A free zero-range process (FZRP) is a simple stochastic process describing the dynamics of a gas of particles hopping between neighboring nodes of a network. We discuss three different cases of increasing complexity: (a) FZRP on a rigid geometry where the network is fixed during the process, (b) FZRP on a random graph chosen from a given ensemble of networks, (c) FZRP on a dynamical network whose topology continuously changes during the process in a way which depends on the current distribution of particles. The case (a) provides a very simple realization of the phenomenon of condensation which manifests as the appearance of a condensate of particles on the node with maximal degree. A particularly interesting example is the condensation on scale-free networks. Here we will model it by introducing a single-site inhomogeneity to a k-regular network. This simplified situation can be easily treated analytically and, on the other hand, shows quantitatively the same behavior as in the case of scale-free networks. The case (b) is very interesting since the averaging over typical ensembles of graphs acts as a kind of homogenization of the system which makes all nodes identical from the point of view of the FZRP. In effect, the partition function of the steady state becomes invariant with respect to the permutations of the particle occupation numbers. This type of symmetric systems has been intensively studied in the literature. In particular, they undergo a phase transition to the condensed phase, which is caused by a mechanism of spontaneous symmetry breaking. In the case (c), the distribution of particles and the dynamics of network are coupled to each other. The strength of this coupling depends on the ratio of two time scales: for changes of the topology and of the FZRP. We will discuss a specific example of that type of interaction and show that it leads to an interesting phase diagram. The case (b) mentioned above can be viewed as a limiting case where the typical time scale of topology fluctuations is much larger than that of the FZRP.

Keywords: Balls-in-boxes model, zero-range process, condensation, dynamical rewirings

I. INTRODUCTION

In the last decade great progress has been achieved in our understanding of the structure and topology of complex networks. Complex networks became a sui generis branch of research on the interface between physics, probability and graph theory, geometry, computer science and many other disciplines and they attracted a lot of researchers. Probably the main interest in networks comes from the fact that in many applications they can be viewed as a skeleton of complex systems, along which various signals and information propagates between various components of the system. The dynamics of this propagation is responsible for the functionality of complex systems and depends on the structure of the underlying network. In some situations the structure can flexibly adjust during the evolution of the system to optimize the propagation and functionality, leading to a feedback of the dynamics on the structure of a network. In the most general case, the dynamics of degrees of freedom propagating on the complex network and the dynamics of the network topology are intertwined.

In this paper we shall discuss a model describing the dynamics of some particles propagating through the network. We shall analyze three cases of gradually increasing complexity: starting from the model on a given network, through the model averaged over networks, and ending with a model where the dynamics of particles and of network topology are coupled to each other. It is clear that to realize this plan one has to consider a relatively simple model to be able to make some predictions analytically and, on the other hand, complicated enough to reveal some interesting properties. A very good candidate is a gas of identical particles (or balls) hopping between neighboring nodes of the network. At each time step, one particle hops from every non-empty node, to one of its neighboring nodes which is chosen at random. It is a particular example of a more general process called zero-range process (ZRP), where the hopping rate only depends on the number of particles at the departure node. Since in our case the hopping rate is identical for all nodes and particles hop unconditionally we call this process free zero-range process (FZRP). Despite its simplicity the model has many interesting features. The system of particles has a steady state and it undergoes a phase transition from a liquid state to the state where a condensate of particles is formed on a single node. The statics of the steady state is so simple that it can be analyzed analytically. Also many aspects of its dynamics can
be well understood by analytical methods. It is probably the simplest model with non-trivial behavior where many aspects of the underlying network structure and its interaction with the degrees of freedom defined on the network become pronounced and play an essential role.

The paper is organized as follows. We first recall the main results on the FZRP on a static network, then we discuss FZRPs averaged over random networks from a given ensemble. We pay a special attention to the effect of homogenization and discuss the relation between the node degree distribution of the random network and the particle distribution of the steady state of the FZRP, defined on this network. Finally, we discuss a simple model of FZRP on a dynamical random network with an explicit coupling between the dynamics of particles and of network topology.

At the end of paper we shortly summarize our main results.

II. STATICS AND DYNAMICS IN FREE ZRP

Consider a system of $M$ identical indistinguishable particles and a graph with $N$ nodes. The dynamics is driven by a stochastic process defined in such a way that at the given moment of time one particle jumps from each non-empty node to a randomly chosen neighboring node on the network. In general one can think of a synchronous dynamics but in practice when one wants to realize the process by a simple Monte Carlo computer simulation, one picks up nodes at random, one by one, and if the given node is not empty, one moves a particle to a neighboring node chosen with probability $1/k_i$. Here $k_i$ is the number of neighbors of the node $i$, called its degree. On average, the outflow of particles along each link is equal to $1/k_i$ if there is at least one ball at that node. The full state of the system is given by the distribution of particles $\{m_1, \ldots, m_N\}$, where $m_i$ is the number of particles at the node $i$. The total number of particles $M = m_1 + \ldots + m_N$ is conserved during the process. The model described above is a special case of the zero-range process (ZRP) with no point-interaction between balls.

