AN ASYMPTOTIC EXPANSION FOR THE EXPECTED NUMBER OF REAL ZEROS OF KAC-GERONIMUS POLYNOMIALS

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ABSTRACT. Let \( \{\varphi(z;\alpha)\}_{i=0}^{\infty} \) corresponding to \( \alpha \in (-1,1) \), be orthonormal Geronimus polynomials. We study asymptotic behavior of the expected number of real zeros, say \( E_n(\alpha) \), of random polynomials
\[
P_n(z) := \sum_{i=0}^{n} \eta_i \varphi(z;\alpha),
\]
where \( \eta_0, \ldots, \eta_n \) are i.i.d. standard Gaussian random variables. When \( \alpha = 0 \), \( \varphi(z;0) = z^i \) and \( P_n(z) \) are called Kac polynomials. In this case it was shown by Wilkins that \( E_n(0) \) admits an asymptotic expansion of the form
\[
E_n(0) \sim \frac{2}{\pi} \log(n+1) + \sum_{p=0}^{\infty} A_p(n+1)^{-p}
\]
(Kac himself obtained the leading term of this expansion). In this work we obtain a similar expansion of \( E(\alpha) \) for \( \alpha \neq 0 \). As it turns out, the leading term of the asymptotics in this case is \((1/\pi) \log(n+1)\).

1. INTRODUCTION AND MAIN RESULTS

Random polynomials is a relatively old subject with initial contributions by Bloch and Pólya, Littlewood and Offord, Erdős and Offord, Arnold, Kac, and many other authors. An interested reader can find a well referenced early history of the subject in the books by Bharucha-Reid and Sambandham [3], and by Farahmand [12]. In [15], Kac considered random polynomials
\[
P_n(z) = \eta_0 + \eta_1 z + \cdots + \eta_n z^n,
\]
where \( \eta_i \) are i.i.d. standard real Gaussian random variables. He has shown that \( E_n(\Omega) \), the expected number of zeros of \( P_n(z) \) on a measurable set \( \Omega \subset \mathbb{R} \), is equal to
\[
E_n(\Omega) = \frac{1}{\pi} \int_{\Omega} \frac{\sqrt{1-h_{n+1}^2(x)}}{|1-x^2|} \, dx,
\]
from which he proceeded with an asymptotic formula
\[
E_n(\mathbb{R}) = \frac{2+o(1)}{\pi} \log(n+1) \quad \text{as} \quad n \to \infty.
\]
It was shown by Wilkins [25], after some intermediate results cited in [25], that there exist constants \( A_p, p \geq 0 \), such that \( E_n(\mathbb{R}) \) has an asymptotic expansion of the form
\[
E_n(\mathbb{R}) \sim \frac{2}{\pi} \log(n+1) + \sum_{p=0}^{\infty} A_p(n+1)^{-p},
\]
where
\[
A_0 = \frac{2}{\pi} \left( \log 2 + \int_0^1 \frac{f(t)}{t} \, dt + \int_1^\infty \frac{f(t) - 1}{t} \, dt \right), \quad f(t) := \sqrt{1 - \left( \frac{2t}{e^t - e^{-t}} \right)^2}.
\]

Many subsequent results on random polynomials are concerned with relaxing the conditions on random coefficients, see, for example, [13, 18, 10], or the behavior of the counting measures of zeros of random polynomials as in [21, 6, 14, 5, 19, 2, 20, 17, 4, 9]. Our primary interest lies in studying the expected number of real zeros when the basis is a family of orthogonal polynomials in the spirit of [7, 8, 26, 16]. More precisely, Edelman and Kostlan [11] considered random functions of the form
\[
P_n(z) = \eta_0 f_0(z) + \eta_1 f_1(z) + \cdots + \eta_n f_n(z),
\]
where \( \eta_i \) are certain real random variables and \( f_m(z) \) are arbitrary functions on the complex plane that are real on the real line. Using a beautiful and simple geometrical argument they have shown\(^\text{1}\) that if \( \eta_0, \ldots, \eta_n \) are elements of a multivariate real normal distribution with mean zero and covariance matrix \( C \) and the functions \( f_m(z) \) are differentiable on the real line, then
\[
E_n(\Omega) = \int_\Omega \rho_n(x) \, dx, \quad \rho_n(x) = \frac{1}{\pi} \frac{s^2}{\partial s \partial t} \log \left( (v(s)^T Cv(t)) \right)_{|z=x},
\]
where \( v(x) = (f_0(x), \ldots, f_n(x))^T \). If random variables \( \eta_i \) in (6) are again i.i.d. standard real Gaussians, then the above expression for \( \rho_n(x) \) specializes to
\[
\rho_n(x) = \frac{1}{\pi} \sqrt{K_{n+1}(x,x)} K_{n+1}^{(1,0)}(x,x) - K_{n+1}^{(1,0)}(x,x)^2 \]
\[
\text{(this formula was also independently rederived in [16, Proposition 1.1] and [24, Theorem 1.2]), where } K_{n+1}(x,y) := K_{n+1}^{(0,0)}(x,y) \text{ and}
\]
\[
K_{n+1}^{(l,k)}(x,y) := \sum_{i=0}^n f_i^{(l)}(x) f_i^{(k)}(y).
\]

We are interested in the case where the spanning functions in (6) are taken to be orthonormal polynomials on the unit circle. Recall [23, Theorem 1.5.2] that monic orthogonal polynomials, say \( \Phi_m(z) \), satisfy the recurrence relations
\[
\begin{cases}
\Phi_{m+1}(z) = z \Phi_m(z) - \overline{\alpha_m} \Phi'_m(z), \\
\Phi'_m(z) = \Phi'_m(z) - \alpha_m z \Phi_m(z),
\end{cases}
\]
where the recurrence coefficients \( \{ \alpha_m \} \) belong to the unit disk \( \mathbb{D} \) and are uniquely determined by the measure of orthogonality. Furthermore, the orthonormal polynomials, which we denote by \( \varphi_m(z) \), are given by
\[
\varphi_m(z) = \rho_m^{-1} \Phi_m(z), \quad \rho_m := \prod_{i=0}^{m-1} \sqrt{1 - |\alpha_i|^2}.
\]

Since the functions \( f_m(z) \) in (6) must be real-valued on the real line, we are only interested in real recurrence coefficients, i.e., \( \alpha_m \in (-1, 1) \) for all \( m \geq 0 \). It is known [27] that when \( \alpha_m \) is a bounded sequence for some \( p > 3/2 \), estimate (3) remains valid for random polynomials (6) with \( f_m(z) = \varphi_m(z) \) given by (8)–(9). Moreover, if the recurrence coefficients decay exponentially, it was shown by the authors in [1] that the expected number of real zeros has a full asymptotic expansion of the form (4) with the constant term still given by (5).

