A numerical point of view at the Gurov-Reshetnyak inequality on the real line

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Abstract

A "norm" of power function in the Gurov-Reshetnyak class on the real line is computed. Moreover, a lower bound for the norm of the operator of even extension from the semi-axis to the whole real line in the Gurov-Reshetnyak class is obtained from numerical experiments.

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1 Introduction

Let us consider functions \( f : R \mapsto \mathbb{R}^+ \) where \( R \) is an interval of \( \mathbb{R} \). In what follows, \( R \) is the real line \( \mathbb{R} \) or the semi-axis \( \mathbb{R}^+ = [0, \infty) \). We also assume that the function \( f \) is locally summable on \( R \), i.e. it is summable on each bounded subinterval \( I \) of \( R \).

The mean value of the function \( f \) on a bounded interval \( I \) is defined by

\[
 f_I = \frac{1}{|I|} \int_I f(x) \, dx,
\]

and the mean oscillation of this function is

\[
 \Omega(f; I) = \frac{1}{|I|} \int_I |f(x) - f_I| \, dx,
\]
where \(| \cdot |\) denotes the Lebesgue measure.

For a given \(\varepsilon \in (0, 2]\) the Gurov–Reshetnyak class \(\mathcal{GR} = \mathcal{GR}(\varepsilon) = \mathcal{GR}_R(\varepsilon)\) is defined as the set of all non-negative functions \(f\) which are locally summable on \(R\) and such that the Gurov–Reshetnyak condition

\[
\Omega(f; I) \leq \varepsilon f_I
\]

is satisfied on all bounded intervals \(I \subset R\) (see [7]). Note that since any non-negative function \(f\) on any interval \(I\) satisfies the inequality \(\Omega(f; I) \leq 2f_I\), the class \(\mathcal{GR}_R(2)\) is trivial and it coincides with the class of all functions locally summable on \(R\). However, if \(\varepsilon \in (0, 2)\) then \(\mathcal{GR}_R(\varepsilon)\) is a non-trivial class (see [10, P. 112], [15]). If \(I\) is a subinterval of \(R\), then the expression \(\langle f \rangle_I = \Omega(f; I)/f_I\) is called the relative oscillation of the function \(f\) on the interval \(I\). Further, the term \(\langle f \rangle_R = \sup_{I \subset R} \langle f \rangle_I\)

is called the "norm" of function \(f\) in the Gurov-Reshetnyak class \(\mathcal{GR}_R\).

One of the main properties of functions from the Gurov–Reshetnyak class consists in the possibility to improve their summability exponents. This property lays the foundation for numerous applications of this class of functions. More precisely, for any \(\varepsilon \in (0, 2)\) there are \(p^+_R = p^+_R(\varepsilon) > 1\) and \(p^-_R = p^-_R(\varepsilon) < 0\), such that the condition \(f \in \mathcal{GR}_R(\varepsilon)\) implies the local summability of the function \(f^p\) for any \(p \in (p^-_R, p^+_R)\) (see. [1], [3], [4], [7], [5], [8], [16], [18]). For \(R = \mathbb{R}^+\), the exact limiting value \(p^+_R = p^+_R(\varepsilon) > 1\) of the positive summability exponent \(p\) is the root of the equation

\[
\frac{p}{(p-1)^{p-1}} = \frac{2}{\varepsilon},
\]

and \(p^-_R = 1 - p^+_R < 0\). The sharpness of the values \(p^-_R\) and \(p^+_R\) can be verified by the use of the power functions \(g(x) = x^{1/(p-1)}\) and \(h(x) = x^{-1/p}\) \((x \in \mathbb{R}_+, p > 1)\), respectively. Thus

\[
\varepsilon_R(p) \equiv \langle g \rangle_{R^+} = \langle h \rangle_{R^+} = \frac{2(p-1)^{p-1}}{p^p},
\]

[10] pp. 131, 144], [11], [12], [14], [15]. These examples also show that for functions \(f \in \mathcal{GR}_R(\varepsilon)\), the function \(f^p\) is not necessarily locally summable in the limiting cases \(p = p^-_R(\varepsilon) < 0\) or \(p = p^+_R(\varepsilon) > 1\).

