AN OBSTRUCTION TO CONTRACTIBILITY OF SPACES OF DIRECTED PATHS

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Abstract. Defining homotopy or collapsibility in directed topological spaces is surprisingly difficult. In the directed setting, the spaces of directed paths between fixed initial and terminal points are the defining feature for distinguishing different directed spaces. The simplest case is when the space of directed paths is homotopy equivalent to that of a single path; we call this the trivial space of directed paths. Surprisingly, directed spaces that are trivial topologically may have non-trivial spaces of directed paths, which means that information is lost when the direction of these topological spaces is ignored. We define a notion of directed contractibility and directed collapsibility in the setting of a directed Euclidean cubical complex, using the spaces of directed paths of the underlying directed topological space, relative to an initial or a final vertex. In addition, we give a sufficient condition for a directed Euclidean cubical complex to have contractible spaces of directed paths from a fixed initial vertex.

1. Introduction

Spaces that are equipped with a direction have recently been given more attention from a topological point of view, and there is still very much that is unknown. One reason for studying directed spaces is their application to concurrent programming where standard algebraic topology does not provide the tools needed [2]. These directed spaces are models for concurrent programs and paths respecting the (time) directions represent executions of programs. In these models, executions are equivalent if their execution paths are homotopic (through directed paths) and this observation has already led to new insight and algorithms. For instance, verification of concurrent programs is simplified by verifying one execution from each connected component of the space of directed paths; see [3] and [2].

While equivalence of executions is clearly stated in this model, equivalence of the directed topological spaces themselves is not well understood. In particular, directed versions of homotopy groups and homology groups are not agreed upon.

Moreover, equivalence of directed topological spaces is not easily defined. Directed homeomorphism is too strong, whereas directed homotopy equivalence is often too weak to preserve the properties of the concurrent programs. In classical (undirected) topology, the concept of simplifying a space by a sequence of collapses goes back to J.H.C. Whitehead [12], and has been studied in [4][11], among others. Defining homotopy or collapsibility in directed topological spaces is surprisingly difficult. In the directed setting, the spaces of directed paths are the defining feature for distinguishing different directed spaces.

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Figure 1. The Swiss flag. We model the set of all executions of two processes that take unit time. The product $I \times I$ is called the state-space. We include only valid regions of this state space (see the shaded regions). A bi-monotone path through this space is a possible execution. Progress in the positive x-direction is the first process, and along the y-axis is the second. There are two unit-capacity resources one of which is needed by process 1 in the middle 3/5 of execution and by process 2 in the middle 1/5 of its execution - this results in a horizontal white rectangle of non-valid states. The other resource is needed symmetrically and results in the vertical white rectangle. Then, there are two regions in the state space that are of particular interest: no path will enter the solid black region; and no path that enters the gray region can complete. Points in the black region are called unreachable, the upper right hand corner of the grey region is referred to as a deadlock and all points in the grey region are doomed. As illustrated by red curves, this directed space has two possible paths from the bottom left to the top right.

In this article, we consider the space of directed paths of Euclidean cubical complexes, which correspond to concurrent programming without loops. We always consider spaces of directed paths relative to a pair of endpoints.

Consider the example in Figure 1, where two processes are vying for the same shared resources. Here, we have two distinct paths in the directed space, up to homotopy equivalence: one corresponding to the first process using the shared resource first, and the other corresponding to the second process using the shared resource first.

Outline. In Section 2 we introduce the notions of a space of directed paths and Euclidean cubical complexes. Given the directed structure of these Euclidean cubical complexes, we do not consider simply the link of a vertex, but the past (or future) link of a vertex. In Section 3 we give results on the topology of the space of directed paths in terms of past links. In Theorem 3.1 sufficient conditions on the past links of every vertex of a complex are given so that the space of directed paths is contractible. Theorem 3.2 gives conditions that are sufficient for the space of directed paths to be connected.

In Section 4 we describe a method of collapsing one complex into another with the purpose of preserving the respective directed path spaces. It involves taking a pair of simplices $(\tau, \sigma)$ from a Euclidean complex $K$ with certain conditions on the nearby links and then collapsing $K$ into a simpler complex by removing $\tau, \sigma$, and all simplicies in between. Then, in Section 5 we discuss how this directed collapse compares to classical homotopy theory, which includes the well-known example of a contractible space that is not collapsible, Bing’s House. Lastly, we put our notion of directed collapse into the context of directed homotopy theory in Section 6.
2. Past Links as Obstructions

In this section, we introduce the notions of spaces of directed paths and Euclidean cubical complexes. The (relative) past link of a vertex of a Euclidean cubical complex is defined as a simplicial complex and a non-contractible past link is recognized as an obstruction to contractibility of certain spaces of directed paths.

**Definition 2.1 (d-space).** A d-space is a pair \((X, \overrightarrow{P}(X))\), where \(X\) is a topological space and \(\overrightarrow{P}(X) \subseteq P(X) \coloneqq X^{[0,1]}\) is a family of paths on \(X\) (called dipaths) that is closed under non-decreasing reparametrizations and concatenations, and contains all constant paths.

For every \(x, y \in X\), let \(\overrightarrow{P}^y_x(X)\) be the family of dipaths from \(x\) to \(y\):

\[
\overrightarrow{P}^y_x(X) := \{ \alpha \in \overrightarrow{P}(X) : \alpha(0) = x \text{ and } \alpha(1) = y \}.
\]

A continuous map \(f : X \to Y\) is a dimap if \(\gamma \in \overrightarrow{P}(X) \Rightarrow f \circ \gamma \in \overrightarrow{P}(Y)\).

