DYNAMICAL ALTERNATING GROUPS, STABILITY, PROPERTY GAMMA, AND INNER AMENABILITY

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Abstract. We prove that the alternating group of a topologically free action of a countably infinite group \( \Gamma \) on the Cantor set has the property that all of its \( \ell^2 \)-Betti numbers vanish and, in the case that \( \Gamma \) is amenable, is stable in the sense of Jones and Schmidt and has property Gamma (and in particular is inner amenable). We show moreover in the realm of amenable \( \Gamma \) that there are large classes of such alternating groups which are simple, finitely generated, and nonamenable, and in many cases also \( C^* \)-simple. Among the tools used in constructing some of these classes is a topological version of Austin’s result on the invariance of measure entropy under bounded orbit equivalence.

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1. Introduction

A \( \Pi_1 \) factor is said to have property Gamma if it has an asymptotically central net of unitaries with trace zero. This concept was introduced by Murray and von Neumann in order to show that there exists a \( \Pi_1 \) factor, in this case the group von Neumann algebra \( \mathcal{L}F_2 \) of the free group on two generators, which is not isomorphic to the hyperfinite \( \Pi_1 \) factor \([39]\). While the stronger property of injectivity was later identified as the von Neumann algebra analogue of amenability for groups, and indeed as being equivalent to this amenability in the case of group von Neumann
algebras, Murray and von Neumann’s argument nevertheless hinted at a connection to amenability through its use of almost invariant measures and paradoxicality in a way that, as they pointed out, paralleled Hausdorff’s paradoxical decomposition of the sphere. In fact property Gamma has come, perhaps surprisingly, to play a significant role in the study of operator-algebraic amenability. It appears as a technical tool in Connes’s celebrated work on the classification of injective factors [9], and its C*-algebraic analogue has very recently been used to establish the equivalence of finite nuclear and \( \mathcal{Z} \)-stability in the Toms–Winter conjecture [5]. The explicit connection to group-theoretic amenability was made by Effros [17], who showed that, for an ICC discrete group \( G \), if the group von Neumann algebra \( \mathcal{L}G \) (which in this case is a II\(_1\) factor) has property Gamma then \( G \) is inner amenable, meaning that there exists an atomless finitely additive probability measure on \( G \setminus \{1_G\} \) which is invariant under the action of \( G \) by conjugation. The converse of this implication turns out to be false, however, as was shown by Vaes in [48].

Let us say for brevity that an ICC discrete group \( G \) has property Gamma if \( \mathcal{L}G \) has property Gamma. When such a group is nonamenable it exhibits the kind of behavior that brings it under the compass of Popa’s deformation/rigidity program. Indeed Peterson and Sinclair showed in [42] that if \( G \) is countable and nonamenable and has property Gamma then each of its weakly mixing Gaussian actions, including its Bernoulli action over a standard atomless base, is \( \mathcal{U}_{\text{fin}} \)-cocycle superrigid, i.e., every cocycle taking values in a Polish group that embeds as a closed subgroup of the unitary group of some II\(_1\) factor, and in particular every cocycle into a countable discrete group, is cohomologous to a homomorphism. This implies that if \( G \) has no nontrivial finite normal subgroups then each of its weakly mixing Gaussian actions is orbit equivalence superrigid, i.e., any orbit equivalence with another ergodic p.m.p. action of a countable discrete group implies that the actions are conjugate modulo an isomorphism of the groups. In [47] Ioana and the second author strengthened Peterson and Sinclair’s result by showing that the conclusion holds more generally if property Gamma is replaced by inner amenability. In Corollary D of [7] it was shown that inner amenability implies the vanishing of the first \( \ell^2 \)-Betti number, which is a necessary condition for \( \mathcal{U}_{\text{fin}} \)-cocycle superrigidity (Corollary 1.2 of [42]) and hence can also be derived from [47].

Inner amenability is also related to Jones and Schmidt’s notion of stability for p.m.p. equivalence relations [22], which is an analogue of the McDuff property for II\(_1\) factors. We say that an ergodic p.m.p. equivalence relation is JS-stable if it is isomorphic to its product with the unique ergodic hyperfinite p.m.p. equivalence relation. A countable group is said to be JS-stable if it admits a free ergodic p.m.p. action whose orbit equivalence relation is JS-stable. Just as McDuff implies property Gamma in the II\(_1\) factor setting, JS-stability for a group implies inner amenability (Proposition 4.1 of [22]). However, there exist ICC groups with property Gamma which are not JS-stable, as shown by Kida [28]. Moreover, while JS-stability for an action implies that the crossed product is McDuff, JS-stability for a group does not
imply that the group has property Gamma \[30\]. We refer the reader to \[14\] for more
an extensive discussion of the relations between all of these properties.

In this paper we show that actions of amenable groups on the Cantor set provide a
rich source of groups which have property Gamma and are JS-stable, and that large
classes of these groups are simple, finitely generated, and nonamenable, and often
even C\(^*\)-simple.

As mentioned above, inner amenability implies the vanishing of the first \(\ell^2\)-Betti
number. It is also known that JS-stability implies the vanishing of all \(\ell^2\)-Betti num-
bers, for in this case the group is measure equivalent to a product of the form \(H \times \mathbb{Z}\)
for some group \(H\), which has the property that all of its \(\ell^2\)-Betti numbers vanish
\[6\], and the vanishing of all \(\ell^2\)-Betti numbers is an invariant of measure equivalence
\[19\]. We will begin by proving in Theorem 3.2 that for every action \(\Gamma \acts \times X\) of a
countably infinite group (amenable or not) on the Cantor set, every infinite subgroup
of the topological full group containing the alternating group has the property that
all of its \(\ell^2\)-Betti numbers vanish. We then show in Theorem 4.7 that, for an ac-
tion \(\Gamma \acts \times X\) of a countable amenable group on the Cantor set, every such subgroup of the
topological full group is JS-stable. To carry this out we use the stability sequence
criterion for JS-stability due to Jones and Schmidt in the free ergodic case \[22\] and
to Kida more generally \[29\] along with an invariant random subgroup generalization
of a result from \[47\] that yields a stability sequence from the existence of fiberwise
stability sequences in an equivariant measure disintegration of an action whose orbit
equivalence relation is hyperfinite (Theorem 4.4).

In Theorem 5.5 we establish property Gamma for the topological full group, and
every subgroup thereof containing the alternating group \(A(\Gamma, X)\), of a topologically
free action \(\Gamma \acts \times X\) of a countable amenable group on the Cantor set (the ICC condi-
tion is automatic in this case by Proposition 5.1). As a corollary these subgroups of
the topological full group are inner amenable, and in Section 6 we provide two direct
proofs of this inner amenability whose techniques may be of use in other contexts. One
of these proofs does not require topological freeness and yields inner amenabil-
ity merely assuming that the subgroup is infinite, which is automatic if \(A(\Gamma, X)\) is
nontrivial.

Perhaps most interesting is the situation when the action \(\Gamma \acts \times X\) is minimal
and expansive, as the first of these conditions implies that the alternating group is
simple and the second implies, under the assumption that \(\Gamma\) is finitely generated, that
the alternating group finitely generated \[40\]. Under these additional hypotheses it is
possible for the alternating group not to be amenable, as Elek and Monod constructed
in \[18\] a free minimal expansive \(\mathbb{Z}^2\)-action whose topological full group contains a copy
of \(F_2\), and in this case the alternating group coincides with the commutator subgroup
of the topological full group and hence is nonamenable. We thereby obtain an example
of a simple finitely generated nonamenable group with property Gamma, and in
particular an example of a simple finitely generated nonamenable inner amenable
group, which answers a question that was posed to the second author by Olshanskii.
It has also recently been shown by Szőke that every infinite finitely generated group which is not virtually cyclic admits a free minimal expansive action on the Cantor set whose topological full group contains $F_2$.

We push this line of inquiry further by constructing large classes of minimal expansive actions $\Gamma \curvearrowright X$ whose alternating groups are nonamenable and in many cases even $\mathbb{C}^*$-simple. In one direction, we combine the kind of construction used by Elek and Monod with methods of Thomas from [46] to show that the relation of continuous orbit equivalence on the space of free minimal Toeplitz subshifts of $\{0, \ldots, 35\}^\mathbb{Z}$ is not smooth (Corollary 7.7). This implies that the isomorphism relation on the space of all finitely generated simple nonamenable groups with property Gamma is not smooth (Corollary 7.8).

In another direction, we show that many torsion-free countable amenable groups, including those that are residually finite and possess a nontorsion element with infinite conjugacy class, admit an uncountable family of topologically free minimal actions on the Cantor set whose alternating groups are $\mathbb{C}^*$-simple (and in particular nonamenable) and pairwise nonisomorphic (Theorem 9.9). These actions are constructed so as to have different values of topological entropy, so that the pairwise nonisomorphism follows by combining the following two facts:

(i) the alternating groups of two minimal actions are isomorphic if and only if the actions are continuously orbit equivalent,

(ii) for topologically free actions of countable amenable groups which are not virtually cyclic, topological entropy is an invariant of continuous orbit equivalence.

The first of these follows from a reconstruction theorem of Rubin [43] and an observation of Li [33] (see Theorem 2.4). The second is a topological version of a theorem of Austin on measure entropy and bounded orbit equivalence [2]. In the setting of finitely generated groups and genuinely free actions, and in some other situations relevant to our application (see the introduction to Section 8), assertion (ii) is a consequence of Austin’s result and the variational principle. We will need however the more general version, which we establish in Section 8 using the same geometric ideas underlying Austin’s approach in the measure setting. The $\mathbb{C}^*$-simplicity (i.e., simplicity of the reduced $\mathbb{C}^*$-algebra, which implies nonamenability) is verified by applying a criterion of Le Boudec and Matte Bon [32] for groups of homeomorphisms which relies on a characterization of $\mathbb{C}^*$-simplicity in terms of uniformly recurrent subgroups due to Kennedy [26]. Le Boudec and Matte Bon applied their criterion to show, among other things, that the full group of a free minimal action of a nonamenable group on the Cantor set is $\mathbb{C}^*$-simple. The novelty in our case is that we are working within the regime of amenable acting groups. The construction itself relies on the kind of recursive blocking technique that was used in [34] to construct minimal $\mathbb{Z}$-actions with prescribed mean dimension.
It should be noted that, for actions of countable amenable groups on the Cantor set, the topological full group and its alternating subgroup are often amenable. A celebrated theorem of Juschenko and Monod says that this is the case for all minimal $\mathbb{Z}$-actions on the Cantor set \cite{23}. It is also the case if the action is free, minimal, and equicontinuous (Theorem 1.3 of \cite{12}) or if the action is free and minimal and the acting group is virtually cyclic \cite{45}. On the other hand, the alternating group of the shift action $\mathbb{Z} \curvearrowright \{1, \ldots, q\}^\mathbb{Z}$ for $q \geq 4$ contains a copy of $F_2$. In fact the strategy of the construction in the proof of Theorem 9.7 involves the embedding of such a shift along some nontorsion direction in the acting group, which can be done in a way that ensures minimality as long as the group is not virtually cyclic. Using an idea from \cite{49} as in the construction of Elek and Monod (see also the discussion in Section 3.7 of \cite{11}), one then exploits the shift structure to show that the free product $\mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2$, which contains a copy of $F_2$, embeds into the alternating group.

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2. Dynamical alternating groups and continuous orbit equivalence

Throughout the paper group actions on compact spaces will always be understood to be continuous. Let $\Gamma$ be a countable discrete group and $\Gamma \curvearrowright X$ an action on a compact metrizable space. We will express the action using the concatenation $\Gamma \times X \ni (s,x) \mapsto sx$, and in the occasional case that we need to notationally distinguish the action from others by giving it a name $\alpha$ we may also write $(s,x) \mapsto \alpha_s x$. The action is said to be topologically free if there is a dense set of points in $X$ with trivial stabilizer. Two actions $\Gamma \curvearrowright X$ and $\Gamma \curvearrowright Y$ of $\Gamma$ on compact metrizable spaces are topologically conjugate if there exists a homeomorphism $\phi : X \to Y$ such that $\phi(sx) = s\phi(x)$ for all $s \in \Gamma$ and $x \in X$. Such a homeomorphism is called a topological conjugacy.

The topological full group $[[\Gamma \curvearrowright X]]$ of the action is the group of all homeomorphisms $T : X \to X$ with the property that for each $s \in \Gamma$ there exists a clopen partition $\{A_1, \ldots, A_n\}$ of $X$ and $s_1, \ldots, s_n \in G$ such that $Tx = s_ix$ for every $i = 1, \ldots, n$ and $x \in A_i$. This group is countable since $\Gamma$ is countable and there are at most countably many clopen partitions of $X$.

Let $d \in \mathbb{N}$. Consider the collection of all group homomorphisms $\varphi$ from the symmetric group $S_d$ on $\{1, \ldots, d\}$ to $[[\Gamma \curvearrowright X]]$ with the property that there are pairwise disjoint clopen sets $A_1, \ldots, A_d \subseteq X$ such that for every $\sigma \in S_d$ the homeomorphism $\varphi(\sigma)$ acts as the identity off of $A_1 \sqcup \cdots \sqcup A_d$ and $\varphi(\sigma)A_i = A_{\sigma(i)}$ for all $i = 1, \ldots, d$. Define $S_d(\Gamma, X)$ to be the subgroup of $[[\Gamma \curvearrowright X]]$ generated by the images of all of these homomorphisms, and $A_d(\Gamma, X)$ to be the subgroup of $[[\Gamma \curvearrowright X]]$ generated by
the images of the alternating group \( A_d \subseteq S_d \) under all of these homomorphisms. If the action has a name \( \alpha \) then we also write \([\alpha]\).

**Definition 2.1** ([40]). The *symmetric group* \( S(\Gamma, X) \) and *alternating group* \( A(\Gamma, X) \) of the action \( \Gamma \curvearrowright X \) are defined to be \( S_2(\Gamma, X) \) and \( A_3(\Gamma, X) \), respectively. If the action has a name \( \alpha \) then we also write \( S(\alpha) \) and \( A(\alpha) \).

If the action has no finite orbits then \( S(\Gamma, X) = S_d(\Gamma, X) \) for every \( d \geq 2 \) and \( A(\Gamma, X) = A_d(\Gamma, X) \) for every \( d \geq 3 \) (Corollary 3.7 of [40]).

The alternating group \( A(\Gamma, X) \) is contained in the commutator subgroup \( [[\Gamma \curvearrowright X]]' \) of \( [[\Gamma \curvearrowright X]] \) and is equal to it when the action is almost finite [36, 40] (see the discussion in Section 4 of [40]).

Recall that the action \( \Gamma \curvearrowright X \) is *expansive* if, fixing a compatible metric \( d \) on \( X \), there is an \( \varepsilon > 0 \) such that for all distinct \( x, y \in X \) there is an \( s \in \Gamma \) for which \( d(sx, sy) \geq \varepsilon \). When \( X \) is zero-dimensional, expansivity is equivalent to the action being topologically conjugate to a right subshift action over a finite alphabet, i.e., the restriction of the right shift action \( \Gamma \curvearrowright \{1, \ldots, n\}^\Gamma \) for some \( n \in \mathbb{N} \), given by \((sx)_t = x_{ts}\), to a closed \( \Gamma \)-invariant subset. The following facts were established by Nekrashevych in Theorem 4.1, Proposition 5.5, and Theorem 5.6 of [40].

**Theorem 2.2** ([40]).

(i) If the action \( \Gamma \curvearrowright X \) is minimal then \( A(\Gamma, X) \) is simple.

(ii) If \( \Gamma \) is finitely generated and the action \( \Gamma \curvearrowright X \) is expansive and has no orbits of cardinality less than 5 then \( A(\Gamma, X) \) is finitely generated.

Next we define continuous orbit equivalence and record its relationship to the symmetric and alternating groups.

**Definition 2.3.** Two actions \( G \curvearrowright X \) and \( H \curvearrowright Y \) on compact metrizable spaces are **continuously orbit equivalent** if there are a homeomorphism \( \Phi : X \to Y \) and continuous maps \( \kappa : G \times X \to H \) and \( \lambda : H \times Y \to G \) such that

\[
\Phi(\alpha_s x) = \beta_{\kappa(s,x)} \Phi(x),
\Phi^{-1}(\beta_t y) = \alpha_{\lambda(t,y)} \Phi^{-1}(y)
\]

for all \( s \in G, x \in X, t \in H, \) and \( y \in Y \).

It is clear that topological conjugacy implies continuous orbit equivalence. If the action \( \beta \) is topologically free, then the map \( \kappa \) is uniquely determined by the first line of the above display since \( \Phi \) is continuous, and one can moreover verify the cocycle identity

\[
\kappa(rs, x) = \kappa(r, \alpha_s x)\kappa(s, x)
\]

for all \( r, s \in G \) and \( x \in X \). If \( \alpha \) and \( \beta \) are both topologically free, then we have

\[
\lambda(\kappa(s, x), \Phi(x)) = s
\]
for all \( s \in G \) and \( x \in X \), and \( \lambda \) is uniquely determined by this identity.

As pointed out in Section 3.3 of [10], the equivalences (i)\( \iff \) (iii)\( \iff \) (iv) in the following theorem are a consequence of a reconstruction theorem of Rubin [43]. The equivalence (i)\( \iff \) (ii) was also established by Matui in Theorem 3.10 of [37]. The equivalence (i)\( \iff \) (v) was observed by Li in Theorem 1.2 of [33].

**Theorem 2.4.** Let \( \Gamma_1 \actson X_1 \) and \( \Gamma_2 \actson X_2 \) be minimal actions of countable groups on compact metrizable spaces. Then the following are equivalent:

(i) \( X_1 \rtimes \Gamma_1 \) and \( X_2 \rtimes \Gamma_1 \) are isomorphic as topological groupoids,

(ii) the groups \([\Gamma_1 \actson X_1]\) and \([\Gamma_2 \actson X_2]\) are isomorphic,

(iii) the groups \( A(\Gamma_1, X_1) \) and \( A(\Gamma_2, X_2) \) are isomorphic,

(iv) the groups \( S(\Gamma_1, X_1) \) and \( S(\Gamma_2, X_2) \) are isomorphic,

(v) the actions \( \Gamma_1 \actson X_1 \) and \( \Gamma_2 \actson X_2 \) are continuously orbit equivalent.

