Constraints on GUT 7-brane Topology in F-theory

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Abstract

We study the relation between phenomenological requirements and the topology of the surfaces that GUT 7-branes wrap in F-theory compactifications. In addition to the exotic matter free condition in the hypercharge flux scenario of $SU(5)_{\text{GUT}}$ breaking, we analyze a new condition that comes from a discrete symmetry aligning the contributions to low-energy Yukawa matrices from a number of codimension-three singularity points. We see that the exotic matter free condition excludes Hirzebruch surfaces (except $F_0$) as the GUT surface, correcting an existing proof in the literature. We further find that the discrete symmetry for the alignment of the Yukawa matrices excludes del Pezzo surfaces and a rational elliptic surface as the GUT surface. Therefore, some GUT 7-brane surfaces are good for some phenomenological requirements, but sometimes not for others, and this aspect should be kept in mind in geometry search in F-theory compactifications.
1 Introduction and Summary

In recent years, F-theory compactifications have been studied for the construction of the realistic models \[1, 2, 3\]. A four dimensional low energy effective field theory realized as a non-Abelian gauge theory on 7-branes wrapping on a surface can be a candidate for a supersymmetric Grand Unified Theory (GUT). Since some of important properties of low-energy effective theory in the visible sector are determined only by geometry of local neighbourhood along the GUT surface in Type IIB / F-theory compactifications \[4\], correspondence between geometry around the GUT complex surface and low-energy physics can be established to some extent without referring to global geometry. Some class of complex surfaces with certain topology may even be ruled out \[5\], when various phenomenological requirements are imposed. Reference \[5\] argued, for example, that the condition of no exotic matter fields in a hypercharge flux scenario of \(SU(5)\)GUT breaking can exclude Hirzebruch surfaces for the GUT 7-brane. References \[6, 7, 8\] also made it clear that the low-energy Yukawa matrices receive independent contributions from enhanced singularity points whose numbers are determined by intersection numbers on the GUT complex surface. In this article, we further pursue this program, and report some constraints on the topology of GUT complex surfaces following from phenomenological conditions.

In section 2.1 we take a moment to have a brief look at the exotic matter free condition in the hypercharge flux scenario of \(SU(5)\)GUT breaking. Requiring that chiral multiplets in the off-diagonal components of the \(5 \times 5\) matrix of \(SU(5)\)GUT be absent below the Kaluza–Klein scale, Ref. \[5\] already concluded that all the Hirzebruch surfaces \(\mathbb{F}_n\) (except \(\mathbb{F}_0\)) are ruled out in this scenario. In fact, there is an error in the calculation. We see that the proof can still be fixed by fully exploiting the exotic matter free condition.

In section 2.2 we move on to study consequences of a new topological condition motivated by phenomenology. Because the number of \(E_6\) enhancement points is always even in \(SU(5)\) models in F-theory \[7, 8\], the low-energy up-type Yukawa matrix consists of contributions from two or more \(E_6\) type points, and the approximately rank-1 structure of individual contributions \[9, 10\] is lost generically. One of the solutions to this problem is to impose a discrete symmetry \(\Gamma\) \[7\], so that the contributions from multiple enhanced singularity points are aligned and the approximate rank-1 structure is maintained even after all these contributions are summed up. This discrete symmetry solution sets constraints on the number of enhanced singularity points. We find that both del Pezzo surfaces and a rational elliptic surface are excluded by the constraints when \(\Gamma = \mathbb{Z}_2\) (which can be used as \(R\)-parity simultaneously). We also argue that a less general condition for the number of codimension-three singularity points can eliminate
del Pezzo surfaces and a rational elliptic surface for all the discrete symmetry $\Gamma$.

To summarize, the $SU(5)_{\text{GUT}}$ breaking by the hypercharge flux excludes Hirzebruch surfaces (except $F_0$), and the discrete symmetry solution for the alignment problem of the Yukawa matrices excludes del Pezzo surfaces and a rational elliptic surface. In other words, if one chooses Hirzebruch surfaces as the $SU(5)_{\text{GUT}}$ GUT surface, one cannot use the hypercharge flux scenario for $SU(5)_{\text{GUT}}$ breaking, and if one chooses del Pezzo surfaces or a rational elliptic surface as the $SU(5)_{\text{GUT}}$ GUT surface, one cannot use a discrete symmetry solution for the alignment problem of the low-energy Yukawa matrices. Note that those results hold also when one embeds the GUT surface in a global Calabi–Yau fourfold. The global embedding of the $SU(5)_{\text{GUT}}$ GUT surface has been recently discussed in [6, 11, 12], and the geometry search has been performed [13, 14]. Our results can constrain the possible choices for the GUT surface in the geometry search if one relies on either one of [or on both of] the two scenarios.

