On cosmological solutions in a spontaneously
broken gauge theory

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Abstract
We consider solutions of the Yang-Mills-Higgs system coupled to gravity in asymptotically de Sitter spacetime. The basic features of two classes of solutions are discussed, one of them corresponding to magnetic monopoles, the other one to sphalerons. We find that although the total mass within the cosmological horizon of these configurations is finite, their mass evaluated at timelike infinity generically diverges. Also, no solutions exist in the absence of a Higgs potential.

Introduction.– Some time ago it has been found that spontaneously broken gauge theories admit classical, particle-like solutions. The monopole [1] and the sphaleron [2] are the best known examples and physically the most relevant. The magnetic monopoles inevitably arise in grand unification theories and are stabilized by a quantum number of topological origin, corresponding to their magnetic charge. Although the sphaleron solutions are unstable, they play an important role in electroweak theory, fixing the energy barrier separating topologically inequivalent vacua.

The effects of the gravitational self-interaction on magnetic monopoles and sphalerons have been addressed by many authors (see [3] for a review). Gravitating solutions exist up to some maximal value of the coupling constant $\alpha$ of the theory (which is proportional to the ratio of

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the vector meson mass and Planck mass). Apart from the solutions with a regular origin, there are also nonabelian “coloured” black holes, parametrized by their event horizon radius.

However, most of the investigations in the literature have been carried out on the assumption that spacetime is asymptotically flat. Less is known when the theory is modified to include a cosmological constant $\Lambda$ which greatly changes the asymptotic structure of spacetime \[4\]. While Einstein-Yang-Mills (EYM) configurations in asymptotic anti-de Sitter (AdS) space present a variety of new qualitative features \[5\], the solutions of a spontaneously broken gauge theory in AdS are rather similar to the asymptotically flat counterparts \[6\], \[7\]. Nontrivial solutions exist for any $\Lambda < 0$; as a new feature, one finds a complicated power decay of the fields at infinity and a decrease of the maximal allowed vacuum expectation value of the Higgs field.

For a positive cosmological constant, the natural ground state of the theory corresponds to de Sitter (dS) spacetime. This spacetime has gained a huge interest in theoretical physics recently for a variety of reasons. First of all, the observational evidence accumulated in the last years \[8\] is in favour of the idea that the physical universe has an accelerated expansion. The most common explanation is that the expansion is driven by a small positive vacuum energy (i.e. a cosmological constant $\Lambda > 0$). Furthermore, dS spacetime plays a central role in the theory of inflation. Another motivation for studying dS spacetime is connected with the proposed holographic duality between quantum gravity in dS spacetime and a conformal field theory on the boundary of dS spacetime (see \[9\] for a recent review of this subject).

In view of these developments, an examination of the classical solutions of gravitating fields in asymptotically dS spacetimes seems appropriate. The physical relevant case of a spontaneously broken nonabelian gauge theory is particularly interesting, since it presents particle like solutions with the same causal structure as dS spacetime. Here we argue that the features of these configurations are rather different as compared to the asymptotically flat of AdS counterparts.

The model.— We consider the action principle

$$S = \int \sqrt{-g} d^4x \left( \frac{1}{16\pi G} (R - 2\Lambda) + \mathcal{L} \right),$$

(1)
describing Einstein gravity with a cosmological term coupled to a Yang-Mills-Higgs (YMH) theory with compact gauge group $\mathcal{G}$ defined by the Lagrangian:

$$\mathcal{L} = -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (D_\mu \Phi)(D^\mu \Phi) - V(\Phi).$$

(2)

Here $F_{\mu\nu} \equiv F^{a}_{\mu\nu} T_a = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ is the gauge field strength, with the gauge field $A_\mu = A^a_\mu T_a$, $T_a$ being the anti-Hermitian gauge group generators and $g$ the gauge coupling constant. The Higgs field $\Phi$ is a vector in the representation space of $\mathcal{G}$ where the generators $T_a$ acts, with the covariant gauge derivative $D_\mu \Phi = (\partial_\mu + A_\mu) \Phi$.

For simplicity we restrict ourselves to $\mathcal{G} = \text{SU}(2)$, and a double-well Higgs potential $V(\phi) = \frac{1}{8} \lambda (\phi^2 - v^2)^2$. There are two cases to be considered, leading to rather different different types of solutions. The Higgs field can be chosen to be either in the real triplet representation, in which case $(T_a)_{ik} = -\epsilon_{aik}$ and we find monopole solutions, or in the complex doublet representation with $(T_a) = \tau_a/2i$ ($\tau_a$ being the Pauli matrices), with sphaleron solutions.
We consider spherically symmetric configurations, with a line element

\[ ds^2 = \frac{dr^2}{N(r)} + r^2(d\theta^2 + \sin^2 \theta d\varphi^2) - \sigma^2(r)N(r)dt^2 \tag{3} \]

where

\[ N(r) = 1 - \frac{2m(r)}{r} - \frac{\Lambda}{3}r^2, \tag{4} \]

and a gauge field ansatz

\[ A = w(r)(-T_2d\theta + T_1 \sin \theta d\varphi) + T_3 \cos \theta d\varphi. \tag{5} \]

For the scalar field, we take \( \Phi^a = \delta^a_3 \phi(r) \) for a Higgs triplet, and \( \Phi^a = \xi^a \phi(r) \), with some constant spinor \( \xi \), for a Higgs field in the doublet representation.

