Type-Decomposition of a Synaptic Algebra

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Abstract

A synaptic algebra is a generalization of the self-adjoint part of a von Neumann algebra. In this article we extend to synaptic algebras the type-I/II/III decomposition of von Neumann algebras, AW*-algebras, and JW-algebras.

1 Introduction

Soon after laying rigorous Hilbert-space based foundations for quantum mechanics in his celebrated book [22], John von Neumann wrote in an unpublished letter to Garrett Birkhoff,

“I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space any more.”

As is authoritatively documented in [25], by the time this letter was written (1935), von Neumann had begun to focus on what is now called a type II1 factor as the appropriate mathematical basis for quantum mechanics. Later von Neumann’s advocacy of type II1 factors was supplemented by the discovery that type III factors occur naturally in relativistic quantum field theory [15].

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In this article we are going to study the type I/II/III decomposition theory for a so-called synaptic algebra, which is a proper generalization of the self-adjoint part of a von Neumann algebra. We believe that our work casts considerable light on just what makes a type I/II/III decomposition work, not only in von Neumann algebras, but in many related algebraic structures as well. We note that a synaptic algebra can host the probability measures that were a main concern of von Neumann [25 §2] and it can serve as a value algebra for quantum-mechanical observables.

A synaptic algebra (from the Greek sunaptein, meaning to join together) [5, 8, 13, 24] unites the notions of an order-unit normed space [1, p. 69], a special Jordan algebra [21], a convex effect algebra [14], and an orthomodular lattice [2, 19]. The generalized Hermitian algebras, introduced and studied in [7, 11], are special cases of synaptic algebras, and numerous additional examples can be found in the papers cited above.

The JW-algebras of D. Topping [27] are important special cases of synaptic algebras and they will motivate much of our work in this article. One of the significant ways in which synaptic algebras are more general than JW-algebras is that, whereas the orthomodular lattice (OML) of projections in a JW-algebra is complete [27 Theorem 4], the OML of projections in a synaptic algebra need not be complete.

Our purpose in this article is twofold: First Project: To show that a synaptic algebra with a complete OML of projections has sufficiently many properties in common with a JW-algebra to enable Topping’s proof of his version of a type-I/II/III decomposition theorem [27 Theorem 13]. Second Project: To show how the type-decomposition theory developed in [10] applies to a synaptic algebra with a projection lattice satisfying the much weaker central orthocompleteness condition. For both projects, our main tool will be the notion of a type determining (TD) subset of the projection lattice (Section 6 below).

2 Some basic properties of a synaptic algebra

To carry out our two type-decomposition projects, we shall need only a portion of the theory of synaptic algebras as developed in [5, 8, 13, 24], and as a convenience for the reader, we devote this and the next two sections to a sketch of some of the parts of this theory that we shall require and to some of the corresponding notation and nomenclature. We use the notation := for
“equals by definition” and “iff” abbreviates “if and only if.”

2.1 Standing Assumption. In this article we assume that $A$ is a synaptic algebra with enveloping algebra $R$ [3, Definition 1.1].

To help fix ideas, the reader might want to keep in mind the case in which $R$ is a von Neumann algebra and $A$ is the self-adjoint part of $R$. Some of the important properties of $A$ and $R$ are as follows:

- $R$ is a real or complex linear associative algebra with unit element 1 and $A$ is a real vector subspace of $R$. To avoid trivialities, we shall assume that $0 \neq 1$.
- $A$ is a partially ordered real vector space under $\leq$ and $0 \leq 1 \in A$.
- Let $a, b \in A$. Then the product $ab$ as calculated in $R$ may or may not belong to $A$. We write $aCb$ iff $a$ and $b$ commute (i.e. $ab = ba$) and we define $C(a) := \{b \in A : aCb\}$. If $B \subseteq A$, then $C(B) := \bigcap_{b \in B} C(b)$.
- If $a \in A$, then $0 \leq a^2 \in A$. Thus, $A$ is a special Jordan algebra under the Jordan product $a \circ b := \frac{1}{2}((a + b)^2 - a^2 - b^2) = \frac{1}{2}(ab + ba) \in A$ for all $a, b \in A$. Therefore, if $a, b \in A$, then $ab + ba = 2(a \circ b) \in A$ and $aCb \Rightarrow ab = ba = a \circ b = b \circ a \in A$. Also, if $ab = 0$, then $aCb$ and $ba = 0$.
- With the operations and partial order inherited from $A$, the set $C(A)$, called the center of $A$, is a synaptic algebra with unit element 1. As such, it is a commutative associative partially-ordered normed real linear algebra and it is its own enveloping algebra. We call $A$ a commutative synaptic algebra iff $A = C(A)$.
- An element $p \in A$ is called a projection iff $p^2 = p$, and the set of all projections in $A$ is denoted by $P$. Under the partial order inherited from $A$, $P$ is an orthomodular lattice (OML) [2, 19] with smallest element 0, largest element 1, and $p \mapsto p^\perp := 1 - p$ as the orthocomplementation.
- If $p \in P$, then, with the operations and partial order inherited from $A$, $pAp := \{pap : a \in A\}$ is a synaptic algebra with $pRp$ as its enveloping algebra and with $p$ as its unit element. The OML of projections in $pAp$ is $pAp \cap P = \{q \in P : q = qpq\} = \{q \in P : q = qp\} = \{q \in P : q \leq p\}$.

An arbitrary cartesian product of synaptic algebras is again a synaptic algebra with coordinatewise operations and relations and with the cartesian product of the enveloping algebras of the factors as its enveloping algebra.
3 Orthomodular lattices

In this section we review some facts about orthomodular lattices (OMLs) that we shall need in our study of the orthomodular projection lattice \( P \) of the synaptic algebra \( A \). More details in regard to OMLs can be found in [2, 19].

3.1 Standing Assumption. In this section, we assume that \( L \) is an OML with smallest element \( 0 \), largest element \( 1 \), and \( p \mapsto p^\perp \) as its orthocomplementation.

Let \( p, q \in L \). We say that \( q \) dominates \( p \), or equivalently, that \( p \) is a subelement of \( q \) iff \( p \leq q \). If \( p \leq q \) and \( p \neq q \), we write \( p < q \). As usual, \( p \lor q \) and \( p \land q \) will denote the supremum (least upper bound) and the infimum (greatest lower bound), respectively, of \( p \) and \( q \) in \( L \). The two elements \( p, q \in L \) are said to be orthogonal, in symbols \( p \perp q \), iff \( p \leq q^\perp \).

The \( p \)-interval in \( L \), defined and denoted by \( L[0, p] := \{ q \in L : q \leq p \} \) is a sublattice of \( L \) and it is an OML in its own right with \( q \mapsto q^\perp_p := p \land q^\perp \) as its orthocomplementation. If \( Q \subseteq L[0, p] \), then \( Q \) has a supremum in \( L \) iff it has a supremum in \( L[0, p] \), and the two suprema, if they exist, coincide. Likewise for infima, provided that \( Q \) is not empty. Therefore, if the OML \( L \) is complete (i.e., every subset of \( L \) has a supremum and an infimum in \( L \)) then the OML \( L[0, p] \) is also complete.

The elements \( p, q \in L \) are called (Mackey) compatible iff there are elements \( p_1, q_1, d \in L \) such that \( p_1 \perp q_1 \), \( p_1 \perp d \), \( p_2 \perp d \), \( p = p_1 \lor p_2 \), and \( q = q_1 \lor q_2 \). For instance, if \( p \leq q \), or if \( p \perp q \), then \( p \) and \( q \) are compatible. The elements \( p \) and \( q \) are compatible iff \( p \) can be written as \( p = x \lor y \) with \( x \in L[0, q] \) and \( y \in L[q, q^\perp] \). The set of all elements in \( L \) that are compatible with every element in \( L \) is called the center of \( L \). The center of \( L \) is a sublattice of \( L \), closed under orthocomplementation, and as such it is a boolean algebra (a complemented distributive lattice). Computations in \( L \) are facilitated by the fact that, if one of the elements \( p, q, r \in L \) is compatible with the other two, then the distributive relations \( p \lor (q \land r) = (p \lor q) \land (p \lor r) \) and \( p \land (q \lor r) = (p \land q) \lor (p \land r) \) hold [4].

If \( c \) belongs to the center of \( L \) and \( p \in L \), then \( p \land c \) belongs to the center of \( L[0, p] \). If, conversely, for every \( p \in L \), every element in the center of \( L[0, p] \) has the form \( p \land c \) for some \( c \) in the center of \( L \), then \( L \) is said to have the relative center property [3].
3.2 Remark. The OML $L$ can be regarded as a lattice effect algebra \cite{6,26} by defining the orthosum $p \oplus q$ for $p, q \in L$ iff $p \perp q$, in which case $p \oplus q := p \lor q$. Then the partial order on $L$ coincides with the effect-algebra partial order, the orthocomplementation on $L$ coincides with the effect-algebra orthosupplementation, and the structure of $L$ as an effect algebra determines its structure as an OML. In this way the theory of effect algebras \cite{6,9,10,12,14,18,5,26} can be applied to $L$.

A family $(p_i)_{i \in I} \subseteq L$ in $L$ is said to be pairwise orthogonal iff, for all $i, j \in I$, $i \neq j \Rightarrow p_i \perp p_j$. Regarding $L$ as an effect algebra and using standard effect-algebra terminology (e.g., \cite{9} p. 286), we have the following: A family in $L$ is orthogonal iff it is pairwise orthogonal, such an orthogonal family is orthosummable iff it has a supremum in $L$, and if the family is orthosummable, then its orthosum is its supremum.

