The Generalized Spherical Radon Transform and Its Application in Texture Analysis

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Abstract

The generalized spherical Radon transform associates the mean values over spherical tori to a function $f$ defined on $S^3 \subset H$, where the elements of $S^3$ are considered as quaternions representing rotations. It is introduced into the analysis of crystallographic preferred orientation and identified with the probability density function corresponding to the angle distribution function $W$. Eventually, this communication suggests a new approach to recover an approximation of $f$ from data sampling $W$. At the same time it provides additional clarification of a recently suggested method applying reproducing kernels and radial basis functions by instructive insight in its involved geometry. The focus is on the correspondence of geometrical and group features but not on the mapping of functions and their spaces.

1 Motivation and Introduction

In texture analysis, i.e. the analysis of preferred crystallographic orientation, the orientation probability density function $f$ representing the probability law of random orientations of crystal grains by volume is a major issue. In x-ray or neutron diffraction experiments spherical intensity distributions are measured which can be interpreted in terms of spherical probability distributions of distinguished crystallographic axes. In texture analysis, they are referred to as pole probability density functions. Generally, if $f$ is the orientation probability density function
of a random rotation represented by its random quaternion variable $Q$, then the probability density function of the random direction $QhQ^*$, where $h \in S^2 \subset \mathbb{R}^3$ is a fixed crystallographic direction, is provided by the 1-dimensional spherical Radon transform, integrating $f$ over all 1-dimensional great circles $C \subset S^3$ representing all rotations mapping $h$ onto a direction $r \in S^2$.

Extracting pole densities from raw intensity data collected in diffraction experiments requires several steps of processing, in particular defining the “background intensity” and correspondingly normalizing the intensity data. Any step in this process is largely subject to primary experimental errors. The influence of experimental errors can easily be reduced by considering mean intensities resulting from an additional rotation of the specimen during the measurements (Nikolayev, 2004). These mean values correspond to integration of spherical pole density functions over small circles. Thus, they correspond to mean values of $f$ over 2-dimensional spherical tori $T \subset S^3$ with core $C$ (Meister and Schaeben, 2004) which are called generalized spherical Radon transform (Helgason, 1994; 1999). Here, it is identified with the angle density function $W$ (cf. Bunge, 1969, 1982). Independently of the new experimental possibilities, they have proven to be instrumental for the inverse spherical Radon transform (Muller et al., 1981; Helgason, 1994; 1999).

2 Geometry of Rotations

One of the most beautiful features of quaternions is the role they play in the representation of the rotations of the low dimensional spaces $\mathbb{R}^3$ and $\mathbb{R}^4$. The description here is based on the book by Delanghe, Sommen and Souček (1992).

2.1 Rotations and Quaternions

The skew-field of quaternions $\mathbb{H}$ is generated by the elements $e_i$, $i = 1, 2, 3$, which fulfil the relations

(i) $e_i^2 = -1$, $i = 1, 2, 3$;
(ii) $e_1e_2 = e_3$, $e_2e_3 = e_1$, $e_3e_1 = e_2$;
(iii) $e_ie_j + e_je_i = 0$, $i, j = 1, 2, 3$; $i \neq j$. 

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The unit element of \( \mathbb{H} \) should be denoted by \( e_0 \). Then an arbitrary quaternion \( q \in \mathbb{H} \) can be represented as

\[
q = q_0 + q = q_0 e_0 + \sum_{i=1}^{3} q_i e_i = \text{Sc} \, q + \text{Vec} \, q,
\]

where \( q_0 = \text{Sc} \, q \) is the scalar part and \( q = \text{Vec} \, q \) is the vector part of \( q \). If \( \text{Sc} \, q = 0 \), then \( q \) is called a pure quaternion, the subset of all pure quaternions is denoted by \( \text{Vec} \, \mathbb{H} \). The subset of all quaternions with vanishing vector part may be denoted by \( \text{Sc} \, \mathbb{H} \). In this way \( \mathbb{R}^3 \) and \( \mathbb{R}^4 = \mathbb{R}^3 \oplus \mathbb{R} \) may be identified with \( \mathbb{H} \).

The quaternion

\[
q^* = q_0 - q = q_0 e_0 - \sum_{i=1}^{3} q_i e_i = \text{Sc} \, q - \text{Vec} \, q
\]

is called the conjugate of \( q \). The conjugation is an anti-involution, i.e. \((pq)^* = q^*p^*\), and

\[
qq^* = q^*q = ||q||^2 = \sum_{j=0}^{3} q_j^2,
\]

where \( ||q|| \) is the norm of the quaternion \( q \) which coincides with the norm of \( q \) in the Euclidean space \( \mathbb{R}^4 \). We denote the set of all unit quaternions, i.e. all quaternions with norm 1, by \( \mathbb{S}^3 \).

A linear transform \( T \) in \( \mathbb{R}^3 \) or \( \mathbb{R}^4 \) is called a rotation if and only if it leaves the scalar product in the Euclidean space \( \mathbb{R}^3 \) and the orientation invariant. Of course we have that \( T \) is invertible with \( \det T = 1 \). The set of all rotations (in the previous sense) in \( \mathbb{R}^3 \) and \( \mathbb{R}^4 \) is given by the special orthogonal group \( \text{SO}(3) \) and \( \text{SO}(4) \) respectively.

**Proposition 1.** Let \( q \in \mathbb{S}^3 \) and define the linear transform \( T_q : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) by

\[
T_q(x) = qxq^*.
\]

Then \( T \in \text{SO}(3) \) if and only if \( T = T_q \) for some \( q \in \mathbb{S}^3 \).

Thus any rotation in \( \mathbb{R}^3 \) can be represented by a unit quaternion \( q \) via \( qxq^* \). As it turns out, quaternions can also be used to represent rotations in \( \mathbb{R}^4 \), i.e. the group \( \text{SO}(4) \).
Proposition 2. Let \( q, p \in S^3 \) be unit quaternions and define the maps
\[
\Phi_q, \Psi_p : \mathbb{R}^4 \to \mathbb{R}^4 \quad \text{by} \quad \Phi_q(x) = qx \quad \text{and} \quad \Psi_p(x) = xp.
\]
Then \( T \in SO(4) \) if and only if there exist \( q, p \in S^3 \) such that \( T = \Theta_{q,p^*} \), where
\[
\Theta_{q,p^*} = \Phi_q \circ \Psi_{p^*} = \Psi_{p^*} \circ \Phi_q.
\]
Thus any rotation \( T \in SO(4) \) may be represented by \( T(x) = qxp^* \), where \( q, p \in S^3 \).

Proposition 3. Let be \( SO(4) \ni T = \Theta_{q,q^*} = T_q \). Then \( T \in SO(3) \).

Obviously,
\[
T(e_0) = \Theta_{q,q^*}(e_0) = T_q(e_0) = qe_0q^* = qq^* = 1 = e_0,
\]
i.e. \( T \) fixes the North Pole \( e_0 \) and hence \( T \in SO(3) \).