The reduced EYMH action can be expressed as

\[ S = \int dt \, dr \, \sigma \left[ \frac{m'}{4\pi G} - 4\pi \left( \frac{1}{g^2}(Nw'^2 + \frac{(1-w^2)^2}{2r^2}) + \frac{1}{2}N\sigma^2\phi'^2 + r^2V(\phi) + U(w,\phi) \right) \right], \tag{6} \]

with

\[ U(w,\phi) = w^2\phi^2 \quad \text{resp.} \quad U(w,\phi) = \frac{1}{4}(w+1)^2\phi^2 \tag{7} \]

for the triplet respectively doublet Higgs. It is important to notice that apart from the cosmological constant, the theory contains three mass scales, the Planck mass \( M_{Pl} = 1/\sqrt{G} \), the mass \( M_W = g\nu \) of the YM field and the mass \( M_H = \sqrt{\lambda}\nu \) of the Higgs field.

Varying the reduced action one obtains the EYMH equations

\[ m' = 4\pi G\left( \frac{1}{g^2}(\omega^2N + \frac{(\omega^2 - 1)^2}{2r^2}) + \frac{1}{2}N\sigma^2\phi'^2 + U + Vr^2 \right), \]

\[ \sigma' = \frac{8\pi G\sigma}{r}\left( \frac{1}{g^2}\omega^2 + \frac{1}{2}\phi'^2r^2 \right), \quad (N\sigma\omega')' = \sigma\left( \frac{\partial U}{\partial \phi} + r^2\frac{dV}{d\phi} \right), \tag{8} \]

where the prime indicates the derivative with respect to \( r \).

Restricting to solutions with a regular origin, we want the metric (3) to describe a nonsingular, asymptotically de Sitter spacetime outside a cosmological horizon located at \( r = r_c > 0 \). Here \( N(r_c) = 0 \) is only a coordinate singularity where all curvature invariants are finite. A nonsingular extension across this null surface can be found just as at the event horizon of a black hole, the Carter-Penrose conformal diagram being qualitatively identical to the de Sitter solution [10].

**Mass definition and asymptotic expansion.**— The computation of the mass of asymptotically dS monopoles and sphalerons is a difficult task due to the absence of spatial infinity and the globally timelike Killing vector. Also, these particle-like solutions typically strongly deform the dS geometry inside the cosmological horizon. Therefore, the perturbative approach
measuring the energy of fluctuations around the dS background proposed by Abbott and Deser may not be appropriate in this case [11]. However, these obstacles can be avoided by using the prescription proposed in [13] in which case the quasilocal tensor of Brown and York (augmented by the AdS/CFT inspired counterterms [14]), is evaluated on the Euclidean surfaces at future/past timelike infinity $I^\pm$. The conserved charge associated with the Killing vector $\partial/\partial t$ - now spacelike outside the cosmological horizon- is interpreted as the conserved mass-energy $\mathcal{M}$ [14]. This allows also a discussion of the thermodynamics of the asymptotically dS solutions outside the event horizon, the efficiency of this approach being demonstrated in a broad range of examples.

When applying this prescription to our case, we find that the asymptotic value of the metric function $m(r)$ determines the mass-energy of the monopole and sphaleron solutions, $\mathcal{M} = -\lim_{r \to \infty} m(r)$.

Following [12], one may also define a total mass $M_c$ inside the cosmological horizon. This can be done by integrating the Killing identity $\nabla^\mu \nabla_\nu K_\mu^\nu = R_{\nu \mu} K^\rho$, for the Killing field $K = \partial/\partial t$ on a spacelike hypersurface $\Sigma$ from the origin to $r_c$ to get the Smarr-type formula

$$M_c = \frac{1}{4\pi G} \int \nabla_\mu K^\mu d\Sigma_{\nu} = \frac{1}{4\pi G} \int \Lambda K^\mu d\Sigma_{\nu} + \int (2T_{\mu \nu} - T g_{\mu \nu}) K^\mu d\Sigma_{\nu}. \quad (9)$$

It is natural to identify the left-hand side as the total mass within the cosmological horizon. $M_c$ can also be rewritten as $M_c = -\kappa_c A_c/4\pi G = -r_c^2 \sigma(r_c) N'(r_c)/2G$, where $\kappa_c, A_c$ are the cosmological horizon surface gravity and area, respectively.

