INVERSE LIMIT SLENDER GROUPS

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ABSTRACT. Classically, an abelian group \( G \) is said to be slender if every homomorphism from the countable product \( \mathbb{Z}^\mathbb{N} \) to \( G \) factors through the projection to some finite product \( \mathbb{Z}^n \). Various authors have proposed generalizations to non-commutative groups, resulting in a plethora of similar but not completely equivalent concepts. In the first part of this work we present a unified treatment of these concepts and examine how they are related. In the second part of the paper we study slender groups in the context of co-small objects in certain categories, and give several new applications including the proof that certain homology groups of Barratt-Milnor spaces are cotorsion groups and a universal coefficients theorem for Čech cohomology with coefficients in a slender group.

1. INTRODUCTION

In this introductory section we describe prior work on slender groups and give a brief summary of our contributions.

1.1. Prior work. Origins of the theory of slender slender groups can be traced back to Baer’s [1] proof that the group \( \mathbb{Z}^\mathbb{N} \) is not free and Specker’s [30] remarkable result that every homomorphism from \( \mathbb{Z}^\mathbb{N} \) (and from many of its subgroups) to \( \mathbb{Z} \) factors through a projection to some finite product \( \mathbb{Z}^n \). This may be restated as

\[
\text{Hom}(\mathbb{Z}^\mathbb{N}, \mathbb{Z}) \cong \bigoplus_{n \in \mathbb{N}} \mathbb{Z},
\]

i.e., the dual of a countable product of groups \( \mathbb{Z} \) is their countable sum. A few years later \( \text{Łoś} \) called a group \( G \) slender if it is torsion-free abelian, and if every homomorphism from \( \mathbb{Z}^\mathbb{N} \) to \( G \) factors through some finite product (see Fuchs [17]). He also proved that the class of slender groups is closed under subgroups, extensions and arbitrary direct sums, and that groups containing \( \mathbb{Q} \) are not slender. In particular, for a slender group \( G \) we have the relation

\[
\text{Hom}(\mathbb{Z}^\mathbb{N}, G) \cong \bigoplus_{n \in \mathbb{N}} G. \tag{*}
\]

Sasiada [27] proved that a countable abelian group is slender if, and only if, it is torsion-free and does not contain a copy of \( \mathbb{Q} \). Finally, without restrictions on the cardinality, Nunke [28] showed that an abelian group is slender if, and only if, it is torsion-free and does not contain \( \mathbb{Q}, \mathbb{Z}^\mathbb{N} \), or \( p \)-adic integers for any \( p \). The theory of slender abelian groups has a natural generalization to modules over arbitrary rings, and there is a wide body of work on slender modules and rings, but in this paper...
we will be mainly interested into generalizations of slenderness to non-commutative groups.

Göbel [22] studied non-abelian groups $G$ that satisfy Łoś’s definition, and proved that $G$ is slender if, and only if, every abelian subgroup of $G$ is slender. However, the generalization of formula $(*)$ is not immediate, because for non-commutative groups $\text{Hom}(\mathbb{Z}^N, G)$ is only a pointed set. Göbel considered countable products of arbitrary groups and showed that $G$ is slender if, and only if, the set $\text{Hom}(\prod_{n \in \mathbb{N}} A_n, G)$ can be expressed as a subset of the product $\prod_{n} \text{Hom}(A_n, G)$, consisting of elements with finite support (see [22, Theorem 4.3] for precise formulation).

A further step ahead was taken by Eda [13] who defined non-commutatively slender (or $n$-slender) groups by replacing the product $\mathbb{Z}^N$ and its projections to finite factors $\mathbb{Z}^n$ with the so-called free complete product of cyclic groups, denoted $\mathbb{H}$, and its projections to free groups on $n$-letters $F_n$. Group $\mathbb{H}$ has a convenient topological description as the fundamental group of the Hawaiian earring (a countable one-point union of a sequence of circles whose radii decrease to zero), and the projections to $F_n$ are induced for each $n$ by the projection of the Hawaiian earring to the first $n$ circles (see [13, Appendix]). Eda proved that $n$-slender groups are torsion-free and are preserved by taking subgroups, free products and restricted direct products. He also showed that $n$-slender is a strictly stronger condition than slender, but that for abelian groups the two concepts coincide.

Slenderness can be also seen as an automatic continuity condition in the sense of Dudley [12]. Let us endow $\mathbb{Z}$ and $G$ with the discrete topology and $\mathbb{Z}^N$ with the product topology. Then it is easy to see that $G$ is slender precisely when every homomorphism from $\mathbb{Z}^N$ to $G$ has open kernel (in other words, it is continuous when $G$ is endowed with the discrete topology). This lead Conner and Corson [5] to define a discrete group $G$ to be $\text{lch}$-slender (resp. $\text{cm}$-slender) if every homomorphism from a locally compact Hausdorff (resp. completely metrizable) group $A$ to $G$ has open kernel. They showed that many interesting groups, including free groups, free abelian groups and torsion-free word-hyperbolic groups, are $\text{lch}$-slender and $\text{cm}$-slender and that for abelian groups we recover the usual slender groups. Several types and properties of automatically continuous were studied in [10, 24]. Most recently, Corson and Varghese [11] strengthened the connection between slender groups and automatic continuity by showing that a group is $\text{lch}$-slender if it is torsion-free and it does not contain $\mathbb{Q}$ or $p$-adic integers for any $p$.

1.2. Our contribution. A common feature of the above examples is that the groups used to test slenderness are related to inverse limits: $\mathbb{Z}^N$ can be represented as an inverse limit of $\mathbb{Z}^n$, $\mathbb{H}$ is a dense subgroup of the inverse limit of free groups $F_n$, every connected locally compact Hausdorff group is an inverse limit of Lie groups. This (and some isolated results for maps from inverse limits) motivated our study of groups that are slender with respect to inverse limits of groups.

Recall that every inverse sequence of groups

$$H_1 \leftarrow H_2 \leftarrow H_3 \leftarrow \cdots$$

gives rise to a limit group $\varprojlim H_n$, together with projections $p_i : \varprojlim H_k \to H_i$. A group $G$ is inverse limit slender (or $\text{lim}$-slender) if for every inverse sequence with
surjective bonding maps, every homomorphism from $\varprojlim H_i$ to $G$ factors through some projection $p_i$.

Our first result states that slenderness with respect to inverse limits can be checked on a single test group. Let $\hat{F}$ denote the inverse limit of an inverse sequence of finite rank free groups (see Definition 2.4 for a formal description). This characterization allows us to prove that inverse limit slender groups are preserved by several basic group constructions.

**Theorem 1.** (Theorem 3.6 and Theorem 3.7) A group $G$ is inverse limit slender if, and only if, every homomorphism from $\hat{F}$ to $G$ factors through a projection to a free group of finite rank.

The class of inverse limit slender groups is closed under taking subgroups and extensions, and under direct sums and free products of arbitrary cardinality.

The relation between different types of slender groups is given by the following theorem.

**Theorem 2.** (Theorem 3.10 and Corollary 3.11) The class of $n$-slender groups is contained in the class of inverse limit slender groups, the class of inverse limit slender groups is contained in the class of slender groups, and the class of slender groups is contained in the class of $\ell$ch-slender groups. All containments are strict. Furthermore, among abelian groups the $n$-slender, inverse limit slender and slender groups coincide.

Interestingly, all examples that distinguish various versions of slenderness are uncountable groups, so we state the following

**Conjecture:** For countable groups the classes of slender groups, $n$-slender groups, inverse limit slender groups, $cm$-slender groups and $\ell$ch-slender groups coincide.

Inverse limit slender groups bring to light a connection with an important concept from category theory. In a category $\mathcal{C}$ that admits direct limits an object $X$ in $\mathcal{C}$ is said to be small if for every direct sequence with monomorphic bonding maps $(X^i)$ there exists a natural bijection $\varinjlim \text{Hom}(X, X^i) = \text{Hom}(X, \varprojlim X^i)$. For example, finitely generated groups are small object in the category of groups. Dually, a co-small object in a category $\mathcal{C}$ that admits inverse limits is an object $X$, such that for every inverse sequence $(X_i)$ with epimorphic bonding maps there is a natural bijection $\varprojlim \text{Hom}(X_i, X) = \text{Hom}(\varprojlim X_i, X)$. Clearly, a group $G$ is inverse limit slender if, and only if it is a co-small object in the category of groups. In Propositions 4.5 and 4.6 we give analogous characterizations for slender and $n$-slender groups.

