ON CLASSIFICATION AND CONSTRUCTION OF ALGEBRAIC FROBENIUS MANIFOLDS

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Abstract. We develop the theory of generalized bi-Hamiltonian reduction. Applying this theory to a suitable loop algebra we recover a generalized Drinfeld-Sokolov reduction. This gives a way to construct new examples of algebraic Frobenius manifolds.

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1. INTRODUCTION

This work was intended as an attempt to prove the Dubrovin conjecture [14] (see also [16]).

The conjecture: Massive irreducible algebraic Frobenius manifolds with positive degrees $d_i$ correspond to primitive conjugacy classes in Coxeter groups.

A Frobenius manifold is a manifold $M$ with the structure of Frobenius algebra on the tangent space $T_t$ at any point $t \in M$ with certain compatibility conditions [14]. We say $M$ is massive if $T_t$ is semisimple for generic

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This structure locally correspond to a potential $F(t^1, ..., t^n)$ satisfying the WDVV equations

\begin{equation}
\partial_i \partial_j \partial_k F(t) \eta^{kp} \partial_p \partial_q \partial_r F(t) = \partial_r \partial_j \partial_k F(t) \eta^{kp} \partial_p \partial_q \partial_i F(t)
\end{equation}

where $(\eta^{-1})_{ij} = \partial_n \partial_i \partial_j F(t)$ is a constant matrix. Here we assume that the quasihomogeneity condition takes the form

\begin{equation}
\sum_{i=1}^{n} d_i t_i F(t) = (3 - d) F(t)
\end{equation}

where $d_n = 1$. This condition defines the degrees $d_i$ and the charge $d$ of $M$. If $F(t)$ is an algebraic function we call $M$ an \textbf{algebraic Frobenius manifold}.

A Coxeter group is a finite group of linear transformations acting on an Euclidean space generated by reflections [23]. Irreducible Coxeter groups are classified by a set of reflections, called \textbf{simple reflections}, that generate all the group. The Weyl group of a simple Lie algebra is an irreducible Coxeter group. A \textbf{primitive conjugacy class} in a Coxeter group is a conjugacy class such that writing any representative of the class as a product of reflections, these reflections generate all the group [3]. The set of simple reflections of a Coxeter group defines a conjugacy class called \textbf{Coxeter conjugacy class}. The conjugacy class is called \textbf{regular} if it has a regular eigenvector, i.e. an eigenvector which is not fixed by any element of the group [29].

The Dubrovin conjecture arises from algebraic solutions to equations of isomonodromic deformation of algebraic Frobenius manifolds [14]. The classification of finite orbits of the braid group action on tuple of reflections obtained by Stefanov [30] (see also [26]) means that algebraic Frobenius manifold leads to a primitive conjugacy class in Coxeter groups. Thus it remains the problem of constructing an algebraic Frobenius manifold for any primitive conjugacy class in Coxeter groups.

Hertling [22] proved that any irreducible massive \textbf{polynomial Frobenius manifold} with positive degrees $d_i$ is isomorphic to the Frobenius structure defined by Dubrovin on the orbit spaces of a Coxeter group [13]. The polynomial Frobenius manifold corresponds to Coxeter conjugacy class in the group [14].

Our main idea is to use the theory of infinite dimensional bi-Hamiltonian manifolds to construct all massive algebraic Frobenius manifolds. A \textbf{bi-Hamiltonian manifold} is a manifold endowed with two Poisson tensors $P_1$ and $P_2$ such that $P_\lambda = P_2 + \lambda P_1$ is a Poisson tensor for any constant $\lambda$. The dispersionless limit of a bi-Hamiltonian structure on the loop space $\mathcal{L}(M)$ of a finite dimensional manifold $M$ (if it exists) always gives a bi-Hamiltonian structure of hydrodynamic type:

\begin{equation}
\{ t^i(x), t^j(y) \}_{1,2} = g_{ij}^1(t(x)) \delta '(x - y) + \Gamma_{1,2;k}^{ij}(t(x)) t^k_2 \delta (x - y),
\end{equation}
defined on the loop space $\mathcal{L}(M)$. This in turn gives a flat pencil of metrics $g_{ij}^{1,2}$ on $M$ which under some assumptions corresponds to a Frobenius structure on $M$ [13].

We use the theory of Lie algebras to associate a bi-Hamiltonian structure to a conjugacy class in Weyl groups. Let $\mathfrak{g}$ be a simple Lie algebra. Denote by $W_\mathfrak{g}$ the Weyl group of $\mathfrak{g}$. Let $[w] \subset W_\mathfrak{g}$ be a primitive regular conjugacy class in $W_\mathfrak{g}$. From [11] there exists a nilpotent orbit $O_e \subset \mathfrak{g}$ corresponding to $[w]$ in the sense of [29]. We define a compatible Lie-Poisson brackets on the loop space $\mathcal{L}(\mathfrak{g})$ depending on a nilpotent element $e \in O_e$. Fix a transversal subspace $M$ to $O_e$ at the point $e$. Then we construct a bi-Hamiltonian structure on $\mathcal{L}(M)$ by using a generalized bi-Hamiltonian reduction.

The bi-Hamiltonian structure on $\mathcal{L}(M)$ does not always admit a dispersionless limit. In this case we perform a Dirac reduction [25] on a suitable submanifold $N \subset M$ to obtain a new bi-Hamiltonian structure on the loop space $\mathcal{L}(N)$. The bi-Hamiltonian structure on $\mathcal{L}(N)$ admits a dispersionless limit.

A generalized Drinfeld-Sokolov reduction is another procedure to obtain a bi-Hamiltonian structure on $\mathcal{L}(M)$. We prove the two reductions are equivalent in the sense that both of them satisfy the hypotheses of Marsden-Ratiu theorem and give the same reduced structures. The equivalence was obtained by Pedroni [28] in the special case of the reduction associated with the principal nilpotent element.

The classical $W$-algebras and their primary fields were obtained in [1] by using the generalized Drinfeld-Sokolov reduction. We obtain this result by applying the generalized bi-Hamiltonian reduction. In [5] they construct the classical $W_n$-algebra of the Lie algebra of type $A_n$ by studying the relation between bi-Hamiltonian and Drinfeld-Sokolov reductions [28]. Our method is more straightforward and does not depend on a particular type of the Lie algebra.

The paper is divided into four parts. In the next section we develop the theory of generalized bi-Hamiltonian reduction. The idea goes back to [7] where a bi-Hamiltonian reduction is given for every bi-Hamiltonian manifold, using Marsden-Ratiu theorem, by taking a level surface $S$ of all the Casimirs of the first Poisson brackets and a distribution $D$ defined by the second Poisson bracket.

Section 3.1 contains a brief summary of the theory of nilpotent elements and gradings on a simple Lie algebra and we set up notations and terminology. In section 3.2 we apply the generalized bi-Hamiltonian reduction to Lie-Poisson bracket on a loop algebra of simple Lie algebra. We give a general bi-Hamiltonian reduction for any nilpotent element $e$ with an associated good grading (in the sense of [19]). In section 3.3 we indicate how the bi-Hamiltonian reduction associated with a nilpotent element may be used to obtain the primary fields of classical $W$-algebras.
Section 4 provides a detailed exposition of a generalized Drinfeld-Sokolov reduction which is a special case of the more general Drinfeld-Sokolov reduction scheme given in [20]. In section 4.1 we establish the equivalence between the generalized bi-Hamiltonian and generalized Drinfeld-Sokolov reductions.

Finally section 5 is devoted to our main aim which is constructing new examples of algebraic Frobenius manifolds. We write in section 5.1 the formulas for Dirac reduction of infinite dimensional Poisson bracket on a loop space \( \mathcal{L}(M) \) to \( \mathcal{L}(N) \) where \( N \) is a suitable submanifold of \( M \). In section 5.2 we apply bi-Hamiltonian reduction to the distinguished nilpotent elements in the Lie algebra of type \( F_4 \). The reduced bi-Hamiltonian structures give (after Dirac reduction) four algebraic Frobenius manifolds in agreement with Dubrovin’s conjecture in the sense that the degrees and charge of the algebraic Frobenius manifold can be read from the eigenvalues and order of the corresponding regular conjugacy class.

2. Bi-Hamiltonian Reduction and Transversal Manifolds

We recall the Marsden-Ratiu reduction theorem for Poisson manifolds. For more information one can consult [24].

**Theorem 2.1.** Let \( M \) be a Poisson manifold with Poisson bracket \( \{.,.\}_M \). Let \( S \) be a submanifold of \( M \) with \( i_s : S \to M \) the canonical immersion of \( S \) in \( M \). Assume \( D \) is a distribution on \( M \) satisfying:

1. \( E = D \cap TS \) is an integrable distribution of \( S \).
2. The foliation induced by \( E \) on \( S \) is regular, so that \( N = S/E \) is a manifold and \( \pi : S \to N \) is a submersion.
3. The Poisson bracket of functions of \( F, G \) that are constant along \( D \), is constant along \( D \).
4. \( P(D^0) \subset TS + D \), where \( D^0 \) is the annihilator of \( D \).

Then \( N \) is a Poisson manifold with bracket \( \{.,.\}_N \) given by

\[
\{f, g\}_N \circ \pi = \{F, G\}_M \circ i_s,
\]

where \( F, G \) are functions on \( M \) extending \( f, g \) and constant along \( D \).

