Opinion formation in laggard societies

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Abstract – We introduce a statistical-physics model for opinion dynamics on random networks where agents adopt the opinion held by the majority of their direct neighbors only if the fraction of these neighbors exceeds a certain threshold, \(p_u\). We find a transition from total final consensus to a mixed phase where opinions coexist amongst the agents. The relevant parameters are the relative sizes in the initial opinion distribution within the population and the connectivity of the underlying network. As the order parameter we define the asymptotic state of opinions. In the phase diagram we find regions of total consensus and a mixed phase. As the “laggard parameter” \(p_u\) increases the regions of consensus shrink. In addition we introduce rewiring of the underlying network during the opinion formation process and discuss the resulting consequences in the phase diagram.

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Many decisions of human beings are often strongly influenced by their social surroundings, e.g. the opinion of friends, colleagues or the neighborhood. Only a few types of decisions in few individuals emerge from absolute norms and firm convictions which are independent of the opinion of others. Much more common is the situation where some sort of social pressure leads individuals to conform to a group, and take decisions which minimize conflict within their nearest neighborhood. For example, if a large fraction of my friends votes for one party, this is likely to influence my opinion on whom to vote for; if I observe my peers realizing huge profits by investing in some stock this might have an influence on my portfolio as well; and if the fraction of physicist friends (coauthors) publishing papers on networks exceeds a certain threshold, I will have to reconsider and do the same; the social pressure would otherwise be just unbearable. Lately, the study of opinion formation within societies has become an issue of more quantitative research. In first attempts agents were considered as nodes on a lattice, and opinion dynamics was incorporated by the so-called voter model (VM) [1,2] (only two neighbors influence each other at one timestep), the majority rule (MR) [3–5] (each member of a group adopts the state of the local majority), or the Axelrod model [6] (two neighbors influence themselves on possibly more than one topic with the objective to become more similar in their sets of opinions). In addition to this variety of interaction rules the underlying network topology was found to play a prominent role in the emergence of collective phenomena. Most observed structures of real-world networks belong to one of three classes: Erdős-Rényi (ER) [7], scale-free [8] or small-world networks [9]. This has been accounted for the VM [10–12] as well as for the MR on different topologies [13,14]. For a review of further efforts in this directions see [15] and citations therein. Aiming at a coherent description of the co-evolution of topologies and opinions, network structure itself has been modeled as a dynamical process [16–20]. An alternative approach to model social interaction—which is not necessarily based on interpreting agents as some sort of Ising spins—was developed out of the notion of catalytic sets [21] (evolutionary approach), leading to an unanimity rule (UR) model [22] on arbitrary networks in an irreversible formulation.

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There are basically two types of social influence [23,24] which the model presented here should be able to capture. Conformity can arise as a consequence of informational influence. Here an individual assumes that others have more information on a given issue and is happy to accept the majority’s opinion. On the other hand, with normative social influence, the mechanism of peer pressure can force an individual to publicly comply with the majority. Aside from the actual size of the majority subgroup, the question whether an agent is likely to conform or not depends also on other determinants [25], such as the social status or prestige of neighbors, the importance of the decision or the prepotency of the group’s induced response. To take these dependencies into account, in the tradition of statistical physics we present a reversible generalization to the UR and MR models introducing an arbitrary threshold governing updates (“laggard” parameter). The UR and MR are extremal cases of the model. In [26] the idea of a threshold was introduced in the context of global cascades in ER networks of “early adopters”. In contrast to this work, where updates were only allowed in one direction (irreversible), the following model is fully reversible in the sense that two opinions compete against each other in a fully symmetric way.

Each individual $i$ is represented as a node in a network. The (binary) state of the node represents its opinion on some subject, yes/no, 0/1, Bush/Mother Theresa, etc. Linked nodes are in contact with each other, i.e. they “see” or know each others opinion. The opinion formation process of node $i$ is a three-step process (see fig. 1): Suppose $i$ is initially in state “0” (“1”).

- Check the state of all nodes connected to $i$.
- If the fraction of state “1” (“0”) -nodes of $i$’s neighbors exceeds a threshold $p_u$, $i$ adopts opinion “1” (“0”).
- Otherwise $i$ remains in state “0” (“1”).