3. \( \ell^2 \)-Betti numbers

A sequence \( (c_n)_{n \in \mathbb{N}} \) of elements of a group \( H \) is said to be asymptotically central in \( H \) if each \( h \in H \) commutes with all but finitely many of the \( c_n \), and nontrivial if \( \{c_n : n \in \mathbb{N}\} \) is infinite. The existence of a nontrivial asymptotically central sequence in \( H \) is equivalent to the centralizer \( C_H(F) \) of each finite set \( F \subseteq H \) being infinite. If there exist two asymptotically central sequences \( (c_n)_{n \in \mathbb{N}} \) and \( (d_n)_{n \in \mathbb{N}} \) in \( H \) which are noncommuting, i.e., which satisfy \( c_n d_n \neq d_n c_n \) for all \( n \in \mathbb{N} \), then it is easy to see that both of these sequences must be nontrivial. The existence of two noncommuting asymptotically central sequences in \( H \) is equivalent to \( C_H(F) \) being non-Abelian for every finite subset \( F \) of \( H \).

Now let \( \Gamma \actson X \) be an action of a countable group on the Cantor set. Let \( G \) be any subgroup of \([\Gamma \actson X]\) containing \( A(\Gamma, X) \). For a finite set \( F \subseteq X \) we write \( G_F \) for the set of all elements in \( G \) which fix a neighborhood of every \( x \in F \). When \( F \) is a singleton \( \{x\} \) we simply write \( G_x \).

**Lemma 3.1.** Let \( \Gamma \actson X \) be an action of a countable group on the Cantor set.

(i) If there is a uniform finite bound on the size of \( \Gamma \)-orbits, then the group \([\Gamma \actson X]\) is locally finite.

(ii) Let \( X_0 \) denote the set of points \( x \in X \) with the property that for every open neighborhood \( U \) of \( x \) and for every \( n \in \mathbb{N} \) there exists some \( y \in X \) such that \( |GY \cap U| \geq n \). The set \( X_0 \) is closed and \( \Gamma \)-invariant, and \( X_0 = \emptyset \) if and only if there is a uniform finite bound on the size of \( \Gamma \)-orbits.

(iii) Suppose that \( X_0 \neq \emptyset \). Let \( x \in X_0 \) and let \( F \) be a finite subset of \( X \) containing \( x \). Let \( G \) be any subgroup of \([\Gamma \actson X]\) containing \( A(\Gamma, X) \). Then \( G_x \) contains two noncommuting asymptotically central sequences, and these sequences can moreover be taken to lie in \( G_F \) and \( A(\Gamma, X) \).

**Proof.** Suppose that every \( \Gamma \)-orbit has size at most \( m \). If \( H \) is a finitely generated subgroup of \([\Gamma \actson X]\), then the intersection \( N \) of all subgroups of \( H \) which have
Definition 4.1. An ergodic discrete p.m.p. equivalence relation $\mathcal{R}$ is said to be JS-stable if $\mathcal{R}$ is isomorphic to the product $\mathcal{R} \otimes \mathcal{R}_0$. A countable group $G$ is said to be JS-stable if it admits a free ergodic p.m.p. action whose orbit equivalence relation is JS-stable.

Theorem 3.2. Let $\Gamma \curvearrowright X$ be an action of a countable group $\Gamma$ on the Cantor set and suppose that $G$ is an infinite subgroup of $[[\Gamma \curvearrowright X]]$ containing $A(\Gamma, X)$. Then all $\ell^2$-Betti numbers of $G$ vanish.

Proof. If there is a uniform finite bound on the size of $\Gamma$-orbits then $G$ is locally finite by (i) of Lemma 3.1 so all $\ell^2$-Betti numbers of $G$ vanish. Assume now that there is no uniform bound on the size of $\Gamma$-orbits. Then the set $X_0$ from (ii) of Lemma 3.1 is nonempty. We first show that for each finite subset $F$ of $X$ which meets $X_0$, the $\ell^2$-Betti numbers of $G(F)$ all vanish. By (iii) of Lemma 3.1 the group $G(F)$ has a nontrivial asymptotically central sequence $(c_n)_{n \in \mathbb{N}}$. After moving to a subsequence we may assume additionally that the $c_n$ for $n \in \mathbb{N}$ pairwise commute, and that the group they generate, which we will call $A$, is an infinite Abelian subgroup of $G(F)$. Note that, for any finite subset $Q$ of $G(F)$, the intersection $\bigcap_{g \in Q} gAg^{-1}$ is infinite since it contains all but finitely many of the $c_n$. It therefore follows from Corollary 1.5 of [3] that all $\ell^2$-Betti numbers of $G(F)$ vanish.

Now fix any $x \in X_0$. Then for any finite subset $Q$ of $G$, letting $F_Q := Qx \subseteq X_0$ we have $\bigcap_{g \in Q} gG(x)g^{-1} = G(F_Q)$, and so all $\ell^2$-Betti numbers of this intersection vanish. We can therefore apply Theorem 1.3 of [3] to conclude that all $\ell^2$-Betti numbers of $G$ vanish. \qed

4. Jones–Schmidt stability

We continue to use the notation and terminology from the last section. In what follows $\mathcal{R}_0$ denotes the unique ergodic hyperfinite type $\Pi_1$ equivalence relation.

Definition 4.1. An ergodic discrete p.m.p. equivalence relation $\mathcal{R}$ is said to be JS-stable if $\mathcal{R}$ is isomorphic to the product $\mathcal{R} \otimes \mathcal{R}_0$. A countable group $G$ is said to be JS-stable if it admits a free ergodic p.m.p. action whose orbit equivalence relation is JS-stable.
Let $G \rtimes (Z, \eta)$ be a p.m.p. action of a countable discrete group. We denote by $[G \rtimes (Z, \eta)]$ the full group of the action, i.e., the collection of all measurable maps $T : Z \to G$ with the property that the transformation $T^0 : Z \to Z$, defined by $T^0(z) := T(z)z$, is an automorphism of the measure space $(Z, \eta)$.

**Definition 4.2.** Let $G \rtimes (Y, \nu)$ be a p.m.p. action of a countable discrete group. A **stability sequence** for the action is a sequence $(T_n, A_n)_{n \in \mathbb{N}}$ with $T_n \in [G \rtimes (Y, \nu)]$ and $A_n$ a measurable subset of $Y$ such that:

1. $\nu(\{y \in Y : T_n(gy) = gT_n(y)g^{-1}\}) \to 1$ for all $g \in G$,
2. $\nu(T_n^0(B)\triangle B) \to 0$ for all measurable $B \subseteq Y$,
3. $\nu(gA_n\triangle A_n) \to 0$ for all $g \in G$,
4. $\nu(T_n^0(A_n)\triangle A_n) \geq \frac{1}{2}$ for all $n$.

The following theorem was originally established by Jones and Schmidt in [22] under the addition hypotheses of freeness and ergodicity, which were later removed by Kida [29].

**Theorem 4.3** ([22, 29]). Suppose that $G$ has a p.m.p. action $G \rtimes (Y, \nu)$ which admits a stability sequence. Then $G$ is JS-stable.

The following generalizes Theorem 17 of [47], by replacing the coamenable normal subgroup used in that theorem with an invariant random subgroup which is coamenable in an appropriate sense.

**Theorem 4.4.** Let $G$ be a countable discrete group and let $\pi : (Y, \nu) \to (X, \mu)$ be a $G$-equivariant measure preserving map between p.m.p. actions $G \rtimes (Y, \nu)$ and $G \rtimes (X, \mu)$. Let $(Y, \nu) = \int_X (Y_x, \nu_x) \, d\mu$ denote the disintegration of $(Y, \nu)$ via $\pi$, so that $Y_x = \pi^{-1}(x)$, and let $G_x$ denote the stabilizer subgroup at a point $x \in X$. Suppose that

(a) the orbit equivalence relation of $G \rtimes (X, \mu)$ is $\mu$-hyperfinitie, and

(b) the action $G_x \rtimes (Y_x, \nu_x)$ admits a stability sequence for $\mu$-almost every $x \in X$.

Then the action $G \rtimes (Y, \nu)$ admits a stability sequence, and hence $G$ is JS-stable.

**Proof.** Let $S$ denote the orbit equivalence relation of $G \rtimes (X, \mu)$. Since $\pi$ is $G$-equivariant and measure preserving, uniqueness of measure disintegrations implies that $\nu_{gx} = g_*\nu_x$ for all $g \in G$ and $\mu$-a.e. $x \in X$. After discarding a null set we may assume that this holds for all $x \in X$, that $S$ is hyperfinite, and also that $G_x \rtimes (Y_x, \nu_x)$ admits a stability sequence for all $x \in X$. Let $F_0 \subseteq F_1 \subseteq \cdots$ be an increasing sequence of finite subsets of $G$ which exhaust $G$. Let $\mathcal{B}_0 \subseteq \mathcal{B}_1 \subseteq \cdots$ be an increasing sequence of finite Boolean algebras of Borel subsets of $Y$ whose union $\mathcal{B} := \bigcup_n \mathcal{B}_n$ generates the Borel $\sigma$-algebra on $Y$. Let $S_0 \subseteq S_1 \subseteq \cdots$ be an increasing sequence of finite Borel subequivalence relations of $S$ with $S = \bigcup_n S_n$. For each $n$ let $X_n \subseteq X$ be a Borel transversal for $S_n$, and for $x \in X$ let $\theta_n(x)$ be the unique element of
Let \( \varphi_n : X \to G \) be any measurable map with \( \varphi_n(x)\theta_n(x) = x \). Let \( \rho_n : G \times X \to G \) be given by \( \rho_n(g, x) := \varphi_n(gx)^{-1}g\varphi_n(x) \), and note that if \( (gx, x) \in S_n \) then \( \varphi_n(gx)^{-1}g\varphi_n(x)\theta_n(x) = \theta_n(x) \), and hence \( \rho_n(g, x) \in G_{\theta_n(x)} \).

For each \( n \) and \( x_0 \in X_n \), since \( G_{x_0} \ind (Y_{x_0}, \nu_{x_0}) \) admits a stability sequence we may find an \( S_n, x_0 \in [G_{x_0} \ind (Y_{x_0}, \nu_{x_0})] \) and a measurable subset \( D_{n, x_0} \) of \( Y_{x_0} \) satisfying:

(i) \( \nu_{x_0}(W_{n, x_0}) > 1 - 1/n \), where \( W_{n, x_0} \) is the set of all \( y \in Y_{x_0} \) such that for all \( g \in F_n \) and \( x \in X \) with \( x, gx \in [x_0]_{S_n} \) one has

\[
S_{n, x_0}(\rho_n(g, x)y) = \rho_n(g, x)S_{n, x_0}(y)\rho_n(g, x)^{-1},
\]

(ii) \( \nu_{x_0}(S_{n, x_0}(\varphi_n(x)^{-1}B) \triangle \varphi_n(x)^{-1}B) < 1/n \) for all \( B \in B_n \) and \( x \in [x_0]_{S_n} \),

(iii) \( \nu_{x_0}(\rho(g, x)^{-1}D_{n, x_0} \triangle D_{n, x_0}) < 1/n \) for all \( g \in F_n \) and \( x \in X \) with \( x, gx \in [x_0]_{S_n} \),

(iv) \( \nu_{x_0}(S_{n, x_0}(D_{n, x_0}) \triangle D_{n, x_0}) \geq 1/2 \).

By a standard measurable selection argument that requires discarding a null set in \( X \), we may additionally assume without loss of generality that the maps \( x_0 \mapsto S_{n, x_0} \) and \( x_0 \mapsto D_{n, x_0} \) are Borel in the sense that the function \( y \mapsto S_{n, \theta_n(\pi(y))}(y) \) from \( Y \) to \( G \) is Borel and \( \{y \in Y : y \in D_{n, \theta_n(\pi(y))}\} \) is a Borel subset of \( Y \).

For each \( n \in \N \) define \( T_n : Y \to G \) and \( A_n \subseteq Y \) by

\[
T_n(y) := \varphi_n(x)S_{n, \theta_n(x)}(\varphi_n(x)^{-1}y)\varphi_n(x)^{-1} \quad \text{for } y \in Y_x, \ x \in X
\]

\[
A_n := \{y \in Y : y \in \varphi_n(x)D_{n, \theta_n(x)} \}, \ \text{where } x = \pi(y).
\]

We will show that \( \{T_n, A_n\}_{n \in \N} \) is a stability sequence for \( G \ind (Y, \nu) \). First note that the map \( T_n^0 : y \mapsto T_n(y)y \) is indeed an automorphism of \( (Y, \nu) \), since for every \( x \in X \) the restriction \( T_n^0 \) to \( Y_x \) is an automorphism of \( (Y_x, \nu_x) \). We are using that \( \varphi_n(x)^{-1}N_{\theta_n(\pi)} = \nu_{\varphi_n(x)^{-1}N_{\theta_n(\pi)}} = \nu_x \). It remains to check that properties (1) through (4) of Definition 1.2 hold for \( \{T_n, A_n\}_{n \in \N} \).

(1): Let \( g \in G \). Given \( x \in X \), for all large enough \( n \) we have \( g \in F_n \) and \( (gx, x) \in S_n \) so that \( \theta_n(gx) = \theta_n(x) \), and hence for all \( y \in \varphi_n(x)W_{n, \theta_n(x)} \subseteq Y_x \) we have

\[
T_n(gy) = \varphi_n(gx)S_{n, \theta_n(x)}(\varphi_n(gx)^{-1}gy)\varphi_n(gx)^{-1}
= \varphi_n(gx)S_{n, \theta_n(x)}(\rho_n(g, x)\varphi_n(x)^{-1}y)\varphi_n(gx)^{-1}
= \varphi_n(gx)\rho_n(g, x)S_{n, \theta_n(x)}(\varphi_n(x)^{-1}y)\rho_n(g, x)^{-1}\varphi_n(gx)^{-1}
= g\varphi_n(x)S_{n, \theta_n(x)}(\varphi_n(x)^{-1}y)\varphi_n(x)^{-1}g^{-1}
= gT_n(y)g^{-1}.
\]

Since by (i) we have \( \nu_x(\varphi_n(x)W_{n, \theta_n(x)}) = \nu_{\theta_n(x)}(W_{n, \theta_n(x)}) > 1 - 1/n \), it follows that (1) holds.

(2): It suffices to show (2) for \( B \in B \). Let \( n \) be large enough so that \( B \in B_n \). For \( x \in X \) we have \( T_n^0(B \cap Y_x) = \varphi_n(x)S_{n, \theta_n(x)}(\varphi_n(x)^{-1}B) \cap Y_x \), and hence by (ii) applied
to $x_0 = \theta_n(x)$ we have
\[
\nu_x(T_n^0(B) \triangle B) = \nu_{\theta_n(x)}(S_{n,\theta_n(x)}^0(\varphi_n(x)^{-1}B) \triangle \varphi_n(x)^{-1}B) < 1/n.
\]

It follows that $\nu(T_n^0(B) \triangle B) < 1/n$ for all large enough $n$, and hence (2) holds.

(3): Let $g \in G$. Given $x \in X$, for all large enough $n$ we have $g \in F_n$ and $(gx, x) \in S_n$ and hence for $y \in Y_x$ we have $gy \in A_n$ if and only if $y \in g^{-1} \varphi_n(gx)D_{n,\theta_n(gx)} = \varphi_n(x)\rho_n(g, x)^{-1}D_{\theta_n(x)}$. Therefore
\[
\nu_x(g^{-1}A_n \triangle A_n) = \nu_x(\varphi_n(x)\rho_n(g, x)^{-1}D_{\theta_n(x)} \triangle \varphi_n(x)D_{\theta_n(x)})
\]
\[
= \nu_{\theta_n(x)}(\rho_n(g, x)^{-1}D_{\theta_n(x)} \triangle D_{\theta_n(x)}) < 1/n
\]
by (iii). Since $\nu(g^{-1}A_n \triangle A_n) = \int_X \nu_x(g^{-1}A_n \triangle A_n) \, d\mu(x)$, property (3) follows.

(4): For each $x \in X$, we have
\[
(T_n^0(A_n) \triangle A_n) \cap Y_x = \varphi_n(x)(S_{n,\theta_n(x)}^0(D_{n,\theta_n(x)} \triangle D_{n,\theta_n(x)}))
\]
and hence $\nu_x(T_n^0(A_n) \triangle A_n) = \nu_{\theta_n(x)}(S_{n,\theta_n(x)}^0(D_{n,\theta_n(x)} \triangle D_{n,\theta_n(x)})) \geq 1/2$ by (iv). This implies (4). $\square$

In what follows $\lambda$ denotes the Lebesgue measure on $[0, 1]$.

**Lemma 4.5.** Let $H$ be a countable discrete group, let $H^* := H \setminus \{1_H\}$, and consider the generalized Bernoulli action $H \curvearrowright ([0, 1]^{H^*}, \lambda^{H^*})$, associated to the conjugation action $H \curvearrowright H^*$. Suppose that there exist sequences $(c_n)_{n \in \mathbb{N}}$, $(d_n)_{n \in \mathbb{N}}$ in $H$ which are asymptotically central in $H$ and satisfy $c_n d_n \neq d_n c_n$ for all $n$. Then the action $H \curvearrowright ([0, 1]^{H^*}, \lambda^{H^*})$ admits a stability sequence.

**Proof.** Define $S_n : [0, 1]^{H^*} \to H$ to be the constant map $S_n(y) := c_n$, and define $D_n := \{ y \in [0, 1]^{H^*} : y(d_n) \in [0, 1/2] \}$ (note that $d_n \neq 1_G$ since $c_n d_n \neq d_n c_n$, and so this definition makes sense). Then it is easily verified that $(S_n, D_n)$ is a stability sequence for $H \curvearrowright ([0, 1]^{H^*}, \lambda^{H^*})$ (see the proof of Proposition 9.8 in [24]). $\square$

**Lemma 4.6.** Let $p : (W, \rho) \to (X, \mu)$ be a $G$-equivariant measure preserving map between p.m.p. actions $G \curvearrowright (W, \rho)$ and $G \curvearrowright (X, \mu)$. Let $(W, \rho) = \int_X (W_x, \rho_x) \, d\mu$ denote the disintegration of $(W, \rho)$ via $p$, so that $W_x = p^{-1}(x)$, and let $G_x$ denote the stabilizer subgroup at $x \in X$. Suppose that

(a) the action $G \curvearrowright (X, \mu)$ generates a $\mu$-amenable orbit equivalence relation, and

(b) for $\mu$-a.e. $x \in X$ the action of $G_x$ on $(W_x, \rho_x)$ generates a $\rho_x$-amenable orbit equivalence relation.

Then the action $G \curvearrowright (W, \rho)$ generates a $\rho$-amenable orbit equivalence relation.

