Musielak-Orlicz BMO-Type Spaces Associated with Generalized Approximations to the Identity

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Abstract Let $X$ be a space of homogenous type and $\phi : X \times [0, \infty) \to [0, \infty)$ a growth function such that $\phi(\cdot, t)$ is a Muckenhoupt weight uniformly in $t$ and $\phi(x, \cdot)$ an Orlicz function of uniformly upper type 1 and lower type $p \in (0, 1]$. In this article, the authors introduce a new Musielak-Orlicz BMO-type space $BMO_\phi^A(X)$ associated with the generalized approximation to the identity, give out its basic properties and establish its two equivalent characterizations, respectively, in terms of the spaces $BMO_{A, \text{max}}(X)$ and $\widetilde{BMO}_{A}(X)$. Moreover, two variants of the John-Nirenberg inequality on $BMO_\phi^A(X)$ are obtained. As an application, the authors further prove that the space $BMO_\phi^A(\sqrt{\Delta}(\mathbb{R}^n))$, associated with the Poisson semigroup of the Laplace operator $\Delta$ on $\mathbb{R}^n$, coincides with the space $BMO_\phi(\mathbb{R}^n)$ introduced by L. D. Ky.

1 Introduction

The classical $BMO(\mathbb{R}^n)$ space (the space of functions with bounded mean oscillation), originally introduced by John and Nirenberg [24], plays an important role in partial differential equations and modern harmonic analysis (see, for example, [24, 18]). Recall that a locally integrable function $f$ on the $n$-dimensional Euclidean space $\mathbb{R}^n$ is said to be in the space $BMO(\mathbb{R}^n)$, if

$$
\|f\|_{BMO(\mathbb{R}^n)} := \sup_{B \subset \mathbb{R}^n} \frac{1}{|B|} \int_B |f(x) - f_B| \, dx < \infty,
$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ and $f_B := \frac{1}{|B|} \int_B f(x) \, dx$. It is well known that many operators such as Carleson-Zygmund singular integral operators are not bounded on the Lebesgue space $L^\infty(\mathbb{R}^n)$, but they are bounded from $L^\infty(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$ (see, for example, [20]). Therefore, the space $BMO(\mathbb{R}^n)$ is considered as a natural substitute for $L^\infty(\mathbb{R}^n)$ when studying the boundedness of operators on function spaces. Moreover, $BMO(\mathbb{R}^n)$ plays a significant role in the interpolation theory of linear operators. Precisely, if a linear operator $T$ is bounded from $L^q(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for some $q \in [1, \infty)$ and bounded from $L^\infty(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$, then $T$ is also bounded from $L^p(\mathbb{R}^n)$...
to $L^p(\mathbb{R}^n)$ for all $p \in [q, \infty)$ (see, for example, [20]). Furthermore, Fefferman and Stein [18] proved that $\text{BMO}(\mathbb{R}^n)$ is the dual space of the Hardy space $H^1(\mathbb{R}^n)$.

Recently, Ky [28] introduced Musielak-Orlicz BMO-type spaces $\text{BMO}^\varphi(\mathbb{R}^n)$, which generalize the classical space $\text{BMO}(\mathbb{R}^n)$, the weighted BMO space $\text{BMO}_\omega(\mathbb{R}^n)$ (see, for example, [37, 38, 5]) and the Orlicz BMO-type spaces $\text{BMO}_\rho(\mathbb{R}^n)$ (see, for example, [43, 23, 46]). Here, $\varphi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ is a growth function such that $\varphi(\cdot, t)$ is a Muckenhoupt weight uniformly in $t$, and $\varphi(x, \cdot)$ is an Orlicz function of uniformly upper type 1 and lower type $p \in (0, 1]$ (see Subsection 2.1 below for the definitions of uniformly upper and lower types).

Recall that the Musielak-Orlicz BMO-type space $\text{BMO}^\varphi(\mathbb{R}^n)$ is defined as the set of all locally integrable functions $f$ on $\mathbb{R}^n$ such that

$$\|f\|_{\text{BMO}^\varphi(\mathbb{R}^n)} := \sup_{B \subset \mathbb{R}^n} \frac{1}{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}} \int_B |f(x) - f_B| \, dx < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$, $\chi_B$ is the characteristic function of $B$ and

$$\|\chi_B\|_{L^\varphi(\mathbb{R}^n)} := \inf \left\{ \lambda \in (0, \infty) : \int_{\mathbb{R}^n} \varphi \left( \frac{x, \chi_B(x)}{\lambda} \right) \, dx \leq 1 \right\}.$$

Notice that Nakai and Yabuta [40] proved that the class of pointwise multipliers for $\text{BMO}(\mathbb{R}^n)$ is just the space of $L^\infty(\mathbb{R}^n) \cap \text{BMO}^{\log}(\mathbb{R}^n)$, where $\text{BMO}^{\log}(\mathbb{R}^n)$ denotes the Musielak-Orlicz BMO-type space $\text{BMO}^\varphi(\mathbb{R}^n)$ related to the growth function

$$\varphi(x, t) := \frac{t}{\ln(e + |x|) + \ln(e + t)}$$

for all $x \in \mathbb{R}^n$ and $t \in [0, \infty)$. Furthermore, Ky [28] proved that $\text{BMO}^\varphi(\mathbb{R}^n)$ is the dual space of the Musielak-Orlicz Hardy space $H^\varphi(\mathbb{R}^n)$, which was also introduced in [28] and includes both the Orlicz-Hardy space $H_\varphi(\mathbb{R}^n)$ in [43, 23] and the weighted Hardy space $H_\varphi^\omega(\mathbb{R}^n)$ with $p \in (0, 1]$ and $\omega \in A_\infty(\mathbb{R}^n)$ in [19, 44]. Here, $A_\omega(\mathbb{R}^n)$, $\omega \in [1, \infty]$, denotes the class of Muckenhoupt weights. Moreover, more interesting applications of these spaces were also presented in [1, 3, 34, 4, 28, 27, 29, 30, 31]. Notice that Musielak-Orlicz functions are the natural generalization of Orlicz functions which may vary in the spatial variable (see, for example, [13, 14, 28, 39]). The motivation to study function spaces of Musielak-Orlicz type is due to that they have wide applications to many branches of physics and mathematics (see, for example, [2, 3, 4, 13, 14, 28, 33, 47]).

Moreover, Duong and Yan [17] introduced a new BMO-type function space on a space $\mathcal{X}$ of homogeneous type in the sense of Coifman and Weiss [9, 10], which is associated with a generalized approximation to the identity and generalizes the classical BMO spaces in another way. Precisely, let $\{A_t\}_{t \geq 0}$ be a class of integral operators, defined by kernels $\{a_t\}_{t \geq 0}$ (which decay fast enough) in the sense that, for all $x \in \mathcal{X}$ and functions $f$ satisfying some growth condition on $\mathcal{X}$,

$$A_tf(x) := \int_\mathcal{X} a_t(x, y)f(y) \, d\mu(y).$$

Duong and Yan [17] first introduced the suitable function set $\mathcal{M}(\mathcal{X})$ such that, for all $f \in \mathcal{M}(\mathcal{X})$ and all $t, s \in (0, \infty)$, $A_tf < \infty$ and $A_s(A_tf) < \infty$ almost everywhere. Then
the BMO-type space $\text{BMO}_A(\mathcal{X})$ is defined as the set of all $f \in \mathcal{M}(\mathcal{X})$ such that
\[
\|f\|_{\text{BMO}_A(\mathcal{X})} := \sup_{B \subset \mathcal{X}} \frac{1}{\mu(B)} \int_B |f(x) - A_B f(x)| \, d\mu(x) < \infty,
\]
where the supremum is taken over all balls $B$ in $\mathcal{X}$ and $t_B := r_B^m$ with $r_B$ being the radius of the ball $B$ and $m$ a positive constant. Duong and Yan [17] gave out some basic properties of $\text{BMO}_A(\mathcal{X})$ including a variant of the John-Nirenberg inequality and further proved that the space $\text{BMO}_\varphi\sqrt{\Delta}(\mathbb{R}^n)$, associated with the Poisson semigroup of the Laplace operator $\Delta$ on $\mathbb{R}^n$, and $\text{BMO}(\mathbb{R}^n)$ coincide with equivalent norms. Tang [45] introduced the Morrey-Campanato type spaces $\text{Lip}_A(\alpha, \mathcal{X})$ associated with the generalized approximation to the identity $\{A_t\}_{t>0}$ and established the John-Nirenberg inequality on these spaces. Furthermore, Deng, Duong and Yan [12] established a new characterization of the classical Morrey-Campanato space on $\mathbb{R}^n$ by using an appropriate convolution to replace the minimizing polynomial of a function $f$ in the Morrey-Campanato norm. Moreover, a similar characterization for the Morrey space on $\mathbb{R}^n$ was also obtained by Duong, Xiao and Yan in [16]. Yang and Zhou [49] introduced some generalized approximations to the identity with optimal decay conditions in the sense that these conditions are sufficient and necessary for these generalized approximations to the identity to characterize $\text{BMO}(\mathcal{X})$, which was introduced by Long and Yang [36]. Furthermore, a new John-Nirenberg-type inequality associated with the generalized approximations to the identity on $\text{BMO}(\mathcal{X})$ was also established in [49]. Recently, Bui and Duong [6] introduced the weighted BMO space $\text{BMO}_A(\mathcal{X}, \omega)$ associated to the generalized approximations to the identity, $\{A_t\}_{t>0}$, and also obtained the John-Nirenberg inequality on these spaces.

Let $\mathcal{X}$ be a space of homogeneous type with degree $(\alpha_0, n_0, N_0)$ (see Remark 2.5 below for its definition), where $\alpha_0, n_0$ and $N_0$ are as in (2.7), (2.3) and (2.5) below, respectively. Let $\varphi : \mathcal{X} \times [0, \infty) \to [0, \infty)$ be a growth function such that $\varphi(\cdot, t)$ is a Muckenhoupt weight uniformly in $t$, and $\varphi(x, \cdot)$ is an Orlicz function of uniformly upper type 1 and lower type $p \in (0, 1]$. Motivated by [28, 17, 45], in this article, we first introduce the Musielak-Orlicz type space $\text{BMO}_\varphi^\alpha(\mathcal{X})$ by a way similar to that used in [28], and then introduce the new Musielak-Orlicz BMO-type space $\text{BMO}_A^\varphi(\mathcal{X})$, via replacing the mean value $f_B$ (see (2.11)) in the definition of $\text{BMO}_\varphi^\alpha(\mathcal{X})$ by $A_t f$, motivated by Duong and Yan [17], which generalizes the space $\text{BMO}_A(\mathcal{X})$ associated with the generalized approximation to the identity $\{A_t\}_{t>0}$ in [17], the Morrey-Campanato type space $\text{Lip}_A(\alpha, \mathcal{X})$ associated with $\{A_t\}_{t>0}$ in [45], and the weighted BMO space $\text{BMO}_A(\mathcal{X}, \omega)$ associated with $\{A_t\}_{t>0}$ in [6]. Then we give out some basic properties of $\text{BMO}_A^\varphi(\mathcal{X})$ and establish its two equivalent characterizations, respectively, in terms of the spaces $\text{BMO}_{A, \text{max}}^\varphi(\mathcal{X})$ and $\text{BMO}_{A, \text{sup}}^\varphi(\mathcal{X})$. Moreover, two variants of the John-Nirenberg inequality are obtained on $\text{BMO}_A^\varphi(\mathcal{X})$, which generalize the John-Nirenberg inequalities established in [17, 45, 6]. As an application, we further prove that the space $\text{BMO}_\sqrt{\Delta}(\mathbb{R}^n)$, associated with the Poisson semigroup of the Laplace operator on $\mathbb{R}^n$, and $\text{BMO}(\mathbb{R}^n)$ coincide with equivalent norms, which means that the new Musielak-Orlicz BMO-type spaces $\text{BMO}_A^\varphi(\mathbb{R}^n)$ also generalize $\text{BMO}(\mathbb{R}^n)$ introduced by Ky [28].

Precisely, this article is organized as follows. In Section 2, we first recall notions of spaces of homogenous type and growth functions $\varphi$ considered in this article. We then give out several examples of growth functions as well as their basic properties. After
recalling the Musielak-Orlicz space $L^\varphi(\mathcal{X})$, we then introduce the Musielak-Orlicz BMO-type space $\text{BMO}^\varphi(\mathcal{X})$ on the space $\mathcal{X}$ of homogeneous type and further give out some useful properties for $L^\varphi(\mathcal{X})$ (see Lemma 2.13 below) and $\text{BMO}^\varphi(\mathcal{X})$ (see Proposition 2.14 below), which are needed in establishing the equivalence between $\text{BMO}^\varphi(\mathcal{X})$ and the new Musielak-Orlicz BMO-type space $\text{BMO}^\varphi_{d}(\mathcal{X})$ introduced in the next section.

In Section 3, we introduce the generalized approximation to the identity $\{A_t\}_{t>0}$ with kernels $\{a_t\}_{t>0}$, which satisfy appropriate decay conditions related to the growth function $\varphi$ (see (3.5) and (3.1) below), and the class $\mathcal{M}(\mathcal{X})$ of functions in which functions have proper growth condition (see (3.2) below) and are suitable to $\{A_t\}_{t>0}$ (see Lemma 3.4 below). Based on this, we introduce the new Musielak-Orlicz BMO-type space $\text{BMO}^\varphi_{d}(\mathcal{X})$ associated with $\{A_t\}_{t>0}$ (see Definition 3.3 below). We prove that, if $\{A_t\}_{t>0}$ satisfies that, for all $t \in (0, \infty)$, $A_t(1) = 1$ almost everywhere, then $\text{BMO}^\varphi(\mathcal{X}) \subset \text{BMO}^\varphi_{d}(\mathcal{X})$ (see (3.5) below). We also give out some useful properties for $\text{BMO}^\varphi_{d}(\mathcal{X})$ (see Propositions 3.7 and 3.9 below), including some size estimates for functions in $\text{BMO}^\varphi_{d}(\mathcal{X})$, which play an important role in the study for $\text{BMO}^\varphi_{d}(\mathcal{X})$. Moreover, we also introduce the Musielak-Orlicz BMO-type spaces, $\text{BMO}^\varphi_{d, \max}(\mathcal{X})$ (see Definition 3.11) and $\text{BMO}^\varphi_{d}(\mathcal{X})$ (see Definition 3.14), associated with $\{A_t\}_{t>0}$, and further prove that, when $\{A_t\}_{t>0}$ satisfies an additional size condition (see (3.22) below), these spaces are equivalent with $\text{BMO}^\varphi_{d}(\mathcal{X})$ (see Theorems 3.12 and 3.15 below). We point out that Theorems 3.12 and 3.15 completely cover, respectively, [17, Theorem 3.1] by taking

\begin{equation}
\varphi(x, t) := t \quad \text{for all } x \in \mathcal{X} \text{ and } t \in [0, \infty),
\end{equation}

and Theorem 3.15 completely covers [45, Proposition 2.4] by taking

\begin{equation}
\varphi(x, t) := t^{1/(1 + \beta)}, \quad \text{with } \beta \in (0, \infty), \quad \text{for all } x \in \mathcal{X} \text{ and } t \in [0, \infty)
\end{equation}

(see Remark 3.16 below).

In Section 4, we establish two variants of the John-Nirenberg inequality on $\text{BMO}^\varphi_{d}(\mathcal{X})$. The first one (see Theorem 4.2 below) is closer to the John-Nirenberg inequalities established in [17, 45]. We remark that Theorem 4.2 completely covers [17, Theorem 3.1] and [45, Theorem 3.1] by taking $\varphi$, respectively, as in (1.1) and (1.2) (see Remark 4.3 below). While the second one (see Theorem 4.9 below) is closer to the John-Nirenberg inequalities on the weighted BMO-type spaces obtained in [38, 6, 32]. It is worth pointing out that Theorem 4.9(i) completely covers [17, Theorem 3.1] and [45, Theorem 3.1] by taking $\varphi$, respectively, as in (1.1) and (1.2), and [6, Theorem 3.6] by taking

\begin{equation}
\varphi(x, t) := \omega(x)t, \quad \text{with } \omega \in A_1(\mathcal{X}), \quad \text{for all } x \in \mathcal{X} \text{ and } t \in [0, \infty).
\end{equation}

Moreover, Theorem 4.9(ii) is new even when

\begin{equation}
\varphi(x, t) := \omega(x)t \text{ for all } x \in \mathcal{X} \text{ and } t \in [0, \infty) \text{ with } \omega \in A_\infty(\mathcal{X}) \text{ satisfying } p_\omega \leq 1 + \frac{1}{t_\omega},
\end{equation}

where $p_\omega$ and $r_\omega$ denote the critical indices of the weight $\omega$, which are defined by a way similar to that used in (2.9) and (2.10) below and $r_\omega'$ denotes the conjugate index of
in [38, Theorem 3] and [32, Theorem 1], we establish the second John-Nirenberg inequalities in Remark 4.10(ii) below when \( \varphi \in \mathcal{A}_1(\mathcal{X}) \).

For \( \tilde{p} \in [1, \infty) \), we also introduce the Musielak-Orlicz BMO-type spaces \( \text{BMO}^{\varphi, \tilde{p}}_{\mathcal{A}}(\mathcal{X}) \) and \( \text{BMO}^{\varphi, \tilde{p}}_{\mathcal{A}}(\mathcal{X}) \) (see Definitions 4.5 and 4.11 below). As applications of these John-Nirenberg inequalities on \( \text{BMO}^{\varphi}_{\mathcal{A}}(\mathcal{X}) \) obtained in Theorems 4.2 and 4.9, we further prove that, for any \( \tilde{p} \in [1, \infty) \), the spaces \( \text{BMO}^{\varphi, \tilde{p}}_{\mathcal{A}}(\mathcal{X}) \), \( \text{BMO}^{\varphi, \tilde{p}}_{\mathcal{A}}(\mathcal{X}) \) and \( \text{BMO}^{\varphi}_{\mathcal{A}}(\mathcal{X}) \) coincide with equivalent norms (see Theorems 4.6 and 4.12 below). We remark that Theorem 4.6 completely covers [17, Theorem 3.4] and [45, Theorem 3.4] by taking \( \varphi \), respectively, as in (1.1) and (1.2). Moreover, Theorem 4.12 is also new even when \( \varphi \) is as in (1.4).

In Section 5, as applications of Theorems 4.6 and 4.12, the boundedness of the classical Littlewood-Paley \( g \)-function on \( L^2(\mathbb{R}^n) \) and the \( \varphi \)-Carleson measure characterization of \( \text{BMO}^{\varphi}(\mathbb{R}^n) \) obtained in [22] (see also Lemma 5.3 below), we prove that the space \( \text{BMO}^{R\varphi}(\mathbb{R}^n) \), associated with the Poisson semigroup of the Laplace operator on \( \mathbb{R}^n \), and \( \text{BMO}^{\varphi}(\mathbb{R}^n) \) coincide with equivalent norms (see Theorem 5.5 below), which completely covers [17, Theorem 2.14] by taking \( \varphi \) as in (1.1) (see Remark 5.6 below). By a similar way, we also prove that the space \( \text{BMO}^{R\varphi}_{\mathcal{A}}(\mathbb{R}^n) \), associated with the heat semigroup of the Laplace operator on \( \mathbb{R}^n \), and \( \text{BMO}^{\varphi}(\mathbb{R}^n) \) coincide with equivalent norms (see Theorem 5.7 below), which, together with Theorems 3.12, 3.15 and 5.5, implies that the spaces

\[
\text{BMO}^{\varphi}(\mathbb{R}^n), \text{BMO}^{R\varphi}(\mathbb{R}^n), \text{BMO}^{R\varphi}_{\mathcal{A}}(\mathbb{R}^n), \text{BMO}^{R\varphi}_{\mathcal{A}, \text{max}}(\mathbb{R}^n), \text{BMO}^{R\varphi}_{\mathcal{A}}(\mathbb{R}^n), \text{BMO}^{R\varphi}_{\mathcal{A}}(\mathbb{R}^n),
\]

\( \text{BMO}^{R\varphi}_{\mathcal{A}, \text{max}}(\mathbb{R}^n) \) and \( \text{BMO}^{R\varphi}_{\mathcal{A}}(\mathbb{R}^n) \) coincide with equivalent norms (see Corollary 5.8 below). We point out that Theorems 5.5, 5.7 and Corollary 5.8 completely cover, respectively, [17, Theorems 2.14, 2.15 and Corollary 2.16] by taking \( \varphi \) as in (1.1). Moreover, Theorems 5.5 and 5.7 and Corollary 5.8 are also new even when \( \varphi \) is as in (1.2).

We remark that the key points of the above approach are to establish the basic properties of \( \text{BMO}^{\varphi}(\mathcal{X}) \) and \( \text{BMO}^{R\varphi}(\mathcal{X}) \) (see Propositions 2.14, 3.7 and 3.9 below), and the John-Nirenberg inequalities on the space \( \text{BMO}^{R\varphi}_{\mathcal{A}}(\mathcal{X}) \) (see Theorems 4.2 and 4.9 below). To this end, we first give out some basic properties of growth functions \( \varphi \) (see Lemmas 2.12 and 2.13 below). Moreover, the essential difficulty to establish Proposition 3.7 comes from the inseparability of the space variable \( x \) and the time variable \( t \) appeared in the grown function \( \varphi(x, t) \). To overcome this difficulty, we first clarify, in (3.11) below, the relation between the degree \( (\alpha_0, n_0, N_0) \) of \( \mathcal{X} \), the uniformly lower type critical index \( \varphi \) (see (2.8) below), the uniformly Muckenhoupt weight critical index \( p(\varphi) \) (see (2.9) below) and the uniformly reverse Hölder critical index \( q(\varphi) \) (see (2.10) below) of \( \varphi \), and the decay order \( M \) for the kernels \( \{a_t\}_{t>0} \) of the generalized approximation to the identity \( \{A_t\}_{t>0} \) (see (3.1)). In the proof of Proposition 3.7, we also need to use dyadic cubes in \( \mathcal{X} \) established by Christ [7] (see also Lemma 3.8 below) and borrow some ideas from Duong and Yan [17] to deal with the time parameter \( t \) appeared in \( \{A_t\}_{t>0} \). Using these properties of \( \text{BMO}^{R\varphi}_{\mathcal{A}}(\mathcal{X}) \), we establish two variants of the John-Nirenberg inequality on the space \( \text{BMO}^{R\varphi}_{\mathcal{A}}(\mathcal{X}) \). Precisely, we obtain the first John-Nirenberg inequality on \( \text{BMO}^{R\varphi}_{\mathcal{A}}(\mathcal{X}) \), in Theorem 4.2 below, by borrowing some ideas from the proof of [17, Theorem 3.1] and using some delicate estimates of the growth function \( \varphi \). Furthermore, following the ways in [38, Theorem 3] and [32, Theorem 1], we establish the second John-Nirenberg inequal-
ity on $\text{BMO}_A^\infty(\mathcal{X})$ in Theorem 4.9 below, via using the Whitney decomposition established in [9, Chapter III, Theorem 1.3] and some basic properties of $\varphi$. Here we also borrow some ideas from the John-Nirenberg inequality on the Musielak-Orlicz Campanato spaces $L_{\varphi,1,\infty}(\mathbb{R}^n)$ established by Liang and Yang [35] and choose the time variant $t := \|\chi_B\|_{L_{\varphi,t}^\infty(\mathcal{X})}^{-1}$ to overcome some essential difficulties caused by the inseparability of the space variable $x$ and the time variable $t$ appeared in $\varphi(x,t)$.

Finally we make some conventions on notation. Throughout the article, we denote by $C$ a positive constant which is independent of the main parameters, but it may vary from line to line. The symbol $A \lesssim B$ means that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$. The symbol $[s]$ for $s \in \mathbb{R}$ denotes the maximal integer not more than $s$. For any given normed spaces $\mathcal{A}$ and $\mathcal{B}$ with the corresponding norms $\| \cdot \|_{\mathcal{A}}$ and $\| \cdot \|_{\mathcal{B}}$, the symbol $\mathcal{A} \subset \mathcal{B}$ means that, for all $f \in \mathcal{A}$, then $f \in \mathcal{B}$ and $\|f\|_\mathcal{B} \lesssim \|f\|_\mathcal{A}$. For any subset $E$ of the space $\mathcal{X}$ of homogeneous type, we denote by $E^c$ the set $\mathcal{X} \setminus E$ and by $\chi_E$ its characteristic function. We also set $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. For any index $q \in [1, \infty]$, we denote by $q'$ its conjugate index, namely, $1/q + 1/q' = 1$. Also, for any $\alpha \in (0, \infty)$ and ball $B := B(x_B, r_B) := \{x \in \mathcal{X} : d(x, x_B) < r_B\}$ with $x_B \in \mathcal{X}$ and $r_B \in (0, \infty)$, we denote by $\alpha B$ the ball $B(x_B, \alpha r_B)$.

## 2 Spaces of homogeneous type, growth functions and Musielak-Orlicz BMO-type spaces

In this section, we introduce the Musielak-Orlicz BMO-type spaces $\text{BMO}^\varphi(\mathcal{X})$ on RD-spaces $\mathcal{X}$. To this end, we first recall some notions on spaces of homogeneous type, RD-spaces and growth functions considered in this article. Then we state some properties of the growth functions. Finally, we give out a basic property for $\text{BMO}^\varphi(\mathcal{X})$.

