An Approximate Bayesian Approach to Surprise-Based Learning

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Abstract

Surprise-based learning allows agents to adapt quickly in non-stationary stochastic environments. Most existing approaches to surprise-based learning and change point detection assume either implicitly or explicitly a simple, hierarchical generative model of observation sequences that are characterized by stationary periods separated by sudden changes. In this work we show that exact Bayesian inference gives naturally rise to a surprise-modulated trade-off between forgetting and integrating the new observations with the current belief. We demonstrate that many existing approximate Bayesian approaches also show surprise-based modulation of learning rates, and we derive novel particle filters and variational filters with update rules that exhibit surprise-based modulation. Our derived filters have a constant scaling in observation sequence length and particularly simple update dynamics for any distribution in the exponential family. Empirical results show that these filters estimate parameters better than alternative approximate approaches and reach comparative levels of performance to computationally more expensive algorithms. The theoretical insight of casting various approaches under the same interpretation of surprise-based learning, as well as the proposed filters, may find useful applications in reinforcement learning in non-stationary environments and in the analysis of animal and human behavior.

1 Introduction

Animals, humans, and similarly reinforcement learning agents may safely assume that the world is stochastic and stationary during some intervals of time marked by change points. The exact position and orientation of leaves on a tree, a stock market index, or the time it takes to travel from A to B in a crowded city may be well captured by stationary stochastic processes for extended periods of time. Then sudden changes may happen and the distribution of leaf positions becomes different due to a storm, the stock market index is affected by the enforcement of a new law, or a blocked road causes additional traffic jams. The violation of an agent’s expectation caused by such sudden changes is perceived by the agent as surprise, which can be seen as a measure of how much the agent’s current belief differs from reality. Surprise, with its physiological manifestations in pupil dilation [1, 2] and EEG signals [3, 4], is believed to modulate learning, potentially through the release of specific neurotransmitters [5, 6], to allow animals and humans to adapt quickly to sudden changes.

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The bulk of work on surprise-based learning has focused more on biological plausibility than accurate learning [2, 5, 7–12]. On the other hand, exact and approximate Bayesian online [13, 14] methods for change point detection and parameter estimation have been developed without any focus on biological plausibility [15, 16]. In this work, we take a top-down approach to surprise-based learning: we start with a generative model of change points and observations and derive approximate online methods that contain a surprise-modulated learning rate. Our goal was to find approximate methods that are computationally efficient and biologically plausible while sacrificing only marginally the learning accuracy. Additionally, we sought to provide theoretical insights on commonalities and differences among existing surprise-based and approximate Bayesian approaches.

2 General Framework and Related Work

2.1 The Generative Model

In order to study learning in an environment that exhibits occasional and abrupt changes we consider the following hierarchical generative model in discrete time. At each time point \( t \), the observation \( Y_t \) comes from a probability distribution with parameter \( \Theta_t \). Abrupt changes of the environment correspond to sudden changes of this parameter. At every time \( t \), there is a change probability \( p_c \in (0, 1) \) for the parameter \( \Theta_t \) to be drawn from its prior distribution \( \pi^{(0)} \) independently of its previous value, and a probability \( 1 - p_c \) to stay the same as \( \Theta_{t-1} \). A change at time \( t \) is specified by the event \( \Delta H_t = 1 \); otherwise \( \Delta H_t = 0 \). We sample \( \Theta_1 \) from the prior \( \pi^{(0)} \), and for \( t \geq 2 \) the generative model is

\[
\Delta H_t \sim \text{Bernoulli}(p_c), \quad \Theta_{t-1} \quad \xrightarrow{\Delta H_{t-1}} \quad \Theta_t \quad \xrightarrow{\Theta_t} \quad Y_t
\]

\[
P(\Theta_t = \theta | \Delta H_t = \Delta h_t, \Theta_{t-1} = \theta') = \begin{cases} 
\delta(\theta - \theta') & \text{if } \Delta h_t = 0, \\
\pi^{(0)}(\theta) & \text{if } \Delta h_t = 1,
\end{cases}
\]

\[
P(Y_t = y | \Theta_t = \theta) = P_Y(y | \theta).
\]

Random variables are indicated by capital letters, and values by small letters. \( P \) stands for either probability density function (for the continuous variables) or probability mass function (for the discrete variables), and \( \delta \) is the Dirac or Kronecker delta distribution respectively. \( P_Y \) is the time-invariant likelihood function.

Given a sequence of observations \( Y_{1:t} = y_{1:t} \equiv (y_1, \ldots, y_t) \), the agent’s belief \( \pi(t)(\theta) \) about the parameter \( \theta \) at time \( t \) is defined as the posterior probability distribution \( P(\Theta_t = \theta | Y_{1:t} = y_{1:t}) \) of the parameter \( \Theta_t \). In the online learning setting studied here, the agent’s goal is to update the belief \( \pi(t)(\theta) \) to the new belief \( \pi(t+1)(\theta) \), or an approximation thereof, upon observing \( Y_{t+1} = y_{t+1} \).

2.2 Contributions

First we demonstrate that exact Bayesian inference on the generative model in Eq. 1 leads to an explicit trade-off between integrating the new observations with the old belief into a distribution \( \pi^\text{integration} \) and forgetting the past observations, so as to restart with the belief \( \pi^\text{reset} \)

\[
\pi^\text{new}(\theta) = (1 - \gamma)\pi^\text{integration}(\theta | y^\text{new}, \pi^\text{old}) + \gamma \pi^\text{reset}(\theta | y^\text{new}, \pi^0).
\]  

This trade-off is governed by a surprise modulated learning rate

\[
\gamma(S, m) = \frac{m S}{1 + m S} \in [0, 1],
\]

where \( S \geq 0 \) has the natural interpretation of the surprise of the most recent observation, and \( m \geq 0 \) is a parameter controlling the effect of surprise on learning. The exact definitions of \( \pi^\text{integration}, \pi^\text{reset} \), and \( S \) will be given in the Results section.

Second, we propose two approximate algorithms (Particle Filtering and Variational SMiLe) which inherit the explicit trade-off and its surprise modulated learning rate from the exact Bayesian approach. Our methods are computationally efficient and biologically plausible; Particle Filtering is shown
to have a neuronal implementation in its general form [17], whereas Variational SMiLe can be implemented with simple update rules for the exponential family of distributions. Moreover, empirical results show that the performance of the two approximate algorithms is comparable to and more robust across environments than other state-of-the-art approximations.

Finally, we interpret existing related algorithms in a unifying way, under the light of this surprise-modulated trade-off in Eq. 2.

2.3 Related Work

Exact Bayesian Inference. For the generative model in Eq. 1, it is possible to find an exact online Bayesian update of the belief using a message passing algorithm [13, 14]. This algorithm’s space and time complexity increases quadratically with t, which makes it unsuitable for a continual learning setting. However, simple approximations like dropping messages below a certain threshold [13] or stratified resampling [14] allow to reduce the computational complexity. Interpretation of these last approaches under our theoretical work, as well as their relationship to our algorithms are discussed in the Supplementary Material.

Leaky Integration and Variations of Delta-Rules. In order to estimate some sufficient statistic, leaky integration of new observations is a particularly simple form of trade-off between integrating and forgetting. After a transient phase, the update of a leaky integrator takes the form of a delta-rule that can be seen as an approximation of corresponding exact Bayesian updates [18–20]. This update rule was found to be biologically plausible and consistent with human behavioral data [18, 20]. However, [12, 19] demonstrated that in some situations, the exact Bayesian model is significantly better than leaky integration in explaining human behavior. The inflexibility of leaky integration with a single, constant leak parameter can be overcome by a weighted combination of multiple leaky integrators [21], where the weights are updated in a similar fashion as in the exact online methods [13, 14], or by considering an adaptive leak parameter [2, 7]. The latter [2, 7] bear close connections to our work, which are further discussed in the Supplementary Material.

Other approaches. Learning in the presence of abrupt changes has also been considered without an explicit assumption about the underlying generative model. One approach uses a surprise modulated learning rate [8] similar to Eq. 3. Other approaches use different generative models, e.g. conditional sampling of the parameters also when there is a change [5], deeper hierarchy without fixed change probability p_c [16], or models with drift in the parameters [22, 23]. In the signal processing literature we find further approaches to address the problem of learning in nonstationary environments with abrupt changes (see [15] for a review, and [24, 25] for two recent examples).

3 Theoretical Results

3.1 Recursive Bayesian Inference

Using Bayes’ rule, our aim is to find a rule to update the belief $\pi^{(t)}(\theta) \equiv P(\Theta_t = \theta | Y_{1:t} = y_{1:t})$ to the new belief

$$
\pi^{(t+1)}(\theta) = \frac{P(Y_{t+1} = y_{t+1} | \Theta_{t+1} = \theta)P(\Theta_{t+1} = \theta | Y_{1:t} = y_{1:t})}{P(Y_{t+1} = y_{t+1} | Y_{1:t} = y_{1:t})}.
$$

The first term in the numerator is the likelihood of the current observation given its parameter, and the second term is the agent’s estimated probability distribution of $\Theta_{t+1}$ before observing $Y_{t+1} = y_{t+1}$. Since there is always the possibility of an abrupt change, the second term is not the agent’s previous belief $\pi^{(t)}$, but $P(\Theta_{t+1} = \theta | Y_{1:t} = y_{1:t}) = (1-p_c)\pi^{(t)}(\theta) + p_c\pi^{(0)}(\theta)$ (see Supplementary Material). As a result, it is possible to find a recursive formula for updating the belief. For the derivation of this recursive rule, we define the following terms.

**Definition 1** The probability of observing $Y_t = y_t$ with the belief $\pi^{(t)}$ is

$$
P(y_t; \pi^{(t)}) = \int P_Y(y_t | \theta) \pi^{(t)}(\theta) d\theta,
$$
Definition 2 Under the assumption of no change $\Delta h_{t+1} = 0$, and using the most recent belief $\pi^{(t)}$ as prior, the exact Bayesian update for $\pi^{(t+1)}$ is

$$\pi_{B}^{(t+1)}(\theta) = \frac{\pi^{(t)}(\theta)P_{Y}(y_{t+1}|\theta)}{P(y_{t+1};\pi^{(t)})}.$$  \hspace{1cm} (6)

Note that $\pi_{B}^{(t+1)}(\theta)$ corresponds to the term $\pi^{t\text{integration}}$ of Eq. 2; it is the incorporation of the new information into the current belief via Bayesian updating.

Definition 3 The “Generative Model Surprise” of the observation $Y_{t+1} = y_{t+1}$ is defined as the ratio of the probability of observing $Y_{t+1} = y_{t+1}$ given $\Delta h_{t+1} = 1$ (i.e. when there is a change), to the probability of observing $Y_{t+1} = y_{t+1}$ given $\Delta h_{t+1} = 0$ (i.e. when there is no change), i.e.

$$S_{GM}(y_{t+1};\pi^{(t)}) = \frac{P(y_{t+1};\pi^{(0)})}{P(y_{t+1};\pi^{(t)})}.$$  \hspace{1cm} (7)

This definition of surprise measures how much more probable the current observation is under the naive prior $\pi^{(0)}$ relative to the current belief $\pi^{(t)}$ (see Supplementary Material for further discussion and interpretation). We emphasize that this definition is not arbitrary, but it is a term that allows us to write the exact inference on the generative model in a recursive form.

Using the above definitions and Eq. 4, we find the recursive update rule (see Supplementary Material)

$$\pi^{(t+1)}(\theta) = (1 - \gamma_{t+1})\pi_{B}^{(t+1)}(\theta) + \gamma_{t+1}P(\theta|y_{t+1}),$$  \hspace{1cm} (8)

where $P(\theta|y_{t+1}) = P(\Theta_{t+1} = \theta|Y_{t+1} = y_{t+1})$, and $\gamma_{t+1} = \gamma(S,m)$ as in Eq. 3, with $S = S_{GM}(y_{t+1};\pi^{(t)})$ as in Eq. 7 and $m = \frac{p_{c}}{1-p_{c}}$. The recursive formula of Eq. 8 shows an explicit trade-off between integrating the new sample with the old information and forgetting the previous observations. The weight $\gamma_{t+1}$ of this convex sum is modulated by surprise in light of the new observation. Since the parameter of modulation $m$ is equal to $\frac{p_{c}}{1-p_{c}}$, the effect of surprise on learning increases when the environment is more volatile, i.e. when the change probability $p_{c}$ increases.

