IRREDUCIBLE REPRESENTATIONS OF LEAVITT ALGEBRAS

ROOZBEH HAZRAT, RAIMUND PREUSSER, AND ALEXANDER SHCHEGOLEV

Abstract. For a weighted graph $E$, we construct representation graphs $F$, and consequently, $L_K(E)$-modules $V_F$, where $L_K(E)$ is the Leavitt path algebra associated to $E$, with coefficients in a field $K$. We characterise representation graphs $F$ such that $V_F$ are simple $L_K(E)$-modules. We show that the category of representation graphs of $E$, $\mathbf{RG}(E)$, is a disjoint union of subcategories, each of which contains a unique universal object $T$ which gives an indecomposable $L_K(E)$-module $V_T$ and a unique irreducible representation graph $S$, which gives a simple $L_K(E)$-module $V_S$.

Specialising to graphs with one vertex and $m$ loops of weight $n$, we construct irreducible representations for the celebrated Leavitt algebras $L_K(n,m)$. On the other hand, specialising to graphs with weight one, we recover the simple modules of Leavitt path algebras constructed by Chen via infinite paths or sinks and give a large class of non-simple indecomposable modules.

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1. Introduction

In a series of papers [21, 22, 23], William Leavitt studied algebras that are now denoted by $L_K(n, n + k)$ and have been coined Leavitt algebras. Let $X = (x_{ij})$ and $Y = (y_{ji})$ be $n \times (n + k)$ and $(n + k) \times n$ matrices consisting of symbols $x_{ij}$ and $y_{ji}$, respectively. Then for a field $K$, $L_K(n, n + k)$ is the unital $K$-algebra generated by all $x_{ij}$ and $y_{ji}$ subject to the relations $XY = I_n$ and $YX = I_{n+k}$. Leavitt established that these algebras are of type $(n, k)$. Recall that a ring $A$ is of type $(n, k)$ if $n$ and $k$ are the least positive integers such that $A^n \cong A^{n+k}$ as right $A$-modules. He further showed that $L_K(1, k + 1)$ are (purely infinite) simple rings and $L_K(n, n + k)$, $n \geq 2$ are domains (see also [11]).

A Leavitt path algebra, first introduced in [1, 8], is a certain quotient of the path algebra of a directed graph. When the graph consists of one vertex, $k + 1$-loops, and their doubles, its Leavitt path algebra is $L_K(1, k + 1)$. The Leavitt path algebras $L_K(E)$ associated to the graphs $E$ with coefficients in the field $K$ in general and the algebras $L_K(1, k + 1)$ in particular turned out to be a very rich and interesting class of algebras whose studies so far have constitute over 150 research papers and counting. A comprehensive treatment of the subject can be found in the book [2].

There have been a substantial number of papers devoted to (irreducible) representations of a Leavitt path algebra $L_K(E)$, i.e., (simple) $L_K(E)$-modules. Starting with a path algebra, the celebrated theorem of Gabriel characterises graphs $E$ whose path algebras $KE$ have a finite number of isomorphism classes of indecomposable representations. For Leavitt path algebras, Ara and Brustenga [5, 6] studied their finitely presented modules, proving that the category of finitely presented modules over a Leavitt path algebra $L_K(E)$ is equivalent to a quotient category of the corresponding category of modules over the path algebra $KE$. A similar statement for graded modules over a Leavitt path algebra was established by Paul Smith [32], allowing for making a bridge to study finite dimensional algebras with radical square zero. On the other hand, by introducing the notion of branching systems for a graph $E$, Gonçalves, and Royer [14] could construct representations for $L_K(E)$, providing sufficient conditions when these representations are faithful.

In [10], starting from a one-sided infinite path $p$ in a graph $E$, Chen considered the $K$-vector space $V_{[p]}$ with basis given by the infinite paths tail equivalent to $p$. Observing that this vector space is endowed with a natural action of $L_K(E)$, he proved that $V_{[p]}$ is a simple $L_K(E)$-module. A similar construction was also given for paths ending on a sink vertex.

Numerous work followed, noteworthy the work of Ara-Rangaswamy and Rangaswamy [30, 31, 9] producing new simple modules associated to infinite emitters and characterising those algebras which have countably (finitely) many distinct isomorphism classes of simple modules. Abrams, Mantese and Tonolo [3] studied the projective resolutions for these simple modules. The recent work of Ánh and Nam [4] provides another way to describe the so-called Chen and Rangaswamy simple modules. We note that these simple modules can also be recovered via the setting of Steinberg algebras (see [7, 25, 33]).

The algebras $L_K(n, n + k)$, for $n > 1$, $k \geq 0$, can not be obtained via Leavitt path algebras. For this reason, the weighted Leavitt path algebras were introduced in [17] which give $L_K(n, n + k)$ for a weighted graph with one vertex and $n + k$ loops of weight $n$ (Example 18). If the weights of all the edges are 1 (i.e., the graph is unweighted), then the weighted Leavitt path algebras reduce to the usual Leavitt path algebras (Example 19).
In this note for a weighted graph $E$ we construct a representation graph $F$, and consequently a representation $V_F$ for the weighted Leavitt path algebra $L_K(E)$. We characterise the representation graphs $F$ such that $V_F$ are irreducible representations, i.e., they are simple $L_K(E)$-modules. A graph $F$ is a representation graph for $E$, if there is a graph homomorphism $F \to E$ such that, roughly speaking, each edge of $E$ ‘uniquely’ lifts to $F$ (Definition 2). Then $V_F$ is generated as a $K$-vector space by vertices of $F$ as the basis and the action of $L_K(E)$ on each vertex $v$ is uniquely determined by moving $v$ on the graph $F$ along the edges determined by the monomials of $L_K(E)$.

Specialising to graphs with one vertex and $m$ loops of weight $n$, we can construct irreducible representations for the classical Leavitt algebras $L_K(n,m)$. As an example, the algebra $L_K(2,3)$ can be obtained as a Leavitt path algebra of a weighted graph with one vertex and two loops of weight 3:

$$f_1f_2f_3 \rightarrow v \rightarrow e_1e_2e_3.$$  

The representation graph $F$ below gives rise to a simple $L(2,3)$-module $V_F$, as follows:

$$F : \begin{array}{c}
v_0 \xrightarrow{f_2} v_1 \xrightarrow{f_1} v_2 \xrightarrow{f_3} v_3 \xrightarrow{f_2} v_4 \xrightarrow{f_1} v_5 \xrightarrow{f_3} v_6 \xrightarrow{f_2} v_7 \xrightarrow{f_1} v_8 \xrightarrow{f_3} v_9 \xrightarrow{f_2} v_{10} \ldots \end{array}$$

Here $V_F$ is a $K$-vector space with basis $\{v_i\}_{i \in \mathbb{N}_0}$ with the action of $e_1, e_1^* \in L_K(2,3)$ on $v_k$ defined by $v_k e_1 = r(e_i)$ and $v_k e_1^* = s(e_i)$ and similarly for $f_1$'s. Here $e_1$ and $v_k$, uniquely determine an edge $e_i$ in $F$ with $s(e_i) = v_k$ and thus $v_k e_1$ slides $v_k$ along the edge $e_i$ to $r(e_i)$. Therefore $v_5 e_3 = v_9$ and $v_6 f_2^* = v_4$.

On the other hand, specialising to graphs with weight one, we recover simple modules of Leavitt path algebras constructed by Chen via infinite paths. Our approach gives a completely new way to represent simple modules of these algebras. Besides being more visual, this approach allows for carrying calculus on these modules with ease.

As an example, for the graph

$$\begin{array}{c}
\xrightarrow{e} f \\
\xrightarrow{g}
\end{array}$$

the infinite paths $p = efgefg\ldots$ and $q = eef^2ef^3\ldots$, give rise to Chen simple $L_K(E)$-modules $V_{[p]}$ and $V_{[q]}$. Using our approach, the representation graphs $F$ and $G$ below give rise to simple $L_K(E)$-modules $V_F$ and $V_G$ such that $V_F \cong V_{[p]}$ and $V_G \cong V_{[q]}$.

$$F : \begin{array}{c}
v_0 \xrightarrow{f} v_1 \xrightarrow{g} v_2 \xrightarrow{f} v_3 \xrightarrow{g} v_4 \xrightarrow{e} v_5 \xrightarrow{g} v_6 \xrightarrow{e} v_7 \xrightarrow{g} v_8 \xrightarrow{e} v_9 \xrightarrow{g} \ldots \end{array}$$
Here $V_F$ is the $K$-vector space with basis $\{v_i \mid i \in \mathbb{N}\}$ and the action of edges slides the vertices along the graph $F$ such as $v_1 ef g = v_1$ and $v_6 e gef^* = v_6$.

We will next study the functor from the category of representation graphs of the graph $E$ (see §2.3) to the category of (right) $L_K(E)$-modules, arising from our construction:

$$V : \text{RG}(E) \longrightarrow \text{Mod } L_K(E),$$

$$(F, \phi) \mapsto V_F.$$

We show that $\text{RG}(E)$ can be written as a disjoint union of certain subcategories, each of which contains a unique universal representation and a unique irreducible representation of $E$, up to isomorphism. We show that the unique universal representation $T$ of each of these subcategories gives an indecomposable $L_K(E)$-module $V_T$, whereas the irreducible representation $S$, gives a simple $L_K(E)$-module $V_S$.

Next in Section 5 we describe branching systems for weighted graphs and show how the representation graphs give rise to examples of branching systems. Branching systems for Leavitt path algebras were systematically studied by Gonçalves and Royer ([14, 13, 12]).

As expounded by Green [16], one can describe the module category of a certain class of quotient path algebras $A_K(E, r) := KE/\langle r \rangle$, where $KE$ is the path algebra of the finite graph $E$ with coefficients in the field $K$ and $r$ is a set of certain relations, via the following equivalence of categories

$$\text{Mod } A_K(E, r) \longrightarrow \text{Rep}(E, r).$$

Here $\text{Mod } A_K(E, r)$ is the category of right $A_K(E, r)$-modules and $\text{Rep}(E, r)$ is the category of representations of the graph $E$ with relations as described in [16]. The objects of the category $\text{Rep}(E, r)$ consist of placing arbitrary $K$-vector spaces on the vertices of the graph $E$ and assigning linear transformations to the edges that satisfy the relations $r$ (see Appendix A). Since (weighted) Leavitt path algebras are among this class, this gives a justification of why branching systems would give representations for (weighted) Leavitt path algebras. However it is not clear how this general machinery of (5) can be used to systematically describe irreducible or indecomposable representations of such algebras as the delicate case of acyclic graphs with no relations, which gives the celebrated Gabriel theorem of indecomposability, shows.

As such we believe that the notion of representation graphs of this paper allows us, for the first time, to produce irreducible and indecomposable representations for a wide class of algebras, such as Leavitt algebras $L_K(n, m)$.

Throughout the paper $K$ denotes a field and $K^* := K \setminus \{0\}$. By a $K$-algebra we mean an associative (but not necessarily commutative or unital) $K$-algebra. The semigroup of positive integers is denoted by $\mathbb{N}$ and the monoid of non-negative integers by $\mathbb{N}_0$. 

\[4\]
2. Representation graphs of a given graph

A directed graph $E$ is a tuple $(E^0, E^1, r, s)$, where $E^0$ and $E^1$ are sets and $r, s$ are maps from $E^1$ to $E^0$. We think of each $e \in E^1$ as an edge pointing from vertex $s(e)$ to vertex $r(e)$. In this paper all directed graphs are assumed to be row-finite, i.e. no vertex emits infinitely many edges.

A weighted graph is a directed graph $E$ equipped with a map $w : E^1 \rightarrow \mathbb{N}$. If $e \in E^1$, then $w(e)$ is called the weight of $e$. We write $(E, w)$ to emphasise that the graph is weighted. Throughout we develop our concepts in the setting of weighted graphs. When the weight map is the constant map $1$, i.e., $w(e) = 1$ for any $e \in E^1$, we retrieve the notions in the classical case of directed graphs.

The main notion introduced and studied in this paper in relation with the theory of Leavitt path algebras, is the notion of a representation graph of a given weighted graph. The concept is closely related to the theory of covering and immersions in graph theory [34, 20]. We start by recalling these notions in the setting of weighted graphs.

2.1. Covering and immersions. For weighted graphs $E$ and $G$, a weighted graph homomorphism $\phi : E \rightarrow G$, consists of two maps $\phi^0 : E^0 \rightarrow G^0$ and $\phi^1 : E^1 \rightarrow G^1$ such that for any $e \in E^1$, $s(\phi^1(e)) = \phi^0(s(e))$, $r(\phi^1(e)) = \phi^0(r(e))$ and $w(\phi^1(e)) = w(e)$. Namely, $\phi$ is a homomorphism of graphs which preserve the weight. For a vertex $v \in E^0$, we define$$w(v) := \max\{w(e) \mid e \in s^{-1}(v)\}.$$If $v$ is a sink we set $w(v) = 0$.

**Definition 1.** Let $E$ and $T$ be weighted graphs and $\phi = (\phi^0, \phi^1) : T \rightarrow E$ a homomorphism of weighted graphs.

1. The pair $(T, \phi)$ is called an immersion in $E$, if for any $v \in T^0$, the map $\phi : s^{-1}(v) \rightarrow s^{-1}(\phi^0(v))$ is injective.

2. The pair $(T, \phi)$ is called a covering of $E$, if the following hold:
   
   (i) The morphism $\phi$ is onto, i.e. $\phi^0$ and $\phi^1$ are surjective.
   
   (ii) For any $v \in T^0$, the map $\phi^1 : r^{-1}(v) \rightarrow r^{-1}(\phi^0(v))$ and $\phi : s^{-1}(v) \rightarrow s^{-1}(\phi^0(v))$ are bijective.

Putting it another way, a weighted graph immersion or covering is a classical immersion or covering which preserves the weights.

Let $(E, w)$ be a weighted graph. The directed graph $\hat{E} = (\hat{E}^0, \hat{E}^1, \hat{s}, \hat{r})$, where $\hat{E}^0 = E^0$, $\hat{E}^1 := \{e_1, \ldots, e_{w(e)} \mid e \in E^1\}$, $\hat{s}(e_i) = s(e)$ and $\hat{r}(e_i) = r(e)$, is called the directed graph associated to $(E, w)$. If $e_i \in \hat{E}^1$, then $\text{tag}(e_i) := i$ is called the tag of $e_i$ and $\text{st}(e_i) := e$ is called the structure edge of $e_i$. There is a forgetful homomorphism

\[ \hat{E} \rightarrow E, \]

\[ u \mapsto u \]

\[ e_i \mapsto e \] (6)

relating these two graphs.

It is easy to see that if $\phi : T \rightarrow E$ is an immersion or a covering of weighted graphs, so is the graph homomorphism $\hat{\phi} : \hat{T} \rightarrow \hat{E}$ defined by $\hat{\phi}(u) = \phi(u)$, $u \in \hat{T}$, and $\hat{\phi}(e_i) := \phi(e_i)$, $1 \leq i \leq w(e)$.

We use the convention that a (finite) path $p$ in a weighted graph $E$ is either a single vertex $p = v \in E^0$ or a sequence $p = e_1 e_2 \cdots e_n$ of edges $e_i$ in $E$ such that $r(e_i) = s(e_{i+1})$ for $1 \leq i \leq n - 1$. 
We define $s(p) = s(e_i)$, and $r(p) = r(e_n)$. We denote by Path$(E)$ the set of all finite paths in $E$. Moreover, if $u,v \in E^0$, then we denote by $\text{Path}_u(E)$ the set of all finite paths starting in $u$, by $\text{Path}_v(E)$ the set of all finite paths ending in $v$ and by $\text{Path}_{u,v}(E)$ the intersection of $\text{Path}(E)$ and $\text{Path}(E)$. 

Given a weighted graph $E$, we define the double graph $E_d$ of $E$ as weighted graph with $E^0_d = E^0$, $E^1_d = \{ e, e^* \mid e \in E^1 \}$ with $w(e^*) = w(e)$, where $e^*$ has orientation the reverse of that of its counterpart $e \in E^1$ (see [2, p. 6]). Note that for a weighted graph $E$, one can identify $(\hat{E})_d$ and $(E_d)$. 

A path $p = x_1 \ldots x_n \in \text{Path}(E_d)$ is called backtracking if there is a $1 \leq j \leq n - 1$ such that $x_jx_{j+1} = ee^*$ or $x_jx_{j+1} = e^*e$ for some $e \in E^1$. We say $p$ is reduced if it is not backtracking. We also use the following convention: when we say $p$ is a reduced path in $E$, we mean $p$ is not backtracking path of $E_d$. This is in line with literature in graph theory.

Let $E$ be a connected weighted graph. Fix a base point $v \in E^0$. The universal covering graph of $E$ is a directed weighted graph $T = T(E,v)$ constructed as follows: let $T^0$ be the set of all reduced path in $E$ starting from $v$, $T^1 = \{ (a,e) \in T^0 \times E^1 \mid r(a) = s(e) \}$ and put $s(a,e) = a$, $r(a,e) = ae$. Furthermore for $(a,e) \in T^1$ we set $w(a,e) = w(e)$ and identify $(a,e)^*$ with $(ae,e^*)$. If the weight of the graph is 1, we obtain the classical case of universal covering. Similar to the classical case, one can show that $T$ is a tree and the isomorphism class of $T = T(E,v)$ is independent of the choice of base point $v$ (see [20]). The notion of universal covers allows us to easily construct representation graphs for a given weighted graph (see Lemma 4 and Example 5). We note that if $T$ is a universal cover of weighted graph $E$, $\hat{T}$ is not necessarily the universal cover for $\hat{E}$.

For a connected (weighted) graph $E$, we denote by $\pi(E)$ its fundamental group (which is independent of the choice of base-point). We define a length map 

$$\| : \pi(\hat{E}) \to \mathbb{Z}^n,$$

$$p \mapsto |p|$$

where $n = \max\{ w(e) \mid e \in E^1 \}$, as follows: For $\{ e_1, \ldots e_{w(e)} \mid e \in E^1 \}$ and $\{ e_1^*, \ldots e_{w(e)}^* \mid e \in E^1 \}$ and $v \in E^0$, set $|v| = 0$, $|e_i| = (0, \ldots , 0, 1, 0, \ldots)$ and $|e_i^*| = (0, \ldots , 0, -1, 0, \ldots) \in \mathbb{Z}^n$, where $1 \leq i \leq w(e)$ and 1 and $-1$ are in the $i$-th component, respectively. One can extend this to a well-defined map (7) by counting the length of the path $p$ which is a homomorphism of groups. Note that $|\pi(\hat{E})| = 0$ if and only if for paths $p,q \in \hat{E}$ with $s(p) = s(q)$ and $r(p) = r(q)$ we have $|p| = |q|$. This will be used to give a criterion when a representation graph gives rise to a graded representation (Theorem 25).

### 2.2. Representation graphs.

We are in a position to define the main notion of this paper, namely a representation graph of a given weighted graph $E$. Roughly, a graph $F$ is a representation graph of the graph $E$ if “locally” $F$ covers all the structure edges arriving at a vertex and all the tags emitting from a vertex in $E$. For the next definition, recall that to a weighted graph $E$ one can associate a directed graph $\hat{E}$ (see §2.1).

**Definition 2.** Let $(E,w)$ be a weighted graph. A pair $(F,\phi)$, where $F = (F^0, F^1, s_F, r_F)$ is a directed graph and $\phi = (\phi^0, \phi^1) : F \to \hat{E}$ is a homomorphism of directed graphs is called a representation graph of $E$, if the following hold:

- (1) For any $v \in F^0$ and $1 \leq i \leq w(\phi^0(v))$, there is precisely one $f \in s_{\phi^1}^{-1}(v)$ such that $\text{tag}(\phi^1(f)) = i$;
- (2) For any $v \in F^0$ and $e \in r^{-1}(\phi^0(v))$, there is precisely one $f \in r_{\phi^1}^{-1}(v)$ such that $\text{st}(\phi^1(f)) = e$. 

In the definition above, using (6) we identify vertices of \( \hat{E} \) with those of \( E \) when needed. Throughout the paper, we simply denote a representation graph \((F, \phi)\) of \((E, w)\) by \(F\) if there is no cause for confusion in context. Some examples of representation graphs are given in Introduction (1) and (3).

Let \( E \) be a weighted graph, \( \hat{E} \) the directed graph associated to \( E \) and \((F, \phi)\) a representation graph for \( E \). Let \( \hat{E}_d \) and \( F_d \) be the double graphs of \( E \) and \( F \), respectively. Clearly the homomorphism \( \phi : F \to \hat{E} \) induces a map \( \text{Path}(F_d) \to \text{Path}(\hat{E}_d) \), which we also denote by \( \phi \). We call two vertices \( u, v \in F^0 \) in \( F \) connected if \( \text{Path}_u(F_d) \neq \emptyset \). A representation graph \( F \) is called connected if any \( u, v \in F^0 \) are connected. This is equivalent to say that the graph \( F \) is connected as an undirected graph. If \( C \) is a connected component of \( F \), then \((C, \phi|_{C})\) is again a representation graph for \( E \).

