ON FINITE TYPE INVARIANTS OF WELDED STRING LINKS
AND RIBBON TUBES

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Abstract. Welded knotted objects are a combinatorial extension of knot theory, which can be used as a tool for studying ribbon surfaces in 4-space. A finite type invariant theory for ribbon knotted surfaces was developed by Kanenobu, Habiro and Shima, and this paper proposes a study of these invariants, using welded objects. Specifically, we study welded string links up to $w_k$-equivalence, which is an equivalence relation introduced by Yasuhara and the second author in connection with finite type theory. In low degrees, we show that this relation characterizes the information contained by finite type invariants. We also study the algebraic structure of welded string links up to $w_k$-equivalence. All results have direct corollaries for ribbon knotted surfaces.

1. Introduction

In the study of knotted surfaces, i.e. smooth embeddings of 2-dimensional manifolds in 4-space, the class of ribbon surfaces has proved to be of particular interest. These are the analogue of ribbon knots in 3-space, as defined by Fox in the sixties, in the sense that such knotted surfaces bound immersed 3-manifolds with only one fixed topological type of so-called ribbon singularities. Ribbon knotted surfaces were extensively studied from the early days higher dimensional knot theory, notably through the work of Yajima [18, 19] and Yanagawa [21, 22, 20]. One nice feature of ribbon knotted surfaces is that they admit a natural notion of 'crossing change'. Indeed, one can always arrange such surfaces so that the only crossings occur along circles of double points: swapping the over/under information along such a circle then yields a new ribbon surface. Just like the usual crossing change can be used to define Vassiliev knot invariants, Kanenobu, Habiro and Shima used this local move to introduce a theory of finite type invariants for ribbon knotted surfaces [6, 8]. For ribbon 2-knots, Habiro and Shima further showed that finite type type invariants are completely determined by the (normalized) Alexander polynomial.

This paper aims at characterizing, in a similar way, finite type invariants of ribbon tubes, which are ribbon knotted annuli in the 4-ball whose boundary is given by fixed copies of the unlink. These objects were introduced in [1], as an higher dimensional analogue of string links. For 1-component ribbon tubes, the situation is strictly the same as for ribbon 2-knots, in the sense that all finite type invariants come from the coefficients $\alpha_k$ ($k \geq 2$) of the normalized Alexander polynomial [13]. For ribbon tubes of more components, a number of finite type invariants can be derived from this polynomial, as follows. Given a sequence $R$ of possibly overlined indices, there is a canonical procedure to connect the various components of a ribbon tube into a single annulus, and evaluating $\alpha_k$ on the latter yields a degree $k$ finite type invariant $I_{R,k}$ of the initial ribbon tube, called a closure invariant. Another family of finite type invariants of ribbon tubes is given by the higher-dimensional Milnor invariants defined in [1]. The first of these invariants is Milnor invariant $\mu(ij)$, which in effect is the (nonsymmetric) linking number of component $i$ with component $j$. The main result of this paper can be stated as follows.
Theorem. Let $T$ and $T'$ be two $n$-component ribbon tubes. The following are equivalent, for $k \in \{2; 3\}$:

1. $T$ and $T'$ are $RC_k$-equivalent;
2. For any finite type invariant $\nu$ of degree $< k$, we have $\nu(T) = \nu(T')$;
3. $T$ and $T'$ cannot be distinguished by the following invariants:
   - If $k = 2$: linking numbers $\mu_{ij}$, for all $i \neq j$.
   - If $k = 3$: linking numbers $\mu_{ij}$ and closure invariants $I(\tau_{i,j,k})$ and $I(\tau_{i,j})$, $I(\tau_{i,j,k})$, $I(\tau_{i,j})$, $I(\tau_{i,j})$, $I(\tau_{i,j})$, $I(\tau_{i,j})$, for all pairwise distinct $i, j, k$ such that $j < k$.

Here, the $RC_k$-equivalence is an equivalence relation introduced by Watanabe in [17] which, by the above, characterizes the information contained by finite type invariants of ribbon tubes of degree $< 3$. This relation, discussed in Section 5, is an analogue in higher dimension of Habiro’s $C_k$-equivalence for usual knotted objects [5]. Watanabe showed that for ribbon 2-knots, the $RC_k$-equivalence characterizes the information contained by all finite type invariants of degree $< k$ [17, Thm. 1.1].

Note that according to the Theorem, any degree 2 invariant of ribbon tubes can be expressed as a combination of linking numbers and closure invariants: we give such an explicit formula for Milnor invariant $\mu(123)$ in Proposition 4.7.

In Appendix A, we investigate degree 3 invariants in the 2-component case. The classification result is only given modulo a conjectured relation, but it appears already at this stage that closure invariants no longer suffice to generate all degree 3 invariants, since the classification also involves length 4 Milnor invariants. See Remark A.8. An extensive study of degree 3 invariants of ribbon tubes, and discussions on the perspectives that it opens, can be found in [2].

The results of this paper on ribbon knotted surfaces are all obtained as consequences of diagrammatic results. Another remarkable feature of ribbon surfaces is indeed that they can be described and studied using welded objects, which are a quotient of virtual knot theory, see Section 2.1. Early works of Yajima [18] highlighted the fact that all relations in the fundamental group of the complement of a ribbon 2-knot can be encoded using the usual diagrammatics of knot theory. This key observation was later completed and formalized by Satoh, as a surjective map from welded objects to ribbon knotted surfaces [16]. Hence the core of the present paper is a characterization of finite type invariants of welded string links. The theory of finite type invariants for welded objects is based on the virtualization move, which replaces a classical crossing by a virtual one, and the above-mentioned closure and Milnor invariants do have a strict analogue for welded string links, with the same finite type properties, see Section 2.3.

Our main tool will be the arrow calculus developed in [13], which is a welded analogue of Habiro’s clasper calculus [5]. In particular, a family of finer and finer equivalence relations on welded objects called $wk$-equivalence is defined in [13], which is closely related to finite type theory. It is indeed known that two $wk$-equivalent welded objects cannot be distinguished by any finite type invariant of degree $< k$. The converse implication also holds for welded (long) knots, thus fully characterizing the information contained by finite type invariants of these objects, and it is conjectured that such an equivalence also holds for welded string links (Conjecture 3.3). This is a natural analogue of the Goussarov-Habiro conjecture for finite type invariants of string links and homology cylinders [5] and, as a matter of fact, our results amount to verify this conjecture at low degree. Indeed, our main diagrammatical results are classifications of welded string links up to $wk$-equivalence for $k = 2$ and 3 by finite type invariants (Corollaries 4.2 and 4.6), which imply the above Theorem as seen in Section 5. Several general results are also given on the
set of \(w_k\)-equivalence classes of welded string links, showing in particular that these form a finitely generated, non abelian group, see Section 3.3.

We conclude this introduction by mentioning the recent work of Colombari, who gave a complete classification of welded string links up to \(w_k\)-concordance, for all \(k\), in [3]. This equivalence relation, generated by \(w_k\)-equivalence and welded concordance, turns out to completely characterize welded string links (hence, classical string links) having same Milnor invariants of length \(\leq k\). This result also shows that all finite type concordance invariants of welded string links are given by Milnor invariants.

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2. **Welded objects and arrow calculus**

2.1. **Welded knotted objects.** Recall that a virtual diagram is a planar immersion of some 1-dimensional manifold; the singular set is a finite collection of transverse double points endowed with a decoration, either as a classical or as a virtual crossing. In figures, classical crossings are represented as in usual knot diagrams, while virtual crossings are simply drawn as double points (we do not follow the customary convention using circled double points).

**Definition 2.1.** A welded knotted object is the equivalence class of a virtual diagram modulo the generalized Reidemeister moves and the OC move. Here the generalized Reidemeister moves consist of the three usual Reidemeister moves (involving classical crossings), and the Detour move shown in Figure 2.1. The OC move is shown on the right-hand side of the same figure.

![Figure 2.1. The Detour and OC moves](image)

Recall that usual (string) links inject into welded (string) links, in the sense that two diagrams without virtual crossings, that are related by a sequence of generalized Reidemeister and OC moves, represent isotopic objects, see [4, Thm 1.B]. Welded objects are also intimately related to ribbon knotted surfaces in 4-space, via Satoh’s Tube map [16], as further developed in Section 5.

**Remark 2.2.** Virtual diagrams modulo generalized Reidemeister move give rise to virtual knotted objects, that were first introduced in the early nineties by L. Kauffman in [9], and M. Goussarov, M. Polyak and O. Viro in [4].