One can obtain a higher dimensional analogue of the Hawaiian earring by taking for each $n$ the space $\mathcal{H}_n$ defined as a countable one-point union of $n$-dimensional spheres in $\mathbb{R}^{n+1}$ whose radii decrease to zero, i.e. the one-point compactification of a countable collection of open $n$-balls. These spaces were first studied by Barratt and Milnor [2] who proved a remarkable fact that, although $\mathcal{H}_n$ is $n$-dimensional, there are infinitely many indices $i$ for which the singular homology groups $H_i(\mathcal{H}_n)$ are uncountably infinite. As a step toward a precise description of these homology groups we prove the following

**Theorem 3.** (Theorem 5.5) For $l > n > 1$, the image of the Hurewicz map
\[ \pi_l(\mathcal{H}_n) \to H_l(\mathcal{H}_n) \] is cotorsion. In particular, \( H_{n+1}(\mathcal{H}_n) \) is cotorsion for every \( n > 1 \).

Together with Barratt and Milnor’s results this proves that \( H_3(\mathcal{H}_2) \) is an uncountable cotorsion group. The proof of the above result is based on the ability to choose arbitrarily small representatives of homology classes (see Lemma 5.2), so we believe that a more general result is true:

**Conjecture:** For \( l > n > 1 \), the groups \( H_l(\mathcal{H}_n) \) are cotorsion.

As mentioned before, part of our motivation for this work is of topological nature. In particular, our study of inverse limits of covering spaces naturally requires good understanding of homomorphisms from inverse limits groups (see Conner, Herfort, Kent and Pavesic [6, 7] for further information). In the final part of this paper we present two applications of our work on (shape) homotopy groups and (Čech) cohomology groups of Peano continua (see Section 6 for relevant definitions).

A well-known theorem of Shelah [28] states that the fundamental group of a Peano continuum is either finite presented or uncountable and this implies that the shape fundamental group \( \tilde{\pi}_1(X) \) of a Peano continuum \( X \) may be very complicated. In fact, it is often uncountable and can even be locally free with uncountable free subgroups. Nevertheless, we will show that \( \tilde{\pi}_1(X) \) admits relatively few homomorphisms into slender groups.

**Corollary 4.** (Corollary 6.1) If \( X \) is a Peano continuum and \( G \) is a a countable inverse limit slender group, then there are only countably many homomorphisms from the first shape group \( \tilde{\pi}_1(X) \) to \( G \).

Our final application is a Universal Coefficients Theorem that expresses Čech cohomology groups with coefficients in a slender abelian group in terms of Čech homology groups. The theorem relies on the equivalence between slender and inverse limit slender abelian groups.

**Theorem 5.** (Theorem 6.5) If \( G \) is a slender abelian group, then there is a short exact sequence

\[
0 \longrightarrow \lim_{i \to \infty} \text{Ext}(\tilde{\pi}_1(X_i), G) \longrightarrow \tilde{\pi}_1(X; G) \longrightarrow \text{Hom}(\tilde{\pi}_1(X), G) \longrightarrow 0.
\]

1.3. Outline. In Section 2 we introduce the notation and terminology of categorical limits. Section 3 contains the main results on inverse limit slender groups and on their relation to other types of slender groups. In Section 4 we examine categorical aspects of slender groups, viewed as co-small objects in corresponding categories. In Section 5 we study algebraic compactness and the cotorsion property of homology groups of Barratt-Milnor spaces. Finally, in Section 6 we give some applications of inverse limit slender groups to shape fundamental groups and Čech cohomology groups.

2. Preliminaries on inverse and direct limits

In this section we set the basic notation on direct and inverse sequences and their limits, and refer to Geoghegan [21 Section 11.2] for a more detailed exposition.
Definition 2.1. Let \( \mathcal{C} \) be a category. An inverse sequence \( (X_i, p_i^j) \) in \( \mathcal{C} \) consists of a set of objects \( \{X_i \mid i \in \mathbb{N}\} \) in \( \mathcal{C} \) and a set of morphisms (called bonding maps) \( \{p_i^j : X_j \to X_i \mid i < j\} \) in \( \mathcal{C} \), satisfying \( p_i^j \circ p_j^k = p_i^k \) whenever \( i < j < k \). We often depict an inverse system as a diagram
\[
X_1 \leftarrow p_1^2 : X_2 \leftarrow \cdots
\]
In the categories that are of interest to us (sets, pointed sets, abelian groups, groups, topological spaces, pointed topological spaces, topological groups) every inverse sequence \( (X_i, p_i^j) \) has an inverse limit, i.e., an object \( \varprojlim(X_i, p_i^j) \) in \( \mathcal{C} \) together with morphisms \( p_i : \varprojlim(X_i, p_i^j) \to X_i \) in \( \mathcal{C} \) called projections satisfying \( p_i = p_i^j \circ p_{j}^{k} \) whenever \( i < j < k \) and the usual universal property (see [21, p.235]). In fact, in the above-mentioned categories we have standard models for the inverse limit whose underlying set is given
\[
\varprojlim(X_i, p_i^j) = \left\{ (x_i) \in \prod_{i \in \mathbb{N}} X_i \mid x_i = p_i^j(x_j) \right\}
\]
and projections \( p_i \) are obtained by projecting each sequence to its \( i \)-th component. For categories with algebraic or topological structure the underlying sets are equipped with component-wise algebraic operations or/and product topology.

We next define a special, well-behaved class of inverse system.

Definition 2.2. An inverse sequence \( \{X_j, p_i^j\} \) satisfies Mittag-Leffler condition (or is an ML-sequence) if for every \( i \) there exists \( j > i \) such that \( p_i^k(X_k) = p_i^j(X_j) \) for every \( k \geq j \).

Clearly, every inverse sequence with surjective bonding maps is an ML-sequence. On the other hand, a sequence, whose bonding maps are injective is an ML-sequence only if it is eventually constant.

Inverse sequences and their limits have categorical duals in the form of direct sequences and direct limits.

Definition 2.3. A direct sequence \( (X^i, u_i^j) \) in a category \( \mathcal{C} \) consists of objects \( \{X^i \mid i \in \mathbb{N}\} \) and bonding maps \( \{u_i^j : X^j \to X^i \mid i < j\} \) in \( \mathcal{C} \), satisfying \( u_j^k \circ u_i^j = u_i^k \) whenever \( i < j < k \). The corresponding diagram is
\[
\cdots \to X^1 \xrightarrow{u_1^2} X^2 \xrightarrow{u_2^3} X^3 \to \cdots
\]
A direct limit of such sequence is an object \( \varinjlim(X^i, u_i^j) \) in \( \mathcal{C} \), together with morphisms \( u_i : X^i \to \varinjlim(X^i, u_i^j) \), satisfying \( u_i = u_j \circ u_i^j \) whenever \( i < j \) and the usual universal property (see [21, p.238]). In all of the above-mentioned categories a model for the direct limit can be obtained by taking a disjoint union of underlying sets, modulo certain equivalence relation, and endowing it with a suitable algebraic or topological structure:
\[
\varinjlim(X^i, u_i^j) = \bigsqcup_{i \in \mathbb{N}} X^i / \sim,
\]
where \( \sim \) is the equivalence relation, generated by \( x_i \sim u_i^j(x_j) \) for all \( i \in \mathbb{N} \) and \( x_i \in X^i \). In particular, if all bonding maps in a direct sequence are injective, then its direct limit is simply the union of its terms.
The main objective of this section is to describe the properties of inverse limit slender groups and relate it to other types of slender groups. We begin by recalling

**Definition 2.4.** Let \( \{x_n \mid n \in \mathbb{N}\} \) be a countable set, and for every \( i \in \mathbb{N} \) let \( F_i \) be the free group on the alphabet \( \{x_1, \ldots, x_i\} \). For \( i < j \) there are natural inclusions \( u_i^j : F_i \to F_j \) and projections \( p_i^j : F_j \to F_i \), which are related by \( p_i^j u_i^j = 1_{F_i} \). Thus we obtain a direct system \( (F_i, u_i^j) \) and an inverse system \( (F_i, p_i^j) \). The direct limit of the former is \( F_{\infty} := \lim \to (F_i, u_i^j) \), the free group generated by \( \{x_n \mid n \in \mathbb{N}\} \), while the inverse limit of the latter is \( \hat{F} = \lim \leftarrow (F_i, p_i^j) \), which is sometimes called *unrestricted free product* of infinite cyclic groups. In addition, we will denote by \( \hat{F}(k) \) the subgroup of \( \hat{F} \) obtained as inverse limit of free groups on the subset \( \{x_n \mid n > k\} \), so that \( \hat{F}(0) = \hat{F} \) and \( \hat{F}(k) \subseteq \ker p_k \).

