On Summability of Random Fourier-Jacobi Series
associated with Stable Process

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Abstract

Let $X(t, \omega)$, $t \in R$ be a symmetric stable process with index $\alpha \in (1, 2]$ and $a_n$ be the Fourier-Jacobi coefficients of $f \in L^p$, where $p \geq \alpha$. For $\gamma, \delta > 0$, $t \in [-1, 1]$, define $A_n(\omega) = \int_{-1}^{1} P_n^{(\gamma, \delta)}(t) \rho^{(\gamma, \delta)} dX(t, \omega)$ where $P_n^{(\gamma, \delta)}(t)$ are orthogonal Jacobi polynomials. The $A_n(\omega)$ exists in the sense of mean. In this paper, it is shown that the random Fourier-Jacobi series $\sum_{n=0}^{\infty} a_n A_n(\omega) P_n^{(\gamma, \delta)}(y)$ converges to the stochastic integral $\int_{-1}^{1} f(y, t) \rho^{(\gamma, \delta)} dX(t, \omega)$ in the sense of mean and the sum function is weakly continuous in probability if the index $\alpha \in (1, 2]$ and $f \in L^p$ where $p \geq \alpha$. However, it is shown that if the index $\alpha$ is one and $f$ is in the weighted space of continuous function $C^{(\eta, \tau)}(-1, 1)$, for $\eta, \tau \geq 0$, then the random Fourier-Jacobi series is $(C, 1)$ summable in probability to the stochastic integral $\int_{-1}^{1} f(y, t) \rho^{(\gamma, \delta)} dX(t, \omega)$.

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1 Introduction

The classical Jacobi polynomials plays an important role in theoretical as well as in applied mathematical analysis and also extensively used in practical applications [2, 11, 16, 18, 19]. The general Jacobi polynomials and their special cases are very helpful for applications in various directions. For example, they have been extensively useful in spectral methods for solving ordinary and partial differential equations of both integers and fractional orders [6, 1]. Now a days, the Jacobi polynomials are playing prominent role in spectral approximations mainly for problems with degenerated or singular coefficients [3]. Also, there has been vast studied on Fourier series using orthogonal Jacobi polynomials [7, 12].

P. Marian and T. Marian [11] studied on power series involving orthogonal polynomials which occur in some problems in quantum optics. They considered Gegenbauer or Laguerre polynomials multiplied by binomial coefficients are the coefficients of the power series. At the present days, the use of orthogonal polynomials in the power series in the field of physics is an active area of research [11]. Recently, we find application of random Fourier Transform in optics [8, 9] associated with Hermite polynomials. These works has motivated to look at random Fourier (RF) series involving orthogonal Jacobi polynomial.

The literatures on random Fourier-Stieltjes (RFS) series associated with stochastic processes provide a way to deal with RFS series involving the orthogonal polynomials. An extensive study on RFS series

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has also been done. It is known that, if $X(t, \omega)$, for $t \in R$ is a continuous stochastic process with independent increments and $f$ is a continuous function in $[a, b]$, then the stochastic integral $\int_0^h f(t)dX(t, \omega)$ is defined in the sense of probability and is a random variable (c.f. Lukacs [10] p. 148). Further, if $X(t, \omega)$ is a symmetric stable process of index $\alpha \in (1, 2]$, then the stochastic integral $\frac{1}{2\pi} \int_0^{2\pi} f(t)dX(t, \omega)$ is defined in the sense of probability for $f \in L^p[0, 2\pi]$, $p \geq \alpha$ (c.f. [14]). It is also defined in the sense of mean (c.f. [14]).

In particular,

$$A_n(\omega) = \int_1^- P_n^{(\gamma, \delta)}(t)\rho^{(\gamma, \delta)} dX(t, \omega)$$

exists, where $P_n^{(\gamma, \delta)}(t)$ are the orthogonal Jacobi polynomials associated with the Jacobi weight function $\rho^{(\gamma, \delta)} = \rho^{(\gamma, \delta)}(t) = (1-t)^\gamma (1+t)^\delta \geq 0$, for $\gamma, \delta > 0$ and $t \in [-1, 1]$. $A_n(\omega)$ are called the random Fourier-Jacobi coefficients of $X(t, \omega)$.

