Constraint Satisfaction Problems around Skolem Arithmetic

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Abstract. We study interactions between Skolem Arithmetic and certain classes of Constraint Satisfaction Problems (CSPs). We revisit results of Glaßer et al. [17] in the context of CSPs and settle the major open question from that paper, finding a certain satisfaction problem on circuits to be decidable. This we prove using the decidability of Skolem Arithmetic. We continue by studying first-order expansions of Skolem Arithmetic without constants, \((\mathbb{N}; \times)\), as CSPs. We find already here a rich landscape of problems with non-trivial instances that are in P as well as those that are NP-complete.

1 Introduction

A constraint satisfaction problem (CSP) is a computational problem in which the input consists of a finite set of variables and a finite set of constraints, and where the question is whether there exists a mapping from the variables to some fixed domain such that all the constraints are satisfied. When the domain is finite, and arbitrary constraints are permitted in the input, the CSP is NP-complete. However, when only constraints from a restricted set of relations are allowed in the input, it can be possible to solve the CSP in polynomial time. The set of relations that is allowed to formulate the constraints in the input is often called the constraint language. The question which constraint languages give rise to polynomial-time solvable CSPs has been the topic of intensive research over the past years. It has been conjectured by Feder and Vardi [13] that CSPs for constraint languages over finite domains have a complexity dichotomy: they are either in P or NP-complete. This conjecture remains unsettled, although dichotomy is now known on substantial classes (e.g. structures of size \(\leq 3\) [34][10] and smooth digraphs [21][2]). Various methods, combinatorial (graph-theoretic), logical and universal-algebraic have been brought to bear on this classification project, with many remarkable consequences. A conjectured delineation for the dichotomy was given in the algebraic language in [11].

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By now the literature on infinite-domain CSPs is also beginning to mature. Here the complexity can be much higher (e.g. undecidable) but on natural classes there is often the potential for structured classifications, and this has proved to be the case for reducts of, e.g. the rationals with order \( \mathbb{Q} \), the random (Rado) graph \( \mathcal{R} \) and the integers with successor \( \mathbb{Z} \): as well as first-order (fo) expansions of linear program feasibility \( [3] \). Skolem Arithmetic, which we take here to be the non-negative integers with multiplication (and possibly constants), represents a perfect candidate for continuation in this vein. These natural classes have the property that their CSPs sit in NP and a topic of recent interest for the second and third authors has been natural CSPs sitting in higher complexity classes.

Meanwhile, a literature existed on satisfiability of circuit problems over sets of integers involving work of the first author \( [17] \), itself continuing a line of investigation begun in \( [36] \) and pursued in \( [38,39,27] \). The problems in \( [17] \) can be seen as variants of certain functional CSPs whose domain is all singleton sets of the non-negative integers and whose relations are set operations of the form: complement, intersection, union, addition and multiplication (the latter two are defined set-wise, e.g. \( A \times B := \{ab : a \in A \land b \in B\} \)). An open problem was the complexity of the problem when the permitted set operators were precisely complement, intersection, union and multiplication. In this paper we resolve that this problem is in fact decidable, indeed in triple exponential space. We prove this result by using the decidability of the theory of Skolem Arithmetic with constants. In studying this problem we are able to bring to light existing results of \( [17] \) as results about their related CSPs, providing natural examples with interesting super-NP complexities.

In the second part of the paper, Skolem Arithmetic takes centre stage as we initiate the study of the computational complexity of the CSPs of its reducts, i.e. those constraint languages whose relations have a fo-definition in \( \langle \mathbb{N}; \times \rangle \). CSP(\( \mathbb{N}; \times \)) is in P, indeed it is trivial. The object therefore of our early study is its fo-expansions. We show that CSP(\( \mathbb{N}; +, \neq \)) is NP-complete, as is CSP(\( \mathbb{N}; \times, c \)) for each \( c > 1 \). We further show that CSP(\( \mathbb{N}; \times, U \)) is NP-complete when \( U \) is any non-empty set of integers greater than 1 such that each has a prime factor \( p \), for some prime \( p \), but omits the factor \( p^2 \). Clearly, CSP(\( \mathbb{N}; \times, U \)) is in P (and is trivial) if \( U \) contains 0 or 1. As a counterpoint to our NP-hardness results, we prove that CSP(\( \mathbb{N}; \times, U \)) is in P whenever there exists \( m > 1 \) so that \( U \supseteq \{m, m^2, m^3, \ldots\} \).

This paper is organised as previewed in this introduction. Several proofs are deferred to the appendix for reasons of space restriction.

**Related work.** Apart from the research on circuit problems mentioned above there has been work on other variants like circuits over integers \( [37] \) and positive natural numbers \( [9] \), equivalence problems for circuits \( [10] \), functions computed by circuits \( [22] \), and equations over sets of natural numbers \( [23,24] \).
2 Preliminaries

Let \( \mathbb{N} \) be the set of non-negative integers, and let \( \mathbb{N}^+ \) be the set of positive integers. For \( m \in \mathbb{N} \), let \( \text{Div}_m \) be the set of factors of \( m \). Finally, let \( \{\mathbb{N}\} \) be the set of singletons \( \{\{x\} : n \in \mathbb{N}\} \). In this paper we use a version of the CSP permitting both relations and functions (and constants). Thus, a constraint language consists of a domain together with functions, relations and constants over that domain. One may thus consider a constraint language to be a first-order structure. A constraint language is a core if all of its endomorphisms are embeddings (equivalently, if the domain is finite, automorphisms). The functional version of the CSP has previously been seen in, e.g., [13]. For a purely functional constraint language, a primitive positive sentence is the existential quantification of a conjunction of term equalities. More generally, and when relations present, we may have positive atoms in this conjunction. The problem CSP(\( \Gamma \)) takes as input a primitive positive sentence \( \varphi \), and asks whether it is true on \( \Gamma \). We will allow that the functions involved on \( \varphi \) be defined on a larger domain than the domain of \( \Gamma \). This is rather unheimlich\(^4\) but it allows the problems of [17] to be more readily realised in the vicinity of CSPs. For example, one such typical domain is \( \{\mathbb{N}\} \), but we will allow functions such as \(-\) (complement), \(\cup\) (union) and \(\cap\) (intersection) whose domain and range is the set of all subsets of \( \mathbb{N} \). We will also recall the operations of set-wise addition \( A + B := \{a + b : a \in A \land b \in B\} \) and multiplication \( A \times B := \{ab : a \in A \land b \in B\} \).

\(^4\) Weird. Thus spake Lindemann about Hilbert’s non-constructive methods in the resolution of Gordon’s problem (see [35]).
there is no path from $v_2$ to $v_1$. Nodes are also called gates. Nodes with indegree 0 are called input gates and $g_C$ is called the output gate. If there is an edge from gate $u$ to gate $v$, then we say that $u$ is a predecessor of $v$ and $v$ is the successor of $u$.

Let $O \subseteq \{\cup, \cap, -, +, \times\}$. An $O$-circuit with unassigned input gates $C = (V, E, g_C, \alpha)$ is a circuit $(V, E, g_C)$ whose gates are labeled by the labeling function $\alpha : V \rightarrow O \cup \mathbb{N} \cup \{\star\}$ such that the following holds: Each gate has an indegree in $\{0, 1, 2\}$, gates with indegree 0 have labels from $\mathbb{N} \cup \{\star\}$, gates with indegree 1 have label $-$, and gates with indegree 2 have labels from $\{\cup, \cap, +, \times\}$. Input gates with a label from $\mathbb{N}$ are called assigned (or constant) input gates; input gates with label $\star$ are called unassigned (or variable) input gates. An $O$-formula is an $O$-circuit that only contains nodes with outdegree one.