Remark 1. It is easily seen that \( S^3 \) is a double covering of \( SO(3) \) and \( S^3 \times S^3 \) is a double covering of \( SO(4) \).

Representing rotations by quaternions yields an instructive and geometrically appealing clarification of the geometry of rotations. Here we summarize the major results of Meister and Schaeben (2004) and add some explicit relationships which will prove helpful for our purposes.

2.2 Geometrical objects of \( S^3 \subset \mathbb{H} \) and of \( S^2 \subset \mathbb{R}^3 \)

Definition 1. Let \( q_1, q_2 \in S^3 \) be two orthogonal unit quaternions. The set of quaternions
\[
q(t) = q_1 \cos t + q_2 \sin t, \quad t \in [0, 2\pi),
\]
constitutes the great-circle denoted \( C(q_1, q_2) \subset S^3 \).

With
\[
q_1 = \frac{1 - rh}{\|1 - rh\|} \quad \text{and} \quad q_2 = \frac{h + r}{\|h + r\|} \quad (1)
\]
for \( h, r \in S^2 \) with \( r \neq -h \) it is obviously
\[
\text{Sc}(q(t)) = \cos \frac{\omega(t)}{2} = \cos \frac{\eta}{2} \cos t.
\]

Obviously, \((h, r)\) and \((-h, -r)\) define the same great circle \( C(q_1, q_2) \subset S^3 \).
Definition 2. Let \( q_1, q_2, q_3, q_4 \in \mathbb{S}^3 \) be four mutually orthonormal quaternions; let \( C(q_1, q_2) \) denote the circle spanned by quaternions \( q_1, q_2, \) and \( C(q_3, q_4) \) the orthogonal circle spanned by \( q_3, q_4. \) The set of quaternions
\[
q(s, t; \Theta) = \left( q_1 \cos s + q_2 \sin s \right) \cos \Theta + \left( q_3 \cos t + q_4 \sin t \right) \sin \Theta,
\]
\( s, t \in [0, 2\pi), \quad \Theta \in [0, \pi/2] \) \((2)\)
constitutes the spherical torus denoted \( T(C(q_1, q_2); \Theta) \subset \mathbb{S}^3 \) with core \( C(q_1, q_2). \)

Definition 3. Let \( r \in \mathbb{S}^2 \) be a unit vector and
\[
r(t) = \cos \frac{t}{2} + r \sin \frac{t}{2}, \quad t \in [0, 2\pi),
\]
represent the rotation about \( r \) by \( t \in [0, 2\pi) \). Then the set of unit vectors
\[
r'(t) = r(t) r'_0 r^*(t), \quad t \in [0, 2\pi), \quad (3)
\]
with \( r'_0 \in \mathbb{S}^2 \) in the plane spanned by \( h \) and \( r \) such that \( r \cdot r'_0 = \cos(\rho), h \cdot r'_0 = \cos(\eta - \rho) \) constitutes the small circle or cone
\[
c(r; \rho) = \{ r' \in \mathbb{S}^2 | r \cdot r' = \cos \rho \} \subset \mathbb{S}^2
\]
with angle \( \rho \) with respect to its centre \( r. \)

Its parametrized form explicitly reads (e.g. Altmann, 1986)
\[
r'(t) = r'_0 \cos t + (r \times r'_0) \sin t + (r \cdot r'_0) r (1 - \cos t), \quad t \in [0, 2\pi).
\]

With \( h \in \mathbb{S}^2 \) and
\[
h(t) = \cos \frac{t}{2} + h \sin \frac{t}{2}, \quad t \in [0, 2\pi),
\]
representing the rotation about \( h \) by \( t \in [0, 2\pi) \) and with \( h'_0 \in \mathbb{S}^2 \) in the plane spanned by \( h \) and \( r \) such that \( h \cdot h'_0 = \cos(\rho), r \cdot h'_0 = \cos(\eta + \rho) \) analogously results in
\[
h'(t) = h(t) h'_0 h^*(t), \quad t \in [0, 2\pi). \quad (4)
\]
2.3 Fibres

The fibre \( G(h, r) \subset SO(3) \) of all rotations with \( gh = r \) is represented by the circle \( C(q_1, q_2) \subset S^3 \) spanned by unit quaternions \( q_1, q_2 \in S^3 \) given in terms of \( h, r \in S^2 \) by Eqs. (1), for example. Therefore, the notation \( C_{h, r} \equiv C(q_1, q_2) \) is used where it is more instructive keeping in mind that \( C_{h, r} \equiv C_{-h, -r} \). Thus the major property of the circle \( C(q_1, q_2) \) is that it consists of all quaternions \( q(t), t \in [0, 2\pi) \), with \( q(t) h q^*(t) = r \) for all \( t \in [0, 2\pi) \), and that it covers the fibre \( G(h, r) \) twice; moreover, it is uniquely characterized by the pair \( (h, r) \in S^2 \times S^2 \) and its antipodally symmetric \((-h, -r)\).

The set of all rotations mapping \( h \) on the small circle \( c(r; \rho) \) is equal to the set of all rotations mapping all elements of the small circle \( c(h; \rho) \) onto \( r \) and represented by the spherical torus \( T(C(q_1, q_2); \frac{\rho}{2}) \subset S^3 \) with core \( C(q_1, q_2) \).

The distance \( d \) of an arbitrary \( q \in S^3 \) from the circle \( C(q_1, q_2) \) is given by

\[
d(q, C(q_1, q_2)) = \frac{1}{2} \arccos(q h q^* \cdot r)
\]

If \( d(q, C(q_1, q_2)) = \rho \), then \( q \) and \( C \) are called \( \rho \)-incident.

Then, the torus \( T(C(q_1, q_2); \frac{\rho}{2}) \) consisting of all quaternions with distance \( \frac{\rho}{2} \) from \( C(q_1, q_2) \) essentially consists of all circles with distance \( \frac{\rho}{2} \) from \( C(q_1, q_2) \) representing all rotations mapping \( u \) on \( c(r; \rho) \) and mapping \( c(h; \rho) \) on \( r \).

The representation of a torus can be factorized in the following way. Let \( (h, u), (r, v) \in S^2 \times S^2 \) with \( u \cdot v = \cos \rho \). Then \( C_1 \equiv C_{h, u} \subset S^3 \) and \( C_2 \equiv C_{r, v} \subset S^3 \) exist such that

\[
\begin{align*}
p_1 h p_1^* &= r_1 \text{ for all } p_1 \in C_1 \\
p_2 h p_2^* &= r_2 \text{ for all } p_2 \in C_2
\end{align*}
\]

Then \( p_2^* p_1 \) maps \( h \) on the small circle with centre \( r \) and angle \( \rho = \arccos(u \cdot v) \) as

\[
p_2^* p_1 h p_1^* p_2 \cdot r = p_1 h p_1 \cdot p_2^* p_2 = u \cdot v,
\]

i.e. \( \{p_2^* p_1 \mid p_i \in C_i, \ i = 1, 2\} \) represents the torus \( T(C; \rho/2) \) with core \( C = C_{h, r} \) and angle \( \rho = \arccos(u \cdot v) \). Let \( p \in T(C_{h, r}; \rho/2) \), then \( p h p^* = x \) with \( x \cdot r = \cos \rho \). Given \( u, v \in S^2 \) with \( u \cdot v = \cos \rho \) there exists a unique
$p_2 \in S^3$ such that $p_2x_2^* = u$ and $p_2r_2^* = v$. For any $p_1 \in C_{h,u}$ it is $p_1^*p_2ph^*p_2^*p_1 = h$ implying $p = p_2^*p_1$.

