Abstract

QCD Laplace sum-rules are briefly reviewed and two sum-rule applications are presented. For scalar gluonium, upper bounds on the lightest state are obtained from ratios of Laplace sum-rules, and the role of instantons and the low-energy theorem on these bounds is investigated. For the light-quark masses, fundamental inequalities for Laplace sum-rules are developed, which lead to a lower bound on the average of the up and down quark masses.

1 Introduction

QCD sum-rules, particularly in the form of Laplace sum-rules [1], are a well-established technique for relating QCD predictions to hadronic physics. In this approach, the non-perturbative aspects of the QCD vacuum are parametrized indirectly through the QCD condensates, or directly by instanton contributions.

In this paper the QCD Laplace sum-rule approach is briefly reviewed in Section 2, and two applications are presented with the common theme of obtaining mass bounds. In Section 3, QCD sum-rules mass bounds on the lightest scalar glueball state will be obtained. Instanton effects in the scalar gluonium sum-rules are seen to be important, and help reconcile a number of conflicting sum-rule results in this channel. Finally, Section 4 develops fundamental inequalities for QCD sum-rules which are then applied to obtain mass bounds for the non-strange light quarks.
2 QCD Laplace Sum-Rules

QCD sum-rules are based upon correlation functions of gauge invariant and renormalization group (RG) invariant composite operators

\[ \Pi_\Gamma \left( Q^2 \right) = i \int d^4x e^{iq \cdot x} \langle O | T [ J_\Gamma (x) J_\Gamma (0) ] | O \rangle, \]

with \( J_\Gamma (x) \) a composite operator with desired quantum numbers \( \Gamma \). These composite operators serve as interpolating fields between hadronic states and the vacuum, appropriate to the quantum numbers of the current.

The correlation function (1) satisfies a dispersion relation with several subtractions appropriate to the large-energy asymptotic behaviour\(^1\) of the correlation function

\[ \Pi_\Gamma \left( Q^2 \right) = \Pi_\Gamma (0) + Q^2 \Pi'_\Gamma (0) + \frac{1}{2} Q^4 \Pi''_\Gamma (0) - Q^6 \frac{1}{\pi} \int_{t_0}^{\infty} \frac{\rho_\Gamma (t)}{t^3 (t + Q^2)} dt, \]

where \( \rho_\Gamma (t) \) is the hadronic spectral function with physical threshold \( t_0 \) appropriate to the quantum numbers of the currents. The dispersion relation serves to relate the (Euclidean momenta) QCD prediction \( \Pi_\Gamma \left( Q^2 \right) \) to hadronic spectral function. The subtraction constants \( \{ \Pi_\Gamma (0), \Pi'_\Gamma (0), \Pi''_\Gamma (0) \} \) are usually unknown, but in certain cases they are determined by low-energy theorems\(^2\).

The dispersion relations are not suitable for applications to light hadronic channels since they receive substantial contributions from excited states and the QCD continuum in the integral of \( \rho_\Gamma (t) \), obscuring the lightest states. These difficulties were resolved by applying the Borel transform operator \( \hat{B} \) to the dispersion relation

\[ \hat{B} \equiv \lim_{N, \frac{Q^2 \to \infty }{N/Q^2 \equiv \tau}} \left( \frac{-Q^2}{\Gamma (N)} \right)^N \left( \frac{d}{dQ^2} \right)^N. \]

This operator has the following properties useful to construction of Laplace sum-rules:

\[ \hat{B} \left[ a_0 + a_1 Q^2 + \ldots + a_m Q^{2m} \right] = 0, \quad (m \text{ finite}) \]

\[ \hat{B} \left[ \frac{Q^{2n}}{t + Q^2} \right] = \tau (-1)^n t^n e^{-\tau} \quad, \quad n = 0, 1, 2, \ldots \quad (n \text{ finite}) \] . \(^{(5)}\)

In general, \( \hat{B} \) is related to the inverse Laplace transform \( \mathcal{L}^{-1} \)

\[ f \left( Q^2 \right) = \int_0^{\infty} F(\tau) e^{-Q^2 \tau} d\tau \equiv \mathcal{L} [ F(\tau) ] \quad \implies \quad \frac{1}{\tau} \hat{B} \left[ f \left( Q^2 \right) \right] = F(\tau) = \mathcal{L}^{-1} \left[ f \left( Q^2 \right) \right] \]

\[ \mathcal{L}^{-1} \left[ f \left( Q^2 \right) \right] = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} f \left( Q^2 \right) e^{Q^2 \tau} dQ^2, \] \(^{(7)}\)

\(^1\)The given form is appropriate to scalar gluonium.
where the real parameter $b$ in the definition (11) of the inverse Laplace transform must be chosen so that $f(Q^2)$ is analytic to the right of the contour of integration in the complex plane.

