Transforms on operator monotone functions

Masato Kawasaki and Masaru Nagisa

Abstract

Let $f$ be an operator monotone function on $[0, \infty)$ with $f(t) \geq 0$ and $f(1) = 1$. If $f(t)$ is neither the constant function 1 nor the identity function $t$, then $h(t) = \frac{(t-a)(t-b)}{(f(t)-f(a))(f(t)-f^2(b))}$ is also operator monotone on $[0, \infty)$, where $a, b \geq 0$ and $f^\#(t) = tf(t)$.

Moreover, we show some extensions of this statement.

1 Introduction

We call a real continuous function $f(t)$ on an interval $I$ operator monotone on $I$ (in short, $f \in \mathcal{P}(I)$), if $A \leq B$ implies $f(A) \leq f(B)$ for any self-adjoint matrices $A, B$ with their spectrum contained in $I$. In this paper, we consider only the case $I = [0, \infty)$ or $I = (0, \infty)$. We denote $f \in \mathcal{P}^+(I)$ if $f \in \mathcal{P}(I)$ satisfies $f(t) \geq 0$ for any $t \in I$.

Let $\mathbb{H}_+$ be the upper half-plane of $\mathbb{C}$, that is,

$$\mathbb{H}_+ = \{ z \in \mathbb{C} \mid \text{Im} z > 0 \} = \{ z \in \mathbb{C} \mid |z| > 0, \ 0 < \text{arg} z < \pi \},$$

where Im $z$ (resp. arg $z$) means the imaginary part (resp. the argument) of $z$. When we choose an element $z \in \mathbb{H}_+$, we consider that its argument satisfies $0 < \text{arg} z < \pi$. As Loewner’s theorem, it is known that $f$ is operator monotone on $I$ if and only if $f$ has an analytic continuation to $\mathbb{H}_+$ that maps $\mathbb{H}_+$ into itself and also has an analytic continuation to the lower half-plane $\mathbb{H}_- (= -\mathbb{H}_+)$, obtained by the reflection across $I$ (see [1],[3]). For an operator monotone function $f(t)$ on $I$, we also denote by $f(z)$ its analytic continuation to $\mathbb{H}_+$.

D. Petz [5] proved that an operator monotone function $f : [0, \infty) \rightarrow [0, \infty)$ satisfying the functional equation

$$f(t) = tf(t^{-1}) \quad t \geq 0$$

is related to a Morozova-Chentsov function which gives a monotone metric on the manifold of $n \times n$ density matrices. In the work [6], the concrete functions

$$f_a(t) = a(1-a)\frac{(t-1)^2}{(t-a-1)(t^2-1)} \quad (-1 < a < 2)$$



1
appeared and their operator monotonicity was proved (see also [2]). V.E.S. Szabo introduced an interesting idea for checking their operator monotonicity in [7]. We use a similar idea as Szabo’s in our argument. M. Uchiyama [8] proved the operator monotonicity of the following extended functions:

\[
\frac{(t-a)(t-b)}{(t^p-a^p)(t^{1-p}-b^{1-p})}
\]

for \(0 < p < 1\) and \(a, b > 0\). It is well known that the function \(t^p\) \((0 \leq p \leq 1)\) is operator monotone as Loewner-Heinz inequality. In this paper, we extend this statement to the following form:

**Theorem 1.** Let \(a\) and \(b\) be non-negative real. If \(f \in \mathcal{P}_{+}[0, \infty)\) and both \(f\) and \(f^\sharp\) are not constant, then

\[
h(t) = \frac{(t-a)(t-b)}{(f(t)-f(a))(f^\sharp(t)-f^\sharp(b))}
\]

is operator monotone on \([0, \infty)\), where

\[
f^\sharp(t) = \frac{t}{f(t)} \quad t \geq 0.
\]

We can also show the operator monotonicity of other functions which have the form related to the above one in Theorem 4.

2 Main result

For \(f \in \mathcal{P}[0, \infty)\), we have the following integral representation:

\[
f(z) = f(0) + \beta z + \int_0^\infty \frac{\lambda z}{z + \lambda} dw(\lambda),
\]

where \(\beta \geq 0\) and

\[
\int_0^\infty \frac{\lambda}{1 + \lambda} dw(\lambda) < \infty
\]

(see [11]). When \(f(0) \geq 0\) (i.e., \(f \in \mathcal{P}_{+}[0, \infty)\)), it holds that \(0 < \arg f(z) \leq \arg z\) for \(z \in \mathbb{H}_{+}\) (i.e., \(0 < \arg z < \pi\)).