Since $q(t)h^*(t) = r$ implies that $q(t)h'^*(t) \cdot r = h' \cdot q^*(t)rq(t) = h'h$ for all $q(t) \in C(q_1,q_2)$, the set \{$(q(t)h'^*(t))$\} represents the small circle (cone) around $r$ with angle $t = \arccos hh'$. In parametrized form employing Eqs. (3), (4)

$$q(t)h'(u)q^*(t) = r'(u+2t) , \ t,u \in [0,2\pi), q(t) \in C(q_1,q_2) \quad \text{(6)}$$

(Meister and Schaeben, 2005), which may be rewritten for quaternions as $q(t) = r(u+2t)$.

Then the distance of $q(t) \in C(q_1,q_2)$ from an arbitrary circle $C_1$ representing all rotations mapping $h_1$ on $r_1$ is given by spherical trigonometry

$$d(q(t), C_1) = \frac{1}{2} \arccos\left(q(t)h_1q^*(t) \cdot r_1\right) \quad \text{(7)}$$

$$= \frac{1}{2} \arccos\left(hh_1 + rr_1 + \sqrt{(1 - (hh_1)^2)(1 - (rr_1)^2)} \cos t\right).$$

Eventually, the set of all circles $C(p_1,p_2) \subset S^3$ with a fixed distance $\frac{\rho}{2}$ of a given $q \in S^3$, i.e. the set of all circles tangential to the sphere $s(q; \rho/2)$ with centre $q$ and radius $\rho/2$, is characterized by

$$\frac{\rho}{2} = d\left(q, C(p_1,p_2)\right) = \frac{1}{2} \arccos\left(qh h^* \cdot r\right), \quad \text{(8)}$$

where $r \in S^2$ is uniquely defined in terms of $h$ and $p_1,p_2$ by $r := p(t)h^*(t)$ for all $p(t) \in C(p_1,p_2)$ and any arbitrary $h \in S^2$, i.e. each circle $C(p_1,p_2)$ represents all rotations mapping some $h \in S^2$ onto an element of the small circle $c(qh q^*; \rho)$. Thus, for each $q \in S^3$ and $\rho \in [0, \pi)$

$$\left\{ C(p_1,p_2) \mid d\left(q, C(p_1,p_2)\right) = \frac{\rho}{2} \right\} = \bigcup_{h \in S^2} \bigcup_{r \in c(qh q^*; \rho)} C_{h,r} \quad \text{(9)}$$

$$= \bigcup_{h \in S^2} \bigcup_{r \in c(qh q^*; \rho)} C(p_1(h,r),p_2(h,r))$$

$$= \bigcup_{h \in S^2_+} \bigcup_{r \in c(qh q^*; \rho)} C(p_1(h,r),p_2(h,r)),$$

where $S^2_+$ denotes the upper hemisphere of $S^2$. The last equation is due to the fact that $(h,r)$ and $(-h,-r)$ characterize the same great circle $C_{h,r} \equiv C_{-h,-r}$. 

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3 Radon transforms

3.1 The spherical Radon and generalized spherical Radon transform

Experimentally accessible crystallographic pole figures $P(h, r)$ of recorded diffracted x-ray or neutron beam intensities are modelled by

$$P(h, r) = \frac{1}{4\pi} \int_{G(h, r) \cup G(-h, r)} f(g) \, dg,$$

and interpreted as the probability density function that the crystallographic axis $\pm h$ statistically coincides with the specimen direction $r$ given the orientation probability density function $f: SO(3) \mapsto \mathbb{R}_+^1$.

Let $C$ denote the set of all $1$–dimensional totally geodesic submanifolds $C \subset S^3$. Each $C \in C$ is a $1$–sphere, i.e. a circle with centre $O$. Each circle is characterized by a unique pair of unit vectors $(h, r) \in S^2 \times S^2$ (and equivalently by its antipodally symmetric) by virtue of $q h q^* = r$ for all $q \in C$. Thus, referring to the quaternion representation, Eq. (10) can be rewritten in a parametric form as

$$P(h, r) = \frac{1}{8\pi} \int_{C \cup C^\perp} f(q) \, d\omega_1(q),$$

where the circle $C \subset S^3$ represents all rotations mapping $h \in S^2$ on $r \in S^2$ and where $C^\perp(q_1, q_2)$ is the orthogonal circle representing all rotations mapping $-h$ on $r$, and where $\omega_1$ denotes the usual one–dimensional circular Riemann measure.

Following Helgason (1994; 1999),

**Definition 4.**

$$\frac{1}{2\pi} \int_C f(q) \, d\omega_1(q) = \int_C f(q) \, dm(q) = (\mathcal{R} f)(C)$$

with the normalized measure $m = \frac{1}{2\pi} \omega_1$ is referred to as $1$–dimensional spherical (totally geodesic) Radon transform of $f$ whenever $f$ is integrable on each great circle.

The Radon transform of $f$ may be represented as the convolution of $f$ with the indicator function of the great circle $C$. It associates with the function $f$ its mean values over great circles $C \in C$. Since each great
circle is uniquely characterized by a pair \((h, r) \in S^2 \times S^2\) and its antipodally symmetric, we also use the notation \((\mathcal{R}f)(h, r) \equiv (\mathcal{R}f)(-h, -r)\) whenever it is more instructive.

It should be noted that no distinction has been made whether \(f\) refers to \(SO(3)\) or \(S^3\), even though the form of \(f\) depends on the representation of \(g\); in particular, with respect to \(S^3\) only even functions \(f\) could be orientation probability density functions as \(q \in S^3\) and \(-q\) represent the same orientation. Then

\[
P(h, r) = \frac{1}{4} \left( (\mathcal{R}f)(C_{h,r}) + (\mathcal{R}f)(C_{-h,r}) \right) = (\mathcal{X}f)(h, r), \quad (11)
\]

where \(\mathcal{X}f\) is referred to as basic crystallographic x-ray transform.

Further following Helgason (1994; 1999) the generalized 1-dimensional spherical Radon transform and the respective dual is well defined.