Despite the simplicity of the recursive formula in Eq. 8, the updated belief $\pi^{(t+1)}$ is generally not in the same family of distributions as the previous belief $\pi^{(t)}$, e.g. the result of averaging two normal distributions is not a normal distribution. Hence it is in general impossible to find a simple and exact update rule for e.g. some sufficient statistic. In the following sections, we investigate two approximations that have simple update rules.

3.2 Particle Filtering

The exact Bayesian update can also be performed by marginalization of $P(\Theta_{t+1}, \Delta h_{1:t+1}|Y_{1:t+1} = y_{1:t+1})$ over $\Delta h_{1:t+1}$. As a result of this marginalization, the agent’s belief is $\pi^{(t+1)}(\theta) = \sum_{\Delta h_{1:t+1}} P(\theta|\Delta h_{1:t+1}, y_{1:t+1})P(\Delta h_{1:t+1}|y_{1:t+1})$, where we dropped the explicit mentioning of the random variables, e.g. $Y_{1:t+1}$, and display only their values, e.g. $y_{1:t+1}$, to shorten notation. The first term is simple to compute, because when $\Delta h_{1:t+1}$ is known, inference depends only on the observations after the last change point. However, since the computation of the term $P(\Delta h_{1:t+1}|y_{1:t+1})$ is difficult and the summation over all hidden states is computationally costly, in this section, we approximate this term via particle filtering [26], i.e.

$$P(\Delta h_{1:t+1}|y_{1:t+1}) \approx \sum_{i=1}^{N} w_{t+1}^{(i)} \delta(\Delta h_{1:t+1} - \Delta h_{1:t+1}^{(i)}),$$  \hspace{1cm} (9)

where $\{\Delta h_{1:t+1}^{(i)}\}_{i=1}^{N}$ is a set of $N$ realizations (particles) drawn from a proposal distribution $Q(\Delta h_{1:t+1}|y_{1:t+1})$ and $\{w_{t+1}^{(i)}\}_{i=1}^{N}$ are their corresponding weights at $t+1$.

Hence the approximated belief is

$$\hat{\pi}^{(t+1)}(\theta) = \sum_{i=1}^{N} w_{t+1}^{(i)} \hat{\pi}^{(t+1)}(\theta) = \sum_{i=1}^{N} w_{t+1}^{(i)} P(\theta|\Delta h_{1:t+1}^{(i)}, y_{1:t+1}),$$  \hspace{1cm} (10)
where $\pi_i^{(t+1)}(\theta)$ is the approximated belief corresponding to particle $i$. The update procedure includes two steps: 1. Updating the weights, and 2. Sampling the new hidden state $\Delta h_{t+1}$ for each particle. The first step amounts to

$$w_{t+1}^{(i)} = \frac{P(y_{t+1}; \hat{\pi}_i^{(t)})}{P(y_{t+1}; \tilde{\pi}_i^{(t)})} w_t^{(i)},$$

and $\gamma_{t+1} = \gamma\left(S_{GM}(y_{t+1}; \hat{\pi}_i^{(t)}), m = \frac{p_e}{1-p_e}\right)$, and $\{w_{t+1}^{(i)}\}_{i=1}^N$ are the weights corresponding to a Bayesian update $\hat{\pi}_B^{(t+1)}$ (Eq. 6; see Supplementary Material for the derivation). As a second step we sample each particle’s hidden state $\Delta h_{t+1}^{(i)}$ from the proposal distribution with the stay probability

$$Q(\Delta h_{t+1}^{(i)} = 0|\Delta h_{t}^{(i)}; y_{1:t+1}) = 1 - \gamma\left(S_{GM}(y_{t+1}; \hat{\pi}_i^{(t)}), m = \frac{p_e}{1-p_e}\right).$$

Interestingly, the above formulas are in the same spirit as Eq. 8. For the weight update there is a trade-off between an exact Bayesian update and keeping the value of the previous time step, controlled by a learning rate modulated exactly in the same way as in Eq. 8. Note that in contrast to Eq. 8, the trade-off between an exact Bayesian update and keeping the value of the previous time step, controlled by a learning rate modulated exactly in the same way as in Eq. 8. Note that in contrast to Eq. 8, the trade-off between an exact Bayesian update and forgetting, since for a particle whose state is changed, the approximated belief $\hat{\pi}_i^{(t+1)}$ is equal to $P(\theta|y_{t+1})$.

In order to avoid degeneracy of the weights, we employed the Sequential Importance Resampling algorithm [26, 27] in our implementation of particle filtering (See Supplementary Material for derivations and more details).

### 3.3 Variational SMiLe Rule

In order to keep the updated belief in the same family as the previous beliefs one possibility is to apply the weighted averaging of the exact Bayesian update rule (Eq. 8) to the logarithm of the beliefs rather than their normal forms, i.e.

$$\log\left(\hat{\pi}_i^{(t+1)}(\theta)\right) = (1 - \gamma_{t+1}) \log\left(\hat{\pi}_i^{(t+1)}(\theta)\right) + \gamma_{t+1} \log\left(P(\theta|y_{t+1})\right) + \text{Const.},$$

where $\gamma_{t+1} = \gamma\left(S_{GM}(y_{t+1}; \hat{\pi}_i^{(t)}), m\right)$ takes the same functional form as for the exact Bayesian update, but $m$ is a positive free parameter which can be tuned to each environment. By doing so, we still have the explicit trade-off of Eq. 8. The advantageous consequence of averaging over logarithms is that, if the initial belief $\pi(0)$ is the conjugate prior of the likelihood function $P_y$, then we always have $\pi(t+1)$ and $\hat{\pi}(t+1)$ in the same family, which applies in particular to distributions from the exponential family. This results in a simple update rule for the parameters of $\hat{\pi}(t+1)$.

One way to interpret this new update rule is to rewrite it as the solution of a constraint optimization problem

$$\hat{\pi}(t+1)(\theta) = \arg\min \mathbf{D}_{KL}\left[q(\theta)||\hat{\pi}_B^{(t+1)}(\theta)\right],$$

subject to $\mathbf{D}_{KL}\left[q(\theta)||P(\theta|y_{t+1})\right] < B_{t+1},$

where $B_{t+1} \in [0, \mathbf{D}_{KL}(\hat{\pi}_B^{(t+1)}(\theta)||P(\theta|y_{t+1})]$ is a decreasing function of $\gamma_{t+1}$ at each timestep (see Supplementary Material for the derivation). According to Eq. 14, the updated belief is a variational approximation of $P(\theta|y_{t+1})$. Because of its similarity to the Surprise Minimization Learning rule “SMiLe” [8], we call this approach “Variational Surprise Minimization Learning” rule, or in short “Variational SMiLe” rule.

### 3.4 Application to the Exponential Family

For both Particle Filtering and Variational SMiLe, we derive compact update rules for $\hat{\pi}(t+1)(\theta)$ when the likelihood function is in the exponential family and $\pi(0)$ is its conjugate prior. The resulting update rules are easy to implement. The pseudocode for Particle Filtering and Variational SMiLe can be found in the Supplementary Material.
3.5 Modifications and extensions of related approaches

In order to enable fair comparisons in simulations and to allow for a comparative discussion from a theoretical point of view, we modified or extended existing related approaches. In the surprise measure defined by [8], the prior \( \pi^{(0)} \) is always a Uniform distribution. We used the generative model prior and simplified the implementation of the modulated learning rate. The algorithms of [2, 7] were specifically developed for the case of a Uniform prior with a range of values much larger than the range of the (Gaussian) likelihood function. We extended their approaches to a more general case where the prior is a Gaussian distribution with arbitrary variance. We implemented the message passing algorithm of [13] and an additional simplified version of it, where we simply keep a fixed number of particles at each time step, the ones with the highest weights. All modifications, extensions and comparative interpretations can be found in the Supplementary Material.

4 Simulations

We evaluated our algorithms on two tasks, a Gaussian and a Categorical estimation task. We compared our algorithms to the online Bayesian Message Passing algorithm [13] (MP Bayes), a simpler variation thereof – inspired by the work of [14] – (MP), the (extended) reduced Bayesian algorithm of [7] (reduced Bayes’10), the (extended) reduced Bayesian algorithm of [2] (reduced Bayes’12), a slightly modified version of SMiLe [8], and a simple Leaky Integrator. The MP Bayes and the MP algorithms come from the field of change point detection. The first has high memory demands and the latter have same memory demands as the Particle Filters we implemented. Note that we also compared to the original algorithm of [14] but found that the simpler MP gave rise to better performance, we therefore report the results of the latter here. The reduced Bayes’10 and ’12 and the SMiLe algorithm come from the human learning literature and are more biologically oriented. More details on the aforementioned algorithms as well as the pseudocode of the modified SMiLe rule can be found in the Supplementary Material.

4.1 Gaussian estimation task

The goal of the agent is to estimate the mean \( \mu_t \) of observed samples, which are drawn from a Gaussian distribution with known variance \( \sigma^2 \), i.e. \( y_{t+1}|\mu_{t+1} \sim \mathcal{N}(\mu_{t+1}, \sigma^2) \). The mean \( \mu_{t+1} \) is itself drawn from a Gaussian distribution \( \mu_{t+1} \sim \mathcal{N}(0, 1) \) whenever the environment changes. An example of the task can be seen in Fig. 1A. We simulated the task for all combinations of \( \sigma \in \{0.1, 0.5, 1, 2, 5\} \) and \( p_c \in \{0.1, 0.05, 0.01, 0.005, 0.001, 0.0001\} \). For each combination of \( \sigma \) and \( p_c \), we first tuned the free parameter of each algorithm, i.e. \( m \) for SMiLe and Variational SMiLe, and the leak parameter \( \alpha \) for the Leaky Integrator, by minimizing the mean squared error on three random initializations of the task. For the Particle Filter (pf), the MP Bayes, the MP, the reduced Bayes’10 and the reduced Bayes’12, the true \( p_c \) of the environment was indeed the value that gave the best performance and we used this value for the simulations.

We evaluated the performance of the algorithms on ten different random task instances for \( 10^5 \) steps each. Note that the parameter \( \sigma \) is known to all algorithms, apart from the Leaky Integrator.