If every orthogonal family in an effect algebra is orthosummable, then the effect algebra is called orthocomplete \cite{18}. By a theorem of S. Holland \cite{17}, $L$ is orthocomplete as an effect algebra iff it is complete as a lattice.

The OML $L$ is said to be modular iff, for all $p, q, r \in L$, $p \leq r \Rightarrow p \lor (q \land r) = (p \lor q) \land r$; it is called locally modular \cite{27} p. 28\] iff, for every nonzero central element $c \in L$, there is a nonzero $p \in L[0,c]$ such that $L[0,p]$ is modular.

If $p, q \in L$, $p \lor q = 1$, and $p \land q = 0$, then $p$ and $q$ are called complements of each other in $L$. For instance, $p$ and $p^\perp$ are complements in $L$. Two elements of $L$ that share a common complement are said to be perspective. If $p$ and $q$ are perspective in the OML $L[0,p \lor q]$, then $p$ and $q$ are called strongly perspective. The transitive closure of the relation of perspectivity is an equivalence relation on $L$ called projectivity; thus $p$ and $q$ are projective iff there is a finite sequence $e_1, e_2, ..., e_n \in L$ such that $p = e_1$, $q = e_n$, and $e_i$ is perspective to $e_{i+1}$ for $i = 1, 2, ..., n - 1$.

3.3 Lemma. Let $r \in L$ and $p, q \in L[0,r]$. Then: (i) If $p$ and $q$ are perspective in $L[0,r]$, then $p$ and $q$ are perspective in $L$. (ii) If $p$ and $q$ are strongly perspective in $L$, then they are perspective in $L$. (iii) If $p$ and $q$ are strongly perspective in $L$, then they are strongly perspective in $L[0,r]$.

Proof. (i) Suppose there exists $x \in L[0,r]$ such that $p \lor x = q \lor x = r$ and $p \land x = q \land x = 0$ and put $y := x \lor r^\perp$. Then $p \lor y = p \lor (x \lor r^\perp) = r \lor r^\perp = 1$, and since $p, x \leq r$, $p \land y = p \land (x \lor r^\perp) = (p \land x) \lor (p \land r^\perp) = 0 \lor 0 = 0$. Likewise $q \lor y = 1$ and $q \land y = 0$. 

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Part (ii) follows from (i) with 
\[ r := p \lor q. \]

(iii) If \( p \) and \( q \) are strongly perspective in \( L \), then they are perspective in \( L[0, p \lor q] = (L[0, r])[0, p \lor q] \), whence they are strongly perspective in \( L[0, r]. \)

3.4 Theorem. The following conditions are mutually equivalent:

(i) \( L \) is modular.

(ii) If \( p, q \in L \) are perspective, then \( p \) and \( q \) are strongly perspective.

(iii) If \( p, q \in L \), \( p \leq q \), and \( p \) is perspective to \( q \), then \( p = q \).

Proof. That (i) \( \Leftrightarrow \) (ii) follows from [16, Theorem 2] and the equivalence (i) \( \Leftrightarrow \) (iii) follows from [27, Lemma 20].

3.5 Theorem. Suppose that \( L \) is both complete and modular. Then:

(i) \( L \) is a continuous geometry [23].

(ii) Perspectivity is transitive on \( L \), i.e., perspectivity coincides with projectivity.

(iii) If \( p, q \in L \), \( p \leq q \), and \( p \) and \( q \) are projective in \( L \), then \( p = q \).

(iv) Any orthogonal family of nonzero elements in \( L \) such that any two of the elements in the family are projective is necessarily finite.

Proof. Part (i) is a classic result of I. Kaplansky [20], (ii) is [23, Theorem 5.16], (iii) is [23, Theorem 4.4], and (iv) is [23, Theorem 3.8].

4 The orthomodular lattice of projections

Owing to the fact that \( P \subseteq A \), the OML \( P \) of projections in \( A \) acquires several special properties, among which are the following:

For all \( p, q \in P \): (i) \( p \leq q \Leftrightarrow pq = p \Leftrightarrow qp = p \Leftrightarrow p = qpq \Leftrightarrow p = pqp \). (ii) If \( p \leq q \), then \( q - p = q \land p^\perp = qp^\perp = p^\perp q \). (iii) If \( pCq \), then \( p \land q = pq = qp \) and \( p \lor q = p + q = pq \). (iv) \( p \perp q \) if and only if \( p + q \leq 1 \) iff \( p + q = p \lor q \) iff \( pq = 0 \). (v) \( p \) and \( q \) are compatible iff \( pCq \). (vi) \( C(A) = C(P) \) [8, p. 242]. (vii) The center of the OML \( P \) is \( P \cap C(P) = P \cap C(A) \) and it coincides with the boolean algebra of projections in the center \( C(A) \) of \( A \). (viii) A projection \( c \in P \) is
central, i.e., it belongs to the center $P \cap C(A) = P \cap C(P)$ of $P$, iff $P = P[0, c] + P[0, c^+] := \{x + y : x \in P[0, c], y \in P[0, c^+]\}$. (ix) If $d \in P \cap C(A)$, then $pd = p \land d$ belongs to the center of $P[0, p]$. (x) If $c \in P \cap C(A)$, then the center of $P[0, c]$ is $\{cd : d \in P \cap C(A)\} = (P \cap C(A))[0, c]$. (xi) If $P$ is complete, then it has the relative center property [13] Theorem 8.7; hence the center of $P[0, p]$ is $\{pd : d \in P \cap C(A)\}$. (xii) The $p$-interval $P[0, p]$ is the OML of projections in the synaptic algebra $pAp$.

If $p_1, p_2, \ldots, p_n$ is a finite orthogonal sequence in $P$, then we refer to $p_1 + p_2 + \cdots + p_n = p_1 \lor p_2 \lor \cdots \lor p_n$ as an orthogonal sum.

4.1 Definition. The family $(p_i)_{i \in I} \subseteq P$ is called centrally orthogonal iff there is an orthogonal family $(c_i)_{i \in I} \subseteq P \cap C(A)$ of projections in the center $C(A) = C(P)$ of $A$ such that $p_i \leq c_i$ for every $i \in I$. We say that $P$ is centrally orthocomplete iff every centrally orthogonal family in $P$ has a supremum in $P$.

Clearly, every centrally orthogonal family is orthogonal, and if $P$ is complete, then it is centrally orthocomplete. If $P$ is centrally orthocomplete, then the center $P \cap C(P) = P \cap C(A)$ is a complete boolean algebra; moreover, for each $a \in A$, there is a smallest central projection $c \in P \cap C(A)$ such that $a = ac$ [13] Lemma 6.5 and Definition 6.6].

4.2 Definition. Suppose that $P$ is centrally orthocomplete. For each $a \in A$, the smallest central projection $c \in P \cap C(A)$ such that $a = ac$ is called the central cover of $a$ and denoted by $\gamma a$.

Thus, if $P$ is centrally orthocomplete, $a \in A$, and $c \in P \cap C(A)$, then $a = ac \iff \gamma a \leq c$. The restriction of the central cover mapping $\gamma : A \to P \cap C(A)$ to $P$ is order preserving, it preserves arbitrary existing suprema in $P$, and if $p \in P$ and $c \in P \cap C(A)$, then $\gamma(p \land c) = \gamma p \land c$ [13] Theorem 6.7.

We note that, if $P$ is centrally orthocomplete, then a family $(p_i)_{i \in I} \subseteq P$ is centrally orthogonal iff the family $(\gamma p_i)_{i \in I}$ of central covers is orthogonal, and it follows that $(p_i)_{i \in I}$ is centrally orthogonal iff it is pairwise centrally orthogonal in the sense that, for $i, j \in I$, $i \neq j$ implies that the pair consisting of $p_i$ and $p_j$ is centrally orthogonal. The following lemma and theorem address the issue of how these notions relativize to an interval $P[0, p]$.

4.3 Lemma. Let $p \in P$, let $(p_i)_{i \in I}$ be a family of projections in $P[0, p]$, and suppose that $(p_i)_{i \in I}$ is centrally orthogonal in $P$. Then $(p_i)_{i \in I}$ is centrally orthogonal in $P[0, p]$. 


Proof. By hypothesis, there exists a pairwise orthogonal family \((c_i)_{i \in I}\) of central projections in \(P\) such that \(p_i \leq c_i\) for all \(i \in I\). Then \((pc_i)_{i \in I}\) is a pairwise orthogonal family of central projections in \(P[0,p]\) and \(p_i \leq pc_i\) for all \(i \in I\). \(\square\)

4.4 Theorem. Suppose that \(P\) is centrally orthocomplete, let \(p \in P\), let \((p_i)_{i \in I}\) be a family of projections in \(P[0,p]\), and suppose that at least one of the following conditions holds: \(P\) is complete or \(p \in P \cap C(A)\). Then: (i) The family \((p_i)_{i \in I}\) is centrally orthogonal in \(P[0,p]\) iff it is centrally orthogonal in \(P\). (ii) \(P[0,p]\) is centrally orthocomplete.

Proof. Assume the hypotheses. If \(P\) is complete, then it has the relative center property, so the center of \(P[0,p]\) is \(\{pc : c \in P \cap C(A)\}\), and the same conclusion holds if \(p \in P \cap C(A)\).

