Stable Stems

Daniel C. Isaksen

DEPARTMENT OF MATHEMATICS, WAYNE STATE UNIVERSITY, DETROIT, MI
48202, USA
E-mail address: isaksen@wayne.edu
Abstract. We present a detailed analysis of 2-complete stable homotopy groups, both in the classical context and in the motivic context over $\mathbb{C}$. We use the motivic May spectral sequence to compute the cohomology of the motivic Steenrod algebra over $\mathbb{C}$ through the 70-stem. We then use the motivic Adams spectral sequence to obtain motivic stable homotopy groups through the 59-stem. In addition to finding all Adams differentials in this range, we also resolve all hidden extensions by 2, $\eta$, and $\nu$, except for a few carefully enumerated exceptions that remain unknown. The analogous classical stable homotopy groups are easy consequences.

We also compute the motivic stable homotopy groups of the cofiber of the motivic element $\tau$. This computation is essential for resolving hidden extensions in the Adams spectral sequence. We show that the homotopy groups of the cofiber of $\tau$ are the same as the $E_2$-page of the classical Adams-Novikov spectral sequence. This allows us to compute the classical Adams-Novikov spectral sequence, including differentials and hidden extensions, in a larger range than was previously known.
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CHAPTER 1

Introduction

One of the fundamental problems of stable homotopy theory is to compute the stable homotopy groups of the sphere spectrum. One reason for computing these groups is that maps between spheres control the construction of finite cell complexes.

After choosing a prime \( p \) and focusing on the \( p \)-complete stable homotopy groups instead of the integral homotopy groups, the Adams spectral sequence and the Adams-Novikov spectral sequence have proven to be the most effective tools for carrying out such computations.

At odd primes, the Adams-Novikov spectral sequence has clear computational advantages over the Adams spectral sequence. (Nevertheless, the conventional wisdom, derived from Mark Mahowald, is that one should compute with both spectral sequences because they emphasize distinct aspects of the same calculation.)

Computations at the prime 2 are generally more difficult than computations at odd primes. In this case, the Adams spectral sequence and the Adams-Novikov spectral sequence seem to be of equal complexity. The purpose of this manuscript is to thoroughly explore the Adams spectral sequence at 2 in both the classical and motivic contexts.

Motivic techniques are essential to our analysis. Working motivically instead of classically has both advantages and disadvantages. The main disadvantage is that the computation is larger and proportionally more difficult. On the other hand, there are several advantages. First, the presence of more non-zero classes allows the detection of otherwise elusive phenomena. Second, the additional motivic weight grading can easily eliminate possibilities that appear plausible from a classical perspective.

The original motivation for this work was to provide input to the \( \rho \)-Bockstein spectral sequence for computing the cohomology of the motivic Steenrod algebra over \( \mathbb{R} \). The analysis of the \( \rho \)-Bockstein spectral sequence, and the further analysis of the motivic Adams spectral sequence over \( \mathbb{R} \), will appear in future work.

This manuscript is a natural sequel to [13], where the first computational properties of the motivic May spectral sequence, as well as of the motivic Adams spectral sequence, were established.

1.1. The Adams spectral sequence program

The Adams spectral sequence starts with the cohomology of the Steenrod algebra \( A \), i.e., \( \text{Ext}_A(\mathbb{F}_2, \mathbb{F}_2) \). There are two ways of approaching this algebraic object. First, one can compute by machine. This has been carried out to over 200 stems [9] [35]. Machines can also compute the higher structure of products and Massey products.
The second approach is to compute by hand with the May spectral sequence. This will be carried out to 70 stems in Chapter 2. See also \[39\] for the classical case. See \[19\] for a detailed Ext chart through the 70-stem.

The $E_\infty$-page of the May spectral sequence is the graded object associated to a filtration on $\text{Ext}_A(F_2,F_2)$, which can hide some of the multiplicative structure. One can resolve these hidden multiplicative extensions with indirect arguments involving higher structure such as Massey products or algebraic squaring operations in the sense of \[30\]. A critical ingredient here is May’s Convergence Theorem \[29\] Theorem 4.1, which allows the computation of Massey products in $\text{Ext}_A(F_2,F_2)$ via the differentials in the May spectral sequence.

The cohomology of the Steenrod algebra is the $E_2$-page of the Adams spectral sequence. The next step is to compute the Adams differentials. This will be carried out in Chapter 3. Techniques for establishing differentials include:

1. Use knowledge of the image of $J$ \[2\] to deduce differentials.
2. Compare to the completely understood Adams spectral sequence for the topological modular forms spectrum $\text{tmf}$ \[15\].
3. Use the relationship between algebraic squaring operations and Adams differentials \[11\] VI.1.
4. Exploit Toda brackets to deduce relations in the stable homotopy ring, which then imply Adams differentials.

We have assembled all previously published results about the Adams differentials in Table 18.

The $E_\infty$-page of the Adams spectral sequence is the graded object associated to a filtration on the stable homotopy groups, which can hide some of the multiplicative structure. The final step is to resolve these hidden multiplicative extensions. This will be carried out in Chapter 4. Analogously to the extensions that are hidden in the May spectral sequence, this generally involves indirect arguments with Toda brackets. We have assembled previously published results about these hidden extensions in Table 24.

The detailed analysis of the Adams spectral sequence requires substantial technical work with Toda brackets. A critical ingredient for computing Toda brackets is Moss’s Convergence Theorem \[34\], which allows the computation of Toda brackets via the Adams differentials. We remind the reader to be cautious about indeterminacies in Massey products and Toda brackets.

### 1.2. Motivic homotopy theory

The formal construction of motivic homotopy theory requires the heavy machinery of simplicial presheaves and model categories \[33\] \[22\] \[12\]. We give a more intuitive description of motivic homotopy theory that will suffice for our purposes.

Motivic homotopy theory is a homotopy theory for algebraic varieties. Start with the category of smooth schemes over a field $k$ (in this manuscript, $k$ always equals $\mathbb{C}$). This category is inadequate for homotopical purposes because it does not possess enough gluing constructions, i.e., homotopy colimits.

In order to fix this problem, we can formally adjoin homotopy colimits. This takes us to the category of simplicial presheaves.
The next step is to restore some desired relations. If \( \{U, V\} \) is a Zariski cover of a smooth scheme \( X \), then \( X \) is the colimit of the diagram

\[
\begin{array}{ccc}
U & \to & U \cap V \\
\downarrow & & \downarrow \\
& V 
\end{array}
\]

in the category of smooth schemes. However, when we formally adjoined homotopy colimits, we created a new object, distinct from \( X \), that served as the homotopy pushout of Diagram (1.1). This is undesirable, so we formally declare that \( X \) is the homotopy pushout of Diagram (1.1) from which we obtain the local homotopy theory of simplicial presheaves. This homotopy theory has some convenient properties such as Mayer-Vietoris sequences.

In fact, one needs to work not with Zariski covers but with Nisnevich covers. See [33] for details on this technical point.

The final step is to formally declare that each projection map \( X \times \mathbb{A}^1 \to X \) is a weak equivalence. This gives the unstable motivic homotopy category.

In unstable motivic homotopy theory, there are two distinct objects that play the role of circles:

1. \( S^{1,0} \) is the usual simplicial circle.
2. \( S^{1,1} \) is the punctured affine line \( \mathbb{A}^1 - 0 \).

For \( p \geq q \), the unstable sphere \( S^{p,q} \) is the appropriate smash product of copies of \( S^{1,0} \) and \( S^{1,1} \), so we have a bigraded family of spheres.

Stable motivic homotopy theory is the stabilization of unstable motivic homotopy theory with respect to this bigraded family of spheres. As a consequence, calculations such as motivic cohomology and motivic stable homotopy groups are bigraded.

Motivic homotopy theory over \( \mathbb{C} \) comes with a realization functor to ordinary homotopy theory. Given a complex scheme \( X \), there is an associated topological space \( X(\mathbb{C}) \) of \( \mathbb{C} \)-valued points. This construction extends to a well-behaved functor between unstable and stable homotopy theories.

We will explain at the beginning of Chapter 3 that we have very good calculational control over this realization functor. We will use this relationship in both directions: to deduce motivic facts from classical results, and to deduce classical facts from motivic results.

One important difference between the classical case and the motivic case is that not every motivic spectrum is built out of spheres, i.e., not every motivic spectrum is cellular. Stable cellular motivic homotopy theory is more tractable than the full motivic homotopy theory, and many motivic spectra of particular interest, such as the Eilenberg-Mac Lane spectrum \( HF_2 \), the algebraic \( K \)-theory spectrum \( KGL \), and the algebraic cobordism spectrum \( MGL \), are cellular. Stable motivic homotopy group calculations are fundamental to cellular motivic homotopy theory. However, the part of motivic homotopy theory that is not cellular is essentially invisible from the perspective of stable motivic homotopy groups.

Although one can study motivic homotopy theory over any base field (or even more general base schemes), we will work only over \( \mathbb{C} \), or any algebraically closed field of characteristic 0. Even in this simplest case, we find a wealth of exotic phenomena that have no classical analogues.
1.3. The motivic Steenrod algebra

The starting point for our Adams spectral sequence work is the description of the motivic Steenrod algebra over \( \mathbb{C} \) at the prime 2, which is a variation on the classical Steenrod algebra. First, the motivic cohomology of a point is \( \mathbb{M}_2 = \mathbb{F}_2[\tau] \), where \( \tau \) has degree \((0, 1)\) \[43\].

The (dual) motivic Steenrod algebra over \( \mathbb{C} \) is \[44\] \[42\] \[7\] Section 5.2

\[ \mathbb{M}_2[\tau_0, \tau_1, \ldots, \xi_1, \xi_2, \ldots] \]

\[ \tau_i^2 = \tau \xi_{i+1} \]

The reduced coproduct is determined by

\[ \tilde{\phi}_s(\tau_k) = \xi_k \otimes \tau_0 + \xi_k^2 \otimes \tau_1 + \cdots + \xi_k^2 \otimes \tau_1 + \cdots + \xi_1^{2^{k-1}} \otimes \tau_{k-1} \]

\[ \tilde{\phi}_s(\xi_k) = \xi_{k-1} \otimes \xi_1 + \xi_1^4 \otimes \xi_2 + \cdots + \xi_k^2 \otimes \xi_1 + \cdots + \xi_1^{2^{k-1}} \otimes \xi_{k-1} \]

The dual motivic Steenrod algebra has a few interesting features. First, if we invert \( \tau \), then we obtain a polynomial algebra that is essentially the same as the classical dual Steenrod algebra. This is a general feature. We will explain at the beginning of Chapter 3 that one recovers classical calculations from motivic calculations by inverting \( \tau \). This fact is useful in both directions: to deduce motivic facts from classical ones, and to deduce classical facts from motivic ones.

Second, if we set \( \tau = 0 \), we obtain a “\( p = 2 \) version” of the classical odd primary dual Steenrod algebra, with a family of exterior generators and another family of polynomial generators. This observation suggests that various classical techniques that are well-suited for odd primes may also work motivically at the prime 2.

1.4. Relationship between motivic and classical calculations

As a consequence of our detailed analysis of the motivic Adams spectral sequence, we recover the analysis of the classical Adams spectral sequence by inverting \( \tau \). We will use known results about the classical Adams spectral sequence from \[3\], \[4\], \[8\], \[26\], and \[40\]. We have carefully collected these results in Tables 18 and 24.

A few of our calculations are inconsistent with calculations in \[23\] and \[24\], and we are unable to understand the exact sources of the discrepancies. For this reason, we have found it prudent to avoid relying directly on the calculations in \[23\] and \[24\]. However, we will follow \[24\] in establishing one particularly difficult Adams differential in Section 5.3.3.

Here is a summary of our calculations that are inconsistent with \[23\] and \[24\]:

1. There is a classical differential \( d_3(Q_2) = \eta \). This means that classical \( \pi_{56} \) has order 2, not order 4; and that classical \( \pi_{57} \) has order 8, not order 16.
2. The element \( h_1g_2 \) in the 45-stem does not support a hidden \( \eta \) extension to \( N \).
3. The element \( C \) of the 50-stem does not support a hidden \( \eta \) extension to \( gn \).
4. \[23\] claims that there is a hidden \( \nu \) extension from \( h_2h_5d_0 \) to \( gn \) and that there is no hidden 2 extension on \( h_0h_3g_2 \). These two claims are incompatible; either both hidden extensions occur, or neither occur. (See Lemma 4.2.31)
The proof of the non-existence of the hidden $\eta$ extension on $h_1g_2$ is particularly interesting because it relies inherently on a motivic calculation. We know of no way to establish this result only with classical tools.

We draw particular attention to the Adams differential $d_2(D_1) = h_0^2h_3g_2$ in the 51-stem. Mark Mahowald privately communicated an argument for the presence of this differential to the author. However, this argument fails because of the Toda bracket calculation in Lemma 4.2.91, which was unknown to Mahowald. Zhouli Xu discovered an independent proof, which is included in this manuscript as Lemma 3.3.13. This settles the order of $\pi_{51}$ but not its group structure. It is possible that $\pi_{51}$ contains an element of order 8.

We also remark on the hidden 2 extension in the 62-stem from $E_1 + C_0$ to $R$ indicated in [24]. We cannot be absolutely certain of the status of this extension because it lies outside the range of our thorough analysis. However, it appears implausible from the motivic perspective. (For entirely different reasons related to $v_2$-periodic homotopy groups, Mark Mahowald communicated privately to the author that he was also skeptical of this hidden extension.)

1.5. Relationship to the Adams-Novikov spectral sequence

We will describe a rigid relationship between the motivic Adams spectral sequence and the motivic Adams-Novikov spectral sequence in Chapter 6. In short, the $E_2$-page of the classical Adams-Novikov spectral sequence is isomorphic to the bigraded homotopy groups $\pi_{*,*}(C\tau)$ of the cofiber of $\tau$. Here $\tau$ is the element of the motivic stable homotopy group $\pi_0, -1$ that is detected by the element $\tau$ of $M_2$. Moreover, the classical Adams-Novikov spectral sequence is identical to the $\tau$-Bockstein spectral sequence converging to stable motivic homotopy groups!

In Chapter 5, we will extensively compute $\pi_{*,*}(C\tau)$. In Chapter 6, we will apply this information to obtain information about the classical Adams-Novikov spectral sequence in previously unknown stems.

However, there are two places in earlier chapters where we use specific calculations from the classical Adams-Novikov spectral sequence. We would prefer arguments that are internal to the Adams spectral sequence, but they have so far eluded us. The specific calculations that we need are:

1. Lemma 4.2.7 shows that a certain possible hidden $\tau$ extension does not occur in the 57-stem. See also Remark 4.1.12. For this, we use that $\beta_{12/6}$ is the only element in the Adams-Novikov spectral sequence in the 58-stem with filtration 2 that is not divisible by $\alpha_1$ [37].

2. Lemma 4.2.35 establishes a hidden 2 extension in the 54-stem. See also Remark 4.1.18. For this, we use that $\beta_{10/2}$ is the only element of the Adams-Novikov spectral sequence in the 54-stem with filtration 2 that is not divisible by $\alpha_1$, and that this element maps to $\Delta^2h_2^2$ in the Adams-Novikov spectral sequence for $tmf$ [51] [37].

The $E_2$-page of the motivic (or classical) Adams spectral sequence is readily computable by machine. On the other hand, there seem to be real obstructions to practical machine computation of the $E_2$-page of the classical Adams-Novikov spectral sequence.

On the other hand, let us suppose that we did have machine computed data on the $E_2$-page of the classical Adams-Novikov spectral sequence. The rigid relationship between motivic stable homotopy groups and the classical Adams-Novikov
spectral sequence could be exploited to great effect to determine the pattern of differentials in both the Adams-Novikov and the Adams spectral sequences. We anticipate that all differentials through the 60-stem would be easy to deduce, and we would expect to be able to compute well past the 60-stem. For this reason, we foresee that the next major breakthrough in computing stable stems will involve machine computation of the Adams-Novikov $E_2$-page.

1.6. How to use this manuscript

The exposition of such a technical calculation creates some inherent challenges. In the end, the most important parts of this project are the Adams charts from [19], the Adams-Novikov charts from [21], and the tables in Chapter 7. These tables contain a wealth of detailed information in a concise form. They summarize the essential calculational facts that allow the computation to proceed. The tables are particularly useful for readers who are looking for information on a specific calculational fact, since the tables include references to more detailed proofs. In fact, the rest of the manuscript merely consists of detailed arguments that support the claims in the tables.

We draw attention to the following charts from [19] and [21] that are of particular interest:

1. A classical Adams $E_2$ chart with differentials.
2. A classical Adams $E_\infty$ chart with hidden extensions by 2, $\eta$, and $\nu$.
3. A motivic Adams $E_2$ chart.
4. A motivic Adams $E_\infty$ chart with hidden $\tau$ extensions.
5. A classical Adams-Novikov $E_2$ chart with differentials.
6. A classical Adams-Novikov $E_\infty$ chart with hidden extensions by 2, $\eta$, and $\nu$.

In each of the charts, we have been careful to document explicitly the remaining uncertainties in our calculations.

We also draw attention to the following tables from Chapter 7 that are of particular interest:

1. Tables 8, 20, 21, and 22 give all of the Adams differentials.
2. Table 19 gives some Massey products in the cohomology of the motivic Steenrod algebra, including indeterminacies.
3. Table 18 summarizes previously known results about classical Adams differentials.
4. Table 19 summarizes previously known results about classical Toda brackets.
5. Table 23 gives some Toda brackets, including indeterminacies.
6. Table 24 summarizes previously known results about hidden extensions in the classical stable homotopy groups.
7. Table 47 gives a correspondence between elements of the classical Adams and Adams-Novikov $E_\infty$ pages.

These tables include specific references to complete proofs of each fact.
1.7. Notation

By convention, we give degrees in the form \((s, f, w)\), where \(s\) is the stem; \(f\) is the Adams filtration; and \(w\) is the motivic weight. An element of degree \((s, f, w)\) will appear on a chart at coordinates \((s, f)\).

We will use the following notation extensively:

1. \(\mathbb{M}_2\) is the mod 2 motivic cohomology of \(\mathbb{C}\).
2. \(A\) is the mod 2 motivic Steenrod algebra over \(\mathbb{C}\).
3. \(A(2)\) is the \(\mathbb{M}_2\)-subalgebra of \(A\) generated by \(\text{Sq}^1, \text{Sq}^2, \text{and} \text{Sq}^4\).
4. \(\text{Ext}\) is the trigraded ring \(\text{Ext}_A(\mathbb{M}_2, \mathbb{M}_2)\).
5. \(A_{\text{cl}}\) is the classical mod 2 Steenrod algebra.
6. \(\text{Ext}_{\text{cl}}\) is the bigraded ring \(\text{Ext}_{A_{\text{cl}}}(\mathbb{F}_2, \mathbb{F}_2)\).
7. \(\pi_{*,*}\) is the 2-complete motivic stable homotopy ring over \(\mathbb{C}\).
8. \(E_r(S^{0,0})\) is the \(E_r\)-page of the motivic Adams spectral sequence converging to \(\pi_{*,*}\). Note that \(E_2(S^{0,0})\) equals \(\text{Ext}\).
9. For \(x\) in \(E_\infty(S^{0,0})\), write \(\{x\}\) for the set of all elements of \(\pi_{*,*}\) that are represented by \(x\).
10. \(\tau\) is both an element of \(\mathbb{M}_2\), as well as the element of \(\pi_{0,-1}\) that it represents in the motivic Adams spectral sequence.
11. \(C\tau\) is the cofiber of \(\tau: S^{0,-1} \to S^{0,0}\).
12. \(H^{*,*}(C\tau)\) is the mod 2 motivic cohomology of \(C\tau\).
13. \(\pi_{*,*}(C\tau)\) are the 2-complete motivic stable homotopy groups of \(C\tau\), which form a \(\pi_{*,*}\)-module.
14. \(E_r(C\tau)\) is the \(E_r\)-page of the motivic Adams spectral sequence that converges to \(\pi_{*,*}(C\tau)\). Note that \(E_r(C\tau)\) is an \(E_r(S^{0,0})\)-module, and \(E_2(C\tau)\) is equal to \(\text{Ext}_A(H^{*,*}(C\tau), \mathbb{M}_2)\).
15. For \(x\) in \(E_2(S^{0,0})\), write \(x\) again (or \(x_{C\tau}\) when absolutely necessary for clarity) for the image of \(x\) under the map \(E_2(S^{0,0}) \to E_2(C\tau)\) induced by the inclusion \(S^{0,0} \to C\tau\) of the bottom cell.
16. For \(x\) in \(E_2(S^{0,0})\) such that \(x\tau = 0\), write \(\pi\) for a pre-image of \(x\) under the map \(E_2(C\tau) \to E_2(S^{0,0})\) induced by the projection \(C\tau \to S^{1,-1}\) to the top cell. There may be some indeterminacy in the choice of \(\pi\). See Section 5.1.3 and Table 40 for further discussion about these choices.
17. \(E_r(S^0; BP)\) is the \(E_r\)-page of the classical Adams-Novikov spectral sequence.
18. \(E_r(S^{0,0}; BPL)\) is the \(E_r\)-page of the motivic Adams-Novikov spectral sequence converging to \(\pi_{*,*}\).
19. \(E_r(C\tau; BPL)\) is the \(E_r\)-page of the motivic Adams-Novikov spectral sequence converging to \(\pi_{*,*}(C\tau)\).

Table II lists some traditional notation for specific elements of the motivic stable homotopy ring. We will use this notation whenever it is convenient. A few remarks about these elements are in order:

1. See [17, p. 28] for a geometric construction of \(\tau\).
2. Over fields that do not contain \(\sqrt{-1}\), the motivic stable homotopy group \(\pi_{0,0}\) contains an element that is usually called \(\epsilon\). Our use of the symbol \(\epsilon\) follows Toda [41]. This should cause no confusion since we are working only over \(\mathbb{C}\).
1. INTRODUCTION

(3) The element \( \eta_4 \) is defined to be the element of \( \{ h_1 h_4 \} \) such that \( \eta^3 \eta_4 \) is zero. (The other element of \( \{ h_1 h_4 \} \) supports infinitely many multiplications by \( \eta \).)

(4) Similarly, \( \eta_5 \) is defined to be the element of \( \{ h_1 h_5 \} \) such that \( \eta^7 \eta_5 \) is zero.

The element \( \theta_{4,5} \) deserves additional discussion. We have perhaps presumptuously adopted this notation for an element of \( \{ h_2 h_3 h_5 \} = \{ h_4 \} \). This element is called \( \alpha \) in [3]. To construct \( \theta_{4,5} \), first choose an element \( \theta'_{4,5} \) in \( \{ h_2 h_3 h_5 \} \) such that \( 4 \theta'_{4,5} \) is contained in \( \{ h_0 h_5 d_0 \} \). If \( \eta \theta'_{4,5} \) is contained in \( \{ h_1 h_5 d_0 \} \), then add an element of \( \{ h_5 d_0 \} \) to \( \theta'_{4,5} \) and obtain an element \( \theta''_{4,5} \) such that \( \eta \theta''_{4,5} \) is contained in \( \{ B_1 \} \). Next, if \( \sigma \theta''_{4,5} \) is contained in \( \{ \tau h_1 h_3 g_2 \} \), then add an element of \( \{ \tau h_1 h_3 g_2 \} \) to \( \theta''_{4,5} \) to obtain an element \( \theta_{4,5} \) such that \( \sigma \theta_{4,5} \) is detected in Adams filtration at least 8. Note that \( \sigma \theta_{4,5} \) may in fact be zero.

This does not specify just a single element of \( \{ h_2 h_3 h_5 \} \). The indeterminacy in the definition contains even multiples of \( \theta_{4,5} \) and the element \( \{ \tau w \} \), but this indeterminacy does not present much difficulty.

In addition, we do not know whether \( \nu \theta_{1,5} \) is contained in \( \{ B_2 \} \). We know from Lemma 4.2.73 that there is an element \( \theta \) of \( \{ h_2 h_5 \} \) such that \( \nu \theta \) is contained in \( \{ B_2 \} \). It is possible that \( \theta \) is of the form \( \theta_{4,5} + \beta \), where \( \beta \) belongs to \( \{ h_5 d_0 \} \). We can conclude only that either \( \nu \theta_{4,5} \) or \( \nu(\theta_{4,5} + \beta) \) belongs to \( \{ B_2 \} \).

For more details on the properties of \( \theta_{4,5} \), see Examples 4.1.6 and 4.1.7, as well as Lemmas 4.2.48 and 4.2.73.

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CHAPTER 2

The cohomology of the motivic Steenrod algebra

This chapter applies the motivic May spectral sequence to obtain the cohomology of the motivic Steenrod algebra through the 70-stem. We will freely borrow results from the classical May spectral sequence, i.e., from [28] and [39]. We will also need some facts from the cohomology of the classical Steenrod algebra that have been verified only by machine [9] [10].

The Ext chart in [19] is an essential companion to this chapter.

Outline. We begin in Section 2.1 with a review of the basic facts about the motivic Steenrod algebra over \( \mathbb{C} \), the motivic May spectral sequence over \( \mathbb{C} \), and the cohomology of the motivic Steenrod algebra.

A critical ingredient is May’s Convergence Theorem [29, Theorem 4.1], which allows the computation of Massey products in \( \text{Ext}^*_A(\mathbb{F}_2, \mathbb{F}_2) \) via the differentials in the May spectral sequence. We will thoroughly review this result in Section 2.2.

Next, in Section 2.3 we describe the main points in computing the motivic May spectral sequence through the 70-stem. We rely heavily on results of [28] and [39], but we must also compute several exotic differentials, i.e., differentials that do not occur in the classical situation.

Having obtained the \( E_\infty \)-page of the motivic May spectral sequence, the next step is to consider hidden extensions. In Section 2.4 we are able to resolve every possible hidden extension by \( \tau \), \( h_0 \), \( h_1 \), and \( h_2 \) through the range that we are considering, i.e., up to the 70-stem. The primary tools here are:

1. shuffling relations among Massey products.
2. squaring operations on Ext groups in the sense of [30].
3. classical hidden extensions established by machine computation [9].

Chapter 7 contains a series of tables that are essential for bookkeeping throughout the computations:

1. Tables 2 and 3 describe the May \( E_2 \)-page in terms of generators and relations and give the values of the May \( d_2 \) differential.
2. Tables 4 through 7 describe the May differentials \( d_r \) for \( r \geq 4 \).
3. Table 8 lists the multiplicative generators of the cohomology of the motivic Steenrod algebra over \( \mathbb{C} \).
4. Table 9 lists multiplicative generators of the May \( E_\infty \)-page that become decomposable in Ext by hidden relations.
5. Table 10 lists all examples of multiplicative generators of the May \( E_\infty \)-page that represent more than one element in Ext. See Section 2.3.6 for more explanation.
6. Tables 11 through 15 list all extensions by \( \tau \), \( 2 \), \( \eta \), and \( \nu \) that are hidden in the May spectral sequence. A few miscellaneous hidden extensions are included as well.
2. THE COHOMOLOGY OF THE MOTIVIC STEENROD ALGEBRA

(7) Table 16 summarizes some Massey products.
(8) Table 17 summarizes some matric Massey products.

Table 16 deserves additional explanation. In all cases, we have been careful to
describe the indeterminacies accurately. The fifth column refers to an argument for
establishing the Massey product, in one of the following forms:

(1) An explicit proof given elsewhere in this manuscript.
(2) A May differential implies the Massey product via May’s Convergence

The last column of Table 16 lists the specific results that rely on each Massey
product. Frequently, these results are just a Toda bracket from Table 23.

Some examples. In this section, we describe several of the computational
intricacies that are established later in the chapter. We also present a few questions
that deserve further study.

Example 2.0.1. An obvious question, which already arose in 13, is to find
elements that are killed by \( \tau^n \) but not by \( \tau^{n-1} \), for various values of \( n \).

The element \( h_2 g^2 \), which is multiplicatively indecomposable, is the first example
of an element that is killed by \( \tau^3 \) but not by \( \tau^2 \). This occurs because of a hidden
extension \( \tau \cdot \tau h_2 g^2 = Ph_1^4 h_5 \). There is an analogous relation \( \tau^2 h_2 g = Ph_4 \) that
is not hidden. We do not know if this generalizes to a family of relations of the form
\( \tau^2 h_2 g^2 = Ph_1^{4k+2}h_{k+4} \).

We will show in Chapter 3 that \( h_2 g^2 \) represents an element in motivic stable
homotopy that is killed by \( \tau^3 \) but not by \( \tau^2 \). This requires an analysis of the motivic
Adams spectral sequence. In the vicinity of \( g^2 \), one might hope to find elements
that are killed by \( \tau^n \) but not by \( \tau^{n-1} \), for large values of \( n \).

Example 2.0.2. Classically, there is a relation \( h_3 \cdot e_0 = h_1 h_4 c_0 \) in the 24-stem
of the cohomology of the Steenrod algebra. This relation is hidden on the \( E_\infty \)-page
of the May spectral sequence. We now give a proof of this classical relation that
uses the cohomology of the motivic Steenrod algebra.

Motivically, it turns out that \( h_2^2 e_0 \) is non-zero, even though it is zero classically.
This follows from the hidden extension \( h_0 \cdot h_2^2 g = h_1^3 h_4 c_0 \) (see Lemma 2.4.10). The
relation \( h_2^2 = h_1^3 h_3 \) then implies that \( h_1^3 h_3 c_0 \) is non-zero. Therefore, \( h_3 e_0 \) is non-zero
as well, and the only possibility is that \( h_3 e_0 = h_1 h_4 c_0 \).

Example 2.0.3. Notice the hidden extension \( h_0 \cdot h_3^2 g^2 = h_1 h_5 c_0 \) (and similarly,
the hidden extension \( h_0 \cdot h_3^2 g = h_1 h_4 c_0 \) that we discussed above in Example 2.0.2).

The next example in this family is \( h_0 \cdot h_3^2 g^3 = h_1^3 D_4 \), which at first does not
appear to fit a pattern. However, there is a hidden extension \( c_0 \cdot i_1 = h_1^3 D_4 \), so
we have \( h_0 \cdot h_3^2 g^3 = h_1^3 c_0 i_1 \). Presumably, there is an infinitely family of hidden
extensions in which \( h_0 \cdot h_3^2 g^k \) equals some power of \( h_1 \) times \( c_0 \) times an element
related to \( Sq^0 \) of elements associated to the image of \( J \).

It is curious that \( c_0 \cdot i_1 \) is divisible by \( h_1^3 \). An obvious question for further study
is to determine the \( h_1 \)-divisibility of \( c_0 \) times elements related to \( Sq^0 \) of elements
associated to the image of \( J \). For example, what is the largest power of \( h_1 \) that
divides \( g^2 i_1 \)?

Example 2.0.4. Beware that \( g^2 \) and \( g^3 \) are not actually elements of the 40-
stem and 60-stem respectively. Rather, it is only \( \tau g^2 \) and \( \tau g^3 \) that exist (similarly,
g does not exist in the 20-stem, but \( \tau g \) does exist). The reason is that there are May differentials taking \( g^2 \) to \( h_5^1 h_5 \), and \( g^3 \) to \( h_9^0 i_1 \). In other words, \( \tau g^2 \) and \( \tau g^3 \) are multiplicatively indecomposable elements. More generally, we anticipate that the element \( g^k \) does not exist because it supports a May differential related to \( \text{Sq}^0 \) of an element in the image of \( J \).

Example 2.0.5. There is an isomorphism from the cohomology of the classical Steenrod algebra to the cohomology of the motivic Steenrod algebra over \( \mathbb{C} \) concentrated in degrees of the form \((2s + f, f, s + f)\). This isomorphism preserves all higher structure, including squaring operations and Massey products. See Section 2.1.3 for more details.

For example, the existence of the classical element \( Ph_2 \) immediately implies that \( h_3 g \) must be non-zero in the motivic setting; no calculations are necessary.

Another example is that \( h_2^{k-1} h_{k+2} \) is non-zero motivically for all \( k \geq 1 \), because \( h_0^{k-1} h_{k+1} \) is non-zero classically.

Example 2.0.6. Many elements are \( h_1 \)-local in the sense that they support infinitely many multiplications by \( h_1 \). In fact, any product of the symbols \( h_1, c_0, P, d_0, e_0, \) and \( g \), if it exists, is non-zero. This is detectable in the cohomology of motivic \( A(2) \) [18].

Moreover, the element \( B_1 \) in the 46-stem is \( h_1 \)-local, and any product of \( B_1 \) with elements in the previous paragraph is again \( h_1 \)-local. We explore \( h_1 \)-local elements in great detail in [14].

Example 2.0.7. The motivic analogue of the “wedge” subalgebra [27] appears to be more complicated than the classical version. For example, none of the wedge elements support multiplications by \( h_0 \) in the classical case. Motivically, many wedge elements do support \( h_0 \) multiplications. The results in this chapter naturally call for further study of the structure of the motivic wedge.

2.1. The motivic May spectral sequence

The following two deep theorems of Voevodsky are the starting points of our calculations.

Theorem 2.1.1 ([43]). \( \mathbb{M}_2 \) is the bigraded ring \( \mathbb{F}_2[\tau] \), where \( \tau \) has bidegree \((0, 1)\).

Theorem 2.1.2 ([42] [44]). The motivic Steenrod algebra \( A \) is the \( \mathbb{M}_2 \)-algebra generated by elements \( \text{Sq}^{2k} \) and \( \text{Sq}^{2k-1} \) for all \( k \geq 1 \), of bidegrees \((2k, k)\) and \((2k - 1, k - 1)\) respectively, and satisfying the following relations for \( a < 2b \):

\[
\text{Sq}^a \text{Sq}^b = \sum_c \binom{b - 1 - c}{a - 2c} \tau^c \text{Sq}^{a+b-c} \text{Sq}^c.
\]

The symbol \( ? \) stands for either 0 or 1, depending on which value makes the formula balanced in weight. See [13] for a more detailed discussion of the motivic Adem relations.

The \( A \)-module structure on \( \mathbb{M}_2 \) is trivial, i.e., every \( \text{Sq}^k \) acts by zero. This follows for simple degree reasons.
It is often helpful to work with the dual motivic Steenrod algebra \( A_{\ast,\ast} \), which equals
\[
\mathbb{M}_2[\tau_0, \tau_1, \ldots, \xi_1, \xi_2, \ldots].
\]
The reduced coproduct in \( A_{\ast,\ast} \) is determined by
\[
\tilde{\phi}_k(\tau_k) = \xi_k \otimes \tau_0 + \xi_{k-1}^2 \otimes \tau_1 + \cdots + \xi_{k-1} \otimes \tau_1 + \cdots + \xi_1^2 \otimes \tau_{k-1}
\]
\[
\tilde{\phi}_k(\xi_k) = \xi_k^2 \otimes \xi_1 + \xi_{k-2}^4 \otimes \xi_2 + \cdots + \xi_{k-1} \otimes \xi_1 + \cdots + \xi_1^{k-1} \otimes \xi_{k-1}.
\]

2.1.1. Ext groups. We are interested in computing \( \text{Ext}_{A_{\ast,\ast}}(\mathbb{M}_2, \mathbb{M}_2) \), which we abbreviate as \( \text{Ext} \). This is a trigraded object. We will consistently use degrees of the form \((s, f, w)\), where:

1. \( f \) is the Adams filtration, i.e., the homological degree.
2. \( s + f \) is the internal degree, i.e., corresponds to the first coordinate in the bidegrees of \( A \).
3. \( s \) is the stem, i.e., the internal degree minus the Adams filtration.
4. \( w \) is the weight.

Note that \( \text{Ext}^{s,0,\ast} = \text{Hom}_{A_{\ast,\ast}}^{s,0}(\mathbb{M}_2, \mathbb{M}_2) \) is dual to \( \mathbb{M}_2 \). We will abuse notation and write \( \mathbb{M}_2 \) for this dual. Beware that now \( \tau \), which is really the dual of the \( \tau \) that we discussed earlier, has degree \((0,0,-1)\). Since \( \text{Ext} \) is a module over \( \text{Ext}^{s,0,\ast} \), i.e., over \( \mathbb{M}_2 \), we will always describe \( \text{Ext} \) as an \( \mathbb{M}_2 \)-module.

The following result is the key tool for comparing classical and motivic computations. The point is that the motivic and classical computations become the same after inverting \( \tau \).

**Proposition 2.1.3** ([13]). There is an isomorphism of rings
\[
\text{Ext} \otimes_{\mathbb{M}_2} \mathbb{M}_2[\tau^{-1}] \cong \text{Ext}_{A_{\ast,\ast}}[\tau^{-1}] \otimes_{\mathbb{F}_2} \mathbb{F}_2[\tau^{-1}].
\]

2.1.2. The motivic May spectral sequence. The classical May spectral sequence arises by filtering the classical Steenrod algebra by powers of the augmentation ideal. The same approach can be applied in the motivic setting to obtain the motivic May spectral sequence. Details appear in [13]. Next we review the main points.

The motivic May spectral sequence is quadruply graded. We will always use gradings of the form \((m, s, f, w)\), where \( m \) is the May filtration, and the other coordinates are as explained in Section 2.1.1.

Let \( \text{Gr}(A) \) be the associated graded algebra of \( A \) with respect to powers of the augmentation ideal.

**Theorem 2.1.4.** The motivic May spectral sequence takes the form
\[
E_2 = \text{Ext}_{\text{Gr}(A)}^{(m,s,f,w)}(\mathbb{M}_2, \mathbb{M}_2) \Rightarrow \text{Ext}_{A_{\ast,\ast}}^{(s,f,w)}(\mathbb{M}_2, \mathbb{M}_2).
\]

**Remark 2.1.5.** As in the classical May spectral sequence, the odd differentials must be trivial for degree reasons.

**Proposition 2.1.6.** After inverting \( \tau \), there is an isomorphism of spectral sequences between the motivic May spectral sequence of Theorem 2.1.4 and the classical May spectral sequence, tensored over \( \mathbb{F}_2 \) with \( \mathbb{F}_2[\tau, \tau^{-1}] \).
2.1. THE MOTIVIC MAY SPECTRAL SEQUENCE

Proof. Start with the fact that $A[\tau^{-1}]$ is isomorphic to $A_{cl} \otimes_{\mathbb{F}_2} \mathbb{F}_2[\tau, \tau^{-1}]$, with the same May filtrations. □

This proposition means that differentials in the motivic May spectral sequence must be compatible with the classical differentials. This fact is critical to the success of our computations.

2.1.3. Ext in degrees with $s + f - 2w = 0$.

Definition 2.1.7. Let $A'$ be the subquotient $\mathbb{M}_2$-algebra of $A$ generated by $\text{Sq}^{2^k}$ for all $k \geq 0$, subject to the relation $\tau = 0$.

Lemma 2.1.8. There is an isomorphism $A_{cl} \to A'$ that takes $\text{Sq}^k$ to $\text{Sq}^{2k}$.

The isomorphism takes elements of degree $n$ to elements of bidegree $(2n, n)$.

Proof. Modulo $\tau$, the motivic Adem relation for $\text{Sq}^2$ takes the form

$$\text{Sq}^2 a \text{Sq}^2 b = \sum_c \binom{2b-1-2c}{2a-4c} \text{Sq}^{2a+2b-2c} \text{Sq}^2 c.$$  

A standard fact from combinatorics says that

$$\binom{2b-1-2c}{2a-4c} = \binom{b-1-c}{a-2c}$$  

modulo $2$. □

Remark 2.1.9. Dually, $A'$ corresponds to the quotient $\mathbb{F}_2[\xi_1, \xi_2, \ldots]$ of $A_{*, *}$, where we have set $\tau$ and $\tau_0, \tau_1, \ldots$ to be zero. The dual to $A'$ is visibly isomorphic to the dual of the classical Steenrod algebra.

Definition 2.1.10. Let $M$ be a bigraded $A$-module. The Chow degree of an element $m$ in degree $(t, w)$ is equal to $t - 2w$.

The terminology arises from the fact that the Chow degree is fundamental in Bloch’s higher Chow group perspective on motivic cohomology [6].

Definition 2.1.11. Let $M$ be an $A$-module. Define the $A'$-module $\text{Ch}_0(M)$ to be the subset of $M$ consisting of elements of Chow degree zero, with $A'$-module structure induced from the $A$-module structure on $M$.

The $A'$-module structure on $\text{Ch}_0(M)$ is well-defined since $\text{Sq}^{2k}$ preserves Chow degrees.

Theorem 2.1.12. There is an isomorphism from $\text{Ext}_{A_{cl}}$ to the subalgebra of $\text{Ext}$ consisting of elements in degrees $(s, f, w)$ with $s + f - 2w = 0$. This isomorphism takes classical elements of degree $(s, f)$ to motivic elements of degree $(2s+f, f, s+f)$, and it preserves all higher structure, including products, squaring operations, and Massey products.

Proof. There is a natural transformation

$$\text{Hom}_A(-, \mathbb{M}_2) \to \text{Hom}_{A'}(\text{Ch}_0(-), \mathbb{F}_2),$$

since $\text{Ch}_0(\mathbb{M}_2) = \mathbb{F}_2$. Since $\text{Ch}_0$ is an exact functor, the derived functor of the right side is $\text{Ext}_{A'}(\text{Ch}_0(-), \mathbb{F}_2)$. The universal property of derived functors gives a natural transformation $\text{Ext}_{A}(-, \mathbb{M}_2) \to \text{Ext}_{A'}(\text{Ch}_0(-), \mathbb{F}_2)$. Apply this natural transformation to $\mathbb{M}_2$ to obtain $\text{Ext}_{A}(\mathbb{M}_2, \mathbb{M}_2) \to \text{Ext}_{A'}(\text{Ch}_0(\mathbb{M}_2), \mathbb{F}_2)$. The left
side is Ext, and the right side is isomorphic to $\text{Ext}_{A_1}$. Since $A'$ is isomorphic to $A_{e1}$ by Lemma 2.1.8.

We have now obtained a map $\text{Ext} \to \text{Ext}_{A_1}$. We will verify that this map is an isomorphism on the part of Ext in degrees $(s, f, w)$ with $s + f - 2w = 0$. Compare the classical May spectral sequence with the part of the motivic May spectral sequence in degrees $(m, s, f, w)$ with $s + f - 2w = 0$. By direct inspection, the motivic $E_1$-page in these degrees is the polynomial algebra over $\mathbb{F}_2$ generated by $h_{ij}$ for $i > 0$ and $j > 0$. This is isomorphic to the classical $E_1$-page, where the motivic element $h_{ij}$ corresponds to the classical element $h_{t_{ij}}$.

**Remark 2.1.13.** Similar methods show that Ext is concentrated in degree $(s, f, w)$ with $s + f - 2w > 0$. The map $\text{Ext} \to \text{Ext}_{A_1}$ constructed in the proof annihilates elements in degrees $(s, f, w)$ with $s + f - 2w > 0$. Thus, $\text{Ext}_{A_1}$ is isomorphic to the quotient of Ext by elements of degree $(s, f, w)$ with $s + f - 2w > 0$.

2.2. Massey products in the motivic May spectral sequence

We will frequently compute Massey products in Ext in order to resolve hidden extensions and to determine May differentials. The absolutely essential tool for computing such Massey products is May’s Convergence Theorem [29, Theorem 4.1]. The point of this theorem is that under certain hypotheses, Massey products in Ext can be computed in the $E_r$-page of the motivic May spectral sequence. For the reader’s convenience, we will state the theorem in the specific forms that we will use. We have slightly generalized the result of [29, Theorem 4.1] to allow for brackets that are not strictly defined. In order to avoid unnecessarily heavy notation, we have intentionally avoided the most general possible statements. The interested reader is encouraged to carry out these generalizations.

**Theorem 2.2.1 (May’s Convergence Theorem).** Let $a_0, a_1, a_2$ be elements of Ext such that the Massey product $\langle a_0, a_1, a_2 \rangle$ is defined. For each $i$, let $a_i$ be a permanent cycle on the $E_{i+1}$-page that detects $a_i$. Suppose further that:

1. there exist elements $a_{01}$ and $a_{12}$ on the May $E_{r+1}$-page such that $d_r(a_{01}) = a_0a_1$ and $d_r(a_{12}) = a_1a_2$.
2. if $(m, s, f, w)$ is the degree of either $a_{01}$ or $a_{12}$; $m' \geq m$; and $m'-t \leq m-r$;

then every May differential $d_i : E_i^{(m', s, f, w)} \to E_i^{(m'-t+1, s-1, f+1, w)}$ is zero.

Then $a_0a_{12} + a_{01}a_2$ detects an element of $\langle a_0, a_1, a_2 \rangle$ in Ext.

The point of condition (1) is that the bracket $\langle a_0, a_1, a_2 \rangle$ is defined in the differential graded algebra $(E_r, d_r)$. Condition (2) is an equivalent reformulation of condition (*) in [29, Theorem 4.1]. When computing $\langle a_0, a_1, a_2 \rangle$, one uses a differential $d_r : E^r_{\langle m, s, f, w \rangle} \to E^r_{\langle m-r+1, s-1, f+1, w \rangle}$. The idea of condition (2) is that there are no later “crossing” differentials $d_i$ whose source has higher May filtration and whose target has strictly lower May filtration.

The proof of May’s Convergence Theorem [2.2.1] is exactly the same as in [29] because every threefold Massey product is strictly defined in the sense that its subbrackets have no indeterminacy.

**Theorem 2.2.2 (May’s Convergence Theorem).** Let $a_0, a_1, a_2,$ and $a_3$ be elements of Ext such that the Massey product $\langle a_0, a_1, a_2, a_3 \rangle$ is defined. For each $i$, let $a_i$ be a permanent cycle on the $E_{i+1}$-page that detects $a_i$. Suppose further that:
Then a theorem has a symmetric version in which the bracket and whose target has strictly lower May filtration.

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\[ \text{[127x507]} \]

(1) there are elements \( a_{01}, a_{12}, \text{ and } a_{23} \) on the \( E_r \)-page such that \( d_r(a_{01}) = a_{01}a_1, d_r(a_{12}) = a_{12}a_2, \text{ and } d_r(a_{23}) = a_{23}a_3. \)

(2) there are elements \( a_{02} \text{ and } a_{13} \) on the \( E_r \)-page such that \( d_r(a_{02}) = a_{01}a_{12} + a_{02}a_3, \text{ and } d_r(a_{13}) = a_{12}a_{23} + a_{13}a_3. \)

(3) if \((m, s, f, w)\) is the degree of \( a_{01}, a_{12}, a_{23}, a_{02}, \text{ or } a_{13}; \text{ and } m' \geq m; \text{ and } m' - t < m - r; \text{ then every differential} \]

\[ d_t : E_t^{(m', s, f, w)} \rightarrow E_t^{(m-t+1, s-1, f+1, w)} \]

is zero.

(4) The subbracket \( \langle \alpha_0, \alpha_1, \alpha_2 \rangle \) has no indeterminacy.

(5) the indeterminacy of \( \langle \alpha_1, \alpha_2, \alpha_3 \rangle \) is generated by elements of the form \( \alpha_1\beta \) and \( \gamma\alpha_3, \text{ where } \beta \text{ and } \gamma \text{ are detected in May filtrations strictly lower than the May filtrations of } a_{23} \text{ and } a_{12} \text{ respectively.} \)

Then \( a_{01}a_{12}a_{23} + a_{02}a_{13} \) detects an element of \( \langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \rangle \) in \( \text{Ext} \).

The point of conditions (1) and (2) is that the bracket \( \langle a_0, a_1, a_2, a_3 \rangle \) is defined in the differential graded algebra \( \langle E_r, d_r \rangle \). Condition (3) is an equivalent reformulation of condition (*) in [29] Theorem 4.1. The point of this condition is that there are no later “crossing” differentials whose source has higher May filtration and whose target has strictly lower May filtration.

Condition (5) does not appear in [29], which only deals with strictly defined brackets. Of course, the theorem has a symmetric version in which the bracket \( \langle \alpha_1, \alpha_2, \alpha_3 \rangle \) has no indeterminacy. It is probably possible to state a version of the theorem in which both threefold subbrackets have non-zero indeterminacy. However, additional conditions are required for such a fourfold bracket to be well-defined [20].

**Proof.** Let \( C \) be the cobar resolution of the motivic Steenrod algebra whose homology is \( \text{Ext} \). Let \( \alpha_0, \alpha_1, \alpha_2, \text{ and } \alpha_3 \) be explicit cycles in \( C \) representing \( a_0, a_1, a_2, \text{ and } a_3 \). As in the proof of [29] Theorem 4.1, we may choose an element \( \tilde{\alpha}_{01} \) of \( C \) such that \( d(\tilde{\alpha}_{01}) = a_{01}\tilde{\alpha}_1 \) and \( \tilde{\alpha}_{01} \) is detected by \( a_{01} \) in the May \( E_r \)-page. We may similarly choose \( \tilde{\alpha}_{12} \) and \( \tilde{\alpha}_{23} \) whose boundaries are \( \tilde{\alpha}_1\tilde{\alpha}_2 \) and \( \tilde{\alpha}_2\tilde{\alpha}_3 \) and that are detected by \( a_{12} \) and \( a_{23} \).

Next, we want to choose \( \tilde{\alpha}_{13} \) in \( C \) whose boundary is \( \tilde{\alpha}_1\tilde{\alpha}_{23} + \tilde{\alpha}_{12}\tilde{\alpha}_3 \) and that is detected by \( a_{13} \). Because of the possible indeterminacy in \( \langle \alpha_1, \alpha_2, \alpha_3 \rangle \), the cycle \( \tilde{\alpha}_1\tilde{\alpha}_{23} + \tilde{\alpha}_{12}\tilde{\alpha}_3 \) may not be a boundary in \( C \). However, since we are assuming that \( \langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \rangle \) is defined, we can add cycles to \( \tilde{\alpha}_{12} \) and \( \tilde{\alpha}_{23} \) to ensure that \( \tilde{\alpha}_1\tilde{\alpha}_{23} + \tilde{\alpha}_{12}\tilde{\alpha}_3 \) is a boundary. When we do this, condition (5) guarantees that \( \tilde{\alpha}_{12} \) and \( \tilde{\alpha}_{23} \) are still detected by \( a_{12} \) and \( a_{23} \). Then we may choose \( \tilde{\alpha}_{13} \) as in the proof of [29] Theorem 4.1.

Finally, we may choose \( \tilde{\alpha}_{02} \) as in the proof of [29] Theorem 4.1. Because \( \langle \alpha_0, \alpha_1, \alpha_2 \rangle \) has no indeterminacy, we automatically know that \( \tilde{\alpha}_0\tilde{\alpha}_{12} + \tilde{\alpha}_{01}\tilde{\alpha}_2 \) is a boundary in \( C \).

For completeness, we will now also state May’s Convergence Theorem for fivefold brackets. This result is used only in Lemma [2.4.23]. The proof is essentially the same as the proof for fourfold brackets.

**Theorem 2.2.3** (May’s Convergence Theorem). Let \( \alpha_0, \alpha_1, \alpha_2, \alpha_3, \text{ and } \alpha_4 \) be elements of \( \text{Ext} \) such that the Massey product \( \langle \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle \) is defined. For
each $i$, let $a_i$ be a permanent cycle on the $E_r$-page that detects $\alpha_i$. Suppose further that:

1. there are elements $a_{01}$, $a_{12}$, $a_{23}$, and $a_{34}$ in $E_r$ such that $d_r(a_{01}) = a_0a_1$, $d_r(a_{12}) = a_1a_2$, $d_r(a_{23}) = a_2a_3$, and $d_r(a_{34}) = a_3a_4$.

2. there are elements $a_{02}$, $a_{13}$, and $a_{24}$ in $E_r$ such that $d_r(a_{02}) = a_0a_2 + a_{01}a_2$, $d_r(a_{13}) = a_1a_3 + a_{12}a_3$, and $d_r(a_{24}) = a_2a_{34} + a_{23}a_4$.

3. there are elements $a_{03}$ and $a_{14}$ in $E_r$ such that $d_r(a_{03}) = a_0a_{14} + a_{01}a_{23} + a_{02}a_3$ and $d_r(a_{14}) = a_1a_{24} + a_{12}a_{34} + a_{13}a_4$.

4. if $(m, s, f, w)$ is the degree of $a_{01}$, $a_{12}$, $a_{23}$, $a_{34}$, $a_{02}$, $a_{13}$, $a_{24}$, $a_{03}$, $a_{14}$; $m' \geq m$; and $m' - t < m - r$; then every differential $d_t : E_t^{(m', s, f, w)} \rightarrow E_t^{(m - t + 1, s - 1, f + 1, w)}$ is zero.

5. the threefold subbrackets $\langle a_0, \alpha_1, \alpha_2 \rangle$, $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$, and $\langle \alpha_2, \alpha_3, \alpha_4 \rangle$ have no indeterminacy.

6. the subbracket $\langle a_0, \alpha_1, \alpha_2, \alpha_3 \rangle$ has no indeterminacy.

7. the indeterminacy of $\langle a_0, \alpha_2, \alpha_3, \alpha_4 \rangle$ is generated by elements contained in $\langle \beta, \alpha_3, \alpha_4 \rangle$, $\langle \alpha_1, \alpha_2, \delta \rangle$, where $\beta, \gamma,$ and $\delta$ are detected in May filtrations strictly lower than the May filtrations of $a_{12}, a_{23},$ and $a_{34}$ respectively.

Then $a_0a_{14} + a_{01}a_{24} + a_{02}a_{34} + a_{03}a_{44}$ detects an element of $\langle a_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$ in Ext.

Although we will use May’s Convergence Theorem to compute most of the Massey For a few Massey products, we also need occasionally the following result [16] 1 Lemma 2.5.4.

**Proposition 2.2.4.** Let $x$ be an element of $\text{Ext}_A(M_2, M_2)$.

1. If $h_0x = 0$, then $\tau h_1x$ belongs to $\langle h_0, x, h_0 \rangle$.
2. If $n \geq 1$ and $h_nx = 0$, then $h_{n+1}x$ belongs to $\langle h_n, x, h_n \rangle$.

We will need the following results about shuffling higher brackets that are not strictly defined.

**Lemma 2.2.5.** Suppose that $\langle a_0, \alpha_1, \alpha_2, \alpha_3 \rangle$ and $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$ are defined and that the indeterminacy of $\langle a_0, \alpha_1, \alpha_2 \rangle$ consists of multiples of $\alpha_2$. Then

$$a_0\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle \subseteq \langle a_0, \alpha_1, \alpha_2, \alpha_3 \rangle \alpha_4.$$

**Proof.** For each $i$, choose an element $a_i$ that represents $\alpha_i$. Let $\beta$ be an element of $a_0\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$. There exist elements $a_{12}, a_{23}, a_{34}, a_{13}$, and $a_{24}$ such that $d(\langle a_{i, i+1} \rangle) = a_0a_{i+1}, d(\langle a_{i, i+2} \rangle) = a_{i,i+1}a_{i+2} + a_{i+1,i}a_{i+2},$ and $\beta$ is represented by

$$b = a_0a_{12}a_{24} + a_{01}a_{23}a_{34} + a_0a_{13}a_4.$$

By the assumption on the indeterminacy of $\langle a_0, \alpha_1, \alpha_2 \rangle$, we can then choose $a_{01}$ and $a_{02}$ such that $d(a_{01}) = a_0a_{12}$ and $d(a_{02}) = a_0a_{12} + a_{01}a_{23}a_{34} + a_0a_{13}a_4$ represents a class in $\langle a_0, \alpha_1, \alpha_2, \alpha_3 \rangle \alpha_4$ that is homologous to $b$. \qed

**Lemma 2.2.6.** Suppose that $\langle a_0, \alpha_1, \alpha_2, \alpha_3 \rangle$ and $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$ are defined, and suppose that $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is strictly zero. Then

$$a_0\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle \cap \langle a_0, \alpha_1, \alpha_2, \alpha_3 \rangle \alpha_4$$

is non-empty.
2.3. The May differentials

2.3.1. The May $E_1$-page. The $E_2$-page of the May spectral sequence is the cohomology of a differential graded algebra. In other words, the May spectral sequence really starts with an $E_1$-page. As described in [13], the motivic $E_1$-page is essentially the same as the classical $E_1$-page. Specifically, the motivic $E_1$-page is a polynomial algebra over $\mathbb{M}_2$ with generators $h_{ij}$ for all $i > 0$ and $j \geq 0$, where:

1. $h_{ij}$ has degree $(i, 2^j - 2, 1, 2^{i-1} - 1)$.
2. $h_{ij}$ has degree $(i, 2^j(2^i - 1) - 1, 1, 2^{i-1}(2^i - 1))$ for $j > 0$.

The $d_1$-differential is described by the formula:

$$d_1(h_{ij}) = \sum_{0 < k < i} h_{kj} h_{i-k, k+j}.$$

2.3.2. The May $E_2$-page. We now describe the $E_2$-page of the motivic May spectral sequence. As explained in [13], it turns out that the motivic $E_2$-page is essentially the same as the classical $E_2$-page. The following proposition makes this precise.

Recall that $\text{Gr}(A)$ is the associated graded object of the motivic Steenrod algebra with respect to powers of the augmentation ideal. Similarly, let $\text{Gr}(A_{cl})$ be the associated graded object of the classical Steenrod algebra with respect to powers of the augmentation ideal.

**Proposition 2.3.1 ([13]).** There are graded ring isomorphisms

(a) $\text{Gr}(A) \cong \text{Gr}(A_{cl}) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\tau]$.
(b) $\text{Ext}_{\text{Gr}(A)}(\mathbb{M}_2, \mathbb{M}_2) \cong \text{Ext}_{\text{Gr}(A_{cl})}(\mathbb{F}_2, \mathbb{F}_2) \otimes_{\mathbb{F}_2} \mathbb{M}_2$.

In other words, explicit generators and relations for the $E_2$-page can be lifted directly from the classical situation [39].

Moreover, because of Proposition 2.1.6, the values of the May $d_2$ differential can also be lifted from the classical situation, except that a few factors of $\tau$ show up to give the necessary weights. For example, classically we have the differential

$$d_2(b_{20}) = h_1^3 + h_0^2 h_2.$$

**Proof.** Choose elements $a_i$ that represent $\alpha_i$. Choose $a_{01}$, $a_{12}$, and $a_{02}$ such that $d(a_{01}) = a_0 a_{11}$, $d(a_{12}) = a_1 a_{22}$, and $d(a_{02}) = a_0 a_{12} + a_{01} a_2$. Also, choose $a_{23}$, $a_{34}$, and $a_{24}$ such that $d(a_{23}) = a_2 a_{33}$, $d(a_{34}) = a_3 a_{44}$, and $d(a_{24}) = a_2 a_{34} + a_{23} a_4$.

Since $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is strictly zero, there exists $a_{13}$ such that $d(a_{13}) = a_1 a_{23} + a_{12} a_3$.

Then

$$a_0 a_{12} a_{24} + a_0 a_{12} a_{34} + a_0 a_{13} a_4$$

represents an element of $\langle \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$, and it is homologous to

$$a_0 a_{13} a_4 + a_1 a_{23} a_4 + a_0 a_{24} a_4,$$

which represents an element of $\langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \rangle \alpha_4$. \hfill \Box

**Lemma 2.2.7.** Suppose that $\langle \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$ and $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \rangle$ are defined and that $\langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \rangle$ is strictly zero. Then

$$a_0 (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \subseteq \langle \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle \alpha_5.$$

**Proof.** The proof is essentially the same as the proof of Lemma 2.2.5. \hfill \Box
Motivically, this does not make sense, since \(b_{20}\) and \(h_2^3 h_2\) have weight 2, while \(h_1^3\) has weight 3. Therefore, the motivic differential must be
\[
d_2(b_{20}) = \tau h_1^3 + h_2^3 h_2.
\]

Table 2 lists the multiplicative generators of the \(E_2\)-page through the 70-stem, and Table 3 lists a generating set of relations for the \(E_2\)-page in the same range. Table 2 also gives the values of the May \(d_2\) differential, all of which are easily deduced from the classical situation [39].

2.3.3. The May \(E_4\)-page. Although the \(E_2\)-page is quite large, the May \(d_2\) differential is also very destructive. As a result, the \(E_4\)-page becomes manageable. We obtain the \(E_4\)-page by direct computation with the \(d_2\) differential.

Remark 2.3.2. As in [39], we use the notation \(B = b_{30} b_{31} + b_{21} b_{40}\).

Having described the \(E_4\)-page, it is now necessary to find the values of the May \(d_4\) differential on the multiplicative generators. Most of the values of \(d_4\) follow from comparison to the classical case [39], together with a few factors of \(\tau\) to balance the weights. There is only one differential that is not classical.

Lemma 2.3.3. \(d_4(g) = h_4^1 h_4\).

Proof. By the isomorphism of Theorem 2.1.12, we know that \(h_4^1 h_4\) cannot survive the motivic May spectral sequence because \(h_3^0 h_3\) is zero classically. There is only one possible differential that can kill \(h_4^1 h_4\).

See also [13] for a different proof of Lemma 2.3.3.

Table 4 lists the values of the \(d_4\) differential on multiplicative generators of the \(E_4\)-page.

2.3.4. The May \(E_6\)-page. We can now obtain the \(E_6\)-page by direct computation with the May \(d_4\) differential and the Leibniz rule.

Having described the \(E_6\)-page, it is now necessary to find the values of the May \(d_6\) differential on the multiplicative generators. Most of these values follow from comparison to the classical case [39], together with a few factors of \(\tau\) to balance the weights. There are only a few differentials that are not classical.

Lemma 2.3.4.

1. \(d_6(x_{56}) = h_5^2 h_5 c_0 d_0\).
2. \(d_6(P x_{56}) = P h_5^2 h_5 c_0 d_0\).
3. \(d_6(B_{23}) = h_5^2 h_5 d_0 e_0\).

Proof. We have the relation \(h_1 x_{56} = c_0 \phi\). The \(d_6\) differential on \(\phi\) then implies that \(d_6(h_1 x_{56}) = h_5^2 h_5 c_0 d_0\), from which it follows that \(d_6(x_{56}) = h_5^2 h_5 c_0 d_0\).

The arguments for the other two differentials are similar, using the relations \(h_1 \cdot P x_{56} = P c_0 \cdot \phi\) and \(h_1 B_{23} = c_0 \phi\).

Lemma 2.3.5. \(d_6(c_0 g^2) = h_1^0 D_4\).

Proof. Lemma 2.4.24 shows that \(c_0 \cdot i_1 = h_1^4 D_4\). Since \(h_1^0 i_1 = 0\), we conclude that \(h_1^0 D_4\) must be zero in Ext. There is only one possible differential that can hit \(h_1^0 D_4\).
2.3. THE MAY DIFFERENTIALS

Remark 2.3.6. The value of $d_6(\Delta h_2^0 Y)$ given in [39, Proposition 4.37(c)] is incorrect because it is inconsistent with machine computations of $\text{Ext}_{E_4}$. The value for $d_6(\Delta h_2^0 Y)$ given in Table [5] is the only possibility that is consistent with the machine computations.

Table [5] lists the values of the May $d_6$ differential on multiplicative generators of the $E_6$-page.

2.3.5. The May $E_8$-page. We can now obtain the May $E_8$-page by direct computation with the May $d_6$ differential and the Leibniz rule. Once we reach the $E_8$-page, we are nearly done. There are just a few more higher differentials to deal with.

Having described the $E_8$-page, it is now necessary to find the values of the May $d_8$ differential on the multiplicative generators. Once again, most of these values follow from comparison to the classical case [39], together with a few factors of $\tau$ to balance the weights. There are only a few differentials that are not classical.

Lemma 2.3.7.

(1) $d_8(g^2) = h_1^8 h_5$.
(2) $d_8(w) = Ph_1^5 h_5$.
(3) $d_8(\Delta c_0 g) = Ph_1^5 h_5 c_0$.
(4) $d_8(Q_3) = h_1^4 h_5^2$.

Proof. It follows from Theorem 2.1.12 that $h_1^8 h_5$ must be zero in Ext, since $h_8^0 h_4$ is zero classically. There is only one differential that can possibly hit $h_1^8 h_5$.

We now know that $Ph_1^5 h_5 = 0$ in Ext since $h_1^8 h_5 = 0$. There is only one differential that can hit this. This shows that $d_8(w) = Ph_1^5 h_5$.

Using the relation $c_0 w = h_1 \cdot \Delta c_0 g$, it follows that $d_8(h_1 \cdot \Delta c_0 g) = Ph_1^5 h_5 c_0$, and then that $d_8(\Delta c_0 g) = Ph_1^5 h_5 c_0$.

Since $h_1^4 h_5^2$ is zero classically, it follows from Theorem 2.1.12 that $h_1^4 h_5^2$ must be zero in Ext. There is only one differential that can possibly hit $h_1^4 h_5^2$. □

Table [6] lists the values of the May $d_8$ differential on multiplicative generators of the $E_8$-page.

