Decoherence as a relative phenomenon: a generalization of the spin-bath model

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In this paper we propose a closed-system perspective to study decoherence. From this perspective we analyze the spin-bath model as presented in the literature, and a natural generalization of that model. On the basis of the results obtained from that analysis, we argue that decoherence may be understood as a phenomenon relative to the partition of a closed system, selected in each particular case. This viewpoint frees the decoherence program from certain conceptual difficulties derived from its open-system perspective. We also argue that the usual picture of decoherence in terms of energy dissipation is misguided.

I. INTRODUCTION

In the literature on quantum mechanics, ‘decoherence’ refers to the quantum process that turns a coherent pure state into a dephased mixed state, which is diagonal in a well defined basis. The phenomenon of decoherence is essential in the account of the emergence of classicality from quantum behavior, since it explains how interference vanishes in an extremely short decoherence time.

The orthodox explanation of the phenomenon is given by the so-called environment-induced decoherence (EID) approach [1, 2, 3, 4], according to which decoherence is a process resulting from the interaction of an open quantum system and its environment. In particular, by studying different physical models, it is proved that the reduced state \( \rho_S(t) = \text{Tr}_E \rho_{SE}(t) \) of the open system rapidly diagonalizes in a well defined pointer basis, defined case by case but not in general, which identifies the candidates for classical states. The EID approach has been extensively applied to many areas of physics, such as atomic physics, quantum optics and condensed matter. In particular, the study of decoherence has acquired a great importance in quantum computation, where the phenomenon of decoherence represents a major obstacle to the implementation of information processing hardware that takes advantage of superpositions. In spite of its impressive practical success, from a conceptual viewpoint the EID approach still faces a difficulty derived from its open-system perspective: the problem of defining the system that decoheres.

From the einselection view, the split of the Universe into the degrees of freedom that are of direct interest to the observer—the system—and the remaining degrees of freedom—the environment—is absolutely essential for decoherence. However, since the environment may be external (a “bath” of particles interacting with the system of interest), internal (such as collections of phonons or other internal excitations) or a combination of both cases, the EID approach offers no general criterion for deciding where to place the “cut” between system and environment. In many cases, the lack of such a general criterion leads to the need of assuming in advance the observables that will behave classically. For instance, in cosmology the usual strategy consists in splitting the Universe into some degrees of freedom representing the “system” of interest, and the remaining degrees of freedom that are supposed to be non accessible and, therefore, play the role of an internal environment. In quantum field theory, when it is known that the background field follows a simple classical behavior, the scalar field is decomposed according to \( \phi = \phi_\text{b} + \phi_\text{q} \), where the background field \( \phi_\text{b} \) plays the role of the system and the fluctuation field \( \phi_\text{q} \) plays the role of the environment (see [5]). Zurek concedes that this absence of a general criterion to discriminate between system and environment is a serious difficulty of his proposal: “In particular, one issue which has been often taken for granted is looming big, as a foundation of the whole decoherence program. It is the question of what are the “systems” which play such a crucial role in all the discussions of the emergent classicality. This issue was raised earlier, but the progress to date has been slow at best” (see [6], p.122; for a discussion of this point, see [7]).

The main purpose of this paper is to argue that this “looming big” problem is actually a pseudo-problem, which is simply dissolved by the fact that the split of a closed quantum system into an open subsystem and its environment is just a way of selecting a particular space of relevant observables of the whole closed system. But there are many different spaces of relevant observables depending on the observational viewpoint adopted. Therefore, the same closed system can be decomposed in many different ways: each decomposition represents a decision about which degrees of freedom are relevant and which can be disregarded in each case. Since there is no privileged or “essential” decomposition, there is no need of an unequivocal criterion for deciding where to place the cut between “the” open system and “the” environment. On this basis, we will show that the usual picture of decoherence in terms of energy dissipation from the open system to the environment can no longer be sustained. Summing up, decoherence is a phenomenon relative to the relevant observables selected in each particular case. The only essential physical fact is that, among all the observational viewpoints that may be adopted to study a quantum system, some of them
determine subspaces of relevant observables for which the system decoheres.

For the purpose of this argumentation, the paper is organized as follows. In Section II, by means of the concept of tensor product space, it will be shown how the split of a whole closed system into an open system and its environment can be understood as the selection of a space of relevant observables. In Sections III, IV and V, the well-known spin-bath model studied by the EID approach is presented from the perspective of the previous subsection, and physically relevant numerical simulations are obtained. In Sections VI, VII and VIII a generalization of the spin-bath model is presented and solved by computer simulations; this task will allow us to compare the results obtained for different ways of splitting the entire closed system into an open system and its environment. In Section IX, the results obtained in the previous sections are discussed from a conceptual viewpoint in order to argue for the relative nature of decoherence—which dissolves Zurek’s “looming big” problem—and for the rejection of the usual description of decoherence in terms of energy dissipation. Finally, in Section X we draw our conclusions.

II. SELECTING THE RELEVANT OBSERVABLES

As it is well-known in the discussions about irreversibility, whenever a classical or quantum state evolves unitarily, it cannot follow an irreversible evolution. Therefore, if a non-unitary evolution is to be accounted for, a further element has to be added, precisely, the split of the maximal information about the system into a relevant part and an irrelevant part; whereas the irrelevant part is discarded, the relevant part may evolve non-unitarily. This idea can be rephrased in operators language. Since the maximal information about the system is given by the space \( \mathcal{O} \) of all its possible observables, then we restrict that maximal information to a relevant part by selecting a subspace \( \mathcal{O}_R \subset \mathcal{O} \) of relevant observables. The irreversible evolution is the non-unitary evolution viewed from the perspective of those relevant observables.

As emphasized by Omnès (8, 9), decoherence is a particular case of irreversible process. Then, the selection of the subspace \( \mathcal{O}_R \subset \mathcal{O} \) is always required in decoherence. In the case of the EID approach, the selection of \( \mathcal{O}_R \) amounts to the partition of the whole closed system \( U \) into the open system \( S \) and its environment \( E \) (see 10). In fact, let us consider the Hilbert space \( \mathcal{H} \) of the closed system \( U \), \( \mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E \), where \( \mathcal{H}_S \) is the Hilbert space of \( S \) and \( \mathcal{H}_E \) the Hilbert space of \( E \). The corresponding von Neumann-Liouville space of \( U \) is \( \mathcal{L} = \mathcal{H} \otimes \mathcal{H} = \mathcal{L}_S \otimes \mathcal{L}_E = \mathcal{O} \), where \( \mathcal{L}_S = \mathcal{H}_S \otimes \mathcal{H}_S \) and \( \mathcal{L}_E = \mathcal{H}_E \otimes \mathcal{H}_E \). In the EID approach, the relevant observables are those corresponding to the open system \( S \):

\[
\mathcal{O}_R = \mathcal{O}_S \otimes 1_E \subset \mathcal{O}_R \subset \mathcal{O}
\]

where \( \mathcal{O}_S \subset \mathcal{L}_S \) and \( 1_E \) is the identity operator in \( \mathcal{L}_E \). The reduced density operator \( \rho_S(t) \) of \( S \) is defined by tracing over the environmental degrees of freedom,

\[
\rho_S(t) = \text{Tr}_E \rho(t)
\]

The EID approach studies the time-evolution of \( \rho_S(t) \) governed by an effective master equation; it proves that, under certain definite conditions, \( \rho_S(t) \) converges to a stable state \( \rho_{S*} \):

\[
\rho_S(t) \rightarrow \rho_{S*}
\]

But we also know that the expectation value of any \( O_R \in \mathcal{O}_R \) in the state \( \rho(t) \) of \( U \) can be computed as

\[
\langle O_R \rangle_{\rho(t)} = \text{Tr} \rho(t)(O_S \otimes 1_E)) = \text{Tr} \rho_S(t)O_S = \langle O_S \rangle_{\rho_S(t)}
\]

Therefore, the convergence of \( \rho_S(t) \) to \( \rho_{S*} \) implies the convergence of the expectation values:

\[
\langle O_R \rangle_{\rho(t)} = \langle O_S \rangle_{\rho_S(t)} \rightarrow \langle O_S \rangle_{\rho_{S*}} = \langle O_R \rangle_{\rho_{S*}}
\]

where \( \rho_{S*} \) is a final diagonal state of the closed system \( U \), such that \( \rho_{S*} = \text{Tr}_E \rho_{S*} \) (see 2; for details, see 10). This means that, although the off-diagonal terms of \( \rho(t) \) never vanish through the unitary evolution, decoherence obtains because it is a coarse-grained process: the system decoheres from the observational point of view given by any observable belonging to the space \( \mathcal{O}_R \).

When viewed from this closed-system perspective, the discrimination between system and environment turns out to be the selection of the relevant observables. By following papers 11 and 12, we will use the expression ‘tensor product structure’ (TPS) to call any factorization \( \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \) of a Hilbert space \( \mathcal{H} \), defined by the set of observables \( \{O_A \otimes 1_B, 1_A \otimes O_B\} \), such that the eigenbases of the sets \( \{O_A\} \) and \( \{O_B\} \) are bases of \( \mathcal{H}_A \) and \( \mathcal{H}_B \) respectively. If \( \mathcal{H} \) corresponds to a closed system \( U \), the TPS \( \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \) can be viewed as representing the decomposition of \( U \) into
two open systems $S_A$ and $S_B$, corresponding to the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. In turn, given the space $\mathcal{O} = \mathcal{H} \otimes \mathcal{H}$ of the observables of $U$, such a decomposition identifies the spaces $\mathcal{O}_A = \mathcal{H}_A \otimes \mathcal{H}_A$ and $\mathcal{O}_B = \mathcal{H}_B \otimes \mathcal{H}_B$ of the observables of the open systems $S_A$ and $S_B$, such that $\mathcal{O}_A \otimes \mathbb{I}_B \subset \mathcal{O}$ and $\mathbb{I}_A \otimes \mathcal{O}_B \subset \mathcal{O}$. Once these concepts are considered, the selection of the space $\mathcal{O}_R$ of relevant observables in the EID approach turns out to amount to the selection of a particular TPS, $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$, such that $\mathcal{O}_R = \mathcal{O}_S \otimes \mathcal{H}_E \subset \mathcal{O} = \mathcal{H} \otimes \mathcal{H}$.

In this paper we will consider the particular case where the closed system $U$ is composed of $n$ spin-1/2 particles $P_i$, each represented in its Hilbert space $H_i$:

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_n = \bigotimes_{i=1}^{n} \mathcal{H}_i$$

(6)

It is quite clear that the system $U$ can be decomposed into two subsystems $S$ and $E$ in different ways, depending on which particles are considered as the open system $S$. For instance, if the particle $P_1$ is the open system $S$, the corresponding TPS reads

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E = (\mathcal{H}_1) \otimes \left( \bigotimes_{i=2}^{n} \mathcal{H}_i \right)$$

(7)

In turn, if the particle $P_k$, with $1 < k < n$, is viewed as the system $S$, the corresponding TPS is

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E = (\mathcal{H}_k) \otimes \left( \bigotimes_{i \neq k}^{n} \mathcal{H}_i \right)$$

(8)

But we can also define the system $S$ as composed of more than a single particle; for instance, if the particles $P_j$, with $1$ to $m < n$, are the system $S$, the TPS in this case reads

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E = \left( \bigotimes_{j=1}^{m} \mathcal{H}_j \right) \otimes \left( \bigotimes_{i=m+1}^{n} \mathcal{H}_i \right)$$

(9)

In the following sections we will study the phenomenon of decoherence for different partitions of the whole closed system $U$.