Families of Laplace sum-rules are now obtained by applying $\hat{B}$ to (2) weighted by integer powers of $Q^2$, which will involve the theoretically-determined quantity

$$L_k^\Gamma(\tau) \equiv \frac{1}{\tau} \hat{B} \left[ (-1)^k Q^{2k} \pi (Q^2) \right] = L^{-1} \left[ (-1)^k Q^{2k} \pi (Q^2) \right].$$

(8)

For $k \geq 0$, this results in the following Laplace sum-rules relating QCD to hadronic physics

$$L_k^\Gamma(\tau) = \frac{1}{\pi} \int_{t_0}^{\infty} t^k e^{-\tau t} \rho_\Gamma(t) \, dt , \quad k \geq 0 .$$

(9)

Compared with the dispersion relation (4), the exponential weight in (9) suppresses the (large-energy) QCD continuum and excited states in the Laplace sum-rule, emphasizing the lightest resonances in a particular channel. The quantity $\tau$ can truly be thought of as a scale which probes the spectral function, since in addition to controlling the region of integration on the hadronic (right-hand) side of (9), RG improvement of the QCD (left-hand) side of (9) implies that the renormalization scale is $\nu^2 = 1/\tau$.

Analysis of the QCD Laplace sum-rules is achieved through a resonance(s) plus continuum model, where hadronic physics is locally dual to QCD for energies above the continuum threshold $t = s_0$

$$\rho_\Gamma(t) = \rho_\Gamma^{\text{had}}(t) + \theta(t - s_0) \text{Im}\Pi^{\text{QCD}}_\Gamma(t) .$$

(10)

The QCD continuum contribution denoted by

$$c_k^\Gamma(\tau, s_0) = \frac{1}{\pi} \int_{s_0}^{\infty} t^k e^{-\tau t} \text{Im}\Pi^{\text{QCD}}_\Gamma(t) \, dt ,$$

(11)

is combined with $L_{k,\text{cont}}(\tau)$ since both are theoretically determined

$$R_k^\Gamma(\tau, s_0) \equiv L_k^\Gamma(\tau) - c_k^\Gamma(\tau, s_0) ,$$

(12)

resulting in the final form of the Laplace sum-rules relating QCD to hadronic physics phenomenology

$$R_k^\Gamma(\tau, s_0) = \frac{1}{\pi} \int_{t_0}^{\infty} t^k e^{-\tau t} \rho_\Gamma^{\text{had}}(t) \, dt , \quad k \geq 0 .$$

(13)

After a hadronic model for $\rho_\Gamma^{\text{had}}(t)$ is established, its resonance parameters and the continuum threshold $s_0$ are determined by fitting the two sides of (13) over a range of the Laplace scale $\tau$ probing the theory and phenomenology.

\begin{footnote}
2The case $k < 0$ is relevant to scalar gluonium, and will be discussed in Section 3.
\end{footnote}
3 Mass Bounds on the Lightest Scalar Gluonium State

Scalar gluonium is studied through the current

\[ J_s(x) = -\frac{\pi^2}{\alpha\beta_0} \beta(\alpha) G_{\mu\nu}^a(x)G_{\mu\nu}^a(x) \]

which is RG invariant for massless quarks \[5\], and where the current normalization has been chosen so that to lowest order in \( \alpha \)

\[ \beta(\alpha) = \nu^2 \frac{d}{d\nu^2} \left( \frac{\alpha(\nu)}{\pi} \right) = -\beta_0 \left( \frac{\alpha}{\pi} \right)^2 - \beta_1 \left( \frac{\alpha}{\pi} \right)^3 + \ldots \]

\[ \beta_0 = \frac{11}{4} - \frac{1}{6} n_f \quad , \quad \beta_1 = \frac{51}{8} - \frac{19}{24} n_f . \]

The correlation function of \( J_s \) has the interesting property that in the chiral limit of \( n_f \) quarks, it satisfies the following low-energy theorem (LET) \[2\]

\[ \Pi_s(0) = \lim_{Q \to 0} \Pi_s(Q^2) = \frac{8\pi}{\beta_0} \langle J_s \rangle . \]

This correlation function satisfies a dispersion relation identical in form to (2) (\( \Gamma \) is identified with \( s \)), implying that the first subtraction constant \( \Pi_s(0) \) is determined by the LET. This property allows extension of the sum-rule (9) to include \( k = -1 \) at the expense of introducing the LET term:

\[ L_{\tau}^{(s)}(\tau) = -\Pi_s(0) + \frac{1}{\pi} \int_{t_0}^\infty \frac{1}{t} e^{-\tau t} \rho_s(t) \, dt \]

\[ L_k^{(s)}(\tau) = \frac{1}{\pi} \int_{t_0}^\infty t^k e^{-\tau t} \rho_s(t) \, dt \quad , \quad k \geq 0 . \]

The only appearance of the \( \Pi_s(0) \) term is in the \( k = -1 \) sum-rule, and as first noted in \[3\], this LET term comprises a significant contribution in this sum-rule. From the significance of this \( (\tau) \) scale-independent term one can ascertain the important qualitative role of the LET in sum-rule phenomenology. To see this role, we first model the hadronic contributions \( \rho_s^{\text{had}}(t) \) using the narrow resonance approximation

\[ \frac{1}{\pi} \rho_s^{\text{had}}(t) = \sum_r F_r^2 m_r^2 \delta(t - m_r^2) , \]

\[ ^3 \text{For } k \leq -2 \text{ the sum-rules would have dependence on subtraction constants such as } \Pi'^2(0) \text{ not determined by the LET.} \]
where the sum over \( r \) represents a sum over resonances of mass \( m_r \). The quantity \( F_r \) is the coupling strength of the resonance to the vacuum through the gluonic current \( J_s(0) \), so the sum-rule for scalar gluonic currents probes scalar gluonium states. In the narrow-width approximation the Laplace sum-rules become

\[
R_{-1}^{(s)}(\tau, s_0) + \Pi_s(0) = \sum_r F_r^2 e^{-m_r^2 \tau} \\
R_k^{(s)}(\tau, s_0) = \sum_r F_r^2 m_r^{2k+2} e^{-m_r^2 \tau}, \quad k \geq 0.
\]

