ROSENTHAL FAMILIES, PAVINGS AND GENERIC CARDINAL INVARIANTS

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Abstract. Following D. Sobota we call a family $F$ of infinite subsets of $\mathbb{N}$ a Rosenthal family if it can replace the family of all infinite subsets of $\mathbb{N}$ in classical Rosenthal’s Lemma concerning sequences of measures on pairwise disjoint sets. We resolve two problems on Rosenthal families: every ultrafilter is a Rosenthal family and the minimal size of a Rosenthal family is exactly equal to the reaping cardinal $r$. This is achieved through analyzing nowhere reaping families of subsets of $\mathbb{N}$ and through applying a paving lemma which is a consequence of a paving lemma concerning linear operators on $\ell_1$ due to Bourgain. We use connections of the above results with free set results for functions on $\mathbb{N}$ and with linear operators on $c_0$ to determine the values of several other derived cardinal invariants.

1. Introduction

Recall that the reaping number $r$ is the minimal cardinality of a family $D$ of infinite subsets of $\mathbb{N}$ (denoted by $[\mathbb{N}]^\omega$) which is not split by a single subset of $\mathbb{N}$, i.e., such that there is no $A \subseteq \mathbb{N}$ such that $A \cap D$ and $D \setminus A$ are both infinite for all $D \in D$ (see [3] for more details).

A family $D$ of infinite subsets of $\mathbb{N}$ will be called dense if for every infinite $A \subseteq \mathbb{N}$ there is $B \in D$ such that $B \subseteq A$. Let $D$ be a collection of dense sets. A family $G \subseteq [\mathbb{N}]^\omega$ is called a generic family for $D$ if $G \cap D \neq \emptyset$ for all $D \in D$. This terminology agrees with the standard one for the partial order $([\mathbb{N}]^\omega, \subseteq)$ but it should be stressed that our generic families need not to be filters. We define the generic cardinal number $\text{gen}(D)$ for $D$ to be the minimal cardinality of a generic family for $D$. It is clear that if $D$ is as above, then $\text{gen}(D) \leq c$ as $[\mathbb{N}]^\omega$ is a generic family for $D$.

As an example consider a sequence $(x_n)_{n \in \mathbb{N}}$ of zeros and ones. Let $\text{Conv}^{01}_{(x_n)_{n \in \mathbb{N}}}$ be the family of all $A \in [\mathbb{N}]^\omega$ such that $(x_n)_{n \in A}$ converges and let $\text{Conv}_{01}$ be the collection of all such sets $\text{Conv}^{01}_{(x_n)_{n \in \mathbb{N}}}$. Then $\text{gen}(\text{Conv}_{01}) = r$ (Theorem 3.7 of [3]). Here the Bolzano-Weierstrass theorem plays the role of a density lemma i.e., a result asserting that the families in question are dense. If $(x_n)_{n \in \mathbb{N}}$ above is an arbitrary bounded sequence of reals and $\text{Conv}$ is the family of all analogous dense sets $\text{Conv}_{(x_n)_{n \in \mathbb{N}}}$, then $\text{gen}(\text{Conv}) = r_\sigma$, where $r_\sigma$ is a modified version of $r$ (see Section 3 of [3]).

Another example of a density lemma, dense sets, generic families and the generic cardinal invariant is the following: Let $f : [\mathbb{N}]^2 \to \{0, 1\}$ and let $\text{Hom}_f$ be the family of all infinite subsets of $\mathbb{N}$ which are homogeneous for $f$. The Ramsey theorem as a density lemma yields the density of each set $\text{Hom}_f$. If $\text{Hom}$ is the collection of all such families $\text{Hom}_f$ for all functions $f$ as above, then $\text{gen}(\text{Hom}) = \max(\mathfrak{d}, r_\sigma)$.
(Theorem 3.10 of [3]), where the dominating number $d$ is the minimal size of a family of functions from $\mathbb{N}$ to $\mathbb{N}$ which eventually dominate any such function.

Moreover, it is proved in 4.7 and 4.9 of [4] that an ultrafilter is generic for Conv if and only if it is p-point and it is generic for Hom if and only if it is selective. In particular, by a theorem of S. Shelah it is consistent that there are no generic ultrafilters for Conv and Hom ([27], cf. [9]). This means that there are no generic families for these collections of dense sets which satisfy the strong finite intersection property (i.e., intersection of every finite subfamily is infinite) as any ultrafilter extending such families would need to be a p-point or a selective ultrafilter by the results of [4].

The topic of this paper falls into the category of the results described above. Originally we were motivated by a classical density lemma frequently used in several parts of mathematics concerning sequences of measures (for the discussion of its classical forms and uses see Subsection 2.3) which can be stated in the following equivalent combinatorial form:

**Lemma 1** (Rosenthal’s lemma; [23], [24]). Suppose that $M = (m_{k,n})_{k,n \in \mathbb{N}}$ is a matrix of non-negative reals, where the set of sums $\{\sum_{n \in \mathbb{N}} m_{k,n} : k \in \mathbb{N}\}$ is bounded. For every $\varepsilon > 0$ and every infinite $B \subseteq \mathbb{N}$ there is an infinite $A \subseteq B$ such that for every $k \in A$ we have

$$\sum_{n \in A \setminus \{k\}} m_{k,n} \leq \varepsilon.$$

Matrices $M$ as above will be called **Rosenthal matrices**, the set of all of them will be denoted by $\mathbb{M}$. The supremum of the set $\{\sum_{n \in \mathbb{N}} m_{k,n} : k \in \mathbb{N}\}$ will be called a norm of $M$ and will be denoted $\|M\|_\infty$. If the condition (*) is satisfied, we will say that $M$ is $\varepsilon$-fragmented by $A$. So, Rosenthal’s lemma is a density lemma which asserts that $\text{Ros}_{M,\varepsilon} = \{A \in [\mathbb{N}]^{\omega} : M$ is $\varepsilon$-fragmented by $A\}$ is dense for every $M \in \mathbb{M}$ and every $\varepsilon > 0$. Generic families for $\text{Ros} = \{\text{Ros}_{M,\varepsilon} : M \in \mathbb{M}, \varepsilon > 0\}$ were introduced by D. Sobota in [26] and called **Rosenthal families**. It was proved in [20] that a basis of a selective ultrafilter is a Rosenthal family and the following cardinal invariant which is the generic cardinal invariant $\text{gen}(\text{Ros})$ was defined:

$$\text{ros} = \min\{|F| : F$ is a Rosenthal family$\}.$$  

It was determined in [26] that $\text{cov}(\mathcal{M}) \leq \text{ros} \leq u_\varepsilon$, where $u_\varepsilon$ stands for the minimal size of a base for a selective ultrafilter if there is one, and $\varepsilon$ otherwise. In particular, it was proved in [26] that $\text{ros}$ can be arbitrarily big on the scale of alephs and that it can be strictly smaller than the continuum. The role of selective ultrafilters here is natural as S. Todorcevic has shown that all of them are $([\mathbb{N}]^{\omega}, \subseteq^*)$-generic over $L[\mathbb{R}]$, at least under a suitable large cardinal assumption (see 4.4. of [13]). It should also be noted that the value of $\text{ros}$ in various models of set theory is not a mere curiosity. As the Rosenthal lemma is a practical tool used for proving properties of Boolean algebras, compact spaces, sequences of measures or Banach spaces, the value of $\text{ros}$ tells us what are the sizes of the objects whose constructions require the use of the Rosenthal lemma with the output in the constructed structure. This is related to the topic of sizes of Boolean algebras or densities of Banach spaces with the Grothendieck property or with the Nikodym property ([24]). The first main result of this paper is the following:

**Theorem 26**. The Rosenthal number $\text{ros}$ is equal to the reaping number $r$. 

So we obtain quite a full picture of the relationships between $\tau \preceq \mathfrak{r}$ and other cardinal invariants from the Cichoń’s and van Douwen’s diagrams which is well known for $\tau$ and can be found e.g. in [3]. In particular $\mathfrak{r}$ is bounded below by $\max(\text{cov}(M), \text{cov}(N), b)$ and above by the ultrafilter number $u$ i.e., the minimal size of a base of a nonprincipal ultrafilter on $\mathbb{N}$. Moreover, it is consistent that the value of $\mathfrak{r}$ is strictly bigger or strictly smaller than many other known cardinal invariants.