The following sections depend on this corollary which replaces the study of the quotient manifold \( N \) with the study of a submanifold in \( S \).

**Corollary 2.2.** Replace the condition (2) by the condition

(2) There exist a transversal submanifold \( Q \) to the distribution \( D \) on \( S \), i.e. at any point \( q \in Q \) we have

\[
T_q S = E_q \oplus T_q Q
\]

Then \( Q \) is a Poisson manifold with bracket \( \{.,.\}_Q \) defined by

\[
\{f, g\}_Q = \{F, G\}_M \circ i_Q
\]

where \( i_Q : Q \hookrightarrow M \) is the canonical immersion and \( F, G \) are functions on \( M \) extending \( f, g \) and constant along \( D \).
A bi-Hamiltonian manifold \( M \) is a manifold endowed with two Poisson tensors \( P_1 \) and \( P_2 \) such that \( P_\lambda = P_2 + \lambda P_1 \) is a Poisson tensor for any constant \( \lambda \). The Jacobi identity for \( P_\lambda \) gives the relation

\[
\{\{F,G\}_1, H\}_2 + \{\{G,H\}_1, F\}_2 + \{\{H,F\}_1, G\}_2 + \{\{F,G\}_2, H\}_1 + \{\{G,H\}_2, F\}_1 + \{\{H,F\}_2, G\}_1 = 0
\]

for any functions \( F, G \) and \( H \) on \( M \). The main implication of this identity is that the set of Casimirs of \( P_1 \) is a Lie algebra with respect to \( P_2 \). Our basic assumption is the following. There is a set

\[
\Xi = \{K_1, K_2, ..., K_n\}
\]

of independent Casimirs of \( P_1 \) (\( n \) is not necessary equal to the corank of \( P_1 \)) closed with respect to \( P_2 \). Let us denote by \( S \) a level set of \( \Xi \) and define the integrable distribution \( D \) on \( M \) generated by the Hamiltonian vector fields

\[
X_{K_i} = P_2(dK_i), \quad i = 1, ..., n.
\]

The following lemma says that \( P_\lambda, S \) and \( D \) verify the hypotheses of Marsden-Ratiu theorem \( 2.1 \) except condition (2).

**Lemma 2.3.** For any constant \( \lambda \)

1. The functions which are constant along \( D \) form a Lie subalgebra with respect to \( P_\lambda \).
2. \( v \in D^0 \) if and only if \( P_\lambda(v) \in TS \). Here \( D^0 \subset T^*M \) is the annihilator of \( D \).

**Proof.** The first condition is easily deduced from the relation \( 2.2 \) and Jacobi identity for \( P_2 \). Since \( P_1(T^*M) \subset TS \) the statement (2) is equivalent to proving that

\[
v \in D^0 \text{ if and only if } P_2(v) \in TS.
\]

To this end, let \( v \in D^0 \). Then

\[
(v, D) = 0 \iff (v, P_2(dK_i)) = 0, \quad i = 1, ..., n \iff (P_2(v), dK_i) = 0, \quad i = 1, ..., n \iff P_2(v) \in TS,
\]

and the proof is complete. \( \square \)

In the remainder of this section we assume there is a submanifold \( Q \subset S \) transversal to \( E = D \cap TS \), i.e.

\[
T_qS = E_q \oplus T_qQ, \quad \text{for all } q \in Q.
\]

Following [7], \( Q \) has a natural bi-Hamiltonian structure \( P_1^Q, P_2^Q \) from \( P_1, P_2 \) respectively (see also corollary \( 2.2 \)). Let \( i : Q \hookrightarrow M \) be the canonical immersion. Then the pencil \( P_\lambda^Q \) is defined, for any functions \( f, g \) on \( Q \), by

\[
\{f, g\}_\lambda^Q = \{F, G\} \circ i
\]

where \( F, G \) are functions on \( M \) extending \( f, g \) and constant along \( D \).
Our next purpose is to find a way to write the reduced Poisson pencil tensor. Here the advantage of having a transversal manifold \( Q \) becomes clear.

**Lemma 2.4.** For any \( q \in Q \) and \( w \in T_q^*Q \) there exists \( v \in T_q^*M \) such that:

1. \( v \) is an extension of \( w \), i.e. \((v, \dot{q}) = (w, \dot{q})\) for any \( \dot{q} \in T_qQ \).
2. \( P_\lambda(v) \in T_qQ \), i.e. \((v, P_\lambda(TQ)^0) = 0\).

Then the Poisson tensor \( P_\lambda^Q(w) \) is given by

\[
P_\lambda^Q w = P_\lambda v
\]

for any extension \( v \) satisfying conditions (1) and (2).

**Proof.** \( w \in T_q^*Q \) has an extension \( v \in T_qM \) satisfying (2) if \( T_qQ \cap P_\lambda(TQ)^0 = 0 \). Assume \( \dot{q} \in T_qQ \cap P_\lambda(TQ)^0 \). Then \( \dot{q} = P_\lambda(r) \) for \( r \in (TQ)^0 \). Since \( P_\lambda(r) \in T_qS \), by lemma 2.3 we have that \( r \in D^0 \). Then

\[
r \in (TQ)^0 \cap D^0 \subset (TQ + D)^0 \subset (TS)^0.
\]

This implies \( P_\lambda(r) \in D \) which gives \( \dot{q} \in E \). But \( \dot{q} \in T_qQ \). This shows that \( \dot{q} = 0 \) and proves the first part. Let \( v_1, v_2 \in T_q^*M \) be extensions of \( w_1, w_2 \in T_q^*Q \) satisfying the condition (2). Then

\[
(w_1, P_\lambda^Q w_2) = (v_1, P_\lambda v_2)
\]

where the first equality is obtained by definition and the second one follows from condition (2). \( \square \)

3. **Examples for Lie-Poisson brackets**

3.1. **Nilpotent elements and gradings in Lie algebras.** Here we introduce some notations and basic facts from the theory of nilpotent elements in simple Lie algebras.

Let \( \mathfrak{g} \) be a simple Lie algebra over complex numbers with a nondegenerate invariant bilinear form \( \langle ., . \rangle \). For a vector subspace \( V \subset \mathfrak{g} \) we denote by \( V^\perp \) its orthogonal complement and by \( \mathcal{L}(V) \) its loop space, i.e. the space of smooth maps from the circle \( S^1 \) to \( V \).

Introduce the following bilinear form on the loop algebra \( \mathcal{L}(\mathfrak{g}) \):

\[
(u|v) = \int_{S^1} \langle u(x)|v(x) \rangle dx, \quad u, v \in \mathcal{L}(M).
\]

Then identify \( (\mathcal{L}(\mathfrak{g}))^* \) with \( \mathcal{L}(\mathfrak{g}) \) using this bilinear form. For a functional \( F \) on \( \mathcal{L}(\mathfrak{g}) \) we define the gradient \( \delta H(q) \) to be the unique element in \( \mathcal{L}(\mathfrak{g}) \) such that

\[
\frac{d}{d\theta} F(q + \theta s) \big|_{\theta = 0} = \int_{S^1} \langle \delta F|s \rangle dx \text{ for all } s \in \mathcal{L}(\mathfrak{g}).
\]
We introduce the following Poisson tensors
\begin{align}
P_2(v) &= v_x + [q, v] \\
P_1(v) &= [a, v]
\end{align}
given at a point \( q \in \mathfrak{L}(\mathfrak{g}) \) and for every \( v \in (\mathfrak{L}(\mathfrak{g}))^* \), here \( a \in \mathfrak{g} \) is constant element. It is well known that the pair in (3.3) defines a bi-Hamiltonian structure on \( \mathfrak{L}(\mathfrak{g}) \) [25]. Our first examples of bi-Hamiltonian reduction will be constructed from (3.3) by choosing an appropriate element \( a \in \mathfrak{g} \) and a set of Casimirs of \( P_1 \).

Let
\begin{equation}
\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i
\end{equation}
be a \( \mathbb{Z} \)-grading on \( \mathfrak{g} \), i.e. \([\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}\) (We will omit the letter \( \mathbb{Z} \) since any grading considered here is a \( \mathbb{Z} \)-grading). Since all derivations of \( \mathfrak{g} \) are inner this grading is defined as eigenspaces of \( \text{ad} \tilde{h} \) for some element \( \tilde{h} \).

Let
\begin{equation}
\mathfrak{g}_i = \{ a \in \mathfrak{g} | \text{ad} \tilde{h}(a) = ia \};
\end{equation}
Hence \( \tilde{h} \) is semisimple.

An element \( e \in \mathfrak{g}_2 \) is called good if it satisfies the following condition;
\begin{equation}
\text{ad } e : \mathfrak{g}_j \to \mathfrak{g}_{j+2} \text{ is injective for } j \leq -1.
\end{equation}
A grading is called good if it admits a good element. All good gradings on simple Lie algebras up to conjugation are classified in [19].

