Algorithmically Optimal Outer Measures

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Abstract

We investigate the relationship between algorithmic fractal dimensions and the classical local fractal dimensions of outer measures in Euclidean spaces. We introduce global and local optimality conditions for lower semicomputable outer measures. We prove that globally optimal outer measures exist. Our main theorem states that the classical local fractal dimensions of any locally optimal outer measure coincide exactly with the algorithmic fractal dimensions. Our proof uses an especially convenient locally optimal outer measure  defined in terms of Kolmogorov complexity. We discuss implications for point-to-set principles.

1 Introduction

Algorithmic fractal dimensions, which quantify the density of algorithmic information in individual points [17, 1, 21], have recently been used to prove new theorems [25, 23, 26, 24, 20] about their classical forerunners, the Hausdorff and packing dimensions of sets. Since algorithmic fractal dimensions are products of the theory of computing, and since the aforementioned new theorems are entirely classical (not involving logic or the theory of computing), these developments call for a more thorough investigation of the relationships between algorithmic and classical fractal dimensions. One significant facet of this investigation, initiated by Orponen [31], is to look for purely classical proofs of these new classical theorems.

In this paper, taking a different approach, we establish direct connections between algorithmic and classical fractal dimensions. Aside from the presence versus absence of algorithms, the most striking difference between algorithmic fractal dimensions and classical fractal dimensions is that the algorithmic dimensions are usefully defined for individual points in Euclidean space, while the classical Hausdorff and packing dimensions vanish on individual points. To bridge this gap, we examine the classical local dimensions (also called pointwise dimensions) of outer measures at individual points in Euclidean spaces [10]. These local fractal dimensions have been studied at least since the 1930s and are essential tools in multifractal analysis [11, 9]. Outer measures and the algorithmic and local dimensions are defined precisely in Section 2 below.

Outer measures, introduced by Carathéodory [4] in the “prehistory” of Hausdorff dimension [12] (defining what later became known as the 1-dimensional Hausdorff measure), are now best known for their role in Carathéodory’s program [5] to generalize Lebesgue measure to a wide variety of settings [35]. However, it is the role of outer measures in local fractal dimensions that are of interest here.

The second author observed [22] that a particular, very nonclassical outer measure  defined in terms of Kolmogorov complexity, has the property that the classical local fractal dimensions of  coincide exactly with the algorithmic fractal dimensions at every point in . This property of  is analogous to Levin’s coding theorem [13, 15], which pertains to a particular, very nonclassical subprobability measure  on strings. Levin’s theorem says that if we substitute  for  in the definition of the classical 1-dimensional Hausdorff measure, then the resulting measure has the property that the classical local fractal dimensions of coincide exactly with the algorithmic fractal dimensions at every point in .
for the probability measure $p$ in the classical Shannon self-information $\log 1/p(x)$, then the resulting quantity $\log 1/m(x)$ is essentially the prefix Kolmogorov complexity (i.e., the algorithmic information content) of the string $x$.

Levin defined $m$ as an optimal lower semicomputable subprobability measure, so the above analogy leads us to investigate here the algorithmic optimality properties of $\kappa$ and other outer measures on Euclidean spaces.

We first investigate outer measures that are *globally optimal*, a property that is closely analogous to the optimality property of Levin’s $m$. In Section 3 we prove that globally optimal outer measures on $\mathbb{R}^n$ exist.

As it turns out, the outer measure $\kappa$ is not globally optimal. In Section 4 we prove this fact, and we introduce and investigate the more general and more subtly defined class of *locally optimal* outer measures on $\mathbb{R}^n$. Our main theorem establishes that every locally optimal outer measure $\mu$ on a Euclidean space $\mathbb{R}^n$ has the property that the classical local fractal dimensions of $\mu$ coincide exactly with the algorithmic dimensions at every point in $\mathbb{R}^n$.

In Section 5 we discuss implications of our results, especially for the point-to-set principles that have enabled the new classical results mentioned in the first paragraph of this introduction.

## 2 Algorithmic and Local Fractal Dimensions

This section reviews the algorithmic fractal dimensions and the classical local fractal dimensions.

Following standard practice [30, 8, 16], we fix a universal prefix Turing machine $U$ and define the (prefix) *Kolmogorov complexity* of a string $w \in \{0,1\}^*$ to be $\mathcal{K}(w) = \min \{|\pi| \mid \pi \in \{0,1\}^* \text{ and } U(\pi) = w\}$, i.e., the minimum number of bits required to cause $U$ to output $w$. By standard binary encodings, we extend this from $\{0,1\}^*$ to other countable domains. In particular, the Kolmogorov complexity $\mathcal{K}(q)$ of a rational point $q \in \mathbb{Q}^n$ is well defined.

The *Kolmogorov complexity* of a point $x \in \mathbb{R}^n$ at a precision $r \in \mathbb{N}$ is $\mathcal{K}_r(x) = \min \{\mathcal{K}(q) \mid q \in \mathbb{Q}^n \text{ and } |q - x| < 2^{-r}\}$, where $|q - x|$ is the Euclidean distance from $q$ to $x$.

We now define the algorithmic fractal dimensions of points in $\mathbb{R}^n$.

**Definition** (17, 28, 31). Let $x \in \mathbb{R}^n$.

1. The *algorithmic dimension* of $x$ is

$$\dim(x) = \liminf_{r \to \infty} \frac{\mathcal{K}_r(x)}{r}.$$  \hfill (2.1)

2. The *strong algorithmic dimension* of $x$ is

$$\text{Dim}(x) = \limsup_{r \to \infty} \frac{\mathcal{K}_r(x)}{r}.$$  \hfill (2.2)

See [32, 29, 19] for surveys of these notions.

The classical local fractal dimensions are local properties of outer measures. An *outer measure* on a set $X$ is a function $\mu : \mathcal{P}(X) \to [0,\infty]$ (where $\mathcal{P}(X)$ is the power set of $X$) with the following three properties.
(i) (vanishes on empty set) \( \mu(\emptyset) = 0 \).

(ii) (monotonicity) For all \( E, F \subseteq X \),

\[
E \subseteq F \implies \mu(E) \leq \mu(F).
\]

(iii) (countable subadditivity) For all \( E_0, E_1, \ldots \subseteq X \),

\[
\mu \left( \bigcup_{n=0}^{\infty} E_n \right) \leq \sum_{n=0}^{\infty} \mu(E_n).
\]

An outer measure \( \mu \) is finite if \( \mu(\mathbb{R}^n) < \infty \).

**Definition** ([10]). If \( \mu \) is a finite outer measure on \( \mathbb{R}^n \), then the lower and upper local (or pointwise) dimensions of \( \mu \) at a point \( x \in \mathbb{R}^n \) are

\[
\dim_{\text{loc}} \mu(x) = \liminf_{r \to \infty} \frac{\log \frac{1}{\mu(B(x, 2^{-r}))}}{r},
\]

and

\[
\text{Dim}_{\text{loc}} \mu(x) = \limsup_{r \to \infty} \frac{\log \frac{1}{\mu(B(x, 2^{-r}))}}{r},
\]

respectively. (The logarithms here are base-2, and \( B(x, \varepsilon) \) is the open ball of radius \( \varepsilon \) about \( x \) in \( \mathbb{R}^n \).)

