STRONG SZEGŐ THEOREM ON A JORDAN CURVE

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Dedicated to the memory of Harold Widom 1932–2021

ABSTRACT. We consider certain determinants with respect to a sufficiently regular Jordan curve \( \gamma \) in the complex plane that generalize Toeplitz determinants which are obtained when the curve is the circle. This also corresponds to studying a planar Coulomb gas on the curve at inverse temperature \( \beta = 2 \). Under suitable assumptions on the curve we prove a strong Szegő type asymptotic formula as the size of the determinant grows. The resulting formula involves the Grunsky operator built from the Grunsky coefficients of the exterior mapping function for \( \gamma \). As a consequence of our formula we obtain the asymptotics of the partition function for the Coulomb gas on the curve. This formula involves the Fredholm determinant of the absolute value squared of the Grunsky operator which equals, up to a multiplicative constant, the Loewner energy of the curve. Based on this we obtain a new characterization of curves with finite Loewner energy called Weil-Petersson quasicircles.

1. Introduction and results

1.1. Definitions and results. Let \( \gamma \) be a Jordan curve in the complex plane and \( g : \gamma \mapsto \mathbb{C} \) a given function on the curve. Define the determinant

\[
D_n[g] = \det \left( \int_{\gamma} \zeta_j \bar{\zeta}_k e^{g(\zeta)} |d\zeta| \right)_{0 \leq j,k < n}
\]

assuming that all integrals exist. In the case \( \gamma = T \), the unit circle, this is \((2\pi)^n\) times the Toeplitz determinant with symbol \( e^g \). These determinants are related to orthogonal polynomials on the curve \( \gamma \) when the weight function \( w = e^g \) is positive, see [17, Sec. 16.2]. Note that our definition is different from the one in [17]; the \( D_n \) there is our \( D_n + 1 / L_n + 1 \) where \( L \) is the length of the curve. In the case when \( g = 0 \) these orthogonal polynomials were introduced by Szegő in [16], and some properties of the polynomials and the determinants, like (1.15) below, were investigated. By Andrieff’s identity, we have the integral formula

\[
D_n[e^g] = \frac{1}{n!} \int_{\gamma} \prod_{1 \leq \mu \neq \nu \leq n} |\zeta_\mu - \zeta_\nu| \prod_{\mu = 1}^{n} e^{g(\zeta_\mu)} \prod_{\mu = 1}^{n} |d\zeta_\mu|.
\]

(1.2)

Note that \( D_n[1] \) is the partition function for a planar Coulomb gas on the curve \( \gamma \),

\[
Z_n(\gamma) = D_n[1] = \frac{1}{n!} \int_{\gamma^n} \exp \left[ - \sum_{1 \leq \mu \neq \nu \leq n} \log |\zeta_\mu - \zeta_\nu|^{-1} \right] \prod_{\mu = 1}^{n} |d\zeta_\mu|.
\]

(1.3)

In the case of Toeplitz determinants the strong Szegő limit theorem gives a precise asymptotic formula for \( D_n[e^g] / D_n[1] \) where \( D_n[1] = Z_n(T) = (2\pi)^n \), see [2], [15] for background on, and proofs of, this theorem. In this case the partition function is easy to compute which is not the case for other curves. We want to generalize the strong Szegő theorem to the case of a more general Jordan curve and also understand the asymptotics of the partition function. Asymptotic properties of the determinant \( [11] \) were studied in [16] and [6, Sec. 6.2], where the asymptotics for a quotient

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of consecutive determinants was given, see [15]. A strong Szegö theorem for $D_n[e^g]/D_n[1]$ was proved in [9, Sec. III]. In this paper we prove a precise asymptotic formula for $D_n[e^g]$ under a somewhat weaker, but certainly not optimal, condition on the curve $\gamma$, and our assumption on the function $g$ is optimal. See Section [1.2] below for further comments and background. The asymptotics of the partition function $Z_n(\gamma)$ turns out to be being interesting and we will discuss its asymptotics under optimal conditions in Section [1.3]. We will not prove any results for the $\beta$-ensemble corresponding to (1.2), but we give a heuristic discussion in Subsection 1.4 and Section 5.

We can expect that the leading order asymptotics of $D_n[e^g]$ should be given by $\exp(-n^2V(\gamma))$, where $V(\gamma)$ is the logarithmic energy of $\gamma$. The logarithmic energy is defined by

$$V(\gamma) = \inf_\mu \int_\gamma \int_\gamma \log |\zeta_1 - \zeta_2|^{-1} \, d\mu(\zeta_1) d\mu(\zeta_2),$$

where the infimum is over all probability measures $\mu$ on $\gamma$. The logarithmic capacity of $\gamma$ can be defined by $\text{cap}(\gamma) = \exp(-V(\gamma))$. Let $\Omega$ be the unbounded component of the complement of $\gamma$ and let $\phi : \{ z ; |z| > 1 \} \mapsto \Omega$ be the exterior mapping function with the expansion,

$$\phi(z) = z + \phi_0 + \phi_1 z^{-1} + \ldots$$

around infinity. If $|z| > 1, |\zeta| > 1$, we have the expansion

$$\log \frac{\phi(\zeta) - \phi(z)}{\zeta - z} = - \sum_{k,\ell=1}^{\infty} a_{k\ell} \zeta^{-k} z^{-\ell},$$

where $a_{k\ell} = a_{\ell k} \in \mathbb{C}$ are the Grunsky coefficients, see e.g. [13, Sec. 3.1]. If $\gamma$ is a quasicircle, i.e. it is the image of the unit circle under a quasiconformal mapping of the plane, there is a constant $\kappa < 1$ such that

$$\sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} |\sqrt{k \ell} a_{k\ell} w_\ell|^2 \leq \kappa^2 \sum_{k=1}^{\infty} |w_k|^2,$$

(1.6)

and

$$\left| \sum_{k,\ell=1}^{\infty} \sqrt{k \ell} a_{k\ell} w_k w_\ell \right| \leq \kappa \sum_{k=1}^{\infty} |w_k|^2,$$

(1.7)

called the Grunsky inequalities, see [13, Sec. 9.4]. Write

$$b_{k\ell} = \sqrt{k \ell} a_{k\ell} = b_{k\ell}^{(1)} + i b_{k\ell}^{(2)},$$

(1.8)

where $b_{k\ell}^{(j)} \in \mathbb{R}$ and $i = \sqrt{-1}$. Consider the Grunsky operator $B$ and its real and imaginary parts,

$$B = (b_{k\ell})_{k,\ell \geq 1}, \quad B^{(j)} = (b_{k\ell}^{(j)})_{k,\ell \geq 1}, j = 1, 2,$$

(1.9)

which are bounded operators on $\ell^2(\mathbb{C})$ by (1.6) with norm $\leq \kappa < 1$. Define the operator $K$ on $\ell^2(\mathbb{C}) \oplus \ell^2(\mathbb{C})$, by

$$K = \begin{pmatrix} B^{(1)} & B^{(2)} \\ B^{(2)} & -B^{(1)} \end{pmatrix}.$$ 

Note that $K$ is real and symmetric.

Expand the function $g(\phi(e^{i\theta}))$ in a Fourier series,

$$g(\phi(e^{i\theta})) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\theta + b_k \sin k\theta,$$

(1.10)
where \(a_k, b_k \in \mathbb{C}\) and we assume that \(g(\phi(e^{i\theta}))\) is integrable. Define the infinite column vector in \(\ell^2(\mathbb{C}) \oplus \ell^2(\mathbb{C})\),

\[
g = \begin{pmatrix} (\frac{1}{2}\sqrt{Ea_k})_{k \geq 1} \\ (\frac{1}{2}\sqrt{Eb_k})_{k \geq 1} \end{pmatrix}.
\]

(1.11)

We can now state our main theorem which gives the asymptotics for the determinant \(D_n\) as \(n \to \infty\).

**Theorem 1.1.** Assume that the Jordan curve \(\gamma\) is \(C^{5+\alpha}\), \(\alpha > 0\), and that

\[
\sum_{k=1}^{\infty} k(|a_k|^2 + |b_k|^2) < \infty.
\]

(1.12)

We then have the asymptotic formula

\[
D_n[e^\theta] = \frac{(2\pi)^n \text{cap}(\gamma)^n}{\sqrt{\det(I + K)}} \exp\left(\frac{n\alpha_0}{2} + g^i(I + K)^{-1}g + o(1)\right),
\]

(1.13)

as \(n \to \infty\).

The theorem will be proved in the next section. We see that the geometry of the curve enters via the operator \(K\), and also directly via \(g\) since we have the composition of \(g\) with \(\phi\) in (1.10). If \(\gamma\) is the unit circle, we have \(K = 0\), \(\text{cap}(\gamma) = 1\), and we get the usual strong Szegő limit theorem. For the partition function \(\text{Z}_n(\gamma) = D_n[1] = \exp\left(n^2 \log \text{cap}(\gamma) + n \log 2\pi - \frac{1}{2} \log \det(I + K) + o(1)\right)\),

(1.14)

as \(n \to \infty\).

1.2. Discussion. It was proved in [6] Sec. 6.2] that if \(\gamma\) is an analytic curve then

\[
\lim_{n \to \infty} \text{cap}(\gamma)^{-2n-1} \frac{D_{n+1}[e^\theta]}{D_n[e^\theta]} = 2\pi \exp\left(\frac{1}{2\pi} \int_{-\pi}^\pi g(\phi(e^{i\theta})) \, d\theta\right),
\]

(1.15)

which is a consequence of (1.13) above. In [9] Sec. III] the following relative Szegő type theorem was proved. If \(g\) is \(C^{1+\alpha}\) on \(\gamma\) and \(\gamma\) is \(C^{10+\alpha}\) for some \(\alpha > 0\), then

\[
\frac{D_n[e^\theta]}{D_n[1]} = \exp[n\alpha_0/2 + g^i(I + K)^{-1}g + o(1)]
\]

(1.16)

as \(n \to \infty\). The expression for the constant term in the exponent in the right side was less clear in [9], see Theorem 7.1 and its Corollary. The form give here for the constant term is more elegant and satisfactory. A calculation shows that they are identical. We see that Theorem 1.1 is a strengthening of the earlier result. In particular it is not a relative Szegő theorem since we do not divide by \(D_n[1]\), and hence we also get asymptotics for the partition function as in (1.14). The condition (1.12) in the theorem on \(g\) is natural since \((I + K)^{-1}\) is an operator on \(\ell^2(\mathbb{C})\) we have to require that \(g \in \ell^2(\mathbb{C})\) which is exactly (1.12). Thus the condition on \(g\) is optimal. The condition in Theorem 1.1 that \(\gamma\) is \(C^{5+\alpha}\) is certainly not optimal although it is not immediately clear what the optimal condition is. If we consider the case when \(g = 0\) we can say more about the optimal condition on \(\gamma\). This is the topic of the next subsection. At the end of that subsection we give a conjecture on the the optimal condition on \(\gamma\) in the theorem.

If \(\gamma\) has a cusp it is not a quasicircle and the Grunsky inequality (1.6) no longer holds with \(\kappa < 1\). It would be interesting to see what the effect of a cusp is on the asymptotics of the determinant. Another question is to generalize the result to an arc instead of a Jordan curve. There are results in case when \(\gamma\) is an interval in \(\mathbb{R}\) since in that case we get a Hankel determinant, see [7], [8], [9]. In the case of an arc on the unit circle there is an asymptotic formula due to Widom in [22]. See also [11] when we have Fisher-Hartwig singularities, and [3] for a relative Szegő theorem.
1.3. Convergence of the partition function and Weil-Petersson quasicircles. Note that since \( \det(I + K) = \det(I - B^*B) \), see (2.28), it follows that (1.14) can be written as

\[
\lim_{n \to \infty} \log \frac{Z_n(\gamma)}{\text{cap}(\gamma)n^2} = \lim_{n \to \infty} \log \frac{Z_n(\gamma)}{(2\pi)^n \text{cap}(\gamma)n^2} = -\frac{1}{2} \log \det(I - B^*B),
\]

(1.17) since \( \text{cap}(T) = 1 \). Interestingly, the right side of (1.17) has occurred in other contexts. In [18] it appears, up to a multiplicative constant, under the name universal Liouville action which is a Kähler potential for the Weil-Petersson metric on the \( T_0(1) \) component of the universal Teichmüller space. It is also the so called Loewner energy of the Jordan curve, see [14], [19]. Motivated in part by connections to the Schramm-Loewner Evolution, the Loewner energy has been further studied in [20], [21]. Curves with the property that the Grunsky operator is a Hilbert-Schmidt operator are called Weil-Petersson quasicircles. There are many different characterizations and possible definitions of Weil-Petersson quasicircles, see [1] and [19, Sect. 8]. It is therefore natural to conjecture that (1.17) holds if and only if \( \gamma \) is a Weil-Petersson quasicircle. We will prove this but in order to state a theorem let us be a bit more precise.

