Holonomy algebras of Einstein pseudo-Riemannian manifolds

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Abstract
The holonomy algebras of Einstein not Ricci-flat pseudo-Riemannian manifolds of arbitrary signature are classified. As illustrating examples, the cases of Lorentzian manifolds, pseudo-Riemannian manifolds of signature $(2, n)$ and the para-quaternionic-Kählerian manifolds with non-zero scalar curvature are considered. Einstein not Ricci-flat metrics of signature $(2, n)$ with all possible holonomy algebras are given.

Keywords: holonomy algebra; Einstein pseudo-Riemannian manifold; para-quaternionic-Kählerian manifold.

1. Introduction
The holonomy group of a pseudo-Riemannian manifold gives rich information about the geometry of the manifold. This motivates the classification problem for holonomy groups. For simplicity one usually restricts the attention to the connected component of the holonomy group, then it is enough to consider the corresponding Lie algebra, called the holonomy algebra. Even in this case the classification problem in arbitrary signature seems to be unsolvable. The complete solution exists only in the Riemannian case \cite{8, 1, 10, 23, 13} and in the Lorentzian case \cite{5, 25, 17}. For an arbitrary signature, only the classification of irreducible holonomy algebras is known \cite{8, 13}. The general case cannot be reduced to the irreducible one unless the metric is of the Riemannian signature. For pseudo-Riemannian manifolds of signature different from Riemannian and Lorentzian ones, only some partial case are considered \cite{6, 7, 11, 12, 16, 19, 21, 24}. There is also a classification of connected irreducible holonomy groups of torsion-free affine connections \cite{26}; the groups corresponding to the Ricci-flat case are found in \cite{4}.

In this paper we restrict our attention to the holonomy algebras of Einstein not Ricci-flat pseudo-Riemannian manifolds. By now the results have been known only in the case of irreducible holonomy algebras \cite{13} and in the Lorentzian signature \cite{20}.

We show that the holonomy algebra of an Einstein not Ricci-flat pseudo-Riemannian manifold contains a significant reductive subalgebra, which is again the holonomy algebra of an Einstein pseudo-Riemannian manifold and which is known. This allows us to get a complete description of the holonomy algebras of such manifolds in arbitrary signature. For that it is enough to combine some modules of the reductive part of the holonomy algebra. As the illustration, we classify holonomy algebras of Einstein not Ricci-flat pseudo-Riemannian manifolds of signature $(2, n)$ as well as of para-quaternionic-Kählerian manifolds with non-zero scalar curvature. In signature $(2, n)$, we construct examples of Einstein not Ricci-flat metrics with all possible holonomy algebras. We also discuss the construction of Einstein metrics in other signatures; this construction is technical and it should be given elsewhere.

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2. Background

The theory of holonomy algebras of pseudo-Riemannian manifolds can be found e.g. in [10, 13, 23]. Here we collect some facts that will be used below.

Let \((M, g)\) be a connected pseudo-Riemannian manifold of signature \((p, q)\). The holonomy group of \((M, g)\) at a point \(x \in M\) is a Lie group that consists of the pseudo-orthogonal transformations given by the parallel displacements along piece-wise smooth loops at the point \(x\), and it can be identified with a Lie subgroup of the pseudo-orthogonal Lie group \(O(p, q)\). The corresponding Lie subalgebra of \(so(p, q)\) is called the holonomy algebra. If the manifold \(M\) is simply connected, then the holonomy group is connected and it is uniquely defined by the holonomy algebra.

Let \(g \subset so(p, q)\) be a subalgebra. The space of curvature tensors of type \(g\) is defined as follows

\[
\mathcal{R}(g) = \left\{ R \in \wedge^2 (\mathbb{R}^{p,q})^\ast \otimes g \mid \begin{array}{l}
R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0 \\
\text{for all } X, Y, Z \in \mathbb{R}^{p,q}
\end{array} \right\}.
\]

The above identity is called the first Bianchi identity. Let \(L(\mathcal{R}(g)) \subset g\) be the ideal spanned by the images of the elements form \(\mathcal{R}(g)\). The subalgebra \(g \subset so(p, q)\) is called a Berger subalgebra if \(L(\mathcal{R}(g)) = g\). From the Ambrose-Singer Theorem it follows that the holonomy algebra \(g \subset so(p, q)\) of a pseudo-Riemannian manifold of signature \((p, q)\) is a Berger subalgebra.

A subalgebra \(g \subset so(p, q)\) is called weakly irreducible if it does not preserve any proper non-degenerate subspace of the pseudo-Euclidean space \(\mathbb{R}^{p,q}\). By the Wu Theorem [28], any pseudo-Riemannian manifold whose holonomy algebra is not weakly irreducible, can be decomposed (at least locally) in the product of a flat pseudo-Riemannian manifold and of pseudo-Riemannian manifolds with weakly irreducible holonomy algebras. In particular, a pseudo-Riemannian manifold is locally indecomposable if and only if its holonomy algebra is weakly irreducible. The following lemma is the well-known algebraic counterpart of the Wu theorem.

**Lemma 1.** If a Berger subalgebra \(g \subset so(p, q)\) is not weakly irreducible, then there exists an orthogonal decomposition

\[
\mathbb{R}^{p,q} = V_0 \oplus V_1 \oplus \cdots \oplus V_r
\]

into a direct sum of pseudo-Euclidean subspaces and a decomposition

\[
g = g_1 \oplus \cdots \oplus g_r
\]

into a direct sum of ideals such that \(g_i\) annihilates \(V_j\) if \(i \neq j\) and \(g_i \subset so(V_i)\) is a weakly irreducible Berger subalgebra.

Consider the vector space

\[
\mathcal{R}^\nabla(g) = \left\{ S \in (\mathbb{R}^{p,q})^\ast \otimes \mathcal{R}(g) \mid \begin{array}{l}
S_X(Y, Z) + S_Y(Z, X) + S_Z(X, Y) = 0 \\
\text{for all } X, Y, Z \in \mathbb{R}^{p,q}
\end{array} \right\}.
\]

If a Berger algebra \(g \subset so(p, q)\) satisfies \(\mathcal{R}^\nabla(g) = 0\), then \(g\) is called a symmetric Berger algebra. Any pseudo-Riemannian manifold with such holonomy algebra is automatically locally symmetric. The list of irreducible holonomy algebras of locally symmetric pseudo-Riemannian manifolds can be found in [9].

M. Berger classified irreducible subalgebras \(g \subset so(p, q)\) that satisfy \(L(\mathcal{R}(g)) = g\) and \(\mathcal{R}^\nabla(g) \neq 0\) in [8]. Later this classification was corrected [1], and it was proved that all the obtained algebras can be realized as the holonomy algebras of pseudo-Riemannian manifolds [13, 23]. Here is the list of these algebras:

\[
so(p, q), so(p, \mathbb{C}) \subset so(p, p),
\]
Consider now the Einstein condition. For a subalgebra \( g \subset so(p, q) \), consider the set
\[
\mathcal{R}_1(g) = \{ R \in \mathcal{R}(g) | \text{Ric}(R)(X, Y) = g(X, Y), \ X, Y \in \mathbb{R}^{p,q} \}.
\]
This is an affine space with the corresponding vector space
\[
\mathcal{R}_0(g) = \{ R \in \mathcal{R}(g) | \text{Ric}(R) = 0 \}.
\]
Let \( L(\mathcal{R}_1(g)) \subset g \) be the vector subspace spanned by the images of the elements of \( R \in \mathcal{R}_1(g) \).
We say that a subalgebra \( g \subset so(p, q) \) is a Berger subalgebra of Einstein type if \( L(\mathcal{R}_1(g)) = g \).
From the Ambrose-Singer Theorem it follows that the holonomy algebra \( g \subset so(p, q) \) of an Einstein not Ricci-flat pseudo-Riemannian manifold of signature \((p, q)\) is a Berger subalgebra of Einstein type. Note that in the Einstein not Ricci-flat case, in Lemma 1 it holds \( V_0 = 0 \) (see the proof of Lemma 5 below).

In Section 3 we will explain the classification of irreducible holonomy algebras of Einstein not Ricci-flat pseudo-Riemannian manifolds.
Now, the problem is to classify weakly irreducible and not irreducible holonomy algebras of Einstein not Ricci-flat pseudo-Riemannian manifolds. Until now this has been done only for the Lorentzian signature in [20].

Let \( g \subset so(p, q) \) be a weakly irreducible not irreducible subalgebra. Then \( g \) preserves a totally isotropic subspace \( V \subset \mathbb{R}^{p,q} \) of dimension \( m \). Let \( (p, q) = (m + r, m + s) \). Let \( p_1, ..., p_m \) be a basis of \( V \). Choose linearly independent vectors \( q_1, ..., q_m \) such that \( g(q_1, q_2) = 0 \) and \( g(p_1, q_1) = \delta_{ij} \). Note that there is no canonical choice of these vectors. The vector space spanned by \( q_1, ..., q_m \) can be identified with \( V^* \). We will denote \( V \) by \( \mathbb{R}^m \). Let \( L \subset \mathbb{R}^{m+r,m+s} \) be the orthogonal complement to \( \mathbb{R}^m \). This subspace can be identified with \( \mathbb{R}^{r,s} \). Let \( e_1, ..., e_{r+s} \) be an orthonormal basis of \( \mathbb{R}^{r,s} \). The basis \( p_1, ..., p_m, e_1, ..., e_{r+s}, q_1, ..., q_m \) is called a Witt basis of \( \mathbb{R}^{m+r,m+s} \). The maximal subalgebra \( so(m + r, m + s)_{\mathbb{R}^m} \subset so(m + r, m + s) \) preserving the subspace \( \mathbb{R}^m \subset \mathbb{R}^{m+r,m+s} \) has the following matrix form:
\[
so(m + r, m + s)_{\mathbb{R}^m} = \left\{ \begin{pmatrix} B & -X^tE_{r,s} & C \\ 0 & A & X \\ 0 & 0 & -B^t \end{pmatrix} \middle| B \in gl(m, \mathbb{R}), A \in so(r, s), \ X \in \mathbb{R}^m \otimes \mathbb{R}^{r,s}, \ C \in \wedge^2 \mathbb{R}^m \right\}.
\]
Denote the element given by the above matrix by \((B, A, X, C)\). The non-zero Lie brackets are the following:
\[
[(B_1, A_1, 0, 0), (B_2, A_2, 0, 0)] = ([B_1, B_2]_{gl(m, \mathbb{R})}, [A_1, A_2]_{so(r, s)}, 0, 0),
\]
\[
[(B, A, 0, 0), (0, 0, X, C)] = (0, 0, AX + XB^t, BC + CB^t),
\]
\[
[(0, 0, X, 0), (0, 0, Y, 0)] = (0, 0, -X^tE_{r,s}Y + Y^tE_{r,s}X).
\]
We see that \( gl(m, \mathbb{R}) \) and \( so(r, s) \) are subalgebras in \( so(m + r, m + s)_{\mathbb{R}^m} \). The vector subspace
\[
\mathcal{N} = \{(0, 0, X, 0) | X \in \mathbb{R}^m \otimes \mathbb{R}^{r,s} \} \subset so(m + r, m + s)_{\mathbb{R}^m}
\]
is isomorphic to the \( gl(m, \mathbb{R}) \oplus so(r, s) \)-module \( \mathbb{R}^m \otimes \mathbb{R}^{r,s} \). The ideal
\[
\mathcal{C} = \{0, 0, 0, C | C \in \wedge^2 \mathbb{R}^m \} \subset so(m + r, m + s)_{\mathbb{R}^m}
\]
is isomorphic to the \( gl(m, \mathbb{R}) \)-module \( \wedge^2 \mathbb{R}^m \). Note that \( \mathcal{N} \times \mathcal{C} \subset so(m + r, m + s)_{\mathbb{R}^m} \) is a solvable ideal. We get the decomposition
\[
so(m + r, m + s)_{\mathbb{R}^m} = gl(m, \mathbb{R}) \oplus so(r, s) \times (\mathcal{N} \times \mathcal{C}).
\]
We will consider projections with respect to this decomposition. Any subalgebra \( g \subset \mathfrak{so}(m + r, m + s) \) preserving a totally isotropic subspace of dimension \( m \) is conjugated to a subalgebra of \( \mathfrak{so}(m + r, m + s)_{\mathbb{R}^m} \).

If the holonomy algebra of a pseudo-Riemannian manifold \((M, g)\) of signature \((m + r, m + s)\) preserves a totally isotropic subspace of the tangent space of dimension \( m \), then locally on \( M \) there exists the Walker \([27]\) coordinates \( v_1, ..., v_m, x_1, ..., x_n, u_1, ..., u_m \) \((n = r + s)\) such that the metric \( g \) is of the form

\[
g = \sum_{a=1}^{m} 2dv_adu_a + h + \sum_{a=1}^{m} \sum_{b=1}^{n} 2A_{ab}dx_adu_b + \sum_{a,b=1}^{m} H_{ab}du_adu_b,
\]

where \( h = \sum_{a,b=1}^{n} h_{ab}(x_1, ..., x_n, u_1, ..., u_m)dx_adx_b \) is a family of pseudo-Riemannian metrics of signature \((r, s)\) depending on the parameters \( u_1, ..., u_m; A_{ab} \) are functions of \( x_1, ..., x_n, u_1, ..., u_m \), and \( H_{ab} \) are functions of all coordinates.