Thus if the (constant) LET term is a significant contribution on the theoretical side of (22), then the left-hand side of (22) will exhibit reduced \( \tau \) dependence relative to other theoretical contributions. To reproduce this diminished \( \tau \) dependence, the phenomenological (i.e. right-hand) side must contain a light resonance with a coupling larger than or comparable to the heavier resonances. By contrast, the absence of the \( \Pi_s(0) \) (constant) term in \( k > -1 \) sum-rules leads to stronger \( \tau \) dependence which is balanced on the phenomenological side by suppression of the lightest resonances via the additional powers of \( m_r^2 \) occurring in (23). Thus if \( \Pi_s(0) \) is found to dominate \( R_{-1}^{(s)}(\tau, s_0) \), then one would expect qualitatively different results from analysis of the \( k = -1 \) and \( k > -1 \) sum-rules.

Such distinct conclusions drawn from different sum-rules can be legitimate. In the pseudoscalar quark sector, the lowest sum-rule is dominated by the pion, and the low mass of the pion is evident from the minimal \( \tau \) dependence in the lowest sum-rule. By contrast, the first subsequent sum-rule has an important contribution from the \( \Pi(1300) \) [7], as the pion contribution is suppressed by its low mass, resulting in the significant \( \tau \) dependence of the next-to-lowest sum-rule.

In the absence of instantons [8], explicit sum-rule analyses of scalar gluonium [6, 9] uphold the above generalization—those which include the \( k = -1 \) sum-rule find a light (less than or on the order of the \( \rho \) mass) gluonium state, and those which omit the \( k = -1 \) sum-rule find a state with a mass greater than 1 GeV. The prediction of a light gluonium state would have interesting phenomenological consequences as a state which could be identified with the \( f_0(400 - 1200) \) (or \( \sigma \)) meson [10].

Work by Shuryak [11] in the instanton liquid model [12] has indicated how a large-energy asymptotic expression for the instanton contribution to the \( k = -1 \) sum-rule may serve to compensate for that sum-rule’s LET component and bring the predicted scalar gluonium mass in line with subsequent lattice estimates (\( \sim 1.6 \text{ GeV} \) [13]). Recent work by Forkel [14] has addressed in detail instanton effects on scalar gluonium mass predictions from higher-weight \( (k \geq 0) \) sum-rules and has also corroborated lattice estimates. The overall consistency of the LET-sensitive \( k = -1 \) sum-rule and the LET-insensitive \( k \geq 0 \) sum-rules has been addressed in [15], and will be reviewed in the remainder of this section.

The field-theoretical (QCD) calculation of \( \Pi(Q^2) \) consists of perturbative (logarithmic) corrections known to three-loop order \((\overline{\text{MS}} \text{ scheme})\) in the chiral limit of \( n_f = 3 \) massless quarks [16]. QCD vacuum effects of infinite correlation length parameterized by the power-law contributions from the QCD vacuum condensates [1, 17] and QCD vacuum effects of finite correlation length

\[^4\text{The calculation of one-loop contributions proportional to } \langle J_s \rangle \text{ in [17] have been extended non-trivially to } n_f = 3\]
devolving from instantons \[18]\)

\[\Pi_s(Q^2) = \Pi_s^{pert}(Q^2) + \Pi_s^{cond}(Q^2) + \Pi_s^{inst}(Q^2) \quad .\]  

(24)

The perturbative contribution (ignoring divergent terms proportional to \(Q^4\)) to (24) is

\[\Pi_s^{pert}(Q^2) = Q^4 \log \left( \frac{Q^2}{\nu^2} \right) \left[ a_0 + a_1 \log \left( \frac{Q^2}{\nu^2} \right) + a_2 \log^2 \left( \frac{Q^2}{\nu^2} \right) \right] \]  

(25)

\[a_0 = -2 \left( \frac{\alpha}{\pi} \right)^2 \left[ 1 + \frac{659 \alpha}{36 \pi} + 247.480 \left( \frac{\alpha}{\pi} \right)^2 \right], \quad a_1 = 2 \left( \frac{\alpha}{\pi} \right)^3 \left[ \frac{9}{4} + 65.781 \frac{\alpha}{\pi} \right], \]  

\[a_2 = -10.1250 \left( \frac{\alpha}{\pi} \right)^4 , \]  

the condensate contributions to (24) are

\[\Pi_s^{cond}(Q^2) = \left[ b_0 + b_1 \log \left( \frac{Q^2}{\nu^2} \right) \right] \langle J_s \rangle + c_0 \frac{1}{Q^2} \langle \mathcal{O}_6 \rangle + d_0 \frac{1}{Q^4} \langle \mathcal{O}_8 \rangle \]  

