1. Introduction

In this paper we prove a decomposition formula for generalized theta functions which is motivated by what in conformal field theory is called the factorization rule.

A rational conformal field theory associates a finite dimensional vector space (called the space of conformal blocks) to a pointed nodal projective algebraic curve over \( \mathbb{C} \) whose marked points are labeled by elements of a certain finite set. If we choose a singular point \( p \) in such a labeled pointed curve \( X \), then we get a new pointed curve \( \tilde{X} \) by taking the partial normalization at \( p \) and marking all points which lie either over one of the marked points of \( X \) (old points) or over the singularity \( p \) (two new points). The factorization rule gives a canonical direct sum decomposition of the space of conformal blocks associated to \( X \) with its labeled marked points, such that the summands appearing in that decomposition are spaces of conformal blocks associated to the pointed curve \( \tilde{X} \) whose old marked points are labeled by the same elements as the corresponding points of \( X \). The direct sum runs over a certain finite set of labeling of the two new points.

In the case of a Wess-Zumino-Witten conformal field theory associated to a simply connected semi-simple algebraic group \( G \) and a natural number \( \kappa \geq 1 \) Tsuchiya, Ueno and Yamada have given a mathematical definition of the spaces of conformal blocks in terms of the representation theory of the affine Lie algebra associated to \( G \) and they have shown that these spaces satisfy the factorization rule ([TUY], [U], cf. also [So] for an overview). It has
been conjectured by physicists and later proved by various mathematicians that the spaces of conformal blocks of Tsuchiya, Ueno, Yamada have an algebro-geometric interpretation: They can be identified with spaces of global sections (called generalized theta functions) of certain line bundles on the moduli space (or moduli stack) of $G$-bundles with parabolic structures at the marked points ([F1], [BL], [LS], [T]). It should be noted that in the case of singular curves the moduli spaces of $G$-bundles is non-compact. Nevertheless, for semi-simple $G$ the space of global sections of a line bundle on these moduli spaces is still finite-dimensional. This has been pointed out to me by the referee and follows e.g. from [T], Theorem 3.

Generalized theta functions can also be defined on moduli of principal $GL_n$-bundles (or equivalently of vector bundles) on a smooth curve. For singular curves however the theta line bundle over the non-compact moduli space of vector bundles carries too many sections, so one has to compactify it, to get a reasonable notion of generalized theta functions. There exist at least two approaches to compactify the moduli space of vector bundles (of given rank and degree, say) on a singular curve. One construction uses torsion free sheaves ([Sc1], [N], [F2]), the other one works with certain vector bundles on modifications on the singular curve and has been introduced by Gieseker [G] in the rank two case and has been generalized to arbitrary rank by Nagaraj and Seshadri [NS] and myself [K2]. The torsion free sheaves approach works for arbitrary singularities; Gieseker’s approach has up to now been carried out only for the case where the curve is irreducible with only one ordinary double point.

A version of the factorization rule for generalized theta functions has been formulated and proved by Narasimhan, Ramadas and Sun in the framework of moduli varieties of semi-stable torsion free sheaves of fixed rank and degree ([NR], [R], [Sim1], [Sim2]). They also prove (at least for rank=2 or genus $\geq 4$) that in case of a one-dimensional family of curves which is generically smooth and degenerates at one point to an irreducible nodal curve with one singularity, the spaces of generalized theta functions form the fibers of a finite rank vector bundle on the one-dimensional base.

In the present paper we prove a factorization rule for generalized theta functions on the moduli stack of Gieseker vector bundles on an irreducible curve with one node (a stack version of Gieseker’s approach which I have constructed in [K2]). Our result is somewhat stronger than the analogous result of Narasimhan, Ramadas and Sun, since we obtain a canonical decomposition, whereas the decomposition proved by those authors is non-canonical.

In the last chapter we show that generalized theta functions on the moduli stack of Gieseker vector bundles behave well in families. However, since we are dealing with Artin-stacks which are neither separated nor of finite type over the base, we can not argue by cohomological flatness, but have to make an explicit dimension calculation using the Verlinde formula for $SL_n$. So our result cannot be regarded as an alternative to the representation theoretic approach to the Verlinde factorization. Rather it shows that the Gieseker type “compactification” of the moduli stack of vector bundles on a singular curve leads to the “correct selection rules” for the sections of the determinant line bundle.

Here is the main result of the paper (cf. Theorem 7.5):

**Theorem:** Let $k$ be an algebraically closed field of characteristic zero. Let $C_0$ be an irreducible projective algebraic curve over $k$ with one ordinary double point, let $\tilde{C}_0$ be its normalization and let $p_1, p_2$ be the two points of $\tilde{C}_0$ which are mapped onto the singular point of $C_0$. Let $\Theta$ be the theta line bundle on $GVB$, the moduli stack of rank $n$ Gieseker vector bundles
on $C_0$ and let $\kappa$ be a positive integer. Then there is a canonical isomorphism of $k$-vector spaces

$$H^0(GVB, \Theta^\kappa) \sim \bigoplus_{(a,b) \in A'} H^0(PB, \Theta^\kappa_{PB}(a,b)).$$

Here $PB$ is the moduli stack parametrizing vector bundles on $\tilde{C}_0$ together with full flags in the fibers at the points $p_1$ and $p_2$ and $A'$ is a finite set (depending on $\kappa$) which parametrizes a set of line bundles $\Theta^\kappa_{PB}(a,b)$ on $PB$.

The main ingredient of the proof of Theorem 7.5 is a result from my earlier paper [K2], which says that $GVB$ has normal crossing singularities and that there is a diagram of algebraic $k$-stacks:

$$\begin{array}{ccc}
VB & \overset{f}{\longrightarrow} & GVBD \\
\downarrow & & \downarrow \\
\nu & \longrightarrow & GVB
\end{array}$$

where $VB$ is the moduli stack of rank $n$ vector bundles on $\tilde{C}_0$, the morphism $f$ is a locally trivial fibration whose standard fiber is a certain canonical compactification $KGL_n$ of the general linear group $GL_n$ and the morphism $\nu$ identifies the stack $GVBD$ with the normalization of $GVB$.

In view of that diagram the strategy of the proof of the theorem is quite straightforward: We identify the space $H^0(GVB, \Theta^\kappa)$ with the subspace of $H^0(GVBD, \nu^*\Theta^\kappa)$ consisting of sections of $\nu^*\Theta$ whose values coincide at points which map onto the same point of $GVB$. We show that the line bundle $\nu^*\Theta$ is naturally isomorphic to $f^*\tilde{\Theta} \otimes \Delta$ where $\tilde{\Theta}$ is the theta line bundle on $VB$ and $\Delta$ is a line bundle whose restriction to a fiber of $f$ is a fixed line bundle on the compactification $KGL_n$. We then apply the result from [K3] where we have decomposed the cohomology of line bundles on $KGL_n$ in terms of irreducible representations of $GL_n \times GL_n$. This yields a canonical decomposition

$$H^0(GVBD, \nu^*\Theta^\kappa) \sim \bigoplus_{(a,b) \in A(\Delta^k)} H^0(PB, \Theta^\kappa_{PB}(a,b)).$$

Finally we determine how the subspace $H^0(GVB, \Theta^\kappa)$ of $H^0(GVBD, \nu^*\Theta^\kappa)$ behaves with respect to this decomposition. It turns out that the composite morphism

$$H^0(GVB, \Theta^\kappa) \hookrightarrow H^0(GVBD, \nu^*\Theta^\kappa) \sim \bigoplus_{(a,b) \in A(\Delta^k)} H^0(PB, \Theta^\kappa_{PB}(a,b)) \longrightarrow \bigoplus_{(a,b) \in A'} H^0(PB, \Theta^\kappa_{PB}(a,b))$$

is an isomorphism. The last arrow in this diagram is simply the projection induced by the inclusion of the finite sets $A' \subset A(\Delta^k)$.

The greater part of this work has been carried out during a stay at the Tata Institute of Fundamental Research in Bombay. Its hospitality is gratefully acknowledged. I am deeply indebted to Don Zagier, who provided me with an ingenious proof of Lemma 8.5.

2. COMPLEMENTS ON MODIFICATIONS OF POINTED NODAL CURVES

Let $S$ be an arbitrary scheme or algebraic stack. In this section we present some constructions which yield two-pointed nodal curves over $S$.

**Definition 2.1.** (1) A **nodal curve** over $S$ is a morphism $\pi : \mathcal{C} \to S$ which is projective, finitely presented and flat and whose geometric fibers are reduced curves with only
IVAN KAUSZ

ordinary double points as singularities. We require furthermore that for each point
\( z \in S \) we have \( H^0(C_z, \mathcal{O}_{C_z}) = \kappa(z) \), where \( C_z \) denotes the fiber of \( \pi \) at \( z \) and \( \kappa(z) \) is
the residue field of the point \( z \).

(2) A one-pointed nodal curve over \( S \) is a tuple \((C, \pi, s)\), where \( \pi : C \to S \) is a nodal
curve and where \( s \) is a section of \( \pi \), whose image is contained in the smooth locus of
\( \pi \) such that for each point \( z \in S \) we have \( H^0(C_z, \mathcal{O}_{C_z}(-s(z))) = (0) \). The morphism
\( \pi \) will often be omitted from the notation.

(3) A two-pointed nodal curve over \( S \) is a tuple \((s_1, C, \pi, s_2)\), where \( \pi : C \to S \) is a nodal
curve and where \( s_1 \) and \( s_2 \) are disjoint sections of \( \pi \) such that \((C, \pi, s_1)\) and \((C, \pi, s_2)\)
are one-pointed nodal curves. We will often write \((s_1, C, s_2)\) instead of \((s_1, C, \pi, s_2)\).

(4) A morphism \((s_1, C, \pi, s_2) \to (s'_1, C', \pi', s'_2)\) of two-pointed nodal curves over \( S \) is an
\( S \)-morphism \( f : C \to C' \) with \( s'_i = f \circ s_i \) for \( i = 1, 2 \).

**Definition 2.2.** Let \((s_1, C, s_2)\) and \((t_1, D, t_2)\) be two-pointed nodal curves over \( S \). Then we
define the two-pointed nodal curve

\[
(s_1, C, s_2) \perp (t_1, D, t_2) := (r_1, B, r_2) ,
\]

where \( B \) is the curve \((C \cup D)/(s_2 = t_1)\) and the sections \( r_1, r_2 \) are defined by \( r_1 : S \to C \setminus \{s_2(S)\} \hookrightarrow B \) and \( r_2 : S \twoheadrightarrow D \setminus \{t_1(S)\} \hookrightarrow B \) respectively.

**Definition 2.3.** Let \( L_1 \) and \( L_2 \) be two line bundles on \( S \). We denote by \(|L_1, L_2|\) the two-
pointed nodal curve \((s_1, C, \pi, s_2)\), where \( C := \mathbb{P}(L_1 \oplus L_2) \) and \( s_i : S \to C \) is defined by the
invertible quotient \( L_1 \oplus L_2 \to L_i \) for \( i = 1, 2 \).

**Lemma 2.4.** Let \( L \) and \( M \) be two line bundles on \( S \) and let \((s, C, \pi, t) := |L, M|\). Let
\( \pi^*(L \oplus M) \to \mathcal{O}_C(1) \) be the tautological invertible quotient on \( C \). Then there are canonical
isomorphisms

(1) \( s^*\mathcal{O}_C(s) = L \otimes M^{-1} \) and \( t^*\mathcal{O}_C(t) = L^{-1} \otimes M \),
(2) \( \mathcal{O}_C(1) = \mathcal{O}_C(s) \otimes \pi^*M = \mathcal{O}_C(t) \otimes \pi^*L \).

**Proof.** This follows easily from the universal property of \( \mathbb{P}(L \oplus M) \). \( \square \)

For the convenience of the reader I will now recall a definition and a result from [K2] §5
which are needed in the following.

**Definition 2.5.** Let \( S \) be a scheme and \((C, \pi, s)\) a one-pointed nodal curve. A simple
modification \((C', f, \pi', s')\) of \((C, \pi, s)\) is a diagram

\[
\begin{array}{ccc}
C' & \xrightarrow{f} & C \\
\downarrow{\pi'} & & \downarrow{\pi} \\
S & \xrightarrow{s'} & S \\
\end{array}
\]

with the following properties:

(1) The triple \((C', \pi', s')\) is again a one-pointed nodal curve over \( S \).
(2) The diagram is commutative in the sense that \( \pi \circ f = \pi' \) and \( f \circ s' = s \).
(3) The morphism \( f \) is proper and finitely presented.
(4) Let $z \in S$ be a point. Then there are two possibilities for the induced morphism $f_z : C'_z \to C_z$ of fibers over $z$: Either $f_z$ is an isomorphism, or $C'_z$ arises from $R \cong \mathbb{P}^1_{\kappa(z)}$ and $C_z$ by the identification of a point in $R(\kappa(z))$ with $s(z) \in C_z$, and $f_z$ contracts $R$ to the point $s(z)$.

**Remark 2.6.** We will most often use a shorter expression by saying that “$(C', s')$ is a simple modification of $(C, s)$”, the data $\pi', \pi$ and $f$ being understood. The definition implies, that $f$ induces an isomorphism $C' \setminus f^{-1}(s(S)) \sim C \setminus s(S)$ In particular, $C \setminus s(S)$ can be considered as an open subscheme of $C'$.

**Proposition 2.7.** Let $S$ be a scheme, and let $(C, \pi, s)$ be a one-pointed nodal curve over $S$. Then there is a canonical isomorphism of groupoids:

$$
\begin{array}{c}
\{ \text{Simple modifications of } (C, \pi, s) \text{ in the sense of definition 2.5} \} \\
\sim \\
\{ \text{Pairs } (M, \mu), \text{ where } M \text{ is an invertible } \mathcal{O}_S\text{-module and } \\
\mu : \mathcal{O}_S \to M \text{ is a global section of } M \}
\end{array}
$$

**Sketch of proof 2.8.** For a proof I refer the reader to [K2] §5. Here we only need the following details of the correspondence $(C', \pi', f', s') \leftrightarrow (M, \mu)$:

1. Let $(C', f, \pi', s')$ be a simple modification of $(C, \pi, s)$. Then $(M, \mu)$ is constructed as follows. First of all we have $M = (s')^* \mathcal{O}_{C'}(-s') \otimes s^* \mathcal{O}_C(s)$. For the section $\mu$ consider the following exact diagram

$$
\begin{array}{ccc}
0 & \to & \mathcal{O}_{C'}(-s') & \to & \mathcal{O}_{C'} & \to & s'_* s^* \mathcal{O}_C & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & 0 \\
0 & \to & M & \to & f^* \mathcal{O}_C(s) & \to & s'_* s^* \mathcal{O}_C(s) & \to & 0
\end{array}
$$

where $M := \mathcal{O}_{C'}(-s') \otimes f^* \mathcal{O}_C(s)$ and where the vertical arrows are induced by the morphism $\mathcal{O}_{C'} \to f^* \mathcal{O}_C(s)$ obtained by applying the functor $f^*$ to the natural injection $\mathcal{O}_C \hookrightarrow \mathcal{O}_C(s)$. Since the right vertical arrow obviously vanishes, the middle vertical arrow factorizes as indicated by the dotted arrow. Applying $(s')^*$ th the morphism $\mathcal{O}_{C'} \to M$ thus obtained, yields the section $\mu : \mathcal{O}_S \to M$.

2. Let $(M, \mu)$ be an invertible $\mathcal{O}_S$-module, together with a section. The associated nodal curve $C'$ is canonically isomorphic to $\mathbb{P}(\mathcal{J})$, where $\mathcal{J}$ is the $\mathcal{O}_C$-module defined by the following exact sequence

$$
\begin{array}{ccc}
0 & \to & \pi^* M^{-1} (i_* - \mu) (\mathcal{O}_C(s) \otimes \pi^* M^{-1}) \oplus \mathcal{O}_C & \to & \mathcal{J} & \to & 0
\end{array}
$$

where $i$ is the morphism induced by the inclusion $\mathcal{O}_C \hookrightarrow \mathcal{O}_C(s)$ and $-\mu$ is the negative of the morphism induced by $\mu : \mathcal{O}_S \to M$.

