Abstract

This paper introduces an intrinsic theory of Thermodynamic Formalism for Iterated Functions Systems with general positive continuous weights (IFSw). We study the spectral properties of the Transfer and Markov operators and one of our first results is the proof of the existence of at least one eigenprobability for the Markov operator associated to a positive eigenvalue. Sufficient conditions are provided for this eigenvalue to be the spectral radius of the transfer operator and we also prove in this general setting that positive eigenfunctions of the transfer operator are always associated to its spectral radius.

We introduce variational formulations for the topological entropy of holonomic measures and the topological pressure of IFSw’s with weights given by a potential. A definition of equilibrium state is then natural and we prove its existence for any continuous potential. We show, in this setting, a uniqueness result for the equilibrium state requiring only the Gâteaux differentiability of the pressure functional. We also recover the classical formula relating the powers of the transfer operator and the topological pressure and establish its uniform convergence. In the last section we present some examples and show that the results obtained can be viewed as a generalization of several classical results in Thermodynamic Formalism for ordinary dynamical systems.

1 Introduction

Thermodynamic Formalism has its roots in the seminal paper by David Ruelle [Rue68], where the transfer operator was introduced. In that work the transfer operator was used to prove the uniqueness of the Gibbs measures for some long-range Statistical Mechanics models in the one-dimensional lattice. In [Sin72],

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Yakov Sinai established a deep connection between one-dimensional Statistical Mechanics and Hyperbolic Dynamical Systems on compact manifolds by using what he called “Markov Partition”. Since then the Thermodynamic Formalism has attracted a lot of attention in the Dynamical Systems community because of its numerous important applications. The transfer operator is a major tool in this theory and has remarkable applications to topological dynamics, meromorphy of dynamical zeta functions and multifractal analysis, just to name a few, see [Bar11, Bow08, Bow79, Man90, MM83, PP90, Rue02, Sin72] and references therein.

In the last decades, an interest in bringing the techniques developed in Thermodynamic Formalism for classical dynamical systems to the theory of Iterated Function Systems (IFS) has surfaced, but although Thermodynamic Formalism in the context of IFS has been discussed by many authors the set of results obtained thus far lack an uniform approach. The best effort has been exerted via Multifractal analysis and Ergodic optimization, however, in the majority of the papers surrounding this matter, hypothesis on the IFS self-maps and the weights such as contractiveness, open set condition (OSC), non-overlapping, conformality and Hölder or Lipschitz regularity are required.

A first version of the Ruelle-Perron-Frobenius theorem for contractive IFS, via shift conjugation, was introduced in [FL99] — the authors study the Hausdorff dimension of the Gibbs measure under conformal and OSC conditions, but several other important tools of Thermodynamic Formalism were not considered. Their proofs rely strongly on contractiveness to built a conjugation between the attractor and a code space, which is the usual shift, bringing back the classic results of the Ruelle-Perron-Frobenius theory.

Subsequently, [MU00] considered the Topological Pressure, Perron-Frobenius type operators, conformal measures and the Hausdorff dimension of the limit set, making use of hypothesis such as conformality, OSC and bounded distortion property (BDP).

Two years later, in [HMU02], a developed model of Thermodynamic Formalism arises and results for Conformal IFS and on Multifractal Analysis were obtained. The authors consider $X \subset \mathbb{R}^d$, a countable index set $I$ and a collection of uniform contractive injections $\tau_i : X \to X$. They define a conformal infinite IFS $(X, (\tau_i)_{i \in I})$ and study it by using a coding map $\pi : I^\infty \to X$, given by $\pi(w) = \cap_{n \geq 0} \tau_{w_0} \cdots \tau_{w_{n-1}}(X)$. The contraction hypotheses on that work are paramount to identify the state space $X$ with a suitable symbolic space. As usual, OSC, conformality, BDP and other properties are assumed. Their main tool is the topological pressure, defined as $P(t) = \lim_{n \to \infty} \frac{1}{n} \ln \sum \|t^n\|$, where the sum is taken over the words of length $n$ in $I^\infty$ and $\| \cdot \|$ is the norm of the derivative of the IFS maps. Under these conditions the remarkable formula $HD(J \equiv \pi(I^\infty)) = \inf\{t \geq 0 | P(t) \leq 0\}$ is obtained. They also deduce a variational formulation for the topological pressure for a potential, which is

$$P(F) = \sup_{\mu} \left\{ h_\mu(\sigma) + \int f(w) d\mu(w) \right\},$$

where $h_\mu(\sigma)$ is the usual entropy for $\mu$ as a shift invariant measure and $f$ is an
amalgamated potential in $P^\infty$: $f(w) = f_{w_0}(\pi(w))$ induced by $F = \{f_i : X \to \mathbb{C}\}_{i \in I}$ a family H"older exponential weight functions. The generalization of the classical formula for the pressure is obtained in this context, i.e.

$$P(F) = \inf_{n \geq 1} \left\{ \frac{1}{n} \ln \sum_{|w|=n} \|Z_n(F)\| \right\},$$

where $Z_n(F)$ is suitable partition function (see [HMU02] for details). From this, equilibrium states are studied.

Some of the main results of this paper are inspired by three recent works [LMMS15, K04, CvER17]. In [LMMS15] Thermodynamic Formalism is developed for H"older potentials with the dynamics given by the left-shift mapping acting on an infinite cartesian product of compact metric spaces, which can be seen as an infinite contractive IFS. In [K04] the author developed a Thermodynamical Formalism for finite iterated function system, including self-affine and self-conformal maps. The topological pressure is defined for cylinder functions, but the measures are defined on an ad-hoc symbolic space and not on the state space itself, and contractiveness is assumed on the IFS to project it onto the code space. A strong control of overlapping cannot be dismissed. One of our main results Theorem 4 is inspired by an analogous result in [CvER17], for symbolic dynamics on uncountable alphabets.

In order to construct a truly intrinsic theory of Thermodynamic Formalism for IFS, conditions such as contractiveness or conformality should be avoided and, ideally, only the continuity of the self-maps ought to be requested. To have a theory where phase transition phenomenon can manifest, continuity (or a less restrictive condition) on the weights is a very natural condition to be considered.

Aiming to have an intrinsic theory, we extended some of the results obtained in [LO09]. The main achievement of that paper was to introduce the idea of holonomic probability measure as the natural replacement of invariance for an IFS. The authors considered a variational notion of entropy and topological pressure in terms of the transfer operator, leading to the existence of a very natural variational principle. However, abundant information on the spectral properties of the transfer operator was required and strong hypothesis on the IFS were needed. Here we show how to avoid such requirements for finite continuous IFS with positive continuous weights, and remark that no coding spaces are used in our work and only intrinsic structures of the IFS are necessary to define and study equilibrium states as probability measures on the state space.

This work adds to the remarkable effort that has been done by several authors in the last few years to produce more general results for IFS, regarding fundamental results such as the existence of attractors and attractive measures for the Markov operator. See the recent works [AJS17] for weakly hyperbolic IFS, [Mel16] for $P$-weakly hyperbolic IFS and [MD16] for non-hyperbolic IFS.

