ORBIFOLD DECONSTRUCTION: A COMPUTATIONAL APPROACH

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Abstract. We present a general deconstruction procedure aimed at recognizing whether a given conformal model may be obtained as an orbifold of another one, and to identify the twist group and the original model in terms of some readily available characteristics. The ideas involved are illustrated on the maximal deconstruction of the Ashkin-Teller model $\mathcal{AT}_{16}$.

1. Introduction

Orbifold compactifications, i.e. the consideration of string propagation on quotients of Minkowski space by some discrete group action, have been introduced a long time ago [17, 18] as a practical mean to generate string models that could be compatible at low energies with the observed behavior of elementary particles. While propagation of pointlike particles on such singular spaces (with singularities corresponding to the fixed points of the group action) could be problematic, it was argued that these difficulties would not manifest themselves when considering one-dimensional strings. Moreover, since these models arise from free string models by the gauging of discrete symmetries, their analysis should be relatively simple and have a nice group theoretic description, in contrast with e.g. Calabi-Yau compactifications [29], which were not amenable to exact description in those times. The idea was to find the relation between the two-dimensional conformal models describing the inner degrees of freedom of a string propagating on Minkowski space and on its quotients by some group action.

From a vantage point of view, one could say that orbifolding, i.e. the gauging of discrete symmetries of a two-dimensional conformal model, is a most important construction procedure in CFT, leading to a host of new (rational) models from known ones\(^1\). At the algebraic level,

\(^1\)It has even been argued [37] that all rational conformal models could be obtained as GKO coset models or orbifolds thereof.
this amounts to the consideration of the fixed-point subalgebra $V^G$ of a Vertex Operator Algebra $V$, cf. [32, 35], under some (discrete) group $G < \text{Aut}(V)$ of automorphisms, the so-called twist group of the model: in this respect, orbifolding was an essential ingredient in the construction of the famous Moonshine module [22]. Unfortunately, the analysis of such models is in general pretty hard, due on one hand to the need to include ‘twisted sectors’ (twisted modules in VOA language), whose structure depends heavily on the precise nature of the action of the twist group, and on the other because of the difficulties associated with the so-called fixed-point resolution procedure [24]. For this reason, only special varieties of orbifolds lend themselves to a general description: toroidal ones [27], when the original conformal model consists of (compactified) free bosons, holomorphic orbifolds [14, 15, 13, 2, 3], when the original model is self-dual, and finally permutation orbifolds [34, 10, 5, 7, 9], when the twist group acts by permuting identical copies of one and the same conformal model. For more generic cases only some basic properties of the orbifold construction are known, usually insufficient to identify unambiguously the resulting model, so ad hoc techniques are required to fill in the details [16, 30].

The aim of the present note is to go in the opposite direction: given a conformal model, recognize whether it is an orbifold of another one, and if so, identify this original model and the corresponding twist group [8]. Of course, the effectiveness of such a deconstruction procedure depends heavily on the amount of knowledge needed to characterize the different models. As we shall see, very limited information is needed for deconstruction: the fusion rules and chiral characters of the primaries [21] usually suffice. That such a procedure could exist should not come as a big surprise, for simple current extensions [25, 6] are nothing but the deconstructions of abelian orbifolds, but the exact details of their generalization to a non-commutative setting are far from being obvious.

In the next section we’ll introduce our basic objects, twisters and their twist classes, and discuss their most important properties. Then we move on to the study of blocks and their characteristics. Section 4 explains the relation of these concepts to orbifolding in general, while Section 5 describes the deconstruction algorithm. An explicit example of deconstruction is presented in Section 6, illustrating some subtle points of the process. Finally, we give an outlook on questions whose study might be worth pursuing.
It should be stressed that our approach in this note is a computational one, focusing on the algorithmic problems related to the deconstruction procedure. Nowhere in the text shall we provide formal proofs of our assertions. For most part such arguments might be readily supplied (we give hints for a few of them), but there are some statements whose actual proof would require much more elaboration, while their truth is evidenced by the fact that the deconstruction algorithm presented in Section 5 leads to the expected results whenever those can be obtained by alternate means, e.g. for holomorphic or permutation orbifolds, while providing at the same time a natural non-commutative generalization of (integer spin) simple current extensions, shedding new light on some delicate aspects of the latter. We have no doubt that the whole subject could be described elegantly in more abstract terms along the lines of [33, 23], but the relevant techniques seem (at least to us) less amenable to practical computations.

