Hausdorff Moment Sequences Induced by Rational Functions

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Abstract
We study the Hausdorff moment problem for a class of sequences, namely \((r(n))_{n \in \mathbb{Z}^+}\), where \(r\) is a rational function in the complex plane. We obtain a necessary condition for such sequence to be a Hausdorff moment sequence. We found an interesting connection between Hausdorff moment problem for this class of sequences with finite divided differences and convolution of complex exponential functions. We provide a sufficient condition on the zeros and poles of a rational function \(r\) so that \((r(n))_{n \in \mathbb{Z}^+}\) is a Hausdorff moment sequence. G. Misra asked whether the module tensor product of a subnormal module with the Hardy module over the polynomial ring is again a subnormal module or not. Using our necessary condition we answer the question of G. Misra in negative. Finally, we obtain a characterization of all real polynomials \(p\) of degree up to 4 and a certain class of real polynomials of degree 5 for which the sequence \((1/p(n))_{n \in \mathbb{Z}^+}\) is a Hausdorff moment sequence.

Keywords Moment problem · Positive definite kernel · Module tensor product · Subnormality

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1 Introduction

Let $H_K$ be a reproducing kernel Hilbert space consisting of holomorphic functions on the unit disc $\mathbb{D}$ with reproducing kernel $K$. Thus $K(z, w)$ is a complex valued function defined on $\mathbb{D} \times \mathbb{D}$, which is holomorphic in the first variable and anti holomorphic in the second variable and is positive definite in the sense that $(K(z_i, z_j))$ is positive definite for every finite subset $\{z_1, z_2, \ldots, z_n\}$ of unit disc $\mathbb{D}$, see [3,10]. If the operator $M_z$ of multiplication by the coordinate function on the Hilbert space $H_K$, is a bounded operator, then $H_K$ is a Hilbert module over the polynomial ring $\mathbb{C}[z]$ with the module action given by

$$(p, f) : \mathbb{C}[z] \times H_K \mapsto (p(M_z))(f) \in H_K.$$ 

See [8] for more discussion on Hilbert modules. If the operator $M_z$ is a contraction, that is, $\|M_z\|_{op} \leq 1$, then $K$ is said to be a contractive kernel. In such case, using Von-Neuman inequality, we have that the Hilbert module $H_K$ is also contractive, that is, $\|pf\|_{H_K} \leq \|p\|_\infty \|f\|_{H_K}$, where $\|p\|_\infty$ denotes the supremum norm of $p$ on the unit disc $\mathbb{D}$. It is well known that a reproducing kernel $K$ is contractive if and only if the function $(1 - z\bar{w})K(z, w)$ is positive definite, see [10, Theorem 5.21].

If the operator $M_z$ is a bounded subnormal operator, that is, the operator $M_z$ has a normal extension to a larger Hilbert space $K$ containing $H_K$, then the Hilbert module $H_K$ is said to be a subnormal module or equivalently $K$ is said to be a subnormal kernel, see [7] for more details on subnormal operators.

Given a reproducing kernel $K$ it is often difficult to determine whether it is a subnormal kernel or not. But there is a simple characterization of contractive subnormal kernels in terms of Hausdorff moment sequences if the kernel $K$ has the following diagonal form:

$$K(z, w) = \sum_{j=0}^{\infty} a_j(z\bar{w})^j, \quad z, w \in \mathbb{D},$$

where $a_j > 0$ for all $j \in \mathbb{Z}_+$. In such case the set $\{\sqrt{a_n}z^n : n \in \mathbb{Z}_+\}$ forms an orthonormal basis for $H_K$ and the operator $M_z$ is a unilateral weighted shift associated with the weight sequence $\left(\sqrt{\frac{a_n}{a_{n+1}}}\right)_{n \in \mathbb{Z}_+}$, see [10, Theorem 4.12].

A sequence $(x_n)_{n \in \mathbb{Z}_+}$ of positive numbers is said to be a Hausdorff moment sequence if there exists a positive Radon measure $\mu$ supported on the interval $[0, 1]$ such that

$$x_n = \int_0^1 t^n d\mu(t), \quad n \in \mathbb{Z}_+.$$ 

(1.2)

Since every continuous function on the interval $[0, 1]$ is a uniform limit of polynomials, it follows that if such a measure $\mu$ exists satisfying (1.2), then it must be unique (cf. [14, Corollary 4.2]). In that case the measure $\mu$ is called the representing measure for the sequence $(x_n)_{n \in \mathbb{Z}_+}$. F. Hausdorff characterized the Hausdorff moment sequences by complete monotonicity [9].
Theorem 1.1 (F. Hausdorff) A sequence \((x_n)_{n \in \mathbb{Z}^+}\) of positive numbers is a Hausdorff moment sequence if and only if the sequence \((x_n)_{n \in \mathbb{Z}^+}\) is completely monotone, that is,

\[
\sum_{j=0}^{m} (-1)^j \binom{m}{j} x_{n+j} \geq 0, \quad m \in \mathbb{N}, \; n \in \mathbb{Z}^+.
\]

The following theorem characterizes all contractive subnormal kernels of the form (1.1) in terms of Hausdorff moment sequences, see [7, Theorem 6.10].

Theorem 1.2 A reproducing kernel \(K(z, w) = \sum_{j=0}^{\infty} a_j (z \bar{w})^j\), defined on unit disc \(\mathbb{D}\) is a contractive subnormal kernel if and only if the sequence \((\frac{1}{a_n})_{n \in \mathbb{Z}^+}\) is a Hausdorff moment sequence.

It is well known that if \(K_1\) and \(K_2\) are two reproducing kernels then their product \(K_1 K_2\) is also a reproducing kernel, see [3]. In 1988, N. Salinas introduced the notion of sharp reproducing kernels and relates the functional Hilbert space \(\mathcal{H}_{K_1} \otimes \mathbb{C}[z] \otimes \mathcal{H}_{K_2}\) over the algebra of polynomials, see [11]. In the same article, N. Salinas asked whether \(\mathcal{H}_{K_1} \otimes \mathbb{C}[z] \otimes \mathcal{H}_{K_2}\) is again subnormal if \(K_1\) and \(K_2\) are subnormal kernels.

Let \(S(z, w) = (1 - z \bar{w})^{-1}\) be the Hardy kernel on the unit disc \(\mathbb{D}\), which is a contractive subnormal kernel. It is well known that the module tensor product \(\mathcal{H}_S \otimes \mathcal{H}_{K_2}\) is contractive if \(K\) is a sharp kernel, see [10, Theorem 5.21] and [11]. G. Misra asked a special version of the question by N. Salinas. He asked whether the module tensor product \(\mathcal{H}_S \otimes \mathcal{H}_{K_2}\) is again subnormal or not if the kernel \(K\) is a subnormal kernel. Anand and Chavan answer the question of N. Salinas in negative, see [1]. They consider the following class of contractive subnormal kernel functions:

\[
K_s(z, w) = \sum_{j=0}^{\infty} (sj + 1)(z \bar{w})^j, \quad z, w \in \mathbb{D}, \; s > 0.
\]

They obtain the following characterizations of Hausdorff moment sequences induced by degree 3 polynomials with real coefficients. For a complex number \(z\) we denote the real part of \(z\) by \(\Re(z)\), that is, for \(z = a + ib\) with \(a, b \in \mathbb{R}\), we have \(\Re(z) = a\).

Theorem 1.3 (Anand-Chavan) Let \(p\) be a real polynomial of the form \(p(z) = (z + 1)(z - \alpha)(z - \beta)\). Then the sequence \((1/p(n))_{n \in \mathbb{Z}^+}\) is a Hausdorff moment sequence if and only if one of the following holds:

1. \(\alpha < 0, \beta < 0\), when \(\alpha, \beta \in \mathbb{R}\),
2. \(\Re(\alpha) \leq -1\), when \(\alpha\) is a non real complex number and \(\beta = \bar{\alpha}\).

Using this characterization they completely determine when the product kernel \(K_{s_1} K_{s_2}\) is again a subnormal kernel and produce a family of counterexamples to answer the question of N. Salinas in negative. But this class fails to answer the question of G. Misra. It turns out that the kernel function \(SK_s\) is again a contractive subnormal
kernel for every $s > 0$. Note that if the reproducing kernel $K$ is of the form (1.1) then the product kernel $SK$ is given by

$$SK(z, w) = S(z, w)K(z, w) = \sum_{n=0}^{\infty} \left( \sum_{j=0}^{n} a_j \right) (z \bar{w})^n, \quad z, w \in \mathbb{D}.$$  

It follows that the kernel function $SK$ is always a contractive kernel, see [10, Theorem 5.21]. Thus in a special case the question of G. Misra reduces to the following equivalent question:

**Question 1.4** If the sequence $(1/a_n)_{n \in \mathbb{Z}^+}$ is a Hausdorff moment sequence, then does it follow that the sequence $(1/(a_0 + \cdots + a_n))_{n \in \mathbb{Z}^+}$ is also a Hausdorff moment sequence?

A similar kind of transformation of Hausdorff moment sequences had been discussed in [6] by Berg and Durán. They show that

**Theorem 1.5** (Berg-Duran) If $(a_n)_{n \in \mathbb{Z}^+}$ is a Hausdorff moment sequence, then $(1/(a_0 + \cdots + a_n))_{n \in \mathbb{Z}^+}$ is also a Hausdorff moment sequence.

In this article we answer the question of G. Misra in negative. Motivated by the classification result, namely the Theorem 1.3, we concentrate our focus on a special class of sequences, namely, the sequences induced by rational functions,

$$x_n = \frac{q(n)}{p(n)}, \quad n \in \mathbb{Z}^+, \quad (1.3)$$

where $q, p$ are polynomials with real coefficients such that $q$ and $p$ have no common zeros and $\deg(q) < \deg(p)$. Note that a Hausdorff moment sequence is necessarily a decreasing sequence of positive numbers and hence a bounded sequence. It follows that if a sequence $(x_n)_{n \in \mathbb{Z}^+}$ of the form (1.3) is a Hausdorff moment sequence, then necessarily we have $\deg(q) \leq \deg(p)$. Our aim is to describe all such polynomials $q, p$ (in terms of their zeros) so that the sequence $(x_n)_{n \in \mathbb{Z}^+}$ is a Hausdorff moment sequence.

In order to study joint subnormality of the spherical Cauchy dual of a balanced joint $q$-isometry multishift, Anand and Chavan started investigating the class of sequences $(1/p(n))_{n \in \mathbb{Z}^+}$, where $p$ is a polynomial satisfying $p(\mathbb{R}_+) \subseteq \mathbb{R}_+$, see [2]. There they draw upon an interesting connection between Hausdorff moment problem, Hermite interpolation and divided differences of exponential functions. Here we have obtained a natural generalization of those results to the case of rational functions.

