MEMORY EFFECTS IN MEASURE TRANSPORT EQUATIONS

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ABSTRACT. Transport equations with a nonlocal velocity field have been introduced as a continuum model for interacting particle systems arising in physics, chemistry and biology. Fractional time derivatives, given by convolution integrals of the time-derivative with power-law kernels, are typical for memory effects in complex systems. In this paper we consider a nonlinear transport equation with a fractional time-derivative. We provide a well-posedness theory for weak measure solutions of the problem and an integral formula which generalizes the classical push-forward representation formula to this setting.

1. Introduction. The measure-valued formulation of nonlinear transport equations has received an increasing interest both from a theoretical perspective [2, 3] and in various applications, as a continuum model for interacting particle systems (e.g. in crowd motion, population dynamics, bacterial chemotaxis, kinetic theory, social systems, etc., see [4, 5, 6] and reference therein).

Recently, anomalous transport problems describing processes deviating from the usual Gaussian behaviour have been observed in different fields (see [12]). In these phenomena, the standard diffusive behaviour is replaced by a subdiffusive one, in which the mean square displacement of the diffusing particles is of order $t^\beta$ with $\beta < 1$. Corresponding models lead to the study of differential equations where the time-derivative is replaced by a fractional one (see for example [1, 8, 13]). The stochastic dynamics driven by fractional differential equations is usually referred to as motion in a non-homogeneous medium. The particles can speed up or slow down according to a random clock which is the inverse to a stable subordinator (time-change of processes) and the dynamic is not Markovian, see [10]. Fractional derivatives are also considered as a typical approach to add a memory effect to a complex system. The memory term introduced by the convolution operator in time finds applications in demography, viscoelastic and biomaterials, biological processes and, in general, in the study of constitutive relations depending on the history of the state variables (see [9, 17]).
In this paper we consider a measure solution approach to the nonlinear transport equation

\[
\begin{cases}
\partial_{\beta(t)}^\beta \mu + \text{div}(v[\mu_t] \mu) = 0 & (x,t) \in \mathbb{R}^d \times (0,T), \\
\mu_{t=0} = \mu_0 & x \in \mathbb{R}^d,
\end{cases}
\]

(1)

where $\partial_{\beta(t)}^\beta \cdot$ is a nonlocal time-differential operator given by the Caputo fractional derivative of order $\beta \in (0,1)$

\[
\partial_{\beta(t)}^\beta \phi(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{d\phi(\tau)}{d\tau} \frac{1}{(t-\tau)^\beta} d\tau, \quad t \in (0,T).
\]

Since a fractional derivative at time $t$ depends on the values of the time-derivative in the interval $(0, t)$ with a progressively decreasing weight on the past history, the problem is nonlocal in time. The coefficient $v[\mu](x)$ is a nonlinear velocity field which depends on the solution itself, for example $v[\mu](x) = \int_{\mathbb{R}^d} K(x-y) d\mu(y)$. An appropriate choice of kernel $K$ allows to describe different phenomena of the physical model such as aggregation, repulsion and diffusion ([3, 4, 5]). The datum $\mu_0$ is a given positive measure on $\mathbb{R}^d$ representing the initial distribution of the population. Observe that measure solutions allow to describe within a unified approach the evolution both for continuous and discrete populations.

For $\beta = 1$, the fractional derivative $\partial_{\beta(t)}^\beta \cdot$ coincides with the standard derivative $\partial_t \cdot$. In this case the measure solution of problem (1) is given by the push-forward $\mu_t = \Phi_t^\beta \# \mu_0$ of the initial datum $\mu_0$ by means of the flow map $\Phi_t^\beta$ associated to the velocity field $v[\mu]$. In this paper, we extend the previous results to the fractional case, providing a well-posedness theory for measure solutions of the problem (1) and an integral formula which generalizes the classical push-forward representation formula to this setting. Indeed, we prove that for $\beta \in (0,1)$ a measure solution of (1) is given by the integral formula

\[
\mu_t(dx) = \int_0^\infty m_s^\mu(dx) h_{\beta}(s,t) ds = \int_0^\infty \Phi_s^\mu \# \mu_0(dx) h_{\beta}(s,t) ds,
\]

(2)

where $m^\mu$ is the solution of the linear transport equation

\[
\begin{cases}
\partial_s m_s + \text{div}(\tilde{v}^\mu(x,s) m_s) = 0 & (x,s) \in \mathbb{R}^d \times \mathbb{R}^+, \\
m_{s=0} = \mu_0 & x \in \mathbb{R}^d,
\end{cases}
\]

with the velocity field given by $\tilde{v}^\mu(x,s) = \int_0^\infty v[\mu_r] g_{\beta}(r,s) dr$ and $\Phi^\mu$ the associated flow. The kernels $g_{\beta}(s,t)$, $h_{\beta}(s,t)$ are the probability density functions (PDFs in short) associated to the time-increasing, continuous stochastic processes $D_t$, $E_t$, which are a $\beta$-stable subordinator and its inverse ([7, 12, 10]). Note that, as in the classical case, (2) defines implicitly the solution of (1) since the velocity field $\tilde{v}^\mu(x,s)$ depends on the solution $\mu$ of the problem.

The paper is organized as follows. In Section 2, we review definitions and some basic properties of the subordinator process, of its inverse and of the associated PDFs $g_{\beta}$, $h_{\beta}$. In Section 3, we study a linear transport equation with Caputo time-derivative. In Section 4 we prove the well-posedness of weak measure solutions to problem (1).
2. Preliminary definitions and properties. Throughout the paper, we always assume that \( \beta \in (0, 1) \). For \( f : \mathbb{R}^+ \rightarrow \mathbb{R} \), the Riemann-Liouville fractional integral of order \( \beta \) is defined by

\[
I_{[0,t]}^{\beta} f(t) := \frac{1}{\Gamma(\beta)} \int_{0}^{t} f(\tau) \frac{1}{(t-\tau)^{1-\beta}} d\tau, \quad t \in \mathbb{R}^+,
\]

and the Caputo fractional derivative of order \( \beta \) by

\[
D_{[0,t]}^{\beta} f(t) := \frac{1}{\Gamma(1-\beta)} \int_{0}^{t} \frac{d}{dt} f(\tau) \frac{1}{(t-\tau)^{\beta}} d\tau, \quad t \in \mathbb{R}^+,
\]

(for a complete account of the theory of fractional derivatives, we refer to [15]).

Fractional derivatives appear in the study of differential equations which govern the evolution of probability density functions for a class of Lévy processes, called stable subordinators ([7, 10, 11]). For \( \beta \in (0, 1) \), a \( \beta \)-stable subordinator is a one-dimensional, non-decreasing Lévy process \( D_t \) starting at 0 which is self-similar, i.e. \( \{D_t, t \geq 0\} \) has the same finite dimensional distribution as \( \{t^{1/\beta}D_{t^{1/\beta}}, t \geq 0\} \), and such that the Laplace transform of \( D_t \) is given by \( \mathbb{E}(e^{-sD_1}) = e^{-s^\beta} \) for \( s \geq 0 \). The inverse stable process \( \{E_t\}_{t \geq 0} \), defined as the first passage time of the process \( D_t \) over the level \( t \), i.e.