As a substrate network we chose random graphs [7], i.e. $N$ nodes are randomly linked with $L$ undirected links (self-interactions are forbidden), the average connectivity being $k = L/N$. We do so to keep results most clear and exclude influences from complex-network topologies. The update threshold $p_u$ has to be higher than 0.5 in order to be meaningful in the above sense. The update is carried out asynchronously. In a network containing $N$ nodes, at time $t$, there are $A^0_t$ nodes with opinion “0” and $A^1_t$ nodes with opinion “1”. The relative number of nodes are $a^0_t/A^0_t = A^0_t/N$. One time step is associated with applying the update procedure $N$ times, i.e. each node gets updated once per timestep on average. As time goes to infinity, the relative population of nodes with opinion 0/1 will be denoted by $a^0/1$.

To derive a master equation for the evolution of this system, we calculate opinion-transition probabilities via combinatorial considerations in an iterative fashion, motivated by [21]. A master equation for $a^0_t$ is found explicitly, the situation for $a^1_t$ is completely analogous. At $t=0$, we have a fraction of $a^0_0$ nodes in state “0”. The probability that at time $t$ one node belonging to $a^0_t$ will flip its opinion to “1” is denoted by $p^{0\rightarrow1}_t$. This probability is nothing but the sum over all combinations where more than a fraction of $p_u$ of the neighbors are in state “1”, weighted by the probabilities for the neighboring nodes to be either from $a^0_t$ or $a^1_t = (1-a^0_t)$,

$$p^{0\rightarrow1}_t = \sum_{i=\lceil kp_u \rceil}^k \binom{k}{i} (1-a^0_t)^i (a^1_t)^{k-i},$$

where $\lceil \cdot \rceil$ denotes the ceiling function, i.e. the nearest integer being greater or equal. The same consideration leads to an expression for the opposite transition $p^{1\rightarrow0}_t$, where 1 and 0 are exchanged in eq. (1). The probability for a node to be switched from “0” to “1”, $\Delta^0_{t\rightarrow 1}$, is the product of the transition probability, $p^{0\rightarrow1}_t$, and the probability to be originally in the fraction $a^0_0$, i.e. $\Delta^0_{t\rightarrow 1} = p^{0\rightarrow1}_t a^0_0$. The same reasoning gives $\Delta^1_{t\rightarrow 0} = p^{1\rightarrow0}_t (1-a^0_0)$ and provides the master equation for the first time step (i.e. updating each node once on average),

$$a^0_0 = a^0_0 + \Delta^0_{t\rightarrow 0} - \Delta^0_{t\rightarrow 1}.$$  

Let us now examine some special cases.

Low connectivity. If $k$ and the update threshold $p_u$ are chosen such that $[kp_u] = k$ holds, the system arrives at a frozen state after one iteration. Here the update rule is effectively the unanimity rule in the sense that all linked nodes have to be in the same internal state to allow for an update. This can be either checked by direct inspection.
or by considering the following: Assume that after the first iteration no consensus has been reached which is equivalent to saying that we can find two neighboring nodes with different internal states, say agent $i$ is in state 0, $j$ holds state 1. To let agent $i$ conform, each of his neighbors ought to be in state 1. But then $i$ could not be in 0. Either the update to 1 would have already occurred or there is an agent $k$ in $i$’s neighborhood which also holds state 0 and will not conform because of his connectedness with $i$. The dynamics of the system freezes after the first iteration. Note that it is crucial that we carry out the simulation of this system (on a regular 1D circle) in a recursive way yields the master equation

$$a_{t+1}^0 = a_0^0 + p_1^{1 \rightarrow 0} (1 - a_0^0) - p_t^{0 \rightarrow 1} a_0^0.$$  \hspace{1cm} (4)

Due to symmetry the same master equation is obtained starting from the other case, $\Delta^t_{0 \rightarrow 1}$. Again, theoretical predictions of eq. (4) agree perfectly with numerical findings, fig. 2(b). Three regimes can be distinguished: two of them correspond to a network in full consensus. Between these there is a mixed phase where no consensus can be reached.

**High connectivity limit.** For the fully connected network the asymptotic population sizes can easily be derived: if $a_0^0 > p_u$ or $a_0^0 < 1 - p_u$ consensus is reached. For $1 - p_u < a_0^0 < p_u$ the system is frustrated and no update will take place, giving rise to a diagram like fig. 2(b). Compared to fig. 2(a) a sharp transition between the consensus phases and the mixed phase has appeared. We now try to understand the origin of this transition.

**Intermediate regime.** The transition between the smooth solution for the final populations as a function of $a_0^0$ and the sharp one for higher connectivities becomes discontinuous when the possibility for an individual node to get updated in a *later timestep* ceases to play a negligible role. In eq. (4) we do not assume any kind of correlations between configurations at different time steps, i.e. the configurations are assumed to be maximally random w.r.t. the constraining population sizes. The fact that eq. (4) coincides with the numerical results for high connectivities justifies the assumption for high $k$. However, as explained above when we have an unanimity rule the correlation is so strong that no updates take place on subsequent iterations. It is intuitively clear that there exists a regime in between where the no-correlation hypothesis loses its validity and evolution does not freeze after one iteration.