2.3.6. The May $E_\infty$-page. Most of the higher May differentials are zero through the 70-stem. The exceptions are the May $d_{12}$ differential, the May $d_{16}$ differential, and the May $d_{32}$ differential. All of the non-zero values of these differentials are easily deduced by comparison to the classical case [39].

Table [7] lists the values of these higher differentials on multiplicative generators of the higher pages. There are no more differentials to consider in our range, and we have determined the May $E_\infty$-page.

The multiplicative generators for the $E_\infty$-page through the 70-stem break into two groups. The first group consists of generators that are still multiplicative generators in Ext after hidden extensions have been considered; these are listed in Table [8]. The second group consists of multiplicative generators of the $E_\infty$-page that become decomposable in Ext because of a hidden extension; these are listed in Table [9].

It is traditional to use the same symbols for elements of the $E_\infty$-page and for the elements of Ext that they represent. Generally, there is no ambiguity with this abuse of notation, but there are several exceptions. These exceptions occur when a
multiplicative generator for the $E_\infty$-page lies in the same degree as another element of the $E_\infty$-page with lower May filtration.

The first such example occurs in the 18-stem, where the element $f_0$ of the $E_\infty$-page represents two elements of Ext because of the presence of the element $\tau h_0^2h_4$ of lower May filtration. This particular example does not cause much difficulty. Just arbitrarily choose one of these elements to be the generator of Ext. The element disappears quickly from further analysis because $f_0$ supports an Adams $d_2$ differential.

However, later examples involve more subtlety and call for a careful distinction between the possibilities. There are no wrong choices, but it is important to be consistent with the notation in different arguments. For example, the element $u'$ of the $E_\infty$-page represents two elements of Ext because of the presence of $\tau h_0^2$. One of these elements is killed by $\tau$, while the other element is killed by $h_0$. Sloppy notation might lead to the false conclusion that there is a multiplicative generator of Ext in that degree that is killed by both $\tau$ and by $h_0$.

Table 10 lists all such examples of multiplicative generators of the $E_\infty$-page that represent more than one element in Ext. In many of these examples, we have given an algebraic specification of one element of Ext to serve as the multiplicative generator, sometimes by comparing to $\text{Ext}^{A(2)}$. In some examples, we have not given a definition because an algebraic description is not readily available, and also because it does not seem to matter for later analysis. The reader is strongly warned to be cautious when working with these undefined elements.

The example $\tau Q_3$ deserves an additional remark. Here we have defined the element in terms of an Adams differential. This is merely a matter of convenience for later work with the Adams spectral sequence in Chapter 3.

2.4. Hidden May extensions

In order to pass from the $E_\infty$-page to Ext, we must resolve some hidden extensions. In this section, we deal with all possible hidden extensions by $\tau$, $h_0$, $h_1$, and $h_2$. We will use several different tools, including:

1. Classical hidden extensions [9].
2. Shuffle relations with Massey products.
3. Squaring operations in the sense of [30].
4. Theorem 2.1.12 for hidden extensions among elements in degrees $(s, f, w)$ with $s + f - 2w = 0$.

2.4.1. Hidden May $\tau$ extensions. By exhaustive search, the following results give all of the hidden $\tau$ extensions.

Proposition 2.4.1. Table 11 lists all of the hidden $\tau$ extensions through the 70-stem.

Proof. Many of the extensions follow by comparison to the classical case as described in [9]. For example, there is a classical hidden extension $h_0 \cdot e_0g = h_0^4x$. This implies that $\tau^2 \cdot h_0e_0g = h_0^4x$ motivically.

Proofs for the more subtle cases are given below. \qed

Lemma 2.4.2.
1. $\tau \cdot \tau h_0^2g^2 = Ph_1^4h_5$.
2. $\tau \cdot \tau h_0^2g^3 = Ph_1^4h_5e_0$. 

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PROOF. Start with the relation \( h_1 \cdot \tau g + h_2 f_0 = 0 \), and apply the squaring operation \( \text{Sq}^1 \). One needs that \( \text{Sq}^1(\tau g) = Ph_2^2 h_5 \). The result is the first hidden extension.

For the second, multiply the first hidden extension by \( e_0 \).

\[ \square \]

**Lemma 2.4.3.**

\begin{enumerate}
\item \( \tau \cdot B_8 = Ph_5 d_0 \).
\item \( \tau \cdot h_4^2 B_{21} = Ph_5 e_0 d_0 \).
\item \( \tau \cdot B_8 d_0 = h_3^4 X_3 \).
\end{enumerate}

**Proof.** There is a classical hidden extension \( c_0 \cdot B_1 = Ph_1 h_5 d_0 \). Motivically, there is a non-hidden relation \( c_0 \cdot B_1 = h_1 B_8 \). It follows that \( \tau \cdot h_1 B_8 = Ph_1 h_5 d_0 \) motivically.

For the second hidden extension, multiply the first hidden extension by \( d_0 \). Note that \( c_0 B_8 = h_4^2 B_{21} \) is detected in the \( E_\infty \)-page of the May spectral sequence.

For the third hidden extension, multiply the first hidden extension by \( d_0 \), and observe that \( Ph_5 d_0^2 = h_3^4 X_3 \), which is detected in the \( E_\infty \)-page of the May spectral sequence.

**Lemma 2.4.4.**

\begin{enumerate}
\item \( \tau \cdot Pu' = h_0^5 R_1 \).
\item \( \tau \cdot P^2 u' = h_0^5 R_1 \).
\item \( \tau \cdot P^3 u' = h_0^5 R_1 \).
\end{enumerate}

**Proof.** We first compute that \( \langle \tau, u', h_0^3 \rangle = \{ Q', Q' + \tau Pu \} \). One might try to apply May’s Convergence Theorem with the May differential \( d_4(b_{20} b_3 h_0(1)) = \tau u' \), but condition (2) of the theorem is not satisfied because of the May differential \( d_4(P \Delta h_0 h_4) = P^2 h_0 h_4^2 \).

Instead, note that \( h_0 \cdot u' = \tau h_0 d_0 l \) by comparison to \( \text{Ext}_{A(2)} \), so we have that \( \langle \tau, u', h_0^3 \rangle = \langle \tau, \tau h_0 d_0 l, h_0^3 \rangle \). The latter bracket is given in Table 11.

Next, Table 11 shows that \( Pu' = \langle u', h_0^3, h_0 h_3 \rangle \), with no indeterminacy. Use the previous paragraph and a shuffle to get that \( \tau \cdot Pu' = \tau h_0 h_3 Q' \). Finally, there is a classical hidden extension \( h_3 \cdot Q' = h_1^3 R_1 \), which implies that the same formula holds motivically.

The argument for the second hidden extension is similar, using the shuffle

\[ \tau \cdot P^2 u' = \tau \langle u', h_0^3, h_0^3 h_4 \rangle = \langle \tau, u', h_0^3 \rangle h_0^3 h_4 = h_0^3 h_4 Q'. \]

The first equality comes from Table 11. Also, we need the classical hidden extension \( h_4 \cdot Q' = h_0^5 R_1 \), which implies that the same formula holds motivically.

The argument for the third hidden extension is also similar, using the shuffle

\[ \tau \cdot P^3 u' = \tau \langle u', h_0^3, h_0^3 i \rangle = \langle \tau, u', h_0^3 \rangle h_0^3 i = h_0^3 i Q'. \]

The first equality comes from Table 11. Also, we need the classical hidden extension \( i \cdot Q' = h_0^3 R_1 \).

**Lemma 2.4.5.** \( \tau \cdot k_1 = h_2 h_5 n \).

**Proof.** First, Table 11 shows that \( k = \langle d_0, h_3, h_2^2 h_3 \rangle \), with no indeterminacy. It follows from Table 11 that \( \text{Sq}^0 k = \langle \text{Sq}^0 d_0, \text{Sq}^0 h_3, \text{Sq}^0 h_2^2 h_3 \rangle \), with no indeterminacy. In other words, \( \text{Sq}^0 k = \langle \tau^2 d_1, h_4, \tau^2 h_2^2 h_4 \rangle \). From the classical calculation \[11\], \( \text{Sq}^0 k \) also equals \( \tau^3 h_2 h_5 n \).
On the other hand, Table 10 show that \( k_1 = (d_1, h_4, h_2^2 h_4) \), with no indeterminacy. This shows that \( \tau^4 \cdot k_1 = \tau^3 h_2 h_5 n \) in Ext, from which it follows that \( \tau \cdot k_1 = h_2 h_5 n \).

Remark 2.4.6. In the 46-stem, \( \tau \cdot u' \) does not equal \( \tau^2 d_{0l} \). Similarly, in the 49-stem, \( \tau \cdot v' \) does not equal \( \tau^2 e_{0l} \). This is true by definition; see Table 10.

2.4.2. Hidden May \( h_0 \) Extensions. By exhaustive search, the following results give all of the hidden \( h_0 \) extensions.

**Proposition 2.4.7.** Table 12 lists all of the hidden \( h_0 \) extensions through the 70-stem.

**Proof.** Many of the extensions follow by comparison to the classical case as described in [9]. For example, there is a classical hidden extension \( h_0 \cdot r = s \). This implies that \( h_0 \cdot r = s \) motivically as well.

Several other extensions are implied by the hidden \( \tau \) extensions established in Section 2.4.1. For example, the extensions \( \tau \cdot Pu' = h_0^2 S_1 \) and \( \tau \cdot \tau h_0 d_{0j}^2 = h_0^5 S_1 \) imply that \( h_0 \cdot Pu' = \tau h_0 d_{0j}^2 \).

Proofs for the more subtle cases are given below.

**Lemma 2.4.8.**
1. \( h_0 \cdot u' = \tau h_0 d_{0l} \).
2. \( h_0 \cdot v' = \tau h_0 e_{0l} \).
3. \( h_0 \cdot P v' = \tau h_0 d_{0j}^2 k \).
4. \( h_0 \cdot P^2 v' = \tau h_0 d_{0j}^3 l \).

**Proof.** These follow by comparison to Ext\(_A(2)\) [18].

**Lemma 2.4.9.**
1. \( h_0 \cdot h_2^2 g = h_3^3 h_4 c_0 \).
2. \( h_0 \cdot h_2^2 g^2 = h_1^3 h_5 c_0 \).
3. \( h_0 \cdot h_2^2 g^3 = h_1^3 D_4 \).

**Proof.** For the first hidden extension, use the shuffle
\[
\langle h_1^3 h_4 (h_1, h_0, h_2^2) \rangle = \langle h_1^3 h_4, h_1, h_0 \rangle h_2^2.
\]
Similarly, for the second hidden section, use the shuffle
\[
\langle h_1^7 h_5 (h_1, h_0, h_2^2) \rangle = \langle h_1^7 h_5, h_1, h_0 \rangle h_2^2.
\]

For the third hidden extension, there is a hidden extension \( c_0 \cdot i_1 = h_1^3 D_4 \) that will be established in Lemma 2.4.24. Use this relation to compute that
\[
h_1^3 D_4 = h_5^2 i_1 (h_1, h_2^2, h_0) = \langle h_5^2 i_1, h_1, h_2^2 \rangle h_0.
\]
Finally, Table 10 shows that \( h_2^2 g^3 = \langle h_5^2 i_1, h_1, h_2^2 \rangle \).

**Lemma 2.4.10.**
1. \( h_0 \cdot gr = Ph_2^3 h_5 c_0 \).
2. \( h_0 \cdot lm = h_1^5 X_1 \).
3. \( h_0 \cdot m^2 = h_1^5 c_0 Q_2 \).

**Remark 2.4.11.** The three parts may seem unrelated, but note that \( lm = e_0 gr \) and \( m^2 = g^2 r \) on the \( E_8 \)-page of the May spectral sequence.
2.4. Hidden May Extensions

Proof. Table 16 shows that $e_0r = \langle \tau^2g^2, h_2^2, h_0 \rangle$. Next observe that

$$h_2 \cdot e_0r = \langle \tau^2g^2, h_2^0, h_0 \rangle h_2 = \langle \tau^2g^2, h_2^0, h_0h_2 \rangle = \langle \tau^2h_2g^2, h_2, h_0h_2 \rangle.$$ 

None of these brackets have indeterminacy.

Use the relation $Ph_1^3h_5 = \tau^2h_2g^2$ from Lemma 2.4.2 to write

$$h_2 \cdot e_0r = \langle Ph_1^3h_5, h_2, h_0h_2 \rangle = Ph_1^3h_5\langle h_1, h_2, h_0h_2 \rangle = Ph_1^3h_5c_0.$$ 

The last step is to show that $h_2 \cdot e_0r = h_0 \cdot gr$. This follows from the calculation

$$h_0 \cdot gr = h_0\langle h_1h_4, h_1r \rangle = \langle h_0, h_1h_4, h_1r \rangle r = h_2c_0 \cdot r,$$

where the brackets are given in Table 16. This finishes the proof of part (1).

For part (2), we will prove below in Lemma 2.4.14 that the only possible hidden extension is

$$\langle \tauh_1, h_2^2, h_0 \rangle.$$ 

This implies that there is a motivic hidden extension

$$h_1 \cdot x = \tauh_2^2d_1.$$ 

For part (3), we will prove below in Lemma 2.4.14 that $h_1X_1 = Ph_5c_0e_0$. So we wish to show that $h_0 \cdot lm = Ph_5h_5c_0e_0$. This follows immediately from part (1), using that $lm = e_0gr$.

The proof of part (3) is similar to the proof of part (1). First, $lm$ equals $\langle \tau^2g^3, h_2^3, h_0 \rangle$. As above, this implies that $h_2lm = \langle \tau^2h_2g^3, h_2, h_0h_2 \rangle$. Now use the (not hidden) relation $\tau^2h_2g^3 = h_1^2Q_2$ to deduce that $h_2lm = h_1^2c_0Q_2$. The desired formula now follows since $h_2l = h_0m$.

Lemma 2.4.12. $h_0 \cdot h_2^0B_{22} = Ph_1h_5c_0d_0$.

Proof. This follows from the hidden $\tau$ extension $\tau \cdot h_1B_{22} = Ph_1h_5c_0d_0$ that follows from Lemma 2.4.13 together with the relation $\tau h_1^2 = h_2^3h_2$.

2.4.3. Hidden May $h_1$ Extensions. By exhaustive search, the following results give all of the hidden $h_1$ extensions.

Proposition 2.4.13. Table 16 lists all of the hidden $h_1$ extensions through the 70-stem.

Proof. Many of the extensions follow by comparison to the classical case as described in 9. For example, there is a classical hidden extension $h_1 \cdot x = h_2^2d_1$. This implies that there is a motivic hidden extension $h_1 \cdot x = \tau h_2^2d_1$.

Proofs for the more subtle cases are given below.

Lemma 2.4.14.

(1) $h_1 \cdot \tau h_1G = h_5c_0e_0$.

(2) $h_1 \cdot h_1B_3 = h_5d_0c_0$.

(3) $h_1 \cdot \tau Ph_1G = Ph_5c_0e_0$.

(4) $h_1 \cdot h_1^2X_3 = h_5c_0d_0c_0$.

Proof. Table 16 shows that $\tau h_1G = \langle h_5, h_2g, h_0^3 \rangle$. Shuffle to obtain

$$h_1 \cdot \tau h_1G = \langle h_5, h_2g, h_0^2h_1 \rangle h_1 = h_5\langle h_2g, h_0^2, h_1 \rangle.$$ 

Finally, Table 16 shows that $c_0e_0 = \langle h_2g, h_0^2, h_1 \rangle$. This establishes the first hidden extension.

For the second hidden extension, Table 16 shows that $\langle h_5c_0e_0, h_0, h_2^2 \rangle$ equals $h_1h_5d_0c_0$. From part (1), this equals $\langle \tau h_1^2G, h_0, h_2^2 \rangle$, which equals $h_1^2\langle \tau G, h_0, h_2^2 \rangle$ because there is no indeterminacy. This shows that $h_5d_0c_0$ is divisible by $h_1$. The only possible hidden extension is $h_1 \cdot h_1B_3 = h_5d_0c_0$.

For the third hidden extension, start with the relation $h_1 \cdot \tau PG = Ph_1 \cdot \tau G$ because there is no possible hidden relation. Therefore, using part (1),

$$h_1^3 \cdot \tau PG = Ph_1 \cdot h_1^2 \cdot \tau G = Ph_1h_5c_0e_0.$$
It follows that $h_1 \cdot \tau P h_1 G = P h_5 c_0 e_0$.

For the fourth hidden extension, use part (1) to conclude that $h_5 c_0 d_0 e_0$ is divisible by $h_1$. The only possibility is that $h_1 \cdot h_1^2 X_3 = h_5 c_0 d_0 e_0$. \hfill \square

**Lemma 2.4.15.** $h_1 \cdot h_1^2 B_5 = \tau h_2^2 d_1 g$.

**Proof.** Table 16 shows that $d_1 g = (d_1, h_1^3, h_1 h_4)$. Using the hidden extension $h_1 \cdot x = \tau h_2^2 d_1 g$ [9], it follows that $\tau h_2^2 d_1 g = \langle h_1 x, h_1^3, h_1 h_4 \rangle$, which equals $h_1 \langle x, h_1^3, h_1 h_4 \rangle$ because there is no indeterminacy. Therefore, $\tau h_2^2 d_1 g$ is divisible by $h_1$, and the only possibility is that $h_1 \cdot h_1^2 B_5 = \tau h_2^2 d_1 g$. \hfill \square

**Lemma 2.4.16.** $h_1 \cdot h_1 D_{11} = \tau^2 c_1 g^2$.

**Proof.** Begin by computing that $h_1 D_{11} = \langle y, h_1^2, h_1^3 h_4 \rangle$, using May’s Convergence Theorem 2.2.2 the May differential $d_4 (g) = h_1^2 h_4$, and the relation $\Delta h_2^2 g = \Delta h_1^2 d_1$. Also recall the hidden extension $h_1 \cdot y = \tau^2 c_1 g$, which follows by comparison to the classical case [9].

It follows that

$$h_1^2 D_{11} = \langle h_1 y, h_1^4, h_4 \rangle = \langle \tau^2 c_1 g, h_1^4, h_4 \rangle$$

because there is no indeterminacy. Finally, Table 16 shows that $\tau^2 c_1 g^2$ equals $\langle \tau^2 c_1 g, h_1^4, h_1^3 h_4 \rangle$. \hfill \square

**Lemma 2.4.17.** $h_1 \cdot C_0 = 0$.

**Proof.** The only other possibility is that $h_1 \cdot C_0$ equals $h_0 h_3 f$. First compute that $C_0$ belongs to $\langle h_0 h_3^3, h_0, h_1, \tau h_1 g_2 \rangle$ using May’s Convergence Theorem 2.2.2 and the May differentials $d_4 (v) = h_0^2 h_3^2$ and $d_4 (x_{47}) = \tau h_1^2 g_2$. The subbracket $\langle h_0 h_3^2, h_0, h_1 \rangle$ is strictly zero. On the other hand, the subbracket $\langle h_0, h_1, \tau h_1 g_2 \rangle$ equals $\{0, \tau h_0 h_2 g_2 \}$. Condition (5) of May’s Convergence Theorem 2.2.2 is satisfied because the May filtration of $\tau h_2 g_2$ is less than the May filtration of $x_{47}$.

Because $\langle h_1, h_0 h_3^3, h_0 \rangle$ is zero, the hypothesis of Lemma 2.2.5 is satisfied. This implies that $h_1 \cdot C_0$ belongs to $\langle h_1, h_0 h_3^3, h_0, h_1 \rangle \tau h_1 g_2$. For degree reasons, the bracket $\langle h_1, h_0 h_3^3, h_0, h_1 \rangle$ consists of elements spanned by $f_0$ and $\tau h_3^2 h_4$. But the products $h_1 \cdot f_0$ and $h_1 \cdot \tau h_3^2 h_4$ are both zero, so $\langle h_1, h_0 h_3^3, h_0, h_1 \rangle \tau h_1 g_2$ must be zero. Therefore, $h_1 \cdot C_0$ is zero. \hfill \square

**Lemma 2.4.18.** $h_1 \cdot r_1 = s$.

**Proof.** This follows immediately from Theorem 2.1.12 and the classical relation $h_0 \cdot r = s$. \hfill \square

**Lemma 2.4.19.** $h_1 \cdot h_1^2 q_1 = h_1^4 X_3$.

**Proof.** Apply $Sq^6$ to the relation $h_2 r = h_1 q$ to obtain that $h_3 r^2 = h_1^2 Sq^5 (q)$. Next, observe that $Sq^5 (q) = h_1 q_1$ by comparison to the classical case [10].

By comparison to the classical case, there is a relation $h_3 r = h_1^2 x + \tau h_2^2 n$, so $h_3 r^2 = h_1^4 x$. Finally, use the hidden extension $h_0 \cdot r = s$ and the non-hidden relation $sx = h_0^3 X_3$. \hfill \square
2.4.4. Hidden May $h_2$ extensions. By exhaustive search, the following results give all of the hidden $h_2$ extensions.

**Proposition 2.4.20.** Table [4] lists all of the hidden $h_2$ extensions through the $70$-stem.

**Proof.** Many of the extensions follow by comparison to the classical case as described in [9]. For example, there is a classical hidden extension $h_2 \cdot Q_2 = h_5 k$. This implies that the same formula holds motivically.

Also, many extensions are implied by hidden $h_0$ extensions that we already established in Section 2.4.2. For example, there is a hidden extension $h_0 \cdot h_2^g = h_1^3 h_4 c_0$. This implies that there is also a hidden extension $h_2 \cdot h_0 h_2^g = h_1^3 h_4 c_0$.

Proofs for the more subtle cases are given below.

**Remark 2.4.21.** We established the extensions

1. $h_2 \cdot e_0 r = P h_1^3 h_5 c_0$
2. $h_2 \cdot l m = h_1^5 c_0 Q_2$

in the proof of Lemma 2.4.10. The extension $h_2 \cdot k m = h_1^6 X_1$ follows from Lemma 2.4.10 and the relation $h_2 k = h_0$.

**Lemma 2.4.22.** $h_2 \cdot h_2 B_2 = h_1 h_5 c_0 d_0$.

**Proof.** Table [10] shows that $h_2 B_2 = \langle g_2, h_0^3, h_2^3 \rangle$, with no indeterminacy. Then $h_2 \cdot h_2 B_2$ equals $\langle g_2, h_0^3, h_2^3 \rangle$, because there is no indeterminacy. This bracket equals $\langle g_2, h_0^3, h_1^3 h_3 \rangle$, which equals $\langle g_2, h_0^3, h_1 \rangle h_1 h_3$ since there is no indeterminacy. Table [10] also shows that the bracket $\langle g_2, h_0^3, h_1 \rangle$ equals $B_1$.

We have now shown that $h_2 \cdot h_2 B_2$ equals $h_1 h_3 \cdot B_1$. It remains to show that there is a hidden extension $h_3 \cdot B_1 = h_5 c_0 d_0$. First observe that $B_1 \cdot \tau h_1^2 d_0 = h_1^3 B_{21}$ by a non-hidden relation. This implies that $B_1 \cdot \tau h_1^2 d_0 = P h_1 h_5 c_0 d_0$ by Lemma 2.4.3.

Now there is a hidden extension $h_3 \cdot P h_1 = \tau h_1^2 d_0$, so $B_1 \cdot h_3 \cdot P h_1 = P h_1 h_5 c_0 d_0$. The only possibility is that $h_3 \cdot B_1 = h_5 c_0 d_0$.

□

**Lemma 2.4.23.** $h_2 \cdot B_6 = \tau e_1 g$.

**Proof.** Table [10] shows that $\langle \tau, B_6, h_1^2 h_3 \rangle = h_2 C_0$ with no indeterminacy. This means that $\langle \tau, B_6, h_2^3 \rangle = h_2 C_0$. If $h_2 \cdot B_6$ were zero, then this would imply that $\langle \tau, B_6, h_2^3 \rangle = h_2 C_0$. However, $h_2 C_0$ cannot be divisible by $h_2^3$.

□

2.4.5. Other hidden May extensions. We collect here a few miscellaneous extensions that are needed for various arguments.

**Lemma 2.4.24.**

1. $c_0 \cdot i_1 = h_1^4 D_4$.
2. $P h_1 \cdot i_1 = h_1^5 Q_2$.
3. $c_0 \cdot Q_2 = P D_4$.

**Proof.** Start by computing that $h_1^2 D_4$ belongs to $\langle c_0, h_1^2, h_3, h_1^3, h_1 h_3 \rangle$; we will not need to worry about the indeterminacy. One can use May’s Convergence Theorem 2.2.3 and the May $d_2$ differential to make this computation. All of the threefold subbrackets are strictly zero, and one of the fourfold subbrackets is also strictly zero. However, $\langle c_0, h_1^2, h_3, h_1^3 \rangle$ equals $\{0, h_1^2 h_5 c_0 \}$. Condition (7) of May’s Convergence
Theorem 2.2.8 is satisfied because \( h_2^2h_5c_0 = \langle h_5c_0, h_3, h_1^1 \rangle \), and the May filtration of \( h_5c_0 \) is less than the May filtration of \( h_1h_0(1, 3) \).

The hypothesis of Lemma 2.2.7 is satisfied because \( \langle h_3, h_1^1, h_1h_3, h_1^2 \rangle \) is strictly zero. Therefore, \( h_1^4D_4 \) is contained in \( c_0(h_2^2, h_3, h_1^1, h_1h_3, h_1^2) \). The main point is that \( h_1^4D_4 \) is divisible by \( c_0 \). The only possibility is that \( c_0 \cdot i_1 = h_1^4D_4 \). This establishes the first formula.

For the second formula, compute that \( h_1Q_2 \) equals \( \langle h_4, h_2^2h_4, h_4, Ph_1 \rangle \) with no indeterminacy, using May’s Convergence Theorem 2.2.2 and the May differentials \( d_4(\nu_1) = h_1^2h_4^2 \) and \( d_4(\Delta h_1) = Ph_1h_4 \). The subbracket \( \langle h_1^2h_4, h_4, Ph_1 \rangle \) equals \( \{0, Ph_1^2h_5 \} \). Condition (5) of May’s Convergence Theorem 2.2.2 is satisfied because the May filtration of \( h_1^2h_5 \) is less than the May filtration of \( \nu_1 \).

Next, compute that \( i_1 = \langle h_4^1, h_4, h_3^2h_4, h_4 \rangle \) with no indeterminacy, using May’s Convergence Theorem 2.2.2 and the May differentials \( d_4(g) = h_1^4h_4^2 \) and \( d_4(\nu_1) = h_1^2h_4^2 \). The subbracket \( \langle h_1^4, h_4, h_3^2h_4 \rangle \) equals \( \{0, h_6h_5 \} \). Condition (5) of May’s Convergence Theorem 2.2.2 is satisfied because the May filtration of \( h_1^2h_5 \) is less than the May filtration of \( \nu_1 \).

The hypothesis of Lemma 2.2.6 is satisfied because \( \langle h_3, h_1^1, h_1h_3, h_1^2 \rangle \) is strictly zero. Therefore,

\[
h_1^4\langle h_4, h_2^2h_4, h_4, Ph_1 \rangle = \langle h_1^4, h_4, h_2^2h_4, h_4 \rangle Ph_1,
\]

and \( h_1^2Q_2 = Ph_1 \cdot i_1 \). This establishes the second formula.

The third formula now follows easily. Compute that \( Ph_1 \cdot c_0 \cdot i_1 \) equals \( Ph_1 \cdot h_1^4D_4 \) and also \( c_0 \cdot h_1^2Q_2 \).

**Remark 2.4.25.** Part (3) of Lemma 2.4.24 shows that the multiplicative generator \( PD_4 \) of the \( E_\infty \)-page becomes decomposable in \( \text{Ext} \) by a hidden extension.

**Lemma 2.4.26.** \( c_0 \cdot B_6 = h_1^3B_3 \).

**Proof.** Table 16 shows that \( h_1^3Q_2 = \langle \tau, B_6, h_1^4 \rangle \). This bracket has no indeterminacy. It follows that \( h_1^3c_0Q_2 = \langle \tau, B_6 \cdot c_0, h_1^4 \rangle \), since this bracket also has no indeterminacy.

The element \( h_1^3c_0Q_2 \) is non-zero by part (3) of Lemma 2.4.24. Therefore, \( \langle h_1^3, B_6 \cdot c_0, \tau \rangle \) is not zero, so \( B_6 \cdot c_0 \) is non-zero. The only possibility is that it equals \( h_1^3B_3 \).

**Lemma 2.4.27.** \( c_0 \cdot G_3 = Ph_1^3h_5e_0 \).

**Proof.** Start with the relation \( h_2^3G_3 = h_2gr \). This implies that \( h_2^3d_0G_3 = h_2d_0gr \), which equals \( h_1^3X_1 \) by Table 13 Therefore, \( c_0^2G_3 \) is non-zero, which means that \( c_0G_3 \) is also. The only possibility is that \( c_0G_3 \) equals \( Ph_1^3h_5e_0 \).

**Lemma 2.4.28.** \( h_2^3B_4 + \tau h_1B_{21} = g_2^4 \).

**Proof.** On the \( E_\infty \)-page, there is a relation \( h_2^3B_4 + \tau h_1B_{21} = 0 \). The hidden extension follows from the analogous classical hidden relation 9.

**Remark 2.4.29.** Through the 70-stem, Lemma 2.4.28 is the only example of a hidden relation of the form \( h_0 \cdot x + h_1 \cdot y, h_0 \cdot x + h_2 \cdot y, \) or \( h_1 \cdot x + h_2 \cdot y \).
CHAPTER 3

Differentials in the Adams spectral sequence

The main goal of this chapter is to compute the differentials in the motivic Adams spectral sequence. We will rely heavily on the computation of the Adams $E_2$-page carried out in Chapter 2. We will borrow results from the classical Adams spectral sequence where necessary. Tables 18 and 19 summarize previously established results about the classical Adams spectral sequence, including differentials and Toda brackets. The tables give specific references to proofs. The main sources are [3], [4], [8], [26], [40], and [41].

The Adams charts in [19] are essential companions to this chapter.

The motivic Adams spectral sequence. We refer to [13], [17], and [32] for background on the construction and convergence of the motivic Adams spectral sequence over $\mathbb{C}$. In this section, we review just enough to proceed with our computations in later sections.

**Theorem 3.0.1** ([13] [17] [32]). The motivic Adams spectral sequence takes the form

$$E_2^{s,f,w} = \text{Ext}_{A}^{s,f,w}(M_2, M_2) \Rightarrow \pi_{s,w},$$

with differentials of the form $d_r : E_r^{s,f,w} \to E_r^{s-1, f+r,w}$.

We will need to compare the motivic Adams spectral sequence to the classical Adams spectral sequence. The following proposition is implicit in [13] Sections 3.2 and 3.4.

**Proposition 3.0.2.** After inverting $\tau$, the motivic Adams spectral sequence becomes isomorphic to the classical Adams spectral sequence tensored over $\mathbb{F}_2$ with $M_2[\tau^{-1}]$.

In particular, Proposition 3.0.2 implies that motivic differentials and motivic hidden extensions must be compatible with their classical analogues. This comparison will be a key tool.

**Outline.** A critical ingredient is Moss’s Convergence Theorem [34], which allows the computation of Toda brackets in $\pi_{s,*}$ via the differentials in the Adams spectral sequence. We will thoroughly review this result in Section 3.1.

Section 3.2 describes the main points in establishing the Adams differentials. We postpone the numerous technical lemmas to Section 3.3.

Chapter 7 contains a series of tables that summarize the essential computational facts in a concise form. Tables 8, 20, 21, and 22 give the values of the motivic Adams differentials. The fourth columns of these tables refer to one argument that establishes each differential, which is not necessarily the first known proof. This takes one of the following forms:

(1) An explicit proof given elsewhere in this manuscript.
“image of $J$” means that the differential is easily deducible from the structure of the image of $J$. 2.

“$tmf$” means that the differential can be detected in the Adams spectral sequence for $tmf$. 15.

“Table 18” means that the differential is easily deduced from the analogous classical result.

“[11 VI.1]” means that the differential can be computed using the relationship between squaring operations and Adams differentials.

Table 23 summarizes some calculations of Toda brackets. In all cases, we have been careful to describe the indeterminacies accurately. The fifth column refers to an argument for establishing this differential, in one of the following forms:

1. An explicit proof given elsewhere in this manuscript.
2. A Massey product (which appears in Table 16) implies the Toda bracket via Moss’s Convergence Theorem 3.1.1 with $r = 2$.
3. An Adams differential implies the Toda bracket via Moss’s Convergence Theorem 3.1.1 with $r > 2$.

The last column of Table 23 lists the specific results that rely on each Toda bracket.

3.1. Toda brackets in the motivic Adams spectral sequence

We will frequently compute Toda brackets in the motivic stable homotopy groups in order to resolve hidden extensions and to determine Adams differentials. The absolutely essential tool for computing such Toda brackets is Moss’s Convergence Theorem [34, Theorem 1.2]. The point of this theorem is that under certain hypotheses, Toda brackets can be computed via Massey products in the $E_r$-page of the motivic Adams spectral sequence. For the reader’s convenience, we will state the Convergence Theorem in the specific forms that we will use.

The $E_2$-page of the motivic Adams spectral sequence possesses Massey products, since it equals the cohomology of the motivic Steenrod algebra. Moreover, since $(E_r, d_r)$ is a differential graded algebra for $r \geq 2$, the $E_{r+1}$-page of the motivic Adams spectral sequence also possesses Massey products that are computed with the Adams $d_r$ differential. When necessary for clarity, we will use the notation $\langle a_0, \ldots, a_n \rangle_{E_{r+1}}$ to refer to Massey products in the $E_{r-1}$-page in this sense. Similarly, $\langle a_0, \ldots, a_n \rangle_{E_2}$ indicates a Massey product in Ext.

**Theorem 3.1.1 (Moss’s Convergence Theorem).** Let $\alpha_0$, $\alpha_1$, and $\alpha_2$ be elements of the motivic stable homotopy groups such that the Toda bracket $\langle \alpha_0, \alpha_1, \alpha_2 \rangle$ is defined. Let $a_i$ be a permanent cycle on the Adams $E_r$-page that detects $\alpha_i$ for each $i$. Suppose further that:

1. the Massey product $\langle a_0, a_1, a_2 \rangle_{E_r}$ is defined (in Ext when $r = 2$, or using the Adams $d_{r-1}$ differential when $r \geq 3$).
2. if $(s, f, w)$ is the degree of either $a_0 a_1$ or $a_1 a_2$; $f' < f - r + 1$; $f'' > f$; and $t = f'' - f'$; then every Adams differential $d_t : E_t^{(s+1, f', w)} \to E_t^{(s, f', w)}$ is zero.

Then $\langle a_0, a_1, a_2 \rangle_{E_r}$ contains a permanent cycle that detects an element of the Toda bracket $\langle \alpha_0, \alpha_1, \alpha_2 \rangle$.

Condition (2) is an equivalent reformulation of condition (1.3) in [34, Theorem 1.2]. When computing $\langle a_0, a_1, a_2 \rangle$, one uses a differential $d_{r-1} : E_r^{(s-1, f-r+1, w)} \to$
The idea of condition (2) is that there are no later “crossing” differentials $d_i$ whose source has strictly lower Adams filtration and whose target has strictly higher Adams filtration.

**Example 3.1.2.** Consider the differential $d_2(h_4) = h_0h_3^2$. This shows that $\langle \eta, 2, \sigma^2 \rangle$ intersects $\{h_1h_4\}$. In fact, Table 2 shows that the bracket equals $\{h_1h_4\} = \{\eta_4, \eta_4 + \eta_15\}$.

**Example 3.1.3.** Consider the Massey product $\langle h_2, h_3, h_3^2h_4 \rangle$. Using the May differential $d_4(\nu) = h_0^2h_2^3$ and May’s Convergence Theorem 2.2.1, this Massey product contains $f_0$ with indeterminacy $\tau h_1h_4$. However, this calculation tells us nothing about the Toda bracket $\langle \nu, \sigma, 4\sigma \rangle$. The presence of the later Adams differential $d_3(h_0h_4) = h_0d_0$ means that condition (2) of Moss’s Convergence Theorem 3.1.1 is not satisfied.

**Example 3.1.4.** Consider the Toda bracket $\langle \theta_4, 2, \sigma^2 \rangle$. The relation $h_1^4 + h_2^3h_5 = 0$ and the Adams differentials $d_2(h_3) = h_0h_2^3$ and $d_4(h_4) = h_0h_3^2$ show that the expression $\langle h_2^4h_0, h_3^2h_5 \rangle$, is zero. This implies that $\langle \theta_4, 2, \sigma^2 \rangle$ consists entirely of elements of Adams filtration strictly greater than 3. In particular, the Toda bracket is disjoint from $\{h_3^2h_5\}$. See Lemma 4.2.91 for more discussion of this Toda bracket.

One case of Moss’s Convergence Theorem 3.1.1 says that Massey products in $\text{Ext}_A(M_2, M_2)$ are compatible with Toda brackets in $\pi_{r,s}$, assuming that there are no interfering Adams differentials. Thus, we will use many Massey products in $\text{Ext}_A(M_2, M_2)$, most of which are computed using May’s Convergence Theorem 2.2.1.

We will also need the following lemma.

**Lemma 3.1.5.** If $2\alpha$ is zero, then $\tau \eta \alpha$ belongs to $\langle 2, \alpha, 2 \rangle$.

**Proof.** The motivic case follows immediately from the classical case, which is proved in [41].

**3.1.1. Toda brackets and cofibers.** The purpose of this section is to establish a relationship between Toda brackets of the form $\langle \alpha_0, \ldots, \alpha_n \rangle$ and properties of the stable homotopy groups of the cofiber $C\alpha_0$ of $\alpha_0$. This relationship is well-known to those who use it. See [41] Proposition 1.8] for essentially the same result.

Suppose given a map $\alpha_0 : S^{p,q} \to S^{0,0}$. Then we have a cofiber sequence

$$S^{p,q} \xrightarrow{\alpha_0} S^{0,0} \xrightarrow{j} C\alpha_0 \xrightarrow{q} S^{p+1,q} \xrightarrow{\alpha_0} S^{1,0}$$

where $j$ is the inclusion of the bottom cell, and $q$ is projection onto the top cell. Note that $\pi_{*,*}(C\alpha_0)$ is a $\pi_{*,*}$-module.

**Proposition 3.1.6.** Let $\alpha_0, \alpha_1,$ and $\alpha_2$ be elements of $\pi_{*,*}$ such that $\alpha_0 \alpha_1$ and $\alpha_1 \alpha_2$ are zero. Let $\overline{\alpha_1}$ be an element of $\pi_{*,*}(C\alpha_0)$ such that $q_*(\overline{\alpha_1}) = \alpha_1$. In $\pi_{*,*}(C\alpha_0)$, the element $\overline{\alpha_1} \cdot \alpha_2$ belongs to $j_*(\langle \alpha_0, \alpha_1, \alpha_2 \rangle)$.

**Proof.** The proof is described by the following diagram. The composition $\overline{\alpha_1} \alpha_2$ can be lifted to $S^{0,0}$ because $\alpha_1 \alpha_2$ was assumed to be zero. This shows that $\overline{\alpha_1} \cdot \alpha_2$ is equal to $j_*(\beta)$. Finally, $\beta$ is one possible definition of the Toda bracket.
3. DIFFERENTIALS IN THE ADAMS SPECTRAL SEQUENCE

\langle \alpha_0, \alpha_1, \alpha_2 \rangle.

\[ S_{0,0} \xrightarrow{\beta} C_{\alpha_0} \]

\[ S_{*,*} \xrightarrow{\alpha_2} S_{*,*} \xrightarrow{q} S^{p+1,q} \alpha_0 \xrightarrow{\alpha_1} S_{1,0} \]

Remark 3.1.7. We have presented Proposition 3.1.6 in the context of stable motivic homotopy groups, but the proof works in the much greater generality of a stable model category. For example, the same result holds for Massey products, where one works in the derived category of a graded algebra \( A \), and maps correspond to elements of Ext groups over \( A \).

Remark 3.1.8. Proposition 3.1.6 can be generalized to higher compositions. Suppose that \( \langle \alpha_0, \ldots, \alpha_n \rangle \) is defined. Then the bracket \( \langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle \) is contained in \( j_* (\langle \alpha_0, \ldots, \alpha_n \rangle) \). The proof is similar to the proof of Proposition 3.1.6 using the definition of higher Toda brackets [38, Appendix A].

### 3.2. Adams differentials

The \( E_2 \)-page of the motivic Adams spectral sequence is described in Chapter 2 (see also [13]). See [19] for a chart of the \( E_2 \)-page through the 70-stem. A list of multiplicative generators for the \( E_2 \)-page is given in Table 8.

Our next task is to compute the Adams differentials. The main point is to compute the Adams \( d_r \) differentials on the multiplicative generators of the \( E_r \)-page. Then one can compute the entire Adams \( d_r \) differential using that \( d_r \) is a derivation.

#### 3.2.1. Adams \( d_2 \) differentials

Most of the Adams \( d_2 \) differentials are lifted directly from the classical situation, in the sense of Proposition 3.0.2. We provide a few representative examples of this phenomenon.

**Example 3.2.1.** The classical differential \( d_2(h_4) = h_0h_2^2 \) immediately implies that there is a motivic differential \( d_2(h_4) = h_0h_2^2 \).

**Example 3.2.2.** Unlike the classical situation, the elements \( h_1^k d_0 \) and \( h_1^k e_0 \) are non-zero in the \( E_2 \)-page for all \( k \geq 0 \). The classical differential \( d_2(e_0) = h_1^2 d_0 \) implies that there is a motivic differential \( d_2(e_0) = h_1^2 d_0 \), from which it follows that \( d_2(h_1^k e_0) = h_1^{k+2} d_0 \) for all \( k \geq 0 \). Technically, these are "exotic" differentials, although we will soon see subtler examples.

**Example 3.2.3.** Consider the classical differential \( d_2(h_0c_2) = h_1^2 e_1 \). Motivically, this formula does not make sense because the weights of \( h_0c_2 \) and \( h_1^2 e_1 \) are 22 and 23 respectively. It follows that there is a motivic differential \( d_2(h_0c_2) = \tau h_1^2 e_1 \). Then \( h_1^2 e_1 \) is non-zero on the \( E_3 \)-page.

**Proposition 3.2.4.** Table 8 lists some values of the motivic Adams \( d_2 \) differential. The motivic Adams \( d_2 \) differential is zero on all other multiplicative generators of the \( E_2 \)-page, through the 70-stem.
Proof. Table 8 cites one possible argument (but not necessarily the earliest published result) for each non-zero differential on a multiplicative generator of the $E_2$-page. These arguments break into several types:

1. Some differentials are consequences of the image of $J$ calculation [2].
2. Some differentials follow by comparison to the Adams spectral sequence for $tmf$ [15].
3. Some differentials follow by comparison to an analogous classical result.
4. One differential follows from the relationship between Adams differentials and algebraic squaring operations [11, VI.1].
5. The remaining differentials are proved in Section 3.3.1.

For the differentials whose values are zero, Section 3.3.1 includes proofs for the cases that are not obvious. □

In order to maintain the flow of the narrative, we have collected the technical computations of miscellaneous $d_2$ differentials in Section 3.3.1.

The $E_2$ chart in [19] indicates the Adams $d_2$ differentials, all of which are implied by the calculations in Table 8.

Remark 3.2.5. Lemma 3.3.3 establishes three differentials $d_2(h_3g) = h_0h_2^2g$, $d_2(h_3g^2) = h_0h_2^2g^2$, and $d_2(h_3g^3) = h_0h_2^2g^3$. Presumably there is an infinite family of exotic differentials of the form

$$d_2(h_3g^k) = h_0h_2^2g^k.$$  

Remark 3.2.6. The differential $d_2(X_1) = h_0^2 B_1 + \tau h_1 B_{21}$ is inconsistent with the results of [24].

Remark 3.2.7. In the 51-stem, we draw particular attention to the Adams differential $d_2(D_1) = h_0^2 h_3 g_2$. Mark Mahowald privately communicated an argument for the presence of this differential to the author. However, this argument fails because of the Toda bracket calculation in Lemma 4.2.91 which was unknown to Mahowald. Zhouli Xu discovered an independent proof, which is included in Lemma 3.3.13.

Remark 3.2.8. As noted in Table 10, the element $\tau Q_3$ is defined in Ext such that $d_2(\tau Q_3) = 0$.

Remark 3.2.9. Quite a few of the $d_2$ differentials in this section follow by comparison to the Adams spectral sequence for $tmf$, i.e., the Adams spectral sequence whose $E_2$-page is the cohomology of the subalgebra $A(2)$ of the Steenrod algebra. See [15] for detailed computations with this spectral sequence.

Presumably, there is a “motivic modular forms” spectrum that is the motivic analogue of $tmf$. If such a motivic spectrum existed, then the $E_2$-page of its Adams spectral sequence would be the cohomology of motivic $A(2)$, as described in [18]. Such a spectral sequence would help significantly in calculating the differentials in the motivic Adams spectral sequence for $S^{0,0}$ that we are considering here.

3.2.2. Adams $d_3$ differentials. See [19] for a chart of the $E_3$-page. This chart is complete through the 70-stem; however, the Adams $d_3$ differentials are complete only through the 65-stem.

The next step is to compute the Adams $d_3$ differential on the multiplicative generators of the $E_3$-page.
Proposition 3.2.10. Table 27 lists some values of the motivic Adams $d_3$ differential. The motivic Adams $d_3$ differential is zero on all other multiplicative generators of the $E_3$-page, through the 65-stem, except that $d_3(D_3)$ might equal $B_3$.

Proof. Table 20 cites one possible argument (but not necessarily the earliest published result) for each non-zero differential on a multiplicative generator of the $E_3$-page. These arguments break into several types:

1. Some differentials are consequences of the image of $J$ calculation [2].
2. Some differentials follow by comparison to the Adams spectral sequence for $tmf$ [15].
3. Some differentials follow by comparison to an analogous classical result.
4. The remaining differentials are proved in Section 3.3.2.

For the differentials whose values are zero, Section 3.3.2 includes proofs for the cases that are not obvious. $\blacksquare$

In order to maintain the flow of the narrative, we have collected the technical computations of miscellaneous $d_3$ differentials in Section 3.3.2.

The $E_3$ chart in [19] indicates the Adams $d_3$ differentials, all of which are implied by the calculations in Table 20. The differentials are complete only through the 65-stem. Beyond the 65-stem, there are a number of unknown differentials.

Remark 3.2.11. The chart in [24] indicates a differential $d_3(D_3) = B_3$. However, we have been unable to independently verify this differential. Because of the relation $h_1B_3 = h_4B_1$ and because $\{B_1\}$ contains $\eta \theta_{4,5}$, we know that $h_1B_3$ detects $\langle \eta \theta_{4,5}, \sigma^2, 2 \rangle$, as shown in Table 23. It follows that $B_3$ detects $(\theta_{4,5}, \sigma^2, 2)$ and that $h_1B_3$ detects $\eta \theta_{4,5}$. We have so far been unable to show that either $\langle \theta_{4,5}, \sigma^2, 2 \rangle$ or $\eta \theta_{4,5}$ is zero.

Remark 3.2.12. We draw attention to the differential $d_3(h_1h_5e_0) = h_1^2B_1$. This can be derived from its classical analogue, which is carefully proved in [8]. Lemma 3.3.30 provides an independent proof. This proof originates from an algebraic hidden extension in the $h_1$-local cohomology of the motivic Steenrod algebra [14].

Remark 3.2.13. The differential $d_3(Q_2) = \tau^2g_1$ given in Lemma 3.3.37 is inconsistent with the chart in [24]. We do not understand the source of this discrepancy.

Remark 3.2.14. We claim that $d_3(r_1)$ is zero; this is tentative because our analysis is incomplete in the relevant range. The only other possibility is that $d_3(r_1)$ equals $h_1^2X_2$. However, we show in Lemma 3.2.12 that $h_1^2X_2$ supports a hidden $\tau$-extension and must therefore be non-zero on the $E_\infty$-page.

3.2.3. Adams $d_4$ differentials. See [19] for a chart of the $E_4$-page. This chart is complete through the 65-stem. Beyond the 65-stem, because of unknown earlier differentials, the actual $E_4$-page is a subquotient of what is shown in the chart.

The next step is to compute the Adams $d_4$ differentials on the multiplicative generators of the $E_4$-page.

Proposition 3.2.15. Table 27 lists some values of the motivic Adams $d_4$ differential. The motivic Adams $d_4$ differential is zero on all other multiplicative generators of the $E_4$-page, through the 65-stem, with the possible exceptions that:

1. $d_4(\tau h_1X_1)$ or $d_4(R)$ might equal $\tau^2d_0e_0r$. 

For the differentials whose values are zero, Section 3.3.2 includes proofs for the cases that are not obvious. $\blacksquare$
(2) $d_4(C')$ or $d_4(\tau X_2)$ might equal $h_2B_{21}$ or $\tau h_2B_{21}$ respectively.

**Proof.** Table [21] cites one possible argument (but not necessarily the earliest published result) for each non-zero differential on a multiplicative generator of the $E_4$-page. These arguments break into several types:

1. Some differentials are consequences of the image of $J$ calculation [2].
2. Some differentials follow by comparison to the Adams spectral sequence for tmf [15].
3. Some differentials follow by comparison to an analogous classical result.
4. The remaining differentials are proved in Section 3.3.3.

For the differentials whose values are zero, Section 3.3.3 includes proofs for the cases that are not obvious. □

The $E_4$ chart in [19] indicates the Adams $d_4$ differentials, all of which are implied by the calculations in Table [21]. The differentials are complete only through the 65-stem. Beyond the 65-stem, there are a number of unknown differentials.

**Remark 3.2.16.** The chart in [24] indicates a classical differential $d_4(h_1X_1) = d_6e_0r$. However, we have been unable to independently verify this differential.

Because of the differential $d_5(\tau P h_5 e_0) = \tau d_0 z$ from Lemma 3.3.55, we strongly suspect that $\tau^2 d_6 e_0 r$ is hit by some differential, but there is more than one possibility.

Note that $\tau^2 d_6 e_0 r$ detects $\tau^2 \eta^3$.

**Remark 3.2.17.** The chart in [24] indicates a classical differential $d_4(C') = h_2B_{21}$. However, we have been unable to independently verify this differential.

Because $B_{21}$ detects $\kappa \theta_{4.5}$, we know that $h_2B_{21}$ detects $\nu \kappa \theta_{4.5}$. If we could show that $\nu \kappa \theta_{4.5}$ is zero, then we could conclude that there is a differential $d_4(C') = h_2B_{21}$.

### 3.2.4. Adams $d_5$ differentials

Because the $d_4$ differentials are relatively sparse, [19] does not provide a separate chart for the $E_5$-page.

The next step is to compute the Adams $d_5$ differentials on the multiplicative generators of the $E_5$-page.

**Proposition 3.2.18.** Table [22] lists some values of the motivic Adams $d_5$ differential. The motivic Adams $d_5$ differential is zero on all other multiplicative generators of the $E_5$-page, through the 65-stem, with the possible exceptions that:

1. $d_5(A')$ might equal $\tau h_1 B_{21}$.
2. $d_5(\tau h_1 H_1)$ might equal $\tau h_2 B_{21}$.
3. $d_5(\tau h_1^2 X_1)$ might equal $\tau^3 d_6 e_0^2$.

**Proof.** The differential $d_5(h_0^2 h_6) = P^6 d_0$ follows from the calculation of the image of $J$ [2]. The differential $d_5(h_1 h_6) = 0$ follows from the existence of the classical element $\eta_6$ [25].

The remaining cases are computed in Section 3.3.4. □

The chart of the $E_4$-page in [19] indicates the very few $d_5$ differentials along with the $d_4$ differentials.

**Remark 3.2.19.** The chart in [24] indicates a classical differential $d_5(A') = h_1 B_{21}$. However, we have been unable to independently verify this differential.
Because $B_{21}$ detects $\kappa \theta_{1.5}$, we know that $h_1 B_{21}$ detects $\eta \kappa \theta_{1.5}$. We have so far been unable to show that $\eta \kappa \theta_{1.5}$ is zero.

Remark 3.2.20. We suspect that $d_5(\tau h_1 H_1)$ equals zero, not $\tau h_2 B_{21}$. This would follow immediately if we knew that $d_4(C') = h_2 B_{21}$ (see Proposition 3.2.15 and Remark 3.2.17).

Remark 3.2.21. We show in Lemma 3.3.58 that $\tau^3 d_2^2 c_0^2$ is hit by some differential. We suspect that $d_5(\tau h_1^2 X_1)$ equals $\tau^4 d_2^2 c_0^2$. The other possibilities are $d_9(\tau X_2)$ and $d_{10}(\tau h_1 H_1)$.

3.2.5. Higher Adams differentials. At this point, we are almost done.

Proposition 3.2.22. Through the 59-stem, the $E_6$-page equals the $E_\infty$-page.

Proof. The only possible higher differential is that $d_6(h_5 c_1)$ might equal $Ph_1^2 h_5 c_0$. However, we will show in the proof of Lemma 3.3.45 that $Ph_1^2 h_5 c_0$ cannot be hit by a differential.

The calculations of Adams differentials lead immediately to our main theorem.

Theorem 3.2.23. The $E_\infty$-page of the motivic Adams spectral sequence over $\mathbb{C}$ is depicted in the chart in 19 through the 59-stem. Beyond the 59-stem, the actual $E_\infty$-page is a subquotient of what is shown in the chart.

3.3. Adams differentials computations

In this section, we collect the technical computations that establish the Adams differentials discussed in Section 4.2.

3.3.1. Adams $d_2$ differentials computations. The first two lemmas establish well-known facts from the classical situation. However, explicit proofs are not readily available in the literature, so we supply them here.

Lemma 3.3.1. $d_2(P_0 c_0) = h_1^2 P_0 d_0$.

Proof. Because of the relation $2 \kappa = 0$, there must be a differential $d_2(\beta) = h_0 d_0$ in the Adams spectral sequence for tmf. Here $\beta$ is the class in the 15-stem as labeled in 15. Then $d_2(h_2 \beta) = h_0^2 c_0$.

Now $f_0$ maps to $h_2 \beta$, so it follows that $d_2(f_0) = h_0^2 c_0$ in the classical Adams spectral sequence for the sphere. The same formula must hold motivically.

The relation $h_0 f_0 = \tau h_1 c_0$ then implies that $d_2(c_0) = h_1^2 d_0$. This establishes the formula for $k = 0$.

The argument for larger values of $k$ is similar, using that $d_2(P_0 h_2 \beta) = P_0 h_0^2 c_0$ in the Adams spectral sequence for tmf; $P_0 h_0 j$ maps to $P_0 h_0^2 h_2 \beta$; and $P_0 h_0^2 j = \tau P_0 h_1 c_0$.

Lemma 3.3.2. $d_2(l) = h_0 d_0 c_0$.

Proof. The differential $d_2(k) = h_0 d_0^2$ follows by comparison to the Adams spectral sequence for tmf. The relation $h_2 k = h_0 l$ then implies that $d_2(l) = h_0 d_0 c_0$.

Lemma 3.3.3.

1) $d_2(h_3 g) = h_0 h_2 h_2 g$.

2) $d_2(h_3 g^2) = h_0 h_2 h_2 g^2$. 


The argument for the second differential is essentially the same. The product $\eta^6 c \eta_5$ is detected by $h_1^1 h_5 c_0$. Since $\eta^3 \sigma = \eta^2 \epsilon$, we get that $\eta^3 \sigma c \eta_5$ is also detected by $h_1^1 h_5 c_0$. However, $\eta^2 \epsilon$ is zero, so $h_1^1 h_5 c_0$ must be hit by some differential.

For the third differential, Table 13 shows that $c_0 i_1 = h_1^1 D_4$ on the $E_2$-page. This implies that $\eta^4 \{i_1\}$ is contained in $\{h_0 h_2^2 g^3\}$ in $\pi_{60,40}$. Using that $\eta^3 \sigma = \eta^2 \epsilon$, we get that $\eta^6 \sigma \{i_1\}$ is contained in $\{h_0 h_2^2 g^3\}$. However, $\eta^6 \{i_1\}$ equals zero, so some differential must hit $h_0 h_2^2 g^3$.

**Lemma 3.3.4.** $d_2(e_0 g) = h_1^2 c_0^2$.

**Proof.** First note that $Ph_1 \cdot e_0 g = h_1 d_0^2 c_0 + h_1^4 v$; this is true in the May $E_\infty$-page. Now apply $d_2$ to this formula to get

$$Ph_1 \cdot d_2(e_0 g) = h_1^3 d_0^2 + h_1^6 u.$$ 

In particular, it follows that $d_2(e_0 g)$ is non-zero. The only possibility is that $d_2(e_0 g) = h_1^2 e_0^2$. 

**Lemma 3.3.5.**

1. $d_2(u') = \tau h_0 d_0^2 c_0$.
2. $d_2(Pu') = \tau Ph_0 d_0^2 c_0$.
3. $d_2(P^2 u') = \tau P^2 h_0 d_0^2 c_0$.
4. $d_2(P^3 u') = \tau P^3 h_0 d_0^2 c_0$.
5. $d_2(v') = h_1^2 u' + \tau h_0 d_0 e_0^2$.
6. $d_2(Pv') = Ph_1^2 u' + \tau P h_0 d_0^2$.
7. $d_2(P^2 v') = P^2 h_1^2 u' + \tau P h_0 d_0^2$.

**Proof.** The first four formulas follow easily from the relations $h_0 u' = \tau h_0 d_0 l$, $h_0 \cdot Pu' = \tau d_0^2 j$, $h_0 \cdot P^2 u' = \tau Ph_0 d_0^2 j$, and $h_0 \cdot P^3 u' = \tau P^2 h_0 d_0^2 j$.

For the fifth formula, start with the relation $c_0 v = h_1 v'$, which holds already in the May $E_\infty$-page. Apply $d_2$ to obtain $h_1^2 c_0 u = h_1 d_2(v')$. We have $c_0 u = h_1 u'$ (also from the May $E_\infty$-page), so $h_1 d_2(v') = h_1^3 u'$. It follows that $d_2(v')$ equals either $h_1^2 u'$ or $h_1 u' + \tau h_0 d_0 e_0^2$. Because of the relation $h_0 v' = \tau h_0 d_0 l$, it must be the latter.

The proofs of the sixth and seventh formulas are essentially the same, using the relations $Ph_1 \cdot v' = h_1 \cdot P v'$, $P^2 h_1 \cdot v' = h_1 \cdot P^2 v'$, $h_0 \cdot P v' = \tau h_0 d_0^2 k$, and $h_0 \cdot P^2 v' = \tau h_0 d_0^3 l$.

**Lemma 3.3.6.** $d_2(G_3) = h_0 g r$.

**Proof.** The argument is similar to the proof of Lemma 3.3.3.

Let $\alpha$ be an element of $\{Ph_1 h_5\}$ such that $\eta^3 \alpha$ is contained in $\nu(\tau^2 g^2)$. Now $\epsilon \alpha$ is contained in $\{Ph_1 h_5 c_0\}$. Using that $\eta^2 \epsilon = \eta^2 \epsilon$ from Table 13, we get that $\eta^2 \epsilon \alpha = \eta^3 \alpha$, which is contained in $\nu \sigma(\tau^2 g^2)$. This is zero, since $\nu \sigma$ is zero.

This means that $Ph_1^2 h_5 c_0 = h_0 g r$ must be zero on the $E_\infty$-page of the Adams spectral sequence, but there are several possible differentials. We cannot have
$d_2(\tau gn) = h_0 gr$, since $\tau g \cdot n$ is the product of two permanent cycles. We cannot have $d_3(h_2 B_2) = h_0 gr$, since we will show later in Lemma 3.3.20 that $B_2$ does not support a $d_3$ differential. We cannot have $d_4(h_3^2 h_3 g_2) = h_0 gr$, $d_5(h_3 h_3 g_2) = h_0 gr$, or $d_6(h_3 g_2) = h_0 gr$, since we will show later in Lemma 3.3.51 that $g_2$ is a permanent cycle.

There is just one remaining possibility, so we conclude that $d_2(G_3) = h_0 gr$. □

**Lemma 3.3.7.** $d_2(B_6) = 0$.

**Proof.** The only other possibility is that $d_2(B_6)$ equals $h_1 h_5 c_0 d_0$. If this were the case, then $d_2(B_6)$ would equal $h_1 h_5 c_0 d_0$ in the motivic Adams spectral sequence, since we will show later in Lemma 3.3.30 that $h_1 h_5 c_0 d_0$ must survive to the $E_3$-page. □

**Lemma 3.3.8.** $d_2(i_1) = 0$.

**Proof.** The only other possibility is that $d_2(i_1) = h_1 h_5 c_0$. However, we will see below in Lemma 3.3.30 that $h_1 h_5 c_0$ must survive to the $E_3$-page. □

**Lemma 3.3.9.** $d_2(gm) = h_0 c_0^2 g$.

**Proof.** This follows easily from the relation $h_0 gm = h_2 c_0 m$ and the differential $d_2(m) = h_0 c_0^2$. □

**Lemma 3.3.10.**

1. $d_2(Q_1) = \tau h^2_1 x'$.
2. $d_2(U) = Ph^2_1 x'$.
3. $d_2(R_2) = h_0 U$.
4. $d_2(G_{11}) = h_0 d_0 x'$.

**Proof.** First note that $d_2(R_1) = h_0^2 x'$, which follows from the classical case as shown in Table 18. Then the relation $h_2 R_1 = h_1 Q_1$ implies that $d_2(Q_1) = \tau h^2_1 x'$. This establishes the first formula.

Next, there is a relation $\tau h_1 U = Ph_1 Q_1$, which is not hidden in the motivic May spectral sequence. Therefore, $\tau h_1 d_2(U) = \tau Ph^2_1 x'$. It follows that $d_2(U) = Ph^2_1 x'$. This establishes the second formula.

For the third formula, start with the relation $h_0^2 R_2 = \tau h_1 U$. This implies that $h_0^2 d_2(R_2)$ equals $\tau Ph^2_1 x'$, which equals $h_0^3 U$. Therefore, $d_2(R_2)$ equals $h_0 U$.

For the fourth formula, start with the relation $h_0 G_{11} = h_2 R_2$. This implies that $h_0 d_2(G_{11})$ equals $h_0 h_2 U$, which equals $h_0^2 d_0 x'$. Therefore, $d_2(G_{11})$ equals $h_0 d_0 x'$. □

**Lemma 3.3.11.**

1. $d_2(H_1) = B_7$.
2. $d_2(D_4) = h_1 B_6$.

**Proof.** First note that classically $h_3^2 H_1$ equals $h_4 A'$ [9]. Therefore, $h_3 d_2(H_1)$ equals $h_0 h_3^2 A' + h_4 d_2(A')$ classically, which equals $h_0 h_3^2 A'$ because $h_4 d_2(A')$ must be zero. This implies that $d_2(H_1)$ is non-zero classically. The only motivic possibility is that $d_2(H_1)$ equals $B_7$. This establishes the first formula.

Next, consider the relation $h_3^2 H_1 = h_3 D_4$, which is not hidden in the motivic May spectral sequence. It follows that $h_3 \cdot d_2(D_4) = h_7 B_7$. The only possibilities are that $d_2(D_4)$ equals $h_1 B_6$ or $h_1 B_6 + \tau h_1^2 G$. 


Table 3.3.8 gives the hidden extension $c_0 \cdot i_1 = h_1^1 D_4$. Since $d_2(i_1) = 0$ from Lemma 3.3.8, it follows that $d_2(h_1^1 D_4) = 0$. Then $d_2(D_4)$ cannot equal $h_1 B_6 + \tau h_1^2 G$ since $h_1^1 \cdot \tau h_1^2 G$ is non-zero. This establishes the second formula. □

**Lemma 3.3.12.**

1. $d_2(X_1) = h_3^0 B_4 + \tau h_1 B_{21}$.
2. $d_2(G_{21}) = h_0 X_3$.
3. $d_2(\tau G) = h_5 c_0 d_0$.

**Proof.** First consider the relation $h_3 R_1 = h_3^0 X_1$ [9]. Table 3.3.1 shows that $d_2(R_1) = h_3^0 x'$, so $h_3^0 d_2(X_1) = h_3^0 h_3 x'$. There is another relation $h_3^0 h_3 x' = h_4 B_4$, which is not hidden in the May spectral sequence. It follows that $d_2(X_1)$ equals either $h_3^0 B_4$ or $h_3^0 B_4 + \tau h_1 B_{21}$.

Next consider the relation $h_3^1 X_1 = h_3 Q_1$ [9]. We know from Lemma 3.3.10 that $d_2(Q_1) = \tau h_1 x'$, so $h_3^0 d_2(X_1)$ equals $\tau h_1^2 h_3 x'$. There is another relation $\tau h_1^2 h_3 x' = \tau h_1^2 B_{21}$, which is not hidden in the May spectral sequence. It follows that $d_2(X_1)$ equals either $\tau h_1 B_{21}$ or $\tau h_1 B_{21} + h_3^0 B_4$.