3. The case of irreducible \( g \subset \mathfrak{so}(p, q) \)

The list of irreducible holonomy algebras \( g \subset \mathfrak{so}(p, q) \) of Einstein not Ricci-flat pseudo-Riemannian manifolds of signature \((p, q)\) that are not locally symmetric is the following (see e.g. \([13]\)):

\[
\mathfrak{so}(p, q), \mathfrak{so}(p, \mathbb{C}) \subset \mathfrak{so}(p, p), \mathfrak{u}(r, s) \subset \mathfrak{so}(2r, 2s), \\
\mathfrak{sp}(r, s) \oplus \mathfrak{sp}(1) \subset \mathfrak{so}(4r, 4s), \mathfrak{sp}(r, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \subset \mathfrak{so}(2r, 2r), \\
\mathfrak{sp}(r, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \subset \mathfrak{so}(4r, 4r).
\]

Consider a simply connected pseudo-Riemannian symmetric space \((M, g)\) with an irreducible holonomy algebra \( g \subset \mathfrak{so}(p, q) \). Let \( \mathfrak{k} \) be the Lie algebra of the Lie group of transvections of \((M, g)\). The irreducibility of \( g \) implies that \( \mathfrak{k} \) is simple. Note that if \( g \subset \mathfrak{so}(n, n) \) preserves two complementary isotropic subspaces and acts irreducibly on these subspaces, then \( \mathfrak{k} \) is simple also in that case (we will consider this situation in the next section).

Consider the symmetric decomposition

\[
\mathfrak{k} = \mathfrak{g} \oplus \mathfrak{m},
\]

where the subspace \( \mathfrak{m} \) is identified with the tangent space of \( M \) at some point. The Ricci tensor at that point is given by the restriction of the Killing form of \( \mathfrak{k} \) to \( \mathfrak{m} \). Consequently the Ricci tensor is non-degenerate. From that and the Schur lemma it follows that the space of symmetric bilinear tensor fields on \((M, g)\) is either one-dimensional or it is two-dimensional. Since the Ricci tensor is parallel and non-degenerate, in the first case the manifold is Einstein and not Ricci-flat. In the second case, there is a parallel \( g \)-symmetric complex structure \( I \) on \((M, g)\). This implies that \( q = p \) and \( \mathfrak{g} \subset \mathfrak{so}(p, \mathbb{C}) \subset \mathfrak{so}(p, p) \). Any parallel symmetric bilinear tensor field is of the form

\[
ag(\cdot, \cdot) + bg(I\cdot, \cdot).
\]

If \( a^2 + b^2 \neq 0 \), then such a form is non-degenerate and it defines the same Levi-Civita connection as \( g \). Thus the metric on \( M \) may be chosen to be Einstein and not Ricci-flat.

4. The case of \( g \subset \mathfrak{so}(n, n) \) preserving two complementary isotropic subspaces

Let \( g \subset \mathfrak{so}(n, n) \) and suppose that \( g \) preserves two complementary isotropic subspaces \( V, V' \subset \mathfrak{so}(n, n) \). Using the metric on \( \mathbb{R}^{n,n} \), the space \( V' \) can be identified with the dual space \( V^* \). If we fix a Witt basis \( p_1, ..., p_n, q_1, ..., q_n \) such that \( p_1, ..., p_n \in V \), and \( q_1, ..., q_n \in V^* \), then \( g \) is
contained in the maximal subalgebra preserving $V$ and $V^*$:

$$g \subset so(n, n)_{\mathbb{R}^n} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix} \mid A \in gl(n, \mathbb{R}) \right\} \subset so(n, n).$$

The matrices $A$ define a subalgebra of $gl(n, \mathbb{R})$, which is isomorphic to $g$ and we write $g \subset gl(n, \mathbb{R})$.

Conversely, if we start with a subalgebra $g \subset gl(n, \mathbb{R})$ we may consider it as the subalgebra $g \subset so(n, n) = so(\mathbb{R}^n \oplus \mathbb{R}^{n^*})$ that preserves $\mathbb{R}^n, \mathbb{R}^{n^*} \subset \mathbb{R}^n \oplus \mathbb{R}^{n^*}$. The pseudo-Euclidean metric on $\mathbb{R}^n \oplus \mathbb{R}^{n^*}$ is given by the pairing.

Holonomy algebras $g \subset so(n, n)$ of this type are studied in [7, 9]. Let us formulate some facts from these papers. Let $R \in \mathcal{R}(g \subset so(n, n))$. If $X, Y \in \mathbb{R}^n$, or $X, Y \in \mathbb{R}^{n^*}$, then $R(X, Y) = 0$.

Next, suppose that $g$ is a reductive Lie algebra. Lemma 1 applied to this case can be formulated in the following way.

**Lemma 2.** Let $g$ be a reductive Lie algebra and suppose that $g \subset so(n, n)$ is a Berger subalgebra preserving two complementary totally isotropic subspaces of $\mathbb{R}^{n,n}$. If $g \subset gl(n, \mathbb{R})$ is not irreducible, then there exists a decomposition

$$\mathbb{R}^n = V_0 \oplus V_1 \oplus \cdots \oplus V_r$$

and a decomposition

$$g = g_1 \oplus \cdots \oplus g_r$$

into a direct sum of ideals such that $g_i$ annihilates $V_j$ if $i \neq j$, $g_i \subset gl(V_i)$ is irreducible, and $g_i \subset so(V_i \oplus V_i^*)$ is a Berger subalgebra.

Let us restrict the attention to the case when $g \subset gl(n, \mathbb{R})$ is irreducible. In [7] it is shown that if $g \subset so(n, n)$ is a Berger subalgebra, then $g \subset gl(n, \mathbb{R})$ is either one of the following:

$$gl(n, \mathbb{R}), \quad sl(n, \mathbb{R}), \quad sp(2m, \mathbb{R}), \quad gl(m, \mathbb{C}), \quad sl(m, \mathbb{C}), \quad sp(2k, \mathbb{C}), \quad 2m = 4k = n,$$

or $g \subset so(n, n)$ is a symmetric Berger subalgebra, and it can be found in the list from [9]. This result may be deduced from the list of irreducible subalgebras $g \subset gl(n, \mathbb{R})$ with $g^{(1)} \neq 0$ given in Table B from [13].

Let us apply now the Einstein condition. Let $R \in \mathcal{R}(g \subset so(n, n))$ be as above; let $X \in V, Y \in V^*$, then it is easy to calculate

$$\text{Ric}(R)(X, Y) = \text{tr}(pr_{gl(n, \mathbb{R})} R(X, Y)). \quad (4.1)$$

This shows that if $\mathcal{R}_1(g) \neq 0$, then $g$ is not contained in $sl(n, \mathbb{R})$. Consequently, if $g \subset gl(n, \mathbb{R})$ is irreducible, then $g$ is either one of

$$gl(n, \mathbb{R}), \quad gl(m, \mathbb{C}) \quad (n = 2m),$$

or $g \subset so(n, n)$ is a weakly irreducible symmetric Berger algebra. Note that, as in the previous section, it can be shown that the corresponding symmetric spaces may be Einstein and not Ricci-flat. Moreover, $gl(n, \mathbb{R}) \subset so(n, n)$ and $gl(m, \mathbb{C}) \subset so(2m, 2m)$ may appear as the
holonomy algebras of symmetric spaces. Thus, we have found all Berger algebras of Einstein type $g \subset \mathfrak{so}(n, n)$ such that $g \subset \mathfrak{gl}(n, \mathbb{R})$ is irreducible. Moreover, all these Berger algebras are holonomy algebras of Einstein not Ricci-flat pseudo-Riemannian manifolds of signature $(n, n)$. We write down that list in Table 1.

**Table 1.** Irreducible subalgebras $g \subset \mathfrak{gl}(n, \mathbb{R}) = \mathfrak{gl}(V)$ defining Berger algebras of Einstein type $g \subset \mathfrak{so}(n, n)$

| $g$ | $V$ | restriction |
|-----|-----|-------------|
| $\mathfrak{gl}(m, \mathbb{R})$ | $\mathbb{R}^m$ | $m \geq 1$ |
| $\mathfrak{gl}(m, \mathbb{C})$ | $\mathbb{C}^m$ | $m \geq 1$ |
| $\mathbb{R} \oplus \mathfrak{so}(p, q)$ | $\mathbb{R}^{p+q}$ | $p + q \geq 3$ |
| $\mathbb{C} \oplus \mathfrak{so}(p, \mathbb{C})$ | $\mathbb{C}^p$ | $p \geq 3$ |
| $\mathbb{R} \oplus \mathfrak{sl}(p, \mathbb{R}) \oplus \mathfrak{sl}(q, \mathbb{R})$ | $\mathbb{R}^{pq}$ | $p \geq q \geq 2, (p, q) \neq (2, 2)$ |
| $\mathbb{C} \oplus \mathfrak{sl}(p, \mathbb{C}) \oplus \mathfrak{sl}(q, \mathbb{C})$ | $\mathbb{C}^{pq}$ | $p \geq q \geq 2, (p, q) \neq (2, 2)$ |
| $\mathbb{R} \oplus \mathfrak{sl}(p, \mathbb{H}) \oplus \mathfrak{sl}(q, \mathbb{H})$ | $\mathbb{R}^{4pq}$ | $p \geq q \geq 1, (p, q) \neq (1, 1)$ |
| $\mathfrak{gl}(p, \mathbb{R})$ | $\mathbb{R}^{p(p+1)/2} \simeq S_p(\mathbb{R})$ | $p \geq 3$ |
| $\mathfrak{gl}(p, \mathbb{C})$ | $\mathbb{C}^{p(p+1)/2} \simeq S_p(\mathbb{C})$ | $p \geq 3$ |
| $\mathfrak{gl}(p, \mathbb{H})$ | $\mathbb{R}^{p(2p+1)} \simeq S_p(\mathbb{H})$ | $p \geq 2$ |
| $\mathfrak{gl}(p, \mathbb{R})$ | $\mathbb{R}^{p(p-1)/2} \simeq A_p(\mathbb{R})$ | $p \geq 5$ |
| $\mathfrak{gl}(p, \mathbb{C})$ | $\mathbb{C}^{p(p-1)/2} \simeq A_p(\mathbb{C})$ | $p \geq 5$ |
| $\mathfrak{gl}(p, \mathbb{H})$ | $\mathbb{R}^{p(2p-1)} \simeq A_p(\mathbb{H})$ | $p \geq 3$ |
| $\mathbb{R} \oplus \mathfrak{spin}(5, 5)$ | $\mathbb{R}^{16}$ | |
| $\mathbb{R} \oplus \mathfrak{spin}(1, 9)$ | $\mathbb{R}^{16}$ | |
| $\mathbb{C} \oplus \mathfrak{spin}(10, \mathbb{C})$ | $\mathbb{C}^{16}$ | |
| $\mathbb{R} \oplus E_6^1$ | $\mathbb{R}^{27}$ | |
| $\mathbb{R} \oplus E_6^3$ | $\mathbb{R}^{27}$ | |
| $\mathbb{C} \oplus E_6^5$ | $\mathbb{C}^{27}$ | |

Ending this section we note that from [4,1] follows the following general statement.

**Proposition 1.** Let $(M, g)$ be a pseudo-Riemannian manifold of signature $(n, n)$ such that its holonomy algebra $g \subset \mathfrak{so}(n, n)$ preserves two complementary totally isotropic subspaces of the tangent space. Then $(M, g)$ is Ricci-flat if and only if $g \subset \mathfrak{sl}(n, \mathbb{R})$.

5. Some lemmas

**Lemma 3.** Let $g \subset \mathfrak{so}(r, s)$ be an irreducible Berger subalgebra of Einstein type, then $g$ contains its centralizer in $\mathfrak{so}(r, s)$.
Proof. This immediately follows from the classification of irreducible holonomy algebras \( g \subset \mathfrak{so}(r, s) \) of Einstein pseudo-Riemannian manifolds. \( \square \)

**Lemma 4.** Let \( g \subset \mathfrak{gl}(n, \mathbb{R}) \) be an irreducible subalgebra such that \( g \subset \mathfrak{so}(n, n) \) is a Berger subalgebra of Einstein type, then \( g \) contains its centralizer in \( \mathfrak{so}(n, n) \).

Proof. As the \( g \)-module, the Lie algebra \( \mathfrak{so}(n, n) \) decomposes as
\[
\mathfrak{so}(n, n) = \mathfrak{gl}(n, \mathbb{R}) \oplus \wedge^2 \mathbb{R}^n \oplus \wedge^2 \mathbb{R}^n^*.
\]
The classification shows that the centralizer of \( g \) in \( \mathfrak{gl}(n, \mathbb{R}) \) is contained in \( g \). Next, \( g \) contains \( \text{id}_{\mathbb{R}^n} \) that acts as multiplication by 2 and -2 in \( \wedge^2 \mathbb{R}^n \) and \( \wedge^2 \mathbb{R}^n^* \), respectively. \( \square \)

**Lemma 5.** Let \( g \subset \mathfrak{so}(r, s) \) be an irreducible subalgebra and \( \mathcal{R}_1(g) \neq 0 \), then \( g \subset \mathfrak{so}(r, s) \) is a Berger subalgebra of Einstein type.