Let $u_1 < \cdots < u_n$ be the unassigned inputs in $C$ and $x_1, \ldots, x_n \in \mathbb{N}$. By assigning value $x_i$ to the input $u_i$, we obtain an $O$-circuit $C(x_1, \ldots, x_n)$ whose input gates are all assigned. In this circuit, each gate $g$ computes the following set $I(g)$: If $g$ is an assigned input gate where $\alpha(g) \neq \star$, then $I(g) = \{\alpha(g)\}$. If $g = u_k$ is an assigned input gate, then $I(g) = \{x_k\}$. If $g$ has label $-$ and predecessor $g_1$, then $I(g) = \mathbb{N} - I(g_1)$. If $g$ has label $\circ \in \{\cup, \cap, +, \times\}$ and predecessors $g_1$ and $g_2$, then $I(g) = I(g_1) \circ I(g_2)$. Finally, let $I(C(x_1, \ldots, x_n)) = I(g_C)$ be the set computed by the circuit $C(x_1, \ldots, x_n)$.

**Definition 1 (membership, equivalence, and satisfiability problems of circuits and formulas).**

Let $O \subseteq \{\cup, \cap, -, +, \times\}$.

- $MC_O(\mathbb{N}) = \{(C, b) | C$ is an $O$-circuit without unassigned inputs and $b \in I(C)\}$
- $EC_O(\mathbb{N}) = \{(C_1, C_2) | C_1$ and $C_2$ are $O$-circuits without unassigned inputs and we have $I(C_1) = I(C_2)\}$
- $SC_O(\mathbb{N}) = \{(C, b) | C$ is an $O$-circuit with unassigned inputs $u_1 < \cdots < u_n$ and there exist $x_1, \ldots, x_n \in \mathbb{N}$ such that $b \in I(C(x_1, \ldots, x_n))\}$

$MF_O(\mathbb{N})$, $EF_O(\mathbb{N})$, and $SF_O(\mathbb{N})$ are the variants that deal with $O$-formulas instead of $O$-circuits.

When an $O$-circuit is used as input for an algorithm, then we use a suitable encoding such that it is possible to verify in deterministic logarithmic space whether a given string encodes a valid circuit.

In Section 3 for $i \in \mathbb{N}$, we often identify $\{i\}$ with $i$, where this can not cause a harmful confusion.

## 3 Circuit Satisfiability and functional CSPs

We investigate the computational complexity of functional CSPs. In many cases we can translate known lower and upper bounds for membership, equivalence, and satisfiability problems of arithmetic circuits \cite{2,16,18} to CSPs. Our main result is the decidability of $SC_{\mathbb{N}}(\cap, \cup, \cap, \times)$ and $CSP(\{\mathbb{N}\}; \cap, \cup, \cap, \times)$, which solves
the main open question of the paper [13]. Table I summarizes the results obtained in this section and shows open questions. In particular, we would like to improve the gap between the lower and upper bounds for CSP(\{N\}; \emptyset), where \emptyset contains \cup and exactly one arithmetic operation (+ or \times).

We start with the observation that the equivalence of arithmetic formulas reduces to functional CSPs. This yields several lower bounds for the CSPs.

**Proposition 1.** For \( \emptyset \subseteq \{\neg, \cup, \cap, +, \times\} \) it holds that\( EF_{n}(\emptyset) \leq_{m}^{\log} \text{CSP}(\{N\}; \emptyset) \).

**Corollary 1.**

1. \( \text{CSP}(\{N\}; \{\neg, \cup, +\}) \) and \( \text{CSP}(\{N\}; \{\neg, \cup, \times\}) \) are \( \leq_{m}^{\log} \)-hard for PSPACE.
2. \( \text{CSP}(\{N\}; \{\cup, +\}), \text{CSP}(\{N\}; \{\cup, \times\}), \text{CSP}(\{N\}; \{+, \cup\}) \), and
   \( \text{CSP}(\{N\}; \{\cup, +\}) \) are \( \leq_{m}^{\log} \)-hard for \( \Pi_{2}^{P} \).

CSPs with + and \times can express diophantine equations, which implies the Turing-hardness of such CSPs.

**Proposition 2.** \( \text{CSP}(\{N\}; +, \times) \) is \( \leq_{m}^{\log} \)-hard for \( \Sigma_{1} \).

**Proposition 3.** \( \text{CSP}(\{N\}; \cup, +, \times) \in \Sigma_{1} \).

**Proposition 4.** \( \text{CSP}(\{N\}; \neg, \cup, +, \times) \in \Sigma_{2} \).

We show that the decidability of Skolem arithmetic [15] can be used to decide the satisfiability of arithmetic circuits without +. This solves the main open question of the paper [13] and at the same time implies the decidability of corresponding CSPs.

**Theorem 1.** \( SC_{n}(\neg, \cup, \times) \in \text{3EXPSPACE} \).

**Proof.** Let \( C = C(x_{1}, \ldots, x_{n}) \) be a circuit with gates \( g_{1}, \ldots, g_{r} \), where \( g_{1}, \ldots, g_{n} \) are the input gates and \( g_{r} \) is the output gate. Without loss of generality we may assume that \( C \) does not have \( \cap \)-gates. For every gate \( g_{k} \) we define a formula \( \varphi_{k} := \varphi_{k}(x_{1}, \ldots, x_{n}, i_{k}, v_{k}, b_{k}) \) in Skolem arithmetic such that the following holds.

\( \ast \) For \( a_{1}, \ldots, a_{n}, v \in \mathbb{N}, b \in \{0, 1\} \), and \( i = 1, \ldots, k \) it holds that \( \varphi_{k}(a_{1}, \ldots, a_{n}, 0, v, b) \) is true and

- \( \varphi_{k}(a_{1}, \ldots, a_{n}, i, v, b) \) is true IFF
- \( (b = 1) \iff C(a_{1}, \ldots, a_{n}) \) produces at \( g_{i} \) a set that contains \( v \).)

Let \( \varphi_{0} := b_{0} \lor \neg b_{0} \lor (x_{1} \cdot \ldots \cdot x_{n} \cdot i_{0} \cdot v_{0} = 0) \), which is always true. For \( 1 \leq k \leq n \), the formula \( \varphi_{k} \) which corresponds to the \( k \)-th input gate is defined as

\[
\varphi_{k} := \exists i_{k-1}, v_{k-1}, b_{k-1} \left[ (i_{k} = k \land b_{k} = 0) \rightarrow (x_{k} \neq v_{k} \land i_{k-1} = 0) \right] \land \\
\left[ (i_{k} = k \land b_{k} = 1) \rightarrow (x_{k} = v_{k} \land i_{k-1} = 0) \right] \land \\
\left[ i_{k} \neq k \rightarrow (i_{k-1} = i_{k} \land v_{k-1} = v_{k} \land b_{k-1} = b_{k}) \right] \right] \land \\
\varphi_{k-1}.
\]
Observe that the free variables of $\varphi_k$ are variables are $x_1, \ldots, x_n, i_k, v_k, b_k$, i.e., $\varphi_k = \varphi_k(x_1, \ldots, x_n, i_k, v_k, b_k)$. Moreover, an induction on $k$ shows that (*) holds for all $\varphi_k$ where $0 \leq k \leq n$.

Now define the formulas $\varphi_k$ for the inner gates $g_k$ where $n < k \leq r$. Here $d_k, e_k, f_k, f'_k, h_k, \text{ and } h'_k$ are used as auxiliary variables.