For any \(f \in \mathcal{P}_{+}[0, \infty)\) \((f \neq 0)\), we define \(f^t\) as follows:

\[
f^t(t) = \frac{t}{f(t)} \quad t \in [0, \infty).
\]

Then it is well-known that \(f^t \in \mathcal{P}_{+}[0, \infty)\).

**Proposition 2.** Let \(f\) be an operator monotone function on \((0, \infty)\) and \(a\) be positive real.

(1) When \(f(t)\) is not constant, the function

\[
g_1(t) = \frac{t-a}{f(t)-f(a)}
\]

is operator monotone on \([0, \infty)\).
(2) When \( f(t) \geq 0 \) for \( t \geq 0 \), the function
\[
g_2(t) = \frac{f(t)(t-a)}{tf(t) - af(a)}
\]
is operator monotone on \([0, \infty)\).

**Proof.** (1) It follows from Theorem 2.1 in [8].

(2) Since \( f \in \mathbb{P}_+[0, \infty) \), we have \( 0 < \arg zf(z) < 2\pi \) for any \( z \in \mathbb{H}_+ \). So we can define
\[
g_2(z) = \frac{zf(z)(z-a)}{zf(z) - af(a)}, \quad z \in \mathbb{H}_+
\]
and \( g_2(z) \) is holomorphic on \( \mathbb{H}_+ \). Because \( g_2([0, \infty)) \subset [0, \infty) \) and \( g_2(z) \) is continuous on \( \mathbb{H}_+ \cup [0, \infty) \), it suffices to show that \( g_2(\mathbb{H}_+) \subset \mathbb{H}_+ \). By the calculation
\[
g_2(z) = \frac{zf(z) - af(a) + af(a) - zf(z)}{zf(z) - af(a)} = 1 - \frac{a(f(z) - f(a))}{zf(z) - af(a)}
\]
we have
\[
\text{Im}g_2(z) = -\text{Im} \frac{a}{z + f(a)g_1(z)} = \text{Im} \frac{a(z + f(a)g_1(z))}{|z + f(a)g_1(z)|^2}.
\]
When \( z \in \mathbb{H}_+ \), \( \text{Im}g_1(z) > 0 \) by (1) and \( \text{Im}g_2(z) > 0 \). So the function \( g_2(t) \) belongs to \( \mathbb{P}_+[0, \infty) \).

For any \( z = e^{i\theta} \) \( (0 < \theta < \pi) \) and any integer \( n \geq 2 \), we set
\[
w = \frac{\sin \theta}{\sin \frac{\pi + (n-1)\theta}{n}} e^{i(n-1)\theta}/n.
\]
Since \( \text{Im}z = \text{Im}w, \ l = z - w > 0 \). Then we can get
\[
\sup \{ l \mid 0 < \theta < \pi \} = \lim_{\theta \to \pi - 0} \frac{\sin \frac{\pi - \theta}{n}}{\sin \frac{(n-1)(\pi - \theta)}{n}} = \frac{1}{n - 1}.
\]
So we have the following:

**Lemma 3.** For any \( z \in \mathbb{H}_+ \) and a positive integer \( n \geq 2 \), we have
\[
\arg z < \arg(z - l) < \frac{\pi + (n-1)\arg z}{n} \quad \text{if} \quad 0 < l \leq \frac{|z|}{n - 1}.
\]

Now we can prove the following theorem and we remark that Theorem 1 easily follows from this:

**Theorem 4.** Let \( n \) be a positive integer, \( a, b, b_1, \ldots, b_n \geq 0 \) and \( f, g, g_1, \ldots, g_n \) be non-constant, non-negative operator monotone functions on \([0, \infty)\).
(1) If \( \frac{f(t)g(t)}{t} \) is operator monotone on \([0, \infty)\), then the function

\[
h(t) = \frac{(t - a)(t - b)}{(f(t) - f(a))(g(t) - g(b))}
\]

is operator monotone on \([0, \infty)\) for any \(a, b \geq 0\).

