Noether symmetry for non-minimally coupled fermion fields*

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Received 12 July 2008, in final form 31 August 2008
Published 23 October 2008
Online at stacks.iop.org/CQG/25/225006

Abstract
A cosmological model where a fermion field is non-minimally coupled with the gravitational field is studied. By applying Noether symmetry the possible functions for the potential density of the fermion field and for the coupling are determined. Cosmological solutions are found showing that the non-minimally coupled fermion field behaves as an inflaton describing an inflationary scenario, whereas the minimally coupled fermion field describes a decelerated period, behaving as a standard matter field.

PACS numbers: 98.80.−k, 98.80.Cq, 95.35.+d

1. Introduction
The problems of flatness of the universe, isotropy of the cosmic microwave radiation background and of unwanted relics can be solved by considering an inflationary era where a scalar field—the so-called inflaton—is responsible for the rapid accelerated expansion of the primordial universe [1]. After the initial period, the universe goes into a decelerated era dominated by matter fields and the recent astronomical observations [2] indicate that the universe has returned to another accelerated period. The most common theories which explain the present accelerated era postulate the existence of an exotic fluid with a negative pressure—the so-called dark energy.

Nowadays, the search for models that satisfactorily explain the past and present acceleration periods is the object of intense investigation. The most popular models are those which consider a scalar field—minimally or non-minimally coupled with the gravitational field—playing the role of inflaton or dark energy fields [3]. Recently, modified gravitational theories have been exploited by considering a \( f(R) \) term into the Einstein–Hilbert action where the field equations are derived via Noether symmetry and whose cosmological solutions can describe the accelerated periods of the universe [4].

* Dedicated to Professor Luis P Chimento on the occasion of his sixtieth birthday.

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Models which consider fermion fields as sources of gravitational fields were also investigated in the literature (see, e.g., [5]). Recently, some authors analyzed the possibility that fermion fields could be responsible for accelerated regimes, playing the role of the inflaton in the primordial universe or of the dark energy in the present evolution of the universe [6].

Several works in the literature applied Noether symmetry in order to search for the forms of couplings and potential densities for scalar fields (see, e.g., [7]), but, to the best of our knowledge, none have analyzed fermion fields.

The objective of the present work is the study of a general model of a fermion field coupled to the gravitational field. The search for the possible forms for the function which is coupled to the gravitational field and for the fermion field self-interaction potential density is obtained via Noether symmetry for the point-like Lagrangian derived from the non-minimally coupled general model. This approach is very important because it allows us to select the potentials and the couplings compatible with the symmetry which implies conserved quantities. Thus the violation of conservation laws is automatically ruled out. In this way, Noether symmetry can be seen as a physical criterion working as a first principle instead of choosing forms of couplings and potentials without any fundamental justification.

The model analyzed here describes a spatially flat, homogeneous and isotropic universe composed by a fermion field. The evolution equations of the universe follow from Einstein’s field equations and Dirac’s equation, which are solved for the couplings and potential densities derived from Noether symmetry. The cosmological solutions obtained show that (i) a non-minimally coupled fermion field behaves as an inflaton and can describe an inflationary period and (ii) a minimally coupled fermion field acts as a matter field describing a decelerated phase of the universe. However, Noether symmetry imposes that accelerated solutions for the present era are not possible, i.e., the fermion field cannot behave as dark energy.

This work is organized as follows: in section 2, Einstein and Dirac equations are derived from a point-like Lagrangian in a spatially flat Friedman–Robertson–Walker metric, which is obtained from an action for a fermion field non-minimally coupled to the gravitational field. The search for the existence of Noether symmetry for the point-like Lagrangian is the subject of section 3, where the possible forms of the coupling and of the potential density are determined. In section 4, the field equations are solved for couplings and potential densities found in the previous section and the cosmological solutions are determined. The final remarks and conclusions are the subject of the last section. The signature of the metric adopted is $(+,-,-,-)$ and natural units $8\pi G = c = \hbar = 1$ are used throughout this work.