We know that the Fourier-Jacobi series $\sum_{n=0}^{\infty} a_n P_n^{(\gamma, \delta)}(t)$ converges to $f$ in $L^p$, for $p > 1$, if $a_n = \int_1^- f(t)P_n^{(\gamma, \delta)}(t)\rho^{(\gamma, \delta)} dt$ are the Fourier-Jacobi coefficients of $f$. This paper deals with random series of the type

$$\sum_{n=0}^{\infty} a_n A_n(\omega)P_n^{(\gamma, \delta)}(y),$$

where $A_n(\omega)$ are defined as in (1.1) and $P_n^{(\gamma, \delta)}(y)$ are the orthogonal Jacobi polynomials. The random series (1.2) is called as random Fourier-Jacobi (RFJ) series. The aim of this paper is to investigate its convergence, summability and continuity of the sum function and also shown that the mode of convergence of random Fourier-Jacobi series depends on the index $\alpha$ of the random variable $A_n(\omega)$ associated with the symmetric stable process $X(t, \omega)$. Theorem (3.3) establishes that the RFJ series (1.2) converges in mean to the stochastic integral $\int_1^- f(y, t)dX(t, \omega)$, if the random variable $A_n(\omega)$ are associated with stable process $X(t, \omega)$ of index $\alpha \in (1, 2]$ and $a_n$ are the Fourier-Jacobi coefficients of $f \in L^p[-1, 1]$, $p \geq \alpha$. However, if $a_n$ are the Fourier-Jacobi coefficients of $f \in C(\eta, \tau)(-1, 1)$, $\eta, \tau \geq 0$, where $C(\eta, \tau)(-1, 1)$ is a weighted space of continuous functions and random variables $A_n(\omega)$ are associated with stable process of index $\alpha = 1$, then the series (1.2) is $(C, 1)$ summable. (see theorem (3.7))

2 Definitions

2.1 Definition

A random function $f(t, \omega)$ is said to be weakly continuous in probability at $t = t_0$, if for all $\epsilon > 0$, $\lim_{h \to 0} P(|f(t_0 + h, \omega) - f(t_0, \omega)| > \epsilon) = 0$. If a function $f(t, \omega)$ is weakly continuous at every $t_0 \in [a, b]$, then the function $f(t, \omega)$ is weakly continuous in probability in a closed interval $[a, b]$.

2.2 Definition

A sequence of random variables $X_n$ converges in $(C, 1)$ probability to a random variable $X$ if

$$\lim_{n \to \infty} P(|Y_n - X| \geq \epsilon) = 0,$$

for all $\epsilon > 0$, where

$$Y_n = \frac{X_1 + X_2 + ... + X_n}{n}.$$
3 Results

To prove theorem we need following lemmas -:

**Lemma 3.1** Let, \( f(t) \) be any function in \( L^p[a, b] \), \( p \geq 1 \) and \( X(t, \omega) \) be a symmetric stable process of index \( \alpha \), for \( 1 \leq \alpha \leq 2 \), then for all \( \epsilon > 0 \),

\[
P\left( \left| \int_a^b f(t) dX(t, \omega) \right| > \epsilon \right) \leq \frac{C\epsilon^{\alpha+1}}{(\alpha+1)\epsilon^\alpha} \int_a^b |f(t)|^\alpha dt,
\]

where \( \epsilon' < \epsilon \) and \( C \) is a +ve constant.

**Lemma 3.2** If \( X(t, \omega) \) is a symmetric stable process with independent increment of index \( \alpha \), for \( 1 \leq \alpha \leq 2 \) and \( f \in L^p[a, b], p \geq \alpha \), then the following inequality hold:

\[
E\left( \left| \int_a^b f(t) dX(t, \omega) \right| \right) \leq \frac{4}{\pi(\alpha-1)} \int_a^b |f(t)|^\alpha dt + \frac{2}{\pi} \int_{|u|>1} \frac{1 - \exp\left(-|u|^\alpha \int_a^b |f(t)|^\alpha dt \right)}{u^2} du.
\]

It is known that \( \int_{-1}^1 f(t)\rho(\gamma, \delta) dX(t, \omega) \) is defined in mean for \( f \in L^p[-1, 1] \), for \( p \geq \alpha \), where \( \alpha \in (1, 2] \) and \( \rho(\gamma, \delta) \) is bounded. In particular, for \( f(t) = P_n(\gamma, \delta)(t) \), the stochastic integral \( \int_{-1}^1 P_n(\gamma, \delta)(t)\rho(\gamma, \delta) dX(t, \omega) \) exists and these integral denote as \( A_n(\omega) \).