One of the side products of the proof of inequality $\tau \preceq \mathfrak{r}$ in Theorem 26 which is obtained in Section 4 is the result (Corollary 23) saying that if a Rosenthal family $\mathcal{F}$ has cardinality less then $u$, then it fails to have the strong finite intersection property, i.e. is far from being a generic filter. Note that $\mathfrak{r} \preceq \mathfrak{u}$ is consistent by Theorem 23 and the main result from [14]. The reason why the inequality $\mathfrak{r} \preceq \mathfrak{u}$ holds is that Rosenthal’s lemma has a stronger version, apparently overlooked by many of its users, namely:

**Theorem 18 (5, 6).** For every $\varepsilon > 0$ there is $l(\varepsilon) \in \mathbb{N}$ such that for every Rosenthal matrix $M$ there is a partition $P = \{P_i : 1 \leq i \leq l(\varepsilon)\}$ of $\mathbb{N}$, such that $M$ is $\varepsilon\|M\|_\infty$-fragmented by $P_i$ for every $1 \leq i \leq l(\varepsilon)$.

This has an immediate corollary which answers Question 3.18 of [26]:

**Theorem 19.** Every nonprincipal ultrafilter over $\mathbb{N}$ is a Rosenthal family. In particular, any $\pi$-base of any nonprincipal ultrafilter is a Rosenthal family.

Recall that a $\pi$-base of a nonprincipal ultrafilter $u$ is a family $\mathcal{B} \subseteq [\mathbb{N}]^\omega$ such that for every $A \in u$ there is $B \in \mathcal{B}$ such that $B \subseteq A$. This result is a bit surprising at first sight because, as we mentioned before, it is consistent that there are no ultrafilters which are generic families for Conv or Hom.

A result like Theorem 18 where not only the density is asserted but actually $\mathbb{N}$ can be partition into sets belonging to the dense family in question will be called a paving lemma in the analogy of to the paving conjectures equivalent to the Kadison-Singer problem (7). The remaining results presented in this paper consist of determining generic cardinal invariants for certain collections of dense sets which are natural subfamilies of $\text{Ros}$. Let us discuss these families and the results below.

Let $X, Y$ be some of the Banach spaces $c_0$ or $\ell_p$ for $1 \leq p \leq \infty$. By $\mathcal{B}_0(X, Y)$ we denote the family of bounded linear operators from $X$ into $Y$ with zero diagonal, i.e. such $T : X \to Y$ that $T(1_{\{n\}})(n) = 0$ for every $n \in \mathbb{N}$; we write $\mathcal{B}_0(X) = \mathcal{B}_0(X, X)$. For $T \in \mathcal{B}_0(X, Y)$ we define

$$\text{Ros}_{T, \varepsilon} = \{A \in [\mathbb{N}]^\omega : \|P_A T P_A\| \leq \varepsilon\|T\|\},$$

where $P_A : \ell_\infty \to \ell_\infty$ is given by $P_A(f) = f \cdot 1_A$ for all $f \in \ell_\infty$ and where $1_A$ is the characteristic function of $A$. Let $\text{Ros}(X, Y) = \{\text{Ros}_{T, \varepsilon} : T \in \mathcal{B}_0(X, Y), \varepsilon > 0\}$.

It turns out that $\text{Ros} = \text{Ros}(c_0, \ell_\infty)$ (Proposition 5, 11) because Rosenthal matrices correspond exactly to matrices of operators from $c_0$ to $\ell_\infty$ (Lemma 1) and the $(\varepsilon\|T\|)$-fragmentation corresponds to the condition in the definition of $\text{Ros}_{T, \varepsilon}$ (Lemma 7). In fact the transposed matrices of Rosenthal matrices correspond exactly to matrices of operators on $\ell_1$ (Lemmas 4 and 5) and so we obtain

**Theorem 28** $\mathfrak{r}_o\varepsilon(\ell_1) = \tau$.

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1The relation between the statement of Theorem 13 and Theorem 1.3’ of [6] which is not obvious will be discussed in Section 3 after the proof of Theorem 13.
Clearly $\text{Ros}(c_0)$ is a proper subfamily of $\text{Ros} = \text{Ros}(c_0, \ell_\infty)$ (Proposition 8 (3)). A generic family for $\text{Ros}(c_0)$ will be called a $c_0$-Rosenthal family and the generic cardinal number for $\text{Ros}(c_0)$ will be denoted $\text{ros}(c_0)$. We obtain the following:

**Theorem 31** $\text{ros}(c_0) = \min(\mathfrak{d}, r)$.

The paving result for operators on $\ell_2$ has been recently obtained in [20] which resolved the Kadison-Singer problem. This gives $\text{ros}(\ell_2) \leq r$ but we are left with:

**Question 2.** What is the value of $\text{ros}(\ell_2)$?

It should be noted that the question if the paving lemma holds for $\ell_p$ for $1 < p < \infty$, $p \neq 2$ is a known open problem ([7]). Another natural subcollection $\text{Ros}_{01}$ of $\text{Ros}$ consists of dense families $\text{Ros}_f = \{ A \in [\mathbb{N}]^{\mathbb{N}} : f[A] \cap A = \emptyset \}$, where $f : \mathbb{N} \to \mathbb{N}$ is a function with no fixed points. It can be easily seen that $\text{Ros}_f = \text{Ros}_{M,1/2}$, where $M = (m_{k,n})_{k,n \in \mathbb{N}}$ is a Rosenthal matrix, where $m_{k,f(k)} = 1$ and $m_{k,n} = 0$ if $n \neq f(k)$ (see Section 2.2). A generic family for $\text{Ros}_{01}$ will be called a binary Rosenthal family and the generic cardinal invariant for $\text{Ros}_{01}$ will be denoted by $\text{ros}_{01}$. We obtain:

**Theorem 27** $\text{ros}_{01} = r$.

The last type of generic families we consider comes from combining $c_0$-Rosenthal families and binary Rosenthal families. In fact, Rosenthal matrices which correspond to elements of $\mathcal{B}_0(c_0)$ are exactly Rosenthal matrices whose columns converge to zero (Lemma 6). If we consider binary Rosenthal matrices $M$ such that $\text{Ros}_{01} \supset \text{Ros}_f = \text{Ros}_{M,1/2}$, then one sees that the condition that the columns converge to zero translates to $f$ being finite-to-one. So we define $\text{Ros}_{01}(c_0)$ as the collection of families $\text{Ros}_f$ where $f : \mathbb{N} \to \mathbb{N}$ is a finite-to-one function with no fixed points. A generic family for $\text{Ros}_{01}(c_0)$ will be called a binary $c_0$-Rosenthal family and the generic cardinal invariant for $\text{Ros}_{01}(c_0)$ will be denoted by $\text{ros}_{01}(c_0)$. We obtain:

**Theorem 32** $\text{ros}_{01}(c_0) = \min(\mathfrak{d}, r)$.

However we do not know the answer to the following:

**Question 3.** What is the value of the generic cardinal invariant for the family $\text{Ros}_{01}^{-1} = \{ \text{Ros}_f : f : \mathbb{N} \to \mathbb{N} \text{ is a one-to-one function with no fixed points} \}$?

