NONLINEAR \( \alpha \)-EFFECT IN DYNAMO THEORY

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ABSTRACT

The standard two-scale theory of the dynamo coefficient \( \alpha \) in incompressible isotropic helical MHD turbulence is extended to include nonlinear effects of \( \mathbf{B} \), the large-scale magnetic field. We express \( \alpha \) in terms of statistical quantities that can be calculated from numerical simulations of the case \( \mathbf{B} = 0 \). For large magnetic Reynolds numbers, our formula agrees approximately with that of Kraichnan but disagrees with that of Cattaneo & Hughes.

Subject headings: MHD — turbulence

1. INTRODUCTION

Magnetic fields of galaxies are important in astrophysics and cosmology. In astrophysics, they are important because they enable fast particles to be accelerated and trapped and affect the dynamics of star formation. In cosmology, they are important because, if galactic magnetic fields do not originate in the modern era, they could be relics from the early universe, carrying information about that period.

Since the Parker (1955) paper on the \( \alpha \)-\( \Omega \) dynamo and his application of dynamo theory to the Galaxy (Parker 1971), most workers have attributed the origin of the magnetic fields of disk galaxies to the operation of an \( \alpha \)-\( \Omega \) turbulent dynamo (Ruzmaikin, Shukurov, & Sokoloff 1988). However, the standard theory of the dynamo has always been open to criticism. For example, Piddington (1970, 1972a, 1972b, 1972c, 1975a, 1975b) argued that the small-scale magnetic field produced by the small-scale turbulence required by the theory would rapidly grow to equipartition, quenching dynamo action, and this point has been demonstrated numerically by Kulsrud & Anderson (1992).

Recently a series of papers (Vainshtein & Rosner 1991; Cattaneo & Vainshtein 1991; Vainshtein & Cattaneo 1992; Tao, Cattaneo, & Vainshtein 1993; Cattaneo 1994; Cattaneo & Hughes 1996; see also Seehafer 1994, 1995) has argued that dynamo action is quenched if the large-scale magnetic field \( \mathbf{B} \) in velocity units exceeds a critical value, \( B_c = R_M^{-1/2}v_0 \), where \( R_M = v_0L/\lambda \) is the magnetic Reynolds number of the turbulence, \( v_0 \) is the turbulent velocity at the outer scale of the turbulence \( L \), and \( \lambda = \eta c^2/4\pi \) is the magnetic diffusivity, with \( \eta \) the resistivity. They argue that this result is supported by direct numerical simulations of MHD incompressible turbulence (Cattaneo & Hughes 1996). If they are correct, the classical \( \alpha \)-\( \Omega \) dynamo theory based upon the Parker (1955) paper and developed by the Potsdam group (Krause & Rädler 1980; see also Moffatt 1978; Parker 1979), which applies to weak large-scale magnetic fields, is not applicable to present-day galaxies, as \( \mathbf{B} \) is observed to exceed \( R_M^{-1/2}v_0 \) greatly.

The argument by Cattaneo and his collaborators depends upon their analysis of nonlinear interactions, which are surely present. The effects of such interactions have also been considered independently in a series of papers on MHD turbulence by Pouquet and her collaborators (Frisch et al. 1975; Pouquet, Frisch, & Léorat 1976; Pouquet & Patterson 1978; Meneguzzi, Frisch, & Pouquet 1981) using spectral methods. By and large this work supports the applicability of the classical theory even for large-scale fields, with \( \mathbf{B} \) approaching \( v_0 \), which encompass those observed. Chandran (1997) confirms the results of Pouquet et al. (1976) by use of a different spectral method.

Since \( R_M > 1 \) in most astrophysical situations (as large as \( 10^{20} \) in the interstellar medium of our Galaxy), the difference between the critical field advocated by Cattaneo and his collaborators, \( B_c \sim R_M^{-1/2}v_0 \), and that implied by Pouquet and her collaborators, \( B_c \sim v_0 \), is crucial.

In this paper we extend the calculation of one of the dynamo coefficients, \( \alpha \), of the classical theory to include arbitrary values of \( \mathbf{B} \). Our result contains two terms. One resembles that advocated by Cattaneo and his collaborators in its dependence upon \( R_M \). The other resembles a formula proposed by Kraichnan (1979) on the basis of a simple model incorporating damping by nonlinear interactions. For \( R_M > 1 \), the latter term dominates, so we find that the classical result for \( \alpha \) applies for \( \mathbf{B} \) of the order of \( v_0 \).

We plan to extend the simulations of Cattaneo and his collaborators to other values of \( R_M \) in order to clarify our disagreement with them.

We confine discussion to the alpha effect and omit any discussion of turbulent diffusion, which is also controversial.

Before proceeding to the nonlinear theory, we review the classical linear theory. The classical theory is based on a clear separation of scales between the scale \( L \) of the dominant turbulent motions (here called the \( " outer scale" \)) and the size \( D \) of the system, with \( L \lesssim D \). The large-scale magnetic field \( \mathbf{B} \) satisfies an induction equation,

\[
\partial_t \mathbf{B} = \nabla \times (\mathbf{V} \times \mathbf{B}) + \nabla \times \langle \mathbf{v} \times \mathbf{b} \rangle + \lambda \nabla^2 \mathbf{B},
\]

where \( \mathbf{V} \) is the large-scale velocity field (differential rotation in the case of a galaxy), and \( \langle \mathbf{v} \times \mathbf{b} \rangle \), called the \( " \)turbulent emf\( " \), is the spatial average of the cross product of the small-scale velocity \( \mathbf{v} \) and the small-scale magnetic field \( \mathbf{b} \) over a scale much smaller than \( D \) but much larger than \( L \).

The first term in equation (1) leads to the so-called \( " \)\( \Omega \)-effect\( " \), according to which lines of force of \( \mathbf{B} \) are stretched by the differential rotation, creating a growing toroidal field from a poloidal one. This term is not controversial and will not be discussed further here. Controversy centers on the turbulent emf, \( \langle \mathbf{v} \times \mathbf{b} \rangle \), which enables a growing toroidal field to feed back into the poloidal direction, giving exponential amplification of the large-scale \( \mathbf{B} \).
The evaluation of $\langle v \times b \rangle$ is usually restricted to incompressible isotropic turbulence. On might assume that if the turbulent velocity field $v$ is isotropic, then the small-scale magnetic field $b$ would be isotropic also, so that the $c$-component of the emf would be

$$\langle v \times b \rangle_c = \epsilon_{cde}v_d b_e \epsilon_{cde} = \frac{1}{3} \epsilon_{cde} \delta_{de} \langle v \cdot b \rangle = 0$$  \hspace{1cm} (2)$$

in the light of an identity for isotropic tensors (Krause & Rädler 1980). However, this is not correct, because $\tilde{B}$, being anisotropic, induces a term in $b$ that is not isotropic even though $v$ is isotropic. Evaluating this term requires the induction equation for $b$, which can be found by writing the induction equation for the total magnetic field $B = \tilde{B} + b$ and separating off the small-scale parts to give

$$\partial_t b = \nabla \times (v \times \tilde{B}) + \nabla \times (\tilde{V} \times b) + \nabla \times (v \times b) + \lambda \nabla^2 b.$$  \hspace{1cm} (3)$$

The second term on the right-hand side in equation (3) represents a change of the reference frame to that moving at the mean velocity $\bar{V}$, in which $v$ may be assumed to be isotropic; we assume $V = \text{const.}$, so we make that change and henceforth omit the term. The fourth term on the right-hand side is a large-scale quantity, so its time integral, being a large-scale quantity, will contribute nothing when crossed with $v$ (small scale) and averaged and so is neglected henceforth. The first term is $-\partial_t \tilde{B} + \nabla \times (\tilde{V} \times b)$. As the $v - \nabla b$ term ultimately leads to turbulent diffusion of $\tilde{B}$, which is not our main interest here, we omit it, leaving

$$\partial_t b = \tilde{B} \cdot \nabla v + \nabla \times (v \times b) + \lambda \nabla^2 b.$$  \hspace{1cm} (4)$$

The second term in equation (4) is neglected in the classical discussions in a step referred to as the first-order smoothing approximation (FOSA). Krause & Rädler (1980) showed that this is legitimate if $b$ is large, so $R_M$ is small, because then Ohmic diffusion keeps $b$ small and the first term dominates the second. However, because $R_M$ is large in astrophysics, this case is not relevant here. They also discuss the case when the Strouhal number

$$S = \frac{v_0 \tau}{L}$$  \hspace{1cm} (5)$$

is small, where $\tau$ is the correlation time for eddies at the outer scale of the turbulence. Because $L/v_0 = \tau_{ed}$, the eddy turnover time, $S = \tau_{ed}$, and one might suppose that $S = O(1)$. In fact, it is observed experimentally that $S \approx 0.2$–0.3 in ordinary hydrodynamic turbulence (Pope 1994). Although this value is not as small as one would like, it provides a way to approximate equation (4). To understand how, it is important to distinguish between $b_0$, the small-scale magnetic field when $B = 0$, and $b^{(1)}$, the perturbation to $b$ when a small $\bar{B}$ is present. If the turbulent velocity field $v$ is isotropic, $b_0$ is, and as shown in equation (2), its contribution to $\langle v \times b \rangle$ vanishes. However, $b^{(1)}$ is not isotropic, and its contribution to $\langle v \times b \rangle$ does not vanish.