**III. THE SPIN-BATH MODEL**

**A. Presentation of the model**

The spin-bath model is a very simple model that has been exactly solved in previous papers (see [1]). Here we will study it from the closed-system perspective presented in the previous section.

Let us consider a closed system $U = P \cup P_1 \cup \ldots \cup P_N = P \cup (\bigcup_{i=1}^{N} P_i)$, where (i) $P$ is a spin-1/2 particle represented in the Hilbert space $\mathcal{H}_P$, and (ii) each $P_i$ is a spin-1/2 particle represented in its Hilbert space $\mathcal{H}_i$. The Hilbert space of the composite system $U$ is, then,

$$\mathcal{H} = \mathcal{H}_P \otimes \left( \bigotimes_{i=1}^{N} \mathcal{H}_i \right)$$

(10)

In the particle $P$, the two eigenstates of the spin operator $S_{P,\overrightarrow{v}}$ in direction $\overrightarrow{v}$ are $|\uparrow\rangle$ and $|\downarrow\rangle$:

$$S_{P,\overrightarrow{v}} |\uparrow\rangle = \frac{1}{2} |\uparrow\rangle \quad S_{P,\overrightarrow{v}} |\downarrow\rangle = -\frac{1}{2} |\downarrow\rangle$$

(11)

In each particle $P_i$, the two eigenstates of the corresponding spin operator $S_{i,\overrightarrow{v}}$ in direction $\overrightarrow{v}$ are $|\uparrow_i\rangle$ and $|\downarrow_i\rangle$:

$$S_{i,\overrightarrow{v}} |\uparrow_i\rangle = \frac{1}{2} |\uparrow_i\rangle \quad S_{i,\overrightarrow{v}} |\downarrow_i\rangle = -\frac{1}{2} |\downarrow_i\rangle$$

(12)
Therefore, a pure initial state of \( U \) reads

\[
|\psi_0\rangle = (a |\uparrow\rangle + b |\downarrow\rangle) \otimes \left( \bigotimes_{i=1}^{N} (\alpha_i |\uparrow_i\rangle + \beta_i |\downarrow_i\rangle) \right)
\]

(13)

where \(|a|^2 + |b|^2 = 1 \) and \(|\alpha_i|^2 + |\beta_i|^2 = 1 \). If the self-Hamiltonians \( H_P \) of \( P \) and \( H_i \) of \( P_i \) are taken to be zero, and there is no interaction among the \( P_i \), then the total Hamiltonian \( H \) of the composite system \( U \) is given by the interaction between the particle \( P \) and each particle \( P_i \) (see \([3], [13]\)):

\[
H = \frac{1}{2} \left( |\uparrow\rangle \langle |\uparrow| - |\downarrow\rangle \langle |\downarrow| \right) \otimes \sum_{i=1}^{N} g_i \left( |\uparrow_i\rangle \langle |\uparrow_i| - |\downarrow_i\rangle \langle |\downarrow_i| \right) \otimes \left( \bigotimes_{j \neq i} \mathbb{I}_j \right)
\]

(14)

where \( \mathbb{I}_j = |\uparrow_j\rangle \langle |\uparrow_j| + |\downarrow_j\rangle \langle |\downarrow_j| \) is the identity operator on the subspace \( \mathcal{H}_j \). Under the action of \( H \), the state \(|\psi_0\rangle\) evolves into

\[
|\psi(t)\rangle = a |\uparrow\rangle |\mathcal{E}_\uparrow(t)\rangle + b |\downarrow\rangle |\mathcal{E}_\downarrow(t)\rangle
\]

(15)

where

\[
|\mathcal{E}_\uparrow(t)\rangle = \bigotimes_{i=1}^{N} \left( \alpha_i e^{-ig_i t/2} |\uparrow_i\rangle + \beta_i e^{ig_i t/2} |\downarrow_i\rangle \right)
\]

(16)

**B. Computing the expectation values**

The space \( \mathcal{O} \) of the observables of the composite system \( U \) can be obtained as \( \mathcal{O} = \mathcal{O}_P \otimes (\otimes_{i=1}^{N} \mathcal{O}_i) \), where \( \mathcal{O}_P \) is the space of the observables of the particle \( P \) and \( \mathcal{O}_i \) is the space of the observables of the particle \( P_i \). Then, an observable \( O \in \mathcal{O} = \mathcal{H} \otimes \mathcal{H} \) can be expressed as

\[
\mathcal{O} = \mathcal{O}_P \otimes \left( \bigotimes_{i=1}^{N} \mathcal{O}_i \right)
\]

(17)

where

\[
\mathcal{O}_P = s_{\uparrow \uparrow} |\uparrow\rangle \langle |\uparrow| + s_{\downarrow \downarrow} |\downarrow\rangle \langle |\downarrow| + s_{\uparrow \downarrow} |\uparrow\rangle \langle |\downarrow| + s_{\downarrow \uparrow} |\downarrow\rangle \langle |\uparrow| \in \mathcal{O}_P
\]

(18)

\[
\mathcal{O}_i = \epsilon^{(i)}_{\uparrow \uparrow} |\uparrow_i\rangle \langle |\uparrow_i| + \epsilon^{(i)}_{\downarrow \downarrow} |\downarrow_i\rangle \langle |\downarrow_i| + \epsilon^{(i)}_{\uparrow \downarrow} |\uparrow_i\rangle \langle |\downarrow_i| \in \mathcal{O}_i
\]

(19)

Since the operators \( \mathcal{O}_P \) and \( \mathcal{O}_i \) are Hermitian, the diagonal components \( s_{\uparrow \uparrow}, s_{\downarrow \downarrow}, \epsilon^{(i)}_{\uparrow \uparrow}, \epsilon^{(i)}_{\downarrow \downarrow} \) are real numbers, and the off-diagonal components are complex numbers satisfying \( s_{\uparrow \downarrow} = s_{\downarrow \uparrow}^*, \epsilon^{(i)}_{\uparrow \downarrow} = \epsilon^{(i)*}_{\downarrow \uparrow} \). Then, the expectation value of the observable \( O \) in the state \(|\psi(t)\rangle\) of eq. (15) can be computed as

\[
\langle O \rangle_{\psi(t)} = (|a|^2 s_{\uparrow \uparrow} + |b|^2 s_{\downarrow \downarrow}) \Gamma_0(t) + 2 Re [ab^* s_{\uparrow \downarrow} \Gamma_1(t)]
\]

(20)

where (see [13])

\[
\Gamma_0(t) = \prod_{i=1}^{N} \left[ |\alpha_i|^2 \epsilon^{(i)}_{\uparrow \uparrow} + |\beta_i|^2 \epsilon^{(i)}_{\downarrow \downarrow} + 2 Re (\alpha_i \beta_i^* \epsilon^{(i)}_{\uparrow \downarrow} e^{ig_i t}) \right]
\]

(21)

\[
\Gamma_1(t) = \prod_{i=1}^{N} \left[ |\alpha_i|^2 \epsilon^{(i)}_{\downarrow \uparrow} e^{ig_i t} + |\beta_i|^2 \epsilon^{(i)}_{\downarrow \downarrow} e^{-ig_i t} + 2 Re (\alpha_i \beta_i^* \epsilon^{(i)}_{\uparrow \downarrow}) \right]
\]

(22)

By contrast to the usual presentations, we will study two different decompositions of the whole closed system \( U \) into a relevant part and its environment.
IV. THE SPIN-BATH MODEL: DECOMPOSITION 1

A. Selecting the relevant observables

In the typical situation studied by the EID approach, the open system \( S \) is the particle \( P \), and the remaining particles \( P_i \) play the role of the environment \( E \): \( S = P \) and \( E = \cup_{i=1}^{N} P_i \). Then, the TPS for this case is

\[
\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E = (\mathcal{H}_P) \otimes \left( \bigotimes_{i=1}^{N} \mathcal{H}_i \right)
\]

Therefore, the relevant observables \( O_R \) of the closed system \( U \) are those corresponding to the particle \( P \), and they are obtained from eqs. (17), (18) and (19), by making \( \epsilon_{i1}^{(i)} = \epsilon_{i1}^{(i)} = 1 \) and \( \epsilon_{i1}^{(i)} = 0 \):

\[
O_R = O_S \otimes \mathbb{I}_E = \left( \sum_{s, s' = 0, \neq} s_{ss'} \left| s \right\rangle \langle s' \right\rangle \right) \otimes \left( \bigotimes_{i=1}^{N} \mathbb{1}_i \right)
\]

The expectation value of these observables in the state \( \left| \psi(t) \right\rangle \) of eq. (15) is given by

\[
\langle O_R \rangle_{\psi(t)} = |a|^2 s_{\psi \psi} + |b|^2 s_{\psi \psi} + 2 \text{Re}[ab^* s_{\psi \psi} r(t)]
\]

where

\[
r(t) = \langle E_{\psi}(t) | E_{\psi}(t) \rangle = \prod_{i=1}^{N} \left( |\alpha_i|^2 e^{-ig_it} + |\beta_i|^2 e^{ig_it} \right)
\]

and, then,

\[
|r(t)|^2 = \prod_{i=1}^{N} \left( |\alpha_i|^4 + |\beta_i|^4 + 2|\alpha_i|^2|\beta_i|^2 \cos 2g_it \right)
\]

This means that, in eq. (20), \( \Gamma_0(t) = 1 \) and \( \Gamma_1(t) = r(t) \).

B. Computing the behavior of the relevant expectation values

In order to know the time-behavior of the expectation value of eq. (25), we have to compute the time-behavior of \( r(t) \). If we take \( |\alpha_i|^2 \) and \( |\beta_i|^2 \) as random numbers in the closed interval \([0, 1]\), such that \( |\alpha_i|^2 + |\beta_i|^2 = 1 \), then

\[
\max_t (|\alpha_i|^4 + |\beta_i|^4 + 2|\alpha_i|^2|\beta_i|^2 \cos 2g_it) = \left( (|\alpha_i|^2 + |\beta_i|^2)^2 \right) = 1
\]

\[
\min_t (|\alpha_i|^4 + |\beta_i|^4 + 2|\alpha_i|^2|\beta_i|^2 \cos (2g_it)) = \left( (|\alpha_i|^2 - |\beta_i|^2)^2 \right) = \left( 2|\alpha_i|^2 - 1 \right)^2
\]

Therefore, \( (|\alpha_i|^4 + |\beta_i|^4 + 2|\alpha_i|^2|\beta_i|^2 \cos 2g_it) \) is a random number which, if \( t \neq 0 \), fluctuates between 1 and \( \left( 2|\alpha_i|^2 - 1 \right)^2 \). Let us notice that, when the environment has many particles (that is, when \( N \to \infty \)), the statistical value of the cases \( |\alpha_i|^2 = 1, |\beta_i|^2 = 1, |\alpha_i|^2 = 0 \) and \( |\beta_i|^2 = 0 \) tends to zero. In this situation, eq. (27) for \( |r(t)|^2 \) is an infinite product of numbers belonging to the open interval \((0, 1)\); as a consequence,

\[
\lim_{N \to \infty} r(t) = 0
\]

If we know that, for \( N \to \infty, r(t) = 0 \) for any \( t \neq 0 \) and \( r(0) = 1 \), it can be expected that, for \( N \) finite, \( r(t) \) will evolve in time from \( r(0) = 1 \) to a very small value. In order to obtain the time-behavior of \( r(t) \), different numerical simulations have been performed, where the random \( |\alpha_i|^2 \) were obtained from a random-number generator, and the \( |\beta_i|^2 \) were computed as \( |\beta_i|^2 = 1 - |\alpha_i|^2 \). The value of the \( g_i \) and the time-interval \([0, t_0]\) for the computations were stipulated. The time-interval \([0, t_0]\) was partitioned into intervals \( \Delta t = t_0/200 \), and the function \( |r(t)|^2 \) was computed at times \( t_k = k\Delta t \), with \( k = 0, 1, ..., 200 \), according to eq. (27).
Simulation (a): The computations were performed with $N = 10^7$, $N = 10^8$ and $N = 10^9$. All the $g_i$ were taken to have the same value. The value $g_i = 400\, \text{Hz}$ was selected as a reference value on the basis of the measurement of the coupling constant in typical models of spin interaction ([14]). Figures 1, 2 and 3 show the time-evolution of $|r(t)|^2$ with $g_i = 200\, \text{Hz}$, $g_i = 400\, \text{Hz}$ and $g_i = 800\, \text{Hz}$, respectively.

