INTEGRAL MOMENTS OF AUTOMORPHIC $L$–FUNCTIONS

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Abstract. This paper exposes the underlying mechanism for obtaining second integral moments of $GL_2$ automorphic $L$–functions over an arbitrary number field. Here, moments for $GL_2$ are presented in a form enabling application of the structure of adele groups and their representation theory. To the best of our knowledge, this is the first formulation of integral moments in adele-group-theoretic terms, distinguishing global and local issues, and allowing uniform application to number fields. When specialized to the field of rational numbers $\mathbb{Q}$, we recover the classical results.

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§1. Introduction

For ninety years, the study of mean values of families of automorphic $L$–functions has played a central role in analytic number theory, for applications to classical problems. In the absence of the Riemann Hypothesis, or the Grand Riemann Hypothesis, rather, when referring to general $L$–functions, suitable mean value results often serve as a substitute. In particular, obtaining asymptotics or sharp bounds for integral moments of automorphic $L$–functions is of considerable interest. The study of integral moments was initiated in 1918 by Hardy and Littlewood (see [Ha-Li]) who obtained the asymptotic formula for second moment of the Riemann zeta-function

\[
\int_0^T |\zeta(\frac{1}{2} + it)|^2 \, dt \sim T \log T
\]

About 8 years later, Ingham in [I] obtained the fourth moment

\[
\int_0^T |\zeta(\frac{1}{4} + it)|^4 \, dt \sim \frac{1}{2\pi^2} \cdot T(\log T)^4
\]

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Since then, many papers by various authors have been devoted to this subject. For instance see [At], [H-B], [G1], [M1], [J1]. Most existing results concern integral moments of automorphic $L$–functions for $GL_1(\mathbb{Q})$ and $GL_2(\mathbb{Q})$. No analogue of (1.1) or (1.2) is known over an arbitrary number field. The only previously known results, for fields other than $\mathbb{Q}$, are in [M4], [S1], [BM1], [BM2] and [DG2], all over quadratic fields.

Here we expose the underlying mechanism to obtain second integral moments of $GL_2$ automorphic $L$–functions over an arbitrary number field. Integral moments for $GL_2$ are presented in a form amenable to application of the representation theory of adele groups. To the best of our knowledge, this is the first formulation of integral moments on adele groups, distinguishing global and local questions, and allowing uniform application to number fields. More precisely, for $f$ an automorphic form on $GL_2$ and $\chi$ an idele class character of the number field, let $L(s, f \otimes \chi)$ denote the twisted $L$–function attached to $f$. We obtain asymptotics for averages

$$\sum_{\chi} \int_{-\infty}^{\infty} |L(\frac{1}{2} + it, f \otimes \chi)|^2 M_\chi(t) \, dt$$

for suitable smooth weights $M_\chi(t)$. The sum in (1.3) is over a certain set of idele class characters which is infinite, in general. For general number fields, it seems that (1.3) is the correct structure of the second integral moment of $GL_2$ automorphic $L$–functions. This was first pointed out by Sarnak in [S1], where an average of the above type was studied over the Gaussian field $\mathbb{Q}(i)$; see also [DG2]. From the analysis of Section 2, it will become apparent that this comes from the Fourier transform on the idele class group of the field.

Meanwhile, in joint work [DGG] with Goldfeld, the present authors have found an extension to treat integral moments for $GL_r$ over number fields. We exhibit specific Poincaré series $\mathcal{P}$ giving identities of the form

$$\text{moment expansion} = \int_{Z_{\Delta GL_r(k)} \backslash GL_r(k)} \mathcal{P} \cdot |f|^2 = \text{spectral expansion}$$

for cuspforms $f$ on $GL_r$. The moment expansion on the left-hand side is of the form

$$\sum_F \frac{1}{2 \pi i} \int_{\Re(s) = \frac{1}{2}} |L(s, f \otimes F)|^2 M_F(s) \, ds + \ldots$$

summed over $F$ in an orthonormal basis for cuspforms on $GL_{r-1}$, as well as corresponding continuous-spectrum terms. The specific choice gives a kernel with a surprisingly simple spectral expansion, with only three parts: a leading term, a sum induced from cuspforms on $GL_2$, and a continuous part again induced from $GL_2$. In particular, no cuspforms on $GL_\ell$ with $2 < \ell \leq r$ contribute. Since the discussion for $GL_r$ with $r > 2$ depends essentially on the details of the $GL_2$ results, the $GL_2$ case merits special attention. We give complete details for $GL_2$ here. For $GL_2$ over $\mathbb{Q}$ and square-free level, the average of moments has a single term, recovering the classical integral moment

$$\int_{-\infty}^{\infty} |L(\frac{1}{2} + it, f)|^2 M(t) \, dt$$

As a non-trivial example, consider the case of a cuspform $f$ on $GL_3$ over $\mathbb{Q}$. We construct a weight function $\Gamma(s, w, f_\infty, F_\infty)$ depending upon complex parameters $s$ and $w$, and upon the
archimedean data for both $f$ and cuspforms $F$ on $GL_2$, such that $\Gamma(s, w, f_\infty, F_\infty)$ has explicit asymptotic behavior similar to those in Section 5 below, and such that the moment expansion above becomes

$$\int_{Z_3GL_3(\mathbb{Q})\backslash GL_3(\mathbb{A})} \text{Pé}(g) |f(g)|^2 \, dg = \sum_{F \text{ on } GL_2} \frac{1}{2\pi i} \int_{\Re(s)=\frac{1}{2}} |L(s, f \otimes F)|^2 \cdot \Gamma(s, w, f_\infty, F_\infty) \, ds$$

$$+ \frac{1}{4\pi i} \frac{1}{2\pi i} \sum_{k \in \mathbb{Z}} \int_{\Re(s_1)=\frac{1}{2}} \int_{\Re(s_2)=\frac{1}{2}} |L(s_1, f \otimes E_{1-s_2}^{(k)})|^2 \cdot \Gamma(s_1, w, f_\infty, E_{1-s_2, \infty}^{(k)}) \, ds_2 \, ds_1$$

where

$$L(s_1, f \otimes E_{1-s_2}^{(k)}) = \frac{L(s_1 - s_2 + \frac{1}{2}, f) \cdot L(s_1 + s_2 - \frac{1}{2}, f)}{\zeta(2 - 2s_2)}$$

In the above expression, $F$ runs over an orthonormal basis for all level-one cuspforms on $GL_2$, with no restriction on the right $K_\infty$–type. Similarly, the Eisenstein series $E_s^{(k)}$ run over all level-one Eisenstein series for $GL_2(\mathbb{Q})$ with no restriction on $K_\infty$–type, denoted here by $k$.

The course of the argument makes several points clear. First, the sum of moments of twists of $L$–functions has a natural integral representation. Second, the kernel arises from a collection of local data, wound up into an automorphic form, and the computation proceeds by unwinding. Third, the local data at finite primes is of a mundane sort, already familiar from other constructions. Fourth, the only subtlety resides in choices of archimedean data. Once this is understood, it is clear that Good’s original idea in [G2], seemingly limited to $GL_2(\mathbb{Q})$, exhibits a good choice of local data for real primes. See also [DG1]. Similarly, while [DG2] explicitly addresses only $GL_2(\mathbb{Z}[i])$, the discussion there exhibits a good choice of local data for complex primes. That is, these two examples suffice to illustrate the non-obvious choices of local data for all archimedean places.

The structure of the paper is as follows. In Section 2, a family of Poincaré series is defined in terms of local data, abstracting classical examples in a form applicable to $GL_r$ over a number field. In Section 3, the integral of the Poincaré series against $|f|^2$ for a cuspform $f$ on $GL_2$ is unwound and expanded, yielding a sum of weighted moment integrals of $L$-functions $L(s, f \otimes \chi)$ of twists of $f$ by Größencharakteren $\chi$. In Section 4, we find the spectral decomposition of the Poincaré series: the leading term is an Eisenstein series, and there are cuspidal and continuous-spectrum parts with explicit coefficients. In section 5, we derive an asymptotic formula for integral moments, and observe that the length of the averages involved is suitable for subsequent applications to convexity breaking in the $t$–aspect. The first appendix discusses convergence of the Poincaré series in some detail, proving pointwise convergence from two viewpoints, also proving $L^2$ convergence. The second appendix computes integral transforms necessary to understand the details in the spectral expansion.

For applications, one needs to combine refined choices of archimedean data with extensions of the estimates in [Ho-Lo] and [S2] (or [BR]) to number fields. However, for now, we content ourselves with a formulation that lays the groundwork for applications and extensions. In subsequent papers we will address convexity breaking in the $t$–aspect, and extend this approach to $GL_r$. 
§2. Poincaré series

Before introducing our Poincaré series \( \mathcal{P}(g) \) on \( GL_r \) (\( r \geq 2 \)) mentioned in the introduction, we find it convenient to first fix some notation in this context. Let \( k \) be a number field, \( G = GL_r \) over \( k \), and define the standard subgroups:

\[
P = P^{r-1,1} = \left\{ \begin{pmatrix} (r-1)\text{-by-}(r-1) & \ast \\ 0 & 1\text{-by-1} \end{pmatrix} \right\}
\]

the standard maximal proper parabolic subgroup,

\[
U = \left\{ \begin{pmatrix} I_{r-1} & \ast \\ 0 & 1 \end{pmatrix} \right\} \quad H = \left\{ \begin{pmatrix} (r-1)\text{-by-}(r-1) & 0 \\ 0 & 1 \end{pmatrix} \right\} \quad Z = \text{center of } G
\]

Let \( K_\nu \) denote the standard maximal compact in the \( k_\nu \)–valued points \( G_\nu \) of \( G \).

The Poincaré series \( \mathcal{P}(g) \) is of the form

\[
(2.1) \quad \mathcal{P}(g) = \sum_{\gamma \in Z_k H_k \setminus G_k} \varphi(\gamma g) \quad (g \in G_A)
\]

for suitable functions \( \varphi \) on \( G_A \) described as follows. For \( v \in \mathbb{C} \), let

\[
(2.2) \quad \varphi = \bigotimes_\nu \varphi_\nu
\]

where for \( \nu \text{ finite} \)

\[
(2.3) \quad \varphi_\nu(g) = \begin{cases} 
|\det A|/d^{r-1}\nu & \text{for } g = mk \text{ with } m = \begin{pmatrix} A & 0 \\ 0 & d \end{pmatrix} \in Z_\nu H_\nu \text{ and } k \in K_\nu \\
0 & \text{otherwise}
\end{cases}
\]

and for \( \nu \text{ archimedean} \) require right \( K_\nu \text{-invariance} \) and left equivariance

\[
(2.4) \quad \varphi_\nu(mg) = \left| \frac{\det A}{d^{r-1}} \right|_\nu \cdot \varphi_\nu(g) \quad \text{for } g \in G_\nu \text{ and } m = \begin{pmatrix} A & 0 \\ 0 & d \end{pmatrix} \in Z_\nu H_\nu
\]

Thus, for \( \nu | \infty \), the further data determining \( \varphi_\nu \) consists of its values on \( U_\nu \). The simplest useful choice is

\[
(2.5) \quad \varphi_\nu\left( \begin{pmatrix} I_{r-1} & x \\ 0 & 1 \end{pmatrix} \right) = (1 + |x_1|^2 + \cdots + |x_{r-1}|^2)^{-d_\nu(r-1)w_\nu/2} \begin{pmatrix} x_1 \\ \vdots \\ x_{r-1} \end{pmatrix} \text{ and } w_\nu \in \mathbb{C}
\]

with \( d_\nu = [k_\nu : \mathbb{R}] \). Here the norm \( |x_1|^2 + \cdots + |x_{r-1}|^2 \) is invariant under \( K_\nu \), that is, \( | \cdot | \) is the usual absolute value on \( \mathbb{R} \) or \( \mathbb{C} \). Note that by the product formula \( \varphi \) is left \( Z_k H_k \)-invariant.

We have the following
Proposition 2.6. (Apocryphal) With the specific choice (2.5) of $\varphi_{\infty} = \otimes_{\nu|\infty} \varphi_{\nu}$, the series (2.1) defining $\text{P}b(g)$ converges absolutely and locally uniformly for $\Re(v) > 1$ and $\Re(w_{\nu}) > 1$ for all $\nu|\infty$.

Proof: In fact, the argument applies to a much broader class of archimedean data. For a complete argument when $r = 2$, and $w_{\nu} = w$ for all $\nu|\infty$, see Appendix 1. □

We can give a broader and more robust, though somewhat weaker, result, as follows. Again, for simplicity, we shall assume $r = 2$. Given $\varphi_{\infty}$, for $x$ in $k_{\infty} = \prod_{\nu|\infty} k_{\nu}$, let

$$\Phi_{\infty}(x) = \varphi_{\infty} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

For $0 < \ell \in \mathbb{Z}$, let $\Omega_{\ell}$ be the collection of $\varphi_{\infty}$ such that the associated $\Phi_{\infty}$ is absolutely integrable, and such that the Fourier transform $\hat{\Phi}_{\infty}$ along $k_{\infty}$ satisfies the bound

$$\hat{\Phi}_{\infty}(x) \ll \prod_{\nu|\infty} (1 + |x|_{\nu}^2)^{-\ell}$$

For example, for $\varphi_{\infty}$ to be in $\Omega_{\ell}$ it suffices that $\Phi_{\infty}$ is $\ell$ times continuously differentiable, with each derivative absolutely integrable. For $\Re(w_{\nu}) > 1$, $\nu|\infty$, the simple explicit choice of $\varphi_{\infty}$ above lies in $\Omega_{\ell}$ for every $\ell > 0$.

Theorem 2.7. (Apocryphal) Suppose $r = 2$, $\Re(v)$, $\ell$ sufficiently large, and $\varphi_{\infty} \in \Omega_{\ell}$. The series defining $\text{P}b(g)$ converges absolutely and locally uniformly in both $g$ and $v$. Furthermore, up to an Eisenstein series, the Poincaré series is square integrable on $Z_{A}G_{k}\backslash G_{k}$.

Proof: See Appendix 1. □

The precise Eisenstein series to be subtracted from the Poincaré series to make the latter square-integrable will be discussed in Section 4 (see formula 4.6). For our special choice (2.5) of archimedean data, both these convergence results apply with $\Re(w_{\nu}) > 1$ for $\nu|\infty$ and $\Re(v)$ large.

For convenience, a monomial vector $\varphi$ as in (2.2) described by (2.3) and (2.4) will be called admissible, if $\varphi_{\infty} \in \Omega_{\ell}$, with both $\Re(v)$ and $\ell$ sufficiently large.

§3. Unwinding to Euler product

From now on, we shall assume $r = 2$. Recall the notation made in the previous section, which in the present case reduces to: $G = GL_{2}$ over the number field $k$ together with the standard subgroups

$$P = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}, \quad N = U = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}, \quad M = ZH = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\}$$

Also, for any place $\nu$ of $k$, let $K_{\nu}$ be the standard maximal compact subgroup. That is, for finite $\nu$, we take $K_{\nu} = GL_{2}(O_{\nu})$, at real places $K_{\nu} = O(2)$, and at complex places $K_{\nu} = U(2)$.

With the Poincaré series defined by (2.1), our main goal is to unwind a corresponding global integral to express it as an inverse Mellin transform of an Euler product. For convenience, recall that

$$(3.1) \quad \text{P}b(g) = \sum_{\gamma \in M_{k}\backslash G_{k}} \varphi(\gamma g) \quad (g \in G_{k})$$
where the monomial vector
\[ \varphi = \bigotimes_{\nu} \varphi_\nu \]
is defined by
\[ \varphi_\nu(g) = \begin{cases} \chi_{0,\nu}(m) & \text{for } g = mk, \ m \in M_\nu \text{ and } k \in K_\nu \\ 0 & \text{for } g \not\in M_\nu \cdot K_\nu \end{cases} \quad \text{(for } \nu \text{ finite}) \]
and for \( \nu \) infinite, we do not entirely specify \( \varphi_\nu \), only requiring the left equivariance
\[ \varphi_\nu(mnk) = \chi_{0,\nu}(m) \cdot \varphi_\nu(n) \quad \text{(for } \nu \text{ infinite, } m \in M_\nu, n \in N_\nu \text{ and } k \in K_\nu) \]
Here, \( \chi_{0,\nu} \) is the character of \( M_\nu \) given by
\[ \chi_{0,\nu}(m) = \left| \frac{a}{d} \right|_\nu^{v} \quad \left( m = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in M_\nu, \ v \in \mathbb{C} \right) \]
Then, \( \chi_0 = \bigotimes_{\nu} \chi_{0,\nu} \) is \( M_k \)-invariant, and \( \varphi \) has trivial central character and is left \( M_A \)-equivariant by \( \chi_0 \). Also, note that for \( \nu \) infinite, our assumptions imply that
\[ x \rightarrow \varphi_\nu \left( \begin{array}{cc} 1 & x \\ 0 & 1 \end{array} \right) \]
is a function of \(|x|\) only.