**Definition 2.9.** Let $(s_1, C, s_2)$ be a two-pointed nodal curve over $S$ and let $M$ be a line bundle on $S$ and $\mu$ a global section of $M$. Then we denote by

$$(M, \mu) \vdash (s_1, C, s_2)$$

the two-pointed nodal curve $(r_1, B, r_2)$, where $(B, r_1)$ is the simple modification of $(C, s_1)$ associated to the data $(M, \mu)$ by 2.7 and $r_2$ is defined as the composition $r_2 : S \xrightarrow{s_2} C \setminus \text{correspondence}$.
$s_1(S) \hookrightarrow B$. Similarly, we write

$$(s_1, C, s_2) \vdash (M, \mu)$$

for the two-pointed nodal curve $(t_1, D, t_2)$, where $(D, t_2)$ is the simple modification of $(C, s_2)$ associated to the data $(M, \mu)$ and $t_1$ is defined by the composition $t_1 : S \xrightarrow{s_1} C \setminus s_2(S) \hookrightarrow D$. In situations where no doubts as to $\mu$ are likely to arise, we will sometimes write $M \vdash (s_1, C, s_2)$ and $(s_1, C, s_2) \vdash M$ instead of $(M, \mu) \vdash (s_1, C, s_2)$ and $(s_1, C, s_2) \vdash (M, \mu)$.

**Lemma 2.10.** Let $M$ be a line bundle on $S$ with zero section $0$ and let $(s_1, C, s_2)$ be a two-pointed nodal curve. Then we have canonical isomorphisms of two-pointed nodal curves as follows:

$$|O_S, M \otimes s_1^*O_C(-s_1)| \perp (s_1, C, s_2) = (M, 0) \vdash (s_1, C, s_2)$$

$$(s_1, C, s_2) \perp |M \otimes s_2^*O_C(-s_2), O_S| = (s_1, C, s_2) \vdash (M, 0)$$

**Proof.** Let $M' := M \otimes s_1^*O_C(-s_1)$ and let $(r_1, B, r_2) := |O_S, M'| \perp (s_1, C, s_2)$. Let $f : B \rightarrow C$ be the morphism, whose restriction to $\mathbb{P}(M' \oplus O_S)$ is the structure morphism to $S$ composed with the section $s_1$ and whose restriction to $C$ is the identity morphism. Clearly $(f : B \rightarrow C, r_1)$ is a simple modification of $(C, s_1)$. Furthermore, by 2.4 the $O_S$-module $r_1^*O_B(-r_1) \otimes s_1^*O_C(s_1)$ is canonically isomorphic to $M$ and the canonical morphism $O_S \rightarrow r_1^*O_B(-r_1) \otimes s_1^*O_C(s_1)$ from construction 2.8 (1) vanishes. The canonical isomorphism $|O_S, M'| \perp (s_1, C, s_2) = (M, 0) \vdash (s_1, C, s_2)$ follows now from 2.7. The other isomorphism follows completely analogously. 

**Lemma 2.11.** Let $L, M$ be two line bundles on $S$ and let $\mu$ be a global section of $M$. Then there is a canonical isomorphism of two-pointed nodal curves as follows:

$$|L, O_S| \vdash (M, \mu) = (M, \mu) \vdash |O_S, (L \otimes M)^{-1}|$$

**Proof.** Let

$$(s_1, C_1, \pi_1, t_1) := |L, O_S|$$

$$(s, C, \pi, t) := |L, O_S| \vdash (M, \mu)$$

$$(s_2, C_2, \pi_2, t_2) := |O_S, (L \otimes M)^{-1}|$$

and let $f_1 : C \rightarrow C_1 = \mathbb{P}(L \oplus O_S)$ be the canonical morphism. By 2.8 (2) we have $C = \mathbb{P}(\mathcal{J})$, where $\mathcal{J}$ is the coherent $O_{C_1}$-module defined by the exact sequence

$$0 \longrightarrow \pi_1^*M^{-1} \xrightarrow{(i, -\mu)} (O_{C_1}(t_1) \otimes \pi_1^*M^{-1}) \oplus O_{C_1} \xrightarrow{p} \mathcal{J} \longrightarrow 0$$

Now consider the composite morphism

$$\pi^*(L \otimes M)^{-1} \oplus O_C \xrightarrow{j} (\pi_1^*O_C(t_1) \otimes \pi_1^*M^{-1}) \oplus O_C \xrightarrow{f_1^p} f_1^*\mathcal{J} \xrightarrow{q} O_{C/C_1}(1)$$

where $j : \pi_1^*(L \otimes M)^{-1} \oplus O_{C_1} \rightarrow (O_{C_1}(t_1) \otimes \pi_1^*M^{-1}) \oplus O_{C_1}$ is induced by the injection $\pi_1^*L^{-1} = \pi_1^*t_1^*O_C(t_1) = O_{C_1}(t_1 - s_1) \rightarrow O_{C_1}(t_1)$ and $q$ is the tautological invertible quotient on $C = \mathbb{P}(\mathcal{J})$. It is easy to check that the morphism $p \circ j$ is surjective. Hence the morphism $u := q \circ f_1^*p \circ f_1^*j$ is surjective and defines a morphism

$$f_2 : C \longrightarrow \mathbb{P}(O_S \oplus (L \otimes M)^{-1}) = C_2$$
It is not hard to see that the pull-back by $s$ and $t$ of the epimorphism $u$ identifies with the epimorphism $\mathcal{O}_S \oplus (L \otimes M)^{-1} \to \mathcal{O}_S$ (projection to the first component) and $\mathcal{O}_S \oplus (L \otimes M)^{-1} \to \mathcal{(L \otimes M)^{-1}}$ (projection to the second component) respectively. Therefore we have $s_2 = f_2 \circ s$ and $t_2 = f_2 \circ t$.

By considering the case where $S$ is the spectrum of a field one verifies that $(\mathcal{C}, s)$ is a simple modification of $(\mathcal{C}_2, s_2)$.

Clearly we have $s^* \mathcal{O}_C(-s) \otimes s_2^* \mathcal{O}_{C_2}(s_2) = M$ and by going through the constructions it follows that the canonical morphism $\mathcal{O}_S \to s^* \mathcal{O}_C(-s) \otimes s_2^* \mathcal{O}_{C_2}(s_2)$ from 2.8 (1) identifies with $\mu$.

The lemma now follows from 2.7.

\[ \text{Proposition 2.12.} \quad \text{Let } M_0, \ldots, M_q \text{ be invertible } \mathcal{O}_S\text{-modules and for } i \in [1, q] \text{ let } \mu_i \text{ be a global section of } M_i. \text{ Then the two-pointed nodal curve} \]

\[ |M_0, \mathcal{O}_S| \sqsubset (M_1, \mu_1) \sqsubset (M_2, \mu_2) \sqsubset \cdots \sqsubset (M_q, \mu_q) \]

\[ \text{is canonically isomorphic to the two-pointed nodal curve} \]

\[ (M_1, \mu_1) \sqsubset (M_2, \mu_2) \sqsubset \cdots \sqsubset (M_q, \mu_q) \sqsubset |\mathcal{O}_S, \bigotimes_{i=0}^{q} M_i^{-1}|. \]

\[ \text{Proof.} \quad \text{This follows by a } q\text{-fold application of lemma 2.11.} \]

The next proposition is needed in the proof of proposition 6.1.

\[ \text{Proposition 2.13.} \quad \text{Let } (s_1, \mathcal{C}, s_2) \text{ be a two-pointed nodal curve such that } \mathcal{C} \to S \text{ is smooth} \]

\[ \text{and let } (L_i, \lambda_i)_{i=1,\ldots,q} \text{ and } (M_i, \mu_i)_{i=1,\ldots,r} \text{ be two families of invertible } \mathcal{O}_S\text{-modules with sections. Consider the two-pointed nodal curve} \]

\[ (t_1, \mathcal{B}, t_2) := (M_r, \mu_r) \sqsubset \cdots \sqsubset (M_1, \mu_1) \sqsubset (s_1, \mathcal{C}, s_2) \sqsubset (L_1, \lambda_1) \sqsubset \cdots \sqsubset (L_q, \lambda_q) \]

\[ \text{and let } \Sigma \hookrightarrow \mathcal{B} \text{ be the singular locus of the morphism } \mathcal{B} \to S. \text{ Then we can express } \Sigma \text{ as a disjoint union of closed subschemes of } S \text{ as follows:} \]

\[ \Sigma = \left( \bigcup_{i=1}^{q} \{ \lambda_i = 0 \} \right) \sqcup \left( \bigcup_{i=1}^{r} \{ \mu_i = 0 \} \right). \]

\[ \text{Proof.} \quad \text{The proposition follows from repeated application of the following assertion:} \]

\[ \text{Let } (\mathcal{C}', \pi', f, s') \text{ be a simple modification of the one-pointed nodal curve } (\mathcal{C}, \pi, s) \text{ over } S \]

\[ \text{and let } (M, \mu) \text{ be the corresponding line bundle with section. Let } \Sigma \subset \mathcal{C} \text{ and } \Sigma' \subset \mathcal{C}' \text{ be the locus of non-smoothness of } \pi \text{ and } \pi' \text{ respectively. Then we have} \]

\[ \Sigma' = \Sigma \sqcup Y, \]

\[ \text{where } Y \subset S \text{ is the closed subscheme of } S \text{ defined by the equation } \mu = 0. \]

Since $\Sigma$ and the section $s(S)$ are disjoint closed subschemes of $\mathcal{C}$, and $\mathcal{C}' \setminus f^{-1}(s(S))$ is isomorphic to $\mathcal{C} \setminus s(S)$, we can identify $\Sigma$ with the locus of non-smoothness of $\mathcal{C}' \setminus f^{-1}(s(S))$ $\to$ $S$ and it suffices to show that the locus of non-smoothness of $\mathcal{C}' \setminus \Sigma$ $\to$ $S$ is isomorphic to $Y$. For this we may replace $\mathcal{C}$ by an open affine neighborhood $V = \text{Spec} (\mathcal{B}) \subset \mathcal{C} \setminus \Sigma$ of the section $s(S)$ and $S$ by $U = s^{-1}(V)$. Then $U$ is also affine, say $U = \text{Spec} (A)$ and we
may also assume that $M|_V$ is trivial and that $\mu$ is given by $a \in A$. By $[K2]$, 5.5 we have $f^{-1}(V) = \text{Proj } (R)$ for the graded $B$-algebra

$$R := B[X,Y]/(bX - aY),$$

where $b \in B$ is the regular element generating the ideal associated to the closed subscheme $s(U) \subset V$. Now $W := \text{Proj } (R)$ is the union of the open affine pieces $W_1 = \text{Spec } (R_1)$ and $W_2 = \text{Spec } (R_2)$, where

$$R_1 \cong B[T]/(b - aT) \quad \text{and} \quad R_2 \cong B[T]/(bT - a).$$

A simple calculation shows that the $R_1$-module $\Omega^1_{R_1/A}$ is trivial of rank one and that the first Fitting ideal of the $R_2$-module $\Omega^1_{R_2/A}$ is $(b, T)$. Therefore $\Sigma' \cap W \cong \text{Spec } (R_2/(b, T)) = \text{Spec } (A/a) = Y \cap U$. □

3. Review of a compactification of $GL_n$

Let $k$ be a field and $E$, $F$ two $k$-vector spaces of rank $n$. In $[K1]$ I have studied a certain compactification $KGL(E, F)$ of the scheme $\text{Isom}(E, F)$ of isomorphisms from $E$ to $F$ which has properties similar to the so called wonderful compactification of adjoint linear groups introduced by De Concini and Procesi. In particular, the complement of $\text{Isom}(E, F)$ in $KGL(E, F)$ is a divisor with normal crossings whose irreducible components are smooth. As in $[K1]$, I denote these components by $Y_0, \ldots, Y_{n-1}, Z_0, \ldots, Z_{n-1}$.

More generally, the construction of $KGL(E, F)$ works also in a relative situation (cf. $[K1]$ §9). Thus let $S$ be a scheme and let $E$ and $F$ be two locally free $O_S$-modules of rank $n$. Then there is a natural $S$-scheme $KGL(E, F)$ containing the scheme $\text{Isom}(E, F)$ as an open subscheme such that for each point $z \in S$ the fiber of $KGL(E, F) \to S$ over $z$ is the compactification $KGL(E_z, F_z)$ of $\text{Isom}(E_z, F_z)$. We denote by $Y_0, \ldots, Y_{n-1}, Z_0, \ldots, Z_{n-1}$ the divisors in $KGL(E, F)$ which are fiber-wise the components of the boundary of $\text{Isom}(E_z, F_z)$ in $KGL(E_z, F_z)$. The construction may be extended to the case where $S$ is an algebraic stack.

The main theorem in $[K1]$ is a concrete description of the $T$-valued points of $KGL(E, F)$ (for any $S$-scheme $T$). As we need this result in the sequel, I will recall the necessary definitions.

Let $T$ be an $S$-scheme and let $\mathcal{E}$, $\mathcal{F}$ be two locally free $O_T$-modules of rank $n$. A $bf$-morphism from $\mathcal{E}$ to $\mathcal{F}$ is a tuple $g = (L, \lambda, \mathcal{E} \to \mathcal{F}, \mathcal{F} \to M \otimes \mathcal{E}, r)$ where $L$ is an invertible $O_T$-module, $\lambda$ is a section of $L$, the arrows $\mathcal{E} \to \mathcal{F}$ and $\mathcal{F} \to L \otimes \mathcal{E}$ are $O_T$-module morphisms and $r$ is an integer between 0 and $n$ such that locally on $T$ there exist isomorphisms

$$\mathcal{E} \xrightarrow{\sim} rO_T \oplus (n-r)O_T$$
$$\mathcal{F} \xrightarrow{\sim} rO_T \oplus (n-r)L$$

with the property that via these isomorphisms the morphisms $\mathcal{E} \to \mathcal{F}$ and $\mathcal{F} \to L \otimes \mathcal{E}$ are expressed by the diagonal matrices

$$\begin{bmatrix} \mathbb{I}_r & 0 \\ 0 & \lambda \mathbb{I}_{n-r} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \lambda \mathbb{I}_r & 0 \\ 0 & \mathbb{I}_{n-r} \end{bmatrix}$$
respectively. We will often use the following more suggestive notation for the $\textbf{f}$-morphism $g$:

$$g = \left( \begin{array}{c}
\mathcal{E} \\
\rightarrow
\begin{array}{c}
(L, \lambda)
\end{array}
\end{array}
\right) \mathcal{F}.$$ 

Let $T$, $\mathcal{E}$, $\mathcal{F}$ be as above. A generalized isomorphism from $\mathcal{E}$ to $\mathcal{F}$ is a sequence of $\textbf{f}$-morphisms connected as follows:

$$\mathcal{E} \xrightarrow{0} E_1 \xrightarrow{1} E_2 \cdots E_{n-1} \xrightarrow{n-1} E_n \simeq F_n \xrightarrow{n-1} F_{n-1} \cdots F_2 \xrightarrow{1} F_1 \xrightarrow{0} \mathcal{F}$$

which has properties for which we refer the reader to [K1] 5.2, since they will not be of importance here.

Now let $S$, $E$, $F$ be as in the beginning of this section and let $T$ be an $S$-scheme. The main theorem in [K1] is the following:

**Theorem 3.1.** There is a natural bijection between the set of $T$-valued points of $\text{KGL}(E, F)$ and the set of (equivalence classes of) generalized isomorphisms from $E_T$ to $F_T$, where $E_T$ and $F_T$ denote the pull back of $E$ and $F$ to $T$.

In particular, if $f : \text{KGL}(E, F) \rightarrow S$ denotes the structure morphism, then there exists a universal generalized isomorphism

$$f^*E \xrightarrow{0} E_1 \xrightarrow{1} E_2 \cdots E_{n-1} \xrightarrow{n-1} E_n \simeq F_n \xrightarrow{n-1} F_{n-1} \cdots F_2 \xrightarrow{1} F_1 \xrightarrow{0} f^*F$$

from $f^*E$ to $f^*F$.

