Existence and Stability Results for Impulsive Fractional $q$-Difference Equation

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Abstract
In this paper, we study the boundary value problem for an impulsive fractional $q$-difference equation. Based on Banach’s contraction mapping principle, the existence and Hyers-Ulam stability of solutions for the equation which we considered are obtained. At last, an illustrative example is given for the main result.

Keywords
Impulsive Fractional $q$-Difference Equation, Hyers-Ulam Stability, Existence, $q$-Calculus

1. Introduction
The $q$-calculus or quantum calculus is an old subject that was initially developed by Jackson [1]; basic definitions and properties of $q$-calculus can be found in [2]. The fractional $q$-calculus had its origin in the works by Al-Salam [3] and Agarwal [4]. But the definitions mentioned above about $q$-calculus can’t be applied to impulse points $t_k, k \in \mathbb{Z}$, such that $t_k \in (qt, t)$. In [5], the authors defined the concepts of fractional $q$-calculus by defining a $q$-shifting operator $\Phi_q(m) = qm + (1-q)a, m, a \in \mathbb{R}$. Using the $q$-shifting operator, the fractional impulsive $q$-difference equation was defined. In paper [5] [6] [7], the authors discussed the existence of solutions for the fractional impulsive $q$-difference equation with Riemann-Liouville and Caputo fractional derivatives respectively. Some other results about $q$-difference equations can be found in papers [8]-[16] and the references cited therein. Dumitru Baleanu et al. discussed the stability of non-autonomous systems with the $q$-Caputo fractional derivatives in reference [17]. However, the existence and stability of solutions for the fractional impul-
sive q-difference have not been yet studied.

Motivated greatly by the above mentioned excellent works, in this paper we investigate the following fractional impulsive q-difference equation with q-integral boundary conditions:

\[
\begin{cases}
\bigtriangleup x(t_k) = x(t_k^+) - x(t_k^-) = \varphi_k(x(t_k)), k = 1, 2, \ldots, m, \\
\eta_1 x(0) + \eta_2 x(T) = \mu \sum_{k=0}^{m} q_{t_k}^\beta x(t_{k+1}).
\end{cases}
\]

(1)

where \( c D_{t_k}^{\alpha_k} \) is the fractional \( q_k \)-derivative of the Caputo type of order \( \alpha_k \) on \( J_k \), \( 0 < \alpha_k < 1 \), \( 0 < q_k < 1 \), \( J_0 = [0, t_1], J_k = [0, t_{k+1}], k = 1, 2, \ldots, m, \)

\( \varphi_k \in C(\mathbb{R}, \mathbb{R}), f \in C(J \times \mathbb{R}, \mathbb{R}). \) \( q_{t_k}^\beta \) denotes the Riemann-Liouville \( q_k \)-fractional integral of order \( \beta_k > 0 \) on \( J_k, k = 0, 1, 2, \ldots, m \) and \( \eta, \eta_2, \mu \) are three constants.

2. Preliminaries on q-Calculus and Lemmas

Here we recall some definitions and fundamental results on fractional q-integral and fractional q-derivative, for the full theory for which one is referred to [5] [6] [7].

For \( q \in (0, 1) \), we define a q-shifting operator as \( _a \Phi_q(m) = q^m + (1 - q)a \).

The new power of q-shifting operator is defined as

\[
_q(n-m)^{\nu}_q = \prod_{i=0}^{\nu-1} (1 - \frac{a}{n} \Phi_q(m)), \quad k \in \mathbb{N} \cup \{0\}, \quad n \in \mathbb{R}.
\]

If \( v \in \mathbb{R} \), then

\[
_q(n-m)^{\nu}_q = n^\nu \prod_{i=0}^{\nu-1} \frac{1 - \frac{a}{n} \Phi_q(m)}{1 - \frac{a}{n} \Phi_q^{v+1}(m)}.
\]

The q-derivative of a function \( f \) on interval \( [a, b] \) is defined by

\[
(_a D_q f)(t) = \frac{f(t) - f(\Phi_q(t))}{(1 - q)(t-a)}, t \neq a, (a \bigtriangleup a) f(a) = \lim_{t \to a} (_a D_q f)(t).
\]

The q-integral of a function \( f \) defined on the interval \( [a, b] \) is given by

\[
(_a I_q f)(t) = \int_a^t f(s) ds = (1 - q)(t-a) \sum_{i=0}^{\infty} q_i f(\Phi_q^{i+1}(t)), t \in [a, b].
\]

Some results about operator \( _a D_q \) and \( _a I_q \) can be found in references [5]. Let us define fractional q-derivative and q-integral on interval \( [a, b] \) and outline some of their properties [5] [6] [7].