Proof. Since the representation \( g \subset \mathfrak{so}(r, s) \) is irreducible, the Lie algebra \( g \) is reductive. Consider the ideal \( \mathfrak{k} = L(\mathcal{R}_1(g)) \subset g \). Suppose that \( \mathfrak{k} \neq g \). Let \( \mathfrak{t} \subset \mathfrak{k} \) be the complementary ideal. The subalgebra \( \mathfrak{t} \subset \mathfrak{so}(r, s) \) is a Berger subalgebra, and \( \mathcal{R}_1(\mathfrak{t}) = \mathcal{R}_1(g) \neq 0 \). By Lemma 3, the ideal \( \mathfrak{t} \subset \mathfrak{g} \) is not commutative. If the semi-simple part of \( g \) is simple, then the statement of the lemma follows from Lemma 5. Assume that the semi-simple part of \( g \) is not simple.

Consider the case when the complexified representation \( g \otimes \mathbb{C} \subset \mathfrak{so}(r + s, \mathbb{C}) \) is not irreducible. Then \( g \otimes \mathbb{C} \) preserves two complementary isotropic subspaces \( V, V' \subset \mathbb{C}^{r+s} \), and the induced representations are irreducible. Let
\[
R \in \mathcal{R}(g \otimes \mathbb{C} \subset \mathfrak{so}(r + s, \mathbb{C})) = \mathcal{R}(g \subset \mathfrak{so}(r, s)) \otimes \mathbb{C}.
\]
As in Section 4 for each \( Y \in V' \) it holds
\[
R(Y, \cdot) = (g \subset \mathfrak{gl}(V))^{(1)}.
\]
This shows that
\[
(g \subset \mathfrak{gl}(V))^{(1)} \neq 0.
\]
Using the list of irreducible representations of complex Lie algebras with non-trivial first prolongations (see e.g. [T3]) and the assumption that the semi-simple part of \( g \) is not simple, we get that
\[
g \otimes \mathbb{C} = \mathfrak{gl}(n, \mathbb{C}) \oplus \mathfrak{gl}(m, \mathbb{C}), \quad V = \mathbb{C}^n \otimes \mathbb{C}^m
\]
for some \( n \) and \( m \). This implies that \( g = u(n_1, n_2) \oplus u(m_1, m_2) \), which is a Berger algebra of Einstein type.

Finally suppose that the complexified representation \( g \otimes \mathbb{C} \subset \mathfrak{so}(r + s, \mathbb{C}) \) is irreducible. The representation of \( g \otimes \mathbb{C} \) in \( \mathbb{C}^{r+s} \) is the tensor products of irreducible representations of \( \mathfrak{t} \otimes \mathbb{C} \) and \( \mathfrak{k} \otimes \mathbb{C} \). The subalgebra \( \mathfrak{t} \otimes \mathbb{C} \subset \mathfrak{so}(r + s, \mathbb{C}) \) is a Berger subalgebra that preserves at least two non-degenerate subspaces of \( \mathbb{C}^{r+s} \) and its representation in each of the invariant subspaces is faithful. This is impossible by the complex version of Lemma 4. This proves the current lemma. \( \square \)

Note that in [14] a statement similar to Lemma 5 was proven, stating that if \( \mathcal{R}(g) \neq 0 \) for an irreducible subalgebra \( g \subset \mathfrak{so}(n) \), then either \( g \) is the holonomy algebra of a Riemannian manifold or \( g = \mathfrak{sp}(m) \oplus \mathfrak{u}(1), n = 4m \).
Lemma 6. Let $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$ be an irreducible subalgebra such that for $\mathfrak{g} \subset \mathfrak{so}(n, n)$ it holds $\mathcal{R}_1(\mathfrak{g}) \neq 0$, then $\mathfrak{g} \subset \mathfrak{so}(n, n)$ is a Berger subalgebra of Einstein type.

Proof. In Section 4 we have seen that the condition $\mathcal{R}_1(\mathfrak{g} \subset \mathfrak{so}(n, n)) \neq 0$ implies

$$ (\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R}))^{(1)} \neq 0, \quad \text{and} \quad \mathfrak{g} \not\subset \mathfrak{sl}(n, \mathbb{R}). $$

Irreducible subalgebras $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$ with non-trivial first prolongation may be found in [13]. This allows us to conclude that $\mathfrak{g} \subset \mathfrak{so}(n, n)$ is a Berger subalgebra of Einstein type. □

Lemma 7. Let $p_1, \ldots, p_m, e_1, \ldots, e_n, q_1, \ldots, q_m$ be a Witt basis of $\mathbb{R}^{m+r,m+s}$ ($r+s=n$) and $\xi = (\text{id}_{\mathbb{R}^n}, 0, X, C) \in \mathfrak{gl}(V) \oplus \mathfrak{so}(r, s) \times (\mathbb{R}^m \otimes \mathbb{R}^{r,s} \otimes \wedge^2 \mathbb{R}^m) = \mathfrak{so}(m+r, m+s)\mathbb{R}^m$

with respect to this basis. Then there exists another Witt basis of $\mathbb{R}^{m+r,m+s}$ with the same $p_1, \ldots, p_m$, with respect to that

$$ \xi = (\text{id}_{\mathbb{R}^n}, 0, 0, 0). $$

Proof. Let $p_1, \ldots, p_m, e_1, \ldots, e_n, q_1, \ldots, q_m$ be a Witt basis of $\mathbb{R}^{m+r,m+s}$. Consider the new basis

$$ p_i' = p_i, \quad e_a' = e_a + \sum_{i=1}^{m} D_{ia} p_i, \quad q_i' = q_i + X_i + \sum_{j=1}^{m} A_{ji} p_j, \quad X_i \in E. $$

Let $A = B + C$ be the decomposition of the matrix $A$ into the symmetric and skew-symmetric parts. The condition that we get again a Witt basis is equivalent to the equalities

$$ B_{ij} = -\frac{1}{2} g(X_i, X_j), \quad D_{ia} = -g(X_i, e_a). $$

Let $\eta = (\text{id}_{\mathbb{R}^n}, 0, 0, 0)$ with respect to the first basis, then with respect to the second basis, $\eta = (\text{id}_{\mathbb{R}^n}, 0, -X, -2C)$), where $X_{ai} = -(X_i, e_a)$. It is clear that $X$ and $C$ can be chosen in arbitrary way. This shows that starting with $\xi = (\text{id}_{\mathbb{R}^n}, 0, X, C) \in \mathfrak{g}$, we may choose a new basis in such a way that $\xi = (\text{id}_{\mathbb{R}^n}, 0, 0, 0) \in \mathfrak{g}$. □

6. The general case

Let $(M, g)$ be an Einstein not Ricci-flat pseudo-Riemannian manifold with the holonomy algebra $\mathfrak{g} \subset \mathfrak{so}(p, q)$. We would like to find all possible $\mathfrak{g}$. As it was explained above, the Wu theorem allows us to assume that $\mathfrak{g} \subset \mathfrak{so}(p, q)$ is weakly irreducible. The list of irreducible $\mathfrak{g} \subset \mathfrak{so}(p, q)$ is known, see Section 3 above. Thus we may assume that $\mathfrak{g} \subset \mathfrak{so}(p, q)$ is weakly irreducible and it preserves a degenerate vector subspace of $\mathbb{R}^{p,q}$.

Theorem 1. Let $\mathfrak{g} \subset \mathfrak{so}(r+m, s+m)$ be a weakly irreducible not irreducible holonomy algebra of an Einstein not Ricci-flat pseudo-Riemannian manifold of signature $(r+m, s+m)$. Let $m$ be the maximal dimension of isotropic $g$-invariant subspace. Then with respect to a proper basis of $\mathbb{R}^{r+m,s+m}$, $\mathfrak{g} \subset \mathfrak{so}(r+m, s+m)\mathbb{R}^m$ has the form

$$ \mathfrak{g} = \mathfrak{f} \oplus \mathfrak{h} \times \left( \bigoplus_{i=1}^{k} \mathfrak{N}_{\alpha}^i \right) \mathfrak{C}_{ij}, $$

where

- $\mathfrak{h} \subset \mathfrak{so}(r, s)$ is a reductive subalgebra that is the holonomy algebra of an Einstein not Ricci-flat pseudo-Riemannian manifold of signature $(r, s)$; there exist an orthogonal
decomposition

\[ \mathbb{R}^{r,s} = L_1 \oplus \cdots \oplus L_t, \]

and the corresponding decomposition

\[ \mathfrak{h} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_t \]

such that \( \mathfrak{h}_\alpha \subset \mathfrak{so}(L_\alpha) \) is an irreducible holonomy algebra of an Einstein not Ricci-flat pseudo-Riemannian manifold;

- \( f \subset \mathfrak{gl}(m, \mathbb{R}) \) is a subalgebra that admits a decomposition

\[ f = f_0 \ltimes \hat{f}, \]

where \( f_0 \subset f \) is a reductive subalgebra and \( \hat{f} \subset f \) is a solvable ideal; the subalgebra \( f_0 \subset \mathfrak{gl}(m, \mathbb{R}) \) defines the decompositions

\[ \mathbb{R}^m = V_1 \oplus \cdots \oplus V_k, \]

\[ f_0 = f_1 \oplus \cdots \oplus f_k \]

such that \( f_i \subset \mathfrak{gl}(V_i) \) is irreducible and the corresponding weakly irreducible subalgebra \( f_i \subset \mathfrak{so}(V_i \oplus V_i^*) \) is the holonomy algebra of an Einstein not Ricci-flat pseudo-Riemannian manifold of neutral signature; farther more,

\[ \hat{f} = \oplus_{1 \leq i < j \leq k} f_{ij}, \]

where \( f_{ij} \subset V_i^* \otimes V_j \) is a \( f_i \oplus f_j \)-submodule; thus \( \hat{f} \subset \mathfrak{gl}(m, \mathbb{R}) \) has the following structure:

\[ f = \left\{ \begin{pmatrix} A_1 & A_{12} & \cdots & A_{1k-1} & A_k \\ 0 & A_2 & \cdots & A_{2k-1} & A_{2k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & A_k \end{pmatrix} \middle| A_i \in f_i, A_{ij} \in f_{ij} \right\}; \]

- \( N_{i\alpha} \) is an \( f_i \oplus \mathfrak{h}_{\alpha} \)-submodule of \( V_i \otimes L_{\alpha} \); for each \( i \), there exists \( \alpha \) such that \( N_{i\alpha} \neq 0 \);
- \( C_{ij} \) is an \( f_i \oplus f_j \)-submodule of \( V_i \wedge V_j \) if \( i \neq j \), and \( C_{ii} \subset \wedge^2 V_i \) is an \( f_i \)-submodule;
- it holds

\[ [f_{ii}, f_{ij}] \subset f_{ij}, \quad [f_{ii}, C_{ij}] \subset C_{ij}, \quad [f_{ii}, N_{i\alpha}] \subset N_{i\alpha}, \quad [N_{i\alpha}, N_{j\alpha}] \subset C_{ij}. \]

**Proof of Theorem**

In the proof of the theorem we will use only the property \( \mathcal{R}_1(\mathfrak{g}) \neq 0 \).

The following lemma gives the form of the projection \( \mathfrak{h} = \text{pr}_{\mathfrak{so}(r,s)} \mathfrak{g} \).

**LEMMA 8.** There exists an orthogonal decomposition

\[ \mathbb{R}^{r,s} = L_1 \oplus \cdots \oplus L_t, \]

and the corresponding decomposition

\[ \mathfrak{h} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_t \]

such that \( \mathfrak{h}_i \subset \mathfrak{so}(L_i) \) is an irreducible holonomy algebra of an Einstein not Ricci-flat pseudo-Riemannian manifold.
Proof. Since \( \mathbb{R}^m \subset \mathbb{R}^{r+s+m} \) is the maximal totally isotropic subspace preserved by \( \mathfrak{g} \), \( \mathfrak{h} \) does not preserve any totally isotropic subspace of \( \mathbb{R}^{r,s} \), and consequently \( \mathfrak{h} \) does not preserve any degenerate subspace of \( \mathbb{R}^{r,s} \). Consequently, if \( \mathfrak{h} \) preserves a proper vector subspace of \( \mathbb{R}^{r,s} \), then this subspace is non-degenerate, and \( \mathfrak{g} \) preserves also its orthogonal complement. This shows that \( \mathfrak{h} \) is a reductive Lie algebra. We get an \( \mathfrak{h} \)-invariant orthogonal decomposition \[
abla \mathbb{R}^{r,s} = U_0 \oplus U_1 \oplus \cdots \oplus U_k
\]
such that \( \mathfrak{h} \) annihilates \( U_0 \) and the induced representation of \( \mathfrak{h} \) on \( U_i \) is irreducible for \( 1 \leq i \leq k \). For each \( i \), let \( \mathfrak{h}_i \subset \mathfrak{h} \) be the ideal annihilating \( U_i^\perp \), i.e. \( \mathfrak{h}_i \subset \mathfrak{so}(U_i) \). We obtain the decomposition
\[
\mathfrak{h} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_k \oplus \mathfrak{h}_i,
\]
where \( \mathfrak{h}_i \subset \mathfrak{h} \) is the complementary ideal to \( \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_k \). The elements of \( \mathfrak{h}_i \) act simultaneously in several \( U_i \). Note that \( \mathfrak{h}_i \subset \mathfrak{so}(U_i) \) a priori must not be irreducible.