If we write $b = b_0 + b^{(1)}$, $b^{(0)}$ is governed by

$$\partial_t b^{(0)} = \nabla \times (v^{(0)} \times b^{(0)}) + \lambda \nabla^2 b^{(0)},$$  \hspace{1cm} (6)$$

where $v^{(0)}$ is the isotropic turbulent velocity. Parker (1979; p. 511) shows that if $b$ is small (the astrophysical case), equation (6) leads to exponential growth of $b^{(0)}$ with a time constant $\tau_{S} S^2$ until (in velocity units) it begins to approach $v^{(0)}$. At that point, we expect that back reaction due to the Lorentz force associated with the large value of the small-scale field $b^{(0)}$ will result in a steady state in which the energy driving the turbulence at the outer scale (buoyancy forces in stars, supernova explosions in galaxies) is balanced by a nonlinear cascade to smaller scales, where it is dissipated by viscosity or Joule heating. This expectation is confirmed by Pouquet et al. (1976) and by direct simulations (see, e.g., Cattaneo & Vainshtein 1991; Cattaneo 1994; Cattaneo & Hughes 1996). As we explain later, we will take this steady state as the base state, which is perturbed by $\tilde{B}$. The first term in equation (6) mediates exchange of magnetic energy with kinetic energy in MHD turbulence, as discussed in Appendix A, and it certainly cannot be neglected.

However, we now argue that the corresponding term in the equation for $b^{(1)}$ can indeed be neglected, as follows. Evidently $b^{(1)}$ is governed by the parts of equation (4) that are of first order in $\tilde{B}$, namely

$$\partial_t b^{(1)} = \tilde{B} \cdot \nabla v^{(0)} + \nabla \times (v^{(0)} \times b^{(1)}) + \lambda \nabla^2 b^{(1)},$$  \hspace{1cm} (7)$$

In the rest of this section, we assume that $R_M \gg 1$, so that we may neglect the third term. We formally integrate equation (7) to get

$$b^{(1)}(x, t) = \tilde{B} \cdot \nabla \int_{-\infty}^{t} dt' v^{(0)}(x, t_1) + \tilde{B} \cdot \nabla \int_{-\infty}^{t} dt' v^{(0)}(x, t_1) \times b^{(1)}(x, t_1).$$  \hspace{1cm} (8)$$

This does not seem useful, because the desired quantity, $b^{(1)}$, appears under the integral. However, we can show that the integral is much smaller than $b^{(1)}$ itself, which appears on the left-hand side, if $S \ll 1$.

Since $v^{(0)}$ is a stochastic function of $t$, only those parts of the integral in equation (8) that come from times $t_1$ that differ from $t$ by less than a correlation time $\tau$ will correlate significantly with $v^{(0)}$ in the turbulent emf, $\langle v^{(0)}(x, t_1) \times b^{(1)}(x, t_1) \rangle$. Hence we can replace the lower limit on the integral with $t - \tau$ and then estimate the integral by $\tau$ times the integrand at $t_1 = t$. Since the order of magnitude of $V$ is $L^{-1}$, the order of magnitude of the second term in equation (8) is

$$\frac{\tau v_0^{(0)}}{L} b^{(1)}(t) = S b^{(1)}(t),$$  \hspace{1cm} (9)$$

whose ratio to the magnitude of the left-hand side of equation (8) is $S$. If, as we shall argue later, $S < 1$, we can neglect the second term in equation (8), and hence in equation (7), so, still neglecting diffusion, equation (7) becomes

$$\partial_t b^{(1)} = \tilde{B} \cdot \nabla v^{(0)}.$$  \hspace{1cm} (10)$$

Effectively $b^{(1)} \sim S \tilde{B}$, so if $S < 1$, $b^{(1)} < \tilde{B}$, even when $\tilde{B} \ll v^{(0)}$. However this says nothing about $b^{(0)}$, which as we have stressed, approaches $v^{(0)}$ in value, even if $v^{(0)} \gg \tilde{B}$. Hopefully this discussion clarifies a point that has led to confusion in the past.

The solution to equation (10) can be written in component form as

$$b^{(1)}(t) = \tilde{B} \cdot \nabla \int_{-\infty}^{t} dt' v^{(0)}(t'),$$  \hspace{1cm} (11)$$

so

$$\langle v^{(0)} \times b^{(1)} \rangle = \epsilon_{cde} \langle v^d(t) \tilde{B} \cdot \nabla \int_{-\infty}^{t} dt' v^{(0)}(t') \rangle.$$  \hspace{1cm} (12)$$

Averaging commutes with integration and differentiation,
spatial differentiation commutes with time integration, and $\dot{B}$ is independent of time on the scale $\tau$, so
\begin{equation}
\langle \mathbf{v} \times \mathbf{b} \rangle_\tau = \dot{B}_p \epsilon_{cde} \int_{-\infty}^{\tau} dt' \langle \mathbf{v}_d^{(0)}(t') \mathbf{c}_p \mathbf{v}_e^{(0)}(t') \rangle .
\end{equation}

(13)

Since $\mathbf{v}^{(0)}$ correlates with its derivatives for a time of order $\tau$, the integral is of order $\tau$ times the average taken at a given time (Krause & Rädler 1980). To make further progress in Radler the integral is of order $t$ and $t'$, because the turbulence is assumed to be steady, it can depend only on $t-t'$, so a convenient representation is
\begin{equation}
\langle \mathbf{v}_d^{(0)}(t) \mathbf{c}_p \mathbf{v}_e^{(0)}(t') \rangle = \langle \mathbf{v}_d^{(0)} \mathbf{c}_p \mathbf{v}_e^{(0)} \rangle e^{-t-t'/\tau},
\end{equation}

(14)

where the common time argument in the second average has been omitted because the turbulence is presumed steady. Hence $\langle \mathbf{v} \times \mathbf{b} \rangle$ is given by
\begin{equation}
\langle \mathbf{v} \times \mathbf{b} \rangle_\tau = \dot{B}_p \epsilon_{cde} \langle \mathbf{v}_d^{(0)} \mathbf{c}_p \mathbf{v}_e^{(0)} \rangle .
\end{equation}

(15)

At this point we use the assumption that the velocity $\mathbf{v}^{(0)}$ is distributed isotropically. According to Krause & Rädler (1980), a third-rank isotropic tensor like that in equation (15) can be written
\begin{equation}
\langle \mathbf{v}_d^{(0)} \mathbf{c}_p \mathbf{v}_e^{(0)} \rangle = \frac{1}{3} \epsilon_{dpe} \langle \mathbf{v}^{(0)} \cdot \mathbf{v}^{(0)} \rangle ,
\end{equation}

(16)

so from equation (13),
\begin{equation}
\langle \mathbf{v} \times \mathbf{b} \rangle = \dot{z} \dot{B} ,
\end{equation}

(17)

where
\begin{equation}
z = -\frac{1}{3} \tau \langle \mathbf{v}^{(0)} \cdot \mathbf{v} \times \mathbf{v}^{(0)} \rangle
\end{equation}

(18)

is the classical expression for the dynamo coefficient (Krause & Rädler 1980; Moffatt 1978), but with the additional feature that the turbulent velocities indicated refer to the zero-order state. The quantity in angular brackets, a pseudoscalar, is known as the kinetic helicity of the turbulent flow $\mathbf{v}^{(0)}$.

In an independent development, Pouquet et al. (1976) calculated magnetic energy spectra for MHD turbulence, solving the spectral equations using a closure method known as the eddy-damped quasi-normal Markovian (EDQNM) approximation. They found that if $\langle \mathbf{v}^{(0)} \cdot \mathbf{v} \times \mathbf{v}^{(0)} \rangle$ vanishes, the magnetic energy spectrum $E_k^{E}$ peaks near $k_0$, the wave number at which turbulent energy is injected, and reaches a steady state in which the total energy $E_k$, the sum of the kinetic energy $E_k^{V}$ and the magnetic energy $E_k^{M}$, cascades to higher wave numbers, ultimately to be lost to Ohmic dissipation and/or viscosity.

If, on the other hand, $\langle \mathbf{v}^{(0)} \cdot \mathbf{v} \times \mathbf{v}^{(0)} \rangle \neq 0$, they found that $E_k^{E}$ inverse cascades, accumulating at an ever-decreasing wave number. They attributed this to the turbulent dynamo effect described above, operating in the nonlinear regime. In the special case in which the wave number $k$ of interest is $\ll k_0$, so that there is a clear separation of scales, they find an approximate expression for a quantity $\alpha$, governing the growth of $E_k^{E}$, where
\begin{equation}
\alpha = -\frac{4}{3} \int_{k_0}^{\infty} dq \theta_{kq} \left( H_q^{V} - q^2 H_q^{M} \right) .
\end{equation}

(19)

Here $\alpha$ is a small parameter, $\theta_{kq}$ is an effective correlation time for modes of wave number $q$, and $H_q^{V}$ and $q^2 H_q^{M}$ are the spectra corresponding to the kinetic helicity correlation function $\langle \mathbf{v}^{(0)}(x) \cdot \mathbf{v} \times \mathbf{v}^{(0)}(x + \xi) \rangle$ and the current (Keinigs 1983) helicity correlation function $\langle \mathbf{b}^{(0)}(x) \cdot \mathbf{v} \times \mathbf{b}^{(0)}(x + \xi) \rangle$, respectively. [Here $\mathbf{b}$ is in velocity units, obtained by dividing $\mathbf{b}$ by $(4\pi \rho)^{1/2}$, so that $\mathbf{b}$ is the vector Alfvén velocity.] The term adopted by Pouquet et al. for equation (19), “residual torsality,” has not reappeared in the literature. Here we note that the first term in equation (19) is similar to equation (18), so that classical dynamo theory can be interpreted as an inverse cascade in helical turbulence. The second term does not appear in the classical result, but Montgomery & Chen (1984) verified it using calculations in real rather than $k$ space. Pouquet et al. thus found that dynamo action takes place in the fully nonlinear regime, with no restriction as to the magnitude of $\mathbf{b}$ or $\dot{B}$, and that the classical expression for $\alpha$ must be modified by the addition of the second term in equation (19).

