Scalable Algorithms for Tractable Schatten Quasi-Norm Minimization

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Abstract
The Schatten-$p$ quasi-norm ($0 < p < 1$) is usually used to replace the standard nuclear norm in order to approximate the rank function more accurately. However, existing Schatten-$p$ quasi-norm minimization algorithms involve singular value decomposition (SVD) or eigenvalue decomposition (EVD) in each iteration, and thus may become very slow and impractical for large-scale problems. In this paper, we first define two tractable Schatten quasi-norms, i.e., the Frobenius/nuclear hybrid and bi-nuclear quasi-norms, and then prove that they are in essence the Schatten-$2/3$ and $1/2$ quasi-norms, respectively, which lead to the design of very efficient algorithms that only need to update two much smaller factor matrices. We also design two efficient proximal alternating linearized minimization algorithms for solving representable matrix completion problems. Finally, we provide the global convergence and performance guarantees for our algorithms, which have better convergence properties than existing algorithms. Experimental results on synthetic and real-world data show that our algorithms are more accurate than the state-of-the-art methods, and are orders of magnitude faster.

Introduction
In recent years, the matrix rank minimization problem arises in a wide range of applications such as matrix completion, robust principal component analysis, low-rank representation, multivariate regression and multi-task learning. To solve such problems, Fazel, Hindi, and Boyd; Candes and Tao; Recht, Fazel, and Parrilo (2001; 2010) have suggested to relax the rank function by its convex envelope, i.e., the nuclear norm. In fact, the nuclear norm is equivalent to the $\ell_1$-norm on singular values of a matrix, and thus it promotes a low-rank solution. However, it has been shown in (Fan and Li 2001) that the $\ell_1$-norm regularization over-penalizes large entries of vectors, and results in a biased solution. By realizing the intimate relationship between them, the nuclear norm penalty also over-penalizes large singular values, that is, it may make the solution deviate from the original solution as the $\ell_1$-norm does (Nie, Huang, and Ding 2012; Lu et al. 2015). Compared with the nuclear norm, the Schatten-$p$ quasi-norm for $0 < p < 1$ makes a closer approximation to the rank function. Consequently, the Schatten-$p$ quasi-norm minimization has attracted a great deal of attention in images recovery (Lu and Zhang 2014; Lu et al. 2014), collaborative filtering (Nie et al. 2012; Lu et al. 2015), Robust analysis (Mohan and Fazel 2012) and MRI analysis (Majumdar and Ward 2011). In addition, many non-convex surrogate functions of the $\ell_0$-norm listed in (Lu et al. 2014; Lu et al. 2015) have been extended to approximate the rank function, such as SCAD (Fan and Li 2001) and MCP (Zhang 2010).

All non-convex surrogate functions mentioned above for low-rank minimization lead to some non-convex, non-smooth, even non-Lipschitz optimization problems. Therefore, it is crucial to develop fast and scalable algorithms which are specialized to solve some alternative formulations. So far, Lai, Xu, and Yin (2013) proposed an iterative reweighted least squares (IRuLq) algorithm to approximate the Schatten-$p$ quasi-norm minimization problem, and proved that the limit point of any convergent subsequence generated by their algorithm is a critical point. Moreover, Lu et al. (2014) proposed an iteratively reweighted nuclear norm (IRNN) algorithm to solve many non-convex surrogate minimization problems. For matrix completion problems, the Schatten-$p$ quasi-norm has been shown to be empirically superior to the nuclear norm (Marjanovic and Solo 2012). In addition, Zhang, Huang, and Zhang (2013) theoretically proved that the Schatten-$p$ quasi-norm minimization with small $p$ requires significantly fewer measurements than the convex nuclear norm minimization. However, all existing algorithms have to be solved iteratively and involve SVD or EVD in each iteration, which incurs high computational cost and is too expensive for solving large-scale problems (Cai and Osher 2013; Liu et al. 2014). In contrast, as an alternative non-convex formulation of the nuclear norm, the bilinear spectral regularization as in (Srebro, Rennie, and Jaakkola 2004; Recht, Fazel, and Parrilo 2010) has been successfully applied in many large-scale applications, e.g., collaborative filtering (Mitra, Sheorey, and Chellappan 2010). As the Schatten-$p$ quasi-norm is equivalent to the $\ell_p$ quasi-norm on singular values of a matrix, it is natural to ask the following question: can we design equivalent matrix factorization forms for the cases of the Schatten quasi-norm, e.g., $p = 2/3$ or $1/2$?
In order to answer the above question, in this paper we first define two tractable Schatten quasi-norms, i.e., the Frobenius/nuclear hybrid and bi-nuclear quasi-norms. We then prove that they are in essence the Schatten-2/3 and 1/2 quasi-norms, respectively, for solving whose minimization we only need to perform SVDs on two much smaller factor matrices as contrary to the larger ones used in existing algorithms, e.g., IRNN. Therefore, our method is particularly useful for many “big data” applications that need to deal with large, high dimensional data with missing values. To the best of our knowledge, this is the first paper to scale Schatten quasi-norm solvers to the Netflix dataset. Moreover, we provide the global convergence and recovery performance guarantees for our algorithms. In other words, this is the best guaranteed convergence for algorithms that solve such challenging problems.

**Notations and Background**

The Schatten-$p$ norm (0 < $p$ < $\infty$) of a matrix $X \in \mathbb{R}^{m \times n}$ ($m \geq n$) is defined as

$$
\|X\|_{S_p} \triangleq \left( \sum_{i=1}^{n} \sigma_i^p(X) \right)^{1/p},
$$

where $\sigma_i(X)$ denotes the $i$-th singular value of $X$. When $p = 1$, the Schatten-1 norm is the well-known nuclear norm, $\|X\|_\star$. In addition, as the non-convex surrogate for the rank function, the Schatten-$p$ quasi-norm with $0 < p < 1$ is a better approximation than the nuclear norm (Zhang, Huang, and Zhang 2013) (analogous to the superiority of the $\ell_p$-norm to the $\ell_1$-norm (Daubechies et al. 2010)).

To recover a low-rank matrix from some linear observations $b \in \mathbb{R}^s$, we consider the following general Schatten-$p$ quasi-norm minimization problem,

$$
\min_X \lambda \|X\|_{S_p}^p + f(A(X) - b),
\tag{1}
$$

where $A : \mathbb{R}^{m \times n} \to \mathbb{R}^s$ denotes the linear measurement operator, $\lambda > 0$ is a regularization parameter, and the loss function $f(\cdot) : \mathbb{R}^s \to \mathbb{R}$ generally denotes certain measurement for characterizing $A(X) - b$. The above formulation can address a wide range of problems, such as matrix completion (Marjanovic and Solo 2012; Rohde and Tsybakov 2011) ($A$ is the sampling operator and $f(\cdot) = \|\cdot\|^2_2$, robust principal component analysis (Candes et al. 2011; Wang, Liu, and Zhang 2013; Shang et al. 2014) ($A$ is the identity operator and $f(\cdot) = \|\cdot\|_1$), and multivariate regression (Hsieh and Olsen 2014) ($A(X) = AX$ with $A$ being a given matrix, and $f(\cdot) = \|\cdot\|^2_2$). Furthermore, $f(\cdot)$ may be also chosen as the hinge loss in (Srebro, Rennie, and Jaakkola 2004) or the $\ell_p$ quasi-norm in (Nie et al. 2012).

Analogous to the $\ell_p$ quasi-norm, the Schatten-$p$ quasi-norm is also non-convex for $p < 1$, and its minimization is generally NP-hard (Lai, Xu, and Yin 2013). Therefore, it is crucial to develop efficient algorithms to solve some alternative formulations of Schatten-$p$ quasi-norm minimization (1). So far, only few algorithms (Lai, Xu, and Yin 2013; Mohan and Fazel 2012; Nie et al. 2012; Lu et al. 2014) have been developed to solve such problems. Furthermore, since all existing Schatten-$p$ quasi-norm minimization algorithms involve SVD or EVD in each iteration, they suffer from a high computational cost of $O(n^2m)$, which severely limits their applicability to large-scale problems. Although there have been many efforts towards fast SVD or EVD computation such as partial SVD (Larsen 2005), the performance of those methods is still unsatisfactory for real-life applications (Cai and Osher 2013).

**Tractable Schatten Quasi-Norms**

As in (Srebro, Rennie, and Jaakkola 2004), the nuclear norm has the following alternative non-convex formulations.

**Lemma 1.** Given a matrix $X \in \mathbb{R}^{m \times n}$ with rank($X$) = $r \leq d$, the following holds:

$$
\|X\|_\star = \min_{U \in \mathbb{R}^{m \times r}, V \in \mathbb{R}^{n \times r}} \|U\|_F \|V\|_F = \min_{U,V : X = UV^T} \frac{\|U\|_F^2 + \|V\|_F^2}{2}.
$$

**Frobenius/Nuclear Hybrid Quasi-Norm**

Motivated by the equivalence relation between the nuclear norm and the bilinear spectral regularization (please refer to (Srebro, Rennie, and Jaakkola 2004; Recht, Fazel, and Parrilo 2010)), we define a Frobenius/nuclear hybrid (F/N) norm as follows.