Lemma 3. Let \( E \) be a weighted graph and \((F, \phi)\) a representation of \( E \) with the induced map \( \phi : \text{Path}(F_d) \to \text{Path}(\hat{E}_d) \). Let \( q, q' \in \text{Path}(F_d) \) such that \( \phi(q) = \phi(q') \). If \( s(q) = s(q') \) or \( r(q) = r(q') \), then \( q = q' \).

Proof. First suppose that \( r(q) = r(q') = v \). If one of the paths \( q \) and \( q' \) is trivial (i.e., is a vertex), then the other must also be trivial and we have \( q = v = q' \) as desired. Assume now that \( q \) and \( q' \) are not trivial. Then \( q = x_n \ldots x_1 \) and \( q' = y_n \ldots y_1 \), for some \( n \geq 1 \) and \( x_1, \ldots, x_n, y_1, \ldots, y_n \in F_d \). We proceed by induction on \( n \).

Case \( n = 1 \): Suppose \( \phi(x_1) = \phi(y_1) = e_i \) for some \( e \in E^1 \) and \( 1 \leq i \leq w(e) \). It follows from Definition 2(2) that \( x_1 = y_1 \) and hence \( q = q' \). Suppose now that \( \phi(x_1) = \phi(y_1) = e_i^* \) for some \( e \in E^1 \) and \( 1 \leq i \leq w(e) \). Then it follows from Definition 2(1) that \( x_1 = y_1 \) and hence \( q = q' \).

Case \( n \to n + 1 \): Suppose that \( q = x_{n+1} \ldots x_1 \) and \( q' = y_{n+1} \ldots y_1 \). By the inductive assumption we have \( x_i = y_i \) for any \( 1 \leq i \leq n \). It follows that \( s_{F_d}(x_n) = s_{F_d}(y_n) =: u \). Hence \( x_{n+1}, y_{n+1} \in \text{Path}_u(F_d) \). Now we can apply the case \( n = 1 \) and obtain \( x_{n+1} = y_{n+1} \).

Now suppose that \( s(q) = s(q') \). Then \( r(q^*) = r((q')^*) \). Since clearly \( \phi(q^*) = \phi(q^*) = \phi(q'^*) = \phi((q')^*) \), we obtain \( q^* = (q')^* \). Hence \( q = q' \). \( \Box \)

Since the notion of \( \text{Path}(F) \) of a graph \( F \) plays a prominent role, we remark that in the language of category Lemma 3 takes on a familiar form. Recall that a functor \( F : C \to D \) between two small categories \( C \) and \( D \) is called star injective, if the map

\[
F|_{\text{St}(x)} : \text{St}(x) \to \text{St}(F(x)),
\]

is injective, where

\[
\text{St}(x) = \{ f : x \to y \text{ a morphism in } C \mid y \in C \},
\]

for every object \( x \in C \). Similarly \( F \) is called co-star injective if \( F|_{\text{St}(x)^{op}} \) is injective, where \( \text{St}(x)^{op} = \{ f : y \to x \text{ a morphism in } C \mid y \in C \} \).

One can consider a graph \( F \) as a category with vertices as objects and paths as morphisms. Then the notion \( \text{Path}(F) \) represents the morphisms of the category \( F \). A graph homomorphism \( \phi : F \to G \) gives rise to a functor, called \( \phi \) again, \( \phi : F \to G \) between the categories \( F \) and \( G \). With this interpretation, Lemma 3 states that the functor \( \phi : F_d \to \hat{E}_d \) is star and co-star injective.

The next lemma shows that any representation for a covering graph induces a representation for the graph as well. This allows us to construct many representations for a given graph (see Example 5).

Lemma 4. Let \( E \) be a weighted graph and \( T \) a covering of \( E \). Then any representation graph \( F \) of \( T \) is a representation graph for \( E \). On the other hand, if \( F \) is a representation of \( E \) and \( G \) is a covering of \( F \) then \( G \) is also a representation of \( E \).
Proof. Let \( \psi : T \to E \) be a covering map and \( \phi : F \to \hat{T} \) a representation for \( T \). It is easy to see that there is a covering of unweighted graphs \( \hat{\psi} : \hat{T} \to \hat{E} \) which respects the tags, i.e., if \( \psi(t) = e \) then \( \hat{\psi}(t_1) = e_i \), for \( 1 \leq i \leq w(t) = w(e) \). Consider the following diagram, where \( \sigma \) is the graph homomorphism \( \hat{\psi}\phi \).

\[
\begin{array}{ccc}
F & \xrightarrow{\phi} & \hat{T} \\
\sigma \downarrow & & \downarrow \hat{\psi} \\
E & & \\
\end{array}
\]

We check that \((F, \sigma)\) is a representation graph for \( E \).

Let \( v \in F^0 \), and \( 1 \leq i \leq w(\sigma^0(v)) \). Since \( \phi \) is a representation of \( T \), by Definition 2(1), there is a unique \( f \in s_{F}^{-1}(v) \) such that \( \text{tag}(\phi^1(f)) = i \). Since \( \hat{\psi} \) is a covering, there is a bijection \( s^{-1}(\phi(v)) \to s^{-1}(\sigma(v)) \), preserving tags. Thus \( f \) is also the unique edge such that \( \text{tag}(\sigma^1(f)) = i \).

To see the second condition of a representation graph for the morphism \( \sigma \), let \( v \in F^0 \) and \( e \in r_{F}^{-1}(\sigma^0(v)) \). Since \( \psi \) is a covering, we have a bijection \( \psi : r^{-1}(\phi(v)) \to r^{-1}(\sigma(v)) \). Thus there is a unique \( e' \in r_{F}^{-1}(\phi(v)) \) such that \( \psi(e') = e \). By Definition 2(2) for the representation \( \phi \), there is precisely one \( f \in r_{F}^{-1}(v) \) such that \( \text{st}(\phi^1(f)) = e' \) and consequently \( \text{st}(\hat{\psi}\phi^1(f)) = e \).

The second part of the lemma is easy and we leave it to the reader. \( \square \)

Example 5. Let \( E \) be a weighted graph with one vertex, three loops of weight three.

\[
E : \quad \begin{array}{c}
\circ \quad e_1, e_2, e_3 \\
\circ \quad f_1, f_2, f_3 \\
\circ \quad g_1, g_2, g_3
\end{array}
\]

The universal cover of the weighted graph \( E \) resembles the Cayley graph of the free group \( F_3 \) with three generators (note that in comparison, the universal cover of \( \hat{E} \) is the Cayley graph of the free group \( F_9 \)). Below we show the part of the universal cover \( T \) generated by \( e, f \) and \( g \). For each node, choosing \( e_1, f_2, g_3 \) to emit and the same arrows to arrive at the node, they satisfy Conditions (1) and (2) of Definition 2 of the representation graphs and thus give a representation graph \( F \) for \( T \) and consequently one for \( E \).
We will see in Section 3 that any representation graph such as $F$ in Example 5 will give a module $V_F$ for the weighted Leavitt path algebra $L_K(E)$. However $V_F$ in this example is not a simple module. Next we characterise those representations that the associated module is simple.

**Definition 6.** Let $E$ be a weighted graph and $(F, \phi)$ a representation graph for $E$ with the induced map $\phi: \text{Path}(F_d) \to \text{Path}(\hat{E}_d)$. We say $F$ is an *irreducible representation* for $E$ if the following equivalent conditions hold.

1. The graph $F$ is connected and
   $$\phi(\text{Path}_u(F_d)) \neq \phi(\text{Path}_v(F_d)),$$
   for any $u \neq v \in F^0$.
2. The graph $F$ is connected and
   $$\phi(\text{Path}_u(F_d)) \neq \phi(\text{Path}_v(F_d)),$$
   for any $u \neq v \in F^0$.

Note that the condition of irreducibility in Definition 6 in the categorical language (see (8)) amounts to saying that for $u \neq v \in F^0$, $\phi(\text{St}(u)) \neq \phi(\text{St}(v))$.

**Example 7.** Consider the representation graph $H$ below of the graph $E$ with one vertex and three loops (2). One can check that the conditions of Definition 6 are not satisfied and therefore $H$ is not an irreducible representation for $E$. On the other hand the representation graphs (3) and (4), respectively, are irreducible.

![Diagram of representation graph H](image)

2.3. **The category of the representation graphs of a given graph.** Let $E$ be a weighted graph. Denote by $\text{RG}(E)$ the category of nonempty, connected representation graphs for $E$. A morphism $\alpha: (F, \phi) \to (G, \psi)$ in $\text{RG}(E)$ is a homomorphism $\alpha: F \to G$ of directed graphs such that $\psi \alpha = \phi$. One checks easily that a morphism $\alpha: (F, \phi) \to (G, \psi)$ is an isomorphism if and only if $\alpha^0$ and $\alpha^1$ are bijective. Lemma 4 states that if $T \to E$ is a covering of the weighted graph $E$, then we have a natural functor $\text{RG}(T) \to \text{RG}(E)$.

In this section we show that the category $\text{RG}(E)$ is a disjoint union of certain subcategories, each of which contains a unique universal representation and a unique irreducible representation of $E$. In §3 we will show that the universal representations of $E$ give us indecomposable $L_K(E)$-modules, whereas irreducible representations of $E$ give rise to simple $L_K(E)$-modules.

We need to establish several lemmas which allows us to define equivalence relations on representations.

**Lemma 8.** Let $(F, \phi)$ and $(G, \psi)$ be objects in $\text{RG}(E)$. Let $u \in F^0$ and $v \in G^0$. If $\phi(\text{Path}_u(F_d)) \subseteq \psi(\text{Path}_v(G_d))$, then $\phi(\text{Path}_u(F_d)) = \psi(\text{Path}_v(G_d))$.

**Proof.** Suppose that $\phi(\text{Path}_u(F_d)) \subseteq \psi(\text{Path}_v(G_d))$. It follows that $\phi^0(u) = \psi^0(v)$ since $u \in \text{Path}_u(F_d)$. We have to show that $\psi(\text{Path}_v(G_d)) \subseteq \phi(\text{Path}_u(F_d))$. Let $p \in \text{Path}_v(G_d)$. If $p = v$, ...
then $\psi(p) = \psi^0(v) = \phi^0(u) \in \phi(u\Path(F_d))$. Suppose now that $p$ is nontrivial. Then $p = y_1 \ldots y_n$ for some $y_1, \ldots, y_n \in G_1^1$ where $n \geq 1$. We proceed by induction on $n$.

Case $n = 1$: Suppose that $p = g$ for some $g \in G^1$. Then $\psi^1(g) = e$, for some $e \in s^{-1}(\psi^0(v))$ and $1 \leq i \leq w(e)$. Since $(F, \phi)$ satisfies Condition (1) in Definition 2, there is precisely one $f \in s^{-1}(\psi^0(u))$ such that $\tag{\phi^1(f)} = i$. Hence $\phi^1(f) = e'_i$ for some $e'_i \in s^{-1}(\phi^0(u))$. Assume that $e \neq e'_i$. Since $\phi(u\Path(F_d)) \subseteq \psi(u\Path(G_d))$, there is a $g' \in s^{-1}_G(v)$ such that $\psi^1(g') = \phi^1(f) = e'_i$. But this is absurd since $(G, \psi)$ satisfies Condition (1) in Definition 2. Thus $\psi(g) = e_i = \phi(f) \in \phi(u\Path(F_d))$.

Suppose now that $p = g^*$ for some $g \in G^1$. Then $\psi^1(g) = e_i$ for some $e \in r^{-1}(\psi^0(v))$ and $1 \leq i \leq w(e)$. Since $(F, \phi)$ satisfies Condition (2) in Definition 2, there is precisely one $f \in r^{-1}(\psi^0(v))$ such that $\tag{\phi^1(f)} = i$. Hence $\phi^1(f) = e'_i$ for some $e'_i \in s^{-1}(\phi^0(u'))$. Assume that $i \neq j$. Since $\phi(u\Path(F_d)) \subseteq \psi(u\Path(G_d))$, there is a $g' \in r^{-1}_G(v')$ such that $\psi^1(g') = \phi^1(f) = e_j$. But this is absurd since $(G, \psi)$ satisfies Condition (1) in Definition 2. Thus $\psi(g) = e_i = \phi(f)$ and hence $\psi(y_1 \ldots y_n g) = \phi(x_1 \ldots x_n f) \in \phi(u\Path(F_d))$.

Case $n \to n + 1$: Suppose $p = y_1 \ldots y_n y_{n+1}$. By the induction assumption we know that $\psi(y_1 \ldots y_n) \in \phi(u\Path(F_d))$. Hence $\psi(y_1 \ldots y_n) = \phi(x_1 \ldots x_n)$ for some path $x_1 \ldots x_n \in u\Path(F_d)$. Set $u' := r_F(x_n)$ and $v' := r_G(y_n)$. Clearly $\phi^0(u') = \psi^0(v')$ since $\phi$ and $\psi$ are graph homomorphisms. Suppose that $y_{n+1} = g$ for some $g \in G^1$. Then $\psi^1(g) = e_i$ for some $e \in r^{-1}(\psi^0(v))$ and $1 \leq i \leq w(e)$. Since $(F, \phi)$ satisfies Condition (1) in Definition 2, there is precisely one $f \in r^{-1}(\psi^0(v))$ such that $\tag{\phi(f)} = \psi^1(g') = \phi^1(f) = e'_i$ for some $e'_i \in s^{-1}(\phi^0(u'))$. Assume that $i \neq j$. Since $\phi(u\Path(F_d)) \subseteq \psi(u\Path(G_d))$, there is a $g' \in r^{-1}_G(v')$ such that $\phi(x_1 \ldots x_n f) = \psi(y_1 \ldots y_n g')$, which implies $\psi^1(g') = \phi^1(f) = e'_j$. But this is absurd since $(G, \psi)$ satisfies Condition (1) in Definition 2. Thus $\psi(g) = e_i = \phi(f)$ and hence $\psi(y_1 \ldots y_n g) = \phi(x_1 \ldots x_n f) \in \phi(u\Path(F_d))$.

Proposition 9. Let $\alpha : (F, \phi) \to (G, \psi)$ be a morphism in $\RG(E)$. If $u \in F^0$, then $\alpha(u\Path(F_d)) = \psi(\alpha^0(u)\Path(G_d))$.

Proof. Let $u \in F^0$ and $v \in \phi(u\Path(F_d))$. Then there is a $q \in u\Path(F_d)$ such that $\phi(q) = p$. Clearly $\alpha(q) \in \phi^0(u)\Path(G_d)$. Hence $p = \phi(q) = \psi(\alpha(q)) \in \psi(\phi^0(u)\Path(G_d))$. We have shown that $\phi(u\Path(F_d)) \subseteq \psi(u\Path(G_d))$. It follows from Lemma 8 that $\phi(u\Path(F_d)) = \psi(u^\phi(u)\Path(G_d))$.

Proposition 10. Let $\alpha : (F, \phi) \to (G, \psi)$ be a morphism in $\RG(E)$. Then $\alpha$ is a covering map.

Proof. We first show that $\alpha^0$ is surjective. Choose a $u \in F^0$. If $v \in G^0$, then there is a path $q \in u\Path(G_d)$ since $G$ is connected. By Proposition 9 we have $\phi(u\Path(F_d)) = \psi(u\Path(G_d))$. Hence there is a $q \in u\Path(F_d)$ such that $\phi(p) = \psi(q)$. Since $\phi = \psi\alpha$, we obtain $\psi(\alpha(p)) = \psi(q)$. It follows from Lemma 3 that $\alpha(p) = q$. Thus $\alpha^0(r(p)) = r(\alpha(p)) = r(q) = v$.

We next show that $\alpha^1$ is surjective. Suppose $g \in G^1$ with $v := s(g)$, $w := s(\psi(g))$ and $i := \tag{\psi(g)}$. Since $\alpha^0$ is surjective, there is a $u \in F^0$ such that $\alpha^0(u) = v$. Since $\phi(u) = \psi(\alpha(u)) = w$, by the definition of a representation graph, there is a unique edge $f \in F^1$ with $s(f) = u$ such that $\tag{\psi(g)} = i$. Since $s(\alpha(f)) = s(g)$ and $\tag{\psi(g)} = \tag{\psi(g)} = \tag{\psi(\alpha(f))}$, it follows that $\alpha(f) = g$. This shows that $\alpha^1$ is surjective.
Next we show that that for any \( u \in F^0 \), the map \( \alpha^1 : s^{-1}(u) \to s^{-1}(\alpha^0(u)) \) is injective. Let \( f_1, f_2 \in F^1 \) are distinct, with \( s(f_1) = s(f_2) = u \). By the definition of the representation, \( \operatorname{tag}(\phi(f_1)) \neq \operatorname{tag}(\phi(f_2)) \) and thus \( \operatorname{tag}(\psi(\alpha(f_1))) \neq \operatorname{tag}(\psi(\alpha(f_1))) \). Hence \( \alpha(f_1) \neq \alpha(f_2) \).

A similar argument shows that for any \( u \in F^0 \), the map \( \alpha^1 : r^{-1}(u) \to r^{-1}(\alpha^0(u)) \) is bijective. 

\(\square\)

2.4. **Quotients of representation graphs.** For any object \((F, \phi)\) in \( \text{RG}(E) \) we define an equivalence relation \( \sim \) on \( F^0 \) by \( u \sim v \) if \( \phi(u \text{Path}(F_d)) = \phi(v \text{Path}(F_d)) \). Recall that if \( \sim \) and \( \approx \) are equivalence relations on a set \( X \), then one writes \( \approx \leq \sim \) and calls \( \approx \) finer than \( \sim \) (and \( \sim \) coarser than \( \approx \)) if \( x \approx y \) implies that \( x \sim y \), for any \( x, y \in X \).

**Definition 11.** Let \((F, \phi) \in \text{RG}(E)\) be a representation graph of \( E \). An equivalence relation \( \approx \) on \( F^0 \) is called **admissible** if the following hold:

1. \( \approx \leq \sim \).
2. If \( u \approx v, p \in u \text{Path}_x(F_d), q \in v \text{Path}_y(F_d) \) and \( \phi(p) = \phi(q) \), then \( x \approx y \).

The admissible equivalence relations on \( F^0 \) (with partial order \( \leq \)) form a bounded lattice whose maximal element is \( \sim \) and whose minimal element is the equality relation \( = \).

Let \((F, \phi) \in \text{RG}(E)\) be a representation graph of \( E \) and \( \approx \) an admissible equivalence relation on \( F^0 \). If \( f, g \in F^1 \) we write \( f \approx g \) if \( s(f) \approx s(g) \) and \( \phi(f) = \phi(g) \). This defines an equivalence relation on \( F^1 \). Define a representation graph \((F_\approx, \phi_\approx)\) for \( E \) by

\[
\begin{align*}
F_\approx^0 &= F^0 / \approx, \\
F_\approx^1 &= F^1 / \approx, \\
s([f]) &= [s(f)], \\
r([f]) &= [r(f)], \\
\phi_\approx^0([v]) &= \phi^0(v), \\
\phi_\approx^1([f]) &= \phi^1(f).
\end{align*}
\]

We call \((F_\approx, \phi_\approx)\) a **quotient** of \((F, \phi)\). It is easy to check that \((F_\approx, \phi_\approx)\) is a representation graph of \( E \).

**Theorem 12.** Let \((F, \phi)\) and \((G, \psi)\) be objects in \( \text{RG}(E) \). Then there is a morphism \( \alpha : (F, \phi) \to (G, \psi) \) if and only if \((G, \psi)\) is isomorphic to a quotient of \((F, \phi)\).

**Proof.** (\(\Rightarrow\)) Suppose there is a morphism \( \alpha : (F, \phi) \to (G, \psi) \). If \( u, v \in F^0 \), we write \( u \approx v \) if \( \alpha^0(u) = \alpha^0(v) \). Clearly \( \approx \) defines an equivalence relation on \( F^0 \). Below we check that \( \approx \) is admissible.

(i) Suppose \( u \approx v \). Then \( \phi(u \text{Path}(F_d)) = \psi(\alpha^0(u) \text{Path}(G_d)) = \psi(\alpha^0(v) \text{Path}(G_d)) = \phi(v \text{Path}(F_d)) \) by Proposition 9. Hence \( u \approx v \).

(ii) Suppose \( u \approx v, p \in u \text{Path}_x(F_d), q \in v \text{Path}_y(F_d) \) and \( \phi(p) = \phi(q) \). Clearly \( \alpha(p) \in \alpha^0(u) \text{Path}(\alpha^0(x)) \) (G_d) and \( \alpha(q) \in \alpha^0(v) \text{Path}(\alpha^0(y)) \) (G_d). Moreover, \( \psi(\alpha(p)) = \phi(p) = \phi(q) = \psi(\alpha(q)) \). Since \( \alpha^0(u) = \alpha^0(v) \), it follows from Lemma 3 that \( \alpha(p) = \alpha(q) \). Hence \( \alpha^0(x) = r(\alpha(p)) = r(\alpha(q)) = \alpha^0(y) \) and therefore \( x \approx y \).

