Upper bounds on the private capacity for bosonic Gaussian channels

Kabgyun Jeong\textsuperscript{1,2},\textsuperscript{*}

\textsuperscript{1}Research Institute of Mathematics, Seoul National University, Seoul 08826, Korea
\textsuperscript{2}School of Computational Sciences, Korea Institute for Advanced Study, Seoul 02455, Korea

(Dated: January 7, 2020)

Quite recently, there have been considerable progresses on the bounds of various quantum channel capacities for bosonic Gaussian channels. Especially, several upper bounds on the classical capacity and the quantum capacity on the bosonic Gaussian channels, via a technique known as a quantum entropy power inequality, have been shed light on understanding the mysterious quantum-channel-capacity problems. However, upper bounds for the private capacity on quantum channels are still waiting for the study on certain universal upper bounds. Here, we derive upper bounds on the private capacity for bosonic Gaussian channels involving a general Gaussian-noise case through the conditional quantum entropy power inequality.

PACS numbers: 03.67.-a, 03.65.Ud, 03.67.Bg, 42.50.-p

I. INTRODUCTION

One of most fundamental and challenging tasks in quantum Shannon theory, a quantum analogue of information theory [1], is to determine the channel capacity of a given quantum channel [2, 3]. In general, the quantum channel transmits a quantum state to another quantum state, and mathematically it is given by a completely positive and trace-preserving map. The channel capacity of a quantum channel is also defined as the maximum rate at which certain (classical, private, or quantum) information can be transmitted reliably through the channel in the limit of vanishing errors. Herein, we restrict to the quantum channel described only by Gaussian unitary transforms over bosonic Gaussian systems with an environmental bosonic Gaussian noise [4–6].

The private capacity for a given quantum channel quantifies the ability to transmit a classical private information, and it is maximum rate in the limit of infinitely many uses of the channel and vanishing errors in the presence of noise through the channel [7]. To compute the private capacity, we also need to device the regularization of classical private information, which quantifies the (real) private capacity of the quantum channel. Main difference of the private capacity to the classical capacity is that, in principle, a classical private information cannot be accessible from the third environmental system out of the channel, thus, it is probably applicable for secure quantum communications. Furthermore, it was known that the private capacity is non-additivity [8, 9], which implies that this quantity is extremely hard to compute.

In this reason, we firstly try to calculate reasonable upper bounds on the private capacity from newly posed quantum entropy power inequality. For the private capacity, previously there have been observed few results on the upper bounds via the data-processing inequality [10–12] and the teleportation simulation argument [13], however, our bounds are more intriguing in the Gaussian regime of low-energy powers.

Quantum entropy power inequality (qEPI), first proposed by König and Smith [14], is the central tool in quantum Shannon theory to estimating the output-entropy of a quantum channel. The inequality states that the output-entropy of a bosonic Gaussian channel, such as a beam-splitter (or amplifier), is always increased under two independent input bosonic Gaussian states. Also, the qEPI has been proved several ways with some applications [15–18] and extended to the conditional cases in discrete and Gaussian regimes [19–23]. The power of quantum entropy power inequalities is that those are only carrying the information about von Neumann entropy without details of the quantum state itself. Recently, it has been known that qEPIs have many applications for obtaining upper bounds of the classical capacity [22, 24] as well as the quantum capacity [25] on bosonic Gaussian channels with general Gaussian noises beyond the thermal-noise case. The general noise means that it can be possible to take the environment as in the form of a squeezed (or even non-Gaussian) quantum state [24].

In this paper, we consider the conditional quantum entropy power inequality (CqEPI) on the bosonic Gaussian channels, in which the environmental system can be general Gaussian states as an input noise, in order to calculate upper bounds on the private capacity for those channels. It is not only the first attempt to get a meaningful result on the private capacity using CqEPI, but also gives us an intuition how can we calculate and apply the quantum channel capacity problems on various quantum channels.

This paper is organized as follows. In Section II, we introduce backgrounds to understand our results including the quantum entropy power inequalities. We derive universal upper bounds on the private capacity for general bosonic Gaussian channels, and present a generalized formula in Section III. In Section IV, we give a specific example for the private capacity with a squeezed thermal noise as one of the general noise model, in order to present physical relevance. Finally, we briefly summarize

\textsuperscript{*}Electronic address: kgjeong6@snu.ac.kr
our results, and comment on a few remarks and open problems in Section V.