If the network is connected, the FZRP has a unique steady state. Although it is a non-equilibrium state, one can formally write for it a partition function $Z(N, M, \vec{k})$ which for the given graph depends on the sequence of the node degrees $\vec{k} = \{k_1, \ldots, k_N\}$ but not on other details of the graph topology:

$$Z(N, M, \vec{k}) = \sum_{m_1=0}^M \cdots \sum_{m_N=0}^M \delta_{m_1 + \ldots + m_N = M} \prod_{i=1}^N p_i(m_i),$$

where $p_i(m) = k_i^m$. The only constraint that prevents $Z(N, M, \vec{k})$ from a full factorization is the discrete delta function reflecting the conservation of particles. The partition function has exactly the same form as the balls-in-boxes model. Since, beside the degree sequence, other topological features of the network are irrelevant for the steady state, one can forget about the network and think only of balls hopping between boxes having statistical weights $p_i(m)$, which may be different for different boxes $i$. The knowledge of the partition function allows one to calculate all static quantities of the model and well approximate most of the dynamical quantities describing its behavior out of the steady state. Similarly as in the ZRP, the most interesting quantity which allows one to grasp what happens in the system is the node occupation distribution $\pi_{i,\vec{k}}(m)$ which tells one the probability that the node $i$ has exactly $m$ balls. It can be calculated as

$$\pi_{i,\vec{k}}(m) = \frac{Z(N-1, M-m, \vec{k}_i)}{Z(N, M, \vec{k})} p_i(m),$$

where $Z(N-1, M-m, \vec{k}_i)$ is the partition function for a graph with $N-1$ nodes, $M-m$ particles and degrees $\vec{k}_i = \{k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_N\}$, where $k_i$ is skipped. If one averages Eq. (2) over nodes of the network, one obtains the probability that a randomly chosen node is occupied by $m$ particles:

$$\pi(m) = \frac{1}{N} \sum_{i=1}^N \pi_{i,\vec{k}}(m).$$

We shall see below, that this global quantity can be used as a signature of condensation. If all $k_i$'s are the same, as it is for a $k$-regular graph, the system is said to be homogeneous. In this case all nodes in the partition function are statistically equivalent and the quantities averaged over the nodes are identical to those obtained separately for any node, for instance $\pi_{i}(m) = \pi(m)$ for all $i$'s. Moreover, the degree $k$ appears in Eq. (1) as a factor $k^M$ which can be pulled out in front of the sum as a constant and which cancels out in physical quantities like the distribution of balls. A more interesting case is of course when the degrees $k_i$ vary from node to node and the system is inhomogeneous.
In this case in order to fully characterize the system one should indeed independently determine $\pi_i(m)$ for every node $i$.

We are often interested in the behavior of the system for a given density of particles $\rho = M/N$ in the thermodynamic limit, that is when the size of the system $N$ tends to infinity. What makes the FZRP (or generally the ZRP) so interesting is that by increasing the density one may trigger off the phenomenon of condensation. Above a certain density of balls $\rho_c$, an extensive number of particles, proportional to the surplus above the critical density: $\Delta M = M - \rho_c N$, condenses on a single node – the one with highest degree. The mechanism underlying the condensation can be understood by comparing flows of particles coming in and going out of a node. The average outflow of particles from a node $i$ does not depend on its degree, but the inflow does. It is proportional to $\sum_j 1/k_j \sim k_i/(k_i)$ where the sum runs over all non-empty neighbors of $i$. If $k_i$ is greater than any $k_j$ and balls are uniformly distributed, the outflow is smaller than the inflow. Thus the node $i$ attracts more and more particles, as long as there are enough balls in the vicinity of $i$. When the density of balls in the neighborhood falls below some value, the in- and outflows balance each other. This process leads to a fast local condensation on a few nodes with higher degrees. Then, by exchanging particles through the background, all partial condensates merge into a single one.

A short inspection of the partition function (1) makes it clear that the condensate has the largest chance to form on a node with the largest degree, since the statistical weight of this node $p_1(m) = k_i^m$ favors larger values of $m$ stronger than other nodes. Therefore, while analyzing the static properties, one should concentrate on this node. In order to simplify the discussion we consider first graphs having all but one regular nodes of degree $m$, and a single irregular node of degree $k_i > k$. It is convenient to introduce in this case a parameter $\alpha = k_i/k$ which controls the strength of inhomogeneity. The distribution of balls at the singular node can be calculated from the partition function (1) with $p_1(m) = k_i^m$ and $p_i(m) = k_i^m$ for $i > 1$. The result is