\[
E_t = \inf\{\tau > 0 : D_\tau > t\},
\]

has sample paths which are continuous, non-decreasing and such that \( E_0 = 0, E_t \to \infty \) as \( t \to \infty \). The PDF of the process \( E_t \) is given by \( h_\beta(s,t) = \frac{t}{\beta} s^{-1-\frac{k}{\beta}} G_\beta(s^{-\frac{k}{\beta}}t) \) where \( G_\beta(\cdot) \) is the PDF of \( D_1 \). The function \( h_\beta(\cdot,t) \) is infinitely differentiable in \( \mathbb{R}^+ \), right continuous in 0 with

\[
h_\beta(0^+, t) = \frac{1}{\Gamma(1-\beta)} t^{-\beta} \quad \text{for } t > 0,
\]

and has finite \( \gamma \)-moment for any \( \gamma > 0 \), given by (see [10, Corollary 3.1] and [14, equation (2.7)])

\[
\mathbb{E}[E_t^\gamma] = \int_{0}^{\infty} s^\gamma h_\beta(s,t) ds = C(\beta, \gamma) t^{\gamma \beta}.
\]

with \( C(\beta, \gamma) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\beta+1)} \). In particular, this property implies the identity

\[
\mathbb{E}(e^{\lambda E_t}) = \mathcal{E}_\beta(\lambda t^\beta), \quad \forall \lambda \in \mathbb{R},
\]

where \( \mathcal{E}_\beta(\cdot) = \sum_{k=0}^{+\infty} \frac{s^k}{\Gamma(k+1)} \) is the Mittag-Leffler function of order \( \beta \) (see [15] for some properties of \( \mathcal{E}_\beta \)). Indeed, we have

\[
\mathbb{E}(e^{\lambda E_t}) = \int_{0}^{+\infty} e^{\lambda s} h_\beta(s,t) ds = \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} \int_{0}^{+\infty} s^k h_\beta(s,t) ds
\]

\[
= \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} \frac{\Gamma(k+1)}{\Gamma(k+\beta+1)} t^{\beta k} = \sum_{k=0}^{+\infty} \left( \frac{\lambda t^\beta}{\Gamma(k+\beta+1)} \right)^k = \mathcal{E}_\beta(\lambda t^\beta).
\]

In the next proposition, we give two crucial properties of \( h_\beta \) we will exploit in the following (see [11] for (i) and [7, Lemma 3.2] for (ii)).

**Proposition 2.1.**  
(i) For \( t > 0 \), \( h_\beta \) is a weak solution of

\[
\partial_{[0,t]}^{\beta} h_\beta(r,t) = -\partial_r h_\beta(r,t) - \frac{t^{-\beta}}{\Gamma(1-\beta)} \delta_0(r), \quad r \in \mathbb{R}^+.
\]
(ii) Given $t > 0$, the density $h_β(·, t)$ is bounded and there exist positive constants $C$, $K$, non depending on $r$, such that

$$h_β(r, t) \leq Ce^{-K\frac{1}{r-t}}$$

for $r$ large enough.

Lemma 2.2. Given $L > 0$, $\mathbb{E}[E_t e^{LE_t}]$, where $E_t$ is the inverse stable subordinator, is finite for every $t \geq 0$.

Proof. Since $E_t \equiv 0$ for $t = 0$, we immediately get that $\mathbb{E}[E_0 e^{LE_0}] = 0$. For $t > 0$, we have by definition

$$\mathbb{E}[E_t e^{LE_t}] = \int_0^\infty re^{Lr} h_β(r, t) dr.$$ 

By (5), it follows that $re^{Lr} h_β(s, t) \sim re^{-\frac{1}{1-β}}$. Since $\frac{1}{1-β} > 1$, this implies that $re^{Lr} h_β(r, t) \to 0$ for $r \to +\infty$ and that $\mathbb{E}[E_t e^{LE_t}] < +\infty$. □

We introduce a space of measures with an appropriate norm where we consider solutions to the measure-valued transport equation (we refer to [2, 5] for a comprehensive account of the theory). We denote by $\mathcal{M}(\mathbb{R}^d)$ the space of finite Borel measures on $\mathbb{R}^d$ and by $\mathcal{M}^+(\mathbb{R}^d)$ the convex cone of the positive measures in $\mathcal{M}(\mathbb{R}^d)$. For $μ \in \mathcal{M}(\mathbb{R}^d)$ and a bounded measurable function $ϕ : \mathbb{R}^d \to \mathbb{R}$ we write

$$⟨μ, ϕ⟩ := \int_{\mathbb{R}^d} ϕ dμ.$$ 

Given a Borel measurable vector field $Φ : \mathbb{R}^d \to \mathbb{R}^d$, we denote by $Φ#μ \in \mathcal{M}(\mathbb{R}^d)$ the push-forward of the measure $μ$ under the action of $Φ$, defined by

$$(Φ#μ)(E) := μ(Φ^{-1}(E)), \quad ∀ E \in \mathcal{B}(\mathbb{R}^d),$$

where $\mathcal{B}(\mathbb{R}^d)$ is the class of Borel sets in $\mathbb{R}^d$. Observe that $⟨Φ#μ, ϕ⟩ = ⟨μ, ϕ • Φ⟩$. We denote by $BL(\mathbb{R}^d)$ the Banach space of bounded and Lipschitz continuous functions $ϕ : \mathbb{R}^d \to \mathbb{R}$ equipped with the norm

$$∥ϕ∥_{BL} = ∥ϕ∥_∞ + |ϕ|_L,$$

where $|ϕ|_L$ is the Lipschitz seminorm, and we introduce a norm in $\mathcal{M}(\mathbb{R}^d)$ by taking the dual norm of $∥∥_{BL}$:

$$∥μ∥^*_{BL} := \sup_{ϕ \in BL(\mathbb{R}^d)} \langle μ, ϕ \rangle.$$ 

Indeed, given the norm $∥·∥_{BL}$ on $BL(\mathbb{R}^d)$, the dual norm $∥·∥^*_{BL}$ is, trivially, defined on the dual space $BL(\mathbb{R}^d)^*$. Since the map $I_μ(ϕ) := ⟨μ, ϕ⟩$ defines a linear embedding of $\mathcal{M}(\mathbb{R}^d)$ into $BL(\mathbb{R}^d)^*$, we can induce a norm, denoted by the same symbol, on $\mathcal{M}(\mathbb{R}^d)$. The norm $∥·∥^*_{BL}$ is said to be the dual Bounded Lipschitz norm on $\mathcal{M}(\mathbb{R}^d)$.

The space $(\mathcal{M}(\mathbb{R}^d), ∥·∥^*_{BL})$ is in general not complete, hence it is customary to consider its completion $\overline{\mathcal{M}(\mathbb{R}^d)}$ with respect to the dual norm. However, the cone $\mathcal{M}^+(\mathbb{R}^d)$, which is a closed subset of the completion of $\mathcal{M}(\mathbb{R}^d)$ in the weak topology, is complete, although it is not a Banach space because it is not a vector space. Since we consider only positive measures, we restrict our attention to the complete metric space $(\mathcal{M}^+(\mathbb{R}^d), d_{BL})$ with the distance $d_{BL}$ induced by the norm $∥·∥^*_{BL}$. 
Remark 2.3. In [2, 4], solutions to transport equations are considered in the space of probability measures $P_1(\mathbb{R}^d)$ with the norm induced by the Wasserstein distance $W_1$. Note that, by the Kantorovich’s dual representation theorem, it follows that

$$\|\mu - \eta\|_{BL} \leq W_1(\mu, \eta), \quad \forall \mu, \eta \in P_1(\mathbb{R}^d).$$

Since we consider transport equations with a source term in Remark 4.5 and, in this case, conservation of the mass in general does not hold, we prefer to opt for the dual Bounded Lipschitz norm $\| \cdot \|_{BL}$ to have a unified framework.

