Dark-matter halos and the $M$–$\sigma$ relation for supermassive black holes

Adam C. Larkin* and Dean E. McLaughlin†
Astrophysics Group, Lennard-Jones Laboratories, Keele University, Keele, Staffordshire, ST5 5BG, UK

9 January 2018

ABSTRACT
We develop models of two-component spherical galaxies to establish scaling relations linking the properties of spheroids at $z = 0$ (total stellar masses, effective radii $R_e$, and velocity dispersions within $R_e$) to the properties of their dark-matter halos at both $z = 0$ and higher redshifts. Our main motivation is the widely accepted idea that the accretion-driven growth of supermassive black holes (SMBHs) in protogalaxies is limited by quasar-mode feedback and gas blow-out. The SMBH masses, $M_{BH}$, should then be connected to the dark-matter potential wells at the redshift $z_{qso}$ of the blow-out. We specifically consider the example of a power-law dependence on the maximum circular speed in a protogalactic dark-matter halo: $M_{BH} \propto V_{d,pk}^4$, as could be expected if quasar-mode feedback were momentum-driven. For halos with a given $V_{d,pk}$ at a given $z_{qso} \geq 0$, our model scaling relations give a typical stellar velocity dispersion $\sigma_{ap}(R_e)$ at $z = 0$. Thus, they transform a theoretical “$M_{BH}$–$V_{d,pk}$ relation” into a prediction for an observable $M_{BH}$–$\sigma_{ap}(R_e)$ relation. We find the latter to be distinctly non-linear in log-log space. Its shape depends on the generic redshift-evolution of halos in a ΛCDM cosmology and the systematic variation of stellar-to-dark matter mass fraction at $z = 0$, in addition to any assumptions about the physics underlying the $M_{BH}$–$V_{d,pk}$ relation. Despite some clear limitations of the form we use for $M_{BH}$ versus $V_{d,pk}$, and even though we do not include any SMBH growth through dry mergers at low redshift, our results for $M_{BH}$–$\sigma_{ap}(R_e)$ compare well to data for local early types if we take $z_{qso} \sim 2-4$.

Key words: galaxies: bulges – galaxies: quasars: supermassive black holes – galaxies: elliptical and lenticular – galaxies: halos – galaxies: fundamental parameters

1 INTRODUCTION
The masses $M_{BH}$ of supermassive black holes (SMBHs) at the centres of normal early-type galaxies and bulges correlate with various global properties of the stellar spheroids—see Kormendy & Ho (2013) for a comprehensive review. The strongest relationships include one between $M_{BH}$ and the bulge mass $M_{bulge}$ (either stellar or dynamical, depending on the author; e.g., Magorrian et al. 1998; Marconi & Hunt 2003; Haring & Rix 2004; McConnell & Ma 2013); a scaling of $M_{BH}$ with the (aperture) stellar velocity dispersion $\sigma_{ap}$ averaged inside some fraction of the effective radius $R_e$ of the bulge ($M_{BH} \sim \sigma_{ap}^{1.5}$ if fitted with a single power law: Ferrarese & Merritt 2000; Gebhardt et al. 2000; Ferrarese & Ford 2005; McConnell & Ma 2013); and a fundamental-plane dependence of $M_{BH}$ on a combination of either $M_{bulge}$ and $\sigma_{ap}$ or $\sigma_{ap}$ and $R_e$ (Hopkins et al. 2007b,c). Whether any one correlation is more fundamental than the others is something of an open question, but collectively they are interpreted as evidence for co-evolution between SMBHs and their host galaxies.

This co-evolution likely involved self-regulated feedback in general. Most of the SMBH mass in large galaxies is grown in a quasar phase of Eddington-rate accretion (Yu & Tremaine 2002), driven by a rapid succession of gas-rich mergers at high redshift. Such accretion deposits significant momentum and energy back into the protogalactic gas supply, which can lead to a blow-out that stops further accretion onto the SMBH. In this context, the empirical correlation between $M_{BH}$ and $\sigma_{ap}$ takes on particular importance, as the stellar velocity dispersion should reflect the depth of the potential well from which SMBH feedback had to expel the protogalactic gas. Cosmological simulations of galaxy formation now routinely include prescriptions for the quenching of Eddington-rate accretion by “quasar-mode” feedback, with free parameters that are tuned to give good fits to the SMBH $M$–$\sigma$ relation at $z = 0$. 

* E-mail: a.larkin@keele.ac.uk
† E-mail: d.e.mclaughlin@keele.ac.uk

© 2015 RAS
However, it is not clear in detail how the stellar velocity dispersions in normal galaxies at $z = 0$ relate to the protogalactic potential wells when any putative blow-out occurred and the main phase of accretion-driven SMBH growth came to an end. For most systems, this was presumably around $z \sim 2-3$, when quasar activity in the Universe was at its peak (Richards et al. 2006; Hopkins et al. 2007a). The potential wells in question were dominated by dark matter, and a general method is lacking to connect the stellar $\sigma_{ap}$ in spheroids to the properties of their dark-matter halos, not only at $z = 0$ but at higher redshift as well. Moreover, it is not necessarily obvious what specific property (or properties) of dark-matter halos provides the key measure of potential-well depth in the context of a condition for accretion-driven blow-out. Different simulations of galaxy and SMBH co-evolution with different recipes for quasar-mode feedback appear equally able (with appropriate tuning of their free parameters) to reproduce the observed $M-\sigma$ relation.

Our main goal in this paper is to address the first part of this problem. We develop “mean-trend” scaling relations between the average stellar properties (total masses, effective radii and aperture velocity dispersions) and the dark-matter halos (virial masses and radii, density profiles and circular-speed curves) of two-component spherical galaxies. These scalings are constrained by some data for a representative sample of local early-type galaxies, and by the properties of dark-matter halos at $z = 0$ in cosmological simulations. We then include an analytical approximation to the mass and potential-well growth histories of simulated dark-matter halos, in order to connect the stellar properties at $z = 0$ to halo properties at $z > 0$. We ultimately use these results to illustrate how one particularly simple analytical expression, which gives a critical SMBH mass for protogalactic blow-out directly in terms of the dark-matter potential well at quasar redshifts, translates to a relation between SMBH mass and stellar velocity dispersion at $z = 0$.

1.1 SMBH masses and halo circular speeds

Under the assumption (which we discuss just below) that accretion feedback is momentum-conserving and takes the form of a spherical shell driven outwards by an SMBH wind with momentum flux $dp_{\text{wind}}/dt = L_{\text{Edd}}/c$, McQuillin & McLaughlin (2012) derive a minimum SMBH mass sufficient to expel an initially static and virialised gaseous medium from any protogalaxy consisting of dark matter and gas only. This critical mass is approximately

$$M_{\text{BH}} \simeq \frac{f_{\text{K}} V_{d, pk}^4}{\pi G^2} \approx 1.14 \times 10^9 M_\odot \left( \frac{f_0}{0.2} \right) \left( \frac{V_{d, pk}}{200 \text{ km s}^{-1}} \right)^4,$$

where $\kappa$ is the Thomson-scattering opacity and $f_0$ is the (spatially constant) gas-to-dark matter mass fraction in the protogalaxy. The velocity scale $V_{d, pk}$ refers to the peak value of the circular speed $V_d^2(r) \equiv GM_d(r)/r$ in a dark-matter halo with mass profile $M_d(r)$. Equation (1) holds for any form of the mass profile, just so long as the associated circular-speed curve has a single, global maximum—as all realistic descriptions of the halos formed in cosmological N-body simulations do. Defining a characteristic (dark-matter) velocity dispersion as $\sigma_0 \equiv V_{d, pk}/\sqrt{2}$ turns equation (1) into a critical $M_{\text{BH}}-\sigma_0$ relation, which is formally the same as that obtained by King (2003, 2005), and similar to the earlier result of Fabian (1999), for momentum-driven blow-out from a singular isothermal sphere.

