Adaptive radial basis function generated finite-difference (RBF-FD) on non-uniform nodes using $p$—refinement

Pankaj K Mishra\textsuperscript{a,1}, Leevan Ling\textsuperscript{b,2}, Xin Liu, Mrinal K Sen

\textsuperscript{a} Institute for Geophysics, The University of Texas at Austin
\textsuperscript{b} Department of Mathematics, Hong Kong Baptist University, Hong Kong

\textsuperscript{1} pankajkmishra01@gmail.com, \textsuperscript{2} lling@hkbu.edu.hk

Abstract

Radial basis functions-generated finite difference methods (RBF-FDs) have been gaining popularity recently. In particular, the RBF-FD based on polyharmonic splines (PHS) augmented with multivariate polynomials (PHS+poly) has been found significantly effective. For the approximation order of RBF-FDs’ weights on scattered nodes, one can already find mathematical theories in the literature. Many practical problems in numerical analysis, however, do not require a uniform node-distribution. Instead, they would be better suited if specific areas of domain, where complicated physics needed to be resolved, had a relatively higher node-density compared to the rest of the domain. In this work, we proposed a practical adaptive RBF-FD with a user defined order of convergence with respect to the total number of (possibly scattered and non-uniform) data points $N$. Our algorithm outputs a sparse differentiation matrix with the desired approximation order. Numerical examples are provided to show that the proposed adaptive RBF-FD method yields the expected $N$-convergence even for highly non-uniform node-distributions. The proposed method also reduces the number of non-zero elements in the linear system without sacrificing the accuracy.

1. Introduction

Radial basis functions (RBFs) have been a vital choice in the development of kernel-based meshless methods for solving partial differential equations (PDEs) numerically. The RBF-FD method is one such local meshless method, which has been gaining popularity recently. Numerical evidences collected in recent years by a rapidly growing community of researchers suggest that the RBF-FD method offers numerical stability on irregular node layouts, high
computational speed, high accuracy, easy local adaptive refinement, and excellent opportunities for large-scale parallel computing.

The basic idea of RBF-based differentiation can be traced back to Tolstykh and Shirobokov [2003] and formally as the RBF-FD to Wright [2003]. Since then, a significant amount of research has been dedicated towards the robust development of the RBF-FD [Wright and Fornberg, 2006; Bayona et al., 2010; Fornberg and Lehto, 2011; Fornberg and Flyer, 2015; Flyer et al., 2016; Bayona et al., 2017; Shankar, 2017; Santos et al., 2017; Petras et al., 2018; Mishra et al., 2018] and its application to solve various problems in science and engineering [Chandhini and Sanyasiraju, 2007; Chinchapatnam et al., 2009; Flyer et al., 2012; Shankar et al., 2015; Martin et al., 2015; Martin and Fornberg, 2017; Mishra et al., 2017; Kindelan et al., 2018; Slak and Kosec, 2018; Martin et al., 2018]. A major advantage of the RBF-FD method is that it works with scattered nodes, where each stencil can have a different configuration. This overcomes the fixed grid/element limitation of conventional numerical methods for solving PDEs. Unlike global meshless methods, the weights in the RBF-FD are computed locally using RBFs, which are expanded at some fixed number of nearest nodes only. Once the weights at each node are computed, they can be stored and used for next-step computation, which makes weight computation a pre-processing step in solving PDEs with RBF-FD. Moreover, computations of weights at each node are independent processes, which makes RBF-FD a desired method for parallel computing.

Recently, the combination of polyharmonic spline RBFs augmented with high-order polynomial (PHS+Poly) in the RBF-FD formulation has been shown to be achieve significant robustness [Flyer et al., 2016; Bayona et al., 2017]. Some key benefits of such an approach is that the PHS+Poly approach

1. does not depend on a free shape parameter, which leads to a simplified formulation without any needs of fine-tuning,
2. is significantly cheaper than using stable algorithms on infinitely smooth RBFs to formulate RBF-FD [Santos et al., 2017],
3. does not require ghost-nodes and works efficiently for approximations near domain boundaries, where the stencils become highly one-sided [Bayona et al., 2019], for ensuring the accuracy near boundaries.
4. has the potential of maintaining the accuracy for large-sparse linear systems, and
5. has order of convergence depending mostly on the highest degree of augmented polynomial, which also dictates the stencil size.

