Reconstruction, thermodynamics and stability of the $\Lambda$CDM model in $f(T, \mathcal{T})$ gravity

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Abstract

We reconstruct the $\Lambda$CDM model for $f(T, \mathcal{T})$ theory, where $T$ is the torsion scalar and $\mathcal{T}$ the trace of the energy-momentum tensor. The result shows that the action of $\Lambda$CDM is a combination of a linear term, a constant $(-2\Lambda)$ and a nonlinear term given by the product $\sqrt{-\mathcal{T}}F_\phi[(T^{1/3}/16\pi G) (16\pi G T + T + 8\Lambda)]$, with $F_\phi$ being a generic function. We show that to maintain conservation of the energy-momentum tensor, we should impose that $F_\phi[y]$ must be linear on the trace $\mathcal{T}$. This reconstruction decays in $f(T)$ theory for $F_\phi \equiv Q$, with $Q$ a constant. Our reconstruction describes the cosmological eras to the present time. The model present stability within the geometric and matter perturbations for the choice $F_\phi = y$, where $y = (T^{1/3}/16\pi G)(16\pi G T + T + 8\Lambda)$, except for the geometric part in the de Sitter model. We impose the first and second laws of thermodynamics to $\Lambda$CDM and find the condition where they are satisfied, that is, $T_A, G_{\text{eff}} > 0$, however where this is not possible in the cases that we choose, this leads to a breakdown of positive entropy and Misner–Sharp energy.
1. Introduction

Gravitational interaction can be described in different ways. The main and best known is that using Riemannian geometry as a tool for the formulation of this description. This is known as Einstein’s theory of general relativity (GR) [1].

Riemannian geometry is based on differential geometry which deals with so-called space-time as a differentiable manifold of dimension four, where we have only the effect of curvature. There are other geometries that generalize this idea. A space-time may have curvature, but also torsion in its structure. There are two fundamental concepts in differential geometry. The formulation of this type of geometry has been undertaken and gravitation is known as Einstein–Cartan geometry [2]. In this theory the gravitational interaction is described by both curvature and torsion of space-time, where the torsion is commonly attributed to the inclusion of spin, through fractional spin fields. A very particular case of this theory is when we take the identically zero curvature, and then only have a space-time with torsion. This type of geometry is known as the Weitzenbock geometry [3, 4], where the torsion describes the gravitational interaction. Various analyses can be performed in this type of geometry, which is proven to be dynamically equivalent to GR [5]. In this context, recently a new formulation has been proposed, and generalizes the so-called teleparallel theory (TT), of the space-time where the gravitational interaction is described solely by the torsion.

Through the standard Big Bang [6] theory, ΛCDM models describe very well the evolution of our universe, in a Riemannian geometry within GR. There are several open issues, but the main one today is so-called dark energy. In order to make the cosmological Wilkinson Microwave Anisotropy Probe (WMAP) data fit the theory, it is necessary to introduce an exotic component in the equations of GR, i.e. dark energy. This can be modelled as a perfect fluid with the equation of state $p_{\text{DE}} = \omega_{\text{DE}} \rho_{\text{DE}}$, where the values of $\omega_{\text{DE}}$ must be very close to $-1$ today. An alternative to this, being consistent with the theory, is to modify the geometry. A good review can be seen in [7]. One of the first general possibilities is the well known $f(R)$ theory [8], where $R$ is the scalar curvature, obtained through the double contraction of the Riemann tensor indices. This theory places the analytic function $f(R)$ in the action, where GR can be reobtained in a certain limit, such as from $f(R) = a_1 R + a_2 R^2$, with $a_2 \to 0$ one again obtains GR. This theory has proved effective in simulating the evolution of our universe, in various epochs. Other possibilities have arisen through the generalization of GR, by changing the action. One such change is the case of $f(T, T)$ theory, where $T$ is the trace of the energy-momentum tensor. In this case, the matter content should be taken into account as having a kind of interaction with the geometry.

A direct analogy could be made between theories with only the curvature and torsion. As $f(R)$ theory is a generalization of GR, it is logical to also think of a generalization of TT, where the analogous object to the Riemannian scalar curvature is the torsion scalar $T$, obtained from contractions between the torsion and contorsion tensors. A change in the action of TT is made considering an analytic function $f(T)$ which depends on the torsion scalar. That was first thought of as a theory arising from the Born–Infeld action [9]. Then, several studies have shown the great accordance of this theory with the most varied approaches in gravitation and cosmology [10].
Another recent proposal is to consider not only the torsion scalar in the action, but also the trace of the energy-momentum tensor, as an analogy to \( f(R, T) \) theory. This theory, called \( f(T, T) \), has been formulated recently\footnote{11}, and still requires verification of compatibility with cosmological data and the physical requirements for a good cosmological theory. This is the reason to check the functional form of the action of this theory, such that the \( \Lambda \text{CDM} \) model is valid. For this we use the reconstruction method for modified gravity\footnote{12}. In addition, we conduct a stability analysis for the \( \Lambda \text{CDM} \) model.

A wide interest also exists in studying the thermodynamics of our universe. Various calculus approaches have been taken for guaranteeing the system obeys the first and second laws of thermodynamics. In GR, it has been shown that the first law of thermodynamics can be written as \( dE = T dS + W dV \)\footnote{13}. This law can also be represented in modified versions of gravity, but with an additive content of entropy production, for a non-equilibrium description\footnote{14}. We have conducted here an analysis which yields the conditions for satisfying the classical thermodynamics laws. We adopt the units \( k_B = c = h = 1 \) and the Newton constant \( G^{-1/2} = M_{\text{Plank}} = 1.2 \times 10^{19} \text{ GeV} \)\footnote{15}.

The paper is organized as follows. In section 2 we make a brief description of \( f(T) \) and \( f(T, T) \) theories, with the main elements and definitions necessary for their formulation. In section 3 we analyse the conservation of energy-momentum tensor, which results in strong constraints to the functional form of the action of the theory, leading to function \( f(T, T) \) with a linear dependence on the trace \( T \). In section 4 we use the reconstruction method for the actions to obtain the \( \Lambda \text{CDM} \) model for \( f(T, T) \) theory. The result shows that the action of \( \Lambda \text{CDM} \) is a combination of a linear term, a constant \(-2\Lambda\) and a nonlinear term given by the product \( \sqrt{-T} F_g [T^{1/3}/16\pi G] (16\pi G T + T + 8\Lambda) \), with \( F_g \) being a generic function. In section 5 we conduct a stability analysis of the studied model. The model presents stability within geometric and matter perturbations for the choice \( F_g = y \), where \( y = (T^{1/3}/16\pi G) (16\pi G T + T + 8\Lambda) \), except for the geometric part in the de Sitter model. We conduct a thermodynamic analysis for \( f(T, T) \) theory in section 6, imposing the first and second laws of thermodynamics to \( \Lambda \text{CDM} \) and find the condition where they are satisfied, that is, \( T_h, G_{\text{eff}} > 0 \), however where this is not possible for the cases that we choose, this leads to a breakdown of positive entropy and Misner–Sharp energy. In section 7 we establish the conservation of energy-momentum tensor to previous results, showing an inconsistency for this approach. We make our final considerations in section 8.

2. \( f(T) \) and \( f(T, T) \) gravities

In this section we will present the basic preliminary concepts for the reconstruction of \( f(T) \) and \( f(T, T) \) theories of gravity.