If $g_k$ is a complement gate with predecessor $g_p$, then let

$$\varphi_k := \exists i_{k-1}, v_{k-1}, b_{k-1} \left[ i_k = k \rightarrow (i_{k-1} = p \land v_{k-1} = v_k \land (b_k = 1 \rightarrow b_{k-1} = 0) \land (b_k = 0 \rightarrow b_{k-1} = 1)) \land \\
(i_k \neq k \rightarrow (i_{k-1} = i_k \land v_{k-1} = v_k \land b_{k-1} = b_k)) \land \\
\varphi_{k-1} \right].$$

If $g_k$ is a $\cap$ gate with predecessors $g_p$ and $g_q$, then let

$$\varphi_k := \exists f_k, h_k \forall e_k \exists i_{k-1}, v_{k-1}, b_{k-1} \left[ (i_k = k \land e_k = 0) \rightarrow (i_{k-1} = p \land v_{k-1} = v_k \land b_{k-1} = f_k) \land \\
(i_k = k \land e_k \neq 0) \rightarrow (i_{k-1} = q \land v_{k-1} = v_k \land b_{k-1} = h_k) \land \\
(i_k = k \land b_k = 1) \rightarrow (f_k = 1 \lor h_k = 1) \land \\
(i_k = k \land b_k = 0) \rightarrow (f_k = 0 \land h_k = 0) \land \\
(i_k \neq k \rightarrow (i_{k-1} = i_k \land v_{k-1} = v_k \land b_{k-1} = b_k)) \land \\
\varphi_{k-1} \right].$$

If $g_k$ is a $\times$ gate with predecessors $g_p$ and $g_q$, then let

$$\varphi_k := \exists f_k, f'_k, h_k \forall e_k \exists h'_k \exists d_k \exists i_{k-1}, v_{k-1}, b_{k-1} \left[ (i_k = k \land b_k = 1 \land e_k = 0) \rightarrow (f_k \cdot f'_k = v_k \land i_{k-1} = p \land v_{k-1} = f_k \land b_{k-1} = 1)) \land \\
(i_k = k \land b_k = 1 \land e_k \neq 0) \rightarrow (f_k \cdot f'_k = v_k \land i_{k-1} = q \land v_{k-1} = f'_k \land b_{k-1} = 1) \land \\
(i_k = k \land b_k = 0 \land h_k = 0) \rightarrow (i_{k-1} = p \land v_{k-1} = h_k \land b_{k-1} = 0) \land \\
(i_k = k \land b_k = 0 \land h'_k = v_k \land d_k = 0) \rightarrow (i_{k-1} = q \land v_{k-1} = h'_k \land b_{k-1} = 0) \land \\
(i_k \neq k \rightarrow (i_{k-1} = i_k \land v_{k-1} = v_k \land b_{k-1} = b_k)) \land \\
\varphi_{k-1} \right].$$

Again it holds that $\varphi_k$’s free variables are $x_1, \ldots, x_n, i_k, v_k, b_k$ and an induction on $k$ shows that (*) holds for all $\varphi_k$ where $0 \leq k \leq r$. So for the output gate $g_r$ we obtain

$$(C, v) \in \text{SCN}(\neg, \cup, \cap, \times) \iff \exists a_1, \ldots, a_n \varphi_r(a_1, \ldots, a_n, r, v, 1).$$

The right-hand side is a first-order sentence of Skolem arithmetic. On input $(C, v)$ this sentence can be computed in polynomial time, which shows that $\text{SCN}(\neg, \cup, \cap, \times) \leq_p \text{SCN}(\neg, \cup, \cap, \times) \leq_p \text{SCN}_r(\neg, \cup, \cap, \times)$. The latter is decidable in $\text{3EXPSPACE}$ [14].

**Corollary 2.** $\text{CSP}([N]; \neg, \cup, \cap, \times) \in \text{3EXPSPACE}$

**Proof.** By Theorem 1 it suffices to show $\text{CSP}([N]; \neg, \cup, \cap, \times) \leq_p \text{SCN}(\neg, \cup, \cap, \times)$. We describe the reduction on input of a $\text{CSP}([N]; \neg, \cup, \cap, \times)$-instance $x = \exists y \in [N]^{n_0}$
\[ \mathbb{N}^n \land_{i=0}^m (t_{2i} = t_{2i+1}) \] Observe that
\[ \bigwedge_{i=0}^m (t_{2i} = t_{2i+1}) \iff \bigwedge_{i=0}^m (t_{2i} \cap \overline{t_{2i+1}}) \cup (\overline{t_{2i}} \cap t_{2i+1}) = \emptyset \]
\[ \iff \bigcup_{i=0}^m [(t_{2i} \cap \overline{t_{2i+1}}) \cup (\overline{t_{2i}} \cap t_{2i+1})] = \emptyset \]
\[ \iff 0 \in 0 \cdot \bigcup_{i=0}^m [(t_{2i} \cap \overline{t_{2i+1}}) \cup (\overline{t_{2i}} \cap t_{2i+1})]. \]

So \( x \in \text{CSP}(\{\mathbb{N}\}; -, \cup, \cap, \times) \) if and only if \((C, 0) \in \text{SC}_N(\neg, \cup, \cap, \times)\).

**Corollary 3.** \(\text{CSP}(\{\mathbb{N}\}; -, \cup, \cap, \times) \in \text{3EXPSPACE} \)

**Proof.** By Corollary 2, it suffices to show that we have \(\text{CSP}(\{\mathbb{N}\}; -, \cup, \cap, \times) \leq^p_m \text{CSP}(\{\mathbb{N}\}; -, \cup, \cap, \times)\). Consider a \(\text{CSP}(\{\mathbb{N}\}; -, \cup, \cap, \times)\)-instance
\[ x := \exists y \in \mathbb{N}^n \land_{i=0}^m (t_{2i} = t_{2i+1}). \]

We may assume that 0 and 1 are the only constants that occur in \( x \). We can do this, since constants \( c > 1 \) can be removed as follows: Let \( l = \lfloor \log c \rfloor \), replace \( c \) with a new variable \( z \), and add constraints
\[ (z_0 = \{2\}) \land (z_1 = z_0 + z_0) \land \cdots \land (z_l = z_{l-1} + z_{l-1}) \land (z = \sum_{i \in I} z_i), \]
where \( z_0, \ldots, z_l \) are new variables and
\[ I = \{ i \mid \text{the } i\text{-th bit in } c\text{'s binary representation is } 1\}. \]

Note that removing constants in this way can be done in polynomial time.

Observe that the term \( q := \overline{(\{0, 1\} \cdot \{0, 1\} \cap \{0, 1, 2\})} \cdot \{0\} \cap \{1\} \) generates the set \( \{2^i \mid i \in \mathbb{N}\} \).

For every term \( t \), let \( t' \) be the term that is obtained from \( t \) if every constant \( c \) is replaced with \( 2^c \), every \( + \) operation is replaced with \( \times \), and every complement operation \( \overline{\cdot} \) is replaced with \( (\overline{\cdot} \cap q) \). The computation of \( t' \) is possible in polynomial time, since only the constants 0 and 1 can appear.

The reduction outputs the \(\text{CSP}(\{\mathbb{N}\}; -, \cup, \cap, \times)\)-instance
\[ x' := \exists y \in \mathbb{N}^n \land_{i=0}^m (t'_{2i} = t'_{2i+1}) \land \land_{i=1}^n (y_i \cup q = q). \]

Observe that for all terms \( t \) and all \( e = (e_1, \ldots, e_n) \in \mathbb{N}^n \) it holds that
\[ t'((2^e_1, \ldots, 2^e_n)) = \{2^i \mid i \in t(e_1, \ldots, e_n)\}. \] (1)
It remains to show that $x$ and $x'$ are equivalent.

If $e = (e_1, \ldots, e_n) \in \mathbb{N}^n$ is a satisfying assignment for $x$, then by equation (1), $z = (2^{e_1}, \ldots, 2^{e_n})$ is a satisfying assignment for $x'$ (note that $\bigwedge_{i=1}^n (y_i \cup q = q)$ holds, since $y_i = 2^{e_i} \in q$).

If $z = (z_1, \ldots, z_n) \in \mathbb{N}^n$ is a satisfying assignment for $x'$, then because of the constraints $\bigwedge_{i=1}^n (y_i \cup q = q)$, $z_1 = 2^{e_1}, \ldots, z_n = 2^{e_n}$ for $e = (e_1, \ldots, e_n) \in \mathbb{N}^n$ and by (1), $e$ is a satisfying assignment for $x$.

