NEGATIVE $K$-THEORY AND CHOW GROUP OF MONOID ALGEBRAS

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Abstract. We show, for a finitely generated partially cancellative torsion-free commutative monoid $M$, that $K_i(R) \cong K_i(R[M])$ whenever $i \leq -d$ and $R$ is a quasi-excellent $\mathbb{Q}$-algebra of Krull dimension $d \geq 1$. In particular, $K_i(R[M]) = 0$ for $i < -d$. This is a generalization of Weibel’s $K$-dimension conjecture to monoid algebras. We show that this generalization fails for $X[M]$ if $X$ is not an affine scheme. We also show that the Levine-Weibel Chow group of 0-cycles $\text{CH}_0^{LW}(k[M])$ vanishes for any finitely generated commutative partially cancellative monoid $M$ if $k$ is an algebraically closed field.

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1. Introduction

The monoid algebras are natural generalizations of polynomial and Laurent polynomial algebras over commutative rings. On can often think of them as subalgebras of polynomial or Laurent polynomial algebras generated by monomials. These are as ubiquitous in the study of various properties of rings as the polynomial or Laurent polynomial algebras. A very natural question in algebraic $K$-theory is to find out to what extent various known facts about the $K$-theory of polynomial and Laurent polynomial algebras remain valid for more general monoid algebras.

Gubeladze proved several results on this subject in a series of many papers (see [11], [13], [15] and [16] to name a few). Using the new direction provided by [19] and [3], Cortiñas, Haesemeyer, Walker and Weibel together have made significant advances in the study of algebraic $K$-theory of monoid algebras (see [4], [5], [6] and [7]). An old question on the $K$-theory of monoid algebras was recently settled in [26]. Gubeladze recently settled an old $K$-theoretic question about monoid algebras [18]. The message that comes out of these papers is that even though some properties of the algebraic $K$-theory of polynomial and Laurent polynomial algebras remain valid for monoid algebras, many of them do not directly extend.

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This paper was motivated by our pursuit for those properties of the algebraic $K$-theory of polynomial and Laurent polynomial algebras which may extend to monoid algebras. Our intuition was that while the higher Quillen $K$-theory of monoid algebras may not resemble that of polynomial and Laurent polynomial algebras, the negative $K$-theory and part of $K_0$ should. In this paper, we attempt to understand two such properties, namely, Weibel’s $K$-dimension conjecture and $SK_0$ for monoid algebras.

1.1. Weibel’s conjecture for monoid algebras. Recall that a famous conjecture of Weibel [36] asserts that if $R$ is a commutative Noetherian ring of Krull dimension $d$, then $K_{-d}(R) \cong K_{-d}(R[t_1, \ldots, t_n])$ and $K_i(R[t_1, \ldots, t_n]) = 0$ for $i < -d$ and $n \geq 0$. An affirmative answer to this conjecture was obtained recently by Kerz, Strunk and Tamme [22]. For Noetherian rings containing $\mathbb{Q}$, this was earlier solved by Cortiñas, Haesemeyer, Schlichting and Weibel [3] (see also [10], [23] and [38] for older results in positive characteristics).

The main technical tool that goes into the proof of Weibel’s conjecture in [22] is a pro-cdh-descent theorem for algebraic $K$-theory. However, the final step in the proof of the conjecture using the pro-cdh-descent breaks down in the case of general monoids because it crucially uses the fact that the map $R \to R[M]$ is smooth.

In this paper, we shall use a combination of pro-cdh-descent techniques and the theory of monoid algebras to show that the assertion of Weibel’s conjecture directly extends to more general monoid algebras over some rings. Broadly speaking, we prove the following. We refer to §3 for the precise results including the relevant terms and notations.

**Theorem 1.1.** Let $M$ be a monoid which is finitely generated, commutative, partially cancellative and torsion-free. Let $R$ be a quasi-excellent $\mathbb{Q}$-algebra of Krull dimension $d$. Assume that one of the following conditions is satisfied.

1. $d \geq 1$.
2. $M$ is cancellative and semi-normal.

Then the map $K_i(R) \to K_i(R[M])$ is an isomorphism for all $i \leq -d$. In particular, $K_i(R[M]) = 0$ for all $i < -d$.

One can interpret this result as a generalization of Weibel’s $K_{-d}$-regularity and $K_{< -d}$-vanishing conjectures to monoid algebras.

**Remark 1.2.** We remark that the condition (2) of Theorem 1.1 is essential if $d = 0$. In this case, we can in fact assume that $R$ is a field (see Lemma 2.4). Assuming this, Gubeladze has shown that $K_0(R) \cong K_0(R[M])$ if and only if $M$ is semi-normal (see [15, Theorem 1.3]).

Even if we expect Theorem 1.1 to hold for $\mathbb{F}_p$-algebras, we do not know how to prove it yet. We hope to pursue this in a future work. However, if we allow ourselves to invert the characteristic of the ground field, then we can indeed show that $K_i(R) \cong K_i(R[M])$ for all $i \leq -d$ when $M$ is as in Theorem 1.1. A proof of this is given in §3.

For non-affine schemes, we can prove the following extension of Weibel’s conjecture.

**Theorem 1.3.** Let $M$ be a cancellative torsion-free monoid and $X$ a quasi-excellent $\mathbb{Q}$-scheme of dimension $d$. Then $K_i(X[M]) = 0$ for $i < -d$. If $M$ is semi-normal, then the map $K_i(X) \to K_i(X[M])$ is an isomorphism for all $i \leq -d$.

If $M$ is not semi-normal in Theorem 1.3 then we can show that the monoid extension of Weibel’s $K_{-d}$-regularity conjecture to general monoids is not valid for non-affine schemes. We deduce this failure from the following precise result.
Theorem 1.4. Let $X$ be a connected smooth projective curve over a field $k$ of characteristic zero. Let $M \subset \mathbb{Z}^+$ be the submonoid generated by $\{2, 3\}$. Then the map $K_{-1}(X) \to K_{-1}(X[M])$ is not an isomorphism if the genus of $X$ is positive.

1.2. Levine-Weibel Chow group and $SK_0$ of monoid algebras. Apart from the negative $K$-theory, we also wanted to look at other cohomological properties of polynomial and Laurent polynomial algebras which can generalize to monoid algebras and which are related to the non-negative $K$-theory. One of these cohomology groups is the Levine-Weibel Chow group of 0-cycles $\text{CH}_0^{LW}(X)$ for a singular variety $X$ [27]. This group is the singular analogue of the classical Chow group of 0-cycles on smooth varieties and it is directly related to the $K$-theory of locally free sheaves.

It is well known that the Levine-Weibel Chow group vanishes for polynomial and Laurent polynomial algebras over a field. On the other hand, even though affine toric varieties come up very naturally in algebraic geometry, it was not known yet if their Levine-Weibel Chow group is zero in positive characteristics. Our next result of this paper is the following.

Theorem 1.5. Let $k$ be an algebraically closed field and $M$ a finitely generated commutative partially cancellative monoid of rank at least two. Then $\text{CH}_0^{LW}(k[M]) = 0$.

Note that the lower bound on the rank of the monoid is essential because it is well known that $\text{CH}_0^{LW}(k[t^2, t^3]) \cong k$. Combined with [30, Corollaries 2.7 and 3.4], the vanishing of the Levine-Weibel Chow group for monoid algebras has the following algebraic consequence.

Corollary 1.6. Let $k$ be an algebraically closed field and $M$ a finitely generated commutative partially cancellative monoid of rank $n \geq 2$. Then every local complete intersection ideal of $k[M]$ of height $n$ is a complete intersection.

One of the two main ingredients in our proof of Theorem 1.5 is the following result of independent interest in the algebraic $K$-theory of monoid algebras. When the base ring is regular and the underlying monoid is cancellative and torsion-free, this result is due to Gubeladze [15, Theorem 1.3].

Theorem 1.7. Let $R$ be a commutative Artinian ring and $M$ a finitely generated commutative partially cancellative monoid. Then $SK_0(R[M]) = 0$.

One may recall that in most of Gubeladze’s works (except [18] on monoid algebras, all monoids were cancellative and torsion-free. However, it turns out that for computing the $K$-theory of cancellative monoid algebras using various fundamental results like excision in algebraic $K$-theory, one needs to extend many $K$-theoretic results to partially cancellative torsion-free monoids. These monoids occur very naturally in affine geometry. The algebras over such monoids were first studied by Swan [31, § 15] under the name of discrete Hodge algebras. Higher $K$-theory analogue of Serre’s problem on projective modules for such algebras was studied by Vorst [35] and Gubeladze [13]. More $K$-theoretic importance of these monoid algebras was later exhibited in several papers of Cortiñas, Haesemeyer, Walker and Weibel. See, for instance, [6] and [7].

Remark 1.8. The reader may have observed that the class of monoids considered in Theorem 1.1 is more restrictive than the one in Theorem 1.5 in that we did not allow torsion monoids in Theorem 1.5. The reason for this is the following. If we let $M$ be a finite torsion group of order $n \geq 2$ and $R$ a quasi-excellent $\mathbb{Q}$-algebra of dimension $d \geq 1$ such that $K_{-d}(R) \neq 0$ (for instance, take $R$ to be the coordinate ring of a product of affine nodal curves over $\mathbb{C}$), then we know that $K_{-d}(R[M]) = K_{-d}(R[M]_{\text{red}})$ contains at least two copies of $K_{-d}(R)$ as direct summands one of which is the inclusion
$K_{-d}(R) \subset K_{-d}(R[M])$. In particular, the canonical map $K_{-d}(R) \to K_{-d}(R[M])$ is not an isomorphism. This shows that the $i = -d$ case Theorem 1.1 fails for torsion monoids. We already saw that the map $K_{-d}(X) \to K_{-d}(X[M])$ is not an isomorphism even when $M$ is torsion-free if $X$ is not affine.

However, if we let $i < -d$, then the situation is likely to be different as some of the above results show. We expect this case of Theorem 1.1 to hold for a very general class of monoids and a general class of Noetherian schemes. So we end the discussion of our results with the following.

**Question 1.9.** Let $X$ be a Noetherian separated scheme of Krull dimension $d \geq 0$ and $M$ a finitely generated commutative cancellative monoid. Is $K_i(X[M]) = 0$ for $i < -d$?

### 1.3. Outline of proofs

We give a brief outline of our proofs. Our main strategy for proving Theorem 1.1 is to eventually reduce the proof to the case of cancellative torsion-free semi-normal monoids with the help of several reductions. To prove the theorem in this restrictive case, we need to use the pro-cdh descent results of [22]. The pro-cdh-descent results and the weak resolution of singularities together allow us to reduce to the case when the base scheme $X$ is regular. The proof of the vanishing in this case is done using some classical results of Gubeladze and a Zariski descent argument. In §2, we recall the basic results about monoids and prove our reduction steps that are needed in the proof of Theorem 1.1. Gubeladze’s Milnor square for monoid algebras plays a key role in these steps. The final proofs of Theorems 1.1 and 1.3 using pro-cdh descent are given in §3.

The idea of proving Theorem 1.5 came to us from two known results. The first is a classical result of Gubeladze which says that $SK_0(k(M))$ vanishes if $k$ is a field and $M$ is cancellative torsion-free. The second is an old conjecture of Murthy [30] which says that the Levine-Weibel Chow group of an affine variety over an algebraically closed field is torsion-free. This conjecture was recently settled in [25]. In view of this positive answer to Murthy’s conjecture, we are left with proving Theorem 1.7 which extends Gubeladze’s result to a more general class of monoids.

We prove this extension in Sections 5 and 6. For the proof of Theorem 1.7, we have to establish several reduction steps to reduce the proof to a nicer class of monoids. This is done in §5 using several Milnor squares. A crucial part of our reductions is the recent technique of Gubeladze [18], which tells us how one should deal with cancellative torsion monoids. This reduction technique plays a key role in our proofs. The final proof is obtained in §6. The key step in the final proof is a generalization of Swan’s result [31, Lemma 15.6] on the prime decomposition of radical ideals in cancellative torsion-free monoids to a more general class of monoids. This allows us to reduce the case of partially cancellative monoids to cancellative monoids.

One could now naturally ask if Theorems 1.5 and 1.7 could be valid for non-cancellative and other more general class of monoids. In the hope of dealing with this question in future, we came up with one more proof of Theorem 1.7 for partially cancellative torsion-free monoids. This proof is given in the end of §6. It is substantially $K$-theoretic in flavor and hence has potential to generalize to a broader class of monoids.

### 2. Recollection and reduction steps

In this section, we fix our notations and provide a limited recollection of the basic definitions in the theory of monoids. We then establish our reduction steps for reducing the proof of Theorem 1.1 to the case of positive cancellative torsion-free semi-normal monoids.
Recall that a monoid is same as a semigroup with an identity element. Throughout this paper, we shall assume all monoids to be commutative and finitely generated. We shall assume all our rings to be commutative and Noetherian and all schemes to be separated and Noetherian. The dimension of a ring or a scheme will mean its Krull dimension.