We consider real polynomials $p$ having roots in the left half plane $\mathbb{H}_- := \{ z \in \mathbb{C} : \Re(z) < 0 \}$. Assume that the polynomial $p$ is of the following form.

$$p(z) = \prod_{j=1}^{m} (z - \alpha_j)^{b_j}, \quad (1.4)$$

where $\alpha_j$’s are distinct complex number in $\mathbb{H}_-$ and $b_j \in \mathbb{N}$ for $j = 1, 2, \ldots, m$. Let $q$ be an arbitrary real polynomial with $\deg(q) < \deg(p)$ and $q$ has no common zero
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with \( p \). First we find that there exists a real valued weight function \( w_{q,p}(t) \), which is in \( L^1[0, 1] \) and continuous on \( (0, 1) \) so that

\[
\frac{q(n)}{p(n)} = \int_0^1 t^n w_{q,p}(t) \, dt, \quad n \in \mathbb{Z}_+.
\]

By uniqueness of the representing measure associated to a Hausdorff moment sequence it follows that the sequence \( (q(n)/p(n))_{n \in \mathbb{Z}_+} \) is a Hausdorff moment sequence if and only if the weight function \( w_{q,p}(t) \geq 0 \) for all \( t \in (0, 1) \).

There is a natural connection between the weight function \( w_{q,p}(t) \) and the general divided differences of a certain family of functions. The special case when \( q \equiv 1 \) and \( p \) has all its zeros in \( \mathbb{R}_- \) has been discussed in [2, Proposition 3.7]. Before we state the generalized result let us recall the definition of divided differences of a function. Let \( F \) be a complex valued function on the complex plane and \( x = (x_1, x_2, \ldots, x_m) \in \mathbb{C}^m \), where \( x_1, x_2, \ldots, x_m \) are \( m \) distinct complex numbers. Let \( P \) and \( P_j \) be the function defined by

\[
P(z, x) = \prod_{k=1}^m (z - x_k), \quad P_j(z, x) = \prod_{k=1, k \neq j}^m (z - x_k), \quad j = 1, 2, \ldots, m.
\]

The divided difference \( F[x_1, x_2, \ldots, x_m] \) on \( m \) distinct points \( x_1, x_2, \ldots, x_m \) of an one variable function \( F \) is defined by

\[
F[x_1, x_2, \ldots, x_m] := \sum_{j=1}^m \frac{F(x_j)}{P_j(x_j, x)}.
\]

The general divided difference with repeated arguments of the function \( F \) is defined by

\[
F[x_1, \ldots, x_1, x_2, \ldots, x_2, \ldots, x_m, \ldots, x_m] := \frac{1}{r_1! r_2! \ldots r_m!} \frac{\partial^{r_1+r_2+\cdots+r_m} F[x_1, x_2, \ldots, x_m]}{\partial x_1^{r_1} \partial x_2^{r_2} \ldots \partial x_m^{r_m}}.
\]

We establish the following relationship between the weight function \( w_{q,p}(t) \) and the general divided differences of the family of function \( F_t(z) := q(z) t^{-z-1} \) defined on \( \mathbb{H}_- \).

**Theorem 1.6** The weight function \( w_{q,p}(t) \) is given by

\[
w_{q,p}(t) = F_t[^{b_1 \text{ times}} \alpha_1, \ldots, ^{b_2 \text{ times}} \alpha_2, \ldots, ^{b_m \text{ times}} \alpha_m], \quad t \in (0, 1].
\]

Minimal Hausdorff moment sequences and completely monotonic functions are intimately related, see [13, Chapter 4]. In our case, for any rational function \( r(x) =
Let $p$ be a real polynomial having at least one non-real root and of the form (1.3) to be a Hausdorff moment sequence. Thus \( \{r(n)\}_{n \in \mathbb{Z}^+} \) is a Hausdorff moment sequence if and only if \( r(x) \) is completely monotone function on \([0, \infty)\).

K. Ball provided a sufficient condition on the zeros and poles of a rational function \( r \) of the form \( r(x) = \prod_{i=1}^{m} (x + z_i)/(x + p_i) \) so that \( r(x) \) is a completely monotone function on \([0, \infty)\), see [4, Theorem 1]. Using Theorem 1.6, we find the following sufficient condition on the zeros and poles of a rational function \( r \) so that \( \{r(n)\}_{n \in \mathbb{Z}^+} \) is a Hausdorff moment sequence.

**Proposition 1.7** Let \( p \) be a real polynomial having distinct negative real roots, say \( \alpha_1, \alpha_2, \ldots, \alpha_m \) with no multiplicity and \( q \) be any real polynomial such that \( \deg(q) < \deg(p) \) and \( q \), \( p \) has no common zero. If the divided difference \( q[\alpha_1, \alpha_2, \ldots, \alpha_j] \geq 0 \) for \( j = 1, 2, \ldots, m \), then the sequence \( \{q(n)/p(n)\}_{n \in \mathbb{Z}^+} \) is a Hausdorff moment sequence.

In this article, we obtain the following necessary condition for a sequence \( (x_n)_{n \in \mathbb{Z}^+} \) of the form (1.3) to be a Hausdorff moment sequence.

**Theorem 1.8** Let \( p \) be a real polynomial having at least one non real root and of the form

\[
p(z) = \prod_{i=1}^{s} (z - r_i)^{l_i} \prod_{i=1}^{d} (z - z_i)^{m_i} \prod_{i=1}^{d} (z - \bar{z}_i)^{m_i},
\]

where \( r_i < 0 \), for \( i = 1, 2, \ldots, s \) and \( z_k = x_k + iy_k \) with \( x_k < 0, y_k > 0 \) for \( k = 1, 2, \ldots, d \). Let \( q \) be any other polynomial having no common zeros with \( p \) and \( \deg(q) < \deg(p) \).

If there exists a non real root, say \( z_1 \), (without loss of generality) of \( p \), such that \( \Re(z_1) > r_i \) for every \( i = 1, 2, \ldots, s \) and \( \Re(z_1) > \Re(z_i) \) for every \( i = 2, 3, \ldots, d \) then the sequence \( \{q(n)/p(n)\}_{n \in \mathbb{Z}^+} \) is not a Hausdorff moment sequence.

To answer the question of G. Misra, we consider the following class of kernel functions

\[
K_c(z, w) = \sum_{n=0}^{\infty} (n + c)^6 z^n \bar{w}^n, \quad z, w \in \mathbb{D}, \quad c > 0.
\]

We find that the kernel function \( K_c \) is a contractive subnormal kernel for every \( c > 0 \) and then using the necessary condition in Theorem 1.8, we show that the product kernel \( SK_c \) is not a subnormal kernel for all \( c \) in a neighborhood of 1. This answers the question of G. Misra and as well as the question of N. Salinas in negative.

The following result as shown in [6, Lemma 2.1] is often helpful in determining whether a given sequence is a Hausdorff moment sequence or not.

**Lemma 1.9** Let \( (x_n)_{n \in \mathbb{Z}^+} \) and \( (y_n)_{n \in \mathbb{Z}^+} \) be two Hausdorff moment sequences associated to some Radon measures \( \mu \) and \( \nu \) respectively. Then the sequence \( (x_n y_n)_{n \in \mathbb{Z}^+} \)
is also a Hausdorff moment sequence and associated with the convolution measure \( \mu \otimes \nu \).

It is straightforward to see that the sequence \( \left\{ \frac{1}{n-a} \right\}_{n \in \mathbb{Z}_{+}} \) is a Hausdorff moment sequence associated with the measure \( t^{-a-1} dt \) on \([0, 1]\) for all \( a < 0 \). Now using Lemma 1.9, it follows that, if \( p \) is a reducible polynomial over \( \mathbb{R} \) having zeros in \( \mathbb{H}_{-} \), then the sequence \( \left\{ 1/p(n) \right\}_{n \in \mathbb{Z}_{+}} \) is a Hausdorff moment sequence, see also [2, Theorem 3.1]. This leads us to the question of classifying all real polynomial \( p \), which are not reducible over \( \mathbb{R} \) so that the sequence \( \left\{ 1/p(n) \right\}_{n \in \mathbb{Z}_{+}} \) is a Hausdorff moment sequence.

Using Lemma 1.9, Theorems 1.3, 1.8 and taking into consideration a special case separately, we obtain the complete list of all real polynomial \( p \) up to degree 4 for which the sequence \( \left\{ 1/p(n) \right\}_{n \in \mathbb{Z}_{+}} \) is a Hausdorff moment sequence. For degree 5 polynomial \( p \), we do not have have the complete classification result. Except when \( p \) is of the form

\[
p(z) = (z - r)(z - \alpha)(z - \bar{\alpha})(z - \beta)(z - \bar{\beta}),
\]

where \( \Re(\alpha), \Re(\beta) \leq r < 0 \), we have answer for all other polynomial \( p \) of degree 5. In the very special case when all the roots of \( p \) lie in a vertical line, that is when \( \Re(\alpha) = \Re(\beta) = r < 0 \), we have the following interesting result:

**Theorem 1.10** Let \( p \) be a real polynomial of the form

\[
p(z) = (z - r) \prod_{j=1}^{2} (z - (r + iy_j))(z - (r - iy_j)), \text{ where } r < 0, \text{ and } 0 < y_1 \leq y_2.
\]

Then the sequence \( 1/p(n) \) is a Hausdorff moment sequence if and only if the ratio \( y_2/y_1 \) is a positive integer greater than 1.

The organization of the paper is as follows. In Sect. 2, we obtain a partial decomposition formula for rational functions of the form \( q/p \), where \( q, p \) are polynomial with \( \deg q < \deg p \). Using this decomposition we show that any sequence \( (x_n)_{n \in \mathbb{Z}_{+}} \) of the form (1.3), where all the roots of \( p \) lie in \( \mathbb{H}_{-} \), must be a moment sequence associated with a real measure (not necessarily positive) of the form \( w_{q,p}(t)dt \) on the interval \([0, 1]\) for some continuous function \( w_{q,p}(t) \) on \((0, 1]\).

We established the relationship between the weight function \( w_{q,p}(t) \) and the general divided difference of the function \( F_t(z) := q(z)t^{-z-1} \) defined on \( \mathbb{H}_{-} \) in Sect. 3. In the case when \( q \equiv 1 \), the weight function \( w_p(t) \) is intimately related with the convolution of complex exponential function \( f_j(y) := \exp(\alpha_j y) \) defined on \( \mathbb{R}_{+} \), where \( \alpha_j \)'s are the roots of \( p \) in \( \mathbb{H}_{-} \). We also explore this relationship in Sect. 3.

In Sect. 4, we obtain the necessary condition for a sequence \( (x_n)_{n \in \mathbb{Z}_{+}} \) of the form (1.3) to be a Hausdorff moment sequence, see Theorem 1.8. Using this condition we produce a family of counterexample to answer the questions of G. Misra and N. Salinas in negative. Theorem 4.1 gives us another necessary condition when all the roots of \( p \) are negative real number.
Section 5 is devoted to the discussion of characterizing all real polynomial \( p \) up to degree 5 for which the sequence \( (1/p(n))_{n \in \mathbb{Z}_+} \) is a Hausdorff moment sequence.