\(^\text{1}\)In fact, Edelman and Kostlan derive an expression for the real intensity function for any random vector \((\eta_0, \ldots, \eta_n)\) in terms of its joint probability density function and of \( v(x) \).
The previous works suggest that the constant $\pi/2$ in front of $\log(n+1)$ in (3) and (4) might change if the recurrence coefficients decay slowly or do not decay at all. In this note we support this guess by considering random polynomials of the form

$$P_n(z) = \eta_0 \varphi_0(z; \alpha) + \eta_1 \varphi_1(z; \alpha) + \cdots + \eta_n \varphi_n(z; \alpha),$$

which we call Kac-Geronimus polynomials, where $\eta_i$ are i.i.d. standard real Gaussian random variables and

$$\varphi_m(z; \alpha) = \rho^{-m} \Phi_m(z; \alpha), \quad \rho := \sqrt{1 - \alpha^2},$$

are real Geronimus polynomials, that is, polynomials $\Phi_m(z; \alpha)$ satisfying (8) with $\alpha_m = \alpha \in (-1, 1)$ for all $m \geq 0$. The measure of orthogonality for general Geronimus polynomials, i.e., $\alpha_m = \alpha \in \mathbb{D}$, is explicitly known, see [23, Section 1.6], and is supported by

$$\Delta_\alpha := \left\{ e^{i\theta} : 2 \arcsin(|\alpha|) \leq \theta \leq 2\pi - 2 \arcsin(|\alpha|) \right\}$$

with a possible pure mass point, which is present if and only if $|\alpha + 1/2| > 1/2$. When $\alpha = 0$, one can clearly see from (8) that $\Phi_m(z;0) = z^m$ and therefore Kac-Geronimus polynomials (10) specialize to Kac polynomials (1).

For random polynomials (6) with $f_m(z) = \varphi_m(z)$ given by (8)–(9) it can be easily shown using the Christoffel-Darboux formula, see [27, Theorem 1.1], that (7) can be rewritten as

$$r_n(x) = \frac{1}{\pi} \sqrt{1 - h_{n+1}(x)}, \quad h_{n+1}(x) := \frac{(1 - x^2) b_{n+1}'(x)}{1 - b_{n+1}^2(x)}, \quad b_{n+1}(x) := \frac{\varphi_{n+1}(x)}{\varphi_{n+1}'(x)},$$

where $\varphi_{n+1}'(x) := x^{n+1}\varphi_{n+1}(1/x)$ is the reciprocal polynomial (there is no need for conjugation as all the coefficients are real).

**Theorem 1.** Let $P_n(z)$ be given by (10)–(11) with $\alpha \in (-1, 0) \cup (0, 1)$. Define

$$r(z) := \sqrt{(z-1)^2 + 4\alpha^2z}$$

to be the branch holomorphic in $\mathbb{C} \setminus \Delta_\alpha$ such that $r(z)/z \to 1$ as $z \to \infty$. Then it holds that

$$\lim_{n \to \infty} b_{n+1}(z) = \frac{-2\alpha}{r(z) + 1 - z}$$

locally uniformly in $\mathbb{D}$. Moreover, it holds that

$$h_{n+1}(x) = -\alpha \frac{x + 1}{r(x)} \left( 1 + \Theta \left( (1 - x)^2 (n + 1)e^{-\sqrt{n+1}/\rho} \right) \right),$$

for $-1 + (n + 1)^{-1/2} \leq x \leq 1 - \delta_\alpha^{-1}$, where $\Theta(\cdot)$ does not depend on $n$ and $\delta_\alpha := 0$ when $\alpha < 0$ while $\delta_\alpha := ((1 - \alpha)/(1 + \alpha))^{1/3}$ when $\alpha > 0$.

Observe that $b_{n+1}(1) = h_{n+1}(1) = 1$ for all $n$ and these equalities remain true in the limit when $\alpha < 0$. However, $b(1) = h(1) = -1$ when $\alpha > 0$. This change is due to a single zero of $\varphi_m(z; \alpha)$ that approaches 1 as $m \to \infty$ for every fixed $\alpha > 0$, see Figure 1, and is the reason we need to introduce $\delta_\alpha$ in (15).

Let $\mathbb{E}_n(\alpha)$ be the expected number of real zeros of random polynomials (10)–(11). It is easy to see that $h_m(1/x) = 1/h_m(x)$ and therefore $b_m(1/x) = x^2 h_m'(x)/b_m^2(x)$. Thus, we get from (12) that $\varphi_m(1/x) = h_m(x)$ and therefore

$$\mathbb{E}_n(\alpha) = \frac{2}{\pi} \int_{-1}^1 \frac{\sqrt{1 - h_{n+1}^2(x)}}{1 - x^2} dx.$$

Using this formula we can prove the following theorem that constitutes the main result of this work.
Lemma 1. Where for definiteness we take the branch zero exponentially close to 1 while polynomials $\varphi$ in our case $U$.

Theorem 2. Let $P_n(z)$ be random polynomials given by (10)–(11) with $\alpha \in (-1, 0) \cup (0, 1)$. Then there exist constants $A_0, p \geq 1$, that do depend on the parity of $n$, such that $E_n(\alpha)$, the expected number of real zeros of $P_n(z)$, satisfies

$$E_n(\alpha) = \frac{1}{\pi} \log(n + 1) + A_0 + \sum_{p=1}^{N-1} A_p^{(-1)^p} (n + 1)^{-p} + \Theta_N((n + 1)^{-N})$$

for any integer $N$, all $n$ large, where $\Theta_N(\cdot)$ depends on $N$, but is independent of $n$, and

$$A_0^{\alpha} = \frac{A_0 + 1 + \text{sgn}(\alpha)}{2} + \frac{1}{\pi} \log \frac{2}{|\alpha|}$$

with $A_0$ given by (5) and $\text{sgn}(\alpha) := \alpha/|\alpha|$.

Notice that $A_0^{\alpha} = A_0^{-|\alpha|} + 1$. This is due to the fact that polynomials $\varphi_m(x; |\alpha|)$ have a zero exponentially close to 1 while polynomials $\varphi_m(x; -|\alpha|)$ do not have such a zero.

2. Proof of Theorem 1

Lemma 1. It holds that

$$b_{n+1}(z) = \frac{\phi(z) - 2(1 + \alpha) - \varepsilon^{n+1}(z)(\psi(z) - 2(1 + \alpha))}{\phi(z) - 2(1 + \alpha)z - \varepsilon^{n+1}(z)(\psi(z) - 2(1 + \alpha)z)}$$

where $\phi(z) := z + 1 + r(z)$, $\psi(z) := z + 1 - r(z)$, $\varepsilon(z) := \psi(z)/\phi(z)$, and $r(z)$ was defined in (13). In particular, (14) takes place.

Proof. Let $U_m(y)$ be the degree $m$ Chebyshev polynomial of the second kind, that is,

$$U_m(y) = \frac{(y + \sqrt{y^2 - 1})^{m+1} - (y - \sqrt{y^2 - 1})^{m+1}}{2\sqrt{y^2 - 1}},$$

where for definiteness we take the branch $\sqrt{y^2 - 1} = y + O(1)$ as $y \to \infty$ with the cut along $[-1, 1]$. It has been shown in [22, Theorem 3.1] that

$$\varphi_m(z; \alpha) = z^{m/2} \left( U_m\left( \frac{z + 1 + \alpha}{2p\sqrt{\varepsilon}} \right) - \frac{1 + \alpha}{\rho} U_{m-1}\left( \frac{z + 1}{2p\sqrt{\varepsilon}} \right) \right),$$

where $U_{-1}(y) \equiv 0$ and we take the branch $\sqrt{\varepsilon}$ that is positive for positive reals (of course, in our case $\alpha = \alpha$). Observe that the map

$$y(z) = (z + 1)/(2p\sqrt{\varepsilon})$$
takes $\mathbb{D}$ into $\{\text{Re}(z) > 0\} \setminus [0, 1/\rho]$, the right half-plane with the real segment $[0, 1/\rho]$ removed, and its boundary values on $\Delta_\alpha$ cover the real interval $[0, 1]$ twice. Therefore,