2. Twisters and twist classes

Let’s consider a rational unitary conformal model [21]. We’ll denote by $d_p$ and $h_p$ the quantum dimension and conformal weight of a primary $p$, by $\chi_p(\tau)$ its chiral character, and by $N(p)$ the associated fusion matrix, whose matrix elements are given by the fusion rules

\[ [N(p)]_r^q = N_{rp}^q \]  

(2.1)

Note that, since

\[ N(p) N(q) = \sum_r N_{pq}^r N(r) \]  

(2.2)

the fusion matrices generate a commutative matrix algebra over $C$ (the Verlinde algebra), whose irreducible representations, all of dimension 1, are in one-to-one correspondence, according to Verlinde’s formula [39], with the primaries: to each primary $q$ corresponds an irrep $\rho_q$ which assigns to the generator $N(p)$ the complex value

\[ \rho_q(p) = \frac{S_{pq}}{S_{0q}} \]  

(2.3)

where $S$ denotes the modular $S$ matrix of the model, and $0$ labels its vacuum. In particular, one has $d_p = \rho_0(p)$ and

\[ \sum_r N_{pq}^r \rho_w(r) = \rho_w(p) \rho_w(q) \]  

(2.4)

We call a set $g$ of primaries a twister if it contains the vacuum $0$, if all of its elements have integer conformal weight and quantum dimension
(\h_\alpha, \d_\alpha \in \mathbb{Z} \text{ for all } \alpha \in \mathfrak{g})$, and if \( \mathfrak{g} \) is closed under fusion, i.e. \( \alpha, \beta \in \mathfrak{g} \) and \( N_{\alpha\beta}^p > 0 \) implies \( p \in \mathfrak{g} \); in other words, the fusion matrices \( N(\alpha) \) for \( \alpha \in \mathfrak{g} \) generate a subring \( \hat{\mathfrak{g}} \) of the fusion ring. Note that, taking into account the positivity of quantum dimensions, this last requirement amounts to the equality
\[
\sum_{\gamma \in \mathfrak{g}} N_{\alpha\beta}^\gamma \d_\gamma = \d_\alpha \d_\beta \tag{2.5}
\]
The spread of the twister \( \mathfrak{g} \) is the (positive) rational integer
\[
|\mathfrak{g}| = \sum_{\alpha \in \mathfrak{g}} \d_\alpha^2 \tag{2.6}
\]
A twister is abelian if \( \d_\alpha = 1 \) for all \( \alpha \in \mathfrak{g} \). In particular, the vacuum primary \( 0 \) forms in itself an abelian twister, the trivial twister. Abelian twisters are, in the traditional language of CFT, nothing but groups of integer spin simple currents.

A twist class \( \mathcal{C} \) of a twister \( \mathfrak{g} \) is a (maximal) set of primaries such that, for \( p \in \mathcal{C} \), all the representations \( \rho_p \) of the Verlinde algebra restrict to one and the same representation \( \rho_{\mathcal{C}} \) of the subalgebra \( \hat{\mathfrak{g}} \) generated by the twister. We shall denote by \( \alpha(\mathcal{C}) \) the value assigned to \( \mathfrak{N}(\alpha) \) by this representation \( \rho_{\mathcal{C}} \), so that \( p \in \mathcal{C} \) precisely when \( S_{\alpha p} = \alpha(\mathcal{C}) S_{0p} \) for all \( \alpha \in \mathfrak{g} \). It is immediate that twist classes partition the set of primaries, and that their number equals the cardinality \( |\mathfrak{g}| \) of the twister.

The extent of a twist class \( \mathcal{C} \) is the quantity
\[
|\mathcal{C}| \equiv \frac{1}{\sum_{p \in \mathcal{C}} S_{0p}^2} \tag{2.7}
\]
which may be shown to be a (positive) rational integer dividing the spread \( |\mathfrak{g}| = \sum_{\alpha \in \mathfrak{g}} \d_\alpha^2 \) of the twister. Using Eq.(2.4), one may derive the orthogonality relations (with the bar denoting complex conjugation)
\[
\sum_{\alpha \in \mathfrak{g}} \alpha(\mathcal{C}_1) \overline{\alpha(\mathcal{C}_2)} = |\mathcal{C}_1| \delta_{\mathcal{C}_1 \mathcal{C}_2} \tag{2.8}
\]
and
\[
\sum_{\mathcal{C}} \frac{\alpha(\mathcal{C}) \overline{\alpha(\mathcal{C})}}{|\mathcal{C}|} = \delta_{\alpha \beta} \tag{2.9}
\]
for \( \alpha, \beta \in \mathfrak{g} \), where the last sum runs over all twist classes. It follows from Eq.(2.8) that
\[
\sum_{\alpha \in \mathfrak{g}} |\alpha(\mathcal{C})|^2 = |\mathcal{C}| \tag{2.10}
\]

The twist class containing the vacuum \( 0 \) is the trivial class \( 1 \): note that \( \alpha(1) = \d_\alpha \) by the above definition. Using the modular relation and Eq.(2.4), one may show that all elements of the twister belong to the
trivial class, i.e. $g \subseteq 1$. Obviously, the extent of the trivial class equals the spread of the twister, while its size is given by

$$|1| = \frac{1}{|g|} \sum_{\alpha \in g} d_{\alpha} \text{Tr}(N(\alpha)) \tag{2.11}$$

A most important property of the trivial twist class that follows ultimately from Eq. (2.8) is the product rule: if $p$ belongs to the trivial twist class and $N_{pq}^r > 0$, then $q$ and $r$ belong necessarily to the same twist class.