2.1. Recollection of monoids. Unless specified otherwise, we shall use the additive notation for the operation in a monoid but switch to multiplicative operation when we consider a monoid inside an algebra generated by it over a commutative ring.

For a monoid $M$, we shall let $\text{gp}(M)$ denote the group completion of $M$. We have a natural monoid homomorphism $M \to \text{gp}(M)$. We let $U(M)$ denote largest submonoid of $M$ which is also a group. We say that $M$ is positive if $U(M) = 0$. The rank of the monoid $M$ is the rank of free part of $\text{gp}(M)$. We shall denote it by $\text{rk}(M)$. Given a subset $S \subset M$, we shall let $\langle S \rangle$ denote the submonoid of $M$ generated by $S$. It is the $\mathbb{Z}_+$-linear combination of elements of $S$.

Given a monoid $M$, we can construct another monoid $M_+$ by adding a base-point $\infty$ to $M$ and by letting $m + \infty = \infty$ for all $m \in M_+$. We shall call $M_+$ the augmented monoid associated to $M$. It is a pointed monoid in the language of [6]. We have a canonical inclusion of monoids $M \subset M_+$. We shall usually avoid the usage of more general pointed monoids considered in [6] as we have no need for them. Every monoid homomorphism $f : M \to N$ uniquely extends to a monoid homomorphism $f : M_+ \to N_+$ which fixes the base-point.

A monoid $M$ is called cancellative if for $a, b, c \in M$, the condition $a + b = a + c$ implies that $b = c$. This is equivalent to saying that the group completion map $M \to \text{gp}(M)$ is injective. We say that $M$ is torsion-free if for $a, b \in M$, the condition $na = nb$ for some $n \geq 1$ implies that $a = b$. If $M$ is cancellative, then it is torsion-free if and only if $\text{gp}(M) \cong \mathbb{Z}_+^r$.

A subset $I \subset M$ is called an ideal in $M$ if $I + M = I$. If $I \subset M$ is a proper ideal ($I \neq M$), then $I \cap U(M) = \emptyset$ and $M \setminus U(M)$ is the largest proper ideal of $M$. If $I \subset M$ is an ideal, we shall let $I_*$ denote $I \cup \{0\}$. This is a submonoid of $M$.

If $I \subset M$ is an ideal, the quotient $M/I$ is obtained as follows. We consider the inclusion $I \subset M_+$ and then take the quotient by the equivalence relation $m \sim \infty$ for all $m \in I$. There is a unique addition rule in $M_+/I$ that makes the canonical surjection $M_+ \to M_+/I$ into a morphism of monoids (see [6] § 1)). We let $M/I$ be the image of $M$ under the quotient map $M_+ \to M_+/I$. There is an epimorphism of monoids $M \to M/I$ and a canonical isomorphism $(M/I)_+ \simeq M_+/I$. We allow $I \subset M$ to be the empty set in which case, we identify $M/I$ with $M$. If $I = M$, the quotient $M/I$ is identified with the constant singleton monoid $\{+\}$.

A monoid is called partially cancellative (pc) if there is a cancellative monoid $N$ and an ideal $I \subset N$ (possibly empty) such that $M = N/I$. A monoid $M$ is called partially cancellative torsion-free (pctf) if there is a cancellative torsion-free monoid $N$ and an ideal $I \subset N$ (possibly empty) such that $M = N/I$. Such monoids are called pctf monoids in [7]. We shall also use this notation in this paper.

For a monoid $M$ and a ring $R$, we let $R[M]$ denote the free $R$-module on $M$. Then $R[M]$ is a commutative $R$-algebra in the usual way, with multiplication given by the addition rule for $M$. Since $R$ is Noetherian and $M$ is finitely generated by our assumption, it is easy to check that $R[M]$ is also Noetherian. One also checks that $R[M]$ is an integral domain if and only if $R$ is an integral domain and $M$ is cancellative torsion-free (see [2] Theorem 4.8)]. $R[M]$ is reduced if $R$ is reduced and $M$ is cancellative torsion-free. It can also happen that $R[M]$ is reduced even if $M$ is a torsion monoid. For example, $R[M]$ is reduced if $R$ is reduced and $M$ is cancellative whenever $\mathbb{Q} \subset R$ (see
Theorem 4.19). If $I$ is an ideal of the monoid $M$, then $R[I]$ is an ideal of the ring $R[M]$ and $R[M/I] = R[M]/R[I]$.

Recall that a cancellative monoid $M$ is called normal if $M = \{a \in \text{gp}(M)\mid na \in M$ for some $n > 0\}$. One says that $M$ is semi-normal if $M = \{a \in \text{gp}(M)\mid 2a, 3a \in M\}$. The semi-normalization of a cancellative monoid $M$ is the submonoid $sn(M)$ consisting of all elements $a \in \text{gp}(M)$ such that $2a, 3a \in M$. Given an inclusion of monoids $M \subset N \subset \text{gp}(M)$ and an element $a \in N$, we let $M(a)$ denote the submonoid of $N$ generated by $M$ and $a$. Given a finite set $F = \{a_1, \cdots, a_r\} \subset N$, we can inductively define $M\langle F \rangle \subset N$. It is then easy to see that $sn(M) = \text{colim}_{F \subset \text{sn}(M)}$. One can also check that $sn(M) = \{a \in \text{gp}(M)\mid na \in M$ for all $n \gg 0\}$ (see [31, § 1]).

We shall let $n(M) = \{a \in \text{gp}(M)\mid na \in M$ for some $n\}$ denote the normalization of $M$ in $\text{gp}(M)$. The following result is elementary.

**Lemma 2.1.** Let $M$ be a cancellative monoid and let $M \subset M' \subset n(M)$ be inclusions of monoids. Then the following hold.

1. $M'$ is cancellative.
2. $M'$ is torsion-free if $M$ is so.
3. $M'$ is positive torsion-free if $M$ is so.
4. $M'$ is positive if $M$ is so provided $M' \subset sn(M)$.

**Proof.** We have the inclusions $M \subset M' \subset n(M) \subset \text{gp}(M)$. In particular, all these monoids have same group completions. The parts (1) and (2) of the lemma follow immediately from this. For (3), suppose that $M$ is positive torsion-free and let $a \in U(M')$ be a non-zero element. We then have $a + b = 0$ for some $b \in M'$. We can choose some $n \gg 0$ such that $na, nb \in M$. Since $M$ is torsion-free, we must have $na, nb \neq 0$. Since $na + nb = n(a+b) = 0$, it follows that $na \in U(M) \setminus \{0\}$. This contradicts our assumption that $M$ is positive.

We now prove (4). Suppose $M$ is positive and let $a \in U(M')$ be a non-zero element. We then have $a + b = 0$ for some $b \in M'$. Since $M' \subset sn(M)$, we can choose some $n_0 \gg 0$ such that $na, nb \in M$ for all $n \geq n_0$ (see above). If $n_0a \neq 0$, then $n_0b \neq 0$ and $n_0a + n_0b = 0$, so we get $0 \neq n_0a \in U(M)$, which contradicts our hypothesis. Suppose that $n_0a = 0$. Then $n_0b = 0$. But this means that $(n_0+1)a = a \neq 0$ and $(n_0+1)b = b \neq 0$. On the other hand, we have $(n_0+1)a + (n_0+1)b = (n_0+1)(a+b) = 0$. Since $(n_0+1)a, (n_0+1)b \in M$, we get $0 \neq (n_0+1)a \in U(M)$. This again contradicts our hypothesis. We are therefore done.

**Remark 2.2.** Note that the part (3) of Lemma 2.1 is not true in general if $M$ is not torsion-free. In fact, it is easy to see in this case that $n(M)$ contains the whole torsion subgroup of $\text{gp}(M)$.

2.2. Reduction of positive monoids to semi-normal monoids. In this subsection, we shall establish some reduction steps which will show how to replace a positive cancellative torsion-free monoid in Theorem 1.1 by the one which is positive cancellative torsion-free and semi-normal. Recall that a Cartesian square of commutative rings

$$
\begin{array}{ccc}
A_1 & \xrightarrow{\alpha} & A_2 \\
\phi \downarrow & & \psi \\
B_1 & \xrightarrow{\beta} & B_2
\end{array}
$$

called a Milnor square if one of $\psi$ and $\beta$ is surjective. We shall use the following consequence of the classical results of Bass [1] and Milnor [29] as one important tool.
**Proposition 2.3.** Given a Milnor square (2.1), the map $K_i(A_1, B_1) \to K_i(A_2, B_2)$ of relative $K$-groups is an isomorphism for $i \leq 0$. In particular, there is a long exact sequence sequence of algebraic $K$-groups

$$K_i(A_1) \to K_i(A_2) \oplus K_i(B_1) \to K_i(B_2) \to K_{i-1}(A_1) \to \cdots$$

for $i \leq 1$.

If $\psi$ and $\beta$ are both surjective in the Milnor square, then $K_i(A_1, B_1) \to K_i(A_2, B_2)$ is an isomorphism for $i \leq 1$. In particular, the above sequence is exact for all $i \leq 2$.

**Lemma 2.4.** Let $A$ be a ring and $I$ a nilpotent ideal of $A$. Then $K_i(A) = K_i(A/I)$ for all $i \leq 0$.

**Proof.** See Bass’ book [1 Chapter IX, Proposition 1.3]. □

**Lemma 2.5.** Let $R$ be a ring and $M$ a positive cancellative torsion-free monoid. Then $K_i(R[M]) \simeq K_i(R[sn(M)])$ for all $i \leq -1$.

**Proof.** We can assume $R$ to be reduced by Lemma 2.4. It follows from [2 Theorem 4.19] that $R[M]$ is also reduced. As in §2.1 we can write $sn(M) = \operatorname{colim}_{F \subseteq sn(M)} M(F)$, where $F$ is a finite set of elements $a \in \operatorname{gp}(M)$ such that $2a, 3a \in M$. Since $K$-theory commutes with direct limits, it is enough to prove that $K_i(R[M]) \simeq K_i(R[sn(M)])$ for $i \leq -1$.

Since $M(F)$ is an iterated extension of monoids of the form $N\langle\{a\}\rangle$ such that $a \in \operatorname{gp}(N)$ with $2a, 3a \in N$, it will suffice to prove inductively that $K_i(R[M]) \simeq K_i(R[M(\{a\})])$ for $i \leq -1$, where $a \in \operatorname{gp}(N)$ with $2a, 3a \in M$. Note here that $M(\{a\})$ is a positive cancellative torsion-free monoid if $M$ is so and $a \in sn(M)$ by Lemma 2.1.

We let $A = R[M]$ and $B = R[M(\{a\})]$. Let $F(A)$ denote the total ring of quotients of $A$. Note that $F(A)$ is a product of fields. We can write $B = A[x] \subset F(A)$, where $x^2, x^3 \in A$. We then get a conductor Milnor square

$$\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
A/I & \longrightarrow & B/I
\end{array}$$

with conductor ideal $I = (x^2, x^3)$.

Furthermore, one knows in this case that $(A/I)_{\text{red}} \to (B/I)_{\text{red}}$ is an isomorphism (see, for instance, [31 Proof of Theorem 14.1]). This gives a commutative diagram of long exact sequences

$$\begin{array}{cccccccc}
K_{i+1}(A/I) & \longrightarrow & K_i(A, I) & \longrightarrow & K_i(A) & \longrightarrow & K_i(A/I) & \longrightarrow & K_{i-1}(A, I) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_{i+1}(B/I) & \longrightarrow & K_i(B, I) & \longrightarrow & K_i(B) & \longrightarrow & K_i(B/I) & \longrightarrow & K_{i-1}(B, I).
\end{array}$$

Since $i \leq -1$ and $(A/I)_{\text{red}} \cong (B/I)_{\text{red}}$, it follows from Lemma 2.4 that the first and the fourth (from the left) vertical arrows are isomorphisms. The second and the fifth vertical arrows are isomorphisms by Proposition 2.3. We conclude that the middle vertical arrow is also an isomorphism. □

**2.3. Reduction to positive monoids.** In this subsection, our goal is to prove a reduction step which will allow us to replace a cancellative torsion-free monoid in Theorem 1.1 by the one which is cancellative torsion-free and positive.

**Lemma 2.6.** Let $A$ be a ring of dimension $d \geq 0$. Then the canonical map $K_i(A) \to K_i(A[X_1, \ldots, X_n, Y_1, \ldots, Y_m])$ is an isomorphism for all $i \leq -d$ and $n, m \geq 0$. 