In particular, consider the following directed space: the directed real line \(\overrightarrow{\mathbb{R}}\) is the directed space constructed from the real line whose family of dipaths \(\overrightarrow{P}(\mathbb{R})\) consists of all non-decreasing paths. The Euclidean space \(\overrightarrow{\mathbb{R}^n}\) is the \(n\)-fold product \(\overrightarrow{\mathbb{R}} \times \cdots \times \overrightarrow{\mathbb{R}}\) with family of dipaths the \(n\)-fold product \(\overrightarrow{P}(\mathbb{R}^n) = \overrightarrow{P}(\mathbb{R}) \times \cdots \times \overrightarrow{P}(\mathbb{R})\).

Furthermore, we can solely focus on the family of dipaths in a d-space and endow it with the compact open topology.

**Definition 2.2 (Space of Directed Paths).** In a d-space \((X, \overrightarrow{P}(X))\), the space of directed paths from \(x\) to \(y\) is the family \(\overrightarrow{P}^y_x(X)\) with the compact open topology.

The space of directed paths has a topology. It is now amenable to topological reasoning and comparison. Note that the individual spaces of directed paths \(\overrightarrow{P}^y_x(X)\) do not have directionality, so contractibility and other topological features are defined as in the classical case. Moreover, observe that the set \(\overrightarrow{P}^y_x(X)\) might have cardinality of the continuum, but is considered trivial if it is homotopy equivalent to a point.

The d-spaces we will consider in this article are constructed from Euclidean cubical complexes. Let \(p = (p_1, \ldots, p_n), q = (q_1, \ldots, q_n) \in \mathbb{R}^n\). We will write \(p \preceq q\) if and only if \(p_i \leq q_i\) for all \(i = 1, \ldots, n\). Furthermore, we will denote by \(q - p := (q_1 - p_1, \ldots, q_n - p_n)\) the component-wise difference between \(q\) and \(p\), \(|p| := \sum_{i=1}^n p_i\) is the element-wise sum, or one-norm, of \(p\). Similarly to the one-dimensional case, the interval \([p, q]\) is defined as \(\{x \in \mathbb{R}^n : p \preceq x \preceq q\}\).

**Definition 2.3 (Euclidean Cubical Complex).** Let \(p, q \in \mathbb{R}^n\). If \(q, p \in \mathbb{Z}^n\) and \(q - p \in \{0, 1\}^n\), then the interval \([p, q]\) is an elementary cube in \(\mathbb{R}^n\) of dimension \(|q - p|\). A Euclidean cubical complex \(K \subseteq \mathbb{R}^n\) is the union of elementary cubes.

**Remark 2.4.** A Euclidean cubical complex \(K\) is a subset of \(\mathbb{R}^n\) and there is an associated abstract cubical complex. By a slight abuse of notation, we will not distinguish these.

Every cubical complex \(K\) inherits the directed structure from the Euclidean space \(\overrightarrow{\mathbb{R}^n}\), described after Definition 2.1. An elementary cube of dimension \(d\) is called a \(d\)-cube. The \(m\)-skeleton of \(K\), denoted by \(K_m\), is the union of all elementary cubes contained in \(K\) that have dimension less than or equal to \(m\). The elements of the zero-skeleton are called the vertices of \(K\). A vertex \(w \in K_0\) is said to be minimal (resp., maximal) if \(w \preceq v\) (resp., \(w \succeq v\)) for every vertex \(v \in K_0\).

Following [13], we define the (relative) past link of a vertex of a Euclidean cubical complex as a simplicial complex. Let \(\Delta^{n-1}\) denote the complete simplicial
complex with vertices \( \{1, \ldots, n\} \). Simplices of \( \Delta^{n-1} \) will be identified with elements \( j \in \{0, 1\}^n \). That is, every subset \( S \subseteq \{1, \ldots, n\} \) is mapped to the \( n \)-tuple with entry 1 in the \( k \)-th position if \( k \) belongs to \( S \) and 0 otherwise. The topological space associated to the simplicial complex \( \Delta^{n-1} \) is the one given by its geometric realization.

**Definition 2.5** (Past Link). In a Euclidean cubical complex \( K \) in \( \mathbb{R}^n \), the past link, \( \ell k^- \) \( K, w \), of a vertex \( v \) with respect to a vertex \( w \) is the simplicial subcomplex of \( \Delta^{n-1} \) defined as follows: \( j \in \ell k^- \) \( K, w \) \( (v) \) if and only if \( [v - j, v] \subseteq K \cap [w, v] \).

**Remark 2.6.** While \( K \) is a cubical complex, the past link of a vertex in \( K \) is always a simplicial complex.

**Remark 2.7.** Often the vertex \( w \) and the complex \( K \) are understood. In this case we will denote the past link of \( v \) by \( \ell k^- (v) \).

**Remark 2.8.** Other definitions of the (past) link are found in the literature. Unlike Definition 2.5, these are usually subcomplexes of \( K \). However, they are homeomorphic to the (past) link of Definition 2.5.

In the following example, we show that there exist vertices of a Euclidean cubical complex which have different past links with respect to two different vertices. We will consider as a Euclidean cubical complex the open top box (Figure 2) and the complex which have different past links with respect to two different vertices. We will show that for fixed \( w, v \) the contractibility of all past links \( \ell k^- \) \( K, w \) \( (v) \) is sufficient to guarantee the contractibility of the space of directed paths \( \mathcal{P}_{\ell k, w} (K) \) from \( w \) to \( v \).

