ON PAIRWISE COMPARISONS WITH VALUES IN A GROUP: ALGEBRAIC STRUCTURES

JEAN-PIERRE MAGNOT

Abstract. We describe the algebraic properties of pairwise comparisons matrices with coefficients in an arbitrary group. We provide a vocabulary adapted for the description of main algebraic properties of inconsistency maps, describe an example where the use of a non abelian group is necessary, and describe a generalization of pairwise comparisons matrices and inconsistency maps on a graph.

Keywords: approximate reasoning, inconsistency, pairwise comparisons, group, holonomy, matrix, simplex, graph.

2010 Mathematics Subject Classification: 03F25

INTRODUCTION

Pairwise comparisons are among the classical ways of decision making and information checking. The main idea is simple: assign a score to the comparison of a pair \((s, s')\) of two states which have to be compared, and we say that the “scores” are consistent if, for three states \((s, s', s'')\), the comparison of \(s\) and \(s''\) can be deduced from the comparisons of \(s\) and \(s'\), and of \(s'\) and \(s''\). If not, the comparisons are called inconsistent. These aspects are precised in section 1, and gives so many applications that it is impossible to cite them all. We mention two of them [5, 6, 15] which are applications of deep interest.

This is mostly why some authors have developed ways to quantify inconsistency [11, 13], see e.g. [1], and there exists actually tentatives of axiomatizations of the so-called “inconsistency indicators” [3, 4, 14]. One can also try to deal with partial orders, after [8 [9, 10, 19], motivated by the obvious lack of informations when one tries to express a complex situation only by a score.

This leads us to the main motivation of this work. Dealing with partial orders can turn out to be very complex (see e.g. the tables in [19]), where as (non abelian) groups furnish a minimal setting where composition and inversion are well-defined, with all the necessary properties for comparisons of more complex datas. Let us now describe the contents of this paper.

Section 2 describes pairwise comparisons matrices with coefficients in a group \(G\), indexed by any (finite or infinite) set of states \(I\). We define also what is an inconsistency map. The terminology of inconsistency indicator is reserved to inconsistency maps with additional properties, and the settings developed will be justified by other parts of this work.

Section 3 deals with algebraic properties of consistency. We highlight adjoint action of the group \(G^I\), called gauge group by analogy with differential geometric settings [12], on the set of pairwise comparisons matrices. The key property states
that consistency is characterized by an orbit of this group. The side-properties
are then described, which motivates the vocabulary for properties of inconsistency
maps, and leads to the terminology of inconsistency indicator. As a concluding
property, we show that Koczkodaj’s inconsistency map is an adjoint-invariant in-
consistency indicator. We are aware that a similar study, based on a deep under-
standing of the properties of Saaty’s inconsistency map [18], is actually investigated
by others.

Section 4 is devoted to examples. The first natural example which arises for
the group $G$ is $G = GL_n(\mathbb{R})$ (section 4.1), and show that 3D-perspective can be
understood as inconsistency with coefficients in the affine group of $\mathbb{R}^2$ (section 4.2).

Section 5 deals with generalization on graphs. This happens when two states
cannot be compared by direct comparisons, but only by comparisons with interme-
diate states. This leads to “holes” in the pairwise comparisons matrices, assigned
to the coefficient “0”, and the notion of holonomy enables us to extend Koczkodaj’s in-
consistency maps to a $\mathbb{R}[[X]]$—valued inconsistency map, which is adjoint-invariant.

1. Pairwise comparisons matrices with coefficients in $\mathbb{R}_+$

It is easy to explain the inconsistency in pairwise comparisons when we consider
cycles of three comparisons, called triad and represented here as $(x, y, z)$, which do
not have the “morphism of groupoid” property such as

$$x \cdot z \neq y$$

Evidently, the inconsistency in a triad $(x, y, z)$ is somehow (not linearly) propor-
tional to $y - xz$. In the linear space, the inconsistency is measured by the “approx-
imate flatness” of the triangle. The triad is consistent if the triangle is flat. For
example, $(1, 2, 1)$ and $(10, 101, 10)$ have the difference $y - xz = 1$ but the inconsis-
tency in the first triad is unacceptable. It is acceptable in the second triad. In order
to measure inconsistency, one usually considers coefficients $a_{i,j}$ with values in an
abelian group $G$, with at least 3 indexes $i, j, k$. The use of “inconsistency” has a
meaning of a measure of inconsistency in this study; not the concept itself. The
approach to inconsistency (originated in [11] and generalized in [7]) can be reduced
to a simple observation:

- search all triads (which generate all 3 by 3 PC submatrices) and locate the
  worse triad with a so-called inconsistency indicator $(ii)$,
- $ii$ of the worse triad becomes $ii$ of the entire PC matrix.