In \( f(T) \) theory, the geometry is determined solely by the matrices that transform the metric of space-time into the Minkowski metric. To begin, we define the space-time as a differentiable manifold in which only the torsion is non-zero, that is, the curvature is identically zero, then all Riemann tensor components are zero.

Now we define the line element as

\[
\text{d}S^2 = g_{\mu \nu} \text{d}x^\mu \text{d}x^\nu.
\] (1)

Taking into account that we can define 1-forms in the co-tangent space of the manifold, and introduce Lorentz symmetry in the line element, we can rewrite the line element as...
\[ \mathbf{dS}^2 = g_{\mu\nu} \mathbf{dx}^\mu \mathbf{dx}^\nu = \eta_{\alpha\beta} \theta^\alpha \theta^\beta, \]  
\tag{2}

where \( \theta^\mu = e^\mu_\mu \mathbf{dx}^\mu \) are 1-forms, index \( a = 0, \ldots, 3 \) and \( \{\eta_{\alpha\beta}\} = \text{diag}[1, -1, -1, -1] \) is the Minkowski metric. Here the Latin indices are related to the co-tangent space and the Greeks ones to the space-time. By way of writing the line element (2), we can establish the following relations

\[ \eta_{ab} = e^a_\mu e^b_\nu \eta_{\mu\nu}, \hfill \]  
\[ g_{ab} = e^a_\mu e^b_\nu \eta_{\mu\nu}, \hfill \]  
\[ e^a_\mu e^b_\nu = \delta^a_\nu e^b_\nu = \delta^b_\nu, \hfill \]  
where \( e^a_\mu \) is the inverse of the tetrad matrix \( e^{\alpha}_\mu \).

The connection is chosen such that all the Riemann tensor components are identically zero, and one has the Weitzenbock connection [16]

\[ \Gamma^\alpha_{\nu\mu} = e^\alpha_a \partial_\mu e^a_\nu. \]  
\tag{3}

We can now define a tensor which gives a sense of torsion to the space-time

\[ T^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu} - \Gamma^\alpha_{\mu\nu} = e^\alpha_a (\partial_\mu e^a_\nu - \partial_\nu e^a_\mu). \]  
\tag{4}

Through the components of the torsion tensor, we can define the contortion tensor components and the tensor \( S^\mu_{\alpha\nu} \)

\[ K^\mu_{\alpha\nu} = -\frac{1}{2} (T^\mu_{\alpha\nu} - T^\nu_{\alpha\mu} - T^\nu_{\mu\alpha}), \]  
\tag{5}

\[ S^\mu_{\alpha\nu} = \frac{1}{2} (K^\mu_{\alpha\nu} + \delta^\mu_\sigma T^\sigma_{\alpha\nu} - \delta^\nu_\sigma T^\sigma_{\alpha\mu}). \]  
\tag{6}

We can also define the analogous object to the scalar curvature in GR, the torsion scalar

\[ T = T^\mu_{\mu\nu} S^\nu_{\alpha\mu} = \frac{1}{4} T^\mu_{\mu\nu\sigma} T_{\nu\sigma} + \frac{1}{4} T^\mu_{\nu\sigma} T_{\nu\sigma} - T^\mu_{\nu\sigma} T^{\nu\sigma}. \]  
\tag{7}

It is this object that plays the curvature scalar role in GR, and that should also form the action of \( f(T) \) theory. The action of \( f(T) \) theory is constructed in a way that we have a linear term in the torsion scalar, another containing the correction term to TT and another term related to the material content. Then we write the action as

\[ S = \frac{1}{16\pi G} \int \mathbf{d}^4x [T + f(T)] + \mathcal{L}_m, \]  
\tag{8}

with \( e = \det[e^a_\mu] = \sqrt{-g} = \sqrt{\det[\eta_{\mu\nu}]} \), \( G \) Newton’s constant, and \( c \) set to unity. In \( f(T) \) theory, tetrads are dynamic fields, then doing the functional variation of the action (8) in relation to them, one gets the following equations of motion

\[ (1 + f_T) [e^{-1} \partial_{\mu} (e^\sigma_\alpha S^\nu_{\sigma\mu})] - e^\lambda_\alpha T^\sigma_{\mu\lambda} S^\nu_{\sigma\mu} \]  
\[ + e^\sigma_\alpha S^\nu_{\sigma\mu} \partial_{\mu} T_{TT} + \frac{1}{4} e^\alpha_\nu T = 4\pi Ge^\alpha_\nu \Theta^\sigma_\nu, \]  
\tag{9}

where we use the nomenclature \( f_T = \partial f/\partial T \) and \( f_{TT} = \partial^2 f/\partial T^2 \). \( \Theta^\sigma_\nu \) represents the components of the matter energy-momentum tensor.

Let’s take the example of the Friedmann–Lemaître–Robertson–Walker (FLRW) universe with flat spatial section

\[ \mathbf{dx}^2 = \mathbf{dr}^2 - a^2(t)(\mathbf{dx}^2 + \mathbf{dy}^2 + \mathbf{dz}^2), \]  
\tag{10}

where \( a(t) \) is the scale factor. The Hubble parameter is given by \( H(t) = \dot{a}/a(t) \). Now we specify our choice of tetrads as

\[ [e^a_\mu] = \text{diag}[1, a(t), a(t), a(t)]. \]  
\tag{11}
With this choice, we can represent the line element (2) through a set of 1-forms \( [\theta^0 = dt, \theta^1 = a(t) dx, \theta^2 = a(t) dy, \theta^3 = a(t) dz] \). Thus, the equations of motion (9) for \( f(T) \) theory, taking the material content as a perfect fluid \( \Theta_{\mu}^{\nu} = \text{diag}[\rho_{\text{mat}}, -p_{\text{mat}}, -p_{\text{mat}}, -p_{\text{mat}}] \), are given as follows

\[
H^2 = \frac{8\pi G}{3} \rho_{\text{mat}} - \frac{f}{6} - 2H^2 f_T
\]

\[
\dot{H} = -\frac{4\pi G (\rho_{\text{mat}} + p_{\text{mat}})}{1 + f_T - 12H^2 f_{TT}}.
\]

The subscripts \( \rho_{\text{mat}} \) and \( p_{\text{mat}} \) means the density and pressure of the total matter in the universe. We consider here only the components of the baryonic matter as \( \{\rho_{\text{mat}}, p_{\text{mat}}\} \) and radiation as \( \{\rho_{r}, p_{r}\} \).

Here, the torsion scalar is obtained by definitions (4)–(7) with (11), resulting in

\[
T = -6H^2,
\]

We now see clearly that the equations of motion of \( f(T) \) theory are identical to those of GR, and the equations of Friedmann (flat spatial section), when the nonlinear terms are zero, i.e. making \( f(T) = f_T = f_{TT} = 0 \) in (12)–(13).

Now we can present the most recent generalization of \( f(T) \) theory, following the analogy of the generalization of \( f(R) \) theory to \( f(R, T) \), where \( T \) is the trace of the energy-momentum tensor. Here, we can also introduce in the action an analytic function that depends not only on the torsion scalar, but also on the trace \( T \). For such a theory, the action is then given by

\[
S = \frac{1}{16 \pi G} \int \! d^4x \, e [T + f(T, T) + 16\pi G \mathcal{L}_m],
\]

where \( f(T, T) \) is an arbitrary analytical function of the torsion scalar \( T \) and of the trace \( T \) of the matter energy-momentum tensor \( \Theta_{\mu}^{\nu} \), and \( \mathcal{L}_m \) is the matter Lagrangian density.