For \( R \in \mathcal{R}_1(\mathfrak{g}) \), let
\[
\tilde{R} = pr_{\mathfrak{so}(r,s)} \circ R|_{\Lambda^2 \mathbb{R}^{r,s}}.
\]
Applying the Bianchi identity for the tensor \( R \) to vectors from \( \mathbb{R}^{r,s} \), and taking the projection to \( \mathbb{R}^{r,s} \), we see that \( \tilde{R} \in \mathcal{R}(\mathfrak{h}) \). Let \( \tilde{\mathfrak{h}} \subset \mathfrak{h} \) be the subalgebra generated by the images of such tensors \( \tilde{R} \). Clearly, \( \tilde{\mathfrak{h}} \subset \mathfrak{so}(r,s) \) is a Berger subalgebra.

Let again \( R \in \mathcal{R}_1(\mathfrak{g}) \) and let \( X, Y \in \mathbb{R}^{r,s} \). It holds
\[
\text{Ric}(R)(X, Y) = \sum_{a=1}^m g(R(p_a, X)Y, q_a) + \sum_{a=1}^{r+s} g(R(e_a, X)Y, e_a)g(e_a, e_a) + \sum_{a=1}^m g(R(q_a, X)Y, p_a).
\]
Since \( R \) takes values in \( \mathfrak{g} \), and \( \mathfrak{g} \) preserves the spaces \( \mathbb{R}^m \) and \( \mathbb{R}^m \oplus \mathbb{R}^r \), we get
\[
g(R(q_a, X)Y, p_a) = 0.
\]
Using the standard property of the curvature tensor, we get
\[
g(R(p_a, X)Y, q_a) = g(R(Y, q_a)p_a, X) = 0.
\]
This shows that
\[
g(X, Y) = \text{Ric}(R)(X, Y) = \text{Ric}(\tilde{R})(X, Y),
\]
and \( \tilde{R} \in \mathcal{R}_1(\mathfrak{h}) \). We claim that \( \tilde{\mathfrak{h}} \) does not annihilate any non-degenerate subspace of \( \mathbb{R}^{r,s} \). Indeed, if \( \tilde{\mathfrak{h}} \) annihilates a non-zero vector \( Z \in \mathbb{R}^{r,s} \), then for any \( X \in \mathbb{R}^{r,s} \) it holds
\[
g(X, Z) = \text{Ric}(\tilde{R})(X, Z) = \sum_{a=1}^{r+s} g(R(e_a, X)Z, e_a)g(e_a, e_a) = 0,
\]
which is impossible.

Since \( \tilde{\mathfrak{h}} \subset \mathfrak{so}(r,s) \) is a Berger subalgebra, we obtain the decompositions
\[
\mathbb{R}^{r,s} = W_1 \oplus \cdots \oplus W_k,
\]
\[
\tilde{\mathfrak{h}} = \tilde{\mathfrak{h}}_1 \oplus \cdots \oplus \tilde{\mathfrak{h}}_k,
\]
where each \( \tilde{\mathfrak{h}}_i \subset \mathfrak{so}(W_i) \) is a weakly irreducible Berger subalgebra.

Since \( \tilde{\mathfrak{h}} \subset \mathfrak{h} \), each \( U_i \) is the direct sum of some number of \( W_j \), and the corresponding \( \tilde{\mathfrak{h}}_j \) are contained in \( \mathfrak{h}_i \). In particular, all \( \mathfrak{h}_i \) are non-trivial and do not annihilate any non-trivial subspace of \( U_i \), and \( U_0 = 0 \). More over, \( \mathcal{R}_1(\mathfrak{h}_i) \neq 0 \). Consider the ideal \( t_i = L(\mathcal{R}(\mathfrak{h}_i)) \subset \mathfrak{h}_i \). Let \( \mathfrak{h}_i \subset \mathfrak{h}_i \) be the complementary ideal, then
\[
\mathfrak{h}_i = t_i \oplus \tilde{\mathfrak{h}}_i.
\]
Clearly, \( t_i \subset \mathfrak{so}(U_i) \) is a Berger subalgebra.

Since \( U_i \) is the direct sum of some number of \( W_j \), \( t_i \) contains the corresponding \( \tilde{\mathfrak{h}}_j \), and \( \tilde{\mathfrak{h}}_j \subset \mathfrak{so}(W_j) \) is weakly irreducible, we see that \( t_i \subset \mathfrak{so}(U_i) \) does not annihilate any non-trivial
subspace of \( U_i \). We get the decompositions

\[
U_i = U_{i1} \oplus \cdots \oplus U_{ir}, \quad t_i = t_{i1} \oplus \cdots \oplus t_{ir},
\]

where \( t_{i\alpha} \subset \mathfrak{so}(U_{i\alpha}) \) is a weakly irreducible Berger subalgebra of Einstein type.

By the construction, each \( t_{i\alpha} \) is a reductive Lie algebra, hence either \( t_{i\alpha} \subset \mathfrak{so}(U_{i\alpha}) \) is irreducible, or it preserves two complementary totally isotropic subspaces. By Lemmas 3 and 4 the induced action of \( h_j \) and \( h \) on each \( U_{i\alpha} \) is trivial, i.e. \( h = 0 \) and all \( h_j = 0 \). Hence, \( h = \oplus_{i,\alpha} t_{i\alpha} \). Since \( h \) does not preserve any degenerate subspace of \( \mathbb{R}^{r,s} \), each \( t_{i\alpha} \subset \mathfrak{so}(U_{i\alpha}) \) is irreducible. We obtain the required decompositions \( \mathbb{R}^{r,s} = \oplus_{i,\alpha} U_{i\alpha} \), \( h = \oplus_{i,\alpha} t_{i\alpha} \). □

Now we find the form of the projection \( \mathfrak{f} = \text{pr}_{\mathfrak{gl}(m,R)} \mathfrak{g} \).

**Lemma 9.** There exists a decomposition \( \mathfrak{f} = \mathfrak{f}_0 \ltimes \mathfrak{f}_1 \), where \( \mathfrak{f}_0 \subset \mathfrak{f} \) is a reductive subalgebra and \( \mathfrak{f}_1 \subset \mathfrak{f} \) is a solvable ideal. It holds \( \mathfrak{f}_0 \subset \mathfrak{g} \). Next, there is a decomposition

\[
\mathbb{R}^m = V = V_1 \oplus \cdots \oplus V_k,
\]

(6.5)

and the corresponding decomposition

\[
\mathfrak{f}_0 = \mathfrak{f}_1 \oplus \cdots \oplus \mathfrak{f}_k
\]

(6.6)

such that \( \mathfrak{f}_1 \subset \mathfrak{gl}(V_1) \) is irreducible and the corresponding weakly irreducible subalgebra \( \mathfrak{f}_1 \subset \mathfrak{so}(V_1 \oplus V_1^*) \) is the holonomy algebra of an Einstein not Ricci-flat pseudo-Riemannian manifold of neutral signature.

Moreover, as the vector space,

\[
\mathfrak{f} = \oplus_{1 \leq i < j \leq k} \mathfrak{f}_{ij},
\]

each \( \mathfrak{f}_{ij} \) takes \( V_j \) to \( V_i \) and annihilates \( V_l, l \neq j \).

Proof. Suppose that the representation of \( \mathfrak{f} \) on \( V \) preserves a vector subspace \( V_1 \subset V \). We may assume that \( V_1 \) does not contain any proper invariant subspace, i.e. the induced representation of \( \mathfrak{f} \) on \( V_1 \) is irreducible. Let \( V'_1 \subset V \) be any complementary subspace, i.e. \( V = V_1 \oplus V'_1 \). The matrices of the elements from \( \mathfrak{f} \) are of the form

\[
\begin{pmatrix}
A_1 & A_{12} \\
0 & A_2
\end{pmatrix}.
\]

That is,

\[
\mathfrak{f} \subset \mathfrak{gl}(V_1) \oplus \mathfrak{gl}(V'_1) \ltimes (V'_1)^* \otimes V_1.
\]

Let \( \mathfrak{f}_1 \subset \mathfrak{gl}(V_1) \) be the projection of \( \mathfrak{f} \) with respect to that decomposition. By the construction, \( \mathfrak{f}_1 \subset \mathfrak{gl}(V_1) \) is irreducible. Consequently, \( \mathfrak{f}_1 \) is a reductive Lie algebra. Consider the decomposition

\[
\mathbb{R}^{m+r,m+s} = V_1 \oplus L' \oplus V_1^*,
\]

where \( L' = V'_1 \oplus L \oplus (V'_1)^* \). We get

\[
\mathfrak{g} \subset \mathfrak{gl}(V_1) \oplus \mathfrak{so}(L') \ltimes (V_1 \otimes L' \ltimes \land^2 V_1).
\]

Let \( R \in \mathcal{R}_1(\mathfrak{g}) \) be a non-zero element. Consider

\[
\bar{R} = \text{pr}_{\mathfrak{gl}(V_1)} \circ R|_{V_1 \otimes V_1^*}.
\]

As in the proof of Lemma 3 it is not hard to show that \( \bar{R} \in \mathcal{R}_1(f_1 \subset \mathfrak{so}(V_1 \oplus V_1^*)) \). Also it is easy to see that if \( X \in V_1 \), and \( Y \in V_1^* \), then

\[
g(X,Y) = \text{Ric}(R)(X,Y) = \text{Ric}(\bar{R})(X,Y).
\]
This implies that $\tilde{R} \neq 0$. By Lemma 3, $f_1 \subset \mathfrak{gl}(V_1)$ is a Berger subalgebra of Einstein type; in Section 4 we noted that any such algebra is the holonomy algebra of an Einstein not Ricci-flat pseudo-Riemannian manifold. From results of Section 4 it follows that $f_1$ contains $\text{id}_{V_1}$.

We claim that $\text{pr}_{\mathfrak{gl}(V_1) \oplus \mathfrak{so}(L')} \mathfrak{g} = f_1 \oplus \text{pr}_{\mathfrak{so}(L')} \mathfrak{g}$. In other words, the ideal

$$\mathfrak{t} = f_1 \cap \text{pr}_{\mathfrak{gl}(V_1) \oplus \mathfrak{so}(L')} \mathfrak{g} \subset f_1$$

coincides with $f_1$.

Using the Bianchi identity it is easy to get

$$\text{pr}_{\mathfrak{gl}(V_1)} \circ R|_{\wedge^2 L'} = 0, \quad \text{pr}_{\mathfrak{so}(L')} \circ R|_{V_1 \otimes V_1^*} = 0. \quad (6.7)$$

The first of these equalities shows that the above defined tensor $\tilde{R}$ takes values in $\mathfrak{t}$. In particular, $\mathcal{R}_1(\mathfrak{t} \subset \mathfrak{gl}(V_1 \oplus V_1^*)) \neq 0$. Let $\mathfrak{t}$ be the ideal complementary to $\mathfrak{t}$ in $f_1$.

Suppose that $\mathfrak{t} \subset \mathfrak{gl}(V_1)$ is irreducible. Since $\mathcal{R}_1(\mathfrak{t} \subset \mathfrak{gl}(V_1 \oplus V_1^*)) \neq 0$, we see that by Lemma 3 $\mathfrak{t} \subset \mathfrak{gl}(V_1 \oplus V_1^*)$ is a Berger subalgebra of Einstein type. Note that $\mathfrak{t} \subset \mathfrak{gl}(V_1) \mathfrak{t}$. By Lemma 4 $\mathfrak{t} \subset \mathfrak{t}$. Thus, $\mathfrak{t} = 0$.

Suppose that $\mathfrak{t} \subset \mathfrak{gl}(V_1)$ is not irreducible. Consider the ideal

$$\mathfrak{t} = L(\mathcal{R}(\mathfrak{t} \subset \mathfrak{so}(V_1 \oplus V_1^*)) \subset \mathfrak{t}.$$

We see that $\mathfrak{t}$ is not trivial, $\mathfrak{t} \subset \mathfrak{so}(V_1 \oplus V_1^*)$ is a Berger subalgebra, and

$$\mathcal{R}_1(\mathfrak{t} \subset \mathfrak{so}(V_1 \oplus V_1^*)) \neq 0.$$

By the above arguments with the Ricci tensor, $\mathfrak{t}$ does not annihilate any non-trivial subspace of $V_1$. We obtain the decompositions

$$V_1 = U_1 + \cdots + U_a, \quad \mathfrak{t} = t_1 + \cdots + t_a,$$

where for each $i$, $t_i \subset \mathfrak{gl}(U_i)$ is irreducible and $t_i \subset \mathfrak{so}(U_i \oplus U_i^*)$ is a Berger subalgebra. By the argument with the Ricci tensor, $\mathcal{R}_1(t_i \subset \mathfrak{so}(U_i \oplus U_i^*)) \neq 0$. By Lemma 4, $t_i \subset \mathfrak{so}(U_i \oplus U_i^*)$ is a Berger subalgebra of Einstein type. This implies that $t_i$ contains $\text{id}_{U_i}$. Then $f_1$ contains the elements $\text{id}_{U_1}$ and $\text{id}_{U_2}$, which by the construction commute with $f_1$. This contradicts the Schur Lemma applied to the irreducible representation $f_1 \subset \mathfrak{gl}(V_1)$. Thus, $f_1 = \mathfrak{t}$ and the claim is proved.

