Nonequilibrium effects in superconducting systems have been gaining new attention due to the increased activities in the field of mesoscopic electron transport. In contrast to earlier work on nonequilibrium superconductivity the new experiments show non-local and size-dependent effects by reaching temperatures below the characteristic Thouless energy $E_{\text{Th}} = \mathcal{D}/d^2$. The understanding of this regime is not only of fundamental interest, but also important for nano-electronic applications. The size reduction of electronic devices is accompanied by an increase in operation frequency. For instance, in the system considered below the latter is limited by $E_{\text{Th}}$.

Recently Pothier et al. \cite{Pothier95} probed the quasiparticle properties in short diffusive wires by coupling tunneling contacts to it. They found that in mesoscopic wires the distribution function has a nonequilibrium energy dependence, with a double-step structure at the electrochemical potentials of both reservoirs. From the smearing of this distribution they further drew conclusions about inelastic relaxation processes. In different setups, sketched in Figs. 1a and b, they demonstrated that in a superconductor – normal metal heterostructure the nonequilibrium quasiparticle distribution due to a normal current flow in N can be used to tune the supercurrent between the superconducting electrodes. This opens the perspective to use such devices as ultrafast transistors.

To account for their experimental findings, Morpurgo et al. \cite{Morpurgo95} proposed a qualitative model based on a quasi-equilibrium distribution function with locally enhanced effective electron temperature. This picture is appropriate in the limit of strong electron-electron interactions. However, inelastic processes have only a weak effect in the mesoscopic sample considered here. In fact we find the best performance of the devices in the opposite limit.

In this article we will describe mesoscopic superconductor – normal metal heterostructures as shown in Fig. 1a and b. For a quantitative analysis we use the quasiclassical theory. It accounts well for the relevant physics: (i) The spectral properties in the normal metal are modified by the proximity effect due to the presence of the superconducting electrodes.

(ii) The nonequilibrium quasiparticle distribution function is found as the solution of a kinetic equation. If

\begin{equation}
\pi = \text{Th} \frac{d}{\text{Th}}.
\end{equation}

For best performance as a transistor, in the setup of Fig. 1a, the width of the superconducting contacts $d_S$ should be chosen narrow compared to the width and length of the normal wire, $d$ and $L$, respectively. The first condition assures that only a small fraction of the control normal current is diverted through the superconductors, and accordingly the quasiparticle distribution function in N is little disturbed by the presence of the superconducting electrodes. The second condition assures that the voltage is nearly constant along the superconducting leads, while the total voltage drop responsible for the nonequilibrium and reduction of the supercurrent may be large. Since the effect relies on a deviation of the distribution function from local equilibrium with shifted electro-chemical potential, the normal wire should have mesoscopic dimensions, i.e. the length $L$ should not exceed the inelastic relaxation length $l_{\text{in}}$.

The presence of the superconducting electrodes induces correlations in the normal metal (proximity effect), which are responsible for a supercurrent. Their decay length depends on energy: correlations with en-
ergy $\epsilon \gg E_{Th}$ decay exponentially, while those within the range of order $E_{Th}$ carry the supercurrent. If, in a nonequilibrium situation, these states are occupied and in this way blocked for superconducting correlations, the superconductivity is weakened and the supercurrent reduced. This suppression mechanism will be described in the following, based on the real-time formalism of quasiclassical Green-Keldysh functions in the diffusive limit.

In the first step we describe the proximity effect in the normal metal by analyzing Usadel’s equations. The standard parameterization of normal and anomalous retarded Green functions $G^R = \cosh \alpha$ and $F^R = \sinh \alpha e^{i\chi}$ allows us to write these equations, for the normal metal region between the superconducting electrodes, in the form

$$D\partial_x^2 \alpha = -2i\epsilon \sinh \alpha - (D/2) (\partial_x \chi)^2 \sinh 2\alpha,$$

$$\partial_x j_\epsilon = 0 , \quad j_\epsilon = (\partial_x \chi) \sinh^2 \alpha . \quad (1)$$