As stated in the introduction, our main objective is to identify a class of outer measures that cause the classical local fractal dimensions (2.3) and (2.4) to coincide with the algorithmic fractal dimensions (2.1) and (2.2).

### 3 Global Algorithmic Optimality

The optimality notions that we discuss in this paper concern outer measures with three special properties that we now define.

**Definition.** An outer measure \( \mu \) on \( \mathbb{R}^n \) is finitely supported on \( \mathbb{Q}^n \) if, for every \( \varepsilon > 0 \), there is a finite set \( A \subseteq \mathbb{Q}^n \) such that \( \mu(\mathbb{R}^n \setminus A) < \varepsilon \).

Note that an outer measure \( \mu \) on \( \mathbb{R}^n \) that is finitely supported on \( \mathbb{Q}^n \) is supported on \( \mathbb{Q}^n \) in the usual sense that \( \mu(\mathbb{R}^n \setminus \mathbb{Q}^n) = 0 \). The following example shows that the converse does not hold.

**Example 3.1.** The function \( \mu : \mathcal{P}(\mathbb{R}^n) \to [0, \infty] \) defined by

\[
\mu(E) = \begin{cases} 
1 - 2^{-|E \cap \mathbb{Q}^n|} & \text{if } |E \cap \mathbb{Q}^n| < \infty \\
2 & \text{if } |E \cap \mathbb{Q}^n| = \infty
\end{cases}
\]

is an outer measure on \( \mathbb{R}^n \) that is supported, but not finitely supported, on \( \mathbb{Q}^n \).

**Definition.** An outer measure \( \mu \) on \( \mathbb{R}^n \) is strongly finite if \( \mu \) is supported on \( \mathbb{Q}^n \) and

\[
\sum_{q \in \mathbb{Q}^n} \mu(\{q\}) < \infty.
\]
It is clear that every strongly finite outer measure is finite. The outer measure of Example 3.1 shows that the converse does not hold.

**Definition.** An outer measure on $\mathbb{R}^n$ is *lower semicomputable* if it is finitely supported on $\mathbb{Q}^n$ and there is a computable function

$$\hat{\mu} : \mathcal{P}^{<\omega}(\mathbb{Q}^n) \times \mathbb{N} \rightarrow \mathbb{Q} \cap [0, \infty)$$

(where $\mathcal{P}^{<\omega}(\mathbb{Q}^n)$ is the finite power set of $\mathbb{Q}^n$, i.e., the set of all finite subsets of $\mathbb{Q}^n$) with the following two properties.

(i) For all $A \in \mathcal{P}^{<\omega}(\mathbb{Q}^n)$ and $s, t \in \mathbb{N},$

$$s \leq t \implies \hat{\mu}(A, s) \leq \hat{\mu}(A, t) \leq \mu(A).$$

(ii) For all $A \in \mathcal{P}^{<\omega}(\mathbb{Q}^n),$

$$\lim_{t \to \infty} \hat{\mu}(A, t) = \mu(A).$$

**Lemma 3.2.** Let $\mu$ be a lower semicomputable outer measure on $\mathbb{R}^n$. If $\hat{\mu}$ is a function testifying to the lower semicomputability of $\mu$, then, for all $E \subseteq \mathbb{R}^n,$

$$\lim_{A \not\subseteq E \cap \mathbb{Q}^n} \hat{\mu}(E, t) = \mu(E),$$

meaning that, for all $\varepsilon > 0,$ there exist $A \in \mathcal{P}^{<\omega}(E \cap \mathbb{Q}^n)$ and $t_0 \in \mathbb{N}$ such that, for all $B \in \mathcal{P}^{<\omega}(E \cap \mathbb{Q}^n)$ and $t \in \mathbb{N},$

$$[B \supseteq A \text{ and } t \geq t_0] \implies |\mu(E) - \hat{\mu}(B, t)| < \varepsilon.$$

**Definition.** An outer measure $\mu$ on $\mathbb{R}^n$ is *globally optimal* if the following conditions hold.

(i) $\mu$ is strongly finite and lower semicomputable.

(ii) For every strongly finite, lower semicomputable outer measure $\nu$ on $\mathbb{R}^n$, there is a constant $\beta \in (0, \infty)$ such that, for all $E \subseteq \mathbb{R}^n,$

$$\mu(E) \geq \beta \cdot \nu(E).$$

In proving that globally optimal outer measures exist, we will use a computable enumeration of all strongly finite, lower semicomputable outer measures that take values in $[0, 1]$. Our main technical lemma, which we now state, shows that such an enumeration exists.

**Lemma 3.3.** Let $\Theta$ be the set of all strongly finite, lower semicomputable outer measures $\mu : \mathcal{P}(\mathbb{R}^n) \rightarrow [0, 1].$ There is a computable function

$$\hat{\theta} : \mathbb{N} \times \mathcal{P}^{<\omega}(\mathbb{Q}^n) \times \mathbb{N} \rightarrow \mathbb{Q} \cap [0, 1]$$

such that, if we write $\hat{\theta}_k(A, t) = \hat{\theta}(k, A, t)$ for all $k, t \in \mathbb{N}$ and $A \in \mathcal{P}^{<\omega}(\mathbb{Q}^n)$, and if we define $\theta_k : \mathcal{P}(\mathbb{R}^n) \rightarrow [0, 1]$ by

$$\theta_k(E) = \lim_{A \not\subseteq E \cap \mathbb{Q}^n} \hat{\theta}_k(A, t)$$

for all $E \subseteq \mathbb{R}^n,$ then $\Theta = \{\theta_k : k \in \mathbb{N}\}.$
Proof. Let $M_0, M_1, M_2 \ldots$ be an enumeration of all prefix Turing machines that take inputs in $\mathcal{P}^{<\omega}(\mathbb{Q}^n) \times \mathbb{N}$, give outputs in $\mathbb{Q} \cap [0, 1]$, and satisfy $M_k(\emptyset, t) = 0$ for all $k, t \in \mathbb{N}$. For each $k \in \mathbb{N}$, we define the following functions.