The following propositions transfer the NP-hardness from satisfiability problems for arithmetic circuits to CSP($\{\mathbb{N}\}; \times$) and CSP($\{\mathbb{N}\}; +$).

Proposition 5. CSP($\{\mathbb{N}\}; \times$) is $\leq_{m}^{\text{log}}$-hard for NP.

Proposition 6. CSP($\{\mathbb{N}\}; +$) is $\leq_{m}^{\text{log}}$-hard for NP.

The remaining results in this section show that certain functional CSPs belong to NP. This needs non-trivial arguments of the form: If a CSP can be satisfied, then even with small values. These arguments are provided by the known results that integer programs, existential Presburger arithmetic, and existential Skolem arithmetic are decidable in NP [8,30,33,19].

Proposition 7. CSP($\{\mathbb{N}\}; \neg, \cap, \cup$) is $\leq_{m}^{\text{log}}$-complete for NP.

Proposition 8. CSP($\{\mathbb{N}\}; +$) ∈ NP.

Proof. Consider a CSP($\{\mathbb{N}\}; +$)-instance $\varphi := \exists x_1, \ldots, x_n [s_1 = t_1 \land \cdots \land s_m = t_m]$. Each atom $s_i = t_i$ term can be written as $0 = t_i - s_i = a_{i,1}x_1 + \cdots + a_{i,n}x_n$ where $a_{i,j} \in \mathbb{Z}$. Let

$A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{pmatrix}.$

Hence $\varphi \in$ CSP($\{\mathbb{N}\}; +$) if and only if there exists an $x = (x_1, \ldots, x_n) \in \mathbb{N}^n$ such that $Ax = 0$. The right-hand side is an integer program that can be solved in NP [8,30].

Proposition 9.

1. CSP($\{\mathbb{N}\}; \neg, +$) $\leq_{m}^{\text{NP}}$ CSP($\{\mathbb{N}\}; +, =, \neq$).
2. CSP($\{\mathbb{N}\}; \cap, \times$) $\leq_{m}^{\text{NP}}$ CSP($\{\mathbb{N}\}; \times, =, \neq$).

Corollary 4. CSP($\{\mathbb{N}\}; \neg, +$), CSP($\{\mathbb{N}\}; \cap, \times$) ∈ NP.
Table 1. Upper and lower bounds for $\text{CSP}([\mathbb{N}]; \emptyset)$. All lower bounds are with respect to $\leq_{\text{log}}$-reductions.

| $\emptyset$ | Lower Bound | Upper Bound |
|-------------|-------------|-------------|
| $\neg \cup \cap \times$ | $\Sigma_1$ | $\Sigma_2$ |
| $\neg \cup \cap$ | PSPACE | 3EXPSPACE |
| $\neg \cup \times$ | PSPACE | 3EXPSPACE |
| $\neg \cup$ | NP | NP |
| $\cup \cap \times$ | $\Pi_2^P$ | $\Pi_2^P$ |
| $\cup \cap$ | $\Pi_2^P$ | $\Pi_2^P$ |
| $\cup \times$ | $\Pi_2^P$ | $\Pi_2^P$ |
| $\cap \cap \times$ | $\Sigma_1$ | $\Sigma_1$ |
| $\cap \cap$ | NP | NP |
| $\cap \times$ | NP | NP |
| $\cap \times$ | NP | NP |

4 CSPs over fo-expansions of Skolem Arithmetic

We now commence our exploration of the complexity of CSPs generated from the simplest expansions of $(\mathbb{N}; \times)$. Abandoning our set-wise definitions, we henceforth use $\times$ to refer to the syntactic multiplication of Skolem Arithmetic (which may additionally carry semantic content). When we wish to refer to multiplication in a purely semantic way, we prefer $\cdot$ or $\prod$. We will consider $\times$ as a ternary relation rather than a binary function. We will never use syntactic $\times$ in a non-standard way, i.e. holding on a triple of integers for which it does not already hold in natural arithmetic.

**Proposition 10.** Let $\Gamma$ be a finite signature reduct of $(\mathbb{N}; \times, 1, 2, \ldots)$. Then $\text{CSP}(\Gamma)$ is in NP.

**Proof.** It is known that Skolem Arithmetic admits quantifier-elimination and that the existential theory in this language is in NP. The result follows when one considers that we can substitute quantifier-free definitions for each among our finite set of fo-definable relations.
4.1 Upper bounds

We continue with polynomial upper bounds.

**Lemma 1.** Let $U \subseteq \mathbb{N}$ be non-empty and $U \cap \{0, 1\} = \emptyset$. $\text{CSP}(\mathbb{N}; x, U)$ is polynomial-time reducible to $\text{CSP}(\mathbb{N}^+; x, U)$.

**Proof.** Let $\varphi$ be an arbitrary instance of $\text{CSP}(\mathbb{N}; x, U)$ involving a set of atoms $C$ on variables $V$. We construct a set $V' \subseteq V$ incrementally by repeating the following three steps until a fixed point is reached.

- if $U(v) \in C$, then $V' := V' \cup \{v\}$,
- if $(x \times y = z) \in C$ and $z \in V'$, then $V' := V' \cup \{x, y\}$, and
- if $(x \times y = z) \in C$ and $x, y \in V'$, then $V' := V' \cup \{z\}$.

Note that if $v \in V'$, then any solution to $\varphi$ must satisfy $s(v) \neq 0$. Let $V_0 = V \setminus V'$. Construct $\varphi'$, an instance for $\text{CSP}(\mathbb{N}; x, U, 0)$, with atoms $C'$ and variables $V'$ by replacing each variable $v \in V_0$ with the constant 0. Note the following:

1. if $U(0) \in C'$, then the variable that was replaced by 0 is a member of $V'$ so this case cannot occur.
2. if $(0 \times y = z) \in C'$ or $(x \times 0 = z) \in C'$, then the variable replaced by 0 is not a member of $V'$ while $z$ is a member of $V'$. This situation cannot occur.
3. if $(x \times y = 0) \in C'$, then note that $x, y \in V'$ so the variable that was replaced with 0 also was a member of $V'$. Hence, this case cannot occur.

Thus, 0 can only appear in three cases: $(x \times 0 = 0)$, $(0 \times x = 0)$, and $(0 \times 0 = 0)$. Let $\varphi''$ be an instance for $\text{CSP}(\mathbb{N}^+; x, U)$ built from atoms $C''$ and variables $V'$ where $C''$ is obtained from $C'$ by removing these kinds of constraints. Note that $(\mathbb{N}; x, U, 0) \models \varphi'$ iff $(\mathbb{N}; x, U) \models \varphi''$. Also note that if $\varphi''$ has a solution, then it has a solution $s : V' \rightarrow \mathbb{N}^+$. This implies that $\varphi'$ is satisfiable on $(\mathbb{N}; x, U)$ iff it is satisfiable on $(\mathbb{N}^+; x, U)$

The transformation above can obviously be carried out in polynomial time. In order to prove the lemma, it remains to show that $(\mathbb{N}; x, U) \models \varphi$ iff $(\mathbb{N}; x, U, 0) \models \varphi''$.

Assume first that $\varphi$ has a solution $s : V \rightarrow \mathbb{N}$. Since $C'' \subseteq C$, it follows immediately that $s$ is a solution to $\varphi''$, too.

Assume instead that $\varphi''$ has a solution $s' : V' \rightarrow \mathbb{N}^+$. We claim that the function $s : V \rightarrow \mathbb{N}$ defined by $s(v) = s'(v)$ for $v \in V'$ and $s(v) = 0$ otherwise is a solution to $\varphi$. If the variable $v \in V$ appears in an atom $U(v)$, then $v \in V'$ and $U(v)$ is satisfied by $s$. Consider an atom $(x \times y = z) \in C$. If $(x, y, z) \subseteq V'$, then $s$ satisfies the atom since $(x \times y = z) \in C''$. Assume $x \notin V'$. Then $z \notin V'$, $s(x) = s(z) = 0$ and the atom is satisfied by $s$. The same reasoning applies when $y \notin V'$. Assume finally that $z \notin V'$. Then at least one of $x, y \notin V'$ so $s(x) = 0$ and/or $s(y) = 0$. Combining this with the fact that $s(z) = 0$ implies that the atom is satisfied by $s$. 