**Definition 1** [5] The fractional q-derivative of Riemann-Liouville type of order \( \nu \geq 0 \) on interval \( [a, b] \) is defined by \( (_a D_q^\nu f)(t) = f(t) \) and \( (_a D_q^\nu f)(t) = (_a D_q^\nu I_q^{\nu+1} f)(t), \nu > 0, \)

where \( l \) is the smallest integer greater than or equal to \( \nu \).

**Definition 2** [5] Let \( \alpha \geq 0 \) and \( f \) be a function defined on \( [a, b] \). The
fractional $q$-integral of Riemann-Liouville type is given by $\left( a_qI_q^0 f \right)(t) = f(t)$ and

$$\left( a_qI_q^\alpha f \right)(t) = \frac{1}{\Gamma_q(\alpha)} \int_a^t (t-s)_q^{-\alpha} f(s) d_q s, \alpha > 0, t \in [a,b].$$

**Lemma 1** [5] Let $\alpha, \beta \in \mathbb{R}^+$ and $f$ be a continuous function on $[a,b], a \geq 0$. The Riemann-Liouville fractional $q$-integral has the following semi-group property

$$a_qI_q^\beta a_qI_q^\alpha f(t) = a_qI_q^{\alpha + \beta} f(t).$$

**Lemma 2** [5] Let $f$ be a $q$-integrable function on $[a,b]$. Then the following equality holds

$$a_qD_q^\alpha a_qI_q^\alpha f(t) = f(t), \text{ for } \alpha > 0, t \in [a,b].$$

**Lemma 3** [5] Let $\alpha > 0$ and $p$ be a positive integer. Then for $t \in [a,b]$ the following equality holds

$$a_qI_q^\alpha a_qD_q^p f(t) = a_qD_q^\alpha a_qI_q^p f(t) - \sum_{k=0}^{p-1} \binom{t-a}{\alpha-k} \frac{(-1)^k}{\Gamma_q(\alpha + k + 1)} a_q D_q^k f(a).$$

**Definition 3** [7] The fractional $q$-derivative of Caputo type of order $\alpha \geq 0$ on interval $[a,b]$ is defined by $\left( ^c a_qD_q^\alpha f \right)(t) = f(t)$ and

$$\left( ^c a_qD_q^\alpha f \right)(t) = \left( a_qI_q^{\alpha - n} a_qD_q^n f \right)(t), \alpha > 0,$$

where $n$ is the smallest integer greater than or equal to $\alpha$.

**Lemma 4** [7] Let $\alpha > 0$ and $n$ be the smallest integer greater than or equal to $\alpha$. Then for $t \in [a,b]$ the following equality holds

$$a_qI_q^\alpha ^c a_qD_q^p f(t) = f(t) - \sum_{k=0}^{p-1} \binom{t-a}{\alpha-k} \frac{(-1)^k}{\Gamma_q(\alpha + k + 1)} a_q D_q^k f(a).$$

### 3. Main Results

In this section, we will give the main results of this paper.

Let $PC(J, \mathbb{R}) = \{x: J \to \mathbb{R}, x(t) \text{ is continuous everywhere except for some } t_k \text{ at which } x(t_k^+) \text{ and } x(t_k^-) \text{ exist, and } x(t_k^+) = x(t_k^-), k = 1, 2, \cdots, m \}$. $PC(J, \mathbb{R})$ is a Banach space with the norm

$$\|x\| = \sup \{|x(t)|: t \in J\}.$$

First, for the sake of convenience, we introduce the following notations:

$$\Lambda = \eta_1 + \eta_2 - \mu \sum_{i=0}^{m} \Omega_{\beta_i} \neq 0, \Omega_{\alpha_i} = \frac{(t_{i+1} - t_i)^{\alpha_i}}{\Gamma_q(\sigma_i + 1)},$$

where $\sigma_i \in \{\alpha, \beta, \alpha + \beta\}, q_i \in (0,1), i = 0, 1, 2, \cdots, m$.

To obtain our main results, we need the following lemma.

