SHARP BOUND ON THE LARGEST POSITIVE EIGENVALUE FOR ONE-DIMENSIONAL SCHröDINGER OPERATORS

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Abstract. Let $H = -D^2 + V$ be a Schrödinger operator on $L^2(\mathbb{R})$, or on $L^2(0, \infty)$. Suppose the potential satisfies $\limsup_{x \to \infty} |xV(x)| = a < \infty$. We prove that $H$ admits no eigenvalue larger than $\frac{4a^2}{\pi^2}$. For any positive $a$ and $\lambda$ with $0 < \lambda < \frac{4a^2}{\pi^2}$, we construct potentials $V$ such that $\limsup_{x \to \infty} |xV(x)| = a$ and the associated Schrödinger operator $H = -D^2 + V$ has eigenvalue $\lambda$.

1. Introduction and main results

Let $H = -D^2 + V$ be a one-dimensional Schrödinger operator on $L^2(\mathbb{R})$, or on $L^2(0, \infty)$, where the potential $V$ satisfies $\limsup_{x \to \infty} |xV(x)| = a < \infty$. By the classical Weyl theorem, the essential spectrum $\sigma_{\text{ess}}(H) = [0, \infty)$. In this paper, we are interested in quantitative relation of the largest positive eigenvalue and $a$. It is well known that there is no eigenvalue larger than $a^2$, see paper [5] for example. This implies there is no eigenvalue embedded into the essential spectrum if $V(x) = o(x^{-1})$ as $x$ goes to infinity. As a standard example with Wigner-von Neumann type potentials shows, for any $0 < \lambda < \frac{a^2}{4}$, there exists $V$ with $\limsup_{x} |xV(x)| = a$ such that $\lambda$ is an eigenvalue. Thus $V(x) = o(x^{-1})$ is the spectral transition for existence of eigenvalue embedded into the essential spectrum. Naboko [8] and Simon [10] constructed examples to show that dense point spectra can be embedded into essential spectrum if the potential decreases slightly slower than $V(x) = O(x^{-1})$. The fact that $V(x) = \frac{O(1)}{1+x}$ is the spectral transition for singular continuous embedding into the essential spectrum was established by Kiselev [6].

Kiselev-Last-Simon [7] proved that if $\limsup_{x} |xV(x)| = a$, then the sum of eigenvalues $\lambda_n$ of $-D^2 + V$ is finite, that is $\sum \lambda_n \leq \frac{a^2}{\pi^2}$.

In this paper, we focus on the largest eigenvalue and obtain the following sharp result.

Theorem 1.1. Suppose potential $V$ satisfies

$$\limsup_{x \to \infty} |xV(x)| = a < \infty.$$ 

Then $H$ admits no eigenvalue larger than $\frac{4a^2}{\pi^2}$.

Theorem 1.2. For any positive $a$ and $\lambda$ with $0 < \lambda < \frac{4a^2}{\pi^2}$, there exist potentials $V$ such that $\limsup_{x \to \infty} |xV(x)| = a$ and the associated Schrödinger operator $H = -D^2 + V$ has eigenvalue $\lambda$.

Our proof is based on the modified Prüfer transformation and some basic analysis. See [7] for more details about modified Prüfer transformation.
Remark 1.3. After we finished this paper, we noticed that Theorem 1.2 has been proved by Halvorsen [4]. See [1–3] for more generalization. Later, Remling also addressed the problem [9] and showed that $H$ has purely absolute continuous spectrum in $(\frac{4a^2}{\pi^2}, \infty)$. We refer readers to Simon’s article for full history [11].

2. Proof of Theorems 1.1 and 1.2

Without loss of generality, we only consider the half line case $L^2(0, \infty)$.

Let $\lambda = k^2$ with $k > 0$ and suppose $u$ is a solution of

$$-u''(x) + V(x)u(x) = k^2u(x).$$

Change variables to

$$u'(x) = kR(x) \cos(\theta(x))$$

$$u(x) = R(x) \sin(\theta(x)).$$

We get a pair of equations

$$\frac{d\theta}{dx} = k - \frac{V(x)}{k} \sin^2 \theta \quad (2)$$

$$\frac{d\log R}{dx} = \frac{1}{2k} V(x) \sin 2\theta \quad (3)$$

If $V = 0$, then $\theta(x) = \theta_0 + kx$ and $R(x) = R_0$ is a solution. The following lemma is well known. The proof is basic, the readers can see Lemma 4.2 in [7] for the details.

Lemma 2.1 (Lemma 4.2, [7]). If $u \in L^2(0, \infty)$ is a solution of (1), then $R(\cdot) \in L^2(0, \infty)$.

Proof of Theorem 1.1

Proof. Under the assumption of Theorem 1.1, for any $\epsilon > 0$, there exists $x_0 > 0$ such that for all $x > x_0$

$$|V(x)| \leq \frac{a + \epsilon}{1 + x}.$$

In the following, we always assume $\epsilon > 0$ is sufficiently small and may change even in the same formula. We also assume $x > 0$ is large enough.