2 Partial Fraction Decomposition of Rational Function and Weight Function

We start by establishing a formula for the partial fraction decomposition of a rational function. For a polynomial \( p \), the partial fraction decomposition of \( 1/p \) is obtained in [2, Lemma 3.3], using generalized Hermite’s interpolation formula, see [12]. We obtain a generalized formula for partial fraction decomposition of a rational function. For a holomorphic function \( f \) on the complex plane and for \( k \in \mathbb{Z}_+ \), we use the notation \( f^{(k)}(w) \) to denote the \( k \)th order complex derivative of \( f \) at the point \( w \), that is, \( f^{(k)}(w) = \partial^k \partial_{z^k} f(z) \big|_{z=w} \).

Proposition 2.1 (Partial fraction decomposition formula) Let \( p \) be a polynomial of the form

\[
p(z) = \prod_{j=1}^{m} (z - \alpha_j)^{b_j}, \quad p_k(z) = \prod_{j=1, j\neq k}^{m} (z - \alpha_j)^{b_j}, \quad k = 1, 2, \ldots, m, \quad (2.1)
\]

where \( \alpha_j \)'s are distinct complex number and \( b_j \in \mathbb{N} \) for \( j = 1, 2, \ldots, m \). Let \( q \) be another polynomial such that \( \deg(q) < \deg(p) \) and \( q \) has no common zero with \( p \). Then it follows that

\[
\frac{q(z)}{p(z)} = \sum_{i=1}^{m} \sum_{j=1}^{b_i} A_{i,j} \frac{1}{(z - \alpha_i)^j}, \quad A_{i,j} = \frac{(q/p_i)^{(b_i-j)}(\alpha_i)}{(b_i-j)!}, \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, b_i.
\]

**Proof** Let \( d \) be the degree of the polynomial \( p \). So, \( d = \sum_{i=1}^{m} b_i \). Let \( f_i \) be the polynomial given by

\[
f_i(z) = \sum_{j=1}^{b_i} A_{i,j} \frac{p(z)}{(z - \alpha_i)^j} = p_i(z) \sum_{j=1}^{b_i} A_{i,j} (z - \alpha_i)^{b_i-j} = p_i(z) h_i(z),
\]

where \( h_i(z) = \sum_{j=1}^{b_i} A_{i,j} (z - \alpha_i)^{b_i-j} \) for \( i = 1, 2, \ldots, m \). Let \( r \) be the polynomial given by

\[
r(z) = \sum_{i=1}^{m} f_i(z) = \sum_{i=1}^{m} \sum_{j=1}^{b_i} A_{i,j} \frac{p(z)}{(z - \alpha_i)^j}.
\]
To prove the Proposition it is sufficient to show that \( r(z) \equiv q(z) \). Now we will show

\[
\begin{align*}
    f_i^{(k)}(\alpha_i) &= q^{(k)}(\alpha_i), \quad k = 0, 1, \ldots, b_i - 1, \quad (2.2) \\
    f_j^{(k)}(\alpha_i) &= 0, \quad j \neq i, \quad k = 0, 1, \ldots, b_i - 1. \quad (2.3)
\end{align*}
\]

This will give us

\[
r^{(k)}(\alpha_i) = q^{(k)}(\alpha_i), \quad k = 0, 1, \ldots, b_i - 1, \quad i = 1, 2, \ldots, m.
\]

Note that each \( f_j \) is a polynomial of degree at most \( d - 1 \). So, \( \deg(r) \) is at most \( d - 1 \). By our assumption \( \deg(q) \) is at most \( d - 1 \). Hence by uniqueness of the interpolating polynomial we obtain that \( r(z) \equiv q(z) \).

To prove (2.3) first observe that for an arbitrary but fixed \( i \in \{1, 2, \ldots, m\} \), each \( p_j(z) \) has \( (z - \alpha_i)^{b_i} \) as a factor for all \( j \neq i \). Hence each \( f_j(z) \) has \( (z - \alpha_i)^{b_i} \) as a factor for \( j \neq i \) and for each \( i = 1, 2, \ldots, m \). It follows trivially that \( f_j^{(k)}(\alpha_i) = 0, \quad j \neq i, \quad k = 0, 1, \ldots, b_i - 1 \). The claim in (2.2) will be proved once the following lemma is established.

**Lemma 2.2** Let \( h(z) \) and \( \frac{q(z)}{h(z)} \) be well defined holomorphic function in a neighborhood of \( \alpha \), and \( B_j = \left(\frac{q(z)}{h(z)}\right)^{(n-j)}(\alpha) \right\} \), for \( j = 1, 2, \ldots, n \). Let \( L(z) = h(z)\left(\sum_{j=1}^{n} B_j(z - \alpha)^{n-j}\right) \). Then it follows that

\[
L^{(k)}(\alpha) = q^{(k)}(\alpha), \quad k = 0, 1, \ldots, n - 1.
\]

**Proof** Let \( r(z) = \sum_{j=1}^{n} B_j(z - \alpha)^{n-j} \). Then \( r^{(k)}(\alpha) = k!B_{n-k} \) for \( k = 1, 2, \ldots, n - 1 \). Applying Leibniz’s differentiation rule to the function \( h(z)r(z) \) and to the function \( h(z)\frac{q(z)}{h(z)} \) we get that

\[
L^{(k)}(\alpha) = \sum_{j=0}^{k} \binom{k}{j} h^{(k-j)}(\alpha) r^{(j)}(\alpha) = \sum_{j=0}^{k} \binom{k}{j} h^{(k-j)}(\alpha) B_{n-j}(\alpha) j!
\]

\[
= \sum_{j=0}^{k} \binom{k}{j} h^{(k-j)}(\alpha) \left(\frac{q(z)}{h(z)}\right)^{(j)}(\alpha) = \left(\frac{h}{q}\right)^{(k)}(\alpha) = q^{(k)}(\alpha), \quad k = 0, 1, \ldots, n - 1.
\]

This completes the proof of the Lemma 2.2.

Applying this Lemma to \( f_i(z) = p_i(z)h_i(z) \), for \( i = 1, 2, \ldots, m \), we get our desired claim in (2.2). Hence the proof of the Proposition 2.1 follows.

Let \( p \) and \( q \) be two polynomial with real coefficients such that \( \deg(q) < \deg(p) \). Let \( p \) be of the form (2.1). Let us further assume that \( p \) is a stable polynomial, that is, \( \alpha_i \in \mathbb{H}_- \) for all \( i = 1, 2, \ldots, m \). We also assume that \( q \) and \( p \) have no common root and \( q(n) > 0 \) for all \( n \in \mathbb{Z}_+ \). We consider the sequence \( x_n = (q(n)/p(n)), \quad n \in \mathbb{Z}_+ \).
$\mathbb{Z}_+$. Now we would determine when the sequence $(x_n)_{n \in \mathbb{Z}_+}$ is a Hausdorff moment sequence. For $x \in \mathbb{R}_+$ and $\alpha \in \mathbb{C}$ with $\Re(\alpha) < 0$, it is clear that

$$\frac{1}{x - \alpha} = \int_0^1 t^{x-\alpha-1} \, dt.$$  

Differentiating both side repeatedly with respect to $x$, we get

$$\frac{1}{(x - \alpha)^k} = \int_0^1 t^{x-\alpha-1}(\log(1/t))^{k-1} \frac{1}{(k-1)!} \, dt, \quad k \in \mathbb{N}, \ x \in \mathbb{R}_+.$$  

Now using partial decomposition formula in Proposition 2.1, we obtain that

$$\frac{q(n)}{p(n)} = \int_0^1 t^n w_{q,p}(t) \, dt,$$

where the weight function $w_{q,p}(t)$ is given by

$$w_{q,p}(t) = \sum_{i=1}^{m} \left( \sum_{j=1}^{b_i} A_{i,j} \frac{(\log(1/t))^{j-1}}{(j-1)!} \right) t^{-\alpha_i-1}, \quad t \in (0, 1]. \quad (2.4)$$

Since $p$ is a real polynomial it follows that if $\alpha$ is a complex root of $p$ with multiplicity $k$, then its conjugate $\bar{\alpha}$ is also a root of $p$ with same multiplicity $k$. Let $r_1, r_2, \ldots, r_s$ be the real roots of $p$ with multiplicity $l_1, l_2, \ldots, l_s$ respectively. Let $z_1, z_2, \ldots, z_d$ along with their conjugates be the complex roots of $p$ with multiplicity $m_1, m_2, \ldots, m_d$ respectively. So, we have $m = s + 2d$. Let us enumerate $\alpha_i$’s in (2.1) in the following way. $\alpha_i = r_i$ for $i = 1, 2, \ldots, s$ and $\alpha_{s+i} = z_i, \alpha_{s+d+i} = \bar{z}_i$ for $i = 1, 2, \ldots, d$. Consequently, we have that $b_i = l_i$ for $i = 1, 2, \ldots, s$ and $b_{s+i} = m_i = b_{s+d+i}$ for $i = 1, 2, \ldots, d$. Thus the polynomial $p$ is of the following form

$$p(z) = \prod_{i=1}^{s} (z - r_i)^{l_i} \prod_{i=1}^{d} (z - z_i)^{m_i} \prod_{i=1}^{d} (z - \bar{z}_i)^{m_i}, \quad (2.5)$$

where $r_i < 0$, for $i = 1, 2, \ldots, s$ and $z_k = x_k + iy_k$ with $x_k < 0, y_k > 0$ for $k = 1, 2, \ldots, d$. For such $p$ the weight function $w_{q,p}(t)$ can be rewritten in the following form,

$$w_{q,p}(t) = \sum_{i=1}^{s} \left( \sum_{j=1}^{l_i} \frac{A_{i,j} (\log(1/t))^{j-1}}{(j-1)!} \right) t^{-r_i-1} + 2\Re \left( \sum_{i=1}^{d} \left( \sum_{j=1}^{m_i} \frac{A_{s+i,j} (\log(1/t))^{j-1}}{(j-1)!} \right) t^{-z_i-1} \right).$$
Let \( z_k = x_k + iy_k \) for \( k = 1, 2, \ldots, d \). Let \( \theta_{i,j} \) be the principal argument of \( A_{x+i,j} \) for \( i = 1, 2, \ldots, d \) and \( j = 1, 2, \ldots, m_i \). Thus \( \mathfrak{m}(A_{x+i,j} t^{-z_i-1}) = |A_{x+i,j}| t^{-x_i-1} \cos(\theta_{i,j} - y_i \log t) \). Consequently, the weight function \( w_{q,p}(t) \) takes the following form.

\[
\begin{align*}
\frac{A_{i,j}}{(j-1)!} \left( \log(1/t) \right)^{j-1} t^{-r_{i,j}} \sum_{j=1}^{s_i} & \frac{|A_{s+i,j}|}{(j-1)!} \left( \log(1/t) \right)^{j-1} 2 \cos(\theta_{i,j} - y_i \log t) t^{-x_i-1}.
\end{align*}
\]

Note that \( w_{q,p}(t) \) is a continuous function on \([0, 1]\) and \( w_{q,p}(t) \) is also integrable on \([0, 1]\). Thus the sequence \( (q(n)/p(n))_{n \in \mathbb{Z}^+} \) is a Hausdorff moment sequence if and only if \( w_{q,p}(t) dt \) is a positive measure on \([0, 1]\), i.e., if and only if the weight function \( w_{q,p}(t) \geq 0 \) for all \( t \in [0, 1] \).