Expressing it a bit more formally in terms of triads (the upper triangle of a PC
submatrix $3 \times 3$), we have:

$$Kii(x, y, z) = 1 - \min \left\{ \frac{y}{xz}, \frac{xz}{y} \right\}.$$  

According to [13], it is equivalent to:

$$ii(x, y, z) = 1 - e^{-\left| \ln \left( \frac{y}{xz} \right) \right|}.$$  

The expression $\left| \ln \left( \frac{y}{xz} \right) \right|$ is the distance of the triad $T$ from 0. When this distance
increases, the $ii(x, y, z)$ also increases. It is important to notice here that this
definition allows us to localize the inconsistency in the matrix PC and it is of a
considerable importance for most applications.
Another possible definition of the inconsistency indicator can also be defined (following [13]) as:

\[(1.2) \quad K_{ii_n}(A) = 1 - \min_{1 \leq i < j \leq n} \left( \frac{a_{ij}}{a_{i,i+1}a_{i+1,i+2} \ldots a_{j-1,j}} \right) \]

since the matrix \( A \) is consistent if and only if for any \( 1 \leq i < j \leq n \) the following equation holds:

\[a_{ij} = a_{i,i+1}a_{i+1,i+2} \ldots a_{j-1,j}.\]

It is equivalent to:

\[(1.3) \quad K_{ii_n}(A) = 1 - \max_{1 \leq i < j \leq n} \left( 1 - e^{-\ln \left( \frac{a_{ij}}{a_{i,i+1}a_{i+1,i+2} \ldots a_{j-1,j}} \right) } \right) \]

The first Koczkodaj’s indicator \( K_{ii_3} \) allows us not only find the localization of the worst inconsistency but to reduce the inconsistency by a step-by-step process which is crucial for practical applications. The second Koczkodaj’s indicator \( K_{ii_n} \) is useful when the global inconsistency indicator is needed for acceptance or rejection of the PC matrix. A hybrid of two \( ii \) definitions may be considered in applications, and an abstract unification will be discussed in the last section.

2. Changing the comparisons structure to arbitrary groups

In the previous section, the comparisons coefficients are \( a_{i,j} \) are scaling coefficients. This means that, if the PC matrix \( A \) is coherent, given a state \( s_k \), we can recover all the other states \( s_j \) by something assimilated to scalar multiplication:

\[s_j = a_{j,k}s_k.\]

In other words, even if the states \( s_j \) are driven by more complex rules, we reduce them to a “score” or an “evaluation” in \( \mathbb{R}_+ \). This is useless to say that such an approach is highly reductive: even in video games, the virtual fighters have more than one characteristic: health, speed, strength, mental... and the global design of these characteristics intends to reflect some “complexity” in the game (please note the “ ”). So that, the states \( s_j \) have to belong to a more complex state space \( S \), and in order to have pairwise comparisons, a straightforward study shows that we define \( [16, 17] \) a semi-category \( C_S \), with set of objects \( Ob(C_S) = S \), and such that morphisms \( Hom(C_S) \) must satisfy the following properties:

- there exists an identity morphism
- if \( a \in Hom(C_S) \), then there exists \( a^{-1} \in Hom(C_S) \)
- any morphism acts on any state, which can be rephrased in the language of categories by: the semi-category is total.

Thus, gathering the necessary properties of \( Mor(C_S) \), we get:

**Proposition 2.1.** \( Hom(C_S) \) is a group.

Then we have that the minimal setting for generalizing pairwise comparisons is given by actions on \( S \) by a group \( G \), which leads to the following setting. Let \( I \) be a set of indexes and let \((k,+,.||.||)\) be a field with absolute value and \( V_k \) a normed \( k \)-vector space.

**Definition 2.2.** [12] Let \((G,.)\) be a group. A pairwise comparisons matrix is a matrix

\[A = (a_{i,j})_{(i,j) \in I^2} \]
such that

1. \( \forall (i, j) \in I^2, a_{i,j} \in G. \)
2. \( \forall (i, j) \in I^2, \ a_{j,i} = a_{i,j}^{-1}. \)
3. \( a_{i,i} = 1_G. \)

We note by \( PC_I(G) \) the set of pairwise comparisons matrices indexed by \( I \) and with coefficients in \( G \). When \( G \) is not abelian, there are two notions of inconsistency:

- \( A \) is **covariantly consistent** if and only if \( \forall (i, j, k) \in I^3, a_{i, k} = a_{i, j} a_{j, k} \)
- \( A \) is **contravariantly consistent** if and only if \( \forall (i, j, k) \in I^3, a_{i, k} = a_{j, k} a_{i, j} \).

Contravariant consistency appears in the geometric realization of \( PC_I(G) \) via the holonomy of a connection on a simplex [12], but we give the following easy remark:

**Remark 2.3.** Let \( A = (a_{i,j})_{(i, j) \in I^2} \) be a contravariant PC-matrix. Then the \( G \)-matrix \( B = (b_{i,j})_{(i, j) \in I^2} \) defined by

\[ \forall (i, j) \in I^2, \ b_{i,j} = a_{i,j}^{-1} \]

is a covariant PC matrix.

This shows that the two notions are dual, and we concentrate our efforts on covariant consistency, that we call **consistency**, in the rest of the paper.