This critical mass is based on the simplified description given by King & Pound's (2003) of a Compton-thick wind resulting from accretion at or above the Eddington rate onto an SMBH. In particular, their analysis provides the assumption that the momentum flux in the SMBH wind is simply $L_{\text{Edd}}/c$ (with no pre-factor). The wind from an SMBH with mass greater than that in equation (1) will then supply an outswards force (i.e., $L_{\text{Edd}}/c = 4\pi GM_{\text{BH}}/\kappa$) on a thin, radiative shell of swept-up ambient gas that exceeds the gravitational attraction of dark matter behind the shell (maximum force $f_0 V_{d, pk}^2/G$ if the gas was initially virialised), everywhere in the halo. It is a condition for the clearing of all gas to beyond the virial radius of any non-isothermal halo.

Equation (1) has limitations. Most notably, the protogalactic outflows driven by SMBH winds are in fact expected to become energy-driven (non-radiative) after an initial radiative phase (Zubovas & King 2012; McQuillin & McLaughlin 2013). This may (Silk & Rees 1998; McQuillin & McLaughlin 2013) or may not (Zubovas & Nayakshin 2014) have high wind speeds of up to $\sim 0.1c$ (King 2010). Such “ultrafast outflows” are observed in many local active galactic nuclei and low-redshift quasars accreting at or near their Eddington rates (e.g., Pound et al. 2003; Reeves et al. 2003; Tombesi et al. 2010, 2011).

Figure 1. SMBH mass versus stellar velocity dispersion averaged over an effective radius. Data are from the compilation of McConnell & Ma (2013) for 53 E or S0 galaxies (filled circles) and 19 bulges in late Hubble types (open circles). The dashed line is equation (1) with a protogalactic gas-to-dark matter fraction $f_0 = 0.18$ and $V_{d, pk} \equiv \sqrt{2} \sigma_y(R_e)$ for all galaxies. Improving upon this poorly-justified association between the characteristic stellar and dark-matter velocities in early-type galaxies is one of the goals of this paper.
change the functional dependence of a critical $M_{\text{BH}}$ for blow-out on the dark-matter $V_{\text{1, pk}}$ or any other characteristic halo velocity scale. Beyond this, the equation also assumes a wind moving into an initially static ambient medium, ignoring the cosmological infall of gas and an additional, confining ram pressure that comes with hierarchical (proto-)galaxy formation (Costa et al. 2014). It also neglects the presence of any stars in protogalaxies, which could contribute both to the feedback driving gaseous outflows (e.g., Murray et al. 2005; Power et al. 2011) and to the gravity containing them. (The assumptions of spherical symmetry and a smooth ambient medium are not fatal flaws; see Zubovas & Nayakshin 2014).

However, it is not our intention here to improve equation (1). Rather, we aim primarily to establish a method by which halo properties at $z > 0$ in relations such as equation (1) can be related to the average properties of stellar spheroids at $z = 0$. By doing this, we hope to understand better how expected relationships between SMBH masses and protogalactic dark-matter halos are reflected in the observed $M-\sigma$ relation particularly. Equation (1) is a good test case because it is simple and transparent but still contains enough relevant feedback physics to be interesting, even with the caveats mentioned above. It is also the only such relation we know of, which does not assume that dark-matter halos are singular isothermal spheres.

1.2 Halo circular speeds and stellar velocity dispersions

As a point of reference, Figure 1 shows SMBH mass against the stellar velocity dispersion $\sigma_{\text{ap}}(R_*)$ within an aperture equal to the stellar effective radius, for galaxies and bulges in the compilation of McConnell & Ma (2013). The dashed line shows equation (1) evaluated with a gas-to-dark-matter mass ratio of $f_0 = 0.18$ (the cosmic average; Planck Collaboration 2014) for all protogalaxies at the time of blow-out, and with the naive substitution $V_{\text{1, pk}} \equiv \sqrt{2} \sigma_{\text{ap}}(R_*)$ for all spheroids at $z = 0$. The proximity of this line to the data—first emphasised by King (2003, 2005), who assumed isothermal halos—encourages taking seriously the basic physical ideas behind equation (1), even though (as discussed above) some details must be incorrect at some level.

However, setting $V_{\text{1, pk}} = \sqrt{2} \sigma_{\text{ap}}(R_*)$ is problematic. A $\sqrt{2}$-proportionality between circular speed and velocity dispersion is appropriate only for isothermal spheres, which real dark-matter halos are not. A dark-matter velocity dispersion can be equated to a stellar velocity dispersion only if the dark matter and the stars have the same spatial distribution, which is not true of real galaxies. And $V_{\text{1, pk}}$ in equation (1) refers to a protogalactic halo, which will have grown significantly since the quasar epoch at $z \sim 2-3$.

In §2, we gather results from the literature that we need in order to address these issues. In §3, we combine them to constrain simple models of spherical, two-component galaxies, focussing on scaling relations between the stellar and dark-matter properties at $z = 0$. This is done without any reference to black holes, and the scalings should be of use beyond applications to SMBH correlations. In §4, we make a new, more rigorous comparison of equation (1) to the SMBH $M-\sigma$ data (compare Figure 6 below to Figure 1). Our work could in principle be used to explore the consequences of SMBH–halo relations like equation (1) for other SMBH–bulge correlations as well, but we do not pursue these here. In §5, we summarise the paper.

2 MODEL INGREDIENTS

Equation (1) incorporates an assumption that gas traced the dark matter in protogalaxies before being blown out by quasar-mode accretion feedback at high redshift. However, it does not make any assumptions about the detailed structure of dark-matter halos at any epoch, and it neither requires nor implies that mass follows light in galaxies at $z = 0$.

In this Section, we collect together analytical expressions from the literature for the (different) stellar and dark-matter mass profiles in galaxies, and for some key structural parameters of dark matter halos and their evolution in $\Lambda$CDM simulations of structure formation. We use these to obtain our new results in §3 and §4. Some of these expressions from the literature, and all of the scaling relations we ultimately derive, represent average trends that can have significant scatter around them. We do not attempt in this paper to analyse such scatter or to predict the net scatter around any scaling that comes from combining others.

This Section and §3 do not rely on any ideas about black hole accretion feedback or SMBH–bulge correlations. We focus repeatedly on the peak circular speed $V_{\text{1, pk}}$ in dark-matter halos, because that is what appears in equation (1) for $M_{\text{BH}}$; but we do not actually use the equation until §4.

2.1 Stellar distribution

We use the spherical density profile of Hernquist (1990) to describe the stars in early-type galaxies at $z = 0$. The density in this model can be written in terms of the total stellar mass, $M_{\ast, \text{tot}}$, and the effective radius, $R_e$:

$$\rho_\ast(r) = \frac{\mathcal{M}}{2\pi R_e^2} \left( \frac{r}{R_e} \right)^{-3} \left[ 1 + \frac{\mathcal{R}}{\mathcal{R}} \left( \frac{r}{R_e} \right) \right]^{-3},$$

(2)

where the constant $\mathcal{R} \approx 1.85127$ (see Hernquist 1990). The mass profile, $M_\ast(r) = \int_0^r 4\pi u^2 \rho_\ast(u) du$, is then

$$\frac{M_\ast(r)}{M_{\ast, \text{tot}}} = \left[ 1 - \frac{r}{R_e + \mathcal{R}} \right]^{-2}.$$

(3)

Integrating the Hernquist $\rho_\ast(r)$ along the line of sight gives a surface density profile that closely approximates the classic $R^{1/4}$ law. Thus, it adequately represents the typical light distributions in spheroids of mass $M_{\ast, \text{tot}} \sim 10^{10}-10^{12} M_\odot$, which more generally follow Sérsic (1968) profiles—$I(R) \sim \exp \left[-(R/R_e)^{1/n}\right]$—with indices $n \approx 3-7$ (e.g., see Graham & Colless 1997). These stellar masses correspond to velocity dispersions $\sigma_{\text{ap}}(R_*) \sim 80-350$ km s$^{-1}$ (see Figure 4), which is the range spanned by the local galaxies that define the black hole $M-\sigma$ relation in Figure 1.