$$\pi_1(m) \propto \alpha^{-m} \left( \frac{M + N - m - 2}{M - m} \right).$$

This distribution has a maximum at $m_\ast \approx M - \rho_c N = (\rho - \rho_c)N$, with a critical density $\rho_c = \alpha/(1 - \alpha)$. We see that above $\rho_c$, an extensive number of particles condense at the irregular node. The critical density is larger than zero for $\alpha < 1$, that is when the degree of the irregular node is larger than the degrees of other nodes. When $k_i$ becomes equal to $k$ or smaller, the critical density becomes infinite and the system never enters the condensed phase. One can also show that the occupation distribution for any regular node falls exponentially for $\rho > \rho_c$,

$$\pi_{\text{reg}}(m) \propto \alpha^m,$$

which means that the condensate does not appear on regular nodes. The average distribution of balls is given by $\pi(m) = [(N - 1)\pi_{\text{reg}}(m) + \pi_1(m)]/N$. From Eqs. (4) and (5) we see that for a fixed density the condensation manifests in $\pi(m)$ as a peak whose position moves linearly with the system size $N$. The area under the peak is equal to $1/N$ since the condensate is located at a single node: one out of $N$. On the left-hand side of Fig. 4 we show plots of $\pi(m)$ calculated for different densities. One sees that the position of the peak moves linearly with $M$.

This simple case of condensation on a graph with a single irregular node captures well what happens on general inhomogeneous networks. This is confirmed by numerical simulations on Barabási-Albert (B-A) scale-free graphs [9], with the degree distribution $\Pi(k)$ falling like $\Pi(k) \sim k^{-3}$. The network is built of a small number of high-degree nodes, called hubs, to which many nodes with small degrees are linked. From the point of view of the FZRP the hubs behave similarly as the irregular nodes in the simple model discussed above. On the right-hand side of Fig. 4 we show the occupation distribution $\pi(m)$ for one particular B-A network with $N = 100$ nodes and $L = 198$ links, as obtained in Monte Carlo computer simulations. The maximal degree is $27$. Similarly to the situation for the single inhomogeneity graph, the condensation takes place only at the node with highest degree. The critical density can be estimated from the position of peaks: $\rho_c \approx 0.19$. This value is close to that obtained using the formula derived for the single inhomogeneity graph: $\rho_c = \alpha/(1 - \alpha) \approx 0.17$, if $\alpha$ is assumed to be $k/k_{\text{max}} \approx 4/27$. In short, as anticipated the statics is almost identical to that of the single inhomogeneity model.

We shall now shortly discuss the dynamics of the condensation. The process of condensate formation from a state where all nodes have approximately the same number of balls has been investigated in ZRP’s [2, 10, 11, 12] and can be divided into two steps. First, the surplus of balls is accumulated on some nodes having relatively high degrees in comparison to the rest. Then, these small condensates merge and form a single one. The characteristic time-scale for this process depends on the structure on the network, but in general it grows like a power of the system size $N$. Once formed, the condensate is not a static object but fluctuates and can sometimes melt down to be indistinguishable from the background. It rebuilds however very quickly. One may ask about the typical life-time $\tau$ of the condensate, that is how much time it takes to fall from $\approx \Delta M$ balls at the condensed node to $\approx \rho_c$, being the average occupation for the “background nodes”. In Ref. [10] a mean-field procedure for calculating $\tau$ has been derived. It relies on the assumption that the state of the condensate varies slowly in comparison to the time scale of fluctuations on all
FIG. 1: Left: theoretical distributions of balls $\pi(m)$ for a single inhomogeneity graph with $k_1 = 8, k = 4, N = 20$, for three different $M$: 20 (solid line), 60 (dashed), and 100 (dotted). The critical density is $\rho_c = 1$. Peaks are located at $m_\ast \approx M - \rho_c N$. Right: distributions for a single B-A network of size $N = 100, L = 198$ and $M = 100$ (circles), 150 (diamonds) and 200 (squares), from computer simulations. The three peaks show $\pi(m)$ for the node with the highest degree $k_1 = 27$, while the two left-most lines are for $k_2 = 20$ (the next largest degree) and $k_3 = k_4 = 13$. An exponential decay is clearly seen in $\pi_{2,3,4}(m)$.

remaining nodes in the system. This is a good approximation if the network is compact, that is if its diameter grows slowly with $N$, for example like a logarithm, because then balls travel very fast through the graph. Actually in many complex networks one observes such a slow increase of the diameter. One can therefore distinguish a single “slow” variable giving the number of balls at the node where the condensate is located, and apply a mean-field dynamics treating the rest of the system as a steady-state. In this way one obtains a closed formula for the mean time $\tau_{mn}$ it takes to decrease the occupation of the node in question from $m$ to $n$ particles [8, 10, 13]. It can be in particular used to determine the life-time of the condensate in the model with a single inhomogeneity [13]. The full formula is quite complicated, but the life-time can be estimated qualitatively with the help of the Arrhenius law [14] which states that the average life-time is inversely proportional to the exponent of the barrier of the effective potential. Since the effective potential is equal to $-\log \pi_1(m)$ up to a normalization, the exponent just gives the minimal value of the distribution $\pi_1(m)$. Taking into account the proper normalization of Eq. (4), one gets $\tau \sim (k_1/k)^N$. The time thus grows exponentially with the system size $N$, while the characteristic time for building the condensate from a uniform distribution of particles grows only like a power of $N$. This is different from what one observes for homogeneous systems, where both the times grow like a power, each having a different exponent, of the system size $[10]$.