Let \( \gamma \) be a Jordan curve and let \( \phi(z) \) be the exterior mapping function for \( \gamma \) as above. Set \( \phi_r(z) = \frac{1}{r} \phi(rz) \) for \( |z| \geq \rho^{-1} \), where \( 1 < \rho < r \), and let \( \gamma_r \) be the image of \( T \) under \( \phi_r \). Note that \( \phi_r \) is an analytic curve so in particular \( Z_n(\gamma_r) \) is well-defined. Define the function

\[
E_n(r) = \log \frac{Z_n(\gamma_r)}{(2\pi)^n \text{cap}(\gamma)n^2},
\]

(1.18) which we informally can think of as a finite \( n \) Loewner energy of \( \gamma_r \). As \( n \to \infty \) it converges to a multiple of the Loewner energy by (1.17). The following lemma will be proved in Section 4.

**Lemma 1.2.** The function \( E_n(r) \) is decreasing in \((1, \infty)\) for every \( n \geq 1 \).

The sequence \( E_n(r) \) is also increasing in \( n \) for each fixed \( r > 1 \). This is the content of Lemma 4.1 below.

The determinant in (1.1) is not well-defined for a general Jordan curve. Therefore we define the \( n \)th partition function for the Jordan curve \( \gamma \) by

\[
Z_n(\gamma) = \lim_{r \to 1^+} Z_n(\gamma_r)
\]

(1.19) with value in \( \mathbb{R} \cup \{\infty\} \). The limit exists by Lemma 1.2 possibly equal to infinity. We can now formulate our theorem which gives a new characterization of Weil-Petersson quasicircles and a way to compute their Loewner energy.

**Theorem 1.3.** The Jordan curve \( \gamma \) is a Weil-Petersson quasicircle if and only if

\[
\limsup_{n \to \infty} \frac{Z_n(\gamma)}{(2\pi)^n \text{cap}(\gamma)n^2} < \infty
\]

and in that case we have the limit (1.17). In fact, by Lemma 1.1, the sequence in (1.20) is increasing in \( n \) so we could replace the upper limit with a proper limit. The theorem is proved in Section 4. In view of this theorem it is reasonable to conjecture that the optimal condition on the curve \( \gamma \) in theorem 1.1 is that \( \gamma \) is a Weil-Petersson quasicircle.

1.4. The \( \beta \)-ensemble. Consider the \( \beta \)-ensemble corresponding to (1.2), i.e. consider

\[
D_{n,\beta}[\epsilon^\beta] = \frac{1}{n!} \int_{\gamma^n} \prod_{1 \leq \mu < \nu \leq n} |\zeta_\mu - \zeta_\nu|^{\beta/2} \prod_{\mu=1}^n e^{\beta \epsilon(\zeta_\mu)} \prod_{\mu=1}^n |d\xi_\nu|,
\]

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where \( \beta > 0 \). This is not a determinant when \( \beta \neq 2 \) but is the quantity analogous to (1.1). The corresponding partition function is \( Z_{n,\beta}(\gamma) = D_{n,\beta}[1] \). If \( \gamma = \mathbb{T} \), then
\[
Z_{n,\beta}(\mathbb{T}) = \frac{(2\pi)^n \Gamma(1 + \beta n/2)}{n! \Gamma(1 + \beta/2)^n}.
\]
The expansion (1.5) gives
\[
\log \phi'(z) = -\sum_{k=2}^{\infty} \left( \sum_{j=1}^{k-1} \right) a_{j,k-j} z^{-k}.
\]
Define
\[
g_{\beta} = (\beta/2 - 1) \text{d} + g,
\]
where
\[
\text{d} = \left( \frac{1}{2} \sqrt{\Re \left( \sum_{j=1}^{k-1} a_{j,k-j} \right)} \right)_{k \geq 1} \left( \frac{1}{2} \sqrt{\Im \left( \sum_{j=1}^{k-1} a_{j,k-j} \right)} \right)_{k \geq 1},
\]
comes from \( \log |\phi'(z)| \). Here the \( k = 1 \) component is \( = 0 \), compare with (1.21). We conjecture that, if \( a_0 = 0 \), then
\[
\lim_{n \to \infty} \frac{D_{n,\beta}[e^{i\theta}]}{Z_{n,\beta}(\mathbb{T}) \text{cap}(\gamma)^{\beta n(n-1)/2+n}} = \frac{1}{\sqrt{\det(I + K)}} \exp \left( \frac{2}{\beta} g_{\beta}'(I + K)^{-1} g_{\beta} \right),
\]
under the assumptions in [9].

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## 2. Proof of the main theorem

In this section we will prove Theorem 1.1. An essential ingredient is Lemma 2.2 which is proved in the next section. Assume that \( \gamma \) is a \( C^{5+\alpha} \)-curve for some \( \alpha > 0 \). Then, by a theorem of Kellogg, see e.g. [4, Thm. II 4.3], the map \( \phi \) can be extended to \( |z| \geq 1 \) in such a way that \( \phi \) is \( C^{5+\alpha} \) on \( \mathbb{T} \), and \( \phi'(z) \neq 0 \) when \( |z| \geq 1 \). Clearly, the expansion (1.5) then holds for \( |z|, |\zeta| \geq 1 \). It follows from (1.2), by introducing the parametrization \( \zeta_\mu = \text{cap}(\gamma) \phi(e^{i\theta_\mu}) \), that
\[
D_n[e^{i\theta}] = \frac{\text{cap}(\gamma)^n}{n!} \int_{[-\pi,\pi]^n} \exp \left[ \sum_{\mu \neq \nu} \log |\phi(e^{i\theta_\mu}) - \phi(e^{i\theta_\nu})| + \sum_{\mu} \log |\phi'(e^{i\theta_\mu})| + g(\phi(e^{i\theta_\mu})) \right] d\theta.
\]

(2.1)
Lemma 2.1. Define the real, symmetric $2m \times 2m$ matrix

$$\Lambda = \begin{pmatrix} \Lambda_m & 0 \\ 0 & -\Lambda_m \end{pmatrix},$$

where $\kappa = \Lambda_m(\lambda - 1) < 1$, so that $B^{(1)}_m$ and $B^{(2)}_m$ are real symmetric $m \times m$ matrices. Since $B_m$ is a complex, symmetric matrix there is a unitary matrix $U_m = R_m + iS_m$, with $R_m$ and $S_m$ real, such that

$$B_m = U_m \Lambda_m U_m^*,$$

(2.6)

where $\Lambda_m = \text{diag}(\lambda_{m,1}, \ldots, \lambda_{m,m})$ and $\lambda_{m,k}, 1 \leq k \leq m$, are the singular values of $B_m$, [8 Sec. 4.4].

Define the real, symmetric $2m \times 2m$ matrix

$$K_m = \begin{pmatrix} B^{(1)}_m & B^{(2)}_m \\ B^{(2)}_m^\ast & -B^{(1)}_m \end{pmatrix},$$

and the matrices

$$T_m = \begin{pmatrix} R_m & S_m \\ S_m^\ast & -R_m \end{pmatrix}, \quad \tilde{\Lambda}_m = \begin{pmatrix} \Lambda_m & 0 \\ 0 & -\Lambda_m \end{pmatrix}.$$

Lemma 2.1. The matrix $T_m$ is orthogonal and

$$K_m = T_m \tilde{\Lambda}_m T_m^\ast,$$

(2.7)

so that $K_m$ has eigenvalues $\pm \lambda_{m,k}$. These eigenvalues satisfy

$$|\lambda_{m,k}| \leq \kappa < 1,$$

(2.8)

where $\kappa$ is the constant in the Grunsky inequality [1.4]. Furthermore, if $x = (x_j)_{1 \leq j \leq m}$, $y = (y_j)_{1 \leq j \leq m}$ are real column vectors, then

$$\text{Re} \sum_{k,\ell=1}^m b_{k\ell}(x_k - iy_k)(x_\ell - iy_\ell) = (x^\ast y)^\ast K_m (x^\ast y).$$

(2.9)

Proof. That $T_m$ is orthogonal follows from $U_m U_m^* = I$. The identities (2.5), (2.6) and $U_m = R_m + iS_m$ give

$$B^{(1)}_m = R_m \Lambda_m R_m^\ast - S_m \Lambda_m S_m^\ast,$$
$$B^{(2)}_m = R_m \Lambda_m S_m^\ast + S_m \Lambda_m R_m^\ast,$$

and the matrices

$$T_m = \begin{pmatrix} R_m & S_m \\ S_m^\ast & -R_m \end{pmatrix}, \quad \tilde{\Lambda}_m = \begin{pmatrix} \Lambda_m & 0 \\ 0 & -\Lambda_m \end{pmatrix}.$$

Combining (1.21) with (1.5) leads to the identity

$$\sum_{\mu \neq \nu} \log |\phi(e^{i\theta_\mu}) - \phi(e^{i\theta_\nu})| + \sum_{\mu} \log |\phi'(e^{i\theta_\mu})|$$
$$= \sum_{\mu \neq \nu} |e^{i\theta_\mu} - e^{i\theta_\nu}| - \text{Re} \sum_{k,\ell=1}^\infty a_{k\ell} \left( \sum_{\mu} e^{-ik\theta_\mu} \right) \left( \sum_{\nu} e^{-i\ell\theta_\nu} \right).$$

(2.2)

Let

$$\mathbb{E}_n[\cdot] = \frac{1}{(2\pi)^n} \int_{[-\pi,\pi]^n} \exp \left( \sum_{\mu \neq \nu} \log |e^{i\theta_\mu} - e^{i\theta_\nu}| \right) d\theta,$$

be the expectation over the eigenvalues $e^{i\theta_\mu}$ of a random unitary matrix with respect to normalized Haar measure, [12]. It follows from (2.1), (2.2) and (2.3) that

$$D_n[e^\theta] = (2\pi)^n \text{cap}(\gamma)^n \mathbb{E}_n \left[ \exp \left( -\text{Re} \sum_{k,\ell=1}^\infty a_{k\ell} \left( \sum_{\mu} e^{-ik\theta_\mu} \right) \left( \sum_{\nu} e^{-i\ell\theta_\nu} \right) + \sum_{\mu} g(\phi(e^{i\theta_\mu})) \right) \right].$$

(2.4)

To proceed we will need some linear algebra. Let $b_{k\ell}$ be given by (1.8) and define

$$B_m = (b_{k\ell})_{1 \leq k,\ell \leq m} = B^{(1)}_m + iB^{(2)}_m,$$

(2.5)

so that $B^{(1)}_m$ and $B^{(2)}_m$ are real symmetric $m \times m$ matrices. Since $B_m$ is a complex, symmetric matrix there is a unitary matrix $U_m = R_m + iS_m$, with $R_m$ and $S_m$ real, such that

$$B_m = U_m \Lambda_m U_m^*,$$

(2.6)

where $\Lambda_m = \text{diag}(\lambda_{m,1}, \ldots, \lambda_{m,m})$ and $\lambda_{m,k}, 1 \leq k \leq m$, are the singular values of $B_m$, [8 Sec. 4.4]. Define the real, symmetric $2m \times 2m$ matrix

$$K_m = \begin{pmatrix} B^{(1)}_m & B^{(2)}_m \\ B^{(2)}_m^\ast & -B^{(1)}_m \end{pmatrix},$$

and the matrices

$$T_m = \begin{pmatrix} R_m & S_m \\ S_m^\ast & -R_m \end{pmatrix}, \quad \tilde{\Lambda}_m = \begin{pmatrix} \Lambda_m & 0 \\ 0 & -\Lambda_m \end{pmatrix}.$$
which translates into (2.7). We also see that
\[
Re \sum_{k, \ell=1}^m b_{k \ell} (x_k - iy_k)(x_\ell - iy_\ell) = \sum_{k, \ell=1}^m b_{k \ell}^{(1)} (x_k x_\ell - y_k y_\ell) + b_{k \ell}^{(2)} (x_k y_\ell + y_k x_\ell)
\]
\[
= \begin{pmatrix} x^t \\ y \\ y \end{pmatrix} \begin{pmatrix} B_m^{(1)} & B_m^{(2)} \\ B_m^{(2)} & -B_m^{(1)} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},
\]
which proves (2.9). It follows from (1.7) and (2.9) that
\[
\left| \begin{pmatrix} x \\ y \\ y \end{pmatrix}^t K_m \begin{pmatrix} x \\ y \end{pmatrix} \right| \leq \kappa \left( \begin{pmatrix} x \\ y \\ y \end{pmatrix}^t \begin{pmatrix} x \\ y \end{pmatrix} \right),
\]
which shows that all the eigenvalues of \( K_m \) have absolute value \( \leq \kappa \).