2. Field equations

The action of a model for a fermion field non-minimally coupled with the gravitational field reads

$$ S = \int \sqrt{-g} \, d^4x \left\{ F(\Psi)R + \frac{i}{2} \left( \nabla^\mu \gamma_\mu D_\nu \Psi - (D_\mu \Psi)\Gamma^\mu \gamma_\nu \right) - V(\Psi) \right\}, $$

where $\Psi$ and $\Psi^\dagger = \psi^\dagger \gamma^0$ denote the spinor field and its adjoint, respectively, the dagger represents complex conjugation and $R$ the Ricci scalar. Furthermore, $V$ is the self-interaction potential density of the fermion field and $F$ a generic function which describes the coupling of the fermion and the gravitational fields. In this work, it is supposed that both $V$ and $F$ are only functions of the bilinear $\Psi^\dagger \gamma^0 \Psi$.

For a spatially flat Friedmann–Robertson–Walker metric, one can obtain—after a partial integration of the action (1)—the point-like Lagrangian

$$ \mathcal{L} = 6a \ddot{a} F + 6a^2 \dot{a} \Psi F' + \frac{L}{2} a^3 (\nabla^0 \Psi^\dagger \gamma^0 \Psi - \Psi^\dagger \gamma^0 \dot{\Psi}) + a^3 V. $$

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In the above equation the dot refers to a time derivative whereas the prime to a derivative with respect to the bilinear $\Psi_1$, and $a$ denotes the cosmic scale factor.

From the Euler–Lagrange equation for $\bar{\psi}$ and $\psi$ applied to Lagrangian (2), it follows Dirac’s equations for the spinorial field and its adjoint which are coupled with the gravitational field, namely,

$$\dot{\psi} + \frac{3}{2} H \psi + i \gamma^0 \psi' V' - i 6 (\dot{H} + 2 H^2) \gamma^0 F' = 0,$$

$$\bar{\psi} + \frac{3}{2} H \bar{\psi} - i \bar{\psi} \gamma^0 \bar{V}' + i 6 (\dot{H} + 2 H^2) \bar{\gamma}^0 F' = 0,$$

where $H = \dot{a}/a$ denotes the Hubble parameter.

The acceleration equation follows from the Euler–Lagrange equation for $a$, applied to Lagrangian (2), yielding

$$\ddot{a} a = -\rho_f + 3 p_f,$$

By imposing that the energy function associated with the Lagrangian (2) is zero, i.e.,

$$E_L = \frac{\partial L}{\partial \dot{a}} \dot{a} + \bar{\psi} \frac{\partial L}{\partial \bar{\psi}} + \dot{\psi} \frac{\partial L}{\partial \dot{\psi}} - L = 0,$$

one can obtain Friedmann’s equation

$$H^2 = \frac{\rho_f}{6 F}.$$  

The expressions for the energy density $\rho_f$ and the pressure $p_f$ of the fermion field are given by

$$\rho_f = V - 6 HF' \bar{\psi},$$

$$p_f = [V' - 6 (\dot{H} + 2 H^2) F'] \Psi - V + 2 (F' \bar{\psi} + 2 HF' \bar{\psi} + F' \bar{\psi})^2.$$  

3. Noether symmetry

In terms of the components of the spinor field $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$ and its adjoint $\bar{\psi} = (\psi_1^\dagger, \psi_2^\dagger, -\psi_3^\dagger, -\psi_4^\dagger)^T$, the Lagrangian (2) can be written as

$$\mathcal{L} = 6 a \dot{a} F + 6 a^2 \dot{a} F' \sum_{i=1}^4 \epsilon_i (\psi_i \bar{\psi}_i + \psi_i^\dagger \bar{\psi}_i^\dagger) + \frac{1}{2} a^3 \sum_{i=1}^4 \left( \psi_i \bar{\psi}_i - \psi_i^\dagger \bar{\psi}_i^\dagger \right) + a^3 V,$$

which is only a function of $(a, \psi_i, \bar{\psi}_i, \dot{a}, \dot{\psi}_i, \dot{\bar{\psi}}_i)$.

Noether’s symmetry is satisfied by the condition

$$L_X \mathcal{L} = 0, \quad \text{i.e.,} \quad X \mathcal{L} = 0.$$  

Above, $X$ is the infinitesimal generator of the symmetry defined by

$$X = C_0 \frac{\partial}{\partial a} + C_0 a \frac{\partial}{\partial \dot{a}} + \sum_{i=1}^4 \left( C_i \frac{\partial}{\partial \psi_i} + D_i \frac{\partial}{\partial \bar{\psi}_i} + C_i^T \frac{\partial}{\partial \dot{\psi}_i} + D_i^T \frac{\partial}{\partial \dot{\bar{\psi}}_i} \right)$$  