2 No go theorems for $SU(5)$ GUT models

F-theory compactified on an elliptically fibered Calabi–Yau fourfold realizes a four-dimensional $\mathcal{N} = 1$ supersymmetric theory as a low energy effective field theory. When the elliptic fiber degenerates and generates a singularity of a type $G$ at a codimension one locus $S$ in the base manifold $B_3$, a number of 7-branes wrap on the divisor $S$ and realizes a non-Abelian gauge theory with the gauge group $G$ [15, 16]. Since we are interested in constructing phenomenological models from F-theory compactifications, we focus on $SU(5)_{\text{GUT}}$ for the gauge group $G$. We will argue that phenomenological requirements indeed restrict the possible candidates for the GUT divisor $S$.

2.1 Exotic matter free conditions in $SU(5)_{\text{GUT}}$ breaking

The $SU(5)_{\text{GUT}}$ gauge group should be broken to obtain the Standard Model gauge group $SU(3) \times SU(2) \times U(1)_Y$. As for the breaking scenario\footnote{Two other $SU(5)$ breaking scenarios were also referred to in [5]. One is to use a Wilson line on a GUT divisor $S$ that is not simply connected, and the other is four-dimensional GUT breaking scenario involving $SU(5)_{\text{GUT-adj}}$ chiral multiplets in the effective theory.} we turn on a hypercharge flux $L_Y \equiv (L_Y)^{-5/6} \in H^2(S, \mathbb{Z})$ \cite{17, 18} where the charge against $L_Y$ gives the hypercharge for matter fields. The flux $L_Y$ should satisfy the BPS condition \cite{19, 20, 21}

$$\omega \wedge c_1(L_Y) = 0,$$

where $\omega$ is a Kähler form on the GUT divisor $S$. The hypercharge flux $L_Y$ yields chiral multiplets in the off-diagonal blocks of the $5 \times 5$ matrix of $SU(5)_{\text{GUT}}$. They are
in the representation of \((3, 2)_{-5/6}\) and \((\bar{3}, 2)_{+5/6}\) under \(SU(3) \times SU(2) \times U(1)_Y\). Since the presence of those matter fields under the GUT scale breaks the \(SU(5)\) gauge coupling unification, we require that there are no zero modes for the matter fields in the representation of \((3, 2)_{-5/6}\) and \((\bar{3}, 2)_{+5/6}\). Then, the exotic matter free conditions are [19, 20, 21]

\[ h^0(S, \mathcal{L}^{\pm 1}_Y) = 0, \quad h^i(S, \mathcal{L}^{\pm 1}_Y) = 0, \quad \ker d_2 = 0, \quad \coker d_2 = 0, \quad (2.2) \]

where \(\ker d_2\) and \(\coker d_2\) are understood as their dimension. Here, \(d_2\) is a map

\[ d_2 : H^0(S, \mathcal{L}_Y \otimes \mathcal{O}(K_S)) \to H^2(S, \mathcal{L}_Y). \quad (2.3) \]

The case where the map \(d_2\) is non-trivial has been discussed in [22] in the context of Type IIB string theory.

When the anti-canonical divisor \(-K_S\) is an effective divisor, \(h^0(S, \mathcal{O}(D)) = 0\) for a divisor \(D\) always implies \(h^0(S, \mathcal{O}(D + K_S)) = h^2(S, \mathcal{O}(-D)) = 0\) [20]. Hence, the exotic free conditions \((2.2)-(2.3)\) are equivalent to

\[ h^i(S, \mathcal{L}^{\pm 1}_Y) = 0 \quad \text{for} \ i = 0, \cdots, 2, \quad (2.5) \]

if \(-K_S \geq 0\). This includes the cases of Hirzebruch surfaces, del Pezzo surfaces and rational elliptic surfaces. In such situations, it follows from \((2.5)\) that

\[ \chi(S, \mathcal{L}_Y) = 0, \quad \chi(S, \mathcal{L}_Y^{-1}) = 0. \quad (2.6) \]

