Cosmological flows on hyperbolic surfaces

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Abstract We outline the geometric formulation of cosmological flows for FLRW models with scalar matter as well as certain aspects which arise in their study with methods originating from the geometric theory of dynamical systems. We briefly summarize certain results of numerical analysis which we carried out when the scalar manifold of the model is a hyperbolic surface of finite or infinite area.

Keywords: Differential Geometry, Continuous Dynamical Systems, Cosmology.

1 Introduction

The standard cosmological model of matter coupled to gravity involves an FLRW space-time with flat spatial section, with classical dynamics of matter fields reduced to spatially homogeneous configurations. The evolution of such models with respect to cosmological time is governed by generally covariant systems of ODEs which can be studied \cite{1, 2} using the theory of dynamical systems, which in this context is most powerful when formulated geometrically \cite{3}.

In most such applications up to date, the matter content of the cosmological model was taken to be relatively simple and was assumed not to carry any interesting ‘internal’ geometry. In particular, scalar inflationary models studied in the dynamical systems framework have largely been assumed to have kinetic terms described by scalar manifolds which are simply-connected...
and flat. However, arguments from string theory and supergravity suggest that the matter content present in the universe immediately after the Big Bang could be quite complicated and in particular that it may involve multiple scalar fields whose kinetic terms are described by rather general scalar manifolds, which need not be flat or simply connected. In this context, there is currently substantial interest in so-called multifield \(\alpha\)-attractor models \([4, 5]\), whose most general incarnation involves an arbitrary hyperbolic scalar manifold \([6]\). When applied to such models, the dynamical systems approach leads to highly non-trivial problems which have deep connections with several branches of mathematics, while posing interesting challenges regarding the statistical interpretation of cosmological dynamics. These aspects are spectacularly illustrated already by the case of scalar two-field models whose scalar manifold is a hyperbolic surface. Indeed, the cosmological dynamics of such models relates not only to uniformization theory, Fuchsian groups and number theory (as already pointed out\(^4\) in \([6]\)) but also, at least in certain limits, to the theory of hyperbolic and Morse-Smale flows as well as to certain aspects of ergodic theory and of non-equilibrium classical statistical mechanics. In references \([11]\) and \([12]\), we illustrated the complexity of cosmological dynamics for models whose scalar manifolds are certain hyperbolic surfaces of finite and infinite area and discussed some of the first predictions made by such models for cosmological observables.

2 Cosmological flows

The cosmological equations of an FLRW model can be reduced to equations governing the time evolution of spatially homogeneous field configurations by eliminating the conformal scale factor \(a(t)\) using the Friedmann constraint. The resulting differential equations (which involve only matter variables\(^5\)) define a dynamical system on the relevant space of states, whose flow we call the *cosmological flow* of the model. For models involving only scalar fields, these differential equations form a geometric system of coupled second order ODEs governing the evolution of scalar variables, which is described by a point moving inside the scalar manifold \(\mathcal{M}\). In such models, the cosmological flow is defined on the tangent bundle \(T\mathcal{M}\), which is the space of states of the associated dynamical system.

\(^4\)See \([7, 8, 9, 10]\) for a different but related approach.

\(^5\)The matter variables describe spatially homogeneous configurations of matter fields in the FLRW universe and hence depend only on the cosmological time.
2.1 Scalar triples and cosmological equations of motion

As explained in [6], cosmological scalar field models of FLRW type with flat and simply-connected spatial section are parameterized by the choice of a scalar triple \((\mathcal{M}, \mathcal{G}, V)\). The scalar manifold \((\mathcal{M}, \mathcal{G})\) is a generally non-compact but complete Riemannian manifold which determines the kinetic terms of the scalar fields, while the scalar potential \(V\) is a smooth and everywhere non-negative real-valued function defined on \(\mathcal{M}\). After eliminating the conformal scale factor \(a(t)\) using the Friedmann constraint, the evolution of the model is described by smooth curves \(\varphi : I \rightarrow \mathcal{M}\) (with \(I \subset \mathbb{R}\) an interval) which satisfy the following geometric second order autonomous ODE, where \(t\) is the cosmological time:

\[
\nabla_t \dot{\varphi}(t) + \left[||\dot{\varphi}(t)||^2_{\varphi(t)} + 2V(\varphi(t))\right]^{1/2} \dot{\varphi}(t) + (\text{grad} V)(\varphi(t)) = 0 .
\] (1)