II. PRELIMINARIES

For any Gaussian input state $\rho_A$ and the environmental system $\rho_E$, the bosonic Gaussian channel $\Lambda$ via the isometric map can be represented by

$$\Lambda(\rho_A) = \text{Tr}_F \left[ V_{AE}(\rho_A \otimes \rho_E) V_{AE}^\dag \right],$$

where $V_{AE}$ is a symplectic unitary transformation on the Hilbert space $\text{Sp}(2n_A, \mathbb{R}) \otimes \text{Sp}(2n_E, \mathbb{R})$ with the bosonic input mode $n_A$ and the environmental mode $n_E$, respectively. Conversely, its complementary channel $\Lambda^c$ of the channel $\Lambda$ is naturally defined by

$$\Lambda^c(\rho_A) = \text{Tr}_B \left[ V_{AE}(\rho_A \otimes \rho_E) V_{AE}^\dag \right].$$

We note that the isometric map $V_{AE} := V_{\mu AE}^{BF}$ in Fig. 1 for the beam-splitting and the amplifying channel has the mixing parameters $\tau \in [0, 1]$ and $\kappa \in (1, \infty)$, respectively.

More precisely, we can take two important symplectic unitaries as follows:

$$V_\tau = \exp \left[ \frac{1 - \tau}{\tau} (\hat{a}^\dagger \hat{b} - \hat{b}^\dagger \hat{a}) \right]$$

and

$$V_\kappa = \exp \left[ \frac{\kappa - 1}{\kappa} (\hat{a}^\dagger \hat{b}^\dagger - \hat{b}^\dagger \hat{a}) \right],$$

where $\hat{a}$ and $\hat{b}$ are annihilation operators of the input and the environment satisfying the canonical commutation relation (CCR), and the superscripts $*$ and $T$ denote the complex conjugate and the transpose operation, respectively. By exploiting the CCR algebra, we can also describe the bosonic Gaussian channels in the forms of

$$\hat{c} = \begin{cases} \sqrt{\tau} \hat{a} + \sqrt{1 - \tau} \hat{b}, & \tau \in [0, 1]; \\ \sqrt{\kappa} \hat{a} + \sqrt{\kappa^{-1} - 1} \hat{b}, & \kappa > 1, \end{cases}$$

where $\hat{c}$ denotes the output bosonic Gaussian state of the channel $\Lambda$.

However, it is important to note that, if the environmental system $\rho_E$ is a mixed state, we cannot simply obtain the complementary channel uniquely by this technique. Instead, firstly we need to purify the environmental state, and then find the corresponding symplectic unitary operator in the extended Hilbert space. It can be expressed as

$$\Lambda^c(\rho_A) = \text{Tr}_B \left[ V_{AE} \otimes 1_C (\rho_A \otimes \psi_{EC}) (V_{AE} \otimes 1_C)^\dagger \right],$$

where $\psi_{EC} := |\psi\rangle\langle\psi|_{EC}$ is a quantum purification satisfying $\text{Tr}_C \psi_{EC} = \rho_E$ [26, 27] and the subscript $C$ denotes a reference system. Also, we can define weak-complementary channel $\Lambda^\text{wc}$ as the case for which a mixed state $\rho_E$ is inserted in Eq. (2), and $\Lambda^\text{wc} = \Lambda^c$ when $\rho_E$ is any pure state [28]. In Fig. 1, we figure out the situation, in which the environment is a mixed state $\rho_E$.

Now, we review the private capacity on bosonic Gaussian channels, actually, we need two definitions [29, 30]. First one is the ‘one-shot’ private capacity of a quantum channel $\Lambda$, and it is defined by $(\forall \rho \in \text{Sp}(2, \mathbb{R})$ i.e., a single-mode bosonic Gaussian state)

$$P^{(1)}(\Lambda) = \max_{\chi \in \text{Sp}(2, \mathbb{R})} [\chi(\Lambda) - \chi(\Lambda^c)],$$

where $\chi(\Lambda) = S(\Lambda(\rho)) - \sum_j p_j S(\Lambda(\rho_j))$ is the well-known Holevo capacity [31, 32], and $S(\rho) = -\text{Tr}[\rho \log \rho]$ is the von Neumann entropy. The quantity $\chi(\Lambda) - \chi(\Lambda^c)$ is so-called the classical private information. In bosonic Gaussian regime, it was known that [17, 24] ($\forall \tau \in [0, 1]$ and $\forall \kappa \in (1, \infty)$)

$$\chi(\Lambda) = \begin{cases} g(\tau N + (1 - \tau)N_E) & - g((1 - \tau)N_E) \\ g(\kappa N + (\kappa - 1)N_E) & - g\left(\frac{\kappa - 1}{\kappa} N_E\right), \end{cases}$$

where $N$ and $N_E$ denote the mean photon numbers for the input state and the thermal-noise environment, respectively, and the entropic function $g(x) := (x + 1) \log(x + 1) - x \log x$.