Note that, the relation between maps \( u_i^j \) and \( p_i^j \) implies that the group \( F_{\infty} \) can be naturally identified with a subgroup of \( \hat{F} \). Even more, if we endow \( \hat{F} \) with the product topology, then \( F_{\infty} \) is a countable dense subgroup of \( \hat{F} \).

We conclude this section by recalling certain commutativity relations between direct and inverse limits. Let \( \lim X_i \) be the limit of an inverse sequence \( (X_i, p_i^j) \) in a category \( \mathcal{C} \). We can associate to every morphism \( f : A \to \lim X_i \) the sequence \( (p_i f) \in \prod \text{Hom}(A, X_i) \). One can easily check that this actually determines a natural bijection

\[
\text{Hom}(A, \lim X_i) \xrightarrow{\cong} \lim \text{Hom}(A, X_i).
\]

Dually, by associating to a morphism \( f : \lim X_i \to A \) the sequence of morphisms \( (fu_i) \in \prod \text{Hom}(X^i, A) \), we obtain a natural bijection

\[
\text{Hom}(\lim X^i, A) \xrightarrow{\cong} \lim \text{Hom}(X^i, A).
\]

Analogous relations for morphisms into direct limits and from inverse limits are more complicated. Consider a direct sequence \( (X^i, u^j_i) \) in a category \( \mathcal{C} \). For every object \( A \) in \( \mathcal{C} \) we may apply the covariant functor \( \text{Hom}(A, -) \) and obtain a direct sequence of sets \( \text{Hom}(A, X^i), (u^j_i)_* \). Every element of \( \lim \text{Hom}(A, X^i) \) is represented by some \( h \in \text{Hom}(A, X^i) \). It is easy to check that the correspondence \( h \mapsto u_i \circ h \in \text{Hom}(A, \lim X^i) \) defines a function

\[
\Delta_A : \lim \text{Hom}(A, X^i), (u^j_i)_* \to \text{Hom}(A, \lim X^i, u^j_i).
\]

Note that \( \Delta \) is injective if bonding maps \( u^j_i \) are monomorphisms in \( \mathcal{C} \).

Dually, if \( (X_i, p^j_i) \) is an inverse sequence in \( \mathcal{C} \), then for every object \( A \) in \( \mathcal{C} \) we have a direct sequence of sets \( \text{Hom}(X_i, A), (p^j_i)^* \). By assigning to every \( h \in \text{Hom}(X_i, A) \) the element \( h \circ p_i \in \text{Hom}(X_i, A) \) we obtain a function

\[
\nabla_A : \lim \text{Hom}(X_i, A), (p^j_i)^* \to \text{Hom}(\lim(X_i, p^j_i), A),
\]

which is injective if bonding maps \( p^j_i \) are epimorphisms in \( \mathcal{C} \).

### 3. Inverse Limit Slender Groups

The main objective of this section is to describe the properties of inverse limit slender groups and relate it to other types of slender groups. We begin by recalling
the classical definition of slender groups and by showing that the integers are a slender group.

**Definition 3.1** (cf. Fuchs [18, Ch. XIII]). Let $\mathbb{Z}^n$ be the countable product of infinite cyclic groups. A group $G$ is **slender** if every homomorphism $\varphi : \mathbb{Z}^n \to G$ can be factored through some $\text{pr}_n : \mathbb{Z}^n \to \mathbb{Z}^n$ (projection to the first $n$ factors of the product), i.e., if there exists $n \in \mathbb{N}$ and a homomorphism $\varphi' : \mathbb{Z}^n \to G$ such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{Z}^n & \xrightarrow{\varphi} & G \\
\| & \searrow & \nearrow \\
\mathbb{Z}^n & \xrightarrow{\text{pr}_n} & G
\end{array}
\]

For the sake of completeness, we give a short, self-contained proof of the well-known fact that $\mathbb{Z}$ is slender.

**Proposition 3.2.** $\mathbb{Z}$ is slender.

**Proof.** If there is a homomorphism $\varphi : \mathbb{Z}^n \to \mathbb{Z}$ that does not factor through any finite projection, then for every $i$ there exists $b_i \in \prod_{n \geq i} \{0\} \times \prod_{n \neq i} \mathbb{Z}$ such that $\varphi(b_i) \neq 0$. One can easily check that the formula $\psi((x_i)) := \sum_i x_i b_i$ defines a homomorphism $\psi : \mathbb{Z}^n \to \mathbb{Z}^n$, such that $\varphi(\psi(e_n)) \neq 0$ for all $n$.

Let $a_1 = 1$, and define $a_2, a_3, \ldots$ recursively by letting $a_{i+1}$ be the minimal multiple of $a_i$ that does not divide $\varphi(\psi(a_i e_i))$. The set $\{(x_i) \in \mathbb{Z}^n \mid x_i \in \{0, a_i\}\}$ is uncountable, so there exist sequences $(x_i) \neq (y_i)$ such that $\varphi(\psi((x_i))) = \varphi(\psi((y_i)))$. Let $n$ be the minimal index for which $x_n \neq y_n$. Then $\varphi((x_i) - (y_i) \pm a_n e_n)$ is divisible by $a_{n+1}$ but $\varphi(\psi((x_i) - (y_i) \pm a_n e_n)) = \varphi(\psi(a_n e_n))$ is not, a contradiction. □

One can clearly recognize a recurring pattern that will appear throughout the paper: a group is slender if it cannot certain infinitary operations. Let us now define a type of slenderness that is of principal interest to this work.

**Definition 3.3.** A group $G$ is **inverse limit-slender** (occasionally abbreviated to lim-slender) if for every inverse sequence of groups $(H_i, p_i^j)$ satisfying the Mittag-Leffler condition and for every homomorphism $\varphi : \varprojlim H_i \to G$ there exists a homomorphism $\overline{\varphi} : H_i \to G$, such that $\varphi = \overline{\varphi} \circ p_i$. In other words, every homomorphism from $\varprojlim H_i$ to $G$ factors through some term in the sequence as in the following diagram:

\[
\begin{array}{ccc}
\varprojlim H_i & \xrightarrow{\varphi} & G \\
\| & \searrow & \nearrow \\
H_i & \xrightarrow{p_i} & G
\end{array}
\]

The following example shows that without the assumption that the inverse sequences is Mittag-Leffler even the integers as the archetypal slender group would not satisfy the defining property for a lim-slender group.
Example 3.4. For each prime $p$, let $\mathbb{Q}_{(p)} = \{a/b \in \mathbb{Q} \mid p \nmid b\}$ and consider the following inverse sequence of groups: $H_i = \bigcap_{p \leq i} \mathbb{Q}_{(p)}$ and $p_i^j : G_j \to G_i$ is the inclusion. Then $\lim H_i = \mathbb{Z}$, but the identity map $\lim H_i \to \mathbb{Z}$ clearly cannot factor through any $H_i$. In general, if the sequence is not Mittag-Leffler, then its terms can have relations that do not hold in the inverse limit. In our example every element of every factor can be divided by arbitrarily large integers $n$, which is not the case in $\mathbb{Z}$.

To proceed, we need the following technical lemma. Recall from definition 2.4 that the group $\hat{F}$ contains the free group $F_\infty$ generated by elements $\{x_n \mid n \in \mathbb{N}\}$.

Lemma 3.5. Let $(H_i, p_i^j)$ be an inverse sequence with surjective bonding maps. If a homomorphism $\varphi : \varprojlim H_i \to G$ does not factor through any projection $p_i$, then there exists a homomorphism $\psi : \hat{F} \to \varprojlim H_i$ such that $\varphi(\psi(x_i)) \neq 1$ for all $i$. In particular, $\varphi \circ \psi : \hat{F} \to G$ does not factor through any projection $\hat{F} \to F_i$.