For an integer $n$ and a twist class $C$, there is a unique twist class $C^n$, the $n$-th power of $C$, for which

$$\alpha(C^n) = [C] \sum_{p,q \in C} N_{\alpha p}^q S_{0p} S_{0q} e^{2\pi i (h_p - h_q)} \tag{2.12}$$

for all $\alpha \in g$. The order of a twist class $C$ is the smallest positive integer $n$ such that $C^n$ equals the trivial class, i.e. $\alpha(C^n) = \alpha(1) = d_{\alpha}$ for all $\alpha \in g$. We note that the order of a twist class may be shown to always divide its extent.

Since a twister consists of simple objects of a modular tensor category with integer dimension, trivial twists ($h_\alpha \in \mathbb{Z}$ for $\alpha \in g$) and is closed under fusion, it follows from results on Tannakian categories [12] that there exists a finite group $G$ of order $|G| = \sum_{\alpha \in g} d_{\alpha}^2 = [g]$ whose representation ring coincides with the fusion subring $\hat{g}$ generated by the twister. In particular, to each element $\alpha \in g$ there corresponds an irreducible character $\alpha^\flat \in \text{Irr}(G)$ of degree $\alpha^\flat(1) = d_{\alpha}$, and these satisfy the multiplication rule

$$\alpha^\flat \beta^\flat = \sum_{\gamma \in g} N_{\alpha \beta}^\gamma \gamma^\flat \tag{2.13}$$

from which one can infer [36] the values of the characters $\alpha^\flat \in \text{Irr}(G)$ on the different conjugacy classes of $G$: to each twist class $C$ corresponds a conjugacy class $C^\flat$ of $G$, with the trivial twist class $1$ corresponding to the trivial conjugacy class containing solely the identity element, and this correspondence is such that

$$\alpha(C) = \alpha^\flat(C^\flat) \tag{2.14}$$

This implies, as a consequence of the second orthogonality relations for group characters, that the size of the conjugacy class $C^\flat$ equals

$$|C^\flat| = \frac{[g]}{|C|} \tag{2.15}$$
While the knowledge of the representation ring does determine many properties of the twist group, in particular its character table and normal structure, it does not determine its isomorphism type uniquely. A famous example of this phenomenon is that of the groups $\mathbb{D}_8$ (the dihedral group of order 8, i.e. the symmetry group of a square) and the group $Q$ of unit quaternions, which have identical representation rings but are nevertheless not isomorphic [36]. But in our case one can pin down uniquely the twist group associated to the twister by exploiting the underlying braided monoidal structure, which makes the subring $\hat{\mathfrak{g}}$ a $\lambda$-ring [1], and this extra structure should match that of the representation ring of $G$. Indeed, Eq.(2.12), which may be deduced along the lines of [4, 28] by considering traces of suitable braidings, allows one to define the Adams operation $\Psi^n$ on $\text{Irr}(G)$ via the rule

$$$(\Psi^n \alpha)(\gamma) = \alpha(\gamma^n)$$$(2.16)

and this extra information is usually sufficient to determine $G$ up to isomorphism [31, 36]. Notice that Eqs.(2.12) and (2.16) imply that the (higher) Frobenius-Schur indicators [4, 38] of the primaries $\alpha \in \mathfrak{g}$ agree with those of the corresponding characters $\alpha^* \in \text{Irr}(G)$.

To sum up, to any twister $\mathfrak{g}$ is associated a group $G$ whose representation ring is isomorphic with $\hat{\mathfrak{g}}$ as a $\lambda$-ring, and in particular the elements of $\mathfrak{g}$ correspond to irreps of $G$, while the twist classes of $\mathfrak{g}$ to conjugacy classes of $G$. As we shall explain in Section 4, each $G$-orbifold contains a set of primaries that form a twister such that the associated group is isomorphic with $G$. Orbifold deconstruction is the process of identifying a conformal model whose $G$-orbifold is the conformal model we started with from information related solely to the orbifold and the corresponding twister.

3. Blocks

As we have seen in the previous section, a twister partitions the primaries of a conformal model into twist classes which are in one-to-one correspondence with the conjugacy classes of the twist group. It turns out that a finer partition plays a major role in orbifold deconstruction, the partition of the primaries into blocks.

The primaries $p$ and $q$ belong to the same block with respect to the twister $\mathfrak{g}$ if there exists some $\alpha \in \mathfrak{g}$ such that $N^p_{\alpha q} > 0$. Put differently, this
means that $C_{pq} > 0$ for the non-negative integer matrix

$$G = \sum_{\alpha \in g} d_{\alpha} N(\alpha)$$

(3.1)

In particular, the elements of the twister themselves form a block, the vacuum block $b_0$. It is straightforward that the blocks partition the set of primaries, and it follows from the product rule and the containment $g \subseteq 1$ that two primaries that belong to the same block also belong to the same twist class. Consequently, each twist class is actually a disjoint union of blocks, with the vacuum block contained in the trivial class.

