WHEN ARE TWO SPACES HOMOTOPY EQUIVALENT?
AN ALGORITHMIC VIEW POINT

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ABSTRACT. This paper investigates sufficient and necessary conditions for the existence of a homotopy equivalence between two finite simplicial complexes from an algorithmic point of view. As a result, the conditions are formulated in terms of the underlying data of Postnikov towers for simplicial sets with so-called effective homology.

1. INTRODUCTION

One classical question in mathematics is whether two objects are the same under a certain equivalence relation. If the objects are described in a suitable form for a computer, the question passes into: Is there an algorithm that decides the equivalence of these objects? Furthermore, if such objects are of a more complicated nature, one may try to decompose them into smaller building blocks and want to find equivalence on this level of abstraction. This principle can lead to the existence of certain obstructions, which directly prevents the entire object from being equivalent. On the other hand, the correspondence of all building blocks can lead to the overall equivalence of the objects. With this in mind, we will deal with the question how to decide algorithmically if two finite simplicial complexes are homotopy equivalent.

The current state of development in this field is mainly influenced by papers by Brown [2] and Nabutovsky and Weinberger [13]. Both algorithms work under two natural requirements: the input simplicial complexes are simply connected and finite-dimensional. Earlier-dated Brown’s work is based on a crucial assumption of the finiteness of homotopy groups. Such a restrictive assumption enables the author to rely on exhaustive searches mainly and makes the respective algorithm to be very inefficient. Thus, one expects that transition to a solution under more general conditions requires a powerful idea to put in place. This is the case of an algorithm proposed by Nabutovsky and Weinberger in [13]. Their proposal tries to combine algorithmic Brown’s space decomposition via Postnikov tower and rational homotopy methods together with an algorithmic solver [5] of questions about algebraic matrix groups to handle infinite homotopy groups. However, the paper’s exposition is quite sketchy and requires a more detailed implementation description. This paper is the first step in this direction. Here, we focus on the necessary and sufficient conditions for two simplicial complexes to be homotopy equivalent in terms of Postnikov towers, which are constructed via an effective homology framework described in [4]. This should enable one to decide algorithmically whether two finite simplicial complexes are homotopy equivalent with no restriction on homotopy groups. At this point, let us emphasize that the full implementation details are out of the scope of this paper and they will be the subject of further work. It shows up that this is a more difficult problem than to find in an algorithmic way if two given maps from a finite simplicial complex to another simply connected finite simplicial complex are homotopic which has been solved in [8].

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Plan of the paper. Our aim is to formulate a necessary and sufficient condition for the mentioned decision problem of homotopy equivalence between spaces $X$ and $Y$. This is mainly realized in three sections. After the introductory part, we briefly describe how input topological data are handled via simplicial sets and the benefits of the effective homology toolbox for simplicial sets. The next part is devoted to Postnikov towers. Here, we focus on different constructions of Postnikov towers and their relations from a theoretical point of view. In particular, we recall the main ingredients for an algorithmic construction from [4]. A key notion is Postnikov $k$-invariant, together with its representative called Postnikov cocycle. Finally, the main results of this paper are formulated and proved in the penultimate section. The necessary and sufficient condition (Corollary 4.5) reduces our original problem by induction to the existence of coefficient isomorphisms $\gamma$ and stage homotopy selfequivalences $\alpha$ satisfying a specific system of equations in cohomology of stages of the Postnikov tower of $X$. The necessary condition (Theorem 4.2) is a simplicial analogy of Kahn’s result [9] in the category of topological spaces. The algorithmic version of the sufficient condition (Theorem 4.4) enables us to decide about the homotopy equivalence of given simplicial sets. If we are able to find isomorphisms $\gamma$ and selfequivalences $\alpha$ we can construct homotopy equivalences between the stages of the Postnikov towers by induction. The last section provides a basic version of an algorithm which decides whether two spaces are homotopy equivalent under a certain finiteness certificate, which is more general than Brown’s assumption.

Future work. Unfortunately, there is no immediate way how to find those isomorphisms $\gamma$ and stage selfequivalences $\alpha$ effectively. To this end, Nabutovsky and Weinberger [13] utilized the rational Postnikov tower and its connection to inductive construction of respective minimal Sullivan model of the input spaces $X$ and $Y$. The first open question is to find an algorithmic correspondence between these two structures. As the minimal Sullivan model is derived on a base of a commutative cochain algebra, it is fundamental to find a certain type of multiplicative connection to the effective homology/cohomology framework. The second open question is to relate algorithmically the existence of homotopy equivalence between rational Postnikov towers and original ones. This approach intends to break down the problem first to an infinite but easier problem over rational Postnikov towers and then to a finite problem by transition from rational Postnikov towers to the original ones.

2. PRELIMINARIES ON SIMPLICIAL SETS WITH EFFECTIVE HOMOLOGY

We will work with simplicial sets instead of simplicial complexes since they are more powerful and flexible. For basic concepts on simplicial sets, we refer to comprehensive sources [11, 6, 7]. In this section we show briefly what we need to use an algorithmic approach to the homotopy theory of simplicial sets.

Simplicial sets. The standard $k$-dimensional simplex is denoted $\Delta^k$ and its $n$-simplices (non-degenerate) are $(n+1)$-element subsets of its vertex set $\{0, 1, \ldots, k\}$. In the simplicial set context, the Eilenberg-MacLane simplicial set $K(\pi, n)$ is defined through its standard minimal model in which $k$-simplices are given by cocycles

$$K(\pi, n)_k = Z^n(\Delta^k, \pi).$$

It will be essential that $K(\pi, n)$ is a simplicial group in the mentioned representation and hence a Kan complex. In a similar way, we define a simplicial set $E(\pi, n)$ where its $k$-simplices are given by cochains

$$E(\pi, n)_k = C^n(\Delta^k, \pi).$$

The previous definitions lead to a natural principal fibration known as the Eilenberg-MacLane fibration $\delta: E(\pi, n) \to K(\pi, n + 1)$ for $n \geq 1$. A further powerful property of that fibration is its minimality. Recall that minimal fibrations are stable under pullbacks and composition.
Furthermore, we remind two standard constructions of simplicial sets - a mapping cylinder and a mapping cone. For a simplicial map \( f: X \to Y \) the simplicial mapping cylinder is a simplicial set \( \text{cyl}(f) = (X \times \Delta^1 \cup Y)/\sim \) where the equivalence relation \( \sim \) is induced by \((x,0) \sim f(x)\) for all \( x \in X \). If we identify \( X \) with \( X \times \{1\} \) then the simplicial mapping cone of \( f \) is \( \text{cone}(f) = \text{cyl}(f)/X \).

**Effective homology.** Here we take a look at the basic notions of the effective homology framework. This paradigm was developed by Sergeraert and his coworkers to deal with infinitary objects, see [14] (or [4]) for more details.

A locally effective simplicial set is a simplicial set whose simplices have a specified finite encoding and whose face and degeneracy operators are specified by algorithms.