**Lemma 5** Let $\mu \sum_{i=0}^{m} \Omega_{\beta_i} \neq 0$ and $h(t) \in C(J, \mathbb{R})$. Then for any $t \in J_k$, 

DOI: 10.4236/jamp.2020.87107 1415 Journal of Applied Mathematics and Physics
the solution of the following problem
\[
\begin{aligned}
\int_{q} D_{q}^{\alpha} x(t) &= h(t), \quad t \in J_{k} \subseteq J = [0, T], \quad t \neq t_{k}, \\
\Delta x(t_{k}) &= x(t_{k}^{+}) - x(t_{k}) = \varphi_{k}(x(t_{k})), \quad k = 1, 2, \ldots, m, \\
\eta_{1} x(0) + \eta_{2} x(T) &= \mu \sum_{k=0}^{m} I_{q_{k}}^{\alpha_{k}} x(t_{k+1})
\end{aligned}
\]  

(2)
is given by
\[
\begin{aligned}
x(t) &= \frac{1}{A} \left[ \sum_{i=0}^{m} \left( \mu_{i} I_{q_{i}}^{\alpha_{i}+\beta_{i}} h(t_{i+1}) - \eta_{2,i} I_{q_{i}}^{\alpha_{i}} h(t_{i+1}) \right) \\
&+ \sum_{i=1}^{m} \left[ \mu \left( \sum_{j=1}^{i} \varphi_{j}(x(t_{j})) \right) + \sum_{j=0}^{i-1} \beta_{i} I_{q_{j}}^{\alpha_{j}} h(t_{j+1}) \right] \Omega_{\beta_{i}} - \eta_{2,i} \varphi_{i}(x(t_{i})) \right] \\
&+ \sum_{i=1}^{m} \varphi_{i}(x(t_{i})) + \sum_{j=0}^{i-1} \beta_{i} I_{q_{j}}^{\alpha_{j}} h(t_{i+1}) + \beta_{i} I_{q_{i}}^{\alpha_{i}} h(t)
\end{aligned}
\]  

(3)

**Proof.** Applying the operator \( \int_{q} I_{q}^{\alpha_{i}} \) on both sides of the first equation of (2) for \( t \in J_{0} \) and using Lemma 4, we have
\[
x(t) = x(t_{0}) + \int_{q_{0}} I_{q_{0}}^{\alpha_{0}} h(t)
\]
Then we get for \( t = t_{i} \) that
\[
x(t_{i}) = x(t_{0}) + \int_{q_{0}} I_{q_{0}}^{\alpha_{0}} h(t_{i}).
\]  

(4)

For \( t \in J_{1} \), again taking the \( \int_{q} I_{q}^{\alpha_{i}} \) to (4) and using the above process, we get
\[
x(t) = x(t_{0}) + \int_{q_{i}} I_{q_{i}}^{\alpha_{i}} h(t_{i+1}) + \int_{q_{i}} I_{q_{i}}^{\alpha_{i}} h(t)
\]
Applying the impulsive condition \( x(t_{i}^{+}) = x(t_{i}) + \varphi_{i}(x(t_{i})) \), we get
\[
x(t) = x(t_{0}) + \varphi_{i}(x(t_{i})) + \int_{q_{i}} I_{q_{i}}^{\alpha_{i}} h(t_{i+1}) + \int_{q_{i}} I_{q_{i}}^{\alpha_{i}} h(t)
\]
By the same way, for \( t \in J_{2} \), we have
\[
x(t) = x(t_{0}) + \varphi_{i}(x(t_{i})) + \varphi_{2}(x(t_{i}^{+})) + \int_{q_{1}} I_{q_{1}}^{\alpha_{1}} h(t_{i+1}) + \int_{q_{1}} I_{q_{1}}^{\alpha_{1}} h(t_{i+2}) + \int_{q_{1}} I_{q_{1}}^{\alpha_{1}} h(t_{i+1}) + \int_{q_{1}} I_{q_{1}}^{\alpha_{1}} h(t)
\]
Repeating the above process for \( t \in J_{k} \subseteq J, k = 0, 1, 2, \ldots, m \), we get
\[
x(t) = x(t_{0}) + \sum_{i=1}^{k} \varphi_{i}(x(t_{i})) + \sum_{j=0}^{i-1} \int_{q_{j}} I_{q_{j}}^{\alpha_{j}} h(t_{i+1}) + \int_{q_{i}} I_{q_{i}}^{\alpha_{i}} h(t)
\]  

(5)

