Soliton solutions for a class of quasilinear Schrödinger equations with a parameter*

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Abstract Using variational methods combined with perturbation arguments, we study the existence of nontrivial classical solution for the quasilinear Schrödinger equation

\[ -\Delta u + V(x)u + \frac{\kappa}{2}[\Delta |u|^2]u = l(u), \ x \in \mathbb{R}^N, \]

where \( V : \mathbb{R}^N \to \mathbb{R} \) and \( l : \mathbb{R} \to \mathbb{R} \) are continuous function, \( \kappa \) is a parameter and \( N \geq 3 \). This model has been proposed in plasma physics and nonlinear optics. As a main novelty with respect to some previous results, we are able to deal with the case \( \kappa > 0 \).

Keywords Quasilinear Schrödinger equations; Mountain pass theorem; Standing waves

MSC 35B33; 35J20; 35J60; 35Q55

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1 Introduction

In this paper we consider the following quasilinear Schrödinger equations

$$-\Delta u + V(x)u + \frac{\kappa}{2}[\Delta |u|^2]u = l(u), \ x \in \mathbb{R}^N, \quad (1.1)$$

where $V: \mathbb{R}^N \to \mathbb{R}$ and $l: \mathbb{R} \to \mathbb{R}$ are continuous functions and $\kappa > 0$ is a parameter. Solutions of (1.1) are related to standing wave solutions for the following Schrödinger equations:

$$iz_t = -\Delta z + W(x)z - \rho(|z|^2)z + \frac{\kappa}{2}[\Delta |z|^2]z, \ x \in \mathbb{R}^N, \quad (1.2)$$

where $W: \mathbb{R}^N \to \mathbb{R}$ is a given potential and $\rho: \mathbb{R} \to \mathbb{R}$ is a real function. Quasi-linear Schrödinger equations like (1.2) play an important role in various domains in physics. For $\rho(s) = as^p$, (1.2) appears in various problems in plasma physics and nonlinear optics, e.g. oscillating soliton instabilities during microwave and laser heating of plasma [11, 22]. Moreover, (1.2) is also the basic equation describing oscillations in a superfluid film when $\rho(s) = \alpha - \beta (a + s)^3$ [15]. We refer the readers to [5, 6, 12, 16, 18] for more details on the background.

In this paper, we restrict ourselves to two model cases $\rho(s) = s - 2$ or $\rho(s) = -1 + \frac{1}{(1+s)^3}$ and we are interested in the existence of standing wave solutions, i.e. solutions of the form $z(t, x) = \exp(-iEt)u(x)$, where $E \in \mathbb{R}$ and $u$ is a real function. Putting $z(t, x) = \exp(-iEt)u(x)$ into (1.2), we are led to the following equations

$$-\Delta u + V(x)u + \frac{\kappa}{2}[\Delta |u|^2]u = |u|^{q-2}u, \ x \in \mathbb{R}^N, \quad (1.3)$$

or

$$-\Delta u + V(x)u + \frac{\kappa}{2}[\Delta |u|^2]u = \left[1 - \frac{1}{(1 + |u|^2)^3}\right]u, \ x \in \mathbb{R}^N, \quad (1.4)$$

with $V(x) = W(x) - E$.

For equation (1.3), semilinear case corresponding to $\kappa = 0$ has been studied extensively in recent years, see e.g. [2, 17]. When $\kappa < 0$, this equation has been introduced in [3, 4, 13] to study a model of self-trapped electrons in quadratic or hexagonal lattices and has attracted much attention. For the subcritical case, i.e., $4 < q < 22^*$ in (1.3), the first existence results are, up to our knowledge, due to Poppenberg, Schmitt and Wang in [23]. In [23], the main existence results are obtained, through a constrained minimization argument. Subsequently a general existence result for (1.3) was derived in Liu, Wang and Wang [20]. The idea in [20] is to make a change of variable and reduce the quasilinear problem (1.3) to semilinear one and an Orlicz space framework was used to prove the existence of a positive solutions via Mountain pass theorem. The same method of changing of variable was also used by Colin and Jeanjean in [8], but the usual Sobolev space $H^1(\mathbb{R}^N)$ framework was used as the working space. Precisely, since the energy functional associated to (1.3) is not well defined in $H^1(\mathbb{R}^N)$,
they first make the changing of unknown variables \( v = f^{-1}(u) \), where \( f \) is defined by ODE:

\[
f'(t) = \frac{1}{\sqrt{1 - \kappa f^2(t)}}, \quad t \in [0, +\infty),
\]

and \( f(t) = -f(-t), \quad t \in (-\infty, 0] \). Then, after the changing of variables, to find the solutions of (1.3), it suffices to study the existence of solutions for the following semilinear equation

\[-\Delta v = \frac{1}{\sqrt{1 - \kappa f^2(v)}} (V(x)f(v) + |f(v)|^{q-2}f(v)), \quad x \in \mathbb{R}^N.\]

By using the classical results given by Berestycki and Lions [2], they proved the existence of a spherically symmetric solution. In [21], the authors used a minimization on a Nehari-type constraint to get existence results. Their argument does not depend on any change of variables, so it can be applied to treat more general problems. By minimization under a convenient constraint, Ruiz and Siciliano in [24] discussed the existence of ground states for (1.3) with \( q \in (2, \frac{4N}{N-2}) \). For the critical case, Silva and Vieira in [25] established the existence of solutions for asymptotically periodic quasilinear Schrödinger equations (1.3) with the nonlinearity \(|u|^{q-2}u\) replaced by \( K(x)u^{2(2^*)^{-1}} + g(x,u) \). The existence of multiple solutions were established in [28]. We refer to [7, 19, 28, 29] for more results.

Recently, in [26, 30], the authors introduced the changing of known variables \( s = G^{-1}(t) \) for \( t \in [0, +\infty) \) and \( G^{-1}(t) = -G^{-1}(-t) \) for \( t \in (-\infty, 0) \), where

\[
G(s) = \int_0^s \sqrt{1 - \kappa t^2} dt.
\]

Since \( \kappa < 0 \), integral (1.6) makes sense and the inverse function \( G^{-1}(t) \) exists. Then, using variational methods, they established the existence of nontrivial solutions for (1.3) with subcritical or critical growth.

The main purpose of the present paper is studying the existence of nontrivial solutions for models (1.3) and (1.4) with \( \kappa > 0 \). Unfortunately, we note that at this moment, neither the changing of variables (1.5) nor (1.6) are suitable for dealing with this kind of problem because \( 1 - \kappa t^2 \) may be negative. As far as we know, in the mathematical literature, few results are known on (1.3) and (1.4) with \( \kappa > 0 \). Hereafter, we assume that potential \( V : \mathbb{R}^N \rightarrow \mathbb{R} \) is continuous and satisfies:

(V0) \( V(x) \geq V_0 > 0 \), for all \( x \in \mathbb{R}^N \).

(V1) \( \lim_{|x| \to \infty} V(x) = V_\infty \) and \( V(x) \leq V_\infty \), for all \( x \in \mathbb{R}^N \).

We have the following result:

**Theorem 1.1.** Assume that \( 2 < q < 2^* \), (V0) and (V1). Then, there exists some \( \kappa_0 > 0 \) such that for all \( \kappa \in [0, \kappa_0) \), (1.3) has a solution. Moreover, \( \max_{x \in \mathbb{R}^N} |u(x)| \leq \sqrt{\frac{1}{\kappa}} \).
For equation (1.4), we may state:

**Theorem 1.2.** Assume that \((V_0)\) with \(V_0 \geq 1\) and \((V_1)\). Then, there exists some \(\kappa_1 \in (0, \frac{1}{3})\) such that for all \(\kappa \in (0, \kappa_1)\), (1.4) has a solution. Moreover, \(\max_{x \in \mathbb{R}^N} |u(x)| \leq 1\).

**Remark 1.1.** When \(V(x) \equiv V_\infty\), using the classical results given by Berestycki and Lions [2], Theorems 1.1 and 1.2 are still true. Furthermore, \(u\) also has the following properties:

1. \(u > 0\) on \(\mathbb{R}^N\);
2. \(u\) is spherically symmetric and \(u\) decreases with respect to \(|x|\);
3. \(u \in C^2(\mathbb{R}^N)\);
4. \(u\) together with its derivatives up to order 2 have exponential decay at infinity:

\[ |D^\alpha u| \leq C e^{-\delta |x|}, \quad x \in \mathbb{R}^N, \]

for some \(C, \delta > 0\) and \(|\alpha| \leq 2\).

Therefore, in this paper, we assume that \(V(x) \leq V_\infty\) for all \(x \in \mathbb{R}^N\) but \(V(x) \not\equiv V_\infty\).

**Remark 1.2.** When \(\kappa = 0\), (1.1) has already been studied by many authors, see e.g. [1, 2, 14]. So, in Theorem 1.1, we only consider the case \(\kappa > 0\).

