Explicit computations of Fourier transforms of polyhedral cones

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Abstract

The Fourier transforms of polyhedral cones can be used, via Brion’s theorem, to compute various geometric quantities of polytopes, such as volumes, moments, and lattice-point counts. We present a novel method of computing these conic Fourier transforms by polynomial interpolation. Given the fact that computing volumes of polytopes is \#P-hard (Dyer–Frieze [DF88]), we cannot hope for efficient algorithms in the general case. However, with extra assumptions on the combinatorics of the cone, we demonstrate it is possible to compute its Fourier transform efficiently.

1 Introduction

Fourier analysis is a marvelous tool to tackle problems in polyhedral geometry. It can be used to study continuous quantities such as volumes (Postnikov [Pos09]), moments (Brion–Vergne [BV97]) and polynomial integration (Barvinok [Bar92]). It has also been employed to investigate discrete volumes which include Ehrhart functions (Diaz–Robins [DR97], Barvinok–Pommersheim [BP99]), solid-angle sums (DeSario–Robins [DR11], Diaz–Le–Robins [DLR]) and exponential sums (Barvinok [Bar93]). In a lot of the above use cases, the Fourier transform of a polytope is a central object. One common way to analyze this object is to apply Brion’s theorem to decompose this polyhedral Fourier transform into a sum of the Fourier transforms of the tangent cones at the vertices. The following general version was proved by Alexander Barvinok (1992) [Bar92]

Theorem 1 (Brion’s theorem). For any convex polytope $P \subset \mathbb{R}^d$, we have the decomposition

$$\hat{1}_P(\xi) = \sum_{v \text{ vertex of } P} \hat{1}_{K_P(v)}(\xi),$$

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The Fourier transform is defined as  \( \hat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{2\pi i (x, \xi)} dx \). Next, the indicator function of a set \( S \) is \( 1_S(x) := 1 \) if \( x \in S \) and \( 0 \) if \( x \notin S \). Finally, the tangent cone at the vertex \( x \) is defined by

\[
K_P(v) := \{ v + x : v + tx \in P \text{ for some } t > 0 \}.
\]

Therefore, the Fourier transforms of the cones \( K_P(v) \) can be very useful in the study of the geometry of the polytope \( P \). This is our main object of study.

Let \( K \subset \mathbb{R}^d \) be a strictly convex (i.e. \( K \) does not contain a line) polyhedral cone with apex \( v \) and the set of generators \( W := \{ w_1, \ldots, w_n \} \), with \( n \geq d \). By diagonals of \( K \), we mean \((d - 1)\)-subsets of \( W \). A diagonal is called extremal if it is contained in the boundary \( \partial K \) of \( K \), and interior otherwise.

If \( K \) is simplicial, that is \( n = d \), it is very straightforward to calculate

\[
\hat{1}_K(\xi) = \left| \det(w_1, \ldots, w_d) \right| \cdot e^{2\pi i (v, \xi)}.
\]

(3)

When \( K \) is not necessarily simplicial, we can triangulate \( K \) and sum up the Fourier transforms of the simplicial faces. Thus, we can see that there exists a homogeneous polynomial \( p_K \) of degree \( n - d \) such that

\[
\hat{1}_K(\xi) = p_K(\xi) \cdot \prod_{j=1}^{n} \langle w_i, \xi \rangle \cdot e^{2\pi i (v, \xi)}.
\]

(4)

However, triangulation can be rather complicated. We propose a different method using polynomial interpolation to compute \( \hat{1}_K \) for non-simplicial \( K \), which can be very efficient given some assumptions on the combinatorics of \( K \).

To state our Main Theorem, we need a generalization of the cross product to higher dimensions. Given \((d - 1)\) vectors \( x_1, \ldots, x_{d-1} \) in \( \mathbb{R}^d \), the generalized cross product \( [x_1, \ldots, x_{d-1}] \) is defined such that

\[
\langle [x_1, \ldots, x_{d-1}], x \rangle = \det(x_1, \ldots, x_{d-1}, x),
\]

(5)

for any \( x \in \mathbb{R}^d \). Indeed, the right-hand side gives rise to a linear functional (with variable \( x \)) on \( \mathbb{R}^d \), which corresponds via the standard inner product to a unique vector that is taken to be the generalized cross product.

