Controllable matter-wave switchers with vector Bose-Einstein solitons

Judit Babarro, María J. Paz-Alonso, Humberto Michinel and José R. Salgueiro
Área de Óptica, Facultade de Ciencias de Ourense, Universidad de Vigo, As Lagoas s/n, Ourense, ES-32004 Spain.

David N. Olivieri
Escola Técnica Superior de Engenharia Informática, Universidad de Vigo, As Lagoas s/n, Ourense, ES-32004 Spain.

We show the possibility of producing matter-wave switching devices by using Manakov interactions between matter wave solitons in two-species Bose-Einstein Condensates (BEC). Our results establish the experimental parameters for three interaction regimes in two-species BECs: symmetric and asymmetric splitting, down-switching and up-switching. We have studied the dependence upon the initial conditions and the kind of interaction between the two components of the BECs.

PACS numbers: 05.45.Yv, 42.65.Tg, 03.75.-b

Introduction.- There has been remarkable experimental and theoretical progress in recent years regarding the interaction phenomena of coherent matter waves within Bose-Einstein condensates (BEC) of ultra cold atomic gases. The interaction between the constitutive bosons inside the condensate is defined in terms of the ground state scattering length $a$. When $a > 0$ the interaction between the particles in the condensate is repulsive, whereas for $a < 0$, the interaction is attractive. Experimental preparation of negative scattering length was made possible by using Feshbach resonances to continuously detune the value of $a$ from positive to negative values. This provides new interest for analyzing systems of interacting condensates with an attractive coupling force, provided the number of particles is limited to avoid collapse.

In spite of this serious difficulty, negative scattering length condensates have some peculiarities which make them interesting. For instance, if the trap is removed in one direction and shrunk in the transverse plane, attractive interaction gives rise (for a given number of particles) to a self-confined stationary state. In this case, the cloud forms a soliton and can be controlled as a particle by acting on it with external fields.

On the other hand, results with two-species BECs allow to perform experiments with two-component (vector) matter wave solitons. In this configuration, the interaction between the two species plays a significant role in determining the dynamics of the clouds. In this work, we extend the concept of Bose-Einstein soliton to the case of two-species condensates. We show the possibility of producing Manakov solitons, with matter waves and to use them in the design of matter-wave switching devices. To this aim, we have studied numerically the interaction dynamics of two-species condensates within the confining cylindrical trap for different scattering length regimes and initial conditions.

The theoretical foundations of our work are presented in the first part of this paper, where we describe the model and the experimental configuration which has been assumed. We used a coupled set of nonlinear Schrödinger equations, properly describing Manakov solitons, for modeling the time evolution of the initial wave functions before and after the interaction. Next, we demonstrate that some physical insight into the dynamics can be obtained from a set of equations derived directly from a variational analysis, at least for the case of stationary states, while more complicated scattering phenomena, must be calculated from direct numerical integration of the field equations. We show that interesting switching devices can be implemented with these interactions. Finally, we present our conclusions.

Theoretical Foundations.- The system we have studied consists of two BEC matter waves of different species which interact via 2-body elastic effects. We have assumed an asymmetric cigar trapping potential in which, once individual BEC solitons are formed, they can be given an initial momentum along $x$ with an external force to make them collide. Assuming that the dynamics of the condensate is frozen in the transversal plane $y, z$ and the trap is switched off along the $x$ direction, the behavior of each condensate can be described through a one-dimensional Gross-Pitaevskii equation (GPE), with the addition of a coupling term that takes into account the interaction potential between the two condensate wave functions:

$$\frac{\partial \psi_k}{\partial \tau} + \frac{1}{2} \frac{\partial^2 \psi_k}{\partial \eta^2} + (a_{kk} |\psi_k|^2 + a_{jk} |\psi_j|^2) \psi_k = 0, \quad (1)$$

where $k = 1, 2, j = 2, 1$. We have defined the dimensionless variables $\tau = (\hbar/mL^2) t$ and $\eta = x/L$, together with the normalized wave function (order parameter) $\psi_k(\eta, \tau)$, which gives the number of particles per unit length. Thus, the normalization for $\psi_k$ is $N_k = \int |\psi_k|^2 d\eta$, being $L \approx 1\mu m$ the radial dimension of the trap, $a$ the ground state scattering length (typically $a \approx 3a_0$, with $a_0$ the Bohr radius) and $m$ the mass of the atoms.