**Definition 1.** For any matrix $X \in \mathbb{R}^{m \times n}$ with rank($X$) = $r \leq d$, we can factorize it into two much smaller matrices $U \in \mathbb{R}^{m \times d}$ and $V \in \mathbb{R}^{n \times d}$ such that $X = UV^T$. Then the Frobenius/nuclear hybrid norm of $X$ is defined as

$$
\|X\|_{F/N} := \min_{X = UV^T} \|U\|_\star \|V\|_F.
$$

In fact, the Frobenius/nuclear hybrid norm is not a real norm, because it is non-convex and does not satisfy the triangle inequality of a norm. Similar to the well-known Schatten-$p$ quasi-norm (0 < $p$ < 1), the Frobenius/nuclear hybrid norm is also a quasi-norm, and their relationship is stated in the following theorem.

**Theorem 1.** The Frobenius/nuclear hybrid norm $\|\cdot\|_{F/N}$ is a quasi-norm. Surprisingly, it is also the Schatten-$2/3$ quasi-norm, i.e.,

$$
\|X\|_{F/N} = \|X\|_{S_{2/3}},
$$

where $\|X\|_{S_{2/3}}$ denotes the Schatten-$2/3$ quasi-norm of $X$.

**Property 1.** For any matrix $X \in \mathbb{R}^{m \times n}$ with rank($X$) = $r \leq d$, the following holds:

$$
\|X\|_{F/N} = \min_{U \in \mathbb{R}^{m \times r}, V \in \mathbb{R}^{n \times r} : X = UV^T} \|U\|_\star \|V\|_F = \min_{X = UV^T} \left( \frac{2\|U\|_\star + \|V\|_F}{3} \right)^{3/2}.
$$

The proofs of Property 1 and Theorem 1 can be found in the Supplementary Materials.
Bi-Nuclear Quasi-Norm

Similar to the definition of the above Frobenius/nuclear hybrid norm, our bi-nuclear (BiN) norm is naturally defined as follows.

Definition 2. For any matrix $X \in \mathbb{R}^{m \times n}$ with rank($X$) $= r \leq d$, we can factorize it into two much smaller matrices $U \in \mathbb{R}^{m \times d}$ and $V \in \mathbb{R}^{n \times d}$ such that $X = UV^T$. Then the bi-nuclear norm of $X$ is defined as

$$\|X\|_{\text{BiN}} := \min_{X=UV^T} \|U\|_* \|V\|_*.$$

Similar to the Frobenius/nuclear hybrid norm, the bi-nuclear norm is also a quasi-norm, as stated in the following theorem.

Theorem 2. The bi-nuclear norm $\|\cdot\|_{\text{BiN}}$ is a quasi-norm. In addition, it is also the Schatten-$1/2$ quasi-norm, i.e.,

$$\|X\|_{\text{BiN}} = \|X\|_{S_{1/2}}.$$ 

The proof of Theorem 2 can be found in the Supplementary Materials. Due to the relationship between the bi-nuclear quasi-norm and the Schatten-$1/2$ quasi-norm, it is easy to verify that the bi-nuclear quasi-norm possesses the following properties.

Property 2. For any matrix $X \in \mathbb{R}^{m \times n}$ with rank($X$) $= r \leq d$, the following holds:

$$\|X\|_{\text{BiN}} = \min_{X=UV^T} \|U\|_* \|V\|_* = \min_{X=UV^T} \frac{\|U\|_* + \|V\|_*}{2}^2.$$

The following relationship between the nuclear norm and the Frobenius norm is well known: $\|X\|_{\text{F}} \leq \|X\|_* \leq \sqrt{r} \|X\|_{\text{F}}$. Similarly, the analogous bounds hold for the Frobenius/nuclear hybrid and bi-nuclear quasi-norms, as stated in the following property.

Property 3. For any matrix $X \in \mathbb{R}^{m \times n}$ with rank($X$) $= r$, the following inequalities hold:

$$\|X\|_* \leq \|X\|_{\text{FN}} \leq \sqrt{r} \|X\|_*,$$

$$\|X\|_* \leq \|X\|_{\text{BiN}} \leq \frac{r}{\sqrt{r}} \|X\|_*.$$

The proof of Property 3 can be found in the Supplementary Materials. It is easy to see that Property 3 in turn implies that any low Frobenius/nuclear hybrid or bi-nuclear norm approximation is also a low nuclear norm approximation.

Optimization Algorithms

Problem Formulations

To bound the Schatten-$2/3$ or $-1/2$ quasi-norm of $X$ by $\frac{1}{2}(2\|U\|_* + \|V\|_{\text{F}}^2)$ or $\frac{1}{2}(\|U\|_* + \|V\|_*^2)$, we mainly consider the following general structured matrix factorization problem as in (Haefele, Young, and Vidal 2014).

$$\min_{U, V} \lambda \varphi(U, V) + f(A(UV^T) - b),$$

where the regularization term $\varphi(U, V)$ denotes $\frac{1}{2}(2\|U\|_* + \|V\|_{\text{F}}^2)$ or $\frac{1}{2}(\|U\|_* + \|V\|_*^2)$.

As mentioned above, there are many Schatten-$p$ quasi-norm minimization problems for various real-world applications. Therefore, we propose two efficient algorithms to solve the following low-rank matrix completion problems:

$$\min_{U, V} \frac{1}{2} \left( \lambda(2\|U\|_* + \|V\|_{\text{F}}^2) + \frac{1}{2} \|P_\Omega(UV^T) - P_\Omega(D)\|_{\text{F}}^2 \right),$$

$$\min_{U, V} \frac{1}{2} \left( \lambda(\|U\|_* + \|V\|_*^2) + \frac{1}{2} \|P_\Omega(UV^T) - P_\Omega(D)\|_{\text{F}}^2 \right),$$

where $P_\Omega$ denotes the linear projection operator, i.e., $P_\Omega(D)_{ij} = D_{ij}$ if $(i, j) \in \Omega$, and $P_\Omega(D)_{ij} = 0$ otherwise. Due to the operator $P_\Omega$ in (3) and (4), we usually need to introduce some auxiliary variables for solving them. To avoid introducing auxiliary variables, motivated by the proximal alternating linearized minimization (PALM) method proposed in (Bolte, Sabach, and Teboulle 2014), we propose two fast PALM algorithms to efficiently solve (3) and (4). The space limitation refrains us from fully describing each algorithm, but we try to give enough details of a representative algorithm for solving (3) and discussing their differences.

Updating $U_{k+1}$ and $V_{k+1}$ with Linearization Techniques

Let $g_k(U) := \|P_\Omega(UV^T) - P_\Omega(D)\|_{\text{F}}^2/2$, and then its gradient is Lipschitz continuous with constant $l^u_{k+1}$, meaning that $\|\nabla g_k(U_1) - \nabla g_k(U_2)\|_{\text{F}} \leq l^u_{k+1} \|U_1 - U_2\|_{\text{F}}$ for any $U_1, U_2 \in \mathbb{R}^{m \times d}$. By linearizing $g_k(U)$ at $U_k$ and adding a proximal term, then we have the following approximation:

$$\hat{g}_k(U, U_k) = g_k(U_k) + \langle \nabla g_k(U_k), U - U_k \rangle + \frac{l^u_{k+1}}{2} \|U - U_k\|_{\text{F}}^2.$$

Thus, we have

$$U_{k+1} = \arg\min_U \frac{2\lambda}{3} \|U\|_* + \hat{g}_k(U, U_k)$$

$$= \arg\min_U \frac{2\lambda}{3} \|U\|_* + \frac{l^u_{k+1}}{2} \|U - U_k + \nabla g_k(U_k)\|_{\text{F}}^2.$$ 

Similarly, we have

$$V_{k+1} = \arg\min_V \frac{\lambda}{3} \|V\|_{\text{F}}^2 + \frac{l^v_{k+1}}{2} \|V - V_k + \nabla h_k(V_k)\|_{\text{F}}^2,$$

where $h_k(V) := \|P_\Omega((U_{k+1}V^T) - P_\Omega(D))\|_{\text{F}}^2/2$ with the Lipschitz constant $l^v_{k+1}$. The problems (6) and (7) are known to have closed-form solutions, which of the former is given by the so-called matrix shrinkage operator (Cai, Candés, and Shen 2010). In contrast, for solving (4), $U_{k+1}$ is computed in the same way as (6), and $V_{k+1}$ is given by

$$V_{k+1} = \arg\min_V \frac{\lambda}{2} \|V\|_* + \frac{l^v_{k+1}}{2} \|V - V_k + \nabla h_k(V_k)\|_{\text{F}}^2.$$ 

Updating Lipschitz Constants

Next we compute the Lipschitz constants $l^u_{k+1}$ and $l^v_{k+1}$ at the $(k+1)$-iteration.
Algorithm 1 Solving (3) via PALM

Input: \( P_\Omega(D) \), the given rank \( d \) and \( \lambda \).
Initialize: \( U_0, V_0, \varepsilon \) and \( k = 0 \).
1: while not converged do
2: \( \text{Update } t_{k+1}^U \text{ and } U_{k+1} \text{ by (9) and (10), respectively.} \)
3: \( \text{Update } t_{k+1}^V \text{ and } V_{k+1} \text{ by (9) and (7), respectively.} \)
4: Check the convergence condition, \( \max \{ \| U_{k+1} - U_k \|_F, \| V_{k+1} - V_k \|_F \} < \varepsilon. \)
5: end while
Output: \( U_{k+1}, V_{k+1} \).

\[
\| \nabla g_k(U_1) - \nabla g_k(U_2) \|_F = \| \| P_\Omega(U_1 V_1^T - U_2 V_2^T) \| V_k \|_F \\
\leq \| V_k \|_2^2 \| U_1 - U_2 \|_F, \\
\| \nabla h_k(V_1) - \nabla h_k(V_2) \|_F = \| U_1^T P_\Omega(U_{k+1}(V_1^T - V_2^T)) \|_F \\
\leq \| U_{k+1} \|_2^2 \| V_1 - V_2 \|_F.
\]

Hence, both Lipschitz constants are updated by
\[
t_{k+1}^V = \| V_k \|_2^2 \text{ and } t_{k+1}^U = \| U_{k+1} \|_2^2.
\]

PALM Algorithms

Based on the above development, our algorithm for solving (3) is given in Algorithm 1. Similarly, we also design an efficient PALM algorithm for solving (4). The running time of Algorithm 1 is dominated by performing matrix multiplications. The total time complexity of Algorithm 1, as well as the algorithm for solving (4), is \( O(\text{imd}) \), where \( d \ll m, n \).

