We consider the Wheeler-DeWitt equation as a device for finding eigenvalues of a Sturm-Liouville problem. In particular, we will focus our attention on the electric (magnetic) Maxwell charge. In this context, we interpret the Maxwell charge as an eigenvalue of the Wheeler-DeWitt equation generated by the gravitational field fluctuations. A variational approach with Gaussian trial wave functionals is used as a method to study the existence of such an eigenvalue. We restrict the analysis to the graviton sector of the perturbation. We approximate the equation to one loop in a Schwarzschild background and a zeta function regularization is involved to handle with divergences. The regularization is closely related to the subtraction procedure appearing in the computation of Casimir energy in a curved background. A renormalization procedure is introduced to remove the infinities together with a renormalization group equation.

**INTRODUCTION**

In 1955, John Archibald Wheeler considered the possibility that the gravitational coupled to the electromagnetic field could lead to a sourceless solution termed “geon”. Further studies in this direction gave birth to particular ideas such as “mass without mass” and “charge without charge”, where fluctuations of the gravitational field were thought as responsible of the generation of elementary particles. It is clear that, if such a possibility exists, this is encoded in the Einstein’s field equations. These equations are simply summarized by

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda c g_{\mu\nu} = \kappa T_{\mu\nu}, \]  

(1)

where \( T_{\mu\nu} \) is the energy-momentum tensor of some matter fields, \( \kappa = 8\pi G \) with \( G \) the Newton’s constant and \( \Lambda c \) is the cosmological constant. The idea is to recognize the gravitational field as a fundamental field and see what implications we have on the cosmological constant and on the matter fields. In Ref. [2], we have applied this concept to the cosmological constant. In particular, we have considered the cosmological constant as an eigenvalue of an associated Sturm-Liouville problem, even in presence of a massive graviton. Motivated by this result, in this paper we would like to apply the same approach to the Maxwell charge. To do this, we need to introduce the Wheeler-DeWitt equation (WDW). The WDW equation can be extracted from the Einstein’s field equations with and without matter fields in a very simple way. If we introduce a time-like unit vector \( u^\mu \) such that \( u^\mu u_\mu = -1 \), then after a little rearrangement of Eqs. (1), we get:

\[ \mathcal{H}_0 = (2\kappa) G_{ijkl} \pi^{ij} \pi^{kl} - \frac{1}{2\kappa} \sqrt{g} \pi^{3} R = 0, \]  

(2)

for the sourceless case and in absence of a cosmological term.

\[ \mathcal{H}_\Lambda = (2\kappa) G_{ijkl} \pi^{ij} \pi^{kl} - \sqrt{g} \left( \frac{3}{2\kappa} R - 2\Lambda c \right) = 0, \]  

(3)

for the sourceless case and in presence of a cosmological term.

\[ \mathcal{H}_Q = (2\kappa) G_{ijkl} \pi^{ij} \pi^{kl} - \sqrt{g} \left( \frac{3}{2\kappa} R - \mathcal{H}_M \right) = 0, \]  

(4)

with a matter term and in absence of a cosmological constant. Note the formal similarity between Eqs. (3) and (4). \( G_{ijkl} \) is the supermetric defined as

\[ G_{ijkl} = \frac{1}{2} (g_{ik} g_{jl} + g_{il} g_{jk} - g_{ij} g_{kl}) \]  

(5)

and \( R \) is the scalar curvature in three dimensions. \( \pi^{ij} \) is called the supermomentum. This is the time-time component of Eqs. (1). It represents the invariance under time reparametrization and it works as a constraint at the classical
The classical constraint $\mathcal{H}_Q$ for a Maxwell charge becomes

$$\mathcal{H}_Q = (2\kappa) G_{ijkl} \pi^i \pi^k - \frac{\sqrt{g}}{2\kappa} \left( 3 R - \kappa \sqrt{3 g} T_{\alpha \beta} u^\alpha u^\beta \right) = 0.$$  

If $\mathcal{H}_Q$ is promoted to an operator, then the following quantum constraint

$$\mathcal{H}_Q \Psi = 0$$

is imposed. This is known as the WDW equation. The WDW can be rearranged in such a way to show a more useful aspect. Indeed, if we integrate Eq. (11) over the hypersurface $\Sigma$ obtained with the help of the Arnowitt-Deser-Misner variables (ADM) and we define

$$\hat{Q}_\Sigma = (2\kappa) G_{ijkl} \pi^i \pi^k - \frac{\sqrt{g}}{2\kappa} \left( 3 R - \kappa \sqrt{3 g} T_{\alpha \beta} u^\alpha u^\beta \right),$$

then Eq. (11) can be cast into the following form

$$\frac{\left< \Psi \left[ \int_{\Sigma} d^3 x \hat{Q}_\Sigma \right] \Psi \right>}{\left< \Psi \Psi \right>} = - \frac{\left< \Psi \left[ \int_{\Sigma} d^3 x \left( \sqrt{3 g} T_{\alpha \beta} u^\alpha u^\beta \right) \right] \Psi \right>}{2 \left< \Psi \Psi \right>}.$$