2. THE METHOD OF THIS PAPER

In this paper we extend the classical analysis of the $\alpha$ effect into the regime of large $\dot{B}$. We use the standard two-scale approximation in real space, and at several points we refer to the results obtained by Pouquet et al. (1976) using the spectral closure method. We find that when a clear distinction is made between the quantities $\mathbf{v}$ and $\mathbf{b}$ on the one hand, and their values when $\dot{B} = 0$, $\mathbf{v}^{(0)}$ and $\mathbf{b}^{(0)}$ on the other, nonlinearity due to $\dot{B}$ appears in a straightforward manner. A key feature of our derivation is the assumption that correlations are damped by nonlinear interactions, following Pouquet et al. (1976) and Kraichnan (1979). As shown by Kichanov (1985), such an assumption can be justified by application of renormalization group methods.

Our results depend upon the damping rate at each wave number $k$, $\gamma_k$, the spatial spectra of $\mathbf{v}^{(0)}$ and $\mathbf{b}^{(0)}$, the value of $\dot{B}$, and the value of $\alpha$, the magnetic diffusivity, expressed in terms of the magnetic Reynolds number $R_M$, which for convenience we assume to be equal to the Reynolds number $R$. We find that $\alpha$ is a well-behaved function of the parameters and that the behavior for larger $R_M$ is similar to that predicted by Kraichnan (1979).

We take as our base state fully developed MHD turbulence driven by external forces and in a steady state as a result of a turbulent cascade to large wave numbers, but with $\dot{B} = 0$. As demonstrated by Pouquet et al. (1976), in such turbulence there is approximate equipartition between the kinetic energy $E_k^{V}$ and the kinetic energy $E_k^{V}$ for wave numbers $k \approx 3k_0$, where $L = k_0^{-1}$ is the outer scale of the turbulence. As shown by Parker (1979, p. 513), approach to the steady state occurs on the scale of the eddy turnover time $t_{ed} = L/\mathbf{v}^{(0)}$, which, as stressed by Kulssrud & Anderson (1992), is much shorter than the dynamo growth time. According to the calculations of Pouquet et al., the saturation that occurs at small scales (large $k$) does not prevent the increase of magnetic energy on scales larger than $L$ if the turbulence is helical. It is important to note that the growth of a large-scale field as a consequence of an $\alpha$ effect does not substantially modify the spectra of $E_k^{E}$ calculated by Pouquet et al. (1976) for $k$ of the order of $k_0$ (their Fig. 8), so our concept of a base state independent of $\dot{B}$ is valid. We assume that the properties of the base state can be calculated once and for all by numerical simulations. Our results then allow us to calculate $\alpha$ in terms of those properties.

Following Montgomery & Chen (1984), we present our calculations in terms of the Elsässer variables $\mathbf{z} = \mathbf{v} \pm \mathbf{b}$, (Biskamp 1993) and have checked our results by carrying out the calculation in terms of $\mathbf{v}$ and $\mathbf{b}$. Elsässer variables are naturally adapted to the problem, shortening the calcu-
lution substantially. More importantly, we show in Appendix A that unlike the kinetic energy $E^k$ associated with $v$ and the magnetic energy $E^m$ associated with $b$, both $E^k$ and $E^m$ cascade directly in isotropic turbulence, allowing us to employ a single damping constant $\gamma_k = \gamma_m = \gamma_k$ to describe the effects of the nonlinear terms in the base state.

Our goal is to calculate the turbulent emf, $\langle v \times b \rangle$. Since

$$ v = \frac{1}{2} (z^+ + z^-) \quad (20) $$
and

$$ b = \frac{1}{2} (z^+ - z^-) , \quad (21) $$

we have

$$ \langle v \times b \rangle_e = \frac{1}{2} \langle (z^+ + z^-) \times (z^+ - z^-) \rangle_e = - \frac{1}{2} \langle \delta_{cde} z_d^+ z_e^- \rangle . \quad (22) $$

We use this formula in what follows.

3. EVOLUTION OF THE TURBULENT FIELDS

As explained above, the correlation indicated in equation (22) vanishes for isotropic turbulence but is nonzero when one takes into account the perturbations of $z^+$ and $z^-$ that are caused by $B$. To obtain these, we consider the dynamical equations for $z^\pm$, or equivalently, $v$ and $b$.

When Ohmic dissipation is included, the induction equation for $b$ is, from equation (3),

$$ \partial_t b = - v \cdot \nabla B + B \cdot \nabla v - \nabla \times \mathbf{V} + b \cdot \nabla v + \lambda \nabla^2 b , \quad (23) $$

where $\lambda$ is the magnetic diffusivity. As explained in §1, adopting a frame of reference moving with $\mathbf{V}$ eliminates the third term on the right, and the fourth term can be neglected with respect to the sixth because we assume that the size of the system $D \gg L$. The remaining terms can be written in terms of $B = \mathbf{B} + b$ as

$$ \partial_t b = - v \cdot \nabla B + B \cdot \nabla v + \lambda \nabla^2 b . \quad (24) $$

The classical theory ignores the effect of $B$ on the velocity, on the grounds that the Lorentz force associated with $B$ is of order $B^2$ and hence is negligible in the limit $B \rightarrow 0$. However, as we have explained above, $b^{(0)}$ grows quickly to approximate equipartition, so even in a first-order calculation, a Lorentz force proportional to $B b^{(0)}$ must be included. It is therefore essential to consider the effect of $B$ on $v$; we shall do so to all orders in $B$. To do this, we use the equation of motion for the small-scale velocity $v$:

$$ \partial_t v = - v \cdot \nabla v - \nabla p + B \cdot \nabla B - \nabla \frac{1}{2} v^2 + \nu \nabla^2 v + f , \quad (25) $$

where $f$ is the applied force per unit mass and $v$ is the kinematic viscosity; $B$, the magnetic field divided by $(4\pi\rho)^{1/2}$, is in velocity units. Hence

$$ \partial_t v = - v \cdot \nabla v - \nabla p + B \cdot \nabla B + \nu \nabla^2 v + f , \quad (26) $$

where

$$ P = p + \frac{1}{2} B^2 . \quad (27) $$

We define the Elsässer variables for the field $B$ as

$$ Z^\pm = v \pm B = v \pm b \pm \mathbf{B} = z^\pm + \mathbf{B} , \quad (28) $$

so that

$$ v = \frac{1}{2} (Z^+ + Z^-) \quad (29) $$

and

$$ B = b + \mathbf{B} = \frac{1}{2} (Z^+ - Z^-) . \quad (30) $$

Then equation (24) becomes

$$ \frac{1}{2} \partial_t Z^+ - \frac{1}{2} \partial_t Z^- = - \frac{1}{2} \nabla (Z^+ + Z^-) \cdot \nabla (Z^+ - Z^-) + \frac{1}{2} \lambda \nabla^2 (Z^+ - Z^-) . \quad (31) $$

(Note that we have set $\partial_t \mathbf{B} = 0$ because $B$ varies only on the long timescale.) Equation (26) becomes

$$ \frac{1}{2} \partial_t Z^+ + \frac{1}{2} \partial_t Z^- = - \frac{1}{2} \nabla (Z^+ + Z^-) \cdot \nabla (Z^+ + Z^-) + \frac{1}{2} \lambda \nabla^2 (Z^+ - Z^-) + \nu \nabla^2 (Z^+ + Z^-) - \nabla P + f . \quad (32) $$

Adding and subtracting equations (31) and (32) yields

$$ \partial_t Z^\pm = - Z^\mp \cdot \nabla Z^\pm + \frac{1}{2} (\nu + \lambda) \nabla^2 Z^\pm + \frac{1}{2} \nu \nabla^2 Z^\pm - \nabla P + f . \quad (33) $$

The dissipative coupling between $Z^+$ and $Z^-$ vanishes in the special case in which the magnetic Prandtl number $\lambda/\nu$ equals unity. For simplicity we assume that is the case in what follows; we note that both Pouquet et al. (1976) and Cattaneo & Hughes (1996) also made the same assumption. Chou & Fish (1999) discuss the case $\lambda/\nu \neq 1$.