This paper will mainly deal with the following class

**Definition 2.3.** An \(n\)-component welded string link is the welded class of \(n\) properly immersed copies of the unit interval into \([0, 1] \times [0, 1]\), endowed with \(n\) fixed points on \([0, 1] \times \{\varepsilon\} \ (\varepsilon = 0, 1)\), such that the \(i\)th copy of the interval runs from the \(i\)th fixed point in \([0, 1] \times \{0\}\) to the \(i\)th fixed point in \([0, 1] \times \{1\}\). A 1-component welded string link is also called a welded long knot.

We denote by \(wSL(n)\) the set of welded string links. The stacking product endows \(wSL(n)\) with a monoid structure, whose unit \(1\) is given by the trivial diagram of \(n\) intervals, with no crossing.
2.2. Arrow calculus. We now review the diagrammatic calculus for welded objects developed in [10], called arrow calculus. This is a welded analogue of Habiro’s clasper calculus for usual knotted objects [5] and, as such, it is intimately related to finite type invariants, see Section 2.3.3. Let $L$ be some welded knotted object.

Definition 2.4. A $w$-tree for $L$ is a connected unirevalent tree $T$, immersed into the plane so that

- trivalent vertices are endowed with a cyclic order, are pairwise disjoint and disjoint from $L$;
- univalent vertices are pairwise disjoint and lie in $L \setminus \{\text{crossings of } L\}$;
- edges are oriented, so that each trivalent vertex involves exactly one outgoing edge;
- edges may contain virtual (but not classical) crossings, either with $L$ or with $T$ itself;
- edges may contain decorations $\bullet$, called beads, which are subject to the involutive rule that two consecutive beads do cancel.

We shall call tails and head the endpoints of a $w$-tree, according to the orientation.

Definition 2.5. The degree of a $w$-tree is the number of tails. For $k \geq 1$, we call $w_k$-tree a $w$-tree of degree $k$.

Now, a $w$-tree is an instruction for modifying $L$, according to a process which we abusively call surgery, defined as follow.

Definition 2.6. Let $A$ be a $w_1$-tree for the diagram $L$. The surgery on $L$ along $A$ yields a new welded diagram $L_A$ according to the local rule:

If $A$ crosses (virtually) either $L$ or some other $w$-tree, then the strands of $L_A$ likewise cross the same object virtually.

In general, if $T$ is a $w_k$-tree for $L$, then surgery along $T$ is defined as surgery along the union of $w_1$-trees $E(T)$, called the expansion of $T$ and defined recursively by the rule:

Remark 2.7. In the above figure, the dotted parts represent parallel subtrees, which are parallel copies of the non-depicted part of the initial $w$-tree, that always cross each other virtually – see [10, Conv. 5.1] for a detailed explanation.

A key point is that any welded object $L$ can be represented in this way as a union of some diagram with no classical crossing and some $w$-trees, called an arrow presentation for $L$. Since we shall be concerned with welded string links in this paper, let us define this notion more formally in this particular context.

Definition 2.8. Let $L$ be an $n$-component welded string link. An arrow presentation $(1, T)$ of $L$ consists of the trivial diagram 1, together with a union of $w$-trees $T$ for 1 such that $L = 1_T$. Two arrow presentations are equivalent if they represent equivalent welded diagrams.
By \cite[Prop. 4.2]{13}, any element of $wSL(n)$ admits an arrow presentation; moreover, a complete set of relations is known, that relates any two arrow presentations of a given diagram:

**Theorem 2.9.** \cite[Thm. 5.21]{13} Two arrow presentations are equivalent if and only if they are related by a sequence of the following moves:

1. Any generalized Reidemeister move involving $w$-trees and/or the diagram;
2. Head and Tail reversal:
   
   \[
   \begin{array}{c}
   \text{\includegraphics[width=0.2\textwidth]{move1.png}} \\
   \end{array}
   \]
3. Tails exchange (tails may or may not belong to the same $w$-tree):
   
   \[
   \begin{array}{c}
   \text{\includegraphics[width=0.2\textwidth]{move2.png}} \\
   \end{array}
   \]
4. Isolated arrow:
   
   \[
   \begin{array}{c}
   \text{\includegraphics[width=0.2\textwidth]{move3.png}} \\
   \end{array}
   \]
5. Inversion:
   
   \[
   \begin{array}{c}
   \text{\includegraphics[width=0.2\textwidth]{move4.png}} \\
   \end{array}
   \]
6. Slide:
   
   \[
   \begin{array}{c}
   \text{\includegraphics[width=0.2\textwidth]{move5.png}} \\
   \end{array}
   \]

Moreover, a collection of further operations on arrow presentations can be derived from these moves, as summarized below.

**Proposition 2.10.** \cite[Lemmas 5.14 to 5.18]{13} The following local moves give equivalent arrow presentations.

7. Heads exchange:
   
   \[
   \begin{array}{c}
   \text{\includegraphics[width=0.2\textwidth]{move6.png}} \\
   \end{array}
   \]
8. Head/Tail exchange:
   
   \[
   \begin{array}{c}
   \text{\includegraphics[width=0.2\textwidth]{move7.png}} \\
   \end{array}
   \]
9. Antisymmetry:
   
   \[
   \begin{array}{c}
   \text{\includegraphics[width=0.2\textwidth]{move8.png}} \\
   \end{array}
   \]

---

\footnote{Notice that the other versions of these moves given in \cite{13} can be deduced from those given here using the Reversal moves (2).}
(10) Fork:

\[
\begin{align*}
\text{(Fork)} & \quad \begin{array}{c}
\text{
}\end{array} = \begin{array}{c}
\text{
}\end{array} = \begin{array}{c}
\text{
}\end{array} = \begin{array}{c}
\text{
}\end{array} = \begin{array}{c}
\text{
}\end{array}
\end{align*}
\]

Convention 2.11. In what follows, we will blur the distinction between arrow presentations and the welded diagrams obtained by surgery. Moreover, we shall use the moves of the previous two results by only referring to their numbering (1)-(10).

2.3. Invariants of welded string links. Given a welded string link \(L\), the classical crossings split \(L\) into a collection of arcs, whose endpoints are either component endpoints or underpassing arcs, and which possibly contain a number of virtual crossings. The \textit{welded group} of \(L\) is the group \(G(L)\) abstractly generated by all arcs, subject to the following Wirtinger relations associated with each classical crossing:

\[
\begin{align*}
& c \cdot a \cdot b \cdot c = 1, \\
& a^{-1}b\cdot a\cdot c^{-1} = 1
\end{align*}
\]

As first noted in [9, § 8], it is easily checked that this group is invariant under generalized Reidemeister moves and the OC move. Note also that, if \(L\) is a classical string link, then \(G(L)\) is the fundamental group of the complement.

2.3.1. Alexander polynomial and closure invariants. We now introduce a family of invariants of welded string links, defined by evaluating the normalized Alexander polynomial on some welded long knot built via a closure process.

The normalized Alexander polynomial of welded long knots was defined in [6], as follows. Let \(K\) be a welded long knot, and suppose that its welded group has presentation \(G(K) = \langle g_1, \ldots, g_m | r_1, \ldots, r_k \rangle\) for some \(m, k\). Consider the matrix \(M = (\frac{\partial r_i}{\partial g_j})_{i,j}\), whose \((i,j)\)-entry is given by the Fox derivative of relation \(r_i\) with respect to generator \(g_j\), composed with the map sending all generators to the variable \(t\). The \textit{Alexander polynomial} of \(K\) is the greatest common divisor of all \((m-1)\)-minors of \(M\), which is a well defined element \(\Delta_K(t)\) of \(\mathbb{Z}[t^{\pm 1}]\) up to a unit factor. The \textit{normalized Alexander polynomial} \(\tilde{\Delta}_K(t)\) is chosen so that \(\tilde{\Delta}_K(1) = 1\) and \(\frac{d}{dt}\tilde{\Delta}_K(1) = 0\).

We extract integer valued invariants of the welded long knot \(K\) as follows.

Definition 2.12. The \textit{kth normalized coefficient} of the Alexander polynomial is the coefficient \(\alpha_k(K)\) in the power series expansion of \(\tilde{\Delta}_K(t)\) at \(t = 1\):

\[
\tilde{\Delta}_K(t) = 1 + \sum_{k=2}^{\infty} \alpha_k(K)(1-t)^k
\]

We shall need the following, which is a consequence of the multiplicativity property of the normalized Alexander polynomial.