The fine details of the assumed stellar density or mass profile matter most in our calculations of dimensionless stellar velocity dispersions $\sigma_{\text{ap}}(R_*)/\sqrt{GM_{\ast, \text{tot}}/R_e}$ using the Jeans equation with model dark matter halos included (see §3.5 below). Secondarily, the exact shape of $\rho_\ast(r)$ affects the mass ratio $M_\ast(r_{\text{vir}})/M_{\ast}(r_e)$, which we discuss in §3.4. We examine closely in §3 the consequences of using Hernquist
profiles for all galaxies in our calculations. In general, it exposes us to possible errors at the $\sim 10\%$ level or less.

2.2 Dark matter distributions

Since the dark-matter circular speed $V_{d,pk}$ enters equation (1) through a high power, it is important that we have a good idea of how sensitive our results may be to the details of the dark-matter density profile that we assume. We therefore consider four different models for spherical halos. Each of these is a two-parameter model defined by a mass scale and a radial scale. To treat them uniformly, it is most convenient to normalise all radii to the point mass scale and a radial scale. To treat them uniformly, it is most convenient to normalise all radii to the point $r_{-2}$ where the logarithmic slope of the dark-matter density is $d\ln \rho_{d}/d\ln r = -2$. Masses are then normalised to the mass enclosed within $r < r_{-2}$.

First, the usual NFW profile (Navarro et al. 1996, 1997) has density

$$\rho_{d}(r) \propto \left(\frac{r}{r_{-2}}\right)^{-1} \left(1 + \frac{r}{r_{-2}}\right)^{-2},$$

which yields the mass profile

$$\frac{M_{d}(r)}{M_{d}(r_{-2})} = \frac{\ln (1 + r/r_{-2}) - (r/r_{-2})(1 + r/r_{-2})^{-1}}{\ln(2) - 1/2}. \tag{5}$$

The circular-speed curve of the halo alone, i.e., $V_{d}^{2}(r) = GM_{d}(r)/r$, is then given by

$$\frac{V_{d}^{2}(r)}{V_{d}^{2}(r_{-2})} = \frac{\ln (1 + r/r_{-2}) - (r/r_{-2})(1 + r/r_{-2})^{-1}}{(r/r_{-2}) \ln(2) - 1/2},$$

which peaks at the radius

$$\frac{r_{pk}}{r_{-2}} \simeq 2.16258. \tag{7}$$

The second model is that of Hernquist (1990), which was first fitted to simulated dark-matter halos by Dubinski & Carlberg (1991). This has the same central density cusp ($\rho_{d} \rightarrow r^{-1}$) as an NFW halo, but a steeper large-radius slope ($\rho_{d} \rightarrow r^{−4}$ rather than $r^{−3}$) and hence a finite, rather than divergent, total mass. When written in terms of $r_{-2}$ and $M(r_{-2})$ rather than the effective radius and total mass, the model is

$$\rho_{d}(r) \propto \left(\frac{r}{r_{-2}}\right)^{-1} \left(1 + \frac{1}{2} \frac{r}{r_{-2}}\right)^{-3} \tag{8}$$

and

$$\frac{M_{d}(r)}{M_{d}(r_{-2})} = 9 \left(\frac{r/r_{-2}}{2 + r/r_{-2}}\right)^{2},$$

(9)

giving a circular-speed curve,

$$\frac{V_{d}^{2}(r)}{V_{d}^{2}(r_{-2})} = \frac{9 r_{-2}}{(2 + r/r_{-2})^{2}},$$

with a peak at radius

$$\frac{r_{pk}}{r_{-2}} = 2. \tag{11}$$

The third model is one from the family developed by Dehnen & McLaughlin (2005), which reproduces the universal power-law behaviour of “pseudo” phase-space density profiles, $\rho_{d}(r)/\sigma_{d}^{2}(r)$, in simulated dark-matter halos. This model fits the resolved parts of the density profiles alone better than either the NFW or Hernquist profiles, and about as well as the Einasto (1965) density profiles with $\rho_{d}(r) \sim \exp(-r^{α})$, first advocated in this context by Graham et al. (2006). The Dehnen & McLaughlin density is

$$\rho_{d}(r) \propto \left(\frac{r}{r_{-2}}\right)^{-7/9} \left[1 + \frac{11}{13} \left(\frac{r}{r_{-2}}\right)^{4/9}\right]^{-6}. \tag{12}$$

This has a slightly shallower central cusp than the NFW or Hernquist profiles and a large-radius fall-off, $\rho_{d} \rightarrow r^{-31/9}$, which is steeper than NFW (resulting in a finite total halo mass) but shallower than Hernquist. The mass profile is then

$$\frac{M_{d}(r)}{M_{d}(r_{-2})} = \left[\frac{24 (r/r_{-2})^{4/9}}{13 + 11 (r/r_{-2})^{5/9}}\right]^{5} \tag{13}$$

and the circular-speed curve is

$$\frac{V_{d}^{2}(r)}{V_{d}^{2}(r_{-2})} = \left[\frac{24 (r/r_{-2})^{11/45}}{13 + 11 (r/r_{-2})^{5/9}}\right]^{5}, \tag{14}$$

which reaches its peak value at

$$\frac{r_{pk}}{r_{-2}} = \frac{13^{9/4}}{9} \simeq 2.28732. \tag{15}$$

Finally, the halo model of Burkert (1995) has a constant-density core that appears more suited to the dynamics of some low-mass galaxies (e.g., Burkert & Silk 1997), and a large-radius fall-off that is the same as NFW. Here, the density is

$$\rho_{d}(r) \propto \left(1 + B r/r_{-2}\right)^{-1} \left(1 + B r^{2} r_{-2}^{2}\right)^{-1}, \tag{16}$$

with $B \simeq 1.52138$. The corresponding mass profile is

$$\frac{M_{d}(r)}{M_{d}(r_{-2})} = \frac{\ln \left[1 + B r/r_{-2}\right] \sqrt{1 + B^{2} (r/r_{-2})^{2}} - \tan^{-1}(B r/r_{-2})}{\ln \left[\sqrt{1 + B^{2}}\right] - \tan^{-1}(B)}, \tag{17}$$

which gives a circular-speed curve,

$$\frac{V_{d}^{2}(r)}{V_{d}^{2}(r_{-2})} = \frac{\ln \left[1 + B r/r_{-2}\right] \sqrt{1 + B^{2} (r/r_{-2})^{2}} - \tan^{-1}(B r/r_{-2})}{(r/r_{-2}) \left\{\ln \left[\sqrt{1 + B^{2}}\right] - \tan^{-1}(B)\right\}}, \tag{18}$$

that peaks at

$$\frac{r_{pk}}{r_{-2}} \simeq 2.13433. \tag{19}$$

Figure 2 shows the circular-speed curves of these halos, from equations (6), (10), (14) and (18). Relative to the NFW
profile, the Hernquist curve has a narrower width overall because of its steeper decline beyond the peak, which follows from its steeper density profile and convergent mass as \( r \to \infty \). The Burkert \( V_M^2(r) \) profile is much narrower because of its steeper rise from small \( r \), which is a result of its having a constant-density core rather than a central density cusp. The Dehnen & McLaughlin (2005) halo has the broadest circular-speed curve overall, largely because of its having a constant-density core rather than a central density cusp. The Hernquist curve has the steepest density profile (which depends on \( r^{4/9} \) rather than just \( r \)) rolls over from its central cusp with \( \rho_d(r) \sim r^{-7/9} \) to its power-law behaviour \( \rho_d(r) \sim r^{-31/9} \) at large radii. In the analysis of §3, these features ultimately affect not only the ratio \( V_d, pk/\sigma_{ap}(R_e) \), but also the self-consistent value of \( M_\star(R_e)/M_d(R_e) \), the stellar mass fraction inside the effective radius.