Algorithm Analysis

We now provide the global convergence and low-rank matrix recovery guarantees for Algorithm 1, and the similar results can be obtained for the algorithm for solving (4).

Global Convergence

Before analyzing the global convergence of Algorithm 1, we first introduce the definition of the critical points of a non-convex function given in (Bolte, Sabach, and Teboulle 2014).

Definition 3. Let a non-convex function \( f : \mathbb{R}^n \to (-\infty, +\infty) \) be a proper and lower semi-continuous function, and \( \text{dom}f = \{ x \in \mathbb{R}^n : f(x) < +\infty \} \).

- For any \( x \in \text{dom}f \), the Fréchet sub-differential of \( f \) at \( x \) is defined as
  \( \hat{\partial}f(x) = \{ u \in \mathbb{R}^n : \liminf_{y \to x, y \neq x} \frac{f(y) - f(x) - \langle u, y-x \rangle}{\|y-x\|} \geq 0 \} \),
  and \( \hat{\partial}f(x) = \emptyset \) if \( x \notin \text{dom}f \).
- The limiting sub-differential of \( f \) at \( x \) is defined as
  \( \partial f(x) = \{ u \in \mathbb{R}^n : \exists x^k \to x, f(x^k) \to f(x) \}
  \text{ and } u^k \in \partial f(x^k) \to u \text{ as } k \to \infty. \)
- The points whose sub-differential contains 0 are called critical points. For instance, the point \( x \) is a critical point of \( f \) if \( 0 \in \hat{\partial}f(x) \).

Theorem 3 (Global Convergence). Let \( \{ (U_k, V_k) \} \) be a sequence generated by Algorithm 1, then it is a Cauchy sequence and converges to a critical point of (3).

The proof of the theorem can be found in the Supplemetary Materials. Theorem 3 shows the global convergence of Algorithm 1. We emphasize that, different from the general subsequence convergence property, the global convergence property is given by \( (U_k, V_k) \to (\hat{U}, \hat{V}) \) as the number of iteration \( k \to +\infty \), where \( (\hat{U}, \hat{V}) \) is a critical point of (3). As we have stated, existing algorithms for solving the non-convex and non-smooth problem, such as IRucLq and IRNN, have only subsequence convergence (Xu and Yin 2014). According to (Attouch and Bolte 2009), we know that the convergence rate of Algorithm 1 is at least sub-linear, as stated in the following theorem.

Theorem 4 (Convergence Rate). The sequence \( \{ (U_k, V_k) \} \) generated by Algorithm 1 converges to a critical point \( (\hat{U}, \hat{V}) \) of (3) at least in the sub-linear convergence rate, that is, there exists \( C > 0 \) and \( \theta \in (1/2, 1) \) such that
\[
\| (U_k^T V_k^T) - (\hat{U}^T, \hat{V}^T) \|_F \leq Ck^{-\theta}.
\]

Recovery Guarantee

In the following, we show that when sufficiently many entries are observed, the critical point generated by our algorithm recovers a low-rank matrix “close to” the ground-truth one. Without loss of generality, assume that \( D = Z + E \in \mathbb{R}^{m \times n} \), where \( Z \) is a true matrix, and \( E \) denotes a random gaussian noise.

Theorem 5. Let \( (\hat{U}, \hat{V}) \) be a critical point of the problem (3) with given rank \( d \) and \( m \geq n \). Then there exists an absolute constant \( C_1 \), such that with probability at least \( 1 - 2 \exp(-m) \),
\[
\sqrt{mn} \| Z - \hat{U}^T \|_F \leq \| E \|_F + C_1 \beta \left( \frac{md \log(m)}{|\Omega|} \right)^{1/4} + 2\sqrt{d} \lambda \sqrt{|\Omega|}
\]
where \( \beta = \max_{i,j} |D_{i,j}| \) and \( C_2 = \frac{\| P_\Omega(D - \hat{U}^T V) \|_F}{\| P_\Omega(D - U^T V) \|_F} \).

The proof of the theorem and the analysis of lower-boundedness of \( C_2 \) can be found in the Supplementary Materials. When the samples size \( |\Omega| \gg md \log(m) \), the second and third terms diminish, and the recovery error is essentially bounded by the “average” magnitude of entries of the noise matrix \( E \). In other words, only \( O(md \log(m)) \) observed entries are needed, which is significantly lower than \( O(nr \log^2(m)) \) in standard matrix completion theories (Candes and Recht 2009; Keshavan, Montanari, and Oh 2010; Recht 2011). We will confirm this result by our experiments in the following section.

Experimental Results

We now evaluate both the effectiveness and efficiency of our algorithms for solving matrix completion problems, such as collaborative filtering and image recovery. All experiments were conducted on an Intel Xeon E7-4830V2 2.20GHz CPU with 64G RAM.
The synthetic matrices $Z \in \mathbb{R}^{m \times r}$ with rank $r$ are generated by the following procedure: the entries of both $U \in \mathbb{R}^{m \times r}$ and $V \in \mathbb{R}^{n \times r}$ are first generated as independent and identically distributed (i.i.d.) numbers, and then $Z = UV^T$ is assembled. Since all these algorithms have very similar recovery performance on noiseless matrices, we only conducted experiments on noisy matrices with different noise levels, i.e., $P_{1l}(D) = P_{1l}(Z + nf + E)$, where $nf$ denotes the noise factor. In other worlds, the observed subset is corrupted by i.i.d. standard Gaussian random noise as in (Lu et al. 2014). In addition, only 20% or 30% entries of $D$ are sampled uniformly at random as training data, i.e., sampling ratio (SR) = 20% or 30%. The rank parameter $d$ of our algorithms is set to $\lfloor 1.25r \rfloor$ as in (Wen, Yin, and Zhang 2012).

The average RSE results of 100 independent runs on noisy random matrices are shown in Figure 1, which shows that if $p$ varies from 0.1 to 0.7, IRucLq and IRNN-Lp achieve similar recovery performance as IRNN-SCAD, IRNN-MCP and our algorithms; otherwise, IRucLq and IRNN-Lp usually perform much worse than the other four methods, especially $p = 1$. We also report the running time of all the methods with 20% SR as the size of noisy matrices increases, as shown in Figure 2. Moreover, we present the RSE results of those methods and APGL (Toh and Yun 2010) (which is one of the nuclear norm solvers) with different noise factors. Figure 2 shows that our algorithms are significantly faster than the other methods, while the running time of IRucLq and IRNN increases dramatically when the size of matrices is increased, and they could not yield experimental results within 48 hours when the size of matrices is 50,000 × 50,000. This further justifies that both our algorithms have very good scalability and can address large-scale problems. In addition, with only 20% SR, all Schatten quasi-norm methods significantly outperform APGL in terms of RSE.
Collaborative Filtering

We tested our algorithms on three real-world recommendation system data sets: the MovieLens1M, MovieLens10M and Netflix datasets (KDDCup 2007). We randomly chose 50%, 70% and 90% as the training set and the remaining as the testing set, and the experimental results are reported over 10 independent runs. In addition to the methods used above, we also compared our algorithms with one of the fastest existing methods, LMaFit [Wen, Yin, and Zhang 2012]. The testing RMSE of all these methods on the three data sets is reported in Table 1, which shows that all those methods with non-convex penalty functions perform significantly better than the convex nuclear norm solver, APGL. In addition, our algorithms consistently outperform the other methods in terms of prediction accuracy. This further confirms that our two Schatten quasi-norm regularized models can provide a good estimation of a low-rank matrix. Moreover, we report the average testing RMSE and running time of our algorithms on these three data sets in Figures 3 and 4, where the rank varies from 5 to 20 and SR is set to 70%. Note that IRucLq and IRNN-Lp could not run on the two larger data sets due to runtime exceptions. It is clear that our algorithms are much faster than APGL, IRucLq and IRNN-Lp on all these data sets. They perform much more robust with respect to ranks than LMaFit, and are comparable in speed with it. This shows that our algorithms have very good scalability and are suitable for real-world applications.