Now combine the previous two paragraphs to obtain the first formula.

For the second formula, start with the relation $h_3^0 G_{21} = h_3 X_1 + \tau e_1 r$ from [9]. Then $d_2(h_3 G_{21})$ equals $h_3 d_2(X_1) = h_3^0 h_3^1 B_4$, which equals $h_3^0 X_3$ [9]. The second formula follows.

For the third formula, Table 3.3.1 gives the relation $P h_1 \cdot \tau G = h_3^2 X_1$. The first formula implies that $P h_1 \cdot d_2(\tau G) = \tau h_1^2 B_{21}$, which equals $P h_1 h_3 c_0 d_0$ by Table 11 □

**Lemma 3.3.13.** $d_2(D_1) = h_3^2 h_3 g_2$.

**Proof.** This proof is due to Z. Xu [45].

Start with the Massey product $\tau G = \langle h_1, h_0, D_1 \rangle$. The higher Leibniz rule [34] Theorem 1.1] then implies that $d_2(\tau G) = \langle h_1, h_0, d_2(D_1) \rangle$ because there is no possible indeterminacy. We showed in Lemma 3.3.12 that $d_2(\tau G)$ equals $h_5 c_0 d_0$. This means that $d_2(D_1)$ is non-zero, and the only possibility is that $d_2(D_1)$ equals $h_3^2 h_3 g_2$.

In fact, note that $h_3^2 h_3 g_2 = h_3^2 h_5 d_0$ and that $h_5 c_0 d_0 = \langle h_1, h_0, h_3^2 h_5 d_0 \rangle$, but this is not essential for the proof. □

**Remark 3.3.14.** The proof of Lemma 3.3.13 relies on the Massey product $\tau G = \langle h_1, h_0, D_1 \rangle$. One might attempt to prove this with May’s Convergence Theorem 2.2.1 and the May differential $d_2(h_2 b_{22} b_{40}) = h_0 D_1$. However, there is a later differential $d_4(D_1 h_1) = h_1 h_3 g_2 + h_1 h_5 g$, so the hypotheses of May’s Convergence Theorem 2.2.1 are not satisfied.

This bracket can be computed via the lambda algebra [45]. Moreover, it has been verified by computer calculation.

**Lemma 3.3.15.**

1. $d_2(D_2) = h_0 Q_2$.
2. $d_2(A) = h_0 B_3$.
3. $d_2(A') = h_0 X_2$.

**Proof.** There is a classical relation $e_0 D_2 = h_0 h_3 G_{21}$ [9]. Since $d_2(G_{21}) = h_0 X_3$ by Lemma 3.3.12, it follows that $e_0 d_2(D_2)$ equals $h_3^2 h_3 X_3$, which is non-zero. The only possibilities are that $d_2(D_2)$ equals either $h_0 Q_2$ or $h_3 j$. 


Next, there is a classical relation $iD_2 = 0$ \[9\]. It follows that $id_2(D_2)$ equals $Ph_0d_0D_2$, which is non-zero. The only possibilities are that $d_2(D_2)$ equals either $h_0Q_2$ or $h_0Q_2 + h_0j$.

We obtain a classical differential $d_2(D_2) = h_0Q_2$ by combining the previous two paragraphs. The same formula must hold motivically. This establishes the first claim.

For the second claim, use the first claim together with the relations $h_2D_2 = h_0A$ and $h_2Q_2 = h_0B_3$. For the third claim, use the second claim together with the relations $h_0A'' = h_2(A + A')$ and $h_2B_3 = h_0X_2$. \[□\]

**Lemma 3.3.16.**

(1) $d_2(B_1) = h_0B_{21}$.

(2) $d_2(B_{22}) = h_1^2B_{21}$.

**Proof.** There is a relation $Ph_2B_1 = iB_2$, which is not hidden in the May spectral sequence. It follows that $Ph_2d_2(B_1)$ equals $Ph_0d_0B_2$, which equals $Ph_0h_2B_{21}$. Therefore, $d_2(B_1)$ equals $h_0B_{21}$. This establishes the first formula.

Now consider the relation $h_0h_2B_3 = \tau h_1B_{22}$, which is not hidden in the May spectral sequence. This implies that $\tau h_1d_2(B_{22})$ equals $h_1^2h_2B_{21}$, which equals $\tau h_1^2B_{21}$. It follows that $d_2(B_{22})$ equals $h_1^2B_{21}$. \[□\]

**Lemma 3.3.17.** $d_2(C') = 0$.

**Proof.** First note that $h_1C' = \tau d_1^2$ is a permanent cycle. Therefore, $h_1d_2(C')$ must equal zero, so $d_2(C')$ does not equal $h_1^2B_3$. \[□\]

**Lemma 3.3.18.**

(1) $d_2(X_2) = h_1^3B_3$.

(2) $d_2(D'_3) = h_1X_3$.

**Proof.** Note that $h_1B_3 = h_4B_1$. We will show in Lemma 4.2.48 that $\{B_1\}$ contains $\eta\theta_{4.5}$. As shown in Table 28, $\{h_1B_3\}$ intersects the bracket $\langle \eta\theta_{4.5}, 2, \sigma^2 \rangle$. In fact, $\langle \eta\theta_{4.5}, 2, \sigma^2 \rangle$ is contained in $\{h_1B_3\}$ because all of the possible indeterminacy is in strictly higher Adams filtration. This shows that $\theta_{4.5}\langle \eta, 2, \sigma^2 \rangle$ intersects $\{h_1B_3\}$.

For the first formula, note that $\{h_1^2B_3\}$ intersects $\langle \eta^2\theta_{4.5}, \eta, 2, \sigma^2 \rangle$. This last expression must be zero for degree reasons. Therefore, $h_1^2B_3$ must be killed by some differential. The only possibility is that $d_2(h_1X_2) = h_1^2B_3$, which implies that $d_2(X_2) = h_1^2B_3$.

For the second formula, note that $h_1^3X_3 = h_1c_0B_3 = h_4c_0B_1$. Lemma 4.2.83 says that $h_4c_0$ detects $\sigma_4$. Therefore, $h_1^2X_3$ detects $\eta\sigma\eta\theta_{4.5}$. The Adams filtration of $\eta\theta_{4.5}$ is at least 8; the Adams filtration of $\eta\sigma\eta\theta_{4.5}$ is at least 11; and the Adams filtration of $\eta\sigma\eta\theta_{4.5}$ is at least 12. Since the Adams filtration of $h_1^2X_3$ is 11, it follows that $h_1^2X_3$ must be hit by some differential. The only possibility is that $d_2(h_1D'_3)$ equals $h_1^3X_3$. \[□\]

**Lemma 3.3.19.**

(1) $d_2(\tau G_0) = h_2C_0 + h_1h_3Q_2$.

(2) $d_2(h_2G_0) = h_1C''$.

**Proof.** We work in the motivic Adams spectral sequence for the cofiber of $\tau$ of Chapter 5 where we have the relation $h_1h_3 \cdot \overline{D_3} = \tau G_0$. From Table 39, we know
that $d_2(D_0) = h_1 \cdot B_6 + Q_2$. It follows that $d_2(h_1 h_3 \cdot D_4) = h_2^2 h_3 \cdot B_6 + h_1 h_3 Q_2$. Finally, observe that $h_2^2 h_3 \cdot B_6 = h_3 \cdot B_0 = h_2 C_0$. This establishes the first formula.

For the second formula, use the first formula together with the relations $\tau \cdot h_2 G_0 = h_2 \cdot \tau G_0$ and $h_2^2 C_0 = \tau h_1 C''$.

**Lemma 3.3.20.**

1. $d_2(\tau B_5) = \tau^2 h_0^2 B_{23}$.
2. $d_2(D_2') = \tau^2 h_0^2 B_{23}$.
3. $d_2(P(A + A')) = \tau^2 h_0 h_2 B_{23}$.

**Proof.** Classically, there is a relation $iB_5 = 0$ [9]. Using that $d_2(i) = Ph_0 d_0$, we get that $id_2(B_5)$ equals $Ph_0 d_0 B_5$ classically, which is non-zero. The only possibility is that there is a motivic differential $d_2(\tau B_5) = \tau^2 h_0^2 B_{23}$. Note that the $Ph_{5j}$ term is eliminated because of the motivic weight. This establishes the first formula.

Classically, there is a relation $iD_2' = 0$ [9]. As in the previous paragraph, we get that $id_2(D_2')$ equals $Ph_0 d_0 D_2'$ classically, which is non-zero. However, this time the motivic weights allow for two possibilities. It follows that $d_2(D_2')$ equals either $\tau^2 h_0^2 B_{23}$ or $\tau^2 h_0^2 B_{23} + Ph_{5j}$.

We know from [10] that classically, $Sq^4(q)$ is non-zero, and $Sq^5(q)$ is a multiple of $h_1$. From [11], we have that $d_2(Sq^4(q)) = h_0 Sq^5(q)$, which is zero. From the previous two paragraphs, it follows that $Sq^5(q)$ must be $B_5 + D_2'$ classically, and $d_2(D_2')$ must be $h_0^2 B_{23}$. The motivic formula $d_2(D_2') = \tau^2 h_0^2 B_{23}$ follows immediately. This establishes the second formula.

For the third formula, there is a classical relation $iP(A + A') = 0$ [9]. As before, we get that $id_2(P(A + A'))$ equals $P^2 h_0 d_0 (A + A')$, which is non-zero. It follows that $d_2(P(A + A'))$ equals $\tau^2 h_0 h_2 B_{23}$ or $h_0^2 G_{21}$. The relation $h_2 D_2' = h_2 P(A + A')$ and the calculation of $d_2(D_2')$ in the previous paragraph imply that $d_2(P(A + A'))$ equals $\tau^2 h_0 h_2 B_{23}$.

**Lemma 3.3.21.** $d_2(P^3 v) = P^3 h_1^2 u$.

**Proof.** We will show in Lemma 3.3.40 that $d_3(\tau^2 P^2 d_0 m) = P^3 h_1 u$. Since $h_1 \cdot \tau^2 P^2 d_0 m$ is zero, $h_1 \cdot P^3 h_1 u$ must be zero on the $E_3$-page. Therefore, some $d_2$ differential must hit it. The only possibility is that $d_2(P^3 v) = P^3 h_1^2 u$.

**Lemma 3.3.22.** $d_2(X_3) = 0$.

**Proof.** Start with the relation $h_1 X_3 = B_3 c_0$. This shows that $h_1 d_2(X_3)$ is zero. Therefore, $d_2(X_3)$ cannot equal $h_1 c_0 Q_2$.

**Lemma 3.3.23.** $d_2(R_1') = P^2 h_0 x'$.

**Proof.** First, there is a relation $h_0^2 R_1' = \tau P^3 u'$, as shown in Table [11]. Lemma 3.3.3 says that $d_2(P^3 u') = \tau P^3 h_0 d_0^2 c_0$, so $h_0^2 d_2(R_1')$ equals $\tau^2 P^3 h_0 d_0^2 c_0$. There is another relation $\tau^2 P^3 h_0 d_0^2 c_0 = P^2 h_0 x'$, as shown in Table [11]. It follows that $d_2(R_1')$ equals $P^2 h_0 x'$.

The next lemma computes a few $d_2$ differentials on decomposable elements. In principle, these differentials are consequences of the previous lemmas. However, the results of the calculations are unexpected because of some extensions that are hidden in the motivic May spectral sequence.

**Lemma 3.3.24.**
3. DIFFERENTIALS IN THE ADAMS SPECTRAL SEQUENCE

\(1\) \(d_2(e_0^2g) = h_1^2B_1\).
\(2\) \(d_2(c_0e_0^2g) = h_1^3B_8\).
\(3\) \(d_2(e_0v) = h_1^3x'.\)
\(4\) \(d_2(e_0v') = h_1^3c_0x' + \tau h_0d_0e_0^3\).

**Proof.** In the first formula, we have \(d_2(e_0 \cdot e_0g) = h_1^2d_0 \cdot e_0g + e_0 \cdot h_1^2e_0^2\). This simplifies to \(h_1^2B_1\), as shown in 14.

The second formula follows immediately from the first formula, using that \(B_1c_0 = h_1B_8\).

For the third formula, start with the relation \(Ph_1 \cdot B_1 = h_1^2x'\). Since \(h_1^2B_1\) is hit by a \(d_2\) differential, it follows that \(h_1^3x'\) must also be hit by a \(d_2\) differential. The only possibility is that \(d_2(e_0v) = h_1^3x'\).

For the fourth formula, \(h_1^3c_0x'\) must be hit by the \(d_2\) differential, since \(h_1^3x'\) is hit by the \(d_2\) differential. The only possibility is that \(d_2(h_1c_0v') = h_1^3c_0x'\). Therefore, \(d_2(e_0v')\) equals either \(h_1^3c_0x'\) or \(h_1^3c_0x' + \tau h_0d_0e_0^3\). The extension \(h_0 \cdot e_0v' = \tau h_0d_0e_0m\) implies that the second possibility is correct. \(\square\)

### 3.3.2. Adams \(d_3\) differentials computations.

**Lemma 3.3.25.** \(d_3(h_4c_0) = 0\).

**Proof.** The only other possibility is that \(d_3(h_4c_0)\) equals \(c_0d_0\). Table 39 shows that \(Pd_0\) is hit by a differential in the Adams spectral sequence for the cofiber \(C\tau\) of \(\tau\). Therefore, \(\{Pd_0\}\) must be divisible by \(\tau\) in the homotopy groups of \(S^{0,0}\). The only possibility is that \(c_0d_0\) is a non-zero permanent cycle and that \(\tau \cdot \{c_0d_0\} = \{Pd_0\}\). \(\square\)

**Lemma 3.3.26.**
\(1\) \(d_3(\tau e_0g) = c_0d_0^2\).
\(2\) \(d_3(\tau d_0v) = Ph_1u'\).
\(3\) \(d_3(\tau^2gm) = h_1d_0u\).
\(4\) \(d_3(\tau e_0g^2) = c_0d_0e_0^2\).
\(5\) \(d_3(\tau gv) = h_1d_0u'\).
\(6\) \(d_3(\tau P d_0v) = P^2h_1u'\).

**Proof.** For the first formula, there is a classical differential \(d_4(e_0g) = Pd_0^2\) given in Table 18. Motivically, there must be a differential \(d_4(\tau^2e_0g) = Pd_0^2\). This shows that \(\tau e_0g\) cannot survive to \(E_4\).

The arguments for the remaining formulas are similar, using the existence of the classical differentials \(d_4(d_0v) = P^2u, d_4(gm) = d_0^3j + h_0^2R_1, d_4(e_0g^2) = d_0^3, d_4(gv) = Pd_0u, \) and \(d_4(P d_0v) = P^3u\). All of these classical differentials can be detected in the Adams spectral sequence for \(tmf\) 15, except for the third one, which is an easy consequence of \(d_4(e_0g) = Pd_0^2\). \(\square\)

**Lemma 3.3.27.**
\(1\) \(d_3(\tau Pd_0e_0) = P^2c_0d_0\).
\(2\) \(d_3(\tau P^2d_0e_0) = P^3c_0d_0\).
\(3\) \(d_3(\tau P^3d_0e_0) = P^4c_0d_0\).
\(4\) \(d_3(\tau P^3d_0e_0) = P^5c_0d_0\).

**Proof.** For the first formula, we know that \(d_3(\tau d_0e_0) = P^2c_0d_0\) by comparison to the classical case. Therefore, \(d_3(\tau Ph_1d_0e_0) = P^2h_1c_0d_0\). The desired formula
follows immediately. The arguments for the second, third, and fourth formulas are essentially the same.

**Lemma 3.3.28.** \(d_3(P\eta_5c_0) = 0\).

**Proof.** The only other possibility is that \(d_3(P\eta_5c_0) = \tau d_0l + u'\). However, \(c_0(\tau d_0 + u') = h_1d_0u\) is non-zero, while \(P\eta_5c_0 = P\theta_1h_5d_0 = 0\) since \(P\eta_5d_0 = \tau B_8\) by Table 11.

**Lemma 3.3.29.** \(d_3(B_2) = 0\).

**Proof.** First, \(B_21\) cannot support a \(d_3\) differential, so \(h_2B_21 = d_0B_2\) cannot support a \(d_3\) differential. This implies that \(d_3(B_2)\) cannot equal \(e_0r\), since \(d_0e_0r\) is non-zero on the \(E_2\)-page.

**Lemma 3.3.30.**

1. \(d_3(h_1\eta_5e_0) = h_1^2B_1\).
2. \(d_3(h_5\eta_5e_0) = h_1^2B_8\).
3. \(d_3(P\eta_5e_0) = h_1^2x'\).
4. \(d_3(h_1X_1 + \tau B_2) = c_0x'\).

**Proof.** We pass to the motivic Adams spectral sequence for the cofiber of \(\tau\), as discussed in Chapter 5. Note that \(h_1^4h_5 \cdot h_1^2\eta_5 = \tau e_0g^2\) in the \(E_2\)-page for the cofiber of \(\tau\). Also, \(h_1^4 \cdot h_1^2B_1 = c_0d_0e_0^2\) in the \(E_2\)-page for the cofiber of \(\tau\).

Now \(d_3(\tau e_0g^2) = c_0d_0e_0^2\) on the \(E_3\)-page for \(S^{0,0}\), as shown in Lemma 3.3.26. It follows that \(d_3(h_5 \cdot h_1^2\eta_5) = h_1^2B_1\) on the \(E_3\)-page for the cofiber of \(\tau\), and then \(d_3(h_1^2h_5e_0) = h_1^3B_1\) for \(S^{0,0}\) as well. This establishes the first formula.

After multiplying by \(c_0\), the second and third formulas follow easily from the first.

For the fourth formula, start with the relation \(P\eta_5c_0e_0 = h_1^2X_1\) from Table 15.

Multiply the third formula by \(c_0\) to obtain the desired formula.

**Lemma 3.3.31.**

1. \(d_3(gr) = \tau h_1d_0e_0^2\).
2. \(d_3(m^2) = \tau h_1e_0^4\).

**Proof.** We have \(d_3(\tau gr) = \tau^2h_1d_0e_0^2\) because \(d_3(r) = \tau h_1d_0^2\). The first formula follows immediately.

For the second formula, multiply the first formula by \(\tau g\) and use multiplicative relations from 9 to obtain that \(d_3(\tau m^2) = \tau^2h_1e_0^4\). The second formula follows immediately.

**Lemma 3.3.32.** \(d_3(\tau^2G) = \tau B_8\).

**Proof.** From Table 26 the product \(\tau e_0\) belongs to \(\{Pd_0\}\) in \(\pi_{22,12}\). Since \(h_1h_5c_0d_0 = 0\) in the \(E_\infty\)-page by Lemma 3.3.12 we know that \(\eta_5e_0\) is either zero or represented in \(E_\infty\) in higher filtration. It follows that \(\tau \eta_5e_0 = \eta_5(Pd_0)\) is either zero or represented in \(E_\infty\) in higher filtration. Now \(P\eta_5h_5d_0 = \tau h_1B_8\) by Table 11 so \(\tau h_1B_8\) must be hit by some differential. The only possibility is that \(d_3(\tau^2G) = \tau B_8\).

**Lemma 3.3.33.**

1. \(d_3(e_1g) = h_1gt\).
(2) $d_3(B_6) = \tau h_2 gn$.
(3) $d_3(gt) = 0$.

**Proof.** Start with the differential $d_3(e_1) = h_1t$ from Table 18. Then $d_3(\tau e_1 g)$ equals $\tau h_1tg$, which implies the first formula. The second formula follows easily, using that $\tau e_1 g = h_2 B_6$ from Table 14.

For the third formula, we know that $d_3(\tau gt) = 0$ because $d_3(\tau g) = 0$ and $d_3(t) = 0$. Therefore, $d_3(gt)$ cannot equal $\tau h_2 c_3^2 g$.

**Lemma 3.3.34.**
(1) $d_3(h_5i) = h_0x'$.
(2) $d_3(h_5j) = h_2x'$.

**Proof.** We proved in Lemma 3.3.30 that $d_3(h_5c_0e_0) = h_1 B_8$. This implies that $d_3(h_5c_0e_0) = h_1 B_8$ in the motivic Adams spectral sequence for the cofiber of $\tau$, which is discussed in Chapter 5. The hidden extensions $h_0 \cdot h_5c_0e_0 = h_5j$ and $h_0 \cdot h_1 B_8 = h_2x'$ then imply that $d_3(h_5j) = h_2x'$ for the cofiber of $\tau$, which means that the same formula must hold for $S^{0,0}$. This establishes the second formula.

The first formula now follows easily, using the relation $h_2h_5i = h_0h_5j$.

**Lemma 3.3.35.** $d_3(B_3) = 0$.

**Proof.** The only other possibility is that $d_3(B_3) = B_{21}$. On the $E_3$-page, $h_2B_3$ is zero while $h_2B_{21}$ is non-zero.

**Lemma 3.3.36.** $d_3(\gamma^3g) = h_1^5 B_8$.

**Proof.** Start with the hidden extension $\tau \gamma^2 \cdot \gamma^2g = \{d_0\}$, which follows from the analogous classical extension given in Table 21. This implies that $\tau \gamma^2 \{\gamma^2g\} = \{\tau d_0^2 c_3^2\}$. In particular, $\gamma^2 \{\gamma^2g\}$ must be non-zero.

Either $\gamma^2 g$ or $\gamma^3 + h_1^5 h_5c_0e_0$ survives the motivic Adams spectral sequence. In the first case, there is no possible non-zero value for a hidden extension of the form $\eta^2 \{\gamma^2g\}$. The only remaining possibility is that $\gamma^3 + h_1^5 h_5c_0e_0$ survives, in which case $\eta^2 \{\gamma^3g + h_1^5 h_5c_0e_0\} = \{h_1^5 h_5c_0e_0\}$ is a non-hidden extension.

**Lemma 3.3.37.**
(1) $d_3(C_0) = nr$.
(2) $d_3(E_1) = nr$.
(3) $d_3(Q_2) = \tau^2 gt$.
(4) $d_3(C''') = nm$.

**Proof.** There are relations $C_0 = h_2^2 \cdot B_6$ and $nr = h_2^2 \cdot \tau h_2 gn$ in the motivic Adams spectral sequence for the cofiber of $\tau$, as discussed in Chapter 5. From Lemma 3.3.33, we know that $d_3(B_6) = \tau h_2 gn$. The first formula now follows easily.

For the second formula, note that $gE_1 = gC_0$ classically, and that $nr$ is non-zero on the $E_3$-page [9]. We already know that $d_3(gc_0) = gnr$ classically, so it follows that $d_3(E_1)$ also equals $nr$ classically. The motivic formula is an immediate consequence.

For the third formula, there are classical relations $wQ_2 = g^2 C_0$ and $wgt = g^2 nr$ [9]. We already know that $d_3(g^2C_0) = g^2 nr$, so it follows that $wd_3(Q_2) = w \cdot gt$. The desired formula follows immediately.
For the fourth formula, there is a classical relation $gC''' = rQ_2$ \[9\. The $d_3$ differentials on $r$ and $Q_2$ imply that $d_3(gC''') = grt$ classically, which equals $gmn$ \[9\. The desired formula follows immediately.\]

**Lemma 3.3.38.**

(1) $d_3(\tau h_1 X_1) = 0$.

(2) $d_3(R) = 0$.

**Proof.** We have classical relations $h_1 r X_1 = 0$ and $h_1^2 d_0^2 X_1 = 0$ \[9\. Therefore, $rd_3(h_1 X_1) = 0$ classically. On the other hand, $rc_0 x'$ is non-zero on the $E_3$-page \[9\. This shows that $d_3(h_1 X_1)$ cannot equal $c_0 x'$ classically, which establishes the first formula.

An identical argument works for the second formula, using that $rR = 0$ and $h_1 d_0^2 R = 0$.\]

**Lemma 3.3.39.** $d_3(\tau gw) = h_1^2 c_0 x'$.

**Proof.** There is a hidden extension $\eta \cdot \{ \tau w \} = \{ \tau d_0 l + u' \}$, which follows from the analogous classical extension given in Table \[24\. This implies that $\eta \{ \tau^2 gw \} = \{ \tau^2 d_0 c_0 m \}$. If $gw$ were a permanent cycle, then $\eta \cdot \{ \tau gw \}$ would be a non-zero hidden extension. But there is no possible value for this hidden extension.\]

**Lemma 3.3.40.** $d_3(\tau^2 P^2 d_0 m) = P^3 h_1 u$.

**Proof.** Note that $P^2 d_0 m$ supports a $d_4$ differential in the Adams spectral sequence for $tmf$ \[15\. However, $P^2 d_0 m$ cannot support a $d_4$ differential in the classical Adams spectral sequence for the sphere. Therefore, $P^2 d_0 m$ cannot survive to the $E_4$-page. The only possibility is that there is a classical differential $d_3(P^2 d_0 m) = P^3 h_1 u$, from which the motivic analogue follows immediately.\]

**Lemma 3.3.41.** $d_3(\tau^2 B_3 + D_2') = 0$.

**Proof.** Classically, $(B_3 + D_2')d_0$ is zero while $d_0 gw$ is non-zero on the $E_3$-page \[9\. Therefore $d_3(\tau^2 B_3 + D_2')$ cannot equal $\tau^2 gw$.\]

**Lemma 3.3.42.** $d_3(X_3) = 0$.

**Proof.** The only other possibility is that $d_3(X_3)$ equals $\tau nm$. However, $gmn$ is non-zero on the classical $E_3$-page, while $gX_3$ is zero \[9\.\]

**Lemma 3.3.43.** $d_3(h_2 B_{23}) = 0$.

**Proof.** This follows easily from the facts that $d_3(\tau B_{23}) = 0$ and that $h_2 \cdot \tau B_{23} = \tau \cdot h_2 B_{23}$.\]

**Lemma 3.3.44.**

(1) $d_3(h_2 B_5) = h_1 B_3 d_0$.

(2) $d_3(\tau c_0 x') = Pc_0 x'$.\]

**Proof.** We will show in Lemma \[3.3.48\] that $d_4(\tau h_2 B_5) = h_1 d_0 x'$. This means that $h_2 B_3$ cannot survive to $E_4$. The only possibility is that $d_3(h_2 B_5) = h_1 B_3 d_0$. This establishes the first formula.

The proof of the second formula is similar. We will show in Lemma \[3.3.48\] that $d_4(\tau^2 c_0 x') = P^2 x'$, so $\tau c_0 x'$ cannot survive to $E_4$. The only possibility is that $d_3(\tau c_0 x') = Pc_0 x'$.\]
3.3.3. Adams $d_4$ differentials computations.

Lemma 3.3.45. $d_4(C) = 0$.

Proof. The other possibility is that $d_4(C)$ equals $P h^2 h_5 c_0$. We will show that $P h^2 h_5 c_0$ survives and is non-zero in the $E_\infty$-page.

Let $\alpha$ be an element of $\{P h^2 h_5 c_0\}$. From Table 15, the bracket $\langle \eta^2, \alpha, \epsilon \rangle$ contains the element $\{P h^3 h_5 c_0\}$. In order to compute this bracket, we need the relation $c_0 \cdot G_3 = P h^3 h_5 c_0$ from Table 15. Note that the bracket has indeterminacy generated by $\eta^2 \{D_{11}\}$.

If $P h^2 h_5 c_0$ were hit by a differential, then $\eta \alpha$ would be zero. Then $\eta(\eta, \alpha, \epsilon)$ would equal $(\eta^2, \alpha, \epsilon)$. But $\{P h^2 h_5 c_0\}$ cannot be divisible by $\eta$. By contradiction, $P h^2 h_5 c_0$ cannot be hit by a differential. □

Lemma 3.3.46.

(1) $d_4(h_0 h_5 i) = 0$.

(2) $d_4(C_{11}) = 0$.

Proof. The only non-zero possibility for $d_4(h_0 h_5 i)$ is $\tau d_0 u$. However, $d_0 u$ survives to a non-zero homotopy class in the Adams spectral sequence for tmf 15. This implies that $\tau d_0 u$ survives to a non-zero homotopy class in the motivic Adams spectral sequence. This establishes the first formula.

The proof of the second formula is similar. The only non-zero possibility for $d_4(C_{11})$ is $\tau^3 d_0 e_0 m$, but $d_0 e_0 m$ survives to a non-zero homotopy class in the Adams spectral sequence for tmf 15. □

Lemma 3.3.47. $d_4(\tau^2 e_0 g^2) = d_6^3$.

Proof. First note that $d_4(\tau^2 e_0 g) = P d_6^2$, which follows from its classical analogue given in Table 18. Multiply this formula by $h_1 d_6^2$ to obtain that $d_4(\tau^2 h_1 d_6 c_0^3)$ equals $P h_1 d_6^3$. Finally, note that $\tau^2 h_1 d_6 c_0^3$ equals $P h_1 \cdot \tau^2 e_0 g^2$. The desired formula follows. □

Lemma 3.3.48.

(1) $d_4(\tau^2 h_1 B_{22}) = P h_1 x'$.

(2) $d_4(\tau h_2 B_5) = h_1 d_6 x'$.

(3) $d_4(\tau^2 e_0 x') = P^2 x'$.

Proof. In the classical situation, $d_0 \cdot P h_1 x'$ is non-zero on the $E_4$-page, and $d_0 \cdot h_1 B_{22} = (d_0 e_0 + h_0^2 h_5) \cdot B_3$ [9]. Using that $d_4(d_0 e_0 + h_0^2 h_5) = P^2 d_0$, it follows that there is a classical differential $d_4(h_1 B_{22}) = P h_1 x'$. The motivic differential follows immediately.

The arguments for the second and third differentials are similar. For the second, use that $P d_0 \cdot h_1 d_6 x'$ is non-zero on the $E_4$-page; $P d_0 \cdot h_2 B_5 = h_1 x' \cdot e_0 g$ [9]; and $d_4(e_0 g) = P d_6^2$ classically.

For the third formula, use that $h_1 d_0 \cdot P^2 x'$ is non-zero on the $E_4$-page; $h_1 d_0 \cdot e_0 x' = P d_0 \cdot h_1 B_{22}$ [9]; and $d_4(h_1 B_{22}) = P h_1 x'$ classically from the first part of the lemma. □

Lemma 3.3.49. $d_4(\tau^2 m^2) = d_6^2 z$.

Proof. First note that $d_4(\tau^2 y r) = i j$, which follows by comparison to tmf [15]. Multiply by $\tau g$ to obtain that $d_4(\tau^3 m^2) = \tau d_6^2 z$, using multiplicative relations from [9]. The desired formula follows. □
3.3.4. Adams $d_5$ differentials computations.

**Lemma 3.3.50.** $d_5(g_2) = 0$.

**Proof.** The only non-zero possibility is that $d_5(g_2)$ equals $\tau^2 h_2 g^2$. However, $e_0 \cdot g_2$ is zero in the $E_5$-page, while $e_0 \cdot \tau^3 h_2 g^2$ is non-zero in the $E_5$-page. □

**Lemma 3.3.51.** $d_5(B_2) = 0$.

**Proof.** The only other possibility is that $d_5(B_2)$ equals $h_1 u'$. Recall from Table 24 that there is a classical hidden extension $\eta\{d_0 l\} = \{Pu\}$. This implies that $\eta\{\tau d_0 l + u'\}$ is non-zero motivically. Therefore, $h_1 u'$ cannot be zero in $E_\infty$. □

Our next goal is to show that $d_5(\tau Ph_5 e_0) = \tau d_6$. We will need a few preliminary lemmas. This approach follows [24] Theorem 2.2, but we have corrected and clarified the details in that argument.

**Lemma 3.3.52.** $\langle\{q\}; 2, 8\sigma\rangle = \{0, 2\tau \pi^2\}$.

**Proof.** Table 10 shows that $\langle q; h_0, h_0^2 h_3\rangle$ equals $\tau h_1 u$. Then Moss’s Convergence Theorem 3.1.1 implies that $\langle\{q\}; 2, 8\sigma\rangle$ contains $\{\tau h_1 u\}$. Table 27 shows that $\{\tau h_1 u\}$ equals $2\tau \pi^2$. Finally, use Lemma 3.2.87 to show that $2\tau \pi^2 = \tau\{q\}$ is in the indeterminacy of the bracket. □

**Lemma 3.3.53.** The bracket $\langle 2, 8\sigma, 2, \sigma^2\rangle$ contains $\tau \nu \pi$.

**Proof.** The subbracket $\langle 2, 8\sigma, 2\rangle$ is strictly zero, as shown in Table 23. We will next show that the subbracket $\langle 8\sigma, 2, \sigma^2\rangle$ is also strictly zero. First, the shuffle $(8\sigma, 2, \sigma^2)\eta = 8\sigma \langle 2, \sigma^2, \eta \rangle$ implies that the subbracket is annihilated by $\eta$. This rules out $Pd_0$. Moss’s Convergence Theorem 3.1.1 with the Adams differential $d_3(h_4) = h_0 h_3^2$ implies that the subbracket is detected in Adams filtration strictly greater than 5. This rules out $\tau h_2 c_1$. The only remaining possibility is that the subbracket contains zero, and there is no possible indeterminacy.

We will work in the motivic Adams spectral for the cofiber $C_2$ of 2. We write $E_2(C_2)$ for $\operatorname{Ext}_\lambda(H^{\infty}(C_2), M_2)$, i.e., the $E_2$-page of the motivic Adams spectral sequence for $C_2$. The cofiber sequence

$$S^{0,0} \xrightarrow{2} S^{0,0} \xrightarrow{j} C_2 \xrightarrow{q} \Sigma^{1,0}$$

induces a map $q_* : E_2(C_2) \to \Sigma^{1,0}E_2$. Let $h_0^2 h_3$ be an element of $E_2(C_2)$ such that $q_* \{h_0^2 h_3\}$ equals $h_0 h_3^2$, and let $\{h_0^2 h_3\}$ be the corresponding element in $\pi_{8,4}(C_2)$. Then $j_* \langle 2, 8\sigma, 2, \sigma^2 \rangle$ equals $\{\{h_0^2 h_3\}, 2, \sigma^2\}$ in $\pi_{23,12}(C_2)$ by Remark 3.1.8.

Because of the Adams differential $d_2(h_4) = h_0 h_3^2$, we know that $\langle\{h_0^2 h_3\}, 2, \sigma^2\rangle$ is detected by $h_4 \cdot h_0^2 h_3$ in $E_\infty(C_2)$. Here we are using a slight generalization of Moss’s Convergences Theorem 3.1.1 in which one considers Toda brackets of maps between different objects (see [34] for the classical case).

Finally, we need to compute $h_4 \cdot h_0^2 h_3$ in $E_2(C_2)$. This equals $j_* \langle h_4, h_0^2 h_3, h_0\rangle$ by Remark 3.1.7 (see also Proposition 3.0.1 for an analogous result). Table 16 shows that $\langle h_4, h_0^2 h_3, h_0\rangle$ equals $\tau^2 h_2 g$. □

**Lemma 3.3.54.** $\langle\epsilon, 2, \sigma^2\rangle = \{\sigma \eta h_4, \sigma \eta h_4 + 4\nu \pi\}$. 

3. DIFFERENTIALS IN THE ADAMS SPECTRAL SEQUENCE

PROOF. Using the Adams differential $d_2(h_4) = h_0 h_2^2$ and Moss’s Convergence Theorem 3.1.1, we know that $\langle \epsilon, 2, \sigma^2 \rangle$ intersects $\{ h_4 c_0 \}$.

Lemma 4.2.83 shows that $\sigma h_4$ is contained in $\{ h_4 c_0 \}$. The indeterminacy of $\{ h_4 c_0 \}$ is generated by $\tau h_2 g$, $\tau h_0 h_2 g$, and $P h_1 d_0$. By Lemma 4.2.17 the indeterminacy consists of multiples of $\nu \kappa$. Therefore, $\{ h_4 c_0 \}$ consists of elements of the form $\sigma h_4 + k \nu \kappa$ for $0 \leq k \leq 7$.

Note that $\langle \epsilon, 2, \sigma^2 \rangle^2$ equals $\epsilon \langle 2, \sigma^2, 2 \rangle$, which is zero because $\langle 2, \sigma^2, 2 \rangle$ contains 0 by Table 23. Therefore, if $\sigma h_4 + k \nu \kappa$ belongs to $\langle \epsilon, 2, \sigma^2 \rangle$, then $k$ equals 0 or 4.

We now know that either $\sigma h_4$ or $\sigma h_4 + 4 \nu \kappa$ belongs to $\langle \epsilon, 2, \sigma^2 \rangle$. But $\nu \kappa$ equals $4 \nu \kappa$ by Lemma 4.2.17 and the hidden $\tau$ extension from $h_1 c_0 d_0$ to $P h_1 d_0$ given in Table 25 so $4 \nu \kappa$ belongs to the indeterminacy of the bracket. It follows that both $\sigma h_4$ and $\sigma h_4 + 4 \nu \kappa$ belong to the bracket. □

**Lemma 3.3.55.** $d_5(\tau P h_5 c_0) = \tau d_0 z$.

**PROOF.** By Lemma 4.2.83, $\nu\kappa \kappa^2$ is detected by $d_0 z$. On the other hand, $\nu \{ q \} \kappa$ equals $\nu \kappa \kappa^2$ by Table 23. We will show that $\nu \{ q \} \kappa$ must be zero. It will follow that some differential must hit $\tau d_0 z$, and there is just one possibility.

From Lemma 3.3.53 we know that $\tau \nu \{ q \} \kappa$ is contained in $\nu \{ q \} \langle 2, 8 \sigma, 2, \sigma^2 \rangle$, which is contained in $\langle \alpha, 2, \sigma^2 \rangle$ for some element $\alpha$ in $\langle \{ q \} \rangle$. By Lemma 3.3.52 the two possible values for $\alpha$ are 0 and $2 \tau \kappa^2$.

First suppose that $\alpha$ is zero. Then $\nu \{ q \} \kappa$ is contained in $\langle 0, 2, \sigma^2 \rangle$, which is strictly zero.

Next suppose that $\alpha$ is $2 \tau \kappa^2$. By Lemma 4.2.87 we know that $\epsilon \{ q \}$ equals $\{ h_1 u \}$, which equals $\kappa \kappa^2$ by Table 27. Therefore, the element $\tau \nu \{ q \} \kappa$ is contained in $\langle \tau \epsilon \{ q \}, 2, \sigma^2 \rangle$. This bracket has no indeterminacy, so it equals $\tau \{ q \} \langle \epsilon, 2, \sigma^2 \rangle$. Using that $4 \nu \kappa \cdot \tau \{ q \}$ is zero, Lemma 3.3.54 implies that $\tau \nu \{ q \} \kappa$ equals $\sigma h_4 \cdot \tau \{ q \}$. We will show in the proof of Lemma 4.2.87 that $\sigma \{ q \}$ equals $\nu \{ t \}$. So $\nu \kappa \cdot \tau \{ q \}$ equals $\nu \kappa h_4 \{ t \}$, which is zero because $\nu \kappa h_4$ is zero. □

**Lemma 3.3.56.** $d_5(r_1) = 0$.

**PROOF.** The only other possibility is that $d_5(r_1)$ equals $h_2 B_{22}$. However, $h_2 r_1$ is zero, while $h_2^2 B_{22}$ is non-zero in the $E_5$-page. □

**Lemma 3.3.57.** $d_5(\tau h_2 C') = 0$.

**PROOF.** We do not know whether $d_4(C')$ equals $h_2 B_{21}$, so there are two situations to consider (see Proposition 3.2.15 and Remark 3.2.17).

In the first case, assume that $d_4(C') = 0$. Then $C'$ survives to the $E_5$-page, and $d_5(C')$ must equal zero because there are no other possibilities. It follows that $d_5(\tau h_2 C') = 0$.

In the second case, assume that $d_4(C')$ equals $h_2 B_{21}$. Then $d_4(h_2 C') = h_0 h_2 B_{22}$, but $\tau h_2 C'$ survives to the $E_5$-page.

The only possible non-zero value for $d_5(\tau h_2 C')$ is $\tau^4 g w$. However, $g w$ survives to a non-zero homotopy class in the classical Adams spectral sequence for $tmf$ [15]. Therefore, $g w$ survives to a non-zero homotopy class in the classical Adams spectral sequence for $S^0$. This implies that $\tau^4 g w$ cannot be hit by a motivic differential. □

Our next goal is to show that the element $\tau^3 d_0^2 c_0^2$ is hit by some differential. We include it in this section because it is likely hit by $d_5(\tau h_1^2 X_1)$.

**Lemma 3.3.58.** The element $\tau^3 d_0^2 c_0^2$ is hit by some differential.
Proof. Classically, $\eta^2 \kappa^2$ is detected by $d_0^3$ because of the hidden $\eta$ extension from $g^2$ to $z$ and from $z$ to $d_0^3$ shown in Table 21. This implies that motivically, $\tau^2 \eta^2 \kappa^3$ is detected by $\tau^3 d_0^3 \eta^2$. Therefore, we need to show that $\tau^2 \eta^2 \kappa^3$ is zero.

Compute that $\langle \kappa, 2\nu, \nu \rangle = \{\eta \kappa, \eta \kappa + \nu \nu_4\}$, using Moss’s Convergence Theorem 3.1.1 and the Adams differential $d_2(f_0) = h_0^2 e_0$. It follows that $\langle \kappa, 2, \nu^2 \rangle$ equals either $\eta \kappa$ or $\eta \kappa + \nu \nu_4$. Using Moss’s Convergence Theorem 3.1.1 and the Adams differential $d_3(h_0 h_4) = h_0 d_0$, we get that $\langle \kappa, 2, \nu^2 \rangle$ is detected in Adams filtration strictly greater than 4. Therefore, $\langle \kappa, 2, \nu^2 \rangle$ equals $\eta \kappa$.

This means that $\tau^2 \eta^2 \kappa^3$ equals $\tau^2 \eta \kappa^2 \langle \kappa, 2, \nu^2 \rangle$. We showed in the proof of Lemma 3.3.35 that $\tau^2 \eta \kappa^2$ is zero. Therefore, $\tau^2 \eta^2 \kappa^3$ is contained in $\langle 0, 2, \nu^2 \rangle$, which is strictly zero. □
CHAPTER 4

Hidden extensions in the Adams spectral sequence

The main goal of this chapter is to compute hidden extensions in the Adams spectral sequence. We rely on the computation of the Adams $E_\infty$-page carried out in Chapter 3. We will borrow results from the classical Adams spectral sequence where necessary. Table 24 summarizes previously established hidden extensions in the classical Adams spectral sequence. The table gives specific references to proofs. The main sources are [3], [4], and [26].

The Adams $E_\infty$ chart in [19] is an essential companion to this chapter.

Outline. Section 4.1 describes the main points in establishing the hidden extensions. We postpone the numerous technical proofs to Section 4.2.

Chapter 7 contains a series of tables that summarize the essential computational facts in a concise form. Tables 25, 27, 29, and 31 list the hidden extensions by $\tau$, 2, $\eta$, and $\nu$. The fourth columns of these tables refer to one argument that establishes each hidden extension, which is not necessarily the first known proof. This takes one of the following forms:

1. An explicit proof given elsewhere in this manuscript.
2. "image of $J$" means that the hidden extension is easily deducible from the structure of the image of $J$ [2].
3. "cofiber of $\tau$" means that the hidden extension is easily deduced from the structure of the homotopy groups of the cofiber of $\tau$, as described in Chapter 5.
4. "Table 24" means that the hidden extension is easily deduced from an analogous classical hidden extension.

Tables 33 and 34 give some additional miscellaneous hidden extensions, again with references to a proof.

Tables 26, 28, 30, and 32 give partial information about hidden extensions in stems 59 through 70. These results should be taken as tentative, since the analysis of Adams differentials in this range is incomplete.

4.1. Hidden Adams extensions

4.1.1. The definition of a hidden extension. First we will be precise about the exact nature of a hidden extension. The most naive notion of a hidden extension is a non-zero product $\alpha \beta$ in $\pi_{s,s}$ such that $\alpha$ and $\beta$ are detected in the $E_\infty$-page by $a$ and $b$ respectively and $ab = 0$ in the $E_\infty$-page. However, this notion is too general, as the following example illustrates.

Example 4.1.1. Consider $\{h_3^2\}$ in $\pi_{14,8}$, which consists of the two elements $\sigma^2$ and $\sigma^2 + \kappa$. We have $h_1 h_3^2 = 0$ in $E_\infty$, but $\eta(\sigma^2 + \kappa)$ is non-zero in $\pi_{15,9}$. 

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This type of situation is not usually considered a hidden extension. Because of the non-zero product $h_1d_0$ in $E_\infty$, one can see immediately that there exists an element $\beta$ of $\{h_2^3\}$ such that $\eta\beta$ is non-zero.

However, it is not immediately clear whether $\beta$ is $\sigma^2$ or $\sigma^2 + \kappa$. Distinguishing these possibilities requires further analysis. In fact, $\eta\sigma^2 = 0$ [41].

In order to avoid an abundance of not very interesting situations similar to Example 4.1.1, we make the following formal definition of a hidden extension.

**Definition 4.1.2.** Let $\alpha$ be an element of $\pi_{*,*}$ that is detected by an element $a$ of the $E_\infty$-page of the motivic Adams spectral sequence. A hidden extension by $\alpha$ is a pair of elements $b$ and $c$ of $E_\infty$ such that:

1. $ab = 0$ in the $E_\infty$-page.
2. There exists an element $\beta$ of $\{b\}$ such that $\alpha\beta$ is contained in $\{c\}$.
3. If there exists an element $\beta'$ of $\{b'\}$ such that $\alpha\beta'$ is contained in $\{c\}$, then the Adams filtration of $b'$ is less than or equal to the Adams filtration of $b$.

In other words, $b$ is the element of highest filtration such that there is an $\alpha$ multiplication from $\{b\}$ into $\{c\}$. Consider the situation of Example 4.1.1. Because $\eta\{d_0\}$ is contained in $\{h_1d_0\}$ and the Adams filtration of $d_0$ is greater than the Adams filtration of $h_2^3$, condition (3) of Definition 4.1.2 implies that there is not a hidden $\eta$ extension from $h_2^3$ to $h_1d_0$.

**Remark 4.1.3.** Condition (3) of Definition 4.1.2 implies that $b'$ is not divisible by $a$ in $E_\infty$. This allows one to easily reduce the number of cases that must be checked when searching for hidden extensions.

**Lemma 4.1.4.** Let $\alpha$ be an element of $\pi_{*,*}$. Let $b$ be an element of the $E_\infty$-page of the motivic Adams spectral sequence, and suppose that there exists an element $\beta$ of $\{b\}$ such that $\alpha\beta$ is zero. Then there is no hidden $\alpha$ extension on $b$.

**Proof.** Suppose that there exists some element $\beta'$ of $\{b\}$ such that $\alpha\beta'$ is in $\{c\}$. Then $\alpha(\beta + \beta')$ is also in $\{c\}$, and the Adams filtration of $\beta + \beta'$ is strictly greater than the Adams filtration of $\beta'$. This implies that there is not a hidden $\alpha$ extension from $b$ to $c$. \qed

**Lemma 4.1.5.** Let $\alpha$ be an element of $\pi_{*,*}$ that is detected by an element $a$ of the $E_\infty$-page of the motivic Adams spectral sequence. Suppose that $b$ and $c$ are elements of $E_\infty$ such that:

1. $ab = 0$ in the $E_\infty$-page.
2. $\alpha\{b\}$ is contained in $\{c\}$.

Then there is a hidden $\alpha$ extension from $b$ to $c$.

**Proof.** Let $\beta$ be any element of $\{b\}$, so $\alpha\beta$ is contained in $\{c\}$. Let $b'$ be an element of the $E_\infty$-page, and let $\beta'$ in $\{b'\}$ be an element such that $\alpha\beta'$ is also contained in $\{c\}$.

Since both $\alpha\beta$ and $\alpha\beta'$ are contained in $\{c\}$, their sum $\alpha(\beta + \beta')$ is detected in Adams filtration strictly greater than the Adams filtration of $c$. Therefore, $\beta + \beta'$ is not an element of $\{b\}$, which means that the Adams filtration of $b'$ must be less than or equal to the Adams filtration of $b$. \qed
4.1. Hidden Adams extensions

Example 4.1.6. The conditions of Lemma 4.1.5 are not quite equivalent to the conditions of Definition 4.1.2. The difference is well illustrated by an example. We will show in Lemma 4.2.48 that there is a hidden \( \eta \) extension from \( \beta h_5 \) to \( B_2 \), so there exists an element \( \beta \) of \( \{ \beta h_5 \} \) such that \( \eta \beta \) is contained in \( \{ B_1 \} \). Now let \( \beta' \) be an element of \( \{ h_5 \} \), so \( \beta + \beta' \) is an element of \( \{ \beta h_5 \} \) because the Adams filtration of \( h_5 \) is greater than the Adams filtration of \( \beta h_5 \).

Note that \( \eta \beta' \) is contained in \( \{ h_1 h_5 \} \). The Adams filtration of \( h_1 h_5 \) is less than the Adams filtration of \( B_1 \), so \( \eta(\beta + \beta') \) is contained in \( \{ h_1 h_5 \} \). This shows that the conditions of Lemma 4.1.5 are not satisfied.

The difference between Definition 4.1.2 and Lemma 4.1.5 occurs precisely when there are “crossing” \( \alpha \) extensions. In the chart of the \( E_\infty \)-page in \( \tau \), the straight line from \( \beta h_5 \) to \( B_1 \) crosses the straight line from \( h_5 \) to \( h_1 h_5 \).

Example 4.1.7. Through the 59-stem, the issue of “crossing” extensions occurs in only two other places. First, we will show in Lemma 4.2.48 that there is a hidden extension from \( h_5 h_5 \) to \( B_2 \). In the \( E_\infty \)-chart in \( \tau \), the straight line from \( h_5 h_5 \) to \( B_2 \) crosses the straight line from \( h_5 \) to \( h_2 h_5 \). Therefore there exists an element \( \beta \) of \( \{ h_5 h_5 \} \) such that \( \nu \beta \) is not contained in \( \{ B_2 \} \).

Second, we will show in Lemma 4.2.48 that there is a hidden extension from \( h_1 f_1 \) to \( \tau h_2 c_1 g \). In the \( E_\infty \)-chart in \( \tau \), the straight line from \( h_1 f_1 \) to \( \tau h_2 c_1 g \) crosses the straight line from \( h_5 h_5 c_0 \) to \( h_2 h_5 c_0 \). Therefore, there exists an element \( \beta \) of \( \{ h_1 f_1 \} \) such that \( \eta \beta \) is not contained in \( \{ \tau h_2 c_1 g \} \).

We will thoroughly explore hidden extensions in the sense of Definition 4.1.2. However, such hidden extensions do not completely determine the multiplicative structure of \( \pi_{*,*} \). For example, the relation \( \eta \sigma^2 = 0 \) discussed in Example 4.1.1 does not fit into this formal framework.

Something even more complicated occurs with the relation \( h_5^3 + h_2^2 h_3 = 0 \) in the \( E_\infty \)-page. There is a hidden relation here, in the sense that \( \nu \beta + \eta^2 \sigma \) does not equal zero; rather, it equals \( \eta c \) \( c \). We do not attempt to systematically address these types of compound relations.

4.1.2. Hidden Adams \( \tau \) extensions. For hidden \( \tau \) extensions, the key tool is the homotopy of the cofiber \( C \tau \) of \( \tau \). This calculation is fully explored in Chapter 5. Let \( \alpha \) be an element of \( \pi_{*,*} \). Then \( \alpha \) maps to zero under the inclusion \( S^{0,0} \rightarrow C \tau \) of the bottom cell if and only if \( \alpha \) is divisible by \( \tau \) in \( \pi_{*,*} \).

Proposition 4.1.8. Table 22 shows some hidden \( \tau \) extensions in \( \pi_{*,*} \), through the 59-stem. These are the only hidden \( \tau \) extensions in this range, with the possible exceptions that there might be hidden \( \tau \) extensions:

1. from \( h_1 i_1 \) to \( h_1 B_8 \).
2. from \( j_1 \) to \( B_{21} \).

Proof. Table 22 cites one possible argument for each hidden \( \tau \) extension. These arguments break into two types:

1. In many cases, we know from Chapter 5 that an element of \( \{ \nu \} \) maps to zero in \( \pi_{*,*}(C \tau) \), where \( C \tau \) is the cofiber of \( \tau \). Therefore, this element of \( \{ \nu \} \) is divisible by \( \tau \) in \( \pi_{*,*} \), which implies that there must be a hidden \( \tau \) extension. Usually there is just one possible hidden \( \tau \) extension.
2. Other more difficult cases are proved in Section 4.2.1.
For many of the possible hidden $\tau$ extensions from $b$ to $b'$, we know from Chapter 5 that none of the elements of $\{b'\}$ map to zero in $\pi_{*,*}(C\tau)$. Therefore, none of the elements of $\{b'\}$ is divisible by $\tau$, so none of these possible hidden $\tau$ extensions are actual hidden $\tau$ extensions. A number of more difficult non-existence proofs are given in Section 4.2.1.

In order to maintain the flow of the narrative, we have collected the technical computations of hidden extensions in Section 4.2.1.

Remark 4.1.9. Table 29 shows some additional hidden $\tau$ extensions in stems 60 through 69. These results are tentative because the analysis of the $E_{\infty}$-page is incomplete in this range. Tentative proofs in Section 4.2.1 are clearly indicated.

Remark 4.1.10. We show in Lemma 4.2.6 that there is no hidden $\tau$ extension on $h_1g_2$. This contradicts the claim in [23] that there is a classical hidden $\eta$ extension from $h_1g_2$ to $N$. We do not understand the source of this discrepancy. See also Remark 4.1.22.

Remark 4.1.11. There may be a hidden $\tau$ extension from $h_1i_1$ to $h_1B_8$. This extension occurs if and only if $d_3(h_1i_1)$ equals $h_1B_8$ in the Adams spectral sequence for the cofiber of $\tau$ (see Proposition 5.2.11). If this extension occurs, then it implies that there is a hidden relation $\nu(C) + \tau\{i_1\} = \{B_8\}$.

Remark 4.1.12. We show in Lemma 4.2.7 that there is no hidden $\tau$ extension on $D_{11}$. This proof is different in spirit from the rest of this manuscript because it uses specific calculations in the classical Adams-Novikov spectral sequence. This is especially relevant since Chapter 6 uses the calculations here to derive Adams-Novikov calculations, so there is some danger of circular arguments. We would prefer to have a proof that is internal to the motivic Adams spectral sequence.

Remark 4.1.13. Remark 3.2.17 explains that the following three claims are equivalent:

1. there is a hidden $\tau$ extension from $j_1$ to $B_{21}$.
2. $d_4(j_1) = B_{21}$ in the motivic Adams spectral sequence for the cofiber of $\tau$.
3. $d_4(C') = h_2B_{21}$ in the motivic Adams spectral sequence for the sphere.

4.1.3. Hidden Adams 2 extensions.

Proposition 4.1.14. Table 27 shows some hidden 2 extensions in $\pi_{*,*}$, through the 59-stem. These are the only hidden 2 extensions in this range, with the possible exceptions that there might be hidden 2 extensions:

1. from $h_0h_3g_2$ to $\tau g_n$.
2. from $j_1$ to $\tau^2c_1g^2$.

Proof. Table 27 cites one possible argument (but not necessarily the earliest published result) for each hidden 2 extension. One extension follows from its classical analogue given in Table 24. The remaining cases are proved in Section 4.2.2.

A number of non-existence proofs are given in Section 4.2.2.

In order to maintain the flow of the narrative, we have collected the technical computations of various hidden 2 extensions in Section 4.2.2.
Remark 4.1.15. Table 28 shows some additional hidden 2 extensions in stems 60 through 69. These results are tentative because the analysis of the $E_{\infty}$-page is incomplete in this range. Tentative proofs in Section 4.2.2 are clearly indicated.

Remark 4.1.16. Recall from Table 24 that there is a hidden 4 extension from $h_2h_5$ to $h_0h_5d_0$. It is tempting to consider this as a hidden 2 extension from $h_0h_2^2h_5$ to $h_0h_5d_0$, but this is not consistent with Definition 4.1.2.

Remark 4.1.17. There is a possible hidden 2 extension from $h_0h_3g_2$ to $\tau gn$. We show in Lemma 4.2.31 that this hidden extension occurs if and only if there is a hidden $\nu$ extension from $h_2h_5d_0$ to $\tau gn$. Lemma 4.2.31 is inconsistent with results of [23], which indicates the hidden $\nu$ extension but not the hidden 2 extension. We do not understand the source of this discrepancy.

Remark 4.1.18. We show in Lemma 4.2.35 that there is a hidden 2 extension from $h_0h_5i$ to $\tau^3e_5g$. This proof is different in spirit from the rest of this manuscript because it uses specific calculations in the classical Adams-Novikov spectral sequence. This is especially relevant since Chapter 6 uses the calculations here to derive Adams-Novikov calculations, so there is some danger of circular arguments. We would prefer to have a proof that is internal to the motivic Adams spectral sequence.

We point out one other remarkable property of this hidden extension. Up to the 59-stem, it is the only example of a 2 extension that is hidden in both the Adams spectral sequence and the Adams-Novikov spectral sequence. (There are several $\eta$ extensions and $\nu$ extensions that are hidden in both spectral sequences.)

4.1.4. Hidden Adams $\eta$ extensions.

Proposition 4.1.19. Table 29 shows some hidden $\eta$ extensions in $\pi_{*,*}$, through the 59-stem. These are the only hidden $\eta$ extensions in this range, with the possible exceptions that there might be a hidden $\eta$ extension from $\tau h_1Q_2$ to $\tau B_{21}$.

Proof. Table 29 cites one possible argument (but not necessarily the earliest published result) for each hidden $\eta$ extension. These arguments break into three types:

1. Some hidden extensions follow from the calculation of the image of $J$ [2].
2. Some hidden extensions follow from their classical analogues given in Table 23.
3. The remaining more difficult cases are proved in Section 4.2.3.

A number of non-existence proofs are given in Section 4.2.3. □

In order to maintain the flow of the narrative, we collect the technical results establishing various hidden $\eta$ extensions in Section 4.2.3.

Remark 4.1.20. Table 30 shows some additional hidden $\eta$ extensions in stems 60 through 69. These results are tentative because the analysis of the $E_{\infty}$-page is incomplete in this range. Tentative proofs in Section 4.2.3 are clearly indicated.

Remark 4.1.21. We show in Lemma 4.2.47 that there is no hidden $\eta$ extension on $\tau h_1g_2$. This contradicts the claim in [23] that there is a classical hidden $\eta$ extension from $h_1g_2$ to $N$. We do not understand the source of this discrepancy.
Remark 4.1.22. The element \( \eta^2 \{g_2\} \) is considered in [3, Lemma 4.3], where it is shown to be equal to \( \sigma^2 \{d_1\} \). Our results indicate that both are zero classically; this is consistent with a careful reading of [3, Lemma 4.3].

Motivically, \( \eta^2 \{g_2\} = \sigma^2 \{d_1\} \) is non-zero because they are detected by \( h_1g_2 = h_3d_1 \). However, Lemma 4.2.47 implies that \( \tau \eta^2 \{g_2\} \) and \( \tau \sigma^2 \{d_1\} \) are both zero.

Remark 4.1.23. We show in Lemma 4.2.52 that there is no hidden \( \eta \) extension on \( C \). This contradicts the claim in [23] that there is a classical hidden \( \eta \) extension from \( C \) to \( gn \). We do not understand the source of this discrepancy.

4.1.5. Hidden Adams \( \nu \) extensions.

Proposition 4.1.24. Table 31 shows some hidden \( \nu \) extensions in \( \pi^* \), \( * \), through the 59-stem. These are the only hidden \( \nu \) extensions in this range, with the possible exceptions that there might be hidden \( \nu \) extensions:

(1) from \( h_2h_5d_0 \) to \( gn \).
(2) from \( i_1 \) to \( gt \).

Proof. Table 31 cites one possible argument (but not necessarily the earliest published result) for each hidden \( \nu \) extension. These arguments break into two types:

(1) Some hidden extensions follow from their classical analogues given in Table 31.
(2) The remaining more difficult cases are proved in Section 4.2.4.

A number of non-existence proofs are given in Section 4.2.4. □

In order to maintain the flow of the narrative, we have collected the technical results establishing various hidden \( \nu \) extensions in Section 4.2.4.

Remark 4.1.25. Table 32 shows some additional hidden \( \nu \) extensions in stems 60 through 69. These results are tentative because the analysis of the \( E_\infty \)-page is incomplete in this range. Tentative proofs in Section 4.2.4 are clearly indicated.

Remark 4.1.26. We draw the reader’s attention to the curious hidden \( \nu \) extensions on \( h_2c_1, h_2c_1g, \) and \( N \). These are “exotic” extensions that have no classical analogues. The hidden extension on \( N \) contradicts the claim in [23] that there is a hidden \( \eta \) extension from \( h_1g_2 \) to \( N \). In addition to the proof provided in Lemma 4.2.63 one can also establish these hidden extensions by computing in the motivic Adams spectral sequence for the cofiber of \( \nu \). One can show that \( \{h_1^2h_4c_0\}, \{Ph_1^2h_5c_0\}, \) and \( \{h_1^3h_5c_0\} \) all map to zero in the cofiber of \( \nu \), which implies that they are divisible by \( \nu \).

Remark 4.1.27. There is a possible hidden \( \nu \) extension from \( h_2h_5d_0 \) to \( gn \). We show in Lemma 4.2.31 that this hidden extension occurs if and only if there is a hidden 2 extension from \( h_0h_3g_2 \) to \( gn \). Lemma 4.2.31 is inconsistent with results of [23], which indicates the hidden \( \nu \) extension but not the hidden 2 extension. We do not understand the source of this discrepancy.

4.2. Hidden Adams extensions computations

In this section, we collect the technical computations that establish the hidden extensions discussed in Section 4.1.
4.2.1. Hidden Adams $\tau$ extensions computations.

**Lemma 4.2.1.**

1. There is a hidden $\tau$ extension from $h_1h_3g$ to $d_0^2$.
2. There is a hidden $\tau$ extension from $h_1h_3g^2$ to $d_0c_0^2$.

**Proof.** We will show in Lemma 4.2.85 that $c\kappa = \kappa^2$ in $\pi_{28,16}$. Therefore, $\kappa^2$ is contained in $\pi(2,\nu^2,\eta)$.

Let $C\tau$ be the cofiber of $\tau$, whose homotopy is studied thoroughly in Chapter 4. Therefore, $\langle \kappa \rangle$ cannot belong to $\langle \kappa \rangle$. This completes the proof of the first claim.

Now $\pi_{27,15}(C\tau)$ consists only of the element $\{P^3h_1^3\}$. However, this element cannot belong to $\langle \kappa \rangle$ because $\{P^3h_1^3\}$ supports infinitely many multiplications by $\eta$, while the elements in the bracket cannot. Therefore, $\langle \kappa \rangle$ must be zero, and the image of $\kappa^2$ in $\pi_{28,16}(C\tau)$ is zero.

Therefore, $\kappa^2$ in $\pi_{28,14}$ is divisible by $\tau$, and there is just one possible hidden $\tau$ extension. This completes the proof of the second claim. The proof for the second claim is analogous, using that $\kappa = c\kappa^2$ from Lemma 4.2.85. The bracket $\langle \{\tau g^2\}_C\tau, 2, \nu^2 \rangle$ in $\pi_{47,27}(C\tau)$ must be zero because there are no other possibilities. \hfill $\square$

**Lemma 4.2.2.**

1. There is no hidden $\tau$ extension on $h_1d_1$.
2. There is no hidden $\tau$ extension on $h_1d_1g$.

**Proof.** For the first formula, the only other possibility is that there is a hidden $\tau$ extension from $h_1d_1$ to $h_1g$. We will show that this is impossible.

Proposition 4.2.5 shows that the element $\{d_1\}$ of $\pi_{32,18}$ is detected in Adams-Novikov filtration 4. Therefore, $\{d_1\}$ realizes to zero in $\pi_{32,tmf}$, so $\tau\eta\{d_1\}$ also realizes to zero in $\pi_{32,tmf}$.

On the other hand, $\{h_1c_0^2\}$ realizes to a non-zero element of $\pi_{32,tmf}$. The classical hidden extension $\nu\{g\} = \{h_1c_0^2\}$ given in Table 24 then implies that $\{g\}$ realizes to a non-zero element of $\pi_{32,tmf}$. Then $\{h_1q\}$ also realizes to a non-zero element of $\pi_{32,tmf}$.

This shows that $\tau\eta\{d_1\}$ cannot belong to $\{h_1q\}$, so it must be zero. Now Lemma 4.4.4 establishes the first claim.