\( \square \)

Let
\[
X = \left( \frac{1}{\sqrt{k}} \sum_{\mu} \cos k \theta_\mu \right)_{k \geq 1}, \quad Y = \left( \frac{1}{\sqrt{k}} \sum_{\mu} \sin k \theta_\mu \right)_{k \geq 1},
\]
be infinite column vectors, and let \( P_m \) denote projection onto the first \( m \) components. It follows from (2.7) and (2.9) that
\[
- \operatorname{Re} \sum_{k, \ell=1}^m a_{k \ell} \left( \sum_{\mu} e^{-i k \theta_\mu} \right) \left( \sum_{\nu} e^{-i \nu \theta_\nu} \right) = \begin{pmatrix} P_m X \\ P_m Y \end{pmatrix}^t \begin{pmatrix} -\Lambda_m & 0 \\ 0 & \Lambda_m \end{pmatrix} T_m \begin{pmatrix} P_m X \\ P_m Y \end{pmatrix}.
\]

Define, for \( \zeta \in \mathbb{C} \),
\[
M_m(\zeta) = \begin{pmatrix} \zeta \Lambda_{m}^{1/2} & 0 \\ 0 & \Lambda_{m}^{1/2} \end{pmatrix} T_m \begin{pmatrix} P_m X \\ P_m Y \end{pmatrix},
\]
so that
\[
- \operatorname{Re} \sum_{k, \ell=1}^m a_{k \ell} \left( \sum_{\mu} e^{-i k \theta_\mu} \right) \left( \sum_{\nu} e^{-i \nu \theta_\nu} \right) = M_m(i)^t M_m(i).
\]

Without loss of generality we can assume that \( a_0 = 0 \) in (1.10) by just subtracting off the mean. For \( \zeta \in \mathbb{C} \), we define
\[
a_k(\zeta) = \frac{1}{2} \sqrt{k} (\operatorname{Re} a_k + \zeta \operatorname{Im} a_k), \quad b_k(\zeta) = \frac{1}{2} \sqrt{k} (\operatorname{Re} b_k + \zeta \operatorname{Im} b_k),
\]
and the infinite column vectors
\[
a(\zeta) = (a_k(\zeta))_{k \geq 1}, \quad b(\zeta) = (b_k(\zeta))_{k \geq 1},
\]
which lie in \( \ell^2(\mathbb{C}) \) by the assumption (1.12). Set
\[
g(\zeta) = \begin{pmatrix} a(\zeta) \\ b(\zeta) \end{pmatrix},
\]
so that \( g(i) = g \) given by (1.11). We see that (1.10) can be written
\[
\sum_{\mu} g(\phi(e^{i \theta_{\nu}})) = 2g^t(\begin{pmatrix} X \\ Y \end{pmatrix}).
\]

Define
\[
w_m(\theta_1, \theta_2) = \operatorname{Re} \sum_{k \lor \ell > m} a_{k \ell} e^{-i k \theta_1 - i \ell \theta_2},
\]
and
\[ W_m(\theta) = -\sum_{\mu,\nu} w_m(\theta_\mu, \theta_\nu). \] (2.16)

Combining (2.12), (2.14), (2.15) and (2.16), we obtain the identity
\[ -\text{Re} \sum_{k,\ell=1}^{\infty} a_{k\ell} \left( \sum_{\mu} e^{-ik\theta_\mu} \right) \left( \sum_{\nu} e^{-i\ell\theta_\nu} \right) + \sum_{\mu} g(\phi(e^{i\theta_\mu})) = M_m(i)^t M_m(i) + 2g(i)^t \begin{pmatrix} X \\ Y \end{pmatrix} + W_m(\theta), \]
for every \( m \geq 1 \). Using this identity in (2.4) leads us to define the entire function
\[ G_{m,n}(\zeta) = \mathbb{E}_n[\exp(M_m(\zeta)^t M_m(\zeta) + 2g(\zeta)^t \begin{pmatrix} X \\ Y \end{pmatrix} + W_m(\theta))], \] (2.17)
for \( \zeta \in \mathbb{C} \), so that, for any \( m \geq 1 \),
\[ D_n(e^\theta) = (2\pi)^n \text{cap}(\gamma_n)^2 G_{m,n}(i). \] (2.18)

Note that \( G_{m,n}(i) \) is independent of \( m \), but of course for other \( \zeta \) the function \( G_{m,n}(\zeta) \) does depend on \( m \).

The expression \( M_m(\zeta)^t M_m(\zeta) \) is a quadratic form in \( X \) and \( Y \) and we want instead to have a linear form in \( X \) and \( Y \). This can be achieved by using a Gaussian integral, an idea that was also used in [10, Sec. 6.5]. Let \( u = (u_k)_{1 \leq k \leq m}, \ v = (v_k)_{1 \leq k \leq m} \) be real column vectors. Then,
\[ \exp(M_m(\zeta)^t M_m(\zeta)) = \frac{1}{\pi^m} \int_{\mathbb{R}^m} du \int_{\mathbb{R}^m} dv \exp \left( - \begin{pmatrix} u \\ v \end{pmatrix}^t \begin{pmatrix} u \\ v \end{pmatrix} + 2 \begin{pmatrix} u \\ v \end{pmatrix}^t M_m(\zeta) \right) \]
and Fubini’s theorem in (2.17) to get the formula
\[ G_{m,n}(\zeta) = \frac{1}{\pi^m} \int_{\mathbb{R}^m} du \int_{\mathbb{R}^m} dv \exp\left( - \begin{pmatrix} u \\ v \end{pmatrix}^t \begin{pmatrix} u \\ v \end{pmatrix} \right) \mathbb{E}_n[\exp(2 \begin{pmatrix} u \\ v \end{pmatrix}^t M_m(\zeta) + 2g(\zeta)^t \begin{pmatrix} X \\ Y \end{pmatrix} + W_m(\theta))]. \] (2.19)

From the definition (2.11) we see that
\[ \begin{pmatrix} u \\ v \end{pmatrix}^t M_m(\zeta) = L_m(\zeta)^t \begin{pmatrix} X \\ Y \end{pmatrix}, \]
where
\[ L_m(\zeta) = \begin{pmatrix} P_m & 0 \\ 0 & P_m \end{pmatrix}^t T_m \begin{pmatrix} \zeta \Lambda_m^{1/2} & 0 \\ 0 & \Lambda_m^{1/2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \] (2.20)

Thus,
\[ G_{m,n}(\zeta) = \frac{1}{\pi^m} \int_{\mathbb{R}^m} du \int_{\mathbb{R}^m} dv \exp\left( - \begin{pmatrix} u \\ v \end{pmatrix}^t \begin{pmatrix} u \\ v \end{pmatrix} \right) \mathbb{E}_n[\exp(2(L_m(\zeta) + g(\zeta))^t \begin{pmatrix} X \\ Y \end{pmatrix} + W_m(\theta))]. \] (2.21)

Note that by the definitions
\[ |G_{m,n}(\zeta)| \leq G_{m,n}(\text{Re} \zeta). \] (2.22)

We will now state a lemma that will allow us to prove the theorem. The proof of the lemma will be given in the next section. Define the function
\[ f_{\zeta}(\lambda) = \begin{cases} (1 - \zeta^2 \lambda)^{-1} \zeta^2 \lambda, & \lambda \geq 0 \\ -(1 + \lambda)^{-1} \lambda, & \lambda < 0 \end{cases}, \] (2.23)
on the real line. We can use spectral calculus to define $f_{\zeta}(K)$. Recall that $K$ is a symmetric trace class operator with spectrum in $[-\kappa, \kappa]$. Define
\[
G(\zeta) = \frac{1}{\sqrt{\det(I - \zeta^2 |B|) \det(I - |B|)}} \exp \left( g(\zeta)I + f_{\zeta}(K)g(\zeta) \right),
\]
which is holomorphic in $|\zeta| < \kappa$. Note that $B$ is a trace-class operator by Lemma 3.1 below.

**Lemma 2.2.** Let $G_{m,n}(\zeta)$ be defined by (2.17). Then if $\rho \in (1, 1/\sqrt{\kappa})$ there is a constant $C$ so that
\[
|G_{m,n}(\zeta)| \leq C
\]
for all $|\zeta| \leq \rho$, $n \geq 1$ and $m$ sufficiently large. Also, if $\zeta$ is real and $|\zeta| \leq \rho$, then
\[
\lim_{m \to \infty} \lim_{n \to \infty} G_{m,n}(\zeta) = G(\zeta).
\]

Assume Lemma 2.2. It follows from (2.25) that $\{G_{m,n}(\zeta)\}$ is a normal family in $|\zeta| < \rho$. Let $H_n = G_{m,n}(i)$, which is independent of $m$. By (2.25), $|H_n| \leq C$ for all $n \geq 1$. Let $\{H_{n_i}\}$ be any convergent subsequence. If we can show that
\[
\lim_{i \to \infty} H_{n_i} = G(i),
\]
we are done since,
\[
I + f_i(K) = I - (I + K)^{-1}K = (I + K)^{-1}.
\]

Also,
\[
\det(I + |B|) \det(I - |B|) = \det(I - |B|^2) = \prod_{j=1}^{\infty} (1 - \lambda_j^2) = \det(I + K),
\]
where $\lambda_j$ are the singular values of $B$. The fact that $\{G_{m,n}(\zeta)\}$ is a normal family and a diagonal argument shows that there is a subsequence $\{n_{i,i}\}$ of $\{n_i\}$ such that
\[
\lim_{i \to \infty} G_{m,n_{i,i}}(\zeta) =: G_m(\zeta)
\]
uniformly in $|\zeta| \leq \rho' < \rho$ for each sufficiently large $m$. By (2.25), $\{G_m(\zeta)\}$ is a normal family in $|\zeta| < \rho$. Let $\{m_j\}$ be any sequence such that $G_{m_j}(\zeta)$ converges uniformly in $|\zeta| \leq \rho'$, where $1 < \rho' < \rho$. Then, for $\zeta$ real, $|\zeta| \leq \rho'$,
\[
\lim_{j \to \infty} G_{m_j}(\zeta) = \lim_{j \to \infty} \lim_{i \to \infty} G_{m_j,n_{i,i}}(\zeta) = G(\zeta),
\]
by (2.26). Since $\{m_j\}$ was arbitrary, we see that $\lim_{m \to \infty} G_m(\zeta) = G(\zeta)$ uniformly for $\zeta \in \mathbb{C}$, $|\zeta| \leq \rho'$. For any fixed $m$,
\[
\lim_{i \to \infty} H_{n_i} = \lim_{i \to \infty} H_{n_{i,i}} = \lim_{i \to \infty} G_{m,n_{i,i}}(i) = G_m(i).
\]

We can now let $m \to \infty$ to get (2.27). This completes the proof of the theorem.

