Dynamical objectivity in quantum Brownian motion

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Abstract – Classical objectivity as a property of quantum states — a view proposed to explain the observer-independent character of our world from quantum theory, is an important step in bridging the quantum-classical gap. It was recently derived in terms of spectrum broadcast structures for small objects embedded in noisy photon-like environments. However, two fundamental problems have arisen: a description of objective motion and applicability to other types of environments. Here we derive an example of objective states of motion in quantum mechanics by showing the formation of dynamical spectrum broadcast structures in the celebrated, realistic model of decoherence — Quantum Brownian Motion. We do it for realistic, thermal environments and show their noise-robustness. This opens a potentially new method of studying the quantum-to-classical transition.

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Introduction. – Reconciliation of quantum theory with the classical world of everyday experience has been one of the central problems in our understanding of Nature [1,2], touching such deep questions as: is there any “reality” out there [3]? One of the aspects of this reconciliation has been how to explain the objective character of our world with fragile quantum systems, inevitably disturbed by measurements. As quantum state is to date our most fundamental description of Nature, it is natural to look for an explanation at this level. Indeed, recently specific quantum state structures — spectrum broadcast structures (SBS) [4,5], have been identified as responsible for the perceived objectivity, suggesting that the latter is, in fact, a property of quantum states. Building on the quantum Darwinism idea [2,6] — a realistic form of decoherence theory [2] where the system of interest $S$ interacts with multiple environments $E_1, \ldots, E_N$ and observers acquire information about $S$ through them, it has been shown in [4] (see also [7]) in a model- and dynamics-independent way that the only, in a certain sense, states that encode objective states of the system are precisely the SBS:

$$g_{S,E} = \sum_i p_i|\psi_i\rangle\langle\psi_i| \otimes \rho^{E_1} \otimes \cdots \otimes \rho^{E_N},$$

where $\{\psi_i\}$ is a pointer basis [8], $p_i$ are pointer probabilities, and $\rho^{E_1}, \ldots, \rho^{E_N}$ some states of $E_1, \ldots, E_N$ with orthogonal supports. As is easy to see from (1), by properly measuring their portions of the environment (projecting on the supports of $\rho^{E_k}$), all the observers will obtain the same result $i$ without disturbing neither the system $S$ nor each other. Since “seeing the same by many” without disturbance arguably defines a form of objectivity [4,6], the states $|\psi_i\rangle$ become thus objective in this sense. Our approach is of course connected to the earlier studies based on information redundancy [6], but here we show it directly at the fundamental level of states, rather than using information-theoretical conditions, known so far only to be necessary [4]. A process of formation of a SBS [5] is a weaker form [9] of quantum state broadcasting [10,11].

A question now arises if such structures are indeed formed in realistic models of decoherence. Recently [5], their formation was shown in the emblematic model of decoherence with scattering-type interactions: A small dielectric sphere illuminated by photons, but the resulting broadcast structure, and hence the objective states, were static (described a fixed position) as the central system had no self-dynamics. In this work we study a fully dynamical model where both the system and the environment have their own dynamics and report the formation of objectively existing states of motion for the fundamental to all physics class of harmonic interactions. In one of the universal models of decoherence — Quantum Brownian
Motion (QBM) [2,12,13], which describes a central oscillator \( S \) linearly coupled to a bath \( E \) of oscillators, we show a formation, in the massive central system limit, of novel dynamical spectrum broadcast structures (1) with time-evolving pointer states \( |x,(t)\rangle \). Due to developed correlations, information about this evolution is redundantly encoded in the environment (in time-evolving, mutually orthogonal states \( \hat{P}_E,\hat{P}_S(t) \)), even if the environment is noisy, and in this sense it becomes objective [4,6]. We model the noise as a thermal noise (with a ramification to arbitrary single-mode Gaussian noise) and numerically study the effect as a function of the temperature, showing a certain noise-robustness. Surprisingly, in spite of being probably the most studied model of decoherence for decades [2,12,13], these state structures have not been noticed before (in the previous studies [14,15] information-theoretical conditions were used, known so far to be only necessary with their sufficiency being open [4] and the environment was pure). Moreover, in contrast to the standard approaches [2,13,14], we do not use the continuous approximation to the environment. We will assume \( [14] \) that the central oscillator alone will not decohere the central system very massive, so it is effectively macroscopic, while the environment was pure (with a ramification to arbitrary single-mode Gaussian noise) and numerically model the noise as a thermal noise (with a ramification: see, e.g., [18]). In this approximation, the system \( S \) evolves according to its self-Hamiltonian \( \hat{H}_S \), with this evolution further approximated using classical trajectories \( X(t;X_0) \), while the environment is driven along each of this trajectory. The resulting state is:

\[
\Psi_{S,E} = \int dX_0 \phi_0(X_0)e^{-i\hat{H}_S t}|X_0\rangle \otimes \hat{U}_E(X(t;X_0))|\psi_0\rangle, \tag{5}
\]

acting on the initial state \( |\phi_0\rangle|\psi_0\rangle \). Since \( \hat{H}_S \) is quadratic, the trajectory approximation is actually exact (the semiclassical propagator is exact). For simplicity, we will limit ourselves to trajectories obtained when the system is initially in the squeezed vacuum state (cf. [14,15]): \( |\phi_0\rangle = \hat{S}(\tau)|0\rangle \), where \( \hat{S}(\tau) \equiv e^{(\hat{a}^\dagger - \hat{a})^2/2} \). Especially interesting is a highly momentum squeezed state due to its large coherences in the position. We may than assume that the initial velocity of each trajectory is zero so that \( X(t;X_0) = X_0 \cos(\Omega t) \). The analysis of the high initial position squeezing, for which \( X_0 = 0 \) and \( X(t;X_0) = X_0 \sin(\Omega t) \), will be analogous. We solve for \( \hat{U}_E(X(t;X_0)) \) using \( \hat{U}_E(X(t;X_0)) = \lim_{\Delta t \to 0} (\Pi_{\tau=t}^{\tau=t+\Delta t} \exp[-i\hat{H}_E(t)] \Delta t) \),

\[
\Delta t \equiv t/n, \quad t_r \equiv r\Delta t \quad \text{and obtain}
\]

\[
\hat{U}_E(X(t;X_0)) = \bigotimes_{k=1}^N \hat{U}_k(X_0;t), \tag{7}
\]

\[
\hat{U}_k(X_0;t) \equiv e^{i\zeta_k(t)f_k^2e^{-i\hat{H}_k t}\hat{D}(\alpha_k(t))X_0^2}, \tag{8}
\]

so that (6) has the following form:

\[
\hat{U}_{S,E}(t) = e^{-i\hat{H}_S t} \otimes e^{-i\sum_{k=1}^N \hat{H}_k t} \times \int dX_0|X_0\rangle \otimes e^{i\zeta_k(t)X_0^2} \hat{D}(\alpha_k(t))X_0) \tag{9}
\]

Here \( \hat{D}(\alpha) \equiv e^{\hat{a}^\dagger - \hat{a}^\dagger} \hat{a}^\dagger \hat{a}^\color{red} \) is the displacement operator [19], \( \hat{a}^\dagger, \hat{a} \) the creation and annihilation operators, \( \zeta_k(t) \) is...
a dynamical phase (as we will show irrelevant for our calculations), and
\[
\alpha_k(t) \equiv -\frac{C_k}{2\sqrt{2m_k\omega_k}} \left[ e^{i(\omega_k + \Omega)t} - 1 \right] \left[ e^{i(\omega_k - \Omega)t} - 1 \right]
\]
(10)
for the momentum squeezing and
\[
\alpha_k(t) \equiv -\frac{C_k}{2\sqrt{2m_k\omega_k}} \left[ e^{i(\omega_k + \Omega)t} - 1 \right] \left[ e^{i(\omega_k - \Omega)t} - 1 \right]
\]
(11)
for the position squeezing.