From Lemma 4 it follows that $\text{id}_{V_1} \in f_1$. From this and the above claim it follows that $(\text{id}_{V_1}, 0, X, C) \in \mathfrak{g}$ for some $X \in V_1 \otimes L'$ and $C \in \wedge^2 V_1$. Lemma 7 shows that we may change the subspaces $L', V_1^* \subset \mathbb{R}^{n+r+m+s}$ in such a way that $(\text{id}_{V_1}, 0, 0, 0) \in \mathfrak{g}$. Note that under this change the projection $\text{pr}_{\mathfrak{gl}(V_1)} \mathfrak{g} = f_1$ does not change; the new space $L'$ is isomorphic to the old one, and the projection $\text{pr}_{\mathfrak{so}(L')} \mathfrak{g}$ does not change if we use this isomorphism as the identification. Let $A \in f_1$. Then $(A, 0, X, C) \in \mathfrak{g}$ for some $X \in V_1 \otimes L'$ and $C \in \wedge^2 V_1$. Taking the Lie brackets of this element with $(\text{id}_{V_1}, 0, 0, 0) \in \mathfrak{g}$, we get that $(0, 0, X, 2C) \in \mathfrak{g}$. Taking the Lie brackets again, we get $(0, 0, X, 4C) \in \mathfrak{g}$. We conclude that $f_1 \subset \mathfrak{g}$. Similarly, $\text{pr}_{\mathfrak{so}(L')} \mathfrak{g} \subset \mathfrak{g}$, i.e. $\text{pr}_{\mathfrak{so}(L')} \mathfrak{g} = \mathfrak{g} \cap \mathfrak{so}(L')$; and

$$\mathfrak{g} = f_1 \oplus (\mathfrak{g} \cap \mathfrak{so}(L')) \times ((\mathfrak{g} \cap (V_1 \otimes L')) \times (\mathfrak{g} \cap \wedge^2 V_1)).$$

We consider now the intersection

$$\mathfrak{g} \cap \mathfrak{so}(L') \subset \mathfrak{gl}(V') \oplus \mathfrak{so}(L') \times (V' \otimes L' \times \wedge^2 V_1).$$

As in the proof of Lemma 5 it is easy to show that $\mathcal{R}_1(\mathfrak{g} \cap \mathfrak{so}(L')) \neq 0$. Consider the projection $\text{pr}_{\mathfrak{gl}(V')} \mathfrak{g} \cap \mathfrak{so}(L')) \subset \mathfrak{gl}(V')$. If it is not irreducible, then it preserves a subspace $V_2 \subset V'$ such that the induced representation on $V_2$ is irreducible. We choose a complementary subspace $V_2' \subset V'$. Then we apply the above consideration to this settings. We will get a similar result for $\mathfrak{g} \cap \mathfrak{so}(L')$ as for $\mathfrak{g}$ above. In particular, $f_2 = \text{pr}_{\mathfrak{gl}(V_2)}(\mathfrak{g} \cap \mathfrak{so}(L')) \subset \mathfrak{gl}(V_2)$ is irreducible and it satisfies the same properties as $f_1$. 
We continue this process. At the last stage we will see that \( f \) acts irreducibly on \( V'_{k-1} \) for some \( k \). We set \( V_k = V'_{k-1} \). We get the \( f \)-invariant decomposition

\[
V = V_1 \oplus \cdots \oplus V_k
\]

and irreducible subalgebras \( f_i \subset \mathfrak{gl}(V_i) \) that are contained in \( f \) and \( g \); in particular, the Lie algebras \( f_i \) are reductive.

The representation of \( f \) on \( \mathbb{R}^m \) is given by matrices of the form

\[
A = \begin{pmatrix}
A_1 & * & \cdots & * \\
0 & A_2 & \cdots & * \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_k
\end{pmatrix}, \quad A_i \in f_i \subset g.
\]

Let

\[
f_0 = f_1 \oplus \cdots \oplus f_k \subset f.
\]

Since the Lie algebra \( f_0 \) is reductive, there exists a complementary subspace \( \hat{f} \subset f \) with respect to the adjoint representation of \( f_0 \) on \( f \). The above consideration shows that the elements of \( f \) are given by the stars in the matrix \( A \). In particular, \( \hat{f} \subset f \) is an ideal. Clearly, \( \hat{f} \subset \oplus_{1 \leq i < j \leq k} V_j^* \otimes V_i \).

We get the semidirect sum \( f = f_0 \rtimes \hat{f} \). Consider the representation of \( f_0 \) on \( \oplus_{1 \leq i < j \leq k} V_j^* \otimes V_i \); on each subspace \( V_{ij} = V_j^* \otimes V_i \) it is given by the tensor product of the representations \( f_i \subset \mathfrak{gl}(V_i^*) \) and \( \hat{f} \subset \mathfrak{gl}(V_i) \). Since all subalgebras \( f_i \subset \mathfrak{gl}(V_i) \) are irreducible, we see that different \( V_{ij} \) and \( V_{kl} \) do not contain isomorphic \( f_0 \)-modules. This shows that \( \hat{f} = \oplus_{1 \leq i < j \leq k} f_{ij} \), where \( f_{ij} = f \cap V_{ij} \).

This proves the lemma. □

Now we consider \( pr_{\mathfrak{gl}(m,\mathbb{R}) \oplus \mathfrak{so}(r,s)} \) \( g \).

**Lemma 10.** It holds \( pr_{\mathfrak{gl}(m,\mathbb{R}) \oplus \mathfrak{so}(r,s)} \) \( g \) = \( \hat{f} \oplus \mathfrak{h} \).

**Proof.** Let \( \mathfrak{e} = pr_{\mathfrak{gl}(m,\mathbb{R}) \oplus \mathfrak{so}(r,s)} \) \( g \). The facts that \( f_0 \subset g \), \( \mathfrak{h} \) is a reductive Lie algebra, and \( \hat{f} \) does not contain non-zero elements commuting with \( f_0 \), imply the decomposition

\[
\mathfrak{e} = f_0 \oplus \hat{f} \oplus \mathfrak{h},
\]

i.e. \( \mathfrak{e} = \hat{f} \oplus \mathfrak{h} \). □

**Lemma 11.** It holds \( \hat{f} \oplus \mathfrak{h} \subset g \).

**Proof.** We have already seen that, for each \( i \), it holds \( \det \mathfrak{v}_i \in f_i \subset g \). Hence, \( \det \mathbb{R} \in f_0 \subset g \).

Let \( A \in f \). From Lemma [10] it follows that \((A,0,X,C) \in g \) for some \( X \in \mathcal{N} \) and \( C \in \mathcal{C} \). Taking the Lie brackets of \( \xi \) with that element, we get \((0,0,X,2C) \in g \). Taking the Lie brackets again, we get \((0,0,X,4C) \in g \). We conclude that \((A,0,0,0) \in g \), i.e. \( f \subset g \). Similarly, \( \mathfrak{h} \subset g \). □

From the last two lemmas it follows that \( g = \hat{f} \oplus \mathfrak{h} \ltimes (g \cap (\mathcal{N} \ltimes \mathcal{C})) \). As in the previous lemma, we may show that if \((0,0,X,0) \in g \), then \((0,0,X,0) \in g \) and \((0,0,0,C) \in g \), that is,

\[
g \cap (\mathcal{N} \ltimes \mathcal{C}) = (g \cap \mathcal{N}) \ltimes (g \cap \mathcal{C}).
\]

Next, as \( f_0 \oplus \mathfrak{h} \)-module, \( \mathcal{N} = \oplus_{i,a} V_i \otimes L_a \). It is clear that these \( f_0 \oplus \mathfrak{h} \)-modules are pairwise non-isomorphic, hence, \( g \cap \mathcal{N} = \oplus_{i,a} \mathcal{N}_{ia} \), where \( \mathcal{N}_{ia} = g \cap (V_i \otimes L_a) \). Similarly, as the \( \mathfrak{h} \)-module, \( \mathcal{C} = \oplus_{i,j} V_i \wedge V_j \), and \( g \cap \mathcal{C} = \oplus_{i,j} \mathcal{C}_{ij} \), where \( \mathcal{C}_{ij} = g \cap (V_i \wedge V_j) \).

Finally, if for some \( \alpha \), all \( \mathcal{N}_{ia} \) are zero, then \( g \) preserves the non-degenerate subspace \( L_{\alpha} \subset \mathbb{R}^{r+m,s+m} \), this gives a contradiction, since \( g \) is weakly irreducible. The theorem is proved. □
Let us now show that there are few possibilities for each $f_{ij}, N_{i\alpha}$ and $C_{ij}$. Recall that the tensor product of two irreducible representations of Lie algebras in complex vector spaces is always irreducible. It turns out that this is not the case for the tensor product of real representations and we prove the following (probably known) lemma.

**Lemma 12.** Let $\mathfrak{h}_i \subset \mathfrak{gl}(V_i), i = 1, 2$, be irreducible representations of real Lie algebras in real vector spaces. If the representation of $\mathfrak{h}_1 \oplus \mathfrak{h}_2$ in $V_1 \otimes V_2$ is not irreducible, then each vector space $V_i$ admits a complex structure $J_i$ commuting with $\mathfrak{h}_i$. Moreover, $V_1 \otimes V_2$ is the direct sum of two irreducible subspaces of the form $(V_1 \otimes_{\mathbb{C}} V_2)$, where one considers the tensor product either of the complex vector spaces $(V_1, J_1)$ and $(V_2, J_2)$, or the tensor product of the complex vector spaces $(V_1, -J_1)$ and $(V_2, J_2)$.

**Proof.** Suppose that the tensor product $V_1 \otimes V_2$ is not irreducible, and $V_1$ does not admit a complex structure commuting with $\mathfrak{h}_1$. Since the product $V_1 \otimes V_2$ is not irreducible, there exists a linear automorphism $K$ of $V_1 \otimes V_2$ commuting with $\mathfrak{h}_1 \oplus \mathfrak{h}_2$ and such that $K^2 = \text{id} \neq \pm K$. We may write $K$ in the form

$$K = \sum_{\alpha} A_{\alpha} \otimes B_{\alpha},$$

where $A_{\alpha} \in \mathfrak{gl}(V_1)$, $B_{\alpha} \in \mathfrak{gl}(V_2)$, and the vectors $\{B_{\alpha}\}$ are linearly independent. Let $A \in \mathfrak{h}_1$, $x \in V_1$ and $y \in V_2$, then

$$K(Ax \otimes y) = K(A(x \otimes y)) = A(K(x \otimes y)),$$

i.e.

$$\sum_{\alpha} A_{\alpha}Ax \otimes B_{\alpha}y = \sum_{\alpha} A_{\alpha}A_{\alpha}x \otimes B_{\alpha}y.$$

This implies that $\sum_{\alpha} [A, A_{\alpha}] \otimes B_{\alpha} = 0$ and $[A, A_{\alpha}] = 0$. We conclude that each $A_{\alpha}$ commutes with $\mathfrak{h}_1$. Since there are no complex structures on $V_1$ commuting with $\mathfrak{h}_1$, we see that from the Schur lemma it follows that $A_{\alpha} = c_{\alpha} \text{id}_{V_1}$, $c_{\alpha} \in \mathbb{R}$. Thus,

$$K = \text{id}_{V_1} \otimes B, \quad B \in \mathfrak{gl}(V_2).$$

Since $K^2 = \text{id}_{V_1} \otimes V_2$, we get that $B^2 = \text{id}_{V_2}$. Since $B$ commutes with $\mathfrak{h}_2$, and $\mathfrak{h}_2 \subset \mathfrak{gl}(V_2)$ is irreducible, we see that $B = \pm \text{id}_{V_2}$. Consequently, $K = \pm \text{id}_{V_1} \otimes V_2$ and we get a contradiction. Thus, each $V_i$ admits a complex structure $J_i$ commuting with $\mathfrak{h}_i$.

The two vector subspaces in $V_1 \otimes V_2$ described in the statement of the lemma are invariant, irreducible (as tensor products of irreducible complex subspaces), their intersection is trivial, and the sum of their real dimensions equals to the dimension of $V_1 \otimes V_2$, i.e. $V_1 \otimes V_2$ is the direct sum of these subspaces. $\square$

We see now that there are only three possibilities for each space $f_{ij}$: it may be trivial; it may coincide with $V_j^* \otimes V_i$; it may coincide with $V_j^* \otimes_{\mathbb{C}} V_i$. In the last case, there exist complex structures on $V_i$ and $V_j$ commuting with $f_i$ and $f_j$, respectively. This means that $m_i$ and $m_j$ are even:

$$f_i \subset \mathfrak{gl}(m_i/2, \mathbb{C}) \subset \mathfrak{gl}(m_i, \mathbb{R}), \quad f_j \subset \mathfrak{gl}(m_j/2, \mathbb{C}) \subset \mathfrak{gl}(m_j, \mathbb{R}),$$

and

$$f_i \subset \mathfrak{u}(m_i/2, m_i/2) \subset \mathfrak{so}(m_i, m_i), \quad f_j \subset \mathfrak{u}(m_j/2, m_j/2) \subset \mathfrak{so}(m_j, m_j)$$

are the holonomy algebras of Einstein not Ricci-flat pseudo-Kählerian manifolds. Similarly, there are three possibilities for each space $N_{i\alpha}$ and $C_{ij}$ with $i \neq j$. In order to find the spaces $C_{ii}$ one should, for each possible $f_i$, decompose the $f_i$-module $\Lambda^2 V_i$ into the direct sum of
irreducible submodules. The space $C_{ij}$ is either trivial, or it is the direct sum of some of these submodules.

