Embedding spherical spacelike slices in a Schwarzschild solution

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Abstract

Given a spherical spacelike 3-geometry, there exists a very simple algebraic condition which tells us whether and in which Schwarzschild solution this geometry can be smoothly embedded. One can use this result to show that any given Schwarzschild solution covers a significant subset of spherical superspace and these subsets form a sequence of nested domains as the Schwarzschild mass increases. This also demonstrates that spherical data offer an immediate counterexample to a weak version of the thick sandwich 'theorem'.

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1. Spherical spacelike slices in a Schwarzschild spacetime

It is clear that many spherical spacelike slices can be embedded in a given extended Schwarzschild spacetime. Even though they may appear different from a geometrical viewpoint, they all have the same topology (except for the slices of flat spacetime). In the negative mass Schwarzschild, all the slices look like $\mathbb{R}^3$ (with the point where it runs into the naked singularity removed). In the positive mass Schwarzschild, one has a range of geometries. There are slices which start at one spacelike infinity, run through the middle and go out to the other end. These look like $S^2 \times \mathbb{R}$. There are slices which start at one or other of the $\mathbb{R}^2=0$ singularities and go out to one or other of the infinities. These look like $\mathbb{R}^3$ (again with the 'point' at the singularity removed). Finally, there are slices which start and end on one or other of the singularities. These look like $S^3$ (with two removed points). There are variants of the infinite ones, where instead of going to spacelike infinity, the slice remains within the horizon and runs more or less along one of the $R=0$ singularities and go out to one or other of the infinities. These look like $R^3$ (again with the 'point' at the singularity removed). Finally, there are slices which start and end on one or other of the singularities. These look like $S^3$ (with two removed points). There are variants of the infinite ones, where instead of going to spacelike infinity, the slice remains within the horizon and runs more or less along one of the $R=0$ singularities and go out to one or other of the infinities. These look like $R^3$ (again with the 'point' at the singularity removed). Finally, there are slices which start and end on one or other of the singularities. These look like $S^3$ (with two removed points). There are variants of the infinite ones, where instead of going to spacelike infinity, the slice remains within the horizon and runs more or less along one of the $R=0$ singularities and go out to one or other of the infinities. These look like $R^3$ (again with the 'point' at the singularity removed). Finally, there are slices which start and end on one or other of the singularities. These look like $S^3$ (with two removed points). There are variants of the infinite ones, where instead of going to spacelike infinity, the slice remains within the horizon and runs more or less along one of the $R=0$ singularities and go out to one or other of the infinities. These look like $R^3$ (again with the 'point' at the singularity removed). Finally, there are slices which start and end on one or other of the singularities. These look like $S^3$ (with two removed points). There are variants of the infinite ones, where instead of going to spacelike infinity, the slice remains within the horizon and runs more or less along one of the $R=0$ singularities and go out to one or other of the infinities. These look like $R^3$ (again with the 'point' at the singularity removed). Finally, there are slices which start and end on one or other of the singularities. These look like $S^3$ (with two removed points).
while outside the horizon the area must monotonically increase. These features are, of course, reflected in the nature of the spacetime in which any given slice can be embedded.

Flat spacetime seems to be a special case; here we have many spherical slices of topology $R^3_1$, without a missing point. However, this is not as special as it seems. For example, consider the Painlevé–Gullstrand [1] slice of a Schwarzschild solution. This is the slice where the spatial metric is flat and the extrinsic curvature $K \sim \sqrt{m/R^3}$. The spatial geometry is regular at the $R = 0$ point, only the extrinsic curvature diverges. This behaviour is generic in the sense that we can show that every spherical slice of flat spacetime of topology $R^3_1$ can be embedded in a Schwarzschild solution with only the extrinsic curvature becoming irregular at $R = 0$.

Any spacelike slice embedded in a solution of the Einstein equations will have an intrinsic geometry, given by a 3-metric $g_{ab}$, and an extrinsic curvature, given by a symmetric tensor $K^{ab}$. These are not independent, they must satisfy the constraints

$$ R^{(3)} - K_{ab} K^{ab} + (g_{ab} K^{ab})^2 = 0, \quad (1) $$

$$ \nabla_a (K^{ab} - g^{ab} g_{cd} K^{cd}) = 0, \quad (2) $$

where $R^{(3)}$ is the scalar curvature of $g_{ab}$. Equation (1) is called the Hamiltonian constraint and equation (2) is called the momentum constraint.