Image Recovery

We also applied our algorithms to gray-scale image recovery on the Boat image of size $512 \times 512$, where 50% of pixels in the input image were replaced by random Gaussian noise, as shown in Figure 5(b). In addition, we employed the well known peak signal-to-noise ratio (PSNR) to measure the recovery performance. The rank parameter of our algorithms and IRucLq was set to 100. Due to limited space, we only report the best results (PSNR and CPU time) of APGL, LMaFit, IRucLq and IRNN-Lp in Figure 5, which shows that our two algorithms achieve much better recovery performance than the other methods in terms of PSNR. And impressively, both our algorithms are significantly faster than the other methods except LMaFit and at least 70 times faster than IRucLq and IRNN-Lp.

Conclusions

In this paper we defined two tractable Schatten quasi-norms, i.e., the Frobenius/nuclear hybrid and bi-nuclear quasi-norms, and proved that they are in essence the Schatten-$2/3$ and $1/2$ quasi-norms, respectively. Then we designed two efficient proximal alternating linearized minimization
algorithms to solve our Schatten quasi-norm minimization for matrix completion problems, and also proved that each bounded sequence generated by our algorithms globally converges to a critical point. In other words, our algorithms not only have better convergence properties than existing algorithms, e.g., IRucLq and IRNN, but also reduce the computational complexity from $O(mn^2)$ to $O(ndd)$, with $d$ being the estimated rank ($d \ll m, n$). We also provided the recovery guarantee for our algorithms, which implies that they need only $O(nd \log (m))$ observed entries to recover a low-rank matrix with high probability. Our experiments showed that our algorithms outperform the state-of-the-art methods in terms of both efficiency and effectiveness.

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References

[Attouch and Bolte 2009] Attouch, H., and Bolte, J. 2009. On the convergence of the proximal algorithm for nonsmooth functions involving analytic features. Math. Program. 116(1-2):5–16.

[Bolte, Sabach, and Teboulle 2014] Bolte, J.; Sabach, S.; and Teboulle, M. 2014. Proximal alternating linearized minimization for nonconvex and nonsmooth problems. Math. Program. 146:459–494.

[Cai and Osher 2013] Cai, J.-F., and Osher, S. 2013. Fast singular value thresholding without singular value decomposition. Methods Anal. appl. 20(4):335–352.

[Cai and She 2010] Cai, J.-F.; Candès, E.; and Shen, Z. 2010. A singular value thresholding algorithm for matrix completion. SIAM J. Optim. 20(4):1956–1982.

[Candès and Recht 2009] Candès, E., and Recht, B. 2009. Exact matrix completion via convex optimization. Found. Comput. Math. 9(6):717–772.

[Candès and Tao 2010] Candès, E., and Tao, T. 2010. The power of convex relaxation: Near-optimal matrix completion. IEEE Trans. Inf. Theory 56(5):2053–2080.

[Candès et al. 2011] Candès, E.; Li, X.; Ma, Y.; and Wright, J. 2011. Robust principal component analysis? J. ACM 58(3):1–37.

[Daubechies et al. 2010] Daubechies, I.; DeVore, R.; Fornasier, M.; and Guntuk, C. 2010. Iteratively reweighted least squares minimization for sparse recovery. Commun. Pure Appl. Math. 63:1–38.

[Fan and Li 2001] Fan, J., and Li, R. 2001. Variable selection via nonconcave penalized likelihood and its Oracle properties. J. Am. Statist. Assoc. 96:1348–1361.

[Fazel, Hindi, and Boyd 2001] Fazel, M.; Hindi, H.; and Boyd, S. 2001. A rank minimization heuristic with application to minimum order system approximation. In Proc. IEEE Amer. Control Conf., 4734–4739.

[Haeffele, Young, and Vidal 2014] Haeffele, B. D.; Young, E. D.; and Vidal, R. 2014. Structured low-rank matrix factorization: Optimality, algorithm, and applications to image processing. In Proc. 31st Int. Conf. Mach. Learn. (ICML), 2007–2015.

[Hsieh and Olsen 2014] Hsieh, C.-J., and Olsen, P. A. 2014. Nuclear norm minimization via active subspace selection. In Proc. 31st Int. Conf. Mach. Learn. (ICML), 575–583.

[KDD Cup 2007] KDD Cup. 2007. ACM SIGKDD and Netflix. In Proc. KDD Cup and Workshop.

[Keshavan, Montanari, and Oh 2010] Keshavan, R.; Montanari, A.; and Oh, S. 2010. Matrix completion from a few entries. IEEE Trans. Inf. Theory 56(6):2980–2998.

[Lai, Xu, and Yin 2013] Lai, M.; Xu, Y.; and Yin, W. 2013. Improved iteratively reweighted least squares for unconstrained smoothed $\ell_p$ minimization. SIAM J. Numer. Anal. 51(2):927–957.

[Larsen 2005] Larsen, R. 2005. PROPACK-software for large and sparse SVD calculations.

[Liu et al. 2014] Liu, Y.; Shang, F.; Cheng, H.; and Cheng, J. 2014. A Grassmannian manifold algorithm for nuclear norm regularized least squares problems. In Proc. 30th Conf. Uncert. in Art. Intell. (UAI), 515–524.

[Lu and Zhang 2014] Lu, Z., and Zhang, Y. 2014. Iterative reweighted singular value minimization methods for $\ell_p$ regularized unconstrained matrix minimization. arXiv:1401.0869.

[Lu et al. 2014] Lu, C.; Tang, J.; Yan, S.; and Lin, Z. 2014. Generalized nonconvex nonsmooth low-rank minimization. In Proc. IEEE Conf. Comput. Vis. Pattern Recognit. (CVPR), 4130–4137.

[Lu et al. 2015] Lu, C.; Zhu, C.; Xu, C.; Yan, S.; and Lin, Z. 2015. Generalized singular value thresholding. In Proc. AAAI Conf. Artif. Intell. (AAAI), 1805–1811.

[Majumdar and Ward 2011] Majumdar, A., and Ward, R. K. 2011. An algorithm for sparse MRI reconstruction by Schatten p-norm minimization. Magn. Reson. Imaging 29:408–417.

[Marianovic and Solo 2012] Marjanovic, G., and Solo, V. 2012. On $\ell_p$ optimization and matrix completion. IEEE Trans. Signal Process. 60(11):5714–5724.

[Mitra, Sheorey, and Chellappa 2010] Mitra, K.; Sheorey, S.; and Chellappa, R. 2010. Large-scale matrix factorization with missing data under additional constraints. In Proc. Adv. Neural Inf. Process. Syst. (NIPS), 1642–1650.

[Mohan and Fazel 2012] Mohan, K., and Fazel, M. 2012. Iterative reweighted algorithms for matrix rank minimization. J. Mach. Learn. Res. 13:3441–3473.

[Nie et al. 2012] Nie, F.; Wang, H.; Cai, X.; Huang, H.; and Ding, C. 2012. Robust matrix completion via joint Schatten $p$-norm and $\ell_p$-norm minimization. In Proc. 12th IEEE Int. Conf. Data Min. (ICDM), 566–574.

[Nie, Huang, and Ding 2012] Nie, F.; Huang, H.; and Ding, C. 2012. Low-rank matrix recovery via efficient Schatten $p$-norm minimization. In Proc. AAAI Conf. Artif. Intell. (AAAI), 655–661.

[Recht, Fazel, and Parrilo 2010] Recht, B.; Fazel, M.; and Parrilo, P. A. 2010. Guaranteed minimum-rank solutions of linear matrix equations via nuclear norm minimization. SIAM Rev. 52(3):471–504.

[Recht 2011] Recht, B. 2011. A simpler approach to matrix completion. J. Mach. Learn. Res. 12:3413–3430.

[Rohde and Tsybakov 2011] Rohde, A., and Tsybakov, A. B. 2011. Estimation of high-dimensional low-rank matrices. Ann. Statist. 39(2):887–930.

[Shang et al. 2014] Shang, F.; Liu, Y.; Cheng, J.; and Cheng, H. 2014. Robust principal component analysis with missing data. In Proc. 23rd ACM Int. Conf. Inf. Knowl. Manag. (CIKM), 1149–1158.
[Srebro, Rennie, and Jaakkola 2004] Srebro, N.; Rennie, J.; and Jaakkola, T. 2004. Maximum-margin matrix factorization. In Proc. Adv. Neural Inf. Process. Syst. (NIPS), 1329–1336.

[Toh and Yun 2010] Toh, K.-C., and Yun, S. 2010. An accelerated proximal gradient algorithm for nuclear norm regularized least squares problems. Pac. J. Optim. 6:615–640.

[Wang, Liu, and Zhang 2013] Wang, S.; Liu, D.; and Zhang, Z. 2013. Nonconvex relaxation approaches to robust matrix recovery. In Proc. 23rd Int. Joint Conf. Artif. Intell. (IJCAI), 1764–1770.

[Wen, Yin, and Zhang 2012] Wen, Z.; Yin, W.; and Zhang, Y. 2012. Solving a low-rank factorization model for matrix completion by a nonlinear successive over-relaxation algorithm. Math. Prog. Comp. 4(4):333–361.

