ISOTROPIC LIFSHITZ SCALING IN FOUR DIMENSIONS

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The presence of isotropic Lifshitz points for a O(N)-symmetric scalar theory is investigated with the help of the Functional Renormalization Group. In particular, at the supposed lower critical dimension $d=4$, evidence for a continuous line of fixed points is found for the O(2) theory, and the observed structure presents clear similarities with the properties observed in the 2-dimensional Berezinskii-Kosterlitz-Thouless phase.

Keywords: Lifshitz point; isotropy; Kosterlitz-Thouless transition.

1. Introduction

Lifshitz points were first introduced in [1], by a generalization of the usual Landau-Ginzburg $\phi^4$ model where the space coordinates are treated anisotropically in such a way that the usual kinetic term with the square gradient of the field in one set of coordinates is kept finite, while the square gradient related to the other set of coordinates is suppressed, thus promoting the term with four powers of the gradient of the field to the leading kinetic role. This class of points also includes the case of isotropic Lifshitz points where, for all coordinates, the four gradient term is leading, the square gradient term being suppressed.

These points are associated to the class of tricritical points on the phase diagram, where ordered and disordered phase coexist with a third phase where the order parameter shows a periodic structure with finite wave vector. It has several realisations in condensed matter, such as magnetic systems, but also polymer mixtures, liquid crystals, high-Tc superconductors (for reviews on this subject see [2,3,4]), and also in the formulation of emergent gravity [5,6,7] as well as in the analysis of dense quark matter with the realization of unconventional phases [8,9,10].

In this paper, after reviewing some general properties of the Lifshitz points in Section 2, we discuss in Section 3 the isotropic points for $O(N)$ theories, which show up in the dimensional range $4 < d < 8$, by means of a non-perturbative approach, namely the Functional Renormalization Group (FRG) flow equations [12,13,14], which is especially useful in cases that are out of reach of other approaches such as the $\epsilon$-expansion [12,15]. Then, in Section 4, we concentrate on the case $d = 4$ [16,17].
where we observe very interesting properties associated to the Lifshitz scaling, such as the appearance of a continuous line of fixed points, that present clear similarities with those observed in the $d = 2$ Berezinskii - Kosterlitz - Thouless phase [18,19]. Our conclusions are reported in Section 5.

2. General Properties of Lifshitz Points

The general form of the action $S[\phi]$, suitable for describing a Lifshitz point in $d$ dimension, with $m$-dimensional anisotropic scaling, is [1]:

$$
S = \int d^{D}x_{\perp}d^{m}x_{\parallel}\left[\frac{W}{2}(\partial_{\perp}^{2}\phi)^{2} + \frac{W}{2}(\partial_{\parallel}^{2}\phi)^{2} + \frac{Z_{\parallel}}{2}(\partial_{\parallel}\phi)^{2} + \frac{Z_{\perp}}{2}(\partial_{\perp}\phi)^{2} + V\right]
$$

(1)

where $D = d - m$, and $\phi(x)$ is a $N$-component vector field and the potential $V = V(\phi)$ is a generic function of the field. Equation (1) allows for an anisotropic structure with different scaling properties of the two subsets of coordinates, $x_{\parallel}$ (which is $m$-dimensional) and $x_{\perp}$, (which is $(d-m)$-dimensional) that is realized by taking $Z_{\parallel} = 0$ and $Z_{\perp} = 1$. Then, in the orthogonal directions the leading derivative term is the one with two derivatives of the field and the term proportional to $W_{\perp}$ with four derivatives remains irrelevant, while the absence of a two derivative term in the parallel directions makes the term with four derivatives, proportional to $W_{\parallel}$, the leading kinetic term in this subset of coordinates.

In addition, at the mean field level, a negative $Z_{\parallel} < 0$, with values of the square mass in $V(\phi)$ below a critical value, induces the appearance of a new modulated phase with an oscillating ground state and, in particular, the point $Z_{\parallel} = m^{2} = 0$ corresponds to a tricritical point, indicating the coexistence of this phase with the two other phases: disordered ($\langle \phi \rangle = 0$) and ordered (constant $\langle \phi \rangle \neq 0$).