For each pair of subsets $I, J \subseteq [0, n - 1]$ let $\overline{O}_{I, J} = \overline{O}_{I, J}(E, F)$ be the closed subscheme of $\text{KGL}(E, F)$ defined by the equations $\mu_i = 0$ ($i \in I$) and $\lambda_j = 0$ ($j \in J$). It is non-empty if and only if $\min(I) + \min(J) \geq n$. With this notation we have

$$\overline{O}_{I, J} = \left( \bigcap_{i \in I} Z_i \right) \cap \left( \bigcap_{j \in J} Y_j \right).$$

In particular, we have $\text{KGL}(E, F) = \overline{O}_{\emptyset, \emptyset}$, $Y_i = \overline{O}_{\emptyset, \{i\}}$ and $Z_i = \overline{O}_{\{i\}, \emptyset}$.

4. Review of the cohomology of line bundles on $\text{KGL}(E, F)$

Let $k$ be a field of characteristic zero. Throughout this section, $S$ will denote a $k$-scheme (or more generally an algebraic $k$-stack). We fix two locally free $\mathcal{O}_S$-modules $E$ and $F$ of rank $n$. Let $\text{KGL}(E, F)$ be the compactification of $\text{Isom}(E, F)$ introduced in [8] and denote by $f : \text{KGL}(E, F) \rightarrow S$ the structure morphism. Let

$$f^*E \xrightarrow{0} E_1 \xrightarrow{1} E_2 \cdots E_{n-1} \xrightarrow{n-1} E_n \simeq F_n \xrightarrow{n-1} F_{n-1} \cdots F_2 \xrightarrow{1} F_1 \xrightarrow{0} f^*F$$

be the universal generalized isomorphism from $f^*E$ to $f^*F$. 
Lemma 4.1. We have the following canonical isomorphisms of invertible $O_{KGL(E,F)}$-modules:

$$f^*(\det E) \otimes \bigotimes_{i=0}^{n-1} M_i^{i-n} = \det E_n = \det F_n = f^*(\det F) \otimes \bigotimes_{i=0}^{n-1} L_i^{i-n}$$

Proof. For $i \in [0, n-1]$ we denote by $g_i$ and $h_i$ the bf-morphism

$$E_{i+1} \xrightarrow{g_i} E_i \quad \text{and} \quad F_{i+1} \xrightarrow{h_i} F_i$$

respectively. By Proposition 6.2 in [K1] these induce canonical morphisms

$$\wedge^n g_i : \det(E_i) \longrightarrow M_i^{n-i} \otimes \det(E_{i+1})$$

$$\wedge^n h_i : \det(F_{i+1}) \longrightarrow L_i^{i-n} \otimes \det(E_i)$$

respectively. It follows that we have canonical morphisms

$$g := (\wedge^n g_{n-1}) \circ (\wedge^n g_{n-2}) \circ \ldots \circ (\wedge^n g_0) : f^* \det(E) \longrightarrow \det(E_n) \otimes \bigotimes_{i=0}^{n-1} M_i^{n-i}$$

$$h := (\wedge^n h_0) \circ (\wedge^n h_1) \circ \ldots \circ (\wedge^n h_{n-1}) : \det(F_n) \longrightarrow f^* \det(F) \otimes \bigotimes_{i=0}^{n-1} L_i^{i-n}$$

Let $\varphi : \det(E_n) \xrightarrow{\sim} \det(F_n)$ be the isomorphism induced by the isomorphism $E_n \xrightarrow{\sim} F_n$. By [K1] 6.5 the morphism

$$\wedge^n \Phi = h \circ \varphi \circ g : f^* \det(E) \longrightarrow \bigotimes_{i=0}^{n-1} (M_i^{n-i} \otimes L_i^{i-n})$$

is nowhere vanishing and consequently $g$ and $h$ are isomorphisms. \hfill \square

Recall from [3] that for each pair of subsets $I, J \subseteq [0, n-1]$ with $\min(I) + \min(J) \geq n$ the closed subscheme $O_{I,J}$ of $KGL(E,F)$ is defined as the zero locus of the sections $\mu_i$ ($i \in I$) and $\lambda_j$ ($j \in J$). Let

$$i_{I,J} : O_{I,J} \longrightarrow KGL(E,F)$$

$$f_{I,J} : O_{I,J} \longrightarrow S$$

denote the inclusion morphisms and structure morphisms respectively.

Let $Fl := Fl(E) \times_S Fl(F)$, where $Fl(E)$ and $Fl(F)$ denote the varieties over $S$ which parametrize full flags in $E$ and $F$ respectively. Let $0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_n = E \otimes O_{Fl}$ and $0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = F \otimes O_{Fl}$ be the two universal flags on $Fl$. For $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, $b = (b_1, \ldots, b_n) \in \mathbb{Z}^n$ we define the invertible $O_{Fl}$-module

$$O_{Fl}(a,b) := \bigotimes_{i=1}^{n} (E_i/E_{i-1})^{\otimes a_i} \otimes \bigotimes_{i=1}^{n} (F_i/F_{i-1})^{\otimes b_i} ,$$

which we sometimes abbreviate by $O(a,b)$ if no confusion is likely to arise.
Definition 4.2. Let \( L \) be a line bundle on \( \text{KGL}(E, F) \) of the form

\[
L = \bigotimes_{i=0}^{n-1} (M_{i}^{m_{i}} \otimes L_{i}^{l_{i}}) \otimes f^{*}(\det E)^{e} \otimes f^{*}(\det F)^{d}.
\]

Let \( I, J \subseteq [0, n-1] \) and let \( i_{1} := \min(I), j_{1} := \min(J) \) where it is understood that \( \min(\emptyset) = n \). Assume \( i_{1} + j_{1} \geq n \). We denote by \( A_{IJ}(L) \) the set of all elements \((a, b) \in \mathbb{Z}^{n} \times \mathbb{Z}^{n}\), which have the following properties:

1. \( a_{1} \leq a_{2} \leq \cdots \leq a_{n} \)
2. \( \sum_{j=i+1}^{n}(a_{j} - e) \leq m_{i} \) for all \( i \in [n - j_{1}, n - 1] \) and equality holds for \( i \in I \).
3. \( \sum_{j=1}^{n-i}(a_{j} - e) \geq -l_{i} \) for all \( i \in [n - i_{1}, n - 1] \) and equality holds for \( i \in J \).
4. For all \( i \in [1, n] \) the equality \( a_{i} - e = -b_{n-i+1} + d \) holds.

For abbreviation we will often write \( A(L) \) instead of \( A_{\emptyset, \emptyset}(L) \).

Theorem 4.3. Let \( L \) be a line bundle on \( \text{KGL}(E, F) \) of the form

\[
L = \bigotimes_{i=0}^{n-1} (M_{i}^{m_{i}} \otimes L_{i}^{l_{i}}) \otimes f^{*}(\det E)^{e} \otimes f^{*}(\det F)^{d}
\]

and let \( I, J \subseteq [0, n-1] \) be subsets with \( \min(I) + \min(J) \geq n \). Then the following holds:

1. The \( \mathcal{O}_{S} \)-module \((f_{IJ})_{*}i_{IJ}^{*}L\) is locally free and comes with a canonical decomposition as follows:

\[
(f_{IJ})_{*}i_{IJ}^{*}L = \bigoplus_{(a, b) \in A_{IJ}(L)} (f_{F_{I}})_{*}\mathcal{O}_{F_{I}}(a, b),
\]

where \( f_{F_{I}} : F_{I} \to S \) denotes the structure morphism.

2. The decomposition stated in 1. is compatible with restriction in the sense that the following diagram commutes:

\[
\begin{array}{ccc}
\bigoplus_{(a, b) \in A(L)} (f_{F_{I}})_{*}\mathcal{O}_{F_{I}}(a, b) & \xrightarrow{\text{Res}} & \bigoplus_{(a, b) \in A_{IJ}(L)} (f_{F_{I}})_{*}\mathcal{O}_{F_{I}}(a, b) \\
\downarrow & & \downarrow \\
\bigoplus_{(a, b) \in A(L) \cap A_{I,J}(L)} (f_{F_{I}})_{*}\mathcal{O}_{F_{I}}(a, b) & \xrightarrow{\text{Res}} & \bigoplus_{(a, b) \in A_{I,J}(L)} (f_{F_{I}})_{*}\mathcal{O}_{F_{I}}(a, b)
\end{array}
\]

where the lower arrows are the canonical projection and inclusion morphisms induced by the inclusions \( A(L) \cap A_{I,J}(L) \subseteq A(L) \) and \( A(L) \cap A_{I,J}(L) \subseteq A_{I,J}(L) \) respectively.

3. Let

\[
L' = \bigotimes_{i=0}^{n-1} (M_{i}^{m'_{i}} \otimes L_{i}^{l'_{i}}) \otimes f^{*}(\det E)^{e} \otimes f^{*}(\det F)^{d},
\]
where $m_i' \leq m_i$ and $l_j' \leq l_j$ and equality holds, if $i \in I$ and $j \in J$ respectively. The following diagram commutes:

\[
\begin{array}{ccc}
(f_{I,J})_* i^{*}_{I,J} L' & \otimes & \mu^{m-m'} \otimes \lambda^{l-l'} \rightarrow (f_{I,J})_* i^{*}_{I,J} L \\
\oplus & \oplus & \oplus \\
(a,b) \in A_{I,J}(L') & (a,b) \in A_{I,J}(L') & (a,b) \in A_{I,J}(L')
\end{array}
\]

where the upper horizontal arrow is induced by the section

\[
\left(\mu_{0}^{m_0-m'_0} \otimes \ldots \otimes \mu_{n-1}^{m_{n-1}-m'_{n-1}} \otimes \lambda_{0}^{l_0-l'_0} \otimes \ldots \otimes \lambda_{n-1}^{l_{n-1}-l'_{n-1}}\right)|_{\sigma_{I,J}}
\]

of $i^{*}_{I,J}(L \otimes (L)^{-1})$ and the lower horizontal arrow is induced by the inclusion $A_{I,J}(L') \subseteq A_{I,J}(L)$.

**Proof.** This is an easy consequence of the main result in [K3].

\[\square\]

5. Review of moduli of Gieseker vector bundles

Let $k$ be an algebraically closed field of characteristic zero. Let $C_0$ be an irreducible projective curve over $k$ which is smooth except for one ordinary double point $p \in C_0(k)$. Let $\tilde{C}_0$ be the normalization of $C_0$ and let $p_1, p_2$ be the two $k$-valued points of $\tilde{C}_0$ lying above the singular point $p$.

For an integer $q \geq 1$ let $C_q$ be the curve which arises from $C_0$ by inserting a chain of length $q$ of projective lines at the point $p$:

\[
\begin{array}{ccccccc}
p_1 & \leftarrow & R_1 & \leftarrow & R_2 & \ldots & \leftarrow & R_{m-1} & \leftarrow & R_m \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\tilde{C}_0 & \leftarrow & R_1 & \leftarrow & R_2 & \ldots & \leftarrow & R_{m-1} & \leftarrow & R_m \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
p_2 & \leftarrow & R_1 & \leftarrow & R_2 & \ldots & \leftarrow & R_{m+1} & \leftarrow & R_{m+1}
\end{array}
\]

**Definition 5.1.** 1. A *Gieseker vector bundle of rank $n$ on $C_0$* is a pair $(X \to C_0, F)$, where $X = C_r$ for some $r \in [0, n]$ and $F$ is a vector bundle of rank $n$ on $X$ such that the following holds:

1. The morphism $X \to C_0$ is the identity if $r = 0$, and it contracts the chain of projective lines into the singular point $p$ of $C_0$ if $r \geq 1$.
2. The restriction of $F$ to any of the inserted projective lines $R_i$ is of the form

\[
d_i \mathcal{O}_{R_i}(1) \oplus (n - d_i) \mathcal{O}_{R_i}
\]

for some $d_i \geq 1$. 


If \( r \geq 1 \) let \( R = \bigcup_i R_i \) be the inserted chain of projective lines and denote by \( p_1, p_2 \in R \) the points at which it meets the curve \( \tilde{C}_0 \). Then we have

\[
H^0(R, \mathcal{F}|_R(-p_1 - p_2)) = (0) .
\]

2. Let \( T \) be a \( k \)-scheme. A Gieseker vector bundle of rank \( n \) on \( C_0 \) over \( T \) is a pair \((X \to C_0 \times T, \mathcal{F})\), where \( X \to C_0 \times T \) is a morphism of curves over \( T \) and \( \mathcal{F} \) is a vector bundle on \( X \) such that if \( z \) is a point in \( T \) and if we denote by \( X_z \) the fiber of \( X \to T \) at \( z \), then the pair \((X_z \to C_0 \otimes_k \kappa(z), \mathcal{F}|_{X_z})\) is a Gieseker vector bundle on \( C_0 \otimes_k \kappa(z) \). We will often write \((X, \mathcal{F})\) instead of \((X \to C_0 \times T, \mathcal{F})\).

**Definition 5.2.** A Gieseker vector bundle data of rank \( n \) on \( C_0 \) over a \( k \)-scheme \( T \) is a triple \((X \to C_0 \times T, \mathcal{F}, x)\), where \((X \to C_0 \times T, \mathcal{F})\) is a Gieseker vector bundle data or rank \( n \) on \( C_0 \) and \( x : T \to X \) is a section of \( X \to T \) whose image is in the singular locus of \( X \to T \). We will often write \((X, \mathcal{F}, x)\) instead of \((X \to C_0 \times T, \mathcal{F}, x)\).

**Remark 5.3.** Let \((X \to C_0 \times T, \mathcal{F}, x)\) be a Gieseker vector bundle data over \( T \). Then there is a canonical two-pointed curve \((x_1, \mathcal{C}, x_2)\) over \( T \) such that \( X \) can be constructed from \( \mathcal{C} \) by identifying the two sections \( x_1 \) and \( x_2 \). Indeed, the curve \( \mathcal{C} \) is simply the blow-up of \( X \) along the closed subscheme \( x(T) \) and \( x_1(T) \sqcup x_2(T) \) is the pre-image of \( x(T) \). The composition \( \mathcal{C} \to X \to C_0 \times T \) factorizes naturally through a morphism \( \mathcal{C} \to \tilde{C}_0 \times T \) and the pull back \( \mathcal{F}' \) of \( \mathcal{F} \) to \( \mathcal{C} \) comes with a natural isomorphism \( \varphi : x_1^* \mathcal{F}' \cong x_2^* \mathcal{F}' \). The datum \(((x_1, \mathcal{C}, x_2) \to (p_1, \tilde{C}_0 \times T, p_2), \mathcal{F}', \varphi)\) is equivalent to the datum \((X \to C_0 \times T, \mathcal{F}, x)\).

In \([K2]\) I have shown that there are algebraic moduli stacks \( GVB \) and \( GVBD \) parametrizing Gieseker vector bundles and Gieseker vector bundle data of rank \( n \) on \( C_0 \) respectively. Furthermore, the stack \( GVBD \) is smooth, the stack \( GVB \) has normal crossing singularities and the forgetful morphism

\[
\nu' : \begin{cases} 
GVBD & \to GVB \\
(X, \mathcal{F}, x) & \mapsto (X, \mathcal{F})
\end{cases}
\]

identifies the stack \( GVBD \) with the normalization of \( GVB \).

Let \( VB \) be the moduli stack of vector bundles of rank \( n \) on \( \tilde{C}_0 \) and denote by \( \tilde{\pi} : \tilde{C}_0 \times VB \to VB \) the projection onto the second factor. Let \( E \) and \( F \) be the pull back of the universal vector bundle on \( \tilde{C}_0 \times VB \) via the section of \( \tilde{\pi} \) induced by the point \( p_1 \) and \( p_2 \) respectively. The main result in \([K2]\) is the construction of a canonical isomorphism

\[
\tau : KGL := KGL(E, F) \cong GVBD .
\]

The isomorphism \( \tau \) is defined by a certain family

\[
(\mathcal{C}', \to KGL, \mathcal{C}', s : KGL \to \mathcal{C}')
\]

of Gieseker vector bundle data on \( KGL \). In the remainder of this section I will recall some details of the construction of this family.