**Dynamical spectrum broadcast structure.** – The formation of SBS (1) is equivalent to: i) decoherence and ii) perfect distinguishability of post-interaction environmental states [5]. We study the evolved S : E state under the approximations described in the previous section and after tracing over a fraction \((1 - f)E\), \(f \in (0, 1)\), of the environment that passes unobserved and is necessary for the decoherence: \(g_{S: E(t)} \equiv \text{tr}_{(1-f)E} g_{S: E(t)}\), \(g_{S: E(t)} \equiv \hat{U}_{S: E(t)}(\phi_0) \otimes \otimes_k \delta_0 \hat{U}_{S: E(t)}^\dagger\). We assume the environment to be initially in a thermal state so that all \(\delta_0\)'s are thermal states with the same temperature \(T\) (later we will generalize to arbitrary single-mode Gaussian states). Although (6) is formally written with a continuous distribution of \(X_0\), it in fact stands for a limit of finite divisions \(\Delta\) of the real number \(R\), with \(|X_0|/\Delta\) approximated by orthogonal projectors \(\Pi_{\Delta}\) on the intervals \(\Delta\) (see, e.g., [20]). From (6)–(8) we obtain
\[
\begin{align*}
g_{S: E(t)} &= \sum_{\Delta \neq \Delta'} e^{-i\hat{H}_E t} \Pi_{\Delta}(\phi_0) \Pi_{\Delta'} \otimes_k \delta_0 (X_0; t^f) \\
&+ \sum_{\Delta \neq \Delta'} \Gamma_{X_0, \Delta'}(t) e^{-i\hat{H}_E t} \Pi_{\Delta}(\phi_0) \Pi_{\Delta'} e^{i\hat{H}_E t} \\
&\otimes \otimes_k \hat{U}_k (X_0; t^f) \delta_0 \hat{U}_k (X_0; t^f),
\end{align*}
\tag{12}
\]
where \(f \in N\) denotes the number of observed oscillators, \(X_0\) is some position within \(\Delta\), and
\[
\begin{align*}
\delta_0 (X_0; t) &\equiv \hat{U}_k (X_0; t) \delta_0 \hat{U}_k (X_0; t)^\dagger, \\
\Gamma_{X_0, \Delta'}(t) &\equiv \prod_{k \in (1-f)E} \text{tr} \left[ \hat{U}_k (X_0; t) \delta_0 \hat{U}_k (X_0; t)^\dagger \right].
\end{align*}
\tag{13}
\tag{14}
\]
the latter being the decoherence factor due to the traced fraction \((1 - f)E\) of the environment (for compactness we denote the system’s initial position by \(X\) rather than \(X_0\)). It governs the vanishing of the off-diagonal part in (12) in the trace-norm [5]. A closed formula for \(\Gamma_{X_0, \Delta'}(t)\) for general initial states \(\delta_0\) is possible, using the fact [21] that one can always write \(\delta_0 = (1/\sqrt{\pi}) \int d^2 \alpha P_k(\alpha) \alpha(\alpha)\), where \(|\alpha\rangle\) are the usual coherent states [19] and \(P_k(\alpha)\) is in general a distributional Glauber-Sudarshan P-representation:
\[
\Gamma_{X_0, \Delta'}(t) \equiv \prod_{k \in (1-f)E} e^{-\frac{|\alpha_k(t)|^2}{2}(X-X')^2} \times \left| \int \frac{dqdp}{\pi} P_k(q, p) e^{2i(X-X') [q \text{Im} \alpha_k(t) - p \text{Re} \alpha_k(t)]} \right|.
\tag{15}
\]
(16)
for an initial momentum squeezed state of \(S\) (cf. (10)). For thermal states at temperature \(T\), \(P_k(q, p) = (1/\sqrt{\pi}) e^{-(q^2 + p^2)/\Delta T}\), \(\Delta T = 1/\sqrt{\Delta k}\), \(\beta = T/k_B\) and the corresponding decoherence factor is given by [13]
\[
\left| \Gamma_{X_0, \Delta'}(t) \right| \equiv \prod_{k \in (1-f)E} \text{exp} \left[ -\frac{(X-X')^2}{2} |\alpha_k(t)|^2 \text{cosh} \left( \frac{\beta \omega_k}{2} \right) \right],
\tag{17}
\]
where \(\text{cosh}(\cdot)\) is the hyperbolic cotangent. From (16) it is clear that bands near the resonant mode \(\omega_k \approx \Omega\) would be enough to effectively decohere the system [14,15]. But here we want to study the opposite, more subtle, regime where a single mode has a very small influence on the system’s coherence. This motivates the condition (4). Due to discrete and random \(\omega_k\)’s, \(\Gamma_{X_0, \Delta'}(t)\) is in our study an almost periodic function of time [22]. We analyze it later.