We will work with nonnegatively graded chain complexes of free abelian groups. Such a chain complex is locally effective if elements of the graded module can be represented in a computer and the operations of zero, addition and differential are computable.

In all parts of the paper where we deal with algorithms, all simplicial sets are locally effective and all chain complexes are non-negatively graded locally effective chain complexes of free \( \mathbb{Z} \)-modules. All simplicial maps, chain maps, chain homotopies, etc. are computable.

An effective chain complex is a (locally effective) free chain complex equipped with an algorithm that generates a list of elements of the distinguished basis in any given dimension (in particular, the distinguished bases are finite in each dimension).

**Definition 2.1** ([14]). Let \((C, d_C)\) and \((D, d_D)\) be chain complexes. A triple of mappings \((f: C \to D, g: D \to C, h: C \to C)\) is called a **reduction** if the following holds

i) \( f \) and \( g \) are chain maps of degree 0,
ii) \( h \) is a map of degree 1,
iii) \( fg = \text{id}_D \) and \( \text{id}_C - gf = [d_C, h] = d_C h - h d_C, \)
iv) \( fh = 0, hg = 0 \) and \( hh = 0. \)

The reductions are denoted as \((f, g, h): (C, d_C) \Rightarrow (D, d_D)\).

A strong homotopy equivalence \( C \rightleftharpoons D \) between chain complexes \( C, D \) is the chain complex \( E \) together with a pair of reductions \( C \leftarrow E \Rightarrow D \).

Let \( C \) be a chain complex. We say that \( C \) is equipped with effective homology if there is a specified strong equivalence \( C \rightleftharpoons C^{\text{ef}} \) of \( C \) with some effective chain complex \( C^{\text{ef}} \).

Similarly, we say that a simplicial set has (or can be equipped with) effective homology if its chain complex generated by nondegenerate simplices is equipped with an effective homology.

It is clear that all finite simplicial sets have effective homology. What is essential from the algorithmic point of view is that also many infinite simplicial sets have effective homology. Moreover, there is a way to construct them from the underlying simplicial sets and their effective chain complexes.

**Proposition 2.2** ([4], Section 3). Let \( n \geq 1 \) be a fixed integer and \( \pi \) a finitely generated abelian group. The standard simplicial model of the Eilenberg-MacLane space can be equipped with effective homology.

If \( P \) is a simplicial set equipped with effective homology and \( f: P \to K(\pi, n+1) \) is computable, then the pullback \( Q \) of \( \delta: E(\pi, n) \to K(\pi, n+1) \) along \( f \) can be equipped with effective homology.

The algebraic mapping cone \( \text{cone}(\varphi) \) of the chain map \( \varphi: C \to D \) between chain complexes \((C_*, d_C)\) and \((D_*, d_D)\) is defined as the chain complex \( C_{*-1} \oplus D_* \) with the differential \((x, y) \mapsto (-d_C(x), d_D(y) + \varphi(x))\).
**Proposition 2.3** ([14], Theorem 63). Let $C$ and $D$ be a chain complexes with effective homology $C_{\text{ef}}$ and $D_{\text{ef}}$, respectively, and $\varphi: C \to D$ be a chain map. Then the $\text{cone}(\varphi)$ can be equipped with effective homology $\text{cone}_{\text{ef}} := C_{\text{ef},-1} \oplus D_{\text{ef}}$ in such a way that the strong equivalence $\text{cone}(\varphi) \iff \text{cone}_{\text{ef}}$ restricts to the original string equivalence $D \iff D_{\text{ef}}$.

3. **POSTNIKOV TOWERS**

In this section, we recall the notion of a general Postnikov tower [7, Chapter VI] and the construction of a functorial Postnikov tower for a given Kan complex by Moore [12] and make a comparison of both definitions. Then we describe an algorithmic construction of a Postnikov tower for simplicial sets with effective homology [4], which we will use in the next sections.

**Definition 3.1.** Let $Y$ be a simplicial set. A simplicial Postnikov tower for $Y$ is the following collection of mappings and simplicial sets organized into the commutative diagram

\[
\begin{array}{c}
Y_n \\
p_n \downarrow \\
\vdots \\
p_1 \\
p_0 \\
Y_0
\end{array}
\]

such that for each $n \geq 0$ the map $\varphi_n: Y \to Y_n$ induces isomorphisms $\varphi_n^*: \pi_k(Y) \to \pi_k(Y_n)$ of homotopy groups with $0 \leq k \leq n$, and $\pi_k(Y_n) = 0$ for $k \geq n + 1$. The simplicial set $Y_n$ is called the $n$-th Postnikov stage.

Homotopy groups of simplicial sets in terms of simplicial maps can be defined only for Kan complexes. To other simplicial sets, this notion can be extended using the geometric realization functor $| |$ and its right adjoint simplicial functor $S$ if we put $\pi_n(Y) = \pi_n(S|Y|)$ since the image of $S$ is in the subcategory of Kan complexes. If we suppose that $Y$ is a Kan complex, we can carry out the following construction due to Moore.

**Definition 3.2.** Let $Y$ be a Kan complex. For each $n \in \mathbb{N}_0$ define an equivalence relation $\sim_n$ on the simplices of $Y$ as follows: two $q$-simplices $x, y: \Delta^q \to Y$ are equivalent if their restrictions on $n$-skeleton $x|_{sk_n(\Delta^q)}$ and $y|_{sk_n(\Delta^q)}$ are equal. Define a simplicial set $Y(n) := Y/ \sim_n$ together with induced degeneracy and face maps from $Y$. There are evident maps $p(n): Y(n) \to Y(n - 1)$ and $\varphi(n): Y \to Y(n)$.

It shows up that this collection is really a Postnikov tower for $Y$. Therefore, we will call it **Moore-Postnikov tower** in accordance with [7]. It holds even more:

**Proposition 3.3.** Let $Y$ be a simply connected Kan complex. The tower $\{Y(n)\}_{n \in \mathbb{N}}$ defined above is a Postnikov tower for $Y$ and the maps $p(n)$ are fibrations. Furthermore, if $Y$ is a simply connected minimal Kan complex then $p(n)$ are minimal fibrations and pullbacks of diagrams along a certain maps $k(n - 1)$:

\[
\begin{array}{ccc}
Y(n) & \longrightarrow & E(\pi_n(Y), n) \\
p(n) \downarrow & & \downarrow \delta \\
Y(n - 1) & \longrightarrow & K(\pi_n(Y), n + 1)
\end{array}
\]
Proof. These are Theorems 2.6 and 3.26 in [12]. See also [7], Chapter VI, Theorem 2.5 and Corollary 5.13.

The main advantage of the Moore-Postnikov tower is that it forms a functor from the category of Kan complexes into a category of towers of simplicial sets. However, we would like to construct Postnikov towers directly from simplicial sets, which are not Kan complexes. A common procedure how to do it is to build up the \( n \)-th stage from the \( (n-1) \)-th stage as a pullback of the minimal Eilenberg-MacLane fibration \( \delta: E(\pi_n(Y), n) \to K(\pi_n(Y), n+1) \).