**Remark 1.3.** When \(\kappa = 0\), equation (1.4) turns into the following asymptotically semilinear problem

\[ -\Delta u + V(x)u = \left[1 - \frac{1}{(1 + |u|^2)^\frac{\alpha}{2}}\right]u, \quad x \in \mathbb{R}^N. \]

That is, the nonlinearity \(\rho(t) = \left[1 - \frac{1}{(1 + |t|^2)^\frac{\alpha}{2}}\right]t\) satisfy \(\lim_{t \to \infty} \frac{\rho(t)}{t} = 1\). Although the existence of nontrivial solutions for this type of equation may already be known, we have been unable to find a proper reference.

**Remark 1.4.** We remark that in Theorem 1.2, \(\kappa_0\) is dependent on the value \(2 < q < 2^*\).

**Remark 1.5.** In [3], L. Brüll, H. Lange and Köln studied the one-dimensional quasilinear Schrödinger equations

\[ iz_t = -\partial_x^2 z - |z|^{2p} z + \kappa \partial_x^2 (|z|^2) z, \quad x \in \mathbb{R} \] \tag{1.7}

and

\[ iz_t = -\partial_x^2 z - \left[\mu + \frac{A}{(a + |z|^2)^\frac{\alpha}{2}}\right] z + \kappa \partial_x^2 (|z|^2) z, \quad x \in \mathbb{R}, \] \tag{1.8}

where \(z = z(x, t)\) is the unknown wave function, \(\kappa\) is a real constant, \(p > 0\), \(\mu > 0\) and \(A < 0\). Under some conditions on \(p, \mu\) and \(A\), they proved that if
0 < \kappa < \kappa_2 \text{ (or } 0 < \kappa < \kappa_3 \text{)} with some } \kappa_2, \kappa_3 > 0, \text{ then } (1.7) \text{ (or } (1.8) \text{)} \text{ has a standing wave solution } v(x) \text{ with } v(x) > 0, v(-x) = v(x), v'(x) < 0 \text{ for } x > 0 \text{ and } \lim_{|x| \to \infty} v(x) = 0. \text{ Moreover, this solution is unique up to translation. Here, we generalize their results to higher dimension.}

**Remark 1.6.** For \( \kappa > 0 \), H. Lange, M. Poppenberg and H. Teisnann [10] studied the whole space Cauchy problem for quasilinear Schrödinger equation (1.2) with \( W = 0 \) and \( \rho = 0 \). When \( N = 1 \) and \( z(0, x) = \phi(x) \), they obtained \( L^2 \)-solutions for (1.2) with \( \kappa|\phi(x)| \leq \delta < 1 \). Moreover, for \( 2\kappa||\phi||_{W^{1,\infty}} < 1 \), they also proved the existence of \( H^2 \)-solutions for arbitrary space dimension. We refer to [10] for more details.

Note that (1.3) is the Euler-Lagrange equation associated to the natural energy functional

\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^N} (1 - \kappa u^2)|\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)u^2 dx - \frac{1}{q} \int_{\mathbb{R}^N} |u|^q dx \tag{1.9}
\]

From the variational point of view, the first difficulty that we have to deal with is to find some proper Sobolev space since (1.9) is not well defined in \( H^1(\mathbb{R}^N) \) for \( N \geq 3 \) and \( \kappa \neq 0 \). However, even if this difficulty is set up, there is another one: to guarantee the positiveness of the principal part, i.e. \( 1 - \kappa u^2 > 0 \).

In order to prove our main results, we first establish a nontrivial solution for a modified quasilinear Schrödinger equation

\[-\text{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = l(u), \quad x \in \mathbb{R}^N \tag{1.10}\]

with \( g(t) = \sqrt{1 - \kappa t^2} \) for \( |t| < \sqrt{\frac{1}{3\kappa}} \) for \( \kappa > 0 \), where \( V : \mathbb{R}^N \to \mathbb{R} \) is a continuous function, \( 2 < q < 2^* \), \( N \geq 3 \). Clearly, when \( g(t) = \sqrt{1 - \kappa t^2} \) and \( l(u) = |u|^{q-2}u \), (1.10) turns into (1.3). Then, by using Morse \( L^\infty \) estimate, we prove that there exists \( \kappa_0 > 0 \) such that for all \( \kappa \in [0, \kappa_0) \) the solutions that we have found verify the estimate \( \max_{\mathbb{R}^N} |u| < \sqrt{\frac{1}{3\kappa}} \). Thus, they are solutions of the original problem (1.3). To prove Theorem 1.2 we need further to modify the nonlinearity.

We mention that similar method have been adopted by Alves, Soares and Souto to study a supercritical Schrödinger-Poisson equation [1]. In [1], they mainly modified the nonlinearity and provide an estimate involving the \( L^\infty \)-norm of a solution related to a subcritical problem. However, unlike [1], here we need to modify the principal part first.

The organization of this paper is as follows: In Section 2, we reformulate the problem and study the existence of nontrivial solutions of a modified quasilinear Schrödinger equation (1.10). In Section 3, we provide an estimate involving the \( L^\infty \)-norm of a solution related to (1.10) and we prove Theorems 1.1. Section 4 is devoted to prove Theorem 1.2.

In this paper, \( C, C_i, i = 1, 2, \cdots \) denote positive (possibly different) constant. Moreover, \( \| \cdot \|_p \) denotes the norm of \( L^p(\mathbb{R}^N) \).
2 The modified problem

Hereafter, we shall work on the space $H^1(\mathbb{R}^N)$ endowed with the norm

$$||u|| = \left[ \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx \right]^{\frac{1}{2}}.$$

By $(V_0)$ and $(V_1)$, the above norm is equivalent the usual one on $H^1(\mathbb{R}^N)$.

For equation $(1.10)$, we let $l(t) = |t|^{q-2}t$ for $2 < q < 2^*$ and we will consider $g : [0, +\infty) \to \mathbb{R}$ given by

$$g(t) = \begin{cases} \sqrt{1-\kappa t^2}, & \text{if } 0 \leq t < \sqrt{\frac{1}{3\kappa}}, \\ \frac{1}{3\sqrt{2\kappa t}} + \sqrt{\frac{1}{6}}, & \text{if } \sqrt{\frac{1}{3\kappa}} \leq t. \end{cases}$$

Setting $g(t) = g(-t)$ for all $t \leq 0$, it follows that $g \in C^1(\mathbb{R}, (\sqrt{\frac{1}{6}}, 1))$, $g$ is a even function, increases in $(-\infty, 0)$ and decreases in $[0, +\infty)$.

Note that $(1.10)$ is the Euler-Lagrange equation associated to the natural energy functional

$$I_\kappa(u) = \frac{1}{2} \int_{\mathbb{R}^N} g^2(u)|\nabla u|^2dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|u|^2dx - \frac{1}{q} \int_{\mathbb{R}^N} |u|^qdx, \tag{2.1}$$

Our goal is proving the existence of a nontrivial critical point $u$ of $(2.1)$ satisfying $\sup_{x \in \mathbb{R}^N} |u(x)| \leq \sqrt{\frac{1}{3\kappa}}$, which will be a nontrivial solution of $(1.10)$ with $g(u) = \sqrt{1-\kappa u^2}$, and so, a nontrivial solution of $(1.3)$.

In what follows, we set

$$G(t) = \int_0^t g(s)ds$$

and we observe that inverse function $G^{-1}(t)$ exists and it is an odd function. Moreover, it is very important to observe that $G, G^{-1} \in C^2(\mathbb{R})$.

Next lemma shows important properties involving functions $g$ and $G^{-1}$ which will be used later on.

**Lemma 2.1.**

1. $\lim_{t \to 0} \frac{G^{-1}(t)}{t} = 1$;
2. $\lim_{t \to \infty} \frac{G^{-1}(t)}{t} = \sqrt{6}$;
3. $t \leq G^{-1}(t) \leq \sqrt{6}t$, for all $t \geq 0$;
4. $-\frac{1}{2} \leq \frac{t}{g(t)} g'(t) \leq 0$, for all $t \geq 0$. 


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Proof. By the definition of $g$,  
\[
\lim_{t \to 0} \frac{G^{-1}(t)}{t} = \lim_{t \to 0} \frac{1}{g(G^{-1}(t))} = 1
\]
and  
\[
\lim_{t \to \infty} \frac{G^{-1}(t)}{t} = \lim_{t \to \infty} \frac{1}{g(G^{-1}(t))} = \sqrt{6}.
\]
Thus, (1) and (2) are proved. Since $g(t) > 0$ is decreasing in $[0, \infty)$, then  
\[
\sqrt{\frac{1}{6}} t \leq g(t) t \leq G(t) \leq t
\]
for all $t \geq 0$, which implies (3). By a direct calculation, we get (4).

\[
\square
\]

Now, fixing the change variable  
\[
v = G(u) = \int_0^u g(s)ds,
\]
we observe that functional $I_{\kappa}$ can be written of the following way  
\[
J_{\kappa}(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) |G^{-1}(v)|^2 dx - \frac{1}{q} \int_{\mathbb{R}^N} |G^{-1}(v)|^q dx. \tag{2.3}
\]
From Lemma 2.1, $J_{\kappa}$ is well defined in $H^1(\mathbb{R}^N)$, $J \in C^1(\mathbb{R}^N)$ and  
\[
J'_{\kappa}(v) \psi = \int_{\mathbb{R}^N} \left[ \nabla v \nabla \psi + V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} \psi - \frac{|G^{-1}(v)|^{q-2} G^{-1}(v)}{g(G^{-1}(v))} \psi \right] dx, \tag{2.4}
\]
for all $v, \phi \in H^1(\mathbb{R}^N)$.