Let us describe the structure of of the paper. Section 2 is devoted to proving some of the above claims concerning the relations between Rosenthal matrices and linear bounded operators, functions without fixed points and sequences of measures. In section 3 we discuss versions of Theorem 18 present in the literature and we prove it and conclude Theorem 19. Section 4 is devoted to applications of nowhere reaping families which together with the results of Section 3 give main results on the values of $\text{ros}$ and $\text{ros}_{01}$. In section 5 we calculate $\text{ros}(c_0)$ and $\text{ros}_{01}(c_0)$. Set-theoretic terminology is based on [3]. Terminology concerning linear operators is introduced at the beginning of Section 2.
2. Rosenthal matrices and families

2.1. Rosenthal matrices and linear bounded operators. If $\mathbb{K}$ is either the field $\mathbb{C}$ of complex numbers or the field $\mathbb{R}$ of the reals, we will consider the Banach space $\ell_\infty(\mathbb{K}) = \{ f \in \mathbb{K}^\mathbb{N} : f \text{ is bounded} \}$ with the supremum norm $||f||_\infty = \sup\{|f(n)| : n \in \mathbb{N} \}$ and its subspace $c_0(\mathbb{K}) = \{ f \in \ell_\infty(\mathbb{K}) : \lim_{n \to \infty} f(n) = 0 \}$ as well as the spaces $\ell_p(\mathbb{K}) = \{ f \in \mathbb{K}^\mathbb{N} : \Sigma_{n \in \mathbb{N}} |f(n)|^p < \infty \}$ with the $p$-norm $||f||_p = \left( \Sigma_{n \in \mathbb{N}} |f(n)|^p \right)^{\frac{1}{p}}$, where $1 \leq p < \infty$. We will also mention the finite-dimensional versions $\ell_\infty^n(\mathbb{K})$, $c_0^n(\mathbb{K})$, $\ell_p^n(\mathbb{K})$ for $n \in \mathbb{N}$ and $1 \leq p < \infty$. Note that $\ell_\infty^n$ is the same as $c_0^n$ for each $n \in \mathbb{N}$.

We will skip the specification of the field $\mathbb{K}$, that is we will use $\ell_\infty$, $c_0$, $\ell_p$, etc., as all of our arguments work for both cases. Linear operators $T : X \to Y$ between Banach spaces $(X, ||x||_X), (Y, ||y||_Y)$ will be considered with the operator norm, i.e., $||T|| = \sup\{|T(x)||y|/||x||_X : x \in X \}$. When dealing with finite or infinite matrices we will specify the norms $|| ||_\infty$ or $|| ||_1$ which are defined in the Lemmas 4 and 5. Recall from the introduction that $\chi_A$ denotes the characteristic function of a set $A$.

We will need the following three elementary and well known lemmas on infinite matrices. We provide the proofs for the convenience of the reader:

**Lemma 4.** Every matrix $M = (m_{k,n})_{k,n \in \mathbb{N}}$ satisfying $\sup\{|\Sigma_{n \in \mathbb{N}} m_{k,n} : k \in \mathbb{N} \} = ||M||_\infty < \infty$ defines a linear bounded operator $T_M : c_0 \to \ell_\infty$ satisfying
\[
T((a_n)_{n \in \mathbb{N}})(k) = \Sigma_{n \in \mathbb{N}} a_n m_{k,n}
\]
for each $k \in \mathbb{N}$. The operator $T_M$ has norm $||M||_\infty$. Moreover, each bounded linear operator from $c_0$ into $\ell_\infty$ is of this form.

**Proof.** The requirement concerning $M$ implies that $T_M$ is well-defined on $c_0$ into $\ell_\infty$. It is clear that $T_M$ is linear. If $(a_n)_{n \in \mathbb{N}} \in c_0$, then $|T_M((a_n)_{n \in \mathbb{N}})(k)| \leq M_1 ||(a_n)_{n \in \mathbb{N}}||$, so $T_M$ is a bounded operator and $||T_M|| \leq ||M||_\infty$. Having fixed $k, i \in \mathbb{N}$ by taking numbers $a_n$ such that $a_n m_{k,n} = |m_{k,n}|$ for $n \leq i$ and $a_n = 0$ for $n > i$ we get that $|T_M((a_n)_{n \in \mathbb{N}})|| \geq \Sigma_{n \leq i}|m_{k,n}|$ and so $||M||_\infty \leq ||T_M||$.

Given any $T : c_0 \to \ell_\infty$ define $m_{k,n} = T(\chi_{\{n\}})(k)$. As $T^*(\delta_k) = \delta_k \circ S$ is a linear bounded functional on $c_0$ of norm not bigger than $||T||$, it must be in $\ell_1 = c_0$. So $\sup\{|\Sigma_{n \in \mathbb{N}} m_{k,n} : k \in \mathbb{N} \} = ||M||_\infty \leq ||T||$. As the span of $\{\chi_{\{n\}} : n \in \mathbb{N} \}$ is dense in $c_0$, we obtain $T((a_n)_{n \in \mathbb{N}})(k) = \Sigma_{n \in \mathbb{N}} a_n m_{k,n}$ for each $(a_n)_{n \in \mathbb{N}} \in c_0$. \hfill $\Box$

**Lemma 5.** Every matrix $M = (m_{k,n})_{k,n \in \mathbb{N}}$ satisfying $\sup\{|\Sigma_{k \in \mathbb{N}} m_{k,n} : n \in \mathbb{N} \} = ||M||_1 < \infty$ defines a linear bounded operator $T_M : \ell_1 \to \ell_1$ satisfying
\[
T((a_n)_{n \in \mathbb{N}})(k) = \Sigma_{n \in \mathbb{N}} a_n m_{k,n}
\]
for each $k \in \mathbb{N}$. The operator $T_M$ has norm $||M||_1$. Moreover, each bounded linear operator on $\ell_1$ is of this form.

**Proof.** The requirement concerning $M$ implies that the rows of $M$ have $\ell_\infty$-norms bounded by $||M||_1$ as well, so $T_M$ is well-defined on $\ell_1$ and is sending norm one elements of $\ell_1$ into sequences bounded by $||M||_1$. It is clear that $T_M$ is linear.

\[
\Sigma_{k \in \mathbb{N}} |T_M((a_n)_{n \in \mathbb{N}})(k)| \leq \Sigma_{k \in \mathbb{N}} \Sigma_{n \in \mathbb{N}} |a_n||m_{k,n}| = \\
\leq \Sigma_{n \in \mathbb{N}} |a_n| \Sigma_{k \in \mathbb{N}} |m_{k,n}| \leq ||M||_1 ||(a_n)_{n \in \mathbb{N}}||.
\]

So $T_M : \ell_1 \to \ell_1$ and $||T_M|| \leq ||M||_1$. We have that $||T_M((1_{\{n\}})_{n \in \mathbb{N}})|| = \Sigma_{k \in \mathbb{N}} |m_{k,n}|$ and so $||M||_1 \leq ||T_M||$. \hfill $\Box$
Given any \( T : \ell_1 \to \ell_1 \) define \( m_{k,n} = T(1_{(n)}) (k) \). As \( T_M((1_{(n)})_{n \in \mathbb{N}}) = (m_{k,n})_{n \in \mathbb{N}} \), it follows that \( \sup \{ \Sigma_{k \in \mathbb{N}} |m_{k,n}| : n \in \mathbb{N} \} \leq \| T_M \|. \) As the span of \( \{ 1_{(n)} : n \in \mathbb{N} \} \) is dense in \( \ell_1 \), we obtain \( T((a_n)_{n \in \mathbb{N}})(k) = \Sigma_{n \in \mathbb{N}} a_n m_{k,n} \) for each \((a_n)_{n \in \mathbb{N}} \in \ell_1 \).

**Lemma 6.** Every matrix \( M = (m_{k,n})_{k,n \in \mathbb{N}} \) satisfying \( \sup \{ \Sigma_{n \in \mathbb{N}} |m_{k,n}| : k \in \mathbb{N} \} = \| M \|_{\infty} < \infty \) and \( \lim_{k \to \infty} m_{k,n} = 0 \) for each \( n \in \mathbb{N} \) defines a linear bounded operator \( T_M : c_0 \to c_0 \) satisfying

\[
T((a_n)_{n \in \mathbb{N}})(k) = \Sigma_{n \in \mathbb{N}} a_n m_{k,n}
\]

for each \( k \in \mathbb{N} \). The operator \( T_M \) has norm \( \| M \|_{\infty} \). Moreover, each bounded linear operator on \( c_0 \) is of this form.

**Proof.** By Lemma 4 the operator \( T_M : c_0 \to \ell_\infty \) has norm equal to \( \| M \|_{\infty} \). Let \( M = (m_{k,n})_{k,n \in \mathbb{N}} \in c_0 \) so \( \lim_{k \to \infty} m_{k,n} = 0 \). For the moreover part use again Lemma 4 to conclude that any operator on \( c_0 \) is given as in Lemma 6. The same argument concerning \( T_M((1_{(n)})_{n \in \mathbb{N}}) \) as above yields \( \lim_{k \to \infty} m_{k,n} = 0 \).