### 2.3 Stellar-to-dark matter mass ratios

The global ratio of stellar to dark-matter mass in galaxies is a strong and non-monotonic function of halo mass that changes with redshift. Behroozi et al. (2013) compare several derivations of this function at \( z = 0 \) by different groups using different methods. In this paper, we adopt a parametrisation from Moster et al. (2010).

Moster et al. assign one central galaxy to each virialised halo (which might be a sub-halo within a larger structure having its own central galaxy) in LCDM simulations of structure formation with \( \Omega_{m,0} = 0.26, \Omega_{\Lambda,0} = 0.74 \) and \( H_0 = 72 \) km s\(^{-1}\) Mpc\(^{-1}\). The stellar mass of any central galaxy is determined by the virial mass of its parent halo according to a prescription that is required ultimately to give agreement between the simulations and the observed galaxy luminosity function. They fit their results, for the central-galaxy mass fraction \( M_\star/M_d \) within the virial radius \( r_{\text{vir}} \) at \( z = 0 \), with a double power-law function:

\[
\frac{M_\star(r_{\text{vir}})}{M_d(r_{\text{vir}})} = 0.0564 \left\{ \left[ \frac{M_d(r_{\text{vir}})}{7.66 \times 10^{11} M_\odot} \right]^{-1.06} + \left[ \frac{M_d(r_{\text{vir}})}{7.66 \times 10^{11} M_\odot} \right]^{0.556} \right\}^{-1} \tag{20}
\]

(see their equation [2] and their Table 6). We discuss the virial radii themselves in the next subsection. Stellar mass fractions inside any other radius follow self-consistently from specifications of the stellar and dark-matter density profiles, as §3 will detail.

Equation (20) represents an average trend; scatter around can be expected, for example, as a result of differences in the merger histories of halos with the same mass at \( z = 0 \). Moster et al. (2010) and Behroozi et al. (2013) show that the relation is in good overall agreement with other theoretical work and/or with data, for halo virial masses \( 10^{13} M_\odot \lesssim M_d(r_{\text{vir}}) \lesssim 10^{15} M_\odot \). This corresponds to stellar masses \( 5 \times 10^8 M_\odot \lesssim M_\star(r_{\text{vir}}) \lesssim 10^{12} M_\odot \) for the central galaxies. The brightest galaxies used to define the observed \( M - \sigma \) relation are at the upper end of this range.

Equation (20) does not attempt to account for the total baryonic mass within the virial radius of any halo; it is only for stellar mass, and only that concentrated at the centre. There will be significantly more baryonic mass in large (cluster-sized) halos especially, in the form of intracluster light and X-ray gas, and in the stars of galaxies inside virialised sub-halos. We discuss this further in §3 and conclude that the complication of additional baryons can safely be ignored for our purposes.

### 2.4 Virial radii and cosmological parameters

We use the fitting formula of Bryan & Norman (1998, see their equation [6]) to calculate the overdensity, relative to the critical density, of a virialised sphere at redshift \( z \) in a flat universe with a cosmological constant (\( \Omega_m + \Omega_\Lambda = 1 \)):

\[
\Delta_{\text{vir}}(z) \equiv \frac{2GM(r_{\text{vir}})}{H^2(z)r_{\text{vir}}^3} \approx 18\pi^2 - 82 \frac{1 - \Omega_{m,0}}{[H(z)/H_0]} - 39 \frac{(1 - \Omega_{m,0})^2}{[H(z)/H_0]} \tag{21}
\]

with

\[
\left[ \frac{H(z)}{H_0} \right]^2 = 1 + \Omega_{m,0} \left[ (1 + z)^3 - 1 \right] \tag{22}
\]

Rearranging the definition of \( \Delta_{\text{vir}} \) yields a convenient relationship between virial radius and virial mass at arbitrary redshift:

\[
\left[ \frac{M(r_{\text{vir}})}{M_\odot} \right] \left[ \frac{r_{\text{vir}}}{\text{kpc}} \right]^{-3} = 1166.1 h_9^2 \Delta_{\text{vir}}(z) \left[ \frac{H(z)}{H_0} \right]^2 \tag{23}
\]

---

Figure 2. Normalised circular-speed curves, \( V_M^2(r) = GM_d(r)/r \), for the four dark-matter halo models we consider. The radius \( r_{-2} \) is that where the local density slope is \( d\ln \rho_d/d\ln r = -2 \). The peaks in \( V_d(r) \) occur at radii near \( r_{pk}/r_{-2} \approx 2 \) in all cases (see text). Broken vertical lines show the concentrations \( r_{pk}/r_{-2} \) of halos with virial masses \( M_d(r_{vir}) = 10^{15} M_\odot \) and \( 10^{11} M_\odot \) at \( z = 0 \) (see §2.5). The different widths of the circular-speed curves for the different halos lead to different values for the baryon fraction inside a stellar effective radius (which is typically in the range \( R_e/r_{-2} \approx 0.02-0.1 \); see §3), as well as different ratios \( V_d, pk/\sigma_{ap}(R_e) \).
where $h_0 \equiv H_0/(100 \text{ km s}^{-1} \text{ Mpc}^{-1})$ as usual. This form is also useful for calculating $M/r^3$ of spheres with other overdensities $\Delta$ besides the virial value [e.g., $\Delta(z) \equiv 200$]. Whenever we use any of equations (21)–(23), we take cosmological parameters from the Planck 2013 results (Planck Collaboration 2014): $h_0 = 0.67$ with $\Omega_{m,0} = 0.32$ (which includes a baryon density of $\Omega_{b,0} = 0.049$) and $\Omega_{\Lambda,0} = 0.68$.

2.5 Halo concentrations

By the concentration of a dark-matter halo, we specifically mean the ratio of $r_{\text{vir}}$ (within which, the mean overdensity is given by equation [21]) to $r_{-2}$ (where the slope of the density profile is $d \ln \rho_{\text{c}}/d \ln r = -2$). It is also common in the literature to define concentration as the ratio of $r_{200}$ (within which, the mean overdensity is $\Delta = 200$) to $r_{-2}$. Either way, $N$-body simulations of CDM structure formation consistently show that, at least for low redshifts, more massive halos have lower concentrations on average. We need to take account of this in order to infer the location and the value of the maximum circular speed in any dark-matter halo with a given virial radius and mass.

Dutton & Macciò (2014) give a fitting formula for the concentrations $r_{\text{vir}}/r_{-2}$ of simulated halos with masses $10^{11} M_\odot \lesssim M_{\text{vir}}(r_{\text{vir}}) \lesssim 10^{15} M_\odot$ at redshifts $0 \leq z \leq 5$ in a Planck cosmology. Namely,

$$\log \left( \frac{r_{\text{vir}}}{r_{-2}} \right) \simeq a - b \log \left[ \frac{M_{\text{vir}}(r_{\text{vir}})}{10^{12} h_0^{-1} M_\odot} \right]$$

(24)

with

$$a = 0.537 + 0.488 \exp \left( -0.718 z^{1.08} \right)$$

$$b = 0.097 - 0.024 z .$$

Again, we set $h_0 = 0.67$ whenever we use this equation. Simulated halos scatter around the average trend at the level of a few tens of percent in $r_{\text{vir}}/r_{-2}$ for a fixed virial mass and redshift (Bullock et al. 2001; Dutton & Macciò 2014).