**Main Theorem.** Let \( D = \{ w_i, \ldots, w_{i_{d-1}} \} \) be a diagonal of \( K \). Set \( D^* := [w_i, \ldots, w_{i_{d-1}}] \)

(i) If \( D \) is extremal, then

\[
p_K(D^*) = \varepsilon \prod_{\substack{j=1 \atop j \notin D}}^{n} \det(w_i, \ldots, w_{i_{d-1}}, w_j),
\]

(6)

where \( \varepsilon \) is the sign of any of the determinants on the right-hand side.

(ii) Otherwise, if \( D \) is interior, we have a simple identity:

\[
p_K(D^*) = 0.
\]

(7)
In effect, the Main Theorem gives the values of the homogeneous polynomial \( p_K \) at a lot of points. Therefore, if the combinatorics of the cone \( K \) is sufficiently generic, we can use interpolation to determine \( p_K \) exactly. This gives a novel way to compute the conic Fourier transform \( \hat{1}_K \).

Suppose \( p_K(\xi) = \sum_E c_E \xi^E \), where \( E \) varies in the set \( \mathcal{E}_{d,n} = \{(e_1, \ldots, e_d) : e_i \geq 0, \sum_i e_i = n - d\} \) and \( \xi^E = \xi_1^{e_1} \cdots \xi_d^{e_d} \). Let us write \( c = \{c_E\} \in \mathbb{R}^{(n-d)} \).

Then, Equations 6 and 7 yield an (overdetermined) linear system

\[
A_K x = b_K, \quad (8)
\]

that has \( x = c \) as a solution. Here \( A_K \) is a \( \left( \binom{n}{d-1} \right) \times \left( \binom{n-1}{d-1} \right) \)-matrix and \( b_K \) is a vector of dimension \( \left( \binom{n}{d-1} \right) \).

**Theorem 2.** Suppose the generators \( w_1, \ldots, w_n \) of \( K \) are in general positions such that no \( d \)-subset of them is linear dependent; or equivalently, \( K \) is the cone over a simplicial polytope of dimension \( d - 1 \). Then, \( A_K \) has full rank.

Therefore, when \( K \) is the cone over a simplicial polytope, we can solve System 8 for \( c \), which in turn determines the conic Fourier transform \( \hat{1}_K \). Since random points are almost surely in general positions, we believe our results will have ramifications in the theory of random polytopes.

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2 Evaluations at diagonals

Suppose \( K \subset \mathbb{R}^d \) has apex \( v \) and \( W = \{w_1, \ldots, w_n\} \) is the set of its generators. More concretely,

\[
K = \{v + t_1 w_1 + \cdots + t_n w_n : t_1, \ldots, t_n > 0\}. \quad (9)
\]

We use \( W^{[k]} \) to denote the set of all \( k \)-subsets of \( W \); hence, \( W^{[d-1]} \) is the set of diagonals of \( K \). For any subset \( S \) of \( \mathbb{R}^d \), we denote

\[
\text{lin}(S) := \left\{ \sum_{x \in S} t_x x \text{ with } t_x \in \mathbb{R} \right\},
\]

\[
\text{cone}(S) := \left\{ \sum_{x \in S} t_x x \text{ with } t_x > 0 \right\},
\]

\[
\text{conv}(S) := \left\{ \sum_{x \in S} t_x x \text{ with } 0 < t_x < 1 \right\}.
\]
We note that, using the standard inner product on $\mathbb{R}^d$, one can conceptually think of the generalized cross product $[x_1, \ldots, x_{d-1}]$ either as the projective dual of the hyperplane $\text{lin}(D)$, or as the Hodge dual of the wedge product $x_1 \wedge \cdots \wedge x_{d-1}$. Computationally, the generalized cross product can be calculated by the Gram–Schmidt process, or more straightforwardly, by taking the $(d-1)$-minors of the $d \times (d-1)$-matrix formed by collating the vectors $x_1, \ldots, x_{d-1}$, as in the definition of the 3-dimensional cross product.

**Proof of Main Theorem.** First of all, because the Fourier transform converts a translation (i.e., time shifting) into a modulation (i.e., frequency shifting), we have

$$\hat{1} \left[ S \cdot x_0 \right] (\xi) = e^{-2\pi i (x_0, \xi)} \hat{1}_S (\xi).$$

Thus, we may assume that the apex $v$ of $K$ is at the origin $0$.