(26)

\[b_0 = 4 \pi \frac{\alpha}{\pi} \left[ 1 + \frac{175 \alpha}{36 \pi} \right], \quad b_1 = -9 \pi \left( \frac{\alpha}{\pi} \right)^2, \quad c_0 = 8 \pi^2 \left( \frac{\alpha}{\pi} \right)^2, \quad d_0 = 8 \pi^2 \frac{\alpha}{\pi} \]  

\[\langle \mathcal{O}_6 \rangle = \left\langle gf_{abc}G_{\mu\nu}^aG_{\nu\rho}^bG_{\rho\mu}^c \right\rangle, \quad \langle \mathcal{O}_8 \rangle = 14 \left\langle (\alpha f_{abc}G_{\mu\nu}^aG_{\nu\rho}^b)^2 \right\rangle - \left\langle (\alpha f_{abc}G_{\mu\nu}^aG_{\nu\rho}^b)^2 \right\rangle \]

and finally the instanton contribution to (24) is

\[\Pi_s^{inst}(Q^2) = 32 \pi^2 Q^4 \int \rho^4 \left[ K_2 \left( \rho \sqrt{Q^2} \right) \right]^2 dn(\rho) \quad , \]  

(27)

where \(K_2(x)\) represents a modified Bessel function \[19\]. The instanton contributions represent a calculation with non-interacting instantons of size \(\rho\), with subsequent integration over the instanton density distribution \(n(\rho)\).

The strong coupling constant \(\alpha\) in (23) is understood to be the running coupling at the renormalization scale \(1/\sqrt{\tau}\), and its three-loop, \(n_f = 3\), \(\overline{\text{MS}}\) running form is:

\[\frac{\alpha_s(\nu)}{\pi} = \frac{1}{\beta_0 L} - \frac{\beta_1 \log L}{(\beta_0 L)^2} \left( \log^2 L - \log L - 1 \right) + \frac{1}{(\beta_0 L)^3} \]  

(28)

\[L = \log \left( \frac{\nu^2}{\Lambda^2} \right), \quad \beta_i = \frac{\beta_i}{\beta_0}, \quad \beta_0 = \frac{9}{4}, \quad \beta_1 = 4, \quad \beta_2 = \frac{3863}{384} \]  

(29)

with \(\Lambda_{\overline{\text{MS}}} \approx 300\) MeV for three active flavours, consistent with current estimates of \(\alpha_s(M_T)\) \[20\]. The dimension-six and dimension-eight gluon condensates are referenced to the gluon condensate \(\langle \alpha G^2 \rangle\) through vacuum saturation \[8\, 21\] .

\[\langle \mathcal{O}_8 \rangle = 14 \left\langle (\alpha f_{abc}G_{\mu\nu}^aG_{\nu\rho}^b)^2 \right\rangle - \left\langle (\alpha f_{abc}G_{\mu\nu}^aG_{\nu\rho}^b)^2 \right\rangle = \frac{9}{16} \left( \langle \alpha G^2 \rangle \right)^2 \]  

(30)

from \(n_f = 0\), and the operator basis has been changed from \(\langle \alpha G^2 \rangle\) to \(\langle J_s \rangle\).
and instanton estimates [1, 2]

\[ \langle O_6 \rangle = \langle gf_{abc} G_{\mu \nu}^a G_{\rho \sigma}^b G_{\rho \mu}^c \rangle = (0.27 \text{ GeV}^2) \langle \alpha G^2 \rangle . \] (31)

The Laplace sum-rules can also be partitioned into perturbative, condensate and instanton contributions

\[ L^{(s)}_k (\tau) = L^{\text{pert}}_k (\tau) + L^{\text{cond}}_k (\tau) + L^{\text{inst}}_k (\tau). \] (32)

Calculation of the instanton contributions to these Laplace sum-rules demands particular care, and requires the use of the inverse Laplace transform in the complex \( Q^2 \) plane (8) to calculate the Borel transform. The complete details of this calculation are given in [15], but the essential aspect of the result is that the instanton contributions distinguish between \( k = -1 \) and \( k > -1 \) sum-rules, similar to the unique role played by the LET term in the \( k = -1 \) sum-rule [15].

\[ L^{\text{inst}}_{-1} (\tau) = -128 \pi^2 \int \frac{d \rho (\rho)}{\rho} - 16 \pi^3 \int \frac{d \rho (\rho) \rho^4}{\rho} \int_0^\infty t J_2 (\rho \sqrt{t}) Y_2 (\rho \sqrt{t}) e^{-t \tau} dt \] (33)

\[ L^{\text{inst}}_k (\tau) = -16 \pi^3 \int \frac{d \rho (\rho) \rho^4}{\rho} \int_0^\infty t^{k+2} J_2 (\rho \sqrt{t}) Y_2 (\rho \sqrt{t}) e^{-t \tau} dt , \quad k > -1 \] (34)

These results are interpreted as a natural partitioning of instanton contributions into an instanton continuum portion devolving from

\[ \frac{1}{\pi} \text{Im} \Pi^{\text{inst}}_s (t) = -16 \pi^3 \int \frac{d \rho (\rho) \rho^4 t^2}{\rho} J_2 (\rho \sqrt{t}) Y_2 (\rho \sqrt{t}) , \] (35)

and a contribution which, like the LET, appears only in the \( k = -1 \) sum-rule

\[ -128 \pi^2 \int \frac{d \rho (\rho)}{\rho} . \] (36)

The qualitative role of instanton effects can be investigated in the instanton liquid model [12]

\[ d \rho (\rho) = n_c \delta (\rho - \rho_c) d \rho , \quad n_c = 8 \times 10^{-4} \text{ GeV}^4 , \quad \rho_c = \frac{1}{600 \text{ MeV}} \] (37)