Let \( f : KGL \to VB \) be the structure morphism and let \( \Phi : \)

\[
f^* E \ar[d]_{(M_0, \mu_0)} \ar[r]^{0} & E_1 \ar[r]^{1} & E_2 \ar[r] & \cdots \ar[r]^{n-1} & E_n \ar[r]_{\sim} & F_n \ar[l]_{(L_n, \lambda_n)}^{n-1} \ar[r]_{(L_n, \lambda_n)} & F_{n-1} \ar[r]_{(L_1, \lambda_1)} & \cdots \ar[r]^{1} & F_2 \ar[l]_{(L_1, \lambda_1)}^{1} \ar[r]^{0} & f^* F
\]
be the universal generalized isomorphism from $f^*E$ to $f^*F$. Let $(s_1, \mathcal{B}, s_2)$ be the two-pointed nodal curve (with the notation of [2])

$$(M_{n-1}, \mu_{n-1}) \rightrightarrows (M_0, \mu_0) \rightrightarrows (p_1, \tilde{C}_0 \times \text{KGL}, p_2) \rightrightarrows (L_0, \mu_0) \rightrightarrows \cdots \rightrightarrows (L_{n-1}, \lambda_{n-1})$$

Then $\mathcal{C}'$ is the curve $\mathcal{B}/(s_1 = s_2)$ over KGL and $s : \text{KGL} \to \mathcal{C}$ is the composition

$$\text{KGL} \xrightarrow{s_i} \mathcal{B} \xrightarrow{\nu} \mathcal{C}' \quad \text{for } i = 1 \text{ or } i = 2.$$ 

Let $\tilde{\mathcal{E}}$ be the universal vector bundle on $\tilde{C}_0 \times \text{VB}$ and let $\tilde{\mathcal{E}}' := (\text{id}_{\tilde{C}_0} \times f)^*\tilde{\mathcal{E}}$ be its pull-back to $\tilde{C}_0 \times \text{KGL}$. The generalized isomorphism $\Phi$ together with the vector bundle $\tilde{\mathcal{E}}'$ induce a vector bundle $\mathcal{G}$ of rank $n$ on the two-pointed nodal curve $(s_1, \mathcal{B}, s_2)$. The details of the construction of $\mathcal{G}$ out of $\Phi$ and $\tilde{\mathcal{E}}'$ are given in §7 and §9 of [2]. For the purpose of this paper it suffices to know that $\mathcal{G}$ has the property that

$$\tilde{\mathcal{E}}' = (h_*(\mathcal{G}(-s_1 - s_2))(p_1 + p_2),$$

where $h : \mathcal{B} \to \tilde{C}_0 \times \text{KGL}$ is the canonical projection. Furthermore there are canonical isomorphisms

$$s_1^*\mathcal{G} = E_n \quad \text{and} \quad s_2^*\mathcal{G} = F_n.$$ 

The vector bundle $\mathcal{E}'$ on $\mathcal{C}'$ is constructed from $\mathcal{G}$ by using the isomorphism $E_n \xrightarrow{\sim} F_n$, which is part of the data contained in $\Phi$, to glue together $\mathcal{G}$ along the sections $s_1$ and $s_2$.

For future reference we collect the various curves, bundles, and moduli spaces in the following diagram:

$$
\begin{array}{cccccccc}
\tilde{\mathcal{E}} & \xrightarrow{h} & \mathcal{G} & \xrightarrow{g} & \mathcal{E}' & \xrightarrow{\nu} & \mathcal{E} \\
\tilde{C}_0 \times \text{VB} & \xrightarrow{p_1} & \tilde{C}_0 \times \text{KGL} & \xrightarrow{\pi_1} & \mathcal{B} & \xrightarrow{\pi} & \mathcal{E} & \xrightarrow{s} & \mathcal{C}' \\
\text{VB} & \xrightarrow{f} & \text{KGL} & \xrightarrow{s_1} & \text{KGL} & \xrightarrow{s} & \text{KGL} & \xrightarrow{\nu} & \text{GVB} \\
E_n, F & \xrightarrow{f^*E, f^*F} & E_n, F & \xrightarrow{s^*\mathcal{E}'} & & & & & \\
\end{array}
$$

Here we have set $\nu := \nu' \circ \tau$. The pair $(\mathcal{E}, \mathcal{E})$ is the universal Gieseker vector bundle on $C_0$ over GVB. The outer squares in this diagram are Cartesian. The bundles $\mathcal{E}'$ and $\mathcal{G}$ are the pull back of the bundle $\mathcal{E}$ and the bundle $\tilde{\mathcal{E}}'$ is the pull back of the bundle $\tilde{\mathcal{E}}$. The bundles on the bottom are the pull back by the respective sections of the bundles upstairs.

6. A remarkable set of isomorphisms

We keep the notation from §5. Thus we have a diagram

$$
\begin{array}{cccc}
\text{VB} & \xrightarrow{f} & \text{KGL} & \xrightarrow{\nu} \text{GVB} \\
\end{array}
$$

of algebraic stacks, where GVB is the moduli stack of Gieseker vector bundles of rank $n$ on the singular curve $C_0$, and VB is the moduli stack of vector bundles of rank $n$ on the
normalization $\tilde{C}_0$ of $C_0$. We have $KGL = KGL(E, F)$, where $E$ and $F$ are the pull-back of the universal bundle on $\tilde{C}_0 \times \text{VB}$ along the sections $p_1$ and $p_2$ respectively. Let

$$f^*E \xrightarrow{i_{(M_i, \mu_i)}} E_1 \cdots \xrightarrow{n-1}{E}_n \xrightarrow{n-1}{F}_n \cdots \xrightarrow{1}{F}_2 \xrightarrow{1}{F}_1 \xrightarrow{0}{f^*F}$$

be the universal generalized isomorphism from $f^*E$ to $f^*F$. As in $\mathbb{B}$ let $Y_i$ and $Z_i$ be the divisors in $KGL$ defined by the equations $\lambda_i = 0$ and $\mu_i = 0$ respectively and let

$$i_{Y_i} := i_{0,(i)} : Y_i \hookrightarrow KGL$$
$$i_{Z_i} := i_{1,(i)} : Z_i \hookrightarrow KGL$$

be the respective inclusion morphisms. In the following proposition we compute the fibre product $KGL \times_{\text{GVB}} KGL$. It is the key ingredient in proposition $\mathbb{C}$ where we identify the space of global sections of a line bundle $\mathcal{L}$ on $\text{GVB}$ with a subspace of the space of global sections of its pull-back to $KGL$.

**Proposition 6.1.** After the choice of isomorphisms $m_{\tilde{C}_0,p_i}/m^2_{\tilde{C}_0,p_i} = k$ for $i = 1, 2$ we have:

1. For each $j \in [0, n-1]$ there is a canonical isomorphism $\beta_j : Y_j \cong Z_j$, which makes the following diagram commutative:

$$\begin{array}{ccc}
Y_j & \xrightarrow{\beta_j} & Z_j \\
\nu_{Y_j} \downarrow & & \downarrow \nu_{Z_j} \\
\text{GVB} & & \\
\end{array}$$

where $\nu_{Y_j} := \nu \circ i_{Y_j}$ and $\nu_{Z_j} := \nu \circ i_{Z_j}$.

2. Let $i, j \in [0, n-1]$. Then we have

$$\beta_j^* i_{Z_j}^*(M_i, \mu_i) = \begin{cases} 
\begin{cases} 
i_{Y_i}(M_{n-j+i}, \mu_{n-j+i}) & \text{for } i \in [0, j-1] \\
i_{Y_j}(\otimes_{r=0}^{n-1}(M_r \otimes L_r)^{-1}, 0) & \text{for } i = j \\
i_{Y_j}(L_{n+j-i}, \lambda_{n+j-i}) & \text{for } i \in [j+1, n-1]
\end{cases} &
\end{cases}$$

$$\beta_j^* i_{Z_j}^*(L_i, \lambda_i) = \begin{cases} 
\begin{cases} (O, 1) & \text{for } i \in [0, n-j-1] \\
i_{Y_j}(L_{j+i-n}, \lambda_{j+i-n}) & \text{for } i \in [n-j, n-1]
\end{cases} &
\end{cases}$$

3. The following morphism is an isomorphism:

$$(id, id) \sqcup \bigcup_{i=0}^{n-1} \bigcup_{j=0}^{n-1} (i_{Y_i}, i_{Z_i} \circ \beta_i) \sqcup \bigcup_{i=0}^{n-1} \bigcup_{j=0}^{n-1} (i_{Z_i}, i_{Y_i} \circ \beta_i^{-1}) : KGL \sqcup \bigcup_{i=0}^{n-1} Y_i \sqcup \bigcup_{i=0}^{n-1} Z_i \longrightarrow KGL \times_{\text{GVB}} KGL$$

**Proof.** Let $T$ be a $k$-scheme. A $T$-valued point of $Y_j$ is given by a pair $(\mathcal{F}, \Psi)$, where $\mathcal{F}$ is a vector bundle of rank $n$ on $\tilde{C}_0 \times T$ and $\Psi$ is a generalized isomorphism

$$G_0 \xrightarrow{0} G_1 \xrightarrow{1} G_2 \cdots \xrightarrow{n-1}{G}_n \xrightarrow{0} H_0 \xrightarrow{0} \cdots \xrightarrow{0} H_2 \xrightarrow{1} H_1 \xrightarrow{0} H_0$$

from $G_0 := p_1^* \mathcal{F}$ to $H_0 := p_2^* \mathcal{F}$, such that $b_j = 0$. 
Let \((X \to C_0 \times T, \mathcal{H}, x)\) be the Gieseker vector bundle data associated to \((\mathcal{F}, \Psi)\) by the canonical isomorphism \(\tau : \text{KGL} \to \text{GVBD}\). Recall from \([5]\) that \(X\) is constructed from the two-pointed nodal curve \((x_1, \mathcal{C}, x_2) = (A_{n-1}, a_{n-1}) \oplus \cdots \oplus (A_0, a_0) \oplus (p_1, \tilde{C}_0 \times T, p_2) \oplus (B_0, b_0) \oplus \cdots \oplus (B_{n-1}, b_{n-1})\) by identifying the two sections \(x_1\) and \(x_2\).

We define the two-pointed nodal curves
\[
(r_1, \mathcal{B}, r_2) := (A_{n-1}) \oplus \cdots \oplus (A_0) \oplus (p_1, \tilde{C}_0 \times S, p_2) \oplus B_0 \oplus \cdots \oplus B_{j-1}
\]
\[
(t_1, D, t_2) := \bigoplus_{r=0}^{j} B_r, \mathcal{O}_S |_B \oplus B_{j+1} \oplus \cdots \oplus B_{n-1}
\]

Since the sections \(a_0, \ldots, a_{n-j-1}\) are nowhere vanishing, and since we have \(b_j = 0\) and \(r^2 \mathcal{O}_B(-r_2) = \bigotimes_{r=0}^{j-1} B_r\) (cf. \([2\mathbf{8}]\) here we make use of the identification \(m_{\tilde{C}_0, p_2} / m_{\tilde{C}_0, p_2}^2 = k\), it follows from \([2.10]\) that we have
\[
(x_1, \mathcal{C}, x_2) = (r_1, \mathcal{B}, r_2) \perp (t_1, D, t_2)
\]

Now we define
\[
(x'_1, \mathcal{C}' , x'_2) := (t_1, D, t_2) \perp (r_1, \mathcal{B}, r_2)
\]

Then the curve \(X'\) obtained from \(\mathcal{C}'\) by identifying the sections \(x'_1\) and \(x'_2\) is canonically isomorphic to \(X\). Thus we have a new Gieseker vector bundle data \((X' \to C_0 \times T, \mathcal{H}', x')\), where \(X' = X, \mathcal{H}' = \mathcal{H}\) and \(x'\) is the composition
\[
T \xrightarrow{x'_m} \mathcal{C}' \longrightarrow X' = X \quad (m = 1 \text{ or } m = 2).
\]

Via the isomorphism \(\tau : \text{KGL} \sim \text{GVBD}\) there corresponds to \((X', \mathcal{H}', x')\) a \(T\)-valued point of \(\text{KGL}\) which is given by a pair \((\mathcal{F}', \Psi')\), where \(\mathcal{F}'\) is a vector bundle on \(\tilde{C}_0\) and \(\Psi'\) is a generalized isomorphism from \(p_1^{\mathcal{F}_1}\) to \(p_2^{\mathcal{F}_2}\).

Recall from \([5]\) that if we write \(\Psi' = 
\[
G_0 \bigoplus_{(A_0, a_0)} G'_1 \bigoplus_{(A'_1, a'_1)} G'_2 \cdots \bigoplus_{(A_{n-1}, a_{n-1})} G'_n \sim H_0 \bigoplus_{(B_0, b_0)} H'_1 \bigoplus_{(B'_1, b'_1)} H'_2 \cdots \bigoplus_{(B_{n-1}, b_{n-1})} H'_n
\]

then \((x'_1, \mathcal{C}', x'_2)\) is isomorphic to
\[
(A'_{n-1}, a'_{n-1}) \oplus \cdots \oplus (A'_0, a'_0) \oplus (p_1, \tilde{C}_0 \times S, p_2) \oplus (B'_0, b'_0) \oplus \cdots \oplus (B'_{n-1}, b'_{n-1})
\]

Since by \([2.12]\) we have
\[
(t_1, D, t_2) = B_{j+1} \oplus \cdots \oplus B_{n-1} \oplus \bigoplus_{r=0}^{n-1} \mathcal{O}_S \bigotimes_{r=0} B_r^{-1}
\]

it follows from \([2.10]\) that
\[
(A'_i, a'_i) := \begin{cases} (A_{n-j+i}, a_{n-j+i}) & \text{for } i \in [0, j - 1] \\ \bigotimes_{r=0}^{n-1} (A_r \otimes B_r)^{-1}, 0 & \text{for } i = j \\ (B_{n+j-i}, b_{n+j-i}) & \text{for } i \in [j + 1, n - 1] \end{cases}
\]
(for \( i = j \) we have made use of the identification \( \mathfrak{m}_{C_{0},p_{1}}/\mathfrak{m}_{C_{0},p_{1}}^{2} = k \) and the fact that the \( a_{0}, \ldots, a_{n-j-1} \) are nowhere vanishing) and

\[
(B_{i}, b_{i}) := \begin{cases} \mathcal{O}_{S}, 1 & \text{for } i \in [0, n-j-1] \\ (B_{j-n+i}, b_{j-n+i}) & \text{for } i \in [n-j, n-1] \end{cases}.
\]

In particular, we have \( \alpha' = 0 \) and therefore \((\mathcal{F}', \Psi')\) is in fact a \( T \)-valued point of the closed substack \( Z_{j} \) of KGL. We define \( \beta_{j} : Y_{j} \to Z_{j} \) by the rule

\[ (\mathcal{F}, \Psi) \mapsto (\mathcal{F}', \Psi') \, . \]

Since the inverse of \( \beta_{j} \) can be constructed completely analogously, it is clear that \( \beta_{j} \) is an isomorphism. By construction, we have

\[ \nu_{Y_{j}}(\mathcal{F}, \Psi) = (X \to C_{0} \times T, \mathcal{F}) = (X' \to C_{0} \times T, \mathcal{F}') = \nu_{Z_{j}}(\mathcal{F}', \Psi') = \nu_{Z_{j}} \circ \beta_{j}(\mathcal{F}, \Psi) \, . \]

This shows the first part of the Proposition.

The second part follows from equations (1) and (2) above.

For the third part it is clearly sufficient to show that there exists a commutative diagram of stacks:

\[
\begin{array}{ccc}
KGL \sqcup \bigcup_{i=0}^{n-1} Y_{i} & \xrightarrow{\alpha} & \bigcup_{i=0}^{n-1} Z_{i} \\
\downarrow \id \cup \bigcup_{i=0}^{n-1} \beta_{i} & & \downarrow \bigcup_{i=0}^{n-1} \beta_{i}^{-1} \\
KGL \sqcup \bigcup_{i=0}^{n-1} Z_{i} & \xrightarrow{\alpha'} & \bigcup_{i=0}^{n-1} Y_{i} \\
\end{array}
\]

where the arrows \( \alpha \) and \( \alpha' \) are isomorphisms such that the following holds:

\[
\begin{align}
\text{pr}_{1} \circ \alpha^{-1} &= \text{id} \cup \bigcup_{i=0}^{n-1} i_{Y_{i}} \cup \bigcup_{i=0}^{n-1} i_{Z_{i}} \\
\text{pr}_{2} \circ (\alpha')^{-1} &= \text{id} \cup \bigcup_{i=0}^{n-1} i_{Z_{i}} \cup \bigcup_{i=0}^{n-1} i_{Y_{i}}
\end{align}
\]

Here, \( \text{pr}_{m} : KGL \times_{GVB} KGL \to KGL \) denotes the projection onto the \( m \)-the factor \( (m = 1, 2) \).