### 2.1 Spaces of homogeneous type and growth functions

We first recall the notion of spaces of homogeneous type in the sense of Coifman and Weiss [9, 10].

**Definition 2.1.** A function $d : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ is called a quasi-metric, if it satisfies the following conditions:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in \mathcal{X}$;

(iii) there exists a constant $C_1 \in [1, \infty)$ such that, for all $x, y, z \in \mathcal{X}$,

\[
(d(x, y) \leq C_1[d(x, z) + d(z, y)]).
\]

The quasi-metric $d$ defines a topology for which the balls $B(x, r) := \{y \in \mathcal{X} : d(y, x) < r\}$ for all $x \in \mathcal{X}$ and $r \in (0, \infty)$ form a basis. However, when $C_1 \in (1, \infty)$, the balls need not be open (see, for example, [9]).

**Definition 2.2.** A space of homogeneous type $(\mathcal{X}, d, \mu)$ is a set $\mathcal{X}$ equipped with a quasi-metric $d$ and a nonnegative Borel measure $\mu$ on $\mathcal{X}$ for which there exists a constant $C_2 \in [1, \infty)$ such that, for all balls $B(x, r)$,

\[
\mu(B(x, 2r)) \leq C_2 \mu(B(x, r)) < \infty \quad \text{(Doubling Property)}.
\]
Remark 2.3. (i) The doubling property implies the following strong homogeneity property: there exist positive constants $n$ and $C$ such that, for all $x \in \mathcal{X}$, $r \in (0, \infty)$ and $\lambda \in [1, \infty)$,

$$\mu(B(x, \lambda r)) \leq C\lambda^n \mu(B(x, r)).$$

Let

$$n_0 := \inf\{n \in (0, \infty) : \text{is as in (2.2)}\}.$$

The parameter $n_0$ is a measure of the “dimension” of $\mathcal{X}$. Observe that $n_0 \in [0, \infty)$ and (2.2) may not be true for $n_0$.

(ii) There also exist a positive constant $C$ and $N \in [0, \infty)$ such that, for all $x, y \in \mathcal{X}$ and $r \in (0, \infty)$,

$$\mu(B(y, r)) \leq C \left[1 + \frac{d(x, y)}{r}\right]^N \mu(B(x, r)).$$

Indeed, let $n$ be as in (2.2). When $N = n$, (2.4) is deduced from the quasi-triangle inequality (2.1) of the quasi metric $d$ and the strong homogeneity property (2.2). In the case of Euclidean spaces $\mathbb{R}^n$ and Lie groups of polynomial growth, $N = 0$.

Let

$$N_0 := \inf\{N \in [0, \infty) : \text{is as in (2.4)}\}.$$

Observe that $N_0 \in [0, n_0]$ and (2.4) may not be true for $N_0$.

Now we recall the notion of the RD-space introduced in [21] (see also [50] for more properties of RD-spaces).

Definition 2.4. The triple $(\mathcal{X}, d, \mu)$ is called an RD-space, if there exist a constant $\alpha \in (0, n]$ and $C \in [1, \infty)$ such that, for all $x \in \mathcal{X}$, $r \in (0, 2 \text{diam}(\mathcal{X}))$ and $\lambda \in [1, 2 \text{diam}(\mathcal{X})/r)$,

$$C^{-1}\lambda^\alpha \mu(B(x, r)) \leq \mu(B(x, \lambda r)) \leq C\lambda^\alpha \mu(B(x, r)),$$

where $n$ is as in (2.2) and $\text{diam}(\mathcal{X}) := \sup_{x, y \in \mathcal{X}} d(x, y)$.

Remark 2.5. Obviously, an RD-space is a space of homogeneous type. It is also known that a connected space of homogeneous type is an RD-space (see [50]). Let

$$\alpha_0 := \sup\{\alpha \in [0, n] : \alpha \text{ is as in (2.6)}\}.$$

Obviously, for an RD-space $\mathcal{X}$, $\alpha_0 \in (0, n_0]$ and (2.6) may not be true for $\alpha_0$. If $\mathcal{X}$ is only known to be a space of homogenous type, then (2.6) may hold true only for $\alpha = 0$, namely, $\alpha_0 = 0$ in this case. In what follows, the triple $(\alpha_0, n_0, N_0)$ is called the degree of the space of homogeneous type, $\mathcal{X}$. 
Throughout this article, we always assume that $\mathcal{X}$ is a space of homogeneous type with degree $(a_0, n_0, N_0)$.

Next, we recall that a function $\Phi : [0, \infty) \to [0, \infty)$ is called an Orlicz function if it is nondecreasing, $\Phi(0) = 0$, $\Phi(t) > 0$ for all $t \in (0, \infty)$ and $\lim_{t \to \infty} \Phi(t) = \infty$ (see, for example, [39, 42, 41]). We point out that, different from the classical definition of Orlicz functions, the Orlicz function in this article may not be convex. The function $\Phi$ is said to be of upper type $p$ (resp. lower type $p$) for some $p \in [0, \infty)$ if there exists a positive constant $C$ such that, for all $s \in [1, \infty)$ (resp. $s \in (0, 1)$) and $t \in [0, \infty)$, $\Phi(st) \leq Cs^p \Phi(t)$. If $\Phi$ is of both upper type $p_1$ and lower type $p_2$, then $p_1 \geq p_2$ and $\Phi$ is said to be of type $(p_1, p_2)$.

Let $\mathcal{X}$ be a space of homogeneous type. For a given function $\varphi : \mathcal{X} \times [0, \infty) \to [0, \infty)$ such that, for any $x \in \mathcal{X}$, $\varphi(x, \cdot)$ is an Orlicz function, $\varphi$ is said to be of uniformly upper type $p$ (resp. uniformly lower type $p$) for some $p \in [0, \infty)$, if there exists a positive constant $C$ such that, for all $x \in \mathcal{X}$, $t \in [0, \infty)$ and $s \in [1, \infty)$ (resp. $s \in (0, 1)$),

$$\varphi(x, st) \leq Cs^p \varphi(x, t).$$

Moreover, $\varphi$ is said to be of positive uniformly upper type (resp. uniformly lower type), if it is of uniformly upper type (resp. uniformly lower type) $p$ for some $p \in (0, \infty)$, and let

$$i(\varphi) := \sup \{p \in (0, \infty) : \varphi \text{ is of uniformly lower type } p \}.$$

(2.8) Observe that $i(\varphi)$ may not be attainable, namely, $\varphi$ may not be of uniformly lower type $i(\varphi)$ (see, for example, [47, 48]).

**Definition 2.6.** Let $\mathcal{X}$ be a space of homogeneous type and $\varphi : \mathcal{X} \times [0, \infty) \to [0, \infty)$. The function $\varphi(\cdot, t)$ is said to satisfy the uniformly Muckenhoupt condition for some $p \in [1, \infty)$, denoted by $\varphi \in A_p(\mathcal{X})$, if, when $p \in (1, \infty)$,

$$A_p(\varphi) := \sup_{t \in (0, \infty)} \sup_{B \subset \mathcal{X}} \frac{1}{\mu(B)} \left\{ \int_B \varphi(x, t) \, d\mu(x) \left\{ \frac{1}{\mu(B)} \int_B \varphi(x, t) \, d\mu(x) \right\}^{-\frac{p}{p-1}} \right\} < \infty,$$

or, when $p = 1$,

$$A_1(\varphi) := \sup_{t \in (0, \infty)} \sup_{B \subset \mathcal{X}} \frac{1}{\mu(B)} \left\{ \int_B \varphi(x, t) \, d\mu(x) \right\}^{\frac{1}{\text{ess sup}_{y \in B} [\varphi(y, t)]^{-1}}} < \infty.$$

Here the first supremums are taken over all $t \in (0, \infty)$ and the second ones over all balls $B \subset \mathcal{X}$.

The function $\varphi(\cdot, t)$ is said to satisfy the uniformly reverse Hölder condition for some $q \in (1, \infty]$, denoted by $\varphi \in RH_q(\mathcal{X})$, if, when $q \in (1, \infty)$,

$$RH_q(\varphi) := \sup_{t \in (0, \infty)} \sup_{B \subset \mathcal{X}} \left\{ \frac{1}{\mu(B)} \left\| \varphi(x, t)^q \right\| d\mu(x) \right\}^{1/q} \left\{ \frac{1}{\mu(B)} \int_B \varphi(x, t) \, d\mu(x) \right\}^{-1} < \infty,$$

or, when $q = \infty$,

$$RH_\infty(\varphi) := \sup_{t \in (0, \infty)} \sup_{B \subset \mathcal{X}} \left\{ \text{ess sup}_{y \in B} \varphi(y, t) \right\} \left\{ \frac{1}{\mu(B)} \int_B \varphi(x, t) \, d\mu(x) \right\}^{-1} < \infty.$$
Here the first supremums are taken over all \( t \in (0, \infty) \) and the second ones over all balls \( B \subset \mathcal{X} \).

We point out that, in Definition 2.6, when \( \mathcal{X} := \mathbb{R}^n \), \( A_p(\mathbb{R}^n) \) with \( p \in [1, \infty) \) was introduced by Ky [28] and, moreover, for any metric space \( \mathcal{X} \) with doubling measure, the notions of \( A_p(\mathcal{X}) \), with \( p \in [1, \infty) \), and \( RH_q(\mathcal{X}) \), with \( q \in (1, \infty] \), were introduced in [48].

Let \( A_\infty(\mathcal{X}) := \bigcup_{p \in [1, \infty)} A_p(\mathcal{X}) \) and the critical indices of \( \varphi \) be defined as follows:

\[
(p(\varphi) := \inf \{ p \in [1, \infty) : \varphi \in A_p(\mathcal{X}) \}
\]

and

\[
r(\varphi) := \sup \{ q \in (1, \infty) : \varphi \in RH_q(\mathcal{X}) \}.
\]

Observe that, if \( p(\varphi) \in (1, \infty) \), then \( \varphi \notin A_p(\varphi)(\mathcal{X}) \), and there exists \( \varphi \notin A_1(\mathcal{X}) \) such that \( p(\varphi) = 1 \) (see, for example, [25]). Similarly, if \( r(\varphi) \in (1, \infty) \), then \( \varphi \notin RH_r(\varphi)(\mathcal{X}) \), and there exists \( \varphi \notin RH_\infty(\mathcal{X}) \) such that \( r(\varphi) = \infty \) (see, for example, [8]).

Now we introduce the notion of growth functions.

**Definition 2.7.** Let \( \mathcal{X} \) be a space of homogeneous type. The function \( \varphi : \mathcal{X} \times [0, \infty) \to [0, \infty) \) is called a growth function if the following hold true:

(i) \( \varphi \) is a Musielak-Orlicz function, namely,

\[\begin{align*}
(i)_1 & \text{ the function } \varphi(x, \cdot) : [0, \infty) \to [0, \infty) \text{ is an Orlicz function for all } x \in \mathcal{X}; \\
(i)_2 & \text{ the function } \varphi(\cdot, t) \text{ is a measurable function for all } t \in [0, \infty).
\end{align*}\]

(ii) \( \varphi \in A_\infty(\mathcal{X}) \).

(iii) The function \( \varphi \) is of uniformly upper type 1 and of uniformly lower type \( p \in (0, 1] \).

Clearly, \( \varphi(x, t) := t^p \) for all \( (x, t) \in \mathcal{X} \times [0, \infty) \) with \( p \in (0, 1] \) and, more generally, \( \varphi(x, t) := \omega(x) \Phi(t) \) for all \( (x, t) \in \mathcal{X} \times [0, \infty) \) with \( \omega \in A_\infty(\mathcal{X}) \) and \( \Phi \) being an Orlicz function of upper type 1 and lower type \( p \in (0, 1] \) are growth functions. Let \( x_0 \in \mathcal{X} \).

Another typical and useful growth function is

\[
\varphi(x, t) := \frac{t^s}{[\ln(e + d(x, x_0))]^\beta + \ln(e + t)]^\gamma}
\]

for all \( x \in \mathcal{X} \) and \( t \in [0, \infty) \) with some \( s \in (0, 1], \beta \in [0, \alpha) \) and \( \gamma \in [0, 2s(1 + \ln 2)] \), where \( \alpha \) is as in (2.6). It is easy to show that \( \varphi \in A_1(\mathcal{X}) \), \( \varphi \) is of uniformly upper type \( s \) and \( i(\varphi) = s \) which is not attainable. We also point out that, when \( \mathcal{X} := \mathbb{R}^n \), a similar example of such \( \varphi \) is given by Ky [28] via replacing \( d(x, x_0) \) by \( |x| \), where \( |\cdot| \) denotes the Euclidean distance on \( \mathbb{R}^n \); see, for example, [47, 48] for more examples of growth functions.

### 2.2 Musielak-Orlicz BMO-type spaces \( BMO^\varphi(\mathcal{X}) \)

Let us first recall the Musielak-Orlicz-type space \( L^\varphi(\mathcal{X}) \). Recall that \( \mathcal{X} \) is always assumed to be a space of homogeneous type.
Definition 2.8. Let $\mathcal{X}$ be a space of homogeneous type and $\varphi$ a growth function as in Definition 2.7. The Musielak-Orlicz-type space $L^\varphi(\mathcal{X})$ is defined to be the space of all measurable functions $f$ such that $\int_{\mathcal{X}} \varphi(x, |f(x)|) \, d\mu(x) < \infty$ endowed with the Luxembourg norm

$$\|f\|_{L^\varphi(\mathcal{X})} := \inf \left\{ \lambda \in (0, \infty) : \int_{\mathcal{X}} \varphi \left( x, \frac{|f(x)|}{\lambda} \right) \, d\mu(x) \leq 1 \right\}.$$

Now we are ready to introduce Musielak-Orlicz BMO-type spaces $\text{BMO}^\varphi(\mathcal{X})$ as follows.

Definition 2.9. Let $\mathcal{X}$ be a space of homogeneous type and $\varphi$ a growth function. A locally integrable function $f$ on $\mathcal{X}$ is said to belong to the Musielak-Orlicz BMO-type space $\text{BMO}^\varphi(\mathcal{X})$, if

$$\|f\|_{\text{BMO}^\varphi(\mathcal{X})} := \sup_{B \subset \mathcal{X}} \frac{1}{\|B\|_{L^\varphi(\mathcal{X})}} \int_B |f(x) - f_B| \, d\mu(x),$$

where the supremum is taken over all balls $B \subset \mathcal{X}$ and

$$f_B := \frac{1}{\mu(B)} \int_B f(x) \, d\mu(x).$$

Remark 2.10. (i) When $\varphi$ is as in (1.1), then $\|\chi_B\|_{L^\varphi(\mathcal{X})} = \mu(B)$ and hence $\text{BMO}^\varphi(\mathcal{X})$ is just the space $\text{BMO}(\mathcal{X})$ on the space of homogeneous type, $\mathcal{X}$, introduced by Long and Yang [36]. $\text{BMO}^\varphi(\mathbb{R}^n)$ was introduced by Ky (see [28]). When $\mathcal{X} := \mathbb{R}^n$ and $\varphi$ is as in (1.1), then $\|\chi_B\|_{L^\varphi(\mathbb{R}^n)} = |B|$ and hence $\text{BMO}^\varphi(\mathbb{R}^n)$ is just the classical $\text{BMO}(\mathbb{R}^n)$ space introduced by John and Nirenberg [24]; when $\mathcal{X} := \mathbb{R}^n$ and $\varphi$ is as in (1.4) without the restriction $p_\omega \leq 1 + 1/\omega'$, then $\|\chi_B\|_{L^\varphi(\mathcal{X})} = \omega(B)$ and hence $\text{BMO}^\varphi(\mathbb{R}^n)$ is just the weighted $\text{BMO}$ space $\text{BMO}_\omega(\mathbb{R}^n)$, which was first introduced by Muckenhoupt and Wheeden [37, 38].

(ii) Another typical example of the space $\text{BMO}^\varphi(\mathbb{R}^n)$ is $\text{BMO}^{\log}(\mathbb{R}^n)$, which is related to the growth function $\varphi(x, t) = \ln c + \ln(\epsilon + |x|) + \ln(\epsilon + t)$, $x \in \mathbb{R}^n$ and $t \in [0, \infty)$. Notice that the class of pointwise multipliers for $\text{BMO}(\mathbb{R}^n)$, characterized by Nakai and Yabuta [40], is just the space $L^\infty(\mathbb{R}^n) \cap \text{BMO}^{\log}(\mathbb{R}^n)$ (see [28] for more details).

To give out a basic property of $\text{BMO}^\varphi(\mathcal{X})$, we need the following lemmas concerning growth functions.

Lemma 2.11. Let $\mathcal{X}$ be a space of homogeneous type and $\varphi$ as in Definition 2.7.

(i) It holds true that $\int_{\mathcal{X}} \varphi(x, \frac{|f(x)|}{\|f\|_{L^\varphi(\mathcal{X})}}) \, d\mu(x) = 1$ for all $f \in L^\varphi(\mathcal{X}) \setminus \{0\}$.

(ii) Let $c$ be a positive constant. Then there exists a positive constant $C$, depending on $c$, such that, if $\int_{\mathcal{X}} \varphi(x, \frac{|f(x)|}{\lambda}) \, d\mu(x) \leq c$ for some $\lambda \in (0, \infty)$, then $\|f\|_{L^\varphi(\mathcal{X})} \leq C\lambda$.

Lemma 2.11 when $\mathcal{X} := \mathbb{R}^n$ is just [28, Lemmas 4.2(i) and 4.3(i)] and, moreover, its proof is also similar to those proofs in [28], the details being omitted.

Lemma 2.12. Let $\mathcal{X}$ be a space of homogeneous type and $\varphi$ as in Definition 2.7.

(i) If $\varphi \in \mathbb{A}_p(\mathcal{X})$ with $p \in [1, \infty)$, then there exists a positive constant $C$ such that, for all balls $B_1, B_2 \subset \mathcal{X}$ with $B_1 \subset B_2$ and $t \in [0, \infty)$,

$$\varphi(B_2, t) \leq C \left[ \frac{\mu(B_2)}{\mu(B_1)} \right]^p.

(ii) If $\varphi \in \mathbb{RH}_q(\mathcal{X})$ with $q \in (1, \infty)$, then there exists a positive constant $C$ such that, for all balls $B_1, B_2 \subset \mathcal{X}$ with $B_1 \subset B_2$ and $t \in [0, \infty)$,

$$\varphi(B_2, t) \geq C \left[ \frac{\mu(B_2)}{\mu(B_1)} \right]^\frac{1}{q-1}.$$
The proof of Lemma 2.12 is similar to that of the corresponding conclusions in $\mathbb{R}^n$ (see, for example, [19, 20]), the details being omitted.

**Lemma 2.13.** Let $\mathcal{X}$ be a space of homogeneous type and $\varphi$ as in Definition 2.7 with uniformly lower type $p \in (0, 1)$. Assume that $\varphi \in \mathcal{A}_{p_1}(\mathcal{X})$ with $p_1 \in [1, \infty)$, and $\varphi \in \mathcal{R}\mathcal{H}_q(\mathcal{X})$ with $q \in (1, \infty)$. Then there exists a positive constant $C$ such that, for all balls $B_1, B_2 \subset \mathcal{X}$ with $B_1 \subset B_2$,

$$
\|\chi_{B_2}\|_{L^\varphi(\mathcal{X})} \leq C \left[ \frac{\mu(B_2)}{\mu(B_1)} \right]^\frac{\alpha}{p} \|\chi_{B_1}\|_{L^\varphi(\mathcal{X})}
$$

and

$$
\|\chi_{B_1}\|_{L^\varphi(\mathcal{X})} \leq C \left[ \frac{\mu(B_1)}{\mu(B_2)} \right]^{\frac{\alpha - 1}{q}} \|\chi_{B_2}\|_{L^\varphi(\mathcal{X})}.
$$

**Proof.** We first prove (2.12). By the uniformly lower type property of $\varphi$, Lemmas 2.11(i) and 2.12(i), we know that

$$
\int_{B_2} \varphi \left( x, \left[ \frac{\mu(B_2)}{\mu(B_1)} \right]^\frac{1}{p} \|\chi_{B_1}\|_{L^\varphi(\mathcal{X})} \right) d\mu(x)
\lesssim \left[ \frac{\mu(B_2)}{\mu(B_1)} \right]^{-p} \int_{B_2} \varphi \left( x, \frac{1}{\|\chi_{B_1}\|_{L^\varphi(\mathcal{X})}} \right) d\mu(x) \lesssim \varphi \left( B_1, \frac{1}{\|\cdot\|_{L^\varphi(\mathcal{X})}} \right) \lesssim 1,
$$

which, together with Lemma 2.11(ii), implies that (2.12) holds true.

By using the uniformly upper type 1 property of $\varphi$ and Lemma 2.12(ii), we conclude that (2.13) holds true by a way similar to the above proof of (2.12), the details being omitted, which completes the proof of Lemma 2.13. \qed

**Proposition 2.14.** Let $\mathcal{X}$ be a space of homogeneous type with degree $(\alpha_0, n_0, N_0)$, where $\alpha_0, n_0$ and $N_0$ are as in (2.7), (2.3) and (2.5), respectively. Let $n \in (n_0, \infty)$, $\alpha \in (0, \alpha_0)$ and $\alpha = 0$ when $\alpha_0 = 0$. Assume that $\varphi$ is as in Definition 2.7 with $\varphi \in \mathcal{A}_{p_1}(\mathcal{X})$ and $\varphi$ of uniformly lower type $p$, where $p_1 \in [1, \infty)$ and $p \in (0, 1]$. Then there exists a positive constant $C$ such that, for all $f \in \text{BMO}^\varphi(\mathcal{X})$, balls $B \subset \mathcal{X}$ and $K \in (1, \infty)$,

$$
|f_B - f_{KB}| \leq CK \left[ \frac{\alpha_0}{\alpha} \right] \frac{\|\chi_B\|_{L^\varphi(\mathcal{X})}}{\mu(B)} \|f\|_{\text{BMO}^\varphi(\mathcal{X})},
$$

where $f_B$ is as in (2.11) and $f_{KB}$ defined similarly.

**Proof.** Let $K \in (1, \infty)$. Then there exists $m \in \mathbb{N}$ such that $e^m \leq K < e^{m+1}$. If $\text{diam}(\mathcal{X}) = \infty$, by (2.6) and (2.12), we see that

$$
|f_B - f_{KB}| \leq |f_B - f_{eB}| + |f_{eB} - f_{e^2B}| + \cdots + |f_{e^mB} - f_{KB}|
\leq \frac{\|\chi_{eB}\|_{L^\varphi(\mathcal{X})}}{\mu(B)} \frac{1}{\|\chi_{eB}\|_{L^\varphi(\mathcal{X})}} \int_{eB} |f(x) - f_{eB}| d\mu(x)
+ \frac{\|\chi_{e^2B}\|_{L^\varphi(\mathcal{X})}}{\mu(eB)} \frac{1}{\|\chi_{e^2B}\|_{L^\varphi(\mathcal{X})}} \int_{e^2B} |f(x) - f_{e^2B}| d\mu(x)
$$

and
follows that

\[ |f(x) - f_{KB}| \]

which is desired.

Now we consider the case that \( \text{diam}(\mathcal{X}) < \infty \). Let \( B := B(x_B, r_B) \). Assume that there exists \( m_0 \in \mathbb{Z}_+ \) with \( m_0 < m \) such that \( 2e^{m_0}r_B < \text{diam}(\mathcal{X}) \leq 2e^{m_0+1}r_B \); otherwise, we obtain the desired conclusion by repeating the procedure same as in (2.14). In the case that \( m_0 < m \), it is easy to see that \( \mu(\mathcal{X}) \sim \mu(e^{m_0+1}B) \). From this, (2.6) and (2.12), it follows that

\[
\|f - f_{KB}\|_{L^p(\mathcal{X})} \leq \|f - f_{eB}\| + \cdots + \|f_{e^{m_0}B} - f_{e^{m_0+1}B}\| + \|f_{e^{m_0+1}B} - f_{e^{m_0+2}B}\| + \cdots + \|f_{e^mB} - f_{KB}\|
\]

\[
\leq \frac{\|X_{KB}\|_{L^p(\mathcal{X})}}{\mu(B)} \frac{1}{\|X_{KB}\|_{L^p(\mathcal{X})}} \int_{KB} |f(x) - f_{KB}| \, d\mu(x) + \cdots + \frac{\|X_{m_0+1}\|_{L^p(\mathcal{X})}}{\mu(e^{m_0+1}B)} \frac{1}{\|X_{m_0+1}\|_{L^p(\mathcal{X})}} \int_{e^{m_0+1}B} |f(x) - f_{e^{m_0+1}B}| \, d\mu(x)
\]

\[
+ \frac{\|X_{m_0+2}\|_{L^p(\mathcal{X})}}{\mu(e^{m_0+2}B)} \frac{1}{\|X_{m_0+2}\|_{L^p(\mathcal{X})}} \int_{e^{m_0+2}B} |f(x) - f_{e^{m_0+2}B}| \, d\mu(x)
\]

\[
+ \cdots + \frac{\|X_{m}\|_{L^p(\mathcal{X})}}{\mu(e^mB)} \frac{1}{\|X_{m}\|_{L^p(\mathcal{X})}} \int_{e^mB} |f(x) - f_{e^mB}| \, d\mu(x)
\]

\[
\leq \left[ e^{\frac{np_1}{p}} + e^{m_0\frac{np_1}{p} - \alpha} + e^{m_0\frac{np_1}{p} - \alpha} \right] \frac{\|X_{KB}\|_{L^p(\mathcal{X})}}{\mu(B)} \|f\|_{\text{BMO}^p(\mathcal{X})}
\]

\[
\leq \left\{ e^{\frac{np_1}{p}} + \frac{e^{m_0\frac{np_1}{p} - \alpha}}{e^{\frac{np_1}{p} - \alpha}} \left[ e^{m_0\frac{np_1}{p} - \alpha} - 1 \right] + (m - m_0)e^{m_0\frac{np_1}{p} - \alpha} \right\}
\]

\[
\times \frac{\|X_{m}\|_{L^p(\mathcal{X})}}{\mu(B)} \|f\|_{\text{BMO}^p(\mathcal{X})} \lesssim K^{\frac{np_1}{p} - \alpha} \frac{\|X_{m}\|_{L^p(\mathcal{X})}}{\mu(B)} \|f\|_{\text{BMO}^p(\mathcal{X})},
\]

which, together with (2.14), completes the proof of Proposition 2.14.