From equation (28) the nonlinear term in equation (33) is

$$ Z^\mp \cdot \nabla Z^\pm = (z^\mp \mp B) \cdot \nabla (z^\pm \pm B) = z^\mp \cdot \nabla z^\pm \mp B \cdot \nabla z^\pm , \quad (34) $$

where we have neglected $\nabla B$ for the reasons given previously. Hence equation (33) becomes

$$ \partial_t z^\pm = - z^\mp \cdot \nabla z^\pm + \lambda \nabla^2 z^\pm - \nabla P + f \pm B \cdot \nabla z^\pm , \quad (35) $$

where we have neglected $\partial_t \mathbf{B}$ for the reason given previously.

We adopt a perturbation expansion in $B$, in which the zero-order variables $z^{(0)}$ describe the turbulence exactly if $B$ is zero. Therefore $z^{(0)}$ satisfies

$$ \partial_t z^{(0)} = - z^{(0)} \cdot \nabla z^{(0)} + \lambda \nabla^2 z^{(0)} - \nabla P^{(0)} + f , \quad (36) $$

where

$$ P^{(0)} = P^{(0)} + \frac{1}{2} B^2 \big|_{B=0} = P^{(0)} + \frac{1}{2} (b^{(0)})^2 . \quad (37) $$

If we assume that the driving force is independent of $B$, $f$ has the same value in both the zero-order and the perturbed state.

Equation (36) can be solved numerically for a variety of initial conditions, and so in principle any averages of zero-order quantities required can be computed. Our task is then to compute $\langle v \times b \rangle$, which depends on the perturbations induced by $B$, in terms of averages over zero-order quantities.

We let

$$ z^\pm = z^{(0)} + z^\pm , \quad (38) $$

where $z^\pm$ contains perturbations of all orders in $B$. Although in principle $z^\pm$ can be represented as a power series in $B$, we find that it is not necessary to do so for the special case $\lambda/\nu = 1$. If we eliminate $f$ between equations (35) and (36), we find that

$$ (\partial_t - \lambda \nabla^2) z^\pm = - z^\mp \cdot \nabla z^\pm - z^{(0)} \cdot \nabla z^\pm - z^\mp \cdot \nabla z^\pm - \nabla (P - P^{(0)}) \pm B \cdot \nabla z^\pm . \quad (39) $$
Inspecting equation (39), we see that the left-hand side and the term \( \vec{B} \cdot \nabla z^\pm \) on the right are linear in \( z^\pm \) and hence are easy to deal with. If \( R_M \) is moderately large, we can ignore the term \( \mathcal{N} \|
abla z^\pm \) in assessing the order of magnitude of the remaining terms. Putting aside the pressure term for the moment, we can write equation (39) in the symbolic form

\[
\partial_t z' = L^{-1} \left[ (z')^2 \frac{\partial z'}{\partial (z^0)}, \vec{B}, \frac{\partial z'}{\partial (z^0)} \right],
\]

where the commas separate terms of potentially different orders and where we have combined the first two terms on the right-hand side of equation (39) into one. As in our discussion of the classical case, we take \( \Delta z = \int dt \, \partial_t z' \), the change in \( z' \) after one correlation time \( \tau \), to equal \( z' \) in order of magnitude, by definition. Then equation (40) implies that

\[
z' = \frac{\tau}{t_{ed}} \left[ (z')^2 / z^0, \vec{B}, \frac{\partial z'}{\partial (z^0)} \right],
\]

where \( t_{ed} \equiv L/z^0 \approx L/u^0 \).

In the classical discussion it is assumed that because \( \vec{B} \) is small, back reaction by the Lorentz force can be neglected, so the motions are hydrodynamic in nature. That allowed us to use the fact that \( \tau/t_{ed} \) is small (see below). That is not really true, because, as we discussed earlier, the zero-order state quickly approaches equipartition. We want to argue, however, that even in the nonlinear case \( \tau/t_{ed} \) is a small number. To do so, we appeal to the nonlinear calculations of Pouquet et al. (1976) regarding the zero-order state. They showed that the magnetic energy spectrum \( E_k^M \) is in equipartition with the kinetic energy spectrum \( E_k^M \) for \( k \gtrsim 3k_0 \), where \( k_0 \) is the wave number at which kinetic energy is being injected (the outer scale of the turbulence). In this range, back reaction is a major effect, causing the turbulence to become a field of interacting Alfvén waves. However, these large wave numbers need not concern us, because in such waves, the current is perpendicular to the small-scale field and the vorticity is perpendicular to the small-scale velocity, so the two helicities vanish.

Of crucial importance for us, Pouquet et al. (1976) found that in the range \( k_0 \lesssim k \lesssim 3k_0 \), \( E_k^M < E_k^P \), so the motions are largely hydrodynamic in character, with only a modest back reaction of the magnetic field. This range is important because it contains most of the energy. Pope (1994) states that, in a pure hydrodynamic turbulence, it is experimentally observed that

\[
\frac{\tau}{t_{ed}} = 0.2–0.3.
\]

We assume that equation (42) applies to the energy-containing eddies in MHD turbulence driven like that of Pouquet et al. Although \( \tau/t_{ed} \) is not a very small number, we may take it to be a small parameter \( \epsilon \) in analyzing equation (41). Thus, equation (41) becomes

\[
z' = \epsilon \left[ (z')^2 / z^0, \vec{B}, \frac{\partial z'}{\partial (z^0)} \right].
\]

It seems reasonable to assume that the balance in equation (43) is between \( z' \) and the third term on the right-hand side, so

\[
z' \sim \epsilon \vec{B}.
\]

We check this by evaluating the first two terms under that assumption. We see that the ratio of the first to the third term and the ratio of the second to the fourth term are both \( \epsilon \), so the first and second terms can be neglected as a first approximation. Note, however, that the ratio of the fourth to the third term is

\[
z' / z^0 = \epsilon \vec{B} / z^0 \sim \epsilon \vec{B} / \bar{z}^0,
\]

which cannot be neglected because, although \( \epsilon \) is small, we want a result that is valid to all orders in \( \vec{B}/\bar{z}^0 \). Indeed, this is the source of the nonlinearity in our calculation.

Note that although we have used the short correlation time approximation to simplify our equations for \( z' \), no assumption is made regarding the ratio of \( b \) to \( \vec{B} \). This is a step forward, because the classical discussion has been justly criticized for ignoring back reaction, which amounts to assuming that \( b \) is small.

We now apply the divergence operator \( \nabla \cdot \) to equation (39) with the first three terms on the right-hand side dropped. Because \( \vec{B} \) is constant and \( \nabla \cdot \vec{B} \) commutes with \( \vec{B} \cdot \nabla \), the fact that

\[
\nabla \cdot z^\pm = \nabla \cdot (v \pm b) = 0
\]

for incompressible turbulence then implies that

\[
\nabla^2 (P - P(0)) = 0,
\]

which for a homogeneous system implies that

\[
(P - P(0)) = \text{const}.,
\]

Hence equation (39) becomes

\[
(\partial_t - \lambda \nabla^2 \mp \vec{B} \cdot \nabla) z^\pm = \pm \vec{B} \cdot \nabla z^\pm(0).
\]

We must solve this equation in order to calculate the turbulent emf according to equation (22).

The operator \( \vec{B} \cdot \nabla \) is best handled by introducing the spatial Fourier transform

\[
\tilde{z}^\pm(k, t) = (2\pi)^{-3} \int dx \, e^{-ik \cdot x} z^\pm(x, t),
\]

so that, when written in terms of the components of \( z \), equation (49) becomes

\[
(\partial_t + \lambda k^2 \mp ik \cdot \vec{B}) \tilde{z}^\pm(k, t) = \pm ik \cdot \vec{B} \tilde{z}^\pm(0)(k, t),
\]

to which the solution is

\[
\tilde{z}^\pm(k, t) = \pm ik \cdot \vec{B} \int_{-\infty}^{t} dt_1 \exp \left\{ -[\pm ik \cdot \vec{B} + \lambda k^2](t - t_1) \right\} \tilde{z}^\pm(0)(k, t_1),
\]

where we have put the lower limit equal to \(-\infty\) because we will find that the short correlation time makes values of \( t_1 \) significantly smaller than \( t \) irrelevant. The nonlinear dependence on \( \vec{B} \) is evident here.