**Example 2.9** (Open Top Box). Let \( L \subset \mathbb{R}^3 \) be the Euclidean cubical complex consisting of all of the edges and vertices in the elementary cube \([w, v] \) and five of the six 2-cubes, omitting the elementary two-cube \([0, 0, 1], v \), i.e., the top of the box. Because the elementary 1-cube \([v - (0, 0, 1), v] \subseteq L \cap [w, v] \) \( = L \), \( \ell k^- \) \( v \) \( (L, w) \) \( (v) \) contains the vertex in \( \Delta^2 \) corresponding to \( j = (0, 0, 1) \). Similarly, because the elementary 2-cube \([v - (0, 1, 1), v] \subseteq L \), \( \ell k^- \) \( v \) \( (L, w) \) \( (v) \) contains the edge in \( \Delta^2 \) corresponding to \( j = (1, 1, 0) \). However, because the elementary 2-cube \([v - (1, 1, 0), v] \) \( = 1 \) \( (L, w) \) \( (v) \) does not contain the edge corresponding to \( j = (1, 1, 0) \). Instead taking \( w' = (0, 0, 1) \), we get that \( \ell k^- \) \( L, w' \) \( (v) \) consists of the two vertices corresponding to \( j = (0, 1, 0) \) and \( j' = (1, 0, 0) \). See Figure 2.

3. A Sufficient Condition for Contractibility of Directed Path Spaces

We next show that for fixed \( w, v \) the contractibility of all past links \( \ell k^- \) \( K, w \) \( (s) \) for \( w \preceq s \preceq v \) is sufficient to guarantee the contractibility of the space of directed paths \( \mathcal{P}_{\ell k, w} (K) \) from \( w \) to \( v \).

**Theorem 3.1** (Contractibility). Let \( K \subset \mathbb{R}^n \) be a Euclidean cubical complex such that the minimal vertex of \( K \) is \( 0 \). Suppose for all \( k \) \( K_0 \), the past link \( \ell k^- (k) \) is contractible or empty. Then, all spaces of directed paths \( \mathcal{P}_{\ell k, 0} (K) \) are contractible.

**Proof.** By [13, Proposition 5.3], if \( \mathcal{P}_{\ell k, 0} (K) \) is contractible for all \( j \in \{0, 1\}^n \), \( j \neq 0 \), and \( j \in \ell k^- (k) \), then \( \mathcal{P}_{\ell k, 0} (K) \) is homotopy equivalent to \( \ell k^- (k) \). Hence, it suffices to see that all these spaces are contractible. This follows by structural induction on the partial order on vertices in \( K \). The start is at \( \mathcal{P}_{\ell k, 0} (K) \), where \( e_i \) is the \( i \)-th unit vector, and \( 0 + e_i \in K_0 \). If \([0, 0 + e_i] \in K \), then \( \mathcal{P}_{\ell k, 0 + e_i} (K) \) is contractible. Otherwise, it is empty and the corresponding \( j \) is not in the past link. □
Now we give an analogous sufficient condition for when spaces of directed paths are connected. Again, looking at properties of past links gives us information about properties of the space of directed paths. Two different proofs of Theorem 3.2 are provided in Section A.1. One proof shows how we can use [10, Prop. 2.20] to get our desired result. The second proof uses notions from category theory and is based on the fact that the colimit of connected spaces over a connected category is connected.

**Theorem 3.2** (Connectedness). With $K$ as above, suppose all past links of all vertices are connected. Then, for all $k \in K_0$, all spaces of directed paths $\overrightarrow{P}_k(K)$ are connected.

**Proof.** See Section A.1. □

**Remark 3.3.** Our conjecture is that similar results for $k$-connected past links should follow from the $k$-connected Nerve Lemma.

We finish this section by providing an example where all past links are contractible with respect to the initial vertex, yet there exists a vertex, $w'$, for which the space of directed paths starting at $w'$ and ending at the final vertex is not contractible. Thus, it is important to note that Theorem 3.1 only states that the space of directed paths starting at the initial vertex is contractible.

**Example 3.4** (Open Top Box). We illustrate that the space of directed paths may be non-trivial, when the initial point is not $0$. Let $L$ be the cubical complex with five two-cells introduced in Example 2.9, i.e., an open top box. Then all past links in $L$ with respect to the initial vertex are contractible, but $\overrightarrow{P}_{w'}(L)$, where $w' = (0, 0, 1)$ and $v = (1, 1, 1)$, is not contractible. It is in fact two points. Note, this does not contradict Theorem 3.1 since it only asserts that $\overrightarrow{P}_0(L)$ is contractible. See Figure 2.
4. Directed Collapsibility

To simplify the underlying topological space of a d-space while preserving topological properties of the associated space of directed paths, we introduce the process of directed collapse. The criteria we require to perform directed collapse on Euclidean cubical complexes involves the topology of the past links of the vertices of the complex. We defined the past links as simplicial complexes that are not themselves directed, so our topological criteria are in the usual sense.

**Definition 4.1** (Directed Collapse). Let $K$ be a Euclidean cubical complex with initial vertex $0$. Consider $\sigma, \tau \in K$ such that $\tau \subset \sigma$, $\sigma$ is maximal, and no other maximal cube contains $\tau$. Let $K' = K \setminus \{\gamma \in K | \tau \subseteq \gamma \subseteq \sigma\}$. $K'$ is a directed (cubical) collapse of $K$ if, for all $v \in K'_0$, $lk^{-}_{K'}(v)$ is homotopy equivalent to $lk^{-}_{K}(v)$. The pair $\tau, \sigma$ is then called a collapsing pair.

**Remark 4.2.** As in the simplicial case, when we remove $\sigma$ from the abstract cubical complex, the effect on the geometric realization is to remove the interior of the cube corresponding to $\sigma$.

**Remark 4.3.** Note for finding collapsing pairs, $(\tau, \sigma)$, using Definition 4.1 with the geometric realization of $\sigma$ given by the elementary cube, $[w - j, w]$, it is sufficient to only check $v \in K'_0$ such that $v = w - j'$ where $j - j' > 0$. Otherwise the past links, $lk^{-}_{K}(v)$ and $lk^{-}_{K'}(v)$, are equal.

