Analysis of the global stability boundaries in type 2 PLLs

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Abstract. This paper discusses one of the approaches to analysis of the global stability boundaries of type 2 PLLs.

Introduction
Synchronization is a classical problem in nonlinear control systems. The phase and frequency in electrical circuits are usually synchronized using Phase-Locked Loops (PLLs) [1–3]. The main idea of a PLL is to synchronize phase and frequency of the Voltage-Controlled Oscillator (VCO) to the reference signal using the phase detector and loop filter to extract the phase error.

The following frequency difference concepts are widely used to characterize synchronization properties: a hold-in range (local stability) and a pull-in/acquisition range (global stability) [4–7]. Stability of the first-order and the second-order PLLs is well-studied (see, e.g., [3,5,8,9]). Since the phase-locked loops are described by nonlinear differential and difference equations [3,10,11], analysis of higher-order models is complicated. In this paper, we use Lyapunov ideas and numerical simulation which allow to study stability of the third-order type 2 PLLs having loop filter with exactly one pole at the origin.

1. Mathematical model of a type 2 PLL and its local analysis
Consider the type 2 PLL block diagram in Figure 1 and the corresponding system of differential equations

\[
\begin{align*}
\dot{x}_1 &= K_{PD} \sin \theta_e, \\
\dot{x}_2 &= -\frac{1}{\tau_p} x_2 + \frac{\tau_1 - \tau_p}{\tau_p} K_{PD} \sin \theta_e, \\
\dot{\theta}_e &= \omega_{\text{free}} - K_F K_{vco} \left( x_1 + x_2 + \frac{\tau_1 \tau_2}{\tau_p} K_{PD} \sin \theta_e \right). 
\end{align*}
\]

Here, \(\theta_e(t) = \theta_{\text{ref}}(t) - \theta_{\text{vco}}(t)\) is a phase error, i.e., the difference between a reference phase \(\theta_{\text{ref}}(t) = \omega_{\text{ref}} t + \theta_{\text{ref}}(0)\) and a phase \(\theta_{\text{vco}}(t)\) of the voltage-controlled oscillator (VCO), \(K_{PD} > 0\) is a gain and \(\sin(\cdot)\) is a characteristic of the phase detector; \(x(t) = (x_1(t), x_2(t))\) is a state of the loop filter with transfer function \(F(s) = \frac{K_F (1+s\tau_1)(1+s\tau_2)}{s(1+s\tau_p)}\) where \(K_F > 0, \tau_1 > 0, \tau_2 > 0, \tau_p > 0, \tau_1 \neq \tau_p, \tau_2 \neq \tau_p\); the output of the loop filter is \(v_F(t) = K_F (x_1(t) + x_2(t) + \frac{\tau_1 \tau_2}{\tau_p} K_{PD} \sin[\theta_e(t)])\).

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\( K_{\text{vco}} > 0 \) is a gain of the VCO; \( \omega_{\text{free}} = \omega_{\text{ref}} - \omega_{\text{free}}^\text{vco} \) is a frequency error, i.e., the difference between the input frequency \( \omega_{\text{ref}} \) and a free-running frequency \( \omega_{\text{free}}^\text{vco} \) of the VCO.

To perform the local stability analysis, consider the Jacobian evaluated at the equilibrium point \((\omega_{\text{free}}^\text{vco}/(K_{F}K_{\text{vco}}), 0, \pi k), \ k \in \mathbb{Z}:

\[
J_k = J(\omega_{\text{free}}^\text{vco}/(K_{F}K_{\text{vco}}), 0, \pi k) = \begin{bmatrix}
0 & 0 & 0 \\
0 & 1/\tau_{p} \cdot \frac{(\tau_{z1} - \tau_{p})(\tau_{p} - \tau_{z2})}{\tau_{z1} - \tau_{p}} K_{\text{PD}} \cos(\pi k) \\
-K_{F}K_{\text{vco}} & -K_{F}K_{\text{vco}} & -K_{\text{PD}}K_{F}K_{\text{vco}} \frac{\tau_{z1} \tau_{z2}}{\tau_{p}} \cos(\pi k)
\end{bmatrix}.
\]  

(2)

Stability of the linearized models \( \dot{x} = J_k x \) (local stability of the equilibria of nonlinear model (1)) can be obtained by analyzing the characteristic polynomial:

\[
\chi(s) = s^2(1 + s\tau_{p}) + K_{\text{PD}}K_{F}K_{\text{vco}}(1 + s\tau_{z1})(1 + s\tau_{z2}) \cos(\pi k).
\]  

(3)

Using the Routh-Hurwitz criterion, we get that equilibria \((\omega_{\text{free}}^\text{vco}/(K_{F}K_{\text{vco}}), 0, \pi + 2\pi m)\) are unstable for any \( m \in \mathbb{Z} \) and equilibria \((\omega_{\text{free}}^\text{vco}/(K_{F}K_{\text{vco}}), 0, 2\pi m)\) are asymptotically stable for any \( m \in \mathbb{Z} \) if and only if the following condition is satisfied:

\[
K_{\text{PD}}K_{F}K_{\text{vco}}\tau_{z1}\tau_{z2}(\tau_{z1} + \tau_{z2}) > \tau_{p} - \tau_{z1} - \tau_{z2}.
\]  