**Definition 2.4.** A (non normalized, non covariant) inconsistency map is a map

\[ ii : PC_I(G) \to V_k \]

such that \( ii(A) = 0 \) if \( A \) is consistent. Moreover, we say that \( ii \) is faithful if \( ii(A) = 0 \) implies that \( A \) is consistent.

We note by \( CPC_I(G) \) the set of consistent PC-matrices.

After that, since \( V_k \) is a vector space equipped with a semi-norm, the semi-norm will give us the “score” of inconsistency, as in the previous section. The main feature in applying this setting will be twofold, and these two points are far to be systematically solved with the present work:

- define a comparisons group \( G \) for which we can get at least one comparison coefficient \( a_{i,j} \) between two states \( s_i \) and \( s_j \) (which means that the \( G \)-action needs to be transitive),
- evaluate (and compute!) inconsistency, if possible generalizing the \( \mathbb{R}^+ \)-setting, in a proper way to get safe decision making. This second point is linked with multiscale analysis.

Let us develop examples which can highlight these features, after developing the algebraic setting.

3. **Algebraic properties on \( PC_I(G) \)**

First, we give the following easy proposition:

**Proposition 3.1.** Any morphism of group \( a : G \to G' \) extends to a map \( \bar{a} : PC_I(G) \to PC_I(G') \) by action on the coefficients, and:

- If \( A \in PC_I(G) \) is consistent, then \( \bar{a}(A) \in PC_I(G') \) is consistent.
- If \( \text{Ker}(a) = \{e_G\} \), then \( A \in PC_I(G) \) is consistent, if and only if \( \bar{a}(A) \in PC_I(G') \).
We call $G^I$ the **gauge group** of $G$, following [12]. Then we get the following actions:

- **a left action** $L : G^I \times PC_I(G) \to PC_I(G)$ defined, for $(g_i)_I \in G^I$ and $(a_{i,j})_{I^2} \in PC_I(G)$ by
  $$L_{(g_i)_I}((a_{i,j})_{I^2}) = (b_{i,j})_{I^2}$$
  with
  $$b_{i,j} = \begin{cases} 
    1 & \text{if } i = j \\
    g_i a_{i,j} & \text{if } i < j \\
    a_{i,j} g_j & \text{if } i > j
  \end{cases}$$

- **a right action** $R : PC_I(G) \times G^I \to PC_I(G)$ defined, for $(g_i)_I \in G^I$ and $(a_{i,j})_{I^2} \in PC_I(G)$ by
  $$R_{(g_i)_I}((a_{i,j})_{I^2}) = (b_{i,j})_{I^2}$$
  with
  $$b_{i,j} = \begin{cases} 
    1 & \text{if } i = j \\
    a_{i,j} g_j & \text{if } i < j \\
    g_i^{-1} a_{i,j} & \text{if } i > j
  \end{cases}$$

- **an adjoint action**
  $$Ad_{(g_i)_I} = L_{(g_i)_I} \circ R_{(g_i)_I}^{-1} = R_{(g_i)_I}^{-1} \circ L_{(g_i)_I}$$

- **a coadjoint action**
  $$((a_{i,j})_{I^2}, (g_i)_I) \mapsto Ad_{(g_i)_I}^{-1}.$$  

These actions are obviously invariant under increasing re-indexation. Let us recall the following theorem from [12] prove when $I \subset \mathbb{Z}$:

**Theorem 3.2.** When $I \subset \mathbb{Z}$,

$$\exists (\lambda_i)_{i \in I}, \quad a_{i,j} = \lambda_i \lambda_j^{-1} \iff A \text{ is consistent.}$$

We rephrase it the following way, extending it to any totally ordered set of indexes $I$:

**Theorem 3.3.** Consistent PC-matrices are the orbits of the PC-matrix $(1)_{I^2}$ with respect to the adjoint action.

**Proof.** Let $A = (a_{i,j})_{I^2}$ be a consistent PC matrix. Let $i_0 \in I$ be a fixed index, and set $g_i = a_{i,i_0}$. Since $A$ is consistent,

$$a_{i,j} = a_{i,i_0} a_{i_0,j} = a_{i,i_0} g_j^{-1} = g_{i_0} a_{j,i_0}^{-1} = g_{i_0} g_j^{-1}$$

\[ \square \]

Let us give the following trivial proposition:

**Proposition 3.4.** $L$ and $R$ are effective actions.
One can wonder whether \( L \) and \( R \) are free or transitive. Let us consider the following "layered cake" example:

Let \( \lambda \in \mathbb{R}_+^* - \{1\} \). Let us consider the matrix

\[
A = \begin{pmatrix}
1 & \lambda & \lambda^{-k} \\
\lambda^{-1} & 1 & \lambda \\
\lambda^k & \lambda^{-1} & 1
\end{pmatrix}.
\]