**Definition 3.4.** Let \( Y \) be a simply connected simplicial set. A standard Postnikov tower is a Postnikov tower such that for all \( n \geq 1 \): \( Y_n \) is the pullback of the fibration \( \delta \) along a map \( k_{n-1}: Y_{n-1} \to K(\pi_n(Y), n+1) \).

\[
\begin{array}{ccc}
Y_n & \xrightarrow{r_n} & E(\pi_n(Y), n) \\
\downarrow{p_n} & & \downarrow{\delta} \\
Y_{n-1} & \xrightarrow{k_{n-1}} & K(\pi_n(Y), n+1)
\end{array}
\]

The map \( k_{n-1} \) is called a Postnikov map. Since the Kan fibrations \( \delta \) are minimal, so their pullbacks \( p_n \) are also minimal and all stages \( Y_n \) are minimal Kan complexes.

In the subsequent proposition, we compare general Postnikov towers and standard Postnikov towers with the Moore-Postnikov ones.

A morphism of Postnikov towers \( f_*: X_* \to Y_* \) is called a weak equivalence of Postnikov towers if all \( f_n: X_n \to Y_n \) are weak equivalences, i.e. they induce isomorphisms on all homotopy groups. Similarly, \( f_* \) is an isomorphism of Postnikov towers if all \( f_n \) are isomorphisms of simplicial sets.

**Proposition 3.5.** Let \( Y \) be a simply connected minimal Kan complex. Then, for any Postnikov tower \( \{Y_n, p_n, \varphi_n\} \) for \( Y \) there is a weak equivalence \( f_* \) to the Moore-Postnikov tower \( \{Y(n), p(n), \varphi(n)\} \) for \( Y \) and an isomorphism \( f: Y \to \lim Y(k) \) such that the diagrams

\[
\begin{array}{ccc}
Y & \xrightarrow{f} & \lim Y(k) = Y \\
\downarrow{\varphi_n} & & \downarrow{\varphi(n)} \\
Y_n & \xrightarrow{f_n} & Y(n)
\end{array}
\]

(3.2)

commute. If the maps \( p_n \) are minimal fibrations, then both towers are isomorphic.

Proof. The first part is a special case of Theorem 5.14 in [7]. So we have a weak equivalence \( f_* \) of the Postnikov towers. This collection of maps induces a weak equivalence \( f: Y \to \lim Y(k) = Y \) such that the diagram commutes.

For a simply connected minimal Kan complex \( Y \), the maps \( p(n) \) are minimal Kan fibrations. If \( p_n \) are also minimal fibrations then \( f_n: Y_n \to Y(n) \) and \( f: Y \to Y \) are weak equivalences between minimal Kan complexes and according to Theorem 2.20 in [6] they are isomorphisms.

**Algorithmic construction of Postnikov tower.** The algorithmic approach constructs a standard Postnikov tower by specifying instructions for maps \( k_{n-1}: Y_{n-1} \to K(\pi_n(Y), n+1) \) and \( \varphi_n: Y \to Y_n \). Here we summarize the algorithm from [4, Section 4] emphasizing some facts needed for our purposes.

Consider a simplicial set \( Y \) which is equipped with effective homology \( C_*(Y) \leftrightarrow C_*^{ef}(Y) \) and suppose that we have constructed its Postnikov tower up to the stage \( n-1 \). This means that we have \( Y_{n-1} \) with effective homology \( C_*(Y_{n-1}) \leftrightarrow C_*^{ef}(Y_{n-1}) \) and that a simplicial map \( \varphi_{n-1}: Y \to Y_{n-1} \) has been computed. The Eilenberg-Zilber theorem implies the existence
of a strong equivalence \( M_* := \text{cone}_*(\varphi_{n-1}) \) \( \iff \) \( C_*(\text{cyl}(\varphi_{n-1}), Y) \). Here \( \text{cone}_*(\varphi_{n-1}) \), is the cone of the chain homomorphism \( \varphi_{n-1} \) and \( \text{cyl}(\varphi_{n-1}) \) is the cylinder of the simplicial map \( \varphi_{n-1} \). A relation of certain homology groups of \( M_* \) to homotopy groups of \( Y \) can be derived from the composition of the Eilenberg-Zilber map \( EZ \), the Hurewicz isomorphism \( h \) and a connecting isomorphism \( c \) of the \( n \)-connected pair \( (\text{cyl}(\varphi_{n-1}), Y) \). For \( i \leq n \) we get

\[
H_{i+1}(M_*) \xrightarrow{EZ_{i+1}} H_{i+1}(\text{cyl}(\varphi_{n-1}), Y) \xrightarrow{h_{i+1}} \pi_{i+1}(\text{cyl}(\varphi_{n-1}), Y) \xrightarrow{c_{i+1}} \pi_i(Y). \tag{3.3}
\]

The chain complex \( M_* = \text{cone}(\varphi_{n-1}) \) has effective homology (see Proposition 2.3) since

\[
M_* = \text{cone}(\varphi_{n-1}) = C_{*+1}(Y) \oplus C_*(Y_{n-1}) \iff C^*_{*+1}(Y) \oplus C^*_*(Y_{n-1}) =: M^*_*. \tag{3.4}
\]

For algorithm purposes, we define in accordance with (3.3)

\[
\pi_n^Y := H_{n+1}(M^*_*) \cong H_{n+1}(M_*).
\]

Now we introduce a notation. Denote \( \iota: C_*(Y_{n-1}) \hookrightarrow M_* \) \( \text{and} \) \( \iota^*: C^*(Y_{n-1}) \hookrightarrow M^*_* \) the inclusions from (3.4). The strong equivalence \( C_*(Y_{n-1}) \iff C^*(Y_{n-1}) \) is a part of the strong equivalence \( M_* \iff M^*_* \). Denote \( \xi^*: M_* \rightarrow M^*_* \) \( \text{and} \) \( \xi: C_*(Y_{n-1}) \rightarrow C^*(Y_{n-1}) \) homomorphisms corresponding to these strong equivalences. They make the following diagram commutative:

\[
\begin{array}{ccc}
C^*(Y_{n-1}) & \xrightarrow{\iota^*} & M^*_* \\
\downarrow{\xi} & & \downarrow{\xi^*} \\
C_*(Y_{n-1}) & \xrightarrow{\iota} & M_*
\end{array}
\tag{3.5}
\]

In [4] it was shown that the projection of cycles \( Z_{n+1}(M^*_*) \) onto \( H_{n+1}(M^*_*) \) can be algorithmically extended to chains, so we have a homomorphism \( \rho^*: M^*_{n+1} \rightarrow H_{n+1}(M^*_*) = \pi_n^Y \). Moreover, we can define an analogical homomorphism \( \rho: M_{n+1} \rightarrow H_{n+1}(M_*) \) such that the diagram

\[
\begin{array}{ccc}
M^*_{n+1} & \xrightarrow{\rho^*} & H_{n+1}(M^*_*) \\
\downarrow{\xi^*} & & \downarrow{\xi} \\
M_{n+1} & \xrightarrow{\rho} & H_{n+1}(M_*)
\end{array}
\tag{3.6}
\]

commutes.

