Some properties of generalized \((s, k)\)-Bessel function in two variables

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Abstract

The devotion of this paper is to study the Bessel function of two variables in \(k\)-calculus. we discuss the generating function of \(k\)-Bessel function in two variables and develop its relations. After this we introduce the generalized \((s, k)\)-Bessel function of two variables which help to develop its generating function. The \(s\)-analogy of \(k\)-Bessel function in two variables is also discussed. Some recurrence relations of the generalized \((s, k)\)-Bessel function in two variables are also derived.

Keywords: \(k\)-Bessel function, generalized \((s, k)\)-Bessel function, generalized \((s, k)\)-Bessel function in two variables.

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1. Introduction

Many special functions of mathematical physics have been generalized to a base \(s\) which are known as special \(s\)-functions. The Bessel \(s\)-function is one of the essential special \(s\)-functions which was introduced by Jackson and Swarthrow [33]. Special functions in term of \(k\) were presented by Diaz and Parigaun [2]. Later on, the researchers introduced various types of \(k\)-special functions by following the idea of Diaz and Parigaun [2]. Kokologiannaki [12] investigated further properties of \(k\)-gamma, \(k\)-beta and \(k\)-zeta functions. Mansour [15] introduced the \(k\)-generalized gamma function by functional equation. Krasniqi [13] investigated limits for \(k\)-gamma and \(k\)-beta functions. Merovci [16] gave the power product inequalities for the \(k\)-gamma function. Mubeen and Habibullah [17] proposed the so-called \(k\)-fractional integral based on gamma \(k\)-function and its applications. In [18], Mubeen and Habibullah defined the integral representation of generalized confluent hypergeometric \(k\)-functions and hypergeometric \(k\)-functions by...
utilizing the properties of Pochhammer k-symbols, k-gamma, and k-beta functions. In [17], Mubeen et al. proposed the following second order linear differential equation for hypergeometric k-functions as

\[ k\omega(1-kx)\omega'' + [\gamma - (\alpha + \beta + k) kx] \omega' - \alpha\beta \omega = 0. \]

The solution in the form of the so-called k-hypergeometric series of k-hypergeometric differential equation by utilizing the Frobenius method can be found in the work of Mubeen et al. [23, 22]. Recently, Li and Dong [14] investigated the hypergeometric series solutions for the second-order non-homogeneous k-hypergeometric differential equation with the polynomial term. Rahman et al. [27, 21] proposed the generalization of Wright hypergeometric k-functions and derived its various basic properties.

Furthermore, Mubeen and Iqbal [19] investigated the generalized version of Grüss-type inequalities by considering k-fractional integrals. Agarwal et al. [1] established certain Hermite-Hadamard type inequalities involving k-fractional integrals. Set et al. [32] proposed generalized Hermite-Hadamard type inequalities for Riemann-Liouville k-fractional integral. Ostrowski type k-fractional integral inequalities can be found in the work of Mubeen et al. [20]. Many researchers have established further the generalized version of Riemann-Liouville k-fractional integrals and defined a large numbers of various inequalities via by using different kinds of generalized fractional integrals. The interesting readers may consult [9, 26, 25, 28]. The Hadamard k-fractional integrals can be found in the work of Farid et al. [5]. In [6], Farid proposed the idea of Hadamard-type inequalities for k-fractional Riemann-Liouville integrals. In [10, 35], the authors have introduced inequalities by employing Hadamard-type inequalities for k-fractional integrals. Nisar et al. [24] investigated Gronwall type inequalities by utilizing Riemann-Liouville k- and Hadamard k-fractional derivatives [24]. In [24], they presented dependence solutions of certain k-fractional differential equations of arbitrary real order with initial conditions. Samraiz et al. [31] proposed Hadamard k-fractional derivative and properties. Recently, Rahman et al. [29] defined generalized k-fractional derivative operator. Diaz and Teruel introduced the generalized gamma and beta (s, k)-functions in 2005 [3]. They also proved various identities of gamma and beta (s, k)-functions in two parameter deformation. In this paper, the generalized (s, k)-Bessel function is introduced. Firstly, the Bessel function of two variables at level k is introduced by constructing its generating function and some recurrence relations. Secondly, the generating function of the generalized (s, k)-Bessel function is constructed and some of its recurrence relations are developed. Also, the s-analogy of the generalized k-Bessel function of two variables is given. Finally, the concluding comments on (s, k)-Bessel function are given.

