Stability of an abstract–wave equation with delay and a Kelvin–Voigt damping

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Outline

1 Motivation
   - Problem
   - The idea
   - Stability

2 Existence results

3 The spectral analysis
   - The discrete spectrum
   - The continuous spectrum

4 Asymptotic behavior

5 Proof of the main result

6 Application to the stabilization of the wave equation with delay and a Kelvin–Voigt damping

7 Conclusion

8 References

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1. **Motivation**
   - Problem
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2. **Existence results**

3. **The spectral analysis**
   - The discrete spectrum
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5. Proof of the main result

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7. Conclusion

8. References
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8 References
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5. Proof of the main result

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7. Conclusion

8. References
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7 Conclusion

8 References
Problem

It is well-known that delay equations like the simplest one of parabolic type,

\[ u_t(t; x) = \Delta u(t - \tau; x); \]

with a delay parameter \( \tau > 0 \), or of hyperbolic type,

\[ u_{tt}(t; x) = \Delta u(t - \tau; x); \]

are not well-posed.

- Their instability is given in the sense that there is a sequence of initial data remaining bounded, while the corresponding solutions, at a fixed time, go to infinity in an exponential manner, see Jordan, Dai & Mickens and Dreher, Quintanilla & Racke.
The same phenomenon of instability is given for a general class of problems of the type

\[ \frac{d^n u}{dt^n}(t) = Au(t - \tau); \]

\( n \in \mathbb{N}; \text{ fixed, whenever } (-A) \text{ is linear operator in a Banach space having} \]
a sequence of real eigenvalues \((\lambda_k)_k\) such that \(0 < \lambda_k \to \infty \text{ as } k \to \infty.\)
The so-called $\alpha - \beta$-system with delay,

$$\begin{cases} u_{tt}(t) + aAu(t - \tau) - bA^\beta \theta(t) = 0, \\ \theta_t(t) + dA^\alpha \theta(t) + bA^\beta u_t(t) = 0 \end{cases}$$

for functions $u, \theta : [0; +\infty) \to H$, with $A$ being a self-adjoint operator in the Hilbert space $H$, having a countable complete orthonormal system of eigenfunctions $(\phi_j)_j$ with corresponding eigenvalues $0 < \lambda_j \to \infty$ as $j \to \infty$. The thermoelastic plate equations appear with $\alpha = \beta = \frac{1}{2}$ and $A = (-\Delta_D)^2$. 
It was shown that we have a strong smoothing property for parameters \((\alpha; \beta)\) in the region

\[
\mathcal{A}_{sm} := \{ (\beta, \alpha); 1 - 2\beta < \alpha < 2\beta, \alpha > 2\beta - 1 \}, \ (\tau = 0).
\]

Figure 1.1: Area of smoothing \(\mathcal{A}_{sm}\) (without delay)
The $\alpha - \beta$-system with delay is not well-posed in the region

$$\mathcal{A}^1_{in} := \left\{ (\beta, \alpha); 0 \leq \beta \leq \alpha \leq 1, \alpha \geq \frac{1}{2}, (\beta, \alpha) \neq (1, 1) \right\}.$$
A similar result will hold for the related system

\[
\begin{align*}
\left\{
\begin{array}{l}
  u_{tt}(t) + aAu(t) - bA^\beta \theta(t) = 0, \\
  \theta_t(t) + dA^\alpha \theta(t - \tau) + bA^\beta u_t(t) = 0
\end{array}
\right.
\end{align*}
\]

in the region

\[A_{in}^2 := \{(\beta, \alpha); 0 \leq \beta \leq \alpha \leq 1, (\beta, \alpha) \neq (1, 1)\}.\]
The idea

- Datko: The effect of a small delay

\[
\begin{aligned}
\ddot{w}(t) + A w(t) + BB^* \dot{w}(t - \tau) &= 0, \quad t \geq 0, \\
\dot{w}(0) &= w^0, \quad \dot{w}(0) = w^1, \\
\dot{w}(t) &= f_0(t), \quad t \in (-\tau, 0),
\end{aligned}
\]

\(\tau > 0\) is the time delay.
\begin{align*}
\ddot{w}(t) + A w(t) + \alpha_1 BB^* \dot{w}(t) + \alpha_2 BB^* \dot{w}(t - \tau) &= 0, \quad t \geq 0, \\
\dot{w}(0) &= w^0, \quad \dot{w}(0) = w^1, \\
\dot{w}(t) &= f_0(t), \quad t \in (-\tau, 0),
\end{align*}

\( \tau > 0 \) is the time delay.