Let \( f_1 \) and \( f_2 \) be cuspforms on \( G_A \). Eventually we will take \( f_1 = f_2 \), but for now merely require the following. At all \( \nu \), require (without loss of generality) that \( f_1 \) and \( f_2 \) have the same right \( K_\nu \)-type, that this \( K_\nu \)-type is irreducible, and that \( f_1 \) and \( f_2 \) correspond to the same vector in the \( K \)-type (up to scalar multiples). Schur’s lemma assures that this makes sense, insofar as there are no non-scalar automorphisms. Suppose that the representations of \( G_\mathfrak{A} \) generated by \( f_1 \) and \( f_2 \) are irreducible, with the same central character. Last, require that each \( f_i \) is a special vector locally everywhere in the representation it generates, in the following sense. Let
\[ f_i(g) = \sum_{\xi \in \mathbb{Z}_k \setminus M_k} W_i(\xi g) \]
be the Fourier expansion of \( f_i \), and let
\[ W_i = \bigotimes_{\nu \leq \infty} W_{i,\nu} \]
be the factorization of the Whittaker function \( W_i \) into local data. By [JL], we may require that for all \( \nu < \infty \) the Hecke type local integrals
\[ \int_{a \in k_\nu^*} W_{i,\nu} \left( \begin{array}{cc} a & 0 \\ 0 & 1 \end{array} \right) \left| a \right|_\nu^{-s - \frac{1}{2}} da \]
differ by at most an exponential function from the local \( L \)-factors for the representation generated by \( f_i \). Eventually we will take \( f_1 = f_2 \), compatible with these requirements.
The integral under consideration is (with notation suppressing details)

\[(3.6)\quad I(x_0) = \int_{Z_kG_k \backslash G_k} \text{P} \hat{e}(g) f_1(g) \hat{f}_2(g) \, dg\]

For \(\chi_0\) (and archimedean data) in the range of absolute convergence, the integral unwinds (via the definition of the Poincaré series) to

\[\int_{Z_kM_k \backslash G_k} \varphi(g) f_1(g) \hat{f}_2(g) \, dg\]

Using the Fourier expansion

\[f_1(g) = \sum_{\xi \in Z_k \backslash M_k} W_1(\xi g)\]

this further unwinds to

\[(3.7)\quad \int_{Z_k \backslash G_k} \varphi(g) W_1(g) \hat{f}_2(g) \, dg\]

Let \(C\) be the idele class group \(GL_1(\mathbb{A})/GL_1(k)\), and \(\hat{C}\) its dual. More explicitly, by Fujisaki’s Lemma (see Weil [W1], page 32, Lemma 3.1.1), the idele class group \(C\) is a product of a copy of \(\mathbb{R}^+\) and a compact group \(C_0\). By Pontryagin duality, \(\hat{C} \approx \mathbb{R} \times \hat{C}_0\) with \(\hat{C}_0\) discrete. It is well-known that, for any compact open subgroup \(U_{\text{fin}}\) of the finite-prime part in \(C_0\), the dual of \(C_0/U_{\text{fin}}\) is finitely generated with rank \([k : \mathbb{Q}] - 1\). The general Mellin transform and inversion are

\[(3.8)\quad f(x) = \int_C \int_C f(y) \chi(y) \, dy \chi^{-1}(x) \, d\chi = \sum_{\chi' \in \hat{C}_0} \frac{1}{2\pi i} \int_{\mathbb{R}(s) = \sigma} \int_C f(y) \chi'(y) |y|^s \, dy \chi'^{-1}(x) |x|^{-s} \, ds\]

for a suitable Haar measure on \(C\).

To formulate the main result of this section, we need one more piece of notation. For \(\nu\) infinite and \(s \in \mathbb{C}\), let

\[K_\nu(s, \chi_0, \chi) = \int_{Z_\nu \backslash M_\nu \backslash N_\nu} \int_{Z_\nu \backslash M_\nu} \varphi_\nu(m_\nu n_\nu) W_{1, \nu}(m_\nu n_\nu)\]

\[\cdot \mathcal{W}_{2, \nu}(m_\nu n_\nu) \chi_\nu(m_\nu) |m_\nu|^{s-\frac{s}{2}} \chi_\nu(m_\nu)^{-1} |m_\nu|^{\frac{s}{2}-s} \, dm_\nu \, dn_\nu \, dm_\nu\]

and set

\[(3.10)\quad K_\infty(s, \chi_0, \chi) = \prod_{\nu | \infty} K_\nu(s, \chi_0, \chi)\]

Here \(\chi_0 = \bigotimes_\nu \chi_{0, \nu}\) is the character defining the monomial vector \(\varphi\), and \(\chi = \bigotimes_\nu \chi_\nu \in \hat{C}_0\). When the monomial vector \(\varphi\) is admissible, the integral (3.9) defining \(K_\nu\) converges absolutely for \(\Re(s)\) sufficiently large. We are especially interested in the choice

\[(3.11)\quad \varphi_\nu(n) = \begin{cases} (1 + x^2)^{-\frac{s}{2}} & \text{for } \nu | \infty \text{ real, and } n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in N_\nu \\ (1 + |x|^2)^{-w} & \text{for } \nu | \infty \text{ complex, and } n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in N_\nu \end{cases} \quad (v, w \in \mathbb{C})\]
The monomial vector $\varphi$ generated by this choice is admissible for $\Re(w) > 1$ and $\Re(v)$ sufficiently large. This choice will be used in Section 5 to derive an asymptotic formula for the $GL_2$ integral moment over the number field $k$. The main result of this section is

**Theorem 3.12.** For $\varphi$ an admissible monomial vector as above, for suitable $\sigma > 0$, 

$$I(\chi_0) = \sum_{\chi \in \widehat{C}_0} \frac{1}{2\pi i} \int_{\Re(s) = \sigma} L(\chi_0 \cdot \chi^{-1} \cdot |1-s|, f_1) \cdot L(\chi \cdot |s|, \hat{f}_2) K_\infty(s, \chi_0, \chi) \, ds$$

Let $S$ be a finite set of places including archimedean places, all absolutely ramified primes, and all finite bad places for $f_1$ and $f_2$. Then the sum is over a set $\widehat{C}_{0,S}$ of characters unramified outside $S$, with bounded ramification at finite places, depending only upon $f_1$ and $f_2$.

**Proof:** Applying (3.8) to $\hat{f}_2$ via the identification

$$\left\{ \begin{pmatrix} a' & 0 \\ 0 & 1 \end{pmatrix} : a' \in C \right\} \approx C$$

and using the Fourier expansion

$$f_2(g) = \sum_{\xi \in Z_k \setminus M_k} W_2(\xi g)$$

the integral (3.7) is

$$\int_{Z_k \setminus G_k} \varphi(g) W_1(g) \left( \int_{\hat{C}} \int_{C} \hat{f}_2(m'g) \chi(m') \, dm' \, d\chi \right) \, dg$$

$$= \int_{\hat{C}} \left( \int_{Z_k \setminus G_k} \varphi(g) W_1(g) \sum_{\xi \in Z_k \setminus M_k} \bar{W}_2(\xi m'g) \chi(m') \, dm' \, d\chi \right) \, d\chi$$

$$= \int_{\hat{C}} \left( \int_{Z_k \setminus G_k} \varphi(g) W_1(g) \int_{\hat{J}} \bar{W}_2(m'g) \chi(m') \, dm' \, d\chi \right) \, d\chi$$

where $\hat{J}$ is the ideles. The interchange of order of integration is justified by the absolute convergence of the outer two integrals. (The innermost integral cannot be moved outside.) This follows from the rapid decay of cuspforms along the split torus.

For fixed $f_1$ and $f_2$, the finite-prime ramification of the characters $\chi \in \widehat{C}$ is bounded, so there are only finitely many bad finite primes for all the $\chi$ which appear. In particular, all the characters $\chi$ which appear are unramified outside $S$ and with bounded ramification, depending only on $f_1$ and $f_2$, at finite places in $S$. Thus, for $\nu \in S$ finite, there exists a compact open subgroup $U_\nu$ of $\mathfrak{o}_k^\times$ such that the kernel of the $\nu^{th}$ component $\chi_{\nu}$ of $\chi$ contains $U_\nu$ for all characters $\chi$ which appear.

Since $f_1$ and $f_2$ generate irreducibles locally everywhere, the Whittaker functions $W_i$ factor

$$W_i(\{g_\nu : \nu \leq \infty\}) = \Pi_\nu W_{i,\nu}(g_\nu)$$

Therefore, the inner integral over $Z_k \setminus G_k$ and $\hat{J}$ factors over primes, and

$$I(\chi_0) = \int_{\hat{C}} \Pi_\nu \left( \int_{Z_k \setminus G_k} \int_{k^\times} \varphi_{\nu}(g_\nu) W_{1,\nu}(g_\nu) \bar{W}_{2,\nu}(m'_\nu g_\nu) \chi_{\nu}(m'_\nu) \, dm'_\nu \, dg_\nu \right) \, d\chi$$
Let $\omega_\nu$ be the $\nu^{th}$ component of the central character $\omega$ of $f_2$. Define a character of $M_\nu$ by
\[
\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mapsto \omega_\nu \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \chi_\nu \begin{pmatrix} a/d & 0 \\ 0 & 1 \end{pmatrix}
\]
Still denote this character by $\chi_\nu$, without danger of confusion. In this notation, the last expression of $I(\chi_0)$ is
\[
I(\chi_0) = \int_C \Pi_\nu \left( \int_{Z_\nu \backslash G_\nu} \int_{Z_\nu \backslash M_\nu} \varphi_\nu(g_\nu) W_{1,\nu}(g_\nu) \overline{W}_{2,\nu}(m'_\nu g_\nu) \chi_\nu(m'_\nu) \, dm'_\nu \, dg_\nu \right) \, d\chi
\]
Suppressing the index $\nu$, the $\nu^{th}$ local integral is
\[
\int_{Z \backslash G} \int_{Z \backslash M} \varphi(g) W_1(g) \overline{W}_2(m'g) \chi(m') \, dm' \, dg
\]
Take $\nu$ finite such that both $f_1$ and $f_2$ are right $K_\nu$–invariant. Use a $\nu$–adic Iwasawa decomposition $g = mnk$ with $m \in M$, $n \in N$, and $k \in K$. The Haar measure is $d(mnk) = dm \, dn \, dk$ with Haar measures on the factors. The integral becomes
\[
\int_{Z \backslash MN} \int_{Z \backslash M} \varphi(mn) W_1(mn) \overline{W}_2(m'mn) \chi(m') \, dm' \, dn \, dm
\]
To symmetrize the integral, replace $m'$ by $m'm^{-1}$ to obtain
\[
\int_{Z \backslash MN} \int_{Z \backslash M} \varphi(mn) W_1(mn) \overline{W}_2(m'n) \chi(m') \chi(m)^{-1} \, dm' \, dn \, dm
\]
The Whittaker functions $W_i$ have left $N$–equivariance
\[
W_1(n g) = \psi(n) W_1(g) \quad \text{(fixed non-trivial $\psi$)}
\]
so
\[
W_1(mn) = W_1(mnm^{-1} m) = \psi(mnm^{-1}) W_1(m)
\]
and similarly for $W_2$. Thus, letting
\[
X(m, m') = \int_N \varphi(n) \psi(mnm^{-1}) \overline{\psi}(m'n'm'^{-1}) \, dn
\]
the local integral is
\[
\int_{Z \backslash M} \int_{Z \backslash M} \chi_0(m) W_1(m) \overline{W}_2(m') \chi(m') \chi^{-1}(m) X(m, m') \, dm' \, dm
\]
We claim that for $m$ and $m'$ in the supports of the Whittaker functions, the inner integral $X(m, m')$ is constant, independent of $m$, $m'$, and it is $1$ for almost all finite primes. First, $\varphi(mn)$ is $0$, unless $n \in M \cdot K \cap N$, that is, unless $n \in N \cap K$. On the other hand,
\[
\psi(mnm^{-1}) \cdot W_1(mk) = \psi(mnm^{-1}) \cdot W_1(m) = W_1(mn) = W_1(m) \quad \text{(for $n \in N \cap K$)}
\]
Thus, for $W_1(m) \neq 0$, necessarily $\psi(mnm^{-1}) = 1$. A similar discussion applies to $W_2$. So, up to normalization, the inner integral is 1 for $m, m'$ in the supports of $W_1$ and $W_2$. Then

$$\int_{Z\setminus M} \int_{Z\setminus M} \chi_0(m) W_1(m) \overline{W_2(m')} \chi(m') \chi^{-1}(m) \, dm \, dm'$$

$$= \int_{Z\setminus M} (\chi_0 \cdot \chi^{-1})(m) W_1(m) \, dm \cdot \int_{Z\setminus M} \chi(m') \overline{W_2(m')} \, dm'$$

$$= L_\nu(\chi_0, \chi^{-1}) \cdot |\nu^{1/2}| \cdot f_1 \cdot L_\nu(\chi, |\nu^{1/2}|, f_2)$$

i.e., the product of local factors of the standard $L$–functions in the theorem (up to exponential functions at finitely many finite primes) by our assumptions on $f_1$ and $f_2$.

For non-trivial right $K$–type $\sigma$, the argument is similar but a little more complicated. The key point is that the inner integral over $N$ (as above) should not depend on $mk$ and $m'k$, for $mk$ and $m'k$ in the support of the Whittaker functions. Changing conventions for a moment, look at $V_\sigma$–valued Whittaker functions, and consider any $W$ in the $\nu^\text{th}$ Whittaker space for $f$, having right $K$–isotype $\sigma$. Thus,

$$W(gk) = W(g)$$

(for $g \in G$ and $k \in K$)

For $\varphi(mn) \neq 0$, again $n \in N \cap K$. Then

$$\sigma(k) \cdot \psi(mnm^{-1}) \cdot W(m) = W(mnk) = \sigma(k) \cdot W(mn) = \sigma(k) \cdot \sigma(n) \cdot W(m)$$

where in the last expression $n$ comes out on the right by the right $\sigma$–equivariance of $W$. For $m$ in the support of $W$, $\sigma(n)$ acts by the scalar $\psi(mnm^{-1})$ on $W(mk)$, for all $k \in K$. Thus, $\sigma(n)$ is scalar on that copy of $V_\sigma$. At the same time, this scalar is $\sigma(n)$, so is independent of $m$ if $W(m) \neq 0$. Thus, except for a common integral over $K$, the local integral falls into two pieces, each yielding the local factor of the $L$–function. The common integral over $K$ is a constant (from Schur orthogonality), non-zero since the two vectors are collinear in the $K$–type. $\square$

At this point the archimedean local factors of the Euler product are not specified. The option to vary the choices is essential for applications.

§4. Spectral decomposition of Poincaré series

The objective now is to spectrally decompose the Poincaré series defined in (3.1). Throughout this section, we assume that $\varphi$ is admissible, in the sense given at the end of Section 2. As we shall see, in general $\text{Pé}(g)$ is not square-integrable. However, choosing the archimedean part of the monomial vector $\varphi$ to have enough decay, and after an obvious Eisenstein series is subtracted, the Poincaré series is not only in $L^2$ but also has sufficient decay so that its integrals against Eisenstein series converge absolutely, by explicit computation. In particular, if the archimedean data is specialized to (3.11), the Poincaré series $\text{Pé}(g)$ has meromorphic continuation in the variables $v$ and $w$. This is achieved via spectral decomposition and meromorphic continuation of the spectral fragments. See [DG1], [DG2] when $k = \mathbb{Q}$, $\mathbb{Q}(i)$.

