Further one-loop results in $O(a)$ improved lattice QCD

Stefan Sint$^a$ and Peter Weisz$^b$

$^a$SCRI, The Florida State University, Tallahassee, Florida 32306–4130
$^b$Max-Planck-Institut für Physik, Föhringer Ring 6, D-80805 München, Germany

Using the Schrödinger functional we have computed a variety of renormalized on-shell correlation functions to one-loop order of perturbation theory. By studying their approach to the continuum limit we have determined the $O(a)$ counterterms needed to improve the quark mass and a number of isovector quark bilinear operators.

1. INTRODUCTION

Recent work by the ALPHA collaboration has focused on non-perturbative renormalization and on-shell $O(a)$ improvement of lattice QCD with Wilson quarks [1–7]. Various improvement coefficients could be determined as functions of the bare coupling $g_0$, including the coefficient of the Sheikholeslami-Wohlert term in the lattice action \[ \int \left( \bar{\psi} \gamma_\mu \frac{1}{2} \tau^a \gamma^\mu \psi \right) \mathrm{d}^4x \, . \]

While a non-perturbative determination of the improvement coefficients is clearly preferable perturbative estimates are nevertheless useful. First of all, any non-perturbative determination should establish contact with perturbation theory at sufficiently small values of the bare coupling constant. This provides a non-trivial check for the chosen strategy and criteria to assess the quality of a given improvement condition. Secondly, for those coefficients which account for lattice effects due to non-zero quark masses, perturbative estimates may indeed be satisfactory provided the quark masses are small when measured in lattice units.

In this contribution we present our one-loop results for the on-shell $O(a)$ improved isovector composite operators which are bilinear in the quark fields. The computational strategy and most of the results have already been published in refs. \cite{ref1,ref2}. In addition we here also include the results for the improved isovector tensor and scalar densities.

2. DEFINITIONS

We consider lattice QCD with $N_f \geq 2$ degenerate quark flavours of bare mass $m_0$ and shall assume that the action has already been on-shell improved \[ \bar{\mathcal{S}}^{\text{ren}} = \bar{\mathcal{S}}^{\text{bare}} + \mathcal{S}_\text{counter} \, . \] We are interested in the improvement of the following isovector operators,

\begin{align}
V^a_\mu(x) &= \bar{\psi}(x)\gamma_\mu \frac{1}{2} \tau^a \psi(x), \\
A^a_\mu(x) &= \bar{\psi}(x)\gamma_\mu \gamma_5 \frac{1}{2} \tau^a \psi(x), \\
P^a(x) &= \bar{\psi}(x)\gamma_5 \frac{1}{2} \tau^a \psi(x), \\
S^a(x) &= \bar{\psi}(x)\frac{1}{2} \tau^a \psi(x), \\
T^{a\mu \nu}_\mu(x) &= i\bar{\psi}(x)\sigma_{\mu \nu} \frac{1}{2} \tau^a \psi(x). \tag{5}
\end{align}

Here $\tau^a (a = 1, 2, 3)$ are the usual Pauli matrices acting in any two-flavour subspace and our conventions for the Dirac matrices are as in ref. \cite{ref3}.

In a mass independent renormalization scheme the renormalized (at renormalization scale $\mu$) and $O(a)$ improved counterparts of the above fields all take the form \[ (X = V, A, P, S, T) \, . \]

\[ X_R = Z_X(\bar{g}_0^2, a\mu) [1 + b_X(\bar{g}_0^2)am_q] X_I, \tag{6}\]

where $X_I$ stands for

\begin{align}
(V_I)_{\mu}^a &= V^a_\mu + c_V(\bar{g}_0^2)\alpha \partial_\mu T^{a\mu \nu}_\nu, \\
(A_I)_{\mu}^a &= A^a_\mu + c_A(\bar{g}_0^2)\alpha \partial_\mu P^a, \\
(T_I)_{\mu \nu}^a &= T^{a\mu \nu}_\mu + c_T(\bar{g}_0^2)\alpha(\partial_\mu V^a_\nu - \partial_\nu V^a_\mu), \tag{9}\end{align}

and otherwise $X_I = X$. Here $\partial_\mu$ denotes the symmetric lattice derivative and the parameter $\bar{g}_0$ is connected to the bare coupling $g_0$ through

\[ \bar{g}_0^2 = g_0^2[1 + b_\delta(\bar{g}_0^2)am_q], \tag{10}\]
where \( m_q = m_0 - m_c \) and \( m_c \) is the critical bare quark mass. Similarly one defines

\[
\bar{m}_q = m_q \left[ 1 + b_m (g_0^2) am_q \right],
\]

and the renormalized O(\(a\)) improved coupling and quark mass are then related to these parameters by

\[
\bar{g}_R^2 = \bar{g}_0^2 Z_{\bar{g}}(\bar{g}_0^2, a\mu),
\]

\[
\bar{m}_R = \bar{m}_q Z_m(\bar{g}_0^2, a\mu).
\]

The improvement coefficients \( b_g, b_m, b_X \) can be expanded in perturbation theory,

\[
b = b^{(0)} + b^{(1)} g_0^2 + O(g_0^4),
\]

and an analogous expansion exists for the coefficients \( c_A, c_V, c_T \).