Next, we turn to the diagonal part in (12), reverting to the continuum limit. We group the observed environment \(fE\) into \(\mathcal{M}\) macro-fractions of an equal size of \(\mathcal{N}/\mathcal{M}\) oscillators each [4,5] and show that there is a regime where the states of each macro-fraction (cf. (13)) \(\varrho_{\text{mac}}(X; t) \equiv \otimes_{k \in \text{mac}} \delta_0 (X; t)\) become perfectly distinguishable for different \(X \in \text{mac}\) running through the oscillators in a given macro-fraction \(\text{mac}\). We use the generalized overlap [23]:
\[
B(\varrho_1, \varrho_2) \equiv \text{tr} \sqrt{\varrho_1 \varrho_2 \varrho_1 \varrho_2},
\tag{18}
\]
as the most convenient measure of distinguishability (cf. (11)): \(\varrho_1\) and \(\varrho_2\) are perfectly distinguishable, \(\varrho_1 \varrho_2 = 0\), if and only if \(B(\varrho_1, \varrho_2) = 0\). A calculation for thermal \(\varrho_0\)’s gives (see appendix)
\[
B_{X, X'}(t) = \prod_{k \in \text{mac}} \text{exp} \left[ -\frac{(X-X')^2}{2} |\alpha_k(t)|^2 \text{cosh} \left( \frac{\beta \omega_k}{2} \right) \right],
\tag{19}
\]
where \(B_{X, X'}(t) \equiv B[\varrho_{\text{mac}}(X; t), \varrho_{\text{mac}}(X'; t)]\) measures the distinguishability of the system’s initial positions \(X, X'\) as recorded into macro-fractions. Note, however, that the states \(\varrho_{\text{mac}}(X; t)\) depend not only on \(X\), but on the whole classical motion through (6). From (17), (19) \(\lim_{t_\rightarrow \infty} \Gamma_{X_0, \Delta'}(t) = 0\), i.e. hot environments decohere the central system better, as but \(\lim_{t_\rightarrow \infty} B_{X, X'}(t) = 1\) they are unable to discriminate its positions, irrespectively of the observed macro-fraction size —hot environments are too noisy (the initial states \(\varrho_0\) are too close to the maximally mixed state) to store any information (cf. (13)). Note that the factor \(\text{th}(\beta \omega_k/2)\), appearing in both (17), (19), is nothing else but the purity \(\text{tr}(\varrho_{0k}^2)\).
Numerical analysis. – We first analyze the case when the system \( S \) initially in a momentum squeezed state. Both \( |\Gamma_{X,X'}(t)| \) and \( B_{X,X'}^{\text{mac}}(t) \) depend on the same almost periodic function of time (16), too complicated for an immediate analytical study. In this work we analyze it numerically. We set \( M = 10^{-5} \text{kg} \), \( \Omega = 3 \times 10^8 \text{s}^{-1} \), \( \omega_k \)'s independently, identically and uniformly distributed in the interval \( 3 \ldots 6 \times 10^9 \text{s}^{-1} \) to satisfy (4), and \( |X-X'| = 10^{-15} \text{m} \). We assume that \( C_k \) depend only on the masses \( C_k \equiv 2\sqrt{(Mm_\gamma\gamma_0)/\pi} \), and \( \gamma_0 = 0.33 \times 10^{19} \text{s}^{-4} \) is a constant. We assume a symmetric situation: The size of the traced macro-fraction \( (1-f)|E\rangle \) in (17) is the same as the size of the observed one \( \text{mac} \) in (19). Intuitively, for large enough macro-fractions for a given \( T \), \( |\Gamma_{X,X'}(t)| \) and \( B_{X,X'}^{\text{mac}}(t) \) should decay rapidly and have small typical fluctuations due to the large amount of random phases in (16), indicating decoherence and perfect distinguishability. This is confirmed in fig. 1. From figs. 1(b), (d) we see that for 30 oscillators both functions decay rapidly, while for 10 oscillators they do not —the macro-fraction is too small for a given \( T \).