From (5), we find that
\[
x(T) = x(t_{0}) + \sum_{i=1}^{k} \varphi_{i}(x(t_{i})) + \sum_{j=0}^{i-1} \int_{q_{j}} I_{q_{j}}^{\alpha_{j}} h(t_{i+1}) + \int_{q_{i}} I_{q_{i}}^{\alpha_{i}} h(T)
\]
From the boundary condition of (2), we get
\[
x(t_{0}) = \frac{1}{A} \left[ \sum_{i=0}^{m} \left( \mu_{i} I_{q_{i}}^{\alpha_{i}+\beta_{i}} h(t_{i+1}) - \eta_{2,i} I_{q_{i}}^{\alpha_{i}} h(t_{i+1}) \right) \\
+ \sum_{i=1}^{m} \left[ \mu \left( \sum_{j=1}^{i} \varphi_{j}(x(t_{j})) \right) + \sum_{j=0}^{i-1} \beta_{i} I_{q_{j}}^{\alpha_{j}} h(t_{j+1}) \right] \Omega_{\beta_{i}} - \eta_{2,i} \varphi_{i}(x(t_{i})) \right]
\]  

(6)

Substituting (6) to (5), we obtain the solution (3). This completes the proof.
We define an operator \( \mathcal{G} : PC(J, \mathbb{R}) \rightarrow PC(J, \mathbb{R}) \) as follows:

\[
\mathcal{G}x(t) = \frac{1}{\Lambda} \left\{ \sum_{i=0}^{m} \left( \mu_i L_{q_i}^{\alpha_i^{+}, \beta_i} f(s,x)(t_{i+1}) - \eta_{2,i} L_{q_i}^{\alpha_i} f(s,x)(t_{i+1}) \right) + \sum_{j=1}^{n} \left[ \mu \left( \sum_{i=0}^{m} \varphi_j(x(t_{j})) + \sum_{j=1}^{k_{\alpha}} L_{q_j}^{\alpha} f(s,x)(t_{j+1}) \right) \Omega_{\beta_i} - \eta_{2,i} \varphi_j(x(t_{j})) \right] \right\} + \sum_{i=1}^{k} \varphi_i(x(t_{i}))) + \sum_{i=0}^{k_{\alpha}} L_{q_i}^{\alpha} f(s,x)(t_{i+1}) + \sum_{i=0}^{k_{\alpha}} L_{q_i}^{\alpha} f(s,x)(t).
\]

(7)

Then, the existence of solutions of system (1) is equivalent to the problem of fixed point of operator \( \mathcal{G} \) in (7).

**Theorem 1** Let \( f : J \times \mathbb{R} \rightarrow \mathbb{R} \) and \( \varphi_k : \mathbb{R} \rightarrow \mathbb{R}, k = 1, 2, \cdots, m \) be continuous functions. Assume that \( \mu \sum_{j=0}^{k_{\alpha}} \Omega_{\beta_j} \neq \eta_1 + \eta_2 \) and the following conditions are satisfied:

(H1) There exists a positive constant \( L \) such that \( \left| \varphi_k(x) - \varphi_k(y) \right| \leq L \left| x - y \right| \) for each \( x, y \in \mathbb{R} \) and \( k = 1, 2, \cdots, m \).

(H2) There exists a function \( M(t) \in C(J, \mathbb{R}^+) \) such that

\[
\left| f(t,x) - f(t,y) \right| \leq M(t) \left| x - y \right|, \forall t \in J, x, y \in \mathbb{R}.
\]

(H3) \( \Delta < 1 \).

Then problem (1) has a unique solution on \( J \), where \( M = \sup_{t \in J} |M(t)| \) and

\[
\Delta = \frac{1}{\Lambda} \sum_{j=0}^{k_{\alpha}} \left( \mu \Omega_{\alpha_j} + (\eta_1 + M) \Omega_{\alpha_j} + \mu \sum_{j=0}^{k_{\alpha}} \Omega_{\alpha_j, \beta_j} + \mu L \Omega_{\beta_j} \right)
+ \frac{1}{\Lambda} \left( \mu \Omega_{\alpha_0} + \eta_2 \Omega_{\alpha_0} \right) + mL \left( \frac{1}{\Lambda} \eta_2 + 1 \right).
\]

**Proof.** The conclusion will follow once we have shown that the operator \( \mathcal{G} \) defined (7) is a contraction with respect to a suitable norm on \( PC(J, \mathbb{R}) \).