From equations (22) and (38), the term of zero order in \( z^\pm \) drops out according to equation (2), leaving

\[
\langle v \times b \rangle_c = -\frac{1}{2} \langle \epsilon_{cde} z^d(0)(x, t) z^e(0)(x, t) \rangle
\]
\[
- \frac{1}{2} \langle \epsilon_{cde} z^d(0)(x, t) z^e(0)(x, t) \rangle
\]
\[
- \frac{1}{2} \langle \epsilon_{cde} z^d(0)(x, t) z^e(0)(x, t) \rangle.
\]
Inverting the Fourier transforms, we see from equation (50) that
\[ z_d^{(0)}(x, t) = \int dk \, e^{ik \cdot x} \tilde{z}^{(0)}(k, t), \tag{54} \]
and, from equation (52),
\[ z_e^{\pm}(x, t) = \pm i \int dk \, e^{ik \cdot x} \tilde{z}^{(0)}(k, t) dt_1 \times \exp \left\{ -[\mp ik \cdot B + \lambda k^2](t - t_1) \right\} \tilde{z}_e^{\pm}(k', t_1). \tag{55} \]
Hence the first term in equation (53) is
\[ \frac{1}{2} i \epsilon_{cde} \int dk \int dk' \, e^{ik + k'} \cdot \tilde{z}^{(0)}(k', t) \times \exp \left\{ -[ik' \cdot B + \lambda k'^2](t - t_1) \right\} \tilde{z}_e^{(0)}(k', t_1). \tag{56} \]
Because averaging commutes with integration, this can be written
\[ \frac{1}{2} i \int dk \int dk' \, e^{ik + k'} \cdot \tilde{z}^{(0)}(k', t) \times \exp \left\{ -[ik' \cdot B + \lambda k'^2](t - t_1) \right\} \epsilon_{cde} \tilde{z}_d^{(0)}(k, t) \tilde{z}_e^{(0)}(k', t_1). \tag{57} \]
The indicated correlation decreases rapidly to zero as \( k - k' \) and \( t - t_1 \) go to zero. In Appendix B we show that the following representation is a reasonable one:
\[ \langle \epsilon_{cde} \tilde{z}_d^{(0)}(k, t) \tilde{z}_e^{(0)}(k', t_1) \rangle = \epsilon_{cde} R_d(-k) \delta(k + k') e^{-\gamma_k |t - t_1|}. \tag{58} \]
In principle, \( \gamma_k \) could be different for \( z^+ \) and \( z^- \) modes, but we show in Appendix A that in turbulence in which the cross helicity \( K = 1/2 \langle \mathbf{v} \cdot \mathbf{B} \rangle \), an ideal invariant, vanishes, the associated energies \( E^+ \) and \( E^- \) cascade directly and are equal. It is therefore reasonable to assume that if \( K = 0 \), \( \gamma_+ = \gamma_- \). Following Pouquet et al. we assume that \( K = 0 \) and thus that \( \gamma_+ = \gamma_- \). Note that because \( \gamma_k \) is a parameter of the zero-order state, it does not depend upon \( B \). From Krause & Rädler (1980, p. 75) an isotropic tensor like \( R_d(k) \) can be expressed in the form
\[ R_d(k) = \langle A - k^{-1} \tilde{\partial}_k B \rangle \delta_{de} - \langle \tilde{\partial}^2_k B - k^{-1} \tilde{\partial}_k B \rangle k^2 d_k e + i k^{-1} \tilde{\partial}_k C \epsilon_{def} k_f, \tag{59} \]
where \( A, B, \) and \( C \) are functions of \( k \) alone. Therefore
\[ \epsilon_{cde} R_d(-k) = -2ik^{-1} \tilde{\partial}_k C \tag{60} \]
is nonvanishing only for turbulence lacking reflection symmetry, so equation (58) becomes
\[ \langle \epsilon_{cde} \tilde{z}_d^{(0)}(k, t) \tilde{z}_e^{(0)}(k', t_1) \rangle = -2ik^{-1} \tilde{\partial}_k C e^{-\gamma_k |t - t_1|}. \tag{58} \]
The integration over \( k' \) converts \( k' \) to \(-k\), so \((k + k') \cdot x = 0\). Since \(|t - t_1| = t - t_1\) in the inte-
so
\[
\int_{-\infty}^{t} dt_1(\gamma) = \left(\frac{1}{ik \cdot B - \lambda k^2}\right) \left(\frac{1}{ik \cdot B - \lambda k^2 - \gamma_k}\right),
\] (69)
where \((\ )\) refers to the function following \(dt_1\) in equation (67). This expression can be written in the equivalent form
\[
\int_{-\infty}^{t} dt_1(\gamma)
= \gamma_k \left[ \frac{ik \cdot B + \lambda k^2}{(k \cdot B)^2 + (\lambda k^2 + \gamma_k)} - \frac{ik \cdot B + (\lambda k^2 + \gamma_k)}{(k \cdot B)^2 + (\lambda k^2 + \gamma_k)^2} \right],
\] (70)
so equation (67) becomes
\[-i \int dk \left( k \cdot \mathbf{B} \right)^2 k \cdot \hat{\epsilon}_k C \gamma_k^{-1} \]
\[
\times \left[ \frac{ik \cdot B + \lambda k^2}{(\lambda k^2)^2 + (k \cdot B)^2} - \frac{ik \cdot B + (\lambda k^2 + \gamma_k)}{(\lambda k^2 + \gamma_k)^2 + (k \cdot B)^2} \right].
\] (71)
We show below that the contributions of the terms of order \(B^2\) in the numerator vanish, so equation (71) becomes
\[
\int dk \left( k \cdot \mathbf{B} \right)^2 k \cdot \hat{\epsilon}_k C \gamma_k^{-1}
\times \left[ \frac{1}{(\lambda k^2)^2 + (k \cdot B)^2} - \frac{1}{(\lambda k^2 + \gamma_k)^2 + (k \cdot B)^2} \right].
\] (72)
Combining equations (72) and (64) yields the following expression for equation (53):
\[
\langle \mathbf{v} \times \mathbf{b} \rangle_c = -2 \int dk k \cdot \hat{\epsilon}_k C
\times \left[ \frac{k \cdot \mathbf{B}(\lambda k^2 + \gamma_k)}{(\lambda k^2 + \gamma_k)^2 + (k \cdot B)^2} - \frac{1}{2}(k \cdot \mathbf{B})^3 \gamma_k^{-1} \right]
\times \left[ \frac{1}{(\lambda k^2)^2 + (k \cdot B)^2} - \frac{1}{(\lambda k^2 + \gamma_k)^2 + (k \cdot B)^2} \right].
\] (73)
To proceed, we adopt a coordinate system \((1, 2, 3)\) such that
\[
\mathbf{B} = (\hat{B}, 0, 0);
\] (74)
has only one component, \(\hat{B}_1 = B\), so that
\[
k \cdot \mathbf{B} = k_1 \hat{B}.
\] (75)
Hence
\[
\langle \mathbf{v} \times \mathbf{b} \rangle_c = -2 \int dk k \cdot \hat{\epsilon}_k C
\times \left[ \frac{k_1 \hat{B}(\lambda k^2 + \gamma_k)}{(\lambda k^2 + \gamma_k)^2 + (k_1 \hat{B})^2} - \frac{1}{2}(k_1 \hat{B})^3 \gamma_k^{-1} \right]
\times \left[ \frac{1}{(\lambda k^2)^2 + (k_1 \hat{B})^2} - \frac{1}{(\lambda k^2 + \gamma_k)^2 + (k_1 \hat{B})^2} \right].
\] (76)
Note that \(\langle \mathbf{v} \times \mathbf{b} \rangle\) vanishes if \(\hat{B} = 0\); as explained earlier, finite \(\hat{B}\) is required to break rotational symmetry.
In our coordinate system
\[
dk = k^2 \sin \theta \, d\theta \, d\phi \, dk,
\] (77)
where \(\theta = \angle \mathbf{n}\), and \(\varphi\) is the azimuth in the 2-3 plane. If \(c = 2\) or 3 in equation (76), \(k = k \sin \theta \sin \varphi\) or \(k \sin \theta \cos \varphi\), and as the rest of the integrand is independent of \(\varphi\), the integral over \(\varphi\) vanishes. We are left with \(c = 1\), in which case the integrand is an even function of \(k_1\). The integral over \(\varphi\) gives \(2\pi\), so
\[
dk \rightarrow -2\pi k^2 \, dk \, d(\cos \theta) = -2\pi k \, dk \, dk_1,
\] (78)
where we have changed variables from \((k, \theta)\) to \((k, k \cos \theta) = (k, k_1)\). As the integral over \(k_1\) is from \(k\) to \(-k\), and the integrand is even in \(k_1\),
\[
dk \rightarrow 4\pi k \, dk \, dk_1,
\] (79)
with \(k_1\) varying from 0 to \(k\). As claimed previously, the terms of order \(B^2\) in equation (71) would have contributed terms of order \(k^3\) to the integrand in equation (72); as this is odd in \(k_1\), their contribution would have vanished when integrated from \(-k\) to \(k\). Hence
\[
\langle \mathbf{v} \times \mathbf{b} \rangle_c = -2 \int_{k=0}^{k} \int_{k_1=0}^{k_1} dk \, k \cdot \hat{\epsilon}_k C
\times \left\{ \frac{k_1 \hat{B}(\lambda k^2 + \gamma_k)}{(\lambda k^2 + \gamma_k)^2 + (k_1 \hat{B})^2} - \frac{1}{2}(k_1 \hat{B})^3 \gamma_k^{-1} \right\}
\times \left[ \frac{1}{(\lambda k^2)^2 + (k_1 \hat{B})^2} - \frac{1}{(\lambda k^2 + \gamma_k)^2 + (k_1 \hat{B})^2} \right].
\] (80)
Hence \(\langle \mathbf{v} \times \mathbf{b} \rangle\) is parallel to \(\mathbf{B}\) as in the result of the classical theory, equation (17), and the dynamo coefficient \(\alpha\) is
\[
\alpha = -8\pi \int_{k=0}^{k} \int_{k_1=0}^{k_1} \frac{1}{2}k_1^2 \eta_1 \gamma_k^{-1} \left[ \frac{1}{(\lambda k^2)^2 + (k_1 \hat{B})^2} - \frac{1}{(\lambda k^2 + \gamma_k)^2 + (k_1 \hat{B})^2} \right].
\] (81)
To carry out the integration over \(k_1\) in equation (81), in the first term in braces we set
\[
\xi = \frac{k \hat{B}}{\lambda k^2 + \gamma_k},
\] (82)
so that
\[
(\lambda k^2 + \gamma_k) \int_{k_1=0}^{k_1} dk_1 \frac{k_1^2}{(\lambda k^2 + \gamma_k)^2 + (k_1 \hat{B})^2}
= k^3(\lambda k^2 + \gamma_k)^{-1} \frac{\xi - \tan^{-1} \xi}{\xi^3}.
\] (83)
In the first term in square brackets we set
\[
\eta = \frac{k \hat{B}}{\lambda k^2},
\] (84)
so that
\[
-\frac{1}{2}B^2 \gamma_k^{-1} \int_{0}^{k} \frac{k_1^4 \, dk_1}{(\lambda k^2)^2 + (k_1 \hat{B})^2}
= -\frac{1}{2} \gamma_k^{-1} k^3 \left( \frac{1}{3} \eta^3 - \eta + \tan^{-1} \eta \right). \] (85)
In the second term in square brackets, we again use equation (82), so that
\[
\frac{1}{2} B^2 \gamma_k^{-1} \int_{0}^{k} \frac{k^4 dk}{(\lambda k^2 + \gamma_k)^2 + (k_1 B)^2} = \frac{1}{2} \gamma_k^{-1} k^3 \left( \frac{1}{3 \xi^3} - \xi + \tan^{-1} \frac{1}{\xi} \right). \tag{86}
\]