The paper is organized as follows. In Section 2 we introduce the notion of IFS with weights, generalizing the basic setting where the idea of potential is present. We also study the associated transfer operator and present a characterization of its spectral radius. In Section 3 we study the Markov operator.
from an abstract point of view and prove a very general result regarding the existence of eigenmeasures associated to a positive eigenvalue. The holonomic measures are introduced in Section 4 and their existence, disintegration and the compatibility with IFS are carefully discussed. In the subsequent section, the notions of entropy for a holonomic probability measure and topological pressure for a continuous potential are introduced and we show that, for any continuous potential and IFS, there is always an equilibrium state. We also prove that the Gâteaux differentiability of the topological pressure is linked to the uniqueness problem of equilibrium states. By imposing extra conditions on the IFS and the weights, we provide a constructive approach to the existence of an equilibrium state by using the maximal eigendata of the transfer operator in Section 6.

2 IFS with weights - Transfer and Markov Operators

In this section we set up the basic notation and present a fundamental result about the eigenspace associated to the maximal eigenvalue (or spectral radius) of transfer operator.

Throughout this paper, $X$, $Y$ and $\Omega$ are general compact metric spaces. The Borel $\sigma$-algebra of $X$ is denoted by $\mathcal{B}(X)$ and similar notation is used for $Y$ and $\Omega$. The Banach space of all real continuous functions equipped with supremum norm is denoted by $C(X, \mathbb{R})$. Its topological dual, as usual, is identified with $\mathcal{M}_s(X)$, the space of all finite Borel signed measures endowed with total variation norm. We use the notation $\mathcal{M}_1(X)$ for the set of all Borel probability measures over $X$ supplied with the weak-$*$ topology. Since we are assuming that $X$ is compact metric space then we have that the topological space $\mathcal{M}_1(X)$ is compact and metrizable.

An Iterated Function System with weights, IFSw for short, is an ordered triple $R_q = (X, \tau, q)$, where $\tau \equiv \{\tau_0, ..., \tau_{n-1}\}$ is a collection of $n$ continuous functions from $X$ to itself and $q \equiv \{q_0, ..., q_{n-1}\}$ is a finite collection of continuous weights from $X$ to $[0, \infty)$. An IFSw $R_q$ is said to be normalized if for all $x \in X$ we have $\sum_{i=0}^{n-1} q_i(x) = 1$. A normalized IFSw with nonnegative weight is sometimes called an IFS with place dependent probabilities (IFSpdp).

Definition 1. Let $R_q = (X, \tau, q)$ be an IFSw. The Transfer and Markov operators associated to $R_q$ are defined as follows:

1. The Transfer Operator $B_q : C(X, \mathbb{R}) \to C(X, \mathbb{R})$ is given by

$$B_q(f)(x) = \sum_{i=0}^{n-1} q_i(x)f(\tau_j(x)), \quad \forall x \in X.$$ 

2. The Markov Operator $L_q : \mathcal{M}_s(X) \to \mathcal{M}_s(X)$ is the unique bounded
linear operator satisfying
\[ \int_X f \, d[\mathcal{L}_q(\mu)] = \int_X B_q(f) \, d\mu, \]
for all $\mu \in \mathcal{M}_s(X)$ and $f \in C(X, \mathbb{R})$.

Note that the isomorphism $C(X, \mathbb{R})^* \simeq \mathcal{M}_s(X)$ allow us to look at the Markov operator $L_q$ as the Banach transpose of $B_q$, i.e., $B_q^* = L_q$.

**Proposition 1.** Let $R_q = (X, \tau, q)$ be a continuous IFSw. Then for the $N$-th iteration of $B_q$ we have
\[ B_q^N(1)(x) = \sum_{w_0 \ldots w_{N-1}=0}^{N-1} P_x^q(w_0, \ldots, w_{N-1}) \]
where, $P_x^q(w_0, \ldots, w_{N-1}) = \prod_{j=0}^{N-1} q_{w_j}(x_j)$, $x_0 = x$ and $x_{j+1} = \tau_{w_j} x_j$.

**Proof.** This expression can be obtained by proceeding a formal induction on $N$. \qed

In the sequel we prove the main result of this section which is Lemma 1. Roughly speaking it states that any possible positive eigenfunctions of the transfer operator $B_q$ lives in the eigenspace associated to the spectral radius of this operator acting on $C(X, \mathbb{R})$. Beyond this nice application the lemma will be used, in the last section, to derive an expression for the topological pressure (see Definition 11) involving only the transfer operator and its powers.

**Lemma 1.** Let $R_q = (X, \tau, q)$ be a continuous IFSw and suppose that there are a positive number $\rho$ and a strictly positive continuous function $h : X \to \mathbb{R}$ such that $B_q(h) = \rho h$. Then the following limit exists
\[ \lim_{N \to \infty} \frac{1}{N} \ln \left( B_q^N(1)(x) \right) = \ln \rho(B_q) \]  \label{eq:topological-pressure}
the convergence is uniform in $x$ and $\rho = \rho(B_q)$, the spectral radius of $B_q$ acting on $C(X, \mathbb{R})$.

**Proof.** From the hypothesis we get a normalized continuous IFSdp $R_p = (X, \tau, p)$, where the weights are given by
\[ p_j(x) = q_j(x) \frac{h(\tau_j(x))}{\rho h(x)}, \quad j = 0, \ldots, n - 1. \]

Note that $P_x^q(w_0, \ldots, w_{N-1})$ and $P_x^p(w_0, \ldots, w_{N-1})$ are related in the following
way

\[ P^q_x(w_0, \ldots, w_{N-1}) = \prod_{j=0}^{N-1} q_{w_j}(x_j) = \prod_{j=0}^{N-1} p_{w_j}(x_j) \frac{\rho h(x_j)}{h(x_{j+1})} \]

\[ = \rho^N \frac{h(x_0)}{h(x_N)} \prod_{j=0}^{N-1} p_{w_j}(x_j) \]

\[ = P^p_x(w_0, \ldots, w_{N-1}) \rho^N \frac{h(x_0)}{h(x_N)}. \]

Since \( X \) is compact and \( h \) is strictly positive and continuous, we have for some positive constants \( a \) and \( b \) the following inequalities \( 0 < a \leq h(x_0)/h(x_N) \leq b \).