For any functions \( x, y \in PC(J, \mathbb{R}) \), we have

\[
\left\| \mathcal{G}x(t) - \mathcal{G}y(t) \right\| \leq \frac{1}{\Lambda} \left\{ \sum_{i=0}^{m} \left( \mu_i L_{q_i}^{\alpha_i^{+}, \beta_i} \left| f(s,x)(t_{i+1}) - f(s,y)(t_{i+1}) \right| + \eta_{2,i} L_{q_i}^{\alpha_i} \left| f(s,x) - f(s,y)(t_{i+1}) \right| \right) + \sum_{j=1}^{n} \left[ \mu \left( \sum_{i=0}^{m} \varphi_j(x(t_{j})) - \varphi_j(y(t_{j})) \right) + \sum_{j=1}^{k_{\alpha}} L_{q_j}^{\alpha} \left| f(s,x) - f(s,y)(t_{j+1}) \right| \right] \Omega_{\beta_j}
+ \eta_{2,i} \left| \varphi_i(x(t_{i})) - \varphi_i(y(t_{i})) \right| \right\} + \sum_{i=1}^{k} \left| \varphi_i(x(t_{i}))) - \varphi_i(y(t_{i})) \right| \right\}
+ \sum_{i=0}^{k_{\alpha}} L_{q_i}^{\alpha} \left| f(s,x) - f(s,y)(t_{i+1}) \right| + \sum_{i=0}^{k_{\alpha}} L_{q_i}^{\alpha} \left| f(s,x) - f(s,y)(t) \right|.
\]

By conditions (H1) and (H2), we get

\[
\left\| \mathcal{G}x(t) - \mathcal{G}y(t) \right\| \leq \frac{1}{\Lambda} \left\{ \sum_{i=0}^{m} \left( \mu_i L_{q_i}^{\alpha_i^{+}, \beta_i} \left( M \left| x - y \right| \right)(t_{i+1}) + \eta_{2,i} L_{q_i}^{\alpha_i} \left( M \left| x - y \right| \right)(t_{i+1}) \right) + \sum_{j=1}^{n} \left[ \mu \left( \sum_{i=0}^{m} L_{q_i}^{\alpha} \left( M \left| x - y \right| \right) \Omega_{\beta_j} + \eta_{2,i} L \left| x - y \right| \right) \right\}
+ \sum_{i=0}^{k_{\alpha}} L_{q_i}^{\alpha} \left( M \left| x - y \right| \right)(t_{i+1}) + \sum_{i=0}^{k_{\alpha}} L_{q_i}^{\alpha} \left( M \left| x - y \right| \right)(t) \right\}
\]
\[ + \sum_{i=0}^{m} L \| x - y \| + \sum_{i=1}^{m-1} \int_{t_i}^{t_{i+1}} \left( M \| x - y \| \right) (t_i, t_{i+1}) + \int_{t_m}^{t_{m+1}} \left( y(t_m) \right) \left( t_m, t_{m+1} \right) \]
\[ \leq \left\{ \frac{1}{\Lambda} \sum_{i=1}^{m} \left( \mu M \Omega_{a_i, \beta_i} + \eta_z \Omega_{a_i} + \mu L \Omega_{\beta_i} + \mu M \sum_{j=0}^{i-1} \Omega_{a_j, \beta_j} + M \Omega_{a_i} \right) \right\} \]
\[ + \frac{1}{\Lambda} \left( \mu \Omega_{a_0, \beta_0} + \eta_2 \Omega_{a_0} \right) + mL \left( \frac{1}{\Lambda} \eta_z + 1 \right) \| x - y \|. \]

which implies that
\[ \| Gx - Gy \| \leq \Delta \| x - y \|. \]

Thus the operator \( G \) is a contraction in view of the condition (H3). By Banach’s contraction mapping principle, the problem (1) has a unique solution on \( J \). This completes the proof.

In the following, we study the Hyers-Ulam stability of impulsive fractional \( q \)-difference Equation (1). Let \( \epsilon > 0, \epsilon > 0 \) and \( \delta : [0, T] \to \mathbb{R} \) be a continuous function. Consider the inequalities:
\[ \left\| x(t) - \bar{x}(t) \right\| \leq \delta(t) \epsilon, \quad t \in J \subseteq \left[ 0, T \right], \quad t \neq t_k, \quad k = 1, 2, \ldots, m, \]
\[ \left\| \Delta x(t_k) - \phi_k(\bar{x}(t_k)) \right\| \leq \epsilon \delta, \quad k = 1, 2, \ldots, m, \]
\[ \eta_1 \bar{x}(0) + \eta_2 \bar{x}(T) = \mu \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \bar{x}(t_k) \]
\[ \text{Now, we give out the definition of Hyers-Ulam stability of system (1).} \]

**Definition 4** System (1) is Hyers-Ulam stable with respect to system (8), if there exists \( A_j > 0 \) such that
\[ \| x - \bar{x} \| \leq A_j \epsilon \]
for all \( t \in J \), where \( \bar{x} \) is the solution of (8), and \( x \) of the solution for system (1).