Let us calculate the orbit of \( A \) with respect to the left action (with the special case \( G \) is abelian.) Let \( g \in \mathbb{R}^3 \). \( L_g(A) \in CPC_3(\mathbb{R}_+^*) \).

\[
L_g(A) = \begin{pmatrix}
g_1 \lambda & g_1 \lambda^{-k} \\
g_1^{-1} \lambda^{-k} & 1 \\
g_1^{-1} \lambda^k & g_2 \lambda
\end{pmatrix}
\]

In this example, \( g_3 \) is not acting, so that \( L \) is not free. Let us solve

\[
L_g(A) = \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\]

we get the incompatible equations:

\[
\begin{cases}
g_1 = \lambda^{-1} \\
g_2 = \lambda^{-1} \\
g_1 = \lambda^k
\end{cases}
\]

So that \( A \) is not in the orbit of \( \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix} \) for the left action, and hence the left action is not transitive. However, one could wonder whether the orbits of the left action intersect \( CPC_I \). With the same example,

let us try to solve "\( L_g(A) \) is consistent", we get

\[
g_1 g_2 \lambda^2 = g_1 \lambda^{-k}
\]

which gives

\[
g_2 = \lambda^{-2-k}.
\]

This gives a one parameter family of solutions

\[
(g_1, g_2) = (t; \lambda^{-2-k}), \quad t \in \mathbb{R}_+^*.
\]

Generalizing this, we give:

**Theorem 3.5.** If \( I = \mathbb{N}_3 \), any orbit for the left action intersects \( CPC_3(G) \). If \( \text{card}(I) > 3 \), there exists orbits for the left action which do not intersect \( CPC_1(G) \).

**Proof.** Let \( A \in PC_3(G) \) and \( g = (g_1, g_2, g_3) \in G^3 \)

\[
L_g(A) = \begin{pmatrix}
1 & g_1 a_{1,2} & g_1 a_{1,3} \\
1 & 1 & g_2 a_{2,3} \\
1 & a_3,1 g_1^{-1} & a_3,2 g_2^{-1} \end{pmatrix}
\]

We want to find \( g \) in order to make \( L_g(A) \) consistent. We get the following relation, among others:

\[
\begin{cases}
g_1 a_{1,2} g_2 a_{2,3} = g_1 a_{1,3} \\
a_2,1 a_{1,3} = a_3,2 g_2^{-1}
\end{cases}
\]
which gives
\[ g_2 = a_2,1 a_1,3 a_{3,2} \]

This condition gives the consistent PC-matrix:
\[
L_g(A) = \begin{pmatrix}
1 & g_1 a_{1,2} & g_1 a_{1,3} \\
a_{2,1} g_1^{-1} & 1 & a_{2,1} a_{1,3} \\
a_{3,1} g_1^{-1} & a_{3,1} a_{1,2} & 1
\end{pmatrix}.
\]

Let us now consider \( A \in PC_4(G) \) and \( g = (g_1, g_2, g_3, g_4) \in G^4 \). Then
\[
L_g(A) = \begin{pmatrix}
1 & g_1 a_{1,2} & g_1 a_{1,3} & g_1 a_{1,4} \\
a_{2,1} g_1^{-1} & 1 & g_2 a_{2,3} & g_2 a_{2,4} \\
a_{3,1} g_1^{-1} & a_{3,2} g_2^{-1} & 1 & g_3 a_{3,4} \\
a_{4,1} g_1^{-1} & a_{4,2} g_2^{-1} & a_{4,2} g_3^{-1} & 1
\end{pmatrix}.
\]

We then apply the procedure given for 3 \( \times \) 3 PC matrices on the diagonal 3 \( \times \) 3 blocks. This gives, for \( i \in \{2; 3\} \):
\[ g_i = a_{i,i-1} a_{i-1,i+1} a_{i+1,i} \]

and reporting this equality in the matrix, we get
\[
L_g(A) = \begin{pmatrix}
1 & g_1 a_{1,2} & g_1 a_{1,3} & g_1 a_{1,4} \\
a_{2,1} g_1^{-1} & 1 & a_{2,1} a_{1,3} & a_{2,1} a_{1,3} a_{3,2} a_{2,4} \\
a_{3,1} g_1^{-1} & a_{3,1} a_{1,2} & 1 & a_{3,2} a_{2,4} \\
a_{4,1} g_1^{-1} & a_{4,2} a_{2,3} a_{3,1} a_{1,2} & a_{4,2} a_{2,3} & 1
\end{pmatrix}.
\]

So that, consistency now depends on the first line and the first column, and we get the relations:
\[
(3.1) \quad g_1 a_{1,2} a_{2,1} a_{1,3} a_{3,2} a_{2,4} = g_1 a_{1,4}
\]
\[
(3.2) \quad g_1 a_{1,3} a_{3,2} a_{2,4} = g_1 a_{1,4}
\]

After simplifying \( g_1 \), we gather the two lines give the same condition
\[
(3.3) \quad a_{1,4} = a_{1,3} a_{3,2} a_{2,4}.
\]

This condition is not fulfilled, unless in very special cases. For an arbitrary \( PC_I(G) \), with \( \text{card}(I) > 4 \), we extract a 4 \( \times \) 4–PC-matrix to get the same result. \( \square \)

Let us now turn to other properties inconsistency maps.