Thus in the \( k = -1 \) sum-rule [13] a cancellation occurs between LET and the LET-like instanton contribution (36)

\[ \frac{-\Pi_s (0) + 128 \pi^2 \int d \rho (\rho)}{\Pi_s (0)} \approx -1 + \frac{128 \pi^2 n_c}{\Pi_s (0)} \leq 0.29 \pm 0.16 \] (38)

where we have used

\[ \Pi_s (0) = \frac{32 \pi}{9} \langle J_s \rangle \geq \frac{32 \pi}{9} \langle \alpha G^2 \rangle . \] (39)
in conjunction with a recent determination of the of the gluon condensate $\langle \alpha G^2 \rangle$ \[22\]

$$\langle \alpha G^2 \rangle = (0.07 \pm 0.01) \text{ GeV}^4 .$$ \hspace{1cm} (40)

Thus we see that instantons could suppress the role of the LET by 55–85%, reducing the discrepancy between gluonium mass determinations in the $k = -1$ and $k \geq 0$ sum-rule analyses \[15\].

This qualitative conclusion is borne out in more detailed analysis. Within the instanton liquid model, the instanton contribution (33) becomes \[15\]

$$L_{\text{inst}}^{1-1}(\tau) = -64\pi^2 n_c \left[ a^2 e^{-a} [a + 1] K_0(a) + ae^{-a} [a^2 + 2a + 2] K_1(a) \right] , \quad a = \frac{\rho_c^2}{2\tau} . \hspace{1cm} (41)$$

The large-energy ($a \gg 1$) limit of (41) agrees with the asymptotic form

$$L_{\text{inst}}^{1-1}(\tau) \rightarrow -16\pi^2 n_c e^{-\frac{\rho_c^2}{2\tau}} \rho_c^5 \tau^{-\frac{5}{2}} \left[ 1 + O \left( \frac{\tau}{\rho_c^4} \right) \right] , \quad \tau \ll \rho_c^2 . \hspace{1cm} (42)$$

given in \[11\]. However, numerical comparison with (41) reveals that the higher-order terms omitted in (42) are significant in the region $a < 2$ ($\tau > 0.54 \text{ GeV}^2$), an energy region important to a sum-rule analysis, so it is necessary to use the full expression (41) for an accurate analysis of instanton effects. Conversely, the small $a$ limit of (41)

$$L_{\text{inst}}^{1-1}(\tau) \rightarrow -128\pi^2 n_c + 8\pi^2 n_c \left[ \rho_c^4 + O \left( \frac{\rho_c^6}{\tau^3} \right) \right] \hspace{1cm} (43)$$

upholds the interpretation of the partitioning of the instanton contribution into an LET-like contribution [the first term on the right-hand side of (13)], while remaining a good numerical approximation to the full expression (11) in the region $\tau \gtrsim 0.85 \text{ GeV}^{-2}$.

Ratios of Laplace sum-rules provide a bound on the mass $m$ of the lightest state contributing to the spectral function. For scalar gluonium, some of these possible ratios are\[6\]

$$\frac{L_1^{(s)}(\tau)}{L_0^{(s)}(\tau)} \geq m^2 \hspace{1cm} (44)$$

$$\frac{L_0^{(s)}(\tau)}{L_{-1}^{(s)}(\tau) + \Pi_s(0)} \geq m^2 . \hspace{1cm} (45)$$

In the absence of instantons the mass bounds obtained from the ratio (44) containing the LET are much lower than the bound obtained from the ratio (44) where the LET term is absent. However, the full form of the instanton contributions raises the mass bound from the LET-dependent ratio

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5 It should be noted that there is some discrepancy between \[22\] and the smaller value $\langle \alpha G^2 \rangle = (0.047 \pm 0.014) \text{ GeV}^4$ found in \[23\].

6 Derivation of these bounds, along with the complete sum-rule expressions needed to analyze these ratios are given in ref. \[15\].
and lowers the mass bound from the LET-independent ratio \((14)\), leading to consistent predictions from the entire family of sum-rules \([13]\). With the inclusion of instanton effects, the final mass bounds from the sum-rule ratios are approximately \([13]\)

\[
m \lesssim 1.25 \text{ GeV} ,
\]

a result consistent with a recent Gaussian sum-rule analysis of scalar gluonium \([24]\). These results suggest that there exists a (possibly small) gluonic component of the PDG-listed states \(f_0(400–1200), f_0(980), \text{or } f_0(1370)\).

## 4 Bounds on the Light Quark Masses

The light quark masses are fundamental parameters of QCD, and determination of their values is of importance for high-precision QCD phenomenology and lattice simulations involving dynamical quarks. In this paper the development of Hölder inequalities for QCD Laplace sum-rules \([25]\) is briefly reviewed. These techniques are then used to obtain bounds on the non-strange (current) quark masses \(m_n = (m_u + m_d)/2\) evaluated at 2 GeV in the MS scheme, updating and extending the Hölder inequality results of ref. \([26]\).

Although it is possible to obtain quark mass ratios in various contexts \([27]\), the only methods which have been able to determine the absolute non-strange quark mass scales are the lattice (see \([28]\) for recent results with two dynamical flavours) and QCD sum-rules \([4, 7, 26, 29, 30, 31, 32]\).