Let $k$ be a number field, $G = GL_2$ over $k$, and $\omega$ a unitary character of $Z_k \setminus Z_k$. Recall the decomposition

$$L^2(Z_k G_k \setminus G_k, \omega) = L^2_{\text{cusp}}(Z_k G_k \setminus G_k, \omega) \oplus L^2_{\text{cusp}}(Z_k G_k \setminus G_k, \omega)$$

$$= L^2(Z_k G_k \setminus G_k, \omega)$$

$$\oplus L^2_{\text{cusp}}(Z_k G_k \setminus G_k, \omega)$$

$$\oplus L^2_{\text{cusp}}(Z_k G_k \setminus G_k, \omega)$$
The orthogonal complement

\[ L_{\text{cusp}}^2(Z_hG_k \backslash G_k, \omega) \cong \{1 - \text{dimensional representations}\} \oplus \int_{(GL_1(k) \backslash GL_1(\mathbb{A}))^-} \bigotimes_{\nu} \text{Ind}_{P_{\nu}}^{G_{\nu}}(\chi_{\nu} \delta_{\nu}^{1/2}) \ d\chi \]

where \(\delta\) is the modular function on \(P_h\), and the isomorphism is via Eisenstein series. Using this, we shall explicitly decompose our Poincaré series as

\[ \text{Pé} = \text{Eisenstein series} + \text{discrete part} + \text{continuous part} \quad \text{(with } \omega = 1) \]

The projection to cusps is straightforward componentwise. We have

**Proposition 4.1.** Let \(f\) be a cuspidal representation on \(G_k\) generating a spherical representation locally everywhere, and suppose \(f\) corresponds to a spherical vector everywhere locally. In the region of absolute convergence of the Poincaré series \(\text{Pé}(g)\), the integral

\[ \int_{Z_hG_k \backslash G_k} \tilde{f}(g) \text{Pé}(g) \ dg \]

is an Euler product. At finite \(\nu\), the corresponding local factors are, up to a constant depending on the set of absolutely ramified primes in \(k\), \(L_{\nu}(\chi_{0,\nu} \cdot | \cdot |^{1/2} \tilde{f})\).

**Proof:** The computation uses the same facts as the Euler factorization in the previous section. Using the Fourier expansion

\[ f(g) = \sum_{\xi \in Z_k \backslash M_k} W(\xi g) \]

unwind

\[ \int_{Z_hG_k \backslash G_k} \tilde{f}(g) \text{Pé}(g) \ dg = \int_{Z_hM_k \backslash G_k} \sum_{\xi} \overline{W}(\xi g) \varphi(g) \ dg = \int_{Z_h \backslash G_k} \overline{W}(g) \varphi(g) \ dg \]

\[ = \prod_{\nu} \left( \int_{Z_{\nu} \backslash G_{\nu}} \overline{W}_{\nu}(g_{\nu}) \varphi_{\nu}(g_{\nu}) \ dg_{\nu} \right) \]

where the local Whittaker functions at finite places are normalized as in [JL] to give the correct local \(L\)-factors.

At finite \(\nu\), suppressing the subscript \(\nu\), the integrand in the \(\nu\text{-th}\) local integral is right \(K_{\nu}\)-invariant, so we can integrate over \(MN\) with left Haar measure. The \(\nu\text{-th}\) Euler factor is

\[ \int_{Z_h \backslash Z} \overline{W}(mn) \varphi(mn) \ dn \ dm = \int_{Z_h \backslash Z} \overline{\psi}(mm^{-1}) \overline{W}(m) \chi_0(m) \varphi(n) \ dn \ dm \]

for all finite primes \(\nu\). The integral over \(n\) is

\[ \int_N \overline{\psi}(mm^{-1}) \varphi(n) \ dn \]
For \( \varphi(n) \) to be non-zero requires \( n \) to lie in \( M \cdot K \), which further requires, as before, that \( n \in N \cap K \). Again, \( W(m) = 0 \) unless 
\[
m(N \cap K)m^{-1} \subset N \cap K
\]
The character \( \psi \) is trivial on \( N \cap K \). Thus, the integral over \( N \) is really the integral of 1 over \( N \cap K \). Thus, at finite primes \( \nu \), the local factor is 
\[
\int_{Z \setminus M} \overline{W}(m) \chi_0(m) \, dm = L_\nu(\chi_{0,\nu} | \cdot |_\nu^{1/2}, f)
\]

Of course, the spectral decomposition of a right \( K_A \)-invariant automorphic form can only involve everywhere locally spherical cuspforms.

Assume that \( \varphi \) is given by (3.11). Taking \( \Re(v) > 1 \) and \( \Re(w) > 1 \) to ensure by Proposition 2.6 absolute convergence of \( P\ell(g) \), the local integral in Proposition 4.1 at infinite \( \nu \) is 
\[
\int_{Z_\nu \setminus G_\nu} \overline{W}_\nu(g_\nu) \varphi_\nu(g_\nu) \, dg_\nu = \mathcal{G}_\nu(\frac{1}{2} + i \bar{\mu}_{\nu}; v, w)
\]
where, up to a constant, 
\[
\mathcal{G}_\nu(s; v, w) = \pi^{-v} \frac{\Gamma(\frac{v+1-s}{2})\Gamma(\frac{v+w-s}{2})\Gamma(\frac{v+s}{2})\Gamma(\frac{v+w+s-1}{2})}{\Gamma(\frac{w}{2})\Gamma(v+w)}
\]
for \( \nu |\infty \) real, and 
\[
\mathcal{G}_\nu(s; v, w) = (2\pi)^{-2v} \frac{\Gamma(v+1-s)\Gamma(v+w-s)\Gamma(v+s)\Gamma(v+w+s-1)}{\Gamma(w)\Gamma(2v+w)}
\]
for \( \nu |\infty \) complex. In the above expression \( i\mu_{\nu,v} \) and \( -i\mu_{\nu,v} \) are the local parameters of \( f \) at \( \nu \). These expressions as ratios of products of gamma functions are obtained by standard computations (see [DG1] and [DG2]), from the normalizations
\[
W_\nu \left( \begin{array}{c} a \\ 1 \end{array} \right) = \begin{cases} |a|^{1/2} K_{i\mu_{\nu,v}}(2\pi |a|) & \text{if } \nu \approx \mathbb{R} \\ |a| K_{2i\mu_{\nu,v}}(4\pi |a|) & \text{if } \nu \approx \mathbb{C} \end{cases}
\]
and invocation of local multiplicity-one of Whittaker models. Then, with respect to an orthonormal basis \( \{F\} \) of everywhere locally spherical cuspforms, it is natural to consider the spectral sum 
\[
\sum_F \bar{\rho}_F \mathcal{G}_{F^\infty}(v, w) L(v + \frac{1}{2}, F) \cdot F
\]
where 
\[
\mathcal{G}_{F^\infty}(v, w) = \prod_{\nu |\infty} \mathcal{G}_\nu(\frac{1}{2} + i \bar{\mu}_{\nu,v}; v, w)
\]
with \( \mathcal{G}_\nu \) defined in (4.2) and (4.3). Here we absorbed all the ambiguous constants at infinite places into \( \bar{\rho}_F \). Traditionally, the constant \( \nu \) is denoted by \( \rho_F(1) \) being considered the first Fourier coefficient of \( F \). As mentioned at the beginning of this section, and as we shall shortly see, the
Poincaré series $Pé(g)$ is up to an Eisenstein series a square-integrable function. It will then be clear that the above spectral sum represents the discrete part of $Pé$.

By considering the usual integral representation against an Eisenstein series of the completed $GL_2 \times GL_2$ Rankin-Selberg $L$-function $Λ(s,F \otimes \bar{F})$ [J] (for the general case $GL_m \times GL_n$, see the review of the literature in [CPS2]), and then taking the residue at $s = 1$, one obtains

$$\text{non-zero constant} = |ρ_F|^2 \cdot L_∞(1,F \otimes \bar{F}) \cdot \text{Res}_{s=1} L(s,F \otimes \bar{F}) \quad (\text{for } ||F|| = 1)$$

the constant on the left being independent of $F$. The local factors of $L(s,F \otimes \bar{F})$ on the right obtained from the integral representation may differ from those of the correct convolution $L$-function obtained from the local theory at only the absolutely ramified primes in $k$. Comparing the gamma factors $L_∞(1,F \otimes \bar{F})$ with $G_{F_∞}^{∞}(v,w)$, we deduce that $\bar{ρ}_F G_{F_∞}^{∞}(v,w)$ has exponential decay in the local parameters of $F$. Combining this with standard estimates and the Weyl’s Law [LV] (see also [Do] for an upper bound, which suffices for us), it follows that the above spectral sum is absolutely convergent for $(v,w) ∈ C^2$, apart from the poles of $G_{F_∞}^{∞}(v,w)$.

For the remaining decomposition, subtract (as in [DG1], [DG2]) a finite linear combination of Eisenstein series from the Poincaré series, leaving a function in $L^2$ with sufficient decay to be integrated against Eisenstein series. The correct Eisenstein series to subtract becomes visible from the dominant part of the constant term of the Poincaré series (below).

Write the Poincaré series as

$$Pé(g) = \sum_{γ ∈ M_k \backslash G_k} ϕ(γg) = \sum_{γ ∈ P_k \backslash G_k} \sum_{β ∈ N_k} ϕ(βγg)$$

By Poisson summation

$$Pé(g) = \sum_{γ ∈ P_k \backslash G_k} \sum_{ψ ∈ (N_k \backslash N_k)^∞} \hat{ϕ}_{γg}(ψ) \quad (4.4)$$

where, $ϕ_g(n) = ϕ(ng)$, and $\hat{ϕ}$ is Fourier transform along $N_k$. The trivial−ψ (that is, with $ψ = 1$) Fourier term

$$\sum_{γ ∈ P_k \backslash G_k} \hat{ϕ}_{γg}(1) \quad (4.5)$$

is an Eisenstein series, since the function

$$g \longrightarrow \hat{ϕ}_g(1) = \int_{N_k} ϕ(ng) \, dn$$

is left $M_k$−equivariant by the character $δχ_0$, and left $N_k$−invariant.

For $ξ ∈ M_k$,

$$\hat{ϕ}_{ξg}(ψ) = \int_{N_k} \overline{ψ}(n) \, ϕ(nξg) \, dn = \int_{N_k} \overline{ψ}(ξ \cdot ξ^{-1} nξ \cdot g) \, dn = \int_{N_k} \overline{ψ}(ξnξ^{-1}) \, ϕ(n \cdot g) \, dn = \hat{ϕ}_g(ψξ)$$
\[ \psi(\xi n \xi^{-1}) = \psi(n) \]

by replacing \( n \) by \( \xi n \xi^{-1} \), using the left \( M_k \)-invariance of \( \varphi \), and invoking the product formula to see that the change-of-measure is trivial. Since this action of \( Z_k \) is transitive on non-trivial characters on \( N_k \), for a fixed choice of non-trivial character \( \psi \), the sum over non-trivial characters can be rewritten as a more familiar sort of Poincaré series

\[
\sum_{\gamma \in P_k \backslash G_k} \sum_{\gamma \in Z_k \backslash M_k} \hat{\varphi}_{\gamma g}(\psi) = \sum_{\gamma \in Z_k \backslash N_k \backslash G_k} \hat{\varphi}_{\gamma g}(\psi)
\]

Denote this version of the original Poincaré series, with the Eisenstein series subtracted, by

\[
P^*_{\gamma}(g) = \sum_{\gamma \in Z_k \backslash N_k \backslash G_k} \hat{\varphi}_{\gamma g}(\psi) = P_{\gamma}(g) - \sum_{\gamma \in P_k \backslash G_k} \hat{\varphi}_{\gamma g}(1)
\]

**Remark:** With (4.6), the square integrability part of the Poincaré series in Theorem 2.7 can be precisely formulated as follows. For \( \varphi \) admissible, the modified Poincaré series \( P^*_{\gamma}(g) \) is in \( L^2(Z_k \backslash G_k) \).

Now we describe the continuous part of the spectral decomposition. At every place \( \nu \), let \( \eta_{\nu} \) be the spherical vector in the (non-normalized) principal series \( \text{Ind}_{P_{\nu}}^{G_{\nu}} \chi_{\nu} \), normalized by \( \eta_{\nu}(1) = 1 \). Take \( \eta = \bigotimes_{\nu \leq \infty} \eta_{\nu} \). The corresponding Eisenstein series is

\[
E_{\chi}(g) = \sum_{\gamma \in P_k \backslash G_k} \eta(\gamma g)
\]

For any left \( Z_k \backslash G_k \)-invariant and right \( K_k \)-invariant square-integrable \( F \) on \( G_k \), write

\[
\langle F, E_{\chi} \rangle = \int_{Z_k \backslash G_k} F(g) \overline{E_{\chi}(g)} \, dg
\]

With suitable normalization of measures,

\[
\text{continuous spectrum part of } F = \int_{\mathbb{R}(\chi) = \frac{1}{2}} \langle F, E_{\chi} \rangle E_{\chi} \, d\chi
\]

Explicitly, let

\[
(4.7) \quad \kappa = \text{meas}(\mathbb{J}^1/k^\times)
\]

the measure on \( \mathbb{J}^1/k^\times \) being the image of the measure \( \gamma \) on \( \mathbb{J} \) defined in [W2], page 128. It is well-known (see [W2], page 129, Corollary) that the residue of the Dedekind zeta-function of \( k \) at \( s = 1 \) is

\[
\text{Res}_{s=1} \zeta_k(s) = \frac{\kappa}{|D_k|^\frac{1}{2}}
\]
where $D_k$ denotes the discriminant of $k$. Then, the continuous part of $F$ can be written as

$$\text{continuous spectrum part of } F = \frac{1}{4\pi i \kappa} \sum_{\chi} \int_{\Re(s) = \frac{1}{2}} \langle F, E_{s,\chi} \rangle \cdot E_{s,\chi} \, ds$$

where the sum is over all absolutely unramified characters $\chi \in \hat{C}_0$. Here $E_{s,\chi}$ stands for $E_{\chi} \cdot |s|$ defined above. In general, this formula requires isometric extensions to $L^2$ of integral formulas that converge literally only on a smaller dense subspace (pseudo-Eisenstein series). However, in our situation, since $P\epsilon^*(g)$ has sufficient decay, its integrals against Eisenstein series (with parameter in a bounded vertical strip containing the critical line) converge absolutely. Furthermore, as $P\epsilon^*$ is sufficiently smooth, with derivatives of sufficient decay, its continuous part of the spectral decomposition also converges. Then by Theorem 2.7 (see also the above remark), Proposition 4.1 and (4.6), with respect to an orthonormal basis $\{F\}$ of everywhere locally spherical cuspforms, we have the spectral decomposition

\begin{equation}
\tag{4.8} P\epsilon = \left( \int_{N_\infty} \varphi_\infty \right) \cdot E_{v+1} + \sum_F \left( \int_{Z_\infty \setminus G_\infty} \varphi_\infty \cdot \overline{W}_{F,v} \right) \cdot L(v + \frac{1}{2}, F) \cdot F \n + \frac{1}{4\pi i \kappa} \sum_{\chi} \int_{\Re(s) = \frac{1}{2}} \langle P\epsilon^*, E_{s,\chi} \rangle \cdot E_{s,\chi} \, ds
\end{equation}

where we set $E_s := E_{s,1}$. To compute the inner product $\langle P\epsilon^*, E_{s,\chi} \rangle$ in the continuous part, first consider an Eisenstein series

$$E(g) = \sum_{\gamma \in P_k \setminus G_k} \eta(\gamma g)$$

for $\eta$ left $P_k$–invariant, left $M_k$–equivariant and left $N_k$–invariant. The Fourier expansion of this Eisenstein series is

$$E(g) = \sum_{\psi^* \in (N_k \setminus N_k)^\psi} \int_{N_k \setminus N_k} \overline{\psi}(n) \, E(n g) \, dn$$

For a fixed non-trivial character $\psi$, the $\psi^\text{th}$ Fourier term is

$$\int_{N_k \setminus N_k} \overline{\psi}(n) \, E(n g) \, dn = \int_{N_k \setminus N_k} \overline{\psi}(n) \sum_{\gamma \in P_k \setminus G_k} \eta(\gamma ng) \, dn$$

$$= \sum_{w \in P_k \setminus G_k} \int_{(N_k \cap w^{-1} P_k w) \setminus N_k} \overline{\psi}(n) \eta(w g) \, dn$$

$$= \int_{N_k \setminus N_k} \overline{\psi}(n) \eta(n g) \, dn + \int_{N_k} \overline{\psi}(n) \eta(n_0 ng) \, dn$$

$$= 0 + \int_{N_k} \overline{\psi}(n) \eta(n_0 ng) \, dn \quad \left(\text{where } n_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$$

\[1\text{There is no residual contribution to the spectral decomposition of } P\epsilon^*(g), \text{ as can be easily verified.} \]
because $\psi$ is non-trivial and $\eta$ is left $N_A$-invariant. Denote the $\psi^\text{th}$ Fourier term by

$$W^E(g) = W^E_{\psi,\psi}(g) = \int_{N_A} \overline{\psi}(n) \eta(w,ng) \, dn$$

We have the following

**Proposition 4.10.** Fix $s \in \mathbb{C}$ such that $\Re(s) > 1$, and suppose $\varphi_\infty \in \Omega_\ell$ with $\Re(\nu)$, $\ell$ sufficiently large. Then,

$$\langle \hat{P}^*, E_{s,\chi} \rangle = \bar{\chi}(\mathfrak{o}) \left( \int_{Z_\infty \setminus G_\infty} \varphi_\infty \cdot \overline{W}^E_{s,\chi,\infty} \right) \frac{L(v + \bar{\nu}, \chi) \cdot L(v + 1 - \bar{\nu}, \chi)}{L(2\bar{\nu}, \chi^2)} \cdot |\mathfrak{o}|^{-(v-\bar{s}+1/2)}$$

where $\mathfrak{o}$ denotes a differential idele (see [W2], page 113, Definition 4) with component 1 at archimedean places.