$\tau_k : \mathcal{P}^{<\omega}(\mathbb{Q}^n) \times \mathbb{N} \to [0, \infty]$ is given by

$$\tau_k(A, t) = \min \max \{s \leq t \mid \text{for all } s' \leq s, \text{ } M_k \text{ halts on input } (B, s') \text{ within } t \text{ steps} \}.$$

$\eta_k : \mathcal{P}^{<\omega}(\mathbb{Q}^n) \times \mathbb{N} \times \mathbb{N} \to \mathbb{Q} \cap [0, 1]$ is given by

$$\eta_k(A, t, T) = \max \{M_k(B, s) \mid \text{M}_k \text{ halts on input } (B, s) \text{ within } T \text{ steps} \}.$$

$\hat{\theta}_k : \mathcal{P}^{<\omega}(\mathbb{Q}^n) \times \mathbb{N} \to \mathbb{Q} \cap [0, 1]$ is given by

$$\hat{\theta}_k(A, t) = \min \left\{ \sum_{i=0}^{\ell-1} \eta_k(A_i, \tau_k(A, t), t) \mid A_0, \ldots, A_{\ell-1} \subseteq A \text{ and } A \subseteq \bigcup_{i=0}^{\ell-1} A_i \right\}.$$

Now fix $k \in \mathbb{N}$. It is immediate from our construction that $\theta_k$ takes values in $[0, 1]$ and is finitely supported on $\mathbb{Q}^n$, and that $\hat{\theta}_k$ is computable. To prove that $\theta_k \in \Theta$, then, it remains to show that $\hat{\theta}_k$ satisfies conditions (i) and (ii) from the definition of lower semicomputable outer measures, and that $\theta_k$ is countably subadditive.

For condition (i), let $A \in \mathcal{P}^{<\omega}(\mathbb{Q}^n)$ and $s, t \in \mathbb{N}$ with $s \leq t$. Then $\tau_k(A, s) \leq \tau_k(A, t)$, and for all $T \in \mathbb{N}$, $\eta_k(A, s, T) \leq \eta_k(A, t, T)$, so $\hat{\theta}_k(A, s) \leq \hat{\theta}_k(A, t)$.

For condition (ii), let $A, B \in \mathcal{P}^{<\omega}(\mathbb{Q}^n)$ with $A \subseteq B$, let $t \in \mathbb{N}$, and let $T \geq t$ be such that $\tau_k(B, T) \geq \tau_k(A, t)$. Then

$$\hat{\theta}_k(B, T) = \min \left\{ \sum_{i=0}^{\ell-1} \eta_k(A_i, \tau_k(B, T), T) \mid A_0, \ldots, A_{\ell-1} \subseteq B \text{ and } B \subseteq \bigcup_{i=0}^{\ell-1} A_i \right\}$$

$$\geq \min \left\{ \sum_{i=0}^{\ell-1} \eta_k(A_i, \tau_k(A, t), t) \mid A_0, \ldots, A_{\ell-1} \subseteq A \text{ and } A \subseteq \bigcup_{i=0}^{\ell-1} A_i \right\}$$

$$= \hat{\theta}_k(A, t),$$

so

$$\lim_{t \to \infty} \hat{\mu}(A, t) \leq \lim_{t \to \infty} \hat{\mu}(B, t).$$

To prove that $\theta_k$ is countably subadditive, we first show that, for all $t \in \mathbb{N}$, $\hat{\theta}_k(\cdot, t)$ is finitely subadditive on $\mathcal{P}^{<\omega}(\mathbb{Q}^n)$. Fix $t \in \mathbb{N}$, let $A \in \mathcal{P}^{<\omega}(\mathbb{Q}^n)$, and let $A_0, \ldots, A_{\ell-1} \subseteq A$ be such that $A \subseteq \bigcup_{i=0}^{\ell-1} A_i$ and

$$\hat{\theta}_k(A, t) = \sum_{i=0}^{\ell-1} \eta_k(A_i, \tau_k(A, t), t).$$
Suppose that, for some $0 \leq i < \ell$, we have $\eta_k(A_i, \tau_k(A, t)) > \hat{\theta}_k(A_i, t)$. Then there are sets $A_{i,0}, \ldots, A_{i,m-1} \subseteq A_i$ such that

$$\eta_k(A_{i,j}, \tau_k(A, t), t) > \sum_{j=0}^{m-1} \eta_k(A_i, \tau_k(A, t), t),$$

which contradicts the minimality of our choice of $A_0, \ldots, A_{\ell-1}$. Hence, for all $0 \leq i < \ell$, $\eta_k(A_i, \tau_k(A, t)) \leq \hat{\theta}_k(A_i, t)$, so

$$\hat{\theta}(A, t) = \sum_{i=0}^{\ell-1} \eta_k(A_i, \tau_k(A, t), t) \leq \sum_{i=0}^{\ell-1} \hat{\theta}_k(A_i, t).$$

Now let $E, E_0, E_1, E_2, \ldots \subseteq \mathbb{R}^n$ be such that $E \subseteq \bigcup_{i=0}^{\infty} E_i$. Let $\varepsilon > 0$, and let $A \in \mathcal{P}^{<\omega}(E \cap \mathbb{Q}^n)$ and $t \in \mathbb{N}$ be such that

$$\hat{\theta}_k(A, t) > \theta_k(E) - \varepsilon.$$

For each $i \in \mathbb{N}$, let $A_i = E_i \cap A$. Since $A$ is a finite set, there is some $\ell \in \mathbb{N}$ such that $A \subseteq \bigcup_{i=0}^{\ell-1} A_i$. By Lemma [3.2] and the finite subadditivity of $\hat{\theta}_k(\cdot, t)$,

$$\theta_k(E) - \varepsilon < \hat{\theta}_k(A, t)$$

$$\leq \sum_{i=0}^{\ell-1} \hat{\theta}_k(A_i, t)$$

$$\leq \sum_{i=0}^{\ell-1} \theta_k(E_i).$$

Letting $\varepsilon \to 0$, we have

$$\theta_k(E) \leq \sum_{i=0}^{\infty} \theta_k(E_i),$$

and we conclude that $\theta_k \in \Theta$, so $\{\theta_k \mid k \in \mathbb{N}\} \subseteq \Theta$.

For the converse, let $\mu \in \Theta$, and let $\hat{\mu}$ be a witness to the lower semicomputability of $\mu$. Then $\hat{\mu}$ is computable and $\hat{\mu}(\emptyset, t) = 0$ for all $t \in \mathbb{N}$, so there is some $k \in \mathbb{N}$ such that, for all $(A, t) \in \mathcal{P}^{<\omega}(\mathbb{Q}^n) \times \mathbb{N}$, we have $M_k(A, t) = \hat{\mu}(A, t)$.