Dutton & Macciò obtain equation (24) by fitting NFW density profiles to their simulated halos in order to measure the radius $r_{-2}$. They also investigate the use of Einasto (1965) profiles instead (which are more like the Dehnen & McLaughlin 2005 halos that we explore) to fit for $r_{-2}$ in estimating the alternative concentration $r_{200}/r_{-2}$. Their results suggest that concentration values depend on the choice of model for the dark-matter density profile, but only at the $\leq 10\%$ level for halos with $M_{\text{vir}}(r_{\text{vir}}) \gtrsim 10^{12} M_\odot$ at $z = 0$. We apply equation (24) in our models regardless of what model we assume for $\rho_{\text{c}}(r)$ and simply accept that there is a modest uncertainty associated with doing so.

The two vertical lines in Figure 2 show the concentrations according to equation (24) for halos with virial masses at $z = 0$ of $M_{\text{vir}}(r_{\text{vir}}) = 10^{13} M_\odot$ (having $r_{\text{vir}}/r_{-2} = 13.8$) and $10^{15} M_\odot$ (having $r_{\text{vir}}/r_{-2} = 5.64$). Equation (20) gives the corresponding stellar masses of the central galaxies as $M_* (r_{\text{vir}}) \equiv 6.3 \times 10^8 M_\odot$ and $1.0 \times 10^{12} M_\odot$. This emphasises the degree to which $V_{d, pk}$—the key predictor of self-limited SMBH masses in the simple feedback model behind equation (1)—reflects conditions far outside the stellar distributions of normal galaxies (generally, $R_e/r_{-2} \sim 0.02–0.1$; see §3).

Equation (24) has been derived from simulations of strictly baryon-free halos. This is not an issue for our modelling, precisely because the equation describes halos on large scales $r > r_{-2} \gg R_e$ well away from any regions that might have been altered significantly by the presence of stars.

2.6 Halo progenitors

If the central black hole in a protogalaxy ended its main, quasar phase of accretion growth at a redshift $z > 0$, with a mass $M_{\text{BH}}$ determined by the circular speed $V_{d, pk}$ in the dark-matter halo at that time, then we need to relate that earlier $V_{d, pk}$ to the value at $z = 0$ (in order ultimately to link it and $M_{\text{BH}}$ to a stellar velocity dispersion at $z = 0$).

From $N$-body simulations and merger trees of $\Lambda$CDM halos with virial masses at $z = 0$ in the range $10^{14} M_\odot \lesssim M_{\text{vir}}(r_{\text{vir}}) \lesssim 10^{15} M_\odot$, van den Bosch et al. (2014) extract for each halo the redshift $z_{1/2}$ at which its most massive progenitor had a virial mass $M_{\text{vir}}(z_{1/2}) = 0.5 M_{\text{vir}}(0)$. Given the bottom-up nature of structure formation in CDM cosmologies, $z_{1/2}$ is a decreasing function of $M_{\text{vir}}(0)$ in general. We have fitted the median dependence shown in Figure 4 of van den Bosch et al. with the function

$$z_{1/2} = 2.05 \left[ \frac{M_{\text{vir}}(0)}{10^{12} h_0^{-1} M_\odot} \right]^{-0.055} - 1 ,$$

(25)

again taking $h_0 = 0.67$ from the Planck cosmology. Once again, there is intrinsic scatter around this overall trend.

Given $z_{1/2}$, we then approximate the virial mass of the most massive progenitor of a halo at any other redshift by the exponential function (see also, e.g., Zhao et al. 2009),

$$\frac{M_{\text{vir}}(z)}{M_{\text{vir}}(0)} = \exp \left[ - \frac{\ln(2)}{z_{1/2}} z \right] .$$

(26)

Equations (25) and (26) together give curves of $M_{\text{vir}}(z)/M_{\text{vir}}(0)$ versus $M_{\text{vir}}(0)$ that, for redshifts $z \lesssim 5$, compare well to the curves plotted by van den Bosch et al. (2014) directly from the simulations they analyse (e.g., see their Figure 2).

To obtain the evolution of the peak circular speed in the most massive progenitor of a halo, we first write (for any $z$)

$$\frac{V_{d, pk}^2(z)}{V_{d,\text{vir}}^2(r_{-2})} = \frac{g(r_{pk}/r_{-2})}{g(r_{pk}/r_{-2}}$$

(27)

where $g(r/r_{-2})$ is one of the normalised circular-speed curves shown in Figure 2 and written on the right-hand sides of equations (6), (10), (14) and (18) above. Then, since the ratio $r_{pk}/r_{-2}$ is independent of redshift (it is fixed by assuming a basic form for the dark-matter density profile), we have

$$\frac{V_{d, pk}^2(z)}{V_{d, pk}^2(0)} = \frac{g \left( (r_{\text{vir}}/r_{-2})_{z=0} \right)}{g \left( (r_{\text{vir}}/r_{-2})_{z=0} \right)} \times \frac{V_{d, \text{vir}}^2(z)}{V_{d, \text{vir}}^2(0)}$$

$$= \frac{g \left( (r_{\text{vir}}/r_{-2})_{z=0} \right)}{g \left( (r_{\text{vir}}/r_{-2})_{z=0} \right)} \times \left[ \frac{M_{\text{vir}}(z)}{M_{\text{vir}}(0)} \right]^{2/3} \left[ \frac{\Delta_{\text{vir}}(z)}{\Delta_{\text{vir}}(0)} \right]^{1/3} \left[ \frac{H(z)}{H_0} \right]^{2/3} ,$$

(28)

where the last line uses the fact that $V_{d}^2(r) \propto M_d(r)/r$ and brings in equation (23). For any choice of dark-matter halo
the $z = 0$ virial mass, calculated using equation (28). These curves depend on the halo density profile. For clarity, we only show results assuming either a Dehnen & McLaughlin (2005) or a Hernquist (1990) density profile, so $g(r/r_{-2})$ is given either by equation (14) or by equation (10).

It is worth noting here the gradual flattening towards higher masses of the curves for $M_{\text{d,vir}}(z)$ versus $M_{\text{d,vir}}(0)$ in the middle panel of Figure 3, and how the flattening sets in at more modest halo masses for larger $z$. This is a generic feature of structure formation by hierarchical merging. Halos in any given mass range at $z = 0$ have progenitors drawn from increasingly narrow mass ranges, on average, at increasingly high redshift; and this narrowing is more pronounced as a function of $z$ for higher-mass halos, because more of their growth has occurred more recently.

Precise numbers—such as the possible value of a maximum mass for the largest progenitors suggested by the $z = 5$ curve in Figure 3—are specific to the dependence of $z_{1/2}$ on $M_{\text{d,vir}}(0)$ in our equation (25). That and equation (26) only give an approximation to the numerical results of van den Bosch et al. (2014) for the median most-massive progenitors of halos with $10^{11}M_\odot \lesssim M_{\text{d,vir}}(0) \lesssim 10^{12}M_\odot$. Fine details following from them are not definitive, especially at the highest end of the $z = 0$ mass range. However, the flattening of $M_{\text{d,vir}}(z)$ as a function of $M_{\text{d,vir}}(0)$ is qualitatively robust. It ultimately has some implications for the shape of the black hole $M$–$\sigma$ relation at high $\sigma$-values, which we discuss further in §4.

In the bottom panel of Figure 3, at any fixed redshift the different halo models give greater differences in $V_{\text{d,vir}}(z)/V_{\text{d,vir}}(0)$ for lower virial masses. This is because lower-mass halos generally have higher concentrations $r_{\text{vir}}/r_{-2}$, and therefore higher ratios of $r_{\text{vir}}/\sigma_{\text{tot}}$ (see equation [24]). Thus, the ratio $V_{\text{d,vir}}(z)/V_{\text{d,vir}}(0)$ is more sensitive in lower-mass halos to the model-dependent steepness of the circular-speed curve at radii $r > r_{\text{pk}}$. But $V_{\text{d,vir}}/V_{\text{d,vir}}(z)/r_{\text{vir}}(z)$ is independent of the halo density profile, and so only $V_{\text{d,vir}}(0)$ is actually model-dependent. Since NFW and Burkert (1995) halos have circular-speed curves that are intermediate in steepness to Dehnen & McLaughlin and Hernquist models beyond $r_{\text{pk}}$ (see Figure 2), the curves for $V_{\text{d,vir}}(z)/V_{\text{d,vir}}(0)$ versus $M_{\text{d,vir}}(0)$ in these other models lie between the two shown in Figure 3.