Here we must consider that the Lagrangian density \( \mathcal{L}_m \) depends only on tetrads and not on its derivatives. For the energy-momentum tensor of a perfect fluid we have the following trace

\[
T = \Theta_{\mu}^{\nu} = \rho_{\text{mat}} - 3p_{\text{mat}}.
\]

We can then make the functional variation of the action (15) with respect to the tetrads, resulting in the following equations of motion [11]

\[
(1 + f_T) [e^{-1} \partial_{\mu}(e_{\alpha}^{\nu} S_{\alpha}^{\mu}) - e_{\alpha}^{\nu} T_{\alpha}^{\mu\nu} S_{\mu}^{\alpha}] + (f_T T_{\nu}^{\alpha\nu} + f_T T_{\nu}^{\alpha} T_{\nu}^{\beta} T_{\nu}^{\gamma} T_{\nu}^{\delta}) e_{\alpha}^{\nu} S_{\mu}^{\alpha} e_{\mu}^{\nu}
\]

\[
+ e_{\mu}^{\nu} \left( \frac{f_T + T}{4} - f_T \left( e_{\alpha}^{\nu} \Theta_{\alpha}^{\mu} + p_{\text{mat}} e_{\mu}^{\nu} \right) \frac{2}{2} \right) = 4\pi G e_{\alpha}^{\nu} \Theta_{\alpha}^{\mu}.
\]

where \( f_T = \partial f / \partial T \) and \( f_{TT} = \partial^2 f / \partial T \partial T \). Here it is evident that in the particular case where the function \( f \) depends only on the torsion scalar \( T \), i.e., \( f \equiv f(T) \)

\[
\text{7 Then, } f_T = f_{TT} \equiv 0.
\]

\[
3H^2 = 8\pi G \rho_{\text{mat}} - \frac{1}{2} (f + 12H^2 f_T) + f_T (\rho_{\text{mat}} + p_{\text{mat}}).
\]
\[ H = -4\pi G (\rho_{mt} + p_{mt}) - H (f_{T} - 12H^{2}f_{TT}) - H (\rho_{mt} - 3 p_{mt}) f_{TT} = f_{T} \left( \frac{\rho_{mt} + p_{mt}}{2} \right). \]  
(19)

In the next section, we propose the reconstruction of the $\Lambda$CDM model and study the stability of the de Sitter and power law solutions.

3. Conservation laws to $f(T, T)$ theory

This section is devoted to establishing a conservation law for $f(T, T)$ theory. For this, we must describe the equations of motion in a covariant form. Let us multiply the equation of motion (17) by $g_{\mu\nu} e_{\mu}^{\alpha} e_{\nu}^{\beta}$ (to have the indicies $\mu$ and $\nu$ free in the end) and using the identity

\[ g_{\mu\nu} e^{\alpha}_{\mu} [e^{\gamma}_{\nu}(e_{\gamma}^{\alpha} S_{\alpha}^{\gamma}) - e_{\nu}^{\alpha} T_{\gamma}^{\lambda} S_{\lambda}^{\nu}] = (1/2) [G_{\mu\nu} - (1/2) g_{\mu\nu} T], \]

where $G_{\mu\nu}$ is the Einstein tensor, we can rewrite our equation of motion as

\[ \frac{1}{2} \left[ 1 + f_{T} \right] \left( G_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) + S_{\gamma}^{\lambda}(f_{TT} \partial_{\gamma} T + f_{TT} \partial_{\gamma} T)
+ \frac{1}{4} g_{\mu\nu}(f + T) - \frac{1}{2} f_{T} (\Theta_{\mu\nu} + g_{\mu\nu} p_{mt}) = 4\pi G \Theta_{\mu\nu}. \]  
(20)

Now we take the divergence $\nabla_{\mu}$ for both sides and isolate $\nabla_{\mu} \Theta_{\mu}^{\nu}$, taking into account that $\nabla_{\rho} G_{\mu\nu}^{\rho} \equiv 0$, what gives us

\[ \nabla_{\mu} \Theta_{\mu}^{\nu} = \frac{1}{(4\pi G + (1/2)f_{T})} \left\{ (f_{TT} \partial_{\gamma} T + f_{TT} \partial_{\gamma} T) e_{\mu}^{\alpha} [e^{\gamma}_{\nu}(e_{\gamma}^{\alpha} S_{\alpha}^{\gamma}) - e_{\nu}^{\alpha} T_{\gamma}^{\lambda} S_{\lambda}^{\nu}] 
- \frac{1}{4} \left[ 1 + f_{T} \right] \partial_{\nu} T + \nabla_{\nu} S_{\nu}^{\mu\lambda}(f_{TT} \partial_{\gamma} T + f_{TT} \partial_{\gamma} T)
+ \frac{1}{4} (f_{TT} \partial_{\nu} T + f_{TT} \partial_{\nu} T + \partial_{\nu} T) T
+ \frac{1}{4} (f_{TT} \partial_{\nu} T + f_{TT} \partial_{\nu} T + \partial_{\nu} T) T
+ \frac{1}{4} (f_{TT} \partial_{\nu} T + f_{TT} \partial_{\nu} T + \partial_{\nu} T) T
+ \frac{1}{4} (f_{TT} \partial_{\nu} T + f_{TT} \partial_{\nu} T + \partial_{\nu} T) T
+ \frac{1}{4} (f_{TT} \partial_{\nu} T + f_{TT} \partial_{\nu} T + \partial_{\nu} T) T
+ \frac{1}{4} (f_{TT} \partial_{\nu} T + f_{TT} \partial_{\nu} T + \partial_{\nu} T) T
+ \frac{1}{4} (f_{TT} \partial_{\nu} T + f_{TT} \partial_{\nu} T + \partial_{\nu} T) T \right\}. \]  
(21)

This equation gives us two constraints, one for $\nu = 0$ and other for $\nu = 1, 2, 3$, as follows

\[ \dot{\rho}_{mt} + 3H (\rho_{mt} + p_{mt}) = \frac{1}{4} \left\{ 2(\rho_{mt} + p_{mt}) \left( f_{TT} \frac{dT}{dt} + f_{TT} \frac{dT}{dt} \right) - f_{T} \frac{dT}{dt} + 2f_{T} \frac{dT}{dt} \right\}, \]  
(22)

\[ 0 = p_{mt} \left[ f_{TT} \frac{dT}{dt} + f_{TT} \frac{dT}{dt} \right]. \]  
(23)

The second constraint results in

\[ \frac{d}{dt} [f_{T}] = 0, \]  
(24)

fixing $f(T, T)$ as a linear function of the trace $T$ or a constant, which is in agreement with the results obtained recently in [32] for the analogous $f(R, T)$ theory.
constraint, if we are to maintain the conservation of energy-momentum tensor, we must impose
\[ 2(\rho_m + p_m)\left(f_T \frac{dT}{dt} + f_{TT} \frac{dT^2}{dt^2}\right) - f_T \frac{dT}{dt} + 2f_r \frac{dp_m}{dt} = 0. \] (25)

When we do the same analysis in \( f(R, T) \) gravity, we also have a restriction on the functional form of the action \([31]\).

We will guard these results and reconstruct the \( \Lambda \)CDM model taking into account these two constraints.