We now show that for all $A \in \mathcal{P}^{<\omega}(\mathbb{Q}^n)$,

$$\lim_{t \to \infty} \hat{\theta}_k(A, t) = \lim_{t \to \infty} \hat{\mu}(A, t).$$
Let $A \in \mathcal{P}^{<\omega}(Q^n)$ and $t \in \mathbb{N}$. Then
\[
\hat{\theta}_k(A, t) \leq \eta_k(A, \tau_k(A, t), t)
\]
\[= \max_{B \subseteq A} \left\{ M_k(B, s) \mid M_k \text{ halts on input } (B, s) \text{ within } t \text{ steps} \right\}
\]
\[= \max_{B \subseteq A} M_k(B, s)
\]
\[\leq \max_{B \subseteq A} M_k(B, s)
\]
\[= M_k(A, t)
\]
\[= \hat{\mu}(A, t).
\]

Now let $\varepsilon > 0$, let $t \in \mathbb{N}$ be such that, for all $B \subseteq A$,
\[\hat{\mu}(B, t) \geq \mu(B) - \varepsilon/2^{|A|},\]
and let $T \in \mathbb{N}$ be such that $\tau_k(A, T) \geq t$. Then for each $B \subseteq A$,
\[\eta_k(A, \tau_k(A, T), T) = \max_{C \subseteq B} \left\{ M_k(C, s) \mid M_k \text{ halts on input } (C, s) \text{ within } T \text{ steps} \right\}
\]
\[= \max_{C \subseteq B} M_k(C, s)
\]
\[= M_k(B, \tau_k(A, T))
\]
\[\geq M_k(B, t)
\]
\[= \hat{\mu}(B, t).
\]

It follows that
\[
\hat{\theta}_k(A, T) = \min \left\{ \sum_{i=0}^{\ell-1} \eta_k(A_i, \tau_k(A, T), T) \mid A_0, \ldots, A_{\ell-1} \subseteq A \text{ and } A \subseteq \bigcup_{i=0}^{\ell-1} A_i \right\}
\]
\[\geq \min \left\{ \sum_{i=0}^{\ell-1} \hat{\mu}(A_i, t) \mid A_0, \ldots, A_{\ell-1} \subseteq A \text{ and } A \subseteq \bigcup_{i=0}^{\ell-1} A_i \right\}
\]
\[\geq \min \left\{ \sum_{i=0}^{\ell-1} \left( \mu(A_i) - \frac{\varepsilon}{2^{|A|}} \right) \mid A_0, \ldots, A_{\ell-1} \subseteq A \text{ and } A \subseteq \bigcup_{i=0}^{\ell-1} A_i \right\}
\]
\[\geq \mu(A) - \varepsilon,
\]
by the countable subadditivity of $\mu$.

We have shown that for every $A \in \mathcal{P}^{<\omega}(Q^n)$, every $\varepsilon > 0$, and every sufficiently large $t \in \mathbb{N}$,
\[\hat{\theta}_k(A, t) \leq \hat{\mu}(A, t) \text{ and there exists some } T \in \mathbb{N} \text{ such that } \hat{\theta}_k(A, T) \geq M_k(A, t) - \varepsilon.
\]
This implies that for all $A \in \mathcal{P}^{<\omega}(Q^n)$,
\[\lim_{t \to \infty} \hat{\theta}_k(A, t) = \lim_{t \to \infty} \hat{\mu}(A, t),
\]
and therefore for all $E \subseteq \mathbb{R}^n$,

$$\theta_k(E) = \lim_{A \uparrow E \cap \mathbb{Q}^n} \hat{\theta}_k(A, t)$$

$$= \lim_{t \to \infty} \hat{\mu}(A, t)$$

$$= \mu(A).$$

We conclude that $\mu \in \{\theta_k \mid k \in \mathbb{N}\}$, so $\Theta \subseteq \{\theta_k \mid k \in \mathbb{N}\}$. \qed

**Theorem 3.4.** Globally optimal outer measures exist.

**Proof.** Define the strongly finite outer measure $\theta : \mathcal{P}(\mathbb{R}^n) \to [0, 1]$ by

$$\theta(E) = \sum_{k=0}^{\infty} \frac{\theta_k(E)}{2^{k+1}},$$

where $\theta_k$ is defined as in Lemma 3.3. This outer measure is supported on $\mathcal{P}^<\omega(\mathbb{Q}^n)$, and the function $\hat{\theta} : \mathcal{P}^<\omega(\mathbb{Q}^n) \times \mathbb{N} \to [0, 1]$ given by

$$\hat{\theta}(A, t) = \sum_{k=0}^{t} \frac{\hat{\theta}_k(A, t)}{2^{k+1}}$$

is a witness to the lower semicomputability of $\theta$.

Let $\mu : \mathcal{P}(\mathbb{R}^n) \to [0, \infty)$ be any lower semicomputable outer measure that is strongly finite and supported on $\mathcal{P}^<\omega(\mathbb{Q}^n)$, and let

$$h = \max_{E \subseteq \mathbb{R}^n} [\mu(E)].$$

Then the function $\hat{\mu} : \mathcal{P}(\mathbb{R}^n) \to [0, 1]$ given by

$$\hat{\mu}(E) = \mu(E)/h$$

belongs to $\Theta$. By Lemma 3.3 there is some $k \in \mathbb{N}$ such that, for all $E \subseteq \mathbb{R}^n$,

$$\theta_k(E) = \hat{\mu}(E).$$

We have, for all $E \subseteq \mathbb{R}^n$,

$$\mu(E) = h \cdot \hat{\mu}(E) = h \cdot \theta_k(E) \leq h \cdot 2^k \cdot \theta(E),$$

so $\theta$ is globally optimal. \qed

### 4 Local Algorithmic Optimality

This paper’s investigation of algorithmic optimality is primarily driven by a specific outer measure $\kappa$. To define $\kappa$, we first define the Kolmogorov complexity of a set $E \subseteq \mathbb{R}^n$ to be

$$K(E) = \min \{K(q) \mid q \in E \cap \mathbb{Q}^n\}.$$ 

That is, $K(E)$ is the minimum number of bits required to cause the universal prefix Turing machine $U$ to print some rational point in $E$. (Shen and Vereschagin 341 introduced a similar notion for a different purpose.)
Definition (22). Define the function $\kappa : \mathcal{P}(\mathbb{R}^n) \to [0, 1]$ by

$$\kappa(E) = 2^{-K(E)}$$

for all $E \subseteq \mathbb{R}^n$.

Observation 4.1 (22). $\kappa$ is an outer measure on $\mathbb{R}^n$.

Our primary interest in $\kappa$ is the following connection between classical local fractal dimensions and algorithmic fractal dimensions.

Observation 4.2 (22). For all $x \in \mathbb{R}^n$,

$$\dim_{\text{loc}}(\kappa)(x) = \dim(x)$$

and

$$\text{Dim}_{\text{loc}}(\kappa)(x) = \text{Dim}(x).$$

Proof. By (2.1)–(2.4), it suffices to note that, for all $x \in \mathbb{R}^n$,

$$\log \frac{1}{\kappa(B(x, 2^{-r}))} = K_r(x).$$

We next investigate the algorithmic properties of the outer measure $\kappa$.

Observation 4.3. $\kappa$ is strongly finite and lower semicomputable.

Proof. It suffices to show three things.