On the other hand, for \(R = \mathbb{R}\) the sharp limiting summability exponents \(p^-_R(\varepsilon) < 0\) and \(p^+_R(\varepsilon) > 1\) of functions \(f \in \mathcal{GR}_R(\varepsilon)\) are not known. It is clear that \(p^-_R(\varepsilon) \leq p^-_R(\varepsilon) \leq p^+_R(\varepsilon) \geq p^+_R(\varepsilon)\). Similarly to \(R = \mathbb{R}^+\), it is only natural to assume that for \(R = \mathbb{R}\) the power functions \(f_\alpha(x) = |x|^{\alpha} \,(x \in \mathbb{R}, \alpha > -1)\) with \(\alpha = 1/(p-1)\) and \(\alpha = -1/p \,(p > 1)\) are also extremal ones. However, the computation of the corresponding Gurov-Reshetnyak "norms" \(\varepsilon_R(p) \equiv \langle f_{1/(p-1)} \rangle_R\) and \(\varepsilon_R^+(p) \equiv \langle f_{-1/p} \rangle_R\) in this case is not as simple as for \(R = \mathbb{R}^+\). Nevertheless, it is shown in [11] that \(\varepsilon_R(p) > \varepsilon_R^+(p)\) and \(\varepsilon_R^+(p) > \varepsilon_R(p)\).
One of the main results of the present work is the computation of the "norm" $\langle f \rangle_{R}^{\alpha}$ of the function $f_{\alpha}$ in the Gurov-Reshetnyak class on the real line $\mathbb{R}$ (see Theorem 2.1 below). In particular, this theorem implies the equation $\varepsilon_{R}(p) = \varepsilon_{R}^{+}(p) (p > 1)$ (cf. Corollary 2.3).

The above problem can be reformulated as follows: If a monotone function $f$ belongs to the class $GR_{R}^{+}(\varepsilon)$ for an $\varepsilon \in (0, 2)$, then its even extension to $\mathbb{R}$, which is also denoted by $f$, belongs to the Gurov-Reshetnyak class $GR_{R}(\varepsilon')$ with an $\varepsilon' \in [\varepsilon, 2)$ (see Lemma 2.1). Therefore, one can also ask a question about the norms $\| T \|_{GR}^{(\varepsilon)} = \sup_{0<\varepsilon<2} \| T \|_{GR}^{(\varepsilon)}$ of the operator $T$ of the even extension of monotone functions $f \in GR_{R}^{+}(\varepsilon)$ to the real line $\mathbb{R}$.

In the Table 1 below we report the results of numerical calculations of the values of $\varepsilon_{R}(p)$ for various $p > 1$. Comparing these results with known values of $\varepsilon_{R}^{+}(p)$, one obtains lower bounds for the norms $\| T \|_{GR}^{(\varepsilon)}$ and $\| T \|_{GR}$.

An analogous problem, concerning the norm of the operator of the even extension for the class $BMO$ of monotone functions with bounded mean oscillation, has been considered in [9]. Some estimates for the norm of such an extension have been obtained in [17]. It is remarkable that the lower estimate presented in Remark 3.1 for the norm of the operator of the even extension $\| T \|_{GR}$ of the present work coincides with the lower estimate $\| T \|_{BMO}$ obtained in [2] for the corresponding operator of the even extension of monotone functions $f \in BMO$ (see Remark 3.2).

2 Gurov-Reshetnyak inequality for power functions on the real line

Let us recall that the mean value $f_{I} = \gamma$ of function $f$ on a subinterval $I$ is uniquely defined by the condition

$$\int_{I_{(f \geq \gamma)}} (f(x) - \gamma) \, dx = \int_{I_{(f \leq \gamma)}} (\gamma - f(x)) \, dx.$$

It is easily seen that

$$\Omega(f; I) = \frac{2}{|I|} \int_{I_{(f \geq f_{I})}} (f(x) - f_{I}) \, dx = \frac{2}{|I|} \int_{I_{(f \leq f_{I})}} (f_{I} - f(x)) \, dx.$$

$^{1}$By $E(P)$ we denote the set of all points $x \in E$, satisfying the condition $P = P(x)$. 

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According to [11], the Gurov-Reshetnyak "norm" of any monotone function \( f \) on \( \mathbb{R}_+ \) can be computed by the formula
\[
\langle \langle f \rangle \rangle_{\mathbb{R}_+} = \sup_{b > 0} \langle f \rangle_{(0,b)} ,
\]
and this fact will be used in what follows.

Let us show that the even extension of monotone functions from a non-trivial Gurov-Reshetnyak class \( GR_{\mathbb{R}_+} \) belong to a non-trivial class \( GR_\mathbb{R} \).

**Lemma 2.1** For any \( \varepsilon \in (0,2) \) there is an \( \varepsilon' \in (0,2) \) such that if a monotone function \( f \) belongs to \( GR_{\mathbb{R}_+}(\varepsilon) \), then the even extension of \( f \) from \( \mathbb{R}_+ \) to \( \mathbb{R} \) belongs to \( GR_\mathbb{R}(\varepsilon') \).

**Proof.** The proof of this lemma can be split into three steps.

**Step 1.** Consider an even function \( f \in GR_{\mathbb{R}_+}(\varepsilon) \), \( 0 < \varepsilon < 2 \). Then for any interval \( I \subset \mathbb{R}_+ \) the Gehring inequality holds\(^2\)
\[
\left( \frac{1}{|I|} \int_I f^q(x) \, dx \right)^{1/q} \leq B \cdot \frac{1}{|I|} \int_I f(x) \, dx ,
\]
where \( q > 1 \) and \( B > 1 \) depend on the parameter \( \varepsilon \) only [10, P. 131], [12].