In Fig. 1B we show the mean squared estimation error of each algorithm for \( n \) steps after a change in the environment, over multiple changes, for two exemplar task settings. The Particle Filter with 10 and 20 particles (pf10 and pf20), and the reduced Bayes’12 have a performance very close to that of the MP Bayes algorithm, with much lower memory requirements. The MP algorithm with 10 and 20 particles (MP10 and MP20) is the closest to MP Bayes for low \( \sigma \) (Fig. 1B, left panel), but its performances deteriorates for the case of high \( \sigma \) and low \( p_c \) levels (Fig. 1B, right panel). Variational SMiLe exhibits very good performance as well. It sometimes outperforms the other algorithms early after an environmental change (Fig. 1B, right panel), but shows slightly higher error values at later phases. For the Leaky Integrator we observe a trade-off between good performance in the transient phase and the stationary phase; a fixed \( \alpha \) value cannot fulfil both requirements. The Modified SMiLe rule, by construction, never narrows its belief \( \hat{\pi}(\theta) \) below some minimal value, which allows it to have a very low – sometimes the lowest – error immediately after a change, but leads to high errors subsequently. The reduced Bayes’10 performs sufficiently well for lower \( \sigma \), but not for higher values. The Particle Filter with 1 particle is in expectation similar to reduced Bayes’10 and reduced Bayes’12 (See Supplementary Material for derivation and discussion), and its performance is governed by the noise that the sampling of a single particle entails. It therefore performs worse than the two reduced
Figure 1: **Gaussian estimation task.** A. At each timestep an observation (depicted as black dots) is drawn from a Gaussian distribution with changing mean $\mu_t$ (marked in blue) and known variance $\sigma^2$. At every change of the environment (marked with red lines) a new mean $\mu_t$ is drawn from a standard Normal distribution. In this example: $\sigma = 1$ and $p_c = 0.01$. B. Mean squared error for the estimation of $\mu_t$ at each timestep after an environmental change, for $\sigma = 0.1$, $p_c = 0.1$ (left panel) and $\sigma = 5$, $p_c = 0.01$ (right panel). The shaded area corresponds to the standard error of the mean. C. Mean squared error of the MP Bayes algorithm for each combination of $\sigma$ and $p_c$. D. Difference between the mean squared error of each algorithm from the MP Bayes (of panel C.). The colorbar of panel C applies to this panel as well.
Figure 2: **Categorical estimation task.** A. At each timestep the agent sees one out of 5 possible categories (marked in black) drawn from a Categorical distribution with parameters $p_t$. Occasional abrupt changes happen with probability $p_c$ and are marked with red lines. After each change a new $p_t$ vector is drawn from a Dirichlet distribution with stochasticity parameter $s$. In this example: $s = 1$ and $p_c = 0.01$. B. Mean squared error for the estimation of $p_t$ at each timestep after an environmental change, for $s = 0.14$, $p_c = 0.01$ (left panel) and $s = 5$, $p_c = 0.005$ (right panel). The shaded area corresponds to the standard error of the mean. C. Mean squared error of the MP Bayes algorithm for each combination of environmental parameters $s$ and $p_c$. D. Difference between the mean squared error of each algorithm from the MP Bayes (of panel C.). The colorbar of panel C applies to this panel as well.
In this task, the goal of the agent is to estimate the occurrence probability of five possible states. Each observation $y_{t+1} \in \{1, \ldots, 5\}$ is drawn from a Categorical distribution with parameters $p_{t+1}$, i.e. $y_{t+1} \mid p_{t+1} \sim \text{Cat}(y_{t+1} \mid p_{t+1})$. When there is a change $\Delta H_{t+1} = 1$ in the environment, the parameters $p_{t+1}$ are drawn from a Dirichlet distribution $\text{Dir}(s \cdot 1)$, where $s \in (0, \infty)$ is the stochasticity parameter. An illustration of this task is depicted in Fig. 2A. We considered all combinations of stochasticity levels $s \in \{0.01, 0.1, 0.14, 0.25, 1, 2, 5\}$ and change probability levels $p_c \in \{0.1, 0.05, 0.01, 0.005, 0.001, 0.0001\}$. The algorithms of [2, 7] were specifically developed for a Gaussian estimation task and their extension to a Categorical task is not be straightforward. Similarly
We plot the mean regret for the following parameter combinations: A. $s = 0.14$ and $p_c = 0.04$, B. $s = 0.14$ and $p_c' = 0.004$, C. $s = 5$ and $p_c = 0.04$, D. $s = 5$ and $p_c' = 0.004$.

As before, the Particle Filter pf20 and the MP20 have a performance closest to that of MP Bayes (Fig. 2B); Particle Filtering performs better for high $s$ and MP20 performs better for low $s$. The MP10 performs also very well. Variational SMiLe is the next in the ranking, with a behavior after a change similar to the Gaussian task. For all algorithms, except for the MP10 and MP20, the highest deviations from the MP Bayes are observed for medium stochasticity levels (Fig. 2D). When the environment is nearly deterministic (e.g. $s = 0.001$ so that the parameter vectors $p$ have almost all mass concentrated in one component), or highly stochastic (e.g. $s > 1$ so that nearly uniform Categorical distributions are more likely to be sampled), these algorithms achieve higher performance, while the Particle Filter is the one that is most resilient against choice of the stochasticity parameter $s$. For the Variational SMiLe in particular, the lowest mean error is achieved for the extreme cases of high $s$ with high $p_c$ and low $s$ with low $p_c$. In summary, for the same memory demands MP10 and MP20 are less robust across stochasticity levels compared to pf10 and pf20.

**4.3 Robustness against suboptimal parameter choice**

To investigate the robustness of the algorithms to a mismatch between the assumed and the actual probability of change points, we first tuned each algorithm’s parameter for an environment with a change probability $p_c$, and then tested the algorithms in environments with different change probabilities, while keeping the parameter fixed. For each new environment with a different change probability, we calculated the difference between the mean squared error of these fixed parameters and the minimum possible mean squared error of the MP Bayes algorithm, i.e. the resulting mean squared error for the case that the MP Bayes’ parameter is tuned for the actual $p_c$. More precisely, if we denote as $E_a(m_{p'}, p_c)$ the mean squared error of an algorithm $a$ with parameters $m_{p'}$ – i.e. parameters tuned for an environment with $p_c'$ – applied in an environment with $p_c$, we calculated the
quantity $E_{0}(m_{p_{c}'}, p_{c}) - E_{\text{MP Bayes}}(m_{p_{c}}, p_{c})$, for each algorithm $a$. We call this quantity mean regret. The lower the values and the flatter the curve of the mean regret are, the better the performance and the robustness of the algorithm in the face of lacking knowledge of the environment. The flatness of the curve indicates the degree of deviations of the performance as we move away from the optimally tuned parameter. We ran three random (and same for all algorithms) task initializations for each $p_{c}$ level.

In Fig. 3 we plot the mean regret for each algorithm for the Gaussian task for 4 pairs of $s$ and $p_{c}'$ levels. For $\sigma = 0.1$ and $p_{c}' = 0.04$ (Fig. 3A) MP Bayes and the MP algorithms show the highest robustness (smaller regret) and are closely followed by the Particle Filter, the Variational SMiLe, and the reduced Bayes’12 (note the regret’s small range of values). The lowest the actual $p_{c}$, the highest the regret, but still the changes are very small. The curve for the SMiLe is also quite flat, but the mean regret is much higher. The same holds for the Leaky Integrator. For $\sigma = 0.1$ and $p_{c}' = 0.004$ (Fig. 3B) MP Bayes, MP, Particle Filtering and Variational SMiLe have very similar robustness levels. The robustness for the Leaky Integrator deteriorates a lot as the actual $p_{c}$ increases. In Fig. 3C and Fig. 3D we plot the mean regret for $\sigma = 5$, and $p_{c}' = 0.04$ and $p_{c}' = 0.004$ respectively. For this high stochasticity level the optimal values for the parameter of the Leaky Integrator were around $0.98 - 0.99$ regardless of the $p_{c}'$ level. This means that in a highly stochastic environment the optimal behavior for the Leaky Integrator is to constantly integrate new observations to its belief, i.e. to act like a Perfect Integrator. This feature makes it blind to the $p_{c}$ and therefore very robust against the lack of knowledge of it (Fig. 3C). The rest of the algorithms are more sensitive to $p_{c}$ changes. The Particle Filter is more robust than the MP algorithms, especially for lower $p_{c}'$, as we saw in the previous subsections. The MP algorithms exhibit high fluctuations in their performance, likely because they are biased estimators. The reduced Bayes’12 is quite robust in this $\sigma$ level (Fig. 3C and D). Overall for MP Bayes, Particle Filtering, Variational SMiLe and reduced Bayes’12, a mismatch of the assumed $p_{c}$ from the actual one does not deteriorate the performance dramatically for $\sigma = 5$, $p_{c}' = 0.004$ (Fig. 3D). The MP Bayes is the most robust for low $p_{c}'$ if $p_{c} < p_{c}'$ (Fig. 3D). If $p_{c} > p_{c}'$, the reduced Bayes’10 seems to be slightly more robust, likely for reasons similar to the case of Leaky Integrator.

In summary, most of the time, the mean regret for MP Bayes, MP10, and MP20 is less than or equal to the mean regret for pf10 and pf20. However, the variability in the mean regret for pf10 and pf20 is smaller, and their curves are flatter across $p_{c}$ levels, which makes their performance more predictable. The results for the Categorical estimation task are similar to the Gaussian task (Fig. 4).

5 Conclusion

We have shown that performing exact Bayesian inference on the generative model of interest naturally leads to a definition of surprise and a surprise modulated adaptive learning rate, which is similar to one that has previously been proposed in the neuroscience literature with heuristic arguments [2, 7, 8]. We have proposed two approximate algorithms for learning in non-stationary environments, which exhibit the surprise modulated learning rate of the exact Bayesian approach. Empirically we observed that our algorithms achieve levels of performance comparable to approximate Bayesian methods with higher memory demands, and are more resilient across different environments compared to methods with similar memory demands. Our methods may find application in a model-based reinforcement learning setting, where it is desirable to have computationally efficient methods with low approximation errors. Our definition of surprise may be of interest for the active field of research on quantitative measures of surprise [28–30, 8] (See Supplementary Material for further discussion on connections between the Generative Model Surprise and other surprise measures). Building on the body of literature on three-factor learning rules [6], where a third factor indicating reward or surprise enables a synaptic change or a belief update [5, 31], our theoretical results may offer interesting interpretations of behavioral and neurophysiological data.

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6 Supplementary Material

6.1 Derivation of the Recursive Bayesian Formula

We denote the posterior given a change in the environment as follows:

\[ \pi^{(t+1)}(\theta) = \frac{P(Y_{t+1} = y_{t+1}|\Theta_{t+1} = \theta)P(\Theta_{t+1} = \theta|Y_{1:t} = y_{1:t})}{P(Y_{t+1} = y_{t+1}|Y_{1:t} = y_{1:t})} \]  

(S1)

The second term in the numerator of Eq. S1 can be written as

\[ P(\Theta_{t+1} = \theta|Y_{1:t} = y_{1:t}) = \sum_{i \in \{0, 1\}} P(\Theta_{t+1} = \theta|Y_{1:t} = y_{1:t}, \Delta H_{t+1} = i)P(\Delta H_{t+1} = i|Y_{1:t} = y_{1:t}) \]

(S2)

The denominator in Eq. S1 can be written as

\[ P(Y_{t+1} = y_{t+1}|Y_{1:t} = y_{1:t}) = \int P_Y(y_{t+1}|\theta)P(\Theta_{t+1} = \theta|Y_{1:t} = y_{1:t})d\theta \]

(S3)

\[ = (1 - p_c)\int P_Y(y_{t+1}|\theta)\pi^{(t)}(\theta)d\theta + p_c \int P_Y(y_{t+1}|\theta)\pi^{(0)}(\theta)d\theta \]

where we used the definition in Eq. 4 from the main text. Using these two expanded forms, Eq. S1 can be rewritten, exploiting the definition of Eq. 6 from the main text:

\[ \pi^{(t+1)}(\theta) = \frac{P(Y_{t+1} = y_{t+1}|\Theta_{t+1} = \theta)P(\Theta_{t+1} = \theta|Y_{1:t} = y_{1:t})}{(1 - p_c)\int P_Y(y_{t+1}|\theta)\pi^{(t)}(\theta)d\theta + p_c \int P_Y(y_{t+1}|\theta)\pi^{(0)}(\theta)d\theta} \]

(S4)

We denote the posterior given a change in the environment as follows:

\[ P(\theta|y_{t+1}) = P(\theta|y_{t+1}, \text{change}) = \frac{\pi^{(0)}(\theta)P_Y(y_{t+1}|\theta)}{1 + \frac{\gamma_{t+1}}{1 - p_c}P(\theta|y_{t+1})} \]

(S5)

We can then write Eq. S4:

\[ \pi^{(t+1)}(\theta) = \frac{(1 - p_c)P(y_{t+1}; \pi^{(0)})\pi^{(t+1)}(\theta) + p_c P(y_{t+1}; \pi^{(t)})P(\theta|y_{t+1})}{1 + \frac{\gamma_{t+1}}{1 - p_c}P(\theta|y_{t+1})} \]

(S6)

where

\[ \gamma_{t+1} = \gamma \left( S_{GM}(y_{t+1}; \pi^{(t)}), m = \frac{p_c}{1 - p_c} \right) \]

(S7)

with \( S_{GM} \) as defined in Eq. 7 of the main text.