1. $\kappa$ is finitely supported on $\mathbb{Q}^n$. For this, let $\varepsilon > 0$. Let

$$A = \left\{ q \in \mathbb{Q}^n \mid K(q) \leq \log \frac{1}{\varepsilon} \right\}.$$ 

Then $A$ is a finite subset of $\mathbb{Q}^n$, and

$$K(\mathbb{R}^n \setminus A) = \min\{K(q) \mid q \in \mathbb{Q}^n \setminus A\} > \log \frac{1}{\varepsilon},$$

so $\kappa(\mathbb{R}^n \setminus A) < \varepsilon$.

2. $\sum_{q \in \mathbb{Q}^n} \kappa(\{q\}) < \infty$. For this, just note that

$$\sum_{q \in \mathbb{Q}^n} \kappa(\{q\}) = \sum_{q \in \mathbb{Q}^n} 2^{-K(q)} \leq 1,$$

by the Kraft inequality for prefix Kolmogorov complexity.

3. $\kappa$ is lower semicomputable. This follows immediately from the well known upper semicomputability of the Kolmogorov complexity function.

\[\square\]

Lemma 4.4. $\kappa$ is not globally optimal.
Proof. Define the function $\nu : \mathcal{P}(\mathbb{R}^n) \to [0, \infty]$ by

$$\nu(E) = \sum_{q \in E \cap \mathbb{Q}^n} 2^{-K(q)}.$$ 

We now show that $\nu$ is a strongly finite, lower semicomputable outer measure on $\mathbb{R}^n$. It is clear that $\nu$ is an outer measure on $\mathbb{R}^n$. It thus suffices to prove that $\nu$ has the properties 1, 2, and 3 proven for $\kappa$ in the proof of Observation 4.3. For properties 2 and 3, the proofs for $\nu$ are identical to those for $\kappa$. For property 1, that $\nu$ is finitely supported on $\mathbb{Q}^n$, let $\varepsilon > 0$. By the Kraft inequality for prefix Kolmogorov complexity,

$$\sum_{q \in \mathbb{Q}^n} 2^{-K(q)} \leq 1,$$

so there is a finite set $A \subseteq \mathbb{Q}^n$ such that

$$\nu(\mathbb{R}^n \setminus A) = \sum_{q \in \mathbb{Q}^n \setminus A} 2^{-K(q)} < \varepsilon.$$

Hence, to prove the lemma, it suffices to exhibit a set $E \subseteq \mathbb{R}^n$ such that, for all $\beta \in (0, \infty)$,

$$\kappa(E) < \beta \nu(E). \tag{4.1}$$

Let $\beta \in (0, \infty)$. Let $c$ be a constant such that, for all $m \in \mathbb{N}$,

$$K(m, 0, \ldots, 0) < \log(m) + 2\log\log(m) + c.$$

Let $\alpha \in (0, \infty)$ be some parameter, let $\gamma = 2 + 2\log(\alpha + 2) + c$, and define the set

$$E = \{q \in \mathbb{Q}^n \mid K(q) > \alpha\}.$$

Then $\kappa(E) < 2^{-\alpha}$, and

$$\nu(E) = \sum_{q \in E \cap \mathbb{Q}^n} 2^{-K(q)} \geq \sum_{q \in \mathbb{Q}^n} 2^{-K(q)} = 2^{-\gamma} \cdot |\{q \in \mathbb{Q}^n \mid K(q) \in (\alpha, \alpha + \gamma)\}|.$$

There are fewer than $2^{\alpha+1}$ rational points $q$ with $K(q) \leq \alpha$. For all $m \in \mathbb{N}$ such that $m \leq 2^{\alpha+2}$,

$$K(m, 0, \ldots, 0) < \alpha \left(2+2\log(m)\right) + c = \alpha + \gamma,$$

so there are at least $2^{\alpha+1}$ rational points $q$ with $K(q) \in (\alpha, \alpha + \gamma)$, and we have $\nu(E) > 2^{-\gamma}$. Thus,

$$\kappa(E) < 2^{-\alpha}\nu(E).$$

Choosing $\alpha$ such that

$$\gamma - \alpha = 2 + 2\log(\alpha + 2) + c - \alpha < \log\beta$$

yields (4.1). \qed
Notwithstanding Lemma 4.4, \( \kappa \) does have an optimality property, which we next define. For each \( m = (m_1, \ldots, m_n) \in \mathbb{Z}^n \), let

\[
Q_m = [m_1, m_1 + 1) \times \cdots \times [m_n, m_n + 1)
\]

be the unit cube at \( m \). For each such \( m \) and each \( r \in \mathbb{N} \), let

\[
Q_m^{(r)} = 2^{-r}Q_m = \{2^{-r}x \mid x \in Q_m\}
\]

be the \( r \)-dyadic cube with address \( m \). Note that each \( Q_m^{(r)} \) is “half-closed, half-open” in such a way that, for each \( r \in \mathbb{N} \), the family

\[
Q^{(r)} = \{Q_m^{(r)} \mid m \in \mathbb{Z}^n\}
\]

is a partition of \( \mathbb{R}^n \).

**Definition.** Let \( \mu \) and \( \nu \) be outer measures on \( \mathbb{R}^n \), and let \( A = (A^{(r)} \mid r \in \mathbb{N}) \) be a sequence of families \( A^{(r)} \subseteq \mathcal{P}(\mathbb{R}^n) \) of subsets of \( \mathbb{R}^n \). We say that \( \mu \) dominates \( \nu \) on \( A \) if there is a function \( \beta : \mathbb{N} \to (0, \infty) \) such that \( \beta(r) = 2^{-o(r)} \) as \( r \to \infty \) and, for every \( r \in \mathbb{N} \) and every set \( E \in A^{(r)} \),

\[
\mu(E) \geq \beta(r) \cdot \nu(E).
\]

We say that \( \mu \) dominates \( \nu \) on dyadic cubes if \( \mu \) dominates \( \nu \) on \( Q = (Q^{(r)} \mid r \in \mathbb{N}) \). We say that \( \mu \) dominates \( \nu \) on balls if \( \mu \) dominates \( \nu \) on \( B = (B^{(r)} \mid r \in \mathbb{N}) \), where \( B^{(r)} \) is the set of all open balls of radius \( 2^{-r} \) in \( \mathbb{R}^n \).

**Definition.** An outer measure \( \mu \) on \( \mathbb{R}^n \) is locally optimal if the following two conditions hold.

(i) \( \mu \) is strongly finite and lower semicomputable.

(ii) For every strongly finite, lower semicomputable outer measure \( \nu \) on \( \mathbb{R}^n \), \( \mu \) dominates \( \nu \) on dyadic cubes.