In fact, Ref. [5] has used the conditions \((2.6)\) to set constraints on Hirzebruch surfaces \(\mathbb{F}_n\). In the following combinations, Ref [5] has obtained

\[ 0 = \chi(\mathbb{F}_n, \mathcal{L}_Y) - \chi(\mathbb{F}_n, \mathcal{L}_Y^{-1}) = c_1(\mathbb{F}_n) \cdot c_1(\mathcal{L}_Y) = b(n + 2) + 2a - 2bn^2, \quad (2.7) \]

\[ 0 = \frac{1}{2}(\chi(\mathbb{F}_n, \mathcal{L}_Y) + \chi(\mathbb{F}_n, \mathcal{L}_Y^{-1})) = 1 + \frac{1}{2}c_1(\mathcal{L}_Y)^2 = 1 + \frac{1}{2}(2ab - b^2n^2). \quad (2.8) \]

Here, \(\mathcal{L}_Y\) is chosen as

\[ \mathcal{L}_Y = \mathcal{O}(af + b\sigma), \quad (2.9) \]

where \(a\) and \(b\) are integers, and \(f\) and \(\sigma\) are generators of the effective curves in \(\mathbb{F}_n\) which satisfy the intersection

\[ f \cdot f = 0, \quad f \cdot \sigma = 1, \quad \sigma \cdot \sigma = -n. \quad (2.10) \]

Under the results of the calculation \((2.7)\) and \((2.8)\), only Hirzebruch surfaces \(\mathbb{F}_n\) with \(n = 0, 1\) are allowed. All the other Hirzebruch surfaces, \(n \geq 2\), were excluded for this reason.
However, the correct results for the computation are

\[ 0 = \chi(\mathbb{F}_n, \mathcal{L}_Y) - \chi(\mathbb{F}_n, \mathcal{L}_Y^{-1}) = c_1(\mathbb{F}_n) \cdot c_1(\mathcal{L}_Y) = 2a + 2b - bn, \quad (2.11) \]

\[ 0 = \frac{1}{2}(\chi(\mathbb{F}_n, \mathcal{L}_Y) + \chi(\mathbb{F}_n, \mathcal{L}_Y^{-1})) = 1 + \frac{1}{2}c_1(\mathcal{L}_Y)^2 = 1 + \frac{1}{2}(2ab - b^2n), \quad (2.12) \]

from the intersection numbers \((2.10)\). Then, the only constraints that we can derive from \((2.11)\) and \((2.12)\) are \(n = \text{even}, \ b = \pm 1\) and \(a = b(n - 2)/2\). Therefore, the conditions \((2.11)\) and \((2.12)\) are not enough to set a bound on the “\(n\)” of Hirzebruch surfaces \(\mathbb{F}_n\).

However, we can indeed exclude all the Hirzebruch surfaces except for \(\mathbb{F}_0\) by exploiting all of \((2.5)\), not just \((2.6)\). It will be straightforward to see that\(^2\)

\[ h^0(S, \mathcal{L}_Y) > 0 \quad (2.13) \]

for \(b = 1\) and \(a = (n - 2)/2\) for \(S = \mathbb{F}_n\) with an even \(n \geq 2\). After using all of the exotic matter free condition \((2.5)\),

\[ \mathbb{F}_0 \text{ with } \mathcal{L}_Y = f - \sigma, \text{ or } -f + \sigma \quad (2.14) \]

is the only solution in the \(S = \mathbb{F}_n\) series in the hypercharge flux scenario of \(SU(5)_{\text{GUT}}\) breaking.

### 2.2 Discrete symmetry for the alignment of Yukawa matrices

As we have already explained in Introduction, the desirable flavor structures can also set constraints on the GUT divisor in a certain scenario. If one could achieve a configuration such that there are only one \(E_6\) and one \(D_6\) singularity points, the up-type and the down-type Yukawa couplings have an approximate rank one structure \([9, 10]\). In generic F-theory compactifications, however, one can show that the number of \(E_6\) singularity points are always even and the approximate rank one structure is generically broken \([7, 8]\). Hence, one has to set some constraints which realize a situation where only one Yukawa point dominantly contributes to a Yukawa coupling in a low energy effective field theory.