**Proof.** Let $S$ and $\mathcal{R}$ be the orbit equivalence relations generated by the actions of $G$ on $X$ and $W$ respectively, and for each $x \in X$ let $\mathcal{R}_x$ be the restriction of $\mathcal{R}$ to $W_x$, so that $\mathcal{R}_x$ is generated by the action of $G_x$ on $W_x$. Let $\alpha : \mathcal{R} \curvearrowright M$ be a Borel action of $\mathcal{R}$ by fiberwise homeomorphisms on a Borel bundle of compact metric spaces over $W$. For each $x \in X$, by restricting to $\mathcal{R}_x$ we obtain the action $\alpha_x : \mathcal{R}_x \curvearrowright M_x$,
where \( M_x := \bigcup_{w \in W_x} M_w \) is a Borel bundle of compact metric spaces over \( W_x \). For \( w \in W \) let \( P_w \) denote the space of all probability measures on \( M_w \), and for \( x \in X \) let \( P_x := \bigcup_{w \in W_x} P_w \).

Let \( \mathcal{L}_x \) denote the collection of all measurable sections \( W_x \to P_x \), \( w \mapsto \lambda_w \in P_w \) which are \( \rho_x \)-a.e. \((\mathcal{R}_x, \alpha_x)\)-invariant, i.e., which satisfy \( \lambda_{w_1} = \alpha_x(w_1, w_0) \lambda_{w_0} \) for a.e. \((w_1, w_0) \in \mathcal{R}_x \), and where we identify two such sections if they agree \( \rho_x \)-almost everywhere. The assumption \( (b) \) implies that the set \( \mathcal{L}_x \) is nonempty for a.e. \( x \in X \). The set \( \mathcal{L}_x \) is naturally a compact metrizable space since it can be identified with a weak* compact subset of the dual of the separable Banach space \( L^1((W_x, \rho_x), (C(M_w))_{w \in W_x}) \), of all measurable assignments \( w \mapsto f_w \in C(M_w) \) with \( \int_W \|f_w\|_\infty d\rho_x(w) < \infty \) (where two such assignments are identified if they agree \( \rho_x \)-almost everywhere).

Then \( \mathcal{L} := \bigcup_{x \in X} \mathcal{L}_x \) is naturally a Borel bundle of compact metric spaces over \( X \), and we have a natural Borel action \( \beta : \mathcal{S} \acts \mathcal{L} \) by fiberwise homeomorphisms with \( \beta(x, gx) : \mathcal{L}_{gx} \to \mathcal{L}_x \) being given by \( \beta(x, gx) \lambda_w := \alpha(w, gw) \lambda_{gw} \) for \( x \in X \), \( g \in G \), and \( w \in W_x \). This action is well defined, for if \( g_0 x = g_1 x \), then \( g_1 g_0^{-1} \in G_{g_0} \), so that \( \alpha(g_1 w, g_0 w) \lambda_{g_0 w} = \lambda_{g_1 w} \), and hence \( \alpha(w, g_1) \lambda_{g_1 w} = \alpha(w, g_0) \lambda_{g_0 w} \).

Since \( \mathcal{S} \) is \( \mu \)-amenable, there exists a measurable section \( x \mapsto \omega_x \in \text{Prob}(\mathcal{L}_x) \) which is \( \mu \)-a.e. \((\mathcal{S}, \beta_x)\)-invariant. Let \( \nu_x \in \mathcal{L}_x \) denote the barycenter of \( \omega_x \). Then \( x \mapsto \nu_x \) is \( \mu \)-a.e. \((\mathcal{S}, \beta)\)-invariant, and the map \( w \mapsto (\nu_{\mathcal{p}(w)})_w \) is \( \rho \)-a.e. \((\mathcal{R}, \alpha)\)-invariant. This shows that \( \mathcal{R} \) is \( \rho \)-amenable.

**Theorem 4.7.** Let \( \Gamma \acts X \) be an action of a countable amenable group on the Cantor set. Let \( G \) be an infinite subgroup of \([\Gamma \acts X]\) which contains \( \mathbb{A}(\Gamma, X) \). Then \( G \) is \( JS \)-stable.

**Proof.** If there is a uniform finite bound on the size of \( \Gamma \)-orbits then \( G \) is locally finite by \( (i) \) of Lemma 3.1 hence \( JS \)-stable. We may therefore assume that no such bound exists and hence, by \( (ii) \) of Lemma 3.1 that the set \( X_0 \) from Lemma 3.1 is nonempty. Since \( X_0 \) is closed and \( \Gamma \)-invariant, and \( \Gamma \) is amenable, there exists a \( \Gamma \)-invariant Borel probability measure \( \mu \) on \( X \) with \( \mu(X_0) = 1 \). The measure \( \mu \) is then invariant under the group \( G \) as well. The orbit equivalence relation of the action \( G \acts (X, \mu) \) is exactly the orbit equivalence relation of \( \Gamma \), and hence is \( \mu \)-hyperfinite by [11]. We break the remainder of the proof into two cases.

**Case 1:** \( G_x = G_{\mathcal{p}(x)} \) for each \( x \in X \). By \( (iii) \) of Lemma 3.1 for \( \mu \)-almost every \( x \in X \) (namely, every \( x \in X_0 \)) the group \( H = G_x \) satisfies the hypothesis of Lemma 4.3. For each \( x \in X \) let \( (Y_x, \nu_x) = ([0, 1]^{G_x}, \lambda^{G_x}) \). Let \( Y := \bigsqcup_{x \in X} Y_x \), let \( \pi : Y \to X \) denote the projection, and equip \( Y \) with the \( \sigma \)-algebra generated by \( \pi \) along with the maps \( y \mapsto y(g) \) from \( \pi^{-1}\{x : x \in G_x\} \) to \([0, 1]\) for \( g \in G \), which is easily seen to be standard Borel (since \( Y \) can be identified with a Borel subset of \( X \times ([0, 1] \cup \{2\})^{G_x} \)).

Then \( \nu := \int_X \nu_x d\mu \) defines a Borel probability measure on \( Y \). We have a natural action \( G \acts (Y, \nu) \) for \( g \in G \) and \( y \in Y_x \), we define \( gy \in Y_{gx} \) by \((gy)(h) := y(g^{-1}hg)\). Observe that \( g_* \nu_x = \nu_{gx} \), so this action preserves the measure \( \nu_x \), and it makes \( \pi \) a \( G \)-equivariant map, with \( x \mapsto \nu_x \) the associated disintegration. In addition, for
each }x \in X_0\text{, this action restricted to }G_x \curvearrowright (Y_x, \nu_x)\text{ coincides with the generalized Bernoulli action associated to conjugation }G_x \curvearrowright G_x^*\text{, and so by Lemma 4.3 it admits a stability sequence. Therefore, by Theorem 4.4 the action }G \curvearrowright (Y, \nu)\text{ admits a stability sequence and hence }G\text{ is JS-stable.}

Case 2: The general case. Let }\mu_0\text{ denote the measure on the space of all subgroups of }G\text{ obtained as the pushforward of the measure }\mu\text{ under the map }x \mapsto G(x).\text{ By [11 [13], we may find a p.m.p. action }G \curvearrowright (W, \rho)\text{ with the property that }\mu_0\text{ is the pushforward of the measure }\rho\text{ under the stabilizer map }w \mapsto G_w.\text{ We let }G \curvearrowright (\tilde{X}, \tilde{\mu}) := (X, \mu) \otimes_{\mu_0} (W, \rho)\text{ denote the relatively independent joining of the actions on }X\text{ and }W.\text{ Thus }\tilde{X} = \{(x, w) \in X \times W : G(x) = G_w\}\text{, and the stabilizer of }(x, w) \in \tilde{X}\text{ is }G_{(x, w)} = G_x \cap G_w = G_x \cap G(x) = G(x).\text{ Let }\tilde{S}\text{ denote the orbit equivalence relation of the action of }G\text{ on }\tilde{X}.\text{ Since the orbit equivalence relation of the action }G \curvearrowright (X, \mu)\text{ is }\mu\text{-amenable by the amenability of }\Gamma\text{ and the groups }G_x/G(x)\text{ for }x \in X\text{ are amenable, we deduce that }\tilde{S}\text{ is }\tilde{\mu}\text{-amenable by Lemma 4.6 and consequently }\tilde{\mu}\text{-hyperfinite by the Connes–Feldman–Weiss theorem [10]. The action of }G\text{ on }\tilde{X}, \tilde{\mu}\text{ thus satisfies all of the properties which allow us to proceed exactly as in Case 1 and conclude that }G\text{ is JS-stable.}\)

5. Property Gamma

A II_1 factor }M\text{ with trace }\tau\text{ is said to have property Gamma if for every }\varepsilon > 0\text{ and finite set }\Omega \subseteq M\text{ there exists a unitary }u \in M\text{ with }\tau(u) = 0\text{ such that }||[a, u]||_2 < \varepsilon\text{ for all }a \in \Omega.\text{ As the proof of Theorem 2.1 in [9] shows, this is equivalent to the existence, for every }\varepsilon > 0\text{ and finite set }\Omega \subseteq M\text{, of a projection }p \in M\text{ such that }\tau(p) = \frac{1}{2}\text{ and }||a, p||_2 < \varepsilon\text{ for all }a \in \Omega.\text{ One may also equivalently require that }|\tau(p) - \frac{1}{2}| < \varepsilon\text{, as one can replace }p\text{ by a subprojection or superprojection with trace }\frac{1}{2}\text{ and adjust }\varepsilon\text{ accordingly. We say that an ICC countable discrete group }G\text{ has property Gamma if its group von Neumann algebra }\mathcal{L}G\text{ (which is a II_1 factor in this case by the ICC condition) has property Gamma.}

We begin by observing the following.

Proposition 5.1. Let }\Gamma \curvearrowright X\text{ be an action of a group on the Cantor set. Assume that the set of points whose orbit contains at least four points is dense in }X.\text{ Then every subgroup of }[[\Gamma \curvearrowright X]]\text{ containing }\mathcal{A}(\Gamma, X)\text{ is ICC.}

Proof. Let }U\text{ denote the set of points whose orbit contains at least four points, so that }U\text{ is open, and by hypothesis it is dense. Given a nonidentity }g \in [[\Gamma \curvearrowright X]],\text{ it suffices to show that }\{hgh^{-1} : h \in \mathcal{A}(\Gamma, X)\}\text{ is infinite. The support }\text{supp}(g),\text{ of }g,\text{ is a nonempty open set, so we may find an infinite sequence }\{(F_n)_{n \in \mathbb{N}}\}\text{ of pairs of pairwise disjoint subsets of }U \cap \text{supp}(g)\text{ of size }|F_n| = 4\text{ along with }x_n \in F_n, n \in \mathbb{N},\text{ such that }gx_n \in F_n, F_n \subseteq \Gamma x_n\text{ for every }n \in \mathbb{N}.\text{ This ensures that for each }n \in \mathbb{N}\text{ we may find some }h_n \in \mathcal{A}(\Gamma, X)\text{ satisfying }h_nx_i = x_i\text{ for all }i \leq n, h_ngx_i = gx_i\text{ for all }
We may therefore assume that $m$ is a random variable by the central limit theorem, and so by a simple approximation argument we see that the last expression in the above display must converge to zero. □

Next we establish Lemma 5.2, which is a local version of the implication (i) $\Rightarrow$ (ii) of Theorem 1.2 in [25] and follows from the same argument, which we reproduce here. See Section 4 of [25] for more details.

Given a set $Y$ and an inclusion of nonempty sets $L \subseteq K$, we write $\pi_L$ for the coordinate projection map $Y^K \to Y^L$, or simply $\pi_q$ if $L$ is a singleton $\{q\}$. We view $\text{Sym}(K)$ as acting on $Y^K$ via left shifts, i.e., $(\sigma y)(t) = y(\sigma^{-1}t)$ for all $y \in Y^K$, $t \in K$, and $\sigma \in \text{Sym}(K)$, where we use the action notation $t \mapsto \sigma^{-1}t$ for permutations $\sigma$ of $K$.

**Lemma 5.2.** Let $I = [-1, 1]$ and let $\nu$ be a Borel probability measure on $I$ which is centered at 0 and does not give 0 full measure. Let $\epsilon > 0$. Then there is a $\delta > 0$ such that if $Q = J \sqcup K \sqcup L$ is a partition of a finite set $Q$ with $|J| \geq (1 - \delta)|Q|$ and $|K| = |L|$ then the three random variables

$$U = \sum_{q \in J} \pi_q, \quad V = \sum_{q \in K} \pi_q, \quad W = \sum_{q \in L} \pi_q$$

on $I^Q$ with the product measure $\nu^Q$ satisfy

$$\Pr(W \leq -U < V) < \epsilon.$$

**Proof.** Suppose that no such $\delta$ as in the lemma statement exists. Then for every $n \in \mathbb{N}$ we can find a partition $Q_n = J_n \sqcup K_n \sqcup L_n$ of a finite set $Q_n$ such that $|J_n| \geq (1 - 1/n)|Q_n|$ and the three random variables

$$U_n = \sum_{q \in J_n} \pi_q, \quad V_n = \sum_{q \in K_n} \pi_q, \quad W_n = \sum_{q \in L_n} \pi_q$$

on $I^Q$ with the product measure $\nu^Q$ satisfy

$$\Pr(W \leq -U < V) \geq \epsilon.$$

The random variables $U_n$, $V_n$, and $W_n$ each have expectation zero. We also note, writing $\sigma^2 = \int_I x^2d\nu(x)$, $M_n = |J_n|$, and $m_n = |K_n| = |L_n|$, that the variance of $U_n$ is equal to $M_n\sigma^2$ while the variance of each of $V_n$ and $W_n$ is equal to $m_n\sigma^2$.

By passing to a subsequence, we may assume that the sequence $(m_n)$ is either bounded or tends to $\infty$. If it has a bound $R$ then $|V_n|, |W_n| \leq R$ for all $n$ and so

$$\Pr(W_n \leq -U_n < V_n) \leq \Pr(-R < U \leq R)$$

$$= \Pr(-R/(\sigma \sqrt{M_n}) < U_n/(\sigma \sqrt{M_n}) \leq R/(\sigma \sqrt{M_n})).$$

Now $R/(\sigma \sqrt{M_n}) \to 0$ and $U_n/(\sigma \sqrt{M_n})$ converges in distribution (to a standard normal random variable) by the central limit theorem, and so by a simple approximation argument we see that the last expression in the above display must converge to zero. We may therefore assume that $m_n \to \infty$. 



By the central limit theorem, the random variables $U_n/(\sigma \sqrt{M_n})$, $V_n/(\sigma \sqrt{m_n})$, and $W_n/(\sigma \sqrt{m_n})$ all converge in distribution to a standard normal random variable $Z$. Setting $\lambda_n = \sqrt{m_n}/M_n$ we have $\lambda_n \to 0$ and so $\lambda_n V_n/(\sigma \sqrt{m_n})$ and $\lambda_n W_n/(\sigma \sqrt{m_n})$ converge in probability to zero, as is straightforward to check. But then, like in the previous case, the quantity $\lambda_n V_n/(\sigma \sqrt{m_n})$ must converge to zero. This yields a contradiction, thus establishing the lemma.

**Lemma 5.3.** Let $(Y, \nu)$ be a standard probability space which does give any singleton full measure. Let $\varepsilon > 0$. Then there exists a $\delta > 0$ such that for every nonempty finite set $F$ and every $E \subseteq F$ with $|E| \geq (1 - \delta)|F|$ there exists a set $A \subseteq Y^F$ with $A = \pi_E^{-1}(\pi_E(A))$ such that $|\nu^F(A) - \frac{1}{2}| < \varepsilon$ and $\nu^F(\sigma A \Delta A) < \varepsilon$ for all $\sigma \in \text{Sym}(F)$.

**Proof.** We may assume that $Y = [-1, 1]$ and that $\nu$ is centered at 0. Let $\delta > 0$ be as given by Lemma 5.2 with respect to $\varepsilon/2$ and let us show that $\delta/2$ does the required job. Let $F$ be a nonempty finite set and let $E$ be a subset of $F$ with $|E| \geq (1 - \delta/2)|F|$. Let $\sigma \in \text{Sym}(F)$. Define $A = \{y \in Y^F : \sum_{s \in E} y_s > 0\}$. Then $A = \pi_E^{-1}(\pi_E(A))$.

Define the three random variables
\[ U = \sum_{s \in E \cap E} \pi_s, \quad V = \sum_{s \in E \setminus \sigma E} \pi_s, \quad W = \sum_{s \in \sigma E \setminus E} \pi_s. \]

Since $|\sigma E \cap E| = |E| - |E \setminus \sigma E| - |E \setminus E| = |E| - 2|E \setminus E| \geq (1 - \delta)|F|$, by our choice of $\delta$ via Lemma 5.2 we then have
\[ \nu^F(\sigma A \Delta A) = \nu^F(A \setminus \sigma A) + \nu^F(\sigma A \setminus A) = 2 \nu^F(A \setminus \sigma A) = 2 \mathbb{P}(W \leq -U < V) < \varepsilon. \]

For a group $\Gamma$, finite sets $F, T \subseteq \Gamma$, and a $\delta > 0$, we say that $F$ is $(T, \delta)$-invariant if $|F \cap \bigcap_{s \in T} s^{-1}F| \geq (1 - \delta)|F|$.

**Lemma 5.4.** Let $\Gamma$ be a countably infinite amenable group. Let $T$ be a finite subset of $\Gamma$ containing $1_\Gamma$ and let $\delta > 0$. Let $F$ be a $(T, \frac{1}{2})$-invariant nonempty finite subset of $\Gamma$. Then there exists a $C \subseteq \Gamma$ with $|C| \geq |F|/(2|T|^2)$ such that the sets $Tc$ for $c \in C$ are pairwise disjoint and contained in $F$.

**Proof.** Write $F' = \bigcap_{s \in T} s^{-1}F$. Take a maximal set $C \subseteq F'$ with the property that the sets $Tc$ for $c \in C$ are pairwise disjoint. Note that $TC \subseteq F$. We also have $F' \subseteq T^{-1}TC$, for otherwise taking a $d \in F' \setminus T^{-1}TC$ we would have $Td \cap TC = \emptyset$ for all $c \in C$, contradicting maximality. Finally, since $F$ is $(T, \frac{1}{2})$-invariant we have $|F| \leq 2|F'| \leq 2|T|^2|C|$, and so $C$ does the job.