2. preliminaries

In this section, we present certain well-known definition and mathematical preliminaries.

**Definition 2.1** ([4]). The s-factorial is defined by

\[ [n]_s! = \frac{(s; s)_n}{(1 - s)_n}, \tag{2.1} \]

where \( n \) is any positive integer and \( 0 < s < 1 \). Replacing \( n \) by \( n + k \) in (2.1), we get

\[ [n + k]_s! = \frac{(s; s)_{n+k}}{(1 - s)_{n+k}}, \tag{2.2} \]

**Definition 2.2** ([3]). The generalized (s, k)-gamma function is defined as

\[ \Gamma_{s,k}(t) = \frac{(1 - s^k)_s^{-1}}{(1 - s)_t^{-1}}, \quad t > 0, \]

where \( k \) is any positive real number and \( 0 < s < 1 \).
After changing of variable \( t \) by \( nk \), we get

\[
\Gamma_{s,k}(nk) = \prod_{j=1}^{n-1} [j]_s = \prod_{j=1}^{n-1} \frac{(1-s^j)}{(1-s)} = \frac{(1-s^k)^{n-1}}{(1-s)^{n-1}}.
\]

**Definition 2.3** ([34]). The \( s \)-Bessel function of two variables \( x \) and \( y \) is given by

\[
J_{\nu,\mu}(x, y; s) = \sum_{m,n=0}^\infty \frac{(-1)^{m+n}(\frac{s}{n})^{2m+n} (\frac{y}{2})^{2n+\mu}}{m!n!\Gamma_k(\nu + mk + k)\Gamma_k(\mu + nk + k)}, \tag{2.3}
\]

where \( \nu, \mu \) are not negative integers.

**Definition 2.4.** The relation between \( s \)-gamma function and \((s, k)\)-gamma function is given by

\[
\lim_{s \to 1} \Gamma_{s,k}(nk) = \lim_{s \to 1} \Gamma_{s,k}(n) = k^{n-1}\Gamma(n),
\]

where \( k > 0, 0 < s < 1 \) and \( n \) is positive real number.

### 3. The \( k \)-Bessel function and generalized \((s, k)\)-Bessel function in two variables

In this section, we introduce \( k \)-Bessel function and generalized \((s, k)\)-Bessel function in two variables.

**Definition 3.1.** The \( k \)-Bessel function in two variables is defined as

\[
J_{\nu,\mu}^k(x, y) = \sum_{m,n=0}^\infty \frac{(-1)^{m+n}(\frac{s}{n})^{2m+n} (\frac{y}{2})^{2n+\mu}}{m!n!\Gamma_k(\nu + mk + k)\Gamma_k(\mu + nk + k)}. \tag{3.1}
\]

If \( \nu \) and \( \mu \) are not negative integers, then we have

\[
J_{-\nu,-\mu}^k(x, y) = (-1)^{\nu+\mu} J_{\nu,\mu}^k(x, y). \tag{3.2}
\]

**Definition 3.2.** The generalized \((s, k)\)-Bessel function of two variables \( x, y \) is defined by

\[
J_{\nu,\mu}^k(x, y; s) = \sum_{m,n=0}^\infty \frac{(-1)^{m+n}(\frac{s}{n})^{2m+n} (\frac{y}{2})^{2n+\mu}}{m!n!\Gamma_{s,k}(\nu + mk + k)\Gamma_{s,k}(\mu + nk + k)},
\]

where \( k \) is any positive real number, \( 0 < s < 1 \) and \( \nu, \mu \) are non negative integers.

**Remark 3.3.** If we let \( k = 1 \), then the generalized \((s, k)\)-Bessel function reduces to \( s \)-Bessel function (2.3).

**Remark 3.4.** If we let \( s = 1 \), then the generalized \((s, k)\)-Bessel function reduces to \( k \)-Bessel function (3.1).