Proposition 2.13. [13 Cor. 6.6] Let \(K\) and \(K'\) be two welded long knots, and let \(k\) be some integer. If \(\alpha_j(K) = 0\) for all \(j < k\), we have \(\alpha_k(K \cdot K') = \alpha_k(K) + \alpha_k(K')\).

We now proceed with defining the general closure process underlying closure invariants.

Definition 2.14. Let \(n \in \mathbb{N}\). A \textit{list} of length \(k\) \((k < n)\) is a sequence of pairwise distinct, possibly overlined integers in \(\{1, \cdots, n\}\).
A list is an instruction for closing an \( n \)-component welded string link into a welded long knot.

**Definition 2.15.** Let \( L \) be an \( n \)-component welded string link, and let \( R = (r_1, \ldots, r_k) \) be a list of length \( k < n \). Then \( Cl_R(L) \) is the welded long knot obtained as follows:

- delete all components of \( L \) whose index does not appear in \( R \), with or without overlining;
- reverse the orientation of all components whose index \( r_i \) is overlined in \( R \);
- connect these \( k \) oriented strands endpoints, following the order of the list \( R \) and the orientation of each strand, with arbitrary arcs that cross virtually the rest of the diagram.

Observe that this process is well-defined thanks to the Detour move. Note also that the process extends naturally to arrow presentations, by closing the trivial diagram \( \mathbf{1} \) as instructed by the list \( R \).

**Example 2.16.** Consider the arrow presentation for \( L \in wSL(2) \) shown on the left-hand side below.

Then the closures \( Cl_{(1,2)}(L) \) and \( Cl_{1,2}(L) \) yield the welded long knot \( K_1 \) represented in the middle of the figure, while \( Cl_{(2,1)}(L) \) and \( Cl_{(2,1)}(L) \) give the welded long knot \( K_2 \) on the right-hand side. By the Fork move (10), \( Cl_R(L) \) is the trivial long knot for any other list \( R \).

We can now define closure invariants of welded string links.

**Definition 2.17.** Let \( R \) be a list, and let \( k \geq 2 \) be some integer. The **closure invariant** \( I_{R,k} \) is the welded string link invariant defined by \( I_{R,k}(L) = \alpha_k(Cl_R(L)) \).

In particular, the closure invariant \( I_{(1),k} \) simply computes the normalized Alexander coefficient \( \alpha_k \) of the \( i \)th component of a welded string link.

It is straightforwardly checked that closure invariants indeed are invariants of welded string links. This family of invariants should be compared to the closure invariants of classical string links of [11], and further developed in [12, § 5.1]; note however that in these latter works, the closure operations introduce classical crossings.

### 2.3.2. Welded linking numbers and Milnor invariants

Given an \( n \)-component welded string link \( L \) and two distinct indices \( i, j \in \{1, \cdots, n\} \), the **welded linking number** \( \mu(ij) \) is given by

\[
\mu_L(ij) = \sum_{c \in C_{i,j}} \text{sign}(c),
\]

where the sum runs over the set \( C_{i,j} \) of classical crossings where component \( i \) passes over component \( j \), and where the sign of the crossing is given by the usual rule:

\[
\begin{align*}
\begin{array}{c}
  \frac{i}{+1} \\
  j \\
\end{array} & = +1 \\
\begin{array}{c}
  \frac{j}{-1} \\
  i \\
\end{array} & = -1
\end{align*}
\]
It is quite straightforward to verify that this indeed defines a welded invariant. If \( L \) is a classical string link, then clearly we have that \( \mu(ij) = \mu(ji) \) is the usual linking number.

These invariants were first introduced in [4], under the name of virtual linking numbers. Just like usual linking numbers were widely generalized into Milnor invariants in [14], there is a welded extension of Milnor invariants \( \mu(I) \) for any sequence of indices \( I \), which generalizes the welded linking numbers. This extension was first given in [1, Sec. 6] using a topological approach and the Tube map (see Section 5), and a purely diagrammatic version was later provided in [15].

2.3.3. Finite type invariants. We now recall the definition of finite type invariants of welded objects, and observe that the above invariants all fall in this category.

Recall that a virtualization move on a welded diagram is the replacement of a classical crossing by a virtual one. Given a welded diagram \( L \), and a subset \( S \) of the set of classical crossings of \( L \), we denote by \( L_S \) the welded diagram obtained by applying the virtualization move to all crossings in \( S \).

**Definition 2.18.** Let \( \nu \) be an invariant of welded string links, taking values in some abelian group. Then \( \nu \) is a finite type invariant of degree \( \leq k \) if, for any \( L \in wSL(n) \) and any set \( S \) of \( k + 1 \) classical crossings in \( L \), we have

\[
\sum_{S' \subset S} (-1)^{|S'|} \nu(L_{S'}) = 0.
\]

This is a finite type invariant of degree \( k \) if, moreover, it is not of degree \( \leq k - 1 \).

This definition was first given in [4, Sec. 2.3] in the context of virtual knots and links. Actually, in that same paper, the authors further identified the first nontrivial invariants of the theory:

**Lemma 2.19.** [4] For all \( i, j \in \{1, \cdots, n\} \), the welded linking number \( \mu(ij) \) is a degree 1 finite type invariant of welded string links.

There are finite type invariants in any degree. Indeed, we have the following.

**Lemma 2.20.** [6] For all \( k \geq 2 \), \( \alpha_k \) is a degree \( k \) finite type invariant of welded long knots.

As an immediate consequence, we have:

**Corollary 2.21.** For all \( k \geq 2 \) and all list \( R \), the closure invariant \( I_{R,k} \) is a degree \( k \) finite type invariant of welded string links.

We note that Lemma 2.19 generalizes to all Milnor invariants, in the sense that for a sequence \( I \) of \( k \geq 2 \) indices in \( \{1, \cdots, n\} \), the welded Milnor invariant \( \mu(I) \) is a degree \( k - 1 \) finite type invariant of welded string links. A complete proof of this fact can be found in the Appendix of [2].

3. \( w_k \)-equivalence

In this section, we review the family of equivalence relations introduced in [13], called \( w_k \)-equivalence, and recall how it can be used as a tool for studying finite type invariants. Although the definition can be made in the general context of welded objects, we shall restrict ourselves below to welded string links; in particular, we investigate the algebraic properties of the group of \( w_k \)-equivalence classes of welded string links.
3.1. Definition and relation to finite type invariants. Let \( k \) be a positive integer.

**Definition 3.1.** Two welded string links \( L, L' \) are \( \text{w}_k \)-equivalent, denoted \( L \xrightarrow{k} L' \), if there exists a finite sequence \( (L_i)_{0 \leq i \leq n} \) of elements of \( \text{wSL}(n) \) such that \( L_0 = L, L_n = L' \) and for each \( i, \) \( L_{i+1} \) is obtained from \( L_i \) either by surgery along a \( \text{w}_l \)-tree for some \( l \geq k \) or by a generalized Reidemeister or OC move.

Using the Expansion move (E), one sees that the \( \text{w}_k \)-equivalence becomes finer as \( k \) increases. This notion turns out to be closely related to finite type theory.

**Proposition 3.2.** [[13, Prop. 7.5]] For \( k \geq 2 \), two welded (string) links that are \( \text{w}_k \)-equivalent, share all finite type invariants of degree \( < k \).

Furthermore, it is proved in [13] that the converse holds for welded knots and welded long knots. For welded long knots, finite type invariants are completely determined by the normalized Alexander coefficients, a result which was first proved by Habiro and Shima [7]. For welded string links, we are naturally led to the following.

**Conjecture 3.3.** Two welded string links are \( \text{w}_k \)-equivalents if and only if they cannot be distinguished by finite type invariants of degree \( < k \).

This can be seen as a welded analogue of the Goussarov-Habiro conjecture, see [5]. As a matter of fact, Corollaries 4.2 and 4.6 validate this conjecture at low degree.

3.2. Refined arrow calculus. When working up to \( \text{w}_k \)-equivalence, the arrow calculus can be further refined: the point is that working up to \( \text{w}_k \)-equivalence allows for operations 'up to higher order terms'. Indeed, in addition to the ten moves of Theorem 2.9 and Proposition 2.10, we have a number of extra operations at our disposal for manipulating arrow presentations. Some of these operations are summarized in Lemma 3.4 below, whose proof can be found in [13, Sec. 7.4]. They are given in terms that are slightly stronger than \( \text{w}_k \)-equivalence, as follows. Given two arrow presentations \( (1, T) \) and \( (1, T') \) and some integer \( k \geq 1 \), we denote by

\[
(1, T) \xrightarrow{k} (1, T')
\]

the fact that \( (1, T) = (1, T' \cup T'') \) for some union \( T'' \) of \( \text{w} \)-trees of degree \( \geq k \). Note that \( (1, T) \xrightarrow{k} (1, T') \) implies that \( 1_T \xrightarrow{k} 1_{T'} \).