Note that by Lemma 3 we have \( f \approx g \) if and only if \( \alpha^1(f) = \alpha^1(g) \), for any \( f, g \in F^1 \). Define a graph homomorphism \( \beta : F_\approx \to G \) by \( \beta^0([v]) = \alpha^0(v) \) and \( \beta^1([f]) = \alpha^1(f) \). Clearly \( \psi \beta = \phi_\approx \) and therefore \( \beta : (F_\approx, \phi_\approx) \to (G, \psi) \) is a morphism. In view of Proposition 10, \( \beta^0 \) and \( \beta^1 \) are bijective and hence \( \beta \) is an isomorphism.
Suppose now that \( (G, \psi) \cong (F_\approx, \phi_\approx) \) for some admissible equivalence relation \( \approx \) on \( F^0 \).
In order to show that there is a morphism \( \alpha : (F, \phi) \to (G, \psi) \) it suffices to show that there is a morphism \( \beta : (F, \phi) \to (F_\approx, \phi_\approx) \). But this is obvious (define \( \beta^0(v) = [v] \) and \( \beta^1(f) = [f] \)).

2.5. The subcategories \( \text{RG}(E)_C \). Let \( (F, \phi) \) and \( (G, \psi) \) be objects in \( \text{RG}(E) \). We write \( (F, \phi) \cong (G, \psi) \) if there is a \( u \in F^0 \) and a \( v \in G^0 \) such that \( \phi(u, \text{Path}(F_d)) = \psi(v, \text{Path}(G_d)) \). One checks easily that \( \cong \) defines an equivalence relation on \( \text{Ob}(\text{RG}(E)) \). If \( C \) is an \( \equiv \)-equivalence class, then we denote by \( \text{RG}(E)_C \) the full subcategory of \( \text{RG}(E) \) such that \( \text{Ob}(\text{RG}(E)_C) = C \). If \( \alpha : (F, \phi) \to (G, \psi) \) is a morphism in \( \text{RG}(E) \), then \( (F, \phi) \equiv (G, \psi) \) by Proposition 9. Thus \( \text{RG}(E) \) is the disjoint union of the subcategories \( \text{RG}(E)_C \), where \( C \) ranges over all \( \equiv \)-equivalence classes.

Fix an \( \equiv \)-equivalence class \( C \), a representation graph \( (F, \phi) \in C \) and a vertex \( u \in F^0 \). We denote by \( \phi(u, \text{Path}(F_d))_{\text{nb}} \) the set of all paths in \( \phi(u, \text{Path}(F_d)) \) that are reduced (see §2.1). Define a representation graph \( (T, \xi) = (T_C, \xi_C) \) for \( E \) by

\[
T^0 = \{ v_p \mid p \in \phi(u, \text{Path}(F_d))_{\text{nb}}, p \neq \phi(u) \},
T^1 = \{ e_p \mid p \in \phi(u, \text{Path}(F_d))_{\text{nb}}, p = \phi(u) \},
\]

\[
s(e_{x_1 \ldots x_n}) = \begin{cases} v_{x_1 \ldots x_{n-1}} & \text{if } x_n \in \hat{E}^1 \setminus \{\phi(u)\}, \\ v_{x_1 \ldots x_n} & \text{if } x_n \in (\hat{E}^1)^*, \end{cases}
\]

\[
r(e_{x_1 \ldots x_n}) = \begin{cases} v_{x_1 \ldots x_n} & \text{if } x_n \in \hat{E}^1 \setminus \{\phi(u)\}, \\ v_{x_1 \ldots x_{n-1}} & \text{if } x_n \in (\hat{E}^1)^*. \end{cases}
\]

\[
\xi^0(v_{x_1 \ldots x_n}) = \begin{cases} \phi(u) & \text{if } n = 1 \text{ and } x_1 = \phi(u), \\ r_{\hat{E}^1}(x_n) & \text{if } x_n \in \hat{E}^1 \setminus \{\phi(u)\}, \\ s_{\hat{E}^1}(x_n^*) & \text{if } x_n \in (\hat{E}^1)^*. \end{cases}
\]

\[
\xi^1(e_{x_1 \ldots x_n}) = \begin{cases} x_n & \text{if } x_n \in \hat{E}^1 \setminus \{\phi(u)\}, \\ x_n^* & \text{if } x_n \in (\hat{E}^1)^*. \end{cases}
\]

Here we use the convention that if \( x_1 \ldots x_n \in \phi(u, \text{Path}(F_d))_{\text{nb}} \), where \( n = 1 \), then \( x_1 \ldots x_{n-1} = \phi(u) \).
Clearly \( T \) is nonempty and connected.

Example 13. Suppose \( E \) is a weighted graph with one vertex and two loops of weight 2:

\[
E : e_1, e_2 \cup v \cup f_1, f_2
\]

and \( (F, \phi) \) is a representation graph for \( E \), where \( F \) is given by

\[
F : g \cup u \cup h
\]

and \( \phi : F \to \hat{E} \) is defined by \( \phi^0(u) = v, \phi^1(g) = e_1, \phi^1(h) = f_2 \). Then \( u, \text{Path}(F_d) = \text{Path}(F_d) \) and \( \phi(u, \text{Path}(F_d))_{\text{nb}} \) consists of all nonbacktracking paths in \( \text{Path}(\hat{E}_d) \) whose letters come from the
alphabet \{v, e_1, e_1^*, f_2, f_2^*\}. Let \( C \) be the \( \Rightarrow \)-equivalence class of \((F, \phi)\). Then \( T_C \) is the graph

\[
\begin{array}{c}
\text{\(e_1 e_1\)} & \text{\(e_1 f_2\)} & \text{\(e_2 f_2\)} \\
\text{\(v e_1\)} & \text{\(v f_2\)} & \text{\(v f_2\)}
\end{array}
\]

Proposition 14. If \((G, \psi) \in C\), then there is a morphism \(\alpha : (T_C, \xi_C) \rightarrow (G, \psi)\).

Proof. Since \((G, \psi) \Rightarrow (F, \phi)\), there is a \( v \in G^0 \) such that \(\phi(u \text{Path}(F_d)) = \psi(u \text{Path}(G_d))\). Define a homomorphism \(\alpha : T_C \rightarrow G\) as follows. Let \(x_1 \ldots x_n \in \phi(u \text{Path}(F_d))_\text{nb}\). Since \(\phi(u \text{Path}(F_d)) = \psi(u \text{Path}(G_d))\), there is a (uniquely determined) path \(y_1 \ldots y_n \in \text{Path}(G_d)\) such that \(\psi(y_1 \ldots y_n) = x_1 \ldots x_n\). Define \(\alpha^0(v_{x_1 \ldots x_n}) = r(y_n), \alpha^1(e_{x_1 \ldots x_n}) = y_n\) if \(y_n \in G^1\) and respectively \(\alpha^1(e_{x_1 \ldots x_n}) = y_n^*\) if \(y_n \in (G^1)^*\). We leave it to the reader to check that \(\alpha\) is a graph homomorphism and that \(\psi \alpha = \xi_C\).

We are in a position to show that in each equivalence class \( C \) of the representation graphs of the weighted graph \( E \), there is only one irreducible representation graph up to isomorphism.

Corollary 15. Up to isomorphism the representation graphs in \( C \) are precisely the quotients of \((T_C, \xi_C)\), and consequently

\[ (S_C, \xi_C) := ((T_C)_{\sim}, (\xi_C)_{\sim}) \]

is the unique irreducible representation graph in \( C \).

Proof. The first statement follows from Theorem 12 and Proposition 14. The second statement now follows since a quotient \(((T_C)_{\sim}, (\xi_C)_{\sim})\) satisfies the equivalent conditions in Definition 6 if and only if \(\approx\) equals \(\sim\).

Recall that an object \( X \) in a category \( C \) is called repelling (resp. attracting) if for any object \( Y \) in \( C \) there is a morphism \( X \rightarrow Y \) (resp. \( Y \rightarrow X \)). By Proposition 14 \((T_C, \xi_C)\) is a repelling object in \( C \). On the other hand, if \((G, \psi)\) is an object in \( C \), then clearly \((S_C, \xi_C)\) is isomorphic to a quotient of \((G, \psi)\). It follows from Theorem 12 that \((S_C, \xi_C)\) is an attracting object in \( C \).

Example 16. Suppose \( E \) is a weighted graph with one vertex and two loops of weight 2:

\[
E : e_1 e_1 \bigcup v \bigcup e_1 e_2.
\]

Consider the representation graphs \((F_1, \phi_1), \ldots, (F_7, \phi_7)\) for \( E \) given below.
All the representation graphs \((F_i, \phi_i)\) \((1 \leq i \leq 7)\) lie in the same \(\sim\)-equivalence class \(C\). One
checks easily that \((F_1, \phi_1) \cong (T_C, \xi_C)\) and \((F_7, \xi_7) \cong (S_C, \zeta_C)\) (cf. Example 13). Moreover, we have

\[
\begin{align*}
(F_1, \xi_1) & \xrightarrow{} (F_2, \xi_2) \\
(F_3, \xi_3) & \xrightarrow{} (F_4, \xi_4) \\
(F_5, \xi_5) & \xrightarrow{} (F_6, \xi_6) \\
(F_7, \xi_7) & \xrightarrow{} (F_8, \phi_8)
\end{align*}
\]

where an arrow \((F_i, \phi_i) \rightarrow (F_j, \phi_j)\) means that \((F_j, \phi_j)\) is a quotient of \((F_i, \phi_i)\).

3. Weighted Leavitt path algebras and their representations

Weighted Leavitt path algebras were introduced in [17], as a generalisation of Leavitt path algebras. They are a model for the algebras \(L_K(n, m)\) which could not be obtained via the classical theory of Leavitt path algebras. The structure of weighted Leavitt path algebras was further investigated in [18, 27, 26, 29]. Further generalisations of these algebras were also carried out in [28, 24].

We recall the notion of weighted Leavitt path algebras.

**Definition 17.** Let \((E, w)\) be a weighted graph. The \(K\)-algebra \(L_K(E, w)\) presented by the generating set \(\{v, e_i, e_i^* \mid v \in E^0, e \in E^1, 1 \leq i \leq w(e)\}\) and the relations

1. \(uv = \delta_{uv}u \quad (u, v \in E^0),\)
2. \(s(e)e_i = e_i = e_ir(e), \quad r(e)e_i^* = e_i^* = e_i^*s(e) \quad (e \in E^1, 1 \leq i \leq w(e)),\)
3. \(\sum_{1 \leq i \leq w(v)} e_i f_i = \delta_{ij}r(e) \quad (v \in E^0, e, f \in s^{-1}(v))\) and
4. \(\sum_{e \in s^{-1}(v)} e_i e_i^* = \delta_{ij}v \quad (v \in E^0, 1 \leq i, j \leq w(v))\)

is called the weighted Leavitt path algebra of \((E, w)\). In relations (3) and (4) we set \(e_i\) and \(e_i^*\) zero whenever \(i > w(v)\).

Throughout the paper, we continue to denote a weighted graph \((E, w)\) by \(E\) and the weighted Leavitt path algebra \(L_K(E, w)\) by \(L_K(E)\). We also interchangeably use the terminology the Leavitt path algebra of a weighted graph instead of a weighted Leavitt path algebra. If all the edges of the weighted graph \(E\) have weight one, then \(L_K(E, w)\) coincides with the classical Leavitt path algebra \(L_K(E)\) (see Example 19).

Weighted Leavitt path algebras are involutary graded rings with unit if \(E^0\) is finite and local units otherwise. In fact, the weighted Leavitt path algebra \(L_K(E)\) is a \(\mathbb{Z}^n\)-graded ring, where \(n = \max\{w(e) \mid e \in E\}\). The grading is defined as follows: Consider the free ring \(F_E\) generated by \(\{v \mid v \in E^0\}, \{e_1, \ldots, e_{w(e)} \mid e \in E^1\}\) and \(\{e_1^*, \ldots, e_{w(e)}^* \mid e \in E^1\}\), with the coefficients in \(K\), set for \(v \in E^0\), \(\deg(v) = 0\), for \(e \in E^1\), \(1 \leq i \leq w(e)\), \(\deg(e_i) = (0, \ldots, 0, 1, 0, \ldots)\) and \(\deg(e_i^*) = (0, \ldots, 0, -1, 0, \ldots)\) \(\in \mathbb{Z}^n\), where 1 and \(-1\) are in the \(i\)-th component, respectively. This defines a
\( \mathbb{Z}^n \)-grading on the free ring \( F_E \). (Note that \( n \) could be infinite). Since all the relations in Definition 17 involve homogeneous elements, so the quotient of \( F_E \) by the homogeneous ideal generated by these relations, i.e., \( L_K(E) \) is also a \( \mathbb{Z}^n \)-graded ring.

**Example 18.** The Leavitt path algebra of a weighted graph consisting of one vertex and \( n+k \) loops of weight \( n \) is isomorphic to the Leavitt algebra \( L_K(n, n+k) \). To show this, let \( E^1 = \{y_1, \ldots, y_{n+k}\} \) with \( w(y_i) = n, 1 \leq i \leq n+k \). Denote the \( n \) edges corresponding to the (structure) edge \( y_i \in E^1 \) by \( \{y_1, \ldots, y_n\} \). We visualise this data as follows:

\[
y_{1+n+k}, \ldots, y_{n+n+k}
\]

Set \( x_{st} = y_{rs}^* \) for \( 1 \leq r \leq n \) and \( 1 \leq s \leq n+k \) and arrange the \( y \)'s and \( x \)'s in the matrices

\[
Y = \begin{pmatrix}
y_{11} & y_{12} & \cdots & y_{1,n+k} \\
y_{21} & y_{22} & \cdots & y_{2,n+k} \\
\vdots & \vdots & \ddots & \vdots \\
y_{n1} & y_{n2} & \cdots & y_{n,n+k}
\end{pmatrix}, \quad X = \begin{pmatrix}
x_{11} & x_{12} & \cdots & x_{1n} \\
x_{21} & x_{22} & \cdots & x_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
x_{n+k,1} & x_{n+k,2} & \cdots & x_{n+k,n}
\end{pmatrix}
\]

Then Condition (3) of Definition 17 precisely says that \( Y \cdot X = I_{n,n} \) and Condition (4) is equivalent to \( X \cdot Y = I_{n+n+k,n+k} \) which are the generators of \( L_K(n, n+k) \).

**Example 19.** Let \( (E, w) \) be a weighted graph where \( w : E \to \mathbb{N} \) is the constant map \( w(e) = 1 \), for all \( e \in E \). Then \( L_K(E, w) \) coincides with the usual Leavitt path algebra \( L_K(E) \).

**Example 20.** Consider a weighted graph with one vertex and one edge \( e \) of weight \( n \) and an unweighted graph \( F \) with one vertex and \( n \) edges \( \{e_1, \ldots, e_n\} \). Then the map

\[
L_K(E, w) \to L_K(F), \quad e_i \mapsto e_i^*
\]

induces an isomorphism, i.e.,

\[
L_K\left( \begin{array}{c}
e_1, \ldots, e_n \\
\end{array} ; w \right) \cong L_K\left( \begin{array}{c}
e_n \\
e_1 \\
e_2 \\
e_3
\end{array} ; e_1 \right).
\]

Note that this isomorphism is not graded as \( L(E, w) \) is \( \mathbb{Z}^n \)-graded, whereas \( L(F) \) is just \( \mathbb{Z} \)-graded.

### 3.1. Representations of weighted Leavitt path algebras.

Let \( E \) be a weighted graph and \( L_K(E) \) the weighted Leavitt path algebra associated to \( E \). In this section, for a representation graph \((F, \phi)\) of \( E \), we construct a \( L_K(E) \)-module \( V_F \). This gives rise to a functor from the category of representations of the graph \( E \) (see §2.3) to the category of (right) \( L_K(E) \)-modules:

\[
V : \text{RG}(E) \to \text{Mod} L_K(E), \quad (F, \phi) \mapsto V_F.
\]

We will then investigate the functor \( V \). In Theorem 28, we show that \( V_F \) is a simple \( L_K(E) \)-module if and only if \( F \) is an irreducible representation. In Theorem 43 we further show that a universal representation \( T \) of \( E \) gives an indecomposable \( L_K(E) \)-module \( V_T \).
Lemma 21. Suppose now that there is no \( u \in E^0 \) such that \( \sigma_u, \sigma_{e_i}, \sigma_{e_i^*} \in \text{End}_K(V_F) \) by

\[
\sigma_u(v) = \begin{cases} v, & \text{if } \phi^0(v) = u \\ 0, & \text{else} \end{cases},
\]

\[
\sigma_{e_i}(v) = \begin{cases} r_F(f), & \text{if } \exists f \in s_F^{-1}(v): \phi^1(f) = e_i \\ 0, & \text{else} \end{cases},
\]

\[
\sigma_{e_i^*}(v) = \begin{cases} s_F(f), & \text{if } \exists f \in r_F^{-1}(v): \phi^1(f) = e_i \\ 0, & \text{else} \end{cases},
\]

where \( v \in F^0 \). It follows from the universal property of \( L_K(E) \) that there is an algebra homomorphism

\[
\pi : L_K(E) \longrightarrow \text{End}_K(V_F)^{\text{op}},
\]

such that \( \pi(u) = \sigma_u, \pi(e_i) = \sigma_{e_i} \) and \( \pi(e_i^*) = \sigma_{e_i^*} \). We refer to this representation, the representation of \( L_K(E) \) defined by \( (F, \phi) \). Clearly \( V_F \) becomes a right \( L_K(E) \)-module by defining \( x \cdot a := \pi(a)(x) \), for any \( a \in L_K(E) \) and \( x \in V_F \).

The following lemma describes the action of monomial elements of the weighted Leavitt path algebra \( L_K(E) \) on the \( K \)-vector space \( V_F \). Note that by Lemma 3, for any \( p \in \text{Path}(\hat{E}_d) \) and \( u \in F^0 \) there is at most one \( v \in F^0 \) such that \( p \in \phi(u\text{Path}_v(F_d)) \).

**Lemma 21.** Let \( E \) be a weighted graph and \((F, \phi)\) a representation of \( E \) with the induced map \( \phi : \text{Path}(F_d) \rightarrow \text{Path}(\hat{E}_d) \). If \( p \in \text{Path}(\hat{E}_d) \) and \( u \in F^0 \), then

\[
u \cdot p = \begin{cases} v, & \text{if } p \in \phi(u\text{Path}_v(F_d)), \text{ for some } v \in F^0, \\ 0, & \text{otherwise}. \end{cases}
\]

**Proof.** Let \( p \in \text{Path}(\hat{E}_d) \) and \( u \in F^0 \). Suppose that \( p = u' \) for some \( u' \in E^0 \). If \( p \in \phi(u\text{Path}_v(F_d)) \), for some \( v \in F^0 \), then clearly \( u = v \) and \( u' = \phi^0(v) \). Hence \( u \cdot p = u = v \) as desired. If there is no \( v \in F^0 \) such that \( p \in \phi(u\text{Path}_v(F_d)) \), then clearly \( u' \neq \phi^0(v) \). Hence \( u \cdot p = 0 \) as desired.

Suppose now that \( p \) is nontrivial, say \( p = x_1 \ldots x_n \) where \( n \geq 1 \) and \( x_1, \ldots, x_n \in \hat{E}_d \). We proceed by induction on \( n \).

Case \( n = 1 \): Suppose \( p = e_i \) for some \( e \in E^1 \) and \( 1 \leq i \leq w(e) \). If \( p \in \phi(u\text{Path}_v(F_d)) \), for some \( v \in F^0 \), then clearly \( p = \phi^1(f) \) for some \( f \in F^1 \) such that \( s_f(f) = u \) and \( r_f(f) = v \). Hence \( u \cdot p = r_f(f) = v \) as desired. If there is no \( v \in F^0 \) such that \( p \in \phi(u\text{Path}_v(F_d)) \), then clearly there is no \( f \in s_f^{-1}(v) \) such that \( \phi^1(f) = e_i \). Hence \( u \cdot p = 0 \) as desired.

Suppose \( p = e_i^* \) for some \( e \in E^1 \) and \( 1 \leq i \leq w(e) \). If \( p \in \phi(u\text{Path}_v(F_d)) \), for some \( v \in F^0 \), then clearly \( e_i = \phi^1(f) \) for some \( f \in F^1 \) such that \( s_f(f) = v \) and \( r_f(f) = u \). Hence \( u \cdot p = s_f(f) = v \) as desired. If there is no \( v \in F^0 \) such that \( p \in \phi(u\text{Path}_v(F_d)) \), then clearly there is no \( f \in r_f^{-1}(u) \) such that \( \phi^1(f) = e_i \). Hence \( u \cdot p = 0 \) as desired.

Case \( n \rightarrow n + 1 \): Suppose \( p = x_1 \ldots x_n x_{n+1} \). If \( p \in \phi(u\text{Path}_v(F_d)) \), for some \( v \in F^0 \), then clearly \( x_1 \ldots x_n \in \phi(u\text{Path}_v(F_d)) \) for some \( u' \in F^0 \) and moreover \( x_{n+1} \in \phi(u'\text{Path}_v(F_d)) \). By the induction assumption we have \( v \cdot x_1 \ldots x_n = u' \). It follows from the case \( n = 1 \) that \( u \cdot p = u' \cdot x_{n+1} = v \).