First we define the isomorphism \( \alpha \). Let \( T \) be a scheme. A \( T \)-valued point of \( KGL \times_{GVB} KGL \) is a pair \((\xi, \xi')\) of \( T \)-valued points of KGL such that \( \nu(\xi) \cong (X, \mathcal{H}) \cong \nu(\xi') \) for some Gieseker vector bundle \((X, \mathcal{H})\) over \( T \). The datum \((\xi, \xi')\) is equivalent to the datum \((X, \mathcal{H}, x, x')\), where \( x, x' : T \to X \) are two sections of \( X \to T \) whose image is contained in the singular locus of \( X \to T \) such that \((X, \mathcal{H}, x) = \tau(\xi)\) and \((X, \mathcal{H}, x') = \tau(\xi')\). There are two cases:

**First case:** If \( x = x' \), then \((X, \mathcal{H}, x, x')\) is nothing else but a \( T \)-valued point of GVB \( \cong \) KGL.

**Second case:** If \( x \neq x' \), then let \((x_{1}, C, x_{2})\) be the two-pointed nodal curve over \( T \) which is the partial normalization of \( X \) along \( x \). The datum of \((X, \mathcal{H}, x, x')\) is clearly equivalent to the datum \((\xi, T \to C)\), where \( T \to C \) is a section of \( \mathcal{C} \to T \) whose image is in the singular locus of \( \mathcal{C} \to T \) such that the composition \( T \to \mathcal{C} \to X \) is the section \( x' \). But the datum \((\xi, T \to C)\) describes precisely a \( T \)-valued point of the closed substack \( \Sigma \) of the curve \( \mathfrak{B} \) which is the locus of non-smoothness of the
morphism $\mathcal{B} \to \text{KGL}$. From the definition of $\mathcal{B}$ and from 2.13 it follows that we have

$\Sigma = \left( \bigsqcup_{i=0}^{n-1} Z_i \right) \sqcup \left( \bigsqcup_{i=0}^{n-1} Y_i \right)$

Thus in this case $(X, \mathcal{H}, x, x')$ is equivalent to a $T$-valued point of the disjoint union of the $Y_i$ and $Z_i$.

Thus we have established an equivalence between $T$-valued points of the product $\text{KGL} \times_{\text{GVB}} \text{KGL}$ and $T$-valued points of the disjoint union of the stacks $\text{KGL}$, $Y_i$ and $Z_i$. This defines the isomorphism $\alpha$. It is clear from the construction that equation (4) holds.

The isomorphism $\alpha'$ is constructed similarly, with the only difference that in the case $x \neq x'$ we take the partial normalization $(x'_1, C'x'_2)$ of $X$ along $x'$ and then define a $T$-valued point of $\Sigma$ by the datum $(\xi', T \to C')$ where $T \to C'$ is induced by $x$. Again the equation (5) is clear.

The commutativity of the diagram (3) is clear from the construction of the $\beta_i$. □

7. Decomposition of generalized theta functions

We keep the notation from the end of §5. In particular, $(\mathcal{C}, \mathcal{E})$ denotes the universal Gieseker vector bundle over $\text{GVB}$ and $\pi : \mathcal{C} \to \text{GVB}$ is the projection onto the base. $\tilde{\mathcal{E}}$ is the universal vector bundle on $\tilde{C}_0 \times \text{VB}$ and $\tilde{\pi} : \tilde{C}_0 \times \text{VB} \to \text{VB}$ is the projection onto the second factor. Let $\Theta := \det R\pi_* \mathcal{E}$ and $\tilde{\Theta} := \det R\tilde{\pi}_* (\tilde{\mathcal{E}})$ be the theta line bundle on $\text{GVB}$ and on $\text{VB}$ respectively. Our convention for the determinant of the cohomology is such that for a curve $X$ and a vector bundle $\mathcal{F}$ on $X$ we have $\det H^0(X, \mathcal{F}) = (\det H^0(X, \mathcal{F}))^{-1} \otimes \det H^1(X, \mathcal{F})$.

We fix a positive integer $\kappa$. Our aim is to decompose the space $H^0(\text{GVB}, \Theta^\kappa)$ canonically into a direct sum, where the summands are related to $\tilde{\Theta}$. The following proposition tells us that we can regard $H^0(\text{GVB}, \Theta^\kappa)$ as a subspace of $H^0(\text{KGL}, \nu^*(\Theta^\kappa))$.

**Proposition 7.1.** Let $\mathcal{L}$ be a line bundle on $\text{GVB}$. Then the canonical homomorphism

$H^0(\text{GVB}, \mathcal{L}) \to H^0(\text{KGL}, \nu^* \mathcal{L})$

is injective. A global section $\theta \in H^0(\text{KGL}, \nu^* \mathcal{L})$ is in the image of this homomorphism, if and only if for each $j \in [0, n-1]$ the equality

$\beta_j^* i_{Y_j}^*(\theta) = i_{Z_j}^*(\theta)$

holds, where $i_{Y_j}^*$ and $i_{Z_j}^*$ denote the restriction homomorphisms

$i_{Y_j}^* : H^0(\text{KGL}, \nu^* (\mathcal{L})) \to H^0(Y_j, \nu_{Y_j}^* (\mathcal{L}))$

$i_{Z_j}^* : H^0(\text{KGL}, \nu^* (\mathcal{L})) \to H^0(Z_j, \nu_{Z_j}^* (\mathcal{L}))$

induced by $i_{Y_j}$ and $i_{Z_j}$ respectively, and $\beta_j^*$ denotes the pull-back isomorphism

$\beta_j^* : H^0(Z_j, \nu_{Z_j}^* (\mathcal{L})) \to H^0(Y_j, \nu_{Y_j}^* (\mathcal{L}))$

induced by $\beta_j$. 

We need the following lemma, which is probably well-known, but for which I did not find a reference.

**Lemma 7.2.** Let $k$ be a field, let $X$ be a smooth $k$-scheme and let $X_0 \subset X$ be a divisor with normal crossings. Let $X_1 \to X_0$ be the normalization of $X_0$ and let $\mathcal{L}_0$ be an invertible $\mathcal{O}_{X_0}$-module. Then the following sequence is exact:

$$0 \to H^0(X_0, \mathcal{L}_0) \to H^0(X_1, \mathcal{L}_1) \to H^0(X_2, \mathcal{L}_2)$$

Here $X_2$ denotes the fiber product $X_1 \times_{X_0} X_1$ and $\mathcal{L}_i$ is the pull back of $\mathcal{L}_0$ by the morphism $X_i \to X_0$ for $i = 1, 2$. The arrows are the obvious ones.

**Proof.** 

**Step 1:** Assume $X = \text{Spec} \,(R)$, where $R$ is a regular local ring and $X_0 = \text{Spec} \,(R_0)$ where $R_0 = R/(\prod_{i=1}^r x_i)$, the elements $x_1, \ldots, x_m$ form a regular system of parameters for $R$ and $r \in [1, m]$.

By [EGA] II 6.3.8 we have $X_1 = \text{Spec} \,(R_1)$ where $R_1 = \prod_{i=1}^r R/(x_i)$ and it follows that $X_2 = \text{Spec} \,(R_2)$ where $R_2 = \prod_{i,j=1}^r R/(x_i, x_j)$. We have to show the exactness of the sequence

$$0 \to R_0 \to R_1 \to R_2.$$ 

Since a regular local ring is a unique factorization domain, any element of $R$ which is divisible by all $x_i$ for $i \in [1, r]$ is also divisible by the product $\prod_{i=1}^r x_i$. This implies the injectivity of $R_0 \to R_1$.

Let $(f_i)_{i \in [1, r]}$ be a family of elements in $R$ with $f_i \equiv f_j \mod (x_i, x_j)$ for $i, j \in [1, r]$. We have to show that there exists an element $f$ of $R$ with $f \equiv f_i \mod (x_i)$ for all $i \in [1, r]$. If $r = 1$, this statement is trivial, so assume that $r > 1$ and that there exists $f' \in R$ with $f' \equiv f_i \mod (x_i)$ for $i \in [1, r-1]$. By assumption we have $f' - f_r \equiv g_i x_i \mod (x_r)$ for $i \in [1, r-1]$ and suitable $g_i \in R$. But the ring $R/(x_r)$ is regular local and thus a unique factorization domain and $x_1, \ldots, x_{r-1}$ represent prime elements in $R/(x_r)$. Therefore it follows that $f' - f_r = g \prod_{i=1}^{r-1} x_i - h x_r$ for suitable $g, h \in R$. The element $f := f' - g \prod_{i=1}^{r-1} x_i = f_r - h x_r$ has the required property.

**Step 2:** Assume that $X_0 \subset X$ is a divisor with strict normal crossings.

This means (cf. [SGA1] XIII 2.1) that there exists a family $(f_i)_{i \in I}$ of global sections of $\mathcal{O}_X$ indexed by a finite set $I$, such that $X_0 = \text{div} \,(\prod_{i \in I} f_i)$ and such that for every $x \in X_0$ the closed subscheme of $X$ cut out by the ideal $(f_i)_{i \in I(x)}$ is smooth of codimension equal to the cardinality of $I(x) := \{i \in I \mid f_i(x) = 0\}$.

Let $\pi_i$ denote the morphism $X_i \to X_0$ $(i = 1, 2)$. By the first step we have an exact sequence of $\mathcal{O}_{X_0}$-modules:

$$0 \to \mathcal{O}_{X_0} \to (\pi_1)_* \mathcal{O}_{X_1} \to (\pi_2)_* \mathcal{O}_{X_2}.$$ 

Tensoring with $\mathcal{L}_0$ and taking global sections yields the desired result.

**Step 3:** General case.

There exists an etale covering $X' \to X$ such that the pull back $X'_0 \subset X'$ of $X_0$ is a divisor with strict normal crossings. Let $X'' := X' \times_X X'$, denote by $X'_1$ and $X'_1''$ the normalization of $X'_0$ and $X''_0$ respectively (which may be identified with the fiber product $X_1 \times_{X_0} X'_0$ and $X_1 \times_{X_0} X''_0$ respectively) and let $X'_2 := X'_1 \times_{X'_0} X'_1$, $X''_2 := X''_1 \times_{X''_0} X''_1$. 

Proposition 7.3. We have a canonical isomorphism of line bundles on \( KGL \):

\[
\nu^*(\Theta) = \Delta \otimes f^*\tilde{\Theta}
\]
where \( f \) is the morphism \( KGL \to VB \) and where

\[
\Delta := \left( \bigotimes_{i=0}^{n-1} \mathcal{M}_i^{-1} \right) \otimes f^*(\det F) = \left( \bigotimes_{i=0}^{n-1} \mathcal{L}_i^{-1} \right) \otimes f^*(\det E) .
\]

**Proof.** Recall from the end of \([5]\) that we have a diagram of curves over \( VB \), \( KGL \) and \( GVB \) together with vector bundles as follows:

\[
\begin{array}{ccccccccc}
\tilde{\mathcal{E}} & \xrightarrow{\mathcal{E}'} & \mathcal{G} & \mathcal{E}' & \xrightarrow{} & \mathcal{E} \\
\tilde{C}_0 \times VB & \leftarrow & \tilde{C}_0 \times KGL & \xrightarrow{h} & \mathcal{B} & \xrightarrow{g} & \mathcal{E}' & \xrightarrow{} & \mathcal{E} \\
\xrightarrow{p_1} & & \xrightarrow{p_1} & & \xrightarrow{p_1} & & \xrightarrow{s_1} & & \xrightarrow{s_1} \\
& \mathcal{E} & & \mathcal{E}' & & \mathcal{E} & & \mathcal{E} & & \mathcal{E} \\
& \xrightarrow{\nu} & & \xrightarrow{\nu} & & \xrightarrow{\nu} & & \xrightarrow{\nu} & & \xrightarrow{\nu} \\
& \xrightarrow{f} & & \xrightarrow{f} & & \xrightarrow{f} & & \xrightarrow{f} & & \xrightarrow{f} \\
& VB & & KGL & & KGL & & KGL & & GVB \\
& \xrightarrow{E, F} & & \xrightarrow{E, F, F} & & \xrightarrow{E_n, F_n} & & s^* \mathcal{E}' \\
\end{array}
\]

and that the bundles \( \mathcal{E}' := (\text{id}_{\tilde{C}_0} \times f)^* \tilde{\mathcal{E}} \) and \( \mathcal{G} \) are related by the equation

\[
\tilde{\mathcal{E}}' = (h_* \mathcal{G}(-s_1 - s_2))(p_1 + p_2) .
\]

(1)

It is clear that we have \( \nu^* \Theta = \det(R\pi'_* \mathcal{E}') \). From the canonical exact sequence of \( \mathcal{O}_\mathbb{C} \)-modules

\[
0 \to \mathcal{E}' \to g_* \mathcal{G} \to s_* s^* \mathcal{E}' \to 0
\]

and the fact that \( R^1 g_* \mathcal{G} = 0 \) (cf. \([Kn]\), Cor 1.5) we get the canonical isomorphism

\[
\det(R\pi'_* \mathcal{E}') = \det(R\rho_* \mathcal{G}) \otimes \det(s^* \mathcal{E}') .
\]

(2)

The canonical exact sequence of \( \mathcal{O}_\mathbb{B} \)-modules

\[
0 \to \mathcal{G}(-s_1 - s_2) \to \mathcal{G} \to (s_1)_* s^*_1 \mathcal{G} \oplus (s_2)_* s^*_2 \mathcal{G} \to 0
\]

yields the canonical isomorphism

\[
\det(R\rho_* \mathcal{G}) = \det(R\rho_* \mathcal{G}(-s_1 - s_2)) \otimes (\det s^*_1 \mathcal{G})^{-1} \otimes (\det s^*_2 \mathcal{G})^{-1} .
\]

(3)

From equation (1) it follows that there is a canonical exact sequence of \( \mathcal{O}_{\tilde{C}_0 \times KGL} \)-modules

\[
0 \to h_* \mathcal{G}(-s_1 - s_2) \to \tilde{\mathcal{E}}' \to (p_1)_* p^*_1 \tilde{\mathcal{E}}' \oplus (p_2)_* p^*_2 \tilde{\mathcal{E}}' \to 0 .
\]

From this sequence, the fact that \( R^1 h_* \mathcal{G}(-s_1 - s_2) = 0 \) (cf. \([Kn]\), Cor 1.5) and the equalities

\[
p_1^* \tilde{\mathcal{E}}' = f^* E \quad \text{and} \quad p_2^* \tilde{\mathcal{E}}' = f^* F
\]

it follows that we have canonically:

\[
\det(R\rho_* \mathcal{G}(-s_1 - s_2)) = \det(R\tilde{\pi}'_* \tilde{\mathcal{E}}') \otimes f^* \det E \otimes f^* \det F .
\]

(4)

Putting together the identifications (2)–(4) and making use of the fact that \( \det(R\tilde{\pi}'_* \tilde{\mathcal{E}}') = f^* \tilde{\Theta} \) and that the \( \mathcal{O}_{KGL} \)-modules \( s^* \mathcal{E}', s^*_1 \mathcal{G}, s^*_2 \mathcal{G}, E_n, F_n \) are all canonically isomorphic, we finally get

\[
\nu^* \Theta = \det(R\pi'_* \mathcal{E}') = f^* \tilde{\Theta} \otimes (\det E_n)^{-1} \otimes f^* (\det E) \otimes f^* (\det F) .
\]

The proposition now follows from \([4,1]\) \qed
Now we will apply the results from \S4. Notation is as in \S3 and \S4 with $S$ replaced by the stack $VB$ and vector bundles $E = p_1^*\mathcal{E}$, $F = p_2^*\mathcal{E}$, on $VB$ as above. Furthermore I will write $PB$ instead of $Fl$ for the product $Fl(E) \times_{VB} Fl(F)$. The letters $PB$ stand of course for parabolic bundles. Let $f_{PB} : PB \to VB$ be the canonical projection and let

$$\begin{align*}
\nu_{I,J} &:= \nu \circ i_{I,J} : O_{I,J} \to GVB \\
f_{I,J} &:= f \circ i_{I,J} : O_{I,J} \to VB
\end{align*}$$

**Proposition 7.4.** Let $I, J \subseteq [0, n - 1]$ with $\min(I) + \min(J) \geq n$. Then the following holds:

1. We have a canonical isomorphism

$$H^0(O_{I,J}, \nu_{I,J}^*(\Theta^\kappa)) = \bigoplus_{(a,b) \in A_{I,J}(\Delta^\kappa)} H^0(PB, \Theta_{PB}^*(a,b)),$$

where $\Theta_{PB}^*(a,b) := f_{PB}^*(\Theta^\kappa) \otimes O(a,b)$ and where $A_{I,J}(\Delta^\kappa)$ is the set of all $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ with the property that $b_i = \kappa - a_{n-i+1}$ for $i \in [1, n]$ and

$$0 = a_1 = \cdots = a_{n-j_1} \leq a_{n-j_1+1} \leq \cdots \leq a_i \leq a_{i+1} = \cdots = a_n = \kappa,$$

where $i_1 := \min(I)$ and $j_1 := \min(J)$.