4. Reconstruction of the \( \Lambda \)CDM model in \( f(T, T) \) theory

In this section we use the reconstruction method, through a particular model, for obtaining what should be the functional form of the function \( f(T, T) \). This method basically consists in choosing a model consistent with the cosmological data and, using the imposition of the characteristic equation for this specific model, must be satisfied at any time of the evolution of our universe. So with an imposition, we integrate the equation of motion in order to make the model valid. This results in a functional form of a fixed function that makes up the action of the theory. The use of this method in some cases of modified gravity can be seen in \([12, 17, 19]\).

Now, for reconstructing the \( \Lambda \)CDM model, it is necessary to impose an equation of motion of this model
\[ 3H^2 = 8\pi G \rho_m + \Lambda, \] (26)
where \( \Lambda \) is the cosmological constant. In this model, matter is described through a perfect fluid formulation, where pressure satisfies the following equation of state \( p_m = \omega_m \rho_m + \omega_r \rho_r = (1/3)\rho_r \), with \( \rho_m \) and \( \rho_r \) being the matter and the radiation densities. The trace of the energy-momentum tensor \([16]\) is
\[ T = \rho_m - 3p_m = (1 - 3\omega_m)\rho_m + (1 - 3\omega_r)\rho_r = \rho_m. \] (27)

By inverting the relation \([27]\) for the energy matter density in terms of the trace and using \([14]\), imposing the \( \Lambda \)CDM model from equation \([26]\)\(^8\), we can substitute in the equation of motion \([18]\) of \( f(T, T) \) theory, yielding
\[ \Lambda = -\frac{1}{2} (f - 2Tf_T) - \frac{1}{3}f_T \left[ T + \frac{1}{2\pi G} \left( \frac{T}{2} + \Lambda \right) \right]. \] (28)

Integrating this differential equation we obtain the following action function
\[ f(T, T) = -2\Lambda + \sqrt{-T} F_x \left[ \frac{T^{1/3}}{16\pi G} (16\pi GT + T + 8\Lambda) \right]. \] (29)
where \( F_x \) is a generic arbitrary function of its argument \( x \). Taking into account constraint \([24]\), we see that \( F_x \) is a linear function in \( x \). The function \( F_x \) can also take a constant value. We see that the reconstruction of the action \([15]\) is given by

\(^8\) We can extract from the imposition of the \( \Lambda \)CDM model the following relationship \( \rho_r = -(1/8\pi G)(T/2) + \Lambda \).
\[ S = \frac{1}{16\pi G} \int d^4x \{ T + f(T, \mathcal{T}) + 16\pi G \mathcal{L}_m \} \]
\[ = \frac{1}{16\pi G} \int d^4x \left\{ T - 2\Lambda + \sqrt{-T} \left[ \frac{T^{1/3}}{16\pi G} (16\pi G T + 8\Lambda) \right] + 16\pi G \mathcal{L}_m \right\}. \quad (30) \]

We can now see that this action generalizes \( f(T) \) theory, because we are able to have terms representing the interaction between the torsion and matter, generated by the function \( F_\phi \) in (29). We also see clearly that this generalization is not for any functional form between \( T \) and \( \mathcal{T} \), but only by the product given in the third term of the above action. This heavily restricts the functional form of the theory, when it is imposes the validity of the \( \Lambda \)CDM model, as we have here. Another important observation is that this action lies in \( f(T) \) when we properly choose \( F_\phi [T^{1/3}/16\pi G] (16\pi G T + 8\Lambda) \equiv Q \), where \( Q \) is a constant given in [18]. In the next section, we will use a model for this generic function, where \( F_\phi [y] = y \), with \( y = (T^{1/3}/16\pi G) (16\pi G T + 8\Lambda) \).

Here there are some important observations. Our model is able to reproduce well the eras that the evolution of our universe goes through. For example, with the imposition (26) we see clearly that the era of radiation, where we write now \( 3H^2 \approx 8\pi G \rho_r + \Lambda \), and of matter, where \( 3H^2 \approx 8\pi G \rho_m + \Lambda \), are now well reproduced. In the radiation era, as the term of the radiation density dominates over the matter term, the generic function is approximated by \( F_\phi [T^{1/3}/16\pi G] (T + 8\Lambda) \). Already in the matter era, as now the dominant term is matter density, we have the same functional form of the action (30). In the dark energy era, its approach is \( 3H^2 \approx \Lambda \), or equivalently \( -(T/2) \approx \Lambda \), which brings us to the same functional form of the radiation era for \( F_\phi \), but now \( T \approx -2\Lambda \). Here inflation can be described similarly, whereas the component that dominates is the vacuum energy density.

As seen, our reconstruction is compatible with the most varied eras of cosmological evolution of our universe, leading us to believe that this model should also be compatible with cosmological experimental data. We know at least that the acceleration of the universe at this stage is consistent with our model, for the very reconstruction is done to satisfy this condition. And we still have compatibility with the measures of type IA supernovae, where \( \omega_{\text{DE}} \sim -1 \) [36]. We can at least see that there is compatibility with ultra-stiff fluid where \( \omega_m = 1 \), made in a recent analysis [35].

One last important observation is that for the reconstruction to satisfy the first constraint (25), we have to fix the trace in terms of the torsion scalar as follows
\[ T = \frac{1}{16\pi G} \left[ (11 + 4608G^2\pi^2) T - 8\Lambda \right]. \]
\[ \quad (31) \]

This shows us that the validity of the reconstruction of the \( \Lambda \)CDM model is subject to fixing the energy-momentum tensor trace, given in (31). This show us that \( f(T, \mathcal{T}) \) theory should always be considered a theory in which the trace of the energy-momentum tensor depends linearly on the torsion scalar. However, this does not permit us to conclude that \( f(T, \mathcal{T}) \) gravity must always fall back on \( f(T) \) gravity, because for \( f(T, \mathcal{T}) \) to fall back on \( f(T) \), it is necessary that \( F_\phi \equiv Q \) in (29), and does not have a linear dependence on the trace, as we have here.

It may be possible to escape from the result that the action should have a linear dependence on the trace \( \mathcal{T} \), which is a direct implication of the conservation of the energy-momentum tensor given in (24). We can think about the possibility of non-conservation of the energy-momentum tensor, arising from the non-minimal coupling between matter and torsion, similar to the case of theories modifying only curvature [33, 34]. This should be interpreted as a possibility that the movement of material content is no longer geodesic, having the
appearance of an extra force term. This can also come to modify the bending of light and add a term in acceleration exerted by gravity. Astrophysical interpretations may have to help in models of dark matter. This approach goes beyond our analysis here and should be done in future work.

In the next section, we will show the stability of the ΛCDM model reconstructed here.