By using the Proposition 1 and the above equality, we get for any fixed \( N \in \mathbb{N} \) the following expression

\[ \frac{1}{N} \ln \left( B^N_q(1)(x) \right) = \frac{1}{N} \ln \left( \sum_{w_0 \ldots w_{N-1} = 0}^{n-1} P^q_x(w_0, \ldots, w_{N-1}) \right) \]

\[ = \frac{1}{N} \ln \left( \sum_{w_0 \ldots w_{N-1} = 0}^{n-1} P^p_x(w_0, \ldots, w_{N-1}) \rho^N \frac{h(x_0)}{h(x_N)} \right) \]

\[ = \ln \rho + O(1/N) + \frac{1}{N} \ln \left( \sum_{w_0 \ldots w_{N-1} = 0}^{n-1} P^p_x(w_0, \ldots, w_{N-1}) \right) \]

\[ = \ln \rho + O(1/N), \]

where the term \( O(1/n) \) is independent of \( x \). Therefore for every \( N \geq 1 \) we have

\[ \sup_{x \in X} \left| \frac{1}{N} \ln \left( B^N_q(1)(x) \right) - \ln \rho \right| = O(1/N). \]

which proves (1). From the above inequality and Gelfand’s formula for the spectral radius we have

\[ |\ln \rho(B_q) - \ln \rho| = \left| \ln \left( \lim_{N \to \infty} \|B^N_q\| \right) - \ln \rho \right| = \lim_{N \to \infty} \left| \frac{1}{N} \ln \|B^N_q\| - \ln \rho \right| \]

\[ \leq \lim_{N \to \infty} \sup_{x \in X} \left| \frac{1}{N} \ln \left( B^N_q(1)(x) \right) - \ln \rho \right| \]

\[ \leq \lim_{N \to \infty} \sup_{N \to \infty} C = 0. \]

Remark 2. In the special case, where there is a positive continuous function \( \psi : X \to \mathbb{R} \) (potential) such that \( q_i(x) \equiv \psi(\tau_i(x)) \) for all \( i = 0, 1, \ldots, n-1 \) and \( x \in X \) we will show later that \( \ln \rho(B_q) \) is the topological pressure of \( \psi \) (Definition 1).
In the following example, Lemma 1 is used to show that lack of contractiveness and existence of a continuous eigenfunction associated to a positive eigenvalue, in general, impose very strong restrictions on the weights.

**Example 3.** We take $X = [0, 1]$ with its usual topology and $\tau_i(x) := (-1)^i x + i$, for $i = 0, 1$. For any choice of a continuous weight $q$, we have that $R_q = (X, \tau, q)$ is a continuous IFSw. Since the derivative of $\tau_i(x)$ is equal to $(-1)^i$, for any point $x \in (0, 1)$, no hyperbolicity exists in this system. Moreover, for each $t \in [0, 1/2]$ the set $A_t = [t, 1-t]$ is fixed by the Hutchinson-Barnsley operator $A \mapsto F_R(A) \equiv \tau_0(A) \cup \tau_1(A)$, so this IFS has no attractor. Note that $\cap_n \tau_{w_n} \circ \cdots \circ \tau_{w_0}([0, 1])$ is never a singleton, so this example do not fit the recent theory of weakly hyperbolic sequences developed in [MD16, Mel16] neither [AJS17].

For $i = 0, 1$ consider the weight $q_i(x) = \psi(\tau_i(x))$, where $\psi : X \to \mathbb{R}$ a continuous and strictly positive function. In this case, the transfer operator is given by $B_q(1)(x) = \psi(x) + \psi(1-x)$ and a simple induction on $N$, shows that $B_q^N(1)(x) = (\psi(x) + \psi(1-x))^N$ for all $N \geq 1$. Therefore

$$\lim_{N \to \infty} \frac{1}{N} \ln (B_q^N(1)(x)) = \lim_{N \to \infty} \frac{1}{N} \ln ((\psi(x) + \psi(1-x))^N) = \ln(\psi(x) + \psi(1-x)).$$

Because of Lemma 1 unless $q$ is chosen such that $\psi(x) + \psi(1-x) = \rho(B_q), \forall x \in X$, no positive continuous eigenfunction for $B_q$, associated to a positive eigenvalue, can exist.

On the other hand, if $\psi(x) + \psi(1-x) \equiv c > 0, \forall x \in X$, then $B_q(1) = c \cdot 1$, i.e., the constant function $h \equiv 1$ is a continuous positive eigenfunction and $c = \rho(B_q)$.

![Figure 1: The graph of a continuous potential $\psi : [0, 1] \to \mathbb{R}$ satisfying the condition $\psi(x) + \psi(1-x) \equiv \rho(B_q)$](image)

For further discussion on this matter see Example 3 in [CR90].
3 Markov Operator and its Eigenmeasures

If \( \mathcal{R}_q = (X, \tau, q) \) is a normalized IFS\(\ell\), then \( \mathcal{L}_q \) maps \( \mathcal{M}_1(X) \) to itself. Since \( \mathcal{M}_1(X) \) is a convex and compact Hausdorff space we can apply the Tychonoff-Schauder fixed point theorem to ensure the existence of at least one fixed point for \( \mathcal{L}_q \). In [BDEG88] (also [Hut81] or [Bar88]) is shown, under suitable contraction hypothesis, that \( \mathcal{L}_q \) has a unique fixed point \( \mu \). Such probability measure \( \mu \) is called the Barnsley-Hutchinson measure for \( \mathcal{R}_q \).

The aim of this section is to present a generalization of the above result for a non-normalized IFS\(\ell\). The central result is Theorem 4.

**Theorem 4.** Let \( \mathcal{R}_q = (X, \tau, q) \) be a continuous IFS\(\ell\) with positive weights. Then there exists a positive number \( \rho \leq \rho(B_q) \) so that the set \( \mathcal{G}^*(q) = \{ \nu \in \mathcal{M}_1(X) : \mathcal{L}_q \nu = \rho \nu \} \) is not empty.

**Proof.** Notice that the mapping \( \mathcal{M}_1(X) \ni \gamma \mapsto \mathcal{L}_q(\gamma)(X) \) sends \( \mathcal{M}_1(X) \) to itself. From its convexity and compactness, in the weak topology which is Hausdorff when \( X \) is metric and compact, it follows from the continuity of \( \mathcal{L}_q \) and the Tychonov-Schauder Theorem that there is at least one probability measure \( \nu \) satisfying \( \mathcal{L}_q(\nu)(X) = (\mathcal{L}_q(\nu)(X)) \nu \).

We claim that

\[
\min_{j \in \{1, \ldots, n\}} \inf_{x \in X} q_j(x) \leq \mathcal{L}_q(\gamma)(X) \leq \max_{j \in \{1, \ldots, n\}} \{\|q_j\|_{\infty}\} \tag{2}
\]

for every \( \gamma \in \mathcal{M}_1(X) \). Indeed,

\[
\min_{j \in \{1, \ldots, n\}} \inf_{x \in X} q_j(x) \leq \int_X B_q(1) \, d\gamma = \int_X 1 \, d[\mathcal{L}_q \gamma] = \mathcal{L}_q(\gamma)(X)
\]

and similarly

\[
\max_{j \in \{1, \ldots, n\}} \{\|q_j\|_{\infty}\} \geq \int_X B_q(1) \, d\gamma = \int_X 1 \, d[\mathcal{L}_q \gamma] = \mathcal{L}_q(\gamma)(X).
\]

From the inequality \( 2 \) follows that

\[
0 < \rho \equiv \sup \{\mathcal{L}_q(\nu)(X) : \mathcal{L}_q(\nu) = (\mathcal{L}_q(\nu)(X)) \nu \} < +\infty.
\]

By a compactness argument one can show the existence of \( \nu \in \mathcal{M}_1(X) \) so that \( \mathcal{L}_q \nu = \rho \nu \). Indeed, let \( (\nu_n)_{n \in \mathbb{N}} \) be a sequence such that \( \mathcal{L}_q(\nu_n)(X) \uparrow \rho \), when \( n \) goes to infinity. Since \( \mathcal{M}_1(X) \) is compact metric space in the weak topology we can assume, up to subsequence, that \( \nu_n \rightharpoonup \nu \). This convergence together with the continuity of \( \mathcal{L}_q \) provides

\[
\mathcal{L}_q \nu = \lim_{n \to \infty} \mathcal{L}_q \nu_n = \lim_{n \to \infty} \mathcal{L}_q(\nu_n)(X) \nu_n = \rho \nu.
\]
thus showing that the set $\mathcal{G}^*(q) \equiv \{ \nu \in \mathcal{M}_1(X) : \mathcal{L}_q \nu = \rho \nu \} \neq \emptyset$.