**Definition 4.4** (Past Link Obstruction). Let $w \in K_0$. A past link obstruction (type-$\infty$) in $K$ with respect to $w$ is a vertex $v \in K_0$ such that $lk^{-}_{K,w}(v)$ is not contractible. A past link obstruction (type-$0$) in $K$ with respect to $w$ is a vertex $v \in K_0$ such that $lk^{-}_{K,w}(v)$ is not connected.

We now see what properties of the space of directed paths are preserved by directed collapse.

**Corollary 4.5.** If there are no type-$\infty$ past link obstructions, then all spaces of directed paths from the initial point are contractible. If there are no type-$0$ past link obstructions, all spaces of directed paths from the initial point are connected.

**Proof.** Contractibility is a direct consequence of Theorem 3.2. Likewise, connectedness follows from Theorem 3.2. \qed

**Corollary 4.6** (Invariants of Directed Collapse). If we have a sequence of directed collapses from $K$ to $K'$, then there are no obstructions in $K$ iff there are no obstructions in $K'$.

**Remark 4.7** (Past Link Obstructions are Inherently Local). The past link of a vertex is constructed using local (rather than global) information from the cubical complex. Therefore, a past link obstruction is also a local property, which is not dependent on the global construction of the cubical complex.

Below, we provide a few motivating examples for our definition of directed collapse. In general, we want our directed collapses to preserve all spaces of directed paths between the initial vertex and any other vertex in our cubical complex. According to Definition 4.1, $\tau$ from Definition 4.4 is a free face of $K$. Performing a directed collapse with an arbitrary free face of a directed space $K$ with minimal element $0 \in K_0$ and maximal element $1 \in K_0$ can modify the individual spaces of directed paths $\vec{\mathcal{P}}^v_0(K)$ and $\vec{\mathcal{P}}^v_1(K)$ for $v \in K_0$.

When $\vec{\mathcal{P}}^v_0(K) = \emptyset$, we call $v$ a deadlock. When $\vec{\mathcal{P}}^v_0(K) = \emptyset$, we call $v$ unreachable. Deadlocks and unreachable vertices are in a sense each others opposites.

Notice if we take the same directed space $K$ yet reverse the direction of all dipaths,
then deadlocks become unreachable vertices and vice versa. However, as Example 4.8 and Example 4.9 illustrate, the creation of an unreachable vertex in the process of a directed collapse might result in a past link obstruction at a neighboring vertex while the creation of a deadlock does not.

**Example 4.8 (3 x 3 Grid, Deadlocks & Unreachability).** Let $K$ be the Euclidean cubical complex in $\mathbb{R}^2$ comprised of vertices $v_{i,j} = (i,j) \in \mathbb{Z}^2$ with $i,j \in \{0,1,2,3\}$ and all elementary two-cubes $[v_{i,j}, v_{i+1,j+1}]$ for $i,j \in \{0,1,2\}$. Consider the Euclidean cubical complexes $K'$ and $K''$ obtained by removing $(\tau,\sigma)$ with $\tau = [v_{1,3}, v_{2,3}], \sigma = [v_{1,2}, v_{2,3}]$ and $(\tau',\sigma')$ with $\tau' = [v_{1,0}, v_{2,0}], \sigma' = [v_{1,0}, v_{2,1}]$, respectively. While $K'$ is a directed collapse of $K$, $K''$ is not a directed collapse of $K$; this is because $K''$ introduces a past link obstruction at $v_{2,1}$. So, $(\tau,\sigma)$ is a collapsing pair while $(\tau',\sigma')$ is not. Collapsing $K$ to $K'$ creates a deadlock at $v_{1,3}$ but this does not change the space of directed paths from the designated start vertex $v_{0,0}$ to any of the vertices between $v_{0,0}$ and the designated end vertex $v_{3,3}$ (see $K'$ in Figure 3). However, collapsing $K$ to $K''$ creates an unreachable vertex $v_{2,0}$ from the start vertex $v_{0,0}$ (see $K''$ in Figure 3) which does change the space of directed paths from $v_{0,0}$ to $v_{2,0}$ to be empty. Hence not all spaces of directed paths starting at $v_{0,0}$ are preserved. This motivates our definition of directed collapse.

![Figure 3](image-url)  
**Figure 3.** Illustrating Example 4.8. On the left: the cubical complex $K$ with initial vertex 0 and final vertex 3. In the center: The cubical complex $K'$ which is a directed collapse of $K$. The deadlock in blue does not change the space of directed paths from $v_{0,0}$ to any of the vertices between $v_{0,0}$ and $v_{3,3}$. On the right: the cubical complex $K''$ which is not a directed collapse of $K$. The space of directed paths into the unreachable red vertex, $v_{2,0}$, becomes empty. This is reflected in the topology of the past link of the red vertex $v_{2,1}$ (see Example 4.9).

Our next example shows how directed collapses can be performed with collapsing pairs $(\tau,\sigma)$ when $\tau$ is of codimension one and greater.

