Consistent gravitational anomalies for chiral bosons

Pietro Menotti

Dipartimento di Fisica, Università di Pisa and
INFN, Sezione di Pisa, Largo B. Pontecorvo 3, I-56127
# Introduction

Anomalies: very important role both at the phenomenological and at the fundamental level
Recent application to Hawking radiation: S. Robinson and F. Wilczek (2005)

Pure gravitational anomalies: only in dimension $2 + 4n$.
L. Alvarez-Gaumé and E. Witten (1984) Bertlmann and Kohprath (2001), H. Leutwyler, S. Mallik (1985)

The simplest instance of pure gravitational anomaly is the one due to the presence of a chiral fermion in two dimensions.

There are also gravitational anomalies produced by boson fields i.e. by the self-dual and anti self-dual fields which are realized in the simplest instance by the chiral bosons in dimension 2.

Purpose of present work: provide with two very simple methods for computing exact consistent anomalies for scalar bosons.
S. Giaccari and P.M. Phys.Rev. D79, 065015 (2009)
The action

The coupling of (anti) self-dual tensors to gravity was given by Henneaux and Teitelboim (1987) which generalizes to curved backgrounds the Floreanini-Jackiw action (1987) for chiral bosons in two dimensions. On flat background such action gives rise to the chirality condition

\[ \partial_0 \varphi + \partial_1 \varphi = 0 \]  

(1)

provided some boundary conditions are satisfied. On curved background the chirality condition becomes (Bastianelli and van Nieuwenhuizen 1989)

\[ E_+^\mu \partial_\mu \varphi = 0 \]  

(2)

where \( E_+^\mu \) are inverse zweibeins. In the following the key role will be played by the adimensional function

\[ K = \frac{E_+^1}{E_+^0} = \frac{N}{\sqrt{h}} - N^1 = \frac{\sqrt{-g} - g_{01}}{g_{11}}. \]  

(3)

Right moving particles are described by \( K > 0 \). The action is provided by [5]

\[ S = -\frac{1}{2} \int d^2 x \, \partial_1 \varphi (\partial_0 \varphi + K \partial_1 \varphi) = \frac{1}{2} \int d^2 x \, \varphi \partial_1 (\partial_0 + K \partial_1) \varphi. \]  

(4)
Lack of explicit invariance under diffeomorphisms: origin of the anomaly (e.g. PV regularization).

The variation of $S$ w.r.t $\varphi$ gives the equation of motion

$$\partial_1 (\partial_0 + K \partial_1) \varphi = 0 \quad (5)$$

and the action vanishes on the equation of motion. Under a diffeomorphism the function $K$ undergoes the following passive transformation of projective nature

$$K(x) \rightarrow K'(x') = \frac{\partial x^{'1}}{\partial x^1} K(x) + \frac{\partial x^{'0}}{\partial x^0} \frac{\partial x^0}{\partial x^1} K(x) + \frac{\partial x^{'0}}{\partial x^0}$$

$$+ \frac{\partial x^{'0}}{\partial x^1} K(x) + \frac{\partial x^0}{\partial x^0} \frac{\partial x^1}{\partial x^1} K(x). \quad (6)$$

More relevant in the following will be the active transformation $K(x) \rightarrow K'(x)$ under which the action is invariant under diffeomorphisms and that in the infinitesimal case takes the form (Bastianelli and van Nieuwenhuizen)

$$\delta_\xi \varphi = \Xi \partial_1 \varphi ; \quad \delta_\xi K = - \partial_0 \Xi - \partial_1 \Xi K + \Xi \partial_1 K \quad (7)$$

where

$$\Xi = \xi^1 - K \xi^0. \quad (8)$$
As $K$ is the only function which appear in the action we have that a single vector (a single ghost) describes all diffeomorphisms.

First noticed by C. Becchi (1988). Main simplifying feature.
Pauli-Villars regulators

PV regulators are very apt to perform perturbative calculation as one can remain in dimension two.

Adaptation of standard calculation of the e.m. tensor for a scalar.