3. **Proof of Lemma 2.2**

We start by giving a technical lemma that we will use below.

**Lemma 3.1.** Assume that $\gamma$ is a $C^{5+\alpha}$ curve for some $\alpha > 0$ so that the extended exterior mapping function $\phi$ is $C^{4+\alpha}$ on $\mathbb{T}$. Let the operator $B$ on $\ell^2(\mathbb{C})$ be defined by (1.9). Then $B$ is a trace class operator. Also, if $\delta_m$ is defined by
\[
\delta_m = \left( \sum_{k \neq \ell ; m} (k\ell)^{2+\epsilon} |b_{k\ell}|^2 \right)^{1/2},
\]
(3.1)
we have that $\delta_m \to 0$ as $m \to \infty$. Furthermore there is a constant $C$ so that
\[
\sum_{k \vee \ell > m} (k^2 + \ell^2)|a_{k\ell}| \leq C
\]  
for all $m \geq 1$.

Proof. Since $\phi$ is $C^{5+\alpha}$ it follows from the definition of the Grunsky coefficients that there is a constant $C$ such that
\[
|a_{k\ell}| \leq \frac{C}{k^{2+\epsilon} \ell^{2+\epsilon}}, \tag{3.3}
\]
\[
|a_{k\ell}| \leq \frac{C}{k^{3+\epsilon} \ell^{1+\epsilon}}, \tag{3.4}
\]
for some $\epsilon > 0$. Consider the operators given by $K = (k^{1+\epsilon/2} \ell^{1+\epsilon/2} a_{k\ell})_{k,\ell \geq 1}$ and $D = (k^{-1/2-\epsilon/2} \delta_{k\ell})_{k,\ell \geq 1}$. Then $K$ and $D$ are Hilbert-Schmidt operators, and since $B = DKD$ we see that $B$ is a trace class operator.

We see from (1.8), (3.1), and (3.3) that
\[
\delta_m^2 = \sum_{k \vee \ell > m} k^{2+\epsilon} \ell^{2+\epsilon} |b_{k\ell}|^2 \leq \sum_{k \vee \ell > m} \frac{C}{k^{1+\epsilon} \ell^{1+\epsilon}},
\]
which $\to 0$ as $m \to \infty$. Also, since $a_{k\ell}$ is symmetric (3.4) gives the estimate
\[
\sum_{k \vee \ell > m} (k^2 + \ell^2)|a_{k\ell}| = 2 \sum_{k \vee \ell > m} k^2|a_{k\ell}| \leq C
\]
for all $m \geq 1$. $\square$

We turn now to the proof of the estimate (2.25). This proof will also give us an upper bound in (2.26). After that we will prove a lower bound in (2.26) which will coincide with the upper bound and hence prove the limit. First, we need an estimate of $W_m(\theta)$ defined by (2.16). Note that
\[
W_m(\theta) = \sum_{k \vee \ell > m} \sum_{k \wedge \ell \leq M} |b_{k\ell}||X_k - iY_k||X_\ell - iY_\ell| = \lim_{M \to \infty} \sum_{k \vee \ell > m} \sum_{k \wedge \ell \leq M} |b_{k\ell}||X_k - iY_k||X_\ell - iY_\ell|.
\]

Let $\epsilon > 0$ be fixed. By the Cauchy-Schwarz' inequality
\[
\sum_{k \vee \ell > m} |b_{k\ell}||X_k - iY_k||X_\ell - iY_\ell| \leq \left( \sum_{k \vee \ell > m} (k\ell)^{2+\epsilon} |b_{k\ell}|^2 \right)^{1/2} \left( \sum_{k \vee \ell > m} \frac{1}{k^{2+\epsilon}} |X_k - iY_k|^2 \frac{1}{\ell^{2+\epsilon}} |X_\ell - iY_\ell|^2 \right)^{1/2}
\]
\[
\leq \delta_m \left( \sum_{k=1}^M \frac{1}{k^{2+\epsilon}} (X_k^2 + Y_k^2) \right) \tag{3.5}
\]

We know from Lemma 3.1 that our assumptions on $\gamma$ imply that $\delta_m \to 0$ as $m \to \infty$. Define the $2M$ times $2M$ matrix $D_{m,M}$ by
\[
D_{m,M}^{-1} = \begin{pmatrix} \text{diag} \left( \frac{\delta_m}{k^{2+\epsilon}} \right)_{1 \leq k \leq M} & 0 \\ 0 & \text{diag} \left( \frac{\delta_m}{\ell^{2+\epsilon}} \right)_{1 \leq \ell \leq M} \end{pmatrix}.
\]
Because of the inequality (2.22) we can assume that $\zeta \in \mathbb{R}$ which we will do from now on. We see from (2.21), Fatou’s lemma and (3.5) that

$$G_{m,n}(\zeta) \leq \frac{1}{\pi^m} \int_{\mathbb{R}^m} du \int_{\mathbb{R}^m} dv \exp\left(-\begin{pmatrix} u \\ v \end{pmatrix}^t \begin{pmatrix} X \\ Y \end{pmatrix}\right)$$

\[ \times \lim_{M \to \infty} \mathbb{E}_n \left[ \exp\left(2(L_m(\zeta)^t + g(\zeta))^t \begin{pmatrix} X \\ Y \end{pmatrix} + \sum_{k \not\in \ell \geq m \atop k \not\in \ell \leq M} |b_{k\ell}||X_k - iY_k||X_\ell - iY_\ell|\right) \right] \]

\[ \leq \frac{1}{\pi^m} \int_{\mathbb{R}^m} du \int_{\mathbb{R}^m} dv \exp\left(-\begin{pmatrix} u \\ v \end{pmatrix}^t \begin{pmatrix} u \\ v \end{pmatrix}\right) \]

\[ \times \lim_{M \to \infty} \mathbb{E}_n \left[ \exp\left(2(L_m(\zeta)^t + g(\zeta))^t \begin{pmatrix} X \\ Y \end{pmatrix} + \left(P_M X\right)^t D_{m,M}^{-1} \left(P_M Y\right)\right)\right]. \]

We now use the Gaussian integral

$$\exp\left(\left(P_M X\right)^t D_{m,M}^{-1} \left(P_M Y\right)\right) = \frac{1}{\pi^M} \left(\prod_{k=1}^{M} \frac{k^{2+\epsilon}}{\delta_m}\right) \int_{\mathbb{R}^M} dp \int_{\mathbb{R}^M} dq \exp\left(-\begin{pmatrix} p \\ q \end{pmatrix}^t D_{m,M} \begin{pmatrix} p \\ q \end{pmatrix}\right)$$

\[ + 2 \left(P_M p\right)^t X \right), \]

where $p$ and $q$ are column vectors in $\mathbb{R}^M$. If we use this identity in (3.6), we obtain the estimate

$$G_{m,n}(\zeta) \leq \frac{1}{\pi^{m+M}} \left(\prod_{k=1}^{M} \frac{k^{2+\epsilon}}{\delta_m}\right) \int_{\mathbb{R}^M} dp \int_{\mathbb{R}^M} dq \int_{\mathbb{R}^m} du \int_{\mathbb{R}^m} dv$$

\[ \times \lim_{M \to \infty} \mathbb{E}_n \left[ \exp\left(2 \left(L_m(\zeta)^t + g(\zeta)\right)^t \begin{pmatrix} X \\ Y \end{pmatrix}\right)\right] \left(\begin{pmatrix} P_M p \\ P_M q \end{pmatrix}^t \begin{pmatrix} X \\ Y \end{pmatrix}\right)\right]. \]

We will now make use of the following upper bound which is a consequence of the strong Szegő limit theorem, [9, p. 268], or the Geronimo-Case-Borodin-Okounkov identity, [10] Lemma 2.3.

**Lemma 3.2.** We have the estimate

$$\mathbb{E}_n \left[ \exp\left(2 \begin{pmatrix} c \\ d \end{pmatrix}^t \begin{pmatrix} X \\ Y \end{pmatrix}\right)\right] \leq \exp\left(\begin{pmatrix} c \\ d \end{pmatrix}^t \begin{pmatrix} c \\ d \end{pmatrix}\right), \quad (3.8)$$

for infinite column vectors $c = (c_k)_{k \geq 1}$, $d = (d_k)_{k \geq 1}$ in $\ell^2(\mathbb{R})$.

The estimate (3.8) gives

$$\mathbb{E}_n \left[ \exp\left(2 \left(L_m(\zeta) + g(\zeta)\right)^t \begin{pmatrix} X \\ Y \end{pmatrix}\right)\right] \leq \exp\left(\left(L_m(\zeta) + g(\zeta)\right)^t \left(L_m(\zeta) + g(\zeta)\right) + \left(P_M P\right)^t \left(P_M P\right) + \left(P_M q\right)^t \left(P_M q\right)\right) \quad (3.9)$$

$$= \exp\left(\left(L_m(\zeta) + g(\zeta)\right)^t \left(L_m(\zeta) + g(\zeta)\right) + 2 \left(L_m(\zeta) + g(\zeta)\right)^t \left(P_M P\right) + \left(P_M q\right)^t \left(P_M q\right)\right).$$
Inserting this into (3.7), the $pq$-integral becomes

\[
\frac{1}{\pi^M} \left( \prod_{k=1}^{M} \frac{k^{2+\epsilon}}{\delta_m} \right) \int_{\mathbb{R}^M} dp \int_{\mathbb{R}^M} dq \exp \left( -\left( \frac{p}{q} \right)^t (D_{m,M} - I) \left( \frac{p}{q} \right) \right)
\]

\[+ 2 \left( L_m(\zeta) + g(\zeta) \right)^t \left( \begin{array}{cc} 0 & 0 \\ P_M & P_M \end{array} \right)^t \left( \frac{p}{q} \right) \]

\[= \left( \prod_{k=1}^{M} \frac{k^{2+\epsilon}/\delta_m}{k^{2+\epsilon}/\delta_m - 1} \right) \exp \left( (L_m(\zeta) + g(\zeta))^t \left( \begin{array}{cc} 0 & 0 \\ P_M & P_M \end{array} \right)^t (D_{m,M} - I)^{-1} \left( \begin{array}{cc} 0 & 0 \\ P_M & P_M \end{array} \right) (L_m(\zeta) + g(\zeta)) \right) \]

\[\leq \left( \prod_{k=1}^{M} \frac{1}{1 - \delta_m/k^{2+\epsilon}} \right) \exp \left( \frac{\delta_m}{1 - \delta_m} (L_m(\zeta) + g(\zeta))^t (L_m(\zeta) + g(\zeta)) \right), \]

where the last inequality follows from the fact that all entries in $(D_{m,M} - I)^{-1}$ are $\leq \delta_m(1 - \delta_m)^{-1}$.