Proof. The proof is by induction on \( m \). The case \( m = 0 \) is due to [22] Theorem B(ii). Let \( m \geq 1 \) and assume that \( K_i(A) \to K_i(A[X_1, \ldots, X_n, Y_1^\pm, \ldots, Y_{m-1}^\pm]) \) is an isomorphism for all \( n \geq 0 \).

We let \( B = A[X_1, \ldots, X_n, Y_1^\pm, \ldots, Y_{m-1}^\pm] \). The induction hypothesis implies that \( K_i(A) \cong K_i(B) \) for all \( i \leq -d \) and all \( n \geq 0 \). Hence by [22] Theorem B(i), we get \( K_{i-1}(B) = 0 \) for all \( i \leq -d \).

We now consider the commutative diagram of \( K \)-groups

\[
\begin{array}{c}
K_i(B) \oplus K_i(B) \rightarrowtail \beta_i K_i(B) \\
\downarrow \xi_i \quad \quad \downarrow \alpha_i \\
K_i(B[Y_m]) \oplus K_i(B[Y_m^{-1}]) \rightarrowtail K_i(B[Y_m^\pm]) \rightarrowtail K_{i-1}(B) = 0,
\end{array}
\]

where the bottom row is the Bass fundamental exact sequence. The map \( \beta_i \) is defined by \( \beta_i(a, b) = a - b \). This is clearly surjective. By induction, we have

\[
K_i(B) \cong K_i(A) \cong K_i(A[X_1, \ldots, X_n, Y_m, Y_1^\pm, \ldots, Y_{m-1}^\pm]) = K_i(B[Y_m]),
\]

where the second isomorphism holds by replacing \( n \) by \( n + 1 \). By the same reason, we get \( K_i(B) \cong K_i(A) \cong K_i(B[Y_m^{-1}]) \) for all \( i \leq -d \). Hence, \( \xi_i \) is an isomorphism for all \( i \leq -d \). A diagram chase shows that \( \alpha_i \) is surjective. Since this is already split injective, we are done.

**Lemma 2.7.** Let \( M \) be a monoid and \( N = M \setminus U(M) \). Let \( R \) be a ring of dimension \( d \geq 0 \). If \( K_i(R) \cong K_i(R[N_i]) \) for all \( i \leq -d \), then \( K_i(R) \cong K_i(R[U(M)]) \) for all \( i \leq -d \).

**Proof.** By [18] Lemma 6.1 (the proof thereof), there is a Milnor square

\[
(2.3) \quad \begin{array}{c}
R[N_i] \rightarrowtail R[M] \\
\downarrow \quad \quad \downarrow \\
R \rightarrowtail R[U(M)],
\end{array}
\]

where \( \phi \) is split surjective because the composite \( R[U(M)] \to R[M] \to R[U(M)] \) is identity (see also [31 § 15]).

Since (2.3) is a commutative diagram of \( R \)-algebras, the trivial Milnor square

\[
(2.4) \quad \begin{array}{c}
R \rightarrowtail R \\
\downarrow \quad \quad \downarrow \\
R \rightarrowtail R
\end{array}
\]

(where all maps are identity) maps to this square. It follows from Proposition (2.3) that there is a commutative diagram of short exact sequences

\[
(2.5) \quad \begin{array}{c}
0 \rightarrowtail K_i(R) \rightarrowtail K_i(R) \oplus K_i(R) \rightarrowtail K_i(R) \rightarrowtail 0 \\
\downarrow \quad \quad \downarrow \\
0 \rightarrowtail K_i(R[N_i]) \rightarrowtail K_i(R) \oplus K_i(R[M]) \rightarrowtail K_i(R[U(M)]) \rightarrowtail 0.
\end{array}
\]

The left vertical arrow is an isomorphism by our assumption and the right vertical arrow is an isomorphism by Lemma (2.6) since \( U(M) \) is torsion-free (and hence free). This implies the desired assertion. \( \square \)
2.4. Cancellative to partially cancellative monoids. We shall now show how we can reduce the proof of Theorem 1.1 to the case of cancellative torsion-free monoids. The result that we shall use is the following.

Lemma 2.8. Let $M$ be a partially cancellative torsion-free monoid and $R$ a ring of dimension $d \geq 0$. Assume that $K_i(R) \to K_i(R[N])$ is an isomorphism for all $i \leq -d$ and for all cancellative torsion-free monoids $N$. Then $K_i(R) \to K_i(R[M])$ is an isomorphism for all $i \leq -d$.

Proof. By definition of partially cancellative torsion-free monoids, we can write $M = N/I$, where $N$ is a cancellative torsion-free monoid. This yields a Milnor square

$$\begin{CD}
R[I_*] @>>> R[N] \\
@VVV @VVV \\
R @>>> R[N/I].
\end{CD}$$

Mimicking the proof of Lemma 2.7 we get a commutative diagram of long exact sequences

$$\begin{CD}
K_i(R) @>>> K_i(R) \oplus K_i(R) @>>> K_i(R) @>>> K_{i-1}(R) \oplus K_{i-1}(R) \\
K_i(R[N]) @>>> K_i(R) \oplus K_i(R[N]) @>>> K_i(R[N/I]) @>>> K_{i-1}(R[I_*]) @>>> K_{i-1}(R) \oplus K_{i-1}(R[N]).
\end{CD}$$

All vertical arrows in this diagram except possibly the middle one are isomorphisms because $N$ and $I_*$ are cancellative torsion-free monoids. It follows that the middle vertical arrow is an isomorphism too. This finishes the proof. $\square$

An identical argument also proves the following variant of Lemma 2.8. We shall not use this here but it may be useful in answering Question 1.9.

Lemma 2.9. Let $M$ be a partially cancellative monoid and $R$ a ring of dimension $d \geq 0$. Assume that $K_i(R) \to K_i(R[N])$ is an isomorphism for all $i \leq -d$ and for all cancellative monoids $N$. Then $K_i(R) \to K_i(R[M])$ is an isomorphism for all $i \leq -d$.

We shall also use the following result in the proof of Theorem 1.1.

Lemma 2.10. Let $R$ be a regular ring and $M$ a partially cancellative torsion-free monoid. Then $K_i(R[M]) = 0$ for all $i \leq -1$.

Proof. If $M$ is cancellative torsion-free, the lemma follows from [15, Theorem 1.3]. In this case, we can also prove it using Lemma 2.5. This lemma allows us to assume that $M$ is semi-normal. Now, by the Gubeladze–Swan theorem (see [31, Corollary 1.4]), we have $K_0(R) \cong K_0(R[M \oplus \mathbb{Z}])$. Together with the fundamental exact sequence, this implies that $K_i(R[M]) = 0$ for all $i \leq -1$.

In case $M$ is partially cancellative torsion-free, we can write $M = N/I$, where $N$ is cancellative torsion-free. Hence, by similar arguments as in Lemma 2.8 we get a Milnor square

$$\begin{CD}
R[I_*] @>>> R[N] \\
@VVV \phi V \\
R @>>> R[N/I]
\end{CD}$$
Using Proposition 2.3 this yields the long exact sequence

\[
\cdots \longrightarrow K_i(R) \oplus K_i(R[N]) \longrightarrow K_i(R[N/I]) \longrightarrow K_{i-1}(R[I]) \longrightarrow \cdots
\]

for \( i \leq -1 \). Note that \( I^* \) is a cancellative torsion-free monoid. Hence, we have \( K_i(R[I]) = 0 \) for all \( i \leq -1 \). Since \( K_i(R[N]) \) and \( K_i(R) \) are both zero for \( i \leq -1 \), we get \( K_i(R[N/I]) = 0 \) for \( i \leq -1 \). This finishes the proof. \[\square\]

3. The monoid version of Weibel’s conjecture

The goal of this section is to prove Theorem 1.1. We fix a field \( k \). We let \( \mathbf{Sch}_k \) denote the category of separated schemes which are essentially of finite type over \( k \) and let \( \mathbf{Sm}_k \) denote the full subcategory of \( \mathbf{Sch}_k \) consisting of schemes which are regular. We shall denote the product of \( X, Y \in \mathbf{Sch}_k \) over \( k \) by \( X \times Y \). Let \( \mathbf{Sch}_k \) denote the category of separated Noetherian schemes over \( k \). We let \( \mathbf{Sch}_k/zar \) denote the Grothendieck site on \( \mathbf{Sch}_k \) given by the Zariski topology. We shall consider all cohomology groups with respect to the Zariski topology in this paper.

3.1. Quasi-excellent schemes and resolution of singularities. Recall from [28, Chapter 32, p. 260] that a (Noetherian) ring \( A \) is called excellent if the following hold.

1. The fibers of the map \( A_m \to \hat{A}_m \) are geometrically regular for every maximal ideal \( m \in A \).
2. The regular locus \( X_{reg} \) of every finite type affine scheme \( X \) over \( A \) is open in \( X \).
3. \( A \) is universally catenary.

If \( A \) satisfies only (1) and (2), it is called quasi-excellent. A Noetherian separated scheme \( X \) is called (quasi-) excellent if is it covered by the spectra of (quasi-) excellent rings. One important property of (quasi-) excellent schemes we shall use in this paper is that if \( X \) is a (quasi-) excellent scheme and \( X' \to X \) is an essentially of finite type morphism, then \( X' \) is also (quasi-) excellent. In particular, all objects of \( \mathbf{Sch}_k \) are excellent. This is also true if we replace \( k \) by any complete local ring.

Let \( \mathcal{C}_k \) be a subcategory of \( \mathbf{Sch}_k \). We shall say that \( \mathcal{C}_k \) admits weak resolution of singularities if the following hold.

1. If \( X \in \mathcal{C}_k \) and \( Y \to X \) is a finite type morphism in \( \mathbf{Sch}_k \), then \( Y \in \mathcal{C}_k \).
2. Given a reduced scheme \( X \in \mathcal{C}_k \), there exists a Cartesian square of schemes

\[
\begin{array}{ccc}
\hat{Y} & \longrightarrow & \hat{X} \\
\downarrow & & \downarrow \\
Y & \longrightarrow & X
\end{array}
\]

such that \( \hat{X} \to X \) is a proper, the horizontal arrows are nowhere dense closed immersions, \( \hat{X} \in \mathbf{Sm}_k \) and \( \hat{X} \setminus \hat{Y} \to X \setminus Y \) is an isomorphism. Note that \( \hat{X}, Y \) and \( \hat{Y} \) belong to \( \mathcal{C}_k \) by (1).

We shall use the following celebrated result of Hironaka [20, Theorem 1*] and its extension to quasi-excellent schemes by Temkin [32, Theorem 1.1].

**Theorem 3.1.** The category of quasi-excellent schemes over \( \mathbb{Q} \) admits weak resolution singularities.
3.2. The spectrum $C^MK$. We let $X \mapsto K(X)$ denote the non-connective Thomason-Trobaugh algebraic $K$-theory presheaf of spectra on $\text{Sch}_k$ (denoted by $K^B(X)$ in (3.3)). Given a monoid $M$, we have an augmented $k$-algebra $k[M]$. For any $X \in \text{Sch}_k$, we let $X[M] = X \times_k \text{Spec}(k[M]) \cong X \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(\mathbb{Z}[M])$. The augmentation of $k[M]$ yields natural maps $X \xrightarrow{\sigma} X[M] \xrightarrow{\text{Spec}} X$ whose composite is identity. In particular, there is functorial decomposition $K(X[M]) \cong K(X) \amalg C^MK(X)$, where $C^MK(X) = \text{hofiber}(K(X[M]) \xrightarrow{\sigma} K(X))$. It follows that $X \mapsto C^MK(X)$ is a presheaf of spectra on $\text{Sch}_k$. For any $i \in \mathbb{Z}$, we shall let $C^MK_i(X)$ denote the stable homotopy group of $\pi_i(C^MK(X))$. We let $C^MK_{i,X}$ denote the Zariski sheaf on $X$ associated to the presheaf $U \mapsto \pi_i(C^MK(U))$.

If $f : Y \to X$ is a morphism in $\text{Sch}_k$, we shall let $C^MK_i(X,Y)$ denote the homotopy fiber of the map $f^* : C^MK(X) \to C^MK(Y)$. We let $C^MK_{i,Y}$ denote the homotopy groups of $C^MK(X,Y)$. We let $C^MK_{i,X}^{(X,Y)}$ denote the Zariski sheaf on $X$ associated to the presheaf $U \mapsto C^MK_i(U,X \times_X Y)$.