(i) Suppose that \((p_i)_{i \in I}\) is centrally orthogonal in \(P[0,p]\). Then there exists a family \((c_i)_{i \in I}\) of central projections in \(P\) such that \(pc_i \perp pc_j\) for \(i, j \in I\) with \(i \neq j\) and \(p_i \leq pc_i\) for all \(i \in I\). Then, for \(i \neq j\), \(p_i \leq pc_i \leq c_j^\perp c_i \in P \cap C(A)\) and \(p_j \leq pc_j \leq c_i^\perp c_j \in P \cap C(A)\) with \(c_j^\perp c_i \perp c_j^\perp c_j\); hence \((p_i)_{i \in I}\) is pairwise centrally orthogonal in \(P\). Since \(P\) is centrally orthocomplete, \((p_i)_{i \in I}\) is centrally orthogonal in \(P\). The converse follows from Lemma 4.3, and (i) is proved. Since \(P\) is centrally orthocomplete, (ii) follows from (i). \(\square\)

Let \(c_1, c_2, \ldots, c_n \in P \cap C(A)\) be a finite sequence of central projections with \(c_i \perp c_j\) for \(i \neq j\) and \(c_1 + c_2 + \cdots + c_n = 1\). Then \(P\) is the (internal) direct sum of the OMLs \(P[0,c_i]\), in symbols

\[
P = P[0,c_1] \oplus P[0,c_2] \oplus \cdots \oplus P[0,c_n],
\]

in the sense that (1) every projection \(p \in P\) can be written uniquely as an orthogonal sum \(p = p_1 + p_2 + \cdots + p_n = p_1 \lor p_2 \lor \cdots \lor p_n\) with \(p_i \in P[0,c_i]\) for \(i = 1, 2, \ldots, n\) and (2) all operations and relations for \(P\) can be computed “coordinatewise” in the obvious sense. This direct sum decomposition of \(P\) is reflected by a corresponding direct sum decomposition \(A = c_1 A \oplus c_2 A \oplus \cdots \oplus c_n A\) of the synaptic algebra \(A\) into the direct summands \(c_i A = c_i Ac_i = Ac_i\), where again every \(a \in A\) can be written uniquely as \(a = a_1 + a_2 + \cdots + a_n\) with \(a_i \in c_i A\) for \(i = 1, 2, \ldots, n\) and all synaptic operations and relations can be computed “coordinatewise.” In this case, \(P\) is isomorphic as an OML to the cartesian product \(P[0,c_1] \times P[0,c_2] \times \cdots \times P[0,c_n]\) and \(A\) is isomorphic as a synaptic algebra to \(c_1 A \times c_2 A \times \cdots \times c_n A\).
We note that, if \( c \in P \cap C(A) \), then \( P = P[0,c] \oplus P[0,c^\perp] \) and \( A = cA \oplus c^\perp A \). Thus, the direct summands of \( P \) (respectively, of \( A \)) are of the form \( P[0,c] \) (respectively, \( cA \)) for central projections \( c \in P \cap C(A) \).

The OML \( P \), is called \textit{irreducible}, and the synaptic algebra is said to be a \textit{factor}, iff \( P \cap C(A) = \{0,1\} \). Thus \( A \) is a factor iff it admits no nontrivial direct-sum decomposition. It can be shown that \( A \) is a factor iff the center \( C(A) \) is the set of all real multiples of the unit element 1.

By regarding \( P \) as an effect algebra, we obtain the following.

4.5 \textbf{Theorem} (\cite[Theorem 6.14]{13}). \textit{Suppose that \( P \) is centrally orthocomplete, let \( (p_i)_{i \in I} \) be a centrally orthogonal family in \( P \) with \( p := \bigvee_{i \in I} p_i \), and let \( X \) be the cartesian product \( X := \prod_{i \in I} P[0,p_i] \) organized into an OML with coordinatewise operations and relations. Define the mapping \( \Phi: X \to P[0,p] \) by \( \Phi((e_i)_{i \in I}) := \bigvee_{i \in I} e_i \) for every \( (e_i)_{i \in I} \in X \). Then \( \Phi \) is an OML-isomorphism of \( X \) onto \( P[0,p] \) and for \( q \in P[0,p] \), \( \Phi^{-1}(q) = (q \land \gamma p_i)_{i \in I} = (q \land p_i)_{i \in I} \).}

5 \textbf{Symmetries and equivalence of projections}

By a \textit{symmetry} in \( A \) we mean an element \( s \in A \) such that \( s^2 = 1 \) [13]. Two projections \( p, q \in P \) are said to be \textit{exchanged} by a symmetry \( s \in A \) iff \( sps = q \), or equivalently, iff \( sqs = p \). We note that \( p \) and \( q \) are exchanged by a symmetry \( s \in A \) iff \( p^\perp = 1 - p \) and \( q^\perp = 1 - q \) are exchanged by \( s \).

5.1 \textbf{Theorem}. \textit{Let \( p, q \in P \). Then: (i) If \( p \) and \( q \) are exchanged by a symmetry in \( A \), then \( p \) and \( q \) are strongly perspective in \( P \). (ii) If \( p \) and \( q \) are strongly perspective in \( P \), then \( p \) and \( q \) are perspective in \( P \). (iii) If \( p \) and \( q \) are perspective in \( P \), then there are symmetries \( s, t \in A \) such that \( tspst = q \).}

\textit{Proof.} Part (i) follows from [13, Theorem 5.11], (ii) is a consequence of Lemma 3.3 (ii), and (iii) follows from [13, Theorem 5.12 (i)]. \hspace{1cm} \Box

5.2 \textbf{Corollary}. \textit{Let \( p, q \in P \) with \( p \perp q \). Then the following conditions are mutually equivalent: (i) There are symmetries \( s, t \in A \) such that \( tspst = q \). (ii) \( p \) and \( q \) are exchanged by a symmetry in \( A \). (iii) \( p \) and \( q \) are strongly perspective in \( P \). (iv) \( p \) and \( q \) are perspective in \( P \).}

\textit{Proof.} That (i) \( \Rightarrow \) (ii) follows from [13, Theorem 5.12 (ii)] and (ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (iv) \( \Rightarrow \) (i) follows from Theorem 5.1. \hspace{1cm} \Box
5.3 Lemma. Let \( r \in P \), let \( p, q \in P[0, r] \). Then \( p \) and \( q \) are exchanged by a symmetry in \( A \) iff \( p \) and \( q \) are exchanged by a symmetry in \( rAr \).

**Proof.** Let \( p, q \in P[0, r] \), let \( s \) be a symmetry in \( A \) with \( sps = q \), and put \( t := (p \lor q)s(p \lor q) \). Clearly, \( tpt = q, tgt = p \) and \( rt = tr = t \). Also, since \( s(p \lor q)s = sps \lor sqs = q \lor p = p \lor q \), it follows that \( t^2 = p \lor q \), whence \( t^3 = (p \lor q)t = t \). Put \( s_1 := t + r - t^2 \). Then \( s_1 = rs_1r \in rAr \) and \( s_1^2 = r \), whence \( s_1 \) is a symmetry in \( rAr \). Moreover, as \( rp = pr = p \) and \( t^2p = pt^2 = p \), we have \( s_1ps_1 = tpt = q \). Conversely, by a straightforward calculation, if \( p \) and \( q \) are exchanged by a symmetry \( u \) in \( rAr \), then \( s := u + 1 - r \) is a symmetry in \( A \) that exchanges \( p \) and \( q \).

5.4 Theorem (Generalized Comparability). Suppose that \( P \) is complete and let \( e, f \in P \). Then there exists a symmetry \( s \in S \) and a central projection \( c \in P \cap C(A) \) such that \( secs \leq fc, \ sfc^\perp s \leq ec^\perp, \ se^\perp c^\perp s \leq f^\perp c^\perp \), and \( sf^\perp cs \leq e^\perp c \).

**Proof.** Since \( P \) is complete, [13, Theorem 8.6] applies, so there exists \( c \in P \cap C(A) \) and a symmetry \( s \in S \) such that \( secs \leq fc \) and \( sfc^\perp s \leq ec^\perp \). Thus, \( fc^\perp \leq sec^\perp s \), so \( se^\perp c^\perp s = s(c^\perp - ec^\perp)s = c^\perp - sec^\perp s \leq c^\perp - fc^\perp = f^\perp c^\perp \). By a similar computation, \( sf^\perp cs \leq e^\perp c \).

If \( x = s_1s_{n-1} \cdots s_1 \in R \) is a finite product of symmetries \( s_n, s_{n-1}, \ldots, s_1 \in A \) then we define \( x^* := s_1s_2 \cdots s_n \in R \) to be the product of the same symmetries, but in the reverse order. We note that \( xx^* = x^*x = 1 \) and, for any \( a \in A \), \( xax^* \in A \).

Let \( p, q \in P \). Then by definition, \( p \) and \( q \) are equivalent, in symbols, \( p \sim q \), if there is a finite sequence of projections \( e_1, e_2, \ldots, e_n \in P \) such that \( p = e_1, q = e_n \), and for each \( i = 1, 2, \ldots, n - 1 \), the projections \( e_i \) and \( e_{i+1} \) are exchanged by a symmetry \( s_i \in A \). Clearly, \( p \sim q \) iff there is a finite product \( x \in R \) of symmetries in \( A \) such that \( q = xpx^* \).

5.5 Lemma. If \( p, q \in P \), then \( p \sim q \) iff \( p \) and \( q \) are projective in \( P \).

**Proof.** As a consequence of parts (i) and (ii) of Theorem 5.4 if \( p \sim q \), then \( p \) and \( q \) are projective, and the converse follows from Theorem 5.4 (iii).