Given a diagonal $D = \{w_{i_1}, \ldots, w_{i_{d-1}}\}$, recall that $D^* = [w_{i_1}, \ldots, w_{i_{d-1}}]$. We write $\det(D, x) := \det(w_{i_1}, \ldots, w_{i_{d-1}}, x) = \langle D^*, x \rangle$, for brevity. Clearly, $\langle D^*, w_{i_1} \rangle = \cdots = \langle D^*, w_{i_{d-1}} \rangle = 0$.

For Case (i), take a triangulation $K = K_1 \cup \cdots \cup K_m$ such that one of the simplicial cones, say $K_1$, generated by $D$, together with an extra vector $w_{i_d}$. Since $K_1$ is simplicial, Equation 1 gives

$$\hat{1}_{K_1} (\xi) = \frac{|\det(D, w_{i_d})|}{\prod_{k=1}^d \langle \xi, w_{i_k} \rangle} \prod_{1 \leq j \leq n} \frac{\langle \xi, w_j \rangle}{\langle \xi, w_{i_j} \rangle} \prod_{j \in D} \langle \xi, w_j \rangle.$$  

(11)

The numerator of the last fraction, evaluated at $\xi = D^*$, is equal to

$$\varepsilon \prod_{1 \leq j \leq n} \frac{\det(D, w_j)}{\prod_{j \notin D} \langle \xi, w_j \rangle},$$

(12)

where $\varepsilon$ is the sign of $\det(D, w_{i_d}) = \det(w_{i_1}, \ldots, w_{i_d})$. We note that this sign does not depend on the triangulation. Indeed, because $K$ is convex, all other generators lie on one side of the hyperplane $\text{lin}(D)$. Thus, all determinants in Equation 12 have the same sign, which implies the sign $\varepsilon$ is independent of the triangulation.

Take $p \geq 2$. Since $D$ is extremal, the simplicial cone $K_p$ cannot contain $D$. Therefore, when we equate the denominator of the conic Fourier transform $\hat{1}_{K_p}$ to $\prod_{j=1}^n \langle \xi, w_j \rangle$, the numerator must contain one of the factors $\langle \xi, w_{i_1} \rangle, \ldots, \langle \xi, w_{i_{d-1}} \rangle$, which vanish when evaluated at $\xi = D^*$. This completes our proof in Case (i).

For Case (ii), take a triangulation $K = K_1 \cup \cdots \cup K_m$ such that two of the simplicial cones, say $K_1$ and $K_2$, have $w_{i_1}, \ldots, w_{i_{d-1}}$ as generators. Let $w^*_{i_d}$ and $w^*_{i_{d+1}}$ be the other generators in $K_1$ and $K_2$, respectively. We have
\[ \hat{I}_{K_1}(\xi) = \frac{|\det(D, w_{i_1})|}{\langle \xi, w_{i_1} \rangle \prod_{k=1}^{d-1} \langle \xi, w_k \rangle} \cdot \prod_{1 \leq j \leq n} \langle \xi, w_j \rangle \]

\[ \hat{I}_{K_2}(\xi) = \frac{|\det(D, w_{i_2})|}{\langle \xi, w_{i_2} \rangle \prod_{k=1}^{d-1} \langle \xi, w_k \rangle} \cdot \prod_{1 \leq j \leq n} \langle \xi, w_j \rangle \]

Observe that, because \( K_1 \) and \( K_2 \) have disjoint interiors, two vectors \( w_{i_1} \) and \( w_{i_2} \) lie on the opposite sides of the hyperplane \( \text{lin}(D) \). Therefore, \( \det(D, w_{i_1}) \) and \( \det(D, w_{i_2}) \) have opposite signs, which implies

\[ \langle D^*, w_{i_1} \rangle \cdot |\det(D, w_{i_1})| + \langle D^*, w_{i_2} \rangle \cdot |\det(D, w_{i_2})| = \det(D, w_{i_1}) \cdot |\det(D, w_{i_1})| + \det(D, w_{i_2}) \cdot |\det(D, w_{i_2})| = 0. \] (15)

Hence, the numerators of the fractions in Equations 13 and 14, when evaluated at \( \xi = D^* = [w_{i_1}, \ldots, w_{i_{d-1}}] \), sum up to 0.