**Lemma 3.4.** Let \( k, k' \) be integers strictly greater than 1.

- (11) **Twist:** For any \( \text{w}_k \)-tree containing a bead, we have

  \[
  \begin{array}{c}
  \text{W} \\
  \xrightarrow{k+1}
  \end{array}
  \]

- (12) **Generalized Head/Tail exchange:** We have

  \[
  \begin{array}{c}
  W \\
  \xrightarrow{k+k'+1}
  \\
  W'
  \end{array}
  \]

  where \( W \) and \( W' \) are \( \text{w} \)-trees of degree \( k \) and \( k' \), respectively, so that \( T \) is a \( \text{w}_{k+k'} \)-tree.

As with arrow moves (1)-(10), we shall use the relations of Lemma 3.4 by only referring to their numbering.

**Remark 3.5.** Combining the Twist relation (11) with the with the reversal move (2) and the Antisymmetry move (9), we have that the two welded long knots \( K_1 \) and \( K_2 \) of Example 2.16 are \( \text{w}_3 \)-equivalent.
Combining relation (12) with the previous exchange moves (3) and (7), we have the following.

Corollary 3.6. [13, Cor. 7.13] Let $T$ and $T'$ be $w$-trees of degree $k$ and $k'$, respectively. One can freely exchange the relative position of two adjacent univalent vertices (head or tail) of $T$ and $T'$ at the cost of extra $w$-trees, all of degree $\geq k+k'$.

This can be used to rearrange any arrow presentation of an element of $wSL(n)$ into a product of elementary pieces, each obtained by surgery along a single $w$-tree, as follows (see [13, Lem. 7.5] for a proof of a slightly more general statement).

Lemma 3.7. Let $L$ be an $n$-component welded string link, and let $k \geq 1$. We have

$$L \overset{k+1}{\sim} \prod_{i=1}^{k} L_i,$$

where, for each $i \in \{1, \ldots, k\}$, the welded string link $L_i$ is a product

$$L_i = 1_{T_i^{(i)}} \cdots 1_{T_{N_i}^{(i)}}$$

for some integer $N_i$, such that each $T_{N_j}^{(i)}$ is a $w_j$-tree; $j \in \{1, \ldots, N_i\}$.

Another noteworthy consequence of Corollary 3.6 is the following additivity property for closure invariants.

Proposition 3.8. Let $k$ and $k'$ be two integers, and let $R$ be a list. Let $W$ and $W'$ be unions of $w$-trees for the trivial diagram 1, of degree $\geq k$ and $\geq k'$, respectively. For all $d < k + k'$, we have

$$\mathcal{I}_{R,d}(1_W \cdot 1_{W'}) = \mathcal{I}_{R,d}(1_W) + \mathcal{I}_{R,d}(1_{W'}).$$

Proof. Let us denote by $D_1$ the union of $w$-trees obtained by stacking the arrow presentations for $Cl_R(1_W)$ and $Cl_R(1_{W'})$. The union of $w$-trees $D_2$ given by $Cl_R(1_W 1_{W'})$ is obtained by intermeshing the endpoints of $W$ and $W'$ along the oriented interval. Starting with $D_2$, we can apply repeatedly Corollary 3.6 until we completely separate all $w$-trees in $W'$ from those in $W$, thus producing $D_1$. When applying Corollary 3.6, extra $w$-trees of degree $\geq k + k'$ may appear, so we obtain that $Cl_R(1_W) \cdot Cl_R(1_{W'}) = 1_{D_1}^{k+k'} 1_{D_2} = Cl_R(1_W 1_{W'})$. Now, since $\alpha_d$ is a finite type invariant of degree $d$ by Proposition 2.20, we have $\alpha_d(1_{D_1}) = \alpha_d(1_{D_2})$, and the result follows from the additivity property of $\alpha_d$ as given in Proposition 2.13.

3.3. The group of welded string links up to $w_k$-equivalence. Let $n, k \in \mathbb{N}^*$. We denote by $wSL(n)_k$ the set of $w_k$-equivalence classes of $n$-component welded string links.

As already observed in [13] § 7.2, $wSL(n)_1$ is the trivial group for all $n \geq 1$. It is also known that $wSL(1)_k$ is a finitely generated abelian group for all $k \geq 1$ [13] Cor. 8.8. In the general case, we have the following results.

Theorem 3.9. For $n, k \in \mathbb{N}^*$, $wSL(n)_k$ is a finitely generated group.

Proof. Let us first prove the group structure. By Lemma 3.7 we can represent any element of $wSL(n)_k$ as a product of welded string links, each obtained from 1 by surgery along a single $w$-tree of degree $< k$. Hence it suffices to show that, for any integer $l$ such that $1 \leq l < k$, and for any $w_l$-tree $T$ for 1, there exists some $X \in wSL(n)$ such that

$$1_T \cdot X \overset{k}{\sim} 1.$$
To this end, consider a parallel copy $T'$ of $T$, which only differs by a bead $\bullet$ next to the head. By the Inversion move (5), the union of these two $w$-trees in ‘parallel’ configuration (as in the figure for the said move) is equivalent to the empty diagram. Now, we can use Corollary 3.6 to move $T'$ above a disk $D$ containing $T$; this introduces a union of $w$-trees $W$ of degree $\geq 2l$, which may intersect $D$. Next $W$ can in turn be moved above the disk $D$ using Corollary 3.6 and this introduces another union of $w$-trees, each of degree $\geq 3l$. Iterating this process, we eventually obtain in this way that the trivial diagram $1$ is $w_k$-equivalent to a product $1_T \cdot 1_W$, where $W$ is a union of $w$-trees, disjoint from $D$. We have thus desired inverse of $1_T$ up to $w_k$-equivalence, by setting $X = 1_W$ in (3.1).

Now, it remains to observe that the group $wSL(n)_k$ is finitely generated. This follows immediately from Lemma 3.7, since there are only finitely many $w$-trees in each finite degree. □

Remark 3.10. A consequence of this proof, in the case $k = l + 1$, is the following. If $T$ is a $w_l$-tree for $1$, then $1_T \cdot 1_T \sim 1$, where $T^\bullet$ is obtained by inserting a bead $\bullet$ near the head of $T$.

Lemma 3.11. The group $wSL(n)_k$ is not abelian for $n \geq 2$ and $k \geq 2$.

Proof. Consider the 2-component welded string links $D$ and $D'$ shown on the left-hand side of Figure 3.1. On one hand, by the Reversal moves (2), the closure $\text{Cl}_{(1,2)}(D)$ is the welded long knot $K$ shown on the right-hand side of the figure. On the other hand, by the Tails exchange and Isolated arrow moves (3) and (4), the welded long knot $\text{Cl}_{(1,2)}(D')$ is trivial. It is easily checked that $\alpha_2(K) = 1$, thus proving that the closure invariant $I_{(1,2),2}$ distinguishes $D$ and $D'$. By Proposition 2.20, this proves that $D$ and $D'$ are not $w_2$-equivalent, hence not $w_k$-equivalent for any $k \geq 2$. □

4. CHARACTERIZATION OF LOW DEGREE INVARIANTS OF WELDED STRING LINKS

Proving the main results of this paper will amount to completely describe the group $wSL(n)_k$ for low values of $k$. As a warmup, we classify in Subsection 4.1 welded string links up to $w_2$-equivalence, and characterize their finite type invariants of degree 1. This will illustrate, in this relatively simple case, the strategy of proof that will be used again in Subsection 4.2 for the next degree case.

4.1. Degree 1 invariants of welded string links. Given two distinct indices $i, j$ in $\{1, \cdots, n\}$, denote by $Z_{i,j}$ and $Z_{i,j}^{-1}$ the following element of $wSL(n)$:

\[
Z_{i,j} = \begin{array}{c}
\circ \\
\bullet \\
\circ
\end{array} \quad Z_{i,j}^{-1} = \begin{array}{c}
\circ \\
\bullet \\
\circ
\end{array}.
\]

Recall from Remark 3.10 that we have $Z_{i,j} \cdot Z_{i,j}^{-1} \sim 1$. For an integer $n \geq 0$, we also denote by $(Z_{i,j})_{i,j}^{\sim n}$ the product of $n$ copies of $Z_{i,j}^{-1}$. 
Theorem 4.1. Let $L$ be an $n$-component welded string link. We have

$$L \cong \prod_{i \neq j} (Z_{i,j})^{\mu_L(ij)},$$

where the product is taken according to the lexicographic order on $(i,j)$.