One way to achieve the desired configuration is to impose a discrete symmetry globally \([7]\). If the discrete symmetry relates all the codimension-three singularity points of the

\(^2\) One will also arrive at the same conclusion by using the BPS condition \((2.1)\) with a Kähler form strictly in the interior of the Kähler cone, instead of using the condition \(h^0(S, \mathcal{L}_Y^{-1}) = 0\). If the Kähler form is on the boundary of the cone, at least one has to be careful in discussing the validity of the Katz–Vafa style effective field theory description on 7+1 dimensions \([23, 19, 20]\).

\(^3\) There are two other solutions to recover the approximate rank one Yukawa matrices. One is to factorize matter curves and make completely independent wavefunctions localize along the separated
same type, all the Yukawa matrices generated from the codimension-three singularities are aligned and have the same structures. Hence, one can recover the approximate rank one structure in that situation.

Introducing a discrete symmetry has another phenomenological motivation. Imposing $\mathbb{Z}_2$ symmetry has been considered in [25, 7] for a solution for the prohibition of the renormalizable R-parity violating operators. In order to forbid the dimension-5 proton decay operators also, the $\mathbb{Z}_2$ symmetry is not enough and one has to extend it to a larger discrete symmetry (e.g., [29]). Therefore, imposing a discrete symmetry may lead to realize the desirable flavor structure and also forbid the dangerous proton decay operators simultaneously.

Then, let us see how the discrete symmetry for the desirable flavor structure sets topological constraints on the GUT divisor $S$ of $SU(5)$ GUT models. Let the symmetry group be $\Gamma$. Imagine a case where all the elements of the symmetry group $\Gamma$ acts on the global geometry non-trivially so that they act on the $E_6$ singularity points [and $D_6$ type points] faithfully and transitively. It then follows that

$$\#E_6 = |\Gamma|, \quad \#D_6 = |\Gamma|. \quad (2.15)$$

More generally, there could be a normal subgroup $H_{E_6}$ or $H_{D_6}$ of $\Gamma$ which acts non-trivially on the neighborhood of the $E_6$ or $D_6$ singularity points but acts trivially on the points respectively. In that case, the requirement (2.15) is relaxed to

$$\#E_6 = \frac{|\Gamma|}{|H_{E_6}|}, \quad \#D_6 = \frac{|\Gamma|}{|H_{D_6}|}. \quad (2.16)$$

### 2.2.1 Del Pezzo surfaces

First, we apply the constraint (2.15) to del Pezzo surfaces, $dP_n$ and $\mathbb{P}^1 \times \mathbb{P}^1 (= F_0)$, assuming that $H_{E_6} = H_{D_6} = \phi$. The numbers of the $E_6$ and $D_6$ singularity enhancement points on the GUT divisor $S$ are [30, 21]

$$\#E_6 = (5K_S + \eta) \cdot (4K_S + \eta), \quad \#D_6 = (5K_S + \eta) \cdot (3K_S + \eta), \quad (2.17)$$

matter curves [6, 7, 8]. Then, one could realize a situation such that only single $E_6$ codimension-three singularity point contributes to the up-type Yukawa coupling. In this case, the matter wavefunctions need to be separated at the level of factorization of spectral surfaces, not at the level of matter curves [24, 25]. In models without an $SO(10)$ or $SU(6)$ GUT divisors, this factorization condition needs to be lifted to a constraint on the global 4-fold geometry [25, 26, 27, 28]. The other is to tune the complex structure moduli of the matter curve for $10$ representation and make the wavefunctions of $10$ matter fields localize along the matter curve [7]. Then, one codimension-three singularity point makes a dominant contribution to the Yukawa matrix in the four-dimensional effective theory, and recovers the approximate rank one structure of the Yukawa matrices in the Standard Model.
where $\eta$ is a divisor in $S$ which satisfies $c_1(N_{S|B_3}) = 6K_S + \eta$. Then, we have
\begin{equation}
0 = \#D_6 - \#E_6 = (5K_S + \eta) \cdot (-K_S),
\end{equation}
because of (2.15). Since $[5K_S + \eta]$ is a class of the matter curve for the 10 representation, $5K_S + \eta$ is an effective divisor. In the case of del Pezzo surfaces, then, one can see that Eq. (2.18) contradicts the ampleness of $-K_{dP}$. This follows from Nakai’s ampleness criterion, which states that the necessary and sufficient conditions for an ample divisor $D$ on a non-singular projective surface $S$ are
\begin{equation}
D \cdot D > 0, \quad C \cdot D > 0
\end{equation}
for any curve $C$ in $S$. Therefore, any del Pezzo surfaces do not satisfy the constraints (2.15), (2.18).