and $L_X$ is Lie’s derivative of $\mathcal{L}$ with respect to the vector $X$ which is defined in the tangent space. Furthermore, $C_0$, $C_i$ and $D_i$ are arbitrary functions of $(a, \psi_i, \bar{\psi}_i)$.
The condition (11) when applied to the Lagrangian (10) leads to an equation which depends explicitly on \( \dot{a}^2, \dot{a} \dot{\psi}, \dot{a} \dot{\psi}_i, \dot{\psi}_i, \dot{\psi}_n, \dot{\psi}_n, \dot{\psi}_i \) and \( \dot{\psi}_i \). By equating the coefficients of the above terms to zero, one obtains the following system of coupled differential equations:

\[
C_0 F + 2a \frac{\partial C_0}{\partial a} F + a^2 F \sum_{j=1}^{4} \left( \frac{\partial C_j}{\partial \psi_j} \epsilon_j \psi_j + \frac{\partial D_j}{\partial \psi_j} \dot{\psi}_j \right) + a F \sum_{j=1}^{4} \left( C_j \dot{\psi}_j + D_j \dot{\psi}_j \right) = 0,
\]

(13)

\[
F' \left( 2C_0 + a \frac{\partial C_0}{\partial a} \right) + a F' \sum_{j=1}^{4} \left( C_j \epsilon_j \psi_j + D_j \epsilon_j \dot{\psi}_j \right) + a F' \sum_{j=1}^{4} \left( C_j \epsilon_j \psi_j + D_j \epsilon_j \dot{\psi}_j \right) = 0,
\]

(14)

\[
F' \left( 2C_0 + a \frac{\partial C_0}{\partial a} \right) + a F' \sum_{j=1}^{4} \left( C_j \epsilon_j \psi_j + D_j \epsilon_j \dot{\psi}_j \right) + a F' \sum_{j=1}^{4} \left( C_j \epsilon_j \psi_j + D_j \epsilon_j \dot{\psi}_j \right) = 0,
\]

(15)

\[
F' \left( \frac{\partial C_0}{\partial \psi_j} \epsilon_j \psi_j + \frac{\partial C_0}{\partial \psi_j} \epsilon_j \dot{\psi}_j \right) = 0, \quad F' \left( \frac{\partial C_0}{\partial \psi_j} \epsilon_j \psi_j + \frac{\partial C_0}{\partial \psi_j} \epsilon_j \dot{\psi}_j \right) = 0,
\]

(16)

\[
F' \left( \frac{\partial C_0}{\partial \psi_j} \epsilon_j \psi_j + \frac{\partial C_0}{\partial \psi_j} \epsilon_j \dot{\psi}_j \right) = 0, \quad \sum_{j=1}^{4} \left( \frac{\partial C_j}{\partial \psi_j} \psi_j - \frac{\partial D_j}{\partial \psi_j} \dot{\psi}_j \right) = 0,
\]

(17)

\[
3C_0 \psi_j + aD_j + a \sum_{i=1}^{4} \left( \frac{\partial C_i}{\partial \psi_j} \psi_i - \frac{\partial D_i}{\partial \psi_j} \dot{\psi}_j \right) = 0,
\]

(18)

\[
3C_0 \dot{\psi}_j + aC_j + a \sum_{i=1}^{4} \left( \frac{\partial C_i}{\partial \psi_j} \psi_i - \frac{\partial D_i}{\partial \psi_j} \dot{\psi}_j \right) = 0.
\]

(19)

There remains a rest equality which is used for the determination of the potential density, namely

\[
3C_0 V + a \sum_{j=1}^{4} \left( C_j \dot{\psi}_j + D_j \dot{\psi}_j \right) = 0.
\]

(20)

In equations (13)–(20) the following symbol was introduced:

\[
\epsilon_i = \begin{cases} 
+1 & \text{for } i = 1, 2, \\
-1 & \text{for } i = 3, 4.
\end{cases}
\]

In the following the coupled system of 55 differential equations (13)–(20) will be examined. First, one infers from equations (16) and (17) that one has two possibilities \( F' = 0 \) or \( F' \neq 0 \). Let us analyze the two cases separately.

