Partial Derivation
of
Transformation Properties of Quarks and Leptons

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Abstract

Under the assumptions that $SU(3)_c \times U(1)_Y \times G'$ with $G'$ simple is a local symmetry group at high energies, that color is parity-conserving, and the $Y$-charges are irreducible, we show that anomaly constraints imply the minimal set of fermions is fifteen in number. Given this minimal set, we further show that $G'$ must be $SU(2)$ and the unbroken gauge symmetry is either color or the product of color with electric charge.
In discussing the standard model[1], one usually postulates the known set of fermions \((u^\alpha, d^\alpha, e, \nu)\) for each generation (where the superscript \(\alpha = 1, 2, 3\) denotes the color index), then one assumes \(SU(3)_c \times SU(2)_L \times U(1)_Y\) to be the gauge group and assigns the gauge quantum numbers for the fermions to match their observed properties. In view of the extraordinary success of this model, it is interesting to ask whether it is possible to reproduce it starting from a more economical set of assumptions. Such results will not only shed light on the fundamental structure of the standard model but may also be useful in pointing the way towards new physics. The present Letter is an attempt in that direction.

Our basic tool will be the freedom from Adler-Bell-Jackiw anomalies[2] required for the consistent renormalization and unitarity of a gauge field theory. Such considerations have already been successfully used in the past[3] to show that there is no need to pre-assign the \(Y\)-charges of the fermions of the one-generation standard model but rather they can be determined by the constraints of anomaly freedom once the fermion content and the gauge group are given, if QED is assumed to be vector-like. Our goal here is to go further using the same anomaly constraints. Our starting assumptions are that the local symmetry of electroweak and strong interactions at high energies is given by \(SU(3)_c \times U(1)_Y \times G'\), that \(SU(3)_c\) is vector-like and that the set of \(Y\)-charges of the fermions is irreducible. The first two assumptions are self-explanatory but the third needs to be explained. By \textit{irreducible} set of \(Y\)-charges we mean that no fermion has \(Y = 0\), no pair of fermions have equal and opposite \(Y\)-charges, and more generally \textit{no subset of the fermions}
separately satisfies all of the anomaly constraints. One motivation behind this assumption is that if a pair of fermions have equal and opposite Y-charges and are vector-like or neutral under color and $G'$, then one can form a gauge invariant mass term for them and \textit{a priori} this mass has no reason to be below the electroweak scale implying that these fermions will decouple from the low energy spectrum of the theory.

We find that the above assumptions combined with the requirement that the gauge group be anomaly-free leads to the following conclusions:

(i) The minimal number of fermions that leads to an anomaly-free theory is 15, which is precisely the number of fermions in the one-generation standard model;

(ii) The maximal allowed simple $G'$ is $SU(2)$ which must be parity-violating;

(iii) The unbroken gauge symmetry has to be \textit{either} color alone \textit{or} the product of color with electric charge.

We believe that the set of assumptions we have made are more economical than those made in the usual construction of the standard model and we are able to reproduce three key ingredients of the one-generation standard model i.e. the number of fermions, the weak gauge group and the correct quantum numbers of the fermions.

In deriving these results we make use of the fact that charged chiral fermions can acquire mass only if vector-like with respect to the corresponding unbroken gauge symmetries. Gauge symmetries which are parity-violating must generally be broken to avoid exact masslessness of charged
chiral fermions. This is why the electroweak gauge symmetry $SU(2) \times U(1)$ must be spontaneously broken either completely or to leave only the $U(1)$ of electric charge.

In order to prove our assertion, let us say that the number of fermions is $N$ and is divided into two groups called quarks and leptons, the quarks being defined as triplets or antitriplets under color and leptons being singlets. The assumed vector nature of QCD requires the number of triplets and antitriplets to be the same; let this number be equal to $Q/2$ where $Q$ is hence an even number. Denoting the number of leptons by $L$, one has $N = 3Q + L$. Let the leptons and quarks have $Y$-charges $y_i \ (1 \leq i \leq L)$ and $z_j \ (1 \leq j \leq Q)$ respectively. The three anomaly constraints arising from $U(1)[Gravity]^2$, $U(1)[SU(3)_c]^2$ and $[U(1)_Y]^3$ lead to the following equations:

\[ \sum_1^L y_i = 0; \]  
\[ \sum_1^Q z_j = 0; \]  
\[ \sum_1^L y_i^3 + 3 \sum_1^Q z_j^3 = 0; \]

As mentioned, we assume all $Y$-values to be rational and we will normalize them such that all $y_i$ and $z_j$ are non-vanishing positive or negative integers. We will be interested in finding the smallest $N$ for which the $y_i$ and $z_j$ will satisfy Eqs. (1)-(3). The assumption of irreducibility implies that $L \geq 3$. Further since $Q$ is even it can be $2, 4, 6$ etc. If $Q = 2$, by Eq. (2) it
has to be a reducible set and is excluded by our assumptions. The smallest value of Q is therefore 4 leading to $N = 15$. This proves our first assertion above. Note that in order to derive this result, nowhere have we used the anomaly constraints for the group $G'$.

Let us now show that the maximal group $G'$ is $SU(2)$. First we show that $G' \neq SU(3)$. Since this $SU(3)$ must be orthogonal to color $SU(3)_c$, the three leptons must be a triplet under it in which case, Eq.(1) cannot be satisfied. This leaves as the only possible simple group $G' = SU(2)$.