$$\sqrt{y(z)^2 - 1} = r(z)/(2\rho \sqrt{z}), \quad z \in \mathbb{D}.$$  

In particular, it follows from (18) that (17) holds. Observe that

$$|\varepsilon(z)| = \left|\frac{y - \sqrt{y^2 - 1}}{y + \sqrt{y^2 - 1}}\right| = \left|\frac{y + \sqrt{y^2 - 1}}{y - \sqrt{y^2 - 1}}\right|^{-2} < 1$$

for $|z| < 1$. Hence, $b_{n+1}(z)$ converges pointwise and therefore locally uniformly ($|b_{n+1}(z)| < 1$ for $z \in \mathbb{D}$) to

$$\frac{z - (1 + 2\alpha) + r(z)}{1 - (1 + 2\alpha)z + r(z)} = \frac{z - (1 + 2\alpha) + r(z)}{1 - (1 + 2\alpha)z + r(z)} \frac{z - (1 + 2\alpha) - r(z)}{z - (1 + 2\alpha) - r(z)} = -\frac{2\alpha}{r(z) + 1 - z}. \quad \Box$$

**Lemma 2.** Let $h(x) := -\alpha(x + 1)/r(x)$. It holds that

$$h_{n+1}(x) = h(x) \left(1 - e^{n+1}(x) \frac{\alpha + (1-x)}{(1 - e^{n+1}(x))(S(x) + R(x)e^{n+1}(x))}\right),$$

where $R(x) := r(x) + \alpha(1+x)$ and $S(x) := r(x) - \alpha(1+x)$.

**Proof.** It follows from (17) that

$$b_{n+1}(x) = 1 - \lambda \frac{(1-x)(1-e^{n+1}(x))}{D(x)},$$

where $\lambda := 2(1 + \alpha)$ and $D(x) := \phi(x) - \lambda x - e^{n+1}(x)(\psi(x) - \lambda x)$. It can be readily checked that

$$1 - b_{n+1}^2(x) = 2\lambda \frac{(1-x)(1-e^{n+1}(x))(S(x) + R(x)e^{n+1}(x))}{D^2(x)}.$$  

Observe that

$$D'(x) = \phi'(x) - \lambda - (n+1)e^n(x)e'(x)(\psi(x) - \lambda x) - e^{n+1}(x)(\psi'(x) - \lambda).$$

It further holds that

$$b_{n+1}'(x) = \lambda \frac{D(x)(1-e^{n+1}(x) + (n+1)(1-x)e^n(x)e'(x)) + D'(x)(1-x)(1-e^{n+1}(x))}{D^2(x)}$$

$$= \lambda \frac{N_1(x) + (n+1)(1-x)e^n(x)e'(x)N_2(x) + N_3(x)e^{n+1}(x) + N_4(x)e^2(n+1)(x)}{D^2(x)},$$

where $N_3(x), N_4(x)$ do not contain terms with $e'(x)$. We have that

$$N_1(x) = \phi(x) - \lambda x + (1-x)(\phi'(x) - \lambda) = -2\alpha + r(x) + r'(1-x) = -2\alpha + 2\alpha^2(1+x)/r(x) = -2\alpha S(x)/r(x).$$

Furthermore, we have that

$$N_2(x) = D(x) - (\psi(x) - \lambda x)(1-e^{n+1}(x)) = 2r(x) = R(x) + S(x).$$

It also holds that

$$N_3(x) = -(\phi(x) - \lambda x) - (\psi(x) - \lambda x) - (1-x)(\psi'(x) - \lambda + \phi'(x) - \lambda) = 4\alpha.$$  

Finally, similarly to $N_1(x)$, we have that

$$N_4(x) = \psi(x) - \lambda x + (1-x)(\psi'(x) - \lambda) = -2\alpha(R(x)/r(x)).$$

Since

$$e'(x) = ((1-x)/x)(e(x)/r(x)),$$  

(21)
it follows from (12) that
\[
\frac{h_{n+1}(x) = h(x)}{(1 - e^{n+1}(x))(S(x) - R(x)e^{n+1}(x)) - \frac{n+1}{\rho} \frac{(1-x)^2}{2} r(x)e^{n+1}(x)}
\]
from which the desired claim easily follows. \(\square\)

**Lemma 3. Formula (15) takes place.**

**Proof.** It can be readily checked that the function \(|y + \sqrt{y^2 - 1}|\) is an increasing function of \(t\) for \(y = t, t \in [1, \infty)\) and \(y = \pm i t, t \in [0, \infty)\). Since \(e^{1} = (1 - |\alpha|)/(|1 + |\alpha|)\), it therefore holds that
\[
\max_{x \in [-1+(n+1)^{-1/2}, 1]} |\varepsilon(x)|^n = |\varepsilon(-1 + (n+1)^{-1/2})|^n
\]
(22)
\[
= \left(1 - (n+1)^{-1/2}/\rho + \delta \left((n+1)^{-1}\right)\right)^n \leq C_{1} e^{-\sqrt{n+1}/\rho}
\]
for some absolute constant \(C_{1} > 0\).

Assume that \(\alpha < 0\). Then \(|S(x)| \geq 2|\alpha|\rho\) for \(x \in [-1, 1]\). Also, since \(|h(x)|\) is an increasing function on \([-1, 1]\), we have that \(|h(x)| \leq 1\) for \(x \in [-1, 1]\). Thus, we get from (20) and (22) that
\[
|h_{n+1}(x) - h(x)| \leq C_{2}(n+1)e^{-\sqrt{n+1}/\rho} \left(|1-x|^2 / |R(x)|\right)
\]
(23)
\[
\leq C_{3}(1-x)^2(n+1)e^{-\sqrt{n+1}/\rho}
\]
for some absolute constants \(C_{2}, C_{3}\), where one needs to observe that \(\varepsilon(0) = 0\) and (24)
\[
S(x)R(x) = \rho^2(1-x)^2.
\]
This proves the lemma in the case \(\alpha < 0\).

Suppose that \(\alpha > 0\). It is quite easy to see that estimate (23) remains valid on \([-1 + (n+1)^{-1/2}, 0]\). Observe also that \(\varepsilon(x) > 0\) and is increasing for \(x \in (0, 1]\), see (21), and \(0 < R(x) < 4\) on \([-1, 1]\). Then by using (24) again, we get that
\[
(1 - e^{n+1}(x))(S(x) + R(x)e^{n+1}(x)) \geq S(x) - R(x)e^{2(n+1)}(x)
\]
\[
\geq (\rho^2/4)(1-x)^2 - 4\varepsilon^{2(n+1)}(1)
\]
for \(x \in [0, 1]\). Notice \(\delta_{\alpha} = \varepsilon^{1/3}(1)\). Then
\[
(\rho^2/4)(1-x)^2 - 4\varepsilon^{2(n+1)}(1) > (\rho^2/8)\delta_{\alpha}^{2(n+1)}
\]
for \(x \in [0, 1 - \delta_{\alpha}^{(n+1)}]\) and \(n\) sufficiently large. Therefore, similarly to (23), it again follows from (24) that there exists a constant \(C_{4}\) such that
\[
|h_{n+1}(x) - h(x)| \leq C_{4}(1-x)^2(n+1)\varepsilon(1)/\delta_{\alpha}^{n+1} = C_{4}(1-x)^2(n+1)e^{2(n+1)/3}(1)
\]
for \(x \in [0, 1 - \delta_{\alpha}^{(n+1)}]\). Since \(\varepsilon(1) < 1\), the desired estimates follows. \(\square\)