In a structured node-layout, the order of the convergence has been numerically established; however, the convergence of the RBF-FD over non-uniform nodes needs some careful considerations. We propose an adaptive RBF-FD algorithm subject to a user-defined global order of convergence with respect to the number of nodes in the domain. In particular, we provide an understanding the order of convergence in a (potentially highly) non-uniform node-layout with significantly large mesh ratio\(^1\). In the light of above listed benefits, we will

\(^1\)The ratio of fill-distance over minimum separating distance, whose role is similar to the mesh-ratio in the context of finite element methods
use the PHS+Poly approach adaptively in the present work so that we can control the order of convergence with a single variable, viz., the degree of augmented polynomial. From here and on, by the term RBF-FD, we mean the RBF-FD, formulated using PHS + polynomials approach. To generate such non-uniform and adaptive nodes we use a downloadable package—NodeLab [Mishra, 2019] by the present author.

The rest of the paper is organized as follows. We discuss the general formulation of the RBF-FD, polynomial augmentation and some insights for its stable implementation in Section 2. In Section 3, we give a precise definition of what we meant by the global order of the convergence in an non-uniform node-layout. We also provide a formulation that connects the global order of convergence to the required degree of polynomial at a local scale depending on the local fill distance. This is followed by a discussion of the weight computation approach through the adaptive RBF-FD method. Finally, We provide numerical examples to support the advantage of the proposed method followed by conclusions.

2. RBF-FD approximation

As a review, in the following subsections, we discuss the standard RBF-FD discretization and some insights to obtain its stable formulation.

Consider a computational domain Ω, within which we have distinctly scattered nodes $X := \{x_1, x_2, ..., x_N\} \in \mathbb{R}^d$. For some linear differential operator $L$ and $1 \leq c \leq N$, we want to compute RBF-FD weights $w_c := \{w_c^1, w_c^2, ..., w_c^N\}$ for approximating the values of $Lu$ at some center point $x_c \in X$ using function values of $u$ at some subset of $X$, that is commonly referred to as the stencil for $x_c$ consisting neighbor nodes in $X$ near and including $x_c$. The number of nonzero elements in $w_c$, denoted by $n_c^w$, is the stencil-size of $x_c$. With an index function, we denote the stencil of $x_c$ by $X_c := \{x_i^c, ..., x_n^c\} \subseteq X$ and we want to have

$$Lu(x_c) \approx \sum_{k=1}^{n_c^w} w_c^k u(x_c^k). \tag{1}$$

At this stage, the task is to compute the weight $w_c$ that gives a good approximation in the sense of (1).

With some radial basis function $\phi : [0, \infty) \to \mathbb{R}$ chosen, the standard RBF-FD approach is to compute these weights by solving the following linear system

$$\begin{bmatrix} \phi(||x_i^c - x_j^c||) \end{bmatrix}_{1 \leq i, j \leq n_c^w} w_c = \begin{bmatrix} L\phi(||x - x_i^c||)|_{x=x_c} \end{bmatrix}_{1 \leq i \leq n_c^w} =: A_c w_c,$$

$$=: L\phi|_{x=x_c}. \tag{2}$$

If $\phi$ is symmetric positive definite (SPD), then the interpolation matrix $A_c$ must also be SPD and hence (2) is uniquely solvable. It is shown in Schaback [2014] that (2) is error optimal in the native space norm corresponding to the radial basis function $\phi$. We refer readers to the article for details.
2.1. RBF-FD (PHS+Poly) approach

Unlike global meshless methods, the local nature of the RBF-FD rules out the exponential\(^2\) convergence even with infinitely smooth RBFs. With the motivations discussed in Schaback [2017] and earlier in this paper, using a conditionally positive definite kernel with finite order of smoothness, like PHS (\(\phi(r) = r^{2m+1}; m = 1, 2, 3, \ldots\)), is reasonable in the RBF-FD method. We also augment polynomial basis of sufficient degrees in order to ensure exact reproduction of low orders polynomials. For computing RBF-FD weights, (1) needs to be modified for the corresponding linear expansion

\[
I(x) = \sum_{k=1}^{n^c} \lambda_k \phi(||x - x_k||) + \sum_{k=1}^{n^c} \gamma_k p_k(x)
\]

subject to the constraints

\[
\sum_{k=1}^{n^c} \gamma_k p_l(x_k) = 0, \quad l = 1, 2, 3, \ldots, n^c_p.
\]