**Remark 3.5.** If we let \( s = k = 1 \), then the generalized \((s, k)\)-Bessel function reduces to the following Bessel function in two variables

\[
J_{\nu,\mu}(x, y) = \sum_{m,n=0}^\infty \frac{(-1)^{m+n}(\frac{s}{n})^{2m+n} (\frac{y}{2})^{2n+\mu}}{m!n!\Gamma(\nu + m + 1)\Gamma(\mu + n + 1)},
\]

where \( \nu, \mu \) are non negative integers.

### 4. Properties of Bessel \( s \)-function and Bessel \((s, k)\)-function in two variables

The study of Bessel function and \( s \)-Bessel function of two variables in \( k \)-calculus gives important theories in the filed of analysis. We discuss some important results about \( k \)-Bessel function and \((s, k)\)-Bessel function in two variables. We derive the generating function of \( k \)-Bessel function of two variables, and also discuss the \( s \)-analogy of the generalized \( k \)-Bessel function of two variables in the form of theorems.
Lemma 4.1. The relation between Bessel function and k-Bessel function in two variables is given by

\[ J_{νk}(x, y) = k^{\frac{(y+μ)}{2k}} J_{\frac{νy}{\sqrt{k}}, \frac{ν}{\sqrt{k}}}(x, y) \]  

(4.1)

or counter part is

\[ J_{νk}(x, y) = k^{\frac{(y+μ)}{2k}} J_{νk}(x\sqrt{k}, y\sqrt{k}), \]  

(4.2)

where \( ν, μ \) are non negative integers and \( k \) is any positive real number.

Proof. By definition of k-Bessel function, we have

\[ J_{νk}(x, y) = \sum_{m,n=0}^{∞} \frac{(-1)^{m+n}(\frac{x}{2})^{2m+\frac{ν}{k}(\frac{μ}{2})^{2n+\frac{ν}{k}}}{m!!n!!Γ_k(ν+mk+k)Γ_k(μ+nk+k)} \]

(4.3)

After replacing \( x \) by \( x\sqrt{k} \) and \( y \) by \( y\sqrt{k} \) in equation (4.3), we get (4.2).

Lemma 4.2. The following relation holds for Bessel function and k-Bessel function of two variables

\[ J_{νk,μk}(x, y) = k^{\frac{(y+μ)}{2k}} J_{ν,μ}(x\sqrt{k}, y\sqrt{k}) \]

(4.4)

or its counter part is

\[ J_{νk,μk}(x, y) = k^{\frac{(y+μ)}{2k}} J_{νk,μk}(x\sqrt{k}, y\sqrt{k}), \]

where \( ν, μ \) are integers and \( k \) is any positive real number.

Proof. Consider the definition of k-Bessel function in two variables, we have

\[ J_{νk,μk}(x, y) = \sum_{m,n=0}^{∞} \frac{(-1)^{m+n}(\frac{x}{2})^{2m+\frac{ν}{k}(\frac{μ}{2})^{2n+\frac{ν}{k}}}{m!!n!!Γ_k(ν+mk+k)Γ_k(μ+nk+k)} \]

(4.5)

After replacing \( ν \) by \( νk \) and \( μ \) by \( μk \) in equation (4.4), we have

\[ J_{νk,μk}(x, y) = k^{\frac{(y+μ)}{2k}} J_{νk,μk}(x\sqrt{k}, y\sqrt{k}), \]

(4.6)

Lemma 4.3. The k-Bessel function in two variables satisfies

\[ J_{ν,-μ}^k(x, y) = (-k)^{\frac{νy}{\sqrt{k}}} J_{ν,μ}(x, y), \]

(4.6)

where \( ν, μ \) are non negative integers and \( k \) is any positive real number.
Proof. Replacing the value of $\nu$, $\mu$ by $-\nu$, $-\mu$ in equation (4.1), and resulting equation is as follows

$$J^k_{-\nu,-\mu}(x,y) = k^{\frac{\nu+\mu}{2}}J^\nu_{-\frac{x}{\sqrt{k}},-\frac{y}{\sqrt{k}}}.$$  