Suppose now that there is no \( v \in F^0 \) such that \( p \in \phi(u\text{Path}_v(F_d)) \). If \( x_1 \ldots x_n \in \phi(u^{-1}u'\text{Path}_v(F_d)) \) for some \( u' \in F^0 \), then there is no \( v \in F^0 \) such that \( x_{n+1} \in \phi(u'\text{Path}_v(F_d)) \). Hence, by the induction
assumption, \( u \cdot p = u' \cdot x_{n+1} = 0 \) as desired. If there is no \( u' \in F^0 \) such that \( x_1 \ldots x_n \in \phi(\alpha \Path_{wp}(F_d)) \), then, by the induction assumption, \( u \cdot p = (u \cdot x_1 \ldots x_n) \cdot x_{n+1} = 0 \cdot x_{n+1} = 0 \) as desired. \( \square \)

**Corollary 22.** If \( a = \sum_{p \in \Path(\hat{E}_d)} k_p p \in L_K(E) \) and \( u \in F^0 \), then

\[
u \cdot a = \sum_{v \in F^0} \left( \sum_{p \in \phi(\alpha \Path_{wp}(F_d))} k_p \right) v.
\]

Let \( T \to E \) be a covering of the weighted graph \( E \) and \( F \) a representation for \( T \). By Lemma 4, the \( L_K(T) \)-module \( V_F \) is also a \( L_K(E) \)-module. In fact, for a particular kind of coverings, namely universal coverings, we can say more.

**Corollary 23.** Let \( E \) be a weighted graph, \( T \) the universal cover of \( E \) and \( F \to \hat{T} \) a connected representation graph of \( T \). Then any \( L_K(T) \)-submodule \( A \subseteq V_F \) is also an \( L_K(E) \)-module.

**Proof.** Since by Lemma 4, \( F \) is a representation for \( E \), \( V_F \) is an \( L_K(E) \)-module as well as an \( L_K(T) \)-module. Let \( A \subseteq V_F \) be an \( L_K(T) \)-submodule and \( a = \sum k_i v_i \in A \). It is enough to show that for an edge \( e \in L_K(E) \), we have \( ae, ae^* \in A \). We have \( ae = \sum k_i v_i e = \sum k_i r_i(f_i) \), for those \( i \) such that \( \sigma(f_i) = e \). Since \( F \to T \), we then have \( af_i = k_i r_i(f_i) \in A \) as \( A \) is a \( L_K(T) \)-module. Thus \( ae = \sum k_i r_i(f_i) \in A \). The argument for \( ae^* \) is similar. \( \square \)

Recall that a weighted Leavitt path algebra is a \( \mathbb{Z}^n \)-graded ring, where \( n \) is the maximum weight in the graph (see §3). In particular, Leavitt path algebras (associated to graphs of weight one) are \( \mathbb{Z} \)-graded. This grading has played a crucial role in determining the algebraic structure of these algebras. We next characterise the representation graphs whose associated \( L_K(E) \)-modules are graded. To do this we will use the following statement whose proof is easy and we will leave it to the reader.

**Lemma 24.** Let \( A \) be a \( \Gamma \)-graded ring and \( M \) a right \( A \)-module, where \( \Gamma \) is a totally ordered abelian group. If there is a homogeneous element \( a \in A \) with \( \deg(a) \neq 0 \) and \( 0 \neq m \in M \) such that \( ma = m \), then \( M \) cannot be \( \Gamma \)-graded.

Let \( F \) be a connected representation graph for the weighted graph \( E \). Since \( F \) is an immersion into \( \hat{E} \), we have a monomorphism \( \pi(F) \to \pi(\hat{E}) \). Combining this with the length map \( (7) \), we obtain a homomorphism \( | | : \pi(F) \to \mathbb{Z}^n \). This allows us to give a criterion on when the \( L_K(E) \)-module \( V_F \) is a graded module.

**Theorem 25.** Let \( E \) be a weighted graph and \( F \) a connected representation graph of \( E \). Then the \( L(E) \)-module \( V_F \) is graded if and only if for (reduced) paths \( p, q \) of \( F \) with \( s(p) = s(q) \) and \( r(p) = r(q) \), we have \( |p| = |q| \).

**Proof.** Suppose for any paths \( p, q \) of \( F \), with \( s(p) = s(q) \) and \( r(p) = r(q) \) we have \( |p| = |q| \). Choose a base vertex \( v \) in \( F \) and assign \( \deg(v) = 0 \). Define a map \( \deg : F^0 \to \mathbb{Z}^n \), by \( \deg(v) = |q| \), where \( q \) is a (reduced) path with \( s(q) = v \) and \( r(q) = w \). Since \( F \) is connected and the length of paths connecting \( v \) to \( w \) are the same, the map is well-defined. For \( \alpha \in \mathbb{Z}^n \), define

\[
(V_F)_\alpha := \left\{ \sum k_i v_i \mid \deg(v_i) = \alpha \right\}.
\]

Clearly \( V_F = \bigoplus_{\alpha \in \mathbb{Z}^n} (V_F)_\alpha \). Next we check that \( (V_F)_\alpha L_K(E)_\beta \subseteq (V_F)_{\alpha + \beta} \), where \( \alpha, \beta \in \mathbb{Z}^n \). It is enough to show that \( wp \in (V_F)_{\alpha + \beta} \), where \( w \in (V_F)_\alpha \cap F^0 \) and \( p \) is a monomial in \( L_K(E)_\beta \). Either \( wp = 0 \) or by the definition of representation, one can lift \( p \) to a unique path \( q \) in \( F \) such that \( |q| = |p| \).
and by Lemma 21, \( wq = z \), where \( z = r(q) \). Since there is a path \( t \) with \( s(t) = v, r(t) = w \) and \(|t| = \alpha\), we have \( s(tq) = v, r(tq) = z \). Since \(|tq| = |t| + |q|\) we have \( z \in (V_F)_{\alpha + \beta} \).

Conversely, suppose \( V_F \) is a graded \( L_K(E) \)-module. If there are paths \( p,q \) in \( F \) with \( s(p) = s(q) \) and \( r(p) = r(q) \) but \(|p| \neq |q|\), then \( s(p) \neq pq^* \in \pi_1(F, s(p)) \). Let \( t \) be the image of \( pq^* \) in \( E \). Then by the definition of representation, \( vt = v \). Since \(|pq^*| \neq 0\), it follows from Lemma 24 that \( V_F \) is not graded, which is a contradiction.

**Example 26.** Consider the weighted graph \( e_1, e_2 \cup v \) from Example 16. Theorem 25 will then give the \( L_K(2,2) \)-modules \( V_{F_1} \) and \( V_{F_2} \) are \( \mathbb{Z}^2 \)-graded whereas \( V_{F_3}, V_{F_4}, V_{F_5}, V_{F_6} \) and the simple module \( V_{F_5} \) are not graded.

Consider the graph \( E = \begin{tikzpicture} \node at (0,0) {\circ}; \node at (1,0) {\circ}; \node at (0,0.5) {\circ}; \node at (1,0.5) {\circ}; \draw (0,0) -- (1,0); \draw (0,0.5) -- (1,0.5); \draw (0,0) -- (0,0.5); \draw (1,0) -- (1,0.5); \end{tikzpicture} \) from the Introduction and its representation graphs \( F \) and \( G \) (see (3) and (4)). Theorem 25 will then give the \( L_K(E) \)-module \( V_G \) is \( \mathbb{Z} \)-graded whereas \( V_F \) is not a graded module.

**Corollary 27.** If \( T \) is a universal representation of \( E \) then \( V_T \) is graded.

**Proof.** If \( T \) is a universal cover, then \( \pi(T) = 1 \). This implies the condition of Theorem 25. \( \square \)

### 3.2. Irreducible representations of weighted Leavitt path algebras

In this section we characterise those representation graphs that induce simple modules on the level of Leavitt path algebras. We continue to denote by \( E \) a weighted graph, \((F, \phi)\) a representation graph for \( E \) and \( V_F \) the \( L_K(E) \)-module defined by \((F, \phi)\). In the following theorem we show that a representation graph \( F \) is irreducible (see Definition 6) if and only if \( V_F \) is a simple \( L_K(E) \)-module.

**Theorem 28.** Let \( E \) be a weighted graph, \((F, \phi)\) a representation graph for \( E \) and \( V_F \) the \( L_K(E) \)-module defined by \( F \). Then the following are equivalent.

(i) The \( L_K(E) \)-module \( V_F \) is simple.

(ii) The representation graph \( F \) is connected and

\[ \text{for any } x \in V_F \setminus \{0\}, \text{ there is } a \in L_K(E), \text{ such that } x \cdot a \in F^0. \]

(iii) The representation graph \( F \) is connected and

\[ \text{for any } x \in V_F \setminus \{0\}, \text{ there is } a k \in K \text{ and } p \in \text{Path}(\hat{E}_d), \text{ such that } x \cdot kp \in F^0. \]

(iv) The representation graph \( F \) is an irreducible representation.

**Proof.** (i) \( \implies \) (iv). If \( C \) is a connected component of \( F \), then the subspace of \( V_F \) generated by \( C^0 \) is clearly invariant under the action of \( L := L_K(E) \). Hence \( F \) must be connected. Now assume that there are \( u \neq v \in F^0 \) such that \( \phi(u \text{Path}(F_d)) = \phi(v \text{Path}(F_d)) \). Consider the submodule \( (u - v)L \subseteq V_F \). Since \( V_F \) is simple by assumption, we have \( (u - v)L = V_F \). Hence there is an \( a \in L \) such that \( (u - v)a = v \). Clearly there is an \( n \geq 1 \), \( k_1, \ldots, k_n \in K^\times \) and pairwise distinct \( p_1, \ldots, p_n \in \text{Path}(\hat{E}_d) \) such that \( a = \sum_{s=1}^n k_s p_s \). We may assume that \( (u - v)p_s \neq 0 \), for any \( 1 \leq s \leq n \). It follows from Lemma 21 that \( p_s \in \phi(u \text{Path}(F_d)) = \phi(v \text{Path}(F_d)) \), for any \( s \) and moreover, that \( (u - v)p_s = u_s - v_s \) for some distinct \( u_s, v_s \in F^0 \). Hence

\[ v = (u - v)a = (u - v)(\sum_{s=1}^n k_s p_s) = \sum_{s=1}^n k_s (u_s - v_s) \]

which contradicts Lemma 63.
Let $x \in V_F \setminus \{0\}$. Then there is an $n \geq 1$, pairwise disjoint $v_1, \ldots, v_n \in F^0$ and $k_1, \ldots, k_n \in K^\times$ such that $x = \sum_{s=1}^n k_s v_s$. If $n = 1$, then $x \cdot k_1^{-1} \phi_0(v_1) = v_1$. Suppose now that $n > 1$. By assumption, we can choose a $p_1 \in \phi(v_1, \text{Path}(F_d))$ such that $p_1 \not\in \phi(v_2, \text{Path}(F_d))$. Clearly $x \cdot p_1 \not= 0$ is a linear combination of at most $n - 1$ vertices from $F^0$. Proceeding this way, we obtain paths $p_1, \ldots, p_m$ such that $x \cdot p_1 \cdots p_m = kv$ for some $k \in K^\times$ and $v \in F^0$. Hence $x \cdot k^{-1} p_1 \cdots p_m = v$.

(iii) $\implies$ (ii). Trivial.

(ii) $\implies$ (i). Let $U \subseteq V_F$ be a nonzero $L_K(E)$-submodule and $x \in U \setminus \{0\}$. By assumption, there is an $a \in L_K(E)$ and a $v \in F^0$ such that $v = x \cdot a \in U$. Let now $v'$ be an arbitrary vertex in $F^0$. Since by assumption $F$ is connected, there is a $p \in \text{Path}_{V'}(F_d)$. It follows from Lemma 21 that $v' = v \cdot \phi(p) \in U$. Hence $U$ contains $F^0$ and thus $U = V_F$. $\square$

3.3. The fullness of the functor $V$. Let $\alpha : (F, \phi) \to (G, \psi)$ be a morphism in $\text{RG}(E)$. Then it induces a surjective $L_K(E)$-module homomorphism $V_\alpha : V_F \to V_G$ such that $V_\alpha(u) = \alpha^0(u)$, for any $u \in F^0$.

The example below shows that $V$ is not full, namely, there can be $L_K(E)$-module homomorphisms $V_F \to V_G$ that are not induced by a morphism $(F, \phi) \to (G, \psi)$.

**Example 29.** Suppose $E$ is the weighted graph

$$E : u \quad u, v.$$ 

Consider the representation graphs $(F, \phi)$ and $(G, \psi)$ for $E$ given below.

$$(F, \phi) : \quad \quad \quad \quad (G, \psi) : \quad \quad \quad \quad$$

Since $(G, \psi)$ is a quotient of $(F, \phi)$, by Theorem 12, there is a morphism $\alpha : (F, \phi) \to (G, \psi)$, which induces a homomorphism $V_\alpha : V_F \to V_G$. Although $(F, \phi)$ is not a quotient of $(G, \psi)$, there is an $L_K(E)$-module homomorphism $\sigma : V_G \to V_F$ in the opposite direction such that $\sigma(u) = u_1 + u_2$ and $\sigma(v) = v_1 + v_2$. Note that $V_\alpha$ and $\sigma$ are not inverse to each other.

**Question 30.** Can it happen that $V_F \cong V_G$ as $L_K(E)$-modules although $(F, \phi) \not\cong (G, \psi)$ in $\text{RG}(E)$?

The authors do not know the answer to Question 30. But we will show that if $V_F \cong V_G$, then $(F, \phi)$ and $(G, \psi)$ must be equivalent.

**Lemma 31.** Let $(F, \phi)$ and $(G, \psi)$ be objects in $\text{RG}(E)$ and let $\sigma : V_F \to V_G$ be a $L_K(E)$-module homomorphism. Let $u \in F^0$ and $\sigma(u) = \sum_{s=1}^n k_s v_s$, where $n \geq 1$, $k_1, \ldots, k_n \in K^\times$ and $v_1, \ldots, v_n$ are pairwise distinct vertices from $G^0$. Then $\phi(u, \text{Path}(F_d)) = \psi(v_s, \text{Path}(G_d))$, for any $1 \leq s \leq n$. 
Proof. Let \( p \in \text{Path}(\tilde{E}_d) \) such that \( p \not\in \phi(u \text{Path}(F_d)) \). Then
\[
0 = \sigma(0) = \sigma(u \cdot p) = \sigma(u) \cdot p = \sum_{s=1}^{n} k_s v_s \cdot p = \sum_{s=1}^{n} k_s (v_s \cdot p)
\]
by Lemma 21. One more application of Lemma 21 gives that \( v_s \cdot p = 0 \), for any \( 1 \leq s \leq n \), whence \( p \not\in \psi(v_s \text{Path}(G_d)) \) for any \( 1 \leq s \leq n \). Hence we have shown that \( \phi(u \text{Path}(F_d)) \supseteq \psi(v_s \text{Path}(G_d)) \), for any \( 1 \leq s \leq n \). It follows from Lemma 8 that \( \phi(u \text{Path}(F_d)) = \psi(v_s \text{Path}(G_d)) \), for any \( 1 \leq s \leq n \). \( \square \)

**Proposition 32.** Let \((F, \phi)\) and \((G, \psi)\) be objects in \( \text{RG}(E) \). If there is a nonzero \( L_K(E)\)-module homomorphism \( \sigma : V_F \to V_G \), then \((F, \phi) \Rightarrow (G, \psi)\).

**Proof.** The proposition follows from Lemma 31 and the definition of the equivalence \( \Rightarrow \). \( \square \)

**Proposition 33.** Let \((F, \phi)\) and \((G, \psi)\) be irreducible representation graphs for the weighted graph \( E \). Then \( V_F \cong V_G \) as \( L_K(E)\)-modules if and only if \((F, \phi) \cong (G, \psi)\).

**Proof.** Clearly isomorphic objects in \( \text{RG}(E) \) yield isomorphic \( L_K(E)\)-modules. Hence we only have to show that \( V_F \cong V_G \) implies \((F, \phi) \cong (G, \psi)\). Suppose that \( V_F \cong V_G \). Then \((F, \phi) \cong (G, \psi)\) by Proposition 32, i.e., \((F, \phi)\) and \((G, \psi)\) are in the same \( \Rightarrow \)-equivalence class \( C \). Since they are irreducible, it follows from Corollary 15 that \((F, \phi) \cong (S_C, \zeta_C) \cong (G, \psi)\). \( \square \)

### 3.4. Indecomposability of \( L_K(E)\)-modules \( V_F \)

Recall that for a ring \( R \), an \( R\)-module is called **indecomposable** if it is non-zero and cannot be written as a direct sum of two non-zero submodules. It is easy to see that an \( R\)-module \( M \) is indecomposable if and only if the only idempotent elements of the endomorphism ring \( \text{End}_R(M) \) are 0 and 1.

We will show in Theorem 43 that for any universal representation \( T \) of the graph \( E \), the \( L_K(E)\)-module \( V_T \) is indecomposable. However, Example 34 below shows that in general the indecomposability of \( L_K(E)\)-module \( V_F \), for a representation \((F, \phi)\), depends on the ground field \( K \).

**Example 34.** Let \( E \) be the graph
\[
\text{•} \quad \bigcup_{e} \quad \text{•}
\]
and the graph \( F \) below the representation graph for \( E \):

If \( \epsilon \in \text{End}_L(V_F) \), then by Lemma 31 there are \( k, l \in K \) such that \( \epsilon(v_1) = kv_1 + lv_2 = v_1 \cdot (kv + le) \). Conversely, if \( k, l \in K \), then there is a uniquely determined endomorphism \( \epsilon \in \text{End}_L(V_F) \) such that \( \epsilon(v_1) = kv_1 + lv_2 \). This yields a bijection between \( \text{End}_L(V_F) \) and \( K \times K \).

Let now \( \epsilon \in \text{End}_L(V_F) \) be the endomorphism corresponding to a pair \((k, l) \in K \times K \). Since \( v_1 \) generates the \( L_K(E)\)-module \( V_F \), the endomorphism \( \epsilon \) is idempotent if and only if \( \epsilon(v_1) = \epsilon^2(v_1) \). Clearly
\[
\epsilon^2(v_1) = v_1 \cdot (kv + le)^2 = v_1 \cdot (k^2v + 2kle + l^2e^2) = (k^2 + l^2)v_1 + 2klv_2.
\]
Thus $\epsilon$ is idempotent if and only if
\[
k = k^2 + l^2 \text{ and } l = 2kl.
\] (11)

If $2 = 0$ in $K$, then the only solutions for (11) are $(k, l) = (0, 0)$ and $(k, l) = (1, 0)$. The corresponding endomorphisms are $\epsilon = 0$ and $\epsilon = \text{id}_{V_p}$. Thus $V_F$ is an indecomposable $L_K(E)$-module in this case. If $2 \neq 0$, then there are two more solutions for (11), namely $(k, l) = (1/2, 1/2)$ and $(k, l) = (1/2, -1/2)$. Thus $V_F$ is not indecomposable in this case.

Fix an $\equiv$-equivalence class $C$ and define $(T, \xi) = (T_C, \xi_C)$ and $(S, \zeta) = (S_C, \zeta_C)$ as in Subsection 2.5. Since the $L_K(E)$-module $V_S$ is simple, it is indecomposable. We will show that the $L_K(E)$-module $V_T$ is indecomposable too.

Let $p, p' \in \text{Path}(\hat{E}_d)$ be non-backtracking paths such that $r(p) = s(p') = v$. We define a non-backtracking path $p \ast p' \in \text{Path}(\hat{E}_d)$ as follows. If $p, p' \in \text{Path}(\hat{E}_d) \setminus \{v\}$ let $p \ast p'$ be the element of $\text{Path}(\hat{E}_d)$ one gets by removing all subwords of the form $e_i e_i^*$ and $e_i^* e_i$ from the juxtaposition $pp'$ (if $p' = p^*$, then $p \ast p' := v$). Moreover define $v \ast p := p$ and $p \ast v := p$.

Define the group
\[
G := \{ p \in \phi(u\text{Path}(F_d))_{nb} \mid v_p \sim v_{\phi(u)} \}
\] (12)

In Proposition 39, we will show that $(G, \ast)$ is a free group. We need several lemmas.

**Lemma 35.** If $p \in \phi(u\text{Path}(F_d))_{nb}$, then $v_p = v_{\phi(u)} \ast p$.

**Proof.** Clearly $v_p = v_{\phi(u)} \ast p$ if $p = \phi(u)$. Suppose now that $p = x_1 \ldots x_n$ for some $x_1, \ldots, x_n \in \hat{E}^1 \cup (\hat{E}^1)^*$. For $1 \leq i \leq n$ define
\[
\hat{e}_{x_1 \ldots x_i} := \begin{cases} e_{x_1 \ldots x_i}, & \text{if } x_i \in \hat{E}^1, \\ e_{x_1 \ldots x_i}^*, & \text{if } x_i \in (\hat{E}^1)^*. \end{cases}
\]

One checks easily that $q := \hat{e}_{x_1} \hat{e}_{x_2} \ldots \hat{e}_{x_{n+1}} \in v_{\phi(u)}\text{Path}_p(T_d)$ and $\xi(q) = p$. It follows from Lemma 21 that $v_{\phi(u)} \ast p = v_{p}$.