For the second claim, Table 23 shows that $\{d_1g\} = \langle\{d_1\}, \eta^3, \eta_4\rangle$, again with no indeterminacy. Now shuffle to obtain $\tau\eta\{d_1g\} = \langle\eta_1, \{d_1\}, \eta^3\rangle\eta_4$. The element $\{\tau\eta_1c_0^2\}$ is the only non-zero element that could possibly be contained in $\langle\eta_1, \{d_1\}, \eta^3\rangle$. In any case, $\langle\eta_1, \{d_1\}, \eta^3\rangle\eta_4$ is zero. This shows that $\tau\eta\{d_1g\}$ is zero. Lemma 4.4.4 establishes the second claim. \hfill $\square$

**Lemma 4.2.3.** There is a hidden $\tau$ extension from $\tau_0g^2$ to $h_1u$.

**Proof.** Classically, there is a hidden 2 extension from $g^2$ to $h_1u$ given in Table 24. This implies that there is a motivic hidden 2 extension from $\tau^2g^2$ to $h_1u$. The desired hidden $\tau$ extension follows. \hfill $\square$

**Lemma 4.2.4.**

1. There is a hidden $\tau$ extension from $\tau h_1g$ to $z$.
2. There is a hidden $\tau$ extension from $\tau h_1c_0g$ to $d_0z$.
PROOF. There is a classical hidden \( \eta \) extension from \( g^2 \) to \( z \) given in Table 24. It follows that there is a motivic hidden \( \eta \) extension from \( \tau^3 g^2 \) to \( z \). The first claim follows immediately.

For the second claim, multiply the first hidden extension by \( d_0 \).

\[ \text{Lemma 4.2.5. There is no hidden } \tau \text{ extension on } h^2_1 g_2. \]

PROOF. We will show in Lemma 4.2.47 that there is no hidden \( \eta \) extension on \( \tau h_1 g_2 \). This implies that there is no hidden \( \tau \) extension on \( h^2_1 g_2 \). \( \Box \)

\[ \text{Lemma 4.2.6. There is no hidden } \tau \text{ extension on } \tau h_2 d_1 g. \]

PROOF. The only other possibility is that there is a hidden \( \tau \) extension from \( \tau h_2 d_1 g \) to \( d_0 z \). However, we showed in Lemma 4.2.4 that there is a hidden \( \tau \) extension from \( \tau h_1 e_0^0 \) to \( d_0 z \). Since \( \{ \tau h_1 e_0^0 \} \) is contained in the indeterminacy of \( \{ \tau h_2 d_1 g \} \), there exists an element of \( \{ \tau h_2 d_1 g \} \) that is annihilated by \( \tau \). Lemma 4.1.4 finishes the proof. \( \Box \)

\[ \text{Lemma 4.2.7. There is no hidden } \tau \text{ extension on } D_{11}. \]

PROOF. This proof is different in spirit from the rest of the manuscript because it relies on specific calculations in the classical Adams-Novikov spectral sequence.

There is an element \( \beta_{12/6} \) in the Adams-Novikov spectral sequence in the 58-stem with filtration 2 \([37]\). Using Proposition 6.2.7, if this class survives, then it would correspond to an element of \( \pi_{58,30} \) that is not divisible by \( \tau \). By inspection of the \( E_{\infty} \)-page of the motivic Adams spectral sequence, there is no such element in \( \pi_{58,30} \). Therefore, \( \beta_{12/6} \) must support a differential in the Adams-Novikov spectral sequence.

Using the framework of Chapter 6, an Adams-Novikov \( d_{2r+1} \) differential on \( \beta_{12/6} \) would correspond to an element of \( \pi_{57, r+30} \) that is not divisible by \( \tau \); that is not killed by \( \tau^r \); and that is annihilated by \( \tau^r \). By inspection of the \( E_{\infty} \)-page of the motivic Adams spectral sequence, the only possibility is that \( r = 1 \), and the corresponding element of \( \pi_{57,31} \) is detected by \( D_{11} \).

\[ \text{Lemma 4.2.8. There is no hidden } \tau \text{ extension on } h^2_2 g_2. \]

PROOF. The only other possibility is that there is a hidden \( \tau \) extension from \( h^2_2 g_2 \) to \( h_1 Q_2 \). However, \( \eta \{ h_1 Q_2 \} \) equals \( \{ h^2_2 Q_2 \} \), which is non-zero. On the other hand, \( \{ h^2_2 g_2 \} \) contains the element \( \sigma^2 \{ g_2 \} \). This is annihilated by \( \eta \) because \( \eta \sigma^2 = 0 \) \([41]\). It follows that \( \tau \{ h^2_2 g_2 \} \) cannot intersect \( \{ h_1 Q_2 \} \).

\[ \text{Lemma 4.2.9. There is no hidden } \tau \text{ extension on } h_3 d_1 g. \]

PROOF. The only other possibility is that there is a hidden \( \tau \) extension from \( h_3 d_1 g \) to \( Ph^3 h_5 e_0 \). We will argue that this cannot occur.

Let \( \alpha \) be an element of \( \{ h_3 d_1 g \} \). Note that \( \alpha \) equals either \( \sigma \{ d_1 g \} \) or \( \sigma \{ d_1 g \} + \nu \{ g t \} \). In either case, \( \eta \alpha \) equals \( \eta \sigma \{ d_1 g \} \), which is a non-zero element of \( \{ h_1 h_3 d_1 g \} \). We know from Lemma 4.2.2 that \( \tau \eta \{ d_1 g \} \) is zero, so \( \tau \eta \alpha \) is zero.

On the other hand, \( \eta \{ Ph^3 h_5 e_0 \} \) equals \( \{ \tau^2 h_0 g^3 \} \), which is non-zero. Therefore, \( \tau \alpha \) cannot equal \( \{ Ph^3 h_5 e_0 \} \).

\[ \text{Lemma 4.2.10. There is a hidden } \tau \text{ extension from } Ph^3 h_5 e_0 \text{ to } \tau d_0 w. \]
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**Proof.** Table 23 shows that \( \langle \tau, \nu \eta^2, \eta \rangle \) in \( \pi_{45,24} \) contains the element \( \{ \tau w \} \). This bracket has indeterminacy generated by \( \tau \eta \{ g_2 \} \). Table 23 also shows that the bracket \( \langle \nu \eta^2, \eta, \eta \kappa \rangle \) equals \( \{ P h^4_1 h^5_{50} \} \), with no indeterminacy.

Now use the shuffle \( \tau \langle \nu \eta^2, \eta, \eta \kappa \rangle = \langle \tau, \nu \eta^2, \eta \rangle \eta \kappa \) to conclude that \( \tau \{ P h^4_1 h^5_{50} \} \) equals \( \eta \kappa \{ \tau w \} \).

There is a classical extension \( \eta \{ w \} = \{ d_0l \} \), as shown in Table 24. It follows that there is a motivic relation \( \eta \kappa \{ \tau w \} = \{ \tau d_0^5 + d_0 w' \} \); in particular, it is non-zero.

We have shown that \( \tau \{ P h^4_1 h^5_{50} \} \) is non-zero. But this equals \( \tau \eta \{ P h^4_1 h^5_{50} \} \), so \( \tau \{ P h^4_1 h^5_{50} \} \) is also non-zero. There is just one possible non-zero value. \( \square \)

**Lemma 4.2.11.** There is no hidden \( \tau \) extension on \( \tau^2 c_1 g^2 \).

**Proof.** The only other possibility is that there is a hidden \( \tau \) extension from \( \tau^2 c_1 g^2 \) to \( \tau d_0 w \). We showed in Lemma 4.2.10 that there is a hidden \( \tau \) extension from \( P h^4_1 h^5_{50} \) to \( \tau d_0 w \). Since \( \{ P h^4_1 h^5_{50} \} \) is contained in the indeterminacy of \( \{ \tau^2 c_1 g^2 \} \), there exists an element of \( \{ \tau^2 c_1 g^2 \} \) that is annihilated by \( \tau \).

**Lemma 4.2.12.** Tentatively, there is a hidden \( \tau \) extension from \( h^2_1 X_2 \) to \( \tau B_{23} \).

**Proof.** The claim is tentative because our analysis of Adams differentials is incomplete in the relevant range.

There exists an element of \( \{ \tau B_{23} \} \) that maps to zero in the homotopy groups of the cofiber of \( \tau \), which is described in Chapter 5. Therefore, this element of \( \{ \tau B_{23} \} \) is divisible by \( \tau \). The only possibility is that there is a hidden \( \tau \) extension from \( h^2_1 X_2 \) to \( \tau B_{23} \). \( \square \)

**Lemma 4.2.13.** Tentatively, there is a hidden \( \tau \) extension from \( h^4_1 X_2 \) to \( B_8 d_0 \).

**Proof.** The claim is tentative because our analysis of Adams differentials is incomplete in the relevant range.

This follows from the hidden \( \tau \) extension from \( h^2_1 X_2 \) to \( \tau B_{23} \) given in Lemma 4.2.12 and the hidden \( \eta \) extension from \( \tau h_1 B_{23} \) to \( B_8 d_0 \) given in Lemma 4.2.60. \( \square \)

**Lemma 4.2.14.** Tentatively, there is a hidden \( \tau \) extension from \( B_8 d_0 \) to \( d_0 x' \).

**Proof.** The claim is tentative because our analysis of Adams differentials is incomplete in the relevant range.

This follows immediately from the hidden \( \tau \) extension from \( B_8 \) to \( x' \) given in Table 25. \( \square \)

### 4.2.2. Hidden Adams 2 extensions computations.

**Lemma 4.2.15.** There is no hidden 2 extension on \( h^2_3 h^4_4 \).

**Proof.** The only other possibility is that there is a hidden 2 extension from \( h^2_3 h^4_4 \) to \( \tau h_1 g \). However, we will show later in Lemma 4.2.39 that there is a hidden \( \eta \) extension on \( \tau h_1 g \). \( \square \)

**Lemma 4.2.16.** There is no hidden 2 extension on \( h_4 c_0 \).

**Proof.** We showed in Lemma 4.2.83 that \( \sigma \eta_4 \) belongs to \( \{ h_4 c_0 \} \), and \( 2 \eta_4 \) equals zero. Now use Lemma 4.1.4 to finish the claim. \( \square \)

**Lemma 4.2.17.**

1. There is a hidden 2 extension from \( h_0 h_2 g \) to \( h_1 c_0 d_0 \).
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(2) There is a hidden 2 extension from $\tau h_0 h_2 g$ to $Ph_1 d_0$.
(3) There is a hidden 2 extension from $h_0 h_2 g^2$ to $h_1 c_0 e_0^3$.
(4) There is a hidden 2 extension from $\tau h_0 h_2 g^2$ to $h_1 d_0^2$.

Proof. Recall that there is a classical hidden 2 extension from $h_0 h_2 g$ to $Ph_1 d_0$, as shown in Table 24. This immediately implies the second claim. The first formula now follows from the second, using the hidden $\tau$ extension from $h_1 c_0 d_0$ to $Ph_1 d_0$ given in Table 25.

For the last two formulas, recall that there is a classical hidden extension $\eta^2 = \{d_0^3\}$, as shown in Table 24. This implies that there is a motivic hidden extension $\tau \eta^2 \{\tau g^2\} = \{h_1 d_0^3\}$. Use the relation $\tau \eta^2 = 4 \nu$ to deduce the fourth formula.

The third formula follows from the fourth, using the hidden $\tau$ extension from $h_1 c_0 e_0^3$ to $h_1 d_0^2$ given in Table 25.

Lemma 4.2.18.
(1) There is no hidden 2 extension on $h_1 h_5$.
(2) There is no hidden 2 extension on $h_1 h_3 h_5$.

Proof. For the first claim, the only other possibility is that there is a hidden 2 extension from $h_1 h_5$ to $\tau d_1$. Table 23 shows that the Toda bracket $\langle \eta, 2, \theta_4 \rangle$ intersects $\{h_1 h_5\}$. Shuffle to obtain

$$2\langle \eta, 2, \theta_4 \rangle = \langle 2, \eta, 2 \rangle \theta_4.$$ 

This expression equals $\tau \eta^2 \theta_4$ by Table 24, which must be zero. Lemma 4.1.4 now finishes the first claim.

The second claim follows easily since $\sigma \eta_5$ is contained in $\{h_1 h_3 h_5\}$.

Lemma 4.2.19. There is no hidden 2 extension on $p$.

Proof. The only other possibility is that there is a hidden 2 extension from $p$ to $h_1 g$. Table 24 shows that $\nu \theta_4$ is contained in $\{p\}$. Also, $2 \theta_4$ is zero. Lemma 4.1.4 now finishes the proof.

Lemma 4.2.20.
(1) There is no hidden 2 extension on $h_2 d_1$.
(2) There is no hidden 2 extension on $h_3 d_1$.
(3) There is no hidden 2 extension on $h_2 d_1 g$.
(4) There is no hidden 2 extension on $h_3 d_1 g$.

Proof. These follow immediately from Lemma 4.1.4 together with the facts that $\nu \{d_1\}$ is contained in $\{h_2 d_1\}$; $\sigma \{d_1\}$ is contained in $\{h_3 d_1\}$; $\nu \{d_1 g\}$ is contained in $\{h_2 d_1 g\}$; $\sigma \{d_1 g\}$ is contained in $\{h_3 d_1 g\}$; and $2 \{d_1\}$ and $2 \{d_1 g\}$ are both zero.

Lemma 4.2.21. There is no hidden 2 extension on $h_5 c_0$.

Proof. The only other possibility is that there is a hidden 2 extension from $h_5 c_0$ to $\tau^2 c_1 g$. Table 23 shows that $\langle \epsilon, 2, \theta_4 \rangle$ intersects $\{h_5 c_0\}$. Now shuffle to obtain that

$$2\langle \epsilon, 2, \theta_4 \rangle = \langle 2, \epsilon, 2 \rangle \theta_4.$$ 

By Table 23 this equals $\tau \eta \epsilon \theta_4$, which must be zero. Lemma 4.1.4 now finishes the argument.
Lemma 4.2.22. There is no hidden 2 extension on $Ph_1h_5$.

Proof. The other possibilities are hidden 2 extensions to $\tau^3g^2$ or $h_1u$. We will show that neither can occur.

We already know from Table 24 that there is a hidden 2 extension from $\tau^3g^2$ to $h_1u$. Therefore, there cannot be a hidden 2 extension from $Ph_1h_5$ to $h_1u$.

Table 29 shows a hidden $\eta$ extension from $\tau^3g^2$ to $z$. This implies that $\tau^3g^2$ cannot be the target of a hidden 2 extension.

Lemma 4.2.23. There is no hidden 2 extension on $h_0^5h_5d_0$.

Proof. As shown in Table 24, $\tau w$ supports a hidden $\eta$ extension. Therefore, it cannot be the target of a hidden 2 extension.

Lemma 4.2.24. There is no hidden 2 extension on $h_2g_2$.

Proof. From the relation $h_2g_2 + h_3f_1 = 0$ in the $E_2$-page, we know that $\{h_2g_2\}$ contains $\sigma\{f_1\}$. Also, $\{f_1\}$ contains an element that is annihilated by 2, so $\{h_2g_2\}$ contains an element that is annihilated by 2. Lemma 4.1.4 finishes the argument.

Lemma 4.2.25. There is no hidden 2 extension on $Ph_5c_0$.

Proof. The element $Ph_5c_0$ detects $p_{15}\eta_5$ [Lemma 2.5]. Also, $2\eta_5$ is zero, so $\{Ph_5c_0\}$ contains an element that is annihilated by 2. Lemma 4.1.4 finishes the proof.

Lemma 4.2.26.

1. There is a hidden 2 extension from $e_0r$ to $h_1u'$.
2. There is a hidden 2 extension from $\tau e_0r$ to $Pu$.

Proof. Table 24 shows that there is a hidden $\eta$ extension from $\tau w$ to $\tau d_0l + u'$. Also, from Table 26, there is a hidden $\tau$ extension from $h_1u'$ to $Pu$. Therefore, $\tau\eta^2\{\tau w\} = \{Pu\}$.

Recall from Table 23 that $\tau\eta^2 = \langle 2, \eta, 2 \rangle$. Since $2\{\tau w\}$ is zero, we can shuffle to obtain

$$\tau\eta^2\{\tau w\} = \langle 2, \eta, 2 \rangle\{\tau w\} = 2\{\eta, 2, \{\tau w\}\}.$$ 

This shows that $\{Pu\}$ is divisible by 2.

By Lemmas 4.2.24 and 4.2.23, the only possibility is that there is a hidden 2 extension from $\tau e_0r$ to $Pu$. This establishes the second claim.

The first claim now follows from the second, using the hidden $\tau$ extension from $h_1u' \to Pu$ given in Table 26.

Lemma 4.2.27. There is no hidden 2 extension on $h_2h_5d_0$.

Proof. There is an element of $2\{h_5d_0\}$ that is divisible by 4, as shown in Table 24. Therefore, there is an element of $2\nu\{h_5d_0\}$ that is divisible by 4. However, zero is the only element of $\pi_{48,26}$ that is divisible by 4.

Lemma 4.2.28. There is no hidden 2 extension on $h_0B_2$.

Proof. We will show later in Lemma 4.2.73 that there exists an element $\alpha$ of $\{h_0^2h_5\}$ such that $\nu\alpha$ belongs to $\{B_2\}$. (We do not know whether $\alpha$ equals $\theta_{1,5}$, but that does not matter here. See Section 1.17 for further discussion.) Therefore, $2\nu\alpha$ belongs to $\{h_0B_2\}$.
Now \(2 \cdot 2\alpha\) equals \(\tau_3\alpha\). There is a classical relation \(\eta_3\alpha = 0\) [3, Lemma 3.5], which implies that \(\eta_3\alpha\) equals zero motivically as well. □

**Lemma 4.2.29.**

1. There is no hidden 2 extension on \(h_5c_1\).
2. There is no hidden 2 extension on \(h_2h_5c_1\).

**Proof.** Table 23 show that \(\langle \sigma, 2, \theta_4 \rangle\) intersects \(\{h_5c_1\}\). Now shuffle to obtain \(2\langle \sigma, 2, \theta_4 \rangle = \langle 2, \sigma, 2 \rangle \theta_4\).

Table 23 shows that \(\langle 2, \sigma, 2 \rangle\) consists of multiples of 2, and \(2\theta_4\) is zero. Therefore, \(2\langle \sigma, 2, \theta_4 \rangle\) is zero. Lemma 4.1.4 now establishes the first claim. □

**Lemma 4.2.30.** There is no hidden 2 extension from \(h_0h_3g_2\) to \(h_2B_2\).

**Proof.** The element \(h_0h_3g_2\) detects \(2\sigma\{g_2\}\). We will show later in Lemma 4.2.75 that \(h_2B_2\) supports a hidden \(\nu\) extension. Therefore, none of the elements of \(\{h_2B_2\}\) are divisible by \(\sigma\). □

**Lemma 4.2.31.** There is a hidden 2 extension on \(h_0h_3g_2\) if and only if there is a hidden \(\nu\) extension on \(h_2h_5d_0\).

**Proof.** Let \(\beta\) be an element of \(\{h_5d_0\}\). Table 23 shows that \(\langle 2, \eta, \eta\beta \rangle\) intersects \(\{h_2h_5d_0\}\), and \(\langle \eta, \eta\beta, \nu \rangle\) intersects \(\{h_0h_3g_2\}\). Now consider the shuffle \(2\langle \eta, \eta\beta, \nu \rangle = \langle 2, \eta, \eta\beta \rangle \nu\).

The indeterminacy here is zero. □

**Lemma 4.2.32.** There is no hidden 2 extension on \(i_1\).

**Proof.** The only other possibility is that there is a hidden 2 extension from \(i_1\) to \(h_1^2G_3\). Because of the hidden \(\tau\) extension from \(h_1^2G_3\) to \(d_0u\) given in Table 23, this would imply a hidden 2 extension from \(\tau i_1\) to \(d_0u\).

However, \(\tau i_1\) detects \(\nu\{C\}\), and \(2\{C\}\) is zero. Therefore, \(\{\tau i_1\}\) contains an element that is annihilated by 2. Lemma 4.1.4 implies that there cannot be a hidden 2 extension on \(\tau i_1\). □

**Lemma 4.2.33.**

1. There is no hidden 2 extension on \(B_8\).
2. There is no hidden 2 extension on \(x'\).

**Proof.** We will show in Lemma 4.2.88 that \(B_8\) detects \(\beta_{4,5}\). Since \(2\epsilon\) is zero, it follows that \(\{B_8\}\) contains an element that is annihilated by 2. Lemma 4.1.4 establishes the first claim.

The second claim follows from the first, using the hidden \(\tau\) extension from \(B_8\) to \(x'\) given in Table 23. □

**Lemma 4.2.34.** There is no hidden 2 extension on \(h_2gn\).

**Proof.** Note that \(\nu\{gn\}\) is contained in \(\{h_2gn\}\), and \(2\{gn\}\) is zero. Therefore, \(\{h_2gn\}\) contains an element that is annihilated by 2, and Lemma 4.1.4 finishes the proof. □
Lemmas 4.2.35. There is a hidden 2 extension from \( h_0 h_5 i \) to \( \tau^4 e_0^g \).

Proof. This proof is different in spirit from the rest of the manuscript because it relies on specific calculations in the classical Adams-Novikov spectral sequence.

The class \( h_0 h_5 i \) detects an element of \( \pi_{34,28} \) that is not divisible by \( \tau \). By Proposition 4.2.34, this corresponds to an element in the classical Adams-Novikov spectral sequence in the 54-stem with filtration 2. The only possibility is the element \( \beta_{10/2} \).

The image of \( \beta_{10/2} \) in the Adams-Novikov spectral sequence for \( tmf \) is \( \Delta^2 h_2^2 \). Since there is no filtration shift, this is detectable in the chromatic spectral sequence. In the Adams-Novikov spectral sequence for \( tmf \), there is a hidden 2 extension from \( \Delta^2 h_2^2 \) to the class that detects \( \kappa \eta^2 \). Therefore, in the Adams-Novikov spectral sequence for the sphere, there must also be a hidden 2 extension from \( \beta_{10/2} \) to the class that detects \( \kappa \eta^2 \).

Since \( \beta_{10/2} \) corresponds to \( h_0 h_5 i \), it follows that in the Adams spectral sequence, there is a hidden 2 extension from \( h_0 h_5 i \) to \( \tau^4 e_0^g \).

Lemma 4.2.36. There is no hidden 2 extension on \( B_{21} \).

Proof. We showed in Lemma 4.2.34 that \( B_{21} \) detects a multiple of \( \kappa \). Since \( 2\kappa \) is zero, it follows that \{\( B_{21} \)\} contains an element that is annihilated by 2. Lemma 4.2.34 finishes the proof.  

Lemma 4.2.37.  
(1) Tentatively, there is a hidden 2 extension from \( \tau^4 g^3 \) to \( d_0 u + \tau d_0^l \).
(2) Tentatively, there is a hidden 2 extension from \( \tau h_0 h_2 g^3 \) to \( h_1 d_0^2 e_0^2 \).
(3) Tentatively, there is a hidden 2 extension from \( \tau c_0 g r \) to \( d_0^2 u \).

Proof. The claims are tentative because our analysis of Adams differentials is incomplete in the relevant range.

The first formula follows immediately from the hidden \( \tau \) extension from \( \tau^2 h_0 g^3 \) to \( d_0 u + \tau d_0^l \) given in Table 20. The second formula follows immediately from the hidden \( \tau \) extension from \( h_0 h_2 g^3 \) to \( h_1 d_0^2 e_0^2 \) given in Table 20. The third formula follows immediately from the hidden \( \tau \) extension from \( h_0 c_0 g r \) to \( d_0^2 u \) given in Table 20.

4.2.3. Hidden Adams \( \eta \) extensions computations.

Lemma 4.2.38. There is no hidden \( \eta \) extension on \( c_1 \).

Proof. The only other possibility is that there is a hidden \( \eta \) extension from \( c_1 \) to \( h_2^g \). We will show in Lemma 4.2.62 that there is a hidden \( \nu \) extension on \( h_2^g \). Therefore, it cannot be the target of a hidden \( \eta \) extension.

Lemma 4.2.39.  
(1) There is a hidden \( \eta \) extension from \( \tau h_1 g \) to \( c_0 d_0 \).
(2) There is a hidden \( \eta \) extension from \( \tau^2 h_1 g \) to \( P d_0 \).
(3) There is a hidden \( \eta \) extension from \( \tau h_1 g^2 \) to \( c_0 e_0^2 \).
(4) There is a hidden \( \eta \) extension from \( z \) to \( \tau d_0^3 \).

Proof. There is a classical hidden \( \eta \) extension from \( h_1 g \) to \( P d_0 \), as shown in Table 20. This implies that there is a motivic hidden \( \eta \) extension from \( \tau^2 h_1 g \) to \( P d_0 \). This establishes the second claim.
The first claim follows from the second claim, using the hidden \( \tau \) extension from \( c_9d_0 \) to \( Pd_0 \) given in Table 25.

Next, there is a classical hidden \( \eta \) extension from \( z \) to \( d_0^3 \), as shown in Table 24. This implies that there is a motivic hidden \( \eta \) extension from \( z \) to \( \tau d_0^3 \). This establishes the fourth claim.

The third claim follows from the fourth, using the hidden \( \tau \) extensions from \( \tau^2h_1g^2 \) to \( z \) and from \( c_9d_0^3 \) to \( d_0^3 \) given in Table 25.

**Lemma 4.2.40.** There is no hidden \( \eta \) extension on \( p \).

**Proof.** Classically, \( \nu\theta \) belongs to \( \{p\} \), as shown in Table 24, so the same formula holds motivically. Therefore, \( \{p\} \) contains an element that is annihilated by \( \eta \). Lemma 4.1.4 finishes the proof.

**Lemma 4.2.41.** There is a hidden \( \eta \) extension from \( h_0^2h_3h_5 \) to \( \tau^2c_1g \).

**Proof.** First, \( \tau^2c_1g \) equals \( h_1y \) on the \( E_2 \)-page [9]. Use Moss’s Convergence Theorem 5.1.1 together with the Adams differential \( d_2(y) = h_0^2x \) to conclude that \( \{\tau^2c_1g\} \) intersects \( \langle \eta, 2, \alpha \rangle \), where \( \alpha \) is any element of \( \{h_0^2x\} \).

However, the later Adams differential \( d_4(h_0h_3h_5) = h_0^2x \) implies that 0 belongs to \( h_0^2x \). Therefore, \( \{\tau^2c_1g\} \) intersects \( \langle \eta, 2, 0 \rangle \). In other words, there exists an element of \( \{\tau^2c_1g\} \) that is a multiple of \( \eta \). The only possibility is that there is a hidden \( \eta \) extension from \( h_0^2h_3h_5 \) to \( \tau^2c_1g \).

**Remark 4.2.42.** Lemma 4.2.41 shows that \( \eta\{h_0^2h_3h_5\} \) is a hidden \( \eta \) extension from \( h_0^2h_3h_5 \) to \( \tau^2c_1g \). However, \( \{\tau^2c_1g\} \) contains two elements because \( u \) is in higher Adams filtration. The sum \( \eta\{h_0^2h_3h_5\} + \tau\sigma \) is either zero or equal to \( \{u\} \). Both \( \{h_0^2h_3h_5\} \) and \( \sigma \) map to zero in \( \pi_{*,*}(tmf) \), while \( \{u\} \) is non-zero in \( \pi_{*,*}(tmf) \). Therefore, \( \eta\{h_0^2h_3h_5\} + \tau\sigma \) must be zero. We will need this observation in Lemmas 5.3.4 and 5.3.8.

**Lemma 4.2.43.** There is no hidden \( \eta \) extension on \( \tau h_3d_1 \).

**Proof.** We know that \( \sigma\{\tau d_1\} \) is contained in \( \{\tau h_3d_1\} \), and there exists an element of \( \{\tau d_1\} \) that is annihilated by \( \eta \). Therefore, \( \{\tau h_3d_1\} \) contains an element that is annihilated by \( \eta \). Lemma 4.1.4 finishes the proof.

**Lemma 4.2.44.** There is no hidden \( \eta \) extension on \( c_1g \).

**Proof.** Since \( \nu\{t\} \) is contained in \( \{\tau c_1g\} \), Lemma 4.1.4 implies that there is no hidden \( \eta \) extension on \( \tau c_1g \). In particular, there cannot be a hidden \( \eta \) extension from \( \tau c_1g \) to \( \tau h_3g^2 \). Therefore, there cannot be a hidden \( \eta \) extension from \( c_1g \) to \( h_0^2g^2 \).

**Lemma 4.2.45.** There is no hidden \( \eta \) extension on \( \tau h_1h_5c_0 \).

**Proof.** Table 29 shows a hidden \( \eta \) extension from \( \tau^3g^2 \) to \( z \). Therefore, there cannot be a hidden \( \eta \) extension from \( \tau h_1h_5c_0 \) to \( z \).

**Lemma 4.2.46.** There is a hidden \( \eta \) extension from \( h_1f_1 \) to \( \tau h_2c_1g \).

**Proof.** Note that \( \{\tau h_2c_1g\} \) contains \( \nu^2\{t\} \). Table 23 shows that \( \nu^2 = \langle \eta, \nu, \eta \rangle \). Shuffle to compute that

\[
\nu^2\{t\} = \langle \eta, \nu, \eta \rangle\{t\} = \eta\{\nu, \eta, \{t\}\},
\]

so \( \nu^2\{t\} \) is divisible by \( \eta \). The only possibility is that there is a hidden \( \eta \) extension from \( h_1f_1 \) to \( \tau h_2c_1g \).
Lemma 4.2.47. There is no hidden $\eta$ extension on $\tau h_2 g_2$.

Proof. We will show in Lemma 4.2.63 that $N$ supports a hidden $\nu$ extension. Therefore, $N$ cannot be the target of a hidden $\eta$ extension.

For degree reasons, $\eta^3 \{g_2\}$ must be zero. Therefore, $\tau \eta^3 \{g_2\}$ must be zero. This implies that the target of a hidden $\eta$ extension on $\tau h_2 g_2$ cannot support an $h_1$ multiplication. Hence, there cannot be a hidden $\eta$ extension from $\tau h_2 g_2$ to $B_1$ or to $\tau d_0 l + u'$.

Lemma 4.2.48. There is a hidden $\eta$ extension from $h_5^2 h_5$ to $B_1$.

Proof. This follows immediately from the analogous classical hidden extension given in Table 24, but we repeat the interesting proof from [40] here for completeness.

First, Table 13 shows that $B_1 = \langle h_1, h_0, h_5^2 g_2 \rangle$. Then Moss’s Convergence Theorem 3.1.1 implies that $\{B_1\}$ intersects $\langle \eta, 2, \{h_5^2 g_2\} \rangle$.

Next, the classical product $\sigma \theta_4$ belongs to $\{x\}$, as shown in Table 24. Since $h_5^2 = h_5^2 g_2$ on the $E_2$-page [9], it follows that $\sigma^2 \theta_4$ equals $\{h_5^2 g_2\}$. The same formula holds motivically.

Now $\langle \eta, 2, \sigma^2 \theta_4 \rangle$ is contained in $\langle \eta, 2 \sigma^2, \theta_4 \rangle$, which equals $\langle \eta, 0, \theta_4 \rangle$. Therefore, $\{B_1\}$ contains an element of the form $\theta_4 \alpha + \eta \beta$.

The possible non-zero values for $\alpha$ are $\eta_4$ or $\eta \rho_{15}$. In the first case, $\theta_4 \alpha$ equals $\theta_4 \langle 2, \sigma^2, \eta \rangle$, which equals $\langle \theta_4, 2, \sigma^2 \rangle \eta$. Therefore, in either case, $\theta_4 \alpha$ is a multiple of $\eta$, so we can assume that $\alpha$ is zero.

We have now shown that $\{B_1\}$ contains a multiple of $\eta$. Because of Lemma 4.2.37, the only possibility is that there is a hidden $\eta$ extension from $h_5^2 h_5$ to $B_1$.

Lemma 4.2.49.

1. There is no hidden $\eta$ extension on $h_1 h_5 d_0$.
2. There is no hidden $\eta$ extension on $N$.

Proof. We showed in Lemma 4.2.26 that there is a hidden 2 extension on $e_0 r$. Therefore, $e_0 r$ cannot be the target of a hidden $\eta$ extension.

Lemma 4.2.50. There is no hidden $\eta$ extension on $h_1 B_1$.

Proof. Classically, there is no hidden $\eta$ extension on $h_1 B_1$ [3] Theorem 3.1(i) and Lemma 3.5]. Therefore, there cannot be a motivic hidden $\eta$ extension from $h_1 B_1$ to $\tau d_0 e_0^2$.

Lemma 4.2.51. There is no hidden $\eta$ extension on $h_5 c_1$.

Proof. Table 25 shows that $\{h_5 c_1\}$ is contained in $\langle \nu, \sigma, \sigma \eta_5 \rangle$. Next compute that

$$\eta \langle \nu, \sigma, \sigma \eta_5 \rangle = \langle \eta, \nu, \sigma \rangle \eta_5,$$

which equals zero because $\langle \eta, \nu, \sigma \rangle$ is zero. Therefore, $\{h_5 c_1\}$ contains an element that is annihilated by $\eta$, so Lemma 3.1.4 says that there cannot be a hidden $\eta$ extension on $h_5 c_1$.

Lemma 4.2.52. There is no hidden $\eta$ extension on $C$. 

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Proof. Table 23 shows that \( \{ C \} \) equals \( \langle \nu, \eta, \tau \eta \alpha \rangle \), where \( \alpha \) is any element of \( \{ g_2 \} \). Compute that

\[
\eta \langle \nu, \eta, \tau \eta \alpha \rangle = \langle \eta, \nu, \eta \rangle \tau \eta \alpha = \nu^2 \cdot \tau \eta \alpha = 0.
\]

This shows that \( \{ C \} \) contains an element that is annihilated by \( \eta \), so Lemma 4.1.4 implies that there cannot be a hidden \( \eta \) extension on \( C \).

Lemma 4.2.53. There is a hidden \( \eta \) extension from \( \tau^2e_0m \) to \( d_0u \).

Proof. This follows immediately from the hidden \( \tau \) extensions from \( h_1G_3 \) to \( \tau^2e_0m \) and from \( h_1^2G_3 \) to \( d_0u \) given in Table 25.

Lemma 4.2.54. There is no hidden \( \eta \) extension on \( \tau i_1 \).

Proof. Suppose that there exists an element \( \alpha \) of \( \{ \tau i_1 \} \) such that \( \eta \alpha \) belongs to \( \{ \tau^2e_0^2g \} \). Using the hidden \( \tau \) extension from \( \tau^2h_1e_0^2g \) to \( d_0z \) given in Table 23, this would imply that \( \tau \eta^2 \alpha \) equals \( \{ d_0z \} \).

Recall from Table 23 that \( \tau \eta^2 = (2, \eta, 2) \). Then the shuffle

\[
\tau \eta^2 \alpha = (2, \eta, 2)\{ \tau i_1 \} = 2\langle \eta, 2, \alpha \rangle
\]

does not show that \( \{ d_0z \} \) is divisible by 2. However, this is not possible.

Lemma 4.2.55. There is a hidden \( \eta \) extension from \( \tau^3e_0^2g \) to \( d_0z \).

Proof. This follows immediately from the hidden \( \tau \) extension from \( \tau^3h_1e_0^2g \) to \( d_0z \) given in Table 25.

Lemma 4.2.56. There is no hidden \( \eta \) extension on \( h_1x' \).

Proof. We already showed in Lemma 4.2.55 that there is a hidden \( \eta \) extension from \( \tau^3e_0^2g \) to \( d_0z \). Therefore, there cannot be a hidden \( \eta \) extension from \( h_1x' \) to \( d_0z \).

Lemma 4.2.57. There is no hidden \( \eta \) extension on \( h_3^2g_2 \).

Proof. Note that \( \sigma^2\{ g_2 \} \) is contained in \( \{ h_3^2g_2 \} \), and \( \eta \sigma^2 \) is zero [41]. Therefore, \( \{ h_3^2g_2 \} \) contains an element that is annihilated by \( \eta \), and Lemma 4.1.4 implies that there cannot be a hidden \( \eta \) extension on \( h_3^2g_2 \).

Lemma 4.2.58. There is no hidden \( \eta \) extension from \( \tau h_1Q_2 \) to \( \tau^2d_0w \).

Proof. Classically, \( d_0w \) maps to a non-zero element in the \( E_\infty \)-page of the Adams spectral sequence for \( \text{tmf} \). In \( \text{tmf} \), this class cannot be the target of an \( \eta \) extension.

Lemma 4.2.59.

1. Tentatively, there is a hidden \( \eta \) extension from \( \tau d_0w \) to \( d_0u' + \tau d_0^3l \).
2. Tentatively, there is a hidden \( \eta \) extension from \( \tau^3g^3 \) to \( d_0e_0r \).
3. Tentatively, there is a hidden \( \eta \) extension from \( \tau^2h_1g^3 \) to \( d_0^2e_0^2g \).
4. Tentatively, there is a hidden \( \eta \) extension from \( d_0e_0r \) to \( \tau^2d_0^2e_0^2 \).
5. Tentatively, there is a hidden \( \eta \) extension from \( \tau^2g^w \) to \( \tau^2d_0e_0m \).
6. Tentatively, there is a hidden \( \eta \) extension from \( \tau^2d_0e_0m \) to \( d_0^u \).
PROOF. The claims are tentative because our analysis of Adams differentials is incomplete in the relevant range.

For the first formula, use the hidden $\tau$ extensions from $Ph_1^3h_5c_0$ to $\tau d_0 w$ given in Lemma 4.2.10 and from $\tau^2 h_0 g^3$ to $d_0 w'$ given in Table 26. The second formula follows immediately from the hidden $\tau$ extension from $\tau^2 h_1 g^2$ to $d_0 e_0 r$ given in Table 26. The third formula follows immediately from the hidden $\tau$ extension from $h_1^2 h_5 c_4 e_0$ to $d_0^2 c_1^2$ given in Table 26. The fourth formula follows immediately from the hidden $\tau$ extensions from $\tau^2 h_1 g^2$ to $d_0 e_0 r$ and from $h_1^2 h_5 c_4 e_0$ to $d_0^2 c_1^2$ given in Table 26. The fifth formula follows immediately from the hidden $\tau$ extension from $h_1^2 X_1$ to $\tau^2 d_0 e_0 m$ given in Table 26. The sixth formula follows immediately from the hidden $\tau$ extensions from $h_1^2 X_1$ to $\tau^2 d_0 e_0 m$ and from $h_0 e_0 r$ to $d_0^2 u$ given in Table 26.

**Lemma 4.2.60.**

1. Tentatively, there is a hidden $\eta$ extension from $\tau h_1 B_{23}$ to $B_3 d_0$

2. Tentatively, there is a hidden $\eta$ extension from $\tau^2 h_1 B_{23}$ to $d_0 x'$

**Proof.** The claims are tentative because our analysis of Adams differentials is incomplete in the relevant range.

The first formula follows from the hidden $\eta$ extension from $\tau h_1 g$ to $c_0 d_0$ given in Lemma 4.2.39 using that $\theta_{4,5}\{\tau h_1 g\}$ is contained in $\{\tau h_1 B_{23}\}$ by Lemma 4.2.84 and that $\theta_{4,5}\{c_0 d_0\}$ is contained in $\{B_3 d_0\}$ by Lemma 4.2.88. The second formula follows from the first, using the hidden $\tau$ extension from $B_3 d_0$ to $d_0 x'$ given in Lemma 4.2.14.

**4.2.4. Hidden Adams $\nu$ extensions computations.**

**Lemma 4.2.61.** There is no hidden $\nu$ extension on $h_0 h_2 h_4$.

**Proof.** This follows immediately from Lemma 4.2.35 where we showed that there is no hidden 2 extension on $h_2^2 h_4$.

**Lemma 4.2.62.**

1. There is a hidden $\nu$ extension from $h_5^2 g$ to $h_1 c_0 d_0$.

2. There is a hidden $\nu$ extension from $\tau h_5 g$ to $P h_1 d_0$.

3. There is a hidden $\nu$ extension from $h_5^2 g^2$ to $h_1 c_0 e_0^2$.

4. There is a hidden $\nu$ extension from $\tau h_5^2 g^2$ to $h_1 d_0^2$.

**Proof.** These follow immediately from the hidden 2 extensions established in Lemma 4.2.17.

**Lemma 4.2.63.**

1. There is a hidden $\nu$ extension from $h_2 c_1$ to $h_1^2 h_4 c_0$.

2. There is a hidden $\nu$ extension from $h_2 c_1 g$ to $h_1^2 h_3 c_0$.

3. There is a hidden $\nu$ extension from $N$ to $P h_2^2 h_5 c_0$.

**Proof.** Table 10 shows that $\langle h_2, h_2 c_1, h_1\rangle$ equals $h_3 g$. This Massey product contains no permanent cycles because $h_3 g$ supports an Adams differential by Lemma 4.3.3. Therefore, (the contrapositive of) Moss’s Convergence Theorem 3.1.3 implies that the Toda bracket $\langle \nu, \nu, \nu, \eta \rangle$ is not well-defined. The only possibility is that $\nu^2 \eta$ is non-zero. This implies that there is a hidden $\nu$ extension on $h_2 c_1$, and the only possible target for this hidden extension is $h_1 h_4 c_0$. This finishes the first claim.
The proof of the second claim is similar. Table 10 show that \( \langle h_2, h_2c_1g, h_1 \rangle \) equals \( h_3g^2 \). Since \( h_3g^2 \) supports a differential by Lemma 3.3.3, Moss’s Convergence Theorem 3.1.1 implies that the Toda bracket \( \langle \nu, \alpha, \eta \rangle \) is not well-defined for any \( \alpha \) in \( \{ h_2c_1g \} \). This implies that there is a hidden \( \nu \) extension on \( h_2c_1g \), and the only possible target is \( h_4^0h_5^0c_0 \). This finishes the second claim.

For the third claim, we will first compute \( \langle h_2, N, h_1 \rangle \) on the \( E_2 \)-page. The May differential \( d_2(\Delta b_{21} h_1(1)) = h_2N \) and May’s Convergence Theorem 2.2.2 imply that \( \langle h_2, N, h_1 \rangle \) equals an element that is detected by \( G_3 \) in the \( E_{\infty} \)-page of the May spectral sequence. Because of the presence of \( \tau gn \) in lower May filtration, the bracket equals either \( G_3 \) or \( G_3 + \tau gn \). In any case, both of these elements support an Adams \( d_2 \) differential by Lemma 3.3.0 because \( \tau gn \) is a product of permanent cycles. Moss’s Convergence Theorem 3.1.1 then implies that the Toda bracket \( \langle \nu, \alpha, \eta \rangle \) is not well-defined for any \( \alpha \) in \( \{ N \} \). This implies that there is a hidden \( \nu \) extension on \( N \), and the only possible target is \( Ph_1^2h_5^0c_0 \). This finishes the third claim.

\[ \text{Lemma 4.2.64.} \]

1. There is a hidden \( \nu \) extension from \( \tau h_2^2g \) to \( h_1d_0^2 \).
2. There is a hidden \( \nu \) extension from \( \tau h_2^2g^2 \) to \( h_1d_0c_0^2 \).

\[ \text{Proof.} \] These follow from the hidden \( \tau \) extensions from \( h_1^2h_3g \) to \( h_1d_0^2 \) and from \( h_1^2h_3g^2 \) to \( h_1d_0c_0^2 \) given in Table 26. \( \square \)

\[ \text{Lemma 4.2.65.} \] There is no hidden \( \nu \) extension on \( h_1h_5 \).

\[ \text{Proof.} \] The only other possibility is that there is a hidden \( \nu \) extension from \( h_1h_5 \) to \( \tau^2h_1c_0^2 \). We know from Table 24 that \( \{ \tau^2h_1c_0^2 \} \) contains \( \nu \{ g \} \). Since \( \{ g \} \) belongs to the indeterminacy of \( \{ h_1h_5 \} \), there exists an element of \( \{ h_1h_5 \} \) that is annihilated by \( \nu \). Lemma 4.1.1 finishes the proof. \( \square \)

\[ \text{Lemma 4.2.66.} \] There is no hidden \( \nu \) extension on \( p \).

\[ \text{Proof.} \] Recall that there is a hidden \( \nu \) extension from \( h_4^2 \) to \( p \), as shown in Table 24. If there were a hidden \( \nu \) extension from \( p \) to \( t \), then \( \nu^4 \theta_4 \) would belong to \( \{ \tau h_2c_1g \} \). This is impossible since \( \nu^4 \) is zero. \( \square \)

\[ \text{Lemma 4.2.67.} \] There is no hidden \( \nu \) extension on \( x \).

\[ \text{Proof.} \] Recall from Table 24 that there is a classical hidden \( \sigma \) extension from \( h_4^2 \) to \( x \). Therefore, \( \sigma \theta_4 \) belongs to \( \{ x \} \) motivically as well, so \( x \) cannot support a hidden \( \nu \) extension. \( \square \)

\[ \text{Lemma 4.2.68.} \] There is no hidden \( \nu \) extension on \( h_2^2h_3h_5 \).

\[ \text{Proof.} \] The only other possibility is that there is a hidden \( \nu \) extension from \( h_2^2h_3h_5 \) to \( Ph_1^2h_5^0 \) or to \( z \). However, from Lemma 4.2.39 both \( \{ Ph_1^2h_5^0 \} \) and \( \{ z \} \) support multiplications by \( \eta \). Therefore, neither \( Ph_1^2h_5^0 \) nor \( z \) can be the target of a hidden \( \nu \) extension. \( \square \)

\[ \text{Lemma 4.2.69.} \]

1. There is no hidden \( \nu \) extension on \( h_1h_3h_5 \).
2. There is no hidden \( \nu \) extension on \( h_3d_1 \).
3. There is no hidden \( \nu \) extension on \( \tau^2c_1g \).
4. There is no hidden \( \nu \) extension on \( h_3g_2 \).
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PROOF. In each case, the possible source of the hidden extension detects an element that is divisible by \( \sigma \). Therefore, each possible source cannot support a hidden \( \nu \) extension.

Note that \( h_3q = \tau^2c_1g \) on the \( E_2 \)-page \( \mathbf{9} \).

**Lemma 4.2.70.** There is no hidden \( \nu \) extension on \( h_5c_0 \).

**Proof.** First, Table 23 shows that \( \{h_5c_0\} \) contains \( \langle \epsilon, 2, \theta_4 \rangle \). Then shuffle to obtain
\[ \nu(\epsilon, 2, \theta_4) = \langle \nu, \epsilon, 2 \rangle \theta_4. \]
Since \( \langle \nu, \epsilon, 2 \rangle \) is zero, there is an element of \( \{h_5c_0\} \) that is annihilated by \( \nu \). Lemma 4.1.4 finishes the proof. \( \square \)

**Lemma 4.2.71.**

(1) There is a hidden \( \nu \) extension from \( u \) to \( \tau d_0^2 \).

(2) There is a hidden \( \nu \) extension from \( \tau w \) to \( \tau^2d_0 e_0^2 \).

**Proof.** First shuffle to compute that
\[ \nu(\eta, \nu, \{\tau^2c_0^2\}) = \langle \nu, \eta, \nu \rangle \{\tau^2c_0^2\} = (\epsilon + \eta \sigma) \{\tau^2c_0^2\}. \]
This last expression equals \( \tau^2\{c_0^2\} \), which equals \( \{\tau d_0^2\} \) because of the hidden \( \tau \) extension from \( c_0^2 \) to \( d_0^2 \) given in Table 25.

Therefore, \( \{\tau d_0^2\} \) is divisible by \( \nu \). Lemmas 4.2.69 and 4.2.70 eliminate most of the possibilities. The only remaining possibility is that there is a hidden \( \nu \) extension on \( u \). This establishes the first claim.

The proof of the second claim is similar. Shuffle to compute that
\[ \nu(\eta, \nu, \tau c_0^2) = \langle \nu, \eta, \nu \rangle \tau c_0^2 = (\epsilon + \eta \sigma) \tau c_0^2 = \tau \epsilon c_0^2. \]
By Lemma 4.2.85, this last expression is detected by \( \tau^2d_0 e_0^2 \).

Therefore, \( \{\tau^2d_0 e_0^2\} \) is divisible by \( \nu \). Because of Lemmas 4.2.72 and 4.2.73, the only possibility is that there is a hidden \( \nu \) extension from \( \tau w \) to \( \tau^2d_0 e_0^2 \). \( \square \)

**Lemma 4.2.72.**

(1) There is no hidden \( \nu \) extension on \( Ph_0h_2h_5 \).

(2) There is no hidden \( \nu \) extension on \( h_0g_2 \).

(3) There is no hidden \( \nu \) extension on \( h_0h_5d_0 \).

**Proof.** These follow immediately from Lemmas 4.2.28, 4.2.29, and 4.2.71. \( \square \)

**Lemma 4.2.73.**

(1) There is a hidden \( \nu \) extension from \( h_0^2h_5 \) to \( B_2 \).

(2) There is a hidden \( \nu \) extension from \( h_0h_5^2h_5 \) to \( h_0B_2 \).

**Proof.** The proof is similar in spirit to the proof of Lemma 4.2.48. Table 11 shows that \( \langle h_2, h_0^2g_2, h_0 \rangle \) equals \( \{B_2, B_2 + h_0^2h_5c_0\} \). Then Moss’s Convergence Theorem 3.1.1 implies that \( \langle \nu, \sigma^2 \theta_4, 2 \rangle \) intersects \( \langle B_2 \rangle \). Here we are using that \( \sigma^2 \theta_4 \) belongs to \( \{h_0^2g_2\} \), as shown in the proof of Lemma 4.2.48.

This bracket contains \( \langle \nu, \sigma^2, 2 \theta_4 \rangle \), which contains zero since \( 2 \theta_4 \) is zero. It follows that \( \{B_2\} \) contains an element in the indeterminacy of \( \langle \nu, \sigma^2 \theta_4, 2 \rangle \). The only possibility is that there is a hidden \( \nu \) extension from \( h_0^2h_5 \) to \( B_2 \). This finishes the proof of the first hidden extension.

The second hidden extension follows immediately from the first. \( \square \)
Lemma 4.2.74. There is no hidden \( \nu \) extension on \( B_1 \).

Proof. We showed in Lemma 4.2.48 that \( \{ B_1 \} \) contains an element that is divisible by \( \eta \). Therefore, \( B_1 \) cannot support a hidden \( \nu \) extension. \( \square \)

Lemma 4.2.75.

(1) There is a hidden \( \nu \) extension from \( h_2B_2 \) to \( h_1B_8 \).

(2) There is a hidden \( \nu \) extension from \( \tau h_2B_2 \) to \( h_1x' \).

Proof. As discussed in Section 1.7, \( \sigma \theta_{4,5} \) is detected in Adams filtration greater than 6. Thus, \( \eta^2\sigma \theta_{4,5} \) is zero, even though \( \sigma \theta_{4,5} \) itself could possibly be detected by \( \tau^2d_1g \) or \( \tau^2e_{0m} \).

Recall from Table 24 that \( \eta^2\sigma + \nu^4 = \eta \epsilon \). Therefore, \( \nu^4\theta_{4,5} \) equals \( \eta \epsilon \theta_{4,5} \). Lemma 4.2.88 implies that \( \eta \epsilon \theta_{4,5} \) is detected by \( h_1B_8 \), so \( \{ h_1B_8 \} \) contains an element that is divisible by \( \nu \). The only possibility is that there must be a hidden \( \nu \) extension from \( h_2B_2 \) to \( h_1B_8 \). This establishes the first claim.

The second claim follows easily from the first, using the hidden \( \tau \) extension from \( h_1B_8 \) to \( h_1x' \) given in Table 25. \( \square \)

Lemma 4.2.76.

(1) There is a hidden \( \nu \) extension from \( h_1G_3 \) to \( \tau^2h_1e_{0g}^2 \).

(2) There is a hidden \( \nu \) extension from \( \tau^2e_{0m} \) to \( d_0z \).

Proof. Table 25 shows that \( \langle \{ q \}, \eta^3, \eta_4 \rangle \) equals \( \{ h_1G_3 \} \), and \( \langle \{ \tau^2h_1e_{0}^2 \}, \eta^3, \eta_4 \rangle \) equals \( \{ \tau^2h_1e_{0}^2g \} \). Neither Toda bracket has indeterminacy; for the second bracket, one needs that \( \eta_4 \{ t \} \) is contained in

\[
\langle \eta, \sigma^2, 2 \rangle \{ t \} = \eta \langle \sigma^2, 2, \{ t \} \rangle,
\]

which must be zero.

Now compute that

\[
\nu \{ h_1G_3 \} = \nu \{ \{ q \}, \eta^3, \eta_4 \} = \langle \{ q \}, \eta^3, \eta_4 \rangle = \langle \{ \tau^2h_1e_{0}^2 \}, \eta^3, \eta_4 \rangle = \{ \tau^2h_1e_{0}^2g \}.
\]

Here we are using that none of the Toda brackets has indeterminacy, and we are using Table 24 to identify \( \nu \{ q \} \) with \( \{ \tau^2h_1e_{0}^2 \} \). This establishes the first claim.

The second claim follows easily from the first, using the hidden \( \tau \) extensions from \( h_1G_3 \) to \( \tau^2e_{0m} \) and from \( \tau^2h_1e_{0}^2g \) to \( d_0z \) given in Table 25. \( \square \)

Lemma 4.2.77. There is no hidden \( \nu \) extension on \( \tau^2d_1g \).

Proof. We showed in Lemma 4.2.70 that there is a hidden \( \nu \) extension from \( \tau^2e_{0m} \) to \( d_0z \). Therefore, there cannot be a hidden \( \nu \) extension from \( \tau^2d_1g \) to \( d_0z \). \( \square \)

Lemma 4.2.78. There is a hidden \( \nu \) extension from \( h_1^6h_5e_0 \) to \( h_2e_{0}^2g \).

Proof. This follows immediately from the hidden \( \tau \) extension from \( h_1^6h_5e_0 \) to \( \tau e_{0}^2g \) given in Table 25. \( \square \)

Lemma 4.2.79. Tentatively, there is a hidden \( \nu \) extension from \( h_0h_2h_5i \) to \( \tau^2d_0^2 \).
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PROOF. The claim is tentative because our analysis of Adams differentials is incomplete in the relevant range.

As explained in the proof of Lemma 4.2.35 the class \( h_0 h_5 i \) detects an element of classical \( \pi_{54} \) that maps to an element of \( \pi_{54} \text{tmf} \) that is detected by \( \Delta^2 h_2^5 \) in the Adams-Novikov spectral sequence for \( \text{tmf} \). Then \( h_0 h_2 h_5 i \) detects an element in \( \pi_{57} \) that maps to an element of \( \pi_{57} \text{tmf} \) that is detected by \( \Delta^2 h_2^5 \) in the Adams-Novikov spectral sequence for \( \text{tmf} \).

In the classical Adams-Novikov spectral sequence for \( \text{tmf} \), there is a hidden \( \nu \) extension from \( \Delta^2 h_2^5 \) to \( 2 g^3 [5] \). Therefore, the corresponding hidden extension must occur in the motivic Adams spectral sequence as well. \( \square \)

**Lemma 4.2.80.**

1. Tentatively, there is a hidden \( \nu \) extension from \( Ph_1^3 h_5 e_0 \) to \( \tau d_0^2 e_0^2 \).
2. Tentatively, there is a hidden \( \nu \) extension from \( \tau d_0 w \) to \( \tau^2 d_0^2 e_0^2 \).
3. Tentatively, there is a hidden \( \nu \) extension from \( \tau g w + h_1^4 X_1 \) to \( \tau^2 e_0^2 \).

PROOF. The claims are tentative because our analysis of Adams differentials is incomplete in the relevant range.

The second formula follows from the hidden \( \nu \) extension from \( \tau w \) to \( \tau^2 d_0 e_0^2 \) given in Lemma 4.2.71. The first formula then follows using the hidden \( \tau \) extension from \( Ph_1^3 h_5 e_0 \) given in Lemma 4.2.10.

For the third formula, start with the hidden \( \nu \) extension from \( \tau w \) to \( \tau^2 d_0 e_0^2 \). Multiply by \( \tau g \) to obtain a hidden \( \nu \) extension from \( \tau^2 g w \) to \( \tau^3 e_0^2 \). The third formula follows immediately. \( \square \)

**Lemma 4.2.81.** Tentatively, there is a hidden \( \nu \) extension from \( \tau h_5^2 g^3 \) to \( h_1 d_0^2 e_0^2 \).

PROOF. The claim is tentative because our analysis of Adams differentials is incomplete in the relevant range.

This follows immediately from the hidden \( \tau \) extension from \( h_5^2 h_2 g^3 \) to \( h_1 d_0^2 e_0^2 \) given in Table 26. \( \square \)

**Lemma 4.2.82.** Tentatively, there is a hidden \( \nu \) extension from \( h_2 c_1 g^2 \) to \( h_4^5 D_4 \).

PROOF. The claim is tentative because our analysis of Adams differentials is incomplete in the relevant range.

The argument is essentially the same as the proof of Lemma 4.2.63. Table 16 shows that \( \langle h_2, h_2 c_1 g^2, h_4 \rangle \) equals \( h_3 g^3 \). Since \( h_3 g^3 \) supports a differential by Lemma 3.3.3 Moss’s Convergence Theorem 3.1.1 implies that the Toda bracket \( \langle \nu, \alpha, \eta \rangle \) is not well-defined for any \( \alpha \) in \( \{ h_2 c_1 g^2 \} \). This implies that there is a hidden \( \nu \) extension on \( h_2 c_1 g^2 \), and the only possible target is \( h_4^5 D_4 \). \( \square \)

### 4.2.5. Miscellaneous Adams hidden extensions

In this section, we include some miscellaneous hidden extensions. They are needed at various points for technical arguments, but they are interesting for their own sakes as well.

**Lemma 4.2.83.** There is a hidden \( \sigma \) extension from \( h_4 h_4 \) to \( h_4 e_0 \).

PROOF. The product \( \eta \sigma \eta \) is contained in \( \{ h_1^2 h_4 e_0 \} \). Now recall the hidden relation \( \eta = \eta^2 \sigma + \nu^3 \) from Table 6. Also \( \nu \eta \) is zero because there is no other possibility. Therefore, \( \eta^2 \sigma \eta \) is contained in \( \{ h_1^2 h_4 e_0 \} \). It follows that \( \sigma \eta \) is contained in \( \{ h_4 e_0 \} \). \( \square \)
Lemma 4.2.84. There is no hidden \( \sigma \) extension on \( h_1h_3h_5 \).

Proof. The element \( \sigma \eta_5 \) belongs to \( \{h_1h_3h_5\} \). We will show that \( \sigma^2 \eta_5 \) is zero and then apply Lemma 4.1.4.

Table 28 shows that \( \eta_5 \) belongs to \( \langle \eta, 2, \theta_4 \rangle \). Then \( \sigma^2 \eta_5 \) belongs to \( \sigma^2 \langle \eta, 2, \theta_4 \rangle = \langle \sigma^2, \eta, 2 \rangle \theta_4 \).

Finally, we must show that \( \langle \sigma^2, \eta, 2 \rangle \) is zero in \( \pi_{16,9} \). First shuffle to obtain \( \langle \sigma^2, \eta, 2 \rangle \eta = \sigma^2 \langle \eta, 2, \eta \rangle \).

Table 28 shows that \( \langle \eta, 2, \eta \rangle \) equals \( \{2\nu, 6\nu\} \), so \( \sigma^2 \langle \eta, 2, \eta \rangle \) is zero. Since multiplication by \( \eta \) is injective on \( \pi_{16,9} \), this shows that \( \langle \sigma^2, \eta, 2 \rangle \) is zero. \( \square \)

Lemma 4.2.85.

1. There is a hidden \( \epsilon \) extension from \( \tau g \) to \( d_0^2 \).
2. There is a hidden \( \epsilon \) extension from \( \tau g^2 \) to \( d_0^2 \).

Proof. Table 28 shows that \( \epsilon \) is contained in \( \{2\nu, \nu, \eta\} \). Therefore, \( \eta \eta \pi \) equals \( \langle 2\nu, \nu, \eta \rangle \eta \pi \) with no indeterminacy. This expression equals \( \langle 2\nu, \nu, \eta \rangle ^2 \pi \) because the latter still has no indeterminacy.

Lemma 4.2.39 tells us that we can rewrite this bracket as \( \langle 2\nu, \nu, \{c_0d_0\} \rangle \), which equals \( \langle 2\nu, \nu, \epsilon \rangle \kappa \). Table 28 shows that \( \{2\nu, \nu, \epsilon \} \) equals \( \eta \kappa \).

Therefore, \( \eta \kappa \pi \) equals \( \eta \kappa^2 \). It follows that \( \epsilon \kappa \pi \) equals \( \kappa^2 \). This establishes the first claim.

The argument for the second claim is essentially the same. Start with \( \eta \kappa \pi \) equals \( \langle 2\nu, \nu, \{c_0d_0\} \rangle \). This shows that \( \eta \kappa \pi \) equals \( \eta \{d_0e_0^2\} \), so \( \epsilon \kappa \pi \) equals \( \{d_0e_0^2\} \). \( \square \)

Remark 4.2.86. Based on the calculations in Lemma 4.2.85 one might expect that there is a hidden \( \epsilon \) extension from \( \tau g^3 + h_1^2h_5c_0e_0 \) to \( e_0^4 \).

Lemma 4.2.87. There is a hidden \( \epsilon \) extension from \( q \) to \( h_1u \).

Proof. This proof follows the argument of the proof of Lemma 2.1, which we include for completeness.

First, recall from Table 28 that \( \epsilon + \eta \sigma \) equals \( \{\nu, \eta, \nu\} \). Then \( \{\epsilon + \eta \sigma\} \{q\} \) equals \( \langle \epsilon + \eta \sigma \rangle \{q\} \) equals \( \{\nu, \eta, \nu\} \{q\} \), which is contained in \( \langle \nu, \eta, \nu, q \rangle \). It follows from Table 24 that \( \nu \{q\} \) equals \( \tau \eta \kappa \pi \), so \( \{\epsilon + \eta \sigma\} \{q\} \) belongs to \( \langle \nu, \eta, \nu, \tau \eta \kappa \pi \rangle \).

On the other hand, this bracket contains \( \langle \nu, \eta, \tau \eta \kappa \rangle \pi \). Table 28 shows that \( \langle \nu, \eta, \tau \eta \kappa \rangle \) equals \( \{\tau h_0g\} = \{2\pi, 6\pi\} \), and \( 4\pi^2 \) is zero. Therefore, \( \langle \nu, \eta, \tau \eta \kappa \rangle \pi \) equals \( 2\pi^2 \).

This shows that the difference \( \{\epsilon + \eta \sigma\} \{q\} \) equals \( 2\pi^2 \) is contained in the indeterminacy of the bracket \( \langle \nu, \eta, \tau \eta \kappa \rangle \pi \). The indeterminacy of this bracket consists of multiples of \( \nu \).

Each of the terms in \( \{\epsilon + \eta \sigma\} \{q\} \) equals \( 2\pi^2 \) is in Adams filtration at least 9, and there are no multiples of \( \nu \) in those filtrations. Therefore, \( \{\epsilon + \eta \sigma\} \{q\} \) equals \( 2\pi^2 \).

We now need to show that \( \eta \sigma \{q\} \) is zero. Because \( h_3g = h_2t \) in Ext, we know that \( \sigma \{q\} + \nu \{t\} \) either equals zero or \( \{u\} \). Note that \( \kappa \{\sigma \{q\} + \nu \{t\}\} \) is zero, while \( \kappa \{u\} = \{d_0u\} \) is non-zero. Therefore, \( \sigma \{q\} + \nu \{t\} \) equals zero, and \( \eta \sigma \{q\} \) is zero as well. \( \square \)

Lemma 4.2.88. There is a hidden \( \epsilon \) extension from \( h_3^2h_5 \) to \( B_8 \).
PROOF. First, there is a relation \( h_1 B_8 = c_0 B_1 \) on the \( E_2 \)-page, which is not hidden in the May spectral sequence. Since \( B_1 \) detects \( \eta \theta_{4,5} \) by definition of \( \theta_{4,5} \) (see Section 4.7), we get that \( h_1 B_8 \) detects \( \eta \theta_{4,5} \) and that \( B_8 \) detects \( \epsilon \theta_{4,5} \). □

On the \( E_\infty \)-page, we have the relation \( h_3^3 h_5 = h_1^2 h_3 h_5 \) in the 40-stem. We will next show that this relation gives rise to a compound hidden extension that is analogous to Toda’s relation \( \nu^3 + \eta^2 \sigma = \eta \epsilon \) (see Table 34). The Note that the element \( \epsilon \eta \) is detected by \( h_1 h_5 c_0 \), whose Adams filtration is higher than the Adams filtration of \( h_3^3 h_5 = h_1^2 h_3 h_5 \).