Using the value of the equation (3.2), we get

$$J^k_{-\nu,-\mu}(x,y) = (-1)^{\frac{\nu+\mu}{2}}k^{\frac{\nu+\mu}{2}}J^\nu_{-\frac{x}{\sqrt{k}},-\frac{y}{\sqrt{k}}} = (-k)^{\frac{\nu+\mu}{2}}J^\nu_{-\frac{x}{\sqrt{k}},-\frac{y}{\sqrt{k}}} = (-k)^{\frac{\nu+\mu}{2}}J^k_{\nu,\mu}(x,y).$$

\[\square\]

**Theorem 4.4.** For $t \neq 0, w \neq 0$ and $t, w \in \mathbb{C}$, then the generating function of $k$-Bessel function in two variables is

$$\exp\left[\frac{x}{2\sqrt{k}}\left(\frac{t}{\sqrt{k}} - \sqrt{\frac{w}{k}}\right) + \frac{yp(x)}{2\sqrt{k}} \left(\frac{w}{\sqrt{k}} - \sqrt{\frac{w}{k}}\right)\right] = \sum_{\nu,\mu=-\infty}^{\infty} t^\nu w^\mu J^k_{\nu,k}(x,y),$$

where $\nu, \mu$ are non-negative integers and $k$ is any positive real number.

**Proof.** Let

$$A \equiv \sum_{\nu,\mu=-\infty}^{\infty} t^\nu w^\mu J^k_{\nu,k}(x,y) = \sum_{\nu,\mu=-\infty}^{\nu,\mu=0} t^\nu w^\mu J^k_{\nu,k}(x,y) + \sum_{\nu,\mu=0}^{\infty} t^\nu w^\mu J^k_{\nu,k}(x,y).$$ (4.7)

After replacing $\nu$ by $-\nu - 1$ and $\mu$ by $-\mu - 1$ in first summation of the equation (4.7), we have

$$A \equiv \sum_{\nu,\mu=0}^{\infty} t^{-\nu-1} w^{-\mu-1} J^k_{(\nu+1)k,(-\mu+1)k}(x,y) + \sum_{\nu,\mu=0}^{\infty} t^\nu w^\mu J^k_{\nu,k}(x,y).$$ (4.8)

By using equation (4.6) in equation (4.8), we have

$$A \equiv \sum_{\nu,\mu=0}^{\infty} t^{-\nu-1} w^{-\mu-1} (-k)^{\nu+\mu+1} J^k_{(\nu+1)k,(-\mu+1)k}(x,y) + \sum_{\nu,\mu=0}^{\infty} t^\nu w^\mu J^k_{\nu,k}(x,y)$$

$$\equiv \sum_{\nu,\mu=0}^{\infty} \sum_{m,n=0}^{\infty} t^{-\nu-1} w^{-\mu-1} (-k)^{\nu+\mu+2} \frac{(-1)^{m+n} (\frac{x}{2})^{2m+\nu+1} (\frac{yp(x)}{2})^{2n+\mu+1}}{m!n!f_k(mk + (\nu + 1)k + k) f_k(nk + (\mu + 1)k + k)}.$$ (4.9)

After replacing $\nu$ by $\nu - 2m$ and $\mu$ by $\mu - 2n$ in the equation (4.9), we have

$$A \equiv \sum_{\nu=0}^{\infty} \sum_{m=0}^{\nu} t^{-\nu-2m-1} (-1)^{\nu-m+1} k^{2m+1} (\frac{x}{2})^{\nu+1} \frac{m!f_k(mk + (\nu - 2m)k + 2k)}{m!f_k(nk + (\nu - 2n)k + 2k)}$$

$$+ \sum_{\mu=0}^{\infty} \sum_{n=0}^{\mu} \frac{w^{-\mu-2n-1} (-1)^{\mu-n+1} k^{2n+1} \frac{yp(x)}{2}^{\mu+1}}{n!f_k(nk + (\nu - 2n)k + 2k)}$$