\(^2\)for the approximation of a function lying in a certain native space
Polynomial degree \((p)\) & Polynomial basis & \(n_p = C(p + 2, 2) = \frac{(p+2)!}{p!2}!\) \\
\hline
0 & 1 & 1 \\
1 & 1, x, y & 3 \\
2 & 1, x, y, x^2, y^2, xy & 6 \\
3 & 1, x, y, x^2, y^2, xy, x^3, y^3, x^y, xy^2 & 10 \\
4 & \ldots & 15 \\
5 & \ldots & 21 \\
\vdots & \vdots & \vdots \\
\hline

Table 1: Understanding the number of polynomial terms based the degree of polynomial to be augmented.

where \(n_s^c\) is again the stencil-size of \(x_c\) and \(n_p^c\) is number augmented multivariate polynomial basis, see Table 1 and Flyer et al. [2016] for more details in 1D and 3D. The augmented version of the linear system (2) is given as

\[
\begin{bmatrix}
A_c & P_c^T & O \\
P_c & O
\end{bmatrix}
\begin{bmatrix}
w \\
\end{bmatrix} =
\begin{bmatrix}
L\phi_{|x_c} \\
Lp_{|x_c}
\end{bmatrix},
\]  

(5)

where \(A_c\) and \(L\phi_{|x_c}\) were as defined in (2). The \(n_s^c \times n_p^c\) matrix \(P_c\) and \(n_p^c \times 1\) vector \(Lp_{|x_c}\) are given as

\[
P_c = [p_j(x^c_i)]_{1 \leq i \leq n_s^c, 1 \leq j \leq n_p^c}, \quad \text{and} \quad Lp_{|x_c} = [Lp_k(x)|_{x=x_c}]_{1 \leq k \leq n_p^c}.
\]

From here and on, we drop the super/subscript “\(c\)” from our notations for simplicity and all computations need to go center by center for all \(x_c \in X\). The number of polynomial basis \((n_p)\) for the space of \(d\)-dimensional polynomials with up to and including degree \(p\) is given by

\[n_p = C(p + d, d) = \frac{(p + d)!}{p!d!}.
\]

Table 1 lists out the number of polynomial terms for different degrees of polynomials, for a two-dimensional case \((d = 2)\). For example, \(n_p = 3\) in 2D, corresponds to appending polynomial up to first order. Equation (5) can be interpreted as an equality-constrained quadratic programming problem (see Bayona et al. [2017] for more details on this).

2.2. Stencil-size and Stagnation error

It has been found [Flyer et al., 2016; Bayona et al., 2017] that, the order of accuracy of RBF-FD was governed by the degree of augmented polynomial \((p)\); while the order of the PHS RBF has a marginal effect on the accuracy and no effect on the order of convergence. Moreover, when the augmented polynomial is of high-degree (and fixed), increasing \(n_s\) — beyond what is required\(^4\), does not improve the accuracy or the order of convergence, significantly. In case the augmented polynomial is of lower-order, increasing \(n_s\) could lead to

\(^4\)to support the bases of the augmented polynomial (i.e. \(n_s = n_p\))
marginal improvement in the accuracy; however, the convergence order practically remains unaffected. Such an specific independence on the stencil size becomes vital for the adaptivity. That is, in sub-domains having higher node-density, we could use a smaller degree of augmented polynomial (and therefore smaller $n_s$) — locally, without any significant loss in the convergence.

The above criterion for choosing the stencil-size, however, is recommended for node-distributions having ghost-nodes outside the boundary. The recommended rule of thumb for ensuring the stability of RBF-FD, without any special boundary treatment is: $n_s \geq 2n_p$

to:

1. get the expected order of accuracy everywhere in the domain including the evaluation points near boundaries, where stencils are highly one-sided with respect to the evaluation point.
2. maintain the numerical-stability for the sparse linear system for large-scale problems.