Theorem 3.2. ([33 Theorem 10.3]) Given $X \in \text{Sch}_k$ and a closed subscheme $Z \subset X$, there exists a strongly convergent spectral sequence

\begin{equation}
E_2^{p,q} = H^p(X,C^MK_{q,(X,Z)}) \Rightarrow C^MK_{q-p}(X,Z).
\end{equation}

Proof. This is proved by repeating the argument of [33 Theorem 10.3] verbatim with the aid of [33 Proposition 3.20.2] and the fact that if $U = \text{Spec}(A)$ is an affine open in $X$ with a prime ideal $p \subset A$ such that $A_p = \lim_{i} A[f_i^{-1}]$, then $A_p \otimes_k k[M] \cong \lim_{i} (A[M])[f_i^{-1}]$.

We leave out the details. \hfill \Box

Lemma 3.3. Let $X \in \text{Sch}_k$ be of dimension $d$ and $X_{\text{red}}$ denote the reduced subscheme of $X$. Let $M$ be any monoid. The $C^MK_i(X,X_{\text{red}}) = 0$ for $i < -d$. If $M$ is cancellative torsion-free, then $C^MK_i(X,X_{\text{red}}) = 0$ for $i \leq -d$. In particular, $C^MK_i(X) \cong C^MK_i(X_{\text{red}})$ for all $i \leq -d$.

Proof. The first assertion is an immediate consequence of the spectral sequence (3.2) and Lemma 2.4. We show the second part.

Using the spectral sequence (3.2), it suffices to show that $C^MK_i(A,I) = 0$ for $i \leq 0$ if $A$ is the local ring of a Zariski point of $X$ and $I$ is its nil-radical. We shall prove the stronger result that $K_i(A[M],I[M]) = 0$ for $i \leq 0$. Using Lemma 2.4 and the long exact sequence for relative $K$-theory, we only need to show that the map $K_1(A[M]) \to K_1(A/I[M])$ is surjective.

We now consider the commutative diagram of short exact sequences

\begin{equation}
0 \to SK_1(A[M]) \to K_1(A[M]) \to (A[M])^\times \to 0
\end{equation}

\begin{equation}
0 \to SK_1(A/I[M]) \to K_1(A/I[M]) \to (A/I[M])^\times \to 0.
\end{equation}

It follows from our assumption on $M$ and [2 Theorem 4.19] that $A/I[M] = (A[M])_{\text{red}}$. We conclude from [1 Chap. IX, Propositions 3.10 and 3.11] (see also [39 Chapter III, Lemma 2.4]) that the left vertical arrow in (3.3) is an isomorphism.
To prove that the right vertical arrow in (3.3) is surjective (which will finish the proof), we consider the commutative diagram

\[
\begin{array}{ccc}
A^\times \times U(M) & \longrightarrow & (A[M])^\times \\
\downarrow & & \downarrow \\
(A/I)^\times \times U(M) & \cong & (A/I[M])^\times.
\end{array}
\]

Since \(A/I\) and \(A/I[M]\) are reduced (observed above) and \(A\) is local, it follows from [2] Proposition 4.20 that the lower horizontal arrow is an isomorphism. Since the left vertical arrow is clearly surjective (uses again that \(A\) is local), we conclude that the right vertical arrow must also be surjective, as desired. \(\square\)

3.3. The main result. We are now ready to prove Theorem 1.1 which extends the assertion of Weibel’s \(K\)-dimension conjecture from polynomial to monoid algebras. The following result is a refinement of [22] Proposition 6.1. We do not use it here but include it because it may be useful in the generalization of Theorem 1.1 for non-affine schemes. We fix a field \(k\).

**Lemma 3.4.** Let \(M\) be a monoid and \(X \in \text{Sch}_k\). Assume that \(C^M K_i(O_{X,x}) = 0\) for all \(i \leq -\dim(O_{X,x})\) and all points \(x \in X\). Then \(C^M K_i(X) = 0\) for all \(i \leq -\dim(X)\).

**Proof.** For \(i < -\dim(X)\), this is stated in [22] Proposition 6.1 and proven in [21] Proposition 3. However, the proof given there works in the modified case as well with no change. We briefly explain it.

Let \(d\) denote the dimension of \(X\). Using the spectral sequence (3.2), it suffices to show that \(H^q(X, C^M K_q,X) = 0\) whenever \(q - p \leq -d\). Suppose first that \(q + d \leq 0\). In this case, it suffices to show that \(C^M K_q,X = 0\). But this follows from our assumption because \(\dim(O_{X,x}) \leq d \leq -q\) for all \(x \in X\).

We now fix \(0 \leq p \leq d\) and \(q - p \leq -d\) (equivalently, \(p > q + d - 1\)) such that \(q + d > 0\). By [21] Lemma 4 (where we take \(r = q + d - 1\)), it suffices to show that if \(x \in X\) is a Zariski point with \(\dim(\{x\}) \geq q + d\), then \(C^M K_q(O_{X,x}) = 0\). But the condition \(\dim(\{x\}) \geq q + d\) implies that \(\dim(O_{X,x}) \leq -q\). Our hypothesis implies again that \(C^M K_q(O_{X,x}) = 0\). This finishes the proof. \(\square\)

**Lemma 3.5.** Let \(M\) be a cancellative torsion-free semi-normal monoid and \(X\) a regular Noetherian scheme of dimension \(d \geq 0\). Then \(C^M K_i(X) = 0\) for \(i \leq -d\).

**Proof.** Since \(X\) is regular, we can assume that it is connected. We first assume that \(X\) is affine. In this case, it follows from [31] Corollary 1.4 that \(C^M K_0(X) = 0\). The same holds if we replace \(X\) by \(X[T^{\pm 1}]\) because the latter scheme is also affine and regular.

On the other hand, the Bass fundamental exact sequence yields a surjective map \(C^M K_i(X[T^{\pm 1}]) \to C^M K_i-1(X)\). We conclude inductively that \(C^M K_i(X) = 0\) for \(i \leq 0\). Using the spectral sequence (3.2) (see the proof of Lemma 3.3), we now conclude that \(C^M K_i(X) = 0\) for \(i \leq -d\) if \(X\) is any regular Noetherian scheme of dimension \(d\). \(\square\)

**Lemma 3.6.** Let \(\mathcal{C}_k\) be a subcategory of \(\text{Sch}_k\) which admits weak resolution of singularities. Let \(X \in \mathcal{C}_k\) be of dimension \(d \geq 0\). Let \(M\) be a cancellative torsion-free semi-normal monoid. Then \(K_i(X) \to K_i(X[M])\) is an isomorphism for all \(i \leq -d\).

**Proof.** The lemma is equivalent to showing that \(C^M K_i(X) = 0\) for \(i \leq -d\). By Lemma 3.3, we can assume that \(X\) is reduced. Consequently, we can assume that \(X[M]\) is reduced by [2] Theorem 4.19 as \(M\) is cancellative torsion-free. Note that \(X[M] \in \mathcal{C}_k\) as \(M\) is finitely generated. We shall prove the lemma by induction on \(d\). The \(d = 0\) case follows from Lemma 3.5. So we assume that \(d \geq 1\).
Due to the assumption of weak resolution of singularity, we have an abstract blow-up square (see [3, Introduction] for definition) as in (3.1). Adjoining our monoid, this yields another abstract blow-up square in $\mathcal{C}_k$:

\[
\begin{array}{ccc}
\tilde{Y}[M] & \longrightarrow & \tilde{X}[M] \\
\downarrow & & \downarrow \\
Y[M] & \longrightarrow & X[M].
\end{array}
\]

For any integer $n \geq 1$, we let $nY$ denote the infinitesimal thickening of $Y$ inside $X$ defined by the sheaf of ideals $I^n_Y$, where $I_Y$ is the sheaf of ideals on $X$ defining $Y$. We define $n\tilde{Y}$ analogously. Then it is easy to see that $nY[M]$ (resp. $n\tilde{Y}[M]$) is an infinitesimal thickening of $Y[M]$ (resp. $\tilde{Y}[M]$) inside $X[M]$ (resp. $\tilde{X}[M]$).

Denote

\[
\begin{array}{ll}
\{C^M K_{i+1}(n\tilde{Y})\} & \rightarrow C^M K_i(X) \rightarrow \{C^M K_i(nY)\} \oplus C^M K_i(\tilde{X}).
\end{array}
\]

Since $Y \subset X$ and $\tilde{Y} \subset \tilde{X}$ are nowhere dense closed subsets, we see that $\dim(Y)$ and $\dim(\tilde{Y})$ are less than $d$. Using induction on $d$ and regularity of $\tilde{X}$, we see that the end terms of this exact sequence vanish. We conclude that $C^M K_i(X) = 0$ for $i \leq -d$. □

Combining Theorem 3.1 and Lemma 3.6, we get:

**Corollary 3.7.** Let $M$ be a cancellative torsion-free semi-normal monoid and $X$ a quasi-excellent scheme over $\mathbb{Q}$ of dimension $d \geq 0$. Then $K_i(X) \rightarrow K_i(X[M])$ is an isomorphism for all $i \leq -d$.

We now state our main result on the extension of Weibel’s conjecture to monoid algebras.

**Theorem 3.8.** Let $M$ be a partially cancellative torsion-free monoid and let $\mathcal{C}_k$ be a subcategory of $\tilde{\text{Sch}}_k$ which admits weak resolution of singularities. Let $X \in \mathcal{C}_k$ be affine of dimension $d \geq 0$. Assume that one of the following holds.

1. $d \geq 1$.
2. $M$ is cancellative and semi-normal.

Then the map $K_i(X) \rightarrow K_i(X[M])$ is an isomorphism for all $i \leq -d$.

**Proof.** In view of the main results of [22], the theorem is equivalent to proving that $C^M K_i(X) = 0$ for $i \leq -d$. Lemma 3.3 allows us to assume that $X$ is reduced. By Lemma 2.8, we can assume that $M$ is a cancellative torsion-free monoid. Using Lemma 2.7, we can further assume that $M$ is positive. If $d \geq 1$, we can use Lemma 2.9 to assume that $M$ is a cancellative torsion-free semi-normal monoid. If $d = 0$, then $M$ is already given to be cancellative torsion-free semi-normal.

We have therefore reduced the proof of the theorem to the case where $M$ is a cancellative torsion-free semi-normal monoid and $X \in \mathcal{C}_k$ a reduced affine scheme of dimension $d \geq 0$. We can therefore apply Lemma 3.6 to conclude the proof. □

Combining Corollary 3.7 and Theorem 3.8, we get:

**Corollary 3.9.** Let $M$ be a partially cancellative torsion-free monoid and let $X$ be a quasi-excellent affine scheme of dimension $d \geq 0$ over $\mathbb{Q}$. Assume that one of the following holds.

1. $d \geq 1$.
2. $M$ is cancellative and semi-normal.
Then the map $K_i(X) \to K_i(X|M)$ is an isomorphism for all $i \leq -d$.

Remark 3.10. One can easily check from the proof of Lemma 3.6 that the above proof of Theorem 3.8 remains valid (without any change) for all $d$-dimensional schemes over any ground field $k$ which admits weak resolution of singularities for schemes of dimensions up to $d$. Since the resolution of singularities is known to hold in dimension up to three over any ground field (see [8]), we see that the assertion of Theorem 3.8 remains valid for affine schemes over any arbitrary field as long as $d \leq 3$.

We also remark that by the same reason as above, Theorem 3.8 is also valid if $k$ is any field and $X$ is either an affine normal crossing scheme or an affine toric variety over $k$. This is because $X$ admits resolution of singularities in both cases.

3.4. Vanishing of $K_{< -d}(X|M)$. In this subsection, we shall prove a vanishing result for $K_{< -d}(X|M)$ if $X$ is a quasi-excellent $\mathbb{Q}$-scheme of dimension $d$ and $M$ is a cancellative torsion-free monoid. We need the following extension of Lemma 3.5.

Lemma 3.11. Let $M$ be a cancellative torsion-free monoid and $X$ a regular Noetherian scheme of dimension $d \geq 0$. Then $K_i(X|M) = 0$ for $i < -d$.

Proof. Since $X$ is regular, we can assume that it is connected. We first assume that $X$ is affine. In this case, it follows from [15, Theorem 1.3] that $K_i(X|M) = 0$ for $i \leq -1$. Suppose now that $X$ is any Noetherian scheme of dimension $d$. Using [22, Theorem B], it suffices to show that $C^M K_i(X) = 0$ for $i < -d$. But this follows immediately from the spectral sequence (3.2) (see the proof of Lemma 3.3). \□

Theorem 3.12. Let $M$ be a cancellative torsion-free monoid and $X$ a quasi-excellent $\mathbb{Q}$-scheme of dimension $d$. Then $K_i(X|M) = 0$ for $i < -d$.

Proof. This proof is identical to that of Lemma 3.6. We give the sketch. We prove the theorem by induction on $d$. The $d = 0$ case follows from Lemma 3.11. So we assume that $d \geq 1$.