**Theorem 5.5.** Let $\Gamma \curvearrowright X$ be a topologically free action of a countably infinite amenable group on the Cantor set. Then every subgroup $[[\Gamma \curvearrowright X]]$ containing $A(\Gamma, X)$ has property Gamma.
Proof. Let $\Omega$ be a finite symmetric subset of $[\Gamma \curvearrowright X]$. By the definition of the topological full group, we can find a clopen partition $\{P_1, \ldots, P_m\}$ of $X$ such that for each $h \in \Omega$ there exist $s_{h,1}, \ldots, s_{h,m} \in G$ for which $hx = s_{h,i}x$, for every $i = 1, \ldots, m$ and $x \in P_i$. Let $K$ be the collection of all of these $s_{h,i}$.

Let $\delta > 0$ be as given by Lemma 5.3 with respect to $\varepsilon$. By amenability there exists a $(K, \delta)$-invariant nonempty finite set $T \subseteq \Gamma$ containing $1$. Again by amenability there is a $(T, \frac{1}{2})$-invariant nonempty finite set $F \subseteq \Gamma$ where, since $\Gamma$ is infinite, we can take to have cardinality greater than $6|K^{\Omega \times T}||T|^2$. Then by Lemma 5.4 we can find a $C \subseteq \Gamma$ with $|C| \geq |F|/(2|T|^2) > 3|K^{\Omega \times T}|$ such that the sets $Tc$ for $c \in C$ are pairwise disjoint and contained in $F$.

By topological freeness there exists an $x_0 \in X$ such that the map $t \mapsto tx_0$ from $F$ to $X$ is injective. By continuity we can then find a clopen neighborhood $B$ of $x_0$ such that the sets $tB$ for $t \in F$ are pairwise disjoint and for each $t \in F$ there is an $1 \leq i \leq m$ such that $tB \subseteq P_i$.

For every $c \in C$ we have a function $\theta_c \in K^{\Omega \times T}$ such that $htcx = \theta_c(h,t)tcx$ for all $h \in \Omega$, $t \in T$, and $x \in B$. Since $|C| > 3|K^{\Omega \times T}|$, by the pigeonhole principle there are a $\theta \in K^{\Omega \times T}$ and a $D \subseteq C$ with $|D| = 3$ such that $\theta_c = \theta$ for each $c \in D$.

Set $T' = \bigcap_{s \in K} s^{-1}T$. For each $s \in K$ we have $sT' \subseteq T$ and so for every $h \in \Omega$ we can find a $\sigma_h \in \text{Sym}(T)$ such that $\sigma_h t = \theta(h,t)$ for all $t \in T'$. Consider the alternating group $A(D)$ on the $3$-element set $D$. This is a cyclic group of order $3$. We regard the product $H = A(D)^T$ as a subgroup of $A(\Gamma, X)$ with an element $(\omega_t)_{t \in T}$ in $H$ acting by $t cx \mapsto t \omega_t(c)x$ for all $x \in B$, $t \in T$, and $c \in D$ and by $x \mapsto x$ for all $x \in X \setminus TDB$.

Write $Y$ for the spectrum of the commutative von Neumann algebra $\mathcal{L}A(D)$, and identify $\mathcal{L}A(D)$ with $C^\gamma$. We have $|Y| = 3$. Write $\nu$ for the uniform probability measure on $Y$. We regard the commutative von Neumann algebra $\mathcal{L}H \cong \mathcal{L}A(D)^\otimes T$ as $C^\gamma T \cong (C^\gamma)^{\otimes T}$ via the canonical identifications. The canonical tracial state on $\mathcal{L}H$ is then given by integration with respect to $\nu^T$.

For $h \in \Omega$ we consider $\sigma_h$ as acting on $Y^T$ by $(\sigma_h x)_t = x_{\sigma_h^{-1}t}$ for all $t \in T$. According to our invocation of Lemma 5.3 above, there exists a set $A \subseteq Y^T$ with $A = \pi_{T'}^{-1}(\pi_{T'}(A))$ such that $|\nu^T(A) - \frac{1}{2}| < \varepsilon$ and $\nu^T(\sigma_h A \Delta A) < \varepsilon$ for all $h \in \Omega$.

Write $p$ for the indicator function of $A$ in $C^\gamma T$, which we view as a projection in $\mathcal{L}A(\Gamma, X)$ under the canonical inclusion of $\mathcal{L}H \cong C^\gamma T$ in $\mathcal{L}A(\Gamma, X)$. Since the canonical tracial state $\tau$ on $\mathcal{L}A(\Gamma, X)$ restricts to integration with respect to $\nu^T$ on $\mathcal{L}H$ we have $|\tau(p) - \frac{1}{2}| < \varepsilon$.

Now let $h \in \Omega$. Given a $g = (g_t)_{t \in T}$ in $A(D)^T$ with $g_t = 1_{A(D)}$ for all $t \in T \setminus T'$ and viewing $A(D)^T$ as subgroup of $A(\Gamma, X)$, the element $hgh^{-1}$ also belongs to $A(D)^T$, with its value at a given coordinate $t \in T'$ being equal to $g_{\sigma_h^{-1}t}$. Thus for every elementary tensor $a = \otimes_{t \in T} a_t$ in $\mathcal{L}A(D)^{\otimes T}$ satisfying $a_t = 1$ for all $t \in T'$ we have $uhua_h^{-1} = \otimes_{t \in T} a_{\sigma_h^{-1}t}$. Since $A = \pi_{T'}^{-1}(\pi_{T'}(A))$, the projection $p$ is a sum of minimal
projections in $\mathcal{L}A(D)^{\otimes T}$ which can be expressed as such elementary tensors and so $u_h p u_h^{-1}$ is equal to the indicator function of $\sigma_h A$ in $C^{X^T}$ under the identification of $C^{X^T}$ with $\mathcal{L}A(D)^{\otimes T}$. It follows that

$$\|u_h p u_h^{-1} - p\|_2 = \nu^T(\sigma_s A \Delta A)^{1/2} < \varepsilon^{1/2}.$$  

We conclude that $\mathcal{L}A(\Gamma, X)$ has property Gamma.

Remark 5.6. By [4], a countable group $G$ has no normal amenable subgroups if and only if its reduced $C^*$-algebra $C^*_r(G)$ has a unique tracial state (the canonical one). In Sections 7 and 9 we will construct large classes of topologically free minimal actions $\Gamma \curvearrowright X$ of countable amenable groups on the Cantor set whose alternating groups $A(\Gamma, X)$ are simple and nonamenable. It follows that for such actions the reduced $C^*$-algebra $C^*_r(A(\Gamma, X))$ has a unique tracial state, and hence has the $C^*$-algebraic version of property Gamma [21, 5] as a consequence Theorem 5.5 and the fact that the von Neumann algebra of a discrete group is the weak operator closure of the reduced group $C^*$-algebra under the GNS representation of the canonical tracial state.

Question 5.7. Can one strengthen the conclusion of Theorem 5.5 by replacing property Gamma with the property that the group von Neumann algebra is McDuff?

6. Inner amenability

A mean on a set $X$ is a finitely additive probability measure defined on the collection of all subsets of $X$. An action of a discrete group $G$ on a set $X$ is called amenable if there exists a $G$-invariant mean on $X$. We say that a discrete group $G$ is inner amenable if there exists a conjugation invariant mean $m$ on $G$ which is atomless, i.e., $m(F) = 0$ for every finite subset $F$ of $G$. This definition of inner amenability is slightly stronger than Effros’s original one in [17] and coincides with it if the group is ICC (the main situation of interest in [17]), but has become standard in the setting of general groups.

A mean on a set $X$ give rise via integration to a unital positive linear functional $\ell^\infty(X) \to \mathbb{C}$ (also called a mean), and conversely each such functional produces a mean on $X$ by evaluation on indicator functions. Thus inner amenability for a discrete group $G$ can be expressed as the existence of a mean $\sigma : \ell^\infty(G) \to \mathbb{C}$ which vanishes on $c_0(G)$ and is invariant under the action of $G$ that composes a function with the action $(s,t) \mapsto s^{-1}ts$ of $G$ on itself by conjugation. It is moreover the case that $G$ is inner amenable if and only if for every finite set $F \subseteq G$ and $\varepsilon > 0$ there exist finite sets $W \subseteq G$ of arbitrarily large cardinality such that $|sWs^{-1} \Delta W| < \varepsilon|W|$ for all $s \in F$. The backward implication is established by viewing the normalized indicator functions of the sets $W$ as means on $\ell^\infty(G)$ via the canonical inclusion $\ell^1(G) \subseteq \ell^\infty(G)^*$ and then taking a weak* cluster point over the net of pairs $(F, \varepsilon)$ to produce a mean on $\ell^\infty(G)$ that witnesses inner amenability.

By [17], if $G$ is an ICC discrete group with property Gamma then it is inner amenable. We thereby derive the following corollary from Theorem 5.5.
Corollary 6.1. Let $\Gamma \curvearrowright X$ be a topologically free action of an infinite amenable group on the Cantor set. Then every subgroup $G$ of $[\Gamma \curvearrowright X]$ containing $A(\Gamma, X)$ is inner amenable.

This corollary can also be proved more directly, and we will now describe two methods for doing this that may be useful for establishing inner amenability in other contexts. The first of these will actually give us the conclusion even without the assumption that the action is topologically free, but just under the (necessary) assumption that the group $G$ is infinite, which always holds when $A(\Gamma, X)$ is nontrivial.

The following two folklore facts can be verified as exercises.

Lemma 6.2. Suppose that $G \curvearrowright X$ is an amenable action of a group $G$ on a set $X$. Let $G \curvearrowright Y$ be another action of $G$ and assume that $G_x \curvearrowright Y$ is amenable for every $x \in X$. Then $G \curvearrowright Y$ is amenable.

Lemma 6.3. Let $G$ be a group and let $H$ be a subgroup of $G$. Suppose that $H$ is inner amenable and the action $G \curvearrowright G/H$ is amenable. Then $G$ is inner amenable.

Let $\Gamma \curvearrowright X$ be an action of a countable amenable group on the Cantor set and let $G$ be an infinite subgroup of $[\Gamma \curvearrowright X]$ containing $A(\Gamma, X)$. As usual, for a finite set $F \subseteq X$ we write $G_{(F)}$ for the set of all elements in $G$ which fix a neighborhood of every $x \in F$, and $G_{(x)}$ when $F$ is a singleton $\{x\}$.

Lemma 6.4. Let $F$ be a finite subset of $X$. Then the action of $G$ on $G/G_{(F)}$ is amenable.

Proof. Let $G_F = \bigcap_{x \in F} G_x$ be the pointwise stabilizer of $F$. Then the group $G_F/G_{(F)}$ is amenable, since it is isomorphic to a subgroup of $\prod_{x \in F} \Gamma_x/\Gamma_{(x)}$, which is amenable since $G$ is amenable. It follows that the action $G_F \curvearrowright G/G_{(F)}$ is amenable, and hence the action $gG_F g^{-1} \curvearrowright G/G_{(F)}$ is amenable for each $g \in G$. Therefore, by Lemma 6.2 to show that the action of $G$ on $G/G_{(F)}$ is amenable it is enough to show that the action $\Gamma \curvearrowright G/G_F$ is amenable. Let $F_0 \subseteq F$ be the subset of points of $F$ whose $\Gamma$-orbit is infinite. Then $G_F$ has finite index in $G_{F_0}$, so it is enough to show that the action $G \curvearrowright G/G_{F_0}$ is amenable, since any $G$-invariant mean $m_0$ on $G/G_{F_0}$ lifts to a $G$-invariant mean $m$ on $G/G_F$ defined on $f \in \ell^\infty(G/G_F)$ by

$$\int_{G/G_F} f(gG_F) \, dm(gG_F) = \frac{1}{[G_{F_0} : G_F]} \int_{G/G_{F_0}} \sum_{gG_F \subseteq hG_{F_0}} f(gG_F) \, dm_0(hG_{F_0}).$$

Enumerate the elements of $F_0$ as $z_0, z_1, \ldots, z_{n-1}$. Consider the diagonal action of $G$ on $X^n$ given by $g(x_0, \ldots, x_{n-1}) := (gx_0, \ldots, gx_{n-1})$. This action is by homeomorphisms which are contained in the topological full group of the action of $\Gamma^n$ on $X^n$. Let $Y \subseteq X^n$ denote the set of $n$-tuples of pairwise distinct points. Then the $G$-orbit of $(z_0, \ldots, z_{n-1})$ coincides with the intersection of $Y$ with the $\Gamma^n$-orbit, $\prod_{i<n} \Gamma_{z_i}$, of $(z_0, \ldots, z_{n-1})$. The action of $G$ on $G/G_{F_0}$ is then conjugate to the action of $G$ on the intersection of $Y$ with $\prod_{i<n} \Gamma_{z_i}$. For each $i < n$ let $m_i$ be a mean on $\Gamma_{z_i}$ which is
invariant under the action of $\Gamma$. Then, arguing by induction on $n$, we see that the mean $m$ on $\prod_{i<n} \Gamma z_i$, defined on $f \in L^\infty(\prod_{i<n} \Gamma z_i)$ by

$$\int f(x_0, \ldots, x_{n-1}) \, dm = \int_{\Gamma z_0} \cdots \int_{\Gamma z_{n-1}} f(x_0, \ldots, x_{n-1}) \, dm_{n-1}(x_{n-1}) \cdots dm_0(x_0),$$

is $\Gamma^n$-invariant and satisfies $m(Y \cap \prod_{i<n} \Gamma z_i) = 1$ since each orbit $\Gamma z_i$ is infinite. Since the action of $G$ on $Y \cap \prod_{i<n} \Gamma z_i$ is by piecewise translations of elements of $\Gamma^n$, the mean $m$ witnesses that this action of $G$ is amenable. Therefore the action of $G$ on $G/G_{F_0}$ is amenable.

Observe that a group which admits a nontrivial asymptotically central sequence $(c_n)_{n \in \mathbb{N}}$ is inner amenable, since any atomless mean on the set $\{c_n : n \in \mathbb{N}\}$ will be conjugation invariant. We can thus assert the following in view of (iii) of Lemma 3.1.

**Lemma 6.5.** Let $X_0$ denote the set of points $x \in X$ with the property that for every open neighborhood $U$ of $x$ and every $n \in \mathbb{N}$ there exists some $y \in X$ such that $|\Gamma y \cap U| \geq n$. Let $F$ be a finite subset of $X$ containing a point from $X_0$. Then $G_{(F)}$ is inner amenable.

We can now obtain the inner amenability of $G$ as follows. If there is a uniform finite bound on the size of $\Gamma$-orbits in $X$ then $G$ is locally finite by (i) of Lemma 3.1 and hence $G$ is inner amenable in this case, since it is infinite and amenable. So assume that there is no such uniform bound. Then the set $X_0$ is nonempty by Lemma 3.1. Therefore, fixing $x \in X_0$, the group $G_{(x)}$ is inner amenable by Lemma 6.5 so by taking $F$ in Lemma 6.4 to be $\{x\}$, and applying Lemma 6.3 with $H = G_{(x)}$, we conclude that $G$ is inner amenable.

Another way to verify the inner amenability in Corollary 6.1 is as follows. Let $\Omega$ be a finite symmetric subset of $[\alpha]$ and $\varepsilon > 0$. It is enough to construct a finite set $W \subseteq A(\alpha)$ of cardinality at least 2 such that $|hWh^{-1} \Delta W|/|W| < \varepsilon$ for all $h \in \Omega$.

By the definition of the topological full group, we can find a clopen partition $\{P_1, \ldots, P_m\}$ of $X$ such that for each $h \in \Omega$ there exist $s_{h,1}, \ldots, s_{h,m} \in G$ for which $hx = s_{h,i}x$ for every $i = 1, \ldots, m$ and $x \in P_i$. Let $\varepsilon' > 0$, to be determined. By the amenability of $G$ there exists a nonempty finite set $F \subseteq G$ such that the set $F' := F \cap \bigcap_{h \in \Omega} \bigcap_{i=1}^m (s_{h,i}F \cap s_{h,i}^{-1}F)$ satisfies $|F'| \geq (1 - \varepsilon')|F|$. Since $G$ is infinite, we may also assume that $|F|$ is large enough for a purpose below. As the action is topologically free there exist an $x \in X$ such that the map $t \mapsto tx$ from $F$ to $X$ is injective. By continuity we can then find a clopen neighborhood $B$ of $X$ such that the sets $tB$ for $t \in F$ are pairwise disjoint and for each $t \in F$ there is an $1 \leq i \leq m$ such that $tB \subseteq P_i$.

Write $A_F$ for the alternating group on the set $F$. To each $\sigma \in A_F$ we associate an element $g_\sigma \in A(\alpha)$ by setting $g_\sigma sx = \sigma(s)x$ for all $s \in F$ and $x \in B$ and $g_\sigma x = x$ for all $x \in X \setminus FB$. Write $W$ for the set of $g_\sigma$ such that $\sigma$ is a 3-cycle in $A_F$, and $W'$ for
the set of $g_\sigma$ such that $\sigma$ is a 3-cycle in $A_F$ with support contained in $F'$. Then

$$|W'|/|W| = 3^{|F'|} = 3^{|F'|(|F'|−1)(|F'|−2)}/|F'|(|F'|−1)(|F'|−2),$$

and so by taking $\varepsilon'$ small enough and requiring $|F|$ to be large enough we may ensure that $|W'|/|W| > 1 - \varepsilon'/2$.

Now let $h \in V$. Let $\sigma = (t_1 t_2 t_3)$ be a 3-cycle in $A_F$ whose support $\{t_1, t_2, t_3\}$ is contained in $F'$. Let $1 \leq i_1, i_2, i_3 \leq m$ be such that $t_k B \subseteq P_k$ for $k = 1, 2, 3$. Then

$$hg_\sigma h^{-1}(s_{h, i_1} t_1 B) = hg_\sigma h^{-1}(ht_1 B) = ht_2 t_1^{-1}(t_1 B) = ht_2 B = s_{h, i_2} t_2 B$$

and similarly $hg_\sigma h^{-1}(s_{h, i_2} t_2 B) = s_{h, i_3} t_3 B$ and $hg_\sigma h^{-1}(s_{h, i_3} t_3 B) = s_{h, i_1} t_1 B$. The elements $s_{h, i_1} t_1, s_{h, i_2} t_2,$ and $s_{h, i_3} t_3$ belong to $F$ and hence are distinct given that the sets $tB$ for $t \in F$ are pairwise disjoint and $h$ is bijective. Moreover, $hg_\sigma h^{-1}$ acts as the identity off of $s_{h, i_1} t_1 B \sqcup s_{h, i_2} t_2 B \sqcup s_{h, i_3} t_3 B$. Thus $hg_\sigma h^{-1} = g_\omega$ where $\omega$ is the 3-cycle in $A_F$ which cyclically permutes the elements $s_{h, i_1} t_1, s_{h, i_2} t_2,$ and $s_{h, i_3} t_3$ in that order. We have thus shown that $hW' h^{-1} \subseteq W$, so that

$$|hW' h^{-1} \Delta W| \leq |(hW' h^{-1} \cup W) \setminus hW' h^{-1}| \leq |hW' h^{-1}| - 2|hW' h^{-1}|$$

$$= 2(|W| - |W'|)$$

$$< \varepsilon |W|,$$

as desired.