**Definition 5.** The generalized 1-dimensional spherical Radon transform of a real function \(f : S^3 \mapsto \mathbb{R}^1\) is defined as

\[
(\mathcal{R}^{(\rho)}f)(C) = \frac{1}{4\pi^2 \sin \rho} \int_{d(q,C) = \rho} f(q) \, dq.
\]

The generalized Radon transform of \(f\) may be represented as the convolution of \(f\) with the indicator function of the torus \(T\). It associates with \(f\) its mean values over the torus \(T(C, \rho)\) with core \(C\) and radius \(\rho\), Eq. (2).

Following Berens et al. (1968), Freeden et al. (1998, p. 64),

**Definition 6.** The spherically generalized translation of a real function \(F \in C(S^2)\) or \(F \in L^p(S^2), 1 \leq p < \infty\), is defined as

\[
(\mathcal{T}^{(\rho)}F)(r) = \frac{1}{2\pi \sqrt{1 - \cos^2 \rho}} \int_{rr' = \cos \rho} F(r') \, dr'.
\]

where \(2\pi \sqrt{1 - \cos^2 \rho}\) is the length of the circle \(c(r; \rho)\) centered at \(r\) with radius \(\cos \rho\).

It can be determined by

\[
(\mathcal{T}^{(\rho)}F)(r) = \frac{1}{2\pi \sqrt{1 - \cos^2 \rho}} \int_0^{2\pi} F(r'(t)) \, dt,
\]
with $r'(t)$ given according to Eq. (4).

When the translation $\mathcal{T}^{(\rho)}$ is applied to the Radon transform with respect to one of its arguments, then the geometry of rotations represented by quaternions amounts to

$$\left(\mathcal{T}^{(\rho)}[Rf]\right)(h, r) = \frac{1}{2\pi \sin \rho} \int_{c(h;\rho)} (Rf)(h', r) dh'$$

$$= \frac{1}{2\pi \sin \rho} \int_{c(r;\rho)} (Rf)(h, r') dr'$$ (12)

$$= \frac{1}{4\pi^2 \sin \rho} \int_{c(r;\rho)} \int_{C(q_1(h, r'), q_2(h, r'))} f(q) d\omega_1(q) dr'$$

$$= \frac{1}{4\pi^2 \sin \rho} \int_{T(C(q_1(h, r), C(q_2(h, r)); \xi))} f(q) dq$$ (13)

$$= \frac{1}{4\pi^2 \sin \rho} \int_{d(q, C(q_1(h, r), q_2(h, r))) = \frac{\pi}{2}} f(q) dq$$

$$= (R^{(\rho/2)}f)(C_h r).$$

Eq. (12), cf. (Bunge, 1969, p. 47; Bunge, 1982, p. 76), is an Ásgeirsson–type mean value theorem (cf. Ásgeirsson, 1937; John, 1938) justifying the application of $\mathcal{T}^{(\rho)}$ to $Rf$ regardless of the order of its arguments, and Eq. (13) is instrumental to the inversion of the spherical Radon transform (Helgason, 1994; 1999). We have just accomplished

**Proposition 4.** The generalized 1–dimensional spherical Radon transform is equal to the translated spherical Radon transform

$$\left(\mathcal{T}^{(\rho)}[Rf]\right)(h, r) = (R^{(\rho/2)}f)(C_h r)$$

and it can be identified with the angle density function

$$(Af)(h, r; \rho) := \frac{1}{2\pi \sin \rho} \int_{c(r;\rho)} (Rf)(h, r') dr'$$ (15)

The angle density function $(Af)(h, r; \rho)$ has been introduced in (Bunge, 1969, p. 44; Bunge, 1982, p. 74) (with a false normalization). It is the
probability density that the crystallographic direction $h$ statistically encloses the angle $\rho$, $0 \leq \rho \leq \pi$, with the specimen direction $r$ given the orientation probability density function $f$. It should be noted that

\[(A_f)(h, r; 0) = (R_f)(h, r), \quad (A_f)(h, r; \pi) = (R_f)(h, -r).\] (16)

Finally, with respect to the diffraction experiment, it should be noticed that

\[(T^\rho[X_f])(h, r) = \frac{1}{4}\left((R^{(\rho/2)}f)(C) + (R^{(\rho/2)}f)(C^\perp)\right)\]

\[= \frac{1}{4}\left((Af)(h, r; \rho) + (Af)(-h, r; \rho)\right)\]

\[= (W)(h, r; \rho)\] (17)

Thus with respect to a diffraction experiment, $(W)(h, r; \rho)$ is at once accessible if the specimen rapidly rotates around the specimen direction $r$ during the measurements.

### 3.2 Kernels and their twofold Radon transform

Now let $K$ be a kernel function $S^3 \times S^3 \mapsto \mathbb{R}^1$. Then we may apply the Radon transform twice in the sense that we apply it once with respect to the first and once with respect to the second variable (cf. Boogaart et al., 2005), i.e.

\[(R[K(\omega_1, \omega_2)])(C_1) = \frac{1}{2\pi} \int_{C_1} K(p_1, p_2) d\omega_1(p_1) = F(C_1, p_2)\]

and

\[(R[(R[K(\omega_1, \omega_2)])(C_1)])(C_2) = (R[F(C_1, \omega_2)])(C_2)\]

\[= \frac{1}{4\pi^2} \int_{C_2} \int_{C_1} K(p_1, p_2) d\omega_1(p_1) d\omega_1(p_2) = G(C_1, C_2),\]

where $C_1$ denotes the great circle representing all rotations mapping $h_1$ on $r_1$, and analogously $C_2$ with respect to $h_2$ and $r_2$.  

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In case the kernel $K$ depends only on the quaternion product $p_1^*p_2$, then

$$\text{(R}[(\mathcal{R}[K(\circ_1^*\circ_2)])(C_1)])(C_2) = \frac{1}{4\pi^2} \int_{p_1h_1p_1^*=r_1} \int_{p_2h_2p_2^*=r_2} K(p_1^*p_2) \, d\omega_1(p_1) \, d\omega_1(p_2)$$

$$= \frac{1}{4\pi^2 \sqrt{1 - (\mathbf{r}_1 \cdot \mathbf{r}_2)^2}} \int_{(\rho_{h_1^*h_2})=(\mathbf{r}_1, \mathbf{r}_2)} K(p) \, d\rho$$

$$= \frac{1}{4\pi^2 \sin(\rho/2)} \int_{(\mathcal{T}C;\rho/2)} K(p) \, d\rho,$$

with the core $C = C_{h_1,h_2}$ and $\rho = \arccos(\mathbf{r}_1 \cdot \mathbf{r}_2)$ according to Eq. (4).

In case the kernel $K$ depends only on $\omega/2 = \arccos(\text{Sc}(p_1^*p_2))$, then the function $F$ depends only on $\arccos(p_2h_1p_1^* \cdot \mathbf{r}_1) = d(p_2, C_1)$. As a function of the variable $p_2$ with parameters $h_1, \mathbf{r}_1$ implicitly provided by $C_1$, $F$ may be referred to as fibre function. They share most of the characteristics of ridge functions initially defined in linear spaces and discussed in (Donoho, 2000). Then the function $G$ depends only on $\arccos(h_1^*h_2)$ and $\arccos(\mathbf{r}_1 \cdot \mathbf{r}_2)$, cf. (Boogaart et al., 2005).

