and Ragoucy E 2009 Bethe ansatz for the universal weight function *Ann. Henri Poincaré* 10 513–48
[18] Belliard S, Pakuliak S, Ragoucy E and Slavnov N A 2013 Bethe vectors of quantum integrable models with GL(3) trigonometric $R$-matrix *SIGMA* 9 058
[19] Belliard S, Pakuliak S, Ragoucy E and Slavnov N A 2012 Highest coefficient of scalar products in SU(3)-invariant models *J. Stat. Mech.* P09003
[20] Chari V and Pressley A 1994 *A Guide to Quantum Groups* (Cambridge: Cambridge University Press)