In these figures we can see that, as expected, (i) for any given value of $g_i$, decoherence is faster as $N$ increases, and (ii) decoherence is faster as $g_i$ increases, that is, as the interaction between the particle $P$ and each particle $P_i$ is stronger.

Simulation (b): Again, the computations of $|r(t)|^2$ were performed with $N = 10^7$, $N = 10^8$ and $N = 10^9$, but now the values of the $g_i$ were obtained from a random-number generator in the interval $[0, 800\, \text{Hz}]$. The results are shown in Figure 4 for a given distribution of the values of the $g_i$ (since the plots obtained for different distributions were indistinguishable).

Again, the figure shows that decoherence is faster as $N$ increases. Moreover, if we compare Figure 4 ($g_i \in [0, 800\, \text{Hz}]$) with Figure 2 ($g_i = 400\, \text{Hz}$), we can see that the random character of the $g_i$ improves the “efficiency” of decoherence: the decoherence time in the case of random $g_i$ is shorter than in the case of constant $g_i$.

Simulation (c): In order to obtain a physically meaningful value of the decoherence time, we performed the computation for $N = 10^{20}$ (closer to the Avogadro number) with the following strategy:

- All the $g_i$ were taken to have the same value, under the reasonable assumption that the particles of the environment are all of the same nature and, therefore, all of them interact with the particle $P$ in the same way. As explained above, the value of $g_i = 400\, \text{Hz}$ was selected on the basis of the typical interaction between spins.

- The upper limit $t_0$ of the time interval $[0, t_0]$ was taken as $2 \times 10^{-8} \, \text{s}$, in order to show the physical dynamics of the phenomenon.
- Of course, the running time of the computing process for $N = 10^{20}$ is unattainable. Nevertheless, different runs of the computing process (with different random values of the $|\alpha_i|^2$ and the $|\beta_i|^2$) show that, for $N = 10^{10}$, the resulting plots are completely indistinguishable. Therefore, the result for $N = 10^{20}$ can legitimately be computed by multiplying the result obtained for $N = 10^{10}$ by itself $10^{10}$ times.

It is interesting to see how the decoherence time decreases as $N$ increases. For this purpose, we have used the strategy of multiplying $|r(t)|^2$ corresponding to $N = 10^{10}$ by itself $10^a$ times: $(|r(t)|^2)^{10^a}$ corresponds to the number of particles $N = 10^{10+a}$. Figure 5 shows the time-evolution of $(|r(t)|^2)^{10^a}$, corresponding to $N = 10^{10+a}$, for $a = 0, 1, 2$ and 3.

The decoherence time was obtained by fitting the curve for $N = 10^{20}$ to an exponential, and by computing the characteristic time of the exponential. The decoherence time so obtained was $t_D = 10^{-13} \text{s}$, as empirically measured (see [14]). Let us notice in eq. (27) that $|r(t)|^2$ comes back to its initial value when $2g_i t = 2\pi$; then, the recurrence–Poincaré time is $t_P = \pi/g_i$. Although there is no strict final relaxation due to the discrete nature of the model, the relaxation time can legitimately be taken as $t_R \leq t_P/2 \sim 10^{-3} \text{s}$. This means that, as expected, the decoherence time is many orders of magnitude shorter than the relaxation time.

V. THE SPIN-BATH MODEL: DECOMPOSITION 2

A. Selecting the relevant observables

Although in the usual presentations of the model the open system of interest is $P$, we can conceive different ways of splitting the whole closed system $U$ into an open system $S$ and its environment $E$. For instance, we can decide to observe a particular particle $P_j$ of what was previously considered the environment, and to consider the remaining particles as the new environment, in such a way that $S = P_j$ and $E = P \cup (\cup_{i=1,i\neq j}^N P_i)$. The total Hilbert space of
the closed composite system $U$ is still given by eq. (10), but in this case the corresponding TPS is

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E = (\mathcal{H}_j) \otimes \left( \mathcal{H}_P \otimes \left( \bigotimes_{i=1, i \neq j}^N \mathcal{H}_i \right) \right)$$

and the relevant observables $O_R$ of the closed system $U$ are those corresponding to the particle $P_j$:

$$O_R = O_S \otimes I_E = O_P \otimes \left( I_P \otimes \left( \bigotimes_{i=1, i \neq j}^N I_i \right) \right)$$

where (see eq. (19))

$$O_{P_j} = \epsilon_{11}^{(j)} | \uparrow \rangle \langle \uparrow | + \epsilon_{11}^{(j)} | \downarrow \rangle \langle \downarrow | + \epsilon_{11}^{(j)} | \uparrow \rangle \langle \downarrow | + \epsilon_{11}^{(j)} | \downarrow \rangle \langle \uparrow |$$

$I_P$ is the identity operator on the subspace $\mathcal{H}_P$, and the coefficients $\epsilon_{11}^{(j)}$, $\epsilon_{11}^{(j)}$, $\epsilon_{11}^{(j)}$ are now generic. The expectation value of the observables $O_R$ in the state $|\psi(t)\rangle$ of eq. (15) is given by

$$\langle O_R | \psi(t) \rangle = \langle \psi(t) | O_{R_j} | \psi(t) \rangle = |\alpha_j|^2 \epsilon_{11}^{(j)} + |\beta_j|^2 \epsilon_{11}^{(j)} + 2 \text{Re} \left( \alpha_j \beta_j^* \epsilon_{11}^{(j)} e^{igt} \right)$$
B. Computing the behavior of the relevant expectation values

In order to know the time-behavior of the expectation value of the \( O_R \), we have to compute the time-behavior of the third term of eq. (33), which can be rewritten as

\[
2 \text{Re} \left( \alpha_j \beta_j^* \epsilon_{\downarrow \uparrow}^{(j)} e^{i g_j t} \right) = 2 \left| \alpha_j \beta_j^* \epsilon_{\downarrow \uparrow}^{(j)} \right| \cos \left( g_j t + \arg \left( \alpha_j \beta_j^* \epsilon_{\downarrow \uparrow}^{(j)} \right) \right)
\]  

(34)

In this case, a numerical simulation is not necessary to see that eq. (34) is an oscillating function which, as a consequence, has no limit for \( t \to \infty \). This result is not surprising, but completely reasonable from a physical point of view. In fact, with the exception of the particle \( P \), the remaining particles of the environment \( E \) are uncoupled to each other: each \( P_i \) evolves as a free system and, for this reason, \( E \) is unable to reach a final stable state.

VI. A GENERALIZED SPIN-BATH MODEL

A. Presentation of the model

Let us consider a closed system \( U = A \cup B \) where:

(i) The subsystem \( A \) is composed of \( M \) spin-1/2 particles \( A_i \), with \( i = 1, 2, ..., M \), each one of them represented in its Hilbert space \( \mathcal{H}_{A_i} \). In each \( A_i \), the two eigenstates of the spin operator \( S_{A,i} \) in direction \( \vec{v} \) are \( |\uparrow_i\rangle \) and \( |\downarrow_i\rangle \):

\[
S_{A_i} |\uparrow_i\rangle = \frac{1}{2} |\uparrow_i\rangle \quad S_{A_i} |\downarrow_i\rangle = -\frac{1}{2} |\downarrow_i\rangle
\]  

(35)
FIG. 5: Figure 5: Evolution of $|r(t)|^2$ for $g_i = 400\text{Hz}$ and $a = 0$ (solid line), $a = 1$ (dot line), $a = 2$ (dash line), $a = 3$ (dot-dash line), with $t_0 = 2.10^{-8}\text{s}$.

The Hilbert space of $A$ is $\mathcal{H}_A = \bigotimes_{i=1}^M \mathcal{H}_{A_i}$. Then, a pure initial state of $A$ reads

$$|\psi_A\rangle = \bigotimes_{i=1}^M (a_i |\uparrow_i\rangle + b_i |\downarrow_i\rangle), \text{ with } |a_i|^2 + |b_i|^2 = 1$$

(ii) The subsystem $B$ is composed of $N$ spin-1/2 particles $B_k$, with $k = 1, 2, ..., N$, each one of them represented in its Hilbert space $\mathcal{H}_{B_k}$. In each $B_k$, the two eigenstates of the spin operator $S_{B_k, \overrightarrow{v}}$ in direction $\overrightarrow{v}$ are $|\uparrow_k\rangle$ and $|\downarrow_k\rangle$:

$$S_{B_k, \overrightarrow{v}} |\uparrow_k\rangle = \frac{1}{2} |\uparrow_k\rangle \quad S_{B_k, \overrightarrow{v}} |\downarrow_k\rangle = -\frac{1}{2} |\downarrow_k\rangle$$

The Hilbert space of $B$ is $\mathcal{H}_B = \bigotimes_{k=1}^N \mathcal{H}_{B_k}$. Then, a pure initial state of $B$ reads

$$|\psi_B\rangle = \bigotimes_{k=1}^N (\alpha_k |\uparrow_k\rangle + \beta_k |\downarrow_k\rangle), \text{ with } |\alpha_k|^2 + |\beta_k|^2 = 1$$

The Hilbert space of the composite system $U = A \cup B$ is, then,

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B = \left( \bigotimes_{i=1}^M \mathcal{H}_{A_i} \right) \otimes \left( \bigotimes_{k=1}^N \mathcal{H}_{B_k} \right)$$
composite system
A
interaction among the particles
B
(white circles): (a) original spin-bath model (\(M = 1\)), and (b) generalized spin-bath model (\(M \neq 1\)).