**Definition 3.6.** Let \( ii \) be an inconsistency map. It is called:
- **normalized** if \( \forall A \in PC_I(G), ||ii(A)|| \leq 1 \).
- **Ad-invariant** if \( \forall A \in PC_I(G), \forall g \in G^I, ii(Ad_g(A)) = ii(A) \).
- **norm invariant** if \( ||ii(.)|| \) is Ad-invariant.

Let \( F_I(G) \) be the quotient space for the Adjoint action of the gauge group on \( PC_I(G) \). Next result is a classical factorization theorem:

**Theorem 3.7.** An Ad-invariant inconsistency map \( ii \) factors in an unique way through the maps
\[ ii = f \circ \pi \]

where
- \( \pi : PC_I(G) \rightarrow F_I(G) \) is the quotient projection
- \( f \in V_{f_I(G)}^k \).
We give also the following easy proposition:

**Proposition 3.8.** Morphisms of groups are acting by pull-back on inconsistency indicators. Moreover, the pull-back of a normalized (resp. Ad–invariant) inconsistency map is a a normalized (resp. Ad-invariant) inconsistency map.

According to [14], we give now the following definition:

**Definition 3.9.** An inconsistency indicator $\mathbb{ii}$ on $PC_I(G)$ is a faithful, normalized inconsistency map with values in $\mathbb{R}^+$ such that there exists an inconsistency map $\mathbb{ii}_3$ on $PC_3(G)$ that defines $\mathbb{ii}$ by the following formula

$$\mathbb{ii}(A) = \sup \{ \mathbb{ii}_3(B) \mid B \subset A, B \in PC_3(G) \}.$$

We remark here that since $\mathbb{ii}$ is faithful, it is in particular (trivially) Ad–invariant on $CPC_I(G)$, but we do not require it to be Ad–invariant. Moreover, with such a definition, to show that $\mathbb{ii}$ is Ad–invariant, it is sufficient to show that $\mathbb{ii}_3$ is Ad–invariant. However, we give the example driven by Koczkodaj’s approach.

This is already proved that $K\mathbb{ii}_3$ generates an inconsistency indicator [14] and we complete this result by the following property:

**Proposition 3.10.** Let $n \geq 3$. Koczkodaj’s inconsistency maps $K\mathbb{ii}_3$ and $K\mathbb{ii}_n$ generate is Ad–invariant inconsistency maps on $PC_n(\mathbb{R}^+)$. 

**Proof.** This follows from straightforward computations of the type:

$$\frac{\lambda_1 a_{1,3} \lambda_3^{-1}}{\left(\lambda_1 a_{1,2} \lambda_2^{-1}\right) \left(\lambda_2 a_{2,3} \lambda_3^{-1}\right)} = \frac{a_{1,3}}{a_{1,2} a_{2,3}}.$$ 

\[\square\]

4. **Examples**

4.1. **$GL_n$-comparisons.** Let $S = \mathbb{R}^\infty$ be the inductive limit of the family $\{\mathbb{R}^n; n \in \mathbb{N}^\ast\}$ such that the inclusions $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ is the canonical inclusion with respect to the first coordinates. Here, $S$ is an object of the category of vector spaces. With this setting, we get $G = GL_\infty(\mathbb{R})$, which is the inductive limit of the family $\{GL_n(\mathbb{R}); n \in \mathbb{N}^\ast\}$. If $I$ is a finite set of indexes (e.g. $I = \mathbb{N}_k$ for some $k \in \mathbb{N}^\ast$), $\sup_{i \in I} \dim(s_i) < +\infty$ and setting $n = \sup_{i \in I} \dim(s_i)$, we work with the restricted setting $S = \mathbb{R}^n$ and $G = GL_n(\mathbb{R})$.

**Remark 4.1.** Even if there exists an index $i \in I$ such that $n_i < n$, we have to consider the inclusion $\mathbb{R}^{n_i} \subset \mathbb{R}^n$ because there is no linear isomorphism from $\mathbb{R}^{n_i}$ to $\mathbb{R}^n$ (by the theorem of dimension).

With this construction, we get a first family of inconsistency maps. The determinant map

$$\det : GL_n(\mathbb{R}) \to \mathbb{R}^\ast$$

is a group morphism.

**Proposition 4.2.** Let $\mathbb{ii}_{\mathbb{R}^\ast}$ be a ($\mathbb{R}^\ast$–valued) inconsistency map on $PC_I(\mathbb{R}^+)$. Then the map

$$\mathbb{ii}_{\det} : PC_I(GL_n(\mathbb{R})) \to \mathbb{R}^+$$

$$A = (a_{i,j})_{i \in I} \mapsto \mathbb{ii}_{\mathbb{R}^\ast}\left(\left(|\det(a_{i,j})|\right)_{(i,j) \in I^2}\right)$$


defined as a composed map

\[ ii_{det} = ii_R \circ |\det(.)|, \]

is a non-faithful, Ad-invariant inconsistency operator on \( PC_1(GL_n(\mathbb{R}))) \).