**Proof:** Fix a non-trivial character $\psi$ on $N_k \setminus N_A$. When $\Re(\nu)$ and $\ell$ are both large, the modified Poincaré series $\hat{P}^*(g)$ has sufficient (polynomial) decay, and therefore, we can unwind it to obtain (see (4.6)):

$$\int_{Z_k \setminus G_k \setminus G_k} \hat{P}^*(g) \overline{E}_{s,\chi}(g) \, dg = \int_{Z_k N_k \setminus G_k} \int_{N_k \setminus N_k} \hat{\varphi}_n(\psi) \overline{E}_{s,\chi}(ng) \, dn \, dg$$

$$= \int_{Z_k N_k \setminus G_k} \hat{\varphi}_n(\psi) \int_{N_k \setminus N_k} \varphi(n) \overline{E}_{s,\chi}(ng) \, dn \, dg = \int_{Z_k N_k \setminus G_k} \hat{\varphi}_n(\psi) \overline{W}^E_{s,\chi}(g) \, dg$$

Since

$$\hat{\varphi}_n(\psi) = \int_{N_k} \overline{\psi}(n) \varphi(ng) \, dn$$

the last integral in (4.11) is

$$\int_{Z_k N_k \setminus G_k} \int_{N_k} \overline{\psi}(n) \varphi(ng) \overline{W}^E_{s,\chi}(g) \, dn \, dg = \int_{Z_k N_k \setminus G_k} \int_{N_k} \varphi(ng) \overline{W}^E_{s,\chi}(ng) \, dn \, dg$$

$$= \int_{Z_k \setminus G_k} \varphi(g) \overline{W}^E_{s,\chi}(g) \, dg$$

(4.12)

The Whittaker function of the Eisenstein series does factor over primes, into local factors depending only upon the local data at $\nu$

$$W^E_{s,\chi} = \prod_{\nu} W^E_{s,\chi,\nu}$$

Thus, by (4.11) and (4.12),

$$\langle \hat{P}^*, E_{s,\chi} \rangle = \left( \int_{Z_\infty \setminus G_\infty} \varphi_\infty \cdot \overline{W}^E_{s,\chi,\infty} \right) \cdot \prod_{\nu < \infty} \int_{Z_\nu \setminus G_\nu} \varphi_\nu(g_\nu) \overline{W}^E_{s,\chi,\nu}(g_\nu) \, dg_\nu$$
At finite $\nu$, using an Iwasawa decomposition and the vanishing of $\varphi_\nu$ off $M_\nu K_\nu$ (see (3.2)), as in the integration against cuspforms, the local factor is

$$\int_{k_\nu^\times} |a|_\nu^\nu \mathcal{W}_{s,\chi,\nu}^E \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} da$$

However, for Eisenstein series, the natural normalization of the Whittaker functions differs from that used for cuspforms, instead presenting the local Whittaker functions as images under intertwining operators. Specifically, define the normalized spherical vector for data $s, \chi_\nu$

$$\eta_\nu(pk) = |a/d|_\nu^{s} \cdot \chi_\nu(a/d) \quad \text{(for } p = \begin{pmatrix} a & * \\ d & \end{pmatrix} \in P_\nu \text{ and } k \in K_\nu)$$

The corresponding spherical local Whittaker function for Eisenstein series is the integral$^2$ (see (4.9))

$$W_{s,\chi,\nu}^E(g) = \int_{N_\nu} \overline{\psi}_\nu(n) \eta_\nu(w_\circ ng) \, dn$$

The Mellin transform of the Eisenstein-series normalization $W_{s,\chi,\nu}^E$ compares to the Mellin transform of the usual normalization as follows. Let $d_\nu \in k_\nu^\times$ be such that

$$(o_\nu^*)^{-1} = d_\nu \cdot o_\nu$$

Let $\mathfrak{d}$ be the idele with $\nu^{\text{th}}$ component $d_\nu$ at finite places $\nu$ and component 1 at archimedean places. Then for finite $\nu$ the $\nu^{\text{th}}$ local integral is (see Appendix 2 for details),

$$\int_{k_\nu^\times} |a|_\nu^\nu \mathcal{W}_{s,\chi,\nu}^E \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} da = |\mathfrak{d}_\nu|_{\nu}^{1/2} \cdot \frac{L_\nu(v + \bar{s}, \chi_\nu) \cdot L_\nu(v + 1 - \bar{s}, \chi_\nu)}{L_\nu(2\bar{s}, \chi_\nu^2)} \cdot |\mathfrak{d}_\nu|_{\nu}^{(v+1-s)} \chi_\nu(\mathfrak{d}_\nu)$$

and the proposition follows. $\square$

Accordingly, the spectral decomposition (4.8) is

$$(4.13)\quad \mathcal{P}^c = \left( \int_{N_\infty} \varphi_\infty \right) \cdot E_{v+1} + \sum_F \left( \int_{Z_\infty \backslash G_\infty} \varphi_\infty \cdot \mathcal{W}_{F, \infty} \right) \cdot L(v + \frac{1}{2}, F) \cdot F$$

$$+ \sum_{\chi} \frac{\zeta(s)}{4\pi^{\text{rk}}} \int_{\Re(s) = \frac{1}{2}} \left( \int_{Z_\infty \backslash G_\infty} \varphi_\infty \cdot \mathcal{W}_{1-s, \chi, \infty}^E \right) \frac{L(v + 1 - s, \chi_\infty) \cdot L(v + s, \chi)}{L(2 - 2s, \chi^2)} |\mathfrak{d}|^{-(v+s-1/2)} \cdot E_{s, \chi} \, ds$$

where we replaced $\bar{s}$ by $1 - s$, for $\Re(s) = \frac{1}{2}$, to maintain holomorphy of the integrand. The archimedean-place Whittaker functions can be expressed in terms of the usual $K$–Bessel function as follows. Let

$$\eta_\nu(nmk) = \eta_{s, \nu}(nmk) = |a/d|_\nu^{s} \quad \text{(for } n \in \mathbb{N}_\nu, m = \begin{pmatrix} a \\ d \end{pmatrix} \in M_\nu, k \in K_\nu)$$

$^2$This integral only converges nicely for $\Re(s) \gg 0$, but admits a meromorphic continuation in $s$ by various means. For example, the algebraic form of Bernstein’s continuation principle applies, since the dimension of the space of intertwining operators from the principal series to the Whittaker space is one-dimensional.
The normalization of the Whittaker function is

\[ W_{s,\nu}^E(g) = \int_{N_\nu} \overline{\psi}_\nu(n) \eta_\nu(w_\nu n g) \, dn \quad \text{(for } \Re(s) \gg 0 \text{ and fixed non-trivial } \psi) \]

Then, for \( \nu \) archimedean and fixed non-trivial character \( \psi_{0,\nu} \) on \( k_\nu \)

\[ W_{s,\nu}^E \left( \begin{array}{cc} a & 0 \\ 1 & 1 \end{array} \right) = \int_{k_\nu} \overline{\psi}_{0,\nu}(x) \left| \frac{a}{aa^t + xx^t} \right|^s dx = |a|^{1-s} \int_{k_\nu} \overline{\psi}_{0,\nu}(ax) \frac{1}{|1 + xx^t|_n^\nu} dx \]

by replacing \( x \) by \( ax \), where \( \iota \) is the complex conjugation for \( \nu \approx \mathbb{C} \) and the identity map for \( \nu \approx \mathbb{R} \).

The usual computation shows that

\[ W_{s,\nu}^E \left( \begin{array}{cc} a & 0 \\ 1 & 1 \end{array} \right) = \frac{|a|^{1/2}}{\pi^{-s} \Gamma(s)} \int_0^\infty e^{-\pi(t+\frac{1}{t})|a|} t^{s-\frac{1}{2}} \frac{dt}{t} = \frac{2 |a|^{1/2} K_{s-1/2}(2\pi|a|)}{\pi^{-s} \Gamma(s)} \]

and, similarly,

\[ W_{s,\nu}^E \left( \begin{array}{cc} a & 0 \\ 1 & 1 \end{array} \right) = \frac{|a|}{(2\pi)^{-2s+1} \Gamma(2s)} \int_0^\infty e^{-2\pi(t+\frac{1}{t})|a|} t^{2s-1} \frac{dt}{t} = \frac{2 |a| K_{2s-1}(4\pi|a|)}{(2\pi)^{-2s} \Gamma(2s)} \]

To simplify the integral over \( Z_\infty \setminus G_\infty \) in the continuous part of (4.13), let

\[ \Phi_{\nu}(x) = \varphi_{\nu} \left( \begin{array}{cc} 1 & x \\ 0 & 1 \end{array} \right) \quad \text{(for } \nu \text{ archimedean)} \]

Using the right \( K_\nu \)-invariance and an Iwasawa decomposition,

\[ \int_{Z_\nu \setminus G_\nu} \varphi_{\nu} \cdot W_{s,\nu}^E = \int_{k_\nu} \int_{k_\nu} |a|_v^\nu \Phi_{\nu}(x) W_{s,\nu}^E \left( \begin{array}{cc} a & 0 \\ 1 & 1 \end{array} \right) \psi_{0,\nu}(ax) dx \, da \]

\[ = \int_{k_\nu} |a|_v^\nu \tilde{\Phi}_{\nu}(a) W_{s,\nu}^E \left( \begin{array}{cc} a & 0 \\ 1 & 1 \end{array} \right) da \]

For \( \chi \in \hat{C}_0 \) absolutely unramified, we have \( W_{s,\nu}^E = W_{s+it_\nu,\nu}^E \), where \( t_\nu \in \mathbb{R} \) is the parameter of the local component \( \chi_{\nu} \) of \( \chi \). Then all archimedean integrals in the continuous part of the spectral decomposition of \( P^E \) are given by (4.14) with \( s \) replaced by \( 1 - s - it_\nu \).

In particular, if \( \varphi_{\nu} \) is specialized to (3.11) we have

\[ \int_{Z_\nu \setminus G_\nu} \varphi_{\nu} \cdot W_{s,\nu}^E = \begin{cases} \frac{G_\nu(s; v, w)}{\pi^{-s} \Gamma(s)} & \text{if } \nu \approx \mathbb{R} \\ \frac{G_\nu(s; v, w)}{(2\pi)^{-2s-1} \Gamma(2s)} & \text{if } \nu \approx \mathbb{C} \end{cases} \]

where \( G_\nu(s; v, w) \) is given in (4.2) and (4.3). Furthermore, with these choices of \( \varphi_{\nu} \),

\[ \int_{N_\nu} \varphi_{\nu} = \begin{cases} \sqrt{\pi} \frac{\Gamma(w-1)}{\Gamma(w)} & \text{if } \nu \approx \mathbb{R} \\ 2\pi (w-1)^{-1} & \text{if } \nu \approx \mathbb{C} \end{cases} \]

As usual, let \( r_1 \) and \( r_2 \) denote the number of real and complex embeddings of \( k \), respectively. Following [DG1], Proposition 5.10, we now prove

---

3The appropriate measure is double the usual in this case.
**Theorem 4.17.** Assume $\varphi$ is defined by (3.11). The Poincaré series $P\epsilon(g)$ has meromorphic continuation to a region in $\mathbb{C}^2$ containing $v = 0, w = 1$. As a function of $w$, for $v = 0$, it is holomorphic in the half-plane $\Re(w) > 11/18$, except for $w = 1$ where it has a pole of order $r_1 + r_2 + 1$.

**Proof:** Let $P\epsilon^{*}_{\text{cusp}}$ and $P\epsilon^{*}_{\text{cont}}$ be, respectively, the discrete and continuous parts of $P\epsilon^{*}$. Then the spectral decomposition (4.13) is

$$P\epsilon = R(w) \cdot E_{v+1} + P\epsilon^{*}_{\text{cusp}} + P\epsilon^{*}_{\text{cont}}$$

where $R(w) = \int_{N_{\infty}} \varphi_{\infty}$

the integral being computed by (4.16). As mentioned before (see the discussion after the proof of Proposition 4.1), the series giving $P\epsilon^{*}_{\text{cusp}}$

$$\sum_F \tilde{\rho}_F \mathcal{G}_{\infty}(v,w) L(v + \frac{1}{2}, F) \cdot F$$

converges absolutely for $(v,w) \in \mathbb{C}^2$, apart from the poles of $\mathcal{G}_\nu(\frac{1}{2} + i\mu; v,w)$. The fact that $P\epsilon^{*}_{\text{cusp}}$ equals the above spectral sum is justified by the square integrability of $P\epsilon^{*}$ for $\Re(w) > 1$ and large $\Re(v)$ (see Theorem 2.7, (4.6) and Appendix 1). Furthermore, using (4.2), (4.3) and the Kim-Shahidi bound of the local parameters $|\Re(i\mu_{f,v})| < 1/9$ (see [K], [KS]), the cuspidal part $P\epsilon^{*}_{\text{cusp}}$ is holomorphic for $v = 0$ and $\Re(w) > 11/18$.

To deal with the continuous part, first note that by (4.15) the expression under the vertical line integral in (4.13) has enough decay in the parameters to ensure the absolute convergence of the integral and sum over $\chi$. Also, note that $P\epsilon^{*}_{\text{cont}}$ is holomorphic for $\Re(v) > \frac{1}{2}$ and $\Re(w) > 1$. Aiming to analytically continue to $v = 0$, first take $\Re(v) = 1/2 + \varepsilon$, and move the line of integration from $\sigma = 1/2$ to $\sigma = 1/2 - 2\varepsilon$. This picks up the residue of the integrand corresponding to $\chi$ trivial due to the pole of $\zeta_k(v+s)$ at $v+s = 1$, that is, at $s = 1 - v$. Its contribution is

$$\frac{1}{2} Q(v;v,w) \cdot |\rho|^{1/2} \cdot |\rho|^{-1/2} \cdot E_{1-v} \cdot E_{1-v}$$

where

$$Q(s;v,w) = \int_{\mathbb{Z}_\infty \setminus \mathcal{G}_{\infty}} \varphi_{\infty} \cdot W_{s,\infty}^{E}$$

stands for the ratio of products of gamma functions computed by (4.15). This expression of $P\epsilon^{*}_{\text{cont}}$ is holomorphic in $v$ in the strip

$$\frac{1}{2} - \varepsilon \leq \Re(v) \leq \frac{1}{2} + \varepsilon$$

Now, take $v$ with $\Re(v) = 1/2 - \varepsilon$, and then move the vertical integral from $\sigma = 1/2 - 2\varepsilon$ back to $\sigma = 1/2$. This picks up $(-1)$ times the residue at the pole of $\zeta_k(v+1-s)$ at 1, that is, at $s = v$, with another sign due to the sign of $s$ inside this zeta function. Thus, we pick up the residue

$$\frac{1}{2} Q(1-v;v,w) \cdot \frac{\zeta_k(2v)}{\zeta_k(2-2v)} \cdot |\rho|^{-2v+1} \cdot E_v = \frac{1}{2} Q(1-v;v,w) \cdot \frac{\zeta_{\infty}(2-2v)}{\zeta_{\infty}(2v)} \cdot E_{1-v}$$

where the last identity was obtained from the functional equation of the Eisenstein series $E_v$. Since $\mathcal{G}_\nu(s;v,w)$ defined in (4.2) and (4.3) is invariant under $s \rightarrow 1 - s$, it follows by (4.15) that the
above residues are equal. Note that the part of $P\delta^*_{\text{cont}}$ corresponding to the vertical line integral and the sum over $\chi$ is now holomorphic in a region of $\mathbb{C}^2$ containing $v = 0$, $w = 1$. In particular, for $v = 0$, this part of the continuous spectrum is holomorphic in the half-plane $\Re(w) > 1/2$.

On the other hand, by direct computation, the apparent pole of $R(w)E_{v+1}$ at $v = 0$ (independent of $w$) cancels with the corresponding pole of $Q(v; v, w)E_{1-v}$. To establish that the order of the pole at $w = 1$, when $v = 0$, is $r_1 + r_2 + 1$, consider the most relevant terms (recall (4.15), (4.16)) in the Laurent expansions of $R(w)E_{v+1}$ and $Q(v; v, w)E_{1-v}$. Putting them together, we obtain an expression

$$
\frac{1}{v} \cdot \left[ \frac{c_1}{(w-1)^{r_1+r_2}} - \frac{c_2}{(2v + w - 1)^{r_1+r_2}} \right]
$$

for some constants $c_1$, $c_2$. As there is no pole at $v = 0$, we have $c_1 = c_2$. Canceling the factor $1/v$, and then setting $v = 0$, the assertion follows.