Therefore, from eqs. (36) and (38), a pure initial state of \(U\) reads

\[
|\psi_0\rangle = |\psi_A\rangle \otimes |\psi_B\rangle = \left( \bigotimes_{i=1}^{M} (a_i |\uparrow_i\rangle + b_i |\downarrow_i\rangle) \right) \otimes \left( \bigotimes_{k=1}^{N} (\alpha_k |\uparrow_k\rangle + \beta_k |\downarrow_k\rangle) \right)
\]

(40)

As in the original spin-bath model, the self-Hamiltonians \(H_{A_i}\) and \(H_{B_k}\) are taken to be zero. In turn, there is no interaction among the particles \(A_i\) nor among the particles \(B_k\). As a consequence, the total Hamiltonian \(H\) of the composite system \(U\) is given by (see eq. (14))

\[
H = H_A \otimes H_B = \left( \sum_{i=1}^{M} \left[ \frac{1}{2} (|\uparrow_i\rangle \langle \uparrow_i| - |\downarrow_i\rangle \langle \downarrow_i|) \otimes \left( \bigotimes_{j \neq i}^{M} I_{A_j} \right) \right] \right) \otimes \left( \sum_{k=1}^{N} \left[ g_k (|\uparrow_k\rangle \langle \uparrow_k| - |\downarrow_k\rangle \langle \downarrow_k|) \otimes \left( \bigotimes_{l \neq k}^{N} I_{B_l} \right) \right] \right)
\]

(41)

where \(I_{A_i} = |\uparrow_i\rangle \langle \uparrow_i| + |\downarrow_i\rangle \langle \downarrow_i|\) is the identity on the subspace \(H_{A_j}\) and \(I_{B_l} = |\uparrow_l\rangle \langle \uparrow_l| + |\downarrow_l\rangle \langle \downarrow_l|\) is the identity on the subspace \(H_{B_l}\). Let us notice that the eq. (14) of the original model is the particular case of eq. (41) for \(M = 1\).

This Hamiltonian describes a situation where the particles of \(A\) do not interact to each other, the same holds for the particles of \(B\), but each particle of \(A\) interacts with all the particles of \(B\) and vice versa, as shown in Figure 6.

In eq. (41), \(H\) is written in its diagonal form; then, the energy eigenvectors are

\[
|\uparrow_1 ... \uparrow_i ... \uparrow_{M-1} |\uparrow_M\rangle |\uparrow_1 ... \uparrow_k ... \uparrow_{N-1} \rangle |\downarrow_N\rangle
\]

\[
|\uparrow_1 ... \uparrow_i ... \uparrow_{M-1} |\downarrow_M\rangle |\uparrow_1 ... \uparrow_k ... \uparrow_{N-1} \rangle |\downarrow_N\rangle
\]

\[
... |\uparrow_j ... \uparrow_i ... \uparrow_{M-1} |\downarrow_M\rangle |\downarrow_1 ... \downarrow_k ... \downarrow_{N-1} \rangle |\uparrow_N\rangle
\]

(42)

In turn, the eigenvectors of \(H_A\) form a basis of \(H_A\). In order to simplify the expressions, we will introduce a particular arrangement into the set of those vectors, by calling them \(|\mathcal{A}_i\rangle\): the set \(\{ |\mathcal{A}_i\rangle \}\) is an eigenbasis of \(H_A\) with \(2^M\) elements. The \(|\mathcal{A}_i\rangle\) will be ordered in terms of the number \(l \in \mathbb{N}_0\) of particles of \(A\) having spin \(|\uparrow\rangle\). Then, we have that:

- \(l = 0\) corresponds to the unique state with all the particles with spin \(|\uparrow\rangle\):

\[
|\mathcal{A}_1\rangle = |\uparrow, \uparrow, ..., \uparrow, \uparrow\rangle \Rightarrow H_A |\mathcal{A}_1\rangle = \frac{M}{2} |\mathcal{A}_1\rangle
\]

(43)

- \(l = 1\) corresponds to the \(M\) states with only one particle with spin \(|\downarrow\rangle\). Since the order of the eigenvectors with the same eigenvalue will be irrelevant for the computations, we will order these states in an arbitrary way:

\[
|\mathcal{A}_j\rangle = |\uparrow, \uparrow, ..., \uparrow, \downarrow, \uparrow, ..., \uparrow\rangle \Rightarrow H_A |\mathcal{A}_j\rangle = \frac{M - 2}{2} |\mathcal{A}_j\rangle
\]

with \(j = 2, 3, ..., M + 1\)

(44)
\[ \cdot l = 2 \text{ corresponds to the } \frac{(M-1)M}{2} \text{ states with two particles with spin } |\downarrow\rangle. \text{ Again, we will order these states in an arbitrary way:} \]
\[ |A_j⟩ = |⇑, ⇑, ⋯, ⇑, \downarrow, ⇑, ⋯, ⇑, \downarrow, ⇑, ⋯, ⇑⟩ \implies H_A |A_j⟩ = \frac{M-4}{2} |A_j⟩ \]
with \[ j = M + 2, M + 3, ⋯, M + 1 + \frac{(M-1)M}{2} \] (45)

\[ \cdot \text{ For the remaining values of } l, \text{ the procedure is analogous.} \]

Consequently, we have:

- 1 eigenvector with eigenvalue \( \frac{M}{2} \)
- \( M \) eigenvectors with eigenvalue \( \frac{M-2}{2} \)
- \( \vdots \)
- \( \frac{M!}{(M-l)!!} \) eigenvectors with eigenvalue \( \frac{M-2l}{2} \) (46)

with \( l = 0, 1, \ldots M \). Then, it is clear that \( H_A \) is degenerate: it has \( 2^M \) eigenvectors but only \( M \) different eigenvalues. Therefore, a generic state \( |A⟩ \) of the system \( A \) can be written in the basis \( \{|A_i⟩\} \) as
\[ |A⟩ = \sum_{i=1}^{2^M} C_i |A_i⟩ \in H_A \quad \text{with} \quad \sum_{i=1}^{2^M} |C_i|^2 = 1 \] (47)

By introducing eq. (47) into eq. (40), a pure initial state of the composite system \( U = A \cup B \) reads
\[ |ψ_0⟩ = \left( \sum_{i=1}^{2^M} C_i |A_i⟩ \right) \otimes \left( \bigotimes_{k=1}^{N} (α_k |⇑_k⟩ + β_k |⇓_k⟩) \right) \] (48)

If we group the degrees of freedom of \( B \) in a single ket \( |B(0)⟩ \), \( |ψ_0⟩ \) results
\[ |ψ_0⟩ = \sum_{i=1}^{2^M} C_i |A_i⟩ \otimes |B(0)⟩ \] (49)

The time-evolution of \( |ψ(t)⟩ \) is ruled by the time-evolution operator \( U(t) = e^{-iHt} = e^{-i(H_A ⊗ H_B)t} \):
\[ |ψ(t)⟩ = U(t)|ψ_0⟩ = \sum_{i=1}^{2^M} C_i e^{-i(H_A ⊗ H_B)t} |A_i⟩ \otimes |B(0)⟩ = \sum_{i=1}^{2^M} C_i e^{-iH_A t} |A_i⟩ \otimes e^{-iH_B t} |B(0)⟩ \] (50)

If we use \( Λ_k \) to denote the eigenvalue of \( H_A \) corresponding to the eigenvector \( |A_k⟩ \), then
\[ |ψ(t)⟩ = \sum_{i=1}^{2^M} C_i |A_i⟩ \otimes e^{-iΛ_k H_B t} |B(0)⟩ = \sum_{i=1}^{2^M} C_i |A_i⟩ \otimes |B(t)⟩ \] (51)

where (see eq. (41))
\[ |B(t)⟩ = e^{-iΛ_k H_B t} |B(0)⟩ = \exp \left[ -i Λ_k \sum_{j=1}^{N} g_j (|↑_j⟩⟨↑_j| - |↓_j⟩⟨↓_j|) t \right] |B(0)⟩ \] (52)
Since the number of the eigenstates of $H_A$ with the same eigenvalue is given by eqs. (40), the terms of $|\psi(t)\rangle$ can be arranged as

$$
|\psi(t)\rangle = (C_1 \, |A_1\rangle \, |B_0(t)\rangle) + \left( \sum_{\lambda=1}^{M+1} C_\lambda \, |A_\lambda\rangle \, |B_1(t)\rangle \right) + \left( \sum_{\lambda=M+2}^{M+1+(M-1)M} C_\lambda \, |A_\lambda\rangle \, |B_2(t)\rangle \right) + ... + \left( \sum_{\lambda=1+\sum_{p=0}^{l-1} (\rho^p)} \sum_{\lambda=1+\sum_{p=0}^{l-1} (\rho^p)} C_\lambda \, |A_\lambda\rangle \, |B_l(t)\rangle \right) + \ldots (C_{2M} \, |A_{2M}\rangle \, |B_{M}(t)\rangle)
$$

(53)

where

$$
|B_l(t)\rangle = \bigotimes_{k=1}^{N} \left( \alpha_k e^{i(2\pi/M)g_k t} \, |\uparrow_k\rangle + \beta_k e^{-i(2\pi/M)g_k t} \, |\downarrow_k\rangle \right)
$$

(54)

If we compare eq. (54) with eq. (16), we can see that $|E_\phi\rangle$ and $|E_\psi\rangle$ are the particular cases of $|B_l(t)\rangle$ for $M = 1$ and, then, $l = 0, 1$. Let us recall that $l$ is the number of particles of the system $A$ having spin $|\psi\rangle$. Then, with $M = 1$ and $l = 0$, $|B_1(t)\rangle = |E_\phi(t)\rangle$, and with $M = 1$ and $l = 1$, $|B_1(t)\rangle = |E_\psi(t)\rangle$.

If we define the function

$$
f(l) = \begin{cases} 
\sum_{p=0}^{l} (\rho^p) & \text{if } l = 0, 1, \ldots, M \\
0 & \text{otherwise}
\end{cases}
$$

(55)

then eq. (53) can be rewritten as

$$
|\psi(t)\rangle = \sum_{l=0}^{M} \sum_{\lambda=f(l-1)+1}^{f(l)} C_\lambda \, |A_\lambda\rangle \, |B_l(t)\rangle
$$

(56)

and the state operator $\rho(t) = |\psi(t)\rangle \langle \psi(t)|$ reads

$$
\rho(t) = \sum_{l,
\lambda, l', \lambda'}^{M} \sum_{f(l-1)+1}^{f(l)} C_\lambda C_{\lambda'}^* \, |A_\lambda\rangle \, |B_l(t)\rangle \, \langle B_{l'}(t)| \, \langle A_{\lambda'}|
$$

(57)

B. Computing the expectation values

An observable $O \in \mathcal{O} = \mathcal{H} \otimes \mathcal{H}$ of the closed system $U = A \cup B$ can be expressed as

$$
O = \left( \sum_{\lambda, \lambda'}^{2M} s_{\lambda, \lambda'} \, |A_\lambda\rangle \, \langle A_{\lambda'}| \right) \otimes \left( \bigotimes_{i=1}^{N} \left( \epsilon_{11}^{(i)} |\uparrow_i\rangle \langle \uparrow_i| + \epsilon_{11}^{(i)} |\downarrow_i\rangle \langle \downarrow_i| + \epsilon_{11}^{(i)} |\uparrow_i\rangle \langle \downarrow_i| + \epsilon_{11}^{(i)} |\downarrow_i\rangle \langle \uparrow_i| \right) \right)
$$

(58)

