Eigen Spherical Fuzzy Set and its Application in Decision Making Problem

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Abstract

Eigen fuzzy set of a fuzzy relation often occurs to be invariant under different computational aspects. The present communication introduces the novel concept of eigen spherical fuzzy set of spherical fuzzy relation along with various composition operators for the first time. We have proposed two algorithms to determine the greatest eigen spherical fuzzy sets and least eigen spherical fuzzy sets using the max – min and min – max composition operators respectively and illustrated the steps with the help of flow charts. Further, two numerical examples related to different fields of decision-making problems have been taken into account for illustrating the proposed methodology. The scope of future work in the field of image information retrieval, genetic algorithm for image reconstruction and notion of eigen spherical fuzzy soft sets/matrices has been duly outlined. The comparative remarks and advantages of the proposed eigen spherical fuzzy sets have also been included for a better readability.

Keywords: Eigen fuzzy set, Spherical fuzzy set, Fuzzy relation, Composition operators, Decision making.

1 Introduction

The researchers in the field of fuzzy sets and information are well aware that various generalizations of the notion of Fuzzy Sets (FS) [1] and Intuitionistic Fuzzy sets (IFS) [2] have taken place to model the uncertainties and the hesitancy inherent in many practical circumstances with a broader range in various applications, particularly in the study of patterns and information systems. Essentially, such generalizations provides a formal way of dealing with real life problems in which the source of impreciseness is the lack of sharply defined criteria of class membership instead of having the random variables. Yager [3] revealed that the existing structures of fuzzy set and intuitionistic fuzzy set is not capable enough to depict the human opinion in more practical/broader sense and introduced the notion of Pythagorean fuzzy sets which effectively enlarged
the span of information by introducing the new conditional constraint. Various other operations over inter-valued Pythagorean fuzzy sets are given by Peng et al. [4]. The concept of membership/belongingness (yes), non-membership/non-belongingness (no) and indeterminacy/neutral (abstain) have been well described by the definition of Intuitionistic Fuzzy Sets as well as by the Pythagorean Fuzzy Sets. Consider an example of voting system where voters can be categorized into four different classes – one who votes for (yes), one who votes against (no), one who neither vote for nor against (abstain), one who refused for voting (refusal). It may be noted that the concept of ‘refusal’ is not being taken into account by any of the sets stated above. In order to deal with such circumstances and to develop a concept which would be sufficiently close to human’s nature of flexibility, Cuong [5] [6] introduced the concept of Picture fuzzy set in which all the four parameters, i.e., degree of membership, degree of indeterminacy (neutral), degree of nonmembership and the degree of refusal have been taken into account.

Recently, Mahmood et al. [7] introduced the notion of Spherical Fuzzy Set (SFS) and $T$-spherical Fuzzy Set (TSFS) which give additional strength to the idea of picture fuzzy sets by broadening/enlarging the space for the grades of all the four parameters. Next, Kifayat et al. [8] studied the geometrical comparison of Fuzzy sets, Intuitionistic fuzzy sets, Pythagorean fuzzy sets, Picture fuzzy sets with Spherical and $T$-spherical fuzzy sets. Also, they studied various existing similarity measures for IFS & Picture fuzzy sets have some limitations and could not be applied in the broader setup of spherical fuzzy environment. Further, they proposed various types of similarity measures for $T$-spherical fuzzy sets with their useful applications in various fields. The evolution process of the generalizations and extensions of a fuzzy set has been summarized in Figure 1.

In the field of pure and applied sciences, the mathematical notion of relation plays a key role to establish the connections between objects, states and events. Fuzzy relations are the generalizations of the concept of binary relations. The notion of fuzzy relation was first introduced by Zadeh [9], with fuzzy equivalence (similarity) relation, and provided the concept of fuzzy ordering along with some basic properties. Sanchez [10] [11] described the role of invariants fuzzy sets linked with a given fuzzy relation using the composition of fuzzy relation and introduced the notion of Eigen Fuzzy Set of a fuzzy relation. Further, Sanchez used $max - min$ composition to determine the Greatest Eigen Fuzzy Set (GEFS) associated with fuzzy relation by providing three major algorithms. Several practical and successful applications of eigen fuzzy sets in the field of image analysis (image reconstruction, image information retrieval, image decomposition) [12] [13] [14], genetic algorithm [15], medicine (drug effectiveness levels) [16], Fuzzy Markov Chain and decision making [17] [18] etc.
In order to span the flexibility of human opinions with revised conditional constraints, we propose an extension as a new paradigm termed as eigen spherical fuzzy sets of spherical fuzzy relations. For application purposes, two new methods for determining the Greatest Eigen Spherical Fuzzy Sets and Least Eigen Spherical Fuzzy Sets by using the $\max - \min$ and $\min - \max$ composition have been proposed.

The rest of the paper is organized as follows: A brief literature review related to the eigen fuzzy sets and their applications has been presented in Section 2. Some fundamental background and definitions are being studied in Section 3. The notion of spherical fuzzy relations and functioning of their composition operators have been introduced in Section 4. In Section 5, the novel concept of eigen spherical fuzzy set of spherical fuzzy relation has been proposed along with the algorithms to determine the greatest eigen spherical fuzzy sets and least eigen spherical fuzzy sets using the well-defined composition operators. This has been well illustrated with the help of examples. In Section 6, the implementation of the proposed algorithms has been shown by solving two examples related to decision making problem. In Section 7, some potential directions and guidelines for future works in different application fields have been provided in brief. The comparative remarks and the advantages of the proposed eigen spherical fuzzy sets have been discussed in Section 8. Finally, the paper has been concluded in Section 9.

2 Literature Review

In literature, various researchers have studied fuzzy relational calculus as an application of fuzzy relation to obtain the possible solutions of the fuzzy relation equations [19] [20] [21] and proposed the notion of eigen fuzzy set being invariant under the associated fuzzy relation. Consequently, the problem of finding the greatest eigen fuzzy set associated with a relation has been dealt. This has generated a considerable amount of interest for researchers for further investigations. Goetschel & Voxman [22] extended the concept and results of eigen fuzzy set in context with the eigen fuzzy numbers by making slight modification in the definition of fuzzy number given by Dubois & Prade [23]. Fernandez et al. [24] generalized the results of Sanchez [19] to determine the greatest $T$-eigen fuzzy set of fuzzy relation and studied some algebraic properties.

The applications of eigen fuzzy sets have successfully been carried out by researchers in the second half of the 1970s which boosted the reader’s interest significantly in this area. Amagaza & Tazaki [25] studied the application in heuristic structure synthesis using eigen fuzzy sets. Cao [26] presented an algorithm for finding eigen fuzzy sets of a fuzzy matrix. The concept of general finite state fuzzy Markov chains in connection with
the greatest eigen fuzzy set of the transition matrix was introduced by Avrachenkov & Sanchez [17] and finally linked it with fuzzy Markov decision making processes.

Nobuhara & Hirota [14] studied the greatest eigen fuzzy set (GEFS) of max−min composition and an adjoint concept of GEFS, i.e., the smallest eigen fuzzy set of adjoint max−min composition of a fuzzy relation using the principal component analysis of images. Di Martino et al. [13] introduced the least eigen fuzzy set (LEFS) based on the min−max composition and established that both GEFS and LEFS are useful in image information retrieval. Further, they compared the original image with the retrieved images by introducing a similarity measure based on GEFS and LEFS. Next, Nobuhara et al. [12] proposed two algorithms for image reconstruction process based on the convex combination of eigen fuzzy sets of max−min & min−max compositions and using the eigen fuzzy sets generated by a permutation matrix, where the images have been treated as fuzzy relations. Based on eigen fuzzy set of fuzzy relation, Di Martino et al. [15] proposed a genetic algorithm for image reconstruction where GEFS and LEFS were used to calculate the highest value of the fitness.

In pharmaceutical applications, the evaluation of medicine action levels have been studied by using the eigen fuzzy sets [27]. Further, by establishing fuzzy relations between the possible symptoms, Andersson [16] utilized the greatest and the least eigen fuzzy sets to measure the drug effectiveness levels.