[Xu and Yin 2014] Xu, Y., and Yin, W. 2014. A globally convergent algorithm for nonconvex optimization based on block coordinate update. arXiv:1410.1386.

[Zhang, Huang, and Zhang 2013] Zhang, M.; Huang, Z.; and Zhang, Y. 2013. Restricted $p$-isometry properties of nonconvex matrix recovery. IEEE Trans. Inf. Theory 59(7):4316–4323.

[Zhang 2010] Zhang, C. H. 2010. Nearly unbiased variable selection under minimax concave penalty. Ann. Statist. 38(2):894–942.
Supplementary Materials for “Scalable Algorithms for Tractable Schatten Quasi-Norm Minimization”

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In this supplementary material, we give the detailed proofs of some properties and theorems.

More Notations
\[ \mathbb{R}^n \] denotes the \( n \)-dimensional Euclidean space, and the set of all \( m \times n \) matrices with real entries is denoted by \( \mathbb{R}^{m \times n} \).

Given two matrices, \( X \) and \( Y \in \mathbb{R}^{m \times n} \), the inner product is defined by \( \langle X, Y \rangle := \text{Tr}(X^TY) \), where \( \text{Tr}(\cdot) \) denotes the trace of a matrix. \( \|X\|_2 \) is the spectral norm and is equal to the maximum singular value of \( X \). \( I \) denotes an identity matrix.

For any vector \( x \in \mathbb{R}^n \), its \( l_p \) quasi-norm for \( 0 < p < 1 \) is defined as
\[
\|x\|_p = \left( \sum_i |x_i|^p \right)^{1/p}.
\]
In addition, the \( l_1 \)-norm and the \( l_2 \)-norm of \( x \) are \( \|x\|_1 = \sum_i |x_i| \) and \( \|x\|_2 = \sqrt{\sum_i x_i^2} \), respectively.

For any matrix \( X \in \mathbb{R}^{m \times n} \), the singular values of \( X \) are ordered as \( \sigma_1(X) \geq \sigma_2(X) \geq \cdots \geq \sigma_r(X) > \sigma_{r+1}(X) = \cdots = \sigma_{\min(m,n)}(X) = 0 \), where \( r = \text{rank}(X) \).

By writing \( X = U \Sigma V^T \) in its standard singular value decomposition (SVD), we can extend \( X = U \Sigma V^T \) to the following definitions.

The Schatten-\( p \) quasi-norm (\( 0 < p < 1 \)) of a matrix \( X \in \mathbb{R}^{m \times n} \) is defined as follows:
\[
\|X\|_{S_p} = \left( \sum_{i=1}^{\min(m,n)} (\sigma_i(X))^p \right)^{1/p}.
\]
The nuclear norm (also called the trace norm or the Schatten-1 norm) of \( X \) is defined as
\[
\|X\|_* = \sum_{i=1}^{\min(m,n)} \sigma_i(X).
\]
The Frobenius norm (also called the Schatten-2 norm) of \( X \) is defined as
\[
\|X\|_F = \sqrt{\text{Tr}(X^TX)} = \sqrt{\sum_{i=1}^{\min(m,n)} \sigma_i^2(X)}.
\]

Proofs of Theorem 1 and Property 1:

In the following, we will first prove that the Frobenius/nuclear hybrid norm \( \| \cdot \|_{FN} \) is a quasi-norm.

Proof. By the definition of the F/N norm, for any \( a, a_1, a_2 \in \mathbb{R} \) and \( a = a_1 a_2 \), we have
\[
\|aX\|_{FN} = \min_{aX=U(a_1U_1+a_2U_2)^T} \|a_1U\|_* \|a_2V\|_F = \min_{X=UVT} \|\sigma_1(U)\|_* \|\sigma_2(V)\|_F = \|a\| \min_{X=UVT} \|U\|_* \|V\|_F = \|a\| \|X\|_{FN}.
\]

Next we will prove that \( \|X + Y\|_{FN} \leq \beta (\|X\|_{FN} + \|Y\|_{FN}) \), where \( \beta \geq 1 \). By Lemma 1, i.e., \( \|X\|_* = \min_{X=UVT} \|U\|_{F} \|V\|_{F} \), there must exist both matrices \( \hat{U} \) and \( \hat{V} \) such that \( \|X\|_* = \|\hat{U}\|_{F} \|\hat{V}\|_{F} \) with the constraint \( X = \hat{U} \hat{V} \).

According to the definition of the F/N-norm and the fact that \( \|X\|_* \leq \sqrt{\text{rank}(X)} \|X\|_F \), we have
\[
\|X\|_{FN} = \min_{X=UVT} \|U\|_* \|V\|_F \leq \|\hat{U}\|_{F} \|\hat{V}\|_{F} = \sqrt{\text{rank}(X)} \|X\|_* = \alpha \|X\|_*,
\]
where \( \alpha = \sqrt{\text{rank}(U)} \). If \( X \neq 0 \), we can know that \( \alpha \geq 1 \). On the other hand, we also have
\[
\|X\|_* \leq \|X\|_{FN}.
\]
By the above properties, there exists a constant \( \beta \geq 1 \) such that the following holds for all \( X, Y \in \mathbb{R}^{m \times n} \):
\[
\|X + Y\|_{FN} \leq \beta (\|X\|_{FN} + \|Y\|_{FN}) 
\]
\[
\leq \beta (\|X\|_* + \|Y\|_*).
\]
Furthermore, \( \forall X \in \mathbb{R}^{m \times n} \) and \( X = UVT \), we have
\[
\|X\|_{FN} = \min_{X=UVT} \|U\|_* \|V\|_F \geq 0.
\]
In addition, if \( \|X\|_{FN} = 0 \), we have \( \|X\|_* \leq \|X\|_{FN} = 0 \), i.e., \( \|X\|_* = 0 \). Hence, \( X = 0 \). In short, the F/N-norm \( \| \cdot \|_{FN} \) is a quasi-norm. \( \square \)
Before giving the complete proofs for Theorem 1 and Property 1, we first present and prove the following important lemma.

**Lemma 2.** Suppose that $Z \in \mathbb{R}^{m \times n}$ is a matrix of rank $r \leq \min(m, n)$, and we denote its SVD by $Z = L \Sigma Z R^T$, where $L \in \mathbb{R}^{m \times r}$, $R \in \mathbb{R}^{n \times r}$ and $\Sigma \in \mathbb{R}^{r \times r}$. For any matrix $A \in \mathbb{R}^{r \times r}$ satisfying $AA^T = A^T A = I_{r \times r}$, and the given $p (0 < p < 1)$, then $(\Sigma Z A^T)_{k,k} \geq 0$ for all $k = 1, \ldots, r$, and

$$\text{Tr}^p(A\Sigma Z A^T) \geq \text{Tr}^p(\Sigma Z) = \|Z\|_{S_p}^p,$$

where $\text{Tr}^p(B) = \sum_i B_{ii}^p$.

**Proof.** For any $k \in \{1, \ldots, r\}$, we have $(A\Sigma Z A^T)_{k,k} = \sum_i a_{ki}^2 \sigma_i \geq 0$, where $\sigma_i \geq 0$ is the $i$-th singular value of $Z$. Then

$$\text{Tr}^p(A\Sigma Z A^T) = \sum_k \left( \sum_i a_{ki}^2 \sigma_i \right)^p \geq \sum_k \sum_i a_{ki}^2 \sigma_i^p = \sum_i \sigma_i^p = \text{Tr}^p(\Sigma Z) = \|Z\|_{S_p}^p.$$  \hfill (10)

According to the above inequality and $\sum_k a_{ki}^2 = 1$ for any $i \in \{1, \ldots, r\}$, (10) can be rewritten as

$$\text{Tr}^p(A\Sigma Z A^T) = \sum_k \left( \sum_i a_{ki}^2 \sigma_i \right)^p \geq \sum_k \sum_i a_{ki}^2 \sigma_i^p = \sum_i \sigma_i^p = \text{Tr}^p(\Sigma Z) = \|Z\|_{S_p}^p.$$  \hfill (11)

This completes the proof. \hfill \(\square\)

**Proof of Theorem 1:**

Assume that $U = L_U \Sigma_U R_U^T$ and $V = L_V \Sigma_V R_V^T$ are the thin SVDs of $U$ and $V$, respectively, where $L_U \in \mathbb{R}^{m \times d}$, $L_V \in \mathbb{R}^{n \times d}$, and $R_U, \Sigma_U, R_V, \Sigma_V \in \mathbb{R}^{d \times d}$. Let $X = L_X \Sigma_X R_X^T$, where the columns of $L_X \in \mathbb{R}^{m \times d}$ and $R_X \in \mathbb{R}^{n \times d}$ are the left and right singular vectors associated with the top $d$ singular values of $X$ with rank at most $r$ ($r \leq d$), and $\Sigma_X = \text{diag}([\sigma_1(X), \ldots, \sigma_r(X), 0, \ldots, 0]) \in \mathbb{R}^{d \times d}$.