**Theorem 2** Assume \( f : J \times \mathbb{R} \to \mathbb{R} \) satisfy assumption (H2), \( \phi_i : \mathbb{R} \to \mathbb{R}, i = 1, 2, \ldots, m \) are continuous functions and satisfy assumption (H1) and the condition (H3) holds, \( \sup \delta(t) \leq 1 \). Then the system (1) is Hyers-Ulam stable with respect to system (8).

**Proof:** Let \( \int_{t_k}^{t_{k+1}} x(t) = f(t, \bar{x}(t)) + g(t), k = 0, 1, \ldots, m \) and \( \Delta x(t_k) = \phi_k(\bar{x}(t_k)) + g_k, k = 1, 2, \ldots, m \). Consider the system
\[ \int_{t_k}^{t_{k+1}} x(t) = f(t, \bar{x}(t)) + g(t), \quad t \in J_k \subseteq \left[ 0, T \right], \quad t \neq t_k, \]
\[ \Delta x(t_k) = \phi_k(\bar{x}(t_k)) + g_k, \quad k = 1, 2, \ldots, m. \]
\[ \eta_1 \bar{x}(0) + \eta_2 \bar{x}(T) = \mu \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \bar{x}(t_k) \]
\[ \text{Similarly to the system in Theorem 1, system (9) is equivalent to the following integral equation in Lemma 5.} \]
\[ \bar{x}(t) = \frac{1}{\Lambda} \left( \mu \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \left( f(s, \bar{x}) + g(s) \right) (t_j, t_{j+1}) - \eta_2 \int_{t_0}^{t} \left( f(s, \bar{x}) + g(s) \right) (t_{j+1}, t_j) \right) \]
\[ + \sum_{j=0}^{n} \mu \left( \int_{t_j}^{t_{j+1}} \phi_j(\bar{x}(t_j)) + g_j \right) + \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \phi_j(\bar{x}(t_j)) + g_j \]
\[ \text{DOI: 10.4236/jamp.2020.87107 1418 Journal of Applied Mathematics and Physics} \]
Now, we define the operator $\bar{G}$ as following

$$
\bar{G}x = \frac{1}{\Lambda} \left\{ \sum_{i=0}^{m} \left( \mu_i I_{\alpha_i}^{\beta_i} f(s, x)(t_{i+1}) - \eta_2 I_{\alpha_i}^{\beta_i} f(s, x)(t_{i+1}) \right) + \sum_{j=1}^{k} \left( \sum_{i=0}^{m} g_j + \sum_{j=0}^{k-1} I_{\alpha_j}^{\beta_j} g(t_{j+1}) \right) + \sum_{i=1}^{m} g_i + \sum_{i=0}^{k-1} I_{\alpha_i}^{\beta_i} g(t_{i+1}) \right\} \right\} \Omega_{\beta_i} - \eta_2 \bar{G}_i \}
$$

$$
= Gx + G(t).
$$

where

$$
G(t) = \frac{1}{\Lambda} \left\{ \sum_{i=0}^{m} \left( \mu_i I_{\alpha_i}^{\beta_i} g(t_{i+1}) - \eta_2 I_{\alpha_i}^{\beta_i} g(t_{i+1}) \right) + \sum_{j=1}^{k} \left( \sum_{i=0}^{m} g_j + \sum_{j=0}^{k-1} I_{\alpha_j}^{\beta_j} g(t_{j+1}) \right) + \sum_{i=1}^{m} g_i + \sum_{i=0}^{k-1} I_{\alpha_i}^{\beta_i} g(t_{i+1}) \right\} \Omega_{\beta_i} - \eta_2 g_i \}
$$

Note that

$$
\| \bar{G}x - \bar{G}y \| = \| Gx - Gy \|.
$$

Then the existence of a solution of (1) implies the existence of a solution to (9), it follows from Theorem 1 that $\bar{G}$ is a contraction. Thus there is a unique fixed point $\bar{x}$ of $\bar{G}$, and respectively $\bar{x}$ of $G$.