Note that, according to the above definition, the fusion matrices $N(\alpha)$ for $\alpha \in g$ can be simultaneously brought into a block-diagonal form after a suitable rearrangement of the primaries:

$$N(\alpha) = \bigoplus_b N_b(\alpha)$$

(3.2)

where $b$ runs over the blocks of $g$, and in particular $G = \bigoplus_b G_b$ with each $G_b$ a positive integer matrix. This implies that the blocks correspond to integral representations of $\hat{g}$, since the corresponding fusion matrices $N_b(\alpha)$ have non-negative integer entries. On the other hand, we know that the irreducible complex representations of $\hat{g}$ are among the $\rho_C$, hence the integral representation corresponding to a block $b$ decomposes into a direct sum of the latter, with the multiplicity of $\rho_C$ given by the overlap

$$\langle b, C \rangle = \frac{1}{|g|} \sum_{\alpha \in g} \frac{\alpha(C)}{\text{Tr} N_b(\alpha)} = \frac{1}{|g|} \sum_{p \in C} \sum_{q \in b} |S_{pq}|^2$$

(3.3)

of the block $b$ and the twist class $C$, which is always a non-negative rational integer according to the above. As a consequence of the symmetry of the matrix $S$, the overlap satisfies the reciprocity relation

$$\sum_{b \subseteq C_1} \langle b, C_2 \rangle = \sum_{b \subseteq C_2} \langle b, C_1 \rangle$$

(3.4)

for any two twist classes $C_1$ and $C_2$.

A consequence of Eq.(2.10) is that the overlap of the vacuum block $b_0$ with any twist class is 1. A similar result follows from Perron’s theorem [26] applied to the positive matrix $G_b$: the overlap of any block $b$ with the trivial twist class $1$ is equal to 1

$$\langle b, 1 \rangle = 1$$

(3.5)
Exploiting the unitarity of $S$, summing Eq.(3.3) over all blocks gives

$$|C| = \sum_b \langle b, C \rangle$$

(3.6)

for the number of primaries in the twist class $C$, while summing over twist classes leads to

$$|b| = \sum_C \langle b, C \rangle$$

(3.7)

for the cardinality of the block $b$.

Combining Eq.(3.5) with Eq.(3.4), one gets that the number of different blocks contained in a given twist class can be expressed as

$$\# \{ b \mid b \subseteq C \} = \sum_{b \subseteq C} \langle b, 1 \rangle = \sum_{b \subseteq \mathbf{1}} \langle b, C \rangle$$

(3.8)

i.e. it equals the sum of overlaps of the twist class with all blocks contained in the trivial class. Summing Eq.(3.8) over all twist classes and taking into account Eq.(3.7) gives

$$\sum_C \# \{ b \mid b \subseteq C \} = \sum_C \sum_{b \subseteq \mathbf{1}} \langle b, C \rangle = \sum_{b \subseteq \mathbf{1}} |b| = |\mathbf{1}|$$

(3.9)

i.e. the total number of blocks with respect to $G$ is equal to the size (cardinality) of the trivial twist class.

Using Perron’s theorem [26] for the matrix $G_b$, combined with some elementary Galois theory, one can show that the ratio of quantum dimensions inside a block $b$ are always rational numbers, hence there exists a largest algebraic integer $D_b$ such that $d_p \in D_b \mathbb{Z}$ for all $p \in b$. As a consequence, the ratio

$$\mu_b = \frac{1}{D_b^2} \sum_{p \in b} d_p^2$$

(3.10)

is a rational integer, which may be shown to divide the extent of the twist class containing $b$. We note that $D_b$ equals in most cases the minimum $\min\{d_p \mid p \in b\}$ of the quantum dimensions of primaries from $b$.

The above results have a nice representation theoretic interpretation. To each block $b$ is associated a subgroup (more precisely, a conjugacy class of subgroups) of the twist group $G$ - the inertia group $I_b$ of the block - and a 2-cocycle $\vartheta_b \in \mathbb{Z}^2(I_b)$. Denoting by $e_b$ the multiplicative order of the cohomology class of $\vartheta_b$, there is a one-to-one correspondence $p \leftrightarrow \xi_p$ between primaries $p \in b$ and (projective) irreducible characters $\xi_p \in \text{Irr}(I_b \mid \vartheta_b)$ of the inertia subgroup $I_b$ with cocycle $\vartheta_b$, such that

$$\xi_p(1) = e_b \frac{d_p}{D_b}$$

(3.11)
and

\[ \sum_{q \in b} N_{\alpha p}^q \xi_q = \alpha_b^* \xi_p \]  

(3.12)

for all \( \alpha \in g \), where \( \alpha_b^* \) denotes the restriction to \( I_b \) of the irreducible character \( \alpha^* \) of the twist group \( G \) associated to the primary \( \alpha \in g \). It follows from (3.11) and Burnside’s theorem [31, 36] that the order of the inertia subgroup is given by

\[ |I_b| = \sum_{p \in b} \xi_p(1)^2 = e_b^2 \mu_b \]  

(3.13)

while the overlap \( \langle b, c \rangle \) counts the number of \( \vartheta_b \)-regular conjugacy classes\(^2\) of \( I_b \) meeting the conjugacy class \( C^c \) of \( G \).