It has been shown in [1] that for a sequence \((1/p(n))_{n \in \mathbb{Z}^+}\), where \( p \) is a degree 3 polynomial with real coefficients, to be a Hausdorff moment sequence it is necessary that all the roots of \( p \) must lie in \( \mathbb{H}_- \) (see also [2]). We believe that the result remains true for an arbitrary polynomial with real coefficients, that is, for a sequence \((x_n)_{n \in \mathbb{Z}^+}\) of the form (1.3) to be a Hausdorff moment sequence it is necessary that all the roots of \( p \) must lie in \( \mathbb{H}_- \). But we were unable to prove it. From now on throughout the article, we will only consider the sequences \((x_n)_{n \in \mathbb{Z}^+}\) of the form (1.3), where all the roots of \( p \) lies in \( \mathbb{H}_- \).

### 3 Weight Function, Finite Divided Differences and Convolution

Now we will provide an alternative expression of the weight function \( w_{q,p}(t) \) in terms of finite divided differences of the function \( F_t(z) := q(z) t^{-z-1} \) defined on \( \mathbb{H}_- \). In [2, Proposition 3.7], this already has been established in the case when all the roots of \( p \) lies in \( \mathbb{R}_- \) and \( q \equiv 1 \). We obtain a natural generalization of their result. First we compute the general divided differences for the family of function \( \{F_t(z) : t \in (0,1)\} \) given by \( F_t(z) = q(z) t^{-z-1} \) for \( \mathfrak{m}(z) < 0 \), where \( q \) is an arbitrary real polynomial with \( \deg(q) < \deg(p) \) and \( q, p \) have no common zeros.

**Proof of Theorem 1.6** Let \( p \) and \( p_i \) be the polynomials of the form (2.1), that is,

\[
p(z) = \prod_{j=1}^{m} (z - \alpha_j)^{b_j}, \quad p_i(z) = \prod_{j=1, j \neq i}^{m} (z - \alpha_j)^{b_j}, \quad i = 1, 2, \ldots, m.
\]
Lemma 3.1

\[ F_t[\alpha_1, \ldots, \alpha_1, \alpha_2, \ldots, \alpha_2, \ldots, \alpha_m, \ldots, \alpha_m] \]

\[ \frac{b_1 \text{ times}}{b_2 \text{ times}} \frac{b_m \text{ times}}{\text{ }} \]

\[ = m \left( \sum_{i=1}^{\frac{b_1}{k}} \left( \sum_{j=1}^{\frac{b_2}{k}} \frac{(\log(1/t))^{\frac{b_3-1}{k}}}{(b_i-j)!(j-1)!} \left( \frac{q(z)}{P_t(z)} \right)^{(b_i-j)}(\alpha_i) \right) t^{-\alpha_i-1} \right). \]

Proof Since \( F_t(z) = q(z) t^{-z-1} \), we have that

\[ F_t[\alpha_1, \alpha_2, \ldots, \alpha_m] = \sum_{i=1}^{m} \frac{q(\alpha_i)}{P_t(\alpha_i, \alpha)} t^{-\alpha_i-1}. \]  

(3.1)

First note that \( P_t(\alpha_i, \alpha) = \prod_{j=1}^{m} (\alpha_i - \alpha_j) \) for \( i = 1, 2, \ldots, m \). It follows that

\[ \frac{\partial^k}{\partial^{\alpha_i} P_t(\alpha_i, \alpha)} = \frac{\partial^k}{\partial z} \frac{q(z)}{P_t(z, \alpha)} \bigg|_{z=\alpha_i} = \left( \frac{q(z)}{P_t(z, \alpha)} \right)^{(k)}(\alpha_i), \]

\[ \frac{\partial^k}{\partial^{\alpha_i} t^{-\alpha_i-1}} P_t(\alpha_i, \alpha) = k! \frac{q(\alpha_i)}{P_t(\alpha_i, \alpha)} t^{-\alpha_i-1}. \]

Using Leibniz’s rule we obtain

\[ \frac{1}{k!} \frac{\partial^k}{\partial^{\alpha_i} \left( \frac{q(\alpha_i)}{P_t(\alpha_i, \alpha)} t^{-\alpha_i-1} \right)} \]

\[ = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} \left( \frac{q(z)}{P_t(z, \alpha)} \right)^{(k-j)}(\alpha_1) (\log(1/t))^j t^{-\alpha_1-1}, \]

\[ = \sum_{j=0}^{k} \frac{(\log(1/t))^j}{(k-j)!j!} \left( \frac{q(z)}{P_t(z, \alpha)} \right)^{(k-j)}(\alpha_1) t^{-\alpha_1-1}, \]

\[ = \sum_{j=1}^{k+1} \frac{(\log(1/t))^{j-1}}{(k+1-j)!(j-1)!} \left( \frac{q(z)}{P_t(z, \alpha)} \right)^{(k+1-j)}(\alpha_1) t^{-\alpha_1-1}. \]

Thus differentiating the both side of (3.1) w.r.t the variable \( \alpha_1 \) we get that,

\[ F_t[\alpha_1, \ldots, \alpha_1, \alpha_2, \ldots, \alpha_3, \ldots, \alpha_m] = \frac{1}{(b_1-1)!} \frac{\partial^{(b_1-1)}}{\partial^{(b_1-1)} \alpha_1} F_t[\alpha_1, \alpha_2, \ldots, \alpha_m] \]

\[ = \sum_{j=1}^{b_1} \frac{(\log(1/t))^{j-1}}{(b_1-j)!(j-1)!} \left( \frac{q(z)}{P_t(z, \alpha)} \right)^{(b_1-j)}(\alpha_1) t^{-\alpha_1-1}. \]
Hausdorff Moment Sequences Induced by Rational Functions

\[ + \sum_{i=2}^{m} \frac{q(\alpha_i)}{(\alpha_i - \alpha_1)^{b_1} \prod_{r=2, r \neq i}^{m} (\alpha_i - \alpha_r)} \cdot t^{-\alpha_i - 1} \]

Observe that the differentiation in \( \alpha_2 \) of each term in the above summation are as follows,

\[
\frac{1}{k!} \frac{\partial^k}{\partial \alpha_2^k} \left( \frac{q(z)}{P_1(z, \alpha)} \right)^{(b_1-j)} \frac{q(\alpha_1)}{P_1(\alpha_1, \alpha)} = \frac{1}{k!} \frac{\partial^k}{\partial \alpha_2^k} \frac{q(\alpha_1)}{P_1(\alpha_1, \alpha)} = \frac{1}{k!} \frac{\partial^k}{\partial \alpha_2^k} \left( \frac{q(\alpha_1)}{P_1(\alpha_1, \alpha)} \right) = \frac{1}{k!} \frac{\partial^k}{\partial \alpha_2^k} \left( \frac{q(z)}{(\alpha_1 - \alpha_2)^{(k+1)} \prod_{r=3}^{m} (\alpha_1 - \alpha_r)} \right) = \frac{1}{k!} \frac{\partial^k}{\partial \alpha_2^k} \left( \frac{q(z)}{(\alpha_2 - \alpha_1)^{b_1} \prod_{r=3}^{m} (\alpha_2 - \alpha_r)} t^{-\alpha_2 - 1} \right) = \sum_{j=1}^{k+1} (\log(1/r))^{-1} \left( \frac{q(z)}{(\alpha_2 - \alpha_1)^{b_1} \prod_{r=3}^{m} (\alpha_2 - \alpha_r)} \right)^{(k+1-j)} t^{-\alpha_2 - 1},
\]

and

\[
\frac{1}{k!} \frac{\partial^k}{\partial \alpha_2^k} \left( \frac{q(\alpha_1)}{(\alpha_i - \alpha_1)^{b_1} \prod_{r=2, r \neq i}^{m} (\alpha_i - \alpha_r)} t^{-\alpha_i - 1} \right) = \frac{q(\alpha_1)}{(\alpha_i - \alpha_1)^{b_1} (\alpha_i - \alpha_2)^{(k+1)} \prod_{r=3}^{m, r \neq 3} (\alpha_i - \alpha_r)} t^{-\alpha_i - 1}, i = 3, 4, \ldots, m.
\]

Hence we get that

\[
F_i[\alpha_1, \ldots, \alpha_{b_1}, \alpha_2, \ldots, \alpha_{b_1}, \alpha_3, \alpha_4, \ldots, \alpha_m] = \frac{1}{(b_1 - 1)!(b_2 - 1)!} \frac{\partial^{(b_2-1)}}{\partial \alpha_2^{(b_2-1)}} \frac{\partial^{(b_1-1)}}{\partial \alpha_1^{(b_1-1)}} F_i[\alpha_1, \alpha_2, \ldots, \alpha_m],
\]

\[
= \sum_{j=1}^{b_1} (\log(1/r))^{-1} \left( \frac{q(z)}{(\alpha_2 - \alpha_1)^{b_2} \prod_{r=3}^{m} (\alpha_2 - \alpha_r)} \right)^{(b_2-j)} t^{-\alpha_2 - 1},
\]
\[ + \sum_{j=1}^{b_2} \frac{(\log(1/t))^{j-1}}{(b_2 - j)!} \left( \frac{q(z)}{(z - \alpha_1)^{b_1}} \prod_{r=3}^{m}(z - \alpha_r) \right)^{(b_2-j)} \alpha_2 \right)^{(b_2-j)} t^{-\alpha_2 - 1} \]

\[ + \sum_{i=3}^{m} \frac{q(\alpha_i)}{(\alpha_i - \alpha_1)^{b_1}(\alpha_i - \alpha_2)^{b_2}} \prod_{r=3}^{m}(\alpha_i - \alpha_r) t^{-\alpha_i - 1}. \]

Continuing this way it is straightforward to see that

\[ F_t[\alpha_1, \ldots, \alpha_1, \alpha_2, \ldots, \alpha_2, \ldots, \alpha_m, \ldots, \alpha_m] \]

\[ = \sum_{i=1}^{m} \left( \sum_{j=1}^{b_1} \frac{(\log(1/t))^{j-1}}{(b_1 - j)!} \left( \frac{q(z)}{p_i(z)} \right)^{(b_1-j)} \alpha_i \right)^{(b_1-j)} t^{-\alpha_i - 1}. \]