3 Preliminaries

In this section, we recall and present some fundamental concepts in connection with spherical fuzzy set and fuzzy relations, which are well known in literature. The following notions explain the generalization process from intuitionistic fuzzy sets to spherical fuzzy sets:

Let $U$ be the universe of discourse with $\mu_A : U \rightarrow [0, 1]$ and $\nu_A : U \rightarrow [0, 1]$ being the degree of membership and degree of non-membership respectively. The set $A = \{ < x, \mu_A(x), \nu_A(x) > | x \in U \}$ is called

- **Intuitionistic Fuzzy Set** [2] $A$ in $U$ if it satisfies the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ with the degree of indeterminacy given by
  \[
  \pi_I(x) = 1 - \mu_I(x) - \nu_I(x).
  \]

- **Pythagorean Fuzzy Set** [3] or **Intuitionistic Fuzzy Set of second type** [28] $A$ in $U$ if it satisfies the condition $0 \leq \mu_A^2(x) + \nu_A^2(x) \leq 1$ with the degree of indeterminacy given by
  \[
  \pi_A(x) = \sqrt{1 - \mu_A^2(x) - \nu_A^2(x)}.
  \]
In order to have further generalization, we consider the universe of discourse $U$ with $\mu_A : U \rightarrow [0, 1]$, $\eta_A : U \rightarrow [0, 1]$ and $\nu_A : U \rightarrow [0, 1]$ being the degree of membership, degree of neutral membership (abstain) and degree of non-membership respectively. The set $A = \{x, \mu_A(x), \eta_A(x), \nu_A(x) \mid x \in U\}$ is called as

- **Picture Fuzzy Set** [5] $A$ in $U$ if it satisfies the condition $\mu_A(x) + \eta_A(x) + \nu_f(x) \leq 1$ with the degree of refusal given by
  
  $$r_A(x) = 1 - (\mu_A(x) + \eta_A(x) + \nu_A(x)).$$

- **Spherical Fuzzy Set** [7] $A$ in $U$ if it satisfies the condition $\mu_A^2(x) + \eta_A^2(x) + \nu_A^2(x) \leq 1$ with the degree of refusal given by
  
  $$r_A(x) = 1 - (\mu_A(x) + \eta_A(x) + \nu_A(x)).$$

Throughout this paper, we denote $SFS(U)$ as the set of all the spherical fuzzy sets on $U$.

**Definition 1 (Fuzzy Relation)** [9] A fuzzy relation $R$ on a fuzzy set $X$ is a fuzzy subset of $X \times X$, i.e.,

$$R = \{(x_1, x_2), \mu_R(x_1, x_2) \mid x_1, x_2 \in X\},$$

such that $0 \leq \mu_R + \nu_R \leq 1$ & $\mu_R, \nu_R \in [0, 1]$. We denote $FR(X \times X)$ as a collection of all the fuzzy relations on $X$.

**Definition 2 (Eigen Fuzzy Set)** [10] Let $R$ be a fuzzy relation on the elements of a fuzzy set $X$, i.e., $R \in FR(X \times X)$. Consider $A \subseteq X$. Then $A$ is said to be an eigen fuzzy set associated with the relation $R$ if $A = \{x, \mu_A(x)\}$, satisfies the condition $A \oplus R = A$ with $\mu_A(x) \in [0, 1]$; where $\oplus$ is any composition operator.

## 4 Spherical Fuzzy Relation and Composition Operators

In this section, we are proposing various composition operators for spherical fuzzy relations. Here, we first define the spherical fuzzy relation [7] as follows:

A spherical fuzzy relation $R$ between two spherical fuzzy sets $X$ and $Y$ is a spherical fuzzy subset of $X \times Y$, given by

$$R = \{(x, y), \mu_R(x, y), \eta_R(x, y), \nu_R(x, y) \mid x \in X, y \in Y\},$$
such that $0 \leq \mu_R^2 + \eta_R^2 + \nu_R^2 \leq 1$ & $\mu_R, \eta_R, \nu_R \in [0,1]$. In this paper, we denote $SFR(X \times Y)$ as the set of all the spherical fuzzy relations between $X$ and $Y$.

Suppose $R_1 \in SFR(X \times Y)$ and $R_2 \in SFR(Y \times Z)$ be two spherical fuzzy relations. We define various composition operators for the spherical fuzzy relations $R_1$ and $R_2$ as follows:

- **Max-Min Composition of Spherical Fuzzy Relations:**
  The $max-min$ composition relation of $R_1$ and $R_2$, denoted by $R_1 \circ R_2 \in SFR(X \times Z)$, defined as
  \[
  R_1 \circ R_2 = \{(x, z), \mu_{R_1 \circ R_2}(x, z), \eta_{R_1 \circ R_2}(x, z), \nu_{R_1 \circ R_2}(x, z) \mid x \in X, z \in Z\},
  \]
  where,
  \[
  \mu_{R_1 \circ R_2} = \max\{\min(\mu_{R_1}(x, y), \mu_{R_2}(y, z))\};
  \]
  \[
  \eta_{R_1 \circ R_2} = \min\{\max(\eta_{R_1}(x, y), \eta_{R_2}(y, z))\};
  \]
  \[
  \nu_{R_1 \circ R_2} = \min\{\max(\nu_{R_1}(x, y), \nu_{R_2}(y, z))\}.
  \]

- **Min-Max Composition of Spherical Fuzzy Relations:**
  The $min-max$ composition relation of $R_1$ and $R_2$, denoted by $R_1 \bullet R_2 \in SFR(X \times Z)$, defined as
  \[
  R_1 \bullet R_2 = \{(x, z), \mu_{R_1 \bullet R_2}(x, z), \eta_{R_1 \bullet R_2}(x, z), \nu_{R_1 \bullet R_2}(x, z) \mid x \in X, z \in Z\},
  \]
  where,
  \[
  \mu_{R_1 \bullet R_2} = \min\{\max(\mu_{R_1}(x, y), \mu_{R_2}(y, z))\};
  \]
  \[
  \eta_{R_1 \bullet R_2} = \min\{\max(\eta_{R_1}(x, y), \eta_{R_2}(y, z))\};
  \]
  \[
  \nu_{R_1 \bullet R_2} = \max\{\min(\nu_{R_1}(x, y), \nu_{R_2}(y, z))\}.
  \]

- **Max-Min Average Composition of Spherical Fuzzy Relations:**
  The $max-min$ average composition relation of $R_1$ and $R_2$, denoted by $R_1 \Phi R_2 \in SFR(X \times Z)$, defined as
  \[
  R_1 \Phi R_2 = \{(x, z), \mu_{R_1 \Phi R_2}(x, z), \eta_{R_1 \Phi R_2}(x, z), \nu_{R_1 \Phi R_2}(x, z) \mid x \in X, z \in Z\},
  \]
  \[
  \mu_{R_1 \Phi R_2} = \max\left\{\frac{\mu_{R_1}(x, y) + \mu_{R_2}(y, z)}{2}\right\};
  \]
  \[
  \eta_{R_1 \Phi R_2} = \min\left\{\frac{\eta_{R_1}(x, y) + \eta_{R_2}(y, z)}{2}\right\};
  \]
  \[
  \nu_{R_1 \Phi R_2} = \min\left\{\frac{\nu_{R_1}(x, y) + \nu_{R_2}(y, z)}{2}\right\}.
  \]
- Min-Max Average Composition of Spherical Fuzzy Relations:
The \( \text{min} - \text{max} \) average composition relation of \( R_1 \) and \( R_2 \), denoted by \( R_1 \Psi R_2 \in SFR(X \times Z) \), defined as
\[
R_1 \Psi R_2 = \{(x, z) \mid \mu_{R_1 \Psi R_2}(x, z), \eta_{R_1 \Psi R_2}(x, z), \nu_{R_1 \Psi R_2}(x, z) \mid x \in X, z \in Z \},
\]
\[
\mu_{R_1 \Psi R_2} = \min \left\{ \frac{\mu_{R_1}(x, y) + \mu_{R_2}(y, z)}{2} \right\};
\]
\[
\eta_{R_1 \Psi R_2} = \min \left\{ \frac{\eta_{R_1}(x, y) + \eta_{R_2}(y, z)}{2} \right\};
\]
\[
\nu_{R_1 \Psi R_2} = \max \left\{ \frac{\nu_{R_1}(x, y) + \nu_{R_2}(y, z)}{2} \right\}.
\]

Remark: Klement et al.\([29], [30]\) studied some basic triangular norm (\( t\)-norm) & triangular conorm (\( t\)-conorm), their types and various properties. Various other operators may also be defined over spherical fuzzy relations. In the present work, the combination of maximum operator (\( t\)-conorm) & minimum operator (\( t\)-norm) has been taken into account. Various other combinations using other types of \( t\)-norm and \( t\)-conorm may also be utilized in future.