We further analyze, fig. 2, the time averages \( \langle|\Gamma_{X,X'}|\rangle = (1/\tau) \int_0^\tau \textrm{d}t|\Gamma_{X,X'}(t)| \), \( \langle B_{X,X'}^{\text{mac}}(t) \rangle \) as functions of the temperature \( T \) with \( \tau \) taken large (\( \sim 1 \text{s} \)): Since both functions are non-negative, the vanishing of their time averages is a good indicator of the functions having small typical fluctuations above zero. From fig. 2(a) one sees that, in the chosen parameter range, there is no formation of the broadcast state for a macro-fraction of 10 oscillators: While \( \langle|\Gamma_{X,X'}|\rangle \approx 0 \) (the lower trace) for \( T \approx 10^{-1} \text{K} \), \( \langle B_{X,X'}^{\text{mac}} \rangle \approx 0.6 \) (the upper trace). The state decoheres, but at too high a temperature to store a perfect record of the system’s position. From (12), the post-interaction partial state is then of a, so-called, Classical-Quantum (CQ) type [11]. However, increasing the size to 30 oscillators both traces become practically zero up to \( T \approx 10^{-2} \text{K} \), as one sees from fig. 2(b) (cf. fig. 1(b), (d)). This serves as a numerical evidence of the formation of the spectrum broadcast structure (1), and hence objectivisation [4], in the Quantum Brownian Motion model with a massive central system, initiated in a highly momentum-squeezed state, \( i.e. \) possessing large coherences in the position. This is our main result.

The situation with initial position squeezing, for which the trajectories are given by \( X(t); X_0 = X_0 \sin(\Omega t) \) is quite different. Under exactly the same conditions as above there is no decoherence neither orthogonalization for macro-fractions of both 10 and 30 oscillators as fig. 3 shows. Actually the plots suggest that both functions are periodic in time (even increasing the macro-fraction size to 100), so there is a periodic revival of coherence. This in general agrees with the findings of [15].

Dynamical objectivity. – Let us assume that a SBS is formed, \( i.e. \) both \( |\Gamma_{X,X'}(t)| \) and \( B_{X,X'}^{\text{mac}}(t) \) approach zero. Then from (12) (taking the usual continuum limit of the sum):

\[
\varrho_{S;E(t)} = \int \text{d}X_0 \ |\langle X_0 |\phi_0 \rangle|^2 
\times |\langle X(t) \rangle| \langle X(t) \rangle \otimes \varrho_{\text{mac}}(X_0; t) \otimes \cdots \otimes \varrho_{\text{mac}}(X_N; t),
\]

and

\[
|\langle X(t) \rangle| \equiv e^{-iH_{\text{SBS}}(X_0)},\]

we have grouped \( |E \) into \( \mathcal{M} \) macro-fractions and \( \varrho_{\text{mac}}(X_0; t) \) have orthogonal supports (for large enough \( t \); cf., e.g., fig. 1(d)). What appears in (20) is a novel, compared to the previous studies [4,5],

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dynamical spectrum broadcast structure (dSBS). Because the system now has its own dynamics, the pointers \(|X(t)|\) are now states of motion—they evolve on a time scale \(t_2 \sim 2 \pi/\Omega\), rather than being static as in [5], and a time-dependent SBS is formed with a reference to these evolving pointers. For the example studied in the previous section, the respective time scales are \(t_2 \sim 2 \times 10^{-8} \text{s}\) and from fig. 1(b), (d) \(t_{\text{SBS}} \sim 2 \times 10^{-10} \text{s}\) so that the SBS is formed two orders of magnitude faster than the intrinsic system evolution. Thanks to it, all the observers will measure the same initial position (= the oscillation amplitude) \(X_0\), leaving the (by now decohered) system undisturbed in its state of motion. But the traces of this motion are present in the environment not only through \(X_0\)—each state \(\varrho_{\text{mac}}(X_0; t)\) depends on the whole trajectory \(X(t; X_0)\) (cf. (5)).