**Theorem 4.5.** The outer measure \( \kappa \) is locally optimal.

**Proof.** We rely on machinery created for a different purpose by Case and the first author [6]. Just as we have “lifted” Kolmogorov complexity from \( \{0, 1\}^* \) to \( \mathbb{Q}^n \) via routine encoding, we lift Levin’s optimal lower semicomputable subprobability measure \( m \) [14, 15] from \( \{0, 1\}^* \) to \( \mathbb{Q}^n \) so that, for all \( q \in \mathbb{Q}^n \),

\[
m(q) = \sum_{U(\pi) = q} 2^{\lvert \pi \rvert}.
\]

We also set

\[
m(E) = \sum_{q \in E \cap \mathbb{Q}^n} m(q)
\]

for all \( E \subseteq \mathbb{R}^n \). The LDS coding theorem of [6] is a mild generalization of Levin’s coding theorem [14, 15] that tells us that there is a constant \( c \in \mathbb{N} \) such that, for all \( r \in \mathbb{N} \) and \( Q \in Q^{(r)} \),

\[
K(Q) \leq \log \frac{1}{m(Q)} + K(r) + c.
\]

(4.2)

To prove the present theorem, it suffices by Observation 4.3 to prove that \( \kappa \) satisfies condition (ii) of the definition of local optimality. For this, let \( \nu \) be a strongly finite, lower semicomputable outer-
measure on $\mathbb{R}^n$. Define $p_\nu : \mathcal{Q}^n \rightarrow [0, \infty]$ by $p_\nu(q) = \nu(\{q\})$ for all $q \in \mathcal{Q}^n$. Then $p_\nu$ is lower semicomputable and $\sum_{q \in \mathcal{Q}^n} p_\nu(q) < \infty$, so the optimality property of $m$ tells us that there is a constant $\alpha \in (0, \infty)$ such that, for all $q \in \mathcal{Q}^n$,

$$m(q) \geq \alpha p_\nu(q).$$

(4.3)

Define $\beta : \mathbb{N} \rightarrow (0, \infty)$ by

$$\beta(r) = \alpha 2^{-(K(r)+c)}$$

for all $r \in \mathbb{N}$. Then

$$\lim_{r \to \infty} \frac{\log \frac{1}{\beta(r)}}{r} = \lim_{r \to \infty} \frac{\log \frac{1}{\alpha(r)} + K(r) + c}{r} = 0,$$

so $\beta(r) = 2^{-\alpha(r)}$ as $r \to \infty$. Also, for all $r \in \mathbb{N}$ and $Q \in \mathcal{Q}^{(r)}$, (4.2), (4.3), and the countable subadditivity of $\nu$ tell us that

$$\kappa(Q) = 2^{-K(Q)} \geq 2^{-(K(r)+c)} m(Q) = 2^{-(K(r)+c)} \sum_{q \in Q \cap \mathcal{Q}^n} m(q) \geq \beta(r) \sum_{q \in Q \cap \mathcal{Q}^n} p_\nu(q) = \beta(r) \sum_{q \in Q \cap \mathcal{Q}^n} \nu(\{q\}) \geq \beta(r) \nu(Q).$$

This shows that $\kappa$ dominates $\nu$ on dyadic cubes, confirming that $\kappa$ is locally optimal. \qed

**Corollary 4.6.** A strongly finite, lower semicomputable outer measure on $\mathbb{R}^n$ is locally optimal if and only if it dominates $\kappa$ on dyadic cubes.

**Lemma 4.7.** There is a constant $c \in \mathbb{N}$ such that, for every $r \in \mathbb{N}$, every $r$-dyadic cube $Q \in \mathcal{Q}^{(r)}$, and every open ball $B \subseteq \mathbb{R}^n$ of radius $2^{-r}$ that intersects $Q$,

$$|K(B) - K(Q)| \leq K(r) + c.$$

**Proof.** Lemma 3.5 of [6] gives us a constant $c_1 \in \mathbb{N}$ such that, for all $r$, $Q$, and $B$ as in the present lemma,

$$K(B) \leq K(Q) + K(r) + c_1.$$

Hence it suffices to show that there is a constant $c_2$ such that, for all $r$, $Q$, and $B$ as in the present lemma,

$$K(Q) \leq K(B) + K(r) + c_2.$$

Let $M$ be a prefix Turing machine that, on input $\pi_1 \pi_2 \pi_3$ where $U(\pi_1) = r \in \mathbb{N}$, and $U(\pi_2) = k \in \mathbb{N}$, and $U(\pi_3) = (q_1, \ldots, q_n) \in \mathcal{Q}^n$, outputs the lexicographically $k^{th}$ point in the product set

$$\prod_{i=1}^{n} \{2^{-r}(\lfloor 2^r q_i \rfloor - 2), 2^{-r}(\lfloor 2^r q_i \rfloor - 1), 2^{-r}[2^r q_i], 2^{-r}(\lfloor 2^r q_i \rfloor + 1), 2^{-r}(\lfloor 2^r q_i \rfloor + 2) \}.$$
Let \( r, Q, \) and \( B \) be as in the present lemma. Let \( q \in B \cap \mathbb{Q}^n \) be such that \( K(q) = K(B) \). Then there is some point \( p = (p_1, \ldots, p_n) \in Q \cap B \cap \mathbb{Q}^n \) such that \( |p - q| < 2^{1-r} \). Hence \( Q \) is the \( r \)-dyadic cube with address \( ([2^r p_1], \ldots, [2^r p_n]) \), and for each \( 1 \leq i \leq n \),

\[
|[2^r q_i] - [2^r p_i]| \leq 2.
\]

That is, the address of \( Q \) belongs to the product set

\[
\prod_{i=1}^n \{[2^r q_i] - 2, [2^r q_i] - 1, [2^r q_i], [2^r q_i] + 1, [2^r q_i] + 2\}.
\]

Let \( k \leq 5^n \) be the lexicographical index of \( Q \)'s address within this set, and let \( \pi_1, \pi_2, \) and \( \pi_3 \) be witnesses to \( K(r), K(k), \) and \( K(q) \), respectively. Then \( M(\pi_1 \pi_2 \pi_3) \in Q \), so letting \( c_M \) be an optimality constant for the machine \( M \), we have

\[
K(Q) \leq |\pi_1| + |\pi_2| + |\pi_3| + c_M = K(r) + K(k) + K(q) + c_M = K(B) + K(r) + K(k) + c_M.
\]

Since \( k \leq 5^n \), there is some constant \( c_3 \) such that \( K(k) \leq 2 \log(5^n) + c_3 \), so the constant

\[
c_2 = c_M + 2 \log(5^n) + c_3
\]

affirms the lemma. \( \square \)

**Lemma 4.8.** A strongly finite, lower semicomputable outer measure \( \mu \) dominates \( \kappa \) on balls if and only if it dominates \( \kappa \) on dyadic cubes.

**Proof.** Suppose that \( \mu \) dominates \( \kappa \) on balls. Then for every \( x \in \mathbb{R}^n \),

\[
\mu(B(x, 2^{-r})) = 2^{-o(r)} \kappa(B(x, 2^{-r})).
\]

Let \( r \in \mathbb{Z} \), and let \( Q \) be an \( r \)-dyadic cube with center \( q \), and let \( B = B(q, 2^{-r-1}) \), so that \( B \subseteq Q \). Then, applying Lemma 4.7

\[
\mu(Q) \geq \mu(B) = 2^{-o(r)} \kappa(B) = 2^{-K(B) - o(r)} = 2^{-K(Q) - o(r)} = 2^{-o(r)} \kappa(Q),
\]

so \( \mu \) dominates \( \kappa \) on dyadic cubes.