By (3), one has

$$\log R(x) \geq \log R(x_0) - \frac{a + \epsilon}{2k} \int_{x_0}^{x} \frac{\sin 2\theta(y)}{y + 1} dy \quad (4)$$

Now we will estimate $\int_{x_0}^{x} \frac{\sin 2\theta(y)}{y + 1} dy$. Let $i_0$ be the largest positive integer such that $2\pi i_0 < \theta(x_0)$. By (2), there exist $x_0 < x_1 < x_2 < \cdots < x_n < x < x_{n+1}$ such that

$$\theta(x_i) = 2\pi i_0 + i \frac{\pi}{2}$$

for $i = 1, 2, \cdots, n, n + 1$.

By (2), one has

$$|x_{i+1} - x_i| = \frac{\pi}{2k} + \frac{O(1)}{x_i + 1}.$$ 

Then, one has

$$n \leq \frac{2k}{\pi} + \epsilon x \quad (5)$$

I would like to thank Barry Simon for telling me the full history of the problem.
and

\((6) \quad (\frac{\pi}{2k} - \epsilon)i < x_i - x_0 < (\frac{\pi}{2k} + \epsilon)i.\)

For \(y \in [x_i, x_{i+1})\), we have

\[
\theta(y) = 2\pi i_0 + \frac{\pi}{2} + k(y - x_i) + \frac{O(1)}{1 + x_i}
= 2\pi i_0 + \frac{\pi}{2} + k(y - x_i) + \frac{O(1)}{1 + i}.
\]

Which implies

\[
\int_{x_i}^{x_{i+1}} \left| \sin 2\theta(y) \right| dy = \int_{0}^{\frac{\pi}{2k}} \sin(2ky) dy + \frac{O(1)}{1 + i}
= \frac{1}{k} + \frac{O(1)}{1 + i}.
\]

(7)

By (5), (6) and (7), we obtain

\[
\int_{x_0}^{x} \left| \sin 2\theta(y) \right| \frac{dy}{y + 1} \leq \sum_{i=1}^{n+1} \left( \frac{1}{k} + \frac{O(1)}{1 + i} \right) \frac{2k}{(\pi - \epsilon)i} + O(1)
\leq \left( \frac{2}{\pi} + \epsilon \right) \ln x + O(1).
\]

Combining with (4), we have

\[
\log R(x) \geq -\left( \frac{a}{k\pi} + \epsilon \right) \ln x + O(1).
\]

Assume \(k > \frac{2a}{\pi}\), one has

\[
\frac{a}{k\pi} + \epsilon < \frac{1}{2}.
\]

Thus

\[
R(x) \geq \frac{1}{\sqrt{x}}
\]

for large \(x\). By Lemma 2.1, \(u \not\in L^2(0, \infty)\). Thus \(\lambda = k^2\) is not an eigenvalue.

\[\square\]

**Proof of Theorem 1.2**

Proof. We define

\[
V(x) = -\frac{a}{1 + x} \text{sgn}(\sin 2\theta(x)),
\]

where \(\text{sgn}(\cdot)\) is the sign function. Substitute (8) into (2), and solve the nonlinear system for \(\theta\) with the initial condition \(\theta(0) = \theta_0\). It is not difficult to see that (2) has a unique piecewise smooth global solution by a standard ODE existence and uniqueness theorem. Thus \(V(x)\) is well defined and

\[
\frac{d}{dx} \ln R = -\frac{a}{2k} \frac{\sin 2\theta}{1 + x}.
\]
Under the notations in the proof of Theorem 1.1 we have
\[
\int_1^x \frac{\left| \sin 2\theta(y) \right|}{y+1} \, dy \geq n+1 \sum_{i=1}^{n+1} \left( \frac{1}{k} + O\left(\frac{1}{i} \right) \right) \left( \frac{2k}{\pi + \epsilon i} \right) + O(1) \\
\geq \left( \frac{2}{\pi} - \epsilon \right) \ln x + O(1).
\]
Combining with (9), it is easy to see
\[
\log R(x) \leq -\left( \frac{a}{k\pi} - \epsilon \right) \ln x + O(1).
\]
Assume \( k < \frac{2a}{\pi} \), then
\[
\frac{a}{k\pi} - \epsilon > \frac{1}{2} + \epsilon.
\]
Thus
\[
R(x)^2 \leq \frac{1}{x^{1+\epsilon}},
\]
for large \( x \). This yields \( u \in L^2(0, \infty) \), so \( \lambda = k^2 \) is an eigenvalue.

\[\square\]

Acknowledgments

The paper is inspired by Fan Yang’s talk at ergodic Schrödinger operators weekly graduate seminar at UCI, where she presented paper \([7]\). I am appreciated her fascinating presentation. I would like to thank Svetlana Jitomirskaya for comments on earlier versions of the manuscript. I also thank Barry Simon for telling me the full history of the problem (see footnote 1). The author was supported by the AMS-Simons Travel Grant 2016-2018 and NSF DMS-1700314. This research was also partially supported by NSF DMS-1401204.

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