(2) If \( \prod_{i=1}^{n} g_i(t) \) is operator monotone on \([0, \infty)\), then the function

\[
h(t) = \frac{(t - a)(t - b)}{(f(t) - f(a)) \prod g_i(t)(t - b_i)}
\]

is operator monotone on \([0, \infty)\) for any \(a, b \geq 0\).

Proof. (1) By \( f, g \in \mathbb{P}_+[0, \infty) \) and Proposition 2 (1),

\[
\frac{t - a}{f(t) - f(a)} \text{ and } \frac{t - b}{g(t) - g(b)}
\]

are operator monotone on \([0, \infty)\). Therefore

\[
h(z) = \frac{(z - a)(z - b)}{(f(z) - f(a))(g(z) - g(b))}
\]

is holomorphic on \(\mathbb{H}_+\), continuous on \(\mathbb{H}_+ \cup [0, \infty)\) and satisfies \(b([0, \infty)) \subseteq [0, \infty)\) and

\[
\arg h(z) = \arg \frac{z - a}{f(z) - f(a)} + \arg \frac{z - b}{g(z) - g(b)} > 0 \text{ for } z \in \mathbb{H}_+.
\]

We assume that \(f(z)\) and \(g(z)\) are continuous on the closure \(\overline{\mathbb{H}_+}\) of \(\mathbb{H}_+\) and

\[
f(t) - f(a) \neq 0 \text{ and } g(t) - g(b) \neq 0 \text{ for any } t \in (-\infty, 0).
\]

Then \(h(z)\) is continuous on \(\overline{\mathbb{H}_+}\).

In the case \(z \in (-\infty, 0)\), i.e., \(|z| > 0\) and \(\arg z = \pi\), we have

\[
\arg h(z) = \arg(z - a) - \arg(f(z) - f(a)) + \arg(z - b) - \arg(g(z) - g(b)) \\
\leq \pi - \arg f(z) + \pi - \arg g(z) \\
\leq 2\pi - \arg z = \pi \quad \text{(since } \arg f(z) + \arg g(z) - \arg z \geq 0).\]

So it holds \(0 \leq \arg h(z) \leq \pi\).

In the case that \(z \in \mathbb{H}_+\) satisfying \(|z| > \max\{a, b\}\), it holds that

\[
\arg(z - a), \arg(z - b) < \frac{\pi + \arg z}{2}
\]

by Lemma 3. Since

\[
\arg h(z) = \arg(z - a) - \arg(f(z) - f(a)) + \arg(z - b) - \arg(g(z) - g(b)) \\
\leq \frac{\pi + \arg z}{2} - \arg f(z) + \frac{\pi + \arg z}{2} - \arg g(z) \\
= \pi + \arg z - \arg f(z) - \arg g(z) \leq \pi,
\]
we have $0 < \arg h(z) < \pi$.

For $r > 0$, we define $H(r) = \{ z \in \mathbb{C} \mid |z| \leq r, \text{Im} z \geq 0 \}$. Whenever $r > l = \max\{a, b\}$, we can get

$$0 \leq \arg h(z) \leq \pi$$
on the boundary of $H(r)$. Since $h(z)$ is holomorphic on $H(r)$, Im$h(z)$ is harmonic on $H(r)$. Because Im$h(z) \geq 0$ on the boundary of $H(r)$, we have $h(H(r)) \subset \mathbb{H}_+$ by the minimum principle of harmonic functions. This implies

$$h(\overline{\mathbb{H}_+}) = h(\bigcup_{r > l} H(r)) \subset \bigcup_{r > l} h(H(r)) \subset \mathbb{H}_+,$$

and $h \in \mathbb{P}_+[0, \infty)$.

In general case, we set

$$\frac{f(t)g(t)}{t} = F(t) \text{ and } \tilde{f}(t) = \frac{f(t)}{F(t)} \quad (t \geq 0).$$