5.6 Lemma. Let \( p, q \in P \) and let \( x \) be a finite product of symmetries in \( A \). Then: (i) For \( a \in A \), the mapping \( a \mapsto xax^* \) is a linear, order, and Jordan automorphism of \( A \). (ii) For \( p \in P \), \( p \sim xpx^* \) and the mapping \( p \mapsto xpx^* \) is
an OML-automorphism of $P$. (iii) If $p \in P$, then for $r \in P[0, p]$, the mapping $r \mapsto xx^*r$ is an OML-isomorphism of $P[0, p]$ onto $P[0, xpx^*]$. (iv) If $p, q \in P$ and $p \sim q$, then $P[0, p]$ is isomorphic as an OML to $P[0, q]$.

Proof. Parts (i) and (ii) follow from [13, Theorem 5.3 (i) and (ii)]. Parts (iii) and (iv) follow from (ii).

The projections $p$ and $q$ are said to be related iff there are nonzero sub-projections $0 \neq p_1 \leq p$ and $0 \neq q_1 \leq q$ such that $p_1 \sim q_1$; otherwise they are unrelated. If there exists a projection $q_1 \leq q$ such that $p \sim q_1$, we say that $p$ is subequivalent to $q$, in symbols, $p \preceq q$. A projection $h \in P$ is called invariant iff it is unrelated to its orthocomplement $h^\perp$.

5.7 Lemma. Suppose that $P$ is centrally orthocomplete and let $p, q, h \in P$. Then: (i) If $p \sim q$, then $\gamma p = \gamma q$. (ii) If $p \preceq q$, then $\gamma p \leq \gamma q$. (iii) $h$ is invariant iff it is central. (iv) If $P$ is complete, then $\gamma p = \bigvee\{q \in P : q \preceq p\}$. (v) If $\gamma p \perp \gamma q$, then $p$ and $q$ are unrelated. (vi) If $P$ is complete, then $p$ and $q$ are unrelated iff $\gamma p \perp \gamma q$.

Proof. Assume that $P$ is centrally orthocomplete, so the central cover mapping $\gamma$ exists. To prove (i), it will be sufficient to prove that, if $p$ and $q$ are exchanged by a symmetry $s \in A$, then $\gamma p = \gamma q$. So assume that $sps = q$. Since $\gamma p \in C(A)$, it follows that $q\gamma p = sps\gamma p = sp(\gamma p)s = sps = q$, whence $\gamma q \leq \gamma p$. Likewise, $\gamma p \leq \gamma q$, and (i) is proved. Part (ii) is an immediate consequence of (i) and the fact that, for $e, f \in P$, $e \leq f \Rightarrow \gamma e \leq \gamma f$. Part (iii) follows from [13, Theorem 7.5], (iv) is a consequence of [13, Theorem 7.7], (v) follows from [13, Corollary 7.6], and (vi) follows from [13, Corollary 7.8].

We denote the set of natural numbers by $\mathbb{N} := \{1, 2, 3, \ldots\}$.

5.8 Lemma (Cf. [27, Lemma 21]). If $e_1, e_2, e_3, \ldots$ is an infinite orthogonal sequence of projections and if $e_i$ and $e_{i+1}$ are exchanged by a symmetry $s_i \in A$ for all $i \in \mathbb{N}$, then for all $i, j \in \mathbb{N}$, the projections $e_i$ and $e_j$ are exchanged by a symmetry in $A$.

Proof. It will be sufficient to prove by induction on $n \in \mathbb{N}$ that, if $i \in \mathbb{N}$ and $i \leq n$, then $e_i$ and $e_n$ are exchanged by a symmetry in $A$. For $n = 1$, this is obvious. Assume that it is true for $n$ and suppose that $i \in \mathbb{N}$ with $i \leq n + 1$. We have to prove that $e_i$ and $e_{n+1}$ are exchanged by a symmetry in $A$. 

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Obviously, we can assume that \( i \leq n \), whence by the induction hypothesis, there is a symmetry \( s \in A \) such that \( se_is = e_n \). But \( sne_ns = e_{n+1} \), so \( sne_isn = e_{n+1} \). Since \( e_i \perp e_{n+1} \), we infer from Corollary 5.2 that \( e_i \) and \( e_{n+1} \) are exchanged by a symmetry in \( A \).

**5.9 Lemma.** Suppose \( s \) and \( t \) are symmetries in \( A \), \( f \in P \), and \( tsf < f \). Define \( f_1 := f \) and \( f_n := (ts)^{n-1}f(st)^{n-1} \) for \( 2 \leq n \in \mathbb{N} \). Then:

(i) \( f = f_1 > f_2 > f_3 > \cdots \). (ii) The sequence \( e_1, e_2, e_3, \ldots \in P[0,f] \) defined by \( e_n := f_n - f_{n+1} > 0 \) for \( n \in \mathbb{N} \) is orthogonal and for all \( i, j \in \mathbb{N} \), \( e_i \) and \( e_j \) are exchanged by a symmetry in \( A \).

**Proof.** For \( n \in \mathbb{N} \), we have \( f_{n+1} = tsf_nst \).

(i) We prove by induction on \( n \in \mathbb{N} \) that \( f_{n+1} < f_n \). For \( n = 1 \), we have \( f_2 = tsf < f = f_1 \). Assume by the induction hypothesis that \( n > 1 \) and \( f_n < f_{n-1} \). Then \( f_{n+1} = tsf_nst < tsf_{n-1}st = f_n \).

(ii) Suppose \( i, j \in \mathbb{N} \) with \( i < j \). Then \( f_j \leq f_{i+1} \) and \( f_{j-1} \leq f_i \), whence \( e_i + e_j \leq e_i + e_j + f_{i+1} - f_j = f_i - f_{j+1} \leq f_i \leq 1 \), and it follows that \( e_i \perp e_j \). Also, for all \( n \in \mathbb{N} \), \( tse_nst = ts(f_n - f_{n+1})st = tsf_nst - tsf_{n+1}st = f_{n+1} - f_{n+2} = e_{n+1} \), and since \( e_n \perp e_{n+1} \), Corollary 5.2 implies that \( e_n \) and \( e_{n+1} \) are exchanged by a symmetry. Therefore, byLemma 5.8, \( e_i \) and \( e_j \) are exchanged by a symmetry for all \( i, j \in \mathbb{N} \).

\[ \square \]

**6 Type-determining sets, orthodensity, and faithful projections**

Material in this section is adapted from [10, §3, §4].

**6.1 Standing Assumption.** Henceforth in this article, we assume that the OML \( P \) is centrally orthocomplete. Therefore the center \( P \cap C(A) \) is a complete boolean algebra and the central cover mapping \( \gamma: A \to P \cap C(A) \) exists.

**6.2 Definition.** Let \( Q \subseteq P \). Then:

1. The set of all suprema of centrally orthogonal families of projections in \( Q \) is denoted by \([Q]\). We understand that \([\emptyset] = \{0\}\).

2. \( Q^+ := \{q \wedge c : q \in Q, c \in P \cap C(A)\}\).

3. \( Q^i := \bigcup_{q \in Q} P[0,q] \). If \( Q^i \subseteq Q \neq \emptyset \), then \( Q \) is called an order ideal.
(4) $Q$ is type determining (TD) iff $[Q] \subseteq Q$ and $Q^\gamma \subseteq Q$.

(5) $Q$ is strongly type determining (STD) iff $[Q] \subseteq Q$ and $Q^\downarrow \subseteq Q$.

(6) $Q$ is projective iff for all $q \in Q$, if $q$ is projective to $p \in P$, then $p \in Q$.

(7) $Q$ is orthodense in $P$ iff every projection in $P$ is the supremum of an orthogonal family of projections in $Q$.

(8) $Q$ is an OML-ideal if $Q$ is an order ideal and $p, q \in Q \Rightarrow p \lor q \in Q$.

An OML-ideal is a $p$-ideal if it is projective [19 p. 75].

We note that $Q \subseteq P$ is TD (respectively, STD) if $Q = [Q] = Q^\gamma$ (respectively, if $Q = [Q] = Q^\downarrow$). Clearly, STD $\Rightarrow$ TD, and the intersection of TD subsets (respectively, STD subsets, projective subsets) of $P$ is again TD (respectively, STD, projective). Since $[\emptyset] = \{0\}$, 0 belongs to every TD set.

If $p \in P$, then the $p$-interval $P[0, p]$ is both an OML ideal and an STD subset of $P$, but it is projective iff $p \in C(A)$. Also, the center $P \cap C(P)$ is a projective TD subset of $P$, but it is STD if $P$ is boolean. By [19] Theorem 4.1, $[Q^\gamma]$ is the smallest TD subset of $P$ that contains $Q$, and $[Q^\downarrow]$ is the smallest STD subset of $P$ that contains $Q$.

Since two projections are projective iff they are equivalent (Lemma 5.5), it follows that $Q \subseteq P$ is projective iff, for all $p \in P$, $p \sim q \in Q \Rightarrow p \in Q$.

Clearly, $Q$ is projective iff, for every symmetry $s \in A$, we have $sQs \subseteq Q$.

6.3 Lemma. Let $p \in P$ and suppose that one of the following conditions holds: $P$ is complete or $p \in P \cap C(A)$. Then, if $Q \subseteq P$ is TD (respectively, STD, projective) it follows that $Q \cap P[0, p]$ is TD (respectively, STD, projective) both in $P$ and in the projection lattice $P[0, p]$ of $pAp$.