3.1. Case \( F' = 0 \)

If \( F' = 0 \) it follows that \( F = \text{constant} \) and equations (16) and (17) are identically satisfied. Furthermore, from equations (14) and (15) one obtains \( C_0 = C_0(a) \) so that \( C_0 \) can be
determined from equation (13), yielding

\[ C_0 = \frac{k}{a^{1/2}}, \tag{21} \]

where \( k \) is a constant. From the remaining coupled equations (17) through (19) one can determine the other two functions of the infinitesimal generator of symmetry \( C_j \) and \( D_j \), namely

\[ C_j = -\frac{3}{2} k \frac{\psi_j^\dagger}{a^{3/2}} + \beta \epsilon_j \psi_j^\dagger \quad \text{and} \quad D_j = -\frac{3}{2} k \frac{\psi_j}{a^{3/2}} - \beta \epsilon_j \psi_j, \tag{22} \]

where \( \beta \) is a constant.

Now from the rest equality (20) one obtains that the potential density is a linear function of the bilinear \( \Psi \), i.e.,

\[ V = \lambda \Psi, \tag{23} \]

where \( \lambda \) is a constant.

### 3.2. Case \( F' \neq 0 \)

From equation (20) one can write

\[ \sum_{j=1}^{4} (C_j \epsilon_j \psi_j^\dagger + D_j \epsilon_j \psi_j^\dagger) = -3 \frac{C_0}{a} \frac{V}{V'}, \tag{24} \]

whose differentiation with respect to \( a \) furnishes

\[ \sum_{j=1}^{4} \left( \frac{\partial C_j}{\partial a} \epsilon_j \psi_j + \frac{\partial D_j}{\partial a} \epsilon_j \psi_j^\dagger \right) = 3 \left( \frac{C_0}{a^2} + \frac{1}{a} \frac{\partial C_0}{\partial a} \right) \frac{V}{V'}. \tag{25} \]

Now, the insertion of equations (24) and (25) into (13) and by recalling that \( F \) and \( V \) are only functions of \( \Psi \), leads to

\[ \frac{a}{C_0} \frac{\partial C_0}{\partial a} = \frac{V' F}{3 V F' - 2 V' F} = s, \tag{26} \]

where \( s \) is a constant.

For \( F' \neq 0 \) equations (16) and (17) imply also that \( C_0 = C_0(a) \), then one can determine \( C_0 \) from equation (26), yielding

\[ C_0 = k a^s, \tag{27} \]

with \( k \) being a constant.

From equations (17) through (19) one obtains that the functions \( C_j \) and \( D_j \) read

\[ C_j = -\frac{3}{2} k \frac{\psi_j^\dagger}{a^{3/2}} + \beta \epsilon_j \psi_j^\dagger, \quad D_j = -\frac{3}{2} k \frac{\psi_j}{a^{3/2}} - \beta \epsilon_j \psi_j. \tag{28} \]

From the rest equality (20), it follows also that the potential density is a linear function of the bilinear \( \Psi \) so that its expression is given by equation (23). Then, from equations (23) and (26) one concludes that the coupling is given by a power law \( F = \alpha \Psi^p \) where \( \alpha \) is a constant. The exponent of the power law for the coupling that follows from equations (14), (15) and (26) must satisfy the two relationships below from which one can determine \( p \) and \( s \), namely,

\[ \begin{cases} 3 p = 1 + 2 s, \\ 3 p = 2 + s, \end{cases} \]

which implies \( (s, p) = \{ (1, 1), (-1, 1/3) \} \).
Hence, when $F' \neq 0$ the admissible solutions according to Noether symmetry are:

(a)
\[
\begin{aligned}
C_0 &= k a, \\
C_j &= -\frac{3}{2} k \psi_j + \beta \epsilon_j \psi_j, \\
D_j &= -\frac{3}{2} k \psi_j - \beta \epsilon_j \psi_j, \\
F &= \alpha \Psi.
\end{aligned}
\] (30)

(b)
\[
\begin{aligned}
C_0 &= \frac{k}{a}, \\
C_j &= -\frac{3}{2} k \frac{\psi_j}{a^2} + \beta \epsilon_j \psi_j, \\
D_j &= -\frac{3}{2} k \frac{\psi_j}{a^2} - \beta \epsilon_j \psi_j, \\
F &= \alpha \Psi^{1/3}.
\end{aligned}
\] (31)

4. Cosmological solutions

Note that one cannot distinguish physically the potential density $V$ given by (23) from a massive term in the action (1), since $V$ is linear in the bilinear $\Psi$. Then one can consider $V = \lambda \Psi = m \Psi$, where $m$ is the mass of the fermion field. From now on, the coefficient of the potential density will be denoted by $m$.