Second definition is the ‘regularized’ private capacity, which is normally said to be the private capacity. The (regularized) private capacity of a bosonic Gaussian channel $\Lambda$ under an energy-constraint with the input mean photon number $N$ is given by

$$P(\Lambda) = \lim_{n \to \infty} \frac{1}{n} P^{(1)}(\Lambda^{\otimes n}, \rho_n),$$

where $\Lambda^{\otimes n}$ is the independent $n$-copy of the Gaussian channel and $\rho_n$ is any input state (i.e., possibly in the entangled state over the channels) in $n$-tensor product of the input Hilbert space. Also we point out that the maximum in the one-shot capacity is taken over all $\rho_n$ such that $E(\rho_n) \leq nN$. If $P^{(1)}(\Lambda) = P(\Lambda)$, then we say the private capacity is additive, however, generally it is not true [8, 9].
The linear relations of qEPI [14, 17] are described as follows:

\[
S(\rho_{X_1} \boxplus \tau \rho_{X_2}) \geq \tau S(\rho_{X_1}) + (1 - \tau) S(\rho_{X_2}) \tag{9}
\]

\[
S(\rho_{X_1} \boxplus \kappa \rho_{X_2}) \geq \frac{\kappa}{2\kappa - 1} S(\rho_{X_2}) + \frac{\kappa - 1}{2\kappa - 1} S(\rho_{X_2}) + \ln(2\kappa - 1), \tag{10}
\]

where \(\rho_{X_1}\) and \(\rho_{X_2}\) are independent input bosonic Gaussian states, and \(\boxplus\tau\) and \(\boxplus\kappa\) represent beam-splitter and amplifier operations with the mixing parameters \(\tau \in [0, 1]\) and \(\kappa \in (1, \infty]\), respectively.

Finally, we introduce two linear versions of conditional quantum entropy power inequality (CqEPI) over two independent input bosonic Gaussian states. For any product states given in the form of \(\rho_{X_1, Z_1} \otimes \rho_{X_2, Z_2}\), the CqEPIs for bosonic Gaussian channels are expressed as [19, 21]

\[
S(\rho_{X_1} \boxplus \tau \rho_{X_2}|Z_1, Z_2) \geq \tau S(\rho_{X_1}|Z_1) + (1 - \tau) S(\rho_{X_2}|Z_2) \tag{11}
\]

\[
S(\rho_{X_1} \boxplus \kappa \rho_{X_2}|Z_1, Z_2) \geq \frac{\kappa}{2\kappa - 1} S(\rho_{X_1}|Z_1) + \frac{\kappa - 1}{2\kappa - 1} S(\rho_{X_2}|Z_2) + \ln(2\kappa - 1), \tag{12}
\]

where the quantum conditional entropy is defined by \(S(\rho_X|Z) := S(\rho_{XZ}) - S(\rho_Z)\). For convenience, we will consider the quantum entropy power inequalities about not the exponential versions but the linear forms only.

### III. UPPER BOUNDS ON THE PRIVATE CAPACITY

Now, we make use of the notation for a bosonic Gaussian channel with the beam-splitter and the amplifier as \(\Lambda_{\tau, \rho E}\) and \(\Lambda_{\kappa, \rho E}\), respectively, in which the environmental system can be any general bosonic Gaussian state with the mean photon number \(N_E\). As mentioned above, there exist upper bounds for the private capacity on the thermal-noise Gaussian channels [10–13]. Most novel contributions of this work are we exploit the conditional quantum entropy power inequality as well as considering the general Gaussian-noise model as an environmental system, but already known results have been used another technique such as the data processing inequalities.