**Lemma 4.2.89.** \( \nu \{ h_3^3 h_5 \} + \eta \sigma \eta_5 \) equals \( \epsilon \eta_5 \).

**Proof.** Table 23 shows that \( \langle 2 \nu^2, 2, \theta_4 \rangle \) equals \( \{ h_3^3 h_5 \} \). Note that \( \{ x \} \) belongs to the indeterminacy, since there is a hidden \( \sigma \) extension from \( h_2^2 \) to \( x \) as shown in Table 24.

Similarly, \( \langle 2 \nu^3, 2, \theta_4 \rangle \) intersects \( \{ h_3 h_5 \} \), with no indeterminacy. In order to compute the indeterminacy, we need to know that \( \eta \mu_9 \theta_4 \) is zero. This follows from the calculation

\[
\nu \theta_4 \langle \eta, 2, 8 \sigma \rangle = \langle \eta \theta_4, \eta, 2 \rangle 8 \sigma = 0.
\]

Table 23 also shows that \( \langle \eta, 2, \theta_4 \rangle \) equals \( \{ \eta_5, \eta_5 + \eta \rho_3 \} \).

With these tools, compute that

\[
\nu \{ h_3^3 h_5 \} = \nu \langle 2 \nu^2, 2, \theta_4 \rangle = \langle 2 \nu^3, 2, \theta_4 \rangle
\]

because there is no indeterminacy in the last bracket. This equals \( \langle \eta^2 \sigma + \epsilon \eta, 2, \theta_4 \rangle \), which equals \( \langle \eta \sigma + \epsilon \rangle \eta, 2, \theta_4 \rangle \), again because there is no indeterminacy. Finally, this last expression equals \( \eta \sigma \eta_5 + \epsilon \eta_5 \).

**Lemma 4.2.90.** There is a hidden \( \nu_4 \) extension from \( h_2^4 \) to \( h_2 h_5 d_0 \).

**Proof.** Table 23 shows that \( \langle \sigma, \nu, \sigma \rangle \) consists of a single element \( \alpha \) contained in \( \{ h_2 h_4 \} \). Then \( \alpha \) must be of the form \( k \nu_4 \) or \( k \nu_4 + \eta \mu_7 \) where \( k \) is odd. Since \( 2 \theta_4 \) and \( \eta \mu_9 \theta_4 \) are both zero, we conclude that \( \langle \sigma, \nu, \sigma \rangle \theta_4 \) equals \( \nu_4 \theta_4 \).

Table 24 shows that \( \sigma \theta_4 \) equals \( \{ x \} \). Therefore, \( \nu_4 \theta_4 \) is contained in \( \langle \sigma, \nu, \{ x \} \rangle \).

Next compute that \( h_2 h_5 d_0 = \langle h_3, h_2, x \rangle \) with no indeterminacy. This follows from the shuffle

\[
h_2 \langle h_3, h_2, x \rangle = \langle h_2, h_3, h_2 \rangle x = h_3^2 x = h_2 h_5 d_0.
\]

Then Moss’s Convergence Theorem 5.2.14 implies that the Toda bracket \( \langle \sigma, \nu, \{ x \} \rangle \) intersects \( \{ h_2 h_5 d_0 \} \). The indeterminacy in \( \langle \sigma, \nu, \{ x \} \rangle \) is concentrated in Adams filtration strictly greater than 6, so \( \langle \sigma, \nu, \{ x \} \rangle \) is contained in \( \{ h_2 h_5 d_0 \} \). This shows that \( \nu_4 \theta_4 \) is contained in \( \{ h_2 h_5 d_0 \} \).

**Lemma 4.2.91.** \( \langle \theta_4, 2, \sigma^2 \rangle \) is contained in \( \{ h_0 h_3^3 h_5 \} \), with indeterminacy generated by \( \rho_3 \theta_4 \) in \( \{ h_0^2 h_5 d_0 \} \).

**Proof.** Table 23 shows that \( \nu_4 \) is contained in \( \langle 2, \sigma^2, \nu \rangle \). Therefore, \( \nu_4 \theta_4 \) is contained in \( \langle \theta_4, 2, \sigma^2 \rangle \nu \). On the other hand, Lemma 4.2.90 says that \( \nu_4 \theta_4 \) is contained in \( \{ h_2 h_5 d_0 \} \).

We have now shown that \( \langle \theta_4, 2, \sigma^2 \rangle \) contains an element \( \alpha \) such that \( \nu \alpha \) belongs to \( \{ h_2 h_5 d_0 \} \). In particular, \( \alpha \) has Adams filtration at most 5. In addition, we know that \( 2 \alpha \) is zero because of the shuffle

\[
\langle \theta_4, 2, \sigma^2 \rangle 2 = \theta_4 \langle 2, \sigma^2, 2 \rangle = 0.
\]
Here we have used Table 23 for the bracket $\langle 2, \sigma, 2 \rangle$. The only possibility is that $\alpha$ belongs to $\{h_0h_5^2\}$. The indeterminacy follows immediately from Corollary 2.8.

**Lemma 4.2.92.** There is a hidden $\eta$ extension from $h_0^2$ to $h_1h_5d_0$.

**Proof.** Table 23 shows that $\eta_4$ belongs to the Toda bracket $\langle \eta, \sigma, 2 \rangle$. Then $\eta_4\theta_4$ belongs to $\eta\langle \sigma, 2, \theta_4 \rangle$. Recall from the proof of Lemma 4.2.91 that $\langle \sigma, 2, \theta_4 \rangle$ consists of elements $\alpha$ in $\{h_0h_3h_5\}$ of order 2. Table 24 shows that there is a hidden 4 extension from $h_2^3h_5$ to $h_1h_5d_0$. It follows that each $\alpha$ must be of the form $2\gamma - \beta$, where $\gamma$ belongs to $\{h_2^3h_5\}$ and $\beta$ belongs to $\{h_5d_0\}$. Then $\eta\alpha = \eta\beta$ must belong to $\{h_1h_5d_0\}$. This shows that $\eta_4\theta_4$ belongs to $\{h_1h_5d_0\}$.

**Lemma 4.2.93.** There is a hidden $\kappa$ extension from either $h_0^2h_5$ or $h_5d_0$ to $B_{21}$.

**Proof.** The element $\tau h_1B_{21}$ may be the target of an Adams differential. Regardless, the element $h_1B_{21}$ is non-zero on the $E_\infty$-page. Note that $h_1B_{21}$ equals $d_0B_1$ on the $E_2$-page 9. Since $B_1$ detects $\eta\theta_4$ by definition of $\theta_4$ (see Section 1.7), $d_0B_{21}$ detects $\eta\kappa\theta_4$. This implies that $B_{21}$ detects $\kappa\theta_4$. Therefore, $B_{21}$ must be the target of a hidden $\kappa$ extension. The possible sources of this hidden extension are $h_2^3h_5$ or $h_5d_0$.

**Lemma 4.2.94.** Tentatively, there is a hidden $\pi$ extension from either $h_0^3h_5$ or $h_5d_0$ to $\tau B_{23}$.

**Proof.** The claim is tentative because our analysis of Adams differentials is incomplete in the relevant range.

The element $\tau h_1B_{23}$ equals $\tau gB_1$ on the $E_2$-page 9. Since $B_1$ detects $\eta\theta_4$ by definition of $\theta_4$ (see Section 1.7), $\tau gB_1$ detects $\eta\pi\theta_4$. Therefore, $\tau B_{23}$ detects $\pi\theta_4$.

It follows that $\tau B_{23}$ is the target of a hidden $\pi$ extension. The possible sources for this hidden extension are $h_2^3h_5$ or $h_5d_0$.

**Remark 4.2.95.** Lemma 4.2.94 is tentative because there are unknown Adams differentials in the relevant range.
CHAPTER 5

The cofiber of $\tau$

The purpose of this chapter is to compute the motivic stable homotopy groups of the cofiber $C\tau$ of $\tau$. We obtain nearly complete results up to the 63-stem, and we have partial results up to the 70-stem. The Adams charts for $C\tau$ in 19 are essential companions to this chapter.

The element $\tau$ realizes to 1 in the classical stable homotopy groups. Therefore, $C\tau$ is an “entirely exotic” object in motivic stable homotopy, since it realizes classically to the trivial spectrum.

There are two main motivations for this calculation. First, it is the key to resolving hidden $\tau$ extensions that were discussed in Section 4.1.2. Second, we will show in Proposition 6.2.5 that the motivic homotopy groups of $C\tau$ are isomorphic to the classical Adams-Novikov $E_2$-page. Thus the calculations in this chapter will allow us to reverse-engineer the classical Adams-Novikov spectral sequence.

The computational method will be the motivic Adams spectral sequence for $C\tau$, which takes the form

$$E_2 = \text{Ext}_{A}(H^*\tau; \mathbb{M}_2) \Rightarrow \pi_\ast\ast(C\tau).$$

We write $E_2(C\tau)$ for this $E_2$-page $\text{Ext}_{A}(H^*\tau; \mathbb{M}_2)$. See 17 for convergence properties of this spectral sequence.

Outline. The first step in executing the motivic Adams spectral sequence for $C\tau$ is to algebraically compute the $E_2$-page, i.e., $\text{Ext}_{A}(H^*\tau; \mathbb{M}_2)$. We carry this out in Section 5.1 using the long exact sequence

$$\text{Ext}_{A}(\mathbb{M}_2, \mathbb{M}_2) \xrightarrow{\tau} \text{Ext}_{A}(\mathbb{M}_2, \mathbb{M}_2) \xrightarrow{\text{Ext}_{A}(H^*\tau; \mathbb{M}_2)} .$$

Some additional work is required in resolving hidden extensions for the action of $\text{Ext}_{A}(\mathbb{M}_2, \mathbb{M}_2)$ on $E_2(C\tau)$.

The next step is to compute the Adams differentials. In Section 5.2 we use a variety of methods to obtain these computations. The most important is to borrow results about differentials in the motivic Adams spectral sequence for $S^{0,0}$ from Tables 8, 20, 21, and 22. In addition, there are several computations that require analyses of brackets and hidden extensions.

The complete understanding of the Adams differentials allows for the computation of the $E_\infty$-page of the motivic Adams spectral sequence for $C\tau$. The final step, carried out in Section 5.3 is to resolve hidden extensions by 2, $\eta$, and $\nu$ in $\pi_\ast\ast(C\tau)$.

Chapter 7 contains a series of tables that summarize the essential computational facts in a concise form. Tables 35, 36, and 37 give extensions by $h_0$, $h_1$, and $h_2$ in $E_2(C\tau)$ that are hidden in the long exact sequence that computes $E_2(C\tau)$. The fourth columns of these tables refer to one argument that establishes each hidden extension. This takes one of the following forms:
5. THE COFIBER OF $\tau$

(1) An explicit proof given elsewhere in this manuscript.

(2) A May differential that computes a Massey product of the form $\langle h_i, x, \tau \rangle$ via May’s Convergence Theorem 2.2.1. This Massey product implies the hidden extension in $E_2(C\tau)$ by Proposition 5.0.1.

Table 38 gives some additional miscellaneous hidden extensions in $E_2(C\tau)$, again with references to a proof.

Table 39 lists the generators of $E_2(C\tau)$ as a module over $\text{Ext}_A(M_2, M_2)$. Table 40 lists all examples of generators of $E_2(C\tau)$ for which there is some ambiguity. See Section 5.1.5 for more explanation.

Tables 39 and 41 provide the values of $d_2$ and $d_3$ differentials in the Adams spectral sequence for $C\tau$. The fourth columns of these tables refer to one argument that establishes each differential. This takes one of the following forms:

(1) An explicit proof given elsewhere in this manuscript.

(2) “top cell” means that the differential is detected by projection $E_r(S^0, 0) \to E_r(C\tau)$ to the top cell.

(3) Some differentials can be established with an algebraic relation to another differential that is detected by the inclusion $E_r(S^0, 0) \to E_r(C\tau)$ of the bottom cell.

Table 42 describes the part of the projection $\pi_{*,*}(C\tau) \to \pi_{*,*}$ to the top cell that are hidden by the map $E_\infty(C\tau) \to E_\infty(S^0, 0)$ of Adams $E_\infty$-pages. See Proposition 5.2.26 for more explanation.

Table 43 gives the extensions by 2, $\eta$, and $\nu$ in $\pi_{*,*}(C\tau)$ that are hidden in $E_\infty(C\tau)$. The fourth column refers to one argument that establishes each hidden extension. This takes one of the following forms:

(1) An explicit proof given elsewhere in this manuscript.

(2) “top cell” means that the hidden extension is detected by the projection $\pi_{*,*}(C\tau) \to \pi_{*,*}(S^0, 0)$ to the top cell.

(3) “bottom cell” means that the hidden extension is detected by the inclusion $\pi_{*,*} \to \pi_{*,*}(C\tau)$ of the bottom cell.

Massey products and cofibers. We will rely heavily on Massey products and Toda brackets, using the well-known relationship between Toda brackets and hidden extensions in the homotopy groups of a cofiber. See Proposition 3.1.6 for an explicit statement. We will also need a similar result for Massey products.

**Proposition 5.0.1.** Let $y$ and $z$ belong to $E_2(S^0, 0)$ such that $\tau y$ and $zy$ are both zero. In $E_2(C\tau)$, there is a hidden extension

$$z \cdot \overline{y} \in \langle z, y, \tau \rangle,$$

where the Massey product is computed in $E_2(S^0, 0)$ and then pushed forward along the map $E_2(S^0, 0) \to E_2(C\tau)$.

**Proof.** The proof is identical to the proof of Proposition 3.1.6 except that we work in the derived category of chain complexes of $A$-modules instead of the motivic stable homotopy category. In this derived category, the cofiber of $\tau : M_2 \to M_2$ is $H^{*,*}(C\tau)$.

$\square$
5.1. The Adams $E_2$-page for the cofiber of $\tau$

The main tool for computing $E_2(C\tau) = \text{Ext}_A(H^{*,*}(C\tau), M_2)$ is the long exact sequence

$$\cdots \longrightarrow E_2(S^{0,0}) \xrightarrow{\tau} E_2(S^{0,0}) \longrightarrow E_2(C\tau) \longrightarrow \cdots$$

associated to the cofiber sequence

$$S^{0,-1} \xrightarrow{\tau} S^{0,0} \longrightarrow C\tau \longrightarrow S^{1,-1}.$$

This yields a short exact sequence

$$0 \longrightarrow \text{coker}(\tau) \longrightarrow E_2(C\tau) \longrightarrow \ker(\tau) \longrightarrow 0.$$

The desired $E_2(C\tau)$ is almost completely described by the previous short exact sequence. It only remains to compute some hidden extensions.

5.1.1. Hidden extensions in the Adams $E_2$-page for the cofiber of $\tau$.

We will resolve all possible hidden extensions by $h_0$, $h_1$, and $h_2$ through the 70-stem. The reader should refer to the charts in [19] in order to make sense of the following results.

**Theorem 5.1.1.** Tables 35, 36, and 37 give some hidden extensions by $h_0$, $h_1$, and $h_2$ in $E_2(C\tau)$. Through the 70-stem, all other possible hidden extensions by $h_0$, $h_1$, and $h_2$ are either zero or are easily implied by extensions in the tables, with the possible exceptions that:

1. $h_2 \cdot \tau h_1 g^2$ might equal $\tau w$.
2. $h_0 \cdot c_0 Q_2$ and $h_2 \cdot c_0 Q_2$ are either both zero, or equal $D'_2$ and $P(A + A')$ respectively.
3. $h_3 c_0 \cdot D_4$ equals either $h_2 B_5$ or $h_2 B_5 + h_1 X_3$.

**Example 5.1.2.** In the 14-stem, there is a hidden extension $h_2 \cdot h_1 c_0 = h_0 d_0$, which does not appear in Table 37. This is easily implied by the hidden extension $h_0 \cdot h_1 c_0 = P h_2$, which does appear in Table 35.

**Proof.** Most of these hidden extensions are established with Proposition 5.0.1 so we just need to compute Massey products of the form $\langle h_1, x, \tau \rangle$ in $E_2(S^{0,0})$. Most of these Massey products are computed using May’s Convergence Theorem 2.2.1. The fourth columns of Tables 35, 36, and 37 indicate which May differentials are relevant for computing each bracket.

A few hidden extensions require more complicated proofs. These proofs are given in the following lemmas.

5.1.2. Hidden $h_0$ extensions in the Adams $E_2$-page for the cofiber of $\tau$.

**Lemma 5.1.3.**

1. $h_0 \cdot c_0 c_0 = j$.
2. $h_0 \cdot P^k c_0 c_0 = P^k j$.
3. $h_0 \cdot c_0 c_0 g = d_0 l$. 
PROOF. We prove the first formula. The proofs for the other formulas are essentially the same.

By Proposition 5.0.1, we must compute \( \langle h_0, c_0 e_0, \tau \rangle \) in \( E_2(S^{0,0}) \). We may attempt to compute this bracket using May’s Convergence Theorem 2.2.1 with the May differential \( d_2(h_0 h_0(1)^2) = \tau_0 e_0 \). However, the hypothesis of May’s Convergence Theorem 2.2.1 is not satisfied because of the later May differential \( d_4(\Delta h_1^3) = P h_1^3 h_4 \).

Instead, note that \( h_3^2 \langle h_0, c_0 e_0, \tau \rangle \) equals \( h_2^2 \langle h_0, h_0, c_0 e_0 \rangle \tau \). Table 16 shows that the last bracket equals \( h_1 d_6 e_0 \).

Therefore, \( h_3^2 \langle h_0, c_0 e_0, \tau \rangle \) equals \( h_1 d_6 e_0 \). It follows that \( \langle h_0, c_0 e_0, \tau \rangle \) equals \( j \).

**Lemma 5.1.4.** \( h_0 \cdot h_1 d_1 g = h_1 h_5 c_0 d_0 \).

**Proof.** By Proposition 5.0.1, we must compute \( \langle h_0, h_1 d_1 g, \tau \rangle \) in \( E_2(S^{0,0}) \). Because there is no indeterminacy, we have
\[
\langle h_0, h_1 d_1 g, \tau \rangle = \langle h_0, d_1, \tau h_1 g \rangle = \langle h_0, d_1, h_2 f_0 \rangle = \langle h_0, d_1, f_0 \rangle h_2.
\]
Table 16 shows that \( h_2 B_2 = \langle h_0, d_1, f_0 \rangle \). Finally, use that \( h_2 \cdot h_2 B_2 = h_1 h_5 c_0 d_0 \)
from Table 14.

**Lemma 5.1.5.** \( h_0 \cdot h_1^2 B_8 = h_2 x' \).

**Proof.** By Proposition 5.0.1, we must compute the bracket \( \langle h_0, h_1^2 B_8, \tau \rangle \), which equals \( \langle h_0, h_1, \tau h_1 B_8 \rangle \) because there is no indeterminacy. Table 16 shows that \( \langle h_0, h_1, \tau h_1 B_8 \rangle \) equals \( h_2 x' \). Note that \( \tau h_1 B_8 = P h_1 h_5 d_0 \) from Table 11.

**5.1.3. Hidden \( h_1 \) extensions in the Adams \( E_2 \)-page for the cofiber of \( \tau \).**

**Lemma 5.1.6.** \( h_1 \cdot \tau h_0 c_0^3 = d_0 u \).

**Proof.** Using Proposition 5.0.1, we wish to compute the bracket \( \langle h_1, \tau h_0 c_0^3, \tau \rangle \) in \( E_2(S^{0,0}) \). We may attempt to use May’s Convergence Theorem 2.2.1 with the May differential \( d_4(\Delta h_0^3) = \tau_2 h_0 c_0^3 \). However, the conditions of May’s Convergence Theorem 2.2.1 are not satisfied because of the later May differential \( d_6(\Delta^2 h_1^3) = P h_1^3 h_5 \).

Instead, Table 36 shows that \( h_1 \cdot \tau h_0 d_0 c_0^2 \) equals \( P v \). Next, observe that \( d_0 \cdot \tau h_0 c_0^3 + c_0 \cdot \tau h_0 d_0 c_0^2 \) is either zero or \( h_1^2 U \). In either case, \( h_1 d_0 \cdot \tau h_0 c_0^3 \) must be non-zero. It follows that \( h_1 \cdot \tau h_0 c_0^3 \) is also non-zero, and there is just one possible non-zero value.

**Lemma 5.1.7.** \( h_1 h_5 \cdot c_0 d_0 = P h_5 e_0 \).

**Proof.** Table 36 shows that
\[
h_1^2 \cdot c_0 d_0 + d_0 \cdot h_1^2 c_0 = P e_0,
\]
which means that
\[
h_1^2 h_5 \cdot c_0 d_0 + h_1^3 h_5 d_0 \cdot h_1^2 c_0 = P h_1^3 h_5 e_0.
\]
But \( h_1^3 h_5 d_0 = 0 \), so \( h_1^2 h_5 \cdot c_0 d_0 = P h_1^3 h_5 e_0 \), from which the desired formula follows.

**Lemma 5.1.8.** \( h_1^2 \cdot h_5 d_0 e_0 = \tau B_{23} + c_0 Q_2 \).
PROOF. Because of Proposition 5.0.1 we wish to compute the Massey product \( \langle h_1, h_1 h_5 d_0 e_0, \tau \rangle \) in \( E_2(S^{0,0}) \). We may attempt to use May’s Convergence Theorem 2.2.1 with the May differential \( d_6(B_{23}) = h_5^2 h_5 d_0 e_0 \).

However, there is a subtlety here. The element \( \tau B_{23} \) belongs to the May \( E_\infty \)-page for \( E_2(S^{0,0}) \). It represents two elements in \( E_2(S^{0,0}) \) because of the presence of \( PD_4 \) with lower May filtration. Thus, we have only determined so far that \( h_1^2 \cdot h_5 d_0 e_0 \) equals either \( \tau B_{23} \) or \( \tau B_{23} + c_0 Q_2 \).

This ambiguity is resolved essentially by definition. In Table 10 the element \( \tau B_{23} \) in \( E_2(S^{0,0}) \) is defined such that \( \langle h_1, h_1 h_5 d_0 e_0, \tau \rangle \) equals \( \tau B_{23} + c_0 Q_2 \).

\[ \text{Lemma 5.1.9.} \quad h_1^5 \cdot h_1^2 Q_2 = \tau gw + h_1^4 X_1. \]

PROOF. Because of Proposition 5.0.1 we wish to compute the Massey product \( \langle h_1^5, h_1^2 Q_2, \tau \rangle \). We may attempt to use May’s Convergence Theorem 2.2.1 with the May differential \( d_4(\Delta h_1 g^2) = h_1 Q_2 \).

As in the proof of Lemma 5.1.8 there is a subtlety here. The element \( \tau gw \) belongs to the May \( E_\infty \)-page for \( E_2(S^{0,0}) \). It represents two elements in \( E_2(S^{0,0}) \) because of the presence of \( Ph_1 h_5 d_0 e_0 \) with lower May filtration. Recall that Table 10 defines \( \tau gw \) to be the element of \( E_2(S^{0,0}) \) such that \( h_1 \cdot \tau gw = 0 \).

We have determined so far that \( h_1^5 \cdot h_1^2 Q_2 \) equals either \( \tau gw \) or \( \tau gw + h_1^4 X_1 \).

Table 20 gives a non-zero value for the Adams differential \( d_3(\tau gw) \). On the other hand, \( d_4(h_1^2 Q_2) \) is zero. Therefore, \( h_1^5 \cdot h_1^2 Q_2 \) cannot equal \( \tau gw \).

\[ \text{Remark 5.1.10.} \quad \text{The proof of Lemma 5.1.9 is not entirely algebraic in the sense that it relies on Adams differentials. We would prefer a purely algebraic proof, but it has so far eluded us.} \]

\[ \text{Lemma 5.1.11.} \quad h_1^3 c_0 \cdot \overline{D_4} \text{ equals either } h_2 B_5 \text{ or } h_2 B_5 + h_1^2 X_3. \]

PROOF. Because of Proposition 5.0.4 we wish to compute \( \langle h_1^3 c_0, D_4, \tau \rangle \). We may attempt to use May’s Convergence Theorem 2.2.1 with the May differential \( d_4(\Delta g) = h_1 Q_2 \).

As in the proof of Lemma 5.1.8 there is a subtlety here. The element \( h_2 B_5 \) belongs to the May \( E_\infty \)-page for \( E_2(S^{0,0}) \). It represents two elements in \( E_2(S^{0,0}) \) because of the presence of \( h_1^2 X_3 \) with lower May filtration (see Table 10).

\[ \text{5.1.4. Other extensions in the Adams } E_2 \text{-page for the cofiber of } \tau. \]

We finish this section with some additional miscellaneous hidden extensions.

\[ \text{Lemma 5.1.12.} \quad h_1^3 \cdot B_6 + h_2 \cdot \overline{\tau h_2 d_1 g} = h_1^2 Q_2. \]

PROOF. Table 13 gives the hidden extension \( h_1 \cdot h_1^2 B_6 = \tau h_2^2 d_1 g \) in \( E_2(S^{0,0}) \).

This means that \( h_1^3 \cdot B_6 + h_2 \cdot \overline{\tau h_2 d_1 g} \) belongs to the image of \( E_2(S^{0,0}) \to E_2(\overline{C} \tau) \).

Next, compute that \( h_1^3 Q_2 = \langle h_1^1, B_6, \tau \rangle \) using May’s Convergence Theorem 2.2.1 with the May differentials \( d_2(b_{ab} b_{ab} b_{ab} b_{ab} b_{ab} b_{ab}) = \tau B_6 \) and \( d_2(h_1^2 b_{ab} b_{ab} b_{ab} + h_1^2 b_{ab} b_{ab}) = h_1^2 B_6 \).

Therefore, \( h_1^4 \cdot \overline{B_6} = h_1^2 Q_2 \) by Proposition 5.0.1. The desired formula now follows.

\[ \text{Remark 5.1.13.} \quad \text{Through the 70-stem, Lemma 5.1.12 is the only example of a hidden relation of the form } h_0 \cdot \overline{\tau} + h_1 \cdot \overline{\tau}, h_0 \cdot \overline{\tau} + h_2 \cdot \overline{\tau}, \text{ or } h_1 \cdot \overline{\tau} + h_2 \cdot \overline{\tau} \text{ in } E_2(\overline{C} \tau). \]

\[ \text{Lemma 5.1.14.} \quad Ph_1 \cdot \overline{B_6} = h_1 q_1. \]
Consider the map $h_1 q_1$ and its Massey product $\langle P h_1, B_4, \tau \rangle$ in $E_2(S^{0,0})$, using May’s Convergence Theorem 2.2.1 with the May differentials $d_2(b_{30} b_{12} q_1(1)) = \tau B_6$ and $d_2(\Delta B h_4^2) = P h_1 \cdot B_6$. The bracket has indeterminacy generated by $\tau^2 h_0 B_{23}$, so it equals $\{h_1 q_1, h_1 q_1 + \tau^2 h_0 B_{23}\}$.

Push forward this bracket into $E_2(C \tau)$, where it collapses to the single element $h_1 q_1$ since $\tau^2 h_0 B_{23}$ maps to zero in $E_2(C \tau)$. Proposition 5.0.1 now gives the desired result. \hfill \Box

**Lemma 5.1.15.**

1. $h_1^2 \cdot c_0 d_0 + d_0 \cdot h_1^2 c_0 = P e_0$.
2. $c_0 \cdot h_1^2 e_0 + e_0 \cdot h_1^2 c_0 = d_0^\prime$.
3. $h_1^2 \cdot h_1 d_0 u + d_0 \cdot h_1^3 u = P v^\prime$.

**Proof.** These formulas have essentially the same proof. We prove only the first formula.

Table 17 shows that there is a matric bracket

$$P e_0 = \left\langle \begin{bmatrix} h_1^2 & d_0 \\ c_0 d_0 \\ h_1^2 c_0 \end{bmatrix}, \tau \right\rangle.$$ 

A matric version of Proposition 5.0.1 gives the desired hidden extension. \hfill \Box

Before considering the next hidden extension, we need a bracket computation.

**Lemma 5.1.16.** $h_1^2 d_0^2 = \langle c_0 e_0, \tau, h_1^4 \rangle$.

**Proof.** The bracket cannot be computed directly with May’s Convergence Theorem 2.2.1 because of the the later May differential $d_4(\Delta h_4^2) = P h_1^3 h_4$. Therefore, we must follow a more complicated route.

Begin with the computation $h_1 c_0 e_0 = \langle d_0, h_3, h_4^4 \rangle$ from Table 16. Therefore,

$$\langle h_1 c_0 e_0, \tau, h_4^4 \rangle = \langle \langle d_0, h_3, h_4^4 \rangle, \tau, h_4^4 \rangle,$$

which equals $d_0 \langle h_3, h_4^4, \tau, h_4^4 \rangle$ by a standard formal property of Massey products since there are no indeterminacies.

Next, compute that $h_1^2 d_0 = \langle h_3, h_4^4, \tau, h_4^4 \rangle$ using May’s Convergence Theorem 2.2.2 with the May differentials $d_2(h_1 b_{20}) = \tau h_4^4$, $d_2(h_1^2 b_{21}) = h_4^3 h_3$, and $d_2(h_1 b_{30}) = \tau h_4^2 b_{21} + h_4 b_{30}$. Note that both subbrackets $\langle h_4^4, \tau, h_4^4 \rangle$ and $\langle h_3, h_4^4, \tau \rangle$ are strictly zero.

We have now shown that $\langle h_1 c_0 e_0, \tau, h_4^4 \rangle$ equals $h_1^2 d_0^2$. The desired formula now follows immediately. \hfill \Box

**Lemma 5.1.17.**

1. $h_1^2 \cdot c_0 e_0 + e_0 \cdot h_1^2 c_0 = d_0^2$.
2. $d_0 \cdot c_0 e_0 + e_0 \cdot c_0 d_0 = h_1 u$.

**Proof.** For the first formula, by a matric version of Proposition 5.0.1, we wish to compute that

$$d_0^2 = \left\langle \begin{bmatrix} h_1^2 & e_0 \\ c_0 e_0 \\ h_1^2 c_0 \end{bmatrix}, \tau \right\rangle.$$

One might attempt to compute this with a matric version of May’s Convergence Theorem 2.2.1. However, the hypotheses of May’s Convergence Theorem 2.2.1 do not apply because of the presence of the later May differential $d_4(\Delta h_4^2) = P h_1^3 h_4$. 

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Instead, we will show that
\[
\left\langle \left[ h_1^2 \ e_0 \right], \left[ \frac{c_0 e_0}{h_1^2 c_0} \right], \tau \right\rangle h_1^4
\]
equals \(h_1^4 d_0^2\), from which the desired bracket follows immediately. Shuffle to obtain
\[h_1^2(c_0 e_0, \tau, h_1^4) + e_0(h_1^2 c_0, \tau, h_1^4)\].

By Table 16, the expression equals \(h_1^4 d_0^2\) as desired. This completes the proof of the first formula.

The proof of the second formula is similar. We wish to compute that
\[
h_1 u = \left\langle \left[ d_0 \ e_0 \right], \left[ \frac{c_0 e_0}{c_0 d_0} \right], \tau \right\rangle.
\]
Again, the hypotheses of May’s Convergence Theorem 2.2.1 do not apply.

Instead, we will show that
\[
\left\langle \left[ d_0 \ e_0 \right], \left[ \frac{c_0 e_0}{c_0 d_0} \right], \tau \right\rangle h_1^4
\]
equals \(h_1^4 u\), from which the desired bracket follows immediately. Shuffle to obtain
\[d_0(c_0 e_0, \tau, h_1^4) + e_0(c_0 d_0, \tau, h_1^4)\].

By Table 16 this expression equals \(h_1^2 d_0^3 + h_1^2 e_0 \cdot Pe_0\). Note that \(e_0 \cdot Pe_0\) equals \(d_0^3 + h_1^3 u\); this is already true in the May \(E_\infty\)-page. Therefore, \(h_1^2 d_0^3 + h_1^2 e_0 \cdot Pe_0\) equals \(h_1^4 u\), as desired.

**Lemma 5.1.18.** \(h_1^2 e_0^2 \cdot h_1^4 e_0 + d_0 e_0 g \cdot h_1^4 + h_1^6 \cdot h_1^4 B_1 = c_0 d_0 e_0^2\).

**Proof.** The relation \(e_0^3 + d_0 \cdot e_0 g = h_1^5 B_1\) is hidden in the May spectral sequence [14].

By Proposition 5.0.1 we wish to compute that
\[c_0 d_0 e_0^2 = \left\langle \left[ h_1^2 e_0^2 \ d_0 e_0 g \ h_1^4 \right], \left[ \frac{h_1^2 e_0}{h_1^4} \ h_1^4 \ h_1^4 B_1 \right], \tau \right\rangle.
\]
This will follow if we can show that \(h_1^4 c_0 d_0 e_0^2\) equals
\[
\left\langle \left[ h_1^2 e_0^2 \ d_0 e_0 g \ h_1^4 \right], \left[ \frac{h_1^2 e_0}{h_1^4} \ h_1^4 \ h_1^4 B_1 \right], \tau \right\rangle h_1^4.
\]
This expression equals
\[h_1^2 e_0^2 (h_1^4 e_0, \tau, h_1^4) + d_0 e_0 g (h_1^4, \tau, h_1^4) + h_1^4 (h_1^4 B_1, \tau, h_1^4)\].

The first two terms can be computed with Table 16. The possible non-zero values for the third bracket are multiples of \(h_0\), which means that the third term is zero in any case.

The desired formula now follows.

**Lemma 5.1.19.**
5.1.5. The Adams $E_2$-page for the cofiber of $\tau$.

Having resolved hidden extensions, we can now state our main theorem about $E_2(C\tau)$.

**Theorem 5.1.20.** The $E_2$-page of the Adams spectral sequence for $C\tau$ is depicted in [19] through the 70-stem. Table 39 lists the $E_2(S^{0,0})$-module generators of $E_2(C\tau)$ through the 70-stem.

For most of the generators in Table 39 the notation $\pi$ is unambiguous. In other words, in each relevant degree, there is just a single element $\pi$ of $E_2(S^{0,0})$ that projects to $x$ in $E_2(C\tau)$. However, there are several cases in which there is a choice of representative for $\pi$ because of the presence of an element in the same degree in the image of the map $E_2(S^{0,0}) \to E_2(C\tau)$. One such example occurs in the 56-stem with $\tau h_0gm$. The presence of $h_2x'$ means that there are actually two possible choices for $\tau h_0gm$.

Table 40 lists all such examples of $E_2(S^{0,0})$-module generators of $E_2(C\tau)$ for which there is some ambiguity. In some cases, we have given an algebraic specification of one element of $E_2(C\tau)$ to serve as the generator. These choices are essentially arbitrary, but it is important to be consistent with the notation between different arguments.

In some cases, we have not given a definition because an algebraic description is not readily available, and also because it does not seem to matter for later analysis. The reader is strongly warned to be cautious when working with these undefined elements.

The generator $h_{111}$ deserves one additional comment. In this case, the presence of $\tau h_1G$ and $B_0$ means that there are four possible choices for this generator. We have given two algebraic specifications for $h_{111}$, which determines a unique element from these four.

5.2. Adams differentials for the cofiber of $\tau$

We have now computed the $E_2$-page of the Adams spectral sequence for $C\tau$. See [19] for a chart of $E_2(C\tau)$ through the 70-stem.

The next step is to compute the Adams differentials. The main point is to compute the Adams $d_r$ differentials on the $E_r(S^{0,0})$-module generators of $E_r(C\tau)$. Then one can compute the Adams $d_r$ differential on any element, using the Adams $d_r$ differentials for $E_r(S^{0,0})$ given in Tables 8, 20, 21, and 22.

5.2.1. Adams $d_2$ differentials for the cofiber of $\tau$.

**Proposition 5.2.1.** Table 39 lists some values of the motivic Adams $d_2$ differential for $C\tau$. The motivic Adams $d_2$ differential is zero on all other $E_2(S^{0,0})$-module generators of $E_2(C\tau)$, through the 70-stem, with the possible exceptions that:

1. $d_2(h_{1111})$ might equal $h_1h_5c_0d_0$.
2. $d_2(h_{1111})$ might equal $\tau h_1G_0$.

**Proof.** We use several different approaches to establish the Adams $d_2$ differentials:

1. From an Adams differential $d_2(x) = y$ in $E_2(S^{0,0})$, push forward along the inclusion $S^{0,0} \to C\tau$ of the bottom cell to obtain the same formula in $E_2(C\tau)$. 
(2) From an Adams differential $d_2(x) = y$ in $E_2(S^{0,0})$, use the projection $C\tau \to S^{1,-1}$ and pull back to $d_2(\tau) = \gamma$ in $E_2(C\tau)$, up to a possible error term that belongs to the image of the inclusion $E_2(S^{0,0}) \to E_2(C\tau)$ of the bottom cell.

(3) Push forward a differential from $E_2(S^{0,0})$ as in (1), and then use a hidden extension in $E_2(C\tau)$. For example, $d_2(\gamma_0d_0) = Pd_0$ because $h_0 \cdot c_0d_0 = i$ in $E_2(C\tau)$ and $d_2(i) = Ph_0d_0$ in $E_2(S^{0,0})$.

(4) Work $h_1$-locally. For example, consider the hidden extensions $h_1^2 \cdot c_0e_0 + e_0 \cdot h_1^2c_0 = d_0^2$ and $h_1^2 \cdot c_0d_0 + d_0 \cdot h_1^2c_0 = Pe_0$ from Table 8. It follows that $d_2(\gamma_0e_0) = h_1^2 \cdot c_0d_0 + Pe_0$.

Most of the differentials are computed with straightforward applications of these techniques. The remaining cases are computed in the following lemmas. \hfill $\square$

The chart of $E_2(C\tau)$ in \cite{19} indicates the Adams $d_2$ differentials, all of which are implied by the calculations in Tables 8 and 9.

**Lemma 5.2.2.** $d_2(h_1^2e_0g) = h_1^2e_0 \cdot h_1^2c_0 + c_0d_0e_0$.

**Proof.** Table 8 gives the differential $d_2(h_1^2e_0g) = h_1^2e_0^2$ in $E_2(S^{0,0})$. Therefore, $d_2(h_1^2e_0g)$ is either $h_1^2e_0 \cdot h_1^2c_0$ or $h_1^2e_0 \cdot h_1^2c_0 + c_0d_0e_0$. However, $d_2(h_1^2e_0 \cdot h_1^2c_0) = h_1^2c_0d_0^2$, so $h_1^2e_0 \cdot h_1^2c_0$ cannot be the target of a $d_2$ differential. \hfill $\square$

**Lemma 5.2.3.** $d_2(\tau^2h_1g^2) = z$.

**Proof.** We will argue that $z$ must be zero in $E_\infty(C\tau)$. There is only one possible differential that can kill it.

Table 12 gives a classical extension $\eta \cdot \{g^2\} = \{z\}$ in $\pi_{41}$. This implies that there must be a hidden relation $\tau \cdot \{\tau^2h_1g^2\} = \{z\}$ in $\pi_{41,22}$. In particular, $\{z\}$ is divisible by $\tau$ in $\pi_{*,*,*}$. This means that $\{z\}$ maps to zero in $\pi_{41,22}(C\tau)$. \hfill $\square$

**Lemma 5.2.4.**

(1) $d_2(h_1d_0u) = Pu'$.

(2) $d_2(P\gamma_1d_0u) = P^2u'$.

**Proof.** Table 8 implies that $d_2(d_0 \cdot \gamma_1v) = d_0 \cdot \gamma_1u$. By Lemma 5.1.10 this equals $h_1^2 \cdot h_1d_0u + Pu'$. Therefore, $h_1^2 \cdot d_2(h_1d_0u) = d_2(Pu')$. By Table 8, $d_2(Pu') = Ph_1u' + \tau h_0d_0^2$ in $E_2(S^{0,0})$. Therefore, $d_2(Pu') = Ph_1u'$ in $E_2(C\tau)$. It follows that $d_2(h_1d_0u)$ must equal $Pu'$. This establishes the first formula.

The second formula follows by multiplying the first formula by $Ph_1$. \hfill $\square$

**Lemma 5.2.5.** $d_2(D_4) = h_1 \cdot B_6 + Q_2$.

**Proof.** Pull back the differential $d_2(D_4) = h_1B_6$ from $E_2(S^{0,0})$ to conclude that $d_2(D_4) = h_1 \cdot B_6$ modulo a possible error term that comes from pushing forward from $E_2(S^{0,0})$. To establish the error term, use that $h_0 \cdot B_4 = D_2$ and that $d_2(D_2) = h_0Q_2$. \hfill $\square$

**Lemma 5.2.6.** $d_2(h_1c_0x^2) = Ph_1x'$.

**Proof.** Table 8 implies that $d_2(c_0 \cdot \gamma') = h_1^2c_0 \cdot \gamma + h_1^2d_0 \cdot \gamma + e_0 \cdot \tau h_0d_0e_0^2$. Recall from \cite{14} the relation $e_0u' + d_0u' = h_1^2c_0x'$, which is hidden in the May spectral sequence. This implies that $d_2(c_0 \cdot \gamma')$ equals $h_1^2 \cdot h_1c_0x' + e_0 \cdot \tau h_0d_0e_0^2$. 

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There is a hidden extension $h_1 e_0 \cdot \tau h_0 d_0 e_0 = P e_0 v$. Therefore, $d_2(h_1 e_0 \cdot v^2)$ equals $h_1^2 \cdot h_1 e_0 x + P e_0 v$, so $h_1^2 \cdot d_2(h_1 e_0 x)$ must equal $d_2(P e_0 v)$.

By Table 3 $d_2(P e_0 v) = Ph^2_1 d_0 v + Ph^2_3 e_0 u$ in $E_2(S^{0,0})$. This equals $Ph^2_1 x'$. \[\Box\]

**Lemma 5.2.7.** $d_2(e_0 Q^2) = 0$.

**Proof.** Start with the relation $h_1 \cdot e_0 Q^2 = P h_1 \cdot D_0$, which follows from Lemma 24.21 Using Lemma 5.2.5 it follows that $h_1 \cdot d_2(e_0 Q^2) = Ph^2_0 \cdot B_0 + P h_1 Q^2$. We know from 9 that $P h_1 Q^2 = h^2_1 q_1$, and we know from Lemma 5.1.14 that $P h^2_1 \cdot B_0 = h^2_1 q_1$. \[\Box\]

**Remark 5.2.8.** We emphasize the calculation $d_2(e_0 g \cdot h^2_1 e_0) = h^6_1 h^6_1 B_1 + c_0 d_0 e_0^2$, which follows from the Leibniz rule and Lemma 5.1.18 This implies that $h^6_1 \cdot h^6_1 B_1$ equals $c_0 d_0 e_0^2$ in $E_3(C\tau)$. This formula is critical for later Adams differentials.

**5.2.2. Adams $d_3$ differentials for the cofiber of $\tau$**. See 19 for a chart of $E_3(C\tau)$. This chart is complete through the 70-stem; however, the Adams $d_3$ differentials are complete only through the 64-stem.

**Remark 5.2.9.** There are a number of classes in $E_2(S^{0,0})$ that do not survive to $E_3(S^{0,0})$, but their images in $E_2(C\tau)$ do survive to $E_3(C\tau)$. The first few examples of this phenomenon are $h_0 y$, $h_0 c_2$, and $h_5^6 Q'$. These elements give rise to $E_3(S^{0,0})$-module generators of $E_3(C\tau)$.

**Remark 5.2.10.** Note the class in the 55-stem labeled “?””. This class is either $h_1 i_1$ or $h_1 i_1 + \tau h_1 G$, depending on whether $d_2(h_1 i_1)$ is zero or non-zero. In the first case, we have that $h^6_1 \cdot h_1 i_1 = \tau g^3$, from which $d_3(h_1 i_1)$ would equal $h_1 B_8$. In the second case, we have that $h^6_1 \cdot (h_1 i_1 + \tau h_1 G) = \tau g^3 + h^6_1 h_5 c_0 e_0$, from which $d_3(h_1 i_1 + \tau h_1 G)$ would equal zero.

The next step is to compute Adams $d_3$ differentials on the $E_3(S^{0,0})$-module generators of $E_3(C\tau)$.

**Proposition 5.2.11.** Table 7 lists some values of the motivic Adams $d_3$ differential for $C\tau$. The motivic Adams $d_3$ differential is zero on all other $E_3(S^{0,0})$-module generators of $E_3(C\tau)$, through the 65-stem, with the possible exception that $d_3(h_1 i_1)$ equals $h_1 B_8$, if $h_1 i_1$ survives to $E_3(C\tau)$.

**Proof.** The techniques for establishing these differentials are the same as in the proof of Proposition 5.2.1 for $d_2$ differentials, except that the $h_1$-local calculations are no longer useful. The few remaining cases are computed in the following lemmas.

The chart of $E_3(C\tau)$ in 19 indicates the Adams $d_3$ differentials, all of which are implied by the calculations in Tables 20 and 41. The differentials are complete only through the 64-stem. Beyond the 64-stem, there are a number of unknown differentials.

**Lemma 5.2.12.**

1. $d_3(h_1 h_3 g) = d_0 e_0^2$.
2. $d_3(h_1 h_3 g^2) = d_0 e_0^2$. 


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Proof. We showed in Lemma 4.2.1 that both $\{d^2_0\}$ and $\{d_0^3e^2_0\}$ are divisible by $\tau$ in $\pi_{*,*}$. Therefore, the classes $d^2_0$ and $d_0^3e^2_0$ of $E_\infty(S^{0,0})$ must map to zero in $E_\infty(C\tau)$. For each element, there is just one possible differential that can hit it. 

Lemma 5.2.13. $d_3(h^2_1g_2) = 0$.

Proof. The only other possibility is that $d_3(h^2_1g_2)$ equals $N$. We showed in Lemma 4.2.5 that the elements of $\{N\}$ are not divisible by $\tau$ in $\pi_{*,*}$. Therefore, $\{N\}$ maps to $\pi_{*,*}(C\tau)$ non-trivially. The only possibility is that $N$ is non-zero in $E_\infty(C\tau)$.

Lemma 5.2.14. $d_3(h_1G_3) = \tau h_0e^3_0$.

Proof. Table 16 shows that $h_1G_3 = \langle h_3, h^3_1, Ph^3_1h_5 \rangle$. It follows that $c_0 \cdot h_1G_3 = \langle c_0, h_3, h^3_1 \rangle Ph^3_1h_5$. Table 16 shows that $\langle c_0, h_3, h^3_1 \rangle = h^3_1e_0$.

We have now shown that $c_0 \cdot h_1G_3 = h^2_1 \cdot Ph^3_1h_5e_0$ or $c_0 \cdot h_1G_3 = h^2_1 \cdot Ph^3_1h_5e_0 + h^3_1B_21$. In either case, $c_0 \cdot h_1G_3 =$ $h^2_1 \cdot Ph^3_1h_5e_0$ in $E_3(C\tau)$ since $h^3_1B_21$ is hit by an Adams $d_2$ differential.

Since $d_3(h^3_1 \cdot Ph^3_1h_5e_0) = d_3u'$ is non-zero, we conclude that $d_3(h_1G_3)$ is also non-zero, and there is just one possible non-zero value.

Lemma 5.2.15. $d_3(h_1d_1g) = 0$.

Proof. The only other possibility is that $d_3(h_1d_1g) = h^2_1G_3$. If this were the case, then $\{h^2_1G_3\}$ in $\pi_{5,3}$ would be divisible by $\tau$. If $\{h^2_1G_3\}$ were divisible by $\tau$, then the only possibility would be that $\tau \{h_1d_1g\} = \{h^2_1G_3\}$. However, $\tau \{h_1d_1g\}$ is zero by Lemma 4.2.2.

Lemma 5.2.16. $d_3(h^3_1D_4) = h_1B_21$.

Proof. Recall from Lemma 5.1.11 that $h^3_1c_0 \cdot D_4$ equals either $h_2B_5$ or $h_2B_5 + h^2_1X_3$. It follows that $c_0 \cdot h^3_1D_4$ equals either $h_2B_5$ or $h_2B_5 + h^2_1X_3$. However, these two elements are equal in $E_3(C\tau)$ since $h^3_1X_3$ is the target of an Adams $d_2$ differential.

We know that $d_3(h_2B_5) = h_1B_8d_0$ by Table 20. It follows that $d_3(h^3_1D_4)$ is non-zero, and there is just one possibility.

Lemma 5.2.17. $d_3(Ph_5c_0e_0) = h^2_1c_0x^7 + U$.

Proof. First note that either $h_1 \cdot Ph_5c_0e_0 = Ph_1 \cdot h_5c_0e_0$ or $h_1 \cdot Ph_5c_0e_0 = Ph_1 \cdot h^2_1c_0e_0 + h^3_1q_1$. In either case, $d_3(h_1 \cdot Ph_5c_0e_0) = Ph_1 \cdot h^2_1B_8$ since $d_3(h^2_1c_0e_0) = h^3_1B_8$ and $d_3(h^3_1q_1) = 0$.

Finally, we must compute that $Ph_1 \cdot h^3_1B_8 = h_1 \cdot h^2_1c_0x^7 + h_1U$. Because of the relation $B_8 \cdot Ph_1 = c_0x^7$, either $Ph_1 \cdot h^2_1B_8 = h^3_1 \cdot h^2_1c_0x^7$ or $Ph_1 \cdot h^2_1B_8 = h^2_1 \cdot h^3_1c_0x^7 + h_1U$. The second case must be correct because this is the element that survives to $E_3(C\tau)$.

5.2.3. Adams $d_4$ differentials for the cofiber of $\tau$. See [19] for a chart of $E_4(C\tau)$. This chart is complete through the 64-stem. Beyond the 64-stem, because of unknown earlier differentials, the actual $E_4$-page is a subquotient of what is shown in the chart.
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The next step is to compute Adams $d_4$ differentials on the $E_4(S^{0,0})$-module generators of $E_4(C\tau)$.

**Proposition 5.2.18.** The motivic Adams $d_4$ differential for the cofiber of $\tau$ is zero on all $E_4(S^{0,0})$-module generators of $E_4(C\tau)$ through the 63-stem, except that:

1. $d_4(h_0^{15}h_6) = \tau P^2 h_0 d_5^2 e_0$.
2. $d_4(j_1)$ might equal $B_{21}$.

**Proof.** For degree reasons, there are very few possible differentials. The only difficult cases are addressed in Lemmas 5.2.21 and 5.2.22. □

The chart of $E_4(C\tau)$ in [19] indicates the Adams $d_4$ differentials, all of which are implied by the calculations in Proposition 5.2.18 and Table 21. The differentials are complete only through the 63-stem. Beyond the 63-stem, there are a number of unknown differentials.

**Remark 5.2.19.** Recall that $h_0^{15}h_6$ does not survive to $E_4(S^{0,0})$, so this element is an $E_4(S^{0,0})$-module generator of $E_4(C\tau)$. This is the reason that the formula for $d_4(h_0^{15}h_6)$ appears in the statement of Proposition 5.2.18.

**Remark 5.2.20.** The possible differential $d_4(C') = h_2 B_{21}$ in $E_4(S^{0,0})$ mentioned in Proposition 3.2.15 occurs if and only if $d_4(j_1) = B_{21}$ in $E_4(C\tau)$. This follows immediately from the relation $h_2 \cdot j_1 = C'$.

**Lemma 5.2.21.** $d_4(h_0 D_2) = 0$.

**Proof.** We showed in Lemma 4.2.7 that $h_0 h_2 h_5 j$ detects an element $\alpha$ of $\pi_{57,30}$ that is not divisible by $\tau$. Therefore, $\alpha$ maps to a non-zero element of $\pi_{57,30}(C\tau)$. The only possibility is that this element of $\pi_{57,30}(C\tau)$ is detected by $h_1 Q_1$. In particular, $h_1 Q_1$ cannot equal $d_4(h_0 D_2)$. □

**Lemma 5.2.22.** $d_4(h_3 d_1 g) = 0$.

**Proof.** The only other possibility is that $d_4(h_3 d_1 g)$ equals $Ph_3 h_5 e_0$. We showed in Lemma 4.2.8 that the element $\{Ph_3 h_5 e_0\}$ of $\pi_{59,33}$ is not divisible by $\tau$. Therefore, $Ph_3 h_5 e_0$ is not hit by a differential in the Adams spectral sequence for $C\tau$.

**5.2.4. Higher Adams differentials for the cofiber of $\tau$.** At this point, we are nearly done. There is just one more differential to compute.

**Lemma 5.2.23.** $d_5(h_2 h_5) = 0$.

**Proof.** The only other possibility is that $d_5(h_2 h_5)$ equals $h_1 q$. We showed in Lemma 4.2.2 that the element $\{h_1 q\}$ of $\pi_{33,18}$ is not divisible by $\tau$. Therefore, $h_1 q$ cannot be hit by a differential in the Adams spectral sequence for the cofiber of $\tau$. □

The $E_4(C\tau)$ chart in [19] indicates the very few $d_5$ differentials along with the $d_4$ differentials.
5.3. Hidden Adams Extensions for the Cofiber of $\tau$

5.2.5. The Adams $E_\infty$-page for the cofiber of $\tau$. Using the Adams differentials given in Table 39 Table 41 and Proposition 5.2.18 as well as the Adams differentials for $S^0.0$ given in Tables 5 20 21 and 22 we can now directly compute the $E_\infty$-page of the Adams spectral sequence for $C\tau$.

**Theorem 5.2.24.** The $E_\infty$-page of the Adams spectral sequence for $C\tau$ is depicted in [19]. This chart is complete through the 63-stem. Beyond the 63-stem, $E_\infty(C\tau)$ is a subquotient of what is shown in the chart.

Through the 63-stem, all unknown differentials are indicated as dashed lines. Beyond the 63-stem, there are a number of unknown differentials.

In a range, we now have a complete understanding of $E_\infty(C\tau)$, which is the associated graded object of $\pi_{\ast\ast}(C\tau)$ with respect to the Adams filtration. In order to better understand $\pi_{\ast\ast}(C\tau)$ itself, we would like to compute the maps of homotopy groups induced by the inclusion $j : S^0.0 \rightarrow C\tau$ of the bottom cell and the projection $q : C\tau \rightarrow S^{31\cdash}$ to the top cell.

**Proposition 5.2.25.** The map $j_\ast : \pi_{\ast\ast} \rightarrow \pi_{\ast\ast}C\tau$ induced by the inclusion of the bottom cell is described as follows, through the 59-stem. Let $\alpha$ be an element of $\pi_{\ast\ast}$ detected by a in $E_\infty(S^{0.0})$.

1. If a does not equal $h_0h_2h_5i$, then $j_\ast(\alpha)$ is detected by $j_\ast(a)$ in $E_\infty(C\tau)$.
2. If a equals $h_0h_2h_5i$, then $j_\ast(\alpha)$ is detected by $h_1Q_1$ in $E_\infty(C\tau)$.

**Proof.** This is a straightforward calculation, using that there is an induced map $E_\infty(S^{0.0}) \rightarrow E_\infty(C\tau)$. \hfill $\Box$

It is curious that the Adams filtration hides so little about the map $j_\ast$.

**Proposition 5.2.26.** The map $q_\ast : \pi_{\ast\ast}C\tau \rightarrow \pi_{\ast\ast\ast\ast+1}$ induced by the projection to the top cell is described as follows, through the 59-stem.

1. An element of $\pi_{\ast\ast}(C\tau)$ in the image of $j_\ast : \pi_{\ast\ast} \rightarrow \pi_{\ast\ast}(C\tau)$ (as described by Proposition 5.2.25) maps to 0 in $\pi_{\ast\ast\ast\ast+1}$.
2. An element of $\pi_{\ast\ast}(C\tau)$ detected by $\mathbf{7}$ in $E_\infty(C\tau)$ maps to an element of $\pi_{\ast\ast\ast\ast+1}$ detected by $x$ in $E_\infty(S^{0.0})$.
3. The remaining possibilities are described in Table 42.

**Proof.** The part of $q_\ast$ that is not hidden by the Adams filtration is described in (1) and (2). The part of $q_\ast$ that is hidden by the Adams filtration is described in Table 42. These are the only possible values that are compatible with the long exact sequence

$$\cdots \rightarrow \pi_{\ast\ast\ast+1} \rightarrow \pi_{\ast\ast} \rightarrow \pi_{\ast\ast}(C\tau) \rightarrow \pi_{\ast\ast\ast\ast+1} \rightarrow \cdots$$

$\Box$

5.3. Hidden Adams extensions for the cofiber of $\tau$

Finally, we will consider hidden extensions by 2, $\eta$, and $\nu$ in the motivic stable homotopy groups $\pi_{\ast\ast}(C\tau)$ of the cofiber of $\tau$. We will show in Lemma 6.2.3 that there are no hidden $\tau$ extensions in $\pi_{\ast\ast}(C\tau)$.

Recall from Proposition 3.1.6 that a hidden extension by $\alpha$ in $\pi_{\ast\ast}(C\tau)$ is the same as a Toda bracket in $\pi_{\ast\ast}$ of the form $\langle \tau, \beta, \alpha \rangle$. Many such Toda brackets are detected in Ext by a corresponding Massey product of the form $\langle \tau, b, a \rangle$. In this circumstance, the extension by $\alpha$ is already detected in $E_\infty(C\tau)$. 

However, there are some Toda brackets of the form $\langle \tau, \beta, \alpha \rangle$ that are not detected by Massey products in Ext. In this section, we will study such Toda brackets methodically.

**Proposition 5.3.1.** Table 43 shows some hidden extensions by $2, \eta, \nu$ in $\pi_{*,*}(C\tau)$. Through the 59-stem, there are no other hidden extensions by $2, \eta, \nu$, except that:

1. there might be a hidden 2 extension from $h_4i_1g$ to $h_1B_8$.
2. there might be a hidden 2 extension from $Q_2$ to $h_1Q_1$.
3. there might be a 2 extension from $h_2D_4$ to $Ph_1h_5e_0$.
4. there might be a 2 hidden extension from $h_2g_2$ to $h_0B_2$.
5. there might be a hidden $\nu$ extension from $B_8$ to $h_1D_{11}$.
6. there might be a hidden $\nu$ extension from $C_{\tau}$ to $B_{21}$.
7. if $h_1i_1 + \tau h_1G$ survives to $E_\infty(C\tau)$, then there might be a hidden $\nu$ extension from $h_1i_1 + \tau h_1G$ to $h_1D_{11}$.
8. if $j_1$ survives to $E_\infty(C\tau)$, then there might be a hidden 2 extension from $j_1$ to $h_1h_3G_3$.

**Proof.** Some of the extensions are detected by the projection $q : C\tau \to S^{1, -1}$ to the top cell, and some of the extensions are detected by the inclusion $j : S^{0, 0} \to C\tau$ of the bottom cell. The remaining cases are established in the following lemmas.

**Remark 5.3.2.** The possible hidden $\eta$ extension on $h_2^2g_2$ is connected to some of the other uncertainties in our calculations. Suppose that there is a hidden $\tau$ extension from $h_1i_1$ to $h_1B_8$ in $\pi_{*,*}$ (see Remark 4.1.11). Then $\nu\{C\} + \tau\{i_1\}$ is detected by $B_8$, and there is a hidden $\nu$ extension in $\pi_{*,*}(C\tau)$ from $C$ to $B_8$. If $\{h_2^2g_2\} \eta$ were zero, then we could further compute that

$$\{B_8\} = \{h_2^2g_2\} \nu^2 = \{h_2^2g_2\} \eta, \nu, \eta = \langle\{h_2^2g_2\}; \eta, \nu\rangle$$

in $\pi_{*,*}(C\tau)$. However, $\{B_8\}$ cannot be divisible by $\eta$ in $\pi_{*,*}(C\tau)$. Therefore, $\{h_2^2g_2\} \eta$ would be non-zero in $\pi_{*,*}(C\tau)$.

**Lemma 5.3.3.** There is no hidden $\nu$ extension on $h_1h_5$.

**Proof.** The only other possibility is that there is a hidden $\nu$ extension from $h_1h_5$ to $u$. We will show that the Toda bracket $\langle \eta, \eta, \eta_5, \nu \rangle$ does not contain $\{u\}$.

The bracket contains $\langle \eta \eta^5, \eta_5, \nu \rangle$, which equals $\langle 4\nu, \eta_5, \nu \rangle$. This bracket contains $4\langle \nu, \eta_5, \nu \rangle$. Note that $\langle \nu, \eta_5, \nu \rangle$ intersects $\{h_1h_5f_5\}$, but $4\langle \nu, \eta_5, \nu \rangle$ is zero.

Finally, the bracket $\langle \tau, \eta, \eta_5, \nu \rangle$ has indeterminacy generated by $\tau\{h_3d_1\}$ and $\tau^2\{c_1g\}$. Therefore, $\{u\}$ is not in the bracket.

**Lemma 5.3.4.** There is a hidden $\eta$ extension from $h_0y$ to $u$.

**Proof.** Table 42 shows that projection to the top cell maps $\{h_0y\}$ to $\{\tau h_2e_0^\alpha\}$ in $\pi_{37, 21}$. The bracket $\langle \tau, \{\tau h_2e_0^\alpha\}, \eta \rangle$ contains $\{\tau^2e_0^\alpha\}, \nu, \eta\}$ which contains $\{u\}$ by Table 23.

The indeterminacy of $\langle \tau, \{\tau h_2e_0^\alpha\}, \eta \rangle$ is generated by $\tau\sigma\{d_3\}$, and $\eta\{h_0h_1h_5\}$. Note that $\tau\sigma\eta$ is equal to $\eta\{h_0h_1h_5\}$, as shown in Remark 4.2.42. Since $\{u\}$ is not in the indeterminacy, the bracket does not contain zero.

**Lemma 5.3.5.** There is no hidden $\nu$ extension on $h_0c_2$.
5.3. HIDDEN ADAMS EXTENSIONS FOR THE COFIBER OF $\tau$

PROOF. According to Table 42, the projection to the top cell takes the elements of $\pi_{41,22}(C\tau)$ detected by $h_0c_2$ to elements of $\pi_{40,23}$ that are detected by $h_1h_3d_1$. These elements of $\pi_{40,23}$ must also be annihilated by $\tau$, so they must be $\eta\sigma\{d_1\}$ and $\eta\sigma\{d_1\} + \{\tau h_0g^2\}$.

It remains to compute the Toda bracket $\langle \tau, \eta\sigma\{d_1\}, \nu \rangle$. This bracket contains $\langle \tau, \eta\{d_1\}, 0 \rangle$, which equals zero. □

**Lemma 5.3.6.** There is no hidden 2 extension on $h_0c_2$.

**Proof.** We showed in Lemma 5.3.5 that there is no hidden $\nu$ extension on $h_0c_2$. Therefore, there cannot be a hidden 2 extension from $h_0c_2$ to $\tau h_0g^2$.

There are no other possible hidden 2 extensions on $h_0c_2$. □

**Lemma 5.3.7.** There is no hidden 2 extension on $h_3 \cdot h_2^5 g$.

**Proof.** The projection to the top cell detects that $h_3 \cdot h_2^5 g$ is the target of a hidden $\eta$ extension from $h_0c_2$. Therefore, $h_3 \cdot h_2^5 g$ cannot support a hidden 2 extension. □

**Lemma 5.3.8.** There is no hidden $\eta$ extension on $\tau h_2c_1 g$.

**Proof.** The projection to the top cell takes the element $\{\tau h_2c_1 g\}$ of $\pi_{43,23}(C\tau)$ to an element of $\pi_{42,24}$ that is detected by $\tau h_2c_1 g$. Two elements of $\pi_{42,24}$ are detected by $\tau h_2c_1 g$, but only one element is killed by $\tau$. The relation $\eta\{h_0^2h_3h_5\} = \tau \sigma k$ from Remark 4.2.42 implies that $\nu \sigma k$ is the element of $\pi_{42,24}$ that is killed by $\tau$ and detected by $\tau h_2c_1 g$. Therefore, the top cell detects that there is no hidden $\eta$ extension on $\tau h_2c_1 g$. □

**Lemma 5.3.9.** There is a hidden $\nu$ extension from $d_0r$ to $h_1u'$.

**Proof.** The inclusion of the bottom cell shows that there is a hidden 2 extension from $e_0r$ to $h_1u'$ in $\pi_{47,26}(C\tau)$. The hidden $\nu$ extension on $d_0r$ is an immediate consequence. □
CHAPTER 6

Reverse engineering the Adams-Novikov spectral sequence

In this chapter, we will show that the classical Adams-Novikov $E_2$-page is identical to the motivic stable homotopy groups $\pi_{*,*}(C\tau)$ of the cofiber of $\tau$ computed in Chapter 5. Moreover, the classical Adams-Novikov differentials and hidden extensions can also be deduced from prior knowledge of motivic stable homotopy groups. We will apply this program to provide detailed computational information about the classical Adams-Novikov spectral sequence in previously unknown stems.