**Theorem 3.3** Let, \( X(t, \omega), t \in R \) be a symmetric stable process of index \( \alpha \in (1, 2] \) and \( A_n(\omega) = \int_{-1}^1 P_n(\gamma, \delta)(t)\rho(\gamma, \delta) dX(t, \omega) \), where \( P_n(\gamma, \delta)(t) \) are Jacobi polynomials with weight function \( \rho(\gamma, \delta) = \rho(\gamma, \delta)(t) = (1-t)^\gamma(1+t)^\delta \) on \([-1, 1]\) with \( \gamma, \delta > 0 \). Then, (a) the RFJ series

\[
\sum_{n=0}^\infty a_n A_n(\omega) P_n(\gamma, \delta)(y)
\]

converges in the mean to the stochastic integral

\[
\int_{-1}^1 f(y, t)\rho(\gamma, \delta) dX(t, \omega), \quad (3.1)
\]

for some \( f \in L^p[-1, 1], p \geq \alpha \) where \( a_n \) are the Fourier-Jacobi coefficients of \( f \). (b) the sum function is weakly continuous in probability.

**Proof.** (a) Let,

\[
S_n(y, \omega) = \sum_{k=0}^n a_k A_k(\omega) P_k(\gamma, \delta)(y)
\]

be the nth partial sum of the RFJ series

\[
f(y, w) = \sum_{n=0}^\infty a_n A_n(\omega) P_n(\gamma, \delta)(y). \quad (3.2)
\]

Consider, the partial sum of Fourier-Jacobi series as

\[
f_n(t) = \sum_{k=0}^n a_k P_k(\gamma, \delta)(t)
\]
and it converges to \( f(t) \) (c.f. [13]).

Denote
\[
f_n(y, t) = \sum_{k=0}^{n} a_k P_k^{(\gamma, \delta)}(y) P_k^{(\gamma, \delta)}(t).
\]

Now
\[
S_n(y, \omega) = \sum_{k=0}^{n} a_k A_k(\omega) P_k^{(\gamma, \delta)}(y)
\]
\[
= \int_{-1}^{1} f_n(y, t) \rho^{(\gamma, \delta)} dX(t, \omega).
\]

By using lemma (3.2) we get,
\[
E \left( \left| \int_{-1}^{1} f(y, t) \rho^{(\gamma, \delta)} dX(t, \omega) - S_n(y, \omega) \right| \right)
\]
\[
= E \left( \left| \int_{-1}^{1} f(y, t) \rho^{(\gamma, \delta)} dX(t, \omega) - \int_{-1}^{1} f_n(y, t) \rho^{(\gamma, \delta)} dX(t, \omega) \right| \right)
\]
\[
\leq \frac{4}{\pi(\alpha - 1)} \int_{-1}^{1} \left| (f(y, t) - f_n(y, t)) \rho^{(\gamma, \delta)} \right|^{\alpha} dt
\]
\[
+ \frac{2}{\pi} \int_{|u|>1} \frac{1 - \exp \left( - |u|^{\alpha} \int_{-1}^{1} \left| (f(y, t) - f_n(y, t)) \rho^{(\gamma, \delta)} \right|^{\alpha} dt \right)}{u^2} du. \tag{3.3}
\]

It is known that (c.f. Zygmund [20] p. 266)) for \( f \in L^p[-1, 1] \), \( p > 1 \),
\[
\lim_{n \to \infty} \int_{-1}^{1} \left| (f(t) - s_n(t)) \right|^{p} dt = 0,
\]
where \( s_n(t) \) is the FJ series of \( f \).

Further, the integrand in the second integral right hand side of (3.3) is dominated by the integrable function \( \frac{1}{u^2} \) in the interval \(( -\infty, -1] \) and \([1, \infty) \).

Since, the weight function \( \rho^{\gamma, \delta} \) of the Jacobi polynomial is bounded, for \( \gamma, \delta \geq 0 \).

Also,
\[
\lim_{n \to \infty} \int_{-1}^{1} \left| (f(t) - f_n(t)) \rho^{(\gamma, \delta)} \right|^{p} dt = 0,
\]
where \( f_n(t) \) is the partial sum of the FJ series of \( f(t) \).

