REMARKS ON D-DIMENSIONAL TSP OPTIMAL TOUR LENGTH BEHAVIOUR

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Abstract. The well-known $O(n^{1-1/d})$ behaviour of the optimal tour length for TSP in d-dimensional Cartesian space causes breaches of the triangle inequality. Other practical inadequacies of this model are discussed, including its use as basis for approximation of the TSP optimal tour length or bounds derivations, which I attempt to remedy.

1. Introduction

Studies of the behaviour of the optimal tour value in respect of the TSP has been in place since the fifties. Considerate work has been done with the $O(n^{1-1/d})$ model of behaviour based on results of Beardwood, Halton and Hammersley. The approach taken here is one of simple real analysis and some of the intermediate results correspond with those of Snyder and Steele and Bern and Eppstein.

As pointed out by Bern and Eppstein, an a priori bound on a geometric graph is a bound that depends on the assumption that all vertices lie within a given container. Rather than bounds, I consider the behaviour of the optimal tour length of a geometric graph in a 2D rectangular region, which is transformed to a square and then simplified (extended) to d-dimensional cube. This is compared to the behaviour of the MST length in the same container space.

The problem of studying the behaviour or approximation of the optimal tour length of TSP instances ($\parallel O \parallel$) can be illustrated by an example: Given two TSP instances with substantially different number of vertices have the same optimal tour length.

It is shown that the general behaviour of $\parallel O \parallel$ does not depend on the number of vertices. I also give experimental evidence in support of this using a small set of TSPLIB instances, and conclude that the commonly cited $O(n^{1-1/d})$ behaviour is inadequate in both description of behaviour (nor useful bounds) of $\parallel O \parallel$. A simple alternative is suggested.

The present is an extended version of a mention in [9].

2. Terms and Definitions

The Minimum TSP in Cartesian space is a special case of the Minimum Weight Hamiltonian Cycle problem. Both are specified as follows:

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\section*{P\# 1 (Minimum Hamiltonian Cycle)}

Given $G = (V, E)$ with a cost function $d : E \to R$, find solution $O = (V_O, E_O) \subseteq G$ such that

(a) $O$ is a Hamiltonian cycle;
(b) $\|O\| = \sum_{e \in E_O} d(e)$ is minimum;

\section*{P\# 2 (Minimum TSP in Cartesian Space)}

\(P\# 1\) and

(a) $\forall v : V \cdot \text{coords}(v) \in R^d$ and
(b) $\forall (u, v) : E \cdot d(u, v) = |\text{coords}(u) - \text{coords}(v)|$
where $\text{coords} : V \to R^d$ are Cartesian.

I (unnecessarily) split the problem specification this way for my personal convenience in what follows (separating construction and geometric aspects of the problems), and for the following observations (from $P\# 1$):

(a) $|V| = |V_O| = |E_O|$
(b) Solution bounds:

\begin{equation}
\text{(1)} \quad nw_0 \leq \|O\| \leq nw_1
\end{equation}

where $n = |E_O|$, $w_0 = \min \{d(e) | e \in E\} \land w_1 = \max \{d(e) | e \in E\}$

The notation used is fairly standard. I have attempted to maintain consistent naming, e.g. $Oabc_i$ is the value (approximate) of the exact $\|O\|$, and the subscript $i$ is the number of 'alternative' formulation of $Oabc$, e.g. $Ob_1$ is an alternative to $Ob_2$.

\section*{3. Two Similar Formulations of d-Dimensional $\|O\|$ Behaviour}

Consider a rectangle (being a container space of the TSP graph and respective tour) and that we have a uniform grid tight within (and on) that rectangle with size $a$ by $b$ points, assuming without loss that $a$ is even. The optimal tour value and the diagonal of the rectangle are:

\begin{align*}
\text{(2)} & \quad \|O\| = a \cdot b \cdot w_0 = n \cdot w_0 \\
\text{(3)} & \quad w_1 = w_0 \sqrt{(a - 1)^2 + (b - 1)^2}
\end{align*}

Dividing:

\begin{equation}
\text{(4)} \quad \frac{\|O\|}{w_1} = \frac{a \cdot b}{\sqrt{(a - 1)^2 + (b - 1)^2}}
\end{equation}

If $a = b$ than we have a square, and $\|O\|/w_1 = n/((\sqrt{n} - 1) \sqrt{2})$. To carry this to Cartesian d-dimensions, (2) and (3) are changed to $O_1 = w_0 \prod_{i=1}^{d} a_i = n \cdot w_0$ and $w_1 = w_0 \left(\sum_{i=1}^{d} (a_i - 1)^2\right)^{1/2}$. Assuming $\forall i, j : 1 \ldots d \cdot a_i = a_j$ then

\begin{equation}
\frac{O_1}{w_1} = \frac{n}{(n^{1/d} - 1)\sqrt{d}}
\end{equation}

Note that $O_1$ differs from the commonly cited $O(n^{1-1/d})$ optimal tour length behaviour, which can be formulated by taking $w_1 = w_0 \left(\sum_{i=1}^{d} a_i^2\right)^{1/2}$ (under the assumption that $a$ is large), then:

\begin{equation}
\frac{O_2}{w_1} = \frac{n^{1-1/d}}{\sqrt{d}}
\end{equation}
It should be noted that in the unit d-dimensional cube \( w_1 = \sqrt{d} \).