Now suppose that \( \mu \) dominates \( \kappa \) on dyadic cubes. Then for every \( r \)-dyadic cube \( Q \),

\[
\mu(Q) = 2^{-o(r)} \kappa(Q).
\]

Let \( r \in \mathbb{Z}, x \in \mathbb{R}^n \), and \( B = B(x, 2^{-r}) \). Let \( Q \) be the \((r + \left\lceil \log \sqrt{n} \right\rceil)\)-dyadic cube containing \( x \), so that \( Q \subseteq B \). Applying Lemma 4.7

\[
\mu(B) \geq \mu(Q) = 2^{-o(r)} \kappa(Q) = 2^{-K(Q) - o(r)} = 2^{-K(B) - o(r)} = 2^{-o(r)} \kappa(B),
\]

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Corollary 4.9. For every strongly finite, lower semicomputable outer measure \( \mu \) on \( \mathbb{R}^n \), the following three conditions are equivalent.

1. \( \mu \) is locally optimal.
2. \( \mu \) dominates \( \kappa \) on balls.
3. For every strongly finite, lower semicomputable outer measure \( \nu \) on \( \mathbb{R}^n \), \( \mu \) dominates \( \nu \) on balls.

We now have everything we need to prove our main theorem, which is the following generalization of Observation 4.2.

Theorem 4.10. If \( \mu \) is any locally optimal outer measure on \( \mathbb{R}^n \), then for all \( x \in \mathbb{R}^n \),

\[
\dim_{\text{loc}} \mu(x) = \dim(x) \tag{4.4}
\]

and

\[
\text{Dim}_{\text{loc}} \mu(x) = \text{Dim}(x). \tag{4.5}
\]

Proof. Let \( \mu \) be any locally optimal outer measure on \( \mathbb{R}^n \). By Corollary 4.9, \( \kappa \) dominates \( \mu \) on balls, and \( \mu \) dominates \( \kappa \) on balls. That is, there exist two functions \( \beta_1, \beta_2 : \mathbb{N} \to (0, \infty) \) such that \( \beta_1(r) = 2^{-o(r)} \) and \( \beta_2(r) = 2^{-o(r)} \) as \( r \to \infty \), and, for every \( r \in \mathbb{N} \) and \( x \in \mathbb{R}^n \),

\[
\kappa(B(x, 2^{-r})) \geq \beta_1(r) \mu(B(x, 2^{-r}))
\]

and

\[
\mu(B(x, 2^{-r})) \geq \beta_2(r) \kappa(B(x, 2^{-r})).
\]

Letting \( \beta(r) = \min\{\beta_1(r), \beta_2(r)\} \), we have \( \beta(r) = 2^{-o(r)} \) as \( r \to \infty \) and, for every \( r \in \mathbb{N} \) and \( x \in \mathbb{R}^n \),

\[
\left| \log \frac{1}{\mu(B(x, 2^{-r}))} - \log \frac{1}{\kappa(B(x, 2^{-r}))} \right| \leq \log \frac{1}{\beta(r)}.
\]

It follows that, for all \( x \in \mathbb{R}^n \),

\[
\left| \log \frac{1}{\mu(B(x, 2^{-r}))} - K_r(x) \right| = o(r)
\]

as \( r \to \infty \), whence (4.4) and (4.5) follow from (2.1)–(2.4). \( \square \)

5 Point-to-Set Principles and Dimensions of Measures

Local dimensions of measures give rise to global dimensions of measures, which we now briefly comment on. In classical fractal geometry, the global dimensions of Borel measures play a substantial role in studying the interplay between local and global properties of fractal sets and measures. The material in this section is from [22].
**Definition** ([9]). For any locally finite Borel measure \( \mu \) on \( \mathbb{R}^n \), the *lower* and *upper Hausdorff and packing dimension* of \( \mu \) are

\[
\dim_H(\mu) = \sup \{ \alpha \mid \mu(\{x \mid \dim_{\mu}(x) < \alpha\}) = 0\}
\]

\[
\Dim_H(\mu) = \inf \{ \alpha \mid \mu(\{x \mid \dim_{\mu}(x) > \alpha\}) = 0\}
\]

\[
\dim_P(\mu) = \sup \{ \alpha \mid \mu(\{x \mid \Dim_{\mu}(x) < \alpha\}) = 0\}
\]

\[
\Dim_P(\mu) = \inf \{ \alpha \mid \mu(\{x \mid \Dim_{\mu}(x) > \alpha\}) = 0\},
\]

respectively.

Extending these definitions to outer measures, we may consider global dimensions of the outer measure \( \kappa \). Since \( \kappa \) is supported on \( \mathcal{P}^{<\omega}(\mathbb{Q}^n) \) and \( \dim(x) = 0 \) for all \( x \in \mathcal{P}^{<\omega}(\mathbb{Q}^n) \),

\[
\dim_H(\kappa) = \Dim_H(\kappa) = \dim_P(\kappa) = \Dim_P(\kappa) = 0.
\] (5.1)

The *point-to-set principle* ([IS]) expresses classical Hausdorff and packing dimensions in terms of *relativized* algorithmic dimensions. That is, algorithmic dimensions in which the underlying universal Turing machine \( U \) is an oracle machine with access to some oracle \( A \subseteq \mathbb{N} \). We write \( \dim^A(x) \) and \( \Dim^A(x) \) to denote the algorithmic dimension and strong algorithmic dimension of a point \( x \in \mathbb{R}^n \) relative to \( A \).

**Theorem 5.1** ([IS]). For every \( E \subseteq \mathbb{R}^n \),

\[
\dim_H(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \dim^A(x)
\]

and

\[
\dim_P(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \Dim^A(x).
\]

In light of Theorem 5.10, this principle may be considered a member of the family of results, such as Billingsley’s lemma [2] and Frostman’s lemma [11], that relate the local decay of measures to global properties of measure and dimension. Useful references on such results include [3, 13, 27].

Among classical results, this principle is most directly comparable to the *weak duality principle* of Cutler [7] (see also [9]), which expresses Hausdorff and packing dimensions in terms of lower and upper pointwise dimensions of measures. For nonempty \( E \subseteq \mathbb{R}^n \), let \( \Delta(E) \) be the collection of Borel probability measures on \( \mathbb{R}^n \) such that the \( E \) is measurable and has measure 1, and let \( \overline{E} \) be the closure of \( E \).