The distance induced in $\mathcal{M}(\mathbb{R}^d)$ by the total variation norm:

$$\|\mu\|_{TV} := \sup_{\varphi \in C_c(\mathbb{R}^d)} (\mu, \varphi),$$

where $C_c(\mathbb{R}^d)$ is the space of bounded continuous functions on $\mathbb{R}^d$, is another metric frequently used for measures. However, as observed in [4], it may not be fully suited to transport problems where one wants to measure the distance between flowing mass distributions.

3. Linear transport equations with Caputo time derivative. In order to explain the construction of a solution to (1) in a simpler setting, in this section we consider the case of a linear transport equation with a Caputo time-derivative posed in measure theoretic sense. We start by introducing a notion of measure solution to (6).

Definition 3.1. A solution to (6) is a map $\mu \in C([0, T], \mathcal{M}(\mathbb{R}^d))$ such that $\mu_{t=0} = \mu_0$ and for any $f \in C_c(\mathbb{R}^d)$ and for almost any $t \in [0, T]$,

$$\partial_{(0, t)}^{\beta} f(x) \mu_t(dx) = \int_{\mathbb{R}^d} Df(x) \cdot v(x, t) \mu_t(dx).$$

Equivalently, since $I^{\beta}_{(0, t)} \partial_{(0, t)}^{\beta} \phi(t) = \phi(t) - \phi(0)$ for an absolutely continuous function $\phi$, $\mu_t$ is a solution to (6) if

$$\int_{\mathbb{R}^d} f(x) \mu_t(dx) = \int_{\mathbb{R}^d} f(x) \mu_0(dx) + I^{\beta}_{(0, t)} \left[ \int_{\mathbb{R}^d} Df(x) \cdot v(x, t) \mu_t(dx) \right].$$

We assume that $v : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ is a given vector field satisfying

(H1) $v$ is bounded by $V_0 > 0$, measurable in $t$ and there exists $L \in \mathbb{R}^+$ such that, for any $x_1, x_2 \in \mathbb{R}^d$ and $t \in \mathbb{R}^+$, it holds

$$|v(x_1, t) - v(x_2, t)| \leq L|x_1 - x_2|.$$

Associated to (6), we consider the linear problem with standard time-derivative

$$\left\{ \begin{array}{ll}
\partial_t m + \text{div}(\bar{v}(x, t) m) = 0 & (x, t) \in \mathbb{R}^d \times \mathbb{R}^+,
\quad & x \in \mathbb{R}^d,
m_{t=0} = \mu_0
\end{array} \right.$$  

(7)

where the vector field $\bar{v} : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ is defined by

$$\bar{v}(x, t) := \mathbb{E}[v(x, D_t)] = \int_0^\infty v(x, s) g_\beta(s, t) ds$$  

(8)
(the process $D_t$ is the $\beta$-stable subordinator defined in Section 2 and $g_\beta$ the corresponding PDF). The vector field $\tilde{v}$ is continuous in $t$ since
\[
g_\beta(s,t) = \frac{1}{t^{1/\beta}} G_\beta\left(\frac{s}{t^{1/\beta}}\right)
\]
and $G_\beta$ is infinitely differentiable in $\mathbb{R}^+$ (see [10]). For any $x_1, x_2 \in \mathbb{R}^d$ and $t \in \mathbb{R}^+$, we have
\[
|\tilde{v}(x_1,t) - \tilde{v}(x_2,t)| \leq \int_0^\infty |v(x_1,s) - v(x_2,s)|g_\beta(s,t)ds \leq L|x_1 - x_2|.
\]
Hence the corresponding flow
\[
\Phi_t(x,0) := x + \int_0^t \tilde{v}(\Phi_s(x,0), s) ds,
\]
giving the trajectory issuing from the point $x$ at time 0 and arriving at the point $\Phi_t(x,0)$ at time $t$, is well defined. It is well known that the unique measure solution to (7) is defined by the push-forward $m_t = \Phi_t#\mu_0$ of the initial datum $\mu_0$, i.e. $\langle m_t, f \rangle = \int_{\mathbb{R}^d} f(\Phi_t(x,0))\mu_0(dx)$ for any $f \in C^\infty_c(\mathbb{R}^d)$ (see for example [4, 16]). Because problem (7) is well posed in measure theoretic sense, we get the corresponding result for problem (6).

**Proposition 3.2.** Assume ($H_1$). For any $T > 0$, the Cauchy problem (6) admits a solution $\mu \in C([0,T], M^+(\mathbb{R}^d))$ given by $\mu_t(dx) = \mathbb{E}[m_{E_t}(dx)]$, i.e.
\[
\mu_t(dx) = \int_0^\infty m_s(dx)h_\beta(s,t)ds = \int_0^\infty \Phi_s#\mu_0(h_\beta(s,t)ds),
\]
where $m \in C(\mathbb{R}^+, M^+(\mathbb{R}^d))$ is the solution of the linear transport problem (7). Moreover, let $\mu^1, \mu^2$ be two solutions of (6) corresponding to the initial data $\mu^1_0, \mu^2_0$. Then, there is a constant $C = C(T)$ such that
\[
\sup_{[0,T]}d_{BL}(\mu^2_t, \mu^1_t) \leq C_{dBL}(\mu^2_0, \mu^1_0).
\]

**Proof.** By assumption ($H_1$), the flow $\Phi_s$ exists for any $s \in \mathbb{R}^+$ and the push-forward $m_s = \Phi_s#\mu_0$ is globally well defined. Therefore also formula (10) is well defined for any $t \in [0,T]$. Moreover $\mu_t$, for any $t \in \mathbb{R}^+$, is a finite measure on $\mathbb{R}^d$ since we have
\[
\mu_t(\mathbb{R}^d) = \int_{\mathbb{R}^d} \mu_t(dx) = \int_0^\infty \int_{\mathbb{R}^d} m_s(dx)h_\beta(r,t)dr
\]
\[
= \int_0^\infty \int_{\mathbb{R}^d} \mu_0(dx)h_\beta(r,t)dr = \mu_0(\mathbb{R}^d).
\]
We claim that (10) defines a weak solution to (6). Since $h_\beta$ satisfies (4), (5), we have
\[
\partial_{[0,t]}^\beta \left( \int_{\mathbb{R}^d} f(x) \mu_t(dx) \right) = \partial_{[0,t]}^\beta \left( \int_0^\infty \int_{\mathbb{R}^d} f(x) m_s(dx)h_\beta(r,t)dr \right)
\]
\[
= \int_0^\infty \left( \int_{\mathbb{R}^d} f(x) m_s(dx) \right) \partial_{[0,t]}^\beta h_\beta(r,t)dr
\]
\[
= - \int_0^\infty \left( \int_{\mathbb{R}^d} f(x) m_s(dx) \right) \partial_r h_\beta(r,t)dr
\]
\[
- \left( \int_{\mathbb{R}^d} f(x) m_s(dx) \right) \frac{t^{-\beta}}{\Gamma(1-\beta)} \delta_0(r)
\]
Recalling that $m$ and $\beta$ are continuous with respect to $x$, we have

$$\int_0^\infty \frac{d}{dr} \left( \int_{\mathbb{R}^d} f(x) m_r(dx) \right) h_\beta(r, t) dr - \left[ h_\beta(r, t) \int_{\mathbb{R}^d} f(x) m_r(dx) \right]_{r=0}^{r=\infty} - \frac{t^{-\beta}}{(1-\beta)} \int_{\mathbb{R}^d} f(x) m_0(dx)$$

$$= \int_0^\infty \frac{d}{dr} \left( \int_{\mathbb{R}^d} f(x) m_r(dx) \right) h_\beta(r, t) dr.$$  