The presence of two different kinetic terms leads to two different scaling regimes with scales $\kappa_{\perp}$ and $\kappa_{\parallel}$ of the two subsets of coordinates, for instance in the two-point function of the theory, and therefore to two different anomalous dimensions, $\eta_{2}$ and $\eta_{d}$. These, in turn connect the the scales $\kappa_{\perp}$ and $\kappa_{\parallel}$ through the anisotropy parameter $\theta = (2 - \eta_{2})/(4 - \eta_{d})$, according to the relation $\kappa_{\parallel} = \kappa_{\perp}^{\theta}$. Moreover, this twofold scaling implies the following scaling dimension of the field $d_{\phi} = (d - m + \theta(m - 4 + \eta_{d}))/2$ in units of $\kappa_{\perp}$ and other operators appearing in Eq. (1), transform accordingly.

Another peculiar aspect of the Lifshitz point is given by the region of the $(m, d)$ plane where quantum corrections do not show singular behavior. In order to have a basic indication on this region, one can determine the intervals of $d$ and $m$ where the one loop integral that contributes to the two point function ($D = d - m$)

$$
I_{d,m}(p, q) = \int \frac{d^{m}q}{(2\pi)^{m}}\frac{d^{D}p'}{(2\pi)^{D}}\frac{1}{p'^{2} + q'^{4}}\frac{1}{|p + p'|^{2} + |q + q'|^{4}}
$$

(2)

does not show any pathology either in the infrared or in the ultraviolet region [20]. One easily realizes that the relevant region corresponds to the quadrilateral region delimited by a segment of the straight line $d = 4 + m/2$ on the
upper and one of the line $d = 2 + m/2$ on the lower side, plus the segment $2 < d < 4$ on the $d$ axis on the left, and a segment on the line $d = m$ on the right side.

Therefore, Fig. 1 shows the lowest order indications for the upper and lower critical dimensions of the anisotropic Lifshitz point extracted by a simple dimensional analysis and which depend both on $d$ and $m$. Clearly this picture is modified by higher order corrections and in particular by the effective change in the dimensions induced by a non-vanishing anomalous dimension that has been neglected so far.

We now focus on the specific case of interest, namely the isotropic Lifshitz point, defined by $m = d$, i.e. the parallel subspace coincides with the full $d$-dimensional space and no orthogonal subspace is left:

$$ S = \int d^d x_\| \left[ \frac{W_\|}{2} (\partial_\|^2 \phi)^2 + \frac{Z_\|}{2} (\partial_\phi)^4 + V \right]. $$

In this case the rotational symmetry in the $d$ dimensional space is fully recovered but, unlike the standard case where the scaling is set by the two derivative term, in Eq. (3) the scaling is determined by the four derivative term. In fact all coordinates now scale as $\kappa_\parallel^{-1}$ and the anomalous dimension associated with $\kappa_\perp$ is vanishing, $\eta_\parallel = 0$, so that the anisotropy parameter reduces to $\theta = 2 / (4 - \eta)$, where $\eta \equiv \eta_4$.

Therefore, it is more convenient to define the scaling dimensions of the various operators with respect to $\kappa_\parallel$ and then the scaling dimension of the field is $d_\phi = ((d - 4 + \eta))/2$, while the dimensions of $V$, $W_\|$, $Z_\|$, are respectively $d_\parallel - \eta$, $2 - \eta$. Accordingly, a square mass operator $m^2$ and a quartic coupling $u$ appearing in the potential $V(\phi)$, have dimensions respectively $4 - \eta$ and $8 - d - 2\eta$.