**Lemma 36.** Let $p, p' \in G$. Then $p \ast p'$ is defined and $p \ast p' \in \phi(u\text{Path}(F_d))_{nb}$.

**Proof.** Let $r, r' \in u\text{Path}(F_d)$ such that $\phi(r) = p$ and $\phi(r') = p'$. By the proof of Lemma 35 there are (uniquely determined) paths $q \in v_{\phi(u)}\text{Path}_p(T_d)$ and $q' \in v_{\phi(u)}\text{Path}_{p'}(T_d)$ such that $\xi(q) = p$ and $\xi(q') = p'$. Let $\alpha : (T, C) \to (F, \phi)$ be the morphism defined in the proof of Proposition 14. One checks easily that $\alpha(q) = r$ and $\alpha(q') = r'$. Since $v_p \sim v_{\phi(u)}$, there is a $q'' \in v_p\text{Path}(T_d)$ such that $\xi(q'') = p' = \xi(q')$. Set $r'' := \alpha(q'')$. Clearly $s(r'') = s(\alpha(q'')) = \alpha(s(q'')) = \alpha(v_{p'}) = \alpha(r(q)) = r(\alpha(q)) = r(r)$. Hence $rr'' \in u\text{Path}(F_d)$. It follows that $pp' = \phi(r r'') \in \phi(u\text{Path}(F_d))$. In particular $r(p) = s(p') = \phi(u)$ and hence $p \ast p'$ is defined.

Write $pp' = x_1 \ldots x_n$ and $rr'' = y_1 \ldots y_n$. Suppose that $x_j x_{j+1} = e_i e_i^*$ for some $1 \leq j \leq n - 1$, $e \in E^1$ and $1 \leq i \leq w(e)$. Since $\phi(y_j) = x_j$ and $\phi(y_{j+1}) = x_{j+1}$, there are $f, g \in F^1$ such that $y_j = f$ and $y_j = g^*$. Moreover $r(f) = r(g)$ and $\phi(f) = e_i = \phi(g)$. It follows from Definition 2(2) that $f = g$. Hence $y_1 \ldots y_{j-1} y_j y_{j+1} \ldots y_n \in u\text{Path}(F_d)$ and $\phi(y_1 \ldots y_{j-1} y_j y_{j+1} \ldots y_n) = x_1 \ldots x_{j-1} x_{j+1} \ldots x_n$. Similarly, if $x_j x_{j+1} = e_i^* e_i$ for some $1 \leq j \leq n - 1$, $e \in E^1$ and $1 \leq i \leq w(e)$, then $y_1 \ldots y_{j-1} y_j y_{j+1} \ldots y_n \in u\text{Path}(F_d)$ and $\phi(y_1 \ldots y_{j-1} y_j y_{j+1} \ldots y_n) = x_1 \ldots x_{j-1} x_{j+1} \ldots x_n$ (follows from Definition 2(1)). Thus $p \ast p' \in \phi(u\text{Path}(F_d))_{nb}$.

**Lemma 37.** Let $p, p' \in G$. Then $p \ast p' \in G$.

\[Q.E.D.\]
Proof. By the previous lemma $p \ast p'$ is defined and $p \ast p' \in \phi(u \, \text{Path}(F_d))_D$. It remains to show that $v_{pp'} \sim v_{\phi(u)}$. By the proof of Lemma 35 there are paths $q \in v_{\phi(u)} \, \text{Path}_{v_p}(T_d)$ and $q' \in v_{\phi(u)} \, \text{Path}_{v_{pp'}}(T_d)$ such that $\xi(q) = p$ and $\xi(q') = p'$. Since $v_p \sim v_{\phi(u)}$, there is a path $q'' \in v_p \, \text{Path}(T_d)$ such that $\xi(q'') = p' = \xi(q')$. By Lemmas 35 and 21 we have

$$v_{pp'} = v_{\phi(u)} \cdot (p \ast p') = v_{\phi(u)} \cdot (pp') = r(qq'') = r(q'')$$

and hence $q'' \in v_p \, \text{Path}_{v_{pp'}}(T_d)$.

We will show that $v_{pp'} \sim v_{p'}$ which implies $v_{pp'} \sim v_{\phi(u)}$. Let $p'' \in \xi(v_{p'} \, \text{Path}(T_d))$. Then there is an $r \in v_p \, \text{Path}(T_d)$ such that $\xi(r) = p''$. Since $q' r \in v_{\phi(u)} \, \text{Path}(T_d)$, we have $p'' p' = \xi(q' r) \in \xi(v_{\phi(u)} \, \text{Path}(T_d))$. Since $v_{\phi(u)} \sim v_p$, there is an $s \in v_p \, \text{Path}(T_d)$ such that $\xi(s) = p' p''$. Write $s = s_1 s_2$ such that $\xi(s_1) = p'$ and $\xi(s_2) = p''$. Clearly $s_1 = v_p$. It follows from Lemma 3 that $s_1 = q''$. Since $q'' \in v_p \, \text{Path}_{v_{pp'}}(T_d)$, we obtain $s_2 \in v_{pp'} \, \text{Path}(T_d)$. Hence $p'' = \xi(s_2) \in \xi(v_{pp'} \, \text{Path}(T_d))$. We have shown that $\xi(v_{pp'} \, \text{Path}(T_d)) \subseteq \xi(v_{\phi(u)} \, \text{Path}(T_d))$. It follows from Lemma 8 that $\xi(v_{p'} \, \text{Path}(T_d)) = \xi(v_{\phi(u)} \, \text{Path}(T_d))$ and thus $v_{pp'} \sim v_{p'}$.

Lemma 38. Let $p \in G$. Then $p^* \in G$.

Proof. By the proof of Lemma 35 there is a path $q \in v_{\phi(u)} \, \text{Path}_{v_p}(T_d)$ such that $\xi(q) = p$. Clearly $q^* \in v_p \, \text{Path}_{v_{pp'}}(T_d)$ and $\xi(q^*) = p^*$. Since $v_p \sim v_{\phi(u)}$, there is a path $q' \in v_{\phi(u)} \, \text{Path}(T_d)$ such that $\xi(q') = p^*$. Let $\alpha : (T, \xi) \to (F, \phi)$ be the morphism defined in the proof of Proposition 14. Then $\alpha(q') \in u \, \text{Path}(F_d)$ and $\phi(\alpha(q')) = \xi(q') = p^*$. Hence $p^* \in \phi(u \, \text{Path}(F_d))$. Moreover, $p^*$ is not backtracking since $p$ is not backtracking.

It remains to show that $v_{p^*} \sim v_{\phi(u)}$. Let $p' \in \xi(v_{p^*} \, \text{Path}(T_d))$. Then there is an $r \in v_{p^*} \, \text{Path}(T_d)$ such that $\xi(r) = p'$. By the proof of Lemma 35 there is a path $q'' \in v_{\phi(u)} \, \text{Path}_{v_{p^*}}(T_d)$ such that $\xi(q'') = p^*$. Clearly $q'' r \in v_{\phi(u)} \, \text{Path}(T_d)$ and hence $p^* p' = \xi(q'' r) \in \xi(v_{\phi(u)} \, \text{Path}(T_d))$. Since $v_{\phi(u)} \sim v_p$, there is an $s \in v_p \, \text{Path}(T_d)$ such that $\xi(s) = p^* p'$. Write $s = s_1 s_2$ such that $\xi(s_1) = p^*$ and $\xi(s_2) = p'$. Clearly $s_1 \in v_{p^*} \, \text{Path}(T_d)$. It follows from Lemma 3 that $s_1 = q^*$. Since $q' \in v_p \, \text{Path}_{v_{\phi(u)}}(T_d)$, we obtain $s_2 \in v_{\phi(u)} \, \text{Path}(T_d)$. Hence $p' = \xi(s_2) \in \xi(v_{p^*} \, \text{Path}(T_d))$. We have shown that $\xi(v_{p^*} \, \text{Path}(T_d)) \subseteq \xi(v_{\phi(u)} \, \text{Path}(T_d))$. It follows from Lemma 8 that $\xi(v_{p^*} \, \text{Path}(T_d)) = \xi(v_{\phi(u)} \, \text{Path}(T_d))$ and thus $v_{p^*} \sim v_{\phi(u)}$.

Proposition 39. The group $(G, \ast)$ defined in (12) is a free group.

Proof. By Lemma 37, $\ast$ defines a binary operation on $G$. Clearly this operation is associative. Moreover, $\phi(u) \ast p = p \ast \phi(u)$ for any $p \in G$. If $p \in G$, then $p^* \in G$ by Lemma 38. Clearly $pp^* = p^* p = \phi(u)$. Hence $(G, \ast)$ is a group.

Let $F$ be the free group on $X := \{e_i \mid e \in E^1, 1 \leq i \leq w(e)\}$. We identify $F$ with the set of all reduced words over the alphabet $X \cup X^{-1}$ (a word over $X \cup X^{-1}$ is reduced if it does contain any subwords of the form $xx^{-1}$ or $x^{-1}x$ where $x \in X$). The product of two reduced words $w$ and $w'$ is the reduced word one gets by removing all subwords of the form $xx^{-1}$ or $x^{-1}x$ from the juxtaposition $ww'$.

Define a map $\theta : G \to F$ as follows. If $p = x_1 \ldots x_n \in G \setminus \{\phi(u)\}$, let $\theta(p)$ be the word one gets by replacing any letter $e_i^+$ by $e_i^{-1}$. Clearly $\theta(p)$ is a reduced word since $p$ is not backtracking. Moreover, we define $\theta(\phi(u))$ as the identity element of $F$, namely the empty word. Clearly $\theta$ is a group homomorphism. Moreover, $\theta$ is injective and hence $G$ is isomorphic to a subgroup of a free group. It follows from the Nielsen-Schreier theorem that $G$ is a free group.
Let \( W \subseteq V_T \) be the linear span of the \( \sim \)-equivalence class of \( v_{\phi(u)} \), i.e. \( W = \langle v_p \mid p \in G \rangle \). Let \( A \) be the subalgebra of \( L_K(E) \) generated by the image of the group \( G \) in \( L_K(E) \).

**Lemma 40.** The \( K \)-vector space \( W \) is a right \( A \)-module where the action of \( A \) on \( W \) is induced by the action of \( L_K(E) \) on \( W \).

**Proof.** It suffices to show that if \( p \in G \) and \( v_{p'} \sim v_{\phi(u)} \), then \( v_{p'} \cdot p = v_{p''} \) for some \( v_{p''} \sim v_{\phi(u)} \). Since \( v_{p'} \sim v_{\phi(u)} \), we have \( p' \in G \). It follows from Lemmas 35 and 36 that
\[
v_{p'} \cdot p = (v_{\phi(u)} \cdot p') \cdot p = v_{\phi(u)} \cdot p' p = v_{\phi(u)} \cdot p' * p = v_{p''p'}.
\]
By Lemma 37 we have \( p * p' \in G \). Thus \( v_{p*p'} \sim v_{\phi(u)} \).

Since \( W \) is a right \( A \)-module, there is a \( K \)-algebra homomorphism \( \delta : A \to \text{End}_K(W)^{\text{op}} \) defined by \( \delta(a)(w) = w \cdot a \). Define \( \bar{A} := A / \ker(\delta) \). Let \( \bar{\delta} : \bar{A} \to \bar{A} \) be the canonical \( K \)-algebra homomorphism and \( \bar{\delta} : \bar{A} \to \text{End}_K(W)^{\text{op}} \) the \( K \)-algebra homomorphism induced by \( \delta \). Then the following diagram commutes.

\[
\begin{array}{ccc}
A & \xrightarrow{\delta} & \text{End}_K(W)^{\text{op}} \\
\downarrow & & \\
\bar{A} & \xrightarrow{\bar{\delta}} & \\
\end{array}
\]

\( W \) becomes a right \( \bar{A} \)-module by defining \( w \cdot \bar{a} = \bar{\delta}(\bar{a})(w) = \delta(a)(w) = w \cdot a \).

**Proposition 41.** The algebra \( \bar{A} \) is isomorphic to the group algebra \( K[G] \).

**Proof.** Let \( a \in A \). Then \( a = \sum_{i=1}^n k_ip_{i1} \cdots p_{im} \) for some \( n, m_1, \ldots, m_n \in \mathbb{N} \), \( k_1, \ldots, k_n \in K \) and \( p_{i,j} \in G \) \((1 \leq i \leq n, 1 \leq j \leq m_i)\). Clearly \( p_{i1} \cdots p_{im} - p_{i1} \cdots p_{im} \in \ker(\delta) \) for any \( 1 \leq i \leq n \). Hence \( a \equiv \sum_{i=1}^n k_ip_{i1} \cdots \cdot p_{im} \) \( \mod \ker(\delta) \). We have shown that any element of \( \bar{A} \) has a representative of the form \( \sum_{p \in G} k_p p \).

Suppose that \( \sum_{p \in G} k_p p \equiv \sum_{p \in G} l_p p \mod \ker(\delta) \). Then \( \delta(\sum_{p \in G} (k_p - l_p)p) = 0 \) and hence
\[
0 = \delta(\sum_{p \in G} (k_p - l_p)p)(v_{\phi(u)}) = v_{\phi(u)} \cdot \left( \sum_{p \in G} (k_p - l_p)p \right) = \sum_{p \in G} (k_p - l_p)v_p
\]
by Lemma 35. Since the \( v_p \) are linearly independent, we obtain \( k_p = l_p \) for any \( p \in G \). Hence any element of \( \bar{A} \) has precisely one representative of the form \( \sum_{p \in G} k_p p \).

Define a map \( \eta : \bar{A} \to K[G] \) by \( \eta(\sum_{p \in G} k_p p) = \sum_{p \in G} k_p p \). By the previous two paragraphs \( \eta \) is well-defined. Moreover, \( \eta \) is clearly bijective. We leave it to the reader to check that \( \eta \) is a \( K \)-algebra homomorphism.

**Proposition 42.** The \( A \)-module \( W \) is free of rank 1 as a \( \bar{A} \)-module.

**Proof.** Let \( \bar{a} \in \bar{A} \). Then \( \bar{a} = \sum_{p \in G} k_p p \) for some \( k_p \in K \) (that was shown in the proof of Proposition 41). Hence
\[
v_{\phi(u)} \cdot \bar{a} = v_{\phi(u)} \cdot \sum_{p \in G} k_p p = v_{\phi(u)} \cdot \sum_{p \in G} k_p p = \sum_{p \in G} k_p v_p
\]
by Lemma 35. It follows that \( \{ v_{\phi(u)} \} \) is a basis for the \( \bar{A} \)-module \( W \).

We are in position to prove the main result of this section.

**Theorem 43.** The \( L_K(E) \)-module \( V_T \) defined by \( (T, \xi) = (T_C, \xi_C) \) is indecomposable.
**Proof.** Let \( \epsilon \in \text{End}_L(V_T) \) be an idempotent endomorphism. It follows from Lemma 31 that \( \epsilon(W) = W \). Hence \( \epsilon|_W \in \text{End}_K(W) \). Clearly \( \epsilon(w \cdot a) = \epsilon(w \cdot a) = \epsilon(w) \cdot a = \epsilon(w) \cdot \bar{a} \) for any \( a \in A \) and \( w \in W \). Hence \( \epsilon|_W \in \text{End}_K(W) \). By Proposition 41 and 42 we have \( \text{End}_K(W) \cong \bar{A} \cong K[G] \). Since \( G \) is free by Proposition 39, the group ring \( K[G] \) has no zero divisors by [19, Theorem 12] (note that as explained in the paragraph before [19, Theorem 12], any free group is indicable throughout). It follows that 0 and 1 are the only idempotents of \( K[G] \). Hence \( \epsilon|_W = 0 \) or \( \epsilon|_W = \text{id}_W \).

Let \( v_p \in T^0 \). Then \( \epsilon(v_p) = \epsilon(v_{\phi(u)} \cdot p) = \epsilon(v_{\phi(u)}) \cdot p \) by Lemma 35. Hence \( \epsilon = 0 \) if \( \epsilon|_W = 0 \) and \( \epsilon = \text{id} \) if \( \epsilon|_W = \text{id}_W \). Thus we have shown that \( V_T \) is indecomposable. \( \square \)

4. IRREDUCIBLE REPRESENTATIONS OF LEAVITT PATH ALGEBRAS

Let \( K \) be a field, \( E \) a directed graph and \( L_K(E) \) its associated Leavitt path algebra. For an infinite path \( p \) in \( E \), Xiao-Wu Chen constructed the left \( L_K(E) \)-module \( V[p] \) of the space of infinite paths tail-equivalent to \( p \) and proved that it is an irreducible representation of the Leavitt path algebra (see [10, Theorem 3.3]). A similar construction was also given for paths ending on a sink vertex. In this section we recover these irreducible modules via the representation graphs of \( E \). This gives another way to express these modules. Besides being more visual, this approach allows for carrying calculus on these modules with ease and produce indecomposable \( L_K(E) \)-modules via universal representations.

4.1. Chen simple modules via representation graphs. We briefly recall the construction of simple modules via infinite paths following the paper of Chen [10].

The set of all right-infinite paths in \( E \) is denoted by \( E^\infty \). If \( p = p_1p_2 \cdots \in E^\infty \) and \( n \geq 0 \), then we set \( \tau_{\leq n}(p) := p_1 \cdots p_n \in E^n \) and \( \tau_{> n}(p) := p_{n+1}p_{n+2} \cdots \in E^\infty \) (if \( n = 0 \), then \( \tau_{\leq n}(p) = s(p) \) and \( \tau_{> n}(p) = p \)). Two right-infinite paths \( p, q \in E^\infty \) are called tail-equivalent, denoted by \( p \sim q \), if there are \( m, n \geq 0 \) such that \( \tau_{> m}(p) = \tau_{> n}(q) \). This defines an equivalence relation on \( E^\infty \). We denote by \( \bar{E}^\infty \) the set of tail-equivalence classes, and for a path \( p \in E^\infty \) denote the corresponding class by \( [p] \).

A right-infinite path \( p \in E^\infty \) is called cyclic if \( p = ccc \ldots \) where \( c \) is a (finite) closed path. A \( p \in E^\infty \) is called rational if \( p \) is tail-equivalent to a cyclic path. Otherwise \( p \) is called irrational. The class \( [p] \) is called rational if \( p \) is rational and irrational if \( p \) is irrational.

If \( [p] \in \bar{E}^\infty \), then the corresponding Chen module is the \( K \)-vector space \( V_p \) with basis \([p]\). One can make the vector space \( V_p \) a right \( L_K(E) \)-module as follows. For any \( v \in E^0 \), \( e \in E^1 \) and \( q \in [p] \) define

\[
q \cdot v = \begin{cases} 
q, & \text{if } v = s(q) \\
0, & \text{else}
\end{cases} \\
q \cdot e = \begin{cases} 
\tau_{> 1}(q), & \text{if } e = \tau_{\leq 1}(q) \\
0, & \text{else}
\end{cases} \\
q \cdot e^* = \begin{cases} 
eq, & \text{if } r(e) = s(q) \\
0, & \text{else}
\end{cases}
\]

The \( K \)-linear extension of this action to all of \( V_p \) endows \( V_p \) with the structure of a right \( L_K(E) \)-module. Chen has proven that the module \( V_p \) is simple and that \( V_p \simeq V_q \) as right \( L_K(E) \)-modules if and only if \([p] = [q]\) [10].

Let \( u \in E^0 \) be a sink, i.e. a vertex that emits no edges. The corresponding Chen module is the \( K \)-vector space \( N_u \) with basis the set of finite paths ending in \( u \). The \( K \)-vector space \( N_u \) becomes
a right \( L_K(E) \)-module as follows. For a finite path \( q = q_1 \ldots q_l \in E^l \) and \( 0 \leq n \leq l \) we define \( \tau_{\leq n}(q) := q_1 \ldots q_n \in E^n \) and \( \tau_{> n}(q) := q_{n+1} \ldots q_l \in E^{l-n} \) (if \( n = 0 \), then \( \tau_{\leq n}(q) = s(q) \) and if \( n = l \), then \( \tau_{> n}(q) = r(q) \)). For a \( v \in E^0 \) set \( \tau_{\leq 1}(v) := v \) and \( \tau_{> 1}(v) := v \). For any \( v \in E^0 \), \( e \in E^1 \) and finite path \( q \) ending in \( u \) define \( q \cdot v, q \cdot e \) and \( q \cdot e^* \) as in the previous paragraph. The \( K \)-linear extension of this action to all of \( \mathcal{N}_u \) endows \( \mathcal{N}_u \) with the structure of a right \( L_K(E) \)-module. Chen has proven that the module \( \mathcal{N}_u \) is simple and that \( \mathcal{N}_u \simeq \mathcal{N}_v \) as right \( L_K(E) \)-modules if and only if \( u = v \). Moreover, if \( p \in E^\infty \) and \( u \) is a sink, then \( V_{[p]} \not\simeq \mathcal{N}_u \) as right \( L_K(E) \)-modules [10].