5 Eigen Spherical Fuzzy Set and Algorithms for GESFS and LESFS

In this section, we introduce the notion of Eigen Spherical Fuzzy Set (ESFS) and provide necessary steps of appropriate methods for finding the Greatest Eigen Spherical Fuzzy Set (GESFS) and the Least Eigen Spherical Fuzzy Set (LESFS) with the help of numerical example.

Suppose \( R \) be a spherical fuzzy relation between two spherical fuzzy sets \( X \) and \( Y \), i.e., \( R \in SFR(X \times Y) \) and \( S \in SFS(U) \) be a spherical fuzzy set. The composition of \( R \) and \( S \) using a composition operator would generate a new spherical fuzzy set, say \( T \in SFS(U) \), denoted by,
\[
S \oplus R = T; \text{ where } \oplus \text{ is any composition operator.}
\]

Consequently, we propose the definition of Eigen Spherical Fuzzy Set as follows:

Definition 3 [Eigen Spherical Fuzzy Set] Let \( R \) be a spherical fuzzy relation on a spherical fuzzy set \( X \in SFS(U) \), i.e., \( R \in SFR(X \times X) \). A spherical fuzzy set \( S \in SFS(U) \) is said to be an eigen spherical fuzzy set associated with the relation \( R \) if \( S = \{x, \mu_S(x), \eta_S(x), \nu_S(x) \mid x \in X \} \), satisfies the condition \( S \oplus R = S \) with \( \mu_S(x), \eta_S(x), \nu_S(x) \in [0, 1] \).
Further, we outline the methods for finding the greatest eigen spherical fuzzy set and the least eigen spherical fuzzy set associated with the spherical fuzzy relation $R$.

### 5.1 Algorithms for Finding Greatest Eigen Spherical Fuzzy Set

In order to obtain the greatest eigen spherical fuzzy set associated with the spherical fuzzy relation $R$, we apply the max – min composition operator for spherical fuzzy relations.

Let $S_1$ be the spherical fuzzy set, i.e., $S_1 \in SFS(U)$, in which the membership value is the greatest of all the elements of the column of relation $R$, the neutral membership and the non-membership values are the smallest of all the elements of the columns of $R$, i.e.,

$$
\begin{align*}
\mu_{S_1}(x') &= \max_{x \in X} \mu_R(x, x'), \text{ for all } x' \in Y; \\
\eta_{S_1}(x') &= \min_{x \in X} \eta_R(x, x'), \text{ for all } x' \in Y; \\
\nu_{S_1}(x') &= \min_{x \in X} \nu_R(x, x'), \text{ for all } x' \in Y.
\end{align*}
$$

We initiate the process by taking $S_0$ as a constant spherical fuzzy set with a value equal to the minimum element of the set $S_1$. It is easy to verify that $S_0$ is an eigen spherical fuzzy set, but not the greatest eigen spherical fuzzy set always. To overcome this difficulty, we define the following sequence of spherical fuzzy sets $S_n$ such that

$$
S_1 \circ R = S_2 \\
S_2 \circ R = S_1 \circ R^2 = S_3 \\
\vdots \\
S_n \circ R = S_1 \circ R^n = S_{n+1}
$$

It may be observed that the obtained sequence $S_n$ is a decreasing sequence and bounded by $S_0$ and $S_1$, i.e., $S_0 \subseteq \ldots \subseteq S_{n+1} \subseteq S_n \subseteq \ldots \subseteq S_3 \subseteq S_2 \subseteq S_1$. Next, we present two fundamental algorithms along with numerical example for the determination of the GESFS as follows:

**Algorithm I (GESFS):**

- **Step 1.** Find the set $S_1$ from $R$ as directed by Equation (1).
- **Step 2.** Set the index $n = 1$ and calculate $S_{n+1} = S_n \circ R$.
- **Step 3.** If $S_{n+1} \neq S_n$, then return to Step 2.
- **Step 4.** If \( S_{n+1} = S_n \), then \( S_n \) is the GESFS associated with \( R \).

The proposed algorithm is being presented using the flow chart given in Figure 2.

**Example 1** Let \( X = \{x_1, x_2, x_3, x_4\} \) be a spherical fuzzy set and \( R \) be the spherical fuzzy relation on \( X \) represented as follows:

\[
R = \begin{pmatrix}
    x_1 & x_2 & x_3 & x_4 \\
    (0.7, 0.1, 0.2) & (0.8, 0.1, 0.2) & (0.5, 0.3, 0.4) & (0.8, 0.1, 0.1) \\
    (0.9, 0.1, 0.2) & (0.5, 0.4, 0.2) & (0.8, 0.2, 0.2) & (0.6, 0.6, 0.1) \\
    (0.5, 0.4, 0.3) & (0.6, 0.2, 0.3) & (0.9, 0.1, 0.3) & (0.5, 0.4, 0.2) \\
    (0.7, 0.5, 0.1) & (0.4, 0.5, 0.2) & (0.6, 0.4, 0.1) & (0.2, 0.6, 0.4)
\end{pmatrix}
\]

The computational steps to find the greatest eigen spherical fuzzy set are as:

- **Step 1.** The set \( S_1 \) is given by

\[
S_1 = \left[ (0.9, 0.1, 0.1), (0.8, 0.1, 0.2), (0.9, 0.1, 0.1), (0.8, 0.1, 0.1) \right].
\]

- **Step 2.** For \( n = 1 \), we have

\[
S_2 = S_1 \circ R = \left[ (0.9, 0.1, 0.1), (0.8, 0.1, 0.2), (0.9, 0.1, 0.1), (0.8, 0.1, 0.1) \right] \circ \left[ (0.7, 0.1, 0.2) & (0.8, 0.1, 0.2) & (0.5, 0.3, 0.4) & (0.8, 0.1, 0.1) \\
(0.9, 0.1, 0.2) & (0.5, 0.4, 0.2) & (0.8, 0.2, 0.2) & (0.6, 0.6, 0.1) \\
(0.5, 0.4, 0.3) & (0.6, 0.2, 0.3) & (0.9, 0.1, 0.3) & (0.5, 0.4, 0.2) \\
(0.7, 0.5, 0.1) & (0.4, 0.5, 0.2) & (0.6, 0.4, 0.1) & (0.2, 0.6, 0.4) \right] 
\]

i.e., \( S_2 = [(0.8, 0.1, 0.1), (0.8, 0.1, 0.2), (0.9, 0.1, 0.1), (0.8, 0.1, 0.1)]. \)

- **Step 3.** Since \( S_2 \neq S_1 \), we set \( n = 2 \) in step 2 and compose \( S_2 \) with \( R \) to get \( S_3 \), i.e.,

\[
S_3 = S_2 \circ R = \left[ (0.8, 0.1, 0.1), (0.8, 0.1, 0.1), (0.8, 0.1, 0.1) \right] \circ \left[ (0.7, 0.1, 0.2) & (0.8, 0.1, 0.2) & (0.5, 0.3, 0.4) & (0.8, 0.1, 0.1) \\
(0.9, 0.1, 0.2) & (0.5, 0.4, 0.2) & (0.8, 0.2, 0.2) & (0.6, 0.6, 0.1) \\
(0.5, 0.4, 0.3) & (0.6, 0.2, 0.3) & (0.9, 0.1, 0.3) & (0.5, 0.4, 0.2) \\
(0.7, 0.5, 0.1) & (0.4, 0.5, 0.2) & (0.6, 0.4, 0.1) & (0.2, 0.6, 0.4) \right] 
\]

i.e., \( S_3 = [(0.8, 0.1, 0.1), (0.8, 0.1, 0.2), (0.9, 0.1, 0.1), (0.8, 0.1, 0.1)]. \)

- **Step 4.** Since \( S_3 = S_2 \), therefore, \( S_2 \) is the greatest eigen spherical fuzzy set associated with \( R \).

**Algorithm II (GESFS):**

- **Step 1.** Find the set \( S_1 \) from \( R \) as directed by Equation (1).
- **Step 2.** Using successive composition of $R$, say, $R^{n+1} = R \circ R \circ R \ldots \circ R$, compute $S_{n+1}$ from $R^{n+1}$ using the Equation 1.