Proof. Observe that the kernels $\ker(p_i : \varprojlim H_i \to H_i)$ form a decreasing sequence of nested normal subgroups of $\varprojlim H_i$ and that $\varphi$ factors through $p_i$ if, and only if, $\ker(p_i) \subseteq \ker(\varphi)$. Thus, by our assumptions, $\ker(p_i - \ker \varphi = \emptyset$ for all $i$.

Pick $u_1 \in \ker(p_1) - \ker(\varphi)$. Since $u_1 \neq 1$ (because $\varphi(u_1) \neq 1$), there exists a minimal index $k_1$, such that $p_{k_1}(u_1) \neq 1$. Pick $u_2 \in \ker(p_{k_1}) - \ker(\varphi)$. As before, there exists a minimal $k_2$, such that $p_{k_2}(u_2) \neq 1$. By continuing the process we obtain a sequence of elements $u_i \in \ker(p_{k_{i-1}} - \ker \varphi$, $i = 1, 2, \ldots$, satisfying $p_{k_i}(u_i) \neq 1$.

For each $n$, define $\psi_n : F_n \to H_{k_n}$ by $\psi_n(x_i) = p_{k_n}(u_i)$ for $i \in \{1, \cdots, n\}$. It is then easy to check that the diagrams

$$
\begin{array}{ccc}
F_n & \xrightarrow{\psi_n} & H_{k_n} \\
\downarrow & & \downarrow p_{k_n}^n \\
F_{n-1} & \xrightarrow{\psi_{n-1}} & H_{k_{n-1}}
\end{array}
$$

commute for every $n$, so we get an induced map $\hat{\psi} : \hat{F} \to \varprojlim H_{k_i} = \varprojlim H_i$, with the property that $\varphi \circ \hat{\psi} = 1$.

We have the following characterization of inverse limit slender groups

Theorem 3.6. The following statements for a group $G$ are equivalent.

1. $G$ is inverse limit slender.
2. Every homomorphism $\varphi : \hat{F} \to G$ factors through some projection $\hat{F} \to F_i$.
3. For every homomorphism $\varphi : \hat{F} \to G$, there exists $i \in \mathbb{N}$, such that $\varphi(x_j) = 1$ for $j > i$.
4. For every (non-necessarily Mittag-Leffler) inverse sequence $(H_i, p_i^j)$ and every homomorphism $\varphi : \varprojlim H_i \to G$ there exists an index $i$ such that $\varphi$ factors through the projection $\varprojlim H_i \to p_i(\varprojlim H_i)$.
Proof. (1) implies (2) because the inverse sequence that defines \( \hat{F} \) is Mittag-Leffler. Also, (2) trivially imply (3).

That (3) implies (4) follows from lemma 3.6. In fact, \( \lim H_i \) can be viewed as the inverse limit of the sequence \( (p_i(\lim H_i), p_i') \) with surjective bonding maps. If there exists a homomorphism \( \varphi: \lim H_i \to G \) that does not factor through any projection to \( p_i(\lim H_i) \), then by lemma 3.5 there exists a homomorphism from \( \hat{F} \) that does not satisfy condition (3).

Finally, to show that (4) implies (1) consider a Mittag-Leffler sequence \( (H_i, p_i') \) and a homomorphism \( \varphi: \lim H_i \to G \). By (3) \( \varphi \) factors through the projection to some \( p_j(\lim H_i) \). By the Mittag-Leffler property there exists some \( j > i \), such that \( p_j'(H_j) = p_j(\lim H_i) \) but that means that \( \varphi \) factors through the projection \( p_j \).

We now show that inverse limit slenderness is preserved by several basic operations.

**Theorem 3.7.** The class of inverse limit slender groups is closed under taking subgroups, extensions, direct sums and free products.

**Proof.** That a subgroup of a lim-slennder group is also lim-slennder is obvious.

To see that an extension of lim-slennder groups is lim-slennder, let \( A \) be a normal subgroup of \( G \) with \( q: G \to G/A \) the projection to the quotient group, and assume that both \( A \) and \( G/A \) are lim-slennder. Clearly, for every homomorphism \( \varphi: \hat{F} \to G \) there exists \( n \) such that \( \hat{F}^{(n)} \subseteq \ker(q \circ \varphi) \), therefore \( \varphi(\hat{F}^{(n)}) \subseteq A \). Since \( A \) is also slender, there exists \( m \geq n \), such that \( \varphi(x_i) = 1 \) for \( i \geq m \).

It is obvious that a finite direct sum of lim-slennder groups is trivially lim-slennder. We will show that a direct sum of an arbitrary family \( \{ G_{\lambda} \mid \lambda \in \Lambda \} \) of lim-slennder groups is also lim-slennder.

If \( \bigoplus \lambda G_{\lambda} \) is not lim-slennder, then there exists a homomorphism \( \varphi: \hat{F} \to \bigoplus \lambda G_{\lambda} \) that does not factor through any projection. By applying Lemma 3.6 if necessary, we may assume that \( \varphi(x_i) \neq 1 \) for all \( i \in \mathbb{N} \). By definition of direct sum, there exists a finite subset \( \Lambda_1 \subseteq \Lambda \), such that \( \varphi(x_i) \) is contained in \( \bigoplus \lambda G_{\lambda} \). Since \( \bigoplus \lambda G_{\lambda} \) is lim-slennder, there exists \( k_1 \in \mathbb{N} \), such that \( \varphi(x_i) \in \ker(p_{\Lambda_1}: \bigoplus \lambda G_{\lambda} \to \bigoplus \lambda G_{\lambda}) \) for \( i \geq k_1 \). Similarly, there exists a finite subset \( \Lambda_2 \subseteq \Lambda \), such that \( \varphi(x_{k_1}) \in \bigoplus \lambda G_{\lambda} \), and an integer \( k_2 \geq k_1 \), such that \( \varphi(x_i) \in \ker(p_{\Lambda_2}) \) for \( i \geq k_2 \). Continuing this procedure, we obtain a sequence of finite subsets \( \Lambda_1 \subseteq \Lambda_2 \subseteq \cdots \Lambda \) and elements \( x_{k_0} = x_1, x_{k_1}, x_{k_2} \) of \( \hat{F} \), such that \( \varphi(x_{k_i}) \in \bigoplus \lambda G_{\lambda} \) for \( i \leq n \), and \( \varphi(x_{k_i}) \in \ker(p_{\Lambda_n}) \) for \( i > n \).

Consider \( \varphi(w) \) for \( w := x_{k_0} x_{k_1} x_{k_2} \cdots \in \hat{F} \). There exists \( n \), so that \( p_{\lambda}(\varphi(w)) = 1 \) for \( \lambda \in (\bigcup \Lambda_k) - \Lambda_n \). It is easy to check that for \( w' := (x_{k_0} \cdots x_{k_n})^{-1} w = x_{k_{n+1}} x_{k_{n+2}} \cdots \) and \( w'' := (x_{k_0} \cdots x_{k_{n+1}})^{-1} w = x_{k_{n+3}} x_{k_{n+4}} \cdots \) we have \( p_{\lambda}(\varphi(w')) = p_{\lambda}(\varphi(w'')) = 1 \) for all \( \lambda \in \bigcup \Lambda_k \). But this implies that \( \varphi(x_{n+1}) = \varphi(w'(w'')^{-1}) = 1 \), which is a contradiction.

Finally, that free products of lim-slennder groups is lim-slennder follows from [29] Theorem 1.5] because, when endowed with the product topology, \( \hat{F} \) becomes a complete topological group.
We are now going to compare inverse limit slender groups with classical slender group, n-slender groups, lch-slender groups and cm-slender groups (see Introduction for the relevant definitions).

Since the group \( \mathbb{Z}^N \) is the limit of an inverse sequence of groups \( \mathbb{Z}^n \) with surjective bonding maps, it immediately follows that every inverse limit slender group is slender. The inclusion is proper, because the group \( \mathbb{F} \) is slender (by [22, Theorem 4.3], since \( \mathbb{F} \) is locally free), but it is not inverse limit slender, because the identity homomorphism on \( \mathbb{F} \) does not factor through any of the projections).