$$+ \sum_{\nu=0}^{\infty} \sum_{m=0}^{\nu} \frac{t^{-\nu-2m} \frac{x}{2}^{\nu}}{m!f_k(mk + (\nu - 2m)k + 2k)} + \sum_{\mu=0}^{\infty} \sum_{n=0}^{\mu} \frac{w^{-\mu-2n} (-1)^{n} \frac{yp(x)}{2}^{\mu}}{n!f_k(nk + (\mu - 2n)k + 2k)}.$$ (4.10)
After replacing $\nu$ by $\nu - 1$ and $\mu$ by $\mu - 1$ in first and second summation of the equation (4.10), we have

$$A = \sum_{v=1}^{\infty} \sum_{m=0}^{\infty} \frac{t^{-v+2m}(-1)^{v-m}k^{-2m} \left(\frac{x}{2}\right)^{v}}{m!\Gamma(v-m+k)} + \sum_{v=0}^{\infty} \sum_{n=0}^{\infty} \frac{w^{\mu-2n}(-1)^{n} \left(\frac{yp(x)}{2}\right)^{\mu}}{n!\Gamma(\mu-k+n+k)} + 2$$

By rearranging the terms, we have

$$A = 2 + \sum_{v=1}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{v-m}k^{-m} \left(\frac{x}{2}\right)^{v}}{m!\Gamma(v-m)!} + \sum_{v=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{n} \left(\frac{w}{\sqrt{k}}\right)^{\mu-n-n} \left(\frac{yp(x)}{2\sqrt{k}}\right)^{\mu}}{n!(\mu-n)!}$$

By using [30, Lemma 12, page 112], we have

$$A = 2 + \sum_{v=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{m} \left(\frac{t}{\sqrt{k}}\right) \left(\frac{x}{2\sqrt{k}}\right)^{v}}{m!\Gamma(v-m)!} + \sum_{v=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{n} \left(\frac{w}{\sqrt{k}}\right)^{\mu-n-n} \left(\frac{yp(x)}{2\sqrt{k}}\right)^{\mu}}{n!(\mu-n)!}$$

which is required generating function of $k$-Bessel function in two variables.

**Lemma 4.5.** The $(s,k)$-Bessel function of two variables satisfies the relation

$$J_{\nu,\mu}^{k}(-x, y; s) = (-1)^{\frac{\nu}{2}} J_{\nu,\mu}^{k}(x, y; s),$$

where $\nu, \mu$ are integers, $k$ is any real number and $0 < s < 1$.

**Proof.** Since $(s,k)$-Bessel function in two variables is

$$J_{\nu,\mu}^{k}(x, y; s) = \sum_{m,n=0}^{\infty} \frac{(-k)^{m+n}(x)^{2m+\frac{\nu}{2}}(\frac{yp(x)}{2})^{2n+\frac{\mu}{2}}}{s_{k}[\nu+m+k][m]_{s,k}[\mu+n+k][n]_{s,k}}$$
by changing $x$ by $-x$ in the above, we get

$$J^k_{\nu, \mu}(-x, y; s) = \sum_{m, n=0}^{\infty} \frac{(-k)^{m+n} (-x)^{2m+\frac{\mu}{2}} (\frac{\mu p(x)}{2})^{2n+\frac{\mu}{2}}}{\Gamma_s,k[\nu + mk + k][m]_s \Gamma_s,k[\mu + nk + k][n]_s}.$$  

$$= (-1)^{2m+\frac{\mu}{2}} \sum_{m, n=0}^{\infty} \frac{(-k)^{m+n} (-x)^{2m+\frac{\mu}{2}} (\frac{\mu p(x)}{2})^{2n+\frac{\mu}{2}}}{\Gamma_s,k[\nu + mk + k][m]_s \Gamma_s,k[\mu + nk + k][n]_s}.$$  

Here, $(-1)^{2m}$ is positive for all values of $m$. Therefore, $(-1)^{2m} = 1$, then we have

$$J^k_{\nu, \mu}(-x, y; s) = (-1)^{\frac{\mu}{2}} \sum_{m, n=0}^{\infty} \frac{(-k)^{m+n} (-x)^{2m+\frac{\mu}{2}} (\frac{\mu p(x)}{2})^{2n+\frac{\mu}{2}}}{\Gamma_s,k[\nu + mk + k][m]_s \Gamma_s,k[\mu + nk + k][n]_s} = (-1)^{\frac{\mu}{2}} J^k_{\nu, \mu}(x, y; s).$$

\[\square\]