By $X = UV^T$, i.e., $L_X \Sigma_X R_X^T = L_U \Sigma_U R_U^T R_V \Sigma_V L_V^T$, then $\exists O_1, \hat{O}_1 \in \mathbb{R}^{d \times d}$ satisfy $L_X = L_U O_1$ and $L_U = L_X \hat{O}_1$, i.e., $O_1 = L_U^T L_X$ and $\hat{O}_1 = L_X^T L_U$. Thus, $O_1 = \hat{O}_1^T$. Since $L_X = L_U O_1 = L_X \hat{O}_1 O_1$, we have $\hat{O}_1 O_1 = O_1^T O_1 = I_{d \times d}$. Similarly, we have $O_1 \hat{O}_1 = O_1^T O_1 = I_{d \times d}$. In addition, $\exists O_2 \in \mathbb{R}^{d \times d}$ satisfies $R_X = L_V O_2$ with $O_2^T O_2 = O_2 O_2^T = I_{d \times d}$. Let $O_3 = O_2^T O_3^T \in \mathbb{R}^{d \times d}$, then we have $O_3 O_3^T O_3^T O_3 = I_{d \times d}$, i.e., $\forall i, j \in \{1, 2, \ldots, d\}$, where $a_{ij}$ denotes the element of the matrix $A$ in the $i$-th row and the $j$-th column. Furthermore, let $O_4 = R_U^T R_V$, we have $\sum_i (O_4)_{ij}^2 \leq 1$ and $\sum_j (O_4)_{ij}^2 \leq 1$ for $\forall i, j \in \{1, 2, \ldots, d\}$.

By the above analysis, then we have $O_2^T \Sigma U O_4 \Sigma_V = O_3 \Sigma U O_4 \Sigma_V$. Let $\tau_1$ and $\varphi_j$ denote the $i$-th and the
j-th diagonal elements of $\Sigma_U$ and $\Sigma_V$, respectively. By Lemma 2, and $p = 2/3$, we have

$$
\|X\|_{S_{2/3}} \leq \left( \text{Tr} \left( O_2 \Sigma_X O_2^T \right) \right)^{3/2} = \left( \text{Tr} \left( O_2 \Sigma_U O_2 \Sigma_V \right) \right)^{3/2} = \left( \text{Tr} \left( O_3 \Sigma_U O_3 \Sigma_V \right) \right)^{3/2}
$$

$$
\leq \left( \sum_{i=1}^{d} \sum_{j=1}^{d} \tau_j (O_3)_{ij} (O_4)_{ji} \right)^{3/2}
$$

$$
\leq \left( \sum_{i=1}^{d} \sum_{j=1}^{d} \tau_j (O_3)_{ij} (O_4)_{ji} \right)^{3/2}
$$

$$
\leq \left( \sum_{i=1}^{d} \sum_{j=1}^{d} \tau_j (O_3)_{ij} (O_4)_{ji} \right)^{3/2}
$$

where the inequality $\leq$ holds due to the Hölder’s inequality [Mitrović 1970], i.e., $\sum_{k=1}^{n} |x_k y_k| \leq (\sum_{k=1}^{n} |x_k|^{p})^{1/p} (\sum_{k=1}^{n} |y_k|^{q})^{1/q}$ with $1/p + 1/q = 1$, and here we set $p = 3$ and $q = 3/2$ in the inequality $\leq$; the inequality $\leq$ follows from the basic inequality $xy \leq \frac{x^2 + y^2}{2}$ for any real numbers $x$ and $y$; the inequality $\leq$ relies on the facts $\sum_{i=1}^{d} (O_3)_{ij}^2 = 1$ and $\sum_{i=1}^{d} (O_4)_{ji}^2 \leq 1$; the inequality $\leq$ holds due to the fact $\sqrt{x_1 x_2 x_3} \leq (|x_1| + |x_2| + |x_3|)/3$ and the inequality $\leq$ holds due to the Jensen’s inequality for the concave function $f(x) = x^{1/2}$. Thus, for any matrices $U \in \mathbb{R}^{m \times d}$ and $V \in \mathbb{R}^{n \times d}$ satisfying $X = UV^T$, we have

$$
\|X\|_{S_{2/3}} \leq \|U\|_*\|V\|_F \leq \left( \frac{2\|U\|_* + \|V\|_F^2}{3} \right)^{3/2}
$$

On the other hand, let $U_* = L_X \Sigma_X^{2/3}$ and $V_* = R_X \Sigma_X^{1/3}$, where $\Sigma^p$ is entry-wise power to $p$, then we have $X = U_* V_*^T$ and

$$
\|X\|_{S_{2/3}} = \left( \text{Tr}^{2/3}(\Sigma_X) \right)^{3/2} = \|U_*\|_*\|V_*\|_F = \left( \frac{2\|U_*\|_* + \|V_*\|_F^2}{3} \right)^{3/2}.
$$

Therefore, under the constraint $X = UV^T$, we have

$$
\|X\|_{S_{2/3}} = \min_{X=UV^T} \|U\|_*\|V\|_F = \min_{X=UV^T} \left( \frac{2\|U\|_* + \|V\|_F^2}{3} \right)^{3/2} = \|X\|_{F/N}.
$$

This completes the proof. \qed
Proof of Theorem 2:

In the following, we will first prove that the bi-nuclear norm \( \| \cdot \|_{\text{BiN}} \) is a quasi-norm.

**Proof.** By the definition of the bi-nuclear norm, for any \( a, a_1, a_2 \in \mathbb{R} \) and \( a = a_1 a_2 \), we have
\[
\|aX\|_{\text{BiN}} = \min_{aX=(a_1U)(a_2V^T)} \|a_1U\|_* \|a_2V\|_*
\]
\[
= \min_{X=UV^T} |a| \|U\|_* \|V\|_*
\]
\[
= |a| \min_{X=UV^T} \|U\|_* \|V\|_* = |a| \|X\|_{\text{BiN}}.
\]

Since \( \|X\|_* = \min_X \|U\|_F \|V\|_F \), and by Lemma 6 in (Mazumder, Hastie, and Tibshirani 2010), there exist both matrices \( \tilde{U} = U_X \Sigma_X^{1/2} \) and \( \tilde{V} = V_X \Sigma_X^{1/2} \) such that \( \|X\|_* = \|\tilde{U}\|_F \|\tilde{V}\|_F \) with the SVD of \( X \), i.e., \( X = U_X \Sigma_X V_X^T \). By the fact that \( \|X\|_* \leq \sqrt{\text{rank}(X)} \|X\|_F \), we have
\[
\|X\|_{\text{BiN}} = \min_{X=UV^T} \|U\|_* \|V\|_* 
\leq \sqrt{\text{rank}(X)} \sqrt{\text{rank}(X)} \|\tilde{U}\|_F \|\tilde{V}\|_F 
\leq \text{rank}(X) \|X\|_*.
\]
If \( X \neq 0 \), then \( \text{rank}(X) \geq 1 \). On the other hand, we also have
\[
\|X\|_* \leq \|X\|_{\text{BiN}}.
\]
By the above properties, there exists a constant \( \alpha \geq 1 \) such that the following holds for all \( X, Y \in \mathbb{R}^{m \times n} \)
\[
\|X + Y\|_{\text{BiN}} \leq \alpha \|X + Y\|_* 
\leq \alpha (\|X\|_* + \|Y\|_*) 
\leq \alpha (\|X\|_{\text{BiN}} + \|Y\|_{\text{BiN}}).
\]
\[
\forall X \in \mathbb{R}^{m \times n} \text{ and } X = UV^T, \text{ we have}
\|X\|_{\text{BiN}} = \min_{X=UV^T} \|U\|_* \|V\|_* \geq 0.
\]
In addition, if \( \|X\|_{\text{BiN}} = 0 \), we have \( \|X\|_* \leq \|X\|_{\text{BiN}} = 0 \), i.e., \( \|X\|_* = 0 \). Hence, \( X = 0 \). In short, the bi-nuclear norm \( \| \cdot \|_{\text{BiN}} \) is a quasi-norm. \( \square \)

**Proof of Theorem 2:**

Proof. To prove this theorem, we use the same notations as in the proof of Theorem 1, for instance, \( X = L_X \Sigma_X R_X^T, U = L_U \Sigma_U R_U^T \) and \( V = L_V \Sigma_V R_V^T \) denote the SVDs of \( X, U \) and \( V \), respectively. By Lemma 2, and \( p = 1/2 \), we have
\[
\|X\|_{S_{1/2}} \leq \left( \text{Tr}^{1/2}(O_2 \Sigma_X O_2^T) \right)^2 = \left( \text{Tr}^{1/2}(O_2 \Sigma_U O_2^T \Sigma_U \Sigma_V) \right)^2 = \left( \text{Tr}^{1/2}(O_3 \Sigma_U O_3 \Sigma_V) \right)^2
\]
\[
= \left( \sum_{i=1}^d \sum_{j=1}^d \tau_j (O_3)_{ij} (O_4)_{ji} \right)^2 = \left( \sum_{i=1}^d \tau_i \sum_{j=1}^d \tau_j (O_3)_{ij} (O_4)_{ji} \right)^2
\]
\[
\leq \sum_{i=1}^d \sum_{j=1}^d \tau_j (O_4)_{ij} (O_4)_{ji}
\]
\[
\leq \sum_{i=1}^d \sum_{j=1}^d \frac{(O_3)_{ij}^2 \tau_j + (O_4)_{ij}^2 \tau_j}{2}
\]
\[
\leq \sum_{i=1}^d \sum_{j=1}^d \tau_j
\]
\[
= \|U\|_* \|V\|_* \leq \left( \frac{\|U\|_* + \|V\|_*}{2} \right)^2,
\]
where the inequality $a \leq \|x\|_{s_1/2}$ holds due to the Cauchy–Schwartz inequality, the inequality $b \leq \|x\|_{s_1/2}$ follows from the basic inequality $xy \leq \frac{x^2 + y^2}{2}$ for any real numbers $x$ and $y$, and the inequality $c \leq \|x\|_{s_1/2}$ relies on the facts $\sum (O_3)^2_{ij} = 1$ and $\sum (O_4)^2_{ij} \leq 1$. Thus, we have
\[
\|X\|_{s_1/2} \leq \|U\|_s \|V\|_s \leq \left( \frac{\|U\|_s + \|V\|_s}{2} \right)^2.
\]