If we assume that $m$ is so large that $\delta_m \leq 1/2$ then there is a constant $C_\epsilon$ so that

\[\prod_{k=1}^{M} \frac{1}{1 - \delta_m/k^{2+\epsilon}} \leq e^{C_\epsilon \delta_m}.\]

Thus, (3.7), (3.9) and (3.10) give

\[G_{m,n}(\zeta) \leq \frac{e^{C_\epsilon \delta_m}}{\pi^m} \int_{\mathbb{R}^m} du \int_{\mathbb{R}^m} dv \exp \left( -\left( \frac{u}{v} \right)^t \left( \frac{u}{v} \right) + \frac{1}{1 - \delta_m} (L_m(\zeta) + g(\zeta))^t (L_m(\zeta) + g(\zeta)) \right). \]

We now insert the definition (2.20) of $L_m(\zeta)$ into the right side. After some computation we get the estimate

\[G_{m,n}(\zeta) \leq \frac{e^{C_\epsilon \delta_m}}{\pi^m} \int_{\mathbb{R}^m} du \int_{\mathbb{R}^m} dv \exp \left( -\left( \frac{u}{v} \right)^t \left( I - \left( \begin{array}{cc} \zeta^2 \Lambda_m/2 & 0 \\ 0 & \Lambda_m^{1/2} \end{array} \right) \right) \left( \frac{u}{v} \right) \right)
\]

\[+ 2 \frac{(1 - \delta_m)}{1 - \delta_m} \left( \frac{u}{v} \right)^t \left( \zeta^2 \Lambda_m/2 + \Lambda_m^{1/2} \right) \left( \begin{array}{cc} P_m & 0 \\ 0 & P_m \end{array} \right) \left( \frac{u}{v} \right)^t \left( \frac{u}{v} \right) \left( \frac{1}{1 - \delta_m} g(\zeta) \right).
\]

Since $0 \leq \lambda_{m,k} \leq \kappa$, we see that if $|\zeta| \leq \rho < 1/\sqrt{\kappa}$ and $m$ is so large that $\rho^2 \kappa/(1 - \delta_m) < 1$, then the matrix $I - \left( \begin{array}{cc} \zeta^2 \Lambda_m/2 & 0 \\ 0 & \Lambda_m^{1/2} \end{array} \right) \left( \begin{array}{cc} P_m & 0 \\ 0 & P_m \end{array} \right) \left( \frac{u}{v} \right)^t \left( \frac{u}{v} \right)$ is positive definite. Hence, we can compute the Gaussian integral in (3.11) to get the estimate

\[G_{m,n}(\zeta) \leq \frac{e^{C_\epsilon \delta_m}}{\sqrt{\det(I - \frac{\zeta^2}{1 - \delta_m} \Lambda_m) \det(1 - \frac{\zeta^2}{1 - \delta_m} \Lambda_m)}} \exp \left( \frac{1}{1 - \delta_m} \left( \frac{P_m a(\zeta)}{P_m b(\zeta)} \right)^t T_m \right)
\]

\[\times \left( \begin{array}{cc} (I - \frac{\zeta^2}{1 - \delta_m} \Lambda_m)^{-1} & \frac{\zeta^2}{1 - \delta_m} \Lambda_m \\ 0 & (I - \frac{\zeta^2}{1 - \delta_m} \Lambda_m)^{-1} \frac{\zeta^2}{1 - \delta_m} \Lambda_m \end{array} \right) \left( \begin{array}{cc} P_m a(\zeta) & P_m b(\zeta) \end{array} \right) \right) \]

for all $\zeta \in [-\rho, \rho]$ and $m$ sufficiently large. Since $T_m$ is an orthogonal matrix we see that the $\ell^2(\mathbb{R}) \oplus \ell^2(\mathbb{R})$-norm of $T_m \left( \begin{array}{cc} P_m a(\zeta) \\ P_m b(\zeta) \end{array} \right)$ is

\[\leq \sum_{k=1}^{m} a(\zeta)_k^2 + b(\zeta)_k^2 \leq \rho^2 \sum_{k=1}^{\infty} k(|a_k|^2 + |b_k|^2) < \infty,
\]
by the assumption \([1.12]\). Since \(|\zeta| \leq \rho, \lambda_{mk}\), and \(\rho^2\kappa/(1 - \delta_m) < 1, \delta_m < 1/2\) for \(m\) sufficiently large, we see that the expression in the exponent in the right side of (3.12) is bounded by a constant. Note that it is proved in Lemma \([3.1]\) that \(\delta_m \to 0\) as \(m \to \infty\) and hence \(C_1\delta_m \leq 1\) if \(m\) is sufficiently large. Also, since \(B\) is trace class

\[
\det(I - \frac{\zeta^2}{1 - \delta_m} |B_m|) \to \det(I - \zeta^2|B|)
\]
as \(m \to \infty\) for \(|\zeta| \leq \rho\). This proves \((2.25)\) in Lemma \([2.2]\). If we recall \((2.23)\), we see that (3.12) gives

\[
\lim_{n \to \infty} G_{m,n}(\zeta) \leq \frac{e^{C_1\delta_m}}{\sqrt{\det(I - \frac{\zeta^2}{1 - \delta_m} |B_m|) \det(I - \frac{1}{1 - \delta_m} |B_m|)}} \times \exp \left( \frac{1}{1 - \delta_m} \left( \frac{P_m(a(\zeta))}{P_m(b(\zeta))} \right)^t f_\zeta \left( \frac{1}{1 - \delta_m} \frac{K_m}{P_m(a(\zeta))} \right) + \frac{1}{1 - \delta_m} g(\zeta)^t g(\zeta) \right),
\]
for \(\zeta \in \mathbb{R}, |\zeta| \leq \rho\). We can let \(m \to \infty\) in the right side to conclude

\[
\lim_{m \to \infty} \lim_{n \to \infty} G_{m,n}(\zeta) \leq \frac{\exp \left( (g(\zeta)^t (I + f_\zeta(K)) g(\zeta) \right)}{\sqrt{\det(I - \zeta^2 |B|) \det(I - |B|)}} = G(\zeta),
\]
for \(\zeta \in \mathbb{R}, |\zeta| \leq \rho\).

In order to prove \((2.26)\) we also need a lower bound. Fix \(D > 0\) and let \(\zeta \in \mathbb{R}\). We see from \([2.21]\) that

\[
G_{m,n}(\zeta) \geq \frac{1}{\pi^m} \int_{[-D,D]^m} du \int_{[-D,D]^n} dv \exp(- \left( \frac{u}{v} \right)^t \left( \frac{u}{v} \right)) \times \mathbb{E}_m \left[ \exp \left( 2(L_m(\zeta) + g(\zeta))^t \left( \frac{X}{Y} \right) + W_m(\theta) \right) \right].
\]

Let \(f(\theta)\) be such that

\[
2(L_m(\zeta) + g(\zeta))^t \left( \frac{X}{Y} \right) = \sum_\mu f(\theta_\mu).
\]
Note that \(f\) is real-valued since \(\zeta \in \mathbb{R}\). We want to estimate

\[
\mathbb{E}_n[\exp(\sum_\mu f(\theta_\mu) + W_m(\theta))] \]
from below. To do this we will use an idea from \([9\, \text{Lemma 2.3}]\). Let \(h(\theta)\) be a given, smooth \(2\pi\)-periodic, real-valued function, and let \(C(h)\) denote a positive constant, whose exact meaning will change, that depends only on \(h\) but not on \(n\) or \(\theta\). Where it occurs below it can be bounded by \(||h||_\infty, ||h'||_\infty\) and \(||h''||_\infty\). Write

\[
S_n(\theta) = \sum_\mu f(\theta_\mu) - \frac{1}{n} h(\theta_\mu),
\]
\[
U_n(\theta) = -\frac{1}{n} \sum_\mu \sum_{\mu \neq \nu} \frac{1}{2} \cot(\frac{\theta_\mu - \theta_\nu}{2})(h(\theta_\mu) - h(\theta_\nu)),
\]
\[
V_n(\theta) = -\frac{1}{n} \sum_\mu h'(\theta_\mu) - \frac{1}{n} \sum_{\mu \neq \nu} \frac{(h(\theta_\mu) - h(\theta_\nu))^2}{\sin^2(\theta_\mu - \theta_\nu)}.\]
If we let

\[
\phi_\mu = \theta_\mu - \frac{1}{n} h(\theta_\mu),
\]
a Taylor expansion gives

\[
\sum_{\mu \neq \nu} \log |e^{i\phi_\mu} - e^{i\phi_\nu}| + \sum_\mu f(\phi_\mu) = \sum_{\mu \neq \nu} \log \left| 2\sin \frac{\theta_\mu - \theta_\nu - \frac{1}{n}(h(\theta_\mu) - h(\theta_\nu))}{2} \right| + \sum_\mu f(\theta_\mu - \frac{1}{n}h(\theta_\mu))
\]

(3.16)

= \sum_{\mu \neq \nu} \log |e^{i\theta_\mu} - e^{i\theta_\nu}| + S_n(\theta) + U_n(\theta) + V_n(\theta) + \frac{1}{n} \sum_\mu h'(\theta_\mu) + R_n^{(1)}(\theta),

where

\[
|R_n^{(1)}(\theta)| \leq \frac{C(h)}{n}.
\]

(3.17)

We see from (3.14), (3.15) and (3.16) that

\[
G_{m,n}(\zeta) \geq \frac{1}{\pi^m} \int_{[-D,D]^m} du \int_{[-D,D]^m} dv \exp(- \left( \begin{array}{c} u \\ v \end{array} \right)^t \left( \begin{array}{c} u \\ v \end{array} \right))
\]

\[
\times \mathbb{E}_n \left[ \exp \left( S_n(\theta) + U_n(\theta) + V_n(\theta) + \sum_{\mu,\nu} w_m(\theta_\mu - \frac{1}{n}h(\theta_\mu), \theta_\nu - \frac{1}{n}h(\theta_\nu)) + R_n^{(2)}(\theta) \right) \right],
\]

(3.18)

where

\[
R_n^{(2)}(\theta) = R_n^{(1)}(\theta) + \sum_\mu \log(1 - \frac{1}{n}h'(\theta_\mu)) + \frac{1}{n}h'(\theta_\mu)
\]

by (3.17) satisfies

\[
|R_n^{(2)}(\theta)| \leq \frac{C(h)}{n}.
\]

(3.19)

The 1- and 2-point marginal densities for \( \mathbb{E}_n[\cdot] \) are given by, [12],

\[
p_{1,n}(\theta) = \frac{1}{2\pi}
\]

(3.20)

\[
p_{2,n}(\theta_1, \theta_2) = \frac{1}{4\pi^2 n(n-1)} \left[ n^2 + \sum_{j=0}^{n-1} e^{ij(\theta_1 - \theta_2)} \right]^2.
\]

These formulas can be used to show that

\[
\mathbb{E}_n[U_n(\theta)] = 0, \text{ and } \mathbb{E}_n[V_n(\theta)] \geq - \sum_{k=1}^{\infty} k|h_k|^2,
\]

(3.21)

where \( h_k \) are the complex Fourier coefficients of \( h \). Hence, we can use Jensen’s inequality in (3.18) to get the estimate

\[
G_{m,n}(\zeta) \geq \frac{1}{\pi^m} \int_{[-D,D]^m} du \int_{[-D,D]^m} dv \exp(- \left( \begin{array}{c} u \\ v \end{array} \right)^t \left( \begin{array}{c} u \\ v \end{array} \right))
\]

\[
\times \exp \left( \mathbb{E}_n[S_n(\theta)] - \sum_{k=1}^{\infty} k|h_k|^2 - \mathbb{E}_n \left[ \sum_{\mu,\nu} w_m(\theta_\mu - \frac{1}{n}h(\theta_\mu), \theta_\nu - \frac{1}{n}h(\theta_\nu)) \right] - \frac{C(h)}{n} \right).
\]

(3.22)
Define
\[ T_n^{(1)} = \mathbb{E}_n[S_n(\theta)] = \frac{n}{2\pi} \int_{-\pi}^{\pi} f(\theta - \frac{1}{n} h(\theta)) \, d\theta, \quad (3.23) \]
\[ T_n^{(2)} = -\frac{n^2}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} w_m(\theta_1, \theta_2 - \frac{1}{n} h(\theta_1), \theta_2 - \frac{1}{n} h(\theta_2)) \, d\theta_1 d\theta_2, \]
\[ T_n^{(3)} = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} w_m(\theta_1, \theta_2 - \frac{1}{n} h(\theta_1), \theta_2 - \frac{1}{n} h(\theta_2)) \left| \sum_{j=0}^{n-1} e^{ij(\theta_1-\theta_2)} \right|^2 \, d\theta_1 d\theta_2 \]
\[-\frac{n}{2\pi} \int_{-\pi}^{\pi} w_m(\theta - \frac{1}{n} h(\theta), \theta - \frac{1}{n} h(\theta)) \, d\theta. \]

Then, using (3.20) and (3.22), we find that for \( \zeta \in \mathbb{R} \),
\[ G_{m,n}(\zeta) \geq \frac{1}{m} \int_{[-D,D]^m} du \int_{[-D,D]^m} dv \exp \left( -\left( \frac{u}{v} \right)^t \left( \frac{u}{v} \right) + T_n^{(1)} + T_n^{(2)} + T_n^{(3)} - \sum_{k=1}^{\infty} k|l_k|^2 - \frac{C(h)}{n} \right) \quad (3.24) \]