- **Step 3.** If $S_{n+1} \neq S_n$, then return to step 2.

- **Step 4.** If $S_{n+1} = S_n$, then $S_n$ is the GESFS associated with the relation $R$.

We consider the same example, i.e., Example 1 for the illustration of the computational steps of Algorithm II as below:

- **Step 1.** Using Equation 1, the set $S_1$ is given by
  \[ S_1 = \begin{bmatrix} (0.9, 0.1, 0.1), (0.8, 0.1, 0.2), (0.9, 0.1, 0.1), (0.8, 0.1, 0.1) \end{bmatrix}. \]

- **Step 2.** To find $S_2$, we compose $R$ with itself.
  
  \[
  R^2 = R \circ R \\
  = \begin{bmatrix} (0.7, 0.1, 0.2) & (0.8, 0.1, 0.2) & (0.5, 0.3, 0.4) & (0.8, 0.1, 0.1) \\ (0.9, 0.1, 0.2) & (0.5, 0.4, 0.2) & (0.8, 0.2, 0.2) & (0.6, 0.6, 0.1) \\ (0.5, 0.4, 0.3) & (0.6, 0.2, 0.3) & (0.9, 0.1, 0.3) & (0.5, 0.4, 0.2) \\ (0.7, 0.5, 0.1) & (0.4, 0.5, 0.2) & (0.6, 0.4, 0.1) & (0.2, 0.6, 0.4) \end{bmatrix} \\
  \circ \\
  \begin{bmatrix} (0.7, 0.1, 0.2) & (0.8, 0.1, 0.2) & (0.5, 0.3, 0.4) & (0.8, 0.1, 0.1) \\ (0.9, 0.1, 0.2) & (0.5, 0.4, 0.2) & (0.8, 0.2, 0.2) & (0.6, 0.6, 0.1) \\ (0.5, 0.4, 0.3) & (0.6, 0.2, 0.3) & (0.9, 0.1, 0.3) & (0.5, 0.4, 0.2) \\ (0.7, 0.5, 0.1) & (0.4, 0.5, 0.2) & (0.6, 0.4, 0.1) & (0.2, 0.6, 0.4) \end{bmatrix}
  \]

  Therefore, $S_2 = \begin{bmatrix} (0.7, 0.1, 0.1), (0.8, 0.1, 0.2), (0.9, 0.1, 0.1), (0.8, 0.1, 0.1) \end{bmatrix}$.

- **Step 3.** Since $S_2 \neq S_1$, therefore we find $S_3$ by further composing $R^2$ with $R$.

  \[
  R^3 = R^2 \circ R \\
  = \begin{bmatrix} (0.8, 0.1, 0.1) & (0.7, 0.1, 0.1) & (0.8, 0.1, 0.2) & (0.8, 0.1, 0.2) \\ (0.7, 0.1, 0.1) & (0.8, 0.1, 0.2) & (0.8, 0.1, 0.1) & (0.8, 0.1, 0.2) \\ (0.6, 0.1, 0.2) & (0.6, 0.1, 0.2) & (0.9, 0.1, 0.2) & (0.6, 0.1, 0.3) \\ (0.7, 0.1, 0.2) & (0.7, 0.1, 0.2) & (0.6, 0.1, 0.2) & (0.7, 0.1, 0.1) \end{bmatrix} \\
  \circ \\
  \begin{bmatrix} (0.7, 0.1, 0.2) & (0.8, 0.1, 0.2) & (0.5, 0.3, 0.4) & (0.8, 0.1, 0.1) \\ (0.9, 0.1, 0.2) & (0.5, 0.4, 0.2) & (0.8, 0.2, 0.2) & (0.6, 0.6, 0.1) \\ (0.5, 0.4, 0.3) & (0.6, 0.2, 0.3) & (0.9, 0.1, 0.3) & (0.5, 0.4, 0.2) \\ (0.7, 0.5, 0.1) & (0.4, 0.5, 0.2) & (0.6, 0.4, 0.1) & (0.2, 0.6, 0.4) \end{bmatrix}
  \]

  Therefore, $S_3 = \begin{bmatrix} (0.8, 0.1, 0.1), (0.8, 0.1, 0.2), (0.9, 0.1, 0.1), (0.8, 0.1, 0.1) \end{bmatrix}$.

- **Step 4.** Since $S_3 = S_2$, therefore, $S_2$ is the greatest eigen spherical fuzzy set associated with $R$. 

10
5.2 Algorithms for Finding Least Eigen Spherical Fuzzy Set

In order to obtain the least eigen spherical fuzzy set associated with the spherical fuzzy relation \( R \), we apply the \( \min - \max \) composition operator for spherical fuzzy relations.

Let \( S_1 \) be the spherical fuzzy set, i.e., \( S_1 \in SFS(U) \), in which the membership & the neutral membership values are the smallest of all the elements of the column of relation \( R \) and the non-membership value is the greatest of all the elements of the columns of \( R \), i.e.,

\[
\begin{align*}
\mu_{S_1}(x') &= \min_{x \in X} \mu_R(x, x'), \text{ for all } x' \in Y; \\
\eta_{S_1}(x') &= \min_{x \in X} \eta_R(x, x'), \text{ for all } x' \in Y; \\
\nu_{S_1}(x') &= \max_{x \in X} \nu_R(x, x'), \text{ for all } x' \in Y.
\end{align*}
\]  

(2)

Here, we initiate the process by taking \( S_0 \) as a constant spherical fuzzy set with a value equal to the minimum element of the set \( S_1 \). It is easy to verify that \( S_0 \) is an eigen spherical fuzzy set, but not the least eigen spherical fuzzy set always. To overcome this difficulty, we define the following sequence of spherical fuzzy sets \( S_n \) such that

\[
\begin{align*}
S_1 \cdot R &= S_2 \\
S_2 \cdot R &= S_1 \cdot R^2 = S_3 \\
& \vdots \\
S_n \cdot R &= S_1 \cdot R^n = S_{n+1}
\end{align*}
\]

Next, we present two fundamental algorithms along with numerical example for the determination of the LESFS as follows:

**Algorithm I (LESFS):**

- **Step 1.** Find the set \( S_1 \) as directed by Equation 2.
- **Step 2.** Set the index \( n = 1 \) and calculate \( S_{n+1} = S_n \cdot R \).
- **Step 3.** If \( S_{n+1} \neq S_n \), then return to step 2.
- **Step 4.** If \( S_{n+1} = S_n \), then \( S_n \) is the LESFS associated with \( R \).

We consider the same example again, i.e., Example 1 for the illustration of the computational steps of Algorithm I (LESFS) as below:

- **Step 1.** The set \( S_1 \) is given by

\[
S_1 = \left[ (0.5, 0.1, 0.3), (0.4, 0.1, 0.3), (0.5, 0.1, 0.4), (0.2, 0.1, 0.4) \right].
\]
- **Step 2.** For \( n = 1 \), we have
\[
S_2 = S_1 \cdot R
\]
\[
= \left[ (0.5, 0.1, 0.3), (0.4, 0.1, 0.3), (0.5, 0.1, 0.4), (0.2, 0.1, 0.4) \right] \circ \left[ 
\begin{array}{cccc}
0.7 & 0.1 & 0.2 & 0.8 \\
0.9 & 0.1 & 0.2 & 0.8 \\
0.5 & 0.4 & 0.3 & 0.9 \\
0.7 & 0.5 & 0.1 & 0.4
\end{array}
\right]
\]
\[
i.e., S_2 = \left[ (0.5, 0.1, 0.3), (0.4, 0.1, 0.3), (0.5, 0.1, 0.3), (0.2, 0.1, 0.4) \right].
\]

- **Step 3.** Since \( S_2 \neq S_1 \), we set \( n = 2 \) in step 2 and compose \( S_2 \) with \( R \) to get \( S_3 \), i.e.,
\[
S_3 = S_2 \cdot R
\]
\[
= \left[ (0.5, 0.1, 0.3), (0.4, 0.1, 0.3), (0.5, 0.1, 0.3), (0.2, 0.1, 0.4) \right] \circ \left[ 
\begin{array}{cccc}
0.7 & 0.1 & 0.2 & 0.8 \\
0.9 & 0.1 & 0.2 & 0.8 \\
0.5 & 0.4 & 0.3 & 0.9 \\
0.7 & 0.5 & 0.1 & 0.4
\end{array}
\right]
\]
\[
i.e., S_3 = \left[ (0.5, 0.1, 0.3), (0.4, 0.1, 0.3), (0.5, 0.1, 0.3), (0.2, 0.1, 0.4) \right].
\]

- **Step 4.** Since \( S_3 = S_2 \), therefore, \( S_2 \) is the least eigen spherical fuzzy set associated with \( R \).