**Example 4.9 (3 x 3 grid, Edge & Vertex Collapses).** Consider again the Euclidean cubical complex $K$ from Example 4.8. If we allow a collapsing pair $(\tau,\sigma)$ with $\tau$ of dimension greater than 0, we may introduce deadlocks or unreachable vertices. In particular, collapsing the free edge $\tau = [v_{1,3}, v_{2,3}]$ of the top blue square $\sigma = [v_{1,2}, v_{2,3}]$ in Figure 4 changes the space of directed paths $\overrightarrow{P}_{v_{1,3}}^v(K)$ from being trivial to empty in $K \setminus \{\gamma \mid \tau \subseteq \gamma \subseteq \sigma\}$. Yet we care about preserving the space of directed paths from our designated start vertex $v_{0,0}$ to any of the vertices $v_{i,j}$ with $0 \leq i,j \leq 3$ since we ultimately are interested in preserving the path space $\overrightarrow{P}_{v_{0,0}}^v(K)$. Because of this, such collapses should be allowed in our directed setting. Note that, in these cases, the past link of all vertices remains contractible. However,
collapsing the free edge $\tau' = [v_{1,0}, v_{2,0}]$ of the bottom red square $\sigma' = [v_{1,0}, v_{2,1}]$ in Figure 4 changes the path space $\mathcal{P}^{v_{1,0}}(K)$ from being trivial to empty. This is reflected in the non-contractible past link of $v_{2,1}$ in $K\{\gamma|\tau' \subseteq \gamma \subseteq \sigma'\}$ that consists of the two vertices $j = (1,0)$ and $j' = (0,1)$ but not the edge $j'' = (1,1)$ connecting them. Restricting our collapsing pairs to only include $\tau$ of dimension 0 allows for only two potential collapses, the corner vertices $v_0, 3$ and $v_3, 0$ into the yellow squares $[v_{0,2}, v_{1,3}]$ and $[v_{2,0}, v_{3,1}]$, respectively. Neither of these collapses create deadlocks or unreachable vertices and the contractibility of the past link at all vertices is preserved. Performing these corner vertex collapses exposes new free vertices that can be a part of subsequent collapses.

![Figure 4](image)

**Figure 4.** Illustrating Example 4.9. On the left: the collapsing of the free edge in the blue squares is an admitted directed collapse. The collapsing of the free edge in the red squares is not an admitted directed collapse. On the right: the collapsing of the free vertex in the yellow squares is an admitted directed collapse.

5. Comparisons to Regular Homotopy Theory and Collapsibility

We wish to compare our results to regular homotopy theory. Recall- given two topological spaces $X, Y$, two maps $f$ and $g : X \to Y$ are said to be homotopic, denoted $f \simeq g$, if there exists a continuous function $H : X \times I \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$. We say that $X$ is homotopy equivalent to $Y$ if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$.

5.1. Simplicial Collapse. The process of simplicial collapse was studied by J.H.C. Whitehead [12], as well as the idea of a simple homotopy equivalence, an equivalence relation generated by simplicial collapse, defined below.

**Definition 5.1.** Let $X$ be a simplicial complex, and $\alpha$ be a face of $X$. If there is no $\beta \in X$ such that $\alpha \in \partial \beta$, then $\alpha$ is a maximal face in $X$.

**Definition 5.2 (Free Face).** Let $\alpha, \beta$ be faces of $X$, with $\alpha \in \partial \beta$. If $\alpha$ is not a face of any other $\beta' \in X$, then $\alpha$ is a called a free face of $\beta$. Note that if $\beta$ contains a free face, then $\beta$ is maximal.

**Theorem 5.3.** If $\alpha$ is a free face of $\beta$, then $X$ is homotopy equivalent to $X\setminus (int(\alpha)) \cup int(\beta))$.

**Definition 5.4 (Simplicial Collapse).** Let $\alpha$ be a free face of $\beta$. The deformation retract $X\setminus (int(\alpha)) \cup int(\beta))$ is called the simplicial collapse of $X$. If $X$ and $Y$ are related by a finite sequence of these simplicial collapses, then $X$ and $Y$ are said to
be *simple homotopy equivalent*. If a space is simple homotopy equivalent to a point, then we say that space is *collapsible*.

Although the idea of a simplicial collapse is easy to state, what is not so easy is how to determine whether a simplicial complex is collapsible. A discrete Morse function on a simplicial complex prescribes a certain number of simplicial collapses according to the gradient vector field corresponding to that Morse function. All non-critical cells can be collapsed to produce a complex homotopy equivalent to a complex with one cell of dimension $p$ for each critical cell of dimension $p$. However, one is unable to determine the efficiency of this discrete Morse function with respect to collapsibility - meaning, there may be cells that are critical that could be collapsed while still preserving the homotopy type. In special cases, critical cells of adjacent dimensions can be "cancelled" (Theorem 9.1 [4]), leaving a complex with fewer critical cells.

**Remark 5.5.** There are other types of simplicial collapse, including a notion of strong collapse, which places a restriction of the link of a collapsing vertex. We leave discussion of this type of collapse to Section 7.

5.2. **Cubical Collapse and Directed Collapse.** Analogous to simplicial collapse, defined below is the idea of cubical collapse, developed in [7].

**Definition 5.6.** Let $[p_1, q_1], [p_2, q_2]$ be elementary cubes. If $[p_1, q_1] \subseteq [p_2, q_2]$, then $[p_1, q_1]$ is a *face* of $[p_2, q_2]$. If $[p_1, q_1] \neq [p_2, q_2]$, then $[p_1, q_1]$ is a *proper face* of $[p_2, q_2]$.

**Definition 5.7 (Maximal Face).** Let $K$ be a Euclidean cubical complex. If a $m$-cube $Q \in K$ is not a proper face of some other $m'$-cube $P \in K$, then it is called a *maximal face*.

**Definition 5.8 (Free Face of a Euclidean Cubical Complex).** Let $Q \in K$. If $Q$ is a proper face of exactly one $P \in K$, then $Q$ is a called a *free face* in $K$. Note, if $P$ contains a free face $Q$, then $P$ is maximal, and $\dim Q = \dim P - 1$. (Lemma 2.63 [7]).

**Definition 5.9 (Elementary Cubical Collapse).** Let $Q$ be a free face in $K$ and let $P$ be the unique cube such that $Q$ is a proper face of $P$. Then $K' := K \{\text{int}(Q), \text{int}(P)\}$ is the cubical space obtained from $K$ via *elementary cubical collapse of $P$ by $Q$*.

As in the case of simplicial collapse, any two spaces that are related by a sequence of cubical collapses are homotopy equivalent, as a cubical collapse is a strong deformation retract.