To finish the proof we observe that by using any $\nu \in \mathcal{G}^*(q)$, we get the following inequality

$$\rho_N = \int_X B_q^N(1) \, d\nu \leq \| B_q^N \|.$$  

From this inequality and Gelfand’s Formula follows that $\rho \leq \rho(B_q)$. \qed

**Theorem 5.** Let $\mathcal{R}_q = (X, \tau, q)$ be a continuous IFSw and suppose $B_q$ has a positive continuous eigenfunction. Then the eigenvalue $\rho$ of the Markov operator, provided by Theorem 4, satisfies $\rho = \rho(B_q)$.

**Proof.** Note that we can apply Lemma 1 to ensure that $N^{-1} \ln (B_q^N(1)(x)) \to \ln \rho(B_q)$, uniformly in $x$ when $N \to \infty$. By this convergence, the Lebesgue Dominated Convergence Theorem and the Jensen Inequality we get

$$\log \rho(B_q) = \lim_{N \to \infty} \frac{1}{N} \int_X \ln B_q^N(1) \, d\nu \leq \lim_{n \to \infty} \frac{1}{N} \ln \int_X B_q^N(1) \, d\nu = \log \rho,$$

where $\nu \in \mathcal{G}^*(q)$. Since $\rho$ is always less than or equal to $\rho(B_q)$ follows from the above inequality that $\rho = \rho(B_q)$. \qed

### 4 Holonomic Measure and Disintegrations

An invariant measure for a classical dynamical system $T : X \to X$ on a compact space is a measure $\mu$ satisfying for all $f \in C(X, \mathbb{R})$

$$\int_X f(T(x)) \, d\mu = \int_X f(x) \, d\mu, \quad \text{equivalently} \quad \int_X f(T(x)) - f(x) \, d\mu = 0.$$

From the Ergodic Theory point of view the natural generalization of this concept for an IFS $\mathcal{R} = (X, \tau)$ is the concept of holonomy.

Consider the cartesian product space $\Omega = X \times \{0, ..., n-1\}$ and for each $f \in C(X, \mathbb{R})$ its “discrete differential” $df : \Omega \to \mathbb{R}$ defined by $[df](i) \equiv f(\tau_i(x)) - f(x)$.

**Definition 6.** A measure $\hat{\mu}$ over $\Omega$ is said holonomic, with respect to an IFS $\mathcal{R}$ if for all $f \in C(X, \mathbb{R})$ we have

$$\int_{\Omega} [df](i) \, d\hat{\mu}(x, i) = 0.$$

Notation, $\mathcal{H}(\mathcal{R}) \equiv \{ \hat{\mu} | \hat{\mu} \text{ is a holonomic probability measure with respect to } \mathcal{R} \}$.

Since $X$ is compact the set of all holonomic probability measures is obviously convex and compact. It is also not empty because $X$ is compact and any average

$$\hat{\mu}_N \equiv \frac{1}{N} \sum_{j=0}^{N-1} \delta(x_j, i_j),$$

where $\{x_j, i_j\}_{j=0}^{N-1}$ is an $N$-periodic orbit of $\mathcal{R}$.
where \( x_{j+1} = \tau_j(x_j) \) and \( x_0 \in X \) is fixed, will have their cluster points in \( \mathcal{H}(\mathcal{R}) \). Indeed, for all \( N \geq 1 \) we have the following identity

\[
\int_{\Omega} [d_x f](i) \, d\hat{\mu}_N(x, i) = \frac{1}{N} \sum_{j=0}^{N-1} [d_x f](i_j) = \frac{1}{N} (f(\tau_{i-1}(x_{N-1})) - f(x_0)).
\]

From the above expression is easy to see that if \( \hat{\mu} \) is a cluster point of the sequence \( (\hat{\mu}_N)_{N \geq 1} \), then there is a subsequence \( (N_k)_{k \to \infty} \) such that

\[
\int_{\Omega} [d_x f](i) \, d\hat{\mu}(x, i) = \lim_{k \to \infty} \int_{\Omega} [d_x f](i) \, d\hat{\mu}_{N_k}(x, i)
= \lim_{k \to \infty} \frac{1}{N_k} (f(\tau_{i-1}(x_{N_k-1})) - f(x_0)) = 0.
\]

**Theorem 7** (Disintegration). Let \( X \) and \( Y \) be compact metric spaces, \( \hat{\mu} : \mathcal{B}(Y) \to [0, 1] \) a Borel probability measure, \( T : Y \to X \) a Borel measurable function and for each \( A \in \mathcal{B}(X) \) define a probability measure \( \mu(A) \equiv \hat{\mu}(T^{-1}(A)) \). Then there exists a family of Borel probability measures \( (\mu_x)_{x \in X} \) on \( Y \), uniquely determined \( \mu \)-a.e., such that

1. \( \mu_x(Y \setminus T^{-1}(x)) = 0 \), \( \mu \)-a.e;

2. \( \int_Y f \, d\hat{\mu} = \int_X \left[ \int_{T^{-1}(x)} f(y) \, d\mu_x(y) \right] \, d\mu(x). \)

This decomposition is called the disintegration of \( \hat{\mu} \), with respect to \( T \).

**Proof.** For a proof of this theorem, see [DM78] p.78 or [AGS08], Theorem 5.3.1. \( \square \)

In this paper we are interested in disintegrations in cases where \( Y \) is the cartesian product \( \Omega = X \times \{0, \ldots, n-1\} \) and \( T : \Omega \to X \) is the projection on the first coordinate. In such cases if \( \hat{\mu} \) is any Borel probability measure on \( \Omega \), then follows from the first conclusion of Theorem 7 that the disintegration of \( \hat{\mu} \) provides for each \( x \in X \) a unique probability measure \( \mu_x \) (\( \mu \)-a.e.) supported on the finite set \( \{x, 0, \ldots, (x, n-1)\} \). So we can write the disintegration of \( \hat{\mu} \) as \( d\hat{\mu}(x, i) = d\mu_x(i) \, d\mu(x) \), where here we are abusing notation identifying \( \mu_x(\{(x, j)\}) \) with \( \mu_x(\{j\}) \).