Moreover, since $D_{E_t} = t$ for any $t$ and recalling that $h_\beta(\cdot, t)$, $g_\beta(\cdot, t)$ are the PDFs of $E_t$, respectively, we have

$$\int_{\mathbb{R}^d} Df(x) \cdot v(x, t) \mu_t(dx) = \mathbb{E} \left[ \int_{\mathbb{R}^d} Df(x) \cdot v(x, D_{E_t}) m_{E_t}(dx) \right]$$

$$= \mathbb{E} \left[ \int_0^\infty \int_{\mathbb{R}^d} Df(x) \cdot v(x, D_r) m_r(dx) h_\beta(r, t) dr \right]$$

$$= \int_0^\infty \int_{\mathbb{R}^d} Df(x) \cdot \mathbb{E} [v(x, D_r)] m_r(dx) h_\beta(r, t) dr$$

$$= \int_0^\infty \left[ \int_{\mathbb{R}^d} Df(x) \cdot \tilde{v}(x, r) \right] m_r(dx) h_\beta(r, t) dr.$$

Replacing the previous identities in Definition 3.1, we get that $\mu$ is a solution if

$$\int_0^\infty \left( \frac{d}{dr} \int_{\mathbb{R}^d} f(x) m_r(dx) - \int_{\mathbb{R}^d} Df(x) \cdot \tilde{v}(x, r) m_r(dx) \right) h_\beta(r, t) dr = 0.$$

Recalling that $m$ is a weak solution to (7), i.e. for any $f \in C_c^\infty(\mathbb{R}^d)$ and for almost any $t \in [0, T]$

$$\frac{d}{dr} \int_{\mathbb{R}^d} f(x)m_r(dx) = \int_{\mathbb{R}^d} Df(x) \cdot \tilde{v}(x, r) m_r(dx),$$

we get the claim. To prove that $\mu$ is continuous with respect to $t$, we estimate $d_{BL}(\mu_t, \mu_{t'}$) for $0 \leq t' < t$. For $f \in BL(\mathbb{R}^d)$ such that $\|f\|_{BL} \leq 1$ we have

$$\langle \mu_t - \mu_{t'}, f \rangle = \int_0^\infty (\langle m_r, f \rangle h_\beta(r, t) - \langle m_r, f \rangle h_\beta(r, t')) dr$$

$$= \int_0^\infty \langle m_r, f \rangle (h_\beta(r, t) - h_\beta(r, t')) dr. \quad (12)$$

By the estimate for $r, r' \in \mathbb{R}^+$,

$$\langle m_r - m_{r'}, f \rangle = \int_{\mathbb{R}^d} \left( f(\Phi_r(x, 0)) - f(\Phi_{r'}(x, 0)) \right) \mu_0(dx)$$

$$\leq \int_{\mathbb{R}^d} \left| \int_0^r \tilde{v}(\Phi_s(x, 0), s) - \int_0^{r'} \tilde{v}(\Phi_s(x, 0), s) \right| \mu_0(dx)$$

$$\leq V_0 \mu_0(\mathbb{R}^d) |r - r'|,$$

we get for $0 \leq t' < t$

$$\int_0^\infty \langle m_r, f \rangle (h_\beta(r, t) - h_\beta(r, t')) dr = \mathbb{E} \left[ \langle m_{E_t} - m_{E_{t'}}, f \rangle \right]$$

$$\leq V_0 \mu_0(\mathbb{R}^d) \mathbb{E} [E_t - E_{t'}] = V_0 \mu_0(\mathbb{R}^d) C(\beta, 1)(t^{\beta} - t'^{\beta})$$

$$\leq V_0 \mu_0(\mathbb{R}^d) C(\beta, 1) |t - t'|^{\beta}. \quad (13)$$
Replacing the previous estimates in (12), we obtain for the arbitrariness of \( f \)
\[
d BL (\mu, \mu') \leq C(\beta, 1)V_0\mu_0(\mathbb{R}^d)|t - t'|^\beta.
\]
We finally prove (11). Let \( f \in BL(\mathbb{R}^d) \) be such that \( \|f\|_{BL} \leq 1 \). By Gronwall’s inequality, the function \( f(\Phi_s(x,0))e^{-Ls} \) is 1-Lipschitz. Then
\[
\langle \mu_t^2 - \mu_1^1, f \rangle = \int_0^\infty \langle m_s^2 - m_s^1, f \rangle h_\beta(s,t)ds
\]
\[
= \int_0^\infty \int_{\mathbb{R}^d} f(\Phi_s(x,0))(\mu^2_0(dx) - \mu^1_0(dx))h_\beta(s,t)ds
\]
\[
\leq d_{BL}(\mu_0^2, \mu_0^1) \int_0^\infty e^{Ls}h_\beta(s,t)ds
\]
\[
= E_\beta(Lt^\beta)d_{BL}(\mu_0^2, \mu_0^1)
\]
where we have used (3) in the last equality.

**Remark 3.3.** If \( \mu_0 = \delta_{x_0} \), then the solution of (7) is given by \( \delta_{\Phi_t(x_0,0)} \), while the solution of (6) by \( \mu_t = \mathbb{E} \left[ \delta_{\Phi_t(x,0)} \right] \). We can interpret this formula in the following way: for the single particle the standard time \( t \) is replaced by an internal clock \( E_t \).
The sample of the process \( E_t \) can be constant on some interval, corresponding to trapping events in the motion, and assume arbitrarily large values, but with a probability decaying exponentially to 0 (see (5)). The solution of the transport equation (6) is obtained by averaging with respect to the internal clock \( E_t \).

**Remark 3.4.** Assume that the velocity \( v \) is a given vector field in \( L^1(\mathbb{R}^+; L^\infty(\mathbb{R}^d))^d \) satisfying the One-Sided Lipschitz (OSL) condition
\[
(v(x, t) - v(y, t)) \cdot (x - y) \leq \alpha(t)|x - y|
\]
for \( \alpha \in L^\infty(\mathbb{R}^+) \). Because of the weak regularity of \( v \), it is natural to consider the characteristic flow \( \Phi \) associated to \( v \) in the sense of Filippov. In [16] it is proved that the push-forward of the initial datum by means of the Filippov flow \( \Phi \) gives a unique measure solution to the corresponding transport problem with a standard time-derivative.

Since the velocity \( \tilde{v} \) defined in (8) satisfies the same assumptions as \( v \), including the OSL condition, the solution to problem (7) is well defined for any \( s \in \mathbb{R}^+ \). Therefore, also in this weaker setting, formula (10) gives the solution to (6).

### 4. Nonlinear transport equations with Caputo time derivative

In this section we consider the Cauchy problem for a nonlinear transport equation with Caputo time-derivative, i.e.
\[
\begin{aligned}
\partial^\beta_{(0,t)} & \mu_t + \text{div}(v[\mu_t]) \mu_t = 0 & (x,t) \in \mathbb{R}^d \times \mathbb{R}^+,
\mu_{t=0} & = \mu_0 & x \in \mathbb{R}^d.
\end{aligned}
\]

Also in this case, we consider solutions in the measure theoretic sense.

