The Mermin Fixed Point

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The most efficient known method for solving certain computational problems is to construct an iterated map whose fixed points are by design the problem’s solution. Although the origins of this idea go back at least to Newton, the clearest expression of its logical basis is an example due to Mermin. A contemporary application in image recovery demonstrates the power of the method.

1 INTRODUCTION

Fixed points arise naturally in the study of physical phenomena, be they invariants with respect to time evolution (dynamical systems) or rescaling (the renormalization group). On the practical side, fixed points form the basis of iteration schemes specifically engineered to solve particular computational problems. One of the simplest applications of this idea is Mermin’s solution\(^1\) of the self-referential digit counting puzzle:

In this sentence,
the digit 0 appears ____ times;
the digit 1 appears ____ times;
the digit 2 appears ____ times;
the digit 3 appears ____ times;
the digit 4 appears ____ times;
the digit 5 appears ____ times;
the digit 6 appears ____ times;
the digit 7 appears ____ times;
the digit 8 appears ____ times;
the digit 9 appears ____ times.

The object is to fill in all the blanks with decimal numerals to make the statement correct. Our instinct is to treat this as an exercise in logic. Mermin noted that considerably less mental energy is required by an iterative procedure, where the iterates are the vectors of ten integers which fill in the blanks: \(x = (x_0, \ldots, x_9)\).
Starting with an initial vector, say \(x(0) = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1)\), successive iterates are computed by treating the current vector as a tentative solution and then counting the actual occurrences of the ten digits; thus \(x(1) = (1, 11, 1, 1, 1, 1, 1, 1, 1, 1)\). When the map encounters a fixed point, the tentative solution is confirmed to be an actual solution. For the choice of initial vector given above, one finds a Mermin fixed point already at iterate \(x(3) = (1, 11, 2, 1, 1, 1, 1, 1, 1, 1)\). As this example reveals, Mermin fixed points are a powerful and insufficiently explored strategy for solving a very broad range of complex problems.

2 IMAGE RECOVERY

When a monochromatic plane wave is weakly and elastically scattered by an object, the resulting diffracted wave has an angular intensity variation derived from the Fourier modes of the object’s scattering density. In this scenario and many others\(^2\), access to the contrast variations within an object is available only via the modulus of its Fourier transform. Since phase information is usually not available, a direct inverse Fourier transform cannot be used to recover the object. On the other hand, relatively mundane properties of the object such as positivity and size, in combination with the Fourier modulus, may be sufficient to “retrieve” the unknown phases and recover the object.

As an example, consider the speckle intensity pattern shown in Figure 1, formed by diffraction from an object in two dimensions. The data is sampled on an array measuring 448 \(\times\) 448 pixels; we lack the corresponding array of phases, with which a discrete Fourier transform could be used to directly produce a

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1 The reader is invited to find another Mermin fixed point of this map. There is also a 2-cycle, which solves the even harder problem of finding two such sentences, each of which has “this” replaced by “that.”
Figure 2: Projections applied to the arbitrarily chosen image $\rho_{WA}$ shown on the left. Middle: Fourier modulus projection, $\pi_F(\rho_{WA})$. The image shown is the minimal modification of $\rho_{WA}$ such that its Fourier transform has the modulus shown in Figure 1. Right: Fourier modulus projection followed by support/positivity projection, $\pi_S \circ \pi_F(\rho_{WA})$.

448 × 448 pixel image of the object.

By the convolution theorem, the Fourier transform of the speckle gives the object’s autocorrelation, shown on the right in Figure 1. Close examination of the autocorrelation shows its support is bounded: specifically, it is negligibly small outside a rectangle measuring $350 \times 440$ pixels. Assuming the object is positive, a well known theorem\(^{(3)}\) allows one to conclude that the object’s support is bounded by a rectangle measuring half the dimensions of the bound on the autocorrelation support, or $175 \times 220$ pixels. This bound on the object’s support, we will see, is sufficient additional information to recover an image of the object.

3 CONSTRAINTS AND PROJECTIONS

The Fourier modulus data (Fig. 1) and the bound on the object’s support are constraints that the image we are trying to recover must satisfy. Given an arbitrary image (for example the image on the left in Figure 2), the mathematical operation that restores a particular constraint, while minimizing the modification of the image, is called a *projection*\(^{(4)}\). A natural choice of image metric is the Euclidean metric in the space of pixel values, since it is invariant with respect to unitary transformations into the Fourier domain. Thus the projection $\pi_F$ that restores the Fourier moduli consists of three operations: (1) transformation of the image to the Fourier domain, (2) rescaling each complex-valued pixel of the Fourier transform to the measured modulus (projection onto a circle), and (3) transformation back to the image domain. Another projection, $\pi_S$, restores positivity and the support constraint by setting pixels outside the support, and negative pixels within the support, equal to zero. The actions of $\pi_F$ and $\pi_S$ are illustrated in Figure 2.

Uniqueness in image recovery requires that the number of constraints outnumber the free variables\(^{(5,6)}\). Since a real-valued image of $N$ pixels has approximately $N/2$ unknown phases in its Fourier transform, a bound on the support of the object measuring less than half the image area is normally sufficient to ensure
uniqueness. This condition is easily satisfied by the example introduced earlier (Fig. 1): there are 100350 independent continuous phases in a $448 \times 448$ pixel Fourier transform, while the bound on the object support constrains a larger number of pixels, $448 \times 448 - 175 \times 220 = 162204$, to be zero.