**Notation 3.1.** For any good grading we introduce the following subalgebras;
\( \mathfrak{b}^- = \bigoplus_{i \leq 0} \mathfrak{g}_i, \mathfrak{b}^+ = \bigoplus_{i \geq 0} \mathfrak{g}_i, \mathfrak{n}^- = \bigoplus_{i \leq -1} \mathfrak{g}_i, \mathfrak{g}^- = \bigoplus_{i \leq -2} \mathfrak{g}_i \) and \( \mathfrak{n}^+ = \bigoplus_{i \geq 1} \mathfrak{g}_i \).

A subspace \( \mathfrak{C} \subset \mathfrak{g} \) is called a transversal subspace of \( e \) if
\begin{equation}
\text{ad } e (\mathfrak{g}^-) \oplus \mathfrak{C} = \mathfrak{b}^-.
\end{equation}

Let \( e \in \mathfrak{g} \) be an arbitrary nilpotent element. By Jacobson-Morozov theorem there exist \( h \) and \( f \in \mathfrak{g} \) such that \( \{ e, h, f \} \) is a \( sl_2 \)-triple, that is,
\begin{align}
[h, e] &= 2e, \quad [h, f] = -2f, \quad [e, f] = h.
\end{align}
From representation theory of \( sl_2 \) it is easy to see that \( h \) defines a grading on \( \mathfrak{g} \) with \( e \) a good element. We call the grading thus obtained the Dynkin grading. A nilpotent orbit is the conjugacy class of a nilpotent element under the action of the adjoint group. It turns out that two nilpotent elements are conjugate if and only if they have the same Dynkin grading. See [9] for more information and the classification tables of the nilpotent orbits which are given in the form of weighted Dynkin diagrams. A nilpotent element is called distinguished iff \( \dim(\mathfrak{g}_0) = \dim(\mathfrak{g}_2) \) in the Dynkin grading associated to \( e \). It follows then that \( \mathfrak{g}_1 = 0 \) [19].

**Example 3.2.** The principal nilpotent orbit in \( \mathfrak{g} \) is the unique nilpotent orbit of codimension \( r(=\text{rank } \mathfrak{g}) \). Any representative \( e \) of this nilpotent orbit is regular, i.e. the centralizer of \( e \) in \( \mathfrak{g} \) is abelian and of dimension \( r \).
The Dynkin grading is the only good grading associated to \(e\). The principal nilpotent orbit is a distinguished nilpotent orbit.

Throughout the paper we assume all gradings are good with a fixed good element denoted by \(e\).

### 3.2. Bi-Hamiltonian reduction for a nilpotent element.

Take on \(\mathcal{L}(g)\) the Poisson pencil \([3.3]\) with \(a \in C\) a homogenous element of minimal degree. Let \(\Xi\) be the subset of the set of Casimirs of \(P_1\) corresponding to \(\mathcal{L}(\mathfrak{n}^-) \subset \text{Ker} P_1\). Since \(\mathfrak{n}^-\) is a Lie subalgebra, it is easy to verify that \(\Xi\) is closed under \(P_2\). Following Drinfeld and Sokolov \([12]\) we take as a level surface the affine space

\[
S := \mathcal{L}(\mathfrak{b}^-) + e.
\]

The following proposition gives a nice Lie algebra theoretic meaning to the distribution \(E\) on \(S\) which is defined by

\[
E := P_2(\mathcal{L}(\mathfrak{n}^-)) \cap \mathcal{L}(\mathfrak{b}^-).
\]

**Proposition 3.3.**

\[
E = P_2(\mathcal{L}(\mathfrak{g}^-)).
\]

**Proof.** From

\[
E = P_2(\{v \in \mathcal{L}(\mathfrak{n}^-) : v_x + [q, v] + [e, v] \in \mathcal{L}(\mathfrak{b}^-) \text{ for } q \in \mathcal{L}(\mathfrak{b}^-)\})
\]

and the gradation \([3.4]\), it is obvious that \(v \in E\) if and only if \([e, v] \in \mathcal{L}(\mathfrak{b}^-)\).

Since \(v \in \mathcal{L}(\mathfrak{n}^-)\) and \(\text{ad } e\) is injective we have \(v \in \mathcal{L}(\mathfrak{g}^-)\).

Fix a transversal space \(\mathcal{C}\) and define the submanifold

\[
Q := e + \mathcal{L}(\mathcal{C})
\]

of \(S\).

**Lemma 3.4.** The manifold \(Q\) is transversal to \(E\) on \(S\).

**Proof.** We must prove that at any point \(q \in \mathcal{L}(\mathcal{C})\) and \(\dot{s} \in \mathcal{L}(\mathfrak{b}^-)\) there are \(v \in \mathcal{L}(\mathfrak{g}^-)\) and \(\dot{w} \in \mathcal{L}(\mathcal{C})\) such that

\[
\dot{s} = P_2(v) + \dot{w}.
\]

We write this equation using the gradation \([3.4]\) of \(g\). We obtain

\[
\dot{s}_i = v'_i + [e, v_{i-2}] + \dot{w}_i + \sum_k [q_k, v_{i-k}].
\]

Then for \(i = 0\) we have

\[
\dot{s}_0 = [e, v_{-2}] + \dot{w}_0
\]

which can be solved uniquely since

\[
\mathcal{L}(\mathcal{C}) \oplus [e, \mathcal{L}(\mathfrak{g}^-)] = \mathcal{L}(\mathfrak{b}^-).
\]

Inductively in this way for \(i < 0\) we obtain a recursive relation to determine \(v\) and \(\dot{s}\) uniquely. \(\square\)
Let us explain the procedure of finding the reduced Poisson pencil following [8]. We first choose a basis $\xi_1, \ldots, \xi_n$ for $\mathfrak{g}$ with $\xi_1, \ldots, \xi_m$ a basis for $\mathcal{C}$ for $m < n$. Let $\xi^*_1, \ldots, \xi^*_n \in \mathfrak{g}$ be a dual basis satisfying $\langle \xi_i | \xi^*_j \rangle = \delta_{ij}$. Then a point in the space $Q$ will have the form $q = q^i \xi_i + e$. For a covector $w = (w_1, \ldots, w_m) \in T^*_Q Q$ a lift $v \in T^*_q \mathcal{L}(\mathfrak{g})$ satisfies the first condition in lemma 2.4 if and only if

$$\langle \xi_i | v \rangle = w_i, \quad i = 1, \ldots, m. \quad (3.18)$$

From lemma 2.3 the second condition gives the constraint

$$P_\lambda(v) \in \mathcal{L}(\mathcal{C}). \quad (3.19)$$

Using the grading we can prove this lift is unique. Then the Poisson pencil $P_\lambda^Q$ is given by

$$\dot{q}^i := \langle P_\lambda(v)|\xi^*_i \rangle. \quad (3.20)$$

Its independence from the choice of a basis follows from lemma 2.4.

**Example 3.5. (Fractional KdV)** Consider $\mathfrak{g} = sl_3$ with its standard representation. We denote by $e_{i,j}$ the fundamental matrix defined by $(e_{i,j})_{s,t} = \delta_{i,s} \delta_{j,t}$. Take the minimal nilpotent element $e := e_{1,3}$. It is a good element for the grading (non Dynkin grading) defined by

$$\tilde{h} := \frac{4}{3} e_{1,1} - \frac{2}{3} e_{2,2} - \frac{2}{3} e_{3,3} \quad (3.21)$$

Take the Poisson tensors (3.3) with $a = e_{2,1}$. Here $\mathfrak{n}^-$ is generated by $\{e_{2,1}, e_{3,1}\}$ and a point $b \in S$ will have the form

$$b = \begin{pmatrix} * & 0 & 1 \\ * & * & * \\ * & * & * \end{pmatrix}. \quad (3.22)$$

We define a transversal space $\mathcal{C}$ such that a point $q \in Q$ takes the form

$$q = \begin{pmatrix} (\alpha - \beta) q_1 & 0 & 1 \\ q_2 & -\alpha q_1 & 0 \\ q_4 & q_3 & \beta q_1 \end{pmatrix}. \quad (3.23)$$
for arbitrary $q_1, \ldots, q_4$ and nonzero constants $\alpha, \beta$. Then the reduced Poisson pencil $P^Q_\lambda$ has the following form

$$\{q_1(x), q_1(y)\}_\lambda = \frac{2 \delta'(x - y)}{3 \alpha^2}$$

$$\{q_1(x), q_2(y)\}_\lambda = -\frac{(\lambda + q_2(x)) \delta(x - y)}{\alpha}$$

$$\{q_1(x), q_3(y)\}_\lambda = q_3(x) \delta(x - y)$$

$$\{q_1(x), q_4(y)\}_\lambda = -\frac{(\alpha - 2 \beta)^2 q_1' \delta(x - y)}{3 \alpha^2} - \frac{(\alpha - 2 \beta)^2 q_1(x) \delta'(x - y)}{3 \alpha^2} - \frac{(\alpha - 2 \beta) \delta''(x - y)}{\alpha}$$

$$\{q_2(x), q_3(y)\}_\lambda = (2 \alpha^2 + \alpha \beta - \beta^2) q_1^2(x) - q_4(x) - (\alpha + \beta) \alpha q_1' \delta(x - y) - 3 \alpha q_1(x) \delta'(x - y) + \delta''(x - y)$$