In sum-rule and lattice approaches, the pseudoscalar or scalar channels are used since they have the strongest dependence on the quark masses. This is exemplified by the correlation function \(\Pi_5 (Q^2)\) of renormalization-group (RG) invariant pseudoscalar currents with quantum numbers of the pion:

\[
J_5(x) = \frac{1}{\sqrt{2}} (m_u + m_d) \left[ \bar{u}(x)i\gamma_5 u(x) - \bar{d}(x)i\gamma_5 d(x) \right] .
\]

The particular form of \((9)\) that will be used in obtaining mass bounds is

\[
\mathcal{L}_0^{(5)}(\tau) = \frac{1}{\pi} \int \frac{\rho_5(t) e^{-\tau t} \, dt}{4m_n^2} .
\]

where \(\rho_5(t)\) is the hadronic spectral function appropriate to the pion quantum numbers.

Perturbative contributions to \(\mathcal{L}_0^{(5)}(\tau)\) are known up to four-loop order in the MS scheme \([29, 34]\). Infinite correlation-length vacuum effects in \(\mathcal{L}_0^{(5)}(\tau)\) are represented by the (non-perturbative) QCD condensate contributions \([1, 29, 35]\). In addition to the QCD condensate contributions, the pseudoscalar (and scalar) correlation functions are sensitive to finite correlation-length vacuum

\(^7\)An overview of selected lattice and sum-rule results for both non-strange and strange masses can be found in \([33]\).
effects described by direct instantons [36] in the instanton liquid model [12]. Combining all these results, the total result for \( L_0^{(5)} (\tau) \) to leading order in the light-quark masses is [26]

\[
L_0^{(5)} (\tau) = \frac{3m_n^2}{8\pi^2\tau^2} \left( 1 + 4.821098 \frac{\alpha}{\pi} + 21.97646 \left( \frac{\alpha}{\pi} \right)^2 + 53.14179 \left( \frac{\alpha}{\pi} \right)^3 \right) \\
+ m_n^2 \left( -\langle m\bar{q}q \rangle + \frac{1}{8\pi} \langle \alpha G^2 \rangle + \frac{1}{4} \pi \langle O_6 \rangle \tau \right) \\
+ m_n^2 \frac{3 \rho_c^2}{8\pi^2\tau^3} \exp \left( -\frac{\rho_c^2}{2\tau} \right) \left[ K_0 \left( \frac{\rho_c^2}{2\tau} \right) + K_1 \left( \frac{\rho_c^2}{2\tau} \right) \right],
\]

where \( \alpha \) and \( m_n = (m_u + m_d) / 2 \) are the \( \overline{\text{MS}} \) running coupling and quark masses at the scale \( 1/\sqrt{\tau} \), and \( \rho_c = 1/(600 \text{ MeV}) \) represents the instanton size in the instanton liquid model [12]. SU(2) symmetry has been used for the dimension-four quark condensates (i.e. \( (m_u + m_d) \langle \bar{u}u + \bar{d}d \rangle \equiv 4m \langle \bar{q}q \rangle \)), and \( \langle O_6 \rangle \) denotes the dimension six quark condensates

\[
\langle O_6 \rangle \equiv \alpha_s \left[ 2\langle \bar{u}\sigma_{\mu\nu}\gamma_5 T^a u \bar{u}\sigma^{\mu\nu}\gamma_5 T^a u \rangle + u \rightarrow d \right] - 4\langle \bar{u}\sigma_{\mu\nu}\gamma_5 T^a u \bar{d}\sigma^{\mu\nu}\gamma_5 T^a d \rangle \\
+ \frac{2}{3}\langle \bar{u}\gamma_\mu T^a u + \bar{d}\gamma_\mu T^a d \rangle \sum_{u,d,s} \bar{q}\gamma^\mu T^a q \right].
\]

The vacuum saturation hypothesis [1] will be used as a reference value for \( \langle O_6 \rangle \)

\[
\langle O_6 \rangle = f_{vs} \frac{448}{27} \alpha \langle \bar{q}q\bar{q}q \rangle = f_{vs} 3 \times 10^{-3} \text{GeV}^6
\]

where \( f_{vs} = 1 \) for exact vacuum saturation. Larger values of effective dimension-six operators found in [23, 37] imply that \( f_{vs} \) could be as large as \( f_{vs} = 2 \). The quark condensate is determined by the GMOR (PCAC) relation

\[
(m_u + m_d) \langle \bar{u}u + \bar{d}d \rangle = 4m \langle \bar{q}q \rangle = -2f_\pi^2 m_\pi^2
\]

where \( f_\pi = 93 \text{ MeV} \).

Note that all the theoretical contributions in (49) are proportional to \( m_n^2 \), demonstrating that the quark mass sets the scale of the pseudoscalar channel. This dependence on the quark mass can be singled out as follows:

\[
L_0^{(5)} (\tau) = \left[ m_n \left( 1/\sqrt{\tau} \right) \right]^2 G_5 (\tau)
\]

where \( G_5 \) is independent of \( m_n \) and is trivially extractable from (49). Higher-loop perturbative contributions in (49) are thus significant since they can effectively enhance the quark mass with increasing loop order.