### 3. Proof of Theorem 2

To prove Theorem 2 we shall use the following straightforward facts. If \(F(y)\) is analytic around the origin, then
\[
F\left(\frac{t}{n+1}\right) = \sum_{\rho=0}^{N-1} \frac{F_{\rho}p^{\rho}}{(n+1)^{\rho}} + \frac{\tilde{F}_{\rho}(t)p^{N}}{(n+1)^{N}}, \quad |\tilde{F}_{\rho}(t)| \leq C_{F}^{N+1},
\]
for \(t \in I_{n} := [0, \sqrt{n+1}]\) and all \(n \geq n_{F}\), where \(F_{\rho} = F^{(\rho)}(0)/\rho!\), the last estimate follows from the extended Cauchy integral formula, and \(C_{F}\) is independent of \(n, N\). Further, if functions \(u(t), v(t)\) satisfy
\[
g(t) = \sum_{\rho=0}^{N-1} \frac{B_{\rho}(g; t)}{(n+1)^{\rho}} + \frac{\tilde{B}_{\rho}(g; t)}{(n+1)^{N}}.
\]

with $g \in \{u, v\}$, then so does their product and

$$B_p(uv; t) = \sum_{k=0}^{p} B_k(u; t) B_{p-k}(v; t)$$

for $p \leq N - 1$, while

$$\tilde{B}_N(uv; t) = \sum_{l=0}^{N} \frac{1}{(n+1)^l} \sum_{k=m=N-l, k \leq N} B_N,k(u; t) B_N,m(v; t)$$

with $B_N,k(t) = B_k(t)$ for $k < N$ and $B_N,N(t) = \tilde{B}_N(t)$. Finally, let $F(y)$ be as in (25) and $g(t)$ be as in (26) with $B_0(g; t) = 0$. Assume that the values of $g(t)$ lie the domain of holomorphy of $F(y)$ for all $n = n_0$. Then

$$F(g(t)) = F(0) + \sum_{p=1}^{N-1} \frac{B_p(F \circ g; t)}{(n+1)^p} + \tilde{B}_N(F \circ g; t),$$

with

$$B_p(F \circ g; t) = \sum_{m_1, \ldots, m_{N-1}} \frac{F(m)(0)}{m_1! \cdots m_{N-1}!} \prod_{k=1}^{N-1} B_k^{m_k}(g; t)$$

where $m = m_1 + \cdots + m_{N-1}$ and the sum is taken over all partitions $p = \sum_{i=1}^{N-1} m_i$, $m_i \geq 0$, and

$$\tilde{B}_N(F \circ g; t) = \sum_{l=0}^{N(N-1)} \frac{1}{(n+1)^l} \sum_{m_1, \ldots, m_N} \frac{F(m)(0)}{m_1! \cdots m_N!} \prod_{k=1}^{N} B_k^{m_k}(g; t)$$

where $m = m_1 + \cdots + m_N$, the inner sum is taken over all partitions $l + N = \sum_{i=1}^{N} m_i$, $m_i \geq 0$, and $B_N,g(t)$ has the same meaning as in (28).

**Lemma 4.** Let $t \in I_n = [0, \sqrt{n+1}]$. Then it holds for all $N \geq 1$ that

$$r \left( -1 + \frac{t}{n+1} \right) = 2 \bar{r} \left( \sum_{p=0}^{N-1} \frac{r_p t^p}{(n+1)^p} + \tilde{r}_N(t) t^N \right)$$

for some constants $r_p$ and functions $\tilde{r}_N(t)$ that obey estimate in (25). In particular, $r_0 = 1$, $r_1 = -1/2$, $r_2 = (1 - p^2)/(8p^2)$. Moreover, for $\varepsilon(z)$, defined in Lemma 1, it holds that

$$\varepsilon^{n+1} \left( 1 + \frac{t}{n+1} \right) = (-1)^{n+1} e^{-t/\rho} \left( 1 + \sum_{p=1}^{N} \frac{\varepsilon_p(t)}{(n+1)^p} + \tilde{\varepsilon}_N(t) t^N \right),$$

where $\varepsilon_p(t)$ is a polynomial of degree $p - 1$ independent of $n, N$, in particular, $\varepsilon_1(t) \equiv -1/(2\rho)$, and $|\tilde{\varepsilon}_N(t)|$ is bounded above on $I_n$ by a polynomial of degree $N - 1$ whose coefficients depend only on $N$.

**Proof.** Observe that for $y > 0$ it follows from (13) and the choice of the branch of $r(z)$ that

$$r(-1 + y) = 2 \rho \sqrt{1 - y + y^2/(4\rho^2)},$$

where the root in right-hand side of the above equality is principal. Since the right-hand side above is analytic around the origin, expansion (32) follows from (25). An absolutely analogous argument yields the expansion

$$\log \left( \frac{1}{-\rho} \right) = \sum_{p=1}^{N} \frac{\varepsilon_p(t)}{(n+1)^p} + \tilde{\varepsilon}_N(t) t^N,$$

where $|\tilde{\varepsilon}_N(t)|$ has an upper bound as in (25). Since we can write

$$\varepsilon^{n+1} \left( 1 + \frac{t}{n+1} \right) = (-1)^{n+1} e^{-t/\rho} \exp \left( \frac{(n+1)}{\rho} \left( -1 + \frac{t}{n+1} \right) + \frac{1}{\rho} \frac{t}{n+1} \right),$$
it follows from (29)–(31) that (33) holds, where \( e_p(t) \) is a polynomial of degree \( p - 1 \) independent of \( n,N \) (notice that always \( m \leq p \) in (30)) and \( |\tilde{e}_N(t)| \) is bounded above on \( I_n \) by a polynomial of degree \( N - 1 \) whose coefficients depend only on \( N \) (again, we use that \( m \leq t + N \) in (31) and that \( t^N \leq (n + 1)^m \) on \( I_n \)).

**Lemma 5.** Set \( \gamma(s) := 2s/(e^s - e^{-s}) \) and let \( x = -1 + t/(n+1), \ t \in I_n \). It holds that

\[
h_{n+1}(x) = h(x) - (-1)^{n+1} \left( \frac{1-x}{2} \right)^2 \gamma(t/p)(1+\Gamma_{n+1}(t))
\]

with \( \Gamma_{n+1}(t) \) having an expansion of the form

\[
\Gamma_{n+1}(t) = \sum_{p=1}^{N-1} \frac{H_p(t)}{(n+1)^p} + \tilde{H}_N(t)
\]

for any \( N \geq 2 \), where \( H_1(t) = t - (-1)^{n+1}(\alpha/2p)t + O(t^2), H_p(t) = O(t^2), \ p \geq 2, \) and \( \tilde{H}_N(t) = O(t^2) \) as \( t \to 0 \), \( |H_p(t)| \) is bounded above by a polynomial of degree \( 2p \) independent of \( n,N \), while \( |\tilde{H}_N(t)| \) is bounded above on \( I_n \) by a polynomial of degree \( 2N \) whose coefficients depend on \( N \) but not on \( n \).