$$\{q_2(x), q_4(y)\}_\lambda = -\frac{(2 (\alpha^2 - \alpha \beta + \beta^2) q_1(x) (\lambda + q_2(x)) - \alpha q_4') \delta(x - y)}{\alpha} + \frac{(\alpha + \beta) (\lambda + q_2(x)) \delta'(x - y)}{\alpha}$$

$$\{q_3(x), q_4(y)\}_\lambda = \frac{2 (\alpha^2 - \alpha \beta + \beta^2) q_1(x) q_3(x) + \alpha q_3' \delta(x - y)}{\alpha}$$

$$\{q_4(x), q_4(y)\}_\lambda = \frac{(2 (\alpha - 2 \beta)^2 (\alpha^2 - \alpha \beta + \beta^2) q_1(x) q_1') \delta(x - y)}{3 \alpha^2} + \frac{(3 \alpha^2 q_4' - 2 (\alpha^3 - 3 \alpha^2 \beta + 3 \alpha \beta^2 - 2 \beta^3) q_1'') \delta(x - y)}{3 \alpha^2} + 2 \frac{(3 \alpha^2 q_4(x) - 2 (\alpha^3 - 3 \alpha^2 \beta + 3 \alpha \beta^2 - 2 \beta^3) q_1') \delta'(x - y)}{3 \alpha^2} + 2 \frac{(\alpha - 2 \beta)^2 (\alpha^2 - \alpha \beta + \beta^2) q_1^2(x) \delta'(x - y)}{3 \alpha^2} - 2 \frac{(\alpha^2 - \alpha \beta + \beta^2) \delta^{(3)}(x - y)}{3 \alpha^2}.$$

The vector field defined by a covector $w \in T^*_4Q$ is written in the form

$$\{q_\lambda = [v, L]$$

where $v$ is an extension of $w$ and $L$ is the matrix operator

$$L = \partial_x + \begin{pmatrix} \alpha - \beta & 0 & 0 \\ q_2 & -\alpha q_1 & 0 \\ q_4 & -\beta q_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$
In the case $\alpha = 2\beta$, $q_4(x)$ is a Virasoro density, i.e.

$$\{q_4(x), q_4(y)\}_2 = 2q_4(x)\delta'(x-y) + \delta(x-y)q_4' - \frac{\delta^{(3)}(x-y)}{2}$$

and the second Poisson bracket is the $W^{2}_3$-algebra (see e.g. [10]).

**Remark 3.6.** Perform the bi-Hamiltonian reduction on $sl_3$ by taking the symplectic leaf of $P_1$ defined by setting $a = e_{2,1} + e_{2,3}$ in (3.3) and fixing the transversal manifold to have the form (3.23). The reduced second Poisson tensor on this manifold is equal to the one of the example above, (see e.g. [6]). The form of the operator $L$ will change to

$$L := \partial_x + \begin{pmatrix} (\alpha - \beta)q_1 & 0 & 0 \\ q_2 & -\alpha q_1 & 0 \\ q_4 & q_3 & \beta q_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ \lambda & 0 & 0 \\ 0 & \lambda & 0 \end{pmatrix}$$

which is the Lax operator considered in [2] to obtain integrable hierarchy associated to $W^{2}_3$-algebra.

### 3.3. Classical $W$-Algebras from bi-Hamiltonian reduction.

Classical $W$-algebras and their primary fields will be obtained for the principal nilpotent element with an appropriate choice of a basis and a transversal subspace constructed using representation theory of $sl_2$-algebra.

Let $e$ be a principal nilpotent element and $\{h, e, f\}$ is the associated $sl_2$-triple. We denote by $A \subset \mathfrak{g}$ the subalgebra generated by this triple. Then we have a decomposition of $\mathfrak{g}$ as irreducible $A$-submodules:

$$\mathfrak{g} = A \oplus \bigoplus_{\alpha=1}^m V_{\alpha}. $$

Let $n_{\alpha} + 1$ be the dimension of $V_{\alpha}$. Fix a basis in $V_{\alpha}$

$$X_{\alpha}^{j}, \quad j = 0, ..., n_{\alpha}, \quad \alpha = 1, ..., m. $$

From the representation theory of $sl_2$ these vectors satisfy the following commutation relations

$$[h, X_{\alpha}^{j}] = (n_{\alpha} - 2j)X_{\alpha}^{j}, $$

$$[f, X_{\alpha}^{j}] = (j + 1)X_{\alpha}^{j+1}, $$

$$[e, X_{\alpha}^{j}] = (n_{\alpha} - j + 1)X_{\alpha}^{j-1}. $$

It is easy to prove that $C = \text{Ker(ad } f\text{)}$ is a transversal subspace associated to $e$. We will apply the generalized bi-Hamiltonian reduction using this transversal subspace. Let

$$v := v_{j}^{\alpha}X_{\alpha}^{j} + v_{h}h + v_{e}e + v_{f}f $$

be a general covector in $\mathcal{L}(\mathfrak{g})^*$ and

$$q := q_{\alpha}^{\alpha}X_{\alpha}^{n_{\alpha}} + q_{f}f + e$$
a point in $Q$. Using (3.30) the second Poisson tensor at $q \in Q$ reads
\[
P_2(v) = \left[ \frac{d}{dx} + q^\alpha X^n_{\alpha} + q_f f + e, v^\alpha_j X^j_{\alpha} + v_h h + v_e e + v_f f \right]
\]
(3.33)
\[= \Psi + (v^\alpha_j X^j_{\alpha} + (v_h)_x h + (v_e)_x e + (v_f)_x f +
 n_\alpha q^\alpha v_h X^n_{\alpha} - q^\alpha v_e X^n_{\alpha-1} + (j + 1) q_f v^\alpha_j X^j_{\alpha+1} + 2v_h q_f f
 - q_f v_e h + (n_\alpha - j + 1) v^\alpha_j X^j_{\alpha-1} - 2v_h e + v_f h
\]
where
\[
\Psi = q^\alpha v^\lambda_j [X^n_{\alpha}, X^j_{\lambda}].
\]
(3.34)
To find the reduced Poisson tensor $P^Q_2$ one must solve the recursion relations equating to zero the coefficients of $X^j_{\alpha}$, for $j = 0, \ldots, n_\alpha - 1$ and of $e$ and $h$ (using the procedure explained after lemma 3.4).

**Proposition 3.7.** The brackets with $q_f$ will be given as follows
\[
\{q_f(x), q_f(y)\} = -c_1 \left( \frac{1}{2} \delta'''(x - y) + 2q_f(x)\delta'(x - y) \right)
+ (q_f)_x \delta(x - y)
\]
(3.35)
\[
\{q^\alpha(x), q_f(y)\} = c^\alpha \left( q^\alpha_x \delta(x - y) + \frac{(n_\alpha + 2)}{2} q^\alpha(x)\delta'(x - y) \right),
\]
(3.36)
where $c_1$ and $c^\alpha$ are some constants depending on the choice of the basis (they are unique up to multiplication of $X^0_{\alpha}$ by a nonzero constant).

*Proof.* The main idea of the proof is to study the contribution of $v_e$ and its derivatives on the solutions of (3.33). This gives the Poisson brackets with $q_f(x)$. First we put $v^\alpha_0 = 0, \alpha = 1, \ldots, m$. It follows easily from Dynkin grading that
\[
v^\alpha_i = 0, \quad i = 1, \ldots, \frac{n_\alpha}{2}, \forall \alpha
\]
(recall $n_\alpha \forall \alpha$ is even for principal nilpotent elements). It follows that the expansion of $\Psi$ does not contain $h, f$ or $e$. Therefore equating the coefficient of $e$ to zero we have
\[
v_h = 1/2(v_e)_x
\]
(3.37)
and the coefficient of $h$ gives
\[
v_f = v_e q_f - 1/2(v_e)_{xx}
\]
(3.38)
and the coefficient of $f$ reads
\[
(v_f)_x + 2v_h q_f =
-1/2(v_e)_{xxx} + 2(v_e)_x q_f + v_e (q_f)_x
\]
(3.39)
Observe that $v_e$ appears explicitly only when equating the coefficient of $X^{n_\alpha - 1}_{\alpha}$ to zero which gives
\[
v^\alpha_{n_\alpha} = q^\alpha v_e + \text{other terms}
\]
(3.40)
The next step is using the fact that \( \ker(\text{ad } f) \) is abelian subalgebra (since \( f \) is a principal nilpotent element) to rewrite

\[
(3.41) \quad \Psi = q^\alpha v^j \{ X_{n^\alpha}^n, X_j \}, \quad j \neq n^\alpha.
\]

Thus solving the equation (3.33) recursively we have

\[
v_j^\alpha = 0, \quad j = n^\alpha + 1, \ldots, n^\alpha.
\]

Then we have

\[
(3.42) \quad v_{n^\alpha}^\alpha = q^\alpha v_e
\]

Finally the coefficient of \( X_{n^\alpha}^n \) leads to the expression

\[
(3.43) \quad (v_{n^\alpha}^\alpha)_x + n^\alpha q^\alpha v_h = q^\alpha v_e + \frac{(n^\alpha + 2)}{2} q^\alpha (v_e)_x
\]

We substitute \( \delta^k(x−y) \) for \( \partial^k_x(v_e) \), and the proof is complete. \( \square \)

Thus we proved the reduced second Poisson brackets to be the classical \( W \)-algebra as defined in [1] where \( q_f(x) \) is a Virasoro density and \( q^\alpha(x) \) are primary fields of weights \( \frac{c^\alpha}{2}(n^\alpha + 2) \).

