Relationship between the degree exponent and its associated timeseries’ hurst exponent of a static scale-free network — Under the viewpoint of mutual representation between timeseries and network graph

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Abstract. During the past decade, bridging the gap between time series and complex networks, which aims to investigate the usage of network features to depict geometry of different type of timeseries becomes a hot topic. In this brief paper, the internal relationship between a typical static scale-free networks and its associated nonlinear timeseries (under the viewpoint of duality analysis of networks and time series) was investigated. A ubiquitous process to discuss such a problem are introduced. It is found that, the scale-free degree exponent has linear relationship with its associated timeseries’ hurst exponent. Research in this direction will provide new methods and powerful tools for analysis of the causal relationships between complex structures and temporal behaviors of a complex dynamical system. Also, it is useful for further investigation on structure and function analysis of complex networks via classical timeseries analysis methods.

1. Introduction
In recent several years, several types of new mapping methods had been made to bridge the gap between time series and complex networks, which aims to study the usage of network features to depict geometric features belongs to different types of timeseries. Some important studies have shown that, different types of timeseries are corresponding to different types of topological structures. What relationship between these topology characters (ie.,the network characteristic quantities) and the original time series, is still unclear. How to apply the network analysis methods to depict the characteristics of the original timeseries’ statistics (vice versa the inverse process of such a research), therefore, become extremely urgent nowadays. After some recent published literatures showed that, different types of time series is corresponding to different types of topological structure, this theme quickly becomes a hot point in nonlinear science and network science research field [1]-[7].

What relationship between these topology characters (ie.,the network characteristic quantities) and the original time series, is still unclear. How to apply the network analysis method to depict the characteristics of the original time series’ statistics (vice versa the inverse process of such a research) therefore become extremely urgent nowadays.
Under the viewpoint of duality analysis of networks and timeseries, we will addressed on the internal relationship between a typical static scale-free network and its associated nonlinear timeseries in this brief paper. We will discuss on the relationship between scale-free degree exponent and its associated timeseries’ hurst exponent.

The rest of this paper is arranged as follows: in section 2, some backgrounds will recalled firstly, including the static model with adjustable degree exponents, the concept of hurst exponent, and constructing associated timeseries methods by using recurrence plot matrix and classical multi-dimensional scale analysis method. In section 3, we will give main results by using the ubiquitous process for discussing such problem proposed in section 2. In section 4, brief conclusions are finally concluded. Research in this direction will provide new methods and powerful tools, for analysis of the causal relationships between complex structures and temporal behaviors of a complex system, and as well as for structure and function analysis of a complex networks.

2. Static scale-free network with scalable scale exponent, hurst exponent and duality transfer method based on recurrence plot matrix

In this section, we will firstly introduce the static scale-free network with scalable scale index [8], Hurst index [12], and duality transformation method [1]-[7], which will be used in following discussion.

2.1. Recall the related static scale-free network model(SM)

In Ref. [8], a static model (registered as SM in the following context) was firstly introduced as an non-increased network algorithm to generate static network (i.e. not growing) with typical scale-free index (with any desired degree exponent $\gamma$ larger than or equal to 2). Firstly, let’s recall the model defined as follows [9]:

The SM is started from $N$ disconnected vertices indexed by an integer number $i \in \mathbb{N}$, taking the values $i = 1, \ldots, N$. A normalized probability $p_i$ is assigned to each vertex given as function of the index $i$ by

$$p_i = \frac{i^{-\alpha}}{\sum_{j=1}^{N} j^{-\alpha}},$$

where $\alpha$ is a real number in the range $\alpha \in [0, 1]$. The following rules are obeyed by the algorithm:

- The SM network is constructed iterating two different vertices $i$ and $j$ are randomly selected from the set of $N$ vertices, with probability $p_i$ and $p_j$, respectively. If there exists an edge between these two vertices, they are discarded and a new pair is randomly drawn. Otherwise, an edge is created between vertices $i$ and $j$.

- Repeated such a process until $E = mN$ edges are created in the network, accounting for a fixed average degree $\langle k \rangle = 2E/N = 2m$.

In this algorithm, a static network was generated with no self-connections (a vertex joined to itself) not multiple connections (two vertices connected by more than one edge). By means of a simple mean-field argument mentioned in [8] and [9], one can analyze the corresponding degree distribution of this SM easily.