5. Stability of de Sitter and power-law solutions

In order to establish the validity of the ΛCDM model, let us conduct a simple stability analysis. To do so, we will do a small perturbation in the geometry and matter

\[ H(t) = h(t)[1 + \delta(t)], \quad \rho_{\text{m}}(t) = \rho_{\text{m}_0}(t)[1 + \delta_{\text{m}}(t)], \]  

where \( h(t), \rho_{\text{m}_0}(t) \) is an exact solution of the equations of motion for call background (obeying (17) and (18)) and \( \delta(t), \delta_{\text{m}}(t) \ll 1 \). With this, the torsion scalar is given by

\[ T = -6H^2 = -6h^2(1 + \delta)^2 = T_0(t)[1 + 2\delta(t)], \]

where we have \( T_0(t) = -6h^2(t) \). The trace of the energy-momentum tensor is given by

\[ T(t) = \rho_{\text{m}}(t) - 3\rho_{\gamma}(t) = (1 - 3\omega_{\gamma})(\rho_{\gamma}(t) + (1 - 3\omega_{\gamma})\rho_{\nu}(t) \]

\[ = \rho_{\text{m}}(t) = T_0(t)[1 + \delta_{\text{m}}(t)], \]

where \( T_0(t) = \rho_{\text{m}_0}(t) \). The perturbations of the torsion scalar and the trace of the energy-momentum tensor are

\[ \delta T(t) = T - T_0 = 2T_0(t)\delta(t), \quad \delta T(t) = T - T_0 = T_0(t)\delta_{\text{m}}(t). \]

Let us express the function \( f(T, T) \) in Taylor series up to first order around the point \([T_0(t), T_0(t)]\), by

\[ f(T, T) \approx f(T_0, T_0) + \left[ \frac{\partial f}{\partial T} \right]_{T=T_0} \delta T + \left[ \frac{\partial \delta f}{\partial T} \right]_{T=T_0} \delta T, \]

its derivatives, that are obtained deriving (36), can be expressed as

\[ f_T(T, T) \approx f_T(T_0, T_0) + \left[ \frac{\partial^2 f}{\partial T^2} \right]_{T=T_0} \delta T + \left[ \frac{\partial \delta f}{\partial T^2} \right]_{T=T_0} \delta T, \]

\[ f_T(T, T) \approx f_T(T_0, T_0) + \left[ \frac{\partial^2 \delta f}{\partial T^2} \right]_{T=T_0} \delta T + \left[ \frac{\partial \delta^2 f}{\partial T^2} \right]_{T=T_0} \delta T. \]

By substituting these expressions into (17), collecting the term up to first order, we get the following result

\[ 3\delta^2(1 + 2\delta) = 8\pi G \rho_{\text{m}_0}(1 + \delta_{\text{m}}) - \frac{1}{2} \{ [f_0 + f_{\gamma} \delta T + f_{\nu} \delta T] + 12h^2(1 + 2\delta) \]

\[ \times \{ f_{\gamma} + f_{\nu} \delta T + f_{\nu} \delta T \} \}

\[ + [f_{\gamma} + f_{\nu} \delta T + f_{\nu} \delta T] \left( \rho_{\text{m}_0} + \frac{4}{3} \rho_{\text{b}} \right)(1 + \delta_{\text{m}}) \]

(39)
that replacing $\rho_{\text{rh}} = T_0$, $h_0 = -T_0/6$ and $\rho_{\text{rh}} = -[(1/8\pi G)((T_0/2) + \Lambda) + T_0]$ becomes

$$-T_0 \left\{ 1 + f_{T_0} + 2T_0 f_{T_0,T_0} - \frac{2}{3} \left[ T_0 + \frac{1}{2\pi G} (T_0/2 + \Lambda) \right] f_{T_0,T_0} \right\} \delta(t)$$

$$= \left\{ \frac{T_0}{2} + \Lambda - \frac{1}{6} \left[ 5T_0 + \frac{1}{\pi G} \left( \frac{T_0}{2} + \Lambda \right) \right] f_{T_0} + T_0 f_{T_0, T_0} \right\} \delta_m(t)$$

$- \frac{1}{3} T_0 \left[ T_0 + \frac{1}{2\pi G} \left( \frac{T_0}{2} + \Lambda \right) \right] f_{T_0,T_0} \delta_m(t).$ \hfill (40)

We maintain this result for the time being. Now we have to express the perturbation $\delta(t)$ in terms of $\delta_m(t)$. To do so, let us take the perturbation of the conservation equation of the energy-momentum tensor

$$\dot{\delta}_m + 3H (\rho_{\text{rh}} + p_{\text{rh}}) = 0$$

$$\dot{\rho}_{\text{rh}}(1 + \delta_m) + \rho_{\text{rh}} \delta_m + 3h(1 + \delta) \left( \rho_{\text{rh}} + \frac{4}{3} \rho_{\text{rh}} \right) (1 + \delta) = 0$$ \hfill (41)

$$\delta(t) = - \frac{\rho_{\text{rh}}}{3h (\rho_{\text{rh}} + p_{\text{rh}})} \delta_m(t) = - \frac{\sqrt{6}}{8\pi G \sqrt{-T_0} \left[ (1/8\pi G)(T_0/2) + \Lambda \right]} \delta_m(t).$$ \hfill (42)

Here, the ‘dot’ denotes the derivative with respect to time $t$. Now, by inserting (42) in (40), one gets

$$\delta_m = \delta_m^{(0)} \exp \left[ \int \frac{F_1(t)}{F_2(t)} dt \right].$$ \hfill (43)

$$F_1(t) = \frac{T_0}{2} + \Lambda - \frac{1}{6} \left[ 5T_0 + \frac{1}{\pi G} \left( \frac{T_0}{2} + \Lambda \right) \right] f_{T_0} + T_0 f_{T_0,T_0}$$

$$- \frac{1}{3} T_0 \left[ T_0 + \frac{1}{2\pi G} \left( \frac{T_0}{2} + \Lambda \right) \right] f_{T_0,T_0},$$ \hfill (44)

$$F_2(t) = \sqrt{-T_0} \left( \sqrt{6} T_0 + 2\Lambda \right) \left( 1 + f_{T_0} + 2T_0 f_{T_0,T_0} \right)$$

$$+ (4\pi G T_0 + T_0 + 2\Lambda f_{T_0,T_0}) / [24\pi G \times (4\pi G T_0 + T_0 + 2\Lambda)].$$ \hfill (45)

This equation is valid for specific models, the ones we have chosen, the de Sitter and power-law models. We will substitute the characteristic of these models in the above equation.

Let us first analyse the stability of the de Sitter solution, where the characteristic is given by $h = h_0$, yielding through the equation of continuity

$$\rho_{\text{rh}}(t) = \rho_0 \exp \left[ -3h_0 t \right].$$ \hfill (46)

The torsion scalar is constant and is given by $T_0 = -6h_0^2$. The trace of the energy-momentum tensor is given by

$$T_0 = \rho_0 \exp \left[ -3h_0 t \right].$$ \hfill (47)

Now we have to use the reconstruction of $f(T, T)$ for $\Lambda$CDM in (29), specifying the case of the de Sitter solution. We first consider the time dependent torsion scalar, i.e. variable
torsion scalar, and then fix for the de Sitter solution. We can then correctly calculate the
derivatives of the function \( f(T, \mathcal{T}) \).

For the de Sitter solution, we take one case for action (30). We make the case where
\( F_{y}(y) = f_{y}, \) for the de Sitter solution. The parameters are chosen as \( \{N = 20, \ h_{0} = 2.1 \times 0.7 \times 10^{-42}, \ \delta_{m}^{(0)} = 1, \ \rho_{0} = 0.1 \times 10^{-12}, \ G = (1.2)^{2} \times 10^{-38}, \ \Lambda = 10^{-42}\\} \).

