ALTERNATING AUGMENTATIONS OF LINKS

RYAN C. BLAIR

ABSTRACT. We show that one can interweave an unknot into any non-alternating connected projection of a link so that the resulting augmented projection is alternating.

In this paper, we will use the term link to mean a tame link embedded in $S^3$. A link projection $D$ is the image of a link under a regular projection into $S^2$. $D$ is a finite four-valent graph in $S^2$. If $D$ is connected then each complement component is a disk; we call the closure of one of these disks a region. Define a labelling of the edges of $D$ in the manner of Fig. 1. Every edge of $D$ receives two labels, one corresponding to each end. An alternating edge is labelled with a plus and a minus while a non-alternating edge receives two pluses or two minuses. Hence, an alternating projection is one in which every edge is labelled with a plus and a minus. We call a non-alternating edge labelled $++$ a positive non-alternating edge and a non-alternating edge labelled $--$ a negative non-alternating edge. Note that a labelled knot projection is equivalent to the usual knot diagram only with the crossing information stored as edge labels. In the spirit of Adams\[1\], we define an augmented link projection of $D$ to be the union of $D$ with an unlink that projects to disjoint simple closed curves in $S^2$.

We begin with a proposition originally outlined by Thistlethwaite\[2\].

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Figure 1.}
\end{figure}

**Proposition 1.** Any non-alternating connected link projection can be augmented so that it becomes alternating.

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Proof. Choose a region $R$ of the non-alternating projection $D$. Let $n$ be the number of positive non-alternating edges of $R$. Since each vertex of $\partial R$ contributes one + and one - label to the edges of $\partial R$ and each edge receives exactly two labels, then $\partial R$ contains exactly $n$ negative non-alternating edges. Let $\alpha$ be a non-alternating edge of $\partial R$. After choosing an orientation of $\partial R$ and again counting the labels contributed by the vertices, the sign of the next non-alternating edge in the direction of the orientation must be opposite that of $\alpha$. The slogan here is that the sign of the non-alternating edges of $\partial R$ alternate.

With this knowledge, we can construct an augmented alternating projection $G$ from $D$ as follows. Introduce vertices into the interiors of all non-alternating edges of $D$. In every non-alternating region $R$, we join pairs of such points together by edges which are disjoint, lie in the interior of $R$ and join consecutive non-alternating edges as depicted in Fig. 2. We call these new edges augmenting edges. Since the augmenting edges connect non-alternating edges of opposite sign then there is a consistent alternating labelling of the edges of $G$. In particular, the end of an augmenting edge which bisects a positive non-alternating edge of $D$ receives a + label; the end that bisects a negative non-alternating edge receives a – label. Since the augmenting edges never cross themselves, the closure of $G-D$ ($cl(G-D)$) is a disjoint collection of simple closed curves embedded in $S^2$. Because this process converts every non-alternating edge of $D$ into two alternating edges of $G$ by interweaving alternating unknots into $D$, $G$ is an alternating augmented projection of $D$. 

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Figure 3 illustrates an operation on link projections we call a Type I move.

**Lemma 2.** Given an alternating connected projection of a link, a Type I move results in an alternating link projection.

![Type I Move](image)

**Figure 3.** Type I Move

*Proof.* Let $D$ be the link projection and $\alpha$ and $\beta$ be the edges involved in the Type I move. $\alpha$ and $\beta$ are boundary edges of a common region $R$. Since $D$ is alternating, all edges of $R$ are labelled with both a plus and a minus. Given two consecutive edges of $\partial R$ their shared vertex contributes a plus label to one edge and a minus label to the other. Thus, a choice of label for a single edge determines the label of all the edges of $\partial R$. In this way, an alternating label for $\alpha$ determines the label for $\beta$, giving rise to the following two possibilities.

![Possible Alternations](image)

**Figure 4.**

In each case, the type I move preserves alternation. □

Figure 5 illustrates an operation on link projections we call a Type II move.

**Lemma 3.** Given an alternating connected projection of a link, a Type II move (after choosing labels incident to the new vertices) results in an alternating link projection.
Proof. Let $D$ be the link projection and $\alpha$ and $\beta$ be the edges involved in the Type II move. We again use the fact that $\alpha$ and $\beta$ are boundary edges to a common region in $D$ to deduce that an alternating label for $\alpha$ determines the label for $\beta$. Hence, we have only the two following possibilities for the labels of $\alpha$ and $\beta$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure6.png}
\caption{Type II Move}
\end{figure}

In each case, as shown in Figure 6, we may choose a labelling of the ends of the edges incident to the new vertices so that the resulting diagrams are alternating. \hfill $\square$

**Theorem 4.** Given any connected projection of a non-alternating link, we can augment the projection by adding a single unknotted component so that the resulting link projection is alternating.

Proof. Let $D$ be a regular projection of a non-alternating link. Create $G$, an alternating augmented projection of $D$, as described in Prop. 1. Let $\text{cl}(G - D) = \bigcup_{1 \leq i \leq n} C_i$ where each $C_i$ is a simple closed curve in $S^2$. If $n = 1$ then there is nothing to prove. If $n \geq 2$ then there is a path component $A$ of $S^2 - \text{cl}(G - D)$ whose closure has at least two distinct boundary components $C_i$ and $C_j$.

If $C_i$ and $C_j$ contain boundary edges of a common region in $G$ then we may use a type I move to join $C_i$ and $C_j$ into a single simple closed curve in $S^2$. By Lemma 2 the resulting projection is alternating.

If $C_i$ and $C_j$ do not contain boundary edges of a common region in $G$, then consider a path $\mu$ in $A$ transverse to $G$ so that $\partial \mu = \{a, b\}$ for $a \in C_i$ and $b \in C_j$. We can propagate $C_i$ along $\mu$ using type II moves.
as depicted in Fig. 7 until $C_i^*$ (the image of $C_i$ under type II moves) and $C_j$ contain boundary components of a common region $R$. Call this projection $G^*$. Since a type II move is an isotopy of $C_i$ in $S^2$ and $\mu$ was restricted to $A$, $C_i^*$ is a simple closed curve which does not intersect the other $C_j$. Hence, $G^*$ is an augmented link of $D$. By Lemma 3, $G$ is alternating implies we may choose labels at the new vertices so that $G^*$ is alternating. We then use the type I move to connect sum the disjoint simple closed curves $C_i^{**}$ and $C_j$ into a single simple closed curve. Call the resulting projection $G^{**}$. Since $G^*$ is alternating so is $G^{**}$, by Lemma 2. Hence, $G^{**}$ is an alternating augmented link of $D$ with one less augmenting component than $G$. Repeat this process until there is an alternating augmented projection of $D$ with exactly one augmenting component, proving the theorem.

\[ \square \]

References

[1] C. Adams, "Augmented alternating link complements are hyperbolic," *London Mathematical Society Lecture Notes Series, 112: Low Dimensional Topology and Kleinian Groups*, pp.115-130, Cambridge University Press, Cambridge, 1986.

[2] Thistlethwaite, Morwen B, "An upper bound for the breadth of the Jones polynomial," *Math. Proc. Cambridge Philos. Soc.* 103 (1988), no. 3, 451–456.