Combining equations (83), (85), and (86), we find that in equation (81)
\[
\int_{k_1 = 0}^{k} dk_1 {\cal J} \left\{ \frac{k^4}{(\lambda k_1^2 + \gamma_k)^2} \right\} = \frac{1}{2} k^3 \gamma_k^{-1} \left[ \frac{\gamma_k - \lambda k^2}{\gamma_k + \lambda k^2} f(\xi) + f(\eta) \right], \tag{87}
\]
where
\[
f(\xi) = \frac{\xi - \tan^{-1} \frac{1}{\xi}}{\xi^3}, \tag{88}
\]
and the braces (“ { }”) refer to the term in braces in equation (81). Hence equation (81) becomes
\[
\alpha = -\int_{k = 0}^{\infty} 4\pi k^2 dk \hat{k} C \gamma_k^{-1} \left[ \frac{\gamma_k - \lambda k^2}{\gamma_k + \lambda k^2} f(\xi) + f(\eta) \right]. \tag{89}
\]
Since the integrand is a function of k alone, 4\pi k^2 dk = dk and this can be written
\[
\alpha = -\int dk \hat{k} C \gamma_k^{-1} \left[ \frac{\gamma_k - \lambda k^2}{\gamma_k + \lambda k^2} f(\xi) + f(\eta) \right]. \tag{90}
\]

To interpret the quantity k\hat{k} C, we multiply the expression on the left-hand side of equation (58), taken for t_0 = t, by ik to get
\[
\left\langle \epsilon_{e\delta e} \dot{z}_e^{+(0)}(x) ik' \dot{z}_e^{-}(0)(k') \right\rangle = -\left\langle \dot{z}_e^{+(0)}(k)[ik' \times \dot{z}_e^{-}(0)(k')] \right\rangle = -\left\langle \dot{z}_e^{+(0)}(k) \cdot \dot{w}^{-}(0)(k) \right\rangle, \tag{91}
\]
where
\[
\dot{w}^{-}(0)(x) \equiv \nabla \times \dot{z}^{-}(0)(x). \tag{92}
\]
From Leslie (1973; eq. [2.16]), equation (91) equals
\[
-\hat{h}(k) \delta(k + k'), \tag{93}
\]
where \hat{h}(k) is the transform of h(\xi), the correlation function
\[
h(\xi) = \left\langle z_e^{+(0)}(x) \cdot w^{-}(0)(x + \xi) \right\rangle. \tag{94}
\]
From equation (61) with t_0 = t, we see that equation (91) is also equal to
\[
\left\langle \epsilon_{e\delta e} \dot{z}_e^{+(0)}(k) ik' \dot{z}_e^{-}(0)(k') \right\rangle = -ik_1 (2ik_1 k^{-1} \partial \delta C) \delta(k + k')
\]
\[
= -2k \partial \delta C \delta(k + k'). \tag{95}
\]
Equating equation (93) with equation (95), we see that
\[
k \partial \delta C = \frac{1}{2} \hat{h}(k), \tag{96}
\]
so that equation (90) becomes
\[
\alpha = -\frac{1}{2} \int dk \hat{h}(k) \gamma_k^{-1} \left[ \frac{\gamma_k - \lambda k^2}{\gamma_k + \lambda k^2} f(\xi) + f(\eta) \right], \tag{97}
\]
which is our principal result, where \hat{h}(k), f, \xi, and \eta are defined by equations (94), (88), (82), and (84), respectively.

To compare equation (97) with the classical result equation (18), we take the limit \bar{B} \rightarrow 0, which according to equations (82) and (84) corresponds to \xi \rightarrow 0, \eta \rightarrow 0, in which case \int f(\xi) = \int f(\eta) = \frac{1}{2}, and the term in square brackets in equation (97) becomes
\[
\frac{1}{2} \left( \frac{\gamma_k - \lambda k^2}{\gamma_k + \lambda k^2} + 1 \right) = \frac{1}{2} \frac{\gamma_k}{\gamma_k + \lambda k^2}, \tag{98}
\]
so
\[
\alpha(\bar{B} \rightarrow 0) = -\frac{1}{2} \int dk \hat{h}(k) (\gamma_k + \lambda k^2)^{-1}. \tag{99}
\]
As the integral of the power spectrum over k is the correlation at zero lag,
\[
\int dk \hat{h}(k) = \left\langle \zeta_e^{+(0)}(x, t) \cdot \nabla \times \zeta_e^{-}(0)(x, t) \right\rangle
\]
\[
= \left\langle \langle v^{(0)} \cdot \nabla \rangle \cdot \nabla \times \langle v^{(0)} \rangle \right\rangle - \left\langle \langle b^{(0)} \cdot \nabla \rangle \cdot \nabla \times \langle b^{(0)} \rangle \right\rangle
\]
\[
+ \left\langle \nabla \cdot \langle v^{(0)} \rangle \right\rangle \left\langle \nabla \times \langle b^{(0)} \rangle \right\rangle
\]
\[
= \left\langle \langle v^{(0)} \rangle \cdot \nabla \times \langle v^{(0)} \rangle \right\rangle - \left\langle \langle b^{(0)} \rangle \cdot \nabla \times \langle b^{(0)} \rangle \right\rangle, \tag{100}
\]
because \langle v^{(0)} \cdot \nabla \rangle = 0 according to equation (2); here we have suppressed the arguments for clarity.

From equation (99) we conclude that if \lambda is small (but not too small—see below), and in the special case in which \gamma_k = \gamma_0 is independent of k,
\[
\alpha(\bar{B} \rightarrow 0) = -\frac{1}{2} \int dk \hat{h}(k) (\gamma_0)^{-1}
\]
\[
= -\frac{1}{2} \gamma_0^{-1} \left[ \langle v^{(0)}(x, t) \cdot \nabla \rangle \cdot \langle v^{(0)}(x, t) \rangle - \langle b^{(0)}(x, t) \cdot \nabla \rangle \cdot \langle b^{(0)}(x, t) \rangle \right]. \tag{101}
\]
The first term agrees with classical result (eq. [18]) if the damping constant \gamma_0 is identified with \tau^{-1}. The second term has the sign and form expected from the work of Pouquet et al. (1976), as indicated in equation (19).

Our more general result for \bar{B} \rightarrow 0, equation (99), differs from the classical result in three ways:

1. a term proportional to the current helicity is included in the form derived by Pouquet et al. (1976);
2. allowance is made for variation of the damping constant with k;
3. and the effect of finite \lambda is included.

In the opposite limit \bar{B} \rightarrow \infty, f(u) \rightarrow u^{-2}, so the term in square brackets in equation (97) becomes
\[
\frac{\gamma_k - \lambda k^2}{\gamma_k + \lambda k^2} \left( \frac{1}{u^2} + \frac{1}{\gamma_k + \lambda k^2} \right) = \frac{\gamma_k}{\gamma_k + \lambda k^2}, \tag{102}
\]
independent of \lambda. Thus,
\[
\alpha(\bar{B} \rightarrow \infty) = -\frac{1}{2\bar{B}^2} \int dk \hat{h}(k) k^{-2} \gamma_k. \tag{103}
\]
An inverse dependence on \bar{B}^2 was cited by Krause & Rädler (1980), but with a different factor.

Finally, we consider the simplification introduced when the spectral density C(k) is effectively concentrated at some wave number k_*$, which if \kappa_1 \sim 2k_0 is a crude representa-
so the dependence upon $B$ is entirely in the multiplicative factor. If we define a magnetic Reynolds number by

$$ \beta = \frac{B}{B_0}, $$

(109)

can we write the bracket in equation (107) in the form,

$$ F(R_M, \beta) = \frac{R_M - 1}{R_M + 1} f\left(\frac{\beta R_M}{R_M + 1}\right) + f(\beta R_M). $$

(110)

As $\beta \to 0$, this approaches

$$ F(R_M, \beta \to 0) = \frac{1}{3} \left(\frac{R_M - 1}{R_M + 1} + 1\right) = \frac{2}{3} \frac{\gamma_*}{\gamma_* + \lambda k_*^2}, $$

(111)

which, when inserted into equation (107), yields agreement with equation (101), our previous result for $B \to 0$.

