Analytical Solutions of the Dielectrophoretic and Travelling Wave Forces Generated by Interdigitated Electrode Arrays

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Abstract. In AC electrokinetics, the application of an AC electric field to a suspension of particles results in the manipulation and separation of the particles also the movement of the fluid. One application is dielectrophoresis (DEP). The second effect is travelling wave dielectrophoresis (twDEP). This paper presents the analytical solutions of the dielectrophoretic and travelling wave forces for the interdigitated electrode arrays energised with either a two- or four-phase signal, respectively. The torque that rotates the particle in the four-phase travelling wave arrays is also analytically solved.

1. Introduction
The study of the movement of the biological particles in the non-uniform alternating current electric field is known as AC electrokinetics [1]. The most frequent used methods include dielectrophoresis (DEP), traveling wave dielectrophoresis (twDEP) and electrorotation (ROT). In the DEP case, two AC voltage signals with phases $0^\circ$ and $180^\circ$ are connected to the bar electrodes, alternatively. In the twDEP case, four signals with phase shift of $90^\circ$ are applied to consecutive electrodes. However, even for this simple bar electrode arrays, there is no strict analytical solution for either the electric field or dielectrophoretic force analysis in the two cases. Several analytical approximations have been demonstrated using series expansions, i. e. Green’s theorem [2] and Fourier series [3]. Later, the numerical solutions of the potential, electric field, DEP force in the DEP and twDEP electrode arrays as well as the electrorotational torque in the twDEP case have been reported by Green et al [4].

In this paper, we present the analytical solutions for the electric field distribution and DEP force, twDEP force in the two cases as well as the ROT torque in the twDEP case. The solutions are based on the accurate analytical electric field expressions in the system [5].

2. Theory
According to AC electrokinetics [1], the DEP force, twDEP force and ROT torque are proportional to the electric field components in the system:

$$\langle F_{\text{DEP}} \rangle \propto \nabla |\mathbf{E}|^2$$ (1)
$$\langle F_{\text{twDEP}} \rangle \propto \nabla \times \left( \text{Re}[\mathbf{E}] \times \text{Im}[\mathbf{E}] \right)$$ (2)
$$\langle \Gamma \rangle \propto \text{Re}[\mathbf{E}] \times \text{Im}[\mathbf{E}]$$ (3)
where $|\mathbf{E}|^2$ is the magnitude of the electric field phasor. $\text{Re}[]$ and $\text{Im}[]$ indicates the real and imaginary part of, respectively. Equation (1) indicates that the DEP force experienced in a non-uniform electric field. Equation (2) shows the twDEP force experienced in a spatially varying phase electric field. Equation (3) shows that the ROT will occur in any electric field with a spatially dependent phase.

3. The Dielectrophoretic Array

Due to the symmetry, a basic cell in the DEP array has been selected for analysis, as shown in figure 1 (a). The analytical solution of the electric field distribution in the selected cell is solved by using Schwarz-Christoffel mapping method [5].

$$
E_{zd} = \frac{V}{h} \frac{K(k_{d1})}{K'(k_{d2})} \left[ t_a(t_E-t_B)(t-1) \right]^{1/2} - \left[ t_B(t_E-1)(t-t_a) \right]^{1/2}
$$

where $K(k_{d1})$ and $K'(k_{d2})$ are the modulus of the elliptic function. $K(k_{d1})$ and $K(k_{d2})$, $K'(k_{d2})$ and $K'(k_{d2})$ are the complete elliptic integral of the first kind with $k_{d1} = \sqrt{1-k^2_{d1}}$ and $k_{d2} = \sqrt{1-k^2_{d2}}$. $\text{sn}(\ldots, \ldots)$ and $\text{cn}(\ldots, \ldots)$ are the Jacobian elliptic functions. The special elliptical functions can be numerically solved by using MATLAB.