Let us explain now, how to list the Lie algebras $g$ obtained in Theorem 1. Fix a signature $(p, q)$, $p, q \geq 1$. Fix numbers $r, s, m$ such that $p = r + m$, $q = s + m$, $m \geq 1$. Choose decompositions

$$m = m_1 + \cdots + m_t, \quad t \geq 1,$$

$$r = r_1 + \cdots + r_k, \quad s = s_1 + \cdots + s_k, \quad k \geq 1.$$

Fix irreducible subalgebras $\mathfrak{h}_\alpha \subset \mathfrak{so}(r_\alpha, s_\alpha)$ and $f_1 \subset \mathfrak{gl}(m_1, \mathbb{R})$ as in Sections 3 and 4 respectively. Choose $f_i \oplus f_j$-modules $f_{ij}$ such that all conditions (6.1) are satisfied. Choose $f_i \oplus \mathfrak{h}_\alpha$-modules $N_{i\alpha}$ such that (6.3) are satisfied. Choose $f_i \oplus f_j$-modules $C_{ij}$ and $f_i$-modules $C_{ii}$ that satisfy (6.2) and (6.4). We obtain a Lie subalgebra $g \subset \mathfrak{so}(r + m, s + m)$. One may check if the obtained subalgebra is weakly irreducible using the following proposition.

**Proposition 2.** A Lie subalgebra $g \subset \mathfrak{so}(r + m, s + m)$ described in Theorem 1 is not weakly irreducible if and only if there are subsets $T_1, T_2 \subset \{1, \ldots, t\}$, $K_1, K_2 \subset \{1, \ldots, k\}$ such that

$$T_1 \cap T_2 = \emptyset, \quad T_1 \cup T_2 = \{1, \ldots, t\}, \quad K_1 \cap K_2 = \emptyset, \quad K_1 \cup K_2 = \{1, \ldots, k\},$$

one of the subsets $T_1 \subset \{1, \ldots, t\}$ and $K_1 \subset \{1, \ldots, k\}$ is proper, and for each $i \in T_1$, $j \in T_2$, $\alpha \in K_1$, $\beta \in K_2$, it holds

$$f_{ij} = 0, \quad N_{i\beta} = 0, \quad C_{ij} = 0.$$

**Proof of the proposition.** If the above sets of indices exist, then $g$ preserves the proper non-degenerate vector subspace

$$\oplus_{i \in K_1} (V_i \oplus V_i^*) \bigoplus_{\alpha \in T_1} L_{\alpha} \subset \mathbb{R}^{r + m, s + m}.$$

If $g \subset \mathfrak{so}(r + m, s + m)$ is not weakly irreducible, then it can be decomposed into the direct sum of two Berger algebras of Einstein type. Applying Theorem 1 to both algebras, we will get the required sets of indices. □

We are left with the problem to construct an Einstein not Ricci-flat pseudo-Riemannian manifold with each of the obtained weakly irreducible holonomy algebras $g \subset \mathfrak{so}(r + m, s + m)$. For manifolds of signatures $(1, n)$ and $(2, n)$ this is done in Sections 7 and 8 below. The general case is discussed in Section 10.

7. **Lorentzian manifolds**

From Theorem 1 it follows that any weakly irreducible not irreducible holonomy algebra $g \subset \mathfrak{so}(1, n + 1)$ of an Einstein not Ricci-flat Lorentzian manifold of dimension $n + 2$ has the form

$$g = \mathfrak{gl}(1, \mathbb{R}) \oplus \mathfrak{h} \ltimes \mathbb{R}^n,$$

where $\mathfrak{h} \subset \mathfrak{so}(n)$ is the holonomy algebra of an Einstein not Ricci-flat Riemannian manifold. This result for the first time was obtained in [20], for that was used the classification from [25]. Note that the result of the present paper provides an independent proof of the results from [25] in the Einstein case. In [20], the just described algebras $g$ are realized as the holonomy algebras of Einstein not Ricci-flat Lorentzian manifolds. Let $\Lambda \neq 0$. The required metric is the following

$$g = 2dvdu + h + (\Lambda v^2 + H_0)(du)^2,$$
where \(v, x^1, \ldots, x^n, u\) are coordinates on an open subset of \(\mathbb{R}^{n+2}\),

\[
h = h_{ij}(x^1, \ldots, x^n)dx^i dx^j
\]

is an Einstein Riemannian metric with the holonomy algebra \(\mathfrak{h} \subset \mathfrak{so}(n)\) and cosmological constant \(\Lambda\), and \(H_0 = H_0(x^1, \ldots, x^n)\) is a function satisfying \(\Delta_h H_0 = 0\), where \(\Delta_h\) is the Laplacian of the metric \(h\). It is required that \(H_0\) has non-zero Hessian in order make the metric to be indecomposable.

The idea of this construction is the following. Let \(g_0\) be the metric \(g\) with \(H_0\) set to zero. Then \(g_0\) is the direct product of the Einstein Lorentzian manifold of dimension 2 with the holonomy algebra \(\mathfrak{gl}(1, \mathbb{R}) = \mathfrak{so}(1, 1)\) preserving two complement isotropic lines and of a Riemannian manifold with the holonomy algebra \(\mathfrak{h}\); the cosmological constant of each of these manifolds is \(\Lambda\). The function \(H_0\) is used in order to curve the product metric and to add the subspace \(\mathbb{R}^n\) to the holonomy algebra. The curvature tensor of \(g\) equals to the curvature tensor of \(g_0\) with the additional term that takes values in \(\mathbb{R}^n \subset \mathfrak{g}\). This additional term is given by a symmetric endomorphism of the Euclidean space \(\mathbb{R}^n\), the Ricci tensor of the additional term must be zero, and this gives the equation \(\Delta_h H_0 = 0\).

8. Pseudo-Riemannian manifolds of index 2

In this section we give a complete classification of the holonomy algebras of Einstein not Ricci-flat pseudo-Riemannian manifolds of signature \((2, n)\), \(n \geq 2\).

We start with the signature \((2, 2)\).

**Theorem 2.** Weakly irreducible not irreducible holonomy algebras of Einstein not Ricci-flat pseudo-Riemannian manifolds of signature \((2, 2)\) are exhausted by the following subalgebras of \(\mathfrak{so}(2, 2)\):

1) \(\mathfrak{g}_1 = (\mathfrak{gl}(1, \mathbb{R}) \oplus \mathfrak{gl}(1, \mathbb{R})) \ltimes \wedge^2 \mathbb{R}^2\).
2) \(\mathfrak{g}_2 = \mathfrak{gl}(2, \mathbb{R})\);
3) \(\mathfrak{g}_3 = \mathfrak{gl}(2, \mathbb{R}) \ltimes \wedge^2 \mathbb{R}^2\);
4) \(\mathfrak{g}_4 = \mathfrak{gl}(1, \mathbb{C}) \subset \mathfrak{u}(1, 1)\).
5) \(\mathfrak{g}_5 = \mathfrak{gl}(1, \mathbb{C}) \ltimes \wedge^2 \mathbb{R}^2 \subset \mathfrak{u}(1, 1)\).
6) \(\mathfrak{g}_6 = \mathfrak{j} \ltimes \wedge^2 \mathbb{R}^2\), where

\[
\mathfrak{j} = \left\{ \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \bigg| a, b, c \in \mathbb{R} \right\} \subset \mathfrak{gl}(2, \mathbb{R}).
\]

**Proof.** The holonomy algebras of pseudo-Riemannian manifolds of signature \((2, 2)\) were classified in \([7, 22]\), see also \([19]\). Comparing this result with Theorem 1 we see that weakly irreducible not irreducible Berger algebras \(\mathfrak{g} \subset \mathfrak{so}(2, 2)\) of Einstein type are exhausted by the algebras form Theorem 2 (note that the algebra \(\mathfrak{j}\) is conjugated to the algebra \(\mathfrak{g}_1\), by this reason it does not appear in the theorem). In order to complete the proof of the theorem we construct Einstein not-Ricci flat metrics realizing the algebras from the statement of the theorem as the holonomy algebras.

Consider on \(\mathbb{R}^4\) the coordinates \(v_1, v_2, u_1, u_2\) and the Walker metric

\[
g = 2dv_1 du_1 + 2dv_2 du_2 + H_1(du_1)^2 + 2H_3 du_1 du_2 + H_2(du_2)^2,
\]

where \(H_1, H_2\) and \(H_3\) are functions. Let \(\Lambda\) be a non-zero real number. We claim that the following choice of the functions provides Einstein metrics with the cosmological constant \(\Lambda\) and with the corresponding holonomy algebras from the statement of the theorem:

1) \(H_1 = \Lambda v_1^2 + u_2^2, H_2 = \Lambda v_2^2, H_3 = 0\);
2) \( H_1 = \frac{2\Lambda}{3}v_1^2 \), \( H_2 = \frac{2\Lambda}{3}v_2^2 \), \( H_3 = \frac{2\Lambda}{3}v_1v_2 \);

3) \( H_1 = \frac{2\Lambda}{3}v_1^3 + v_1u_1u_2^2 \), \( H_2 = \frac{2\Lambda}{3}v_2^3 - v_2u_1^2u_2 \), \( H_3 = \frac{2\Lambda}{3}v_1v_2 \);

4) \( H_1 = \alpha(v_1^2 - v_2^2), H_2 = -H_1, H_3 = \Lambda v_1v_2 \);

5) \( H_1 = \beta(v_1^2 - v_2^2), H_2 = -H_1, H_3 = \Lambda v_1v_2 + u_1u_2 \);

6) \( H_1 = \Lambda v_1^2 + \text{ArcTan}(v_2), H_2 = \Lambda v_2^2 + \Lambda, H_3 = 0 \).

We use a computer program to compute the Christoffel symbols, the curvature tensor and the Ricci tensor of these metrics. We check that the obtained metrics are Einstein with the cosmological constant \( \Lambda \). Let us explain, how to check that the holonomy algebras of the metrics are the corresponding algebras from the statement of the theorem. Since each metric under the consideration is analytical, its holonomy algebra \( g \subset \mathfrak{so}(2, 2) \) at the point 0 is generated by the elements of the form

\[
\nabla_{X_1} \ldots X_r R_0(X, Y),
\]

where \( r \geq 0 \), and \( X, Y, X_1, \ldots, X_r \) are tangent vector at the point 0.

1) The vector fields \( \partial_{\alpha_1} \) and \( \partial_{\alpha_2} \) are recurrent, hence \( g \) is contained in \( g_1 \). The form of the curvature tensor shows that the dimension of \( g \) is at least three. This shows that \( g = g_1 \).

2) The elements \( R_0(X, Y) \), where \( X \) and \( Y \) are tangent vectors at 0, span \( g_2 \), and it holds \( \nabla R = 0 \).

3) The subalgebra \( g_3 \subset \mathfrak{so}(2, 2) \) it the maximal subalgebra preserving a fixed two-dimensional isotropic subspace and its dimension is 5. The elements of the form \( R_0(X, Y) \) and \( \nabla Z R_0(X, Y) \) generate a space of dimension 5, consequently \( g = g_3 \).

4) The considerations are the same as in the case 2).

5) The elements \( R_0(X, Y) \) span \( g_5 \). Denote the coordinates on \( \mathbb{R}^4 \) by \( x_1, \ldots, x_4 \). For each \( i, \ 1 \leq i \leq 4 \), consider the matrix \( \Gamma_i = (\Gamma_i^j)_{j=1}^4 \). The values and all partial derivatives at 0 of \( R(\partial_i, \partial_j) \) and \( \Gamma_k \) belong to \( g_5 \). Since the covariant derivatives of \( R \) are expressed in terms of the partial derivatives and the Lie brackets of \( R(\partial_i, \partial_j) \) and \( \Gamma_k \), it holds \( g = g_5 \).

6) The vector field \( \partial_{\alpha_1} \) is recurrent. The subalgebra \( g_6 \subset \mathfrak{so}(2, 2) \) it the maximal subalgebra preserving a fixed two-dimensional isotropic subspace and a line in this subspace; its dimension is 4. The elements of the form \( R_0(X, Y) \) and \( \nabla Z R_0(X, Y) \) generate a space of dimension 4, consequently \( g = g_6 \).

The theorem is proved. \( \square \)

Consider now the general case. We use the notation of Theorem 11.