\( 0 < \alpha_2 < \alpha_1 \).
$u_{tt}(x, t) - \Delta u(x, t) + au_t(x, t - \tau) = 0, \quad x \in \Omega, \ t > 0, \quad (1)$

$u(x, t) = 0, \quad x \in \Gamma_0, \ t > 0 \quad (2)$

$\frac{\partial u}{\partial \nu}(x, t) = -ku_t(x, t), \quad x \in \Gamma_1, \ t > 0 \quad (3)$

$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \quad (4)$

$u_t(x, t) = g(x, t), \quad x \in \Omega, \ t \in (-\tau, 0), \quad (5)$

where $\nu$ stands for the unit normal vector of $\partial \Omega$ pointing towards the exterior of $\Omega$ and $\frac{\partial u}{\partial \nu}$ is the normal derivative. Moreover, the constant $\tau > 0$ is the time delay, $a$ and $k$ are two positive numbers and the initial data are taken in suitable spaces.
Theorem (A-Nicaise-Pignotti)

For any $k > 0$ there exist positive constants $a_0, C_1, C_2$ such that

$$E(t) \leq C_1 e^{-C_2 t} E(0),$$

for any regular solution of problem (29)-(31) with $0 \leq a < a_0$. The constants $a_0, C_1, C_2$ are independent of the initial data but they depend on $k$ and on the geometry of $\Omega$. 
The opposite problem, that is to contrast the effect of a time delay in the boundary condition with a velocity term in the wave equation, is still, as far as we know, open and it seems to be much harder to deal with. However, there is a positive answer by Datko, Lagnese and Polis [6] in the one dimensional case for the problem

\[
\begin{align*}
    u_{tt}(x, t) - u_{xx}(x, t) + 2au_t(x, t) + a^2 u(x, t) &= 0, \quad 0 < x < 1, \quad t > 0, \\
    u(0, t) &= 0, \quad t > 0 \\
    u_x(1, t) &= -ku_t(1, t - \tau), \quad t > 0;
\end{align*}
\]
with $a, k$ positive real numbers. Indeed, through a careful spectral analysis, in [6] the authors have shown that, for any $a > 0$, if $k$ satisfies

$$0 < k < \frac{1 - e^{-2a}}{1 + e^{-2a}},$$

(11)

then the spectrum of the system (7)–(9) lies in $\Re \omega \leq -\beta$, where $\beta$ is a positive constant depending on the delay $\tau$. 
Stabilization by switching time-delay

\[ \ddot{w}(t) + Aw(t) = 0, \quad 0 \leq t \leq T_0, \quad (12) \]
\[ \ddot{w}(t) + Aw(t) + \mu_1 BB^* \dot{w}(t) = 0, \quad (2i + 1)T_0 \leq t \leq (2i + 2)T_0, \quad (13) \]
\[ \ddot{w}(t) + Aw(t) + \mu_2 BB^* \dot{w}(t - T_0) = 0, \quad (2i + 2)T_0 \leq t \leq (2i + 3)T_0, \quad (14) \]
\[ w(0) = w_0, \dot{w}(0) = w_1, \quad (15) \]

where \( T_0 > 0 \) is the time delay, \( \mu_1, \mu_2 \) are real numbers and the initial datum \((w_0, w_1)\) belongs to a suitable space.
Examples

• Pointwise stabilization:

\[ u_{tt}(x, t) - u_{xx}(x, t) = 0, \quad (0, \ell) \times (0, 2\ell), \quad (16) \]

\[ u_{tt}(x, t) - u_{xx}(x, t) + a u_t(\xi, t - 2\ell) \delta_\xi = 0, \quad (0, \ell) \times (2\ell, +\infty), \quad (17) \]

\[ u(0, t) = 0, \quad u_x(\ell, t) = 0, \quad (0, +\infty), \quad (18) \]

\[ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad (0, \ell), \quad (19) \]
Boundary stabilization:

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2}(x, t) - \frac{\partial^2 u}{\partial x^2}(x, t) &= 0 \quad (0, \ell) \times (0, +\infty), \\
\frac{\partial u}{\partial t}(0, t) &= 0, \quad (0, +\infty), \\
\frac{\partial u}{\partial x}(\ell, t) &= 0, \quad (0, 2\ell), \\
\frac{\partial u}{\partial x}(\ell, t) &= \mu_1 \frac{\partial u}{\partial t}(\ell, t), \quad (2(2i + 1)\ell, 2(2i + 2)\ell), \forall i \in \mathbb{N}, \\
\frac{\partial u}{\partial x}(\ell, t) &= \mu_2 \frac{\partial u}{\partial t}(\ell, t - 2\ell), \quad (2(2i + 2)\ell, 2(2i + 3)\ell), \forall i \in \mathbb{N}, \\
u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x), \quad (0, \ell),
\end{align*}
\]

where \( \ell > 0, \mu_1, \mu_2, a \) and \( \xi \in (0, \ell) \) are constants.
Theorem (A-Nicaise-Pignotti)

We suppose that \( \xi = \frac{\ell}{2} \). Then for any \( a \in (0, 2) \) \( \exists C_1, C_2 > 0 \) s.t. for all initial data in \( \mathcal{H} \), the solution of (29)-(30) satisfies

\[
E_p(t) \leq C_1 e^{-C_2 t}.
\] (26)

The constant \( C_1 \) depends on the initial data, on \( \ell \) and on \( a \), while \( C_2 \) depends only on \( \ell \) and on \( a \).
For any $\mu_1, \mu_2$ satisfying one of the following conditions
\[ 1 < \mu_2 < \mu_1, \mu_1 < \mu_2 < 1, \]
\[ \exists C_1, C_2 > 0 \text{ s.t. for all initial data in } \mathcal{H}, \text{ the solution of (20)–(25) satisfies} \]
\[ E_b(t) \leq C_1 e^{-C_2 t}. \]

Where $E_p(t) = E_b(t) = \frac{1}{2} \int_0^\ell \{|u_x(x, t)|^2 + |u_t(x, t)|^2\} dx,$
and $\mathcal{H} = \{ u \in H^1(0, \ell), u(0) = 0 \} \times L^2(0, \ell).$
Stability of an abstract–wave equation with delay and Kelvin-Voigt damping

- We consider a stabilization problem for an abstract wave equation with delay and a Kelvin–Voigt damping.
- We prove an exponential stability result for appropriate damping coefficients by using a frequency–domain approach.
Our main goal is to study the internal stabilization of a delayed abstract wave equation with a Kelvin–Voigt damping. More precisely, given a constant time delay $\tau > 0$, we consider the system given by:

\[
  u''(t) + a BB^* u'(t) + BB^* u(t - \tau) = 0, \quad \text{in} \ (0, +\infty), \\
  u(0) = u_0, \quad u'(0) = u_1, \\
  B^* u(t - \tau) = f_0(t - \tau), \quad \text{in} \ (0, \tau),
\]
where \( a > 0 \) is a constant, \( B : D(B) \subset H_1 \to H \) is a linear unbounded operator from a Hilbert space \( H_1 \) into another Hilbert space \( H \) equipped with the respective norms \( \| \cdot \|_{H_1}, \| \cdot \|_H \) and inner products \((\cdot, \cdot)_{H_1}, (\cdot, \cdot)_H\), and \( B^* : D(B^*) \subset H \to H_1 \) is the adjoint of \( B \). The initial datum \((u_0, u_1, f_0)\) belongs to a suitable space. We suppose that the operator \( B^* \) satisfies the following coercivity assumption: there exists \( C > 0 \) such that

\[
\| B^* v \|_{H_1} \geq C \| v \|_H, \quad \forall v \in D(B^*). \tag{32}
\]

We set \( V = D(B^*) \) and we assume that it is closed with the norm \( \| v \|_V := \| B^* v \|_{H_1} \) and that it is compactly embedded into \( H \).
To restitute the well-posedness character and its stability we propose to add the Kelvin–Voigt damping term $a BB^* u'$.