This gives us that the weight function in (2.4) can be written in the following form

\[ w_{q, p}(t) = F_t[\alpha_1, \ldots, \alpha_1, \alpha_2, \ldots, \alpha_2, \ldots, \alpha_m, \ldots, \alpha_m], \quad (3.2) \]

where the family of function \( \{F_t : t \in (0, 1]\} \) is given by \( F_t(z) = q(z)t^{-z-1}, \Re(z) < 0. \)

Let us denote the weight function \( w_{q, p}(t) \) by \( w_p(t) \) in the case when \( q \equiv 1. \) In such case we provide an alternative expression for the weight function \( w_p(t) \) in terms of convolution of some exponential functions. Let \( \alpha_1, \alpha_2, \ldots, \alpha_m \) be an arbitrary \( m \) points (not necessarily distinct) in \( \mathbb{H} \). Let \( p \) be the polynomial given by

\[ p(z) = \prod_{j=1}^{m}(z - \alpha_j). \]

Laplace transform of a measurable function \( f : [0, \infty) \to \mathbb{C} \) is defined by

\[ (\mathcal{L} f)(s) = \int_{0}^{\infty} \exp(-s y) f(y) dy, \]

for those \( s \in \mathbb{C} \) for which the integral make sense. Consider functions

\[ f_j(y) = \exp(\alpha_j y), \quad j = 1, 2, \ldots, m. \]

Their Laplace transforms are

\[ (\mathcal{L} f_j)(s) = \frac{1}{s - \alpha_j}, \quad j = 1, 2, \ldots, m, \Re(s) > 0. \]
The convolution of two measurable functions $g$ and $h$ is defined by
\[
(g * h)(y) = \int_0^y g(x)h(y - x)dx,
\]
where $g, h : [0, \infty) \to \mathbb{C}$ are two complex valued functions so that the integrals mentioned above are finite for all $y \in [0, \infty)$. The Laplace transform of $g * h$ is
\[
(L(g * h))(s) = (Lg)(s)(Lh)(s).
\]
Let $f$ be the function defined by $f = f_1 * f_2 * \cdots * f_m$. Then it follows that
\[
\frac{1}{p(s)} = \int_0^\infty \exp(-sy)f(y)dy = \int_0^1 t^{s-1} f(\log(1/t))dt, \quad s > 0.
\]
Thus the weight function has the following form
\[
w_p(t) = \frac{1}{t}(f_1 * f_2 * \cdots * f_m)(\log(1/t)), \quad t \in (0, 1]. \tag{3.3}
\]
Since each $f_j$ is an exponential function, the convolution of those $f_j$’s has the following simple expression.
\[
(f_1 * f_2 * \cdots * f_m)(y) = y^{m-1} \int_{\Delta_{m-1}} \exp \left( y \sum_{j=1}^m \lambda_j \alpha_j \right) d\lambda, \tag{3.4}
\]
where $\Delta_{m-1} = \{ (\lambda_1, \lambda_2, \ldots, \lambda_m) \in [0, 1]^m : \sum_{j=1}^m \lambda_j = 1 \}$ is the $(m-1)$ dimensional simplex and $d\lambda$ is the usual Lebesgue measure on the simplex $\Delta_{m-1}$.

**Proof of (3.4)** $m = 2$:
\[
(f_1 * f_2)(y) = \int_0^y \exp(\alpha_1 x) \exp(\alpha_2 (y - x))dx \\
= y \int_0^1 \exp(\alpha_1 sy) \exp(\alpha_2 (y - sy))ds, \\
= y \int_0^1 \exp(y(s\alpha_1 + (1-s)\alpha_2))ds = y \int_{\Delta_1} \exp \left( y \sum_{j=1}^2 \lambda_j \alpha_j \right) d\lambda,
\]
Now we use induction to prove for an arbitrary $m$. Let us assume that the statement is true for $m = k$. So by definition of integral over the simplex $\Delta_{k-1}$ we have
\[
(f_1 * f_2 * \cdots * f_k)(x) = x^{k-1} \int_0^1 \int_0^{1-\lambda_1} \ldots \int_0^{1-\lambda_1 \cdots - \lambda_{k-2}} \exp \left( x \sum_{j=1}^k \lambda_j \alpha_j \right) d\lambda_1 \cdots d\lambda_{k-1},
\]
where λ_k = 1 − λ_1 − ⋯ − λ_k−1. So, \( (f_1 * f_2 * \cdots * f_{k+1})(y) = \int_0^1 (f_1 * f_2 * \cdots * f_k)(y-x)dx \). We apply a change of variable \( x = sy \) to obtain that

\[
g(y) = (f_1 * f_2 * \cdots * f_{k+1})(y) = y^k \int_0^1 \frac{1}{s^{k-1}} \int_0^1 \int_0^{s-\lambda_1} \cdots \int_0^{s-\lambda_1-\cdots-\lambda_{k-1}} \exp \left( y \left( \sum_{j=1}^k s\lambda_j \alpha_j + (1-s)\alpha_{k+1} \right) \right) d\lambda_1 \cdots d\lambda_{k-1} ds.
\]

Now we again apply a change of variable, namely \( s\lambda_j = c_j \) for \( j = 1, 2, \ldots, k-1 \), and we obtain that

\[
g(y) = y^k \int_0^1 \int_0^1 \int_0^{s-c_1} \cdots \int_0^{s-c_1-\cdots-c_{k-2}} \exp \left( y \left( \sum_{j=1}^k c_j \alpha_j + (1-s)\alpha_{k+1} \right) \right) dc_1 \cdots dc_{k-1} ds,
\]

where \( c_k = s - c_1 - \cdots - c_{k-1} \). Finally we apply the change of variable \( s = 1 - c_0 \) and we get that

\[
g(y) = y^k \int_0^1 \int_0^{1-c_0} \int_0^{1-c_0-c_1} \cdots \int_0^{1-c_0-c_1-\cdots-c_{k-2}} \exp \left( y \left( c_0 \alpha_{k+1} + \sum_{j=1}^k c_j \alpha_j \right) \right) dc_1 \cdots dc_{k-1} dc_0,
\]

where \( c_k \) satisfies the relationship \( c_k = 1 - \sum_{j=0}^{k-1} c_j \). Hence it follows that

\[
(f_1 * f_2 * \cdots * f_{k+1})(y) = y^k \int_{\Delta_k} \exp \left( y \sum_{j=1}^{k+1} \lambda_j \alpha_j \right) d\lambda,
\]

where \( \Delta_k = \{(\lambda_1, \lambda_2, \ldots, \lambda_{k+1}) \in [0, 1]^m : \sum_{j=1}^{k+1} \lambda_j = 1 \} \) is the \( k \) dimensional simplex and \( d\lambda \) is the usual Lebesgue measure on the simplex \( \Delta_k \).

Hence using (3.3) we get that the weight function \( w_p(t) \) has the following form

\[
w_p(t) = \frac{1}{t} (\log(1/t))^{m-1} \int_{\Delta_{m-1}} \left( \frac{1}{t} \right)^{\sum_{j=1}^m \lambda_j \alpha_j} d\lambda, \quad t \in (0, 1]. \tag{3.5}
\]

As a corollary we observe that the Hausdorff moment property is preserved when the roots are shifted or rescaled; it could be used to prove Theorem 5.1 below and
simplify some of our proofs. This is also consequence of the known relation between the Hausdorff moment sequence and completely monotone functions on \([0, \infty)\), see the paragraph after Theorem 1.6. Here we find relationship between the weight function for the shifted and rescaled rational function. More precisely,

**Lemma 3.2** Let \(p\) be a polynomial with real coefficient having all its roots in \(\mathbb{H}_-\). Let \(\hat{p}(z) = p(z + c)\) and \(\hat{p}(z) = p(z/d)\), where \(c\) is any real number such that roots of \(\hat{p}\) lies in \(\mathbb{H}_-\) and \(d > 1\). Then it follows that if the sequence \((1/p(n))_{n \in \mathbb{Z}_+}\) is a Hausdorff moment sequence, then both the sequences \((1/\hat{p}(n))_{n \in \mathbb{Z}_+}\) and \((1/\hat{p}(n))_{n \in \mathbb{Z}_+}\) is also Hausdorff moment sequence with weight functions given by

\[
\begin{align*}
    w_{\hat{p}}(t) &= \frac{1}{t^n} w_p(t), \\
    \hat{w}(t) &= \frac{1}{b} \left( \log(1/t) \right)^{m-1} (\log(b/t))^{1-m} w_p(t/b), \quad t \in (0, 1].
\end{align*}
\]

### 4 A Necessary Condition

In this section we start with the proof of Theorem 1.8 which gives us a necessary condition on the zeros of \(p\) so that \((q(n)/p(n))_{n \in \mathbb{Z}_+}\) is a Hausdorff moment sequence.

**Proof of Theorem 1.8** We have \(x_1 > r_i\) for \(i = 1, 2, \ldots, s\) and \(x_1 > x_i\) for \(i = 2, 3, \ldots, d\). From (2.6) we obtained that \(q(n)/p(n) = \int_0^1 t^n w_{q,p}(t) dt\), where the weight function \(w_{q,p}(t)\) is given by

\[
\begin{align*}
    w_{q,p}(t) &= \sum_{i=1}^{s} \left( \sum_{j=1}^{l_i} \frac{A_{i,j}(\log(1/t))^{j-1}}{(j-1)!} \right) t^{-r_i} \left( 1 - t^{x_i-1} \right) + \sum_{i=1}^{d} \left( \sum_{j=1}^{m_i} \frac{|A_{i,j}|(\log(1/t))^{j-1}2 \cos(\theta_{i,j} - y_i \log t)}{(j-1)!} \right) t^{-x_i}.
\end{align*}
\]

The sequence \(q(n)/p(n)\) is a Hausdorff moment sequence if and only if \(w_{q,p}(t) \geq 0\) for all \(t \in (0, 1]\). It is straightforward to see that \(t^{x_1+1} w_{q,p}(t)\) will take the following form

\[
\begin{align*}
    t^{x_1+1} w_{q,p}(t) &= \sum_{i=1}^{s} \left( \sum_{j=1}^{l_i} \frac{A_{i,j}(\log(1/t))^{j-1}}{(j-1)!} \right) t^{x_1-r_i} \left( 1 - t^{x_1-1} \right) + \sum_{i=1}^{d} \left( \sum_{j=1}^{m_i} \frac{|A_{i,j}|(\log(1/t))^{j-1}2 \cos(\theta_{i,j} - y_i \log t)}{(j-1)!} \right) t^{x_1-x_i}.
\end{align*}
\]