For this case, we have the following integral in (43)
\[
\delta_{m}(t) = \delta_{m}^{(0)} \exp \left\{ -\frac{4 \times 6^{1/3} (h_{0}^{5/3} + 6^{1/6} \pi G)(3h_{0}^{2} - \Lambda) \left[(3h_{0}^{2} - \Lambda) t + \frac{2\pi G \rho_{0} e^{-3h_{0} t}}{3h_{0}} \right]}{h_{0}^{5/3} (6^{5/6} h_{0}^{2} - 6\pi G h_{0}^{4/3})(3\sqrt{6} h_{0}^{2} - \Lambda)} \right\} 
\]
and from (42)
\[
\delta(t) = -\delta_{m}(t) \frac{4 \times 6^{1/3} (h_{0}^{5/3} + 6^{1/6} \pi G)(3h_{0}^{2} - \Lambda)[3h_{0}^{2} - \Lambda - 2\pi G \rho_{0} e^{-3h_{0} t}]}{(6^{5/6} h_{0}^{2} - 6\pi G h_{0}^{4/3})(3\sqrt{6} h_{0}^{2} - \Lambda)}. 
\]

We draw a numerical graph representing the temporal evolution of these perturbations in figure 1. We can see that the perturbations rapidly decrease to zero, showing possible stability for the de Sitter solution. There is a serious problem here. The graphical representation to \( \delta(t) \) in figure 1 shows that despite this function quickly dropping to zero, the initial values are of the order to \( 10^{-41} \), thus contradicting the initial assumption for perturbation which is \( \delta(t) \ll 1 \). This shows us that the stability of the de Sitter model is impossible, due precisely to the initial values for perturbation of the geometric part.

Doing the same procedure for the case of power law, where now \( h_{0}(t) = \alpha/t, \) equation (43), for the case \( F_{y} = y \) with \( y = (T_{0}^{1/3} / 16\pi G)(16\pi G T_{0} + T_{0} + 8\Lambda) \), becomes
\[
\delta_{m}(t) = \delta_{m}^{(0)} \exp \left\{ \int \frac{F_{p1} \rho_{1}}{F_{p2}} \left( -\frac{4 \times 6^{1/3} (h_{0}^{5/3} + 6^{1/6} \pi G)(3h_{0}^{2} - \Lambda)[3h_{0}^{2} - \Lambda - 2\pi G \rho_{0} e^{-3h_{0} t}]}{(6^{5/6} h_{0}^{2} - 6\pi G h_{0}^{4/3})(3\sqrt{6} h_{0}^{2} - \Lambda)} \right) \right\} 
\]
We numerically represent the evolution of perturbations \( \delta_m(t) \) and \( \delta(t) \) in figure 2. Again we can see that the perturbations quickly decay to zero, showing the stability of this solution.

In the next section, we will study the two laws of thermodynamics for the \( \Lambda \)CDM model.

6. Thermodynamic laws according to the \( \Lambda \)CDM model

By analogy to the thermodynamics of black holes, the thermodynamics of cosmological models can be realized, verifying if these models satisfy classical thermodynamics. This has been done in the context of GR [20]. It has also been of wide interest to study these laws for theories of modified gravities [21]. In the context of modified gravity it is common to use a description of non-equilibrium thermodynamics for representing the physical system [22]. We will use this representation here.

In this section we will investigate the conditions of satisfying the first and second laws of thermodynamics.
6.1. First law

We will redefine now the action of the theory \( f(T, T) \) as

\[
S = \int d^4x [f(T, T) + 16\pi G L_m].
\]  

(54)

This is necessary to make possible the definition of an effective Newton constant \( G_{\text{eff}} \). Now the equations of motion are rewritten as

\[
3H^2 = 8\pi G_{\text{eff}} (\rho_{\text{mt}} + \rho_{\text{DE}}),
\]  

(55)

\[
\dot{H} = -4\pi G_{\text{eff}} (\rho_{\text{mt}} + p_{\text{mt}} + \rho_{\text{DE}} + p_{\text{DE}}),
\]  

(56)

with

\[
G_{\text{eff}} = \frac{G}{f_T} \left( 1 + \frac{f_T}{16\pi G} \right),
\]  

(57)

\[
\rho_{\text{DE}} = \frac{1}{16\pi G_{\text{eff}} f_T} \left( f_T p_{\text{mt}} - \frac{1}{2} f \right),
\]  

(58)

\[
p_{\text{DE}} = \rho_{\text{mt}} + p_{\text{mt}} - \rho_{\text{DE}} - \frac{1}{4\pi G_{\text{eff}} f_T} \left[ 12H^2 f_T T - f_T T H (\dot{\rho}_{\text{mt}} - 3\ddot{\rho}_{\text{mt}}) \right].
\]  

(59)

We impose the conservation of the matter sector \( \rho_{\text{mt}} + 3H (\rho_{\text{mt}} + p_{\text{mt}}) = 0 \), as discussed in sections 3 and 4. But as we are doing a non-equilibrium thermodynamics description, the dark energy sector is not conserved,

\[
\dot{\rho}_{\text{DE}} + 3H (\rho_{\text{DE}} + p_{\text{DE}}) = \frac{3H^2}{8\pi} \frac{d}{dr} \left( \frac{1}{G_{\text{eff}}} \right),
\]  

(60)

where we used (55)–(59). This is in accordance with the theory analogue to \( f(T, T) \), in [23], and also for the particular case where \( f \equiv f(T) \) in [14]. We still should compare our results with the ones of \( f(R) \) theory, in [24]. It is also clear here that, when \( G_{\text{eff}} = G \) (in the particular case \( f(T, T) = T \) the dark energy sector is conserved), the TT is recovered, or the GR analogously, within an equilibrium description of thermodynamics.

Now, we establish the basic tools for the first law of thermodynamics. The line element (10) can be rewritten as

\[
dS^2 = h_{\mu\nu} dx^\mu dx^\nu + \tilde{f}^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\]  

(61)

where we have the metric of two-dimensional space \([h_{\mu\nu}] = [1, -a^2(t)]\), for \( \mu, \nu = 0, 1 \) and \( x^0 = t, x^1 = r \), a new radial coordinate \( \tilde{r} = ra(t) \), with \( r \) being the usual radial coordinate \( r^2 = x^2 + y^2 + z^2 \). Here space-time can be decomposed in a two-dimensional space with the metric \( h_{\mu\nu} \) and other two-dimensional spaces with a 2-sphere, with the line element \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \).

Through this line element (61), we can calculate the apparent horizon\(^9\) and the associated temperature to this. The apparent horizon is obtained from the expression \( h^{\mu\nu} \partial_\mu \partial_\nu \tilde{r}_A \partial_\nu \tilde{r}_A = 0 \), which results in

\(^9\) We choose to work the apparent horizon according to the fact that the event horizon does not satisfy the second law of thermodynamics, contrary to the apparent one where the law is satisfied [26].
The temperature is calculated by the expression
\[
T_A = \frac{1}{2\pi} \ln\left| \partial_{\mu}[\sqrt{-h} h^{\mu\nu}\partial_{\nu} \hat{\rho}_A] \right|
\]
where \( h = \det[h_{\mu\nu}] \). The temperature is given by
\[
T_A = \frac{\hat{\rho}_A}{4\pi}(H + 2H^2),
\]
where, from now, we impose \( H + 2H^2 > 0 \) for getting a definite positive temperature.

Using (62) and (56), we can calculate the derivative of \( \hat{r}_A \)
\[
\frac{d\hat{r}_A}{dt} = 4\pi G_{\text{eff}} \hat{r}_A^2 (\rho_{\text{tot}} + p_{\text{tot}}),
\]
with \( \rho_{\text{tot}} = \rho_{\text{mt}} + \rho_{\text{DE}} \) and \( p_{\text{tot}} = \rho_{\text{mt}} + p_{\text{DE}} \). Now, we will take the entropy related to the apparent horizon, for modified gravity theories [25], that is \( S_A = A/(4G_{\text{eff}}) \), with \( A = 4\pi \hat{r}_A^2 \).