The stagnation error corresponds to the disrupt in the convergence under node-refinement, which is a limitation in the RBF-FD formulation with pure RBF without any polynomial augmentation. The stagnation error, however, has different interpretations with respect to GA and PHS RBF as explained in Flyer et al. [2016]. In a purely PHS-based RBF-FD, the stagnation error is mainly due to the large boundary errors. However, if one follows above thumb rules explained in this subsection, with proper polynomial augmentation, the RBF-FD formulation remains free from stagnation error.

3. Non-uniform nodes & Convergence

For quasi-uniform node distribution, the convergence of an RBF-FD method can be estimated in a straightforward manner, that is, by plotting the numerical error against the fill-distance ($h$) of $X$. According to numerical tests performed in [Flyer et al., 2016; Bayona et al., 2017], for a quasi-uniform nodes the order of convergence of a RBF-FD method is $O\left(h^{p-k+1}\right)$, where $p$ is the polynomial degree\(^4\) and $k$ is the order of operator being approximated. However, for a highly non-uniform node-distribution, the relationship of $h$ and $N$ is different from case to case and determining the order of the convergence is not as straightforward. In what follows, we shall establish a definition of a global convergence with respect to an effective fill distance by formulating an adaptive RBF-FD approach.

Given a set of points $X = \{x_1, \ldots, x_N\} \subset \mathbb{R}^d$, its fill distance is given by

$$h_{X, \Omega} := h_X = \sup_{x \in \Omega} \min_{x_j \in X} \|x - x_j\|_{L_2(\mathbb{R}^d)},$$

the separation distance is

$$q_X := \frac{1}{2} \min_{i \neq j} \|x_i - x_j\|_{L_2(\mathbb{R}^d)},$$

\(^4\)In our context, this degree refers to the fact that all polynomial basis with degree up to $p$ are augmented
and $\rho_X = h_X/q_X \geq 1$ is the mesh ratio of $X$. Quasi-uniform means that, as $X$ refines, $\rho_X$ is bounded above for all $N$. A usual estimate is: $N \sim q_X^d$ for quasi-uniform sets; notation $\sim$ means proportional with some constant.

Adaptive node distributors (we proposed, or generally) end up with (a sequence of) quasi-uniform sets of data points. These sets are expected to have large mesh ratio $\rho$, which, to users’ eyes, are nonuniform and being denser in some problematic spatial region. By the definition of the fill distance, there must be at least one stencil $(S_i)$ in $\Omega$ in the coarse region with local fill distance being exactly $h_{X,\Omega_i} = h_{X,\Omega}$. If we employ a $p$-order FD approximation scheme there, the resulting local error will refine as

$$\varepsilon \sim h_X^p = (\rho_X q_X)^p \sim \rho_X^p N_X^{-p/d}.$$  

For uniform node (with $\rho_X = 1$), the last factor $N_X^{-p/d} (\sim h_e)$ is exactly what we expect from a $p$-order scheme. In cases when $\rho_X \gg 1$, the $N$-asymptotic convergence rate remains $p$ but come with a huge constant $\rho_X^p$. We propose to improve convergence rate with $N_X$ non-constant orders $p_i$ for each node so that

$$h_{X,\Omega_i}^p \approx h_{e,\Omega}^g,$$

where $h_e$ is the effective fill distance, that is, as number of nodes per unit volume as given by

$$h_{e,\Omega} = \left( \frac{Vol(\Omega)}{N} \right)^{1/d},$$

where $N$ is total number of nodes. Our next task is to determined the degree of augmented polynomials $p_i = p_i(h_{X,\Omega}, h_{e,\Omega}, g)$ to be used in $\Omega_i$ based on the local fill distance ($h_{X,\Omega_i}$), effective fill distance ($h_{e,\Omega}$), global order of convergence ($g$), and the order of the operator ($k$).
3.1. Adaptive RBF-FD approach

Adaptive methods are quite popular in the context of Finite/Spectral Element Methods (FEM), which have give many version of the algorithm such as $h$-FEM, $p$-FEM, or $hp$-FEM. The idea behind adaptivity is to vary one or more parameters affecting the accuracy of the algorithm, locally, to meet a certain order of accuracy throughout the domain.