**Algorithm II (LESFS):**

- **Step 1.** Find the set \( S_1 \) from \( R \) as directed by Equation (2).

- **Step 2.** Using successive composition of \( R \), say, \( R^{n+1} = R \cdot R \cdot R \cdots R \), compute \( S_{n+1} \) from \( R^{n+1} \) using the Equation 2.

- **Step 3.** If \( S_{n+1} \neq S_n \), then return to step 2.

- **Step 4.** If \( S_{n+1} = S_n \), then \( S_n \) is the least eigen spherical fuzzy set associated with the relation \( R \).

The proposed algorithm is being presented using the flow chart given in Figure 3.

We consider the same example, i.e., Example 1 for the illustration of the computational steps of Algorithm II (LESFS) as below:

- **Step 1.** Using Equation 2, the set \( S_1 \) is given by
\[
S_1 = \left[ (0.5, 0.1, 0.3), (0.4, 0.1, 0.3), (0.5, 0.1, 0.4), (0.2, 0.1, 0.4) \right].
\]
- Step 2. To find $S_2$, we compose $R$ with itself.

$$R^2 = R \cdot R$$

$$= \begin{bmatrix}
(0.7, 0.1, 0.2) & (0.8, 0.1, 0.2) & (0.5, 0.3, 0.4) & (0.8, 0.1, 0.1) \\
(0.9, 0.1, 0.2) & (0.5, 0.4, 0.2) & (0.8, 0.2, 0.2) & (0.6, 0.6, 0.1) \\
(0.5, 0.4, 0.3) & (0.6, 0.2, 0.3) & (0.9, 0.1, 0.3) & (0.5, 0.4, 0.2) \\
(0.7, 0.5, 0.1) & (0.4, 0.5, 0.2) & (0.6, 0.4, 0.1) & (0.2, 0.6, 0.4)
\end{bmatrix} \cdot \begin{bmatrix}
(0.7, 0.1, 0.2) & (0.8, 0.1, 0.2) & (0.5, 0.3, 0.4) & (0.8, 0.1, 0.1) \\
(0.9, 0.1, 0.2) & (0.5, 0.4, 0.2) & (0.8, 0.2, 0.2) & (0.6, 0.6, 0.1) \\
(0.5, 0.4, 0.3) & (0.6, 0.2, 0.3) & (0.9, 0.1, 0.3) & (0.5, 0.4, 0.2) \\
(0.7, 0.5, 0.1) & (0.4, 0.5, 0.2) & (0.6, 0.4, 0.1) & (0.2, 0.6, 0.4)
\end{bmatrix}$$

$$= \begin{bmatrix}
(0.5, 0.1, 0.3) & (0.6, 0.1, 0.3) & (0.7, 0.1, 0.3) & (0.5, 0.1, 0.2) \\
(0.7, 0.1, 0.2) & (0.5, 0.1, 0.2) & (0.6, 0.1, 0.2) & (0.6, 0.1, 0.2) \\
(0.7, 0.1, 0.3) & (0.5, 0.1, 0.3) & (0.5, 0.1, 0.3) & (0.5, 0.1, 0.2) \\
(0.6, 0.1, 0.2) & (0.4, 0.1, 0.2) & (0.6, 0.1, 0.2) & (0.2, 0.1, 0.4)
\end{bmatrix}.$$  

Therefore,

$$S_2 = \begin{bmatrix}
(0.5, 0.1, 0.3), (0.4, 0.1, 0.3), (0.5, 0.1, 0.3), (0.2, 0.1, 0.4)
\end{bmatrix}.$$  

- Step 3. Since $S_2 \neq S_1$, therefore we find $S_3$ by further composing $R^2$ with $R$.

$$R^3 = R^2 \cdot R$$

$$= \begin{bmatrix}
(0.5, 0.1, 0.3) & (0.6, 0.1, 0.3) & (0.7, 0.1, 0.3) & (0.5, 0.1, 0.2) \\
(0.7, 0.1, 0.2) & (0.5, 0.1, 0.2) & (0.6, 0.1, 0.2) & (0.6, 0.1, 0.2) \\
(0.7, 0.1, 0.3) & (0.5, 0.1, 0.3) & (0.5, 0.1, 0.3) & (0.5, 0.1, 0.2) \\
(0.6, 0.1, 0.2) & (0.4, 0.1, 0.2) & (0.6, 0.1, 0.2) & (0.2, 0.1, 0.4)
\end{bmatrix} \cdot \begin{bmatrix}
(0.7, 0.1, 0.2) & (0.8, 0.1, 0.2) & (0.5, 0.3, 0.4) & (0.8, 0.1, 0.1) \\
(0.9, 0.1, 0.2) & (0.5, 0.4, 0.2) & (0.8, 0.2, 0.2) & (0.6, 0.6, 0.1) \\
(0.5, 0.4, 0.3) & (0.6, 0.2, 0.3) & (0.9, 0.1, 0.3) & (0.5, 0.4, 0.2) \\
(0.7, 0.5, 0.1) & (0.4, 0.5, 0.2) & (0.6, 0.4, 0.1) & (0.2, 0.6, 0.4)
\end{bmatrix}$$

$$= \begin{bmatrix}
(0.7, 0.1, 0.3) & (0.5, 0.1, 0.3) & (0.5, 0.1, 0.3) & (0.5, 0.1, 0.2) \\
(0.6, 0.1, 0.2) & (0.5, 0.1, 0.2) & (0.6, 0.1, 0.2) & (0.6, 0.1, 0.2) \\
(0.5, 0.1, 0.3) & (0.5, 0.1, 0.3) & (0.6, 0.1, 0.3) & (0.5, 0.1, 0.2) \\
(0.6, 0.1, 0.2) & (0.4, 0.1, 0.2) & (0.6, 0.1, 0.2) & (0.2, 0.1, 0.4)
\end{bmatrix}.$$  

Therefore,

$$S_3 = \begin{bmatrix}
(0.5, 0.1, 0.3), (0.4, 0.1, 0.3), (0.5, 0.1, 0.3), (0.2, 0.1, 0.4)
\end{bmatrix}.$$  

- Step 4. Since $S_3 = S_2$, therefore, $S_2$ is the least eigen spherical fuzzy set associated with $R$.

6 Multi-Criteria Decision Making Using GESFS and LESFS

In a decision making problem, because of fuzziness in the human thinking, there is always a kind of complexity and uncertainty which is inherit in the available alternatives and laid down criteria. Therefore, it is apparently difficult to evaluate the parameters of the decision process with a desired preciseness. In view of these, the problem of multi-criteria decision making has been widely dealt and utilized in various applications [17] [18] [31] [32] [33]. On the basis of GESFS and LESFS, similar applications in the above stated field of decision making may also be studied. However, in order to illustrate the proposed methodology, we consider the following examples.
6.1 Example Related to Insurance Company

We consider an insurance company where the satisfaction/abstain/non-satisfaction levels of the customers are being taken into account for formulating the multiple-criteria decision making problem.

Example 2 Suppose an insurance company collects the information from 10 of its prime customers/experts about the important features of the company. Let the features are listed as

- \( F_1 \) : Customer Friendly Policies
- \( F_2 \) : Amplitude of Financial Benefits
- \( F_3 \) : Post Insurance Services

We may figure out the customer’s feedback on the basis of a survey. However, in order to illustrate the proposed methodology, we assume a set of data presented below without an exhaustive survey.

For evaluating some concluding remarks in view of the insurance company, we assume the each customer’s feedback as a spherical fuzzy information in a relative fashion among all the available/provided features and tabulate them in Table 1, Table 2 and Table 3.