**Proposition 5.** For a kernel function of the form $K(p_1, p_2) = K(p_1^*p_2)$, its angle density function, i.e. its generalized spherical Radon transform, is identical with the twofold application of the spherical Radon transform with respect to the two components of its argument.

### 3.3 The dual spherical Radon and dual generalized spherical Radon transform

Following again Helgason (1994, 1999) we have two more definitions.

**Definition 7.** The dual 1–dimensional spherical Radon transform of a real continuous function $\varphi : \mathcal{C} \mapsto \mathbb{R}^1$ is defined as

$$({\tilde{\mathcal{R}}} \varphi)(q) = \int_{C \ni q} \varphi(C) \, d\mu(C),$$

where $\mu$ denotes the unique measure on the compact space $C \in \mathcal{C} : q \in C$, invariant under all rotations around $q$, and having total measure 1.
Thus \((\tilde{R}_R \varphi)(q)\) is the mean value of \(\varphi\) over the set of circles \(C \in \mathcal{C}\) passing through \(q\).

**Definition 8.** The dual generalized 1–dimensional spherical Radon transform of a real continuous function \(\varphi : \mathcal{C} \rightarrow \mathbb{R}^1\) is defined as

\[
(\tilde{R}^{(\rho)}\varphi)(q) = \int_{\{C \in \mathcal{C} : d(q,C) = \rho\}} \varphi(C) \, d\tilde{\mu}(C).
\]

With the normalized measure \(\tilde{\mu} = A^{-1}(\rho)d\mu\), \(A(\rho) = 4\pi \sin^2 \rho\), it is the mean value of \(\varphi\) over the set of all circles \(C \in \mathcal{C}\) with distance \(\rho\) from \(q\).

Again, geometrical reasoning, Eqs. (8), (9), yields that \((\tilde{R}^{(\rho)}[RF])(q)\) is the mean value of \((RF)\) over the set of all circles \(C \in \mathcal{C}\) with distance \(\rho\) from \(q\), i.e. tangential to the sphere \(s(q;\rho)\).

More specifically, with the usual two–dimensional spherical Riemann measure \(\omega_2\),

\[
(\tilde{R}^{(\rho)}[RF])(q) = \int_{d(q,C) = \rho} (RF)(C_{h,r}) \, d\mu(C_{h,r})
\]

\[
= \frac{1}{4\pi \sin(\rho/2)} \int_{S^2} \int_{c(qh,q^{\ast};\rho/2)} (RF)(C_{h,r}) \, dr \, d\omega_2(h)
\]

\[= \frac{1}{2} \int_{S^2} (Af)(h, qhq^{\ast}; \rho/2) \, d\omega_2(h), \quad (18)\]

\[= \frac{1}{2} \int_{S^2} (R^{(\rho)}f)(C_{h,qh^{\ast}}) \, d\omega_2(h)
\]

\[= \left(\tilde{R}[RF](\varphi)\right)(q), \quad (19)\]

which is instrumental for the inversion of the spherical Radon transforms.

In particular,

\[
\left(\tilde{R}[RF](\varphi)\right)(q) = \frac{1}{2} \int_{S^2} (RF)(h, qhq^{\ast}) \, d\omega_2(h), \quad (20)
\]

Next we want to describe the generalized dual Radon transform in a more group-theoretical way (cf. Helgason, 1994; 1999). Let \(C_0 \in \mathcal{C}\) be a fixed great circle with distance \(\rho\) from \(q\), i.e. \(d(q, C_0) = \rho\). Let \(\Theta_{r,s}\)
be the rotation that maps the North Pole $e_0$ to $q$, i.e. $q = re_0s^* = rs^*$. Since the distance on $S^3$ is rotational invariant we obtain
\[ d(q, C_0) = d(rs^*, C_0) = d(e_0, r^*C_0s). \]

Since all rotations in $SO(3)$ leave the North Pole $e_0$ invariant we conclude
\[ d(q, C_0) = d(e_0, kr^*C_0sk^*) = d(re_0s^*, rkr^*C_0sk^*s^*) = d(q, rkr^*C_0sk^*s^*). \]

Thus the set of all great circles with distance $\rho$ from $q$ is given by
\[ \{ rkr^*C_0sk^*s^*, k \in S^3, q = rs^* \}. \]

The previous consideration amounts to a new description of the generalized dual Radon transform:
\[
(\tilde{R}(\rho) \varphi)(q) = \int_{\{ C \in \mathcal{C} : d(q, C) = \rho \}} \varphi(C) d\mu(C) \\
= \int_{SO(3)} \varphi(rkr^*C_0sk^*s^*) dk,
\]
where $dk = \frac{1}{8\pi^2} \sin \beta d\alpha d\beta d\gamma$ is the invariant Haar measure on $SO(3)$, $C_0 \in \mathcal{C}$ a fixed great circle such that $d(q, C_0) = \rho$ and $q = rs^*$.

Analogously to the spherically generalized translation for function on $S^2$, we have

**Definition 9.** The spherically generalized translation of a real function $f \in C(S^3)$ or $F \in L^p(S^3), 1 \leq p < \infty$, is defined by the mean value operator
\[
(T(\rho)f)(q) = \frac{1}{A(\rho)} \int_{q^*p = \cos \rho} f(p) dp,
\]
where $A(\rho)$ denotes the surface area of the sphere $s(q; \rho)$ centered at $q \in S^3$ with radius $\cos \rho$. Then the following proposition can be shown.

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Proposition 6.

\[
\left( \tilde{\mathcal{R}}^{(\rho)} \left[ (\mathcal{R}f) (\circ) \right] \right) (q) = \int_{\rho \in C : d(q,\rho) = \rho} (\mathcal{R}f) (C) \, d\tilde{\mu}(C)
\]

\[
= \int_{C_0} (T^{d(q,p)} f) (q) \, dm(p), \quad (23)
\]

where \( C_0 \) is a fixed great circle with distance \( d(q, C_0) = \rho \) from \( q \).

\[
\left( \tilde{\mathcal{R}}^{(\rho)} \left[ (\mathcal{R}f) (\circ) \right] \right) (q) = \int_{\rho \in C : d(q,\rho) = \rho} (\mathcal{R}f) (C) \, d\tilde{\mu}(C)
\]

\[
= \int_{SO(3)} (\mathcal{R}f) (rkr^*C_0s^*k^*s^*) \, dk
\]

\[
= \int_{SO(3)} \int_{C_0} f(rkr^*ps^*k^*s^*) \, dm(p) \, dk
\]

\[
= \int_{C_0} \int_{SO(3)} f(rkr^*ps^*k^*s^*) \, dk \, dm(p)
\]