Since $t \in [0, T]$ and $\sup_{t \in J} \delta(t) \leq 1$, we obtain

$$
\| G \| = \max_{t \in J} \| G(t) \|
$$

$$
\leq \max_{t \in J} \left\{ \frac{1}{\Lambda} \left\{ \sum_{i=0}^{m} \left( \mu_i I_{\alpha_i}^{\beta_i} g(t_{i+1}) - \eta_2 I_{\alpha_i}^{\beta_i} g(t_{i+1}) \right) + \sum_{j=1}^{k} \left( \sum_{i=0}^{m} g_j + \sum_{j=0}^{k-1} I_{\alpha_j}^{\beta_j} g(t_{j+1}) \right) + \sum_{i=1}^{m} g_i + \sum_{i=0}^{k-1} I_{\alpha_i}^{\beta_i} g(t_{i+1}) \right\} \Omega_{\beta_i} - \eta_2 g_i \right\}.
$$
\begin{align*}
+ \sum_{i=1}^{n} g_i + \sum_{j=0}^{n-1} I_{t_j}^{g_j} g(t) &= \left(\frac{1}{\Lambda} \sum_{j=0}^{n-1} \left( \mu \Omega_{\alpha_i + \beta_i} + \eta_2 \Omega_{\alpha_k} + \mu \eta \Omega_{\beta_i} + \mu \sum_{j=0}^{n-1} \Omega_{\alpha_j} \Omega_{\beta_j} + \Omega_{\alpha_j} \right) \right) \\
&+ \frac{1}{\Lambda} \left( \mu \Omega_{\alpha_k + \beta_k} + \eta_2 \Omega_{\alpha_k} \right) + mc \left( \frac{1}{\Lambda} \eta_2 + 1 \right) \varepsilon.
\end{align*}

Then, we get
\begin{align*}
\| \mathbf{x} - \bar{\mathbf{x}} \| &= \left\| \mathbf{G} - \mathbf{G} \mathbf{x} + G(t) \right\| \leq \left\| \mathbf{G} \mathbf{x} \right\| + \left\| G(t) \right\| \\
&\leq \Delta \left\| \mathbf{x} - \bar{\mathbf{x}} \right\| + \frac{1}{\Lambda} \sum_{j=0}^{n-1} \left( \mu \Omega_{\alpha_i + \beta_i} + \eta_2 \Omega_{\alpha_k} + \mu \eta \Omega_{\beta_i} + \mu \sum_{j=0}^{n-1} \Omega_{\alpha_j} \Omega_{\beta_j} + \Omega_{\alpha_j} \right) \\
&+ \frac{1}{\Lambda} \left( \mu \Omega_{\alpha_k + \beta_k} + \eta_2 \Omega_{\alpha_k} \right) + mc \left( \frac{1}{\Lambda} \eta_2 + 1 \right) \varepsilon.
\end{align*}

By condition (H3), we have
\begin{align*}
\| \mathbf{x} - \bar{\mathbf{x}} \| &\leq (1 - \Delta)^{-1} \left\{ \frac{1}{\Lambda} \sum_{j=0}^{n-1} \left( \mu \Omega_{\alpha_i + \beta_i} + \eta_2 \Omega_{\alpha_k} + \mu \eta \Omega_{\beta_i} + \mu \sum_{j=0}^{n-1} \Omega_{\alpha_j} \Omega_{\beta_j} + \Omega_{\alpha_j} \right) \\
&+ mc \left( \frac{1}{\Lambda} \eta_2 + 1 \right) \varepsilon.
\end{align*}

Let
\begin{align*}
A_j &= (1 - \Delta)^{-1} \left\{ \frac{1}{\Lambda} \sum_{j=0}^{n-1} \left( \mu \Omega_{\alpha_i + \beta_i} + \eta_2 \Omega_{\alpha_k} + \mu \eta \Omega_{\beta_i} + \mu \sum_{j=0}^{n-1} \Omega_{\alpha_j} \Omega_{\beta_j} + \Omega_{\alpha_j} \right) \\
&+ mc \left( \frac{1}{\Lambda} \eta_2 + 1 \right) \varepsilon,
\end{align*}
then
\begin{align*}
\| \mathbf{x} - \bar{\mathbf{x}} \| &\leq A_j \varepsilon.
\end{align*}

This completes the proof.