To compare n-slender and inverse slender group we prove the following result.

**Proposition 3.8.** Every n-slender group is inverse limit slender.

**Proof.** Assume \( G \) is not inverse limit slender. By theorem 3.6 there exists a homomorphism \( \varphi: \mathbb{F} \to G \), such that \( \varphi(x_i) \neq 1 \) for infinitely many indices \( i \). But then the restriction \( \varphi|_H: H \to G \) of \( \varphi \) to the free complete product of cyclic groups \( H \) clearly cannot factor through any projection to a free group of finite rank, so \( G \) is not n-slender. \( \Box \)

On the other hand Eda [14, Theorem 1.2] proved that the group \( \mathbb{H} \) is slender but it is obviously not n-slender (because the identity homomorphism from \( \mathbb{H} \) to itself does not factor through any of the projections). Note that \( \mathbb{H} \) can be viewed as a dense subgroup of \( \mathbb{F} \) if we equip the latter with the inverse limit topology (as a subspace of the product of discrete groups \( \mathbb{F}_n \)). Thus every continuous homomorphism from \( \mathbb{F} \) to a topological group \( G \) is uniquely determined by its restriction to \( \mathbb{H} \). This fact can be extended to all subgroups of \( \mathbb{F} \) that contain the infinitely generated free group \( \mathbb{F}_\infty \).

**Theorem 3.9.** Let \( G \) be an inverse limit slender group and let \( H \) be any subgroup of \( \mathbb{F} \) containing \( \mathbb{F}_\infty \). Then the restriction function

\[
\text{res}: \text{Hom}(\mathbb{F}, G) \to \text{Hom}(H, G)
\]

is injective.

**Proof.** Note that we can turn every group into a topological group by endowing it with the discrete topology. If \( G \) is inverse limit slender, then by 4.6(2) every homomorphism \( \varphi: \mathbb{F} \to G \) factors as \( \varphi = \psi \circ p_i \) for some homomorphism \( \psi: F_i \to G \). If we endow \( F_i \) and \( G \) with the discrete topology and \( \mathbb{F} \) with the product topology, then \( \psi \) is continuous because \( F_i \) is discrete and \( p_i \) is continuous by definition of product topology. We conclude that \( \varphi \) is continuous, being a composition of continuous functions. Since \( p_i(F_\infty) = F_i \) it follows that \( F_\infty \) is dense in \( \mathbb{F} \), and the same holds for every subgroup \( H \leq \mathbb{F} \) containing \( F_\infty \). Therefore, if \( \varphi, \psi: \mathbb{F} \to G \) are homomorphisms satisfying \( \varphi|_H = \psi|_H \), then \( \varphi = \psi \), so the restriction homomorphism is injective. \( \Box \)

Note, that \( \text{res}: \text{Hom}(\mathbb{F}, G) \to \text{Hom}(\mathbb{H}, G) \) need not be injective for an arbitrary group \( G \). In fact, the normal closure \( N \) of \( \mathbb{H} \) in \( \mathbb{F} \) is a proper subgroup of \( \mathbb{F} \), so the quotient homomorphism \( q: \mathbb{F} \to \mathbb{F}/N \) is a non-trivial element of \( \text{Hom}(\mathbb{F}, \mathbb{F}/N) \) that restricts to a trivial element of \( \text{Hom}(\mathbb{H}, \mathbb{F}/N) \).
In a recent paper Corson and Varghese [11] proved that a group $G$ is lh-slender if, and only if, $G$ is torsion-free and it does not contain subgroups isomorphic to $\mathbb{Q}$ or to $p$-adic integers for any prime $p$. This coincides with the characterization of stout groups in Göbel [22, Theorem 4.1]. Following Göbel [22] a group $G$ is stout if $\text{Hom}(\prod H_i, G)$ is trivial whenever $\{H_i \mid i \in \mathbb{N}\}$ is a set of groups with trivial $\text{Hom}(H_i, G)$. He proved that every slender group is stout, and that $\mathbb{Z}^\mathbb{N}$ is stout but not slender.

We may summarize the above discussion in the following theorem.

**Theorem 3.10.** The class of n-slender groups is contained in the class of inverse limit slender groups, which is contained in the class of slender groups, which is in turn contained in the class of lh-slender groups. All containments are strict.

Furthermore, the class of cm-slender groups is contained in the class of inverse limit slender groups.

The world of slender groups is quite rich and varied, see Remark 4.4. Eda [13, Theorem 3.3] proved that an abelian group is n-slender if, and only if, it is slender. Since the class of inverse limit slender groups is intermediate, they must all coincide. On the other hand, the group $\mathbb{Z}^\mathbb{N}$ is lh-slender by [11] but is not slender.

**Corollary 3.11.** For abelian groups the class of slender, inverse limit slender and n-slender groups coincide, and are strictly contained in the class of lh-slender abelian groups.

Incidentally, the class of abelian lh-slender groups coincides with the class of cotorsion-free groups (i.e., groups that do not contain any cotorsion group in the sense of Fuchs [18, Ch.9].

All examples of groups that distinguish the above classes are uncountable, and we have not been able to find analogous examples of countable groups. Thus we state the following conjecture.

**Conjecture 1.** The classes of slender, inverse limit slender, n-slender, lh-slender and cm-slender groups coincide for all countable groups.

### 4. Slender as a co-small object

The concept of inverse limit slender groups leads to a connection between slender groups and the categorical notion of small and co-small objects that we examine in this section.

As explained in Section 2 in any category there are natural bijections

$$\text{Hom}(A, \lim X_i) \cong \lim \text{Hom}(A, X_i)$$

and

$$\text{Hom}(\lim X_i, A) = \lim \text{Hom}(X_i, A)$$

whenever suitable direct and inverse limit exist. In addition, there are natural functions

$$\Delta_A : \lim \text{Hom}(A, X^i) \rightarrow \text{Hom}(A, \lim X^i)$$

and

$$\nabla_A : \lim \text{Hom}(X_i, A) \rightarrow \text{Hom}(\lim X_i, A)$$
again provided the existence of suitable limits. In addition, if the bonding maps in the direct sequence are monomorphisms then $\Delta$ is injective, and if the bonding maps in the inverse sequence are epimorphisms then $\nabla$ is injective.

**Definition 4.1.** An object $A$ of the category $C$ is small if $\Delta_A$ is a bijection for every convergent direct sequence in $C$ with monomorphic bonding maps. Dually, an object $A$ of the category $C$ is co-small if $\nabla_A$ is a bijection for every convergent inverse sequence in $C$ with epimorphic bonding maps.

**Remark 4.2.** Although small objects and related concepts are extensively studied in category theory, the terminology and precise definitions are far from being standardized. They are known variously as finitely generated objects, finitely presentable objects, compact objects, small-projective objects, tiny objects etc., and the definitions may require preservation of arbitrary co-limits, filtered co-limits, co-limits with monomorphic bonding maps, co-products and several other variants. The dual concept is usually known as co-small or co-compact object, and there is also a variety of definitions.

It is not difficult to see that small objects in the category of groups are precisely the finitely generated groups. To understand which objects are co-small in the category of groups observe that for a given inverse sequence $(H_i, p^i_j)$ the image of $\nabla_A$ consists precisely of homomorphisms from $\lim_{\leftarrow} H_i$ to $A$ that can be factored through some projection $p_i: \lim_{\leftarrow} H_i \rightarrow H_i$. Thus we obtain the following restatement of Theorem 3.6.

**Theorem 4.3.** The following statements are equivalent:

1. The group $G$ is a co-small object in the category of groups.
2. $G$ is an inverse slender group.
3. $\nabla_G: \lim_{\leftarrow} \operatorname{Hom}(F_n, G) \rightarrow \operatorname{Hom}(\hat{F}, G)$ is a bijection.
4. For every inverse sequence of groups $(H_i, p^i_j)$ (non necessarily with surjective bonding maps) $\nabla_G$ induces a bijection between $\operatorname{Hom}(\lim_{\leftarrow} H_i, G)$ and $\lim_{\leftarrow} \operatorname{Hom}(\operatorname{Im} p_i, G)$.