**Step 2.** Let us show that the function \( f \) satisfies the Gehring inequality on \( \mathbb{R} \). It suffices to consider only the intervals of the form \( I = (-a,b) \), where \( 0 < a < b \). Since \( f \) is an even function, one can apply the inequality (2) on the interval \( (0,b) \) and obtain
\[
\left( \frac{1}{|I|} \int_I f^q(x) \, dx \right)^{1/q} \leq \left( \frac{2}{a + b} \int_0^b f^q(x) \, dx \right)^{1/q} \leq \left( \frac{2b}{a + b} \right)^{1/q} B \frac{1}{b} \int_0^b f(x) \, dx \leq 2^{1/q} \left( \frac{b}{a + b} \right)^{1/q-1} B \frac{1}{a + b} \int_{-a}^b f(x) \, dx \leq 2B \frac{1}{|I|} \int_I f(x) \, dx .
\]

**Step 3.** Now it remains to use the fact that the Gehring inequality implies the Gurov-Reshetnyak inequality with an \( \varepsilon' \equiv \varepsilon'(\varepsilon) \) (see [10, P. 114], [13]), which completes the proof.

**Remark 2.1** The above obtained value of \( \varepsilon' \equiv \varepsilon'(\varepsilon) \) is not exact since at each step of the proof of Lemma 2.1 the parameters used are overestimated.

\(^2\)Recall the Gehring inequality first originated in [6].
Remark 2.2 For $\varepsilon < 1$ there is a simpler proof of Lemma 2.1. In this case one can employ the simple inequality (2)

$$\Omega(f; (-\delta b, b)) \leq \frac{2}{1+\delta} \Omega(f; (0, b)) \quad (0 \leq \delta \leq 1, \ b > 0).$$

Indeed, since

$$f(-\delta b, b) = \frac{1}{(1+\delta) b} \int_{-\delta b}^{b} f(x) \, dx \geq \frac{1}{1+\delta b} \int_{0}^{b} f(x) \, dx = \frac{1}{1+\delta} f(0, b),$$

then

$$\frac{\Omega(f; (-\delta b, b))}{f(-\delta b, b)} \leq (1+\delta) \frac{\Omega(f; (-\delta b, b))}{f(0, b)} \leq (1+\delta) \frac{2}{1+\delta} \frac{\Omega(f; (0, b))}{f(0, b)} \leq 2 \|f\|_{R^+}.$$ 

Other details of the proof are left to the reader.

Remark 2.3 It follows from the proof in Remark 2.2 that if $\varepsilon \in (0, 1)$ then $\|T\|_{GR}^{(\varepsilon)} \leq 2$. However, since $\varepsilon'$ in Lemma 2.1 satisfies the relation $\varepsilon' \leq 2$ the inequality $\|T\|_{GR}^{(\varepsilon)} \leq 2$ remains valid for $\varepsilon \in [1, 2)$. Therefore, one also has $\|T\|_{GR} \leq 2$. On the other hand, a lower bound for $\|T\|_{GR}$, obtained in numerical experiments, is presented in Remark 3.1 below.

Further, we will compute the "norm" of power function in the Gurov-Reshetnyak class. Using a linear transformation one can check that for the function $f_{\alpha}(x) = |x|^\alpha \ (x \in \mathbb{R}, \alpha > -1)$ the following relations

$$\langle f_{\alpha}\rangle_{R} = \sup_{0 \leq \eta \leq 1} \langle f_{\alpha}\rangle_{(-\eta, 1)}, \quad \langle f_{\alpha}\rangle_{R^+} = \langle f_{\alpha}\rangle_{(0, 1)} = \frac{2|\alpha|}{(\alpha + 1)(\alpha + 1)/\alpha}$$

hold.

Theorem 2.1 If $\alpha > -1, \alpha \neq 0$, then

$$\langle f_{\alpha}\rangle_{R} = \langle f_{\alpha}\rangle_{R^+} \cdot \max_{0 \leq \eta \leq 1} \psi(\alpha, \eta),$$

where

$$\psi(\alpha, \eta) = \begin{cases} \frac{(1+\eta^{\alpha+1})^{1/\alpha}}{(1+\eta)(\alpha+1)/\alpha} + \frac{(\alpha+1)(\alpha+1)/\alpha}{\alpha} \left[ \frac{1}{1+\eta^{\alpha+1}} - \frac{1}{1+\eta} \right], & \text{if } 0 \leq \eta \leq \eta_1, \\
2 \frac{(1+\eta^{\alpha+1})^{1/\alpha}}{(1+\eta)(\alpha+1)/\alpha}, & \text{if } \eta_1 \leq \eta \leq 1, \\
\end{cases}$$

and $\eta_1(\alpha) \in (0, 1)$ is the root of the equation

$$\eta^{\alpha} = \frac{1}{1+\alpha(\eta+1)}. \quad (3)$$
Proof. For any fixed $\eta \in [0, 1]$ we set $I = I(\eta) = (-\eta, 1)$. Then
\[
(f_\alpha)_I = \frac{1}{1 + \eta} \int_{-\eta}^{1} |x|^{\alpha} \, dx = \frac{1}{\alpha + 1} \cdot \frac{1 + \eta^{\alpha+1}}{1 + \eta}.
\]
Let $\eta_1 \in (0, 1)$ be the root of the equation $\eta = ((f_\alpha)_I)^{1/\alpha}$, cf. (3). It is easily seen that this equation is solvable and the root is unique.