**Lemma 2.2.** If $v \in C^2(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ is a critical point of $J_{\kappa}$, then $u = G^{-1}(v) \in C^2(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ and it is a classical solution for (1.10).  

**Proof.** By using the fact that $G^{-1} \in C^2(\mathbb{R})$ together with Lemma 2.1, a direct computation gives $u = G^{-1}(v)$ belongs to $C^2(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$.  

If $v$ is a critical point for $J_{\kappa}$, we have that  
\[
\int_{\mathbb{R}^N} \left[ \nabla v \nabla \psi + V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} \psi - \frac{|G^{-1}(v)|^{q-2} G^{-1}(v)}{g(G^{-1}(v))} \psi \right] dx = 0, \quad \forall \psi \in H^1(\mathbb{R}^N).
\]
For each $\varphi \in C_0^\infty(\mathbb{R}^N)$, we consider $\psi = g(u)\varphi \in C_0^2(\mathbb{R}^N) \subset H^1(\mathbb{R}^N)$ in (2.4), to get  
\[
\int_{\mathbb{R}^N} [g^2(u) \nabla u \nabla \varphi + g(u) g'(u) |\nabla u|^2 \varphi + V(x) u \varphi - |u|^{q-2} u \varphi] dx = 0
\]
or equivalently,  
\[
\int_{\mathbb{R}^N} [-\text{div}(g^2(u) \nabla u) + g(u) g'(u) |\nabla u|^2 + V(x) u - |u|^{q-2} u] \varphi dx = 0 \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N),
\]

showing that \( u \) is a classical solution of
\[
- \text{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = |u|^{q-2}u \quad \text{in} \quad \mathbb{R}^N.
\]

Therefore, in order to find a nontrivial solutions of (1.10), it suffices studying the existence of nontrivial solutions of the following equation
\[
- \Delta v + V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))} - \frac{|G^{-1}(v)|^{q-2}G^{-1}(v)}{g(G^{-1}(v))} = 0 \quad \text{in} \quad \mathbb{R}^N. \tag{2.5}
\]

Next, we establish the geometric hypotheses of the Mountain Pass Theorem for \( J_\kappa \).

**Lemma 2.3.** For \( 2 < q < 2^* \), there exist \( \rho_0, a_0 > 0 \), such that \( J_\kappa(v) \geq a_0 \) for \( \|v\| = \rho_0 \). Moreover, there exists \( e \in H^1(\mathbb{R}^N) \) such that \( J_\kappa(e) < 0 \).

**Proof.** By Lemma 2.1 (3) and Sobolev embedding,
\[
J_\kappa(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^2 dx - \frac{1}{q} \int_{\mathbb{R}^N} |G^{-1}(v)|^q dx \\
\geq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|v|^2 dx - \frac{6^{q/2}}{q} \int_{\mathbb{R}^N} |v|^q dx \\
\geq \frac{1}{2} \|v\|^2 - C\|v\|^q.
\]

Thereby, by choosing \( \rho_0 \) small, we get
\[
a_0 = \frac{1}{2} \rho_0^2 - C\rho_0^q > 0,
\]
and so,
\[
J_\kappa(v) \geq a_0 \quad \text{for} \quad \|v\| = \rho_0.
\]

In order to prove the existence of \( e \in H^1(\mathbb{R}^N) \) such that \( J_\kappa(e) < 0 \), we fix \( \varphi \in C_0^\infty(\mathbb{R}^N, [0,1]) \) with \( \text{supp} \varphi = \bar{B}_1 \) and show that \( J_\kappa(t\varphi) \to -\infty \) as \( t \to \infty \), because the result follows taking \( e = t\varphi \) with \( t \) large enough. By Lemma 2.1 (3),
\[
J_\kappa(t\varphi) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla \varphi|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|G^{-1}(t\varphi)|^2 dx - \frac{1}{q} \int_{\mathbb{R}^N} |G^{-1}(t\varphi)|^q dx \\
\leq 3t^2 \int_{\mathbb{R}^N} (|\nabla \varphi|^2 + V_\infty \varphi^2) dx - \frac{6^{q/2}}{q} \int_{\mathbb{R}^N} \varphi^q dx.
\]

Since \( q > 2 \), it follows that \( J_\kappa(t\varphi) \to -\infty \) as \( t \to \infty \).

In consequence of Lemma 2.3 and Ambrosetti–Rabinowitz Mountain Pass Theorem [27], for the constant
\[
c_\kappa = \inf_{\gamma \in \Gamma_\kappa} \sup_{t \in [0,1]} J_\kappa(\gamma(t)) \geq a_0 > 0, \tag{2.6}
\]
where

$$\Gamma_\kappa = \{ \gamma \in C([0,1], H^1(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(1) \neq 0, J_\kappa(\gamma(1)) < 0 \},$$

there exists a Palais-Smale sequence at level $c_\kappa$, that is,

$$J_\kappa(v_n) \to c_\kappa \quad \text{and} \quad J'_\kappa(v_n) \to 0 \quad \text{as} \quad n \to \infty.$$

**Lemma 2.4.** For $2 < q < 2^*$, the Palais-Smale sequence $\{v_n\}$ is bounded.

**Proof.** Since $\{v_n\} \subset H^1(\mathbb{R}^N)$ is a Palais-Smale sequence, then

$$J_\kappa(v_n) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^2 dx - \frac{1}{q} \int_{\mathbb{R}^N} |G^{-1}(v_n)|^q dx - \frac{q-2}{2} \int_{\mathbb{R}^N} \nabla v_n \cdot \nabla (v_n) dx$$

and for any $\psi \in H^1(\mathbb{R}^N)$, $J'_\kappa(v_n) \psi = o(1)||\psi||$, that is,

$$\int_{\mathbb{R}^N} \left[ \nabla v_n \nabla \psi + V(x)G^{-1}(v_n)\psi - \frac{G^{-1}(v_n)^q - 2G^{-1}(v_n)\psi}{qG^{-1}(v_n)} G^{-1}(v_n) \psi \right] dx = o(1)||\psi||. \quad (2.7)$$

Fixing $\psi = G^{-1}(v_n)g(G^{-1}(v_n))$, it follows from Lemma 2.1-(4),

$$|\nabla(G^{-1}(v_n))g(G^{-1}(v_n))| \leq \left[ 1 + \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} g'(G^{-1}(v_n)) \right] |\nabla v_n| \leq |\nabla v_n|. \quad (2.8)$$

On the other hand, by Lemma 2.1-(3),

$$|G^{-1}(v_n)g(G^{-1}(v_n))| \leq \sqrt{6}|v_n|. \quad (2.9)$$

Combining 2.9 and 2.10, we have $\psi \in H^1(\mathbb{R}^N)$ with $||\psi|| \leq 6||v_n||$. Thus, by using $\psi = G^{-1}(v_n)g(G^{-1}(v_n))$ as a test function in (2.8), we derive that

$$o(1)||v_n|| = J'_\kappa(v_n)G^{-1}(v_n)g(G^{-1}(v_n))$$

$$= \int_{\mathbb{R}^N} \left[ \left( 1 + \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} g'(G^{-1}(v_n)) \right) |\nabla v_n|^2 + V(x)|G^{-1}(v_n)|^2 \right.$$

$$\left. - |G^{-1}(v_n)|^q \right] dx$$

$$\leq \int_{\mathbb{R}^N} \left[ |\nabla v_n|^2 + V(x)|G^{-1}(v_n)|^2 - |G^{-1}(v_n)|^q \right] dx. \quad (2.11)$$

Therefore, by (2.7), (2.8) and (2.11),

$$qc_n + o(1) + o(1)||v_n|| = qJ_\kappa(v_n) - J'_\kappa(v_n)G^{-1}(v_n)g(G^{-1}(v_n))$$

$$\geq \frac{(q-2)}{2} \int_{\mathbb{R}^N} \left[ |\nabla v_n|^2 + V(x)|G^{-1}(v_n)|^2 \right] dx$$

$$\geq \frac{(q-2)}{2} ||v_n||^2,$$

showing the boundedness of $\{v_n\}$. \qed
Since \( \{v_n\} \) is a bounded sequence and \( H^1(\mathbb{R}^N) \) is a separable Hilbert space, there exists \( v_\kappa \in H^1(\mathbb{R}^N) \) and a subsequence of \( \{v_n\} \), still denoted by itself, such that

\[
v_n \to v_\kappa \text{ in } H^1(\mathbb{R}^N), v_n \to v_\kappa \text{ in } L^q_{loc}(\mathbb{R}^N) \text{ for } q \in [2, 2^*) \text{ and } v_n \to v_\kappa \text{ a.e. on } \mathbb{R}^N.
\]

**Theorem 2.1.** The weak limit \( v_\kappa \) of \( \{v_n\} \) is a nontrivial critical point of \( J_\kappa \) and \( J_\kappa(v_\kappa) \leq c_\kappa \).