### 3. COMPUTATIONAL STRATEGY

To determine the improvement coefficients we chose to compute a number of on-shell correlation functions derived from the Schrödinger functional (SF). The SF is the Euclidean functional integral for QCD on a finite space-time manifold where the (spatially periodic) quantum fields satisfy Dirichlet boundary conditions in the time direction [11,12]. For proper choice of the boundary conditions it can be shown that the lattice action has a unique absolute minimum [13]. The saddle point expansion about this minimum is then straightforward (albeit technically involved), and zero modes do not appear.

The gauge field boundary conditions imply that only global gauge transformations are allowed at the boundaries. Therefore, gauge invariant correlation functions can be defined where the quark and antiquark fields at the boundaries are separately projected onto their zero spatial momentum components. This is convenient because the perturbative expansion of such a correlation function starts with tree diagrams. Furthermore, exactly the same correlation functions can be used in numerical simulations.

In order to take the continuum limit in a finite space-time volume one fixes the time extent \(T\), the renormalized O(\(a\)) improved quark mass and all other dimensionful parameter in units of \(L\), the spatial extent of the space-time manifold. As a result O(\(a\)) lattice artefacts always appear as \(a/L\) effects and can be identified by varying the lattice size. In each renormalized correlation function these effects are cancelled by an a priori different linear combination of O(\(a\)) improvement coefficients. The finite volume provides a great flexibility here because many different renormalized correlation functions can be obtained by simple changes of the boundary conditions.

For completeness we mention that Dirichlet boundary conditions cause additional divergences and O(\(a\)) artefacts localised at the boundaries. These can be absorbed by renormalizing the boundary quark and antiquark fields in the same way as the composite fields in eq. (6), and by including additional O(\(a\)) boundary counterterms in the lattice action [11,12,3].

For details of our computational strategy and the definitions of most of the correlation functions the reader should consult refs. [9,10]. We have treated the case of the isovector tensor density in complete analogy to the improved vector current, and the computation of \(b_S\) involved a correlation function similar to \(f_V\) of ref. [5], where the vector current was replaced by the isovector scalar density.

| \(X\) | \(b_X^{(0)}\) | \(b_X^{(1)}\) | ref. |
|------|--------|--------|-----|
| \(g\) | 0 | 0.012000(2) \(\times \) \(N_f\) | [13] |
| \(m\) | \(-\frac{1}{2}\) | \(-0.07217(2) \times C_F\) | [13] |
| \(V\) | 1 | 0.11492(4) \(\times C_F\) | [13] |
| \(A\) | 1 | 0.11414(4) \(\times C_F\) | [13] |
| \(P\) | 1 | 0.11484(2) \(\times C_F\) | [13] |
| \(S\) | 1 | 0.14434(5) \(\times C_F\) | [13] |
| \(T\) | 1 | 0.10434(4) \(\times C_F\) | [13] |

Table 1. Improvement coefficients \(b\)

### 4. RESULTS

To one-loop order of perturbation theory we have carried out many consistency checks and...
thus confirm the general framework of $O(a)$ improvement as described in ref. [3].

Numerically we obtain ($C_F = (N^2 - 1)/2N$ for $N$ colours),

\[
\begin{align*}
  c_A &= -0.005680(2) \times C_F g_0^2 + O(g_0^4), \quad (15) \\
  c_V &= -0.01225(1) \times C_F g_0^2 + O(g_0^4), \quad (16) \\
  c_T &= 0.00896(1) \times C_F g_0^2 + O(g_0^4). \quad (17)
\end{align*}
\]

The coefficient $c_A$ has first been obtained in ref. [9], and $c_V$ was given in ref. [10]. The results and references for the $b$-coefficients are collected in table 1. Note that to order $g_0^2$ we find, within errors,

\[
b_S = -2b_m. \tag{18}
\]

In fact, it can be shown that eq. (18) is an exact identity in quenched QCD and furthermore the isoscalar scalar operator has the same $b$-coefficient as the isovector [14].

Comparison with the non-perturbative results for $c_A$ [11] and $c_V$ [12] shows that in these cases perturbative estimates are not accurate at large values of the bare coupling constant. In the case of $b_V$ we can compare with the non-perturbative result of ref. [13]. In figure 1 we see that the non-perturbative values (represented by the solid line) are quite a bit higher than the perturbative estimate (dotted line), even when Parisi’s boosted coupling [15] is used (crosses). However, using the boosted perturbative one-loop estimate at $g_0 = 1$ an error of only 1 per cent is induced in the normalisation factor $1 + b_V a m_q$ [cf. eq. (18)], provided $am_q \leq 0.05$.

It is of course not clear whether similar conclusions can be drawn for the other $b$-coefficients. Further non-perturbative results would obviously be welcome and some progress in this direction has been reported at this conference [16,17].

This work is part of the ALPHA collaboration research programme. We would like to thank M. Lüscher, S. Sharpe and U. Nierste for discussions. S. Sint acknowledges support by the U.S. Department of Energy (contracts DE-FG05-85ER250000 and DE-FG05-96ER40979).

REFERENCES

[1] K. Jansen et al., Phys. Letts. B372 (1996) 275