Our aim is to show that the function \(t^{x_1+1} w_{q,p}(t)\) is strictly negative for some \(t_0\) in \((0, 1]\). For the sake of simplicity we write \(t^{x_1+1} w_{q,p}(t) = f_1(t) + f_2(t), \quad t \in (0, 1],\)
where $f_1$ and $f_2$ is given by

$$
f_1(t) = \sum_i \sum_j l_i \frac{A_{i,j} (\log(1/t))^{i-1}}{(j-1)!} t^{x_1 - r_i},
$$

$$
+ \sum_i \sum_j m_j |A_{s+i,j}| (\log(1/t))^{j-1} 2 \cos(\theta_i - y_i \log t) \frac{t^{x_1 - x_i}}{(j-1)!},
$$

$$
f_2(t) = \sum_j |A_{s+1,j}| (\log(1/t))^{j-1} 2 \cos(\theta_1 - y_1 \log t) \frac{t^{x_1 - x_i}}{(j-1)!}.
$$

For any positive number $c > 0$ and $j \in \mathbb{N}$, we have $t^c (\log(1/t))^j \to 0$ as $t \to 0$. It follows that $f_1(t) \to 0$ as $t \to 0$. Consider the sequence $t_k = e^{\theta_1,m_i - (2k+1)\pi}$, $k \in \mathbb{N}$, so that $\cos(\theta_1,m_1 - y_1 \log t_k) = -1$. Note that $t_k \to 0$ as $k \to \infty$ and we have that

$$
f_2(t_k) = \left( \sum_j |A_{s+1,j}| (\log(1/t_k))^{j-1} 2 \cos(\theta_1 - y_1 \log t_k) \frac{t_k^{x_1 - x_i}}{(j-1)!} \right)
$$

$$
- 2 |A_{s+1,m_1}| (\log(1/t_k))^{m_1-1} \frac{t_k^{x_1 - x_i}}{(m_1!)}.
$$

Now two cases arises, namely $m_1 = 1$ and $m_1 > 1$.

**Case 1: $m_1 = 1$.** In this case $f_2(t_k) = -2 |A_{s+1,1}| < 0$. Since $f_1(t_k) \to 0$ as $k \to \infty$, it follows that there exists $k_0 \in \mathbb{N}$ such that $t_k^{x_1+1} w_{q,p}(t_k) = f_1(t_k) + f_2(t_k)$ is strictly negative for all $k \geq k_0$. By continuity $w_{q,p}(t) < 0$ in some interval containing $t_{k_0}$. Thus the measure $w_{q,p}(t) dt$ is not a positive measure, and the sequence $q(n)/p(n)$ is not a Hausdorff moment sequence.

**Case 2: $m_1 > 1$.** Consider the real polynomial $u(x)$ defined by

$$
u(x) = \left( \sum_j |A_{s+1,j}| \frac{t_k^{x_1 - x_i}}{(j-1)!} x^{j-1} \right) - 2 |A_{s+1,m_1}| \frac{t_k^{x_1 - x_i}}{(m_1-1)!} x^{m_1-1}, \ x \in \mathbb{R}.
$$

Since $A_{s+1,m_1} \neq 0$, $u(x)$ $\to -\infty$ as $x \to \infty$. Thus for an arbitrary large $M > 0$, there exists a $x_0 \in \mathbb{R}$ such that $u(x) < -M$ for all $x > x_0$. Note that $f_2(t) < u(\log 1/t)$ for all $t \in (0, 1]$ and we have $\log(1/t_k) \to \infty$ as $k \to \infty$. Consequently there exists $k_0 \in \mathbb{N}$ such that $f_2(t_k) < u(\log 1/t_k) < -M$ for all $k \geq k_0$. Since $f_1(t_k) \to 0$ as $k \to \infty$, it follows that there exist $N \in \mathbb{N}$ such that $t_k^{x_1+1} w_{q,p}(t_k) = f_1(t_k) + f_2(t_k)$ is strictly negative for $k \geq N$. Hence $w_{q,p}(t) < 0$ in some interval containing $t_N$. Thus the sequence $q(n)/p(n)$ is not a Hausdorff moment sequence.

We use this result to answer the question of G. Misra. We show that the Schur product $SK(z, w)$ of a subnormal kernel $K(z, w)$ with the Szego kernel $S(z, w) = (1 - z\bar{w})^{-1}$ on the unit disc need not necessarily be a subnormal kernel.
Let us consider the following family of kernel functions:

\[ K_c(z, w) = \sum_{n=0}^{\infty} (n + c)^6(z \bar{w})^n, \quad c > 0, \ z, w \in \mathbb{D}. \quad (4.1) \]

It is straightforward to see that for every \( c > 0 \), the sequence \( \{(n + c)^{-6}\}_{n \in \mathbb{Z}_+} \) is a Hausdorff moment sequence and associated with the measure \( \frac{1}{2\pi} t^{-1} (\log(1/t))^5 dt \) on the unit interval \((0, 1]\). Hence the kernel function \( K_c \) is a contractive and subnormal kernel. The Schur product of \( K_c \) with the Szegő kernel \( S(z, w) = (1 - z \bar{w})^{-1} = \sum_{n=0}^{\infty} (z \bar{w})^n \) on the unit disc \( \mathbb{D} \) is given by

\[ SK_c(z, w) = \sum_{n=0}^{\infty} \left( \sum_{j=0}^{n} (j + c)^6 \right) (z \bar{w})^n, \quad z, w \in \mathbb{D}. \]

Let \( p_c(n) = \sum_{j=0}^{n} (j + c)^6 \). Since the kernel \( SK_c \) is already a contractive kernel, it follows that the kernel function \( SK_c \) is subnormal if and only if the sequence \( \{1/p_c(n)\}_{n \in \mathbb{Z}_+} \) is a Hausdorff moment sequence. Using the known formulas for the power sums \( \sum_{j=0}^{n} j^k \), it is easy to see that for every \( c > 0 \) the term \( p_c(n) \) is a real polynomial in \( n \) of degree 7. For \( c = 1 \) we use Faulhaber’s formula for sums of powers of integers (see [5]) to obtain

\[ p_1(n) = \sum_{j=0}^{n} (j + 1)^6 = \frac{(n + 1)(n + 2)(2n + 3)(3(n + 1)^4 + 6(n + 1)^3 - 3(n + 1) + 1)}{42} \]

a polynomial of degree 7. That is, \( p_1(z) = (z + 1)(z + 2)(2z + 3)q(z) \) has three real roots \(-2, -\frac{3}{2}, -1\), with \( q(z) = \frac{3(z+1)^3 + 6(z+1)^3 - 3(z+1) + 1}{42} \) a polynomial of degree 4. All roots of any polynomial of degree 4 can be found explicitly, in particular we can get all roots of \( q \) and then of \( p_1 \). We perform numerically computations instead and find that the complex roots are approximately given by \(-0.62 \pm 0.16i, -2.38 \pm 0.16i\). As \( \Re(-0.62 \pm 0.16i) = -0.62 > -1 \), it follows from Theorem 1.8 that there exists a \( t_0 \in (0, 1) \) such that the weight function \( w_{p_1}(t_0) < 0 \) and that \( \{1/p_1(n)\}_{n \in \mathbb{Z}_+} \) is not a Hausdorff moment sequence. By continuity in \( c \), it follows that the weight function \( w_{p_c}(t_0) < 0 \) for all \( c \) in some neighborhood of \( c = 1 \). Consequently \( \{1/p_c(n)\}_{n \in \mathbb{Z}_+} \) is not a Hausdorff moment sequence for all \( c \) in some neighborhood of 1. Thus we produce a family of subnormal kernel \( K \) for which \( SK \) is not a subnormal.

The choice of power 6 in the expression of (4.1) is optimal in the following sense. Let us consider the kernel function \( K_{c, j}(z, w) \) defined by

\[ K_{c, j}(z, w) = \sum_{n=0}^{\infty} (n + c)^j(z \bar{w})^n, \quad c > 0, \ j \in \mathbb{N}, \ z, w \in \mathbb{D}. \]
The kernel function $K_{c,j}(z, w)$ is contractive and subnormal for each $c > 0$ and $j \in \mathbb{N}$. In our previous counter example, we find that the kernel function $SK_{1,6}$ is not subnormal. A similar kind of computation can be performed to verify that for each $j$ satisfying $1 \leq j \leq 5$, the product kernel $SK_{1,j}$ is again a contractive and subnormal kernel.

Now we will discuss few special cases of rational function $q(z)/p(z)$. First let us assume that all the roots of $p$ are real and negative. Let $p$ be the polynomial of the form

$$p(x) = \prod_{j=1}^{m} (x - \alpha_j)^{b_j},$$

where $\alpha_j$'s are distinct negative real numbers. We order the roots so that $\alpha_k < \alpha_{k-1}$, for $k = 2, 3, \ldots, m$ and $\alpha_1 < 0$. Let $q$ be any polynomial with real coefficients such that $\deg(q) < \deg(p)$. In this case, using (2.4) we have,

$$w_{q,p}(t) = \sum_{i=1}^{m} \left( \sum_{j=1}^{b_i} A_{i,j} \frac{(\log(1/t))^{j-1}}{(j-1)!} \right) t^{-\alpha_i-1}, \ t \in (0, 1],$$

where $A_{i,j} = \frac{a_i}{(b_j - 1)!} (\alpha_i), \ i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, b_i$.

So, in this case $t^{\alpha_1+1}w_{q,p}(t)$ takes the following form

$$t^{\alpha_1+1}w_{q,p}(t) = \sum_{j=1}^{b_1} \frac{A_{1,j} (\log(1/t))^{j-1}}{(j-1)!} + \sum_{i=2}^{m} \left( \sum_{j=1}^{b_i} \frac{A_{i,j} (\log(1/t))^{j-1}}{(j-1)!} \right) t^{\alpha_1-\alpha_i},$$

where $f_1(t) = f_2(t), \ t \in (0, 1]$,

where $f_1$ and $f_2$ is given by

$$f_1(t) = \sum_{j=1}^{b_1} \frac{A_{1,j} (\log(1/t))^{j-1}}{(j-1)!},$$

$$f_2(t) = \sum_{i=2}^{m} \left( \sum_{j=1}^{b_i} \frac{A_{i,j} (\log(1/t))^{j-1}}{(j-1)!} \right) t^{\alpha_1-\alpha_i}, \ t \in (0, 1].$$

Since $f_1(t)$ is a polynomial in $\log(1/t)$ and $A_{1,b_1} = \frac{q}{p_1}(\alpha_1) \neq 0$, it follows that $f_1(t) \to +\infty$ as $t \to 0$ if $A_{1,b_1} > 0$ and $f_1(t) \to -\infty$ as $t \to 0$ if $A_{1,b_1} < 0$. Also we have $f_2(t) \to 0$ as $t \to 0$. So, if $w_{q,p}(t) \geq 0$ on $(0, 1]$, then necessarily we must have $A_{1,b_1} > 0$, that is $q(\alpha_1) > 0$. Thus we get a necessary condition, in this case, for $(q(n)/p(n))_{n \in \mathbb{Z}^+}$ to be a Hausdorff moment sequence.