**Definition 4.1.** A solution to (14) is a map \( \mu \in C(\mathbb{R}^+, \mathcal{M}^+(\mathbb{R}^d)) \) such that \( \mu_{t=0} = \mu_0 \) and for any \( f \in C_c(\mathbb{R}^d) \) and for almost any \( t \in \mathbb{R}^+ \)
\[
\partial^\beta_{(0,t)} \int_{\mathbb{R}^d} f(x) \mu_t(dx) = \int_{\mathbb{R}^d} Df(x) \cdot v[\mu_t] \mu_t(dx).
\]

Along this section we assume that
(H2) $v$ is bounded by $V_0 > 0$ and Lipschitz continuous, i.e. there exists $L > 0$ such that for any $x_1, x_2 \in \mathbb{R}^d$, $\mu_1, \mu_2 \in \mathcal{M}^+(\mathbb{R}^d)$,

$$|v[\mu_1](x_1) - v[\mu_2](x_2)| \leq L(|x_1 - x_2| + d_{BL}(\mu_1, \mu_2)).$$

(H3) $\mu_0 \in \mathcal{M}^+(\mathbb{R}^d)$ and, for given constants $C_k$,

$$\int_{\mathbb{R}^d} |x|^k \mu_0(dx) \leq C_k, \quad \text{for } k = 1, 2.$$ 

The previous assumptions are standard in the framework of the nonlinear transport theory (see [4]). For fixed $\mu \in C(\mathbb{R}^+, \mathcal{M}^+(\mathbb{R}^d))$, we consider the linear problem

$$\begin{cases} 
\partial_t m + \text{div}(\tilde{v}^\mu(x,t)m) = 0 & (x,t) \in \mathbb{R}^d \times \mathbb{R}^+, \\
m_t = \mu_0 & x \in \mathbb{R}^d,
\end{cases}$$

(15)

where the linear velocity field $\tilde{v}^\mu : \mathbb{R}^d \times \mathbb{R}^+ \rightarrow \mathbb{R}^d$ is defined by

$$\tilde{v}^\mu(x,t) := E[v[\mu_{D_t}](x)] = \int_0^\infty v[\mu_s](x) g_\beta(s,t)ds.$$  

(16)

Observe that $\tilde{v}^\mu$ is bounded by $V_0$. Moreover, we have

$$|\tilde{v}^\mu(x_1,t) - \tilde{v}^\mu(x_2,t)| \leq \int_0^\infty |v[\mu_s](x_1) - v[\mu_s](x_2)| g_\beta(s,t)ds$$

$$\leq L|x_1 - x_2| \int_0^\infty g_\beta(s,t)ds = L|x_1 - x_2|,$$

and therefore $\tilde{v}^\mu$ is Lipschitz continuous in $x$. Moreover, by (9) and since the function $G_\beta$ is infinitely differentiable in $\mathbb{R}^+$, $\tilde{v}^\mu$ is continuous in $t$. Hence, for fixed $\mu$, the flow

$$\Phi^\mu_t(x,0) := x + \int_0^t \tilde{v}^\mu(\Phi^\mu_s(x,0),s) \, ds,$$  

(17)

is well defined and the measure solution to the linear problem (15) is given by $m^\mu_t = \Phi^\mu_t \# \mu_0$.

In the next theorem, we prove the well-posedness of problem (14), showing existence and uniqueness of the solution in weak measure sense, and we provide an integral formula which generalizes the classical push-forward representation formula to this setting.

**Theorem 4.2.** Assume (H2)-(H3). Then, the Cauchy problem (14) admits a unique solution $\mu \in C(\mathbb{R}^+, \mathcal{M}^+(\mathbb{R}^d))$, satisfying the formula $\mu_t(dx) = E[m^\mu_{E_t}]$ or, equivalently, by

$$\mu_t(dx) = \int_0^\infty m^\mu_s(dx) h_\beta(s,t)ds = \int_0^\infty \Phi^\mu_s \# \mu_0(dx) h_\beta(s,t)ds,$$  

(18)

where $m^\mu \in C(\mathbb{R}^+, \mathcal{M}^+(\mathbb{R}^d))$ is the solution of the linear transport problem (15).

In the next lemma, we prove existence of a solution to (14) when the initial datum $\mu_0$ has compact support and the velocity is null outside a given ball.

**Lemma 4.3.** Besides (H2)-(H3), assume that $\text{supp} \{\mu_0\} \subset B(0, R)$ and $v[\mu](x) \equiv 0$ for all $\mu \in \mathcal{M}^+(\mathbb{R}^d)$, $x \in \mathbb{R}^d \setminus B(0, 2R)$, for some positive constant $R$. Then the Cauchy problem (14) admits a solution $\mu \in C(\mathbb{R}^+, \mathcal{M}^+(\mathbb{R}^d))$ implicitly defined by the integral formula (18).
Proof. Let \( \{T_n\}_{n \in \mathbb{N}} \) be an increasing sequence such that \( \lim_{n \to \infty} T_n = +\infty \) and, for any \( n \in \mathbb{N} \), define the map \( \Psi_n : C(\mathbb{R}^+, \mathcal{M}^+(\mathbb{R}^d)) \to C(\mathbb{R}^+, \mathcal{M}^+(\mathbb{R}^d)) \) which associates to \( \nu \in C(\mathbb{R}^+, \mathcal{M}^+(\mathbb{R}^d)) \) a measure-valued curve \( \mu \in C(\mathbb{R}^+, \mathcal{M}^+(\mathbb{R}^d)) \), defined for every \( t \in \mathbb{R}^+ \) as

\[
\mu_t(dx) = \int_0^{T_n} m_r^\nu(dx) h_\beta(r,t) dr,
\]

where \( m^\nu \) is the solution of the linear transport equation

\[
\begin{align*}
\partial_t m_r + \text{div}(\tilde{v}^\nu(x,r)m_r) &= 0, \quad (x,r) \in \mathbb{R}^d \times \mathbb{R}^+, \\
m_{r=0} &= \mu_0, \quad x \in \mathbb{R}^d,
\end{align*}
\]

and \( \tilde{v}^\nu \) is defined as in (16) with \( \nu \) in place of \( \mu \) and \( \Phi^\nu \) is the associated flow.

Let us check that, for any \( n \in \mathbb{N} \), \( \Psi_n \) is well-defined and continuous from the set

\[
C := \left\{ \mu \in C(\mathbb{R}^+, \mathcal{M}^+(\mathbb{R}^d)) : \sup_{t \neq t'} \frac{d_{BL}(\mu_t, \mu_{t'})}{|t-t'|^\beta} \leq R_0, \sup_{t \in \mathbb{R}^+} \int_{\mathbb{R}^d} |x|^k \mu_t(dx) \leq R_0, \quad k = 1, 2 \right\}
\]

into itself, for an appropriate constant \( R_0 \) to be chosen later. Note that the set \( C \) is non-empty, since it contains any constant in time measure-valued map. Moreover, it is convex since a convex combination of maps in \( C \) preserves the Hölder condition and boundedness of the moments. Lastly, the compactness is a consequence of Ascoli-Arzela’s Theorem in space of measures (see [2, Section 3.3 and Lemma 7.1.5] and [4, pag.148]). Indeed, given a sequence \( \{\mu^n\}_{n \in \mathbb{N}} \subset C \), then for a given \( T > 0 \), by Ascoli-Arzela’s Theorem we can find a converging subsequence in \([0, T]\). Iterating the same argument in \([kT, (k+1)T]\), \( k \in \mathbb{N} \), we can find a subsequence \( \mu^{n_k} \) converging to \( \mu \in C \).