Let \( j \geq 3 \). Since both sides of \( \text{cone}(D) \) have been covered by \( K_1 \) and \( K_2 \), the cone \( K_j \) cannot contain \( D \). Using the same argument as in Case (i), we see that the contribution of \( K_j \) (\( j \geq 3 \)) to the polynomial \( p_K \) (of Equation 4) evaluates to 0 when \( \xi = D^* \). Thus, overall, we have proved \( p_K(D^*) = 0 \) and finished the proof in Case (ii).

### 3 Veronese–Vandermonde determinants

The rows of the matrix \( A_K \) in System 8 remind us of the Veronese map and the dependence of \( A_K \) on the positions of the generators \( w_i \) of the cone \( K \) resembles the Vandermonde determinant. In this section, we will flesh out these connections.

The classical Vandermonde determinant is

\[ \det \left( x_i^{k-1} \right)_{i,k=1}^{m} = \prod_{1 \leq i < j \leq m} (x_j - x_i). \] (16)

We would like to generalize this identity to higher dimensions using the Veronese map. Recall that the Veronese map \( \nu_k : \mathbb{R}^d \to \mathbb{R}^{(d+k-1)} \) of degree \( k \) is
defined by taking all monomials of degree $k$ on $d$ variables and then evaluating all these monomials at each point of $\mathbb{R}^d$. For instance,

$$
\nu_2(x_1, x_2, x_3) = (x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2).
$$

The $i$-th row of the Vandermonde matrix can be thought of as the image of the point $(x_i, 1) \in \mathbb{R}^2$ under the Veronese map $\nu_{m-1}$. Similarly, in System 8, each row of the $\binom{n}{d-1} \times \binom{n-1}{d-1}$-matrix $A_K$ equals to $\nu_{n-d}(D^\ast)$, for a diagonal $D$ of $K$.

A maximal minor (of order $\binom{n-1}{d-1}$) of $A_K$ is given by a choice of $\binom{n-1}{d-1}$ diagonals of $K$, i.e., by an element $D \in (W^{[d-1]})\binom{[n-1]}{[d-1]}$. We write $\mu_D(A_K)$ to be the maximal minor of $A_K$ associated to $D$.

If we consider elements of $W^{[p]}$ as simplices of dimension $p - 1$, we can think of elements of $(W^{[p]})^{[q]}$ as subcomplexes of the complete simplicial complex $\Sigma^{p-1}(W)$ of dimension $p - 1$ with vertices in $W$. Therefore, we can regard $\mathcal{D}$ as a subcomplex of $\Sigma^{d-2}(W)$.

Given a $(d - 1)$-simplex $E \in W^{[d]}$, we define the multiplicity $\text{mult}_D(E)$ of the complex $\mathcal{D}$ at $E$ to be the number of facets of $E$ contained in $\mathcal{D}$, or equivalently, the number of elements $D$ of $\mathcal{D}$ such that $D \subset E$. We say that the subcomplex $\mathcal{D}$ fills $\Sigma^{d-2}(W)$ if, for any $(d - 1)$-simplex $E$ in $\Sigma^{d-1}(W)$, at least one facet of $E$ belongs to $\mathcal{D}$. In other words, $\mathcal{D}$ fills $\Sigma^{d-2}(W)$ if and only if $\text{mult}_D(E) \geq 1$ for all $E \in W^{[d]}$.

**Proposition 3** (Veronese–Vandermonde determinants). Suppose the generators $w_i$ of $K$ are in general positions. Let $\mathcal{D}$ be a family of $\binom{n-1}{d-1}$ diagonals of $K$.

(i) If $\mathcal{D}$ does not fill the complete simplicial complex $\Sigma^{d-2}(W)$, then

$$
\mu_D(A_K) = 0.
$$

(ii) If $\mathcal{D}$ fills $\Sigma^{d-2}(W)$, then the minor of $A_K$ associated to $\mathcal{D}$ is

$$
\mu_D(A_K) = \pm \prod_{E \in W^{[d]}} \text{det}(E)^{\text{mult}_D(E)-1}.
$$

**Proof of Theorem 2.** Because we can always choose a family $\mathcal{D}$ that fills the complete simplicial complex $\Sigma^{d-2}(W)$, Proposition 3 implies that $A_K$ has a nonzero maximal minor, and thus, it has full rank. \hfill \Box

This proposition seems related to Theorem 4.15 in Ben Yaacov [Yaa14], but we have not been able to figure out the connection. Since we do not need such generality as in [Yaa14], we will provide an elementary proof resembling that of the Vandermonde determinant.