In the DEP array case, there only exists the real part of the electric field phasor [5]. Therefore, the vector $\nabla|\mathbf{E}|^2$ for the dielectrophoretic force component in the DEP array case is calculated as

$$
\nabla |E_{zd}|^2 = \nabla \left( E_{zd,x}^2 + E_{zd,y}^2 \right)
$$

Figure 1 (a) Diagram showing the dielectrophoretic electrode array with boundary conditions and geometrical parameters. (b) The vectors of the dielectrophoretic force component in the basic cell of DEP array case.
4. The Travelling Wave Array

Similarly to the DEP case, due to the symmetry, a basic cell in the twDEP array has been selected for analysis, as shown in figure 2(a). The real part of the field distribution is given as below [5].

\[ E_{Z_0} = \frac{V}{h} K(k_{11}) \left[ \frac{t_{E}(t_{E} - t_{E}')(t - t_{E}' - 1)/(t - t_{E}')}{t_{E}'-1} \right]^{1/2} \]  

(6)

With

\[ \frac{K(k_{11})}{K(k_{12})} = \frac{w + g}{h}, \quad k_{11} = \frac{t_{E}(t_{E} - 1)}{t_{E}'(t_{E} - 1)}, \quad k_{12} = \frac{t_{E}(t_{E} - t_{E}')}{\sqrt{t_{E}'(t_{E} - 1)}} \]

\[ t_{E}' = \frac{t_{E}' t_{E} \text{cn}^{2} \left( \frac{(w + 2g)K(k_{11})}{2h}i_{i_{1}} \right)}{t_{E} - t_{E}' \text{sn}^{2} \left( \frac{(w + 2g)K(k_{11})}{2h}i_{i_{1}} \right)} \]

\[ t_{E}' = \frac{t_{E}' t_{E} \text{sn}^{2} \left( \frac{wK(k_{11})}{2h}i_{i_{1}} \right)}{t_{E} - t_{E}' \text{cn}^{2} \left( \frac{wK(k_{11})}{2h}i_{i_{1}} \right)} \]

where \( E_{Z_0} \) is the real part of the electric field distribution in the selected cell for the twDEP array case.

In the twDEP array case, there exits both the real and imaginary part of the electric field phasor. The imaginary part of the electric field is the mirror image of the real part about the centre of the gap. Thus, the components for the DEP and twDEP force and the ROT torque are calculated as:

DEP Force Component:

\[ \nabla |E_{Z_0}|^2 = \nabla \left( E_{Z_0}^2 + E_{Z_0}^2 \right) = \nabla \left( E_{Z_0}^2 + E_{Z_0}^2 + E_{Z_0}^2 + E_{Z_0}^2 \right) \]  

(7)

ROT Torque Component:

\[ (E_{Z_0} \times E_{Z_0}) = E_{Z_0} \times E_{Z_0} - E_{Z_0} \times E_{Z_0} \]  

(8)

twDEP Force Component:

\[ \nabla \times (E_{Z_0} \times E_{Z_0}) = \frac{\partial}{\partial \gamma} \left( E_{Z_0} \times E_{Z_0} - E_{Z_0} \times E_{Z_0} \right) - \frac{\partial}{\partial \alpha} \left( E_{Z_0} \times E_{Z_0} - E_{Z_0} \times E_{Z_0} \right) \]  

(9)

where \( E_{Z_0} \) and \( E_{Z_0} \) are the real part of the electric field vectors in the 2D plane along the \( x \)-axis and \( y \)-axis, respectively. \( E_{Z_0} \) and \( E_{Z_0} \) are the imaginary part of the electric field vectors in the 2D plane along the \( x \)-axis and \( y \)-axis, respectively. In equation (8), the ROT torque component exists in the third direction, which is vertical to the 2D plane. The first term on the right side of equation (9) is the force component along the \( x \)-axis direction and the second term is the force component along the \( y \)-axis direction. The results in equation (7), (8) and (9) are numerically evaluated in MATLAB 7.0, as shown in the figure 2 (b), (c) and (d).
Figure 2 (a) Diagram showing the travelling wave dielectrophoretic electrode array with boundary conditions and geometrical parameters. (b) The vector of the dielectrophoretic force in the basic cell of twDEP array case. (c) The magnitude of the electrorotation torque in the basic cell of twDEP array case. (d) The vector of the traveling wave dielectrophoretic force in the basic cell of twDEP array case.

5. Conclusion

In this paper, as far as we have concerned, it is for the first time to analytically solve the DEP force, twDEP force, ROT torque in the interdigitated electrode arrays without linear boundary condition assumptions, which is used in the Fourier series. Compared to the numerical solutions, the accuracy of the solutions is not influenced by the numerical error.

References

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