**Proof.** Our first goal is proving that \( v_\kappa \) is a weak solution. To this end, it suffices showing that

\[
J'_\kappa(v_\kappa) \psi = 0 \quad \forall \psi \in H^1(\mathbb{R}^N)
\]

or equivalently,

\[
\int_{\mathbb{R}^N} \left( \nabla v_n \nabla \psi + V(x) \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} \psi - \frac{|G^{-1}(v_n)|^q G^{-1}(v_n)}{g(G^{-1}(v_n))} \right) dx = 0 \quad \forall \psi \in H^1(\mathbb{R}^N).
\]

Once that \( C_{0}^{\infty}(\mathbb{R}^N) \) is dense in \( H^1(\mathbb{R}^N) \), it is sufficient to show the last equality only for functions belonging to \( C_{0}^{\infty}(\mathbb{R}^N) \).

In what follows, for each \( R > 0 \) we consider \( \psi_R \in C_{0}^{\infty}(\mathbb{R}^N) \) verifying

\[
0 \leq \psi_R(x) \leq 1 \quad \forall x \in \mathbb{R}^N, \quad \psi_R(x) = 1 \quad \forall x \in B_R(0) \quad \text{and} \quad \psi_R(x) = 0 \quad \forall x \in B_{2R}(0).
\]

By \([31]\),

\[
|v_n| \leq |z(x)| \quad \text{for every } n \text{ with } z \in L^q(B_{2R}(0)).
\]

Consequently,

\[
\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} v_n \to \frac{G^{-1}(v_\kappa)}{g(G^{-1}(v_\kappa))} v_\kappa \text{ a.e. on } B_{2R}(0), \quad \text{as } n \to \infty,
\]

and

\[
\frac{|G^{-1}(v_n)|^q G^{-1}(v_n)}{g(G^{-1}(v_n))} v_n \to \frac{|G^{-1}(v_\kappa)|^q G^{-1}(v_\kappa)}{g(G^{-1}(v_\kappa))} v_\kappa \text{ a.e. on } B_{2R}(0), \quad \text{as } n \to \infty.
\]

Moreover, by Lemma \([21]\)

\[
\left| V(x) \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} v_n \psi \right| \leq 6V_\infty |v_n|^2 |\psi| \leq 6V_\infty |z(x)|^2 |\psi|
\]

and

\[
\left| V(x) \frac{|G^{-1}(v_n)|^q G^{-1}(v_n)}{g(G^{-1}(v_n))} \psi \right| \leq 6^{\frac{q+4}{2}} V_\infty |v_n|^q |\psi| \leq 6^{\frac{q+4}{2}} V_\infty |z(x)|^q |\psi|.
\]

Hence, by Lebesgue Dominated Theorem

\[
\int_{\mathbb{R}^N} V(x) \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} v_n \psi dx \to \int_{\mathbb{R}^N} V(x) \frac{G^{-1}(v_\kappa)}{g(G^{-1}(v_\kappa))} v_\kappa \psi dx \quad (2.12)
\]
Palais-Smale sequence for functional $J$

Since $v \in H_{\text{loc}}^1$ showing that $\varphi$ supposing that $v

Indeed, since $V(v_n) = o_n(1)$ and $J'_\kappa(v_n)(v_n \psi) = o_n(1)$ give

\[
\int_{\mathbb{R}^N} |\nabla v_n - \nabla v|_R^2 \, dx \to 0,
\]

from where it follows that

\[
\int_{B_R(0)} |\nabla v_n - \nabla v|_R^2 \, dx \to 0.
\]

Once that $R$ is arbitrary and $v_n \to v_\kappa$ in $L^2_{\text{loc}}(\mathbb{R}^N)$, we conclude that $v_n \to v_\kappa$ in $H^1_{\text{loc}}(\mathbb{R}^N)$. Thereby,

\[
J'_\kappa(v_n) \psi \to J'_\kappa(v_\kappa) \psi \quad \forall \psi \in C^\infty_0(\mathbb{R}^N).
\]

Since $J'_\kappa(v_n) \psi = o_n(1)$, the last limit yields $J'_\kappa(v_\kappa) \psi = 0$ for all $\psi \in C^\infty_0(\mathbb{R}^N)$, showing that $v_\kappa$ is a critical point for $J_\kappa$.

Now, next step is showing that $v_\kappa \neq 0$. To prove this, we argue by contradiction supposing that $v_\kappa = 0$. We claim that in this case, $\{v_n\}$ is also a Palais-Smale sequence for functional $J_{\kappa, \infty} : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

\[
J_{\kappa, \infty}(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 \, dx + \frac{1}{2} V_\infty \int_{\mathbb{R}^N} |G^{-1}(v)|^2 \, dx - \frac{1}{q} \int_{\mathbb{R}^N} |G^{-1}(v)|^q \, dx.
\]

Indeed, since $V(x) \to V_\infty$ as $|x| \to \infty$, $|G^{-1}(s)| \leq \sqrt{6}|s|$ and $v_n \to 0$ in $L^2_{\text{loc}}(\mathbb{R}^N)$, we have

\[
J_\kappa(v_n) - J_{\kappa, \infty}(v_n) = \frac{1}{2} \int_{\mathbb{R}^N} [V(x) - V_\infty] |G^{-1}(v_n)|^2 \, dx \to 0.
\]

On the other hand, recalling $\frac{|G^{-1}(s)|}{g(G^{-1}(s))} \leq 6|s|$, we have

\[
\sup_{\|\psi\| \leq 1} |J'_\kappa(v_n) - J'_{\kappa, \infty}(v_n, \psi)| = \sup_{\|\psi\| \leq 1} \left| \int_{\mathbb{R}^N} [V(x) - V_\infty] \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} \psi \, dx \right| \to 0.
\]
Next, we claim that for all $R > 0$, the following vanishing cannot occur:

$$\lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |v_n|^2 dx = 0. \quad (2.19)$$

Suppose by contradiction that (2.19) occurs, then by a Lions’ compactness lemma [17], $v_n \to 0$ in $L^q(\mathbb{R}^N)$ for any $q \in (2, 2^*)$. So,

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} |G^{-1}(v_n)|^q dx \leq 6^{\frac{q}{2}} \lim_{n \to \infty} \int_{\mathbb{R}^N} |v_n|^q dx \to 0$$

and

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|G^{-1}(v_n)|^{q-2}G^{-1}(v_n)}{g(G^{-1}(v_n))} v_n dx \leq 6^{\frac{q}{2}} \lim_{n \to \infty} \int_{\mathbb{R}^N} |v_n|^q dx \to 0.$$ 

Once that,

$$\lim_{s \to 0} \frac{1}{s^2} \left[ |G^{-1}(s)|^2 - \frac{G^{-1}(s)}{g(G^{-1}(s))} s \right] = \lim_{s \to \infty} \frac{1}{|s|^2} \left[ |G^{-1}(s)|^2 - \frac{G^{-1}(s)}{g(G^{-1}(s))} s \right] = 0,$$

we derive

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \left[ |G^{-1}(v_n)|^2 - \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} v_n \right] dx = 0.$$

Therefore, we deduce that

$$2c_k + o(1) = 2J_k(v_n) - J'_k(v_n)v_n$$

$$= \int_{\mathbb{R}^N} \left[ |G^{-1}(v_n)|^2 - \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} v_n \right] dx$$

$$- \frac{2}{q} \int_{\mathbb{R}^N} |G^{-1}(v_n)|^q dx + \int_{\mathbb{R}^N} \frac{|G^{-1}(v_n)|^{q-2}G^{-1}(v_n)}{g(G^{-1}(v_n))} v_n dx \to 0,$$

which is a contradiction, because $c_k \geq c_0 > 0$.