Here the norm, covariant derivative and gradient are taken with respect to the scalar manifold metric \(\mathcal{G}\). The second term arising in the left hand side of (1) (known as the Hubble friction term) involves the Hubble function \(H_{\varphi}\) of the curve, which is defined through:

\[
H_{\varphi}(t) \overset{\text{def}}{=} H(\varphi(t), \dot{\varphi}(t)) ,
\] (2)

where the absolute Hubble function \(H\) of the scalar triple \((\mathcal{M}, \mathcal{G}, V)\) is defined on \(T\mathcal{M}\) through the formula:

\[
H(u) \overset{\text{def}}{=} \frac{1}{3} \left[||u||^2_{\pi(u)} + 2V(\pi(u))\right]^{1/2} = \frac{1}{3} \sqrt{2E(u)} \quad (u \in T\mathcal{M}) .
\] (3)

Here \(\pi : T\mathcal{M} \rightarrow \mathcal{M}\) is the bundle projection and \(E = \frac{9}{2} \mathcal{H}^2 \in C^\infty(\mathcal{M}, \mathbb{R})\) is the Hubble energy:

\[
E(u) \overset{\text{def}}{=} \frac{1}{2} ||u||^2_{\pi(u)} + V(\pi(u)) .
\] (4)

The Hubble energy and absolute Hubble function are continuous on \(T\mathcal{M}\) and smooth on the slit tangent bundle \(\dot{T}\mathcal{M} \overset{\text{def}}{=} T\mathcal{M} \setminus 0\) (where 0 denotes the image of the zero section of \(T\mathcal{M}\)). The following one-parameter deformation of (1):

\[
\epsilon \nabla_t \dot{\varphi}(t) + \left[||\dot{\varphi}(t)||^2_{\varphi(t)} + 2V(\varphi(t))\right]^{1/2} \dot{\varphi}(t) + (\text{grad} V)(\varphi(t)) = 0
\] (5)

interpolates between the geodesic equation \(\nabla_t \dot{\varphi}(t) = 0\) of \((\mathcal{M}, \mathcal{G})\) (which is recovered for \(\epsilon \rightarrow \infty\)) and the equation \(\left[||\dot{\varphi}(t)||^2_{\varphi(t)} + 2V(\varphi(t))\right]^{1/2} \dot{\varphi}(t) +...
\((\text{grad} V)(\varphi(t)) = 0\) (which is obtained for \(\epsilon \to 0\)). As explained in [6], the latter is equivalent with the gradient flow equation \(\frac{d\varphi(u)}{dq} + (\text{grad} V)(\varphi(q)) = 0\) of the scalar triple \((\mathcal{M}, \mathcal{G}, V)\) through the change of parameter \(t \to q\) defined through:

\[\frac{dt}{dq} = 3H\varphi(q)dq.\]  

Hence the cosmological equation of motion (which corresponds to \(\epsilon = 1\)) ‘sits between’ the geodesic and gradient flow equations. Unlike the geodesic equation (which is the Euler-Lagrange equation of the free particle Lagrangian \(L(u) \text{ def.} = \frac{1}{2}||u||^2_{\pi(u)}\)), the cosmological equation of motion need not admit a Lagrangian description (see [13] for a discussion of this point in the simple case of one-field models). Using the interpolation provided by (5), one can develop two perturbation expansions for the dynamics of the model, namely the \textit{gradient flow expansion} (whose leading approximation was already discussed in [6]) and the \textit{geodesic flow expansion}. Besides these two, there are numerous other perturbation expansions which can be considered for such models.

### 2.2 The cosmological semispray and cosmological flow

Since the second order ODE (1) is geometric, it is equivalent with the flow equation:

\[\dot{\gamma}(t) = S(\gamma(t))\quad(\text{where } \gamma : I \to T\mathcal{M})\]  

of a vector field \(S\) defined on \(T\mathcal{M}\) which satisfies the \textit{semispray condition} (see [14]):

\[\pi_u(S_u) = u, \quad \forall u \in T\mathcal{M}\]  

and which we call the \textit{cosmological semispray} of the scalar triple \((\mathcal{M}, \mathcal{G}, V)\).