Here, appears a new restriction for the thermodynamic system, \( G_{\text{eff}} > 0 \) for getting a definite positive entropy \( S_A \geq 0 \). Considering this entropy, and using \( G_{\text{eff}} \) in (57) and (65), we get the following differential
\[
dS_A = 2\pi \hat{r}_A \left[ 4\pi \hat{r}_A^2 (\rho_{\text{tot}} + p_{\text{tot}}) dt + \frac{1}{2} \hat{r}_A \left( \frac{1}{G_{\text{eff}}} \right) \right]
\]
Now we take the Misner-Sharp energy for modified gravities \( E_{\text{MS}} = \hat{r}_A/(2G_{\text{eff}}) = V \rho_{\text{tot}} \)
[27], with \( V = (4/3)\pi \hat{r}_A^3 \), where using (65), the differential gives rise to
\[
dE_{\text{MS}} = 2\pi \hat{r}_A^2 (\rho_{\text{tot}} + p_{\text{tot}}) dt + \frac{1}{2} \hat{r}_A \left( \frac{1}{G_{\text{eff}}} \right)
\]
Using (62) in (64), considering (65), we can rewrite the temperature as
\[
T_A = \frac{\hat{r}_A}{4\pi} \left[ \frac{d}{dr} \left( \frac{1}{\hat{r}_A} \right) + 2 \left( \frac{1}{\hat{r}_A} \right)^2 \right] = \frac{1}{4\pi \hat{r}_A} \left( 1 - \frac{1}{2} \frac{d\hat{r}_A}{dt} \right)
\]
Making the product (68) for (66) we have
\[
T_A dS_A = -dE_{\text{MS}} + 2\pi \hat{r}_A^2 (\rho_{\text{tot}} - p_{\text{tot}}) d\hat{r}_A + 6\pi \hat{r}_A^2 (\rho_{\text{tot}} + p_{\text{tot}}) dt - 4\pi \hat{r}_A^2 \rho_{\text{tot}} d\hat{r}_A
\]
\[
+ \frac{1}{2} \hat{r}_A (1 + 2\pi \hat{r}_A T_A) d \left( \frac{1}{G_{\text{eff}}} \right)
\]
Defining the work density given as \( W = (1/2)(\rho_{\text{tot}} - p_{\text{tot}}) \) [14], using (55) and (65), we can rewrite (69) as
\[
T_A dS_A + T_A dS_p = -dE_{\text{MS}} + W dV,
\]
\[
T_A dS_p = -\frac{1}{2} \hat{r}_A (1 + 2\pi \hat{r}_A T_A) d \left( \frac{1}{G_{\text{eff}}} \right)
\]
where the term \( T_A dS_p \) can be interpreted as coming from an entropy production. Considering \( T_A > 0 \), we eliminate the singularity in the entropy production in \( H = -2H^2 \) \((T_A = 0)\). Here,

\[10\] Once again we have to restrict \( G_{\text{eff}} > 0 \) for obtaining definite positive energy. This has been indicated in [28], in order to avoid ghost structure in the quantized theory.
we see that the result is in accord with the particular case where \( f \equiv f(T) \), in [14]; also in the analogue theory \( f(R,T) \) in [23]. This entropy production \( dS_p \) is directly linked to the non-conservation of the dark sector in \((60)\). When we fix the particular case \( f(T,T) = T \) \((G_{\text{eff}} = G)\), we recover a theory without entropy production, i.e. \( TT \).

We take the effective constant \((57)\) to the power law solution, which provides

\[
G_{\text{eff}}(t) = 6G \left( \frac{\alpha}{t_0} \right)^{3\alpha} \left[ 8 \times 6^{1/3} \pi G t_0^2 \left( \frac{\alpha}{t_0} \right)^{1/3} - 3\alpha^2 \right] \left[ 20 r^2 \left( \frac{t_0}{t} \right)^{3\alpha} + 2\pi G \rho_0 - 33\alpha^2 \left( \frac{t_0}{t} \right)^{3\alpha} \right]
\]

We represent the temporal evolution of the effective Newton constant \( G_{\text{eff}}(t) \) in figure 3. We show what for the case of the power law, \( G_{\text{eff}} \) assumes negative values, precluding a definition of Misner–Sharp energy and entropy positives.

So we can conclude that this type of behavior suggests (we have not tested all possibilities of cases) that \( f(T,T) \) gravity does not have well-established thermodynamics, with all the thermodynamic quantities physical and well-defined in every time interval.

6.2. Second law

We will impose the second law of thermodynamics to \( f(T,T) \). To do so, let us consider the Gibbs equation in terms of matter fluid and total energy given by

\[
T_{\text{tot}} dS_{\text{tot}} = (\rho_{\text{tot}} + p_{\text{tot}}) dV + V d\rho_{\text{tot}},
\]

where \( T_{\text{tot}} \) and \( S_{\text{tot}} \) are the temperature and total entropy associated with the system, respectively. According to the second law of thermodynamics, we must have

\[
\frac{dS_{\text{tot}}}{dt} + \frac{dS_A}{dt} + \frac{dS_p}{dt} \geq 0.
\]

Knowing that \( E_{\text{MS}} = V \rho_{\text{tot}} \) and using \((70)\) and \((73)\) we can write the left side of \((74)\) as

\[
\frac{dS_{\text{tot}}}{dt} + \frac{dS_A}{dt} + \frac{dS_p}{dt} = \frac{1}{T_{\text{tot}} T_{\text{tot}}} \left[ (\rho_{\text{tot}} + p_{\text{tot}}) \frac{dV}{dt} \left( T_A - T_{\text{tot}} \right) + V \frac{d\rho_{\text{tot}}}{dt} (T_A - T_{\text{tot}}) \right].
\]

Figure 3. A graph showing the temporal evolution of \( G_{\text{eff}}(t) \) for the case \( f = f(y) \), with \( y = \left( T^{1/3}_{\text{tot}} / 16\pi G \right) \left( 16\pi G T_0 + T_0 + 8\Lambda \right) \), for the particular case of the power law solution. The parameters are chosen as \( \{ \alpha = 2, t_0 = 3, h_0 = 2.1 \times 0.7 \times 10^{-42}, \delta_{\text{eff}}^0 = 1, \rho_0 = 0.1 \times 10^{-121}, G = (1.2)^2 \times 10^{-38}, \Lambda = 10^{-42} \} \).
For $T_a = T_{\text{tot}}$, using \((56)\) and \(dV = 4\pi r_A^2 \, dr\) we get, for \((75)\),
\[
\frac{dS_{\text{tot}}}{dt} + \frac{dS_A}{dt} + \frac{dS_p}{dt} = \frac{1}{2T_A G_{\text{eff}}} \left( \frac{dF_A}{dr} \right)^2,
\]
with $G_{\text{eff}} > 0$, as previously done. The observation is that the temperature of the apparent horizon in \((64)\) depends on the components of the universe, through \((55)\) and \((56)\), including the photons of the cosmic microwave background (CMB) with 2.73 K. This assumes that the total temperature of the system $T_{\text{tot}}$ is equal to the one of the apparent horizon, i.e. this temperature also includes the photons of CMB.

We have presented the expressions of the second law similar to particular cases, or analogues $f(T)$ and $f(R, T)$, but we are still restricted to the problem of the previous section, in which we cannot set $G_{\text{eff}}$ always positive, leading to a possibility of negative entropy, which is physically inconsistent for the cases studied here.