**Theorem 4.1** Let $p$ be a polynomial with all negative real roots. Let $\alpha_1$ be the largest among all roots of $p$. Let $q$ be another polynomial with real coefficients such that
deg(q) < deg(p) and have no common roots with p. If \(\{q(n)/p(n)\}_{n \in \mathbb{Z}_+}\) is a Hausdorff moment sequence, then necessarily we have \(q(\alpha_1) > 0\). This condition is also sufficient if \(q\) is a monic polynomial of degree 1.

The sufficient part follows from the following observation. Let \(q\) be the polynomial \(q(x) = (x - \beta)\) for some \(\beta \in \mathbb{R}\). The necessity condition \(q(\alpha_1) > 0\) gives us \(\beta < \alpha_1\). In that case

\[
\frac{n - \beta}{n - \alpha_1} = 1 + \frac{\alpha_1 - \beta}{n - \alpha}, \quad n \in \mathbb{Z}_+.
\]

This gives us that the sequence \(\{\frac{n - \beta}{n - \alpha_1}\}_{n \in \mathbb{Z}_+}\) is a Hausdorff moment sequence. Now it is straightforward to see that the sufficiency part of the Theorem 4.1 follows using the Lemma 1.9.

We have obtained an expression for the weight function \(w_{q, p}\) in terms of finite divided differences in (3.2). This expression can be used to provide a sufficient condition on \(q\) so that the sequence \(\{q(n)/p(n)\}_{n \in \mathbb{Z}_+}\) becomes a Hausdorff moment sequence.

**Proof of Proposition 1.7** Using (3.2), in this case, we have

\[
w_{q, p}(t) = F_t[\alpha_1, \alpha_2, \ldots, \alpha_m], \quad t \in (0, 1],
\]

where \(F_t(z) = q(z)t^{-z-1}\), for \(\Re(z) < 0\). Now we apply Leibniz’s rule for finite divided difference of product of two functions to obtain

\[
w_{q, p}(t) = \sum_{j=1}^{m} (q(z)[\alpha_1, \alpha_2, \ldots, \alpha_j])(t^{-z-1}[\alpha_j, \alpha_{j+1}, \ldots, \alpha_m]), \quad t \in (0, 1].
\]

Since \(\alpha_j\)’s are all real, applying mean value theorem for finite divided differences, we obtain that there exists a \(\zeta_j\) in \(\mathbb{R}\), determined by \(\{\alpha_j, \ldots, \alpha_m\}\) so that

\[
t^{-z-1}[\alpha_j, \alpha_{j+1}, \ldots, \alpha_m] = \frac{1}{(m-j)!} \frac{\partial^{m-j}}{\partial z^{m-j}} t^{-z-1}(\zeta_j),
\]

\[
= \frac{1}{(m-j)!} t^{\zeta_-1}(\log(1/t))^{m-j} \geq 0, \quad t \in (0, 1].
\]

Thus \(w_{q, p}(t) \geq 0\) for all \(t \in (0, 1]\) if \(q[\alpha_1, \alpha_2, \ldots, \alpha_j] \geq 0\) for \(j = 1, 2, \ldots, m\).

Let \(r\) be a rational function of the form \(q(x)/p(x)\), where \(\deg(q) = \deg(p)\) and assume that all the roots of \(q, p\) are distinct negative real number. In [4, Theorem 1], Ball described a sufficient condition on the zeros and poles of \(r\) so that \(r\) is a completely monotone function on \([0, \infty)\). In particular these conditions ensure that \(\{r(n)\}_{n \in \mathbb{Z}_+}\) is a Hausdorff moment sequence. Using Proposition 1.7, we obtain an alternative proof of the result of Ball in the case when \(\deg(q) = \deg(p)\) is at most 3. Let \(r\) be a rational function given by
We will show that if \( p_1 < b_1 \) and \( p_1 + p_2 < b_1 + b_2 \) then \((r(n))_{n \in \mathbb{Z}_+} \) is a Hausdorff moment sequence.

Let \( q, p \) be the polynomial given by \( q(x) = (x + b_1)(x + b_2) \) and \( p(x) = (x + p_1)(x + p_2) \). From our assumption we get that the divided differences of \((q - p)\)

\[
(q - p)\left[-p_1\right] = (b_1 - p_1)(b_2 - p_1) > 0,
\]

\[
(q - p)\left[-p_1, -p_2\right] = (b_1 + b_2) - (p_1 + p_2) > 0.
\]

Using Proposition 1.7, we get that \( r(n) - 1 = (q - p)(n)/p(n) \) is a Hausdorff moment sequence and consequently \((r(n))_{n \in \mathbb{Z}_+} \) is a Hausdorff moment sequence. In a similar manner, for a rational function \( r(x) \) of the form

\[
r(x) = \frac{(x + b_1)(x + b_2)(x + b_3)}{(x + p_1)(x + p_2)(x + p_3)}, \quad 0 < b_1 < b_2 < b_3, \quad 0 < p_1 < p_2 < p_3,
\]

it can be shown that if \( p_1 < b_1, \ p_1 + p_2 < b_1 + b_2, \ p_1 + p_2 + p_3 < b_1 + b_2 + b_3 \), then the divided differences \((q - p)[−p_1], (q - p)[−p_1, −p_2]\) and \((q - p)[−p_1, −p_2, −p_3]\) are strictly positive, where

\[
q(x) = \prod_{j=1}^{3} (x + b_j), \quad p(x) = \prod_{j=1}^{3} (x + p_j).
\]

Consequently, we will have \((r(n))_{n \in \mathbb{Z}_+} \) is a Hausdorff moment sequence. It might be interesting to find necessary and sufficient conditions for this class of rational functions \( r(x) \) so that \((r(n))_{n \in \mathbb{Z}_+} \) is a Hausdorff moment sequence.

5 Sequences Induced by Polynomials

In this section first we describe all polynomials \( p \) with real coefficients up to degree 4 and having roots in \( \mathbb{H}_- \) so that the sequence \( \{1/p(n)\}_{n \in \mathbb{Z}_+} \) is a Hausdorff moment sequence. As we have seen that if \( p \) is reducible over \( \mathbb{R} \), then for such \( p \) the sequence \( \{1/p(n)\}_{n \in \mathbb{Z}_+} \) is always a Hausdorff moment sequence, see Lemma 1.9. So, now on we consider only polynomials \( p \) which are not reducible over \( \mathbb{R} \), that is, \( p \) has at least one pair of non real roots.

The necessary condition in Theorem 1.8 gives us the answer in the case of degree 2 polynomial \( p \).

Let \( p \) be a real polynomial of degree 3 with a non real root, say \( \alpha \). Assume that \( p \) is of the form

\[
p(z) = (z - r)(z - \alpha)(z - \bar{\alpha}),
\]

where \( r, \Re(\alpha) < 0 \). In the case of \( r = -1 \), Theorem 1.3 shows that for such polynomial \( p \) the sequence \( \{1/p(n)\}_{n \in \mathbb{Z}_+} \) is a Hausdorff moment sequence if and only if \( \Re(\alpha) \leq 1 \).
Their proof can be modified to get the answer for an arbitrary \( r < 0 \); it can also be proved using our Lemma 3.2 and Theorem 1.8. We state this as following:

**Theorem 5.1** (Anand and Chavan) For \( p(z) = (z - r)(z - \alpha)(z - \bar{\alpha}) \) with \( r, \Re(\alpha) < 0, \Im(\alpha) \neq 0 \), the sequence \( \{1/p(n)\}_{n \in \mathbb{Z}^+} \) is a Hausdorff moment sequence if and only if \( \Re(\alpha) \leq r \).

To get a complete classification for all real polynomial \( p \) of degree 4 for which \( \{1/p(n)\} \) is a Hausdorff moment sequence, we need to consider different cases separately.

**Theorem 5.2** Let \( p \) be a real polynomial of the form \( p(z) = \prod_{j=1}^{2}(z - (r + iy_j)) \prod_{j=1}^{2}(z - (r - iy_j)) \), where \( r < 0 \) and \( 0 < y_1 < y_2 \). Then the sequence \( \{1/p(n)\}_{n \in \mathbb{Z}^+} \) is never a Hausdorff moment sequence.

**Proof** In this case the weight function \( w_p(t) \) is given by

\[
w_p(t) = t^{-r-1} \left( \frac{\sin(y_1 \log t)}{y_1(y_1^2 - y_2^2)} + \frac{\sin(y_2 \log t)}{y_2(y_2^2 - y_1^2)} \right)
= \frac{t^{-r-1}}{y_2(y_2^2 - y_1^2)} \left( \frac{y_2}{y_1} \sin(y_1 \log(1/t)) - \sin \left( \frac{y_2}{y_1} y_1 \log(1/t) \right) \right), \quad t \in (0, 1).
\]

Since \( \frac{y_2}{y_1} > 1 \), we get that \( w_p(t) < 0 \) whenever \( y_1 \log(1/t) = \frac{3\pi}{2} \). Hence \( \{1/p(n)\} \) is not a Hausdorff moment sequence. \( \square \)

Thus in view of Theorems 1.9, 1.8, 5.1 and 5.2, we obtain the following classification of all all real polynomials \( p \) of degree 4 for which \( \{1/p(n)\} \) is a Hausdorff moment sequence.

**Theorem 5.3** Let \( p \) be a real polynomial of degree 4 having a non real root, say \( \alpha \). Assume all the roots of \( p \) lies in \( \mathbb{H}_- \). Then the sequence \( \{1/p(n)\}_{n \in \mathbb{Z}^+} \) is a Hausdorff moment sequence if and only if there exists a real root, say \( r \), of \( p \) such that \( \Re(\alpha) \leq r \).

Although we could not classify all real polynomial \( p \) of degree 5 but we have a complete classification result for a special class of degree 5 polynomials, namely polynomials whose roots lies in a vertical line.

**Proof of Theorem 1.10** We will divide the proof in the following two cases.

**Case 1**: \( y_1 = y_2 \); In this case we have \( p(z) = (z - r)(z - (r + iy_1))^2(z - (r - iy_1))^2 \). Using partial decomposition formula of \( 1/p(z) \) from Proposition 2.1 we obtain

\[
\frac{1}{p(z)} = \frac{A_1}{z - r} + \frac{A_2}{z - (r + iy_1)} + \frac{A_3}{z - (r - iy_1)}
+ \frac{A_2}{(z - (r + iy_1))^2} + \frac{A_3}{(z - (r - iy_1))^2},
\]

where \( A_1 = \frac{1}{y_1^2}, A_2 = -\frac{1}{2y_1^2}, A_3 = \frac{i}{4y_1} \).
Thus from (2.6) we get that $\frac{1}{p(n)} = \int_0^1 t^n w_p(t) dt$, where the weight function $w_p(t)$ is given by
\[
w_p(t) = \frac{1}{y_1^4} t^{-r-1} - \frac{2 \cos(y_1 \log t)}{2y_1^4} t^{-r-1} + \frac{\log(\frac{1}{t}) 2 \cos\left(\frac{\pi}{2} - y_1 \log t\right)}{4y_1^3} t^{-r-1},
\]
\[
eq \frac{t^{-r-1}}{y_1^4} \left(1 - \cos(y_1 \log t) - \frac{y_1 \log t}{2} \sin(y_1 \log t)\right).
\]

Consider the sequence $t_k = \exp(-\frac{(4k+1)\pi}{2y_1})$, $k \in \mathbb{N}$. Note that $w_p(t_k) = \frac{t_k^{-r-1}}{y_1^4} (1 - \frac{(4k+1)\pi}{4}) < 0$ for all $k \in \mathbb{N}$. Hence the sequence $\frac{1}{p(n)}$ is not a Hausdorff moment sequence.