This completes the proof. $\square$

§5. Asymptotic formula

Let $k$ be a number field with $r_1$ real embeddings and $2r_2$ complex embeddings. Assume that $\varphi$ is specialized to (3.11). By Theorem 3.12, for $\Re(v)$ and $\Re(w)$ sufficiently large, the integral $I(\chi_0) = I(v, w)$ defined by (3.6) is

$$
(5.1) \quad I(v, w) = \sum_{\chi \in \hat{C}_0, s} \frac{1}{2\pi i} \int \left. L(\chi^{-1} \cdot |v+1-s|, f_1) \cdot L(\chi \cdot |s|, \bar{f}_2) \right|_{\Re(s) = \sigma} K_\infty(s, v, w, \chi) ds
$$

where $K_\infty(s, v, w, \chi)$ is given by (3.9) and (3.10), and where the sum is over $\chi \in \hat{C}_0$ unramified outside $S$ and with bounded ramification, depending only on $f_1$ and $f_2$.

By Theorem 4.17, it follows that $I(v, w)$ admits meromorphic continuation to a region in $\mathbb{C}^2$ containing the point $v = 0$, $w = 1$. In particular, if $f_1 = f_2 = \bar{f}$, then $I(0, w)$ is holomorphic for $\Re(w) > 11/18$, except for $w = 1$ where it has a pole of order $r_1 + r_2 + 1$.

We will shift the line of integration to $\Re(s) = \frac{1}{2}$ in (5.1) and set $v = 0$. To do so, we need some analytic continuation and reasonable decay in $|\Im(s)|$ for the kernel function $K_\infty(s, v, w, \chi)$. In fact, it is desirable for applications to have precise asymptotic formulæ as the parameters $s, v, w, \chi$ vary. By the decomposition (3.10), the analysis of the kernel $K_\infty(s, v, w, \chi)$ reduces to the corresponding analysis of the local component $K_v(s, v, \chi_v)$, for $\nu|\infty$. When $\nu$ is complex, one can use the asymptotic formula already established in [DG2], Theorem 6.2. For coherence, we include a simple computation matching, as it should, the local integral (3.9), for $\nu$ complex, with the integral (4.15) in [DG2].

Fix a complex place $\nu|\infty$. An irreducible unitary representation of $GL_2(\mathbb{C})$ always contains a spherical vector. Therefore, since we are interested only in the finite-prime part of the $L$–function associated to a cuspidal representation, we may as well suppose that $f_1$ and $f_2$ are spherical at all complex places. Also, recall that any character $\chi_v$ of $\mathbb{Z}_v \backslash M_v \approx \mathbb{C}^\times$ has the form

$$
\chi_v(m_v) = |z_v|^\ell_v \cdot e^{\ell_v \cdot t_v} \quad (m_v = \begin{pmatrix} z_v & 0 \\ 0 & 1 \end{pmatrix}, t_v \in \mathbb{R}, \ell_v \in \mathbb{Z})
$$
Then, the local integral (3.9) at \( \nu \) is

\[
K_\nu(s, v, w, \chi_\nu) = \int_0^\infty \int_0^\infty \int_C \int_C \int_\pi^{2\pi} \int_\pi^{2\pi} \frac{(x^n)^{3/2}}{(e^{\pi/2} + 1)^w} e^{2\pi i \cdot \text{Tr}_C/(a, x e^{i\theta_1}, a_x e^{i\theta_2})} \nu^{2v-2} \nu^{2w-2} K_{2i\mu_1}(4\pi a_1) a_2^{2+2it_\nu-1} K_{2i\mu_2}(4\pi a_2) e^{i\ell_1} e^{-i\ell_2} d\theta_1 d\theta_2 dx da_1 da_2
\]

Replacing \( x \) by \( x/a_1 \), we obtain

\[
K_\nu(s, v, w, \chi_\nu) = \int_0^\infty \int_0^\infty \int_C \int_C \int_\pi^{2\pi} \int_\pi^{2\pi} \frac{(x^n)^{3/2}}{(e^{\pi/2} + 1)^w} e^{2\pi i \cdot \text{Tr}_C/(x e^{i\theta_1} - a_x e^{i\theta_2})} \nu^{2v-2} \nu^{2w-2} K_{2i\mu_1}(4\pi a_1) a_2^{2+2it_\nu-1} K_{2i\mu_2}(4\pi a_2) e^{i\ell_1} e^{-i\ell_2} d\theta_1 d\theta_2 dx da_1 da_2
\]

If we further substitute

\[
a_1 = r \cos \phi \quad x_1 = r \sin \phi \cos \theta \quad x_2 = r \sin \phi \sin \theta \quad a_2 = u \cos \phi
\]

with \( 0 \leq \phi \leq \frac{\pi}{2} \) and \( 0 \leq \theta \leq 2\pi \), then

\[
K_\nu(s, v, w, \chi_\nu) = \int_0^\infty \int_0^\frac{\pi}{2} \int_0^{2\pi} \int_0^{2\pi} (\cos \phi)^{2w-2} e^{2\pi i \cdot \text{Tr}_C/(r \sin \phi \cdot e^{i(\theta_1 + \theta_2)} - u \sin \phi \cdot e^{i(\theta_1 + \theta_2)})} \nu^{2v-2} \nu^{2w-2} K_{2i\mu_1}(4\pi r \cos \phi) u^{2+2it_\nu-1} K_{2i\mu_2}(4\pi u \cos \phi) e^{i\ell_1} e^{-i\ell_2} \sin \phi d\theta_1 d\theta_2 dr d\phi du
\]

Using the Fourier expansion

\[
e^{i\ell \sin \theta} = \sum_{k=-\infty}^{\infty} J_k(t) e^{ik\theta}
\]

we obtain

\[
K_\nu(s, v, w, \chi_\nu) = (2\pi)^3 \int_0^\infty \int_0^\frac{\pi}{2} \int_0^{2\pi} \int_0^{2\pi} K_{2i\mu_1}(4\pi r \cos \phi) K_{2i\mu_2}(4\pi u \cos \phi) J_{\ell_1}(4\pi r \sin \phi) J_{\ell_2}(4\pi u \sin \phi) \nu^{2v-2} \nu^{2w-2} K_{2i\mu_1}(4\pi r \cos \phi) K_{2i\mu_2}(4\pi u \cos \phi) e^{i\ell_1} e^{-i\ell_2} \sin \phi dr d\phi du
\]

In the notation of [DG2], equation (4.15), this is essentially \( K_{\ell_0}(2s + 2it_\nu, 2v, 2w) \). It follows that \( K_\nu(s, v, w, \chi_\nu) \) is analytic in a region \( \mathcal{D} : \Re(s) = \sigma > \frac{1}{2} - \varepsilon_0, \Re(v) > -\varepsilon_0 \) and \( \Re(w) > \frac{3}{4} \), with a fixed (small) \( \varepsilon_0 > 0 \), and moreover, we have the asymptotic formula

\[
K_\nu(s, v, w, \chi_\nu) = \pi^{-2v+1} A(v, w, \mu_1, \mu_2) \cdot \left( 1 + \varepsilon_0^2 + 4(t + t_\nu)^2 \right)^{-w} \nu^{2v-2} \nu^{2w-2} K_{2i\mu_1}(4\pi r \cos \phi) K_{2i\mu_2}(4\pi u \cos \phi) e^{i\ell_1} e^{-i\ell_2} \sin \phi \frac{d\theta_1 d\theta_2}{ru}
\]

(5.2)

where \( A(v, w, \mu_1, \mu_2) \) is the ratio of products of gamma functions.
\[(5.3) \quad 2^{4w-4v-4} \frac{\Gamma(w + v + i\mu_1 + i\bar{\mu}_2) \Gamma(w + v - i\mu_1 + i\bar{\mu}_2) \Gamma(w + v + i\mu_1 - i\bar{\mu}_2) \Gamma(w + v - i\mu_1 - i\bar{\mu}_2)}{\Gamma(2w + 2v)} \]

For \(\nu\) real, the corresponding argument (including the integrals that arise from the (anti-) holomorphic discrete series) is even simpler (see [DG1] and [Zh2]). In this case, the asymptotic formula of \(K_\nu(s,v,w,\chi_\nu)\) becomes

\[(5.4) \quad K_\nu(s,v,w,\chi_\nu) = B(v,w,\mu_1,\mu_2) \cdot \left[ 1 + \mathcal{O}_{\sigma,v,w,\mu_1,\mu_2} \left( (1 + |t + t_\nu|)^{-\frac{3}{2}} \right) \right] \]

where \(B(v,w,\mu_1,\mu_2)\) is a similar ratio of products of gamma functions.

It now follows that for \(\Re(w)\) sufficiently large,

\[(5.5) \quad I(0,w) = \sum_{\chi \in C_0, S} \frac{1}{2\pi} \int_{-\infty}^{\infty} L(\chi^{-1} : \frac{1}{2} - it, f_1) \cdot L(\chi : \frac{1}{2} + it, \bar{f}_2) K_\infty(\frac{1}{2} + it, 0, w, \chi) dt \]

Since \(I(0,w)\) has analytic continuation to \(\Re(w) > 11/18\), a mean value result can already be established by standard arguments. For instance, assume \(f_1 = f_2 = \bar{f}\), and choose a function \(h(w)\) which is holomorphic and with sufficient decay (in \(|\Im(w)|\)) in a suitable vertical strip containing \(\Re(w) = 1\). For example, one can choose a suitable product of gamma functions. Consider the integral

\[(5.6) \quad \frac{1}{i} \int_{\Re(w) = L} I(0,w) h(w) T^w dw \]

with \(L\) a large positive constant. Assuming \(h(1) = 1\), we have the asymptotic formula

\[(5.7) \quad \sum_{\chi \in C_0, S} \int_{-\infty}^{\infty} |L(\chi : \frac{1}{2} + it, f \otimes \chi)|^2 \cdot M_{\chi, \tau}(t) dt \sim AT (\log T)^{r_1 + r_2} \]

for some computable positive constant \(A\), where

\[(5.8) \quad M_{\chi, \tau}(t) = \frac{1}{2\pi i} \int_{\Re(w) = L} K_\infty(\frac{1}{2} + it, 0, w, \chi) h(w) T^w dw \]

For a character \(\chi \in \hat{C}_0\), put

\[(5.9) \quad \kappa_\chi(t) = \prod_{\nu \approx \mathbb{R}} (1 + |t + t_\nu|) \cdot \prod_{\nu \approx \mathbb{C}} (1 + \ell_\nu^2 + 4(t + t_\nu)^2) \quad (t \in \mathbb{R}) \]

where \(it_\nu\) and \(\ell_\nu\) are the parameters of the local component \(\chi_\nu\) of \(\chi\). Since \(\chi\) is trivial on the positive reals,

\[\sum_{\nu \approx \mathbb{R}} d_\nu t_\nu = 0\]
with \( d_\nu = [k_\nu : \mathbb{R}] \) the local degree. Then, the main contribution to the asymptotic formula (5.7) comes from terms for which \( \kappa_\nu(t) \ll T \). For applications, it might be more convenient to work with a slightly modified function \( Z(w) \) defined by

\[
Z(w) = \sum_{\chi \in \mathcal{C}_0,S} \int_{-\infty}^{\infty} |L(\frac{1}{2} + it, f \otimes \chi)|^2 \cdot \kappa_\nu(t)^{-w} \, dt
\]

(5.10)

It is obtained from the function \( I(0,w) \) by essentially picking off just the main terms in the asymptotic formulae (5.2) and (5.4) of the local components \( K_\nu(s,0,w,\chi_\nu) \). Its analytic properties can be transferred (via some technical adjustments) from those of \( I(0,w) \). As an illustration of this fact, we show that the right-hand side of (5.10) is absolutely convergent for \( \Re(w) > 1 \).

To see the absolute convergence of the defining expression (5.5) for \( I(0,w) \), first note that the triple integral expressing \( K_\nu(s,v,w,\chi_\nu) \) can be written as

\[
K_\nu(\frac{1}{2} + it,0,w,\chi_\nu) = (2\pi)^3 \int_0^{\pi} (\cos \phi)^{2w-1} \sin \phi \cdot |V_{f,v,\chi_\nu}(t,\phi)|^2 \, d\phi \quad \text{ (for } \nu \approx \mathbb{C})
\]

(5.11)

when \( v = 0 \) and \( \Re(s) = \frac{1}{2} \), where

\[
V_{f,v,\chi_\nu}(t,\phi) = \int_0^\infty u^{2i(t_\nu + t)} K_{2i\mu_{f,v}}(4\pi u \cos \phi) J_{|t_\nu|}(4\pi u \sin \phi) \, du
\]

(5.12)

Here we also used the well-known identity \( J_{-t}(z) = (-1)^t J_t(z) \). The convergence of the last integral is justified by 6.576, integral 3, page 716 in [GR]. For \( \nu \approx \mathbb{R} \), the local integral (3.9) has a similar form, when \( v = 0 \) and \( \Re(s) = 1/2 \), as it can be easily verified by a straightforward computation.

The form of the integral (5.11) allows us to adopt the argument used in the proof of Landau’s Lemma to our context giving the desired conclusion. We shall follow [C], proof of Theorem 6, page 115.

Choose a sufficiently large real number \( a \) such that the right-hand side of (5.5) is convergent at \( w = a \). Since \( I(0,w) \) is holomorphic for \( \Re(w) > 1 \), its Taylor series

\[
\sum_{j=0}^{\infty} \frac{(w-a)^j}{j!} I^{(j)}(0,a)
\]

(5.13)

has radius of convergence \( a - 1 \). Using the structure of (5.11) and its analog at real places, we have that

\[
(w-a)^j \cdot K_\nu^{(j)}(\frac{1}{2} + it,0,a,\chi) \geq 0 \quad \text{ (for } w \leq a)
\]
Having all terms non-negative in (5.13) when \( w < a \), we can interchange the first sum with the second and the integral. Since

\[
K_\infty\left(\frac{1}{2} + it, 0, w, \chi\right) = \sum_{j=0}^{\infty} \frac{(w - a)^j}{j!} K_\infty^{(j)}\left(\frac{1}{2} + it, 0, a, \chi\right)
\]

the absolute convergence of (5.10) for \( \Re(w) > 1 \) follows.

Setting \( w = 1 + \varepsilon \), then for arbitrary \( T > 1 \),

\[
\sum_{\chi \in \mathcal{C}_0, s} \int_{I_\chi(T)} |L(\frac{1}{2} + it, f \otimes \chi)|^2 \cdot T^{-1-\varepsilon} \, dt < Z(1 + \varepsilon) \ll_{\varepsilon} 1
\]

where \( I_\chi(T) = \{ t \in \mathbb{R} : \kappa_\chi(t) \leq T \} \), and hence

\[
\sum_{\chi \in \mathcal{C}_0, s} \int_{I_\chi(T)} |L(\frac{1}{2} + it, f \otimes \chi)|^2 \, dt \ll_{\varepsilon} T^{1+\varepsilon}
\]

Only finitely many characters contribute to the left-hand sum. This estimate is compatible with the convexity bound, in the sense that it implies for example that

\[
\int_{0}^{T} |L(\frac{1}{2} + it, f)|^2 \, dt \ll_{\varepsilon} T^{k|Q|+\varepsilon}
\]

Therefore, the function \( Z(w) \) defined by (5.10) leads to averages of reasonable size suitable for applications. We return to a further study of the analytic properties of this function in a forthcoming paper.

**Concluding remarks:** The specific choice (3.11) of the data \( \varphi_\nu \) at archimedean places was made for no reason other than simplicity, enabling us to illustrate the non-vacuousness of the structural framework. Specifically, this choice allowed us to show that asymptotic formulas can be obtained, and that the averaging is not too long, i.e., compatible with the convexity bound. This choice sufficed for our purposes, which, again, were to stress generality, leaving aside the more technical aspects necessary in obtaining sharper results. Its use allowed us to quickly understand the size of the averages via the pole at \( w = 1 \), and dispensed with unnecessary complications.

The function \( I(v, w) \) in (5.1) is analytic for \( v \) in a neighborhood of 0 and \( \Re(w) \) sufficiently large. This follows easily from the analytic properties of \( K_\infty(s, v, w, \chi) \) discussed in Section 5. By computing \( I(v, w) \) using (4.13), this simple observation can be used to find the value of the constant \( \kappa \) given in (4.7).