For a metric (non-topological) graph \( |O| \geq 2w_1 \) must hold. Both \( O_1 \) and \( O_2 \) fail to preserve the triangle inequality when \( d = 1 \), due to the “spatial” nature of the derivations (no container). It is also evident that \( O_1 \) preserves the triangle inequality for any \( n \geq 2 \), whereas \( O_2 \) fails in doing so when \( n^{1-1/d} \leq 2\sqrt{d} \). (There is nothing to prevent such instances from occurring.) The ratio of the two estimates as \( d \to \infty \) is:

\[
e(n, d) = \frac{O_2}{O_1} = 1 - \frac{1}{n^{1/d}} \to 0
\]

The divergent behaviour of \( O_1 \) and \( O_2 \) is fixed by imposing the condition \( n \geq 2d + k \), \( k \geq 0 \) in order to have a valid d-dimensional cube, and \( 1/2 \leq e(n, d) \leq 1 \). Either \( O_1 \) or \( O_2 \) will fail in the description (if they are assumed to be descriptive) of the optimal tour length behaviour (see Fig.1).

The MST length behaviour (\( ||T|| \)) is analogous to that of the TSP in a grid: \( ||T|| = (n - 1) \cdot w_0 \) and \( T_1/w_1 = (n - 1) / \left( (n^{1/d} - 1)\sqrt{d} \right) \). \( ||O|| \) can be expressed in terms of \( ||T|| \) (treated as known value computed exactly by Prim’s algorithm, say):

\[
O_{t1} = ||T|| + \frac{w_1}{(n^{1/d} - 1)\sqrt{d}}
\]

\[
O_{t2} = ||T|| + \frac{w_1}{n^{1/d}\sqrt{d}}
\]

As a matter of fact:

\[
||O|| \geq ||T|| + w_0
\]

\[
(2||T|| \geq ||O|| \geq ||T|| \frac{n}{n-1})
\]

It is the case that for any metric graph \( ||T|| \) accounts for at least 50% of \( ||O|| \) and can be considered a major term in any attempted approximation. It is obvious that the second term in \( O_{t1} \) may well lead to \( w_0 > w_1 / \left( (n^{1/d} - 1)\sqrt{d} \right) \), as \( n \to \infty \), and ultimately \( O_{t1} \to ||T|| \) (so will \( O_{t1} \)), which cannot be. This suggests that \( ||O|| \) does not depend on the number of vertices. Moreover, the \( e(n, d) \) term also remains in the relative difference in estimates for the remaining 50% of \( ||O|| \), as seen from \( O_{t1} \) and \( O_{t2} \). In other words, if \( ||O|| \) were to depend on the number of vertices the following must be true:

\[
e(n, d) = \frac{O_{t2} - ||T||}{O_{t1} - ||T||} = \frac{O_2}{O_1} \wedge ||T|| \neq 0
\]

Obviously this is absurd, unless \( O_{t1} = O_{t2} \wedge O_1 = O_2 \), which is not the case. Thus, it is difficult to contemplate using \( O_1 \) or \( O_2 \) (see Fig.3) as basis for behaviour description of \( ||O|| \). The \( O_{t1} \)’s are slightly better (see Fig.3), exhibiting an almost “bounding” behaviour, for which we already have \( \bar{O}_1 \) and \( \bar{O}_2 \) and the HK bound (see Fig.3).

From the above one can conclude that \( n \) can do more harm than good in the analysis of \( ||O|| \). Setting \( n = 2^d \) (see above), yet another approximation (see Fig.4 and Table.3) to \( ||O|| \) is obtained:

\[
O_{tc1} = ||T|| + \frac{w_1}{\sqrt{d}} = O_{tc2} + \frac{w_1}{2\sqrt{d}}
\]
3.1. **Test Data.** The test data consist of 85 instances in total of sizes $4 \leq n \leq 2560$, I have used 17 instances of my own and 68 instances from TSPLIB \cite{5} of type EUC\_2D, CEIL and ATT. The three norms were treated as EUC\_2D without rounding. ATT are treated in the same way, as tour values were multiplied by $\sqrt{10}$ to bring them in line with unrounded EUC\_2D. The HK-bound values are taken from \cite{3}. There were 5 TSPLIB with published shortest tour, for which I have been able to find shorter tours. These are given in Table\,3. The HK bound is greater than the actual tour length for tsp225 and att48 by 0.55\% and 0.01\% (see Table\,3), respectively, possibly due to a typographical error (or the use of initial rounding in computing the HK relaxation). I do not consider accuracy (see Table\,2) in this analysis of major importance, as for the 50 instances for which the optimal tour itself is not published, the data can be fixed (by point displacement) so the reported values are accurate (This is not the case with HK bound).