**Remark 4.4.** The class of groups satisfying the statements of the above theorem is by no means small. In fact, by combining Proposition 3.8 with results of [5] and [9] one can see that the above hold for any group in the following list:

1. Free groups,
2. Free abelian groups,
3. Torsion-free word hyperbolic groups,
4. Torsion-free one-relator groups,
5. Baumslag-Solitar groups,
6. Countable diagram groups over some semigroup presentation,
7. Thompson’s group $\hat{F}$.

Other types of slender groups can be characterized in a similar way, which allows to compare them in a unified way. So, for slender groups we have the following result, where the first statement is essentially the definition and the second statement follows from [22] Theorem 4.3].
Proposition 4.5. A group $G$ is slender, if, and only if
\[ \nabla_G : \lim\to \text{Hom}(\mathbb{Z}^n, G) \to \text{Hom}(\mathbb{Z}^N, G) \]
is a bijection, or equivalently, if the natural inclusion
\[ \bigcup_{i \in \mathbb{N}} \text{Hom}(\prod_{i=1}^n H_i, G) \to \text{Hom}(\prod_{i \in \mathbb{N}} H_i, G) \]
is a bijection for every sequence of groups $\{H_i \mid i \in \mathbb{N}\}$.

The characterization of $n$-slender groups is slightly more complicated, and it can be expressed in terms of the restriction of homomorphisms along the inclusion $H \hookrightarrow \hat{F}$.

Proposition 4.6. A group $G$ is $n$-slender if, and only if, the composition
\[ \lim\to \text{Hom}(F_n, G) \xrightarrow{\nabla_G} \text{Hom}(\hat{F}, G) \xrightarrow{\text{res}} \text{Hom}(\mathbb{H}, G) \]
is surjective.

Proof. If $G$ is $n$-slender, then for every homomorphism $\varphi : \mathbb{H} \to G$ there exists a homomorphism $\overline{\varphi} : F_i \to G$, such that $\overline{\varphi} \circ p_i = \varphi$ or in other words, $\text{res}(\nabla_G(\overline{\varphi})) = \varphi$. The converse statement is obvious. \qed

We can use the above to characterize $n$-slender groups among inverse limit slender groups.

Corollary 4.7. Let $G$ be an inverse limit slender group. Then $G$ is $n$-slender if, and only if the restriction $\text{Hom}(\hat{F}, G) \to \text{Hom}(\mathbb{H}, G)$ is a bijection.

Proof. Since $G$ is inverse limit slender, $\nabla_G : \lim\to \text{Hom}(F_n, G) \to \text{Hom}(\hat{F}, G)$ is bijective by Theorem 4.5 and $\text{res} : \text{Hom}(\hat{F}, G) \to \text{Hom}(\mathbb{H}, G)$ is injective by Theorem 3.9. It is thus sufficient to observe that, by Proposition 4.6 $G$ is $n$-slender if, and only if, $\text{res}$ is surjective. \qed

5. Homology of Barratt-Milnor Examples

Let $\mathcal{H}_n$ be the one-point union of a sequence of spheres in $\mathbb{R}^{n+1}$ whose radii converge to zero (so that $\mathcal{H}_1$ is just the Hawaiian earring), and let $S^n$ denote the unit sphere in $\mathbb{R}^{n+1}$. Eda and Kawamura showed that for $n > 1$ the group $\pi_n(\mathcal{H}_n)$ is isomorphic to $\mathbb{Z}^N$ by a homomorphism taking $e_i \in \mathbb{Z}^N$ to the homotopy class of embedding of $S^n$ onto the $i$-th sphere of $\mathcal{H}_n$.

Barratt and Milnor showed that $H_l(\mathcal{H}_n)$ was uncountable for infinitely many $l$. Here we will further investigate this group by showing that they are cotorsion.

Definition 5.1. An abelian group $G$ is cotorsion if it is a direct summand of every extension by a torsion-free group, i.e. $\text{Ext}(A, G) = 0$ for all torsion-free groups $A$. A group $G$ is Higman-complete if for any sequence $(g_i)$ of elements of $G$ and for a given sequence of words $(w_i)$ in two symbols, there exists a sequence $(h_i)$ of elements of $G$ such that all the equations $h_i = w_i(f_i, h_{i+1})$ hold simultaneously.

Herfort and Hojka proved that the class of cotorsion groups is the same as the class of abelian Higman-complete groups [23, Theorem 3].
Lemma 5.2. For $l > n > 1$, the retraction $q_k : \mathcal{H}_n \to \mathcal{H}_n$ that collapses the first $k$ spheres to the base point induces the identity map on $H_l(\mathcal{H}_n)$.

Proof. Let $A := \mathcal{H}_n \setminus \{p_1, \ldots, p_k\}$ where $p_i$ is any point other than the base point of the $i$-th sphere in $\mathcal{H}_n$. Notice that $A$ deformation retracts to the image of $q_k$. Let $B := \bigcup_{i=1}^{k} S_i \setminus x_0$ where $S_i$ is the $i$-th sphere in $\mathcal{H}_n$ and $x_0$ is the base point. Then the Mayer-Vietoris sequence gives us the following exact sequence

$$\cdots H_i(A \cap B) \to H_i(A) \oplus H_i(B) \to H_i(A \cup B) \to H_{i-1}(A \cap B) \to \cdots.$$ 

Since $A \cap B$ is homotopy equivalent to the disjoint union of spheres with dimension $n - 1$, we have that $H_i(A \cap B), H_{i-1}(A \cap B)$ are trivial. Thus the middle homomorphism is an isomorphism. Since $H_i(B)$ is trivial, the middle isomorphism is induced by inclusion. Thus the retract $q_k$ induces the identity homomorphism. □

Let us denote by $h : \pi_l(\mathcal{H}_n) \to H_l(\mathcal{H}_n)$ the standard Hurewicz homomorphism.

Lemma 5.3. Let $l > n > 1$. For every sequence of elements $(b_i)$ in $\pi_l(\mathcal{H}_n)$ there exits a homomorphism $\varphi : \mathbb{Z}^\mathbb{N} \to H_l(\mathcal{H}_n)$ such that $\varphi(e_i) = h(b_i)$.

Proof. Let $(b_i)$ be a sequence of elements in $\pi_l(\mathcal{H}_n)$ and for each $i$ let $f_i : \mathbb{S}^l \to \mathcal{H}_n$ be a representative of $b_i$. Then we can define a continuous map $p : \mathcal{H}_l \to \mathcal{H}_n$ by $p|_{S_i} = q_i \circ f_i$, where $S_i$ is the $i$-th sphere of $\mathcal{H}_l$. Since $\lim(q_i)$ converge to the wedge point, $p$ is a continuous map and induces a homomorphism $p_* : \mathbb{Z}^\mathbb{N} \to \pi_l(\mathcal{H}_n)$. If $i_1 : \mathbb{S}^l \to \mathcal{H}_l$ is the embedding that corresponds to $e_i \in \mathbb{Z}^\mathbb{N}$ under the identification of $\pi_l(\mathcal{H}_l)$ with $\mathbb{Z}^\mathbb{N}$, then

$$h \circ p_* ([i_1]) = h([p \circ i_1]) = h([q_i \circ f_i])$$

$$= h \circ q_* ([f_i]) = q_* \circ h([f_i]) = h([f_i])$$

where the equality follow from Lemma 5.2 and from the naturality of $q_i$ and $h$. Thus $h \circ p_*$ sends $e_i$ to $h(b_i)$ as claimed. □

Theorem 5.4. For $l > n > 1$ and let $G$ be a lim-slander group. Then the homomorphism

$$\text{Hom}(H_l(\mathcal{H}_n), G) \to \text{Hom}(\pi_l(\mathcal{H}_n), G),$$

induced by the Hurewicz homomorphism, is trivial.

In particular, $\text{Hom}(H_{n+1}(\mathcal{H}_n), G)$ is trivial for all $n$.

Proof. Suppose that $\psi : \pi_l(\mathcal{H}_n) \to G$ is a nontrivial homomorphism to a lim-slander group $G$ that factors through the Hurewicz homomorphism $h$. Then there exists a homomorphism $\overline{\psi} : H_l(\mathcal{H}_n) \to G$ such that $\overline{\psi}(h(b)) \neq 1$ for some $b \in \pi_l(\mathcal{H}_n)$. Then $\overline{\psi} \circ \phi(e_i) \neq 1$ for all $i$ where $\phi$ is the homomorphism from Lemma 5.3 for the constant sequence $(b)$. This contradicts the assumption that $G$ is lim-slander. Thus the Hurewicz homomorphism induces the trivial homomorphism from $\text{Hom}(H_l(\mathcal{H}_n), G)$ to $\text{Hom}(\pi_l(\mathcal{H}_n), G)$ for any lim-slander group $G$.