For $p_u = 0.8$ the sharp transition arises for values of $k$ around 10. Figure 2(c) shows simulation data for ER graphs with $N = 10^4$ nodes and $k = 10$ with $p_u = 0.8$. Here we already find two regimes with consensus and an almost linear regime in between. The curve obtained from numerical summations of eq. (4) resembles the qualitative behavior of the simulations up to deviations.

Fig. 2: Asymptotic population sizes of the “0”-state fraction, $a_0^0$, as a function of its initial size, $a_0^0$, for $N = 10^4$, $p_u = 0.8$. (a) $k = 2$ for all nodes (1D circle), (b) ER graph with $k = 9000$ and (c) ER graph with $k = 10$. 
due to the no-correlation assumption. The dynamics of the system is shown in the phase diagram, fig. 3(a). It illustrates the size of the respective regimes and their dependence on the parameters $a_0^0$ and connectedness $\bar{k}/N$. The order parameter is $a_0^\infty$. Along the dotted lines a smooth transition takes place, solid lines indicate discontinuous transitions from the consensus phase to the mixed phase. The change from smooth to sharp appears at $\bar{k}/N \approx 0.01$. For larger $p_u$, the regions of consensus shrink towards the left and right margins of the figure.

So far we assumed static networks. However, this is far from being realistic, as social ties fluctuate. We now check the robustness of the phase diagram when stochastically perturbing the underlying network structure, i.e. allowing links to randomly rewire with the rewirement process taking place on a larger time scale than the opinion update. Let us assume that the number of rewired links per rewirement-timestep is fixed to $L'$. Then perturb the system by a rewirement step and randomly rewire $L'$ links among the $N$ nodes ($N$ and $L$ are kept constant over time), increase the time-unit for the rewirement steps by one and let the system relax into a (converged) opinion configuration. Iterate this procedure. Note that this process can be viewed as a dynamical map of the curves shown in figs. 2(a)–(c).

The evolution of opinions in a network at $T \neq 0$ is as follows: We fix a network and perform the same dynamics as for $T = 0$, until the system has converged and no further updates occur. Then perturb the system by a rewirement step and randomly rewire $L'$ links among the $N$ nodes ($N$ and $L$ are kept constant over time), increase the time-unit for the rewirement steps by one and let the system relax into a (converged) opinion configuration. Iterate this procedure. Note that this process can be viewed as a dynamical map of the curves shown in figs. 2(a)–(c). With this view it becomes intuitively clear that consensus will be reached for a wider range of parameters, where the time to arrive there crucially depends on the value of $k$.

To incorporate the temperature effect in the master equation we introduce the second timescale and denote the population in state “0” as $a_0^t$. Here $t$ is the time for the update process as before and $\bar{t}$ is the time step on the temperature time scale, i.e. counts the number of rewirement steps. We use $a_0^0 \equiv a_0^0, a_0^\infty$ can be obtained from $a_0^\infty = \lim_{t \to \infty} (a_0^t + \Delta_{0,0}^1 - \Delta_{1,0}^0)$ for high $\bar{k}$, and from eq. (2) for low $\bar{k}$, when we only observe updates during the first iteration. This evolution is nothing but a dynamical map. The probabilities to find a configuration of neighbors allowing an update are no longer given only by $\Delta_{0,0}^1$ and $\Delta_{1,0}^0$, instead we have to count the ones constituted by a rewiring, which happens with probability $T$. That is why we can consider this kind of evolution as a dynamical map of the former process, with $a_0^\infty$ as the initial population for the first rewirement step evolving to $a_{\infty,0}^0$, and so on. The transition probabilities are now given by $T \Delta_{0,0}^1$ and $T \Delta_{1,0}^0$, since only new configurations can give rise to an update. We thus assume the master equation for a system at $T \neq 0$ after the first rewiring to be

$$a_{\infty,t+1}^0 = \lim_{t \to \infty} \left( a_{t,f}^0 + T \left( \Delta_{1,0}^0 - \Delta_{0,0}^1 \right) \right).$$