**Theorem 5.2** ([7]). For every nonempty \( E \subseteq \mathbb{R}^n \),

\[
\dim_H(E) = \inf_{\mu \in \Delta(E)} \sup_{x \in E} \dim_{\mu}(x)
\]

and

\[
\dim_P(E) = \inf_{\mu \in \Delta(E)} \sup_{x \in E} \Dim_{\mu}(x).
\]

By letting \( A = \{\kappa^A \mid A \subseteq \mathbb{N}\} \) and invoking Observation 4.2, Theorem 5.1 can be restated even more similarly as

\[
\dim_H(E) = \inf_{\mu \in A} \sup_{x \in E} \dim_{\mu}(x)
\]

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and
\[
dimp(E) = \inf_{\mu \in \mathcal{A}} \sup_{x \in E} \text{Dim}_{\text{loc}} \mu(x).
\]

Notice, however, that the collections over which the infima are taken in these two results, \(\mathcal{A}\) and \(\Delta(E)\), are disjoint and qualitatively very different. In particular, \(\mathcal{A}\) does not depend on \(E\). Whereas the global dimensions of the measures in \(\Delta(E)\) are closely tied to the dimensions of \(E\) \([9]\), \((5.1)\)
shows that the outer measures in \(\mathcal{A}\) all have trivial global dimensions.

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References

[1] Krishna B. Athreya, John M. Hitchcock, Jack H. Lutz, and Elvira Mayordomo. Effective strong dimension in algorithmic information and computational complexity. *SIAM Journal on Computing*, 37(3):671–705, 2007.

[2] Patrick Billingsley. Hausdorff dimension in probability theory II. *Illinois Journal of Mathematics*, 5(2):291–298, 1961.

[3] Christopher J. Bishop and Yuval Peres. *Fractals in Probability and Analysis*. Cambridge University Press, 2017.

[4] Constantin Carathéodory. Über das lineare maß von punktmengen—eine verallgemeinerung des längenbegriffs. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, pages 404–426, 1914. Translated as On the Linear Measure of Point Sets—a Generalization of the Concept of Length. In *Classics on Fractals*, Gerald A. Edgar (Ed.). Addison Wesley, 1993.

[5] Constantin Carathéodory. *Vorlesungen über reelle Funktionen*. Teubner, Leipzig, 1918.

[6] Adam Case and Jack H. Lutz. Mutual dimension. *ACM Transactions on Computation Theory*, 7(3):12, 2015.

[7] Colleen D. Cutler. Strong and weak duality principles for fractal dimension in Euclidean space. *Mathematical Proceedings of the Cambridge Philosophical Society*, 118:393–410, 1995.

[8] Rod Downey and Denis Hirschfeldt. *Algorithmic Randomness and Complexity*. Springer-Verlag, 2010.

[9] Kenneth J. Falconer. *Techniques in Fractal Geometry*. Wiley, 1997.

[10] Kenneth J. Falconer. *Fractal Geometry: Mathematical Foundations and Applications*. Wiley, third edition, 2014.

[11] Otto Frostman. Potential d’équilibre et capacité des ensembles avec quelques applications à la théorie des fonctions. *Meddelanden från Lunds Universitets Matematiska Seminarium*, 3:1–118, 1935.
[12] Felix Hausdorff. Dimension und äusseres Mass. *Mathematische Annalen*, 79:157–179, 1919.

[13] Michael Hochman. Lectures on dynamics, fractal geometry, and metric number theory. *Journal of Modern Dynamics*, 8(3–4):437–497, 2014.

[14] Leonid A. Levin. On the notion of a random sequence. *Soviet Mathematics Doklady*, 14(5):1413–1416, 1973.

[15] Leonid A. Levin. Laws of information conservation (nongrowth) and aspects of the foundation of probability theory. *Problem Peredachi Informatsii*, 10(3):30–35, 1974.

[16] Ming Li and Paul M. B. Vitányi. *An Introduction to Kolmogorov Complexity and its Applications*. Springer-Verlag, Berlin, fourth edition, 2019.

[17] Jack H. Lutz. The dimensions of individual strings and sequences. *Information and Computation*, 187(1):49–79, 2003.

[18] Jack H. Lutz and Neil Lutz. Algorithmic information, plane Kakeya sets, and conditional dimension. *ACM Transactions on Computation Theory*, 10(2), 2018.

[19] Jack H. Lutz and Neil Lutz. Who asked us? how the theory of computing answers questions about analysis. In Dingzhu Du and Jie Wang, editors, *Complexity and Approximation: In Memory of Ker-I Ko*, pages 48–56. Springer, 2020.

[20] Jack H. Lutz, Neil Lutz, and Elvira Mayordomo. The dimensions of hyperspaces. arXiv:2004.07798, 2020.

[21] Jack H. Lutz and Elvira Mayordomo. Dimensions of points in self-similar fractals. *SIAM Journal on Computing*, 38(3):1080–1112, 2008.

[22] Neil Lutz. *Algorithmic Information, Fractal Geometry, and Distributed Dynamics*. PhD thesis, Rutgers University, 2017.

[23] Neil Lutz. Fractal intersections and products via algorithmic dimension. In *42nd International Symposium on Mathematical Foundations of Computer Science August 21–25, 2017, Aalborg, Denmark, Proceedings*, 2017.

[24] Neil Lutz. Fractal intersections and products via algorithmic dimension. arXiv:1612.01659v4 [cs.CC], 2019.

[25] Neil Lutz and D. M. Stull. Bounding the dimension of points on a line. *Proceedings of the 14th Annual Conference on Theory and Applications of Models of Computation, TAMC 2017, Bern, Switzerland*, 2017.

[26] Neil Lutz and D. M. Stull. Projection theorems using effective dimension. In *43rd International Symposium on Mathematical Foundations of Computer Science, MFCS 2018, August 27-31, 2018, Liverpool, UK*, pages 71:1–71:15, 2018.

[27] Pertti Mattila. *Geometry of sets and measures in Euclidean spaces: fractals and rectifiability*. Cambridge University Press, 1995.

[28] Elvira Mayordomo. A Kolmogorov complexity characterization of constructive Hausdorff dimension. *Inf. Process. Lett.*, 84(1):1–3, 2002.
[29] Elvira Mayordomo. Effective fractal dimension in algorithmic information theory. In S. Barry Cooper, Benedikt Löwe, and Andrea Sorbi, editors, New Computational Paradigms: Changing Conceptions of What is Computable, pages 259–285. Springer-Verlag, 2008.

[30] Andre Nies. Computability and Randomness. Oxford University Press, 2009.

[31] Tuomas Orponen. Combinatorial proofs of two theorems of Lutz and Stull. arXiv:2002.01743 [math.CA], 2020.

[32] Jan Reimann. Computability and fractal dimension. PhD thesis, Heidelberg University, 2004.

[33] Claude E. Shannon. A mathematical theory of communication. Bell System Technical Journal, 27(3–4):379–423, 623–656, 1948.

[34] Alexander Shen and Nikolai K. Vereshchagin. Logical operations and Kolmogorov complexity. Theoretical Computer Science, 271(1–2):125–129, 2002.

[35] Terence Tao. An Introduction to Measure Theory. Graduate Studies in Mathematics. American Mathematical Society, 2011.