§ **Appendix 1. Convergence of Poincaré series**

The aim of this appendix is to discuss the proofs of Proposition 2.6 and Theorem 2.7. Given the lack of complete arguments in the literature, we have given a full account here, applicable more
generally. For a careful discussion of some aspects of \( GL(2) \), see [GJ] and [CPS1]. Note that the latter source needs some small corrections in the inequalities on pages 28 and 29.

We first prove the absolute convergence of the Poincaré series, uniformly on compacts on \( G_k \), for \( G = GL_2 \) over a number field \( k \) with ring of integers \( \mathfrak{o} \), for \( \Re(v) > 1 \) and \( \Re(w) > 1 \). Second, we recall the notion of norm on a group, to prove convergence in \( L^2 \) for admissible data (see the end of Section 2), also reproving pointwise convergence by a more broadly applicable method.

Toward our first goal, we need an elementary comparison of sums and integrals under mild hypotheses. Let \( V_1, \ldots, V_n \) be finite-dimensional real vector spaces, with fixed inner products, and put

\[
V = V_1 \oplus \ldots \oplus V_n \quad \text{(orthogonal direct sum)}
\]

with the natural inner product. Fix a lattice \( \Lambda \) in \( V \), and let \( F \) be a period parallelogram for \( \Lambda \) in \( V \), containing 0. Let \( g \) be a real-valued function on \( V \) with \( g(\xi) \geq 1 \), such that \( 1/g \) has finite integral over \( V \), and is multiplicatively bounded on each translate \( \xi + F \), in the sense that, for each \( \xi \in \Lambda \),

\[
\sup_{y \in \xi + F} \frac{1}{g(y)} \ll \inf_{y \in \xi + F} \frac{1}{g(y)} \quad \text{(with implied constant independent of \( \xi \))}
\]

For a differentiable function \( f \), let \( \nabla_i f \) be the gradient of \( f \) in the \( V_i \) variable. Then,

\[
\sum_{\xi \in \Lambda} |f(\xi)| \ll \int_V |f(\xi)| \, d\xi + \sum_i \sup_{\xi \in V} (g(\xi) \cdot |\nabla_i f(\xi)|)
\]

with the implied constant independent of \( f \).

The following calculus argument gives this comparison (Abel summation). Let \( \text{vol}(\Lambda) \) be the natural measure of \( V/\Lambda \). Certainly,

\[
\text{vol}(\Lambda) \cdot \sum_{\xi \in \Lambda} |f(\xi)| = \sum_{\xi \in \Lambda} |f(\xi)| \cdot \int_{\xi + F} dx
\]

and

\[
f(\xi) \int_{\xi + F} dx = \int_{\xi + F} (f(\xi) - f(x)) \, dx + \int_{\xi + F} f(x) \, dx
\]

The sum over \( \xi \in \Lambda \) of the latter integrals is obviously the integral of \( f \) over \( V \), as in the claim. The differences \( f(\xi) - f(x) \) require further work. For \( i = 1, \ldots, n \), let \( x_i \) and \( y_i \) be the \( V_i \)-components of \( x, y \in V \), respectively. Let

\[
d_i(F) = \sup_{x, y \in F} |x_i - y_i|
\]

By the Mean Value Theorem, we have the easy estimate

\[
|f(\xi) - f(x)| \leq \sum_{i=1}^n d_i(F) \cdot \sup_{y \in \xi + F} |\nabla_i f(y)|
\]

Then,
\[
\sum_{\xi \in \Lambda} \int_{\xi + F} |f(\xi) - f(x)| \, dx \ll \sum_{\xi \in \Lambda} \sum_{i=1}^{n} \sup_{y \in \xi + F} |\nabla_i f(y)|
\]

\[
= \sum_{i=1}^{n} \sum_{\xi \in \Lambda} \sup_{y \in \xi + F} \left( \frac{1}{g(y)} g(y) |\nabla_i f(y)| \right) \leq \sum_{i=1}^{n} \sum_{\xi \in \Lambda} \left( \sup_{y \in \xi + F} \frac{1}{g(y)} \right) \cdot \left( \sup_{y \in \xi} |\nabla_i f(y)| \right)
\]

\[
\ll \int_{V} \frac{du}{g(u)} \cdot \sum_{i} \sup_{y \in V} (g(y) |\nabla_i f(y)|) \ll \sum_{i} \sup_{y \in V} (g(y) |\nabla_i f(y)|)
\]

This gives the indicated estimate.

The above estimate will show that the Poincaré series with parameter \( v \) is dominated by the sum of an Eisenstein series at \( v \) and an Eisenstein series at \( v + 1 + \varepsilon \) for every \( \varepsilon > 0 \), under mild assumptions on the archimedean data. Such an Eisenstein series converges absolutely and uniformly on compacts for \( \Re(v) > 1 \), either by Godement’s criterion, in classical guise in [B], or by more elementary estimates that suffice for \( GL_2 \). Thus, the Poincaré series converges absolutely and uniformly for \( \Re(v) > 1 \).

The assumptions on the archimedean data

\[
\Phi_{\infty}(x) = \varphi_{\infty} \left( \begin{array}{c} 1 \\ x \\ 1 \end{array} \right)
\]

are that

\[
\int_{k_{\infty}} |\Phi_{\infty}(\xi)| \, d\xi < +\infty
\]

and, letting \( \nabla_\nu \) be the gradient along the summand \( k_\nu \) of \( k_{\infty} \), that, for some \( \varepsilon > 0 \), for each \( \nu |_{\infty} \),

\[
\sup_{\xi \in k_{\infty}} (1 + |\nabla_\nu \Phi_{\infty}(\xi)|) < \infty
\]

The comparison argument is as follows. To make a vector from which to form an Eisenstein series, left-average the kernel

\[
\varphi \left( \begin{array}{c} a \\ d \\ \frac{1}{d} \end{array} \right) \left( \begin{array}{c} 1 \\ x \\ 1 \end{array} \right) = |a/d|^\nu \cdot \Phi(x) \quad \text{(extended by right } K_A\text{–invariance)}
\]

for the Poincaré series over \( N_k \). That is, form

\[
\tilde{\varphi}(g) = \sum_{\beta \in N_k} \varphi(\beta \cdot g)
\]

This must be proven to be dominated by a vector (or vectors) from which Eisenstein series are formed. The usual vector for standard spherical Eisenstein series is

\[
\eta_s \left( \begin{array}{c} a \\ * \\ d \end{array} \right) = |a/d|^s
\]

extended to \( G_A \) by right \( K_A\)–invariance. We claim that

\[
\tilde{\varphi} \ll \eta_v + \eta_{v+1+\varepsilon} \quad \text{(for all } \varepsilon > 0)\]
Since all functions $\varphi$, $\tilde{\varphi}$ and $\eta$ are right $K_A$-invariant and have trivial central character, it suffices to consider $g = nh$ with $n \in N_A$ and

$$h = \begin{pmatrix} y \\ 1 \end{pmatrix} \in H_A$$

Let

$$n_t = \begin{pmatrix} 1 \\ t \\ 1 \end{pmatrix}$$

We have

$$\varphi(n_\xi \cdot n_x h) = \varphi(h \cdot h^{-1} n_\xi n_x h) = \varphi(h \cdot h^{-1} n_\xi + x h) = |y|^{\nu} \cdot \Phi \left( \frac{1}{y} \cdot (\xi + x) \right)$$

Thus, to dominate the Poincaré series by Eisenstein series, it suffices to prove that

$$\sum_{\xi \in k} \Phi \left( \frac{1}{y} \cdot (\xi + x) \right) \ll 1 + |y|$$

(uniformly in $x \in N_A$, $y \in J$)

Since $\tilde{\varphi}$ is left $N_k$-invariant, it suffices to take $x \in A$ to lie in a set of representatives $X$ for $A/k$, such as

$$X = k_\infty/\mathfrak{o} \oplus \prod_{\nu < \infty} \mathfrak{o}_\nu$$

where, by abuse of notation, $k_\infty/\mathfrak{o}$ refers to a period parallelogram for the lattice $\mathfrak{o}$ in $k_\infty$. As $\varphi$ and $\tilde{\varphi}$ are left $H_k$-invariant, so we can adjust $y$ in $J$ by $k^\times$. Since $\mathbb{J}^1/k^\times$ is compact, we can choose representatives in $\mathbb{J}$ for $\mathbb{J}/k^\times$ lying in $C' \cdot (0, +\infty)$ for some compact set $C' \subset J^1$, with $(0, +\infty)$ embedded in $J$ as usual by

$$t \rightarrow (t^{1/n}, t^{1/n}, \ldots, t^{1/n}, 1, 1, \ldots, 1, \ldots)$$

(non-trivial entries at archimedean places)

where $n = [k : \mathbb{Q}]$. Further, for simplicity, we may adjust the representatives $y$ such that $|y|_\nu \leq 1$ for all finite primes $\nu$. The compactness of $C'$ implies that

$$|y| \leq \prod_{\nu | \infty} |y_\nu|_\nu \ll |y|$$

(with implied constant depending only on $k$)

Likewise, due to the compactness, the archimedean valuations of representatives have bounded ratios.

At a finite place, $\Phi_\nu \left( \frac{1}{y} \cdot (x + \xi) \right)$ vanishes unless

$$\frac{1}{y} \cdot (x + \xi) \in \mathfrak{o}_\nu$$

That is, since we want a uniform bound in $x \in \mathfrak{o}_\nu$, this vanishes unless

$$\xi \in \mathfrak{o}_\nu + y \cdot \mathfrak{o}_\nu \subset \mathfrak{o}_\nu$$

since we have taken representatives $y$ with $y_\nu$ integral at all finite $\nu$. Thus, the sum over $\xi$ in $k$ reduces to a sum over $\xi$ with archimedean part in the lattice $\Lambda = \mathfrak{o} \subset k_\infty$. 

Setting up a comparison as above, let
\[ V = k_\infty = \bigoplus_{\nu|\infty} k_\nu \]

Let \( \text{vol}(\Lambda) \) be the volume of \( \Lambda \). For archimedean place \( \nu \) let \( \nabla_\nu \) be the gradient along \( k_\nu \), and \( d_\nu(\Lambda) \) the maximum of \( |x_\nu - y_\nu|_\nu \) for \( x, y \in F \), a fixed period parallelogram for \( \Lambda \) in \( k_\infty \). We have
\[
\sum_{\xi \in \Lambda} \Phi \left( \frac{1}{y} \cdot (\xi + x) \right) \ll \int_{k_\infty} \Phi_\infty \left( \frac{1}{y} \cdot (\xi + x) \right) \, d\xi + \sum_{\nu|\infty} \sup_{\xi \in k_\infty} \left( g(\xi) \cdot |\nabla_\nu \Phi_\infty(\xi)| \right)
\]
for any suitable weight function \( g \). In the integral, replace \( \xi \) by \( \xi - x \), and then by \( \xi \cdot y \), to see that
\[
\int_{k_\infty} \Phi_\infty \left( \frac{1}{y} \cdot (\xi + x) \right) \, d\xi = |y|_\infty \cdot \int_{k_\infty} \Phi_\infty(\xi) \, d\xi \ll |y| \cdot \int_{k_\infty} \Phi_\infty(\xi) \, d\xi
\]
with the implied constant depending only upon \( k \), using the choice of representatives \( y \) for \( J/k^\times \).

To estimate the sum, for \( x \in k_\infty \), fix \( \varepsilon > 0 \) and take weight function \( g(\xi) = \prod_{\nu|\infty} (1 + |\xi|_\nu^2)^{\frac{1}{2} + \varepsilon} \).

This is readily checked to have the \textit{multiplicative boundedness} property needed: the function \( g \) is continuous, and for \( |\xi| \geq 2|x| \), we have the elementary
\[
\frac{1}{2} \cdot |\xi| \leq |\xi - x| \leq 2 \cdot |\xi|
\]
from which readily follows the bound for \( g(\xi) \).

What remains is to compute the indicated suprema with attention to their dependence on \( y \).

At an archimedean place \( \nu \),
\[
\sup_{\xi \in k_\infty} \left( g(\xi) \cdot |\nabla_\nu \Phi_\nu \left( \frac{1}{y} \cdot (\xi + x) \right)| \right) = \sup_{\xi \in k_\infty} \left( g(\xi - x) \cdot |\nabla_\nu \Phi_\nu \left( \frac{1}{y} \cdot \xi \right)| \right)
\ll \sup_{\xi \in k_\infty} \left( g(\xi) \cdot |\nabla_\nu \Phi_\nu \left( \frac{1}{y} \cdot \xi \right)| \right)
\]
by using the boundedness property of \( g \). Then replace \( \xi \) by \( \xi \cdot y \), to obtain
\[
\sup_{\xi \in k_\infty} \left( g(\xi \cdot y) \cdot |\nabla_\nu \Phi_\nu(\xi)| \right)
\]
Since
\[
(1 + |y|_\nu^2 |\xi|_\nu^2) \leq (1 + |y|_\nu^2) \cdot (1 + |\xi|_\nu^2) \quad \text{(for all \( \nu|\infty \))}
\]
we have \( g(\xi \cdot y) \leq g(y) \cdot g(\xi) \), and
\[
\sup_{\xi \in k_\infty} \left( g(\xi \cdot y) \cdot |\nabla_\nu \Phi_\nu(\xi)| \right) \leq g(y) \cdot \sup_{\xi \in k_\infty} \left( g(\xi) \cdot |\nabla_\nu \Phi_\nu(\xi)| \right)
\]
Here the weighted supremums of the gradients appear, which we have assumed finite.
Finally, estimate
\[ g(y) = \prod_{\nu \mid \infty} (1 + |y|^2_{\nu}) \]
with \( y \) in our specially chosen set of representatives. For these representatives, for any two archimedean places \( \nu_1 \) and \( \nu_2 \), we have
\[ |y|_{\nu_1}^{n_{\nu_1}} \ll |y|_{\nu_2}^{n_{\nu_2}} \]
where the \( n_{\nu_i} \) are the local degrees \( n_{\nu_i} = [k_{\nu_i} : \mathbb{R}] \). Therefore,
\[ |y|_{\nu} \ll |y|^{n_{\nu}/n} \]
where \( n = \sum_{\nu} n_{\nu} \) is the global degree. Thus,
\[ \prod_{\nu \mid \infty} (1 + |y|^2_{\nu}) \ll 1 + |y|^2 \]
Then,
\[ \prod_{\nu \mid \infty} (1 + |y|^2_{\nu})^{\frac{1}{2} + \varepsilon} \ll (1 + |y|^2)^{\frac{1}{2} + \varepsilon} \]
Putting this all together, for every \( \varepsilon > 0 \)
\[ \bar{\varphi} \begin{pmatrix} y & * \\ 0 & 1 \end{pmatrix} \ll |y|^v \cdot (1 + |y|^2)^{\frac{1}{2} + \varepsilon} = |y|^v + |y|^{v+1+2\varepsilon} = \eta_v \begin{pmatrix} y & * \\ 0 & 1 \end{pmatrix} + \eta_{v+1+2\varepsilon} \begin{pmatrix} y & * \\ 0 & 1 \end{pmatrix} \]
which is the desired domination of the Poincaré series by a sum of Eisenstein series.

For the particular choice of archimedean data
\[ \Phi_{\infty}(\xi) = \prod_{\nu \mid \infty} \frac{1}{(1 + |\xi|^2_{\nu})^{w/2}} \]
the integrability condition is met when \( \Re(w) > 1 \). Similarly, the weighted supremums of gradients are finite for \( \Re(w) > 1 \).

Altogether, this particular Poincaré series is absolutely convergent for \( \Re(v) > 1 + 2\varepsilon \) and \( \Re(w) > 1 + \varepsilon \), for every \( \varepsilon > 0 \). This proves Proposition 2.6.

