HOMOGENEOUS RICCI SOLITONS ARE ALGEBRAIC

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ABSTRACT. In this short note, we show that homogeneous Ricci solitons are algebraic. As an application, we see that the generalized Alekseevskii conjecture is equivalent to the Alekseevskii conjecture.

1. INTRODUCTION

A Riemannian manifold \((M, g)\) is said to be a Ricci soliton if it satisfies the equation

\[
\text{ric}_g = cg + L_X g
\]

for some \(c \in \mathbb{R}\) and some smooth vector field \(X \in \mathfrak{X}(M)\). Such metrics are of interest as they correspond to self-similar solutions of the Ricci flow

\[
\frac{\partial}{\partial t} g = -2 \text{ric}_g
\]

That is, \(g\) is the initial value of a solution to the Ricci flow of the form \(g_t = c(t) \varphi_t^* g\), where \(c(t) \in \mathbb{R}\) and \(\varphi_t \in \text{Diff}_{\text{geo}}(M)\). In this way, Ricci solitons are geometric fixed points of the flow and so are special metrics.

Homogeneous Ricci solitons arise naturally as limits under the Ricci flow \cite{15, 14} and, independently, hold a distinguished place apart from other homogeneous metrics. For example, nilmanifolds cannot admit Einstein metrics, but do often admit Ricci solitons \cite{9, 7}, Ricci solitons on nilmanifolds are precisely the minima of a natural geometric functional \cite{13}, and Ricci solitons are metrics of maximal symmetry on certain solvmanifolds \cite{4}.

One natural kind of example arises as follows. Consider a homogeneous space \(G/K\) where \(K\) is closed and connected. For every derivation \(D \in \text{Der}(\mathfrak{g})\) such that \(D : \mathfrak{k} \to \mathfrak{k}\), we have a well-defined map \(D_{\mathfrak{g}/\mathfrak{k}} : \mathfrak{g}/\mathfrak{k} \to \mathfrak{g}/\mathfrak{k}\). Denote such derivations of \(\mathfrak{g}\) by \(\text{Der}(\mathfrak{g}/\mathfrak{k})\). A homogeneous Ricci soliton \((G/K, g)\) is called \(G\)-semi-algebraic if the \((1, 1)\) Ricci tensor is of the form

\[
\text{Ric} = c\text{Id} + \frac{1}{2}(D_{\mathfrak{g}/\mathfrak{k}} + D_{\mathfrak{g}/\mathfrak{k}}^t)
\]

on \(\mathfrak{g}/\mathfrak{k} \cong T_e G/K\), for some \(c \in \mathbb{R}\) and some \(D \in \text{Der}(\mathfrak{g}/\mathfrak{k})\). This definition is motivated by the idea of taking our family of diffeomorphisms \(\{\varphi_t\}\) above to come from automorphisms of the group \(G\) which leave \(K\) invariant, see \cite{6} or \cite{12} for more details.

If our semi-algebraic Ricci soliton satisfies the seemingly stronger condition that \(D_{\mathfrak{g}/\mathfrak{k}}\) is symmetric, then it is called a \(G\)-algebraic Ricci soliton. Up to this point, all known examples of semi-algebraic Ricci solitons were in fact algebraic and isometric to solvmanifolds. (This follows from \cite{6} together with \cite{11}.) Further, it was known that every homogeneous Ricci soliton must be semi-algebraic relative to its full isometry group \cite{6}. We now present our main result.

**Theorem 1.** Every \(G\)-semi-algebraic Ricci soliton is necessarily \(G\)-algebraic.

**Corollary 2.** Let \((M, g)\) be a homogeneous Ricci soliton. There exists a transitive group \(G\), of isometries, such that \(M = G/K\) is a \(G\)-algebraic Ricci soliton.