More precisely, a curve \(\gamma : I \to T\mathcal{M}\) is a solution of (7) if and only if it coincides with the complete lift\(^6\) \(\tilde{\varphi}\) of a solution \(\varphi : I \to \mathcal{M}\) of (1). One can show that the following relation holds:

\[S = S_0 + 3HC + (\text{grad} V)^v,\]  

where \(S_0\) is the geodesic spray of \((\mathcal{M}, \mathcal{G})\), \(C\) is the Liouville vector field of \(T\mathcal{M}\) and \(X^v \in \mathcal{X}(T\mathcal{M})\) denotes the vertical lift of a vector field \(X \in \mathcal{X}(\mathcal{M})\). Hence the classical dynamics of the model is described by the \textit{cosmological dynamical system} defined by the vector field (9) on \(T\mathcal{M}\), whose flow we call the \textit{cosmological flow} of the scalar triple \((\mathcal{M}, \mathcal{G}, V)\). The state space \(T\mathcal{M}\) of

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\(^6\)The complete lift of a curve \(\varphi : I \to \mathcal{M}\) is the curve \(\tilde{\varphi} : I \to T\mathcal{M}\) defined through \(\tilde{\varphi}(t) \text{ def.} = (\varphi(t), \dot{\varphi}(t)) \in T\mathcal{M}\) for any \(t \in I\).
this dynamical system is the space of positions and velocities of the spatially-
homogeneous scalar field distributions described by the curve $\varphi$. When $H$
and $\text{grad} V$ are small, the cosmological semispray (9) can be viewed as a
vertical perturbation of the geodesic spray of $(\mathcal{M}, \mathcal{G})$.

The cosmological flow of a general scalar triple can be extremely complica-
ted and its proper study requires deep ideas from the theory of dynamical
systems, asymptotic analysis and singular perturbation theory. One useful
perspective on this flow is provided by scale analysis, a.k.a. by ‘renormaliza-
tion group’ techniques. In the models discussed here, one can show that
the IR limit corresponds to a reparameterized gradient flow, while the UV
limit corresponds to the geodesic flow. In particular, the regime of most
interest for inflation (which is the ‘slow motion’, i.e. the IR regime) can be
understood by studying the gradient flow and gradient flow expansion. The
UV and IR regimes of the model are markedly different, as illustrated quite
dramatically by the case $\dim \mathcal{M} = 2$.

3 Cosmological flows on hyperbolic surfaces

As mentioned above, the cosmological flow of a general scalar triple
$(\mathcal{M}, \mathcal{G}, V)$ can be extremely complicated, so it is useful to consider the case
of two-dimensional scalar manifolds. In this situation, $\mathcal{M}$ is a (generally non-
compact and not simply connected) smooth surface which we shall denote
by $\Sigma$.

The special case when $\mathcal{G} = 3\alpha G$ with $\alpha$ a positive parameter and $G$ a
complete hyperbolic metric on $\Sigma$ of Gaussian curvature $-1$ produces a gen-
eralized two-field $\alpha$-attractor model in the sense of reference [6]. Such mod-
els are particularly interesting since, under mild assumptions on the scalar
potential, they have universal behavior for certain special cosmological tra-
jectories which are close to the Freudenthal ends of $\mathcal{M}$. More precisely, it
was shown in [6] that for scalar potentials having ‘good’ asymptotic behav-
ior at the ends and in the slow-roll approximation for certain cosmological tra-
jectories $\varphi$ located close to the ends, the naive one-field truncation pro-
duces the same values of the spectral index $n_s$ and tensor to scalar ratio $r$ as
ordinary one-field $\alpha$-attractors, thus being in good agreement with current
observations:

$$n_s \approx 1 - \frac{2}{N}, \quad r \approx \frac{12\alpha}{N^2},$$

(10)

\footnote{In our context, scale analysis is applied to a classical dynamical system, unlike Wilson’s
famous application of such methods to quantum field theories.}
where $N \overset{\text{def}}{=} \int_{t_i}^{t_f} H_\varphi(t)dt$ is the number of eefolds. In fact, much stronger universality arguments can be made for such models using dynamical systems techniques.

Models based on hyperbolic surfaces are also interesting from a mathematical perspective, given their deep connection to Fuchsian groups and number theory (which stems from Poincaré’s uniformization theorem) and the special behavior of their geodesic flow. For example, it is well-known that the geodesic flow of a hyperbolic surface of finite area is ergodic and mixing.

In [11] and [12], we performed a numerical study of cosmological flows for certain non-compact hyperbolic surfaces. Reference [11] considered cosmological flows on elementary hyperbolic surfaces (namely the Poincaré disk $D$, the hyperbolic punctured disk $D^*$ and the hyperbolic annuli $A(R)$), while [12] studied the case of the hyperbolic triply-punctured sphere $Y(2)$.