(a). If $\eta \leq \eta_1$, then
\[
\varphi_0(\alpha, \eta) \equiv \langle f_\alpha \rangle_I = \frac{\Omega(f_\alpha; I)}{(f_\alpha)_I} = \frac{2 \cdot \text{sign } \alpha}{1 + \eta} \left[ \frac{\alpha}{\alpha + 1} ((f_\alpha)_I)^{(\alpha+1)/\alpha} - (f_\alpha)_I + \frac{1}{\alpha + 1} \right].
\]
Set
\[
\phi_0(\alpha, \eta) \equiv \langle f_\alpha \rangle_I = \frac{\Omega(f_\alpha; I)}{(f_\alpha)_I} = \frac{2 \cdot \text{sign } \alpha}{1 + \eta} \left[ \frac{\alpha}{\alpha + 1} ((f_\alpha)_I)^{(\alpha+1)/\alpha} - (f_\alpha)_I + \frac{1}{\alpha + 1} \right].
\]
Taking into account that
\[
\varphi_0(\alpha, 0) = \langle f_\alpha \rangle_{(0,1)} = \frac{2|\alpha|}{(\alpha + 1)^{(\alpha+1)/\alpha}},
\]
one obtains
\[
\psi_0(\alpha, \eta) \equiv \frac{\langle f_\alpha \rangle_I}{\langle f_\alpha \rangle_{(0,1)}} = \frac{(1 + \eta^{\alpha+1})^{1/\alpha}}{(1 + \eta)^{(\alpha+1)/\alpha}} + \frac{(\alpha + 1)^{(\alpha+1)/\alpha}}{\alpha} \left[ \frac{1}{1 + \eta^{\alpha+1}} - \frac{1}{1 + \eta} \right].
\]

(b). On the other hand, if $\eta \geq \eta_1$, then
\[
\Omega(f_\alpha; I) = \frac{2 \cdot \text{sign } \alpha}{1 + \eta} \cdot 2 \int_{0}^{1} ((f_\alpha)_I - x^{\alpha}) \, dx = \frac{4}{1 + \eta} \frac{|\alpha|}{\alpha + 1} ((f_\alpha)_I)^{(\alpha+1)/\alpha}.
\]
Consider now the expression
\[ \varphi_1(\alpha, \eta) \equiv \langle f_\alpha \rangle_I = \frac{\Omega(f_\alpha; I)}{(f_\alpha)_I} = \frac{4|\alpha|}{(\alpha + 1)(\alpha + 1/\alpha)} \left(1 + \eta^{\alpha+1}\right)^{1/\alpha}. \]

Since
\[ \varphi_1(\alpha, 1) = \frac{2|\alpha|}{(\alpha + 1)(\alpha + 1/\alpha)} = \varphi_0(\alpha, 0) = \langle f_\alpha \rangle_{(0,1)}, \]
then
\[ \psi_1(\alpha, \eta) \equiv \frac{\langle f_\alpha \rangle_I}{\langle f_\alpha \rangle_{(0,1)}} = \frac{\varphi_1(\alpha, \eta)}{\varphi_1(\alpha, 1)} = 2 \left(\frac{1 + \eta^{\alpha+1}}{1 + \eta}\right)^{1/\alpha}. \]

Let \( \psi \) be the function defined by
\[ \psi(\alpha, \eta) = \begin{cases} 
\psi_0(\alpha, \eta), & 0 \leq \eta \leq \eta_1, \\
\psi_1(\alpha, \eta), & \eta_1 \leq \eta \leq 1.
\end{cases} \]

Then
\[ \frac{\langle f_\alpha \rangle_R}{\langle f_\alpha \rangle_{R+}} = \max_{0 \leq \eta \leq 1} \psi(\alpha, \eta), \]
and the proof is completed.

**Corollary 2.1** Let \( \psi \) be the function defined in Theorem 2.1. Then
\[ \psi \left( -\frac{\alpha}{\alpha + 1}, \eta^{\alpha+1} \right) = \psi(\alpha, \eta) \quad (\alpha > -1, \ 0 \leq \eta \leq 1). \]

**Proof.** Straightforward calculations. \( \blacksquare \)

For \( p > 1 \) we set \( \alpha = 1/(p-1) \). Then
\[ \alpha + 1 = p/(p-1), \quad -\alpha/(\alpha + 1) = -1/p, \]
and Corollary 2.1 can be rewritten in the following form.