All these implies,
\[
E \left( \left| \int_{-1}^{1} f(y, t) \rho^{(\gamma, \delta)} dX(t, \omega) - S_n(y, \omega) \right| \right)
\]
converges to 0 as \( n \to \infty \), for \( p \geq \alpha \) which proves (a).

(b) The RFJ series (3.2) converges in mean, then converges in probability for \( \alpha \in (1, 2] \).
Now, by using lemma (3.1)
\[ P \left( \left| \int_{-1}^{1} f(x,t)\rho^{(\gamma,\delta)}dX(t,\omega) \right| - \int_{-1}^{1} f(y,t)\rho^{(\gamma,\delta)}dX(t,\omega) \right| > \epsilon \right) \]
\[ \leq \frac{C^{2\alpha+1}}{(\alpha + 1)\epsilon^\alpha} \int_{-1}^{1} \left| \left( f(x,t) - f(y,t) \right)\rho^{(\gamma,\delta)} \right|^\alpha dt, \]
which converges to 0 as \( n \to \infty \) (c.f. Zygmund [20, p. 37]). This confirms the weak continuity of the sum function of the RFJ series (3.2).

(C.1) Summability of RFJ series
To show Cesaro summability of RFJ series we define weighted spaces of continuous functions as follows—
for \( \eta, \tau \geq 0 \), consider
\[ C^{(\eta,\tau)} = C^{(\eta,\tau)}(-1,1) = \{ f \in C(-1,1) | \lim_{|t| \to 1} (f^{(\eta,\tau)}(t)) = 0 \}. \]
If \( \eta = 0, \tau > 0 \) or \( (\eta > 0, \tau = 0) \), then \( C^{(\eta,\tau)} \) denote the spaces of continuous functions on \((-1,1)\) for which
\[ \lim_{t \to 1} (f^{(\eta,\tau)}(t)) = 0 \quad \text{or} \quad \lim_{t \to -1} (f^{(\eta,\tau)}(t)) = 0. \]
In the case \( \eta = \tau = 0 \) i.e. \( \rho^{(\eta,\tau)} = 1 \), then \( C^{(\eta,\tau)} = C(-1,1) \).
\( C^{(\eta,\tau)} \) is a linear space over \( R \) the norm defined on it as
\[ ||f||_{\infty,(\eta,\tau)} = ||f||^{(\eta,\tau)}_{\infty} = \sup_{|t| \leq 1} |(f^{(\eta,\tau)})(t)|, \]
and \( C^{(\eta,\tau)} = (C^{(\eta,\tau)}, ||\cdot||_{\infty,(\eta,\tau)}) \) is a Banach space.
Let, \( \theta \) be the summation matrix defined as
\[ \theta = \begin{bmatrix} \theta_{0,1} & \theta_{1,2} & \theta_{2,3} \\ \theta_{0,1} & \theta_{1,2} & \theta_{2,3} \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix} \]
where \( \theta_{k,n} \)'s are real numbers.
Let, the \( n \)th \( \theta \)-sum of the Fourier-Jacobi series of a function \( f \in C^{(\eta,\tau)} \) be
\[ f_n^\theta(y) = \sum_{k=0}^{\infty} \theta_{k,n} a_k P_k^{(\gamma,\delta)}(y), \quad y \in [-1,1], \quad n \in N. \quad (3.4) \]
Consider, the following conditions
\( T_1 : \lim_{n \to +\infty} (1 - \theta_{k,n}) = 0, \quad \text{for all fixed } k \in N. \)
\( T_2 : \theta_{n-1,n} = O\left(\frac{1}{n}\right), \quad n \in N. \)
\( T_3 : \triangle^2 \theta_{k-1,n} = O\left(\frac{1}{n^2}\right), \quad (k = 1, 2, \ldots, n - 1, n \in N). \)
\( T_4 : \triangle^2 \theta_{k-1,n}(k = 1, 2, \ldots, n - 1, n \in N) \text{ is a constant sign.} \)
\( T_5 : sgn \triangle^2 \theta_{k-1,n} = sgn \theta_{n-1,n}(k = 1, 2, \ldots, n - 1, n \in N), \)
Lemma 3.6 Suppose that 

\[ \triangle^2 \theta_{k,n} = \triangle \theta_{k+1,n} - \triangle \theta_{k,n}, \quad \triangle \theta_{k,n} = \theta_{k+1,n} - \theta_{k,n}, \quad (\theta_{n,n}) = 0. \]

Use the notations 

(\Xi_1) if (T_1), (T_2) and (T_3) hold, 

(\Xi_2) if (T_1), (T_2) and (T_3) hold, 

(\Xi_3) if (T_1), (T_3) hold.