Determinations of the non-strange quark mass \( m_n \) using the sum-rule (48) require input of a phenomenological model for the spectral function \( \rho_\pi (t) \). The mass \( m_n \) can then be determined by fitting to find the best agreement between the phenomenological model and the theoretical
prediction respectively appearing on the right- and left-hand sides of (18). For example, the simple resonance(s) plus continuum model
\[
\frac{1}{\pi} \rho_5(t) = 2 f_\pi^2 m_\pi^4 \left[ \delta \left( t - m_\pi^2 \right) + \frac{F_{1\Pi}^2 M_{1\Pi}^4}{f_\pi^2 m_\pi^4} \delta \left( t - M_{1\Pi}^2 \right) \right] + \Theta (t - s_0) \frac{1}{\pi} \rho_{QCD}(t)
\]
represents the pion pole \((m_\pi)\), a narrow-width approximation to the pion excitation \((M_{1\Pi})\) such as the \(\Pi(1300)\), and a QCD continuum above the continuum threshold \(t = s_0\). Of course more detailed phenomenological models can be considered which take into account possible width effects for the pion excitation, further resonances, resonance(s) enhancement of the \(3\pi\) continuum etc. This leads to significant model dependence which partially accounts for the spread of theoretical estimates in [4, 31, 32]. Since the common phenomenological portion of all these models is the pion pole, it is valuable to extract quark mass bounds which only rely upon the input of the pion pole on the phenomenological side of (18).

The existence of such bounds is easily seen by separating the pion pole out from \(\rho_5(t)\), in which case (18) becomes
\[
\left[ m_n \left( \frac{1}{\sqrt{\tau}} \right) \right]^2 G_5(\tau) = 2 f_\pi^2 m_\pi^4 + \frac{1}{\pi} \int_9 m_\pi^2 \rho_5(t) e^{-t\tau} \, dt . \tag{55}
\]
Since \(\rho_5(t) \geq 0\) in the integral appearing on the right-hand side of (55), a bound on the quark mass is obtained:
\[
m_n \left( \frac{1}{\sqrt{\tau}} \right) \geq \sqrt{\frac{2 f_\pi^2 m_\pi^4}{G_5(\tau)}} . \tag{56}
\]
Analysis of these bounds following from simple positivity of the “residual” portion on the right-hand side of (55) was studied in [4, 29]. Other quark mass bounds have been obtained from dispersion relation inequalities [30].

Improvements upon the positivity bound of (56) are achieved by developing more stringent inequalities based on the positivity of \(\rho_5(t)\). Since \(\rho_5(t) \geq 0\), the right-hand (phenomenological) side of (18) must satisfy integral inequalities over a measure \(d\mu = \rho_5(t) \, dt\). In particular, Hölder’s inequality over a measure \(d\mu\) is [38]
\[
\left| \int_{t_1}^{t_2} f(t)g(t) \, d\mu \right| \leq \left( \int_{t_1}^{t_2} |f(t)|^p \, d\mu \right)^{\frac{1}{p}} \left( \int_{t_1}^{t_2} |g(t)|^q \, d\mu \right)^{\frac{1}{q}}, \quad \frac{1}{p} + \frac{1}{q} = 1 ; \quad p, \, q \geq 1 , \tag{57}
\]
which for \(p = q = 2\) reduces to the familiar Schwarz inequality, implying that the Hölder inequality is a more general constraint. The Hölder inequality can be applied to Laplace sum-rules by identifying \(d\mu = \rho_5(t) \, dt\), and defining
\[
S_5(\tau) = \frac{1}{\pi} \int_{\mu_{th}}^{\infty} \rho_5(t) e^{-t\tau} \, dt \tag{58}
\]
where $\mu_{th}$ will later be identified with $9m_\pi^2$. Suitable choices of $f(t)$ and $g(t)$ in the Hölder inequality \[(57)\] yield the following inequality for $S_5(t)$ \[(58)\]:

\begin{equation}
S_5(\tau + (1 - \omega)\delta\tau) \leq [S_5(\tau)]^\omega [S_5(\tau + \delta\tau)]^{1-\omega}, \quad \forall 0 \leq \omega \leq 1.
\end{equation}

In practical applications of this inequality, $\delta\tau \leq 0.1$ GeV$^{-2}$ is used, in which case this inequality analysis becomes local (depending only on the Borel scale $\tau$ and not on $\delta\tau$) \[(59)\].

To employ the Hölder inequality \[(59)\] we separate out the pion pole by setting $\mu_{th} = 9m_\pi^2$ in \[(58)\]

\begin{equation}
S_5(\tau) = \mathcal{L}^{(5)}_0(\tau) - 2f_\pi^2m_\pi^4 = \int_{9m_\pi^2}^{\infty} \rho_\pi(t) e^{-t\tau} \, dt
\end{equation}

which has a right-hand side in the standard form \[(58)\] for applying the Hölder inequality. Note that simple positivity of $\rho_\pi(t)$ gives the inequality

\begin{equation}
S_5(\tau) \geq 0
\end{equation}

which simply rephrases \[(50)\]. Lower bounds on the quark mass $m_n$ can now be obtained by finding the minimum value of $m_n$ for which the Hölder inequality \[(59)\] is satisfied. Introducing further phenomenological contributions (e.g. three-pion continuum) give a slightly larger mass bound as will be discussed later. However, if only the pion pole is separated out, then the analysis is not subject to uncertainties introduced by the phenomenological model.