In addition, we remark that the values of the upper and lower critical dimensions, corresponding respectively to a vanishing scaling dimension of $u$: $d_u = 8 - 2\eta$ and to a vanishing scaling dimension of $\phi$: $d_\parallel = 4 - \eta$, contain the corrections, due to the anomalous dimension, to the lowest order indications coming from Fig. 1 with $d = m$, where one finds $d_u = 8$ and $d_\parallel = 4$. We also find an additional relevant operator, with positive scaling dimension, namely $Z_\parallel$, that has no counterpart in the usual picture of the standard dimensional analysis.
3. Functional Renormalization Group Analysis

We shall now briefly summarize some results concerning the determination of the Lifshitz point and in particular of the corresponding anomalous dimension, obtained with the help of the Functional Renormalization Group flow equations, that can be regarded as an alternative non-perturbative tool to complement the known perturbative techniques such as the $\epsilon$-expansion (for the Lifshitz case, $\epsilon = 8 - d$) \[12,15\], or the $1/N$-expansion (for a $O(N)$ theory with large number of fields) \[21\].

The functional differential equation that determines the FRG flow is \[14\]:

$$ k \partial_k \Gamma_k[\phi] = \frac{1}{2} \int_q \partial_\tau R_k(q) \left[ \Gamma_k^{(2)}[q, -q; \phi] + R_k(q) \right]^{-1} $$

(4)

$\Gamma_k[\phi]$ being the running effective action at scale $k$, $\Gamma_k^{(2)}[q, -q; \phi]$ its second functional derivative with respect to the field, and $R_k(q)$ a suitable regulator that suppresses the modes with $q \ll k$ and allows to integrate those with $q \gg k$. The flow runs from an ultraviolet fixed action, that is taken as a boundary condition of the differential equation at $k = \Lambda$, down to the deep infrared region, $k = 0$, where the solution of the differential equation, $\Gamma_k=0[\phi]$ results in the full effective action, i.e. the 1PI diagram generator of the theory. The fixed points of the theory appear as stationary solutions of the flow equation, properly expressed in terms of dimensionless quantities.

In order to solve the flow equation, one has to introduce some approximation scheme, as well as some specific ansatz of the running effective action (as, for instance, the explicit form given in Eq. (3), with $k$-dependent parameters $W_i$, $Z_i$, $V$). Then, the full flow reduces to a set of flow equations for the $k$-dependent parameters.

![Fig. 2. The anomalous dimension $\eta$ as obtained in the large $N$ expansion at order $1/N$ (dot-dashed) and in the LPA’ approximation (solid).](image)

The case of a scalar $O(N)$ symmetric theory, was considered long ago in the framework of the $1/N$-expansion and in particular the $1/N$ order computation of the anomalous dimension of the isotropic Lifshitz point as a function of the dimension $d$, was carried out \[21,22\]. The result is reported in Fig. 2.
In addition, in Fig. 2 it is shown the output of the FRG determination of the anomalous dimension at order $1/N$, obtained in \cite{16} by first determining the flow equations for the various parameters to order $1/N$ and then by solving these equations at two different orders of approximation, namely the Local Potential Approximation (LPA) (where $W_\parallel = 1$, $Z_\parallel = 0$, $\eta = 0$ are kept fixed and only one flow equation for the potential $V_k(\phi)$ is solved), and the improved approximation LPA’ (where again $W_\parallel = 1$, $Z_\parallel = 0$ and their flow is neglected but, together with the flow of the potential, one allows for a non-vanishing anomalous dimension $\eta$).

The results of the LPA indicate the existence of a nontrivial Lifshitz point when $4 < d < 8$, as expected from the analysis discussed in the previous Section. The result of the LPA’ for $\eta$ are shown in Fig. 2. As discussed above, $\eta$ from the LPA’ is not the complete $1/N$ determination of the anomalous dimension, as it comes from the LPA’ approximation to the flow equation at order $1/N$. In spite of that, the agreement of the two determinations in Fig. 2 is qualitatively good and they become coincident when approaching $\eta = 0$ in proximity of $d = 4$ and $d = 8$ \cite{16}.

In conclusion, the expectations of a Lifshitz point in the range $4 < d < 8$ (as $\eta$ vanishes at the two end points) at large $N$ are confirmed both by the full $O(1/N)$ computation and by the LPA’ approximation to the $O(1/N)$ FRG flow equations. Remarkably, $\eta$ switches sign from positive to negative, with a zero around $d = 6$.