Before we note the relations between various generic families we need one more observation:

**Lemma 7.** Suppose that \( M \) is a Rosenthal matrix with zero diagonal and \( A \subseteq \mathbb{N} \) is infinite. \( M \) is \( \varepsilon \)-fragmented by \( A \) if and only if \( \| M_A M M_A \|_{\infty} \leq \varepsilon \), where \( M_A = (m_{k,n})_{k,n \in \mathbb{N}} \) is the diagonal matrix satisfying \( m_{k,k} = 1 \) if \( k \in A \) and \( m_{k,k} = 0 \) otherwise.

**Proof.** \( M \) is \( \varepsilon \)-fragmented by \( A \) if and only if \( \Sigma_{k \in A} m_{k,n} \leq \varepsilon \) for each \( k \in A \) if and only if \( \| M_A M M_A \|_{\infty} \leq \varepsilon \).

**Proposition 8.**

1. \( \text{Ros} = \text{Ros}(c_0, \ell_\infty) \)
2. \( \text{Ros}(c_0, \ell_\infty) = \text{Ros}(\ell_1) \)
3. \( \text{Ros}(c_0) \subseteq \text{Ros} \)

**Proof.** Let \( M \) be a Rosenthal matrix and \( \varepsilon > 0 \). Let \( M' = (m'_{k,n})_{k,n \in \mathbb{N}} \) be obtained from \( M \) by replacing the diagonal entries by zeros. We will see that

\[
\text{Ros}_{M,\varepsilon} = \text{Ros}_{TM',\varepsilon/\|TM'\|} = \{ A \in [\mathbb{N}]^\omega : \| PAT_M'PA \| \leq \varepsilon \},
\]

where \( TM' \) is the operator defined in Lemma 4 for \( M' \). Let \( A \subseteq \mathbb{N} \) be infinite. \( M \) is \( \varepsilon \)-fragmented by \( A \) if and only if \( M' \) is \( \varepsilon \)-fragmented by \( A \) if and only if \( \| M_A M' M_A \|_{\infty} \leq \varepsilon \) (by Lemma 4) if and only if \( \| P_A T_M' P_A \| \leq \varepsilon \) by Lemma 4.

Now let \( T : c_0 \to \ell_\infty \) has zeros on the diagonal, i.e., \( T((1_{(n)})_n) = 0 \) for each \( n \in \mathbb{N} \). Define a matrix \( M = (m_{k,n})_{k,n \in \mathbb{N}} \) given by \( m_{k,n} = |S((1_{(n)})_n)(k)| \) for all \( k,n \in \mathbb{N} \). Lemma 4 implies that it is a Rosenthal matrix. Let \( A \subseteq \mathbb{N} \) be infinite.

\[
\| P_A T_M P_A \| \leq \varepsilon \| T \| \text{ if and only if } \| M_A M M_A \| \leq \varepsilon \| T \| \text{ if and only if } M \text{ is } \varepsilon \| T \|\text{-fragmented by } A \text{ by Lemma 4.}
\]

So \( \text{Ros}_{M,\varepsilon/\|T\|} = \text{Ros}_{T,\varepsilon} \) which proves (1).

For (2) we note that by Lemmas 3 and 4 a matrix \( M \) defines an operator \( T_M \) from \( c_0 \) into \( \ell_\infty \) if and only if its transpose \( M' \) defines an operator \( T_{M'} \) on \( \ell_1 \). Moreover \( \| M \|_{\infty} = \| M' \|_1 \). So \( \| P_A T_M P_A \| = \| P_A M' P_A \|_{\infty} = \| P_A M' P_A \|_1 = \| P_A T_{M'} P_A \| \) and consequently (2) is proved.

(3) follows from the fact that operators on \( c_0 \) form a subclass of operators from \( c_0 \) into \( \ell_\infty \).
So we immediately obtain:

**Proposition 9.**

1. \( t_{\text{ros}} = t_{\text{ros}}(f_1) \).
2. \( t_{\text{ros}}(c_0) \leq t_{\text{ros}} \).

### 2.2. Rosenthal families and free sets.

From a combinatorial point of view, a natural special kind of matrices \((m_{k,n})_{k,n \in \mathbb{N}}\) as in Definition 1 is defined by requiring that it is binary (i.e., \(m_{k,n} \in \{0, 1\}\) for all \(k, n \in \mathbb{N}\)), antidiagonal (i.e., \(m_{k,k} = 0\) for all \(k \in \mathbb{N}\)) and each row has a nonzero entry, i.e., there is a function \(f : \mathbb{N} \to \mathbb{N}\) with no fixed points such that \(m_{k,f(k)} = 1\) for each \(k \in \mathbb{N}\) and \(m_{k,n} = 0\) for each \(n \neq f(k)\). We will denote such a matrix by \(M_f\).

**Lemma 10.** Suppose that \(f : \mathbb{N} \to \mathbb{N}\) has no fixed points and \(A \subseteq \mathbb{N}\) is infinite.

\(M_f\) is 1/2-fragmented by \(A\) if and only if \(f[A] \cap A = \emptyset\).

**Proof.** \(M_f\) is 1/2-fragmented by \(A\) if and only if \(\Sigma_{k \in A} m_{k,n} \leq 1/2\) for each \(k \in A\) if and only if \(f(k) \not\in A\) for each \(k \in A\) if and only if \(f[A] \cap A = \emptyset\). \(\square\)

A set \(A\) satisfying \(f[A] \cap A = \emptyset\) is called free for \(f\) following a well-established combinatorial terminology (e.g. [18]) according to which, more generally given a set mapping \(f : X \to \wp(X)\) a set \(Y \subseteq X\) is called free if \(y \not\in f(y')\) for any \(y, y' \in Y\).

**Proposition 11.**

1. \( t_{\text{ros}}(c_0) \leq t_{\text{ros}} \).
2. \( t_{\text{ros}}(c_0) \leq t_{\text{ros}}(c_0) \).
3. \( t_{\text{ros}}(c_0) \leq t_{\text{ros}}(c_0) \).

**Proof.** For (1) we note that by Lemma 10 we have \(\text{Ros}_f = \text{Ros}_{M_f, 1/2}\), where \(f : \mathbb{N} \to \mathbb{N}\) has no fixed points. So we have \(\text{Ros}_{c_0} \subseteq \text{Ros}\) and this implies (1).

For (2) we need to note that \(\text{Ros}_{c_0}(c_0) \subseteq \text{Ros}_{c_0}\) which follows from the inclusion of finite to one functions with no fixed points in all functions with no fixed points.

For (3) we need to note that \(\text{Ros}_{c_0}(c_0) \subseteq \text{Ros}(c_0)\) which follows from the fact that if \(f : \mathbb{N} \to \mathbb{N}\) has no fixed points and is finite to one, then \(M_f\) is a Rosenthal matrix whose columns have only finitely many non-zero entries and so by Lemma 10 the matrix \(M_f\) corresponds to a linear bounded operator \(T_{M_f}\) on \(c_0\). Moreover, by Lemmas 7 and 10 the conditions \(\|P_A T_{M_f} P_A\| \leq 1/2\) corresponds to \(f[A] \cap A = \emptyset\) for any infinite \(A \subseteq \mathbb{N}\). \(\square\)

### 2.3. Rosenthal families and sequences of measures.

In this section we show that the generic cardinal invariant \(t_{\text{ros}}\) corresponding to the combinatorial version of Rosenthal’s lemma (Lemma 11) is the same for the families of dense sets corresponding to both of the classical versions of Rosenthal’s lemma (Lemma 12). H. Rosenthal proved in [23] and [24] the following:

\[\text{original references}\]
Lemma 12. Let $\mathcal{A}$ be a Boolean algebra and $\mu_k : \mathcal{A} \to \mathbb{R}_+ \cup \{0\}$ be finitely additive measures on $\mathcal{A}$ for each $k \in \mathbb{N}$ which are uniformly bounded i.e., $\mu_k(1_\mathcal{A}) \leq \rho$ for some $\rho \geq 0$, where $1_\mathcal{A}$ is the unit of $\mathcal{A}$. Let $(A_n)_{n \in \mathbb{N}}$ be pairwise disjoint elements of $\mathcal{A}$ and $\varepsilon > 0$. Then there is an infinite $A \subseteq \mathbb{N}$ such that for every $k \in A$ we have

$$\sum_{n \in A \setminus \{k\}} \mu_k(A_n) \leq \varepsilon.$$  

Moreover, if $\mathcal{A}$ is a $\sigma$-complete Boolean algebra (but still the measures are assumed only to be finitely additive), then $A$ above may be chosen to satisfy the following stronger requirement for each $k \in A$:

$$\mu_k \left( \bigvee_{n \in A \setminus \{k\}} A_n \right) \leq \varepsilon,$$

where $\bigvee$ denotes the supremum in $\mathcal{A}$.