We now borrow the following slight simplification of Lemma 6 from [25].

**Lemma 2 (Scalability [25]).** Let \( \Gamma \) be a finite signature constraint language with domain \( \mathbb{R} \), whose relations are quantifier-free definable in +, ≤ and <, such that the following holds.

- Every satisfiable instance of \( \text{CSP}(\Gamma) \) is satisfied by some rational point.
- For each relation \( R \in \Gamma \), it holds that if \( \mathbf{r} := (x_1, x_2, \ldots, x_k) \in R \), then \( (ax_1, ax_2, \ldots, ax_k) \in R \) for all \( a \in \{ y : y \in \mathbb{R}, y \geq 1 \} \).
- \( \text{CSP}(\Gamma) \) is in \( P \).

Then \( \text{CSP}(\Delta) \) is in \( P \), where \( \Delta \) is obtained from \( \Gamma \) by substituting the domain \( \mathbb{R} \) by \( \mathbb{Z} \).

**Lemma 3.** Arbitrarily choose \( m > 1 \) and \( U \subseteq \mathbb{N}^+ \) such that \( \{ m, m^2, m^3, \ldots \} \subseteq U \). Then, \( \text{CSP}(\mathbb{N}^+; \times, U) \) is in \( P \).

**Proof.** Define \( h(1) = 1 \) and \( h(x) = m^\ell(x) \) when \( x > 1 \). Let \( D = \{ 1, m, m^2, m^3, \ldots \} \). The function \( h \) is a homomorphism from \( (\mathbb{N}^+; \times, U) \) to \( (D; \times, U \cap D) \). Clearly, \( h(U) = U \cap D \). Suppose \( a \cdot b = c \) where \( a, b, c \in \mathbb{N}^+ \). We see that

\[
    h(a) \cdot h(b) = m^{\ell(a)} \cdot m^{\ell(b)} = m^{\ell(a)+\ell(b)} = m^{\ell(c)} = h(c).
\]

Define \( h'(1) = 0 \) and \( h'(m^k) = k \). Note that \( h' \) is a homomorphism from \( (D; \times, U \cap D) \) to \( (\mathbb{N}; +, x \geq 1) \). We know that that \( \text{CSP}(\mathbb{R}; +, x \geq 0, x \geq 1) \) is in \( P \) (via linear programming) and this implies tractability of \( \text{CSP}(\mathbb{N}; +, x \geq 1) \) through \( \text{CSP}(\mathbb{Z}; +, x \geq 1, x \geq 0) \) by Lemma 2.

**Proposition 11.** Arbitrarily choose \( m > 1 \) and \( U \subseteq \mathbb{N} \) such that \( \{ m, m^2, m^3, \ldots \} \subseteq U \). Then, \( \text{CSP}(\mathbb{N}; \times, U) \) is in \( P \).

**Proof.** Combine Lemma 1 with Lemma 3.

### 4.2 Cores

We say that an integer \( m > 1 \) has a degree-one factor \( p \) if and only if \( p \) is a prime such that \( \frac{p}{m} \) and \( \frac{p^2}{m} \) are contained in \( (\mathbb{N}; \times, m) \) by pp-definable \( \exists y \ x \times y = m \). Let \( \text{Div}_m \) be the set of divisors of \( m \), pp-definable in \( (\mathbb{N}; \times, m) \) since \( x = 1 \) iff \( x \times x = x \) (recalling \( 0 \notin \text{Div}_m \)). It follows that \( \{ 1, m \} \) are contained in the core of \( (\text{Div}_m; \times, m) \).

**Lemma 4.** Let \( m > 1 \) be an integer that has a degree-one factor \( p \). Then \( (\text{Div}_m; \times, m) \) has a two-element core.

**Proof.** Consider the function \( e : \text{Div}_m \to \text{Div}_m \) uniquely defined by \( e(1) = 1 \), \( e(p) = m \), \( e(p_1) = \cdots = e(p_k) = 1 \) (i.e. all the other prime divisors map to 1), and the rule \( e(x \cdot x') = e(x) \cdot e(x') \). We claim that \( e \) is an endomorphism of \( (\text{Div}(m); \times, m) \). Clearly, \( e(m) = m \). Arbitrarily choose a tuple \((x, y, z) \in (x \times y = \cdots = x_{k+1}) \). We have that \( e(x \cdot x') = e(x) \cdot e(x') \) as required.
$z$). Let $x = x_1^{\alpha_1} \cdots x_a^{\alpha_a}$ and $y = y_1^{\beta_1} \cdots y_b^{\beta_b}$ be prime factorizations. Note that at most one of $x_1, \ldots, x_a, y_1, \ldots, y_b$ can equal $p$ and, if so, the corresponding exponent must equal one. If none of the factors equal $p$, then $e(x) = e(y) = e(z) = 1$ and $e(x) \times e(y) = e(z)$. Otherwise, assume without loss of generality that $x_1 = p$. Then we have $e(x) = e(z) = m$ and $e(y) = 1$. Once again $e(x) \times e(y) = e(z)$ and $e$ is indeed an endomorphism of $(\text{Div}_m; \times, m)$. It follows that $(\{1, m\}; \times, m)$ is the core of $(\text{Div}_m; \times, m)$.

**Lemma 5.** Let $m$ be an integer that does not have a degree-one factor and let $D$ contain the divisors of $m$. Then $(\text{Div}_m; \times, m)$ does not have a two-element core.

**Proof.** Assume $m$ has the prime factorization $m = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ and note that $\alpha_1, \ldots, \alpha_k > 1$. Assume $e : \text{Div}_m \rightarrow \{a, b\}$ is an endomorphism to a two-element core, i.e. the range of $e$ is $\{1, m\}$. Since multiplication is determined by the action of the primes, we can see that for one prime $p \in \{p_1, \ldots, p_k\}$ we must have $e(p) = m$. Consider $p \times p = p^2$. If we apply the endomorphism $e$ to this tuple, we end up with $e(p) \times e(p) = e(p)^2$ which is not possible. Hence, $e$ does not exist and $(\text{Div}_m; \times, m)$ does not admit a two-element core.

### 4.3 Lower bounds

We now move to lower bounds of NP-completeness.

**Proposition 12.** $\text{CSP}(\mathbb{N}; \neq, \times)$ is NP-complete.

**Proof.** NP membership follows from Proposition 10. For NP-hardness, we will encode the CSP of a certain Boolean constraint language, i.e. with domain $\{0, 1\}$, with two relations: $\neq$ and $R_1 := \{0, 1\}^3 \setminus \{(1, 1, 1)\}$. This CSP is NP-hard because $\neq$ omits constant and semilattice polymorphisms and $R_1$ omits majority and minority polymorphisms (an algebraic reformulation of Schaefer’s Theorem in the spirit of [22]).

To encode our Boolean CSP, ensure all variables $v$ satisfy $v \times v = v$, which enforces the domain $\{0, 1\}$. Consider 0 to be false and 1 to be true. 0 is pp-definable by $x \times x = x \land \exists y y \neq x \land x \times y = x$. For $\{0, 1\}^3 \setminus \{(1, 1, 1)\}$ take $R_1(x, y, z)$ to be $\exists w x \times y = z \land w \times z = 0$; and for $\neq$ take $\neq$. The reduction may now be done by local substitution and the result follows.