**Corollary 2.2** Let \( p > 1 \) and let \( \psi \) be the function defined in Theorem 2.1. Then
\[ \psi \left( -\frac{1}{p}, \eta^{p/(p-1)} \right) = \psi \left( \frac{1}{p-1}, \eta \right) \quad (0 \leq \eta \leq 1). \]

Recall that the "norms" on the Gurov-Reshetnyak class are denoted by \( \varepsilon_R^-(p) \equiv \langle f_{1/(p-1)} \rangle_R \) and \( \varepsilon_R^+(p) \equiv \langle f_{-1/p} \rangle_R, \ p > 1 \). However, one has
\[ \langle f_{1/(p-1)} \rangle_{R+} = \langle f_{-1/p} \rangle_{R+} \equiv \varepsilon_R^+(p), \]
and Theorem 2.1 and Corollary 2.2 lead to the following result.
Corollary 2.3 If \( p > 1 \), then

\[
\langle f_{1/(p-1)} \rangle_R = \langle f_{-1/p} \rangle_R \equiv \varepsilon_R(p) = \varepsilon_{R+}(p) \cdot \max_{0 \leq \eta \leq 1} \psi \left( \frac{1}{p-1}, \eta \right).
\]

The next corollary allows us to improve Theorem 2.1 by using a better description of the set where the function \( \psi \) can attain its maximum.

Corollary 2.4 Let \( \alpha > -1 \) and \( \alpha \neq 0 \). Then

\[
\max_{0 \leq \eta \leq 1} \psi(\alpha, \eta) = \max_{0 \leq \eta \leq \eta_1} \psi_0(\alpha, \eta),
\]

where the function \( \psi_0(\alpha, \eta) \) is defined in the proof of Theorem 2.1 and the number \( \eta_1 = \eta_1(\alpha) \in (0, 1) \) is derived from the equation (3).

**Proof.** Since the function \( \psi(\alpha, \eta) \) is continuous on the interval \([0, 1]\) in the variable \( \eta \), it suffices to show that for any fixed \( \alpha \) the function \( \psi_1(\alpha, \eta) \), defined in the proof of Theorem 2.1, is decreasing on the interval \([\eta_1, 1]\). For this we compute the derivative

\[
\frac{\partial}{\partial \eta} \psi_1(\alpha, \eta) = 2 \frac{\alpha + 1}{\alpha} \frac{(1 + \eta^{\alpha+1})^{1/\alpha} - 1}{(1 + \eta)^{2+1/\alpha}} \psi_1^\alpha - 1",
\]

and observe that if \( 0 < \eta < 1 \), then the derivative is negative. This completes the proof. \( \blacksquare \)

Corollaries 2.2 and 2.4 imply the following result:

Corollary 2.5 If \( p > 1 \), then

\[
\max_{0 \leq \eta \leq \eta_1(1/(p-1))} \psi_0 \left( \frac{1}{p-1}, \eta \right) = \max_{0 \leq \eta \leq \eta_1(-1/p)} \psi_0 \left( \frac{-1}{p}, \eta \right).
\]

Let us compute the derivative of the function \( \psi_0(\alpha, \eta) \). Thus

\[
\frac{\partial}{\partial \eta} \psi_0(\alpha, \eta) = \frac{(\alpha + 1)^{\alpha+1} / \alpha}{1 + \eta} \frac{\alpha + 1}{1 + \eta^{\alpha+1}} \frac{1}{1 + \eta}
\]

\[
\times \left\{ \left[ 1 - \left( \frac{1}{\alpha + 1} + \eta^{\alpha+1} \right)^{1/\alpha} \right] \left[ \frac{1}{\alpha + 1} + \eta^{\alpha+1} \right] - \eta^\alpha \right\}
\]

\[
- \left[ \frac{\alpha}{\alpha + 1} \left( \frac{1}{\alpha + 1} + \eta^{\alpha+1} \right)^{(\alpha+1)/\alpha} \right] - \frac{1}{\alpha + 1} + \eta^{\alpha+1} + \frac{1}{\alpha + 1}
\]

\[
\times \eta^\alpha \left( \frac{\alpha + 1}(1 + \eta) \right)^{\alpha(\alpha+1)}.\]

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Note that for $\eta = \eta_1$ the second factor in the second line of the formula is equal to zero. Moreover, the expression in the third line is nothing else but $(1+\eta)\Omega(f_\alpha; I)/2\text{sign }\alpha$, and elementary computations lead to the following representation for the derivative of the function $\psi_0(\alpha, \eta)$:

$$
\frac{\partial}{\partial \eta} \psi_0(\alpha, \eta) = \frac{\alpha + 1}{\alpha} \frac{1}{(1 + \eta)^{2+1/\alpha} (1 + \eta^{\alpha+1})^2} \times \left[ (1 + \eta^{\alpha^2+1})^{(\alpha+1)/\alpha} (\eta^\alpha - 1) + (\alpha + 1)^{1/\alpha} \left( 1 + \eta^{\alpha+1} \right)^2 (1 + \eta)^{1/\alpha} - (\alpha + 1)^{(\alpha+1)/\alpha} \eta^\alpha (1 + \eta)^{2+1/\alpha} \right].
$$