Thus, $\{v_n\}$ does not vanish and there exist $\alpha, R > 0$, and $\{y_n\} \subset \mathbb{R}^N$ verifying

$$\lim_{n \to \infty} \int_{B_R(y_n)} |v_n|^2 dx \geq \alpha > 0. \quad (2.20)$$

Define $\bar{v}_n(x) = v_n(x + y_n)$. Since $\{v_n\}$ is a Palais-Smale sequence for $J_{k, \infty}$, $\{\bar{v}_n\}$ is also a Palais-Smale sequence for $J_{k, \infty}$. Arguing as in the case of $\{v_n\}$, we get that $\bar{v}_n \to \bar{v}_k$ in $H^1_{loc}(\mathbb{R}^N)$ and $J'_{k, \infty}(\bar{v}_k) = 0$. Moreover, by (2.20), we also have $\bar{v}_n \not\equiv 0$. Henceforward, without loss of generality, we assume that

$$\bar{v}_n(x) \to \bar{v}_k(x) \text{ and } \nabla \bar{v}_n(x) \to \nabla \bar{v}_k(x) \text{ a.e. on } \mathbb{R}^N.$$
The last limits together with Fatou's Lemma lead to

$$2c_\kappa = \lim_{n \to \infty} \left[ 2J_{\kappa,\infty}(\bar{v}_n) - J'_{\kappa,\infty}(\bar{v}_n)G^{-1}(v_n)g(G^{-1}(\bar{v}_n)) \right]$$

$$= -\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{G^{-1}(\bar{v}_n)g'(G^{-1}(\bar{v}_n))}{g(G^{-1}(\bar{v}_n))} |\nabla \bar{v}_n|^2 dx$$

$$- \frac{(2-q)}{q} \lim_{n \to \infty} \int_{\mathbb{R}^N} |G^{-1}(\bar{v}_n)|^q dx$$

$$\leq -\int_{\mathbb{R}^N} \frac{G^{-1}(\bar{v}_k)g'(G^{-1}(\bar{v}_k))}{g(G^{-1}(\bar{v}_k))} |\nabla \bar{v}_k|^2 dx - \frac{(2-q)}{q} \int_{\mathbb{R}^N} |G^{-1}(\bar{v}_k)|^q dx$$

$$= 2J_{\kappa,\infty}(\bar{v}_k) - J'_{\kappa,\infty}(\bar{v}_k)G^{-1}(\bar{v}_k)g(G^{-1}(\bar{v}_k))$$

$$= 2J_{\kappa,\infty}(\bar{v}_k),$$

that is, $J_{\kappa,\infty}(\bar{v}_k) \leq c_\kappa$. Now, as in [13], we define

$$\tilde{v}_{\kappa,t}(x) = \begin{cases} \bar{v}_\kappa(x/t), & \text{if } t > 0, \\ 0, & \text{if } t = 0. \end{cases}$$

Then,

$$\int_{\mathbb{R}^N} |\nabla \tilde{v}_{\kappa,t}|^2 dx = t^{N-2} \int_{\mathbb{R}^N} |\nabla \bar{v}_\kappa|^2 dx, \quad \int_{\mathbb{R}^N} |G^{-1}(\tilde{v}_{\kappa,t})|^2 dx = t^N \int_{\mathbb{R}^N} |G^{-1}(\bar{v}_\kappa)|^2 dx,$$

and

$$\int_{\mathbb{R}^N} |G^{-1}(\tilde{v}_{\kappa,t})|^q dx = t^N \int_{\mathbb{R}^N} |G^{-1}(\bar{v}_\kappa)|^q dx.$$

Since $J'_{\kappa,\infty}(\bar{v}_k) = 0$, elliptic regularity implies that $\bar{v}_\kappa \in C^2(\mathbb{R}^N)$. Hence,

$$\frac{d}{dt} J_{\kappa,\infty}(\bar{v}_{\kappa,t}) \bigg|_{t=1} = 0,$$

leading to

$$\frac{(N-2)}{2N} \int_{\mathbb{R}^N} |\nabla \bar{v}_\kappa|^2 dx = -\frac{V_\infty}{2} \int_{\mathbb{R}^N} |G^{-1}(\bar{v}_\kappa)|^2 dx + \frac{1}{q} \int_{\mathbb{R}^N} |G^{-1}(\bar{v}_\kappa)|^q dx. \quad (2.22)$$

Setting $\gamma(t)(x) = \tilde{v}_{\kappa,t}(x)$, we see that

$$J_{\kappa,\infty}(\gamma(t)) = \frac{t^{N-2}}{2} \int_{\mathbb{R}^N} |\nabla \bar{v}_\kappa|^2 dx - t^N \left[ -\frac{V_\infty}{2} \int_{\mathbb{R}^N} |G^{-1}(\bar{v}_\kappa)|^2 dx \right.$$ 

$$+ \left. \frac{1}{q} \int_{\mathbb{R}^N} |G^{-1}(\bar{v}_\kappa)|^q dx \right].$$

Thus $\gamma \in C([0, \infty), H^1(\mathbb{R}^N))$ and

$$\frac{d}{dt} J_{\kappa,\infty}(\gamma(t)) = \frac{N-2}{2} t^{N-3} \int_{\mathbb{R}^N} |\nabla \bar{v}_\kappa|^2 dx - N t^{N-1} \left[ -\frac{V_\infty}{2} \int_{\mathbb{R}^N} |G^{-1}(\bar{v}_\kappa)|^2 dx \right.$$ 

$$+ \left. \frac{1}{q} \int_{\mathbb{R}^N} |G^{-1}(\bar{v}_\kappa)|^q dx \right]$$

$$= \left( \frac{N-2}{2} t^{N-3} - \frac{1}{2} \right) \int_{\mathbb{R}^N} |\nabla \bar{v}_\kappa|^2 dx.$$
So, \( \frac{d}{dt} J_{\kappa, \infty}(\gamma(t)) > 0 \) for \( t \in (0, 1) \) and \( \frac{d}{dt} J_{\kappa, \infty}(\gamma(t)) < 0 \) for \( t > 1 \) implying that

\[
\max_{t \geq 0} J_{\kappa, \infty}(\gamma(t)) = J_{\kappa, \infty}(\tilde{v}_\kappa).
\]

Furthermore, \( J_{\kappa, \infty}(\gamma(L)) < 0 \) for sufficiently large \( L > 1 \), showing that \( \tilde{\gamma}(t) = \gamma(Lt) \) belongs to \( \Gamma_\kappa \). Thereby

\[
c_\kappa \leq \max_{t \in [0, 1]} J_\kappa(\tilde{\gamma}(t)) < J_{\kappa, \infty}(\gamma(L)) < 0,
\]

which is a contradiction. This way, \( v_\kappa \) is a nontrivial critical point for \( J \). Moreover, repeating the same type of arguments explored in (2.21), we have that \( J_\kappa(v_\kappa) \leq c_\kappa \).

### 3 \( L^\infty \) estimate of the solution

In this section, we will establish an \( L^\infty \) estimate for solution \( v_\kappa \) obtained in Theorem 2.1. Indeed, by standard elliptic regularity estimate [10], \( v_\kappa \in L^\infty(\mathbb{R}^N) \). However, this boundedness is not enough to prove our results. In the following, we will prove an \( L^\infty \) estimate dependent on \( \kappa > 0 \). To this end, firstly we need to give an uniform boundedness of the Sobolev norm independent on \( \kappa > 0 \) for \( v_\kappa \).

**Lemma 3.1.** The solution \( v_\kappa \) satisfies \( \|v_\kappa\|^2 \leq \frac{2qc_\kappa}{q-2} \).

**Proof.** Using the hypothesis that \( v_\kappa \) is a critical point of \( J_\kappa \),

\[
qc_\kappa = qJ(v_\kappa) - J'(v_\kappa)G^{-1}(v_\kappa)g(G^{-1}(v_\kappa))
\]

\[
\geq \frac{(q-2)}{2} \int_{\mathbb{R}^N} |\nabla v_\kappa|^2 dx + \frac{(q-2)}{2} \int_{\mathbb{R}^N} V(x)|G^{-1}(v_\kappa)|^2 dx,
\]

from where it follows that,

\[
\|v_\kappa\|^2 \leq \frac{2qc_\kappa}{q-2}.
\]

From now on, we consider the functional

\[
P_\infty(v) = 3 \int_{\mathbb{R}^N} (|v|^2 + V_\infty v^2) dx - \frac{1}{q} \int_{\mathbb{R}^N} |v|^q dx
\]

and the set

\[
\Gamma^\infty = \{ \gamma \in C([0, 1], H^1(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(1) \neq 0, P_\infty(\gamma(1)) < 0 \}.
\]

By Lemma 2.1–(3), we have \( J_\kappa(v) \leq P_\infty(v) \) and thus \( \Gamma^\infty \subset \Gamma_\kappa \). Therefore

\[
c_\kappa = \inf_{\gamma \in \Gamma_\kappa} \sup_{t \in [0, 1]} J_\kappa(\gamma(t)) \leq \inf_{\gamma \in \Gamma^\infty} \sup_{t \in [0, 1]} J_\kappa(\gamma(t)) \leq \inf_{\gamma \in \Gamma^\infty} \sup_{t \in [0, 1]} P_\infty(\gamma(t)) = d_\infty.
\]
where $d_\infty$ is independent on $\kappa$. Consequently, by Lemma 4.4 the solution $v_\kappa$ must satisfy the estimate
\[ \| v_\kappa \|_2 \leq \frac{2qd_\infty}{q-2}. \] (3.1)

**Proposition 3.1.** There exists a constant $C_0 > 0$ independent on $\kappa$, such that
\[ \| v_\kappa \|_\infty \leq C_0\kappa^{-\frac{1}{4}} \] for $\kappa \leq 6\frac{q}{q-2}$.