Suppose that an input bosonic Gaussian state \(\rho\) of each channel has the mean photon number \(N\) (thus, the multiple input \(\rho_n\) has the energy-constraint bounded above by \(nN\)), then we can describe an upper bound of the regularized private capacity for \(\Lambda_{\tau, \rho E}\) is given by

\[
\mathcal{P}(\Lambda_{\tau, \rho E}, N) := \frac{1}{n} \mathcal{P}(\Lambda_{\tau, \rho E}^\otimes n, \theta_n)
\]

\[
= \lim_{n \to \infty} \max_{E(\rho_n) \leq N} \frac{1}{n} \left[ S(\Lambda_{\tau, \rho E}^\otimes n(\rho_n)) - \sum_j p_j S(\Lambda_{\tau, \rho E}^\otimes n(p_j)) - \sum_j p_j S(\Lambda_{\rho E}^\otimes n(p_j)) \right]
\]

\[
\leq \lim_{n \to \infty} \max_{\rho_n} \frac{1}{n} \left[ S(\Lambda_{\tau, \rho E}^\otimes n(\rho_n)) + S(\Lambda_{\rho E}^\otimes n(\rho_n)) \right] = \lim_{n \to \infty} \min_{\rho_n} \frac{1}{n} \left[ S(\Lambda_{\tau, \rho E}^\otimes n(\rho_n)) + S(\Lambda_{\rho E}^\otimes n(\rho_n)) \right]
\]

\[
= \lim_{n \to \infty} \min_{\rho_n} \frac{1}{n} \left[ S(\Lambda_{\tau, \rho E}^\otimes n(\rho_n)) + S(\Lambda_{\rho E}^\otimes n(\rho_n)) \right]
\]

where the first inequality in Eq. (13) comes from the minimization of all quantum state \(\rho_n\), and the second inequality in Eq. (14) from the sub-additivity of the von Neumann entropy. We know the upper bounds of the first two terms of Eq. (15) follow from the fact that bosonic Gaussian states always fulfill maximal entropies for given first and second moments [33]; that is, we have

\[
\max_{\rho} S(\Lambda_{\tau, \rho E}(\rho)) = \max_{\rho} S(\Lambda_{\tau, \rho E}^\otimes n(\rho)) = g(\tau N + (1 - \tau) N_E),
\]

where we take roughly the maximal entropy of the complementary channel \(\Lambda_{\rho E}^\otimes n\) as equivalent to the maximal entropy of the channel \(\Lambda_{\tau, \rho E}\) in Eq. (15). Further, we note that, for first two terms in Eq. (15), i.e., \(2g(\tau N + (1 - \tau) N_E)\), we call it as ‘maximal capacity’ for the private capacity \(\mathcal{P}(\Lambda_{\tau, \rho E}, N)\) on the bosonic Gaussian channels.

The main problem is how can we calculate the minimal values of the last two terms, which are extremely hard to compute. Actually, this is a challenging topic in quantum Shannon theory, and it has a special name such as ‘(Gaussian) minimum output entropy conjecture’. In order to obtain a useful bound on the last two terms, we need to use qEPI in Eq. (9) and CqEPI in Eq. (11) simultaneously.
Before the detailed proof, we divide the last two terms as follows:

[D.1] \[ S_{\text{MOE}}(\Lambda_\tau) = \lim_{n \to \infty} \min_{\rho_n} \frac{1}{n} \left[ S(\Lambda_{\tau,\rho_E}^{\otimes n}(\rho_n)) \right] \]

[D.2] \[ S_{\text{MOE}}(\Lambda_\tau^c) = \lim_{n \to \infty} \min_{\rho_n} \frac{1}{n} \left[ S(\Lambda_{\tau,\rho_E}^{\otimes n}(\rho_n)) \right] , \]

where it was conjectured that \( S_{\text{MOE}}(\Lambda_\tau) = g(1 - \tau)N_E \) for [D.1] [34, 35] and, without loss of generality, we also assume that \( S_{\text{MOE}}(\Lambda_\tau^c) = g(1 - \tau)N_E \) roughly for [D.2] case. However, in our case, the environment and the output of complementary channel are conditioned by the purifying system \( C \), while the input and the environmental systems are initially to be a product state by the definition of the channel. Therefore, we need two-track strategy to bound the private capacity, not only for the channel \( \Lambda_{\tau,\rho_E} \) but also for the complementary channel \( \Lambda_{\tau,\rho_E}^c \).