**Case 2:** $y_1 < y_2$: In this case, using partial decomposition formula of $1/p(z)$ from Proposition 2.1 we obtain that
\[
\frac{1}{p(z)} = \frac{A_1}{z-r} + \frac{A_2}{z-(r+iy_1)} + \frac{\overline{A_2}}{z-(r-iy_1)} + \frac{A_3}{z-(r+iy_2)} + \frac{\overline{A_3}}{z-(r-iy_2)},
\]
where $A_1 = \frac{1}{y_1^4 y_2^2}$, $A_2 = \frac{1}{2y_1^2 (y_1^2 - y_2^2)}$, $A_3 = \frac{1}{2y_2^2 (y_2^2 - y_1^2)}$.

Again from (2.6) we get that $\frac{1}{p(n)} = \int_0^1 t^n w_p(t) dt$, where the weight function $w_p(t)$ is given by
\[
w_p(t) = \frac{1}{y_1^4 y_2^2} t^{-r-1} + \frac{2 \cos(y_1 \log t)}{2y_1^2 (y_1^2 - y_2^2)} t^{-r-1} + \frac{2 \cos(y_2 \log t)}{2y_2^2 (y_2^2 - y_1^2)} t^{-r-1},
\]
\[
eq \frac{t^{-r-1}}{y_1^4 y_2^2 (y_2^2 - y_1^2)} \left((y_2^2 - y_1^2) - y_2^2 \cos(y_1 \log t) + y_1^2 \cos(y_2 \log t)\right),
\]
\[
= \frac{t^{-r-1}}{y_1^4 y_2^2 (y_2^2 - y_1^2)} \left(\frac{y_2^2}{y_1^2} - 1 - \frac{y_2^2}{y_1^2} \cos(y_1 \log t) + \cos\left(\frac{y_2}{y_1} y_1 \log t\right)\right).
\]

Let $u = \frac{y_2}{y_1}$ so that $u > 1$. Our aim is to show that $w_p(t) \geq 0$ for all $t \in (0, 1]$ if and only $u$ is an integer greater than 1. We apply a change of variable $x = y_1 \log(1/t)$. As $t \in (0, 1]$, we have $x \in [0, \infty)$. Let $g$ be the function on $[0, \infty)$ defined by
\[
g(x) = g_u(x) = u^2 - 1 - u^2 \cos(x) + \cos(ux), \ x \in [0, \infty).
\]

Then $w_p(t) = \frac{t^{-r-1}}{y_1^4 y_2^2 (y_2^2 - y_1^2)} g(y_1 \log(1/t))$ for all $t \in (0, 1]$. It is now sufficient to show that the function $g(x) \geq 0$ for all $x \in [0, \infty)$ if and only if $u$ is an integer.

First observe that $g(2\pi) = \cos(2u\pi) - 1$. Hence $g(2\pi) < 0$ if $u$ is not an integer.
Now let us assume $u$ is a positive integer. We will show in this case $g(x) \geq 0$ for all $x \in [0, \infty)$. We have

$$g'(x) = u(u \sin(x) - \sin(ux)),$$
$$g''(x) = u^2(\cos(x) - \cos(ux)).$$

As $u$ is a positive integer, $g$ is a periodic function with period $2\pi$. We have $g(0) = g(2\pi) = 0$. So, it is enough to show that $g(x) \geq 0$ for all $x \in [0, 2\pi]$.

Since $\cos(x)$ is decreasing function on $[0, \pi]$, we obtain that $\cos(x) > \cos(ux)$ for $x \in [0, \frac{\pi}{u}]$. Thus $g''(x) > 0$ for $x \in [0, \frac{\pi}{u}]$. Consequently, $g'(x)$ is strictly increasing on $[0, \frac{\pi}{u}]$. Also $g'(0) = 0$. Thus we obtain that $g'(x) > 0$ for $x \in [0, \frac{\pi}{u}]$. As $\sin(x)$ is increasing on $[0, \frac{\pi}{2}]$, we get that $u \sin(x) - \sin(ux) \geq u(\frac{\pi}{2u}) - 1 = g'\left(\frac{\pi}{2u}\right) > 0$ for all $x \in [\frac{\pi}{2u}, \frac{\pi}{2}]$. Thus

$$g'(x) > 0, \quad x \in \left[0, \frac{\pi}{2}\right],$$

and consequently $g(x)$ is strictly increasing function on $[0, \frac{\pi}{2}]$. So, $g(x) > g(0) = 0$ for $x \in (0, \frac{\pi}{2}]$.

**Sub case 1:** $u = 2m + 1$ is an odd integer, $m \in \mathbb{N}$ : In this case it is straight forward to verify that $g'(\frac{\pi}{2} - x) = g'(\frac{\pi}{2} + x)$ for all $x$. Since we already have $g'(x) > 0$ for $x \in [0, \frac{\pi}{2}]$, it follows that $g'(x) > 0$ for $x \in [0, \pi]$. Thus $g(x)$ is strictly increasing function on $[0, \pi]$. So, $g(x) > g(0) = 0$ for $x \in (0, \pi]$.

As $u$ is an odd integer, it follows that $g'(\pi + x) = -g'(x)$ for all $x$. Since $g'(x) > 0$ for $x \in (0, \pi]$, we get that $g'(x) < 0$ for all $x \in (\pi, 2\pi]$. Thus $g(x)$ is strictly decreasing function on $(\pi, 2\pi]$. So, $g(x) > g(2\pi) = 0$ for all $x \in [\pi, 2\pi]$. Hence $g(x) \geq 0$ for all $x \in (0, 2\pi]$.

**Sub case 2:** $u = 2m$ : is an even integer $2m$ for some $m \in \mathbb{N}$ : From (5.1) we have that $g'(x) > 0$ for $x \in [0, \frac{\pi}{2}]$. Now we will show that $g'(x) > 0$ for $x \in [0, \pi]$. It is sufficient to show that $g'(\pi - x) > 0$ for $x \in (0, \frac{\pi}{2}]$.

As $u$ is an even integer, $g'(\pi - x) = u(u \sin(x) + \sin(ux))$. So, $g'(\pi - x) > 0$ for $x \in (0, \frac{\pi}{2}]$. Since $\sin(x)$ is increasing on $[0, \frac{\pi}{2}]$ and $g'(\frac{\pi}{2u}) = u(u \sin(\frac{\pi}{2u}) - 1) > 0$, we obtain that

$$u \sin(x) + \sin(ux) \geq u \sin(x) - 1 > u \sin(\frac{\pi}{2u}) - 1 = \frac{1}{u}g'(\frac{\pi}{2u}) > 0, \quad x \in \left[\frac{\pi}{2u}, \frac{\pi}{2}\right].$$

Thus $g'(\pi - x) > 0$ for $x \in (0, \frac{\pi}{2}]$. Hence $g'(x) > 0$ for $x \in [0, \pi]$. It follows that $g(x)$ is an increasing function on $[0, \pi]$ and $g(x) > g(0) = 0$ for all $x \in (0, \pi]$.

Now we will show $g'(x) < 0$ for $x \in (\pi, 2\pi]$. Since $u$ is an even integer, it is straightforward to see that $u^{-1}g'(\pi + x) = -(u \sin(x) + \sin(ux)) = -u^{-1}g'(\pi - x)$ for all $x$. As we already have $g'(\pi - x) > 0$ for $x \in (0, \frac{\pi}{2}]$, we get that $g'(x) < 0$ for $x \in (\pi, \frac{3\pi}{2}]$. Also note that $u^{-1}g'(2\pi - x) = -(u \sin(x) - \sin(ux)) = -u^{-1}g'(x)$. Using (5.1) we get that $g'(x) < 0$ for $x \in \left[\frac{3\pi}{2}, 2\pi\right)$. Hence $g'(x) < 0$ for $x \in (\pi, 2\pi)$. So, $g(x)$ is strictly decreasing function on $(\pi, 2\pi]$ and $g(x) > g(2\pi) = 0$ for all $x \in [\pi, 2\pi]$. Thus $g(x) \geq 0$ for all $x \in [0, 2\pi]$. \qed
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References

1. Anand, A., Chavan, S.: Module tensor product of subnormal modules need not be subnormal. J. Funct. Anal. 272(11), 4752–4761 (2017)
2. Anand, A., Chavan, S.: A moment problem and joint $q$-isometry tuples. Complex Anal. Oper. Theory 11(4), 785–810 (2017)
3. Aronszajn, N.: Theory of reproducing kernels. Trans. Am. Math. Soc. 68(3), 337–404 (1950)
4. Ball, K.: Completely monotonic rational functions and Hall’s marriage theorem. J. Comb. Theory, Ser. B 61(1), 118–124 (1994)
5. Beardon, A.F.: Sums of powers of integers. Am. Math. Mon. 103(3), 201–213 (1996)
6. Berg, C., Durán, A.J.: Some transformations of Hausdorff moment sequences and harmonic numbers. Can. J. Math. 57(5), 941–960 (2005)
7. Conway, J.B.: The Theory of Subnormal Operators, vol. 36. American Mathematical Society, Providence (1991)
8. Douglas, R.G., Paulsen, V.I.: Hilbert Modules Over Function Algebras. Longman Sc & Tech, Harlow (1989)
9. Hausdorff, F.: Momentprobleme für ein endliches intervall. Math. Z. 16(1), 220–248 (1923)
10. Paulsen, V.I., Raghupathi, M.: An Introduction to the Theory of Reproducing Kernel Hilbert Spaces, vol. 152. Cambridge University Press, Cambridge (2016)
11. Salinas, N.: Products of Kernel Functions and Module Tensor Products, Topics in Operator Theory, Operator Theory Advances and Applications, vol. 32, pp. 219–241. Birkhäuser, Basel (1988)
12. Spitzbart, A.: A generalization of Hermite’s interpolation formula. Am. Math. Mon. 67, 42–46 (1960)
13. Widder, D.V.: Laplace Transform. Princeton University Press, Princeton (2015)
14. Schmüdgen, K.: The Moment Problem, vol. 9. Springer, Berlin (2017)

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