Now we can define the cocycle

\[
\kappa^*_{n-1} := \rho^* \circ \iota^*: C^*_*(Y_{n-1}) \rightarrow \pi_n^Y.
\]

We will call it an effective Postnikov cocycle. Its cohomology class \( [\kappa^*_{n-1}] \in H^{n+1}(C^*_*(Y_{n-1}); \pi_n^Y) \) will be called an effective Postnikov class. Next, we define the Postnikov cocycle \( \kappa_{n-1} \) as

\[
\kappa_{n-1} := \kappa^*_{n-1} \circ \xi = \xi^*(\kappa^*_{n-1}) \in Z^{n+1}(C_*(Y_{n-1}), \pi_n^Y)
\]
and the Postnikov class as its cohomology class \([\kappa_{n-1}] \in H^{n+1}(C_*(Y_{n-1}); \pi^n_Y)\). The previous definitions can now be summarized in the commutative diagram:

\[
\begin{array}{ccccccc}
C_{n+1}^{\text{ef}}(Y_{n-1}) & \xrightarrow{\epsilon_n} & M_{n+1}^{\text{ef}} & \xrightarrow{\rho_n} & H_{n+1}^{\text{ef}}(M_n^{\text{ef}}) & \xrightarrow{\text{id}} & \pi^n_Y \\
\xi_{n+1} & \uparrow & \xi_{n+1} & \uparrow & \xi_n & \uparrow & \xi_n \\
C_{n+1}(Y_{n-1}) & \xrightarrow{\iota_n} & M_{n+1} & \xrightarrow{\rho} & H_{n+1}(M_n) & \xrightarrow{k_n-1} & \pi^n_Y \\
\end{array}
\]

(3.7)

For every abelian group \(\pi\), the evaluation \(ev: \text{SMap}(Y, K(\pi, k)) \to Z^k(Y, \pi)\) from simplicial maps into \(k\)-cocycles on \(Y\) is the map such that for a simplicial map \(f: Y \to K(\pi, k)\) the cocycle \(ev(f)\) assigns to every \(k\)-simplex \(\sigma \in Y_k\) the value of the cocycle \(f(\sigma) \in K(\pi, k)_k = Z^k(\Delta^k, \pi)\) on the unique nondegenerate \(k\)-simplex of \(\Delta^k\). We will write \(ev(f)(\sigma)\) for this value in \(\pi\). The evaluation is a bijection and enables us to define the simplicial map \(k_{n-1}\) corresponding to the Postnikov cocycle \(\kappa_{n-1}\). This will be the Postnikov map for our construction. Using it, we can construct the \(n\)-th Postnikov stage \(Y_n\) as the pullback along this map (see the definition of the standard Postnikov tower). This pullback has effective homology. To complete the construction of the \(n\)-th stage we have to find a simplicial map \(\varphi_n: Y \to Y_n\) which fits into diagram (3.1) of the Postnikov tower. Since \(Y_n\) is the pullback, the required map is given by the couple of maps \(\varphi_{n-1}: Y \to Y_{n-1}\) and \(l_n: Y \to E(\pi^n_Y, n)\). The map \(l_n\) corresponds via bijective evaluation \(\text{SMap}(Y, E(\pi^n_Y, n)) \to C^n(Y; \pi^n_Y)\) to the cochain \(\lambda_n\) given by the composition

\[C_n(Y) \to M_n \xrightarrow{\xi_n} M_n^{\text{ef}} \xrightarrow{\rho_n} \pi^n_Y.\]

In [4] it is proved that \(k_{n-1}, Y_n\) and \(\varphi_n\) satisfy all properties in the definition of the standard Postnikov tower.

**Definition 3.6.** The Postnikov tower constructed above will be called an effective Postnikov tower of the simplicial set \(Y\).

Let us recall the notion of the fundamental class in \(K(\pi, k)\). Let \(\iota_k: K(\pi, k)_k \to \pi\) be the mapping assigning to each \(\pi\)-valued cocycle \(\sigma \in Z^k(\Delta^k, \pi)\) its value on the unique \(k\)-simplex of \(\Delta^k\), i.e. \(\iota_k = ev(\text{id}_{K(\pi, k)})\). The cohomology class \([\iota_k] \in H^k(K(\pi, k); \pi)\) is called the fundamental class. Next lemma provides the relations between fundamental classes and the Postnikov class \([\kappa_{n-1}]\) constructed above. These relations play a crucial role in the proof of Theorem 4.2.

**Lemma 3.7.** Denote \(\tau^K\) and \(\tau\) the transgressions in the Serre long exact sequences of fibrations \(K(\pi^n_Y, n) \to E(\pi^n_Y, n) \to K(\pi^n_Y, n + 1)\) and \(K(\pi^n_Y, n) \to Y_n \to Y_{n-1}\), respectively. Then

\[\tau^K([l_n]) = [l_{n+1}] \quad \text{and} \quad \tau([l_n]) = k^{*-1}_{n-1}([l_{n+1}]) = [\kappa_{n-1}].\]

**Proof.** Using the fact that \(H^k(K(\pi^n_Y, k); \pi^n_Y) \cong \text{Hom}(H_k(K(\pi^n_Y, k), \pi^n_Y)\) the transgression \(\tau^K\) is dual to a homomorphism \(H_{n+1}(K(\pi^n_Y, n + 1)) \to H_n(K(\pi^n_Y, n))\) which can be represented by

---
a homomorphism $\tau_K$ on the level of chain complexes:

$$
\begin{array}{ccc}
C_n(Y, n) & \xrightarrow{pr^*} & C_{n+1}(E(Y, n+1), n+1) \\
\downarrow & & \downarrow \\
C_n(K(Y, n), n) & \xrightarrow{\delta^*} & C_{n+1}(K(Y, n+1))
\end{array}
$$

To get a precise definition of $\tau_K$, one needs to specify the left inverse $\varphi$ of $(\delta \cdot pr)_*$. Take a simplex $\sigma$ of $K(Y, n+1)$ uniquely determined by its value $ev(id_{K(Y, n+1)})(\sigma) \in \pi^n_Y$. Define $\varphi(\sigma)$ as the cochain in $C^n(\Delta^{n+1}, \pi^n_Y)$ (i.e. an element of $C_{n+1}(E(\pi^n_Y, n+1), n+1)$) satisfying $\varphi(\sigma)(d^0) = ev(id_{K(Y, n+1)})(\sigma)$ and $\varphi(\sigma)(d^i) = 0$ for $i \geq 1$ where $d^i : \Delta^n \to \Delta^{n+1}$ are the standard faces in $\Delta^{n+1}$. Then $\partial(\varphi(\sigma))$ is a simplex in $K(Y, n)$ with the property that

$$ev(id_{K(Y, n)})(\partial(\varphi(\sigma))) = ev(id_{K(Y, n+1)})(\sigma).$$

Hence $\tau_K([\iota_n]) = [\iota_{n+1}]$.