Given a spherical slice in a spherical spacetime, both the 3-geometry and the extrinsic curvature are spherically symmetric. There are several ‘natural’ choices of coordinates that are used to write down the spherical 3-metric. The one we favour is the ‘proper distance gauge’ where one writes the 3-metric as

$$ ds^2 = dl^2 + R(l)^2 d\Omega^2, \quad (3) $$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$ is the round 2-metric on a sphere, $R(l)$ is the (Schwarzschild) radius of the isometry 2-spheres (expressed as a function of $l$) and $l$ is the proper distance between surfaces of constant $R$. Any spherical 3-metric can be expressed in this form. For the slices which go from one infinity to another in the positive mass Schwarzschild, it seems natural to choose $l \in (-\infty, \infty)$; if the slice starts from a singular point and goes to infinity, one could choose $l \in (0, \infty)$; and if the slice starts and ends on a singularity, one could choose $l \in (-a, +b)$, where $a + b$ is the proper distance between the points where $R \to 0$. Finally, of course, we can consider ‘patches’ of any one of these slices where on either (or both) end(s) $R$ is finite and nonzero. Here again we would have $l$ belonging to some finite interval.

A spherically symmetric 2-tensor has only two independent components. We can write the extrinsic curvature in a form consistent with spherical symmetry as [2]

$$ K^{ab} = n^a n^b K_l + (g^{ab} - n^a n^b) K_R, \quad (4) $$

where $K_l$ and $K_R$ are two scalars and $n^a$ is the outward-pointing unit normal to the 2-surfaces of constant $R$. Any spherical 3-metric can be expressed in this form. For the slices which go from one infinity to another in the positive mass Schwarzschild, it seems natural to choose $l \in (-\infty, \infty)$; if the slice starts from a singular point and goes to infinity, one could choose $l \in (0, \infty)$; and if the slice starts and ends on a singularity, one could choose $l \in (-a, +b)$, where $a + b$ is the proper distance between the points where $R \to 0$. Finally, of course, we can consider ‘patches’ of any one of these slices where on either (or both) end(s) $R$ is finite and nonzero. Here again we would have $l$ belonging to some finite interval.

The Hamiltonian and momentum constraints can now be written in the special case of spherical symmetry as [2]

$$ K_R [K_R + 2K_l] - \frac{1}{R^2} [2RR'' + R^2 - 1] = 0 \quad (5) $$

and

$$ K_R' + \frac{K_R}{R} [K_R - K_l] = 0, \quad (6) $$

where $'$ represents the derivative with respect to $l$. There is a first integral of the constraints [2]

$$ m = \frac{R_l^3 K_R^2}{2} + \frac{R}{2} [1 - R^2], \quad (7) $$

where $R_l$ is the scalar curvature of $g_{ab}$. Equation (1) is called the Hamiltonian constraint and equation (2) is called the momentum constraint.
where \( m \) is the Schwarzschild mass. This is the Misner–Sharp, Hawking et al mass formula. An immediate consequence of equation (7) is that if we define
\[
M = \max \frac{R}{2} [1 - R^{'2}],
\]
where the maximum is taken over the particular 3-geometry we are considering, then
\[
m \geq M.
\]
This is our key equation, it is an algebraic relation between a global spacetime quantity, \( m \), and a quantity, \( M \), which depends only on the specific 3-geometry of any spacelike slice (or patch thereof) embedded in a given spacetime.

In this paper, we will show that essentially the converse of condition (9) holds. More precisely, given a 3-geometry in the proper distance gauge, one can always compute the quantity \( M \) as defined by equation (8). Let us assume that it is finite. This only requires that \( R \sim l \) as \( l \to \pm \infty \). We will show that this spacelike spherical geometry can be embedded in any Schwarzschild solution whose Schwarzschild mass satisfies
\[
m > M.
\]
The quantity \( M \) is really a coordinate-independent object. \( 4\pi R^2 \) is the area of the two-dimensional isometry spheres and \( 2R^{'}/R \) is the mean extrinsic curvature of the isometry 2-spheres as embedded surfaces in the 3-geometry.