First, we can modify the bound on the definition [D.1] case, that is,

\[
S(\Lambda_{\tau,\rho_E}^{\otimes n}(\rho_n)) \geq \tau S(\rho_n) + (1 - \tau)S(\rho_E^{\otimes n})
\]

\[
= \tau S(\rho_n) + n(1 - \tau)S(\rho_E)
\]

\[
\geq n(1 - \tau)g(N_E),
\]

where the inequality, Eq. (16), follows from the qEPI and the equality in Eq. (17) comes from independent and identically distributed (i.i.d.) assumption for environmental noise \( \rho_E \), and, in Eq. (18), we make use of the non-negativity of the entropy.

Second, the bound on [D.2] can be similarly derived by CqEPI. In this case, the CqEPI for the bosonic Gaussian complementary channel is given by

\[
S(\Lambda_{\tau,\rho_E}^{\otimes n}(\rho_n)|c) \geq \tau S(\rho_E^{\otimes n}|c) + (1 - \tau)S(\rho_n|c),
\]

however, the input state \( \rho_n \) is initially separable from the reference system \( C \), and we also take an assumption of independent and identically distributed (i.i.d.) for the environmental noise \( \rho_E \). Then,

\[
S(\Lambda_{\tau,\rho_E}^{\otimes n}(\rho_n)|c) := S(\Lambda_{\tau,\rho_E}^{\otimes n}(\rho_n)) - nS(\rho_C)
\]

\[
\geq \tau S(\rho_E^{\otimes n}|c) + (1 - \tau)S(\rho_n)
\]

\[
= (1 - \tau)S(\rho_n|c) - n\tau S(\rho_E)
\]

\[
\geq -n\tau g(N_E),
\]

where the first inequality follows from the CqEPI, the second equality comes from i.i.d. assumption for each \( \rho_E \)’s and \( S(\rho_{EC}) = 0 \), so that \( S(\rho_E^{\otimes n}|c) = -S(\rho_C) \), and the last inequality is obtained from the non-negativity of the von Neumann entropy. Finally, we get the inequality as \( S(\Lambda_{\tau,\rho_E}^{\otimes n}(\rho_n)) \geq n(1 - \tau)S(\rho_E) \) from \( S(\rho_E) = S(\rho_C) \). Notice that if the environmental system has a Gaussian thermal noise, then \( S(\rho_E) = g(N_{\text{th}}) \), where \( N_{\text{th}} \) is the mean thermal photon number of the environment, i.e., \( \sum_j \frac{\nu_j - 1}{\nu_j} \), \( \forall j \) for the symplectic eigenvalues \( \nu_j \) of a given covariance matrix. For general noise cases, \( N_{\text{th}} \equiv g^{-1}(S(\rho_E)) \).

Thus, we can conclude that

\[
S_{\text{MOE}}(\Lambda_\tau) + S_{\text{MOE}}(\Lambda_\tau^c) \geq 2(1 - \tau)g(N_E),
\]

where the dependences of \( n \to \infty \) and \( \rho_n \) (i.e., \( \bar{E}(\rho_n) \leq nN_E \)) can be dropped. Now, let us combine above Eq. (21) into Eq. (15), we can take an upper bound for the private capacity as in the form of

\[
\mathcal{P}(\Lambda_{\tau,\rho_E}, N) \leq 2 [g(\tau N + (1 - \tau)N_E) - (1 - \tau)g(N_E)].
\]

We here observe that the upper bound on the private capacity for the beam-splitter channel has double value compare to the upper bound on the quantum capacity case [25], i.e., \( \mathcal{P}(\Lambda_{\tau,\rho_E}, N) \approx 2Q(\Lambda_{\tau,\rho_E}, N) \).

Similarly, we can also get the upper bound on the amplifier, which follows from Eq. (12) above,

\[
\mathcal{P}(\Lambda_{\kappa,\rho_E}, N) \leq 2 \left[ g((\kappa N + (\kappa - 1)(N_E + 1))
\right.
\left. - \kappa - 1 \right) \frac{1}{2k - 1} g(N_E) - \ln(2k - 1) \right].
\]

It is worth to mention that the upper bound increases as the mean photon number \( N_E \) of the environment increases. However, it doesn’t mean that actual private capacity always depends on the environmental energy.