Chirpi [4] has studied the convergence of the FJ series 

\[ \sum_{k=0}^{n} \theta_{k,n} a_k P_k^{(\gamma,\delta)}(y), \]

where \( a_k \) are the FJ coefficients of \( f \in C^{(\eta,\tau)} \), which is stated in the following lemma.

**Lemma 3.4** [4] Suppose that \( \gamma, \delta \geq \frac{1}{2}, \) and \( \eta, \tau \geq 0 \) satisfy the inequalities

\[ \frac{\gamma}{2} - \frac{1}{4} < \eta < \frac{\delta}{2} + \frac{3}{4} \quad \text{and} \quad \frac{\delta}{2} - \frac{1}{4} < \tau < \frac{\delta}{2} + \frac{3}{4}. \]

Then (\( \Xi_1 \)) or (\( \Xi_2 \)) or (\( \Xi_3 \)) imply

\[ \lim_{n \to +\infty} ||f - f_n^{(\eta)}(y)||_{\infty,(\eta,\tau)} = 0, \]

for all \( f \in C^{(\eta,\tau)} \).

Let us, look into random Fourier-Jacobi series

\[ \sum_{k=0}^{\infty} \theta_{k,n} a_k A_k(\omega) P_k^{(\gamma,\delta)}(y), \]

where \( a_k \) are the FJ coefficient of \( f \in C^{(\eta,\tau)}(-1,1) \) and \( A_k \) are defined as in (3.1). The following theorem establishes that the series (3.6) converges in probability to the stochastic integral

\[ \int_{-1}^{1} f(y,t) \rho^{(\gamma,\delta)} dX(t,\omega). \]

**Theorem 3.5** Let, \( X(t,\omega), t \in R \) be a symmetric stable process of index \( \alpha = 1 \) and \( A_n(\omega) = \int_{-1}^{1} P_n^{(\gamma,\delta)}(t) \rho^{(\gamma,\delta)} dX(t,\omega) \), where \( P_n^{(\gamma,\delta)}(t) \) are Jacobi polynomials with weight function \( \rho^{(\gamma,\delta)} = \rho^{(\gamma,\delta)}(t) = (1-t)^\gamma (1+t)^\delta \) on \([-1,1]\) with \( \gamma, \delta > 0 \). Then, the RFJ series of (3.6) converges in the probability to the stochastic integral

\[ \int_{-1}^{1} f(y,t) \rho^{(\gamma,\delta)} dX(t,\omega), \]

for some \( f \in C^{(\eta,\tau)}(-1,1), \eta, \tau \geq 0 \).

**Proof.** It is similar to the proof of the theorem (3.3) by using lemma (3.4) and lemma (3.1) stated below.

The following lemma is required to prove the (\( C,1 \)) summability of the series (3.6).

**Lemma 3.6** [4] Suppose \( \gamma, \delta \geq \frac{1}{2} \) and \( \eta, \tau \geq 0 \) satisfy the conditions (3.3). Then the Cesaro summations \( (C,\mu), \mu \geq 1 \) are uniformly convergent in the space \( C^{(\eta,\tau)} \).

Now we will prove the (\( C,1 \)) summability of the RFJ series (3.6).

**Theorem 3.7** Let, \( X(t,\omega) \) be a stable process of index \( \alpha = 1 \), with \( A_n(\omega) \) and \( a_n \) have the same meaning as in theorem (3.3), with \( f \in C^{(\eta,\tau)} \). Suppose that \( \gamma, \delta \geq \frac{1}{2} \) and \( \eta, \tau \geq 0 \) satisfy the inequalities (3.3) and (\( \Xi_1 \)) or (\( \Xi_2 \)) or (\( \Xi_3 \)), then the series (3.6) is (\( C,1 \)) summable to the stochastic integral \( \int_{-1}^{1} f(y,t) \rho^{(\gamma,\delta)} dX(t,\omega) \) in probability.
Proof. Let, denote the $n$th sum of the RFJ series of a function $f \in C^{(\eta,\tau)}$ as

$$S_{n}^{\theta}(y, \omega) = \sum_{k=0}^{\infty} \theta_{k,n} a_k(\omega) F_k^{(\gamma,\delta)}(y),$$

and

$$\sigma_{n}^{'}(y, \omega) = \frac{S_{0}^{\theta}(y, \omega) + S_{1}^{\theta}(y, \omega) + \ldots + S_{n-1}^{\theta}(y, \omega)}{n},$$

where $f_{n}^{\theta}(t)$ has the same meaning in equation (3.5).