By the relation $\tilde{f}(t)g(t) = t$, we have $\tilde{f} \in \mathbb{P}_+[0, \infty)$. We define the function $f_p, \tilde{f}_p$ and $g_p \ (0 < p < 1)$ as follows:

$$f_p(z) = f(z^p), \quad \tilde{f}_p(z) = \tilde{f}(z^p),$$

and

$$g_p(z) = (\tilde{f}_p)'(z) = \frac{z}{\tilde{f}_p(z)} = \frac{zF(z^p)}{F(z^p)} = z^{1-p}g(z^p)$$

for $z \in \overline{\mathbb{H}_+}$. Then we have $f_p, g_p \in \mathbb{P}_+[0, \infty)$ and

$$h_p(z) = \frac{(z - a)(z - b)}{(f_p(z) - f_p(a))(g_p(z) - g_p(b))}$$

is holomorphic on $\mathbb{H}_+$ and continuous on $\overline{\mathbb{H}_+}$. By the fact $\frac{f_p(t)g_p(t)}{t} = F(t^p)$ is operator monotone on $[0, \infty)$, $h_p(t)$ becomes operator monotone on $[0, \infty)$.

Since

$$h_p(t) = \frac{(t - a)(t - b)}{(f_p(t) - f_p(a))(g_p(t) - g_p(b))} = \frac{(t - a)(t - b)}{(f(t^p) - f(a^p))(t^{1-p}g(t^p) - t^{1-p}g(a^p))} \quad \text{for } t \geq 0,$$

we have

$$\lim_{p \to 1^-} h_p(t) = h(t).$$

So we can get the operator monotonicity of $h(t)$.

(2) We show this by the similar way as (1). By Proposition 2,

$$\frac{t - a}{f(t) - f(a)} \quad \text{and} \quad \frac{g_i(t) - g_i(1)}{t g_i(t) - b_i g_i(b_i)} \quad (i = 1, 2, \ldots, n)$$

are operator monotone on $[0, \infty)$. So we have that

$$h(z) = \frac{z - a}{f(z) - f(a)} \prod_{i=1}^{n} \frac{g_i(z) - b_i}{z g_i(z) - b_i g_i(b_i)}$$

5
is holomorphic on $\mathbb{H}_+$, continuous on $\mathbb{H}_+ \cup [0, \infty)$ and satisfies $h([0, \infty)) \subset [0, \infty)$ and
\[
\arg h(z) = \arg \frac{z - a}{f(z) - f(a)} + \sum_{i=1}^{n} \arg \frac{g_i(z) - b_i}{zg_i(z) - b_i} > 0
\]
for $z \in \mathbb{H}_+$.

We assume that $f(z)$ and $g_i(z)$ ($i = 1, 2, \ldots, n$) are continuous on $\mathbb{H}_+$ and
\[
f(t) - f(a) \neq 0 \quad \text{and} \quad tg_i(t) - b_i, g_i(b_i) \neq 0 \quad \text{for any } t \in (-\infty, 0).
\]
Then $h(z)$ is continuous on $\mathbb{H}_+$.

In the case $z \in (-\infty, 0)$, i.e., $|z| > 0$ and $\arg z = \pi$, we have
\[
\arg h(z)
= \arg(z - a) + \sum_{i=1}^{n} \arg g_i(z)(z - b_i) - \arg(f(z) - f(a)) - \sum_{i=1}^{n} \arg(zg_i(z) - b_i)\]
\[
\leq \pi + \sum_{i=1}^{n} \arg g_i(z) + n\pi - \arg f(z) - n\pi \quad \text{(since } \arg(zg_i(z) - b_i) \geq \pi)\]
\[
\leq \pi \quad \text{(since } \arg f(z) = \sum_{i=1}^{n} \arg g_i(z) \geq 0)\).
\]
So it holds $0 \leq \arg h(z) \leq \pi$.

In the case $z \in \mathbb{H}_+$ satisfying $|z| > n \max\{a, b_1, b_2, \ldots, b_n\}$, it holds that
\[
\arg(z - a), \arg(z - b) < \frac{\pi + n \arg z}{n + 1}
\]
by Lemma 3. We may assume that there exists a number $k$ ($1 \leq k \leq n$) such that
\[
\arg(zg_i(z)) \leq \pi \quad (i \leq k), \quad \arg(zg_i(z)) > \pi \quad (i > k).
\]
Since
\[
\arg h(z)
= \arg(z - a) + \sum_{i=1}^{n} \arg(z - b_i) + \sum_{i=1}^{n} \arg g_i(z)
\]
\[
- \arg(f(z) - f(a)) - \sum_{i=1}^{n} \arg(zg_i(z) - b_i)\]
\[
\leq \frac{\pi + n \arg z}{n + 1} \times (n + 1) + \sum_{i=1}^{n} \arg g_i(z)
\]
\[
- \arg f(z) - \sum_{i=1}^{k} \arg zg_i(z) - (n - k)\pi
\]
\[
= \pi + n \arg z + \sum_{i=k+1}^{n} \arg g_i(z) - \arg f(z) - k \arg z - (n - k)\pi
\]
\[
\leq \pi + (n - k) \arg z - (n - k)\pi \leq \pi,
\]
we have $0 \leq \arg h(z) \leq \pi$.