The constraint (2.15) is in fact stronger than the general constraint (2.16). However, in the case of $\Gamma = \mathbb{Z}_2$, one can show that any del Pezzo surfaces cannot satisfy the general constraint (2.16). Note that the number of the $D_6$ singularity points is always larger than the number of the $E_6$ singularity points, since
\begin{equation}
\#D_6 - \#E_6 = (5K_{dP} + \eta) \cdot (-K_{dP}) > 0.
\end{equation}
Here, we use the fact that $5K_{dP} + \eta$ is an effective divisor and $-K_{dP}$ is an ample divisor. Therefore, the constraint (2.16) is satisfied if and only if $\#D_6 = 2$ and $\#E_6 = 1$. However, the number of the $E_6$ singularity points is always even [7, 8]. Then, the only possibility of $\#D_6 = 2$ and $\#E_6 = 1$ cannot be realized. Namely, it is impossible to satisfy the constraint (2.16) when $\Gamma = \mathbb{Z}_2$.

2.2.2 Rational elliptic surface

We next apply the constraint (2.15) to a rational elliptic surface. Since the anti-canonical divisor of a rational elliptic surface is not ample, one cannot use the discussion of del Pezzo surfaces for the constraints from the discrete symmetry solution in the case of a rational elliptic surface. The condition (2.18) in the case of a rational elliptic surface implies that the matter curve of the class $5K_S + \eta$ is parallel to the elliptic fiber (i). In the case of special complex structure moduli of a rational elliptic surface, the elliptic fiber can degenerate into smaller irreducible pieces at some special points on the base $\mathbb{P}^1$. In such a case, the matter curve can be one of the components (ii). In these cases, the
number of $E_6$ singularity points are
\[
\#E_6 = (5K_S + \eta) \cdot (-2K_S) + (5K_S + \eta) \cdot (6K_S + \eta),
\]
\[
= \deg K_{\bar{c}_{10}} = 2g - 2,
\]
\[
= 0 \quad \text{for (i),} \tag{2.21}
\]
\[
or -2 \quad \text{for (ii),} \tag{2.22}
\]
where $\bar{c}_{10}$ is a curve in the class $|5K_S + \eta|$. Hence, neither case realizes the configuration $\#E_6 > 0$. The negative result for case (ii) means that the condition $\eqref{2.15}$ will never be satisfied. In case (i), the condition $\eqref{2.15}$ means that there is no $E_6$ type singularity point in the GUT complex surface $S$, and the up-type Yukawa couplings need to be generated through D-brane instantons [31, 32].

The general constraint $\eqref{2.16}$ cannot be satisfied in the case of $\Gamma = \mathbb{Z}_2$ for a rational elliptic surface, either. Given the fact that $\#E_6$ is even, while $\#E_6 = \#D_6$ cannot be realized, as we have seen above, the only possibility under $\Gamma = \mathbb{Z}_2$ is $\#E_6 = 2$ and $\#D_6 = 1$. This implies that
\[
\#D_6 - \#E_6 = (5K_S + \eta) \cdot (-K_S) = -1. \tag{2.23}
\]
Since $5K_S + \eta$ is an effective divisor, the curve class can be written as $af + \sum_i b_i C_i$ where $a$ and $b_i$ are positive integer. $f$ is the elliptic fiber class and $C_i$’s are curves which satisfy $C_i^2 = -1, C_i \cdot f = 1$. Those are the generators of the cone for the effective curves. Due to the intersection number, $(af + \sum_i b_i C_i) \cdot (-K_S) = \sum_i b_i > 0$, it is impossible to satisfy $\eqref{2.23}$.

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