### 7. Isomorphism on the space of finitely generated simple nonamenable groups with property Gamma

Recall that a Borel equivalence relation $E$ on a standard Borel space $X$ is said to be smooth if there is a standard Borel space $Y$ and a Borel map $\varphi : X \to Y$ such that for all $x_1, x_2 \in X$ one has $x_1 E x_2$ if and only if $\varphi(x_1) = \varphi(x_2)$. The prototype for nonsmoothness is the equivalence relation $E_0$ on $\{0, 1\}^\mathbb{N}$ under which $x \sim y$ whenever there exists an $m \in \mathbb{N}$ such that $x_n = y_n$ for all $n \geq m$. An obstruction to smoothness for the relation $E$, which we will use below in the proof of Theorem 7.4, is the existence of a Borel homomorphism from $X$ to $E_0$ (i.e., a Borel map $\varphi : X \to \{0, 1\}^\mathbb{N}$ such that $x_1 E x_2$ implies $\varphi(x_1) E_0 \varphi(x_2)$ for all $x_1, x_2 \in X$) which is injective. In the case that $E$ is countable, i.e., each class is countable, the Glimm–Effros dichotomy says that this is the only obstruction to smoothness.

The main goal of this section is to establish the nonsmoothness of the isomorphism relation on the space of finitely generated simple nonamenable groups with property Gamma (Corollary 7.8). This will be done by first establishing the nonsmoothness of the relation of topological conjugacy, and consequently also of the relation of continuous orbit equivalence, on the space of free minimal Toeplitz $\mathbb{Z}^2$-subshifts over a symbol set of cardinality 36 which have the property that their topological full group (equivalently, the commutator subgroup thereof) is nonamenable (Theorem 7.4 and
Corollary 7.7. Corollary 7.8 will then follow with the help of Theorem 2.4. Note that free actions of \( \mathbb{Z}^d \) on the Cantor set are almost finite (see Lemma 6.3 of [36]) and so in this setting the alternating group of the action coincides with the commutator subgroup of the topological full group.

Throughout the section we identify the natural number \( n \) with the discrete topological space \( \{0, 1, \ldots, n-1\} \). Let \( \Gamma \) be a countable group and let \( n \geq 2 \) be a natural number. We refer to closed subsets of \( n^\Gamma \) which are invariant under the left shift action \((x_0, \ldots, x_n) \mapsto (x_0, \ldots, x_{n-1}, x_n)\) of \( \Gamma \) as subshifts, and think of these simultaneously as actions of \( \Gamma \) via restriction of the left shift action. Let \( \text{Subsh}(n^\Gamma) \) denote the space of all subshifts of \( n^\Gamma \). We equip \( \text{Subsh}(n^\Gamma) \) with the Vietoris topology, so that it is a compact Polish space. Let \( \text{FMSubsh}(n^\Gamma) \) denote the subspace of all free minimal subshifts of \( n^\Gamma \), which is a Borel subset of \( \text{Subsh}(n^\Gamma) \). For a given \( d \geq 1 \) and a faithful minimal subshift \( X \in \text{FMSubsh}(n^\mathbb{Z}^d) \) it is shown in Theorem 1 of [8] that the commutator subgroup \( [[\mathbb{Z}^d \curvearrowright X]]' \) is finitely generated, and the authors describe an explicit process for obtaining a finite generating set which yields the following proposition.

**Proposition 7.1.** Let \( d \geq 1 \) and \( n \geq 2 \). There exists a Borel map \( X \mapsto (G_X, T_X) \) from the standard Borel space \( \text{FMSubsh}(n^\mathbb{Z}^d) \) to the Polish space \( \mathcal{G}_{fg} \) of marked finitely generated groups such that for each \( X \in \text{FMSubsh}(n^\mathbb{Z}^d) \) the group \( G_X \) is isomorphic to \( [[\mathbb{Z}^d \curvearrowright X]]' \) and \( T_X \) is a finite generating set for \( G_X \).

By Proposition 9.2.5 of [20], topological conjugacy is a countable Borel equivalence relation on \( \text{FMSubsh}(n^\Gamma) \). If \( \Gamma \) is finitely generated, then the same holds for continuous orbit equivalence:

**Proposition 7.2.** Let \( \Gamma \) be a finitely generated group. Then continuous orbit equivalence is a countable Borel equivalence relation on \( \text{FMSubsh}(n^\Gamma) \).

**Proof.** Let \( S_\Gamma \) be a finite symmetric generating set for \( \Gamma \). For each continuous function \( u : S_\Gamma \times n^\Gamma \to \Gamma \) let \( X_u^0 \) be the subset of \( n^\Gamma \) on which \( u \) defines a cocycle for the action of \( \Gamma \); more precisely, \( X_u^0 \) consists of all those \( x \in n^\Gamma \) with the property that, for every pair \( ((s_n, \ldots, s_0), (s'_m, \ldots, s'_0)) \) of finite sequences of elements of \( S_\Gamma \) satisfying \( s_n \cdots s_1 s_0 = s'_m \cdots s'_1 s'_0 \), we have \( \bar{u}((s_n, \ldots, s_0), x) = \bar{u}((s'_m, \ldots, s'_0), x) \), where \( \bar{u} \) is defined by

\[
\bar{u}((s_n, \ldots, s_0), x) = u(s_n, s_{n-1} \cdots s_0 x) \cdots u(s_2, s_1 s_0 x) u(s_1, s_0 x) u(s_0, x).
\]

The set \( X_u^0 \) is closed since for each fixed sequence \( (s_n, \ldots, s_0) \) the function \( x \mapsto \bar{u}((s_n, \ldots, s_0), x) \) is continuous. It is clear that \( X_u^0 \) is invariant under each \( s \in S_\Gamma \) and hence is invariant under the entire group \( \Gamma \). The restriction \( u|_{S_\Gamma \times X_u^0} \) naturally extends to a well-defined continuous cocycle \( u^0 : \Gamma \times X_u^0 \to \Gamma \) given by \( u^0(s_n \cdots s_0, x) := \bar{u}((s_n, \ldots, s_0), x) \). Let \( X_u \) be the set of all those \( x \in X_u^0 \) with the property that the map \( u^0_x : \Gamma \to \Gamma \) given by \( s \mapsto u^0(s, x) \) is bijective. Then \( X_u \) is easily seen to be a \( G_\delta \) subset of \( n^\Gamma \). For each continuous function \( f : n^\Gamma \to n \) let \( \varphi_{(u, f)} : X_u \to n^\Gamma \)
be the function defined by \( \varphi_{(u,f)}(x)(u^0(s,x)^{-1}) = f(sx) \). Then \( \varphi_{(u,f)} \) is a continuous function satisfying \( \varphi_{(u,f)}(sx) = u^0(s,x)\varphi_{(u,f)}(x) \) for all \( s \in \Gamma \) and \( x \in X_u \).

**Claim 7.3.** Let \( X, Y \in \text{FMSubsh}(n^\Gamma) \). Then \( X \) and \( Y \) are continuously orbit equivalent if and only if there exist continuous functions \( u, v : S_\Gamma \times n^\Gamma \to \Gamma \) and \( f, g : n^\Gamma \to n \) such that

1. \( X \subseteq X_u \) and \( Y \subseteq X_v \),
2. \( \varphi_{(u,f)}(X) = Y \) and \( \varphi_{(v,g)}(Y) = X \),
3. \( (\varphi_{(v,g)} \circ \varphi_{(u,f)})|_X = \text{id}_X \) and \( (\varphi_{(u,f)} \circ \varphi_{(v,g)})|_Y = \text{id}_Y \).

**Proof of Claim 7.3.** If such functions \( u, v, f, g \) exist then it is clear that \( \varphi_{(u,f)}|_X \) is a continuous orbit equivalence from \( X \) to \( Y \), with inverse \( \varphi_{(v,g)}|_Y \) and associated cocycles \( u^0 \) and \( v^0 \). Conversely, suppose that \( \varphi : X \to Y \) is a continuous orbit equivalence with associated continuous cocycles \( u_\varphi : \Gamma \times X \to Y \) and \( v_\varphi : \Gamma \times Y \to X \). Let \( u : S_\Gamma \times n^\Gamma \to Y \) and \( v : S_\Gamma \times n^\Gamma \to X \) be any continuous maps extending the restrictions \( u_\varphi|_{S_\Gamma \times X} \) and \( v_\varphi|_{S_\Gamma \times Y} \), respectively, and let \( f : n^\Gamma \to n \) and \( g : n^\Gamma \to n \) be any continuous maps extending the maps \( x \mapsto \varphi(x)(1_\Gamma) \) from \( X \) to \( n \) and \( y \mapsto \varphi^{-1}(y)(1_\Gamma) \) from \( Y \) to \( n \), respectively. Since \( u_\varphi \) and \( v_\varphi \) are cocycles associated to a continuous orbit equivalence, we have \( X \subseteq X_u \), \( Y \subseteq X_v \), \( u^0|_{\Gamma \times X} = u_\varphi \), and \( v^0|_{\Gamma \times Y} = v_\varphi \). We also have, for \( x \in X \) and \( s \in \Gamma \),

\[
\varphi_{(u,f)}(x)(u^0(s,x)^{-1}) = f(sx) = \varphi(sx)(1_\Gamma) = (u_\varphi(s,x)\varphi(x))(1_\Gamma) = \varphi(x)(u^0(s,x)^{-1}),
\]

and hence \( \varphi_{(u,f)}|_X = \varphi \). Likewise, \( \varphi_{(v,g)}|_Y = \varphi^{-1} \). It follows that (1), (2), and (3) hold. \( \square \)[Claim 7.3]

Note that there are only countably many tuples \((u, v, f, g)\) of continuous functions \( u, v : S_\Gamma \times n^\Gamma \to \Gamma \) and \( f, g : n^\Gamma \to n \). Therefore, for a fixed \( X \in \text{FMSubsh}(n^\Gamma) \) the set of all \( Y \in \text{FMSubsh}(n^\Gamma) \) which are continuously orbit equivalent to \( X \) is countable, since each such \( Y \) is of the form \( \varphi_{(u,f)}(X) \) for some such tuple \((u, v, f, g)\) satisfying (1), (2), and (3) of the claim. Finally, to see that continuous orbit equivalence is Borel it suffices to show that for a fixed tuple \((u, v, f, g)\) the set of all pairs \((X, Y)\) satisfying (1), (2), and (3) in the claim is Borel. This is a straightforward exercise (see Proposition 9.2.5 of [20] for a similar computation). \( \square \)

Let \( \Gamma \) be a residually finite group. Following [31], we call a function \( x : \Gamma \to n \) Toeplitz if it is continuous for the profinite topology on \( \Gamma \), i.e., the topology generated by all cosets of finite index subgroups of \( \Gamma \). A subshift \( X \) of \( n^\Gamma \) is Toeplitz if it is the closure of the \( \Gamma \)-orbit of some Toeplitz function \( x \in n^\Gamma \). Toeplitz subshifts are always minimal [31] and, as explained in [44], the set \( \text{ToepSubsh}(n^\Gamma) \) of all free Toeplitz subshifts of \( n^\Gamma \) is a Borel subset of \( \text{FMSubsh}(n^\Gamma) \). We let \( \text{NAToepl}(n^{\mathbb{Z}^2}) \) denote the Borel subset of \( \text{ToepSubsh}(n^{\mathbb{Z}^2}) \) consisting of all free Toeplitz subshifts \( X \subseteq n^{\mathbb{Z}^2} \) with the property that the topological full group \([[\mathbb{Z}^2 \curvearrowright X]]\) is nonamenable.
(equivalently, the commutator subgroup $[[Z^2 \rtimes X]]'$ is nonamenable). We will meld the arguments and strategies used in [16] and [18] to show the following.

**Theorem 7.4.** Let $N = 36$. Then the relation of topological conjugacy on the space NAToepSubsh($N^{Z^2}$) is not smooth.

**Proof.** We define the group $\Delta$ to be the free product of three involutions $a_0, a_1, a_2$. We also fix a six-letter alphabet $\Sigma = \{a_0, a_1, a_2, d, 0, 1\}$ containing the three generators of $\Delta$.

Let $\mathcal{E}$ denote the undirected edge set of the usual Cayley graph of $Z^2$. Since we are dealing with undirected edges, each edge $e \in \mathcal{E}$ has the form either of a vertical edge $e = \{(n, m), (n, m + 1)\}$ or of a horizontal edge $e = \{(n, m), (n + 1, m)\}$ for some $(n, m) \in Z^2$. The natural translation action of $Z^2$ on $\mathcal{E}$ then has two orbits, one consisting of vertical edges and one consisting of horizontal edges, and is isomorphic to the disjoint union of two copies of the translation action of $Z^2$ on itself. There is therefore a natural isomorphism $\Phi : \Sigma^\mathcal{E} \to (6^2)Z^2 = N^{Z^2}$ from the shift action $Z^2 \rtimes \Sigma^\mathcal{E}$ to the shift action $Z^2 \rtimes N^{Z^2}$. Let $S$ denote the closed $Z^2$-invariant subspace of $\Sigma^\mathcal{E}$ consisting of all labelings $\lambda$ of $\mathcal{E}$ by elements of $\Sigma$ that satisfy the following:

- $\lambda$ assigns no two adjacent vertical edges the same label, and $\lambda$ assigns every vertical edge a label from the set $\{a_0, a_1, a_2, d\}$,
- $\lambda$ assigns every horizontal edge a label from the set $\{0, 1\}$.

The group $\Delta$ acts on $S$ by homeomorphisms contained in the topological full group of the action of $Z^2$ on $S$. Specifically, the generator $a_i \in \Delta$ acts on $\lambda \in S$ according to the homeomorphism $\varphi_{a_i}$ given as follows: If $\lambda$ assigns the label $a_i$ to one (and hence exactly one) of the vertical edges $e = \{(0, 0), (0, j)\}$ incident to the origin (so $j \in \{-1, 1\}$), then $\varphi_{a_i}(\lambda)$ is defined to be the translate of $\lambda$ in the direction taking $(0, j)$ to $(0, 0)$, i.e., $\varphi_{a_i}(\lambda) = (0, -j) \cdot \lambda$. If on the other hand $\lambda$ assigns the color $a_i$ to no vertical edge incident to the origin, then we define $\varphi_{a_i}(\lambda) = \lambda$. Since $\varphi_{a_i}$ is an involution, the assignment $a_i \mapsto \varphi_{a_i}$ extends to a group homomorphism from $\Delta$ to the topological full group of the action $Z^2 \rtimes S$.

We construct an injective Borel homomorphism from $E_0$ to topological conjugacy on NAToepSubsh($N^{Z^2}$) in the following manner. We will define a Borel function $2^N \to S, z \mapsto \lambda_z$, such that $\Phi(\lambda_z) \in N^{Z^2}$ is a Toeplitz sequence and, letting $Y_z$ denote the closure of the $Z^2$-orbit of $\lambda_z$, such that the topological full group $[[Z^2 \rtimes Y_z]]$ contains an isomorphic copy of $\Delta$. This will ensure that the Borel map $z \mapsto \Phi(Y_z)$ takes values in NAToepSubsh($N^{Z^2}$). Using [16] we will ensure that the map $z \mapsto \Phi(Y_z)$ is an injective homomorphism from $E_0$ to topological conjugacy.

We will define the edge labelings $\lambda_z$ for $z \in 2^N$ so that they all coincide on vertical edges. We begin with the definition of the common labeling $\lambda_0$ of the vertical edges, which is essentially a particular instance of the construction in [18].

We fix a surjective function $g : Z \to \Delta \setminus \{1_\Delta\}$ which is continuous for the profinite topology on $Z$. For example, if $\alpha$ is any element of the profinite completion $\hat{Z}$ of
which does not belong to \( \mathbb{Z} \), then, since \( \hat{\mathbb{Z}} \) is homeomorphic to a Cantor set, \( \hat{\mathbb{Z}} \setminus \{ \alpha \} \) can be partitioned into infinitely many nonempty clopen sets, which we can assume are indexed by \( \Delta \setminus \{ 1_\Delta \} \) and hence, after restricting to \( \mathbb{Z} \), define a function \( g \) with the desired properties. With this \( g \), we proceed exactly as in [18] to define \( \lambda_0 \) on vertical edges: we view each nonidentity \( w \in \Delta \) as a nonempty word in the alphabet \( \{ a_0, a_1, a_2 \} \), and for each \( n \in g^{-1}(w) \) we color the edges along the vertical line \( \{(n, m) : m \in \mathbb{Z}\} \), so that the line spells out the bi-infinite word \( \cdots wdwdw \cdots \) when read from top to bottom, and also so that the edge \( \{(n, -1), (n, 0)\} \) is assigned the label \( d \). Doing this for all \( w \in \Delta \setminus \{ 1_\Delta \} \) and all \( n \in g^{-1}(w) \) defines \( \lambda_0 \) on all vertical edges.

As in [18], the construction ensures that if \( \lambda \in \mathcal{S} \) is any labeling extending \( \lambda_0 \) then the action of \( \Delta \) restricted to the \( \mathbb{Z}^2 \)-orbit of \( \lambda \) is faithful: if \( w \in \Delta \setminus \{ 1_\Delta \} \) and \( n \in g^{-1}(w) \), then \( \varphi_w((-n, 0) \cdot \lambda) \neq (-n, 0) \cdot \lambda \) since in the labeling \( (-n, 0) \cdot \lambda \) the edge \( \{(0, 0), (0, 1)\} \) is labeled by the rightmost letter of \( w \), whereas \( \varphi_w((-n, 0) \cdot \lambda) \) labels this edge by the letter \( d \).

We will use the following result of Thomas from §4 of [46].