An operation $t : D^k \rightarrow D$ is a weak near-unanimity operation if $t$ is idempotent and satisfies the equations

$$t(y, x, \ldots, x) = t(x, y, x, \ldots, x) = \cdots = t(x, \ldots, x, y).$$

**Theorem 2 ([11]).** Let $\Gamma$ be a constraint language over a finite set $D$. If $\Gamma$ is a core and does not have a weak near-unanimity polymorphism, then $\text{CSP}(\Gamma)$ is NP-hard.
Lemma 6. Arbitrarily choose an $m > 1$ such that $m \neq k^n$ for all $k, n > 1$ together with a finite set $\{1, m\} \subseteq S \subseteq \mathbb{N} \setminus \{0\}$. If $(\text{Div}_m; \times, m)$ is a core, then $\text{CSP}(S; \times, m)$ is NP-hard.

Proof. Assume $(S; \times, m)$ admits a weak near-unanimity operation $t : S^k \to S$. The relation $\prod_{i=1}^k x_i = x_{k+1}$ is pp-definable in $(S; \times, m)$ and so is the relation

$$R = \{(x_1, \ldots, x_k) \in S^k \mid \prod_{i=1}^k x_i = m\}.$$

The relation $R$ contains the tuples

\begin{align*}
(m, 1, \ldots, 1) \\
(1, m, 1, \ldots, 1) \\
\vdots \\
(1, \ldots, 1, m).
\end{align*}

Applying $t$ component-wise (i.e. vertically) to these tuples yields a tuple $(a, \ldots, a)$ for some $a \in D$. However, $R$ does not contain a tuple $(a, \ldots, a)$ for any $a \in D$ since this would imply that $m = a^k$ for some $a, k > 1$. We conclude that $\text{CSP}(S; \times, m)$ is NP-hard by Theorem 2.

Note that the proof of this last lemma was eased by our assumption that $\times$ is a relation and not a function. Had it been a function we would have to prove the domain $S$ would be closed under it.

Theorem 3. $\text{CSP}(\mathbb{N}; \times, m)$ is NP-hard for every integer $m > 1$.

Proof. If $m = k^n$ for some $k, n > 1$, then we can pp-define the constant relation $\{k\}$ since $x = k \iff \prod_{i=1}^k x = m$. Hence, we assume without loss of generality that $m \neq k^n$ for all $k, n > 1$.

We further know that $\text{Div}_m$ is pp-definable in $(\mathbb{N}; \times, m)$, i.e. there is polynomial time reduction from $(\text{Div}_m; \times, m)$ to $(\mathbb{N}; \times, m)$. The core of $(\text{Div}_m; \times, m)$ is some $(S; \times, m)$, where $\{1, m\} \subseteq S \subseteq \text{Div}_m$ and the result follows from Lemma 6.

Theorem 4. Let $U$ be any subset of $\mathbb{N} \setminus \{0, 1\}$ so that every $x \in U$ has a degree-one factor. Then $\text{CSP}(\mathbb{N}; x, U)$ is NP-hard.

Proof. From Lemma 4, for each $x \in U$, the core of $(\mathbb{N}; x, x)$ is the same (up to isomorphism). Fix some $m \in U$. We claim there is a polynomial time reduction from $\text{CSP}(\text{Div}_m; x, m)$ to $\text{CSP}(\mathbb{N}; x, U)$, whereupon the result follows from Theorem 3.

To see the claim, take an instance $\varphi$ of $\text{CSP}(\text{Div}_m; x, m)$ and build an instance $\psi$ of $\text{CSP}(\mathbb{N}; x, U)$ by adding an additional variable $v_m$, now substituting instances of $m$ for $v_m$, and adding the constraint $U(v_m)$. Correctness of the reduction is easy to see and the result follows.
For $x \in \mathbb{N} \setminus \{0, 1\}$, define its minimal exponent, min-exp($x$), to be the smallest $j$ such that $x$ has a factor of $p^j$, for some prime $p$, but not a factor of $p^{j+1}$. Thus an integer with a degree-one factor has minimal exponent 1. Call $x \in \mathbb{N} \setminus \{0, 1\}$ square-free if it omits all repeated prime factors. For a set $U \subseteq \mathbb{N} \setminus \{0, 1\}$, define its basis, basis($U$) to be the set \{min-exp($x$) : $x \in U$\}.

**Lemma 7.** Let $U \subseteq \mathbb{N} \setminus \{0, 1\}$, so that basis($U$) is finite and basis($U$) \neq \{1\}. There is some set $X$ pp-definable in $(\mathbb{N}; \times, U)$ so that basis($X$) = \{1\}.

Proof. Let $r = \max$(basis($U$)) and take an element $x$ that witnesses this, of the form $q^r \cdot p_1^{a_1} \cdots p_k^{a_k}$, where $p_1, \ldots, p_k$ are prime and each is coprime to $q$ (which is square-free), and where $a_1, \ldots, a_k > r$. Set

$$\xi(y) := \exists z, x_1, \ldots, x_k \; U(z) \land q^r \cdot x_1^{a_1} \cdots x_k^{a_k} = z.$$  

We claim that $\xi$ defines a set of integers $X$ so that basis($X$) has the desired property. The non-emptiness is clear since 1 \in basis($X$) by construction.

Firstly, we will by contradiction argue that 0 \notin basis($X$). Assume 0 \in basis($X$). This implies that 1 \in $X$. Hence, $\exists z, x_1, \ldots, x_k \; U(z) \land 1^r \cdot x_1^{a_1} \cdots x_k^{a_k} = z$, so $x_1^{a_1} \cdots x_k^{a_k} \in U$. It follows that $d = \min\{a_1, \ldots, a_k\} \in$ basis($U$). Now, $a_1, \ldots, a_k > r$ so $d > r$. This contradicts the fact that $r = \max$(basis($U$)).

We will now argue by contradiction that 1 < $s \notin$ basis($X$). Assume $s \in$ basis($X$). Then there exists a $t = q^s \cdot p_1^{a_1} \cdots p_k^{a_k} \in X$ where $s < a_1, \ldots, a_k$. Since $t \in X$, we know that $\exists z, x_1, \ldots, x_k \; U(z) \land t^s \cdot x_1^{a_1} \cdots x_k^{a_k} = z$. Let $e = t^r \cdot x_1^{a_1} \cdots x_k^{a_k} \in U$ as above. Let’s expand $t$: $e = (q^s \cdot p_1^{a_1} \cdots p_k^{a_k})^r \cdot x_1^{a_1} \cdots x_k^{a_k}$. We see that min-exp($e$) > $r$ which contradicts the choice of $r$.

**Example 1.** We provide an example of the construction of the previous lemma in vivo. Let $U := \{p^2, p^2q^3, p^4q^4r^8 : p, q, r$ primes\}, so that basis($U$) = \{2, 4\}. Then $\xi(y) := \exists z, x \; U(z) \land y^4 \cdot x^8 = z$. We can now deduce $X := \{p, pq^2 : p, q$ primes\} and basis($X$) = \{1\}.

**Theorem 5.** Let $U \subseteq \mathbb{N} \setminus \{0, 1\}$ be so that basis($U$) is finite. Then CSP($\mathbb{N}; \times, U$) is NP-complete.

Proof. Membership of NP follows from Proposition [10]. We use the construction of the previous lemma to pp-define $X$ with basis($X$) = \{1\}. This allows us to polynomially reduce CSP($\mathbb{N}; \times, X$) to CSP($\mathbb{N}; \times, U$) by local substitution. NP-hardness for the former comes from Theorem [14] and the result follows.