III. FREE ZRP ON QUENCHED RANDOM NETWORKS

In the previous section we have shown results of Monte Carlo simulations for a single B-A scale-free network. One can ask what happens if one considers FZRP not on a single network, but rather on the whole set of networks. We define therefore a statistical ensemble of networks, where each graph has a certain probability of occurrence. All physical quantities, as for instance the occupation probability, have to be now averaged over this ensemble. Because static properties of the ZRP depend only on the sequence of degrees $\vec{k}$, we can reduce the problem to averaging with weights $P(\vec{k})$ giving probabilities of occurrence of networks with a degree sequence $\vec{k}$. The effective occupation distribution of balls averaged over the ensemble then reads

$$\pi(m) = \frac{1}{N} \sum_{\vec{k}} \sum_{k_1,...,k_N} P(\vec{k}) \pi_{i,\vec{k}}(m), \quad (6)$$

where $\pi_{i,\vec{k}}(m)$ is given by Eq. (2) for each individual sequence $\vec{k}$. It is important to realize that, for typical ensembles of graphs, the probability distribution $P(\vec{k})$ is invariant with respect to a permutation of degrees $k_i$ in the sequence $\vec{k} = \{k_1,...,k_N\}$. This is because labeling of nodes plays often only an auxiliary role and is unphysical. The invariance with respect to permutations means that each term in the sum over $i$ in the last equation is identical and independent
of $i$. It is therefore sufficient to calculate it for the first node and average over the ensemble:

$$\pi(m) = \sum_{k_1, \ldots, k_N} P(\vec{k}) \pi_{1,\vec{k}}(m).$$  \hspace{1cm} (7)

The system becomes now homogeneous and the distribution of balls is independent of the node. The partition function $Z(N, M)$ has now the following form:

$$Z(N, M) = \sum_{k_1, \ldots, k_N} P(\vec{k}) Z(N, M, \vec{k}),$$  \hspace{1cm} (8)

where $Z(N, M, \vec{k})$ is given by Eq. (1). In general, $P(\vec{k})$ could have a complicated form. We shall restrict here to ensembles of quenched networks with a product measure sometimes called uncorrelated networks. One can find explicit canonical and grand-canonical realization of such ensembles for which the probability $P(\vec{k}) = \Pi(k_1) \cdots \Pi(k_N)$ factorizes in the limit $N \to \infty$, where $\Pi(k)$ denotes the probability distribution of the node degrees. The difference between the canonical ensemble, having a fixed number of edges, and grand-canonical one disappears in the thermodynamic limit if the node degree distribution $\Pi(k)$ falls with $k$ faster than any power-law \cite{16}. This factorization partly breaks down for scale-free networks which we shall not discuss here. The factorization allows us to rewrite the formula for $Z(N, M)$ in the form of Eq. (1) with $p_1(m) = \cdots = p_N(m) = \mu(m)$,

$$Z(N, M) = \sum_{m_1=0}^{M} \cdots \sum_{m_N=0}^{M} \delta_{m_1+\cdots+m_N} Z(N, M) = \prod_{i=1}^{N} \mu(m_i),$$  \hspace{1cm} (9)

where $\mu(m)$ is $m$-th moment of the degree distribution $\Pi(k)$,

$$\mu(m) = \sum_{k=1}^{\infty} \Pi(k) k^m.$$  \hspace{1cm} (10)

In contrast to Eq. (1) the partition function is invariant under permutations of the ball occupation numbers $m_i$. We see that the problem of finding the distribution of balls in the ensemble of uncorrelated networks reduces to the ZRP on homogeneous networks, with weights given by Eq. (10). Zero-range processes on homogeneous networks with identical weight functions for all nodes were intensively investigated in the past. In this paper we shall cite some of the most important results and compare them to those on inhomogeneous graphs described in the previous section. We will also use them to solve the specific case where the weights are given by the moments of the distribution $\Pi(k)$. It is quite surprising that the degree sequence of the underlying network entirely determines the distribution of particles and other properties of the steady state of the FRZP defined on these networks.