In the vicinity of the origin, the solutions resemble the well known flat space configurations, with $w(0) = 1, \phi(0) = 0$ and $m(0) = 0$. The existence of a regular cosmological event horizon at $r = r_c$ leads to the following conditions

$$m(r_c) = \frac{r_c}{2}(1 - \frac{\Lambda r_c^2}{3}), \quad (N' \sigma \omega') \bigg|_{r_c} = \sigma \left( \frac{\omega(\omega^2 - 1)}{r^2} + \frac{g^2}{2} \frac{\partial U}{\partial w} \right) \bigg|_{r_c}, \quad N' \sigma r^2 \phi' \bigg|_{r_c} = \sigma \left( \frac{\partial U}{\partial \phi} + r^2 \frac{dV}{d\phi} \right) \bigg|_{r_c}. \quad (10)$$

The boundary conditions at $r \to \infty$ are fixed by the requirements that the spacetime is asymptotically dS. When discussing the pure EYM system with $\Lambda > 0$, there are no restrictions on the asymptotic value of the gauge potential [10]. However, in the presence of a Higgs field, we find that the gauge field should approach asymptotically a fixed value $\omega_0$, which is zero for monopoles and $-1$ for sphalerons, while the Higgs field reaches its vacuum expectation value. This set of boundary conditions is shared also by asymptotically flat or AdS configurations.

However, the situation for $\Lambda > 0$ is more subtle, since the cosmological constant enters in a nontrivial way the solutions’ expression as $r \to \infty$. The analysis of the scalar field asymptotics is standard; Strominger’s mass bound is $M_S^2 = 3\Lambda/4$ [19] and separates the infinite energy solutions from solutions which may present a finite mass (this would depend also on the gauge field behaviour). For small enough values of the Higgs field mass, $M_H < M_S$ the scalar field decays as

$$\phi(r) \sim v + c_1 r^{-\frac{2}{3}}(1 + \sqrt{1 - M_H/M_S}), \quad (10)$$

which assures a finite contribution to the total mass-energy $\mathcal{M}$. 

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For a Higgs mass exceeding Strominger’s bound, the scalar field behaves asymptotically as
\[ \phi(r) \sim v + c_2 r^{-3/2} \sin \left( \frac{3}{2} \sqrt{M_H^2/M_S^2 - 1} \log r + c_3 \right) \]
which leads to a logarithmic divergence in the asymptotic expression of the mass function \( m(r) \). One may think that this bound may be circumvented by solutions with a vanishing Higgs potential. However, by rewriting the Higgs field equation in the form
\[ \frac{1}{2} (N r^2 (\phi^2)')' = \sigma (N r^2 \phi^2 + \phi \frac{\partial U}{\partial \phi} + \phi \frac{dV}{d\phi}), \]
and integrating it between the origin and the cosmological horizon, it can easily be proven that no nontrivial solutions exist for \( V(\phi) = 0 \) or for a convex potential.

A similar analysis reveals that a positive cosmological constant sets another mass bound for the gauge sector, which is \( M_b = \sqrt{\Lambda/12} \) for monopoles and \( M_b = \sqrt{\Lambda/3} \) for sphalerons. Asymptotically dS solutions with a finite mass-energy exist for \( M_W < M_b \), in which case the expression of the gauge field as \( r \to \infty \) is
\[ w(r) \sim w_0 + c_4 r^{-3/2} \left( 1 + \sqrt{1 - M_W^2/M_b^2} \right), \]
which contrasts with the exponential decay found in an asymptotically flat spacetime. For \( M_W > M_b \), the large \( r \) behaviour of the gauge field is \( w(r) \sim w_0 + c_5 r^{-1/2} \sin \left( \frac{1}{2} \sqrt{M_W^2/M_b^2 - 1} \log r + c_6 \right) \) which leads to an infinite mass-energy \( \mathcal{M} \) of the configurations (the constants \( c_i \) which enter the above relations are free parameters). The solutions with \( M_H = M_S, M_W = M_b \) saturate these bounds and lead also to infinite mass configurations. Once we know the asymptotics of the matter fields, the corresponding expression for the metric functions results straightforwardly from the equations (8).

**Numerical solutions.**— The solutions of the equations (8) are evaluated numerically. With the boundary conditions discussed above, the procedure is to integrate separately between the origin and cosmological horizon and from the cosmological horizon to infinity, matching the solutions at \( r = r_c \).