Corollary 4.2. The following are equivalent.

1. Two welded string links $L$ and $L'$ are $w_2$-equivalent;
2. For any finite type invariant $\nu$ of degree 1, we have $\nu(L) = \nu(L')$;
3. For any pair $(i,j)$ of distinct integers, we have $\mu_L(ij) = \mu_{L'}(ij)$.

Proof. The fact that (1) implies (2) is given by Proposition 3.2, and since the welded linking numbers are degree 1 invariants (Lemma 2.19), we have implication (2) $\Rightarrow$ (3). Theorem 4.1 shows that (3) implies (1). □

Proof of Theorem 4.1. By Lemma 3.7, we have that $L$ is $w_2$-equivalent to a product of welded string links, each obtained from $1$ by surgery along a single $w_1$-tree, and any two such factors commute up to $w_2$-equivalence. By the Isolated arrow move (3), any such $w_1$-tree with both endpoints on the same component can be deleted. Now for each $w_1$-trees with endpoints on components $i$ and $j$ for some $i \neq j$, we can use the involutivity of beads and the Reversal moves (2) to deform it into a copy of either $Z_{i,j}$ or $Z_{i,j}^{-1}$. By Remark 3.10, we thus have

$$L \cong \prod_{i \neq j} (Z_{i,j})^{e_{i,j}},$$

for some coefficients $e_{i,j} \in \mathbb{Z}$. Observe that the welded linking numbers are clearly additive under stacking, and recall that they are $w_2$-equivalence invariants by Lemma 2.19 and Proposition 3.2. Hence, applying $\mu(kl)$ (for some $k \neq l$) to this equivalence gives

$$\mu_L(kl) = \sum_{i \neq j} e_{i,j} \times \mu_{Z_{i,j}}(kl).$$

Observe that the welded diagram obtained by surgery along $Z_{i,j}$ is as follows:

```
  i   j
```

An elementary computation then gives that $\mu_{Z_{i,j}}(kl) = \delta_{ik}\delta_{jl}$. We thus obtain that $e_{i,j} = \mu_L(ij)$ for any pair $(i,j)$ of distinct integers, as desired. □

4.2. Degree 2 invariants of welded string links. Let us proceed with the characterization of degree 2 invariants. As in the preceding section, this will fall as a consequence of a normal form result for welded string links up to $w_3$-equivalence. Let us introduce the 'building blocks' for this normal form. For pairwise distinct indices $i, j, k$, let $E_i, A_{i,j}, B_{i,j}, C_{i,j}, D_{i,j}$ and $G_{i,j,k}$ be the following welded string links:
For each of the above elements $X$ of $wSL(n)$, we also denote by $X^{-1}$ the welded string link obtained by inserting a bead near the head of the defining $w_2$-tree; note that by Remark 3.10 this defines the inverse of $X$ in $wSL(n)$. For $n \geq 0$, we also denote by $X^{-n}$ the product of $n$ copies of $X^{-1}$.

There are a number of relations among the above elements. As direct consequences of the Antisymmetry move (9) and the Twist relation (11), together with the involutivity of beads, we have the following equivalences, for all pairwise distinct $i, j, k \in \{1, \cdots, n\}$:

$$A_{i,j} \sim C_{j,i}^{-1}, \quad B_{i,j} \sim D_{j,i}^{-1} \quad \text{and} \quad G_{i,k,j} \sim G_{i,j,k}^{-1}.$$ (4.1)

Moreover, we have the following much less intuitive relation.

**Proposition 4.3.** For all pairwise distinct $i, j \in \{1, \cdots, n\}$, we have

$$A_{i,j} \cdot B_{i,j} \cdot C_{i,j} \cdot D_{i,j} \sim 1.$$

**Proof.** This is shown by the following sequence of equivalences:

$$A_{i,j} \cdot B_{i,j} \cdot C_{i,j} \cdot D_{i,j} \sim 1.$$

Here, the second equality is the Expansion move (E) applied to $A_{i,j}$. The third equality is given by the Slide move (6), and the equivalence is then given by the Head/Tail exchange move (8) on the $i$th component of 1, followed by the Twist relation (11). The $w_2$-tree introduced in this last step is a copy of $C_{i,j}^{-1}$ by the Head reversal move (2), and which can also be isolated from the rest of the diagram using Corollary 3.6. This gives the first equivalence below:

$$A_{i,j} \sim C_{i,j}^{-1}, \quad B_{i,j} \sim D_{i,j}^{-1} \quad \text{and} \quad G_{i,k,j} \sim G_{i,j,k}^{-1}.$$ (4.1)

The second equivalence above is given by exchanging, on component $i$, the bottom head and tail by move (8). This introduces a $w_2$-tree which is a copy of $D_{i,j}^{-1}$ by the Head reversal move (2), and which can also be isolated from the rest of the diagram by Corollary 3.6. This is the third equivalence above. Now, exchanging
the bottom head and tail on component \( j \) gives the first equivalence below:

\[
A_{i,j} \sim 3 \cdot (C_{i,j} \cdot D_{i,j})^{-1}.
\]

Again, this move introduces a \( w_2 \)-tree, which is this time a copy of \( B_{i,j}^{-1} \), and which can be isolated by Corollary \ref{cor:isolation}. The rest of the diagram can then be deleted using the Inversion move (5). Proposition \ref{prop:isolation} then follows.

We next set the notation for the invariants involved in our normal form result.

**Notation** \ref{notation:invariants}. For pairwise distinct indices \( i, j, k \in \{1, \ldots, n\} \), we set

\[(i). \phi_{i,j,k} = I_{i,j,k}^2 - I_{i,j,k}I_{i,j} - I_{i,j} - I_{i,j,k} - \mu(j)\mu(k); \]
\[(ii). \alpha_{i,j} = I_{j,i,j} - I_{j,i,j} - I_{j,i,j} - I_{j,i,j} - \mu(j); \]
\[(iii). \beta_{i,j} = I_{j,i} - I_{j,i} - I_{j,i} - \mu(j); \]
\[(iv). \gamma_{i,j} = I_{j,j} - I_{j,j} - \mu(j). \]

**Theorem** \ref{thm:main}. Let \( L \) be an \( n \)-component welded string link. We have

\[L \sim L_1 \cdot L_2,\]

where \( L_1 = \prod_{i \neq j} (Z_{i,j})^{\mu_L(i,j)} \) as in Theorem \ref{thm:invariants} and

\[L_2 = \prod_{1 \leq i \leq n} E_i^{I_{i,j}^2(L)} \prod_{1 \leq i < j \leq n} A_{i,j}^{a_{i,j}(L)} B_{i,j}^{b_{i,j}(L)} C_{i,j}^{c_{i,j}(L)} \prod_{1 \leq i \leq n} G_{i,j}^{d_{i,j,k}(L)}.\]

Since all invariants involved in this result are finite type invariants, as with Corollary \ref{cor:finite_type} we have the following immediate consequence.

**Corollary** \ref{cor:equivalence}. The following are equivalent.

1. Two welded string links \( L \) and \( L' \) are \( w_3 \)-equivalent;
2. For any finite type invariant \( \nu \) of degree at most 2, we have \( \nu(L) = \nu(L') \);
3. For all pairwise distinct indices \( i, j, k \) such that \( j < k \), \( L \) and \( L' \) have same invariants \( I_{j,i}^2, I_{j,j}^2, I_{j,j}^2, I_{j,i}^2 \) and \( \mu(i) \).

**Proof of Theorem** \ref{thm:main}. By Lemma \ref{lem:product} \( L \) is \( w_3 \)-equivalent to a product of factors, each obtained from \( \mathbf{1} \) by surgery along a single \( w_1 \)-tree (\( i = 1, 2 \)), and these factors can be ordered by their degree. Using the arguments of the proof of Theorem \ref{thm:invariants} we can assume that the product of terms given by \( w_1 \)-trees is the welded string link \( L_1 \) of the statement. Let us now focus on the terms given by \( w_2 \)-trees. Consider a \( w_2 \)-tree \( T \) for \( \mathbf{1} \), such that \( \mathbf{1}_T \) is a factor of the above product.