To each block \( b \) is associated the non-negative matrix

\[ N(b) = \frac{1}{e_b \mu_b} \sum_{p \in b} d_p N(p) \]  

(3.14)

These matrices may be shown to form a ring, i.e. the product of any two of them may be expressed as a sum

\[ N(a) N(b) = \sum_c \eta_{ab}^c N(c) \]  

(3.15)

with integer **block-fusion coefficients** given by

\[ \eta_{ab}^c = \frac{\epsilon_c}{e_a \mu_a e_b \mu_b} \sum_{p \in a} \sum_{q \in b} \sum_{r \in c} N_{pq}^r d_p \frac{d_q}{D_b} \frac{d_r}{D_c} \]  

(3.16)

If the block \( b \) is contained in a twist class \( C \) of order \( n \), i.e. such that \( \alpha(C^n) = \alpha(1) = d_\alpha \) for all \( \alpha \in g \), then one may show, using some elementary character theory, that the conformal weights of all primaries from \( b \) differ from each other by integer multiples of \( \frac{1}{n} \). In other words, denoting by \( h_b = \min \{ h_p \mid p \in b \} \) the minimal conformal weight inside the block \( b \), one has \( n(h_p - h_b) \in \mathbb{Z}_+ \) for all \( p \in b \). As a consequence, the character

\[ \chi_b(\tau) = \frac{\Phi_b}{D_b} \sum_{p \in b} d_p \chi_p(\tau) \]  

(3.17)

of the block \( b \) will have a Puiseux expansion

\[ \chi_b(\tau) = q^{h_b - \frac{1}{n}} \sum_{k=0}^{\infty} a_k q^{\frac{k}{n}} \]  

(3.18)

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\(^2\)Recall that an element \( x \in I_b \) belongs to a \( \vartheta_b \)-regular conjugacy class if \( \vartheta_b(x, y) = \vartheta_b(y, x) \) whenever \( y \in I_b \) commutes with \( x \), cf. [31, 36].
in terms of the variable \( q = \exp(2\pi i \tau) \), with non-negative integer coefficients \( a_k \in \mathbb{Z}_+ \). In particular, the conformal weights inside blocks contained in the trivial twist class \( 1 \) may only differ by integers, and the corresponding expansions will be in integral powers of \( q \) (apart from an overall factor coming from the conformal weight). Let’s note that, using the matrix introduced in Eq.\((3.1)\), one has the following expression for the modulus squared of the character (where, as usual, the bar denotes complex conjugation)

\[
|\chi_b(\tau)|^2 = \frac{\sum_{p,q \in b} g_{pq} \chi_p(\tau) \overline{\chi_q(\tau)}}{\| g \|}
\]

\((3.19)\)

4. Relation to orbifolding

Consider a rational conformal model with associated Vertex Operator Algebra \( V \), cf. [22, 32, 35], and some (finite) group \( G < \text{Aut}(V) \) of automorphisms. The \( G \)-orbifold has associated VOA \( V^G \), the fixed-point subalgebra of \( V \). For well behaved \( V \) (rational, \( C_2 \)-cofinite, etc.) the representation theory of \( V^G \) may be reconstructed from the knowledge of the \( g \)-twisted modules of \( V \) for \( g \in G \), cf. [19, 20]. More precisely, there is a natural action of \( G \) on the set of all \( G \)-twisted modules under which an element \( h \in G \) sends a \( g \)-twisted module to a \( hgh^{-1} \)-twisted module, and this leads to a partition of the set of twisted modules into sectors labeled by the conjugacy classes of \( G \), with each such sector being itself a union of \( G \)-orbits. Since the twisted modules inside a \( G \)-orbit are related by an automorphism of \( V \), they have pretty similar properties while still differing from each other, e.g. their trace functions coincide.

The stabilizer \( G_M = \{ h \in G \mid h(M) \cong M \} \) of a given \( g \)-twisted module \( M \) consists of those elements \( h \in G \) for which \( h(M) \) is isomorphic to \( M \). Clearly, \( G_M \) is a subgroup of the centralizer \( C_G(g) \), and the stabilizers of different modules belonging to the same \( G \)-orbit are conjugate subgroups of \( G \). By the above, there is an action of the stabilizer \( G_M \) on \( M \), but it should be stressed that this is usually not a linear representation, but only a projective one, with an associated 2-cocycle \( \theta_M \in Z^2(G_M, \mathbb{C}) \).

In particular, the untwisted sector corresponding to the trivial conjugacy class of \( G \) always contains a \( G \)-orbit of length 1 that consists solely of the vacuum. Its stabilizer subgroup is the whole of \( G \), represented linearly, hence the vacuum decomposes into sectors corresponding to the irreducible representations of \( G \). Each such sector will correspond to a primary field of the orbifold having integral conformal weight and
quantum dimension (the later being equal to the dimension of the corresponding representation of \(G\)), and with fusion rules corresponding to tensor products of irreps of the twist group \(G\). This means that these primaries of the orbifold will form a twister \(\hat{g}\), with associated subring \(\hat{\mathbb{g}}\) coinciding with the representation ring of the twist group. Exploiting Eq.(2.16), the twister does even determine the representation ring as a \(\lambda\)-ring, which is essential for the identification of the twist group \(G\) from the above data.