While \( k \) varies in \( SO(3) \), the set \( \{rkr^*ps^*k^*s^*, k \in SO(3)\} \) is a small sphere around \( q \) containing \( p \):

\[
d(q, rkr^*ps^*k^*s^*) = d(rs^*, rkr^*ps^*k^*s^*) = d(e_0, rkr^*ps^*)
\]

\[
= d(k^*e_0k, r^*ps) = d(e_0, r^*ps) = d(rs^*, p) = d(q, p),
\]

and \( dk = A^{-1}(\rho) \, dp \) is the normalized Riemann measure of the small sphere \( s(q; \rho) \) with center \( q \) and radius \( \rho = d(q, p) \). Hence

\[
\left( \tilde{\mathcal{R}}^{(\rho)} \left[ (\mathcal{R}f) (\circ) \right] \right) (q) = \int_{C_0} \int_{SO(3)} f(rkr^*ps^*k^*s^*) \, dk \, dm(p)
\]

\[
= \int_{C_0} \frac{1}{A(\rho)} \int_{s(p)\rho(q)} f(p) \, dp \, dm(p)
\]

\[
= \int_{C_0} (T^{d(q,p)} f) (q) \, dm(p). \quad (24)
\]

Remark 2. A special situation occurs when \( f \) is a central function. Assuming that \( f \) is central with respect to \( q_0 \), \( (\mathcal{R}f) \) is constant on the set of all great circles \( C \in \mathcal{C} \) for which \( q_0 h_C q_0^* \cdot r_C \), where \( h_C, r_C \) denote
the unit vectors associated to the great circle $C$. This is true for the set of all great circles with $d(q_0, C) = \rho$ as they are characterized by Eq. (8). Thus we can drop the outer integration and find for the right hand side of Eq. (23)

$$\left( \tilde{R}(\rho) \left[(\mathcal{R}f)(\circ)\right] \right)(q_0) = (\mathcal{R}f)(C) \text{ for some } C \text{ with } d(q_0, C) = \rho.$$ 

Moreover, in this case, $(\mathcal{T}^\rho f)(q_0) \equiv f(q_0)$ for all $\rho$. Thus the left hand side of Eq. (23)

$$\int_{d(q_0, C) = \rho} (\mathcal{T}^d(q_0, p) f)(q_0) \, dm(p) = (\mathcal{R}f)(C) \text{ for some } C \text{ with } d(q_0, C) = \rho.$$ 

Parameterizing the great circle $C_0$ by polar coordinates and assume $f$ even ($f(q) = f(-q)$) Eq. (24) takes the form

$$\left( \tilde{R}(\rho) \left[(\mathcal{R}f)(\circ)\right] \right)(q) = 4 \int_0^{\pi/2} (\mathcal{T}^d(q, p) f)(q) \, d\tau,$$

(25)

Denote by $q_0 \in C_0$ a point with minimum distance from $q$, i.e. $d(q_0, q) = \rho$. Using spherical trigonometry with respect to the triangle $qq_0p$ we obtain

$$\cos d(q, p) = \cos d(q_0, q) \cos d(q_0, p).$$

Fix $q$ and set $v = \cos d(q_0, q)$ and $u = v \cos d(q_0, p) = \cos d(q, p)$,

$$F(u) = (\mathcal{T}^d(q_0, p) f)(q), \quad \hat{F}(v) = \left( \tilde{R}(\rho) \left[(\mathcal{R}f)(\circ)\right] \right)(q).$$

Then Eq. (26) becomes an Abel’s integral equation

$$\hat{F}(v) = 4 \int_0^v \frac{F(u)}{\sqrt{v^2 - u^2}} \, du,$$

which is inverted by

$$F(u) = \frac{1}{2\pi} \frac{d}{du} \int_0^u \hat{F}(v) \frac{v}{\sqrt{u^2 - v^2}} \, dv.$$ 

Since $F(1) = \lim_{u \to 1} (\mathcal{T}^{\arccos u} f)(q) = f(q)$, we get the following inversion formula (cf. Helgason (1999))

$$f(q) = \frac{1}{2\pi} \left[ \frac{d}{du} \int_0^u \left( \tilde{R}^{\arccos v} \left[(\mathcal{R}f)(\circ)\right] \right)(q) \frac{v}{\sqrt{u^2 - v^2}} \, dv \right]_{u=1},$$

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and with Eq. (19) we obtain

\[ f(q) = \frac{1}{4\pi} \left[ \frac{1}{u} \int_0^u (Af)(h, qh^*; 2 \text{arccos } v) \, dv \right]_{u=1}, \]

which will be further transformed. First, we substitute \( t = v^2 \), getting

\[ \frac{1}{8\pi} \frac{1}{du} \int_0^{u^2} \int_{S^2} (Af)(h, qh^*; 2 \text{arccos } t) \, dh \frac{1}{\sqrt{u^2 - t}} \, dt \bigg|_{u=1} \]

Next, we put \( s = u^2 \) to get

\[ \frac{1}{4\pi} \frac{1}{ds} \int_0^{s} \int_{S^2} (Af)(h, qh^*; 2 \text{arccos } \sqrt{s - w}) \, dh \frac{1}{\sqrt{w}} \, dw \bigg|_{s=1}. \]

To shift the singularity, we set \( w = s - t \) which gives

\[ \frac{1}{4\pi} \frac{1}{ds} \int_0^{s} \int_{S^2} (Af)(h, qh^*; 2 \text{arccos } \sqrt{s - w}) \, dh \frac{1}{\sqrt{w}} \, dw \bigg|_{s=1}. \]

Now, differentiation gives

\[ = \frac{1}{4\pi} \left( \int_{S^2} (Af)(h, qh^*; \pi) \, d\omega_2(h) \right. \\
\quad + \left. \int_0^{s} \int_{S^2} \frac{d}{ds} (Af)(h, qh^*; 2 \text{arccos } \sqrt{s - w}) \, dh \frac{1}{\sqrt{w}} \, dw \right) \bigg|_{s=1} \]

Using \( \frac{d}{dt}(Af) = -\frac{d}{dw}(Af) \) and taking into account \( (Af)(h, qh^*; \pi) = (Rf)(h, -qh^*) \) (cf. Eq. (16)) and \( s = 1 \) result in

\[ = \frac{1}{4\pi} \left( (Rf)(h, -qh^*) \, d\omega_2(h) \right. \\
\quad - \left. \int_0^1 \int_{S^2} \frac{d}{dw} ((Af)(h, qh^*; 2 \text{arccos } \sqrt{s - w})) \, dh \frac{1}{\sqrt{w}} \, dw \right) \]

and finally put \( 2w = 1 - \cos \theta \) to obtain

\[ = \frac{1}{4\pi} \left( (Rf)(h, -qh^*) \, d\omega_2(h) \\
\quad + 2 \int_0^{\pi} \int_{S^2} \frac{d}{d\cos \theta} ((Af)(h, qh^*; \theta)) \, d\omega_2(h) \cos \frac{\theta}{2} \, d\theta \right). \]
4 Inversion of the generalized spherical Radon transform

For \( f \in C^\infty(S^3) \) the spherical Radon transformation \( \mathcal{R} : f \mapsto Rf \) has a kernel consisting of the odd functions, i.e. the functions satisfying \( f(q) + f(-q) = 0 \). Since \( q \) and \( -q \) represent the same rotation, and since \( f \) is a probability density function, we are interested in even functions \( f : S^3 \mapsto \mathbb{R}_+ \) only. In practical texture analysis, we are especially interested to recover \( f \) from data \( w_i = w(h_i, r_i; \rho_i) \) originating in sampling \( (Wf)(h, r; \rho) \) for discrete values of \( h, r \in S^2 \) and \( \rho \in (0, \pi/2) \). Obviously, due to the additional symmetry Eqs. (11), introduced by Friedel’s law, \( f \) cannot completely be recovered from integral intensity data \( \iota_i = P(h_i, r_i) \) nor by mean integral intensity data \( w_i = w(h_i, r_i; \rho_i) \).