Note that \( f \) depends on \( u \) and \( v \), and we can choose \( h \) to depend on \( u \) and \( v \) also. Hence \( T_n^{(j)} \) depends on \( u \) and \( v \). Define, for \( n \) sufficiently large (depending on \( h \)),
\[ r_n(\theta) = \theta - \frac{1}{n} h(\theta), \quad \text{and} \quad s_n(\theta) = r_n^{-1}(\theta). \quad (3.25) \]

Then, if we write
\[ s_n'(\theta) = 1 + \frac{1}{n} h'(\theta) + \frac{1}{n^2} H_n(\theta), \quad (3.26) \]
we have the bound
\[ |H_n(\theta)| \leq C(h). \quad (3.27) \]

By (3.23), (3.25), (3.26), and the fact that \( f \) has zero mean,
\[ T_n^{(1)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)h'(\theta) \, d\theta + \frac{1}{2\pi n} \int_{-\pi}^{\pi} f(\theta)H_n(\theta) \, d\theta \]
\[ = -i \sum_{k \in \mathbb{Z}} k f_k h_{-k} + e_n^{(1)}, \quad (3.28) \]
where
\[ |e_n^{(1)}| \leq \frac{C(h)}{n} ||f||_1. \quad (3.29) \]

Also,
\[ T_n^{(2)} = -\frac{n^2}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} w_m(\theta_1, \theta_2) s_n'(\theta_1) s_n'(\theta_2) \, d\theta_1 d\theta_2 \]
\[ = -\operatorname{Re} \sum_{k \neq \ell \geq m} a_{k\ell} \frac{n^2}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{-ik\theta_1 - i\ell\theta_2} s_n'(\theta_1) s_n'(\theta_2) \, d\theta_1 d\theta_2. \]

By (3.26) and (3.27),
\[ \frac{n^2}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{-ik\theta_1 - i\ell\theta_2} s_n'(\theta_1) s_n'(\theta_2) \, d\theta_1 d\theta_2 = -k\ell h_k h_{\ell} + e_n^{(2)}(k, \ell), \]
where
\[ |e_n^{(2)}(k, \ell)| \leq \frac{C(h)}{n}. \]
Thus
\[ T_n^{(2)} = \text{Re} \sum_{k \vee \ell > m} k\ell a_{k\ell} h_{k\ell} - e_n^{(2)}, \]
(3.30)
where
\[ |e_n^{(2)}| = \left| \text{Re} \sum_{k \vee \ell > m} a_{k\ell} e_n^{(2)}(k, \ell) \right| \leq \frac{C(h)}{n}, \]
(3.31)
by Lemma 3.1. We now consider \( T_n^{(3)} \). By Taylor’s theorem
\[ w_m(\theta_1 - \frac{1}{n} h(\theta_1), \theta_2 - \frac{1}{n} h(\theta_2)) = \text{Re} \sum_{k \vee \ell > m} a_{k\ell} e^{-ik\theta_1 - i\ell \theta_2} \]
\[ - \frac{1}{n} (h(\theta_1) + h(\theta_2)) \text{Re} \sum_{k \vee \ell > m} -ika_{k\ell} e^{-ik\theta_1 - i\ell \theta_2} + e_n^{(3)}(\theta_1, \theta_2), \]
where
\[ |e_n^{(3)}(\theta_1, \theta_2)| \leq \frac{C(h)}{n^2} \sum_{k \vee \ell > m} (k^2 + \ell^2)|a_{k\ell}| \leq \frac{C(h)}{n^2}, \]
(3.32)
by Lemma 3.1. Thus, by the definition of \( T_n^{(3)} \),
\[ T_n^{(3)} = \text{Re} \sum_{k \vee \ell > m} a_{k\ell} \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{-ik\theta_1 - i\ell \theta_2} \left| \sum_{j=0}^{n-1} e^{ij(\theta_1 - \theta_2)} \right|^2 d\theta_1 d\theta_2 \]
(3.33)
\[ - \frac{1}{n} \text{Re} \sum_{k \vee \ell > m} -ika_{k\ell} \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} (h(\theta_1) + h(\theta_2)) e^{-ik\theta_1 - i\ell \theta_2} \left| \sum_{j=0}^{n-1} e^{ij(\theta_1 - \theta_2)} \right|^2 d\theta_1 d\theta_2 \]
\[ + 2\text{Re} \sum_{k \vee \ell > m} a_{k\ell} \frac{1}{2\pi} \int_{-\pi}^{\pi} h(\theta)(-ike^{-i(k+\ell)\theta}) d\theta + e_n^{(3)} =: I_1 + I_2 + I_3 + e_n^{(3)}, \]
where
\[ e_n^{(3)} = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e_n^{(3)}(\theta_1, \theta_2) \left| \sum_{j=0}^{n-1} e^{ij(\theta_1 - \theta_2)} \right|^2 d\theta_1 d\theta_2 - \frac{n}{2\pi} \int_{-\pi}^{\pi} e_n^{(3)}(\theta, \theta) d\theta. \]
Since
\[ \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left| \sum_{j=0}^{n-1} e^{ij(\theta_1 - \theta_2)} \right|^2 d\theta_1 d\theta_2 = n \]
it follows from the estimate 3.32 that
\[ |e_n^{(3)}| \leq \frac{C(h)}{n}. \]
(3.34)
Now,
\[ I_1 = \text{Re} \sum_{k \vee \ell > m} a_{k\ell} \sum_{j_1, j_2=0}^{n-1} \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{-ik\theta_1 - i\ell \theta_2 + i(j_1 - j_2)(\theta_1 - \theta_2)} d\theta_1 d\theta_2 \]
(3.35)
\[ = \text{Re} \sum_{k \vee \ell > m} a_{k\ell} \sum_{j_1, j_2=0}^{n-1} \delta_{k,j_1-j_2} \delta_{\ell,j_2-j_1} = 0, \]
(3.36)
We see that
\[ I_2 = -\frac{1}{n} \text{Re} \sum_{k \land \ell > m} -ika_{k\ell} \sum_{j_1,j_2=0}^{n-1} (h_{k+j_2-j_1} \delta_{\ell,j_2-j_1} + h_{\ell+j_2-j_1} \delta_{k,j_1-j_2}) \]
\[ = -\frac{1}{n} \text{Re} \sum_{k \land \ell > m} -ika_{k\ell} (2n - (k + \ell)) h_{k+\ell}. \]

Finally,
\[ I_3 = 2\text{Re} \sum_{k \land \ell > m} -ika_{k\ell} h_{k+\ell} \]
and thus
\[ I_2 + I_3 = -\frac{1}{2n} \sum_{k \land \ell > m} i(k + \ell)^2 a_{k \ell} h_{k+\ell}. \]

Using Lemma 3.1 we see that
\[ |I_2 + I_3| \leq \frac{C(h)}{n}, \]
and thus by (3.33), (3.34) and (3.35),
\[ |T_n^{(3)}| \leq \frac{C(h)}{n}. \]

We have shown that
\[ T_n^{(1)} + T_n^{(2)} + T_n^{(3)} \geq -i \sum_{k \in \mathbb{Z}} k f_k h_{-k} + \text{Re} \sum_{k \land \ell > m} k \ell a_{k\ell} h_k h_\ell - \frac{C(h)}{n} (1 + ||f||_1), \]
and inserting this estimate into (3.24) gives
\[ G_{n,n}(\zeta) \geq \frac{1}{\pi^m} \int_{[-D,D]^m} du \int_{[-D,D]^m} dv \exp \left( - \left(\begin{array}{c} u \\ v \end{array}\right)^t \left(\begin{array}{c} u \\ v \end{array}\right) - \sum_{k=1}^{\infty} k h_k h_{-k} - i \sum_{k \in \mathbb{Z}} k f_k h_{-k} \right) \sum_{k \land \ell > m} k \ell a_{k\ell} h_k h_\ell - \frac{C(h)}{n} (1 + ||f||_1) \right). \]

We now choose \( h_k = -\text{sgn}(k) f_k, 1 \leq |k| \leq m, h_k = 0 \) if \( |k| > m \) or \( k = 0 \), so that \( h \) is a cut-off of the Fourier series for the conjugate function to \( f \). Then
\[ -i \sum_{k \in \mathbb{Z}} k f_k h_{-k} = 2 \sum_{k=1}^{m} k|f_k|^2, \quad \text{and} \quad \sum_{k=1}^{\infty} k h_k h_{-k} = \sum_{k=1}^{m} k|f_k|^2, \]
and
\[ \text{Re} \sum_{k \land \ell > m} k \ell a_{k\ell} h_k h_\ell = 0. \]

Hence, from (3.37) we see that for \( \zeta \in \mathbb{R} \),
\[ G_{n,n}(\zeta) \geq \frac{1}{\pi^m} \int_{[-D,D]^m} du \int_{[-D,D]^m} dv \exp \left( - \left(\begin{array}{c} u \\ v \end{array}\right)^t \left(\begin{array}{c} u \\ v \end{array}\right) + \sum_{k=1}^{m} k|f_k|^2 - \frac{C(h)}{n} (1 + ||f||_1) \right). \]

With \( m \) fixed and for \( u, v \in [-D,D]^m \) with \( D \) fixed, and \( |\zeta| \leq \rho \), we see that \( C(h) \) is bounded and thus
\[ \lim_{n \to \infty} G_{n,n}(\zeta) \geq \frac{1}{\pi^m} \int_{[-D,D]^m} du \int_{[-D,D]^m} dv \exp \left( - \left(\begin{array}{c} u \\ v \end{array}\right)^t \left(\begin{array}{c} u \\ v \end{array}\right) + \sum_{k=1}^{m} k|f_k|^2 \right). \]
We see from (3.15) that
\[
\sum_{k=1}^{m} |k| f_k|^2 = (L_m(\zeta) + P_m g(\zeta))^t (L_m(\zeta) + P_m g(\zeta)).
\]
In (3.38) we can let \( D \to \infty \) so that the integration in the right side is over \( \mathbb{R}^m \) and compute the Gaussian integral. The same computations that led to (3.12) then give
\[
\lim_{n \to \infty} G_{m,n}(\zeta) \geq \frac{1}{\sqrt{|\det(I - \zeta^2 B_m)| \det(I - \Lambda_m)}} \exp\left(\left(P_m a(\zeta)\right)^t T_m \left(P_m a(\zeta)\right) + \left(P_m b(\zeta)\right)^t \left(P_m b(\zeta)\right)\right).
\]
We can now let \( m \to \infty \), and the same computations as previously then give
\[
\lim_{m \to \infty} \lim_{n \to \infty} G_{m,n}(\zeta) \geq G(\zeta),
\]
for \( \zeta \in [-\rho, \rho] \) which is what we wanted to prove.

4. PROOF OF THEOREM 1.3

Without loss of generality we can assume that \( \text{cap}(\gamma) = 1 \) and we will do so in this section. Consider the function \( E_n(r) \) defined by (1.18). We have the following lemma.

Lemma 4.1. The sequence of functions \( E_n(r) \), \( n \geq 1 \) is increasing.

The lemma will be proved below. We will now prove Theorem 1.3 using Lemma 1.2 and Lemma 4.1.