2. We have $A_{I,J}(\Delta^\kappa) \subseteq A(\Delta^\kappa)$ and the following diagram commutes:

$$\begin{array}{ccc}
H^0(KGL, \nu^*(\Theta^\kappa)) & \xrightarrow{i_{I,J}^*} & H^0(O_{I,J}, \nu_{I,J}^*(\Theta^\kappa)) \\
\bigoplus_{(a,b) \in A(\Delta^\kappa)} H^0(PB, \Theta_{PB}^*(a,b)) & \longrightarrow & \bigoplus_{(a,b) \in A_{I,J}(\Delta^\kappa)} H^0(PB, \Theta_{PB}^*(a,b))
\end{array}$$

Here, $i_{I,J}^*$ is the restriction morphism induced by the inclusion $i_{I,J} : O_{I,J} \hookrightarrow KGL$ and the lower horizontal arrow is the projection map induced by the inclusion $A_{I,J}(\Delta^\kappa) \subseteq A(\Delta^\kappa)$.

**Proof.** By Theorem 4.3 we have a canonical decomposition

$$(f_{I,J})_*\Delta^\kappa = \bigoplus_{(a,b) \in A_{I,J}(\Delta^\kappa)} (f_{PB})_* O(a,b).$$

This, together with 7.3 implies the isomorphism stated in the first part of the proposition. The concrete description of the set $A_{I,J}(\Delta^\kappa)$ follows easily from definition 4.2. The second part of the proposition is immediate from 4.3.2. \qed

The main result of this paper is the following

**Theorem 7.5.** There is a canonical isomorphism

$$H^0(GVB, \Theta^\kappa) \cong \bigoplus_{(a,b) \in A'} H^0(PB, \Theta_{PB}^*(a,b)),$$

where $A'$ is the set of all $(a, b) \in \mathbb{Z}^n \times \mathbb{Z}^n$ with $0 \leq a_1 \leq \cdots \leq a_n \leq \kappa - 1$ and $b_i = \kappa - a_{n-i+1}$ for $i \in [1, n]$.

The remaining of this section is devoted to the proof of theorem 7.5.
Definition 7.6. For \( p, q \in [0, n] \) with \( p + q \geq n \) we set
\[
A_{p,q} := \{ (a, b) \in A(\Delta^\kappa) \mid a_i = 0 \text{ for } i \in [1, n-q] \text{ and } a_i = \kappa \text{ for } i \in [p+1, n] \}
\]
\[
A'_{p,q} := \{ (a, b) \in A_{p,q} \mid a_i \leq \kappa - 1 \text{ for } i \in [1, p] \}
\]
\[
V_{p,q} := \bigoplus_{(a,b) \in A_{p,q}} H^0(\text{PB}, \Theta_{\text{PB}}(a, b))
\]
\[
V'_{p,q} := \bigoplus_{(a,b) \in A'_{p,q}} H^0(\text{PB}, \Theta_{\text{PB}}(a, b))
\]

Remark 7.7. 1. Let \( I, J \subseteq [0, n-1] \), \( p := \min(I) \) and \( q := \min(J) \), and assume \( p + q \geq n \). Then by 7.3 we have a canonical isomorphism
\[
V_{p,q} = H^0(\Theta_{\text{PB}}) \quad .
\]

2. For \( p, p', q, q' \in [0, n] \) with \( p \leq p' \), \( q \leq q' \) and \( p + q \geq n \) we have \( A_{p,q} \subseteq A_{p',q'} \). Furthermore we have \( A_{n,n} = A(\Delta^\kappa) \), and \( A'_{n,n} = A' \) is the set which appears in theorem 7.3.

3. Let \( p, q \in [0, n] \) with \( p + q \geq n \). Then \( A_{p,q} \) is the disjoint union of the sets \( A'_{i,q} \), where \( i \) runs through \([n-q, p]\). Therefore we have \( V_{p,q} = \bigoplus_{i=n-q} V_{i,q} \). It follows that \( V_{p,q} = V'_{p,q} \), if \( p + q = n \) and \( V_{p,q} = V'_{p,q} \oplus V_{p-1,q} \), if \( p + q > n \).

Definition 7.8. 1. Let \( p, p', q, q' \in [0, n] \) with \( p \leq p' \), \( q \leq q' \) and \( p + q \geq n \). Then we denote by
\[
\sigma_{p,q}^{p',q'} : V_{p',q'} \rightarrow V_{p,q} \quad \text{and} \quad \tau_{p,q}^{p',q'} : V_{p',q'} \rightarrow V'_{p,q}
\]
the projection morphisms induced by the inclusions \( A_{p,q} \hookrightarrow A_{p',q'} \) and \( A'_{p,q} \hookrightarrow A'_{p',q'} \) respectively.

2. For \( p, q \in [0, n] \) with \( p + q \geq n \) we denote by
\[
\pi_{p,q} : V_{p,q} \rightarrow V'_{p,q}
\]
the projection induced by the inclusion \( A'_{p,q} \hookrightarrow A_{p,q} \).

3. Let \( p \in [0, n-1] \). We denote by
\[
\beta_{p,n}^{p,n} : V_{p,n} = H^0(\text{Z}_p, \nu^{\kappa}_{\text{Z}_p}) \sim \rightarrow H^0(Y_p, \nu_{Y_p}^\kappa) = V_{n,p}
\]
the isomorphism induced on cohomology by \( \beta_p : Y_p \sim \rightarrow \text{Z}_p \) (cf. 7.1) via the identification 7.1. For convenience, we define \( \beta_{p,n}^{n,n} \) to be the identity morphism on \( V_{n,n} \).

Remark 7.9. 1. By 7.4 the morphisms \( i_{\text{Z}_p}^* \), \( i_{\text{Y}_p}^* \), and \( \beta_p^\bullet \) from 7.1 are equal (via the identification 7.1) to the morphisms \( \sigma_{p,n}^{n,n} \), \( \sigma_{n,p}^{n,n} \) and \( \beta_{n,p}^{n,n} \) respectively. Thus by 7.4 the space \( H^0(\text{GVB}, \Theta^\kappa) \) can be identified with the subspace of all \( \theta \in V_{n,n} \) which have the property that
\[
\beta_{n,p}^{n,n} \sigma_{p,n}^{n,n} \theta = \sigma_{n,p}^{n,n} \theta
\]
for every \( p \in [0, n-1] \).

2. The following equalities are trivially verified:
\[
\sigma_{p,q}^{p',q'} \circ \sigma_{p',q'}^{p''} = \sigma_{p,q}^{p'',p'} \quad \text{and} \quad \pi_{p,q} \circ \sigma_{p,q}^{p',q'} = \tau_{p,q} \circ \pi_{p,q}^{p',q'}. 
\]
Lemma 7.10. Let $p \in [0, n]$. Then the isomorphism
\[ \beta_{p,n}^p : V_{p,n} \xrightarrow{\sim} V_{n,p} \]
maps the subspace $V_{p,n}'$ onto the subspace $V_{n,p}'$.

Proof. For $p = n$ the assertion is trivial, so let $p \in [0, n - 1]$. Assume for a moment that there exist line bundles
\[ \mathcal{M} = \bigotimes_{i=0}^{n-1} M_i^{m_i} \quad \text{and} \quad \mathcal{M}' = \bigotimes_{i=0}^{n-1} M_i'^{m_i'} \]
on KGL with the following properties:

1. The $m_i$ and $m_i'$ are non-negative and $m_i = 0$ for $i \in [p, n - 1]$.
2. $\beta_p^* i_{Z_p}^* (\mathcal{M}, \mu_0^{m_0} \otimes \ldots \otimes \mu_{n-1}^{m_{n-1}}) = i_{Y_p}^* (\mathcal{M}', \mu_0^{m_0'} \otimes \ldots \otimes \mu_{n-1}^{m_{n-1}'})$
3. $A'_{p,n} = A_{(p), \emptyset}(\Delta^\kappa \otimes \mathcal{M}^{-1})$
   $A'_{n,p} = A_{\emptyset, (p)}(\Delta^\kappa \otimes (\mathcal{M}')^{-1})$

By property 1 multiplication with $\mu_0^{m_0} \otimes \ldots \otimes \mu_{n-1}^{m_{n-1}}$ induces an injection
\[ \nu_{Z_p}^* \Theta^\kappa \otimes i_{Z_p}^* \mathcal{M}^{-1} \hookrightarrow \nu_{Y_p}^* \Theta^\kappa \]
It follows from property 2 that application of the functor $\beta_p^*$ to this injection yields the injection
\[ \nu_{Y_p}^* \Theta^\kappa \otimes i_{Y_p}^* (\mathcal{M}')^{-1} \hookrightarrow \nu_{Y_p}^* \Theta^\kappa \]
induced by multiplication with $\mu_0^{m_0'} \otimes \ldots \otimes \mu_{n-1}^{m_{n-1}'}$. Therefore we have a commutative diagram:

\[ \begin{array}{ccc}
H^0(Z_p, \nu_{Z_p}^* \Theta^\kappa) & \xrightarrow{\simeq} & H^0(Y_p, \nu_{Y_p}^* \Theta^\kappa) \\
\downarrow & & \downarrow \\
H^0(Z_p, \nu_{Z_p}^* \Theta^\kappa \otimes i_{Z_p}^* \mathcal{M}^{-1}) & \xrightarrow{\simeq} & H^0(Y_p, \nu_{Y_p}^* \Theta^\kappa \otimes i_{Y_p}^* (\mathcal{M}')^{-1})
\end{array} \]

where the horizontal arrows are induced by the isomorphism $\beta_p$. By property 3 and this diagram may be identified with a diagram of the form

\[ \begin{array}{ccc}
V_{p,n} & \xrightarrow{\beta_{p,n}^p} & V_{n,p} \\
\downarrow & & \downarrow \\
V_{p,n}' & \xrightarrow{\sim} & V_{n,p}'
\end{array} \]

where the vertical arrows are induced by the inclusions $A'_{p,n} \hookrightarrow A_{p,n}$ and $A'_{n,p} \hookrightarrow A_{n,p}$. This clearly implies the lemma.

Thus it remains only to prove the existence of $\mathcal{M}$ and $\mathcal{M}'$. For this, let $c, c' \in \mathbb{Z}^n$ be defined by
\[ c_i := \begin{cases} 
\kappa - 1 & \text{if } i \in [1, p] \\
\kappa & \text{if } i \in [p + 1, n]
\end{cases} \quad \text{and} \quad c'_i := \begin{cases} 
0 & \text{if } i \in [1, n-p] \\
\kappa - 1 & \text{if } i \in [n-p+1, n]
\end{cases} \]
and let
\[ m_i := \kappa(n - i) - \sum_{j=i+1}^{n} c_j \quad \text{and} \quad m'_i := \kappa(n - i) - \sum_{j=i+1}^{n} c'_j \]
for \( i \in \{0, n\} \). A simple calculation shows that the line bundles \( \mathcal{M} \) and \( \mathcal{M}' \) formed with this choice of \( m_i \) and \( m'_i \) have the properties 1 and 3. Property 2 follows easily from proposition 6.13.

After these preparations we now come to the proof of theorem 7.31. I claim that the composite morphism
\[ H^0(\text{GVB}, \Theta^\kappa) \to H^0(\text{KGL}, \nu^*\Theta^\kappa) = V_{n,n} \xrightarrow{\pi_{n,n}} V'_{n,n} = \bigoplus_{(a,b) \in A'} H^0(\text{PB}, \Theta^\kappa_{\text{FB}}(a, b)) \]
is an isomorphism.

To prove injectivity, let \( \theta \in V_{n,n} \) be an element in the kernel of \( \pi_{n,n} \), which satisfies the condition stated in 7.9.1. We have to show that \( \theta = 0 \).

Since
\[ V_{n,n} = \bigoplus_{p=0}^{n} V'_{p,n} \]
it suffices to show that \( \pi_{p,n} \sigma_{p,n} \theta = 0 \) for all \( p \in \{0, n\} \). We do this by induction on \( p \).

For \( p = 0 \) we have \( \beta_{0,n,0} \sigma_{0,n} \theta = \sigma_{0,n} \theta = \pi_{0,n} \sigma_{0,n} \theta = \tau_{n,n} \pi_{n,n} \theta = 0 \), which implies \( \pi_{0,n} \sigma_{0,n} \theta = \sigma_{0,n} \theta = 0 \). Now let \( p = 1 \) and assume \( \pi_{q,n} \sigma_{q,n} \theta = 0 \) for all \( q \in \{0, p-1\} \). This implies that \( \sigma_{n,n} \theta \) is in fact contained in \( V'_{p,n} \). Therefore by 7.10 we have that \( \beta_{p,n} \sigma_{p,n} \theta = \sigma_{n,n} \theta \) is contained in \( V'_{n,p} \). But this implies \( \beta_{p,n} \sigma_{p,n} \theta = \pi_{n,p} \sigma_{n,n} \theta = \tau_{n,n} \pi_{n,n} \theta = 0 \) and thus \( \pi_{p,n} \sigma_{p,n} \theta = \sigma_{p,n} \theta = 0 \).

It remains to prove surjectivity. Let \( \theta' \) be an element of \( V'_{n,n} \). For \( p \in \{0, n\} \) let \( \theta'_p \in V'_{p,n} \) be defined inductively by the property
\[ \beta_{p,n} \theta'_p = \tau_{n,p} \theta' - \sum_{q=0}^{p-1} \pi_{n,p} \beta_{p,n} \theta'_q \]
and let \( \theta_p := \sum_{q=0}^{p} \theta'_q \) and \( \theta := \theta_n \). Clearly, we have \( \pi_{n,n} \theta = \theta'_n = \theta' \). Therefore it suffices to show that \( \theta \) is an element of \( H^0(\text{GVB}, \Theta^\kappa) \).

By 7.9.1 this amounts to proving that \( \beta_{n,q} \theta_q = \sigma_{n,q} \theta \) for \( q \in \{0, n\} \). Since we have
\[ V_{n,q} = \bigoplus_{p=n-q}^{n} V'_{p,q} \]
this is equivalent to the statement that
\[ \pi_{p,q} \sigma_{p,q} \beta_{n,q} \theta_q = \pi_{p,q} \sigma_{p,q} \theta \]
for all \( p, q \in \{0, n\} \) with \( p + q \geq n \).

But this equality is clear, since
\[
\begin{align*}
\pi_{p,q} \sigma_{p,q} \beta_{n,q} \theta_q &= \tau_{p,q} \pi_{n,q} \beta_{n,q} \theta_q + \theta_{q-1} = \\
&= \tau_{p,q} \pi_{n,q} \beta_{n,q} \theta_q + \beta_{n,q} \theta_{q-1} = \\
&= \tau_{p,q} \theta' = \pi_{p,q} \sigma_{p,q} \theta
\end{align*}
\]
Remark 7.11. The decomposition given in Theorem 7.5 is not symmetric with respect to the two points \( p_1 \) and \( p_2 \). Indeed, interchanging the role of the two points means interchanging \( a \) and \( b \) in \( (a, b) \), but then the set \( A' \) changes to the set \( A'' \) consisting of all \( (b, a) \in \mathbb{Z} \times \mathbb{Z} \) with \( 1 \leq b_1 \leq \cdots \leq b_n \leq \kappa \) and \( a_i = \kappa - b_{i-1} + 1 \). At first sight this seems strange, since the decomposition should certainly not depend on how we numerate the points \( p_1 \) and \( p_2 \). The answer to this riddle is that in our proof of Theorem 7.5 we have made a choice between two possibilities. In fact, one can equally well show that the composite morphism

\[
H^0(GVB, \Theta^\kappa) \otimes_{(a,b) \in A(\Delta^\kappa)} H^0(PB, \Theta^\kappa_{PB}(a, b)) \otimes_{(a,b) \in A''} H^0(PB, \Theta^\kappa_{PB}(a, b))
\]

is an isomorphism, where the last arrow is the projection morphism induced by the inclusion \( A'' \subset A(\Delta^\kappa) \).