2. Embedding slices in a Schwarzschild solution

Let us assume that we are given a spherical 3-metric, say in the form of equation (3). It could be a complete manifold or just a patch. We wish to consider this as the metric of (patch of) a slice embedded in a spherical vacuum solution of the Einstein equations. Therefore, we need to find a spherical extrinsic curvature \( (K^a_b) \) expressed, say, in the form of equation (4), such that the combination \( (g_{ab}, K^a_b) \) satisfies both the Hamiltonian and momentum constraints, equations (1) and (2) (or, rather, equations (5) and (6)). A priori, at least, this seems to be a reasonable task. We have two free functions, \( K_R \) and \( K_l \), and we need to pick them so as to satisfy two scalar equations. However, we do not have a completely free choice. Given the metric, we can compute \( M \) via equation (8) and if we succeed in embedding this in a Schwarzschild solution of mass \( m \), we must have that \( m \geq M \). In addition to being a necessary condition, it is almost a sufficient condition. More precisely, we can prove

**Proposition.** Given a spherical Riemannian 3-geometry with finite \( M \) as defined by equation (8) and any (extended) vacuum Schwarzschild solution whose mass \( m \) satisfies \( m > M \), then a spacelike slice (or patch thereof) can be found in this spacetime which is isometric to the given 3-geometry.

**Proof.** We start off with the mass expression, equation (7), and write it as
\[
K_R = \sqrt{\frac{2m}{R^3} - \frac{1}{R^2}[1 - R^{'2}]}.
\]
The quantity under the square root is positive definite since \( m > M \) and so the equation makes sense. We can choose either the positive or the negative root. With either choice we get a well-defined function \( K_R(l) \) which does not change sign. It may well be that if \( R \to 0 \) then \( |K_R| \to \infty \). All this does reflect the fact that the slice goes into the Schwarzschild singularity.
Given that $K_R$ is bounded away from zero, we can now solve the Hamiltonian constraint, equation (5), for $K_l$. More precisely, we rewrite equation (5) as

$$2K_K K_l + \frac{2m}{R^3} - \frac{2R''}{R} = 0$$

and manipulate this to get

$$K_l = \frac{1}{K_R} \left( \frac{R''}{R} - \frac{m}{R^3} \right).$$

This is clearly well defined and finite except possibly, again, as $R \to 0$. It is very straightforward to show that $K_R$ from equation (11) and $K_l$ from equation (13) satisfy the momentum constraint, equation (6).

We have only shown that we can construct spherically symmetric initial data that satisfy the vacuum Einstein constraints. However, the Einstein evolution equations allow us to propagate these data so as to construct at least a patch of spacetime (which will be spherically symmetric) and which satisfy the vacuum Einstein equations. In turn, Birkhoff’s theorem guarantees that this must be part of the Schwarzschild solution with given mass $m$. \qed

3. Concrete examples

We wish to embed the ‘static’ (moment-of-time-symmetry) slice from one Schwarzschild solution (of mass $m_1$) in another Schwarzschild solution (of mass $m_2$). We assume $m_2 > m_1$. The 3-metric we are given, written in Schwarzschild coordinates, is

$$ds^2 = \frac{dR^2}{1 - \frac{2m_1}{R}} + R^2 d\Omega^2.$$  \hspace{1cm} (14)

To convert to the proper distance gauge we would need to integrate

$$l(r) = \int_{2m_1}^{r} \frac{dR}{\sqrt{1 - \frac{2m_1}{R}}}.$$  \hspace{1cm} (15)

but there is no real need to do so; all we really use is

$$R' = \frac{dR}{dl} = \sqrt{1 - \frac{2m_1}{R}} \Rightarrow \frac{R}{2} \left[ 1 - R^2 \right] = m_1.$$  \hspace{1cm} (16)

When this is substituted into equation (11), we get

$$K_R = -\sqrt{\frac{2(m_2 - m_1)}{R^3}}.$$  \hspace{1cm} (17)

We choose the negative root to get the slice in the upper half-plane.

We can use the Hamiltonian constraint, equation (5), remembering that the scalar curvature of the metric given by equation (14) vanishes, to get

$$K_l = -\frac{K_R}{2} = \sqrt{\frac{m_2 - m_1}{2R^3}}.$$  \hspace{1cm} (18)

It is easy to confirm that this choice of extrinsic curvature satisfies the momentum constraint, equation (6).