Proof. Assume the hypotheses. Then, as in the proof of Theorem 4.4, the center of $P[0, p]$ is $\{pc : c \in P \cap C(A)\}$; moreover, by Theorem 4.4, $P[0, p]$ is centrally orthocomplete and a family in $Q \cap P[0, p]$ is centrally orthogonal in $P[0, p]$ iff it is centrally orthogonal in $P$. Consequently, $Q \cap P[0, p]$ is closed under the formation of suprema of centrally orthogonal families. If $d$ belongs to the center of $P[0, p]$, then $d = pc$ for some $c \in P \cap C(A)$, whence, for any $q \in Q \cap P[0, p]$, we have $qd = qpc = qc \in Q \cap P[0, p]$, and therefore $Q \cap P[0, p]$ is TD in $P[0, p]$. If $Q$ is STD, it is clear that $Q \cap P[0, p]$ is STD in $P[0, p]$. Suppose $Q$ is projective, let $q \in Q \cap P[0, p]$, and suppose $q$ and a projection $r \in P[0, p]$ are exchanged by a symmetry in $pAp$. Then, by Lemma 5.3, $q$ and $r$ are exchanged by a symmetry in $A$, whence $r \in Q \cap P[0, p]$. \qed
6.4 Definition. A nonempty class \( L \) of OMLs is called an OML type class iff the following conditions are satisfied: (1) If \( L \in L \) and \( c \) belongs to the center of \( L \), then \( L[0, c] \in L \). (2) \( L \) is closed under the formation of arbitrary cartesian products. (3) If \( L_1 \) and \( L_2 \) are isomorphic OMLs and \( L_1 \in L \), then \( L_2 \in L \). If, in addition to (2) and (3), \( L \) satisfies (1') if \( L \in L \), then \( p \in L \Rightarrow L[0, p] \in L \), then \( L \) is called a strong OML type class.

Some examples of strong OML type classes are the following: The class of all boolean algebras, all modular OMLs, all complete OMLs, all \( \sigma \)-complete OMLs, and all atomic OMLs. Obviously, the intersection of (strong) OML type classes is again a (strong) OML type class. For instance, the class of all complete modular OMLs is a strong OML type class. The class of all locally modular OMLs provides an example of an OML type class that is not strong; however the class of all complete locally modular OMLs is a strong OML type class.

6.5 Theorem. If \( Q \) is a OML type class (respectively, a strong OML type class), then \( Q := \{q \in P : P[0, q] \in Q\} \) is a projective TD set (respectively, a projective STD set).

Proof. Assume that \( Q \) is a OML type class and \( Q := \{q \in P : P[0, q] \in Q\} \). Suppose that \((q_i)_{i \in I}\) is a centrally orthogonal family in \( Q \). Since \( P \) is centrally orthocomplete (Assumption 6.1), \( q := \bigvee_{i \in I} q_i \) exists in \( P \). For every \( i \in I \), \( P[0, q_i] \in Q \), whence \( X := \prod_{i \in I} P[0, q_i] \in Q \). By Theorem 4.5, \( X \) is isomorphic as an OML to \( P[0, q] \), so \( P[0, q] \in Q \), and therefore \( q \in Q \). Thus, \( \gamma Q \subseteq Q \).

Let \( q \in Q \) and \( c \in P \cap C(A) \). Then \( P[0, q] \in Q \) and, \( q \land c = qc \) belongs to the center of \( P[0, q] \), whence \( P[0, qc] = (P[0, q])[0, qc] \in Q \), and so \( qc \in Q \). This proves that \( \gamma Q \subseteq Q \), so \( Q \) is a TD-set. To prove that \( Q \) is projective, let \( s \in A \) be a symmetry. Then \( P[0, sqs] \) is isomorphic as an OML to \( P[0, q] \in Q \), whereupon \( P[0, sqs] \in Q \), and we have \( sqs \in Q \).

To complete the proof, suppose that \( Q \) is a strong OML type class, let \( q \in Q \) and suppose \( p \in P[0, q] \). Then \( P[0, p] \in Q \), and it follows that \( (P[0, p])[0, q] = P[0, q] \in Q \), whence \( p \in Q \).

If \( Q \subseteq P \), we understand that \( \gamma Q := \{\gamma q : q \in Q\} \). The following theorem is an adaptation to our present context of [10, Theorem 4.5 and Corollary 4.6].
6.6 Theorem. Let $Q \subseteq P$ be a TD set. Then: (i) $Q \cap \gamma(Q) = Q \cap C(A) \subseteq \gamma(Q) \subseteq P \cap C(A)$. (ii) There is a unique central projection $c_Q \in P \cap C(A)$ such that $\gamma(Q) = (P \cap C(A))[0, c_Q]$. (iii) There is a unique central projection $c_{Q \cap C(A)} \in P \cap C(A)$ such that $Q \cap \gamma(Q) = Q \cap C(A) = (P \cap C(A))[0, c_{Q \cap C(A)}]$. (iv) Both $\gamma(Q)$ and $Q \cap \gamma(Q) = Q \cap C(A)$ are TD subsets of $P$.

6.7 Definition (Cf. [10, Definition 4.7]). Let $Q$ be a TD subset of $P$. Then the central projection $c_Q$ in Theorem 6.6(ii) is called the type-cover of $Q$, and the central projection $c_{Q \cap C(A)}$ in Theorem 6.6(iii) is called the restricted type-cover of $Q$.

The type cover $c_Q$ and the restricted type cover $c_{Q \cap C(A)}$ will play significant roles in Sections 8 and 9 below.

6.8 Lemma ([10, Lemma 4.8]). Let $Q \subseteq P$ be a TD set. Then: (i) $c_Q$ is the largest projection in $\gamma(Q)$ and every central subprojection of $c_Q$ belongs to $\gamma(Q)$. (ii) $c_{Q \cap C(A)}$ is the largest central projection in $Q$. (iii) $c_{Q \cap C(A)} \leq c_Q$. (iv) The smallest central projection $c \in P \cap C(A)$ such that $Q \subseteq P[0, c]$ is $c = c_Q$. (v) The smallest central projection $d \in P \cap C(A)$ such that $Q \cap C(A) \subseteq P[0, d]$ is $d = c_{Q \cap C(A)}$. (vi) If $c \in P \cap C(A)$, then $c \perp c_Q$ iff $Q \cap P[0, c] = \{0\}$. (vii) If $c \in P \cap C(A)$, then $c \perp c_{Q \cap C(A)}$ iff $Q \cap (P \cap C(A))[0, c] = \{0\}$.

6.9 Definition. A projection $f \in P$ is faithful iff $\gamma f = 1$.

Clearly, $f \in P$ is faithful iff the only central projection $c \in P \cap C(A)$ such that $f \leq c$ is $c = 1$. The next lemma clarifies how faithfulness relativizes to a direct summand of $P$.

6.10 Lemma ([10, Lemma 3.5]). Let $c \in P \cap C(A)$ and let $f \in P[0, c]$. Then the following conditions are mutually equivalent: (i) $f$ is faithful in the projection lattice $P[0, c]$ of $cA$. (ii) $\gamma f = c$. (iii) $\gamma(P[0, f])$ is the center of $P[0, c]$. (iv) $f$ has a nonzero component in every nonzero direct summand of $P[0, c]$, i.e., if $0 \neq d \in (P \cap C(A))[0, c]$, then $fd \neq 0$.

6.11 Lemma. Let $Q \subseteq P$ and let $c := \bigvee \gamma(Q)$. Then $Q \subseteq P[0, c]$ and the following conditions are mutually equivalent: (i) If $p \in P$, $d \in \gamma(Q)$, and $pd \neq 0$, then $Q \cap P[0, p] \neq \{0\}$. (ii) If $0 \neq p \in P[0, c]$, then $Q \cap P[0, p] \neq \{0\}$. (iii) $Q$ is orthodense in $P[0, c]$.
Proof. If \( q \in Q \), then \( q \leq \gamma q \leq c \), so \( Q \subseteq P[0, c] \). The rest of the lemma follows from \( [12, \text{Lemma 5.3}] \) by taking \( \eta = \gamma \).

6.12 Theorem (Cf. \( [27, \text{Propositions 13 and 16}] \)). Suppose that \( P \) is complete, \( Q^\perp \subseteq Q \subseteq P \), \( Q \) is projective, and \( c = \bigvee \{ \gamma q : q \in Q \} \). Then (i) \( Q \subseteq P[0, c] \). (ii) \( Q \) is orthodense in \( P[0, c] \). (iii) \( c = \bigvee Q \). (iv) If \( Q \) is TD, then \( c = c_Q \in \gamma(Q) \).

Proof. Assume the hypotheses. Part (i) follows from Lemma [6.11]. To prove (ii), it will be sufficient to show that part (i) of Lemma [6.11] holds. Thus, suppose that \( p \in P \), \( q \in Q \), and \( p \wedge \gamma q \neq 0 \). Since \( P \) is complete, \( [13, \text{Corollary 7.8}] \) applies, whence \( p \) and \( q \) are related, i.e., there are nonzero subprojections \( 0 \neq p_1 \leq p \) and \( 0 \neq q_1 \leq q \) such that \( p_1 \sim q_1 \). As \( q_1 \in Q^\perp \subseteq Q \) and \( Q \) is projective, it follows that \( p_1 \in Q \), and (ii) is proved. By (ii), \( c \) is the supremum of an orthogonal family in \( Q \), by (i), \( c \) is an upper bound for \( Q \), whence (iii) holds. If \( Q \) is TD, then by Lemma [6.8] (i), \( c_Q \) is the largest projection in \( \gamma(Q) \), whence \( c_Q = \bigvee \gamma(Q) = c \).