Given an infinite Boolean algebra $\mathcal{A}$ by $ac(\mathcal{A})$ we will denote the class of all infinite pairwise disjoint sequences $\overline{A} = (A_n)_{n \in \mathbb{N}}$ of elements of $\mathcal{A}$. By $\mu_\infty(\mathcal{A})$ we will denote the class of all uniformly bounded sequences of finitely additive measures on $\mathcal{A}$. Given $\varepsilon > 0$, a Boolean algebra $\mathcal{A}$, $\overline{A} \in ac(\mathcal{A})$ and $\overline{\mu} \in \mu_\infty(\mathcal{A})$ we can consider

$$Ros_{\mathcal{A},\overline{A},\overline{\mu},\varepsilon} = \{ A \in [\mathbb{N}]^\omega : \forall k \in A \sum_{n \in A \setminus \{k\}} \mu_k(A_n) \leq \varepsilon \}.$$  

Proposition 13. The generic families for the collection of dense sets of the form $Ros_{\mathcal{A},\overline{A},\overline{\mu},\varepsilon}$, where $\mathcal{A}$ is a Boolean algebra, $\overline{A} \in ac(\mathcal{A})$, $\overline{\mu} \in \mu_\infty(\mathcal{A})$ and $\varepsilon > 0$ are exactly Rosenthal families. Consequently the generic cardinal invariant for these collections is $109$.

Proof. It is clear that $Ros_{\mathcal{A},\overline{A},\overline{\mu},\varepsilon} = Ros_{M,\varepsilon}$, where $M = (m_{k,n})_{k,n \in \mathbb{N}}$ is a Rosenthal matrix defined by $m_{k,n} = \mu_k(A_n)$ for every $k, n \in \mathbb{N}$. So the corresponding notions of generic families and the generic cardinal invariants are the same for the above collection of dense sets and for $Ros$.

On the other hand if $M = (m_{k,n})_{k,n \in \mathbb{N}}$ is a Rosenthal matrix, then we can define a finitely additive measure $\mu_k(A) = \Sigma_{n \in A} m_{k,n}$ for any finite or cofinite $A \subseteq \mathbb{N}$. Now $Ros_{\mathcal{A},\overline{\mathcal{A}},\overline{\mu},\varepsilon} = Ros_{M,\varepsilon}$, where $\mathcal{A}$ is the algebra of finite or cofinite subsets of $\mathbb{N}$, $\overline{\mathcal{A}} = \{ \{n\} \}_{n \in \mathbb{N}}$.  

Dense sets corresponding to the second version of Rosenthal’s lemma above are of the form

$$Ros^\sigma_{\mathcal{A},\overline{A},\overline{\mu},\varepsilon} = \{ A \in [\mathbb{N}]^\omega : \forall k \in A \mu_k \left( \bigvee_{n \in A \setminus \{k\}} A_n \right) \leq \varepsilon \},$$

where $\mathcal{A}$ is a $\sigma$-complete Boolean algebra. To show that the generic cardinal invariant corresponding to the dense sets from the second version of Rosenthal’s lemma (Proposition 15) we need the following:

Proposition 14. Suppose that $(C_\xi)_{\xi < \omega_1}$ is an almost disjoint family of infinite subsets of $\mathbb{N}$. If $\mathcal{A}$ is a Boolean algebra and $\mu_k$s for $k \in \mathbb{N}$ are finitely additive positive measures on $\mathcal{A}$ whose norms are bounded by $\rho \in \mathbb{R}$, and if $(A_n)_{n \in \mathbb{N}}$ are
pairwise disjoint elements of \( A \), then for all but countably many \( \xi < \omega_1 \) for every \( k \in \mathbb{N} \)

\[
\mu_k \left( \bigvee_{n \in B} A_n \right) = \sum_{n \in B} \mu_k(A_n)
\]

whenever \( B \subseteq C_\xi \) and \( \bigvee_{n \in B} A_n \) exists in \( A \).

**Proof.** This is basically the argument from [23]. Consider the Stone space \( K_A \) of the Boolean algebra \( A \) and its clopen sets \( [A] \) which are those ultrafilters of \( A \) which contain \( A \). The measures \( \mu_k \)'s define linear functionals of norm not bigger than \( \rho \) on the subspace of \( C(K) \) consisting of continuous functions with finitely many values. By the Hahn-Banach theorem they extend to the entire \( C(K) \) preserving the norm, and by the Riesz representation theorem the extensions can be associated with countably additive Borel regular measures on \( K \). We will denote these extensions by the same letters \( \mu_k \).

Suppose that the lemma fails, so there is an uncountable set \( X \subseteq \omega_1 \), \( k \in \mathbb{N} \) and infinite \( B_\xi \subseteq C_\xi \) such that \( \bigvee_{n \in B_\xi} A_n \) exists in \( A \) but \( \mu_k \left( \Delta_{B_\xi} \right) > 0 \) for each \( \xi \in X \), where

\[
\Delta_B = \left( \bigvee_{n \in B} A_n \right) \setminus \bigcup_{n \in B} [A_n]
\]

for any \( B \subseteq \mathbb{N} \) for which the supremum \( \bigvee_{n \in B} A_n \) exists in \( A \). Now one notes that \( \left[ \bigvee_{n \in B} A_n \right] \) is a disjoint union of \( \left[ \bigvee_{n \in B} A_n \setminus F \right] \) and \( \bigcup_{n \in F} [A_n] \) for any finite \( F \subseteq B \).

With this we conclude that \( \Delta_B = \Delta_{B'} \) if \( B \) and \( B' \) differ by a finite set, and so \( \Delta_{B_\xi} \cap \Delta_{B'_{\xi'}} = 0 \) if \( \xi \neq \xi' \). But a bounded Radon measure cannot be nonzero on uncountably many pairwise disjoint sets, so there is \( \xi < \omega_1 \) such that \( \mu_k(\Delta_\xi) = 0 \) for each \( k \). This contradicts the choice of \( B_\xi \). \( \square \)

**Proposition 15.** The generic cardinal invariant for the collection of all dense sets of the form \( \text{Ros}^\sigma_{A, \overrightarrow{A}, \overrightarrow{\mu}, \varepsilon} \), where \( A \) is a \( \sigma \)-complete Boolean algebra, \( \overrightarrow{A} \in \text{ac}(A) \), \( \overrightarrow{\mu} \in \mu_\infty(A) \) are equal to \( \text{ros} \).

**Proof.** Let \( F \) be a generic family for the collection of the dense sets as in the proposition. We will show that it is a Rosenthal family. Let \( M = (m_{k,n})_{k,n} \) be a Rosenthal matrix and \( \varepsilon > 0 \). Define measures \( \mu_k(A) = \Sigma_{n \in A} m_{k,n} \) for \( A \subseteq \wp(\mathbb{N}) \). As in the \( \sigma \)-complete Boolean algebra \( \wp(\mathbb{N}) \) the suprema are infinite unions and the above measures are \( \sigma \)-additive we have

\[
\text{Ros}_{M, \varepsilon} = \text{Ros}^\sigma_{\wp(\mathbb{N}), \overrightarrow{A}, \overrightarrow{\mu}, \varepsilon},
\]

where \( \overrightarrow{A} = (\{ n \})_{n \in \mathbb{N}} \) and \( \overrightarrow{\mu} = (\mu_k)_{k \in \mathbb{N}} \) and hence \( F \) is a Rosenthal family.