Now we take \( \hat{\mu} \in \mathcal{H}(\mathcal{R}) \) and \( f : \Omega \to \mathbb{R} \) as being any bounded continuous function, depending only on its first coordinate. From the very definition of holonomic measures we have the following equations

\[
\int_{\Omega} [d_x f](i) \, d\hat{\mu}(x, i) = 0 \iff \int_{\Omega} f(\tau_i(x)) \, d\hat{\mu}(x, i) = \int_{\Omega} f(x) \, d\hat{\mu}(x, i)
\]

by disintegrating both sides of the second equality above we get that

\[
\int_X \int_{\{0, \ldots, n-1\}} f(\tau_i(x)) \, d\mu_x(i) \, d\mu(x) = \int_X \int_{\{0, \ldots, n-1\}} f(x) \, d\mu_x(i) \, d\mu(x).
\]
Recalling that $\mu_x$ is a probability measure follows from the above equation that
\[
\int_X \sum_{i=0}^{n-1} \mu_x(i) f(\tau_i(x)) \, d\mu(x) = \int_X f(x) \, d\mu(x).
\]

The last equation establishes a natural link between holonomic measures for an IFS $R$ and disintegrations. Given an IFS $R = (X, \tau)$ and $\hat{\mu} \in \mathcal{H}(R)$ we can use the previous equation to define an IFSpd $R_q = (X, \tau, q)$, where the weights $q_i(x) = \mu_x(\{i\})$. If $B_q$ denotes the transfer operator associated to $R_q$ we have from the last equation the following identity
\[
\int_X B_q(f) \, d\mu(x) = \int_X f(x) \, d\mu(x).
\]
Since in the last equation $f$ is an arbitrary bounded measurable function, depending only on the first coordinate, follows that the Markov operator associated to the IFSpd $R_q$ satisfies
\[
\mathcal{L}_q(\mu) = \mu.
\]
In other words the “second marginal” $\mu$ of a holonomic measure $\hat{\mu}$ is always an eigemeasure for the Markov operator associated to the IFSpd $R_q = (X, \tau, q)$ defined above.

Reciprocally. Since the last five equations are equivalent, given an IFSpd $R_q = (X, \tau, q)$ such that the associated Markov operator has at least one fixed point, i.e., $\mathcal{L}_q(\mu) = \mu$, then it is possible to define a holonomic probability measure $\hat{\mu} \in \mathcal{H}(R)$ given by $d\hat{\mu}(x,i) = d\mu_x(i) \, d\mu(x)$, where $\mu_x(i) \equiv q_i(x)$. This Borel probability measure on $\Omega$ will be called the holonomic lifting of $\mu$, with respect to $R_q$.

5 Entropy and Pressure for IFSw

In this section we introduce the notions of topological pressure and entropy. We adopted variational formulations for both concepts because it allows us to treat very general IFSw. These definitions introduced here are inspired, and generalizes, the recent theory of such objects in the context of symbolic dynamics of the left shift mapping acting on $X = M^\mathbb{N}$, where $M$ is an uncountable compact metric space.

As in the previous section the mapping $T : \Omega \to X$ denotes the projection on the first coordinate. Even when not explicitly mentioned, any disintegrations of a probability measure $\hat{\nu}$, defined over $\Omega$, will be from now considered with respect to $T$.

We write $B_1$ to denote the transfer operator $B_q$, where the weights $q_i(x) \equiv 1$ for all $x \in X$ and $i = 0, \ldots, n - 1$.

**Definition 8 (Average and Variational Entropies).** Let $R$ be an IFS, $\hat{\nu} \in \mathcal{H}(R)$ and $d\hat{\nu}(x,i) = d\nu_x(i) \, d\nu(x)$ a disintegration of $\hat{\nu}$, with respect to $T$. The
variational and average entropies of \( \hat{\nu} \) are defined, respectively, by

\[
h_v(\hat{\nu}) \equiv \inf_{g \in C(X, \mathbb{R})} \left\{ \int_X \ln \frac{B_1(g)(x)}{g(x)} d\nu(x) \right\}
\]

and

\[
h_a(\hat{\nu}) \equiv -\int_X \sum_{i=0}^{n-1} q_i(x) \ln q_i(x) d\nu(x),
\]

where \( q_i(x) \equiv \nu_x(i) \), for all \( x \in X \) and \( i = 0, \ldots, n-1 \).

**Definition 9** (Optimal Function). Let \( \mathcal{R} \) be an IFS, \( \hat{\nu} \in H(\mathcal{R}) \), \( d\hat{\nu}(x, i) = dv_x(i) d\nu(x) \) a disintegration of \( \hat{\nu} \), with respect to \( T \) and \( q_i(x) = \nu_x(i) \), for all \( x \in X \) and \( i = 0, \ldots, n-1 \). We say that a positive function \( g \in C(X, \mathbb{R}) \) is optimal, with respect to the the IFSpdp \( \mathcal{R}_q = (X, \tau, q) \) if for all \( i = 0, \ldots, n-1 \) we have

\[
q_i(x) = \frac{g(\tau_i(x))}{B_1(g)(x)}.
\]

As the reader probably already noted, the expression of the average entropy \( h_a \) is a familiar one. In the sequel we prove a theorem establishing some relations between the two previous defined concepts of entropies. This result is a useful tool when doing some computations regarding the pressure functional, which will be defined later. Before state the theorem we recall a fundamental inequality regarding non-negativity of the conditional entropy of two probability vectors. See, for example, the reference [PP90] for a proof.

**Lemma 2.** For any probability vectors \((a_0, \ldots, a_{n-1})\) and \((b_0, \ldots, b_{n-1})\) we have

\[
-\sum_{i=0}^{n-1} a_i \ln(a_i) \leq -\sum_{i=0}^{n-1} a_i \ln(b_i)
\]

and the equality is attained iff \( a_i = b_i \).

**Theorem 10.** Let \( \mathcal{R} \) be an IFS, \( \hat{\nu} \in H(\mathcal{R}) \), \( d\hat{\nu}(x, i) = dv_x(i) d\nu(x) \) a disintegration of \( \hat{\nu} \), with respect to \( T \) and \( \mathcal{R}_q = (X, \tau, q) \) the IFSpdp with \( q_i(x) = \nu_x(i) \) for all \( x \in X \) and \( i = 0, \ldots, n-1 \). Then

1. \( 0 \leq h_a(\hat{\nu}) \leq h_v(\hat{\nu}) \leq \ln n \);
2. if there exists some optimal function \( g^* \), with respect to \( \mathcal{R}_q \), then

\[
h_a(\hat{\nu}) = h_v(\hat{\nu}) = \int_X \ln \frac{B_1(g^*)(x)}{g^*(x)} d\nu,
\]

**Proof.** We first prove item 1. Since \( 0 \leq q_i(x) = \nu_x(i) \leq 1 \) follows from the definition of average entropy that \( h_a(\hat{\nu}) \geq 0 \).
From the definition of variational entropy we obtain
\[ h_v(\hat{\nu}) = \inf_{g \in C(X, \mathbb{R})} \left\{ \int_X \ln \frac{B_1(g)}{g} \, d\nu \right\} \leq \int_X \frac{B_1(1)}{1} \, d\nu = \ln n. \]