3 Musielak-Orlicz BMO-type spaces \( \text{BMO}^\varphi_A(\mathcal{X}) \) associated with generalized approximations to the identity

In this section, we first introduce Musielak-Orlicz BMO-type spaces \( \text{BMO}^\varphi_A(\mathcal{X}) \) associated with generalized approximations to the identity, \( \{A_t\}_{t>0} \), and then give out their
basic properties and two equivalent characterizations in terms of the space $\text{BMO}^\varphi_A(\mathcal{X})$ (see Definition 3.11 below) and the space $\widehat{\text{BMO}}^\varphi_A(\mathcal{X})$ (see Definition 3.14 below).

### 3.1 Definition of $\text{BMO}^\varphi_A(\mathcal{X})$

Let $\mathcal{X}$ be a space of homogeneous type with degree $(\alpha_0, n_0, N_0)$, where $\alpha_0$, $n_0$ and $N_0$ are as in (2.7), (2.3) and (2.5), respectively. Let $x_0 \in \mathcal{X}$,

$$M > n_0[1 + p(\varphi)/i(\varphi)] - \alpha_0,$$

where $n_0$, $p(\varphi)$, $i(\varphi)$ and $\alpha_0$ are, respectively, as in (2.3), (2.9), (2.8) and (2.7), and $\beta \in (0, M - n_0[1 + p(\varphi)/i(\varphi)] + \alpha_0)$. A function $f \in L^1_{\text{loc}}(\mathcal{X})$ is said to be of type $(x_0, \beta)$, if there exists a positive constant $C$ such that

$$\int_{\mathcal{X}} \frac{|f(x)|}{[1 + d(x_0, x)]^{\alpha_0 + \beta} \mu(B(x_0, 1 + d(x_0, x))} \, d\mu(x) \leq C. \tag{3.2}$$

Moreover, denote by $\mathcal{M}_{(x_0, \beta)}(\mathcal{X})$ the collection of all function of type $(x_0, \beta)$. The norm of $f$ in $\mathcal{M}_{(x_0, \beta)}(\mathcal{X})$ is defined by

$$\|f\|_{\mathcal{M}_{(x_0, \beta)}(\mathcal{X})} := \inf\{C \in (0, \infty) : \text{ (3.2) holds true}\}.$$

For a fixed $x_0 \in \mathcal{X}$, it is easy to see that $\mathcal{M}_{(x_0, \beta)}(\mathcal{X})$ is a Banach space under the norm $\|\cdot\|_{\mathcal{M}_{(x_0, \beta)}(\mathcal{X})}$. Moreover, it is easy to show that, for any $x_1 \in \mathcal{X}$, $\mathcal{M}_{(x_1, \beta)}(\mathcal{X}) = \mathcal{M}_{(x_0, \beta)}(\mathcal{X})$ with equivalent norms. Let

$$\mathcal{M}(\mathcal{X}) := \bigcup_{x_0 \in \mathcal{X}} \bigcup_{\beta : 0 < \beta < M - n_0 - N_0} \mathcal{M}_{(x_0, \beta)}(\mathcal{X}), \tag{3.3}$$

where $n_0$, $N_0$ and $M$ are as in (2.3), (2.5) and (3.1), respectively.

To give the definition of the space $\text{BMO}^\varphi_A(\mathcal{X})$, we also need to recall the notion of the generalized approximation to the identity, $\{A_t\}_{t > 0}$. In this article, we always assume that, for any $t \in (0, \infty)$, the operator $A_t$ is defined by the kernel $a_t$ in the sense that

$$A_t f(x) := \int_{\mathcal{X}} a_t(x, y) f(y) \, d\mu(y)$$

for all $f \in \mathcal{M}(\mathcal{X})$ and $x \in \mathcal{X}$.

We further assume that, for any $t \in (0, \infty)$, the kernel $a_t$ satisfies that, for all $x, y \in \mathcal{X}$,

$$|a_t(x, y)| \leq h_t(x, y),$$

where $h_t(x, y)$ is given by setting, for all $x, y \in \mathcal{X}$,

$$h_t(x, y) := \frac{1}{\mu(B(x, t^{1/m}))^m} g \left( \frac{|d(x, y)|^m}{t} \right), \tag{3.4}$$

in which $m$ is a positive constant and $g$ a positive, bounded, decreasing function satisfying that

$$\lim_{r \to \infty} r^M g(r^m) = 0. \tag{3.5}$$
where $M$ is as in (3.1).

It is easy to prove that there exists a positive constant $C$ such that, for all $x \in X$ and $t \in (0, \infty)$,

$$C^{-1} \leq \int_X h_t(x, y) \, d\mu(y) \leq C \quad \text{and} \quad C^{-1} \leq \int_X h_t(y, x) \, d\mu(y) \leq C$$

(see also [15]).

Then we have the following technical lemma.

**Lemma 3.1.** Let $X$ be a space of homogeneous type with degree $(\alpha_0, n_0, N_0)$, where $\alpha_0$, $n_0$ and $N_0$ are as in (2.7), (2.3) and (2.5), respectively. Assume that $\varphi$ is as in Definition 2.7 and $\{A_t\}_{t>0}$ a generalized approximation to the identity satisfying (3.4) and (3.5).

(i) If $f \in \text{BMO}^\varphi(X)$, then $f \in \mathcal{M}(X)$.

(ii) For any $t \in (0, \infty)$ and $f \in \mathcal{M}(X)$, it holds true that $|A_t f(x)| < \infty$ for almost every $x \in X$.

(iii) For any $t, s \in (0, \infty)$ and $f \in \mathcal{M}(X)$, it holds true that $|A_t(A_s f)(x)| < \infty$ for almost every $x \in X$.

Moreover, if, for almost every $x, y \in X$,

$$a_{t+s}(x, y) = \int_X a_t(x, z) a_s(z, y) \, d\mu(z),$$

then, for any $f \in \mathcal{M}(X)$, $A_{t+s}f = A_t(A_s f)$ almost everywhere.

**Proof.** Let $f \in \text{BMO}^\varphi(X)$. For any $x_0 \in X$, fix a ball $B := B(x_0, 1)$ centered at $x_0$ and of radius 1. Let $\beta$ be as in (3.2). By the definitions of $n_0$, $\alpha_0$, $p(\varphi)$ and $i(\varphi)$, respectively, as in (2.3), (2.7), (2.9) and (2.8), we know that there exist $n \in [n_0, \infty)$, $\alpha \in [0, \alpha_0]$, $p_1 \in [p(\varphi), \infty)$ and $p \in (0, i(\varphi)]$ such that $X$ satisfies (2.2) and (2.6), respectively, for $n$ and $\alpha$, $\varphi \in \mathcal{A}_{p_1}(X)$, $\varphi$ is of uniformly lower type $p$ and $\frac{n_0 p(\varphi)}{i(\varphi)} - \alpha_0 + \beta > \frac{np_1}{p} - \alpha$. From this and Proposition 2.14, it follows that, for all $k \in \mathbb{N}$,

$$|f_B - f_{2^k B}| \lesssim 2^{k \frac{np_1}{p} - \alpha} \frac{\|\chi_B\|_{L^p(X)}}{\mu(B)} \|f\|_{\text{BMO}^\varphi(X)},$$

which, together with (2.12) and $\frac{np_1(\varphi)}{i(\varphi)} - \alpha_0 + \beta > \frac{np_1}{p} - \alpha$, implies that

$$\int_X \frac{|f(y) - f_B|}{[1 + d(x_0, y)]^{\frac{np_1(\varphi)}{i(\varphi)} - \alpha_0 + \beta} \mu(B(x_0, 1 + d(x_0, y)))} \, d\mu(y) \leq \sum_{k=0}^{\infty} \int_{2^kB \setminus 2^{k-1}B} \frac{|f(y) - f_B|}{[1 + d(x_0, y)]^{\frac{np_1(\varphi)}{i(\varphi)} - \alpha_0 + \beta} \mu(B(x_0, 1 + d(x_0, y)))} \, d\mu(y) \lesssim \sum_{k=0}^{\infty} 2^{-k \frac{np_1(\varphi)}{i(\varphi)} - \alpha_0 + \beta} \left\{ (\mu(2^kB))^{-1} \int_{2^kB} |f(y) - f_{2^kB}| \, d\mu(y) + |f_B - f_{2^kB}| \right\} \lesssim \sum_{k=0}^{\infty} 2^{-k \frac{np_1(\varphi)}{i(\varphi)} - \alpha_0 + \frac{np_1}{p} + \alpha + \beta} \frac{\|\chi_B\|_{L^p(X)}}{\mu(B)} \|f\|_{\text{BMO}^\varphi(X)}.$$
where $2^{-1}B := \emptyset$, $f_B$ is as in \eqref{eq:2.11} and $f_{2^kB}$ defined similarly. Moreover, it is easy to see that

$$C_{x_0} := \int_{\mathcal{X}} \frac{1}{[1 + d(x_0, y)]^{\frac{\alpha p(y)}{\beta} - \alpha + \beta} \mu(B(x_0, 1 + d(x_0, y))} d\mu(y) < \infty.$$  

Thus, we have

$$\|f\|_{\mathcal{M}_{(x_0, \beta)}(\mathcal{X})} \lesssim \frac{\|\chi_B\|_{L^{p}(\mathcal{X})}}{\mu(B)} \|f\|_{\text{BMO}^{\varphi}_{\mathcal{X}}(\mathcal{X})} + C_{x_0} |f_B| < \infty,$$

which implies that $f \in \mathcal{M}_{(x_0, \beta)}(\mathcal{X})$, and hence $f \in \mathcal{M}(\mathcal{X})$.

The proofs of (ii) and (iii) are similar to that of [17, Lemma 2.3], the details being omitted, which completes the proof of Lemma 3.1.}

\[
\text{Remark 3.2.} \text{ Recall that, if a generalized approximation to the identity } \{A_t\}_{t>0} \text{ satisfies } (3.6), \text{ then } \{A_t\}_{t>0} \text{ is said to have the semigroup property.}
\]

We now introduce the space $\text{BMO}^{\varphi}_{\mathcal{A}}(\mathcal{X})$ associated with the generalized approximation to the identity $\{A_t\}_{t>0}$.

\[
\text{Definition 3.3.} \text{ Let } \mathcal{X} \text{ be a space of homogeneous type, } \varphi \text{ as in Definition 2.7 and } \{A_t\}_{t>0} \text{ a generalized approximation to the identity satisfying (3.4) and (3.5). The Musielak-Orlicz BMO-type space } \text{BMO}^{\varphi}_{\mathcal{A}}(\mathcal{X}) \text{ associated with } \{A_t\}_{t>0} \text{ is defined to be the space of all functions } f \in \mathcal{M}(\mathcal{X}) \text{ such that}
\]

$$\|f\|_{\text{BMO}^{\varphi}_{\mathcal{A}}(\mathcal{X})} := \sup_{B \subset \mathcal{X}} \frac{1}{\|\chi_B\|_{L^{p}(\mathcal{X})}} \int_B |f(x) - A_{t_B} f(x)| \, d\mu(x) < \infty,$$

where the supremum is taken over all balls $B \subset \mathcal{X}$, $t_B := r_B^m$, $r_B$ is the radius of ball $B$ and $m$ as in (3.4).

\[
\text{Remark 3.4.} \text{ (i) We point out that } (\text{BMO}^{\varphi}_{\mathcal{A}}(\mathcal{X}), \| \cdot \|_{\text{BMO}^{\varphi}_{\mathcal{A}}(\mathcal{X})}) \text{ is a seminormed vector space, with the seminorm vanishing on the space } \mathcal{K}_{\mathcal{A}}(\mathcal{X}), \text{ which is defined by}
\]

$$\mathcal{K}_{\mathcal{A}}(\mathcal{X}) := \{ f \in \mathcal{M}(\mathcal{X}) : A_t f(x) = f(x) \text{ for } \mu\text{-almost every } x \in \mathcal{X} \text{ and all } t \in (0, \infty) \}.$$  

Then, it is customary to think $\text{BMO}^{\varphi}_{\mathcal{A}}(\mathcal{X})$ to be modulo $\mathcal{K}_{\mathcal{A}}(\mathcal{X})$.

(ii) When $\varphi$ is as in \eqref{eq:1.1}, then $\|\chi_B\|_{L^{p}(\mathcal{X})} = \mu(B)$ and the space $\text{BMO}^{\varphi}_{\mathcal{A}}(\mathcal{X})$ is just the space $\text{BMO}^{\varphi}_{\mathcal{A}}(\mathcal{X})$ associated with $\{A_t\}_{t>0}$ introduced by Duong and Yan [17]; when $\varphi$ is as in \eqref{eq:1.2}, then $\|\chi_B\|_{L^{p}(\mathcal{X})} = [\mu(B)]^{1 + \beta}$ and the space $\text{BMO}^{\varphi}_{\mathcal{A}}(\mathcal{X})$ is just the Morrey-Campanato type spaces $\text{Lip}_{\mathcal{A}}^\beta(\mathcal{X})$ introduced by Tang [45]; when $\varphi$ is as in \eqref{eq:1.4} without the restriction $p_\omega \leq 1 + 1/r_\omega$, $\|\chi_B\|_{L^{p}(\mathcal{X})} = \omega(B)$ and the space $\text{BMO}^{\varphi}_{\mathcal{A}}(\mathcal{X})$ is just the weighted BMO space $\text{BMO}_{\mathcal{A}}(\mathcal{X}, \omega)$ introduced by Bui and Duong [6].

Now we establish a relation between the spaces $\text{BMO}^{\varphi}_{\mathcal{A}}(\mathcal{X})$ and $\text{BMO}^{\varphi}_{\mathcal{A}}(\mathcal{X})$.

\[
\text{Proposition 3.5.} \text{ Let } \mathcal{X} \text{ be a space of homogeneous type with degree } (\alpha_0, n_0, N_0), \text{ where } \alpha_0, n_0 \text{ and } N_0 \text{ are as in (2.7), (2.3) and (2.5), respectively. Assume that } \varphi \text{ is as in Definition 2.7, } \{A_t\}_{t>0} \text{ a generalized approximation to the identity satisfying (3.4) and (3.5), and, for any } t \in (0, \infty), \text{ } A_t(1) = 1 \text{ almost everywhere, namely, } \int_\mathcal{X} a_t(x, y) \, d\mu(y) = 1.
\]
for almost every $x \in X$. Then, $\text{BMO}^\varphi(X) \subset \text{BMO}^\varphi_A(X)$ and there exists a positive constant $C$ such that, for all $f \in \text{BMO}^\varphi(X)$,

\begin{equation}
\|f\|_{\text{BMO}^\varphi_A(X)} \leq C\|f\|_{\text{BMO}^\varphi(X)}.
\end{equation}

However, the reverse inequality does not hold true in general.

**Proof.** Let $M$ be as in (3.1). By $M > n_0[1 + p(\varphi)/i(\varphi)] - \alpha_0$, we see that there exist $n \in [n_0, \infty)$, $\alpha \in [0, \alpha_0]$, $p_1 \in \left[p(\varphi), \infty\right)$ and $p \in (0, i(\varphi)]$ such that $X$ satisfies (2.2) and (2.6), respectively, for $n$ and $\alpha$, $\varphi \in A_{p_1}(X)$, $\varphi$ is of uniformly lower type $p$ and $M > n(1 + \frac{p_1}{p}) - \alpha$.

Let $f \in \text{BMO}^\varphi(X)$, $B := B(x_B, r_B)$ and $t_B := r_B$. Then, by $A_4(1) = 1$, we conclude that

\begin{equation}
\frac{1}{\|\chi_B\|_{L^p(X)}} \int_B |f(x) - A_{t_B}f(x)| \, d\mu(x)
\leq \frac{1}{\|\chi_B\|_{L^p(X)}} \int_B \int_X h_{t_B}(x, y)|f(x) - f(y)| \, d\mu(y) \, d\mu(x)
= \frac{1}{\|\chi_B\|_{L^p(X)}} \int_B \int_{2B} h_{t_B}(x, y)|f(x) - f(y)| \, d\mu(y) \, d\mu(x)
+ \sum_{k=1}^{\infty} \frac{1}{\|\chi_B\|_{L^p(X)}} \int_B \int_{2^{k+1}B \setminus 2^k B} \cdots =: \text{I} + \text{II}.
\end{equation}

We first estimate $\text{I}$. Since $x \in B$, by (2.4), we know that $\mu(B) \lesssim \mu(B(x, r_B))$, which, together with (2.2), (3.4) and the decreasing property of $g$, implies that, for all $y \in 2B$,

\[ h_{t_B}(x, y) = \frac{g([d(x, y)]^m t_B^{-1})}{\mu(B(x, r_B))} \lesssim \frac{g(0)}{\mu(B)} \lesssim \frac{1}{\mu(2B)}. \]

From this and (2.12), we deduce that

\begin{equation}
\text{I} \lesssim \frac{1}{\|\chi_B\|_{L^p(X)} \mu(2B)} \int_B \int_{2B} |f(x) - f(y)| \, d\mu(y) \, d\mu(x)
\lesssim \frac{1}{\|\chi_B\|_{L^p(X)} \mu(2B)} \int_B \int_{2B} |f(x) - f_{2B}| + |f_{2B} - f(y)| \, d\mu(y) \, d\mu(x)
\lesssim \frac{1}{\|\chi_B\|_{L^p(X)} \mu(2B)} \int_{2B} |f(x) - f_{2B}| \, d\mu(x) \lesssim \|f\|_{\text{BMO}^\varphi(X)}.
\end{equation}

Regarding $\text{II}$, for $x \in B$ and $y \in 2^{k+1}B \setminus 2^k B$, we see that $d(x, y) \geq 2^{k-1}r_B$. Then, by (2.2), (3.4) and the decreasing property of $g$, we conclude that

\[ h_{t_B}(x, y) = \frac{g([d(x, y)]^m t_B^{-1})}{\mu(B(x, r_B))} \lesssim \frac{g(2^{(k-1)m})}{\mu(B)} \lesssim \frac{g(2^{(k-1)m})2^{(k+1)n}}{\mu(2^{k+1}B)}, \]

which further implies that

\begin{equation}
\text{II} \lesssim \sum_{k=1}^{\infty} 2^{kn} \frac{1}{\|\chi_B\|_{L^p(X)} \mu(2^{k+1}B)} \int_B \int_{2^{k+1}B} |f(x) - f(y)| \, d\mu(y) \, d\mu(x).
\end{equation}
Moreover, from Proposition 2.14, it follows that, for each \( k \in \mathbb{N} \),
\[
\frac{1}{\| \chi_B \|_{L^\varphi(x)} \mu(2^{k+1}B)} \int_B \int_{2^{k+1}B} |f(x) - f(y)| \, d\mu(y) \, d\mu(x)
\leq \frac{\mu(B)}{\| \chi_B \|_{L^\varphi(x)} \mu(2^{k+1}B)} \int_{2^{k+1}B} |f(y) - f_{2^{k+1}}| \, d\mu(y)
+ \frac{1}{\| \chi_B \|_{L^\varphi(x)} \mu(2^{k+1}B)} \int_B |f(x) - f_{2^{k+1}}| \, d\mu(x)
\leq \frac{\mu(B)}{\| \chi_B \|_{L^\varphi(x)} \mu(2^{k+1}B)} \frac{1}{\| \chi_{2^{k+1}B} \|_{L^\varphi(x)}} \int_{2^{k+1}B} |f(y) - f_{2^{k+1}}| \, d\mu(y)
+ \frac{1}{\| \chi_B \|_{L^\varphi(x)} \mu(2^{k+1}B)} \int_B |f(x) - f_B| \, d\mu(x) + \frac{\mu(B)}{\| \chi_B \|_{L^\varphi(x)}} (|f_B - f_{2^B}|)
+ \cdots + |f_{2^k} - f_{2^{k+1}}| \|f\|_{\text{BMO}^\varphi(x)}.
\]
By this, (3.10), (3.5) and \( M > n(1 + p_1/p) - \alpha \), we find that
\[
\|f\|_{\text{BMO}^\varphi(x)} \lesssim \|f\|_{\text{BMO}^\varphi(x)},
\]
which, together with (3.8) and (3.9), implies that (3.7) holds true.

Finally, we show that the converse inequality of (3.7) does not hold true in general. We consider \( \mathbb{R} \) with the Lebesgue measure \( dx \) and the approximation of the identity, \( \{ A_t \}_{t > 0} \), given by the kernels
\[
a_t(x, y) := \frac{1}{2\kappa m} \chi_{(x-t, x+t)}(y) \quad \text{for all } x, y \in \mathbb{R}.
\]
Let \( f(x) = x \) for all \( x \in \mathbb{R} \). For every \( t \in (0, \infty) \), \( A_t f(x) = x \) and \( \|f\|_{\text{BMO}^\varphi(x)} = 0 \), but \( \|f\|_{\text{BMO}^\varphi(x)} \neq 0 \). Thus, the converse inequality of (3.7) does not hold true in general, which completes the proof of Proposition 3.5.

\textbf{Remark 3.6.} We remark that the assumption \( A_t(1) = 1 \) almost everywhere is necessary for (3.7). Indeed, let \( f(x) := 1 \) for all \( x \in \mathcal{X} \). Then (3.7) implies that \( \|1\|_{\text{BMO}^\varphi(x)} = 0 \) and hence, for every \( t \in (0, \infty) \), \( A_t(1) = 1 \) almost everywhere.

\subsection{Some basic properties of \( \text{BMO}^\varphi_A(\mathcal{X}) \)}

From now on, we \textit{always need} the following assumption on the generalized approximation to the identity, \( \{ A_t \}_{t > 0} \).

\textbf{Assumption SP.} Let \( \mathcal{X} \) be a space of homogeneous type, \( \varphi \) as in Definition 2.7 and \( \{ A_t \}_{t > 0} \) a generalized approximation to the identity satisfying (3.4) and (3.5). Assume that \( A_0 \) is the identity operator \( I \) and the operators \( \{ A_t \}_{t \geq 0} \) have the semigroup property, namely, for any \( t, s \in [0, \infty) \) and \( f \in \mathcal{M}(\mathcal{X}) \), \( A_t A_s f = A_{t+s} f \) for almost every \( x \in \mathcal{X} \) (see also Remark 3.2). \qed
Then we have the following property for \( \{A_t\}_{t>0} \) on \( \text{BMO}^\varphi_A(\mathcal{X}) \), which is essential for developing the theory of \( \text{BMO}^\varphi_A(\mathcal{X}) \).

**Proposition 3.7.** Let \( \mathcal{X} \) be a space of homogeneous type with degree \((\alpha_0, n_0, N_0)\), where \( \alpha_0, n_0 \) and \( N_0 \) are as in (2.7), (2.3) and (2.5), respectively. Assume that \( \varphi \) is as in Definition 2.7 and \( \{A_t\}_{t>0} \) satisfies Assumption SP with

\[
M > n + \frac{np_1}{p} + N_0 - \frac{n[r(\varphi) - 1]}{r(\varphi)},
\]

where \( N_0 \) and \( r(\varphi) \) are, respectively, as in (2.5) and (2.10), \( n \in [n_0, \infty) \), \( p_1 \in [p(\varphi), \infty) \) and \( p \in (0, i(\varphi)) \) with \( p(\varphi) \) and \( i(\varphi) \) being, respectively, as in (2.7) and (2.8) such that \( \mathcal{X} \) satisfies (2.2) for \( n, \varphi \in A_{p_1}(\mathcal{X}) \) and \( \varphi \) is of uniformly lower type \( p \). Then there exists a positive constant \( C \), depending on \( n, \alpha, p \) and \( p_1 \), such that, for all \( f \in \text{BMO}^\varphi_A(\mathcal{X}) \), \( t \in (0, \infty) \), \( K \in (1, \infty) \) and almost every \( x \in \mathcal{X} \),

\[
|A_t f(x) - A_K f(x)| \leq CK \frac{1}{m} \left( \frac{np_1}{p} - \alpha \right) \frac{\|\varphi\|_{L^p(\mathcal{X})}}{\mu(B(x, t^{1/m}))} \|f\|_{\text{BMO}^\varphi_A(\mathcal{X})},
\]

where \( \alpha \in [0, \alpha_0] \) such that \( \mathcal{X} \) satisfies (2.6) for \( \alpha \).

To prove Proposition 3.7, we first recall a result of Christ \cite[Theorem 11]{christ}, which gives an analogue of Euclidean dyadic cubes.