We know that maximal slices which run from one end to the other of a Schwarzschild solution cannot approach the singularity closer than $R = 3m/2$ \cite{7}. If we relax the maximal condition and replace by the requirement that the 3-scalar curvature be non-negative we get no such restriction; we can approach the singularity as closely as we wish.
If we let $m_1 \to 0$, we get the well-known flat slice of the Schwarzschild solution (the Painlevé–Gullstrand representation). Now $R \to 0$, so the flat slice runs into the singularity and the extrinsic curvature is given by

$$K_R = -2K_i = -\sqrt{\frac{2m_1}{R^3}}. \quad (19)$$

We can also embed the ‘static’ slices of the negative mass Schwarzschild solution in flat spacetime. These slices have intrinsic metric

$$ds^2 = \frac{dR^2}{1 + \frac{2m_1}{R} + R^2} + d\Omega^2 \quad (20)$$

and extrinsic curvature

$$K_R = -2K_i = -\sqrt{\frac{2m_1}{R^3}}. \quad (21)$$

This slice is regular everywhere except at the origin.

4. The significance of $m > M$

We seek the maximum of $\frac{R}{2}(1 - R^2)$. If we wish to embed this slice in a negative mass Schwarzschild solution (or in flat spacetime), we need $\frac{R}{2}(1 - R^2) < 0$ over the entire slice. This implies $R' \geq 1$. The areal radius starts off at $R = 0$ and monotonically increases; there cannot be a throat. Not only that but it must increase rapidly. It must look like a trumpet. This is completely in agreement with the requirement that the topology of slices embedded in the negative mass Schwarzschild must be $R^3$.

In general, however, the maximum of $\frac{R}{2}(1 - R^2)$ will be positive and therefore this slice can only be embedded in a positive mass Schwarzschild solution. For example, if the area is not monotonic then the function $R(l)$ has either a local maximum or minimum (or both). Thus, the slice has at least one point where $R' = 0$, and if the areal radius equals $R_0$ at that point then $\frac{R}{2}(1 - R^2) = R_0/2$ there. Thus, we have that $M \geq R_0/2$ and a necessary condition that this slice be embeddable in a Schwarzschild solution of mass $m$ is that $2m \geq R_0$. This means that if we have an extremum of the area we can only embed this slice in a positive mass Schwarzschild solution and also that all maxima and minima of the areal radius must occur inside the horizon.

If we have a local maximum of $R(l)$, i.e., a point where $R' = 0$, $R'' \leq 0$, this point is also a local maximum of $\frac{R}{2}(1 - R^2)$. This is easy to show: all one needs to note is that both $R/2$ and $[1 - R^2]$ have a maximum at that point. No equivalent statement can be made about a point where $R(l)$ is a local minimum. One could have a maximum or minimum of $\frac{R}{2}(1 - R^2)$ at that point or nothing at all. It is also worth noting that the maximum of $\frac{R}{2}(1 - R^2)$ need not occur at an extremum of $R(l)$. It may occur at a ‘large’ value of $R$ where $R'$ may be small without ever going to zero. One can then choose an $m > M$ such that this point is outside the horizon.

5. The marginal case: what happens when $m = M$

If we have a spherical slice in a Schwarzschild solution we know that $m \geq M = \max \frac{R}{2}(1 - R^2)$. Conversely, we have shown that if we have a spherically symmetric Riemannian 3-metric we can embed it in a Schwarzschild solution with mass $m$ if $m > M$. In this section, we would like to discuss the issue of when and if one can embed a given spherical slice in a Schwarzschild solution satisfying $m = M$. 
Very useful tools for analysing spherical spacetimes are the optical scalars, \( \Theta_\pm \) (see e.g. \cite{2, 3}). These are defined by

\[
\Theta_\pm = \frac{2}{R} (R' \pm RK_R).
\]  

These are the divergences of the future-pointing and past-pointing outward radial light rays from the isometry spheres. Both are defined so as to be positive on flat space and near infinity. Note that the definition of extrinsic curvature used agrees with Wald \cite{4} and not with Misner, Thorne and Wheeler \cite{5}.