**Proof of Proposition 3.** In Case (i), we will prove the matrix of $\mu_D(A_K)$ admits a nonzero null vector. Because the subcomplex $\mathcal{D}$ does not fill $\Sigma^{d-2}(W)$, there exists a $(d - 1)$-simplex $E \in W^{[d]}$ such that $\mathcal{D} \not\subseteq E$ for all $D \in \mathcal{D}$. Therefore, any $D$ in $\mathcal{D}$ must intersect $F = W \setminus E$. Note that each row of $\mu_D(A_K)$ is of the
form $\nu_{n-d}(D^*)$ for some $D \in \mathcal{D}$. In Lemma 5, we construct from $F$ a nonzero vector $F^+$ such that $(F^+, \nu_{n-d}(D^*)) = 0$, which means the matrix of $\mu_{\mathcal{D}}(A_K)$ admits a nonzero null vector. Therefore, $\mu_{\mathcal{D}}(A_K) = 0$, as desired.

In Case (ii), we think of $W$ as a matrix of $nd$ variables $w_{ij}$ for $1 \leq i \leq n$ and $1 \leq j \leq d$. Since the generalized cross product can be computed by taking minors, the entries of $\mu_{\mathcal{D}}(A_K)$ are polynomials in $w_{ij}$, which implies that the determinant $\mu_{\mathcal{D}}(A_K)$ is also a polynomial in $w_{ij}$.

As a sanity check, let us compare the degrees of the two sides of Equation 18 as polynomials in $w_{ij}$. For each $D \in \mathcal{D}$, the coordinates of $D^*$ are determinants of $(d-1) \times (d-1)$-matrices and have degree $(d-1)$. The application of the Veronese map $\nu_{n-d}$ raises the degrees to $(d-1)(n-d)$. Hence, the matrix of $\mu_{\mathcal{D}}(A_K)$ have entries with degrees $(d-1)(n-d)$ and its size is $\binom{n-1}{d-1}$. Therefore, the total degree of the left-hand side of Equation 18 is $(d-1)(n-d)\binom{n-1}{d-1}$. On the other hand, the total degree of the right-hand side is

$$d \left( \sum_{E \in W^{[d]}} \text{mult}_{\mathcal{D}}(E) - \binom{n}{d} \right) = d \left( \sum_{D \in \mathcal{D}} (n \# D) - \binom{n}{d} \right)$$

$$= d \left( \binom{n-1}{d-1}(n - d + 1) - \binom{n}{d} \right) = \binom{n-1}{d-1}(d(n-d+1)-n)$$

$$= \frac{(n-1)(d-1)(d+1)(n-d)}{d-1}$$

as expected.

For $D = \{w_{i1}, \ldots, w_{id-1}\}$, the defining equation (Equation 5) of the generalized cross product infers that, if the vectors $D$ are dependent, then $D^* = 0$. Also, if $w_i$ is dependent on $D$ and $D'$ is obtained by replacing a vector in $D$ with $w_i$, then $D'^*$ is parallel to $D$, that is $D'^* = \lambda D^*$ for some $\lambda \in \mathbb{R}$. Therefore, $\nu_{n-d}D'^* = \lambda^{n-d}\nu_{n-d}D^*$ are parallel.

Let $E \in W^{[d]}$ be a $(d-1)$-simplex. Suppose $\text{mult}_{\mathcal{D}}(E) \geq 2$. Let $D_1, D_2 \in \mathcal{D}$ such that $D_1, D_2 \subset E$. Without loss of generality, let us say $D_1 = \{w_{j1}, \ldots, w_{jd-2}, w_{j1}\}$ and $D_2 = \{w_{j1}, \ldots, w_{jd-2}, w_{j2}\}$. If $\det(E) = 0$, then $w_{j2}$ is dependent on $D_1$. Therefore, as noted above, $D_1^*$ and $D_2^*$ are parallel, and so are $\nu_{n-d}D_1^*$ and $\nu_{n-d}D_2^*$.