Remark 7.12. Let \( \mathcal{X} \) be an algebraic \( k \)-stack and let \( \mathcal{F} \) be a sheaf on the smooth-étale site of \( \mathcal{X} \) (cf. [LM] §12). By definition (loc. cit. (12.5.3)), the set of global sections of \( \mathcal{F} \) is the set of all families \( s_{(U,u)} \) of sections of \( \mathcal{F} \) over \( (U, u) \in \text{ob} \text{Lis-ét}(\mathcal{X}) \) such that \( \text{res}_{\varphi s_{(V,v)}} = s_{(U,u)} \) for all arrows \( \varphi : (U, u) \to (V, v) \) in \( \text{Lis-ét}(\mathcal{X}) \). Now assume in particular that \( \mathcal{F} \) is an \( \mathcal{O}_\mathcal{X} \)-module and that for each object \( (U, u) \in \text{ob} \text{Lis-ét}(\mathcal{X}) \) there is a homomorphism \( k^x \to \text{Aut}_{\text{Lis-ét}(\mathcal{X})}(U, u), a \mapsto \varphi_a \). Assume furthermore that there is a number \( \chi \in \mathbb{Z} \) such that for each \( (U, u) \) and \( a \in k^x \) the morphism \( \text{res}_{\varphi_a} : \mathcal{F}(U, u) \to \mathcal{F}(U, u) \) is multiplication with the \( \chi \)-th power of \( a \). Then it is clear that unless \( \chi = 0 \), the only global section of \( \mathcal{F} \) is the zero section.

The stack \( \text{GVB} \) is the disjoint union of open closed substacks \( \text{GVB}_d \) parametrizing Gieseker vector bundles of degree \( d \) \((d \in \mathbb{Z})\). By the above consideration it follows that \( H^0(\text{GVB}_d, \Theta^\kappa) \) vanishes unless the Euler characteristic

\[
\chi = d + n(1 - g)
\]

of a bundle of rank \( n \) and degree \( d \) on a curve of genus \( g \) is zero, i.e. unless \( d = n(g - 1) \). Therefore we have \( H^0(\text{GVB}, \Theta^\kappa) = H^0(\text{GVB}_{n(g - 1)}, \Theta^\kappa) \). A similar remark applies to the groups \( H^0(PB, \Theta^\kappa_{PB}(a, b)) \) appearing on the right hand side of the isomorphism in 7.5.

8. Degeneration

Let \( B \) be the spectrum of a discrete valuation ring and let \( C \to B \) be a projective relative curve of genus \( g \geq 1 \) over \( B \), whose generic fiber \( C_\eta \) is smooth and whose special fiber \( C_0 \) is irreducible with one ordinary double point \( p \). In [K2] I have shown that there is a flat algebraic moduli stack \( \text{GVB}(C/B) \) over \( B \) whose generic fiber \( \text{VB}(C_\eta) \) parametrizes vector bundles on \( C_\eta \) and whose special fiber \( \text{GVB}(C_0) \) parametrizes Gieseker vector bundles of rank \( n \) on \( C_0 \). Let

\[
(\pi_{C/B} : \mathcal{C}_{C/B} \to \text{GVB}(C/B), \mathcal{E}_{C/B})
\]

be the universal Gieseker vector bundle over \( \text{GVB}(C/B) \) and let

\[
\Theta(C/B) := \det R(\pi_{C/B})_* \mathcal{E}_{C/B}
\]

be the determinant line bundle on \( \text{GVB}(C/B) \).

The rest of this section is dedicated to the proof of the following result which shows that the model \( \text{GVB}(C/B) \) of \( \text{VB}(C_\eta) \) defines the “correct selection rules” for generalized theta functions.
Theorem 8.1. The $B$-module $H^0(GVB(C/B), \Theta(C/B)^\kappa)$ is locally free of finite rank.

It is clear that there is a decomposition into a disjoint union:

$$GVB(C/B) = \bigsqcup_{d \in \mathbb{Z}} GVB_d(C/B),$$

where $GVB_d(C/B)$ is the open substack of $GVB(C/B)$ parametrizing vector bundles of degree $d$. As in [7,12] it follows that there are no non-vanishing sections of $\Theta(C/B)^\kappa$ over $GVB_d(C/B)$ unless $d = n(g-1)$. From now on we let $d := n(g-1)$ and we will restrict our attention to the open substack $GVB_d(C/B)$. Its closed and special fiber over $B$ will be denoted by $GVB_d(C_0)$ and $VB_d(C_\eta)$ and the restriction of $\Theta(C/B)$ to these by $\Theta(C_0)$ and $\Theta(C_\eta)$ respectively.

We have to show that for $\kappa \geq 1$ the vector spaces $H^0(GVB_d(C_0), \Theta(C_0)^\kappa)$ and $H^0(VB_d(C_\eta), \Theta(C_\eta)^\kappa)$ are finite dimensional and that the equality

$$\dim H^0(GVB_d(C_0), \Theta(C_0)^\kappa) = \dim H^0(VB_d(C_\eta), \Theta(C_\eta)^\kappa)$$

(1)

holds. Since the stacks $GVB_d(C_0)$ and $VB_d(C_\eta)$ are not separated and not of finite type over their respective base fields, we cannot apply general results from [LM] §15 or [F3] to prove finite dimensionality of cohomology. By the same reason we cannot argue by cohomological flatness to show that the dimensions coincide, even if we could assume that the relevant theorems in [EGA] III hold in the context of Artin stacks. Our strategy therefore is to compute the dimensions of $H^0(GVB_d(C_0), \Theta(C_0)^\kappa)$ and of $H^0(VB_d(C_\eta), \Theta(C_\eta)^\kappa)$ individually by relating them to the dimensions of spaces of generalized theta functions for (parabolic) $SL_n$-bundles and using the Verlinde formula.

Definition 8.2. (i) We fix once and for all a line bundle $L_\eta$ of degree $d$ on $C_\eta$. We denote by $SVB(C_\eta)$ the closed substack of $VB_d(C_\eta)$ which parametrizes vector bundles whose determinant is isomorphic to $L_\eta$. We write $\Theta_{SVB}(C_\eta)$ for the restriction of $\Theta(C_\eta)$ to $SVB(C_\eta)$.

(ii) Also we fix once and for all a line bundle $\tilde{L}_0$ of degree $d$ on $C_0$. As in the previous paragraphs we let $VB$ denote the moduli stack of rank $n$ vector bundles on $C_0$. Let $VB_d = VB_d(C_0)$ be the open substack of $VB$ which parametrizes bundles of degree $d$ and let $SVB = SVB(C_0)$ be the closed substack of $VB$ which parametrizes vector bundles whose determinant is isomorphic to $\tilde{L}_0$. Recall from [7] that $PB$ denotes the stack of vector bundles on $C_0$ together with full flags in the fibers over the points $p_1$ and $p_2$. We define

$$PB_d := PB \times_{VB} VB_d, \quad SPB := PB \times_{VB} SVB$$

and for $(a,b) \in \mathbb{Z}^n \times \mathbb{Z}^n$ and $\kappa \in \mathbb{Z}$ we denote by $\Theta_{PB}, \mathcal{O}_{SPB}(a,b)$ and $\Theta_{SPB}(a,b)$ the restriction of the line bundles $\Theta_{PB} = f_{PB}^* \tilde{\Theta}$, $\mathcal{O}_{PB}(a,b)$ and $\Theta_{PB}(a,b) = f_{PB}^* \tilde{\Theta} \otimes \mathcal{O}_{PB}(a,b)$ respectively to the closed substack $SPB$ of $PB$.

Proposition 8.3. (i) The dimensions of the vector spaces $H^0(SVB(C_\eta), \Theta_{SVB}(C_\eta)^\kappa)$ and $H^0(VB_d(C_\eta), \Theta(C_\eta)^\kappa)$ are finite and we have

$$\dim H^0(VB_d(C_\eta), \Theta(C_\eta)^\kappa) = \binom{k}{n}^g \cdot \dim H^0(SVB(C_\eta), \Theta_{SVB}(C_\eta)^\kappa).$$
(ii) Let \((a, b) \in \mathbb{Z}^n \times \mathbb{Z}^n\) and let \((a', b') \in \mathbb{Z}^n \times \mathbb{Z}^n\) be defined by \(a_i' := a_i - a_1, b_i' := b_i - b_1\). Then the vector spaces \(H^0(\text{PB}_d, \Theta^\kappa(a, b))\) and \(H^0(\text{SPB}, \Theta^\kappa_{\text{SPB}}(a', b'))\) are finite dimensional and we have
\[
\dim H^0(\text{PB}_d, \Theta^\kappa(a, b)) = \left(\frac{\kappa}{n}\right)^{g-1} \cdot \dim H^0(\text{SPB}, \Theta^\kappa_{\text{SPB}}(a', b')) .
\]

Proof. The finiteness of \(H^0(\text{SVB}(C)_\eta, \Theta^\kappa_{\text{SVB}(C)_\eta})\) and \(H^0(\text{SPB}, \Theta^\kappa_{\text{SPB}}(a', b'))\) follows from the interpretation of these vector spaces as spaces of conformal blocks (cf. [BL] and [P]).

The analogous equality to \((i)\) in the context of coarse moduli spaces of semi-stable bundles has been proved by Donagi and Tu in [DT]. Since their proof works almost identically in our situation, we will concentrate on the second equation, of which \((i)\) can anyway be considered a special case (namely the case \(a = b = 0\)).

For the proof of \((ii)\) consider the following Cartesian diagram

\[
\begin{array}{ccc}
\text{SPB} \times \mathcal{J}_0 & \xrightarrow{\sigma} & \text{PB}_d \\
\downarrow f_{\text{SPB} \times \text{id}} & & \downarrow f_{\text{PB}} \\
\text{SVB} \times \mathcal{J}_0 & \xrightarrow{\tau} & \text{VB}_d \\
\downarrow \text{pr} & & \downarrow \text{det} \\
\mathcal{J}_0 & \xrightarrow{\rho} & \mathcal{J}_d \\
\downarrow \text{L} \rightarrow \text{L}^\times \otimes \text{L}_0 & &
\end{array}
\]

where \(\mathcal{J}_0\) and \(\mathcal{J}_d\) are the moduli stacks of line bundles on \(\tilde{C}_0\) of degree 0 and \(d\) respectively. Similarly as in [DT], the sought-for equality follows from computing the space of global sections of the line bundle \(\sigma^* \Theta^\kappa_{\text{PB}}(a, b)\) on \(\text{SPB} \times \mathcal{J}_0\) in two different ways.

**First way:** Let \(\mathcal{E}_{\text{PB}}, \mathcal{E}_{\mathcal{J}_0}\), etc. be the universal vector bundle on \(\tilde{C}_0 \times \text{PB}, \tilde{C}_0 \times \mathcal{J}_0\), etc.. Let \(\pi_{\text{PB}}, \pi_{\mathcal{J}_0}\), etc. be the projection from \(\tilde{C}_0 \times \text{PB}, \tilde{C}_0 \times \mathcal{J}_0\) etc. to the second factor and denote by \(p_1, p_2\) the sections of \(\pi_{\text{PB}}, \pi_{\mathcal{J}_0}\) etc. induced by the points \(p_1, p_2 \in \tilde{C}_0\). We have
\[
\Theta^\kappa_{\text{PB}}(a, b) = (\det p_2^* \mathcal{E}_{\text{PB}} \otimes \det R\pi_{\text{PB}*} \mathcal{E}_{\text{PB}})^\kappa \\
\otimes (\det p_2^* \mathcal{E}_{\text{PB}})^{a_1+b_1-\kappa} \otimes \mathcal{O}_{\text{PB}}(a', b') \\
\otimes (\det p_1^* \mathcal{E}_{\text{PB}} \otimes (\det p_2^* \mathcal{E}_{\text{PB}})^{-1})^{a_1}.
\]

Since \(\det p_2^* \mathcal{E}_{\text{PB}} \otimes \det R\pi_{\text{PB}*} \mathcal{E}_{\text{PB}} = f_{\text{PB}}^* \det R\tilde{\pi}_* \tilde{\mathcal{E}}(-p_2)\), it follows as in [DT], Cor 6 that we have
\[
\sigma^*(\det p_2^* \mathcal{E}_{\text{PB}} \otimes \det R\pi_{\text{PB}*} \mathcal{E}_{\text{PB}}) = \Theta^\kappa_{\text{SVB}} \otimes \Theta^\kappa_{\mathcal{J}_0} \\
\otimes (\det p_2^* \mathcal{E}_{\text{PB}})^{a_1+b_1-\kappa} \otimes \mathcal{O}_{\text{PB}}(a', b') \\
\otimes (\det p_1^* \mathcal{E}_{\text{PB}} \otimes (\det p_2^* \mathcal{E}_{\text{PB}})^{-1})^{a_1}.
\]

and
\[
\sigma^*(\det p_1^* \mathcal{E}_{\text{PB}} \otimes (\det p_2^* \mathcal{E}_{\text{PB}})^{-1}) = \mathcal{O}_{\text{SPB}} \otimes (p_1^* \mathcal{E}_{\mathcal{J}_0} \otimes p_2^* \mathcal{E}_{\mathcal{J}_0}^{-1})^{a_1}.
\]

Summarizing, we have
\[
\sigma^* \Theta^\kappa_{\text{PB}}(a, b) = \Theta^\kappa_{\text{SPB}}(a', b') \otimes (\Theta^\kappa_{\mathcal{J}_0} \otimes (p_1^* \mathcal{E}_{\mathcal{J}_0} \otimes p_2^* \mathcal{E}_{\mathcal{J}_0}^{-1})^{a_1}) .
\]
Using the techniques from [DT] §5 we see that
\[ \Theta_{N}^{m} \otimes (p_{1}^{*}E_{J_{0}} \otimes p_{2}^{*}E_{J_{0}}^{-1})^{\sum a_{i}} = \tau_{M}^{*} \Theta_{N}^{m} \]
where \( M \in \text{Pic}^{0}(\tilde{C}_{0}) \) is a \( \kappa n \)-th root of the line bundle \( \mathcal{O}_{\tilde{C}_{0}}((\sum_{i=1}^{\kappa n} a_{i})p_{2} - p_{1}) \) and \( \tau_{M} : J_{0} \to J_{0} \) is the translation by \( M \). Thus we have
\[ \sigma^{*} \Theta_{\text{PB}}^{e}(a, b) = \Theta_{\text{PB}}^{e}(a', b') \otimes \tau_{M}^{*} \Theta_{N}^{m} \]
and consequently
\[ H^{0}(\text{SPB} \times J_{0}, \sigma^{*} \Theta_{\text{PB}}^{e}(a, b)) \cong H^{0}(\text{SPB}, \Theta_{\text{SPB}}^{e}(a', b')) \otimes H^{0}(J_{0}, \Theta_{N}^{m}) \quad (*) \]