In fact, the classical Adams-Novikov spectral sequence appears to be identical to the $\tau$-Bockstein spectral sequence converging to stable motivic homotopy groups. We have only a computational understanding of this curious phenomenon. Our work calls for a more conceptual study of this relationship.

The simple pattern of weights in the motivic Adams-Novikov spectral sequence is the key idea that allows this program to proceed. See Theorem 6.1.4 for more explanation. For example, for simple degree reasons, there can be no hidden $\tau$ extensions in the motivic Adams-Novikov spectral sequence. Also for simple degree reasons, there are no “exotic” Adams-Novikov differentials; each non-zero motivic differential corresponds to a classical non-zero analogue.

Outline. Section 6.1 describes the motivic Adams-Novikov spectral sequence in general terms. Section 6.2 deals with specific properties of the motivic Adams-Novikov spectral sequence for the cofiber of $\tau$. The main point is that this spectral sequence collapses. Section 6.3 carries out the translation of information about $\pi_{*,*}(C\tau)$ into information about the classical Adams-Novikov spectral sequence.

Chapter 7 contains a series of tables that summarize the essential computational facts in a concise form. Tables 44, 45, and 46 list the extensions by 2, $\eta$, and $\nu$ that are hidden in the Adams-Novikov spectral sequence.

Table 47 gives a correspondence between elements of the classical Adams $E_\infty$-page and elements of the classical Adams-Novikov $E_\infty$-page. When possible, the table also gives an element of $\pi_*$ that is detected by these $E_\infty$ elements.

Tables 48 and 49 list the classical Adams-Novikov elements that are boundaries and that support differentials respectively. The tables list the corresponding elements of $\pi_{*,*}(C\tau)$.

Classical Adams-Novikov inputs. The point of this chapter is to deduce information about the Adams-Novikov spectral sequence from prior knowledge of the motivic stable homotopy groups obtained in Chapters 3, 4, and 5. To avoid circularity, Chapters 3, 4, and 5 intentionally avoid use of the Adams-Novikov spectral sequence whenever possible. However, we need a few computational facts about the Adams-Novikov spectral sequence in Chapter 4.
(1) Lemma \[4.2.7\] shows that a certain possible hidden \( \tau \) extension does not occur in the 57-stem. See also Remark \[4.1.12\]. For this, we use that \( \beta_{12/6} \) is the only element in the Adams-Novikov spectral sequence in the 58-stem with filtration 2 that is not divisible by \( \alpha_1 \). \[37\].

(2) Lemma \[4.2.35\] establishes a hidden 2 extension in the 54-stem. See also Remark \[4.1.18\]. For this, we use that \( \beta_{10/2} \) is the only element of the Adams-Novikov spectral sequence in the 54-stem with filtration 2 that is not divisible by \( \alpha_1 \), and that this element maps to \( \Delta^2 h_2^3 \) in the Adams-Novikov spectral sequence for \( \text{tmf} \). \[5\] \[37\].

Some examples.

Example 6.0.1. Consider the element \( \{h_1^2 h_3 g\} \) of \( \pi_{29,18} \). This element is killed by \( \tau^2 \) but not by \( \tau \).

The Adams-Novikov element \( \alpha_1 z_{28} \) detects \( \{h_1^2 h_3 g\} \) (see the charts in \[21\]). Therefore, \( \tau^2 \alpha_{1z_{28}} \) must be hit by some Adams-Novikov differential. This implies that there is a classical Adams-Novikov \( d_5 \) differential from the 30-stem to the 29-stem. This differential is well-known \[36\].

Example 6.0.2. Consider the element \( \{h_1^7 h_5 e_0\} \) of \( \pi_{55,33} \). This element is killed by \( \tau^4 \) but not by \( \tau^3 \).

The Adams-Novikov element \( \alpha_1 z_{54,10} \) detects \( \{h_1^7 h_5 e_0\} \) (see the charts in \[21\]). Therefore, \( \tau^4 \alpha_{1z_{54,10}} \) must be hit by some Adams-Novikov differential. This implies that there is a classical Adams-Novikov \( d_6 \) differential from the 56-stem to the 55-stem. This differential lies far beyond previous calculations.

6.1. The motivic Adams-Novikov spectral sequence

We adopt the following notation for the classical Adams-Novikov spectral sequence.

Definition 6.1.1. Let \( E_r(S^0;BP) \) (and \( E_\infty(S^0;BP) \)) be the pages of the classical Adams-Novikov spectral sequence for \( S^0 \). We write \( E^r_{s,f}(S^0;BP) \) for the part of \( E_r(S^0;BP) \) in stem \( s \) and filtration \( f \).

The even Adams-Novikov differentials \( d_{2r} \) are all zero, so we will only consider \( E_r(S^0;BP) \) when \( r \) is odd (or is \( \infty \)).

We now describe the motivic Adams-Novikov spectral sequence. Recall that \( BPL \) is the motivic analogue of the classical Brown-Peterson spectrum \( BP \).

Definition 6.1.2. Let \( E_r(S^{0,0};BPL) \) (and \( E_\infty(S^{0,0};BPL) \)) be the pages of the motivic Adams-Novikov spectral sequence for the motivic sphere \( S^{0,0} \). We write \( E^r_{s,f,w}(S^{0,0};BPL) \) for the part of \( E_r(S^{0,0};BPL) \) in stem \( s \), filtration \( f \), and weight \( w \).

Our goal is to describe the motivic Adams-Novikov spectral sequence in terms of the classical Adams-Novikov spectral sequence, as in \[17\] Theorem 8 and Section 4.

Definition 6.1.3. Define the tri-graded object \( \overline{E}_2(S^{0,0};BPL) \) such that:

1. \( \overline{E}^r_{2,f,w}(S^{0,0};BPL) \) is isomorphic to \( E^r_{2,s,f}(S^0;BP) \).
2. \( \overline{E}^s_{2,f,w}(S^{0,0};BPL) \) is zero if \( w \neq \frac{s+f}{2} \).
The following theorem completely describes the motivic $E_2(S^{0,0};BPL)$-page in terms of the classical $E_2(S^0;BP)$-page.

**Theorem 6.1.4.** [17] Theorem 8 and Section 4] The $E_2(S^{0,0};BPL)$-page of the motivic Adams-Novikov spectral sequence is isomorphic to the tri-graded object $E_2(S^{0,0};BPL) \otimes_{\mathbb{Z}^2} \mathbb{Z}_2[\tau]$, where $\tau$ has degree $(0, 0, -1)$.

In other words, in order to produce the motivic $E_2$-page, start with the classical $E_2$-page. At degree $(s, f)$, replace each copy of $\mathbb{Z}_2$ or $\mathbb{Z}/2^n$ with a copy of $\mathbb{Z}_2[\tau]$ or $\mathbb{Z}/2^n[\tau]$, where the generator has weight $\frac{s+f}{2}$.

We will now compare the classical and motivic Adams-Novikov spectral sequences. As we have seen in earlier chapters, $\tau$-localization corresponds to passage from the motivic to classical situations.

**Theorem 6.1.5.** After inverting $\tau$, the motivic Adams-Novikov spectral sequence is isomorphic to the classical Adams-Novikov spectral sequence tensored over $\mathbb{Z}_2$ with $\mathbb{Z}_2[\tau \pm 1]$.

**Proof.** The proof is analogous to the corresponding result for the motivic and classical Adams spectral sequences. See Proposition 3.0.2 and [19] Sections 3.2 and 3.4. □

6.2. The motivic Adams-Novikov spectral sequence for the cofiber of $\tau$

We will now study the motivic Adams-Novikov spectral sequence that computes the homotopy groups of the cofiber $C_{\tau}$ of $\tau$.

**Definition 6.2.1.** Let $E_*(C_{\tau};BPL)$ (and $E_\infty(C_{\tau};BPL)$) be the pages of the motivic Adams-Novikov spectral sequence for $C_{\tau}$. We write $E_2^{s,f,w}(C_{\tau};BPL)$ for the part of $E_2(C_{\tau};BPL)$ in stem $s$, filtration $f$, and weight $w$.

**Lemma 6.2.2.** $E_2(C_{\tau};BPL)$ is isomorphic to $E_2(S^{0,0};BPL)$.

**Proof.** The cofiber sequence

\[
S^{0,-1} \xrightarrow{\tau} S^{0,0} \xrightarrow{C_{\tau}} S^{1,-1}
\]

induces a long exact sequence

\[
\cdots \rightarrow E_2(S^{0,0};BPL) \xrightarrow{\tau} E_2(S^{0,0};BPL) \rightarrow E_2(C_{\tau};BPL) \rightarrow \cdots.
\]

Theorem 6.1.4 tells us that the map $\tau : E_2(S^{0,0};BPL) \rightarrow E_2(S^{0,0};BPL)$ is injective, so $E_2(C_{\tau};BPL)$ is isomorphic to the cokernel of $\tau$. Theorem 6.1.4 tells us that this cokernel is isomorphic to $E_2(S^{0,0};BPL)$. □

**Lemma 6.2.3.** There are no differentials in the motivic Adams-Novikov spectral sequence for $\tau$.

**Proof.** Lemma 6.2.2 tells us that $E_2(C_{\tau};BPL)$ is concentrated in tridegrees $(s, f, w)$ where $s + f - 2w$ equals zero. The Adams-Novikov $d_r$ differential increases $s + f - 2w$ by $r - 1$. Therefore, all differentials are zero. □

**Lemma 6.2.4.** There are no hidden $\tau$ extensions in $E_\infty(C_{\tau};BPL)$.
PROOF. Let \( x \) and \( y \) be two elements of \( E_\infty(C\tau; BPL) \) of degrees \( (s, f, w) \) and \( (s', f', w') \) with \( f' > f \). Then \( w' > w \) since \( w = \frac{s+f}{2} \) and \( w' = \frac{s'+f'}{2} \). For degree reasons, it is not possible that there is a hidden \( \tau \) extension from \( x \) to \( y \) because \( \tau \) has degree \((0, -1)\). \( \square \)

PROPOSITION 6.2.5. There is an isomorphism \( \pi_{s,w}(C\tau) \rightarrow E_2(S^0; BP) \) that takes the group \( \pi_{s,w}(C\tau) \) into \( E_2^s,2w-s(S^0; BP) \).

PROOF. Lemma 6.2.2 and Definition 6.1.8 say that \( E_2(C\tau; BPL) \) is isomorphic to \( E_2(S^0; BP) \). Lemma 6.2.3 implies that \( E_\infty(C\tau; BPL) \) is also isomorphic to \( E_2(S^0; BP) \). As in the proof of Lemma 6.2.4 for degree reasons there cannot be hidden extensions of any kind. Therefore, \( \pi_{s,w}(C\tau) \) is also isomorphic to \( E_2(S^0; BP) \). \( \square \)

6.3. Adams-Novikov calculations

We will now provide explicit calculations of the classical Adams-Novikov spectral sequence. The charts in [21] are an essential companion to this section.

6.3.1. The classical Adams-Novikov \( E_2 \)-page. We use the traditional notation for elements of the \( \alpha \) family, as described in [36]. We draw particular attention to \( \alpha_1 \) in degree \((1, 1)\) and \( \alpha_{2/2} \) in degree \((3, 1)\). These elements detect \( \eta \) and \( \nu \) respectively.

For elements not in the \( \alpha \) family, we have labelled decomposable elements as products whenever possible. For elements that are not known to be products, we use arbitrary symbols of the form \( z_{s,f} \) and \( z'_{s,f} \) for elements in the \( s \)-stem with filtration \( f \). When there is no ambiguity, we simplify this to \( z_s \) and \( z'_s \).

Our notation is unfortunately arbitrary and does not necessarily convey deeper structure. However, at least it allows us to give names to every element in the spectral sequence. Our notation is not compatible with the standard notation for elements of the Adams-Novikov spectral sequence [36].

EXAMPLE 6.3.1. Consider the elements in degree \((46, 4)\) in the Adams-Novikov \( E_2 \) chart in [21]. From left to right, they are \( \alpha_1^2z_{44,2} \), \( \alpha_1z_{45} \), \( \alpha_1z'_{45} \), and \( \alpha_{2/2}z_{43,3} \).

THEOREM 6.3.2. The \( E_2 \)-page of the classical Adams-Novikov spectral sequence is depicted through the 59-stem in the chart in [21]. The chart is complete except for the uncertainties described in Propositions 6.3.3 and 6.3.4, and the following:

1. \( \alpha_1z_{47,3} \) might equal \( 2\alpha_{2/2}z'_{45} \).
2. If \( \alpha_1z_{8}z'_{45} \) is non-zero, then \( 2z_{54,6} \) might equal \( \alpha_1z_{8}z'_{45} \).
3. \( \alpha_{2/2}z_{53} \) might equal \( \alpha_1^2z_{54,6} \).
4. \( 2z_{57} \) might equal \( \alpha_1z_{56,2} \).
5. \( \alpha_{2/2}z_{55} \) or \( \alpha_{2/2}^2z'_{55} \) might equal \( \alpha_1^2z_{56,4} \).
6. \( \alpha_{2/2}z_{56,4} \) might equal \( z_{59,5} \).
7. If \( \alpha_{60,4} \) is non-zero, then \( 2z_{60,4} \) might equal \( \alpha_1^2z_{58,2} \).

PROOF. This follows immediately from Proposition 6.2.3 and the calculation of \( \pi_{s,w}(C\tau) \) given in Chapter [5]. The uncertainties are consequences of uncertainties in the structure of \( \pi_{s,w}(C\tau) \). \( \square \)

PROPOSITION 6.3.3. Modulo elements of the form \( \alpha_k^\nu \alpha_{k/6} \), in the 53-stem, 54-stem, and 55-stem, either case (1) or case (2) occurs.
Therefore, \( \alpha^2 \) corresponds to the element \( E_0 \) corresponds in differential hitting proposition corresponds to the possibility that this differential does occur. Case (2) of the proposition corresponds to the possibility that this differential does not occur. Case (2) of Proposition 6.3.3 says that \( z_6 \) is isomorphic to \( \mathbb{Z}/2 \oplus \mathbb{Z}/2 \) with generators \( z_{59,5} \) and \( z'_{59,5} \);

and \( \alpha^2_{2/2}z_{47,3} \) is zero in \( E_2^{53,5}(S^0; BP) \).

(2) \( E_2^{54,6}(S^0; BP) \) has order two;

\( E_2^{55,5}(S^0; BP) \) has order two;

and \( \alpha^2_{2/2}z_{47,3} = z_{58}z_{45}' \) in \( E_2^{53,5}(S^0; BP) \).

**Proof.** In the motivic Adams spectral sequence for \( C_7 \), there is a possible \( d_3 \) differential hitting \( h_1B_8 \) discussed in Proposition 5.2.11 Case (1) of the proposition corresponds to the possibility that this differential does not occur. Case (2) of the proposition corresponds to the possibility that this differential does occur.

**Proposition 6.3.4.** Modulo elements of the form \( \alpha^n \alpha_k \), in the 59-stem and 60-stem, either case (1) or case (2) occurs.

(1) \( E_2^{59,5}(S^0; BP) \) is isomorphic to \( \mathbb{Z}/2 \oplus \mathbb{Z}/2 \), with generators \( \alpha^2_{2/2}z_{57} \) and \( z_{59,5} \);

and \( E_2^{60,4}(S^0; BP) \) has order four, containing two distinct non-zero elements \( z_{60,4} \) and \( \alpha^2_{2/2}z_{57} \).

(2) the only non-zero element of \( E_2^{59,5}(S^0; BP) \) is \( \alpha^2_{2/2}z_{57} \);

and \( E_2^{60,4}(S^0; BP) \) has order two.

**Proof.** In the motivic Adams spectral sequence for \( C_7 \), there is a possible \( d_4 \) differential hitting \( B_{21} \) discussed in Proposition 5.2.13 Case (1) of the proposition corresponds to the possibility that this differential does not occur. Case (2) of the proposition corresponds to the possibility that this differential does occur.

**Lemma 6.3.5.** Assume that case (1) of Proposition 6.3.3 occurs. Then \( \alpha z_{47,3} \) equals \( 2\alpha_{2/2}z_{45}' \).

**Proof.** Case (1) of Proposition 6.3.3 says that \( \alpha_{2/2}z_{47,3} \) is not divisible by \( \alpha_1 \). If \( \alpha z_{47,3} \) were zero, then we could shuffle Massey products to obtain

\[ \alpha^2_{2/2}z_{47,3} = (\alpha_1, \alpha_{2/2}, \alpha_1)z_{47,3} = \alpha_1(\alpha_{2/2}, \alpha_1, z_{47,3}). \]

Therefore, \( \alpha z_{47,3} \) must be non-zero.

Under the isomorphism of Proposition 6.2.20 the element \( z_{47,3} \) of \( E_2^{47,3}(S^0; BP) \) corresponds to the element \( h_{1/2}^7g_2 \) in \( \pi_{47,25}(C_7) \). We showed in Proposition 5.3.1 that the only possible hidden \( \eta \) extension on \( h_{1/2}^7g_2 \) takes the value \( h_0B_2 \), which corresponds in \( E_2^{48,26}(S^0; BP) \) to \( 2\alpha_{2/2}z_{45}' \).

**6.3.2. Adams-Novikov differentials.** Having obtained the Adams-Novikov \( E_2 \)-page, we next compute differentials.

**Theorem 6.3.6.** The differentials in the classical Adams-Novikov spectral sequence are depicted through the 59-stem in the chart in [21]. The chart is complete except for the following:

(1) if \( z_{55}' \) exists in \( E_2^{55,5}(S^0; BP) \), then \( d_3(z_{55}') = \alpha z_{53} \).

(2) if \( z_{60,4} \) exists in \( E_2^{60,4}(S^0; BP) \), then \( d_3(z_{60,4}) = z_{59,7}' \).
Theorem 6.3.7. The $E_\infty$-page of the classical Adams-Novikov is depicted in the chart in Figure 21 through the 59-stem. The chart includes all hidden extensions by 2, $\eta$, and $\nu$. The chart is complete except for the uncertainties described in Propositions 6.3.9 and 6.3.10, and the following:

1. There might be a hidden $\nu$ extension from $\alpha_2/2z_{45}$ to $z_{51}$.
2. There might be a hidden 2 extension from $2\alpha_4/4z_{44}$ to $z_{51}$.

Proof. The $E_\infty$-page can be computed directly from Theorems 6.3.2 and 6.3.6 because we know the $E_2$-page and all differentials up to some specified uncertainties.

The hidden extensions by 2, $\eta$, and $\nu$ all follow from extensions in $\pi_{*,*}(C\tau)$, as computed in Chapter 3.

Tables 14, 15, and 16 list all of the hidden extensions by 2, $\eta$, and $\nu$ in the motivic Adams-Novikov spectral sequence.

Remark 6.3.8. From Lemma 6.3.31, the possible extension (1) in Theorem 6.3.7 occurs if and only if the possible extension (2) occurs.

Proposition 6.3.9. In the 53-stem, 54-stem, and 55-stem, either case (1) or case (2) occurs.

1. $\alpha_1z_8z_{45}'' = \alpha_2^3/2z_{45}''$ is a non-zero element of $E^{54,6}_\infty(S^0; BP)$; $E^{54,6}_\infty(S^0; BP)$ is zero;
   $\alpha_2/2z_{50}$ is zero in $E^{53,5}_\infty(S^0; BP)$;
   and there is a hidden $\nu$ extension from $z_{50}$ to $z_{53}$.

2. $E^{54,6}_\infty(S^0; BP)$ is zero;
   $\alpha_1z_{53}$ is a non-zero element of $E^{54,8}_\infty(S^0; BP)$;
   $\alpha_2/2z_{50} = z_8z_{45}''$ in $E^{53,5}_\infty(S^0; BP)$;
   and there is a hidden $\nu$ extension from $\alpha_2^2/2z_{45}'$ to $\alpha_1z_{53}$.

Proof. The two cases are associated with the two cases of Proposition 6.3.3. See also the first uncertainty in Theorem 6.3.6.

Proposition 6.3.10. In the 59-stem, either case (1) or case (2) occurs.

1. $z_{59,5}$ is the only non-zero element of $E^{59,5}_\infty(S^0; BP)$; $z_{59,7}$ is the only non-zero element of $E^{59,7}_\infty(S^0; BP)$.

2. $E^{59,5}_\infty(S^0; BP)$ is zero;
   and $E^{59,7}_\infty(S^0; BP)$ has two generators $z_{59,7}$ and $z_{59,7}'$.

Proof. The two cases are associated with the two cases of Proposition 6.3.4. See also the second uncertainty in Theorem 6.3.6.
Table 1: Notation for $\pi_{*,*}$

| element | $(s, w)$ | Ext | definition |
|---------|---------|-----|------------|
| $\tau$  | $(0, -1)$ | $\tau$ |  |
| $2$     | $(0, 0)$ | $h_0$ |  |
| $\eta$  | $(1, 1)$ | $h_1$ |  |
| $\nu$   | $(3, 2)$ | $h_2$ |  |
| $\sigma$| $(7, 4)$ | $h_3$ |  |
| $\epsilon$ | $(8, 5)$ | $c_0$ |  |
| $\mu_{8k+1}$ | $(1, 1) + k(8, 4)$ | $P_k h_1$ |  |
| $\zeta_{8k+3}$ | $(3, 2) + k(8, 4)$ | $P_k h_2$ |  |
| $\kappa$ | $(14, 8)$ | $d_0$ |  |
| $\rho_{15}$ | $(15, 8)$ | $h_3^3 h_4$ |  |
| $\eta_4$ | $(16, 9)$ | $h_1 h_4$ | $\eta^3 \cdot \eta_4 = 0$ |
| $\nu_4$ | $(18, 10)$ | $h_2 h_4$ | $\nu_4 = \langle 2\sigma, \sigma, \nu \rangle$ |
| $\bar{\sigma}$ | $(19, 11)$ | $c_1$ |  |
| $\bar{\pi}$ | $(20, 11)$ | $\tau g$ |  |
| $\rho_{23}$ | $(23, 12)$ | $h_5^2 i$ |  |
| $\theta_4$ | $(30, 16)$ | $h_4^2$ |  |
| $\rho_{31}$ | $(31, 16)$ | $h_0^{10} h_5$ |  |
| $\eta_{5}$ | $(32, 17)$ | $h_1 h_5$ | $\eta_5 \in \langle \eta, 2, \theta_4 \rangle$, $\eta^7 \cdot \eta_5 = 0$ |
| $\theta_{4.5}$ | $(45, 24)$ | $h_4^3$ | $4\theta_{4.5} \in \{h_0 h_5 d_0\}$, $\eta \theta_{4.5} \in \{B_1\}$ |
|          |         |     | $\sigma \theta_{4.5} \notin \{\tau h_1 h_3 g_2\}$ |

Table 2: May $E_2$-page generators

| $(m, s, f, w)$ | $d_2$ | description |
|---------------|-------|-------------|
| $h_0$ | $(1, 0, 1, 0)$ | $h_{10}$ |
| $h_1$ | $(1, 1, 1, 1)$ | $h_{11}$ |
| $h_2$ | $(1, 3, 1, 2)$ | $h_{12}$ |
| $b_{20}$ | $(4, 4, 2, 2)$ | $\tau h_1^3 + h_0^2 h_2$ | $h_{20}$ |
| $h_3$ | $(1, 7, 1, 4)$ | $h_{13}$ |
| $h_{0(1)}$ | $(4, 7, 2, 4)$ | $h_0 h_2^2$ | $h_{20} h_{21} + h_{11} h_{30}$ |
| $b_{21}$ | $(4, 10, 2, 6)$ | $h_3^2 + h_1^2 h_3$ | $h_{21}^2$ |
| $b_{30}$ | $(6, 12, 2, 6)$ | $\tau h_1 b_{21} + h_3 b_{20}$ | $h_{30}$ |
| $h_4$ | $(1, 15, 1, 8)$ | $h_{14}$ |
7. TABLES

Table 2: May $E_2$-page generators

| $(m, s, f, w)$ | $d_2$ | description |
|---------------|-------|-------------|
| $h_1(1)$     | $(4, 16, 2, 9)$ | $h_1^3 + h_3^2$ | $h_2h_{22} + h_{12}h_{31}$ |
| $b_{22}$     | $(4, 22, 2, 12)$ | $h_3^3 + h_2^2h_4$ | $h_{22}^2$ |
| $b_{31}$     | $(6, 26, 2, 14)$ | $h_4b_{21} + h_3b_{22}$ | $h_3^3$ |
| $b_{30}$     | $(8, 28, 2, 14)$ | $h_4b_{30} + \tau h_1b_{31}$ | $h_{30}^2$ |
| $h_5$        | $(1, 31, 1, 16)$ | | $h_{15}$ |
| $h_2(1)$     | $(4, 34, 2, 18)$ | $h_2h_5^2$ | $h_{22}^2h_{23} + h_{13}h_{32}$ |
| $h_{0}(1, 3)$| $(7, 38, 3, 20)$ | $h_4^3h_0(1) + h_0h_2h_2(1)$ | $h_{50}^2h_{11}h_{13} + h_{40}h_{11}h_{23} + h_{20}h_{41}h_{13} + h_{20}h_{31}h_{23}$ |
| $b_{23}$     | $(4, 46, 2, 24)$ | $h_4^3 + h_3^2h_5$ | $h_{23}^2$ |
| $h_{0}(1, 2)$| $(9, 46, 3, 24)$ | $h_3h_0(1, 3)$ | $h_{30}h_{31}h_{32} + h_{30}h_{41}h_{22} + h_{40}h_{21}h_{32} + h_{40}h_{41}h_{12} + h_{50}h_{21}h_{22} + h_{50}h_{31}h_{12}$ |
| $b_{32}$     | $(6, 54, 2, 28)$ | $h_5b_{22} + h_3b_{23}$ | $h_{32}^2$ |
| $b_{41}$     | $(8, 58, 2, 30)$ | $h_5b_{31} + h_2b_{32}$ | $h_{31}^2$ |
| $b_{30}$     | $(10, 60, 2, 30)$ | $h_5b_{40} + \tau h_1b_{41}$ | $h_{50}^2$ |
| $h_6$        | $(1, 63, 1, 32)$ | | $h_{16}$ |
| $h_3(1)$     | $(4, 70, 2, 36)$ | $h_3h_5^2$ | $h_{23}h_{24} + h_{14}h_{43}$ |

Table 3: May $E_2$-page relations

| relation          | $(m, s, f, w)$ |
|-------------------|---------------|
| $h_{0}h_1$        | $(2, 1, 2, 1)$ |
| $h_{1}h_2$        | $(2, 4, 2, 3)$ |
| $h_{2}b_{20} = h_{0}h_{0}(1)$ | $(5, 7, 3, 4)$ |
| $h_{2}h_3$        | $(2, 10, 2, 6)$ |
| $h_{2}h_{0}(1) = h_{0}b_{21}$ | $(5, 10, 3, 6)$ |
| $h_{2}h_{0}(1)$   | $(5, 14, 3, 8)$ |
| $h_{0}(1)^2 = h_{2}b_{21} + h_{1}^2h_{30}$ | $(8, 14, 4, 8)$ |
| $h_{0}h_{1}(1)$   | $(5, 16, 3, 9)$ |
| $h_{3}b_{21} = h_{1}h_{1}(1)$ | $(5, 17, 3, 10)$ |
| $b_{20}h_{1}(1) = h_{1}h_{3}b_{30}$ | $(8, 20, 4, 11)$ |
| $h_{3}h_{4}$     | $(2, 22, 2, 12)$ |
| $h_{3}h_{1}(1) = h_{1}b_{22}$ | $(5, 23, 3, 13)$ |
| $h_{0}(1)h_{1}(1)$ | $(8, 23, 4, 13)$ |
| $b_{20}b_{22} = h_{0}^2b_{31} + h_{3}^2b_{30}$ | $(8, 26, 4, 14)$ |
| $b_{22}h_{0}(1) = h_{0}h_{2}b_{31}$ | $(8, 29, 4, 16)$ |
| $h_{4}h_{1}(1)$  | $(5, 31, 3, 17)$ |
| $h_{1}(1)^2 = h_{2}b_{22} + h_{2}^2b_{31}$ | $(8, 32, 4, 18)$ |
| $h_{1}h_{2}(1)$  | $(5, 35, 3, 19)$ |
| $h_{4}b_{22} = h_{2}h_{2}(1)$ | $(5, 37, 3, 20)$ |
| $b_{20}b_{2}(1) = h_{0}h_{0}(1, 3)$ | $(8, 38, 4, 20)$ |
| $h_{2}h_{0}(1, 3) = h_{0}h_{4}b_{31}$ | $(8, 41, 4, 22)$ |
| $h_{0}(1)h_{2}(1) = h_{0}h_{4}b_{31}$ | $(8, 41, 4, 22)$ |
7. Tables

Table 3: May $E_2$-page relations

| relation | $(m, s, f, w)$ |
|----------|----------------|
| $b_{21}b_2(1) = h_2h_4b_{31}$ | (8, 44, 4, 24) |
| $h_0(1)h_0(1, 3) = h_2^2 h_4b_{40} + h_4b_{20}b_{31}$ | (11, 45, 5, 24) |
| $h_4h_5$ | (2, 46, 2, 24) |
| $b_{30}h_2(1) = h_0h_0(1, 2) + h_2h_4b_{40}$ | (10, 46, 4, 24) |
| $b_{21}h_0(1, 3) = h_2^2 h_0(1, 2) + h_4b_{21}h_0(1)$ | (11, 48, 5, 26) |
| $h_4h_2(1) = h_2b_{23}$ | (5, 49, 3, 26) |
| $h_1(1)h_2(1)$ | (8, 50, 4, 27) |
| $b_{30}h_0(1, 3) = b_{20}h_0(1, 2) + h_4b_{20}h_0(1)$ | (13, 50, 5, 26) |
| $b_{23}h_0(1) = h_4h_0(1, 3)$ | (8, 53, 4, 28) |
| $h_0(1)h_0(1, 2) = h_4b_{40}b_{21} + h_4b_{30}b_{31}$ | (13, 53, 5, 28) |
| $h_1(1)h_0(1, 3) = h_1h_3h_0(1, 2)$ | (11, 54, 5, 29) |
| $b_{21}b_{23} = h_1^2 b_{32} + h_3^2 b_{31}$ | (8, 56, 4, 30) |
| $b_{30}b_{23} = b_{20}b_{32} + h_3^2 h_{11} + h_3^2 b_{40}$ | (10, 58, 4, 30) |
| $b_{22}h_0(1, 3) = h_2^2 h_0(1, 2) + h_0b_{31}h_2(1)$ | (11, 60, 5, 32) |
| $b_{32}h_0(1) = h_4h_0(1, 2) + h_0h_2b_{41}$ | (10, 61, 4, 32) |
| $b_{23}h_1(1) = h_1h_3b_{32}$ | (8, 62, 4, 33) |
| $h_5h_2(1)$ | (5, 65, 3, 34) |
| $b_{22}b_{23} = h_2(1)^2 + h_3^2 b_{32}$ | (8, 68, 4, 36) |
| $h_5h_0(1, 3)$ | (8, 69, 4, 36) |

Table 4: The May $d_4$ differential

| $(m, s, f, w)$ | description | $d_4$ |
|---------------|-------------|-------|
| $P$ (8, 8, 4, 4) | $b_2^2$ | $h_3^4h_5$ |
| $\nu$ (7, 15, 3, 8) | $h_2b_{30}$ | $h_2^3 h_3^3$ |
| $g$ (8, 20, 4, 12) | $b_{31}^2$ | $h_4^4 h_3$ |
| $\Delta$ (12, 24, 4, 12) | $b_{30}^2$ | $\tau_2^2 h_2 g + Ph_4$ |
| $\nu_1$ (7, 33, 3, 18) | $h_3b_{31}$ | $h_2^6 h_4^2$ |
| $x_{34}$ (7, 34, 5, 18) | $h_4^3 h_2(1) + h_0h_2^2 b_{20}$ | $x_{34}$ |
| $x_{35}$ (10, 35, 4, 18) | $h_0h_3b_{40}$ | $\tau h_2^3 g_2$ |
| $x_{47}$ (13, 47, 5, 25) | $h_2b_{40}h_1(1)$ | $h_0h_3c_2$ |
| $x_{49}$ (10, 49, 4, 26) | $h_2h_0(1, 2)$ | $h_2^3 h_3 c_2$ |
| $\Delta_1$ (12, 52, 4, 28) | $b_{31}^2$ | $h_5 g + h_3 g_2$ |
| $\Gamma$ (16, 56, 4, 28) | $b_{40}^2$ | $\Delta h_5 + \tau^2 \Delta h_2$ |
| $x_{59}$ (19, 59, 7, 31) | $h_2b_{30}b_{40}h_1(1)$ | $\tau^2 e_1 g$ |
| $x_{63}$ (20, 63, 8, 33) | $h_1b_{20}h_{30}h_0(1, 2) + \tau h_4 h_2^2 b_{31}h_0(1)$ | $\tau h_5 d_0 e_0$ |
| $x_{65}$ (20, 65, 8, 34) | $h_0h_3b_{20}b_{31}b_{40}$ | $h_0^3 A''$ |
| $x_{68}$ (16, 68, 6, 36) | $h_2^3 b_{31}b_{40} + \tau h_4^2 h_3 h_1(1)$ | $\tau s_1$ |
| $\nu_2$ (7, 69, 3, 36) | $h_4b_{32}$ | $h_2^3 h_4^2$ |
| $x_{69}$ (13, 69, 5, 36) | $h_3 b_{40}h_2(1)$ | $h_5^2 d_2$ |
Table 5: The May $d_6$ differential

| $(m, s, f, w)$ | description | $d_6$ |
|---------------|-------------|-------|
| $P_i$         | $P^2\nu$    | $h_0^3s$ |
| $P_r$         | $P\Delta h_2^2$ | $h_0^6x$ |
| $Y$           | $Bh_0(1)$    | $h_0^5g_2$ |
| $\phi$        | $Bh_1b_2$    | $h_1^3h_5d_0$ |
| $X$           | $Bh_0b_2h_3$ | $P_{h_0h_5d_0}$ |
| $PQ'$         | $P^2\Delta h_2^2\nu$ | $h_0^6X$ |
| $x_{56}$      | $Bh_1b_2h_0(1)$ | $h_1^2h_5c_0d_0$ |
| $x'_{56}$     | $PBB_{23}$   | $P_{h_1^2h_5d_0}$ |
| $\phi'$       | $Bh_0b_{30}h_0(1)$ | $P_{h_0h_5c_0}$ |
| $P_{x_{56}}$  | $P_{h_1^2h_5c_0d_0}$ |
| $P_{x'_{56}}$ | $P_{h_2h_5d_0}$ |
| $B_{23}$      | $Yg$         | $h_1^2h_5d_0c_0$ |
| $P\phi'$      | $P_{h_0h_5c_0}$ |
| $c_0g^3$      | $h_1^{10}D_4$ |
| $\Delta h_0^2Y$ | $\Delta h_0^5g_2 + h_0h_5d_0i$ |

Table 6: The May $d_8$ differential

| element      | $(m, s, f, w)$ | description | $d_8$ |
|--------------|---------------|-------------|-------|
| $P^2$        | $(16, 16, 8, 8)$ | $h_0^3h_4$ |
| $\Delta h_3$ | $(13, 31, 5, 16)$ | $h_1^3h_5$ |
| $g^2$        | $(16, 40, 8, 24)$ | $h_1^2h_5$ |
| $w$          | $(21, 45, 9, 25)$ | $\Delta h_1g$ | $P_{h_1^2h_5}$ |
| $\Delta^2$   | $(24, 48, 8, 24)$ | $P^2h_5^2$ |
| $\Delta c_0g$| $(25, 52, 11, 29)$ | $P_{h_1^2h_5c_0}$ |
| $Q_3$        | $(13, 67, 5, 36)$ | $\Delta_1h_4$ | $h_1^3h_5^2$ |
| $\Gamma h_0h_3^2$ | $(19, 70, 7, 36)$ | $h_1^3p^i$ |

Table 7: Higher May differentials

| element      | $(m, s, f, w)$ | $d_r$ | value   |
|--------------|---------------|-------|---------|
| $P^2Q'$      | $(45, 63, 21, 32)$ | $d_{12}$ | $P_{h_0^3}h_5i$ |
| $P^4$        | $(32, 32, 16, 16)$ | $d_{16}$ | $h_0^{16}h_5$ |
| $\Delta^2h_4$ | $(25, 63, 9, 32)$ | $d_{16}$ | $h_0^8h_5^2$ |
| $P^8$        | $(64, 64, 32, 32)$ | $d_{32}$ | $h_0^{32}h_6$ |
| element | \( (m, s, f, w) \) | May description | \( d_2 \) | reference |
|---------|------------------|-----------------|----------|----------|
| \( h_0 \) | \( (1, 0, 1, 0) \) | \( h_1 h_0(1) \) | \( h_0 h_3^2 \) | image of \( J \) |
| \( h_1 \) | \( (1, 1, 1, 1) \) | \( h_2 \) | \( h_1^2 d_0 \) | Lemma 3.3.1 |
| \( h_2 \) | \( (1, 3, 1, 2) \) | \( h_3 \) | \( h_2 h_1(1) \) | |
| \( h_3 \) | \( (1, 7, 1, 4) \) | \( d_0 \) | \( h_0(1)^2 \) | |
| \( c_0 \) | \( (5, 8, 3, 5) \) | \( e_0 \) | \( b_{21} h_0(1) \) | |
| \( Ph_1 \) | \( (9, 9, 5, 5) \) | \( f_0 \) | \( h_2 \nu \) | \( h_0^2 e_0 \) | \( \text{tmf} \) |
| \( Ph_2 \) | \( (9, 11, 5, 6) \) | \( c_1 \) | \( h_2 h_1(1) \) | |
| \( d_1 \) | \( (13, 16, 7, 9) \) | \( e_0 \) | \( b_{21}^2 \) | |
| \( e_0 \) | \( (8, 17, 4, 10) \) | \( P^2 h_1 \) | \( h_0 b_{30} h_0(1)^2 \) | \( Ph_0 e_0 \) | \( \text{tmf} \) |
| \( P^2 h_2 \) | \( (17, 19, 9, 10) \) | \( f_0 \) | \( h_0 h_2^2 \) | |
| \( \tau g \) | \( (8, 20, 4, 11) \) | \( h_2 g \) | \( h_0 d_0^2 \) | \( \text{tmf} \) |
| \( \tau g \) | \( (8, 20, 4, 11) \) | \( i \) | \( P \nu \) | \( Ph_0 d_0 \) | \( \text{tmf} \) |
| \( P^2 c_0 \) | \( (21, 24, 11, 13) \) | \( j \) | \( \nu h_4 \) | \( Ph_0^2 d_0 \) | Lemma 3.3.1 |
| \( P^3 c_0 \) | \( (25, 25, 13, 13) \) | \( k \) | \( h_3 y \) | \( h_0^2 \) | \( \text{Lemma 3.3.3} \) |
| \( P^3 h_1 \) | \( (25, 25, 13, 13) \) | \( l \) | \( h_0 d_0 \) | \( \text{tmf} \) |
| \( \Delta h_2 \) | \( (15, 29, 7, 16) \) | \( m \) | \( \nu h_4 \) | \( \text{image of} \ J \) |
| \( \Delta h_1 h_3 \) | \( (14, 30, 6, 16) \) | \( n \) | \( h_2 b_{30} h_1(1) \) | |
| \( \Delta h_1 h_2 \) | \( (24, 30, 12, 16) \) | \( d_1 \) | \( h_1(1)^2 \) | |
| \( \Delta h_1 h_3 \) | \( (15, 32, 7, 18) \) | \( q \) | \( e_0 \) | \( h_0 d_0 e_0 \) | Lemma 3.3.2 |
| \( \nu h_1 \) | \( (29, 32, 15, 17) \) | \( P^1 e_0 \) | \( h_0 \) | \( Ph_0 e_0 \) | \( \text{Lemma 3.3.4} \) |
| \( \nu h_1 \) | \( (8, 33, 4, 18) \) | \( P^2 e_0 \) | \( h_0 \nu_1 \) | \( \text{Lemma 3.3.1} \) |
| \( \nu h_1 \) | \( (24, 33, 12, 18) \) | \( P^2 h_1 \) | \( P^2 h_2 \) | \( \text{Lemma 3.3.1} \) |
| \( \nu h_1 \) | \( (33, 33, 17, 17) \) | \( P^2 h_1 \) | \( P^2 h_2 \) | \( \text{Lemma 3.3.1} \) |
| \( \nu h_1 \) | \( (33, 33, 17, 17) \) | \( P^2 h_2 \) | \( \text{Lemma 3.3.1} \) |
| \( \nu h_1 \) | \( (33, 35, 17, 18) \) | \( P^4 h_2 \) | \( \text{Lemma 3.3.1} \) |
| \( \tau b_{21}^2 h_1(1) + h_1^2 b_{22} b_{30} \) | \( (12, 36, 6, 20) \) | \( t \) | \( \tau b_{21}^2 h_1(1) \) | |
| \( h_2^2 b_{22} b_{30} + h_2^2 b_{40} \) | \( (11, 37, 5, 20) \) | \( x \) | \( \tau b_{21}^2 h_1(1) \) | |
| \( h_1^2 e_0^2 \) | \( (16, 37, 8, 22) \) | \( e_{0g} \) | \( h_2^2 e_0 \) | \( \text{Lemma 3.3.4} \) |
| \( b_{22} h_1(1) \) | \( (8, 38, 4, 21) \) | \( e_1 \) | \( b_{22} h_1(1) \) | |
Table 8: Adams $E_2$ generators

| element | $(m, s, f, w)$ | May description | $d_2$ | reference |
|---------|---------------|-----------------|-------|-----------|
| $y$     | (14, 38, 6, 20) | $\Delta h^3_3$  | $h^3_0 x$ | Table 18 |
| $P^3 d_0$ | (32, 38, 16, 20) |                | | |
| $c_1 g$  | (13, 39, 7, 23) |                | | |
| $u$      | (21, 39, 9, 21) | $\Delta h_1 d_0$ | | |
| $P^2 j$  | (31, 39, 15, 20) |                | | |
| $f_1$    | (8, 40, 4, 22)  | $h_3 v_1$      | $P^3 h_0 d_0$ | tmf |
| $\tau^g_2$ | (16, 40, 8, 23) |                | | |
| $P^4 c_0$ | (37, 40, 19, 21) |                | | |
| $c_2$    | (5, 41, 3, 22)  | $h_3 h_2(1)$   | $h_0 f_1$ | Table 18 |
| $z$      | (22, 41, 10, 22) | $\Delta h^2_5 e_0$ | | |
| $P^3 e_0$ | (32, 41, 16, 22) |                | $P^3 h^2_1 d_0$ | Lemma 3.3.1 |
| $P^5 b_1$ | (41, 41, 21, 21) |                | | |
| $v$      | (21, 42, 9, 23)  | $\Delta h_1 e_0$ | $h^3_1 u$ | Table 18 |
| $P^2 j$  | (31, 42, 15, 22) |                | $P^3 h_0 e_0$ | tmf |
| $h^2 g^2$ | (17, 43, 9, 26)  |                | | |
| $P^5 h_2$ | (41, 43, 21, 22) |                | | |
| $g_2$    | (8, 44, 4, 24)   | $b^2_2$         | | |
| $\tau w$ | (21, 45, 9, 24)  | $\tau \Delta h_1 g$ | | |
| $B_1$    | (17, 46, 7, 25)  | $Y h_1$         | | |
| $N$      | (18, 46, 8, 25)  | $\Delta h_2 c_1$ | | |
| $u'$     | (25, 46, 11, 25) | $\Delta e_0 d_0$ | $\tau h_0 d^2_0 e_0$ | Lemma 3.3.5 |
| $h^3 g^2$ | (17, 47, 9, 28)  |                | $h_0 h^3 g^2$ | Lemma 3.3.3 |
| $P^4 d_0$ | (40, 46, 20, 24) |                | | |
| $Q'$     | (29, 47, 13, 24) | $P \Delta h^0_0 e_0$ | $h_0 v^2$ | Table 18 |
| $P u$    | (29, 47, 13, 25) | $P \Delta h_1 d_0$ | | |
| $B_2$    | (17, 48, 7, 26)  | $Y h_2$         | | |
| $P^5 c_0$ | (45, 48, 23, 25) |                | | |
| $v'$     | (25, 49, 11, 27) | $\Delta e_0 e_0$ | $h^3_1 u' + \tau h_0 d_0 e_0^2$ | Lemma 3.3.5 |
| $P^4 e_0$ | (40, 49, 20, 26) |                | $P^4 h^2_1 d_0$ | Lemma 3.3.1 |
| $P^6 h_1$ | (49, 49, 25, 25) |                | | |
| $C$      | (14, 50, 6, 27)  | $h_2 x_{47}$    | | |
| $gr$     | (22, 50, 10, 28) | $\Delta h^2 g$ | | |
| $P v$    | (29, 50, 13, 27) | $P \Delta h_1 e_0$ | $Ph^2_1 u$ | Table 18 |
| $P^3 j$  | (39, 50, 19, 26) |                | $P^4 h_0 e_0$ | tmf |
| $G_3$    | (21, 51, 9, 28)  | $\Delta h_3 g$ | $h_0 g r$ | Lemma 3.3.6 |
| $gn$     | (19, 51, 9, 29)  |                | | |
| $P^6 h_2$ | (49, 51, 25, 26) |                | | |
| $D_1$    | (11, 52, 5, 28)  | $h_2 x_{49}$    | $h^2_0 h_3 g_2$ | Lemma 3.3.13 |
| $d_1 g$  | (16, 52, 8, 30)  |                | | |
| $i_1$    | (15, 53, 7, 30)  | $g v_1$         | | Lemma 3.3.3 |
| $B_8$    | (21, 53, 9, 29)  | $Y c_0$         | | |
| $x'$     | (24, 53, 10, 28) | $P Y$           | | |
| $\tau G$ | (14, 54, 6, 29)  | $\tau \Delta h_1^2$ | $h_5 g_0 d_0$ | Lemma 3.3.12 |
| $R_1$    | (26, 54, 10, 28) | $\Delta^2 h^2_2$ | $h^2_0 x'$ | Table 18 |
Table 8: Adams $E_2$ generators

| element | $(m, s, f, w)$ | May description | $d_2$ | reference |
|---------|----------------|----------------|-------|-----------|
| $P^3u'$ | $(33, 54.1, 29)$ | $P \Delta c_0 d_0$ | $\tau P h_0 d_0^2 e_0$ | Lemma 3.3.3.5 |
| $P^5d_0$ | $(48, 54, 24, 29)$ | | | | |
| $B_6$ | $(17, 55, 7, 30)$ | $B h_1 h_1(1)$ | | Lemma 3.3.7 |
| $g$ | $(23, 55, 11, 32)$ | $g^2 \nu$ | $h_0 \epsilon_{0}^2$ | Lemma 3.3.4 |
| $P_{c_2}^4$ | $(37, 55, 17, 29)$ | | | | |
| $P_{d_0}^4i$ | $(47, 55, 23, 28)$ | | $P^5 h_0 d_0$ | tmf |
| $g t$ | $(20, 56, 10, 32)$ | | | | |
| $Q_1$ | $(26, 56, 10, 29)$ | $\Delta^2 h_1 h_3$ | $\tau h_1^3 x'$ | Lemma 3.3.11 |
| $P_{c_0}^6e_0$ | $(53, 56, 27, 29)$ | | | | |
| $D_4$ | $(14, 57, 6, 31)$ | $h_1 b_2 h_0(1, 2)$ | $h_0 B_6$ | Lemma 3.3.11 |
| $Q_2$ | $(19, 57, 7, 30)$ | $\Delta \nu_3$ | | | |
| $D_{11}$ | $(21, 57, 9, 31)$ | $\Delta h_1 d_1$ | | | |
| $\nu'$ | $(33, 57, 15, 31)$ | $P \Delta c_0 e_0$ | $P h_1^4 u' + P^5 h_1^2 d_0^4$ | Lemma 3.3.5 |
| | | | + $\tau h_0 d_0^4$ | | |
| $P^5 e_0$ | $(48, 57, 24, 30)$ | | $P^5 h_1^2 d_0$ | Lemma 3.3.4 |
| $P^7 h_1$ | $(57, 57, 29, 29)$ | | | | |
| $D_2$ | $(16, 58, 6, 30)$ | $h_0 b_3 h_0(1, 2)$ | $h_0 Q_2$ | Lemma 3.3.15 |
| $e_1 g$ | $(16, 58, 8, 33)$ | | | | |
| $P^2 e_0$ | $(37, 58, 17, 31)$ | $h_0 b_3 h_0(1, 2)$ | | | |
| $P^4 j$ | $(47, 58, 23, 30)$ | $P^2 h_1^2 u$ | Table 18 | | |
| $j_1$ | $(15, 59, 7, 33)$ | $h_1 b_2 b_2 b_3 b_3$ | | | |
| $B_{21}$ | $(24, 59, 10, 32)$ | $Y d_0$ | | | |
| $c_3 g^2$ | $(21, 59, 11, 35)$ | | | | |
| $P^7 h_2$ | $(57, 59, 29, 30)$ | | | | |
| $B_{3}$ | $(17, 60, 7, 32)$ | $Y h_4$ | | | |
| $B_{4}$ | $(23, 60, 9, 32)$ | $Y \nu$ | $h_0 B_{21}$ | Lemma 3.3.10 |
| $\tau g^3$ | $(24, 60, 12, 35)$ | | | | |
| $h_0 g^3$ | $(25, 60, 13, 36)$ | | | | |
| $D_{3}$ | $(10, 61, 4, 32)$ | $h_4 h_0(1, 2)$ | | | |
| $A$ | $(16, 61, 6, 32)$ | $h_2 b_3 h_0(1, 2)$ | $h_0 B_{3}$ | Lemma 3.3.15 |
| $A'$ | $(16, 61, 6, 32)$ | $h_2 b_3 h_0(1, 2)$ | + $h_0 h_3 b_3 b_3 b_0$ | | |
| $B_{7}$ | $(17, 61, 7, 33)$ | $B h_1 b_2$ | | | |
| $X_1$ | $(23, 61, 9, 32)$ | $\Delta x$ | $h_0^2 B_{4} + \tau h_1 B_{21}$ | Lemma 3.3.12 |
| $h_0 x_0$ | $(33, 61, 7, 33)$ | | | | |
| $C_{2}$ | $(20, 62, 8, 33)$ | $h_2 x_5$ | | | |
| $E_{2}$ | $(20, 62, 8, 33)$ | $\Delta e_1$ | | | |
| $B_{22}$ | $(24, 62, 10, 34)$ | $Y e_0$ | $h_0^2 B_{21}$ | Lemma 3.3.10 |
| $R$ | $(26, 62, 10, 32)$ | $\Delta^2 h_3^2$ | $\tau P^2 h_0 d_0^4 e_0$ | Lemma 3.3.5 |
| $P^2 u'$ | $(41, 62, 19, 33)$ | | | | |
| $P^6 d_0$ | $(56, 62, 28, 32)$ | | | | |
| $h_6$ | $(1, 63, 1, 32)$ | | $h_0 h_5^2$ | image of $J$ |
| $C'$ | $(17, 63, 7, 34)$ | $h_2 b_3 h_1(1)^2$ | | Lemma 3.3.17 |
| $X_2$ | $(17, 63, 7, 34)$ | $\tau h_1 b_2 b_3^2 + h_0^2 B_{3}$ | | Lemma 3.3.18 |
Table 8: Adams $E_2$ generators

| element | $(m, s, f, w)$ | May description | $d_2$ | reference |
|---------|----------------|-----------------|-------|-----------|
| $h_{2g}^3$ | (25, 63, 13, 38) | $+h_1^2 h_3 b_{31} b_{40}$ | $h_0 X_2$ | Lemma 3.3.19 |
| $P^3 u$ | (45, 63, 21, 33) | | | |
| $A''$ | (14, 64, 6, 34) | $h_0 b_{31} h_0(1, 3)$ | $h_0 U$ | Lemma 3.3.10 |
| $q_1$ | (26, 64, 10, 33) | $\Delta^2 h_1 h_4$ | | |
| $U$ | (34, 64, 14, 34) | $\Delta^2 h_1^2 d_0$ | $Ph_1^2 x'$ | Lemma 3.3.10 |
| $P^7 e_0$ | (61, 64, 31, 33) | | | |
| $k_1$ | (15, 65, 7, 36) | $d_1 \nu_1$ | | |
| $\tau B_{23}$ | (24, 65, 10, 35) | | | |
| $R_2$ | (33, 65, 13, 34) | $\Delta^2 h_0 e_0$ | $h_0 U$ | Lemma 3.3.10 |
| $\tau gw$ | (29, 65, 13, 36) | | | |
| $P^2 v'$ | (41, 65, 19, 35) | $P^2 h_1^2 u'$ + $+\tau Ph_0 d_0^4$ | | |
| $P^6 e_0$ | (56, 65, 28, 34) | $P^6 h_1^2 d_0$ | | Lemma 3.3.1 |
| $P^8 h_1$ | (65, 65, 33, 33) | | | |
| $r_1$ | (14, 66, 6, 36) | $\Delta_1 h_3^2$ | | |
| $\tau G_0$ | (17, 66, 7, 35) | $\tau h_2^2 h_0(1, 2)$ + $+h_1 h_3 b_{30} h_0(1, 2)$ | $h_2 C_0 +$ $+h_1 h_3 Q_2$ | Lemma 3.3.19 |
| $\tau B_5$ | (24, 66, 10, 35) | $\tau h_2^2 B_{23}$ | | Lemma 3.3.20 |
| $D'_2$ | (24, 66, 10, 34) | $PD_2 + \Delta h_0 h_3 x_{35}$ | $\tau h_2^2 B_{23}$ | Lemma 3.3.20 |
| $P^3 v$ | (45, 66, 21, 35) | | | |
| $P^5 j$ | (55, 66, 27, 34) | | | |
| $n_1$ | (11, 67, 5, 36) | $h_3 b_{31} h_2(1)$ | $h_0 r_1$ | [11] VI.1 |
| $\tau Q_3$ | (13, 67, 5, 35) | | | |
| $h_0 Q_3$ | (14, 67, 6, 36) | | | |
| $C''$ | (21, 67, 9, 37) | $g_4 x_{47}$ | | |
| $X_3$ | (21, 67, 9, 36) | $\Delta_1 h_3^2 v + \tau \Delta_1 h_1 d_0$ | | Lemma 3.3.22 |
| $C_{11}$ | (29, 67, 11, 35) | $\Delta_2 c_1$ | | |
| $h_3 g^3$ | (25, 67, 13, 40) | | $h_0 h_2^2 g^3$ | Lemma 3.3.3 |
| $P^8 h_2$ | (65, 67, 33, 34) | | | |
| $d_2$ | (8, 68, 4, 36) | $h_2(1)^2$ | | |
| $G_{21}$ | (20, 68, 8, 36) | $\Delta g_2$ | $h_0 X_3$ | Lemma 3.3.12 |
| $h_{2 B_{23}}$ | (25, 68, 11, 38) | | | |
| $G_{11}$ | (33, 68, 13, 36) | $\Delta^2 h_2 e_0$ | $h_0 d_0 x'$ | Lemma 3.3.10 |
| $p'$ | (8, 69, 4, 36) | $h_0 \nu_2$ | | |
| $D'_3$ | (18, 69, 8, 37) | $h_1 b_{30} b_{31} h_0(1, 3)$ + $+h_1^2 b_{40} h_0(1, 3)$ | $h_1 X_3$ | Lemma 3.3.18 |
| $h_2 G_0$ | (18, 69, 8, 38) | | $h_1 C''$ | Lemma 3.3.19 |
| $P(A + A')$ | (24, 69, 10, 36) | | | |
| $h_{2 B_{5}}$ | (25, 69, 11, 38) | | | |
| $\tau W_1$ | (33, 69, 13, 36) | $\tau \Delta^2 h_1 g$ | | |
| $P^2 v'$ | (40, 69, 18, 36) | | | |
| $p_1$ | (8, 70, 4, 37) | $h_1 \nu_2$ | | |
Table 8: Adams $E_2$ generators

| element | $(m, s, f, w)$ | May description | $d_2$ | reference |
|---------|----------------|----------------|-------|-----------|
| $h_2Q_3$ | (14,70,6,38) | | | |
| $R_1^i$ | (41,70,17,36) | $\Delta^2Ph_0d_0$ | $P^2h_0x'$ | Lemma 3.3.23 |
| $P^3u'$ | (49,70,23,37) | | | Lemma 3.3.5 |
| $P^7d_0$ | (64,70,32,36) | | | |

Table 9: Temporary May $E_\infty$ generators

| element | $(m, s, f, w)$ | description |
|---------|----------------|-------------|
| $s$     | (13,30,7,16)  | $Ph_4h_0(1) + h_0^3b_{20}b_{31}$ |
| $P^2s$  | (29,46,15,24) | | |
| $S_1$   | (25,54,11,28) | $h_0^2X$ |
| $g_2'$  | (23,60,11,32) | $P\Delta_1h_0^3$ |
| $\tau PG$ | (22,62,10,33) | $\tau P\Delta_1h_1^2$ |
| $Ph_0i$ | (24,62,12,32) | | |
| $P^4s$  | (45,62,23,32) | | |
| $PD_4$  | (22,65,10,35) | | |
| $s_1$   | (13,67,7,37)  | $h_5b^2_{21}h_1(1) + h_1^2b_{21}b_{32}$ |
| $Ph_2^2$ | (10,70,6,36)  | | |
| $\tau P^2G$ | (30,70,14,37) | | |
| $P^2S_1$ | (41,70,19,36) | | |

Table 10: Ambiguous Ext generators

| element | $(m, s, f, w)$ | ambiguity | definition |
|---------|----------------|-----------|------------|
| $f_0$   | (8,18,4,10)   | $\tau h_1^3h_4$ | | |
| $y$     | (14,38,6,20)  | $\tau^2 h_2^3d_1$ | | |
| $f_1$   | (8,40,4,22)   | $h_1^2h_3h_5$ | | |
| $u'$    | (25,46,11,25) | $\tau d_0l$ | $\tau \cdot u' = 0$ |
| $B_2$   | (17,48,7,26)  | $h_0^2h_5e_0$ | | |
| $v'$    | (25,49,11,27) | $\tau e_0l$ | $\tau \cdot v' = 0$ |
| $G_3$   | (21,51,9,28)  | $\tau gn$ | $h_2 \cdot G_3 = 0$ |
| $R_1$   | (26,54,10,28) | $\tau Ph_1h_5d_0$ | | |
| $P_{u'}$ | (33,54,15,29) | $\tau d_0^3j$ | | |
| $B_6$   | (17,55,7,30)  | $\tau b_1G$ | $\tau \cdot B_6 = 0$ |
| $Q_1$   | (26,56,10,29) | $\tau^3gt$ | | |
| $P_{v'}$ | (33,57,15,31) | $\tau d_0^3k$ | $\tau \cdot P_{v'} = 0$ |
| $B_3$   | (23,60,9,32)  | $h_0h_5k$ | | |
| $\tau g^3$ | (24,60,12,35) | $h_1^4h_5c_0e_0$ | $h_1^2 \cdot \tau g^3 = 0$ |
| $H_1$   | (13,62,5,33)  | $h_1D_3$ | | |
| $R$     | (26,62,10,32) | $\tau^2B_{22}, \tau^2PG$ | $h_1 \cdot R = 0$ |
| $P^2u'$ | (41,62,19,33) | $\tau Pd_0^3j$ | | |
| $q_1$   | (26,64,10,33) | $\tau^3h_1^4E_1$ | | |
### Table 10: Ambiguous Ext generators

| element | $(m, s, f, w)$ | ambiguity | definition |
|---------|---------------|-----------|------------|
| $U$     | $(34, 64, 14, 34)$ | $\tau^2 km$ | $\Delta^2 h_0^2 d_0$ in $\text{Ext}_{A(2)}$ |
| $\tau B_{23}$ | $(24, 65, 10, 35)$ | $PD_4$ | $c_0 Q_2 + (h_1, h_1 h_5 d_0 e_0, \tau)$ |
| $R_2$  | $(33, 65, 13, 34)$ | $\tau^3 gw$ | $0$ in $\text{Ext}_{A(2)}$ |
| $\tau gw$ | $(29, 65, 13, 36)$ | $\tau h_3 h_5 c_0 e_0$ | $h_1 \cdot \tau gw = 0$ |
| $P^2 v'$ | $(41, 65, 19, 35)$ | $\tau d_3^2 j$ | $\tau \cdot P^2 v' = 0$ |
| $\tau G_0$ | $(17, 66, 7, 35)$ | $\tau h_0 r_1$ | $(\tau, h_1^2 H_1, h_1)$ |
| $n_1$  | $(11, 67, 5, 36)$ | $h_1^2 h_6$ | $h_1 \cdot n_1 = 0$ |
| $\tau Q_3$ | $(13, 67, 5, 35)$ | $\tau n_1$ | Admas $d_2(\tau Q_3) = 0$ |
| $h_0 Q_3$ | $(14, 67, 6, 36)$ | $h_0 n_1$ | $\tau \cdot h_0 Q_3 = h_0 \cdot \tau Q_3$ |
| $C_{11}$ | $(29, 67, 11, 35)$ | $\tau h_2^3 X_3$ | $h_0 \cdot C_{11} = 0$ |
| $G_{21}$ | $(20, 68, 8, 36)$ | $\tau h_3 B_7$ | |
| $G_{11}$ | $(33, 68, 13, 36)$ | $h_3^6 G_{21}$ | |
| $h_2 B_5$ | $(25, 69, 11, 38)$ | $h_2^3 X_3$ | $h_1 \cdot h_2 B_5 = 0$ |
| $P^2 x'$ | $(40, 69, 18, 36)$ | $d_3^2 z$ | $0$ in $\text{Ext}_{A(2)}$ |
| $R'_{11}$ | $(41, 70, 17, 36)$ | $\tau^3 d_3^2 v$ | $(\tau, P_{c1x'}, h_0)$ |
| $P^3 u'$ | $(49, 70, 23, 37)$ | $\tau P^2 d_3^2 j$ | $(u', h_0^3, h_0^3 i)$ |

### Table 11: Hidden May $\tau$ extensions

| $(s, f, w)$ | $x$ | $\tau \cdot x$ | reference |
|-------------|-----|----------------|-----------|
| $(30, 11, 16)$ | $P c_0 d_0$ | $h_1^3 s$ | classical |
| $(37, 9, 20)$ | $\tau h_1 e_0 g$ | $h_1^4 x$ | classical |
| $(37, 10, 20)$ | $\tau h_2^3 e_0 g$ | $h_1^3 x$ | classical |
| $(41, 5, 22)$ | $h_1 f_1$ | $h_1^2 c_2$ | classical |
| $(43, 9, 24)$ | $\tau h_2 g^2$ | $P h_1^1 h_5$ | Lemma 2.4.4 |
| $(46, 19, 24)$ | $P^3 c_0 d_0$ | $P^2 h_0^8$ | classical |
| $(53, 9, 28)$ | $B_8$ | $P h_5 d_0$ | Lemma 2.4.3 |
| $(53, 17, 28)$ | $\tau P h_0 d_0^2 e_0$ | $h_0^6 x'$ | classical |
| $(53, 18, 28)$ | $\tau P h_0^2 d_0^2 e_0$ | $h_0^6 x'$ | classical |
| $(54, 10, 29)$ | $h_1 B_8$ | $P h_1 h_5 d_0$ | $\tau \cdot B_8$ |
| $(54, 15, 28)$ | $P u'$ | $h_1^4 S_1$ | Lemma 2.4.4 |
| $(54, 16, 28)$ | $\tau h_0 d_0^2 j$ | $h_0^6 S_1$ | classical |
| $(54, 17, 28)$ | $\tau^2 P h_1 d_0^2 e_0$ | $h_0^6 S_1$ | classical |
| $(58, 8, 30)$ | $h_1 Q_2$ | $h_0^3 D_2$ | classical |
| $(60, 13, 34)$ | $\tau h_0 g^3$ | $P h_1^1 h_5 e_0$ | Lemma 2.4.2 |
| $(61, 12, 33)$ | $h_1^2 B_{21}$ | $P h_5 c_0 d_0$ | Lemma 2.4.3 |
| $(61, 13, 32)$ | $x' e_0$ | $h_1^4 X_1$ | classical |
| $(62, 13, 34)$ | $h_1^3 B_{21}$ | $P h_1 h_5 c_0 d_0$ | $\tau \cdot h_1^2 B_{21}$ |
| $(62, 19, 32)$ | $P^2 u'$ | $P h_0^2 h_5 i$ | Lemma 2.4.4 |
| $(62, 20, 32)$ | $\tau P h_0 d_0^2 j$ | $P h_0^2 h_5 i$ | classical |
| $(62, 21, 32)$ | $\tau^2 P^2 h_1 d_0^2 e_0$ | $P h_0^2 h_5 i$ | classical |
| $(62, 27, 32)$ | $P^5 c_0 d_0$ | $P^4 h_0^8$ | classical |
| $(64, 8, 34)$ | $h_1 X_2$ | $h_0^2 A''$ | classical |
### Table 11: Hidden May $\tau$ extensions

| $(s, f, w)$   | $x$       | $\tau \cdot x$ | reference          |
|--------------|-----------|-----------------|--------------------|
| (65, 7, 35)  | $k_1$     | $h_2 h_5 n$     | Lemma 2.4.5       |
| (65, 9, 34)  | $h_1 h_3 Q_2$ | $h_2^2 h_3 D_2$ | classical          |
| (66, 12, 34) | $h_1^2 q_1$ | $h_2^2 D'_2$     | classical          |
| (67, 13, 36) | $B_8 d_0$  | $h_1^2 X_3$     | Lemma 2.4.3       |
| (68, 12, 36) | $\tau h_0 h_2 B_{23}$ | $h_3^4 G_21 + h_5 d_0 i$ | classical          |
| (69, 7, 36)  | $\tau h_2^2 Q_3$ | $h_0^3 p^3$     | classical          |
| (69, 25, 36) | $\tau P^3 h_0 d_0^2 c_0$ | $P^3 h_0^2 x'$   | classical          |
| (69, 26, 36) | $\tau P^3 h_0 d_0^2 c_0$ | $P^2 h_0^2 x'$   | classical          |
| (70, 6, 36)  | $\tau h_2 Q_3$ | $P h_0^2$       | classical          |
| (70, 7, 36)  | $\tau h_0 h_2 Q_3$ | $P h_0 h_2^2$   | classical          |
| (70, 23, 36) | $P^3 u'$   | $P^2 h_0 S_1$   | Lemma 2.4.3       |
| (70, 24, 36) | $\tau P^2 h_0 d_0^2 j$ | $h_0^2 S_1$     | classical          |
| (70, 25, 36) | $\tau P^2 h_0 d_0^2 j$ | $h_0^2 S_1$     | classical          |