**Proof.** In what follows, we denote $v_\kappa$ by $v$. For each $m \in \mathbb{N}$ and $\beta > 1$, let $A_m = \{ x \in \mathbb{R}^N : |v|^{\beta-1} \leq m \}$ and $B_m = \mathbb{R}^N \setminus A_m$. Define
\[ v_m = \begin{cases} v |v|^{2(\beta-1)} & \text{in } A_m, \\ m^2 v & \text{in } B_m. \end{cases} \]

Note that $v_m \in H^1(\mathbb{R}^N)$, $v_m \leq |v|^{2\beta-1}$ and
\[ \nabla v_m = \begin{cases} (2\beta-1)|v|^{2(\beta-1)} \nabla v & \text{in } A_m, \\ m^2 \nabla v & \text{in } B_m. \end{cases} \] (3.2)

Using $v_m$ as a test function in (2.5), we deduce that
\[ \int_{\mathbb{R}^N} \left[ \nabla v \nabla v_m + V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} v_m \right] dx = \int_{\mathbb{R}^N} \frac{|G^{-1}(v)|^q G^{-1}(v)}{g(G^{-1}(v))} v_m dx. \] (3.3)

By (3.3),
\[ \int_{\mathbb{R}^N} \nabla v \nabla v_m dx = (2\beta-1) \int_{A_m} |v|^{2(\beta-1)} |\nabla v| dx + m^2 \int_{B_m} |\nabla v|^2 dx. \] (3.4)

Let
\[ w_m = \begin{cases} v |v|^{\beta-1} & \text{in } A_m, \\ m^2 v & \text{in } B_m. \end{cases} \]

Then $w_m^2 = vv_m \leq |v|^{2\beta}$ and
\[ \nabla w_m = \begin{cases} \beta |v|^{\beta-1} \nabla v & \text{in } A_m, \\ m \nabla v & \text{in } B_m. \end{cases} \]

Hence,
\[ \int_{\mathbb{R}^N} |\nabla w_m|^2 dx = \beta^2 \int_{A_m} |v|^{2(\beta-1)} |\nabla v|^2 dx + m^2 \int_{B_m} |\nabla v|^2 dx. \] (3.5)

Then, from (3.4) and (3.5),
\[ \int_{\mathbb{R}^N} (|\nabla w_m|^2 - \nabla v \nabla v_m) dx = (\beta-1)^2 \int_{A_m} |v|^{2(\beta-1)} |\nabla v|^2 dx. \] (3.6)
Combing (3.3), (3.4) and (3.6), since $\beta > 1$, we have
\[
\int_{\mathbb{R}^N} |\nabla w_m|^2 \, dx \leq \left[ \frac{(\beta - 1)^2}{2\beta - 1} + 1 \right] \int_{\mathbb{R}^N} \nabla v \nabla w_m \, dx
\]
\[
\leq \beta^2 \int_{\mathbb{R}^N} \left[ \nabla v \nabla w_m + V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} v_m \right] \, dx
\]
\[
= \beta^2 \int_{\mathbb{R}^N} \frac{|G^{-1}(v)|^{\gamma - 2} G^{-1}(v)}{g(G^{-1}(v))} v_m \, dx.
\]
Choosing $\kappa \frac{\beta^2}{\beta^2 - 1} \leq \sqrt{\frac{q}{\delta}}$, we have $g(t) \geq \kappa \frac{\beta^2}{\beta^2 - 1}$. Setting $\theta = \frac{2^\gamma - 2}{4}$, by Sobolev inequality and Lemma (2.1)–(3),
\[
\left( \int_{A_m} |w_m|^2^\gamma \, dx \right)^{(N - 2)/N} \leq S \int_{\mathbb{R}^N} |\nabla w_m|^2 \, dx \leq 6 \frac{\beta^2}{\beta^2 - 1} S \beta^2 \kappa^{-\theta} \int_{\mathbb{R}^N} |v|^{\gamma - 2} w_m^2 \, dx.
\]
By Hölder inequality, we have
\[
\left( \int_{A_m} |w_m|^2^\gamma \, dx \right)^{(N - 2)/N} \leq 6 \frac{\beta^2}{\beta^2 - 1} S \beta^2 \kappa^{-\theta} \|v\|_{2^\gamma} \left( \int_{\mathbb{R}^N} |w_m|^2 \, dx \right)^{1/q_1}
\]
where $1/q_1 + (q - 2)/2^\gamma = 1$. Since $|w_m| \leq |v|^\beta$ in $\mathbb{R}^N$ and $|w_m| = |v|^\beta$ in $A_m$, we have
\[
\left( \int_{A_m} |v|^{2^\gamma} \, dx \right)^{(N - 2)/N} \leq 6 \frac{\beta^2}{\beta^2 - 1} S \beta^2 \kappa^{-\theta} \|v\|_{2^\gamma} \left( \int_{\mathbb{R}^N} |v|^{2 \beta q_1} \, dx \right)^{1/q_1}
\]
By Monotone Convergence Theorem, letting $m \to \infty$, we have
\[
\|v\|_{2^\gamma} \leq \beta^{1/\beta} \left( 6 \frac{\beta^2}{\beta^2 - 1} S \kappa^{-\theta} \|v\|_{2^\gamma} \right)^{1/(2^\gamma)} \|v\|_{2^\omega_1}.
\]  
(3.7)

Setting $\sigma = 2^\gamma/(2q_1)$ and $\beta = \sigma$ in (3.7), we obtain $2q_1 \beta = 2^\gamma$ and
\[
\|v\|_{\sigma^{2^\gamma}} \leq \sigma^{1/\sigma} \left( 6 \frac{\beta^2}{\beta^2 - 1} S \kappa^{-\theta} \|v\|_{2^\gamma} \right)^{1/(2^\sigma)} \|v\|_{2^\sigma}.
\]  
(3.8)

Taking $\beta = \sigma^2$ in (3.7), we have
\[
\|v\|_{\sigma^{2^\gamma}} \leq \sigma^{2/\sigma^2} \left( 6 \frac{\beta^2}{\beta^2 - 1} S \kappa^{-\theta} \|v\|_{2^\gamma} \right)^{1/(2^{\sigma^2})} \|v\|_{\sigma^2}.
\]  
(3.9)

From (3.8) and (3.9),
\[
\|v\|_{\sigma^{2^\gamma}} \leq \sigma^{1/\sigma + 2/\sigma^2} \left( 6 \frac{\beta^2}{\beta^2 - 1} S \kappa^{-\theta} \|v\|_{2^\gamma} \right)^{1/(2(1/\sigma + 1/\sigma^2))} \|v\|_{2^\gamma}.
\]

Taking $\beta = \sigma^i$ ($i = 1, 2, \ldots$) and iterating (3.7), we get
\[
\|v\|_{\sigma^{2^\gamma}} \leq \sigma^{1/\sigma + 2/\sigma^2} \left( 6 \frac{\beta^2}{\beta^2 - 1} S \kappa^{-\theta} \|v\|_{2^\gamma} \right)^{1/(2(1/\sigma + 1/\sigma^2))} \|v\|_{2^\gamma}.
\]

Therefore, by Sobolev inequality, (3.1) and taking the limit of $j \to +\infty$, we get
\[
\|v\|_\infty \leq \sigma^{1/\sigma + 2/\sigma^2} \left( 6 \frac{\beta^2}{\beta^2 - 1} S \kappa^{-\theta} \|v\|_{2^\gamma} \right)^{1/(2(1/\sigma + 1/\sigma^2))} \|v\|_{2^\gamma} = C_0 \kappa^{-\frac{1}{2}}
\]
for $\kappa \leq 6 \frac{\beta^2}{\beta^2 - 1}$, where $C_0 > 0$ is independent of $\kappa > 0$. This ends the proof. \qed
3.1 Proof of Theorem 1.1

Combining the arguments in Section 2 and Proposition 3.1, the solution \( v_\kappa \) of (1.10) established in Theorem 2.1 satisfies \( \| v_\kappa \|_\infty \leq C_0\kappa^{-\frac{1}{2}} \) for \( \kappa \leq 6^{\frac{1}{2}}q^{-2} \star \).

Choosing \( \kappa_0 = \min \left\{ 6^{\frac{1}{2}}q^{-2} \star, \frac{1}{C_0\sqrt{18}} \right\} \), it follows that

\[
\| G^{-1}(v_\kappa) \|_\infty \leq \sqrt{6}\| v_\kappa \|_\infty < \sqrt{\frac{1}{3\kappa}} \quad \forall \kappa \in [0, \kappa_0).
\]

From this, \( u = G^{-1}(v_\kappa) \) is a classical solution of (1.3).