### 3 Galaxy and Halo Scalings at $z = 0$

A two-component model for a spherical galaxy is formally defined by four parameters: $R_\ast$ and $M_{\ast,\text{tot}}$ for the stars, which we assume here to follow Hernquist (1990) density profiles (summarised in §2.1), plus $r_{-2}$ and $M_d(r_{-2})$ for a dark-matter profile (described in §2.2). However, there are interdependences between these parameters: $R_\ast$ and $M_{\ast,\text{tot}}$ are correlated (discussed just below), while the radii and masses of dark-matter halos are connected to each other and to $M_{\ast,\text{tot}}$ by cosmological simulations (the stellar mass fractions in §2.3 and the concentrations in §2.5). These dependences allow the models to be put in terms of a single independent parameter, which we choose to be $M_{\ast,\text{tot}}$.

Figure 4 shows the average trends for various galaxy properties versus $M_{\ast,\text{tot}}$ at $z = 0$, together in some cases with data from the literature. In this Section we detail the
procedures leading to these plots. In §4, we fold in the redshift evolution of \(V_d, pk\) (from §2.6) to apply equation (1) for predicted black hole masses and consider the empirical correlation between \(M_{BH}\) and the stellar \(\sigma_{200}(R_e)\).

Our goal here is to establish representative trend-line relationships between various stellar and halo properties. Scatter around the trends is inevitable, and it can contain physical information, but in this paper we set aside the task of characterising or explaining any scatter in detail.

### 3.1 Stellar masses and effective radii

Panel (a) of Figure 4 plots effective radius against total stellar mass for local early-type galaxies in two datasets: 258 systems from the ATLAS3D survey (squares: Cappellari et al. 2011, 2013a,b) and 100 from the ACS Virgo Cluster Survey (ACSVCS; triangles: Côté et al. 2004; Chen et al. 2010).

In each case, the effective radii are tabulated by the original authors, either in kpc directly or as angular sizes along with the distances to individual galaxies. To estimate the stellar masses, we have taken integrated luminosities provided by the authors and calculated mass-to-light ratios using the single-burst population-synthesis models of Maraston (1998, 2005) assuming stellar ages of 9 Gyr and a Kroupa (2001) stellar initial mass function (IMF). The masses in these \(M/L\) ratios include both luminous stars and dark remnants. We have also used Bruzual & Charlot (2003) models to confirm that extended star formation lasting as long as 6 Gyr gives the same \(M/L\) values, to within \(\lesssim 5\%\), when the mean stellar age is 9 Gyr.

Cappellari et al. (2011) give \(K\)-band absolute magnitudes for galaxies in the ATLAS3D survey. At an age of 9 Gyr and for metallicities \(-1.7 \leq [\text{Z}/\text{H}] \leq +0.3\), the mass-to-light ratios tabulated by Maraston (2005) are 0.93 \(\gtrsim M_*/L_K \gtrsim 0.82 \, M_\odot \, L_\odot^{-1}\). We therefore adopt a constant \(M_*/L_K = 0.88 \, M_\odot \, L_\odot^{-1}\) for all of the ATLAS3D galaxies. This value changes by approximately \(\pm 15\%\) if the mean age of the stars is changed by \(\pm 2\) Gyr.

Chen et al. (2010) give \(g\)-band apparent magnitudes and \((g - z)\) colours for the ACSVCS galaxies. Combining these with surface-brightness fluctuation distances from Blakeslee et al. (2009) allows us to calculate absolute \(z\)-band magnitudes. Then, for metallicities \(-1.7 \leq [\text{Z}/\text{H}] \leq +0.3\), a Kroupa IMF and an age of 9 Gyr, the Maraston models give \(1.40 \lesssim M_*/L_z \lesssim 2.0 \, M_\odot \, L_\odot^{-1}\). We have used a single \(M_*/L_z = 1.7 \, M_\odot \, L_\odot^{-1}\) for all of the ACSVCS galaxies to plot the points in panel (a) of Figure 4. Again, this changes by \(\pm 15\%\)-\(\pm 20\%\) if the assumed age is changed by \(\pm 2\) Gyr.

The line going through the \(R_e\) vs \(M_\text{tot}\) data in Figure 4 is a parametrisation of the average correlation,

\[
\frac{R_e}{\text{kpc}} = 1.5 \left( \frac{M_\text{tot}}{2 \times 10^{10} \, M_\odot} \right)^{0.1} \left[ 1 + \left( \frac{M_\text{tot}}{2 \times 10^{10} \, M_\odot} \right)^{2} \right]^{0.1},
\]

which we decided by eye. Roughly equal numbers of ATLAS3D + ACSVCS data points lie above and below this line. A \(\pm 20\%\) change in adopted mass-to-light ratios (whether due to a different assumed mean age or a different star formation history) results in a \(\pm 20\%\) change to the mass scale in equation (29).

The ATLAS3D sample covers the full range of stellar masses, \(10^{10} \, M_\odot \lesssim M_\text{tot} \lesssim 10^{12} \, M_\odot\), of the local galaxies that define the black hole \(M - \sigma\) relation. As mentioned in §2.1, the light profiles in this mass range can generally be fitted by Sérsic (1968) models with indices \(n \approx 3 - 7\), all of which can be approximated adequately, for our purposes, by a Hernquist (1990) profile in projection. The ACSVCS galaxies include many with \(M_\text{tot} < 10^{10} \, M_\odot\), where surface-brightness profiles are increasingly better fitted by lower-index Sérsic functions tending towards exponentials. We have included these systems mainly to ensure that our analysis incorporates the change in slope that they show in the \(R_e\) vs \(M_\text{tot}\) correlation. In all of what follows, we address with some care the extent to which our results might (or may not) be put in error by assuming Hernquist stellar-density profiles for all systems.

### 3.2 Virial radii and halo virial masses

For any value of \(M_\text{tot}\), equation (29) gives a typical value for \(R_e\). Assuming a Hernquist density profile for the stars we can then write, for the ratio of stellar-to-dark matter mass within the virial radius of a galaxy,

\[
f_{\text{svir}} \equiv \frac{M_\text{svir}}{M_\text{dm(\text{vir})}} = \frac{M_\text{tot}}{M_\text{dm(\text{vir})}} \left( \frac{r_{\text{vir}}/R_e}{r_{\text{vir}}/R_e + 1/\Delta} \right)^{-2}
\]

with \(\Delta \simeq 1.81527\) (see equation [3]). Understanding the dark-matter mass to be that of the main halo centred on the stars in the galaxy, \(f_{\text{svir}}\) is additionally constrained by cosmological simulations, as discussed in §2.3 and represented by equation (20) above from Moster et al. (2010). Repeating this for convenience, at \(z = 0\) we have

\[
f_{\text{svir}} = 0.0564 \left\{ \frac{M_{\text{dm(\text{vir})}}}{7.66 \times 10^{11} \, M_\odot} \right\}^{-1.06} + \left[ \frac{M_{\text{dm(\text{vir})}}}{7.66 \times 10^{11} \, M_\odot} \right]^{-0.556}^{-1}.
\]

Finally, if the total mass within \(r_{\text{vir}}\) is simply the sum of the dark matter plus the stars in the central galaxy, i.e., \(M(r_{\text{vir}}) = M_{\text{dm(\text{vir})}}(1 + f_{\text{svir}})\), then the definition of \(r_{\text{vir}}\) in equation (23) gives (at \(z = 0\) for the 2013 Planck cosmological parameters)

\[
f_{\text{svir}} = 0.0544 \left( \frac{r_{\text{vir}}}{100 \, \text{kpc}} \right)^3 \left( \frac{M_{\text{dm(\text{vir})}}}{10^{12} \, M_\odot} \right)^{-1} - 1.
\]

Solving equations (30)–(32) for all of \(f_{\text{svir}}, r_{\text{vir}}\) and \(M_{\text{dm(\text{vir})}}\) as functions of \(M_\text{tot}\) gives the curves shown in panels (b), (c) and (d) of Figure 4. These are independent of any assumptions about the internal density profiles of the halos.