By Lemma 3.3, we can assume that $X$ is reduced. Consequently, we can assume that $X|M$ is reduced by [2, Theorem 4.19] as $X$ is a $\mathbb{Q}$-scheme.

By Theorem 3.1, we have an abstract blow-up square as in (3.1). Adjoining our monoid, this yields another abstract blow-up square:

\[
\begin{array}{ccc}
\mathcal{Y}[M] & \to & \mathcal{X}[M] \\
\downarrow & & \downarrow \\
\mathcal{Y}[M] & \to & X[M].
\end{array}
\]

(3.7)

Now, as we did in the proof of Lemma 3.6, we can apply the pro-cdh-descent theorem [22, Theorem A] to get an exact sequence of pro-abelian groups

\[
\{K_{i+1}(n\mathcal{Y}[M])\} \to K_i(X|M) \to \{K_i(n\mathcal{Y}[M])\} \oplus K_i(\mathcal{X}[M]).
\]

(3.8)

Since $Y \subset X$ and $\mathcal{Y} \subset \mathcal{X}$ are nowhere dense closed subsets, we see that $\dim(Y)$ and $\dim(\mathcal{Y})$ are less than $d$. Using induction on $d$ and regularity of $\mathcal{X}$, we see that the end terms of this exact sequence vanish. We conclude that $K_i(X|M) = 0$ for $i < -d$. \□

3.5. The positive characteristic case. The weak resolution of singularities in all dimensions is yet unknown if the ground field $k$ has positive characteristic. Nevertheless, we can show that Theorem 3.8 is valid in this case too if we invert $\text{char}(k)$. Using the reduction steps of §2, this turns out to be actually an easy consequence of the homotopy invariance property of Weibel’s homotopy $K$-theory. More precisely, we can prove the following.
**Theorem 3.13.** Let $k$ be a field of characteristic $p > 0$. Let $X \subset \widetilde{\text{Spec}}(R)$ be an affine scheme of dimension $d \geq 0$. Assume that one of the following holds.

1. $d \geq 1$.
2. $M$ is cancellative and semi-normal.

Then the map $K_i(X)[\frac{1}{p}] \to K_i(X[M])[\frac{1}{p}]$ is an isomorphism for all $i \leq -d$.

**Proof.** In this proof, we shall work only with $\mathbb{Z}[\frac{1}{p}]$-modules. In particular, $K_i(X)$ will mean $K_i(X)[\frac{1}{p}]$ for simplicity of notation. In this case, we can replace the spectrum $K(X)$ by $K\text{H}(X)$ (see Exercise 9.11(h)), where the latter is Weibel’s homotopy $K$-theory.

Now, we can assume $M$ to be cancellative torsion-free and positive by using the $K\text{H}$-analogues of Lemmas 2.8 and 2.7. If we now write $X = \text{Spec}(R)$, it follows that $R[M]$ is a positively graded $R$-algebra. Therefore, by the homotopy invariance property [37, Theorem 1.2], we get $K\text{H}_i(X) \cong K\text{H}_i(X[M])$ for $i \leq -d$.

4. **Counterexample for non-affine schemes**

Recall that Weibel’s $K_{-d}$-regularity conjecture is true for all Noetherian separated schemes [22]. We saw in the previous section that this holds also for cancellative torsion-free semi-normal monoids. However, we shall show in this section that the monoid extension of $K_{-d}$-regularity conjecture is not valid for non-affine schemes if $M$ is not semi-normal. This shows that the extension of Weibel’s $K_{-d}$-regularity conjecture to monoids is a subtle question.

We let $M \subset \mathbb{Z}_+$ be the submonoid generated by $\{2, 3\}$. It is clear that $M$ is cancellative torsion-free, but not semi-normal. It is also clear that the inclusion $R[M] \subset R[\mathbb{Z}_+]$ is same as the inclusion $R[x^2, x^3] \subset R[x]$ for any ring $R$. We fix a field $k$ of characteristic zero and let $C = \text{Spec}(k[M])$.

**Theorem 4.1.** Let $X$ be a connected smooth projective curve over $k$ of positive genus. Then the map $K_{-1}(X) \to K_{-1}(X[M])$ is not an isomorphism.

**Proof.** Since $K_{-1}(X) = 0$, the theorem is equivalent to showing that $K_{-1}(X[M]) \neq 0$.

We consider the conductor square

\[
\begin{array}{ccc}
S & \overset{u}{\longrightarrow} & T \\
\downarrow f' & & \downarrow f \\
\overset{T'}{C} & \overset{f}{\longrightarrow} & \overset{T'}{C} \\
S & \overset{u}{\longrightarrow} & T \\
\downarrow f' & & \downarrow f \\
\overset{T'}{C} & \overset{f}{\longrightarrow} & \overset{T'}{C} \\
\end{array}
\]

where the square on the right is obtained by the one on the left by the base change via the map $X \to \text{Spec}(k)$. The map $f$ is the normalization map, $S \cong \text{Spec}(k)$ and $T \cong \text{Spec}(k[x]/(x^2))$. The inclusion $u : S \to T$ is induced by the augmentation $k[x]/(x^2) \to k$.

We write $X_T = X \times Y$ for any $k$-scheme $Y$. Note that $X_S \cong X$. For any $n \geq 1$, we have a commutative diagram of relative $K$-theory exact sequences:

\[
\begin{array}{ccc}
K_0(X_C) & \longrightarrow & K_0(nX) \longrightarrow K_{-1}(X_C, nX) \overset{\alpha_n}{\longrightarrow} K_{-1}(X_C) \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
K_0(X_{A_k^n}) & \longrightarrow & K_0(nX_T) \overset{\beta_n}{\longrightarrow} K_{-1}(X_{A_k^n}, nX_T) \longrightarrow 0,
\end{array}
\]

where the vertical arrows are induced by $f$. Note that for a closed immersion $W \subset Y$ defined by the sheaf of ideals $\mathcal{I}_W$, the subscheme $nW \subset Y$ is defined by $\mathcal{I}_n^W$. The map $\alpha_n$ is surjective because its cokernel will otherwise map injectively into $K_{-1}(nX)$. But
this latter term is isomorphic to $K_{-1}(X)$ by Lemma 3.3 which is zero. The map $\beta_n$ is surjective because $K_{-1}(X_{k^n}) = 0$.

When $n = 1$, we have the situation

\[
\begin{array}{c}
\xymatrix{
K_0(X) & K_0(X_C) & K_0(X) \\
\ar[d]_{p^*} & \ar[l]_{\iota^*} & \ar[d]_{f^*} \\
K_0(X_{T_{k^n}}) & K_0(X_T) & K_{-1}(X_{k^n}, X_T),
}
\end{array}
\]

where $p^*$ is induced by the projection $X_C \to X$. It follows that the composite horizontal arrow on the top is identity. The indicated isomorphism is by the homotopy invariance. A diagram chase shows that $\iota^*$ is split surjective and $K_{-1}(X_{k^n}, X_T) \cong \text{Coker}(f^*) = \text{Coker}(f'^*).$ Using this, we also see that $\alpha_1$ is an isomorphism. In particular, the theorem is equivalent to showing that $K_{-1}(X_C, X) \neq 0$.

We now consider the inclusion of inductive systems of closed pairs $\{(X_{k^n}, X_T)\} \hookrightarrow \{(X_{k^n}, nX_T)\}$. We make the following

**Claim.** The induced map of pro-abelian groups $\{K_{-1}(X_{k^n}, X_T)\} \to K_{-1}(X_{k^n}, X_T)$ is surjective.

To prove the claim, it suffices to show that the map $K_{-1}(X_{k^n}, X_T) \to K_{-1}(X_{k^n}, X_T)$ is surjective for all $n \geq 1$.

We fix $n \geq 1$ and consider the commutative diagram

\[
\begin{array}{c}
\xymatrix{
K_0(nX_T) & K_{-1}(X_{k^n}, nX_T) \\
\ar[d] & \ar[d] \\
K_0(X_T) & K_{-1}(X_{k^n}, X_T).
}
\end{array}
\]

Because $K_{-1}(X_{k^n}) = 0$, the relative $K$-theory exact sequence tells us that the horizontal arrows are surjective. It suffices therefore to show that the left vertical arrow is surjective.

Since $\dim(X) = 1$, it is easy to check using the Thomason-Trobaugh spectral sequence [33, Theorem 10.3] that there is a functorial split exact sequence

\[
0 \to H^1(nX_T, O_{nX_T}^\times) \to K_0(nX_T) \to H^0(X, \mathbb{Z}) \to 0.
\]

To compute the left term, we consider the short exact sequence of sheaves

\[
0 \to (1 + \mathcal{I}_X) \to O_{nX_T}^\times \to O_X^\times \to 0,
\]

where $\mathcal{I}_X$ is the sheaf of ideals on $nX_T$ defining $X$. Since $nX_T \cong X \times \text{Spec}(k[x]/(x^{2n}))$ and $\text{char}(k) = 0$, this sequence is split and the exponential map $\exp : O_X \otimes_k (x)/(x^{2n}) \to (1 + \mathcal{I}_X)$ is an isomorphism of the sheaves of abelian groups. We thus have a commutative diagram of the split exact sequences of sheaves of abelian groups

\[
\begin{array}{c}
\xymatrix{
0 & O_X \otimes_k (x)/(x^{2n}) & O_{nX_T}^\times \\
\ar[d] & \ar[l] & \ar[d] \\
0 & O_X \otimes_k (x)/(x^2) & O_X^\times.
}
\end{array}
\]

This yields an associated commutative diagram of split exact sequences of the first cohomology groups. On the other hand, the left vertical arrow in (4.6) is split surjective because $(x)/(x^{2n}) \cong \bigoplus_{i=1}^{2n-1} (x^i)/(x^{i+1})$. It follows that the map $H^1(nX_T, O_{nX_T}^\times) \to$
$H^1(X_T, \mathcal{O}^\times_{X_T})$ is split surjective. Using (4.5), we conclude that the map $K_0(nX_T) \to K_0(X_T)$ is surjective. We have thus proven the claim.

Since $\mathcal{O}_X \cong \mathcal{O}_X \otimes_k (x)/(x^2)$, it also follows from (4.5) and (4.6) that $K_0(X_T) \cong H^1(X, \mathcal{O}_X) \oplus K_0(X)$. Using this in (4.3), we get

$$K_{-1}(X_{\mathbb{A}^1_k}, X_T) \cong \text{Coker}(f^*) \cong \text{Coker}(i^*) \cong H^1(X, \mathcal{O}_X).$$

In the final step, we consider the commutative diagram of pro-abelian groups

$$
\begin{array}{ccc}
\{K_{-1}(X_C, nX)\} & \xrightarrow{\psi} & K_{-1}(X_C, X) \\
f^* & & f^* \\
\{K_{-1}(X_{\mathbb{A}^1_k}, nX_T)\} & \xrightarrow{\psi'} & K_{-1}(X_{\mathbb{A}^1_k}, X_T),
\end{array}
$$

where $\psi$ is induced by the inclusion $(X_C, X) \hookrightarrow (X_C, nX)$. The arrow $\psi'$ is similarly defined. By the pro-cdh-descent theorem [22, Theorem A] (note that [24, Theorem 1.1] suffices for the present case), applied to the abstract blow-up square on the right of (4.1), we see that the left vertical arrow in (4.8) is an isomorphism. The bottom horizontal arrow is surjective by the above claim. It follows that $f^* \circ \psi = \psi' \circ f^*$ is surjective.

We conclude that the map $f^*: K_{-1}(X_C, X) \to K_{-1}(X_{\mathbb{A}^1_k}, X_T)$ is surjective. We can now apply (4.7) to see that $K_{-1}(X_C, X)$ can not be zero if the genus of $X$ is positive. The proof of the theorem is complete.

5. $SK_0$ of monoid algebras: Some reductions

Our goal in the next two sections is to prove Theorems [1.3] and [1.7]. In this section, we establish some reduction steps which go into the proof of Theorem [1.7].

5.1. Structure of positive cancellative semi-normal monoids. Let $M$ be a cancellative monoid with possible torsion. We shall denote the torsion part of $\text{gp}(M)$ by $t(M)$. We let $\overline{M} := \text{image}(M \to \text{gp}(M)/t(M))$. There is an identification $\text{gp}(M) = \text{gp}(\overline{M}) \times t(M)$.

Suppose now that $M$ is a cancellative torsion-free monoid and let $\text{gp}(M) \cong \mathbb{Z}^r$. Recall from [31, § 5] that the interior of $M$ is its subset consisting of all elements $a \in M$ such that for all $b \in M$, there is an integer $n > 0$ and $c \in M$ such that $na = b + c$. We denote this set by $\text{Int}(M)$. Note that if $M$ is generated by a finite set $\{x_1, \ldots, x_n\}$, then $\sum_{i=1}^n x_i \in \text{Int}(M)$. In particular, $\text{Int}(M) \neq \emptyset$.