**Theorem 7.5** ([46]). There exists a Borel map \( z \mapsto \tilde{z} \) from \( 2^N \) to \( 2^\mathbb{Z} \) with the following properties:

(i) \( \tilde{z} \) is a Toeplitz sequence on \( \mathbb{Z} \) for every \( z \in 2^N \),

(ii) for each \( z \in 2^N \), denoting by \( X_z \) the closure of the \( \mathbb{Z} \)-orbit of \( \tilde{z} \) one has \( X_z \in \text{ToepSubsh}(2^\mathbb{Z}) \), and for any \( (z_0, z_1) \in E_0 \) there exists a topological conjugacy from \( X_{z_0} \) to \( X_{z_1} \) taking \( \tilde{z}_0 \) to \( \tilde{z}_1 \),

(iii) the map \( 2^N \to \text{ToepSubsh}(2^\mathbb{Z}) \) given by \( z \mapsto X_z \) is injective and Borel.

In particular, the map \( z \mapsto X_z \) is an injective Borel homomorphism from \( E_0 \) to topological conjugacy on \( \text{ToepSubsh}(2^\mathbb{Z}) \).

We are now ready to define the map \( 2^N \to \mathcal{S}, z \mapsto \lambda_z \), using Thomas's map given by Theorem 7.5. Given \( z \in 2^N \), we extend \( \lambda_0 \) to the labeling \( \lambda_z \in \mathcal{S} \) by copying \( \tilde{z} \) onto every horizontal line, i.e., define

\[
\lambda_z(\{(n, m), (n + 1, m)\}) = \tilde{z}(n)
\]

for each horizontal edge \( \{(n, m), (n + 1, m)\} \in \mathcal{E} \).

It is clear that the assignment \( z \mapsto \lambda_z \) is Borel (in fact continuous). For each \( z \in 2^N \) let \( Y_z \) denote the closure of the \( \mathbb{Z}^2 \)-orbit of \( \lambda_z \).

**Claim 7.6.** The assignment \( z \mapsto \lambda_z \) has the following properties:

(I) For each \( z \in 2^N \) the function \( \Phi(\lambda_z) \in N^{\mathbb{Z}^2} \) is Toeplitz, and \( \Phi(Y_z) \in \text{NAToepSubsh}(N^{\mathbb{Z}^2}) \).

(II) Suppose \( (z_0, z_1) \in E_0 \). Then there exists a topological conjugacy from \( Y_{z_0} \) to \( Y_{z_1} \) taking \( \lambda_{z_0} \) to \( \lambda_{z_1} \).

(III) The map \( 2^N \to \text{NAToepSubsh}(N^{\mathbb{Z}^2}) \) given by \( z \mapsto \Phi(Y_z) \) is Borel and injective.
Proof of Claim 7.6. (I) Fix $z \in \mathbb{Z}^2$ and let $(n, m) \in \mathbb{Z}^2$. Let $w = g(n)$ and let $l$ be the length of the word $ud$ (i.e., $l - 1$ is the length of $w$). By our choice of $g$, there is some $k_0 \in \mathbb{Z}$ such that $g$ takes the constant value $w$ on the set $n + k_0 \mathbb{Z}$. Since all the vertical lines with horizontal coordinate in $n + k_0 \mathbb{Z}$ are labeled in the same way by $\cdots wdw \cdots$, which is periodic with period $l$, the top and bottom vertical edges incident with any $(i, j) \in (n, m) + (k_0 \mathbb{Z} \times l \mathbb{Z})$ have the same respective labels as the top and bottom vertical edges incident with $(n, m)$. Since $\tilde{z}$ is a Toeplitz function, there is some $k_1 \in \mathbb{N}$ such that $\tilde{z}$ is constant on $n + k_1 \mathbb{Z}$. Since $\lambda_z$ labels all horizontal lines in the same way by $\tilde{z}$, the left and right horizontal edges incident with any $(i, j) \in (n, m) + (k_1 \mathbb{Z} \times \mathbb{Z})$ have the same respective colors as the left and right horizontal edges incident with $(n, m)$. Thus, the configuration of $\lambda_z$-labels of all four edges incident to any element of $(n, m) + (k_1 k_0 \mathbb{Z} \times l \mathbb{Z})$ coincides with the configuration of $\lambda_z$-labels of the four edges incident to $(n, m)$. This shows that $\Phi(\lambda_z)$ is a Toeplitz function on $\mathbb{Z}^2$.

Since $\Phi(\lambda_z)$ is a Toeplitz function, the action of $\mathbb{Z}^2$ on $\Phi(Y_z)$ is minimal. Since $\mathbb{Z}^2$ acts faithfully on the orbit of $\lambda_z$, it follows that the action of $\mathbb{Z}^2$ on $Y_z$, and hence on $\Phi(Y_z)$, is free (Lemma 3 of [S]). As remarked above, since $\lambda_z$ extends $\lambda_0$, the action of $\Delta$ on the $\mathbb{Z}^2$-orbit (and hence the orbit closure) of $\lambda_z$ is faithful. Thus, the nonamenable group $\Delta$ embeds in the topological full group $[[\mathbb{Z}^2 \curvearrowright Y_z]]$, which is isomorphic to $[[\mathbb{Z}^2 \curvearrowright \Phi(Y_z)]]$. Therefore $\Phi(Y_z) \in \text{NATopSubsh}(N\mathbb{Z}^2)$.

(II) By (ii) of Theorem 7.5, there is a topological conjugacy $\varphi : X_{z_0} \to X_{z_1}$ with $\varphi(z_0) = z_1$. Let $\psi : \mathcal{S} \to \mathbb{Z}^2$ be given by

$$\psi(\lambda)(n) = \lambda(\{(n, 0), (n + 1, 0)\}).$$

Then $\psi$ is continuous, and $\psi((n, m) \cdot \lambda_{z_0}) = n \cdot z_0$ for each $(n, m) \in \mathbb{Z}^2$. It follows that $\psi(Y_{z_0}) = X_{z_0}$. Suppose that the sequence $((n_i, m_i) \cdot \lambda_{z_0})_{i \in \mathbb{N}}$ converges to some $\lambda \in Y_{z_0}$. Then $((n_i, m_i) \cdot \lambda_{z_1})(e) \to \lambda(e)$ for each vertical edge $e$, since $\lambda_{z_0}$ and $\lambda_{z_1}$ agree on all labels of vertical edges. For each horizontal edge $e = \{(n, m), (n + 1, m)\}$ we have

$$((n_i, m_i) \cdot \lambda_{z_1})(e) = (n_i \cdot z_1)(n) = \varphi(n_i \cdot z_0)(n) = \varphi(\psi((n_i, m_i) \cdot \lambda_{z_0}))(n) \to \varphi(\psi(\lambda))(n)$$

as $i \to \infty$. This shows that $(n_i, m_i) \cdot \lambda_{z_1}$ converges in $Y_{z_1}$ to the labeling $\varphi_0(\lambda)$ given by

$$\varphi_0(\lambda)(e) = \begin{cases} \lambda(e) & \text{if } e \text{ is a vertical edge,} \\ \varphi(\psi(\lambda))(n) & \text{if } e = \{(n, m), (n + 1, m)\}. \end{cases}$$

The map $\varphi_0 : Y_{z_0} \to Y_{z_1}$ is continuous since $\varphi$ and $\psi$ are continuous, and it is $\mathbb{Z}^2$-equivariant since it extends the equivariant map $(n, m) \cdot \lambda_{z_0} \mapsto (n, m) \cdot \lambda_{z_1}$. Using $\varphi^{-1}$ instead of $\varphi$, we analogously obtain the continuous equivariant map $\varphi_1 : Y_{z_1} \to Y_{z_0}$. Then $\varphi_1 \circ \varphi_0$ fixes $\lambda_{z_0}$, and so by equivariance and continuity it is the identity map.
on \( Y_{z_0} \). Likewise, \( \varphi_0 \circ \varphi_1 \) is the identity map on \( Y_{z_1} \). This shows that \( \varphi_0 \) is the desired conjugacy.

(III) It is clear that \( z \mapsto \Phi(Y_z) \) is Borel. For injectivity, it is enough to show that the map \( z \mapsto Y_z \) is injective. Let \( \psi : S \to 2^N \) be the map defined by (1) in the proof of (II). Since \( z \mapsto X_z \) is injective, and \( \psi(Y_z) = X_z \), it follows that \( z \mapsto Y_z \) is injective. \( \square \) [Claim 7.6]

Claim 7.6 shows that \( z \mapsto \Phi(Y_z) \) is an injective Borel homomorphism from \( E_0 \) to topological conjugacy on \( NAToepSubsh(N\mathbb{Z}^2) \), and so the latter equivalence relation is not smooth. This completes the proof of Theorem 7.4. \( \square \)

Corollary 7.7. Let \( N = 36 \). Then the relation of continuous orbit equivalence on the space \( NAToepSubsh(N\mathbb{Z}^2) \) is not smooth.

Proof. This relation is a countable Borel equivalence relation by Proposition 7.2. It is thus nonsmooth since it contains a nonsmooth Borel subequivalence relation by Theorem 7.4. \( \square \)

Corollary 7.8. The isomorphism relation restricted to the subset \( I \) of \( \mathcal{S}_{fg} \) consisting of all finitely generated simple nonamenable groups with property Gamma is not smooth.

Proof. Let \( d = 2 \) and \( N = 36 \) and consider the Borel map \( X \mapsto (G_X, T_X) \) given by Proposition 7.1 but with its domain restricted to the subspace \( NAToepSubsh(N\mathbb{Z}^2) \). For \( X \in NAToepSubsh(N\mathbb{Z}^2) \), the finitely generated group \( G_X \), being isomorphic to \([\mathbb{Z}^2 \acts X]\)'s, is nonamenable, simple (by [36]), and has property Gamma (by Theorem 5.5), and hence \((G_X, T_X) \in I\). By Theorem 2.4 the groups \([\mathbb{Z}^2 \acts X]\) and \([\mathbb{Z}^2 \acts Y]\) are isomorphic if and only if the actions \( \mathbb{Z}^2 \acts X \) and \( \mathbb{Z}^2 \acts Y \) are continuously orbit equivalent. Therefore, the map \( X \mapsto (G_X, T_X) \) is a Borel reduction from continuous orbit equivalence on \( NAToepSubsh(N\mathbb{Z}^2) \) to isomorphism on \( I \). Corollary 7.7 therefore implies that isomorphism on \( I \) is not smooth. \( \square \)

8. Entropy and continuous orbit equivalence

In order to show the pairwise nonisomorphism of the alternating groups in Theorem 9.7 of the next section, we will need Theorem 8.6 which says that topological entropy is an invariant of continuous orbit equivalence for topologically free actions of countable amenable groups which are not locally virtually cyclic. When the groups are finitely generated and the actions are free, this result is a consequence of Austin’s work on measure entropy and bounded orbit equivalence [2] in conjunction with the variational principle, as freeness guarantees that the ergodic p.m.p. actions appearing in the variational principle are free, so that [2] applies. It also follows from [2] when the groups are finitely generated and torsion-free and the actions are minimal and have nonzero topological entropy, for in this case the ergodic p.m.p. actions with nonzero entropy appearing in the variational principle are again guaranteed to be free, this time by [51, 38] (in both of these cases one doesn’t need to assume that
the groups are not virtually cyclic, in accord with [2]). This second situation is what we need for our application in Theorem 9.9 when the acting group is finitely generated, but since we don’t assume finite generation there we will require the more general version. The argument will proceed by directly applying the geometric ideas in Austin’s approach to the topological framework.

We begin by recalling the definition of topological entropy. Let $G \curvearrowright X$ be an action on a compact metrizable space. Let $d$ be a compatible metric on $X$. For an open cover $\mathcal{U}$ of $X$ we write $N(\mathcal{U})$ for the minimum cardinality of a subcover, and for a nonempty finite set $F \subseteq G$ we write $\mathcal{U}^F$ for the join $\bigvee_{s \in F} s^{-1}\mathcal{U}$, defined as the collection of all sets of the form $\bigcap_{s \in S} s^{-1}U_s$ where $U_s \in \mathcal{U}$ for each $s \in F$. For a finite set $F \subseteq G$ and $\varepsilon > 0$, a set $A \subseteq X$ is said to be $(d, \alpha, F, \varepsilon)$-separated if for all distinct $x, y \in A$ there exists an $s \in F$ such that $d(x, y) > \varepsilon$. Write $\text{sep}_d(\alpha, F, \varepsilon)$ for the maximum cardinality of an $(d, \alpha, F, \varepsilon)$-separated subset of $X$.

Let $(F_n)$ be a Følner sequence for $G$. The topological entropy of $\alpha$ is defined by

$$h_{\text{top}}(\alpha) = \sup_{\mathcal{U}} \lim_{n \to \infty} \frac{1}{|F_n|} \log N(\mathcal{U}^F_n)$$

where $\mathcal{U}$ ranges over all finite open covers of $X$. The limit in the above expression exists due to the subadditivity of the function $\mathcal{U} \mapsto N(\mathcal{U})$ with respect to joins. One also has the following alternative formulation using separated sets (see Section 9.9 of [27]):

$$h_{\text{top}}(\alpha) = \sup_{\varepsilon > 0} \limsup_{n \to \infty} \frac{1}{|F_n|} \log \text{sep}_d(\alpha, F_n, \varepsilon).$$

Lemma 8.1. Let $G \curvearrowright X$ be an action on a compact metrizable space. Let $\eta > 0$. Then there is a finite set $K \subseteq G$ such that for every subgroup $H \subseteq G$ containing $K$ the restriction $H \curvearrowright X$ of $\alpha$ satisfies $h_{\text{top}}(\alpha') \geq h_{\text{top}}(\alpha) - \eta$.

Proof. Take a finite open cover $\mathcal{U}$ of $X$ such that for every sufficiently left invariant nonempty finite set $F \subseteq G$ one has $|F|^{-1} \log N(\mathcal{U}^F) \geq h_{\text{top}}(\alpha) - \eta$. Choose a Lebesgue number $\varepsilon > 0$ for $\mathcal{U}$. Then $N(\mathcal{U}^F) \leq \text{sep}_d(F, \varepsilon)$ for every finite set $F \subseteq G$, and so there exist a finite set $K \subseteq G$ and $\delta > 0$ such that every $(K, \delta)$-invariant nonempty finite set $F \subseteq G$ satisfies $|F|^{-1} \log \text{sep}_d(\alpha, F, \varepsilon) \geq h_{\text{top}}(\alpha) - \eta$. This $K$ then does the required job.

As in Section 3 for finite sets $E, K \subseteq G$ and a $\delta > 0$ we say that $E$ is $(K, \delta)$-invariant if $|E \cap \bigcap_{s \in K} s^{-1}E| \geq (1 - \delta)|E|$, which is equivalent to $|\{s \in E : Ks \subseteq E\}| \geq (1 - \delta)|E|$. 

Lemma 8.2. Let $G \curvearrowright X$ and $H \curvearrowright Y$ be free continuous actions of amenable groups on compact metrizable spaces and $\Phi : X \to Y$ a continuous orbit equivalence between $\alpha$ and $\beta$. Let $\kappa$ and $\lambda$ be as in Definition 2.3 with respect to $\Phi$. Let $L$ be a finite
subset of $H$ and $\delta > 0$. Let $E$ be a nonempty $(\lambda(L,Y),\delta)$-invariant finite subset of $G$. Then for every $x \in X$ the subset $\kappa(E,x)$ of $H$ is $(L,\delta)$-invariant.

\textit{Proof.} We may assume, by conjugating $\beta$ by $\Phi$, that $Y = X$ and $\Phi = \text{id}_X$. Set $K = \lambda(L,X)$, which is a finite subset of $G$ since $\lambda$ is continuous. Let $x \in X$, and set $F = \kappa(E,x)$. By the properties of $\kappa$ and $\lambda$, for every $g \in G$ and $t \in L$ we have

$$\beta_{\kappa(g,x)}x = \beta_t \beta_{\kappa(g,x)}x = \beta_t \alpha_g x = \alpha_{\lambda(t,\alpha_g)x} \alpha_g x = \alpha_{\lambda(t,\alpha_g)x} \alpha_g x \in \alpha_K g x$$

and hence $\beta_{L \kappa(g,x)} x \subseteq \alpha_K g x$. By freeness the map $g \mapsto \kappa(g, x)$ from $G$ to $H$ is bijective, and so we obtain

$$\left| \{ h \in F : L h \subseteq F \} \right| \geq \left| \{ g \in E : K g \subseteq E \} \right| \geq (1 - \delta)|E| = (1 - \delta)|F|,$$

that is, $F$ is $(L,\delta)$-invariant. \hfill $\square$

Suppose now that $G$ finitely generated and $S$ is a symmetric generating set for $G$. We equip $G$ with the word-length metric associated to $S$, whose value for a pair $(s,t) \in G \times G$ is by definition the least $r \in \mathbb{N}$ such that $s^{-1} t \in S^r$.

\textbf{Definition 8.3.} Let $r \in \mathbb{N}$ and $V \subseteq G$. We say that $V$ is $r$-\textit{connected} if for all $v, w \in V$ there exist $s_1, \ldots, s_n \in S^r$ such that $s_1 = v$, $s_n s_{n-1} \cdots s_1 = w$, and $s_k s_{k-1} \cdots s_1 \in V$ for every $k = 2, \ldots, n - 1$. In the case $r = 1$ we also simply say that $V$ is \textit{connected}.

We say that $V$ is $r$-\textit{separated} if the distance between any two distinct elements of $V$ is greater than $r$. A subset $C$ of $V$ is said to be $r$-\textit{spanning} for $V$ if for every $v \in W$ there is a $w \in C$ such that the distance between $v$ and $w$ is at most $r$.

The following is Lemma 3.3 of \cite{2}.

\textbf{Lemma 8.4.} For every finite set $F \subseteq G$ and $\delta > 0$ there exists a connected $(F,\delta)$-invariant finite subset of $G$.

The following is a slight variation of Corollary 8.6 of \cite{2} and follows by the same argument, which we reproduce below mutatis mutandis.

\textbf{Lemma 8.5.} Suppose that $G$ is not virtually cyclic. Then there is a constant $b > 0$ such that for every $r \in \mathbb{N}$ and every connected finite set $F \subseteq G$ with $|S^r F| \leq 2|F|$ there exists a $2r$-spanning tree in $F$ with at most $b |F| / r$ vertices.