Eq. (13) has been obtained by multiplying Eq. (11) by $\Psi^* [g_{ij}]$ and by functionally integrating over the three spatial metric $g_{ij}$. To fix the ideas, let us consider the electric case. Thus, if we substitute the expression (8) into Eq. (13), we get

$$\frac{\left< \Psi \left[ \int_{\Sigma} d^3 x \hat{Q}_\Sigma \right] \Psi \right>}{\left< \Psi \Psi \right>} = - \frac{1}{2} \int_{\Sigma} d^3 x \sqrt{g} \rho_e.$$

It is immediately clear that a classical solution is represented by the Reissner-Nordström (RN) metric, whose form is

$$ds^2 = - f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right),$$

where the gravitational potential is expressed by

$$f(r) = 1 - \frac{2MG}{r} + \frac{G(Q_e^2 + Q_m^2)}{r^2}.$$
Nevertheless, we are not interested in finding corrections to the RN metric, rather we want to use Eq. (14) as a device to find consistent solutions of an electric/magnetic charge generated by quantum fluctuations of the gravitational field. This approach is not completely new, it appears naturally when black hole mass quantization is discussed in some specific models even including a charge[6, 7, 8]. Nevertheless, in this approach we will never deal with RN black holes. On the other hand, it appears that our approach seems to be closer to Ref.[9], where a quantum analysis of Wheeler’s geons[1] was carried out, even if a real quantum computation seems to be absent. The semi-classical procedure followed in this work relies heavily on the formalism outlined in Refs.[2, 3]. The computation was realized through a variational approach with Gaussian trial wave functionals. A zeta function regularization is used to deal with the divergences, and a renormalization procedure is introduced, where the finite one loop is considered as a self-consistent source for traversable wormholes. Rather than reproduce the formalism, we shall refer the reader to Refs.[2, 3] for details, when necessary.

ONE LOOP CHARGE EVALUATION

We can write the background metric in the following form

\[ ds^2 = -N^2(r) dt^2 + \frac{dr^2}{1 - \frac{br}{r}} + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \]  

where \( r \in [r_t, +\infty) \) and

\[ b(r_t) = r_t. \]  

(17)

(18)

\( r_t \) is termed the throat. \( N(r) \) is the “lapse function” playing the role of the “redshift function”, while \( b(r) \) is termed “shape function”. We take into account the total regularized one loop energy given by

\[ E^\text{TT} = 2 \int_{r_0}^{\infty} dr \frac{r^2}{\sqrt{1 - b(r)/r}} [\rho_1(\varepsilon) + \rho_2(\varepsilon)]. \]  

\[ (19) \]

The energy densities, \( \rho_i(\varepsilon) \) (with \( i = 1, 2 \)), are defined as

\[ \rho_i(\varepsilon) = \frac{1}{4\pi} \mu^{2\varepsilon} \int_{U_i(r)} d\tilde{E}_i \frac{\tilde{E}_i^2}{\left[ \tilde{E}_i^2 - m_i^2(r) \right]^{\varepsilon - 1/2}} \]

\[ = -\frac{m_i^4(r)}{64\pi^2} \left[ \frac{1}{\varepsilon} + \ln \left( \frac{\mu^2}{m_i^2(r)} \right) + 2\ln 2 - \frac{1}{2} \right]. \]

\[ (20) \]

where we have defined two r-dependent effective masses \( m_1^2(r) \) and \( m_2^2(r) \), which can be cast in the following form

\[ \begin{cases} 
    m_1^2(r) = m_L^2(r) + m_{1,S}^2(r) \\
    m_2^2(r) = m_L^2(r) + m_{2,S}^2(r)
\end{cases} \]  

\[ (21) \]

where

\[ m_L^2(r) = \frac{6}{r^2} \left( 1 - \frac{b(r)}{r} \right) \]  

\[ (22) \]

and

\[ \begin{cases} 
    m_{1,S}^2(r) = \frac{3}{2\pi} b'(r) - \frac{1}{2\pi} b(r) \\
    m_{2,S}^2(r) = \frac{3}{2\pi} b'(r) + \frac{1}{2\pi} b(r)
\end{cases} \]  

\[ (23) \]

We refer the reader to Refs. [2, 3] for the deduction of these expressions in the Schwarzschild case. The zeta function regularization method has been used to determine the energy densities, \( \rho_i \). It is interesting to note that this method is identical to the subtraction procedure of the Casimir energy computation, where the zero point energy in different
term by the fact that this is the most simple spherically symmetric solution of the Einstein's field equations

Using Eqs. (22), we impose that

Substituting Eq. (29) into Eq. (27) we find

It is essential to renormalize the divergent energy by absorbing the singularity in the classical quantity, by redefining the bare classical charge \( Q_e \) as

Using this, Eq. (26) takes the form

To avoid the dependence on the arbitrary mass scale \( \mu \) in Eq. (27), we adopt the renormalization group equation and we impose that

Solving it we find that the renormalized squared charge \( Q^{\text{\scriptsize eff}}_{e,0} (\mu, r) \) should be treated as a running one in the sense that it varies provided that the scale \( \mu \) is changing