The intuitive picture is that while the system rotates on its intrinsic time scale, the environment follows this movement and past the transient period a spectrum broadcast structure is being continuously formed, leading to a perception of objective position at each moment of time. Of course due to the neglected back reaction on the system, the structure (20) is only a first approximation to this situation, as e.g. there is no dynamical production of coherences in the system’s position. The next logical step would be to include the back reaction.

**General Gaussian initial states.**—We recall [24] that an arbitrary single-mode Gaussian state can be parametrized as: 
\[ \varrho = e^{i\varphi a^\dagger a} D(\gamma) S(\xi) \varrho_T S(\xi)^\dagger \tilde{D}(\gamma) \tilde{e}^{i\varphi a^\dagger a}, \]
where \(S(\xi) \equiv e^{(\xi^2 - \xi a^2)/2}, \xi \equiv re^{i\vartheta}\), and \(\varrho_T\) is some thermal state. Parametrizing each \(\varrho_{\text{th}}\) as above leads to the same expressions (17), (19) but with \(\alpha_k(t)\) (cf. (10)) substituted by \(\tilde{\alpha}_k(t) \equiv \cosh r \left[ e^{-i\varphi \alpha_k(t)} - e^{i(\varphi + \vartheta)\alpha_k(t)} \right] \tanh r \). The introduction of a squeezing increases the temperature range where a dynamical SBS can be formed via increasing the informational capacity of the environment, e.g. for \(r = 5\), the temperature range is increased up to \(T = 1\) K [25].

**Concluding remarks.**—Our findings generally agree with that of [14,15] in that there is a parameter range in QBM such that objectivity appears, but it has been obtained with a deeper analysis directly on quantum states, uncovering previously unnoticed dynamical spectrum broadcast structures. Our method, although developed here in a specific model, is in fact much more universal and can be generalized to test other decoherence models for a presence of dynamical forms of objectivity: One checks if states of the type (20) are formed during the evolution. One immediate generalization is to allow for other trapping potentials than harmonic and other couplings than linear (see, e.g., [26]). Another, is to study finite-dimensional systems, e.g., spins [27], but a far more challenging generalization would be the application to quantum fields, leading to objective dynamical classical fields. Finally, a possible connection between Markovianity/non-Markovianity of the evolution and the formation of broadcast structures can also be studied [28].

**The generalized overlap \(B_{\text{mac}}^{\text{mic}}(X'; t)\) for thermal environment states.**—We calculate
\[ B_{X,X'}^{\text{mac}}(t) \equiv \text{tr} \varrho_{\text{mac}}(X; t) \varrho_{\text{mac}}(X'; t), \]
for \(\varrho_{\text{mac}}(X; t) \equiv \bigotimes \varrho_k U(X_k; t)\varrho_k U(X_k; t)^\dagger\) and \(\varrho_k\) thermal. The above distinguishability measure [23] scales with the tensor product \(B(\bigotimes \varrho_k, \bigotimes \varrho_k) = \prod_k B(\varrho_k, \varrho_k)\), so that it is enough to calculate it for a single environment.