4 Proof of Theorem 1.2

In this section, we fix \( 0 < \kappa < \frac{1}{3} \) and for equation (1.10), we let

\[
g(t) = \begin{cases} 
\sqrt{1 - \kappa t^2}, & \text{if } 0 \leq t < \sqrt{\frac{1}{3\kappa}}, \\
\frac{\kappa}{t\sqrt{1 - \kappa}} + \frac{1 - 2\kappa}{\sqrt{1 - \kappa}} & \text{if } t \geq 1.
\end{cases}
\]

Setting \( g(t) = g(-t) \) for all \( t \leq 0 \), clearly \( g \in C^1(\mathbb{R}, (\frac{1-2\kappa}{\sqrt{1 - \kappa}}, 1)) \) and \( g \) increases in \((\infty, 0)\) and decreases in \([0, +\infty)\).

We further modify the nonlinearity of equation (1.4) as follows:

\[
f(t) = \begin{cases} 
0, & \text{if } t \leq 0, \\
\left[ 1 - \frac{1}{(1 + t^2)^3} \right] t, & \text{if } 0 \leq t \leq 1, \\
\frac{7}{8} t^{q-1}, & \text{if } t \geq 1,
\end{cases}
\]

where \( 2 < q < \min\{\frac{14}{5}, 2\star\} \) and fix \( l(t) = f(t) \) in (1.10). We note that \( f \) is continuous and satisfies the following conditions:

\begin{itemize}
  \item[(f_1)] \( f(0) = 0 \);
  \item[(f_2)] \( \lim_{t \to 0} \frac{f(t)}{t} = 0 \);
  \item[(f_3)] \( \lim_{t \to +\infty} \frac{f(t)}{t^{q-1}} = \frac{7}{8} \);
  \item[(f_4)] \( \lim_{t \to +\infty} \frac{f(t)}{t} = +\infty \);
  \item[(f_5)] \( 2F(t) - f(t)t \leq 0 \) with \( t \in \mathbb{R} \), where \( F(t) = \int_0^t f(s)ds \);
  \item[(f_6)] \( f(t) \leq 7t^{q-1} \) with \( t \geq 0 \).
\end{itemize}
We make change of variables
\[ v = G(u) = \int_0^u g(t) dt. \]

Then, at this moment, the inverse function \( G^{-1}(t) \) satisfies the following properties:

**Lemma 4.1.**

1. \( \lim_{t \to 0} \frac{G^{-1}(t)}{t} = 1; \)
2. \( \lim_{t \to \infty} \frac{G^{-1}(t)}{t} = \frac{\sqrt{1 - \kappa}}{1 - 2\kappa}; \)
3. \( t \leq G^{-1}(t) \leq 3t, \) for all \( t \geq 0; \)
4. \( -\frac{3}{2} \leq \frac{t}{g(t)} g'(t) \leq 0, \) for all \( t \geq 0. \)

**Proof.** By the definition of \( g \) and since \( \kappa < \frac{1}{3}, \)
\[
\lim_{t \to 0} \frac{G^{-1}(t)}{t} = \lim_{t \to 0} \frac{1}{g(G^{-1}(t))} = 1
\]
and
\[
\lim_{t \to \infty} \frac{G^{-1}(t)}{t} = \lim_{t \to \infty} \frac{1}{g(G^{-1}(t))} = \frac{\sqrt{1 - \kappa}}{1 - 2\kappa} \leq 3.
\]

Thus, (1) and (2) are proved. Since \( g(t) > 0 \) is decreasing in \([0, \infty), \) then
\[
\frac{1}{3} t \leq \frac{1 - 2\kappa}{\sqrt{1 - \kappa}} t \leq g(t) t \leq G(t) \leq t \text{ for all } t \geq 0, \text{ which implies (3).}
\]
By direct calculation, we get (4).

Next, we consider the equation
\[
-\Delta v + V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} - \frac{f(G^{-1}(v))}{g(G^{-1}(v))} = 0 \text{ in } \mathbb{R}^N. \quad (4.1)
\]

We establish the geometric hypotheses of the Mountain Pass Theorem for the following energy functional corresponding to (4.1):
\[
\tilde{J}_\kappa(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^2 dx - \int_{\mathbb{R}^N} F(G^{-1}(v)) dx. \quad (4.2)
\]

From Lemma 4.1 and (f1)-(f3), \( \tilde{J}_\kappa \) is well defined in \( H^1(\mathbb{R}^N) \) and \( \tilde{J}_\kappa \in C^1(H^1(\mathbb{R}^N), \mathbb{R}). \)

**Lemma 4.2.**

1. There exist \( \rho_1, a_1 > 0, \) such that \( \tilde{J}_\kappa(v) \geq a_1 \) for \( \|v\| = \rho_1. \)
2. There exists \( \phi \in H^1(\mathbb{R}^N) \) such that \( \tilde{J}_\kappa(\phi) < 0. \)
Proof. By \((f_2)\) and \((f_3)\),
\[
|F(t)| \leq \frac{1}{4} V_0 |t|^2 + C|t|^q.
\]
Thus, by Lemma \[4.1\]-(3) and Sobolev embedding inequality, we have
\[
\tilde{J}_\kappa(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) |G^{-1}(v)|^2 dx - \int_{\mathbb{R}^N} F(G^{-1}(v)) dx
\]
\[
\geq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) |G^{-1}(v)|^2 dx - \frac{1}{4} V_0 \int_{\mathbb{R}^N} |G^{-1}(v)|^2 dx
\]
\[
- C \int_{\mathbb{R}^N} |G^{-1}(v)|^q dx
\]
\[
\geq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{4} V_0 \int_{\mathbb{R}^N} v^2 dx - C \int_{\mathbb{R}^N} |v|^q dx
\]
\[
\geq \frac{1}{4} \|v\|^2 - C\|v\|^q.
\]
Therefore, by choosing \(\rho_1\) small, we get (1) for \(\|v\| = \rho_1\).

To prove (2), we choose some \(\varphi \in C_0^\infty(\mathbb{R}^N, [0, 1]) \setminus \{0\}\) with \(supp \varphi = B_1(0)\).
We will show that \(\tilde{J}_\kappa(t\varphi) \to -\infty\) as \(t \to \infty\), which will prove the result if we take \(\phi = t\varphi\) with \(t\) large enough. By \((f_4)\),
\[
\lim_{t \to +\infty} \frac{F(t)}{t^2} = +\infty.
\]
Thus, given \(A = \frac{9}{2} V_0 \int_{\mathbb{R}^N} \varphi^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} |\nabla \varphi|^2 dx + 1\), there exists \(D > 0\) such that
\[
F(t) \geq At^2 - D \quad \forall t \geq 0.
\]
Hence,
\[
\tilde{J}_\kappa(t\varphi) \leq -t^2 + D|B_1(0)| \to -\infty \text{ as } t \to +\infty,
\]
which implies (2). \(\square\)

In consequence of Lemma \[4.1\] and of a special version of the Mountain Pass Theorem found in \[2\], for the constant
\[
\tilde{c}_\kappa = \inf_{\gamma \in \tilde{\Gamma}_\kappa} \sup_{t \in [0, 1]} \tilde{J}_\kappa(\gamma(t)) \geq a_0 > 0,
\]
where
\[
\tilde{\Gamma}_\kappa = \{ \gamma \in C([0, 1], H^1(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(1) \neq 0, \tilde{J}_\kappa(\gamma(1)) < 0 \},
\]
there exists a Cerami sequence at level \(\tilde{c}_\kappa\), that is,
\[
\tilde{J}_\kappa(v_n) \to c \quad \text{and} \quad (1 + \|v_n\|)\|\tilde{J}_\kappa'(v_n)\| \to 0.
\]