Let us notice that eq. (57) (a generic observable in the original spin-bath model) is a particular case of this eq. (58), with only four terms in the first factor. Analogously to that case, the diagonal components $s_{\lambda, \lambda}$, $\epsilon_{11}^{(i)}$ are real numbers, and the off-diagonal components are complex numbers satisfying $s_{\lambda, \lambda'} = s_{\lambda', \lambda}^*$, $\epsilon_{11}^{(i)} = \epsilon_{11}^{(i)*}$. Then, the expectation value of the observable $O$ in the state $\rho(t)$ of eq. (57) can be computed as

$$
\langle O \rangle_{\rho(t)} = Tr (O \rho(t)) = \sum_{l, l', \lambda, \lambda'}^{M} \sum_{f(l-1)+1}^{f(l')} B_{\lambda, \lambda'} T_{l,l'}(t)
$$

(59)

where

$$
T_{l,l'}(t) = \prod_{j=1}^{N} \left[ |\alpha_j|^2 \, \epsilon_{11}^{(j)} \, e^{i(g_j, l-g_j, l')} + |\beta_j|^2 \, \epsilon_{11}^{(j)} \, e^{-i(g_j, l-g_j, l')} + 2 \text{Re} \left( \alpha_j \beta_j^* \epsilon_{11}^{(j)} \, e^{i(g_j, l+g_j, l')} \right) \right]
$$

(60)
and
\[ g_{j,t} = (2l - M) g_j, \quad B_{\lambda,\lambda'} = C_{\lambda} C_{\lambda'}^* s_{\lambda,\lambda'} \] (61)

Since the exponents in eq. (60) are of the form \( g_{j,t} \pm g_{j,t'} \), in some cases they are zero. So, we can write
\[ \langle O \rangle_{\rho(t)} = \sum_{l=0}^{M} \sum_{\lambda = f(l-1) + 1}^{M} B_{\lambda,\lambda'} T_{l,0}(t) + \sum_{l=0}^{M} \sum_{\lambda = f(l-1) + 1}^{M} B_{\lambda,\lambda'} 2 \text{Re} \left( T_{l,M-1}(t) \right) + \sum_{l=0}^{M} \sum_{\lambda = f(l-1) + 1}^{M} B_{\lambda,\lambda'} T_{l,t'}(t) \] (62)

where
\[ \tilde{M} = \begin{cases} \frac{M-2}{2} & \text{if } M \text{ is even} \\ \frac{M-1}{2} & \text{if } M \text{ is odd} \end{cases} \] (63)

\[ T_{l,t}(t) = \prod_{j=1}^{N} \left[ |\alpha_j|^2 \epsilon_{11}^{(j)} + |\beta_j|^2 \epsilon_{11}^{(j)} + 2 \text{Re} \left( \alpha_j \beta_j^* \epsilon_{11}^{(j)} e^{ig_{j,1}t} \right) \right] \] (64)

\[ T_{l,M-1}(t) = \prod_{j=1}^{N} \left[ |\alpha_j|^2 \epsilon_{11}^{(j)} e^{ig_{j,1}t} + |\beta_j|^2 \epsilon_{11}^{(j)} e^{-ig_{j,1}t} + 2 \text{Re} \left( \alpha_j \beta_j^* \epsilon_{11}^{(j)} \right) \right] \] (65)

Let us notice that eqs. (64) and (65) are analogous to eqs. (21) and (22) for \( \Gamma_0(t) \) and \( \Gamma_1(t) \), respectively, in the original model, with \( g_{j,t} = (2l - M) g_j \) instead of \( g_j \). In particular, when \( M = 1 \) and, so, \( l = 0, 1 \), then \( T_{l,0}(t) = \Gamma_0(t) \) and \( T_{l,1}(t) = \Gamma_1(t) \).

As in the case of the original spin-bath model, here we will consider different meaningful ways of selecting the relevant observables.

### VII. GENERALIZED SPIN-BATH MODEL: DECOMPOSITION 1

#### A. Selecting the relevant observables

In this case \( A \) is the open system \( S \) and \( B \) is the environment \( E \). This is a generalization of Decomposition 1 in the original spin-bath model. The only difference with respect to that case is that here the system \( S \) is composed of \( M \geq 1 \) particles instead of only one. Then, the TPS for this case is
\[ \mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E = \left( \bigotimes_{i=1}^{M} \mathcal{H}_{A_i} \right) \otimes \left( \bigotimes_{k=1}^{N} \mathcal{H}_{B_k} \right) \] (66)

Therefore, the relevant observables \( O_R \) of the closed system \( U \) are those corresponding to \( A \), and they are obtained from eq. (58) by making \( \epsilon_{11}^{(i)} = \epsilon_{11}^{(i)} = 1, \epsilon_{11}^{(i)} = 0 \) (compare with eq. (21) in the original spin-bath model):
\[ O_R = O_S \otimes I_E = \left( \sum_{\lambda,\lambda' = 0}^{2M} s_{\lambda,\lambda'} |A_{\lambda}\rangle \langle A_{\lambda'}| \right) \otimes \left( \bigotimes_{i=1}^{N} I_i \right) \] (67)

With this condition, the expectation values of these observables are given by eq. (62), with
\[ T_{l,t}(t) = \prod_{j=1}^{N} \left( |\alpha_j|^2 + |\beta_j|^2 \right) = 1 \] (68)
\[ T_{l,M-1}(t) = \prod_{j=1}^{N} \left( |\alpha_j|^2 e^{ig_{j,1}t} + |\beta_j|^2 e^{-ig_{j,1}t} \right) \] (69)
\[ T_{l,t'}(t) = \prod_{j=1}^{N} \left( |\alpha_j|^2 e^{ig_{j,1}t} + |\beta_j|^2 e^{-ig_{j,1}t} \right) \] (70)
If we define the functions $R_l(t) = |T_{l,M-l}(t)|^2$ and $R_{ll'}(t) = |T_{l,l'}(t)|^2$, they result

$$R_l(t) = \prod_{j=1}^{N} \left( |\alpha_j|^4 + |\beta_j|^4 + 2 |\alpha_j|^2 |\beta_j|^2 \cos (2 (2l - M) g_j t) \right)$$

(71)

$$R_{ll'}(t) = \prod_{j=1}^{N} \left( |\alpha_j|^4 + |\beta_j|^4 + 2 |\alpha_j|^2 |\beta_j|^2 \cos (2 (l - l') g_j t) \right)$$

(72)

We can see that $|r(t)|^2$ of eq. (27) in the original model is the particular case of $R_l(t)$ for $M = 1$.

**B. Computing the behavior of the relevant expectation values**

The expectation value given by eq. (62) has three terms, $\langle O_R \rangle_{\rho(t)} = \Sigma^{(1)} + \Sigma^{(2)} + \Sigma^{(3)}$, which can be analyzed separately:

- From eq. (68), the first term reads

$$\Sigma^{(1)} = \sum_{l=0}^{M} \sum_{\lambda, \lambda' = f(l-1)+1} f(l) B_{\lambda,\lambda'} = \sum_{l=0}^{M} \sum_{\lambda, \lambda' = f(l-1)+1} C_{\lambda} C_{\lambda'}^* s_{\lambda',\lambda} \neq \Sigma^{(1)}(t)$$

(73)

It is clear that this first term does not evolve with time.

- The time-dependence of the second term is given by $T_{l,M-l}(t)$:

$$\Sigma^{(2)}(t) = \sum_{l=0}^{M} \sum_{\lambda = f(l-1)+1} f(l) \prod_{\lambda' = f(l-1)+1} B_{\lambda,\lambda'} 2 \Re \{T_{l,M-l}(t)\}$$

(74)

Then, in order to obtain the limit of this term, we have to compute the limit of $R_l(t) = |T_{l,M-l}(t)|^2$ of eq. (71). As in the case of the original spin-bath model, here we take $|\alpha_j|^2$ and $|\beta_j|^2$ as random numbers in the closed interval $[0, 1]$, such that $|\alpha_j|^2 + |\beta_j|^2 = 1$. Then

$$\max_{l} \left( |\alpha_j|^4 + |\beta_j|^4 + 2 |\alpha_j|^2 |\beta_j|^2 \cos (2 (2l - M) g_j t) \right) = 1$$

(75)

$$\min_{l} \left( |\alpha_j|^4 + |\beta_j|^4 + 2 |\alpha_j|^2 |\beta_j|^2 \cos (2 (2l - M) g_j t) \right) = \left( 2 |\alpha_j|^2 - 1 \right)^2$$

(76)

Therefore, $\left[ |\alpha_j|^4 + |\beta_j|^4 + 2 |\alpha_j|^2 |\beta_j|^2 \cos (2 (2l - M) g_j t) \right]$ is a random number which, if $t \neq 0$, fluctuates between 1 and $\left( 2 |\alpha_j|^2 - 1 \right)^2$. Again, when the environment has many particles (that is, when $N \to \infty$), the statistical value of the cases $|\alpha_j|^2 = 1, |\beta_j|^2 = 1, |\alpha_j|^2 = 0$ and $|\beta_j|^2 = 0$ tends to zero. In this situation, eq. (71) for $R_l(t)$ is an infinite product of numbers belonging to the open interval $(0, 1)$. As a consequence, when $N \to \infty$, $R_l(t) \to 0$.

- The time-dependence of the third term is given by $T_{l,l'}(t)$:

$$\Sigma^{(3)}(t) = \sum_{l,l' = 0}^{M} \sum_{\lambda = f(l-1)+1} f(l) f(l') B_{\lambda,\lambda'} T_{l,l'}(t)$$

(77)

with the restrictions on $l$ and $l'$: $l \neq l'$ and $l' \neq M - l$. As in the second term, we have to compute the limit of $R_{ll'}(t) = |T_{l,l'}(t)|^2$ of eq. (72) and, on the basis of an analogous argument, the result is the same as above: when $N \to \infty$, $R_{ll'}(t) \to 0$. 

If we want now to evaluate the limit of $\langle O_R \rangle_{\rho(t)}$ for $t \to \infty$, we have to compute the limits of the second and the third terms (since the first term, as we have seen, is time-independent). Here we have to distinguish three cases: $M \ll N$, $M \gg N$ and $M \simeq N$.

**Case (a):** $M \ll N$

This case is similar to Decomposition 1 in the original spin-bath model, since in both cases $M \ll N$: the only difference is that in the original model $M = 1$ whereas here $M \geq 1$.