Since $\text{gp}(M) \cong \mathbb{Z}^r$, we can view $M$ as the set of of integral points in the vector space $\mathbb{R}^r$. We let $\mathbb{R}_+M$ denote the set of non-negative $\mathbb{R}$-linear combinations of elements in $M$. In this case, we have $\text{Int}(M) = \text{Int}(\mathbb{R}_+M) \cap \mathbb{Z}^r$, where $\text{Int}(\mathbb{R}_+M)$ is the topological interior of the cone $\mathbb{R}_+M$.

If $M$ is positive cancellative but not necessarily torsion-free, then $\overline{M}$ is a positive cancellative torsion-free monoid. We shall let $F(M) := \{F_0, \ldots, F_r\}$ denote the set of faces of the cone $\mathbb{R}_+\overline{M}$ including 0 and $\mathbb{R}_+\overline{M}$. We index $F(M)$ so that $\dim(F_i) < \dim(F_j)$ implies $i < j$ for all $F_i, F_j \in F(M)$. Let $\text{rk}(M) = m$ so that $\dim(\mathbb{R}_+\overline{M}) = m$. Given two monoids $L, N$, we define $L \times N := L \times N \setminus \{(0, n) | n \in N \setminus \{0\}\}$.

We shall use the following description of the semi-normalization of $M$, due to Gubeladze [18, Lemma 9.1].

**Lemma 5.1.** Let $M$ be a positive cancellative monoid (possibly with torsion). For every $F \in F(M)$, there is a subgroup $T_F \subset t(M)$ such that
(1) \( sn(M) = \bigcup_{F(M)} (\text{Int}(n(M \cap F)) \times T_F). \)

(2) \( T_{R+M} = t(M). \)

(3) \( T_{F_1} \subset T_{F_2} \text{ for } F_1 \subset F_2. \)

Recall from [31 § 5] that a submonoid \( E \) of any monoid \( M \) is called extremal if it is non-empty and \( a + b \in E \Rightarrow a, b \in E \) for all \( a, b \in M \). The following is from [31 Theorem 5.4].

**Lemma 5.2.** Let \( M \) be a cancellative monoid. Then a submonoid \( E \) of \( M \) is extremal if and only if there is a monoid homomorphism \( \phi : M \rightarrow \mathbb{N} \) such that \( \phi^{-1}(0) = E \).

We note down some properties of the monoid \( \overline{M} \) in the following lemma for later use.

**Lemma 5.3.** Let \( M \) be a positive cancellative semi-normal monoid. Then the following hold.

1. \( \overline{M} \) is a positive cancellative torsion-free semi-normal monoid.
2. Each \( \overline{M} \cap F_i \) is an extremal submonoid of \( \overline{M} \).
3. Each \( \overline{M} \cap F_i \) is a cancellative torsion-free semi-normal monoid.
4. If \( N \) is a submonoid of \( \overline{M} \) such that \( \text{Int}(N) \cap (\overline{M} \cap F_i) \) is non-empty, then \( N \subset \overline{M} \cap F_i. \)

**Proof.** We have already observed that \( \overline{M} \) is a positive cancellative torsion-free monoid. To see semi-normality, let \( a \in gp(\overline{M}) \) such that \( 2a, 3a \in \overline{M} \). Let \( sn(M) \) and \( sn(\overline{M}) \) denote the semi-normalizations of \( M \) and \( \overline{M} \), respectively. Using Lemma 5.1, we get a commutative diagram

\[
\begin{array}{ccc}
M & \rightarrow & sn(M) = \bigcup_{F(M)} (\text{Int}(n(\overline{M} \cap F)) \times T_F) \\
\phi \downarrow & & \downarrow \psi \\
\overline{M} & \rightarrow & sn(\overline{M}) = \bigcup_{F(M)} (\text{Int}(n(\overline{M} \cap F))
\end{array}
\]

where \( \psi \) is induced from the projection on each face. Clearly, \( \psi \) is a surjective map.

Since \( \overline{M} \) is cancellative, the map \( \overline{M} \rightarrow gp(\overline{M}) \) is injective. Hence the map \( \overline{M} \rightarrow sn(\overline{M}) \) is injective. Since \( a, 2a, 3a \in sn(\overline{M}) \) (see § 2.1), we can lift them to \( sn(\overline{M}) \) using (5.1) and use the fact that \( M = sn(M) \) to conclude that \( a \in \overline{M} \). Hence, we have \( \overline{M} = sn(M) \).

The part (2) follows from [2] Remark 2.6(c), Exercise 2.3(a) [see also 31 § 5, Remark]. To prove (3), we only need to show that \( \overline{M} \cap F_i \) is semi-normal. For this, let \( a \in gp(\overline{M} \cap F_i) \) such that \( 2a, 3a \in \overline{M} \cap F_i. \) Since \( \overline{M} \) is semi-normal, we must have \( a \in \overline{M}. \) Since \( F_i \) is an extremal submonoid of \( \overline{M} \) by (2) and \( a + a \in F_i, \) we must have \( a \in F_i. \) This proves (3).

For (4), let \( z \in N \) and \( x \in \text{Int}(N) \cap (\overline{M} \cap F_i). \) By definition of \( \text{Int}(N), \) there exists an integer \( n > 0 \) such that \( nx = z + y \) for some \( y \in N. \) Since \( F_i \) is extremal and \( z + y = nx \in F_i, \) it follows that \( z, y \in F_i. \) In particular, \( z \in N \cap F_i \subset \overline{M} \cap F_i. \)

**Lemma 5.4.** Let \( M \) be a positive cancellative semi-normal monoid. With the above notations, let \( I_k := \bigcup_{j \geq k} (\text{Int}(n(\overline{M} \cap F_j))) \times T_{F_j} \) be subsets of \( M \) for \( 0 \leq k \leq r. \) Then \( I_k \) is an ideal of \( M \) for each \( k. \)

**Proof.** Since \( M \) is semi-normal, Lemma 5.3(1) implies that \( \overline{M} \) is also semi-normal. It follows from Lemma 5.3(3) that each \( \overline{M} \cap F_j \) is also a cancellative torsion-free semi-normal monoid. Therefore, we deduce from [31 Lemma 6.6] that \( \text{Int}(n(\overline{M} \cap F_j)) = \text{Int}(\overline{M} \cap F_j). \)

We need to show that \( I_k + M \subset I_k \) for each \( k. \) For this, we note that \( (\text{Int}(\overline{M} \cap F_j)) \times T_{F_j} = \text{Int}(\overline{M} \cap F_j) \times T_{F_j} \) for each \( j. \) Furthermore, \( R_+ \overline{M} \cap M = \bigcup_{j} \text{Int}(\overline{M} \cap F_j) \) is a disjoint
union (see \[31\], Lemma 5.3). Hence, Lemma 5.1 implies that $M = \Pi_j \text{Int}(M \cap F_j) \times T_{F_i}$ is a disjoint union, where we identify $gp(M) = gp(M) \times t(M)$ and look at $M$ as a submonoid of $gp(M)$.

We now let $a \in I_k$ and $b \in M$. Then $a = (a_1, a_2) \in \text{Int}(M \cap F_j) \times T_{F_i}$ for some $j \geq k$ and $b = (b_1, b_2) \in \text{Int}(M \cap F_i) \times T_{F_i}$ for some $0 \leq i \leq r$. Then $a + b = (a_1 + b_1, a_2 + b_2) \in M$ and hence it must belong to $\text{Int}(M \cap F_i) \times T_{F_i}$ for some $l$. We have to show that $l \geq k$ to finish the proof.

To show this, we note that $M \cap F_i$ is an extremal submonoid of $M$. It follows from this that $a_1, b_1 \in M \cap F_i$. In particular, we get $a_1 \in \text{Int}(M \cap F_i) \cap \text{Int}(M \cap F_i)$. Lemma 5.3(4) therefore implies that $M \cap F_j \subseteq M \cap F_i$. Since each face of $R_+ M$ is its intersection with finitely many hyperplanes, each of which must either be non-negative or non-positive on $R_+ M$, it follows that $F_j \subseteq F_i$. This in turn implies that $l \geq j$. As $j \geq k$, we get $l \geq k$, as desired.

\[5.2.\] Milnor square associated to positive torsion monoids. Let $R$ be a ring and $M$ a positive cancellative seminormal monoid as in §5.1. We let $A_k = R[M/I_k] = R[M/I_k]$ for $0 \leq k \leq r$. There is a sequence of surjective $R$-algebra homomorphisms $A_{r+1} := R[M] \twoheadrightarrow A_r \rightarrow \cdots \rightarrow A_k \rightarrow A_{k-1} \rightarrow \cdots \rightarrow A_1 \twoheadrightarrow A_0 = R$. Let $\phi_k : A_k \rightarrow A_{k-1}$ be the quotient map for $1 \leq k \leq r + 1$.

**Lemma 5.5.** The following hold for $1 \leq k \leq r + 1$.

1. $\text{Ker}(\phi_k) = R(\text{Int}(M \cap F_{k-1}) \times T_{F_{k-1}})$.
2. There is a Milnor square

\[
\begin{array}{ccc}
R(\text{Int}(M \cap F_{k-1}) \times T_{F_{k-1}}) & \longrightarrow & A_k \\
\downarrow \delta_{k-1} & & \downarrow \phi_k \\
R & \longrightarrow & A_{k-1}.
\end{array}
\]

**Proof.** Let $I_{k-1}/I_k$ be the image of $I_{k-1}$ in $M/I_k$ under the quotient map $M \rightarrow M/I_k$. Then it is clear from Lemma 5.4 that $I_{k-1}/I_k$ is an ideal in $M/I_k$ and the monoid $M/I_{k-1}$ is obtained from $M/I_k$ by collapsing $I_k/I_{k-1}$. It follows that

\[
0 \rightarrow R[I_{k-1}/I_k] \rightarrow R[M/I_k] \xrightarrow{\phi_k} R[M/I_{k-1}] \rightarrow 0
\]

is an exact sequence of $R$-modules (see §2.11). The first assertion now follows as $R[I_{k-1}/I_k] = R[\text{Int}(M \cap F_{k-1}) \times T_{F_{k-1}}]$.

To prove (2), we first note that $\delta_k$ is the canonical inclusion $R \hookrightarrow A_k$. Moreover, there is a commutative diagram

\[
\begin{array}{ccc}
R & \xrightarrow{\delta_k} & R[I_{k-1}/I_k] \\
\downarrow & & \downarrow \\
\text{Int}(M \cap F_{k-1}) \times T_{F_{k-1}} & \longrightarrow & A_k \\
\downarrow \delta_{k-1} & & \downarrow \phi_k \\
R & \longrightarrow & A_{k-1}.
\end{array}
\]

Using this diagram, we conclude immediately from (1) that (5.2) is Cartesian. Since $\phi_k$ is surjective, it follows that this is also a Milnor square.

\[\square\]
5.3. **Reduction to positive semi-normal monoids.** In this subsection, we shall prove some lemmas to reduce the proof of Theorem 14.7 to the case of positive semi-normal monoids. We begin by recalling the definition of $SK_1$ and $SK_0$ of rings.

Let $R$ be a ring and let $H_0(R)$ denote the set of all continuous functions from $\text{Spec}(R) \to \mathbb{Z}$ with respect to the Zariski topology on $\text{Spec}(R)$ and the discrete topology on $\mathbb{Z}$. It is easy to verify that this is a ring. There is a group homomorphism $rk : K_0(R) \to H_0(R)$ such that $rk([P]) = \text{rank}(P)$ if $P$ is a projective $R$-module. We define $\bar{K}_0(R) := \ker(rk)$. There is a map $det : K_0(R) \to \text{Pic}(R)$ which sends $[P]$ to $[\wedge^r(P)]$, where $r$ is the rank of $P$. Its restriction yields a group homomorphism $det : \bar{K}_0(R) \to \text{Pic}(R)$. We define $SK_0(R) := \ker(det)$. Let $SK_1(R) = SL(R)/E(R)$ so that there is a canonical decomposition $K_1(R) = SK_1(R) \oplus U(R)$. The following lemma is elementary.

**Lemma 5.6.** $SK_0(R) = 0$ if and only if for every projective $R$-module $P$, one has $P \oplus R^s \cong \wedge^r(P) \oplus R^t$, where $r = \text{rk}(P)$.

**Proof.** Suppose first that $SK_0(R) = 0$. Let $P$ be a projective $R$-module of rank $r \geq 1$. Then it is easy to see that $[P] = [\wedge^r(P)]$ in $K_0(R)$. But this implies that $P \oplus R^s \cong \wedge^r P \oplus R^t$, as is well known. The converse is obvious. \hfill $\square$

The following result which connects $SK_1$ with $SK_0$, is due to Bass [11, Corollary 5.12].