The last statement follows from the well-known fact that the Hurewicz homomorphism $h : \pi_{n+1}(\mathcal{H}_n) \to H_{n+1}(\mathcal{H}_n)$ is surjective. □
Theorem 5.5. For $l > n > 1$, the image of the Hurewicz map from $\pi_l(\mathcal{H}_n)$ to $H_l(\mathcal{H}_n)$ is cotorsion. In particular, $H_{n+1}(\mathcal{H}_n)$ is cotorsion for $n > 1$.

Proof. By [25, Section 2] it is sufficient to show that for every sequence of elements $(b_i)$ in $h(\pi_l(\mathcal{H}_n))$ and every sequence of natural numbers $(n_i)$, the infinite system of equations

$$x_i = b_i + n_i x_{i+1} \quad i = 1, 2, 3, \ldots$$

has a solution in $h(\pi_l(\mathcal{H}_n))$ (note that the above equations are the abelianization of the Higman equations). By Lemma 5.3 we can find a homomorphism $\varphi: \mathbb{Z}^\mathbb{N} \to H_l(\mathcal{H}_n)$ such that $\varphi(e_i) = b_i$. It is then easy to check that for $a_i := (0, \ldots, 0, 1, n_i, n_i n_{i+1}, n_i n_{i+2}, \ldots)$, the sequence $(\varphi(a_i))$ is a solution to the above system of equations in $h(\pi_l(\mathcal{H}_n))$. \hfill \Box

As we already mentioned in the Introduction, the groups $H_l(\mathcal{H}_n)$ are often very big. The precise statement, proved by Barratt and Milnor [2] is that the image of the rational Hurewicz homomorphism $\pi_l(\mathcal{H}_n) \to H_l(\mathcal{H}_n; \mathbb{Q})$ is uncountable whenever $l \equiv 1 \mod (n - 1)$. Thus the above theorem (in particular the case $l = n + 1$) lead us to believe that more is true:

Conjecture 2. For $l > n > 1$ all groups $H_l(\mathcal{H}_n)$ are cotorsion.

6. Applications to shape groups and Čech cohomology

Classical homotopy and (co)homology groups are best suited for the study of simplicial or CW-complexes. For more general spaces like metric compacta or Peano continua with bad local properties these invariants fail to give useful information about their homotopy type. Classical examples include Warsaw circle, Sierpinski gasket, Hawaiian Earring, $p$-adic solenoids, attractors of dynamical systems, boundaries of groups and many others. The main problem is that there may not be sufficiently many maps from polyhedra (e.g. spheres) to such spaces to be able to distinguish between them. A standard way around this problem is to approximate a given space by a sequence of nicer spaces like polyhedra or absolute neighbourhood retracts. This is the so-called shape theory approach, for which we refer to the classical monograph [25] by Mardesic and Segal.

Our main interest will be in compact, connected metric spaces that are also locally connected. Such spaces are classically known as Peano continua and form a quite general class of spaces, which are still sufficiently nice to allow a rich structure theory. For every Peano continuum $X$ there exists an inverse sequence of finite polyhedra $(X_i, p^i_k)$, such that $X = \lim_i (X_i, p^i_k)$. These approximations are unique up to a homotopy equivalence of inverse sequences (see [25, Appendix 1]) and can be used to define homotopy invariants of $X$.

Let $x_0$ be a base-point in $X = \lim_i X_i$ and let $k \geq 1$. Then for every $i$ take $p_i(x_0)$ as a base-point in $X_i$ and define the $k$-th shape homotopy group of $X$ as $ar{\pi}_k(X, x_0) := \lim_i \pi_k(X_i, p_i(x_0))$. This definition make sense for $k = 0$ as well, but $\bar{\pi}_0(X, x_0)$ is in general only a pointed set. The shape fundamental group $\bar{\pi}_1(X, x_0)$ can be a non-commutative group, while all higher shape homotopy groups are
abelian. Note that there is a natural homomorphism 
\[ \pi_1(p, X, x_0) \to \pi_1(p, X, x_0) \]
which is often injective (e.g., if \( X \) is 1-dimensional), and that for \( \pi_1(p, X, x_0) \) there holds a famous Shelah’s dichotomy \[28\]: they are either finitely presented or uncountable.

In spite of that we have the following result.

**Theorem 6.1.** Let \( X \) be a Peano continuum with a representation \( X = \lim X_i \) as a limit of an inverse sequence of finite polyhedra, and let \( G \) be an inverse limit slender group. Then \( \text{Hom}(\tilde{\pi}_1(X), G) = \lim \text{Hom}(\pi_1(X_i), G) \). In particular, if \( G \) is countable, then \( \text{Hom}(\tilde{\pi}_1(X), G) \) is also countable.

**Proof.** Since \( X \) is locally path-connected, the inverse sequence of fundamental groups \( (\pi_1(X_i), (p_i^j)_2) \) satisfies the Mittag-Leffler condition \([25\text{ Sec. II.7.2}], \) see also \[27\ Cor. 3.2\]. Then Theorem 4.3(4) implies the formula \( \text{Hom}(\tilde{\pi}_1(X), G) = \lim \text{Hom}(\pi_1(X_i), G) \). Finally, if \( G \) is countable, then \( \text{Hom}(\pi_1(X_i), G) \) are countable for every \( i \) (because fundamental groups of finite polyhedra are finitely presented), and so their inverse limit \( \text{Hom}(\tilde{\pi}_1(X), G) \) must be countable as well. \( \square \)

The last theorem can be extended to higher shape groups for compact and connected metric spaces that satisfy a local connectivity condition called \( LC^n \) for suitable \( n \) (see \[25\text{ Sec. II.7.2}]\). We leave the details to the reader.

Our final application is a universal coefficient theorem for Čech cohomology groups with coefficients in a slender abelian group. Every compact metric space \( X \) can be represented as a limit of an inverse sequence of finite polyhedra \( X = \lim X_i \). Such representation are unique up to homotopy equivalence of inverse sequences, so one can define Čech homology groups with coefficients in an abelian group \( G \) as
\[
\tilde{H}_n(X, G) := \lim H_n(X_i; G).
\]

Since the cohomology is a contravariant functor, it turns inverse sequences into direct sequences, so one can also define Čech cohomology groups with coefficients in \( G \) as
\[
\tilde{H}^n(X, G) := \lim H^n(X_i; G).
\]
Čech cohomology groups are a tool of choice for the study of spaces with bad local properties. On the other hand, Čech homology is much less used, because in general it does not form long exact sequences of a pair, and is thus not a homology theory in the usual sense (note however, that it satisfies the exactness axiom if we take coefficients in a finite group or in a field \[19\ Ch. IX]\). Nevertheless, we may use our results on inverse limit slender groups to derive a Universal coefficient theorem for Čech cohomology.

We first need some algebraic lemmas.

**Lemma 6.2.** Let \( H \) be a finitely generated free abelian group and \( B \) a subgroup. Then there exists a unique maximal subgroup \( C \) of \( H \) containing \( B \) such that \( C/B \) is finite and \( C \) is a free abelian factor of \( H \).

**Sketch of proof.** Let \( C = \{ a \in H \mid a^k \in B \text{ for some } k \in \mathbb{Z}\setminus\{0\} \} \). It is trivial to show that \( C \) is actually a subgroup. Since \( C/B \) is an abelian, finitely generated, torsion group it is finite. It is an exercise to see that if \( a^l \in C \) for some \( a \in H \) and some
nonzero integer \( l \), then \( a \in C \). Thus \( H/C \) is a free abelian group and \( C \) is a free abelian factor of \( H \).