4 THE DIFFERENCE MAP

Short of having a projection that directly recovers the image by simultaneously restoring, from an arbitrary input, both the Fourier modulus and support/positivity constraints, one can hope to use the projections $\pi_F$ and $\pi_S$ in an iterative fashion such that the solution can be extracted from an appropriate Mermin fixed point. One such approach is the difference map$^6$:

$$\rho \mapsto D(\rho) = \rho + \beta \left[ \pi_S \circ f_F(\rho) - \pi_F \circ f_S(\rho) \right].$$  \hspace{1cm} (1)

The action of $D$ is to add to the current iterate $\rho$ the difference of projections (composed with two additional maps) scaled by a parameter $\beta$. To see how a Mermin fixed point of $D$, $\rho^*$, provides the solution, $\rho_{sol}$, we observe that $D(\rho^*) = \rho^*$ implies the difference of projections vanishes. In other words, the same image, now identified as $\rho_{sol}$, was produced by each of the two projections and therefore satisfies both sets of constraints:

$$\pi_S \circ f_F(\rho^*) = \rho_{sol} = \pi_F \circ f_S(\rho^*).$$  \hspace{1cm} (2)

The maps $f_F$ and $f_S$ have so far not been specified but must be chosen with care in order to make the Mermin fixed point attractive. Reference 6 makes the choice

$$f_F = (1 + \gamma_F \pi_F - \gamma_F)$$
$$f_S = (1 + \gamma_S \pi_S - \gamma_S),$$  \hspace{1cm} (3)

and finds $\gamma_F = \beta^{-1}, \gamma_S = -\beta^{-1}$ as optimal parameter values.$^2$

The difference map is superior to the naive alternating projection map $A = \pi_S \circ \pi_F$ because of stagnation caused by fixed points of $A$ which do not satisfy the Fourier modulus constraint (and have no simple relationship to the solution which does, in contrast to eq. 2). To overcome stagnation, Fienup(7) introduced the hybrid input-output map which, interestingly, is obtained as a special case of the difference map for the parameter values $\gamma_F = \beta^{-1}, \gamma_S = -1$. Although the hybrid input-output map has been the main tool for phase retrieval for nearly twenty years, its fixed point properties in the geometrical setting of projections has come to light only recently$^{(6,8)}$.

5 A PHASE RETRIEVAL EXAMPLE

When implemented with the fast Fourier transform, the difference map can be computed in a time that grows only quasi-linearly with the number of pixels in the image. Although there is as yet no comprehensive theory of the number of iterations required to reach a Mermin fixed point, numerical experiments indicate that progress toward the solution, measured by the norm of the difference,

$$\epsilon = \| \pi_S \circ f_F(\rho) - \pi_F \circ f_S(\rho) \|,$$  \hspace{1cm} (4)

is systematic though not strictly monotone.

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$^2$When $f_F$ and $f_S$ are identity maps ($\gamma_F = \gamma_S = -1$) the Mermin fixed point is found to be repulsive$^6$. 
Mermin’s example\(^{(1)}\) is the inspiration for iterative solutions to problems considerably beyond Newton’s square root and descendants. Phase retrieval belongs to the class of feasibility problems, usually posed in the context of linear programming with convex constraints\(^{(8)}\). Since the Fourier modulus constraint is nonconvex, standard algorithms either do not apply or have no guarantee of convergence. The iterative difference map algorithm\(^{(6)}\), also without a bound on the number of iterations, is currently the most efficient method for solving a large class of phase retrieval problems. This includes a highly simplified version of phase retrieval: the problem of recovering a fixed length binary sequence from its cyclic autocorrelation\(^{(9)}\). The latter appears to be comparable in difficulty, and has a mathematical kinship to, the problem of factoring.
Figure 4: Iterates $\rho(1), \rho(5), \rho(50)$ and $\rho(500)$ of the difference map; the initial image was $\rho(0) = \rho_{WA}$ (Fig. 2).
integers.

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3One is interested in factoring elements in the ring of cyclotomic integers, with each of the two factors known a priori to have binary coefficients and related as algebraic conjugates.
References

[1] N.D. Mermin, private communication and old Physics 209 homework problem (unpublished).

[2] R.P. Millane, “Phase retrieval in crystallography and optics,” *J. Opt. Soc. Am. A* **7**, 394-411 (1990).

[3] J.R. Fienup, T.R. Crimmins, and W. Holsztynski, “Reconstruction of the support of an object from the support of its autocorrelation,” *J. Opt. Soc. Am.* **72**, 610-624 (1982).

[4] H. Stark and Y. Yang, *Vector space projections* (John Wiley & Sons, 1998).

[5] J. Miao, D. Sayre, and H.N. Chapman, “Phase retrieval from the magnitude of the Fourier transforms of nonperiodic objects,” *J. Opt. Soc. Am. A* **15**, 1662-1669 (1998).

[6] V. Elser, “Phase retrieval by iterated projections,” preprint (2001).

[7] J.R. Fienup, “Phase retrieval algorithms: a comparison,” *Appl. Opt.* **21**, 2758-2769 (1982).

[8] H.H. Bauschke, P.L. Combettes and D.R. Luke, “Phase retrieval, Gerchberg-Saxton algorithm, and Fienup variants: A view from convex optimization,” preprint (2001).

[9] M. Zwick, B. Lovell, and J. Marsh, “Global optimization studies on the 1-D phase problem,” *International Journal of General Systems* **25**, 47-59 (1996).