Hence the stabilization of problem (29)–(31) is performed using a frequency domain approach combined with a precise spectral analysis.
We introduce the auxiliary variable

$$z(\rho, t) = B^* u(t - \tau \rho), \quad \rho \in (0, 1), \ t > 0. \quad (33)$$

Then, problem (29)–(31) is equivalent to

$$u''(t) + a B B^* u'(t) + Bz(1, t) = 0, \quad \text{in} \ (0, +\infty), \quad (34)$$

$$\tau z_t(\rho, t) + z_\rho(\rho, t) = 0 \quad \text{in} \ (0, 1) \times (0, +\infty), \quad (35)$$

$$u(0) = u_0, \quad u'(0) = u_1, \quad (36)$$

$$z(\rho, 0) = f_0(-\rho \tau), \quad \text{in} \ (0, 1), \quad (37)$$

$$z(0, t) = B^* u(t), \quad t > 0. \quad (38)$$
If we denote
\[ U := (u, u', z)\top, \]
then
\[ U' := (u', u'', z_t)\top = (u', -aBB^* u' - Bz(1, t), -\tau^{-1} z_\rho)\top. \]

Therefore, problem (34)–(38) can be rewritten as
\[
\begin{aligned}
U' &= AU, \\
U(0) &= (u_0, u_1, f_0(-\cdot \tau))\top,
\end{aligned}
\] (39)
where the operator $\mathcal{A}$ is defined by

\[
\mathcal{A} \begin{pmatrix} u \\ v \\ z \end{pmatrix} := \begin{pmatrix} v \\ -aB B^* v - Bz(\cdot, 1) \\ -\tau^{-1} z_\rho \end{pmatrix},
\]

with domain

\[
D(\mathcal{A}) := \left\{ (u, v, z)^T \in D(B^*) \times D(B^*) \times H^1(0, 1; H_1) : aB^* v + z(1) \in D(B), B^* u = z(0) \right\},
\]

in the Hilbert space

\[
\mathcal{H} := D(B^*) \times H \times L^2(0, 1; H_1),
\]

equipped with the standard inner product

\[
((u, v, z), (u_1, v_1, z_1))_{\mathcal{H}} = (B^* u, B^* u_1)_{H_1} + (v, v_1)_H + \xi \int_0^1 (z, z_1)_{H_1} \, d\rho,
\]

where $\xi > 0$ is a parameter fixed later on.
We will show that $\mathcal{A}$ generates a $C_0$ semigroup on $\mathcal{H}$ by proving that $\mathcal{A} - cld$ is maximal dissipative for an appropriate choice of $c$ in function of $\xi, \tau$ and $a$. We prove the next result.

**Lemma**

*If $\xi > \frac{2\tau}{a}$, then there exists $a^* = \left(\frac{1}{a} + \frac{\xi}{2\tau}\right)^{-1} > 0$ such that $\mathcal{A} - a^{-1}_* ld$ is maximal dissipative in $\mathcal{H}$.***
We have then the following result.

**Proposition**

*The system (29)–(31) is well-posed. More precisely, for every $(u_0, u_1, f_0) \in \mathcal{H}$, there exists a unique solution $(u, v, z) \in C(0, +\infty, \mathcal{H})$ of (39). Moreover, if $(u_0, u_1, f_0) \in D(A)$ then $(u, v, z) \in C(0, +\infty, D(A)) \cap C^1(0, +\infty, \mathcal{H})$ with $v = u'$ and $u$ is indeed a solution of (29)–(31).*
The spectral analysis

- As $D(B^*)$ is compactly embedded into $H$, the operator $BB^* : D(BB^*) \subset H \rightarrow H$ has a compact resolvent.