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This work was supported in part by NSF grant DMS-1105647.
The theorem above resolves questions raised by Lafuente-Lauret [12] and He-Petersen-Wylie [4]. In these works, it was shown that one can always extend a simply-connected, algebraic soliton to an Einstein metric on a larger homogeneous space. The goal was to relate the classical Alekseevskii conjecture on Einstein metrics to a more general version for Ricci solitons. More precisely, they showed that (among simply-connected manifolds) the Alekseevskii conjecture for Einstein metrics is equivalent to the (apriori) more general conjecture in the case of algebraic Ricci solitons. We state these conjectures for completeness.

**Alekseevskii Conjecture:** Every homogeneous Einstein metric with negative scalar curvature is isometric to a simply-connected solvmanifold.

**Generalized Alekseevskii Conjecture:** Every expanding homogeneous Ricci soliton is isometric to a simply-connected solvmanifold.

Until now, it was not clear if these conjectures were equivalent. Applying [12] or [4] in the simply-connected case together with [8] and the results here, we now know the following.

**Theorem 3.** The generalized Alekseevskii conjecture is equivalent to the Alekseevskii conjecture.

**Remark.** It is important to note that the Alekseevskii conjecture stated above is a more modern, geometric version than that given in [2]. The version given in [2] has the weaker, topological conclusion that a non-compact, homogeneous, Einstein space is only diffeomorphic to \( \mathbb{R}^n \). It is still an open question as to whether the classical version stated in [2] is equivalent to the stronger version we pose above.

**Acknowledgments:** It is our pleasure to thank Ramiro Lafuente for providing useful comments on a draft of this manuscript.

## 2. Ricci solitons by type

The analysis of (homogeneous) Ricci solitons varies depending on which of the following categories the metric falls into. A Ricci soliton is called shrinking, steady, or expanding (respectively) if the cosmological constant \( \lambda \) appearing in Eqn. 1.1 satisfies \( \lambda > 0 \), \( \lambda = 0 \), or \( \lambda < 0 \) (respectively).

**Shrinking solitons.** The simplest example of a non-Einstein, homogeneous, shrinker is obtained by considering a compact homogeneous Einstein space \( M' \) (which necessarily has positive scalar curvature) and taking a product with \( \mathbb{R}^n \), i.e. \( M = M' \times \mathbb{R}^n \). Here the vector field \( X \in \mathfrak{X}(M) \) appearing in Eqn. 1.1 generates a family of diffeomorphisms which simply dilate the \( \mathbb{R}^n \) factor. Examples of this type are called trivial Ricci solitons and a result of Petersen-Wylie [16] says that every homogeneous shrinking Ricci soliton is finitely covered by a trivial one. Observe that such spaces are algebraic Ricci solitons.

**Steady solitons.** A homogeneous steady soliton is necessarily flat. This well-known fact is proved as follows. Along the Ricci flow of any homogeneous manifold, the scalar curvature \( sc \) evolves by the ODE 

\[
\frac{d}{dt} sc = 2|\text{Ric}|^2
\]

As the scalar curvature of a steady soliton does not change along the flow, we see that the homogeneous, steady solitons are Ricci flat and so flat by [1]. Such spaces are trivially algebraic Ricci solitons.
Expanding solitons. Every homogeneous, expanding Ricci soliton is necessarily non-compact, non-gradient and all known examples of such spaces are isometric to solvable Lie groups with left-invariant metrics. While there is no characterization in this case as nice as the previous two cases, new structural results have recently appeared in [12]. The results obtained there are essential in our proof and we briefly recall those which we need. We first observe that it suffices to prove the theorem for simply-connected manifolds. Now consider a simply-connected, expanding, semi-algebraic Ricci soliton on $G/K$. As $G/K$ is endowed with a $G$-invariant metric, $Ad(K)$ is contained in a compact subgroup of $Aut(G)$ and so we have a decomposition $g = p \oplus \mathfrak{k}$, where $p$ is an $Ad(K)$-complement to $\mathfrak{k}$. We fix the point $p = eK \in M = G/K$ and naturally identify $p$ with $T_p M$ as follows

$$X \in p \leftrightarrow \frac{d}{ds} \bigg|_{s=0} \exp(sX) \cdot p = \frac{d}{ds} \bigg|_{s=0} \exp(sX)K.$$ 

Although there is more than one choice of $p$ that one can make, we apply the work [12] in the sequel and so we choose, as they do, to have $B(\mathfrak{k}, p) = 0$, where $B$ is the Killing form of $g$.