Furthermore, one expects the existence of a critical value $k_c$, below which the intermediate regime (mixed state) will disappear. This will occur whenever there is no chance that a configuration of neighbors can be found leading to an update. The value for $k_c$ can be easily estimated: Say we have a node in state “1” and ask if an update to state “0” is possible under the given circumstances. For a given $\bar{k}$ this requires that there are at least $\lceil k/N \rceil$ neighbors in state “0” present in the set $A_0^t$. If $\bar{k}$ is above the critical value $k_c$ it occurs that even if all nodes from $A_0^t$ were...
neighbors of the node in state “1”, there are still too other neighboring nodes (which are then necessarily in state “1”) to exceed the update threshold. This means that we cannot have updates if \( k p_u > A_0^0 \), and we get \( k_c = \frac{A_0^0}{p_u} \). For \( p_u = 0.8 \) and \( A_0^0 = 0.5 \), \( k_c \approx 0.61 N \). We next consider the time-to-convergence in the system. To this end, we measure the half-life time \( \tau \), of initial populations at \( A_0^0 = 0.5 \) for different connectivities \( k \), see fig. 3(c). The figure suggests that the observed scaling of \( \tau \) could be of power law type, with a pole at \( k_c/N \), i.e. \( \tau \propto \left( \frac{2 - k_c}{k_c} \right)^{-\gamma} \). The estimated critical exponent \( \gamma \approx 7.4 \) seems to be independent of temperature. Note, that the estimate is taken rather far from the pole at \( k_c \), which suggests to interpret the actual numbers with some care.

The phase diagram for the \( T \neq 0 \) system is shown in fig. 3(b). There are still three regimes, which are arranged in a different manner than before. Consensus is found for a much wider range of order parameters; the mixed phase is found for high connectivities, i.e. \( k \gg k_c \). The value of \( k_c \) at \( A_0^0 = 0.5 \), as found in fig. 3(b), is 0.63, slightly above the prediction of 0.61. This mismatch is because we used networks with inhomogeneous degree distributions (Poisson). Whether a network allows for an update or not is solely determined by the node with the lowest degree \( k \), which explains why we can still observe updates when the average degree \( \bar{k} \) is near to but already above \( k_c \). Systems in the mixed phase are frustrated. \( k_c \) is linear in \( A_0^0 \) which we confirm by finding a straight line separating the frustrated from the consensus phase. For larger \( p_u \) the regions of consensus shrink. The solutions depicted in fig. 3(b) are independent of \( T \). Here we do not assume a (dis)assortative mixing scheme (linking preferences) and focus on stochastic perturbations instead. The explicit type of perturbation plays no role in this mechanism so we chose the simplest possible.

Summarizing we presented a model bridging the gap between existing MR and UR models. The conceptual novelty of this work is that we interpret opinion formation as a special case of the evolution of catalytic systems [21,22]. This different perspective places opinion formation problems in a more general framework with respect to previous extensions and modifications of the “Ising model” type in the literature, recently called the “Ising paradigm” [27]. On a technical level this results in an algorithmically more feasible and straightforward way to actually solve opinion formation models —for special cases even in closed form. In particular we studied opinion dynamics on static random networks where agents adopt the opinion held by the majority of their direct neighbors only if the fraction of neighbors exceeds a pre-specified laggard threshold, \( p_u \). This system shows two phases, full consensus and a mixed phase where opinions coexist. We studied the corresponding phase diagram as a function of the initial opinion distribution and the connectivity of the underlying networks. As the laggard parameter \( p_u \) increases the regions of full consensus shrink. Opinion formation models can be categorized by whether consensus is the only frozen state, as for the voter model, MR, Sznajd model, etc., [27], or models allowing for a continuum of stationary solutions, as bounded confidence, the UR, or the model presented here. The reported richness of stationary solutions arises from the interplay between the update threshold \( p_u \) and the random sequential update procedure only, and cannot be attributed to network topology effects. For this reason we restricted this work to random networks. We introduced rewiring of the underlying network during the opinion formation process (“social temperature”). For \( T > 0 \) the coexistence phase vanishes, the system can escape the frozen state \( A_0^{1/2} \neq 1 \), and global consensus is reached. In the case of “usual” temperature (opinions of nodes switch randomly) [14], a different behavior is expected. For low temperature, the system also can escape the frozen state, however for higher values of \( T \) the system undergoes a transition from an ordered to an unordered phase, where \( a_\infty = 1/2 \). In the formation of public opinion one can find two scenarios [28]: a trend toward consensus or a coexistence of different opinions. From our findings we can speculate that this difference could be related to our concept of social temperature. Even though laggards sometimes enjoy a bad reputation as being slow and backward-oriented, societies of laggards are shown to have remarkable levels of versatility as long as they are not forced to interact too much.

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