The mass expression, equation (7), can be rewritten as

\[
\frac{2m}{R} = 1 - \frac{\Theta_+ \Theta_- R^2}{4}.
\]

If \( m \leq 0 \), then \( \Theta_+ \) and \( \Theta_- \) must always remain positive. If \( m > 0 \) and if \( 2m/R > 1 \), i.e., if we are inside the horizon, then one or other of the optical scalars must be negative. In the upper half of the extended Schwarzschild solution it is \( \Theta_+ \), which is negative and in the lower half it is \( \Theta_- \), which is negative. Equation (11) is double valued because we can choose either sign when we take the square root. This means that we have two slices with the same intrinsic geometry. These two solutions are just reflections of each other about the \( t = 0 \) plane.

Say we are given a spherical 3-geometry with finite \( M \) and try to embed this in a Schwarzschild spacetime with \( m = M \). This means that we no longer, from equation (11), get \( K_R > 0 \). Rather we get \( K_R = 0 \) at the point(s) where \( M \) achieves its maximum. At such a point, since \( \left( \frac{R}{2}[1 - R^2] \right) \) is at its maximum, we know that

\[
0 = \frac{\partial}{\partial \ell} \left( \frac{R}{2}[1 - R^2] \right) = \frac{R^2 R'}{4} \times \frac{2}{R^2} [1 - R^2 - 2RR''].
\]

Therefore, at the point(s) where \( K_R = 0 \) we are guaranteed that either \( R' = 0 \) or \( \frac{1}{R^2}[1 - R^2 - 2RR''] = 0 \). From equation (5) it is clear that we need the second expression (which is nothing else but the 3-scalar curvature, \( R^{(3)} \), of the 3-geometry) to vanish whenever \( K_R = 0 \). Hence, we need to distinguish between the case where the maximum of \( \frac{R}{2}[1 - R^2] \) occurs at a point where \( R' = 0 \) (which is bad, except if simultaneously \( R^{(3)} = 0 \) there, which cannot be guaranteed) and the case where the maximum occurs at a point where \( R' \neq 0 \) (which is good because \( R^{(3)} = 0 \) there). This question is trivial in the case where \( M \leq 0 \) because there cannot be any point(s) with \( R' = 0 \) in the geometry and so we automatically have \( R^{(3)} = 0 \) at the point where \( M \) achieves its maximum.

It is also clear that we are unable to use equation (13) directly to evaluate \( K_l \) at the point(s) where \( K_R = 0 \). However, we can return to the momentum constraint equation (6) and rewrite it as

\[
K_l = \frac{RK_R'}{R'} + K_R.
\]

This shows us that \( K_l \) is finite and well defined at the point(s) where \( K_R = 0 \), so long as \( R' \neq 0 \) at those points. This is consistent with equation (13) because the term in square brackets in this equation is essentially \( R^{(3)} \), so equation (13) becomes \( K_l = 0/0 \) and use of l'Hospital’s rule gives us a finite value. Therefore, the only case we need to worry about is when the maximum of \( M \) coincides with \( R' = 0 \). We cannot, in general, hope to solve the constraints with \( m = M \) in this situation. Nevertheless, there are special cases.
It is clear that if there exists a point where $K_R = 0$ and $R' = 0$ then both optical scalars, $\Theta_\pm$, vanish simultaneously. In the Schwarzschild solution, the only place where this can happen is at the bifurcation ‘point’, where both horizons cross. Note again that $R' = 0$ is only possible with $m > 0$. If the given 3-geometry has the property that the maximum of $\frac{2m}{R} - R^2$ occurs at a point where $R' = 0$ and we try $m = M$, then clearly $K_R = 0$ and $R' = 0$ at this point; if this slice is to be embedded in this Schwarzschild solution, it must pass through the bifurcation point. This places both local and global restrictions on the geometry of the slice. This point where $R' = 0$ must be a local minimum of the area, i.e., $R'' > 0$. More than that, we require that the scalar curvature vanishes at that point, hence $2R'' = -1/R$. Further, it must be the global minimum. The area cannot oscillate. If we had a local maximum of $R$, then the value of $\frac{2m}{R} - R^2$ at this point would be larger than the value at the ‘throat’, where it is supposed to be the maximum. Therefore, we can only have one minimum.