As $G/K$ admits an expanding Ricci soliton, we know from [12] that the group $G$ decomposes as $N \times U$ where $N$ is the nilradical and $U$ is a reductive subgroup which contains the stabilizer $K$. Thus the underlying manifold of $M$ may be considered as $N \times U/K$ and we naturally identify the point $p = eK \in G/K$ with $(e, eK) \in N \times U/K$. The subalgebra $\mathfrak{u}$ contains a subspace $\mathfrak{h}$ which is complementary to $\mathfrak{k}$, and so we have $T_p M \cong p = \mathfrak{n} \oplus \mathfrak{h}$. Furthermore, $\mathfrak{n}$ and $\mathfrak{h}$ are orthogonal subspaces of $T_p M$. For more details, see [12].

Denote the restriction of our metric $g$ to $p \cong T_p G/K$ by $\langle \cdot, \cdot \rangle$. Denote by $H \in \mathfrak{p}$ the ‘mean curvature vector’ of $G/K$ defined by

$$\langle H, X \rangle = tr(ad X) \quad \text{for all } X \in \mathfrak{p}.$$ 

Observe that $H \in \mathfrak{h}$. It is a useful fact that the subspace $\mathfrak{h}$ of $\mathfrak{u}$ is $(ad H)$-stable [12, Prop. 4.1]. If $D$ is the soliton derivation appearing Eqn. [12] then we have

$$D = -ad H + D_1$$

where $D_1$ is the derivation which vanishes on $\mathfrak{u}$ and restricts to the nilsoliton derivation on $\mathfrak{n}$.

In [12, Prop. 4.14], several conditions are given for when a semi-algebraic Ricci soliton is actually algebraic. One of those conditions is

$$(2.1) \quad S(ad H|_{\mathfrak{h}}) = 0$$

where $S(A) = \frac{1}{2}(A + A^t)$. This is the technical result that we will prove, from which the theorem follows.

3. The proof of theorem [11]

The soliton inner product $\langle \cdot, \cdot \rangle$ on $T_p M$ above gives rise to a natural inner product on the endomorphisms of $T_p M$ given by $\langle A, B \rangle = tr(AB^t)$, where $B^t$ denotes the metric adjoint of $B$ relative to $\langle \cdot, \cdot \rangle$.

**Lemma 4.** Using the above inner product on endomorphisms we have

$$\langle (0, ad H|_{\mathfrak{h}}), Ric \rangle = 0$$

where $(0, ad H|_{\mathfrak{h}})$ is the map on $T_p M$ defined as 0 on $\mathfrak{n}$ and $ad H|_{\mathfrak{h}}$ on $\mathfrak{h}$.

**Remark.** As has been observed by R. Lafuente [10], our proof of the lemma holds more generally. In fact, one simply needs the group to satisfy $G = U \times N$ with $N$ nilpotent, $U$ reductive, and $K < U$, the metric to satisfy $N \perp U/K$ at $eK$, and the element $H$ may be replaced by any $Y \in \mathfrak{u}$ satisfying $[Y, \mathfrak{k}] \subset \mathfrak{k}$. 

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Before proving the lemma, we use it to verify that Eqn. 2.1 holds.