The partition function of a homogeneous system is in general given by Eq. (11) with $p_i(m) = p(m)$, where $p(m)$ is some arbitrary weight function. The critical properties of the model depend on the asymptotic behavior of $p(m)$. First, one should notice that if one rescales the weight as follows: $p(m) \to e^{a+b m} p(m)$ then the partition function changes by a multiplicative constant: $Z(M, N) \to e^{N a+b M} Z(M, N)$. Such a constant prefactor in the partition function does not change physical quantities. Therefore, choosing $p(m) \to e^{a+b m} p(m)$ one can get rid of exponential growth or decay and consider only the remaining large $m$ asymptotics of $p(m)$. The asymptotic behavior can be classified into three groups: $p(m)$ falls to zero faster than any power of $m$, falls like a power $\sim m^{-b}$, or approaches a positive constant.

In the last case the system belongs to the same universality class as a system of non-interacting balls and thus it is equivalent to a random walk of $M$ particles on a homogeneous graph. There is no condensation in this case. Similarly, there is no condensation when the weight $p(m)$ decreases with the number of balls $m$ faster than a power-law since there exists an effective repulsive force between balls preventing them from occupying a single site. Balls tend to distribute on the whole graph and the critical density is $\rho_c = \infty$. The most interesting case is when $p(m)$ falls like a power of $m$: $p(m) \propto m^{-b}$. There are two subcases. In the first one, $0 < b < 2$ and the attraction between particles is still too weak to form a condensate. In the second and the most interesting one, $2 < b < \infty$ and a qualitatively different picture emerges. The critical density is finite in this case. The attraction is strong enough to trigger the condensation if the density $\rho$ is larger than $\rho_c$. At the critical density, the distribution of balls $\pi(m) \sim m^{-b}$ falls like a power-law. When $\rho$ exceeds $\rho_c$, the distribution has the same shape as the critical one but it additionally develops a peak which departs from the ‘bulk’ distribution when $N$ goes to infinity. The area $1/N$ under the peak tells us that the condensate occupies a single site (one of $N$) almost all time. The site is chosen randomly from all $N$ nodes by spontaneous symmetry breaking, in contrast to inhomogeneous systems, where the symmetry is explicitly broken. Sometimes the condensate melts and then rebuilds at another node. The characteristic life-time $\tau$ can be estimated
but accidentally it resembles a quasi-power-law for small
$m$ or intermediate values of $N$. In the limit of
one can show that one has to do with a visual artifact caused by looking only on less than two decades in
can be estimated that the leading term in $\mu$ density
in the inhomogeneous networks from the previous section. Si nce it takes some number of balls, it raises the critical
nodes except the one with largest degree. On the most inhomog eneous node the condensation occurs similarly like
$N$ left-hand side of Fig. 2, we show this effect for
$\phi$ scales linearly with
Eq. (12) is valid only for infinite networks, for which
$\phi$ is reproduced in the thermodynamic limit at the critical den sity
is the opposite question: can we determine the distribut ion of particles of the FZRP on a random network with
one is the opposite question: can we determine the distribut ion
$π$ for trees and $Π(k) = e^{−k(k−1)}$ for trees and $Π(k) = e^{−k(k−1)}$ for
E-R graphs, and are effectively uncorrelated even for small networks, thus our method should perform well. On
the right-hand side of Fig. 2 we see results of numerical expe riments. A quick glance on the figure would suggest
that there is again a power-law behavior in $π(m)$. This is, however, not the case. By a simple analytic calculation
one can show that one has to do with a visual artifact caused by looking only on less than two decades in $m$. It can be estimated that the leading term in $μ(m)$ is proportional to $\exp(m \log m)$ and grows for both the types of networks, so in the limit of $N \to \infty$ with a fixed density of particles there is always a condensation in these models. For intermediate values of $M$, the dependence of the moments $μ(m)$ on their order $m$ falls faster than exponentially but accidentally it resembles a quasi-power-law for small $m$. 

\[\Pi(k) \propto (\phi - k)^{b-1}\]

for $k < \phi$ and zero for $k > \phi$, the following power-law in $π(m)$,

\[π(m) \propto \Gamma(m + 1)/\Gamma(m + b + 1) \sim m^{-b},\]

is reproduced in the thermodynamic limit at the critical density $ρ_c = 1/(b-2)$. The value $\phi$ plays a role of the maximal
degree in the network. It depends approximately linearly on the average degree: $\phi \approx (b+1)k$. Strictly speaking, Eq. (12) is valid only for infinite networks, for which $\phi \to \infty$. For finite $\phi$, the distribution $π(m)$ has a cut-off which scales linearly with $\phi$. There is also another effect which disturbs the pure power law for finite networks. The power
law in $π(m)$ comes from a superposition of exponential decays for nodes of different degrees: $π_i(m) \sim (k_i/\phi)^m$ for all
nodes except the one with largest degree. On the most inhomogeneous node the condensation occurs similarly like
in the inhomogeneous networks from the previous section. Since it takes some number of balls, it raises the critical
density $ρ_c$ for which the power law is observed, and leads to the appearance of an additional peak in $π(m)$. On the
left-hand side of Fig. 2 we show this effect for $N = M = 400$ and $\phi = 30$. In Ref. [17] it is shown that these finite-size
effects becomes less and less important for larger networks.