Given \( \nu \in C \), we set \( \mu = \Psi_n(\nu) \). Since \( m_r^\nu = \Phi^\nu_r \# \mu_0(dx) \), we have

\[
\mu_t(\mathbb{R}^d) = \int_{\mathbb{R}^d} \mu_t(dx) = \int_0^{T_n} \int_{\mathbb{R}^d} m_r^\nu(dx) h_\beta(r,t) dr \\
\leq \int_0^{\infty} \int_{\mathbb{R}^d} \mu_0(dx) h_\beta(r,t) dr = \mu_0(\mathbb{R}^d).
\]

To estimate the first and second order moments of \( \mu_t \), observe that since \( \nu(\nu)(x) \equiv 0 \) in \( \mathbb{R}^d \setminus B(0, 2R) \), then \( \tilde{v}^\nu(x,s) \equiv 0 \) for all \( (x,s) \in (\mathbb{R}^d \setminus B(0, 2R)) \times \mathbb{R}^+ \). Therefore, if \( x \in B(0, R) \), it follows that \( \Phi^\nu_r(x, 0) \in B(0, 2R) \) for all \( r \in \mathbb{R}^+ \). Hence, for \( k = 1, 2 \), we get

\[
\int_{\mathbb{R}^d} |x|^k \mu_t(dx) = \int_0^{T_n} \int_{\mathbb{R}^d} |x|^k m_r^\nu(dx) h_\beta(r,t) dr \\
\leq \int_0^{\infty} \int_{\mathbb{R}^d} |\Phi^\nu_r(x,0)|^k \mu_0(dx) h_\beta(r,t) dr \leq (2R)^k \mu_0(\mathbb{R}^d) \quad (19)
\]

We estimate \( d_{BL}(\mu_t, \mu_{t'}) \) for \( 0 \leq t' < t \). Arguing as in the estimates (12)-(13), for \( f \in \text{BL}(\mathbb{R}^d) \) such that \( \|f\|_{BL} \leq 1 \) we have

\[
\langle \mu_t - \mu_{t'}, f \rangle = \int_0^{T_n} \langle m_r^\nu, f \rangle h_\beta(r,t) - \langle m_r^\nu, f \rangle h_\beta(r,t') \rangle dr \\
= \int_0^{T_n} \langle m_r^\nu, f \rangle (h_\beta(r,t) - h_\beta(r,t')) dr
\]

(20)
\[ \Delta \leq V_0 \mu_0(\mathbb{R}^d) C(\beta, 1)|t - t'|^\beta. \]

By (19) and (20) (which are uniform in \( n \)), it follows that the map \( \Psi_n \) is well defined. Moreover if \( R_0 \) in the definition of \( C \) is greater than the constants appearing in the estimates (19) and (20), then \( \Psi_n \) maps \( C \) into itself.

We now prove that \( \Psi_n \) is continuous. We first estimate the distance between the flows corresponding to different measures. Given \( \mu, \nu \in C(\mathbb{R}^+, \mathcal{M}^+(\mathbb{R}^d)) \), by (H2) we have

\[
|\tilde{v}^\mu(x, s) - \tilde{v}^\nu(x, s)| = \left| \int_0^\infty (v[\mu_r] - v[\nu_r]) g_\beta(r, s) dr \right| = |\mathbb{E}[v[\mu_D, s](x) - v[\nu_D, s](x)]| \leq L \mathbb{E}[d_{BL}(\mu, \nu)].
\]

Hence,

\[
|\Phi^\mu_s(x, 0) - \Phi^\nu_s(x, 0)| = \int_0^s |\tilde{v}^\mu(\Phi^\mu_r(x, 0), r) - \tilde{v}^\nu(\Phi^\nu_r(x, 0), r)| dr.
\]

Therefore, by Gronwall’s inequality, we have the estimate

\[
|\Phi^\mu_s(x, 0) - \Phi^\nu_s(x, 0)| \leq L \mathbb{E}[d_{BL}(\mu, \nu)] s \leq L \mathbb{E}[d_{BL}(\mu, \nu)].
\]

Let \( \{\nu_k\}_{k \in \mathbb{N}} \subset C \) be a sequence converging to \( \nu \in C \) and set \( \mu_k = \Psi(\nu_k), \mu = \Psi(\nu) \). Denoted with \( \Phi^\nu, \Phi^\nu \) the corresponding flows, we have for \( f \in BL(\mathbb{R}^d) \)

\[
\langle \mu_k - \mu, f \rangle = \int_0^T \int_{\mathbb{R}^d} (m^\nu_k - m^\nu_s, f) h_\beta(s, t) ds
\]

\[
= \int_0^T \int_{\mathbb{R}^d} \left( f(\Phi^\nu_s(x, 0), s) - f(\Phi^\nu_s(x, 0)) \right) \mu_0(dx) h_\beta(s, t) ds
\]

\[
\leq \mu_0(\mathbb{R}^d) \int_0^T \mathbb{E} \left[ \sup_{r \in [0, T_n]} (d_{BL}(\nu_k, \nu_r)) \right] L \mathbb{E}[h_\beta(s, t) ds]
\]

\[
\leq \mu_0(\mathbb{R}^d) \sup_{r \in [0, T_n]} (d_{BL}(\nu_k, \nu_r)) \mathbb{E}[h_\beta(s, t) ds]
\]

\[
\leq \mu_0(\mathbb{R}^d) T_n L \mathbb{E}[h_\beta(s, t) ds]
\]

Then, the convergence of \( \mu^k \) to \( \mu \), i.e. the continuity of the map \( \Psi_n \).

Applying the Schauder’s Fixed Point Theorem to \( \Psi_n \), we have that, for every \( n \in \mathbb{N} \), there exists \( \mu^n \in C \) such that \( \mu^n = \Psi_n(\mu^n) \), i.e.

\[
\mu^n_t(dx) = \int_0^T m^{\mu^n}_s(dx) h_\beta(s, t) ds, \quad \forall t \in \mathbb{R}^+. \tag{22}
\]

Due to the the uniform in time estimates on the first and second moments and the uniform Hölder estimate (20), by a diagonal argument, we have the existence of a
subsequence \( \{ \mu_{n_k} \}_{k \in \mathbb{N}} \) and \( \mu \in C \) such that \( \mu_{n_k} \rightarrow \mu \) for \( t \in \mathbb{R}^+ \). We prove that for any \( t \in \mathbb{R}^+ \), \( \mu \) satisfies the integral formula (18). Denote with \( \{ \mu^n \}_{n \in \mathbb{N}} \) the (sub-)sequence converging to \( \mu \), for any \( f \in BL(\mathbb{R}^d) \) with \( \| f \|_{BL} \leq 1 \) we have

\[
\int_0^{+\infty} \langle m_{n_k}^n, f \rangle_h(s, t) ds - \int_0^T \langle m_{n_k}^n, f \rangle_h(s, t) ds = \int_{T_n}^{+\infty} \langle m_{n_k}^n, f \rangle_h(s, t) ds + \int_0^T \langle m_{n_k}^n - m_{n_k}^n, f \rangle_h(s, t) ds.
\]

The first term on the right hand side converges to 0 for \( n \rightarrow +\infty \), since

\[
\int_{T_n}^{+\infty} \langle m_{n_k}^n, f \rangle_h(s, t) ds \leq \mu_0(\mathbb{R}^d) \int_{T_n}^{+\infty} \beta_h(s, t) ds \rightarrow 0.
\]

For the second term, by (21) we have

\[
\int_0^T \langle m_{n_k}^n - m_{k}^n, f \rangle_h(s, t) ds \leq \mu_0(\mathbb{R}^d) \int_{T_n}^{+\infty} \beta_h(s, t) ds \rightarrow 0.
\]

Note that by Lemma 2.2, \( \mathbb{E}[E_t e^{tE_T}] < \infty \) for any \( t \geq 0 \). Hence, for \( n \rightarrow +\infty \) and \( t \in \mathbb{R}^+ \), we have

\[
\int_0^T m_{n_k}^n(dx) \beta_h(s, t) ds \rightarrow \int_0^{+\infty} m_{n_k}^n(dx) \beta_h(s, t) ds
\]

and passing to the limit for \( n \rightarrow \infty \) in (22), we get (18). As in Proposition 3.2, it is possible to show that formula (18) defines a weak solution of the problem. Hence the existence of a solution to (14) follows.