**Proof.** Recall (20). Notice that

\[
(1 - e^{n+1}(x))(S(x) + R(x)e^{n+1}(x)) = S(x) + 2\alpha(x+1)e^{n+1}(x) - R(x)e^{2(n+1)}(x).
\]

It follows from (32) that \( S(x) \) and \( R(x) \) have expansions as in (26) with

\[
B_p(S;t) = B_p(R;t) = 2pr_p\rho^p, \ p \neq 1, \quad B_1(S;t) = -(\alpha + \rho)t, \quad B_1(R;t) = (\alpha - \rho)t,
\]

and \( \tilde{B}_N(S;t) = \tilde{B}_N(R;t) = 2\bar{r}_N(t) \) for any \( N \geq 2 \). Therefore, we get from (27)–(28) and (33) that

\[
R(x)e^{2(n+1)}(x) = 2\rho e^{-2t/\rho} \left( 1 + \sum_{p=1}^{N-1} \frac{C_p(t)t^p}{(n+1)^p} + \tilde{C}_N(t)t^N \right)
\]

for any \( N \geq 2 \), where \( C_1(t) = (\alpha - \rho - 2t)/(2\rho) \), \( C_p(t) = r_p + tq_p(t) \) for some polynomial \( q_p(t) \) of degree \( p - 1 \) when \( p \geq 2 \), and \( |\tilde{C}_N(t)| \) is bounded above on \( I_n \) by a polynomial of degree \( N \) independent of \( n \). Consequently, we get that the expression in (36) has an expansion

\[
2\rho \left( 1 - e^{-2t/\rho} \right) \left( 1 + \frac{D_1(t)}{n+1} + \sum_{p=2}^{N-1} \frac{D_p(t)t^p}{(n+1)^p} + \frac{\tilde{D}_N(t)t^N}{(n+1)^N} \right)
\]

for all \( N \geq 2 \), where

\[
D_1(t) = -\alpha \left( \frac{1 - (-1)^{n+1}e^{-t/\rho}}{2} \right)^2 \frac{2t/\rho}{1 - e^{-2t/\rho}} + t \left( \frac{2t/\rho}{e^{2t/\rho} - 1} - 1 \right)
\]

with \( |D_1(t)| \) bounded above by a linear function independent of \( n,N \), and

\[
D_p(t) = r_p + \gamma(t/\rho) \left( (-1)^{n+1} \alpha e_{p-1}(t) - \rho e^{-t/\rho} q_p(t) \right) / 2
\]

for all \( p \geq 2 \), with \( |D_p(t)| \) being bounded above on \([0,\infty)\), and \( |\tilde{D}_N(t)| \) that is bounded on \( I_n \) by a constant that depends on \( N \) but not on \( n \). In particular, we have that

\[
\left| \frac{D_1(t)}{n+1} + \sum_{p=2}^{N-1} \frac{D_p(t)t^p}{(n+1)^p} + \frac{\tilde{D}_N(t)t^N}{(n+1)^N} \right| = \frac{D_1(t)}{n+1} + \frac{\tilde{D}_2(t)t^2}{(n+1)^2} \leq \frac{c_N}{\sqrt{n+1}} < 1
\]
for \( t \in I_n \) and all \( n \geq n_N \), where \( c_N, n_N \) are constants dependent only on \( N \). Thus, it follows from (29)–(31) with \( F(y) = 1/(1 + y) \) that the reciprocal of (36) has an expansion

\[
\frac{1}{2^\rho} \frac{1}{1 - e^{-2t/\rho}} \left( 1 + \sum_{p=1}^{N-1} \frac{E_p(t)}{(n+1)^p} + \tilde{E}_N(t) \right),
\]

for all \( N \geq 2 \), where \( E_1(t) = -D_1(t) \) and more generally

\[
E_p(t) = (-1)^p D_p(t) + \Theta(t^2) = \alpha^p \frac{1 - (-1)^{p+1}}{2} + \Theta(t^2)
\]

as \( t \to 0 \) with \( |E_p(t)| \) bounded above by a polynomial of degree \( p \) independent of \( n, N \), while \( |\tilde{E}_N(t)| \) is bounded above on \( I_n \) by a polynomial of degree \( N \) whose coefficients depend on \( N \) but not on \( n \). Furthermore, observe that

\[
-h(x)e^{n+1}(x) \left( \frac{n+1}{\alpha} \frac{1 - x^2}{x} r(x) + 2R(x) \left( 1 - e^{n+1}(x) \right) \right) =
\]

\[
-(n+1)(1+x)(1-x)^2e^{n+1}(x) \left( \frac{1}{x} - \frac{2\alpha}{n+1} \frac{R(x)}{r(x)} \frac{1 - e^{n+1}(x)}{(1-x)^2} \right).
\]

It follows from an argument similar to the one given in the first part of the lemma that the above expression has an expansion of the form

\[
-(1-x)^2(-1)^{n+1}te^{-t/\rho} \left( 1 + \sum_{p=1}^{N-1} \frac{G_p(t)t^{p-1}}{(n+1)^p} + \tilde{G}_N(t)t^{N-1} \right),
\]

for any \( N \geq 3 \), where

\[
G_1(t) = -\alpha \frac{1 - (-1)^{n+1}}{2} + \left( 1 - (-1)^{n+1} \frac{\alpha}{2\rho} \right) t + \Theta(t^2)
\]

and

\[
G_2(t) = -\alpha \frac{1 - (-1)^{n+1}}{2} \left( 1 + \frac{\alpha}{2\rho} \right) + \Theta(t)
\]

as \( t \to 0 \), \( |G_p(t)| \) is bounded above by a polynomial of degree \( p + 1 \) independent of \( n, N \), while \( |\tilde{G}_N(t)| \) is bounded above on \( I_n \) by a polynomial of degree \( N + 1 \) whose coefficients depend on \( N \) but not on \( n \). We now get from (20), (37), and (39), that (34) and (35) do hold for \( N \geq 3 \) and functions \( H_p(t) \) and \( \tilde{H}_N(t) \) that can be computed via (27)–(28) and whose moduli satisfy the desired bounds. The vanishing of \( H_p(t) \) as \( t \to 0 \) can be verified by using (27), (38), (40), and (41). To see that \( \tilde{H}_N(t) = \Theta(t^2) \), observe that

\[
h_{n+1}(x) = (-(-1)^{n+1} - (-1)^{n+1} (1-t/(n+1)) \tilde{H}_N(t)(n+1)^{-N} + \Theta(t^2)
\]

by what precedes. Thus, we need to show that \( h_{n+1}(x) + (-1)^{n+1} \) is divisible by \((1+x)^2\) (of course, if this were not true, formula (16) would not have made sense). Since \( h_{n+1}(-1) = (-(-1)^{n+1} \) it must hold that \( h'_{n+1}(-1) = 0 \). As was mentioned before (16), \( h_{n+1}(x) = h_{n+1}(1/x) \) and therefore \( x^2h'_{n+1}(1/x) = -h'_{n+1}(1/x) \), which yields the desired claim. Finally, since \( \tilde{H}_2(t) = H_2(t) + \tilde{H}_3(t)(n+1)^{-3} \), we can take \( N = 2 \) in (35) as well.