**Lemma 4.6.** The $(s, k)$-Bessel function of two variables holds

$$J^k_{\nu, \mu}(x, -y; s) = (-1)^{\frac{\mu}{2}} J^k_{\nu, \mu}(x, y; s),$$

where $\nu, \mu$ are non negative integers, $k$ is any real positive number and $0 < s < 1$.

**Proof.** The $(s, k)$-Bessel function is

$$J^k_{\nu, \mu}(x, y; s) = \sum_{m, n=0}^{\infty} \frac{(-k)^{m+n} (\frac{x}{2})^{2m+\frac{\mu}{2}} (\frac{\mu p(x)}{2})^{2n+\frac{\mu}{2}}}{\Gamma_s,k[\nu + mk + k][m]_s \Gamma_s,k[\mu + nk + k][n]_s}.$$  

(4.11)

After replacing $y$ by $-y$ in the equation (4.11), we have

$$J^k_{\nu, \mu}(x, -y; s) = \sum_{m, n=0}^{\infty} \frac{(-k)^{m+n} (\frac{x}{2})^{2m+\frac{\mu}{2}} (\frac{\mu p(x)}{2})^{2n+\frac{\mu}{2}}}{\Gamma_s,k[\nu + mk + k][m]_s \Gamma_s,k[\mu + nk + k][n]_s}.$$  

$$= (-1)^{2n+\frac{\mu}{2}} \sum_{m, n=0}^{\infty} \frac{(-k)^{m+n} (\frac{x}{2})^{2m+\frac{\mu}{2}} (\frac{\mu p(x)}{2})^{2n+\frac{\mu}{2}}}{\Gamma_s,k[\nu + mk + k][m]_s \Gamma_s,k[\mu + nk + k][n]_s}.$$  

For all values of $n$, $(-1)^{2n}$ is positive. Therefore, $(-1)^{2n} = 1$, then we have

$$J^k_{\nu, \mu}(x, -y; s) = (-1)^{\frac{\mu}{2}} \sum_{m, n=0}^{\infty} \frac{(-k)^{m+n} (-x)^{2m+\frac{\mu}{2}} (\frac{\mu p(x)}{2})^{2n+\frac{\mu}{2}}}{\Gamma_s,k[\nu + mk + k][m]_s \Gamma_s,k[\mu + nk + k][n]_s} = (-1)^{\frac{\mu}{2}} J^k_{\nu, \mu}(x, y; s).$$

\[\square\]

**Lemma 4.7.** The $(s, k)$-Bessel function of two variables holds

$$J^k_{\nu, \mu}(-x, -y; s) = (-1)^{\frac{\nu + \mu}{2}} J^k_{\nu, \mu}(x, y; s),$$

where $\nu, \mu$ are non negative integers, $k$ is any real positive number and $0 < s < 1$.

**Proof.** The $(s, k)$-Bessel function of two variables is given by

$$J^k_{\nu, \mu}(x, y; s) = \sum_{m, n=0}^{\infty} \frac{(-k)^{m+n} (\frac{x}{2})^{2m+\frac{\mu}{2}} (\frac{\mu p(x)}{2})^{2n+\frac{\mu}{2}}}{\Gamma_s,k[\nu + mk + k][m]_s \Gamma_s,k[\mu + nk + k][n]_s}.$$
After replacing \( x \) by \(-x\) and \( y \) by \(-y\) in the above, we have

\[
J_{v,\mu}^k(-x, -y; s) = \sum_{m, n=0}^{\infty} \frac{(-k)^{m+n}(\frac{x}{2})^{2m+\frac{\mu}{2}}(\frac{-yp(x)}{2})^{2n+\frac{\mu}{2}}}{\Gamma_{s,k}[\nu + mk + k|n|_s]!} = (1)^{\frac{\mu}{2}} \sum_{m, n=0}^{\infty} \frac{(-k)^{m+n}(\frac{x}{2})^{2m+\frac{\mu}{2}}(\frac{yp(x)}{2})^{2n+\frac{\mu}{2}}}{\Gamma_{s,k}[
u + mk + k|n|_s]!}.
\]