The picture in the case of the single field scalar theory ($N = 1$), is instead less clear, at least in proximity of $d = 4$. In fact a perturbative approach in this case consist in an $\epsilon$-expansion around the upper critical dimension $d = 8$, and therefore a non-perturbative tool is needed at the other endpoint. Here, we just briefly mention the FRG numerical analysis of the coupled equations for $W_\parallel$, $Z_\parallel$, $V$ (see \cite{23}), in the context of the Proper Time Flow, as in this case the differential equation for these three parameters had already been derived in \cite{24}. The output of this analysis is that a non-trivial Lifshitz point with negative anomalous dimension is found in the range $5.5 < d < 8$. Above $d = 8$, as expected, only a gaussian-like Lifshitz point exists, while the lower limit, $d = 5.5$, does not have a physical meaning, as it is only a numerical limit below which the solution becomes very difficult to find, probably because of interference with multicritical solutions that appear when $d < 5.5$.

4. Isotropic Lifshitz points in $d = 4$

The results of the previous Sections show the nice correspondence between the range $2 < d < 4$ for the standard scaling and the range $4 < d < 8$ for the isotropic Lifshitz scaling, due to the modification in the scaling dimension of the field. For the latter scaling regime, at least for large $N$, the presence of a non-trivial Lifshitz point, which disappears both for $d > 8$ and for $d < 4$, is verified in analogy with the Wilson-Fisher fixed point, observed in the former scaling regime with $2 < d < 4$. A relevant difference between the two cases is the change of sign of the Lifshitz anomalous dimension, whereas $\eta > 0$ in the Wilson-Fisher case.

By following the above analogy, one could expect that the Lifshitz scaling shows,
at the lower critical dimension $d = 4$, properties that are similar to those observed at the lower critical dimension of the standard scaling, $d = 2$. In particular in $d = 2$, it is known that despite the Coleman - Mermin - Wagner theorem forbids, for an $O(N)$ theory, an ordered phase with finite order parameter and, therefore, also forbids a typical phase transition from a disordered to an ordered phase, it is still possible to observe, for the $O(2)$ theory, a transition of topological nature from a disordered to a quasi-ordered phase (i.e. with algebraic, rather than exponential, decay of the two-point correlation function at large distance), which is known as Berezinskii - Kosterlitz - Thouless (BKT) transition [18, 19].

Although for the Lifshitz scaling in $d = 4$ there is no equivalent of the Coleman-Mermin-Wagner theorem, we now show that some scaling features observed in the case of the BKT transition, are also reproduced in the Lifshitz case. In fact, a few interesting results are obtained by means of a simple FRG analysis, performed in analogy with a previous study on the two-dimensional BKT transition [25].

Our starting point is the following four dimensional $U(1)$ invariant model:

$$
\Gamma_k[\phi] = \int d^4r \left\{ \frac{u_k}{8} (|\phi|^2 - \alpha_k^2)^2 + \frac{W_k^A}{2} [\partial^2 \phi \partial^2 \phi^*] \\
+ \frac{W_k^B}{8} [\partial^2 |\phi|^2]^2 + \frac{Z_k^A}{2} [\partial \phi \partial \phi^*] + \frac{Z_k^B}{8} [\partial |\phi|^2]^2 \right\}
$$

(5)

where four field derivative as well as two field derivative terms, with field independent parameters, $W_k^A, W_k^B, Z_k^A, Z_k^B$, have been included and the quartic potential is expressed in terms of $u_k, \alpha_k$. Instead of a $O(2)$ symmetric action, in Eq. (5) we took an action invariant under $U(1)$ transformations of the complex field $\phi(r)$, that can be decomposed into a longitudinal and a transverse component, including an expectation value of the longitudinal component $\alpha_k$: $\phi(r) = \alpha + \sigma(r) + i \pi(r)$.