As explained in [6], the equations of motion of any two-field generalized $\alpha$-attractor model can be lifted from $\Sigma$ to the Poincaré half-plane $\mathbb{H}$ by using the covering map $\pi_\mathbb{H} : \mathbb{H} \to \Sigma$ which uniformizes $(\Sigma, G)$ to $\mathbb{H}$. This allows one to determine the cosmological trajectories $\varphi(t)$ by projecting to $\Sigma$ the trajectories $\tilde{\varphi}(t)$ of a “lifted” model defined on $\mathbb{H}$, which is governed by the following system of second order non-linear ODEs:

\begin{align}
\dot{x} - \frac{2}{y} \dot{y} + \frac{1}{M_0} \left[ 3\alpha \frac{\dot{x}^2 + \dot{y}^2}{y^2} + 2\tilde{V}(x, y) \right]^{1/2} \dot{x} + \frac{1}{3\alpha} y^2 \partial_x \tilde{V}(x, y) &= 0 \\
\dot{y} + \frac{1}{y} (x^2 - y^2) + \frac{1}{M_0} \left[ 3\alpha \frac{\dot{x}^2 + \dot{y}^2}{y^2} + 2\tilde{V}(x, y) \right]^{1/2} \dot{y} + \frac{1}{3\alpha} y^2 \partial_y \tilde{V}(x, y) &= 0 .
\end{align}

Here $M_0 = \sqrt{\frac{3}{2}} M$ (where $M$ is the reduced Planck mass), while $x = \text{Re} \tau$, $y = \text{Im} \tau$ are the Cartesian coordinates on the Poincaré half plane with complex coordinate $\tau$ and $\tilde{V} \overset{\text{def}}{=} V \circ \pi_\mathbb{H} : \mathbb{H} \to \mathbb{R}$ is the lifted potential. Let $u = \pi_\mathbb{H}(\tau)$ be the complex coordinate on $\Sigma$. Let $u_0$ be any point of $\Sigma$ and let $\tau_0 \in \mathbb{H}$ be chosen such that $\pi_\mathbb{H}(\tau_0) = u_0$. An initial velocity vector $v_0 = \dot{u}_0 \in T_{u_0} \Sigma$ and its unique lift $\tilde{v}_0 = \tilde{v}_0 \in T_{\tau_0} \mathbb{H}$ through the differential of $\pi_\mathbb{H}$ at $\tau_0$ are related through:

\begin{equation}
v_0 = (\text{d} \tau_0 \pi_\mathbb{H})(\tilde{v}_0) . \tag{12}\end{equation}

Writing $\tau = x + iy$ and $\tau_0 = x_0 + iy_0$, we have $\tilde{v}_0 = \tilde{v}_{0x} + i\tilde{v}_{0y}$. A cosmological trajectory on $\Sigma$ with initial condition $(u_0, \tau_0)$ can be written as $\varphi(t) = \pi_\mathbb{H}(\tilde{\varphi}(t))$, where $\tilde{\varphi}(t) = x(t) + iy(t)$ is the solution of the lifted system (11) with initial conditions:

\begin{equation}
x(0) = x_0 \ , \ y(0) = y_0 \text{ and } \dot{x}(0) = \tilde{v}_{0x} \ , \ \dot{y}(0) = \tilde{v}_{0y} . \tag{13}\end{equation}
In the next subsections, we limit ourselves to presenting examples of trajectories on $\mathbb{D}^*$ and $\mathbb{A}(R)$ found in [11] for the corresponding globally well-behaved scalar potentials which lift to the following smooth function:

$$\hat{V}_+(\psi) = M_0 \cos^2 \frac{\psi}{2}$$

written in spherical coordinates $(\psi, \theta)$ on $S^2$. The trajectories were obtained by numerical computation of solutions (11) on the Poincaré half-plane, followed by projection to $\Sigma$ through the explicitly-known uniformization maps.

### 3.1 Cosmological trajectories on the hyperbolic punctured disk $\mathbb{D}^*$

Figure 1 shows five lifted trajectories on the Poincaré half-plane $\mathbb{H}$ and their projections to $\mathbb{D}^* = \{ u \in \mathbb{C} \mid 0 < |u| < 1 \}$ (which is endowed with its unique complete hyperbolic metric) for the globally well behaved scalar potential:

$$V_+ = M_0 \frac{1}{1 + (\log |u|)^2},$$

with the initial conditions listed in Table 1. Since $V_+$ has a minimum at the cusp end (which corresponds to the center of the disk), it produces an attractive force toward the cusp, which acts as counterbalance to the repulsive effect of the hyperbolic metric. Out of the five trajectories, the one showed in yellow produces 55 efolds, which fits the observationally favored range of 50-60 efolds.