**Soft convergence estimates on Poincaré series:** Now we give a different approach to convergence, more convenient for proving square integrability of Poincaré series. It is more robust, and does also reprove pointwise convergence, but gives a weaker result than the previous more explicit approach. Let \( G \) be a (locally compact, Hausdorff, separable) unimodular topological group. Fix a compact subgroup \( K \) of \( G \). A norm \( g \rightarrow \|g\| \) on \( G \) is a positive real-valued continuous function on \( G \) with properties

* \( \|g\| \geq 1 \) and \( \|g^{-1}\| = \|g\| \)
* Submultiplicativity: \( \|gh\| \leq \|g\| \cdot \|h\| \)
* \( K \)-invariance: for \( g \in G \), \( k \in K \), \( \|k \cdot g\| = \|g \cdot k\| = \|g\| \)
* Integrability: for sufficiently large \( \sigma > 0 \),
\[ \int_G \|g\|^{-\sigma} \, dt < +\infty \]
For a discrete subgroup $\Gamma$ of $G$, for $\sigma > 0$ large enough such that $\|g\|^{-\sigma}$ is integrable on $G$, we claim the corresponding summability:

$$\sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|^{\sigma}} < +\infty$$

The proof is as follows. From

$$\|\gamma \cdot g\| \leq \|\gamma\| \cdot \|g\|$$

for $\sigma > 0$

$$\frac{1}{\|\gamma\|^{\sigma} \cdot \|g\|^{\sigma}} \leq \frac{1}{\|\gamma \cdot g\|^{\sigma}}$$

Invoking the discreteness of $\Gamma$ in $G$, let $C$ be a small open neighborhood of $1 \in G$ such that

$$C \cap \Gamma = \{1\}$$

Then,

$$\int_{C} \frac{dg}{\|g\|^{\sigma}} \cdot \sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|^{\sigma}} \leq \int_{C} \sum_{\gamma \in \Gamma} \frac{1}{\|\gamma \cdot g\|^{\sigma}} dg = \sum_{\gamma \in \Gamma} \int_{C} \frac{1}{\|\gamma\|^{\sigma}} dg \leq \int_{G} \frac{dg}{\|g\|^{\sigma}} < +\infty$$

This gives the indicated summability. Let $H$ be a closed subgroup of $G$, and define a relative norm

$$\|g\|_{H} = \inf_{h \in H \cap \Gamma} \|h \cdot g\|$$

From the definition, there is the left $H \cap \Gamma$–invariance

$$\|h \cdot g\|_{H} = \|g\|_{H} \quad \text{for all } h \in H \cap \Gamma$$

Note that $\| \cdot \|_{H}$ depends upon the discrete subgroup $\Gamma$.

**Moderate increase, sufficient decay.** Let $H$ be a closed subgroup of $G$. A left $H \cap \Gamma$–invariant complex-valued function $f$ on $G$ is of moderate growth modulo $H \cap \Gamma$, when, for sufficiently large $\sigma > 0$,

$$|f(g)| \ll \|g\|_{H}^{\sigma}$$

The function $f$ is rapidly decreasing modulo $H \cap \Gamma$ if

$$|f(g)| \ll \|g\|^{-\sigma}_{H} \quad \text{for all } \sigma > 0$$

The function $f$ is sufficiently rapidly decreasing modulo $H \cap \Gamma$ (for a given purpose) if

$$|f(g)| \ll \|g\|^{-\sigma}_{H} \quad \text{for some sufficiently large } \sigma > 0$$

Since $\|g\|_{H}$ is an infimum, for $\sigma > 0$ the power $\|g\|^{-\sigma}$ is a supremum

$$\frac{1}{\|g\|^{\sigma}} = \sup_{h \in H \cap \Gamma} \frac{1}{\|hg\|^{\sigma}}$$
**Pointwise convergence of Poincaré series:** We claim that, for $f$ left $H \cap \Gamma$–invariant and sufficiently rapidly decreasing mod $H \cap \Gamma$, the Poincaré series

$$P_f(g) = \sum_{\gamma \in (H \cap \Gamma)\setminus \Gamma} f(\gamma \cdot g)$$

converges absolutely and uniformly on compacts. To see this, first note that, for all $h \in H \cap \Gamma$,

$$\|\gamma\|_H \leq \|h \cdot \gamma\| = \|h \cdot \gamma \cdot g^{-1}\| \leq \|h\gamma\| \cdot \|g^{-1}\|$$

Thus, taking the inf over $h \in H \cap \Gamma$,

$$\frac{\|\gamma\|_H}{\|g^{-1}\|} \leq \|\gamma \cdot g\|_H$$

Thus, for $\sigma > 0$,

$$\frac{1}{\|\gamma \cdot g\|_H^\sigma} \leq \frac{\|g\|_H^\sigma}{\|\gamma\|_H^\sigma}$$

and

$$P_f(g) = \sum_{\gamma \in (H \cap \Gamma)\setminus \Gamma} f(\gamma \cdot g) \ll \sum_{\gamma \in (H \cap \Gamma)\setminus \Gamma} \frac{1}{\|\gamma \cdot g\|_H^\sigma} \leq \|g\|_H^\sigma \cdot \sum_{\gamma \in (H \cap \Gamma)\setminus \Gamma} \frac{1}{\|h \cdot \gamma\|_H^\sigma} = \|g\|_H^\sigma \cdot \sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|_H^\sigma} \ll \|g\|_H^\sigma$$

estimating a sup of positive terms by the sum, for $\sigma > 0$ sufficiently large so that the sum over $\Gamma$ converges.

**Moderate growth of Poincaré series:** Next, we claim that Poincaré series are of moderate growth modulo $\Gamma$, namely, that

$$P_f(g) \ll \|g\|_G^\sigma$$

(for sufficiently large $\sigma > 0$)

Indeed, the previous estimate is uniform in $g$, and the left-hand side is $\Gamma$–invariant. That is, for all $\gamma \in \Gamma$,

$$P_f(g) = P_f(\gamma \cdot g) \ll \|\gamma \cdot g\|_H^\sigma$$

(with implied constant independent of $g, \gamma$)

Taking the inf over $\gamma$ gives the assertion.

**Square integrability of Poincaré series:** Next, we claim that for $f$ left $H \cap \Gamma$–invariant and sufficiently rapidly decreasing mod $H \cap \Gamma$, $P_f$ is square-integrable on $\Gamma \setminus G$. Unwind, and use the assumed estimate on $f$ along with the above-proven moderate growth of the Poincaré series:

$$\int_{\Gamma \setminus G} |P_f|^2 = \int_{(H \cap \Gamma) \setminus G} |f| \cdot |P_f| \ll \int_{(H \cap \Gamma) \setminus G} \|g\|_H^{-2\sigma} \cdot \|g\|_H^\sigma \, dg$$
Estimating a sup by a sum, and unwinding further,

\[
\int_{(H \cap \Gamma) \setminus G} \|g\|^{-\sigma} dg \leq \int_{(H \cap \Gamma) \setminus G} \sum_{h \in (H \cap \Gamma) \setminus \Gamma} \|h \cdot g\|^{-\sigma} dg = \int_G \|g\|^{-\sigma} dg < +\infty
\]

for large enough \(\sigma > 0\). This proves the square integrability of the Poincaré series.

Construction of a norm on \(PGL_2(\mathbb{A})\): We want a norm on \(G = PGL_2(\mathbb{A})\) over a number field \(k\) that meets the conditions above, including the integrability, with \(K\) the image in \(PGL_2(\mathbb{A})\) of the maximal compact

\[
\prod_{\nu \approx \mathbb{R}} O_2(\mathbb{R}) \times \prod_{\nu \approx \mathbb{C}} U(2) \times \prod_{\nu < \infty} GL_2(o_{\nu})
\]

of \(GL_2(\mathbb{A})\). We take \(\Gamma\) to be the image in \(PGL_2(\mathbb{A})\) of \(GL_2(k)\). Let \(g\) be the algebraic Lie algebra of \(GL_2\) over \(k\), so that, at each place \(\nu\) of \(k\),

\[
g_{\nu} = \{\text{2-by-2 matrices with entries in } k_{\nu}\}
\]

Let \(\rho\) denote the Adjoint representation of \(GL_2\) on \(g\), namely,

\[
\rho(g)(x) = gxg^{-1} \quad \text{(for } g \in GL_2 \text{ and } x \in g)\]

The kernel of \(\rho\) on \(GL_2\) is the center \(Z\), so the image \(G\) of \(GL_2\) under \(\rho\) is \(PGL_2\). As expected, let

\[
G_{\nu} = \rho(GL_2(k_{\nu})) = GL_2(k_{\nu})/Z_{\nu} \quad K_{\nu} = \rho(GL_2(o_{\nu})) = GL_2(o_{\nu})/(Z_{\nu} \cap GL_2(o_{\nu}))
\]

and

\[
\Gamma = G_k = \rho(GL_2(k)) = GL_2(k)/Z_k
\]

Since \(\Gamma\) is a subgroup of \(GL_k(g_k)\), it is discrete in the adelization of \(GL_k(g_k)\), so is discrete in \(G_{\mathbb{A}}\). Let \(\{e_{ij}\}\) be the 2-by-2 matrices with non-zero entry just at the \((i,j)^{th}\) location, where the entry is 1.

At an archimedean place \(\nu\) of \(k\), put a Hilbert space structure on \(g_{\nu}\) by

\[
\langle x, y \rangle = \text{tr}(y^*x)
\]

where \(y^*\) is \(y\)-transpose for \(\nu\) real, and \(y\)-transpose-conjugate for \(\nu\) complex. We put the usual (sup-norm) operator norm on linear operators \(T\) on \(g_{\nu}\), namely

\[
|T|_{op} = \sup_{|x| \leq 1} |Tx|
\]

By design, since the inner product on \(g_{\nu}\) is \(\rho(K_{\nu})\)-invariant, this operator norm is invariant under \(\rho(K_{\nu})\).

For a non-archimedean local field \(k\) with norm \(|\cdot|_\nu\), and ring of integers \(o\), give \(g_{\nu}\) the sup-norm

\[
|\sum_{ij} a_{ij} e_{ij}| = \sup_{ij} |a_{ij}|_\nu \quad \text{(with } a_{ij} \in k_{\nu})
\]
There is the operator norm on $GL_{k_{\nu}}(g_{\nu})$ given by

$$|g|_{op} = \sup_{x \in V, |x| \leq 1} |g \cdot x|$$

By design, this norm is invariant under $\rho(K_{\nu})$.

**Norms on local groups and adele groups:** For any place $\nu$ of $k$, define a (local) norm $\|g\|_{\nu}$ on the image $G_{\nu} = PGL_{2}(k_{\nu})$ of $GL_{2}(k_{\nu})$ in $GL_{k_{\nu}}(g_{\nu})$ by

$$\|g\|_{\nu} = \max\{|g|_{op}, |g^{-1}|_{op}\}$$

Since the norm on $g_{\nu}$ is $K_{\nu}$–invariant, and $K_{\nu}$ is stable under inverse, the operator norms are left and right $K_{\nu}$–invariant, and the norms $\| \|_{\nu}$ are left and right $K_{\nu}$–invariant. Note that for $\nu < \infty$ the operator norm is 1 on $K_{\nu}$. To prove that $\|g \cdot h\|_{\nu} \leq \|g\|_{\nu} \cdot \|h\|_{\nu}$ use the definition:

$$\|g \cdot h\|_{\nu} = \max\{|gh|_{op}, |h^{-1}g^{-1}|_{op}\} \leq \max\{|g|_{op}, |g^{-1}|_{op}\} \cdot \max\{|h|_{op}, |h^{-1}|_{op}\} = \|g\|_{\nu} \cdot \|h\|_{\nu}$$

For $g = \{g_{\nu}\}$ in the adele group $G_{A}$, let

$$\|g\| = \prod_{\nu} \|g_{\nu}\|_{\nu}$$

The factors in the product are 1 for all but finitely many places. The left and right $K$–invariance for $K = \prod_{\nu} K_{\nu}$ follows from the local $K_{\nu}$–invariance. Invariance under inverse is likewise clear.

**Integrability:** Toward integrability, we explicitly bound the local integrals

$$\int_{G_{\nu}} \|g\|_{\nu}^{-\sigma} \, dg$$

At finite primes, use the $p$–adic Cartan decomposition (here just the elementary divisor theorem) inherited from $GL_{2}(k_{\nu})$ via the quotient map, namely,

$$G_{\nu} = \bigsqcup_{\delta \in A_{\nu}/(A_{\nu} \cap Z_{\nu}K_{\nu})} K_{\nu} \cdot \delta \cdot K_{\nu} \quad (\text{where } A_{\nu} \text{ is diagonal matrices})$$

By conjugating by permutation matrices and adjusting by $Z_{\nu}$, we may assume, further, that

$$\delta = \begin{pmatrix} \delta_{1} & 0 \\ 0 & 1 \end{pmatrix} \quad (\text{with } |\delta_{1}| \geq 1)$$

For any choice $\varpi_{\nu}$ of local parameter for $k_{\nu}$, we may adjust by $A_{\nu} \cap K_{\nu}$ so that $\delta_{1}$ is a power of $\varpi$. On a given $K_{\nu}$ double coset, the norm is

$$\|K_{\nu} \cdot \delta \cdot K_{\nu}\|_{\nu} = \|\delta\|_{\nu} = \max\{||\rho(\delta)||_{op}, |\rho(\delta^{-1})||_{op}\} = \max\{|\delta_{1}|_{\nu}, |\delta_{1}^{-1}|_{\nu}\}$$
and
\[ \text{meas}(K_\nu \delta K_\nu) = \text{meas}(K_\nu) \cdot \text{card}(K_\nu \setminus K_\nu \delta K_\nu) \]

Let \( q = q_\nu \) be the residue field cardinality, and let \( |\delta|_\nu = q^\ell \) with \( \ell \geq 0 \). Then,
\[ \text{card}(K_\nu \setminus K_\nu \delta K_\nu) = \text{card}(K_\nu \cap \delta^{-1} K_\nu \delta \setminus K_\nu) \leq \text{card}(K_\nu(\ell) \setminus K_\nu) \]
where \( K_\nu(\ell) \) is a sort of congruence subgroup, namely,
\[ K_\nu(\ell) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in K_\nu : c \in \varpi^\ell \cdot o_\nu \right\} \]

Let \( K'_\nu = \{ g \in K_\nu : g = I_2 \mod \varpi o \} \), and let \( \mathbb{F}_q \) be the finite field with \( q \) elements. We have an elementary estimate
\[ \left[ K_\nu : K_\nu(1) \right] = \left[ K_\nu : K'_\nu \right] = \text{card}\{\text{lines in } \mathbb{F}_q^2\} = \frac{q^2 - 1}{q - 1} = q + 1 < q^2 \]
and
\[ \left[ K_\nu(\ell) : K_\nu(\ell + 1) \right] = q \quad \text{(for } \ell \geq 1) \]
Thus,
\[ \left[ K_\nu : K_\nu(\ell) \right] \leq q^2 \cdot q^{\ell - 1} \quad \text{(for } \ell \geq 1) \]
Thus, the integral of \( \|g\|_{\nu}^{-\sigma} \) has an upper bound
\[ \int_{G_\nu} \frac{dg}{\|g\|_{\nu}^\sigma} \leq 1 + \sum_{\ell \geq 1} q^{-\sigma \ell} \cdot q^{2 + (\ell - 1)} \leq 1 + q \cdot \sum_{\ell \geq 1} (q^{1 - \sigma})^\ell \]
For \( \sigma > 1 \), the geometric series converges. Thus,
\[ \int_{G_\nu} \frac{dg}{\|g\|_{\nu}^\sigma} \leq 1 + q \cdot \frac{q^{1 - \sigma}}{1 - q^{1 - \sigma}} < \frac{1 + q \cdot q^{1 - \sigma}}{1 - q^{1 - \sigma}} = \frac{1 + q^{2 - \sigma}}{1 - q^{1 - \sigma}} \]
Note that there is no leading constant.