Let us first show that, up to \( w_3 \)-equivalence, \( T \) can be assumed to be one of the terms appearing in \( L_2 \) in the statement of the theorem. Suppose first that all three endpoints of \( T \) are on the same component, say component \( i \). Then by the Fork move (10), \( \mathbf{1}_T \) is nontrivial only if the head of \( T \) is located between both tails of \( T \), and the Reversal moves (2), Antisymmetry move (9) and Twist relation (11) ensure that \( T \) is necessarily a copy of either \( E_i \) or \( E_i^{-1} \). In the case where \( T \) is attached to exactly two components of \( \mathbf{1} \), say \( i \) and \( j \), then the same combinatorial arguments give that \( T \) can be freely assumed to be a copy of \( A_{i,j}^{\pm 1}, B_{i,j}^{\pm 1}, C_{i,j}^{\pm 1} \) or \( D_{i,j}^{\pm 1} \). Moreover, by \ref{thm:invariants} and Proposition \ref{prop:isolation} we can further assume that \( T \) is either \( A_{i,j}^{\pm 1}, B_{i,j}^{\pm 1} \) or \( C_{i,j}^{\pm 1} \) with \( i < j \). Finally, in the case where the three endpoints of \( T \) lie on pairwise distinct components \( i, j, k \), then the same considerations together with \ref{thm:invariants} show that \( T \) is a copy of \( G_{i,j,k}^{\pm 1} \) for pairwise distinct indices \( i, j, k \) such that...
For some elements \( g_{i,j,k}, a_{i,j}, b_{i,j}, c_{i,j} \) and \( e_i \) of \( \mathbb{Z} \), that we shall now determine. In what follows, for convenience we shall call basic factor any factor appearing in the product (4.2).

We now consider the invariant \( \mathcal{I}_{(i):2} \), which is the normalized Alexander coefficient \( \alpha_2 \) of the \( i \)th component. By [13, Lem. 6.4], we have that \( \mathcal{I}_{(i):2}(E_{i}^{\pm 1}) = \pm 1 \) and \( \mathcal{I}_{(i):2} \) vanishes on any other basic factor. Recall that \( \mathcal{I}_{(i):2} \) is an invariant of \( w_3 \)-equivalence (Theorem 3.2). By the additivity property of Proposition 3.8, evaluating \( \mathcal{I}_{(i):2} \) on (4.2) thus gives us that

\[
\mathcal{I}_{(i):2}(L) = \mathcal{I}_{(i):2}(L).
\]

Next we evaluate the closure invariants \( \mathcal{I}_{(i,j)} \) and \( \mathcal{I}_{(i,k)} \) on basic factors:

\[
\mathcal{I}_{(i,j):2} = \begin{pmatrix}
A_{i,j} & B_{i,j} & C_{i,j} & E_i & E_j & \star
\end{pmatrix}
\]

where \( \star \) stands for any other basic factor (we shall use the same convention below).

Using again Theorem 3.2 and Proposition 3.8, evaluating these invariants on (4.2) gives:

\[
\begin{align*}
\mathcal{I}_{(i,j):2}(L) - \mathcal{I}_{(i,k):2}(L_1) &= a_{i,j} - c_{i,j} + e_i + e_j, \\
\mathcal{I}_{(i,j):2}(L) - \mathcal{I}_{(i,k):2}(L_1) &= b_{i,j} + e_i + e_j, \\
\mathcal{I}_{(i,j):2}(L) - \mathcal{I}_{(i,k):2}(L_1) &= a_{i,j} + e_i + e_j.
\end{align*}
\]

Now, the basic factor \( Z_{i,j} \) is mapped to the trivial diagram by the closures \( Cl_{(i,j)} \), \( Cl_{(i,k)} \), and \( Cl_{(i,j)} \), thanks to the Isolated move (4). Hence the three corresponding closure invariants vanish on \( L_1 \). Consequently, we have

(i) \( a_{i,j} = \mathcal{I}_{(i,j):2}(L) - \mathcal{I}_{(i):2}(L) - \mathcal{I}_{(j):2}(L); \)

(ii) \( b_{i,j} = \mathcal{I}_{(i,j):2}(L) - \mathcal{I}_{(i):2}(L) - \mathcal{I}_{(j):2}(L); \)

(iii) \( c_{i,j} = \mathcal{I}_{(i,j):2}(L) - \mathcal{I}_{(i):2}(L). \)

Finally, we have the following computation for the basic factors \( G_{i,j,k}: \)

\[
\mathcal{I}_{R:2}(G_{i,j,k}) = \begin{cases}
1 & \text{if } R = (j, i, k), \\
-1 & \text{if } R = (E, i, j), \\
0 & \text{for any length 3 list } R \text{ starting with either } i \text{ or } 7.
\end{cases}
\]

The closure invariant \( \mathcal{I}_{(i,j,k):2} \) more generally takes the following values on basic factors:

\[
\mathcal{I}_{(i,j,k):2} = \begin{pmatrix}
A_{i,j} & B_{i,j} & C_{i,j} & A_{i,k} & B_{i,k} & C_{i,k} & A_{j,k} & B_{j,k} & C_{j,k} & E_i & E_j & E_k & \star
\end{pmatrix}
\]

Hence by Theorem 3.2 and Proposition 3.8, we have:

\[
\mathcal{I}_{(i,j,k):2}(L) - \mathcal{I}_{(i,j,k):2}(L_1) = g_{i,j,k} + b_{i,j} + a_{i,k} + b_{j,k} + e_i + e_j + e_k.
\]

Here however, unlike in the preceding computation, \( \mathcal{I}_{(i,j,k):2} \) does not vanish on \( L_1 \). Indeed, the closure \( C_l(i,j,k) \) of the following diagram
yields the welded long knot $K$ of Figure 3.1, which satisfies $\alpha_2(K) = 1$. Based on this observation, a computation shows that (see [2, Lem. 2.3.18]):

$$I_{(\bar{j},i,k)}(L) = \mu_L(ji)\mu_L(ki).$$

It follows that

$$g_{i,j,k} = I_{(j,i,k)}(L) - \mu_L(ji)\mu_L(ki) - a_{i,k} - b_{i,j} - b_{j,k} - e_i - e_j - e_k$$

This gives the desired formula. □

A notable observation about Corollary 4.6 is that degree 2 finite type invariants of welded string links are generated by closure invariants – while degree 1 invariants are generated by the welded linking numbers. This means that any other degree 2 invariant can be expressed as a linear combination of (products of) such invariants, and Theorem 4.5 can be used effectively to make this explicit. The next result gives such a formula for the welded Milnor invariant $\mu(123)$.

**Proposition 4.7.** We have

$$\mu(123) = \mu(21)\mu(13) - \mu(12)\mu(23) + I_{(1,3,2)}(L) - I_{(2,3,1)}(L) - I_{(3,1,2)}(L) + I_{(1,2)}(L) + I_{(j,1,2)}(L) - \mu_L(j)\mu_L(ki).$$

Proof. Let $L$ be a 3-component welded string link. Since $\mu(123)$ is a degree 2 invariant, and using the additivity property of [13, Lem. 6.11], evaluating on the $w_3$-equivalence class representative of Theorem 4.5 gives

$$\mu_L(123) = \mu_{L_1}(123) + \mu_{L_2}(123) = \mu_{L_1}(123) + \phi_{3,1,2}(L) \times \mu_{G_{3,1,2}}(123).$$

A direct computation gives

$$\mu_{L_1}(123) = \mu_L(21)\mu_L(13) - \mu_L(12)\mu_L(23),$$

and the desired formula then follows from the definition of the invariant $\phi_{3,1,2}$. □

**Remark 4.8.** The main result of this section, Corollary 4.6, should be seen as a welded analogue of [10, Thm. 4.23], as restated in [11, Thm. 2.2], in the classical case. There, it is shown that two classical string links are $C_3$-equivalent if and only if they have same Vassiliev invariants of degree $< 3$, which is equivalent to having same linking numbers, Milnor’s triple linking numbers, Casson knot invariants of each component, and a closure-type invariant, namely the Casson invariant of the closure $Cl_{1,2}$. Observe that the welded case of Corollary 4.6 involves a significantly greater number of invariants. Now, Proposition 2.10 of [10] expresses the classical triple linking number in terms of closure invariants. This shows that, in the classical setting as well, degree 2 invariants are generated by closure invariants. It follows from the results of [11] and [12] that this remains true at least up to degree 5; see Remark A.8 for the welded case.