This means that it holds

$$0 \leq \arg h(z) \leq \pi$$

if $z$ belongs to the boundary of $H(r) = \{ z \in \mathbb{C} \mid |z| \leq r, \text{Im} z \geq 0 \}$ for a sufficiently large $r$. Using the same argument in (1), we can prove the operator monotonicity of $h$.

In general case, we define functions, for $p (0 < p < 1)$, as follows:

$$f_p(t) = f(t^p), \quad g_{i,p}(t) = g_i(t^p) \quad (i = 1, 2, \ldots, n).$$

Since $f, g_i \in \mathbb{P}_+[0, \infty)$,

$$0 < \arg f_p(z) < \pi, \quad 0 < \arg zg_{i,p}(z) < 2\pi$$

for $z \in \mathbb{H}_+$. This means that $f_p(z)$ and $g_{i,p}(z)$ are continuous on $\mathbb{H}_+$ and

$$f_p(t) - f_p(a) \neq 0 \text{ and } tg_{i,p}(t) - b_i g_{i,p}(b_i) \neq 0 \text{ for any } t \in (\mathbb{R}, 0).$$

Since

$$\prod_{i=1}^{n} f_{i,p}(t) = \frac{f(t^p)}{\prod_{i=1}^{n} g_{i,p}(t)} \quad (0 < p < 1)$$

is operator monotone on $[0, \infty)$, we can get the operator monotonicity of

$$h_p(t) = \frac{t - a}{f_p(t) - f_p(a)} \prod_{i=1}^{n} g_{i,p}(t) (t - b_i)$$

$$= \frac{t - a}{f(t^p) - f(a^p)} \prod_{i=1}^{n} g_{i}(t^p) (t^p - b_i).$$

So we can see that

$$h(t) = \lim_{p \to 1^{-}} h_p(t)$$

is operator monotone on $[0, \infty)$. \hfill \Box

**Remark 5.** Using Proposition 2 and Theorem 4, we can prove the operator monotonicity of the concrete functions in [6]. Since $t^a \ (0 < a < 1)$ and $\log t$ is operator monotone on $(0, \infty)$,

$$f_a(t) = a(1 - a) \frac{(t - 1)^2}{(t^a - 1)(t^{1-a} - 1)} \quad (-1 < a < 2)$$

becomes operator monotone.
**Corollary 6.** Let \( f \in \mathbb{P}_+(0, \infty) \) and both \( f \) and \( f^2 \) be not constant. For any \( a > 0 \), we define

\[
h_a(t) = \frac{(t - a)(t - a^{-1})}{(f(t) - f(a))(f^2(t) - f^2(a^{-1}))} \quad t \in (0, \infty).
\]

Then we have

1. \( h_a \) is operator monotone on \( (0, \infty) \).
2. \( f(t) = t \cdot f(t^{-1}) \) implies \( h_a(t) = t \cdot h_a(t^{-1}) \).
3. \( a = 1 \) and \( f(t^{-1}) = f(t)^{-1} \) implies \( h_1(t) = t \cdot h_1(t^{-1}) \).

**Proof.** We can directly prove (1) from theorem 3. Because

\[
t \cdot h_a(t^{-1}) = \frac{t(t^{-1} - a)(t^{-1} - a^{-1})}{(f(t^{-1}) - f(a))(f^2(t^{-1}) - f^2(a^{-1}))} = \frac{(t-1)(t-a)(t-a^{-1})}{t(f(t^{-1}) - f(a))(f^2(t^{-1}) - f^2(a^{-1}))},
\]

we can compute

\[
t(f(t^{-1}) - f(a))(f^2(t^{-1}) - f^2(a^{-1})) - (f(t) - f(a))(f^2(t) - f^2(a^{-1})
\]

\[
= (f(t^{-1}) - f(a))(1/f(t^{-1}) - t/a f(a^{-1})) - (f(t) - f(a))(t/f(t) - 1/a f(a^{-1}))
\]

\[
= 0
\]

if it holds \( f(t) = t \cdot f(t^{-1}) \) or \( a = 1 \), \( f(t^{-1}) = f(t)^{-1} \). So we have (2) and (3).