In terms of spherical harmonics, only harmonic coefficients with respect to harmonics of even order can be determined from diffraction intensity data (Matthies, 1979).

4.1 Inversion of the spherical Radon transform

In texture analysis, i.e. in material science and engineering, the best known inversion formula dates back right to the beginning of “quantitative” texture analysis (Bunge, 1965; Roe, 1965). The formula may be rewritten in a rather abstract way as

\[
 f = \mathcal{F}_{SO(3)}^{-1} S \mathcal{F}_{S^2 \times S^2} \mathcal{R} f ,
\]

where \( S \) denotes a scaling matrix with entries \( \sqrt{2\ell + 1} \) indicating the ill-posedness of the inverse problem. It states that the harmonic coefficients of \( f \) are up to a scaling equal to the harmonic coefficients of its Radon transform. In the context of texture analysis and experimentally accessible “pole figures”, the former statement is true only for even-order coefficients.

The first analytical inversion formula was contributed by Matthies (1979) and then rewritten in terms of the angle density function (Muller et al., 1981).
In integral geometry the following inversion formulae are known

\[
 f = \frac{1}{4\pi} \int_{S^2} (-2\Delta_{S^2 \times S^2} + 1)^{1/2} (\mathcal{R} f)(h, gh) \, d\omega_2(h)
 = \frac{1}{2\pi} \tilde{R} \left[ (-2\Delta_{S^2 \times S^2} + 1)^{1/2} (\mathcal{R} f) \right]
\]

where \( \tilde{R} \) denotes the adjoint operator of \( \mathcal{R} \) with respect to \( L^2 \), and with respect to Eq. (22)

\[
 f = \frac{1}{2\pi} (-4\Delta_{S^3} + 1)^{1/2} \tilde{R} \mathcal{R} f
 = \frac{1}{2\pi} \left( (-4\Delta_{S^3} + 1)^{1/2} \tilde{R} \right) \mathcal{R} f
 = \frac{1}{2\pi} (-4\Delta_{S^3} + 1)^{1/2} \left( \tilde{R} \mathcal{R} f \right)
\]

cf. (Helgason, 1994; 1999). The equivalence of all these formula is shown in (Bernstein and Schaeben, 2005).

Comparing Eq. (28) with Eq. (29) with respect to Eq. (26) we may ask ourselves: Is

\[
 (-4\Delta_{S^3} + 1)^{1/2} \left( \tilde{R} \mathcal{R} f \right)
\]

a better conditioned inverse problem in practical applications than

\[
 \left( (-4\Delta_{S^3} + 1)^{1/2} \tilde{R} \right) \mathcal{R} f
\]

which is conditioned like

\[
 2\pi \mathcal{F}_{SO(3)}^{-1} \mathcal{S} \mathcal{F}_{S^2 \times S^2}
\]

Obviously, \((-4\Delta_{S^3} + 1)^{1/2}\) is worse conditioned than \((-4\Delta_{S^3} + 1)^{1/2} \tilde{R}\), and \(\tilde{R} \mathcal{R}\) is better conditioned than \(\mathcal{R}\). Thus, the question can be specified whether the differences just cancel out, or does the later improvement dominate? This problem will be pursued elsewhere.
4.2 Harmonic series expansion

In terms of spherical harmonics, Eqs. (22, 15) result in

\[
(Wf)(h, r; \rho) = \frac{1}{2\pi\sqrt{1 - \cos^2 \rho}} \int_{c(r; \rho)} \sum_{\ell=2l} \sum_{m,n} C_{\ell m} Y_{\ell m}^m(h) Y_{\ell m}^n(r') dr'
\]

\[
= \frac{1}{2\pi\sqrt{1 - \cos^2 \rho}} \sum_{\ell=2l} \sum_{m,n} C_{\ell m} Y_{\ell m}^m(h) \int_{c(r; \rho)} Y_{\ell m}^n(r') dr'.
\]

Applying

\[
\frac{1}{2\pi\sqrt{1 - \tau^2}} \int_{rr' = \tau} Y_{\ell m}^m(r') dr' = P_\ell(\tau) Y_{\ell m}^n(r)
\]

(Freeden et al., 1998, p. 64) with \(\tau = \cos \rho\) finally yields

\[
(Wf)(h, r; \rho) = \sum_{\ell=2l} P_\ell(\cos \rho) \sum_{m,n} C_{\ell m} Y_{\ell m}^m(h) Y_{\ell m}^n(r), \quad (30)
\]

(cf. Bunge, 1969, p. 45; Bunge, 1982, p. 74).

In case of an orientation probability function \(f(\arccos(\text{Sc}(q_0^* q)))\) which is radially symmetric with respect to a given \(q_0\), Eq. (30) simplifies further to

\[
(Wf)(h, r; \rho) = \sum_{\ell=2l} P_\ell(\cos \rho) \sum_{m,n} C_{\ell m} Y_{\ell m}^m(h) Y_{\ell m}^n(r)
\]

\[
= \sum_{\ell=2l} \frac{2\ell + 1}{4\pi} C_\ell P_\ell(\cos \rho) P_\ell(\cos \eta) \quad (31)
\]

as the Radon transform becomes a function of \(\eta = \arccos(q_0 h q_0^* \cdot r) = d(q_0, C)\) only, where \(C\) denotes the great circle representing all rotations mapping \(h\) on \(r\). It should be noted that \(\rho\) and \(\eta\) commute, i.e.

\[
(Wf)(h, r; \rho)\big|_{q_0 h q_0^* \cdot r = \cos \eta} = (Wf)(h, r; \eta)\big|_{q_0 h q_0^* \cdot r = \cos \rho}
\]

Integrating the radially symmetric spherical Radon transform over a small circle with angle \(\rho\) with respect to \(r\) with \(q_0 h q_0^* \cdot r = \cos \eta\) is equal to integrating the spherical Radon transform over a small circle with angle \(\eta\) with respect to \(r\) with \(q_0 h q_0^* \cdot r = \cos \rho\). In terms of the generalized spherical Radon transform we find
Proposition 7. For a radially symmetric function \( f \)
\[
(R^{(\rho/2)} f)(C_{h,r}|_{\theta=\cos \eta}) = (R^{(\eta/2)} f)(C_{h,r}|_{\theta=\cos \rho})
\]
Sampling \((Wf)(h,r;\rho)\) for discrete values of \( h, r \in S^2 \) and \( \rho \in (0,\pi/2) \) gives rise to data \( w_i = w(h_i,r_i;\rho_i) \) and a system of linear equations
\[
w_i = \sum_{\ell=2l} P_{\ell}(\cos \rho_i) \sum_{m,n} C_{\ell}^{mn} Y_{\ell}^m(h_i) Y_{\ell}^n(r_i),\quad i = 1, \ldots, I
\]
which may allow a solution for the \( C_{\ell}^{mn} \)-coefficients with even \( \ell \).