**Lemma 6.** Let \( x = -1 + t/(n+1) \), \( t \in I_n \). It holds that

\[
\sqrt{1 - h_{n+1}^2(x)} = \frac{\rho f(t/\rho)}{r(x)} \left( 1 + \gamma(t/\rho) \sum_{p=1}^{N-1} \frac{K_p(t)}{(n+1)^p} + \gamma(t/\rho) \frac{\tilde{K}_N(t)}{(n+1)^N} \right)
\]

for any \( N \geq 2 \), where \( |K_p(t)| \) is bounded above by a polynomial of degree \( 2p \) independent of \( n, N \) while \( |\tilde{K}_N(t)| \) is bounded above on \( I_n \) by a polynomial of degree \( 2N \) whose coefficients depend on \( N \) but not on \( n \).
Furthermore, it clearly holds that

\[
\frac{1 - h_n^2(x)}{1 - h^2(x)} = \gamma(t/\rho) \left( 1 + \Gamma_{n+1}(t) \right) \frac{r^2(x)}{4 \rho^2}.
\]

Since \( h(x)r(x) = -\alpha(1 + x) \), expansions (32), (35) and formulæ (27)–(28) yield that

\[
(-1)^{n+1} \left( 1 + \Gamma_{n+1}(t) \right) h(x) \frac{r^2(x)}{2 \rho^2} = \sum_{p=1}^{N-1} \frac{H_p^*(t)}{(n+1)^p} + \frac{\tilde{H}_N^*(t)}{(n+1)^N},
\]

for any \( N \geq 2 \), where \( H_p^*(t) = (-1)^{n+1}(\alpha/p) t + O(t^2) \), \( H_p^*(t) = O(t^2) \), \( p \geq 2 \), and \( \tilde{H}_N^*(t) = O(t^2) \) as \( t \to 0 \), while \( |H_p^*(t)| \) and \( |\tilde{H}_N^*(t)| \) have similar bounds to \( |H_p(t)| \) and \( |\tilde{H}_N(t)| \). Furthermore, it clearly holds that

\[
\frac{(1-x)^2}{4} = 1 - \frac{t}{n+1} + \frac{1}{4(n+1)^2} \quad \text{and} \quad \frac{r^2(x)}{4 \rho^2} = 1 - \frac{t}{n+1} + \frac{1}{4 \rho^2 (n+1)^2}.
\]

Therefore, we again get from (27)–(28) that

\[
\left( 1 + \Gamma_{n+1}(t) \right) \frac{2}{4} \frac{(1-x)^2}{4} \frac{r^2(x)}{4 \rho^2} = 1 + \sum_{p=1}^{N-1} \frac{H_p^*(t)}{(n+1)^p} + \frac{\tilde{H}_N^*(t)}{(n+1)^N},
\]

for any \( N \geq 2 \), where \( H_p^*(t) = (-1)^{n+1}(\alpha/p) t + O(t^2) \), \( H_p^*(t) = O(t^2) \), \( p \geq 2 \), and \( \tilde{H}_N^*(t) = O(t^2) \) as \( t \to 0 \) while \( |H_p^*(t)| \) and \( |\tilde{H}_N^*(t)| \) have similar bounds to \( |H_p(t)| \) and \( |\tilde{H}_N(t)| \). Altogether, it holds that

\[
\frac{1 - h_n^2(x)}{1 - h^2(x)} = f^2(t/\rho) \left( 1 + \gamma(t/\rho) \sum_{p=1}^{N-1} \frac{J_p(t)}{(n+1)^p} + \gamma(t/\rho) \frac{\tilde{J}_N(t)}{(n+1)^N} \right),
\]

where \( J_p(t) = f^{-2}(t/\rho) (H_p^*(t) - \gamma(t/\rho) H_p^*(t)) \) and a similar formula holds for \( \tilde{J}_N(t) \).

Observe that \( f^2(s) \) is a positive function for \( s > 0 \) that tends to 1 as \( s \to \infty \) and such that \( f^2(s) = s^2/3 + O(s^3) \) as \( s \to 0 \). Therefore, it follows from the corresponding properties of \( H_p^*(t) \), \( H_p^*(t) \), \( \tilde{H}_N^*(t) \), and \( \tilde{H}_N^*(t) \) that \( J_p(t) \) and \( \tilde{J}_N(t) \) have finite value at the origin and have moduli that satisfy similar bounds to \( |H_p(t)| \) and \( |\tilde{H}_N(t)| \). Observe also that there exist \( n_N \) and \( c_N \) such that

\[
\frac{|\gamma(t/\rho)|}{(n+1)^p} \sum_{p=1}^{N-1} \frac{J_p(t)}{(n+1)^p} + \frac{\gamma(t/\rho) \tilde{J}_N(t)}{(n+1)^N} < c_N
\]

for all \( n \geq n_N \). Therefore, the claim of the lemma now follows from (29)–(31) applied with \( F(y) = \sqrt{1+y} \).

**Lemma 7.** Let \( x = -1 + t/(n+1) \), \( t \in I_n \). There exist constants \( O_p \), \( p \geq 1 \), such that

\[
2 \pi \int_0^{\pi/2} \frac{f(t/\rho)}{tr(x)} dt = \frac{1}{2\pi} \log(n+1) + A_0 + \frac{1}{\pi} \log(2\rho) + \frac{1}{\pi} \mathcal{L} \left( -1 + \frac{1}{\sqrt{n+1}} \right) + \sum_{p=1}^{N-1} \frac{O_p}{(n+1)^p} + O_N \left((n+1)^{-N}\right),
\]