6.13 Corollary. Suppose that \( P \) is complete, \( Q \subseteq P \), \( Q \) is projective, and \( Q \) is STD. Then \( c_Q = \bigvee Q \), \( Q \subseteq P[0, c_Q] \), and \( Q \) is orthodense in \( P[0, c_Q] \).

6.14 Theorem. Let \( Q \subseteq P \) be TD and let \( 0 \neq p \in P \). Then the following two conditions are equivalent: (i) There exists \( 0 \neq q \in Q \cap P[0, p] \) such that \( \gamma q = \gamma p \). (ii) For all \( d \in P \cap C(A) \), if \( pd \neq 0 \), then \( Q \cap P[0, pd] \neq \{0\} \).

Proof. (i) \( \Rightarrow \) (ii). Assume (i) and let \( d \in P \cap C(A) \) with \( pd \neq 0 \). By (i), there exists \( 0 \neq q \in Q \cap P[0, p] \) with \( \gamma q = \gamma p \). Put \( q_0 := qd = q \wedge d \leq p \wedge d = pd \). Then \( q_0 \in Q^\perp \subseteq Q \), and since \( d \in P \cap C(A) \), we have \( 0 \neq pd \leq \gamma(pd) = (\gamma p)d = (\gamma q)d = \gamma(qd) = \gamma q_0 \), whence \( q_0 \neq 0 \).

(ii) \( \Rightarrow \) (i). Assume (ii), let \( (q_i)_{i \in I} \) be a maximal centrally orthogonal family in \( Q \cap P[0, p] \), and put \( q := \bigvee_{i \in I} q_i \subseteq [Q] \subseteq Q \). Then \( q \leq p \), so \( \gamma q \leq \gamma p \). Taking \( d = 1 \) in (ii), we find that \( Q \cap P[0, p] \neq \{0\} \), whence \( q \neq 0 \). If \( \gamma q = \gamma p \), then we are done, so, we assume that \( \gamma q < \gamma p \) and this time we put \( d = \gamma p - \gamma q = \gamma p(\gamma q)^\perp \) in (ii). Then, as \( p \leq \gamma p \), we have \( pd = p(\gamma q)^\perp \), whence, if \( pd = 0 \), then \( p \leq \gamma q \), so \( \gamma p \leq \gamma q \leq \gamma p \), contradicting \( \gamma q < \gamma p \). Therefore, \( pd \neq 0 \), and it follows from (ii) that there exists \( 0 \neq q_0 \in Q \cap P[0, pd] \). But then, \( q_0 \leq p \) and \( q_0 \leq d \leq (\gamma q)^\perp \leq (\gamma q_i)^\perp \) for all \( i \in I \), contradicting the maximality of \( (q_i)_{i \in I} \).  

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6.15 Corollary. Let $c \in P \cap C(A)$ and let $Q$ be a TD subset of $P$. Then the following two conditions are equivalent: (i) $c \in \gamma(Q)$. (ii) $Q$ has a nonzero intersection with every nonzero direct summand of $P[0,c]$.

Proof. Assume the hypotheses. If $c = 0$, then (i) and (ii) are both true, so we assume that $c \neq 0$ and put $p := c$ in Theorem 6.14. Then conditions (i) and (ii) in Theorem 6.14 are equivalent to conditions (i) and (ii) in the corollary. 

6.16 Lemma (Cf. [27, Proposition 15]). Suppose that $P$ is complete, $Q \subseteq P$, $Q$ is projective, $Q$ is STD, $c \in \gamma(Q)$, and $0 \neq p \in P[0,c]$. Then there exists $0 \neq q \in Q \cap P[0,p]$ with $\gamma q = \gamma p$.

Proof. Assume the hypotheses and suppose that $d \in P \cap C(A)$ with $pd \neq 0$. By Theorem 6.14, it will be sufficient to prove that $Q \cap P[0,pd] \neq \{0\}$. But, by Theorem 6.6 (ii), $c \leq c_Q$, whence, if $0 \neq pd \in P[0,c]$, then $0 \neq pd \in P[0,c_Q]$, and it follows from Corollary 6.13 that $pd$ is the supremum of an orthogonal family in $Q$. Therefore, since $pd \neq 0$, it follows that $Q \cap P[0,pd] \neq \{0\}$. 

7 Abelian, modular, locally modular, and complete projections

In this section we study some important examples of TD and STD subsets of $P$. Many of the results in this section are generalizations to a synaptic algebra of results due to D. Topping for JW-algebras [27]. Often the proofs of these results are more or less the same as Topping’s proofs, but we include these proofs here in the interest of a more coherent account. The assumption that $P$ is centrally orthocomplete is still in force.

7.1 Definition. Let $p \in P$.

(1) $p$ is abelian (also called boolean [10] p. 1551) iff $P[0,p]$ is a boolean algebra. (We shall regard $P[0,0] = \{0\}$ as a “degenerate” boolean algebra, hence $0$ is an abelian projection in $A$.) We denote the set of all abelian projections in $P$ by $B$.

(2) $p$ is modular iff $P[0,p]$ is a modular OML. We denote the set of all modular projections in $P$ by $M$. 

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(3) \( p \) is locally modular iff \( P[0, p] \) is a locally modular OML. We denote the set of all locally modular projections in \( P \) by \( M_0 \).

(4) \( p \) is complete iff \( P[0, p] \) is a complete OML. We denote the set of all complete projections in \( P \) by \( T \).

7.2 Theorem. (i) \( B \subseteq M \subseteq M_0 \). (ii) The sets \( B, M, \) and \( T \) are projective STD sets. (iii) \( M_0 \) is a projective TD set. (iv) If \( c \in P \cap C(A) \), then \( c \in M_0 \iff c \in \gamma(M) \).

Proof. Part (i) is obvious. Since the class \( B \) of all boolean OMLs, the class \( M \) of all modular OMLs, and the class \( T \) of all complete OMLs are strong OML type classes and the class \( M_0 \) of all locally modular OMLs is an OML type class, (ii) and (iii) follow from Theorem 6.5, and part (iv) follows from Corollary 6.15.

7.3 Lemma. Let \( p, q \in P \). Then: (i) \( p \in B \) iff \( pAp \) is a commutative synaptic algebra. (ii) If \( p \in B \), then \( p \land q \) is an abelian projection in the synaptic algebra \( qAq \). (iii) If \( p \in P[0, q] \), then \( p \in B \) iff \( p \) is an abelian projection in the synaptic algebra \( qAq \).

Proof. (i) By [13, Theorem 4.5], \( pAp \) is a commutative synaptic algebra iff its lattice of projections \( P[0, p] \) is a boolean algebra.

(ii) Suppose \( p \in B \), i.e., \( P[0, p] \) is boolean. Then \( q \land p \in P[0, q] \), which is the lattice of projections in \( qAq \), and \( (P[0, q])[0, q \land p] = P[0, q \land p] \). But, since \( P[0, p] \) is boolean, so is the sublattice \( P[0, q \land p] \subseteq P[0, p] \), whence \( q \land p \) is abelian in \( qAq \).

(iii) If \( p \) is abelian in \( qAq \), then \( (P[0, q])[0, p] = P[0, p] \) is boolean, whence \( p \in B \). Conversely, suppose that \( p \in P[0, q] \), i.e., \( p \leq q \). Then, if \( p \in B \), it follows from (ii) that \( p = p \land q \) is abelian in \( qAq \).

7.4 Theorem (Cf. [27, Theorem 11]). Let \( p \in P \) and consider the following two conditions: (i) Every orthogonal family of nonzero projections in \( P[0, p] \), any two of which are exchanged by a symmetry in \( A \), is necessarily finite. (ii) \( p \in M \). Then (i) \( \Rightarrow \) (ii), and if \( p \in T \), then (ii) \( \Rightarrow \) (i).

Proof. To prove that (i) \( \Rightarrow \) (ii), it will be sufficient to show that if (ii) fails, then (i) fails. So assume that \( P[0, p] \) is not modular. Then by Theorem 3.4, there exist projections \( e, f \in P[0, p] \) such that \( e < f \) and \( e \) is perspective to \( f \) in \( P[0, p] \). Thus by Lemma 3.3 (i), \( e \) is perspective to \( f \) in \( P \), whence
by Theorem 5.1 (iii), there are symmetries \( s, t \in A \) such that \( stes = f \). Therefore, \( tsfes = e < f \), and by Lemma 5.9 (ii), (i) fails.

Conversely, assume that \( P[0, p] \) is complete, that (ii) holds, and that \( (e_i)_{i \in I} \) is an orthogonal family of nonzero projections in \( P[0, p] \) any two of which are exchanged by a symmetry in \( A \). Therefore, by Corollary 5.2, any two projections in \( (e_i)_{i \in I} \) are strongly perspective in \( P \), whence by Lemma 3.3 they are strongly perspective, hence perspective, in \( P[0, p] \). Since the OML \( P[0, p] \) is modular and complete, it follows from Theorem 3.5 (iv) that \( (e_i)_{i \in I} \) is finite. 