**Lemma 3.8.** There exists a collection of open subsets, \( \{Q^k_\alpha \subset \mathcal{X} : k \in \mathbb{Z}, \alpha \in I_k\} \), where \( I_k \) denotes some (possibly finite) index set, depending on \( k \), and constants \( \delta \in (0, 1) \), \( a_0 \in (0, 1) \), and \( D \in (0, \infty) \) such that

(i) \( \mu(\mathcal{X} \setminus \cup_{\alpha} Q^k_\alpha) = 0 \) for all \( k \in \mathbb{Z} \).
(ii) If \( l \geq k \), then either \( Q^l_\beta \subset Q^k_\alpha \) or \( Q^l_\beta \cap Q^k_\alpha = \emptyset \).
(iii) For each \( (k, \alpha) \) and each \( l < k \), there exists a unique \( \beta \) such that \( Q^k_\alpha \subset Q^l_\beta \).
(iv) The diameter of \( Q^k_\alpha \) is not more than \( D \delta^k \).
(v) Each \( Q^k_\alpha \) contains some ball \( B(z^k_\alpha, a_0 \delta^k) \).

Now we prove Proposition 3.7 by using Lemma 3.8.

**Proof of Proposition 3.7.** For any given \( t \in (0, \infty) \), choose \( s \in (0, \infty) \) such that \( \frac{s^m t}{s^m} \leq s \leq \frac{t}{s} \).

With the same notation as in Lemma 3.8, we first fix \( l_0 \) such that

\[
D \delta^{l_0} \leq s^{1/m} < D \delta^{l_0 - 1}.
\]

Fix \( x \in \mathcal{X} \). By (i) and (iv) of Lemma 3.8, we see that there exists a subset \( Q^{l_0}_{\alpha_0} \) such that \( x \in Q^{l_0}_{\alpha_0} \) and \( Q^{l_0}_{\alpha_0} \subset B(x, D \delta^{l_0}) \). For any \( k \in \mathbb{N} \), let

\[
M_k := \{ \beta \in I_{l_0} : Q^{l_0}_{\beta} \cap B(x, D \delta^{l_0 - k}) \neq \emptyset \},
\]

where \( I_{l_0} \) is as in Lemma 3.8. Using (i) and (iv) of Lemma 3.8 again, we know that

\[
B(x, D \delta^{l_0 - k}) \subset \bigcup_{\beta \in M_k} Q^{l_0}_{\beta} \subset B(x, D \delta^{l_0 - (k + k_0)}),
\]
where $k_0$ is an integer such that $\delta^{-k_0} \geq 2C_1$ and $C_1$ is as in (2.1).

In [17], it was proved that there exists a positive constant $C$, independent of $k$, such that the number of open subsets, $\{Q^k_\beta\}_{\beta \in M_k}$, is less than $C\delta^{-k(n+N)}$. Namely,

$$m_k := \#\{Q^k_\beta : \beta \in M_k\} \leq C\delta^{-k(n+N)},$$

where $n$ and $N$ are, respectively, as in (2.2) and (2.4), and $\#E$ denotes the cardinality of the set $E$.

By (3.11), we know that there exist $N \in [N_0, \infty)$ and $q \in (1, r(\varphi))$ such that $\mathcal{X}$ satisfies (2.4) for $N$, $\varphi \in \mathbb{R}^q(\mathcal{X})$ and $M > n + \frac{np}{p} + N - \frac{n(q-1)}{q}$.

We first estimate $|A_t f(x) - A_{t+s} f(x)|$ for the case $\frac{t}{s} \leq s \leq t$. By Assumption SP, we conclude that $A_t f - A_{t+s} f = A_t (f - A_s f)$ almost everywhere. From this and $f \in \text{BMO}^p_\varphi(\mathcal{X})$, we deduce that, for almost every $x \in \mathcal{X}$,

\begin{equation}
|A_t f(x) - A_{t+s} f(x)| \leq \int_{\mathcal{X}} h_t(x,y) |f(y) - A_s f(y)| \, d\mu(y)
\end{equation}

\begin{align*}
&= \frac{1}{\mu(B(x,t^{1/m}))} \int_{\mathcal{X}} g \left( \frac{|d(x,y)|^m}{t} \right) |f(y) - A_s f(y)| \, d\mu(y) \\
&\lesssim \|X_{B(x,t^{1/m})}^\varphi\|_{L^p(\mathcal{X})} \frac{1}{\mu(B(x,t^{1/m}))} \|X_{B(x,t^{1/m})}^\varphi\|_{L^p(\mathcal{X})} \int_{B(x,t^{1/m})} |f(y) - A_s f(y)| \, d\mu(y) \\
&\quad + \frac{1}{\mu(B(x,t^{1/m}))} \int_{B(x,t^{1/m})} g \left( \frac{|d(x,y)|^m}{t} \right) |f(y) - A_s f(y)| \, d\mu(y) \\
&\lesssim \|X_{B(x,t^{1/m})}^\varphi\|_{L^p(\mathcal{X})} \|f\|_{\text{BMO}^p_\varphi(\mathcal{X})} + I,
\end{align*}

where

$$I := \frac{1}{\mu(B(x,t^{1/m}))} \int_{B(x,t^{1/m})} g \left( \frac{|d(x,y)|^m}{t} \right) |f(y) - A_s f(y)| \, d\mu(y).$$

Notice that, for any $y \in B(x, D\delta_0^{-(k+1)}) \setminus B(x, D\delta_0^{-k})$, it holds true that $d(x,y) \geq D\delta_0^{-k}$, which, together with (3.13) and the decreasing property of $g$, implies that

\begin{equation}
\int_{B(x,t^{1/m})} g \left( \frac{|d(x,y)|^m}{t} \right) |f(y) - A_s f(y)| \, d\mu(y)
\end{equation}

\begin{align*}
&\leq \int_{B(x,D\delta_0^{-(k+1)})} g \left( \frac{|d(x,y)|^m}{t} \right) |f(y) - A_s f(y)| \, d\mu(y) \\
&\leq \sum_{k=0}^\infty \int_{B(x,D\delta_0^{-(k+1)}) \setminus B(x,D\delta_0^{-k})} g \left( \frac{|d(x,y)|^m}{t} \right) |f(y) - A_s f(y)| \, d\mu(y) \\
&\leq \sum_{k=0}^\infty g(4^{-1}\delta^{-(k-1)m}) \int_{B(x,D\delta_0^{-(k+1)})} |f(y) - A_s f(y)| \, d\mu(y) \\
&\leq \sum_{k=0}^\infty \sum_{\beta \in M_{k+1}} g_1 (\delta^{-km}) \int_{Q^k_\beta} |f(y) - A_s f(y)| \, d\mu(y),
\end{align*}
where $g_1(\cdot) := g(\delta^m / 4)$ satisfies the same property as in (3.5).

Applying (iv) of Lemma 3.8, we know that $Q_{\beta}^{l_0} \subset B(z_{\beta}^{l_0}, D \delta^{l_0}) \subset B(z_{\beta}^{l_0}, s^{1/m})$. By this, (3.15), $m_k \lesssim \delta^{-k(n+N)}$, (2.12), (2.13) and (2.6) and $M > n + \frac{np}{p} + N - \frac{n(q-1)}{q}$, we see that

\[
I \leq \sum_{k=0}^{\infty} \sum_{\beta \in M_{k+1}} g_1(\delta^{-km}) \frac{1}{\mu(B(x, t^{1/m}))} \int_{Q_{\beta}^{l_0}} |f(y) - A_s f(y)| \, d\mu(y)
\]

\[
\leq \sum_{k=0}^{\infty} \sum_{\beta \in M_{k+1}} g_1(\delta^{-km}) \frac{\|X_{B(z_{\beta}^{l_0}, s^{1/m})}\|_{L^p(\mathcal{X})}}{\mu(B(x, t^{1/m}))} \int_{B(z_{\beta}^{l_0}, s^{1/m})} |f(y) - A_s f(y)| \, d\mu(y)
\]

\[
\leq \sum_{k=0}^{\infty} \sum_{\beta \in M_{k+1}} g_1(\delta^{-km}) \frac{\|X_{B(z_{\beta}^{l_0}, s^{1/m})}\|_{L^p(\mathcal{X})}}{\mu(B(x, t^{1/m}))} \frac{\|X_{B(x, s^{1/m})}\|_{L^p(\mathcal{X})} \|f\|_{BMO_{A}^q(\mathcal{X})}}{\mu(B(x, t^{1/m}))} |f|_{BMO_{A}^q(\mathcal{X})}
\]

This, together with (3.14), (2.13) and (2.2), implies that, when $\frac{t}{4} \leq s \leq t$, for almost every $x \in \mathcal{X}$,

\[
|A_t f(x) - A_{t+s} f(x)| \lesssim \frac{\|X_{B(x, s^{1/m})}\|_{L^p(\mathcal{X})}}{\mu(B(x, t^{1/m}))} |f|_{BMO_{A}^q(\mathcal{X})}
\]

(3.16)

\[
\lesssim \frac{\|X_{B(x, t^{1/m})}\|_{L^p(\mathcal{X})}}{\mu(B(x, t^{1/m}))} |f|_{BMO_{A}^q(\mathcal{X})}.
\]

For the case $0 < s < t/4$, by Assumption SP, we write

\[
A_t f(x) - A_{t+s} f(x) = A_t f(x) - A_{2t} f(x) - A_{t+s} (f - A_{t-s} f)(x)
\]

for almost every $x \in \mathcal{X}$. In this case, $(t+s)/4 < t - s < t + s$. By this observation, together with (3.16) and (2.12), we conclude that, for almost every $x \in \mathcal{X}$,

\[
|A_t f(x) - A_{t+s} f(x)| \lesssim \left\{ \frac{\|X_{B(x, t^{1/m})}\|_{L^p(\mathcal{X})}}{\mu(B(x, t^{1/m}))} + \frac{\|X_{B(x, (t+s)^{1/m})}\|_{L^p(\mathcal{X})}}{\mu(B(x, (t+s)^{1/m}))} \right\} |f|_{BMO_{A}^q(\mathcal{X})}
\]

(3.17)

\[
\lesssim \frac{\|X_{B(x, t^{1/m})}\|_{L^p(\mathcal{X})}}{\mu(B(x, t^{1/m}))} |f|_{BMO_{A}^q(\mathcal{X})}.
\]

In general, for any $K \in (1, \infty)$, let $l$ be an integer such that $2^l \leq K < 2^{l+1}$. If $\text{diam}(\mathcal{X}) = \infty$, from (3.16), (3.17), (2.12) and (2.6), we deduce that, for almost every $x \in \mathcal{X}$,

\[
|A_t f(x) - A_{Kl} f(x)|
\]

(3.18)
\[
\leq \sum_{k=0}^{l-1} |A_{2k+1}f(x) - A_{2k+1}f(x)| + |A_{2l+1}f(x) - A_{2l}f(x)| \\
\leq \sum_{k=0}^{l} \frac{\|\chi_{B(x,(2k+1)m)}\|_{L^p(\mathcal{X})}}{\mu(B(x,(2k+1)m))} \|f\|_{BMO^p_\alpha(\mathcal{X})} \\
\leq \sum_{k=0}^{l} 2^{k - \frac{mp}{p - \alpha}} \frac{\|\chi_{B(x,t^m)}\|_{L^p(\mathcal{X})}}{\mu(B(x,t^m))} \|f\|_{BMO^p_\alpha(\mathcal{X})} \\
\leq K \frac{2^k - \frac{mp}{p - \alpha}}{\mu(B(x,t^m))} \|f\|_{BMO^p_\alpha(\mathcal{X})}.
\]

If \(\text{diam}(\mathcal{X}) < \infty\), we also have the same estimate as in (3.18) via some minor modifications similar to those used in the estimates for (2.15), which completes the proof of Proposition 3.7.

Applying Proposition 3.7, we further prove the following size estimate for functions in \(BMO^p_\alpha(\mathcal{X})\) at infinity.

**Proposition 3.9.** Let \(\mathcal{X}\) be a space of homogeneous type with degree \((\alpha_0, n_0, N_0)\), where \(\alpha_0, n_0\) and \(N_0\) are as in (2.7), (2.3) and (2.5), respectively. Assume that \(\varphi\) is as in Definition 2.7 and \(\{A_t\}_{t>0}\) satisfies Assumption SP. Let \(x_0 \in \mathcal{X}\) and

\[
(3.19) \quad \delta > n_0 + N_0 + \frac{1}{m} \left[ \frac{n_0 p(\varphi)}{i(\varphi)} - \alpha_0 \right] - \frac{n_0 [r(\varphi) - 1]}{r(\varphi)},
\]

where \(m, p(\varphi), i(\varphi)\) and \(r(\varphi)\) are, respectively, as in (3.4) (2.9), (2.8) and (2.10). Then there exists a positive constant \(C(\delta)\), depending on \(\delta\), such that, for all \(f \in BMO^p_\alpha(\mathcal{X})\),

\[
\int_{\mathcal{X}} \frac{|f(x) - A_t f(x)|}{[t^{1/m} + d(x_0,x)]^\delta \|\chi_{B(x,t^{1/m} + d(x_0,x))}\|_{L^p(\mathcal{X})}} d\mu(x) \leq \frac{C(\delta)}{t^{1/m}} \|f\|_{BMO^p_\alpha(\mathcal{X})}.
\]

**Proof.** By (3.19), we know that there exist \(n \in [n_0, \infty)\), \(N \in [N_0, \infty)\), \(\alpha \in [0, \alpha_0]\), \(p \in [p(\varphi), \infty)\), \(i \in (0, i(\varphi))\) and \(q \in (1, r(\varphi))\) such that \(\mathcal{X}\) satisfies (2.2), (2.4) and (2.6), respectively, for \(n, N\) and \(\alpha, \varphi \in A_p(\mathcal{X})\) and \(\varphi\) is of uniformly lower type \(p, \varphi \in \mathcal{R}_q(\mathcal{X})\) and \(\delta > n + N + \frac{1}{m} \left[ \frac{n_0 p(\varphi)}{i(\varphi)} - \alpha_0 \right] - \frac{n_0 [r(\varphi) - 1]}{r(\varphi)}\).

Let \(B := B(x_0, t^{1/m})\) and \(k \in \mathbb{N}\). From Proposition 3.7, (2.12) and (2.13), we deduce that

\[
\frac{1}{\|\chi_{2kB}\|_{L^p(\mathcal{X})}} \int_{2kB} |f(x) - A_t f(x)| d\mu(x) \\
\leq \frac{1}{\|\chi_{2kB}\|_{L^p(\mathcal{X})}} \int_{2kB} |f(x) - A_{2k} f(x)| d\mu(x) \\
+ \frac{1}{\|\chi_{2kB}\|_{L^p(\mathcal{X})}} \int_{2kB} |A_{2k} f(x) - A_t f(x)| d\mu(x) \\
\leq \|f\|_{BMO^p_\alpha(\mathcal{X})} + \frac{\|f\|_{BMO^p_\alpha(\mathcal{X})}}{\|\chi_{2kB}\|_{L^p(\mathcal{X})}} \int_{2kB} \frac{\|\chi_{B(x,t^{1/m})}\|_{L^p(\mathcal{X})}}{\mu(B(x,t^{1/m}))} 2^{k \frac{mp}{p - \alpha}} d\mu(x)
\]
\[ \lesssim \left\{ 1 + 2^{k[n+N+\frac{1}{m}(\frac{p_1}{p} - \alpha) - \frac{n(q-1)}{q}]} \right\} \|f\|_{BMO^\varphi_A(X)}, \]

which, together with \( \delta > n + N + \frac{1}{m}(\frac{np_1}{p} - \alpha) - \frac{n(q-1)}{q} \), implies that

\[
\begin{align*}
& \int_X \frac{|f(x) - A_if(x)|}{[t^{1/m} + d(x_0, x)]^\delta \|\chi_B(x_0, t^{1/m} + d(x_0, x))\|_{L^p(X)}} \ d\mu(x) \\
& \leq \sum_{k=0}^{\infty} \int_{2kB \setminus 2k-1B} \frac{1}{[t^{1/m} + d(x_0, x)]^\delta \|\chi_B(x_0, t^{1/m} + d(x_0, x))\|_{L^p(X)}} \ |f(x) - A_if(x)| \ d\mu(x) \\
& \lesssim \sum_{k=0}^{\infty} (2^k t^{1/m})^{-\delta} \frac{1}{\|\chi_{2kB}\|_{L^p(X)}} \int_{2kB} |f(x) - A_if(x)| \ d\mu(x) \\
& \lesssim \frac{1}{t^{\delta/m}} \left( \sum_{k=0}^{\infty} 2^{-\delta k} + \sum_{k=0}^{\infty} 2^{k[n+N+\frac{1}{m}(\frac{np_1}{p} - \alpha) - \frac{n(q-1)}{q}]} \right) \|f\|_{BMO^\varphi_A(X)} \lesssim \frac{1}{t^{\delta/m}} \|f\|_{BMO^\varphi_A(X)}. 
\end{align*}
\]

This finishes the proof of Proposition 3.9. \( \square \)

### 3.3 Two characterizations of \( \text{BMO}^\varphi_A(X) \)

For the need of the following sections, from now on, we **always assume** that the following assumption on the generalized approximation to the identity, \( \{A_t\}_{t>0} \).

**Assumption A.** Let \( \mathcal{X} \) be a space of homogeneous type with degree \((\alpha_0, n_0, N_0)\), where \( \alpha_0, n_0 \) and \( N_0 \) are as in (2.7), (2.3) and (2.5), respectively. Assume that \( \varphi \) is as in Definition 2.7 and \( \{A_t\}_{t>0} \) satisfies Assumption SP with \( M \) in (3.5) additionally satisfying that

\[
M > n_0 + \frac{2n_0 p(\varphi)}{i(\varphi)} + N_0 - \frac{n_0[r(\varphi) - 1]}{r(\varphi)} - \alpha_0,
\]

where \( p(\varphi), i(\varphi), \) and \( r(\varphi) \) are, respectively, as in (2.9), (2.8) and (2.10). \( \square \)

**Remark 3.10.** If \( M \) is as in Assumption A, we then conclude that \( M \) also satisfies (3.11) in Proposition 3.7. Indeed, if \( M \) is as in Assumption A, by the definitions of \( n_0, N_0, \alpha_0, p(\varphi), i(\varphi) \) and \( r(\varphi) \), we know that there exist \( n \in [n_0, \infty), \ N \in [N_0, \infty), \ \alpha \in [0, \alpha_0], \ p_1 \in [p(\varphi), \infty), \ p \in (0, i(\varphi)] \) and \( q \in (1, r(\varphi)] \) such that \( \mathcal{X} \) satisfies (2.2), (2.4) and (2.6), respectively, for \( n, N \) and \( \alpha, \varphi \in A_{p_1}(\mathcal{X}) \) and \( \varphi \) is of uniformly lower type \( p, \varphi \in \mathbb{RH}_q(\mathcal{X}) \) and

\[
M > n + \frac{2np_1}{p} + N - \frac{n(q-1)}{q} - \alpha,
\]

which, together with \( n \geq \alpha \) and \( p_1 \geq p \), implies the above claim.

From now on, we **always use** the labels \( n, N, \alpha, p_1, p \) and \( q \) as in Remark 3.10 to characterize the space of homogeneous type \( \mathcal{X} \) and the growth function \( \varphi \). We first introduce the space \( \text{BMO}^\varphi_{A,\text{max}}(\mathcal{X}) \).
**Definition 3.11.** Let $\mathcal{X}$ be a space of homogeneous type, $\varphi$ as in Definition 2.7 and \(\{A_t\}_{t>0}\) a generalized approximation to the identity satisfying (3.4) and (3.5). The space $\text{BMO}_{\varphi}^A(\mathcal{X})$ is defined to be the set of all $f \in \mathcal{M}(\mathcal{X})$ such that

\begin{equation}
(3.21) \quad \|f\|_{\text{BMO}_{\varphi}^A(\mathcal{X})} := \sup_{t \in (0, \infty), x \in \mathcal{X}} \frac{\mu(B(x, t^{1/m}))}{\|\chi_{B(x, t^{1/m})}\|_{L^\varphi(\mathcal{X})}} |A_t([f - A_tf])(x)| < \infty.
\end{equation}

We are ready to obtain the first characterization of $\text{BMO}_{\varphi}^A(\mathcal{X})$ with the following extra assumption (3.22) on the kernel $a_t$ of $A_t$.

**Theorem 3.12.** Let $\mathcal{X}$ be a space of homogeneous type with degree $(\alpha_0, n_0, N_0)$, where $\alpha_0$, $n_0$ and $N_0$ are as in (2.7), (2.3) and (2.5), respectively. Assume that $\varphi$ is as in Definition 2.7, \(\{A_t\}_{t>0}\) satisfies Assumption A, and, for any $t \in (0, \infty)$, the kernel $a_t$ of the operator $A_t$ is a nonnegative function satisfying the following lower bound: for all $t \in (0, \infty)$, $x \in \mathcal{X}$ and $y \in B(x, t^{1/m})$,

\begin{equation}
(3.22) \quad a_t(x, y) \geq \frac{C}{\mu(B(x, t^{1/m}))},
\end{equation}

where $C$ is a positive constant independent of $t$, $x$ and $y$. Then the spaces $\text{BMO}_{\varphi}^A(\mathcal{X})$ and $\text{BMO}_{\varphi}^A(\mathcal{X})$ coincide with equivalent norms.

**Proof.** We first prove $\text{BMO}_{\varphi}^A(\mathcal{X}) \subset \text{BMO}_{\varphi}^A(\mathcal{X})$. For any fixed $x \in \mathcal{X}$ and $t \in (0, \infty)$, let $B := B(x, t^{1/m})$. Let $f \in \text{BMO}_{\varphi}^A(\mathcal{X})$. Since $\{A_t\}_{t>0}$ satisfies Assumption A, we then choose $n$, $N$, $\alpha$, $p_1$, $p$ and $q$ as in Remark 3.10 such that (3.20) holds true. From (3.20), we further deduce that $M > \frac{np}{p}$. By this and (3.20), together with Proposition 3.7, Lemma 2.13, and the decreasing property of $g$, we see that

\[
|A_t([f - A_tf])(x)| \\
\leq \int_{\mathcal{X}} |a_t(x, y)||f(y) - A_tf(y)| \, d\mu(y) \\
\leq \sum_{k=0}^{\infty} \frac{1}{\mu(B)} \int_{2^k B \setminus 2^{k-1} B} g \left( \frac{|d(x, y)|^m}{t} \right) |f(y) - A_tf(y)| \, d\mu(y) \\
\lesssim \sum_{k=0}^{\infty} g(2^{(k-1)m}) \frac{1}{\mu(B)} \left[ \int_{2^k B} |f(y) - A_{t2^k B}f(y)| \, d\mu(y) + \int_{2^k B} |A_{t2^k B}f(y) - A_tf(y)| \, d\mu(y) \right] \\
\lesssim \sum_{k=0}^{\infty} g(2^{(k-1)m}) \frac{1}{\mu(B)} \left[ \|\chi_{2^k B}\|_{L^\varphi(\mathcal{X})} + 2^{k(n_1-p_1) - \alpha} \int_{2^k B} \frac{\|\chi_{B(y, t^{1/m})}\|_{L^\varphi(\mathcal{X})}}{q(\mu(B(y, t^{1/m})))} \, d\mu(y) \right] \|f\|_{\text{BMO}_{\varphi}^A(\mathcal{X})} \\
\lesssim \sum_{k=0}^{\infty} \left[ 2^{k(\frac{np}{p} - M)} + 2^{k(n + \frac{2np}{p}) + N - \frac{q(q-1)}{q} - \alpha - M} \right] \frac{\|\chi_B\|_{L^\varphi(\mathcal{X})}}{\mu(B)} \|f\|_{\text{BMO}_{\varphi}^A(\mathcal{X})}
\]
for a given function
functions,
appropriate estimates.
with bounded, real symmetric coefficients on \(\mathbb{R}\)
the condition (3.22) include heat kernels of uniformly divergence form elliptic operators
which, together with the arbitrariness of \(x\) and \(t\in (0, \infty)\), implies that \(f \in \text{BMO}_{A, \max}^\varphi(\mathcal{X})\) and \(\|f\|_{\text{BMO}_{A, \max}^\varphi(\mathcal{X})} \lesssim \|f\|_{\text{BMO}_{A}^\varphi(\mathcal{X})}\).

Conversely, let \(f \in \text{BMO}_{A, \max}^\varphi(\mathcal{X})\) and \(B := B(x, r_B)\) with \(x \in \mathcal{X}\) and \(r_B \in (0, \infty)\). Let \(t_B := r_B^m\). Then, from the assumption (3.22), it follows that

\[
\frac{1}{\|\chi_B\|_{L^p(\mathcal{X})}} \int_B |f(y) - A_{t_B} f(y)| \, d\mu(y) \lesssim \frac{\mu(B(x, t_B^m))}{\|\chi_{B(x, t_B^m)}\|_{L^p(\mathcal{X})}} \int_{B(x, t_B)} a_{t_B}(x, y) |f(y) - A_{t_B} f(y)| \, d\mu(y) \lesssim \|f\|_{\text{BMO}_{A, \max}^{\varphi}(\mathcal{X})},
\]

which implies that \(f \in \text{BMO}_{A}^\varphi(\mathcal{X})\) and \(\|f\|_{\text{BMO}_{A}^\varphi(\mathcal{X})} \lesssim \|f\|_{\text{BMO}_{A, \max}^{\varphi}(\mathcal{X})}\). This finishes the proof of Theorem 3.12.