Now suppose \( F \) is a Rosenthal family. Consider \( F' \subseteq \mathbb{N}^\omega \) such that below each element \( A \in F \) there is in \( F' \) an almost disjoint family of size \( \omega_1 \) of infinite subsets of \( A \). As \( F \) is uncountable, it is easy to construct such \( F' \) of the same uncountable size as \( F \). We will show that \( F' \) meets each dense set as in the proposition.

Let \( A \) be a \( A \) is a \( \sigma \)-complete Boolean algebra, \( \overrightarrow{A} \in \text{ac}(A) \), \( \overrightarrow{\mu} \in \mu_\infty(A) \), \( \varepsilon > 0 \). Using the fact that \( F \) is a Rosenthal family find \( A \in F \cap \text{Ros}^\sigma_{A, \overrightarrow{A}, \overrightarrow{\mu}, \varepsilon} \). By Lemma [14] there is \( A' \in F' \) as in this lemma which implies that \( A' \in F' \cap \text{Ros}^\sigma_{A, \overrightarrow{A}, \overrightarrow{\mu}, \varepsilon} \). \( \square \)
3. Paving lemma for sets fragmenting Rosenthal matrices

The main purpose of this section is to prove Theorem 18 and 19. This is based on a paving lemma which can be concluded from a paving lemma due to Bourgain ([5], [6], see comments below Theorem 18) But we provide our original proof because it is purely combinatorial. We will need the following two lemmas that deal with triangular matrices:

Lemma 16. Let $M = (m_{k,n})_{k,n \in \mathbb{N}}$ be a Rosenthal matrix such that $\|M\|_\infty \leq 1$ and for every $n \in \mathbb{N}$ and every $n \geq k$ we have $m_{k,n} = 0$, then for every positive $l \in \mathbb{N}$ there is a partition $\mathcal{P}_0 = \{ P_i : 1 \leq i \leq l \}$ of $\mathbb{N}$ such that $M$ is $(\frac{1}{l})$-fragmented by $P_i$ for every $1 \leq i \leq l$.

Proof. Fix $M \in \mathbb{M}$ and $l \in \mathbb{N}$ as in the lemma. We will recursively construct a function $f : \mathbb{N} \to \{1, \ldots, l\}$ such that for every $1 \leq i \leq l$ the set $P_i = f^{-1}[i]$, $\frac{1}{l}$-fragments $M$: If $j < j_0$, then let $f(j) = j$. Suppose that $f(j)$ has been constructed for every $j < j_0$, we will construct $f(j_0)$. For $1 \leq i \leq l$ let $P_i^{j_0} = f^{-1}[i]$. Observe that $\{P_i^{j_0} : 1 \leq i \leq l \}$ is a partition of $[1, j_0)$. Using the fact that $M$ has its norm not bigger than 1 and the pigeonhole principle it is possible to pick $1 \leq i \leq l$ such that $\sum_{n \in P_i^{j_0}} m_{j_0,n} \leq \frac{1}{l}$. So we put $f(j_0) = i$. This finishes the construction.

It follows that the partition $\{f^{-1}(i) : 1 \leq i \leq l \}$ is the partition we are looking for.

Lemma 17. Let $M = (m_{k,n})_{k,n \in \mathbb{N}}$ be a Rosenthal matrix such that $\|M\|_\infty \leq 1$ and for every $n \in \mathbb{N}$ and every $n \leq k$ we have $m_{k,n} = 0$, then for every positive $l \in \mathbb{N}$ there is a partition $\mathcal{P}_1 = \{ P_i : 1 \leq i \leq l \}$ of $\mathbb{N}$ such that $M$ is $(\frac{1}{l})$-fragmented by $P_i$ for every $1 \leq i \leq l$.

Proof. Fix $M \in \mathbb{M}$ and $l \in \mathbb{N}$ as in the lemma. Recursively with respect to $|F|$ we will find for each $F \in [\mathbb{N}]^{<\omega}$, a function $f_F : F \to \{1, \ldots, l\}$ such that for every $1 \leq i \leq l$ the set $P_i = f^{-1}[i]$, $\frac{1}{l}$-fragments $M$: Clearly this can be done if $|F| = 1$, so suppose that we already constructed $f_F$ for every $F$ such that $|F| < j$ and let $G \subseteq [\mathbb{N}]^j$. Let $g = \min G$ and let $F = G \setminus \{g\}$. For $1 \leq i \leq l$, let $A_i = f_F^{-1}\{\{i\}\}$. Next, we use the fact that $M$ has its norm not bigger than 1 and the pigeonhole principle to find $1 \leq i \leq l$ such that $\sum_{n \in A_i} m_{g,n} \leq \frac{1}{l}$ and let $f_G = \langle g, i \rangle \cup f_F$. It follows that $f_G$ has the desired properties.

To finish the proof, observe that $\{1, \ldots, l\}^\mathbb{N}$ is a compact metrizable space, so $(f_n)_{n \in \mathbb{N}}$, where $f_n = f_{\{0, \ldots, n\}}$ has a convergent subsequence $(f_{n_k})_{k \in \mathbb{N}}$. It follows that any finite fragment $f|\{0, \ldots, n\}$ of $f$ agrees with some $f_{n_k}$ for some $k \in \mathbb{N}$ on $\{0, \ldots, n\}$, this means that $\{f^{-1}\{i\} : 1 \leq i \leq l \}$ is the partition that we are looking for.

Theorem 18 ([5], [6]). For every $\varepsilon > 0$ there is $l(\varepsilon) \in \mathbb{N}$ such that for every Rosenthal matrix $M$ an there is a partition $\mathcal{P} = \{ P_i : 1 \leq i \leq l(\varepsilon) \}$ of $\mathbb{N}$, such that $M$ is $\|M\|_\infty$-fragmented by $P_i$ for every $1 \leq i \leq l(\varepsilon)$.

Proof. Fix $\varepsilon > 0$ and positive $l \in \mathbb{N}$ such that $2/l < \varepsilon$. Let $M = (m_{k,n})_{k,n \in \mathbb{N}}$. We may assume that it is nonzero. Let $M_0 = (m_{k,n}^0)_{k,n \in \mathbb{N}}$ be the matrix defined by $m_{k,n}^0 = m_{k,n}/\|M\|_\infty$ if and only if $k < n$, otherwise $m_{k,n}^0 = 0$ and let $M_1 = (m_{k,n}^1)_{k,n \in \mathbb{N}}$ be the matrix defined by $m_{k,n}^1 = m_{k,n}/\|M\|_\infty$ if and only if $k > n$,
otherwise \( m_{k,n}^0 = 0 \). Apply Lemma 16 and Lemma 17 for \( M_0 \) and \( M_1 \) respectively to obtain two partitions of \( \mathbb{N} \), namely \( P_0 = \{ P_{i}^0 : 1 \leq i \leq l \} \) and \( P_1 = \{ P_{i}^1 : 1 \leq i \leq l \} \) with the properties stated in those lemmas. Consider the family \( P = \{ P_{i}^0 \cap P_{j}^1 : 1 \leq i \leq l^2 \} \). It is clear that \( M = \| M \|_\infty M_0 + \| M \|_\infty M_1 \) is \((\frac{2}{\varepsilon})\| M \|_\infty\) fragmented by any \( A \in P \).

The sources [5] and [6] contain paving lemmas for operators on \( \ell_1^\infty \). In fact, the number of pieces of the partition there is better than ours. It is also called a matrix-splitting lemma in Section 4.1 of [17]. It is was well known that using a compactness type argument like in the proof of Lemma 17 one can obtain from these versions a paving lemma for infinite dimensional \( \ell_1 \). From this using Lemmas 4 and 5 one can obtain the above paving lemma for Rosenthal matrices. After we proved Theorem 18 and realized that it yields a paving lemma for operators on \( c_0 \) we asked B. Johnson and G. Schechtman if it was already known. We are grateful to P. Komjath for providing us with this reference. In fact the compactness arguments used to pass from the finite to the infinite matrices could be seen as a version of an application of de Bruijn-Erdős theorem which says that the chromatic number of an infinite graph is \( \leq k \) if and only if the chromatic number of every of its finite subgraph is \( \leq k \), where \( k \in \mathbb{N} \) ([8]).

In [20], D. Sobota proved that every selective ultrafilter is a Rosenthal family and asked whether this is the case for ultrafilters in general (Question 3.18). The following is a positive answer to this question.