The behaviour of $K_R$ in the neighbourhood of the point where $K_R = 0$ is worth noting. Let us define $f(l) = \frac{2m}{R} - R^2$ and assume $f = f_0 = m$ at $l = l_0$, $f'_0 = 0$ and $f''_0 < 0$. We have $K^2_R = \frac{2m}{R} - R^2$. Hence, we have $(K^2_R)_0 = 0$, $(K^2_R)'_0 = 0$ and $(K^2_R)''_0 = -\frac{2l}{R^3}_0 > 0$. Now make a Taylor expansion of $K^2_R$ around $l = l_0$ in terms of $x = l - l_0$. In general, we will get $K^2_R(x) = A + Bx + Cx^2 + \cdots$ with $A = B = 0$ and $C > 0$. Thus, we get $K_R(x) = \sqrt{C} + \cdots$. Therefore, $K_R$ will pass through zero at $l = l_0$ with nonzero slope and so must change sign.

There is a further special case. The requirement that $f(l)$ be a maximum at $l = l_0$ only requires $f''_0 \leq 0$. If $f''_0 = 0$, we would have $(K^2_R)'_0 = 0$, $(K^2_R)'_0 = 0$ and $(K^2_R)''_0 = 0$. Now the requirement that we have a maximum of $f$ forces $0''_0 = 0$ and the first nontrivial derivative must be at fourth order. In this case, the Taylor expansion of $K^2_R$ can only start at $x^4$, i.e., $K^2_R = Dx^4 + \cdots$ and $K_R = \sqrt{D}x^2 + \cdots$. In this case, $K_R = 0$ and $K'_R = 0$ at $l = l_0$. Following from this, not only does $K_R$ vanish at $l = l_0$ but also, from equation (25), we have that $K_1 = 0$ at $l_0$.

To summarize, if the maximum of $\frac{2m}{R} - R^2$ occurs at a point where $R' \neq 0$, we can choose $m = M$. This condition of $R' \neq 0$ is trivially satisfied if $M < 0$. If the maximum occurs at a point where $R' = 0$, we can only choose $m = M$ if this point becomes the bifurcation point. This places further requirements (both local and global) on the 3-geometry.

6. Spherical superspace and the thick sandwich theorem

Wheeler identified the configuration space of canonical general relativity as being the space of all spacelike 3-geometries, which he called ‘superspace’. A trajectory of a solution of the Einstein equations in this configuration space corresponds to the sequence of Riemannian 3-geometries generated by a foliation. This is not a unique curve, rather we have ‘a spray of geodesics’ in the language of DeWitt. Every slicing of a spacetime generates a different sequence of 3-geometries and by changing the slicing one changes the curve through superspace. It is interesting to ask what fraction of superspace do all the curves corresponding to one given spacetime pass through. This set of 3-geometries which is the union of all the solution curves is exactly the same as the set of all spacelike 3-geometries that can be embedded in a given spacetime.

Let us define spherical superspace ($SS$) as the space of all spherically symmetric spacelike 3-geometries. Now consider a Schwarzschild solution ($S(m)$), which is defined by its mass $m$. There will be many spacelike spherical 3-geometries that can be embedded in $S(m)$. We define $B_S(m)$ as the collection of all spherically symmetric spacelike 3-geometries that can be embedded in a given $S(m)$. Obviously, $B_S(m) \subset SS$. The analysis so far tells us a great deal about the relationship between $B_S(m)$ and $SS$. 

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It is clear that any single $B_S(m)$ with given mass $m$ covers a large fraction of spherical superspace. We have a measure on spherical spacelike 3-geometries given by $M = \max \{ \pi^2/4 (1 - R^2) \}$ and we have shown that all 3-geometries which satisfy $M < m$ belong to $B_S(m)$. We also know that any 3-geometry which satisfies $M > m$ cannot belong to $B_S(m)$. Thus, $B_S(m)$ is defined by a single algebraic condition. The only uncertainty is about those 3-geometries on the boundary of $B_S(m)$, i.e., those metrics which satisfy $M = m$. If $m \leq 0$ we know that $B_S(m)$ is closed; every geometry which satisfies $M = m$ can be embedded in the appropriate Schwarzschild solution. This is because $M \leq 0 \Rightarrow R' \neq 0$, so, as discussed in the previous section, we have no difficulty in solving for $K_l$ at the point where $K_R = 0$. If $m > 0$, we find that $B_S(m)$ is neither open nor closed. We find that most of the geometries on the boundary can be embedded. However, we have a relatively small class of geometries, those for which the maximum of $M$ occurs at a local maximum of the area, i.e., a point where $R' = 0$, which cannot be embedded in the Schwarzschild solution with mass $m = M$.