In fact, we have seen that $T_{l,M-s}(t)$ is analogous to $\Gamma_l(t)$ in the original model. Moreover, $T_{l,s}(t)$ has the same functional form as $\Gamma_l(t)$. In paper [12] it is shown that $\Gamma_l(t)$ approaches zero for $t \to \infty$. This means that we can infer that $T_{l,M-s}(t)$ and $T_{l,s}(t)$ also approach zero for $t \to \infty$. On the other hand, the terms $\Sigma(3)(t)$ and $\Sigma(3)(t)$ are sums of less than $M$ terms involving $T_{l,M-s}(t)$ and $T_{l,s}(t)$. As a consequence, since in this case $M$ is a small number, the sum of a small number of terms approaching zero for $t \to \infty$ also approaches zero: $\lim_{t \to \infty} \Sigma(3)(t) = 0$ and $\lim_{t \to \infty} \Sigma(3)(t) = 0$. Therefore,

$$\lim_{t \to \infty} \langle O_R \rangle_{\rho(t)} = \lim_{t \to \infty} \left[ \Sigma(1)(t) + \Sigma(2)(t) + \Sigma(3)(t) \right] = \Sigma(1)(t)$$  \hfill (78)

In other words,

$$\lim_{t \to \infty} \langle O_R \rangle_{\rho(t)} = \sum_{l=0}^{M} B_{\lambda,\lambda'} = \sum_{l=0}^{M} C_{\lambda} C_{\lambda'}^* = \langle O_R \rangle_{\rho_s}$$  \hfill (79)

where $\rho_s$ is the final diagonal state of $U$. This result can also be expressed in terms of the reduced density operator $\rho_A$ of the system $A$ as (see eq. (5)):

$$\lim_{t \to \infty} \langle O_R \rangle_{\rho(t)} = \langle O_R \rangle_{\rho_s} = \lim_{t \to \infty} \langle O_A \rangle_{\rho_A(t)} = \langle O_A \rangle_{\rho_{A_s}}$$  \hfill (80)

In the eigenbasis of the Hamiltonian $H_A$ of $A$, the final reduced density operator $\rho_{A_s}$ is expressed by a $2^M \times 2^M$ matrix:

$$\rho_{A_s} = \begin{pmatrix} \rho_{l=0} & 0 & 0 & 0 & \ldots & 0 \\ 0 & \rho_{l=1} & 0 & 0 & \ldots & 0 \\ 0 & 0 & \rho_{l=2} & 0 & \ldots & 0 \\ 0 & 0 & 0 & \rho_{l=3} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & \rho_{l=M} \end{pmatrix}$$  \hfill (81)

where $\rho_{l=0} = |C_1|^2$ and each $\rho_l$ is a matrix of dimension $\frac{M_l}{(M-l)!} \times \frac{M_l}{(M-l)!}$. This result might seem insufficient for decoherence because, since the $\rho_l$ are matrices, $\rho_{A_s}$ seems to be non completely diagonal in the eigenbasis of the Hamiltonian $H_A$. However, we have to recall that all the states $|A_i\rangle$ with same $l$ are degenerate eigenvectors corresponding to the same eigenvalue of $H_A$; then, the basis that diagonalizes $\rho_{A_s}$ (i.e., that diagonalizes all the matrices $\rho_l$) is an eigenbasis of $H_A$. Summing up, the system $S = A$ of $M$ particles in interaction with its environment $E = B$ of $N \gg M$ particles decoheres in the eigenbasis of $\rho_{A_s}$, which is also an eigenbasis of $H_A$.

If we want to compute the time-behavior of $\langle O_R \rangle_{\rho(t)}$, we have to consider that $\Sigma(1)$ is a sum of terms of the form

$$\langle B_{\lambda,\lambda'} | a_j |^2 + B_{\lambda,\lambda'} | b_j |^2 \rangle,$$

that is, terms of the expectation value coming from the diagonal part of $\rho(t)$ in the basis of the Hamiltonian $H$. Therefore, if there is decoherence, the sum $\Sigma(1)(t) = \Sigma(2) + \Sigma(3)$, involving the terms of $\langle O_R \rangle_{\rho(t)}$ coming from the non-diagonal part of $\rho(t)$, has to approach zero for $t \to \infty$.

In order to show an example of the time-behavior of $\langle O_R \rangle_{\rho(t)}$, numerical simulations for $\Sigma(1)(t)$ have been performed, with the following features:

(i) $s_{\lambda',\lambda} = 1$ (see eq. (67)).

(ii) The initial condition for $S = A$ is selected as (see eq. (77)):

$$|A| = \frac{1}{\sqrt{2^M}} \sum_{i=1}^{2^M} |A_i\rangle \implies \forall \lambda, C_{\lambda} = C_{\lambda}^* = \frac{1}{\sqrt{2^M}} C_{\lambda} C_{\lambda}^* = \frac{1}{2^M}$$  \hfill (82)

Then, from (i) and (ii), $B_{\lambda,\lambda'} = 2^{-M}$ (see eq. (61)).
(iii) $|\alpha_i|^2$ is generated by a random-number generator in the interval $[0, 1]$, and $|\beta_i|^2$ is obtained as $|\beta_i|^2 = 1 - |\alpha_i|^2$.

(iv) $g_i = 400\, Hz$: as explained above, the coupling constant in typical models of spin interaction.

(v) As in the original model, the time-interval $[0, t_0]$ was partitioned into intervals $\Delta t = t_0/200$, and the function $\Sigma^{nd}(t)$ was computed at times $t_k = k\Delta t$, with $k = 0, 1, ..., 200$.

(vi) $N = 10^3$, and $M = 1$ and $M = 10$.

Figure 7 shows the time-evolution of $\Sigma^{nd}(t)$.

This result shows that, as expected, a small open system $S = A$ of $M$ particles decoheres in interaction with a large environment $E = B$ of $N \gg M$ particles.

**Case (b): $M \gg N$**

In this case, where the open system $S = A$ has much more particles than the environment $E = B$, the argument of Case (a) cannot be applied: since now $\Sigma^{(2)}(t)$ and $\Sigma^{(3)}(t)$ are no longer sums over a small number of terms, the fact that each term approaches zero does not guarantee that the sums also approach zero. In particular, if $N = 1$, then (see eq. (70))

$$T_{i,i'}(t) = |\alpha_1|^2 e^{i(g_{1,i}-g_{1,i'})t/2} + |\beta_1|^2 e^{-i(g_{1,i}-g_{1,i'})t/2}$$

which clearly has no limit for $t \to \infty$. Nevertheless, it might happen that, with $N$ high but $M$ much higher than $N$, each term of the sums approaches zero. So, in order to know the time behavior of $(O_R)_{i,i'}(t)$, numerical simulations for $\Sigma^{nd}(t)$ have been performed, with the same features as in the previous case, with the exception of condition (vi), which was taken as:

(vi) $M = 10^3$, and $N = 10$ and $N = 100$. 

Figure 8 shows the time-evolution of $\Sigma^{nd}(t)$ in this case.

This result is also what may be expected from the EID perspective: when the open system $S = A$ of $M$ particles is larger that the environment $E = B$ of $N \ll M$ particles, $S$ does not decohere.

**Case (c): $M \approx N$**

In this case, where the numbers of particles of the open system $S = A$ and of the environment $E = B$ do not differ in more than one order of magnitude, the time behavior of $\langle O_R \rangle_{\rho(t)}$ cannot be inferred from the equations. Numerical simulations have been performed, with the same features as in Case (b), with the exception of condition (vi), which was taken as:

(vi) $N = 10^3$, and $M = 10^2$ and $M = 10^3$.

Figure 9 shows the time-evolution of $\Sigma^{nd}(t)$.

Again, this result is not surprising from the viewpoint of the EID approach: if the environment $E = B$ of $N$ particles is not large enough when compared with the open system $S = A$ of $M$ particles, $S$ does not decohere.

Let us notice that, for $N = 10^3$, the system $S = A$ with $M = 10^2$ does not decohere (Figure 9), whereas it does decohere with $M = 10$ (Figure 7). This shows that, in the case of this decomposition, $M \ll N$ means that $N$ is at least two orders of magnitude higher than $M$.

**Summarizing results**

Up to now, in this Decomposition 1 all the arguments were directed to know whether the system $A$ of $M$ particles decoheres or not in interaction with the system $B$ of $N$ particles. But, given the symmetry of the whole system, the same arguments can be used to decide whether the system $B$ of $N$ particles decoheres or not in interaction with the system $A$ of $M$ particles, with analogous results: $B$ decoheres only when $M \gg N$; if $M \ll N$ or $M \approx N$, $B$ does not decohere. Therefore, all the results obtained in this section can be summarized as follows:

(i) If $M \ll N$, $A$ decoheres and $B$ does not decohere.
(ii) If $M \gg N$, $A$ does not decohere and $B$ decoheres.

(iii) If $M \simeq N$, neither $A$ nor $B$ decohere.

This general conclusion completely agrees with the usual reading of the EID approach: the decoherence of an open system $S$ is the result of its interaction with a very large environment $E$. This happens when $S = A$ in (i) and when $S = B$ in (ii), but it does not happen in (iii).

VIII. GENERALIZED SPIN-BATH MODEL: DECOMPOSITION 2

A. Selecting the relevant observables

In this case we decide to observe only one particle of the open system $A$. This amounts to splitting the closed system $U$ into two new subsystems: the open system $S$ is, say, the particle $A_M$ with ket $|\uparrow, \uparrow, \ldots, \uparrow, \downarrow, \downarrow\rangle$, and the environment is $E = (\cup_{i=1}^{M-1} A_i) \cup B = (\cup_{i=1}^{M-1} A_i) \cup (\cup_{k=1}^N B_k)$. Let us notice that the Decomposition 2 of the original spin-bath model is a particular case of this one, for $N = 1$ (see eq. (30), where $N$ plays the role of the $M$ of this case). The TPS for this case is

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E = (\mathcal{H}_{A_M}) \otimes \left( \bigotimes_{i=1}^{M-1} \mathcal{H}_{A_i} \right) \otimes \left( \bigotimes_{k=1}^N \mathcal{H}_{B_k} \right)$$

(84)

Therefore, the relevant observables $O_R$ of the closed system $U$ are those corresponding to the particle $A_M$:

$$O_R = O_S \otimes \mathbb{I}_E = \left( \sum_{\alpha, \alpha'} s_{\alpha, \alpha'} |\alpha\rangle \langle \alpha'| \right) \otimes \left( \bigotimes_{i=1}^{M-1} \mathbb{I}_i \right) \otimes \left( \bigotimes_{k=1}^N \mathbb{I}_k \right)$$

(85)
It is easy to see that the relevant observables selected in this Decomposition 2 form a subspace of the space of the relevant observables selected in Decomposition 1: eq. (85) can be obtained from eq. (67) by making \( s_{\lambda,\lambda'} = 1 \) for \( \lambda = \lambda' \) and \( s_{\lambda,\lambda'} = 0 \) for \( \lambda \neq \lambda' \) in all the terms of the sum except for the terms corresponding to the particle \( A_M \).