**Verification of 2.1.** Consider the mean curvature vector $H \in u$. As $u$ is reductive, $ad H|_u$ is traceless. Furthermore, since $ad H$ vanishes on the stabilizer $\mathfrak{k}$ (see Eqn. 26 of [12]) and $u = \mathfrak{k} \oplus \mathfrak{h}$, we see that $tr ad H|_h = 0$. Together with the above lemma we have

$$
0 = \langle (0, ad H|_h), Ric \rangle = \langle (0, ad H|_h), cId - S(ad H) + D_1 \rangle = \langle ad H|_h, cId|_h - S(ad H|_h) \rangle = c tr(ad H|_h) - tr S(ad H|_h)^2 = 0 - tr S(ad H|_h)^2
$$

Thus $S(ad H|_h) = 0$, as claimed. \qed

We now prove the lemma by considering a certain deformation of the metric $g$ on $M$. As $ad H$ vanishes on $\mathfrak{k}$ and $K$ is connected, the family of automorphisms $\Phi_t = C_{exp(tH)} \in Aut(U)$ is the identity on $K$ and hence gives rise to well-defined diffeomorphisms $\phi_t$ on $U/K$ given by

$$
\phi_t(uK) = \Phi_t(u)K \quad \text{for } u \in U
$$

Note that $(\Phi_t)_* = Ad(exp(tH)) = e^t ad H \in Aut(u)$. On the manifold $M = N \times U/K$, we consider the family of diffeomorphisms given by

$$
\varphi_t = (id, \phi_t) \quad \text{on } N \times U/K
$$

The deformations of $g$ of interest are $g_t = \varphi_t^* g$.

As $\varphi_t$ fixes the point $p := eK = (e, eK) \in M = N \times U/K$, and scalar curvature is an invariant, we have

$$
\frac{d}{dt} \bigg|_{t=0} sc(\varphi_t^* g)_p = 0
$$

We use this in the following general equation which holds for any family of metrics $\{g_t\}$ with variation $h = \frac{\partial g_t}{\partial t}$ (see [3, Lemma 3.7])

$$
\frac{\partial}{\partial t} sc = -\Delta T + div(div h) - \langle h, ric \rangle
$$

where in local coordinates we have

$$
\Delta T = g^{ij} g^{kl} \nabla_i \nabla_j h_{kl}
$$

and

$$
\text{div(div } h) = g^{ij} g^{kl} \nabla_i \nabla_j h_{kl}
$$

Observe, at the point $p := eK = (e, eK) \in M$ we have $\frac{\partial}{\partial t}|_{t=0}(\varphi_t)_* = (0, ad H|_h)$ and so the lemma follows from Eqn. 3.1 (evaluated at $p$) upon showing the terms $\Delta T$ and $\text{div(div } h)$ vanish.

**Remark.** Recall that, in local coordinates, we define the metric inverse $g^{ij}$ as the function satisfying

$$
\delta_i^l = g^{ij} g_{jl}. \quad \text{By choosing a frame which is } g\text{-orthonormal at every point, one would have that both } g_{ij} \text{ and } g^{ij} \text{ are the identity. We make such a choice below.}
$$

To ease computational burden, we build a frame which is $g$-orthonormal at every point and exploits the property that our metric $g$ is $G$-invariant. We start with an orthonormal basis of $T_p M$. As $T_p M = \mathfrak{n} \oplus \mathfrak{h}$, we may choose a basis $\{e_i\}$ which is the union of an orthonormal basis of $\mathfrak{n}$ together with an orthonormal basis of $\mathfrak{h}$. 4
Next, we extend the basis \( \{e_i\} \) to a local frame nearby to \( p \in M \). To do this, we first consider a slice \( \mathcal{G} \) of the right \( K \) action on \( G \) through \( e \in G \). That is, we have a submanifold \( \mathcal{G} \) of \( G \) containing \( e \) such that \( \dim \mathcal{G} = \dim G/K \) and the map

\[
s \mapsto sK \quad s \in \mathcal{G}
\]

is a diffeomorphism of a neighborhood of \( e \in \mathcal{G} \) to a neighborhood of \( eK \in G/K \). Now, for \( q \in M \) nearby to \( p \), there exists \( s \in \mathcal{G} \) such that \( q = s \cdot p \) and we define

\[
e_i(q) = s_* e_i,
\]

where \( s_* \) denotes the differential of the translation \( s : p \mapsto q \). We note that the frame is well-defined as our choice of \( s \in \mathcal{G} \) is unique, since \( \mathcal{G} \) is a slice. Furthermore, our frame is \( g \)-orthonormal as \( g \) is \( G \)-invariant.