6.2 Derivation of the Optimization-Based Formulation of Variational SMiLe Rule

To derive the optimization-based update rule for the Variational SMiLe rule, we used the same approach used in [8].

Consider the general form of the following variational optimization problem:

\[ q^*(\theta) = \arg\min D_{KL}[q(\theta)||p_1(\theta)] \]

(S8)

\[ q(\theta) \text{ s.t. } D_{KL}[q(\theta)||p_2(\theta)] < B \text{ and } E_q[1] = 1, \]

where \( B \in [0, D_{KL}[p_1(\theta)||p_2(\theta)]] \), and on the extremes of \( B \), we will have trivial solutions:

\[ q^*(\theta) = \begin{cases} p_2(\theta) & \text{if } B = 0 \\ p_1(\theta) & \text{if } B = D_{KL}[p_1(\theta)||p_2(\theta)] \end{cases} \]

(S9)
where \( B \) is specified so that it satisfies the constraint. Although Eq. S16 looks very similar to Eq. 8 of [8], and as result \( γ \), and as result \( CC \) and \( CC \), the authors showed that the solution to this optimization problem is:

\[
\hat{\pi}^{(t+1)}(\theta) = \text{argmin} D_{KL}[q(\theta)||P(\theta|y_{t+1})] \quad q(\theta) \text{ s.t. } D_{KL}[q(\theta)||\hat{\pi}^{(t)}(\theta)] < B_{t+1},
\]

where \( B_{t+1} \in [0, D_{KL}[P(\theta|y_{t+1})||\hat{\pi}^{(t)}(\theta)] \] is an arbitrary bound. The authors showed that the solution to this optimization problem is:

\[
\log(\hat{\pi}^{(t+1)}(\theta)) = (1 - \gamma_{t+1}) \log(\hat{\pi}^{(t)}(\theta)) + \gamma_{t+1} \log(P(\theta|y_{t+1})) + \text{Const.,}
\]

where \( \gamma_{t+1} \in [0, 1] \) is specified so that it satisfies the constraint. Although Eq. S16 looks very similar to Eq. 8 of the main text, it signifies a trade-off between the latest belief \( \hat{\pi}^{(t)} \) and the belief updated by only the most recent observation \( P(\theta|y_{t+1}) \), whereas in the approaches we analyzed the trade-off is (Eq. 8 in the main text) between integrating the new observation to the old ones (i.e. \( \pi_{[t+1]}^\prime \) and \( P(\theta|y_{t+1}) \)). To modulate the learning rate by surprise, [8] considered the boundary \( B_{t+1} = B_{\text{max}} \gamma \left( S_{CC}(y_{t+1}; \hat{\pi}^{(t)}), m \right) \), where \( m \) is a free parameter, and \( B_{\text{max}} \) is the maximum value for the boundary, \( D_{KL}[P(\theta|y_{t+1})||\hat{\pi}^{(t)}(\theta)] \). Since [8] mentioned that this choice was arbitrary, to be consistent with our other approaches, we modulate the learning rate of the Modified SMiLe rule similar to the Variational SMiLe rule, but with \( S_{CC}(y_{t+1}; \pi^{(t)}) \) (as opposed to \( S_{GM} \)) as the measure of surprise: \( \gamma_{t+1} = \gamma \left( S_{CC}(y_{t+1}; \pi^{(t)}), m \right) \).

### 6.3 Modified SMiLe Rule

The constraint of the minimization problem for the Variational SMiLe is essentially a modified version of the Confidence Corrected Surprise (See below for the original version) defined by [8]:

\[
S_{CC}(y_{t+1}; \pi^{(t)}) = D_{KL}[\pi^{(t)}(\theta)||P(\theta|y_{t+1})].
\]

In the original version of \( S_{CC} \) (See below), \( \pi^{(0)} \) is always assumed to be a uniform distribution for the computation of \( P(\theta|y_{t+1}) \), which is not well-defined for some types of parameters. With the aim of minimizing the Confidence Corrected Surprise by updating the belief during time, [8] suggested a update rule solving the optimization problem:

\[
\dot{\pi}^{(t+1)}(\theta) = \text{argmin}_{q(\theta)} D_{KL}[q(\theta)||P(\theta|y_{t+1})] \quad q(\theta) \text{ s.t. } D_{KL}[q(\theta)||\hat{\pi}^{(t)}(\theta)] < B_{t+1},
\]

where \( B_{t+1} \in [0, D_{KL}[P(\theta|y_{t+1})||\hat{\pi}^{(t)}(\theta)] \] is an arbitrary bound. The authors showed that the solution to this optimization problem is:

\[
\dot{\pi}^{(t+1)}(\theta) = \text{argmin}_{q(\theta)} D_{KL}[q(\theta)||P(\theta|y_{t+1})] \quad q(\theta) \text{ s.t. } D_{KL}[q(\theta)||\pi^{(t)}(\theta)] < B_{t+1},
\]

where \( \gamma_{t+1} \in [0, 1] \) is specified so that it satisfies the constraint. Although Eq. S16 looks very similar to Eq. 8 of the main text, it signifies a trade-off between the latest belief \( \dot{\pi}^{(t)} \) and the belief updated by only the most recent observation \( P(\theta|y_{t+1}) \), whereas in the approaches we analyzed the trade-off is (Eq. 8 in the main text) between integrating the new observation to the old ones (i.e. \( \pi_{[t+1]}^\prime \) and \( P(\theta|y_{t+1}) \)). To modulate the learning rate by surprise, [8] considered the boundary \( B_{t+1} \) as a function surprise, i.e. \( B_{t+1} = B_{\text{max}} \gamma (S_{CC}(y_{t+1}; \pi^{(t)}), m) \), where \( m \) is a free parameter, and \( B_{\text{max}} \) is the maximum value for the boundary, \( D_{KL}[P(\theta|y_{t+1})||\pi^{(t)}(\theta)] \). Since [8] mentioned that this choice was arbitrary, to be consistent with our other approaches, we modulate the learning rate of the Modified SMiLe rule similar to the Variational SMiLe rule, but with \( S_{CC}(y_{t+1}; \pi^{(t)}) \) (as opposed to \( S_{GM} \)) as the measure of surprise: \( \gamma_{t+1} = \gamma (S_{CC}(y_{t+1}; \pi^{(t)}), m) \).

### 6.4 Original SMiLe Rule

In the original version of the SMiLe rule proposed by [8], the definition of the Confidence Corrected surprise is given by:

\[
S_{CC}(y_{t+1}; \pi^{(t)}) = D_{KL}[\pi^{(t)}(\theta)||\hat{\pi}(\theta|y_{t+1})]
\]

where \( \hat{\pi}(\theta|y_{t+1}) \) is the scaled likelihood defined as

\[
\hat{\pi}(\theta|y_{t+1}) = \frac{P_Y(y_{t+1}|\theta)}{\int P_Y(y_{t+1}|\theta)d\theta}
\]
which potentially can be ill-defined, since the normalization factor can be infinite. The other parts are exactly the same as the modified version except for the modulation procedure. In the original version, the modulation is done over the boundary

$$B_{t+1} = B_{\max} \gamma \left( S_{CC}(y_{t+1}), m \right)$$  \hspace{1cm} (S19)

where \(B_{\max} = D_{KL}[P(\theta | y_{t+1}) | \phi^{(t)}] \),

and then, \(\gamma_{t+1}\) is found by satisfying the constraint of the optimization.

### 6.5 Derivation of Particle Filtering

We derive here the weight update for the particle filter. We start by defining the number of changes from beginning until time \(t\) as the random variable \(H_t = \sum_{k=1}^{t} \Delta H_k\).

The difference in our formalism from a standard derivation [27] is the absence of the Markov property of conditional observations (i.e. \(P(y_{t+1} | h_{t+1}, y_{t}) \neq P(y_{t+1} | h_{t+1})\)). We have

$$w_{t+1}^{(i)} \propto \frac{P(h_{t+1}^{(i)} | y_{t+1})}{Q(h_{t+1}^{(i)} | y_{t+1})} \frac{P(h_{t+1}^{(i)}, y_{t+1} | y_{t}) / P(y_{t+1} | y_{t})}{Q(h_{t+1}^{(i)}, y_{t+1} | y_{t})} \frac{P(h_{t+1}^{(i)} | y_{t+1})}{Q(h_{t+1}^{(i)} | y_{t+1})}.$$

$$= \frac{P(y_{t+1} | h_{t+1}^{(i)}, y_{t}) P(h_{t+1}^{(i)}, y_{t}) P(h_{t+1}^{(i)} | y_{t})}{Q(h_{t+1}^{(i)} | h_{t+1}^{(i)}, y_{t+1}) Q(h_{t+1}^{(i)} | y_{t})}.$$  \hspace{1cm} (S20)

Notice that \(w_{t+1}^{(i)} \propto \frac{P(h_{t+1}^{(i)} | y_{t})}{Q(h_{t+1}^{(i)} | y_{t})}\) are the weights calculated at the previous timestep and that \(P(h_{t+1}^{(i)} | h_{t+1}^{(i)}, y_{t}) = P(h_{t+1}^{(i)} | h_{t+1}^{(i)})\). Therefore

$$w_{t+1}^{(i)} \propto \frac{P(y_{t+1} | h_{t+1}^{(i)}, y_{t}) P(h_{t+1}^{(i)} | h_{t+1}^{(i)})}{Q(h_{t+1}^{(i)} | h_{t+1}^{(i)}, y_{t+1})} w_{t}^{(i)} .$$  \hspace{1cm} (S21)

We use the optimal proposal function in terms of variance of the weights [32]

$$Q(h_{t+1}^{(i)} | h_{t}^{(i)}, y_{t+1}) = P(h_{t+1}^{(i)} | h_{t}^{(i)}, y_{t+1}).$$  \hspace{1cm} (S22)

Since \(P(h_{t+1}^{(i)} | h_{t}^{(i)}, y_{t+1}) = \frac{P(h_{t+1}^{(i)}, y_{t+1} | h_{t}^{(i)})}{P(y_{t+1} | h_{t}^{(i)})}\),

and \(P(y_{t+1} | h_{t}^{(i)}, y_{t+1}) = \frac{P(y_{t+1}, y_{t+1} | h_{t}^{(i)})}{P(y_{t+1} | h_{t}^{(i)})}\),

with Eq. S21 and Eq. S22 we find

$$u_{t+1}^{(i)} \propto \frac{P(y_{t+1} | h_{t+1}^{(i)}, y_{t+1}) P(h_{t+1}^{(i)} | h_{t}^{(i)})}{P(y_{t+1} | h_{t}^{(i)})} \frac{P(h_{t+1}^{(i)} | h_{t}^{(i)})}{P(y_{t+1} | h_{t}^{(i)})} w_{t}^{(i)} ,$$  \hspace{1cm} (S23)

and after cancelling out common terms in the numerator and denominator,

$$w_{t+1}^{(i)} \propto \frac{P(y_{t+1} | h_{t+1}^{(i)}, y_{t+1}) P(h_{t+1}^{(i)} | h_{t}^{(i)})}{P(y_{t+1} | h_{t}^{(i)})} \frac{P(h_{t+1}^{(i)} | h_{t}^{(i)})}{P(y_{t+1} | h_{t}^{(i)})} w_{t}^{(i)} .$$  \hspace{1cm} (S24)