**Example 7.** Using this corollary, we can repeatedly construct an operator monotone function \( h(t) \) on \([0, \infty)\) satisfying the relation

\[
h(t) = t \cdot h(t^{-1}) \quad t > 0.
\]

If we choose \( p \) (0 < \( p < 1 \)) as \( f(t) \) in Corollary 5(3),

\[
h(t) = \frac{(t - 1)^2}{(tp - 1)(t^{1-p} - 1)}.
\]

If we choose \( \frac{(t - 1)^2}{(tp - 1)(t^{1-p} - 1)} \) as \( f(t) \) in Corollary 5(2),

\[
b(t) = \frac{t - a}{(tp - 1)(t^{1-p} - 1)} \cdot \frac{(a - 1)^2}{(a^p - 1)(a^{1-p} - 1)}
\]

\[
\times \frac{t - a^{-1}}{t(t - 1)^2} \cdot \frac{a(a - p - 1)(a^{p-1} - 1)}{(a - 1)^2}
\]

8
for $a > 0$. If we choose $t^p + t^{1-p}$ ($0 < p < 1$) as $f(t)$ in Corollary 5(2),

$$h(t) = \frac{t - a}{p^t + t^{1-p} - a^p + t^{1-p}} \times \frac{1}{t^{p-1} + t^{-p}} - \frac{1}{a^p + t^{-p}}$$

$(a > 0)$

$$= \frac{\sqrt{t} (\cosh(\log t) - \cosh(\log a))}{\cosh(\log \sqrt{t}) - \cosh(\log(\sqrt{t} + t^p + t^{1-p}) - \log(a^p + t^{-p})).}

These functions, $h \in \mathbb{P}_+[0, \infty)$, satisfy the relation (*)

### 3 Extension of Theorem 4

Let $m$ and $n$ be positive integers and $f_1, f_2, \ldots, f_m$, $g_1, g_2, \ldots, g_n$ be non-
constant, non-negative operator monotone functions on $[0, \infty)$. We assume that
the function

$$F(t) = \frac{\prod_{i=1}^m f_i(t)}{\prod_{j=1}^n g_j(t)}$$

is operator monotone on $[0, \infty)$. For non-negative numbers $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n$, we define the function $h(t)$ as follows:

$$h(t) = \prod_{i=1}^m f_i(t) - f_i(a_i) \prod_{j=1}^n g_j(t) - b_j g_j(b_j) \quad (t \geq 0).$$

Then it follows from Propostion 2 that $h(z)$ is holomorphic on $\mathbb{H}_+, h([0, \infty)) \subset [0, \infty)$ and $\arg h(z) > 0$ for any $z \in \mathbb{H}_+$.

**Theorem 8.** In the above setting, we have the followings:

1. When $f_i$ and $g_j$ ($1 \leq i \leq m, 1 \leq j \leq n$) are continuous on $\mathbb{H}_+$ and

   $$f_i(t) - f_i(a_i) \neq 0, \quad g_j(t) - b_j g_j(b_j) \neq 0, \quad t \in (-\infty, 0),$$

   then $h(t)$ is operator monotone on $[0, \infty)$.

2. When there exists a positive number $\alpha$ such that $\alpha \arg z \leq \arg F(z)$ for

   all $z \in \mathbb{H}_+$, $h(t)$ is operator monotone on $[0, \infty)$.

**Proof.** (1) Using the same argument of proof of Theorem 4 (1), it suffices to show that $0 \leq \arg h(z) \leq \pi$ for $z \in \mathbb{R}$ or $z \in \mathbb{H}_+$ whose absolutely value is

sufficiently large.