**Example 5.10 (3x3 grid).** Consider the Euclidean cubical complexes $K, K', K''$ from Example 4.8. All three complexes are homotopy equivalent, and both $K', K''$ can be achieved by elementary cubical collapses of $K$. However, only $K'$ is a direct collapse of $K$ according to the definition we introduce in Definition 4.1.

**Example 5.11 (Open Top Box).** We can collapse $L$ to $L'$ using both cubical collapse and directed collapse (see Figure 5).

When extending the idea of cubical collapse to a directed cube, the added information given by the directedness of the space must be taken into account. The preservation of this information leads to the necessity of the additional past link requirement in the definition of directed collapse, a condition not present in the definition of cubical collapse. For example: the undirected space $L$ in Example 2.9 is cubically collapsible to a point through a series of cubical collapses. Using the definition of directed collapsibility, we can also see that $L$ is directed collapsible to
3 edges and 4 vertices, the 2 free vertices being the initial and end vertex. However, no directed collapses can be performed on the space which is the open box without a bottom, because of a past link obstruction, even though the underlying undirected space remains cubically collapsible to a point. This motivates the following example:

**Example 5.12** (Open Bottom Box). We cannot directed collapse $M$ to $M'$ (see Figure 6 on the top) as a past link obstruction is removed at $v_{1,0}$. We also cannot directed collapse $M$ to $M''$ (see Figure 6 on the bottom) as a past link obstruction is added at $v_{1,0}$. By symmetry, we therefore cannot perform any directed collapses of $M$.

Another issue with applying simplicial collapses to a directed space is the creation of unreachable vertices which alter the space of directed paths. In example **Example 4.8** while $K$ and $K''$ are simple homotopy equivalent, the simplicial collapses that create this simple homotopy equivalence create the unreachable vertex $v_{0,2}$. It is this unreachable vertex which causes the directed path spaces for $K$ and $K''$ to differ, i.e. $\tilde{P}_{v_{0,2}}(K) \neq \tilde{P}_{v_{0,2}}(K')$. 

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**Figure 5.** Illustrating Example 5.11. On the left: the open top box. On the right: the directed collapse of one face of the open top box.

**Figure 6.** Illustrating Example 5.12. On the left: the open bottom box. On the top right: A collapse which is not directed; a past link obstruction is removed at $v_{1,0}$. On the bottom right: A collapse which is not directed; the past link obstruction is added at $v_{1,0}$. 

5.3. Bing’s House with Two Rooms: The Directed and Undirected Cases.
We now discuss a topological space commonly referred to as the “house with two rooms” or “Bing’s house”. This topological space is an example of a space that is contractible but not collapsible in the undirected sense. Here, we show that the path space is non-trivial in the directed setting.

Figure 7. Illustrating Bing’s house (far left) as the union of: (1) three horizontal layers, where the outer two layers are homeomorphic to annuli and one to a two-holed annulus, (2) two rectangular chambers with a support wall that connects the chambers and (3) four outer walls. Note that the vertical chambers do not have a top or bottom. We note that, while not shown explicitly, the faces can be decomposed into elementary cubes (axis-aligned squares).

Bing’s house is a two-dimensional subspace of $\mathbb{R}^3$. We consider a particular realization of Bing’s house illustrated in Figure 7; we call this space $B$ and note that $B \subseteq [0, 4] \times [0, 3] \times [0, 2] \subseteq \mathbb{R}^3$. This space consists of the union of three horizontal layers, where two are homeomorphic to annuli and one to a two-holed annulus, two rectangular chambers with a support wall connecting the two chambers, and four vertical walls [6, Ch. 0]. Bing’s house is contractible, since the three-disk $D^3$ deformation retracts onto $B$. However, Bing’s house is not collapsible, as there are no free faces to begin collapsing (see Definition 5.2).

Now, we describe the path space in the directed setting. In Theorem 5.13, we show that the path space for Bing’s house is determined by the middle horizontal layer which is homeomorphic to a two-holed annulus. The main idea resides using the fact that the combinatorial information of the cubical complex such as the holes and deadlocks gives the sufficient amount of information to compute the space of dipaths up to homotopy equivalence [2, Sec. 7.1.2]. So since no new combinatorial information is introduced from Bing’s house after computing the path space of middle horizontal layer, then the space of directed paths of $B$ is indeed the same as the space of directed paths of the middle horizontal layer.

**Theorem 5.13** (Space of Directed Paths of Bing’s House). *The space of directed paths of $B$ with initial vertex $(0, 0, 0)$ and final vertex $(4, 3, 2)$ is the graph that consists of three edges between two vertices.*

**Proof.** We first compute the space of directed paths of the middle horizontal layer denoted as $M \subseteq B$, along with elementary one-cubes that join $M$ with the initial and final vertex; see Figure 8(left), and then show that this space of directed paths determines the space of directed paths of $B$.

In $M$, there are three classes of dipaths. Dipath class, $[a]$, consists of all dipaths that go up and around the hole on the left. Dipath class, $[b]$, consists of all dipaths that intersect the line between the two holes and lastly, dipath class, $[c]$, consists of all dipaths that go below and around the hole on the right. So the space of directed paths of $M$ union with the elementary one-cubes that join $M$ with the initial and final vertex is the graph of three edges between two vertices.

Any dipath that travels around the top chamber in the clockwise direction and then the top horizontal layer is dihomotopic to a dipath in $[a]$. Similarly, any dipath
that travels on the bottom horizontal layer and then around the bottom chamber
in the counterclockwise direction is dihomotopic to a dipath in \([c]\). Any other new
dipaths introduced are dihomotopic to a dipath in \([b]\). Hence, observe that adding
in the vertical chambers and other two horizontal layers to \(M\) does not affect the
space of directed paths.

We next add the walls. Here, the new dipaths introduced wrap around some
portion of the walls in either the clockwise or counterclockwise direction which are
dihomotopic to \([a]\) if clockwise and either \([b]\) or \([c]\) otherwise. Hence, the space of
directed paths is the graph that consists of three edges between two vertices, as
shown in Figure 8 on the right.