**Proposition 5.7.** If (2.7) is a Milnor square, then we have the 6-term exact sequence $SK_1(A_1) \to SK_1(A_2) \oplus SK_1(B_1) \to SK_1(B_2) \to SK_0(A_1) \to SK_0(A_2) \oplus SK_0(B_1) \to SK_0(B_2)$.

**Lemma 5.8.** Let $R$ be an Artinian ring and $M$ a cancellative torsion-free semi-normal monoid. Then the following hold.

1. $SK_0(R[M]) = 0$.
2. If $M$ is free and positive, then $SK_1(R[M]) = 0$.

**Proof.** By Lemma 2.4, we can assume that $R$ is reduced. In this case, we have $R \cong k_1 \times \cdots \times k_m$, where each $k_i$ is a field. The assertion (1) now follows [15, Theorem 1.3].

Similar to the case of $SK_0(R)$, we have $SK_1(R[M]) = SK_1(R_{red}[M])$ by [11, Chapter IX, Proposition 3.10, 3.11]). So we can assume $R$ to be a field. Since $M \cong \mathbb{Z}_+^r$, by our assumption, we get $SK_1(R[M]) \cong SK_1(R) = 0$, where the first isomorphism is by the homotopy invariance. \hfill $\square$

Recall from [31, § 14] that a ring extension $A \subset B$ is called an elementary subintegral extension if $B = A[x]$, where $x^2, x^3 \in A$. We say that $A \subset B$ is a subintegral extension if it is a filtered union of elementary subintegral extensions.

**Lemma 5.9.** Let $A \subset B$ be a subintegral extension. If $SK_0(B) = 0$, then $SK_0(A) = 0$.

**Proof.** By Lemma 5.6, we need to show that $P \oplus A^s \cong \wedge^r(P) \oplus A^t$ for any projective $A$-module $P$ of rank $r \geq 1$. At any rate, our assumption and Lemma 5.6 together imply that $P_B := P \otimes_A B$ has the property that $P_B \oplus B^s \cong \wedge^r(P_B) \oplus B^t$. Equivalently, we have $(P \oplus A^s)_B \cong (\wedge^r(P) \oplus A^t)_B$. But this implies that $P \oplus A^s \cong \wedge^r(P) \oplus A^t$ by [31, Theorem 14.1]. \hfill $\square$

**Lemma 5.10.** Let $R$ be an Artinian ring and $M$ any monoid. We let $N = M \setminus U(M)$, where $U(M)$ is the group of units of $M$. If $SK_0(R[N_s]) = 0$, then $SK_0(R[M]) = 0$.

**Proof.** Since $M$ is finitely generated, we have $U(M) \cong \mathbb{Z}^r \oplus G$ for some $r \geq 0$, where $G$ is a finite abelian group. Under the hypothesis of the lemma, there is a Milnor square of rings as in (2.3) (see also [31, § 6]). Since $R[G]$ is an Artinian ring, we have $SK_0(R[U(M)]) \cong SK_0(R[G][\mathbb{Z}^r]) = 0$ by Lemma 5.8. As $SK_0(R[N_s]) = 0$, we conclude using Proposition 5.7 that $SK_0(R[M]) = 0$. \hfill $\square$
6. $SK_0$ and the Levine–Weibel Chow group

In this final section, we shall first prove Theorem 1.7 and then deduce Theorem 1.5 using Theorem 1.7 and the affine Roitman torsion theorem 23 for the Levine–Weibel Chow group.

6.1. $SK_0$ of cancellative monoid algebras. We shall first prove our main result for the vanishing of $SK_0$ of monoids algebras when the underlying monoid is cancellative. More precisely, we prove:

Lemma 6.1. Let $R$ be an Artinian ring and $M$ a commutative cancellative monoid. Then $SK_0(R[M]) = 0$.

Proof. We can assume, using Lemmas 5.9 and 5.10 that $M$ is semi-normal and positive. If we further assume that $M$ is torsion-free, then $SK_0(R[M]) = 0$ by Lemma 5.8. So we can assume that $M$ is positive, cancellative and semi-normal but not torsion-free.

We can write $M = sn(M) = \cup F(M)(\int(M \cap F \times T_F))$ by Lemma 5.4. Note that $\int(n(M \cap F)) = \int(M \cap F)$ by Lemma 6.6. We shall prove the lemma by applying Proposition 5.7 to (5.2). We note that $A_0 \cong A_1 \cong R$ and $A_{r+1} = R[M]$. Hence, it suffices to show using an induction argument that $SK_0(A_k) = 0$ for $1 \leq k \leq r + 1$.

Since $SK_0(R) = 0$, the base case for induction is established. It suffices now to prove the lemma for $A_k$ assuming it holds for $A_{k-1}$ for $k \geq 2$. Using Lemma 5.5 and Proposition 5.7, it suffices to show that $SK_0(R[\{(\int(M \cap F_{k-1}) \times T_{F_{k-1}})\}]) = 0$.

Now, we observe that $R[\{(\int(M \cap F_{k-1}) \times T_{F_{k-1}})\} = R[\{(\int(M \cap F_{k-1})) \times T_{F_{k-1}}\}].$ Letting $L = (\int(M \cap F_{k-1}))$, we see that $L$ is a normal positive cancellative torsion-free monoid (see Proposition 2.40). Writing $F_{k-1} = \prod_{i=1}^{m} \mathbb{Z}/n_i$, it suffices to show by induction on $m$ that

(6.1) $SK_0(R[L \times (\prod_{i=1}^{m} \mathbb{Z}/n_i)]) = 0$.

If $m = 0$, then (6.1) is true by our assumption. In general, there is a Milnor square (see Proof of Theorem 1.1, p. 214)

(6.2) $\Lambda \longrightarrow R[L \times (\prod_{i=1}^{m} \mathbb{Z}/n_i)]$ $\downarrow$ $\downarrow$

$R[\mathbb{Z}_+][L \times (\prod_{i=1}^{m-1} \mathbb{Z}/n_i)] \xrightarrow{\pi} R[\mathbb{Z}/n_m][L \times (\prod_{i=1}^{m-1} \mathbb{Z}/n_i)]$,

where

1. $\Lambda = A + B$,
2. $A = R[\mathbb{Z}/n_m][(L \times \mathbb{Z}_+) \times (\prod_{i=1}^{m-1} \mathbb{Z}/n_i)]$,
3. $B = R[t^{n_m} - t, \ldots, t^{2n_m-1} - t^{n_m-1}] \subset R[t] \simeq R[\mathbb{Z}_+]$ and
4. $\pi$ is induced by $t \mapsto x$ for some generator $x \in \mathbb{Z}/n_m$.

By induction on $m$ and Proposition 5.7, it suffices to show that $SK_0(\Lambda) = 0$.

We now consider another Milnor square (see Proof of Theorem 1.1) $A \longrightarrow \Lambda$ $\downarrow$ $\downarrow$

$R \longrightarrow B$. 

\[\Lambda \longrightarrow R[L \times (\prod_{i=1}^{m} \mathbb{Z}/n_i)] \downarrow \downarrow \]

$R[\mathbb{Z}_+][L \times (\prod_{i=1}^{m-1} \mathbb{Z}/n_i)] \xrightarrow{\pi} R[\mathbb{Z}/n_m][L \times (\prod_{i=1}^{m-1} \mathbb{Z}/n_i)],$
Using this square and Proposition 5.7 again, it suffices to prove that $SK_0(A) = 0 = SK_0(B)$.

Since $L \times Z_+$ is a finitely generated positive, cancellative torsion-free normal monoid, it follows by induction on $m$ that $SK_0(A) = 0$. To prove the result for $B$, we can assume $R$ is reduced by Lemma 2.4. We can further assume that $R$ is a field. We now observe that there is a conductor square

$$
\begin{array}{ccc}
B & \rightarrow & R[t] \\
\downarrow & & \downarrow \\
B/C & \rightarrow & R[t]/C,
\end{array}
$$

where $C$ is the conductor ideal of the extension $B \subset R[t]$. Being a subring of the integral domain $R[t]$, $B$ is an integral domain. Since $t(t^m - 1) \in C$ is a non-zero-divisor, we see that the height of $C$ is positive. It follows that $B/C$ and $R[t]/C$ are both Artinian rings. In particular, $SK_1(R[t]/C) = 0$. Since $SK_0(B/C) = SK_0(R[t]) = 0$, it follows from Proposition 5.7 that $SK_0(B) = 0$. \qed

6.2. The final step for Theorem 1.7. Let $M$ be an arbitrary (finitely generated) monoid. Recall from [31 § 15] that an ideal $P \subset M$ is called prime if $P \neq M$ and if $x, y \in M, x + y \in P$ implies $x \in P$ or $y \in P$. This is equivalent to saying that $N = M \setminus P$ is a non-empty submonoid of $M$. In this case, there are monoid algebra morphisms $R[N] \rightarrow R[M] \rightarrow R[N] \cong R[M]/R[P]$ for any ring $R$ whose composite is identity. It is also easy to check that if $p$ is a prime ideal of $R[M]$, then $p \cap M$ is a prime ideal of $M$.

An ideal $I \subset M$ is called a radical ideal if every element $x \in M$ with the property $nx \in I$ for some $n > 0$, belongs to $I$. If $I \subset M$ is an ideal, we let $\sqrt{I} = \{x \in M| nx \in I \text{ for some } n > 0\}$. We call this the radical of $I$. It is easy to check that $\sqrt{I}$ is a radical ideal of $M$. Our key step to finish the proof of Theorem 1.7 is the following result which generalizes [31 Lemma 15.6] to arbitrary monoids.

**Lemma 6.2.** Let $I \subset M$ be a proper radical ideal in a monoid $M$. Then there are prime ideals $p_1, \ldots, p_r$ in $M$ such that $I = \cap_{i=1}^r p_i$.

**Proof.** In this proof, we shall use the multiplicative notation for the monoid operation on $M$. Since $Z[I]$ is an ideal of the Noetherian ring $Z[M]$ (note that $M$ is finitely generated), we have the (irredundant) primary decomposition $Z[I] = \cap_{i=1}^r Q_i$. We let $P_i$ denote the unique associated prime of $Q_i$ in $Z[M]$. Recall that if we write the monoid operation of $M$ multiplicatively, then there is a multiplicative monoid embedding $M \hookrightarrow Z[M]$ which sends $m$ to $(1 \cdot m)$. We let $q_i = Q_i \cap M$ and $p_i = P_i \cap M$ via this embedding. It is easy to check from the definition of $Z[M]$ that $I = Z[I] \cap M$. We therefore get $I = Z[I] \cap M = \cap_{i=1}^r (Q_i \cap M) = \cap_{i=1}^r q_i$. In particular, we have $I = \cap_{i=1}^r q_i \subset \cap_{i=1}^r p_i$. Note that each $p_i$ is a prime ideal of $M$ as we already observed earlier.

Suppose now that there is an element $x \in M$ which lies in $\cap_{i=1}^r p_i$. This implies that $x$ lies in $\cap_{i=1}^r P_i$ inside $Z[M]$. Since this intersection is same as $\sqrt{Z[I]}$ in $Z[M]$, it follows that $x^m \in Z[I]$ for some $m > 0$. Since $x \in M$ and $M$ is multiplicatively closed in $Z[M]$, we also have $x^m \in M$. Consequently, we get $x^m \in Z[I] \cap M = I$. But $I$ is a radical ideal of $M$ and hence we must have $x \in I$. We have therefore shown that $\cap_{i=1}^r p_i \subset I$. This proves the lemma. \qed
The following lemma is elementary.

**Lemma 6.3.** Let \( M \) be any monoid and \( I \subset M \) any ideal. Then \( R[\sqrt{I}] \subseteq R[I] \) for any ring \( R \).

**Proof.** As in the proof of Lemma 6.2 we shall use the multiplicative notation for the monoid structure of \( M \). Let \( u = a_1x_1 + \cdots + a_rx_r \in R[\sqrt{I}] \) with \( a_i \in R \) and \( x_j \in \sqrt{I} \) for each \( i \). Since \( I \subset M \) is an ideal, we can find \( m_0 \gg 0 \) such that \( x_i^{m_0} \in I \) for all \( m \geq m_0 \) and all \( i \geq 1 \). It is then straightforward to check using the multinomial expansion of \( u^m \) that for all \( m \geq rm_0 \), all \( M \)-coefficients of \( u^m \) will lie in \( I \). That is, if we write \( u^m = b_1y_1 + \cdots + b_sy_s \), then each \( y_i \in I \) for \( 1 \leq i \leq s \). But this implies that \( u \in \sqrt{R[I]} \).