However, one must be careful about this procedure if $R_M \approx 1$, as is usually the case, for even if $\beta \ll 1$, $\beta$ may be $\gg R_M^{-1}$, so the second term on the right-hand side of equation (110) would effectively be 0 rather than 1/3. In this case,

$$ F(R_M \to \infty, \beta \to 0) = \frac{1}{3}, $$

(112)

and $\alpha$ is given by $1/3$ the value in equation (101). Evidently, the classical expression for $\alpha$ (as modified by Pouquet et al.) is off by a factor of 2 if $R_M \gg 1$. Figure 1 is a plot of $F(R_M, \beta)$, compared with $F_*(R_M, \beta) = (1 + R_M \beta^2)^{-1}$ from Cattaneo & Hughes (1996). As $F_*(R_M, \beta)$ was not proposed to be accurate for $R_M < 1$, the curves of $F^*$ for $R_M < 1$ should be ignored. In qualitative agreement with $F(R_M, \beta)$,

and a nondimensional mean magnetic field by

$$ \beta = \frac{B}{(\gamma_*/k_*)}, $$

(109)

we can express

$$ F(R_M, \beta) = \frac{R_M - 1}{R_M + 1} f\left(\frac{\beta R_M}{R_M + 1}\right) + f(\beta R_M). $$

(110)
\( F^* \) decreases as \( R_M \) increases, but, of critical importance is that it decreases to 0 rather than remaining finite as \( R_M \to \infty \), the case for the galactic dynamo.

4. DISCUSSION

Our general result for \( \alpha \), equation (97), displays significant features of the \( \alpha \)-effect due to driven helical turbulence. Our discussion of equation (101), above, makes clear that the values of \( \vec{v} \) and \( \vec{b} \) referred to in \( \tilde{h}(k) \) are \( \vec{v}^{(0)} \) and \( \vec{b}^{(0)} \), the values of the small-scale velocity and magnetic fields that apply if \( \vec{B} = 0 \).

As indicated in the text, it is in principle possible to calculate \( \vec{b}^{(0)} \) and \( \vec{b}^{(0)} \), and hence \( \tilde{h}(k) \), from numerical simulations of driven helical turbulence with \( \vec{B} = 0 \). Then, evaluating \( \alpha \) from equation (97) requires the value of \( \gamma_{\vec{b}} \), which can be calculated from equation (61) in terms of the statistics of the zero-order state. One should also calculate \( \alpha \) directly from its definition using simulations with various values of \( \vec{B} \) and then compare the results with equation (97).

Another feature of our work is that the contribution proportional to the current helicity, \( \langle \vec{b}^{(0)} \cdot \vec{V} \times \vec{b}^{(0)} \rangle \), first described by Pouquet et al. (1976), emerges naturally from our work. To give it a physical interpretation, we refer to a derivation we carried out in terms of the variables \( \vec{v}^{(0)} \) and \( \vec{b}^{(0)} \) to check the derivation presented here in terms of \( z^{(0)} \).

Referring to equation (25), we find that, at first order in \( \vec{B} \), the current helicity term is traceable to \( \langle \vec{v}^{(1)} \times \vec{b}^{(0)} \rangle \), as opposed to \( \langle \vec{v}^{(0)} \times \vec{b}^{(1)} \rangle \), which gives rise to the kinetic helicity term, where the velocity perturbation \( \vec{v}^{(1)} \) is due to the first-order Lorentz force \( (\vec{V} \times \vec{b}^{(0)}) \times \vec{B} \) as explained by Pouquet et al. (1976; p. 332).

Current helicity is discussed in the Soviet and Russian literature, Vainshtein (1972) and Vainshtein & Zeldovich (1972) called attention to effects of current helicity (which they referred to as "magnetic gyrotropy"). They argued on physical grounds that, as \( \vec{B} \) grows, it induces current helicity, which, because its sign is opposite to that of \( \vec{v} \times \vec{V} \), reduces \( \alpha \), causing dynamo activity to cease at some finite value of \( \vec{B} \). As we have explained, Pouquet et al. (1976) showed that current helicity is important at all values of \( \vec{B} \) and, on the basis of their spectral equations, found that the effect described by Vainshtein (1972) would be smaller and of the opposite sign than he claimed. Vainshtein & Kichatinov (1983) accept the interpretation of Pouquet et al. (1976).

They also derive a differential equation for the small-scale current helicity from the conservation of magnetic helicity as did Kleeorin & Ruzmaikin (1982); see Kleeorin, Rogachevskii, & Ruzmaikin (1995). We discuss these matters further in a forthcoming paper (Blackman & Field 1999).

The present work relies upon \( t < t_{ed} \) to evaluate the perturbations of \( \vec{v} \) and \( \vec{b} \) due to \( \vec{B} \) (although not in evaluating \( \vec{v}^{(0)} \) and \( \vec{b}^{(0)} \)). This is a reasonable approximation because the dominant contribution to \( \alpha \) comes from wavenumbers \( < 3k_0 \), which are relatively unaffected by magnetic back reaction and therefore should follow the Pope (1994) finding that in hydrodynamic turbulence, \( \tau < t_{ed} \). Unfortunately, the inequality is not as strong as we desire.

5. CONCLUSIONS

We have found an analytic formula for the dynamo coefficient \( \alpha \) based upon a perturbation expansion in \( \vec{B} \), the magnitude of the large-scale magnetic field, which is responsible for perturbing isotropic helical turbulence in such a way as to produce a turbulent emf along \( \vec{B} \). Our formula gives \( \alpha \) in terms of \( \vec{B} \) and the magnetic diffusivity \( \lambda \), together with the spectra of kinetic and current helicities, and a damping coefficient. Both of the latter are calculable from a simulation of incompressible isotropic helical MHD turbulence. For small values of \( \vec{B} \), our results agree with the classical results as modified by Pouquet et al. (1976). For large \( \vec{B} \), we find that two terms contribute to \( \alpha \). The first term is independent of the magnetic diffusivity \( \lambda \) and is similar to the expression proposed by Kraichnan (1979). The second term, which depends on \( \lambda \) in a way reminiscent of the expressions suggested by Cattaneo and his collaborators, vanishes in the limit \( \lambda \to 0 \), as predicted by Cattaneo et al., but, contrary to Cattaneo et al., the first term remains finite, so \( \alpha \) is reduced only by a factor of 2 for large values of \( \vec{B} \).

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APPENDIX A

ENERGY CASCADE

Start with the Navier-Stokes equation, which describes incompressible hydrodynamic (HD) turbulence:

\[
\begin{align*}
\partial_t \vec{v} &= - \vec{v} \cdot \vec{V} - \nabla p + \vec{v} \nabla^2 \vec{v} + \vec{f},
\end{align*}
\]

where \( p \) is equal to the pressure divided by \( \rho \). Define the kinetic energy \( E^V \) by

\[
E^V = \langle \frac{1}{2} \vec{v}^2 \rangle.
\]

Then the scalar product of equation (A1) and \( \vec{v} \) shows that

\[
\partial_t E^V = - \vec{v} \langle \omega^2 \rangle + \langle \vec{f} \cdot \vec{v} \rangle,
\]

where

\[
\omega = \nabla \times \vec{v}.
\]
is the vorticity. Hence in unforced ideal HD, \( E^V \) is conserved. The only effect of the nonlinear term \( -\v v\cdot \v \v \) in equation (A1) is to redistribute \( E^V_k \), the kinetic energy spectrum, over \( k \), keeping \( E^V = \int E^V_k \, dk \) constant. Numerical simulations show that this redistribution takes the form of a direct cascade, that is, from lower \( k \) to higher \( k \).