Since edges are connected to vertices with a probability given by the factor $p_i$, the probability of any edge belongs to the vertex $i$ whose degree equals to $k_i$ is:

$$\frac{k_i}{\sum_j k_j} \sim p_i.$$
For the case of $N \to \infty$, approximating sums by integrals, for $0 < \alpha < 1$, one can see (3),

$$\sum_{j=1}^{N} j^{-\alpha} \sim \int_{1}^{\infty} j^{-\alpha} dj \sim \frac{N^{1-\alpha}}{1-\alpha}. \quad (3)$$

Since $\sum_{j} k_{j} = \langle k \rangle N$, from Eq. (2), one can see the following:

$$k_{i} \sim p_{i} \sum_{j} k_{j} \sim 2m(1-\alpha) \left( \frac{i}{N} \right)^{-\alpha}. \quad (4)$$

By using general arguments from network theory [10], one can obtain that, the degree distribution relationship to characterize the SM network have the form,

$$P(k) \sim k^{-\gamma}$$

with a associated scale free degree exponent as a function of parameter $\alpha$:

$$\gamma = 1 + \frac{1}{\alpha}. \quad (5)$$

Obviously, by tuning the parameter $\alpha$ in the interval $[0, 1]$, the scale index of the generated SM network must behaved as a typical scale free network with a associated degree exponent, in the board range $\gamma \in [2, \infty)$.

Since the above mentioned algorithm generates a scale free networks with built-in degree correlations mechanism, the maximum degree corresponding to the index $i = 1$ can be calculated as:

$$k_{i=1} \sim 2m(1-\alpha)N^{\alpha}. \quad (6)$$

which implies that the cut-off (or maximum expected degree) $k_{c}(N)$ in the SM network [11] scales with the network size as $k_{c}(N) \sim N^{\alpha}$. So, in the case of absence of multiple and self-connections, in order to have no correlations, a scale-free networks with size $N$ must have a cut-off scaling at most as $k_{c}(N) \sim N^{1/2}$ (the so-called structural cut-off) [11].

Therefore, the SM will yield a correlated network for given parameter values $\alpha > 1/2$, i.e., for degree exponents in the interval $2 < \gamma < 3$, which correspond to those values empirically observed in real networks with scale free characters.

In the following Sections, we will discuss the relationship between the scale index and its associated timeseries’s hurst index of a correlated static scale-free network model (with degree exponents in the interval $2 < \gamma < 3$).

2.2. Recall the calculation of the Hurst exponent of timeseries

‘Hurst exponent’ or ‘Hurst coefficient’, derives from Harold Edwin Hurst (1880 – 1978), who was the lead researcher in this direction. The so called ‘Hurst exponent’ is used as a important measure of long-term memory of linear or nonlinear time series [12] [13]. It relates to the autocorrelations of the time series, and the rate at which these decrease as the lag between pairs of values increases. Studies involving the Hurst exponent were originally developed in hydrology for the practical matter of determining optimum dam sizing for the Nile river’s volatile rain and drought conditions that had been observed over a long period of time.

The hurst exponent, $H$, is defined in terms of the asymptotic behaviour of the re-scaled range as a function of the time span of a time series as follows [12]:

$$E\left[ \frac{R(n)}{S(n)} \right] = Cn^{H} \quad (n \to \infty),$$
where $R(n)$ is the range of the first $n$ values, and $S(n)$ is their standard deviation, $E[x]$, is the expected value. Parameter $n$ is the time span of the observation (number of data points in a time series), and $C$ is a constant.

The values of the Hurst exponent vary between 0 and 1, with higher values indicating a smoother trend, less volatility, and less roughness.

A number of estimators of long-range dependence of time series have been proposed in published literatures. The oldest and best well known is the so-called re-scaled range (R/S) analysis popularized by Mandelbrot and Wallis’ milestone work, which was constructed based on previous hydrological findings of Hurst. Alternatives include DFA, periodogram regression, aggregated variances, local Whittle’s estimator, wavelet analysis, both in the time domain and frequency domain. One can consult methods used in different fields from published literatures, i.e., [12]-[18], to name just a few.

### Table 1. Significance of different value range of Hurst index

| $H$     | Significance                                      |
|---------|---------------------------------------------------|
| $H > 0.5$ | indicating a smoother trend                        |
| $H = 0.5$ | indicating a less volatility                        |
| $0 \leq H < 0.5$ | indicating a less roughness                        |

2.3. A typical transfer method to reproduce the associated timeseries of network graph

In refs [2],[3],[4], a method was widely used to reproduce distance matrices and original time series from recurrence plots was proposed by Hirata et al. Their procedure can fulfill the task to convert a recurrence plot matrix to a weighted network graph, and then calculate a distance matrix named by ‘reproduced distance matrix’ between each pair of nodes on this weighted graph. Their method can reproduce the topological shape of original time series fairly well. Such an effective algorithm will be used here, and one can find algorithm details in references [2],[3],[4], and [22].