The free propagator is given by

$$\langle T \varphi(x) \varphi(y) \rangle = i (\partial_1 (\partial_0 + \partial_1))^{-1} \delta^2(x - y)$$

$$= -i \int \frac{d^2p}{(2\pi)^2} \sqrt{2} \frac{p_+}{p_1} \frac{e^{ip \cdot (x-y)}}{(p^2 - i\varepsilon)}$$

$$iW^{(2)}[h] = \frac{1}{2} \int dxdy \langle 0 | TiL_I(x) iL_I(y) | 0 \rangle$$

$$= -\frac{1}{2} \int d^2xd^2y \frac{1}{2} h_{++}(x) \langle T \partial_1 \varphi(x) \partial_1 \varphi(x) \partial_1 \varphi(y) \partial_1 \varphi(y) \rangle \frac{1}{2} h_{++}(y)$$

$$= -\frac{1}{2} \int d^2p \ h_{++}(p) U(p) h_{++}(-p)$$

$$U(p) = \frac{1}{4} \int d^2x \ e^{-ip \cdot x} \langle T \partial_1 \varphi(x) \partial_1 \varphi(x) \partial_1 \varphi(0) \partial_1 \varphi(0) \rangle.$$
For completeness we introduce an IR regularization obtained by introducing a mass $m$

$$\frac{1}{\sqrt{2} p_1 p_+ - i\varepsilon} \rightarrow \frac{1}{\sqrt{2} p_1 p_+ + m^2 - i\varepsilon} \tag{12}$$

while the PV regularization is obtained as usual by weighting the one loop graphs with $m \rightarrow M_i$ with coefficients $c_i$ obeying

$$1 + \sum_{i=1}^{4} c_i = 0 \; ; \; m^2 + \sum_{i=1}^{4} c_i M_i^2 = 0 ;$$

$$\log m^2 + \sum_{i=1}^{4} c_i \log M_i^2 = 0 ; \; m^2 \log m^2 + \sum_{i=1}^{4} c_i M_i^2 \log M_i^2 = 0 \tag{13}$$

where $M_i \rightarrow \infty$ with $c_i$ not diverging in such a limit. This can be achieved by setting $M_1^2 = s \; m^2 \; , \; M_2^2 = s^2 \; m^2 \; , \; M_3^2 = s^3 \; m^2 \; , \; M_4^2 = s^4 \; m^2$ with $s \rightarrow \infty$. At the end we take the IR regulator $m$ to zero. Performing the following change of variables

$$l_1 = k_1 + \frac{k_0}{2} ; \; \mathcal{P}_1 = p_1 + \frac{p_0}{2}$$

$$l_0 = \frac{k_0}{2} ; \; \mathcal{P}_0 = \frac{p_0}{2} \tag{14}$$
which is legal being the regularized expression convergent, we can rewrite the amplitude $U$ in the form

$$U(P, \sqrt{2} m^2) = 2T_{----}$$  \hspace{1cm} (15)

where $T_{----}$ is the not yet regulated amplitude relative to non chiral scalar bosons.

We can also write

$$\delta_\xi W^{(2)} = -\frac{i}{24\pi} \int d^2p \ p_1^3 h_{++}(p)(\xi_1(-p) + \xi_0(-p))$$  \hspace{1cm} (16)

which will bear a strong similarity with the exact result we shall obtain in the following section.
Exact calculation through Schwinger-De-Witt

The generating functional is given by

\[ Z[K] = e^{iW[K]} = \int \mathcal{D}[\phi] \exp \left[ -\frac{i}{2} \int d^2 x \partial_1 \phi (\partial_0 + K \partial_1) \phi \right] \]

\[ = \int \mathcal{D}[\phi] \exp \left[ i \frac{1}{2} \int d^2 x \phi (\partial_1 \partial_0 + \partial_1 K \partial_1) \phi \right] \]

\[ \equiv (\det -i(\partial_1 \partial_0 + \partial_1 K \partial_1))^{-\frac{1}{2}}. \quad (17) \]