Suppose that \( D/B \) is finite. Then for every \( d \in D \) there exits a \( k \in \mathbb{N} \) such that \( d^k \in B \) which implies that \( D \subset C \).

\[ \text{Lemma 6.3.} \quad \text{Every homomorphism from an inverse limit of finitely generated abelian groups to a lim-slender group factors through an inverse limit of finitely generated free abelian groups.} \]

**Proof.** Let \( \hat{H} = \lim\limits_{\leftarrow} (H_i, p_i) \) where \( (H_i, p_i) \) is an inverse sequence of finitely generated abelian groups. Let \( T_i \) be the torsion subgroup of \( H_i \) and \( q_i : H_i \to A_i \) be the quotient homomorphism with kernel \( T_i \). Then \( A_i \) is free abelian. Let \( (A_i, \tilde{p}_i) \) be the inverse sequence of free abelian groups defined by \( \tilde{p}_i \circ q_i = q_j \circ p_j \). Then the maps \( \{q_i\} \) induce a homomorphism \( q : \hat{H} \to \hat{A} \) where \( \hat{A} = \lim\limits_{\leftarrow} (A_i, \tilde{p}_i) \). The short exact sequence \( 0 \to T_i \to H_i \overset{\varphi_i}{\to} A_i \to 0 \) gives rise to the exact sequence \( 0 \to \hat{T} \to \hat{H} \overset{\varphi}{\to} \hat{A} \to \lim^{1}(T_i, p_i) \otimes_{T_i} \). Then by [11 Proposition 11.3.13] the last term is 0, so \( q \) is surjective with kernel \( \hat{T} \).

Let \( \varphi : \hat{H} \to G \) be a homomorphism to a lim-slender group. It is immediate that \( \hat{H} = \lim(p_i(H_i), p_i|_{p_i(H_i)}) \). Since \( G \) is lim-slender, there exists an \( m \) such that \( \ker(p_m) \subset \ker(\varphi) \). Since slender groups are torsion-free, \( \ker(q_m \circ p_m) \subset \ker(\varphi) \). Thus we have that \( \ker(q) = \hat{T} \subset \ker(q_m \circ p_m) \), therefore \( \varphi \) factors through \( q \).

**Lemma 6.4.** Let \( (H_i, p_i) \) be an inverse sequence of finitely generated abelian groups. Then every homomorphism from \( \lim\limits_{\leftarrow} (H_i, p_i) \) to a lim-slender group \( G \) factors through a projection, and the natural homomorphism

\[
\nabla_G : \lim\limits_{\leftarrow} \Hom(H_i, G) \to \Hom(\lim\limits_{\leftarrow} H_i, G)
\]

is bijective.

**Proof.** By Lemma 6.3, we may assume that \( (H_i, p_i) \) is an inverse sequence of finitely generated free abelian groups. Let \( \hat{H} = \lim(H_i, p_i) \) and suppose that \( \varphi : \hat{H} \to G \) is a homomorphism to an lim-slender group \( G \). Then there exist an \( m_0 \) and \( \varphi' : p_{m_0}(\hat{H}) \to G \) such that \( \varphi = \varphi' \circ p_{m_0} \).

By Lemma 6.2 there exists \( H'_i \) a free abelian factor of \( H_i \) such that \( p_i(\hat{H}) \) is a finite index subgroup of \( H'_i \). Let \( q_i : H_i \to H'_i \) be the quotient homomorphism projecting \( H_i \) onto the factor \( H'_i \). Notice that \( \hat{H} = \lim(H'_i, p'_i) \), since \( p_i(\hat{H}) \subset H'_i \).

For any \( i \geq j \geq k \), we have that

\[
[H'_k, p_{j,k}(H'_i)] \leq [H'_k, p_{i,k}(H'_i)] \leq [H'_k, p_k(\hat{H})] < \infty.
\]

Thus for each \( k \), we may choose \( n_k \geq k \) such that \( [H'_k, p_{n_k,k}(H'_n)] = [H'_k, p_{i,k}(H'_i)] \) for all \( i \geq n_k \). Then, for \( i \geq n_k \), the natural quotient map from \( H'_k/p_{n_k,k}(H'_n) \) to \( H'_k/p_{n_k,k}(H'_n) \) is injective since the groups have the same finite cardinality. Thus \( p_{n_k,k}(H'_i) = p_{n_k,k}(H'_n) \) for all \( i \geq n_k \). It is then an exercise to show that \( p_{n_k,k}(H'_n) = p_k(\hat{H}) \) for any \( k \). In particular, \( p_{n_{m_0},m_0}(H'_{n_{m_0}}) = p_{m_0}(\hat{H}) \), which
allows us to define $\tilde{\varphi} : H_{n_{m_0}} \to G$ by $\tilde{\varphi} = \varphi' \circ p_{n_{m_0}m_0} \circ q_{n_{m_0}}$. Since $q_{n_{m_0}}|_{p_{n_{m_0}}(\tilde{B})}$ is the identity it is immediate that $\varphi = \tilde{\varphi} \circ p_{n_{m_0}}$. □

The main point of the last Lemma is that in some cases the assumption that an inverse sequence is Mittag-Leffler is not needed. This is of crucial importance in the next theorem, which is the main result of this section.

**Theorem 6.5 (Universal Coefficients Theorem for Čech Cohomology).** Let $X$ be a compact metric space, represented as a limit of an inverse sequence of finite polyhedra, $X = \lim_{\leftarrow} X_i$. Then for every slender abelian group we have short exact sequences

$$0 \longrightarrow \lim_{i \to \infty} \text{Ext}(H_{n-1}(X_i), G) \longrightarrow \tilde{H}^n(X; G) \longrightarrow \text{Hom}(\tilde{H}_n(X), G) \longrightarrow 0$$

**Proof.** Notice that $(H_n(X_i), p_{i,j*})$ is an inverse sequence of finitely generated abelian groups. Thus by Lemma 6.3 every homomorphism from their inverse limit to $G$ factors through a projection.

The theorem now follows by taking the direct limit of the following short exact sequence

$$0 \longrightarrow \text{Ext}(H_{n-1}(X_i), G) \longrightarrow H^n(X_i; G) \overset{h}{\longrightarrow} \text{Hom}(H_n(X_i), G) \longrightarrow 0$$

□

We conclude with an example that shows that the above result is not valid for groups that are not slender. Let $B_i$ be a bouquet of $i$ circles. Then there is a natural map for $B_i$ to $B_{i-1}$ that collapses one of the circles and is a homeomorphism on the rest of the circles. The inverse limit of this system is the Hawaiian earring, which we will denote by $\mathcal{H}_1$. Notice that $\text{Ext}(H_{n-1}(B_i), G) = 0$ since $H_{n-1}(B_i)$ is free abelian or trivial.

**Proposition 6.6.** If $G$ is a non-slender abelian group $G$, then the homomorphism

$\nabla_G \circ \widetilde{T} : \tilde{H}^1(\mathcal{H}_1; G) \longrightarrow \text{Hom}(\tilde{H}_1(\mathcal{H}_1), G)$

is not surjective, so it cannot be extended to a short exact sequence as in Theorem 6.5

**Proof.** Since $H_0(B_i) = \mathbb{Z}$ it follows that $\text{Ext}(H_0(B_i), G) = 0$. As before we can take the direct limit of the short exact sequences

$$0 \longrightarrow 0 \longrightarrow H^1(B_i; G) \overset{h}{\longrightarrow} \text{Hom}(H_1(B_i), G) \longrightarrow 0$$

to obtain the short exact sequence,

$$0 \longrightarrow 0 \longrightarrow \tilde{H}^1(\mathcal{H}_1; G) \overset{h}{\longrightarrow} \lim_{i \to \infty} \text{Hom}(H_1(B_i), G) \longrightarrow 0.$$

Notice that $\lim_{i \to \infty} \text{Hom}(H_1(B_i), G) = \lim_{i \to \infty} \text{Hom} (\mathbb{Z}^i, G)$ and $\text{Hom}(\lim_{i \to \infty} H_1(B_i), G) = \text{Hom}(\mathbb{Z}^\mathbb{N}, G)$. Thus the natural map $\nabla_G : \lim_{i \to \infty} \text{Hom}(H_1(B_i), G) \to \text{Hom}(\tilde{H}_1(\mathcal{H}_1), G)$ is surjective if and only if the abelian group $G$ is lim-slender. Therefore

$$\tilde{H}_1(\mathcal{H}_1; G) \overset{\nabla_G \circ \widetilde{T}}{\longrightarrow} \text{Hom}(\tilde{H}_1(\mathcal{H}_1), G)$$

is not surjective. □
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