**THEOREM 3.** Weakly irreducible not irreducible holonomy algebras \( g \subset \mathfrak{so}(2, n + 2) \) of Einstein not Ricci-flat pseudo-Riemannian manifolds of signature \((2, n + 2)\), \( n \geq 1 \), are exhausted by the following algebras:

a) \( g_a = \mathfrak{gl}(1, \mathbb{R}) \oplus \mathfrak{h} \times \mathbb{R}^{1, n+1} \subset \mathfrak{so}(2, n + 2)_{\mathbb{R}}, \) where \( \mathfrak{h} \subset \mathfrak{so}(1, n + 1) \) is the holonomy algebra of an Einstein not Ricci-flat Lorentzian manifold that is the direct sum of an \( \mathfrak{so}(1, l + 1) \) and a holonomy algebra of an Einstein not Ricci-flat Riemannian manifold;

b) \( g_b = \mathfrak{gl}(2, \mathbb{R}) \oplus \mathfrak{h} \times (\mathbb{R}^2 \otimes \mathbb{R}^n \times \Lambda^2 \mathbb{R}^2) \subset \mathfrak{so}(2, n + 2)_{\mathbb{R}^2}, \) where \( \mathfrak{h} \subset \mathfrak{so}(n) \) is the holonomy algebra of an Einstein not Ricci-flat Riemannian manifold;

c) \( g_c = \mathfrak{gl}(1, \mathbb{C}) \oplus \mathfrak{h} \times (\otimes_{\alpha=1}^{l+1} N_{\alpha} \times \Lambda^2 \mathbb{R}^2) \subset \mathfrak{so}(2, n + 2)_{\mathbb{R}^2}, \) where \( \mathfrak{h} \subset \mathfrak{so}(n) \) is the holonomy algebra of an Einstein not Ricci-flat Riemannian manifold; \( N_{\alpha} \subset \mathbb{R}^2 \otimes L_{\alpha} \) is a non-trivial subspace; if \( N_{\alpha} \neq \mathbb{R}^2 \otimes L_{\alpha} \), then \( h_{\alpha} \subset u(L_{\alpha}) \) and either \( N_{\alpha} \simeq L_{\alpha} \), or \( N_{\alpha} \simeq L_{\alpha}; \)

d) \( g_d = \mathfrak{gl}(1, \mathbb{R}) \oplus \mathfrak{gl}(1, \mathbb{R}) \oplus \mathfrak{h} \times (\otimes_{\alpha=1}^{l+1} N_{1\alpha} \oplus \otimes_{\alpha=1}^{l+1} N_{2\alpha} \times \Lambda^2 \mathbb{R}^2) \subset \mathfrak{so}(2, n + 2)_{\mathbb{R}^2} \), where \( \mathfrak{h} \subset \mathfrak{so}(n) \) is the holonomy algebra of an Einstein not Ricci-flat Riemannian manifold; each \( N_{1\alpha}, N_{2\alpha} \subset L_{\alpha} \) is either \( L_{\alpha} \) or 0; for each \( \alpha \), at least one of \( N_{1\alpha} \) and \( N_{2\alpha} \) equals to \( L_{\alpha}; \)
\( e) \ g_e = f \oplus h \ltimes (\oplus_{\alpha=1}^{t_1} N_1 \alpha \oplus \oplus_{\alpha=1}^{t_2} N_2 \alpha \ltimes \wedge^2 \mathbb{R}^2) \subset \mathfrak{so}(2, n + 2)_{\mathbb{R}^2}, \) where

\[
f = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}
\]

\( h \subset \mathfrak{so}(n) \) is the holonomy algebra of an Einstein not Ricci-flat Riemannian manifold; for each \( \alpha, N_1 \alpha = L_\alpha \) and \( N_2 \alpha \) is either \( L_\alpha \) or 0.

**Proof.**

1. **Algebraic classification.** Suppose that the maximal dimension of a totally isotropic subspace preserved by \( g \subset \mathfrak{so}(2, n + 2) \) is one, then from Theorem 1 it follows that \( g \) is the first algebra from the statement of the theorem.

In other cases, \( g \) preserves a totally isotropic subspace of dimension 2. Suppose that \( f \subset \mathfrak{gl}(2, \mathbb{R}) \) is irreducible. Then either \( f = \mathfrak{gl}(2, \mathbb{R}), \) or \( g = \mathfrak{gl}(1, \mathbb{C}). \)

Suppose that \( g = \mathfrak{gl}(1, \mathbb{C}). \) By Theorem 1, \( N = \oplus_{\alpha=1}^{\alpha} N_\alpha, \) and each \( N_\alpha \subset \mathbb{R}^2 \otimes L_\alpha \) is a non-trivial \( \mathfrak{gl}(1, \mathbb{C}) \oplus h_\alpha \)-module. The space \( \mathbb{R}^2 \otimes L_\alpha \) can be considered as the complexification \( C \otimes L_\alpha, \) of \( L_\alpha. \) This \( \mathfrak{gl}(1, \mathbb{C}) \oplus h_\alpha = C \oplus h_\alpha \)-module is irreducible if and only if the complexified representation \( C \otimes h_\alpha \subset \mathfrak{gl}(C \otimes L_\alpha) \) is irreducible. This is equivalent to the non-existence of a complex structure on \( L_\alpha. \) In the opposite case, \( C \otimes L_\alpha \simeq L_\alpha \oplus L_\alpha. \) We obtain the third algebra.

If \( g = \mathfrak{gl}(2, \mathbb{R}), \) then each \( \mathfrak{gl}(2, \mathbb{R}) \oplus h_\alpha \)-module \( \mathbb{R}^2 \otimes L_\alpha \) is irreducible. We obtain the second algebra.

Suppose that \( f \subset \mathfrak{gl}(2, \mathbb{R}) \) is not irreducible. Then either \( f = \mathfrak{gl}(1, \mathbb{R}) \oplus \mathfrak{gl}(1, \mathbb{R}), \) or \( f = \mathfrak{gl}(1, \mathbb{R}) \oplus \mathfrak{gl}(1, \mathbb{R}) \oplus f_{12}, \) where \( f_{12} = \mathbb{R}. \)

From Theorem 1 it follows that \( g \cap N = \oplus_{\alpha=1}^{t_1} N_1 \alpha \oplus \oplus_{\alpha=1}^{t_2} N_2 \alpha, \) where each \( N_\alpha \) is either 0, or \( L_\alpha. \) Next, at least one of \( N_1 \alpha \) or \( N_2 \alpha \) is non-zero, i.e. it equals to \( L_\alpha. \) Suppose that \( N_1 \alpha = N_2 \alpha = R_\alpha \) for some \( \alpha, \) then \( [N_1 \alpha, N_2 \alpha] = C \otimes L_\alpha, \)

Suppose that \( g \cap C = 0, \) then, for each \( \alpha, \) either \( N_1 \alpha = 0, \) or \( N_2 \alpha = 0. \) Consequently \( g \) preserves the non-degenerate subspaces \( \mathbb{R}p_1 \oplus \oplus_{\alpha=0} N_2 \alpha = 0 L_\alpha \oplus \mathbb{R}q_1, \) and \( \mathbb{R}p_2 \oplus \oplus_{\alpha=0} N_1 \alpha = 0 L_\alpha \oplus \mathbb{R}q_2. \) This shows that

\( g = \mathfrak{gl}(1, \mathbb{R}) \oplus (\oplus_{\alpha=0} h_\alpha \ltimes L_\alpha) \oplus \mathfrak{gl}(1, \mathbb{R}) \oplus (\oplus_{\alpha=0} h_\alpha \ltimes L_\alpha) \)

is \( Z \)-graded of depth 1.

If \( g \cap C \neq 0, \) then \( g \) is \( Z \)-graded of depth 2, hence it cannot preserve a non-degenerate subspace of Lorentzian signature. Clearly \( g \) cannot preserve any Euclidean subspace. Thus, \( g \) is weakly irreducible. We obtain the algebra d).

Let \( f = \mathfrak{gl}(1, \mathbb{R}) \oplus \mathfrak{gl}(1, \mathbb{R}) \oplus f_{12}. \) Since \( [f_{12}, N_2 \alpha] = N_1 \alpha, \) we see that \( N_1, L_\alpha = L_\alpha \) for all \( \alpha. \) Suppose that \( g \cap C = 0. \) Then \( N_2 \alpha = 0 \) for all \( \alpha. \) In this case \( g \) preserves the isotropic subspace spanned by the vectors \( p_1, \) and \( q_2. \) If we consider the vectors \( p_1, p_2, q_1, p_2 \) instead of \( p_1, p_2, q_1, q_2, \) the we get the algebra d) \( (f_{12} \text{ becomes } C) \).

If \( g \cap C \neq 0, \) then we get the algebra e). The fact that it is weakly irreducible follows from the previous case.

2. **Construction of metrics.** To complete the proof of Theorem we should show that each of the algebras \( g \subset \mathfrak{so}(2, n + 2) \) from the statement of the theorem may appear as the holonomy algebra of an Einstein not Ricci-flat pseudo-Riemannian manifold. The metrics are constructed using the idea from Section 7.

a) For the case of the first algebra, the metric can be taken as in Section 7 with \( h \) to be an Einstein not Ricci-flat Lorentzian metric with the holonomy algebra \( \mathfrak{h} \subset \mathfrak{so}(1, n + 1). \)

Next, let \( h \) be an Einstein not Ricci-flat Riemannian metric defined on \( \mathbb{R}^n \) with the holonomy algebra \( \mathfrak{h} \subset \mathfrak{so}(n) \) and cosmological constant \( \Lambda. \) The de Rham decomposition implies that the
coordinates can be divided into the groups
\[(x^1, \ldots, x^n) = (x_1^\alpha, \ldots, x_n^\alpha)_\alpha=1^t\]
corresponding to the decomposition of \(\mathfrak{h}\) into irreducible parts. Consider the coordinates \(v_1, v_2, \ldots, v_n, u_1, u_2\) on \(\mathbb{R}^{n+4}\).

The algebras \(\mathfrak{g}_b, \mathfrak{g}_c, \mathfrak{g}_d, \mathfrak{g}_e\) may be obtained respectively from the algebras \(\mathfrak{g}_3, \mathfrak{g}_4, \mathfrak{g}_1, \mathfrak{g}_6\) by adding certain subspace \(N_j \subset \mathbb{R}^2 \otimes \mathbb{R}^n\). Fix one of the algebras from Theorem 2 and denote it be \(f\). Denote by \(f\) be the corresponding metric constructed above. The holonomy algebra of the product metric \(f + h\) is \(f \oplus \mathfrak{h}\). We will show that it is possible to twist slightly this metric in order to get an Einstein not Ricci-flat metric with the corresponding holonomy algebra from the statement of the theorem.

Consider the metric\[g = f + h + F_1(du^1)^2 + 2F_2du^1du^2 + F_3(du^2)^2,\]
where\[F_i = \sum_{\alpha=1}^t F_i(\alpha, x_1^\alpha, \ldots, x_n^\alpha), \quad i = 1, 2, 3,\]
are some function that should be chosen in a proper way in order to add to the initial holonomy algebra \(f \oplus \mathfrak{h}\) a subspace \(N_j \subset \mathbb{R}^2 \otimes L_\alpha\). Since the coordinates \(x_1, \ldots, x_n\) are separated into groups, it is enough to find a construction for \(t = 1\).

The non-zero Christoffel symbols of the metric \(f + h\) are some of\[\Gamma_{vi,uj}^{vk}, \quad \Gamma_{ui,uj}^{uk}, \quad \Gamma_{uk,uj}^{xk}, \quad \Gamma_{uk,uj}^{xk}, \quad \Gamma_{uk,uj}^{xk}.\]
The non-zero Christoffel symbols of the metric \(g\) are the following non-zero Christoffel symbols of the metric \(f + h\):
\[\Gamma_{vi,uj}^{vk}, \quad \Gamma_{ui,uj}^{uk}, \quad \Gamma_{uk,uj}^{xk}, \quad \Gamma_{uk,uj}^{xk}, \quad \Gamma_{uk,uj}^{xk}, \quad \Gamma_{uk,uj}^{xk}.\]
In particular, it holds\[\Gamma_{vi,uj}^{v} = \frac{1}{2}\partial_u F_{ij},\]
where we use the denotation \(F_{11} = F_1, F_{12} = F_2, F_{21} = F_3, F_{22} = F_2\). Since we are interested in the projection of the holonomy algebra onto \(\mathbb{R}^2 \otimes \mathbb{R}^n\), the symbols \(\Gamma_{vi,uj}^{xk}\) are the only we should carry about. Similarly, we are interested in the components \(\nabla_i \cdots R_{xk,uj}^{vi}\) of the curvature tensor and its covariant derivatives. It holds\[R_{xk,uj}^{vi} = \frac{1}{2}\partial_x \partial_u F_{ij} - \frac{1}{2} \sum_{c=1}^n \Gamma_{xk,uj}^{x} \partial_{x,} F_{ij}.\]
Note that \(R_{xk,uj}^{vi} = 0\).

Consider now the algebras from the statement of the theorem.

b) We take \(F_2 = F_3 = 0\). The Einstein equation (with the cosmological constant \(\Lambda\)) is equivalent to the condition\[\Delta_h F_1 = 0,\]
where \(\Delta_h\) is the Laplace operator with respect to the metric \(h\). If we take the function \(F_1\) sufficiently general (such that \(R_{xk,uj}^{vi} \neq 0\) for some indices), then the holonomy algebra \(\mathfrak{g}\) of that metric at the point 0 contains a non-trivial intersection with \(\mathbb{R}^2 \otimes \mathbb{R}^n\). Since the holonomy algebra \(\mathfrak{g}\) contains also \(\mathfrak{g}(2, \mathbb{R}) \ltimes \wedge^2 \mathbb{R}\) and \(\mathfrak{h}\), by the proof of the first part of the theorem, \(\mathfrak{g} = \mathfrak{g}_b\).
c) If \( \mathcal{N} = \mathbb{R}^2 \otimes \mathbb{R}^n \), then the construction is the same as in the previous case. Suppose that \( n = 2l, \mathfrak{h} \subset \mathfrak{u}(l) \), and \( \mathcal{N} = \mathcal{C}^l \subset \mathbb{R}^2 \otimes \mathbb{R}^{2l} \). More precisely, \( \mathcal{N} \) consists of the matrices
\[
\begin{pmatrix}
X & -Y \\
Y & X
\end{pmatrix},
\]
where \( X, Y \in \mathbb{R}^l \). Denote by \( J \) the Kählerian structure of the metric \( h \). Let the coordinates \( x_1, \ldots, x_{2l} \) satisfy
\[
J \partial_{x_a} = \partial_{x_{a+1}}, \quad J \partial_{x_{a+l}} = -\partial_{x_a}, \quad a = 1, \ldots, l.
\]
Let \( F_2 = -F_1 \) and
\[
F_1(x_1, \ldots, x_{2l}) = F_3(x_{l+1}, \ldots, x_{2l}, -x_1, \ldots, -x_l).
\]
The Einstein equation is equivalent to the condition \( \Delta_h F_3 = 0 \). We get
\[
\Gamma^{v_1}_{x_a u_j} = \Gamma^{v_2}_{x_{a+l} u_j}, \quad \Gamma^{v_2}_{x_a u_j} = -\Gamma^{v_1}_{x_{a+l} u_j}, \quad a = 1, \ldots, l, \quad j = 1, 2.
\]
This implies
\[
R^{v_1}_{x_a x_{b} u_j} = R^{v_2}_{x_{a+l} x_{b} u_j}, \quad R^{v_2}_{x_a x_{b} u_j} = -R^{v_1}_{x_{a+l} x_{b} u_j}, \quad a, b = 1, \ldots, l, \quad j = 1, 2.
\]
Consequently the same condition satisfy all covariant derivatives of the curvature tensor at the point 0, this implies that \( \mathfrak{g} = \mathfrak{g}_c \).