**Proof of Theorem 4.2.** Given \( R > \frac{1}{2} \), we consider a sequence of initial data given by \( \mu_0^R = \chi_{B(0,R)}(x) \cdot \mu_0 \), where \( \chi_{B(0,R)} \) is the characteristic function of the set \( B(0,R) \), and a sequence of velocity fields \( v^R[\mu](x) = v[\mu](x) \cdot \sigma_R(x) \) where \( \sigma_R : \mathbb{R}^d \rightarrow \mathbb{R} \) is a smooth, non negative function such that \( \sigma_R(x) = 1 \) for \( x \in B(0,2R-1) \), \( \sigma_R(x) = 0 \) for \( x \in \mathbb{R}^d \setminus B(0,2R) \) and \( |D\sigma_R| \leq 1 \). By Lemma 4.3, for any \( R > 1 \), there exists a solution \( \mu^R \) to the Cauchy problem (14) given by the formula

\[
\mu_t^R(dx) = \int_0^\infty m_s^R(dx) \beta_h(s, t) ds = \int_0^\infty \Phi_s^R \# \mu_0^R(dx) \beta_h(s, t) ds
\]

(23)

where \( m^R \) is the solution of (15) corresponding to the velocity

\[
v^R[\mu_t^R](x)g_\beta(r, s) dr
\]

and \( \Phi^R \) the associated flow.

For any \( T > 0 \), we consider the restriction of the sequence \( \{ \mu^R \} \) to the interval \([0, T]\). We estimate the first and second order moment of \( \mu^R_t \) for \( t \in [0, T] \), uniformly
with respect to \( R \) and associated to the measure \( \mu \) to 0 for any \( t \). The previous estimates imply that the sequence \( \mu^t = \mu_0(\mathbb{R}^d) \) for any \( t \in [0, T] \). Moreover estimate (20), which is independent of \( R \), implies that the sequence \( \mu^t \) is also equi-continuous with respect to \( t \). By a diagonal argument, we can extend the measure \( \mu \in C(\mathbb{R}^+, \mathcal{M}^+(\mathbb{R}^d)) \) to be the solution of the Caucho problem (15) associated to the measure \( \mu \), i.e. with velocity field \( \tilde{v}(x, s) = \int_0^\infty v[\nu_\beta](x) g_{\beta}(r, s) dr \) and initial datum \( \mu_0 \). Then, for \( f \in \text{BL}(\mathbb{R}^d) \) such that \( \|f\|_{\text{BL}} \leq 1 \), we have

\[
(m_s^R - m_s^\nu, f) = \int_{\mathbb{R}^d} f(\Phi_s^R(x, 0)) \mu_0^R(dx) - \int_{\mathbb{R}^d} f(\Phi_s^\nu(x, 0)) \mu_0(dx)
\]

\[
= \int_{\mathbb{R}^d} \left( f(\Phi_s^R(x, 0)) - f(\Phi_s^\nu(x, 0)) \right) \mu_0^R(dx)
\]

\[
+ \int_{\mathbb{R}^d} f(\Phi_s^\nu(x, 0))(\mu_0^R(dx) - \mu_0(dx))
\]

\[
\leq \int_{\mathbb{R}^d} \left| \Phi_s^R(x, 0) - \Phi_s^\nu(x, 0) \right| \mu_0^R(dx) + \mu_0(\mathbb{R}^d \setminus B(0, R)).
\]

To estimate the first term on the right hand side, we preliminary observe by (H2)

\[
|\tilde{v}^R(x, s) - \tilde{v}(x, s)| = \left\| \int_0^\infty (v[\nu_\beta](x) - v[\nu_\beta^R](x) \sigma^R(x)) g_{\beta}(r, s) dr \right\|
\]

\[
\leq \int_0^\infty |v[\nu_\beta](x) - v[\nu_\beta^R](x)| g_{\beta}(r, s) dr + \int_0^\infty |v[\nu_\beta^R](x)(1 - \sigma^R(x))| g_{\beta}(r, s) dr
\]

\[
\leq L \|d_{BL}(\mu_{D_s}, \mu_{D_s}^R)\| + V_0 \chi_{\mathbb{R}^d \setminus B(0, 2R - 1)}(x).
\]
Observe that if \( m_{s}^{R}(x,0) = \int_{0}^{s} |\tilde{v}(\Phi_{s}^{R}(x,0),r) - \tilde{v}(\Phi_{s}^{R}(x,0),r)|dr \)
\( \leq \int_{0}^{s} |\tilde{v}(\Phi_{s}^{R}(x,0),r) - \tilde{v}(\Phi_{s}^{R}(x,0),r)|dr \)
\( + \int_{0}^{s} |\tilde{v}(\Phi_{s}^{R}(x,0),r) - \tilde{v}(\Phi_{s}^{R}(x,0),r)|dr \)
\( \leq L \int_{0}^{s} |\Phi_{s}^{R}(x,0) - \Phi_{s}^{R}(x,0)|dr + \int_{0}^{s} \left( \int_{0}^{0} LE[\mu_{D_{s}}, \mu_{D_{s}}^{R}] \right) dr \)
and therefore, by Gronwall’s inequality, we get
\[
|\Phi_{s}^{R}(x,0) - \Phi_{s}^{R}(x,0)| \leq e^{Ls} \int_{0}^{0} \left( \int_{0}^{0} LE[\mu_{D_{s}}, \mu_{D_{s}}^{R}] \right) dr + V_{0}\chi_{R \setminus B(0,2R-1)}(\Phi_{s}^{R}(x,0))dr.
\]
Replacing (25) in (24), we get
\[
\langle m_{s}^{R} - m_{s}^{\mu}, f \rangle \leq e^{Ls} \int_{0}^{0} \left( \int_{0}^{0} LE[\mu_{D_{s}}, \mu_{D_{s}}^{R}] \right) dr + V_{0}\chi_{R \setminus B(0,2R-1)}(\Phi_{s}^{R}(x,0))\mu_{0}(dx)dr + \mu_{0}(\mathbb{R}^{d} \setminus B(0,R)).
\]
and therefore, as in (21), we get
\[
\int_{0}^{0} \langle m_{s}^{R} - m_{s}^{\mu}, f \rangle h_{\beta}(s,t)ds \leq \int_{0}^{0} \sup_{r \in [0,s]} E[\mu_{D_{s}}, \mu_{D_{s}}^{R}] Lse^{Ls} h_{\beta}(s,t)ds
\]
\( + \int_{0}^{0} e^{Ls} \int_{0}^{0} V_{0}\chi_{R \setminus B(0,2R-1)}(\Phi_{s}^{R}(x,0))\mu_{0}(dx)h_{\beta}(s,t)drds
\]
\( + \mu_{0}(\mathbb{R}^{d} \setminus B(0,R)) \leq \sup_{r \in [0,t]} d_{BL}(\mu_{r}, \mu_{r}^{R}) E[LE_{t}e^{LE_{t}}]
\]
\( + \left[ e^{LE_{t}} \int_{0}^{E_{t}} \int_{0}^{E_{t}} V_{0}\chi_{R \setminus B(0,2R-1)}(\Phi_{s}^{R}(x,0))\mu_{0}(dx)dr \right] + \mu_{0}(\mathbb{R}^{d} \setminus B(0,R)).
\]
Observe that if \( x \in B(0,R) \), then
\[
E[\sup_{[0,E_{t}]} |\Phi_{s}^{R}(x,0)|] \leq |x| + V_{0}C(\beta,1)t^{\beta}.
\]
Since inequality (26) holds for any \( f \) and all the terms on the right hand side tend to 0 for \( R \to \infty \), we get
\[
\lim_{R \to \infty} \int_{0}^{0} m_{s}^{R}(dx)h_{\beta}(s,t)ds = \int_{0}^{0} m_{s}^{\mu}(dx)h_{\beta}(s,t)ds.
\]
Passing to the limit for \( R \to \infty \) in (23), we get that \( \mu \) satisfies (18) and therefore it is a weak solution to (14).