To prove the latter statement, consider the pullback diagram of fibrations:

$$
\begin{array}{ccc}
K(Y, n) & \xrightarrow{id} & K(Y, n) \\
\downarrow & & \downarrow \\
Y_n & \xrightarrow{\delta} & E(Y, n) \\
\downarrow & & \downarrow \\
Y_{n-1} & \xrightarrow{k_{n-1}} & K(Y, n+1)
\end{array}
$$

Since the transgression is natural, we get

$$\tau([\iota_n]) = k_{n-1}^*(\tau_K([\iota_n])) = k_{n-1}^*([\iota_{n+1}]) = k_{n-1}^*([ev(id_{K(Y, n+1)}))] = [ev(k_{n-1} \circ id_{K(Y, n+1)}))]

= [ev(k_{n-1})] = [\kappa_{n-1}].$$

\[\square\]

4. NECESSARY AND SUFFICIENT CONDITION

The aim of this section is to establish a necessary and sufficient condition in terms of their effective Postnikov towers to decide if the geometric realizations of two finite simply connected simplicial sets $X$ and $Y$ are homotopy equivalent. First, we move from the geometric realizations of simplicial sets to their substitutions as minimal Kan complexes.

**Lemma 4.1.** Let $X$ and $Y$ be simply connected simplicial sets. Let $\{X_n, p^n_X, \varphi^n_X\}$ and $\{Y_n, p^n_Y, \varphi^n_Y\}$ be their standard Postnikov towers. Put $X' := \lim X_n$ and $Y' := \lim Y_n$. Then there is an isomorphism $[|X|, |Y|] \cong [|X', Y'|]$

Especially, $|X|$ and $|Y|$ are homotopy equivalent if and only if there is a simplicial weak homotopy equivalence $f : X' \to Y'$. 
The desired isomorphism is a composition of two bijections. First, using the induced maps $\varphi^X: X \to X'$ and $\varphi^Y: Y \to Y'$ one can define the isomorphism $[[X], |Y|] \cong [[X'], |Y'|]$. Next, we use a canonical isomorphism $[A, B] \cong [[A], |B|]$ for Kan complexes $A$ and $B$ applied to $A = X'$ and $B = Y'$.

The idea of our algorithm, which should decide if the geometric realizations of two simplicial sets $X$ and $Y$ are homotopy equivalent, is to look for weak equivalences between the stages of the Postnikov towers of both simplicial sets successively from lower to higher stages. The next theorem gives a necessary condition for such a situation. In the category of topological spaces, this condition was derived and used by Kahn in [9] and [10]. However, our algorithmic construction of the Postnikov tower of simplicial sets is different from that given by Kahn and there is no immediate way to restate his result from topological spaces to simplicial sets. In a simplicial context, the key construction ingredient is a minimal model of Eilenberg-MacLane space.

**Theorem 4.2.** Let $X$ and $Y$ be simply connected simplicial sets. Let $\{X_n, p^X_n, \varphi^X_n\}$ and $\{Y_n, p^Y_n, \varphi^Y_n\}$ be their effective Postnikov towers with Postnikov cocycles $\kappa^X_n$, $\kappa^Y_n$, respectively. Let $X' := \varprojlim X_n$ and $Y' := \varprojlim Y_n$. If there is a simplicial map $f: X' \to Y'$, then there are maps $f_n: X_n \to Y_n$ of the Postnikov stage such that all diagrams

\[
\begin{array}{ccc}
X_n & \xrightarrow{f_n} & Y_n \\
\downarrow p^X_n & & \downarrow p^Y_n \\
X_{n-1} & \xrightarrow{f_{n-1}} & Y_{n-1}
\end{array}
\]

and

\[
\begin{array}{ccc}
X' & \xrightarrow{f} & Y' \\
\downarrow \varphi^X' & & \downarrow \varphi^Y' \\
X & \xrightarrow{f_n} & Y
\end{array}
\]

commute strictly. Moreover, the Postnikov classes $[\kappa^X_{n-1}]$ and $[\kappa^Y_{n-1}]$ are in the relation

$$\gamma_n([\kappa^X_{n-1}]) = f_n^*([\kappa^Y_{n-1}])$$

(4.1)

where $\gamma_n: H^{n+1}(X_{n-1}; \pi^X_n) \to H^{n+1}(X_{n-1}; \pi^Y_n)$ is induced by a homomorphism $\gamma: \pi^X_n \to \pi^Y_n$ which captures the map $f$ in a suitable manner.

If $f$ is a homotopy equivalence then all $f_n$ are homotopy equivalences (and even isomorphisms of simplicial sets) and $\gamma$ is an isomorphism.

**Proof.** Start with deriving both diagrams. Note that $\{X_n, \kappa^X_n\}$ and $\{Y_n, \kappa^Y_n\}$ with canonical maps $\varphi^X_n: X' \to X_n$ and $\varphi^Y_n: Y' \to Y_n$ are standard Postnikov towers for $X'$ and $Y'$, respectively. Apply Proposition 3.5 for these Postnikov towers to compare them with the Moore-Postnikov towers of $X'$ and $Y'$, respectively. Those maps between towers together with the functoriality of Moore-Postnikov towers provide the commutative diagram:
The diagram defines horizontal maps \( f_n : X_n \to Y_n \) for all \( n \in \mathbb{Z} \) and gives the commutative diagrams in our theorem. If \( f \) is a weak equivalence then maps \( (hf g^{-1})(n) \) and \( f_n \) are also weak equivalences between minimal Kan complexes, so they are isomorphisms by Theorem 2.20 in [6].

It remains to show that equation (4.1) holds. To complete this task, take the commutative diagram from the previous part of the proof:

\[
\begin{array}{ccc}
K(\pi^X_n, n) & \to & K(\pi^Y_n, n) \\
\downarrow & & \downarrow \\
X_n & \to & Y_n \\
\downarrow p^n & & \downarrow p^n \\
X_{n-1} & \to & Y_{n-1} \\
\end{array}
\]

Consider the homomorphism \( \gamma : \pi^X_n \to \pi^Y_n \) determined by the commutative square:

\[
\begin{array}{ccc}
H_n(K(\pi^X_n, n)) & \xrightarrow{f_n, \cong} & H_n(K(\pi^Y_n, n)) \\
\downarrow \text{ev(id}_{K(\pi^X_n, n)}) & & \downarrow \text{ev(id}_{K(\pi^Y_n, n)}) \\
\pi^X_n & \xrightarrow{\gamma} & \pi^Y_n \\
\end{array}
\]