In this scheme one can compute the flow equation for the various $k$-dependent parameters [17]. By starting the flow at an initial scale $k = \Lambda$, with large values of $\alpha_\Lambda$, one immediately observes for this parameter a power law when $k \to 0$, with exponent $\eta$, according to its scaling dimension: $\alpha_k^2 \propto k^\eta$. At the same time the field renormalization parameter $W_k^A$ shows the inverse scaling $W_k^A \propto k^{-\eta}$, again in agreement with the scaling dimensional analysis. As a consequence, the renormalized square field, i.e. the product $J = W_k^A \alpha_k^2$, remains constant along the flow. When the flow is started at a lower value of $\alpha_\Lambda$, the above picture breaks down and $J$ shows a scale dependence that leads to $J \to 0$ at some finite scale $k$.

This behavior is summarized in Fig. 3, where $J$ is plotted vs. $t = \log(\Lambda/k)$ for different initial values of $\alpha_\Lambda$, with upper (solid) curves corresponding to larger initial values. Each upper curve in Fig. 3 indicates the presence of a fixed point where all dimensionless parameters in our model reach a $k$-independent value and remain constant along the flow, as verified in [17]. In particular, the relevant parameter $Z_k^A$, when its initial value is suitably taken on the critical surface, in the infrared region goes to zero as $Z_k^A \propto k^{2-\eta}$. Therefore, we obtained a continuous line of fixed points
parametrized by $J$ (or equivalently by $\alpha_\Lambda$) that disappears at sufficiently small $J$, as the corresponding curves (dashed in Fig. 3) are no longer constant.

![Flow of $J = W^\Lambda a_1^2$ vs. $t = \log(\Lambda/k)$ for different initial values of $\alpha_\Lambda$. Flat upper solid curves correspond to larger values of $\alpha_\Lambda$, while dashed and dot-dashed to lower values.]

This is analogous to the picture obtained for the BKT transition where $\alpha_\Lambda^2 \to 0$, when $k \to 0$, indicates a vanishing order parameter both in the disordered and in the quasi-ordered phase, while the line of fixed points associated to a large non-vanishing $t$-independent $J$ (in BKT language, the stiffness) indicates the presence of a quasi-ordered phase associated to the algebraic decay of the two-point function.

It is known that the BKT line of fixed points, regardless of the microscopic details of the model considered, ends at the universal value $J = 2/\pi$ and, below it, $J$ is no more scale independent, as $J \to 0$, and the disordered phase replace the quasi-ordered phase. Apparently, this endpoint corresponds, in Fig. 3, to the separation between the upper solid flat curves and the lower dashed curves that show no constant behavior.

Unfortunately, this statement is not exactly true. In fact, there is no clear transition between the two sets of curves and, more specifically, if one continues the flow of the upper set to extremely large values of $t$, one will eventually observe a decrease of $J$ toward zero, meaning that in Fig. 3 strictly speaking, there is no definite value of $J$ signalling a transition and the regime corresponding to the line of fixed points is only approximately reached at large $J$.

This, in turn, does not mean that for the present case of the Lifshitz point, there is no equivalent of the BKT transition, but only that the approximations involved in the FRG analysis could be not sufficient to detect this effect, yielding only an approximate description of it. In fact even for the two dimensional BKT case, the analogous FRG analysis produces similar results [25,26], and attempts to reproduce the correct picture require an enlargement of the ansatz in Eq. 6 and therefore a greater numerical effort [27], or the use of more sophisticated approaches [28,29], not applied so far to the more complicated case of the Lifshitz scaling.

At this point, instead of testing improved versions of the FRG flow, we look
for a possible field configuration that could explain the scaling properties discussed above. To this purpose, we consider Eq. (5) as the starting point, and use polar coordinates \( \phi(r) = \sqrt{\rho(r)} \exp[i\theta(r)] \). Then, in the infrared region a mass term for the radial component \( \rho(r) \) suppresses spatial fluctuations, so that this component reduces to a constant \( \rho(r) \to \rho_0 \). In addition, we neglect relevant operators such as the two field derivative term, that are suppressed in the infrared region, when suitably taken on the critical surface. Therefore, we consider the following effective action depending on the angular fluctuations only,

\[
\Gamma = \frac{K}{2} \int d^4r \left[ \partial^2 \theta(r) \partial^2 \theta(r) \right],
\]

(6)
to provide the correct description of the infrared sector of the original theory. In principle, Eq. (5) produces an additional term proportional to \( (\partial \theta \partial \theta)^2 \) that we neglect here, as it is always possible to include in Eq. (5) an additional operator, proportional to \( \partial \phi \partial \phi^* \), that cancels exactly the quartic term in \( \theta \).