\textit{Proof.} Since $G$ is not virtual cyclic there exists a there exists a $c \geq 0$ such that $|S^r| \geq cr^2$ for all $r \in \mathbb{N}$ (Corollary 3.5 of \cite{3}). Suppose now that we are given an $r \in \mathbb{N}$ and a connected finite set $F \subseteq G$ with $|S^r F| \leq 2|F|$. Take a maximal $r$-separated set $V \subseteq F$. Then

$$|V||S^r| = \left| \bigcup_{g \in V} S^r g \right| \leq |S^r F| \leq 2|F|$$

and so

$$|V| \leq \frac{2|F|}{|S^r|} \leq \frac{2c}{r^2}|F|.$$
By maximality, $V$ is $2r$-spanning for $F$. Consider now the graph on $V$ whose edges are those pairs of points which can be joined, within $F$, by a path of length at most $4r + 1$. This graph is connected because $F$ is connected, and so it contains a spanning tree $(V, E)$. For each pair in $E$ choose a path of length at most $4r + 1$ from $s$ to $t$ and write $E'$ for the collection of all pairs that appear as an edge in one of these paths. As the graph $(V, E)$ is connected, it contains a spanning tree $(V, E')$, and we have

$$|E'| \leq (4r + 1)|E| \leq 5r(|V| - 1) \leq 5r \cdot \frac{2c}{r^2}|F| = \frac{10c}{r}|F|.$$ 

We may thus take $b = 10c$. 

\[\square\]

**Theorem 8.6.** Let $G$ and $H$ be countably infinite amenable groups which are not locally virtually cyclic and let $G \not\bowtie X$ and $H \not\bowtie Y$ be topologically free actions on compact metrizable spaces which are continuously orbit equivalent. Then $h_{t_{\text{top}}} (\alpha) = h_{t_{\text{top}}} (\beta)$.

**Proof.** We show that $h_{t_{\text{top}}} (\alpha) \leq h_{t_{\text{top}}} (\beta)$, which by symmetry is enough to prove the theorem. We may assume that $h_{t_{\text{top}}} (\alpha) > 0$. We will also assume that $h_{t_{\text{top}}} (\alpha) < \infty$, the argument for $h_{t_{\text{top}}} (\alpha) = \infty$ being the same subject to obvious modifications.

We may assume that $Y = X$ and that $\text{id}_X$ is a continuous orbit equivalence by conjugating $\beta$ by a continuous orbit equivalence between $\alpha$ and $\beta$. Let $\kappa : G \times X \to H$ and $\lambda : H \times X \to G$ be the associated cocycles. Let $X_0$ be the dense $G_0$ subset of $X$ consisting of those points which have trivial stabilizer for $\alpha$ and $\beta$. This set is invariant for both $\alpha$ and $\beta$, as these actions share the same orbits. Fix a compatible metric $d$ on $X$.

Set $h = h_{t_{\text{top}}} (\alpha)$ for brevity. Let $\eta > 0$. Let $L$ be a finite subset of $H$ and $\delta > 0$. As $G$ is not locally virtually cyclic, using Lemma 8.1 we can find a finitely generated subgroup $G_0$ of $G$ such that $G_0$ contains $\lambda(L, X)$ and is not virtually cyclic and every nonempty finite set $F \subseteq G_0$ which is sufficiently left invariant, as a subset of $G_0$, satisfies $|F|^{-1} \log \text{sep}_d (\alpha, F, \varepsilon) > h - \eta$.

Fix a finite generating set for $G_0$ and endow $G_0$ with the associated word-length metric. Take a $b > 0$ as given by Lemma 8.5 with respect to this metric. Since $\kappa$ is continuous we can find a clopen partition $\mathcal{P}$ of $X$ such that for every $s \in G$ the map $x \mapsto \kappa(s, x)$ is constant on each member of $\mathcal{P}$. Choose an $r \in \mathbb{N}$ large enough so that $(b/r) \log |\mathcal{P}| \leq \eta$. Take an $(\lambda(L, X), \delta)$-invariant nonempty finite set $F \subseteq G_0$ which is also sufficiently left invariant, as a subset of $G_0$, so that both $|S^r F| \geq 2|F|$ and $|F|^{-1} \log \text{sep}_d (\alpha, F, \varepsilon) > h_{t_{\text{top}}} (\alpha) - \eta$. By Lemma 8.4 we may assume that $F$ is connected.

By uniform continuity there is an $\varepsilon' > 0$ such that if $x, y \in X$ are such that $d(sx, sy) < \varepsilon'$ for all $s$ in the ball of radius $2r$ in $G_0$ then $d(x, y) < \varepsilon$. By our choice of $F$ there exists a $(d, \alpha, F, \varepsilon)$-separated set $A \subseteq X$ of cardinality at least $e^{(h-\eta)|F|}$, and by making small perturbations we assume that $A \subseteq X_0$. 

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By our choice of $b$ and $F$ there exists a $2r$-spanning tree in $F$ whose vertex set $V$ satisfies $|V| \leq b|F|/r$. Root this tree at some arbitrarily chosen vertex $v$. Consider a directed path $(v_0, v_1, \ldots, v_n)$ in the tree with $v_0 = v$. Set $s_k = v_kv_{k-1}^{-1} \in S$ for $k = 2, \ldots, n$. Making repeated use of the cocycle identity we have, for every $x \in X$,

$$\kappa(v_n, x) = \kappa(s_n \cdots s_1, x) = \prod_{k=0}^{n} \kappa(s_{n-k}, s_{n-k-1} \cdots s_1 x).$$

This shows that the collection of all maps from $V$ to $H$ of the form $v \mapsto \kappa(v, x)$ for some $x \in X$ has cardinality at most $|\mathcal{P}|^{|V|}$, which is bounded above by $|\mathcal{P}|^{b|F|/r} = e^{(b/r)(\log|\mathcal{P}|)|F|}$, which in turn is bounded above by $e^{h|F|}$ by our choice of $r$. It follows by the pigeonhole principle that we can find an $A_0 \subseteq A$ with $A_0 \geq e^{-\eta|F|}|A| \geq e^{(h-2\eta)|F|}$ such that the map $v \mapsto \kappa(v, x)$ from $V$ to $H$ is the same for every $x \in A_0$. Write $W$ for the common image of these maps.

Choose an $x_0 \in A_0$ and set $E = \{\kappa(s, x_0) : s \in F\}$. Then $E$ is an $(L, \delta)$-invariant subset of $H$, and $|E| = |F|$ since $\alpha$ and $\beta$ act freely on $X_0$, which contains $A_0$. Since $A_0$ is $(d, \alpha, F, \varepsilon)$-separated and $V$ is $2r$-dense in $F$, the set $A_0$ is $(d, \alpha, V, \varepsilon')$-separated, and hence also $(d, \beta, W, \varepsilon')$-separated. Consequently

$$\frac{1}{|E|} \log \text{sep}_d(\beta, E, \varepsilon') \geq \frac{1}{|E|} \log \text{sep}_d(\beta, W, \varepsilon') \geq \frac{1}{|F|} \log |A_0| \geq h - 2\eta.$$ 

Since $\varepsilon'$ does not depend on $L$ and $\delta$, we thereby obtain $h_{\text{top}}(\beta) \geq h - 2\eta$. Letting $\eta \to 0$ we conclude that $h_{\text{top}}(\beta) \geq h$, as desired. \hfill \Box

9. $C^*$-simple alternating groups

Throughout this section $\Gamma$ is a countable amenable group.

We begin by formulating a property of countable amenable groups, called property $\text{ID}$, that will permit us to carry out the construction in the proof of Theorem 9.7. Among the $\Gamma$ which satisfy this property are those that are residually finite and contain a nontrivial element with infinite conjugacy class, as we show in Theorem 9.5.

By a tiling of $\Gamma$ we mean a finite collection $\{(S_i, C_i)\}_{i \in I}$ of pairs where the $S_i$ are nonempty finite subsets of $\Gamma$ (shapes) and the $C_i$ are subsets of $\Gamma$ (center sets) such that $\Gamma$ partitions as $\bigcup_{i \in I} \bigcup_{c \in C_i} S_i c$. The sets $S_i c$ in this disjoint union are called tiles. The tiling is a monotiling if the index set $I$ is a singleton.

By a tightly nested sequence $(\mathcal{T}_n)$ of tilings we mean that, for every $n > 1$ and $(S, C) \in \mathcal{T}_n$, the shape $S$ can be expressed as a partition $\bigcup_{j \in J} S_j g_j$, where the $S_j$ are shapes of $\mathcal{T}_{n-1}$ and the $g_j$ are elements of $\Gamma$, such that for every $c \in C$ and $j \in J$ the element $g_j c$ is a tiling center for the shape $S_j$ (in which case $\bigcup_{j \in J} S_j g_j c$ is a partition of $Sc$ into tiles of $\mathcal{T}_{n-1}$). Observe that if the sequence $(\mathcal{T}_n)$ is tightly nested then all of its subsequences are as well.
We call \((\mathcal{T}_n)\) a sequence of Følner tilings if for every finite set \(F \subseteq \Gamma\) and \(\delta > 0\) there is an \(N \in \mathbb{N}\) such that for every \(n \geq N\) each shape \(S\) of the tiling \(\mathcal{T}_n\) is \((F, \delta)\)-invariant, i.e., \(|S \cap \bigcap_{t \in F} S t^{-1}| \geq (1 - \delta)|F|\).

As in Section 3 of [16], to each tiling \(\{(S_i, C_i)\}_{i \in I}\) of \(\Gamma\) we associate an action \(\alpha\) of \(\Gamma\) as follows. Write \(Y = \{S_i : i \in I\} \cup \{0\}\) and define \(x \in Y^\Gamma\) by \(x_t = S_i\) if \(t \in C_i\) and \(x_t = 0\) otherwise. The action \(\alpha\) is then the restriction of the right shift action \(\Gamma \acts Y^\Gamma\) to the closed \(\Gamma\)-invariant set \(\overline{\Gamma x}\). The entropy of the tiling is defined to be the topological entropy of this subshift action.

A set \(C \subseteq \Gamma\) is said to be syndetic if there is a finite set \(F \subseteq \Gamma\) for which \(FC = \Gamma\).

**Definition 9.1.** We say that \(\Gamma\) has property ID if there exist an infinite cyclic subgroup \(H \subseteq \Gamma\) of infinite index and a tightly nested sequence \((\mathcal{T}_n)\) of Følner tilings with syndetic center sets and entropies converging to zero such that

(i) for every finite set \(F \subseteq \Gamma\) there is an \(n \in \mathbb{N}\) such that \(F\) is contained in some tile of \(\mathcal{T}_n\),

(ii) for every \(\delta > 0\) there is an \(n_0 \in \mathbb{N}\) such that for all \(n \geq n_0\) one has \(|H \cap T| \leq \delta|T|\) for every tile \(T\) of \(\mathcal{T}_n\), and

(iii) for every \(n \in \mathbb{N}\) there are a shape \(S\) of \(\mathcal{T}_n\) and a subgroup \(H_0\) of \(H\) such that the sets \(Sc\) for \(c \in H_0\) are tiles of \(\mathcal{T}_n\) whose union contains \(H\).

**Example 9.2.** \(\mathbb{Z}^d\) for \(d \geq 2\) is the prototype of a group with property ID.

**Example 9.3.** More generally, if \(\Gamma\) is of the form \(\Gamma_0 \times \mathbb{Z}\) where \(\Gamma_0\) is infinite then \(\Gamma\) has property ID. To verify this we can use tilings which are products (in the obvious sense) of a suitable tiling of \(\Gamma_0\), as given by Theorem 3.5 of [15], and a monotiling of \(\mathbb{Z}\) by translates of an interval of the form \([-n, -n + 1, \ldots, n]\).

In another direction of generalization we have Theorem 9.5 below. To establish this we need the following lemma.

**Lemma 9.4.** Let \(N_1 \supseteq N_2 \supseteq \ldots\) be a decreasing sequence of finite-index normal subgroups of \(\Gamma\) with intersection \(\{1\}\). Let \(a\) be an element of \(\Gamma\) with infinite conjugacy class, and set \(H = \langle a \rangle\). Then \(|\Gamma/N_k|/|HN_k/N_k| \to \infty\) as \(k \to \infty\).

**Proof.** We may naturally identify \(\Gamma\) with a subgroup of the profinite group \(P = \lim \leftarrow G/N_k\), in which case the limit \(\lim_{k \to \infty} |\Gamma/N_k|/|HN_k/N_k|\) coincides with the index of the closure, \(\overline{\Gamma}\), of \(H\) in \(P\). The group \(\overline{\Gamma}\) is abelian since \(H\) is, and so if \(\overline{\Gamma}\) were of finite index in \(P\) then \(a\) would commute with the finite-index subgroup \(\Gamma \cap \overline{\Gamma}\) of \(\Gamma\), which is impossible since the conjugacy class of \(a\) in \(\Gamma\) is infinite. \(\square\)

**Theorem 9.5.** Suppose that \(\Gamma\) is residually finite and that it contains a nontorsion element with infinite conjugacy class. Then \(\Gamma\) has property ID.

**Proof.** Let \(a\) be a nonorsion element of \(\Gamma\) with infinite conjugacy class, and write \(H\) for the subgroup of \(\Gamma\) it generates. Fix an increasing sequence \(\{1\} = E_1 \subseteq E_2 \subseteq \ldots\) of finite subsets of \(\Gamma\) whose union is equal to \(\Gamma\), and a decreasing sequence \(1 = \varepsilon_1 > \varepsilon_2 > \ldots\)
... of strictly positive real numbers tending to zero. We will recursively construct a tightly nested sequence of monotilings $\mathcal{T}_k = \{(S_k, N_k)\}$ such that $N_1 \supseteq N_2 \supseteq \ldots$ is a decreasing sequence of finite-index normal subgroups of $\Gamma$ with intersection $\{1\}$ and for every $k$ the shape $S_k$ is $(E_k, \varepsilon_k)$-invariant and contains $E_k$, the pair $\{(S_k \cap H, N_k \cap H)\}$ is a monotiling of $H$, and $|H \cap S_k c| \leq \varepsilon_k |S_k|$ for every $c \in N_k$.

Since $N_k \cap H$ is a subgroup of $H$ for each $k$, the unique center set of a monotiling is trivially syndetic, and the entropies of a sequence of Følner monotilings must converge to zero (as can be seen by a density argument and Stirling’s formula—see Lemma 3.9 of [16]), this will establish property ID.

For the base case $k = 1$ set $\mathcal{T}_1 = \{\{(1\Gamma), \Gamma\}\}$. Now let $k > 1$ and assume that we have constructed $\mathcal{T}_{k-1} = \{(S_{k-1}, N_{k-1})\}$ with the desired properties. As is readily seen, there exists a $\delta > 0$ with $\delta \leq \varepsilon_k/2$ such that every $(E_k, \delta)$-invariant nonempty finite set $F \subseteq \Gamma$ has the property that every set $F' \subseteq \Gamma$ satisfying $|F'| = |F|$ and $|F' \cap F| \geq (1 - \delta)|F|$ is $(E_k, \varepsilon_k)$-invariant. Since $\Gamma$ is residually finite, by the quasitiling argument in Section 2 of [50], we can find a finite-index normal subgroup $N_k \subseteq \Gamma$ with $N_k \subseteq N_{k-1}$ and $N_k \cap (E_k \setminus \{1\}) = \emptyset$ and a transversal $F$ for $N_k$ in $\Gamma$ which contains $E_k$ and is $(E_k, \delta)$-invariant and is also sufficiently left invariant so that

$$|\{c \in \Gamma : F \cap S_{k-1} c \neq \emptyset \text{ and } (\Gamma \setminus F) \cap S_{k-1} c \neq \emptyset\}| \leq \delta |F|/|S_{k-1}|.$$  

We may moreover assume, in view of Lemma 9.4 that $|HN_k/N_k| \leq \delta |\Gamma/N_k|$, which implies that

$$|HN_k \cap F| \leq \delta |F|.  \tag{3}$$

Define $S_k$ to be the set $S_{k-1}(F \cap N_{k-1})$, which is equal to the disjoint union $\bigcup_{c \in F \cap N_{k-1}} S_{k-1} c$. Since $S_{k-1}$ is a transversal for $N_{k-1}$ in $\Gamma$ and $F \cap N_{k-1}$ is a transversal for $N_k$ in $N_{k-1}$, it follows that $S_k$ is a transversal for $N_k$ in $\Gamma$, and so $\mathcal{T}_k := \{(S_k, N_k)\}$ is a monotiling of $\Gamma$.

For every element $t$ of $F$ not belonging to $S_{k-1}(F \cap N_{k-1})$, the fact that the sets $S_{k-1} c$ for $c \in N_{k-1}$ tile $\Gamma$ implies the existence of a $c_0 \in N_{k-1} \setminus F$ such that $t \in S_{k-1} c_0$, in which case $S_{k-1} c_0$ intersects both $F$ and $\Gamma \setminus F$ (since $1 \Gamma \subseteq S_{k-1}$). In view of (2) this implies that the set of all such $t$ has cardinality less than $\delta |F|$, so that

$$|S_k \cap F| = |S_{k-1}(F \cap N_{k-1}) \cap F| \geq (1 - \delta)|F|. \tag{4}$$

Since $|S_k| = |\Gamma/N_k| = |F|$ it follows by our choice of $\delta$ that $S_k$ is $(E_k, \varepsilon_k)$-invariant.

Finally, observe using (3) and (4) that for all $c \in N_k$ we have

$$|H \cap S_k c| \leq |HN_k \cap N_k| \leq |HN_k \cap F| + |S_k \setminus F| \leq 2\delta |F| \leq \varepsilon_k |F| = \varepsilon_k |S_k|.$$

This completes the recursive construction.

It remains to observe that, since $S_1 \subseteq S_2 \subseteq \ldots$ and the tiling center sets $N_k$ are normal subgroups, the sequence $(\mathcal{T}_k)$ is tightly nested. $\square$

We next aim to prove Theorem 9.7.
Lemma 9.6. Suppose that \( \Gamma \) is torsion-free and let \( \Gamma \curvearrowright X \) be a minimal action on a compact metrizable space with nonzero topological entropy. Then the action is topologically free.

Proof. By the variational principle (Theorem 9.48 of [27]) there is a \( \Gamma \)-invariant Borel probability measure \( \mu \) on \( X \) such that the measure entropy of the action \( \Gamma \curvearrowright (X, \mu) \) is nonzero. By \([11, 38]\) and torsion-freeness, there is a \( \mu \)-nonnull, and in particular nonempty, set of points in \( X \) whose stabilizer is trivial. By minimality this implies that the action \( \Gamma \curvearrowright X \) is topologically free. \( \square \)

Theorem 9.7. Suppose that \( \Gamma \) is torsion-free and has property ID. Let \( \lambda \in (0, \infty) \). Then there is an expansive topologically free minimal action \( \alpha \) of \( \Gamma \) on the Cantor set with topological entropy \( \lambda \) such that \( A(\alpha) \) is \( C^* \)-simple.