As usual the direct computation of (17) is difficult. However we shall be interested in the variation of (17) under an infinitesimal diffeomorphisms which provides us with the anomaly. We have

\[ i\delta_\xi W[K] = \int \mathcal{D}[\phi] e^{i\frac{1}{2} \int d^2 x \phi (\partial_1 (K \partial_1 + \partial_0) \phi} \int d^2 x \frac{i}{2} \phi \partial_1 (\delta_\xi K \partial_1 \phi)/Z[K] \]

\[ = \frac{1}{2} \int d^2 x \delta_\xi H \ G(x, x') |_{x' = x} \quad (18) \]

with

\[ H = \partial_1 (\partial_0 + K \partial_1) \quad (19) \]
and

$$\delta_\xi H = \partial_1 (\Xi H) - H \Xi \partial_1$$  \hspace{1cm} (20)

with $\Xi = \xi^1 - K \xi^0$.

$G(x, x')$ is the exact Green function in the external field $K$. We regularize $G(x, x')$ à la Schwinger-DeWitt

$$G(x, x', \varepsilon) = i \langle x | \int_\varepsilon^\infty e^{iHt} dt | x' \rangle$$ \hspace{1cm} (21)

and thus

$$i \delta_\xi W[K] = \frac{i}{2} \int_\varepsilon^\infty dt \int d^2 x \delta_\xi H \langle x | e^{iHt} | x' \rangle |_{x'=x}. \hspace{1cm} (22)$$

We exploit now the fact that and take advantage of the equation satisfied by $\langle x | e^{iHt} | x' \rangle$ due to the evolution equation

$$\frac{d e^{iHt}}{dt} = iHe^{iHt} = ie^{iHt}H, \hspace{1cm} (23)$$

to rewrite eq.(22) as

$$\delta_\xi W = \frac{i}{2} \int_\varepsilon^\infty dt \frac{d}{dt} \int d^2 x d^2 x' \delta(x - x') \Xi(x') (\partial_1' + \partial_1) \langle x | e^{iHt} | x' \rangle =$$

$$- \frac{i}{2} \int d^2 x d^2 x' \delta(x - x') \Xi(x') (\partial_1' + \partial_1) \langle x | e^{iH\varepsilon} | x' \rangle. \hspace{1cm} (24)$$
We compute the short time behavior of $\langle x | e^{iHt} | x' \rangle$ by using the Schwinger-DeWitt technique. The operator
\[ H = \partial_1 K \partial_1 + \partial_1 \partial_0 \] (25)
is the Laplace-Beltrami operator in the “mathematical” metric
\[ g_{11} = 0; \quad g_{10} = g_{10} = 2; \quad g_{00} = -4K \] (26)
We apply the well known expansion Schwinger-DeWitt expansion with
\[ a_0(x, x') = 1; \quad \partial_1 \Delta(x, x')|_{x' = x} = 0; \quad \partial_1 a_1(x, x')|_{x' = x} = \frac{1}{12} \partial_1 R(x) \] (27)
being $R(x)$ the scalar curvature of the metric (26), we obtain
\[ \partial_1 \langle x | e^{iHt} | x' \rangle = \frac{i}{24\pi} \partial_1 R = -\frac{i}{24\pi} \partial_1^3 K \] (28)
and thus
\[ \delta_\xi W = -\frac{1}{24\pi} \int d^2x \ \Xi(x) \ \partial_1^3 K(x) = \frac{1}{24\pi} \int d^2x \ \partial_1^3 \Xi(x) \ K(x) \equiv G^E[K, \Xi] \] (29)
which is the exact Einstein anomaly.

Very simple calculation.
Wess-Zumino consistency condition and nontriviality of the anomaly

It is useful to introduce the anticommuting diffeomorphism ghosts $v^1, v^2$. Using eq.(7) we can write the BRST variation

$$\delta K = -\partial_0 V - \partial_1 VK + V \partial_1 K$$  \hspace{1cm} (30)

where $V = v^1 - K v^0$. As $K$ is the only function appearing in the theory we see that all diffeomorphisms are described by the single ghost $V$.