4.3 Convolution kernels and radial basis functions
Assume that the orientation probability density function shall be modeled by the superposition of “components” or kernels \( K_i(q) \), i.e.
\[
f(q) = \sum_j \lambda_j K_j(\tilde{q}, q) = \sum_j \lambda_j K(q_j, q).
\]
Usually in practice, we choose radially symmetric kernels which give rise to
\[
f(q) = \sum_j \lambda_j K(\arccos(\text{Sc}(q_j^* q))).
\]
Then the corresponding generalized spherical Radon transform \( Wf \) is the superposition of the correspondingly generalized spherically Radon transformed kernels \( k_j = WK_j \)
\[
(Wf)(h,r;\rho) = \sum_j \lambda_j (WK_j)(h,r;\rho) = \sum_j \lambda_j k_j(h,r;\rho),
\]
which may be fitted to the experimental data
\[
w_i \approx \sum_j \lambda_j^* k(h_i,r_i;\rho_i)
\]
in some sense, e.g. in the sense of a Hilbert-Sobolev norm as developed in (Boogaart et al., 2005). Then, an approximate orientation probability density function explaining the data is given by
\[
f(q) = \sum_j \lambda_j^* K(\arccos(\text{Sc}(q_j^* q))).
\]
5 Examples

In the following we provide some formulae for the Abel–Poisson (in probability: Cauchy) and the de la Vallée Poussin kernel, their Radon transform and their twofold Radon transform.

In texture analysis the Abel–Poisson kernel is referred to as Lorentz standard function (Matthies et al., 1987, p. 98; Matthies et al., 1990, p. 477), and the formulae were actually taken from there. Obviously, they were initially not related to reproducing kernels and their twofold spherical Radon transform. The de la Vallée Poussin kernel has been introduced into texture analysis because of its harmonic series expansion is finite (Schaeben, 1997; 1999).

For the symmetrical kernel $K(p_1, p_2) = K(p^*_1p_2)$ defined on $S^3 \times S^3$, the variable $\omega = 2 \arccos(\text{Sc}(p^*_1p_2))$ denotes the angle of the rotation of $p^*_1p_2$; for the Radon transformed kernel $RK_{S^3}(p, h, r)$ defined on $S^3 \times S^2 \times S^2$, the variable $\eta$ denotes the angle $\angle(p^*hp, r)$. The two variables $\eta_1$ and $\eta_2$ of the twofold Radon transformed kernel $RRK(h_1, r_1, h_2, r_2)$ defined on $(S^2 \times S^2) \times (S^2 \times S^2)$ correspond to the angles $\angle h_1h_2$ and $\angle r_1r_2$. The Gegenbauer respectively the Legendre coefficients of the kernels are denoted $a_\ell$.

In the table we have used the following notations for special functions: B Beta function, $2F_1$ hypergeometric function, and $\Gamma$ Gamma function.

For the Abel–Poisson kernel we have

$$a_\ell = \frac{(2\ell + 1)\kappa^{2\ell}}{2\ell + 1}$$
$$K = \frac{1}{2} \left[ \frac{1 - \kappa^2}{(1 - 2\kappa \cos(\omega/2) + \kappa^2)^2} + \frac{1 - \kappa^2}{(1 + 2\kappa \cos(\omega/2) + \kappa^2)^2} \right]$$
$$RK = \frac{1 - \kappa^4}{(1 - 2\kappa^2 \cos \eta + \kappa^4)^{3/2}}$$
$$RRK = \frac{2}{\pi (C - D)\sqrt{C + D}} \text{E}\left(\frac{2D}{C + D}\right)$$

where $C = 1 - 2\kappa \cos \eta_1 \cos \eta_2 + \kappa^2$ and $D = 2\kappa \sin \eta_1 \sin \eta_2$

Analogously, for the de la Vallée Poussin kernel

$$a_\ell = \frac{(2\ell + 1)\kappa^{2\ell}}{2\ell + 1} \left[ S_\ell(\kappa) - S_{\ell+1}(\kappa) \right]$$
where \( S_\ell(\kappa) = \sum_{k=0}^{\ell} (-1)^k \binom{2\ell}{2k} B(k + \frac{1}{2}, \kappa + \ell - k + \frac{1}{2}) \), and further

\[
K = \frac{B(3/2, 1/2)}{B(3/2, \kappa + 1/2)} \cos(\omega/2)^{2\kappa}
\]

\[
\mathcal{R}K = (1 + \kappa) \cos(\eta/2)^{2\kappa}
\]

\[
\mathcal{R}\mathcal{R}K = \frac{1}{\pi \cos \eta_1 \cos \eta_2} \frac{2^{1-\kappa}}{\Gamma(2+\kappa)} \frac{\Gamma(\frac{3}{2}+\kappa)}{\Gamma(3/2+\kappa)} \left( A^{1+\kappa} F_1\left(\frac{1}{2}, 1+\kappa, \frac{3}{2}+\kappa, \frac{A}{B}\right) - B^{1+\kappa} F_1\left(\frac{1}{2}, 1+\kappa, \frac{3}{2}+\kappa, \frac{B}{A}\right) \right)
\]

where \( A = 1 + \cos(\eta_1 + \eta_2) \) and \( B = 1 + \cos(\eta_1 - \eta_2) \).

6 Conclusions

The essential role of the probability density of the angle distribution for the inverse spherical “totally geodesic” Radon transform has been clarified by purely geometric arguments. It is identified with the generalized spherical Radon transform which in turn is identified with the “spherically translated” spherical Radon transform. Of particular interest is that the twofold spherical Radon transform of a symmetrical kernel function is again its corresponding angle probability density function. Practical methods of inversion in terms of harmonics or radially symmetric basis functions are sketched. Thus, our contribution is also a tribute to the late Hans-Joachim Bunge (1929 – 2004), who introduced the angle distribution into “quantitative texture analysis”. The problem whether the inversion of the generalized spherical Radon transform is better conditioned than the inversion of the spherical radon transform is postponed to a future contribution as it requires a detailed analysis of the experiment to collect integral radiation intensity data.

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