Thermodynamic Properties of Small Localized Black Holes

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In a previous paper, we developed a numerical method to obtain a static black hole localized on a 3-brane in the Randall-Sundrum infinite braneworld, and presented examples of numerical solutions that describe small localized black holes. In this paper we quantitatively analyze the behavior of the numerically obtained black hole solutions, focusing on thermodynamic quantities. The thermodynamic relations show that the localized black hole deviates smoothly from a five-dimensional Schwarzschild black hole, which is a solution in the limit of a small horizon radius. We compare the thermodynamic behavior of these solutions with that of the exact solution on the 2-brane in the 4D braneworld. We find similarities between them.

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I. INTRODUCTION

Black holes in higher dimensions have been studied in many publications. A great deal of attention has recently been focused on black holes in compactified spacetime in the context of the braneworld. This has resulted from the interesting proposal for the possibility of black hole production at colliders \(1,2\) in the scenario of large extra dimensions \(3\). Such black holes in a compactified spacetime have been discussed in many works. Some properties of 4D Schwarzschild black holes in a compactified spacetime are considered in Refs. \(4\) and \(5\) (see also references therein). The black brane is another type of black hole in a compactified spacetime, and its instability, which is known as the Gregory-Laflamme instability \(6,7,8\), has been studied extensively by many authors \(9,11,12,13,14\).

Black holes in a warped spacetime are also interesting. In the Randall-Sundrum (RS) braneworld model \(16,17\), the tension of the brane and the cosmological constant in the bulk are not negligible, and it is difficult to construct black hole solutions \(18\). Based on the Gregory-Laflamme instability, however, physically realistic black holes in the RS infinite braneworld are conjectured to be localized on the brane \(19\). (See Ref. \(20\) for a recent discussion of black string perturbations in AdS space.) The localization of black holes is also hypothesized by gravitational collapse on the brane, where ordinary matter fields are confined.

An exact solution representing a localized black hole was found in the 4D RS braneworld by Emparan, Horowitz and Myers (EHM) \(21\). However, no exact solution representing a physically acceptable localized black hole has been found in the original 5D braneworld model, although many authors have tried to construct one. There is an argument that has been given to explain why the search for a static localized black hole solution has not yet succeeded. It was first conjectured in Ref. \(22\) on the basis of an extensive use of the AdS/CFT correspondence that localized black holes may classically evaporate in the 5D RS model. If this conjecture is correct, there would be no static solution of a black hole that is asymptotically AdS and is sufficiently large compared with the bulk curvature length. As there exist several possibilities for stable solar mass black holes, such evaporation would set an interesting constraint on the allowed value of the bulk curvature scale. Further discussion motivated by a no-go theorem \(23\) and a more explicit study showing that the existence of the EHM solution does not contradict this conjecture are found in Ref. \(24\). (Also see Ref. \(27\) for AdS/CFT in the RS braneworld.)

In contrast with the situation described above, however, recovery of 4D gravity on the 3-brane has been successfully realized in the RS models. It was shown that 4D Einstein gravity is approximately recovered in linear perturbations \(25,26,27,28,29,30\) and also in second-order perturbations \(31,32,33,34\). Moreover, even in the case of highly relativistic stars, 4D Einstein gravity was numerically shown to be a good approximation \(35\). Hence, there are also results that suggest the existence of black hole solutions localized on the 3-brane. If this is the case, then black holes should be produced as a result of gravitational collapse on the brane. Because Birkhoff’s theorem does not hold on the brane \(28\), exterior fields are not static, even during spherical collapse on the brane. However, it is believed that the exterior fields settle down to a static state at late times within the standard picture of 4D general relativity. This is the original interpretation of the no-go theorem \(28\).

In a previous paper \(36\), which is referred to as KTN in the present paper, we developed a numerical method to construct a localized black hole without assuming any artificial conditions, and we presented numerical examples of small localized black holes whose horizon radii are smaller than the AdS curvature radius. In this paper, we study the thermodynamic properties of localized black holes using these numerically obtained solutions. We find that the numerically obtained solutions exhibit a smooth transition from the 5D Schwarzschild black hole and that the thermodynamic relations deviate significantly from those expected on the basis of naive consideration of the 5D Schwarzschild black hole.
This paper is organized as follows. In the next section, we briefly explain the formulation for constructing the local- 
ized black holes numerically. In Section III, we discuss the thermodynamic relations obtained by analyzing the 
numerical solutions, and we compare them with those for the EHM solution. Section IV is devoted to sum-
mary. The relation between the numerical solutions and the conjecture based on the AdS/CFT correspondence is 
also discussed there.