For all values of \( m \) and \( n \), \((-1)^{2m+2n}\) is positive. Therefore, \((-1)^{2m+2n} = 1\), then we have

\[
J_{v,\mu}^k(-x, -y; s) = (1)^{\frac{\mu}{2}} \sum_{m, n=0}^{\infty} \frac{(-k)^{m+n}(\frac{x}{2})^{2m+\frac{\mu}{2}}(\frac{yp(x)}{2})^{2n+\frac{\mu}{2}}}{\Gamma_{s,k}[\nu + mk + k|n|_s]!} = (1)^{\frac{\mu}{2}} J_{v,\mu}(x, y; s).
\]

Now, we construct the generating function of the generalized \((s, k)\)-Bessel function of two variables.

**Theorem 4.8.** Prove that the generating function of the generalized Bessel \( q, k \)-function of two variables is the expansion of

\[
E_k\left[ x \left( \frac{t - \frac{k}{t}}{2} \right) + \frac{y}{2}(w - \frac{k}{w}) \right] = \sum_{\nu=0}^{\infty} \frac{(\frac{x}{2})^\nu}{\Gamma_{s,k}^{\nu}} \sum_{m=0}^{\infty} \frac{(-k)^{m+n}}{\Gamma_{s,k}[\nu + mk + k|n|_s]!} \sum_{\mu=0}^{\infty} \frac{(\frac{yp(x)}{2})^\mu}{\Gamma_{s,k}[\nu + mk + k|n|_s]!} \sum_{n=0}^{\infty} \frac{(-kyp(x))}{\Gamma_{s,k}[\nu + mk + k|n|_s]!}.
\]

Replacing \( \nu \) by \( \frac{r}{k} + m \) and \( \mu \) by \( \frac{\mu}{k} + n \) in the equation (4.14), we have

\[
E_k\left[ x \left( \frac{t - \frac{k}{t}}{2} \right) + \frac{y}{2}(w - \frac{k}{w}) \right] = \sum_{\nu=0}^{\infty} \frac{(\frac{x}{2})^\nu}{\Gamma_{s,k}^{\nu}} \sum_{m=0}^{\infty} \frac{(-k)^{m+n}}{\Gamma_{s,k}[\nu + mk + k|n|_s]!} \sum_{\mu=0}^{\infty} \frac{(\frac{yp(x)}{2})^\mu}{\Gamma_{s,k}[\nu + mk + k|n|_s]!} \sum_{n=0}^{\infty} \frac{(-kyp(x))}{\Gamma_{s,k}[\nu + mk + k|n|_s]!}.
\]

which is required generating function for \((s, k)\)-Bessel function of two variables.
Lemma 4.9. If the parameters $\nu$ and $\mu$ are integers then generalized $(s, k)$-Bessel function satisfies

$$J_{-\nu, \mu}^k(x, y; s) = (-k)^{\frac{y}{2}} J_{\nu, \mu}^k(x, y; s).$$

Proof. Replacing $\nu$ by $-\nu$ in $(s, k)$-Bessel function of two variables we get

$$J_{-\nu, \mu}^k(x, y; s) = \sum_{m, n=0}^{\infty} \frac{(-k)^{m+n} (\frac{x}{2})^{2m+\frac{y}{2} (\frac{y}{2})} 2n+\frac{y}{2}}{\Gamma_{s,k}[\nu + m k + k][m]_{sk}! \Gamma_{s,k}[\mu + n k + k][n]_{sk}!}.$$

Substituting $\nu$ by $-\nu$ in the equation (4.15), we get

$$J_{-\nu, \mu}^k(x, y; s) = \sum_{m, n=0}^{\infty} \frac{(-k)^{m+n} (\frac{x}{2})^{2m-\frac{y}{2} (\frac{y}{2})} 2n+\frac{y}{2}}{\Gamma_{s,k}[\nu - m + 1][m]_{sk}! \Gamma_{s,k}[\mu + n k + k][n]_{sk}!}.$$