Substituting Eq. (29) into Eq. (27) we find

Using Eqs. (22, 23), Eq. (30) becomes

Even if this result is valid for an arbitrary function \( b(r) \), we fix our attention to the Schwarzschild metric, motivated by the fact that this is the most simple spherically symmetric solution of the Einstein’s field equations without a source term. In this case, we get \( b(r) = 2MG = r_t \). It is straightforward to see that for large distances, \( m^2_{L} (r) \) dominates and \( Q^{\text{\scriptsize eff}}_{e,0} (\mu_0, r) \) diverges. So, the validity of this result is confined to stay close to the throat. When we approach \( r_t \), \( m^2_{L} (r) \to 0 \) and \( m^2_{L,S} (r) < 0 \). Precisely, \( m^2_{L,S} (r) \) becomes negative when \( r \in [r_t, \frac{3}{2} r_t] \) and \( m^2_{L,S} (r) = -m^2_{L,S} (r) = -3r_t/2r^3 \). In such a range, Eq. (31) simplifies to

\( Q^{\text{\scriptsize eff}}_{e,0} (\mu_0, r) = -\frac{r^4}{16\pi} \frac{3r_t \sqrt{e}}{2r^3} \ln \left( \frac{3r_t \sqrt{e}}{8\pi^2 \mu_0^2} \right) \).
In order to remove the dependence on \( r \), we compute

\[
\frac{\partial Q_{e,0}^2 (\mu_0, r)}{\partial r} = 0 \quad \Rightarrow \quad \frac{3 r \mu_0}{8 \mu_0^2} = \bar{r}^3
\]  

(33)

and

\[
Q_{e,0}^2 (\mu_0, \bar{r}) = \frac{\bar{r}^4 \mu_0^4}{2 \pi e^2} = \left( \frac{3 r \mu_0}{8 \sqrt{e}} \right)^{\frac{1}{3}} \frac{1}{2 \pi}.
\]  

(34)

Since

\[
r \in \left[ r_t, \frac{5}{4} r_t \right] \quad \Rightarrow \quad \sqrt{\frac{3 e}{8 r_t^2}} \geq \mu_0 \geq \sqrt{\frac{24 e}{125 r_t^2}},
\]  

(35)

we find the following bound

\[
2.2 \times 10^{-2} = \frac{9}{128 \pi} \geq Q_{e,0}^2 (\mu_0, \bar{r}) \geq \frac{9}{200 \pi} = 1.4 \times 10^{-2}.
\]  

(36)

Note that the fine structure constant is \( \frac{1}{\gamma} = 7.3 \times 10^{-21} \). A comment to this result is in order. From Eq. (34) and the bound (36), we infer that once the charge has been created form quantum fluctuations, it never disappears, unless the throat is large as the whole universe. Secondly, the bound is very close to the fine structure constant, but it has not the desired value. This is a good news, because the computation is limited to the graviton and even if it is represented by TT modes which are gauge invariant, they do not represent the whole perturbation, only the leading one. Therefore, it is likely that the new input will correct the value of the bound. In particular, it is expected that trace modes could screen the Maxwell charge because of the opposite sign of the spin 0 term.

**CONCLUSIONS**

In this letter, we have considered the possibility that an electric/magnetic charge be generated by quantum fluctuations of the pure gravitational field. The calculational kit is based on a variational version of the Sturm-Liouville problem, already applied in the cosmological context\[2, 3\]. In this context, as in other contexts examined with this approach, e.g., self-sustained wormholes\[12\], we have put the accent on the fluctuations of the gravitational field which act as a source for the matter fields. This still is in the spirit of the Einstein’s field equations and the idea of unification of the forces, but it is in contrast with the Sakharov’s induced gravity\[13\]. In such a theory, the low-energy effective action \( \Gamma [g] \) is defined as a quantum average of the constituent matter fields \( \Phi \) propagating in a given gravitational background \( g \)

\[
\exp (-\Gamma [g]) = \int \mathcal{D}\Phi \exp (-S [\Phi, g]).
\]  

(37)

The Sakharov’s basic assumption is that the gravitational field becomes dynamical only as the result of quantum effects of the constituents fields. This theory has the pleasant feature of being renormalizable. However, its application is rather limited to some specific problems like black hole entropy\[14\]. It is interesting to note that our approach provides also a nonvanishing magnetic charge. This result could support the idea that at the Planck scale magnetic monopoles could exist. Nevertheless, if this is true, it is straightforward to see that a suppression mechanism at low energy should emerge. Moreover, we have to observe that the choice of the Gaussian wave functional corresponds to a “ground state” functional and consistently we obtain only one eigenvalue. Different choices of the wave functional correspond to different boundary conditions. Moreover, different forms of the Gaussian wave functionals can be considered to form “excited states”. Finally, we want to remark that the Schwarzschild choice has been made because this is the simplest spherically symmetric solution of the Einstein’s field equations without matter fields, which is exactly what we need to generate a charge from gravity.

\[1\] See also Ref.\[11\] for another approach based on the WDW equation.
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