Consider the partial sum of the RFJ series of $f \in C^{(\eta,\tau)}$ as

$$S_{n}^{\theta}(y, \omega) = \int_{-1}^{1} f_{n}^{\theta}(y, t) \rho^{(\gamma,\delta)} dX(t, \omega),$$

and we obtain that

$$\sigma_{n}^{'}(y, \omega) = \int_{-1}^{1} \sigma_{n}(y, t) \rho^{(\gamma,\delta)} dX(t, \omega).$$

Hence,

$$\sigma_{n}^{'}(y, \omega) - \sigma_{m}^{'}(y, \omega) = \int_{-1}^{1} \left( \sigma_{n}(y, t) - \sigma_{m}(y, t) \right) \rho^{(\gamma,\delta)} dX(t, \omega)$$

For taking $\alpha = 1$, in lemma (3.1) we get

$$P \left( \left| \sigma_{n}^{'}(y, \omega) - \sigma_{m}^{'}(y, \omega) \right| > \epsilon \right) \leq \frac{2C}{\epsilon'} \int_{-1}^{1} \left| \left( \sigma_{n}(y, t) - \sigma_{m}(y, t) \right) \rho^{(\gamma,\delta)} \right| dt.$$

It is know from (c.f. Zygmund [20, p. 144]) that

$$\lim_{n,m \to \infty} \int_{-1}^{1} \left| \sigma_{n}(t) - \sigma_{m}(t) \right| dt = 0,$$

and

$$\lim_{n \to \infty} \int_{-1}^{1} \left| \sigma_{n}(y, t) - f(y, t) \right| dt = 0,$$

where $f_{n}$ and $\sigma_{n}$ are the $n$th partial sum and Cesaro sum of the FJ series of $f$, respectively.

Hence, by using lemma (3.6)

$$\lim_{n,m \to \infty} \int_{-1}^{1} \left| \left( \sigma_{n}(y, t) - \sigma_{m}(y, t) \right) \rho^{(\gamma,\delta)} \right| dt = 0,$$

and

$$\lim_{n \to \infty} \int_{-1}^{1} \left| \left( \sigma_{n}(y, t) - f(y, t) \right) \rho^{(\gamma,\delta)} \right| dt = 0,$$

where $f_{n}$ and $\sigma_{n}$ are the $n$th partial sum and Cesaro sum of the RFJ series of $f$ respectively. Since, $f \in C^{(\eta,\tau)}(-1, 1)$ gives that

$$\lim_{n,m \to \infty} \int_{-1}^{1} \left| \left( \sigma_{n}(y, t) - \sigma_{m}(y, t) \right) \rho^{(\gamma,\delta)} \right| dt = 0,$$
which implies the convergence of $\sigma_n$ in probability. Further, by using lemma (3.1) we have

\[
P \left( \left| \sigma'_n(y, \omega) - \int_{-1}^{1} f(y, t) \rho^{(\gamma, \delta)} dX(t, \omega) \right| > \epsilon \right) \\
= P \left( \left| \int_{-1}^{1} \left( \sigma_n(y, t) - f(y, t) \right) \rho^{(\gamma, \delta)} dX(t, \omega) \right| > \epsilon \right) \\
\leq \frac{2C}{\epsilon} \int_{-1}^{1} \left| \left( \sigma_n(y, t) - f(y, t) \right) \rho^{(\gamma, \delta)} \right| dt,
\]

where $\sigma_n(y, t)$ is the $n$th partial sum of the RFJ series of $f$ which belongs to $C^{(\eta, \tau)}(-1, 1)$, we obtain

\[
\lim_{n \to \infty} \int_{-1}^{1} \left| \left( \sigma_n(y, t) - f(y, t) \right) \rho^{(\gamma, \delta)} \right| dt = 0,
\]

which show that Cesaro sum of the series convergence in probability to the stochastic integral $\int_{-1}^{1} f(y, t) \rho^{(\gamma, \delta)} dX(t, \omega)$. Hence the proof of the theorem is complete.

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