Further, we have a nested structure on spherical superspace. Given two Schwarzschild solutions with masses $m_1$ and $m_2$ with $m_1 < m_2$, then every spherical slice that can be embedded in the first solution can also be embedded in the second one. This means that $B_S(m_1) \subset B_S(m_2)$. The foliation freedom in a given solution to the Einstein equations is represented by the fact that one can choose an arbitrary lapse (the shift freedom can be ignored because we are considering the geometries rather than 3-metrics). Starting with spherical data and maintaining the spherical symmetry means that we must have a spherical lapse. Therefore, the foliation choice is represented by one single spherical function. However, as we have seen in equation (3), the freedom in spherical 3-geometries is also represented by one single spherical function. Thus, it should not come as too much of a surprise that any one Schwarzschild solution should cover so much of spherical superspace.

Since this kind of counting argument works so well in the spherical case, it is interesting to try it in more general situations. Let us define axial superspace as the set of all axially symmetric spacelike geometries. The foliation freedom of axially symmetric slicings of an axially symmetric spacetime is represented by a single axially symmetric function. However, to give a general axially symmetric 3-geometry (say written in the Brill conformal gauge [6]), we have to give two axially symmetric functions. Therefore, we expect that a given axially symmetric spacetime will visit only a subset of axially symmetric superspace, corresponding more or less to the square root of the whole. In the general, unsymmetric, case the foliation freedom is represented by a single arbitrary function while the general 3-geometry is represented by three arbitrary functions.

Spherical geometries are also useful in that they offer a simple counterexample to a long-standing idea in canonical gravity, the thick sandwich approach. Wheeler introduced the concept of the ‘thick sandwich theorem’. The plan was to choose an initial and a final point in configuration space (in this case a pair of 3-geometries) and that the equations of motion (the dynamical part of the Einstein equations) would give a natural path joining these two points. In other words, to try and find a unique 4-manifold, satisfying the vacuum Einstein equations, which fills in between the two given 3-geometries.

In attempts to give a mathematically precise meaning to the thick sandwich conjecture, one meets the requirement that the slices be non-intersecting Cauchy hypersurfaces (see, e.g., [8]). This seems far too restrictive a condition to impose in the asymptotically flat case. The ADM mass (which really should be called the ADM energy) is a function of the asymptotic 3-geometry only. Therefore, if we are given two asymptotically flat 3-metrics, the probability that the ADM masses of the two be equal is vanishingly small. The only realistic way that one can hope that these two 3-metrics be embedded in the same spacetime is that one is boosted relative to the other. This means that the slices must cross. Further, consider
an extended Schwarzschild solution. Now consider one spacelike slice which runs from one
infinity to the other and another slice which runs from infinity into one of the singularities.
This forms an immediate counterexample to the version of the thick sandwich posed above
because one geometry is not a Cauchy hypersurface, even though they do (by construction) have a spacetime ‘filling’.

Therefore, we wish to pose a weaker version of the thick sandwich conjecture which at least is consistent with the cases above, i.e., given two spacelike 3-geometries we seek a unique 4-manifold (modulo ‘domain of dependence’ issues) satisfying the vacuum Einstein equations into which both 3-geometries can be embedded. The analysis in this paper shows how ill posed this ‘weak’ version of the ‘thick sandwich’ concept is.

Choose any two spherical 3-geometries. Evaluate $M$ for each geometry. Let them be $M_1$ and $M_2$, respectively. Pick any Schwarzschild solution with mass $m$ which satisfies $m > \max(M_1, M_2)$. Both of the given 3-geometries can simultaneously be embedded in the given Schwarzschild solution. This generates a ‘filling’ (which satisfies the Einstein equations) between the given slices which is highly non-unique. Further, it is impossible to lift this ambiguity by demanding that the given 3-geometries are, with respect to any measure, close to one another. If we find any Schwarzschild solution that interpolates between them, all Schwarzschild solutions with larger mass will also do the job.

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