In the above, we were interested in the question of how to tune the degree distribution $Π(k)$ of the random network
to obtain a desired distribution $π(m)$ of particles in the FZRP on this network. Actually, a simpler and more natural
one is the opposite question: can we determine the distribution of particles of the FZRP on a random network with
given node degree distribution? The answer is again positive. As an example let us discuss what happens for some
popular network ensembles, i.e. for random trees [18] and Erdős-Rényi (E-R) random graphs [15]. Both of them
reveal a faster-than-exponential decay of the degree distribution: $Π(k) = e^{-1/(k-1)}$ for trees and $Π(k) = e^{-k(k-1)}$ for
E-R graphs, and are effectively uncorrelated even for small networks, thus our method should perform well. On
the right-hand side of Fig. 2 we see results of numerical experiments. A quick glance on the figure would suggest
that there is again a power-law behavior in $π(m)$. This is, however, not the case. By a simple analytic calculation
one can show that one has to do with a visual artifact caused by looking only on less than two decades in $m$. It can be estimated that the leading term in $μ(m)$ is proportional to $\exp(m \log m)$ and grows for both the types of networks, so in the limit of $N \to \infty$ with a fixed density of particles there is always a condensation in these models. For intermediate values of $M$, the dependence of the moments $μ(m)$ on their order $m$ falls faster than exponentially but accidentally it resembles a quasi-power-law for small $m$. 

FIG. 2: Left: the distribution of balls for a network with degree distribution $\Pi(k)$ with $b = 3$, which gives $ρ_c = 1$. Circles: computer simulation for $N = M = 400$ and $\phi = 30$. Solid line: asymptotic solution [12] for $\phi \to \infty$, dashed line: the distribution for an infinite network but calculated for the cut-off $\phi = 30$, from Eq. (11) with Eq. (11) inserted. Right: $π(m)$ for E-R graphs with $N = 800, M = 450, k = 8$ (circles). Solid line: theoretical solution, resembling a power-law but only in two decades of $m$. If continued for larger $m$, the curve deviates much from a straight line.
IV. MATTER-NETWORK INTERACTION

Up to now we have considered the situation where the network had a fixed topology and the only dynamical part of the system were balls hopping between nodes. But in fact the model from the last section where we average over random networks, can be regarded as a special kind of a dynamics of balls coupled to a very slowly rearranging network. The characteristic time scale \( t_R \) for these rewirings is, however, much larger than the time scale \( t_B \) for the evolution of balls, so from the ball’s point of view the network is static or “quenched”. One can pose the question, what changes when these two time scales become comparable. In this section we shall discuss this issue using as an example a slightly modified FZRP with an explicit coupling between the two systems.

In this model particles propagate on a dynamically rearranging network. At each time step of the process one of two changes is made: either a particle is moved or the network is locally updated. It is realized by a simple Monte Carlo process where we pick a node \( i \) at random and either with probability \( 1 - r \) move a ball like in Sec. \( \text{[11]} \) or with probability \( r \) we pick up a link \( l \) at random and rewire one of its endpoints \( j \) to \( i \) with a Metropolis probability: \( \min \{1, w(m_i)/w(m_j)\} \). Here \( w(m) \) is an arbitrary weight function which is an external parameter of the model. The function \( w(m) \) encodes the mutual coupling between the network topology and the particles. If for example \( w(m) \) decreases with \( m \), the links tend to avoid nodes with many balls while when it increases, the links tend to condense on such nodes.

The two characteristic time scales \( t_B \propto 1/(1 - r) \) and \( t_R \propto 1/r \) can be tuned by the parameter \( r \). Thus changing \( r \) we can explore different regimes of a matter-network interaction. The model has two obvious limits, for \( r = 0 \) and \( r = 1 \). For \( r = 0 \) we get pure FZRP on a network fixed by the initial condition at the beginning of the process. In the limit of \( r = 1 \) we have only rewirings and the distribution of balls is static. This is similar to hidden-variable models of the evolution of complex networks, see Refs. \( \text{[16, 20]} \). In particular, when \( w(m) = \text{const} \) or the initial distribution of balls is homogeneous, the rewiring process produces the E-R graph because the rewiring is entirely random \( \text{[13]} \). The behavior of the model for \( 0 < r < 1 \) strongly depends on the function \( w(m) \). If \( w(m) \) increases with \( m \), even a small initial inhomogeneity in the degree distribution grows with time, because the inhomogeneity induces a condensation on irregular nodes, which is immediately amplified by the weight \( w(m) \) which causes an attraction of new links to these nodes. This is a kind of avalanche reaction which leads to a condensation of balls and links on a single node.