5 Final remarks

In this paper we have provided a solution to the major open question from [17] as well as begun the investigation of CSPs associated with Skolem Arithmetic. However, the thrust of our work must be considered exploratory and there are two major directions in which more work is necessary.
A perfunctory glance at the results of Section 3 shows that our bounds are not tight, and it would be great to see some natural CSPs in this region manifesting complexities such as Pspace-complete. It is informative to compare our Table 1 with Table 1 in [17]. Our weird formulation of these CSPs belies the fact there are more natural versions where, for \( \mathcal{O} \subseteq \{ -, \cap, \cup, +, \times \} \), we ask about CSP(\( \mathcal{P}(\mathbb{N}); \mathcal{O} \)), where \( \mathcal{P}(\mathbb{N}) \) is the power set of \( \mathbb{N} \), rather than the somewhat esoteric CSP(\{\mathbb{N}\}; \mathcal{O}). \) Indeed, if we replace complement “−” by set difference “\( \setminus \)”, these questions could also be phrased for just the finite sets of \( \mathcal{P}(\mathbb{N}) \). These are all reasonable questions where we are not so sure of the boundary of decidability. Indeed, the complexity might be higher when the domain is \( \mathcal{P}(\mathbb{N}) \), in comparison to \{\mathbb{N}\}.

Meanwhile, the results of Section 4 need to be extended to a classification of complexity for all CSP(\( \Gamma \)), where \( \Gamma \) is a reduct of Skolem Arithmetic (\( \mathbb{N}; \times \)). We anticipate the first stage is to complete the classification for CSP(\( \mathbb{N}; \times, U \)) where \( U \) is fo-definable in (\( \mathbb{N}; \times \)). We conjecture that Theorem 5 is tight in the sense that the outstanding cases are in P (of course this is trivial if 0 or 1 \( \in U \)).

**Conjecture 1.** Let \( U \subseteq \mathbb{N} \setminus \{0, 1\} \) be fo-definable in (\( \mathbb{N}; \times \)), such that basis(\( U \)) is not finite. Then CSP(\( \mathbb{N}; \times, U \)) is in P.

If this can be proved, the task is to extend to all problems of the form CSP(\( \Gamma \)) where \( \Gamma \) is an fo-expansion of (\( \mathbb{N}; \times \)). Since adding disequality already results in an NP-hard problem, we imagine this improvement is achievable. Finally, the task is to consider reducts, and not just fo-expansions, of (\( \mathbb{N}; \times \)). This might be harder, and will surely involve a specialised theorem of the form ‘Petrus’ in [5].

The mechanism through which we derived a polynomial algorithm to solve Proposition 11 belies a close relationship between Skolem Arithmetic and Presburger Arithmetic (integers with addition). This was well-known to Mostowski [29], who was able to prove decidability of Skolem Arithmetic through seeing it as a certain weak direct product of Presburger Arithmetic. Thus, it makes sense to study CSP(\( \Gamma \)), where \( \Gamma \) is a reduct of Presburger Arithmetic (\( \mathbb{N}; + \)), in tandem with the program for Skolem Arithmetic. A first step would be to classify CSP(\( \mathbb{N}; +, U \)) where \( U \) is fo-definable in (\( \mathbb{N}; + \)) and a group in Dresden [5] is working on this problem. Their polynomial algorithms may yet yield the solution to Conjecture 1.

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Appendix

Missing proofs from Section 3

We start with the observation that the equivalence of arithmetic formulas reduces to functional CSPs. This yields several lower bounds for the CSPs.

**Proposition 1** For $\mathcal{O} \subseteq \{ -, \cup, \cap, +, \times \}$ it holds that $\text{EF}_N(\mathcal{O}) \leq_{m} \log m \text{ CSP}(\{N\}; \mathcal{O})$.

**Proof.** An $\text{EF}_N(\mathcal{O})$-instance $(F_1, F_2)$ is mapped to the $\text{CSP}(\{N\}; \mathcal{O})$-instance $F_1 = F_2$.

**Corollary 1**
1. $\text{CSP}(\{N\}; \{-, \cup, \cap, +\})$ and $\text{CSP}(\{N\}; \{-, \cup, \cap, \times\})$ are $\leq_{m} \log$-hard for PSPACE.
2. $\text{CSP}(\{N\}; \{\cup, \cap, +\})$, $\text{CSP}(\{N\}; \{\cup, \cap, \times\})$, $\text{CSP}(\{N\}; \{\cup, +\})$, and $\text{CSP}(\{N\}; \{\cup, \times\})$ are $\leq_{m} \log$-hard for $\Pi^P_2$.

**Proof.** The statements follow from Proposition 1 and the following facts [16]: $\text{EF}_N(\{-, \cup, \cap, +\})$ and $\text{EF}_N(\{-, \cup, \cap, \times\})$ are $\leq_{m} \log$-complete for PSPACE. $\text{EF}_N(\{\cup, \cap, +\})$, $\text{EF}_N(\{\cup, \cap, \times\})$, $\text{EF}_N(\{\cup, +\})$, and $\text{EF}_N(\{\cup, \times\})$ are $\leq_{m} \log$-complete for $\Pi^P_2$.

**Proposition 2** $\text{CSP}(\{N\}; +, \times)$ is $\leq_{m} \log$-hard for $\Sigma_1$.

**Proof.** By the Matiyasevich-Robinson-Davis-Putnam theorem [26,12], there exists an $n \in \mathbb{N}$ and a multivariate polynomial $p$ with integer coefficients such that for every $A \in \Sigma_1$ there exists an $a \in \mathbb{N}$ such that

$$x \in A \iff \exists y \in \mathbb{N}^n, p(a, x, y) = 0.$$  

In the equation $p(a, x, y) = 0$ we can move negative monoms and negative constants to the right-hand side. This yields multivariate polynomials $l$ and $r$ with coefficients from $\mathbb{N}$ such that

$$x \in A \iff \exists y \in \mathbb{N}^n, l(a, x, y) = r(a, x, y).$$

The right-hand side is a $\text{CSP}(\{N\}; +, \times)$-instance. Hence $A \leq_{m} \log \text{ CSP}(\{N\}; +, \times)$ for every $A \in \Sigma_1$.

**Proposition 3** $\text{CSP}(\{N\}; \cup, \cap, +, \times) \in \Sigma_1$.

**Proof.** It is decidable whether a given assignment satisfies a $\text{CSP}(\{N\}; \cup, \cap, +, \times)$-instance. Hence testing the existence of a satisfying assignment is in $\Sigma_1$.

**Proposition 4** $\text{CSP}(\{N\}; -, \cup, \cap, +, \times) \in \Sigma_2$. 
Proof. By Glaßer et al. [10], \( EC_N(\neg, \cup, \cap, +) \in \Delta_2 \). Consider an arbitrary \( \text{CSP}\{\{N\}; \neg, \cup, \cap, +, \times\} \)-instance \( x := \exists y \in N^n \{ t_0 = t_1 \wedge \cdots \wedge \langle t_{2m} = t_{2m+1} \rangle \}. \) It holds that

\[
x \in \text{CSP}\{\{N\}; \neg, \cup, \cap, +, \times\} \iff \exists y \in N^n [EC_N(t_0, t_1) \wedge \cdots \wedge EC_N(t_{2m}, t_{2m+1})].
\]

The right-hand side is a \( \Sigma_2 \) predicate.

**Proposition 5** \( \text{CSP}\{\{N\}; +\} \) is \( \leq_{m}^{\log} \)-hard for NP.

Proof. It is known that 3SAT \( \leq_{m}^{\log} \text{SC}_N(\{\cap, \times\}) \) [13]. The reduction has the additional property that it outputs pairs \((C, b)\) where the circuit \( C \) is connected in the sense that from each gate there exists a path to the output gate. Hence it suffices to construct a \( \leq_{m}^{\log} \)-reduction that works on \( \text{SC}_N(\{\cap, \times\})\)-instances \((C, b)\) where \( C \) is connected.

For such a pair \((C, b)\) we construct a \( \text{CSP}\{\{N\}; +\} \)-instance where each gate \( g \) is represented by the variable \( g \). Moreover, each gate \( g \) causes the following constraints: If \( g \) is an assigned input gate with value \( k \in N \), then we add the constraint \( g = k \). For unassigned input gates no additional constraints are needed. If \( g \) is a \( + \)-gate with predecessors \( g_1 \) and \( g_2 \), then we add the constraint \( g = g_1 \cdot g_2 \). If \( g \) is a \( \cap \)-gate with predecessors \( g_1 \) and \( g_2 \), then we add the constraints \( g = g_1 \) and \( g = g_2 \). If \( g \) is the output gate, then this causes the additional constraint \( g = b \). Finally, if \( g_1, \ldots, g_n \) are the gates in \( C \) and \( c_1, \ldots, c_m \) are the constraints described above, then the reduction outputs the \( \text{CSP}\{\{N\}; +\} \)-instance \( \varphi := \exists g_1, \ldots, g_n [c_1 \wedge \cdots \wedge c_m] \).