In what follows, we propose a strategy for locally choosing the degree of polynomial ($p$) to be augmented at a local scale while approximating a $k^{th}$ order derivative.

\[ p_i = g + k - 1 + \log_{10} \left( \frac{h_{X,\Omega_i}}{h_{e,\Omega}} \right) \]  

Equation (8)

For quasi-random node-layout ($h_{X,\Omega_i} \approx h_{e,\Omega}$), Equation (9) simplifies to

\[ p = g + k - 1 \quad \text{or} \quad g = p - k + 1 \]  

Equation (9)

that is, the order of an RBF-FD algorithm will be $O(h^{p-k+1})$, which is consistent with the existing literature [Flyer et al., 2016; Bayona et al., 2017].

Once we know the required value of $p$ for a local stencil, we can determine the stencil size just enough to make the algorithm stable. Since ‘more than required’ stencil size does not add to the accuracy — in a region with high-node density, reduced value of $p$ allows us to pick a smaller stencil, see Figure (2), without significantly affecting the accuracy or stability of the RBF-FD discretization. Since, the computational cost of the RBF-FD depends on the stencil-size, a smaller stencil size in the dense part of the domain, will correspond to a speed-up in the algorithm. Algorithm 2 is a pseudo-code for computing and using $N$—adaptive RBF-FD weights.
Algorithm 2: Pseudo-code for computing and using $N$-adaptive RBF-FD weights

```plaintext
1: procedure Input($g \in \mathbb{Z}$, $x \in \mathbb{R}^d$) \hspace{1cm} \triangleright Input expected order and nodes
2: Compute total number of nodes $N$
3: Compute $h_{e,\Omega} = \frac{\text{Vol}(\Omega)}{\sqrt{N}}$
4: Fix a maximum stencil size $n_m$
5: Find $n_m$ nearest neighbors for each $x_i$ \hspace{1cm} \triangleright We have used knn-search
6: Index stencil points as a vector $x_s = (x_s, y_s), x_s \in \mathbb{R}^{d \times n_m}$
7: for (loop over all the interior nodes) do
8: \hspace{1cm} $[w] = \text{Weight2D}([x, y], g, (x_s, y_s), h_e)$ \hspace{1cm} \triangleright Computing Weights for one interior node
9: \hspace{1cm} $p_i = g + k - 1 + \log_{10} \left(\frac{h_{X,\Omega}}{h_{e,\Omega}}\right)$ \hspace{1cm} \triangleright Polynomial degree required to maintain order $g$
10: \hspace{1cm} $n_p = C(p + d, d) = \frac{(p + d)!}{p!}$ \hspace{1cm} \triangleright Number of polynomial basis
11: \hspace{1cm} $n_s = 2n_p + 1$ \hspace{1cm} \triangleright Stencil size just enough to support the polynomial $p$
12: \hspace{1cm} $r = \sqrt{(x^2 + y^2)}$
13: \hspace{1cm} $A_0 = r^m$ \hspace{1cm} \triangleright The RBF interpolant
14: \hspace{1cm} $\text{RHS}0 = [\text{rhs}_L]$ \hspace{1cm} \triangleright This is the RHS column in equation (1).
15: \hspace{1cm} $A = A_0 + \text{Poly}(p)$ \hspace{1cm} \triangleright The RBF interpolant after augmenting polynomial of degree $p$
16: \hspace{1cm} $\text{RHS} = \text{RHS}0 + \text{Poly}(p)$ \hspace{1cm} \triangleright Adding corresponding polynomial of degree $p$
17: \hspace{1cm} $w_L = A/RHS$ \hspace{1cm} \triangleright Solve the linear system for weights
18: \hspace{1cm} $w_L = w(1 : n_s)$ \hspace{1cm} \triangleright Ignore the weights corresponding to polynomial augmentation
19: end
20: for (loop over all the interior nodes) do
21: \hspace{1cm} $L(f) = w_L f$
22: end
23: end
```

4. Example: Solving PDEs with adaptive RBF-FD

In this test, we investigate the adaptive RBF-FD for solving a partial differential equation (PDE). The test problem has been set up as described below.