It is easily seen that

$$
\frac{\partial}{\partial \eta} \psi_0(\alpha, 0) = \frac{\alpha + 1}{\alpha} \left[ (\alpha + 1)^{1/\alpha} - 1 \right] > 0 \text{ if } \alpha > 0,
$$

$$
\frac{\partial}{\partial \eta} \psi_0(\alpha, 0+) = +\infty \text{ if } -1 < \alpha < 0.
$$

On the other hand,

$$
\frac{\partial}{\partial \eta} \psi_0(\alpha, \eta_1) = -\frac{(\alpha + 1)^{(3\alpha+1)/\alpha}}{2|\alpha|} \frac{1 + \eta_1}{(1 + \eta_1^{\alpha+1})^2 \eta_1^\alpha \Omega(f_\alpha; I(\eta_1))} < 0.
$$

This leads to the following result.

**Corollary 2.6** For each fixed $\alpha$ the function $\psi_0(\alpha, \eta)$ attains its maximal value on the interval $[0, \eta_1]$ at the inner point $\eta_{\text{max}} = \eta_{\text{max}}(\alpha) \in (0, \eta_1)$, i.e. where

$$
\frac{\partial}{\partial \eta} \psi_0(\alpha, \eta) = 0.
$$

Corollary 2.6 means that $\eta_{\text{max}}$ is the root of the equation

$$
(1 + \eta^{\alpha+1})^{(\alpha+1)/\alpha} (\eta^\alpha - 1) + (\alpha + 1)^{1/\alpha} \left( 1 + \eta^{\alpha+1} \right)^2 (1 + \eta)^{1/\alpha} - (\alpha + 1)^{(\alpha+1)/\alpha} \eta^\alpha (1 + \eta)^{2+1/\alpha} = 0. \quad (4)
$$

However, even in the simplest case $\alpha = 1$ the authors do not know an analytic solution of this equation (see Example 3.1 below).

**Remark 2.4** The numerical study of the behaviour of the function $\psi_0(\alpha, \eta)$ for different values of $\alpha$ shows that the derivative $\frac{\partial}{\partial \eta} \psi_0(\alpha, \eta)$ has a unique root $\eta_{\text{max}} = \eta_{\text{max}}(\alpha)$ in the interval $(0, \eta_1)$. Nevertheless, the authors do not know any rigorous proof of this fact.
3 Numerical experiments, examples, and comments

Fix an \(\varepsilon \in (0,2)\). Set \(p = p(\varepsilon) = p^*_R(\varepsilon) > 1\), \(\alpha = \alpha(\varepsilon) = 1/(p(\varepsilon) - 1)\) and define \(\eta_1 = \eta_1(\varepsilon)\) by (3). According to Theorem 2.1 and Corollary 2.4, one has

\[
\| T \|_{GR}^{(\varepsilon)} \geq \max_{0 \leq \eta \leq \eta_1} \psi_0(\alpha, \eta) \equiv C_{\varepsilon}, \quad \| T \|_{GR} \geq \sup_{0 < \varepsilon < 2} C_{\varepsilon} \equiv C. \tag{5}
\]

Table 1 shows some values of the parameters mentioned obtained in numerical experiments, where the columns 6 and 8 contain the maximum points of the function \(\psi_0(\alpha, \eta)\) for \(\alpha = 1/(p - 1)\) and \(\alpha = -1/p\), correspondingly.

Table 1: The Gurov-Reshetnyak "norms" and extremal points. Numerical results.

|   | \(\varepsilon = \varepsilon_{R+}\) | \(\varepsilon_R\) | \(C_\varepsilon = \frac{\varepsilon_R}{\varepsilon_{R+}}\) | \(\alpha = \frac{1}{p - 1}\) | \(\eta_{\text{max}}^+\) | \(\alpha = -\frac{1}{p}\) | \(\eta_{\text{max}}^-\) |
|---|-----------------|----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1 | 2.00            | 0.5000         | 0.6224          | 1.244737        | 1.0000          | 0.2531          | -0.5000         | 0.0640          |
| 3 | 0.2963          | 0.3726          | 1.257683        | 0.5000          | 0.2001          | -0.3333         | 0.0895          |
| 6 | 0.0000          | 0.0001          | 1.264797        | 0.0000          | 0.1375          | -0.0001         | 0.1374          |

In addition, the results of numerical experiments are reflected in Figures 1 and 2.

Comments to the graphs.

Figure 1. The lower graph in the left part of the Figure 1 shows the dependance of the Gurov-Reshetnyak "norms" \(\varepsilon_{R+}(p)\) on the parameter \(p\). These results are obtained from formula (1). Note that the data are represented in the logarithmic scale and do not include all results from Column 2. The upper graph shows the dependance of the exponents \(\varepsilon_R(p)\) on \(p\), presented in Corollary 2.3.