Once the coupling $F$ is a known function of the bilinear $\Psi$, the search for cosmological solutions is an easy task. Indeed from Dirac’s equations (3) and (4) one can build an evolution equation for the bilinear which reads

\[
\dot{\Psi} + 3H\Psi = 0, \quad \text{so that} \quad \Psi = \frac{\Psi_0}{a^3},
\] (32)

where $\Psi_0$ is a constant.

The time evolution of the cosmic scale factor follows from the Friedmann equation (7) and will be analyzed below for the two cases described in the last section.

4.1. Case $F' = 0$

The choice $F = \text{constant} = 1/2$ refers to a minimally coupling of the fermion and gravitational fields that follows from the normalization of the action (1) in terms of natural units. Hence, Friedmann’s equation (7) furnishes that the time evolution of the cosmic scale factor reads

\[
a(t) = \left[ K(t - t_0) \right]^{2/3}, \quad \text{where} \quad K = \frac{3}{2} \sqrt{\frac{m \Psi_0}{3}}.
\] (33)

One can infer from equation (33) that it describes a decelerated universe dominated by a matter field.

The energy density and pressure of the fermion field follows from (8) and (9), yielding

\[
\rho_f = \frac{m \Psi_0}{a^3}, \quad p_f = 0.
\] (34)

Therefore, in this case the fermion field behaves as a standard pressureless matter field.

4.2. Case $F' \neq 0$

Let us begin with the case (a) where $F = \alpha \Psi$ so that the Friedmann equation (7) becomes

\[
\frac{da}{a} = \sqrt{-\frac{m}{12 \alpha}} dt.
\] (35)

The above equation shows an exponential behavior of the cosmic scale factor which can describe an inflationary era. Hence, the fermion field can be identified with the inflaton and
the cosmic scale factor is given by

$$a(t) = \exp[K(t - t_0)], \quad \text{where} \quad K = \sqrt{-\frac{m}{12\alpha}}.$$  \tag{36}

Here the energy density and the pressure of the fermion field read

$$\rho_f = -\frac{m\Psi_0}{2a^3}, \quad p_f = -\rho_f.$$  \tag{37}

By evoking the weak-energy condition which dictates that the energy density is a non-negative quantity, i.e., \(\rho_f \geq 0\), one infers from equation (37) that \(\Psi_0 < 0\). This last condition imposes that \(\alpha < 0\), since the coupling \(F = \alpha\Psi\) must be a positive quantity. Note also that the condition \(\alpha < 0\) implies that \(K > 0\). Furthermore, from equation (37) one concludes that the pressure of the fermion field is always negative and proportional to its energy density.

In this scenario the time evolution of the bilinear is given by

$$\Psi(t) = \Psi_0 \exp[-3K(t - t_0)]$$  \tag{38}

thanks to equation (32). Moreover, the time evolution of the energy density and pressure of the fermion field read

$$\rho_f(t) = 6\alpha\Psi_0K^2 \exp[-3K(t - t_0)] = -p_f(t).$$  \tag{39}

Although equation (36) predicts an eternal accelerated expansion for the universe, equations (38) and (39) show that the source of the accelerated expansion should come to an end, since the potential density and the energy density of the fermion field tend to zero at a finite time.

For the case (b) where \(F = \alpha\Psi^{1/3}\) the Friedmann equation (7) does not have a solution.

5. Final remarks and conclusions

In this work a classical spinor field was considered which is understood as a set of complex-valued spacetime functions that transform according to the Lorentz group. More details about classical spinors can be found in the work by Armendáriz-Picón and Greene in [5]. About such a consideration one has to observe that: (i) the spinorial field can be treated classically if its state is close to the vacuum and (ii) the expectation value of a spinorial field in a physical state is not a Grassmannian number but a complex number.

In the literature (see, e.g., [5, 6]) varied forms of the potential density and coupling for fermion fields were proposed in order to describe cosmological models with accelerated and decelerated periods of the universe. The results obtained in the present work show that the functions for the potential density and the coupling are very restrictive, once Noether symmetry is satisfied.

Furthermore, it was also shown that: (i) the non-minimally coupling of the fermion and gravitational fields leads to an accelerated expansion describing an inflationary era where the fermion field behaves as an inflaton and (ii) a minimally coupled fermion field implies a decelerated regime where the fermion field acts as a standard matter field.

Acknowledgments

The authors acknowledge support from CNPq (Brazil).
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