Along the Jeong et al.’s formulation on the classical capacity (see Appendix B in Ref. [24]), we can introduce a similar argument on the upper bound of the private capacity for bosonic Gaussian channels. Let us \( \Gamma_G \) be a single-mode covariance matrix (CvM) for a general bosonic Gaussian noise \( \rho_E \) (with the mean photon number \( N_E \)) satisfying

\[
det \Gamma_G = (2N_E + 1)^2,
\]

then we have a generalized formula for the strong upper bound on the private capacity as follows:

**Proposition 1.** Let \( \Lambda_{\kappa,\rho_E} \) be a general bosonic Gaussian noise channel with an input bosonic Gaussian state \( \rho_A \) with the mean photon number \( N \), and the mixing parameter \( \tau \in [0, 1] \). Then the upper bound on the private capacity of the channel is given by

\[
\mathcal{P}(\Lambda_{\tau,\rho_E}, N) \leq 2g(\tau N + (1 - \tau)N_E^*)
\]

\[
- 2(1 - \tau)g(N_E^*),
\]

where the general environmental noise \( \rho_E \) has a mean photon number \( N_E^* := \frac{1}{2} \sqrt{\det \Gamma_G} - 1 \).

We notice that the above upper bound, Eq. (25), is weak in the sense of that it could be diverge in the limit of \( N \to \infty \), thus it only works in the low-energy limit, however, we take a general Gaussian-noise case \( N_{\text{th}} \) not on just thermal-noise case with \( N_E \). In the thermal-noise channel, the private capacity can be suitably upper-bounded as long as \( N \to \infty \) from the data-processing upper bounds [10–12].
Now, we briefly mention about the upper bound on the private capacity for the Gaussian amplifier. For the general amplifier channel $\Lambda_{\kappa,\rho_E}$ with $\kappa \in (1, \infty]$, we have [24]

$$\mathcal{P}(\Lambda_{\kappa,\rho_E}, N) \leq 2g(\kappa N + (\kappa - 1)(N_E^g + 1)) - \frac{2\kappa - 2}{2\kappa - 1}g(N_E^g) - \ln(2\kappa - 1)^2. \quad (26)$$

We can observe that the upper bound on the private capacity has also double value for the upper bound on the classical capacity, $\mathcal{P}(\Lambda_{\tau,\rho_E}, N) \simeq 2\mathcal{C}(\Lambda_{\tau,\rho_E}, N)$. Operationally, those three quantities have a relation so that, for any quantum channel $\Lambda$, $\mathcal{Q}(\Lambda) \leq \mathcal{P}(\Lambda) \leq \mathcal{C}(\Lambda)$, this implies that $\mathcal{P}(\Lambda)$ can be more tighten the upper bound about 1/2 in bosonic Gaussian channels.

IV. PLOTS ON THE BOUNDS FOR THE PRIVATE CAPACITY

In the previous section, we have investigated upper bounds on the private capacity for the bosonic Gaussian noise channel in which an environmental system can be any Gaussian-noise state. We should remember that we can take the lower bound from [D.1] and [D.2] from substituting the thermal-noise to the general Gaussian-noise case as

$$\mathcal{P}(\Lambda_{\tau,\rho_E}, N) \geq 2g(\tau N + (1 - \tau)N_E) - g((1 - \tau)N_E),$$

since we can fix the lower bound roughly. Similarly, the amplifier’s lower bound can be given by

$$\mathcal{P}(\Lambda_{\kappa,\rho_E}, N) \geq 2g(\kappa N + (\kappa - 1)N_E) - g\left(\frac{2\kappa - 2}{2\kappa - 1}N_E\right).$$

We plot the upper and lower bounds of the private capacity for the bosonic Gaussian channels with respect to the input state energy $N$ in Fig. 2. As a special case, it was known that a lower bound for the one-shot private capacity on the thermal-noise channel is given by Eq. (8.9) in [11]

$$P^{(1)}(\Lambda_{\tau,\rho_E}, N) \geq \mathcal{P}_L(\Lambda_{\tau,\rho_E}, N) \equiv I_c(\Lambda_{\tau,\rho_E}, N) - I_c(\Lambda_{\tau,\rho_E}, N^2),$$

where $I_c(\Lambda, \rho) := S(\Lambda(\rho)) - S(\Lambda(\rho'))$ denotes the coherent information for the thermal-noise channel for an input $\rho$ with $N$. (We have plotted the Sharma et al.’s lower bound in Fig. 2 (a). However, we will remain the amplifier case as a future work.)