We define \(m^{k}\) as the timepoint when a new hidden state \(h_{k}^{(i)} = k\) started, i.e \(m^{k} = \min[j \in \{1, ..., t\} | h_{j}^{(i)} = k]\). Respectively we define as \(n^{k} = \max[j \in \{1, ..., t\} | h_{j}^{(i)} = k]\) the timepoint when a new hidden state \(h_{k}^{(i)} = k\) stopped. Every time we have a new state \(h_{k}^{(i)}\) we draw the corresponding parameters \(\theta_{t+1}\) from the prior. The observations given the parameters come from the likelihood function defined by \(\theta_{t+1}\). We group together the observations coming from the same hidden state and drop the conditioning on the hidden states since this information is incorporated in the \(m^{k}, n^{k}\) variables. We keep the conditioning only to explicitly signal the
We now define the weights corresponding to the Bayesian update \( \pi \). When \( N \) we implemented the Sequential Importance Resampling algorithm \([26],[32]\), where in order to avoid the threshold. The effective number of the particles can be computed as \([32],[27]\)

\[
P(y_{t+1}|h_{t+1}, h_t) = \pi_{t-1} \prod_{k=1}^{h_{t-1}} P(y_{m_{k-1}}|h_{t-1})
\]

\[
P(y_{t+1}|h_{t+1}, h_t) = \pi_{t-1} \prod_{k=1}^{h_{t-1}} P(y_{m_{k-1}}|h_{t-1}) + \pi_{t+1} \prod_{k=1}^{h_{t-1}} P(y_{m_{k-1}}|h_{t-1})
\]

\[
P(y_{t+1}|h_{t+1}, h_t) = \pi_{t-1} \prod_{k=1}^{h_{t-1}} P(y_{m_{k-1}}|h_{t-1}) + \pi_{t+1} \prod_{k=1}^{h_{t-1}} P(y_{m_{k-1}}|h_{t-1})
\]

This gives us

\[
w_t(1) \propto \left[ (1 - p_c) \frac{P(y_{m_{t+1}}|h_{t+1})}{P(y_{m_{t+1}}|h_{t-1})} + p_c P(y_{t+1}|h_{t+1}) \right] w_t(1),
\]

and finally

\[
w_t(1) = \left[ (1 - p_c) P(y_{t+1}|y_{m_{t+1}}, h_{t+1}) + p_c P(y_{t+1}|h_{t+1}) \right] w_t(1) / Z
\]

where \( P(y_{t+1}|\pi_t(t)) = P(y_{t+1}|y_{m_{t+1}}, h_{t+1}) \) and \( Z \) is the normalization factor

\[
Z = (1 - p_c) P(y_{t+1}|h_{t+1}) + p_c P(y_{t+1}|\pi_t(t)).
\]

We now define the weights corresponding to the Bayesian update \( \pi_{t+1} \)

\[
w_t(2) \propto \frac{P(y_{t+1}|h_{t+1})}{P(y_{t+1}|h_{t-1})} w_t(1).
\]

Combining Equations S26, S27 and S29 we can then rewrite the weight update rule as

\[
w_t(1) = \left( 1 - p_c \right) P(y_{t+1}|h_{t+1}) w_t(1) + p_c P(y_{t+1}|\pi_t(t)) w_t(1)
\]

\[
w_t(1) = (1 - \gamma_t) w_t(1) + \gamma_t w_t(1)
\]

where \( \gamma_t = E \left( S_{GM}(y_{t+1}, \pi_t(t)), m = \frac{p_c}{1 - p_c} \right) \).

At every time step \( t+1 \) we sample each particle's hidden state \( h_{t+1} \) from the proposal distribution. We calculate the stay probability

\[
Q(h_{t+1} \mid h_t) = \frac{1}{1 + \gamma_{t+1} w_{t+1}}
\]

\[
= \frac{1}{1 + \gamma_{t+1} w_{t+1}} = 1 - \gamma \left( S_{GM}(y_{t+1}, \pi_t(t)), m = \frac{p_c}{1 - p_c} \right).
\]

We implemented the Sequential Importance Resampling algorithm \([26],[32]\), where in order to avoid the problem of degeneracy of the weights, the particles are resampled when their effective number falls below a threshold. The effective number of the particles can be computed as \([32],[27]\)

\[
N_{eff} \approx \frac{1}{\sum_n (w_t(1))^2}.
\]

When \( N_{eff} \) is below a critical threshold, the particles are resampled with replacement from the Categorical distribution defined by their weights, and all their weights are set to \( w_t(1) = 1/N \). We performed resampling when \( N_{eff} \leq N/2 \).

### 6.6 General Formulation for Exponential Family Distributions

In this section, we derive the compact update rules of our methods for \( \hat{\pi}(t+1)(\theta) \) for the exponential family of distributions. A likelihood function belonging to the exponential family of distributions has the form
Algorithm 1 Pseudocode for Particle Filtering

1: Specify $P_Y(y|\theta)$, $P_\pi(\Theta = \theta; \chi, \nu)$, $m = p_c/(1-p_c)$, $N$, and $N_{\text{thre}}$
2: Initialize $\chi_i^{(0)}, \nu_i^{(0)}, w_0^{(i)} \forall i \in \{1...N\}$
3: $t \leftarrow 0$
4: while the sequence is not finished do
5: Observe $y_{t+1}$
6: Compute each particle’s surprise $S_{GM}(y_{t+1}, \hat{\pi}_i^{(t)})$ using Eq. S33 with $\chi_i^{(t)}, \nu_i^{(t)} \forall i \in \{1...N\}$
7: Compute $S_{GM}(y_{t+1}, \hat{\pi}_i^{(t)})$ as the weighted ($w_t^{(i)}$) harmonic mean of $S_{GM}(y_{t+1}, \hat{\pi}_i^{(t)})$
8: Compute modulation factor $\gamma_{t+1} = \gamma(S_{GM}(y_{t+1}, \hat{\pi}_i^{(t)}), m)$
9: Compute each particle’s bayesian weight $w_{B,t+1}^{(i)}$ using Eq. S29
10: $u_{t+1}^{(i)} \leftarrow (1 - \gamma_{t+1})w_{B,t+1}^{(i)} + \gamma_{t+1}w_t^{(i)} \forall i \in \{1...N\}$
11: Sample $\Delta h_{t+1}^{(i)} \sim \text{Bernoulli}(\gamma(S_{GM}^{(i)}(y_{t+1}, m)))$
12: $N_{\text{eff}} \leftarrow (\sum_{i=1}^N w_t^{(i)})^{-1}$
13: If $N_{\text{eff}} \leq N_{\text{thre}}$, resample
14: for $i \in \{1...N\}$ do
15: if $\Delta h_{t+1}^{(i)} = 0$ then
16: $\chi_i^{(t+1)} \leftarrow \chi_i^{(t)} + \phi(y_{t+1})$
17: $\nu_i^{(t+1)} \leftarrow \nu_i^{(t)} + 1$
18: else
19: $\chi_i^{(t+1)} \leftarrow \chi_i^{(0)} + \phi(y_{t+1})$
20: $\nu_i^{(t+1)} \leftarrow \nu_i^{(0)} + 1$
21: $\hat{\pi}_i^{(t+1)}(\theta) = \sum_{i=1}^N u_{t+1}^{(i)} P_{\pi}(\Theta = \theta; \chi_i^{(t+1)}, \nu_i^{(t+1)})$
22: $t \leftarrow t + 1$

Algorithm 2 Pseudocode for Variational SMiLe

1: Specify $P_Y(y|\theta)$, $P_\pi(\Theta = \theta; \chi, \nu)$, and $m$
2: Initialize $\chi^{(0)}, \nu^{(0)}$
3: $t \leftarrow 0$
4: while the sequence is not finished do
5: Observe $y_{t+1}$.
6: Compute surprise $S_{GM}(y_{t+1}, \hat{\pi}(t))$ using Eq. S33
7: Compute modulation factor $\gamma_{t+1} = \gamma(S_{GM}(y_{t+1}, \hat{\pi}(t)), m)$
8: $\chi^{(t+1)} \leftarrow (1 - \gamma_{t+1})\chi^{(t)} + \gamma_{t+1}\chi^{(0)} + \phi(y_{t+1})$
9: $\nu^{(t+1)} \leftarrow (1 - \gamma_{t+1})\nu^{(t)} + \gamma_{t+1}\nu^{(0)} + 1$
10: $\hat{\pi}(t+1)(\theta) = P_{\pi}(\Theta = \theta; \chi^{(t+1)}, \nu^{(t+1)})$
11: $t \leftarrow t + 1$.

6.7 Modified SMiLe for the exponential family of distributions

The pseudocode is written in Algorithm 3, where the Confidence Corrected surprise $S_{CC}$
Algorithm 3 Pseudocode for Modified SMiLe

1: Specify $P_Y(y|\theta), P_\pi(\Theta = \theta; \chi, \nu)$, and $m$
2: Initialize $\chi^{(0)}, \nu^{(0)}$
3: $t \leftarrow 0$
4: while the sequence is not finished do
5:   Observe $y_{t+1}$.
6:   Compute surprise $S_{CC}(y_{t+1}, \hat{\pi}^{(t)})$ using Eq. S35
7:   Compute modulation factor $\gamma_{t+1} = \gamma(S_{CC}(y_{t+1}, \hat{\pi}^{(t)}), m)$
8:   $\chi^{(t+1)} \leftarrow (1 - \gamma_{t+1})\chi^{(t)} + \gamma_{t+1} \left( \chi^{(0)} + \phi(y_{t+1}) \right)$
9:   $\nu^{(t+1)} \leftarrow (1 - \gamma_{t+1})\nu^{(t)} + \gamma_{t+1} (\nu^{(0)} + 1)$
10: $\hat{\pi}^{(t+1)}(\theta) = P_\pi(\Theta = \theta; \chi^{(t+1)}, \nu^{(t+1)})$
11: $t \leftarrow t + 1$

\[ S_{CC}(y_{t+1}; P_\pi(\Theta = \theta; \chi^{(t)}, \nu^{(t)})) = D_{KL} [P_\pi(\theta; \chi^{(t)}, \nu^{(t)})] \| [P_\pi(\theta; \chi^{(0)}, \nu^{(0)} + 1)] \] (S34)

is the Kullback–Leibler divergence between the current belief of time $t$ and the belief that would arise if $y_{t+1}$ is the first sample ever. Evaluation gives:

\[
S_{CC}(y_{t+1}; P_\pi(\Theta = \theta; \chi^{(t)}, \nu^{(t)})) = \log \left( \frac{f(\chi^{(t)}, \nu^{(t)})}{f(\chi^{(0)} + \phi(y_{t+1}), \nu^{(0)} + 1)} \right) \\
+ (\chi^{(t)} - \chi^{(0)} - \phi(y_{t+1}) \right)^T \mathbb{E}_{\theta|\chi^{(t)}, \nu^{(t)}}[\theta] \\
- (\nu^{(t)} - \nu^{(0)} - 1) \mathbb{E}_{\theta|\chi^{(t)}, \nu^{(t)}}[A(\theta)] \] (S35)

6.8 Surprise-Based Interpretation of the Message-Passing Algorithm of [13] and [14]

Let us define the random variable $R_t = \max\{n \in \mathbb{N} : H_{t-\mu+1} = H_t\}$. This is the time window from the last change point. Then the exact Bayesian form for $\pi^{(t)}(\theta)$ can be written as

\[
\pi^{(t)}(\theta) = P(\Theta_{t+1} = \theta|Y_{1:t} = y_{1:t}) \\
= \sum_{r_t=1}^{t} P(\Theta_{t+1} = \theta, R_t = r_t|Y_{1:t} = y_{1:t}) \\
= \sum_{r_t=1}^{t} P(R_t = r_t|Y_{1:t} = y_{1:t}) P(\Theta_{t+1} = \theta|R_t = r_t, Y_{1:t} = y_{1:t}) \] (S36)