Here comes the main step of this algorithm used in our numerical analysis parts:

- **I)** Prepare a recurrence plots (RP) of the considered nonlinear timeseries [19],[20],[21]: Eckmann et al., introduced recurrence plots, which provide a way to visualize the periodic nature of a trajectory through a phase space. It can be used to transfer a timeseries to network graph [22]. Let $x(i)$ be a state at time index $i (1 \leq i \leq n)$. Denote by $R$ a function of two states that tells whether the two states are close or not. If they are close to each other, then $R$ returns 1. Otherwise, it returns 0. Then a RP can be defined as $\{(i, j) : R(x_i, x_j) = 1, i, j = 1, 2, \cdots, n\}$ (For more details on RP analysis method, one can consult references [19],[20],[21], to name just a few).

- **II)** Construct a graph from the RP: In such a graph, each node corresponds to a time index, and an edge exists between nodes $i$ and $j$ if the states $x_i, x_j$ corresponding to the time indices $i, j$ are close, namely, $R(x_i, x_j) = 1$.

- **III)** For each edge, assign a weight: Define $G_i$ by $\{j : R(x_i, x_j) = 1\}$. $G_i$ is the set of time indices to which $i$ is close. For each existing edge between $i$ and $j$, define the weight $w$ by

$$w(i, j) = 1 - \frac{|G_i \cap G_j|}{|G_i \cup G_j|}$$

where $|A|$ shows the number of elements in set $A$, $\cap$ shows the intersection of two sets, and $\cup$ shows the union of two sets.

- **IV)** Calculate the shortest distance between each pair of nodes on the graph weighted by $w(i, j)$. (One can use the fast Dijkstra method for calculating the shortest distance.)
• V) Use classical multidimensional scaling analysis (CMDS) technique, which can arrange a set of points so that these points preserve the given distance matrix, to reproduce the associated timeseries. (The CMDS algorithm can be fulfilled by function cmdscale.m by Matlab software suite.)

In this algorithm, the classical multidimensional scaling method was used. In the reproduction process, the component with the largest eigenvalue are used. This algorithm can be seen as an extended version of classical Isomap method, which was widely used for dimension reduction except the definition of edge weights by Equation (7) is constructed in a new way. This method works fairly well with the condition that the related graph is connected. In the following sections, the above mentioned algorithm from step II to the end will be used for further numerical discussion.

3. Main results
In this section, we begin to discuss the topic we focus on: analysis relationship between the scale free degree exponent and its associated timeseries’s Hurst exponent of a static network model. Since there are not a one-one map published for analyzing the problem theoretically yet. We will discuss the problem numerically. The basic procedure is listed as follows:

• Step I) Create a group of static scale free networks with the same scale \( N \) and average degree \( m \);
• Step II) Create the associated timeseries of step I by using the aforementioned CMDS method;
• Step III) Calculating the scale-free degree exponent and its associated timeseries’s hurst exponent and try to find hidden relationship between these two characters.

Results in details will be given in the followed subsections.

3.1. Numerical results and discussion
In Figure 1, we give the numerical results of relationship between scale index \( \gamma \) and Hurst index \( H \) with \( N = 600 \). It can be seen that, the linear mode relationship does exist there. The related information of this fitting process with \( N=600 \), is listed in Table 2. One can see that the goodness parameter \( R^2 \) of this fit is very close to 1, which means the curve fitting is linear mode.

Table 2. The related information of this fitting process \( N = 600, m = 4 \) for Figure 1(Coefficients with 95% confidence bounds.)

| Linear model Poly1: \( f(x) = p_1 x + p_2 \) | Coefficients : \( p_1 = 0.08006(0.0148, 0.1453), p_2 = 0.2346(0.07579, 0.3934) \) |
| Goodness of fit: SSE: 0.001541, \( R^2: 0.7436 \), Adjusted \( R^2: 0.6795 \), RMSE: 0.01963 |

In Figure 2, we give the numerical results of relationship between scale index \( \gamma \) and Hurst index \( H \) with \( N = 900 \). It can also be seen that, the linear mode relationship exists there, too. The related information of this fitting process \( N=900, m=4 \), was listed in Table 3. One can also see that the goodness parameter \( R^2 \) of this fit is very close to 1, which means the curve fitting is linear mode.

Similarly, in Figure 3, the numerical results of relationship between scale index \( \gamma \) and Hurst index \( H \) with \( N = 1200 \) is given. It can be seen that, the linear mode relationship exists there, too. The related information of this fitting process \( N=1200, m=4 \), are listed in Table 4.
2.1 2.2 2.3 2.4 2.5 2.6 2.7 2.8 2.9 3

\[ H = 1 + \frac{1}{\gamma} \]

\[ \gamma = 1 + \frac{1}{\alpha} \]

Figure 1. Relationship between scale index $\gamma$ and Hurst index $H$, where $N = 600$. (mean of 20 run numerical experiments.)