On the other hand, let $U_* = L_X \Sigma^{1/2}_X$ and $V_* = R_X \Sigma^{1/2}_X$, then we have $X = U_* V_*$ and
\[
\|X\|_{s_1/2} = \left( \|L_X \Sigma^{1/2}_X\|_s + \|R_X \Sigma^{1/2}_X\|_s \right)^2 = \left( \frac{\|U_*\|_s + \|V_*\|_s}{2} \right)^2.
\]

Therefore, under the constraint $X = UV^T$, we have
\[
\|X\|_{s_1/2} = \min_{X = UV^T} \|U\|_s \|V\|_s = \min_{X = UV^T} \left( \frac{\|U\|_s + \|V\|_s}{2} \right)^2 = \min_{X = UV^T} \frac{\|U\|_s^2 + \|V\|_s^2}{2} = \|X\|_{SN}.
\]

This completes the proof. \qed

**Proof of Property 3:**

*Proof.* The proof involves some following properties of the $\ell_p$ quasi-norm, which must be recalled. For any vector $x \in \mathbb{R}^n$ and $0 < p_2 \leq p_1 \leq 1$, we have
\[
\|x\|_{\ell_1} \leq \|x\|_{\ell_{p_1}}, \quad \|x\|_{\ell_{p_1}} \leq \|x\|_{\ell_{p_2}} \leq n^{1/p_2 - 1/p_1} \|x\|_{\ell_{p_1}}.
\]

Suppose $X \in \mathbb{R}^{m \times n}$ is of rank $r$, and denote its SVD by $X = U_{m \times r} \Sigma_{r \times r} V^T_{n \times r}$. By Theorems 1 and 2, and the properties of the $\ell_p$ quasi-norm, we have
\[
\|X\|_s = \|\text{diag}(\Sigma_{r \times r})\|_{\ell_1} \leq \|\text{diag}(\Sigma_{r \times r})\|_{\ell_2/3} = \|X\|_{F/N} \leq \sqrt{r} \|X\|_s,
\]
\[
\|X\|_s \leq \|\text{diag}(\Sigma_{r \times r})\|_{\ell_1} \leq \|\text{diag}(\Sigma_{r \times r})\|_{\ell_{1/2}} = \|X\|_{BN} \leq r \|X\|_s.
\]

Similarly,
\[
\|X\|_{F/N} = \|\text{diag}(\Sigma_{r \times r})\|_{\ell_2/3} \leq \|\text{diag}(\Sigma_{r \times r})\|_{\ell_{1/2}} = \|X\|_{BN}.
\]

This completes the proof. \qed

**Proof of Theorem 3:**

In this paper, the proposed algorithms are based on the proximal alternating linearized minimization (PALM) method for solving the following non-convex problem:
\[
\min_{x,y} \Psi(x, y) := F(x) + G(y) + H(x, y),
\]
where $F(x)$ and $G(y)$ are proper lower semi-continuous functions, and $H(x, y)$ is a smooth function with Lipschitz continuous gradients on any bounded set.

In Algorithm 1, we state that our algorithm alternates between two blocks of variables, $U$ and $V$. We establish the global convergence of Algorithm 1 by transforming the problem (3) into a standard form (11), and show that the transformed problem satisfies the condition needed to establish the convergence. First, the minimization problem (3) can be expressed in the form of (11) by setting
\[
\begin{align*}
F(U) &:= \frac{2\lambda}{3} \|U\|_s, \\
G(V) &:= \frac{\lambda}{3} \|V\|_p^2, \\
H(U, V) &:= \frac{1}{2} \|p_{11}(UV^T) - p_{11}(D)\|_F^2.
\end{align*}
\]

The conditions for global convergence of the PALM algorithm proposed in \cite{BolteSaboTe} are shown in the following lemma.

**Lemma 3.** Let $\{(x_k, y_k)\}$ be a sequence generated by the PALM algorithm. This sequence converges to a critical point of (11), if the following conditions hold:

1. $\Psi(x, y)$ is a Kurdyka–Łojasiewicz (KL) function;
2. \( \nabla H(x, y) \) has Lipschitz constant on any bounded set;

3. \( \{ (x_k, y_k) \} \) is a bounded sequence.

As stated in the above lemma, the first condition requires that the objective function satisfies the KL property (for more details, see [Bolte, Sabach, and Teboulle 2014]). It is known that any proper closed semi-algebraic function is a KL function as such a function satisfies the KL property for all points in \( \text{dom} f \) with \( \varphi(s) = cs^{1-\theta} \) for some \( \theta \in [0, 1) \) and some \( c > 0 \). Therefore, we first give the following definitions of semi-algebraic sets and functions, and then prove that the proposed problem (3) is also semi-algebraic.

**Definition 4** ([Bolte, Sabach, and Teboulle 2014]). A subset \( S \subset \mathbb{R}^n \) is a real semi-algebraic set if there exists a finite number of real polynomial functions \( \phi_{ij}, \psi_{ij} : \mathbb{R}^n \to \mathbb{R} \) such that

\[
S = \bigcup_j \{ u \in \mathbb{R}^n : \phi_{ij}(u) = 0, \psi_{ij}(u) < 0 \}.
\]

Moreover, a function \( h(u) \) is called semi-algebraic if its graph \( \{(u, t) \in \mathbb{R}^{n+1} : h(u) = t \} \) is a semi-algebraic set.

Semi-algebraic sets are stable under the operations of finite union, finite intersections, complementation and Cartesian product. The following are the semi-algebraic functions or the property of semi-algebraic functions used below:

- Real polynomial functions.
- Finite sums and product of semi-algebraic functions.
- Composition of semi-algebraic functions.

**Lemma 4.** Each term in the proposed problem (3) is a semi-algebraic function, and thus the function (3) is also semi-algebraic.

**Proof.** It is easy to notice that the sets \( U = \{ U \in \mathbb{R}^{n \times d} : \| U \|_\infty \leq D_1 \} \) and \( V = \{ V \in \mathbb{R}^{n \times d} : \| V \|_\infty \leq D_2 \} \) are both semi-algebraic sets, where \( D_1 \) and \( D_2 \) denote two pre-defined upper-bounds for all entries of \( U \) and \( V \), respectively.

For the first term \( F(U) = \frac{1}{2\lambda} \| U \|_F^2 \). According to [Bolte, Sabach, and Teboulle 2014], we can know that the \( \ell_1 \)-norm is a semi-algebraic function. Since the nuclear norm is equivalent to the \( \ell_1 \)-norm on singular values of the associated matrix, it is natural that the nuclear norm is also semi-algebraic.

For both terms \( G(V) = \frac{1}{2\ell} \| V \|_F^2 \) and \( H(U, V) = \frac{1}{2} \| P_\Omega( U V^T - D ) \|_F^2 \), they are real polynomial functions, and thus are semi-algebraic functions ([Bolte, Sabach, and Teboulle 2014]). Therefore, (3) is semi-algebraic due to the fact that a finite sum of semi-algebraic functions is also semi-algebraic.

For the second condition in Lemma 3, \( H(U, V) = \frac{1}{2} \| P_\Omega( U V^T - D ) \|_F^2 \) is a smooth polynomial function, and \( \nabla H(U, V) = ( P_\Omega( U V^T - D ) ) V \) is a semi-algebraic function. Moreover, a function \( \nabla H(U, V) \) has Lipschitz constant on any bounded set.

For the final condition in Lemma 3, \( U_k \in U \) and \( V_k \in V \) for any \( k = 1, 2, \ldots \), which implies the sequence \( \{ (U_k, V_k) \} \) is bounded.

In short, we can know that three similar conditions as in Lemma 3 hold for Algorithm 1. According to the above discussion, it is clear that the problem (4) is also a semi-algebraic function. In other words, another proposed algorithm shares the same convergence property as in Theorem 3.

**Proof of Theorem 5:**

According to Theorem 3, we can know that \( (\hat{U}, \hat{V}) \) is a critical point of the problem (3). To prove Theorem 5, we first give the following lemmas.