**Remark 3.13.** It was pointed out by Duong and Yan [17] that examples of \(a_t(x, y)\) satisfy the condition (3.22) include heat kernels of uniformly divergence form elliptic operators with bounded, real symmetric coefficients on \(\mathbb{R}^n\), and the Laplace-Beltrami operator on a complete Riemannian manifold \(M\) with nonnegative Ricci curvature (see also [11, Theorems 3.3.4 and 5.6.1]).

Next, we give another equivalent characterization of \(\text{BMO}_{A}^\varphi(\mathcal{X})\). In other words, the average value \(A_{t_B} f\) in Definition 3.3 can be changed into other value \(f_B\) which satisfies appropriate estimates.

**Definition 3.14.** Let \(\mathcal{X}\) be a space of homogeneous type with degree \((\alpha_0, n_0, N_0)\), where \(\alpha_0, n_0\) and \(N_0\) are as in (2.7), (2.3) and (2.5), respectively. Assume that \(\varphi\) is as in Definition 2.7 and \(\{A_t\}_{t \geq 0}\) a generalized approximation to the identity satisfying (3.4) and (3.5). If, for a given function \(f \in M(\mathcal{X})\), there exist a positive constant \(C\) and a collection of functions, \(\{f_B\}_B\) (in other words, for each ball \(B\), there exists a function \(f_B\)), such that

\[
\sup_{B \subset \mathcal{X}} \frac{1}{\|\chi_B\|_{L^p(\mathcal{X})}} \int_B |f(x) - f_B(x)| \, d\mu(x) \leq C,
\]

\[
|f_{B_2}(x) - f_{B_1}(x)| \leq C \frac{\|\chi_{B(x, r_{B_1})}\|_{L^p(\mathcal{X})}}{\mu(B(x, r_{B_1}))} \left(\frac{r_{B_2}}{r_{B_1}}\right)^{\frac{n_0}{p} - \alpha}
\]

for any two balls \(B_1 \subset B_2\), and

\[
|f_B(x) - A_{t_B} f_B(x)| \leq C \frac{\|\chi_{B(x, r_B)}\|_{L^p(\mathcal{X})}}{\mu(B(x, r_B))}
\]

for almost every \(x \in \mathcal{X}\), where \(t_B = r_B^m\), then it is said that \(f\) belongs to the space \(\text{BMO}_{A}^\varphi(\mathcal{X})\) and

\[
\|f\|_{\text{BMO}_{A}^\varphi(\mathcal{X})} := \inf \{C : \text{C satisfies (3.23), (3.24) and (3.25)}\},
\]

where the infimum is taken over all the constants \(C\) as above and all the functions \(\{f_B\}_B\) satisfying (3.23), (3.24) and (3.25).
We have the following equivalence between the spaces $\text{BMO}_A^\varphi(\mathcal{X})$ and $\widetilde{\text{BMO}}_A^\varphi(\mathcal{X})$.

**Theorem 3.15.** Let $\mathcal{X}$ be a space of homogeneous type with degree $(\alpha_0, n_0, N_0)$, where $\alpha_0$, $n_0$ and $N_0$ are as in (2.7), (2.3) and (2.5), respectively. Assume that $\varphi$ is as in Definition 2.7 and $\{A_t\}_{t>0}$ satisfies Assumption A. The spaces $\text{BMO}_A^\varphi(\mathcal{X})$ and $\widetilde{\text{BMO}}_A^\varphi(\mathcal{X})$ coincide with equivalent norms.

**Proof.** Let $f \in \mathcal{M}(\mathcal{X})$. It is easy to prove that $\text{BMO}_A^\varphi(\mathcal{X}) \subset \widetilde{\text{BMO}}_A^\varphi(\mathcal{X})$ and, for all $f \in \text{BMO}_A^\varphi(\mathcal{X})$, $\|f\|_{\text{BMO}_A^\varphi(\mathcal{X})} \lesssim \|f\|_{\text{BMO}_A^\varphi(\mathcal{X})}$. Indeed, let $f^B(x) := A_{tB}f(x)$ for each ball $B$ and $x \in \mathcal{X}$. Then, by Proposition 3.7, the estimates (3.23), (3.24) and (3.25) hold true with $C$ replaced by $\tilde{C}\|f\|_{\text{BMO}_A^\varphi(\mathcal{X})}$, where $\tilde{C}$ is a positive constant independent of $f$.

Conversely, we need to prove that $\widetilde{\text{BMO}}_A^\varphi(\mathcal{X}) \subset \text{BMO}_A^\varphi(\mathcal{X})$ and, for all $f \in \widetilde{\text{BMO}}_A^\varphi(\mathcal{X})$, $\|f\|_{\text{BMO}_A^\varphi(\mathcal{X})} \lesssim \|f\|_{\text{BMO}_A^\varphi(\mathcal{X})}$. To this end, for any $f \in \text{BMO}_A^\varphi(\mathcal{X})$ and fixed ball $B_0 := B(x_0, r_{B_0})$ with $x_0 \in \mathcal{X}$ and $r_{B_0} \in (0, \infty)$, it suffices to prove that

$$
\frac{1}{\|X_{B_0}\|_{L^\varphi(\mathcal{X})}} \int_{B_0} |f(x) - A_{t_{B_0}}f(x)| \, d\mu(x) \lesssim \|f\|_{\widetilde{\text{BMO}}_A^\varphi(\mathcal{X})},
$$

where $t_{B_0} := r_{B_0}^{m}$. For any $x \in B_0$, by (2.4), we see that $\mu(B_0) \lesssim \mu(B(x, t_{B_0}^{1/m}))$. Notice that $M$ satisfies (3.20) by Assumption A together with Remark 3.10. Thus, $m > \frac{n_p}{p}$. From this, (3.20), (3.23) and (3.24), together with the decreasing property of $g$, we deduce that

$$
|A_{t_{B_0}}(f - f^{B_0})(x)|
\leq \frac{1}{\mu(B(x, t_{B_0}^{1/m}))} \int_{\mathcal{X}} g\left(\frac{[d(x, y)]^m}{t_{B_0}}\right) |f(y) - f^{B_0}(y)| \, d\mu(y)
\lesssim \sum_{k=0}^\infty \frac{1}{\mu(B_0)} \int_{2^k B_0 \setminus 2^{k-1} B_0} g\left(\frac{[d(x, y)]^m}{t_{B_0}}\right) |f(y) - f^{B_0}(y)| \, d\mu(y)
\lesssim \sum_{k=0}^\infty g(2^{(k-2)m}) \frac{1}{\mu(B_0)} \left[ \int_{2^k B_0} |f(y) - f^{2^k B_0}(y)| \, d\mu(y) + \int_{2^k B_0} |f^{2^k B_0}(y) - f^{B_0}(y)| \, d\mu(y) \right]
\lesssim \sum_{k=0}^\infty g(2^{(k-2)m}) \frac{1}{\mu(B_0)} \left[ \|X_{2^k B_0}\|_{L^\varphi(\mathcal{X})} \|f\|_{\text{BMO}_A^\varphi(\mathcal{X})} + 2^k \frac{n_p}{p} \int_{2^k B_0} \frac{\|X_B(y, r_{B_0})\|_{L^\varphi(\mathcal{X})}}{\mu(B(y, r_{B_0}))} \|f\|_{\text{BMO}_A^\varphi(\mathcal{X})} \, d\mu(y) \right]
\lesssim \sum_{k=0}^\infty \left[ 2^k \left( \frac{n_p}{p} - \alpha \right) + 2^k n + 2^{n_p} \frac{1}{p} + N - \frac{n(q-1)}{q} - \alpha + M \right] \frac{\|X_{B_0}\|_{L^\varphi(\mathcal{X})}}{\mu(B_0)} \|f\|_{\text{BMO}_A^\varphi(\mathcal{X})}
\lesssim \frac{\|X_{B_0}\|_{L^\varphi(\mathcal{X})}}{\mu(B_0)} \|f\|_{\text{BMO}_A^\varphi(\mathcal{X})},
$$
which, together with (3.23), (3.25) and (2.12), implies that

\[
\frac{1}{\|\chi_{B_0}\|_{L^\varphi(X)}} \int_{B_0} |f(x) - A_{tB_0}(f)(x)| \, d\mu(x) \\
\leq \frac{1}{\|\chi_{B_0}\|_{L^\varphi(X)}} \int_{B_0} \left[ |f(x) - f(B_0)(x)| + |f(B_0)(x) - A_{tB_0}(f)(B_0)(x)| \right] \\
+ |A_{tB_0}(f - f(B_0))(x)| \, d\mu(x) \\
\lesssim \|f\|_{\widetilde{BMO}_\varphi^\varphi(X)} \left[ 1 + \frac{1}{\|\chi_{B_0}\|_{L^\varphi(X)}} \int_{B_0} \frac{\|\chi_{B(x,rB_0)}\|_{L^\varphi(X)}}{\mu(B(x,rB_0))} \, d\mu(x) \right] \lesssim \|f\|_{\widetilde{BMO}_\varphi(A(X))}.
\]

By this, combined with the arbitrariness of \(B_0 \subset \mathcal{X}\), we then conclude that \(f \in \text{BMO}_\varphi(A(X))\) and \(\|f\|_{\text{BMO}_\varphi(A(X))} \lesssim \|f\|_{\widetilde{BMO}_\varphi(A\varphi(X))}\), which completes the proof of Theorem 3.15.

\[\square\]

Remark 3.16. Theorems 3.12 and 3.15 completely cover, respectively, [17, Propositions 2.10 and 2.12] by taking \(\varphi\) as in (1.1). Moreover, Theorem 3.15 completely covers [45, Proposition 2.4] by taking \(\varphi\) as in (1.2).

4 Two variants of the John-Nirenberg inequality on \(\text{BMO}_\varphi(A(X))\)

In this section, we establish two variants of the John-Nirenberg inequality on \(\text{BMO}_\varphi(A(X))\). We then discuss the relationship between these two John-Nirenberg inequalities in Remark 4.10 when \(\varphi \in \mathcal{A}_1(\mathcal{X})\). Moreover, we also introduce the Musielak-Orlicz BMO-type spaces \(BMO_{\varphi, \overline{p}}(\mathcal{X})\) and \(\widetilde{BMO}_{\varphi, \overline{p}}(\mathcal{X})\) with \(\overline{p} \in [1, \infty)\). As an application of these John-Nirenberg inequalities on \(\text{BMO}_\varphi(A(X))\), we further prove that, for any \(\overline{p} \in [1, \infty)\), the spaces \(BMO_{\varphi, \overline{p}}(\mathcal{X})\), \(\widetilde{BMO}_{\varphi, \overline{p}}(\mathcal{X})\) and \(\text{BMO}_{\varphi, \overline{p}}(\mathcal{X})\) coincide with equivalent norms.

4.1 The first variant of the John-Nirenberg inequality on \(\text{BMO}_\varphi(A(X))\)

In order to establish the first variant of the John-Nirenberg inequality on the space \(\text{BMO}_\varphi(A(X))\), we assume that the growth function \(\varphi\) satisfies the following property: there exists a positive constant \(C\) such that, for all balls \(B_1, B_2 \subset \mathcal{X}\) with \(B_1 \subset B_2\),

\[
(4.1) \quad \frac{\|\chi_{B_1}\|_{L^\varphi(X)}}{\mu(B_1)} \leq C \frac{\|\chi_{B_2}\|_{L^\varphi(X)}}{\mu(B_2)}.
\]

Remark 4.1. Let \(\mathcal{X}\) be a space of homogeneous type with degree \((\alpha_0, n_0, N_0)\), where \(\alpha_0, n_0\) and \(N_0\) are as in (2.7), (2.3) and (2.5), respectively. There exist following nontrivial growth functions \(\varphi\) satisfying (4.1).

1. Let \(\varphi(x, t) := t\overline{p}\) for all \(x \in \mathcal{X}\) and \(t \in [0, \infty)\), where \(\overline{p} \in (0, 1]\). Then, \(\varphi\) satisfies (4.1).
Indeed, it is easy to see that, for any ball $B \subset \mathcal{X}$, $\|\chi_B\|_{L^\varphi(\mathcal{X})} = [\mu(B)]^{\frac{1}{p}}$. Then, we know that, for all balls $B_1, B_2 \subset \mathcal{X}$ with $B_1 \subset B_2$,

$$\frac{\|\chi_{B_1}\|_{L^\varphi(\mathcal{X})}}{\|\chi_{B_2}\|_{L^\varphi(\mathcal{X})}} = \frac{[\mu(B_1)]^{\frac{1}{p}}}{[\mu(B_2)]^{\frac{1}{p}}} \leq \frac{\mu(B_1)}{\mu(B_2)},$$

which implies that (4.1) holds true.

(ii) Let $\varphi(x,t) := \omega(x)t^{\bar{p}}$ for all $x \in \mathcal{X}$ and $t \in [0, \infty)$, where $\bar{p} \in (0, 1]$, $\omega \in A_\infty(\mathcal{X}) \cap RH_{\bar{q}}(\mathcal{X})$ with $\bar{q} \in (1, \infty)$. If $\bar{p} \in (0, \frac{\alpha}{\alpha+1})$, then $\varphi$ satisfies (4.1).

Indeed, it is easy to see that, for any ball $B \subset \mathcal{X}$, $\|\chi_B\|_{L^\varphi(\mathcal{X})} = [\omega(B)]^{\frac{1}{p}}$. Then, by (2.13), we know that, for all balls $B_1, B_2 \subset \mathcal{X}$ with $B_1 \subset B_2$,

$$\frac{\|\chi_{B_1}\|_{L^\varphi(\mathcal{X})}}{\|\chi_{B_2}\|_{L^\varphi(\mathcal{X})}} = \frac{[\omega(B_1)]^{\frac{1}{p}}}{[\omega(B_2)]^{\frac{1}{p}}} \leq \frac{\mu(B_1)}{\mu(B_2)},$$

which also implies that (4.1) holds true.

(iii) Let $x_0 \in \mathcal{X}$ and $\varphi(x,t) := \ln(e + [d(x_0,x)]^\bar{p})$, with $\bar{p} \in (0, 1]$, for all $x \in \mathcal{X}$ and $t \in [0, \infty)$. If $\alpha_0 \in (0, \infty)$ and $\bar{p} \in (0, \frac{\alpha_0}{\alpha_0+1})$, then $\varphi$ satisfies (4.1).

Indeed, $\mathcal{X}$ satisfies (2.4) and (2.6), respectively, for any $\tilde{N} \in (N_0, \infty)$ and $\tilde{\alpha} \in (0, \alpha_0)$. It is easy to see that $\varphi \in A_1(\mathcal{X})$ and

$$\|\chi_B\|_{L^\varphi(\mathcal{X})} \sim \frac{[\mu(B)]^{\frac{1}{p}}}{\ln(e + \mu(B)^{-1}) + \sup_{x \in B} \ln(e + d(x_0,x))}$$

for any ball $B \subset \mathcal{X}$. Then, for any $\tilde{p} \in (0, \frac{\alpha_0}{\alpha_0+1})$, $B_1 := B(x_1, r_1)$ with $x_1 \in \mathcal{X}$ and $r_1 \in (0, \infty)$, $B_2 := B(x_2, r_2)$ with $x_2 \in \mathcal{X}$ and $r_2 \in (0, \infty)$, and $B_1 \subset B_2$, it holds true that

$$\frac{\mu(B_2)}{\mu(B_1)}\|\chi_{B_1}\|_{L^\varphi(\mathcal{X})} \sim \frac{[\mu(B_1)]^{\frac{1}{p}}}{[\mu(B_2)]^{\frac{1}{p}}} \leq \frac{\mu(B_1)}{\mu(B_2)} \leq \frac{\ln(e + [\mu(B_2)]^{-1}) + \sup_{x \in B_2} \ln(e + d(x_0,x))}{\ln(e + [\mu(B_1)]^{-1}) + \sup_{x \in B_1} \ln(e + d(x_0,x))} \leq \left(\frac{r_1}{r_2}\right)^{\tilde{\alpha} (\frac{1}{p} - 1)} \lesssim 1,$$

which implies that (4.1) holds true.

Moreover, we point out that, when $\mathcal{X} := \mathbb{R}^n$ and $\varphi(x,t) = \ln(e + t^{\tilde{p}})$, with $\tilde{p} \in (0, 1]$, for all $x \in \mathbb{R}^n$ and $t \in [0, \infty)$, the Musielak-Orlicz Hardy space $H^{\varphi}(\mathbb{R}^n)$ related to $\varphi$ arises naturally in the study of pointwise products of functions in Hardy spaces $H^{\varphi}(\mathbb{R}^n)$ with functions in BMO$^\varphi(\mathbb{R}^n)$ (see [2]) in the setting of holomorphic functions in convex domains of finite type or strictly pseudoconvex domains in $\mathbb{C}^n$, where the space $H^{\varphi}(\mathbb{R}^n)$ is introduced by Ky [28], and its dual space is BMO$^\varphi(\mathbb{R}^n)$. In this case, (4.1) holds true for $\varphi$ with $\tilde{p} \in (0, 1)$, which needs more precise estimates, the details being omitted. However, when $\tilde{p} = 1$, (4.1) does not hold true. Indeed, we choose $B_1 := B(0,1)$ and $B_2 := B(0,r)$ for $r \in (1, \infty)$. Letting $r \to \infty$, we then find that

$$\frac{|B_2|}{|B_1|} \|\chi_{B_1}\|_{L^\varphi(\mathcal{X})} \sim \frac{\ln(e + |B_2|^{-1}) + \sup_{x \in B_2} \ln(e + |x|)}{\ln(e + |B_1|^{-1}) + \sup_{x \in B_1} \ln(e + |x|)}$$
\[ \sim \frac{\ln(e + r^{-n}|B_1|^{-1}) + \ln(e + r)}{\ln(e + |B_1|^{-1}) + \ln(e + 1)} \to \infty, \]

which implies that (4.1) does not hold true.

Now we establish the first variant of the John-Nirenberg inequality on \( \text{BMO}_A^\alpha(X) \) by using Proposition 3.8 and borrowing some ideas from Duong and Yan [17]. Recall that the Hardy-Littlewood maximal operator \( M \) is defined by setting, for all \( f \in L^1_{\text{loc}}(X) \) and \( x \in X \),

\[ M(f)(x) := \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y)| \, d\mu(y), \]

where the supremum is taken over all balls in \( X \) containing \( x \).

**Theorem 4.2.** Let \( X \) be a space of homogeneous type with degree \( (\alpha_0, n_0, N_0) \), where \( \alpha_0, n_0 \) and \( N_0 \) are as in (2.7), (2.3) and (2.5), respectively. Assume that \( \varphi \) is as in Definition 2.7 and satisfies (4.1). Let \( \{A_t\}_{t > 0} \) satisfy Assumption A. Then, there exist positive constants \( c_1 \) and \( c_2 \) such that, for all \( f \in \text{BMO}_A^\alpha(X) \), balls \( B \) and \( \lambda \in (0, \infty) \),

\[ \mu(\{x \in B : |f(x) - A_{t_B}f(x)| > \lambda\}) \leq c_1 \mu(B) \exp \left\{ - \frac{c_2 \lambda \mu(B)}{\|X\|_{L^\alpha(X)} \|f\|_{\text{BMO}_A^\alpha(X)}} \right\}, \]

where \( t_B := r_B^m \) and \( m \) is as in (3.4).

**Proof.** Since \( \{A_t\}_{t > 0} \) satisfies Assumption A, we then choose \( n, N, \alpha, p_1, p \) and \( q \) as in Remark 3.10 such that (3.20) holds true.

Let \( B := B(x_B, r_B) \), with \( x_B \in X \) and \( r_B \in (0, \infty) \), and \( f \in \text{BMO}_A^\alpha(X) \). In order to prove (4.2), it suffices to consider the case \( \|f\|_{\text{BMO}_A^\alpha(X)} > 0 \). Otherwise, (4.2) holds true obviously. Without loss of generality, we may assume that

\[ \frac{\mu(B)}{\|X\|_{L^\alpha(X)} \|f\|_{\text{BMO}_A^\alpha(X)}} = 1. \]

Otherwise, we replace \( f \) by \( \frac{\mu(B)f}{\|X\|_{L^\alpha(X)} \|f\|_{\text{BMO}_A^\alpha(X)}} \). Thus, we only need to prove that there exist positive constants \( c_1 \) and \( c_2 \) such that, for any \( \lambda \in (0, \infty) \),

\[ \mu(\{x \in B : |f(x) - A_{t_B}f(x)| > \lambda\}) \leq c_1 e^{-c_2 \lambda} \mu(B), \]

where \( t_B := r_B^m \).

It is obvious that, when \( \lambda \in (0, 1) \), (4.4) holds true for \( c_1 := e \) and \( c_2 := 1 \).

Let \( \lambda \in [1, \infty) \) and \( f_0 := (f - A_{t_B}f)\chi_{10C_1^4B} \), where \( C_1 \) is as in (2.1). By Proposition 3.7, (2.2), (2.4), (2.12), (2.13) and (4.3), we know that

\[ \|f_0\|_{L^1(X)} = \int_{10C_1^4B} |f(x) - A_{t_B}f(x)| \, d\mu(x) \]

\[ \leq \int_{10C_1^4B} \left| f(x) - A_{t_{10C_1^4B}}f(x) \right| \, d\mu(x) \]
From the fact that $c$ is a positive constant of radius $r$, we deduce that there exists a collection of balls, \[ B(x, r) \text{ for } x \in \Omega, \]

From [9, Chapter III, Theorem 1.3], we deduce that there exists a collection of balls, \[ \{B_{1,i}\}_{i \in \mathbb{N}}, \]
satisfying that

(i) each point of $\Omega$ is contained in at most a finite number $L$ of the balls $B_{1,i}$;

(ii) there exists $C \in (1, \infty)$ such that $CB_{1,i} \cap F \neq \emptyset$ for each $i$.

By (i), we see that, for any $x \in B \setminus (\bigcup_i B_{1,i})$, \[ |f(x) - A_t f(x)| = |f_0(x)| \chi_F(x) \leq M(f_0)(x) \chi_F(x) \leq \beta. \]

From the fact that $M$ is of weak type $(1,1)$, (i), (ii) and (4.5), it follows that there exists a positive constant $c_3$ such that \[ \sum_i \mu(B_{1,i}) \leq L \mu(\Omega) \leq \frac{1}{\beta} \|f_0\|_{L^1(\mathcal{X})} \leq \frac{c_3}{\beta} \mu(B). \] (4.6)

For any $B_{1,i} \cap B \neq \emptyset$, we denote by $B_{1,i} := B(x_{B_{1,i}}, r_{B_{1,i}})$ the ball centered at $x_{B_{1,i}}$ and of radius $r_{B_{1,i}}$. Notice that $d(x_B, x_{B_{1,i}}) \leq C_1(r_B + r_{B_{1,i}})$. If $r_B < r_{B_{1,i}}$, by (4.6), we know that there exists a positive constant $\tilde{c}_3$ such that \[ \mu(B(x_B, r_{B_{1,i}})) \leq \left[ 1 + \frac{r_B + r_{B_{1,i}}}{r_{B_{1,i}}} \right]^N \mu(B(x_{B_{1,i}}, r_{B_{1,i}})) \leq \frac{\tilde{c}_3 \mu(B)}{\beta}. \] (4.7)

We now choose $\beta > \tilde{c}_3$ and, therefore, $r_B \geq r_{B_{1,i}}$. Otherwise, if $r_B < r_{B_{1,i}}$, then $\mu(B) \leq \mu(B(x_B, r_{B_{1,i}}))$, which contradicts to (4.7).