7.5 Lemma (Cf. [27, Lemma 23]). Suppose that \( p \in T \), but \( p \notin M \). Then there is a projection \( e \in P[0, p] \) with the following properties: (i) \( e \) is the supremum of an infinite sequence of nonzero projections in \( P[0, p] \) any two of which are exchanged by a symmetry in \( A \). (ii) There is a symmetry \( s \in A \) with \( ses \in P[0, p] \) and \( ses \perp e \).

Proof. Assume the hypotheses. Then by Theorem 7.4 there is an infinite sequence \( e_1, e_2, e_3, \ldots \) of nonzero projections in \( P[0, p] \), any two of which are exchanged by a symmetry in \( A \), whence also by a symmetry in \( pAp \) (Lemma 5.3). Putting \( e := \bigvee_{n=1}^{\infty} e_{2n} \), we have (i). To prove (ii), we work in the synaptic algebra \( pAp \) and its complete OML \( P[0, p] \) of projections. Let \( f := \bigvee_{n=1}^{\infty} e_{2n-1} \). Then \( e \perp f \), whence by [13, Theorem 5.15] (a weak form of additivity for exchangeability by symmetries), there is a symmetry \( t \in pAp \) such that \( tet = f \). By Lemma 5.3 again, there is a symmetry \( s \in A \) with \( ses = P[0, p] \) and \( ses \perp e \).

7.6 Theorem (Cf. [27, Theorem 12]). (i) If \( p, q \in M \) and \( p \vee q \in T \), then \( p \vee q \in M \). (ii) If \( P \) is complete, then \( M \) is both a projective STD set and a \( p \)-ideal in \( P \).

Proof. (i) Assuming the hypothesis of (i), we have to prove that \( P[0, p \vee q] \) is modular; hence we may drop down to the synaptic algebra \( (p \vee q)A(p \vee q) \) with complete projection lattice \( P[0, p \vee q] \). Thus, changing notation, we can (and do) assume that \( P \) is complete, that \( p, q \in M \) with \( p \vee q = 1 \), and we have to prove that \( P \) is modular. By [13, Theorem 5.9 (ii)] (the symmetry parallelogram law) \( p^+ = 1 - p = (p \vee q) - p \) is exchanged by a symmetry in \( A \) with the modular projection \( q - (p \land q) \leq q \), so \( p^+ \) is modular.

Now, aiming for a contradiction, we assume that \( P \) is not modular. Therefore by Lemma 7.5 (with \( p=1 \)), there is a projection \( e \in P \) such that \( e \) is
the supremum of an infinite sequence of nonzero projections in $P$ any two of which are exchanged by a symmetry in $A$, and there is a symmetry $t \in A$ with $tet \perp e$. Applying Theorem 5.4 to the pair $e, p$, we find that there is a symmetry $s \in A$ and a central projection $c \in P \cap C(A)$ such that $secs \leq pc$ and $se^\perp e^\perp s \leq p^\perp e^\perp$. From the latter inequality and the fact that $p^\perp \in M$, we infer that $e^\perp c^\perp \in M$. But $tec^\perp t = te^\perp e^\perp \leq e^\perp c^\perp$, whence $ec^\perp \in M$. Moreover, the pair of modular projections $ec$ and $ec^\perp$ is centrally orthogonal, hence $e = ec + ec^\perp \in M$, contradicting Theorem 7.4.

(ii) Part (ii) follows immediately from (i). \qed

7.7 Theorem (Cf. [27, Corollary 21]). Assume that $P$ is complete and let $p, q \in P$. Then:

(i) If $p \in M$ and $p \sim q$, then $q \in M$, there is a projection $r \in M$ such that $p, q \in P[0, r]$, and $p$ is perspective to $q$ in $P[0, r]$.

(ii) If $p \in M$ and $p \sim q$, then $q \in M$ and $p$ and $q$ are perspective in $P$.

(iii) On the set $M$, perspectivity is transitive.

(iv) If $q \leq p \in M$ and $q \sim p$, then $q = p$ (i.e., $p$ is finite [27, p.23]).

(v) If $p, q \in M$, then $p \sim q$ iff $p$ and $q$ are exchanged by a symmetry in $A$.

(vi) If $p, q \in M$, $p \preceq q$, and $q \preceq p$, then $p \sim q$.

Proof. (i) Assume $p \in M$ and $p \sim q$. Since $M$ is projective, $q \in M$. Also there exist projections $p = e_1, e_2, ..., e_n = q$ such that $e_i$ is exchanged by a symmetry in $A$ with $e_{i+1}$ for $i = 1, 2, ..., n - 1$. Since $p \in M$, it follows from Lemma 5.6 (iii) that $e_1, e_2, ..., e_n \in M$, and by Theorem 7.6 $r := e_1 \vee e_2 \vee \cdots \vee e_n \in M$. By Lemma 5.5, for $i = 1, 2, ..., n - 1$, $e_i$ is exchanged with $e_{i+1}$ by a symmetry in $rAr$. Since $P$ is complete, so is $P[0, r]$; hence, we may apply Theorem 7.5 (ii) to $rAr$ and its complete modular projection lattice $P[0, r]$ and infer that $p = e_1$ is perspective to $q = e_n$ in $P[0, r]$.

(ii) By (i) and Lemma 3.3, $p$ and $q$ are perspective in $P$.

(iii) Suppose that $p, q, r \in M$ with $p$ perspective to $q$ and $q$ perspective to $r$ in $P$. Then by Theorem 5.1 (iv), $p \sim q$ and $q \sim r$, so $p \sim r$, and by (ii), $p$ is perspective to $r$ in $P$.

(iv) Assume that $q \leq p \in M$ and $q \sim p$. By (i) there exists $r \in M$ such that $q \leq p \in P[0, r]$ and $p$ is perspective to $q$ in $P[0, r]$; hence $p = q$ by Theorem 3.4 applied to the modular OML $P[0, r]$. 

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(v) Suppose that $p \in M$ $p \sim q$. Then $q \in M$ and applying Theorem 5.4 we infer that there is a symmetry $s \in A$ and a central projection $c \in P \cap C(A)$ such that $spcs \leq qc$ and $sqc^\perp s \leq pc^\perp$. Since $p \sim q$, there is a finite product of symmetries $x$ such that $xpcx^* = q$. Thus, $spcs \leq qc = xpcx^*$, whence $e := x^*spcsx \leq pc$ with $sxex^*s = pc$. Therefore, $e \leq pc$ with $e \sim pc$, and since $pc \in M$, $e = pc$ by (iv), and it follows that $spcs = xex^* = xpcx^* = qc$. Likewise, $f := xsqc^\perp sx^* \leq xpc^\perp x^* = qc^\perp$ with $sx^*fxs = qc^\perp$, and we deduce that $f = qc^\perp$, whence $sqc^\perp s = x^*fx = x^*qc^\perp x = pc^\perp$, so $spc^\perp s = qc^\perp$. Consequently, $spcs = xpcx^* = q$. Conversely, if $p$ and $q$ are exchanged by a symmetry, then $p \sim q$.

(vi) By hypothesis, there are finite products of symmetries $u$ and $x$ such that $q_1 := upux^* \leq q$ and $p_1 := xqx^* \leq p$. Thus, $xq_1x^* \leq xqx^* = p_1 \leq p$ with $xq_1x^* = xupux^* \sim p$. By (iv), $xq_1x^* = p_1$, and therefore $q_1 = x^*px$. Consequently, $q = x^*p_1x \leq x^*px = q_1$, so $q_1 = q$, whence $p \sim q$. \qed

Examination of the results in [27] required for Topping’s proof of his version of the type-I/II/III decomposition theorem for a JW-algebra [27, Theorem 13] now shows that all of these results either have been obtained above (often assuming that $P$ is complete) or follow easily from the results above. Therefore, we claim that our first project has been accomplished. We now focus on our second project.

8 The fundamental direct-decomposition theorem

The assumption that $P$ is centrally orthocomplete is still in force.

8.1 Standing Assumption. In this section and the next, we assume that $Q$ is a TD subset of $P$.

We note that our subsequent results, apart from Theorem 9.5, do not require completeness of the OML $P$, nor do they require that $Q$ is STD. The terminology in the following definition is borrowed from [27, pp. 28–29].

8.2 Definition. Let $c \in P \cap C(A)$. Then:

(1) $c$ is type-$Q$ iff $c \in Q$.

(2) $c$ is locally type-$Q$ iff $c \in \gamma(Q)$. 

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(3) $c$ is purely non-$Q$ iff no nonzero subprojection of $c$ belongs to $Q$.

(4) $c$ is properly non-$Q$ iff no nonzero central subprojection of $c$ belongs to $Q$.

If $c \in P \cap C(A)$ and if $c$ is type-$Q$ (respectively, locally type-$Q$, purely non-$Q$, etc.), one also says that the direct summand $P[0,c]$ of $P$ and the direct summand $cA$ of $A$ are type-$Q$ (respectively, locally type-$Q$, purely non-$Q$, etc.).

We note that, by Theorem 7.2 (iv), for central projections $c \in P \cap C(A)$, the notion of local modularity introduced in Definition 7.1 (3) is consistent with Definition 8.2 (2), i.e., $c \in M_0$ iff $c$ is locally type-$M$.

8.3 Theorem ([10, Theorem 5.2]). Let $c \in P \cap C(A)$. Then:

(i) $c$ is type-$Q$ iff $c \in Q \cap \gamma(Q) = Q \cap C(A)$ iff every central subprojection of $c$ belongs to $Q \cap C(A)$ iff $c \leq c_{Q \cap C(A)}$.

(ii) If $Q$ is STD, then $c$ is type-$Q$ iff $P[0,c] \subseteq Q$.