**Second way:** The morphism \( \sigma \) is a Galois covering with Galois group \( G \) the subgroup of \( n \)-torsion points of \( \text{Pic}^{0}(\tilde{C}_{0}) \). As in [DT], Prop. 4. and Lemma 7. we have
\[ \sigma_{*} \mathcal{O}_{\text{SPB} \times J_{0}} = \bigoplus_{\lambda \in \hat{G}} L_{\lambda} \]
where \( \hat{G} \) is the character group of \( G \) and for each \( \lambda \in \hat{G} \) we have
\[ L_{\lambda} = (\det \circ J_{\text{PB}})^{*} N_{\lambda} \]
for some line bundle \( N_{\lambda} \) of degree zero on \( J_{d} \). By the projection formula we have
\[ \sigma_{*} \sigma^{*} \Theta_{\text{PB}}^{e}(a, b) = \bigoplus_{\lambda \in \hat{G}} \Theta_{\text{PB}}^{e}(a, b) \otimes L_{\lambda} \]
As in [DT] §5 it follows that
\[ \Theta_{\text{PB}}^{e}(a, b) \otimes L_{\lambda} = \tau_{M_{\lambda}}^{*} \Theta_{\text{PB}}^{e}(a, b) \]
for some line bundle \( M_{\lambda} \in \text{Pic}^{0}(\tilde{C}_{0}) \), where \( \tau_{M_{\lambda}} : \text{PB} \to \text{PB} \) is the isomorphism which sends a parabolic bundle \( E \) to \( E \otimes M_{\lambda} \). Therefore we have
\[ H^{0}(\text{SPB} \times J_{0}, \sigma^{*} \Theta_{\text{PB}}^{e}(a, b)) = H^{0}(\text{PB}, \sigma_{*} \sigma^{*} \Theta_{\text{PB}}^{e}(a, b)) \cong \bigoplus_{\lambda \in \hat{G}} H^{0}(\text{PB}, \Theta_{\text{PB}}^{e}(a, b)) \quad (**) \]

**Conclusion:** The sought-for equation follows from (*) and (**) together with the fact that the group \( \hat{G} \) is of order \( n^{2(g-1)} \) and the fact that we have
\[ \dim H^{0}(J_{0}, \Theta_{N}^{m}) = (\kappa n)^{g-1} \]
This last equality is well known for theta functions on the Jacobian variety \( J_{0} \), but since we are dealing here with the stack \( J_{0} \), it requires some further justification. For this let \( K \) be the open subscheme of a Quot-scheme which parametrizes invertible quotients \( L \) of \( \mathcal{O}_{\tilde{C}_{0}}((2-2g)p_{1})^{g-1} \) such that \( H^{1}(\tilde{C}_{0}, L((2g-2)p_{1})) = 0 \) and such that the induced morphism \( k^{g-1} \to H^{0}(\tilde{C}_{0}, L((2g-2)p_{1})) \) is an isomorphism. Let \( E_{K} \) be the universal quotient bundle on \( \tilde{C}_{0} \times K \) and let \( K' \) be the complement of the zero section of (the total space of) the line bundle \( p_{1}^{*}E_{K} \) on \( K \). Then \( GL_{g-1} \) operates in an obvious way on \( K \) and \( K' \) such that the center \( G_{m} \subset GL_{g-1} \) operates trivially on \( K \) and the projection \( K' \to K \) is a \( G_{m} \)-torsor. We have
\[ J_{0} = [K/GL_{g-1}] \quad \text{and} \quad J_{0} = [K'/GL_{g-1}] \]
Since at each point of \( K \) the group \( G_m \) operates trivially on the fiber of the line bundle \( \det R\pi_{K*}(\mathcal{E}_K \otimes N) \) we have
\[
H^0(J_0, \Theta_K^{\kappa n}) = H^0(K, (\det R\pi_{K*}(\mathcal{E}_K \otimes N))^{\kappa n})^{GL_{g-1}} = H^0(J_0, \Theta_N^{\kappa n}) ,
\]
where the superscript \( GL_{g-1} \) means taking invariants under this group. \( \square \)

Since by Theorem 7.5 we have
\[
\dim H^0(GVB_d(C_0), \Theta(C_0)^\kappa) = \sum_{(a,b) \in A'} \dim H^0(PB_d, \Theta^\kappa(a,b))
\]
it follows from 8.3 that the vector space \( H^0(GVB_d(C_0), \Theta(C_0)^\kappa) \) is finite and that the equality (1) is equivalent to the equality
\[
\sum_{(a',b') \in SA'} (\kappa - a'_n) \cdot \dim H^0(SP, \Theta_{SP}(a', b')) = \frac{\kappa}{n} \cdot \dim H^0(SVB(C_\eta), \Theta_{SVB}(C_\eta)^\kappa) ,
\]
where \( SA' \) is the set of all \((a',b') \in \mathbb{Z}^n \times \mathbb{Z}^n\) with the property that \( 0 = a'_1 \leq a'_2 \leq \cdots \leq a'_n \leq \kappa \) and \( b'_i = a'_n - a'_{n-i+1} \).

It is well known that the dimensions of the vector spaces \( H^0(SP, \Theta_{SP}(a', b')) \) and \( H^0(SVB(C_\eta), \Theta_{SVB}(C_\eta)^\kappa) \) are given by the Verlinde formula. To write down the formulas explicitly, we need to introduce some notation.

Let
\[
P = \left( \bigoplus_{i=1}^n \mathbb{Z} \epsilon_i \right) / \left( \sum_{i=1}^n \epsilon_i \right)
\]
be the weight lattice of \( \mathfrak{sl}_n \). Let \(( \ | \ ) : P \times P \to \mathbb{Z}[1/n] \) be the normalized Killing form defined by
\[
(\epsilon_i | \epsilon_j) := \delta_{i,j} - \frac{1}{n} .
\]
Let
\[
R_+ := \{ \epsilon_i - \epsilon_j \mid 1 \leq i < j \leq n \}
\]
be the set of positive roots of \( \mathfrak{sl}_n \). Let
\[
\theta := \epsilon_1 - \epsilon_n ,
\]
\[
\rho := \sum_{i=1}^n (n-i) \epsilon_i
\]
the highest root and the half sum of all positive roots respectively. Let \( P_+ := \{ \lambda \in P \mid (\alpha | \lambda) \geq 0 \text{ for all } \alpha \in R_+ \} \) be the set of dominant weights and let
\[
P_\kappa := \{ \lambda \in P_+ \mid (\theta | \lambda) \leq \kappa \} .
\]
Recall that \( P_+ \) parametrizes the finite dimensional representations of \( \mathfrak{sl}_n \). The Weyl group \( W = S_n \) operates on \( P \) by permuting the generators \( \epsilon_i \). Let \( w_0 : j \mapsto n - j + 1 \) be the longest element in \( W \). Then \( \lambda \mapsto \lambda^* := -w_0 \lambda \) is an involution of the set \( P_+ \) (and \( P_\kappa \)), which corresponds to taking the dual representation.
For $\lambda, \mu \in P$ we define the complex number
\[
J(\lambda, \mu) := \sum_{w \in W} \text{sign}(w) \exp \left( \frac{2\pi i}{n+\kappa} (w(\lambda)|\mu) \right).
\]

**Proposition 8.4.**

(i) We have
\[
\dim H^0(SVB(C_\eta), \Theta_{SVB}(C_\eta)^{\kappa}) = (n(n + \kappa)^{n-1})^{g-1} \cdot \sum_{\mu \in \rho + P_\kappa} |J(\rho, \mu)|^{2(1-g)}
\]

(ii) For $(a', b') \in SA'$ we have
\[
\dim H^0(SPB, \Theta_{SPB}^{\kappa}(a', b')) = (n(n + \kappa)^{n-1})^{g-2} \cdot \sum_{\mu \in \rho + P_\kappa} |J(\lambda, \mu)|^{2} |J(\rho, \mu)|^{2(1-g)},
\]
where $\lambda = \rho + \sum_{i=1}^{n} a'_{n-i+1} \epsilon_i$.

**Proof.** This is a well established fact, only the shape of the formulas is maybe a bit unusual. To explain the formulas we will employ the notation of [B]. Let $X$ be a smooth projective curve over $\mathbb{C}$, let $x_1, \ldots, x_m$ be distinct points on $X$ and let $\lambda_1, \ldots, \lambda_m$ be dominant weights of the Lie-algebra $\mathfrak{sl}_n$. To these data there is associated a finite dimensional vector space $V_X((x_1, \ldots, x_m), (\lambda_1, \ldots, \lambda_m))$, called the space of conformal blocks. As shown in [BL] and [P], we have
\[
\dim H^0(SVB(C_\eta), \Theta_{SVB}(C_\eta)^{\kappa}) = \dim V_Y(\emptyset)
\]
\[
\dim H^0(SPB, \Theta_{SPB}^{\kappa}(a', b')) = \dim V_{C_0}((p_1, p_2), (\lambda, \lambda^*))
\]
where $Y$ is some smooth projective curve of genus $g$, $\lambda$ is the dominant weight $\sum_{i=1}^{n} a'_{n-i+1} \epsilon_i$ and $\lambda^* = -w_0 \lambda = \sum_{i=1}^{n} b'_{n-i+1} \epsilon_i$ is its dual.

By [B], Cor. 9.8 we have
\[
\dim V_Y(\emptyset) = (n(n + \kappa)^{n-1})^{g-1} \cdot \sum_{\mu \in P_\kappa} \frac{1}{\Delta(t_{\mu})^{g-1}}
\]
\[
\dim V_{C_0}((p_1, p_2), (\lambda, \lambda^*)) = (n(n + \kappa)^{n-1})^{g-2} \cdot \sum_{\mu \in P_\kappa} \frac{\text{Tr}_{V_{(\lambda, \lambda^*)}}(t_{\mu})}{\Delta(t_{\mu})^{g-2}}
\]
where $\Delta(t_{\mu}) = |J(\rho, \rho + \mu)|^2$ and $\text{Tr}_{V_{(\lambda, \lambda^*)}}(t_{\mu}) = |J(\rho + \lambda, \rho + \mu)|^2 / \Delta(t_{\mu})$. \qed

From Proposition 8.4 it is immediate that for the proof of equality (2) it is sufficient to show the following lemma, which is elementary in its statement but which I could not prove without the help of Don Zagier.

**Lemma 8.5.** For every $\mu \in \rho + P_\kappa$ we have
\[
\sum_{\lambda \in \rho + P_\kappa} \gamma(\lambda) \cdot |J(\lambda, \mu)|^2 = n(n + \kappa)^{n-1}
\]
where for $\gamma(\lambda) := \kappa + n - 1 - \lambda_1$. 

(3)
Proof. (The proof of this lemma is due to Don Zagier). Let \( m := \kappa + n \) and let \( \zeta_m := \exp(2\pi i/m) \). Observe that the mapping \( \lambda = \sum_i \lambda_i \epsilon_i \mapsto \{\lambda_1, \ldots, \lambda_n\} \) is a bijection from the set \( \rho + \rho_\kappa \) to the set \( Q_0 \) of all subsets \( A \) of \( N := \{0, \ldots, m-1\} \) with \( |A| = n \) and \( 0 \in A \). It follows directly from the definitions that if \( \lambda, \mu \in \rho + \rho_\kappa \) are mapped to \( A, B \in Q_0 \), then we have

\[
|J(\lambda, \mu)|^2 = |\Delta_{A,B}|^2 ,
\]

where \( \Delta_{A,B} \) is the determinant of the \( n \times n \)-sub-matrix of the matrix \( M_m = (\zeta_m^{a,b})_{0 \leq a, b \leq m-1} \) corresponding to the rows and columns with indices in the sets \( A \) and \( B \). Thus the assertion of the lemma holds if and only if

\[
\sum_{A \in Q_0} \gamma(A)|\Delta_{A,B}|^2 = (m - n)m^{n-1} \quad (B \in Q) \tag{4}
\]

where \( \gamma(A) = n - \max(A) - 1 \) and \( Q \) is the set of all subsets \( B \subset N \) with \( |B| = n \). (The condition “\( 0 \in B \)” can be omitted since a translation of the set \( B \) just multiplies every determinant \( \Delta_{A,B} \) by a root of unity and does not have any influence on the expression \( |\Delta_{A,B}|^2 \).) We can further optically simplify the formula by replacing \( M_m \) by \( M_m^* = m^{-1/2}M_m \) and thus \( \Delta_{AB} \) by \( \Delta_{AB}^* = m^{-n/2}\Delta_{AB} \) (which is reasonable since \( M_mM_m^* = m \cdot \Id \) and therefore \( M_m^* \) is unitary):

\[
\sum_{A \in Q_0} \gamma(A)|\Delta_{AB}^*|^2 = \frac{m - n}{m} \quad (B \in Q). \tag{5}
\]

We compute first the left hand side of (5) leaving out the factor \( \gamma(A) \) and the condition “\( 0 \in A \)”. The numbers \( \{\Delta_{AB}^*\}_{A,B \in Q} \) are nothing else but the matrix coefficients of the \( n \)-th exterior product \( \wedge^n(M^*_m) \) of the operator represented by the matrix \( M^*_m \). The fact that \( M^*_m \) is unitary remains true also for \( \wedge^n \) of this matrix; consequently we have \( \wedge^n(M^*_m) \wedge^n(M^*_m) = \Id_{\binom{n}{m}} \) or explicitly \( \sum_{A \in Q} \Delta_{AB}^* \delta_{AB'} = \delta_{BB'} \) for \( B, B' \in Q \). The special case \( B = B' \) of this yields

\[
\sum_{A \in Q} |\Delta_{AB}^*|^2 = 1, \quad (B \in Q). \tag{6}
\]

It remains to show that the left hand side of the equation (5) differs by the factor \( (m-n)/m \) from the left hand side of the equation (6). For this we give yet another description of the set \( Q \). We denote by \( \overline{Q} \) the set of all mappings \( \alpha : \mathbb{Z}/n \to \mathbb{Z}/m \) that are “cyclically strictly decreasing” (i.e. they can be lifted to a mapping \( a : \mathbb{Z} \to \mathbb{Z} \) for which \( a(i) > a(i+1) > \cdots > a(i+n) = a(i) - m \) for all \( i \in \mathbb{Z} \) holds). Let \( \overline{Q}_0 \subset \overline{Q} \) be the subset defined by the additional property \( \alpha(0) = 0 \). We have a diagram of mappings between sets as follows:

\[
\begin{array}{c}
\overline{Q}_0 \searrow \downarrow \nearrow \overline{Q}_0 \\
\overline{Q} \downarrow \nearrow Q
\end{array}
\]

where the map \( \overline{Q} \to Q \) maps \( \alpha \) to the subset \( A \) of \( N \) which corresponds to the image of \( \alpha \) via the obvious bijection \( N \sim \mathbb{Z}/m \) and \( \overline{Q}_0 \to Q_0 \) is the restriction of \( \overline{Q} \to Q \) to the subset \( \overline{Q}_0 \). There is an operation of the cyclic group \( C_n = \mathbb{Z}/n \) on \( \overline{Q} \) defined by \( \alpha \mapsto \alpha(\cdot + j) \) and the mapping \( \overline{Q} \to Q \) identifies \( Q \) with the quotient \( \overline{Q}/C_n \). On the other hand, there is also
an operation of the cyclic group $C_m = \mathbb{Z}/m$ on $Q$, given by $\alpha \mapsto \alpha + k$ and $Q_0$ is a system of representatives for this operation. The mapping $Q_0 \to Q_0$ is a bijection.

For $\alpha \in Q$ with lifting $a : \mathbb{Z} \to \mathbb{Z}$ and $i \in \mathbb{Z}$ the number $\gamma_i(\alpha) := a(i) - a(i + 1) - 1$ depends only on $\alpha$ and on $i \mod n$; clearly if $\alpha_0$ is in $Q$ and $A_0$ the corresponding element in $Q$, then $\sum_{\alpha \to A_0} \gamma_0(\alpha) = \sum_{i \mod n} \gamma_i(\alpha_0) = m - n$. We write $\Delta_{\alpha,B}^* := \Delta_{AB}^*$ where $A \in Q$ is the element corresponding to $\alpha$.

Equation (5) can now be seen as follows:

$$\sum_{A \in Q_0} \gamma(A) |\Delta_{AB}^*|^2 = \sum_{\alpha \in Q_0} \gamma_0(\alpha) |\Delta_{\alpha,B}^*|^2 =$$

$$\overset{(*)}{=} \frac{1}{m} \sum_{\alpha \in Q} \gamma_0(\alpha) |\Delta_{\alpha,B}^*|^2 = \frac{1}{m} \sum_{A \in Q} |\Delta_{AB}^*|^2 \sum_{\alpha \to A} \gamma_0(\alpha) =$$

$$= \frac{1}{m} \sum_{A \in Q} (m - n) |\Delta_{AB}^*|^2$$

$$\overset{(6)}{=} \frac{m - n}{m}. $$

Here $(*)$ follows from the fact that $\gamma_0(\alpha + k)|\Delta_{\alpha + k,B}^*| = \gamma_0(\alpha)|\Delta_{\alpha,B}^*|$ for $k \in C_m$. 

□

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