Here we give specific example in order to consider the physical meanings of our results. The non-trivial example on the private capacity of the channel $\Lambda_{\mu,\rho_E}$ is the beam-splitter ($\mu = \tau$) involving the general thermal-noise ($\rho_E = \rho_{\text{sth}}$), in which the environmental system is the squeezed thermal state $\rho_{\text{sth}}$ as a general Gaussian noise. In general, the squeezed thermal state has the CVM known in the form of

$$\Gamma_{\text{sth}} = (2N_{\text{th}} + 1)\begin{pmatrix} e^{-2r} & 0 \\ 0 & e^{2r} \end{pmatrix}, \quad (27)$$

where $N_{\text{th}}(= N_E)$ is the mean photon number from the thermal noise, and $r \in [0, \infty]$ is the squeezing parameter. Notice that $\det \Gamma_{\text{sth}} = (2N_{E} + 1)^2$ is equivalent to as in the general case $\det \Gamma_G$ in Sec. III. The squeezed thermal state is the most general single-mode Gaussian state when its mean is placed at the origin, which can be always removed by the local symplectic unitary transformation. Consequently, what we are considering here are general bosonic Gaussian channels, and actually the mean photon number of general $\Gamma_{\text{sth}}$ is equivalent to the thermal state, i.e., $N_{\text{sth}} = N_{\text{th}} = N_E$.

V. DISCUSSIONS

Computing the exact channel-capacity on a quantum channel is very hard problem. However, recently it was known that we can effectively obtain the upper bounds through a powerful tool of quantum entropy power inequality. In this study, we have investigated non-trivial upper bounds on the energy-constrained private capacity for bosonic Gaussian channels with mixing parameters $\tau$ and $\kappa$. Here, our principal method is CqEPI, which can be used for obtaining bounds on the output entropy of the complementary bosonic Gaussian channel. Although our results do not allow tighter bounds for the private capacity, those are applicable to more general environmental noise such a squeezed thermal noise, and potentially even at non-Gaussian one.

Our results showed that the upper bounds for the private capacity have a double value comparing to the classical and the quantum capacity for the bosonic Gaussian channels. In other words, we have some reservations about tight bounds for the three quantities in the operational framework of $\mathcal{Q} \leq \mathcal{P} \leq \mathcal{C}$. We expect that our work could be extended for the knowledge of the private capacity, which is still far from reaching fully understanding in quantum Shannon theory.
ACKNOWLEDGMENTS

The author thanks to anonymous referee for valuable comments. This work was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the the Ministry of Education, Korea (NRF-2018R1D1A1B07047512).