**Theorem 19.** Every nonprincipal ultrafilter over \( \mathbb{N} \) is a Rosenthal family. In particular, any \( \pi \)-base of any nonprincipal ultrafilter is a Rosenthal family.

**Proof.** Let \( u \) be a nonprincipal ultrafilter on \( \mathbb{N} \). Fix a matrix \( M = (m_{k,n})_{k,n \in \mathbb{N}} \) and \( \varepsilon > 0 \). Apply Theorem 18 for \( M \) and \( \varepsilon/\| M \|_\infty \) obtaining a partition of \( \mathbb{N} \) consisting of sets which \( \varepsilon \)-fragment \( M \). One element of the partition must be a member of the ultrafilter \( u \).

We note that a paving lemma is not true for an arbitrary bounded linear \( T : \ell_\infty \to \ell_\infty \) satisfying \( T(1_{\{n\}})(n) = 0 \) for each \( n \in \mathbb{N} \). Let \( u \) be a nonprincipal ultrafilter over \( \mathbb{N} \). Define \( T(f)(k) = \lim_{n \in u} f(n) \) for each \( k \in \mathbb{N} \), i.e, the range of \( T \) are constant sequences. It is clear that \( \| T \| = 1 \). Since \( u \) is nonprincipal it follows that \( T(1_{\{n\}}) = 0 \) for each \( n \in \mathbb{N} \). Given any partition \( \{ A_1, ..., A_l \} \) of \( \mathbb{N} \) for some \( l \in \mathbb{N} \) there is \( 1 \leq i \leq l \) such that \( A_i \in u \), so \( P_{A_i} T P_{A_i}(\chi_{A_i}) = P_{A_i}(\chi_{\mathbb{N}}) = \chi_{A_i} \), so \( P_{A_i} T P_{A_i} \) has norm one.

4. The Rosenthal number and the reaping number

**Definition 20.** A family \( \mathcal{F} \) of infinite subsets of \( \mathbb{N} \) is called nowhere reaping if for every \( B \subseteq [\mathbb{N}]^{\omega} \) satisfying \( \{ A \cap B : A \in \mathcal{F} \} \subseteq [\mathbb{N}]^{\omega} \), there is \( C_B \subseteq B \) such that \( A \cap B \cap C_B \) and \( A \cap B \setminus C_B \) are both infinite for all \( A \in \mathcal{F} \).

Note that a subfamily of \( [\mathbb{N}]^{\omega} \) of size smaller than \( \mathfrak{r} \) are nowhere reaping.

**Lemma 21.** If \( \mathcal{F} \subseteq [\mathbb{N}]^{\omega} \) is nowhere reaping, then it is not a binary Rosenthal family.
Proof. Let $\mathcal{F} \subseteq [\mathbb{N}]^\omega$ be nowhere reaping. We will construct $f : \mathbb{N} \to \mathbb{N}$ with no fixed points such that $f[A] = \mathbb{N}$ for every $A \in \mathcal{F}$. This will show that $\mathcal{F}$ is not a binary Rosenthal family.

First, by recursion in $n \in \mathbb{N}$, we construct a pairwise disjoint family $\{B_n : n \in \mathbb{N}\}$ of infinite subsets of $\mathbb{N}$ such that $B_n \cap A$ and $A \setminus (\bigcup_{i \leq n} B_i)$ are infinite for each $A \in \mathcal{F}$ and for every $n \in \mathbb{N}$. The existence of $B_n$ follows from the fact that $\mathcal{F}$ is not reaping. The inductive hypothesis and the fact that $\mathcal{F}$ is nowhere reaping applied below $\mathbb{N} \setminus \bigcup_{i \leq n} B_i$ produces the next set $B_{n+1}$ in the inductive step of the construction of $\{B_n : n \in \mathbb{N}\}$.

This induces an entire function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(k) = n$ if $n \neq k \in B_n$ and $f(k) = k + 1$ if $k$ is not in any $B_n$ or if $k \in B_k$. Clearly $f$ has no fixed points and $f^{-1}\{n\} \supseteq (B_n \setminus \{n\}) \cap A \neq \emptyset$ for every $A \in \mathcal{F}$. It follows that $f[A] = \mathbb{N}$ for every $A \in \mathcal{F}$ as required.

Corollary 22. If $\mathcal{F}$ is a filter and a Rosenthal family, then there is an infinite $A \subseteq \mathbb{N}$ such that $\{F \cap A : F \in \mathcal{F}\} \cup \{\mathbb{N} \setminus A\}$ generates an ultrafilter.

Proof. By Lemma 21 there is infinite $B \subseteq \mathbb{N}$ such that $\{F \cap B : F \in \mathcal{F}\} \subseteq [\mathbb{N}]^\omega$ and there is no infinite $C_B \subseteq B$ which splits $\{F \cap B : F \in \mathcal{F}\}$. So for any $C \subseteq B$ either $C \in \mathcal{F}$ or $B \setminus C \in \mathcal{F}$, so the corollary follows.

Recall that a family $\mathcal{F} \subseteq [\mathbb{N}]^\omega$ has the strong finite intersection property if the intersection of every finite subfamily of $\mathcal{F}$ is infinite.

Corollary 23. No (binary) Rosenthal family of cardinality smaller than $u$ has the strong finite intersection property.

Proof. Suppose that $\mathcal{F}$ of cardinality smaller than $u$ has the strong finite intersection property. Let $\mathcal{F}'$ be the filter generated by $\mathcal{F}$. By Corollary 22 we would obtain a nonprincipal ultrafilter generated by less than $u$ elements.

Proposition 24. $r \leq \tau 01$.

Proof. Suppose that $\mathcal{F}$ has cardinality smaller then $r$. Then it is nowhere reaping and so by Lemma 21 the family $\mathcal{F}$ is not a binary Rosenthal family.

Proposition 25. $\tau 01 \leq r$.

Proof. Let $\mathcal{F} \subseteq [\mathbb{N}]^\omega$, of size $r$, be such a reaping family that for every $A \in \mathcal{F}$, the set $\mathcal{F} \cap \mathcal{P}(A)$ is a reaping family of size $r$ and $\mathcal{F}$ is closed under finite modifications. Such family can be easily constructed (see 3.7 of [10] where such families are called hereditarily reaping).

Let $M \in \mathbb{M}$ and $\varepsilon > 0$. We will see that there is an $F \in \mathcal{F}$ such that $M$ is $\varepsilon$-fragmented by $F$. Let $\{A_1, \ldots, A_l\}$ be a partition of $\mathbb{N}$ such that each piece $\varepsilon$-fragments $M$ which exists by Theorem 18. Using the fact that $\mathcal{F}$ is reaping and closed under finite modifications find $F_1 \in \mathcal{F}$ such that either $F_1$ is disjoint with $A_1$ or $F_1$ is contained in $A_1$. If $F_1 \subseteq A_1$ then $F = F_1$ works. Otherwise it is possible to pick an infinite $F_2 \subseteq F_1$ in $\mathcal{F}$ such that either $F_2 \subseteq A_2$ or $F_2$ is disjoint from $A_1 \cup A_2$. If we follow this process, it is evident that we will eventually find $F_i \in \mathcal{F}$ for $1 \leq i \leq l$ such that $F_i \subseteq A_i$. Then $F = F_i$ is as required. \qed
An alternative proof of Proposition 25 is to use a theorem of Balcar and Simon from [2] which says that the reaping number $τ$ is the minimal size of a $π$-base of a nonprincipal ultrafilter on $\mathbb{N}$. Such a $π$-base is a Rosenthal family by the second part of Theorem 19.

**Theorem 26.** The Rosenthal number $r_0σ$ is equal to the reaping number $τ$.

**Proof.** By Propositions 11 (1), 25 and 24 we have $τ \leq r_0σ ≤ τ$. □

The argument in the above proof also gives:

**Theorem 27.** $r_0σ = τ$.

**Theorem 28.** $r_0σ(ℓ_1) = τ$.

**Proof.** Use Theorem 26 and Proposition 9 (1). □

5. $c_0$-Rosenthal numbers

In this section we calculate the $c_0$-Rosenthal number $r_0σ(c_0)$ and the binary $c_0$-Rosenthal number $r_0σ_1(c_0)$ - see the introduction for the definitions. First let us recall some terminology. If $f, g ∈ 2^{\mathbb{N}}$, then $g$ eventually dominates $f$, denoted by $f \leq^* g$, if there is a $n ∈ \mathbb{N}$ such that for every $k > n$, $f(k) ≥ g(k)$. $D ⊆ 2^{\mathbb{N}}$ is a dominating family if every $f ∈ 2^{\mathbb{N}}$ is dominated by some member of $D$.