Translated to the unweighted setting the definition of a representation graph (Definition 2) reduces to the following.

**Definition 44.** Let \( E \) be a graph. A representation graph for \( E \) is a pair \( (F, \phi) \), where \( F = (F^0, F^1, s_F, r_F) \) is a directed graph and \( \phi = (\phi^0, \phi^1) : F \to E \) is a homomorphism of directed graphs such that the following hold:

1. For any \( v \in F^0 \) such that \( \phi^0(v) \) is not a sink, we have \( |s_F^{-1}(v)| = 1 \).
2. For any \( v \in F^0 \), the map \( \phi^1 : r^{-1}(v) \to r^{-1}(\phi^0(v)) \) is bijective.

**Example 45.** Let \( E \) be a graph with one vertex and two loops:

\[
\begin{array}{c}
\text{\( \infty \)} \\
\bigcup \\
\text{\( v \)} \\
\bigcup \\
\text{\( \infty \)}
\end{array}
\]

Then the universal covering graph of \( E \) is the Cayley graph of the free group with two generators \( \mathbb{F}_2 \).

Let \( (F, \phi) \) be a representation graph for \( E \) and \( V_F \) the \( K \)-vector space with basis \( F^0 \). For any \( v \in E^0 \), \( e \in E^1 \) and \( u \in F^0 \) define

\[
\begin{align*}
\hat{u} \cdot v &= \begin{cases} u, & \text{if } \phi^0(u) = v \\ 0, & \text{else} \end{cases}, \\
\hat{u} \cdot e &= \begin{cases} r_F(f), & \text{if } \exists f \in s_F^{-1}(u) : \phi^1(f) = e \\ 0, & \text{else} \end{cases}, \\
\hat{u} \cdot e^* &= \begin{cases} s_F(f), & \text{if } \exists f \in r_F^{-1}(u) : \phi^1(f) = e \\ 0, & \text{else} \end{cases}.
\end{align*}
\]

The \( K \)-linear extension of this action to all of \( V_F \) endows \( V_F \) with the structure of a right \( L_K(E) \)-module. We call \( V_F \) the \( L_K(E) \)-module defined by \( (F, \phi) \).

We will construct for any Chen module \( V_{[p]} \) (resp. \( \mathcal{N}_u \)) a representation graph \( (F, \phi) \) such that the \( L_K(E) \)-module \( V_F \) is isomorphic to \( V_{[p]} \) (resp. \( \mathcal{N}_u \)). We divide this into three cases:

**4.1.1. The case of \( \mathcal{N}_u \), where \( u \) is a sink.** Let \( u \in E^0 \) be a sink. We denote by \( P \) the set of all nontrivial finite paths ending in \( u \). Define a directed graph \( F \) by

\[
\begin{align*}
F^0 &= \{ v \} \cup \{ v_p \mid p \in P \}, \\
F^1 &= \{ f_p \mid p \in P \}, \\
s_F(f_p) &= v_p, & r_F(f_p) &= \begin{cases} v_{\tau_{> 1}(p)}, & \text{if } |p| \geq 2, \\ v, & \text{if } |p| = 1. \end{cases}
\end{align*}
\]

Define \( \phi^0(v) = u, \phi^0(v_p) = s(p) \) and \( \phi^1(f_p) = \tau_{\leq 1}(p) \). One checks easily that \( \phi = (\phi^0, \phi^1) : F \to E \) is a homomorphism of directed graphs and that Conditions (1) and (2) in Definition 44 are satisfied. Hence \( (F, \phi) \) is a representation graph for \( E \).
Define a map $\alpha : F^0 \to P \cup \{u\}$ by $\alpha(v) = u$ and $\alpha(v_p) = p$. Obviously $\alpha$ is a bijection and hence it induces an isomorphism $\hat{\alpha} : V_F \to \mathcal{N}_u$ of $K$-vector spaces. One checks easily that $\hat{\alpha}$ is an isomorphism of $L_K(E)$-modules.

4.1.2. The case of $V_p$, where $[p]$ is irrational. Let $[p] \in \mathbb{E}^\infty$ be an irrational class. Then

$$\tau_{>m}(p) \neq \tau_{>n}(p) \text{ for any distinct } m,n \geq 0. \quad (13)$$

Write $p = p_1p_2p_3 \ldots$. For any $i \in \mathbb{N}$ we denote by $P_i$ the set of all nontrivial finite paths $q$ such that $r(q) = s(p_i)$ and the last letter of $q$ is not equal to $p_{i-1}$ if $i \geq 2$. Define $F$ by

$$F^0 = \{v_i \mid i \in \mathbb{N}\} \cup \{v_{i,q} \mid i \in \mathbb{N}, q \in P_i\},$$
$$F^1 = \{f_i \mid i \in \mathbb{N}\} \cup \{f_{i,q} \mid i \in \mathbb{N}, q \in P_i\},$$
$$s_F(f_i) = v_i, \quad r_F(f_i) = v_{i+1},$$
$$s_F(f_{i,q}) = v_{i,q}, \quad r_F(f_{i,q}) = \begin{cases} v_{i,\tau_{>|q|}(q)}, & \text{if } |q| \geq 2, \\ v_i, & \text{if } |q| = 1. \end{cases}$$

Define $\phi^0(v_i) = s(p_i)$, $\phi^0(v_{i,q}) = s(q)$, $\phi^1(f_i) = p_i$ and $\phi^1(f_{i,q}) = \tau_{\leq 1}(q)$. One checks easily that $\phi = (\phi^0, \phi^1) : F \to E$ is a homomorphism of directed graphs and that Conditions (1) and (2) in Definition 44 are satisfied. Hence $(F, \phi)$ is a representation graph for $E$.

**Example 46.** For the graph $E$ of (2) and the infinite irrational path $p = eefef^2ef^3 \ldots$, the above construction gives the representation graph (4).

Define a map $\beta : F^0 \to [p]$ by

$$\beta(v_i) = p_ip_{i+1} \ldots \quad \text{and} \quad \beta(v_{i,q}) = qp_ip_{i+1} \ldots.$$ 

**Lemma 47.** The map $\beta$ defined above is bijective.

**Proof.** (Surjectivity) Let $p' \in [p]$. Let $k \geq 0$ be minimal such that $\tau_{>k}(p') = \tau_{>l}(p)$ for some $l \geq 0$. If $k = 0$, then $p' = p_ip_{i+1} \ldots$ for some $i$. Hence $p' = \beta(v_i)$. If $k > 0$, then $p' = p_1' \ldots p_k' p_ip_{i+1} \ldots$ for some $i$. Clearly $p_{i}' \neq p_{i-1}$ because of the minimality of $k$. Hence $q := p_i' \ldots p_k' \in P_i$ and $p' = \beta(v_{i,q})$.

(Injectivity) Because of (13), it cannot happen that $\beta(v_i) = \beta(v_j)$ for some $i \neq j \in \mathbb{N}$. Assume now that $\beta(v_{i,q}) = \beta(v_j)$ where $1 \leq i, j \leq n$ and $q \in P_i$. Then clearly

$$p_ip_{i+1} \ldots = \tau_{>|q|}(\beta(v_{i,q})) = \tau_{>|q|}(\beta(v_j)) = p_jp_{j+1} \ldots$$

for some $r \in \mathbb{N}$. It follows from (13) that $r = i$. Hence

$$q|q|p_ip_{i+1} \ldots = \tau_{>|q|-1}(\beta(v_{i,q})) = \tau_{>|q|-1}(\beta(v_j)) = p_{i-1}p_ip_{i+1} \ldots$$

and hence $q|q| = p_{i-1}$, which contradicts the assumption that $q \in P_i$.

Assume now that $\beta(v_{i,q}) = \beta(v_{i,q'})$ where $1 \leq i, j \leq n$, $q \in P_i$ and $q' \in P_j$. If $|q| \neq |q'|$, then we obtain a contradiction as in the previous paragraph (say $|q| < |q'|$; then consider $\tau_{>|q'|}(\beta(v_{i,q})) = \tau_{>|q'|}(\beta(v_{j,q'}))$). Suppose now that $|q| = |q'|$. It follows that $q = q'$ since $\beta(v_{i,q}) = \beta(v_{j,q'})$. Moreover, (13) implies that $i = j$ as desired. 

Since $\beta$ is a bijection, it induces an isomorphism $\hat{\beta} : V_F \to V_{[p]}$ of $K$-vector spaces. One checks easily that $\hat{\beta}$ is an isomorphism of $L_K(E)$-modules.
4.1.3. The case of $V_0$, where $|p|$ is rational. Let $[p] \in \widehat{E}^\infty$ be a rational class. Then we may assume that $p$ is cyclic, i.e. $p = \text{ccc} \ldots$, where $c$ is a closed path. We also may assume that $c$ is simple, i.e. $c$ is not a power of a shorter closed path.

Suppose that $c = c_1 \ldots c_n$. For any $1 \leq i \leq n$ we denote by $P_i$ the set of all nontrivial finite paths $q$ such that $r(q) = s(c_i)$ and the last letter of $q$ is not equal to $c_{i-1}$, respectively $c_n$ if $i = 1$. Define $F$ by

\[
F^0 = \{v_i \mid 1 \leq i \leq n\} \cup \{v_{i,q} \mid 1 \leq i \leq n, q \in P_i\},
\]
\[
F^1 = \{f_i \mid 1 \leq i \leq n\} \cup \{f_{i,q} \mid 1 \leq i \leq n, q \in P_i\},
\]
\[
s_F(f_i) = v_i, \quad r_F(f_i) = v_{i+1},
\]
\[
s_F(f_{i,q}) = v_{i,q}, \quad r_F(f_{i,q}) = \begin{cases} v_{i,\tau+1(q)}, & \text{if } |q| \geq 2, \\ v_i, & \text{if } |q| = 1. \end{cases}
\]

Here we use the convention $n + 1 = 1$. Define $\phi^0(v_i) = s(c_i), \phi^0(v_{i,q}) = s(q), \phi^1(f_i) = c_i$ and $\phi^1(f_{i,q}) = \tau(q)$. One checks easily that $\phi = (\phi^0, \phi^1) : F \to E$ is a homomorphism of directed graphs and that Conditions (1) and (2) in Definition 44 are satisfied. Hence $(F, \phi)$ is a representation graph for $E$.

**Example 48.** For the graph $E$ of (2) and the infinite rational path $p = efgefg \ldots$, the above construction gives the representation graph (3).

Define a map $\gamma : F^0 \to [p]$ by

$\gamma(v_i) = c_i \ldots c_n$$\text{ccc} \ldots$ and $\gamma(v_{i,q}) = qc_i \ldots c_n$$\text{ccc} \ldots$.

**Lemma 49.** Let $x_1 x_2 x_3 \ldots$ be a right-infinite word over some alphabet. Let $n \in \mathbb{N}$ such that $x_r = x_{r+n}$ for any $r \in \mathbb{N}$. If there is an $n' \in \mathbb{N}$ such that $n' \not< n$ and $x_r = x_{r+n'}$ for any $r \in \mathbb{N}$, then there is an $m \in \mathbb{N}$ such that $m \not< n, m|n$ and $x_r = x_{r+m}$ for any $r \in \mathbb{N}$.

**Proof.** Set $A := \{n' \in \{1, \ldots, n - 1\} \mid x_r = x_{r+n'}$ for any $r \in \mathbb{N}\}$ and $m := \min(A)$. Assume that $m$ does not divide $n$. Then there are $s, t \in \mathbb{N}$ such that $t < m$ and $n = sm + t$. Clearly $x_r = x_{r+n} = x_{r+sm+t} = x_{r+t}$ for any $r \in \mathbb{N}$. Hence $t \in A$, which contradicts the minimalty of $m$. Thus $m|n$. \hfill $\square$

**Lemma 50.** The map $\gamma$ defined above is bijective.

**Proof.** (Surjectivity) Let $p' \in [p]$. Let $k \geq 0$ be minimal such that $\tau_{\geq k}(p') = \tau_{\geq l}(p)$ for some $l \geq 0$. If $k = 0$, then $p' = c_i \ldots c_n$$\text{ccc} \ldots$ for some $i$. Hence $p' = \gamma(v_i)$. If $k > 0$, then $p' = p'_1 \ldots p'_l c_i \ldots c_n$$\text{ccc} \ldots$ for some $i$. Clearly $p'_k \not= c_{i-1}$ because of the minimality of $k$. Hence $q := p'_1 \ldots p'_l \in P_i$ and $p' = \gamma(v_{i,q})$.

(Injectivity) Assume $\gamma(v_i) = \gamma(v_j)$ for some $1 \leq i < j \leq n$. Write $\gamma(v_i) = x_1 x_2 x_3 \ldots$ and $\gamma(v_j) = y_1 y_2 y_3 \ldots$. Clearly $x_r = x_{r+n}$ for any $r \in \mathbb{N}$. Moreover, $y_r = x_{r+j-i}$ and hence $x_r = x_{r+j-i}$ for any $r \in \mathbb{N}$ since by assumption $x_r = y_r$. Obviously $1 \leq j - i < n$. It follows from Lemma 49 that there is an $m \in \mathbb{N}$ such that $m < n, m|n$ and $x_r = x_{r+m}$ for any $r \in \mathbb{N}$. Since $\gamma(v_i) = c_i \ldots c_n$$\text{ccc} \ldots$ we obtain $c = dd \ldots d$ where $d = c_1 \ldots c_m$. But this contradicts the simplicity of $c$. \hfill $\square$
Assume now that \( \gamma(v_{i,q}) = \gamma(v_j) \) where \( 1 \leq i, j \leq n \) and \( q \in P_i \). Then clearly
\[
c_i \ldots c_n \text{ccc} \ldots = \tau_{q} \gamma(v_{i,q}) = \tau_{q} \gamma(v_j) = q \ldots c_n \text{ccc} \ldots
\]
for some \( 1 \leq r \leq n \). If \( r \neq i \), then we obtain a contradiction as in the previous paragraph. Suppose now that \( r = i \). Then
\[
q \gamma(c_i \ldots c_n \text{ccc} \ldots) = \tau_{q} \gamma(v_{i,q}) = \tau_{q} \gamma(v_j) = c_i \ldots c_n \text{ccc} \ldots
\]
and hence \( q \gamma = c_{i-1} \), which contradicts the assumption that \( q \in P_i \).

Assume now that \( \beta(v_{i,q}) = \beta(v_{j,q'}) \) where \( 1 \leq i, j \leq n \), \( q \in P_i \) and \( q' \in P_j \). If \( |q| \neq |q'| \), then we obtain a contradiction as in the previous paragraph (say \( |q| < |q'| \); then consider \( \tau_{q} \gamma(v_{i,q}) = \tau_{q} \gamma(v_{j,q'}) \)). Suppose now that \( |q| = |q'| \). It follows that \( q = q' \) since \( \gamma(v_{i,q}) = \gamma(v_{j,q'}) \). If \( i \neq j \), then we obtain a contradiction as in the paragraph before the previous one. Thus we also get \( i = j \) as desired. \( \square \)

Since \( \gamma \) is a bijection, it induces an isomorphism \( \hat{\gamma} : V_{F} \rightarrow V_{[p]} \) of \( K \)-vector spaces. One checks easily that \( \hat{\gamma} \) is an isomorphism of \( L_{R}(E) \)-modules.

We can recover the properties of Chen simple modules via our machinery of representation graphs.

**Theorem 51.** Let \( E \) be a graph. Let \( p \) and \( q \) be infinite paths. Then

(i) The Chen module \( V_{[p]} \) is a simple \( L_{K}(E) \)-module.

(ii) The \( L_{K}(E) \)-modules \( V_{[p]} \) and \( V_{[q]} \) are isomorphic if and only if \( [p] = [q] \).

(iii) The \( L_{K}(E) \)-module \( V_{[p]} \) is graded if and only if \( p \) is an irrational infinite path.

**Proof.** (i) Suppose first that \( p \) is irrational. Let \( F \) be the representation graph constructed in (4.1.2). Then \( V_{F} \cong V_{[p]} \). Clearly \( F \) is connected. Moreover, \( \phi(\sigma_{n}(\beta(v))) = \{ \tau_{<n}(\beta(v)) \mid n \geq 0 \} \) for any \( v \in F^{0} \). It follows that \( \phi(\sigma_{n}(\beta(F))) = \phi(\sigma_{n}(\beta(F))) \) for any \( u \neq v \in F^{0} \) since the map \( \beta : F^{0} \rightarrow [p] \) is injective by Lemma 47. Clearly this implies that \( \phi(\sigma_{n}(\beta(F))) = \phi(\sigma_{n}(\beta(F))) \) for any \( u \neq v \in F^{0} \) and hence \( F \) satisfies Condition (2) in Definition 6. Thus \( V_{F} \cong V_{[p]} \) is simple by Theorem 28. The case that \( p \) is rational is similar (just replace \( \beta \) by \( \gamma \)).

(ii) Let \( (F, \phi) \) and \( (G, \psi) \) be the representation graphs constructed in (4.1.2) respectively (4.1.3) such that \( V_{F} \cong V_{[p]} \) and \( V_{G} \cong V_{[q]} \). It suffices to show that \( V_{F} \cong V_{G} \) implies \( [p] = [q] \). So suppose that \( V_{F} \cong V_{G} \). Then by Proposition 32 we have \( (F, \phi) \Rightarrow (G, \psi) \). Hence there are vertices \( u \in F^{0} \) and \( v \in G^{0} \) such that \( \phi(\sigma_{n}(\beta(F))) = \psi(\sigma_{n}(\beta(G))) \). Clearly this implies \( \phi(\sigma_{n}(\beta(F))) = \psi(\sigma_{n}(\beta(G))) \). But \( \phi(\sigma_{n}(\beta(F))) = \{ \tau_{<n}(p') \mid n \geq 0 \} \) for some \( p' \in [p] \) and \( \phi(\sigma_{n}(\beta(G))) = \{ \tau_{<n}(q') \mid n \geq 0 \} \) for some \( q' \in [q] \) (compare the previous paragraph). It follows that \( p' = q' \) and thus \( [p] = [p'] = [q] = [q] \).

(iii) Note that by the constructions in (4.1.2) and (4.1.3), if the infinite path \( p \) is irrational, then the representation graph \( F \) is a tree, whereas if \( p \) is rational then the representation graph \( F \) has a cycle. Now the statement follows immediately from Theorem 25 and the fact that \( V_{F} \cong V_{[p]} \) as \( L_{K}(E) \)-modules. \( \square \)

**5. Branching systems of weighted graphs and representations**

Branching systems for Leavitt path algebras were systematically studied by Gonçalves and Royer ([14, 13, 12]). A branching system of a graph gives rise to a representation for its associated Leavitt path algebra. This notion was also generalised to other type of graphs, such as separated graphs and ultragraphs ([15]). Here we adopt this notion for the weighted graphs and show that a representation graph for a weighted graph gives a branching system for this graph. We will show that
under certain assumptions on the field each representation defined by a branching system contains a subrepresentation isomorphic to one given by a representation graph. Conversely, every representation graph defines a branching system that defines the same (isomorphic) representation as the graph itself.

**Definition 52** (Branching system of a weighted graph). Let \((E, w)\) be a weighted graph. Let \(X\) be a set, \(\{R_{e_i} \mid e_i \in \hat{E}^1\}\) and \(\{D_v \mid v \in E^0\}\) families of subsets of \(X\) and \(\{g_{e_i} : R_{e_i} \to D_{r(e_i)} \mid e_i \in \hat{E}^1\}\) a family of injections such that:

1. \(\{D_v \mid v \in E^0\}\) is a partition of \(X\) (i.e. \(D_v \cap D_u = \emptyset\) whenever \(v \neq u\) and \(\bigcup_{v \in E^0} D_v = X\));
2. for each \(v \in E^0\) and \(1 \leq i \leq w(v)\) the family of sets \(\{R_{e_i} \mid e \in s^{-1}(v), w(e) \geq i\}\) forms a partition of \(D_v\);
3. Set \(D_{e_i} = g_{e_i}(R_{e_i})\) for each \(e_i \in \hat{E}^1\). Then for each \(e \in E^1\) the family \(\{D_{e_i} \mid 1 \leq i \leq w(e)\}\) forms a partition of \(D_{r(e)}\).

We call the quadruple \(X = (X, \{R_{e_i}\}, \{D_v\}, \{g_{e_i}\})\) an \((E, w)\)-branching system or simply an \(E\)-branching system.

When \(w(E) = 1\) the definition above reduces to the definition of an \(E\)-algebraic branching system of [14] with the only twist that the bijections \(f_e\) thereof are called \(g_{e_i}^{-1}\) here. Note that we do not assume by default that the sets comprising a branching system are nonempty.