- Hence let $(\lambda_k)_{k \in \mathbb{N}^*}$ be the set of eigenvalues of $BB^*$ repeated according to their multiplicity (that are positive real numbers and are such that $\lambda_k \rightarrow +\infty$ as $k \rightarrow +\infty$) and denote by $(\varphi_k)_{k \in \mathbb{N}^*}$ the corresponding eigenvectors that form an orthonormal basis of $H$. 
Lemma

If $\tau \leq a$, then any eigenvalue $\lambda$ of $A$ satisfies $\Re \lambda < 0$. 
If $a < \tau$, we show that there exist some pairs of $(a, \tau)$ for which the system (29)–(31) becomes unstable. Hence the condition $\tau \leq a$ is optimal for the stability of this system.

**Lemma**

There exist pairs of $(a, \tau)$ such that $0 < a < \tau$ and for which the associated operator $\mathcal{A}$ has a pure imaginary eigenvalue.
Recall that an operator $T$ from a Hilbert space $X$ into itself is called singular if there exists a sequence $u_n \in D(T)$ with no convergent subsequence such that $\|u_n\|_X = 1$ and $Tu_n \to 0$ in $X$.

$T$ is singular if and only if its kernel is infinite dimensional or its range is not closed.
Let $\Sigma := \{ \lambda \in \mathbb{C}; \ a\lambda + e^{-\lambda \tau} = 0 \}$. The following results hold:

**Theorem**

1. If $\lambda \in \Sigma$, then $\lambda I - \mathcal{A}$ is singular.
2. If $\lambda \notin \Sigma$, then $\lambda I - \mathcal{A}$ is a Fredholm operator of index zero.
Lemma

If $\tau \leq a$, then

$$\Sigma \subset \{\lambda \in \mathbb{C} : \Re \lambda < 0\}.$$
Corollary

It holds

\[ \sigma(A) = \sigma_{pp}(A) \cup \Sigma, \]

and therefore if \( \tau \leq a \)

\[ \sigma(A) \subset \{ \lambda \in \mathbb{C} : \Re \lambda < 0 \}. \]
Asymptotic behavior

- We show that if $\tau \leq a$ and $\xi > \frac{2\tau}{a}$, the semigroup $e^{tA}$ decays to the null steady state with an exponential decay rate.

**Theorem (A-Nicaise-Pignotti)**

If $\xi > \frac{2\tau}{a}$ and $\tau \leq a$, then there exist constants $C, \omega > 0$ such that the semigroup $e^{tA}$ satisfies the following estimate

$$\|e^{tA}\|_{\mathcal{L}(\mathcal{H})} \leq C e^{-\omega t}, \forall \ t > 0.$$  

(42)
Proof of the main result

To obtain this, our technique is based on a frequency domain approach and combines a contradiction argument to carry out a special analysis of the resolvent.

We will employ the following frequency domain theorem for uniform stability of a $C_0$ semigroup on a Hilbert space:
Lemma

A $C_0$ semigroup $e^{t\mathcal{L}}$ on a Hilbert space $\mathcal{H}$ satisfies

$$\|e^{t\mathcal{L}}\|_{\mathcal{L}(\mathcal{H})} \leq C e^{-\omega t},$$

for some constant $C > 0$ and for $\omega > 0$ if and only if

$$\Re \lambda < 0, \forall \lambda \in \sigma(\mathcal{L}), \quad (43)$$

and

$$\sup_{\Re \lambda \geq 0} \| (\lambda I - \mathcal{L})^{-1} \|_{\mathcal{L}(\mathcal{H})} < \infty. \quad (44)$$

where $\sigma(\mathcal{L})$ denotes the spectrum of the operator $\mathcal{L}$. 
According to Corollary 9 the spectrum of $\mathcal{A}$ is fully included into $\Re \lambda < 0$, which clearly implies (43). Then the proof of Theorem 10 is based on the following lemma that shows that (44) holds with $\mathcal{L} = \mathcal{A}$.