for any \( N \geq 1 \), where \( O_N(\cdot) \) does not depend on \( n \) and

\[
\mathcal{L}(x) := \log \left( \frac{4\rho}{\rho(1-x) + r(x)} \right).
\]
Proof. Similarly to (32), there exist constants $r_p^*$ such that
\begin{equation}
\frac{2\rho}{r(x)} = 1 + \sum_{p=1}^{N-1} r_p^{*p} + \frac{r_N^*(t)^N}{(n+1)^N},
\end{equation}
for any $N \geq 1$, where $|\tilde{r}_N^*(t)|$ is bounded above on $I_0$ by a constant that depends only on $N$. Then
\begin{equation}
\mathcal{J}_1 := \frac{2\rho}{\pi} \int_0^\rho \frac{f(t/\rho)}{tr(x)} \, dx = \frac{1}{\pi} \int_0^1 \frac{f(t)}{t} \, dt + \sum_{p=1}^{N-1} \frac{L_p}{(n+1)^p} + O_N((n+1)^{-N}),
\end{equation}
where $L_p := (r_p^*\rho^p/\pi) \int_0^1 f(t)^{p-1} \, dt$ and $O_N(\cdot)$ does not depend on $n$. Furthermore, it holds that
\begin{equation}
\mathcal{J}_2 := \frac{2\rho}{\pi} \int_0^{\sqrt{n+1}} \frac{\text{dr}}{tr(x)} = \frac{2\rho}{\pi} \int_{1+\rho/(n+1)}^{-1+1/\sqrt{n+1}} \frac{dx}{(1+x)r(x)}.
\end{equation}
It can be easily verified by differentiation that an antiderivative of $2\rho/((1+x)r(x))$ is $\log(1+x) + \mathcal{L}(x)$. Again, similarly to (32), there exist constants $I_p$ such that
\begin{equation}
\mathcal{L}(x) = \sum_{p=1}^{N-1} \frac{I_p\rho^p}{(n+1)^p} + \frac{I_N^*(t)^N}{(n+1)^N},
\end{equation}
for any $N \geq 1$, where $|\tilde{I}_N^*(t)|$ is bounded above on $I_0$ by a constant that depends only on $N$. Therefore, it holds that
\begin{equation}
\mathcal{J}_2 = \frac{1}{2\pi} \log(n+1) - \frac{1}{\pi} \log \rho + \frac{1}{\pi} \mathcal{L}(-1 + \frac{1}{\sqrt{n+1}}) - \sum_{p=1}^{N-1} \frac{L_p\rho^p}{(n+1)^p} + O_N((n+1)^{-N}),
\end{equation}
where, again, $O_N(\cdot)$ does not depend on $n$. Next, we have from (43) that
\begin{equation}
\mathcal{J}_3 := \frac{2\rho}{\pi} \int_0^{\sqrt{n+1}} \frac{f(t/\rho)}{tr(x)} \, dt = \frac{1}{\pi} \int_1^{\sqrt{n+1}/\rho} \frac{f(t) - 1}{t} \, dt + \sum_{p=1}^{N-1} r_p^{*p} \int_1^{\sqrt{n+1}/\rho} (f(t) - 1)\rho^{p-1} \, dt + \frac{r_N^*\rho^N/\pi}{(n+1)^N} \int_1^{\sqrt{n+1}/\rho} (f(t) - 1)\tilde{r}_N^*(\rho t)^{N-1} \, dt
\end{equation}
for any $N \geq 1$. Notice that
\begin{equation}
0 < 1 - f(t) < t^2 \text{csch}^2(t) < 8t^2 e^{-2t}, \quad t \geq 1.
\end{equation}
Therefore, it holds that
\begin{equation}
0 < \int_1^{\infty} (1 - f(t))\rho^{p-1} \, dt \leq C_p(n+1)^{(p+1)/2}e^{-(2/\rho)\sqrt{n+1}} = o_N((n+1)^{-N})
\end{equation}
for any $p \geq 0$ and $N \geq 1$ and some constant $C_p$ that depends only on $p$, where $o_N(\cdot)$ does not depend on $n$. Moreover, since $|\tilde{r}_N^*(t)|$ is bounded above on $I_0$ by a constant that depends only on $N$, we have that
\begin{equation}
\left| \int_1^{\sqrt{n+1}/\rho} (f(t) - 1)\tilde{r}_N^*(\rho t)^{N-1} \, dt \right| \leq C_N \int_1^{\infty} (1 - f(t))\rho^{N-1} \, dt = C_N^* .
\end{equation}
Thus, we can conclude that
\begin{equation}
\mathcal{J}_3 = \frac{1}{\pi} \int_1^{\sqrt{n+1}/\rho} \frac{f(t) - 1}{t} \, dt + \sum_{p=1}^{N-1} \frac{M_p}{(n+1)^p} + O_N((n+1)^{-N}),
\end{equation}
where $M_p := (r_p^*\rho^p/\pi) \int_1^{\infty} (f(t) - 1)\rho^{p-1} \, dt$ and $O_N(\cdot)$ does not depend on $n$. Since the integral in the statement of the lemma is equal to $\mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3$, the desired claim now follows from the definition of $A_0$ in (5), where $O_p = L_p - l_p\rho^p/\pi + M_p$. \qed
Lemma 8. There exist constants \( T_p \) such that

\[
\frac{2}{\pi} \int_{-1}^{-1+1/\sqrt{nT}} \frac{\sqrt{1-h_{n+1}^2(x)}}{1-x^2} \, dx = \frac{1}{2\pi} \log(n+1) + \frac{A_0}{2} - \frac{1}{\pi} \log(2\rho) + \frac{1}{\pi} \mathcal{O}\left(-1 + \frac{1}{\sqrt{n+1}}\right) + \sum_{p=1}^{N-1} \frac{T_p}{(n+1)^p} + \mathcal{O}_N((n+1)^{-N}),
\]

for any \( N \geq 1 \), where \( \mathcal{O}_N(\cdot) \) does not depend on \( n \).

Proof. Recall (42). It follows from (43) and (27)–(28) that

\[
\frac{2\rho}{r(x)} \left( \sum_{p=1}^{N-1} K_p(t) (n+1)^p + K_N(t) \right) = \sum_{p=1}^{N-1} S_p(t) (n+1)^p + \tilde{S}_N(t) \quad (n+1)^N
\]

for any \( N \geq 2 \), where \( |S_p(t)| \) is bounded above by a polynomial of degree \( 2p \) independent of \( n,N \) while \( |\tilde{S}_N(t)| \) is bounded above on \( I_n \) by a polynomial of degree \( 2N \) whose coefficients depend on \( N \) but not on \( n \). Similar to (45), it holds that \( \gamma(s) \leq 3e^{-x} \) for \( s \geq \log 2 \). Because \( f(s) \to 1 \) as \( s \to \infty \), it holds as in (46) that

\[
0 < \int_{\sqrt{nT}/\rho}^{\infty} |p S_p(pt)| \gamma(t) f(t) \, dt \leq C_p(n+1)^{p+1/2} e^{-\sqrt{nT}/\rho} = o_N((n+1)^{-N})
\]

for any \( p \geq 1 \) and \( N \geq 1 \) and some constant \( C_p \) that depends only on \( p \), where \( o_N(\cdot) \) does not depend on \( n \). Moreover, a similar estimate takes place if \( S_p(t) \) is replaced by \( \tilde{S}_N(t) \). The claim of the lemma now follows by making a substitution \( x = -1 + t/(n+1) \) to get

\[
\frac{2}{\pi} \int_{-1}^{-1+1/\sqrt{nT}} \frac{\sqrt{1-h_{n+1}^2(x)}}{1-x^2} \, dx = \frac{2}{\pi} \int_{0}^{\sqrt{nT}} \frac{\sqrt{1-h_{n+1}^2(x)}}{1-x} \, dx \, dt
\]

and then using Lemmas 6 and 7, where \( T_p = O_p + (p/\pi) \int_0^\infty f(t) \gamma(t) S_p(pt) \, dt \) (since \( T_1/(n+1) = O_N((n+1)^{-1}) \), the claim indeed holds for all \( N \geq 1 \). \( \square \)

Lemma 9. It holds that

\[
\frac{2}{\pi} \int_{-1}^{-1+1/\sqrt{nT}} \frac{\sqrt{1-h_{n+1}^2(x)}}{1-x^2} \, dx = \frac{1}{2\pi} \log(n+1) + \frac{1}{\pi} \log\left(\frac{4\rho}{|\mathcal{A}|}\right) - \frac{1}{\pi} \mathcal{O}\left(-1 + \frac{1}{\sqrt{n+1}}\right) + o_N((n+1)^{-N})
\]

for any integer \( N \geq 1 \), where \( o_N(\cdot) \) is independent of \( n \), but does depend on \( N \).