The integrability condition on the adele group can be verified by showing the finiteness of the product of the corresponding local integrals. Since there are only finitely many archimedean places, it suffices to consider the product over finite places. By comparison to the zeta function of the number field \( k \), a product
\[ \prod_{\nu < \infty} \frac{1 + q^{-a}}{1 - q^{-b}} = \prod_{\nu < \infty} \frac{1 - q^{-2a}}{(1 - q^{-b})(1 - q^{-a})} = \frac{\zeta_k(a) \cdot \zeta_k(b)}{\zeta_k(2a)} \]
converges for \( a > 1 \) and \( b > 1 \). Thus, letting \( G_{\text{fin}} \) be the finite-prime part of the idele group \( G_\Lambda \),
\[ \int_{G_{\text{fin}}} \frac{dg}{\|g\|_{\nu}^\sigma} = \prod_{\nu < \infty} \int_{G_\nu} \frac{dg}{\|g\|_{\nu}^\sigma} < +\infty \]
for \( \sigma > 0 \) sufficiently large, from the previous estimate on the corresponding local integrals.
For integrability locally at archimedean places, exploit the left and right $K_{\nu}$-invariance, via Weyl’s integration formula. Let $A_{\nu}$ be the image under $\text{Ad}$ of the standard maximal split torus from $GL_2(k_{\nu})$, namely, real diagonal matrices. Let $\Phi^+ = \{\alpha\}$ be the singleton set of standard positive roots of $A_{\nu}$, namely

$$\alpha : \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \mapsto a_1/a_2$$

$g_\alpha$ be the $\alpha$-rootspace, and, for $a \in A_{\nu}$, let

$$D(a) = |\alpha(a) - \alpha^{-1}(a)|^{\dim g_\alpha}$$

The Weyl formula for a left and right $K_{\nu}$-invariant function $f$ on $G_{\nu}$ is

$$\int_{G_{\nu}} f(g) \, dg = \int_{A_{\nu}} D(a) \cdot f(a) \, da$$

For $PGL_2$, the dimension $\dim g_\alpha$ is 1 for $k_{\nu} \cong \mathbb{R}$ and is 2 for $k_{\nu} \cong \mathbb{C}$. The norm of a diagonal element is easily computed via the adjoint action on $g_\nu$, namely

$$\|a\|_\nu = \max\{|a_1/a_2|, |a_2/a_1|\}$$

with the usual absolute value on $\mathbb{R}$. Thus,

$$D(a) \ll \|a\|_\nu^{d_{\nu}} \quad (\text{with } d_{\nu} = [k_{\nu} : \mathbb{R}])$$

Thus, the integral over $PGL_2(k_{\nu})$ is dominated by a one-dimensional integral, namely,

$$\int_{G_{\nu}} \frac{dg}{\|g\|_\nu^2} = \int_{A_{\nu}} \frac{D(a)}{\|a\|_\nu^2} \, da \ll \int_{\mathbb{R}^2} (\max(|x|, |x|^{-1}))^{d_{\nu} - \sigma} \, dx \quad (\text{with } d_{\nu} = [k_{\nu} : \mathbb{R}])$$

The latter integral is evaluated in the fashion

$$\int_{\mathbb{R}^2} (\max(|x|, |x|^{-1}))^{-\beta} \, dx = \int_{|x| \leq 1} (|x|^{-1})^{-\beta} \, dx + \int_{|x| \geq 1} |x|^{-\beta} \, dx < +\infty$$

for $\nu$ either real or complex. This gives the desired local integrability for large $\sigma$ at archimedean places, and completes the proof of global integrability.

**Poincaré series for $GL_2$:** Recall the context of Sections 2 and 3. Let $G = GL_2(k)$ over a number field $k$, $Z$ the center of $GL_2$, and $K_{\nu}$ the standard maximal compact in $G_{\nu}$. Let

$$M = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\} \quad N = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$$

To form a Poincaré series, let $\varphi = \bigotimes_{\nu} \varphi_{\nu}$, where each $\varphi_{\nu}$ is right $K_{\nu}$-invariant, $Z_{\nu}$-invariant, and on $G_{\nu}$

$$\varphi_{\nu} \begin{pmatrix} a \\ 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 1 & 1 \end{pmatrix} = |a|_{\nu}^{\varphi} \cdot \Phi_{\nu}(x)$$
where at finite primes $\Phi_\nu$ is the characteristic function of the local integers $\mathfrak{o}_\nu$. At archimedean places, we assume that $\Phi_\nu$ is sufficiently continuously differentiable, and that these derivatives are absolutely integrable. The global function $\varphi$ is left $M_k$–invariant, by the product formula. Then, let

$$f(g) = \int_{N_\mathbb{A}} \overline{\psi(n)} \varphi(ng) \, dn$$

where $\psi$ is a standard non-trivial character on $N_k \backslash N_\mathbb{A} \approx k \backslash \mathbb{A}$. As in (4.6), but with slightly different notation, the Poincaré series of interest is

$$P^* (g) = \sum_{\gamma \in \mathbb{Z} \cdot N_k \backslash G_k} f(\gamma \cdot g)$$

**Convergence uniformly pointwise and in $L^2$:** From above, to show that this converges absolutely and uniformly on compacts, and also that it is in $L^2(Z_kG_k \backslash G_k)$, use a norm on the group $PGL_2 = GL_2 / \mathbb{Z}$, take $\Gamma = PGL_2(k)$, and show that $f$ is *sufficiently rapidly decreasing on $PGL_2(\mathbb{A})$ modulo $N_k$.*

To give the sufficient decay modulo $N_k$, it suffices to prove sufficient decay of $f(nm)$ for $n$ in a well-chosen set of representatives for $N_k \backslash N_\mathbb{A}$, and for $m$ in among representatives

$$m = \begin{pmatrix} a \\ 1 \end{pmatrix}$$

for $M_k / \mathbb{Z}$. For $m \in M_k$ and $n \in N_\mathbb{A}$, the submultiplicativity $\|nm\| \leq \|n\| \cdot \|m\|$ gives

$$\frac{1}{\|n\|^\sigma \cdot \|m\|^\sigma} \leq \frac{1}{\|nm\|^\sigma} \quad \text{(for $\sigma > 0$)}$$

That is, roughly put, it suffices to prove decay in $N_\mathbb{A}$ and $M_\mathbb{A}$ separately. Since $N_k \backslash N_\mathbb{A}$ has a set of representatives $E$ that is *compact*, on such a set of representatives the norm is *bounded*. Thus, it suffices to prove that

$$f(nm) \ll \frac{1}{\|m\|^\sigma} \quad \text{(for $n \in E$, and $m = \begin{pmatrix} a \\ 1 \end{pmatrix}$)}$$

Since $f$ factors over primes, as does $\|m\|$, it suffices to give suitable *local* estimates.

At finite $\nu$, the $\nu$th local factor of $f$ is left $\psi$–equivariant by $N_\nu$, and

$$f_\nu(nm) = \psi(n) \cdot \int_{N_\mathbb{A}} \overline{\psi(n')} \varphi(n'm) \, dn' = \psi(n) \cdot \int_{N_\mathbb{A}} \overline{\psi(n')} \varphi(m \cdot m^{-1} n'm) \, dn'$$

$$= \psi(n) \cdot |a|_\nu \cdot \int_{k_\nu} \overline{\psi_o(x)} |a|_\nu^\nu \cdot \Phi_\nu(x) \, dx$$

where

$$\psi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) = \overline{\psi_o(x)} \quad m = \begin{pmatrix} a \\ 1 \end{pmatrix}$$

Then,

$$|f_\nu(nm)| = |a|_\nu^{\nu+1} \cdot \int_{k_\nu} \overline{\psi_o(x)} \Phi_\nu(x) \, dx = |a|_\nu^{\nu+1} \cdot \tilde{\Phi}_\nu(a)$$
At every finite place \( \nu \), \( \Phi_\nu \) has compact support, and at almost every finite \( \nu \), \( \hat{\Phi}_\nu \) is simply the characteristic function of \( o_\nu \). Thus, almost everywhere,

\[
|f_\nu(nm)| \leq |a|_{\nu}^{v+1} \cdot \hat{\Phi}_\nu(a) \leq (\max\{|a|_{\nu}, |a|_{\nu}^{-1}\})^{-(v+1)} = \|m\|_{\nu}^{-(v+1)}
\]

At the finitely many finite places where \( \hat{\Phi} \) is not exactly the characteristic function of \( o_\nu \), the same argument still gives the weaker estimate

\[
|f_\nu(nm)| \leq |a|_{\nu}^{v+1} \cdot \hat{\Phi}_\nu(a) \ll (\max\{|a|_{\nu}, |a|_{\nu}^{-1}\})^{-(v+1)} = \|m\|_{\nu}^{-(v+1)}
\]

Thus, we have the finite-prime estimate

\[
\prod_{\nu<\infty} |f_\nu(nm)| \ll \prod_{\nu<\infty} \|m\|_{\nu}^{-(v+1)}
\]

At archimedean places, given \( \ell > 0 \), for \( \Phi_\nu \) sufficiently differentiable with absolutely integrable derivatives, ordinary Fourier transform theory implies that

\[
|\hat{\Phi}_\nu(a)| \ll (1 + |a|_{\nu})^{-\ell}
\]

Thus, from the general local calculation above,

\[
|f_\nu(nm)| = |a|_{\nu}^{v+1} \cdot \hat{\Phi}_\nu(a) \ll |a|_{\nu}^{v+1} \cdot (1 + |a|_{\nu})^{-\ell}
\]

This gives the sufficient decay of \( f \) at archimedean places.

In summary, since the local factors \( f_\nu \) of \( f \) have sufficient decay, the function \( f \) has sufficient decay so that the associated Poincaré series \( P_\nu^* = P_f \) converges uniformly on compacts, and is in \( L^2(Z_\ell GL_2(k) \backslash GL_2(\mathbb{A})) \). This proves Theorem 2.7.

§ Appendix 2. Mellin transform of Eisenstein Whittaker functions

The computation discussed in this appendix was needed in the proof of Proposition 4.10. While the details of this computation are given below, we also cite [W2], Chapter VII, for standard facts about the Tate-Iwasawa theory of zeta integrals.

The global Mellin transform of \( W^E \) factors

\[
\int_3 |a|^s W^E_{s, \chi} \left( \frac{a}{0} \frac{0}{1} \right) da = \prod_{\nu} \int_{k_\nu^2} |a|_{\nu}^s W^E_{s, \chi, \nu} \left( \frac{a}{0} \frac{0}{1} \right) da
\]

To compute this, we cannot simply change the order of integration, since this would produce a divergent integral along the way. Instead, we present the vectors \( \eta_\nu \) in a different form. Let \( \Phi_\nu \) be any Schwartz function on \( k_\nu^2 \) invariant under \( K_\nu \) (under the obvious right action of \( GL_2 \)), and put

\[
\eta_\nu'(g) = \chi_\nu(\det g) |\det g|_{\nu}^s \cdot \int_{k_\nu^2} \chi_\nu^2(t) |t|_{\nu}^{2s} \cdot \Phi_\nu(t \cdot e_2 \cdot g) dt
\]

where \( e_2 = e_2, \nu \) is the second basis element in \( k_\nu^2 \). This \( \eta_\nu' \) has the same left \( P_\nu \)-equivariance as \( \eta_\nu \), namely

\[
\eta_\nu' \left( \begin{array}{cc} a & * \\ 0 & d \end{array} \right) = |a/d|_{\nu}^s \cdot \chi_\nu(a/d) \cdot \eta_\nu'(g)
\]
For $\Phi_\nu$ invariant under the standard maximal compact $K_\nu$ of $GL_2(k_\nu)$, the function $\eta'_\nu$ is right $K_\nu$–invariant. By the Iwasawa decomposition, up to constant multiples, there is only one such function, so

$$\eta'_\nu(g) = \eta'_\nu(1) \cdot \eta_\nu(g) \quad \text{(since } \eta_\nu(1) = 1\text{)}$$

and

$$\eta'_\nu(1) = \int_{k_\nu^\times} \chi^2(t)|t|^{2s} \cdot \Phi(t \cdot e_2 \cdot 1) \, dt = \zeta_\nu(2s, \chi^2, \Phi(0, *)) \quad \text{(a Tate-Iwasawa zeta integral)}$$

Thus, it suffices to compute the local Mellin transform of

$$\eta'_\nu(1) \cdot W_\nu^E \psi(x, \nu)(m) = \int_{N_\nu} \overline{\psi}(n) \eta'_\nu(w_\nu nm) \, dn = \chi(a)|a|^s \int_{N_\nu} \overline{\psi}(n) \int_{k_\nu^\times} \chi^2(t)|t|^{2s} \Phi(t x, ta) \, dt \, dx \quad \text{(with } m = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix})$$

At finite primes $\nu$, we may as well take $\Phi$ to be

$$\Phi(t, x) = \operatorname{ch}_{\nu}(t) \cdot \operatorname{ch}_{\nu}(x) \quad \text{(ch}_\nu = \text{characteristic function of set } X)$$

Then $\eta'_\nu(1)$ is exactly an $L$–factor (see [W2], page 119, Proposition 10)

$$\eta'_\nu(1) = \zeta_\nu(2s, \chi^2, \operatorname{ch}_{\nu}) = L_\nu(2s, \chi^2)$$

and (for further details see [W2], page 107, Corollary 1, and page 108, Corollary 3),

$$\eta'_\nu(1) \cdot W_\nu^E \psi(x, \nu) \left( \begin{array}{c} a \\ 0 \\ 1 \end{array} \right) = \chi(a)|a|^s \int_{k_\nu^\times} \overline{\psi}(x) \operatorname{ch}_{\nu}(ta) \int_{k_\nu^\times} \chi^2(t)|t|^{2s} \operatorname{ch}_{\nu}(ta) \, dt \, dx$$

$$= \chi(a)|a|^s \operatorname{meas}(\sigma_\nu) \int_{k_\nu^\times} \operatorname{ch}_{\nu}(1/t)\chi^2(t)|t|^{2s-1} \operatorname{ch}_{\nu}(ta) \, dt$$

$$= |\sigma_\nu|^{1/2} \cdot \chi(a)|a|^s \int_{k_\nu^\times} \operatorname{ch}_{\nu}(1/t)\chi^2(t)|t|^{2s-1} \operatorname{ch}_{\nu}(ta) \, dt$$

where $\sigma_\nu \in k_\nu^\times$ is such that $(\sigma_\nu)^{-1} = \sigma_\nu \cdot a_\nu$. We can compute now the Mellin transform

$$\int_{k_\nu^\times} |a|^v \cdot \left( \chi(a)|a|^s \int_{k_\nu^\times} \operatorname{ch}_{\nu}(1/t)\chi^2(t)|t|^{2s-1} \operatorname{ch}_{\nu}(ta) \, dt \right) \, da$$

Replace $a$ by $a/t$, and then $t$ by $1/t$ to obtain a product of two zeta integrals

$$\left( \int_{k_\nu^\times} |a|^v \cdot \chi(a)|a|^s \operatorname{ch}_{\nu}(a) \, da \right) \cdot \left( \int_{k_\nu^\times} \operatorname{ch}_{\nu}(1/t)\chi(t)|t|^{s-1-v} \, dt \right)$$

$$= \zeta_\nu(v + s, \chi, \operatorname{ch}_{\nu}) \cdot \zeta_\nu(v + 1 - s, \overline{\chi}, \operatorname{ch}_{\nu}^{-1})$$

$$= L_\nu(v + s, \chi) \cdot L_\nu(v + 1 - s, \overline{\chi}) \cdot |\sigma_\nu|^{-(v+1-s)} \chi(\sigma_\nu)$$

---

4From now on, to avoid clutter, suppress the subscript $\nu$ where there is no risk of confusion. For instance, we shall write $| \cdot |, \psi, \chi, $ etc., rather than $| \cdot |_\nu, \psi_\nu, \chi_\nu, $ etc.
Thus, dividing through by $\eta'_\nu(1)$ and putting back the measure constant, the Mellin transform of $W_{E, \chi, \nu}$ is

$$\int_{k^*} |a|^\nu W_{E, \chi, \nu} \left( \begin{array}{ccc} a & 0 & 1 \\ 0 & 1 & 0 \end{array} \right) \, da = |\mathfrak{d}_\nu|^{1/2} \cdot \frac{L_\nu(v+s, \chi) \cdot L_\nu(v+1-s, \overline{\chi})}{L_\nu(2s, \chi^2)} \cdot |\mathfrak{d}_\nu|^{-(v+1-s)} \chi(\mathfrak{d}_\nu)$$

Let $\mathfrak{d}$ be the idele whose $\nu$th component is $\mathfrak{d}_\nu$ for finite $\nu$ and whose archimedean components are all 1. The product over all finite primes $\nu$ of these local factors is

$$\int_{\text{fin}} |a|^\nu W_{E, \chi} \left( \begin{array}{ccc} a & 0 & 1 \\ 0 & 1 & 0 \end{array} \right) \, da = |\mathfrak{d}|^{1/2} \cdot \frac{L(v+s, \chi) \cdot L(v+1-s, \overline{\chi})}{L(2s, \chi^2)} \cdot |\mathfrak{d}|^{-(v+1-s)} \chi(\mathfrak{d})$$

In our application, we will replace $s$ by $1-s$ and $\chi$ by $\overline{\chi}$, giving

$$\int_{\text{fin}} |a|^\nu W_{E, 1-s, \overline{\chi}} \left( \begin{array}{ccc} a & 0 & 1 \\ 0 & 1 & 0 \end{array} \right) \, da = |\mathfrak{d}|^{1/2} \cdot \frac{L(v+1-s, \overline{\chi}) \cdot L(v+s, \chi)}{L(2-2s, \chi^2)} \cdot |\mathfrak{d}|^{-(v+s)} \overline{\chi}(\mathfrak{d})$$

In particular, with $\chi$ trivial,

$$\int_{\text{fin}} |a|^\nu W_{E, 1-s} \left( \begin{array}{ccc} a & 0 & 1 \\ 0 & 1 & 0 \end{array} \right) \, da = |\mathfrak{d}|^{1/2} \cdot \frac{\zeta_k(v+1-s) \cdot \zeta_k(v+s)}{\zeta_k(2-2s)} \cdot |\mathfrak{d}|^{-(v+s)}$$

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