The analytic solution to the set of equations represents a difficult problem. For some particular cases, explicit formulas were given by Manakov. However,
we can obtain approximate results by assuming an initial Gaussian wave function and performing a variational analysis. This provides physical insight into the propagation and elastic collisions of fundamental parameters. Although this procedure breaks down for inelastic scattering when the beams split off, we will use these analytic results as a guide when we perform a full numerical integration of Eqs. (1). To obtain the differential equations for the motion of the centroid and width of the initial Gaussian trial wave functions, we must minimize the Lagrange density, which produces the GP equation, over a set of Gaussian trial wave functions such that:

\[ \psi_k(\eta, \tau) = A_k \exp \left[ -\frac{(\eta - \eta_k)^2}{2\sigma_k^2} + i \left( \eta \alpha_k + \eta^2 \beta_k \right) \right], \quad (2) \]

where \( k = 1, 2 \). Inserting this trial function into the Lagrangian density and taking the variation with respect to \( \eta \), we obtain a set of differential equations with the above parameters, all of them \( \tau \) dependent: \( A_k \) (complex amplitude), \( \sigma_k \) (half width of the cloud), \( \eta_k \) (position of the centroid), \( \alpha_k \) (velocity) and \( \beta_k \) (inverse square root of the beam curvature radius). The equations obtained describe the motion of the centroid and the oscillations of the soliton widths. Thus, taking the variation with respect to \( A_k \) and \( A_k^* \), and operating with them, we obtain the conservation of the number of particles: \( N_k = 0 \).

Taking the variation with respect to the parameter \( \alpha_k \) and defining \( \eta_{kk} = \eta_j - \eta_k \) (distance between centroids), we obtain the evolution of the separation of the beam centers and widths:

\[ \ddot{\eta}_{jk} = -\frac{2a_{jk}(N_j + N_k)\eta_{jk}}{\pi(\sigma_j^2 + \sigma_k^2)^{3/2}} \exp \left[ -\frac{\eta_{jk}^2}{\sigma_j^2 + \sigma_k^2} \right], \quad (3) \]

\[ \dot{\sigma}_j = \frac{1}{\sigma_j^3} \frac{a_{jj}N_j}{2\pi \sigma_j^2} \left[ 1 - \frac{2\eta_{jk}^2}{\sigma_j^2 + \sigma_k^2} \right] \exp \left[ -\frac{\eta_{jk}^2}{\sigma_j^2 + \sigma_k^2} \right]. \quad (4) \]

Eq. (4) can be integrated to obtain the potential \( \Pi_k \) ruling the interaction between the two species:

\[ \Pi_k = -\frac{a_{jk}N_j}{\pi \sqrt{\sigma_k^2 + \sigma_j^2}} \exp \left[ -\frac{\eta_{jk}^2}{\sigma_k^2 + \sigma_j^2} \right], \quad (5) \]

with \( k = 1, 2; \; j = 2, 1 \). Equations (3) and (4) are not exact, are valuable tools for a further detailed numerical exploration. In first place, we must notice that depending on the sign of the coupling constant, the interaction potential will correspond to a barrier \((a_{jk} < 0)\) or to a well \((a_{jk} > 0)\). Thus, we can predict a minimum separation distance \( \eta_{cr} \) as the half width estimate of \( \Pi_k \), which corresponds to a value \( \eta_{cr} \approx \sqrt{\sigma_j^2 + \sigma_k^2} \). This means that two clouds separated more than their width will not interact. Our numerical simulations corroborate all these predictions from the variational analysis.

On the other hand, when the two clouds are in the same initial position \((i.e., \eta_j(0) = \eta_k(0))\) with null velocities, the condition \( \dot{\sigma}_1 = \dot{\sigma}_2 = 0 \), predict the existence of stationary states corresponding to a zero spreading of the wave functions. These vector solitons are formed due to self-trapping from \( a_{jj} \) and \( a_{kk} \) terms in Eq. (4) together with cross-interaction \((a_{jk})\). We have performed a numerical calculation by means of a relaxation method of these stationary states for different values of the scattering lengths. The results are in agreement with the variational calculations. As we will comment below, the computer simulations reveal that turning the value and sign of the \( a_{jk} \) can yield to a dramatic change in the stability of the stationary states. From a physical point of view, it is evident that a repulsive cross-interaction \((a_{jk} < 0)\) opposed to the self-trapping effect could yield to an instabilization of the vector BEC soliton. This situation is also predicted by the variational calculation when \( \dot{\sigma}_j = 0 \) in Eq. (4). In this case, the stationary states will only exist if:

\[ \frac{|a_{jk}|}{a_{jj}} \leq \frac{N_j}{N_k} \left( 1 + \frac{\sigma_j^2}{\sigma_j^2} \right)^{3/2} \frac{1}{2\sqrt{2}}, \quad (6) \]

For the particular case of two condensates with equal number of particles and widths, the above condition takes the form: \( |a_{jk}/a_{jj}| \leq 1 \). The numerical simulation sets this value to \( |a_{jk}/a_{jj}| \leq 0.95 \), which is in very good agreement with the theoretical prediction.

Numerical simulations.- For the integration of Eqs. (1), we utilized a Crank-Nicholson finite difference discretization with 500 points grid. Our simulations consisted, in first place, of comparing different velocities and number of particles of the incident BEC soliton wave functions described, and observing the evolution of the clouds after the interaction, for several scattering lengths. From an exhaustive numerical exploration, we realized that there exist well defined scattering regimes for which the emerging wave functions can be radically different. We also explored the effect of turning the value of the scattering length for several collision processes. We have found that a change of the sign and the values of the \( a_{ij} \) and \( a_{jk} \) terms has a deep influence on the dynamics of the clouds.