(iii) $c$ is locally type-$Q$ iff every central subprojection of $c$ belongs to $\gamma(Q)$ iff $c \leq c_Q$.

(iv) $c$ is purely non-$Q$ iff $Q \cap P[0,c] = \{0\}$ iff $c \leq (c_Q)^\perp$.

(v) $c$ is properly non-$Q$ iff the only central projection in $Q \cap P[0,c]$ is $0$ iff $c \leq (c_{Q \cap C(A)})^\perp$.

8.4 Corollary. Let $c, d \in P \cap C(A)$. Then: (i) If $c$ is type-$Q$, then $c$ is locally type-$Q$. (ii) If $c$ is purely non-$Q$, then $c$ is properly non-$Q$. (iii) If $c$ is both type-$Q$ and properly non-$Q$, then $c = 0$. (iv) If $c$ is both locally type-$Q$ and purely non-$Q$, then $c = 0$. (v) If $c$ is type-$Q$ (respectively, locally type-$Q$, purely non-$Q$, properly non-$Q$), then so is $c \wedge d$. (vi) If both $c$ and $d$ are type-$Q$ (respectively, locally type-$Q$, purely non-$Q$, properly non-$Q$), then so is $c \lor d$.

8.5 Lemma ([10, Lemma 5.5]). (i) There exists a unique central projection $c$, namely $c = c_Q$, such that $A = cA \oplus c^\perp A$, $cA$ is locally type-$Q$, and $c^\perp A$ is purely non-$Q$; moreover, $Q \subseteq P[0,c_Q]$. (ii) There exists a unique central projection $d$, namely $d = c_{Q \cap C(A)}$, such that $A = dA \oplus d^\perp A$, $dA$ is type-$Q$, and $d^\perp A$ is properly non-$Q$; moreover, $Q \cap C(A) \subseteq P[0,c_{Q \cap C(A)}]$. 

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The following theorem results from combining the direct decompositions in parts (i) and (ii) of Lemma 8.5. We regard this theorem as the fundamental direct-decomposition theorem for the synaptic algebra $A$.

8.6 Theorem ([10, Theorem 5.6]). Corresponding to the TD set $Q$, there exist unique pairwise orthogonal central projections $c_1$, $c_2$ and $c_3$, namely $c_1 = c_Q \cap C(A)$, $c_2 = c_Q \wedge (c_Q \cap C(A)) \perp$, and $c_3 = (c_Q) \perp$, such that $c_1 + c_2 + c_3 = 1$;

$$A = c_1 A \oplus c_2 A \oplus c_3 A;$$

$c_1 A$ is type-$Q$; $c_2 A$ is locally type-$Q$, but properly non-$Q$; and $c_3$ is purely non-$Q$. Moreover, $Q \cap C(A) = (P \cap [C(A)]][0, c_1], Q \subseteq P[0, c_1 + c_2], \text{and } (P \cap C(A)][0, c_2 + c_3] \cap Q = \{0\}$.

9 The type-I/II/III decomposition theorem

The assumption that $P$ is centrally orthocomplete is still in force.

9.1 Standing Assumption. In this section, we continue to assume that $Q \subseteq P$ is TD, and we also assume that $K \subseteq P$ is TD and that $Q \subseteq K$.

Since $Q \subseteq K$, we have $c_Q \leq c_K$ and $c_Q \cap C(A) \leq c_K \cap \gamma(K)$.

9.2 Definition. Let $c \in P \cap C(A)$. Then, with respect to the pair of TD sets $Q \subseteq K$:

(1) $c$ is type I iff it is locally type-$Q$.

(2) $c$ is type II iff it is locally type $K$, but purely non-$Q$.

(3) $c$ is type III iff it is purely non-$K$.

(4) $c$ is type $I_K$ (respectively, type $II_K$) iff it is type I (respectively, type II) and also type-$K$.

(5) $c$ is type $I_{\bar{K}}$ (respectively, type $II_{\bar{K}}$) iff it is type I (respectively, type II and also properly non-$K$).

If $c \in P \cap C(A)$ and if $c$ is type I (respectively, type II, type III, etc.), one also says that the direct summand $P[0, c]$ of $P$ and the direct summand $cA$ of $A$ are type I (respectively, type II, type III, etc.).

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9.3 Lemma. Let \( c \in P \cap C(A) \). Then the following conditions are mutually equivalent: (i) \( c \) is type I. (ii) There is a projection \( q \in Q \) such that \( \gamma q = c \). (iii) There is a projection \( q \in Q \cap P[0, c] \) that is faithful in \( P[0, c] \). (iv) Every nonzero direct summand of \( P[0, c] \) contains a nonzero projection in \( Q \). (v) \( c \leq c_Q \).

Proof. (i) \( \iff \) (ii) is the definition of \( c \) being locally type-Q, (ii) \( \iff \) (iii) follows from Lemma 6.10, and (i) \( \iff \) (iv) \( \iff \) (v) follows from Theorem 8.3 (iii). \( \square \)

The following is the type-I/II/III decomposition theorem for synaptic algebras. It is obtained by combining the fundamental direct-decomposition theorems for \( Q \) and for \( K \).

9.4 Theorem ([10, Theorem 6.4]). Corresponding to the pair of TD sets \( Q \) and \( K \) with \( Q \subseteq K \), there are unique pairwise orthogonal central projections \( c_1, c_\Pi \) and \( c_{III} \), namely \( c_1 = c_Q \), \( c_\Pi = c_K \cap (c_Q)^\perp \), and \( c_{III} = (c_K)^\perp \), such that \( c_1 + c_\Pi + c_{III} = 1 \);

\[
A = c_1 A \oplus c_\Pi A \oplus c_{III} A;
\]

and \( c_1 A \), \( c_\Pi A \), and \( c_{III} A \) are of types I, \( \Pi \), and III, respectively. Moreover, there are further decompositions

\[
c_1 A = c_{IK} A \oplus c_{IK} A \quad \text{and} \quad c_\Pi A = c_{IIK} A \oplus c_{IIK} A,
\]

where \( c_{IK}, c_{IK}, c_{IIK}, \) and \( c_{IIK} \) are central projections of types \( I_K, I_K, \) \( \Pi_K \), and \( \Pi_K \), respectively; these decompositions are also unique; and

\[
c_{IK} = c_Q \land (c_K \cap (K))^\perp, \quad c_{IIK} = c_K \land (c_K \cap (K))^\perp \land (c_Q)^\perp;
\]

Furthermore, the type \( I_K \) direct summand decomposes as

\[
c_{IK} A = c_{11} A \oplus c_{21} A,
\]

where \( c_{11} \) and \( c_{21} \) are central projections, \( c_{11} \) is type-Q (hence also of type-K), and \( c_{21} \) is type-K, locally type-Q, but properly non-Q. The latter decomposition is also unique, and

\[
c_{11} = c\cap C(A), \quad c_{21} = c_K \cap (c_Q \cap (A))^\perp.
\]
9.5 Theorem. With the notation of Theorem 9.4:

(i) \( Q \subseteq \mathcal{P}[0, c_I] \) and \( K \subseteq \mathcal{P}[0, c_I + c_{II}] \).

(ii) If \( \mathcal{P} \) is complete, \( Q \) is projective, and \( Q \) is STD, then \( c_I = \bigvee Q \) and \( Q \) is orthodense in \( \mathcal{P}[0, c_I] \).

(iii) If \( \mathcal{P} \) is complete, \( K \) is projective, and \( K \) is STD, then \( c_I + c_{II} = \bigvee K \) and \( K \) is orthodense in \( \mathcal{P}[0, c_I + c_{II}] \).

Proof. (i) As \( c_I = c_Q \), and \( c_I + c_{II} = c_Q + (c_K - c_Q) = c_K \), (i) follows from Lemma 6.8 (iv). In view of part (i), parts (ii) and (iii) follow from Corollary 6.13.

In Theorem 9.4, the unique five-fold direct-sum decomposition

\[
A = c_{I_K}A \oplus c_{I_{\bar{K}}}A \oplus c_{II_K}A \oplus c_{II_{\bar{K}}}A \oplus c_{III}A
\]

of \( A \) into direct summands of types \( I_K, I_{\bar{K}}, II_K, II_{\bar{K}}, \) and III is a generalization of the classic type-I/II/III decomposition for a von Neumann algebra (see Remark 9.6 below); moreover, the additional decomposition \( c_{I_K}A = c_{11}A \oplus c_{21}A \) into direct summands of type-\( Q \) and of type-\( K \), locally type-\( Q \), but properly non-\( Q \) yields a six-fold direct decomposition of \( A \),

\[
A = c_{11}A \oplus c_{21}A \oplus c_{I_K}A \oplus c_{II_K}A \oplus c_{II_{\bar{K}}}A \oplus c_{III}A.
\]

Of course, if \( A \) is a factor, then it is of precisely one of these six types.

9.6 Remark. If \( R \) is a von Neumann algebra and \( A \) is the synaptic algebra of all self-adjoint elements of \( A \), then one obtains the classic type-I/II/III decomposition of \( A \) (and also of \( R \)) by taking \( Q = B \), the TD set of abelian projections in \( A \), and taking \( K \) to be the set of all finite projections in \( A \).

9.7 Remark. If \( A \) is a JW-algebra, regarded as a synaptic algebra, then one obtains Topping’s version of a type-I/II/III decomposition \([27, \text{Theorem 13}]\) by taking \( Q = B \) and \( K = M \).

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