(4)

2. Global analysis and simulation in MATLAB

In this section, the global stability of system (1) is analysed. If any trajectory of system (1) tends to the stationary set, then the system is called globally stable (see, e.g., [12] and the corresponding discussion within). Using the following Lyapunov function

\[
V(x_1, x_2, \theta_e) = \frac{K_{F}K_{\text{vco}}}{2} \left( x_1 - \frac{\omega_{\text{free}}}{K_{F}K_{\text{vco}}} \right)^2 + K_{F}K_{\text{vco}} \frac{\tau_{z1}\tau_{z2}}{2} \frac{(\tau_{z1} + \tau_{z2})(\tau_{z1} + \tau_{z2} - \tau_{p})}{(\tau_{z1} - \tau_{p})^2(\tau_{p} - \tau_{z2})^2} x_2^2 + \int_0^{\theta_e} K_{\text{PD}} \sin \sigma d\sigma
\]  

(5)

and the extension of the direct Lyapunov method for the cylindrical phase space (see, e.g., [12, 13]), the condition of global stability of system (1) was obtained in [14]:

\[
\tau_{z1} + \tau_{z2} - \tau_{p} > 0.
\]  

(6)
Notice that condition (6) is a conservative estimate of the global stability domain in the parameters space. Further, it will be refined numerically by studying self-excited oscillations in system (1). Self-excited oscillations can be localized numerically by computation of a trajectory starting from a point of unstable manifold in a neighborhood of unstable equilibrium.

Analyzing characteristic polynomial (3) at the unstable equilibria

\[ \chi^a(s) = \tau_p s^3 + \left(1 - K_{PD} K_F K_{vco} \tau_1 \tau_2 \right) s^2 - K_{PD} K_F K_{vco} (\tau_1 + \tau_2) s - K_{PD} K_F K_{vco} \]

we state that exactly two roots of the characteristic equation \( \chi^a(s) = 0 \) have negative real parts and the remaining root is positive. Hence, for the linearized system, the unstable manifold \( W^u \left( (\omega_e^1 / (K_F K_{vco}), 0, \pi + 2\pi m) \right) \) of the unstable equilibria is one-dimensional, and it is determined by the eigenvector corresponding to the positive eigenvalue of the Jacobian (2) at the unstable equilibria.

In our numerical procedure, we consider the behaviour of the trajectory with initial data from unstable manifold in a neighborhood of unstable equilibrium \( (\omega_e^1 / (K_F K_{vco}), 0, \pi) \). For the fixed parameters, the simulation stops when the basin of attraction is the basin of attraction of the stable equilibria\(^1\). In order to estimate the basin of attraction of the stable equilibria, consider the following Lyapunov function:

\[ V(x_1, x_2, \theta_e) = K_F K_{vco} \left( x_1 - \frac{\omega_e^1}{K_F K_{vco}} \right)^2 + K_F K_{vco} \frac{\tau_p^2}{(\tau_1 - \tau_p)(\tau_p - \tau_2)} x_2^2 + K_{PD} \theta_e^2. \]

Derivative of \( V \) along the trajectories of system (1) is

\[ \dot{V}(x_1, x_2, \theta_e) = 2K_{PD} K_F K_{vco} (x_1 - \frac{\omega_e^1}{K_F K_{vco}} + x_2) (\sin \theta_e - \theta_e) - \frac{2K_F K_{vco} \tau_p}{(\tau_1 - \tau_p)(\tau_p - \tau_2)} x_2^2 \]

\[ - 2K_{PD} K_F K_{vco} \frac{\tau_1 \tau_2}{\tau_p} \theta_e \sin \theta_e. \]

Observe that

\[ \dot{V}(x_1, x_2, \theta_e) \leq \]

\[ \leq 2K_{PD} K_F K_{vco} (|x_1 - \frac{\omega_e^1}{K_F K_{vco}}| + |x_2| + K_{PD} \frac{\tau_1 \tau_2}{\tau_p} |\theta_e|) |\sin \theta_e - \theta_e| - \frac{2K_F K_{vco} \tau_p}{(\tau_1 - \tau_p)(\tau_p - \tau_2)} x_2^2 \]

\[ - 2K_{PD} K_F K_{vco} (|x_1 - \frac{\omega_e^1}{K_F K_{vco}}| + |x_2| + K_{PD} \frac{\tau_1 \tau_2}{\tau_p} |\theta_e|) \theta_e^2 - 2K_{PD}^2 K_F K_{vco} \frac{\tau_1 \tau_2}{\tau_p} \theta_e^2. \]

Thus, if \( |x_1 - \frac{\omega_e^1}{K_F K_{vco}}| < \delta, \ 0 < |x_2| < \delta, \ |\theta_e| < \delta, \ \text{and} \ \delta < \delta_{est} = \frac{K_{PD} \tau_1 \tau_2}{2 \tau_p + K_{PD} \tau_1 \tau_2}, \) then

\[ \dot{V}(x_1, x_2, \theta_e) < 0. \]

As a result, the domain

\[ D = \{ x_1, x_2, \theta_e \mid |x_1 - \frac{\omega_e^1}{K_F K_{vco}}| < \delta_{est}, \ |x_2| < \delta_{est}, \ |\theta_e| \mod 2\pi < \delta_{est} \} \]

lies in a basin of attraction for the stable equilibria. When a trajectory of system (1) reaches the domain \( D \), the numerical procedure can be stopped.