In order to simplify expressions, in this case it is convenient to introduce a new arrangement for the eigenvectors of the Hamiltonian \( H_A \), by calling them \( \hat{A}_i \): the set \( \{ \hat{A}_i \} \) is an eigenbasis of \( H_A \) with \( 2^M \) elements. The \( \hat{A}_i \) will be ordered by analogy with the binary numbers:

\[
\begin{align*}
\hat{A}_1 &= |\uparrow , \uparrow , \ldots , \uparrow , \uparrow , \uparrow \rangle , & \hat{A}_2 &= |\uparrow , \uparrow , \ldots , \uparrow , \uparrow , \downarrow \rangle , & \hat{A}_3 &= |\uparrow , \uparrow , \ldots , \uparrow , \downarrow , \uparrow \rangle , \hat{A}_4 &= |\uparrow , \uparrow , \ldots , \uparrow , \downarrow , \downarrow \rangle , & \ldots \\
\hat{A}_{2^M} &= |\downarrow , \downarrow , \ldots , \downarrow , \downarrow , \downarrow \rangle
\end{align*}
\]

(86)

According to this arrangement, the \( \hat{A}_i \) with even \( i \) have the spin \( M \) in the state \( |\downarrow \rangle \), and the \( \hat{A}_i \) with odd \( i \) have the spin \( M \) in the state \( |\uparrow \rangle \). So, the relevant observables of eq. (85) can be rewritten in terms of the \( \hat{A}_i \) as

\[
O_R = \left( \sum_{\lambda=1}^{2^M} (\tilde{s}_{\theta \theta} |\hat{A}_{2\lambda} \rangle + \tilde{s}_{\theta \theta} |\hat{A}_{2\lambda-1} \rangle + \tilde{s}_{\theta \theta} |\hat{A}_{2\lambda} \rangle + \tilde{s}_{\theta \theta} |\hat{A}_{2\lambda-1} \rangle) \right) \otimes (\bigotimes_{k=1}^{N} 1_k)
\]

(87)

**B. Computing the behavior of the relevant expectation values**

Here the expectation values of the relevant observables are given by eq. (62), with \( T_{l,l'}(t), T_{l,l}(t) \) and \( T_{l,M-l}(t) \) given by eqs. (60), (64) and (65) respectively, but now replacing \( B_{\lambda,\lambda'} \) with \( \tilde{B}_{\lambda,\lambda'} \),

\[
\langle O_R \rangle_{\rho(t)} = \sum_{l=0}^{M} \sum_{\lambda = f(l-1)+1}^{M} \tilde{B}_{\lambda,\lambda'} + \sum_{l=0}^{M} \sum_{\lambda = f(l-1)+1}^{M} \frac{f(l)}{f(M-l)} \tilde{B}_{\lambda,\lambda'} 2 \text{Re} (T_{l,M-l}(t)) + \sum_{l=0}^{M} \sum_{\lambda = f(l-1)+1}^{M} \frac{f(l)}{f(l')} \tilde{B}_{\lambda,\lambda'} T_{l,l'}(t)
\]

(88)

where the \( \tilde{B}_{\lambda,\lambda'} \) can be written in the basis \( \{ \hat{A}_i \} \) as

\[
\tilde{B}_{\lambda,\lambda'} = \begin{cases} 
C_{\lambda,\lambda'} \tilde{s}_{\theta \theta} & \text{if } \lambda \text{ is an even number and } \lambda' = \lambda \\
C_{\lambda,\lambda'}' \tilde{s}_{\theta \theta} & \text{if } \lambda \text{ is an even number and } \lambda' = \lambda - 1 \\
C_{\lambda,\lambda'} \tilde{s}_{\theta \theta} & \text{if } \lambda \text{ is an odd number and } \lambda' = \lambda + 1 \\
C_{\lambda,\lambda'}' \tilde{s}_{\theta \theta} & \text{if } \lambda \text{ is an odd number and } \lambda' = \lambda \\
0 & \text{otherwise}
\end{cases}
\]

(89)

According to eq. (89), \( \tilde{B}_{\lambda,\lambda'} \neq 0 \) only when

\[
\lambda' = \lambda \quad \text{or} \quad \lambda' = \lambda \pm 1
\]

(90)

Since \( \lambda = f(l-1)+1 \) and \( \lambda' = f(l'-1)+1 \), relations (90) imply that

\[
l' = l \quad \text{or} \quad l' = l \pm 1
\]

(91)

The expectation value given by eq. (88) has again three terms, \( \langle O_R \rangle_{\rho(t)} = \Sigma^{(1)} + \Sigma^{(2)} + \Sigma^{(3)} \), which can be analyzed separately:

- From eqs. (89) and (90), the first term reads

\[
\Sigma^{(1)} = \sum_{l=0}^{M} \sum_{\lambda = f(l-1)+1}^{M} B_{\lambda,\lambda} = \sum_{\lambda = 0}^{2^M-1} \left( |C_{2\lambda} \tilde{s}_{\theta \theta} + |C_{2\lambda+1} |^2 \tilde{s}_{\theta \theta} \right) \neq \Sigma^{(1)}(t)
\]

(92)

Analogously to eq. (73) of Decomposition 1, this first term does not evolve with time.
• The time-dependence of the second term is given by \( T_{l,M-l}(t) \). But with the restrictions of eqs. (90) and (91), \( \Sigma^{(2)} \) has only two terms:

\[
\Sigma^{(2)}(t) = \sum_{l=0}^{M-1} \sum_{\lambda=f(l-1)+1}^{f(l)-1} B_{\lambda,\lambda'} 2 \text{Re}(T_{l,M-l}(t)) =
\]

\[
= C_{f(l)-1}^{f(l)-1} C_{f(f(l)-1)+2}^{f(f(l)-1)+2} (\bar{s}_\theta \bar{g} + \bar{s}_\theta \bar{g}) 2 \text{Re} \left( T_{l,M-l}^{M+1}(t) \right)
\]

Then, in order to obtain the limit of this term, we have to compute the limit of \( T_{M-1}^{M+1}(t) \), which is precisely the \( T_{l,l'}(t) \) of Decomposition 1 in the particular case that \( l = \frac{M-1}{2} \) and \( l' = \frac{M+1}{2} \) (see eq. (70)). But, as we have seen in Case (a) of Decomposition 1, \( T_{l,l'}(t) \) has the same functional form as \( \Gamma_1(t) \) of the original model (see eq. (22)), which approaches zero for \( t \to \infty \) when \( N \gg 1 \). Therefore, for \( N \gg 1 \), \( T_{M-1}^{M+1}(t) \) also approaches zero for \( t \to \infty \), and the same holds for \( \Sigma^{(2)}(t) \) since it is a sum of two terms containing \( T_{M-1}^{M+1}(t) \).

• The time-dependence of the third term is given by \( T_{l,l'}(t) \). But with the restrictions of eqs. (90) and (91), \( \Sigma^{(3)} \) results:

\[
\Sigma^{(3)}(t) = \sum_{l=0}^{M} \sum_{\lambda=f(l-1)+1}^{f(l)-1} (B_{\lambda,\lambda+1} T_{l,t+1}(t) + B_{\lambda,\lambda-1} T_{l,t-1}(t))
\]

Since here \( l' = l \pm 1 \) (see eq. (91)), in this case \( T_{l,t \pm 1}(t) \) is:

\[
T_{l,t \pm 1}(t) = \prod_{j=1}^{N} \left( |\alpha_j|^2 e^{\mp ig_j t} + |\beta_j|^2 e^{|\pm ig_j t}| \right)
\]

If we compare this equation with eq. (26) for \( r(t) \) in the original spin-bath model, we can see that

\[
T_{l,t+1}(t) = r(t) \quad \text{and} \quad T_{l,t-1}(t) = r^*(t)
\]

Then,

\[
\Sigma^{(3)}(t) = (S_+ r(t) + S_- r^*(t))
\]

where \( S_+ \) and \( S_- \) are constants given by

\[
S_\pm = \sum_{l'=\pm 1}^{M} \sum_{\lambda=f(l-1)+1}^{f(l)-1} B_{\lambda,\lambda \pm 1}
\]

On the basis of the simulations of the original model we have seen that, when \( N \gg 1 \), \( r(t) \) approaches zero for \( t \to \infty \). Therefore, in this case we can conclude that, when \( N \gg 1 \), \( \Sigma^{(3)}(t) \) approaches zero for \( t \to \infty \).

Summing up, \( \langle O_R \rangle_{\rho(t)} \) is the sum of three terms: one is time-independent and the other two tend to zero for \( t \to \infty \). In particular, from eq. (22) we know that, for \( N \gg 1 \),

\[
\lim_{t \to \infty} \langle O_R \rangle_{\rho(t)} = \sum_{l=0}^{M} \sum_{\lambda,\lambda'=f(l-1)+1} B_{\lambda,\lambda'} \sum_{l=0}^{M-1} \sum_{\lambda=0}^{M} \left( |C_{2\lambda}|^2 \bar{s}_{\uparrow \uparrow} + |C_{2\lambda+1}|^2 \bar{s}_{\uparrow \downarrow} \right) = \langle O_R \rangle_{\rho_*}
\]

where \( \rho_* \) is the final diagonal state of \( U \). Again, this result can also be expressed in terms of the reduced density operator \( \rho_S = \rho_{AM} \) of the open system \( S = AM \) as (see eq. (80))

\[
\lim_{t \to \infty} \langle O_R \rangle_{\rho(t)} = \langle O_R \rangle_{\rho_*} = \lim_{t \to \infty} \langle O_{AM} \rangle_{\rho_{AM}(t)} = \langle O_{AM} \rangle_{\rho_{AM}^*}
\]
where the final reduced density operator $\rho_{A_M^*}$ in the basis $\{|\uparrow\rangle, |\downarrow\rangle\}$ reads

$$
\rho_{A_M^*} = \begin{pmatrix}
|\alpha_M|^2 & 0 \\
0 & |\beta_M|^2
\end{pmatrix}
$$

This shows that the open system $S = A_M$, composed of a single particle, decoheres in interaction with its environment $E$ of $N + M - 1$ particles when $N \gg 1$, independently of the value of $M$.

In order to illustrate this conclusion, we have computed $\Sigma^{nd}(t) = \Sigma^{(2)}(t) + \Sigma^{(3)}(t)$ by means of numerical simulations with the same features as in Decomposition 1, with the exception of condition (vi), which was taken as:

Figure 10: (vi) $M = 10^3$ and $N = 1$.

Figure 11: (vi) $M = 10^3$ and $N = 10^2$.

Figure 12: (vi) $M = 10^3$ and $N = 10^3$.

**Summarizing results**

As we have seen, in this decomposition of the whole closed system, the open system $S = A_M$ decoheres when $N \gg 1$, independently of the value of $M$. But the particle $A_M$ was selected as $S$ only for computation simplicity; the same argument can be developed for any particle $A_i$ of $A$. Then, when $N \gg 1$ and independently of the value of $M$, any particle $A_i$ decoheres in interaction with its environment $E$ of $N + M - 1$ particles.