Dropping the explicit dependence on \(k\) and denoting a single-system overlap by \(B_{X,X'}^{\text{mic}}(t)\) we obtain \(B_{X,X'}^{\text{mic}}(t) = \text{tr} \sqrt{\varrho_0 U(X'; t)\varrho_0 U(X; t)^\dagger U(X; t)\varrho_0 U(X'; t)^\dagger}\), where we have pulled the extreme left and right unitaries out of both square roots and used the cyclic property of the trace to cancel them out. Thus, modulo phase factors: 
\[ U(X'; t) \varrho_0 U(X; t) \equiv \text{tr} (\psi(0)(X' - X)) \equiv \tilde{D}(\gamma). \]
Next, assuming all the \(\varrho_{\text{th}}\) are thermal with the same temperature, we use the \(P\)-representation for the middle \(\varrho_0\) under the square root in \(B_{X,X'}^{\text{mic}}(t)\): 
\[ \varrho_0 = \int d^2\gamma/(\pi\tilde{n}) \exp \left(-|\gamma|^2/\tilde{n}\right) |\gamma\rangle \langle \gamma|, \quad \tilde{n} = 1/(e^{\varphi_0} - 1), \quad \beta \equiv 1/k_BT. \]
Denoting the Hermitian operator under the square root in \(B_{X,X'}^{\text{mic}}(t)\) by \(\tilde{A}_t\), we obtain \(\varrho_0 = \int d^2\gamma/(\pi\tilde{n}) e^{-|\gamma|^2/\tilde{n}} \sqrt{|\gamma + \eta\rangle \langle \gamma + \eta|} \). To perform the square roots above we use the Fock representation \(\varrho_0 = \sum_n \left( \tilde{n}^n/\sqrt{(n!)(n+1)^{n+1}} \right) |n\rangle \langle n|\), so that
\[ A_t = \int d^2\gamma \pi/\tilde{n} e^{-|\gamma|^2/\tilde{n}} \sum_n \left( \tilde{n}^n/(n+1)^{n+1} \right)^{\tilde{n}} \langle \eta + \eta \rangle^n \langle n\rangle \langle n| \]
and the scalar products above read \(\langle n|\gamma + \eta\rangle = \exp \left(-|\gamma + \eta|^2/2\right) \left(|\gamma + \eta\rangle^n/\sqrt{n!}\right)\). The strategy is now to use this relation and rewrite each sum in (22) as a coherente state but with a rescaled argument, and then try to rewrite (22) as a single thermal state (with a different mean photon number than \(\varrho_0\)). To this end we note that
\[ e^{-\frac{1}{2}(|\gamma + \eta|^2 n)} \sum_n \left( \frac{n}{\tilde{n} + 1} \right)^n \frac{\tilde{n}^n}{\sqrt{n!}} |n\rangle = \]
\[ e^{-\frac{1}{2}(|\gamma + \eta|^2 n)} \sum_n \left( \frac{n}{\tilde{n} + 1} \right)^n \frac{\tilde{n}^n}{\sqrt{n!}} (\gamma + \eta)^n \]
Substituting this into (22) and reordering gives
\[ A_t = \frac{1}{\tilde{n} + 1} e^{-|\gamma|^2/\tilde{n}} \int d^2\gamma \pi/\tilde{n} e^{-|\gamma|^2/\tilde{n}} (|\gamma + \eta|^2 n)^{\tilde{n}} \]
\[ \times \left( \sqrt{\tilde{n} + 1} (\gamma + \eta) \right) \left( \sqrt{\tilde{n} + 1} (\gamma + \eta) \right). \]

Note that since we are interested in \(\text{tr} \sqrt{A}_t\) rather than \(A_t\) itself, there is the freedom of rotating \(A_t\) by a unitary operator, in particular by a displacement. We now
find such a displacement as to turn (24) into the thermal form. Comparing the exponential under the integral in (24) with the thermal form, we see that the argument of the subsequent coherent states should be proportional to $\gamma + (n) / (1 + 2n) \eta$. Simple algebra gives

$$\left| \sqrt{\frac{n}{n+1}} (\gamma + \eta) \right|^2 \simeq \hat{D} \left( \sqrt{\frac{n}{n+1}} \frac{n+1}{1+2n} \eta \right) \left| \sqrt{\frac{n}{n+1}} (\gamma + \frac{n}{1+2n} \eta) \right|^2 ,$$

where we have omitted the irrelevant phase factor as we are interested in the projector on the above state. Inserting the above relation into (24), dropping the displacements, and changing the integration variable: $\gamma \to \sqrt{n} / (n + 1) (\gamma + (1 + 2n) \eta)$ gives

$$B^\text{nic}_{n,X}(t) = e^{\frac{1}{2} \frac{\alpha^2}{1 + 2n} \frac{1}{\sqrt{1 + 2n}}} \sqrt{\text{tr}} \varrho_{th}(n^2/(1 + 2n)),$$

where $\varrho_{th}(\hat{n})$ is a thermal state with the mean photon number $\bar{n}$. We use the Fock expansion for $\varrho_{th}(\hat{n}^2/(1 + 2n))$:

$$B^\text{nic}_{n,X}(t) = e^{\frac{1}{2} \frac{\alpha^2}{1 + 2n} \frac{1}{\sqrt{1 + 2n}}} \left( 1 + \frac{\bar{n}^2}{1 + 2n} \right)^{-\frac{1}{2}} \left( \frac{1 + 2n}{1 + n^2/(1 + 2n)} \right)^\frac{3}{2} \left( \frac{\beta \omega}{2} \right)^{\frac{1}{2}} \exp \left( -\frac{(X - X')^2}{2} |\alpha(t)|^2 \text{th} \left( \frac{\beta \omega}{2} \right) \right),$$

where we have used the definition of $\eta$ and $\bar{n} = 1/(e^{\beta \omega} - 1)$. Coming back to the generalized overlap for macro-fraction states (21) with the help of (18), we finally obtain the desired result (19).