Proof. Fix an infinite cyclic subgroup \( H \subseteq \Gamma \) and a tightly nested sequence \( (\mathcal{T}_n) \) of Følner tilings for \( \Gamma \) as in the definition of Property ID.

Let \( q \) be an integer greater than both 3 and \( e^{2\lambda} \). For a set \( L \subseteq \Gamma \) we will write \( \pi_L \) for the coordinate projection map \( x \mapsto x|_L \) from \( \{1, \ldots, q\}^\Gamma \) to \( \{1, \ldots, q\}^L \). By a box in \( \{1, \ldots, q\}^\Gamma \) we mean a subset of \( \{1, \ldots, q\}^\Gamma \) of the form \( \prod_{s \in \Gamma} A_s \) where \( A_s \subseteq \{1, \ldots, q\} \) for each \( s \in \Gamma \). For every box \( A = \prod_{t \in T} A_t \subseteq \{1, \ldots, q\}^T \) and \( T \subseteq \Gamma \) write

\[
D_T(A) = \{ t \in T : A_t = \{1, \ldots, q\} \}.
\]

Set \( \theta = \lambda / \log q \), and note that \( \theta < 1/2 \) by our choice of \( q \). We will recursively construct an increasing sequence \( 1 = n_0 < n_1 < n_2 < \ldots \) of integers and a decreasing sequence \( A_0 = \{1, \ldots, q\}^\Gamma \supseteq A_1 = \prod_{t \in T} A_t \supseteq A_2 = \prod_{t \in T} A_{t,1} \supseteq \ldots \) of boxes in \( \{1, \ldots, q\}^\Gamma \) such that, writing \( F_k \) for the tile of \( \mathcal{T}_{n_k} \) containing \( 1_\Gamma \), we have the following for every \( k \geq 1 \):

\begin{itemize}
  \item[(i)] \( \theta + 1/2^{k+1} < |D_T(A_k)|/|T| < \theta + 1/2^k \) for every tile \( T \) of \( \mathcal{T}_{n_k} \).
  \item[(ii)] \( |T|/2^{k+4} > 1 \) for every tile \( T \) of \( \mathcal{T}_{n_k} \).
  \item[(iii)] \( H \subseteq D_T(A_k) \).
  \item[(iv)] for every \( (S; C) \in \mathcal{T}_{n_k} \) one has \( A_{k, sc} = A_{k, sd} \) for all \( s \in S \) and \( c, d \in C \).
  \item[(v)] \( \pi_{F_{k-1}}(A_k) = \pi_{F_{k-1}}(A_{k-1}) \).
\end{itemize}

As indicated we take \( n_0 = 1 \) and \( A_0 = \{1, \ldots, q\}^\Gamma \). Suppose now that \( k \geq 1 \) and that we have defined \( n_{k-1} \) and \( A_{k-1} \).

Let \( \{S_j\}_{j \in J} \) be the collection of shapes of the tiling \( \mathcal{T}_{n_{k-1}} \) and for each \( j \) choose a tiling center \( c_j \) for \( S_j \) and set \( W_j = \prod_{s \in S_j} A_{k-1, sc_j} \) (this is independent of the choice of the \( c_j \) since (iv) holds for \( k-1 \)).

Take an integer \( n_k > n_{k-1} \) which is large enough for purposes to be specified. Pick a set \( \mathcal{R} \) of representatives among the tiles of \( \mathcal{T}_{n_k} \) for the relation of having the same shape, and include among these representatives the tile \( F_k \) which contains \( 1_\Gamma \).

Let \( T \in \mathcal{R} \). Then we have \( T = \bigsqcup_{i \in I} T_i \) for some tiles \( T_i \) from \( \mathcal{T}_{n_{k-1}} \). By taking \( n_k \) large enough (independently of the particular \( T \in \mathcal{R} \) at hand, which we can do
because $\mathcal{R}$ is finite) we may ensure that $|T|/2^{k+4} > 1$ and that $I$ can be partitioned as $I' \sqcup I'' \sqcup \bigcup_{j \in J} I_j$ where

(a) the tiles $T_i$ for $i \in I'$ cover a large enough proportion of $T$ so that

$$\sum_{i \in I'} |T_i| > \frac{\theta + 1/2^{k+1}}{\theta + 5/2^{k+3}} |T|,$$

(b) $I''$ is equal to the set of all $i \in I$ such that either $T_i = F_{k-1}$ or $T_i \cap H \neq \emptyset$ and the tiles $T_i$ for $i \in I''$ cover a small enough proportion of $T$ (as is possible by condition (ii) in the definition of property ID) so that

$$\sum_{i \in I''} |T_i| < \frac{1}{2^{k+3}} |T|$$

and also so that we can arrange (c) below,

(c) for each $j \in J$ the tiles $T_i$ for $i \in I_j$ all have shape $S_j$ and there exists a surjection $\varphi_j : I_j \to W_j$ such that for every $i \in I_j$ we have $\varphi_j(i)_s \in A_{k-1,sc}$ for all $s \in S_j$ where $c$ is the tiling center for $T_i$ (this is possible view of (iv) for $k - 1$, since the tiling center sets for $\mathcal{R}_{k-1}$ are syndetic and so $T$ can be chosen to be sufficiently left invariant so that its intersection with each of these tiling center sets has sufficiently large cardinality both to enable the existence of such $\varphi_j$ and to allow for (a), with the tiles $T_i$ for $i \in I''$ being assumed at the same time to cover a small enough proportion of $T$).

Using this partition of $I$ we define the sets $A_{k,t}$ for $t \in T$ in three stages:

(d) For $i \in I''$ and $t \in T_i$ define $A_{k,t} = A_{k-1,t}$.

(e) For each $j \in J$ and $i \in I_j$ define $A_{k,sc} = \varphi_j(i)_s$ for all $s \in S_j$ where $c$ is the tiling center for $T_i$.

(f) For each $i \in I'$ choose a set $T_i' \subseteq D_{T_i}(A_{k-1})$ with

$$(\theta + 5/2^{k+3})|T_i| < |T_i'| < (\theta + 6/2^{k+3})|T_i|,$$

which we can do since $|T_i|/2^{k+3} > 1$ and $|D_{T_i}(A_{k-1})| > (\theta + 1/2^k)|T_i|$ by hypothesis (including the case $k = 1$ since $\theta < 1/2$). We define $A_{k,t} = A_{k-1,t}$ for all $t \in T_i'$, while for $t \in T_i \setminus T_i'$ we define $A_{k,t}$ to be any singleton contained in $A_{k-1,t}$.

Note that by (b) and (f) we have

$$D_T(A_k) < (\theta + 6/2^{k+3}) \sum_{i \in I'} |T_i| + \sum_{i \in I''} |T_i| < (\theta + 1/2^k)|T|$$

while by (a) and (f) we have

$$D_T(A_k) > (\theta + 5/2^{k+3}) \sum_{i \in I'} |T_i| > (\theta + 1/2^{k+1})|T|. $$
Having thereby defined $A_{k,t}$ for all $t$ in each tile in our collection $\mathcal{R}$ of shape representatives, we can now extend the definition of $A_{k,t}$ to all $t \in \Gamma$ in the unique way that enables us to satisfy (iv). Since the sequence of tilings $(\mathcal{R}_n)$ is tightly nested and (iv) holds for $k - 1$, we will have the inclusion $A_{k,t} \subseteq A_{k-1,t}$ for every $t \in \Gamma$. Condition (i) for $k$ is verified by (5) and (6), while conditions (ii) and (v) are built into the construction. Since condition (iii) holds for $k - 1$, it is guaranteed to hold for $k$ by our construction.

Set $A = \bigcap_{k=1}^{\infty} A_k$, which is a decreasing intersection of nonempty closed sets and hence is itself nonempty and closed. Consider the right shift action $\Gamma \curvearrowright \{1, \ldots, q\}^\Gamma$ and set $X = \Gamma A$, which is closed and $\Gamma$-invariant. Let $\Gamma \curvearrowright X$ be the subshift action. It remains to check that $\alpha$ is topologically free and minimal, that its topological entropy is equal to $\lambda$, and that the group $\mathcal{A}(\alpha)$ is $C^*$-simple. Note that minimality, topological freeness, and the infiniteness of $\Gamma$ together imply that the space $X$ has no isolated points and hence, being a compact subset of a metrizable zero-dimensional space, is the Cantor set.

For minimality, let $x, y \in A$. Let $F$ be a nonempty finite subset of $\Gamma$. Given that the sequence $(\mathcal{R}_n)$ came from the definition of property ID, we can find a $k$ such that $F$ is contained in a tile $T$ of $\mathcal{R}_{nk}$. Let $(S, C) \in \mathcal{J}_{nk+1}$. By the construction of the set $A_{k+1}$, for every $c \in C$ there is a $\tilde{c} \in \Gamma$ such that $T\tilde{c} \subseteq Sc$ and $y_{\tilde{c}} = x_t$ for all $t \in T$, in which case $\tilde{c}y_F = x_F$. Also, since $C$ is syndetic by the definition of property ID, there is a finite set $E \subseteq \Gamma$ such that $EC = \Gamma$, in which case the set $\tilde{C} = \{\tilde{c} : c \in C\}$ satisfies $E\tilde{C} = \Gamma$, and hence is syndetic. From these observations we deduce that $x \in \Gamma y$ and, by taking $y = x$ and applying a standard characterization of minimality (Proposition 7.13 of [27]), that the action $\Gamma \curvearrowright \Gamma x$ is minimal. On the other hand, reversing the roles of $x$ and $y$ we get $y \in \Gamma x$ whence $\Gamma A = \Gamma x$ by the minimality of $\Gamma \curvearrowright \Gamma x$, so that $\alpha$ is minimal.

For the entropy calculation, an inductive application of (v) shows that for every $k \in \mathbb{N}$ we have $\pi_{F_k}(A_j) = \pi_{F_k}(A_k)$ for all $j \geq k$ and hence $\pi_{F_k}(A) = \pi_{F_k}(A_k)$, so that, by (i),

$$\frac{1}{|F_k|} \log |\pi_{F_k}(A)| = \frac{|D_{F_k}(A_k)|}{|F_k|} \log q \geq (\theta + 1/2^{k+1}) \log q > \lambda.$$ 

Since $(F_k)_k$ is a Følner sequence for $\Gamma$, it follows that $h_{\text{top}}(\Gamma, X) \geq \lambda$.

Let $\delta > 0$. Take a large enough $k \in \mathbb{N}$ so that the tiling $\mathcal{R}_{nk}$ has entropy less than $\delta$, which we can do by the definition of property ID. Writing $S$ for the union of the shapes of the tiling $\mathcal{R}_{nk}$, any sufficiently left invariant nonempty finite set $F \subseteq \Gamma$ will be such that the set of restrictions to $S^{-1}F$ of elements in the right shift action associated to $\mathcal{R}_{nk}$ as in the definition of tiling entropy has cardinality at most $e^{\delta |F|}$.

Take a set $R$ of representatives in $\Gamma A_k$ for the relation of having the same restriction to $F$. We may assume that $R \subseteq \Gamma A_k$ via perturbation. In view of (i) we then have

$$\log |\pi_F(R)| \leq \log(e^{\delta |F|} q^{(\theta + 1/2^k) |F|}) = \delta |F| + (\theta + 1/2^k)(\log q) |F|.$$

Taking a Følner sequence of such $F$, we deduce that the topological entropy of the action $\Gamma \curvearrowright \mathcal{F}_k$ is less than $(\delta + \theta + 1/2^k) \log q$. Since $X \subseteq \mathcal{F}_k$ it follows by the monotonicity of entropy that $h_{\text{top}}(\Gamma, X) \leq (\delta + \theta + 1/2^k) \log q$. Since we can take $\delta$ arbitrarily small and $k$ arbitrarily large, we infer that $h_{\text{top}}(\Gamma, X) \leq \theta \log q = \lambda$ and hence that $h_{\text{top}}(\Gamma, X) = \lambda$, as desired.

As we now know the topological entropy to be nonzero, we can deduce topological freeness from Lemma 9.6 using the fact that the action is minimal and $\Gamma$ is torsion-free.

Finally we verify $C^*$-simplicity. It is enough to show that for every nonempty open set $U \subseteq X$ the rigid stabilizer $A(\alpha)_U$ (i.e., the group of all homeomorphisms in $A(\alpha)$ which fix every point in the complement of $U$) is nonamenable, for this implies by [32] that every nontrivial uniformly recurrent subgroup of $A(\alpha)$ is nonamenable and hence by [24] that $A(\alpha)$ is $C^*$-simple (Corollary 1.3 of [32]). Let $U$ be such an open set. Since $\Gamma A$ is dense in $X$ there is an $s \in \Gamma$ such that $A \cap sU \neq \emptyset$, and as $A(\alpha)_{sU} = sA(\alpha)_{U}s^{-1}$ we may thus assume, for the purpose of establishing the nonamenability of $A(\alpha)_U$, that $A \cap U \neq \emptyset$. We will proceed by showing that $\mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2$ embeds into $A(\alpha)_U$, which implies the desired nonamenability since this free product contains a free subgroup on two generators (take for instance the product of generators of the first two copies of $\mathbb{Z}_3$ and the product of the generators of the second two copies).

Fix a $z \in A \cap U$. Then there is a finite set $F \subseteq \Gamma$ such that the clopen set $\{x \in X : x|_F = z|_F\}$ is contained in $U$. Fix a generator $a$ of the group $H$. By the definition of property ID, there exist $m, k \in \mathbb{N}$ and a shape $S$ of $\mathcal{T}_n$ containing $F \cup \{1\}$ such that the sets $Sa^m$ for $j \in \mathbb{Z}$ are tiles of $\mathcal{T}_n$ whose union contains $H$. Let $l \in \{1, 2, 3\}$. Write $V_l$ for the set of all $x \in X$ such that

- $x_{a^{-m}} = l$,
- $x_{a^m}, x_{a^{3m}},$ and $x_{a^{5m}}$ are distinct elements of $\{1, 2, 3, 4\} \setminus \{l\}$,
- $x_{a^{2jm}} = z_4$ for all $s \in S$ and $j = 0, 1, 2, 3$.

Then $V_l, a^{2m}V_l, a^{4m}V_l,$ and $a^{6m}V_l$ are pairwise disjoint clopen subsets of $X$ which are contained in $U$ and so we can define an element $g_l$ of $A(\alpha)_U$ by setting

- $g_lx = a^{6m}x$ for all $x \in V_l$,
- $g_lx = a^{-6m}x$ for all $x \in a^{6m}V_l$,
- $g_lx = a^{2m}x$ for all $x \in a^{2m}V_l$,
- $g_lx = a^{-2m}x$ for all $x \in a^{4m}V_l$,
- $g_lx = x$ for all $x \in X \setminus (V_l \cup a^{2m}V_l \cup a^{4m}V_l \cup a^{6m}V_l)$.

Each $g_l$ for $l = 1, 2, 3$ generates a copy of $\mathbb{Z}_2$ inside of $A(\alpha)$ (each is the image of an even permutation under an embedding of the ordinary alternating group $A_4$ into the dynamical alternating group $A_4(\alpha)$ of the kind that appears in the definition of the latter, and $A_4(\alpha)$ is contained in $A(\alpha)$ according to the comment after Definition [2.1]). We will check that these three elements together generate a copy of $\mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2$. 
Let $r \in \mathbb{N}$ and let $(l_1, \ldots, l_r)$ be a tuple in $\{1, 2, 3\}^r$ such that $l_{j+1} \neq l_j$ for each $j = 1, \ldots, r - 1$. As $z \in A$ and each $A_k$ is a box, by (iii), (iv), and (v) we can find an $x \in X$ such that

- $x a^{(6j-1)m} = l_{j+1}$ for every $j = 0, \ldots, r - 1$,
- $x a^{(6j+1)m}$, $x a^{(6j+3)m}$, and $x a^{(6j+5)m}$ are distinct elements of $\{1, 2, 3\} \setminus \{l_{j+1}\}$ for every $j = 0, \ldots, r - 1$,
- $x a^{(6j)m} = z_s$ for every $s \in S$ and $j = 0, \ldots, r$,
- $x a^{(6r-1)m} \neq x a^{-m}$.

Then for each $j = 0, \ldots, r - 1$ we have $g_{l_1} g_{l_2} \cdots g_{l_j} x \in \mathcal{V}_{l_{j+1}}$ and hence $g_{l_1} g_{l_2} \cdots g_{l_r} x = a^{\text{perm}_m x} \neq x$, so that $g_{l_1} g_{l_2} \cdots g_{l_r}$ is not the identity element in $A(\alpha)$. We conclude that the subgroup generated by $g_1, g_2$, and $g_3$ is isomorphic to $\mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2$. \hfill \Box

**Remark 9.8.** If in the conclusion of the theorem we drop the requirement that $A(\alpha)$ be $C^*$-simple then we do not need to assume that $\Gamma$ has property ID. That is, for every torsion-free countable amenable group $\Gamma$ and $\lambda \in (0, \infty)$ there exists an expansive topologically free minimal action $\alpha$ of $\Gamma$ on the Cantor set with topological entropy $\lambda$.

By Theorem 2.4 the dynamical alternating group is an invariant of continuous orbit equivalence. Since a torsion-free group which is locally virtually cyclic must be Abelian, Theorems 9.7 and 8.6 then combine to yield the following. All of the alternating groups below are simple by Theorem 2.2(i), have property Gamma by Theorem 5.5, and are finitely generated when $\Gamma$ is finitely generated by Theorem 2.2(ii).

**Theorem 9.9.** Suppose that $\Gamma$ is torsion-free and has property ID. Then there is an uncountable family of topologically free expansive minimal actions $\alpha$ of $\Gamma$ on the Cantor set such that $A(\alpha)$ is $C^*$-simple and such that the groups $A(\alpha)$ for different $\alpha$ are pairwise nonisomorphic.

In view of Theorem 9.5 and Example 9.3 we obtain the following corollary.

**Corollary 9.10.** Suppose that $\Gamma$ is either (i) torsion-free, ICC, and residually finite or (ii) of the form $\Gamma_0 \times \mathbb{Z}$ where $\Gamma_0$ is nontrivial and torsion-free. Then there is an uncountable family of topologically free expansive minimal actions $\alpha$ of $\Gamma$ on the Cantor set such that $A(\alpha)$ is $C^*$-simple and such that the groups $A(\alpha)$ for different $\alpha$ are pairwise nonisomorphic.

Examples of groups which are torsion-free, ICC, and residually finite are the wreath product $\mathbb{Z} \wr \mathbb{Z}$ and any torsion-free weakly branch group, such as the basilica group.

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