Now the fundamental classes \([\kappa^X_n]\) and \([\kappa^Y_n]\) fit into the commutative diagram:

\[
\begin{array}{ccc}
\text{id}_{K(\pi^X_n, n)} \in \text{SMap}(K(\pi^X_n, n), K(\pi^X_n, n)) & \xrightarrow{\text{ev,} \cong} & Z^n(K(\pi^X_n, n), \pi^X_n) \\
\downarrow & & \downarrow \\
f_n, \cong & & \tau(f_n) \cong \\
\text{id}_{K(\pi^Y_n, n)} \in \text{SMap}(K(\pi^Y_n, n), K(\pi^Y_n, n)) & \xrightarrow{\text{ev,} \cong} & Z^n(K(\pi^Y_n, n), \pi^Y_n) \\
\downarrow & & \downarrow \\
f_n^* \cong & & \tau(f_n^*) \cong \\
\end{array}
\]

Using the naturality of the transgression, Lemma 3.7 and the last column of the previous diagram we obtain:

\[
f_{n-1}^*([\kappa^Y_{n-1}]) = f_{n-1}^*(\tau([\kappa^Y_n])) = \tau(f_n^*([\kappa^X_n])) = \gamma_n([\kappa^X_n]) = \gamma_n([\kappa^X_{n-1}]).
\]

Finally, if \( f_n \) is a weak equivalence then \( \bar{f} \) is a homotopy equivalence by the first diagram in this part. Next, the induced map \( \bar{f}_* \) is an isomorphism (see the second diagram) which implies that \( \gamma \) is an isomorphism as well.

As a consequence, we can formulate known assertions that the homotopy type of a finite dimensional simplicial complex is given only by a finite part of its Postnikov tower. From an algorithmic point of view, this fact plays a crucial role.

**Corollary 4.3.** Let \( X \) and \( Y \) be finite simply connected simplicial sets of dimensions \( \leq d \) with effective Postnikov towers \( \{X_n\} \) and \( \{Y_n\} \), respectively. Then \( |X| \) and \( |Y| \) are homotopy equivalent if and only if there is a weak homotopy equivalence

\[
f_d : X_d \to Y_d.
\]
Theorem.

Proof. If $|X|$ and $|Y|$ are homotopy equivalent, then by Lemma 4.1 there is a weak homotopy equivalence $f' : X' := \lim_{\leftarrow} X_n \to Y' := \lim_{\rightarrow} Y_n$ and by Theorem 4.2 there is a homotopy equivalence $f_d : X_d \to Y_d$.

Conversely, suppose that there is a weak homotopy equivalence $f_d : X_d \to Y_d$. Since $|X|$ is a CW-complex of dimension $d$ and $|\varphi_d^Y| : |X| \to |Y|$ is an isomorphism in homotopy groups $\pi_i$ for $i \leq d$ and an epimorphism for $i = d + 1$, the induced map $|\varphi_d^Y| : |X| \to |Y|$ is a bijection. Let $F : |X| \to |Y|$ be a map which homotopy class corresponds to the homotopy class of $|f_d \circ \varphi_d^Y| : |X| \to |Y_d|$. Then $F$ induces isomorphisms in homotopy groups $\pi_i$ for $i \leq d$. Since $|X|$ and $|Y|$ have dimensions $\leq d$, it induces also isomorphisms in homology groups $H_i(\cdot; \mathbb{Z})$ for all $i$. Therefore $F : |X| \to |Y|$ is a homotopy equivalence by Whitehead Theorem.

Next theorem and its proof show that equation (4.1), where $\gamma$ is an isomorphism, is sufficient for an algorithmic construction of a simplicial map lifting a given homotopy equivalence between Postnikov $(n - 1)$-stages to a homotopy equivalence between $n$-stages.

**Theorem 4.4.** Let $X$ and $Y$ be simply connected simplicial sets with effective homology. Let $\{X_n, p_n^X, \gamma_n\}$ and $\{Y_n, p_n^Y, \gamma_n\}$ be their effective Postnikov towers with Postnikov cocycles $\kappa_n^X$ and $\kappa_n^Y$, respectively. Assume that there are a computable simplicial map $f_{n-1} : X_{n-1} \to Y_{n-1}$ and an isomorphism $\gamma : \pi_n^X \to \pi_n^Y$ such that relation (4.1)

$$
\gamma \ast [\kappa_{(n-1)s}^X] = f_{n-1}^* [\kappa_{(n-1)s}^Y]
$$

holds. Then, there is an algorithm which constructs a map $f_n : X_n \to Y_n$ such that the diagram

$$
\begin{matrix}
X_n & \xrightarrow{f_n} & Y_n \\
p_n^X \downarrow & & \downarrow p_n^Y \\
X_{n-1} & \xrightarrow{f_{n-1}} & Y_{n-1}
\end{matrix}
$$

commutes. If $f_{n-1}$ is a homotopy equivalence, so is $f_n$ (and even an isomorphism of simplicial sets).

Moreover, any two maps $f_n, f'_n : X_n \to Y_n$ making the above diagram commutative (not necessarily constructed by the algorithm) differ by a map $X_n \to K(\pi_n^Y, n)$ in the following sense: Consider the natural action $+ : Y_n \times K(\pi_n^Y, n) \to Y_n$. Then there is a map $d : X_n \to K(\pi_n^Y, n)$ such that

$$
f'_n = f_n + d.
$$

Furthermore, if we assume that the homotopy classes of weak equivalences

$$
f_n \circ i^X, f'_n \circ i^X : K(\pi_n^X, n) \to i^Y(K(\pi_n^Y, n))
$$

are equal then there exists $c : X_{n-1} \to K(\pi_n^X, n)$ satisfying

$$
f'_n \sim f_n + \gamma \circ c \circ p_n^X.
$$

where $\gamma : K(\pi_n^X, n) \to K(\pi_n^Y, n)$ is determined by the isomorphism $\gamma$ and $i^X : K(\pi_n^X, n) \to X_n$ is the fibre inclusion.

**Remark.** Denote $\text{Eq}(X_n, Y_n)$ the set of homotopy classes of homotopy equivalences $X_n \to Y_n$. Then the second part of the previous theorem for a fixed $f_n$ induces a map

$$
m_{\gamma} : H^n(X_{n-1}; \pi_n^X) \to \text{Eq}(X_n, Y_n)
$$

such that $m_{\gamma}(0)$ is the homotopy class of the map $f_n$. If $f'_n : X_n \to Y_n$ is another homotopy equivalence, then there is a class $c \in H^n(X_{n-1}; \pi_n^X)$ such that for homotopy classes of maps we have

$$
[f'_n] = m_{\gamma}(c) = [f_n] + [\gamma \circ p_n^X].
$$
The mapping $m_\gamma$ becomes injective when we take the quotient

$$H^n(X_{n-1}; \pi_n^X) / \text{Stab}([f_n])$$

by the stabilizer of the action of $H^n(X_{n-1}; \pi_n^X)$ on the homotopy class of $f_n$.