Because of the fact that multivariate polynomial rings over fields are unique factorization domains, we see that the polynomial $\det(E)$ divides $\mu_{\mathcal{D}}(A_K)$.

Suppose $D_1, \ldots, D_k$ are all the diagonals in $\mathcal{D}$ that are contained in the simplex $E$, where $k = \text{mult}_{\mathcal{D}}(E)$. Then, the number of appearances of $\det(E)$ as divisors of $\mu_{\mathcal{D}}(A_K)$ is equal to $\text{mult}_{\mathcal{D}}(E) - 1$. This can be seen by fixing $D_1$ and letting the $(k-1)$ vectors $D_2 \setminus D_1, \ldots, D_k \setminus D_1$ independently approach the hyperplane $\text{lin}(D_1)$.

Thus far, we have proved that the RHS of Equation 18 divides $\mu_{\mathcal{D}}(A_K)$. By the sanity check above, the degrees of the two sides are equal, which infers that they differ only by a constant factor. Since the coefficients of $\mu_{\mathcal{D}}(A_K)$ and $\det(E)$ are $\pm 1$, the constant factor is also $\pm 1$. Therefore, we have completed the proof in Case (ii).
We need the following lemma before stating Lemma 5.

**Lemma 4.** Take \( n \geq d > 0 \). Let \( P \) be a \( (\binom{n-1}{d-1}) \times d \)-matrix whose rows are indexed by \( IV_{d,n-d} \), where \( IV_{d,s} := \{ x \in \mathbb{Z}_{\geq 0}^d : x_1 + \cdots + x_d = s \} \), such that

- If \( x_j = 0 \), then the \((x, j)\)-entry of \( P \) is zero.
- For any \( y \in IV_{d,n-d-1} \), the \((y + e_j, j)\)-entries, for \( 1 \leq j \leq d \) are equal, say, to a number \( c_y \). Here \( e_j \) is the \( j \)-th standard vector of dimension \( d \).

Let \( Q \) be an antisymmetric \( d \times d \)-matrix. Then, for any \( v \in \mathbb{R}^d \),

\[
\langle P v, \nu_{n-d}(Qv) \rangle = 0.
\] (19)

Recall that here \( \nu_{n-d} \) is the Veronese map of degree \( n - d \).

**Proof.** Note that the rows of \( \nu_{n-d}(Qv) \) are indexed by \( IV_{d,n-d} \) and equal to \( (Qv)^x \) for \( x \in IV_{d,n-d} \), where \( a^x = a_1^{x_1} \cdots a_d^{x_d} \). We can see that

\[
\langle P v, \nu_{n-d}(Qv) \rangle = \sum_{y \in IV_{d,n-d-1}} c_y (v^T Qv).
\] (20)

But, \( v^T Qv = 0 \) because \( Q \) is antisymmetric. This proves the lemma.

**Lemma 5.** Let \( F \subset \mathbb{R}^d \) be a \((n-d)\)-subset and \( D \subset \mathbb{R}^d \) a \((d-1)\)-subset such that \( D \) and \( F \) intersect. Then, we can construct from \( F \) a nonzero vector \( F^+ \) of dimension \( \binom{n-1}{d-1} \) such that

\[
(F^+, \nu_{n-d}(D^*)) = 0.
\] (21)

**Proof.** Suppose \( F = \{v_1, \ldots, v_m\} \). We will construct \( F^+ \) whose coordinates are indexed by \( x \in IV_{d,n-d} \). Then, we set the \( x \)-th coordinate of \( F^+ \) to be

\[
F^+_x = \sum_{k \in \{1, \ldots, d\}^{n-d}} \prod_{i=1}^{n-d} (v_i)_{k_i},
\] (22)

where \( \sigma(k) = \sum_{i=1}^{n-d} k_i e_i \) with \( e_i \) the \( i \)-th standard vector of dimension \( d \). We can check that \( F^+ \) has the form \( P v_1 \) for a matrix \( P \) as in Lemma 4.

Suppose \( v_1 \in D \cap F \). Then, \( D^* = Q v_1 \) for some antisymmetric \( d \times d \)-matrix. Therefore, by Lemma 4,

\[
(F^+, \nu_{n-d}(D^*)) = \langle P v_1, \nu_{n-d}(Q v_1) \rangle = 0.
\] (23)
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