If \( w(m) \) is constant or decreases with \( m \), then one can prevent the condensation by increasing \( r \). Because these two situations are similar to some extent, for simplicity we stick to \( w(m) = 1 \). The network evolves now independently of balls. Because rewirings are random, the network topology quickly reaches a quasi-equilibrium state which is equivalent to E-R random graphs. The steady-state distribution of degrees \( \Pi(k) \) is Poissonian. From the previous section we know that on the static network of that type, a condensation occurs on the node with the highest degree. Now, however, because the network is continuously rewired it may happen that the largest number of links hops from node to node. If this process is fast enough, one could expect that there is not enough time to form the condensate. For large \( r \), any excess of balls appearing on a particular node could be quickly discharged and the balls became distributed quite uniformly in the network.

We will now present computer simulations. As we will see the results confirm the phenomenological intuitive picture sketched above. Moreover, we will be able to derive a characteristic scaling of the parameter \( r \) which balances the relative time scales of the rewirings and FZRP dynamics and keeps the system on the coexistence line between the condensed and the fluid phase.

In Fig. \( \text{[8]} \) left, we show plots of \( \pi(m) \) for fixed network size \( N \), for various rewiring probabilities \( r \) and number of balls \( M \). We have chosen a relatively large value of the average degree \( \bar{k} = 2L/N = 8 \) to prevent the graph from splitting into disconnected parts during the rewiring. Let us first focus on the dependence of \( \pi(m) \) on \( M \) for \( N, r \) fixed. When \( r \) is small, there exists some critical density of balls \( \rho_c \) above which the condensate appears in the system. The value of \( \rho_c \) can be estimated assuming that the position of the peak in \( \pi(m) \) is given by \( \Delta M = M - N\rho_c \). This gives \( \rho_c \approx 1.6 \) for the network from Fig. \( \text{[8]} \). When \( r \) is large enough, the condensate is absent, regardless of the density of balls. The distribution \( \pi(m) \) becomes more flat for increasing \( M \) but it does not exhibit a peak. It is obvious, that there must be some critical value \( r_c \) which separates those two regimes. This critical point is, strictly speaking, determined by the ratio \( t_R/t_B \) of the two time scales, and thus depends on the size of the system because the characteristic time for condensate formation depends on \( N \). The rewiring rate \( r \) must hence be properly rescaled to get a value independent of \( N \). On the right-hand side of Fig. \( \text{[8]} \) we see what happens when \( r \) is fixed while increasing the size of the system, keeping the average degree and density of balls constant. The condensate, present for small networks, disappears for large \( N \). But if we rescale \( r \) with the system size \( N \) as follows,

\[
r = R/N^2,
\]

where \( R \) is some constant, then the peak stays in \( \pi(m) \) even for large \( N \). Moreover, the straight line on the log-log scale, going through the tops of the peaks for various \( N \) corresponds to a scaling with \( 1/N \) of the peaks’ area, convincing us that the scaling \( \propto 1/N^2 \) indeed indicates one node with the condensate. In Fig. \( \text{[4]} \) we show plots of
FIG. 3: Left: plots of the occupation distribution $\pi(m)$ for fixed number of nodes and links $N = 20, L = 80$, different values of $M = 50, 100, 200, 400, 800$ (curves from left to right) and two different $r = 0.00005$ (solid lines) and $r = 0.01$ (dashed lines). Inset: the difference between $M$ and the position of the peak in $\pi(m)$, for various $M$. This difference is by definition equal to $N\rho_c$ which allows for estimating $\rho_c \approx 1.6$ in the limit $M \to \infty$. Right: plots of $\pi(m)$ for fixed $r = 0.00005, \rho = 10$, average degree $\bar{k} = 8$ and various $N = 20, 40, 80, 160$. The condensation disappears for large networks. Inset: if $r$ scales like $R/N^2$, the condensation is present also for large networks. All peaks for $N = 40, 80, 160, 240$ lie on the same line, $R = 0.16, \rho = 10, \bar{k} = 8$.

FIG. 4: The distribution of balls for $R = 0.16, 0.32, 0.48, 0.64$. Left curves: $N = 40$, right curves: $N = 80$. In both cases $\bar{k} = 8, \rho = 10$. Inset: the point represents an estimated value of $\rho_c(R_c)$ for $R_c \approx 0.72$. The dashed curve is a sketch of the critical line separating the liquid and condensed phase.