We finally prove the uniqueness of the solution to (14). Let \( \mu_{1}^{1}, \mu_{2}^{1} \) be two solutions to (14) with initial conditions \( \mu_{0}^{1}, \mu_{0}^{2} \in M^{+}(\mathbb{R}^{d}) \). For \( f \in BL(\mathbb{R}^{d}) \) with \( \|f\|_{BL} \leq 1 \),
we have
\[
\langle \mu_1 - \mu_2^2, f \rangle = \int_0^{+\infty} \langle m_1^2 - m_2^2, f \rangle h_\beta(s, t)ds
\]
\[
= \int_0^{+\infty} \left( \langle \mu_0, f(\Phi_s^1) - f(\Phi_s^2) \rangle + \langle \mu_0^2, f(\Phi_s^2) \rangle \right) h_\beta(s, t)ds
\]
\[
\leq \int_0^{+\infty} \left( \langle \mu_0, \Phi_s^1 - \Phi_s^2 \rangle + \langle \mu_0^2, f(\Phi_s^2) \rangle \right) h_\beta(s, t)ds.
\]
By the first inequality in (21), i.e.
\[
|\Phi_s^r(x, 0) - \Phi_s^r(x, 0)| \leq e^{Ls} \int_0^s \mathbb{E}[d_{BL}(\mu_{D_r}, \nu_{D_r})]dr,
\]
and since \( f(\Phi_s^2(x, 0))e^{-Ls} \) is 1-Lipschitz, it follows that
\[
\langle \mu_1^2 - \mu_2^2, f \rangle \leq \mu_0^1(\mathbb{R}^d) \int_0^{+\infty} e^{Ls} \left( \int_0^s \Lambda(r)dr \right) h_\beta(s, t)ds
\]
\[
+ d_{BL}(\mu_0^1, \mu_0^2) \int_0^{+\infty} e^{Ls} h_\beta(s, t)ds
\]
\[
= \mu_0^1(\mathbb{R}^d) \int_0^{+\infty} e^{Ls} \left( \int_0^s \Lambda(r)dr \right) h_\beta(s, t)ds + d_{BL}(\mu_0^1, \mu_0^2)\mathbb{E}[e^{LE_1}],
\]
where \( \Lambda(r) = \mathbb{E}[d_{BL}(\mu_{D_r}, \mu_{D_r})] \). For the arbitrariness of \( f \) and recalling that \( h_\beta(\cdot, t) \)
the solution to (14).

\[d_{BL}(\mu_1^1, \mu_2^2) \leq \mu_0^1(\mathbb{R}^d)\mathbb{E}\left[ e^{LE_1} \int_0^{E_1} \Lambda(r)dr \right] + d_{BL}(\mu_0^1, \mu_0^2)\mathbb{E}[e^{LE_1}]. \tag{27}\]

Observe that, by the conservation of the initial mass, we have \( \Lambda(r) \leq \mathbb{E}[\|\mu_1^1\|_{BL} + \|\mu_2^2\|_{BL}] = \|\mu_0^1\|_{BL} + \|\mu_0^2\|_{BL} \) and therefore \( \Lambda(r) \) is finite for any \( r \geq 0 \). Moreover, replacing \( t = D_r \) in (27) and taking the expectation, we get
\[
\Lambda(r) \leq \mu_0^1(\mathbb{R}^d)e^{LE_{D_r}} \left[ e^{LE_{D_r}} \int_0^{E_{D_r}} \Lambda(z)dz \right] + d_{BL}(\mu_0^1, \mu_0^2)\mathbb{E}\left[ e^{LE_{D_r}} \right]
\]
\[
= \mu_0^1(\mathbb{R}^d)e^{Lr} \int_0^{E_{D_r}} \Lambda(z)dz + d_{BL}(\mu_0^1, \mu_0^2)e^{Lr},
\]
since \( E_t \) is the inverse process of \( D_t \). Then, by Gronwall’s inequality, we have
\[
\Lambda(r) \leq e^{Lr}d_{BL}(\mu_0^1, \mu_0^2)e^{\mu_0^1(\mathbb{R}^d)(e^{Lr} - 1)}.
\]
Hence, if we put \( \mu_0^1 = \mu_0^2 \) in the previous inequality, it follows that \( \Lambda(r) = 0 \), i.e.
\[
\int_0^{+\infty} d_{BL}(\mu_0^1, \mu_0^2)g_\beta(s, r)ds = 0.
\]
Since \( g_\beta(s, r) > 0 \) for \( (s, r) \in \mathbb{R}^+ \times \mathbb{R}^+ \) and \( \mu^i \in C(\mathbb{R}^+, \mathcal{M}^+(\mathbb{R}^d)) \), \( i = 1, 2 \), it follows that \( d_{BL}(\mu_0^1, \mu_0^2) = 0 \) for \( s \in \mathbb{R}^+ \), i.e. the uniqueness in measure theoretic sense of the solution to (14).

\[\square\]

**Remark 4.4.** Measure solutions to nonlinear transport equations with standard time derivative are studied in [6] under more general assumptions than the ones considered in this paper. In particular, in assumption (H2), the constant \( L \) is
replaced by function $L(t) \in L^1(\mathbb{R}^+)$. We aim to investigate time-fractional transport equations and the corresponding applications under weaker assumptions in the future.

**Remark 4.5.** Consider a nonlinear transport equation with source term
\[
\begin{aligned}
\partial_t^\beta \mu + \text{div}(\nu[\mu]) \mu &= \gamma_t \quad (x, t) \in \mathbb{R}^d \times (0, T), \\
\mu_{t=0} &= \mu_0 \quad x \in \mathbb{R}^d,
\end{aligned}
\tag{28}
\]
where $\gamma \in C([0, T], M^+(\mathbb{R}^d))$ with bounded first and second order moments. Then the solution of (28) is given by
\[
\mu_t(dx) = E[m_{E_t}(dx)] = \int_0^\infty m^\mu_s(dx) h_\beta(s, t) ds
\]
where $m^\mu \in C(\mathbb{R}^+, M^+(\mathbb{R}^d))$ is the solution of the linear transport problem
\[
\begin{aligned}
\partial_s m + \text{div}(\tilde{v}^\mu(x, s) m) &= \Gamma_s \quad (x, s) \in \mathbb{R}^d \times \mathbb{R}^+, \\
m_{s=0} &= \mu_0 \quad x \in \mathbb{R}^d,
\end{aligned}
\]
with $\tilde{v}^\mu$ defined as in (16) and the source term given by
\[
\Gamma_s(dx) = \int_0^\infty \gamma_r(dx) g_\beta(r, s) ds.
\]
In terms of the push-forward by means of the flow $\Phi^\mu$ associated to $\tilde{v}^\mu$, the solution of (28) can be also written as
\[
\mu_t(dx) = \int_0^\infty \left( \Phi^\mu_s \# \mu_0(dx) + \int_0^s \Phi^\mu_{s-r} \# \Gamma_r(dx) dr \right) h_\beta(s, t) ds.
\]

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