In [1] they obtained the same brackets from the Drinfeld-Sokolov reduction associated to \( e \) and the transversal space \( C \). We will discuss the Drinfeld-Sokolov reduction and its relation with bi-Hamiltonian reduction in the next section.

**Remark 3.8.** In a similar manner one can obtain the Virasoro density using arbitrary distinguished nilpotent element \( e' \) (see also [2]). Our methods fail to produce the primary fields since the Poisson bracket with Virasoro density will depend on the structure constants of the transversal space \( C' = \ker(\text{ad } f') \) where \( \{e', h', f'\} \) is the associated \( sl_2 \)-triple.

## 4. Drinfeld-Sokolov Reduction

In this section we will recall briefly the Drinfeld-Sokolov reduction which is another procedure to obtain a bi-Hamiltonian manifold. In the next section we will show its equivalence to our bi-Hamiltonian reduction. We use the notations and terminology of section 3.1.

Let us denote by \( S \) the manifold consisting of operators of the form

\[
(4.1) \quad L = \frac{d}{dx} + b + e \quad \text{where } b \in \mathfrak{L}(b^-).
\]

The adjoint group \( G^- \) of \( \mathfrak{L}(g^-) \) acts on \( S \) by

\[
(4.2) \quad (n, L) \rightarrow \exp(\text{ad } n) \quad L \quad \text{for all } n \in \mathfrak{L}(g^-) \quad \text{and } L \in S.
\]

**Proposition 4.1.** For any operator \( L \in S \) there is a unique element \( s \in \mathfrak{L}(g^-) \) such that the operator \( \overline{L} = \exp \text{ad } sL \) has the form

\[
(4.3) \quad \overline{L}^s := \frac{d}{dx} + q + e
\]
where \( q \in \mathfrak{L}(C) \). The entries of \( q \) are generators of the ring \( R \) of differential polynomials invariant under the action of the group \( G^- \) on \( S \).

**Proof.** We write \( I^e = \exp(\text{ad } s) \) \( L \) in the grading associated to \( e \). Then inductively for \( i \leq 0 \) the equation has the form

\[
b_i + [e, s_{i-2}] = \ldots
\]

where the right-hand side is a differential expression in \( q \) and \( s \) of the degree greater than \( i \). The result follows by noticing that

\[
\mathcal{C}_i \oplus \text{ad}(g_{-2}^-) = b_i^-.
\]

\( \square \)

From the above lemma we define the space \( \tilde{Q} := S/G^- \). The set \( \mathcal{R} \) of functionals on \( \tilde{Q} \) can be realized as functionals on \( S \) which have densities in the ring \( \mathcal{R} \). Consider the space \( S \) as a subspace of \( \mathfrak{L}(g) \). Then for a functional \( H \) on \( S \) we define the gradient \( \delta H(q) \) to be the unique element in \( \mathfrak{L}(b^+) \) such that

\[
\frac{d}{d\theta} H(q + \theta \dot{s}) |_{\theta=0} = \int_{s^1} \langle \delta H | \dot{s} \rangle \text{ for all } \dot{s} \in \mathfrak{L}(b^-)
\]

and

\[
\int_{s^1} \langle \delta H | \dot{s} \rangle = 0 \text{ for all } \dot{s} \in \mathfrak{L}(n^-).
\]

We define on \( \mathfrak{L}(g) \) the Poisson pencil (3.3) with \( a \in \mathfrak{g} \) a homogenous element of minimal degree. Then this Poisson pencil \( P_\lambda \) is reduced on \( \tilde{Q} \) using the following

**Lemma 4.2.** \( \mathcal{R} \) is a closed subalgebra with respect to the Poisson pencil \( P_\lambda \).

**Proof.** Note that if

\[
L = \frac{d}{dx} + q + e \in S
\]

and

\[
\tilde{L} := \frac{d}{dx} + \tilde{q} + e = \exp(\text{ad } n) \) L
\]

then for \( F \in \mathcal{R} \) we have \( \delta F(q) = \exp(\text{ad } (-n)) \) \( \delta F(q) \). The proof is easily obtained by using any faithful matrix representation of \( \mathfrak{g} \). The result follows by substituting into the bracket

\[
\{F, H\}_\lambda(q) = \int_{s^1} \langle [\delta H, \delta F] | \frac{d}{dx} + q + \lambda a \rangle
\]

and using the invariance of the bilinear form \( \langle ., . \rangle \) under the adjoint action. \( \square \)
4.1. Drinfeld-Sokolov and bi-Hamiltonian reductions. In this section we will be mainly following the spirit of [28].

**Theorem 4.3.** (Marsden-Weinstein reduction) Let $M'$ be a Poisson manifold with Poisson tensor $P'$, let $G'$ be a Lie group and $g'$ its Lie algebra; suppose that $G'$ acts on $M'$ by a Hamiltonian action $\Psi$, with momentum map $J : M' \to g'^*$, i.e. for every $\xi \in g'$ the fundamental vector field $X_\xi$ is a Hamiltonian vector field,

$$X_\xi = P' dH_\xi$$

with Hamiltonian $H_\xi(m) = (J(m), \xi)$. Suppose the momentum map $J$ to be $Ad^*$-equivariant, i.e.

$$J(\Psi_g(u)) = Ad_g^* J(u) \text{ for all } g \in G'.$$

Let $\mu \in g'$ be a regular value of $J$, so that $S' = J^{-1}(\mu)$ is a submanifold of $M'$, and let $D'$ be the tangent distribution to the orbits of $\Psi$. Then the triple $(M', S', D')$ is Poisson-reduced using the Marsden-Ratiu reduction theorem 2.1. The quotient manifold turns out to be $N' = J^{-1}(\mu)/G'_\mu$, where $G'_\mu$ is the isotropy group of $\mu$.

Let $M$ be the space of operators of the form

$$Z = \frac{d}{dx} + q \quad \text{where } q \in \mathfrak{L}(g).$$

The adjoint group $N^-$ of $\mathfrak{L}(n^-)$ acts on $M$ by

$$\Psi_n : q \to nqn^{-1} - n_xn^{-1} \quad q \in \mathfrak{L}(g), n \in N^-.$$

Introduce on $M$ the bi-Hamiltonian structure (3.3) with $a \in \mathcal{C}$ a homogenous element of minimal degree.

**Proposition 4.4.** The action of $N^-$ on $M$ with Poisson tensor $P_\lambda$ is Hamiltonian for all $\lambda$. It admits a momentum map $J$ to be the projection

$$J : g \to n^+.$$

Moreover, $J$ is $Ad^*$-equivariant.

**Proof.** We consider a faithful matrix representation of $g$. Then the action on $M$ has the form

$$\Psi_n : q \to nqn^{-1} - n_xn^{-1} \quad q \in \mathfrak{L}(g), n \in N^-.$$

For $\xi \in \mathfrak{L}(n^-)$ the fundamental vector field will have the form

$$X_{-\xi} = \frac{d}{dt}(\exp(-t\xi) \ q \exp(t\xi) - (\exp(-t\xi))_x \exp(t\xi))$$

$$\quad = \xi_x + [q, \xi].$$

Define the functional

$$H_\xi(q) = \int \langle q, \xi \rangle = \int \langle J(q), \xi \rangle.$$
Then
\[(4.16)\]
\[P_\lambda \delta H_\xi = \xi_x + [\xi, q + \lambda a] = \xi_x + [\xi, q],\]
which proves the action is Hamiltonian. The momentum map is Ad*-equivariant iff
\[J(\Psi_n(q)) = \text{Ad}_n^* J(q).\]
Since the moment map is just the projection we have
\[J(\Psi_n(q)) = J(nqn^{-1} - n_x n^{-1}) = J(nJ(q)n^{-1}) = \text{Ad}_n^* J(q)\]
where the last equality follows from the fact that the coadjoint action of \(N^-\) on \(n^+ \simeq n^-\) is given by
\[(4.17) \quad \text{Ad}_n^* v = J(nvn^{-1}), \quad v \in n^-\]

We take the nilpotent element \(e\) as a regular value of \(J\). Define the space
\[(4.18) \quad S := J^{-1}(e) = \frac{d}{dx} + \mathfrak{L}(\mathfrak{b}^-) + e\]
and let \(D\) denote the distribution defined by the group action
\[(4.19) \quad D := P_\lambda(\mathfrak{L}(\mathfrak{n}^-)) = P_2(\mathfrak{L}(\mathfrak{n}^-)).\]
Let \(E := D \cap TS\). Then \(P_\lambda, S\) and \(D\) satisfy Marsden-Ratiu theorem [24].
According to theorem 4.3, \(P_\lambda\) is reduced on the space
\[(4.20) \quad \tilde{Q} := S/G^-\]
where \(G^- \subset N^-\) is the isotropy subgroup of \(e\) under the action of \(N^-\). From the properties of the grading \(G^-\) is the adjoint group of \(\mathfrak{g}^-\). This obviously leads to Drinfeld-Sokolov reduction.