This equation can be seen as a version of Particle Filtering (Eq. 10) with number of particles equal to $t$. After observing a new sample $Y_{t+1}$, one new particle is generated and added to the set of particles, modelling the possibility of a change point occurring at $t + 1$. To have a formulation similar to the one of Particle Filtering we rewrite the belief as

\[
\pi^{(t)}(\theta) = \sum_{k=0}^{t-1} w_t^{(k)} P(\Theta_{t+1} = \theta|R_t = t - k, Y_{1:t} = y_{1:t}) \] (S37)

where $w_t^{(k)} = P(R_t = t - k|Y_{1:t} = y_{1:t})$ is the weight of the particle $k$ at time $t$. To update the belief after observing $Y_{t+1} = y_{t+1}$, one can use the exact Bayesian recursive formula, for which one needs to compute
This gives us
\[ \pi_{y_{t+1}}^{(t+1)}(\theta) = \frac{\pi^{(t)}(\theta) P_Y(y_{t+1}|\theta)}{P^{(y_{t+1}; \pi^{(t)})} \sum_{k=0}^{t-1} \omega_{t+1}^{(k)} P(\Theta_{t+1} = \theta | R_t = t - k, Y_{1:t} = y_{1:t}) P_Y(y_{t+1}|\theta)} \]
\[ = \frac{1}{P^{(y_{t+1}; \pi^{(t)})}} \sum_{k=0}^{t-1} \omega_{t+1}^{(k)} P(\Theta_{t+1} = \theta | R_t = t - k, Y_{k+1:t} = y_{k+1:t}) P_Y(y_{t+1}|\theta) \]
\[ = \frac{1}{P^{(y_{t+1}; \pi^{(t)})}} \sum_{k=0}^{t-1} \omega_{t+1}^{(k)} \frac{P(X_{1:t} = y_{1:t} | \Theta_{t+1} = \theta, R_t = t - k) \pi^{(0)}(\theta)}{P(Y_{1:t} = y_{1:t} | R_t = t - k)} P_Y(y_{t+1}|\theta) \]
\[ = \frac{1}{P^{(y_{t+1}; \pi^{(t)})}} \sum_{k=0}^{t-1} \omega_{t+1}^{(k)} \prod_{i=k+1}^{t} P_Y(y_i | \theta) \pi^{(0)}(\theta) P_Y(y_{t+1}|\theta) \] (S38)
\[ = \frac{1}{P^{(y_{t+1}; \pi^{(t)})}} \sum_{k=0}^{t-1} \omega_{t+1}^{(k)} \prod_{i=k+1}^{t} P_Y(y_i | \theta) \pi^{(0)}(\theta) \]
and as a result
\[ \pi_{y_{t+1}}^{(t+1)}(\theta) = \frac{1}{P^{(y_{t+1}; \pi^{(t)})}} \sum_{k=0}^{t-1} \omega_{t+1}^{(k)} \frac{P(X_{1:t+1} = y_{1:t+1} | R_t = t - k + 1)}{P(Y_{1:t} = y_{1:t} | R_t = t - k)} \times \]
\[ \times P(\Theta_{t+1} = \theta | R_{t+1} = t - k + 1, Y_{1:t+1} = y_{1:t+1}) \] (S39)
This gives us
\[ \pi_{y_{t+1}}^{(t+1)}(\theta) = \sum_{k=0}^{t-1} \omega_{t+1}^{(k)} \frac{P(y_{t+1}; \pi^{(t)}_{k})}{P^{(y_{t+1}; \pi^{(t)})}} \]
\[ \times P(\Theta_{t+1} = \theta | R_{t+1} = t - k + 1, Y_{1:t+1} = y_{1:t+1}) \] (S40)
and finally
\[ \omega_{t+1}^{(k)} = \frac{P(y_{t+1}; \pi^{(t)}_{k})}{P^{(y_{t+1}; \pi^{(t)})}} \omega_{t+1}^{(k)} \] (S41)
This update is identical to the update of the Particle Filter weights that correspond to a Bayesian update, that we saw in Eq. S29 of the main text. Using the recursive formula, the update rule for the weights for \( 0 \leq k \leq t - 1 \) is
\[ \omega_{t+1}^{(k)} = (1 - \gamma_{t+1}) \omega_{t+1}^{(k)} \]
\[ = (1 - \gamma_{t+1}) \frac{P(y_{t+1}; \pi^{(t)}_{k})}{P^{(y_{t+1}; \pi^{(t)})}} \omega_{t+1}^{(k)} \] (S42)
and for the newly added particle \( t \)
\[ \omega_{t+1}^{(t)} = \gamma_{t+1} \] (S43)
The work of [14] follows the same principle as [13], but employs a different way to eliminate particles with negligible weights, in order to reduce computational complexity and memory requirements. In [13] all (exact Bayes) or all but some particles below a cut-off threshold are kept. Fearnhead and Liu [14] explored different methods to reduce the total number of particles below \( t \). We experienced that the small errors introduced in their resampling step accumulate and lead to a worse performance than e.g. keeping simply the \( N \) particles with the highest weight at each time step, despite the latter being a biased estimation of the distribution. This simple approximation can therefore also be seen as a variation of the [13] (MP Bayes) algorithm, with fixed number of particles and a variable cut-off threshold.

The updates of the equations Eq. S41, Eq. S42 and Eq. S43 are essentially the same as the ones of Particle Filtering, and entail the same surprise modulation and the same trade-off. The only difference is that, while in Particle Filtering the trade-off between integration and reset is accomplished via sampling, in [13, 14] it is accomplished by adding at each time step a new particle with weight \( \gamma_{t+1} \).

6.9 Modified Algorithm of [2, 7]: Adaptation for Gaussian Prior

Let us first consider the case of a stationary regime (i.e. no change points) where observed samples are drawn from a Gaussian distribution with known variance: \( y_{t+1} | \theta \sim \mathcal{N}(\mu_{t+1}, \sigma^2) \), and the parameter \( \theta \) is also drawn from a Gaussian distribution \( \theta \sim \mathcal{N}(\mu_\theta, \sigma^2) \). After having observed samples \( y_1, \ldots, y_{t+1} \), it can be shown that, using Bayes rule, the posterior distribution \( P(\theta | y_{1:t+1}) = \pi_{y_{t+1}}^{(t+1)}(\theta) \) is
\[ P(\theta | y_{1:t+1}) = \mathcal{N}\left(\theta; \mu_{B,t+1} = \frac{1}{\sigma^2} \left( \frac{\mu_\theta}{\sigma_0^2} + \frac{\sum_{i=1}^{t+1} y_i}{\sigma^2} \right), \sigma_{B,t+1}^2 = \frac{1}{\sigma^2} \right). \] (S44)
An estimate of $\theta$ is its expected value $E(\theta|y_{1:t+1}) = \mu_{B,t+1}$.

In a nonstationary regime where, after having observed $y_1, \ldots, y_t$ from the same hidden state, there is the possibility for a change point upon observing $y_{t+1}$, the posterior distribution is

$$P(\theta|y_{1:t+1}) = (1 - \gamma_{t+1})P(\theta|y_{1:t+1}, \Delta h_{t+1} = 0) + \gamma_{t+1}P(\theta|y_{1:t+1}, \Delta h_{t+1} = 1).$$  \hspace{1cm} (S45)

To facilitate notation later in this subsection we note this as

$$P(\theta|y_{1:t+1}) = (1 - \gamma_{t+1})P(\theta|y_{1:t+1}, \text{stay}) + \gamma_{t+1}P(\theta|y_{1:t+1}, \text{change})$$  \hspace{1cm} (S46)

The above is equivalent to what we saw in subsection 3.1, namely

$$\pi(t+1)(\theta) = (1 - \gamma_{t+1})\pi_B(t+1)(\theta) + \gamma_{t+1}P(\theta|y_{t+1}),$$  \hspace{1cm} (S47)

where $\gamma$ is the learning rate we saw in Eq. 3, and it is essentially the probability to change given the new observation, i.e. $P(\text{change}|y_{t+1})$. In [7] this quantity is denoted as $\Omega_{t+1}$. Taking Eq. 34 into account we have

$$E(\theta|y_{1:t+1}, \text{stay}) = \mu_{B,t+1} = \frac{1}{\sigma_0^2 + \frac{rt+1}{\sigma^2}} \left( \frac{\mu_0}{\sigma_0^2} + \frac{\sum_{i=t+1}^{t+1} y_i}{\sigma^2} \right),$$

$$E(\theta|y_{1:t+1}, \text{change}) = \frac{1}{\sigma_0^2 + \frac{1}{\sigma^2}} \left( \frac{\mu_0}{\sigma_0^2} + \frac{y_{t+1}}{\sigma^2} \right),$$

(S48)

where $r_t$ is the time interval of observations coming from the same hidden state, calculated at time $t$. Taking the expectation of Eq. 59 we have

$$\hat{\mu}_{t+1} = (1 - \gamma) \frac{1}{\sigma_0^2 + \frac{r_{t+1}}{\sigma^2}} \left( \frac{\mu_0}{\sigma_0^2} + \frac{\sum_{i=t+1}^{t+1} y_i}{\sigma^2} \right) + \gamma \frac{1}{\sigma_0^2 + \frac{1}{\sigma^2}} \left( \frac{\mu_0}{\sigma_0^2} + \frac{y_{t+1}}{\sigma^2} \right),$$

(S49)

where we dropped the subscript $t+1$ in $\gamma$ to simplify notations. We have

$$\hat{\mu}_{t+1} = (1 - \gamma) \frac{1}{\sigma_0^2 + \frac{r_{t+1}}{\sigma^2}} \left( \frac{\mu_0}{\sigma_0^2} + \frac{\sum_{i=t+1}^{t+1} y_i}{\sigma^2} \right) + \gamma \frac{1}{\sigma_0^2 + \frac{1}{\sigma^2}} \left( \frac{\mu_0}{\sigma_0^2} + \frac{y_{t+1}}{\sigma^2} \right).$$

(S50)

Since $\hat{\mu}_t = \frac{1}{\sigma_0^2 + \frac{1}{\sigma^2}} \left( \frac{\mu_0}{\sigma_0^2} + \frac{\sum_{i=t}^{t-1} y_i}{\sigma^2} \right)$ we have

$$\hat{\mu}_{t+1} = (1 - \gamma) \frac{1}{\sigma_0^2 + \frac{r_t}{\sigma^2}} \left( \frac{\hat{\mu}_t}{\sigma_0^2} + \frac{r_{t+1}}{\sigma^2} + \frac{y_{t+1}}{\sigma^2} \right) + \gamma \frac{1}{\sigma_0^2 + \frac{1}{\sigma^2}} \left( \frac{\mu_0}{\sigma_0^2} + \frac{y_{t+1}}{\sigma^2} \right),$$

$$= (1 - \gamma) \frac{1}{\sigma_0^2 + \frac{r_t}{\sigma^2} + \frac{1}{\sigma^2}} \left( \frac{\hat{\mu}_t}{\sigma_0^2} + \frac{r_{t+1}}{\sigma^2} + \frac{y_{t+1}}{\sigma^2} \right) - \hat{\mu}_t \frac{1}{\sigma_0^2 + \frac{1}{\sigma^2}} \left( \frac{\mu_0}{\sigma_0^2} + \frac{1}{\sigma^2} \right) + \gamma \frac{1}{\sigma_0^2 + \frac{1}{\sigma^2}} \left( \frac{\mu_0}{\sigma_0^2} + \frac{1}{\sigma^2} \right),$$

$$= (1 - \gamma) \hat{\mu}_t + (1 - \gamma) \frac{1}{\sigma_0^2 + \frac{r_t}{\sigma^2} + \frac{1}{\sigma^2}} \left( \frac{y_{t+1} - \hat{\mu}_t}{\sigma_0^2} + \gamma \mu_0 \right) + \gamma \frac{1}{\sigma_0^2 + \frac{1}{\sigma^2}} \left( \frac{y_{t+1} - \mu_0}{\sigma_0^2} \right),$$

$$= (1 - \gamma) \hat{\mu}_t + \gamma \mu_0 + (1 - \gamma) \frac{1}{\sigma_0^2 + \frac{r_t + 1}{\sigma^2}} \left( y_{t+1} - \hat{\mu}_t \right) + \gamma \frac{1}{\sigma_0^2 + \frac{1}{\sigma^2}} \left( \frac{y_{t+1} - \mu_0}{\sigma_0^2} \right).$$

(S51)

We define $\rho = \frac{\sigma^2}{\sigma_0^2}$ and we have

$$\hat{\mu}_{t+1} = (1 - \gamma)\hat{\mu}_t + \gamma \mu_0 + (1 - \gamma) \frac{1}{\rho + r_t + 1} (y_{t+1} - \hat{\mu}_t) + \gamma \frac{1}{\rho + 1} (y_{t+1} - \mu_0),$$