\[
\begin{align*}
\nabla^2 u(x, y) &= f(x, y), & (x, y) &\in \Omega \\
u(x, y) &= \sin(x^2 + y), & (x, y) &\in \Gamma \setminus \Gamma_4 \\
\frac{\partial u(x, y)}{\partial n} &= \cos(x^2 + y), & (x, y) &\in \Gamma_4,
\end{align*}
\]

(10)

where $\Omega = (-1, 1) \times (-1, 1)$ and $\Gamma_4$ is the boundary having $y = 1$. The analytical solution and the source term are given, respectively, as

\[
\begin{align*}
u(x, y) &= \sin(x^2 + y) \\
f(x, y) &= 2 \cos(x^2 + y) - (4x^2 + 1) \sin(x^2 + y)
\end{align*}
\]

(11)

The aim of this numerical test is to demonstrate that, in certain situations, an adaptive node-distribution may provide a more accurate solution with the same number of nodes.
Figure 3: $N = 2500$ (a) quasi-uniform, and (b) non-uniform distributed nodes in $\Omega = (-1, 1)^2$.

and in such cases the proposed RBF-FD will provide an efficient scheme at a relatively lower cost.

Figure 3 shows 2500 number of interior nodes in $\Omega$ having quasi-uniform and non-uniform point-distribution. Non-uniform distribution has been obtained by using the transformation $z = (\sin \frac{\pi}{2}(z))$, in each coordinate. Figure 5 shows the sparsity pattern and eigenvalues of the globally assembled $A$-matrix through RBF-FD discretization of the test problem. Histograms in the Figure 6, show the frequency of polynomial degree chooses during overall weight computation. For quasi-uniform nodes, the polynomial degree remains fixed, which equals to the global order $g$. For non-uniform nodes, however, the polynomial degree was selected from a range 4 to 8, depending on the local node-density.

The adaptive RBF-FD, however decreases the polynomial degree (and hence the stencil size) in the regions having higher node-density, which in turn, reduces the total number of nodes in the $A$-matrix by 43292 (i.e., around 18% cheaper) and yet does not compromise with the accuracy (Figure 4).
Figure 4: Comparison of the convergence of RBF-FD with quasi-uniform nodes and the adaptive RBF-FD with non-uniform nodes. The adaptive RBF-FD gives similar accuracy while maintaining a similar (desired) convergence and at the same time, it leads to a more sparse system matrix. For example, with $N = 2500$, the number of non-zero elements in the system matrix and the corresponding error for the adaptive RBF-FD are $(nz = 188724, E = 6.68e-08)$ while the same for the standard RBF-FD are $(nz = 232016, E = 1.3e-07)$.
Figure 5: Sparsity pattern and eigenvalue spectra of the system matrix of adaptive RBF-FD for uniform (a, b) and non-uniform (c, d) node distribution, respectively.
5. Benchmark

Next, we consider a benchmark test problem, designed to test adaptive FEM algorithms, by US National Institute for Standards and Technology (NIST) [Mitchell, 2013]. The test problem is an elliptic PDE (Poisson equation) in \((0,1)^2\), which has an exponential peak in the domain and non-constant Dirichlet boundary conditions matching the exact solution — given by

\[- \Delta u(x,y) - f(x,y) = 0 \quad \text{in} \quad \Omega = (0,1)^2,\]  

and the exact solution is

\[u(x,y) = e^{-\alpha(x-x_c)^2 + (y-y_c)^2},\]  

where \((x_c, y_c)\) is the location of the peak and \(\alpha\) determines the strength of the peak. A typical value of \(\alpha = 1000\) is suggested to test an adaptive algorithm. The right hand side \((f)\) satisfying the exact solution is given as

\[f(x,y) = 4e^{-\alpha((x-x_c)^2 + (y-y_c)^2)} \left(\alpha^2(x-x_c)^2 + \alpha^2(y-y_c)^2 - \alpha\right)\]  

A reference solution for \(\alpha = 1000\) has been shown in the Figure 7. For PDE-dependent node generation, we used NodeLab. First, we shall try to solve this problem with standard RBF-FD approach over quasi-uniform nodes in the domain and a fixed polynomial degree \((d = 8)\). Figures 8(a)–(c) show the node-distribution \((N = 2470)\), approximated solution and point-wise absolute error in the approximated solution, respectively. It is evident from
the error plot that the approximation error is maximum in the peak zone of the sub-domain, and an adaptive node-distribution, which is relatively finer in this zone would potentially enhance the accuracy in the approximation. Figures 8(d)–(f) show the node-distribution ($N = 2470$), approximated solution, and point-wise absolute error in the approximated solution with an expected global order ($g = 8$).