**Proof.** The only non trivial part is non-faithfulness. For this, let us give a counter-
example. Let \( A \in PC_3(GL_2(\mathbb{R})) \) defined by

\[ a_{1,2} = a_{2,3} = -a_{1,3} = I_2. \]

Then

\[ a_{1,2}a_{2,3} \neq a_{1,3} \]

where as

\[ |\det(A)| = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \in CPC_3(\mathbb{R}^*_+), \]

thus \( ii_{det}(A) = 0 \).

But this class of inconsistency maps is not the only one of interest, even if \( det \)
generates \( Hom(GL_n(\mathbb{R}), \mathbb{R}^*) \) in the category of groups.

4.2. **Perspective in image processing.** Let \( \omega \in \mathbb{R}^3 \setminus [abcd] \), and let \( P_\omega \) be a
(projection) plan, such that \( \omega \notin P_\omega \). In projective perspective, the projection of
\( x \in \mathbb{R}^3 \setminus \{\omega\} \), is \( x_0 \in P_\omega \) such that

\[ \{x_0\} = (\omega, x) \cap P_\omega \quad (\text{if it exists}). \]

Let us recall that projections exist in a “generic” way, that is, \( x_0 \) exists unless \((x, \omega)\)
and \( P_\omega \) are parallel.

Let us consider for simplicity, first, a tetrahedron \([abcd] \subset \mathbb{R}^3\), and let \( a_0, b_0, c_0 \)
and \( d_0 \) the corresponding projections with respect to \( \omega \). Any 2-simplex \([xyz] \) of
\([abcd] \) projects to a 2-simplex \([x_0y_0z_0] \) in \( P_\omega \). For another choice \((\omega', P_\omega')\), we get
other projections \([x_0'y_0'z_0'] \) of \([xyz] \).

**Remark 4.3.** In order to give a differential geometric flavour to this projective
geometric setting, these two choices \((\omega, P_\omega)\) and \((\omega', P_\omega')\) can be understood as two
charts on \( P_2(\mathbb{R}) \).

Now, identifying \( P_\omega \) and \( P_{\omega'} \), with the standard plan \( \mathbb{R}^2 \) by an arbitrary choice
of coordinates, we give the numbers 1,2,3 and 4 to the faces resp. \([abc] \), \([abd] \), \([acd] \)
and \([bcd] \) and, in \( \mathbb{R}^2 \), define \( a_{i,j} \) as the unique affine map \( \mathbb{R}^2 \to \mathbb{R}^2 \) which transforms
the \( i \)-th face to the \( j \)-th face for \( \omega \)-the projection, and \( a'_{i,j} \) the corresponding
coefficient for the \( \omega' \)-projection.

Let us now consider the unique affine maps \( \lambda_i \) which transforms the \( i \)-th face
form the \( \omega \)-projection to the \( \omega' \)-projection. We then have that

\[ a_{i,j} = \lambda_i^{-1} a'_{i,j} \lambda_i \]

for any index, and the PC-matrices \((a_{i,j})\) and \((a'_{i,j})\) are consistent, but the tetra-
hedron is flat if and only if

\[ \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda. \]

Thus, this leads to the following definition:
Definition 4.4. Let $G$ be the group of affine bijections of $\mathbb{R}^2$. We call perspective matrix of $[abcd]$ with respect to $(\omega, P_\omega)$ and $(\omega', P_{\omega'})$ the matrix $(b_{i,j}) \in PC_4(G)$ defined by

$$b_{i,j} = \lambda_i a_{i,j} \lambda_i^{-1},$$

which is defined up to $Ad$-action.

Then we get the trivial proposition, passing through the consistency in $PC_4(G)$:

Proposition 4.5. A matrix $(b_{i,j}) \in PC_n(G)$ encodes a triangulated simplicial complex in $\mathbb{R}^3$ if and only if it is in $CPC_n(G)$.

Thus perspective is an example of inconsistency.

5. Generalization: comparisons on a graph

We consider in this section a family of states $(s_i)_I$ such that any $s_i$ cannot be a priori compared directly with any other $s_j$. This leads us to consider a graph $\Gamma_I$ linking the elements which can be compared. For example, in the previous sections, $\Gamma_I$ was the 1−skeleton of the simplex. For simplicity, we assume that $\Gamma_I$ is a connected graph, and that at most one vertex connects any two states $s_i$ and $s_j$. We note this (oriented) vertex by $<s_i, s_j>$, and the comparison coefficient by $a_{i,j}$. By the way, we get a pairwise comparisons matrix $A$ indexed by $I$ with “holes” (with virtual 0−coefficient) when a vertex does not exist, and for which $a_{j,i}^{-1} = a_{i,j}$.