Here, $D$ is the diffusion coefficient and $x$ the coordinate normal to the NS interfaces. We also introduced the energy-dependent ‘spectral current’ $j_\epsilon$. For simplicity we ignored here a dependence of the spectral quantities on the coordinate $y$ parallel to the interfaces. This dependence should merely lead to a quantitative modification of our results and conclusions. In the realistic limit $\Delta \gg E_{Th}$ and for transparent metallic interfaces the boundary conditions at these interfaces $x = \pm d/2$ read (see \cite{1} and Refs. therein)

$$\alpha(\pm d/2) = -i\pi/2 , \quad \chi(\pm d/2) = \pm \phi/2. \quad (2)$$

In two limits we find analytic solutions of Eqs. (1, 2). For low energies $\epsilon \ll E_{Th}$ we obtain

$$\alpha \simeq -i\pi/2 + (\epsilon/E_{Th}) a(\phi) + O(\epsilon^3)$$

$$\chi \simeq \phi x/d - (\epsilon/E_{Th})^2 b(\phi) + O(\epsilon^3) ,$$

where $a(\phi)$ and $b(\phi)$ are real-valued functions (omitted for brevity), while for high energies $\epsilon \gg E_{Th}$ we obtain

$$F^R = F_0(x-d/2)e^{i\phi/2} + F_0(d/2-x)e^{-i\phi/2} \quad (3)$$

$$F_0(s) = 4q \frac{1+q^2}{(1-q^2)^2} , \quad q(s) = i(\sqrt{2} - 1)e^{-s\sqrt{-2i\epsilon/E_{Th}}},$$

$$\text{Im}(j_\epsilon) = 64 \sin \phi \text{Im} \left( \frac{(1+q^2)(1+6q^2+q^4)}{(1-q^2)^3} q^2 \right) \bigg|_{s=L/2} .$$

In addition we have studied the problem numerically. Combining our results in Fig. 2 we observe the following features of the spectral current $\text{Im}(j_\epsilon)$. It is an odd function of $\epsilon$ and shows a proximity induced mini-gap $\Delta_0 \simeq 3.2E_{Th}$ at $\phi = 0$, below which $\text{Im}(j_\epsilon) = 0$. This gap decreases with increasing $\phi$ and vanishes at $\phi = \pi$. At energies directly above the gap, $\text{Im}(j_\epsilon)$ increases sharply, but rapidly decreases at higher $\epsilon$. At large energies, it changes sign and oscillates around zero with exponentially decaying amplitude.

Next we determine the nonequilibrium quasiparticle distribution in the normal metal between the reservoirs, which are at different electrochemical potentials $\pm eV/2$. In a mesoscopic length wire $L \ll l_{eq}$ in the diffusive limit the distribution function obeys the kinetic equation

$$\partial^2_y f = 0 \quad . \quad (4)$$

In the absence of superconducting contacts its solution

$$f(\epsilon, y) = (1/2 - y/L) f_{eq}^{\epsilon}(\epsilon + eV/2) + (1/2 + y/L) f_{eq}^{\epsilon}(\epsilon - eV/2) \quad (5)$$

has two temperature-rounded steps at the electrochemical potentials of both reservoirs. The step heights depend on the position along the wire; in this way the distribution function interpolates linearly between the boundary conditions at $y = \pm L/2$. This functional dependence had been detected in the experiments of Pothier et al.

Although the distribution function definitely has not a thermal form, a local electrochemical potential of the normal metal, $\mu(y)$, and an effective electron temperature can be defined by the corresponding moments of the distribution function. For instance $\mu(y)$ follows from

$$\int_{-\infty}^{\infty} d\epsilon [f(\epsilon, y) - f_{eq}^{\epsilon}(\epsilon - \mu(y))] = 0 ,$$

where $f_{eq}$ denotes the Fermi function. In the following we will consider the situation where the electrochemical potential of the superconductors coincides with the local value of the normal metal, which guarantees that there is no net current out of the normal metal into the superconductors. Since we further have chosen the size of the superconducting contacts, $d_5$, small compared to the width and length, $d$ and $L$, of the normal wire, the distribution function in N is little disturbed by the contacts.