Then, by recalling the form of the Green function of the Laplacian operator in four dimensions, \( \partial_r^2 [ -1/(r - r')^2 ] = (2\pi)^2 \delta^4(r - r') \), one realizes that the configuration

\[
\theta_{r'}(r) = \int d^4r'' (2\pi)^2 \frac{1}{(r - r'')^2} \frac{1}{(r'' - r')^2} = \frac{1}{4} \ln \left( \frac{R^2}{(r - r')^2} \right)
\]

(7)
which has a singularity at the point \( r' \), does actually minimize \( \Gamma \) in Eq. (6), as the extremum equation gives: \( (\partial_r^2 \partial_r^2)\theta_{r'}(r) = (2\pi)^2 \delta^4(r - r') \), i.e. it vanishes everywhere, but at the singular point \( r' \). The integral in Eq. (7) is then performed by introducing a large distance cut-off \( R \); the output is displayed in the right hand side of (7).

This result resembles the one obtained for the vortex configuration of the BKT problem and, as in that case, it allows to compute the energy associated to the configuration in Eq. (7), provided one introduces also a short distance cut-off (or lattice spacing), \( r_0 \). In fact, by regarding Eq. (6) as the hamiltonian of the angular field \( \theta(r) \) in a 4-dimensional space, one gets the energy for the configuration in Eq. (7):

\[
\Gamma[\theta_{r'}] = \frac{K}{2} \pi^2 \ln \left( R^2/r_0^2 \right).
\]

Then, by noticing that the entropy \( \Sigma \) associated to placing such a configuration (i.e. the singularity \( r' \)) in the four dimensional space delimited at large and small distance respectively by \( R \) and \( r_0 \), is given by \( \Sigma[\theta_{r'}] = \ln \left( R^4/r_0^4 \right) \), one can estimate the free energy \( F \) of the system (\( T \) indicates the temperature):

\[
F = \Gamma[\theta_{r'}] - T \Sigma[\theta_{r'}] = \left( \frac{K}{2} \pi^2 - 2T \right) \ln \left( R^2/r_0^2 \right)
\]

(8)

Then, as for the BKT case, the transition of \( F \) from positive to negative indicates instability with respect to the generation of such configurations and therefore indicates the transition to a disordered phase where free (unpaired) configurations are observed. The transition point in our case is \( K/T = 4/\pi^2 \), and although this specific value directly depends on the particular normalization chosen in (7), it signals the existence of a transition point in \( K/T \), from the quasi-ordered to the disordered phase, that was not detected in the previous FRG analysis.
5. Conclusions

The FRG analysis of the Isotropic Lifshitz scaling shows, more accurately in the case of very large number of fields $N$, the presence of a non-trivial fixed point in the dimensional range $4 < d < 8$ that, to some extent, resembles the properties of the well known Wilson-Fisher fixed point in the range $2 < d < 4$. Exactly as for the latter case in $d = 2$, it is found that the Lifshitz point disappears in $d = 4$.

However, it has been found, again in close analogy with the two-dimensional BKT transition, that for the $O(2)$ symmetric model in $d = 4$, the Lifshitz scaling predicts the presence of a continuous line of fixed points, corresponding to a quasi-ordered phase with algebraic long-distance decay of the correlation function.

Unfortunately, due to the limits of the approximations adopted to solve the FRG flow, no definite picture of a phase transition to the disordered phase is obtained. Instead of attempting to improve on these approximations, we conjectured that a specific configuration of the angular component of the complex field, which is a minimum of the energy, is responsible of the fixed point line and also of the transition to the disordered phase at some finite value of the effective coupling, in the same way as the vortex configurations act in the BKT transition.

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