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REFERENCES

[1] Schilpp P. A. (Editor), Albert Einstein, Philosopher-Scientist: The Library of Living Philosophers, Vol. VII (Open Court, La Salle) 1949.

[2] Joos E. et al., Decoherence and the Appearance of a Classical World in Quantum Theory (Springer, Berlin) 2003; Schlosshauer M., Decoherence and the Quantum-To-Classical Transition (Springer, Berlin) 2007.

[3] Fine A., The Shaky Game (The University Of Chicago Press, Chicago) 1986.

[4] Horodecki R., Korbicz J. K. and Horodecki P., Phys. Rev. A, 91 (2015) 032122.

[5] Korbicz J. K., Horodecki P. and Horodecki R., Phys. Rev. Lett., 112 (2014) 120402.

[6] Zurek W. H., Nat. Phys., 5 (2009) 181.

[7] Brandao F. G. S. L., Piani M. and Horodecki P., Nat. Commun., 6 (2015) 7908.

[8] Zurek W. H., Phys. Rev. D, 24 (1981) 1516.

[9] Korbicz J. K., Horodecki P. and Horodecki R., Phys. Rev. A, 86 (2012) 042319.

[10] Barnum H., Cavies C. M., Fuchs C. A., Jozsa R. and Schumacher B., Phys. Rev. Lett., 76 (1996) 2818.

[11] Piani M., Horodecki P. and Horodecki R., Phys. Rev. Lett., 100 (2008) 090502.

[12] Ullersma P., Physica, 32 (1966) 27.

[13] Breuer H.-P. and Petruccione F., The Theory of Open Quantum Systems (Oxford University Press, Oxford) 2002.

[14] Blume-Kohout R. and Zurek W. H., Phys. Rev. Lett., 10, (2008) 240405.

[15] Paz J. P. and Roncalli A. J., Phys. Rev. A, 80 (2008) 042111.

[16] Zurek W. H., Phys. Rev. D, 26 (1982) 1862.

[17] It is analogous to, e.g., the large wavelength condition in the illuminated sphere model, where photon wavelengths are much larger than the separation of the possible positions of the sphere; see, e.g., [5] and Joos E. and Zeh H. D., Z. Phys. B Condens. Matter, 59 (1985) 223.

[18] Jasper A. W. and Truhlar D. G., in Conical Intersections: Theory, Computation and Experiment, edited by Domcke W., Yarkony D. R. and Köppl H. (World Scientific, Singapore) 2011.

[19] Perelomov A., Generalized Coherent States and Their Applications (Springer, Berlin) 1986.

[20] Galindo A. and Pascual P., Quantum Mechanics, Vol. I (Springer, Berlin) 1990.

[21] Miller M. M. and Misikin E. A., Phys. Rev., 164 (1967) 1610.

[22] Bescovitch A. S., Almost Periodic Functions (Dover) 1954.

[23] Fuchs C. A. and van de Graaf J., IEEE Trans. Inf. Theory, 45 (1999) 1216.

[24] Weedbrook C., Pirandola S., García-Patron R., Cerf N. J., Ralph T. C., Shapiro J. H. and Lloyd S., Rev. Mod. Phys., 84 (2012) 621.

[25] Tuziemski J. and Korbicz J. K., Photonics, 2 (2015) 228.

[26] Massignan P., Lampo A., Wehr J. and Lewenstein M., Phys. Rev. A, 91 (2015) 033627.

[27] Miranowicz P., Horodecki P. and Korbicz J. K., in preparation.

[28] Galve F., Zambrini R. and Maniscalco S., arXiv:1412.3316 (2014).