On the other hand, the quasiparticle distribution function influences the superconducting correlations induced in the normal metal. The supercurrent through the heterostructure is given by

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{The spectral current $\text{Im}(j_\epsilon)$ as a function of energy for different values of the phase difference $\phi$.}
\end{figure}
where \(1/R_d = 2e^2N_0DS/d\), \(S\) is the junction area, and \(y_S\) denotes the position of the superconducting electrodes. It is important to note that the energy is measured relative to the electrochemical potential of the superconductors, which we have chosen to coincide with the local value of the normal metal, \(\mu(y)\). Due to the odd symmetry of \(\text{Im}(j_e)\), only the odd component of a nonequilibrium quasiparticle distribution modifies \(I_S\). Accordingly, the first term in the integral (6) can be written as \(\frac{4}{\pi} - f(\epsilon, y_S) + f(-\epsilon, y_S)\), which displays that an excess number of electron-like or of hole-like excitations have the same effect on the supercurrent.

The largest effect is found when the superconducting electrodes are placed symmetrically between the two normal reservoirs (i.e. at \(y = 0\), see eq. (6)). The resulting modification of the supercurrent across the SNS junction is presented in Fig. 3 as a function of the voltage \(V\) across the normal metal.

\[
I_S = \frac{d}{2R_d} \int_{-\infty}^{\infty} d\epsilon \left[ 1 - 2f(\epsilon, y_S) \right] \text{Im}(j_e),
\]

FIG. 3. The supercurrent as function of control voltage (at \(T = 0\)) and temperature (at \(V = 0\)) for various values of \(\phi\).

At low temperatures \(T \ll eV\), \(f(\epsilon, 0)\) deviates from the equilibrium value only in the window \(-eV/2 < \epsilon < eV/2\). Since the spectral current \(\text{Im}(j_e)\) vanishes for \(\epsilon < \epsilon_g\) there is no modification of \(I_S\) for small voltages \(eV < 2\epsilon_g\). On the other hand, for \(eV > 2\epsilon_g\), extra quasiparticle and hole states with energies \(|\epsilon|\) below \(eV/2\) are occupied and the supercurrent is diminished. Since this energy window increases with increasing \(V\) the supercurrent decays rapidly with voltage (cf. Fig. 3). This is exactly what has been observed in the experiments. Furthermore, at still larger voltage \(eV > 10E_{Th}\), the supercurrent changes sign since the integral in (6) is dominated by the energy interval where \(\text{Im}(j_e) < 0\) (Fig. 2). We thus find a transition to a so-called \(\pi\)-junction controlled by nonequilibrium effects. This effect is rather pronounced, the critical current of the \(\pi\)-junction is approximately 30% of \(I_c\) at \(T = eV = 0\).

For high voltages or temperatures \(eV, T \gg \epsilon_g\) we find \(I_S = I_c \sin \phi\) with critical current

\[
I_c R_d = \frac{64\pi}{3 + 2\sqrt{2}} \exp(-\sqrt{\frac{\Omega V}{E_{Th}}}) \cos \left(\frac{eV}{2\sqrt{\Omega V E_{Th}}} + \varphi_0\right) \times
\]

\[
\times \left\{ \begin{array}{ll}
T & \text{for } eV \gg \epsilon_g, T \ll \epsilon_g \\
\frac{\pi}{2} \times \tan^{-1} \left(\frac{eV}{2\pi T}\right) & \text{for } T \gg \epsilon_g
\end{array} \right.
\]

Here \(\Omega V = \pi T + \sqrt{(\pi T)^2 + (eV/2)^2}\) and

\[
\varphi_0 = \left\{ \begin{array}{ll}
\pi/2 & \text{for } eV \gg \epsilon_g, T \ll \epsilon_g \\
\frac{1}{2} \times \tan^{-1} \left(\frac{eV}{2\pi T}\right) & \text{for } T \gg \epsilon_g
\end{array} \right.
\]

Nonequilibrium effects also influence the current-phase relation \(I_S(\phi)\). The rich variety of different curves at different voltages is displayed in Fig. 4. In contrast a quasi-equilibrium theory would always predict a functional dependence as shown in the inset of the figure.