\(^1\) Since system (1) is \( 2\pi \)-periodic in \( \theta_e \), it is sufficient to determine the basin of attraction of equilibrium \( (\omega_e^1 / (K_F K_{vco}), 0, 0) \) only.
Observe that using the substitution \( z_1(t) = \omega_{\text{free}} - K_F K_{\text{vco}} x_1(t) \) we can exclude \( \omega_{\text{free}} \) from the system. Hence, for the stability analysis of system (1) we can put \( \omega_{\text{free}} = 0 \) without loss of generality.

For the study of global stability boundaries of the third-order type 2 PLL, consider \( K_{PD} = K_F = 1, K_{\text{vco}} = 10, \tau_p = 0.9 \). The stability domains in parameter space are depicted in Fig 2.

![Figure 2.](image)

**Figure 2.** \( K_{PD} = K_F = 1, K_{\text{vco}} = 10, \tau_p = 0.9, \omega_{\text{free}} = 0 \). Thin green line (condition (6)): the domain over this line is a domain of global stability of system (1) provided by inequality (6). Dashed magenta line (condition (4)): for parameters \( \tau_{z_1} \) and \( \tau_{z_2} \) under this line system (1) has unstable equilibria only, otherwise, asymptotically stable equilibria exist. Thick blue line: the numerical procedure for self-excited oscillations.

### 3. Numerical example

Consider simulation of system (1) with parameters indicating the loss of the global stability in MATLAB Simulink (see Fig. 3). A trajectory starting from the unstable manifold in a neighborhood of unstable equilibrium was computed in this example. The simulation results show that the loop filter output \( v_F(t) \) does not synchronize to the carrier (left subfigure) and the PLL baseband model is not locked (the phase error \( \theta_e(t) \) in the right subfigure oscillates).

![Figure 3.](image)

**Figure 3.** Simulation of PLL system (1) with parameters \( \tau_{z_1}, \tau_{z_2} \) corresponding to the boundary of the global stability in MATLAB Simulink. Parameters: \( K_{PD} = K_F = 1, K_{\text{vco}} = 10, \omega_{\text{free}} = 0, \tau_p = 0.9, \tau_{z_1} = 0.12, \tau_{z_2} = 0.5575 \). Initial data: \( x_1(0) = -0.0302, x_2(0) = 0.0074, \theta_e(0) = 3.2366 \).
Figure 4 depicts the behaviour of a trajectory which starts from the unstable manifold in a neighborhood of unstable equilibrium, in the phase space. In the left subfigure, the trajectory tends to the stable equilibrium (the loop filter parameters $\tau_{z1}$ and $\tau_{z2}$ are chosen over the thin blue line in Fig. 2). The right subfigure corresponds to the periodic oscillation shown in Fig. 3: the trajectory tends to the self-excited oscillation.

**Figure 4.** Simulation of PLL system (1) in MATLAB. Left subfigure: the trajectory with initial data $x_1(0) = -0.0302$, $x_2(0) = 0.0074$, $\theta_e(0) = 3.2366$ tends to the stable equilibrium; parameters: $K_{PD} = K_F = 1$, $K_{VCO} = 10$, $\omega^\text{free}_e = 0$, $\tau_p = 0.9$, $\tau_{z1} = 0.12$; $\tau_{z2} = 0.56$. Right subfigure: the trajectory with initial data $x_1(0) = -0.0302$, $x_2(0) = 0.0073$, $\theta_e(0) = 3.2367$ tends to the self-excited oscillation; parameters: $K_{PD} = K_F = 1$, $K_{VCO} = 10$, $\omega^\text{free}_e = 0$, $\tau_p = 0.9$, $\tau_{z1} = 0.12$; $\tau_{z2} = 0.5575$.

**Conclusion**

Although the second-order type 2 analog PLL models are globally stable for any loop parameters, the higher-order models have more complex behaviour. The nonlinear analysis of these systems by Lyapunov functions often gives only sufficient conditions of the global stability and rough estimates. As a result, the analysis of the global stability boundary in the parameters space is a challenging problem. In this paper, the global stability boundaries for the third-order type 2 analog PLL were studied numerically by analyzing the self-excited oscillations. Further analysis and precise determination of the global stability boundaries demand the study of hidden oscillations by application of special analytical-numerical methods [15, 16].

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