As seen in Section 2, twist classes of \(g\) correspond to conjugacy classes of \(G\), which makes possible to associate to each twist class \(C\) the set of all \(g\)-twisted modules of \(V\) with \(g \in \mathcal{C}^*\). In particular, to the trivial twist class \(1\) will correspond the untwisted sector, i.e. the set of all ordinary \(V\) modules. As explained above, these sets of twisted modules are organized into orbits under the outer action of \(G\), and to each block \(b \subseteq C\) corresponds such an orbit of the twist group \(G\), with the stabilizer \(G_M\) of any twisted module \(M\) belonging to it isomorphic to the inertia subgroup \(I_b\) of \(b\) (note that the stabilizers of different modules from the same orbit are conjugate, hence isomorphic subgroups of \(G\), and the 2-cocycles \(\vartheta_M\) and \(\vartheta_b\) belonging to the same cohomology class. Finally, the primaries \(p \in b\) are in one-to-one correspondence with the irreducible projective representations \(\xi_p\) (with cocycle \(\vartheta_b\)) of the inertia subgroup. We note that the above argument leads to the expression

\[
\sum_{b \subseteq C} \left\lfloor \frac{|g|}{e_b^2 \mu_b} \right\rfloor \tag{4.1}
\]

for the number of different \(g\)-twisted modules with \(g \in \mathcal{C}^*\), and in particular

\[
\sum_{b \subseteq 1} \left\lfloor \frac{|g|}{e_b^2 \mu_b} \right\rfloor \tag{4.2}
\]

for the number of different \(V\) modules.

As noticed before, the twisted modules belonging to the same \(G\)-orbit, being related by an automorphism of \(V\), have very similar properties. In particular, the (quantum) dimension \(d_M\) of any twisted module \(M\) belonging to the orbit corresponding to the block \(b\) equals the dimension

\[
d_b = \frac{e_b \mu_b D_b}{|g|} \geq 1 \tag{4.3}
\]

of the block, its Frobenius-Schur indicator \([4]\) is given by

\[
u_b = e_b \sum_{p \in b} \frac{d_p}{D_b} \sum_{q, r \in 1} N_{pq} \tau_{pq} S_{qr} \tau_{qr} e^{4 \pi i (b_r - b_q)} \tag{4.4}\]
while its trace function equals the block’s character

\[ Z_M(\tau) = \text{Tr}_M \left\{ e^{2\pi i \tau (L_0 - c/24)} \right\} = \chi_b(\tau) \] (4.5)

and more generally, one has

\[ \text{Tr}_M \left\{ h e^{2\pi i \tau (L_0 - c/24)} \right\} = \sum_{p \in b} \xi_p(h) \chi_p(\tau) \] (4.6)

for any \( h \in G_M \).

Taking into account multiplicities, we get from Eqs. (4.5) and (3.19) the sum rules

\[ \sum_M \chi_d M(\tau) = \sum_{p \in C} d_p \chi_p(\tau) \] (4.7)

\[ \sum_M |Z_M(\tau)|^2 = \sum_{p, q \in C} G_{pq} \chi_p(\tau) \chi_q(\tau) \] (4.8)

where the sum on the left hand side runs over all \( g \)-twisted modules \( M \) with \( g \in C' \). In particular, the (diagonal) modular invariant partition function of the deconstructed model reads

\[ Z_d(\tau, \overline{\tau}) = \sum_{p, q \in 1} G_{pq} \chi_p(\tau) \chi_q(\tau) \] (4.9)

5. The deconstruction algorithm

Armed with the above, we are now ready to describe the deconstruction procedure in detail. We start from a unitary conformal model for which we know the fusion rules \( N_{r pq} \), conformal weights \( h_p \), quantum dimensions \( d_p \) and chiral characters \( \chi_p(\tau) \) of the primary fields, and we wish to identify it as a non-trivial orbifold of some other conformal model.

The first step is to determine the (non-trivial) twisters of the model. Each twister leads, according to the above, to a deconstruction with a different twist group\(^3\). Of special interest are maximal deconstructions for which the twister is maximal, i.e. not contained in any other twister, for these lead to primitive models, i.e. models that cannot be obtained as a non-trivial orbifold of some other model. Indeed, if a twister \( g_1 \) is contained in a twister \( g_2 \), then the twist group \( G_1 \) corresponding to \( g_1 \) will be a normal subgroup of the twist group \( G_2 \) corresponding to \( g_2 \), and the deconstruction with respect to \( g_1 \) will result in a \( G_2/G_1 \)-orbifold of the deconstruction with respect to \( g_2 \).

\(^3\)Although both the twist groups and the deconstructed models might be isomorphic, but the twist group action would be different.
Once we have chosen a twister $g$, the next step is to determine the corresponding partition of the primaries into twist classes. The knowledge of the twist classes allows to determine at once the character table of the twist group, and by computing powers of twist classes according to Eq.(2.12), one can even determine the power maps of the twist group, thus making possible the precise identification (up to isomorphism) of the latter.