\( \text{d,e) Let } F_3 = 0. \) The Einstein equation is equivalent to the condition \( \Delta_h F_1 = \Delta_h F_2 = 0 \). If we take both \( F_1 \) and \( F_2 \) sufficiently general, then \( \mathcal{N} = \mathbb{R}^2 \otimes \mathbb{R}^n \). If we take \( F_2 = 0 \) and \( F_1 \) sufficiently general, then \( \mathcal{N} = \mathcal{N}_{11} = \mathbb{R}^n \subset \mathbb{R}^2 \otimes \mathbb{R}^n \).

The theorem is proved. \( \square \)

9. Para-quaternionic-Kählerian manifolds

Quaternionic-Kählerian manifolds of non-zero scalar curvature are always Einstein and not Ricci-flat. Holonomy algebras of these manifolds in arbitrary signature are classified in [11]. This result may be also deduced from Theorem 1 in the same way as we do now for the case of para-quaternionic-Kählerian manifolds.

Recall that a para-quaternionic-Kählerian manifold \( [2, 15] \) is a pseudo-Riemannian manifold \( (M, g) \) that admits a parallel three-dimensional subbundle of the bundle of endomorphisms of the target bundle of \( M \) locally spanned by endomorphisms \( I, S, T \) preserving \( g \) and satisfying the relations of the split quaternions
\[
-I^2 = S^2 = T^2 = \text{id}, \quad IS = T = -SI.
\]
Note that the endomorphisms \( I, S, T \) generate the Lie algebra isomorphic to \( \mathfrak{sl}(2, \mathbb{R}) \).

Equivalently, a para-quaternionic-Kählerian manifold is a pseudo-Riemannian manifold with the holonomy group contained in
\[
\text{Sp}(2n, \mathbb{R}) \cdot \text{SL}(2, \mathbb{R}) \subset \text{SO}(2n, 2n).
\]
For the holonomy algebra we get
\[
\mathfrak{g} \subset \mathfrak{sp}(2n, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \subset \mathfrak{so}(2n, 2n).
\]
In [2] it is shown that each para-quaternionic-Kählerian manifold of non-zero scalar curvature is Einstein, locally indecomposable (i.e. with weakly irreducible holonomy algebra), and its holonomy algebra contains the subalgebra \( \mathfrak{sl}(2, \mathbb{R}) \).

Let \( (M, g) \) be a para-quaternionic-Kählerian manifold of non-zero scalar curvature and let \( \mathfrak{g} \subset \mathfrak{sp}(2n, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \) be its holonomy algebra. Suppose that \( \mathfrak{g} \) is not irreducible, i.e. it preserves an isotropic subspace of the tangent space. Since \( \mathfrak{g} \) contains the subalgebra \( \mathfrak{sl}(2, \mathbb{R}) \),
which does not annihilate any subspace of the tangent space, Theorem 11 implies that \( r + s = 0, m = 2n, \mathfrak{t} = \mathfrak{t}_0 = \mathfrak{t}_1 \subset \mathfrak{gl}(2n, \mathbb{R}) \) is irreducible, and

\[
\mathfrak{g} \subset \mathfrak{gl}(2n, \mathbb{R}) \ltimes \wedge^2 \mathbb{R}^{2n}.
\]

Since \( \mathfrak{g} \) stabilizes \( \mathfrak{sl}(2, \mathbb{R}) \), i.e. \( \mathfrak{g}, \mathfrak{sl}(2, \mathbb{R}) \subset \mathfrak{sl}(2, \mathbb{R}) \), we get that

\[
\text{pr}_{\mathfrak{gl}(2n, \mathbb{R})} \mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}),
\]

where the representation of \( \mathfrak{gl}(n, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \) in \( \mathbb{R}^{2n} \) is given by the tensor products of representations: \( \mathbb{R}^{2n} = \mathbb{R}^n \otimes \mathbb{R}^2 \). Next,

\[
\wedge^2 \mathbb{R}^{2n} = (\wedge^2 \mathbb{R}^n \otimes \mathbb{R}^2) \oplus (\mathbb{R}^2 \otimes \wedge^2 \mathbb{R}^n)
\]

as the \( \mathfrak{gl}(n, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \)-module. Note that the \( \mathfrak{sl}(2, \mathbb{R}) \)-module \( \wedge^2 \mathbb{R}^2 \) is trivial. Since \( \mathfrak{g} \) stabilizes \( \mathfrak{sl}(2, \mathbb{R}) \), we get

\[
\text{pr}_{\wedge^2 \mathbb{R}^{2n}} \mathfrak{g} \subset (\mathbb{R}^2 \otimes \wedge^2 \mathbb{R}^n).
\]

Thus,

\[
\mathfrak{g} = \mathfrak{t} \ltimes \mathcal{C}, \quad \mathfrak{t} \subset \mathfrak{gl}(n, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}), \quad \mathcal{C} \subset \mathbb{R}^2 \otimes \mathbb{R}^n \otimes \wedge^2 \mathbb{R}^2.
\]

Let \( \mathfrak{g}_0 = \mathfrak{gl}(n, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \subset \mathfrak{gl}(2n, \mathbb{R}) \). The corresponding subalgebra \( \mathfrak{g}_0 \subset \mathfrak{so}(2n, 2n) \) is the holonomy algebra of the para-quaternionic-Kählerian symmetric space

\[
\text{SL}(n + 2, \mathbb{R})/(\text{GL}(n, \mathbb{R}) \cdot \text{SL}(2, \mathbb{R})).
\]

Let \( R_0 \in \mathcal{R}_1(\mathfrak{g}_0 \subset \mathfrak{so}(2n, 2n)) \) be the corresponding algebraic curvature tensor. From Lemma 2 it follows that

\[
\mathcal{R}(\mathfrak{gl}(n, \mathbb{R}) \subset \mathfrak{so}(2n, 2n)) = 0.
\]

This implies that

\[
\mathcal{R}(\mathfrak{g}_0 \subset \mathfrak{so}(2n, 2n)) = \mathbb{R} R_0.
\]

This and the proof of Lemma 9 show that \( \mathfrak{t} = \mathfrak{g}_0 \). Finally note that the \( \mathfrak{g}_0 \)-module \( \mathbb{R}^2 \otimes \mathbb{R}^n \otimes \wedge^2 \mathbb{R}^2 \) is irreducible, consequently, either \( \mathcal{C} = 0 \), or \( \mathcal{C} = \mathbb{R}^2 \otimes \mathbb{R}^n \otimes \wedge^2 \mathbb{R}^2 \). We have proved the following theorem.

**Theorem 4.** Let \((M, g)\) be a para-quaternionic-Kählerian manifold of non-zero scalar curvature and dimension \( 4n \). If its holonomy algebra \( \mathfrak{g} \) is not irreducible, then it preserves an isotropic subspace of dimension \( 2n \), and it coincides with one of the following subalgebras of \( \mathfrak{so}(2n, 2n) \):

- \( \mathfrak{gl}(n, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \),
- \( (\mathfrak{gl}(n, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})) \ltimes (\mathbb{R}^2 \otimes \mathbb{R}^n \otimes \wedge^2 \mathbb{R}^2) \).

Let us note that the irreducible holonomy algebras of para-quaternionic-Kählerian manifolds of non-zero scalar curvature are exhausted by \( \mathfrak{sp}(2n, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \) and by the holonomy algebras of para-quaternionic-Kählerian symmetric spaces; the list of para-quaternionic-Kählerian symmetric spaces may be found in [2] [15], or it may be deduced from [9]; the only symmetric space with non-irreducible holonomy algebra is \( \text{SL}(n + 2, \mathbb{R})/(\text{GL}(n, \mathbb{R}) \cdot \text{SL}(2, \mathbb{R})) \).

10. **Final remarks**

It is an open problem to construct examples of Einstein not Ricci-flat metrics with the holonomy algebras \( \mathfrak{g} \subset \mathfrak{so}(m + r, m + s) \) from Theorem 11 for signatures different from \((1, N)\).
and \((2, N)\). The following idea generalizing the construction from Section 8 may be used. Let \(\mathfrak{g} \subset \mathfrak{so}(m + r, m + s)\) be an algebra from Theorem 1. Consider the coordinates

\[
(v_1^i, \ldots, v_{m_i}^i, \alpha^i, x_{n_i}^i, u_1^i, \ldots, u_{m_i}^i)
\]

on \(\mathbb{R}^{m+r,m+s}\). Let

\[
f_i = \sum_{a} 2 du_a du_a^i + \sum_{a,b} H_{ab}(v_1^i, \ldots, v_{m_i}^i) du_a^i du_b^i
\]

be an Einstein metric with the cosmological constant \(\Lambda\) and the holonomy algebra \(f_i \subset \mathfrak{so}(m_i, m_i)\). Let

\[
h_{\alpha} = \sum_{a,b=1}^n h_{ab}(x_1^\alpha, \ldots, x_{n_i}^\alpha) dx_a^\alpha dx_b^\alpha
\]

be an Einstein metric with the cosmological constant \(\Lambda\) and the holonomy algebra \(h_{\alpha} \subset \mathfrak{so}(n_{\alpha})\). Such metrics exist, one may consider e.g. metrics of symmetric spaces. The metric

\[
g = f + h = \sum_{i=1}^k f_i + \sum_{\alpha=1}^t h_{\alpha}
\]

is Einstein with the holonomy algebra \(f_0 \oplus h\). This metric is decomposable. Next, if some of \(f_{ij}, N_{i\alpha}\) or \(C_{ij}\) is non-trivial, then we further curve the metric adding to \(g\) the following terms:

\[
f_{ij} = \sum_{a,b=1}^m F_{ij}^{ab}(u_1^i, \ldots, u_{m_i}^i) du_a^i du_b^i, \quad n_{i\alpha} = \sum_{a,b=1}^m N_{i\alpha}^{ab}(x_1^{\alpha}, \ldots, x_{n_i}^{\alpha}) du_a^i du_b^i,
\]

\[
c_{ij} = \sum_{a=1}^m \sum_{b=1}^m C_{ij}^{ab}(u_1^i, \ldots, u_{m_i}^i, u_1^j, \ldots, u_{m_j}^j) du_a^i du_b^j.
\]

The additional terms will give some additional components to the curvature tensor of the metric \(f + h\), these components will take values in \(V_j^* \otimes V_i, V_i \otimes L_\alpha, V_i \otimes V_j\), respectively. The Einstein condition and the condition on the holonomy algebra to coincide with \(\mathfrak{g}\) will give equations on the functions \(F_{ij}^{ab}, N_{i\alpha}^{ab}, C_{ij}^{ab}\). Above we have seen that for each of \(F_{ij}^{ab}, N_{i\alpha}^{ab}, C_{ij}^{ab}\), \(i \neq j\), there are only three possibilities: it can be trivial, coincide with the one of the corresponding spaces \(V_j^* \otimes V_i, V_i \otimes L_\alpha, V_i \otimes V_j\) or with one of the corresponding spaces \(V_j^* \otimes \mathbb{C} V_i, V_i \otimes \mathbb{C} L_\alpha, V_i \otimes \mathbb{C} V_j\). In the first case, the corresponding functions may be chosen to be zero; in the second case, the corresponding functions may be chosen to be harmonic and sufficiently general; in the third case, the corresponding functions may be found in the similar way as in case c) from the proof of Theorem 3. A complication may appear, when one considers the \(f_i\)-modules \(C_{ii}\). Each \(C_{ii}\) is a submodule of \(\wedge^2 V_i\). One should go through the list of the representations \(f_i \subset \mathfrak{gl}(m_i, \mathbb{R})\) from Section 1 and in each case describe all submodules of the wedge product \(\wedge^2 V_i\). It is a challenge to find proper functions \(C_{ij}^{ab}\) for each of these submodules. One example of the module \(C_{ii}\) provides Section 9. Since the rigorous construction and its justification require complicated technical work, they should be done in a separate paper.

We complete the paper by giving two remarks. First, the holonomy algebra of an Einstein not Ricci-flat pseudo-Riemannian symmetric space is reductive [3]. Thus the only holonomy algebras of symmetric spaces appearing in Theorem 1 are contained in \(\mathfrak{so}(n, n)\) and have been discussed in Section 1. Finally, note that in [18] we show that in the general non-Einstein case the subalgebra \(\mathfrak{h} \subset \mathfrak{so}(r, s)\) associated to a holonomy algebra \(\mathfrak{g} \subset \mathfrak{so}(r + m, s + m)\) may be absolutely arbitrary, i.e. the Einstein case is in sharp contrast to the non-Einstein case.
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