To finish the proof of item 1 remains to show that \( h_a(\hat{\nu}) \leq h_v(\hat{\nu}) \). Let \( g : X \to \mathbb{R} \) be continuous positive function and define for each \( x \in X \) a probability vector \((p_0(x), \ldots, p_{n-1}(x))\), where \( p_j(x) = g(\tau_j(x))/B_1(g)(x) \), for each \( j = 0, \ldots, n - 1 \). From Lemma 2 and the properties of the holonomic measures we get the following inequalities for any continuous and positive function \( g \)
\[
\begin{align*}
&h_a(\hat{\nu}) = -\int_X \sum_{j=0}^{n-1} q_j(x) \ln(q_j(x)) \, d\nu \leq -\int_X \sum_{j=0}^{n-1} q_j \ln \left( \frac{g \circ \tau_j}{B_1(g)} \right) \, d\nu \\
&= -\int_X B_q(\ln g) \, d\nu + \int_X \ln(B_1(g)) \, d\nu \\
&= -\int_X \ln g \, d\nu + \int_X \ln(B_1(g)) \, d\nu \\
&= \int_X \frac{B_1(g)}{g} \, d\nu.
\end{align*}
\]
(3)

Therefore
\[
h_a(\hat{\nu}) \leq \inf_{g \in C(X, \mathbb{R})} \left\{ \int_X \ln \frac{B_1(g)}{g} \, d\nu \right\} = h_v(\hat{\nu})
\]
and the item 1 is proved.

Proof of item 2. From Lemma 2 it follows that the equality in (3) is attained for any optimal function with respect to \( R_q \). Since we are assuming the existence of at least one optimal function \( g^* \) we have
\[
h_a(\hat{\nu}) = \int_X \ln \frac{B_1(g^*)}{g^*} \, d\nu \geq h_v(\hat{\nu}).
\]
Since the reverse inequality is always valid we are done.

We now introduce the natural generalization of the concept of topological pressure of a continuous potential.

**Definition 11.** Let \( \psi : X \to \mathbb{R} \) be a positive continuous function and \( R_\psi \equiv (X, \tau, \psi \star \tau) \) an IFSw, where the weights are give by \( (\psi \star \tau)_i(x) \equiv \psi(\tau_i(x)) \).
The topological pressure of $\psi$, with respect to the IFS $\mathcal{R}_\psi$, is defined by the following expression

$$P(\psi) \equiv \sup_{\hat{\nu} \in \mathcal{H}(\mathcal{R})} \inf_{g \in C(X,\mathbb{R})} \left\{ \int_X \ln \frac{B_{\psi \tau}(g)}{g} \, d\nu \right\},$$

where $d\nu_x(i) = d\hat{\nu}(x, i)$ is a disintegration of $\hat{\nu}$, with respect to $\mathcal{R}$.

**Lemma 3.** Let $\psi : X \to \mathbb{R}$ be a positive continuous function and $\mathcal{R}_\psi = (X, \tau, \psi \tau)$ the IFS above defined. Then the topological pressure of $\psi$ is alternatively given by

$$P(\psi) = \sup_{\hat{\nu} \in \mathcal{H}(\mathcal{R})} \{ h_{\hat{\nu}}(\hat{\nu}) + \int_X \ln \psi \, d\nu \}.$$

**Proof.** To get this identity we only need to use the pressure’s definition and the basic properties of the transfer operator as follows

$$P(\psi) = \sup_{\hat{\nu} \in \mathcal{H}(\mathcal{R})} \inf_{g \in C(X,\mathbb{R})} \left\{ \int_X \ln \frac{B_{\psi \tau}(g)}{g} \, d\nu \right\}$$

$$= \sup_{\hat{\nu} \in \mathcal{H}(\mathcal{R})} \left\{ \int_X \ln \psi \, d\nu - \int_X \ln \psi \, d\nu + \int_X \ln \frac{B_{\psi \tau}(g)}{g} \, d\nu \right\}$$

$$= \sup_{\hat{\nu} \in \mathcal{H}(\mathcal{R})} \left\{ \int_X \ln \psi \, d\nu + \inf_{g \in C(X,\mathbb{R})} \int_X \ln \frac{B_{\psi \tau}(g)}{\psi g} \, d\nu \right\}$$

$$= \sup_{\hat{\nu} \in \mathcal{H}(\mathcal{R})} \left\{ \int_X \ln \psi \, d\nu + h_{\hat{\nu}}(\hat{\nu}) \right\}.$$

**Definition 12** (Equilibrium States). Let $\mathcal{R}$ be an IFS and $\hat{\mu} \in \mathcal{H}(\mathcal{R})$. We say that the holonomic measure $\hat{\mu}$ is an equilibrium state for $\psi$ if

$$h_{\hat{\nu}}(\hat{\mu}) + \int_X \ln(\psi(x)) \, d\mu(x) = P(\psi).$$

**Lemma 4.** Let $X$ and $Y$ compact separable metric spaces and $T : Y \to X$ a continuous mapping. Then the push-forward mapping $\Phi_T \equiv \Phi : \mathcal{M}_1(Y) \to \mathcal{M}_1(X)$ given by

$$\Phi(\hat{\mu})(A) = \hat{\mu}(T^{-1}(A)), \quad \text{where } \hat{\mu} \in \mathcal{M}_1(Y) \text{ and } A \in \mathcal{B}(X)$$

is weak-* to weak-* continuous.
Proof. Since we are assuming that $X$ and $Y$ are separable compact metric spaces then we can ensure that the weak-$\ast$ topology of both $\mathcal{M}_1(Y)$ and $\mathcal{M}_1(X)$ are metrizable. Therefore is enough to prove that $\Phi$ is sequentially continuous. Let $(\hat{\mu}_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{M}_1(Y)$ so that $\hat{\mu}_n \rightharpoonup \hat{\mu}$. For any continuous real function $f : X \to \mathbb{R}$ we have from change of variables theorem that

$$\int_X f \, d[\Phi(\hat{\mu}_n)] = \int_Y f \circ T \, d\hat{\mu}_n,$$

for any $n \in \mathbb{N}$. From the definition of the weak-$\ast$ topology follows that the rhs above converges when $n \to \infty$ and we have

$$\lim_{n \to \infty} \int_X f \, d[\Phi(\hat{\mu}_n)] = \lim_{n \to \infty} \int_Y f \circ T \, d\hat{\mu}_n = \int_Y f \circ T \, d\hat{\mu} = \int_X f \, d[\Phi(\hat{\mu})].$$

The last equality shows that $\Phi(\hat{\mu}_n) \rightharpoonup \Phi(\hat{\mu})$ and consequently the weak-$\ast$ to weak-$\ast$ continuity of $\Phi$. \hfill \Box

For any $\hat{\nu} \in \mathcal{H}(\mathbb{R})$ it is always possible to disintegrate it as $d\hat{\nu}(x,i) = d\nu_x(i) d[\Phi(\hat{\nu})](x)$, where $\Phi(\hat{\nu}) \equiv \nu$ is the probability measure on $\mathcal{B}(X)$, defined for any $A \in \mathcal{B}(X)$ by

$$\nu(A) \equiv \Phi(\hat{\nu})(A) \equiv \hat{\nu}(T^{-1}(A)), \quad (4)$$

where $T : \Omega \to X$ is the canonical projection of the first coordinate. This observation together with the previous lemma allow us to define a continuous mapping from $\mathcal{H}(\mathbb{R})$ to $\mathcal{M}_1(X)$ given by $\hat{\nu} \mapsto \Phi(\hat{\nu}) \equiv \nu$.