Now we use corollary 2.2. Define the space
\[(4.21) \quad Q := \frac{d}{dx} + \mathfrak{L}(\mathfrak{c}) + e.\]
Then \(Q\) is transversal to the distribution \(E := D \cap TS = P_2(\mathfrak{L}(\mathfrak{b}^-))\) on \(S\) by lemma 3.4 and 3.12. This gives the generalized bi-Hamiltonian reduction.

Thus we have proved the following

**Theorem 4.5.** The generalized Drinfeld-Sokolov and generalized bi-Hamiltonian reductions are equivalent in the sense that they satisfy the Marden-Ratiu theorem with the same Poisson pencil \(P_\lambda,\) the submanifold \(S\) and the distribution \(D\).

As we mentioned in the introduction, in the special case of principal nilpotent element the equivalence is obtained in [28]. Using generalized bi-Hamiltonian reduction the proof is more simpler even in this case.
5. Applications to Frobenius manifolds

Let $M$ be a manifold with local coordinates $(U^1, ..., U^n)$. On the loop space $\mathfrak{L}(M)$ a local Poisson bracket can be written in the form

\[(5.1) \quad \{U^i(x), U^j(y)\} = \sum_{k=-1}^{\infty} \epsilon^k \{U^i(x), U^j(y)\}^{[k]}.
\]

Here $\epsilon$ is just a parameter and

\[(5.2) \quad \{U^i(x), U^j(y)\}^{[k]} = \sum_{s=0}^{k+1} A^{ij}_{k,s} \delta^{(k-s+1)}(x - y),
\]

where $A^{ij}_{k,s}$ are homogenous polynomials in $\partial^j_x U^i(x)$ of degree $s$ (we assign $\partial^j_x U^i(x)$ degree $j$). The first terms can be written as follows

\[(5.3) \quad \{U^i(x), U^j(y)\}^{[-1]} = F^{ij}(U) \delta(x - y)
\]

\[(5.4) \quad \{U^i(x), U^j(y)\}^{[0]} = g^{ij}(U) \delta'(x - y) + \Gamma^{ij}_k(U) U^k_x \delta(x - y)
\]

where $F^{ij}$, $g^{ij}$ and $\Gamma^{ij}_k$ are smooth functions in $U^i$. The matrix $F^{ij}$ defines a Poisson structure on $M$. If $F^{ij} = 0$ then $\{U^i(x), U^j(y)\}^{[0]}$ defines a Poisson bracket on $\mathfrak{L}(M)$ known as Poisson bracket of hydrodynamic type. By nondegenerate Poisson bracket of hydrodynamic type we mean the metric $g^{ij}$ is nondegenerate. In this case its inverse defines a flat metric on the tangent space $TM$ and $\Gamma^{ij}_k$ are the contravariant Levi-Civita coefficients of $g^{ij}$ [17]. Assume there is a bi-Hamiltonian structure on $\mathfrak{L}(M)$ defined by Poisson tensors $P_1$ and $P_2$. Suppose $P_1$ and $P_2$ admit a nondegenerate Poisson brackets of hydrodynamics type and $\det(g^{ij}_2 - \lambda g^{ij}_1) \neq 0$, for generic $\lambda$. Then by definition [15] $g^{ij}_2$ form a flat pencil of metrics. Under some assumption of quasihomogeneity and regularity a flat pencil of metrics is equivalent to a Frobenius structure on $M$ [15]. In the notations of [11] from a Frobenius structure on $M$ the flat pencil of metric is found from the relations

\[(5.5) \quad \eta^{ij} = g^{ij}_1
\]

\[(5.6) \quad g^{ij}_2 = (d - 1 + d_i + d_j) \eta^{\alpha \beta} \eta^{ij} \partial_\alpha \partial_\beta F
\]

5.1. Dirac reduction. In this section we write the formulas for a Dirac reduction of a Poisson bracket on the loop space $\mathfrak{L}(M)$ to a loop space $\mathfrak{L}(N)$ of a suitable submanifold $N \subset M$. We use the notations introduced in the beginning of this section.

It is well known that the Dirac reduction of Poisson bracket of hydrodynamic types may result a nonlocal Poisson bracket [21]. We obtain a Dirac reduction for a local Poisson bracket which does not admit a dispersionless limit. The resulting Poisson bracket is local.

Let $N$ be a submanifold of $M$ of dimension $m$. Assume $N$ is defined by the equations $U^\alpha = 0$ for $\alpha = m + 1, ..., n$. We introduce three types of indexes;
capital letters $I, J, K, \ldots = 1, \ldots, n$, small letters $i, j, k, \ldots = 1, \ldots, m$ which parameterize the submanifold $N$ and Greek letters $\alpha, \beta, \delta, \ldots = m + 1, \ldots, n$.

**Proposition 5.1.** Assume the matrix $F^{\alpha \beta}$ is nondegenerate. Then Dirac reduction is well defined on $\mathcal{L}(N)$ and gives a local Poisson bracket.

**Proof.** Let $\mathcal{F}$ be a Hamiltonian functional on $\mathcal{L}(M)$. Then the Hamiltonian flows have the equation

$$U^I_t = B^{IJ} \frac{\delta \mathcal{F}}{\delta U^J}$$

where

$$B^{IJ} = \epsilon^k A_k^{IJ} \frac{d^{k-s+1}}{dx^{k-s+1}}.$$

Then Dirac equation on $N$ will have the form

$$U_i^i = B_{ij} \frac{\delta \mathcal{F}}{\delta U^j} + B_{i\beta} (\frac{\delta \mathcal{F}}{\delta U^\beta} + C^\beta)$$

where $C^\beta(y)$ can be found from the equation

$$0 = U_\beta^\alpha = B_{\alpha j} \frac{\delta \mathcal{F}}{\delta U^j} + \int \{U^\beta, U^\beta\} C^\beta(y) dy = \int \{U^\beta, U^\beta\} C^\beta + B_{\alpha \beta} (\frac{\delta \mathcal{F}}{\delta U^\beta} + C^\beta).$$

In powers of $\epsilon$ this equation reads

$$(\epsilon^{-1} B^\alpha_{-1} + B^\alpha_0 + \epsilon B^\alpha_1 + \ldots) \frac{\delta \mathcal{F}}{\delta U^i} =$$

$$(\epsilon^{-1} B^{\alpha \beta} + B^{\alpha \beta}_0 + \epsilon B^{\alpha \beta}_1 + \ldots)(\epsilon^{-1} C^{-1}_\beta + (\frac{\delta \mathcal{F}}{\delta U^\beta} + C^0_\beta)) + \epsilon C^1_\beta + \ldots).$$

We will solve this equation recursively. We depend on the fact that the matrix $B^{\alpha \beta}_{-1} = F^{\alpha \beta}$ is invertible. Then the coefficients of $\epsilon^{-2}$ gives

$$0 = F^{\alpha \beta} C^{-1}_\beta \Rightarrow C^{-1}_\beta = 0$$

and the coefficient of $\epsilon^{-1}$

$$(\frac{\delta \mathcal{F}}{\delta U^\beta} + C^0_\beta) =$$

$$(\frac{\delta \mathcal{F}}{\delta U^\beta} + C^0_\beta) = -F^{\alpha \beta}_\beta F^\alpha_{\beta \alpha} \frac{\delta \mathcal{F}}{\delta U^i}$$

where $F^{\alpha \beta}_\beta$ is the inverse of $F^{\beta \alpha}$. The constant term in $\epsilon$ leads to

$$-B^0_\alpha \frac{\delta \mathcal{F}}{\delta U^i} = B^0_\alpha (\frac{\delta \mathcal{F}}{\delta U^\beta} + C^0_\beta) + F^{\alpha \beta} C^1_\beta$$

$$C^1_\beta = F^{-1}_\beta = -B^0_\alpha \frac{\delta \mathcal{F}}{\delta U^i} + B^0_\alpha \frac{\delta \mathcal{F}}{\delta U^i} (F^{\alpha \gamma} C^1_\beta).$$
Note that $B_0$ are differential operators. We continue in this way to find all the elements $C^i_\beta$ for $i > 1$. In fact

\begin{equation}
C^s_\beta = F^i_\beta ( - B^{a_0}_{s-1} \frac{\delta F}{\delta U^i} - B^{a_0 \phi}_{s-1} ( \frac{\delta F}{\delta U^\phi} + C^0_\phi) - B^{a_0 \phi}_{s-2} C^1_\phi - B^{a_0 \phi}_{s-3} C^2_\phi - ... )
\end{equation}

Therefore we get a differential operators acting on the vector $\delta F / \delta U^i$. Finally, we substitute the values of $C^i_\beta$ in the equations (5.7). We get a local Poisson bracket on $\mathfrak{L}(N)$. This ends the proof. \hfill \Box

The Hamiltonian equations on $N$ read

\begin{equation}
U^i_t = \sum_{k=-1}^{\infty} \epsilon^k B^i_k \frac{\delta F}{\delta U^j} + \left( \sum_{k=-1}^{\infty} \epsilon^k B^i_k \right) (\frac{\delta F}{\delta U^j} + C^0_\beta + C^1 + ...) = \sum_{k=-1}^{\infty} \epsilon^k B^i_k \frac{\delta F}{\delta U^j} + \left( \sum_{k=-1}^{\infty} \epsilon^k B^i_k \right) (-F^i_\beta F^j_\alpha \frac{\delta F}{\delta U^i} + \epsilon F^i_\beta (-B^{a_0}_{0} \frac{\delta F}{\delta U^i} + B^{a_0 \phi}_{0} (F^\phi \gamma \frac{\delta F}{\delta U^i}) + O(\epsilon^2))
\end{equation}

Hence if we write the Poisson bracket on $N$ in the form

\begin{align}
\{U^i(x), U^j(y)\}_{-1} &= \widetilde{F}^i_\beta \delta(x - y) \\
\{U^i(x), U^j(y)\}_0 &= \widetilde{g}^{ij} \delta'(x - y) + \tilde{\Gamma}^i_k U^k \delta(x - y).
\end{align}

We have

\begin{equation}
\widetilde{F}^i_\beta = (F^i - F^i_\beta F^{a_0} F^{a_j})
\end{equation}

and

\begin{equation}
\widetilde{g}^{ij} = g^{ij} - g^{i_\beta} F^{\beta_0} F^{a_\gamma} + F^{i_\beta} F^{\beta_0} g^{a_\phi} F^{\phi_\gamma} F^{\gamma_\delta} - F^{i_\beta} F^{\beta_0} g^{a_\phi}
\end{equation}

from the coefficient of $\epsilon^{-1}$ and of $\epsilon^0$ respectively.