---

2Proposition 4.7 is to be compared to [10, Prop. 2.10] (this formula was later widely generalized in [12]).
5. Application to ribbon knotted surfaces

As mentioned in the introduction, one of the main features of welded theory is that it can serve as a tool for the study of certain surfaces in 4-space, called ribbon surfaces. In this section, we briefly review this connection and draw topological applications of our results on welded string links.

A ribbon immersion of a 3-manifold $M$ into 4-space, is an immersion of $M$ with a single type of singularities, called ribbon singularities, which are disks of double points: one of the two preimages of such a disk is embedded in the interior of $M$, while the other preimage is properly embedded in $M$.

**Definition 5.1.** A smoothly embedded surface in 4-space if a ribbon knotted surface if it is the boundary of a ribbon immersion.

Ribbon knotted surfaces were extensively studied in the sixties. Among these seminal papers, the work of Yajima [18] set the ground for the connection with knot diagrammatics and welded theory: the Tube map, discovered by Satoh in 2000, is a surjective map from welded knotted objects to ribbon knotted surfaces [16]. We shall not review the Tube map here in details. Let us only mention that it relies on the diagrammatic description of knotted surfaces by their broken diagrams, and that it consists in assigning, to each classical crossing in a welded diagram, a local configuration of two knotted annuli, which should be thought of as the neighborhood of (the boundary of) a ribbon singularity:

![Tube](image)

The Tube map then yields a broken surface diagram by connecting these local configurations by unknoted and unlinked annuli, as combinatorially prescribed by the welded diagram.

In the case of a welded string link, the Tube map produces in this way ribbon-knotted annuli, that were introduced in [1] under the name of ribbon tubes, a terminology that we shall also use here. Let us fix in the 3-ball $B^3$, $n$ oriented Euclidian circles $C_1, \ldots, C_n$ bounding pairwise disjointly embedded disks.

**Definition 5.2.** An $n$-component ribbon tube is the image of a smooth embedding

$$\bigsqcup_{i=1}^n (S^1 \times [0,1]), \hookrightarrow B^4$$

of $n$ disjoint copies of the oriented annulus $S^1 \times [0,1]$ into $B^4 = B^3 \times [0,1]$, such that the $i$th annulus is cobounded by $C_i \times \{0\}$ and $C_i \times \{1\}$, with consistent orientations.

Denote by $\mathcal{RT}(n)$ the set of $n$-component ribbon tubes, up to isotopy fixing the boundary. This is a monoid, with composition given by the stacking product, and it follows immediately from the local definition of the Tube map that it induces a monoid homomorphism

$$\text{Tube}: \mathcal{WSL}(n) \longrightarrow \mathcal{RT}(n).$$

As a matter of fact, all definitions and main results given in this paper for welded string links, do translate naturally in the context of ribbon tubes via the Tube map.

Firstly, as already observed in [1] Sec. 2.2.1, the Tube map induces an isomorphism from the welded group of $L \in \mathcal{WSL}(n)$ to the fundamental group of the exterior of $\text{Tube}(L)$, and this isomorphism preserves peripheral elements (meridians and preferred longitudes) from which the invariants used in this paper are extracted. In other words, the normalized Alexander coefficients for ribbon knotted surfaces defined in [6] satisfies $\alpha_k(L) = \alpha_k(\text{Tube}(L))$ for all $k \geq 2$, and since the
closure process applies straightforwardly to ribbon tubes, we have more generally that
\[ \mathcal{I}_{R,k}(L) = \mathcal{I}_{R,k}(\text{Tube}(L)) \]
for any list \( R \) and any integer \( k \geq 2 \).
(Note that here and below, we use the same notation for closure invariants of ribbon tubes). On the other hand, Milnor invariants of ribbon tubes defined in [1, § 5] are likewise compatible with the Tube map, and in particular we have
\[ \mu_L(ij) = \mu_{\text{Tube}(L)}(ij) \]
for any pair of distinct indices \( i, j \).

Secondly, the Tube map is compatible with finite type theory. Such a theory for ribbon knotted surfaces was developped in [6–8]. It relies on the following local move, called \textit{crossing change at crossing circles}:

The figure shows a crossing of two pieces of a knotted surface along a 'crossing circle', and the local move swaps the over/under information at this circle. Let \( v^{(4)} \) be an invariant of ribbon tubes, taking values in some abelian group. Then \( v^{(4)} \) is a \textit{finite type invariant of degree at most} \( k \) if, for any ribbon tube \( T \) and any set \( S \) of \( k + 1 \) crossing circles on \( T \), we have
\[ \sum_{S' \subset S} (-1)^{|S'|} v^{(4)}(T_{S'}) = 0, \]
where \( T_{S'} \) is obtained from \( T \) by changing all crossing circles in \( S' \). Now, let \( v \) be a welded string link invariant, that extends to an invariant \( v^{(4)} \) of ribbon tubes in the sense that \( v^{(4)}(\text{Tube}(L)) = v(L) \) for any \( L \in wSL(n) \). If \( v \) is a finite type invariant of degree \( k \), then so is \( v^{(4)} \). Indeed, if two welded string links differ by a virtualization move, then their images by the Tube map differ by a crossing change at a crossing circle, as the following figure illustrates:

Thirdly, an analogue of the \( w_k \)-equivalence is known for ribbon knotted surfaces. Watanabe introduced in [14] the \( RC_k \)-equivalence, and showed that two \( RC_k \)-equivalent ribbon knotted surfaces cannot be distinguished by finite type invariants of degree \( < k \). Furthermore, Watanabe proves that the converse implication is also true for ribbon 2–knots. This is an analogue in higher dimension of Habiro’s characterization of Vassiliev knot invariants [5]. We shall not recall here the (rather technical) definition of \( RC_k \)-equivalence, but only note the following key fact (see [13]): two welded string links that are \( w_k \)-equivalent, have \( RC_k \)-equivalent images by the Tube map.

Combining these facts on the Tube map with the results of this paper, has several concrete consequences for ribbon tubes.
Using the surjectivity and additivity of the Tube map, we have the following from the results of Section 3.3.

**Corollary 5.3.** The set $rT(n)_k$ of $RC_k$-equivalence classes of $rT(n)$, is a finitely generated group. This group is abelian if and only if $k = 1$ or $n = 1$.

The characterization of degree $< 3$ finite type invariants of ribbon tubes, stated in the introduction, likewise follows immediately from Corollaries 4.2 and 4.6. Of course, we also have analogues of the normal form results, Theorems 4.1 and 4.5, that we shall not state here explicitly. Parallel to Conjecture 3.3, this result in low degree raises the following.

**Conjecture 5.4.** Two ribbon tubes are $RC_k$-equivalents if and only if they cannot be distinguished by finite type invariants of degree $< k$.

**Appendix A.** Towards a $w_4$-classification of welded string links

As indicated in the Introduction, the characterization of degree 3 finite type invariants of welded string links and ribbon tubes, was investigated in details [2], but a complete result is not known. In this appendix, we outline the 2-component case, referring the reader to [2] for the general case. As a matter of fact, even for 2 components, we only show a characterization modulo a conjectural relation (A.2).

We expect that this exploratory section will lay the ground for future works.

Consider the welded long knots $F$ and $F'$, and the 2-component welded string links $A, B, C, D$ and $TO_i, OT_i$ $(i = 1, 2, 3, 4)$ shown below.

For $i = 1, 2$, we denote by $F_i$ the 2-component welded string link obtained from the trivial one by inserting a copy of $F$ on the $i$th component. We also denote with a superscript $-1$ the welded string links obtained from the above ones by inserting a • near the head, which by Remark 3.10 defines the inverse up to $w_4$-equivalence.

**Remark A.1.** Let $T$ be a $w_3$-tree for the 2-component welded string link 1. Similar combinatorial considerations as in the proofs of Theorems 4.1 and 4.5 show that, up to $w_4$-equivalence, $T$ can be freely assumed to be a copy of one of the $w_3$-trees listed above, or its inverse.

Before listing the known relations among these welded string links, let us state the above-mentioned conjectural relation.

**Conjecture A.2.**

$$A \cdot D \not< B \cdot C.$$
This can be seen as an analogue of relation (4.3) up to \( w \)-equivalence. The normal form and classification results Theorem A.6 and Corollary A.7 below are to be understood as statements \textit{modulo} Conjecture A.2.

Now, we have the following known relations.