Since the supercurrent decays exponentially as a function of both \(T\) and \(V\), one might try to describe the system properties by a quasi-equilibrium theory with effective \(V\)-dependent temperature \(T^*\). Exactly this strategy was adopted in Ref. <sup>3</sup> with \(T^* = \sqrt{T^2 + \gamma^2 (eV)^2}\). The best fit to the experimental data was obtained for \(\gamma \approx 6K(mV)^{-1}\). On the other hand, as demonstrated above, the analogy between temperature and voltage is incomplete and even misleading. The distribution function in the normal metal in general cannot be described by the Fermi function with effective temperature \(T^*\). One of the most striking consequences, the transition to a \(\pi\)-junction, is not obtained in the quasi-equilibrium description. Apart from this qualitative difference, the dependence of \(I_c(V, T)\) <sup>2</sup> deviates from that suggested in Ref. <sup>3</sup>. While we find a fairly good agreement between our results and the data <sup>3</sup> at the temperatures of the experiment, we cannot summarize our results in general by
an effective temperature model. It would be desirable to expand the experiments to a parameter range where the difference between the two descriptions become more pronounced.

The configuration of Fig. 1a is but one realization of a mesoscopic system where the supercurrent can be controlled by an externally applied voltage. Another realization, also studied in Ref. 4, is depicted in Fig. 1b. In this case the distribution function in the N-layer between two superconductors is driven out of equilibrium by the normal current flowing parallel to the supercurrent. Provided \( d \ll L \) there is practically no voltage drop across the SNS junction and only dc Josephson effect can be considered. Since the distribution function has the same form \( \tilde{f} \) as before, the previous results for \( I_S(V) \) apply also for the structure of Fig. 1b. Again good agreement with the experimental findings \( \tilde{f} \) is observed.

Yet another system with a voltage controlled supercurrent was studied by Volkov. He considered a SINIS system where the normal metal was thin and separated by low transparency barriers from the superconductors. In this case the superconductors and the normal metal are in equilibrium, except that their electrostatic potentials are shifted relative to each other, with the total voltage drop across the barriers. Although this nonequilibrium situation is very different from the one discussed here, he also finds a voltage-dependent supercurrent reduction as well as a transition to a \( \pi \)-junction.

If \( L \) is not large compared to \( d_S \) and \( d \), the conversion between super- and normal currents in the junction area cannot be neglected and the theoretical analysis has to be extended. For instance the distribution functions depend both on \( x \) and \( y \). A thorough analysis of the microscopic theory \( \tilde{f} \) reveals further that the expression for the current \( \tilde{f} \) has to be extended, and that odd and even component (in energy) of the distribution function, \( f_L \) and \( f_T \), obey two coupled diffusion equations, but with different, position-dependent effective diffusion coefficients. They are coupled by terms of the form \( \text{Im} f_L \cdot \nabla f_T \). In this case the suppression of \( f_T \) influences \( f_L \) and weakens the performance of the transistor, although qualitatively the physical situation remains unchanged.

We now turn to the important practical question: how efficient is this device as a transistor? The control and signal voltage, \( V \) and \( V_S = I_c R_d \), are both of the order of the Thouless energy \( E_{\text{Th}} \). Thus, no voltage gain is obtained. However, the power amplification is proportional to the ratio of the relevant resistors \( R_L/R_d \). Here \( R_L \) is the resistance of the normal metal of length \( L \), while \( R_d \), introduced in \( \tilde{f} \), is the resistance between the superconducting electrodes in a situation where \( I_c \) is low and this transport is dissipative as well. Since both are governed by the same material-dependent conductivity the ratio depends on the relevant lengths, \( R_L/R_d \propto L d_S/d^2 \). Hence, by choosing a sufficiently long control line, \( L \gg d, d_S \), a power amplification can be achieved. The limitation in operation frequency in the mesoscopic regime is also provided by the Thouless energy.

In summary, we have presented a microscopic description of nonequilibrium electronic properties of mesoscopic SNS heterostructures. The distribution function in the normal metal can be driven far from equilibrium by a voltage applied at a distance \( \sim L \) from the junction. This distance is limited only by the inelastic relaxation length \( l_m \). We analyzed how the supercurrent across the sample is reduced by this control voltage. The strongest reduction and, hence, best performance of the device is found in a mesoscopic situation, when the distribution function deviates significantly from a local equilibrium form. We established the connection to experiments and suggested further tests. The possibility to control the supercurrent by an external voltage allows several technical applications, for instance the use as a high-frequency transistor with power gain proportional to the ratio between length and width of the normal wire.

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