By $r_B \geq r_{B_{1,i}}$, together with (2.4), (2.2) and $d(x_B, x_{B_{1,i}}) \leq C_1(r_B + r_{B_{1,i}})$, we find that, for some positive constant $c_4$, \[ \mu(B) \leq \left( \frac{r_B}{r_{B_{1,i}}} \right)^n \mu(B(x_B, r_{B_{1,i}})) \leq \left( \frac{r_B}{r_{B_{1,i}}} \right)^n \left[ 1 + \frac{C_1(r_B + r_{B_{1,i}})}{r_{B_{1,i}}} \right]^N \mu(B(x_{B_{1,i}}, r_{B_{1,i}})) \leq \frac{c_4}{\beta} \left( \frac{r_B}{r_{B_{1,i}}} \right)^{n+N} \mu(B). \] (4.8)

We further choose $\beta > \max\{\tilde{c}_3, c_4(10C_1)^{n+N}, c_3 e\}$. Then, from (4.8) and the fact $r_B > 10C_1 r_{B_{1,i}}$ together with (2.1), we deduce that, for any $B_{1,i} \cap B \neq \emptyset$, $B_{1,i} \subset 2C_1 B$. 

\[ \int_{10C_1^2 B} \left| A_t f(x) - A_t f(x) \right| d\mu(x) \]

\[ \lesssim \|f\|_{\text{BMO}} \left[ \left\| \chi_{10C_1^2 B} \right\|_{L^\beta(\mathcal{X})} + \int_{10C_1^2 B} \frac{\|\chi_B(x, r_B)\|_{L^\beta(\mathcal{X})}}{\mu(B(x, r_B))} d\mu(x) \right] \]

\[ \lesssim \|\chi_B\|_{L^\beta(\mathcal{X})} \|f\|_{\text{BMO}} \sim \mu(B). \]
We claim that there exists a positive constant $c_5$ such that, for any $B_{1,i} \cap B \neq \emptyset$ and almost every $x \in B_{1,i}$,

$$|A_{tB_{1,i}} f(x) - A_B f(x)| \leq c_5 \beta.$$  \hspace{1cm} (4.9)

Indeed, from Assumption A, it follows that, for almost every $x \in X$,

$$A_{tB_{1,i}} f(x) - A_B f(x) = A_{tB_{1,i}} (f - A_B f)(x) + \left[ A_{(tB_{1,i} + t_B)} f(x) - A_B f(x) \right].$$  \hspace{1cm} (4.10)

By the fact that $t_{B_{1,i}} + t_B$ and $t_B$ have comparable sizes, Proposition 3.12, (2.12), (2.13), and (2.4), we find that, for almost every $x \in B_{1,i}$,

$$|A_{(tB_{1,i} + t_B)} f(x) - A_B f(x)| \leq \frac{\|X_{B(x,t_{B_{1}}^{1/m})} \|_{L^p(x)} \|f\|_{BMO^p(x)}}{\mu(B(x,t_{B_{1}}^{1/m}))} \leq \frac{\|X_B \|_{L^p(x)} \|f\|_{BMO^p(x)}}{\mu(B)} \lesssim \beta.$$

From this and (4.10), we deduce that, to prove (4.9), we only need to prove that, for almost every $x \in B_{1,i}$,

$$|A_{tB_{1,i}} (f - A_B f)(x)| \leq \beta.$$  \hspace{1cm} (4.11)

Let $q_i$ be the smallest integer such that $2C_i^2 B \subset 2^{q_i+1} B_{1,i}$ and $2C_i^2 B \cap (2^{q_i} B_{1,i})^C \neq \emptyset$. We claim that $2^{q_i+1} B_{1,i} \subset 10C_i^4 B$. Indeed, by $2C_i^2 B \cap (2^{q_i} B_{1,i})^C \neq \emptyset$, we see that

$$2^{q_i} r_{B_{1,i}} < C_1 [r_{2C_i^2 B} + d(x_B, x_{B_{1,i}})] \leq C_1 [2C_i^2 r_{B} + C_1 (r_B + r_{B_{1,i}})],$$

which, together with (2.1), implies that, for all $z \in 2^{q_i+1} B_{1,i}$,

$$d(x_B, z) \leq C_1 [d(x_B, x_{B_{1,i}}) + d(x_{B_{1,i}}, z)] < C_1 [C_1 (r_B + r_{B_{1,i}}) + 2^{q_i+1} r_{B_{1,i}}]$$

$$\leq C_1^2 [r_B + r_{B_{1,i}} + 4C_i^2 r_B + 2C_1 (r_B + r_{B_{1,i}})] \leq 10C_i^4 r_B.$$

Thus, the above claim holds true. Moreover, we write

$$|A_{tB_{1,i}} (f - A_B f)(x)| \leq \frac{1}{\mu(B_{1,i})} \int_{X} g \left( \frac{[d(x,y)]^m}{t_{B_{1,i}}} \right) |f(y) - A_B f(y)| d\mu(y)$$

$$\leq \frac{1}{\mu(B_{1,i})} \int_{2^{k} B_{1,i} \setminus 2^{k-1} B_{1,i}} g \left( \frac{[d(x,y)]^m}{t_{B_{1,i}}} \right) |f(y) - A_B f(y)| d\mu(y)$$

$$+ \frac{1}{\mu(B_{1,i})} \int_{X \setminus 2^{k+1} B_{1,i}} \cdots =: I + II.$$

It follows immediately from property (iii) of $\{B_{1,i}\}_{i \in \mathbb{N}}$ that, for all $k \in \{0, \ldots, q_i + 1\}$,

$$\frac{1}{\mu(2^k B_{1,i})} \int_{2^k B_{1,i}} |f_0(x)| d\mu(x) \lesssim \beta,$$
which, together with \( f_0 := (f - A_{1B}f)\chi_{10C_1^4B} \) and \( 2^{n+1}B_{1,i} \subset 10C_1^4B \) by the above claim, implies that
\[
(4.13) \quad \frac{1}{\mu(2^kB_{1,i})} \int_{2^kB_{1,i}} |f(x) - A_{1B}f(x)| \, d\mu(x) = \frac{1}{\mu(2^kB_{1,i})} \int_{2^kB_{1,i}} |f_0(x)| \, d\mu(x) \lesssim \beta.
\]

Notice that, for any \( x \in B_{1,i} \), \( y \in 2^kB_{1,i} \setminus 2^{k-1}B_{1,i} \) with \( k \in \mathbb{N} \) and \( k \geq \lfloor \log_2 C_1 \rfloor + 2 \), there exists a positive constant \( c_0 \) such that \( d(y, x) \geq c_0 2^k r_{B_{1,i}} \). This, together with (4.13), (3.5), the decreasing property of \( g \), and \( M > n + \frac{2np}{q} + N - \frac{n(q-1)}{q} - \alpha > n \), implies that
\[
(4.14) \quad \mathcal{I} \lesssim \sum_{k=0}^{[\log_2 C_1] + 1} 2^{kn} \left( \frac{\mu(2^kB_{1,i})}{\mu(2^kB_{1,i})} \right) \left( \int_{2^kB_{1,i}} g(d(x, y)) m \, \frac{|f(y) - A_{1B}f(y)|}{d\mu(y)} \right)
+ \sum_{k=[\log_2 C_1] + 2}^{q_i+1} 2^{kn} g(c_0 m 2^{km}) \left( \frac{\mu(2^kB_{1,i})}{\mu(2^kB_{1,i})} \right) \left( \int_{2^kB_{1,i}} g(d(x, y)) m \, \frac{|f(y) - A_{1B}f(y)|}{d\mu(y)} \right)
\lesssim \beta \sum_{k=0}^{[\log_2 C_1] + 1} 2^{kn} + \beta \sum_{k=0}^{q_i+1} 2^{k(n-M)} \lesssim \beta,
\]
where the second term is vacant, if \( q_i < [\log_2 C_1] + 1 \).

Now we estimate \( \mathcal{II} \). Let \( s_i \) be an integer satisfying \( 2^{s_i} r_{B_{1,i}} \leq r_B < 2^{s_i+1} r_{B_{1,i}} \). Let \( 2^{-1}B_{1,i} = \emptyset \). Then, by (2.4), we know that \( \mu(B(x, r_{B_{1,i}})) \lesssim 2^{N} \mu(B_{1,i}) \). It is easy to see that, for any \( x \in B_{1,i} \) and \( y \in 2^{k+1}B \setminus 2^{k}B \) with \( k \in \mathbb{N} \) and \( k \geq \lfloor \log_2 C_1 \rfloor + 2 \), there exists a positive constant \( c_7 \) such that \( d(y, x) \geq c_7 2^{k+s_i+t_{B_{1,i}}} \). Recall that \( 2C_1 B \subset 2^{q_i+1}B_{1,i} \) and \( M > n + \frac{2np}{q} + N - \frac{n(q-1)}{q} - \alpha > \max\{\frac{np}{p}, n + N\} \). Thus, from these facts, the decreasing property of \( g \), Proposition 3.7, (2.6) and Lemma 2.13, it follows that
\[
\mathcal{II} \lesssim \sum_{k=[2\log_2 C_1] + 1}^{\infty} \left( \frac{1}{\mu(B_{1,i})} \int_{2^{k+1}B \setminus 2^{k}B} g \left( \frac{[d(x, y)]^+}{t_{B_{1,i}}} \right) \frac{|f(y) - A_{1B}f(y)|}{d\mu(y)} \right)
\lesssim \sum_{k=[2\log_2 C_1] + 1}^{\infty} 2^{s_i(N+n)} g(c_7 m 2^{s_i}) \left( \frac{1}{\mu(B)} \right) \left( \int_{2^{k+1}B} \frac{|f(y) - A_{1B}f(y)|}{d\mu(y)} \right)
\lesssim \sum_{k=[2\log_2 C_1] + 1}^{\infty} 2^{s_i(N+n)} g(c_7 m 2^{s_i}) \left( \frac{1}{\mu(B)} \right) \left[ \int_{2^{k+1}B} \frac{|f(y) - A_{1B}f(y)|}{d\mu(y)} \right]
\lesssim \sum_{k=[2\log_2 C_1] + 1}^{\infty} 2^{s_i(N+n)} g(c_7 m 2^{s_i}) \left( \frac{1}{\mu(B)} \right) \left[ \int_{2^{k+1}B} \frac{|f(y) - A_{1B}f(y)|}{d\mu(y)} \right]
\lesssim 2^{s_i(N+n)} g(c_7 m 2^{s_i}) \left( \frac{1}{\mu(B)} \right) \left[ \int_{2^{k+1}B} \frac{|f(y) - A_{1B}f(y)|}{d\mu(y)} \right]
\lesssim \|B(y, t_{B_{1,i}}^1/m^2)\|_{L^\infty(X)} \|\chi_{2^{k+1}B}\|_{L^\infty(X)}
+ \frac{2^{(k+1)}(np_1/p) - \alpha}{2^{k+1}B} \left( \frac{\mu(B(y, t_{B_{1,i}}^1/m^2))}{d\mu(y)} \right)
\lesssim \sum_{k=[2\log_2 C_1] + 1}^{\infty} 2^{s_i(N+n-M)} \left[ \frac{2^{n(p+1-M)} + 2^{n+2np/p + N - \frac{n(q-1)}{q} - \alpha - M}}{2^{(k+1)}(np_1/p) - \alpha} \right]
\[ \times \frac{\|\chi_B\|_{L^p(\mathbb{R})}}{\mu(B)} \|f\|_{\text{BMO}_\lambda^p(\mathbb{R})} \lesssim \beta, \]

which, together with (4.14) and (4.12), implies that (4.11) and hence (4.9) hold true.

We claim that, for each \( B_{1,i} \) with \( B_{1,i} \cap B \neq \emptyset \), \( B_{1,i} \subset 2C_1^2B \). Indeed, notice that
\[ B_{1,i} \cap B \neq \emptyset \quad \text{and} \quad r_B \geq 10C_1r_{B_{1,i}}. \]
Then, by (2.1), we know that, for all \( x \in B_{1,i} \),
\[ d(x_B, x) \leq C_1(2C_1r_{B_{1,i}} + r_B) \leq \frac{9}{8}C_1r_B \leq 2C_1^2r_B. \]
Thus, \( B_{1,i} \subset 2C_1^2B \), namely, the claim holds true. This, together with (4.1) and Lemma 2.12(i), implies that there exists a positive constant \( c_7 \) such that
\[ \frac{\|\chi_{B_{1,i}}\|_{L^p(\mathbb{R})}}{\mu(B_{1,i})} \leq \frac{\|\chi_{2C_1^2B}\|_{L^p(\mathbb{R})}}{\mu(2C_1^2B)} \leq c_6 \frac{\|\chi_B\|_{L^p(\mathbb{R})}}{\mu(B)}. \]

From this and (4.3), we deduce that
\[ \frac{\|\chi_{B_{1,i}}\|_{L^p(\mathbb{R})}}{\mu(B_{1,i})} \|f\|_{\text{BMO}_\lambda^p(\mathbb{R})} \leq c_6 \frac{\|\chi_B\|_{L^p(\mathbb{R})}}{\mu(B)} \|f\|_{\text{BMO}_\lambda^p(\mathbb{R})} \leq c_6. \]

Applying the decomposition in [9, Chapter III, Theorem 1.3] for
\[ f_{1,i} := (f - A_{tB_{1,i}}f)\chi_{10C_1^2B_{1,i}}, \]
with the same value \( \beta \) as above again, we obtain a collection \( \{B_{2,m}\}_{m \in \mathbb{N}} \) of balls satisfying that \( B_{2,j} \cap B_{1,i} \neq \emptyset \) for any \( B_{2,j} \subset \{B_{2,m}\}_{m \in \mathbb{N}} \), \( |f(x) - A_{tB_{1,i}}f(x)| \leq \beta \) for any \( x \in B_{1,i} \setminus (\cup_{m}B_{2,m}) \), and \( \sum_{m} \mu(B_{2,m}) \leq \frac{c_6}{\beta} \mu(B_{1,i}) \). We now further choose \( \beta > \max\{c_3, c_4(10C_1)^{n+N}, c_5\varepsilon\}\max\{1, c_6\} \) and let \( c_6 := \max\{1, c_6\} \). By a method similar to that used in the proof of (4.9), we see that, for almost every \( x \in B_{2,m} \),
\[ |A_{tB_{2,m}}f(x) - A_{tB_{1,i}}f(x)| \leq c_5c_6\beta. \]

Now we put together all families \( \{B_{2,m}\} \) corresponding to different \( B_{1,i} \), which are still denoted by \( \{B_{2,m}\} \). Then, for all \( x \in B \setminus (\cup_{m}B_{2,m}) \), we have
\[ |f(x) - A_{tB}f(x)| \leq |f(x) - A_{tB_{1,i}}f(x)| + |A_{tB_{1,i}}f(x) - A_{tB}f(x)| \leq 2c_5c_6\beta \]
and
\[ \sum_{m} \mu(B_{2,m}) \leq \left( \frac{c_3}{\beta} \right)^2 c_6 \mu(B). \]

Therefore, by induction, we know that, for each \( K \in \mathbb{N} \), there exists a family \( \{B_{K,m}\}_{K \in \mathbb{N}} \) of balls satisfying that, for any \( B_{K+1,m} \), there exists a ball \( B_{K,m} \) satisfying \( B_{K+1,m} \cap B_{K,m} \neq \emptyset \), \( r_{B_{K,m}} > 10C_1r_{B_{K+1,m}} \) and \( B_{K+1,m} \subset 2C_1^2B \). Then, from (4.1), we deduce that
\[ \frac{\|\chi_{B_{K+1,m}}\|_{L^p(\mathbb{R})}}{\mu(B_{K+1,m})} \|f\|_{\text{BMO}_\lambda^p(\mathbb{R})} \leq c_6 \frac{\|\chi_B\|_{L^p(\mathbb{R})}}{\mu(B)} \|f\|_{\text{BMO}_\lambda^p(\mathbb{R})} \leq c_6. \]

Moreover, we also have
\[ |f(x) - A_{tB}f(x)| \leq Kc_5c_6\beta \text{ for almost every } x \in B \setminus \left( \bigcup_{m}B_{K,m} \right), \]
and
\[ \sum_m \mu(B_{K,m}) \leq \left( \frac{c_3c_6}{\beta} \right)^K \frac{\mu(B)}{c_6}. \]

If \( Kc_5c_6\beta \leq \alpha < (K + 1)c_5c_6\beta \) with \( K \in \mathbb{N} \), then, from \( \beta > (c_3c_6)^2 \), we deduce that
\[ \mu(\{ x \in B : |f(x) - A_tBf(x)| > \alpha \}) \leq \sum_m \mu(B_{K,m}) \leq \left( \frac{c_3c_6}{\beta} \right)^K \frac{\mu(B)}{c_6} \leq e^{-K\log \beta/2} \frac{\mu(B)}{c_6} \leq \sqrt{\beta} e^{-\frac{\alpha}{c_5c_6^2\beta}} \mu(B). \]

On the other hand, if \( \alpha < c_5c_6\beta \), we just have the following trivial estimate
\[ \mu(\{ x \in B : |f(x) - A_tBf(x)| > \alpha \}) \leq \mu(B) \leq e^{1-\frac{\alpha}{c_5c_6^2}} \mu(B). \]

Combining both estimates, we then obtain (4.4) for each \( \alpha > 1 \) by choosing
\[ c_1 := \max \left\{ e, \frac{\sqrt{\beta}}{c_6} \right\} \quad \text{and} \quad c_2 := \min \left\{ (\log \beta)/4, 1 \right\}, \]
which completes the proof of Theorem 4.2. \( \Box \)

**Remark 4.3.** (i) When \( \varphi \) is as in (1.1), (4.1) automatically holds true and Theorem 4.2 is just [17, Theorem 3.1].

(ii) When \( \varphi \) is as in (1.2), (4.1) also automatically holds true and Theorem 4.2 is just [45, Theorem 3.1].

As a consequence of Theorem 4.2, we obtain the following conclusion for \( \text{BMO}_A^{\varphi}(X) \).

**Theorem 4.4.** Let \( X \) be a space of homogeneous type with degree \( (\alpha_0, n_0, N_0) \), where \( \alpha_0, n_0 \) and \( N_0 \) are as in (2.7), (2.3) and (2.5), respectively, \( \varphi \) as in Definition 2.7 satisfying (4.1), and \( \{ A_t \}_{t > 0} \) satisfy Assumption A. Assume that \( f \in \text{BMO}_A^{\varphi}(X) \). Then there exist positive constants \( \lambda \) and \( C \) such that
\[ \sup_{B \subset X} \frac{1}{\mu(B)} \int_B \exp \left\{ \frac{\lambda \mu(B)}{\| \chi_B \|_L^\infty(X) \| f \|_{\text{BMO}_A^{\varphi}(X)}} |f(x) - A_tBf(x)| \right\} \, d\mu(x) \leq C, \]
where the supremum is taken over all balls \( B \subset X \) and \( t_B := r_B^m \).

**Proof.** Let \( \lambda := c_2/2 \), where \( c_2 \) is as in Theorem 4.2. Then, by Theorem 4.2, we see that, for any \( B \subset X \),
\[ \int_B \exp \left\{ \frac{\lambda \mu(B)}{\| \chi_B \|_L^\infty(X) \| f \|_{\text{BMO}_A^{\varphi}(X)}} |f(x) - A_tBf(x)| \right\} \, d\mu(x) \]
\[ = \int_0^\infty \mu \left( \left\{ x \in B : \exp \left[ \frac{\lambda \mu(B)}{\| \chi_B \|_L^\infty(X) \| f \|_{\text{BMO}_A^{\varphi}(X)}} |f(x) - A_tBf(x)| \right] > t \right\} \right) \, dt. \]
Definition 4.5. Let $\mathcal{X}$ be a space of homogeneous type, $\varphi$ as in Definition 2.7, $\mathcal{M}(\mathcal{X})$ as in (3.3), and $\{A_t\}_{t > 0}$ a generalized approximation to the identity satisfying (3.4) and (3.5). Let $\tilde{p} \in [1, \infty)$. The space $\text{BMO}^{\varphi, \tilde{p}}(\mathcal{X})$ is defined to be the set of all $f \in \mathcal{M}(\mathcal{X})$ such that

$$
\|f\|_{\text{BMO}^{\varphi, \tilde{p}}(\mathcal{X})} := \sup_{B \subset \mathcal{X}} \frac{\mu(B)}{\|\varphi\|_{L^p(\mathcal{X})}} \left\{ \frac{1}{\mu(B)} \int_B |f(x) - A_{tB}f(x)|^{\tilde{p}} \, d\mu(x) \right\}^{\frac{1}{\tilde{p}}} < \infty,
$$

where the supremum is taken over all balls $B \subset \mathcal{X}$, $t_B := r_B^p$ and $r_B$ denotes the radius of the ball $B$.

By Theorem 4.2, we obtain the following conclusion.

Theorem 4.6. Let $\mathcal{X}$ be a space of homogeneous type with degree $(\alpha_0, n_0, N_0)$, where $\alpha_0$, $n_0$ and $N_0$ are as in (2.7), (2.3) and (2.5), respectively. Assume that $\varphi$ is as in Definition 2.7 satisfying (4.1) and $\{A_t\}_{t > 0}$ satisfies Assumption A. For different $\tilde{p} \in [1, \infty)$, the spaces $\text{BMO}^{\varphi, \tilde{p}}(\mathcal{X})$ coincide with equivalent norms.

Proof. For any $f \in \text{BMO}^{\varphi, \tilde{p}}(\mathcal{X})$ with $\tilde{p} \in [1, \infty)$, by H"older’s inequality, we see that, for any ball $B \subset \mathcal{X}$,

$$
|f(x) - A_{tB}f(x)| \leq \frac{\mu(B)}{\|\varphi\|_{L^p(\mathcal{X})}} \left\{ \frac{1}{\mu(B)} \int_B |f(x) - A_{tB}f(x)|^{\tilde{p}} \, d\mu(x) \right\}^{\frac{1}{\tilde{p}}},
$$

which implies that $f \in \text{BMO}^{\varphi, \tilde{p}}(\mathcal{X})$ and $\|f\|_{\text{BMO}^{\varphi}(\mathcal{X})} \leq \|f\|_{\text{BMO}^{\varphi, \tilde{p}}(\mathcal{X})}$.

Let $f \in \text{BMO}^{\varphi}(\mathcal{X})$ and $\tilde{p} \in [1, \infty)$. From Theorem 4.2, it follows that, for any $B \subset \mathcal{X}$,

$$
\|f\|_{\text{BMO}^{\varphi}(\mathcal{X})} \leq \mu(B) \int_0^\infty \lambda^{0} e^{-\|\varphi\|_{L^p(\mathcal{X})}\|\text{BMO}^{\varphi}(\mathcal{X})\|_{\text{BMO}^{\varphi}(\mathcal{X})}} \, d\lambda
\leq \mu(B) \int_0^\infty \lambda^{0} e^{-\frac{\lambda}{\mu(B)^{\tilde{p}} \mu(B)}} \, d\lambda
\leq \|f\|_{\text{BMO}^{\varphi}(\mathcal{X})} \frac{\|\varphi\|_{L^p(\mathcal{X})}}{\mu(B)^{\tilde{p}}} \mu(B),
$$
which implies that
\[
\frac{\mu(B)}{\|\chi_B\|_{L^\varphi(X)}} \left[ \frac{1}{\mu(B)} \int_B |f(x) - A_B f(x)|^{\tilde{p}(x)} d\mu(x) \right]^{\frac{1}{\tilde{p}(x)}} \lesssim \|f\|_{\text{BMO}_A^p(X)}.
\]
Thus, \( f \in \text{BMO}_A^\varphi(\mathcal{X}) \) and \( \|f\|_{\text{BMO}_A^\varphi(\mathcal{X})} \lesssim \|f\|_{\text{BMO}_A^p(X)} \). This finishes the proof of Theorem 4.6.

\[\square\]

**Remark 4.7.** Theorem 4.6 completely covers [17, Theorem 3.4] and [45, Theorem 3.3] by taking \( \varphi \), respectively, as in (1.1) and (1.2).

### 4.2 The weighted version of the John-Nirenberg inequality on \( \text{BMO}_A^\varphi(\mathcal{X}) \)

In this subsection, we establish a weighted John-Nirenberg inequality on \( \text{BMO}_A^\varphi(\mathcal{X}) \).

We begin with the following Lemma 4.8, whose proof is similar to that of [32, Lemma 2], the details being omitted.

In what follows, for any set \( E \subset X \) and \( t \in (0, \infty) \), let \( \varphi(E, t) := \int_E \varphi(x, t) d\mu(x) \).

**Lemma 4.8.** Let \( X \) be a space of homogeneous type. Assume that \( \varphi \) is as in Definition 2.7 and \( \varphi \in \mathcal{A}_{\overline{p}_1}(\mathcal{X}) \) with \( \overline{p}_1 \in (1, \infty) \). Then there exists a positive constant \( C \) such that, for all balls \( B \subset \mathcal{X} \), \( \lambda \in (0, \infty) \) and \( t \in (0, \infty) \),

\[
\varphi \left( \left\{ x \in B : M_{\varphi(\cdot, t)} \left( \frac{1}{\varphi(\cdot, t)} \chi_B \right) (x) > \lambda \right\} , t \right) \leq C \left[ \frac{\mu(B)}{\lambda \varphi(B, t)} \right]^{\frac{\overline{p}_1}{\overline{p}_1' - 1}} \varphi(B, t),
\]

where \( \frac{1}{\overline{p}_1} + \frac{1}{\overline{p}_1'} = 1 \) and \( M_{\varphi(\cdot, t)} \) denotes the maximal function associated with \( \varphi(\cdot, t) \), namely, for all \( f \in L_{\text{loc}}(\varphi(\cdot, t) d\mu) \) and \( x \in \mathcal{X} \),

\[
M_{\varphi(\cdot, t)}(f)(x) := \sup_{B \ni x} \frac{1}{\varphi(B, t)} \int_B |f(y)| \varphi(x, t) d\mu(y).
\]

Now we give out the weighted John-Nirenberg inequality on \( \text{BMO}_A^\varphi(\mathcal{X}) \) as follows.

**Theorem 4.9.** Let \( X \) be a space of homogeneous type with degree \((\alpha_0, n_0, N_0)\), where \( \alpha_0 \), \( n_0 \) and \( N_0 \) are as in (2.7), (2.3) and (2.5), respectively. Let \( \varphi \) be as in Definition 2.7 and \( \{A_t\}_{t>0} \) satisfy Assumption A.