Thus, we will show in first place the results obtained for the case of attractive scattering lengths coefficients \((i.e., a_{jj} \approx a_{jk} > 0)\). Fig. 1 illustrates different Y-switching processes based upon the characteristics of the incoming wave function. In (a) and (b) we show respectively the square modulus of \( u_1 \) and \( u_2 \). The simulations correspond to the parameters: \( N_1 = N_2 = 4000 \), width of the clouds \( w_1 = w_2 = 10\mu m \), initial separation of the centroids \( x_0 = 15\mu m \), and initial velocities.
functions. The simulations correspond to the parameters: $N_1 = N_2 = 4000$, width of the clouds $10\mu m$, initial separation of the centroids $x_0 = 15\mu m$, and initial velocities $v_{i1} = -v_{i2} = 0.05\mu m/s$ (left), and $v_{i1} = -v_{i2} = 0.5\mu m/s$ (right). Scattering length parameters are: $a_{11} = a_{22} = 1.0$ and $a_{12} = a_{21} = 2.0$.

$v_{i1} = -v_{i2} = 0.05\mu m/s$. The scattering length adimensional coefficients are $a_{11} = a_{22} = 1.0$ and $a_{12} = a_{21} = 2.0$. The two incoming clouds split symmetrically into two mutually trapped vector BEC solitons with the same number of particles. In c) the maximum of $|u_1|^2$ and $|u_2|^2$ are plotted. Figs. (d) and (e) correspond to the same parameters but increasing the initial velocities one order of magnitude: $v_{i1} = -v_{i2} = 0.5\mu m/s$. In this case the emerging distributions are not symmetric, as can be clearly appreciated in Fig. (f) where the evolution of both peaks of $|u_1|^2$ is shown. In both cases the resulting matter waves are stable vector BEC solitons with larger velocities than the input beams. The wave functions take the form $a_1u_1 + a_2u_2$, being $|a_1|^2 + |a_2|^2 = 1$. We have found numerically that the different splitting regimes are limited by the following values: $0 < |v_i| < 0.1\mu m/s$ for the symmetric case and $0.1\mu m/s < |v_i| < 0.9\mu m/s$, for the splitting with different number of particles. For initial velocities $|v_i| > 0.9\mu m/s$ the two wave functions are mutually transparent and arise with any change after the collision.

The collision of two vector BEC solitons constructed by superposition of Gaussian-shaped clouds is shown in Figure 2. In this case it is obtained an up-switching of the peak of both wave functions which emerge fused after the collision. The simulation corresponds to the parameters: number of particles: $N_{1l} = N_{1r} = 8000$ (a) and $N_{2l} = N_{2r} = 3000$, where subindices $l, r$ designate the soliton coming from left or right side (b); cloud widths $5\mu m$; initial velocities $v_l = -v_2 = 0.3\mu m/s$ and initial separation of the centroids $x_0 = 15\mu m$. The scattering length coefficients are: $a_{11} = a_{22} = 1.0$ and $a_{12} = a_{21} = 2.0$. We must stress that the fusion of two condensates only takes place for clouds with different number of particles.

We will now consider the case of repulsive cross-interaction ($a_{jk} < 0$). This situation is very interesting, as it corresponds to condensates of different types of atoms (for instance $^{85}Rb$ and $^{40}K$) as in the experiments from ref. [5]. As it can be appreciated in Fig. (3) the results of the collisions are very different comparing with Fig. (2) which corresponds to attractive cross-interaction. Figs. (4) (a) and (b) show the down-switching of one of the condensates which is almost annihilated by the other, which is repelled. The plot (c) shows the evolution of the peaks of both distributions. The parameters used for the calculation are typical of experiments: $N_1 = N_2 = 4000$, widths of the clouds $5.0\mu m$, initial separation of the centroids $x_0 = 30\mu m$, and input velocities $v_l = -v_2 = 0.8\mu m/s$. The values of adimensional scattering length parameters $a_{11} = 0.6$, $a_{22} = 1.0$, and $a_{12} = a_{21} = -2.3$ were taken form ref. [5]. In figures (d) and (f) we show the same simulation but taking $a_{11} = a_{22} = 1.0$, and $a_{12} = a_{21} = -1$. In this case the effect of the collision is dramatic and both clouds almost completely spread.

Finally we have studied the evolution of two initially
with almost exact number of particles, the existence of the stationary state is not affected by the negative sign of \( a_{jk} \). However, if the number of particles is slightly altered (taking, for instance \( N_1 = 15430 \) and \( N_2 = 15750 \)), the condensate with less population splits off in two solitons and the other condensate remains oscillating, as shown in Fig. 4.

**Conclusions.** We have studied, through an approximate theoretical derivation and through numerical simulations, several consequences of the interaction of composite Bose-Einstein mutually interacting solitons. These systems give rise to a mutual trapping phenomena due to the soliton-soliton interaction forces. As a result of these interactions, we have predicted new phenomena not previously detected for the formation of stable vector matter-wave solitons, after a violent division, and have shown that there exist a critical initial velocity for these phenomena to occur. Depending on the values and signs of the scattering length coefficients, we have found several different switching regimes for matter wave solitons. Finally, we have calculated the effect of a repulsive cross-interaction in the stability of vector Bose-Einstein solitons.

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