After replacing $m$ by $\frac{y}{2} + r$ in equation (4.16), we have

$$J_{-\nu, \mu}^k(x, y; s) = \sum_{r, n=0}^{\infty} \frac{(-k)^{r+n} (\frac{x}{2})^{2r+\frac{y}{2} (\frac{y}{2})} 2n+\frac{y}{2}}{\Gamma_{s,k}[\nu - r + 1][r]_{sk}! \Gamma_{s,k}[\mu + n k + k][n]_{sk}!}.$$

which is required recurrence relation. \qed

Lemma 4.10. If the parameters $\nu$ and $\mu$ are integers, then $(s, k)$-Bessel function of two variables satisfies the relation

$$J_{\nu, -\mu}^k(x, y; s) = (-k)^{\frac{y}{2}} J_{\nu, \mu}^k(x, y; s).$$

Proof. Consider the $(s, k)$-Bessel function of two variables

$$J_{\nu, \mu}^k(x, y; s) = \sum_{m, n=0}^{\infty} \frac{(-k)^{m+n} (\frac{x}{2})^{2m+\frac{y}{2} (\frac{y}{2})} 2n+\frac{y}{2}}{\Gamma_{s,k}[\nu + m k + k][m]_{sk}! \Gamma_{s,k}[\nu + n k + k][n]_{sk}!}.$$

Replacing $\mu$ by $-\mu$ in equation (4.17), we have

$$J_{\nu, -\mu}^k(x, y; s) = \sum_{m, n=0}^{\infty} \frac{(-k)^{m+n} (\frac{x}{2})^{2m+\frac{y}{2} (\frac{y}{2})} 2n+\frac{y}{2}}{\Gamma_{s,k}[\nu + m k + k][m]_{sk}! \Gamma_{s,k}[\nu + n k + k][n]_{sk}!}.$$

Replacing $n$ by $s + \frac{y}{2}$ in equation (4.18), we get

$$J_{\nu, -\mu}^k(x, y; s) = \sum_{m, s=0}^{\infty} \frac{(-k)^{m+s} (\frac{x}{2})^{2m+\frac{y}{2} (\frac{y}{2})} 2s+\frac{y}{2}}{\Gamma_{s,k}[\nu + m k + k][m]_{sk}! \Gamma_{s,k}[s + \frac{y}{2}][s + \frac{y}{2}][s + \frac{y}{2}]_{sk}!}.$$
By taking left hand side of the equation (4.19) and using the equation (2.2), we have

\[ (-k)^{\frac{m}{2}} \sum_{m,n=0}^{\infty} \frac{(-k)^m (\frac{x}{2})^{m+n} (\frac{y}{2})^{n+\frac{m}{2}}}{\Gamma_{s,k}(v + mk + k)[n]_{s,k}!} = (-k)^{\frac{m}{2}} \sum_{q,k=0}^{\infty} \frac{(-k)^m (\frac{x}{2})^{m+n} (\frac{y}{2})^{n+\frac{m}{2}}}{\Gamma_{q,k}(v + mk + k)[n]_{q,k}!} \]

Gaspor [7] has given the relation

\[ (s;s)_{n+r} = (s;s)_r (s^{t+1};s)_n. \] (4.20)

5. Conclusion

In our work, the two parameter deformation of classical Bessel function is introduced. We discussed some important relations between k-Bessel function and simple Bessel function in two variables. Also,
we developed the generating functions which satisfies the k-Bessel function and \((s,k)\)-Bessel function in two variables. Moreover, we established a result in which \(k\) we developed the generating functions which satisfies the

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If \(k = 1\), generalized \((s,k)\)-Bessel function reduces to \(s\)-Bessel function in two variables. By taking \(s = 1\) in \((s,k)\)-Bessel function, we get k-Bessel function in two variables. For \(s = 1, k = 1\), the generalized \((s,k)\)-Bessel function reduces to simple classical Bessel function.

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