In the case $z \in (-\infty, 0)$, i.e., $|z| > 0$ and $\arg z = \pi$, we have

$$\arg h(z)$$

$$= \sum_{i=1}^m \arg(z - a_i) + \sum_{j=1}^n \arg(g_j(z)(z - b_j))$$

$$- \sum_{i=1}^m \arg(f_i(z) - f_i(a_i)) - \sum_{j=1}^n (z g_j(z) - b_j g_j(b_j))$$

$$\leq m\pi + n\pi + \sum_{j=1}^n \arg g_j(z) - \sum_{i=1}^m \arg f_i(z) - n\pi$$

$$= \pi - \arg \frac{\prod_{i=1}^m f_i(z)}{\prod_{j=1}^n g_j(z)} \leq \pi.$$
So it holds $0 \leq \arg h(z) \leq \pi$.
In the case that $z \in \mathbb{H}_+$ satisfies

$$|z| > (m + n - 1) \max \{a_i, b_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}.$$ 

Then it holds that

$$\arg(z - a_i), \arg(z - b_j) < \frac{\pi + (m + n - 1) \arg z}{m + n}$$

by Lemma 3. We may assume that there exists $k$ ($1 \leq k \leq n$) such that

$$\arg(zg_j(z)) \leq \pi \quad (j \leq k), \quad \arg(zg_j(z)) > \pi \quad (j > k).$$

Since

$$\arg h(z) \leq \pi + (m + n - 1) \arg z \times m + \frac{(m + n - 1) \arg z}{m + n} \times n + \sum_{j=1}^{n} \arg g_j(z)$$

we have $0 \leq \arg h(z) \leq \pi$. So $h(t)$ is operator monotone on $[0, \infty)$.

(2) We choose a positive number $p$ as follows:

$$p = \frac{m - 1}{\alpha + m - 1} < 1.$$ 

We define functions $f_{i,p}, g_{j,p}$ as follows:

$$f_{i,p}(z) = f_i(z^p), \quad g_{j,p}(z) = g_j(z^p) \quad (z \in \mathbb{H}_+).$$

Since $f_i, g_j \in P_+[0, \infty)$, $f_{i,p}, g_{j,p}$ are continuous on $\mathbb{H}_+$ and satisfy the condition

$$f_{i,p}(t) - f_{i,p}(a_i) \neq 0, \quad tg_{j,p}(t) - b_{j,g_{j,p}(b_j)} \neq 0, \quad t \in (-\infty, 0).$$

We put

$$F_p(t) = \frac{\prod_{i=1}^{m} f_{i,p}(t)}{\prod_{j=1}^{n} g_{j,p}(t)} = F(t^p) e^{-(m-1)(1-p)}.$$ 

Then $F_p$ is holomorphic on $\mathbb{H}_+$ and satisfies $F_p([0, \infty)) \subset (0, \infty)$. For any $z \in \mathbb{H}_+$, we have

$$\arg F_p(z) = \arg F(z^p) - (m - 1)(1 - p) \arg z \leq \arg F(z^p) \leq \pi.$$
and

\[
\arg F_p(z) \geq \alpha \arg z^p - (m - 1)(1 - p) \arg z
\]

\[
= (\alpha p - (m - 1)(1 - p)) \arg z
\]

\[
= ((\alpha + m - 1)p - (m - 1)) \arg z > 0.
\]

So we can see \( F_p \in \mathbb{P}_+[0, \infty) \). By (1), we can show that

\[
h_p(t) = \prod_{i=1}^{m} \frac{(t - a_i) \cdots (t - a_m)}{(t^{p_1} - a_1^{p_1}) \cdots (t^{p_m} - a_m^{p_m})(t^{1+q_1} - b_1^{1+q_1}) \cdots (t^{1+q_n} - b_n^{1+q_n})}
\]

is operator monotone on \([0, \infty)\). When \( p \) tends to 1, \( h_p(t) \) also tends to \( h(t) \). Hence \( h(t) \) is operator monotone on \([0, \infty)\).