Once the twist classes are known, the next step is to determine the partition of the primaries into blocks. Once we know the blocks, it is straightforward to compute some of their characteristic quantities, like the weights $h_b$, dimensions $D_b$ and characters $\chi_b(\tau)$. On the other hand, the precise determination of the inertia groups $I_b$ and 2-cocycles $\vartheta_b$ is much more involved. Fortunately, for most of the deconstruction process one only needs to know the multiplicative order $e_b$ of $\vartheta_b$, and this can be determined in many cases without the actual knowledge of $\vartheta_b$ itself, by using some simple divisibility properties. In particular, the product $e_b^2 \mu_b$, which is equal to the order of the inertia group $I_b$, always divides the extent $|c|$ of the twist class containing the block $b$, being at the same time a multiple of its order, restricting to a large extent the possible values of $e_b$. In case this is still not enough to determine $e_b$ uniquely, one can restrict further the possible values using Eq.(4.4).

Actually, most of the above considerations are superfluous if one is only interested in the deconstructed model proper, for in that case it is enough to perform the above computations for the blocks contained in the trivial class $1$. To each block $b \subseteq 1$ there will correspond

$$[G:I_b] = \frac{|g|}{e_b^2 \mu_b}$$

different primaries of the deconstructed model, each of conformal weight $h_b$, quantum dimension $d_b$ and character $\chi_b(\tau)$. In many cases this is already enough to identify uniquely the deconstructed model (remember that the central charge does not change during orbifolding/deconstruction). If there is still some ambiguity left, one can use the block-fusion coefficients Eq.(3.16), which characterize to some extent the fusion rules of the deconstructed model, but for one important difference: they do not describe the fusion of the individual primaries belonging to the corresponding blocks, but only that of the direct sum of the modules contained in the orbits corresponding to the relevant blocks.
6. A worked-out example: the Ashkin-Teller model $\mathcal{AT}_{16}$

Ashkin-Teller models are unitary conformal models of central charge $c=1$ obtained by coupling two Ising models through their energy densities. They may be shown to be equivalent to $\mathbb{Z}_2$-orbifolds (with respect to space reflection) of the compactified boson [27] at suitable radii determined by the coupling. At specific values of the coupling corresponding to compactification radii of the form $R_{orb} = \sqrt{N/2}$ with integer $N$, the resulting $\mathcal{AT}_N$ models are rational, with $N+7$ primary fields [11]. In particular, for $N = 16$ we get a total of 23 primaries, whose most important properties are listed in Table 1 (note the unusual labeling $u_+$ of the vacuum).

| label $u_+$ | conformal weight $0$ | dimension $1$ | character $\frac{1}{24}\theta_3(32\tau) + \sqrt{\frac{N}{2^N}}(\tau)$ |
|-------------|----------------------|---------------|------------------------------------------------------------------|
| label $u_-$ | conformal weight $1$ | dimension $1$ | character $\frac{1}{24}\theta_3(32\tau) - \sqrt{\frac{N}{2^N}}(\tau)$ |
| label $\phi_\pm$ | conformal weight $4$ | dimension $1$ | character $\frac{1}{24}\theta_2(32\tau)$ |
| label $\sigma_\pm$ | conformal weight $\frac{1}{16}$ | dimension $4$ | character $\frac{1}{2}\left\{\sqrt{\frac{N}{2^N}}(\tau) + \sqrt{\frac{N}{2^N}}(\tau)\right\}$ |
| label $\tau_\pm$ | conformal weight $\frac{9}{16}$ | dimension $4$ | character $\frac{1}{2}\left\{\sqrt{\frac{N}{2^N}}(\tau) - \sqrt{\frac{N}{2^N}}(\tau)\right\}$ |
| label $\chi_k$ | conformal weight $\frac{k^2}{16}$ | dimension $2$ | character $\frac{1}{24}\theta_3\left[\frac{k}{6}\right](32\tau)$ for $1 \leq k \leq 15$ |

Table 1. Primaries of the Ashkin-Teller model $\mathcal{AT}_{16}$

Here

$$\theta_{[a]\,_{[b]}^{[\alpha]}}(\tau) = \sum_{n \in \mathbb{Z}} e^{i \pi \tau (n-a)^2} e^{-2 \pi i a n} \quad (6.1)$$

for $a, b \in \mathbb{Q}$,

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n) \quad (6.2)$$

is Dedekind’s eta function (with $q=e^{2i\pi \tau}$), while

$$\theta_2 = \theta_{[1]_{[0]}^{[2]}}(\tau) = 2q^{1/8} \prod_{n=1}^{\infty} (1-q^n)(1+q^n)^2$$
$$\theta_3 = \theta_{[0]_{[0]}^{[2]}}(\tau) = \prod_{n=1}^{\infty} (1-q^n)^2 \left(1+q^{n-1/2}\right)^2$$
$$\theta_4 = \theta_{[0]_{[1]}^{[2]}}(\tau) = \prod_{n=1}^{\infty} (1-q^n) \left(1-q^{n-1/2}\right)^2$$

are the classical theta functions of Jacobi.
Besides the integer spin simple currents $u_\pm$ and $\phi_\pm$, the only primary of integer conformal weight and dimension is $\chi_8$. Because $\chi_8$ is fixed by all of these simple currents, the set $g = \{u_+, u_-, \phi_+, \phi_- , \chi_8\}$ is closed under fusion, hence it forms a twister of spread $8$, the unique maximal twister of $AT_{16}$. In the sequel, we shall investigate the maximal deconstruction of $AT_{16}$ with respect to $g$.