(S52)

and after re-arranging the terms

$$\hat{\mu}_{t+1} = (1 - \gamma)\hat{\mu}_t + \frac{1}{\rho + r_t + 1} (y_{t+1} - \hat{\mu}_t) + \gamma \frac{1}{\rho + 1} (y_{t+1} - \mu_0),$$

(S53)

where we added back the dependency of $\gamma$ on time. We can see that the updated mean is a weighted average between incorporating the new observation to the current mean $\hat{\mu}_t$ and incorporating it to the prior mean $\mu_0$, in the same spirit as the other algorithm we considered here. The last equation can also be seen as a weighted sum of two delta rules: one including a prediction error between the new observation and the current mean $y_{t+1} - \hat{\mu}_t$ and one including a prediction error between the observed sample and the prior mean $y_{t+1} - \mu_0$. In order to obtain a form similar to the one of [2, 7] we continue and we spell out the terms that include the quantities $\hat{\mu}_t$, $\mu_0$ and $y_{t+1}$

$$\hat{\mu}_{t+1} = (1 - \gamma)\hat{\mu}_t - (1 - \gamma) \frac{1}{\rho + r_t + 1} \hat{\mu}_t$$

$$+ \gamma \mu_0 - \gamma \frac{1}{\rho + 1} \mu_0$$

$$+ (1 - \gamma) \frac{1}{\rho + r_t + 1} y_{t+1} + \gamma \frac{1}{\rho + 1} y_{t+1},$$

(S54)
Using that \( \frac{1}{\rho_t + r_t + 1} = \frac{1}{\rho_t} - \frac{r_t}{(\rho_t + r_t + 1)^2} \) we have

\[
\dot{\mu}_{t+1} = (1 - \gamma)\dot{\mu}_t - (1 - \gamma) \frac{1}{\rho + 1} \dot{\mu}_t + (1 - \gamma) \frac{r_t}{(\rho + 1)(\rho + r_t + 1)} \dot{\mu}_t
\]

\[
+ \gamma \mu_0 - \gamma \frac{1}{\rho + 1} \mu_0
\]

\[
+ (1 - \gamma) \frac{1}{\rho + 1} y_{t+1} - (1 - \gamma) \frac{r_t}{(\rho + 1)(\rho + r_t + 1)} y_{t+1} + \gamma \frac{1}{\rho + 1} y_{t+1}
\]

\[
\dot{\mu}_{t+1} = (1 - \gamma - (1 - \gamma) \frac{1}{\rho + 1}) \dot{\mu}_t + (1 - \gamma) \frac{r_t}{(\rho + 1)(\rho + r_t + 1)} \dot{\mu}_t
\]

\[
+ \gamma \rho \frac{\dot{\mu}_t - \mu_0}{\rho + 1}
\]

\[
+ \frac{1}{\rho + 1} y_{t+1} - (1 - \gamma) \frac{r_t}{(\rho + 1)(\rho + r_t + 1)} y_{t+1},
\]

and finally

\[
\dot{\mu}_{t+1} = \frac{\rho}{\rho + 1} \left( (1 - \gamma) \dot{\mu}_t + \gamma \mu_0 \right) + \frac{1}{\rho + 1} \left( (1 - \gamma) \frac{r_t}{\rho + r_t + 1} (\dot{\mu}_t - y_{t+1}) + y_{t+1} \right).
\]

If we define \( 1 - \alpha = (1 - \gamma) \frac{r_t}{\rho + r_t + 1} \Rightarrow \alpha = 1 - (1 - \gamma) \frac{r_t}{\rho + r_t + 1} \Rightarrow \alpha = \frac{\rho + r_t + 1}{\rho + r_t + 1} \) and rearrange the terms, we have

\[
\dot{\mu}_{t+1} = \frac{\rho}{\rho + 1} \left( (1 - \gamma) \dot{\mu}_t + \gamma \mu_0 \right) + \frac{1}{\rho + 1} \left( (1 - \alpha) \dot{\mu}_t + \alpha y_{t+1} \right)
\]

\[
\dot{\mu}_{t+1} = \frac{\rho}{\rho + 1} \left( \dot{\mu}_t + \gamma (\mu_0 - \dot{\mu}_t) \right) + \frac{1}{\rho + 1} \left( \dot{\mu}_t + \alpha (y_{t+1} - \dot{\mu}_t) \right).
\]

Adding back the dependency of \( \gamma \) and \( \alpha \) on time we finally have

\[
\dot{\mu}_{t+1} = \frac{\rho}{\rho + 1} \left( \dot{\mu}_t + \gamma (\dot{\mu}_0 - \dot{\mu}_t) \right) + \frac{1}{\rho + 1} \left( \dot{\mu}_t + \alpha (y_{t+1} - \dot{\mu}_t) \right).
\]

We can see that the final update rule takes the form of a weighted average of two delta rules: one including a prediction error between the prior mean and the current mean \( \mu_0 - \dot{\mu}_t \) and one including a prediction error between the observed sample and the current mean \( y_{t+1} - \dot{\mu}_t \).

In [2, 7] the true new mean after a change point is drawn from a Uniform distribution with a range of values much larger than the width of the Gaussian likelihood. The derivations in [2, 7] implicitly follow the approximation of the Uniform distribution with a Gaussian distribution with \( \sigma_0 \gg \sigma \). Note that if \( \sigma_0 \gg \sigma \) then \( \rho \to 0 \), the first term of Eq. S58 disappears, and \( \alpha_{t+1} = \frac{\rho + r_t + 1}{\rho + r_t + 1} \). This results in the delta-rule of [2, 7]: \( \dot{\mu}_{t+1} = \dot{\mu}_t + \alpha_{t+1} (y_{t+1} - \dot{\mu}_t) \). (Note that \( \gamma_{t+1} = \Omega_{t+1} \)).

For the case of a nonstationary regime with a history of change points, the time interval \( r_t \) is not known. The authors in [2, 7] used as an estimate \( \tilde{r}_t \) the expected time interval. We make a distinction here between [2] and [7]:

In [7] \( \tilde{r}_t \) is calculated recursively on each trial in the same spirit as Eq. 8: \( \tilde{r}_{t+1} = (1 - \gamma_{t+1}) (\tilde{r}_t + 1) + \gamma_{t+1} \). That is, at each step there is a probability \( (1 - \gamma_{t+1}) \) that \( \tilde{r}_t \) increments by 1 and a probability \( \gamma_{t+1} \) that it is reset to 1. So \( \tilde{r}_{t+1} \) is the weighted sum of these two outcomes. Thus, Eq. S58 combined with the expected time interval \( \tilde{r}_t \) constitutes a generalization of the update rule of [7] for the case of Gaussian prior \( \mathcal{N}(\mu_0, \sigma_0^2) \).

In [2] the authors calculate first the variance \( \sigma_{t+1}^2 = \text{Var}(\theta|y_{t+1}) \) and based on this compute then \( \tilde{r}_{t+1} \). We derive here these calculations for the case of Gaussian prior. We remind once again that:

\[
P(\theta|y_{t+1}) = (1 - \gamma_{t+1}) P(\theta|y_{t+1}, \text{stay}) + \gamma_{t+1} P(\theta|y_{t+1}, \text{change})
\]

(S59)

For the variance \( \sigma_{t+1}^2 = \text{Var}(\theta|y_{t+1}) \) It can be shown that

\[
\sigma_{t+1}^2 = (1 - \gamma) \sigma_{t+1}^2 + \gamma \sigma_{t+1}^2 + (1 - \gamma) \gamma (\mu_{\text{stay}} - \mu_{\text{change}})^2
\]

(S60)

\[
\sigma_{t+1}^2 = \frac{1}{\sigma_0^2 + \sigma^2} + \gamma \frac{1}{\sigma_0^2 + \sigma^2}
\]

where \( \sigma_{t+1}^2 = \frac{1}{\sigma_0^2 + \sigma^2} \) and \( \sigma_{t+1}^2 = \frac{1}{\sigma_0^2 + \sigma^2} \). We have defined earlier \( \rho = \frac{\sigma^2}{\sigma_0^2} \) and for the first two terms we have:

\[
A = (1 - \gamma) \sigma_{t+1}^2 + \gamma \sigma_{t+1}^2
\]

\[
A = (1 - \gamma) \frac{1}{\sigma_0^2 + \sigma^2} + \gamma \frac{1}{\sigma_0^2 + \sigma^2}
\]

(S61)

\[
A = (1 - \gamma) \frac{\sigma^2}{\rho + r_t + 1} + \gamma \frac{\sigma^2}{\rho + 1}
\]
Using, as before, that \( \frac{1}{\rho + r_t + 1} = \frac{1}{\rho + 1} - \frac{r_t}{(\rho + 1)(\rho + r_t + 1)} \) we have:

\[
A = \sigma^2 \left( (1 - \gamma) \frac{1}{\rho + 1} - (1 - \gamma) \frac{1}{(\rho + 1)(\rho + r_t + 1)} + \gamma \frac{1}{\rho + 1} \right) \\
= \frac{\sigma^2}{\rho + 1} \left( (1 - \gamma) \frac{1}{\rho + r_t + 1} + \gamma \right) \\
= \frac{\sigma^2}{\rho + 1} \left( 1 - (1 - \gamma) \frac{1}{\rho + r_t + 1} \right)
\]  
(S62)

We have defined earlier the learning rate: \( \alpha = 1 - (1 - \gamma) \frac{r_t}{\rho + r_t + 1} \), so we have:

\[
A = \frac{\sigma^2}{\rho + 1} \alpha 
\]  
(S63)

Note that \( \mu_t = \frac{1}{\sigma^2} \left( \frac{\mu_0}{\sigma_0^2} + \frac{\sum_{i=t+1-r_t}^t y_i}{\sigma^2} \right) \), so for the calculation of the last term we have:

\[
B = \mu_{B,t+1} - \mu_{\text{change}}
\]

\[
= \frac{1}{\sigma^2} \left( \mu_0 + \frac{\sum_{i=t+1-r_t}^t y_i}{\sigma^2} \right) - \frac{1}{\sigma_0^2} \left( \mu_0 + \frac{y_{t+1}}{\sigma_0^2} \right) \\
= \frac{1}{\sigma^2} \left( \mu_0 + \frac{\sum_{i=t+1-r_t}^t y_i}{\sigma^2} + \frac{y_{t+1}}{\sigma_0^2} \right) - \frac{1}{\sigma_0^2} \left( \mu_0 + \frac{y_{t+1}}{\sigma_0^2} \right) \\
= \frac{1}{\sigma^2} \left( \mu_0 + \frac{r_t}{\sigma_0^2} + \frac{1}{\sigma_0^2} (y_{t+1} - \mu_t) \right) - \frac{1}{\sigma_0^2} \left( \mu_0 + \frac{y_{t+1}}{\sigma_0^2} \right) \\
= \mu_t + \frac{1}{\sigma_0^2} \left( \mu_0 + \frac{r_t}{\sigma_0^2} + \frac{1}{\sigma_0^2} (y_{t+1} - \mu_t) \right) - \frac{1}{\sigma_0^2} \left( \mu_0 + \frac{y_{t+1}}{\sigma_0^2} \right) \\
= \frac{\sigma^2}{\rho + 1} \alpha + (1 - \gamma) \mu_t + \mu_0 + \frac{r_t}{\sigma_0^2} + \frac{1}{\sigma_0^2} (y_{t+1} - \mu_t) - \frac{1}{\sigma_0^2} \left( \mu_0 + \frac{y_{t+1}}{\sigma_0^2} \right) \\
= \frac{\sigma^2}{\rho + 1} \alpha + (1 - \gamma) \mu_t + \mu_0 + \frac{r_t}{\rho + r_t + 1} (y_{t+1} - \mu_t) - \frac{1}{\rho + 1} (y_{t+1} - \mu_t) \\
= \frac{\sigma^2}{\rho + 1} \alpha + (1 - \gamma) \mu_t + \mu_0 + \frac{r_t}{\rho + r_t + 1} (y_{t+1} - \mu_t) - \frac{1}{\rho + 1} (y_{t+1} - \mu_0) \\
= \mu_t + \left( \frac{1}{\rho + 1} - \frac{r_t}{\rho + r_t + 1} \right) (y_{t+1} - \mu_t) - \mu_0 - \frac{1}{\rho + 1} (y_{t+1} - \mu_0)
\]  
(S64)

So finally we have:

\[
\sigma_{t+1}^2 = \frac{\sigma^2}{\rho + 1} \alpha + (1 - \gamma) \mu_t + \mu_0 + \frac{r_t}{\rho + r_t + 1} (y_{t+1} - \mu_t) - \mu_0 - \frac{1}{\rho + 1} (y_{t+1} - \mu_0)
\]  
(S65)

Then the \( \hat{r}_t \) is calculated at each time point as: \( \hat{r}_t = \frac{\sigma^2}{\sigma_t^2} - \frac{\sigma^2}{\sigma_0^2} \). These two version of calculating \( \hat{r}_t \) give different results and we compare with both in our simulations.