The estimation of the desired levels in the form of the satisfaction/abstain/non-satisfaction levels is possible by considering the spherical fuzzy relation. Each pair of the relation \( R(F_j, F_k) \) has the membership value (satisfied), the indeterminacy value (abstain) and the non-membership value (not satisfied) which range from 0 to 1 is given by

\[
R(F_j, F_k) = \left( \frac{\sum_{p=1,q=1}^{p=m,q=n} \mu_{pq}}{m}, \frac{\sum_{p=1,q=1}^{p=m,q=n} \eta_{pq}}{m}, \frac{\sum_{p=1,q=1}^{p=m,q=n} \nu_{pq}}{m} \right); \quad (3)
\]

and

\[
R(F_i, F_i) = \frac{R(F_i, F_j) + R(F_i, F_k)}{2}; \quad (4)
\]

where \( j, k = 1, 2, \ldots, n \).

Using Equation (3) and Equation (4), the membership, indeterminacy and non-membership values for the different pairs of features have been computed as follows:

\[
R(F_1, F_1) = (0.585, 0.29, 0.285), \quad R(F_1, F_2) = (0.55, 0.30, 0.32), \quad R(F_1, F_3) = (0.62, 0.28, 0.25);
\]
On the basis of computations, we have found that the greatest eigen spherical fuzzy set, i.e., Algorithm I (GESFS) and get

\[ R_{(F_1,F_1)} = (0.59, 0.37, 0.31), \quad R_{(F_2,F_2)} = (0.605, 0.32, 0.255), \quad R_{(F_3,F_3)} = (0.62, 0.27, 0.20); \]

\[ R_{(F_3,F_1)} = (0.58, 0.24, 0.32), \quad R_{(F_3,F_2)} = (0.60, 0.28, 0.20), \quad R_{(F_3,F_3)} = (0.59, 0.26, 0.26). \]

Next, we construct the spherical fuzzy relation \( R \) using the above obtained interdependency of the features as follows:

\[
R = \begin{pmatrix}
F_1 & F_2 & F_3 \\
F_1 & R_{(F_1,F_1)} & R_{(F_1,F_2)} & R_{(F_1,F_3)} \\
F_2 & R_{(F_2,F_1)} & R_{(F_2,F_2)} & R_{(F_2,F_3)} \\
F_3 & R_{(F_3,F_1)} & R_{(F_3,F_2)} & R_{(F_3,F_3)}
\end{pmatrix}.
\]

Setting all the values, we get

\[
R = \begin{pmatrix}
F_1 & F_2 & F_3 \\
F_1 & (0.585, 0.29, 0.285) & (0.55, 0.30, 0.32) & (0.62, 0.28, 0.25) \\
F_2 & (0.59, 0.37, 0.31) & (0.605, 0.32, 0.255) & (0.62, 0.27, 0.20) \\
F_3 & (0.58, 0.24, 0.32) & (0.60, 0.28, 0.20) & (0.59, 0.26, 0.26)
\end{pmatrix}.
\]

Now we use the first proposed algorithm for finding the greatest eigen spherical fuzzy set, i.e., Algorithm I (GESFS) and get

\[
S_1 = [(0.59, 0.24, 0.285), (0.605, 0.28, 0.2), (0.62, 0.26, 0.2)];
\]

\[
S_2 = S_1 \circ R = [(0.59, 0.24, 0.285), (0.605, 0.24, 0.2), (0.605, 0.24, 0.2)];
\]

\[
S_3 = S_2 \circ R = [(0.59, 0.24, 0.285), (0.605, 0.24, 0.2), (0.605, 0.24, 0.2)].
\]

Since, \( S_2 = S_3 \), therefore, we conclude that \( S_2 \) is the greatest eigen spherical fuzzy set.

Further, we use the first proposed algorithm for finding the least eigen spherical fuzzy set, i.e., Algorithm I (LESFS) and get

\[
S_1 = [(0.58, 0.24, 0.32), (0.55, 0.28, 0.32), (0.59, 0.26, 0.26)];
\]

\[
S_2 = S_1 \bullet R = [(0.585, 0.24, 0.31), (0.58, 0.24, 0.32), (0.59, 0.24, 0.26)];
\]

\[
S_3 = S_2 \bullet R = [(0.585, 0.24, 0.31), (0.585, 0.24, 0.31), (0.59, 0.24, 0.26)];
\]

\[
S_4 = S_3 \bullet R = [(0.585, 0.24, 0.31), (0.585, 0.24, 0.31), (0.59, 0.24, 0.26)].
\]

Since, \( S_3 = S_4 \), therefore, we conclude that \( S_3 \) is the least eigen spherical fuzzy set.

**Observations and Results:**

On the basis of computations, we have found that the greatest eigen spherical fuzzy set and the least eigen spherical fuzzy set are given by

\[
GESFS = [(0.59, 0.24, 0.285), (0.605, 0.24, 0.2), (0.605, 0.24, 0.2)].
\]
and

\[ LESFS = \left[ (0.585, 0.24, 0.31), (0.585, 0.24, 0.31), (0.59, 0.24, 0.26) \right] \]

respectively. The values obtained in these sets represents the range of satisfaction/abstain/non-satisfaction levels for the features under consideration for an insurance company.

- Customers are satisfied in the range of 58.5% to 59%, abstain (24%) and unsatisfied in the range of 28.5% to 31% with respect to feature \( F_1 \).
- Customers are satisfied in the range of 58.5% to 60.5%, abstain (24%) and unsatisfied in the range of 20% to 31% with respect to feature \( F_2 \).
- Customers are satisfied in the range of 59% to 60.5%, abstain (24%) and unsatisfied in the range of 20% to 26% with respect to feature \( F_3 \).

It may be noted that the numerical values obtained in the GESFS and LESFS are close to each other. Actually, the illustration of the proposed algorithms has been done through the particular example (Example 2) which has a limited format and have less variability with respect to the dimensions and features involved in it. If we will have a big data with higher dimensionality of features, we may observe a significant variation in the values. However, the closeness in the values is a sign of preciseness in the process of decision making.

6.2 Example Related to E-learning Websites

Recently, Garg et al. [34] proposed a decision support system to educational organizations to develop and access E-learning websites by utilizing a hybrid multi-attribute decision making method for their evaluation and ranking. Jain et al. [35] utilized the weighted distance based approximation for the selection and ranking of E-websites. Based on weighted Euclidean distance, Garg [36] also proposed a computational quantitative model in order to evaluate, select and rank E-learning websites. Garg and Arora [37] evaluated Fraud detection model versus selection criteria as a multi-criteria decision making (MCDM) problem and obtained a validated comprehensive ranking.

We consider three E-learning websites where the satisfaction/abstain/non-satisfaction levels of the users are being taken into account for formulating the decision making problem. We assume the each user’s feedback as a spherical fuzzy information and tabulate the performance ratings of E-learning websites in Table 4, Table 5 and Table 6.
Next, we construct the spherical fuzzy relation $R$ using the above obtained inter-
dependency of the features as follows:

\[
R = \begin{bmatrix}
W_1 & W_2 & W_3 \\
W_1 & R_{(W_1,W_1)} & R_{(W_1,W_2)} & R_{(W_1,W_3)} \\
W_2 & R_{(W_2,W_1)} & R_{(W_2,W_2)} & R_{(W_2,W_3)} \\
W_3 & R_{(W_3,W_1)} & R_{(W_3,W_2)} & R_{(W_3,W_3)}
\end{bmatrix}.
\]

Setting all the values, we get

\[
R = \begin{bmatrix}
W_1 & W_2 & W_3 \\
W_1 & (0.615, 0.255, 0.315) & (0.63, 0.29, 0.33) & (0.62, 0.24, 0.29) \\
W_2 & (0.56, 0.30, 0.36) & (0.59, 0.325, 0.29) & (0.67, 0.35, 0.22) \\
W_3 & (0.55, 0.30, 0.31) & (0.26, 0.36, 0.21) & (0.555, 0.33, 0.26)
\end{bmatrix}.
\]

Next, on the basis of computations, we have found that the greatest eigen spherical
fuzzy set and the least eigen spherical fuzzy set are given by

\[
GESFS = \left[ (0.615, 0.22, 0.31), (0.63, 0.24, 0.22), (0.67, 0.24, 0.22) \right]
\]

and

\[
LESFS = \left[ (0.555, 0.22, 0.33), (0.56, 0.24, 0.33), (0.555, 0.24, 0.29) \right]
\]

respectively.

The values obtained in these sets represents the range of satisfaction/abstain/non-
satisfaction level among the users for the websites under consideration.

- Users are satisfied in the range of $55.5\%$ to $61.5\%$, abstain ($22\%$) and unsatisfied
  in the range of $31\%$ to $33\%$ with respect to the features of website $W_1$.