Example. Let us consider the graph $\Gamma_5$, with 5 states described figure 2.

A PC-matrix on $\Gamma_5$ is of the type:

![Figure 1. Two projections of [abcd]](image)
5.1. Hierarchyless comparisons, “hearsay” evaluation and holonomy on a graph. In this model, the comparison between two states $s_i$ and $s_j$ can be performed by any path between $s_i$ and $s_j$ of any length. This model modelize the propagation of rumours, where validation of information is based on hearsay results. With this approach, the capacity of propagation of an evaluation is not controlled. We note by $<s_1, ..., s_k>$ the composition of paths along vertices. By analogy with the holonomy of a connection, we define:

**Definition 5.1.** Let $s = s_i$ and $s' = s_j$ be two states and let

$$ H_{s,s'} = \{ a_{i_1,i_2}...a_{i_k-1,i_k} | <s,s_{i_2},...,s_{i_k-1},s_{i_k}> \text{ is a path from } s \text{ to } s' \}.$$ 

We note by $H_s$ the set $H_{s,s}$.

By the way, we get the following properties, usual for classical holonomy and with easy proof:

**Proposition 5.2.**

1. Let $s$ be a state, then $H_s$ is a subgroup of $G$. We call it holonomy group at $s$.
2. Let $s$ and $s'$ be two states. Then $H_s$ and $H_{s'}$ are conjugate subgroups of $G$.
3. $H_{s_i,s_j} = a_{i,j}H_{s_j} = H_{s_i}a_{i,j}$.

**Example.** With the graph $\Gamma_5$ of figure 2, $H_1$ is the subgroup of $G$ generated by $a_{1,4}a_{4,5}a_{5,1}$.

**Definition 5.3.** The PC matrix $A$ on the graph $\Gamma_I$ is consistent if and only if there exists a state $s$ such that $H_s = \{1\}$.

5.2. Ranking the trustworthiness of indirect comparisons. The main problem with hierarchyless comparisons of two states $s$ and $s'$ is that paths of any length give comparison coefficients which cannot be distinguished. An indirect comparison, given by a path with 3 vertices, has the same status as a comparison involving a path with 100 vertices. This is why we need to introduce a grading on $H_{s,s'}$ called order. This terminology will be justified by the propositions thereafter.
Definition 5.4. Let \( s \) and \( s' \) be two states.

- Let \( \gamma \) be a path on \( \Gamma_I \) from \( s \) to \( s' \). The length of \( \gamma \), noted by \( l(\gamma) \), is the number of vertices of \( \gamma \), and by \( H(\gamma) \) its holonomy.
- Let \( h \in H_{s,s'} \). The order of \( h \) is defined as 
  \[
  \text{ord}(h) = \min \{ l(\gamma) \mid H(\gamma) = h \}.
  \]

As a trivial consequence of the triangular equality, and as a justification of the terminology, we have:

Proposition 5.5. Let \( s, s', s'' \) three states. Let \( (h, h') \in H_{s,s'} \times H_{s', s''} \). Then
  \[
  \text{ord}(hh') \leq \text{ord}(h) + \text{ord}(h').
  \]

Left action, right action and adjoint action of \( G^I \) extend straightway to PC-matrices on \( \Gamma_I \) setting
  \[
  \forall g \in G, \quad g.0 = 0, g = 0.
  \]

Adapting the proof of Theorem 3.3 we get:

Theorem 5.6. Let \( A = (a_{i,j})_{(i,j) \in I^2} \) be a PC matrix on \( \Gamma \). Then \( A \) is consistent if and only if there exists \( (\lambda_i) \in G^I \) such that
  \[
  a_{i,j} = \lambda_i^{-1} \lambda_j
  \]
when \( a_{i,j} \neq 0 \).

5.3. Inconsistency maps ranked by trustworthiness. Let \( A \) be a PC matrix on \( \Gamma_I \). Inconsistency will be given here by the holonomy of a loop. Let us recall that a trivial holonomy of a loop \( <s_{i_1}, s_{i_2}, \ldots, s_{i_k}, s_{i_1}> \) implies that
  \[
  a_{i_1,i_k} \left(a_{i_1,i_2} \cdots a_{i_{k-1},i_k}\right)^{-1} = 1.
  \]
This relation has to be compared with formula (1.2). The principle of ranking inconsistency with loop length gives the following:

Definition 5.7. Let \( \mathcal{F} : G \to \mathbb{R}_+ \) be a map such that \( I(1) = 0 \). Let \( s \) be a basepoint on \( \Gamma_I \). The ranked Koczkodaj’s inconsistency map associated to \( \mathcal{F} \) the map
  \[
  K_{iiN} = \sum_{n \in \mathbb{N}} a_n X^n
  \]
where
  \[
  a_n = \sup \{ \mathcal{F}(H(\gamma)) \mid \gamma \text{ is a loop at } s \text{ and } l(\gamma) = n \}.
  \]
One can easily see that \( a_n \) generalize \( K_{ii_n} \), and \( K_{iiN} \) is a \( \mathbb{R}[[X]] \)-valued inconsistency map. Adapting Proposition 3.10, we get the following property:

Proposition 5.8. \( K_{ii_n} \) is an Ad-invariant inconsistency map if and only if \( \mathcal{F} \) is Ad-invariant.