Lemma 4.3. The Cerami sequence \(\{v_n\}\) given in \((4.4)\) is bounded.
Proof. For any \( v \in H^1(\mathbb{R}^N) \),
\[
\tilde{J}_n(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^2 dx
\]
\[
- \int_{\{x \in \mathbb{R}^N, |G^{-1}(v(x))| \leq 1\}} \left[ \frac{G^{-1}(v)^2}{2} + \frac{1}{4(1 + G^{-1}(v))^2} - \frac{1}{4} \right] dx
\]
\[
- \frac{7}{8q} \int_{\{x \in \mathbb{R}^N, |G^{-1}(v(x))| \geq 1\}} |G^{-1}(v)|^q dx
\]
and
\[
\tilde{J}_n'(v) G^{-1}(v) g(G^{-1}(v)) = \int_{\mathbb{R}^N} \left[ 1 + \frac{G^{-1}(v)}{g(G^{-1}(v))} g'(G^{-1}(v)) \right] |\nabla v|^2 dx + \int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^2 dx
\]
\[
- \int_{\{x \in \mathbb{R}^N, |G^{-1}(v(x))| \leq 1\}} \left[ G^{-1}(v)^2 - \frac{G^{-1}(v_n)^2}{(1 + G^{-1}(v))^3} \right] dx
\]
\[
- \frac{7}{8} \int_{\{x \in \mathbb{R}^N, |G^{-1}(v(x))| \geq 1\}} G^{-1}(v)^q dx. \tag{4.5}
\]
Now, by previous arguments, we know that there is \( C > 0 \) such that
\[
|G^{-1}(v_n)| g(G^{-1}(v_n))| \leq C \|v_n\| \quad \forall n \in \mathbb{N}.
\]
Thus, the last inequality combined with \( 3.4 \), \( 4.4 \) and \( 4.5 \) implies that
\[
q \tilde{c}_n + o_n(1)
\]
\[
= q \tilde{J}_n(v_n) - \tilde{J}_n'(v_n) G^{-1}(v) g(G^{-1}(v)) \]
\[
\geq \int_{\mathbb{R}^N} \left[ \frac{(q - 2)}{2} - \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} g'(G^{-1}(v_n)) \right] |\nabla v_n|^2 dx +
\]
\[
+ \frac{(q - 2)}{2} \int_{\{x \in \mathbb{R}^N, |G^{-1}(v(x))| \leq 1\}} (V(x) - 1) G^{-1}(v_n)^2 dx
\]
\[
+ \int_{\{x \in \mathbb{R}^N, |G^{-1}(v(x))| \leq 1\}} \left[ \frac{2q|G^{-1}(v_n)|^2 + q|G^{-1}(v_n)|^4}{4(1 + G^{-1}(v_n))^2} - \frac{G^{-1}(v_n)^2}{(1 + G^{-1}(v_n))^3} \right] dx
\]
\[
\geq \frac{(q - 2)}{2} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx.
\]
Once that \( V(x) \geq 1 \) for all \( x \in \mathbb{R}^N, q > 2 \) and \( t \in [0, 1] \),
\[
\frac{2q t^2 + q t^4}{4(1 + t^2)^2} - \frac{t^2}{(1 + t^2)^3} \geq 0 \quad \forall t \in [0, 1],
\]
it follows that
\[
\limsup_{n \to \infty} \|\nabla v_n\|_2^2 \leq \frac{2q \tilde{c}_n}{q - 2}. \tag{4.7}
\]
Recalling that there is $S > 0$ such that
\[ \int_{\mathbb{R}^N} |v|^2^* \leq S \left( \int_{\mathbb{R}^N} |\nabla v|^2 dx \right)^{2^*} \quad \forall v \in H^1(\mathbb{R}^N), \]
we derive that
\[ \limsup_{n \to +\infty} \int_{\mathbb{R}^N} |v_n|^2^* \leq \left( \frac{2q\tilde{c}_\kappa}{q - 2} \right)^{2^*} \quad \forall n \in \mathbb{N}. \quad (4.8) \]

From definition of $f$, given $\epsilon = \frac{\lambda_0}{2}$, there is $C > 0$ such that
\[ f(t) \leq \frac{\lambda_0}{18} t + Ct^{2^*-1} \quad \forall t \geq 0. \]

This together with $J'(v_n)v_n = o_n(1)$ gives
\[ \int_{\mathbb{R}^N} \left[ |\nabla v_n|^2 + V(x) \frac{G^{-1}(v_n) - 1}{G^{-1}(v_n)} v_n \right] dx \leq \int_{\mathbb{R}^N} \left[ \frac{\lambda_0}{18} (G^{-1}(v_n)) + C(G^{-1}(v_n))^{2^*-1} \right] v_n dx. \]

Using Lemma 4.1-3 and the fact that $\frac{1}{q} \leq g(t) \leq 1$ for all $t \in \mathbb{R}$, we get
\[ \frac{\lambda_0}{2} \int_{\mathbb{R}^N} |v_n|^2 dx \leq C \int_{\mathbb{R}^N} |v_n|^2^*. \]
Then, by (4.8),
\[ \limsup_{n \to +\infty} \int_{\mathbb{R}^N} |v_n|^2 dx \leq 2 \frac{C}{\lambda_0} \left( \frac{2q\tilde{c}_\kappa}{q - 2} \right)^{2^*} \quad \forall n \in \mathbb{N}. \quad (4.9) \]

From (4.7) and (4.8), it follows that $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$.

Analogous to the arguments in the end of Section 2, we can assume there is $v_\kappa \in H^1(\mathbb{R}^N)$ and a subsequence of $\{v_n\}$, still denoted by itself, such that
\[ v_n \rightharpoonup v_\kappa \text{ in } H^1(\mathbb{R}^N), \quad v_n \to v_\kappa \text{ in } H^1_{loc}(\mathbb{R}^N) \text{ and } v_n \to v_\kappa \text{ in } L^p_{loc}(\mathbb{R}^N) \quad \forall p \in [1, 2^*]. \]

Moreover, $v_\kappa$ is a nontrivial critical point for $\tilde{J}_\kappa$, $\tilde{J}_\kappa(v_\kappa) \leq \tilde{c}_\kappa$ and by (4.7),
\[ \|\nabla v_\kappa\|_2^2 \leq 2 \frac{q\tilde{c}_\kappa}{q - 2}. \quad (4.10) \]

**Lemma 4.4.** The nontrivial solution $v_\kappa$ of (4.1) verifies $\|\nabla v_\kappa\|_2 \leq \frac{2q d_\infty}{q - 2}$, where $d_\infty$ is independent on $\kappa > 0$.

**Proof.** Setting the functional $Q_\infty : H^1(\mathbb{R}^N) \to \mathbb{R}$ given by
\[ Q_\infty(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{9V_\infty}{2} \int_{\mathbb{R}^N} |v|^2 dx - \int_{\mathbb{R}^N} F(v) dx, \]

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we observe that
\[ \tilde{J}_\kappa(v) \leq Q_\infty(v) \quad \forall v \in H^1(\mathbb{R}^N). \]
As in Lemma 4.2, it follows that \( Q_\infty \) verifies the mountain pass geometry. Therefore, if \( d_\infty \) denotes the mountain pass level associated with \( Q_\infty \), the last inequality yields \( \tilde{c}_\kappa \leq d_\infty \). Hence, by (4.10)
\[ \|\nabla v_\kappa\|_2^2 \leq \frac{2qd_\infty}{q-2}, \]
finishing the proof.

**Proposition 4.1.** There exists a constant \( C_1 > 0 \) independent on \( \kappa \) such that \( \|v_\kappa\|_\infty \leq C_1 \kappa^{\frac{1}{q-2}} \).

**Proof.** Since the proof is similar to Proposition 3.1, we only indicate the necessary changes. Denoting \( v_\kappa \) by \( v \), we now have
\[
\int_{\mathbb{R}^N} |\nabla w_m|^2 dx \leq \left[ \frac{(\beta - 1)^2}{2\beta - 1} + 1 \right] \int_{\mathbb{R}^N} \nabla v \nabla v_m dx
\]
\[
\leq \beta^2 \int_{\mathbb{R}^N} \nabla v \nabla v_m + V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} v_m dx
\]
\[
= \beta^2 \int_{\mathbb{R}^N} f(G^{-1}(v)) \frac{G^{-1}(v)}{g(G^{-1}(v))} v_m dx
\]
\[
\leq 7\beta^2 \int_{\mathbb{R}^N} |G^{-1}(v)|^{q-1} g\left(\frac{G^{-1}(v)}{q}\right) v_m dx.
\]
Since \( \frac{1}{g(t)} \leq \frac{\sqrt{1 - \kappa}}{1 - 2\kappa} \leq 3\sqrt{2\kappa} \) for \( 0 < \kappa \leq \frac{1}{3} \). Then, by Sobolev inequality and Lemma 2.1-(3), we get
\[
\left( \int_{A_m} |w_m|^2 dx \right)^{(N-2)/N} \leq S \int_{\mathbb{R}^N} |\nabla w_m|^2 dx \leq 3\sqrt{2S} \beta^2 \kappa^{\frac{1}{2}} \int_{\mathbb{R}^N} |v|^{q-2} v_m^2 dx.
\]
Therefore,
\[
\|v\|_\infty \leq \sigma \left( 3\sqrt{2S} \beta^2 \kappa^{\frac{1}{2}} S^{1/2} C^{(q-2)/2} \right)^{\frac{1}{2q-2}} S^{1/2} C^{1/2} = C_1 \kappa^{\frac{1}{q-2}},
\]
where \( C_1 > 0 \) is independent of \( \kappa > 0 \), finishing the proof.

**4.1 Proof of Theorem 1.2.**

Combining the above arguments and Proposition 4.1, the solution \( v_\kappa \) of (1.10) satisfies \( \|v_\kappa\|_\infty \leq C_1 \kappa^{\frac{1}{q-2}} \). Fixing \( \kappa_1 = \min \left\{ \frac{1}{3}, \left( \frac{1}{10C_1} \right)^{\frac{1}{q-2}} \right\} \), we have that
\[
\left\| G^{-1}(v_\kappa) \right\|_\infty \leq \sqrt{6} \|v_\kappa\|_\infty < \sqrt{\frac{1}{3\kappa}} \quad \forall \kappa \in [0, \kappa_1).
\]
This implies that \( u = G^{-1}(v_\kappa) \) is a positive solution of (1.4).
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