The usual rescaling \( r \to gvr, \phi \to \phi/v \) reveals the existence of two dimensionless parameters \( \alpha \) and \( \beta \), expressible through the mass ratios \( \alpha = M_W/M_P, \beta = M_H/M_W \). The third parameter of the system is the rescaled cosmological constant \( \Lambda \to \Lambda G/g^2 v^2 \). The configurations with \( \alpha = 0 \) correspond to monopoles and sphalerons in a fixed dS background, and contain already the basic features of the theory.

The equations of motion (8) were solved varying \( \Lambda \) for a range of values of the coupling parameter \( \alpha \) and several values of \( \beta \). While a negative cosmological constant exerts an additional pressure on solitons, causing their typical radius to become thinner [6, 7], a positive \( \Lambda \) has the opposite effect, causing the soliton radius to expand beyond the value it would have in asymptotically flat space. Also, as \( \alpha \) increases, the cosmological horizon shrinks in size. The dS solitons are generally not confined inside the cosmological horizon, with all variables and their first derivatives extending smoothly through the cosmological horizon. The profiles of typical solutions are presented in Figure 1.

When \( \Lambda \) is increased from zero, while keeping \( \alpha, \beta \) fixed, a branch of dS solutions emerges from the corresponding asymptotically flat configurations. This branch ends at a maximal value
Figure 1. Typical asymptotically de Sitter monopole (figure 1a) and sphaleron solutions (figure 1b).

$\Lambda_{max}$. A second branch of solutions always appears at $\Lambda_{max}$, extending backwards in $\Lambda$ to a zero value of the cosmological constant (for monopoles) or to some small $\Lambda_c \neq 0$ for sphalerons. In this limit, the trivial solution $\phi(r) = 0, w(r) = 1$ is approached. The value of $\Lambda_{max}$ depends on the parameters $\alpha, \beta$; for example, for solutions with $\beta = 0.1$ in a fixed dS background, we find $\Lambda_{max} \approx 0.069$ for monopoles and $\Lambda_{max} \approx 0.0506$ for sphalerons. The value of $\Lambda_{max}$ is only slightly affected by changing $\alpha$, e.g. for solutions with $\alpha = 1, \beta = 0.1$ we find $\Lambda_{max} \approx 0.067$ for monopoles and $\Lambda_{max} \approx 0.0505$ for sphalerons.
These statements are illustrated in Figure 2 where some numerical data is plotted as function of $\Lambda$ for the two branches, respectively for dS monopoles and dS sphalerons. The figures are obtained for $\alpha = 1$, $\beta = 0.1$ but they remain qualitatively the same for all gravitating solutions we considered. The maximal values of $\Lambda$ are always below the critical values found for solutions in fixed dS background, and as a result the mass of our solutions measured at timelike infinity always diverges, although the mass $M_c$ within the cosmological horizon stays finite (see also Figure 1). The existence of other disconnected branches of solutions for $\Lambda > \Lambda_{\text{max}}$ appears
unlikely. Note that the EYM theory also presents solutions with dS asymptotics only for values of the cosmological constant up to some $\Lambda_{\text{max}} < 3/4$ [10].

For a given value of $\Lambda < \Lambda_{\text{max}}$, we notice the existence of a maximal value of $\alpha$, which depends on $\beta$. The behavior of solutions as $\alpha \to \alpha_{\text{max}}$ is similar to the asymptotically flat case. The gravitating monopoles separate in this limit into an interior region with a smooth origin and a nontrivial YM field, and an exterior extremal Reissner-Nordström-dS solution with $w = 0$. Different from the monopole case, the sphaleron solutions may be continued all the way back to $\alpha = 0$, where we end up with a cosmological EYM solution.

**Conclusions.**—In this letter we discussed the basic properties of the monopole and sphaleron solutions in an asymptotically dS spacetime. Contrary to the naive expectation that a small $\Lambda$ will not affect the properties of the configurations drastically, we find that the mass of dS solutions evaluated at timelike infinity by using the quasilocal tensor of Brown and York diverges (although the mass within the cosmological horizon stays finite). These features are shared also by the black hole counterparts of the soliton solutions, which can be constructed by using the same techniques.

A divergent ADM mass has been found also for solutions of some theories in asymptotically AdS spacetime. However, in some cases it is still possible to obtain a finite mass by allowing the regularizing counterterms to depend not only on the boundary metric but also on the matter fields on the boundary [17]. It would be interesting to generalize this method to the dS case and to assign a finite mass (evaluated outside the cosmological horizon) to the solutions of a spontaneously broken gauge theory.

We believe that this may lead to further understanding of the rich structure of a field theory in dS space as well as profound implications to the evolution of the early universe.

An extensive analysis of the solutions with variation of the parameters of the theory, as well as black hole configurations, will be presented in a separate publication.

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