We now prove a theorem ensuring the existence of equilibrium states for any continuous positive function $\psi$. Although this theorem has clear and elegant proof and works in great generality it has the disadvantage of providing no description of the set of equilibrium states.

**Theorem 13** (Existence of Equilibrium States). *Let $\mathcal{R}$ be an IFS and $\psi : X \to \mathbb{R}$ a positive continuous function. Then the set of equilibrium states for $\psi$ is not empty.*

**Proof.** As we observed above we can define a weak-$\ast$ to weak-$\ast$ continuous mapping

$$\mathcal{H}(\mathcal{R}) \ni \hat{\nu} \mapsto \nu \in \mathcal{M}_1(X),$$

where $d\hat{\nu}(x,i) = d\nu_x(i) d\nu(x)$ is the above constructed disintegration of $\hat{\nu}$. From this observation follows that for any fixed positive continuous $g$ we have that the mapping $\mathcal{H}(\mathcal{R}) \ni \hat{\nu} \mapsto \int_X \ln(B_1(g)/g) \, d\nu$ is continuous with respect to the weak-$\ast$ topology. Therefore the mapping

$$\mathcal{H}(\mathcal{R}) \ni \hat{\nu} \mapsto \inf_{g \in C(X,\mathbb{R})} \left\{ \int_X \ln \frac{B_1(g)}{g} \, d\nu \right\} \equiv h_\nu(\hat{\nu}).$$
is upper semi-continuous (USC) which implies by standard results that the following mapping is also USC
\[ H(\mathbb{R}) \ni \hat{\nu} \mapsto h_\psi(\hat{\nu}) + \int_X \ln(\psi(x)) \, d\nu(x). \]
Since \( H(\mathbb{R}) \) is compact in the weak-* topology and the above mapping is USC then follows from Bauer maximum principle that this mapping attains its supremum at some \( \hat{\mu} \in H(\mathbb{R}) \), i.e.,
\[ \sup_{\nu \in H(\mathbb{R})} \left\{ \int_X \ln \psi \, d\nu + h_\psi(\nu) \right\} = \int_X \ln \psi \, d\mu + h_\psi(\hat{\mu}) \]
thus proving the existence of at least one equilibrium state.

5.1 Pressure Differentiability and Equilibrium States
In this section we consider the functional \( p : C(X, \mathbb{R}) \to \mathbb{R} \) given by
\[ p(\varphi) = P(\exp(\varphi)). \] (5)
It is immediate to verify that \( p \) is convex and finite valued functional. We say that a Borel signed measure \( \nu \in \mathcal{M}_s(X) \) is a subgradient of \( p \) at \( \varphi \) if it satisfies the following subgradient inequality
\[ p(\eta) \geq p(\varphi) + \nu(\eta - \varphi). \]
The set of all subgradients at \( \varphi \) is called subdifferential of \( p \) at \( \varphi \) and denoted by \( \partial p(\varphi) \). It is well-known that if \( p \) is a continuous mapping then \( \partial p(\varphi) \neq \emptyset \) for any \( \varphi \in C(X, \mathbb{R}) \).

We observe that for any pair \( \varphi, \eta \in C(X, \mathbb{R}) \) and \( 0 < t < s \), follows from the convexity of \( p \) the following inequality
\[ s(p(\varphi + t\eta) - p(\varphi)) \leq t(p(\varphi + s\eta) - p(\varphi)). \]
In particular, the one-sided directional derivative \( d^+ p(\varphi) : C(X, \mathbb{R}) \to \mathbb{R} \) given by
\[ d^+ p(\varphi)(\eta) = \lim_{t \downarrow 0} \frac{p(\varphi + t\eta) - p(\varphi)}{t} \]
is well-defined for any \( \varphi \in C(X, \mathbb{R}) \).

**Theorem 14.** For any fixed \( \varphi \in C(X, \mathbb{R}) \) we have
1. the signed measure \( \nu \in \partial p(\varphi) \) iff \( \nu(\eta) \leq d^+ p(\varphi)(\eta) \) for all \( \eta \in C(X, \mathbb{R}) \); 2. the set \( \partial p(\varphi) \) is a singleton iff \( d^+ p(\varphi) \) is the Gâteaux derivative of \( p \) at \( \varphi \).

**Proof.** This theorem is consequence of Theorem 7.16 and Corollary 7.17 of the reference [AB06]. \( \Box \)

**Theorem 15.** Let \( \mathcal{R} \) be an IFS, \( \psi : X \to \mathbb{R} \) a positive continuous function and \( \Phi \) defined as in (4). If the functional \( p \) defined on (5) is Gâteaux differentiable at \( \varphi \equiv \log \psi \) then
\[ \# \{ \Phi(\hat{\mu}) : \hat{\mu} \text{ is an equilibrium state for } \psi \} = 1. \]
Proof. Suppose that \( \hat{\mu} \) is an equilibrium state for \( \psi \). Then we have from the definition of the pressure that
\[
 p(\varphi + t\eta) - p(\varphi) = P(\psi \exp(t\eta)) - P(\psi)
 \geq h_v(\hat{\mu}) + \int_X \ln \psi \, d\mu + \int_X t\eta \, d\mu - h_v(\hat{\mu}) - \int_X \ln \psi \, d\mu
 = t \int_X \eta \, d\mu.
\]
Since we are assuming that \( p \) is Gâteaux differentiable at \( \varphi \), follows from the above inequality that \( \mu(\eta) \leq \frac{d}{dt} p(\varphi)(\eta) \) for all \( \eta \in C(X, \mathbb{R}) \). From this inequality and Theorem 14 we can conclude that \( \partial p(\varphi) = \{ \mu \} \). Therefore for all equilibrium state \( \hat{\mu} \) for \( \psi \) we have \( \Phi(\hat{\mu}) = \partial p(\varphi) \), thus finishing the proof.

6 Applications and Constructive Approach to Equilibrium States

In this section we show how one can construct equilibrium states using the spectral analysis of the transfer operator. The results present here are based on Theorem 3.27 of [CO17] which is a result about Dynamic Programming. In order to make its statement and this section as self-contained as possible we provided here the needed background.

The Theorem 3.27 mentioned above is a kind of generalized version of the classical Ruelle-Perron-Frobenius theorem. To be more precisely, we consider a sequential decision-making process \( S = (X, A, \xi, f, u, \delta) \) derived from \( R_q = (X, \tau, q) \) by choosing \( A = \{0, ..., n-1\} \) the action set, \( \xi(x) = A \) for any \( x \in X \), \( f(x,i) = \tau_i(x) \) the dynamics, \( u(x,i) = \ln q_i(x) \) and \( \delta \) a discount function so that for some increasing function \( \gamma : [0, \infty) \to [0, \infty) \) with \( \lim_{n \to \infty} \gamma^n(t) = 0 \) we have
\[
|\delta(t_2) - \delta(t_1)| \leq \gamma(|t_2 - t_1|)
\]
for any \( t_1, t_2 \in \mathbb{R} \).