**Proof of Theorem 1.7:** We can assume \( R \) to be reduced by Lemma 2.4. We can then further assume that \( R \) is a field. Let \( N \) be a cancellative monoid and \( I \subset N \) an ideal such that \( M = N/I \). If \( I = N \), then \( R[M] \cong R \) and the theorem is obvious. So we can assume that \( I \subset N \) is a proper ideal. In particular, \( I \cap U(N) = \emptyset \).

Since the image of \( R[\sqrt{I}] \) in \( R[M] \) is nilpotent by Lemma 6.3, we can assume that \( I \) is a radical ideal of \( N \) by Lemma 2.4. In this case, Lemma 6.2 says that we can write \( I = \bigcap_{i=1}^r p_i \), where \( p_1, \ldots, p_r \) are prime ideals in \( N \).

We shall now prove the theorem by induction on \( r \geq 1 \). If \( r = 1 \), then \( I \) is a prime ideal of \( N \) so that \( J = N \setminus I \) is a submonoid of \( N \) and \( R[M] \cong R[J] \). Since \( N \) is cancellative, it follows that \( J \) is also cancellative. So we are done in this case by Lemma 6.1.

In general, we let \( p = p_1 \) and \( q = p_2 \cap \cdots \cap p_r \) with \( r \geq 2 \). Then \( I = p \cap q \) and \( L = p \cup q \) is a proper ideal of \( N \) since \( p, q \subset N \setminus U(N) \). Note here that the union of two ideals in a ring is generally not an ideal, but it is true for ideals in a monoid.

Since \( p/I \cong L/q \), it is easy to check using exact sequences of the type (5.3) that the diagram

\[
\begin{array}{ccc}
R[N]/R[I] & \rightarrow & R[N]/R[q] \\
\downarrow & & \downarrow \\
R[N]/R[p] & \rightarrow & R[N]/R[L]
\end{array}
\]

is Cartesian (see [31, Proof of Theorem 15.1]). Furthermore, if we let \( N' = N \setminus p \) and \( S = N' \cap L = N' \cap q \), then \( R[N']/R[S] \cong R[N]/R[L] \). On the other hand, the inclusion \( N' \hookrightarrow N \) takes \( S \) into \( q \) and induces a map \( R[N']/R[S] \rightarrow R[N]/R[q] \). This shows that the right vertical arrow in (6.3) is a split surjection.

By induction on \( r \), we see that \( SK_0(R[N]/R[p]) = 0 = SK_0(R[N]/R[q]) \). It follows from Proposition 5.7 that the sequence

\[
SK_1(R[N]/R[p]) \oplus SK_1(R[N]/R[q]) \rightarrow SK_1(R[N]/R[L]) \rightarrow SK_0(R[N]/R[I]) \rightarrow 0
\]

is exact. Since the right vertical arrow in (6.3) is a split surjective morphism of \( R \)-algebras, it follows that the first arrow from left in this exact sequence is surjective. This implies that \( SK_0(R[M]) = SK_0(R[N]/R[I]) = 0 \). This finishes the proof of the theorem.

**6.3. The Levine-Weibel Chow group.** Our goal in this subsection is to prove Theorem 1.3. As we explained in §11, our approach is to use the vanishing of \( SK_0 \) of the given monoid algebra and then use the Affine Roitman torsion theorem for the Levine-Weibel Chow group from [25]. Before we give the details, we recall the definition of the Levine-Weibel Chow group from [27] for reader’s reference.
Let $k$ be an algebraically closed field of any characteristic. Let $A$ be a finite type reduced $k$-algebra and let $X = \text{Spec}(A)$ denote the spectrum of $A$. We shall say that a point $x \in X$ is regular if $\mathcal{O}_{X,x}$ is a regular local ring. We let $X_{\text{reg}} \subset X$ denote the regular locus of $X$ so that $x \in X_{\text{reg}}$ if and only if it is a regular point. We let $X_{\text{sing}} = X \setminus X_{\text{reg}}$ denote the singular locus of $X$. A closed subscheme $C \subset X$ is called a Cartier curve if it is a scheme of pure dimension one such that the following hold.

1. No irreducible component of $C$ lies in $X_{\text{sing}}$.
2. For every $x \in C \cap X_{\text{sing}}$, the ideal $I_{C,x}$ of $C$ in the local ring $\mathcal{O}_{X,x}$ is generated by a regular sequence.

For a Cartier curve $C$, let $k(C, X_{\text{sing}})^\times$ denote the group of invertible elements in the ring of total quotients of $C$ which are regular along $C \cap X_{\text{sing}}$.

Let $Z_0(X)$ denote the free abelian group on the set of regular closed points of $X$. Given a Cartier curve $C \subset X$ and $f \in k(C, X_{\text{sing}})^\times$, we have the divisor $\text{div}_C(f)$ of $f$ in the sense of [9 § 1.2] (see also [27, § 1]). Since $f$ is regular and invertible along $X_{\text{sing}}$, it follows that $\text{div}_C(f) \in Z_0(X)$. We let $\text{CH}^1_{LW}(X)$ be the quotient of $Z_0(X)$ by the subgroup $\mathcal{R}_0(X)$, generated by $\text{div}_C(f)$, where $C$ runs over all Cartier curves on $X$ and $f \in k(C, X_{\text{sing}})^\times$. We shall use the notations $\text{CH}^1_{LW}(X)$ and $\text{CH}^1_{LW}(A)$ interchangeably.

When $X$ is regular, $\text{CH}^1_{LW}(X)$ coincides with the classical Chow group of 0-cycles on $X$ (see [9, Chapter 1]). This is however not the case when $X$ has singularity. In this case, it is $\text{CH}^1_{LW}(X)$ which is known to be the correct Chow group of 0-cycles and is supposed to constitute the 0-cycle part of the conjectural full theory of cohomological Chow groups of $X$. Furthermore, it is directly related to theory of vector bundles on $X$ unlike the classical homological Chow group $\text{CH}^1(X)$.

Since the structure sheaf of a regular closed point $x \in X$ has finite tor-dimension over $X$, it follows that this point has a class $\text{cyc}(x)$ in $K_0(X)$. We thus get a cycle class map $\text{cyc} : Z_0(X) \to K_0(X)$. Furthermore, it is shown in [27, Proposition 2.1] that this map kills $\mathcal{R}_0(X)$ so that there is a well-defined cycle class map

$$\text{cyc} : \text{CH}^1_{LW}(X) \to K_0(X).$$

We have the following result about this cycle class map which is supposed to be well known.

**Lemma 6.4.** Suppose that $\dim(X) \geq 2$. Then the cycle class map has the factorization

$$\text{cyc} : \text{CH}^1_{LW}(X) \to SK_0(X).$$

**Proof.** Let $x \in X$ be a regular closed point and let $U = \text{Spec}(A) \setminus \{x\}$. We then have the Thomason-Trobaugh localization exact sequence

$$K_0(\{x\}) \xrightarrow{\cong} K_0\{x\}(X) \to K_0(X) \to K_0(U).$$

Since the image of the middle arrow is the subgroup generated by $\text{cyc}(x)$, it suffices to show that the maps $H^i(X, \mathbb{Z}) \to H^i(U, \mathbb{Z})$ and $\text{Pic}(X) \to \text{Pic}(U)$ are injective. The first assertion is obvious. So we need to prove the second assertion. For this, we let $S = \{x\}$. Using the isomorphism $H^1(X, \mathcal{O}_X^\times) \cong \text{Pic}(X)$ and the exact sequence

$$H^1_{\text{et}}(X, \mathcal{O}_X^\times) \to H^1(X, \mathcal{O}_X^\times) \to H^1(U, \mathcal{O}_U^\times),$$

it suffices to show that $H^1_{\text{et}}(X, \mathcal{O}_X^\times) = 0$. Since $x$ is a regular closed point, we have $H^1_{\text{et}}(X, \mathcal{O}_X^\times) \cong H^1_{\text{reg}}(X, \mathcal{O}_{X_{\text{reg}}}^\times)$ by excision. So we can assume $X$ is regular.

In this case, we have a long exact sequence

$$H^0(X, \mathcal{O}_X^\times) \to H^0(U, \mathcal{O}_U^\times) \to H^1_{\text{et}}(X, \mathcal{O}_X^\times) \to H^1(X, \mathcal{O}_X^\times) \to H^1(U, \mathcal{O}_U^\times).$$

Since $X$ is regular and codimension of $S$ is at least two in $X$, it is well known that the map $H^i(X, \mathcal{O}_X^\times) \to H^i(U, \mathcal{O}_U^\times)$ is an isomorphism for $i \leq 1$. We are therefore done. □
The second key ingredient in the proof of Theorem 1.5 is the following affine Roitman torsion theorem for 0-cycles. This is an old conjecture of Murthy [30] and is now a theorem [25, Corollary 7.6].

**Theorem 6.5.** Let $A$ be a reduced affine algebra over an algebraically closed field $k$. Then the cycle class map $\text{cyc} : \text{CH}_0^L(A) \to K_0(A)$ is injective.

**Proof of Theorem 6.5:** We let $X = \text{Spec}(k[M])$. Using Theorem 1.7 and Lemma 6.4, it suffices to show that the map $\text{cyc} : \text{CH}_0^L(X) \to K_0(X)$ is injective. But this follows from Theorem 6.5. $\square$

6.4. **A different proof of Theorem 1.5 for pcf monoids.** We end our discussion with another proof of Theorem 1.5 in the special case in which the underlying monoid is (partially cancellative and) torsion-free. Note that Theorem 1.5 proves the vanishing of $SK_0(R[M])$ for all partially cancellative monoids which are not necessarily torsion-free. But we decided to include this different proof in the special case because it is more $K$-theoretic in nature and crucially uses negative $K$-theory. Our hope is that this $K$-theoretic approach may be helpful in future generalizations of Theorem 1.5 to more general monoid algebras.

We now begin the proof. So let $M$ be a partially cancellative torsion-free monoid and $R$ an Artinian ring. We want to show that $SK_0(R[M]) = 0$.

If $M$ is cancellative, then the result follows from Lemma 6.1. We can therefore assume that $M$ is torsion-free but only partially cancellative. As in the proof of Lemma 2.8, we can assume that $M = N/I$, where $N$ is a cancellative torsion-free monoid. Associated to the Milnor-square (2.4), there exists a commutative diagram

\[
\begin{array}{c}
SK_0(R) \oplus SK_0(R[N]) \xrightarrow{\alpha} SK_0(R[M]) \\
\downarrow \quad \downarrow \\
K_0(R) \oplus K_0(R[N]) \xrightarrow{\beta} K_0(R[M]) \to K_{-1}(R[I_s]),
\end{array}
\]

where the bottom row is exact by Proposition 2.3. The vertical arrows are clearly injective. Here, the arrows $\alpha$ and $\beta$ are the canonical maps.

Since $R$ is Artinian and $I_s$ is a cancellative torsion-free monoid, it follows from Lemmas 2.3 and 2.10 that the last term of the bottom exact sequence in (6.6) vanishes. A diagram chase shows that that $SK_0(R[M])$ lies in the image of $\beta$. Since $K_0(R)$ is a canonical direct summand of $K_0(R)$, it follows that the image of $SK_0(R[M])$ in $\tilde{K}_0(R[M])$ lies in the image of the map $\tilde{K}_0(R) \oplus \tilde{K}_0(R[N]) \to \tilde{K}_0(R[M])$.

We now consider the commutative diagram

\[
\begin{array}{c}
SK_0(R) \oplus SK_0(R[N]) \xrightarrow{\alpha} SK_0(R[M]) \\
\downarrow \quad \downarrow \\
\tilde{K}_0(R[I_s]) \quad \tilde{K}_0(R) \oplus \tilde{K}_0(R[N]) \xrightarrow{\beta} \tilde{K}_0(R[M]) \to 0 \\
\downarrow \quad \downarrow \quad \downarrow \\
\text{Pic}(R[I_s]) \quad \text{Pic}(R) \oplus \text{Pic}(R[N]) \to \text{Pic}(R[M]) \to 0.
\end{array}
\]

If $A$ is a ring and $L$ is a projective $A$-module of rank one, then $[L] - [A] \in \tilde{K}_0(A)$. Furthermore, the map $\text{det} : \tilde{K}_0(A) \to \text{Pic}(A)$ sends $[L] - [A]$ to $[L]$. It follows that the map $\tilde{K}_0(A) \to \text{Pic}(A)$ is surjective. We conclude that the lower vertical arrows in (6.7) are all surjective. Since the two columns on the right are exact and the two
lower rows are also exact, a diagram chase shows that $\alpha$ is surjective. On the other hand, $SK_0(R) \oplus SK_0(R[H]) = 0$ by Lemma 6.1. It follows that $SK_0(R[M]) = 0$. This finishes the proof. \hfill $\square$

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