Proof of Theorem 1.3. Let \( B_r \), \( r > 1 \) be the Grunsky operator for the curve \( \gamma_r \). Then for each \( r > 1 \), by (1.17),
\[
E(r) := \lim_{n \to \infty} E_n(r) = -\frac{1}{2} \log \det(I - B_r^* B_r) \quad (4.1)
\]
since \( \gamma_r \) satisfies the conditions of Theorem 1.1. Assume first that \( \gamma \) is a Weil-Petersson quasicircle. Then the Grunsky operator \( B \) for \( \gamma \) is a Hilbert-Schmidt operator so \( B^* B \) is a trace-class operator and consequently it follows from (1.1) that
\[
E := \lim_{r \to 1^+} E(r) = -\frac{1}{2} \log \det(I - B^* B) < \infty, \quad (4.2)
\]
which can be seen by expressing the Grunsky coefficients for \( B_r \) in terms of the Grunsky coefficients of \( B \), see below. Since \( E_n(r) \) is increasing in \( n \), we have that \( E_n(r) \leq E(r) \) and combining this with (4.2) we obtain
\[
E_n := \log \frac{Z_n(\gamma)}{(2\pi)^n} = \lim_{r \to 1^+} E_n(r) \leq \lim_{r \to 1^+} E(r) = E < \infty
\]
for all \( n \geq 1 \). Hence
\[
\limsup_{n \to \infty} \frac{Z_n(\gamma)}{(2\pi)^n} \leq E < \infty, \quad (4.3)
\]
which proves (1.20). It remains to prove that we also get the right limit. From the monotonicity in \( r \) we have that \( E_n(r) \leq E_n(r') \) if \( 1 < r < r' \), and letting \( r' \to 1^+ \) gives \( E_n(r) \leq E_n \) for all \( r > 1 \),
$n \geq 1$. Taking the limit $n \to \infty$ gives $E(r) \leq \lim_{n \to \infty} E_n$ for all $r > 1$. Finally, we can let $r \to 1+$ to obtain $E \leq \lim_{n \to \infty} E_n$, which combined with (4.3) gives,

$$\lim_{n \to \infty} \log Z_n(\gamma) = E = -\frac{1}{2} \log \det(I - B^*B),$$

which is what we wanted to prove.

Next we want to prove that if (1.20) holds, then $\gamma$ is a Weil-Petersson quasicircle. It follows from Lemma 1.2 and the definition (1.19) that

$$Z_n(\gamma_r) = 1 \cdot \frac{(2\pi)^n}{n!} = \lim_{n \to \infty} \frac{Z_n(\gamma)}{(2\pi)^n},$$

for any $r > 1$. We can use (1.17) and take the limit $n \to \infty$ in (4.4) to obtain

$$(\det(I - B^*_rB_r))^{-1/2} \leq \lim_{n \to \infty} \frac{Z_n(\gamma)}{(2\pi)^n} =: A < \infty$$

for any $r > 1$. From (1.5) we see that if the Grunsky coefficients for $\gamma$ are $b_{k\ell}$, $k, \ell \geq 1$, then the Grunsky coefficients for $\gamma_r$ are $b_{k\ell}/r^{k+\ell}$, $k, \ell \geq 1$. Let $\lambda_j(r)$ be the singular values of $B_r$. Then (4.5) gives the inequality

$$\prod_{j=1}^{\infty} (1 - \lambda_j(r)^2)^2 \geq A^{-2},$$

so

$$||B_r||_{HS}^2 = \sum_{j=1}^{\infty} \lambda_j(r)^2 \leq -\sum_{j=1}^{\infty} \log(1 - \lambda_j(r)^2) \leq 2 \log A$$

for all $r > 1$. Thus

$$\sum_{k,\ell=1}^{\infty} \frac{|b_{k\ell}|^2}{2r^{k+\ell}} \leq 2 \log A,$$

and letting $r \to 1+$ shows that $B$ is a Hilbert-Schmidt operator, so $\gamma$ is a Weil-Petersson quasicircle.

Lemma 1.2 will follow from the following lemma.

**Lemma 4.2.** The function $E_n(r)$ defined by (1.18) satisfies

$$rE_n''(r) + E_n'(r) \geq 0$$

for all $r > 1$. Furthermore

$$\lim_{r \to \infty} E_n(r) = 0.$$ 

**Proof.** Note that by definition

$$Z_n(\gamma_r) = \frac{1}{n!r^{n(n-1)}} \int_{[-\pi,\pi]^n} \exp \left( \Re \left( \sum_{\mu \neq \nu} \log(\phi(re^{i\mu}) - \phi(re^{i\nu})) + \sum_{\mu} \log \phi'(re^{i\mu}) \right) \right) d\theta,$$

where we used $\phi_r(z) = \phi(rz)/r$ and $\phi_r'(z) = \phi'(rz)$. Making the change of variables $\theta_\mu \mapsto \theta_\mu + \alpha$ for some real $\alpha$ in the right side of (1.8) does not change its value so

$$Z_n(\gamma_r) = \frac{1}{n!r^{n(n-1)}} \int_{[-\pi,\pi]^n} e^{F(r,\alpha,\theta)} d\theta,$$

where

$$F(r, \alpha, \theta) = \sum_{\mu \neq \nu} \log(\phi(re^{i(\theta_\mu + \alpha)}) - \phi(re^{i(\theta_\nu + \alpha)})) + \sum_{\mu} \log \phi'(re^{i(\theta_\mu + \alpha)}).$$
From the definition of $E_n(r)$ we then obtain

$$E_n(r) = -\log((2\pi)^n n!) - n(n - 1) \log r + \log \int_{[-\pi, \pi]^n} e^{F(r, \alpha, \theta)} d\theta. \quad (4.11)$$

Using this formula we can compute the derivatives of $E_n(r)$ which gives

$$E'_n(r) = -\frac{n(n - 1)}{r} + \frac{\int (\text{Re} \partial_r F) e^{Re F} d\theta}{\int e^{Re F} d\theta}, \quad (4.12)$$

and

$$E''_n(r) = \frac{n(n - 1)}{r} + \frac{\int (\text{Re} \partial_r^2 F + (\text{Re} \partial_r F)^2) e^{Re F} d\theta}{\int e^{Re F} d\theta} - \left( \frac{\int (\text{Re} \partial_r F) e^{Re F} d\theta}{\int e^{Re F} d\theta} \right)^2, \quad (4.13)$$

where the integrals are over $[-\pi, \pi]^n$. If we take the derivative with respect to $\alpha$ in (4.11) we get similarly

$$\frac{\int (\text{Re} \partial_\alpha F) e^{Re F} d\theta}{\int e^{Re F} d\theta} = 0, \quad (4.14)$$

and

$$\frac{\int (\text{Re} \partial_\alpha^2 F + (\text{Re} \partial_\alpha F)^2) e^{Re F} d\theta}{\int e^{Re F} d\theta} - \left( \frac{\int (\text{Re} \partial_\alpha F) e^{Re F} d\theta}{\int e^{Re F} d\theta} \right)^2 = 0. \quad (4.15)$$

From the definition (4.10) we see that

$$\partial_r F = \sum_{\mu \neq \nu} \frac{e^{i(\theta_\mu + \alpha)} \phi'(r e^{i(\theta_\mu + \alpha)}) - e^{i(\theta_\nu + \alpha)} \phi'(r e^{i(\theta_\nu + \alpha)})}{\phi(\theta_\nu + \alpha) - \phi(\theta_\nu + \alpha)} + \sum_{\mu} \frac{e^{i(\theta_\mu + \alpha)} \phi''(r e^{i(\theta_\mu + \alpha)})}{\phi'(e^{i(\theta_\mu + \alpha)})},$$

and

$$\partial_r^2 F = \sum_{\mu \neq \nu} \left[ \frac{(e^{i(\theta_\mu + \alpha)} \phi''(r e^{i(\theta_\mu + \alpha)}) - (e^{i(\theta_\nu + \alpha)} \phi''(r e^{i(\theta_\nu + \alpha)}))}{\phi(\theta_\nu + \alpha) - \phi(\theta_\nu + \alpha)} 
- \left( \frac{e^{i(\theta_\mu + \alpha)} \phi'(r e^{i(\theta_\mu + \alpha)}) - e^{i(\theta_\nu + \alpha)} \phi'(r e^{i(\theta_\nu + \alpha)})}{\phi(\theta_\nu + \alpha) - \phi(\theta_\nu + \alpha)} \right)^2 \right]$$

$$+ \sum_{\mu} \left[ \frac{(e^{i(\theta_\mu + \alpha)} \phi''(r e^{i(\theta_\mu + \alpha)})}{\phi'(e^{i(\theta_\mu + \alpha)})} - \left( \frac{e^{i(\theta_\mu + \alpha)} \phi''(r e^{i(\theta_\mu + \alpha)})}{\phi'(e^{i(\theta_\mu + \alpha)})} \right)^2 \right].$$

An analogous computation gives

$$\partial_\alpha F = i r \partial_r F, \quad \text{and} \quad \partial_\alpha^2 F = -r \partial_r F - r^2 \partial_r^2 F.$$  
Consequently,

$$\text{Re} \partial_\alpha F = -r \text{Im} \partial_r F, \quad \text{and} \quad \text{Re} \partial_\alpha^2 F = -r \text{Re} \partial_r F - r^2 \text{Re} \partial_r^2 F.$$  

If we insert these relations into (4.15), we get

$$\frac{\int (-r \text{Re} \partial_r F - r^2 \text{Re} \partial_r^2 F + r^2 (\text{Im} \partial_r F)^2) e^{Re F} d\theta}{\int e^{Re F} d\theta} - \left( \frac{\int (r \text{Im} \partial_r F) e^{Re F} d\theta}{\int e^{Re F} d\theta} \right)^2 = 0,$$
Let \( \Pi \) be the set of all polynomials of degree \( \leq n \) with leading coefficient 1. Then, see [17, Sec. 16.2], [16], we have that

\[
\frac{Z_{n+1}(\gamma_r)}{Z_n(\gamma_r)/(2\pi)^n} = \frac{D_{n+1}(1)/(2\pi)^{n+1}}{D_n(1)/(2\pi)^n} = \frac{1}{k_n^2} = \min_{p \in \Pi_n} \frac{1}{2\pi} \int_{\gamma_r} |p(|\zeta|)|^2 |d\zeta|, \tag{4.18}
\]

which gives

\[
\frac{r^2 \int (\text{Re} \partial^2 F)e^{\text{Re} F} d\theta}{\int e^{\text{Re} F} d\theta} = -\frac{r^2 \int (\text{Im} \partial_r F)e^{\text{Re} F} d\theta}{\int e^{\text{Re} F} d\theta} + \frac{r^2 \int (\text{Im} \partial_r F)^2 e^{\text{Re} F} d\theta}{\int e^{\text{Re} F} d\theta} - r^2 \left( \frac{\int (\text{Im} \partial_r F)e^{\text{Re} F} d\theta}{\int e^{\text{Re} F} d\theta} \right)^2,
\]

This can be written

\[
\frac{1}{r^2} (r^2 E''(r) + rE'(r))(\int e^{\text{Re} F} d\theta)^2 = \frac{1}{2} \int d\theta \int d\theta' [(\text{Im} \partial_r F(r, \alpha, \theta) - \text{Im} \partial_r F(r, \alpha, \theta'))^2 + (\text{Re} \partial_r F(r, \alpha, \theta) - \text{Re} \partial_r F(r, \alpha, \theta'))^2] e^{\text{Re} F(r, \alpha, \theta) + \text{Re} F(r, \alpha, \theta')} \geq 0,
\]

and we have proved the inequality \( (4.6) \).

We have that

\[
Z_n(\gamma_r)/(2\pi)^n = \frac{1}{n!} \int_{[-\pi,\pi]^n} \prod_{\mu \neq \nu} |\phi_r(e^{i\theta_\mu}) - \phi_r(e^{i\theta_\nu})| \prod_{\mu} |\phi'_r(e^{i\theta_\mu})| d\theta. \tag{4.16}
\]

Since the series \( (1.1) \) is absolutely convergent for \(|z| > 1\), there is a constant \( C \) so that \(|\phi_{-k}| \leq C2^k\) for all \( k \geq 1 \). Now, by \( (1.4) \)

\[
\phi_r(z) = z + \sum_{k=0}^\infty \frac{\phi_{-k}}{r^{k+1}z^{k+1}}, \quad \text{and} \quad \phi'_r(z) = 1 + \sum_{k=1}^\infty \frac{k\phi_{-k}}{r^{k+1}z^{k+1}}, \tag{4.17}
\]

and consequently \( \phi_r(z) \to z \) and \( \phi'_r(z) \to 1 \) uniformly for \( z \in \mathbb{T} \) as \( r \to \infty \). Hence, we can take the limit \( r \to \infty \) in \( (4.16) \) to obtain

\[
\lim_{r \to \infty} \frac{Z_n(\gamma_r)}{(2\pi)^n} = \frac{1}{(2\pi)^{n+1}} \int_{[-\pi,\pi]^n} \prod_{\mu \neq \nu} |e^{i\theta_\mu} - e^{i\theta_\nu}| d\theta = 1.
\]

This proves \( (4.7) \) and we are done. \( \square \)

Now we can give the proof of Lemma 4.2. Assume that \( E''_n(r_0) > 0 \) for some \( r_0 > 1 \). From \( (4.6) \) we see that \( rE''_n(r) \) is increasing and hence \( rE''_n(r) \geq r_0 E''_n(r_0) \) for \( r \geq r_0 \). Thus,

\[
E_n(r) \geq E_n(r_0) + r_0 E''_n(r_0) \int_{r_0}^r \frac{ds}{s} = E_n(r_0) + r_0 E''_n(r_0) \log(r/r_0).
\]

If we let \( r \to \infty \) this contradicts \( (4.7) \). Consequently, \( E''_n(r) \leq 0 \) for all \( r > 1 \). \( \square \)

We turn now to the proof of Lemma 4.1.