![Figure 8. Illustrating the space of directed paths of Bing’s house.](image)
The middle horizontal layer determines the space of directed paths
of Bing’s house since no other new combinatorial information is
introduced through the remaining portions of Bing’s house. Since
the path space of the middle horizontal layer (left figure) consists
of three classes of dipaths denoted as \([a]\), \([b]\), \([c]\), then the space of
directed paths of Bing’s house is the graph that consists of three
edges between two vertices (right figure).

### 6. Comparisons to Existing Notions of Dihomotopy Equivalence

In this section, we give some of the notions of dihomotopy equivalence defined
in the literature (see Definition 6.2 below). Our definition of directed collapsibil-
ity in Section 4 above introduces yet another equivalence of Euclidean complexes
that is weaker than any of the other definition of dihomotopy equivalence, in which
more complexes are equivalent. The feature preserved in a directed collapse is
contractibility or connectedness of past links, with a chosen initial point. Our mo-
tivation for preserving this feature comes from our concurrency application men-
tioned in the introduction. Specifically, preserving contractibility and connected-
ness of past links in directed collapses helps us preserve notions of partial executions.
Below, we primarily compare our notion of dihomotopy equivalence with that of
Thomas Kahl \[8\]. We give explicit examples of Euclidean cubical complexes we
are able to collapse in the directed sense but are unable to collapse in the directed
sense using Kahl’s notions.

**Definition 6.1** *(Directed Interval)*. The *directed interval* is the d-space, \(\vec{I} = (I, d\vec{I})\) where \(d\vec{I}\) consists of increasing maps, \(I \to I\).

**Definition 6.2** *(Dihomotopy \[2\])* . Given d-spaces \(X, Y\) and dimaps \(f, g : X \to Y\), we say \(f\) and \(g\) are *dihomotopic* if there is a dimap \(H : X \times I \to Y\) such that
\(H(x, 0) = f(x)\) and \(H(x, 1) = g(x)\). This is denoted by \(f \sim g\), and such a map \(H\)
is called a *dihomotopy*.
We say \( X, Y \) are **dihomotopy equivalent** if there are dimaps \( f : X \to Y \) and \( g : Y \to X \) such that \( f \circ g \sim id_Y \) and \( g \circ f \sim id_X \).

Furthermore, we can have dihomotopies between directed paths. Given a d-space \( X \), a **dihomotopy** between two directed paths, \( f,g : \overrightarrow{I} \to X \) is an end-point preserving homotopy \( h : I \times \overrightarrow{I} \to X \) from \( f \) to \( g \) such that for all \( t \in [0,1] \), the maps, \( p_t : X \to X \) where \( x \mapsto h(t,x) \) is directed.

Additionally, M. Grandis [5] gives the following definition with notation as above:

**Definition 6.3 (D-homotopy).** Let \( f \) and \( g \) be as in Definition 6.2. Then \( f \) and \( g \) are **d-homotopic** if there is a dimap, \( H : X \times \overrightarrow{I} \to Y \). Such a map is called a **d-homotopy**. D-homotopy equivalence is the transitive, symmetric closure of this relation.

In [8, Def. 3.1], Thomas Kahl’s notion of dihomotopy equivalence is more restrictive: \( f : X \to Y \subset X \) is a **dihomotopy equivalence relative to** a subspace \( A \subset Y \subset X \) if there exists a d-map \( g : Y \to X \) such that \( g(a) = a \) for all \( a \in A \), and there exists dihomotopies, \( H_X, \) between \( g \circ f \) and \( id_X \), and \( H_Y, \) between \( f \circ g \) and \( id_Y \) such that \( H_X(-,t) \) coincides with \( g \circ f \) and \( id_X \) on \( A \), and similarly, \( H_Y(-,t) \) coincides with \( f \circ g \) and \( id_Y \) on \( A \). Kahl then goes on to define directed collapses on two-dimensional cubical complexes, which are required to induce relative dihomotopy equivalence with respect to \( A \), a set of initial and final vertices.

Our definition of directed collapse does not require any of the above dihomotopy equivalences. If \( K' \) is a directed collapse of \( K \), there is an inclusion map \( i : K' \hookrightarrow K \), but we do not require even a dimap \( K \to K' \). As a consequence, there is a difference between our notion of collapsing (in the directed sense) and Kahl’s, regarding what cubical complexes can be collapsed in the directed setting. Below are two examples of a directed collapse that does not result in a cubical complex that is dihomotopy equivalent.

**Example 6.4 (3x3 grid).** Consider the Euclidean cubical complexes \( K, K', K'' \) from Example 4.8. Neither \( K' \) nor \( K'' \) are dihomotopy equivalent to \( K \) since there is a directed path from \( v_{1,3} \) to \( v_{2,3} \) in \( K \) but no directed paths from \( v_{1,3} \) to \( v_{2,3} \) in \( K' \), and similarly since there is a directed path from \( v_{1,0} \) to \( v_{2,0} \) in \( K \) but no directed paths from \( v_{1,0} \) to \( v_{2,0} \) in \( K'' \). This shows that directed collapse does not necessarily result in a dihomotopy equivalent complex. However, collapsing the corner vertices \( v_{3,0} \) and \( v_{0,3} \) with the corner squares \([v_{2,0},v_{1,1}]\) and \([v_{0,2},v_{1,3}]\), respectively, does result in complex dihomotopy equivalent to \( K \).

**Example 6.5 (Open Top Box).** Consider the Euclidean cubical complexes \( L \) and \( L' \) from Example 5.11. The open top box, \( L \) contains the directed collapsing pair, \([(1,0,1),(1,1,1)],[(1,0,0),(1,1,1)])\) showing how to get from \( L \) to \( L' \). However, we are not allowed to collapse \( L \) to \( L' \) using Kahl’s notions, since the dihomotopy type is not preserved.