2.1 2.2 2.3 2.4 2.5 2.6 2.7 2.8 2.9 3

\[ H = 1 + \frac{1}{\gamma} \]

\[ \gamma = 1 + \frac{1}{\alpha} \]

Figure 2. Relationship between scale index $\gamma$ and Hurst index $H$, where $N = 900$. (mean of 20 run numerical experiments.)

can also see that the goodness parameter $R^2$ of this fit is very close to 1, which means the curve fitting is linear mode.

Finally, in Figure 4, we gave the numerical results of relationship between scale index $\gamma$ and Hurst index $H$ with $N = 1500$. It can be seen that, the linear relationship exists there, too. The related information of this fitting process $N=1500$, $m=4$, are listed in Table 5. One can also
Table 3. The related information of this fitting process N=900, m=4 for Figure 2 (Coefficients with 95% confidence bounds.)

| Linear model Poly1: f(x) = p_1 x + p_2 |
|-----------------|-----------------|
| Coefficients:   | p_1 = 0.08625(0.05807, 0.1144), p_2 = 0.1952(0.1266, 0.2638) |
| Goodness of fit:| SSE: 0.0002874, R^2: 0.9475, Adjusted R^2: 0.9344, RMSE: 0.008477 |

Figure 3. Relationship between scale index γ and Hurst index H, where N = 1200. (mean of 20 run numerical experiments.)

Table 4. The related information of this fitting process N=1200, m=4 for Figure 3 (Coefficients with 95% confidence bounds.)

| Linear model Poly1: f(x) = p_1 x + p_2 |
|-----------------|-----------------|
| Coefficients:   | p_1 = 0.08088(0.01024, 0.1515), p_2 = 0.2271(0.05526, 0.399) |
| Goodness of fit:| SSE: 0.001805, R^2: 0.7164, Adjusted R^2: 0.6455, RMSE: 0.02125 |

see that the goodness parameter R^2 of this fit is very close to 1, which means the curve fitting is linear mode. We also gave the residuals plot (fitting error bar plot), from which, one can see the fitting behaves fairly well.

Table 5. The related information of this fitting process N=1500, m=4 for Figure 4 (Coefficients with 95% confidence bounds.)

| Linear model Poly1: f(x) = p_1 x + p_2 |
|-----------------|-----------------|
| Coefficients:   | p_1 = 0.1043(0.07738, 0.1313), p_2 = 0.1637(0.09818, 0.2293) |
| Goodness of fit:| SSE: 0.0002627, R^2: 0.9665, Adjusted R^2: 0.9582, RMSE: 0.008104 |

From the above mentioned numerical results, one can see that, with the increasing of network graph scale (i.e., the length of related associated timeseries) the linear mode relationship becomes
more and more clearly (long term tendency). Limited to the processing power of our personal computers, this paper fails to provide large-scale numerical results. But we infer that this linear mode correlation phenomenon might exist there.

Also, we found that, the Hurst exponents are all smaller than 0.5 under different network graph scale, which means that, the associated timeseries created from static scale-free network with adjustable degree exponent $\gamma \in [2, 3]$ are with less roughness. This is coincide with the real character of real nonlinear timeseries which induced the related networks [1], [2].

3.2. Compare the fit tendency

In this section, we further give the fit tendency analysis. In the following figure 5, the relationship of network graph scale and the R-Square parameter $R^2$ is listed. We fitted the tendency by a linear-like mode function $f(x) = a(sin(x - \pi)) + b((x - 10)^2) + c$, where the related coefficients with 95% confidence bounds, are listed as follows, respectively.

- $a = -0.05591 (-2.595, 2.483)$
- $b = 8.051e - 08 (-2.467e - 06, 2.628e - 06)$
- $c = 0.7668 (-2.485, 4.018)$.

One can see that, the $R^2$ parameter is roughly increasing with network graph scale parameter $N$ (the oscillation might be caused by numerical error, it might be smoothed under more times numerical run average), which means that, the larger network graph scale is, the more definitely of existing linear mode relationship between scale-free degree exponents and hurst exponents of its associated timeseries. So, one might deduce that, there do exist some 'one-one map' between network graph and its associated timeseries, and vice versa.

4. Conclusion

Under the viewpoint of duality analysis of networks and time series, the internal relationship between a typical static scale free networks and its associated nonlinear timeseries are investigated in this brief paper. It is found that, the scale-free degree exponent has linear mode
relationship with its associated timeseries’ Hurst exponent. The larger network graph scale is, the more definitely of existing linear mode relationship between scale free degree exponent and Hurst exponent of its associated timeseries. So, one can deduce that, there do exist some ‘one-one map’ between network graph and its associated timeseries, and vice versa. Research in this direction will provide new methods and potential tools, for analysis of the causal relationship between complex structures and temporal behaviors of a complex dynamical system, and as well as for structure and function analysis of a complex network graph. In the future, we will further investigate the relationship between other network characters and the statistical character of the associated timeseries. It might reveal some deep relationship for duality analysis of nonlinear timeseries and complex network graph.

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