**Proposition A.3.**

(A.1) \[ F' \sim F^{-1} \]

(A.2) \[ TO_1 \sim TO_2^{-1} \quad \text{and} \quad TO_3 \sim TO_4^{-1} \]

(A.3) \[ OT_1 \sim OT_2^{-1} \quad \text{and} \quad OT_3 \sim OT_4^{-1} \]

(A.4) \[ A \cdot OT_3 \sim B \cdot OT_2 \quad \text{and} \quad C \cdot TO_3 \sim D \cdot TO_2 \]

We will need the following technical result to prove Proposition A.3.

**Claim A.4.** Let \( T_1 \) and \( T_2 \) be two \( w_k \)-trees for \( 1 \), which are identical except in a disk where they differ as shown below.

![Diagram of T1 and T2](image)

We have \[ 1_{T_1} \cdot 1_{T_2} \sim 1. \]

**Proof.** Let us start with the union \( A \cup V \) of \( w \)-trees shown on the left below. Exchanging the tail of \( A \) with the head of \( V \) by move (8), one can isolate and delete \( A \) by move (4). The exchange move (8) introduces a \( w_k \)-tree, which can be isolated up to \( w_{k+1} \)-equivalence by Corollary 3.6. More precisely, we obtain:

![Diagram of exchange move](image)

where the second equivalence is obtained by exchanging the tails of the \( w_k \)-tree by move (3), followed by the Antisymmetry move (9) and the Twist relation (11).

Now let us return to the union \( A \cup V \), and exchange now the head of \( A \) with the adjacent tail of \( V \) using the exchange relation (12). This introduces a \( w_k \)-tree, which can be isolated by Corollary 3.6 as follows:

![Diagram of exchange relation](image)

As we show below, the resulting union \( L \cup R \) of \( w \)-trees satisfies the second equivalence above, and multiplying by the inverse of \( V \) in \( wSL(n)_{k+1} \) then gives the desired equivalence.

Let us turn to the union \( L \) of \( w \)-trees. Exchanging both heads by move (7), we can delete the resulting \( w_1 \)-tree by move (4). As before, using Lemma 3.6 we then
obtain the equivalence on the left-hand side below.

We now focus on the \( w_k \)-tree \( R \). Using the IHX relation of [13, Lem. 7.16], we have the equivalence shown on the right-hand side. Combining these equivalences for \( L \) and \( R \) indeed provides the desired equivalence, and the proof is complete. □

**Proof of Proposition A.3.** Using Claim A.4, with \( k = 3 \), we immediately obtain relations (A.i) for \( i = 1, 2, 3 \). Let us now prove (A.4): we only show the relation on the left-hand side, since the second relation is proved by the exact same argument. The strategy is very similar to the previous proof, so we only outline the successive operations needed. Consider a union \( U \) of a \( w_1 \) and \( w_2 \)-tree, as shown on the left-hand side below. Applying the generalized Head/Tail exchange relation (12) at the bottom of component 2, we obtain the following equivalence:

On the other hand, exchanging the head and tail on component 1 using relation (12), followed by the IHX relation [13, Lem. 7.16], gives the first equivalence below:

The second equivalence is then obtained by using relation (12) at the top of component 2. This proves that \( A \not\sim B \cdot OT_2 \cdot OT_3^{-1} \). □

In order to state the normal form result for 2-component welded string links up to \( w_4 \)-equivalence, we introduce the following.

**Notation A.5.**

(i). \( \gamma_1 = -\mathcal{I}_{(1,2),3} + \mathcal{I}_{(1),3} + \mathcal{I}_{(2),3} + \mu(1121) \).

(ii). \( \gamma_2 = \mathcal{I}_{(2,1),3} - \mathcal{I}_{(1),3} - \mathcal{I}_{(2),3} + \mathcal{I}_{(1),2} + \mathcal{I}_{(2),2} \).

(iii). \( \gamma_3 = -\mathcal{I}_{(2,1),3} + \mathcal{I}_{(2),3} + \mu(2212) \).

**Theorem A.6.** Let \( L \) be a 2-component welded string link. Assuming Conjecture A.2, we have

\[ L \sim L_{12} \cdot L_3, \]

where

\[ L_{12} := L_1 \cdot L_2 = \mathcal{F}_{1,2}^{(1,2)} \cdot \mathcal{F}_{2,1}^{(21)} \cdot E_1^{\mathcal{I}_{(1),2}}(L) \cdot E_2^{\mathcal{I}_{(2),2}}(L) \cdot A_{1,2}^{\mathcal{I}_{(1),1}}(L) \cdot B_{1,2}^{\mathcal{I}_{(2),1}}(L) \cdot C_{1,2}^{\mathcal{I}_{(1),1}}(L) \]

as in Theorem 4.3, and where \( L_3 \) is given by

\[ (F_1)_{\mathcal{I}_{(1),3}}(L) - \mathcal{I}_{(1),3}(L_{12}) \cdot (F_2)_{\mathcal{I}_{(2),3}}(L) - \mathcal{I}_{(2),3}(L_{12}) \cdot (A)_{\gamma_1(L) - \gamma_1(L_{12})} \cdot (B)_{\gamma_2(L) - \gamma_2(L_{12})} \cdot (C)_{\gamma_3(L) - \gamma_3(L_{12})} \cdot (TO_1)_{\mu(L)(1121) - \mu(L_{12})(1121)} \cdot (OT_1)_{\mu(L)(2212) - \mu(L_{12})(2212)}. \]

As before, we immediately deduce the following characterization result.

**Corollary A.7.** Assuming Conjecture A.2, the following are equivalent.
(1) Two 2-component welded string links \( L \) and \( L' \) are \( w_4 \)-equivalent;
(2) For any finite type invariant \( \nu \) of degree at most 3, we have \( \nu(L) = \nu(L') \);
(3) \( L \) and \( L' \) have same Milnor invariants \( \mu(12) \), \( \mu(21) \), \( \mu(1121) \) and \( \mu(2212) \), and same closure invariants \( I_{(1,2);3} \), \( I_{(2,1);3} \), \( I_{(2);3} \), \( I_{(1);3} \), \( I_{(1,2);2} \), \( I_{(2,1);2} \), \( I_{(1);2} \), \( I_{(2);2} \).

Proof of Theorem A.6. By Lemma 3.7, \( L \) is \( w_4 \)-equivalent to a product of terms, each obtained from \( 1 \) by surgery along a single \( w_i \)-tree (\( i \leq 3 \)), ordered by their degree. Following the proofs of Theorems 4.1 and 4.5, we can further assume that

\[
L \sim L_1 \cdot L_2 \cdot \tilde{L}_3,
\]

where \( L_1 \) and \( L_2 \) are as given in the statement and where \( \tilde{L}_3 \) is a product of terms, each obtained from \( 1 \) by surgery along a single \( w_3 \)-tree. Remark A.1, together with the relations of Proposition A.3, then give us that

\[
L \sim L_1 \cdot L_2 \cdot A^a B^b C^c (TO_1)^1 (OT_1)^t (F_1)^e_1 (F_2)^e_2,
\]

for some coefficients \( a, b, c, t, u, e_1 \) and \( e_2 \) in \( \mathbb{Z} \), that we must now determine. We have the following evaluations of our invariants:

\[
\begin{pmatrix}
\sim & A & B & C & TO_1 & OT_1 & F_1 & F_2 \\
I_{(2,1);3} & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
I_{(1,2);3} & 0 & 1 & -1 & 0 & 1 & 1 & 1 \\
\mu(1211) & -1 & 0 & 0 & 1 & 0 & 1 & 1 \\
\mu(2122) & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\
I_{(1);3} & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\
I_{(2);3} & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
\end{pmatrix}
\]

Note that this matrix has rank 7. Thanks to the additivity properties of closure invariants (Proposition 3.8) and of welded Milnor invariants ([13, Lem. 6.11]), determining the above coefficients then essentially amounts to computing the inverse matrix. (Note that here, unlike in Theorems 4.1 and 4.5, we do not make explicit the evaluations of our invariants on the degree \( \leq 2 \) part \( L_{12} \), as it is not necessary for deriving Corollary A.7.) Details can be found in [2] and are left to the reader.

Remark A.8. Corollary A.7 suggests that, unlike in the degree 2 case, one cannot generate the space of degree 3 finite type invariants of welded string links by only closure invariants. As a matter of fact, further computations show that one cannot replace the classifying invariants \( \mu(1121) \) and \( \mu(2212) \) by any combination of the closure invariants \( I_{R;3} \) with \( R \) a list of length \( \leq 2 \).

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