In the right part of Figure 1 we show the graphs of the inverse relations, i.e. these are values of those \(p\), for which the function \(f_1/(p-1)\) belongs to the class \(G_{R,1}(\varepsilon)\) (the lower line) or to the class \(G_{R}(\varepsilon)\) (the upper line) for a given \(\varepsilon\).
The Gurov - Reshetnyak’s "norms"  

\[ \varepsilon_{R_+}(p) \quad \varepsilon_R(p) \]

The limiting exponent  

\[ p(\varepsilon_{R_+}) \quad p(\varepsilon_{R}) \]

Figure 1: Relations between \( p \) and \( \varepsilon \).

The growth of "norms"  

\[ \left( \frac{\varepsilon_R}{\varepsilon_{R_+}} \right)(p) \]

The lower estimate of \( \| T \|_{GR}^{(\varepsilon)} \)

Figure 2: The growth of the norms during the extension from \( \mathbb{R}_+ \) to \( \mathbb{R} \).

Remark 3.1 As is seen from Table 1, the "norm" of the operator \( T \) can be estimated as follows

\[ \| T \|_{GR} \geq C = \lim_{\varepsilon \to 0^+} C_{\varepsilon} \approx 1.264797, \]

where the constant \( C \) is defined in (5). The corresponding numerical value \( C \) is shown in boldface in the last row of Table 1.

Remark 3.2 For functions \( f \in BMO \) with bounded mean oscillation the question about the sharp value of the norm \( \| f \|_{BMO,\mathbb{R}} = \sup_{I \subset \mathbb{R}} \Omega(f; I) \) of the even extension...
from $\mathbb{R}_+$ to $\mathbb{R}$ of a monotone function on the semi-axis $\mathbb{R}_+$ is posted in [9] and, to the best of our knowledge, it is still open. In [2] the BMO-norm of the function $f_0(x) = \ln(1/|x|)$, which is a typical representative of this class, is found. Thus

$$
\|f_0\|_{BMO, \mathbb{R}} = \frac{2}{e} \cdot \frac{1}{t+1} \left[ \exp \left( \frac{t \ln t}{t+1} \right) + e^{t \ln t} \right],
$$

where $t > 1$ is the root of the equation

$$
\exp \left( \frac{t \ln t}{t+1} \right) = e \left( t - 1 - \frac{t+1}{\ln t} \right).
$$

Since $\|f_0\|_{BMO, \mathbb{R}} = 2/e$, the approximate solution of the equation (6) leads to the following estimate (see [2])

$$
\|T\|_{BMO} = \sup_{f \text{ is even on } \mathbb{R}, \text{ and monotone on } \mathbb{R}_+} \|f\|_{BMO, \mathbb{R}} \geq \frac{\|f_0\|_{BMO, \mathbb{R}}}{\|f_0\|_{BMO, \mathbb{R}_+}} = C_0 \approx 1.264797.
$$

It is remarkable that $C$ and $C_0$ coincide up to 6 significant digits after the point, i.e. the lower bounds of the norms $\|T\|_{GR}$ and $\|T\|_{BMO}$ of the operator of even extension $T$ turned out to be the same within the calculation accuracy.

**Example 3.1** In the simplest case $\varepsilon = 1/2$, $p = 2$ and $\alpha = 1$ or $\alpha = -1/2$, one can find the roots of the equation (3), which are $\eta_1(1) = \sqrt{2} - 1 \approx 0.414$, $\eta_1(-1/2) = 3 - 2\sqrt{2} \approx 0.172$.

The corresponding line in Table 1 is marked out by boldface. In Figure 3 the graphs of the functions $\psi(-1/2, \eta)$ and $\psi(1, \eta)$ are represented by solid lines. Note that $\psi_0(1, \eta_1(1)) = \psi_0(-1/2, \eta_1(-1/2)) = 2\sqrt{2} \left( \sqrt{2} - 1 \right) \approx 1.172$. For these values of the parameters, the equation (4), which is used to find $\eta_{\max} = \eta_{\max}(1) \in (0, \eta_1)$, takes the form

$$
3\eta^5 - 3\eta^4 - 6\eta^3 - 10\eta^2 - \eta + 1 = 0.
$$

The detection of any analytic solution of this equation seems to be difficult. However, one can show that in the interval $(0, \eta_1)$ this equation has a unique solution (see Remark 2.4). Indeed, since the second derivative of the function $\psi_0(1, \eta)$ is

$$
\frac{\partial^2}{\partial \eta^2} \psi_0(1, \eta) = -4 \left( \frac{3\eta}{(1+\eta)^3} + \frac{1-3\eta^2}{(1+\eta^2)^3} \right),
$$

and $1 - 3\eta^2 > 0$ for $0 < \eta < \sqrt{2} - 1$, one obtains that $\frac{\partial^2}{\partial \eta^2} \psi_0(1, \eta) < 0$ on the interval $(0, \eta_1)$. This means that $\frac{\partial}{\partial \eta} \psi_0(1, \eta)$ is strictly decreasing in the interval $(0, \eta_1)$, hence it has a unique root in this interval. By Corollary 2.2, the derivative $\frac{\partial}{\partial \eta} \psi_0(-1/2, \eta)$ also has a unique root in the interval $(0, \eta_1(-1/2))$.