**Proof.** Let $\gamma: \pi_n^X \to \pi_n^Y$ be an isomorphism satisfying equation (4.1). Put

$$k_{n-1}^X := \gamma^{-1} f_{n-1}^* k_{n-1}^Y.$$

Since $[k_{n-1}^X] = [k_{n-1}^Y]$ according to (4.1), the map $k_{n-1}^X = ev^{-1} k_{n-1}^X$ is homotopic to the Postnikov map $k_{n-1}^X = ev^{-1} k_{n-1}^Y$. Let us build an auxiliary stage $X_n$ as the pullback along the new Postnikov map $k_{n-1}^X$:

$$\begin{array}{ccc}
X_n & \longrightarrow & E(\pi_n^X, n) \\
\downarrow p_n^X & & \downarrow \delta \\
X_{n-1} & \longrightarrow & K(\pi_n^X, n + 1)
\end{array}$$

We will construct the required $f_n$ as the composition of maps $j: X_n \to \overline{X}_n$ and $\overline{f}_n: \overline{X}_n \to Y_n$ which make the following diagram commutative.

$$\begin{array}{ccc}
X_n & \xrightarrow{j} & \overline{X}_n & \xrightarrow{\overline{f}_n} & Y_n \\
\downarrow p_n^X & & \downarrow \overline{p}_n^X & & \downarrow p_n^Y \\
X_{n-1} & \xrightarrow{id} & X_{n-1} & \xrightarrow{f_{n-1}} & Y_{n-1}
\end{array}$$

1. **The construction of a map $\overline{f}_n$.**

   The relation between Postnikov cocycles $k_{n-1}^X$ and $k_{n-1}^Y$ means that the diagram

   $$\begin{array}{ccc}
X_{n-1} & \xrightarrow{f_{n-1}} & Y_{n-1} \\
\downarrow k_{n-1}^X & & \downarrow k_{n-1}^Y \\
K(\pi_n^X, n + 1) & \xrightarrow{\gamma_*} & K(\pi_n^Y, n + 1)
\end{array}$$

   is commutative. The mapping $\mu: E(\pi_n^X, n) \to E(\pi_n^Y, n)$ determined by $\gamma: \pi_n^X \to \pi_n^Y$ makes the commutative front square of the following diagram:

   $$\begin{array}{ccc}
\overline{X}_n & \longrightarrow & Y_n \\
\downarrow \overline{p}_n^X & & \downarrow \overline{p}_n^Y \\
X_{n-1} & \xrightarrow{k_{n-1}^X \delta} & Y_{n-1} & \xrightarrow{k_{n-1}^Y \delta} & K(\pi_n^X, n + 1) & \xrightarrow{\gamma_*} & K(\pi_n^Y, n + 1)
\end{array} \quad \text{(4.2)}$$

Since $Y_n$ is the pullback of the right hand square we obtain $\overline{f}_n$ making the back square of diagram (4.2) commutative as the pair of compatible maps

$$\begin{array}{ccc}
\overline{X}_n & \xrightarrow{\overline{p}_n^X} & X_{n-1} & \xrightarrow{f_{n-1}} & Y_{n-1} \\
\rightarrow & & \rightarrow & & \rightarrow \\
\overline{X}_n & \xrightarrow{E(\pi_n^X, n)} & \xrightarrow{\mu} & \xrightarrow{E(\pi_n^Y, n)} & \rightarrow
\end{array}$$

Since $Y_n$ is the pullback of the right hand square we obtain $\overline{f}_n$ making the back square of diagram (4.2) commutative as the pair of compatible maps

$$\begin{array}{ccc}
\overline{X}_n & \xrightarrow{\overline{p}_n^X} & X_{n-1} & \xrightarrow{f_{n-1}} & Y_{n-1} \\
\rightarrow & & \rightarrow & & \rightarrow \\
\overline{X}_n & \xrightarrow{E(\pi_n^X, n)} & \xrightarrow{\mu} & \xrightarrow{E(\pi_n^Y, n)} & \rightarrow
\end{array}$$
2. The construction of \( j \).

The homotopy class of the map \( k_{n-1}^X - k_{n-1}^Y : X_{n-1} \to K(\pi_n^X, n + 1) \) is trivial, so according to Lemma 2.11 in [3] one can compute algorithmically its lift \( \omega : X_{n-1} \to E(\pi_n^X, n) \).

\[
\begin{array}{c}
E(\pi_n^X, n) \\
\downarrow \delta \\
K(\pi_n^X, n + 1)
\end{array}
\xrightarrow{\omega} \begin{array}{c}
X_{n-1} \\
k_{n-1}^X - k_{n-1}^Y
\end{array}
\]

Any two lifts differ by a map \( c : X_{n-1} \to K(\pi_n^X, n) \) to the fiber of the fibration \( \delta \). For a fixed \( \omega : X_{n-1} \to E(\pi_n^X, n) \) and every \( c : X_{n-1} \to K(\pi_n^X, n) \) let us define \( j_c : X_{n-1} \times E(\pi_n^X, n) \to X_{n-1} \times E(\pi_n^X, n) \) by the formula

\[ j_c(x, y) = (x, y + \omega(x) + c(x)). \]

The restriction of this map to \( X_n \) has the image in \( \overline{X}_n \), since for \( \delta y = k_{n-1}^X(x) \) we get

\[ \delta(y) + \delta(\omega + c)(x) = \delta(y) + \delta\omega(x) = k_{n-1}^X(x) + (k_{n-1}^X - k_{n-1}^Y)(x) = k_{n-1}^X(x). \]

Finally, put \( f_n = \overline{f_n} \circ j_c \). If \( f_{n-1} \) is a homotopy equivalence, so is \( f_n \), which follows from the long exact sequences for the fibrations \( K(\pi_n^X, n) \to X_n \to X_{n-1}, K(\pi_n^X, n) \to \overline{X}_n \to X_{n-1} \) and \( K(\pi_n^Y, n) \to Y_n \to Y_{n-1} \) and 5-lemma.

Now, continue to prove the last part of Theorem 4.4. Since \( p_n^Y \circ f_n = p_n^Y \circ f_n' \) and \( p_n^Y \) is a principal \( K(\pi_n^Y, n) \)-fibration, there is a map \( d : X_n \to K(\pi_n^Y, n) \) such that \( f_n' = f_n + d \). Take the long Serre exact sequence of the fibration \( p_n^X \):

\[
\cdots \to 0 \to H^n(X_{n-1}, \pi_n^X) \to H^n(X_n, \pi_n^Y) \to H^n(K(\pi_n^X, n), \pi_n^Y) \to 0 \to \cdots
\]

The class \([d] = [f_n - f_n'] \in H^n(X_n, \pi_n^Y)\) maps to zero in \( H_n(K(\pi_n^X, n); \pi_n^Y)\) as both \( f_n \) and \( f_n' \) induce the same maps up to homotopy on fibres, so \( f_n \) and \( f_n' \) differ by a certain map \( d : X_{n-1} \to K(\pi_n^Y, n) \) up to homotopy, i.e. \( f_n' \sim f_n + d \circ p_n^X \). Since the isomorphism \( \gamma : \pi_n^X \to \pi_n^Y \) induces isomorphism of simplicial sets \( \gamma_* : K(\pi_n^X, n) \to K(\pi_n^Y, n) \), there is a map \( c : X_{n-1} \to K(\pi_n^X, n) \) such that \( f_n' \sim f_n + \gamma_* \circ c \circ p_n^X \).