Although the details are still a matter of dispute, the overall validity of QCD predictions at the tau mass is evidenced by the analysis of the tau hadronic width, hadronic contributions to $\alpha_{EM}(M_Z)$ and the muon anomalous magnetic moment \cite{39}, so we impose the inequality \[(59)\] at the tau mass scale $1/\sqrt{\tau} = M_\tau = 1.77$ GeV. This also has the advantage of minimizing perturbative uncertainties in the running of $\alpha$ and $m_n$, since the PDG reference scale for the light-quark masses is at 2 GeV \cite{10}, in close proximity to $M_\tau$, and the result $\alpha_s(M_\tau) = 0.33 \pm 0.02$ \cite{20} can thus be used to its maximum advantage. For the remaining small energy range in which the running of $\alpha$ and $m_n$ is needed, the four-loop $\beta$-function \cite{11} and four-loop anomalous mass dimension \cite{12} with three active flavours are used, appropriate to the analysis of \cite{20}. This use of the 2 GeV reference scale for $m_n$ combined with input of $\alpha(M_\tau)$ improves upon the perturbative uncertainties in \cite{20} which employed $\alpha(M_Z)$ and a 1 GeV $m_n$ reference scale which necessitated matching through the (uncertain) $b$ and $c$ flavour thresholds.

Further theoretical uncertainties devolve from the QCD condensates as given in \cite{10} and \cite{54} with $1 \leq f_{es} \leq 2$, along with a 15% uncertainty in the instanton liquid parameter $\rho_c$ \cite{12}. The effect of higher-loop perturbative contributions to $\mathcal{L}^{(5)}_0(\tau)$ on the resulting $m_n$ bounds is estimated using an asymptotically-improved Padé estimate \cite{13} of the five-loop term, introducing a 138 $(\alpha/\pi)^3$ correction into \cite{19}. Finally, we allow for the possibility that the overall scale of the instanton is 50% uncertain.

The resulting Hölder inequality bound on the 2.0 GeV $\overline{\text{MS}}$ quark masses, updating the analysis of \cite{20}, is

\begin{equation}
m_n(2 \text{ GeV}) = \frac{1}{2}[m_u(2 \text{ GeV}) + m_d(2 \text{ GeV})] \geq 2.1 \text{ MeV}.
\end{equation}
This final result is identical to previous bounds on $m_n(1 \text{ GeV})$ [26] after conversion to 2 GeV by the PDG [40], indicative of the consistency of perturbative inputs used in the two analyses. The theoretical uncertainties in the quark mass bound (62) from the QCD parameters and (estimated) higher-order perturbative effects are less than 10%, and the result (62) is the absolute lowest bound resulting from the uncertainty analysis. The dominant sources of uncertainty are $\alpha(M_\tau)$ and potential higher-loop corrections. The instanton size $\rho_c$ is the major source of non-perturbative uncertainty, but its effect is smaller than the perturbative sources of uncertainty.

Compared with the positivity inequality (61), as first used to obtain quark mass bounds from QCD sum-rules [4, 29], the Hölder inequality leads to quark mass bounds 50% larger for identical theoretical and phenomenological inputs at $1/\sqrt{t} = M_\tau$, demonstrating that the Hölder inequality provides stringent constraints on the quark mass.

Finally, we discuss the effects of extending the resonance model to include the $3\pi$ continuum calculated using lowest-order chiral perturbation theory [32]

\[
\frac{1}{\pi} \rho_\pi(t) = 2 f_\pi^2 m_\pi^4 \left[ \delta(t - m_\pi^2) + \Theta(t - 9m_\pi^2) \right] \rho_3(t) \frac{t}{18(16\pi^2f_\pi^2)^2} \]

\[
\rho_3(t) = \int_{4m_\pi^2}^{(\sqrt{t}-m_\pi)^2} \frac{du}{t} \sqrt{\lambda(1, \frac{t}{m_\pi^2})} \sqrt{1 - \frac{4m_\pi^2}{u}} \left\{ 5 + \frac{1}{2(t - m_\pi^2)^2} \left[ \frac{4}{3} \left( t - 3 \left( u - m_\pi^2 \right) \right)^2 + \frac{8}{3} \lambda(t, u, m_\pi^2) \left( 1 - \frac{4m_\pi^2}{u} \right) + 10m_\pi^4 \right] \right\} + \frac{1}{t - m_\pi^2} \left[ 3 \left( u - m_\pi^2 \right) - t + 10m_\pi^2 \right] \lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2xz
\]

which becomes $\rho_3(t) \to 3$ in the limit $m_\pi \to 0$. Inclusion of the $3\pi$ continuum (64) is still likely to underestimate the total spectral function since more complicated models of the spectral function involve resonance enhancement of this $3\pi$ continuum [32]. If this limiting form is used up to a cutoff of 1 GeV, then the resulting Hölder inequality quark mass bounds are raised by approximately 10%, and a 14% effect is observed if the cutoff is moved to infinity. Working with the full form (64) complicates the numerical analysis, but the following simple form (with $t$ in GeV units) is easily verified to be a bound on the $3\pi$ continuum in the region below 1 GeV:

\[
\rho_3(t) \geq \frac{4}{3} \left[ \left( \sqrt{t} - m_\pi \right)^2 - 4m_\pi^2 \right] .
\]

This approximate form of the $3\pi$ continuum again raises the resulting quark mass bounds by approximately 10%.

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8 The exponential suppression of the large-$t$ region in the Laplace sum-rule [48] minimizes any errors in this region from this approximation to the $3\pi$ continuum, and also leads to the observed small difference in extending the cutoff to infinity.
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