$\pi(m)$, with $r = R/N^2$, for different values of $R$. The estimated position of $R_c = N^2\rho_c \approx 0.72(3)$ is the same for two different sizes $N = 40, 80$ and $\bar{k} = 8, \rho = 10$. This confirms again that the chosen scaling is correct. Repeating this procedure, one could obtain a phase diagram $\rho(R)$. A sketch of the diagram is shown in the inset of Fig. 4. The critical line $\rho_c(R)$ separates two phases: the liquid one for $\rho < \rho_c$, and the condensed one for $\rho > \rho_c$. The point represents the result of simulations. We know also that $\rho_c(N^2) = \infty$ since in this case the system is homogeneous from the balls’ point of view and no condensation is possible. For $R = 0$, the critical value of $\rho$ depends on the average degree, but in general is non-zero (see plots for E-R graphs from Sec. III).
V. SUMMARY

Despite its simplicity, free zero-range processes on networks exhibit a very rich behavior, being very well suited to address complicated questions concerning dynamics on networks including non-equilibrium effects or interactions between network topology and dynamics of degrees of freedom defined on the network. Such questions are important for a deeper understanding of the relation of the structure and functionality of complex systems. Some of them were explicitly discussed in this paper but some others like a systematic study of the back-reaction of matter and network is still open. Surprisingly, similar questions have already been addressed in a completely different field of research, namely in the field of quantum gravity where random graphs (dynamical triangulations) were used to approximate Riemannian manifolds (Euclidean version of space-time). Such Riemannian manifolds were let to interact with matter fields defined on space-time. It turns out that the dynamical geometry of such manifolds affects critical properties of the models defined on them. For example, the Ising model on two-dimensional random triangulation is known to have different critical indices than its counterpart on a single two-dimensional graph which has the Onsager exponents. However, one of the most surprising effects observed in the Ising model on dynamical triangulation is that not only randomness of the graph influences the matter field but that there is a strong backreaction from the behavior of the matter field on the geometry of the dynamical triangulation. When the matter field becomes critical it strongly influences geometrical properties of the underlying triangulation. Certainly similar effects should also be observed in complex systems where the behavior of degrees of freedom in the system may act collectively to change its skeleton and the underlying network structure on which the signals propagate. We hope to address this issue in the future.

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[1] M. R. Evans, S. N. Majumdar, and R. K. P. Zia, “Canonical analysis of condensation in factorized steady states”, J. Stat. Phys. 123, 357 (2006).
[2] C. Godrèche, “Nonequilibrium phase transition in a non integrable zero-range process”, J. Phys. A: Math. Gen. 39, 9055 (2006); “From urn models to zero-range processes: statics and dynamics”, cond-mat/0604276.
[3] M. R. Evans, S. N. Majumdar, and R. K. P. Zia, “Factorised steady states in mass transport models on an arbitrary graph”, J. Phys. A: Math. Gen. 39, 4859 (2006).
[4] J. D. Noh, G. M. Shim, and H. Lee, “Complete condensation in a zero range process on scale-free networks”, Phys. Rev. Lett. 94, 198701 (2005).
[5] M. R. Evans and T. Hanney, “Nonequilibrium statistical mechanics of the zero-range process and related models”, J. Phys. A: Math. Gen. 38, 195(R) (2005).
[6] P. Bialas, Z. Burda, and D. Johnston, “Condensation in the backgammon model”, Nucl. Phys. B 493, 505 (1997).
[7] P. Bialas, Z. Burda, and D. Johnston, “Phase diagram of the mean field model of simplicial quantum gravity”, Nucl. Phys. B 542, 413 (1999).
[8] B. Waclaw, L. Bogacz, Z. Burda, and W. Janke, “Balls-in-boxes condensation on networks”, submitted to Phys. Rev. E; cond-mat/0701353.
[9] R. Albert and A.-L. Barabási, “Statistical mechanics of complex networks”, Rev. Mod. Phys. 74, 47 (2002).
[10] C. Godrèche and J. M. Luck, “Dynamics of the condensate in zero-range processes”, J. Phys. A: Math. Gen. 38, 7215 (2005).
[11] S. Grosskinsky, G. M. Schütz, and H. Spohn, “Condensation in the zero range process: stationary and dynamical properties”, J. Stat. Phys. 113, 389 (2003).
[12] J. D. Noh, “Stationary and dynamical properties of a zero-range process on scale-free networks”, Phys. Rev. E 72, 056123 (2005).
[13] B. Waclaw, “Statistical mechanics of complex networks”, based on PhD thesis, arXiv:0704.3702v1.
[14] S. Arrhenius, Z. Phys. Chem. 4, 226 (1889).
[15] L. Bogacz, Z. Burda, and B. Waclaw, “Homogeneous complex networks”, Phys. A 366, 587 (2006).
[16] S. N. Dorogovtsev, J. F. F. Mendes, and A. N. Samukhin, “Principles of statistical mechanics of random networks”, Nucl. Phys. B 666, 396 (2003).
[17] B. Waclaw, Z. Burda, and W. Janke, in preparation.
[18] P. Bialas, Z. Burda, J. Jurkiewicz, and A. Krzywicki, “Tree networks with causal structure”, Phys. Rev. E 67, 066106 (2003).

[19] M. Boguna and R. Pastor-Satorras, “Class of correlated random networks with hidden variables”, Phys. Rev. E 68, 036112 (2003).

[20] V. D. P. Servedio, G. Caldarelli, and P. Buttà, “Vertex intrinsic fitness: How to produce arbitrary scale-free networks”, Phys. Rev. E 70, 056126 (2004).