6. Outlook

In [12], pairwise comparisons matrices with coefficients in \( \mathbb{R}_+^* \) were generalized to pairwise comparisons matrices with coefficients in an arbitrary Lie group \( G \) motivated by an analogy with the geometric setting of connections on a flat principal bundle. In this work,
- we have extracted key algebraic properties of these matrices. We have also improved a terminology, that we hope non misleading, and inspired by the terminology commonly used in algebra.

- We have shown how the use of a (non abelian) group $G$, (namely, $G$ is the affine group of $\mathbb{R}^2$) leads to the description of the well-known phenomena of perspective in terms of inconsistency.

- We have shown how this setting fits to indirect comparisons.

- We have developed an inconsistency map which has values in an infinite dimensional vector space, and which arises naturally from a class of classical inconsistency maps already developed.

This work shows that the applications of the settings developed here are non void. From this ground setting, we need now to develop tools of analysis for “transforming comparisons to consistent ones” when $G$ is a topological group. We feel that there exists potential applications of this setting in information theory, image processing and theoretical physics. These applications can help to improve and complete the present setting.

References

[1] Bozóki, S.; Rapcsák, T.; On Saaty’s and Koczkodaj’s inconsistencies of pairwise comparison matrices; J. Glob. Optim. 42 no.2 (2008) 157-175

[2] Brunelli, R.; Template Matching Techniques in Computer Vision: Theory and Practice Wiley (2010)

[3] Brunelli, M.; Fedrizzi, M.; Axiomatic properties of inconsistency indices for pairwise comparisons; J. Operational Research Society, 66 no1 (2015) 1-15

[4] Brunelli, M.; Studying a set of properties of inconsistency indices for pairwise comparisons; ArXiv:1507.08826

[5] Cavallo, B.; DApuzzo, L.; Squillante,M.; Pairwise Comparison Matrices over abelian Linearly Ordered Groups: A Consistency Measure and Weights for the Alternatives; in : ”Multicriteria and Multiagent Decision Making with Applications to Economics and Social Sciences” Studies in Fuzziness and Soft Computing 305 (2010) 49-64

[6] Crawford, G.; Williams, C.; The analysis of subjective judgment matrices; A project AIR FORCE report prepared for united states air force report number R-2572-1-AF (may 1985)

[7] Duszak, Z.; Koczkodaj, W.W., Generalization of a New Definition of Consistency for Pairwise Comparisons, IPL, 52(5): 273-276, 1994.

[8] Janicki, R.; Pairwise Comparisons Based Non-Numerical Ranking; Fundamenta Informaticae 94 no 2 (2009) 197-217

[9] Janicki, R.; Koczkodaj, W.W.; A weak order solution to a group ranking and consistency-driven pairwise comparisons Appl. Math. Comp. 94 (1998) 227-241

[10] Janicki, R.; Zhai, Y.; On a pairwise-based consistent non-numerical ranking; Logic Journal of IGPL 4 no 4 (2011) 1-10

[11] Koczkodaj, W.W.; A new definition of consistency of pairwise comparisons, Math. Comput. Modelling 8 (1993) pp. 79-84

[12] Koczkodaj, W.W.; Magnot, J-P.; A Geometric Framework for the Inconsistency in Pairwise Comparisons; extended version of arXiv:1601.01301 (submitted)

[13] Koczkodaj, W.W.; Szwarz, R.; Axiomatization of Inconsistency Indicators for Pairwise Comparisons, Fundamenta Informaticae, 132(4): 485-500, 2014.

[14] Koczkodaj, W.W.; Sybowski, J.; Axiomatization of Inconsistency Indicators for Pairwise Comparisons Matrices Revisited, arXiv:1509.03781

[15] Lahby, M.; Leghris, C.; Adib, A.; Network Selection Decision based on handover history in Heterogeneous Wireless Networks; International Journal of Computer Science and Telecommunications 3 no 2, (2012) 21-25

[16] Lawvere, W.; Schanuel, S.; Conceptual Mathematics: A First Introduction to Categories. Cambridge: Cambridge University (1997)
Mac Lane, S.; *Categories for the Working Mathematician* Graduate Texts in Mathematics 5 (2nd ed.), Springer-Verlag, (1998)

Saaty, T.; A scaling methods for priorities in hierarchical structures; *J. Math. Psychol.* 15 (1977) 234-281

Zhai, Y.; *Non numerical ranking based on pairwise comparisons* PhD thesis, Mc Master university, Hamilton, Ontario, Canada (2010)

Lycée Jeanne d’Arc, 30 avenue de Grande Bretagne, F-63000 Clermont-Ferrand

*E-mail address*: jean-pierre.magnot@ac-clermont.fr