[1] C. E. Shannon, “A mathematical theory of communication,” Bell Syst. Tech. J. 27, 379–423, 623–656 (1948).
[2] M. A. Nielsen and I. L. Chuang, “Quantum Computation and Quantum Information,” Cambridge University Press (2000).
[3] M. M. Wilde, “Quantum Information Theory,” Cambridge University Press (2017).
[4] A. S. Holevo and R. F. Werner, “Evaluating capacities of bosonic Gaussian channels,” Phys. Rev. A 63, 032312 (2001).
[5] C. Weedbrook, S. Pirandola, R. García-Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, “Gaussian quantum information,” Rev. Mod. Phys. 84, 621–669 (2012).
[6] A. Seraﬁni, “Quantum Continuous Variables: A Primer of Theoretical Methods,” CRC Press, London (2017).
[7] I. Devetak, “The private classical capacity and quantum capacity of a quantum channel,” IEEE Trans. Inf. Theory 51, 44–55 (2005).
[8] G. Smith, J. M. Renes, and J. A. Smolin, “Structured Codes Improve the Bennett-Brassard-84 Quantum Key Rate,” Phys. Rev. Lett. 100, 170502 (2008).
[9] K. Li, A. Winter, X.B. Zou, and G.C. Guo, “Private Capacity of Quantum Channels is Not Additive,” Phys. Rev. Lett. 103, 120501 (2009).
[10] M. Rosati, A. Mari, and V. Giovannetti, “Narrow bounds for the quantum capacity of thermal attenuators,” Nat. Commun. 9, 4339 (2018).
[11] K. Sharma, M. M. Wilde, S. Adhikari, and M. Takeoka, “Bounding the energy-constrained quantum and private capacities of phase-insensitive bosonic Gaussian channels,” New J. Phys. 20, 063025 (2018).
[12] K. Noh, V. V. Albert, and L. Jiang, “Quantum Capacity Bounds of Gaussian Thermal Loss Channels and Achievable Rates With Gottesman-Kitaev-Preskill Codes,” IEEE Trans. Inf. Theory 65, 2563–2582 (2019).
[13] R. Laurenza, S. Tserkis, L. Banchi, S. L. Braunstein, T. C. Ralph, and S. Pirandola, “Tight bounds for private communication over bosonic Gaussian channels based on teleportation simulation with optimal finite resources,” Phys. Rev. A 100, 042301 (2019).
[14] R. König and G. Smith, “The Entropy Power Inequality for Quantum Systems,” IEEE Trans. Inf. Theory 60, 1536–1548 (2014).
[15] R. König and G. Smith, “Limits on classical communication from quantum entropy power inequalities,” Nat. Photonics 7, 142–146 (2013).
[16] R. König and G. Smith, “Classical Capacity of Quantum Thermal Noise Channels to within 1.45 Bits,” Phys. Rev. Lett. 110, 040501 (2013).
[17] G. De Palma, A. Mari, and V. Giovannetti, “A generalization of the entropy power inequality to bosonic quantum systems,” Nat. Photonics 8, 958–964 (2014).
[18] K. Audenaert, N. Datta, and M. Ozols, “Entropy power inequalities for qubits,” J. Math. Phys. 57, 052202 (2016).
[19] R. Koenig, “The conditional entropy power inequality for Gaussian quantum states,” J. Math. Phys. 56, 022201 (2015).
[20] K. Jeong, S. Lee, and H. Jeong, “Conditional quantum entropy power inequality for d-level quantum systems,” J. Phys. A: Math. Theor. 51, 145303 (2018).
[21] G. De Palma and D. Trevisan, “The Conditional Entropy Power Inequality for Bosonic Quantum Systems,” Commun. Math. Phys. 360, 639 (2018).
[22] G. De Palma and S. Huber, “The conditional entropy power inequality for quantum additive noise channels,” J. Math. Phys. 59, 122201 (2018).
[23] S. Huber and R. König, “Coherent state coding approaches the capacity of non-Gaussian bosonic channels,” J. Phys. A: Math. Theor. 51, 184001 (2018).
[24] K. Jeong, H. H. Lee, and Y. Lim, “Universal upper bounds for Gaussian information capacity,” Ann. Phys. 407, 46–56 (2019).
[25] Y. Lim, S. Lee, J. Kim, and K. Jeong, “Upper bounds on the quantum capacity for a general attenuator and amplifier,” Phys. Rev. A 99, 052326 (2019).
[26] R. Jozsa, “Fidelity for Mixed Quantum States,” J. Mod. Opt. 41, 2315–2323 (1994).
[27] K. Jeong and Y. Lim, “Purification of Gaussian maximally mixed states,” Phys. Lett. A 380, 3607–3611 (2016).
[28] F. Caruso, V. Giovannetti, and A. S. Holevo, “One-mode bosonic Gaussian channels: a full weak-degradability classification,” New J. Phys. 8, 310 (2006).
[29] H. Qi and M. M. Wilde, “Capacities of quantum amplifier channels,” Phys. Rev. A. 95, 012339 (2017).
[30] M. M. Wilde and H. Qi, “Energy-Constrained Private and Quantum Capacities of Quantum Channels,” IEEE Trans. Inf. Theory 64, 7802–7827 (2018).
[31] A. S. Holevo, “The capacity of the quantum channel with general signal states,” IEEE Trans. Inf. Theory 44, 269–273 (1998).
[32] B. Schumacher and M. D. Westmoreland, “Sending classical information via noisy quantum channels,” Phys. Rev. A 56, 131 (1997).
[33] M. M. Wolf, G. Giedke, and J. I. Cirac, “Extremality of Gaussian Quantum States,” Phys. Rev. Lett. 96, 080502 (2006).
[34] V. Giovannetti, S. Guha, S. Lloyd, L. Maccone, and J. H. Shapiro, “Minimal output entropy of bosonic channels: a conjecture,” Phys. Rev. A 70, 032315 (2004).
[35] V. Giovannetti, S. Guha, S. Lloyd, L. Maccone, J. H. Shapiro, and H. P. Yuen, “Classical Capacity of the Lossy Bosonic Channel: The Exact Solution,” Phys. Rev. Lett. 92, 027902 (2004).