**Example 9.** Let \( 0 < p_i \leq 1 \) \((i = 1, 2, \ldots, m)\) and \( 0 \leq q_j \leq 1 \) \((j = 1, 2, \ldots, n)\). We put

\[
f_i(t) = t^{p_i}, \quad g_j(t) = t^{q_j} \quad (t \geq 0).
\]

By the calculation

\[
F(t) = \prod_{i=1}^{m} f_i(t) = \frac{t^{\sum_{i=1}^{n} p_i - \sum_{j=1}^{m} q_i - (m - 1)}}{t^{p_1} \cdots t^{p_m} - a_m^{p_m}},
\]

we have, for real numbers \( a_i, b_j \geq 0 \),

\[
h(t) = t^{\sum_{i=1}^{n} q_i} \frac{(t - a_1) \cdots (t - a_m)(t - b_1) \cdots (t - b_n)}{(t^{p_1} - a_1^{p_1}) \cdots (t^{p_m} - a_m^{p_m})(t^{1+q_1} - b_1^{1+q_1}) \cdots (t^{1+q_n} - b_n^{1+q_n})}
\]

is operator monotone on \([0, \infty)\) if it holds

\[
0 \leq \sum_{i=1}^{m} p_i - \sum_{j=1}^{n} q_j - (m - 1) \leq 1,
\]

i.e., \( F(t) \) is operator monotone on \([0, \infty)\).

When \( \sum_{i=1}^{m} p_i = \sum_{j=1}^{n} q_j + (m - 1) \), we can see that

\[
h(t) = \frac{t^{\sum_{i=1}^{n} q_i} (t - 1)^{m+n}}{\prod_{i=1}^{m} (t^{p_i} - 1) \prod_{j=1}^{m} (t^{1+q_j} - 1)}
\]

is operator monotone on \([0, \infty)\) and satisfies the functional equation

\[
h(t) = t \cdot h(t^{-1}). \quad (*)
\]

We can easily check that, if \( h_1, h_2 \in \mathbb{P}_+[0, \infty) \) satisfy the property \((*)\), then the functions

\[
f(t) = h_1(t)^{1/p} h_2(t)^{1-1/p} \quad (0 < p < 1)
\]

\[
g(t) = \frac{t}{h_1(t)}
\]

are operator monotone on \([0, \infty)\) and satisfy the property \((*)\).
Combining these facts, for \( r_i, s_i \ (i = 1, 2, \ldots, n) \) with
\[
0 < r_1, \ldots, r_c \leq 1, \quad 1 \leq r_{c+1}, \ldots, r_n \leq 2
\]
\[
0 < s_1, \ldots, s_d \leq 1, \quad 1 \leq s_{d+1}, \ldots, s_n \leq 2
\]
\[
\sum_{i=1}^{c} r_i = \sum_{i=c+1}^{n} r_i - 1, \quad \sum_{i=1}^{d} s_i = \sum_{i=d+1}^{n} s_j - 1,
\]
we can see that the function
\[
h(t) = \left( t^\gamma \prod_{i=1}^{n} \frac{r_i(t^{s_i} - 1)}{s_i(t^{r_i} - 1)} \right)^{1/n}
\]
is operator monotone on \([0, \infty)\) and satisfies the property (*) and \( h(1) = 1 \), where
\[
\gamma = 1 - c + d + \sum_{i=1}^{c} r_i - \sum_{i=1}^{d} s_i.
\]

Acknowledgements. The authors thank to Professor M. Uchiyama for his useful comments. The works of M. N. was partially supported by Grant-in-Aid for Scientific Research (C)22540220.

References

[1] R. Bhatia, *Matrix Analysis*, Springer, 1996.

[2] L. Cai and F. Hansen, Metric-adjusted skew information: Convexity and restricted forms of superadditivity, Lett. Math. Phys. 93(2010) 1–13.

[3] F. Hiai, Matrix Analysis: Matrix monotone functions, matrix means, and majorization, Interdecip. Inform. Sci. 16 (2010) 139–248.

[4] E.A. Morozova and N.N. Chentsov, Markov invariant geometry on state manifolds, Itogi Nauki i Techniki 36 (1990) 69–102, Translated in J. Soviet Math. 56(1991) 2648–2669.

[5] D. Petz, Monotone metric on matrix spaces, Linear Algebra Appl. 244 (1996) 81–96.

[6] D. Petz and H. Hasegawa, On the Riemannian metric of \( \alpha \)-entropies of density matrices, Lett. Math. Phys. 38 (1996) 221–225.

[7] V.E.S. Szabo, A class of matrix monotone functions, Linear Algebra Appl. 420 (2007) 79–85.

[8] M. Uchiyama, Majorization and some operator monotone functions, Linear Algebra Appl. 432(2010) 1867–1872.

Graduate School of Science
Chiba University
Inage-ku, Chiba 263-8522
Japan
e-mail: nagisa@math.s.chiba-u.ac.jp