Now add a magnetic field \( \v B \), so the momentum equation becomes (eq. [25])

\[
\partial_t \v v = -\v v \cdot \nabla \v v - \nabla p + \v B \cdot \nabla \v B - \nabla \frac{1}{2} B^2 + v \nabla^2 \v v + f,
\]

where

\[ P = p + \frac{1}{2} B^2. \]  

(A5)

The equation for \( E^V \) becomes

\[
\partial_t E^V = \langle \v v \cdot (\v B) \v B \rangle - v \langle \v^2 \v \rangle + \langle f \cdot v \rangle,
\]

(A7)

so \( E^V \) is not conserved in unforced ideal MHD because the first term allows magnetic energy to exchange with kinetic energy. To account for this, we write the induction equation governing \( \v B \) as

\[
\partial_t \v B = -\v v \cdot \nabla \v B + \v B \cdot \nabla \v v + v \nabla^2 \v B,
\]

(A8)

where we have specialized to the case of unit magnetic Prandtl number, \( \lambda/v \). From the scalar product of equation (A8) with \( \v B \), we obtain an equation for the magnetic energy \( E^M = \langle 0.5 B^2 \rangle \),

\[
\partial_t E^M = \langle \v B \cdot (\v B) \v v \rangle - v \langle J^2 \rangle,
\]

(A9)

where

\[ J = \nabla \times \v B. \]  

(A10)

\( E^M \) is not conserved in unforced ideal MHD because of exchange with \( E^V \). However, we note that in equation (A9)

\[
\langle \v B \cdot (\v B) \v v \rangle = \langle \v B \cdot \nabla (\v B) \v v \rangle - \langle \v v \cdot (\v B) \v B \rangle
\]

\[ = -\langle \v v \cdot (\v B) \v B \rangle,
\]

(A11)

since spatial averaging converts the divergence to a vanishing surface integral. Hence

\[
\partial_t E^M = -\langle \v v \cdot (\v B) \v B \rangle - v \langle J^2 \rangle.
\]

(A12)

The energy-exchange term in equation (A12) is the negative of that in equation (A7), indicating that energy gained by \( E^V \) is lost by \( E^M \) and vice versa. Hence if \( v = f = 0 \), the sum of equations (A7) and (A13) yields

\[
\partial_t E = \partial_t (E^V + E^M) = 0
\]

(A13)

for the conservation of the total energy \( E \) in unforced ideal MHD. By analogy with \( E \) in HD, the second nonlinear terms in equations (A5) and (A8) for MHD can only redistribute \( E_k \) over \( k \). Numerical studies like those of Pouquet et al. (1976) show that, in the non helical case, the effect of the nonlinear terms in MHD is a direct cascade of total energy. However, because \( E^V \) and \( E^M \) are not separately conserved, it is not obvious that both \( E^V \) and \( E^M \) individually cascade directly in general.

To answer this question, we turn to the variables for the small-scale field \( z^\pm \), which in the case \( \lambda = v \) satisfy equation (35):

\[
\partial_t z^\pm = -z^\mp \cdot \nabla z^\pm - \nabla P + v \nabla^2 z^\pm + f \pm \v B \cdot \nabla z^\pm.
\]

(A14)

Define

\[ E^\pm = \langle \frac{1}{2} (z^\pm)^2 \rangle. \]  

(A15)

Then the scalar product of equation (A15) with \( z^\pm \) yields

\[
\partial_t E^\pm = -\langle z^\pm \cdot (z^\mp \cdot \nabla z^\pm) \rangle - \langle z^\pm \cdot \nabla P \rangle + v \langle z^\pm \cdot \nabla^2 z^\pm \rangle + \langle f \cdot z^\pm \rangle \pm \langle \v B \cdot \nabla z^\pm \rangle
\]

\[ = -\langle (z^\mp \cdot \nabla) (z^\pm)^2 \rangle - \langle \nabla \cdot (z^\pm)^2 \rangle + v \langle \nabla \cdot [\nabla \cdot z^\pm] \rangle + \langle f \cdot z^\pm \rangle \pm \langle \v B \cdot \nabla^2 (z^\pm)^2 \rangle. \]

(A16)

Since \( \nabla \cdot z^\pm = 0 \), this gives

\[
\partial_t E^\pm = -\langle \nabla \cdot [\frac{1}{2} (z^\pm)^2] \rangle - \langle \nabla \cdot (z^\pm P) \rangle - v \langle z^\pm \cdot \nabla \cdot \omega^\pm \rangle + \langle f \cdot z^\pm \rangle \pm \langle \nabla \cdot [\v B^2 (z^\pm)^2] \rangle,
\]

(A17)
\[ \omega^\pm = \nabla \times z^\pm = \omega \pm j. \]  
(A18)

Since
\[ z^\pm \cdot \nabla \times \omega^\pm = \nabla \cdot (\omega^\pm \times z^\pm) + \omega^\pm \cdot \nabla \times z^\pm \]
\[ = \nabla \cdot (\omega^\pm \times z^\pm) + (\omega^\pm)^2 \]  
(A19)

and averages of divergences vanish,
\[ \partial_t E^\pm = -v\langle (\omega^\pm)^2 \rangle + \langle f \cdot z^\pm \rangle, \]  
(A20)

an equation identical in form to equation (A3). We conclude that if \( \lambda = v \) both \( E^+ \) and \( E^- \) are individually conserved in unforced ideal MHD, and that the effect of nonlinear interactions is to redistribute the two energy spectra \( E^+_k \) and \( E^-_k \) individually. Note that the presence or absence of a mean magnetic field \( B \) does not affect this conclusion.

That leaves the question as to whether the redistribution is a direct cascade. Since
\[ E^\pm = \langle \frac{1}{2}(z^\pm)^2 \rangle = \langle \frac{1}{2}(v \pm B)^2 \rangle \]
\[ = \langle \frac{1}{2}v^2 \rangle + \langle \frac{1}{2}B^2 \rangle \pm \langle v \cdot B \rangle \]
\[ = E^V + E^M \pm 2K = E \pm 2K, \]  
(A21)

where
\[ K = \frac{1}{2}\langle v \cdot B \rangle \]  
(A22)

is the cross helicity (Biskamp 1993, p. 179), it follows from the conservation of \( E^+ \), \( E^- \), and \( E \) that \( K \) is also conserved in unforced ideal MHD if \( \lambda = v \). As pointed out by Pouquet et al. (1976), if the initial state is assumed to be statistically invariant under \( b \rightarrow -b \), it vanishes initially and thus remains equal to zero, and from equation (A21) we have
\[ E^+ = E^- = E. \]  
(A23)

As \( E \) cascades directly, it follows that \( E^+ \) and \( E^- \) cascade directly if \( \lambda = v \).

## APPENDIX B

### CORRELATIONS

Here we show that equation (58) is a reasonable representation. Let
\[ V = \langle \hat{z}^{+0}(k, t) \times z^{-0}(k', t_1) \rangle, \]  
(B1)

so that the average in equation (57) is \( V_c \). From equation (50)
\[ V = (2\pi)^{-6} \int dx \, dx' \, e^{-i(k \cdot x + k' \cdot x')} \langle \hat{z}^{+0}(x, 0) \times z^{-0}(x', \tau) \rangle \]
\[ = (2\pi)^{-6} \int dx \, dr \, e^{-i(k + k') \cdot x - ik' \cdot r} \langle \hat{z}^{+0}(0, 0) \times z^{-0}(r, \tau) \rangle, \]  
(B2)

where we have put \( t_1 = t + \tau \) and \( x' = x + r \) and have used stationarity and homogeneity. Carrying out the integration over \( x \) gives
\[ V = (2\pi)^{-3} \delta(k + k') \hat{h}(k', \tau), \]  
(B3)

where
\[ \hat{h}(k', \tau) = \int dr \, e^{-ik' \cdot r} \langle \hat{z}^{+0}(0, 0) \times z^{-0}(r, \tau) \rangle. \]  
(B4)

If we write
\[ Q(r, \tau) = \langle \psi(0, 0) \times \psi(r, \tau) \rangle, \]  
(B5)
\[ S(r, \tau) = \langle \hat{b}(0, 0) \times \hat{b}(r, \tau) \rangle, \]  
(B6)
\[ P_\psi(r, \tau) = \langle \psi(0, 0) \times \psi(r, \tau) \rangle, \]  
(B7)

and
\[ P_b(r, \tau) = \langle \hat{b}(0, 0) \times \hat{b}(r, \tau) \rangle, \]  
(B8)
where we have omitted superscripts for clarity, then the average in equation (B4) is, from equations (20) and (21),
\[
\langle z^{+0}(0, 0) \times z^{-0}(r, \tau) \rangle = Q(r, \tau) - S(r, \tau) - P_s(r, \tau) + P_p(r, \tau),
\]
so
\[
\hat{I} = \hat{Q}(k', \tau) - \hat{S}(k', \tau) - \hat{P}_s(k', \tau) + \hat{P}_p(k', \tau),
\]
where
\[
\hat{Q}(k', \tau) = \int dr \, e^{-i k' \cdot r} Q(r, \tau),
\]
so \(\hat{I}(k', \tau)\) is an even function of \(\tau\), or, equivalently, a function of \(|\tau|\), which we shall denote \((2\pi)^3 R(k'; |\tau|)\). Hence, from equation (B3),
\[
V_c = \delta(k + k') R_c(k', |\tau|).
\]
Because \(R\) is composed of cross products, we introduce a tensor \(R_{de}(k', |\tau|)\) such that
\[
R_c(k', |\tau|) = \epsilon_{cde} R_{de}(k', |\tau|),
\]
so that
\[
V_c = \epsilon_{cde} \delta(k + k') R_{de}(k', |\tau|).
\]

The last step is to assume that \(R_{de}(k', |\tau|)\) is a function of \(k'\) times a function of \(|\tau|\) (which may depend upon \(k = |k| = |k'|\):
\[
R_{de}(k', |\tau|) = R_{de}(k') f_{de}(|\tau|),
\]

where by taking \(f_d(0) = 1\), we assure that \(R_{de}(k')\) represents the maximum value of the correlation. As a natural choice for \(f_d(|\tau|)\) we take
\[
f_{de}(|\tau|) = e^{-\gamma_k |\tau|},
\]

where \(\gamma_k\) is a \(k\)-dependent inverse correlation time, in the spirit of the eddy-damping approximation of Pouquet et al. (1976) and Kraichnan (1979). Then
\[
V_c = \epsilon_{cde} \delta(k + k') R_{de}(k') e^{-\gamma_k |\tau|},
\]
which is equation (58).

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