There are 5 different twist classes for the maximal twister $g$, whose properties are summarized in Table 2.

| class | fields | extent | order | size |
|-------|--------|--------|-------|------|
| 1     | $u_+, u_-, \phi_+, \phi_- , \chi_8, \chi_{12}$ | 8      | 1     | 7    |
| $C_e$ | $\sigma_+, \tau_+$ | 4      | 2     | 2    |
| $C_o$ | $\sigma_-, \tau_-$ | 4      | 2     | 2    |
| $C_k$ | $\chi_k$ with $k$ odd | 4      | 4     | 8    |
| $C_e$ | $\chi_k$ with $k \equiv 2 \pmod{4}$ | 8      | 2     | 4    |

Table 2. Twist classes of $g$.

From the knowledge of the twist classes we get the following character table for the twist group:

| $u_+^n$ | $u_-^n$ | $\phi_+^n$ | $\phi_-^n$ | $\chi_8^n$ |
|---------|---------|------------|------------|------------|
| $1^n$   | 1       | 1          | 1          | 1          |
| $C_e^n$ | 1       | -1         | 1          | -1         |
| $C_o^n$ | 1       | 1          | -1         | 1          |
| $C_k^n$ | 1       | -1         | -1         | 0          |

There are two non-isomorphic groups with this character table [36], hence two candidates for the twist group: the dihedral group $D_8$ and the quaternion group $Q$, but the latter possibility has 3 different conjugacy classes of elements of order 4, while Eq.(2.12) allows only one such class. We conclude that the twist group is $D_8$.

There are 7 blocks with respect to $g$, collected in Table 3. Notice that $e_6 = 2$ for the second block, providing an explicit example where non-trivial projective representations arise. Thanks to the simple structure of the twist group, it is possible in this case to identify the inertia subgroups of all the blocks from the above information without much difficulty, but we won’t actually need this information.
Table 3. Blocks of $\mathbb{A}T_{16}$ with respect to the maximal twister $g$.

Only the first two blocks are of interest for identifying the deconstructed model, since these are the ones contained in the trivial twist class. Since the inertia groups of these blocks both have order $e_b^u \mu_b = 8$, it follows that the deconstructed model has two different primaries, with respective characters

$$q^{-\frac{1}{2}} \left( 1 + 3q + 4q^2 + 7q^3 + 13q^4 + \cdots \right)$$

and

$$q^{\frac{1}{2}} \left( 2 + 2q + 6q^2 + 8q^3 + 14q^4 + \cdots \right)$$

This already allows to identify the deconstructed model with the $SU(2)$ WZNW model at level 1 (or, what is the same, the free boson compactified on a circle of radius $R = \frac{1}{\sqrt{2}}$) [27].

We conclude that the Ashkin-Teller model $\mathbb{A}T_{16}$ is a $D_8$-orbifold of $SU(2)_1$. Of course, this is a well-known result [11], but we arrived at this conclusion from a totally new approach. We note that, besides identifying unambiguously both the deconstructed model and the twist group (up to isomorphism), we also get non-trivial information on the structure of some of the twisted modules of $SU(2)_1$.

7. Summary and outlook

Orbifold deconstruction, i.e. the inverse process of orbifolding is, as we tried to demonstrate, a well defined effective procedure to recover from some simple data characterizing an orbifold the relevant twist
group and original conformal model. Since in this case we know the result of orbifolding right from the start, this could be particularly helpful in the study of general properties of orbifolds. In particular, the deconstruction procedure provides valuable information on the structure of twisted modules, e.g. their trace functions and tensor products. Since the structure of $g$-twisted modules does only depend on the conjugacy class of $g$ in the whole automorphism group of the deconstructed model, this information pertains to the structure of any orbifold of the deconstructed model whose twist group contains elements conjugate to some element of $G$.

Another interesting aspect of orbifold deconstruction is related to modular invariants. Indeed, the (diagonal) modular invariant of the deconstructed model, cf. Eq.(4.9), is a non-trivial modular invariant of extension type $[21]$ of the orbifold, and it seems likely that many such invariants can be related to suitable deconstructions.

Finally, orbifold deconstruction might prove useful in attempts to classify rational conformal models. For one thing, the classification problem can be reduced to that of primitive models, i.e. the ones that don’t have nontrivial twistings, since all other models, being suitable orbifolds of the primitive ones, can be classified by group theoretic means. Besides this, primitive models can be grouped together if they have identical orbifolds with respect to suitable twist groups, i.e. if they arise from maximal deconstructions of one and the same conformal model, since this points to some close relation between them. In any case, pursuing this line of thought seems to be a worthy undertaking.

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