### Table 12: Hidden May $h_0$ extensions

| $(s, f, w)$   | $x$       | $h_0 \cdot x$ | reference          |
|--------------|-----------|----------------|--------------------|
| (26, 7, 16)  | $h_2^2 g$ | $h_3^1 h_4 c_0$ | Lemma 2.4.9       |
| (30, 7, 16)  | $r$       | $s$            | classical          |
| (46, 11, 28) | $h_2^2 g^2$ | $h_1^2 h_5 c_0$ | Lemma 2.4.9       |
| (46, 12, 25) | $u'$      | $\tau h_0 d_0 l$ | Lemma 2.4.8       |
| (46, 15, 24) | $i^2$     | $P^2 s$        | classical          |
| (49, 12, 27) | $v'$      | $\tau h_0 e_0 l$ | Lemma 2.4.8       |
| (50, 11, 28) | $gr$      | $P h_3^1 h_5 c_0$ | Lemma 2.4.10     |
| (54, 16, 29) | $P u'$   | $\tau h_0 d_0^2 j$ | $\tau \cdot P u'$ |
| (54, 11, 28) | $R_1$     | $S_1$          | classical          |
| (56, 11, 29) | $Q_1$     | $\tau h_2 x'$ | classical          |
| (57, 16, 31) | $P v'$   | $\tau h_0 d_0 c_0 j$ | Lemma 2.4.8     |
| (60, 8, 32)  | $B_3$     | $h_5 k$        | classical          |
| (60, 16, 33) | $d_0 u'$  | $\tau h_0 e_0^2 j$ | $h_0 \cdot u'$      |
| (62, 20, 33) | $P^2 u'$ | $\tau P h_0 d_0^2 j$ | $\tau \cdot P^2 u'$ |
| (62, 12, 32) | $h_0 R$  | $P h_5 i$    | classical          |
| (62, 13, 34) | $h_2^2 B_{22}$ | $P h_1 h_5 c_0 d_0$ | Lemma 2.4.12 |
| (62, 23, 32) | $P^2 j^2$ | $P^4 s$       | classical          |
| (63, 8, 34)  | $X_2$     | $h_5 l$       | classical          |
| (63, 15, 38) | $h_0 h_2 g^3$ | $h_2^2 h_5 c_0 e_0$ | $h_0 \cdot h_2^2 g^2$ |
| (63, 16, 35) | $e_0 u'$ | $\tau h_0 e_0^2 k$ | $h_0 \cdot u'$     |
| (64, 8, 34)  | $h_2 A'$  | $\tau^2 d_3^1$ | classical          |
| (64, 15, 34) | $U$      | $P h_2 x'$    | classical          |
| (65, 20, 35) | $P^2 v'$ | $\tau h_0 d_0^2 i$ | Lemma 2.4.8       |
| (66, 15, 40) | $h_2^2 g^3$ | $h_3^1 D_2$   | Lemma 2.4.9       |
| (66, 16, 37) | $e_0 v'$ | $\tau h_0 e_0^2 l$ | $h_0 \cdot v'$    |
| (67, 14, 38) | $lm$     | $h_1^2 X_1$ | Lemma 2.4.10      |
### Table 12: Hidden May $h_0$ extensions

| $(s, f, w)$ | $x$ | $h_0 \cdot x$ | reference |
|-------------|-----|----------------|-----------|
| (68, 15, 36) | $h_0 G_{11}$ | $\tau h_1 d_0 x'$ | classical |
| (68, 20, 37) | $P d_{0} u'$ | $\tau h_0 d_{0}^{q_0} q_3$ | $h_0 \cdot u'$ |
| (70, 24, 37) | $P^3 u'$ | $\tau P^2 h_0 d_{0}^{q_0} q_3$ | $\tau \cdot P^3 u'$ |
| (70, 15, 40) | $m^2$ | $h_0^3 c_0 Q_2$ | Lemma 2.4.10 |
| (70, 19, 36) | $h_0 R_1'$ | $P^2 S_1$ | classical |

### Table 13: Hidden May $h_1$ extensions

| $(s, f, w)$ | $x$ | $h_1 \cdot x$ | reference |
|-------------|-----|----------------|-----------|
| (38, 6, 21) | $x$ | $\tau h_2^3 d_1$ | classical |
| (39, 7, 21) | $y$ | $\tau^2 c_1 g$ | classical |
| (56, 8, 31) | $\tau h_1 G$ | $h_5 c_0 e_0$ | Lemma 2.4.14 |
| (58, 10, 33) | $h_2^2 B_6$ | $\tau h_2^2 d_1 g$ | Lemma 2.4.15 |
| (59, 11, 33) | $h_1 D_{11}$ | $\tau^2 c_1 g^2$ | Lemma 2.4.16 |
| (62, 9, 34) | $h_1 B_3$ | $h_5 d_0 e_0$ | Lemma 2.4.14 |
| (62, 10, 33) | $X_1$ | $\tau P G$ | classical |
| (64, 8, 35) | $C'$ | $\tau d_1^2$ | classical |
| (64, 12, 35) | $\tau P h_1 G$ | $P h_5 c_0 e_0$ | Lemma 2.4.14 |
| (67, 7, 37) | $r_1$ | $s_1$ | Lemma 2.4.18 |
| (67, 13, 36) | $h_2^2 g_1$ | $h_4 X_3$ | Lemma 2.4.19 |
| (68, 10, 38) | $C''$ | $d_1 t$ | classical |
| (70, 5, 37) | $p'$ | $h_2^2 c_0$ | classical |
| (60, 12, 39) | $h_2^2 X_3$ | $h_5 c_0 d_0 e_0$ | Lemma 2.4.14 |
| (70, 14, 37) | $\tau W_1$ | $\tau P^2 G$ | classical |

### Table 14: Hidden May $h_2$ extensions

| $(s, f, w)$ | $x$ | $h_2 \cdot x$ | reference |
|-------------|-----|----------------|-----------|
| (26, 7, 16) | $h_0 h_2 g$ | $h_3^2 h_4 c_0$ | $h_0 \cdot h_2^2 g$ |
| (46, 11, 28) | $h_0 h_2 g^2$ | $h_3^2 h_5 c_0$ | $h_0 \cdot h_2^2 g^2$ |
| (49, 12, 27) | $u'$ | $\tau h_0 c_0 l$ | $h_0 \cdot u'$ |
| (50, 11, 28) | $e_0 r$ | $P h_4^3 h_5 c_0$ | Remark 2.4.21 |
| (52, 12, 29) | $v'$ | $\tau h_0 e_0 m$ | $h_0 \cdot v'$ |
| (54, 9, 30) | $h_2 B_2$ | $h_1 h_5 c_0 d_0$ | Lemma 2.4.22 |
| (57, 16, 31) | $P u'$ | $\tau h_0 d_0 e_0 j$ | $h_0 \cdot P u'$ |
| (58, 8, 32) | $B_6$ | $\tau e_1 g$ | Lemma 2.4.23 |
| (59, 11, 31) | $Q_1$ | $\tau h_0 B_{21}$ | classical |
| (60, 8, 32) | $Q_2$ | $h_5 k$ | classical |
| (60, 9, 32) | $h_0 Q_2$ | $h_0 h_5 k$ | classical |
| (60, 16, 33) | $P v'$ | $\tau h_0^2 e_0 j$ | $h_0 \cdot P v'$ |
| (62, 13, 34) | $h_2^2 B_{21}$ | $P h_1 h_5 c_0 d_0$ | $h_0 \cdot h_2^2 B_{22}$ |
| (63, 8, 34) | $B_3$ | $h_5 l$ | classical |
### Table 14: Hidden May $h_2$ extensions

| $(s, f, w)$ | $x$ | $h_2 \cdot x$ | reference |
|-------------|-----|----------------|-----------|
| (63, 15, 38) | $h_5^2g^3$ | $h_7^2h_5c_0e_0$ | $h_0 \cdot h_0h_2g^3$ |
| (63, 16, 35) | $d_0u'$ | $\tau h_0c_0^2k$ | $h_2 \cdot u'$ |
| (64, 7, 34) | $A + A'$ | $h_0A''$ | classical |
| (64, 8, 34) | $h_0(A + A')$ | $h_0^2A''$ | classical |
| (64, 8, 35) | $B_7$ | $\tau d_1^2$ | classical |
| (64, 14, 36) | $km$ | $h_1^2X_1$ | Remark 2.4.21 |
| (65, 20, 35) | $P^2u'$ | $\tau h_0\partial_0^3i$ | $h_0 \cdot P^2u'$ |
| (66, 14, 40) | $h_0h_2g^3$ | $h_1^3D_4$ | $h_0 \cdot h_2g^3$ |
| (66, 16, 37) | $e_0u'$ | $\tau h_0c_0^2l$ | $h_2 \cdot u'$ |
| (67, 15, 36) | $U$ | $h_0d_0x'$ | classical |
| (68, 10, 37) | $h_2C_0$ | $\tau d_1t$ | classical |
| (68, 14, 36) | $R_2$ | $\tau h_1d_0x'$ | classical |
| (68, 20, 37) | $P^2v'$ | $\tau h_0d_0^3j$ | $h_0 \cdot P^2v'$ |
| (69, 16, 39) | $e_0v'$ | $\tau h_0c_0^2m$ | $h_2 \cdot v'$ |
| (70, 15, 40) | $lm$ | $h_1^3c_0Q_2$ | Remark 2.4.21 |

### Table 15: Some miscellaneous hidden May extensions

| $(s, f, w)$ | relation | reference |
|-------------|----------|-----------|
| (59, 12, 33) | $c_0 \cdot G_3 = Ph_1^3h_5c_0$ | Lemma 2.4.27 |
| (60, 11, 32) | $h_5^3B_4 + \tau h_1B_{21} = g'_2$ | Lemma 2.4.28 |
| (61, 10, 35) | $c_0 \cdot i_1 = h_1^4D_4$ | Lemma 2.4.24 |
| (62, 12, 35) | $Ph_1 \cdot i_1 = h_1^2Q_2$ | Lemma 2.4.24 |
| (63, 10, 35) | $c_0 \cdot B_6 = h_1^3B_3$ | Lemma 2.4.26 |
| (65, 10, 35) | $c_0 \cdot Q_2 = PD_4$ | Lemma 2.4.24 |
Table 16: Some Massey products in Ext

| $(s, f, w)$ | bracket | contains | indeterminacy | proof | used for |
|------------|---------|----------|---------------|-------|---------|
| (2, 2, 1) | $\langle h_0, h_1, h_0 \rangle$ | $\tau h_1^2$ | | Proposition 2.2.3 | $(2, \eta, 2)$ |
| (3, 2, 2) | $\langle h_1, h_0, h_1 \rangle$ | $h_0 h_2$ | | Proposition 2.2.4 | $(\eta, 2, \eta)$ |
| (6, 2, 4) | $\langle h_1, h_2, h_1 \rangle$ | $h_2^2$ | | Proposition 2.2.8 | $(\eta, \nu, \eta)$ |
| (8, 3, 5) | $\langle h_1, h_2, h_0 h_2 \rangle$ | $c_0$ | $d_2(h_0(1)) = h_0 h_2^2$ | Lemma 2.4.9, 2.4.10 |
| (8, 5, 4) | $\langle h_0, h_0^3 h_3, h_0 \rangle$ | $0$ | | Lemma 5.1.16 | $(2, 8\sigma, 2)$ |
| (9, 4, 5) | $\langle h_0, c_0, h_0 \rangle$ | $\tau h_1 c_0$ | | Proposition 2.2.4 | $(2, \epsilon, 2)$ |
| (9, 7, 7) | $\langle h_1^4, \tau, h_1^4 \rangle$ | $0$ | | Lemma 5.1.16 | |
| (12, 4, 7) | $\langle \tau, h_1^4, h_3 \rangle$ | $0$ | | Lemma 5.1.16 | |
| (14, 2, 8) | $\langle h_2, h_3, h_2 \rangle$ | $h_2^2$ | | Proposition 2.2.4 | |
| (15, 5, 9) | $\langle h_0 h_2, h_2, c_0 \rangle$ | $h_1 d_0$ | $d_2(h_0(1)) = h_0 h_2^2$ | Lemma 5.1.17 | $(2\nu, \nu, \epsilon)$ |
| (15, 8, 10) | $\langle h_1^2 c_0, \tau, h_1^4 \rangle$ | $0$ | | Lemma 2.4.17 | |
| (16, 4, 9) | $\langle h_0 h_2^3, h_0, h_1 \rangle$ | $0$ | | Lemma 5.1.16 | |
| (17, 7, 11) | $\langle h_1^4, \tau, h_1^4, h_3 \rangle$ | $h_1^3 d_0$ | $d_2(h_1 b_{20}) = \tau h_1^4$ | Lemma 5.1.16 | $(\sigma, \nu, \sigma)$ |
| (18, 2, 10) | $\langle h_3, h_2, h_3 \rangle$ | $h_2 h_4$ | $d_2(h_1 b_{21}) = h_1^2 h_3$ | Lemma 5.2.14 | |
| (19, 6, 12) | $\langle c_0, h_3, h_1^4 \rangle$ | $h_1^2 e_0$ | $d_4(g) = h_1^4 h_4$ | Lemma 2.4.10 | $(\sigma, \tau, 2)$ |
| (20, 4, 11) | $\langle h_0, c_1, h_0 \rangle$ | $0$ | | Proposition 2.2.4 | |
| (20, 5, 12) | $\langle h_0, h_1, h_3^2 h_4 \rangle$ | $h_0 g$ | $d_4(g) = h_1^4 h_4$ | Lemma 2.4.10 | |
| (22, 6, 14) | $\langle h_3, h_1^2, h_1 h_3, h_2^2 \rangle$ | $0$ | | Lemma 2.4.24 | |
| (23, 5, 12) | $\langle h_4, h_0^3 h_3, h_0 \rangle$ | $\tau^2 h_2 g$ | $d_4(P) = h_0^2 h_3$ | Lemma 3.3.26 | |
| (24, 9, 15) | $\langle h_1^3 e_0, \tau, h_1^4 \rangle$ | $h_1^3 c_0 d_0$ | $d_2(h_1 b_{30} h_0(1)) = \tau h_1^2 e_0$ | Lemma 5.1.16 | |
| (25, 7, 15) | $\langle h_2 g, h_0^3, h_1 \rangle$ | $c_0 e_0$ | $d_2(h_0(1)^2 b_{21}) = h_3^2 h_0(1)^2$ | Lemma 2.4.14 | |
| (26, 8, 16) | $\langle h_1^4, h_3, d_0 \rangle$ | $h_1 c_0 e_0$ | $d_2(h_1^3 b_{21}) = h_1^2 h_3$ | Lemma 5.1.16 | |
Table 16: Some Massey products in Ext

| $(s, f, w)$ | bracket | contains | indeterminacy | proof | used for |
|------------|---------|----------|--------------|--------|----------|
| (27, 5, 16) | $(h_2, h_2c_1, h_1)$ | $h_3g$ | $d_2(b_2h_1(1)) = h_2^2c_1$ | **Lemma 2.4.24** |
| (27, 10, 16) | $(c_0d_0, f, h_1^4)$ | $Ph_1^2e_0$ | $d_2(b_2d_0h_0(1)) = f \eta e_0$ | **Lemma 2.4.7** |
| (29, 7, 16) | $(d_0, h_3, h_3h_3)$ | $k$ | $d_4(\nu) = h_3^2h_3^2$ | **Lemma 2.4.5** |
| (30, 10, 18) | $(c_0e_0, f, h_1^4)$ | $h_1^2d_0^2$ | | **Lemma 5.1.16** |
| (32, 9, 19) | $(h_2^2, h_0, c_0e_0)$ | $h_1d_0e_0$ | $d_2(h_0(1)) = h_0h_2^2$ | **Lemma 5.1.3** |
| (37, 7, 22) | $(h_2^2, h_4, h_3^2h_4)$ | $0$ | $h_3^2h_5$ | **Lemma 2.4.24** |
| (39, 3, 21) | $(h_2, h_1h_5, h_2)$ | $h_1h_3h_5$ | | **Proposition 2.2.1** |
| (40, 9, 24) | $(h_1^4h_5, h_1, h_0^3)$ | $h_0g^2$ | $d_8(g^2) = h_1^4h_5$ | **Lemma 2.4.9** |
| (40, 10, 21) | $(q, h_0, h_3h_3)$ | $\tau h_1u$ | $d_4(P) = h_0^4h_3$ | **Lemma 3.3.52** |
| (42, 8, 23) | $(h_2^2h_4, h_4, h_1)$ | $0$ | $Ph_1^2h_5$ | **Lemma 2.4.24** |
| (45, 9, 24) | $(\tau, \tau^2h_2g^2, h_1)$ | $\tau w$ | $d_8(w) = Ph_1^5h_5$ | $(\tau, \nu \tau^2, \eta)$ | **Lemma 5.1.17** |
| (46, 7, 25) | $(h_1, h_0, h_0^2g_2)$ | $B_1$ | $d_0(Y) = h_0^4g_2$ | | **Lemma 2.4.24** |
| (46, 7, 25) | $(g_2, h_0^3, h_1)$ | $B_1$ | $d_0(Y) = h_0^4g_2$ | | **Lemma 2.4.24** |
| (47, 6, 25) | $(h_0, h_1, h_1h_2g_2)$ | $0$ | $\tau h_0h_2g_2$ | | **Lemma 2.4.24** |
| (47, 9, 28) | $(h_2, h_2c_1, h_1)$ | $h_3g^2$ | $d_5(b_3^2h_1(1)) = h_2^2c_1$ | | **Lemma 4.2.9** |
| (47, 10, 26) | $(\tau^2g^2, h_2, h_0)$ | $c_0r$ | $d_4(\Delta g_2) = \tau^2h_2^2g_2$ | | **Lemma 2.4.10** |
| (47, 13, 24) | $(\tau, \nu, h_3^0)$ | $Q'$ | $\tau P_\nu$ | | **Lemma 2.4.3** |
| (47, 13, 24) | $(\tau, \tau h_0d_0l, h_0^2)$ | $Q'$ | $\tau P_\nu$ | | **Lemma 2.4.3** |
| (48, 4, 26) | $(h_4, h_2^2h_4, h_4)$ | $0$ | | | **Lemma 2.4.24** |
| (48, 6, 26) | $(h_3, h_2, x)$ | $h_3h_3d_0$ | | | **Lemma 2.4.24** |
| (48, 7, 26) | $(h_2, h_2^2g_2, h_0)$ | $B_2$ | $d_2(Y) = h_0^4g_2$ | | **Lemma 2.4.17** |
| (50, 4, 27) | $(h_2, h_3, h_1h_3h_5)$ | $h_5c_1$ | $d_2(h_5c_1(1)) = h_1h_3h_5$ | $(\nu, \sigma, \sigma h_5)$ | **Lemma 4.2.9** |
| (50, 6, 27) | $(h_2, h_1, h_1h_2g_2)$ | $C$ | $d_4(x_{47}) = \tau h_1^2g_2$ | $(\nu, \eta, \tau h_1^2g_2)$ | **Lemma 2.4.24** |
| (50, 7, 28) | $(h_2^2c_0, h_3, h_1^4)$ | $h_1^2h_5e_0$ | $d_2(h_1b_{21}) = h_3^4h_3$ | | **Lemma 2.4.24** |
| (50, 7, 28) | $(c_0, h_2^2, h_3, h_3^3)$ | $0$ | $h_3^2h_5e_0$ | | **Lemma 2.4.10** |
| (50, 10, 28) | $(h_1^2h_4, h_4, r)$ | $gr$ | $d_4(g) = h_4^4h_4$ | | **Lemma 2.4.10** |
Table 16: Some Massey products in Ext

| $(s, f, w)$ | bracket | contains | indeterminacy | proof | used for |
|-------------|---------|----------|--------------|-------|----------|
| (51, 8, 28) | $(g_2, h_0^3, h_0^2)$ | $h_2B_2$ | $d_6(Y) = h_0^3g_2$ | Lemma 2.4.22 |
| (51, 8, 28) | $(h_0, d_1, f_0)$ | $h_2B_2$ | $d_2(Bh_2b_{21}) = f_0d_1$ | Lemma 5.1.4 |
| (51, 9, 28) | $(h_2, N, h_1)$ | $G_3$ or $G_3 + \tau g_n$ | $d_2(\Delta b_{21}h_1(1)) = h_2N$ | Lemma 12.63 |
| (52, 8, 30) | $(d_1, h_1^3, h_1h_4)$ | $d_1g$ | $d_4(g) = h_1^3h_4$ | Lemma 2.4.13 |
| (52, 10, 29) | $(q, h_0^3, h_1h_4)$ | $h_1G_3$ | $d_4(g) = h_1^3h_4$ | Lemma 2.4.4 |
| (52, 10, 29) | $(h_3, h_0^3, Ph_0^3h_5)$ | $h_1G_3$ | $d_4(P) = h_0^3h_3$ | Lemma 2.4.13 |
| (53, 7, 30) | $(h_0^2, h_3, h_1^3, h_1h_3, h_1)$ | $i_1$ | $d_4(\tau) = \Delta h_3h_5$ | Lemma 5.3.13 |
| (53, 7, 30) | $(h_1^3, h_4, h_1^3h_4, h_4)$ | $i_1$ | $d_4(\tau) = \Delta h_5h_6$ | Lemma 5.3.13 |
| (54, 6, 29) | $(h_1, h_0, D_1)$ | $\tau G$ | $\tau G$ | Lemma 5.4.21 |
| (54, 15, 29) | $(u', h_0^3, h_0h_3)$ | $\tau \tau'G$ | $\tau \tau'G$ | Lemma 5.1.5 |
| (55, 7, 30) | $(h_1h_3, h_0^2)$ | $\tau h_1G$ | $d_4(\Delta h_2) = h_2h_5g$ | Lemma 2.4.13 |
| (55, 13, 31) | $(\tau^2h_1c_0^3, h_1^2, h_1h_4)$ | $\tau^2h_1c_0^2g$ | $d_4(g) = h_1^4h_4$ | Lemma 2.4.13 |
| (56, 11, 30) | $(h_0, h_1, \tau h_1B_3s)$ | $h_2x'$ | $d_6(x_{50}) = Ph_0^3h_5d_0$ | Lemma 2.4.13 |
| (57, 9, 32) | $(x, h_0^2, h_1h_4)$ | $h_1^3B_6$ | $h_1^3B_6$ | Lemma 2.4.16 |
| (58, 8, 31) | $(h_4, h_1^3h_4, h_4, Ph_1)$ | $h_1Q_2$ | $h_1Q_2$ | Lemma 2.4.16 |
| (58, 10, 32) | $(y, h_1^3, h_1^3h_4)$ | $h_1D_1$ | $d_4(\tau) = h_1^4h_4$ | Lemma 2.4.16 |
| (59, 8, 33) | $(c_0, h_0^3, h_3, h_1^3, h_1h_3)$ | $h_1^2D_4$ | $d_4(\tau) = h_1^4h_4$ | Lemma 2.4.16 |
| (59, 11, 35) | $(c_1g, h_1^3h_4^2)$ | $c_1g^2$ | $d_4(\tau) = h_1^4h_4$ | Lemma 2.4.16 |
| (60, 10, 33) | $(\tau, B_6, h_1^2)$ | $h_1^3Q_2$ | $d_3(b_{30}d_{40}h_1(1)) = \tau B_6$ | Lemma 2.4.26 |
| (61, 8, 33) | $(\tau G, h_0, h_0^2)$ | $h_1B_3$ | $d_3(h_1^2d_{30}b_{30}h_1) = h_1^2B_6$ | Lemma 2.4.14 |
| (62, 8, 33) | $(h_0h_0^3, h_0, h_1, \tau h_1g_2)$ | $C_0$ | $d_3(h_0h_1^2) = h_0^3h_3$ | Lemma 2.4.14 |
| (62, 19, 33) | $(u', h_0^3, h_0h_4)$ | $P'\tau'$ | $d_4(\tau^2) = h_0^3h_4$ | Lemma 2.4.14 |
| (63, 10, 35) | $(h_5c_0e_0, h_0, h_0^2)$ | $h_1d_{40}c_0$ | $d_4(\tau) = h_1^4h_4$ | Lemma 2.4.14 |
| (65, 7, 36) | $(d_1, h_1, h_1^3h_4)$ | $k_1$ | $d_4(\tau) = h_1^4h_4$ | Lemma 2.4.14 |
Table 16: Some Massey products in Ext

| (s, f, w) | bracket | contains | indeterminacy proof | used for |
|-----------|---------|----------|---------------------|----------|
| (65, 9, 35) | (τ, B_6, h_1^2h_3) | h_2C_0 | d_2(b_{30}b_{40}h_1(1)) = τB_6 | Lemma 2.4.23 |
| | | | d_2(Bh_1b_{21}h_1(1)) = h_1^2h_3B_6 | |
| (66, 14, 40) | (h_1^3i_1, h_1, h_2^3) | h_2^3g^3 | d_2(b_1^2h_1(1)) = h_2^2c_1g^2 | Lemma 1.2.82 |
| (67, 14, 38) | (τ^2g^3, h_2, h_0h_2) | lm | d_4(∆h_2g^2) = τ^2h_2g^3 | Lemma 2.4.10 |
| (70, 23, 37) | (u', h_0, h_0^3i) | P^3u' | d_4(P^3) = h_0^3i | Lemma 2.4.4 |

Table 17: Some matric Massey products in Ext

| (s, f, w) | bracket | equals | proof | used in |
|-----------|---------|--------|-------|--------|
| (25, 8, 14) | [ h_1^2, d_0 ] , [ c_0d_0 ] , τ | P_{e_0} | d_2(b_{20}h_0(1)) = τh_1^2c_0 | Lemma 5.1.15 |
| | | | d_2(b_{20}b_{21}b_{30}) = τc_0d_0 | |
| (28, 8, 16) | [ c_0, e_0 ] , [ h_1^2c_0 ] , τ | d_0^2 | d_2(b_1b_{30}h_0(1)) = τh_1^2e_0 | Lemma 5.1.15 |
| | | | d_2(b_{20}h_0(1)) = τh_1^2c_0 | |
| (28, 8, 16) | [ h_1^2, e_0 ] , [ c_0e_0 ] , τ | d_0^2 | Lemma 5.1.17 | Lemma 5.1.17 |
| (40, 10, 22) | [ d_0, e_0 ] , [ c_0e_0 ] , τ | h_1u | Lemma 5.1.17 | Lemma 5.1.17 |
| (56, 15, 33) | [ h_1^2e_0 ] , [ h_1^4 ] , [ h_1^2e_0 ] , τ | c_0d_0e_0^2 | Lemma 5.1.18 | Lemma 5.1.18 |
| (57, 15, 31) | [ h_1^2, d_0 ] , [ h_1d_0u ] , τ | P_{u'} | d_2(h_1b_{20}b_{30}h_0(1)^2) = τh_1d_0u | Lemma 5.1.19 |
| | | | d_2(h_1b_{20}b_{30}h_0(1)^2) = τh_1^3u | |
Table 18: Classical Adams differentials

| $(s, f)$ | $d_r$ | $x$ | $d_r(x)$ | reference |
|---|---|---|---|---|
| (30, 6) | $d_3$ | $r$ | $h_1 d_0^3$ | Theorem 2.2.2 |
| (31, 8) | $d_3$ | $d_0 e_0$ | $P_{c_0} d_0$ | Proposition 4.3.1 |
| (31, 8) | $d_4$ | $d_0 e_0 + h_3^2 h_5$ | $P^2 d_0$ | Corollary 4.3.2 |
| (34, 2) | $d_3$ | $h_2 h_5$ | $h_0 p$ | Proposition 3.3.7 |
| (37, 8) | $d_4$ | $e_0 g$ | $P d_0^2$ | Theorem 4.2.1 |
| (38, 2) | $d_4$ | $h_3 h_5$ | $h_0 x$ | Theorem 7.3.7 |
| (38, 4) | $d_3$ | $e_1$ | $h_1 t$ | Theorem 4.1 |
| (38, 6) | $d_2$ | $y$ | $h_3^2 x$ | Theorem 5.1.4 |
| (39, 12) | $d_4$ | $P d_0 e_0$ | $P^3 d_0$ | Corollary 4.3.4 |
| (41, 3) | $d_2$ | $c_2$ | $h_0 f_1$ | Corollary 3.3.6 |
| (42, 9) | $d_2$ | $v$ | $h_2^2 u$ | Proposition 6.1.5 |
| (44, 10) | $d_3$ | $d_0 r$ | $h_1 d_0^3$ | Corollary 4.4.2 |
| (45, 12) | $d_4$ | $d_0 e_0$ | $P^2 d_0^2$ | Theorem 4.2.3 |
| (46, 14) | $d_3$ | $i^2$ | $P^2 h_1 d_0^2$ | Proposition 4.4.1 |
| (47, 13) | $d_2$ | $Q'$ | $P^2 h_0 r$ | p. 540 |
| (47, 16) | $d_4$ | $P^2 d_0 e_0$ | $P^4 d_0$ | Corollary 4.3.4 |
| (47, 18) | $d_3$ | $h_0 Q'$ | $P^4 h_0 d_0$ | p. 540 |
| (49, 6) | $d_3$ | $h_1 h_5 e_0$ | $h_2^2 B_1$ | Corollary 3.6 |
| (49, 10) | $d_3$ | $d_0 m$ | $P h_1 u$ | Proposition 6.1.3 |
| (50, 13) | $d_2$ | $P v$ | $P h_1^2 u$ | Corollary 6.1.4 |
| (53, 16) | $d_4$ | $P d_0^2 e_0$ | $P^3 d_0^2$ | Corollary 4.3.3 |
| (54, 10) | $d_2$ | $R_3$ | $h_0^2 x'$ | Proposition 5.2.3 |
| (55, 20) | $d_4$ | $P^3 d_0 e_0$ | $P^5 d_0$ | Corollary 4.3.4 |
| (56, 13) | $d_4$ | $d_0 v$ | $P^2 u$ | Proposition 6.1.1 |
| (57, 15) | $d_3$ | $P d_0 m$ | $P^2 h_1 u$ | Corollary 6.1.2 |
| (58, 17) | $d_2$ | $P^2 v$ | $P^2 h_2^2 u$ | Corollary 6.1.2 |
| (61, 20) | $d_4$ | $P^2 d_0^2 e_0$ | $P^4 d_0^2$ | Corollary 4.3.3 |
| (64, 25) | $d_4$ | $P^4 h_1 d_0 e_0$ | $P^6 h_1 d_0$ | Corollary 4.3.5 |
| (69, 24) | $d_4$ | $P^3 d_0^2 e_0$ | $P^5 d_0^2$ | Corollary 4.3.3 |
| \( s \) | bracket | contains | indeterminacy | reference |
|---|---|---|---|---|
| 8 | \( \langle 2, \eta, \nu, \eta^2 \rangle \) | \( \epsilon = \{c_0\} \) | | [40] Lemma 1.5 |
| 8 | \( \langle \nu, \eta, \nu \rangle \) | \( \eta \sigma + \epsilon \in \{h_1h_3\} \) | | [41] p. 189 |
| 8 | \( \langle \nu^2, 2, \eta \rangle \) | \( \epsilon = \{c_0\} \) | \( \eta \sigma \in \{h_1h_3\} \) | [41] p. 189 |
| 9 | \( \langle 8\sigma, 2, \eta \rangle \) | \( \mu_9 = \{Ph_1\} \) | | [41] p. 189 |
| | | | \( \eta \in \{h_1c_0\} \) | |
| 11 | \( \langle 2\sigma, 8, \nu \rangle \) | \( \zeta_{11} \in \{Ph_2\} \) | | [41] p. 189 |
| 15 | \( \langle \epsilon, 2, \nu^2 \rangle \) | \( \eta \kappa \in \{h_1d_0\} \) | | [41] p. 189 |
| 16 | \( \langle \eta, 2, \kappa \rangle \) | 0 | \( \eta \rho_{15} = \{Pc_0\} \) | [3] Lemma 2.4 |
| 16 | \( \langle \sigma^2, 2, \eta \rangle \) | \( \eta_4 \in \{h_1h_4\} \) | \( \eta \rho_{15} = \{Pc_0\} \) | [41] p. 189 |
| 17 | \( \langle \mu_9, 2, 8\sigma \rangle \) | \( \mu_{17} = \{P^2h_1\} \) | \( \mu \eta \in \{Ph_1c_0\} \) | [41] p. 189 |
| 18 | \( \langle 2\sigma, \sigma, \nu \rangle \) | \( \nu_4 \in \{h_3h_4\} \) | | [41] p. 189 |
| 18 | \( \langle \sigma, \nu, \sigma \rangle \) | \( 7\nu_4 \in \{h_2h_4\} \) | | [41] p. 189 |
| 19 | \( \langle \zeta_{11}, 8, 2\sigma \rangle \) | \( \zeta_{19} \in \{P^2h_2\} \) | | [41] p. 189 |
| 19 | \( \langle \nu, \sigma, \eta \sigma \rangle \) | \( \varphi \in \{c_1\} \) | | [41] p. 189 |
| 30 | \( \langle \sigma, 2\sigma, \sigma, 2\sigma \rangle \) | \( \theta_4 = \{h_4^2\} \) | | [26] Theorem 8.1.1 |
| 31 | \( \langle \nu, \sigma, \varphi \rangle \) | intersects \( \{\eta \} \) | | [4] Proposition 3.1.1 |
| 31 | \( \langle \sigma^2, 2, \eta_4 \rangle \) | \( \eta \theta_3 = \{h_1h_4^2\} \) | | [4] Proposition 3.2.1 |
| 32 | \( \langle \sigma^2, \eta, \sigma^2, \eta \rangle \) | contained in \( \{d_1\} \) | \( \eta \rho_{31} = \{P^3c_0\} \) | [4] Proposition 3.1.4 |
| 32 | \( \langle \eta, 2, \theta_4 \rangle \) | \( \eta_5 \in \{h_1h_5\} \) | \( \eta \rho_{31} = \{P^3c_0\} \) | [4] Proposition 3.2.2 |
| 32 | \( \langle \eta, \kappa^2, 2, \eta \rangle \) | contains \( \{q\} \) | \( \eta_5 \in \{h_1h_5\} \) | [4] Proposition 3.3.1 |
| 33 | \( \langle \eta_4, \eta_4, 2 \rangle \) | contained in \( \{p\} \) | | [4] Proposition 3.3.3 |
| 34 | \( \langle \sigma, \varphi, \sigma \rangle \) | \( \nu \{\eta \} \in \{h_2n\} \) | | [8] Corollary 4.3 |
| 34 | \( \langle \eta, 2, \eta_5 \rangle \) | contained in \( \{h_0h_2h_5\} \) | \( \eta^2\eta_5 \in \{h_0^2h_2h_5\} \) | [4] Corollary 3.2.3 |
| 35 | \( \langle \sigma, \varphi, \sigma \rangle \) | contained in \( \{h_2d_1\} \) | | [4] Proposition 3.1.2 |
### Table 19: Some classical Toda brackets

| $s$ | bracket | contains | indeterminacy | reference |
|-----|---------|----------|--------------|-----------|
| 35  | $\langle \sigma^2, \eta, \bar{\sigma} \rangle$ | contained in $\{h_2d_1\}$ | | [4 Proposition 3.1.3] |
| 36  | $\langle \nu, \eta, \eta \theta_4 \rangle$ | $\{t\}$ | | [8 Corollary 4.3] |
| 36  | $\langle \nu, \eta_4, \eta_4 \rangle$ | $\{t\}$ | | [8 Corollary 4.3] |
| 36  | $\langle \bar{\sigma}, 2, \eta_4 \rangle$ | 0 | | [8 Corollary 4.3] |
| 36  | $\langle \epsilon + \eta \sigma, \sigma, \kappa \rangle$ | $\{t\}$ | | [8 Section 5] |
| 36  | $\langle \{\eta\}, \eta, \nu \rangle$ | $\{t\}$ | | [4 Proposition 3.1.5] |
| 37  | $\langle \theta_4, 2, \nu^2 \rangle$ | $\{h_5^2h_5\}$ | $\sigma \theta_4 = \{x\}$ | [4 Proposition 3.2.4] |
| 38  | $\langle \rho_{15}, \sigma, 2\sigma, \sigma \rangle$ | contained in $\{h_0^3h_3h_5\}$ | | [10 Proposition 2.9] |
| 39  | $\langle \theta_4, 2, \epsilon \rangle$ | contained in $\{h_5c_0\}$ | | [4 Proposition 3.2.4] |
| 39  | $\langle \eta, \nu, \kappa \bar{\sigma} \rangle$ | $\{u\}$ | $\eta \{h_0^3h_3h_5\} \in \{c_1g\}$ | [4 Proposition 3.4.4] |
| 40  | $\langle \theta_4, 2, \mu_9 \rangle$ | contained in $\{Ph_1h_5\}$ | $\rho_{31}\mu_9 = \{P^4c_0\}$ | [4 Proposition 3.2.4] |
| 44  | $\langle \nu \theta_4, \nu, \sigma \rangle$ | $\{h_0g_9\}$ | $\sigma^2 \theta_4 = \{h_0^2g_2\}$ | [4 Proposition 3.5.3] |
| 45  | $\langle 2, \theta_4, \kappa \rangle$ | intersects $\{h_5d_0\}$ | $2\{h_0^2h_5\} \subseteq \{h_0^2h_5\}$ | [4 Section 4] |
Table 20: Adams $d_3$ differentials

| element              | $(s, f, w)$ | $d_3$         | reference       |
|----------------------|-------------|---------------|-----------------|
| $h_0 h_4$            | (15, 2, 8)  | $h_0 d_0$     | image of $J$    |
| $r$                  | (30, 6, 16) | $\tau h_1 d_0$| Table 18        |
| $h_0^3 h_5$          | (31, 4, 16) | $h_0 r$       | image of $J$    |
| $\tau d_0 e_0$       | (31, 8, 17) | $P c_0 d_0$   | Table 18        |
| $h_0 h_5$            | (34, 2, 18) | $\tau h_1 d_1$| Table 18        |
| $\tau e_0 g$         | (37, 8, 21) | $c_0 d_0^2$   | Lemma 3.3.26    |
| $e_1$                | (38, 4, 21) | $h_1 t$       | Table 18        |
| $\tau P d_0 e_0$     | (39, 12, 21)| $P^2 c_0 d_0$ | Lemma 3.3.27    |
| $i^2$                | (46, 14, 24)| $\tau P^2 h_1 d_0^2$| Table 18  |
| $\tau P^2 d_0 e_0$   | (47, 16, 25)| $P^3 c_0 d_0$ | Lemma 3.3.27    |
| $h_0^3 Q'$           | (47, 18, 24)| $P^4 h_0 d_0$ | Table 18        |
| $h_1 h_5 e_0$        | (49, 6, 27) | $h_1^2 B_1$   | Lemma 3.3.30    |
| $\tau^2 d_0 m$       | (49, 11, 26)| $P h_1 u$     | Table 18        |
| $gr$                 | (50, 10, 28)| $\tau h_1 d_0 c_0^2$ | Lemma 3.3.31    |
| $\tau^2 G$           | (54, 6, 28)| $\tau B_8$   | Lemma 3.3.32    |
| $h_0 i$              | (54, 8, 28)| $h_0 x'$     | Lemma 3.3.34    |
| $B_6$                | (55, 7, 30)| $\tau h_2 gn$| Lemma 3.3.33    |
| $\tau^2 g m$         | (55, 11, 30)| $h_1 d_0 u$  | Lemma 3.3.26    |
| $\tau P^3 d_0 e_0$   | (55, 20, 29)| $P^4 c_0 d_0$ | Lemma 3.3.27    |
| $h_0 c_0 e_0$        | (56, 8, 31)| $h_1^2 B_8$  | Lemma 3.3.30    |
| $P h_1 e_0$          | (56, 9, 30)| $h_1^2 x'$   | Lemma 3.3.30    |
| $\tau d_0 v$         | (56, 13, 30)| $P h_1 u'$   | Lemma 3.3.26    |
| $Q_2$                | (57, 7, 30)| $\tau^2 g t$ | Lemma 3.3.37    |
| $h_5 j$              | (57, 8, 30)| $h_2 x'$     | Lemma 3.3.34    |
| $\tau e_0 g^2$       | (57, 12, 33)| $c_0 d_0 e_0^2$| Lemma 3.3.26    |
| $\tau P d_0 m$       | (57, 15, 30)| $P^2 h_1 u$  | Table 18        |
| $e_1 g$              | (58, 8, 33)| $h_1 g t$    | Lemma 3.3.33    |
| $\tau g^3$           | (60, 12, 35)| $h_1^2 B_8$  | Lemma 3.3.36    |
| $D_3$                | (61, 4, 32)| ?             |                 |
| $C_0$                | (62, 8, 33)| $nr$         | Lemma 3.3.37    |
| $E_1$                | (62, 8, 33)| $nr$         | Lemma 3.3.37    |
| $h_1 X_1 + \tau B_{22}$ | (62, 10, 33)| $c_0 x'$      | Lemma 3.3.30    |
| $\tau q v$           | (62, 13, 34)| $h_1 d_0 u'$ | Lemma 3.3.34    |
| $P^2 i^2$            | (62, 22, 32)| $\tau P^2 h_1 d_0^2$ | Lemma 3.3.26 |
| $h_0^5 h_6$          | (63, 8, 32)| $h_0 R$      | image of $J$    |
| $\tau P^4 d_0 e_0$   | (63, 24, 33)| $P^5 c_0 d_0$ | Lemma 3.3.27    |
| $\tau P d_0 v$       | (64, 17, 34)| $P^2 h_1 u'$ | Lemma 3.3.26    |
| $\tau q v$           | (65, 13, 36)| $h_1^3 c_0 x'$| Lemma 3.3.39    |
| $\tau^2 P^2 d_0 m$   | (65, 19, 34)| $P^3 h_1 u$  | Lemma 3.3.40    |
| $C''$                | (67, 9, 37)| $nm$         | Lemma 3.3.37    |
| $h_2 B_5$            | (69, 11, 38)| $h_1 B_8 d_0$ | Lemma 3.3.34    |
| $\tau W_1$           | (69, 13, 36)| $\tau^4 c_0^3$| Lemma 3.3.34    |
| $m^2$                | (70, 14, 40)| $\tau h_1 e_0$| Lemma 3.3.31    |
| $\tau e_0 x'$        | (70, 14, 37)| $P c_0 x'$   | Lemma 3.3.34    |
### Table 21: Adams $d_4$ differentials

| Element | $(s, f, w)$ | $d_4$ | Reference |
|---------|-------------|-------|-----------|
| $\tau^2 d_0 e_0 + h_0 h_5$ | (31, 8, 16) | $P^2 d_0$ | Table [18] |
| $\tau^2 e_0 g$ | (37, 8, 20) | $P d_0^2$ | Table [18] |
| $h_3 h_5$ | (38, 2, 20) | $h_0 x$ | Table [18] |
| $\tau^2 P d_0 e_0$ | (39, 12, 20) | $P^3 d_0$ | Table [18] |
| $\tau^2 P^2 d_0 e_0$ | (47, 16, 24) | $P^4 d_0$ | Table [18] |
| $\tau^2 g r$ | (50, 10, 26) | $i j$ | tmf |
| $\tau^3 g m$ | (55, 11, 29) | $P u' + \tau d_0^2 j$ | tmf |
| $\tau^2 P^3 d_0 e_0$ | (55, 20, 28) | $P^5 d_0$ | Table [18] |
| $\tau^2 d_0 v$ | (56, 13, 29) | $P^2 u$ | Table [18] |
| $\tau^2 e_0 g^2$ | (57, 12, 32) | $d_0^2$ | Lemma 3.3.47 |
| $\tau^2 d_0^3 r$ | (58, 14, 30) | $P^2 i j$ | tmf |
| $\tau h_1 X_1$ | (62, 10, 32) | ? |
| $R$ | (62, 10, 32) | ? |
| $\tau^2 g v$ | (62, 13, 33) | $P d_0 u$ | tmf |
| $C'$ | (63, 7, 34) | ? |
| $\tau X_2$ | (63, 7, 33) | ? |
| $\tau^2 h_1 B_{22}$ | (63, 11, 33) | $P h_3 x'$ | Lemma 3.3.48 |
| $\tau^3 d_0^5 m$ | (63, 15, 33) | $P^2 u' + \tau P d_0^2 j$ | tmf |
| $h_0^2 h_6$ | (63, 19, 32) | $P^{2} h_0 x'$ | image of $J$ |
| $\tau^2 P d_0 v$ | (64, 17, 33) | $P^3 u$ | tmf |
| $\tau^2 P d_0^2 r$ | (66, 18, 34) | $P^2 i j$ | tmf |
| $\tau h_2 B_3$ | (69, 11, 37) | $h_3 d_0 x'$ | Lemma 3.3.48 |
| $\tau m^2$ | (70, 14, 38) | $d_0^2 z$ | Lemma 3.3.49 |
| $\tau^2 e_0 x'$ | (70, 14, 36) | $P^2 x'$ | Lemma 3.3.48 |

### Table 22: Adams $d_5$ differentials

| Element | $(s, f, w)$ | $d_5$ | Reference |
|---------|-------------|-------|-----------|
| $\tau P h_5 c_0$ | (56, 9, 29) | $\tau d_0 z$ | Lemma 3.3.55 |
| $A'$ | (61, 6, 32) | ? |
| $\tau h_1 H_1$ | (63, 6, 33) | ? |
| $\tau h_1^2 X_1$ | (63, 11, 33) | ? |
| $h_0^2 h_6$ | (63, 23, 32) | $P^6 d_0$ | image of $J$ |
Table 23: Some Toda brackets

| $(s, w)$ | bracket | contains | indeterminacy | proof | used in |
|----------|---------|----------|---------------|-------|---------|
| $(2, 1)$ | $(2, \eta, 2)$ | $\tau \eta^2 = \{\tau h_1^2\}$ | $\tau h_1^2 = \langle h_0, h_1, h_0\rangle$ | Lemmas 4.2.18, 4.2.20 | 4.2.48 |
| $(3, 2)$ | $(\eta, 2, \eta)$ | $\{2\nu, 6\nu\} = \{h_0 h_2\}$ | $4\nu = \{\tau h_1^3\}$ | $h_0 h_2 = \langle h_1, h_0, h_1\rangle$ | Lemmas 4.2.10, 4.2.42 |
| $(6, 4)$ | $(\eta, \nu, \eta)$ | $\nu^2 = \{h_2^2\}$ | $h_2^2 = \langle h_1, h_2, h_1\rangle$ | Lemmas 3.3.53, 4.2.53 |
| $(8, 4)$ | $(2, 8\sigma, 2)$ | $0$ | $0 = \langle h_0, h_3^h, h_0\rangle$ | Lemmas 3.3.53, 4.2.89 |
| $(8, 5)$ | $(2\nu, \nu, \eta)$ | $\epsilon = \{c_0\}$ | $\eta \sigma \in \{h_1 h_3\}$ | $c_0 = \langle h_0 h_2, h_2, h_1\rangle$ | Lemmas 4.2.1, 4.2.85 |
| $(8, 5)$ | $(\nu, \eta, \nu)$ | $\epsilon + \eta \sigma \in \{h_1 h_3\}$ | | | |
| $(9, 5)$ | $(\eta, 2, 8\sigma)$ | $\mu_9 = \{P h_1\}$ | $\tau \eta^2 \sigma \in \{\tau h_1^3 h_3\}$ | Table 10 | Lemmas 4.2.87, 5.3.4 |
| $(9, 5)$ | $(2, \epsilon, 2)$ | $\tau \eta \epsilon \in \{\tau h_1 c_0\}$ | $\tau h_1 c_0 = \langle h_0, c_0, h_0\rangle$ | Lemma 4.2.21 | 4.2.48 |
| $(12, 7)$ | $(\eta, \nu, \sigma)$ | $0$ | | | |
| $(12, 7)$ | $(\nu, \epsilon, 2)$ | $0$ | | | |
| $(15, 8)$ | $(2, \sigma^2, 2)$ | $0$ | $\{2k \rho_{15}\} = \{h_4^3\}$ | Lemma 3.3.15 | Lemmas 3.3.54, 4.2.91 |
| $(15, 9)$ | $(2\nu, \nu, \epsilon)$ | $\eta \kappa = \{h_1 d_0\}$ | $h_1 d_0 = \langle h_0 h_2, h_2, c_0\rangle$ | Lemma 4.2.86 | 4.2.48, 4.2.92 |
| $(16, 9)$ | $(\eta, 2, \sigma^2)$ | $\eta_4 \in \{h_1 h_4\}$ | $\eta \rho_{15} = \{P c_0\}$ | $d_4 (h_4) = \langle h_0 h_3\rangle$ | Lemmas 3.3.18, 3.3.53 |
| $(16, 9)$ | $(\sigma^2, \eta, 2)$ | $0$ | | | |
| $(18, 10)$ | $(2\sigma, \sigma, \nu)$ | $\nu_4 \in \{h_2 h_4\}$ | | | |
| $(18, 10)$ | $(\sigma, \nu, \sigma)$ | intersects $\{h_2 h_4\}$ | $d_4 (h_4) = \langle h_0 h_3\rangle$ | Lemma 4.2.91 | 4.2.84 |
| $(20, 11)$ | $(2, \sigma, 2)$ | $0$ | $\{2k \kappa\} = \{\tau h_0 g\}$ | $0 = \langle h_0, c_1, h_0\rangle$ | Lemma 4.2.29 |
| $(20, 12)$ | $(\nu, \eta, \eta \kappa)$ | $\{h_0 g\}$ | $\nu_2 \kappa = \{h_0^2 g\}$ | $d_4 (c_0) = \langle h_1^2 d_0\rangle$ | Lemma 4.2.87 |
| $(21, 12)$ | $(\kappa, 2, \nu)$ | $\eta \kappa \in \{\tau h_1 g\}$ | | | |
| $(22, 12)$ | $(8\sigma, 2, \sigma^2)$ | $0$ | | | |
| $(23, 12)$ | $(2, 8\sigma, 2, \sigma^2)$ | $\tau \nu \kappa \in \{\tau^2 h_2 g\}$ | $\{2k \rho_{23}\} = \{h_0^3 g\}$ | Lemma 3.3.58 | 3.3.53 |
| | | | | $\{2k \tau^2 \nu \kappa\} \subset \{\tau^2 h_2 h_3 g\}$ | | |
| | | | | $\tau \sigma \eta_4 \in \{\tau h_4 c_0\}$ | | |
Table 23: Some Toda brackets

| (s, w) | bracket | contains | indeterminacy | proof | used in |
|--------|---------|----------|---------------|-------|---------|
| (23, 13) | | | | | |
| (32, 17) | | | | | |
| (37, 20) | | | | | |
| (39, 21) | | | | | |
| (40, 21) | | | | | |
| (41, 23) | | | | | |
| (45, 24) | | | | | |
| (45, 24) | | | | | |
| (47, 25) | | | | | |
| (48, 26) | | | | | |
| (50, 27) | | | | | |
| (50, 27) | | | | | |
| (50, 27) | | | | | |
| (51, 28) | | | | | |
| (52, 29) | | | | | |
| (52, 30) | | | | | |
| (55, 31) | | | | | |

? TABLES
Table 23: Some Toda brackets

| $(s, w)$ | bracket | contains | indeterminacy | proof | used in |
|----------|---------|----------|--------------|-------|---------|
| (59, 33) | $\langle \eta^2, \{Ph_1h_5c_0\}, e \rangle$ | $\{Ph_1^2h_5c_0\}$ | $\eta^2\{D_{11}\} \in \{\tau^2c_1g^2\}$ | $d_2(G_3) = Ph_1^3h_5c_0$ | Lemma 3.3.45  |
| (60, 32) | $\langle \theta_{4, 5}, \sigma^2, 2 \rangle$ | intersects $\{B_3\}$ | ? | $d_2(e_0) = h_1^2d_0$ | Lemma 4.2.11  |
| (60, 34) | $\langle \nu^2, \eta, \eta \rangle$ | $\{Ph_1^4h_5c_0\}$ | | $d_2(h_4) = h_0h_3^2$ | Remark 3.2.11  |
| (61, 33) | $\langle \eta^2, \sigma^2, 2 \rangle$ | intersects $\{h_1B_3\}$ | ? | $d_2(h_4) = h_0h_3^2$ | Lemma 3.3.18  |

$\tau^2h_1e_0^2, h_1^3, h_1h_4$
Table 24: Classical Adams hidden extensions

| $(s, f)$ | type  | from  | to    | reference |
|---------|-------|-------|-------|-----------|
| $(20, 6)$ | $\nu$ | $h_{5}^{2}g$ | $P_{h_{1}d_{0}}$ | [26] Theorem 2.1.1 |
| $(21, 5)$ | $\eta$ | $h_{1}g$ | $P_{d_{0}}$ | [26] Theorem 2.1.1 |
| $(21, 6)$ | 2 | $h_{0}h_{2}g$ | $P_{h_{1}d_{0}}$ | [26] Theorem 2.1.1 |
| $(30, 2)$ | $\nu$ | $h_{4}^{2}$ | $p$ | [4] Proposition 3.3.5 |
| $(30, 2)$ | $\sigma$ | $h_{4}^{2}$ | $x$ | [4] Proposition 3.5.1 |
| $(32, 6)$ | $\nu$ | $g$ | $h_{1}e_{0}^{2}$ | [4] Proposition 3.3.1 |
| $(40, 8)$ | 2 | $g^{2}$ | $h_{1}u$ | [4] Proposition 3.4.3 |
| $(40, 8)$ | $\eta$ | $g^{2}$ | $z$ | [4] Corollary 3.4.2 |
| $(41, 10)$ | $\eta$ | $z$ | $d_{0}^{3}$ | [4] Proposition 3.4.1 |
| $(45, 3)$ | 4 | $h_{4}^{3}$ | $h_{0}h_{5}d_{0}$ | [3] Theorem 3.3(i) |
| $(45, 3)$ | $\eta$ | $h_{4}^{3}$ | $B_{1}$ | [3] Theorem 3.1(i) |
| $(45, 9)$ | $\eta$ | $w$ | $d_{0}l$ | [3] Theorem 3.1(iv) |
| $(46, 11)$ | $\eta$ | $d_{0}l$ | $P_{u}$ | [3] Theorem 3.1(ii) |
| $(47, 10)$ | $\eta$ | $e_{0}r$ | $d_{0}e_{0}^{2}$ | [3] Theorem 3.1(vi) |

Table 25: Hidden Adams $\tau$ extensions

| $(s, f, w)$ | from  | to    | reference |
|-------------|-------|-------|-----------|
| $(22, 7, 13)$ | $c_{0}d_{0}$ | $P_{d_{0}}$ | cofiber of $\tau$ |
| $(23, 8, 14)$ | $h_{1}c_{0}d_{0}$ | $P_{h_{1}d_{0}}$ | cofiber of $\tau$ |
| $(28, 6, 17)$ | $h_{1}h_{3}g$ | $d_{0}^{2}$ | Lemma [4.2.1] |
| $(29, 7, 18)$ | $h_{1}^{2}h_{3}g$ | $h_{1}d_{0}^{2}$ | Lemma [4.2.1] |
| $(40, 9, 23)$ | $\tau h_{0}g^{2}$ | $h_{1}u$ | Lemma [4.2.2] |
| $(41, 9, 23)$ | $\tau^{2}h_{1}g^{2}$ | $z$ | Lemma [4.2.3] |
| $(42, 11, 25)$ | $c_{0}e_{0}^{2}$ | $d_{0}^{3}$ | cofiber of $\tau$ |
| $(43, 12, 26)$ | $h_{1}c_{0}e_{0}^{2}$ | $h_{1}d_{0}^{3}$ | cofiber of $\tau$ |
| $(47, 12, 26)$ | $h_{1}u'$ | $P_{u}$ | cofiber of $\tau$ |
| $(48, 10, 29)$ | $h_{1}h_{3}g^{2}$ | $d_{0}e_{0}^{3}$ | Lemma [4.2.1] |
| $(49, 11, 30)$ | $h_{1}^{2}h_{3}g^{2}$ | $h_{1}d_{0}e_{0}^{3}$ | Lemma [4.2.1] |
| $(52, 10, 29)$ | $h_{1}G_{3}$ | $\tau^{2}e_{0}m$ | cofiber of $\tau$ |
| $(53, 9, 29)$ | $B_{8}$ | $x'$ | cofiber of $\tau$ |
| $(53, 11, 30)$ | $h_{1}^{2}G_{3}$ | $d_{0}u$ | cofiber of $\tau$ |
| $(54, 8, 31)$ | $h_{1}i_{1}$ | ? | cofiber of $\tau$ |
| $(54, 10, 30)$ | $h_{1}B_{8}$ | $h_{1}x'$ | cofiber of $\tau$ |
| $(54, 11, 32)$ | $h_{1}^{2}h_{5}e_{0}$ | $\tau e_{0}^{2}g$ | cofiber of $\tau$ |
| $(55, 12, 33)$ | $h_{1}^{2}h_{5}e_{0}$ | $\tau h_{1}e_{0}^{2}g$ | cofiber of $\tau$ |
| $(55, 13, 31)$ | $\tau^{2}h_{1}e_{0}^{2}g$ | $d_{0}z$ | Lemma [4.2.4] |
| $(59, 7, 33)$ | $f_{1}$ | ? | cofiber of $\tau$ |
| $(59, 12, 33)$ | $P_{h_{1}h_{5}e_{0}}$ | $\tau d_{0}w$ | Lemma [4.2.10] |
## Table 26: Tentative hidden Adams $\tau$ extensions

| $(s, f, w)$ | from | to | reference |
|-------------|------|----|-----------|
| $(60, 13, 34)$ | $\tau^2 h_0 g^3$ | $d_0 u' + \tau d_0^2 l$ | cofiber of $\tau$ |
| $(61, 13, 35)$ | $\tau^2 h_1 g^3$ | $d_0 e_0 r$ | cofiber of $\tau$ |
| $(62, 14, 37)$ | $h_1^3 h_0 e_0$ | $d_0^2 e_0^2$ | cofiber of $\tau$ |
| $(63, 15, 38)$ | $h_0^3 h_2 g^3$ | $h_1 d_0^2 e_0^2$ | cofiber of $\tau$ |
| $(65, 9, 36)$ | $h_2^3 x_2$ | $\tau B_{23}$ | Lemma 4.2.12 |
| $(66, 10, 37)$ | $h_2^3 x_2$ | $\tau h_1 B_{23}$ | Lemma 4.2.12 |
| $(66, 14, 37)$ | $h_2^3 x_1$ | $\tau^2 d_0 e_0 m$ | cofiber of $\tau$ |
| $(67, 11, 38)$ | $h_3^3 x_2$ | $B_{8} d_0$ | Lemma 4.2.13 |
| $(67, 13, 37)$ | $B_{8} d_0$ | $d_0 x'$ | Lemma 4.2.13 |
| $(67, 15, 38)$ | $h_0 e_0 g r$ | $d_0^2 u$ | cofiber of $\tau$ |
| $(68, 14, 41)$ | $h_1 h_3 g^3$ | $e_0^2$ | cofiber of $\tau$ |
| $(69, 15, 42)$ | $h_2^3 h_3 g^3$ | $h_1 e_0^2$ | cofiber of $\tau$ |

## Table 27: Hidden Adams 2 extensions

| $(s, f, w)$ | from | to | reference |
|-------------|------|----|-----------|
| $(23, 6, 14)$ | $h_0 h_2 g$ | $h_1 c_0 d_0$ | Lemma 4.2.17 |
| $(23, 6, 13)$ | $\tau h_0 h_2 g$ | $P h_1 d_0$ | Lemma 4.2.17 |
| $(40, 8, 22)$ | $\tau^2 g^2$ | $h_1 u$ | Table 24 |
| $(43, 10, 26)$ | $h_0 h_2 g^2$ | $h_1 c_0 e_0^2$ | Lemma 4.2.17 |
| $(43, 10, 25)$ | $\tau h_0 h_2 g^2$ | $h_1 d_0^3$ | Lemma 4.2.17 |
| $(47, 10, 26)$ | $c_0 r$ | $h_1 u'$ | Lemma 4.2.26 |
| $(47, 10, 25)$ | $\tau c_0 r$ | $P u$ | Lemma 4.2.26 |
| $(51, 9, 28)$ | $h_0 h_3 g_2$ | $h_4$ | Lemma 4.2.26 |
| $(54, 9, 28)$ | $h_0 h_3 i$ | $\tau^2 e_0^2 g$ | Lemma 4.2.35 |
| $(59, 7, 33)$ | $f_1$ | ? | Lemma 4.2.36 |

## Table 28: Tentative hidden Adams 2 extensions

| $(s, f, w)$ | from | to | reference |
|-------------|------|----|-----------|
| $(60, 12, 33)$ | $\tau^3 g^3$ | $d_0 u' + \tau d_0^2 l$ | Lemma 4.2.34 |
| $(63, 14, 37)$ | $\tau h_0 h_2 g^3$ | $h_1 d_0^2 e_0^2$ | Lemma 4.2.37 |
| $(67, 14, 37)$ | $\tau c_0 g r$ | $d_0^2 u$ | Lemma 4.2.37 |