(i) Assume that \( \varphi \in \mathcal{A}_1(\mathcal{X}) \). Then, there exist positive constants \( c_1 \) and \( c_2 \) such that, for all \( f \in \text{BMO}_A^\varphi(\mathcal{X}) \), balls \( B \) and \( \lambda \in (0, \infty) \),

\[
\varphi \left( \left\{ x \in B : \frac{|f(x) - A_B f(x)|}{\varphi(x, \|\chi_B\|^{-1}_{L^\varphi(\mathcal{X})})} > \lambda \right\} , \|\chi_B\|^{-1}_{L^\varphi(\mathcal{X})} \right) \leq c_1 \exp \left\{ - \frac{c_2 \lambda}{\|\chi_B\|_{L^\varphi(\mathcal{X})} \|f\|_{\text{BMO}_A^\varphi(\mathcal{X})}} \right\},
\]

where \( t_B := \frac{1}{A_t} \) and \( m \) is as in (3.4).
(ii) Assume that \( \varphi \in A_{p_1}(X) \) for some \( p_1 \in (1, \infty) \) and \( p(\varphi) \leq 1 + \frac{1}{[r(\varphi)]'} \), where \( p(\varphi) \) and \( r(\varphi) \) are, respectively, as in (2.9) and (2.10), and \( 1/r(\varphi) + 1/[r(\varphi)]' = 1 \). Then, there exist positive constants \( b_1 \) and \( b_2 \) such that, for all \( f \in BMO^*_A(X) \), balls \( B \) and \( \lambda \in (0, \infty) \),

\[
\varphi \left( \left\{ x \in B : \left( \frac{|f(x) - A_{tB} f(x)|}{\varphi(x, t_0)} > \lambda \right) \right\}, t_0 \right) \leq b_1 \left( \min \left\{ 1, b_2 \left[ \frac{\lambda \varphi(B, \|\chi_X\|^{-\frac{1}{r_1}}(\chi_X)) \|\chi_B\|^{-\frac{1}{r_1}}(\chi_X) \|f\|_{BMO^*_A(X)} \} \right] \right) \right).
\]

Proof. Let \( f \in BMO^*_A(X) \). Fix a ball \( B_0 \subset X \). In order to prove (4.15) and (4.16), it suffices to consider the case \( \|f\|_{BMO^*_A(X)} > 0 \). Otherwise, they hold true obviously. Let \( t_0 := \|\chi_{B_0}\|^{-\frac{1}{r_1}}(\chi_X) \). Without loss of generality, we may assume that

\[
\|f\|_{BMO^*_A(X)} = t_0;
\]

otherwise, we replace \( f \) by \( \frac{tf}{\|f\|_{BMO^*_A(X)}} \). Thus, we only need to prove that there exist positive constants \( c_1, c_2, b_1 \), and \( b_2 \) such that, for any \( \lambda \in (0, \infty) \), when \( \varphi \in A_{1}(X) \),

\[
\varphi \left( \left\{ x \in B_0 : \left( \frac{|f(x) - A_{tB_0} f(x)|}{\varphi(x, t_0)} > \lambda \right) \right\}, t_0 \right) \leq c_1 e^{-c_2 \lambda}
\]

and, when \( \varphi \in A_{p_1}(X) \) with \( p_1 \in (1, \infty) \),

\[
\varphi \left( \left\{ x \in B_0 : \left( \frac{|f(x) - A_{tB_0} f(x)|}{\varphi(x, t_0)} > \lambda \right) \right\}, t_0 \right) \leq b_1 \left( \min \left\{ 1, b_2 \lambda^{-r_1} \right\} \right),
\]

where \( t_B := \frac{r_B^{p_1}}{r_B^{p_1}} \).

It is obvious that, when \( \lambda \in (0, 1) \), (4.18) and (4.19) hold true for \( c_1 := e \), \( c_2 := 1 \), \( b_1 := 1 \) and \( b_2 := 2 \).

Now, let \( \lambda \in [1, \infty) \). Let \( B := B(x_B, r_B) \subset B_0 \) and, for all \( x \in X \),

\[
f_0(x) := \frac{f(x) - A_{tB} f(x)}{\varphi(x, t_0)} \chi_{10C_1B}(x),
\]

where \( C_1 \) is as in (2.1). Similar to the proof of (4.5), by the property of uniformly upper type 1 of \( \varphi \), Lemma 2.11(i) and (4.17), we know that

\[
\|f_0\|_{L^1_{\varphi(x, t_0)}(\chi_X)} := \int_X |f_0(x)| \varphi(x, t_0) d\mu(x)
\]

\[
\lesssim \|\chi_B\|_{L^\varphi(\chi_X)} \|f\|_{BMO^*_A(X)}
\]

\[
\lesssim \frac{\varphi(B, t_0)}{\|\chi_{B_0}\|^{-\frac{1}{r_1}}(\chi_X)} \|\chi_B\|_{L^\varphi(\chi_X)} \|f\|_{BMO^*_A(X)}
\]

\[
\lesssim \frac{\varphi(B, t_0)}{\|\chi_{B_0}\|^{-\frac{1}{r_1}}(\chi_X)} \|\chi_B\|_{L^\varphi(\chi_X)} \|f\|_{BMO^*_A(X)} \sim \varphi(B, t_0).
\]
Let $\beta \in (1, \infty)$, 

$$F := \{ x \in X : M_{\varphi(\cdot, t_0)}(f_0)(x) \leq \beta \} \quad \text{and} \quad \Omega := F^c = \{ x \in X : M_{\varphi(\cdot, t_0)}(f_0)(x) > \beta \}.$$ 

By [9, Chapter III, Theorem 1.3], we know that there exists a collection of balls, $\{B_{1,i}\}_{i \in \mathbb{N}}$, satisfying that

(i) $\cup_i B_{1,i} = \Omega$;

(ii) each point of $\Omega$ is contained in at most a finite number $L$ of the balls $B_{1,i}$;

(iii) there exists $C \in (1, \infty)$ such that $CB_{1,i} \cap F \neq \emptyset$ for each $i$.

From (i), we deduce that, for any $x \in B \setminus (\cup_i B_{1,i})$,

$$|f(x) - A_{t_B} f(x)| = |f_0(x)|\chi_F(x) \leq M_{\varphi(\cdot, t_0)}(f_0)(x)\chi_F(x) \leq \beta.$$ 

By the fact that $M_{\varphi(\cdot, t_0)}$ is of weak type $(1,1)$, (i), (ii) and (4.20), we conclude that there exists a positive constant $c_3$ such that

$$\sum_i \varphi(B_{1,i}, t_0) \leq L\varphi(\Omega, t_0) \lesssim \frac{1}{\beta} \| f_0 \|_{L^1_C(X)} \leq \frac{c_3}{\beta} \varphi(B, t_0). \tag{4.21}$$

For any $B_{1,i} \cap B \neq \emptyset$, we denote by $B_{1,i} := B(x_{B_{1,i}}, r_{B_{1,i}})$ the ball centered at $x_{B_{1,i}} \in X$ and of radius $r_{B_{1,i}} \in (0, \infty)$. Notice that $d(x_B, x_{B_{1,i}}) < C_1(r_B + r_{B_{1,i}})$. If $r_B < r_{B_{1,i}}$, then $d(x_B, x_{B_{1,i}}) < 2C_1 r_B$. By Lemma 2.12(ii), (2.6), Lemma 2.12(i), (2.4), (4.21) and some estimates similar to those used in (4.7), we know that there exists a positive constant $\tilde{c}_3$ such that

$$\varphi(B, t_0) \leq \frac{c_3}{\beta} \left( \frac{r_B}{r_{B_{1,i}}} \right)^{\frac{\alpha(q-1)}{q}} \varphi(B, t_0),$$

which further implies that, when $\beta > \tilde{c}_3$, it holds true $r_B \geq r_{B_{1,i}}$. We now choose $\beta > \tilde{c}_3$ and hence $r_B \geq r_{B_{1,i}}$. By this, together with Lemma 2.12(i), (2.4), (2.2), (4.21), $d(x_B, x_{B_{1,i}}) < 2C_1 r_B$ and some estimates similar to those used in (4.8), we find that there exists a positive constant $c_4$ such that

$$\varphi(B, t_0) \leq \frac{c_4}{\beta} \left( \frac{r_B}{r_{B_{1,i}}} \right)^{2np_1} \varphi(B, t_0). \tag{4.22}$$

We further choose

$$\beta > \max\{c_3, c_4(10C_1)^{2np_1}, 4^{p_1}c_3\}, \tag{4.23}$$

where $1/p_1 + 1/p_1' = 1$. Then, from (4.22), we deduce that $r_B > 10C_1 r_{B_{1,i}}$, which, together with (2.1), implies that, for any $B_{1,i} \cap B \neq \emptyset$, $B_{1,i} \subset 2C_1 B$.

Now we prove (i) and (ii) separately.

(i) When $\varphi \in A_1(X)$, we claim that there exists a positive constant $c_5$ such that, for any $B_{1,i} \cap B \neq \emptyset$ and almost every $x \in B_{1,i}$,

$$\frac{|A_{t_B} f(x) - A_{t_B} f(x)|}{\varphi(x, t_0)} \leq c_5 \beta. \tag{4.24}$$
Indeed, when \( \varphi \in A_1(\mathcal{X}) \), by Definition 2.6, for any ball \( \bar{B} \subset \mathcal{X} \) and \( t \in (0, \infty) \), we see that
\[
\frac{\varphi(\bar{B}, t)}{\mu(\bar{B})} = \frac{1}{\mu(\bar{B})} \int_{\bar{B}} \varphi(x, t) \, d\mu(x) \lesssim \underset{y \in \bar{B}}{\text{ess inf}} \varphi(y, t).
\]

Then, we obtain (4.24) by a procedure similar to that used in the estimates for (4.9).

Let \( b := c_5 \beta \). Then, for any \( \lambda \in (0, \infty) \), we find that
\[
(4.25) \quad \left\{ x \in B : \frac{|f(x) - A_{tB} f(x)|}{\varphi(x, t_0)} > \lambda + b \right\}
\subset \bigcup_{i} \left\{ x \in B_{1,i} : \frac{|f(x) - A_{t_{B_{1,i}}} f(x)|}{\varphi(x, t_0)} > \lambda \right\}
\cup \left\{ x \in B_{1,i} : \frac{|A_{t_{B_{1,i}}} f(x) - A_{tB} f(x)|}{\varphi(x, t_0)} > b \right\}
\subset \bigcup_{i} \left\{ x \in B_{1,i} : \frac{|f(x) - A_{t_{B_{1,i}}} f(x)|}{\varphi(x, t_0)} > \lambda \right\}.
\]

For any \( \lambda \in (0, \infty) \), we define \( \sigma_{f,B}(\lambda) := \varphi(\{ x \in B : \frac{|f(x) - A_{tB} f(x)|}{\varphi(x, t_0)} > \lambda \}, t_0) \) and \( F_f(\lambda) := \sup_{B \subset B_0} \frac{\sigma_{f,B}(\lambda)}{\varphi(B, t_0)} \). Then, by (4.25), we know that, for any \( \lambda \in (0, \infty) \), \( \sigma_{f,B}(\lambda+b) \leq \sum_i F_f(\lambda) \mu(B_{1,i}) \), which, together with (4.21) and (4.23), implies that, for any \( \lambda \in (0, \infty) \),
\[
F_f(\lambda+b) \leq \frac{c_3}{\beta} F_f(\lambda) \leq e^{-1} F_f(\lambda).
\]

By induction, we know that, for all \( n \in \mathbb{N} \), \( F_f(nb) \leq e^{1-n} \). Thus, for any \( n \in \mathbb{N} \) and \( \lambda \in [nb, (n+1)b) \), by the fact that \( F_f(\lambda) \) is non-increasing, we conclude that
\[
(4.26) \quad F_f(\lambda) \leq F_f(nb) \leq e^{1-n} < e^{2-\frac{\lambda}{2}}.
\]

Notice that \( F_f(\lambda) \leq 1 \). It is obvious that (4.26) holds true for \( \lambda \in [1, b) \). Thus, (4.18) always holds true, which completes the proof of Theorem 4.9(i).

(ii) In this case, by Assumption A and \( p(\varphi) \leq 1 + \frac{1}{[\tilde{r}(\varphi)]} \), we see that there exist \( n \in \mathbb{N} \), \( N \in \mathbb{N} \), \( \alpha \in [0, \alpha_0] \), \( p_1 \in [p(\varphi), \infty) \), \( p \in (0, \tilde{r}(\varphi)] \) and \( q \in (1, r(\varphi)] \) such that \( \mathcal{X} \) satisfies (2.2), (2.4) and (2.6), respectively, for \( n \), \( N \) and \( \alpha \), \( \varphi \in A_{p_1(\mathcal{X})} \), \( \varphi \) is of uniformly lower type \( p \), \( \varphi \in \mathbb{R}_q(\mathcal{X}) \), \( p_1 - 1 - \frac{q-1}{q} \leq 0 \) and \( M > n + \frac{2np_1}{p} + N - \frac{n(q-1)}{q} > n p_1 \).

We claim that there exists a positive constant \( \tilde{c}_5 \) such that, for any \( B_{1,i} \cap B \neq \emptyset \) and almost every \( x \in B_{1,i} \),
\[
(4.27) \quad \frac{|A_{t_{B_{1,i}}} f(x) - A_{tB} f(x)|}{\varphi(x, t_0)} \leq \frac{\tilde{c}_5 \beta \varphi(B_{1,i}, t_0)}{\varphi(x, t_0) \mu(B_{1,i})}.
\]

Indeed, from Assumption A, it follows that, for almost every \( x \in \mathcal{X} \),
\[
(4.28) \quad A_{t_{B_{1,i}}} f(x) - A_{tB} f(x) = A_{t_{B_{1,i}}} (f - A_{tB} f)(x) + \left[A_{(t_{B_{1,i}}+tB)} f(x) - A_{tB} f(x)\right].
\]
By some estimates similar to those used in the proof of Proposition 3.8 (see (3.14), (3.15) and (3.16)), (4.17), Lemma 2.12(i), (2.13) and $p_1 - 1 - \frac{q-1}{q} \leq 0$, we find that, for almost every $x \in B_{1,i}$,

\begin{equation}
|A_{t_{B_{1,i}}+t_B}f(x) - A_{t_B}f(x)| \leq \frac{\|\chi_{B(x,t_{B_{1,i}}^{1/m})}\|_{L^p(x)}}{\mu(B(x,t_B^{1/m}))} \|f\|_{\text{BMO}_M^p(x)}
\end{equation}

\begin{equation}
\leq \frac{\|\chi_{B_{1,i}}\|_{L^p(x)} \varphi(B_0,t_0)}{\mu(B)} \frac{\|\chi_{B_0}\|_{L^p(x)}}{\|\mu(B)\|_{L^p(x)}}
\end{equation}

\begin{equation}
\leq \frac{\varphi(B_{1,i},t_0)}{\mu(B_{1,i})} \frac{\mu(B_0)}{\mu(B_{1,i})} \|\chi_{B_0}\|_{L^p(x)} \leq \frac{\varphi(B_{1,i},t_0)}{\mu(B_{1,i})}.
\end{equation}

From this and (4.28), it follows that, to prove (4.27), we only need to show that, for almost every $x \in B_{1,i}$,

\begin{equation}
|A_{t_{B_{1,i}}}(f - A_{t_B}f)(x)| \leq \frac{\beta \varphi(B_{1,i},t_0)}{\mu(B_{1,i})}.
\end{equation}

Following the estimates same as those used in (4.12), we still divide $|A_{t_{B_{1,i}}}(f - A_{t_B}f)(x)|$ into two parts I and II same as in (4.12).

By $M > n + np_1 - \alpha$ and some estimates similar to those used in (4.13) and (4.14), we know that $I \lesssim \frac{\beta \varphi(B_{1,i},t_0)}{\mu(B_{1,i})}$, the details being omitted.

By $M > n + \frac{2np_1}{p} + N - \frac{n(q-1)}{q} - \alpha > np_1$ and the arguments similar to those used in the estimate II of Theorem 4.2, we obtain $II \lesssim \frac{\beta \varphi(B_{1,i},t_0)}{\mu(B_{1,i})}$, the details being omitted again. Thus, (4.27) holds true.

For $\lambda \in (0,\infty)$, we write

\begin{equation}
\left\{ x \in B : \frac{|f(x) - A_{t_B}f(x)|}{\varphi(x,t_0)} > \lambda \right\}
\end{equation}

\begin{equation}
\subset \bigcup_i \left( \left\{ x \in B_{1,i} : \frac{|f(x) - A_{t_{B_{1,i}}}f(x)|}{\varphi(x,t_0)} > \lambda \right\} \right)
\end{equation}

\begin{equation}
\bigcup \left( \left\{ x \in B_{1,i} : \frac{|A_{t_{B_{1,i}}f(x) - A_{t_B}f(x)}|}{\varphi(x,t_0)} > \frac{\lambda}{2} \right\} \right).
\end{equation}

By Lemma 4.8 and (4.27), we know that there exists a positive constant $c_6$ such that, for any $\lambda \in (0,\infty)$,

\begin{equation}
\varphi \left( \left\{ x \in B : \frac{|A_{t_{B_{1,i}}}f(x) - A_{t_B}f(x)|}{\varphi(x,t_0)} > \frac{\lambda}{2} \right\},t_0 \right)
\end{equation}

\begin{equation}
\leq \varphi \left( \left\{ x \in B : \frac{1}{\varphi(x,t_0)} \chi_{B_{1,i}}(x) > \frac{\lambda \mu(B_{1,i})}{2c_6 \beta \varphi(B_{1,i},t_0)} \right\},t_0 \right)
\end{equation}
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\[ \leq c_6 \left( \frac{\beta}{\lambda} \right)^{p_1^*} \varphi(B_{1,i}, t_0). \]

We also define \( \sigma_{f,B}(\lambda) := \varphi(\{ x \in B : \frac{|f(x) - A_{tB}f(x)|}{\varphi(x,t_0)} > \lambda \}, t_0) \) and

\[ F_f(\lambda) := \sup_{B \subset B_0} \frac{\sigma_{f,B}(\lambda)}{\varphi(B, t_0)}. \]

Then, from (4.30), (4.31) and (4.21), it follows that, for any \( \lambda \in (0, \infty) \)

\[ \sigma_{f,B}(\lambda) \leq \left[ F_f \left( \frac{\lambda}{2} \right) + c_6 \left( \frac{\beta}{\lambda} \right)^{p_1^*} \right] \sum_i \varphi(B_{1,i}, t_0) \]

\[ \leq \frac{c_3}{\beta} \left[ F_f \left( \frac{\lambda}{2} \right) + c_6 \left( \frac{\beta}{\lambda} \right)^{p_1^*} \right] \varphi(B, t_0), \]

which, together with (4.23), implies that there exists a positive constant \( c_7 \) such that, for any \( \lambda \in (0, \infty) \), \( F_f(\lambda) \leq 4^{p_1^*} F_f\left( \frac{\lambda}{2} \right) + c_7 \lambda^{p_1^*}. \)

By induction, we see that, for all \( m \in \mathbb{Z}_+ \) and \( \lambda \in (c_7, 2c_7] \),

\[ F_f(2^m \lambda) \leq (2^{c_7})^{p_1^*} (2^m \lambda)^{-p_1^*} \]

which implies that (4.19) holds true and hence completes the proof of Theorem 4.9. \( \square \)

**Remark 4.10.** (i) Theorem 4.9(i) completely covers [17, Theorem 3.1] and [45, Theorem 3.1], respectively, by taking \( \varphi \), respectively, as in (1.1) and (1.2). Moreover, Theorem 4.9(i) completely covers [6, Theorem 3.6] by taking \( \varphi \) as in (1.3).

(ii) Theorem 4.9(ii) is new even when \( \varphi \) is as in (1.4).

(iii) Let \( \varphi \) be as in Definition 2.7 and satisfy (4.1). If \( \varphi \in \mathcal{A}_{1}(\mathcal{X}) \), then (4.15) can be deduced from (4.2).

Indeed, assume that \( f \in \text{BMO}^\varphi_A(\mathcal{X}) \) and (4.2) holds true. Then, by \( \varphi \in \mathcal{A}_{1}(\mathcal{X}) \), we see that, for any ball \( B \subset \mathcal{X} \) and \( t \in (0, \infty) \), \( \frac{\varphi(B,t)}{\mu(B)} \lesssim \text{ess}_{x \in B} \varphi(x,t) \), which, together with (4.2), implies that

\[ \mu \left( \left\{ x \in B : \frac{|f(x) - A_{tB}f(x)|}{\varphi(x,\|\chi_B\|_{L^\varphi(\mathcal{X})}^{-1})} > \lambda \right\} \right) \]

\[ \lesssim \mu \left( \left\{ x \in B : |f(x) - A_{tB}f(x)| > \frac{\lambda \varphi(B,\|\chi_B\|_{L^\varphi(\mathcal{X})})}{\mu(B)} \right\} \right) \]

\[ \lesssim \mu(B)^{\frac{c_2 \lambda}{\|\chi_B\|_{L^\varphi(\mathcal{X})}\|f\|_{\text{BMO}^\varphi_A(\mathcal{X})}}}, \]

where \( c_2 \) is as in (4.2). This, together with Lemma 2.12(ii), implies that (4.15) holds true.

However, when \( \varphi \notin \mathcal{A}_{1}(\mathcal{X}) \), the relationship between Theorems 4.2 and 4.9 is still unclear.
Now, we introduce the space \( \widetilde{\text{BMO}}^{\varphi, \widetilde{p}}_A (\mathcal{X}) \) for \( \widetilde{p} \in [1, \infty) \).

**Definition 4.11.** Let \( \mathcal{X} \) be a space of homogeneous type, \( \varphi \) as in Definition 2.7, \( \mathcal{M}(\mathcal{X}) \) as in (3.3) and \( \{A_t\}_{t>0} \) a generalized approximation to the identity satisfying (3.4) and (3.5). Let \( \widetilde{p} \in [1, \infty) \). The space \( \widetilde{\text{BMO}}^{\varphi, \widetilde{p}}_A (\mathcal{X}) \) is defined as the set of all \( f \in \mathcal{M}(\mathcal{X}) \) such that

\[
\|f\|_{\widetilde{\text{BMO}}^{\varphi, \widetilde{p}}_A (\mathcal{X})} := \sup_{B \subseteq \mathcal{X}} \left\{ \int_B \left| f(x) - A_{tB}f(x) \right|^{\widetilde{p}} \varphi(x, \|\chi_B\|^{-1}_{L^p(\mathcal{X})}) \, d\mu(x) \right\}^{1/\widetilde{p}} < \infty,
\]

where the supremum is taken over all balls \( B \) in \( \mathcal{X} \), \( t_B := r^m_B \) and \( r_B \) denotes the radius of the ball \( B \).

By Theorem 4.9, we obtain the following conclusion.

**Theorem 4.12.** Let \( \mathcal{X} \) be a space of homogeneous type with degree \( (\alpha_0, n_0, N_0) \), where \( \alpha_0, n_0 \) and \( N_0 \) are as in (2.7), (2.3) and (2.5), respectively, \( \varphi \) as in Definition 2.7 and \( \{A_t\}_{t>0} \) satisfy Assumption A.

(i) Assume that \( \varphi \in \mathbb{A}_1(\mathcal{X}) \). For different \( \widetilde{p} \in [1, \infty) \), the spaces \( \widetilde{\text{BMO}}^{\varphi, \widetilde{p}}_A(\mathcal{X}) \) coincide with equivalent norms.

(ii) Assume that \( p(\varphi) \leq 1 + \frac{1}{r(\varphi)} \), where \( p(\varphi) \) and \( r(\varphi) \) are, respectively, as in (2.9) and (2.10). For different \( \widetilde{p} \in [1, [p(\varphi)]'' \) \), the spaces \( \widetilde{\text{BMO}}^{\varphi, \widetilde{p}}_A(\mathcal{X}) \) coincide with equivalent norms.

**Proof.** For any \( f \in \widetilde{\text{BMO}}^{\varphi, \widetilde{p}}_A(\mathcal{X}) \) with \( \widetilde{p} \in [1, \infty) \), by Hölder’s inequality and some estimates similar to those used in the proof of Theorem 4.6, we see that \( f \in \text{BMO}_A^\varphi(\mathcal{X}) \) and \( \|f\|_{\text{BMO}_A^\varphi(\mathcal{X})} \leq \|f\|_{\widetilde{\text{BMO}}^{\varphi, \widetilde{p}}_A(\mathcal{X})} \), the details being omitted.

Conversely, let \( f \in \text{BMO}_A^\varphi(\mathcal{X}) \). Now we prove (i) and (ii) separately.

(i) In this case, \( \varphi \in \mathbb{A}_1(\mathcal{X}) \). From this, Theorem 4.9(i) and some estimates similar to those used in the proof of Theorem 4.6, it follows that \( f \in \widetilde{\text{BMO}}^{\varphi, \widetilde{p}}_A(\mathcal{X}) \) and \( \|f\|_{\widetilde{\text{BMO}}^{\varphi, \widetilde{p}}_A(\mathcal{X})} \lesssim \|f\|_{\text{BMO}_A^\varphi(\mathcal{X})} \), the details being omitted. This finishes the proof of (i).

(ii) In this case, let \( \widetilde{p} \in [1, [p(\varphi)]'' \). By the definition of \( p(\varphi) \), we see that \( \varphi \in \mathbb{A}_{\widetilde{p}'}(\mathcal{X}) \). Moreover, let \( p_1 \in (p(\varphi), \widetilde{p}') \). Then, \( \varphi \in \mathbb{A}_{p_1}(\mathcal{X}) \) and \( \widetilde{p} \leq p_1 < [p(\varphi)]'' \). By Theorem 4.9(ii), we find that, for any \( B \subseteq \mathcal{X} \) and \( t \in (0, \infty) \),

\[
\int_B \left| f(x) - A_{tB}f(x) \right|^{\widetilde{p}} \varphi(x, \|\chi_B\|^{-1}_{L^p(\mathcal{X})}) \, d\mu(x) \leq \frac{1}{t} \int_B \left| f(x) - A_{tB}f(x) \right|^{p_1} \varphi(x, \|\chi_B\|^{-1}_{L^{p_1}(\mathcal{X})}) \, d\mu(x) \lesssim \frac{1}{t} \int_B \left| f(x) - A_{tB}f(x) \right|^{p_1} \varphi(x, \|\chi_B\|^{-1}_{L^{p_1}(\mathcal{X})}) \, d\mu(x),
\]

where the supremum is taken over all balls \( B \) in \( \mathcal{X} \), \( t_B := r^m_B \) and \( r_B \) denotes the radius of the ball \( B \).
the new Musielak-Orlicz BMO-type space $BMO_\varphi$ obtained in Theorems 4.2 or 4.9, we show that, when $g$ in [22], the boundedness of the classical Littlewood-Paley

Poisson semigroup corresponding then we can define the generalized approximation to the identity introduced by Ky [28]. We begin with some notions.