It remains to argue that for connected \( C \) it holds that

\[
(C, b) \in \text{SC}_N(\{\cap, \times\}) \iff \varphi \in \text{CSP}\{\{N\}; +\}.
\]

Assume \((C, b) \in \text{SC}_N(\{\cap, \times\})\) and consider an assignment that produces \( \{b\} \) at the output gate. Since \( C \) is connected, each gate \( g_i \) computes a singleton \( \{a_i\} \). Hence \( a_1, \ldots, a_n \) is a satisfying assignment for \( \varphi \), which shows \( \varphi \in \text{CSP}\{\{N\}; +\} \).

Assume \( \varphi \in \text{CSP}\{\{N\}; +\} \). Let \( a_1, \ldots, a_n \) be a satisfying assignment for \( \varphi \) and let \( l \) be the number of \( C \)'s input gates. The constraints in \( \varphi \) make sure that \( C(a_1, \ldots, a_i) \) produces \( \{a_i\} \) at gate \( g_i \). In particular, \( C(a_1, \ldots, a_i) \) produces \( \{b\} \) at the output gate, which shows \((C, b) \in \text{SC}_N(\{\cap, \times\})\).

**Proposition 6** \( \text{CSP}\{\{N\}; +\} \) is \( \leq_{m}^{\log} \)-hard for NP.

Proof. It suffices to show \( \text{SC}_N(\{+\}) \leq_{m}^{\log} \text{CSP}\{\{N\}; +\} \) [18]. The proof is similar to the proof of Proposition 5 but easier, since we have no \( \cap \)-gates and hence we do not need the assumption that \( C \) is connected.

**Proposition 7** \( \text{CSP}\{\{N\}; \neg, \cap, \cup\} \) is \( \leq_{m}^{\log} \)-complete for NP.
Proposition \[9\]

1. \(\text{CSP}\{\{N\}; \cap, +) \leq_{NP}^{m} \text{CSP}\{\{N\}; +, =, \#}\).
2. \(\text{CSP}\{\{N\}; \cap, \times) \leq_{NP}^{m} \text{CSP}\{\{N\}; \times, =, \#}\).

Proof. We show the first statement, the proof of the second one is analogous.

For a term \(t\), let \(t'\) be the term obtained from \(t\) if every subterm of the form \(s_1 \cap s_2\) is replaced with \(s_1\).

We describe the \(\leq_{NP}^{m}\)-reduction on input of a \(\text{CSP}\{\{N\}; \cap, +)\)-instance

\[
\varphi := \exists x_1, \ldots, x_n[t_0 = t_1 \land \cdots \land t_{2m} = t_{2m+1}].
\]

For each atom \(t_{2i} = t_{2i+1}\), we guess nondeterministically whether \(t_{2i} = t_{2i+1} \in \{N\}\) or \(t_{2i} = t_{2i+1} = \emptyset\). If we guessed \(t_{2i} = t_{2i+1} \in \{N\}\), then replace \(t_{2i}\) with \(t'_{2i}\), replace \(t_{2i+1}\) with \(t'_{2i+1}\), and for every subterm \(s_1 \cap s_2\) that appears in \(t_{2i}\),
or \( t_{2i+1} \). If we guessed \( t_{2i} = t_{2i+1} = \emptyset \), then guess a subterm \( u_1 \cap u_2 \) in \( t_2i \), guess a subterm \( u_3 \cap u_4 \) in \( t_{2i+1} \), remove the atom \( t_2i = t_{2i+1} \), and add the constraints \( u_1' \neq u_2' \) and \( u_3' \neq u_4' \). The obtained formula \( \psi \) is the result of the \( \leq \text{NP} \)-reduction.

We argue that the described \( \leq \text{NP} \)-reduction reduces the \( \text{CSP}(\{N\};\cap,+) \) to the \( \text{CSP}(\{N\};+,=,\neq) \).

Assume \( \varphi \in \text{CSP}(\{N\};\cap,+) \) and fix some satisfying assignment \( a = (a_1, \ldots, a_n) \in \mathbb{N}^n \). Consider the nondeterministic path of the reduction that for all atoms correctly guesses whether \( t_{2i} = t_{2i+1} \in \{N\} \) or \( t_{2i} = t_{2i+1} = \emptyset \), and that for all \( t_{2i} = t_{2i+1} = \emptyset \) guesses subterms \( u_1 \cap u_2 \in t_{2i} \) and \( u_3 \cap u_4 \in t_{2i+1} \) such that \( u_1, u_2, u_3, u_4 \in \{N\}, u_1 \neq u_2, \) and \( u_3 \neq u_4. \) If \( t_{2i} = t_{2i+1} \in \{N\} \), then \( t_{2i}' = \emptyset \) and \( t_{2i+1}' = \emptyset \), and hence the formula is still satisfied after replacing \( t_{2i} \) with \( t_{2i}' \) and \( t_{2i+1} \) with \( t_{2i+1}' \). Moreover, the added constraints \( s_1' = s_2' \) are satisfied, since in \( t_{2i} \) and \( t_{2i+1} \) all subterms \( s_1 \cap s_2 \) must be nonempty. If \( t_{2i} = t_{2i+1} = \emptyset \), then after removing the atom \( t_{2i} = t_{2i+1} \) and after adding the constraints \( u_1' \neq u_2' \) and \( u_3' \neq u_4' \) the formula is still satisfied. So at the described nondeterministic path the reduction outputs a formula \( \psi \in \text{CSP}(\{N\};+,=,\neq) \).

Assume there is a nondeterministic path where the reduction outputs a formula \( \psi \in \text{CSP}(\{N\};+,=,\neq) \). Consider a satisfying assignment \( a \) for \( \psi \), we claim that \( a \) satisfies \( \varphi \). If this is not true, then \( \varphi \) must have an atom \( t_{2i} = t_{2i+1} \) that is not satisfied by \( a \).

Case 1: At the path that produced \( \psi \) we guessed that \( t_{2i} = t_{2i+1} \in \{N\} \). In this case we added the constraints \( s_1' = s_2' \), which ensure that \( t_{2i} = t_{2i}' \) and \( t_{2i+1} = t_{2i+1}' \). Hence under the assignment \( a \) it holds that \( t_{2i} = t_{2i}' = t_{2i+1} = t_{2i+1}' \), which contradicts the assumption that \( t_{2i} = t_{2i+1} \) is not satisfied by \( a \).

Case 2: At the path that produced \( \psi \) we guessed that \( t_{2i} = t_{2i+1} = \emptyset \). Here we added the constraints \( u_1' \neq u_2' \) and \( u_3' \neq u_4' \), which are satisfied by \( a \). Hence under the assignment \( a \) we have \( t_{2i} = t_{2i+1} = \emptyset \), which contradicts the assumption that \( t_{2i} = t_{2i+1} \) is not satisfied by \( a \).

It follows that \( a \) satisfies \( \varphi \) and hence \( \varphi \in \text{CSP}(\{N\};\cap,+) \).

**Corollary 4** \text{CSP}(\{N\};\cap,+) \text{CSP}(\{N\};\cap,\times) \in \text{NP}.

**Proof.** \text{CSP}(\{N\};+,=,\neq)-instances and \text{CSP}(\{N\};\times,=,\neq)-instances are formulas of existential Presburger arithmetic and existential Skolem arithmetic, which are both decidable in \text{NP}. Now the statement follows from Proposition 4.