Proof. Since \( |h_{n+1}(x)|, |h(x)| \leq 1 \) when \( x \in [-1, 1] \), it holds that

\[
\left| \sqrt{1-h^2(x)} - \sqrt{1-h_{n+1}^2(x)} \right| = \frac{|h_{n+1}^2(x) - h^2(x)|}{(1-x^2)(\sqrt{1-h^2(x)} + \sqrt{1-h_{n+1}^2(x)})} \leq \frac{2|h_{n+1}(x) - h(x)|}{(1-x^2)|1-h^2(x)|} = \frac{2}{\rho} \frac{r(x)}{(1+x)} |h_{n+1}(x) - h(x)| \quad (1-x^2)^2.
\]

Since \( r(x) \leq 2, x \in [-1, 1] \), we obtain from (15) that

\[
\left| \sqrt{1-h^2(x)} - \sqrt{1-h_{n+1}^2(x)} \right| \leq C(n+1)^{3/2} e^{-\sqrt{nT}/\rho}
\]
for $-1 + 1/\sqrt{n+1} \leq x \leq 1 - \delta_n^{n+1}$ and some constant $C$. Therefore, it holds that

$$\frac{2}{\pi} \int_{1-\delta_n^{n+1}}^{1} \sqrt{1-h^2(x) - \sqrt{1-h_n^{n+1}(x)}} \frac{dx}{1-x^2} = o_N((n+1)^{-N})$$

for any $N \geq 1$, where $o_N(\cdot)$ is independent of $n$, but does depend on $N$. Furthermore, since $r(x) \geq 2|\alpha|p$ for $x \in [-1, 1]$, it holds that

$$\frac{2}{\pi} \int_{1-\delta_n^{n+1}}^{1} \sqrt{1-h^2(x) \frac{dx}{1-x^2}} \leq \frac{\delta_n^{n+1}}{|\alpha|\pi} = o_N((n+1)^{-N})$$

for any $N \geq 1$ by the very definition of $\delta_n$, where, again, $o_N(\cdot)$ is independent of $n$, but does depend on $N$. The observation made after (44) allows us now to conclude that

$$\frac{2}{\pi} \int_{1-\delta_n^{n+1}}^{1} \rho dx = \frac{1}{2\pi} \log(n+1) + \frac{1}{\pi} \log \left(\frac{4\rho}{|\alpha|}\right) = \frac{1}{\pi} \rho \left(1 - \frac{1}{\sqrt{n+1}}\right),$$

which finishes the proof of the lemma.

**Lemma 10.** When $\alpha > 0$, it holds that

$$\frac{2}{\pi} \int_{1-\delta_n^{n+1}}^{1} \sqrt{1-h^2(x) \frac{dx}{1-x^2}} = 1 + o_N((n+1)^{-N})$$

for any $N \geq 1$, where $o_N(\cdot)$ is independent of $n$, but does depend on $N$.

**Proof.** It follows from (20) and (24) that

$$h_{n+1}(x) = h(x) - h(x) \frac{e^{n+1}(x)X_{n+1}(x)}{(1-x)^2 + e^{n+1}(x)Y_{n+1}(x)},$$

where

$$X_{n+1}(x) := \frac{R(x)}{\alpha \rho^2} \left((n+1)r(x) \frac{(1-x)^2}{x} + 2\alpha R(x) \left(1 - e^{n+1}(x)\right)\right)$$

and

$$Y_{n+1}(x) := \frac{R(x)}{\rho^2} \left(2\alpha(x+1) - R(x) e^{n+1}(x)\right).$$

Therefore, we can write

$$1 - h_n^{n+1}(x) = \rho^2 \frac{(1-x)^2}{r^2(x)} + h^2(x) \frac{(1-x)^2 e^{n+1}(x)X_{n+1}(x)}{(1-x)^2 + e^{n+1}(x)Y_{n+1}(x))^2} \times \left(2 - e^{n+1}(x) \frac{X_{n+1}(x) - 2Y_{n+1}(x)}{(1-x)^2}\right).$$

We have that

$$\frac{X_{n+1}(x) - 2Y_{n+1}(x)}{(1-x)^2} = \frac{R(x)}{\rho^2(1-x)^2} \left(2S(x) + (n+1)\frac{(1-x)^2r(x)}{\alpha x}\right) = 2 + (n+1) \frac{r(x)R(x)}{\alpha \rho^2 x},$$

where we used (24) once more. Hence, it holds that

$$\sqrt{1 - h_n^{n+1}(x)} = \frac{\epsilon^{(n+1)/2}(x) V_{n+1}(x)}{(1-x)^2 + e^{n+1}(x)Y_{n+1}(x)),}$$

where

$$V_{n+1}(x) := - \frac{2 h(x) R(x)}{\rho (1+x)} \sqrt{\left(1 - e^{n+1}(x) \left(1 + (n+1) \frac{r(x)R(x)}{2\alpha \rho^2 x}\right)\right) \times \left(1 - e^{n+1}(x) + (n+1) \frac{(1-x)^2r(x)}{2\alpha x R(x)}\right)} + \left(\frac{\rho^2 (1-x)^2 + e^{n+1}(x)Y_{n+1}(x)}{2e^{(n+1)/2}(x)h(x)r(x)R(x)}\right)^2.$$
(observe that \(-h(x) > 0\)). Recall that \(\delta_0 = e^{1/3}(1)\). In particular, we get from (21) that
\[
\frac{(1-x)^2}{e^{(n+1)/2}(x)} \leq e^{n+1}(1) \left( \frac{e(1)}{e(1-\delta_0^{n+1})} \right)^{\delta_0} = (1 + o(1)) e^{n+1}(1) = o_N ((n+1)^{-N})
\]
for \(1 - \delta_0^{n+1} \leq x \leq 1\). Since
\[
y_{n+1}(x) = (2\alpha/\rho^2)(x+1)R(x) + o_N ((n+1)^{-N})
\]
on any fixed small enough neighborhood of 1, it holds that
\[
V_{n+1}(x) = -\frac{2}{p} \frac{h(x)R(x)}{1+x} + o_N ((n+1)^{-N}) = \frac{4\alpha}{\rho} + o_N ((n+1)^{-N})
\]
uniformly for \(1 - \delta_0^{n+1} \leq x \leq 1\). Let
\[
Z_{n+1}(x) := \sqrt{n+1(x)} - \frac{x-1}{2} \left( (n+1) \frac{1-x}{x} Y_{n+1}(x) + \frac{Y'_{n+1}(x)}{\sqrt{Y_{n+1}(x)}} \right).
\]
It follows from the definition of \(Z_{n+1}(x)\) and an estimate similar to (48) that
\[
\int_{1-\delta_0^{n+1}}^{1} \frac{e^{(n+1)/2}(x)Z_{n+1}(x)dx}{1-x^2 + e^{n+1}(x)Y_{n+1}(x)} = \arctan \left( \frac{x-1}{\sqrt{e^{n+1}(x)Y_{n+1}(x)}} \right) \bigg|_{1-\delta_0^{n+1}}^{1} = \pi - \arctan \left( (1)e^{\frac{n+1}{2}} \right) = \frac{\pi}{2} + o_N ((n+1)^{-N}).
\]
Furthermore, we get from (49), the definition of \(Z_{n+1}(x)\), and (50) that
\[
Z_{n+1}(x) = \frac{4\alpha}{\rho} + o_N ((n+1)^{-N}) = V_{n+1}(x) + o_N ((n+1)^{-N})
\]
uniformly for \(1 - \delta_0^{n+1} \leq x \leq 1\). Therefore, (47) yields that
\[
\frac{2}{\pi} \int_{1-\delta_0^{n+1}}^{1} \frac{1-h_{n+1}(x)}{1-x^2} dx = \frac{2}{\pi} \int_{1-\delta_0^{n+1}}^{1} \frac{e^{(n+1)/2}(x)Z_{n+1}(x) + o_N ((n+1)^{-N})}{1-x^2 + e^{n+1}(x)Y_{n+1}(x)} dx = 1 + o_N ((n+1)^{-N}),
\]
where we used positivity of the integrand for the last estimate. \(\square\)

**Proof of Theorem 2.** The claim follows from formula (16) and Lemmas 8–10. \(\square\)

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