The advantage of using an adaptive RBF-FD is twofold (i) it resolves the physics of the problem with a much better accuracy and (ii) the number of non-zero elements in the $A$ matrix of the resulting system are relatively smaller because it uses smaller stencils (because lower than 8 order of polynomials) in the sub-domains having higher node-density. The advantage (ii) would become more important for solving large-scale problems using sparse linear solvers.
Figure 8: The first row shows the (a) node-distribution, (b) approximated solution, and (c) pointwise absolute error in the approximated solution, respectively with standard PHS+poly approach with $N = 2470$ and polynomial degree $p = 8$. The second row shows the same for adaptive RBF-FD with $N = 2470$ and variable polynomial degree $p$ corresponding to a desired order $g = 8$ in (d)–(f), respectively.
Figure 9: Comparison of convergence of the standard RBF-FD and the Adaptive RBF-FD for a Poisson equation with delta-like exact solution.
6. Conclusions

In this paper we suggest an adaptive version of RBF-FD (PHS+poly) approach, which is best suited for the problems requiring high-order approximation over highly non-uniform node-distributions in the domain. When the node-distribution is uniform, the adaptive RBF-FD method eventually simplifies to the standard RBF-FD approach. We define a global order of convergence of the adaptive RBF-FD over non-uniform node-distributions. Through the numerical tests performed in this paper, it has been shown that, for a fixed number of nodes in the domain, the adaptive RBF-FD with non-uniform nodes distributions, provides a better numerical approximation than the standard RBF-FD with quasi-uniform nodes and at the same time significantly reduces the number of non-zero elements in the resultant linear system, suggesting a cheaper computational cost. The adaptive RBF-FD approximately maintains a user defined global order of convergence even for non-uniform nodes. We are currently applying the proposed approach for seismic modeling problems in geophysics, which will be updated in this preprint in coming weeks.

References

Victor Bayona, Miguel Moscoso, Manuel Carretero, and Manuel Kindelan. RBF-FD formulas and convergence properties. *Journal of Computational Physics*, 229(22):8281–8295, 2010.

Victor Bayona, Natasha Flyer, Bengt Fornberg, and Gregory A Barnett. On the role of polynomials in RBF-FD approximations: II. numerical solution of elliptic PDEs. *Journal of Computational Physics*, 332:257–273, 2017.

Victor Bayona, Natasha Flyer, and Bengt Fornberg. On the role of polynomials in rbf-fd approximations: Iii. behavior near domain boundaries. *Journal of Computational Physics*, 2019. ISSN 0021-9991. doi: https://doi.org/10.1016/j.jcp.2018.12.013.

G Chandhini and YVSS Sanyasiraju. Local RBF-FD solutions for steady convection–diffusion problems. *International Journal for Numerical Methods in Engineering*, 72(3):352–378, 2007.

PBNP Phani Chinchapatnam, K Djidjeli, PB Nair, and M Tan. A compact RBF-FD based meshless method for the incompressible navierstokes equations. *Proceedings of the Institution of Mechanical Engineers, Part M: Journal of Engineering for the Maritime Environment*, 223(3):275–290, 2009.

Natasha Flyer, Erik Lehto, Sébastien Blaise, Grady B Wright, and Amik St-Cyr. A guide to RBF-generated finite differences for nonlinear transport: Shallow water simulations on a sphere. *Journal of Computational Physics*, 231(11):4078–4095, 2012.

Natasha Flyer, Bengt Fornberg, Victor Bayona, and Gregory A Barnett. On the role of polynomials in RBF-FD approximations: I. interpolation and accuracy. *Journal of Computational Physics*, 321:21–38, 2016.

Bengt Fornberg and Natasha Flyer. Solving PDEs with radial basis functions. *Acta Numerica*, 24:215–258, 2015.