- Users are satisfied in the range of $56\%$ to $63\%$, abstain ($22\%$) and unsatisfied
  in the range of $22\%$ to $33\%$ with respect to the features of website $W_2$.

- Users are satisfied in the range of $55.5\%$ to $67\%$, abstain ($24\%$) and unsatisfied
  in the range of $22\%$ to $29\%$ with respect to the features of website $W_3$.

However, the closeness in the obtained values of the range are not that much prominent
as the data is hypothetical and quite small in size but for a large sample data the values
will certainly characterize the attainment level clearly.
7 Scope for Future Work

7.1 Image Information Retrieval

In the field of fuzzy image information analysis, an input original image is considered to be a fuzzy relation by a process of image intensity normalization. We may consider the idea of greatest eigen spherical fuzzy set w.r.t. the $\max - \min$ composition operator and the least eigen spherical fuzzy set w.r.t. the $\min - \max$ composition operator using the spherical fuzzy relation for solving the problems of retrieving the image information. A comparison between the sample image and the retrieved images can be done by using the similarity measure based on the GESFS and LESFS as follows:

$$D(R_1, R_2) = \sum_{x \in X} ((S_1(x) - S_2(x))^2 + T_1(x) - T_2(x))^2;$$

where $P = \{1, 2, \ldots, 256\}$, $S_i, T_i \in SFS(P)$ are the GESFS and LESFS of the relation $R_i \in SFR(P \times P); i = 1, 2$ w.r.t. the $\max - \min$ and $\min - \max$ composition respectively.

In literature, it may be noted that the existing conventional algorithms for obtaining the greatest/least eigen fuzzy sets are not capable enough to deal with the image retrieval problems for a solution significantly of good quality. It has also been observed that the use of the eigen fuzzy sets enhances the quality of the reconstructed image more often.

Using the convex combination of $\max - \min$ and $\min - \max$ composition operator for eigen fuzzy sets equations, various eigen fuzzy sets may be generated. Many eigen fuzzy sets equations can be obtained through the above stated convex combination, i.e., by changing the value of $\lambda$ present in the convex combination. The eigen fuzzy sets equation is given by

$$\lambda \cdot (S \circ R) + (1 - \lambda) \cdot (S \bullet R) = S;$$

where $\lambda$ can be chosen as per the best fit. The detailed and comprehensive study for the image information retrieval can be carried out by implementing the proposed algorithms and technique along with the above outlined directions.

7.2 Genetic Algorithm for Image Reconstruction

Fuzzy relational calculus for image compression is a natural tool for a genetic algorithm depending on the eigen fuzzy sets for image reconstruction. By normalizing the values of the pixels of any image (of size $m \times m$) w.r.t. the length of the gray scale used, it can be
interpreted as a square fuzzy relation \( R \). Here, the interpretation will be as a spherical fuzzy relation. Implementing the genetic algorithm method would lead to a prospective solution to a particular problem on basic data structure problems, e.g., chromosome in reference with the image by utilizing the recombination operators to these structures. Genetic algorithms have been frequently seen as optimization functions. However, the range of the problems to which the genetic algorithms have been applied is broad. The genetic algorithm approach may also be used for the reconstruction of an image by using its GESFS and LESFS of the spherical fuzzy relation in the fitness function of a chromosome. The value of the fitness function is given by

\[
F(R, R_k) = \frac{1}{\text{MSE}_{\text{GESFS} \leftrightarrow \text{LESFS}}(R, R_k)};
\]

where \( k = 1, 2, \ldots, N \) and the mean squared error is given by

\[
\text{MSE}_{\text{GESFS} \leftrightarrow \text{LESFS}}(R, R_k) = \frac{1}{256} \sum_{x \in X} \left( (S(x) - S_k(x))^2 + T(x) - T_k(x))^2 \right);
\]

where \( P = \{1, 2, \ldots, 256\} \), \( S \) and \( T \in \text{SFS}(P) \) are the GESFS and LESFS respectively of the spherical fuzzy relation \( R \) obtained by normalizing the pixels of the input original image.

In particular, we denote \( S_k \) and \( T_k \in \text{SFS}(P) \) as the GESFS and LESFS, respectively, of the spherical fuzzy relation \( R_k \) obtained by normalizing the pixels of the \( k^{th} \) image-chromosome of the population \( (N) \).

### 7.3 Notion of Eigen Spherical Fuzzy Soft Sets/Soft Matrices

In literature, a variety of extensions of soft sets [38] to imprecise and incomplete information have been proposed. In view of the generalizations and extensions of fuzzy sets shown by Figure 1 in the introduction section, we may further propose to extend the notion of eigen spherical fuzzy set to another new concept of eigen spherical fuzzy soft set and eigen spherical complex fuzzy soft set (Refer [39] [40] [41]). Naim and Serdar [42] introduced the concept of soft matrices from the soft sets and on similar lines one can study eigen spherical fuzzy soft matrices and its various properties in future. Since there is kind of parametrization tool involved in the soft sets and consequently in soft matrices, therefore based on this, various related applications e.g. stock management [43], valuation of assets [44], medical diagnosis [45], multi-criteria decision making [46] [47] [48] [49], dimensionality reduction [50] have been studied recently. Hence, introducing the concept of Eigen Spherical Fuzzy Soft Sets/Soft Matrices can lead to a new dimension in the extension of soft set theory and related applications.
8 Comparative Remarks and Advantages of ESFSs

The proposed notion of eigen spherical fuzzy set is a novel concept and an advanced extension of the classical fuzzy set. The eigen spherical fuzzy sets have an added advantage to deal with a wider sense of applicability in uncertain situations. In detail, some important comparative remarks and advantages of utilizing eigen spherical fuzzy set are listed below:

- The existing fuzzy sets, intuitionistic fuzzy sets and picture fuzzy sets have their own limitations that they are not capable to capture the full information specification, i.e., there is a missing additional component of degree of refusal which is addressed by the spherical fuzzy set.

- When the uncertain or imprecise information takes the form of a fuzzy relation then to ensure a kind of invariability in the relation after subsequent transitions, we utilize the concept of eigen fuzzy sets in natural sciences for therapeutic recommendations.

- The drawback in the existing literature of the eigen fuzzy sets is that the condition does not allow the experts/decision makers to allocate the membership values of their own choice (Refer Table 7). Somehow, this makes the decision makers bounded for providing their input in a particular domain. However, the proposed eigen spherical fuzzy sets provide a generalization feature which make a strong impact.

- The discussion over implementing the eigen spherical fuzzy sets and the methodology proposed for insurance problem and E-learning websites problem in Section 6 shows that the proposed work handled the generalized framework in an effective and consistent way. In other words, the GESFSs and the LESFSs of the spherical fuzzy relations have been employed to approximate the optimal level.

9 Conclusions

The concept of spherical fuzzy relation and various composition operators (\( max - min, \min - max, max - min \) average and \( min - max \) average) based on the combination of triangular norm & conorm have been successfully introduced. Further, the formal definition of an eigen spherical fuzzy set of spherical fuzzy relation has been provided followed by two respective algorithms to determine the greatest eigen spherical fuzzy sets and least eigen spherical fuzzy sets using the \( max - min \) and \( min - max \) composition operators. Some numerical examples have also been included for illustrations.
of the proposed algorithms. Utilization of the greatest eigen spherical fuzzy sets and least eigen spherical fuzzy sets in the field of decision making problem has also been successfully presented. The proposed algorithms can also be applied on a data with more variability as well. The directions for future work in the field of image information retrieval, genetic algorithm for image reconstruction and outlines to introduce the notion of Eigen Spherical Fuzzy Soft Sets/Soft Matrices have been briefly stated for further research.

Compliance with ethical standards

Conflict of interest: Authors declare that he/she has no conflict of interest.

Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.