Remark 5.2. As expected the formula of $\widetilde{F}^{i_\beta}$ coincide with Dirac reduction of the finite dimensional Poisson bracket defined by $F^{I_\beta}$ on $M$ to $N$.

Corollary 5.3. Assume $U^i$, $i = 1, ..., n$ are Casimirs of the Poisson bracket $F^{I_\beta}$ on $M$. Then $F^{i_\alpha} = 0$. Hence the reduced Poisson bracket on $\mathfrak{L}(N)$ reads

\begin{align}
\widetilde{F}^{i_\beta} &= 0 \\
\widetilde{g}^{i_\beta} &= g^{i_\beta}.
\end{align}
5.2. Algebraic Frobenius manifolds from Lie algebra $F_4$. Our aim is to obtain a Frobenius manifold from a reduced Lie-Poisson pencil. We apply generalized bi-Hamiltonian reduction to distinguished nilpotent elements of the Lie algebra $F_4$. There are four distinguished nilpotent orbits on $F_4$. They correspond to regular conjugacy classes in Weyl group $W_{F_4}$ [11].

Let $e$ be a distinguished nilpotent element in $F_4$. We apply the bi-Hamiltonian reduction to $e$ with Dynkin grading. Let $P^Q_\lambda$ denote the reduced Poisson pencil on the loop space $\mathcal{L}(Q)$. If $e$ is not principal then the reduced Poisson pencil $P^Q_\lambda$ does not admit a dispersionless limit, i.e. the leading term of $P^Q_\lambda$ of degree $-1$ does not vanish. We apply Dirac reduction to a submanifold $N \subset Q$ such that all the pencil $P^Q_\lambda$ is reduced. The new Poisson pencil $P^N_\lambda$ on $\mathcal{L}(N)$ gives a nondegenerate Poisson pencil of hydrodynamic type. The example below illustrates this procedure in details.

The distinguished nilpotent elements in $F_4$ give four non isomorphic algebraic Frobenius manifolds of dimension 4. Two of them give a polynomial Frobenius manifolds isomorphic to Frobenius structure on the orbit spaces of Coxeter group of type $F_4$ and $B_4$ [13]. One of the remaining distinguished nilpotent orbits is likely to give algebraic Frobenius structure isomorphic to the one found in [27] by applying Drinfeld-Sokolov reduction on Lie algebra of type $D_4$. We obtain the same result by applying the generalized bi-Hamiltonian reduction to $D_4$. We end with one class of nilpotent elements which give a new algebraic Frobenius manifold.

**Example 5.4.** (Algebraic Frobenius manifold) Denote by $\Psi$ the set of roots of the Lie algebra $F_4$. For the following computations we use the minimal representation of $F_4$ [4]. Assume $X_\alpha$ with $H_\alpha \in \mathfrak{h}$, $\alpha \in \Psi$, form Weyl-Chevalley basis of $F_4$. We apply the generalized bi-Hamiltonian reduction with the nilpotent element

$$e := X_{a_2} + X_{a_1+a_2} + X_{a_2+a_3} + X_{a_1+a_2+a_3} + X_{a_2+2a_3} + X_{a_1+a_2+2a_3} + X_{a_4} + X_{a_3+a_4}$$

(5.20)

which is a representative of the nilpotent orbit $F_4(a_2)$ in the notations of [9]. We fix the associated Dynkin grading and define the first Poisson bracket with $a = X_{-2a_1-3a_2-4a_3-2a_4}$. Define the transversal manifold $Q$ to be of the form

$$q = U_2 X_{-a_2-2a_3} + U_3 X_{-a_1-a_2-2a_3} + U_1 X_{-a_1-a_2-a_3} + U_8 X_{-a_1-3a_2-4a_3-2a_4} + U_7 X_{-a_1-3a_2-4a_3-2a_4} + U_5 X_{-a_1-a_2-2a_3-2a_4} + U_4 X_{-a_1-a_2-2a_3-a_4} + U_6 X_{-a_1-2a_2-3a_3-2a_4}$$

(5.21)

Write the reduced Poisson pencil in the notations of (5.2). Then the coefficient $F^i_\lambda$ of $e^{-1}$ does not vanish. Indeed, the coefficient $F^{ij}_1$ of the first
Poisson bracket reads

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{5}{72} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{11}{72} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{4}{9} & 0 & 10U_1 + 23U_2 + 5U_3 & -6U_1 + 43U_2 + U_3 \\
0 & 0 & 0 & -17U_1 - 198U_2 - 18U_3 & -U_4 & -U_4 & 1080 \\
0 & 0 & -17U_1 - 198U_2 - 18U_3 & -U_4 & -U_4 & 1080 \\
0 & 0 & -17U_1 - 198U_2 - 18U_3 & -U_4 & -U_4 & 1080 \\
0 & 0 & -17U_1 - 198U_2 - 18U_3 & -U_4 & -U_4 & 1080 \\
\end{pmatrix}
\]

where

\[
F^{6,7}_1 = \frac{-15U_1^2 - 380U_1U_2 - 700U_2^2 - 60U_1U_3 - 328U_2U_3 - 60U_3^2 + 150U_5}{5760}
\]

\[
F^{6,8}_1 = \frac{69U_1^2 + 772U_1U_2 - 1356U_2^2 - 28U_1U_3 - 424U_2U_3 - 76U_3^2 + 110U_5}{5760}
\]

Our next aim is to use Dirac reduction. For this we introduce the coordinates

\[
W_1 = \frac{U_1}{2} + U_2 + U_3
\]

\[
W_2 = 11U_2 + 5U_3
\]

\[
W_3 = \frac{U_1 (-205U_1^2 + 1908U_1U_2 - 7416U_2^2 + 360(U_1 + 2U_2)U_3 + 360U_3^2)}{1080} + \frac{5U_1U_5}{12} - 2U_2U_5 + U_7 - 5U_8
\]

\[
W_4 = \frac{95U_1^3 - 849U_1^2U_2 + 3948U_1U_2^2 - 195U_1^2U_3 - 120U_1U_2U_3 - 180U_1U_3^2}{720} + \frac{U_1^2}{32} + \left(\frac{-U_1}{12} + \frac{19U_2}{12} + \frac{U_3}{12}\right)U_5 + U_7 + 3U_8
\]

\[
W_5 = U_1
\]

\[
W_6 = U_4
\]

\[
W_7 = U_5
\]

\[
W_8 = U_6
\]
where the first four are the Casimirs of $F_{ij}^j$. We rewrite the Poisson bracket in the new coordinates. Define the submanifold $N$ given by

\[
W_6 = W_8 = 0 \\
W_5 = Z \\
W_7 = \frac{-5 Z^2 + 150 Z W_1 - 100 W_1^2 - 33 Y W_2 + 64 W_1 W_2 - 7 W_2^2}{150}
\]

where $Z(W_1, ..., W_4)$ is a solution of a cubic equation to be given below. It turn out that along $N$ the entries $P_{i\alpha}^\lambda = 0$, $i = 1, ..., 4$, $\alpha = 5, ..., 8$. Hence From corollary $5.3$ the Poisson pencil $P^\lambda_Q$ is reduced along $N$. The new Poisson bracket $P^N_\lambda$ gives a flat pencil of metrics. This result a Frobenius structure on $N$. In a flat coordinates $(s_1, s_2, s_3, s_4)$ the Frobenius structure will have the potential