As a brief digression, suppose $G'$ were not simple but merely a $U(1)$. We shall now show the extra anomaly constraints associated with it imply that it is vectorlike. We can always take linear combinations of the two $U(1)$ charges to define two new $U(1)$'s and call their charges X and Y respectively. By means of an appropriate choice we can make it vanish for one of the leptons. Let us denote the X-charges of quarks to be $x_a$ where $a = 1$ to 4 and those of leptons to be $x_5$, $x_6$, 0. The mixed $U(1)[Gravity]$ and $U(1)[SU(3)]^2$ anomalies then imply:

$$x_1 + x_2 + x_3 + x_4 = 0; \quad (4a)$$

$$x_5 = -x_6; \quad (4b)$$

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0; \quad (4c)$$

Again as before if we choose the X charges also to be rational, then
Eq. (4), combined with the Fermat’s last theorem will imply that \( x_1 = -x_2 \) and \( x_3 = -x_4 \) i.e. \( U(1)_X \) is parity conserving. We are led directly to identify \( G' \) with electric charge and there is no non-abelian nature to the ornamentation of color - there are no weak interactions!

Thus the only parity-violating choice, like the only simple-group choice, is \( G' = SU(2) \), as we henceforth assume.

The next question then arises: how do the quarks and leptons transform under this \( SU(2) \) group? Consistent with our assumptions, there can be at most one doublet among the quarks otherwise Eq. (2) will imply that the Y-charges for the quarks become reducible in conflict with our assumptions. So far we are not using the anomalies associated with the \( SU(2) \) group.

Let us now require that there must be an \( SU(2) \) doublet among the leptons. From this we conclude that two of the \( y_i \)’s must be equal.

Now Eq. (3) implies that

\[
(z_1 + z_2)(z_2 + z_3)(z_3 + z_1) = -2y^3/3; \quad (5)
\]

where \( y_1 = y_2 = y = -y_3/2 \). Eq.(5) dictates that \( y \) is divisible by three and we may take as the simplest possibility \( y_1 = y_2 = -3 \) and \( y_3 = 6 \). Thus

\[
(z_1 + z_2)(z_2 + z_3)(z_3 + z_1) = +18. \quad (6)
\]

There are four independent ways to factor 18 into three integers: (1)(2)(9); (1)(3)(6); (1)(2)(18) and (2)(3). It can be shown that these factorizations correspond to the sets
\( (z_i) = (-3, +4, +5, -6); (-1, +2, +4, -5); (-8, +9, +9, -10) \) and 
\( (+1, +1, +2, -4) \) respectively. Only the last of these allows an unbroken gauge symmetry beyond color. This implies that, there is only one doublet of \( SU(2) \) in the quark sector.

Given that the quarks contain an \( SU(2) \) doublet we can write the \( Y \) charges as follows: \((y_1, y_1, -2y_1; z_1, z_1, z_2, -(2z_1 + z_2))\). This notation makes it clear which particles are in \( SU(2) \) doublets. In this notation, the anomaly constraints in Eqs. (1) and (2) are automatically satisfied and Eq.(3) becomes:

\[
\frac{y_1^3}{3} = -z_1(z_1 + z_2)^2; \tag{7}
\]

We will now show that vector-likeness of the unbroken \( U(1) \) will then determine not only the charge formula but also the individual \( Y \)-charges of the fermions. The generator of the final unbroken \( U(1) \) can be written as:

\[
Q_e = T_3 + \eta Y; \tag{8}
\]

The constraints of vector-like \( Q_e \) are:

\[
-\frac{1}{2} + \eta y_1 = 2\eta y_1; \tag{9a}
\]

\[
+\frac{1}{2} + \eta z_1 = -\eta z_2; \tag{9b}
\]

\[
-\frac{1}{2} + z_1 = +\eta(2z_1 + z_2); \tag{9c}
\]
Equation (9b) and (9c) are actually equivalent; taken together Eqs. (9) imply that \( \eta = -1/2y_1 \) and \( y_1 = z_1 + z_2 \). Putting these relations in eq. (4), we get \( y_1 = -3z_1 \) and \( z_2 = 2z_1 \). With the normalization \( z_1 = 1 \) one confirms that the set of 15 integers \( (1)^6(2)^3(-3)^2(-4)^3(6)^1 \) is a solution of Eqs. (1)-(3) which is irreducible in the sense defined above.

If we now rewrite \( Y \) as \( Y' = -\eta Y \), and call \( Y' \) as the standard model \( Y \), then the electric charge formula becomes

\[
Q_e = T_3 + \frac{Y}{2}.
\]

The values of the new \( Y \) are precisely those of the standard model. Note also that in deriving the above electric charge formula nowhere have we used any assumption about the Higgs structure of the theory. Also, we have derived that if there is an unbroken gauge group beyond color then it must be a \( U(1) \) generated by \( Q_e \) of Eq. (10).

In conclusion, we have given an alternative partial derivation of the one generation standard model starting from a rather economical set of assumptions. We may restate our conclusion in the following words. It is taken as given that the color gauge group \( SU(3)_c \) is unbroken and vector-like with an equal number of triplets and antitriplets. This QCD is then ornamented by an additional would-be-electroweak gauge symmetry \( U(1)_Y \times G' \) with simple \( G' \). Assuming the \( Y \) charges to form an irreducible set we show that the minimal model must have only 15 fermions matching those of the standard model. We then find that \( G' \) is a gauged \( SU(2) \). Finally requiring the unbroken gauge group to be vector-like implies that it be color alone or the product
of color with electric charge. In the latter case, the transformation properties
of quarks and leptons are uniquely derived.

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