**Lemma**

*The resolvent operator of $\mathcal{A}$ satisfies condition*

$$
\sup_{\Re \lambda \geq 0} \| (\lambda I - \mathcal{L})^{-1} \|_{\mathcal{L}(\mathcal{H})} < \infty.
$$

(45)
Suppose that condition (45) is false. By the Banach-Steinhaus Theorem, there exists a sequence of complex numbers $\lambda_n$ such that $\Re \lambda_n \geq 0$, $|\lambda_n| \to +\infty$ and a sequence of vectors $Z_n = (u_n, v_n, z_n)^t \in D(A)$ with

$$\|Z_n\|_H = 1$$

(46)

such that

$$\|(\lambda_n I - A)Z_n\|_H \to 0 \quad \text{as} \quad n \to \infty,$$

(47)

i.e.,

$$\lambda_n u_n - v_n \equiv f_n \to 0 \quad \text{in} \quad D(B^*),$$

(48)

$$\lambda_n v_n + a B(B^* v_n + z_n(1)) \equiv g_n \to 0 \quad \text{in} \quad H,$$

(49)

$$\lambda_n z_n + \tau^{-1} \partial_\rho z_n \equiv h_n \to 0 \quad \text{in} \quad L^2((0, 1); H_1).$$

(50)
Our goal is to derive from (47) that $\|Z_n\|_{\mathcal{H}}$ converges to zero, that furnishes a contradiction. We notice that from (48) we have

$$\|(\lambda_n I - A)Z_n\|_{\mathcal{H}} \geq |\Re ( (\lambda_n I - A)Z_n, Z_n )_{\mathcal{H}} |$$

$$\geq \Re \lambda_n - a_*^{-1} \|B^* u_n\|_{H_1}^2 + \left( \frac{\xi}{2\tau} - \frac{1}{a} \right) \|z_n(1)\|_{H_1}^2 + \frac{a}{2} \|B^* v_n\|_{H_1}^2$$

$$= \Re \lambda_n - a_*^{-1} \left\| \frac{B^* v_n + B^* f_n}{\lambda_n} \right\|_{H_1}^2 + \left( \frac{\xi}{2\tau} - \frac{1}{a} \right) \|z_n(1)\|_{H_1}^2 + \frac{a}{2} \|B^* v_n\|_{H_1}^2 ,$$

where $a^* = \left( \frac{1}{a} + \frac{\xi}{2\tau} \right)^{-1}$. 
Hence using the inequality

$$\| B^* v_n + B^* f_n \|_{H_1}^2 \leq 2 \| B^* v_n \|_{H_1}^2 + 2 \| B^* f_n \|_{H_1}^2,$$

we obtain that

$$\| (\lambda_n I - A) Z_n \|_{\mathcal{H}} \geq \Re \lambda_n - 2a_*^{-1} |\lambda_n|^{-2} \| B^* f_n \|_{H_1}^2 + \left( \frac{\xi}{2 \tau} - \frac{1}{a} \right) \| z_n(1) \|_{H_1}^2$$

$$+ \left( \frac{a}{2} - 2a_*^{-1} |\lambda_n|^{-2} \right) \| B^* v_n \|_{H_1}^2.$$
Hence for $n$ large enough, say $n \geq n^*$, we can suppose that

$$\frac{a}{2} - 2a^{-1}_* |\lambda_n|^{-2} \geq \frac{a}{4}.$$ 

and therefore for all $n \geq n^*$, we get

$$\|(\lambda_n I - A)Z_n\|_{\mathcal{H}} \geq \Re \lambda_n - 2a^{-1}_* |\lambda_n|^{-2} \|B^* f_n\|^2_{H_1} +$$

$$\left(\frac{\xi}{2\tau} - \frac{1}{a}\right) \|z_n(1)\|_{H_1}^2 + \frac{a}{4} \|B^* v_n\|_{H_1}^2.$$  (51)
By this estimate, (47) and (48), we deduce that

\[ z_n(1) \rightarrow 0, \quad B^* v_n \rightarrow 0, \quad \text{in } H_1, \text{ as } n \rightarrow \infty, \]  

(52)

and in particular, from the coercivity (32), that

\[ v_n \rightarrow 0, \quad \text{in } H, \text{ as } n \rightarrow \infty. \]

This implies according to (48) that

\[ u_n = \frac{1}{\lambda_n} v_n + \frac{1}{\lambda_n} f_n \rightarrow 0, \quad \text{in } D(B^*), \text{ as } n \rightarrow \infty, \]  

(53)

as well as

\[ z_n(0) = B^* u_n \rightarrow 0, \quad \text{in } H_1, \text{ as } n \rightarrow \infty. \]  