**Lemma 5** ([Lin, Chen, and Wu 2009]). Let \( H \) be a real Hilbert space endowed with an inner product \( \langle \cdot, \cdot \rangle \) and an associated norm \( \| \cdot \| \), and \( y \in \partial \| x \| \), where \( \partial \| \cdot \| \) denotes the subgradient of the norm. Then \( \| y \| = 1 \) if \( x \neq 0 \), and \( \| y \| \leq 1 \) if \( x = 0 \), where \( \| \cdot \|^* \) is the dual norm of \( \| \cdot \| \). For instance, the dual norm of the nuclear norm is the spectral norm, \( \| \cdot \|_2 \), i.e., the largest singular value.

**Lemma 6** ([Wang and Xu 2012]). Let \( \mathcal{L}(X) = \frac{1}{\sqrt{mn}} \| X - \hat{X} \|_F \) and \( \hat{\mathcal{L}}(X) = \frac{1}{\sqrt{m}} \| P_\Omega( X - \hat{X} ) \|_F \) be the actual and empirical loss function respectively, where \( X, \hat{X} \in \mathbb{R}^{m \times n} \) \( m \geq n \). Furthermore, assume entry-wise constraint \( \max_{i,j} | X_{ij} | \leq \beta_1 \). Then for all rank-\( r \) matrices \( X \), with probability greater than \( 1 - 2 \exp(-m) \), there exists a fixed constant \( C \) such that

\[
\sup_{X \in S_r} | \hat{\mathcal{L}}(X) - \mathcal{L}(X) | \leq C \beta_1 \left( \frac{mr \log(m)}{m} \right)^{1/4},
\]

where \( S_r = \{ X \in \mathbb{R}^{m \times n} : \text{rank}(X) \leq r, \| X \|_F \leq \sqrt{mn} \beta_1 \} \).
Proof of Theorem 5:

Proof. By \( C_2 = \|P_\Omega(D - \hat{U}\hat{V}^T)\hat{V}\|_F / \|P_\Omega(D - \hat{U}\hat{V}^T)\|_F \), we have

\[
\frac{\|D - \hat{U}\hat{V}^T\|_F}{\sqrt{mn}} = \frac{\|D - \hat{U}\hat{V}^T\|_F - \|P_\Omega(D - \hat{U}\hat{V}^T)\hat{V}\|_F}{\sqrt{\|\hat{V}\|_F}} + \frac{\|P_\Omega(D - \hat{U}\hat{V}^T)\hat{V}\|_F}{\sqrt{\|\hat{V}\|_F}}. 
\]

Let \( \tau(\Omega) := \left| \frac{1}{\sqrt{mn}}\|D - \hat{U}\hat{V}^T\|_F - \frac{1}{\sqrt{\|\hat{V}\|}}\|P_\Omega(D - \hat{U}\hat{V}^T)\hat{V}\|_F \right| \), then we need to bound \( \tau(\Omega) \). Since \( \text{rank}(\hat{U}\hat{V}^T) \leq d \) and \( \hat{U}\hat{V}^T \in S_d \), and according to Lemma 6, then with probability greater than \( 1 - 2\exp(-m) \), then there exists a fixed constant \( C_1 \) such that

\[
\sup_{\hat{U}\hat{V}^T \in S_d} \tau(\Omega) = \left| \frac{\|\hat{U}\hat{V}^T - D\hat{V}\|_F}{\sqrt{mn}} - \frac{\|P_\Omega(\hat{U}\hat{V}^T) - P_\Omega(D)\|_F}{\sqrt{\|\hat{V}\|}} \right| 
\leq C_1 \beta \left( \frac{md\log(m)}{|\Omega|} \right)^{\frac{1}{4}}. 
\]  

(12)

We also need to bound \( \|P_\Omega(\hat{U}\hat{V}^T - D)\hat{V}\|_F \). Given \( \hat{V} \), the optimization problem with respect to \( U \) is formulated as follows:

\[
\min_U \frac{2\lambda\|U\|_*}{3} + \frac{1}{2}\|P_\Omega(U\hat{V}^T) - P_\Omega(D)\|_F^2. 
\]  

(13)

Since \((\hat{U}, \hat{V})\) is a critical point of the problem (3), the first-order optimal condition of the problem (13) is written as follows:

\[
P_\Omega(D - \hat{U}\hat{V}^T)\hat{V} = \frac{2\lambda}{3} \theta \|\hat{U}\|_*.
\]  

(14)

Using Lemma 5, we obtain

\[
\|P_\Omega(\hat{U}\hat{V}^T - D)\hat{V}\|_2 \leq \frac{2\lambda}{3},
\]

where \( \|\cdot\|_2 \) is the spectral norm. According to \( \text{rank}(P_\Omega(\hat{U}\hat{V}^T - D)\hat{V}) \leq d \), we have

\[
\|P_\Omega(\hat{U}\hat{V}^T - D)\hat{V}\|_F \leq \sqrt{d}\|P_\Omega(\hat{U}\hat{V}^T - D)\hat{V}\|_2 \leq \frac{2\sqrt{d}\lambda}{3}. 
\]  

(15)

By (12) and (15), we have

\[
\frac{\|Z - \hat{U}\hat{V}^T\|_F}{\sqrt{mn}} \leq \frac{\|E\|_F}{\sqrt{mn}} + \frac{\|D - \hat{U}\hat{V}^T\|_F}{\sqrt{mn}} \leq \frac{\|E\|_F}{\sqrt{mn}} + \frac{\|P_\Omega(D - \hat{U}\hat{V}^T)\hat{V}\|_F}{\sqrt{\|\hat{V}\|}} \leq \frac{\|E\|_F}{\sqrt{mn}} + C_1 \beta \left( \frac{md\log(m)}{|\Omega|} \right)^{\frac{1}{4}} + \frac{2\sqrt{d}\lambda}{3C_2\sqrt{|\Omega|}}.
\]

This completes the proof. \( \square \)

Lower bound on \( C_2 \)

Finally, we also discuss the lower boundedness of \( C_2 \), that is, it is lower bounded by a positive constant. By the characterization of the subdifferentials of norms, we have

\[
\partial\|X\|_* = \{Y \mid \langle Y, X \rangle = \|X\|_*, \|Y\|_2 \leq 1 \}. 
\]  

(16)
Let $Q = \mathcal{P}_\Omega(D - \hat{U} \hat{V}^T)\hat{V}$, and by (14), we have $Q \in \frac{2\lambda}{3} \partial \|\hat{U}^*\|_\ast$. By (16), we have
\[ \left\langle \frac{3}{2\lambda} Q, \hat{U} \right\rangle = \|\hat{U}\|^\ast. \]

Note that $\|X\|^\ast \geq \|X\|_F$ and $\langle X, Y \rangle \leq \|X\|_F \|Y\|_F$ for any matrices $X$ and $Y$ of the same size.
\[ \frac{3}{2\lambda} \|Q\|_F \|\hat{U}\|_F \geq \left\langle \frac{3}{2\lambda} Q, \hat{U} \right\rangle = \|\hat{U}\|^\ast \geq \|\hat{U}\|_F. \]
Since $\|\hat{U}\|_F > 0$ and $\lambda \neq 0$, thus we obtain
\[ \|\mathcal{P}_\Omega(D - \hat{U} \hat{V}^T)\hat{V}\|_F = \|Q\|_F \geq \frac{2\lambda}{3}. \]

$\hat{U}$ is the optimal solution of the problem (13) with the given matrix $\hat{V}$, then
\[ \|\mathcal{P}_\Omega(D - \hat{U} \hat{V}^T)\|_F^2 < \|\mathcal{P}_\Omega(D - \hat{U} \hat{V}^T)\|^\ast + \frac{2\lambda}{3} \|\hat{U}\|^\ast \leq \|\mathcal{P}_\Omega(D)\|_F^2 = \gamma, \]
where $\gamma > 0$ is a constant. Thus,
\[ C_2 = \frac{\|\mathcal{P}_\Omega(D - \hat{U} \hat{V}^T)\hat{V}\|_F}{\|\mathcal{P}_\Omega(D - \hat{U} \hat{V}^T)\|_F} > \frac{2\lambda}{3\sqrt{\gamma}} > 0. \]

**PALM Algorithm for Solving (4)**

We present an efficient proximal alternating linearized minimization (PALM) algorithm for solving the bi-nuclear quasi-norm regularized matrix completion problem (4), as outlined in Algorithm 2. Moreover, Algorithm 2 shares the same convergence property as in Theorems 3 and 4, and has the similar theoretical recovery guarantee as in Theorem 5.

**References**

Bolte, J.; Sabach, S.; and Teboulle, M. 2014. Proximal alternating linearized minimization for nonconvex and nonsmooth problems. *Math. Program.* 146:459–494.

Lin, Z.; Chen, M.; and Wu, L. 2009. The augmented Lagrange multiplier method for exact recovery of corrupted low-rank matrices. Univ. Illinois, Urbana-Champaign.

Mazumder, R.; Hastie, T.; and Tibshirani, R. 2010. Spectral regularization algorithms for learning large incomplete matrices. *J. Mach. Learn. Res.* 11:2287–2322.

Mitrinović, D. S. 1970. *Analytic Inequalities.* Heidelberg: Springer-Verlag.

Wang, Y., and Xu, H. 2012. Stability of matrix factorization for collaborative filtering. In *Proc. 29th Int. Conf. Mach. Learn. (ICML)*, 417–424.