Proof of Lemma 4.1. Let \( \Pi_n \) be the set of all polynomials of degree \( \leq n \) with leading coefficient 1. Then, see [17, Sec. 16.2], [16], we have that

\[
\frac{Z_{n+1}(\gamma_r)/(2\pi)^{n+1}}{Z_n(\gamma_r)/(2\pi)^n} = \frac{D_{n+1}(1)/(2\pi)^{n+1}}{D_n(1)/(2\pi)^n} = \frac{1}{k_n^2} = \min_{p \in \Pi_n} \frac{1}{2\pi} \int_{\gamma_r} |p(|\zeta|)|^2 |d\zeta|, \tag{4.18}
\]
and the minimum is attained if and only if \( p(\zeta) = \pi_n(\zeta) := \frac{1}{n!}p_n(\zeta) = \zeta^n + \ldots \), where \( p_n \) are the orthonormal polynomials with respect to \( \gamma_r \),

\[
\int_{\gamma_r} p_m(\zeta)p_n(\zeta) |d\zeta| = \delta_{mn}.
\]

Hence,

\[
E_{n+1}(r) - E_n(r) = \log \left( \frac{1}{2\pi} \int_{\gamma_r} |\pi_n(\zeta)|^2 |d\zeta| \right). \tag{4.19}
\]

Note that \( \phi_r \) is analytic in \( |z| > \rho^{-1} \) if \( 1 \leq \rho < r \). Fix \( \rho \in (1, \infty) \). Then, by (4.17),

\[
z\phi_r \left( \frac{1}{z} \right) = 1 + \sum_{k=0}^{\infty} \frac{\phi_{-k}}{r^{k+1}} z^{k+1}, \quad \text{and} \quad h_r(z) := \phi_r \left( \frac{1}{z} \right) = 1 + \sum_{k=0}^{\infty} \frac{k\phi_{-k}}{r^{k+1}} z^{k+1}, \tag{4.20}
\]

and these functions are analytic in \( |z| < \rho \). By (4.19) and Jensen’s inequality,

\[
E_{n+1}(r) - E_n(r) \geq \frac{1}{2\pi} \int_{-\pi}^{\pi} 2 \log |\pi_n(\phi_r(e^{i\theta}))| + \log |\phi_r'(e^{i\theta})| d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} 2 \log |\pi_n(\phi_r(e^{-i\theta}))| + \log |\phi_r'(e^{-i\theta})| d\theta.
\]

Note that

\[
|\pi_n(\phi_r(e^{-i\theta}))| = |(e^{i\theta})^n \pi_n(\phi_r(e^{-i\theta}))| = |\psi_{n,r}(e^{i\theta})|,
\]

where

\[
\psi_{n,r}(z) = z^n \pi_n(\phi_r \left( \frac{1}{z} \right)).
\]

If \( \pi_n(z) = \sum_{j=0}^{n} a_j z^j \), with \( a_n = 1 \), then

\[
\psi_{n,r}(z) = z^n \sum_{j=0}^{n} a_j \phi_r \left( \frac{1}{z} \right)^j = \sum_{j=0}^{n} a_j z^{n-j} \left( z \phi_r \left( \frac{1}{z} \right) \right)^j, \tag{4.23}
\]

so we see that \( \psi_{n,r} \) is analytic in \( |z| < \rho \). Hence, \( \log |\psi_{n,r}(z)| \) and \( \log |h_r(z)| \) are subharmonic functions in \( |z| < \rho \), and we see from (4.21) and (4.22) that

\[
E_{n+1}(r) - E_n(r) \geq \frac{1}{2\pi} \int_{-\pi}^{\pi} 2 \log |\psi_{n,r}(e^{i\theta})| + \log |h_r(e^{i\theta})| d\theta \geq 2 \log \psi_{n,r}(0) + \log |h_r(0)|. \tag{4.24}
\]

It follows from (4.20) that \( h_r(0) = 1 \), and from (4.20) and (4.23),

\[
\psi_{n,r}(0) = \sum_{j=0}^{n} a_j 0^{n-j} \cdot 1^j = a_n = 1.
\]

Consequently, (4.24) gives \( E_{n+1}(r) - E_n(r) \geq 0. \)

\[\square\]

5. HEURISTIC ARGUMENT FOR THE \( \beta \)-ENSEMBLE

We will use the same notation, sometimes slightly modified, as in the previous sections and only sketch the argument. Let

\[
\mathbb{E}_{n,\beta}(\cdot) = \frac{1}{Z_{n,\beta}(T)n!} \int_{[-\pi,\pi]^n} \prod_{\mu \neq \nu} |e^{i\theta_{\mu}} - e^{i\theta_{\nu}}|^\beta/2(\cdot) d\theta.
\]
denote expectation with respect to the $\beta$-ensemble on the unit circle. As in (2.20), we see that

$$D_{n,\beta}[e^g] = Z_{n,\beta}(T)\text{cap}(\gamma)^{\frac{\beta - 2}{2}} E_{n,\beta} \left[ \exp \left( -\frac{\beta}{2} \text{Re} \sum_{k,\ell=1}^{\infty} a_{k\ell} \left( \sum_{\mu} e^{-ik\theta_{\mu}} \left( \sum_{\nu} e^{-i\theta_{\nu}} \right) \right) \right. 
$$

$$+ \left. \left( 1 - \frac{\beta}{2} \right) \sum_{\mu} \log |\phi'(e^{i\theta_{\mu}})| + \sum_{\mu} g(\phi(e^{i\theta_{\mu}})) \right]$$

$$= Z_{n,\beta}(T)\text{cap}(\gamma)^{\frac{\beta - 2}{2}} E_{n,\beta} \left[ \exp \left( -\frac{\beta}{2} \text{Re} \sum_{k,\ell=1}^{\infty} a_{k\ell} \left( \sum_{\mu} e^{-ik\theta_{\mu}} \right) \left( \sum_{\nu} e^{-i\theta_{\nu}} \right) + 2g_{\beta}^t \left( \begin{array}{c} X \\ Y \end{array} \right) \right] \right],$$

where we used (1.22). If we write

$$\sum_{k,\ell=1}^{\infty} a_{k\ell} \left( \sum_{\mu} e^{-ik\theta_{\mu}} \right) \left( \sum_{\nu} e^{-i\theta_{\nu}} \right) = \lim_{m \to \infty} \sum_{k,\ell=1}^{m} a_{k\ell} \left( \sum_{\mu} e^{-ik\theta_{\mu}} \right) \left( \sum_{\nu} e^{-i\theta_{\nu}} \right)$$

in (5.1), take the limit outside the expectation and then interchange the order of the $m \to \infty$ and $n \to \infty$ limits, we are led to study the limit

$$\lim_{m \to \infty} \lim_{n \to \infty} E_{n,\beta} \left[ \exp \left( -\frac{\beta}{2} \text{Re} \sum_{k,\ell=1}^{m} b_{k\ell}(X_k - iY_k)(X_\ell - iY_\ell) + 2g_{\beta}^t \left( \begin{array}{c} X \\ Y \end{array} \right) \right) \right].$$

It seems that it is not easy to justify changing the order of the limits but doing so leads, as we will see, to the conjecture (1.22). Set $M_{m,\beta} = \sqrt{\frac{\beta}{2}} M_m(i)$, with $M_m(i)$ as in (2.11). We can then use a Gaussian integral to write

$$E_{n,\beta} \left[ \exp \left( -\frac{\beta}{2} \text{Re} \sum_{k,\ell=1}^{m} b_{k\ell}(X_k - iY_k)(X_\ell - iY_\ell) + 2g_{\beta}^t \left( \begin{array}{c} X \\ Y \end{array} \right) \right) \right]$$

$$= \frac{1}{\pi^m} \int_{\mathbb{R}^m} du \int_{\mathbb{R}^m} dv \exp \left( -\left( \begin{array}{c} u \\ v \end{array} \right)^t \left( \begin{array}{c} u \\ v \end{array} \right) \right) \mathcal{E}_{n,\beta} \left[ \exp \left( 2 \left( \begin{array}{c} u \\ v \end{array} \right)^t L_{m,\beta} + 2g_{\beta}^t \left( \begin{array}{c} X \\ Y \end{array} \right) \right) \right]$$

$$= \frac{1}{\pi^m} \int_{\mathbb{R}^m} du \int_{\mathbb{R}^m} dv \exp \left( -\left( \begin{array}{c} u \\ v \end{array} \right)^t \left( \begin{array}{c} u \\ v \end{array} \right) \right) \mathcal{E}_{n,\beta} \left[ \exp \left( 2(L_{m,\beta} + g_{\beta})^t \left( \begin{array}{c} X \\ Y \end{array} \right) \right) \right],$$

where

$$L_{m,\beta} = \sqrt{\frac{\beta}{2}} L_m(i) = \sqrt{\frac{\beta}{2}} \left( \begin{array}{cc} P_m & 0 \\ 0 & P_m \end{array} \right) T_m \left( \begin{array}{cc} i\Lambda_m^{1/2} & 0 \\ 0 & \Lambda_m^{1/2} \end{array} \right) \left( \begin{array}{c} u \\ v \end{array} \right).$$

We can now use the strong Szegő limit theorem for the $\beta$-ensemble on the unit circle to take the $n \to \infty$ limit in the last expectation in (5.2). This gives the limit

$$\frac{1}{\pi^m} \int_{\mathbb{R}^m} du \int_{\mathbb{R}^m} dv \exp \left( -\left( \begin{array}{c} u \\ v \end{array} \right)^t \left( \begin{array}{c} u \\ v \end{array} \right) \right) \exp \left( \frac{2}{\beta} (L_{m,\beta} + g_{\beta})^t (L_{m,\beta} + g_{\beta}) \right).$$

Now,

$$\frac{2}{\beta} L_{m,\beta} L_{m,\beta} = - \left( \begin{array}{c} u \\ v \end{array} \right)^t \left( I - \left( -\Lambda_m & 0 \\ 0 & \Lambda_m \right) \right) \left( \begin{array}{c} u \\ v \end{array} \right),$$

and

$$\frac{4}{\beta} L_{m,\beta} g_{\beta} = 2 \sqrt{\frac{2}{\beta}} \left( \begin{array}{c} u \\ v \end{array} \right)^t \left( \begin{array}{cc} i\Lambda_m^{1/2} & 0 \\ 0 & \Lambda_m^{1/2} \end{array} \right) T_m \left( \begin{array}{cc} P_m & 0 \\ 0 & P_m \end{array} \right) g_{\beta}.$$
We can now perform the Gaussian integrations in (5.3) to get

$$\frac{1}{\sqrt{\det(I + K_m)}} \exp\left(-\frac{2}{\beta} g^\beta \left( P_m 0 0 \right) (I + K_m)^{-1} K_m \left( P_m 0 \right) g^\beta + \frac{2}{\beta} g^\beta g^\beta \right).$$

If we take the $m \to \infty$ limit of this expression we obtain the right side of (1.22).

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