The peak in \(f_{\text{svir}}\) in panel (b), at a value of \(\approx 0.03\) for \(M_\text{tot} \approx 3.4 \times 10^{10} \, M_\odot\) or \(M_{\text{dm(\text{vir})}} \approx 1.1 \times 10^{12} \, M_\odot\), comes directly from the form of equation (31) taken from Moster et al. (2010). It is intriguing that the mass scale of this peak is close to the mass where the empirical \(R_e - M_\text{tot}\) correlation changes slope (equation [29]), but we do not pursue this issue here. The immediate point is that \(f_{\text{svir}}\) decreases rapidly towards higher masses, such that the halos around central galaxies with \(M_\text{tot} \gtrsim 10^{11} \, M_\odot\) have \(M_{\text{dm(\text{vir})}} \gtrsim 10^{13} \, M_\odot\) and \(r_{\text{vir}} \gtrsim 500 \, \text{kpc}\). They encompass entire groups and clusters.
Figure 4. Model scaling relations for stellar and dark-matter halo properties versus total stellar mass, $M_{\ast,\text{tot}}$, in spherical galaxies at $z = 0$. With the exception of the curve in panel (a), the low-mass extensions of these models to $M_{\ast,\text{tot}} \lesssim 5 \times 10^9 M_{\odot}$ (stellar velocity dispersions $\sigma_{\text{ap}}(R_e) \lesssim 60 \text{ km s}^{-1}$) should be viewed with some caution, as discussed in §3.6.

Panel (a): Stellar effective radius, $R_e$. Data points represent galaxies in the ATLAS$^3$D survey (Cappellari et al. 2011; green squares) and the ACS Virgo Cluster Survey (Chen et al. 2010; magenta triangles). See §3.1 for details. Panel (b): Ratio $f_{\ast, \text{vir}}$ of stellar-to-dark matter mass within the virial radius; see §2.3 and §3.2. Panel (c): Virial radius, $r_{\text{vir}}$; see §3.2. Panel (d): Mass of dark matter within the virial radius, $M_{d, \text{vir}}$; see §3.2. Panel (e): Radius $r_{pk}$ where the dark-matter circular-speed curve peaks. The different coloured curves are for four different models of the dark-matter density profile. See §2.2 and §3.3 for details. Panel (f): Peak value of the dark-matter circular speed, $V_{d, pk}$, assuming each of the four different dark-matter halo models; see §3.3. Panel (g): Ratio $f_{\ast}(R_e)$ of stellar mass to dark matter mass within a sphere of radius $r < R_e$, for each of the four different halo models; see §3.4. Data points are from dynamical modelling by the ATLAS$^3$D survey (Cappellari et al. 2013a,b); arrows at the top of the panel represent galaxies consistent in their analysis with having no dark matter inside $R_e$. Panel (h): Stellar velocity dispersion $\sigma_{\text{ap}}(R_e)$ within an aperture of radius $R_e$. Data points are taken from the ATLAS$^3$D survey. See §3.5 for details.
For the massive systems in particular, there may be baryons that reside in the halos but are not associated directly with the stars of the central galaxy—intracluster light and gas, and the stars in any off-centre satellite galaxies. Equation (32) for the virial radius takes no account of any such "extra" baryons. To do so properly would require additionally constraining the global baryon fraction in galaxy clusters, which is itself a mass-dependent quantity (see, e.g., Giodini et al. 2009; McGaugh et al. 2010; Zhang et al. 2011). However, in no case would the total virial mass be increased by more than $\approx 15\%$ (this being the cosmic average baryon fraction, $\Omega_b/\Omega_m$), and hence the virial radius would not increase by more than $\approx 5\%$. We therefore ignore the complication as far as $r_{\text{vir}}$ is concerned.

Then, over the range of galaxy masses shown in Figure 4, we find that $110 \lesssim r_{\text{vir}}/R_e \lesssim 170$. As a result, the stellar mass inside the virial radius is $M_\ast(r_{\text{vir}}) \gtrsim 0.99 M_{\ast,\text{tot}}$ in all cases, and equation (30) says that $f_{\ast,\text{vir}} \approx M_{\ast,\text{tot}}/M_{\ast,\text{vir}}$ with only a very weak dependence on $r_{\text{vir}}/R_e$. The mass of dark matter alone within $r_{\text{vir}}$ is then determined (through equation [31]) by $M_{\ast,\text{tot}}$ almost independently of $r_{\text{vir}}$. Thus, our values for $M_{\ast,\text{vir}}$ would not be changed discernibly by having additional baryons distributed in the halos outside of the central galaxies.

These conclusions still hold if the stars in the central galaxies are described by Sérsic models that depart significantly from Hernquist profiles in projection, so long as $M_\ast(r)$ still essentially converges within $r \lesssim 100 R_e$. Hence, the curves for $f_{\ast,\text{vir}}$, $r_{\text{vir}}$ and $M_{\ast,\text{vir}}$ versus $M_{\ast,\text{tot}}$ in Figure 4 are insensitive to the choice of stellar density profile.

### 3.3 Peak halo circular speeds

With virial radii and dark-matter virial masses known as functions of $M_{\ast,\text{tot}}$, the scale $r_{\text{pk}}$ follows from equation (24) in §2.5 for the concentration $r_{\text{vir}}/r_{\text{pk}}$ versus $M_{\ast,\text{vir}}$ (Dutton & Macciò 2014), evaluated at $z = 0$. The location of the peak of the dark-matter circular-speed curve then comes from the ratio $r_{\text{pk}}/r_{\text{pk}}$ specific to a choice of $\rho_\ast(r)$ for the dark matter (one of equations [7], [11], [15] or [19] in §2.2). Panel (e) of Figure 4 shows the final curves of $r_{\text{pk}}$ versus $M_{\ast,\text{tot}}$ for all four of the halo profiles we are considering. There is little difference between the curves because we have assumed the same $(r_{\text{vir}}/r_{\text{pk}})$ versus $M_{\ast,\text{tot}}$ relation for all halo models, and because $r_{\text{pk}}/r_{\text{pk}} \approx 2-2.3$ for all of them. They are also essentially independent of the form of the stellar density profile, because the underlying curves of $r_{\text{vir}}$ and $M_{\ast,\text{vir}}$ versus $M_{\ast,\text{tot}}$ are. Ultimately, we have approximately $15 \lesssim r_{\text{pk}}/R_e \lesssim 70$ and $0.14 \lesssim r_{\text{pk}}/r_{\text{vir}} \lesssim 0.40$ for stellar masses in the range $10^8 M_\odot \lesssim M_{\ast,\text{tot}} \lesssim 10^{13} M_\odot$.