Becchi (1988) was the first to point out that a single ghost is sufficient to describe all diffeomorphisms for chiral bosons

Using

$$\delta v^\mu = v^\lambda \partial_\lambda v^\mu$$  \hspace{1cm} (31)

and eq.(30) it is easily proved that

$$\delta V = V \partial_1 V.$$  \hspace{1cm} (32)

The algebra of diffeomorphisms requires

$$\delta^2 K = 0$$  \hspace{1cm} (33)
a relation which can be explicitly verified using eqs.(30,32). We can now verify the Wess-Zumino consistency condition for the found anomaly (29). In fact we have

\[
\delta (\partial_1^3 V K) = \partial_1^3 (V \partial_1 V) K - \partial_1^3 V \delta K = \partial_1 (V \partial_1^3 V K) + \partial_1^3 V \partial_0 V. \tag{34}
\]

The first term on the r.h.s. is a divergence while for the second, integrating by parts and using the anticommutativity of \(V\), we have

\[
\delta G^E[K,V] = \text{const.} \int d^2x \ \partial_1^3 V \partial_0 V = 0. \tag{35}
\]

The Wess-Zumino consistency relation can also be written as

\[
\delta Q^1_2 = -dQ^2_1 \tag{36}
\]

where \(Q^1_2 = V \partial_1^3 K\) and \(Q^2_1\) is a 1-form of degree 2 in the ghost \(V\).

We shall now construct the descending cohomology chain and show algebraically by examining the last term of the chain, that the found anomaly (29) is non trivial.
We write
\[ \delta(V \partial_1^3 K dx^1 \land dx^0) = -dQ_1^2. \] (37)

The 1-form \( Q_1^2 \) can be explicitly found
\[
Q_1^2 = -\frac{1}{2}(\partial_3^3 VV)dx^1 - \left[ \frac{1}{2} \partial_2^2 V \partial_0 V - \frac{1}{2} \partial_1 V \partial_1 \partial_0 V + \frac{1}{2} V \partial_1^2 \partial_0 V \\
- \partial_1 V \partial_1^2 V K - V \partial_1 V \partial_1^2 K + V \partial_1 (\partial_1^2 V K) \right] dx^0. \] (38)

The fact that such a form \( Q_1^2 \) satisfying (36) can be constructed is the content of the Wess-Zumino consistency condition. Using the algebraic Poincaré lemma [13, 14, 15, 16, 21] a 0-form \( Q_0^3 \) must exist such that
\[
\delta Q_1^2 = -dQ_0^3. \] (39)

Such form is easily found
\[ Q_0^3 = \frac{1}{2} V \partial_1 V \partial_1^2 V. \] (40)

It is easily proved that if \( Q_2^1 \) is a trivial anomaly i.e.
\[ Q_2^1 = \delta X_2^0 + dX_1^1 \] (41)
it follows

\[ Q^3_0 = \delta X^2_0. \] (42)

Thus if we show that no \( X^2_0 \) exists satisfying (42) we prove algebraically the non triviality of \( Q^1_2 \). One notices that \( X^2_0 \) must contain two derivatives and thus the most general \( X^2_0 \) is given by

\[
X^2_0 = V \partial^2_1 V g_1 + V \partial_1 \partial_0 V g_2 + V \partial^2_0 V g_3 + \partial_1 V \partial_0 V g_4 + \\
+ V \partial_1 V \partial_1 K g_5 + V \partial_1 V \partial_0 K g_6 + V \partial_0 V \partial_1 K g_7 + V \partial_0 V \partial_0 K g_8. \quad (43)
\]

Putting all the variations together by simple algebra we arrive to the obstruction \( 0 = 1/K \) which proves algebraically the non triviality of the found anomaly.
Cohomological derivation of the anomaly

Very general cohomological treatment of anomalies for conformal invariant theories: Brandt, Dragon, Kreutzer, Barnich, Henneaux, Troost, van Proyen (1990-1996) using two ghosts.

Here exploiting the fact that in our case one can deal with a single ghost and the cohomological procedure will be very straightforward.