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Figure Captions

- Figure 1: Extensions and Generalizations of Fuzzy Set
- Figure 2: Flow Chart for Algorithm I (GESFS)
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Table Captions

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Figure 1: Extensions and Generalizations of Fuzzy Set
Figure 2: Flow Chart for Algorithm I (GESFS)

Figure 3: Flow Chart for Algorithm II (LESFS)

Table 1: Relative Feedback with $F_1$ and $F_2$

| Customers/Experts | $F_1$       | $F_2$       |
|-------------------|-------------|-------------|
| $E_1$             | (0.5, 0.1, 0.6) | (0.8, 0.3, 0.2) |
| $E_2$             | (0.6, 0.6, 0.1) | (0.7, 0.6, 0.2) |
| $E_3$             | (0.7, 0.3, 0.2) | (0.9, 0.2, 0.1) |
| $E_4$             | (0.8, 0.1, 0.2) | (0.6, 0.6, 0.1) |
| $E_5$             | (0.4, 0.6, 0.1) | (0.5, 0.4, 0.4) |
| $E_6$             | (0.3, 0.4, 0.4) | (0.5, 0.5, 0.2) |
| $E_7$             | (0.2, 0.4, 0.6) | (0.4, 0.2, 0.6) |
| $E_8$             | (0.8, 0.2, 0.1) | (0.3, 0.2, 0.7) |
| $E_9$             | (0.3, 0.2, 0.7) | (0.8, 0.3, 0.2) |
| $E_{10}$          | (0.9, 0.1, 0.2) | (0.4, 0.4, 0.4) |
Table 2: Relative Feedback with $F_1$ and $F_3$

| Customers/Experts | $F_1$       | $F_3$       |
|-------------------|-------------|-------------|
| $E_1$             | (0.7, 0.3, 0.2) | (0.5, 0.5, 0.1) |
| $E_2$             | (0.4, 0.5, 0.4) | (0.4, 0.2, 0.6) |
| $E_3$             | (0.6, 0.3, 0.2) | (0.7, 0.3, 0.1) |
| $E_4$             | (0.8, 0.1, 0.2) | (0.8, 0.4, 0.1) |
| $E_5$             | (0.4, 0.4, 0.3) | (0.9, 0.1, 0.2) |
| $E_6$             | (0.9, 0.2, 0.1) | (0.4, 0.2, 0.6) |
| $E_7$             | (0.6, 0.3, 0.2) | (0.2, 0.1, 0.7) |
| $E_8$             | (0.7, 0.4, 0.1) | (0.6, 0.2, 0.4) |
| $E_9$             | (0.4, 0.1, 0.6) | (0.7, 0.1, 0.3) |
| $E_{10}$          | (0.7, 0.2, 0.2) | (0.6, 0.3, 0.1) |

Table 3: Relative Feedback with $F_2$ and $F_3$

| Customers/Experts | $F_2$       | $F_3$       |
|-------------------|-------------|-------------|
| $E_1$             | (0.4, 0.5, 0.1) | (0.9, 0.1, 0.2) |
| $E_2$             | (0.6, 0.2, 0.3) | (0.8, 0.1, 0.1) |
| $E_3$             | (0.8, 0.1, 0.1) | (0.7, 0.3, 0.1) |
| $E_4$             | (0.9, 0.1, 0.2) | (0.6, 0.4, 0.1) |
| $E_5$             | (0.7, 0.4, 0.1) | (0.5, 0.1, 0.5) |
| $E_6$             | (0.8, 0.1, 0.3) | (0.4, 0.3, 0.1) |
| $E_7$             | (0.4, 0.4, 0.3) | (0.6, 0.4, 0.1) |
| $E_8$             | (0.2, 0.7, 0.1) | (0.4, 0.2, 0.4) |
| $E_9$             | (0.9, 0.1, 0.1) | (0.5, 0.5, 0.2) |
| $E_{10}$          | (0.5, 0.1, 0.4) | (0.6, 0.4, 0.2) |

Table 4: Relative Feedback with $W_1$ and $W_2$

| Customers/Experts | $W_1$       | $W_2$       |
|-------------------|-------------|-------------|
| $U_1$             | (0.7, 0.1, 0.6) | (0.8, 0.2, 0.1) |
| $U_2$             | (0.8, 0.2, 0.1) | (0.3, 0.2, 0.7) |
| $U_3$             | (0.7, 0.3, 0.2) | (0.9, 0.2, 0.1) |
| $U_4$             | (0.9, 0.1, 0.2) | (0.4, 0.2, 0.6) |
| $U_5$             | (0.4, 0.6, 0.1) | (0.5, 0.4, 0.4) |
| $U_6$             | (0.6, 0.6, 0.1) | (0.4, 0.6, 0.1) |
| $U_7$             | (0.7, 0.3, 0.2) | (0.8, 0.3, 0.2) |
| $U_8$             | (0.6, 0.2, 0.5) | (0.4, 0.5, 0.4) |
| $U_9$             | (0.2, 0.4, 0.6) | (0.4, 0.2, 0.6) |
| $U_{10}$          | (0.9, 0.2, 0.3) | (0.3, 0.2, 0.5) |
Table 5: Relative Feedback with \( W_1 \) and \( W_3 \)

| Customers/Experts | \( W_1 \)    | \( W_3 \) |
|-------------------|-------------|-------------|
| \( U_1 \)         | (0.6, 0.3, 0.2) | (0.7, 0.3, 0.2) |
| \( U_2 \)         | (0.7, 0.2, 0.3) | (0.5, 0.5, 0.2) |
| \( U_3 \)         | (0.9, 0.1, 0.1) | (0.3, 0.6, 0.2) |
| \( U_4 \)         | (0.6, 0.2, 0.3) | (0.4, 0.3, 0.6) |
| \( U_5 \)         | (0.4, 0.5, 0.4) | (0.6, 0.2, 0.4) |
| \( U_6 \)         | (0.7, 0.2, 0.2) | (0.6, 0.3, 0.1) |
| \( U_7 \)         | (0.9, 0.2, 0.1) | (0.4, 0.2, 0.6) |
| \( U_8 \)         | (0.4, 0.2, 0.5) | (0.6, 0.3, 0.2) |
| \( U_9 \)         | (0.8, 0.1, 0.2) | (0.5, 0.2, 0.4) |
| \( U_{10} \)      | (0.2, 0.2, 0.6) | (0.9, 0.1, 0.2) |

Table 6: Relative Feedback with \( W_2 \) and \( W_3 \)

| Customers/Experts | \( W_2 \)    | \( W_3 \) |
|-------------------|-------------|-------------|
| \( U_1 \)         | (0.9, 0.1, 0.2) | (0.6, 0.2, 0.3) |
| \( U_2 \)         | (0.8, 0.4, 0.1) | (0.4, 0.5, 0.1) |
| \( U_3 \)         | (0.8, 0.2, 0.3) | (0.4, 0.3, 0.1) |
| \( U_4 \)         | (0.2, 0.7, 0.1) | (0.8, 0.2, 0.1) |
| \( U_5 \)         | (0.5, 0.5, 0.2) | (0.5, 0.4, 0.4) |
| \( U_6 \)         | (0.6, 0.4, 0.2) | (0.4, 0.4, 0.3) |
| \( U_7 \)         | (0.2, 0.6, 0.5) | (0.9, 0.2, 0.2) |
| \( U_8 \)         | (0.8, 0.2, 0.1) | (0.4, 0.5, 0.2) |
| \( U_9 \)         | (0.5, 0.2, 0.4) | (0.6, 0.4, 0.1) |
| \( U_{10} \)      | (0.9, 0.2, 0.1) | (0.5, 0.5, 0.3) |

Table 7: Need to address the problem arises in IFSs, PyFSs and PFSs

| \( R \) | \( C_1 \) | \( C_2 \) | \( C_3 \) | \( C_4 \) |
|---------|----------|----------|----------|----------|
| \( C_1 \) | (1.0 + 0.0 + 0.0 = 1) | (0.40 + 0.20 + 0.69 > 1) | (0.36 + 0.19 + 0.79 > 1) | (0.56 + 0.17 + 0.62 > 1) |
| \( C_2 \) | (0.68 + 0.20 + 0.44 > 1) | (1.0 + 0.0 + 0.0 > 1) | (0.40 + 0.24 + 0.56 > 1) | (0.51 + 0.29 + 0.61 > 1) |
| \( C_3 \) | (0.76 + 0.20 + 0.42 > 1) | (0.54 + 0.24 + 0.42 > 1) | (1.0 + 0.0 + 0.0 > 1) | (0.48 + 0.14 + 0.77 > 1) |
| \( C_4 \) | (0.49 + 0.17 + 0.68 > 1) | (0.59 + 0.29 + 0.53 > 1) | (0.77 + 0.17 + 0.38 > 1) | (1.0 + 0.0 + 0.0 > 1) |