Finally, it is worth noting that, as mentioned in subsection 3.1, the updated belief of Eq. 8 does not generally result in the same family of distributions as the belief of the previous time step, i.e. \( P(\theta | y_{1:t+1}) \) is not a Gaussian distribution. The authors in [2, 7] implicitly consider that the posterior belief is approximated by a Gaussian, whose mean is used for the update at the next time step.

### 6.10 Particle Filtering with One Particle and Relation to [2, 7]

In the case of Particle Filtering with only one particle, we sample, at each step, the particle’s hidden state with stay probability \( Q(\Delta h_{t+1}^{(1)} = 0 | \Delta h_{1:t+1}^{(1)}, y_{1:t+1}) = 1 - \gamma_{t+1} \), where \( \gamma_{t+1} = \frac{m_{h_{t+1}}^{(1)}}{m_{h_{t+1}}^{(1)} + m_{h_{t+1}}^{(0)}} \), and update the posterior belief:

\[
\hat{r}_{t+1}^{(1)}(\theta) = \frac{\pi_{t+1}^{(t+1)}(\theta)}{P(\theta | y_{t+1})} = \begin{cases} \pi_B(\theta) & \text{if } \Delta h_{t+1} = 0 \\ P(\theta | y_{t+1}) & \text{if } \Delta h_{t+1} = 1 \end{cases}
\]  
(S66)
We can see that the adapted update of \[7\] we saw in the previous section is at each time step where for computing this shows that these two properties, in addition to the fact that this measure naturally modulates the learning rate show that

\[ S \]

The definition of the Generative Model Surprise

Interestingly, for the defined generative model, it is possible to express (see below for the derivation) the Shannon and Shannon Surprise \[28\] is defined as

\[ Sh(y_{t+1}; \pi^{(l)}) = -\log P(Y_{t+1} = y_{t+1} | Y_{1:t} = y_{1:t}) \] (S72)

where for computing \[P(Y_{t+1} = y_{t+1} | Y_{1:t} = y_{1:t})\], one should know the structure of the generative model. Interestingly, for the defined generative model, it is possible to express (see below for the derivation) the Shannon surprise as a function of modulated learning rate (and therefore of \[S_{GM}\]) as

\[ Sh(y_{t+1}; \pi^{(l)}) = Sh(y_{t+1}; \pi^{(0)}) + \log \left( \frac{\gamma_{t+1}}{p_c} \right) \] (S73)
and the modulated learning rate as

\[ \gamma_{t+1} = p_c \exp \left( \Delta S_{Sh}(y_{t+1}; \pi^{(t)}, \pi^{(0)}) \right), \]  

where \( \Delta S_{Sh}(y_{t+1}; \pi^{(t)}, \pi^{(0)}) = S_{Sh}(y_{t+1}; \pi^{(t)}) - S_{Sh}(y_{t+1}; \pi^{(0)}) \). 

(S74)

The final form shows that the modulated learning rate is not just a function of Shannon surprise upon observing \( Y_{t+1} = y_{t+1} \), but a function of the difference between the Shannon surprise of this observation under the current and under the prior beliefs. This means that a high value of Shannon surprise should not necessarily be a sign for forgetting, and hence should not necessarily modulate the learning rate; it is the above difference that regulates the modulation.

Finally, \( S_{GM} \) can be written as a function of the difference in Shannon surprise as

\[ S_{GM} = \frac{(1 - p_c) \exp \left( \Delta S_{Sh}(y_{t+1}; \pi^{(t)}, \pi^{(0)}) \right)}{1 - p_c \exp \left( \Delta S_{Sh}(y_{t+1}; \pi^{(t)}, \pi^{(0)}) \right)}. \]  

(S75)

which has a compact form for the case of non-volatile environments (i.e. \( p_c = 0 \))

\[ S_{GM} = \exp \left( \Delta S_{Sh}(y_{t+1}; \pi^{(t)}, \pi^{(0)}) \right). \]  

(S76)

**Derivations:**

Given the defined generative model, the Shannon surprise upon observing \( Y_{t+1} = y_{t+1} \) can be written as

\[ S_{Sh}(y_{t+1}; \pi^{(t)}) = \log \left( \frac{1}{P(Y_{t+1} = y_{t+1} | Y_{1:t} = y_{1:t})} \right) \]

\[ = \log \left( \frac{1}{(1 - p_c) P(y_{t+1}; \pi^{(t)}) + p_c P(y_{t+1}; \pi^{(0)})} \right) \]

\[ = \log \left( \frac{1}{P(y_{t+1}; \pi^{(t)})} \right) + \log \left( \frac{1}{p_c} \frac{1}{1 + \frac{1 - p_c}{p_c} \frac{P(y_{t+1}; \pi^{(0)})}{P(Y_{t+1}; \pi^{(0)})}} \right) \]

\[ = \log \left( \frac{1}{P(y_{t+1}; \pi^{(t)})} \right) + \log \left( \frac{1}{p_c} \frac{1}{1 + \frac{1 - p_c}{p_c} S_{GM}(y_{t+1}; \pi^{(t)})} \right) \]

\[ = \log \left( \frac{1}{P(y_{t+1}; \pi^{(t)})} \right) + \log \left( \frac{\gamma_{t+1}}{p_c} \right) \]

\[ = S_{Sh}(y_{t+1}; \pi^{(0)}) + \log \left( \frac{\gamma_{t+1}}{p_c} \right), \]

where \( \gamma_{t+1} = \gamma \left( S_{GM}(y_{t+1}; \pi^{(t)}), m = \frac{p_c}{1 - p_c} \right) \). As a result, the modulated learning rate can be written as in Eq. S74 and the Generative Model Surprise as in Eq. S75.

### 6.13 The Relation Between \( S_{GM} \), Bayesian, and Confidence Corrected Surprise

Bayesian [30, 29] and Confidence Corrected surprise [8] (denoted by \( S_{Ba} \) and \( S_{CC} \) respectively) depend by definition on the shape of the current belief (i.e. whether it is narrow or broad) and not only on the probabilities of events under that belief. This means that even if one event is as unexpected as another based on the (subjective) probabilities, it can have greater \( S_{Ba} \) or \( S_{CC} \) than the other, depending on the shape of the current belief.

This is in contrast to the behavior of \( S_{GM} \) and \( S_{Sh} \), which are functions of only the probabilities of events under the current and the prior beliefs. Therefore, there is not a unique relation between these two types of surprise (\( S_{GM} \) and \( S_{Sh} \) versus \( S_{Ba} \) and \( S_{CC} \)). Indeed taking the shape of belief into account will have its own beneficial aspects (i.e. considering the effects of confidence or information gain) and satisfies the first property of surprise definition from a new perspective. However, the modulation observed in our generative model (\( \gamma_{t+1} \)) cannot be written as a function of these measures of surprise.

**Derivation of Bayesian Surprise:**
Given the defined generative model, the Bayesian surprise corresponding to an observation $Y_{t+1} = y_{t+1}$ is

$$S_{Bo}(y_{t+1}; \pi^{(t)}) = D_{KL} \left[ P(\Theta_{t+1}|Y_{1:t} = y_{1:t}) || P(\Theta_{t+1}|Y_{1:t+1} = y_{1:t+1}) \right]$$

$$= E_{P(\Theta_{t+1}|Y_{1:t} = y_{1:t})} \left[ \log \left( \frac{P(\Theta_{t+1}|Y_{1:t} = y_{1:t})}{P(\Theta_{t+1}|Y_{1:t+1} = y_{1:t+1})} \right) \right]$$

$$= E_{P(\Theta_{t+1}|Y_{1:t} = y_{1:t})} \left[ \log \left( \frac{P(Y_{t+1} = y_{t+1}|\Theta_{t+1})}{P(Y_{t+1} = y_{t+1}|\Theta_{t+1})} \right) \right]$$

$$= E_{P(\Theta_{t+1}|Y_{1:t} = y_{1:t})} \left[ \log \left( \frac{1}{P(Y_{t+1} = y_{t+1}|\Theta_{t+1})} \right) \right] + \log \left( \frac{P(Y_{t+1} = y_{t+1}|Y_{1:t} = y_{1:t})}{P(Y_{t+1} = y_{t+1}|\Theta_{t+1})} \right)$$

$$= E_{P(\Theta_{t+1}|Y_{1:t} = y_{1:t})} \left[ \log \left( \frac{1}{P(Y_{t+1} = y_{t+1}|\Theta_{t+1})} \right) \right] - S_{Sh}(y_{t+1}; \pi^{(t)})$$

$$= p_c E_{\pi^{(0)}} \left[ \log \left( \frac{1}{P(\Theta_{t+1}|y_{1:t})} \right) \right] + (1 - p_c) E_{\pi^{(t)}} \left[ \log \left( \frac{1}{P(\Theta_{t+1}|\Theta_{t+1})} \right) \right] - S_{Sh}(y_{t+1}; \pi^{(t)}).$$

(S78)

Derivation of Confidence Corrected Surprise:

The Confidence Corrected surprise does not have any explicit assumption on the structure of the generative model, so it can be computed completely independent of it:

$$S_{CC}(y_{t+1}; \pi^{(t)}) = D_{KL} \left[ \pi^{(t)}(\theta)||P(\theta|y_{t+1}) \right]$$

$$= E_{\pi^{(t)}} \left[ \log \left( \frac{\pi^{(t)}(\theta)}{P(\theta|y_{t+1})} \right) \right]$$

$$= E_{\pi^{(t)}} \left[ \log \left( \frac{\pi^{(t)}(\theta)}{\pi^{(0)}(\theta)} \right) \right] + E_{\pi^{(t)}} \left[ \log \left( \frac{1}{P(\theta|y_{t+1})} \right) \right] + \log \left( \frac{P(Y_{t+1} = y_{t+1})}{P(Y_{t+1} = y_{t+1}|\Theta_{t+1})} \right)$$

(S79)

$$= D_{KL} \left[ \pi^{(t)}(\theta)||\pi^{(0)}(\theta) \right] + E_{\pi^{(t)}} \left[ \log \left( \frac{1}{P(\theta|y_{t+1})} \right) \right] - S_{Sh}(y_{t+1}; \pi^{(0)}),$$

where it should be mentioned that $S_{Sh}(y_{t+1}; \pi^{(0)})$ does not depend on the structure of the generative model - since it is computed under the prior belief.

However, given the generative model, $S_{CC}$ can be written as a function of Bayesian and Shannon surprise as

$$S_{CC}(y_{t+1}; \pi^{(t)}) = D_{KL} \left[ \pi^{(t)}(\theta)||\pi^{(0)}(\theta) \right] + S_{Bo}(y_{t+1}; \pi^{(t)})$$

$$+ \frac{1}{1 - p_c} \Delta S_{Sh}(y_{t+1}; \pi^{(t)}, \pi^{(0)})$$

$$+ \frac{p_c}{1 - p_c} \Delta S_{Bo}(y_{t+1}; \pi^{(t)}, \pi^{(0)}).$$

(S80)