On the other hand, as in Decomposition 1, here the symmetry of the whole system $U$ allows us to draw analogous conclusions when the system $S$ is one of the particles of $B$, say, $B_N$: $S = B_N$ decoheres when $M \gg 1$, independently of the value of $N$. And, on the basis of the same considerations as above, when $M \gg 1$ and independently of the value of $N$, any particle $B_i$ decoheres in interaction with its environment $E$ of $N + M - 1$ particles.
A. Analyzing results

According to the usual reading of the EID approach, the decoherence of an open system is induced by its interaction with a large environment. Such an interaction is what leads to the dissipation of energy from the open system $S$ to the environment $E$. So, the orthodox view suggests a picture of decoherence where an energy flow from the open system $S$ to the environment $E$ washes out the original coherence and allows the classicality of $S$ to emerge. According to this picture, in the original spin-bath model a spin-1/2 particle $S$ decoheres when immersed in a large bath of spin-1/2 particles $E = \bigcup_i P_i$; $P$ dissipates its energy into $E$ and may acquire a classical nature. However, this reading has to face the “looming big” problem of defining the open systems involved in decoherence, since it does not provide a criterion to identify the open system $S$ and its environment $E$. Now we will discuss the results obtained in the generalized spin-bath model, in order to see how they may contribute to the clarification of the problem.

a) As we have seen, in our generalized model, where $U = A \cup B$, with $A$ of $M$ particles $A_i$ and $B$ of $N$ particles $B_i$, (i) when $M \gg N$ or $M \simeq N$, the subsystem $A$ does not decohere (Decomposition 1 of Section VII), but (ii) the particles $A_i$, considered independently, decohere when $N \gg 1$ (Decomposition 2 of Section VIII). This means that there are physically meaningful situations, given by $M \gg N \gg 1$ or $M \simeq N \gg 1$, where all the $A_i$ decohere although $A$ does not decohere. In other words, in spite of the fact that certain particles decohere and may behave classically, the subsystem composed by all of them retains its quantum nature. This seemingly paradoxical conclusion sounds even more strange when the situation is conceived in terms of energy dissipation. In spite of the fact that all the $A_i$ dissipate their energy into the environment (mainly into the subsystem $B$ due to the interaction among each $A_i$ and all the $B_j$), the composite system $A = \bigcup_i A_i$ (which should dissipate the energy of all the $A_i$) does not decohere.

b) We have also seen that, by symmetry, all the particles $B_i$, considered independently, also decohere when $M \gg 1$. Then, when $M \gg N \gg 1$ or $M \simeq N \gg 1$, the requirement $M \gg 1$ holds and we can conclude that not only all the $A_i$, but also all the $B_i$ decohere. So, all the particles of the closed system $U = (\bigcup_i A_i) \cup (\bigcup_j B_j)$ may become classical.
when considered independently, although the whole system $U$ certainly does not decohere and, therefore, retains its quantum character. Again, the explanation of this result is even more difficult when it is conceived in terms of the energy dissipated from the system that decoheres to its environment, since we are committed to decide which particles give and which receive the dissipated energy.

These difficulties are further consequences of the "looming big" problem of defining the open systems involved in decoherence. The irony of this story is that such a problem is the consequence of what has been considered to be the main advantage of the decoherence program: its open-system perspective. According to this perspective, particles interacting with other particles by exchanging energy are well-defined open systems, and the collections of those particles are open systems too. So, the problem is to decide which one of all these open systems is the system that decoheres or, in other words, where to place the cut between the system $S$ and its environment $E$.

The open-system approach not only leads to the "looming big" problem, but also disregards the well-known holism of quantum mechanics: a quantum system is not the mere collection of its parts and its interactions. In order to retain its holistic nature, a quantum system has to be considered as a whole: the open "subsystems" are only partial descriptions of the whole closed system, given by the selection of particular subspaces of relevant observables. On the basis of this closed-system perspective, we can develop a different conceptual viewpoint for understanding decoherence, which dissolves the problems of the orthodox open-system view.

### B. A different conceptual viewpoint

As we have seen, a TPS expresses the decomposition of the closed system $U$, represented in the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, into two open systems $S_A$ and $S_B$, represented in $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. Such a decomposition amounts to the split of the whole space $\mathcal{O} = \mathcal{H} \otimes \mathcal{H}$ of the observables of $U$ into the subspaces $\mathcal{O}_A = \mathcal{H}_A \otimes \mathcal{H}_A$ and $\mathcal{O}_B = \mathcal{H}_B \otimes \mathcal{H}_B$ such that $\mathcal{O} = \mathcal{O}_A \otimes \mathcal{O}_B$. In particular, the total Hamiltonian of $U$, $H \in \mathcal{O}$, can be expressed as $H = H_A \otimes I_B + I_A \otimes H_B + H_{AB}$, where $H_A \in \mathcal{O}_A$ is the Hamiltonian of $S_A$, $H_B \in \mathcal{O}_B$ is the Hamiltonian of $S_B$, and $H_{AB} \in \mathcal{O}$ is the interaction Hamiltonian, representing the interaction between the systems $S_A$ and $S_B$. 
As stressed in papers [11] and [12], in general a quantum system $U$ admits a variety of TPSs, that is, a variety of different decompositions into $S_A$ and $S_B$, each one defined by the space of observables $O_A$ of $S_A$ and $O_B$ of $S_B$. Among all the possible decompositions of $U$, there is a particular TPS that remains dynamically invariant. This is the case when the interaction Hamiltonian $H_{AB} = 0$: there is no interaction between $S_A$ and $S_B$ and, then,

$$[H_A \otimes I_B, I_A \otimes H_B] = 0 \implies \exp(-iHt) = \exp(-iH_At) \exp(-iH_Bt)$$

(103)

Therefore,

$$\rho_A(t) = \text{Tr}_B(\rho(t)) = e^{iH_A t} \text{Tr}_B(\rho_0) \exp(-iH_A t) = e^{iH_A t} \rho_0 e^{-iH_A t}$$

(104)

$$\rho_B(t) = \text{Tr}_A(\rho(t)) = e^{iH_B t} \text{Tr}_A(\rho_0) \exp(-iH_B t) = e^{iH_B t} \rho_0 e^{-iH_B t}$$

(105)

This means that, even if the initial state $\rho_0$ of $U$ is an entangled state with respect to the TPS $H = H_A \otimes H_B$, the subsystems $S_A$ and $S_B$ are dynamically independent: each one of them evolves unitarily under the action of its own Hamiltonian. As a consequence, the subsystems $S_A$ and $S_B$ resulting from this particular, dynamically invariant TPS do not decohere.

Once we have excluded the dynamically invariant TPS of $U$, all the remaining TPSs define interacting subsystems $S_A$ and $S_B$, such that $H_{AB} \neq 0$. As a result of the interaction, $S_A$ and $S_B$ evolve non-unitarily and, then, depending on the particular interaction between them, they may decohere. But the point to stress here is that there is no privileged non-dynamically invariant decomposition of $U$: each partition of the closed system into $S_A$ and $S_B$ is just a way of selecting the spaces of observables $O_A$ and $O_B$.

When we adopt a closed-system perspective by means of the concept of TPS, it turns out to be clear that, in decoherence, there is no essential criterion for identifying the “open system” and its “environment”. Given the closed system $U$, that identification requires two steps: (i) to select a TPS $H = H_A \otimes H_B$, such that $U = S_A \cup S_B$, and (ii) to decide that one of the systems resulting from the decomposition, say $S_A$, is the open system $S$, and the other, $S_B$, is the environment $E$. Since the TPS is defined by the spaces of observables $O_A$ and $O_B$, the decomposition of $U$ is just the adoption of a descriptive perspective: the identification of $S$ and $E$ amounts to the selection of the observables relevant in each situation. But since the split can be performed in many ways, with no privileged or essential decomposition, there is no need of an unequivocal criterion for deciding where to place the cut between “the” system and “the” environment. Decoherence is not a yes-or-not process, but a phenomenon relative to the chosen decomposition of the whole closed quantum system. When viewed from this closed-system perspective, Zurek’s “looming big problem” does not constitute a real threat to the decoherence program: the supposed challenge dissolves once the relative nature of decoherence is taken into account.

From this perspective, quantum mechanics is a theory whose dynamical postulate refers to closed systems: the time-behavior of the parts resulting from different partitions of the closed system has to be inferred from that postulate. Since the total Hamiltonian rules the dynamical evolution of the closed system, then the time-behavior of its open subsystems depends on the form in which the Hamiltonian is decomposed in each particular partition. This means that decoherence cannot be simply described as the result of an interaction through which a small open system—typically, a particle—dissipates its energy into a large environmental bath. As we have seen in the generalized spin-bath model, this picture of decoherence leads to perplexities: the relationships between the whole closed system and its open subsystems is subtler than that picture suggests (in a future paper we will study those relationships from a theoretical viewpoint in order to draw some general conclusions regarding decoherence). Therefore, the decomposition of the total Hamiltonian has to be studied in detail in each particular case, in order to know whether the system of interest resulting from the partition of the whole closed system decoheres or not under the action of its self-Hamiltonian and the interaction Hamiltonian.

X. CONCLUSIONS

The aim of this paper has been to argue that decoherence can be viewed from a closed-system perspective, which improves the understanding of the phenomenon. For this purpose, we have analyzed the simple spin-bath model by studying the time-behavior of the expectation values of relevant observables belonging to different sets. Then, we have generalized the original model in order to see how decoherence depends on the way in which the relevant observables are selected.

On the basis of the analysis of the two models from a closed-system perspective, we have drawn the following conclusions:

(i) Decoherence is a phenomenon relative to which degrees of freedom of the whole closed system are considered relevant and which are disregarded in each case.
(ii) The explanation of decoherence requires the detailed study of the interaction Hamiltonian resulting from the selected partition of the whole closed system.

(ii) Although it is usually claimed that EID is a dissipative approach to decoherence, the simple account of decoherence in terms of energy dissipation from the open system to its environment is misguided, to the extent that there are situations where all the particles of a closed system decohere when considered independently.

(iv) Since there is no privileged or essential decomposition of the closed system, there is no need of an unequivocal criterion for identifying the systems involved in decoherence. Therefore, the “looming big problem”, which, according to Zurek, poses a serious threat to the whole decoherence program, loses its strength in the light of the relative nature of decoherence.

[1] W. H. Zurek, Phys. Rev. D, 26, 1862, 1982.
[2] W. H. Zurek, Progr. Theor. Phys., 89, 281, 1993.
[3] J. P. Paz and W. H. Zurek, “Environment-induced decoherence and the transition from quantum to classical”, in Dieter Heiss (ed.), Lecture Notes in Physics, Vol. 587, Heidelberg-Berlin: Springer, 2002.
[4] W. H. Zurek, Rev. Mod. Phys., 75, 715, 2003.
[5] E. A. Calzetta, B. L. Hu and F. D. Mazzitelli, Phys. Rep., 352, 459, 2001.
[6] W. H. Zurek, Phil. Trans. Roy. Soc., A356, 1793, 1998. See also arXiv:quant-ph/9805065, 1998.
[7] M. Castagnino and O. Lombardi, Stud. Hist. Phil. Mod. Phys., 35, 73, 2004.
[8] R. Omnès, “Decoherence: an irreversible process” [arXiv:quant-ph/0106006], 2001.
[9] R. Omnès, Phys. Rev. A, 65, 052119, 2002.
[10] M. Castagnino, S. Fortin, R. Laura and O. Lombardi, Class. Quant. Grav., 25, 154002, 2008.
[11] N. L. Harshman and S. Wickramasekara, S., Phys. Rev. Lett., 98, 080406, 2007.
[12] N. L. Harshman and S. Wickramasekara, S., Open Systems & Information Dynamics, 14, 341-351, 2007.
[13] M. Schlosshauer, Phys. Rev. A, 72, 012109, 2005.
[14] D. G. Cory, A. F. Fahmy and T. F. Havel, Proc. Natl. Acad. Sci. USA, 94, 1634, 1997.
I. L. Chuang, L. M. K. Vandersypen, X. Zhou, D. W. Leung and S. Lloyd, “Experimental realization of a quantum algorithm”, arXiv:quant-ph/9801037v2, 1998.
S. S. Somaroo, C. H. Tseng, T. F. Havel, R. Laflamme and D. G. Cory, Phys. Rev. Lett., 82, 5381, 1999.
C. H. Tseng, S. S. Somaroo, Y. Sharf, E. Knill, R. Laflamme, T. F. Havel and D. G. Cory, Phys. Rev. A, 61, 012302, 2000.
G. Teklemariam, E. M. Fortunato, C. C. López, J. Emerson, J. P. Paz, T. F. Havel and D. G. Cory, Phys. Rev. A, 67, 062316, 2003.