II. NUMERICAL SOLUTIONS OF SMALL LOCALIZED BLACK HOLES

To obtain black hole solutions in the RS infinite braneworld model, it is necessary to formulate the prob-
lem as a boundary value problem without assuming any artificial boundary conditions. In KTN, we used and de-
veloped the method that was first applied to obtain a solution representing the gravity of a relativistic star on 
the brane[35] (see also Ref. [10]). In this section, we 
brieﬂy review the method and the notation presented in 
KTN.

Because we wish to consider a static black hole solution that is localized on the brane, we consider the static, D-
dimensional axial symmetric form

\[ ds^2 = \frac{\ell^2}{z^2} ( -T^2 dt^2 + e^{2R} (dr^2 + dz^2) + r^2 e^{2C} d\Omega^2_{D-3} ) . \]  

(1)

where the line element \( d\Omega^2_D \) is that of a unit \( n \)-sphere. The cosmological constant in the bulk is related to the 
bulk curvature length \( \ell \) by \( \Lambda = - (D-1)(D-2)/2\ell^2 \), and the tension of the brane is \( \sigma = 2(D-2)/8\pi G_D \ell \), where \( G_D \) is the D-dimensional Newton constant. If we set \( T = 1 \) and \( R = C = 0 \), this metric becomes the 
AdS metric in the Poincaré coordinates. Because this 
metric form has the residual gauge degrees of freedom of 
conformal transformations in the two-dimensional space \( \{r, z\} \), we can use these conformal degrees of freedom to 
transform the location of the event horizon to

\[ \rho h_{\text{horizon}} = \text{const} = \rho_h , \]

(2)

where we have introduced the polar coordinates

\[ r = \rho \sin \sqrt{\xi} , \]
\[ z = \ell + \rho \cos \sqrt{\xi} . \]

(3)

The Einstein equations yield five non-trivial equations. Three of them yield elliptic equations for the respective 
metric components \( X_i = \{T, R, C\} \),

\[ \Delta X_i = S X_i , \]

(4)

where \( \Delta = \partial_r^2 + \partial_z^2 \) is the Laplace operator, and the 
source \( S X_i \) depend non-linearly on \( X_i \) and its deriva-
tives. The remaining two equations give ‘constraint’ equations, \( \Theta_1 = 0 \) and \( \Theta_2 = 0 \). All the boundary condi-
tions that we need in order to solve the equations are de-
\( \text{vi} \)ed consistently from physical requirements. Because

all the equations and all the boundary conditions can be 
rewritten in terms of the non-dimensional coordinates \( \{t, \hat{\rho}, \hat{\chi}\} = \{t/\ell, \rho/\ell, \chi\} \), we find that the system of equa-
tions is characterized by a single parameter,

\[ L = \frac{\ell}{\rho_h} . \]

(5)

We use this parameter to specify the black hole solutions. In the numerical calculation, we have set \( \rho_h = 1 \) and 
varied \( \ell \) to specify the value of this parameter. The nu-
merically obtained solutions are characterized by \( L > 1 \), 
and then are small localized black holes. Note that vari-
ation of \( \ell \), keeping \( L \) fixed, corresponds to a rescaling of the length scale, because the assumed metric can be 
rewritten as

\[ ds^2 = \ell^2 ds^2_L , \]

where \( ds^2_L \) is the dimensionless part of the line element given by \( \{t, \hat{\rho}, \hat{\chi}\} \).

III. THERMODYNAMIC BEHAVIOR

Now we consider the 5D RS model. In the limit 
\( L = \ell/\rho_h \to \infty \) for fixed \( \rho_h \), the cosmological constant and 
the tension of the brane vanish, and hence the 5D Schwarzschild black hole is the solution in this limit:

\[ T_S = \frac{\rho^2}{\rho^2 + \rho^2_h} , \quad R_S = C_S = \log \left( 1 + \frac{\rho^2}{\rho^2} \right) . \]

(7)

Thus, we see that the 5D Schwarzschild black hole is an 
approximate solution of very small localized black holes, 
and for this reason it is interesting to observe the devia-
tions of the numerically obtained solutions from the 
5D Schwarzschild black hole. To quantify these deviations, we focus on thermodynamic quantities and eluci-
date thermodynamic properties of small localized black 
holes.