In the following discussion we shall denote by $S^n(m)$ the space of terms containing $n$ ghosts and $m$ derivatives, e.g. $V \partial_1^2 V \partial_0 K f(K) \in S^2(3)$. First we prove that the last term $Q_3^3$ in the cohomological chain is unique apart for the addition of a trivial term i.e. a $\delta$ variation, and then, starting from such $Q_3^3$ we prove that the cohomology chain can be climbed up in a unique way leading, in a pure algebraic way to the result (29) apart a multiplicative constant.

The first question is equivalent to the cohomological problem of proving that the sequence

$$S^0(0) \xrightarrow{\delta} S^1(1) \xrightarrow{\delta} S^2(2) \xrightarrow{\delta} S^3(3) \xrightarrow{\delta} S^4(4) \xrightarrow{\delta} 0$$

(44)
differs from an exact sequence only in the penultimate junction due to the presence of the non trivial term $N_0^3 \equiv \text{const.} V \partial_1 V \partial_1^2 V$.

Non trivial to work out due to the large number of dimensions (30-36).

Great simplification performing a change of basis by replacing the basis element $\partial_0 V$ by $W \equiv \delta K \equiv -\partial_0 V - \partial_1 V K + V \partial_1 K$, which is equivalent to it due to the relation (30). Then the algebra we shall use is simply

$$\delta V = V \partial_1 V; \quad \delta K = W; \quad \delta W = 0$$

and that will be sufficient to perform all calculations.

Due to the introduced change of variables, in the proof of the exacteness of the sequences it is sufficient to examine only the “head” on the right hand side, i.e. the terms with the highest number of $V$ as they appear only once.

For example the space $S^1(1)$ has dimension 4 and its elements can be written as

$$\partial_1 V h_1 + W h_2 + V \partial_1 K h_3 + V \partial_0 K h_4.$$  

(46)
In examining the kernel of $\delta$ from $S^1(1)$ into $S^2(2)$ we can gauge fix to zero $h_2$ to zero by adding the variation of $f(K)$ with $f'(K) = -h_2(K)$. It is then easily shown that the kernel on the remaining space is $h_1 = h_3 = h_4 = 0$ as

$$\delta(\partial_1 V h_1) = V \partial_1^2 V h_1 + \ldots$$

$$\delta(V \partial_1 K h_3) = V \partial_1 V \partial_1 K h_3 + \ldots$$

$$\delta(V \partial_0 K h_4) = V \partial_1 V \partial_0 K h_4 + \ldots \quad (47)$$

and the terms written explicitly in each equation have no counterpart in the remaining two equations.

$S^3(3)$ has dimension 8 and gauge fixing to zero $V \partial_1^2 V W f_4$, $V \partial_1 V W \partial_i K f_5^{(i)}$, $V \partial_1 V \partial_i W f_6^{(i)}$ we are left with the terms $V \partial_1 V \partial_1^2 V f_1$, $V W \partial_1 W f_2$, $V W \partial_0 W f_3$ from which, as $\delta(V \partial_1 V \partial_1^2 V) = 0$, we obtain for the kernel $f_1'(K) = 0$, $f_2(K) = f_3(K) = 0$.

Thus we found that the most general solution of $\delta Q_3^3 = 0$ can be written in the form

$$Q_0^3 = N_0^3 + \delta X_0^2 \quad (48)$$

where

$$N_0^3 = \text{const.} V \partial_1 V \partial_1^2 V \quad (49)$$
i.e the short sequence around $S^3(3)$ in (44) is exact except for the element $N_0^3$. Similarly one deals with the other two sequences the exactens of which prove that one can climb the cohomological chain in a unique way.
Conclusions

Two direct derivations of the consistent gravitational anomaly generated by a chiral boson in two dimensions.

1) Schwinger-DeWitt technique.

2) Cohomological methods.

The simplicity of the cohomological method is due to the fact that only one function of the metric appears in the Henneaux-Teitelboim action and that a single ghost generates all the diffeomorphisms.

In such an approach we show the uniqueness of the final term in the cohomological chain and that the cohomological chain can be climbed up in a unique way. This is due to the exactness of three cohomological sequences but for a junction which gives rise to the anomaly.
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