In this setting we consider a parametric family of discount functions \( \delta_n : [0, +\infty) \to \mathbb{R} \), where \( \delta_n(t) \to I(t) = t \), pointwise and the normalized limits \( \lim_{n \to \infty} w_n(x) - \max w_n \) of the fixed points
\[
w_n(x) = \ln \sum_{i=0}^{n-1} \exp \left( u(x,i) + \delta_n(w_n(f(x,i))) \right)
\]
of a variable discount decision-making process \( S_n = (X, A, \psi, f, u, \delta_n) \) defined by a continuous and bounded immediate reward \( u : X \times A \to \mathbb{R} \) and a sequence of discount functions \( (\delta_n)_{n \geq 0} \), satisfying the admissibility conditions:

1. the contraction modulus \( \gamma_n \) of the variable discount \( \delta_n \) is also a variable discount function;
2. \( \delta_n(0) = 0 \) and \( \delta_n(t) \leq t \) for any \( t \in [0, \infty) \);
3. for any fixed \( \alpha > 0 \) we have \( \delta_n(t+\alpha) - \delta_n(t) \to \alpha \), when \( n \to \infty \), uniformly in \( t > 0 \).
Theorem 16 (CO17). Let $R_q = (X, \tau, q)$ be an IFSw, such that the above defined immediate associated return $u$ satisfy:

1. $u$ is uniformly $\delta$-bounded;
2. $u$ is uniformly $\delta$-dominated.

Then there exists a positive and continuous eigenfunction $h_q$ such that $B_q(h_q) = \rho(B_q)h_q$.

As pointed out in [CO17], the hypothesis of this theorem are not so restrictive as it initially looks like. In fact, a lot of variable discount parametric families satisfies our requirements and, the uniformly $\delta$-bounded and $\delta$-dominated property are satisfied for $u$ in the Lipschitz and Hölder spaces provided that the IFS is contractive. So the hypothesis placed on the variable discount allow us to apply this theorem for a large class of weights.

Corollary 1. Under the same hypothesis of Theorem 16, given an IFS $R_q = (X, \tau, q)$ there exists a continuous IFSpdp $R_p = (X, \tau, p)$ called the normalization of $R_q$, where

$$p_j(x) = q_j(x) \frac{h_q(\tau_j(x))}{\rho(B_q)h_q(x)}, \ j = 0, ..., n - 1.$$ 

Theorem 17 (Variational principle). Let $\psi : X \rightarrow \mathbb{R}$ be a positive and continuous function and $R_{\psi * \tau} = (X, \tau, \psi * \tau)$ an IFSw. Assume that $R_{\psi * \tau}$ satisfies the hypothesis of Theorem 16. Consider $R_p = (X, \tau, p)$ be the normalization of $R_{\psi * \tau}$ given by Corollary 1 and $\hat{\mu} = \Phi(\mu) \in \mathcal{H}(\mathcal{R})$ be the holonomic lifting of the fixed probability measure $\mu$ of $L_p$. Then $\hat{\mu}$, is an equilibrium state for $\psi$ and

$$P(\psi) = \ln \rho(B_{\psi * \tau}) = \lim_{N \rightarrow \infty} \frac{1}{N} \ln \left( B_{\psi * \tau}^N(1)(x) \right),$$

where $\rho(B_{\psi * \tau})$ is the spectral radius of the transfer operator $B_{\psi * \tau}$ acting on $C(X, \mathbb{R})$ and the convergence in the above limit is uniform and independent of the choice of $x \in X$.

Proof. From item 2 of Theorem 16 follows that $h_u(\hat{\nu}) \leq h_v(\hat{\nu})$. From Lemma 2 and Corollary 1 we get the following inequality

$$h_u(\hat{\nu}) = - \int_X \sum_{j=0}^{n-1} (\psi \circ \tau_j) \ln q_j \, d\nu \leq - \int_X \sum_{j=0}^{n-1} (\psi \circ \tau_j) \ln p_j \, d\nu$$

$$= \ln \rho(B_{\psi * \tau}) - \int_X \ln \psi \, d\nu,$$

with equality being attained if $\hat{\nu} = \hat{\mu}$, where $\hat{\mu}$ is the holonomic lifting of the fixed point $L_p(\mu) = \mu$. So we have

$$h_u(\hat{\mu}) = h_v(\hat{\mu}) = \ln \rho(B_{\psi * \tau}) - \int_X \ln \psi \, d\mu.$$
From Lemma 3 we have \( \sup \{ h_v(\hat{\nu}) + \int_X \ln \psi \, d\nu : \hat{\nu} \in \mathcal{H}(\mathcal{R}) \} = P(\psi) \), so for every \( \hat{\nu} \in \mathcal{H}(\mathcal{R}) \) we have

\[
P(\psi) \geq h_v(\hat{\nu}) + \int_X \ln \psi \, d\nu \geq h_a(\hat{\nu}) + \int_X \ln \psi \, d\nu.
\]

On the other hand, the identity \( h_a(\hat{\mu}) = h_v(\hat{\mu}) = \ln \rho(B_{\psi^{\ast}}) - \int_X \ln \psi \, d\mu \) is equivalent to both \( h_a(\hat{\mu}) + \int_X \ln \psi \, d\mu = \ln \rho(B_{\psi^{\ast}}) \) and \( h_v(\hat{\mu}) + \int_X \ln \psi \, d\mu = \ln \rho(B_{\psi^{\ast}}) \). Thus showing that \( P(\psi) \geq \ln \rho(B_{\psi^{\ast}}) \). Recall that the pressure is defined by

\[
P(\psi) \equiv \sup_{\hat{\nu} \in \mathcal{H}(\mathcal{R})} \inf_{g \in \mathcal{C}(X, \mathbb{R})} \left\{ \int_X \ln \frac{B_{\psi^{\ast}}(g)}{g} \, d\nu \right\}.
\]

The eigenfunction \( h_{\psi^{\ast}} \) is positive and continuous so we have for any fixed \( \hat{\nu} \in \mathcal{H}(\mathcal{R}) \) the following upper bound

\[
\inf_{g \in \mathcal{C}(X, \mathbb{R})} \left\{ \int_X \ln \frac{B_{\psi^{\ast}}(g)}{g} \, d\nu \right\} \leq \int_X \ln \frac{B_{\psi^{\ast}}(h_{\psi^{\ast}})}{h_{\psi^{\ast}}} \, d\nu = \ln \rho(B_{\psi^{\ast}}).
\]

By taking the supremum over \( \mathcal{H}(\mathcal{R}) \), on both sides of the last inequality, we get that \( P(\psi) \leq \ln \rho(B_{\psi^{\ast}}) \). Since we already shown the reverse inequality the first equality in (6) is proved. The second equality claimed in the theorem statement is now a straightforward application of Lemma 1.

\[\square\]

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