Let \(M\) be the \(K\)-module of all functions \(X \to K\) with respect to point-wise operations. Let \(M_0\) denote the \(K\)-submodule of \(M\) that consists of all functions with finite support. We are going to define a structure of a right \(L_K(E)\)-module on \(M\). In order to simplify notations, we will abuse the notation as follows. Let \(Z \subseteq Y \subseteq X\) and \(\psi : Y \to K\). By \(\chi_Z \cdot \psi\) we denote the function \(X \to K\)

\[
x \mapsto \begin{cases} 
    \psi(x) & \text{if } x \in Z \\
    0 & \text{otherwise}
\end{cases}
\]

Using this convention, set for any \(\phi \in M\), any \(e_i \in \hat{E}^1\) and any \(v \in E^0\)

\[
\begin{align*}
\phi.e_i &= \chi_{D_{e_i}} \cdot (\phi \circ g_{e_i}^{-1}) \\
\phi.e_i^* &= \chi_{R_{e_i}} \cdot (\phi \circ g_{e_i}) \\
\phi.v &= \chi_{D_v} \cdot \phi.
\end{align*}
\] (14)

We will show that the multiplication defined above may be extended by linearity to the structure of a right \(L_K(E)\)-module on \(M\). Prior to that we would like to stress that

- by switching the action of \(e_i\) and \(e_i^*\) on \(M\) we get a left \(L_K(E)\)-module, in case of an unweighted graph \(E\) this module coincides with the one defined in [14];
- the same action of \(L_K(E)\) may be defined on the \(K\)-module \(M_0\) of all functions in \(M\) with finite support. The proof of the next theorem remains valid if \(M\) is replaced by \(M_0\);
- the notion of a branching system generalises easily to the case of a not necessarily row-finite graph \((E, w)\). In this case one should simply drop the assumption that the family of sets \(\{R_{e_i} \mid e \in s^{-1}(v), w(e) \geq i\}\) covers \(D_v\) whenever this family is infinite. However one must still assume that this family is disjoint.

**Theorem 53.** Let \((E, w)\) be a weighted graph and \(X\) an \((E, w)\)-branching system. Then the equalities (14) define a structure of a right \(L_K(E)\)-module on \(M\).
Proof. Due to the universal nature of $L_K(E)$ we only need to check that the standard generators $e_i, e_i^*$ and $v$ of $L_K(E)$ treated as endomorphisms of $M$ satisfy the relations (1)–(4) in Definition 17.

(i) Clearly $v$ is an idempotent for each $v \in E^0$; and for each $v \neq u \in E^0, \phi \in M$ one has $\phi.v.u = 0$ as $D_v$ and $D_u$ are disjoint.

(ii) Note that

$$\phi.s(e).e_i = \chi_{D_{e_i}} \cdot ((\chi_{D_{s(e)}} \cdot \phi) \circ g_{e_i}^{-1}) = \chi_{D_{e_i}} \cdot (\phi \circ g_{e_i}^{-1}) = \phi.e_i,$$

where the second equality follows from the fact that $g_{e_i}^{-1}(D_{e_i}) = R_{e_i} \subseteq D_{s(e)}$. Similarly,

$$\phi.e_i.r(e) = \chi_{D_{r(e)}} \cdot (\chi_{D_{e_i}} \cdot (\phi \circ g_{e_i}^{-1})) = \chi_{D_{e_i}} \cdot (\phi \circ g_{e_i}^{-1}) = \phi.e_i,$$

as $D_{e_i} \subseteq D_{r(e)}$. In the same way,

$$\phi.e_i^*.s(e) = \chi_{D_{s(e)}} \cdot (\chi_{R_{e_i}} \cdot (\phi \circ g_{e_i})) = \chi_{R_{e_i}} \cdot (\phi \circ g_{e_i}) = \phi.e_i^*,$$

for $R_{e_i} \subseteq D_{s(e)}$ and

$$\phi.r(e).e_i^* = \chi_{R_{e_i}} \cdot ((\chi_{D_{r(e)}} \cdot \phi) \circ g_{e_i}) = \chi_{R_{e_i}} \cdot (\phi \circ g_{e_i}) = \phi.e_i^*,$$

as $g_{e_i}(R_{e_i}) = D_{e_i} \subseteq D_{r(e)}$.

(iii) Now let $e, f \in E^1$ such that $s(e) = s(f) = v \in E^0$. Then

$$\sum_{1 \leq i \leq w(v)} \phi.e_i^*.f_i = \sum_{1 \leq i \leq \min(w(e), w(f))} \chi_{D_{f_i}} \cdot ((\chi_{R_{e_i}} \cdot \phi \circ g_{e_i}) \circ g_{f_i}^{-1}).$$

(15)

Recall that $e_i^* = e_i = 0$ whenever $i > w(e)$. Note that if $f \neq e$ the image $g_{f_i}^{-1}(D_{f_i}) = R_{f_i}$ is disjoint with $R_{e_i}$ and thus each summand in (15) equals zero. If $f = e$ then we get

$$((\chi_{R_{e_i}} \cdot \phi \circ g_{e_i}) \circ g_{f_i}^{-1}) = \phi$$

when restricted to $D_{e_i}$. This allows to rewrite the right-hand side of (15) as

$$\sum_{1 \leq i \leq w(e)} \chi_{D_{e_i}} \cdot \phi = \chi_{D_{r(e)}} \cdot \phi = \phi.r(e),$$

where the first equality follows from the fact $\{D_{e_i} \mid 1 \leq i \leq w(e)\}$ is a partition of $D_{r(e)}$.

(iv) Finally, let $v \in E^0$ and $1 \leq i, j \leq w(v)$. Then

$$\sum_{e \in s^{-1}(v)} \phi.e_i.e_j^* = \sum_{e \in s^{-1}(v)} \chi_{R_{e_j}} \cdot (\chi_{D_{e_i}} \cdot \phi \circ g_{e_i}^{-1}) \circ g_{e_j}.$$  

(16)

If $i \neq j$ then $g_{e_j}(R_{e_j}) = D_{e_j}$ which is disjoint with $D_{e_i}$ and each summand above equals zero. Assume $j = i$. As $g_{e_i}(R_{e_i}) = D_{e_i}$, we get

$$((\chi_{D_{e_j}} \cdot \phi \circ g_{e_i}^{-1}) \circ g_{e_j}) = \phi$$

when restricted to $R_{e_i}$. Then the right-hand side of (16) rewrites as

$$\sum_{e \in s^{-1}(v)} \chi_{R_{e_i}} \cdot \phi = \chi_{D_{e_i}} \cdot \phi = \phi.v,$$

as $\{R_{e_i} \mid e \in s^{-1}(v)\}$ partition $D_{s(e)}$. Note that this computation makes sense even if the number of summands in (16) is infinite, as the supports of the summands are disjoint. However, this is unnecessary as the corresponding relation in a weighted Leavitt algebra does not have to hold. □
5.1. Branching systems on an interval. The first series of examples of branching system is a generalization of the one given in [14, Theorem 3.1]. However, in the weighted case it becomes clear that the distinction between sinks and not sinks made there is redundant, as not only source sets, but also range sets must be partitioned. Thus, even in the case of weight 1, we don’t get precisely the same branching system as in [14], but morally the same (in particular, the resulting classes of representations coincide).

Let \((E, w)\) be a at most countable weighted graph, i.e. \(E^0\) and \(E^1\) are both finite or countable. By fixing some linear order on \(E^0\) we may write \(E^0 = \{v^1, v^2, \ldots\}\). For each \(i\) set \(D_{v^i} = [i - 1, i)\). Clearly, such sets are disjoint. Put \(X = \bigcup D_{v^i}\).

Now fix a vertex \(v^i \in E^0\) and \(1 \leq j \leq w(v^i)\). The set \(X_j^i = \{e \in s^{-1}(v^i) \mid w(e) \geq j\}\) is finite (as \(E\) is row-finite). By ordering this set we can rewrite it as \(X_j^i = \{e^{i,1}, e^{i,2}, \ldots\}\). For each \(1 \leq k \leq |X_j^i|\) set

\[ R_{e^j_i} = [i - 1 + \frac{k - 1}{|X_j^i|}, i - 1 + \frac{k}{|X_j^i|})\].

It is clear that the set \(\{R_{e^j_i} \mid e \in X_j^i\}\) forms a partition of \(D_{v^i}\).

In a similar fashion fix some \(e \in E^1\) and let \(v^i = r(e)\). For each \(1 \leq j \leq w(e)\) set

\[ D_{e^j_i} = [i - 1 + \frac{j - 1}{w(e)}, i - 1 + \frac{j}{w(e)})\].

Clearly, the family of sets \(\{D_{e^j_i} \mid 1 \leq j \leq w(e)\}\) forms a partition of \(D_{v^i}\).

Finally, the bijections \(g_{e_i^j} : R_{e^j_i} \rightarrow D_{e^j_i}\) may be chosen arbitrary, for example, a composition of a translation, scaling and another translation. The following theorem is now obvious.

**Theorem 54.** The sets defined above form an \((E, w)\)-branching system.

5.2. Branching systems and representation graphs. Now we will show that every representation graph defines a branching system in a natural way and representations induced by these structures are isomorphic in a reasonable sense.

Let \((E, w)\) be a weighted graph and \((F, \phi)\) a representation graph for \((E, w)\). Put \(X = F^0\) and \(D_v = (\phi^v)^{-1}(v)\) for each \(v \in E^0\). Clearly, \(\{D_v \mid v \in E^0\}\) is a partition of \(X\).

Fix \(v \in E^0\) and a tag \(1 \leq i \leq w(v)\). For each \(e \in s^{-1}(v)\) such that \(i \leq w(e)\) set

\[ R_{e^i} = \{u \in D_v \mid \text{there exists } f \in s^{-1}_v(u) : \phi^f(u) = e_i\} \].

The Condition (1) of Definition 2 of a representation graph translates in this setting as follows: each vertex \(u\) in \(D_v\) is contained in one and only one \(R_{e^i}\), where \(e\) ranges over all edges of weight at least \(i\) emitted by \(v\). In other words, the family \(\{R_{e^i} \mid e \in s^{-1}(v), w(e) \geq i\}\) forms a partition for \(D_v\).

Now fix a vertex \(v \in E^0\) and an edge \(e \in r^{-1}(v)\). For each \(1 \leq i \leq w(v)\) set

\[ D_{e^i} = \{u \in D_v \mid \text{there exists } f \in r^{-1}_v(u) : \phi^f(u) = e_i\} \].

The Condition (2) of Definition 2 of a representation graph guarantees that the family of sets \(\{D_{e^i} \mid 1 \leq i \leq w(E)\}\) forms a partition for \(D_v\).

Finally, fix some \(e_i \in E^1\). We need to define a bijection \(g_{e_i} : R_{e^i} \rightarrow D_{e^i}\). For each \(u \in R_{e^i}\) there exists precisely one edge \(f \in s^{-1}(u) \cap (\phi^i)^{-1}(e_i)\). Set \(g_{e_i}(u) = r(f)\).

**Theorem 55.** The quadruple \(X = (X, \{R_{e^i}\}, \{D_{e^i}\}, \{g_{e_i}\})\) defined above is a \((E, w)\)-branching system. The right \(L_K(E, w)\)-module induced by \(X\) on \(M_0\) is isomorphic to the module defined by \((F, \phi)\).
Proof. It is clear that $X$ is a branching system. Let $\pi$ denote the representation on $K^{E_0}$ induced by the representation graph. The module $M_0$ has a $K$-basis of delta-functions:

$$\delta_x: y \mapsto \begin{cases} 1 & \text{if } y = x \\ 0 & \text{otherwise}, \end{cases}$$

which correspond bijectively to elements of $X$. For the sake of simplicity, we will identify $x$ with $\delta_x$. In order to show that the module structure induced on $M_0$ by $X$ is isomorphic to the module structure defined by $\pi$ it is enough to show that

$$\pi(v)(x) = x.v,$$

$$\pi(e_i)(x) = x.e_i \text{ and }$$

$$\pi(e^*_i)(x) = x.e^*_i,$$

for each $x \in X = F^0$, each $v \in E^0$ and each $e_i \in \hat{E}^1$.

By definition of $\pi$, $\pi(v)(x) = x$ iff $x \in D_v$ and 0 otherwise. On the other hand, $x.v = \chi_{D_v} \cdot \delta_x = \delta_x = x$ iff $x \in D_v$ and 0 otherwise.

Further, let $e_i \in \hat{E}^1$. Then $\pi(e_i)(x) = r(f)$, if there exists a (unique!) edge $f$ in $s^{-1}(x)$ such that $\phi^1(f) = e_i$, and 0 otherwise. Similarly, $x.e_i = \chi_{D_{e_i}} \cdot (\delta_x \circ g_{e_i})$. Note that the restriction of $\delta_x \circ g_{e_i}$ to $D_{e_i}$ coincides with that of $\delta_{g_{e_i}(x)}$. If $x \in R_{e_i}$ then there exits a (again, unique) edge $f$ mapped to $e_i$ under $\phi^1$ and $g_{e_i}$ takes $x$ to $r(f) \in D_{e_i}$. Therefore $x.e_i = r(f)$. If an edge $f$ as above does not exist, then $x$ emits a different edge with tag $i$ and $(x \notin R_{e_i})$. Therefore the supports of $\chi_{D_{e_i}}$ and $\delta_x$ are disjoint and $x.e_i = 0$.

In the same manner one checks that $\pi(e^*_i)(x) = x.e^*_i$ for each $x \in X$. \hfill $\Box$

We will show next that given a $(E, w)$-branching system $X$ it is possible to describe a representation graph that defines the same representation as $X$ induces on $M_0$.

Theorem 56. Let $(E, w)$ be a weighted graph and $V$ be a vector space over $K$ equipped with the structure of a right $L_K(E)$-module. Suppose there exists a $K$-basis $B$ of $V$ such that

(i) for each $a \in B$ and each $e \in \hat{E}^1_d$ holds $a.e \in B \cup \{0\}$;

(ii) for each $a \in B$ and each $v \in E^0$ holds $a.v \in \{a, 0\}$;

(iii) for each $a \in B$ $a.L_K(E) \neq \{0\}$;

(iv) either $\text{char}(K) = 0$ or for each $a \in B$ holds: $a.e^*_if_i = 0$ for each $e \neq f$ and $a.eie^*_j = 0$ for each $i \neq j$.

Then there exists a representation graph that defines an $L_K(E)$-module isomorphic to $V$.

Proof. First, we will show that each $a \in B$ is fixed by precisely one $v \in E^0$ and annihilated by the rest. We will refer to this as $v$-property. By the assumption (iii), there exists an element $x \in L_K(E)$ such that $a.x \neq 0$. Without loss of generality we assume that $x$ is a standard generator of $L_K(E)$, i.e $x \in E^0 \cup \hat{E}^1_d$. Suppose $x$ is an edge. By the property (2) in Definition 17 one has $0 \neq a.x = a.s(x)x$. Therefore $a.s(x) \neq 0$. Similarly, if $x$ is a ghost edge then $0 \neq a.x = a.r(x)x$. Therefore $a.r(x) \neq 0$. Summing up, $a.v \neq 0$ for some $v \in E^0$. By assumption (i), $a.v = a$. Suppose $u \neq v \in E^0$. By the property (1) in Definition 17, $a.u = a.vu = a.0 = 0$. Therefore each vertex except for $v$ annihilates $a$. With this in mind set $F^0 = B$ and for each $a \in B$ set $\phi^0(a) = v$, where $v$ is the only vertex in $E^0$ such that $a.v \neq 0$. 


Note that we have proven a useful fact on the way: if \(a.e = b\) for some edge \(e\) and \(a, b \in B\) then \(a.s(e) = a\) and \(b.r(e) = a.e \cdot r(e) = a.e = b\). Similarly, if \(c.e^* = d\) for some ghost edge \(e^*\) and \(c, d \in B\) then \(c.r(e) = c\) and \(d.s(e) = d\). We shall refer to this as the \(sr\)-property.

Next, we construct \(F^1\) and \(\phi^1\). Start with \(F^1 = \emptyset\). For each \(a \in B\) and \(e_i \in \hat{E}^1\) such that \(a.e_i \neq 0\) add an edge \(g_{a,e_i}\) from \(a\) to \(a.e_i\) (cf. assumption (i)) to \(F^1\) and set \(\phi^1(g_{a,e_i}) = e_i\). For convenience, also set \(\text{tag}(g_{a,e_i}) = i\), \(\text{st}(g_{a,e_i}) = e\) and \(w(a) = w(\phi^0(a))\). Therefore, \(\text{tag}(g) = \text{tag}(\phi^1(g))\) and \(\text{st}(g) = \text{st}(\phi^1(g))\) for each \(g \in F^1\). In order to show that \((F, \phi)\) is a representation graph for \((E, w)\) it is enough to show that each \(a \in F^0\) emits precisely one edge with \(\text{tag} i\) for each \(1 \leq i \leq w(a)\) and receives precisely one edge with the structured edge \(e\) for each \(e \in r_{F}^{-1}(\phi^0(a))\).

We will show existence of such edges first. Suppose there is a vertex \(a\) with \(\phi^0(a) = v\) and an index \(1 \leq i \leq w(v)\) such that for each \(e \in s_{F}^{-1}(v)\) we have \(a.e_i = 0\). But this contradicts the property (4) in Definition 17. Indeed, in this case

\[
a = a.v = a.\left(\sum_{e \in s_{F}^{-1}(v)} e_i e_i^*\right) = \sum_{e \in s_{F}^{-1}(v)} (a.e_i).e_i^* = 0,
\]

where the first equality is by the \(v\)-property. Therefore \(a\) emits an edge with \(\text{tag} i\) for each \(1 \leq i \leq w(v)\) as required. The symmetric property, namely that \(a\) receives an edge \(f\) such that \(\text{st}(f) = e\) for each \(e \in r_{F}^{-1}(a)\), is shown exactly in the same way using the property (3) in Definition 17.

Next we will show that the first alternative in assumption (iv) yields the second one, which in fact tell us that for each relation of form (3) or (4) in Definition 17 with zero right-hand side already each summand acts like zero on each \(a \in B\). Indeed, pick any such relation, for example \(\sum_{e \in s_{F}^{-1}(v)} e_i e_j^* = 0\), where \(i \neq j\). Choose any \(a \in B\). Then \(0 = a.0 = \sum_{e \in s_{F}^{-1}(v)} a.e_i e_j^*\). Note that each summand on the right-hand side of the last expression is either zero or a basis element. When \(\text{char}(K) = 0\) a sum of basis elements is never zero, thus all summands must be zero. The relation (3) in Definition 17 is treated similarly.

Finally, we will show that each \(a \in F^0\) cannot emit two edges with the same tag or receive two edges with the same structured edge. Fix some \(a \in B\) and let \(v = \phi^0(a)\). By the property (4) in Definition 17,

\[
a = a.v = a.\left(\sum_{g \in s_{F}^{-1}(v)} g_i g_i^*\right) = \sum_{g \in s_{F}^{-1}(v)} a.g_i g_i^* \tag{17}
\]

The right-hand side of (17) is a sum of basis elements and zeros and thus must contain the basis element \(a\). Thus, \(a = a.g_i g_i^*\) for some \(g\). By the assumption (iv) we get

\[
a.f_i = a.g_i g_i^* f_i = (a.g_i)(g_i^* f_i) = 0
\]

for each \(f \neq g\). Therefore each vertex in \(F^0\) emits precisely one edge with each tag.

Similarly, for a fixed \(e \in r_{F}^{-1}(v)\) by the property (iii) in Definition 17 we have

\[
a = a.v = a.\left(\sum_{1 \leq i \leq w(e)} e_i e_i^*\right) = \sum_{1 \leq i \leq w(e)} a.e_i^* e_i. \tag{18}
\]

As in the previous case, the right-hand side of (18) is a sum of basis elements and zeros and thus contains the term \(a\). That is \(a = a.e_i^* e_i\) for some \(1 \leq i \leq w(e)\). Then by the assumption (iv),

\[
a.e_i^* \cdot a.e_i e_j^* = (a.e_i^*).e_i e_j^* = 0.
\]
Summing up, \( a.e_i^* \neq 0 \) for some fixed \( i \) and \( a.e_i^* = 0 \) for any \( j \neq i \). Now suppose \( b.e_j = a \) for some \( b \in B \). Then \( (b.e_j).e_i^* = a.e_i^* \neq 0 \). On the other hand if \( j \neq i \) then by assumption (iv) one has \( b.(e_j.e_i^*) = 0 \). Therefore \( j \) must be equal to \( i \). Thus, each vertex in \( F_0 \) cannot receive two edges with the same structured edge.

Summing up \((F, \phi)\) is a representation graph for \((E, w)\). It is clear from the construction of \((F, \phi)\) that the \( L_K(E)\)-module defined by \((F, \phi)\) is isomorphic to the \( L_K(E)\)-module \( V \).