\[
F = \frac{9 Z^2 s_1^5}{44800} + \frac{3 Z^2 s_1^4 s_2}{89600} - \frac{3 Z^2 s_1^3 s_2^2}{89600} - \frac{3 Z^2 s_1^2 s_2^3}{640000} + \frac{153 Z^2 s_1 s_2^4}{89600000} \\
+ \frac{1107 Z^2 s_2^5}{4480000000} + \frac{81 Z^2 s_1^2 s_2^3}{2800} - \frac{243 Z^2 s_1 s_2^3 s_3}{14000} + \frac{729 Z^2 s_2^2 s_3}{2800000} \\
+ \frac{409 Z s_1^6}{2419200} - \frac{191 Z s_1^5 s_2}{1344000} + \frac{187 Z s_1^4 s_2^2}{5376000} + \frac{67 Z s_1^3 s_2^3}{134400000} \\
+ \frac{319 Z s_1^2 s_2^4}{179200000} - \frac{529 Z s_1^2 s_2^3}{1344000000} + \frac{1247 Z s_2^6}{38400000000} + \frac{27 Z s_1^3 s_3}{560} \\
+ \frac{117 Z s_1^2 s_2 s_3}{560000} + \frac{9 Z s_1^2 s_2^3 s_3}{560000000} + \frac{369 Z s_2^3 s_3}{70} + \frac{243 Z s_3^2}{2000} \\
+ \frac{5600}{5600000} + \frac{560000000}{5600000000} + \frac{29459 s_1^6 s_2}{58060800000} - \frac{6089 s_1^5 s_2^2}{276480000} - \frac{580608000000}{348364800000} + \frac{1161216000000}{560000000000} \\
+ \frac{300457 s_1^2 s_2^5}{580608000000000000} - \frac{1973651 s_1 s_2^6}{1741824000000000000} + \frac{292289 s_2^7}{1935360000000000000} + \frac{17 s_1 s_3}{44800} \\
+ \frac{58060800000000000000}{2647 s_1^3 s_2 s_3} + \frac{6059 s_1^2 s_2^2 s_3}{22400000} - \frac{18223 s_1 s_2^3 s_3}{336000000} \\
+ \frac{11443 s_1^7}{1741824000000000000} + \frac{60131 s_1^2 s_2^3 s_3}{134400000000} + \frac{s_1 s_3^2}{20} + \frac{3 s_2 s_3}{200} \\
- 2 s_1 s_3 s_4 + \frac{3 s_2 s_3 s_4}{5} + 2 s_1 s_4^2
\]

where $Z$ is a solution of the cubic equation

\[
Z^3 - Z \left( \frac{s_1^2}{48} + \frac{s_1 s_2}{80} + \frac{3 s_2^2}{1600} \right) \\
- \frac{s_1^3}{96} + \frac{13 s_1^2 s_2}{2880} - \frac{s_1 s_2^2}{28800} - \frac{41 s_2^3}{28800} - \frac{3 s_3}{2} = 0
\]

It is straightforward to check validity of the WDVV equations for this potential. The quasihomogeneity reads

\[
s_4 \partial_1 F(s) + s_3 \partial_3 F(s) + \frac{1}{3} s_2 \partial_2 F(s) + \frac{1}{3} s_1 \partial_1 F(s) = \frac{7}{3} F(s).
\]
The four examples of algebraic Frobenius manifolds obtained from Lie algebra $F_4$ are related to their conjugacy classes in $W_{F_4}$ as follows. If the regular conjugacy class is of order $o_4 + 1$ and the eigenvalues are $\omega^{o_i}, \ i = 1, \ldots, 4$ where $\omega$ is a primitive $(o_4 + 1)$-th roots of unity, then the degrees of the corresponding algebraic Frobenius manifold are $\frac{o_i + 1}{o_4 + 1}$ and the charge is $1 - \frac{o_1 + 1}{o_4 + 1}$.

The examples of algebraic Frobenius manifolds obtained on the Lie algebra $F_4$ suggest that algebraic Frobenius structures exist for all regular primitive conjugacy classes in Weyl groups.

In a subsequent publication we will consider further examples of Frobenius structures and integrable hierarchies on bi-Hamiltonian manifolds produced by applying the reduction methods introduced in this paper.

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References

[1] Balog, J.; Feher, L.; O’Raifeartaigh, L.; Forgacs, P.; Wipf, A., Toda theory and $W$-algebra from a gauged WZNW point of view. Ann. Physics 203, no. 1, 76–136 (1990).

[2] Burroughs, Nigel J.; de Groot, Mark F.; Hollowood, Timothy J.; Miramontes, J. Luis, Generalized Drinfeld-Sokolov hierarchies. II. The Hamiltonian structures. Comm. Math. Phys. 153, no. 1, 187–215 (1993).

[3] Carter, R. W., Conjugacy classes in the Weyl group. Compositio Math. 25, 1–59 (1972).

[4] Carter, R. W., Lie algebras of finite and affine type. Cambridge Studies in Advanced Mathematics, 96. Cambridge University Press, ISBN: 978-0-521-85138-1 (2005).

[5] Casati, Paolo; Falqui, Gregorio; Magri, Franco; Pedroni, Marco Bi-Hamiltonian reductions and $W_n$-algebras. J. Geom. Phys. 26, no. 3-4, 291–310 (1998).

[6] Casati, Paolo; Falqui, Gregorio; Magri, Franco; Pedroni, Marco, A note on fractional KdV hierarchies. J. Math. Phys. 38, no. 9, 4606–4628 (1997).

[7] Casati, Paolo; Magri, Franco; Pedroni, Marco, Bi-Hamiltonian manifolds and $\tau$-function. Mathematical aspects of classical field theory, 213–234 (1992).

[8] Casati, Paolo; Pedroni, Marco Drinfeld-Sokolov reduction on a simple Lie algebra from the bi-Hamiltonian point of view. Lett. Math. Phys. 25, no. 2, 89–101 (1992).
Collingwood, David H.; McGovern, William M., Nilpotent orbits in semisimple Lie algebras. Van Nostrand Reinhold Mathematics Series. ISBN: 0-534-18834-6 (1993).

de Groot, Mark F.; Hollowood, Timothy J.; Miramontes, J. Luis, Generalized Drinfeld-Sokolov hierarchies. Comm. Math. Phys. 145, no. 1, 57–84 (1992).

Delduc, F.; Feher, L., Regular conjugacy classes in the Weyl group and integrable hierarchies. J. Phys. A 28, no. 20, 5843–5882 (1995).

Drinfeld, V. G.; Sokolov, V. V., Lie algebras and equations of Korteweg-de Vries type. (Russian) Current problems in mathematics, Vol. 24, 81–180, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, (1984).

Dubrovin, Boris, Differential geometry of the space of orbits of a Coxeter group. Surveys in differential geometry IV: integrable systems, 181–211 (1998).

Dubrovin, Boris, Geometry of 2D topological field theories. Integrable systems and quantum groups (Montecatini Terme, 1993), 120–348, Lecture Notes in Math., 1620, Springer, Berlin, (1996).

Dubrovin, Boris, Flat pencils of metrics and Frobenius manifolds. Integrable systems and algebraic geometry (Kobe/Kyoto, 1997), 47–72, World Sci. Publ. (1998).

Dubrovin, Boris, Painleve transcendents in two-dimensional topological field theory. The Painlevé property, 287–412, CRM Ser. Math. Phys., Springer, New York (1999).

Dubrovin, B. A.; Novikov, S. P., Poisson brackets of hydrodynamic type. (Russian) Dokl. Akad. Nauk SSSR 279, no. 2, 294–297 (1984).

Dubrovin, B. , Zhang, Y., Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov-Witten invariants, www.arxiv.org [math/0108160].

Elashvili, A. G.; Kac, V. G. Classification of good gradings of simple Lie algebras. Lie groups and invariant theory, 85–104, Amer. Math. Soc. Transl. Ser. 2, 213 (2005).

Feher, L.; O’Raifeartaigh, L.; Ruelle, P.; Tsutsui, I.; Wipf, A. On Hamiltonian reductions of the Wess-Zumino-Novikov-Witten theories. Phys. Rep. 222, no. 1 (1992).

Ferapontov, E. V., Nonlocal Hamiltonian operators of hydrodynamic type: differential geometry and applications. Amer. Math. Soc. Transl. Ser. 2, 33–58, 170 (1995).

Hertling, Claus, Frobenius manifolds and moduli spaces for singularities. Cambridge Tracts in Mathematics, 151. Cambridge University Press, ISBN: 0-521-81296-8 (2002).

Humphreys, James E. Reflection groups and Coxeter groups. Cambridge Studies in Advanced Mathematics, 29. Cambridge University Press, Cambridge, ISBN: 0-521-37510-X (1990).

Marsden, Jerrold E.; Ratiu, Tudor, Reduction of Poisson manifolds. Lett. Math. Phys. 11, no. 2, 161–169 (1986).

Marsden, Jerrold E.; Ratiu, Tudor S., Introduction to mechanics and symmetry. Springer-Verlag, ISBN: 0-387-97275-7; 0-387-94347-1 (1994).

Michel, J. Hurwitz action on tuples of Euclidean reflections. J. Algebra 295, no. 1, 289–292 (2006).

Pavlyk, O., Solutions to WDVV from generalized Drinfeld-Sokolov hierarchies, www.arxiv.org [math-ph/0003020] (2003).

Pedroni, Marco, Equivalence of the Drinfeld-Sokolov reduction to a bi-Hamiltonian reduction. Lett. Math. Phys. 35, no. 4, 291–302 (1995).

Springer, T. A., Regular elements of finite reflection groups. Invent. Math. 25, 159–198 (1974).

Stefanov, A., Finite orbits of the braid group action on sets of reflections, www.arxiv.org [math-ph/0409026] (2004).

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