Using the above choice of frame nearby to \( p \in M \), we now study Eqns. (3.2) and (3.3). We begin by computing the variation \( h \) of \( g_t = \varphi_t g \) in terms of \( \{e_i\} \). For a point \( q \in M \) near \( p \),

\[
(3.4) \quad h_{ij}(q) = \frac{\partial}{\partial t} \bigg|_{t=0} (g_t)_{ij}(q) = \frac{\partial}{\partial t} \bigg|_{t=0} (g_t)(e_i(q), e_j(q)) = \frac{\partial}{\partial t} \bigg|_{t=0} g((\varphi_t)_* e_i(q), (\varphi_t)_* e_j(q))
\]

Next we compute \( (\varphi_t)_* v_q \) for a vector \( v_q \in T_q M \).

As \( G = NU \), there exist \( n \in N \) and \( u \in U \) such that \( s \in \mathcal{G} \) may be written as \( s = nu \) and \( q = (nu) \cdot p \). Furthermore, there exists \( X \in p = n \oplus \mathfrak{h} \) such that \( v_q = (nu)_* \frac{d}{ds} \big|_{s=0} \exp(sX) \cdot p \). To understand Eqn. (3.4), we analyze separately the cases when \( X \) is an element of \( \mathfrak{n} \) or of \( \mathfrak{h} \).

For \( X \in \mathfrak{n} \), we have

\[
(\varphi_t)_* v_q = (\varphi_t)_* (nu)_* X = \frac{d}{ds} \bigg|_{s=0} \varphi_t(nu \exp(sX) \cdot p) = \frac{d}{ds} \bigg|_{s=0} \varphi_t(nu \exp(sX) u^{-1} u \cdot p) = \frac{d}{ds} \bigg|_{s=0} \varphi_t(nu \exp(sX) u^{-1} \cdot uK) = \frac{d}{ds} \bigg|_{s=0} (n \exp(sAd_u X), \Phi_t(u)K) = \frac{d}{ds} \bigg|_{s=0} (n \Phi_t(u) \Phi_t(u)^{-1} \exp(sAd_u X) \Phi_t(u)K) = \frac{d}{ds} \bigg|_{s=0} (n \Phi_t(u) \exp(sAd_{\Phi_t(u)}^{-1} Ad_u X)K) = (n\Phi_t(u))_* Ad_{\Phi_t(u)^{-1} u} X
\]

Here we have used that \( N \) is normal in \( G \). Note also that \( Ad_{\Phi_t(u)^{-1} u} X \in \mathfrak{n} \).
In the case when \( X \in \mathfrak{h} \subset \mathfrak{u} \), we have
\[
(\varphi_t)_* v_q = (\varphi_t)_* (nu)_* X = \left. \frac{d}{ds} \right|_{s=0} \varphi_t(nu \exp(sX) \cdot p) = \left. \frac{d}{ds} \right|_{s=0} \varphi_t(nu \exp(sX) K) = \left. \frac{d}{ds} \right|_{s=0} (n \Phi_t(u \exp(sX)) K) = \left. \frac{d}{ds} \right|_{s=0} (n \Phi_t(u) \exp(s(\Phi_t)_* X)) K)
\]
(3.6)

Observe that since \( \text{ad} \, H \) preserves \( \mathfrak{h} \) (Eqn. 32), \( (\Phi_t)_* X \in \mathfrak{h} \) and so the the last line is consistent with our identification of \( p = \mathfrak{n} \oplus \mathfrak{h} \) with \( T_p \mathfrak{M} \).