| Instances | Size          | O     | $\|O\|$ | HK           |
|-----------|---------------|-------|---------|--------------|
| 17        | $4 \leq n \leq 2560$ | known | exact   | $\_\_\_-$  |
| 18        | $48 \leq n \leq 2392$ | published | exact reported |           |
| 50        | $99 \leq n \leq 2319$ | reported | reported | reported     |

| $\|T\| + w_0$ | $\|T\|_{\frac{n}{n-1}}$ | $\|T\|$ | $\|T\|_{\frac{n}{n-1}}$ |
|---------------|------------------------|---------|--------------------------|
| $X$ | $\min \epsilon$ | $\max \epsilon$ | $\max \epsilon - \min \epsilon$ | $\epsilon_{\text{rms}}$ |
| $O_1$ | 0 | 1668.14 | 1668.14 | 308.96 |
| $O_2$ | $-39.69$ | 1632.05 | 1671.74 | 299.91 |
| $Ot_1$ | $-49.26$ | 0.76 | 50.02 | 14.16 |
| $Ot_2$ | $-49.28$ | 0.75 | 50.02 | 15.00 |
| $Otc_1$ | $-14.64$ | 35.27 | 49.91 | 7.75 |
| $Otc_2$ | $-32.32$ | 17.61 | 49.93 | 11.14 |
| $HK^*$ | $-9.73$ | 0.55 | 10.28 | 2.18 |

Table 1. Relative error ranges for the various formulas. $\epsilon = 100 \cdot (X/\|O\| - 1) \%$. (The largest $HK^*$ value +0.55 is possibly due to a misprint in \cite{3})

4. **Conclusions**

Inadequacies of $O (n^{1-1/d})$ and similar formulations are demonstrated. A probabilistic or statistical \cite{4} argument may show otherwise.
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Instances | $\|O_{\text{reported}}\|$ | HK | $\|O_{\text{published}}\|$ | $\|O_{\text{shorter}}\|$  
--- | --- | --- | --- | ---  
eil51 | 426 | 422.4 | 429.983307 | 429.117879  
st70 | 675 | 670.9 | 678.597412 | 677.194473  
eil101 | 629 | 627.3 | 642.309509 | 640.42177  
ch130 | 6110 | 6074.6 | 6110.859375 | 6110.722217  
ch150 | 6528 | 6486.6 | 6532.282227 | 6530.906184  
att48 | 10628 | 10602.1 | 10601.1282 | =  
tsp225 | 3919 | 3880.3 | 3858.99064 | =  

Table 3. Shorter tours. Values are as reported in [3] and computed from published tours in [5].

The optimal tour length does not depend on the number of vertices - consider again a grid example in a square: Let’s suppose that we have found an optimal tour $S = (V_S, E_S)$ from $G = (V, E)$ (a comb-like shape, say) and insert (in $G$) exactly one zillion equidistant points along each arc in $E_S \subseteq E$ of the optimal tour $S$ to get $G_Z$. If $S_Z$ is an optimal tour of $G_Z$ then $\|S\| = \|S_Z\|$. Such point insertion is not inconsequential - the number of optimal solutions may be reduced, i.e. $G_Z$ has a lesser number of optimal solutions than $G$.

I make no claims as to quality (if used as estimates) of the above given formulae, as I would not use any of them (see Table 3) as an estimate of $\|O\|$. $O_1$ is an upper (easily proven so) bound, for any $n \geq 2$, however running the Nearest town algorithm [6] to get us an upper bound within a factor of two is simpler. A simple derandomisation of the same algorithm yields an approximate solution within a factor of $\sqrt{2}$ (provable under a geometric argument) for P#2 in the plane.

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Figure 1. Plots of $e(2^d + k, d)$, for $k = \log_2 d$, $k = d \log_2 d$, $k = d^d$, and $k = d^{d+1}$, for $d = 1 \ldots 100$.

Figure 2. Point histograms of relative error of $O_1$ and $O_2$ - 85 instances

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Figure 3. Point histograms of relative error of $O_{t1}$, $O_{t2}$ and $HK$ bound (-line used for clarity)

Figure 4. Point histograms of relative error of $O_{tc1}$ and $O_{tc2}$
Figure 5. Point histograms of the lower bounds $\|T\| + w_0$, $\|T\| \frac{n}{n-1}$ and $HK$ bound (-line used for clarity)