The surface gravity on the horizon, or the temperature, 
is determined by

\[ \kappa = e^{-RT/\rho} , \quad (\rho = \rho_h) \]

(8)

and the horizon volume, or entropy, of the black hole is 
given by

\[ A_\text{S} = 2\rho_h^3 \int d\Omega_2 \int_0^{(\pi/2)^2} d\xi \sin^2 \frac{\sqrt{\xi}}{2} \left( \frac{\ell}{\xi} \right)^3 e^{R+2C} . \]

(9)

Here, the factor 2 is due to \( Z_2 \) symmetry. The proper area 
of the intersection between the horizon and the brane is 
also an interesting quantity. It is given by

\[ A_4 = 4\pi \rho^2_h e^{2C} . \quad (\xi = (\pi/2)^2) \]

(10)

For comparison, we list the corresponding thermody-
namic quantities for the 5D Schwarzschild-AdS (SA) 
black hole and the black string (BS):

\[ \kappa_{SA} = \frac{1}{\ell^2} + \frac{2\omega}{\ell^2} , \]

\[ A_{SA} = 2\pi \rho^2_h e^{2C} , \quad A_{BS} = 2\pi \rho^2_h . \]
As mentioned above, variation of \( \mathcal{L} \) keeping \( L \) fixed corresponds to a rescaling of the length scale, and thus the length scales of the solutions differ. Hence, for comparison, we use non-dimensional combinations of the thermodynamic quantities. Note that \( \rho_h \) in our numerical calculation does not have a clear geometrical meaning, and thus it is not a relevant quantity to make a non-dimensional combination.

In Figs. 1 and 2 we display relations between thermodynamic quantities. Figure 1 depicts the relation between the 4D area \( A_4 \) and the surface gravity \( \kappa \), taking an appropriate non-dimensional combination. In the figure, the thermodynamic relations for the 4D and 5D Schwarzschild black hole are also plotted for comparison. We observe that for large \( L \), i.e., for very small black holes, the numerical solutions behave like the 5D Schwarzschild black hole solution, and then, as \( L \) becomes small, they deviate from that solution and tend to behave like the 4D Schwarzschild black hole solution. In Fig. 2 we see the same tendency in the relation between the 5D area and the surface gravity. When the horizon is sufficiently small (\( \kappa \ell \gg 1 \)), the result for the numerical solutions approximately coincides with the line corresponding to the 5D Schwarzschild(-AdS) black hole. However, as the horizon radius increases, the numerical solutions begin to deviate from this line and move to the direction along the black string.

The EHM solution \([21]\) is the exact black hole solution in the 4D braneworld. However, because of the low dimensionality, the black hole is not asymptotically AdS, but has a deficit angle, and it is uncertain whether the properties of the EHM solution reflect general characteristics of a black hole in the original 5D braneworld. Therefore it is interesting to compare the thermodynamic relations for the 5D localized black hole with the relations for the EHM solution. We find similar behavior of these thermodynamic relations for \( \ln \kappa \ell \gtrsim 0 \) in Fig. 3, and this shows that the 4D exact solution gives a good description of the qualitative behavior of the thermodynamic relations for the 5D solution, at least for small horizon radii. In addition, this seems to give insight into a method for extrapolating the thermodynamic relations to larger black hole solutions. However, for large black holes, there is also a big difference between the 5D model and the 4D model, as discussed in Refs. \([22]\) and \([24]\). On one hand, the metric induced on the brane for a large black hole in the 5D model is conjectured to mimic the 4D Schwarzschild metric with some small corrections. On the other hand, there are no 3D black hole solutions without a cosmological constant that correspond to the 4D Schwarzschild black hole in the 5D model. Thus, we need to be careful with speculation based on analogy to the EHM solution.

Let us consider the shape of a small localized black hole in the bulk. In KTN, we derived the ratio of the mean radius in four dimensions, \( \sqrt{A_4} \), (on the brane)
FIG. 3: Thermodynamic relation for the EHM solution in the 4D braneworld model [21] (see Fig. 2). The two dashed lines represent the small and large limits of the black hole, which are given by $A_4 \kappa^2 = \pi$ and $A_4 \kappa^2 = (2^{7/3} \pi / 3)(\kappa \ell)^{4/3}$, respectively.

FIG. 4: Ratio of the 4D entropy $S_4$ to the 3D entropy $S_3$ for the EHM solution (see Fig. 5). $\hat{x}$ specifies the mass parameter of the solution [21]. For comparison, we also plot the ratio for the 4D Schwarzschild black hole (dashed curve).

FIG. 5: Ratio of the 5D entropy $S_5$ to the 4D entropy $S_4$. The dashed curve represents the ratio for the 5D Schwarzschild black hole.

FIG. 6: The relation between the thermodynamic mass and the entropy ($L = 5 \sim 500$). The dashed line and the thin line represent the 5D Schwarzschild black hole and the black string, respectively.