Let $\Delta := -\sum_{i=1}^{n} \partial_{x_i}^2$ denote the Laplace operator on $\mathbb{R}^n$ and $\{e^{-t\sqrt{\Delta}}\}_{t > 0}$ be the corresponding Poisson semigroup. Observe that, if $f$ is a function belonging to the set

$$M_{\sqrt{\Delta}}(\mathcal{X}) := \{f \in L^{1}_{\text{loc}}(\mathbb{R}^n) : f(x)(1 + |x|^{n+1})^{-1} \in L^{1}(\mathbb{R}^n)\},$$

then we can define the generalized approximation to the identity $\{A_t\}_{t > 0}$ by the Poisson integral as follows: for all $t \in (0, \infty)$ and $x \in \mathbb{R}^n$,

$$A_tf(x) := P_tf(x) := \int_{\mathbb{R}^n} p_t(x-y)f(y) \, dy,$$

where, for all $t \in (0, \infty)$ and $x \in \mathbb{R}^n$,

$$p_t(x) := \frac{c_{n,t}}{(t^2 + |x|^2)^{\frac{(n+1)}{2}}} \quad \text{and} \quad c_{n,t} := \frac{\Gamma[(n+1)/2]}{\pi^{(n+1)/2}}.$$

In this case, we have the following assumption for $\varphi$.

**Assumption B.** Let $\varphi$ be as in Definition 2.7 and satisfy

$$\frac{2np(\varphi)}{i(\varphi)} - \frac{n[r(\varphi) - 1]}{r(\varphi)} < n + 1,$$

where $p(\varphi)$, $i(\varphi)$ and $r(\varphi)$ are, respectively, as in (2.9), (2.8) and (2.10). \qed
Remark 5.1. From Assumption B, we deduce that the Poisson kernel satisfies the Assumption A and that
\[ p(\varphi) \leq 1 + \frac{1}{r(\varphi)} \] in Theorem 4.9(ii) automatically holds true. Moreover, \( m \) in (3.4) and (3.5) is equal to 1. Then, it is easy to see that \( \{P_t\}_{t>0} \) satisfies (3.4), (3.5) and (3.22).

For any \( f \in L^p(\mathbb{R}^n) \) with \( p \in [1, \infty] \), \( P_t f = e^{-t\sqrt{\Delta}} f \). We use \( \text{BMO}^\varphi_{\sqrt{\Delta}}(\mathbb{R}^n) \) to denote \( \text{BMO}_A(\mathbb{R}^n) \) space associated with the Poisson semigroup \( \{e^{-t\sqrt{\Delta}}\}_{t>0} \).

Now, we recall the \( \varphi \)-Carleson measure introduced in [22].

Definition 5.2. Let \( \varphi \) be as in Definition 2.7. A measure \( d\mu \) on \( \mathbb{R}^{n+1}_+ \) is called a \( \varphi \)-Carleson measure if
\[
\|d\mu\|_\varphi := \sup_{B \subset \mathbb{R}^n} \frac{|B|^{1/2}}{\|\chi_B\|_{L^p(\mathbb{R}^n)}} \left\{ \int_B |d\mu(x,t)| \right\}^{1/2} < \infty,
\]
where the supremum is taken over all balls \( B \subset \mathbb{R}^n \),
\[
\hat{B} := \{(x, t) \in \mathbb{R}^{n+1}_+: x \in B, \ t \in (0, r_B)\}
\]
and \( r_B \) denotes the radius of the ball \( B \).

Let \( \phi \in \mathcal{S}(\mathbb{R}^n) \) be a radial real-valued function satisfying that
\[ \int_{\mathbb{R}^n} \phi(x)x^\gamma \, dx = 0 \]
for all \( \gamma \in \mathbb{Z}_+^n \) with \( |\gamma| \leq s \), where \( s \in \mathbb{Z}_+ \), \( s \geq \lceil n[p(\varphi)/i(\varphi) - 1]\rceil \) and, for all \( \xi \in \mathbb{R}^n \setminus \{0\} \),
\[ \int_0^{\infty} |\hat{\phi}(t\xi)|^2 \frac{dt}{t} = 1, \tag{5.1} \]
where \( \hat{\phi} \) denotes the Fourier transform of \( \phi \).

The following \( \varphi \)-Carleson measure characterization of \( \text{BMO}^\varphi(\mathbb{R}^n) \) is just [22, Theorem 5.3].

Lemma 5.3. Let \( \varphi \) be as in Definitions 2.7 and \( \phi \) as above.

(i) Assume that \( b \in \text{BMO}^\varphi(\mathbb{R}^n) \) and \( q(\varphi)[r(\varphi)]' \in (1, 2) \). Then
\[
d\mu(x,t) := |\phi_t * b(x)|^2 \frac{dx \, dt}{t}
\]
is a \( \varphi \)-Carleson measure on \( \mathbb{R}^{n+1}_+ \); moreover, there exists a positive constant \( C \), independent of \( b \), such that \( \|d\mu\|_\varphi \leq C\|b\|_{\text{BMO}^\varphi(\mathbb{R}^n)} \).

(ii) Assume that \( np(\varphi) < (n+1)i(\varphi) \). Let \( b \in L^2_{\text{loc}}(\mathbb{R}^n) \) and, for all \( (x, t) \in \mathbb{R}^{n+1}_+ \),
\[
d\mu(x,t) := |\phi_t * b(x)| \frac{dx \, dt}{t}
\]
be a \( \varphi \)-Carleson measure on \( \mathbb{R}^{n+1}_+ \). Then \( b \in \text{BMO}^\varphi(\mathbb{R}^n) \) and, moreover, there exists a positive constant \( C \), independent of \( b \), such that \( \|b\|_{\text{BMO}^\varphi(\mathbb{R}^n)} \leq C\|d\mu\|_\varphi \).
Remark 5.4. Actually, if we replace $p(\varphi)[r(\varphi)]' \in (1,2)$ by (4.1), then Lemma 5.3(i) still holds true. To this end, we need to use a John-Nirenberg inequality on $\text{BMO}^p(\mathbb{R}^n)$ similar to Theorem 4.2, which further leads to Lemma 5.3(i) with the assumption $p(\varphi)[r(\varphi)]' \in (1,2)$ replaced by (4.1). In this way, (4.1) is needed. We omit the details.

The main result of this section is as follows. Recall that the space $K_{\sqrt[n]{\Delta}}(\mathbb{R}^n)$ is defined as in Remark 3.4(i) with $A$ and $X$ replaced, respectively, by $\{e^{-t\sqrt[n]{\Delta}}\}_{t>0}$ and $\mathbb{R}^n$.

Theorem 5.5. Let $\varphi$ satisfy Assumption B.

(i) If $\varphi$ additionally satisfies (4.1) or $p(\varphi)[r(\varphi)]' \in (1,2)$, where $p(\varphi)$ and $r(\varphi)$ are, respectively, as in (2.9) and (2.10), then, for any $f \in \text{BMO}^p_{\sqrt[n]{\Delta}}(\mathbb{R}^n)$,

$$
\left\langle t \frac{\partial}{\partial t} P_t(\mathcal{I} - P_t)f(x) \right\rangle^2 \frac{dxdt}{t} \lesssim \|f\|^2_{\text{BMO}^p_{\sqrt[n]{\Delta}}(\mathbb{R}^n)}.
$$

is a $\varphi$-Carleson measure on $\mathbb{R}^{n+1}$.

(ii) Assume further that $np(\varphi) < (n+1)i(\varphi)$, where $i(\varphi)$ is as in (2.8). Then the spaces $\text{BMO}^p(\mathbb{R}^n)$ and $\text{BMO}^p_{\sqrt[n]{\Delta}}(\mathbb{R}^n)$ (modulo $K_{\sqrt[n]{\Delta}}(\mathbb{R}^n)$) coincide with equivalent norms.

Proof. To prove (i), by Definition 5.2, it suffices to prove that, for any ball $B \subset \mathbb{R}^n$,

$$
\left\langle t \frac{\partial}{\partial t} P_t(\mathcal{I} - P_t)f(x) \right\rangle^2 \frac{dxdt}{t} \lesssim \|f\|^2_{\text{BMO}^p_{\sqrt[n]{\Delta}}(\mathbb{R}^n)}.
$$

Let $B := B(x_B, r_B)$. Recall that, in this case, $m = 1$, where $m$ is as in (3.4) and (3.5). Thus, $t_B = r_B$. Notice that $\mathcal{I} - P_t = (\mathcal{I} - P_t)(\mathcal{I} - P_{t_B}) + (\mathcal{I} - P_t)P_{t_B}$. Then, we have

$$
t \frac{\partial}{\partial t} P_t(\mathcal{I} - P_t)f(x) = t \frac{\partial}{\partial t} P_t(\mathcal{I} - P_t)(\mathcal{I} - P_{t_B}) + t \frac{\partial}{\partial t} P_t(\mathcal{I} - P_t)P_{t_B}
$$

$$
= t \frac{\partial}{\partial t} P_t(\mathcal{I} - P_t)(\mathcal{I} - P_{t_B}) + t \frac{\partial}{\partial t} P_{t_B}((P_{2t+B} + P_{2t+B})/2 - P_{2t+B}/2).
$$

Once we prove that

$$
\left\langle t \frac{\partial}{\partial t} P_t(\mathcal{I} - P_t)(\mathcal{I} - P_{t_B})f(x) \right\rangle^2 \frac{dxdt}{t} \lesssim \|f\|^2_{\text{BMO}^p_{\sqrt[n]{\Delta}}(\mathbb{R}^n)}
$$

and

$$
\left\langle t \frac{\partial}{\partial t} P_{t_B}((P_{2t+B} + P_{2t+B})/2 - P_{2t+B}/2)f(x) \right\rangle^2 \frac{dxdt}{t} \lesssim \|f\|^2_{\text{BMO}^p_{\sqrt[n]{\Delta}}(\mathbb{R}^n)},
$$

by (5.3), we then conclude that (5.2) holds true, which is obvious.

To show (5.4) and (5.5), we borrow some ideas from [17, pp. 1393-1395].

It is easy to see that, for any $f \in \bigcup_{p \geq 1} L^p(\mathbb{R}^n)$,

$$
g(f, x) := \left( \int_0^\infty \left\langle t \frac{\partial}{\partial t} P_t(\mathcal{I} - P_t)f(x) \right\rangle^2 \frac{dt}{t} \right)^{\frac{1}{2}}
$$
is the Littlewood-Paley $g$-function. By [26, p. 80], we know that $g$ is bounded on $L^2(\mathbb{R}^n)$.

Let $b_1 := (I - P_t) f x_{2B}$ and $b_2 := (I - P_t) f x_{(2B)}$. If $\varphi$ satisfies (4.1), by Proposition 3.7, Theorem 4.6, (2.12), (2.13) and the boundedness of $g$ on $L^2(\mathbb{R}^n)$, we find that

$$
(5.6) \quad \frac{|B|}{\|\chi_B\|_{L^p(\mathbb{R}^n)}} \int_B \left| t \frac{\partial}{\partial t} P_t(I - P_t) b_1(x) \right|^2 \frac{dxdt}{t} \\
\leq \frac{|B|}{\|\chi_B\|_{L^p(\mathbb{R}^n)}} \int_{\mathbb{R}^n+} \left| t \frac{\partial}{\partial t} P_t(I - P_t) b_1(x) \right|^2 \frac{dxdt}{t} \\
\leq \frac{|B|}{\|\chi_B\|_{L^p(\mathbb{R}^n)}} \|b_1\|^2_{L^2(\mathbb{R}^n)} \sim \frac{|B|}{\|\chi_B\|_{L^p(\mathbb{R}^n)}} \int_{2B} |(I - P_{t_0}) f(x)|^2 dx \\
\leq \frac{|B|}{\|\chi_B\|_{L^p(\mathbb{R}^n)}} \left\{ \int_{2B} |(I - P_{t_2B}) f(x)|^2 dx + \int_{2B} \left[ \frac{\|\chi_B(x \cap B]\|_{L^p(\mathbb{R}^n)}}{|B|} \|f\|_{\text{BMO}_\infty(\mathbb{R}^n)} \right]^2 dx \right\} \lesssim \|f\|_{\text{BMO}_\infty(\mathbb{R}^n)}^2.
$$

If $p(\varphi)(r(\varphi))' \in (1, 2)$, we see that $[p(\varphi)]' > 2$ and $r(\varphi) > \frac{2[p(\varphi)]'-1}{|p(\varphi)|-2}$. From this and the definition of $r(p)$, we deduce that there exists $\tilde{p} \in (2, [p(\varphi)]')$ such that $r(p) > \frac{2[\tilde{p}]'-1}{\tilde{p}^2}$ and hence $\varphi \in \mathbb{R} \mathbb{A}_{2(\tilde{p})-1}(\mathbb{R}^n)$. By this, Hölder’s inequality and Theorem 4.12, we conclude that

$$
(5.7) \quad \int_{2B} |(I - P_{t_2B}) f(x)|^2 dx \\
= \int_{2B} \left[ \varphi \left( x, \|\chi_B\|^{-1}_{L^p(\mathbb{R}^n)} \right) \right]^2 \left[ \varphi \left( x, \|\chi_{2B}\|^{-1}_{L^p(\mathbb{R}^n)} \right) \right]^\frac{1}{p} f(x) dx \\
= \left\{ \int_{2B} \left[ \varphi \left( x, \|\chi_{2B}\|^{-1}_{L^p(\mathbb{R}^n)} \right) \right]^\tilde{p} f(x) dx \right\} \frac{2[\tilde{p}]'-1}{\tilde{p}} \\
\leq \frac{\|\chi_{2B}\|^2_{L^p(\mathbb{R}^n)}}{|2B|} \|f\|^2_{\text{BMO}_\infty(\mathbb{R}^n)} \lesssim \frac{\|\chi_B\|^2_{L^p(\mathbb{R}^n)}}{|B|} \|f\|^2_{\text{BMO}_\infty(\mathbb{R}^n)}.
$$

On the other hand, by Assumption B, we see that there exist $p_1 \in [p(\varphi), \infty)$, $p \in (0, i(\varphi)]$ and $q \in (1, r(\varphi)]$ such that $\varphi \in \mathbb{A}_{p_1}(\mathbb{X})$, $\varphi$ is of uniformly lower type $p$, $\varphi \in \mathbb{R} \mathbb{H}_q(\mathbb{X})$ and $\frac{2p_1}{p} - \frac{n(q-1)}{q} < n + 1$. From this, we further deduce that $n < \frac{2p_1}{p} < n + 1$, which implies that $\frac{2p_1}{p} - n - 1 < 0$ and $\frac{2p_1}{p} - \frac{n(q-1)}{q} - n - 1 < 0$.

For any $x \in B$ and $y \in (2B)^c$, it holds true that $|x - y|^{n+1} > r_B^{n+1} = t_B^{n+1}$. This, together with Proposition 3.7, (2.12), (2.13), $\frac{np_1}{p} - n - 1 < 0$, $\frac{2np_1}{p} - \frac{n(q-1)}{q} - n - 1 < 0$ and some estimates similar to those used in the proof of Proposition 3.9, implies that

$$
\left| \frac{|B|}{\|\chi_B\|_{L^p(\mathbb{R}^n)}} t \frac{\partial}{\partial t} P_t(I - P_t) b_2(x) \right|.
$$
satisfies that, for all \(x, y\)

By this and (5.8), similar to the proof of (5.4), we see that

Moreover, it is easy to show that the kernel of (5.8)

Now we prove (5.5). For any \(\parallel \chi B \parallel_{L^p(\mathbb{R}^n)}\)

From this, we further deduce that

By this and (5.6) or (5.7), we conclude that (5.4) holds true.

Now we prove (5.5). For any \(t \in (0, r_B)\), it follows, from Proposition 3.7, that, for any \(x \in \mathbb{R}^n\),

Moreover, it is easy to show that the kernel \(k_{t,t_B}\) of the operator \(T_{t,t_B} = t \frac{\partial}{\partial t} P_{(2t+t_B)/2}\) satisfies that, for all \(x, y \in \mathbb{R}^n\),

By this and (5.8), similar to the proof of (5.4), we see that

\[
\frac{|B|}{\|\chi B\|^2_{L^p(\mathbb{R}^n)}} \int_B \left| \frac{\partial}{\partial t} P_t (I - P_t) b_2(x) \right|^2 \frac{dxdt}{t} \\
= \frac{1}{|B|} \int_B \left| \frac{B}{\|\chi B\|_{L^p}} \frac{\partial}{\partial t} P_t (I - P_t) b_2(x) \right|^2 \frac{dxdt}{t} \\
\lesssim \frac{\|f\|^2_{\text{BMO}^\varphi(\mathbb{R}^n)}}{t^2_B |B|} \int_B t \, dxdt \lesssim \|f\|^2_{\text{BMO}^\varphi(\mathbb{R}^n)}.
\]
and, for all 
respectively, by the proof of Theorem 5.5, we obtain the following theorem, the details being omitted. Recall that the space $K_\Delta$ associated with the heat semigroup $(P_t)_t$ be the kernel of the operator 
(5.1) can be replaced by $n/2$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$ in (5.1) can be replaced by $f \in \text{BMO}^{\varphi}(\mathbb{R}^n)$ and $\|f\|_{\text{BMO}^{\varphi}(\mathbb{R}^n)} \lesssim \|f\|_{\text{BMO}^{\varphi}(\mathbb{R}^n) \cap \mathcal{X}}$, which completes the proof of Theorem 5.5. □

Remark 5.6. Theorem 5.5 completely covers [17, Theorem 2.14] by taking $\varphi$ as in (1.1) with $\mathcal{X}$ replaced by $\mathbb{R}^n$. Moreover, Theorem 5.5 is also new even when $\varphi$ is as in (1.2) with $\mathcal{X}$ replaced by $\mathbb{R}^n$.

Next, we consider the space $\text{BMO}^{\varphi}_A(\mathbb{R}^n)$ associated with the generalized approximation to the identity $\{A_t\}_{t>0}$ acting on the function $f$ which satisfies that

$$\int_{\mathbb{R}^n} \left| f(x) e^{-|x|^2} \right| dx < \infty.$$ 

Here, the generalized approximation to the identity $\{A_t\}_{t>0}$ is given by setting, for all $t \in (0, \infty)$ and $x \in \mathbb{R}^n$,

$$A_t f(x) := H_t f(x) := \int_{\mathbb{R}^n} h_t(x-y) f(y) dy$$

and, for all $x, y \in \mathbb{R}^n$,

$$h_t(x-y) := \frac{1}{(4\pi t)^{n/2}} e^{-|x-y|^2/4t}.$$ 

It is easy to see that $\{H_t\}_{t>0}$ satisfies (3.4) and (3.5). Notice that, for any $f \in L^p(\mathbb{R}^n)$ with $p \in [1, \infty]$, $H_t f = e^{-t\Delta} f$. Then, as above, we use $\text{BMO}^{\varphi}_A(\mathbb{R}^n)$ to denote $\text{BMO}^{\varphi}_A(\mathbb{R}^n)$ associated with the heat semigroup $\{e^{-t\Delta} \}_{t>0}$. By making some minor modifications on the proof of Theorem 5.5, we obtain the following theorem, the details being omitted. Recall that the space $K_\Delta(\mathbb{R}^n)$ is defined as in Remark 3.4(i) with $A$ and $\mathcal{X}$ replaced, respectively, by $\{e^{-t\Delta} \}_{t>0}$ and $\mathbb{R}^n$. 

$$\lesssim \frac{1}{t_B^2 |B|} \int_B \frac{|B|}{\|\lambda_B\|_{L^p(\mathbb{R}^n)}} \int_{\mathbb{R}^n} \frac{t_B}{t_B t^2 + |x-y|^2 + 1} (P_{t_B^2} - P_{(2t_B^2)}/2) f(y) dy \left| \frac{\partial}{\partial t} P_{t_B^2} - P_{(2t_B^2)}/2 \right| f(y) dy \right|^2 dx dt \lesssim \frac{1}{t_B^2 |B|} \int_B \frac{|B|}{\|\lambda_B\|_{L^p(\mathbb{R}^n)}} \int_{\mathbb{R}^n} \frac{t_B}{t_B t^2 + |x-y|^2 + 1} (P_{t_B^2} - P_{(2t_B^2)}/2) f(y) dy \left| \frac{\partial}{\partial t} P_{t_B^2} - P_{(2t_B^2)}/2 \right| f(y) dy \right|^2 dx dt \lesssim \frac{1}{t_B^2 |B|} \int_B \frac{|B|}{\|\lambda_B\|_{L^p(\mathbb{R}^n)}} \int_{\mathbb{R}^n} \frac{t_B}{t_B t^2 + |x-y|^2 + 1} (P_{t_B^2} - P_{(2t_B^2)}/2) f(y) dy \left| \frac{\partial}{\partial t} P_{t_B^2} - P_{(2t_B^2)}/2 \right| f(y) dy \right|^2 dx dt \lesssim \frac{1}{t_B^2 |B|} \int_B \frac{|B|}{\|\lambda_B\|_{L^p(\mathbb{R}^n)}} \int_{\mathbb{R}^n} \frac{t_B}{t_B t^2 + |x-y|^2 + 1} (P_{t_B^2} - P_{(2t_B^2)}/2) f(y) dy \left| \frac{\partial}{\partial t} P_{t_B^2} - P_{(2t_B^2)}/2 \right| f(y) dy \right|^2 dx dt \lesssim \frac{1}{t_B^2 |B|} \int_B \frac{|B|}{\|\lambda_B\|_{L^p(\mathbb{R}^n)}} \int_{\mathbb{R}^n} \frac{t_B}{t_B t^2 + |x-y|^2 + 1} (P_{t_B^2} - P_{(2t_B^2)}/2) f(y) dy \left| \frac{\partial}{\partial t} P_{t_B^2} - P_{(2t_B^2)}/2 \right| f(y) dy \right|^2 dx dt \lesssim \frac{1}{t_B^2 |B|} \int_B \frac{|B|}{\|\lambda_B\|_{L^p(\mathbb{R}^n)}} \int_{\mathbb{R}^n} \frac{t_B}{t_B t^2 + |x-y|^2 + 1} (P_{t_B^2} - P_{(2t_B^2)}/2) f(y) dy \left| \frac{\partial}{\partial t} P_{t_B^2} - P_{(2t_B^2)}/2 \right| f(y) dy \right|^2 dx dt.$$
Theorem 5.7. Let $\varphi$ be as in Definition 2.7 and satisfy $np(\varphi) < (n + 1)i(\varphi)$, where $p(\varphi)$ and $i(\varphi)$ are, respectively, as in (2.9) and (2.8). If $\varphi$ additionally satisfies (4.1) or $p(\varphi)[r(\varphi)]' \in (1, 2)$, where $r(\varphi)$ is as in (2.10), then the spaces $\text{BMO}_\varphi(\mathbb{R}^n)$ and $\text{BMO}_\Delta(\mathbb{R}^n)$ (modulo $K_\Delta(\mathbb{R}^n)$) coincide with equivalent norms.

By Theorems 3.12, 3.15, 5.5 and 5.7, we finally obtain the following conclusion.

Corollary 5.8. Let $\varphi$ be as in Definition 2.7 and satisfy $np(\varphi) < (n + 1)i(\varphi)$ and 

$$\frac{2np(\varphi)}{i(\varphi)} - \frac{n[r(\varphi) - 1]}{r(\varphi)} < n + 1,$$

where $n$, $p(\varphi)$, $i(\varphi)$, $N$, $r(\varphi)$ and $\alpha$ are, respectively, as in (2.2), (2.9), (2.8), (2.4), (2.10) and (2.6). If $\varphi$ additionally satisfies (4.1) or $p(\varphi)[r(\varphi)]' \in (1, 2)$, then the spaces 

$$\text{BMO}_\varphi(\mathbb{R}^n), \text{BMO}^{\varphi}_\Delta(\mathbb{R}^n), \text{BMO}^{\varphi}_{\Delta, \text{max}}(\mathbb{R}^n), \text{BMO}^\varphi_\Delta(\mathbb{R}^n), \text{BMO}_\Delta(\mathbb{R}^n),$$

$$\text{BMO}^{\varphi}_{\Delta, \text{max}}(\mathbb{R}^n) \text{ and } \widetilde{\text{BMO}}^{\varphi}_\Delta(\mathbb{R}^n)$$

coincide with equivalent norms.

Remark 5.9. We point out that Theorem 5.7 and Corollary 5.8 completely cover [17, Theorem 2.15 and Corollary 2.16] by taking $\varphi$, respectively, as in (1.1) and (1.2) with $\mathcal{X}$ replaced by $\mathbb{R}^n$.

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