### 7. Comparison with previous results

We noticed a great need to compare our results with the first work in this extension of $f(T)$ gravity. The observation that we have done here is similar to the first construction of $f(T, T)$ theory by Kiani and Nozari [29]. But due to the ambiguity of the choice of the Lagrangian density for the perfect fluid, with the Lagrangian density being $\mathcal{L}_m = \rho$ (our choice) or $\mathcal{L}_m = -\rho$ (see [30]), they use the latter description. This suggests a new approach that is possibly equivalent to the same theory. They conduct an analysis of the stability for the de Sitter model, but as they did not reconstruct the action, they considered the model as a generic one $f(T, T) = k_1 T + k_2 T^{\alpha} T^\beta$.

Here we will verify the consistency of this approach to the conservation of energy-momentum tensor. Taking the equation of motion \((11)\) in [29], considering \((13)\), we multiply by $e^\nu_\mu$ and considering the identity $e^\nu_\mu \left[ e^{-1} \partial_\lambda (e_{\mu \alpha} \partial^\alpha S_\lambda) - e_{\nu \alpha} T^\lambda \gamma^\alpha S_\gamma \right] = (1/2)G_{\nu \mu} (1/2)\delta^\rho_\mu T$, we have
\[
S_{\nu \rho} f_{TT} \partial_\rho T + \frac{1}{2} f_T \left( G_{\nu \mu} - \frac{1}{2} \delta^\nu_\mu T \right) + \delta^\nu_\mu \left( \frac{1}{4} f + \frac{3}{2} \rho_{\text{tot}} f_T \right) = \left( 4\pi G + \frac{5}{4} f_T \right) \Theta_\nu^\mu. \tag{77}
\]
Now we take the total divergence $\nabla_\mu$ of the above equation of motion, and isolating $\nabla_\mu \Theta_\nu^\mu$, taking into account that $\nabla_\mu G_{\nu \mu} \equiv 0$, we have the following result
\[
\nabla_\mu \Theta_\nu^\mu = \frac{1}{(4\pi G + (5/4)f_T)} \left\{ \nabla_\mu S_{\nu \rho} f_{TT} \partial_\rho T + S_{\nu \rho} f_{TT} \nabla_\rho \partial_\mu T + \frac{1}{2} \left[ f_{TT} \partial_\mu T + f_{TT} \partial_\rho T \left( G_{\nu \mu} - \frac{1}{2} \delta^\nu_\mu T \right) \right] + \frac{1}{4} f_T \partial_\mu T + \frac{3}{2} \left[ f_T \partial_\rho T + \rho_{\text{tot}} \left( f_{TT} \partial_\rho T + f_{TT} \partial_\rho T \right) \right] - \frac{5}{4} \left( f_{TT} \partial_\rho T + f_{TT} \partial_\rho T \right) \Theta_\nu^\mu \right\}. \tag{78}
\]
Then arise two constraints here, one for $n = 0$ and another for $n = 1, 2, 3$, 

\[
\rho_m + 3H(\rho_m + p_m) = \frac{1}{4} \left\{ [\rho_m f_{TT} + 12(1 - 2H^2)f_{TT}] \frac{dT}{dt}
\right.
\]

\[+ [\rho_m f_{TT} + f_T + 12(1 - H^2)f_{TT} + 6f_T \rho_m] \right\}, \tag{79}
\]

\[
0 = 3(H^2 - a) \left[ f_{TT} \frac{dT}{dt} + f_T \frac{dT}{dt} \right] \tag{80}
\]

The second constraint is satisfied, in general, to $(d/dt)f_T \equiv 0$, fixing function $f (T, T)$ as a linear function on the torsion scalar $T$. This returns to a trace coupling with TT, which is not the intention here. Thus, the description of the theory as shown in [29], is not consistent with the conservation of energy-momentum tensor presented here. We must then discard here that description by the density of the fluid.

An important observation is that this result does not mean that the description of Lagrangian fluid through the density is incorrect, but that the specific choice of the set of diagonal tetrad, $\{e^\mu_{\nu}\} = [1, a(t), a(t), a(t)]$ means it cannot provide a consistent result. It should be investigated in another future work, as well as the consistency between two description possibilities, one by density and the other by fluid pressure.

8. Conclusion

$f (T, T)$ theory can be a good opportunity to test whether we can describe dark energy by changing the TT equations of motion. To do this, it is first necessary to test the most common models of modern cosmology, the famous $\Lambda$CDM one for example.

We reconstructed the gravitational action of this model and have done a brief analysis of stability. The result is restated as follows.

The reconstruction of the $\Lambda$CDM model gives rise to an action as a combination of a linear term in the torsion scalar, a constant $(-2\Lambda)$ and another, which is the interaction between matter and torsion, described by the product $\sqrt{-T} F_T \{f_T / (16\pi G)(16\pi GT + T + 8\Lambda)\}$. This reconstruction decays in $f(T)$ theory, by appropriately choosing $F_T = Q$, with $Q$ a constant given in [18]. This reconstruction severely restricts the functional form of the action, but reproduces the cosmological era well.

We have analysed energy-momentum tensor conservation through the obtained reconstruction model, resulting in the need for the function $F_T [x]$ to be linear in its argument, so that conservation is satisfied as in the usual $\Lambda$CDM model. This action should not contain terms of higher orders and is restricted to linearity in the trace $T$. This model does not prevent comparison with observational data, particularly for the case of ultra-stiff fluid, where $w_m = 1$ (see [35]).

We have done an analysis of the stability of this model, for a particular choice $F_T = y$, finding that the model is stable for de Sitter and power law solutions, except for the geometric part of the de Sitter one.

We finished by analysing the thermodynamics for the $\Lambda$CDM model. We found that the imposition of the thermodynamic laws are satisfied for the conditions $T_a, G_{ab} > 0$, which cannot be met for the analysis here. We found it impossible to define positive entropy and Misner–Sharp energy in any interval of physical time, as well as prevent ghost structure in the quantized theory [28]. This suggests that the cosmological thermodynamic theory to $f (T, T)$ may not be well established fundamentally.
Our perspectives are listed as follows.

We have used here the reconstruction method for cosmological models, which can then be compared to other known models like the holographic and Chaplygin gas models, for example. These new approaches cannot modify the main result of energy-momentum tensor conservation. The action is linear in trace $\mathcal{T}$, so models should cooperate with the acceleration of the universe at this stage. This will be covered in detail in a later study.

Another possible approach is when we consider the space-time connection, regardless of the set of tetrads, and then one of the dynamic fields. That would be a new version of $f(T,\mathcal{T})$ cosmology in Palatini’s formalism, which may reveal new results in the reconstruction of the $\Lambda$CDM model. This approach may also reveal a different constraint to the conservation of energy tensor-momentum. Using the formalism reconstruction from the LQC model, as seen in [37], we can generalize to a non-minimum coupling with matter. We see in [37] the reconstruction by an inverse method considering also $\rho \equiv \rho(R)$ or yet $\mathcal{R} \equiv \mathcal{R}(T)$, which is obtained in terms of the torsion scalar in (31). We could still think of the non-conservation of the momentum-energy tensor as a consequence of a quantum gravity anomaly [38], then giving the possibility of action containing nonlinear terms in the trace $T$. This should be addressed in a future work. Non-conservation may also arise from quantum effects such as creating particles in the expansion of the universe, which also occurs in Rastall’s cosmology [39].

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