## Table 29: Hidden Adams $\eta$ extensions

| $(s, f, w)$ | from | to | reference |
|-------------|------|----|-----------|
| $(15, 4, 8)$ | $h_0^3 h_4$ | $P c_0$ | image of $J$ |
| $(21, 5, 12)$ | $\tau h_1 g$ | $c_0 d_0$ | Lemma 4.2.39 |
| $(21, 5, 11)$ | $\tau^2 h_1 g$ | $P d_0$ | Lemma 4.2.39 |
| $(23, 9, 12)$ | $h_0^3 i$ | $P^2 c_0$ | image of $J$ |
Table 29: Hidden Adams $\eta$ extensions

| $(s, f, w)$ | from | to | reference |
|-------------|------|----|----------|
| (31, 11, 16) | $h_0^{10}h_5$ | $P^3c_0$ | image of $J$ |
| (38, 4, 20) | $h_0^2h_3h_5$ | $\tau^2c_1g$ | Lemma 4.2.31 |
| (39, 17, 20) | $P^2h_0^3i$ | $P^4c_0$ | image of $J$ |
| (40, 8, 21) | $\tau^3g^2$ | $z$ | Table 24 |
| (41, 5, 23) | $h_1f_1$ | $\tau h_2c_1g$ | Lemma 4.2.46 |
| (41, 9, 24) | $\tau h_1g^2$ | $c_0c_0^2$ | Lemma 4.2.50 |
| (41, 10, 22) | $z$ | $\tau d_0^2$ | Lemma 4.2.30 |
| (45, 3, 24) | $h_3^2h_5$ | $B_1$ | Lemma 4.2.48 |
| (45, 9, 24) | $\tau w$ | $\tau d_0l + u'$ | Table 24 |
| (46, 11, 24) | $\tau^2d_0l$ | $P_u$ | Table 24 |
| (47, 10, 26) | $e_0r$ | $\tau d_0e_0^2$ | Table 24 |
| (47, 20, 24) | $h_0^6Q'$ | $P^5c_0$ | image of $J$ |
| (52, 11, 28) | $\tau^2e_0m$ | $d_0u$ | Lemma 4.2.53 |
| (54, 12, 29) | $\tau^3e_0^2y$ | $d_0z$ | Lemma 4.2.55 |
| (55, 25, 28) | $P^4h_0^5i$ | $P^6c_0$ | image of $J$ |
| (58, 8, 30) | $\tau h_1Q_2$ | $?$ | |
### Table 31: Hidden Adams $\nu$ extensions

| $(s, f, w)$ | from | to | reference |
|------------|------|----|-----------|
| $(42, 8, 25)$ | $h_2 c_1 g$ | $h_5^0 h_5 c_0$ | Lemma 4.2.63 |
| $(45, 3, 24)$ | $h_5^2 h_5$ | $B_2$ | Lemma 4.2.73 |
| $(45, 4, 24)$ | $h_0 h_5^2 h_5$ | $h_5 B_2$ | Lemma 4.2.73 |
| $(45, 9, 24)$ | $\tau w$ | $\tau^2 d_0 c_0^2$ | Lemma 4.2.71 |
| $(46, 7, 25)$ | $N$ | $N h_5 c_0$ | Lemma 4.2.63 |
| $(46, 10, 27)$ | $\tau^2 h_2^2 g^2$ | $h_1 d_0 c_0^2$ | Lemma 4.2.63 |
| $(48, 6, 26)$ | $h_2 h_5 d_0$ | $?$ | Lemma 4.2.75 |
| $(51, 8, 28)$ | $h_2 B_2$ | $h_1 B_8$ | Lemma 4.2.75 |
| $(51, 8, 27)$ | $\tau h_2 B_2$ | $h_1 x'$ | Lemma 4.2.75 |
| $(52, 10, 29)$ | $h_1 G_3$ | $\tau^2 h_1 c_0^2 g$ | Lemma 4.2.76 |
| $(52, 11, 28)$ | $\tau^2 e_0 m$ | $d_0 z$ | Lemma 4.2.76 |
| $(53, 7, 30)$ | $i_1$ | $?$ |Lemma 4.2.81 |
| $(54, 11, 32)$ | $h_1^0 h_5 e_0$ | $h_2 c_0^2 g$ | Lemma 4.2.76 |

### Table 32: Tentative hidden Adams $\nu$ extensions

| $(s, f, w)$ | from | to | reference |
|------------|------|----|-----------|
| $(57, 10, 30)$ | $h_0 h_2 h_5 i$ | $\tau^2 d_0^2 l$ | Lemma 4.2.79 |
| $(59, 13, 32)$ | $\tau d_0 w$ | $\tau^2 d_0^2 c_0^2$ | Lemma 4.2.80 |
| $(59, 12, 33)$ | $P h_1^3 h_5 c_0$ | $\tau^2 d_0^2 c_0^2$ | Lemma 4.2.80 |
| $(60, 14, 35)$ | $\tau h_0^2 g^3$ | $h_1 d_0^2 c_0^2$ | Lemma 4.2.81 |
| $(62, 12, 37)$ | $h_2 c_1 g^2$ | $h_1^4 D_4$ | Lemma 4.2.82 |
| $(65, 13, 36)$ | $\tau g w + h_1^4 X_1$ | $\tau^2 e_0^4$ | Lemma 4.2.80 |

### Table 33: Some miscellaneous hidden Adams extensions

| $(s, f, w)$ | type | from | to | reference |
|------------|------|------|----|-----------|
| $(16, 2, 9)$ | $\sigma$ | $h_1 h_4$ | $h_4 c_0$ | Lemma 4.2.83 |
| $(20, 4, 11)$ | $\epsilon$ | $\tau g$ | $d_0^2$ | Lemma 4.2.85 |
| $(40, 8, 23)$ | $\epsilon$ | $\tau g^2$ | $d_0 c_0^2$ | Lemma 4.2.85 |
| $(32, 6, 17)$ | $\epsilon$ | $g$ | $h_1 u$ | Lemma 4.2.84 |
| $(45, 3, 24)$ | $\epsilon$ | $h_5^3 h_5$ | $B_8$ | Lemma 4.2.88 |
| $(30, 2, 16)$ | $\nu_4$ | $h_5^3$ | $h_2 c_0 h_5 d_0$ | Lemma 4.2.90 |
| $(30, 2, 16)$ | $\eta_4$ | $h_5^3$ | $h_1 h_5 d_0$ | Lemma 4.2.92 |
| $(45, ?, 24)$ | $\kappa$ | $h_5^2 h_5$ or $h_5 d_0$ | $B_21$ | Lemma 4.2.93 |
| $(45, ?, 24)$ | $\pi$ | $h_5^2 h_5$ or $h_5 d_0$ | $\tau B_23$ | Lemma 4.2.93 |

(tentative)
Table 34: Some compound hidden Adams extensions

| \( (s, w) \) | relation | \( \text{reference} \) |
|----------------|-----------------|-----------------|
| \((9, 6)\) | \( \nu^2 + \eta^2 \sigma = \eta \epsilon \) | 41 |
| \((40, 22)\) | \( \nu \{h_2 h_5\} + \eta \sigma \eta_5 = c \eta_5 \) | Lemma 4.2.89 |

Table 35: Hidden \( h_0 \) extensions in \( E_2(Cr) \)

| \((s, f, w)\) | \( \overline{f} \) | \( h_0 \cdot \overline{f} \) | \( \text{reference} \) |
|----------------|----------------|-----------------|-----------------|
| \((11, 4, 6)\) | \( \overline{h_1 c_0} \) | \( Ph_2 \) | \( d_2(b_{20} h_0(1)) = \tau h_1^2 c_0 \) |
| \((11, 4, 6)\) | \( \overline{P^k h_2 c_0} \) | \( P^{k+1} h_2 \) | \( d_2(b_{20}^{2k+1} h_0(1)) = \tau P^k h_2^2 c_0 \) |
| \(+k(8, 4, 4)\) | \( c_0 d_0 \) | \( i \) | \( d_2(b_{20} b_{30} h_0(1)) = \tau c_0 d_0 \) |
| \((26, 14)\) | \( c_0 e_0 \) | \( j \) | | Lemma 5.1.3 |
| \(+k(8, 4, 4)\) | \( \overline{P^k e_0 c_0} \) | \( P^k j \) | | Lemma 5.1.3 |
| \((39, 14, 20)\) | \( \overline{P^2 c_0 d_0} \) | \( P^2 i \) | \( d_2(h_{20}^5 b_{30} h_0(1)) = \tau P^2 c_0 d_0 \) |
| \((41, 9, 22)\) | \( h_0 \cdot \overline{\tau h_0 g^2} \) | \( z \) | \( d_4(\Delta h_0 c_0) = \tau^2 h_0^2 g^2 \) |
| \((44, 9, 24)\) | \( h_2 \cdot \overline{\tau h_0 g^2} \) | \( d_0 r \) | \( d_4(\Delta h_0 g) = \tau^2 h_0 h_2 g^2 \) |
| \((46, 10, 26)\) | \( c_0 e_0 g \) | \( d_0 l \) | | Lemma 5.1.3 |
| \((46, 13, 24)\) | \( h_0 \cdot \overline{\tau h_0 d_0 e_0} \) | \( i^2 \) | \( d_4(P \Delta h_0 d_0) = \tau^2 h_0^2 d_0^2 e_0 \) |
| \((47, 9, 26)\) | \( \overline{\tau h_2 g^2} \) | \( e_0 r \) | \( d_4(\Delta h_2 g) = \tau^2 h_2^2 g^2 \) |
| \((47, 12, 24)\) | \( h_2^2 \cdot \underline{w} \) | \( Q' \) | \( d_4(\Delta h_0 i) = \tau^2 h_0^2 d_0 l \) |
| \((49, 13, 26)\) | \( h_0 \cdot \overline{\tau h_0 d_0 e_0} \) | \( i j \) | \( d_4(P \Delta h_0 e_0) = \tau^2 h_0^2 d_0 e_0 \) |
| \((52, 13, 28)\) | \( h_0 \cdot \overline{\tau h_0 e_0^3} \) | \( i k \) | \( d_4(P \Delta h_2 e_0) = \tau^2 h_2^2 e_0^3 \) |
| \((54, 8, 30)\) | \( k d_0 g \) | \( h_1 h_5 c_0 d_0 \) | | Lemma 5.1.3 |
| \((55, 13, 30)\) | \( h_0 \cdot \overline{\tau h_0 e_0^3 g} \) | \( i l \) | \( d_4(\Delta h_0 d_0 e_0) = \tau^2 h_0^2 e_0^2 g \) |
| \((55, 22, 28)\) | \( \overline{P^2 c_0 d_0} \) | \( P^4 i \) | \( d_2(b_{20}^3 b_{30} h_0(1)) = \tau P^4 c_0 d_0 \) |
| \((56, 10, 30)\) | \( \overline{h_1 B_8} \) | \( h_2 x' \) | | Lemma 5.1.5 |
| \((57, 17, 30)\) | \( h_0 \cdot \overline{\tau h_0 d_0^4} \) | \( P i j \) | \( d_4(P^2 \Delta h_0 e_0) = \tau^2 h_0^2 d_0^4 \) |
| \((58, 5, 30)\) | \( D_4 \) | \( D_2 \) | \( d_2(b_{30} h_0(1, 2)) = \tau D_4 \) |
| \((58, 13, 32)\) | \( h_0 \cdot \overline{\tau h_0 e_0^3 g^2} \) | \( i m \) | \( d_4(\Delta h_0 e_0^2 g) = \tau^2 h_0^2 e_0^2 g^2 \) |
| \((61, 13, 34)\) | \( h_2 \cdot \overline{\tau h_0 e_0^2 g^2} \) | \( j m \) | \( d_4(\Delta h_0 e_0 g) = \tau^2 h_0^2 e_0^2 g \) |
| \((62, 12, 34)\) | \( h_2 \cdot \overline{\tau h_2 g m} \) | \( \tau e_0 w \) | \( d_4(\Delta h_2 g m) = \tau^2 h_2^2 g m \) |
| \((62, 21, 32)\) | \( h_0 \cdot \overline{\tau P^2 h_0 d_0^2 e_0} \) | \( P^2 i j \) | \( d_4(P^3 \Delta h_0 e_0) = \tau^2 P^2 h_0^2 d_0^2 e_0 \) |
| \((65, 21, 34)\) | \( h_0 \cdot \overline{\tau P h_0 d_0^3} \) | \( P^2 i j \) | | \( d_4(P^3 \Delta h_0 e_0) = \tau^2 P h_0^2 d_0^3 \) |
| \((66, 9, 34)\) | \( c_0 Q_2 \) | \( ? \) | | |
| \((67, 13, 38)\) | \( \overline{\tau h_2 g^3} \) | \( l m \) | | \( d_4(\Delta h_2 g^2) = \tau^2 h_2^2 g^3 \) |
| \((68, 3, 6)\) | \( h_3 r_1 \) | \( h_3(A + A') \) | \( d_4(x_{68}) = \tau h_1 r_1 \) |
| \((68, 21, 36)\) | \( h_0 \cdot \overline{\tau P h_0 d_0^3 e_0} \) | \( P^2 i k \) | \( d_4(P^2 \Delta h_0 d_0^3 e_0) = \tau^2 P h_0^2 d_0^3 e_0 \) |
| \((69, 9, 36)\) | \( h_1 x_3 \) | \( P(A + A') \) | \( d_2(h_{20}^2 h_3 b_{31} b_{40} + \tau h_1 b_{20} b_{30} b_{31}^2) = \tau h_1 x_3 \) |
| \((70, 16, 36)\) | \( P c_0 x' \) | \( R_1' \) | \( d_2(P^2 B b_{20} b_{30}) = \tau P c_0 x' \) |
### Table 36: Hidden $h_1$ extensions in $E_2(C\tau)$

| $(s, f, w)$ | $\varpi$ | $h_1 \cdot \varpi$ | reference |
|-------------|----------|-------------------|-----------|
| (35, 5, 19) | $h_1^2 d_1$ | $t$ | $d_2(h_1 b_{30} b_{22}) = \tau h_1^2 d_1$ |
|             |          |                   | $d_2(b_{21}^2 h_1(1)) = h_1^3 d_1$ |
| (39, 7, 22) | $h_1^2 h_5^2 \cdot h_1^2$ | $\tau g^2$ | $d_s(g^2) = h_1^2 h_5$ |
| (41, 8, 22) | $\tau h_0 g^2$ | $\tau v$ | $d_8(\Delta e_0) = \tau^2 h_0 g^2$ |
| (44, 8, 23) | $\tau h_0 d_5^2 e_0$ | $\tau w$ | $d_8(w) = Ph_1^3 h_5$ |
| (46, 12, 24) | $\tau h_0 d_5^2 e_0$ | $P u$ | $d_4(P \Delta d_0) = \tau^2 h_0 d_5^2 e_0$ |
| (49, 12, 26) | $\tau h_0 d_5^2 e_0$ | $P v$ | $d_4(\Delta P e_0) = \tau^2 h_0 d_5^2 e_0$ |
| (52, 12, 28) | $h_1 h_5 \cdot c_0 d_0$ | $d_0 u$ | Lemma 5.1.6 |
| (55, 8, 29) | $h_1^2 c_0 d_0$ | $P h_5 c_0$ | Lemma 5.1.7 |
| (55, 9, 31) | $h_1 \cdot h_1 d_1 g$ | $gt$ | $d_2(h_1 b_{30} b_{22}) = \tau h_1^2 d_1 g$ |
|             |          |                   | $d_2(h_1(1) g^2) = h_1^3 d_1 g$ |
| (56, 8, 30) | $\tau h_2 d_4 g$ | $D_{11}$ | $d_4(\Delta d_1) = \tau^2 h_2 d_4 g$ |
| (56, 11, 32) | $h_1^2 h_5 \cdot h_1^2 e_0$ | $\tau e_0 g^2$ | $d_8(e_0 g^2) = h_1^2 h_5 e_0$ |
| (57, 16, 30) | $\tau h_0 d_5^2 e_0$ | $P^2 v$ | $d_4(P^2 \Delta e_0) = \tau^2 h_0 d_5^2 e_0$ |
| (59, 11, 34) | $h_1^4 \cdot h_1^3 i_1$ | $\tau g^5$ | $d_4(g^5) = h_1^4 i_1$ |
| (60, 10, 32) | $h_1 h_1 \cdot B_6$ | $n r$ | $d_4(\Delta t) = \tau^3 c_1 g^2$ |
| (61, 12, 33) | $h_1^2 h_5 \cdot Ph_1^2 e_0$ | $\tau e_0 w$ | $d_8(e_0 w) = Ph_1^2 h_5 e_0$ |
| (62, 20, 32) | $\tau P^2 h_0 d_5^2 e_0$ | $P^3 u$ | $d_4(P^4 \Delta d_0) = \tau^2 P^2 h_0 d_5^2 e_0$ |
| (64, 34, 31) | $h_1 h_1 \cdot B_6$ | $h_2 C_0$ | $d_2(B h_1 B_{21} h_1(1)) = h_1^2 B_7$ |
|             |          |                   | $d_2(h_1^2 b_{30} b_{22} b_{24}) = \tau h_1^2 B_7$ |
| (64, 9, 34) | $h_1^2 \cdot h_2 d_5 e_0$ | $\tau B_2 + c_0 Q_2$ |Lemma 5.1.8 |
| (64, 12, 25) | $h_1^4 \cdot c_0^2 Q_2$ | $\tau gw + h_1^4 X_1$ | Lemma 5.1.9 |
| (65, 6, 34) | $h_3 \cdot B_4$ | $\tau G_0$ | $d_2(h_3 b_{30} h_0(1, 2)) = \tau h_1^2 H_1$ |
|             |          |                   | $d_2(b_{21}^2 h_0(1, 2)) = h_1^3 H_1$ |
| (65, 20, 34) | $\tau Ph_0 d_5^2$ | $P^3 v$ | $d_4(P^4 \Delta d_0) = \tau^2 P^2 h_0 d_5^2$ |
| (68, 8, 37) | $h_1 h_1 \cdot j_1$ | $h_0 h_2 G_0$ | $d_2(h_1 b_{21}^2 b_{22} b_{31}) = h_1^2 h_3 j_1$ |
|             |          |                   | $d_2(h_1^2 h_1(1) b_{22} b_{30}) = \tau h_1 h_3 j_1$ |
| (68, 10, 37) | $h_1^2 c_0 \cdot T_4$ | $h_2 B_5$ | Lemma 5.1.11 |
|             |          | $h_2 B_5 + h_1^2 X_3$ |     |
| (68, 12, 35) | $\tau B_2 d_0$ | $\tau W_1$ | $d_8(\tau D^2 g) = \tau h_1^2 X_3$ |

### Table 37: Hidden $h_2$ extensions in $E_2(C\tau)$

| $(s, f, w)$ | $\varpi$ | $h_2 \cdot \varpi$ | reference |
|-------------|----------|-------------------|-----------|
| (28, 4, 15) | $h_3 g$ | $n$ | $d_2(b_{30} h_1(1)) = \tau h_3 g$ |
| (42, 8, 22) | $\tau h_2 g$ | $? \tau h_2 g^2$ | ? |
| (43, 7, 23) | $\tau h_2 c_1 g$ | $N$ | $d_4(\Delta c_1) = \tau^2 h_2 c_1 g$ |
| (44, 9, 24) | $h_2 \cdot h_2 g^2$ | $e_0 r$ | $d_4(\Delta h_0 g) = \tau^2 h_0 h_2 g^2$ |
| (47, 5, 25) | $h_2^2 g$ | $C$ | $d_4(h_2 b_{30} h_1(1)) = \tau h_2^2 g$ |
| (48, 8, 27) | $h_3 c_0^2$ | $g n$ | $d_2(b_{21}^2 b_{30} h_1(1)) = \tau h_3 g^2$ |
| (54, 8, 30) | $h_1 d_1 g$ | $h_1^2 B_6$ | $d_2(b_{21}^2 b_{30} b_{22} + h_2^2 b_{21} b_{40}) = \tau h_1 d_1 g$ |
Table 37: Hidden $h_2$ extensions in $E_2(C\tau)$

| ($s,f,w$) | $\mathcal{X}$ | $h_2 \cdot \mathcal{X}$ | reference |
|-----------|---------------|----------------------|-----------|
| (58, 5, 30) | $D_4$ | $A$ | $d_2(b_{30}h_0(1,2)) = \tau D_4$ |
| (58, 10, 31) | $h_2 \cdot \tau h_2 g n$ | $nr$ | $d_4(\Delta h_2 n) = \tau^2 h_2 g n$ |
| (59, 5, 31) | $h_2^2 g_2$ | $h_5 n$ | $d_2(h_5 b_{30}h_1(1)) = \tau h_5 h_3 g$ |
| (59, 7, 31) | $h_2^2 \cdot \overline{B_6}$ | $C_0$ | $d_4(x_{59}) = \tau^2 c_1 g$ |
| (60, 6, 32) | $J_1$ | $C'$ | $d_2(h_1(1)^2 b_{40}) = \tau j_1$ |
| (61, 13, 34) | $h_2 c_0 \cdot \tau h_2 c_0 g$ | $km$ | $d_4(\Delta h_0 c_0 g) = \tau^2 h_2^2 c_0^2 g$ |
| (62, 11, 32) | $P h_5 c_0 d_0$ | $P h_5 j$ | $d_2(P h_5 b_{20} h_0(1) b_{30}) = \tau P h_5 c_0 d_0$ |
| (62, 12, 34) | $h_2 \cdot \tau h_2 g m$ | $\tau g w$ | $d_4(\Delta h_2 g) = \tau^2 h_2 g^2$ |
| (63, 11, 35) | $\tau h_2 c_1 g_2$ | $\nu m$ | $d_4(\Delta c_1 g) = \tau^2 h_2 c_1 g^2$ |
| (64, 13, 36) | $\tau h_0 h_2 g^2$ | $im$ | $d_4(\Delta h_0 g^2) = \tau^2 h_0 h_2 g^3$ |
| (66, 8, 36) | $h_1 d_1^2$ | $h_1 h_3 B_7$ | $d_2(b_{30} b_{22} h_1(1)^2) = \tau h_1 d_1^2$ |
| (66, 9, 34) | $c_0 Q_2$ | $?$ | $d_4(\Delta h_2 g) = \tau^2 h_2 g^2$ |
| (67, 13, 38) | $\tau h_2^2 g^2$ | $m^2$ | $d_4(\Delta h_2 g^2) = \tau^2 h_2^2 g^3$ |

Table 38: Some miscellaneous hidden extensions in $E_2(C\tau)$

| ($s,f,w$) | relation | reference |
|-----------|----------|-----------|
| (25, 8, 14) | $h_2^2 \cdot c_0 d_0 + d_0 \cdot h_2^2 c_0 = P e_0$ | Lemma 5.1.15 |
| (28, 8, 16) | $c_0 \cdot h_2^2 c_0 + e_0 \cdot h_2^2 c_0 = d_0^2$ | Lemma 5.1.15 |
| (28, 8, 16) | $h_2^2 \cdot c_0 c_0 + e_0 \cdot h_2^2 c_0 = d_0^2$ | Lemma 5.1.17 |
| (40, 10, 22) | $d_0 \cdot c_0 c_0 + e_0 \cdot c_0 d_0 = h_1 u$ | Lemma 5.1.17 |
| (56, 15, 33) | $h_2^2 c_0^2 \cdot h_2^2 c_0 + d_0 c_0 g \cdot h_2^2 c_0 + h_1^2 \cdot h_2^2 B_1 = c_0 d_0 c_0^2$ | Lemma 5.1.18 |
| (57, 15, 31) | $h_2^2 \cdot h_1 d_0 u + d_0 \cdot h_1^2 u = P v'$ | Lemma 5.1.15 |
| (59, 9, 32) | $h_3^2 \cdot B_0 + h_2 \cdot \tau h_2 d_1 g = h_3^2 Q_2$ | Lemma 5.1.12 |
| (65, 11, 34) | $P h_1 \cdot B_0 = h_1 q_1$ | Lemma 5.1.14 |

Table 39: $E_2(C\tau)$ generators

| ($s,f,w$) | $x$ | $d_2(x)$ | reference |
|-----------|-----|---------|-----------|
| (5, 3, 3) | $h_2^4$ | $d_2(x)$ | top cell $h_0 \cdot c_0 d_0 = i$ |
| (11, 4, 6) | $h_2^3 c_0$ | $P d_0$ | $d_2(i)$ in bottom cell $h_0 \cdot c_0 d_0 = j$ |
| (5, 3, 3) | $P h_2^4$ | $d_0 \cdot h_1^4$ | \[ \text{top cell} \] $d_0 \cdot h_1^4$ |
| (5, 3, 3) | $+ k(8, 4, 4)$ | $P h_2^4 c_0$ | \[ \text{top cell} \] $d_0 \cdot h_1^4$ |
| (11, 4, 6) | $P h_2^4 c_0$ | $c_0 d_0$ | $h_0 \cdot c_0 d_0 = i$ |
| (20, 5, 11) | $h_2^4 c_0$ | $d_0 \cdot h_1^4$ | \[ \text{top cell} \] $d_2(i)$ in bottom cell $h_0 \cdot c_0 d_0 = j$ |
| (23, 6, 12) | $c_0 d_0$ | $P d_0$ | $d_2(i)$ in bottom cell $h_0 \cdot c_0 d_0 = j$ |
| (26, 6, 14) | $c_0 d_0$ | $h_2^4 c_0 d_0$ | \[ \text{top cell} \] $d_2(i)$ in bottom cell $h_0 \cdot c_0 d_0 = j$ |
Table 39: $E_2(Cτ)$ generators

| $(s, f, w)$ | $x$ | $d_2(x)$ | reference |
|------------|-----|---------|----------|
| (28, 4, 15) | $h^g_3$ | $h^g_3 \cdot h^c_0$ | $d_2(j)$ in bottom cell |
| (20, 5, 11) | $P^{k}h^c_0$ | $P^{k}d_0 \cdot h^c_1$ | $d_2(c_0c_0)$ in top cell |
| $+k(8, 4, 4)$ | | | top cell |
| (26, 6, 14) | $P^{k}c_0c_0$ | $P^{k}h^c_0$, $c_0d_0$ | $h_0 \cdot P^{k}c_0c_0 = P^{k}j$ |
| $+k(8, 4, 4)$ | | $+P^{k+1}c_0$ | $d_2(P^{k}j)$ in bottom cell |
| (39, 14, 20) | $P^3c_0d_0$ | $P^3d_0$ | $d_2(P^{k}c_0c_0)$ in top cell |
| (40, 9, 23) | $h^c_0g_1$ | $h^c_0 \cdot h^c_0$ | $h_0 \cdot P^{k}c_0d_0 = P^2i$ |
| $+c_0d_0c_0$ | + | | $d_2(P^2i)$ in bottom cell |
| (41, 8, 22) | $τh_0g^2$ | $h^c_1u$ | Lemma 5.2.2 |
| (42, 8, 22) | | $τh_1g^2$ | $h_1 \cdot τh_0g^2 = v$ |
| (43, 7, 23) | $τh_2c_1g$ | $z$ | $d_2(v)$ in bottom cell |
| (43, 11, 23) | $h^2u_1$ | | Lemma 5.2.3 |
| (44, 8, 23) | $τ^2h_2g^2$ | | |
| (44, 9, 23) | $h^c_0v$ | $h^c_1u$ | $h_0 \cdot c_0c_0g = d_0l$ |
| (46, 10, 26) | $c_0g_0c_0$ | $h^c_0 \cdot c_0c_0$ | $d_2(d_0l)$ in bottom cell |
| | $+d_0^2c_0$ | | $d_2(c_0c_0g)$ in top cell |
| (46, 12, 24) | $τh_0d_0^2c_0$ | | |
| (47, 5, 25) | $h^c_2g_2$ | | |
| (47, 9, 26) | $τh^2g_2$ | | |
| (47, 10, 24) | $w^c_0$ | $τh_0d_0^2c_0$ | $h_1 \cdot τh_0d_0c_0 = P^2v$ |
| (48, 8, 27) | $h^c_3g^2$ | $h^c_3h_5 \cdot h^c_0$ | $d_2(P^2v)$ in bottom cell |
| (49, 12, 26) | $τh_0d_0^2c_0$ | $Ph_1u$ | |
| (50, 9, 27) | $h^c_1B_1$ | | |
| (50, 10, 26) | $v^c_0$ | $h^c_1 \cdot w^c_0$ | $h_1 \cdot τh_0d_0c_0 = P^2v$ |
| | $+τh_0d_0c_0^2$ | | |
| (51, 15, 27) | $Ph^c_1u$ | | |
| (52, 8, 27) | $G_3$ | $h_1h_5 \cdot Ph^c_1c_0$ | $h_0 \cdot G_3c_0d_0 = P^{4}i$ |
| (52, 12, 28) | $τh_0c_0$ | | |
| (52, 13, 27) | $Ph_1v$ | $Ph^c_1u$ | $d_2(P^{4}i)$ in bottom cell |
| (54, 8, 30) | $h^c_1d_1g$ | | |
| (55, 7, 30) | $h^c_1i$ | | |
| (55, 9, 29) | $τh_2g_0$ | | |
| (55, 13, 29) | $h^c_1d_0u$ | $Pu'$ | Lemma 5.2.4 |
| (55, 22, 28) | $P^2c_0d_0$ | $P^2d_0$ | $h_0 \cdot P^2c_0d_0 = P^{4}i$ |
| | | | $d_2(P^{4}i)$ in bottom cell |
Table 39: $E_2(C\tau)$ generators

| $(s, f, w)$ | $x$ | $d_2(x)$ | reference |
|-------------|-----|---------|-----------|
| $(56, 6, 29)$ | $B_6$ | $\tau h_2 d_1 g$ |  |
| $(56, 8, 30)$ | $\tau h_2 d_1 g$ | $h_0 d_0 \cdot \tau h_0 g^2$ | top cell |
| $(56, 10, 30)$ | $h_2^2 B_s$ | $P^3 h_1 u$ | $h_1 \cdot \tau h_0 d_0^3 = P^2 v$ |
| $(56, 11, 30)$ | $\tau h_0 g m$ | $h_2^2 B_s$ | $d_2(P^2 v)$ in bottom cell |
| $(57, 16, 30)$ | $\tau h_0 d_0^3$ | $P^2 h_1 u$ |  |
| $(58, 5, 30)$ | $D_4$ | $h_1 \cdot B_6 + Q_2$ | Lemma 5.2.5 |
| $(58, 14, 30)$ | $P v'$ | $P h_1^2 \cdot h_0 g^2 + \tau h_0 d_0^4$ | top cell |
| $(58, 19, 31)$ | $P \tau h_2 g m$ | $h_0 e_0 \cdot \tau h_0 g^2$ | top cell |
| $(60, 6, 32)$ | $h_1$ | $P^2 h_1 v$ | $P^2 h_1 v$ in top cell |
| $(60, 8, 31)$ | $h_1^2 Q_2$ | $P^3 h_1 u$ |  |
| $(60, 17, 31)$ | $P h_5 c_0 d_0$ | $P^2 h_1 v$ |  |
| $(62, 11, 32)$ | $h_2^2 d_0 e_0$ | $P h_5 c_0 d_0$ |  |
| $(63, 8, 33)$ | $\tau h_2 g m$ | $h_1 c_0 x'$ | Lemma 5.2.6 |
| $(63, 13, 33)$ | $P h_1 d_0 u$ | $P^2 u'$ | Lemma 5.2.4 |
| $(65, 11, 35)$ | $h_2^2 B_2$ | $B_21 \cdot h_1^2$ | top cell |
| $(65, 20, 34)$ | $P h_0 d_0^3$ | $P^3 h_1 u$ | $h_1 \cdot \tau h_0 d_0^3 = P^3 v$ |
| $(66, 9, 34)$ | $h_5 d_0 e_0$ | $P^2 v'$ |  |
| $(66, 18, 34)$ | $P^2 v'$ | $\tau P h_0 d_0^3 + P^2 h_1^2 \cdot u'$ |  |
| $(67, 13, 39)$ | $\tau h_2 g^2$ | $x' \cdot P h_1^2$ | top cell |
| $(67, 15, 35)$ | $\tau h_2 g^3$ | $x' \cdot P h_1^2$ |  |
| $(67, 23, 35)$ | $P^3 h_1^2 u$ | $\tau h_1$ |  |
| $(68, 6, 36)$ | $h_1 r_1$ | $? \tau h_1$ |  |
| $(68, 12, 35)$ | $\tau B h_0$ | $h_1 d_0$ |  |
| $(68, 12, 39)$ | $h_0 g'$ | $h_1 \cdot D_4$ |  |
| $(68, 21, 35)$ | $P h_1 v$ | $P^3 h_1 u$ |  |
| $(69, 9, 36)$ | $h_1 X_3$ | $P h_1 v$ |  |
| $(69, 17, 37)$ | $h_1 d_0^2 u$ | $P h_1 v$ |  |
| $(70, 16, 36)$ | $P c_0 x'$ | $P^2 x'$ | $h_0 \cdot P c_0 x' = R'_1$ |

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References:
- Lemma 5.2.5
- Lemma 5.2.6
- Lemma 5.2.4
- Lemma 5.2.7
- Lemma 5.2.8

Note: The references are not fully provided in the table.
7. Tables

Table 40: Ambiguous $E_2(C\tau)$ generators

| $(s, f, w)$ | $x$ | ambiguity | definition |
|------------|-----|-----------|------------|
| (55, 7, 30) | $h_1t_1$ | $\tau h_1 G$ | $h_1^3 \cdot h_1 t_1 = \tau g^3$ |
| (56, 11, 30) | $\tau h_0 g m$ | $h_2 x'$ |
| (59, 11, 32) | $\tau h_2 g m$ | $h_0 B_{21}$ |
| (62, 11, 32) | $\frac{P h_5 c_0 d_0}{h_0 R}$ | $h_0 \cdot P h_5 c_0 d_0 = 0$ |
| (68, 6, 36) | $\frac{h_1 r_1}{}\tau h_1 Q_3$ |

Table 41: Adams $d_3$ differentials for $E_3(C\tau)$

| $(s, f, w)$ | $x$ | $d_3(x)$ | reference |
|------------|-----|---------|-----------|
| (29, 5, 16) | $h_1 h_3 g$ | $d_3^0$ | Lemma 5.2.12 |
| (47, 17, 24) | $h_3^2 Q'$ | $P^4 d_0$ | $d_3(h_3^2 Q')$ in bottom cell |
| (49, 9, 28) | $h_1^6 h_3 g^2$ | $d_0 e_0^3$ | Lemma 5.2.12 |
| (51, 6, 27) | $h_1^7 h_5 c_0$ | $h_1^2 B_1$ | $h_1^6 h_5 \cdot h_1^7 c_0 = \tau e_0 g^2$ |
| (53, 9, 28) | $h_1 G_3$ | $\tau h_0 e_0^3$ | $h_1^6 \cdot h_3 B_1 = c_0 d_0 e_0^2$ in $E_3(C\tau)$ |
| (54, 7, 28) | $h_5 c_0 d_0$ | $x'$ | $d_3(h_5 i)$ in bottom cell |
| (55, 7, 30) | $h_1 t_1$ or $h_1 t_1 + \tau h_1 G$ | ? |
| (56, 6, 29) | $B_6$ | $\tau h_2 g m$ | top cell |
| (57, 7, 30) | $h_5 c_0 e_0$ | $h_1^2 B_3$ | top cell |
| (59, 10, 31) | $\frac{P h_5 c_0 d_0}{h_1 B_{21}}$ | $x' \cdot h_1^2$ | top cell |
| (61, 8, 33) | $h_1^2 D_4$ | $h_1 B_{21}$ | Lemma 5.2.10 |
| (65, 11, 34) | $\frac{P h_5 c_0 d_0}{h_1 \cdot h_1 c_0 x' + U}$ | $d_3(h_2 B_5)$ in bottom cell |

Table 42: Projection to the top cell of $C\tau$

| $(s, f, w)$ | element of $E_\infty(C\tau)$ | element of $E_\infty(S^0\Sigma)$ |
|------------|-----------------------------|--------------------------------|
| (30, 6, 16) | $r$ | $h_1 d_0^2$ |
| (34, 2, 18) | $h_2 h_5$ | $h_1 d_1$ |
| (38, 7, 20) | $h_0 y$ | $\tau h_2 e_0^2$ |
| (41, 4, 22) | $h_0 c_2$ | $h_1 h_3 d_1$ |
| (44, 10, 24) | $d_0 r$ | $h_1 d_0^3$ |
| (50, 10, 28) | $g r$ | $h_1 d_0 e_0^2$ |
| (55, 7, 30) | $B_6$ | $h_2 g m$ |
Table 42: Projection to the top cell of $C\tau$

| $(s, f, w)$ | element of $E_\infty(C\tau)$ | element of $E_\infty(S^{0,0})$ |
|-------------|-------------------------------|-------------------------------|
| (56, 10, 29) | $Q_1$ | $d_0z$ |
| (57, 7, 30) | $Q_2$ | $\tau g t$ |
| (58, 7, 30) | $h_0D_2$ | $D_{31}$ |
| (58, 11, 32) | $Ph_7^2h_5e_0$ | $\tau h_2e_3^2g$ |
| (59, 8, 33) | $h_1^2D_4$ | $h_2^2d_1g$ |

Table 43: Hidden Adams extensions in $E_\infty(C\tau)$

| $(s, f, w)$ | type | from | to | reference |
|-------------|------|------|----|-----------|
| (35, 5, 19) | $\eta$ | $h_2h_5$ | $h_3^2g$ | top cell |
| (39, 9, 21) | $\eta$ | $h_0y$ | $u$ | Lemma 5.3.4 |
| (41, 9, 22) | $\nu$ | $h_0y$ | $\tau h_2^2g^2$ | top cell |
| (42, 6, 23) | $\eta$ | $h_0c_2$ | $h_3 \cdot h_2^2g$ | top cell |
| (47, 12, 26) | $\nu$ | $d_0r$ | $h_1u'$ | Lemma 5.3.9 |
| (48, 8, 26) | $\eta$ | $\tau h_2^2g^2$ | ? |
| (54, 10, 30) | 2 | $h_1d_1g$ | ? |
| (57, 11, 30) | 2 | $Q_2$ | ? |
| (57, 11, 30) | $\nu$ | $h_0h_5i$ | $h_1Q_1$ | bottom cell |
| (58, 10, 32) | $\nu$ | $h_1^2 + \tau h_1 G$ | ? |
| (58, 10, 32) | $\nu$ | $B_6$ | ? |
| (59, 10, 32) | $\nu$ | $\tau h_2d_1g$ | ? |
| (59, 9, 31) | $\eta$ | $h_0D_2$ | $h_3G_3$ | top cell |
| (59, 12, 33) | 2 | $h_1^2D_4$ | ? |
| (60, 10, 32) | $\nu$ | $Q_2$ | $h_1 \cdot h_3G_3$ | top cell |
| (60, 10, 32) | 2 | $\tilde{f}_1$ | ? |

Table 44: Hidden Adams-Novikov 2 extensions

| $(s, f, w)$ | from to |
|-------------|--------|
| (3, 1, 2) | $4\alpha_{2/2}$ |
| (11, 1, 6) | $4\alpha_{6/3}$ |
| (18, 2, 10) | $2z_{18}$ |
| (19, 1, 10) | $4\alpha_{10/3}$ |
| (20, 2, 11) | $z_{20/2}$ |
| (27, 1, 14) | $4\alpha_{14/3}$ |
| (34, 2, 18) | $2z_{34/2}$ |
| (35, 1, 18) | $4\alpha_{18/3}$ |
| (40, 6, 23) | $\alpha_{1/2}z_{36}$ |
| (42, 2, 22) | $4z_{42}$ |
| (43, 1, 22) | $4\alpha_{22/3}$ |
Table 44: Hidden Adams-Novikov 2 extensions

| $(s, f, w)$ | from      | to                         |
|------------|-----------|----------------------------|
| (51, 1, 26) | $2\alpha_{26}/3$ | $\tau\alpha_{27}\alpha_{25}$ |
| (51, 5, 28) | $4\alpha_{4}/4\tau\alpha_{44,4}$ | $?\tau\tau_{54,10}$ |
| (54, 2, 28) | $\tau_{54,2}$ | $4\alpha_{30}/3$ $\tau\alpha_{7}\tau_{29}$ |
| (59, 1, 30) | $\tau_{29,7}$ | $?\tau_{59,11}$ |

Table 45: Hidden Adams-Novikov $\eta$ extensions

| $(s, f, w)$ | from      | to                         |
|------------|-----------|----------------------------|
| (37, 3, 20) | $\alpha_{4}/4\tau_{30}$ | $\tau\alpha_{2}/2\tau_{32,4}$ |
| (38, 2, 20) | $\tau_{38}$ | $\tau\tau_{29,7}$ |
| (39, 3, 21) | $\alpha_{1}\tau_{38}$ | $\tau\tau_{40,8}$ |
| (41, 5, 23) | $\alpha_{1}\tau_{40,4}$ | $\tau\alpha_{2}/2\tau_{39,7}$ |
| (47, 5, 26) | $\tau_{47,5}$ | $\tau\tau_{48}$ |
| (58, 6, 32) | $\alpha_{3}\tau_{56,4}$ | $\tau\tau_{59,11}$ |

Table 46: Hidden Adams-Novikov $\nu$ extensions

| $(s, f, w)$ | from      | to                         |
|------------|-----------|----------------------------|
| (0, 0, 0)  | $4\alpha_{1}$ | $\alpha_{3}$ |
| (20, 2, 11) | $\tau_{20,2}$ | $\tau_{23}$ |
| (32, 2, 17) | $\tau_{32,2}$ | $\alpha_{1}\tau_{34,6}$ |
| (36, 4, 20) | $\alpha_{2}\tau_{34,2}$ | $\tau\tau_{39,7}$ |
| (39, 3, 21) | $\alpha_{1}\tau_{38}$ | $\alpha_{1}\tau_{36}$ |
| (40, 6, 23) | $\alpha_{2}\tau_{36}$ | $\tau\tau_{43}$ |
| (45, 3, 24) | $\alpha_{1}\tau_{44,2}$ | $\tau\tau_{48}$ |
| (48, 4, 26) | $\alpha_{2}/2\tau_{45}$ | $?\tau\tau_{50}$ |
| (51, 4, 27) | $\tau_{50}$ | $?\tau\tau_{54,10}$ |
| (52, 6, 29) | $\alpha_{2}/2\tau_{45}$ | $?\tau\tau_{50}$ |
| (56, 8, 32) | $\alpha_{2}\tau_{54,6}$ | $\tau\tau_{59,11}$ |

Table 47: Correspondence between classical Adams and Adams-Novikov $E_{\infty}$

| $s$ | Adams      | Adams-Novikov | detects |
|-----|------------|---------------|---------|
| 0   | $h_0^0$    | $2^k$         | $2^k$   |
| 1   | $h_1$      | $\alpha_1$   | $\eta$  |
| 2   | $h_2^2$    | $\alpha_1^2$ | $\eta^2$ |
| 3   | $h_2$      | $\alpha_2/2$ | $\nu$   |
| 3   | $h_0h_2$   | $2\alpha_2/2$ | $2\nu$ |
Table 47: Correspondence between classical Adams and Adams-Novikov $E_\infty$

| $s$ | Adams | Adams-Novikov | detects |
|-----|-------|---------------|---------|
| 3  | $h_3^2 h_2$ | $\alpha_1^3$ | $4\nu$ |
| 6  | $h_2^3$ | $\alpha_2^{3/2}$ | $\nu^2$ |
| 7  | $h_0 h_3$ | $2\alpha_{4/4}$ | $\sigma$ |
| 8  | $h_1 h_3$ | $\alpha_1 \alpha_{4/4}$ | $\eta \sigma$ |
| 9  | $c_0$ | $\alpha_1^2 + \alpha_2^{3/4} \alpha_{4/4}$ | $\epsilon$ |
| 9  | $h_1 c_0$ | $\alpha_1^2 \alpha_2^{1/4} + \alpha_2^{3/4} \alpha_{4/4}$ | $\eta \epsilon$ |
| 9  | $h_3^2 h_3$ | $\alpha_1^2 \alpha_{4/4}$ | $\eta^2 \sigma$ |
| 8$k + 1$ | $p^k h_1$ | $\alpha_{4k+1}$ | $\mu_{4k+1}$ |
| 8$k + 2$ | $p^k h_1^2$ | $\alpha_1 \alpha_{4k+1}$ | $\eta \mu_{4k+1}$ |
| 8$k + 3$ | $p^k h_1$ | $2\alpha_{4k+2/3}$ | $\xi_{4k+3}$ |
| 8$k + 3$ | $p^k h_0 h_2$ | $4\alpha_{4k+2/3}$ | $2\xi_{4k+3}$ |
| 8$k + 3$ | $p^k h_0^2 h_2$ | $\alpha_1^2 \alpha_{4k+1}$ | $4\xi_{4k+3}$ |
| 14 | $h_3^3$ | $\alpha_2^{3/4}$ | $\sigma^2$ |
| 14 | $d_0$ | $\alpha_2^{1/4}$ | $\kappa$ |
| 15 | $h_0^{k+3} h_4$ | $2^k \alpha_{1/5}$ | $2^k \rho_{15}$ |
| 15 | $h_1 d_0$ | $\alpha_1 \alpha_{1/4}$ | $\eta \kappa$ |
| 16 | $h_1 h_4$ | $\alpha_1 \alpha_{1/4}$ | $\eta_4$ |
| 8$k + 8$ | $p^k c_0$ | $\alpha_1 \alpha_{4k+4/6}$ | $\eta \mu_{8k+7}$ |
| 17 | $h_1^2 h_4$ | $\alpha_1 \alpha_{1/4}$ | $\eta \eta_4$ |
| 17 | $h_2 d_0$ | $\alpha_2^{2/4} \alpha_{1/4}$ | $\nu \kappa$ |
| 8$k + 9$ | $p^k h_1 c_0$ | $\alpha_1^2 \alpha_{4k+4/6}$ | $\eta^2 \mu_{8k+7}$ |
| 18 | $h_2 h_4$ | $\alpha_2^{1/4}$ | $\nu_4$ |
| 18 | $h_0 h_2 h_4$ | $2 \alpha_2^{1/4}$ | $2 \nu_4$ |
| 18 | $h_1 h_4$ | $\alpha_1^2 \alpha_{1/4}$ | $4 \nu_4$ |
| 19 | $c_1$ | $\alpha_2^{1/4}$ | $\sigma$ |
| 20 | $g$ | $\alpha_2^{20/2}$ | $\pi$ |
| 20 | $h_0 g$ | $\alpha_2^{20/4}$ | $2 \pi$ |
| 20 | $h_0^2 g$ | $2 \alpha_2^{20/4}$ | $4 \pi$ |
| 21 | $h_2 h_4$ | $\alpha_2^{2/4} \alpha_{1/4}$ | $\nu \nu_4$ |
| 21 | $h_1 g$ | $\alpha_1 \alpha_{20/2}$ | $\eta \kappa$ |
| 22 | $h_2 c_1$ | $\alpha_2^{2/4} \alpha_{1/4}$ | $\nu \sigma$ |
| 22 | $P d_0$ | $\alpha_2^{2/4} \alpha_{20/2}$ | $\eta^2 \pi$ |
| 23 | $h_4 c_0$ | $\alpha_4 \alpha_{1/4}$ | $\sigma \eta_4$ |
| 23 | $h_2 g$ | $\alpha_2^{20/2}$ | $\nu \pi$ |
| 23 | $h_0 h_2 g$ | $2 \alpha_2^{20/2}$ | $2 \nu \pi$ |
| 23 | $P h_1 d_0$ | $4 \alpha_2^{20/2}$ | $4 \nu \pi$ |
| 23 | $h_0^{k+2} h_1$ | $2^k \alpha_{12/4}$ | $2^k \rho_{23}$ |
| 24 | $h_1 h_4 c_0$ | $\alpha_1 \alpha_{4/4} \alpha_{1/4}$ | $\eta \eta_4$ |
| 26 | $h_2^2 g$ | $\alpha_2^{2/4} \alpha_{23/2}$ | $\nu \pi$ |
| 28 | $d_0^2$ | $\alpha_2^{28}$ | $\kappa^2$ |
| 30 | $h_3^3$ | $\alpha_3^{30}$ | $\theta_4$ |
| 31 | $h_1 h_4^2$ | $\alpha_1 \alpha_{30}$ | $\eta \theta_4$ |
| 31 | $n$ | $\alpha_1 \alpha_{31}$ | $\eta \theta_4$ |
Table 47: Correspondence between classical Adams and Adams-Novikov $E_\infty$

| $s$ | Adams | Adams-Novikov | detects |
|-----|-------|---------------|---------|
| 31  | $h_0^{k+10}h_5$ | $2^k\alpha_{16/6}$ | $2^k\rho_{31}$ |
| 32  | $h_1h_5$ | $z_{32,2}$ | $\eta_5$ |
| 32  | $d_1$ | $z_{32,4}$ |   |
| 32  | $q$ | $z_{32,2}$ |   |
| 33  | $h_1^2h_5$ | $\alpha_1 z_{32,2}$ | $\eta \eta_5$ |
| 33  | $p$ | $\alpha_2 z_{30}$ | $\nu \theta_4$ |
| 33  | $h_1q$ | $\alpha_1 z_{32,2}$ |   |
| 34  | $h_0h_2h_5$ | $2z_{34,2}$ |   |
| 34  | $h_2h_5$ | $\alpha_2 z_{34,2}$ | $\eta^2 \eta_5$ |
| 34  | $h_2n$ | $\alpha_2/2z_{34,1}$ |   |
| 34  | $e_0^2$ | $z_{34,6}$ | $\kappa \pi$ |
| 35  | $h_2d_1$ | $\alpha_2/2z_{32,4}$ |   |
| 35  | $h_1e_0^2$ | $\alpha_1 z_{34,6}$ | $\eta \kappa \pi$ |
| 36  | $t$ | $\alpha_1 z_{34,2}$ |   |
| 37  | $h_2^2h_5$ | $\alpha_2/2z_{34,2}$ |   |
| 37  | $x$ | $\alpha_4/4z_{30}$ | $\sigma \theta_4$ |
| 38  | $h_0^2h_3h_5$ | $z_{38}$ |   |
| 38  | $h_0^2h_3h_5$ | $2z_{38}$ |   |
| 38  | $h_2^2d_1$ | $\alpha_2/2z_{32,4}$ |   |
| 39  | $h_1h_3h_5$ | $z_{39,3}$ |   |
| 39  | $h_5e_0$ | $z_{39,3}$ |   |
| 39  | $h_3d_1$ | $\alpha_4/4z_{32,4}$ |   |
| 39  | $h_2t$ | $z_{39,7}$ |   |
| 39  | $u$ | $\alpha_1 z_{38}$ |   |
| 39  | $p^2h_0^{k+2}i$ | $2^k \alpha_{20/4}$ | $2^k \rho_{39}$ |
| 40  | $h_1^2h_3h_5$ | $\alpha_1 z_{39,3}$ |   |
| 40  | $f_1$ | $z_{40,4}$ |   |
| 40  | $h_1h_5c_0$ | $\alpha_1 z_{40,3}$ |   |
| 40  | $Ph_1h_5$ | $z_{40,2}$ |   |
| 40  | $y_2^2$ | $\alpha_1 z_{36}$ | $\kappa^2$ |
| 40  | $h_1u$ | $z_{40,8}$ | $2\pi^2$ |
| 41  | $h_1f_1$ | $\alpha_1 z_{40,4}$ |   |
| 41  | $Ph_1^2h_5$ | $\alpha_1 z_{40,2}$ |   |
| 41  | $z$ | $\alpha_1 z_{36}$ | $\eta \kappa^2$ |
| 42  | $Ph_2h_5$ | $2z_{42}$ |   |
| 42  | $Ph_0h_2h_5$ | $4z_{42}$ |   |
| 42  | $Ph_1^2h_5$ | $\alpha_1^2 z_{40,2}$ |   |
| 42  | $d_0^3$ | $\alpha_1^2 z_{36}$ | $\eta^2 \kappa^2$ |
| 44  | $g_2$ | $z_{44,4}$ |   |
| 44  | $h_0g_2$ | $2z_{44,4}$ |   |
| 44  | $h_0^2g_2$ | $4z_{44,4}$ |   |
| 45  | $h_1^2h_5$ | $z_{45}^2$ | $\theta_{4,5}$ |
| 45  | $h_0h_3^2h_5$ | $2z_{45}^2$ | $2\theta_{4,5}$ |
Table 47: Correspondence between classical Adams and Adams-Novikov $E_\infty$

| $s$ | Adams | Adams-Novikov | detects |
|-----|-------|---------------|---------|
| 45  | $h_5d_0$ | $z_{45} + 2z'_45$ |         |
| 45  | $h_1g_2$ | $\alpha_1 z_{44,4}$ |         |
| 45  | $h_0h_5d_0$ | $4z'_{45}$ | $4\eta_{4,5}$ |
| 45  | $h_0^2h_5d_0$ | $8z'_{45}$ | $8\eta_{4,5}$ |
| 45  | $w$ | $\alpha_1 z_{44,2}$ |         |
| 46  | $h_1h_5d_0$ | $\alpha_1 z_{45}$ |         |
| 46  | $B_1$ | $\alpha_1 z'_{45}$ | $\eta_{4,5}$ |
| 46  | $N$ | $\alpha_2/2 z_{43,3}$ |         |
| 46  | $d_0l$ | $\alpha_4/2 z_{44,2}$ |         |
| 47  | $h_2g_2$ | $\alpha_2/2 z_{44,4}$ |         |
| 47  | $Ph_5c_0$ | $\alpha_8/5 z'_{52,2}$ |         |
| 47  | $h_1B_1$ | $\alpha_4 z_{45}$ | $\eta^2\theta_{4,5}$ |
| 47  | $e_0r$ | $2z_{47,5}$ |         |
| 47  | $Pu$ | $2z_{47,5}$ |         |
| 47  | $h_0^{k+7}Q'$ | $2^k \alpha_{24,5}$ | $2^k \rho_{47}$ |
| 48  | $h_2h_5d_0$ | $\alpha_2/2 z_{45}$ |         |
| 48  | $B_2$ | $\alpha_2/2 z'_{45}$ | $\nu_{4,5}$ |
| 48  | $h_0B_2$ | $2\nu_{2}/2 z_{45}$ | $2\nu_{4,5}$ |
| 48  | $Ph_1h_5c_0$ | $\alpha_1/8 z_{52,2}$ |         |
| 48  | $d_0e_0^2$ | $z_{48}$ | $\kappa^2\tau$ |
| 50  | $h_5e_1$ | $z_{50}$ |         |
| 50  | $C$ | $z_{50}$ |         |
| 51  | $h_3g_2$ | $\alpha_4/4 z_{44,4}$ |         |
| 51  | $h_0h_3g_2$ | $2\alpha_4/4 z_{44,4}$ |         |
| 51  | $h_2B_2$ | $\alpha_2/2 z'_{45}$ | $\nu^2\theta_{4,5}$ |
| 51  | $gn$ | $z_{51}$ |         |
| 52  | $h_1h_3g_2$ | $\alpha_1/4 z_{44,4}$ |         |
| 52  | $d_1g$ | $z_{52,8}$ |         |
| 52  | $e_0m$ | $z_{52,6}$ |         |
| 53  | $h_2h_5c_1$ | $\alpha_2/2 z'_{50}$ |         |
| 53  | $h_2C$ | $z_{50} z_{45}^\prime$ or $z_{53}$ | $e\theta_{4,5}$ |
| 53  | $x'$ | $z_{50} z_{45}^\prime$ or $z_{53}$ | $e\theta_{4,5}$ |
| 53  | $d_0u$ | $\alpha_1 z_{52,6}$ |         |
| 54  | $h_0h_5i$ | $z_{54,2}$ |         |
| 54  | $h_1x'$ | $\alpha_1 z_{54,5}^\prime$ or $\alpha_1 z_{53,5}$ |         |
| 54  | $e_0^2g$ | $z_{54,10}$ | $\kappa^2\tau_{4,5}$ |
| 55  | $P^4h_0^{k+2}i$ | $2^k \alpha_{28,4}$ | $2^k \rho_{55}$ |
| 57  | $h_0h_2h_5i$ | $\alpha_2/2 z_{54,2}$ |         |
| 58  | $h_1Q_2$ | $\alpha_1 z_{57}$ |         |
| 59  | $B_{21}$ | $z_{59,5}$ or $z_{59,7}$ | $\kappa\theta_{4,5}$ |
| 59  | $d_0w$ | $z_{59,7}$ |         |
Table 48: Classical Adams-Novikov boundaries

| (s, f) | boundary | $\pi_{s,s}(C\tau)$ |
|--------|----------|-------------------|
| (4, 4) + k(1, 1) | $\alpha_1^{k+4}$ | $h_1^{k+4}$ |
| (10, 4) + k(1, 1) | $\alpha_1^{k+3} \alpha_4/4$ | $h_1^{k+2} c_0$ |
| (12, 4) + k(1, 1) | $\alpha_1^{k+3} \alpha_5$ | $h_1^{k+4}$ |
| (18, 4) + k(1, 1) | $\alpha_1^{k+3} \alpha_8/5$ | $h_1^{k+2} P c_0$ |
| (20, 4) + k(1, 1) | $\alpha_1^{k+3} \alpha_9$ | $P^2 h_1^{k+4}$ |
| (26, 4) + k(1, 1) | $\alpha_1^{k+3} \alpha_{12}/4$ | $h_1^{k+2} P^2 c_0$ |
| (28, 4) + k(1, 1) | $\alpha_1^{k+3} \alpha_{13}$ | $P^3 h_1^{k+4}$ |
| (34, 4) + k(1, 1) | $\alpha_1^{k+3} \alpha_{16}/6$ | $h_1^{k+2} P^3 c_0$ |
| (36, 4) + k(1, 1) | $\alpha_1^{k+3} \alpha_{17}$ | $P^4 h_1^{k+4}$ |
| (42, 4) + k(1, 1) | $\alpha_1^{k+3} \alpha_{20}/4$ | $h_1^{k+2} P^4 c_0$ |
| (44, 4) + k(1, 1) | $\alpha_1^{k+3} \alpha_{21}$ | $P^5 h_1^{k+4}$ |
| (50, 4) + k(1, 1) | $\alpha_1^{k+3} \alpha_{24}/5$ | $h_1^{k+2} P^5 c_0$ |
| (54, 4) + k(1, 1) | $\alpha_1^{k+3} \alpha_{25}$ | $P^6 h_1^{k+4}$ |
| (58, 4) + k(1, 1) | $\alpha_1^{k+3} \alpha_{28}/4$ | $h_1^{k+2} P^6 c_0$ |

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Table 48: Classical Adams-Novikov boundaries

| $(s, f)$ | boundary | $\pi_{*,*}(C\tau)$ |
|----------|----------|------------------|
| $(55, 9)$ | $\alpha_2^2 \tau_{52,8}$ | $h_2 d_1 g$ |
| $(55, 9)$ | $\alpha_1^2 253$ | $h_1^2 i_1$ |
| $(55, 11)$ | $\alpha_1 54,10$ | $h_1^2 h_5 e_0$ |
| $(56, 8)$ | $\alpha_1^2 54,6$ | $gt$ |
| $(56, 10)$ | $\alpha_1^2 533$ | $h_1^2 i_1$ |
| $(57, 5)$ | $\alpha_1 56,4$ | $D_{11}$ |
| $(57, 11)$ | $\alpha_1^2 533$ | $h_1^2 i_1$ |
| $(59, 7)$ | $? z_{59, 7}$ | $j_1$ |
| $(59, 9)$ | $\alpha_4 4^2 \tau_{52,8}$ | $h_3 d_1 g$ |
| $(59, 11)$ | $z_{59, 11}$ | $c_1 g^2$ |

Table 49: Classical Adams-Novikov non-permanent classes

| $(s, f)$ | class | $\pi_{*,*}(C\tau)$ |
|----------|-------|------------------|
| $(5, 1) + k(1, 1)$ | $\alpha_k^k \alpha_3$ | $h_1^k \cdot h_1^k$ |
| $(11, 1) + k(1, 1)$ | $\alpha_k^k \alpha_6, 3$ | $h_1^k \cdot h_2^2 c_0$ |
| $(13, 1) + k(1, 1)$ | $\alpha_k^k \alpha_7$ | $h_1^k \cdot P h_1^k i_1 k$ |
| $(19, 1) + k(1, 1)$ | $\alpha_k^k \alpha_{10, 3}$ | $h_1^k \cdot P h_1^k i_1 c_0$ |
| $(21, 1) + k(1, 1)$ | $\alpha_k^k \alpha_{11}$ | $h_1^k \cdot P^2 h_1^k i_1$ |
| $(27, 1) + k(1, 1)$ | $\alpha_k^k \alpha_{14, 3}$ | $h_1^k \cdot P^2 h_1^k i_1 c_0$ |
| $(29, 1) + k(1, 1)$ | $\alpha_k^k \alpha_{15}$ | $h_1^k \cdot P^3 h_1^k i_1$ |
| $(35, 1) + k(1, 1)$ | $\alpha_k^k \alpha_{18, 3}$ | $h_1^k \cdot P^3 h_1^k i_1 c_0$ |
| $(37, 1) + k(1, 1)$ | $\alpha_k^k \alpha_{19}$ | $h_1^k \cdot P^4 h_1^k i_1$ |
| $(43, 1) + k(1, 1)$ | $\alpha_k^k \alpha_{22, 3}$ | $h_1^k \cdot P^4 h_1^k i_1 c_0$ |
| $(45, 1) + k(1, 1)$ | $\alpha_k^k \alpha_{23}$ | $h_1^k \cdot P^5 h_1^k i_1$ |
| $(51, 1) + k(1, 1)$ | $\alpha_k^k \alpha_{26, 3}$ | $h_1^k \cdot P^5 h_1^k i_1 c_0$ |
| $(53, 1) + k(1, 1)$ | $\alpha_k^k \alpha_{27}$ | $h_1^k \cdot P^6 h_1^k i_1$ |
| $(59, 1) + k(1, 1)$ | $\alpha_k^k \alpha_{30, 3}$ | $h_1^k \cdot P^6 h_1^k i_1 c_0$ |
| $(26, 2)$ | $z_{26}$ | $h_1^2 h_4 c_0$ |
| $(30, 2)$ | $z_{30}$ | $r$ |
| $(34, 2)$ | $z_{34, 2}$ | $h_2 h_5$ |
| $(35, 3)$ | $\alpha_1 z_{34, 2}$ | $h_2^2 g$ |
| $(36, 2)$ | $z_{36}$ | $h_1^2 h_5$ |
| $(37, 3)$ | $\alpha_1 z_{36}$ | $h_1 \cdot h_1^2 h_5$ |
| $(38, 2)$ | $z_{38}$ | $h_0 y$ |
Table 49: Classical Adams-Novikov non-permanent classes

| $(s, f)$ | class | $\pi_{s, \ast}(C_T)$ |
|----------|-------|---------------------|
| (38, 4)  | $\alpha_2^2 z_{36}$ | $h_1^2 \cdot h_1^3 h_5$ |
| (39, 5)  | $\alpha_2^2 z_{36}$ | $h_1^3 \cdot h_1^2 h_5$ |
| (41, 3)  | $z_{41}$ | $h_0 c_2$ |
| (41, 3)  | $\alpha_2/2 z_{38}$ | $\tau h_2 g^2$ |
| (42, 2)  | $z_{42}$ | $h_1^2 h_5 c_0$ |
| (42, 4)  | $\alpha_1 z_{41}$ | $h_3 \cdot h_3^2 g$ |
| (43, 3)  | $\alpha_1 z_{42}$ | $h_1 \cdot h_1^2 h_5 c_0$ |
| (44, 2)  | $z_{44}$ | $\tau h_2 c_1 g$ |
| (44, 4)  | $\alpha_1 z_{42}$ | $h_1^2 \cdot h_1^2 h_5 c_0$ |
| (44, 4)  | $z_{44}^*$ | $\tau h_2 h_2 g^2$ |
| (44, 4)  | $2 z_{44}^*$ | $d_0 r$ |
| (45, 5)  | $\alpha_1 z_{42}$ | $h_1^3 \cdot h_1^2 h_5 c_0$ |
| (46, 6)  | $\alpha_1 z_{42}$ | $h_1^4 \cdot h_1^2 h_5 c_0$ |
| (47, 3)  | $z_{47}$ | $h_1^2 g_2$ |
| (47, 5)  | $z_{47}$ | $\tau h_2 g^2$ |
| (50, 2)  | $z_{50}$ | $\tau h_2 d_1 g$ |
| (50, 6)  | $\alpha_2/2 z_{47}$ | $\tau h_2 h_5 c_0$ |
| (54, 6)  | $z_{54}$ | $\tau h_2 d_1 g$ |
| (55, 5)  | $z_{55}$ | $B_6$ |
| (55, 7)  | $\alpha_1 z_{54}$ | $h_1^2 d_1 g$ |
| (55, 5)  | $z_{55}$ | $h_1^2 d_1 g + \tau h_1 G$ |
| (56, 2)  | $z_{56}$ | $Q_1$ |
| (56, 4)  | $z_{56}$ | $\tau h_2 d_1 g$ |
| (56, 6)  | $z_{56}$ | $h_1^2 i_1 + h_5 c_0 e_0$ |
| (57, 3)  | $z_{57}$ | $Q_2$ |
| (57, 7)  | $\alpha_1 z_{56}$ | $h_1 \cdot h_1^2 i_1 + h_1 h_5 c_0 e_0$ |
| (58, 2)  | $z_{58}$ | $h_0 D_2$ |
| (58, 6)  | $z_{58}$ | $\tau h_2 h_5 c_0$ |
| (58, 8)  | $\alpha_1^2 z_{56}$ | $h_1^2 \cdot h_1^2 i_1 + h_1^2 h_5 c_0 e_0$ |
| (59, 3)  | $z_{59}$ | $h_3^2 g_2$ |
| (59, 3)  | $\alpha_1 z_{58}$ | $h_3 G_3$ |
| (59, 7)  | $z_{59}$ | $h_1^2 D_4$ |
| (59, 9)  | $\alpha_1^2 z_{56}$ | $h_1^{3} \cdot h_1^2 i_1 + h_1^{3} h_5 c_0 e_0$ |
| (60, 2)  | $z_{60}$ | $h_1^2 Q_2$ |
| (60, 4)  | $z_{60}$ | $J_1$ |
| (60, 4)  | $\alpha_2/2 z_{57}$ | $h_1 \cdot h_3 G_3$ |
| (60, 6)  | $z_{60}$ | $h_3 d_1 g$ |
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