(54)
By integration of the identity (50), we have

\[ z_n(\rho) = z_n(0) e^{-\tau \lambda_n \rho} + \tau \int_0^\rho e^{-\tau \lambda_n (\rho - \gamma)} h_n(\gamma) \, d\gamma. \tag{55} \]

Hence recalling that \( \mathcal{R} \lambda_n \geq 0 \)

\[
\int_0^1 \|z_n(\rho)\|^2_{H_1} \, d\rho \leq 2\|z_n(0)\|^2_{H_1} + \\
2\tau^2 \int_0^1 \int_0^\rho \|h_n(\gamma)\|^2_{H_1} \, d\gamma \, d\rho \to 0, \text{ as } n \to \infty.
\]

All together we have shown that \( \|Z_n\|_{\mathcal{H}} \) converges to zero, that clearly contradicts \( \|Z_n\|_{\mathcal{H}} = 1 \).
Application to the wave equation

We study the internal stabilization of a delayed wave equation. More precisely, we consider the system given by:

\[
\begin{align*}
    u_{tt}(x, t) - a \Delta u_t(x, t) - \Delta u(x, t - \tau) &= 0, \quad \text{in} \quad \Omega \times (0, +\infty), \\
    u &= 0, \quad \text{on} \quad \partial\Omega \times (0, +\infty), \\
    u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x), \quad \text{in} \quad \Omega, \\
    \nabla u(x, t - \tau) &= f_0(t - \tau), \quad \text{in} \quad \Omega \times (0, \tau),
\end{align*}
\]

where \(\Omega\) is a smooth open bounded domain of \(\mathbb{R}^n\) and \(a, \tau > 0\) are constants.
This problem enters in our abstract framework with

\[ H = L^2(\Omega), \quad B = - \text{div} : D(B) = H^1(\Omega)^n \to L^2(\Omega), \]

\[ B^* = \nabla : D(B^*) = H^1_0(\Omega) \to H_1 := L^2(\Omega)^n, \]

the assumption (32) being satisfied owing to Poincaré’s inequality.
The operator $\mathcal{A}$ is then given by

$$
\mathcal{A} \begin{pmatrix} u \\ v \\ z \end{pmatrix} := \begin{pmatrix} v \\ a\Delta v + \text{div } z(\cdot, 1) \\ -\tau^{-1}z_{\rho} \end{pmatrix},
$$

with domain

$$
D(\mathcal{A}) := \left\{ (u, v, z)^{\top} \in H_{0}^{1}(\Omega) \times H_{0}^{1}(\Omega) \times L^{2}(\Omega; H^{1}(0, 1)) : a\nabla v + z(\cdot, 1) \in H^{1}(\Omega), \nabla u = z(\cdot, 0) \text{ in } \Omega \right\},
$$

in the Hilbert space $\mathcal{H} := H_{0}^{1}(\Omega) \times L^{2}(\Omega) \times L^{2}(\Omega \times (0, 1))$. 
Corollary

If $\tau \leq a$, the system (56)–(59) is exponentially stable in $\mathcal{H}$, namely for $\xi > \frac{2\tau}{a}$, the energy

$$
E(t) = \frac{1}{2} \left( \int_{\Omega} (|\nabla u(x, t)|^2 + |u_t(x, t)|^2) \, dx + \xi \int_{\Omega} \int_{0}^{1} |\nabla u(x, t - \tau \rho)|^2 \, dx \, d\rho \right),
$$

satisfies

$$
E(t) \leq Me^{-\omega t} E(0), \ \forall \ t > 0, \ \forall \ (u_0, u_1, f_0) \in D(A),
$$

for some positive constants $M$ and $\omega$. 
Conclusion

By a careful spectral analysis combined with a frequency domain approach, we have shown that the system (29)–(31) is exponentially stable if $\tau \leq a$ and that this condition is optimal. But from the general form of (29), we can only consider interior Kelvin-Voigt dampings.

Hence an interesting perspective is to consider the wave equation with dynamical Ventcel boundary conditions with a delayed term and a Kelvin-Voigt damping.
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Thank you for your attention