The sufficient condition shows that there can be many possible lifts of a given stage map to the next stage. For all possible homotopy classes of stage maps \( f_{n-1} \), it is necessary to verify the cohomological equation (4.1). To reduce a complexity of the test, one can think about a new parameterization of homotopy classes between \((n-1)\)-stages. This simple observation is the subject of the next statement. Its proof is based on Theorems 4.2 and 4.4.

**Corollary 4.5.** Let \( X \) and \( Y \) be simply connected simplicial sets with effective homology. Let \( \{X_n, p_n^X, \varphi_n^X\} \) and \( \{Y_n, p_n^Y, \varphi_n^Y\} \) be their effective Postnikov towers with Postnikov cocycles \( \kappa_n^X \) and \( \kappa_n^Y \), respectively. Assume that there is a homotopy equivalence \( f_{n-1} : X_{n-1} \to Y_{n-1} \). Then, \( X_n \) and \( Y_n \) are homotopy equivalent if and only if there is a pair of isomorphisms \( \gamma : \pi_n^X \to \pi_n^Y \) and \( \alpha : X_n \to X_n \) satisfying the relation

\[ \gamma_*[\kappa_{(n-1)}^X] = (f_{n-1})^*[\kappa_{(n-1)}^Y]. \]

**Proof.** Suppose that \( X_n \) and \( Y_n \) are homotopy equivalent through an isomorphism \( f : X_n \to Y_n \). Then, Theorem 4.2 provides the commutative diagram
for a certain homotopy equivalence \(g_{n-1}: X_{n-1} \to Y_{n-1}\) as clearly \(X' = X_n\) and \(Y' = Y_n\).
Moreover, there is an available an isomorphism \(\gamma: \pi^X_n \to \pi^Y_n\) such that
\[
\gamma_*[\kappa^X_{n-1}] = (g_{n-1})^*[\kappa^X_{n-1}]_\ast.
\]
Now, it is enough to take \(\alpha := f^{-1}_{n-1}g_{n-1}\) to get (4.3).

For the opposite direction, suppose that (4.3) holds and apply the sufficient condition from Theorem 4.4 on isomorphisms \(f_{n-1}\alpha: X_{n-1} \to Y_{n-1}\) and \(\gamma: \pi^X_n \to \pi^Y_n\). It leads to a homotopy equivalence \(f: X_n \to Y_n\) such that \((f_{n-1}\alpha)p^X_n = p^Y_n f\).

\(\square\)

5. Applications

At the end of the previous section, we have introduced cohomological test (4.3) which is executed only once per stage. However, the core of the test is the computation of a homotopy self-equivalence coming from a complicated algebraic group structure. Thus, such an approach requires to employ a far-reaching algorithm, most likely [5]. A subsequent paper will be devoted to applications of this result. Here, we want to focus on the sufficient condition in Theorem 4.4 and exhaustive search methods which work for a wider family of spaces as opposed to Brown’s algorithm [2] for all spaces with finite homotopy groups.

Going carefully through the sufficient condition, one can spot two key input ingredients – computation of the set \(\text{Eq}(K(\pi^X_n, n), K(\pi^Y_n, n)) \cong \text{Iso}(\pi^X_n, \pi^Y_n)\) of all isomorphisms \(\gamma: \pi^X_n \to \pi^Y_n\) and determination of all elements of \(H^n(X_{n-1}; \pi^X_n)\) for all \(n \leq \dim(X) = \dim(Y)\). For utilizing recursive exhaustive search, both data sets are required to be finite. The first requirement is clearly met for spaces with homotopy groups of a rank at most one. If \(X\) is a finite simplicial set, the finiteness of \(H^n(X_{n-1}; \pi^X_n)\) can be easily verified via the effective homology framework. One concrete example of spaces satisfying both properties are spaces of Lusternik-Schnirelmann category \(\leq 2\) modulo the class of finite abelian groups with homotopy groups of a rank at most one (e.g. spheres). To be more precise, Lusternik-Schnirelmann category \(\leq 2\) spaces can be characterized in two equivalent ways if we restrict to only spaces of homotopy type of a connected CW complex (see [1, Section 2]). One of these definitions directly extends to a relative version of this notion:

**Definition 5.1.** A simply connected simplicial set \(X\) is said to be of Lusternik-Schnirelmann category \(\leq 2\) modulo the class of finite abelian groups \(C\) if there is a map \(\phi: |X| \to |X| \vee |X|\) such that the compositions \(r_j \circ \phi\) are isomorphisms in homology modulo the class \(C\) where the maps \(r_j: |X| \vee |X| \to |X|\) are obvious retractions the \(j\)-th component of the wedge product \(|X| \vee |X|\).

More details are in the subsequent remark.

**Remark.** Berstein [1, Remark 3.1] has shown that spaces of Lusternik-Schnirelmann category \(\leq 2\) modulo the class of finite abelian groups have property that \(\text{coker} \ h_n\) of the Hurewicz homomorphism \(h_n: \pi_n(X) \to H_n(X)\) is finite. Let us prove that it implies that \(H^n(X_{n-1}; \pi^X_n)\) is also finite. Consider the Serre long exact cohomology sequence of the fibration \(K(\pi^X_n, n) \to X_n \to X_{n-1}\) with coefficients \(\pi^X_n \otimes \mathbb{Q}\) which we suppress:

\[
\cdots \to H^{n-1}(K(\pi^X_n, n)) \to H^n(X_{n-1}) \to H^n(X_n) \to H^n(K(\pi^X_n, n)) \to H^{n+1}(X_{n-1}) \to \cdots
\]
If we prove that the epimorphism $i^*$ is also a monomorphism, we get that $H^n(X_{n-1}; \pi_n^X \otimes \mathbb{Q})$ is trivial and $H^n(X_{n-1}; \pi_n^X)$ is finite.

The fibration $X \to X_n$ has an $n$-connected fiber and hence $H_n(X) \cong H_n(X_n)$. Then the cokernel of the Hurewicz map $h_{X_n}^X : \pi_n(X_n) \to H_n(X_n)$ is isomorphic to the cokernel of the Hurewicz map $h_n$, and so it is finite. From the diagram

$$
\begin{array}{ccc}
H_n(K(\pi_n^X, n)) & \xrightarrow{i_*} & H_n(X_n) \\
\downarrow h_n^K \cong & & \downarrow h_{X_n} \\
\pi_n(K(\pi_n^X, n)) & \xrightarrow{\cong} & \pi_n(X_n)
\end{array}
$$

we get that $\text{coker } i_*$ is also finite. Applying $\text{Hom}(-, \pi_n^X \otimes \mathbb{Q})$ and Universal Coefficient Theorem we obtain the exact sequence $0 \to H^n(X_n; \pi_n^X \otimes \mathbb{Q}) \xrightarrow{i_*} H^n(K(\pi_n^X, n); \pi_n^X \otimes \mathbb{Q})$, which shows that $i^*$ is a monomorphism.

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