Notice that the values $\eta_{\max}(1) \approx 0.253$, $\eta_{\max}(-1/2) \approx 0.064$ are approximations of the corresponding roots and $C_{1/2} \approx 1.245$. 

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Example 3.2 Let $\alpha = 1/2$, $p = 3$, and $\varepsilon \approx 0.296$. In Table 1 the numerical results corresponding to this case are shown in the line partially written in boldface. In Figure 3 the graph of the corresponding function $\psi(1/2, \eta)$ is represented by the dashed line.

In this case, the equation (3) takes the form

$$\sqrt{\eta} = \frac{2}{3 + \eta}.$$ 

The solution of this equation

$$\eta_1 \equiv \eta \left( \frac{1}{2} \right) = \sqrt[3]{3 + 2\sqrt{2} + \sqrt[3]{3 - 2\sqrt{2}} - 2} \approx 0.355$$

is obtained by Cardano formulas. We also have

$$\psi_0 \left( \frac{1}{2}, \eta \right) = \left( 1 + \eta^{3/2} / (1 + \eta)^3 \right)^2 + \frac{27}{4} \left[ \frac{1}{1 + \eta^{3/2}} - \frac{1}{1 + \eta} \right],$$

$\psi(1/2, \eta_1) \approx 1.180$. In addition, the equation (4) takes the form

$$\left( 1 + \eta^{3/2} \right)^3 (\eta^{1/2} - 1) + \frac{9}{4} \left( 1 + \eta^{3/2} \right)^2 (1 + \eta)^2 - \frac{27}{8} \eta^{1/2}(1 + \eta)^4 = 0,$$
and its approximate solution is $\eta_{\text{max}}(1/2) \approx 0.200$. Finally, we obtain an approximate value of $C_{0.296}$, namely $C_{0.296} \approx 1.258$.

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References

[1] B. Bojarski, Remarks on stability of the inverse Holder inequalities and quasi-conformal mappings, Ann. Acad. Sci. Fenn. Math. 10, 291–296 (1985)

[2] V. D. Didenko, A. A. Korenovskyi and N. J. Tuah, Mean oscillations of the logarithmic function, Ricerche di Matematica 62(1), 81-90 (2013)

[3] M. Franciosi, Higher integrability results and Hölder continuity, J. Math. Anal. Appl. 150(1), 161–165 (1990)

[4] M. Franciosi, The Gurov–Reshetnyak condition and VMO, J. Math. Anal. Appl. 181(1), 17–21 (1994)

[5] M. Franciosi and G. Moscariello, Higher integrability results, Manuscripta Math. 52(1), 151–170 (1985)

[6] F. W. Gehring, The $L^p$-integrability of the partial derivatives of a quasiconformal mapping, Acta Math. 130, 265–273 (1973)

[7] L. G. Gurov and Y. G. Reshetnyak, An analogue of the concept of functions with bounded mean oscillation, Siberian Math. J. 17(3), 417–422 (1976)

[8] T. Iwaniec, On $L^p$-integrability in PDE’s and quasiregular mappings for large exponent, Ann. Acad. Sci. Fenn. Math. 7(2), 301–322 (1982)

[9] I. Klemes, A mean oscillation inequality, Proc. Amer. Math. Soc. 93(3), 497–500 (1985)

[10] A. A. Korenovskii, Mean Oscillations and Equimeasurable Rearrangements of Functions, Lecture Notes of Unione Mat. Ital. 4, Springer, Berlin (2007)

[11] A. Korenovskyi, The Gurov–Bögel–Reshetnyak inequality on semi-axes, Ann. Mat. Pura Appl. (1923-), Published online: 19 February 2015, DOI 10.1007/s10231-015-0482-2
[12] A. A. Korenovskyy, On the connection between mean oscillation and exact integrability classes of functions, Mat. Sb. 181(12), 1721–1727 (1990)

[13] A. A. Korenovskyy, On the embedding of the Gehring class into the Gurov–Reshetnyak class, Visnyk Odes’kogo Natsional’nogo Universytetu, Matematyka i Mekhanika 8(2), 15–21 (2003)

[14] A. A. Korenovskyy, Relation between the Gurov–Reshetnyak and the Muckenhoupt function classes, Mat. Sb. 194(6), 127–134 (2003)

[15] A. A. Korenovskyy, About the Gurov–Reshetnyak class of functions, Pr. Inst. Mat. Nats. Akad. Nauk Ukr. Mat. Zastos. 1(1), 189–206 (2004)

[16] A. A. Korenovskyy, A. K. Lerner and A. M. Stokolos, A note on the Gurov–Reshetnyak condition, Math. Res. Lett. 9(5–6), 579–583 (2002)

[17] R. V. Shanin, Extension of functions with bounded mean oscillation, J. Math. Sci. (N. Y.) 196(5), 693–704 (2014)

[18] I. Wik, Note on a theorem by Reshetnyak–Gurov, Dep. Math. Univ. Umea (Publ.) 6, 1–7 (1985)