The dominating number $δ$ is the smallest size of a dominating family. Following Definition 2.9 of [3] an interval partition is a partition of $\mathbb{N}$ into (infinitely many) finite intervals $I_n$ where $n ∈ \mathbb{N}$. We will assume that the intervals are numbered in the natural order, so that, if $i_n$ is the left endpoint of $I_n$ then $i_0 = 0$ and $I_n = [i_n, i_{n+1})$. We say that the interval partition $\{I_n : n ∈ \mathbb{N}\}$ dominates another interval partition $\{J_n : n ∈ \mathbb{N}\}$ if for all but finitely many $n ∈ \mathbb{N}$ there is $k ∈ \mathbb{N}$ such that $J_k ⊆ I_n$. By Theorem 2.10 of [3] the dominating number $δ$ is equal to the smallest cardinality of a family of interval partitions dominating all interval partitions. By a $c_0$-matrix we will mean a matrix of a linear operator on $c_0$ in the sense of Lemma 6.

**Proposition 29.** $r_0σ(c_0) ≤ δ$.

**Proof.** Note that there is a family $A = \{A_α : α < δ\}$ of infinite subsets of $\mathbb{N}$ such that for every function $f : \mathbb{N} → \mathbb{N}$ there is $α < δ$ such that $f(k) < n$ whenever $k < n$ are two elements of $A_α$. To prove this assume that $f$ is strictly increasing and $f(0) = 1$ and consider the interval partition $I = \{[f^{2i}(0), f^{2i+2}(0)) : i ∈ \mathbb{N}\}$.

If $J$ is an interval partition that dominates $I$, then for almost all endpoints $k < n$ of intervals in $J$ there is $i ∈ \mathbb{N}$ such that $k ≤ f^{2i}(0) < f^{2i+2}(0) ≤ n$. In particular $f(k) ≤ f^{2i+1}(0) < f^{2i+2}(0) ≤ n$, so if we take as a set $A$ the endpoints of the intervals of $J$ minus some finite set, we obtain that $f(k) < n$ whenever $k < n$ are two elements of $A$. So, as $A = \{A_α : α < δ\}$ we take the family of all finite modifications of the sets of all the endpoints of partitions from a family of interval partitions of cardinality $δ$ which is dominating and which exists by the discussion at the beginning of this Section.

We will see that for each $ε > 0$ each $c_0$-matrix $M = (m_{k,n})_{k,n∈\mathbb{N}}$ is $ε$-fragmented by an element of $A$. Let $\|M\|_∞ = ρ$. Find a function $f_M : \mathbb{N} → \mathbb{N}$ such that for every $n ∈ \mathbb{N} \setminus \{0\}$ we have

$$m_{k,n} ≤ \frac{ε}{2n+2}$$
for all \( k \geq f_M(n) \). Its existence follows from the fact that \( M \) is a \( c_0 \)-matrix, and so \((m_{k,n})_{k \in \mathbb{N}}\) converges to 0 for each \( n \in \mathbb{N} \). Now find a function \( g_M : \mathbb{N} \to \mathbb{N} \) such that for any \( k \in \mathbb{N} \),

\[
\sum_{n \geq g_M(k)} m_{k,n} \leq \frac{\varepsilon}{2}.
\]

Its existence follows from the fact that \( \sum_{n \in \mathbb{N}} m_{k,n} \leq \rho \) for every \( k \in \mathbb{N} \). Now find \( \alpha < d \) such that for any \( i < j \) in \( A_\alpha \) we have \( \max(f_M(i), g_M(i)) < j \).

We claim that \( M \) is \( \varepsilon \)-fragmented by \( A_\alpha \). Let \( k \in A_\alpha \). If \( n \in A_\alpha \) and \( n < k \), then \( f_M(n) < k \), so \( m_{k,n} \leq \frac{\varepsilon}{2m} \) by (1) and therefore \( \sum_{n \in A_\alpha} m_{k,n} \leq \frac{\varepsilon}{2} \). On the other hand, if \( k < n \), then \( g_M(k) < n \) and therefore \( \sum_{n \in A_\alpha \cap k} m_{k,n} \leq \frac{\varepsilon}{2} \) by (2).

Therefore

\[
\sum_{n \in A_\alpha \setminus \{k\}} m_{k,n} \leq \sum_{n \in A_\alpha \cap k} m_{k,n} + \sum_{n \in A_\alpha \cap k} m_{k,n} \leq \varepsilon
\]

which completes the proof. \( \square \)

**Proposition 30.** \( \min\{d, r\} \leq \tau_{\mathcal{S}}(c_0) \).

**Proof.** Let \( \kappa < \min\{d, r\} \) and let \( \mathcal{A} = \{A_\alpha : \alpha \in \kappa\} \) be a family of infinite subsets of \( \mathbb{N} \) closed under finite modifications. We will construct a finite-to-one \( f : \mathbb{N} \to \mathbb{N} \) such that \( f[A_\alpha] \cap A_\alpha \neq \emptyset \) for any \( \alpha < \kappa \).

First, by Theorem 24 because \( \kappa < r = \tau_{\mathcal{S}} \), there is \( g : \mathbb{N} \to \mathbb{N} \) with no fixed points such that \( g[A_\alpha] \cap A_\alpha \neq \emptyset \) for every \( \alpha < \kappa \). For every \( \alpha < \kappa \), recursively construct an increasing function \( f_\alpha \) such that for every \( n \in \mathbb{N} \) we have

\[
g[[f_\alpha(n), f_\alpha(n + 1)] \cap A_\alpha] \cap ([f_\alpha(n), f_\alpha(n + 1)] \cap A_\alpha) \neq \emptyset.
\]

This can be done as \( g[A_\alpha \setminus [0, f_\alpha(n))] \cap (A_\alpha \setminus [0, f_\alpha(n))) \neq \emptyset \) since \( \mathcal{A} \) is closed under finite modifications.

Secondly because \( \kappa < d \) by the discussion at the beginning of this section there is an interval partition \( \mathcal{I} = \{I_n : n \in \mathbb{N}\} \) which dominates each interval partition \( \mathcal{J}^\alpha = \{[f_\alpha(n), f_\alpha(n + 1)) : n \in \mathbb{N}\} \) for \( \alpha < \kappa \). Define \( f : \mathbb{N} \to \mathbb{N} \) as follows:

\[
f(i) = \begin{cases} 
g(i) & \text{if } i, g(i) \text{ are in the same piece of } \mathcal{I}, \\
1 + i & \text{otherwise.}
\end{cases}
\]

Clearly \( f \) is finite-to-one and with no fixed points. To finish the proof note that if \( \alpha < \kappa \), then there is an \( n \in \mathbb{N} \) such that \( C = [f_\alpha(n), f_\alpha(n + 1)) \cap A_\alpha \) is included in a single piece of \( \mathcal{I} \), so if \( i, j \in C \), then \( f(i) = g(i) \) and by (3) we have \( f[A_\alpha] \cap A_\alpha \neq \emptyset \). \( \square \)

**Theorem 31.** \( \tau_{\mathcal{S}}(c_0) = \min\{d, r\} \).

**Proof.** We have \( \tau_{\mathcal{S}}(c_0) \leq \tau_{\mathcal{S}} = r \) by Proposition 20 and by Theorem 26. So Proposition 28 implies \( \tau_{\mathcal{S}}(c_0) \leq \min\{d, r\} \). The other inequality follows from Proposition 31 and Proposition 30. \( \square \)

**Theorem 32.** \( \tau_{\mathcal{S}}(c_0) = \min\{d, r\} \).

**Proof.** Use Proposition 29, Theorem 31 and Proposition 30. \( \square \)

We should add here that \( \min\{d, r\} \) is investigated in [1] where for example it is proved that \( \min\{d, r\} = \min\{d, u\} \).
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