(a) Trajectories for $\tilde{V} = \tilde{V}_+$ on $\mathbb{H}$.  
(b) Projection on $\mathbb{D}^*$ of the trajectories shown at the left.

Figure 1: Numerical solutions for $V = V_+$ and $\alpha = \frac{M_0}{3}$. 

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In all figures, we show in the background the level sets of the potential \( \tilde{V} = \tilde{V}_+ \) on \( \mathbb{H} \), respectively \( V = V_+ \) on \( \Sigma \), where dark green indicates minima of the potential and light brown indicates maxima.

Table 1: Initial conditions for five trajectories on the Poincaré half-plane.

| trajectory | \( \tau_0 \) | \( \tilde{v}_0 \) |
|------------|-------------|-------------|
| orange     | 0.3 + 0.159i | 0           |
| yellow     | 0.01 + 0.009i | 0           |
| red        | 0.1 + 0.2i   | 2 + 3i      |
| blue       | i           | 1 + i       |
| magenta    | 0.1i        | 1.3 + 7i    |

3.2 Cosmological trajectories on the annuli \( \mathbb{A}(R) \)

Figure 2 shows five trajectories (orange, yellow, red, blue and magenta) on the Poincaré half-plane \( \mathbb{H} \) and on the annulus \( \mathbb{A}(R) \) \( \text{def.} = \{ u \in \mathbb{C} \mid \frac{1}{R} < |u| < R \} \) (which is endowed with its complete hyperbolic metric of modulus \( \mu = 2 \log R \)) for \( \alpha = \frac{M_0}{\Lambda}, \ R = e \) and scalar potential:

\[
V = V_+ = M_0 \frac{1}{1 + \left[ \log \frac{R-\frac{1}{R}}{|u|} \right]^2},
\]

(16)

with the same initial conditions of Table 1. Here the potential induces an attractive force toward the inner funnel end, making some trajectories to turn at some point in \( \mathbb{A}(R) \) and evolve back toward the inner funnel end.

Figure 2: Five trajectories for \( V = V_+, \ \alpha = \frac{M_0}{\Lambda} \) and \( R = e \), with initial conditions of Table 1.
In this case the orange and yellow trajectories give 76 and 74 efolds respectively, while the remaining three trajectories do not start in the inflationary region of the chosen potential.

4 Conclusions

Cosmological models with multiple scalar fields described by general scalar triples \((\mathcal{M}, \mathcal{G}, V)\) have not been studied systematically from a global perspective. Such models lead to geometric dynamical systems defined by a certain semispray on the tangent bundle \(T\mathcal{M}\), whose flow ‘interpolates’ in an appropriate sense between the geodesic flow of \((\mathcal{M}, \mathcal{G})\) and the gradient flow of \((\mathcal{M}, \mathcal{G}, V)\). This flow becomes particularly interesting when the complete metric \(\mathcal{G}\) has negative sectional curvature. In particular, it relates to deep aspects of asymptotic analysis and ergodic theory.

When the scalar manifold is a surface \(\Sigma\) endowed with a metric of the form \(\mathcal{G} = 3\alpha G\) with \(\alpha\) a positive parameter and \(G\) a complete hyperbolic metric defined on \(\Sigma\), the associated cosmological model is a two-field generalized \(\alpha\)-attractor model in the sense of [6]. The cosmological flow of such models can already be very intricate, especially when \((\Sigma, G)\) has finite hyperbolic area. Numerical studies as well as arguments based on the gradient flow approximation indicate [11, 12, 15] that such models can be compatible with current observational constraints.

The epistemological falsifiability of scalar cosmological models is limited by the largely arbitrary choice of the scalar potential, a problem which is only compounded in multi-field models. As such, it is natural to look for criteria which could constrain the choice of \(V\). A natural way to achieve this is to require that the model admits a non-generic symmetry. In reference [16], we studied two-field cosmological \(\alpha\)-attractors with \((\Sigma, G)\) an elementary hyperbolic surface, determining those scalar potentials for which such models admit a ‘separated’ Noether symmetry. This approach can be extended to more general hyperbolic surfaces and could serve as one avenue for further constraining such models.

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