ties, and it would be interesting to determine the mass of localized black holes. However, it is difficult to extract the mass asymptotically, for example on the brane, from numerical solutions. To obtain an estimate of the mass, we simply assume the first law of thermodynamics, represented by $dM = \kappa dA / 8 \pi G_5$, and we calculate the thermodynamic mass $M$ by integrating this quantity. In order to carry out this integration, we interpolated the thermodynamic quantities obtained for discrete values of $L$ up to $L = 5$. The smallest numerically obtained black hole ($L = 500$) was smoothly extrapolated to the 5D Schwarzschild black hole. Figure 6 displays the mass-entropy relation. For comparison, the mass-entropy relations for the black string and the 5D Schwarzschild black hole are also plotted. We see that the localized black hole ($L \gtrsim 5$) has greater entropy, at least in the plotted region, than both the black string and the 5D Schwarzschild black hole of the same mass. Extrapolating the curves, however, it seems that the entropy of the black string is likely to become larger than that of the localized black hole as the mass increases. If this is the
case, the localized black hole solution would seem to be unstable from the point of view of the entropy. Of course, we know that the black string in the RS braneworld is dynamically unstable, as long as we add a negative tension brane at some finite but sufficiently distant position $h/\ell < 1$, and therefore the larger entropy does not imply the dynamical stability of the black string. This observation might be suggestive, but we cannot give any conclusive statement about the dynamical stability of the localized black hole, because stability based on entropy does not lead directly dynamical stability.

IV. DISCUSSION

Physically acceptable black hole solutions that represent small black holes localized on the brane in the RS infinite braneworld model were constructed in an approximate manner using a numerical method in KTN. The horizon radius $\rho_h$ of the numerically obtained black holes is small compared to the bulk curvature scale $\ell$. In this paper, we have studied the thermodynamic properties of small localized black holes and found that the thermodynamic relations of the numerically obtained solutions deviate significantly from those expected on the basis of naive consideration of the 5D Schwarzschild(-AdS) black hole as its horizon radius increases (Fig. 4). Comparison of the thermodynamic quantities also suggests that the numerical solutions approach solutions whose induced metric on the brane behaves like that of the 4D Schwarzschild black hole (Fig. 1), although we could not obtain clear evidence for this from the direct comparison of the induced metric. Moreover, small localized black holes have greater entropy than a black string of the same thermodynamic mass (Fig. 4), and a localized black hole undergoes a shape transition in the bulk (Fig. 4 and see also KTN). A black hole tends to flatten as its horizon radius increases.

Although the EHM solution is the solution in lower dimensions, it is interesting to compare our solution with it. We have compared the thermodynamic relations of the EHM solution with those of the 5D numerical solutions and observed similarities between them (Figs. 3 and 4). This observation might be suggestive, but because our calculations are restricted to only small black hole solutions ($\rho_h/\ell < 1$), we cannot state a definite conclusion regarding large black hole solutions ($\rho_h/\ell \gtrsim 1$). Thus, it is an important problem to determine the nature of large black hole solutions. In particular, induced gravity and its deviation from the 4D Schwarzschild black hole at large distances are interesting.

Here we discuss the relation between our results and the classical black hole evaporation conjecture of Du et al. 22,24. Based on the AdS/CFT correspondence, it was argued that there might be no static black hole solution in the RS single brane model. It might be thought that the discovery of small black hole solutions contradicts this conjecture. However, there are two possible consistent scenarios. One is that only small black hole solutions exist in the strict sense. Because the classical evaporation conjecture applies only for sufficiently large black holes, for which the quantum correction in the AdS picture is negligible, the existence of small black hole solutions does not directly contradict the conjecture. However, the relations between thermodynamic quantities suggest that there is no critical point at which the sequence of solutions suddenly ceases to exist. Therefore, we think that this first possibility is less likely. The other possibility is that even small black hole solutions exist only in an approximate sense. Even if there is no static black hole solution in the strict sense, irrespective of its size, the internal inconsistency contained in the setup of the problem might be made irrelevant by numerical errors. If this is the case, the failure to find larger black hole solutions might be a signal of increased effect of the inconsistency. Of course, there also remains the possibility that the classical evaporation conjecture is incorrect. Unfortunately, however, we cannot state anything definite about these points at the moment, because the lack of convergence we encountered in the relaxation scheme could be a purely technical problem with the numerical scheme.

We have to this point discussed the 5D numerical solutions obtained in KTN. The same formulation and numerical method can be applied to obtain localized black holes in higher dimensions ($D \geq 5$). Numerical calculation in higher dimensions has an advantage: The metric in 6 dimensions, for example, dies away faster than that in 5 dimensions, and therefore an asymptotic boundary condition can be imposed at a location closer to the center of the simulation volume. This would reduce significantly the total lattice size, and hence make the numerical calculation more tractable. Actually, using the same numerical scheme, we have performed the calculation in 6 dimensions. The result of the numerical calculations shows that 6D localized black holes have thermodynamic properties similar to those of 5D black holes. Furthermore, we cannot again maintain the convergence of the calculation for $L \lesssim 1$, and only small localized black holes are found. There are many open questions regarding the features of localized black holes. To obtain a thorough understanding of localized black holes, it is crucially important to find large black holes, if they exist. In order to realize this, we need further development of investigational methods.

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