Common to all previous notions of dihomotopy equivalence are dimaps back and forth. On the other hand, in our setting, directed collapsing equivalence is not directed homotopy equivalence in any of the notions listed here. This allows us to collapse more than e.g. Kahl, and by Theorem 4.6 if \( K' \) is a directed collapse of \( K \) with respect to \( v \) and \( K' \) has trivial spaces of directed paths from \( v \), then so does \( K \). Similarly, if all such spaces of directed paths are connected in \( K' \), this is true in \( K \). In summary, unlike other notions mentioned in the literature, our directed collapsing equivalence does not preserve common existing notions of directed homotopy equivalences. Because of this, we are able to collapse more complexes and study more types of concurrent systems than we would be able to otherwise.
7. Discussion

Directed topological spaces have a rich underlying structure and many interesting applications. The analysis of this structure requires tools that are not fully developed, and a further investigation into these methods will lead to a better understanding of directed spaces. In particular, the development of these notions, such as directed collapse, may lead to a better understanding of equivalence of directed spaces and their spaces of directed paths.

There are many future avenues of research that we hope to pursue in the directed topological setting. First, we hope to find necessary and sufficient conditions for a pair of cubical cells \((\tau, \sigma)\) to be a collapsing pair. The key will be to have a better understanding of what removing a cubical cell does to the past link of a complex. In addition to this, as there are many types of simplicial collapse, it would be interesting to see what the directed counterpart to each of these types of collapses might be. For example, is there a notion of strong directed collapse? As strong collapse also considers the link of a vertex, a consideration of how this extends to a directed setting seems natural.

Next, there is more to learn about past link obstructions. It is clear performing a directed collapse will not alter the space of directed paths of a Euclidean cubical complex; however, if we are unable to perform a directed collapse due to a past link obstruction, what does this say about the space of directed paths? In what way are these obstructions realized as disconnected or non-contractible spaces of directed paths?

Another direction of research we hope to pursue is defining a way to compute a directed homology that is collapsing invariant. Even in the two-dimensional setting (where the cubes are at most dimension two), this has proved to be difficult, as adding one two-cell can have various effects depending on the past links of the vertices involved. We would like to classify the spaces where such a dynamic programming approach would work.

Lastly, there are many computational questions on how to implement the collapse of a directed cubical complex. In [9], an example of collapsing a three-dimensional cubical complex is implemented in C++. This could be used as a model when handling the directed complex.

Directed topology is a rich field with lots of interesting theoretical and computational questions. We hope that our research excites others in studying cubical complexes in the directed setting.
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Appendix A. Omitted Proofs

A.1. Two Proofs of Theorem 3.2 In this Section we will give two different proofs of Theorem 3.2.

Theorem. (Theorem 3.2) With $K$ as above, suppose all past links of all vertices are connected. Then, all spaces of directed paths $\overline{P}_k$ are connected.

In the first proof below we show that [10 Prop. 2.20] is an equivalent condition to all past links being connected.

Proof. In [10 Prop. 2.20], a local condition is given that ensures that all spaces of directed paths to a certain final point are connected. Here, we explain how the local condition is equivalent to all past links being connected. Their condition is in terms of the local future; however, we reinterpret this in terms of local past instead of local future. Since we consider all spaces of directed paths from a point (as opposed to to a point), this is the right setting we should look at. The local condition is the following: for each vertex, $v$ and all pairs of edges $[v-e_i,v]$, $[v-e_s,v]$ in $K$, there is a sequence of two-cells $[v-e_{k_i},v]$, $i=1,\ldots,m$ all in $K$ s.t. $l_i=k_{i+1}$, $i=1,\ldots,m-1$, $k_1=r$ and $l_m=s$. Now, we show that this local condition is equivalent to ours. In the past link considered as a simplicial complex, such a sequence of two-cells corresponds to a sequence of edges from the vertex $r$ to the vertex $s$. For $x,y \in lk^{-}(v)$, they are both connected to a vertex via a line. And those vertices are connected. Hence, the past link is connected.

Vice versa: Suppose $lk^{-}(v)$ is connected. Let $p,q$ be vertices in $lk^{-}(v)$ and let $\gamma : I \to lk^{-}(v) \in \Delta^{n-1}$ be a path from $p$ to $q$. The sequence of simplices traversed by $\gamma$, $S_1, S_2, \ldots, S_k$, satisfies $S_i \cap S_{i+1} \neq \emptyset$. Moreover, the intersection is a simplex. Let $p_i \in S_i \cap S_{i+1}$. A sequence of pairwise connected edges connecting $p$ to $q$ is constructed by such sequences from $p_i$ to $p_{i+1}$ in $S_{i+1}$ thus providing a sequence of two-cells similar to the requirement in [10]. Hence, by [10], if all past links of all vertices are connected, then all $\overline{P}_k$ are connected. ∎
This second proof of Theorem 3.2 has a more categorical flavor. The key of the proof is that the colimit of connected spaces over a connected category is connected.

**Proof.** We give a more categorical argument which is closer to the proof of Theorem 3.1. In [11, Prop 2.3 and Equation 2.2], the space of directed paths $P_0^k$ is given as a colimit over $P_0^{k-j}$. The indexing category is $J_K$ with objects $\{j \in \{0,1\}^n : [k-j] \subseteq K\}$ and morphisms $j \to j'$ for $j \geq j'$ given by inclusion of the simplex $\Delta^j \subset \Delta^{j'}$. The geometric realization of the index category is the past link which with our requirements is connected. The colimit of connected spaces over a connected category is connected. Hence, by induction as above, beginning with edges from $0$, $P_0^{k-j}$ are all connected and the conclusion follows. \qed