**Remark 1** Note that (1) has a very general form, as special instances results from (1), when, \( \eta_1 = \eta_2 = 1, \mu = 0 \), (1) reduces to the antiperiodic boundary value problem of the impulsive fractional \( q \)-difference equation:
\begin{align*}
\begin{cases}
{}^q_i D_{t_i}^{\alpha_i} x(t) = f(t, x(t)), & t \in J_1 \subseteq [0, T], t \neq t_k, \\
\Delta x(t_k) = x(t_k) - x(t_k) = \varphi_k(x(t_k)), & k = 1, 2, \cdots, m, \\
x(0) + x(T) = 0.
\end{cases}
\end{align*}

### 4. Example

Consider the following boundary value problem:
\begin{align*}
\begin{cases}
{}^q_i D_{t_i}^{\alpha_i} x(t) = \sin^2 t \frac{2|x(t)|}{t^2 + 50 + 1 + x(t)} + \frac{3t}{4}, & t \in \left[ 0, \frac{3}{2} \right) \setminus \{ t_1, t_2 \}, \\
\Delta x(t_k) = \frac{x(t_k) + 2|x(t_k)|}{1 + x(t_k)} + \frac{k}{5} t_k = \frac{k}{2}, & k = 1, 2, \\
\frac{8}{3} x(0) + x \left( \frac{3}{2} \right) = \frac{1}{2} \sum_{k=0}^{\frac{3}{2}} \frac{1}{1 + k} \frac{2}{4} x(t_k + 1).
\end{cases}
\end{align*}
Corresponding to boundary value problem (1), one see that
\[ \alpha_k = \frac{k+1}{3k+2}, \beta_k = \frac{k+1}{k+2}, q_k = \frac{3k+1}{4k+3}, t_k = \frac{k}{2}, f(t,x) = \frac{\sin^2 t}{t^2 + 50} \frac{2|x(t)|}{1 + |x(t)|} + \frac{3}{4}, \]

\[ \phi_k(x(t_k)) = \frac{1}{200k}\left(\frac{x(t_k)}{1 + |x(t_k)|}\right)^2. \]

Through a simple calculation, we get
\[ |f(t,x) - f(t,y)| \leq \frac{\sin^2 t}{t^2 + 25} |x - y|, M(t) = \frac{\sin^2 t}{t^2 + 25} \leq \frac{1}{25} = M, \]
\[ |\phi_k(x) - \phi_k(y)| \leq \frac{1}{200k} |x - y| \leq \frac{1}{200} |x - y|, L = \frac{1}{200}, \]
\[ \Lambda \triangleq 1.7875 > 0, \Delta \triangleq 0.4873 < 1. \]

From Theorem 1, the problem (16) has a unique solution \( x \) on \( [0, \frac{3}{2}] \). Furthermore, the solution \( x \) is Hyers-Ulam stable with respect to the following system

\[ \left\{ \begin{array}{l}
\dot{x}(t) - \frac{1}{200k}\left(\frac{x(t)}{1 + |x(t)|}\right)^2 = \epsilon, t \in \left[0, \frac{3}{2}\right] \setminus \{t_1, t_2\}, \\
\Delta x(t_k) = \frac{1}{200k}\left(\frac{x(t_k)}{1 + |x(t_k)|}\right)^2 = \epsilon, t_k = \frac{k}{2}, k = 1, 2, \\
8 x(0) + \frac{1}{6} x\left(\frac{3}{2}\right) = \frac{1}{2} \sum_{k=0}^{\frac{k+1}{4k+3}} x(t_{k+1}),
\end{array} \right. \]

where \( \epsilon > 0, \epsilon > 0, \sup_{t \in [0, \frac{3}{2}]} |\delta(t)| < 1. \)

5. Conclusion

In this paper, we study the existence and Hyers-Ulam stability of solutions for impulsive fractional \( q \)-difference equation. We obtain some results as following:
1) Using the \( q \)-shifting operator, the results of existence of solutions for impulsive fractional \( q \)-difference equation with \( q \)-integral boundary conditions are obtained. 2) The Hyers-Ulam stability of the nonlinear impulsive fractional \( q \)-difference equations was obtained.

Funding

This research was supported by Science and Technology Foundation of Guizhou Province (Grant No. [2016] 7075), by the Project for Young Talents Growth of Guizhou Provincial Department of Education under (Grant No. Ky [2017] 133), and by the project of Guizhou Minzu University under (Grant No.16yjrxml002).

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.
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