Bengt Fornberg and Erik Lehto. Stabilization of RBF-generated finite difference methods for convective PDEs. *Journal of Computational Physics*, 230(6):2270–2285, 2011.

Manuel Kindelan, Diego Álvarez, and Pedro Gonzalez-Rodriguez. Frequency optimized RBF-FD for wave equations. *Journal of Computational Physics*, 2018.

Bradley Martin and Bengt Fornberg. Using radial basis function-generated finite differences (RBF-FD) to solve heat transfer equilibrium problems in domains with interfaces. *Engineering Analysis with Boundary Elements*, 79:38–48, 2017.

Bradley Martin, Bengt Fornberg, and Amik St-Cyr. Seismic modeling with radial-basis-function-generated finite differences. *Geophysics*, 80(4):T137–T146, 2015.
Bradley Martin, Atef Elsherbeni, Gregory E Fasshauer, and Mohammed Hadi. Improved FDTD method around dielectric and PEC interfaces using RBF-FD techniques. In Applied Computational Electromagnetics Society Symposium (ACES), 2018 International, pages 1–2. IEEE, 2018.

Pankaj Mishra. Nodelab: A matlab package for meshfree node-generation and adaptive refinement. Journal of Open Source Software, 4(40):1173, 2019.

Pankaj Mishra, Sankar Nath, Gregory Fasshauer, and Mrinal Sen. Frequency-domain meshless solver for acoustic wave equation using a stable radial basis-finite difference (RBF-FD) algorithm with hybrid kernels. In SEG Technical Program Expanded Abstracts 2017, pages 4022–4027. Society of Exploration Geophysicists, 2017.

Pankaj K. Mishra, Gregory E. Fasshauer, Mrinal K. Sen, and Leevan Ling. A stabilized radial basis-finite difference (rbf-fd) method with hybrid kernels. Computers & Mathematics with Applications, 2018. ISSN 0898-1221. doi: https://doi.org/10.1016/j.camwa.2018.12.027.

William F Mitchell. A collection of 2d elliptic problems for testing adaptive grid refinement algorithms. Applied mathematics and computation, 220:350–364, 2013.

Argyrios Petras, Leevan Ling, and Steven J Ruuth. An RBF-FD closest point method for solving PDEs on surfaces. Journal of Computational Physics, 370:43–57, 2018.

LGC Santos, N Manzanares-Filho, GJ Menon, and E Abreu. Comparing RBF-FD approximations based on stabilized Gaussians and on polyharmonic splines with polynomials. International Journal for Numerical Methods in Engineering, 2017.

Robert Schaback. A computational tool for comparing all linear PDE solvers. Advances in Computational Mathematics, 41(2):333–355, 2014. ISSN 1019-7168. doi: 10.1007/s10444-014-9360-5.

Robert Schaback. Error Analysis of Nodal Meshless Methods, pages 117–143. Springer International Publishing, 04 2017. ISBN 978-3-319-51953-1. doi: 10.1007/978-3-319-51954-8_7.

Varun Shankar. The overlapped radial basis function-finite difference (RBF-FD) method: A generalization of RBF-FD. Journal of Computational Physics, 342:211–228, 2017.

Varun Shankar, Grady B Wright, Robert M Kirby, and Aaron L Fogelson. A radial basis function (RBF)-finite difference (FD) method for diffusion and reaction–diffusion equations on surfaces. Journal of scientific computing, 63(3):745–768, 2015.

Jure Slak and Gregor Kosec. Refined RBF-FD solution of linear elasticity problem. In 2018 3rd International Conference on Smart and Sustainable Technologies (SpliTech), pages 1–6. IEEE, 2018.

AI Tolstykh and DA Shirobokov. On using radial basis functions in a “finite difference mode” with applications to elasticity problems. Computational Mechanics, 33(1):68–79, 2003.

Grady B Wright. Radial basis function interpolation: numerical and analytical developments. PhD thesis, University of Colorado at Boulder, 2003.

Grady B Wright and Bengt Fornberg. Scattered node compact finite difference-type formulas generated from radial basis functions. Journal of Computational Physics, 212(1):99–123, 2006.