From Eqs. 3.4, 3.5, and 3.6 we see that
(i) If \( e_i \in \mathfrak{n} \) and \( e_j \in \mathfrak{h} \), then \( g_{ij}(q) = 0 \).
(ii) If \( e_i \in \mathfrak{n} \) and \( e_j \in \mathfrak{h} \), then \( h_{ij}(q) = 0 \).
(iii) If \( e_i, e_j \in \mathfrak{h} \), then \( h_{ij}(q) \) does not depend on \( n \) and \( u \), and so is constant in \( q \).
(iv) If \( e_i, e_j \in \mathfrak{n} \), then \( h_{ij}(q) \) does not depend on \( n \), but does depend on \( u \).

Using these observations, we see that the only possible non-zero terms of
\[
\text{div(div \, h)} = g^{ij} g^{kl} \nabla_i \nabla_k h_{jl}
\]
are when \( e_j, e_l \in \mathfrak{n} \) and \( e_i, e_k \in \mathfrak{h} \). However, \( (g_{\alpha \beta}) = Id \) implies \( (g^{\alpha \beta}) = Id \) and so \( g^{kl} = 0 \). This yields
\[
\text{div(div \, h)} = 0
\]

Next we study \( \Delta \mathcal{P} = g^{ij} g^{kl} \nabla_i \nabla_j h_{kl} \). As above, the only possible non-zero terms occur when \( e_k, e_l \in \mathfrak{n} \) and \( e_i, e_j \in \mathfrak{h} \). Further, as our frame is orthonormal, we have
\[
\Delta \mathcal{P}(q) = g^{ij}(q) g^{kk}(q) (\nabla_i \nabla_i h_{kk})(q) = \sum_i \left( \nabla_i \nabla_i \sum_k h_{kk} \right)(q)
\]
where the first sum is over the frame from \( \mathfrak{h} \) and the second is over the frame from \( \mathfrak{n} \). From Eqs. 3.3 and 3.5 we have
\[
h_{kk}(q) = \left. \frac{\partial}{\partial t} \right|_{t=0} g((\varphi_t)_* e_k(q), (\varphi_t)_* e_k(q)) = \left. \frac{\partial}{\partial t} \right|_{t=0} \langle \text{Ad} \Phi_t(u)^{-1} u(e_k), \text{Ad} \Phi_t(u)^{-1} u(e_k) \rangle
\]
\[
= 2 \langle e_k, (d\Phi_t \bigg|_{t=0}) (\text{Ad} \Phi_t(u)^{-1} u)(e_k) \rangle = 2 \langle e_k, \text{ad} \, M(e_k) \rangle
\]
where \( M = \left. \frac{d}{dt} \right|_{t=0} \Phi_t(u)^{-1} u \). To see that this last line makes sense, observe that \( \Phi_t(u)^{-1} u \) is a curve in \( U \) with \( \Phi_0(u)^{-1} u = e \) and thus \( \left. \frac{d}{dt} \right|_{t=0} \Phi_t(u)^{-1} u = e \).

**Remark.** Although \( M \) is a function of \( u \), we suppress this detail as it does not impact the rest of our proof.
We claim that \( \text{ad } M|_n \) is traceless. To see this, we use that fact that \( U \) being reductive and connected implies \( U = [U, U]Z(U) \), where \( Z(U) \) is the center of \( U \). Thus, we may write \( u = u_1u_2 \) where \( u_1 \in [U, U] \) and \( u_2 \in Z(U) \). As \( u_2 \) is central and \( \Phi_t \) is an inner automorphism, \( \Phi_t(u_2) = u_2 \) and

\[
\Phi_t(u_2)^{-1}u = \Phi_t(u_1)^{-1}u_1 \in [U, U]
\]

This gives \( \text{ad } M \in \text{ad } [u, u] \) from which our claim immediately follows.

Putting the above computations together,

\[
\Delta \mathcal{H}(q) = \sum_i \left( \nabla_i \nabla_i \sum_k h_{kk} \right)(q)
\]

\[
= 2 \sum_i \nabla_i \nabla_i \text{tr } \text{ad } M|_n
\]

\[
= 0
\]

which completes the proof of the lemma.

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