Note that the assumption on \( \text{char}(K) \) in Theorem 56 may be weakened. It is clear from the proof of the theorem that it suffices that \( \text{char}(K) \) exceeds the number of terms of each relation of form (3) or (4) in Definition 17. Moreover, a slight modification of the proof works when \( \text{char}(K) \) exceeds each relation of one of the forms (iii) or (iv). Thus, assumption (iv) of Theorem 56 is fulfilled for any unweighted graph \( E \) or, for example, any weighted chain.

It is clear that any representation of \( L_K(E) \) on \( M_0 \) induced by a branching system satisfies the assumptions (i)—(iii) of Theorem 56 with respect to the basis of delta-functions. Further, the proof of Theorem 53 shows in particular that property (iv) is also satisfied for such representations. Further, the representation induced by a branching system on \( M_0 \) is a subrepresentation of the one induced on \( M \). Summing up, we get the following corollary.

**Corollary 57.** Any representation of a weighted Leavitt path algebra induced by a branching system \( X \) on the module \( M \) of functions \( X \to K \) contains a subrepresentation isomorphic to a representation of a branching system given by a representation graph. This representation is precisely the one induced by the same branching system on the subspace \( M_0 \) of functions with finite support.

It is unclear to the authors, if a basis satisfying the assumptions of Theorem 56 can be chosen for the whole \( L_K(E)\)-module \( M \) defined by a branching system.

### 5.3. Exceptional characteristic.

Here we provide an example that shows that the assumption (iv) in Theorem 56 is nonredundant. We will construct a representation that satisfies the assumptions (i)—(iii) thereof, but not (iv) and thus is not defined by any representation graph or a branching system. As mentioned before, such an example does not exist for a graph of weight 1.

Let \( K = \mathbb{F}_2 \). Consider the weighted graph

\[
E = e_1,e_2 \cup v \cup f_1,f_2.
\]

Introduce the structure of a right \( L_K(E)\)-module on \( K^1 \) by specifying the action of standard generators:

\[
1.e_2 = 1.f_1^* = 0, \quad 1.v = 1.e_1 = 1.e_1^* = 1.e_2^* = 1.f_1 = 1.f_2 = 1.f_2^* = 1.
\]

The easiest way to check that this is a well defined representation is to check all the relations in Definition 17. The relations (i) and (ii) are trivially satisfied and the relations (iii) and (iv) can be checked with a straightforward computation:

\[
\begin{align*}
1.(e_1^*e_1 + e_2^*e_2) &= 1 + 0 = 1, \\
1.(f_1^*f_1 + f_2^*f_2) &= 0 + 1 = 1, \\
1.(e_1^*f_1 + e_2^*f_2) &= 1 + 1 = 0, \\
1.(f_1^*e_1 + f_2^*e_2) &= 0 + 0 = 0,
\end{align*}
\]

\[
\begin{align*}
1.(e_1e_1^* + f_1f_1^*) &= 1 + 0 = 1, \\
1.(e_1e_2^* + f_2f_2^*) &= 0 + 1 = 1, \\
1.(e_1e_2^* + f_1f_2^*) &= 1 + 1 = 0, \\
1.(e_2e_1^* + f_2f_1^*) &= 0 + 0 = 0.
\end{align*}
\]
APPENDIX A. REPRESENTATION GRAPHS FOR QUIVERS WITH RELATIONS

In this Appendix we give a general construction of representation graphs for a directed graph with relations (Definition 61) which in turn gives rise to (simple) modules for path algebras with relations. The case of weighted graphs and thus weighted Leavitt path algebras falls into this general construction. However, the trade off is that this general construction does not give the elegant and easy to use representation graphs that we obtained in the case of (weighted) graphs.

Let $E$ be a directed graph. Consider $E$ as a category whose objects are the vertices and for two vertices $v, w$, the morphism set $\text{Hom}_E(v, w)$ are the paths from $v$ to $w$. A covariant functor from $E$ to $\text{Vec} K$, the category of vector spaces over a field $K$, is called a representation of $E$. By $\text{Rep}(E)$ we denote the functor category whose objects are the representations of $E$ and morphisms are the natural transformations. A relation in $E$ is a formal sum, $\sum_i k_ip_i$, where $k_i \in K \setminus \{0\}$ and $p_i \in \text{Hom}_E(v, w)$, for fixed vertices $v, w$. Let $r$ be a set of relations in $E$. Then $\text{Rep}(E, r)$ is a full subcategory of $\text{Rep}(E)$ which satisfies the relations in $r$, i.e., representations $\rho : E \to \text{Vec} K$ such that $\sum_i k_i\rho(p_i) = 0$, where $\sum_i k_ip_i$ is a relation in $r$. Let $\langle r \rangle$ be the two sided ideal of the path algebra $KE$ generated by the set of relations $r$. Set

$$A_K(E, r) := KE/\langle r \rangle.$$  \hfill (19)

Then, by a result of Green [16, Theorem 1.1], for a finite graph $E$, there exists an exact equivalent functor

$$\text{Mod} A_K(E, r) \longrightarrow \text{Rep}(E, r),$$

where $\text{Mod} A_K(E, r)$ is the category of right $A_K(E, r)$-modules.

We are in a position to define the representation graphs for a given graph in this setting.

**Definition 58.** Let $E$ be a directed graph. A representation graph for $E$ is a pair $(F, \phi)$, where $F$ is a directed graph and $\phi : F \to E$ is a homomorphism such that for any $e \in E^1$ and $u \in F^0$ there is at most one $f \in F^1$ such that $\phi(f) = e$ and $s(f) = u$.

Hence a representation graph for $E$ is an immersion of the graph $E$. The following lemma is easy to check.

**Lemma 59.** Let $E$ be a graph and $\phi : F \to E$ a representation for $E$. If $q \neq q' \in \text{Path}(F)$ and $\phi(q) = \phi(q')$, then $s(q) \neq s(q')$.

Let $(F, \phi)$ be a representation graph for $E$. Define a representation $\rho = \rho_{(F, \phi)}$ of $E$ by

$$\rho(v) := \sum_{u \in \phi^{-1}(v)} K u$$ \hfill (20)

for any $v \in \text{Ob}(E) = E^0$, and

$$\rho(p)(u) := \begin{cases} r(q), & \text{if } \exists q \in \text{Path}(F) : \phi(q) = p \land s(q) = u, \\ 0, & \text{otherwise}, \end{cases}$$

for any $p \in \text{Hom}_E(v, w)$ and $u \in \phi^{-1}(v)$. Note that $\rho(p)$ is well-defined by Lemma 59.

Clearly if $\rho$ is a representation of $E$ defined by a representation graph $(F, \phi)$, then there are linear bases $B_v \subseteq \rho(v)$ ($v \in \text{Ob}(E)$) such that (21) and (22) below are satisfied.
\[ B_v \cap B_w = \emptyset \text{ for any } v \neq w \in \text{Ob}(E). \]  \hfill (21)

For any \( p \in \text{Hom}_E(v, w) \) and \( x \in B_v \) either \( \rho(p)(x) = 0 \) or \( \rho(p)(x) \in \bigcup_{u \in \text{Ob}(E)} B_u \). \hfill (22)

It follows from Theorem 60 below that the representations \( \rho_{(F, \phi)} \) cover precisely the representations \( \rho \) of \( E \) for which there are linear bases \( B_v \subseteq \rho(v) \) (\( v \in \text{Ob}(E) \)) such that (21) and (22) are satisfied.

**Theorem 60.** Let \( \rho \) be a representation of \( E \) such that there are linear bases \( B_v \subseteq \rho(v) \) (\( v \in \text{Ob}(E) \)) for which (21) and (22) are satisfied. Then there is a representation graph \((F, \phi)\) for \( E \) such that \( \rho \) is isomorphic to \( \rho_{(F, \phi)} \).

**Proof.** Define a representation graph \((F, \phi)\) by

\[ F^0 = \bigcup_{u \in \text{Ob}(E)} B_u, \]

\[ F^1 = \{ f_{e,x} | e \in E^1, x \in B_{s(e)}, \rho(e)(x) \neq 0 \}, \]

\[ s(f_{e,x}) = x, \]

\[ r(f_{e,x}) = \rho(e)(x), \]

\[ \phi^0(x) = v \text{ if } x \in B_v, \]

\[ \phi^1(f_{e,x}) = e. \]

Set \( \rho' := \rho_{(F, \phi)} \). Recall that for any \( v \in \text{Ob}(E) \) we have \( \rho'(v) = \sum_{x \in \phi^{-1}(v)} Kx = \sum_{x \in B_v} Kx \). Since \( B_v \) is a basis for \( \rho(v) \), there is an isomorphism \( \eta_v : \rho'(v) \to \rho(v) \) such that \( \eta_v(x) = x \) for any \( x \in B_v \).

We leave it to the reader to check that for any \( p \in \text{Hom}_E(v, w) \) the diagram

\[
\begin{array}{ccc}
\rho'(v) & \xrightarrow{\eta_v} & \rho(v) \\
\rho'(e) & \text{down} & \rho(e) \\
\rho'(w) & \xrightarrow{\eta_w} & \rho(w)
\end{array}
\]

commutes, i.e. that \( \eta : \rho' \to \rho \) is an isomorphism of functors. \( \square \)

Next we consider graphs with relations. Let \( E \) be a directed graph and \( r \) a set of relations in \( E \). Call a path in \( E \) trivial if it has positive length (i.e. it is not a vertex) and nontrivial otherwise. We assume that a coefficient \( k_i \in K \) in a relation \( \sum_{i=1}^n k_ip_i \in r \) equals 1 if \( p_i \) is nontrivial and \(-1\) if \( p_i \) is trivial (note that the defining relations of weighted and unweighted Leavitt path algebras satisfy this condition). Hence a relation \( r \) is either of type

\[
\sum_{i=1}^n p_i \quad \text{(A)}
\]

where \( n \in \mathbb{N}, k_1, \ldots, k_n \in K^\times \) and \( p_1, \ldots, p_n \in \text{Hom}_E(v, w) \) are pairwise distinct nontrivial paths, or of type

\[
\sum_{i=1}^n p_i - v \quad \text{(B)}
\]

where \( n \in \mathbb{N}, k_1, \ldots, k_n \in K^\times \) and \( p_1, \ldots, p_n \in \text{Hom}_E(v, v) \) are pairwise distinct nontrivial paths.
Recall that a representation of \((E,r)\) is a functor \(\rho: E \to \text{Vec} K\) such that \(\sum_i \rho(p_i) = 0\) for any relation \(\sum_i p_i\) of type (A) and \(\sum_i \rho(p_i) = \rho(v)\) for any relation \(\sum_{i=1}^n p_i - v\) of type (B).

**Definition 61.** A representation graph \((F,\phi)\) for \(E\) is called a representation graph for \((E,r)\) if the following hold.

1. If a path \(p\) appears in a relation of type (A), then there is no \(q \in \text{Path}(F)\) such that \(\phi(q) = p\).
2. If \(\sum_{i=1}^n p_i - v\) is a relation of type (B), then for any \(u \in \phi^{-1}(v)\) there is precisely one \(1 \leq j \leq n\) such that there exists a \(q \in \text{Path}(F)\) for which \(s(q) = u\) and \(\phi(q) = p_j\). Moreover, \(r(q) = u\).

Let \((F,\phi)\) be a representation graph for \((E,r)\). Then clearly the representation \(\rho = \rho_{(F,\phi)}\) of \(E\) defined in (20) is a representation for \((E,r)\). Moreover, there are linear bases \(B_v \subseteq \rho(v)\) \((v \in \text{Ob}(E))\) such that (21),(22) and (23),(24) below are satisfied.

1. If a path \(p\) appears in a relation of type (A), then \(\rho(p) = 0\).
2. If \(\sum_{i=1}^n p_i - v\) is a relation of type (B), then for any \(x \in B_v\) there is a \(1 \leq j \leq n\) such that \(\rho(p_j)(x) = x\) and \(\rho(p_i)(x) = 0\) for any \(i \neq j\).

It follows from Theorem 62 below that the representations \(\rho_{(F,\phi)}\) where \((F,\phi)\) is a representation graph for \((E,r)\) cover precisely the representations \(\rho\) of \((E,r)\) for which there are linear bases \(B_v \subseteq \rho(v)\) \((v \in \text{Ob}(E))\) such that (21)-(24) are satisfied.

**Theorem 62.** Let \(\rho\) be a representation of \((E,r)\) and \(B_v \subseteq \rho(v)\) \((v \in \text{Ob}(E))\) linear bases such that (21),(22),(23) and (24) are satisfied. Then there is a representation graph \((F,\phi)\) for \((E,r)\) such that \(\rho\) is isomorphic to \(\rho_{(F,\phi)}\).

**Proof.** Let \((F,\phi)\) be the representation graph for \(E\) defined in the proof of Theorem 60. One checks easily that \((F,\phi)\) is a representation graph for \((E,r)\). In the proof of Theorem 60 it is shown that \(\rho\) is isomorphic to \(\rho_{(F,\phi)}\). \(\square\)

### A.1. Irreducible representation graphs of a graph with relations

Let \(E\) be a graph with a set of relations \(r\). Furthermore, let \((F,\phi)\) be a representation graph for \((E,r)\) and \(\rho\) be the representation of \((E,r)\) defined by \((F,\phi)\). Let \(V_F\) be the \(K\)-vector space with basis \(F^0\). Define the algebra \(A := KE/\langle r \rangle\). Then \(V_F\) becomes a right \(A\) -module by defining

\[ u.p = \begin{cases} r(q), & \text{if } q \in \text{Path}(F) \text{ such that } \phi(q) = p \text{ and } s(q) = u, \\ 0, & \text{otherwise,} \end{cases} \]

for any \(p \in \text{Hom}_E(v,w)\) and \(u \in F^0\).

In order to prove the main theorem of this subsection (Theorem 65), we need the following lemma which is also used in the main text in the proof of Theorem 28.

**Lemma 63.** Let \(W\) be a \(K\)-vector space and \(B\) a linearly independent subset of \(W\). Let \(k_i \in K\) and \(u_i, v_i \in B\), where \(1 \leq i \leq n\). Then \(\sum_{s=1}^n k_s (u_s - v_s) \notin B\).
Proof. Clearly we may assume that \( u_s \neq v_s \) for any \( 1 \leq s \leq n \). Moreover, we may assume that \( n \geq 2 \). Let \( w_1, \ldots, w_m \) be the distinct elements of the set \( \{ u_s, v_s \mid 1 \leq s \leq n \} \). Clearly there are \( l_{ij} \in K \) (\( 1 \leq i < j \leq m \)) such that

\[
\sum_{s=1}^{n} k_s (u_s - v_s) = \sum_{1 \leq i < j \leq m} l_{ij} (w_i - w_j).
\]

(25)

One checks easily that

\[
\sum_{1 \leq i < j \leq m} l_{ij} (w_i - w_j) = \sum_{1 \leq i \leq m} \left( \sum_{i < j \leq m} l_{ij} - \sum_{1 \leq j < i} l_{ji} \right) w_i.
\]

(26)

We prove by induction on \( m \) that

\[
\sum_{1 \leq i \leq m} \left( \sum_{i < j \leq m} l_{ij} - \sum_{1 \leq j < i} l_{ji} \right) = 0.
\]

(27)

Case \( m = 2 \): Clearly \( \sum_{1 \leq i \leq 2} \left( \sum_{i < j \leq 2} l_{ij} - \sum_{1 \leq j < i} l_{ji} \right) = l_{12} - l_{12} = 0 \) as desired.

Case \( m \to m + 1 \): Clearly

\[
\sum_{1 \leq i \leq m+1} \left( \sum_{i < j \leq m+1} l_{ij} - \sum_{1 \leq j < i} l_{ji} \right)
= \sum_{1 \leq i \leq m} \left( \sum_{i < j \leq m+1} l_{ij} - \sum_{1 \leq j < i} l_{ji} \right) - \sum_{1 \leq j < m+1} l_{j,m+1}
= \sum_{1 \leq i \leq m} \left( \sum_{i < j \leq m} l_{ij} - \sum_{1 \leq j < i} l_{ji} \right) + \sum_{1 \leq i \leq m} l_{i,m+1} - \sum_{1 \leq j < m+1} l_{j,m+1}
= \sum_{1 \leq i \leq m} \left( \sum_{i < j \leq m} l_{ij} - \sum_{1 \leq j < i} l_{ji} \right) = 0
\]

by the induction assumption. Hence (27) holds true. Now suppose that \( \sum_{s=1}^{n} k_s (u_s - v_s) \in B \). Then, in view of (25) and (26), precisely one of the coefficients

\[
\sum_{i < j \leq m} l_{ij} - \sum_{1 \leq j < i} l_{ji} \quad (1 \leq i \leq m)
\]

equals 1 and the remaining coefficients are 0. Hence the sum of the coefficients equals 1 which contradicts (27). This completes the proof. \( \square \)

Recall that \( q \neq q' \in \text{Path}(F) \) and \( \phi(q) = \phi(q') \) implies \( s(q) \neq s(q') \) by Lemma 59. We need a definition.

**Definition 64.** Let \( E \) be a graph with the set of relations \( r \) and \((F, \phi)\) a representation for \((E, r)\). We say \((F, \phi)\) is well-behaved if \( q \neq q' \in \text{Path}(F) \) and \( \phi(q) = \phi(q') \) implies \( r(q) \neq r(q') \). We call \( F \) strongly connected if none of the sets \( \text{Path}_{\text{w}}(F) \) \( v, w \in F^0 \) is empty.

**Theorem 65.** Let \( E \) be a graph with the set of relations \( r \) and let \( A = KE/(r) \) be the \( K \)-algebra associated to \((E, r)\). Further suppose \((F, \phi)\) is a well-behaved representation for \((E, r)\). Then the following are equivalent.

(i) The \( A \)-module \( V_F \) is simple.
(ii) $F$ is strongly connected and for any $x \in V_F \setminus \{0\}$ there is an $a \in A$ and a $v \in F^0$ such that $x.a = v$.

(iii) $F$ is strongly connected and for any $x \in V_F \setminus \{0\}$ there is a $k \in K$, a $p \in \Path(E)$ and a $v \in F^0$ such that $x.kp = v$.

(iv) $F$ is strongly connected and $\phi(u_{\Path}(F)) \neq \phi(v_{\Path}(F))$, for any $u \neq v \in F^0$.

Proof. (i)$\Rightarrow$ (iv) Suppose there are $v, w \in F^0$ such that $v_{\Path}(w)(F) = \emptyset$. Then clearly $w \not\in v.A$. Hence $v.A$ is a proper submodule of $V_F$ and therefore $V_F$ is not simple. Thus $F$ must be strongly connected. Now assume there are $u \neq v \in F^0$ such that $\phi(u_{\Path}(F)) = \phi(v_{\Path}(F))$. Consider the submodule $(u - v)A \subseteq V_F$. Since $V_F$ is simple by assumption, we have $(u - v)A = V_F$. Hence there is an $a \in A$ such that $(u - v).a = v$. Clearly there is an $n \geq 1$, $k_1, \ldots, k_n \in K^\times$ and pairwise distinct $p_1, \ldots, p_n \in \Path(E)$ such that $a = \sum_{s=1}^n k_s p_s$. We may assume that $(u - v).p_s \neq 0$ for any $1 \leq s \leq n$. Hence $p_s \in \phi(u_{\Path}(F)) = \phi(v_{\Path}(F))$ for any $s$. Since $(F, \phi)$ is well-behaved, we have $(u - v).p_s = u_s - v_s$ for some distinct $u_s, v_s \in F^0$. Hence

$$v = (u - v).a = (u - v)(\sum_{s=1}^n k_s p_s) = \sum_{s=1}^n k_s (u_s - v_s)$$

which contradicts Lemma 63.

(iv)$\Rightarrow$ (iii). Let $x = V_F \setminus \{0\}$. Then there is an $n \geq 1$, pairwise disjoint $v_1, \ldots, v_n \in F^0$ and $k_1, \ldots, k_n \in K^\times$ such that $x = \sum_{s=1}^n k_s v_s$. If $n = 1$, then $x.k_1^{-1}(v_1) = v_1$. Suppose now that $n > 1$. Since by assumption (iv) holds, we may assume that there is a $p_1 \in \phi(v_1_{\Path}(F))$ such that $p_1 \not\in \phi(v_{\Path}(F))$. Since $(F, \phi)$ is well-behaved, $x.p_1 \neq 0$ is a linear combination of at most $n - 1$ vertices from $F^0$. Proceeding like that we obtain paths $p_1, \ldots, p_m$ such that $x.p_1 \ldots p_m = kv$ for some $k \in K^\times$ and $v \in F^0$. Hence $x.p_1 \ldots p_n k^{-1} = v$.

(iii)$\Rightarrow$ (ii). Trivial.

(ii)$\Rightarrow$ (i). Let $U \subseteq V_F$ be a nonzero $A$-submodule and $x \in U \setminus \{0\}$. Since by assumption (ii) is satisfied, there is an $a \in A$ and a $v \in F^0$ such that $v = x.a \in U$. Let now $v'$ be an arbitrary vertex in $F^0$. Since by assumption $F$ is strongly connected, there is a $p \in v_{\Path}(F)$. Hence $v' = v.\phi(p) \in U$. Hence $U$ contains $F^0$ and thus $U = V_F$.

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