The peak value of the dark-matter circular speed is obtained as

$$V_{\text{d,pk}}^2 = \frac{V_{\text{d},r_{\text{pk}}}^2}{V_{\text{d},r_{\text{vir}}}^2} = \left( \frac{M_{\ast,\text{vir}}}{M_{\ast,\text{tot}}} \right)_{r_{\text{pk}}} G M_{\ast,\text{vir}} \frac{r_{\text{vir}}}{r_{\text{pk}}}. \quad (33)$$

The normalised circular-speed profiles $V_{\text{d}}^2(r)/V_{\text{d},r_{\text{pk}}}^2$ for different halo models are shown in Figure 2 and given in equations (6), (10), (14) and (18) of §2.2. Evaluating the appropriate one of these at $r_{\text{pk}}/r_{\text{pk}}$ and $r_{\text{vir}}/r_{\text{pk}}$ after choosing a density profile $\rho_\ast(r)$, and then folding in the dependences of $M_{\ast,\text{vir}}$ and $r_{\text{vir}}$ on $M_{\ast,\text{tot}}$, yields $V_{\text{d},\text{pk}}$ at any given total stellar mass. The results are shown in panel (f) of Figure 4.

The curves for $V_{\text{d},\text{pk}}$ versus $M_{\ast,\text{tot}}$ are again insensitive to the use of a Hernquist profile for the stellar distributions. The differences between them come from the (small) differences in the values of $r_{\text{pk}}/r_{\text{pk}}$ in the different halo models, and the (larger) differences in the widths of the normalised circular-speed curves between $r_{\text{pk}}$ and $r_{\text{vir}}$, as seen in Figure 2. The differences are greater for systems with smaller $M_{\ast,\text{tot}}$ because those halos are less massive and have higher concentrations on average, with larger ratios $r_{\text{vir}}/r_{\text{pk}}$ and hence ratios $V_{\text{d}}(r_{\text{pk}})/V_{\text{d}}(r_{\text{vir}})$ that are more sensitive to the shape of the circular-speed curve at large radii in a halo.

It is clear that the circular speeds $V_{\text{d},\text{pk}}$ for the most massive model galaxies, which represent those defining the upper end of the observed black hole $M - \sigma$ relation, will far exceed the stellar velocity dispersions measured within $R_e$ in the real systems. This is because the dark-matter halos centred on such massive galaxies correspond to entire clusters. It is also why the naive substitution $V_{\text{d},\text{pk}} = \sqrt{2} \sigma_{\text{tot}}(R_e)$, inspired by the singular isothermal sphere, cannot suffice for a proper comparison of a prediction like equation (1) to the $M - \sigma$ data (cf. Figure 1). At the same time, the most massive halos are the ones that will have grown the most at low redshifts, after the epoch of peak quasar activity that may have mainly determined self-regulated black hole masses. Hence it is essential that $V_{\text{d},\text{pk}}$ be calculated in the progenitors of halos if equation (1) is to be assessed self-consistently.

### 3.4 Stellar mass fractions inside $R_e$

The ratio of stellar mass to dark-matter mass contained within radius $r$ in a galaxy with a specified total stellar mass can be written as

$$f_\ast(r) \equiv \frac{M_\ast(r)}{M_{\ast,\text{tot}}} = f_{\ast,\text{vir}} \frac{M_\ast(r)/M_{\ast,\text{tot}}}{M_\ast(r)/M_{\ast,\text{tot}}} \cdot \quad (34)$$

Here, $f_{\ast,\text{vir}}$ is known from above as a function of $M_{\ast,\text{tot}}$. The normalised stellar mass profile $M_\ast(r)/M_{\ast,\text{tot}}$ comes from equation (3) for a Hernquist density profile and is determined by $M_{\ast,\text{tot}}$ because $R_e$ and $r_{\text{vir}}$ are. Once a dark-matter halo model has been chosen, the mass profile $M_\ast(r)/M_{\ast,\text{tot}}$ follows from one of equations (5), (9), (13) or (17) and is also determined by $M_{\ast,\text{tot}}$ because that fixes the concentration $r_{\text{vir}}/r_{\text{pk}}$.

The function $f_\ast(r)$ enters into the Jeans equation for calculations of the stellar velocity dispersion in §3.5. First, however, we evaluate it specifically at the radius $r = R_e$ for galaxies with a range of stellar masses, in order to compare our results with some additional data.

Cappellari et al. (2013a,b) have used dynamical (Jeans) modelling to estimate the ratio of dark-to-total mass within a sphere of radius $r = R_e$ for each of the ATLAS3D galaxies. This fraction, which they denote $f_{\text{d,ATLAS}}$, is related to our stellar-to-dark mass ratio within $r < R_e$ by $f_\ast(R_e) = f_{\text{d,ATLAS}} - 1$. Although the Cappellari et al. modelling assumes that dark-matter halos have NFW density profiles, their results are not sensitive to this detail, since usually $M_\ast(R_e) < M_{\ast,\text{tot}}(R_e)$ by factors of several in their galaxies—see Cappellari et al. (2013a) for further details.

Panel (g) of Figure 4 shows the $f_\ast(R_e)$ data for 258 ATLAS3D galaxies (arrows at the top of the panel indicate galaxies for which the modelling by Cappellari et al. is consistent with no dark matter inside $r < R_e$). The curves show...
the typical \( f_s(R_e) \) expected at a given \( M_{*,\text{tot}} \) on the basis of our equation (34), for each of the four different dark-matter halo profiles.

These curves depend on the stellar density profile as \( f_s(R_e) \propto M_s(R_e)/M_s(r_{\text{vir}}) \approx M_*(R_e)/M_{*,\text{tot}} \). In the mass range \( M_{*,\text{tot}} \gtrsim 10^{10} M_\odot \), describing the stars by Sérés models with \( 3 \lesssim n \lesssim 7 \) rather than by Hernquist models alters \( M_*(R_e)/M_{*,\text{tot}} \), and hence \( f_s(R_e) \), by less than 5%. Much lower-mass galaxies, which have no \( f_s(R_e) \) data in Figure 4 and are not represented in the empirical \( M-\sigma \) relation, will have closer to exponential surface-brightness profiles. For these, \( M_*(R_e)/M_{*,\text{tot}} \) and \( f_s(R_e) \) are lower than the Hernquist model values, but by no more than \( \approx 20\% \).

The curves are rather more sensitive to the choice of dark-matter halo profile, in particular to how steeply the enclosed mass \( M_\text{v}(r) \) decreases inwards to \( r \to 0 \). This is reflected in the shapes of the circular-speed curves in Figure 2. For a given value of \( M_{*,\text{tot}} \), and hence \( M_\text{v}(r_{\text{vir}}) \), NFW and Hernquist halos have similar values for \( M_*(R_e)/M_\text{v}(r_{\text{vir}}) \), and thus for \( f_s(R_e) \), because of their identical central structures. Dehnen & McLaughlin (2005) halos have higher \( M_\text{v}(R_e)/M_\text{v}(r_{\text{vir}}) \) and lower \( f_s(R_e) \) for the same stellar mass, because they have significantly shallower mass profiles than either NFW or Hernquist halos. The much steeper \( M_\text{v}(r) \) or \( \rho_\text{v}(r) \) profiles in the constant-density cores of Burkert (1995) models put substantially more dark matter at large radii in these halos, giving lower values of \( M_\text{v}(R_e)/M_\text{v}(r_{\text{vir}}) \) and higher \( f_s(R_e) \) for a fixed \( M_{*,\text{tot}} \).

The three dark-matter halos with central density cusps all imply \( f_s(R_e) \) values that are broadly consistent with the data in Figure 4(g) for systems with \( M_{*,\text{tot}} \gtrsim 10^{10} M_\odot \). However, the cored halo of Burkert (1995) is incompatible with these data. This is a valuable check on our calculations, and an argument for not considering Burkert halos further in the context of the black hole \( M-\sigma \) relation for intermediate- and high-mass galaxies. But it is not surprising, since the Burkert model was originally proposed only in connection with dwarf spheroidal galaxies, not regular ellipticals.

3.5 Stellar velocity dispersions

To calculate stellar velocity dispersions, we solve the isotropic Jeans equation including contributions to the gravitational potential from the dark matter, the stars and the accumulated ejecta from stellar winds and supernovae over the lifetime of a galaxy. Assuming that these ejecta are confined to the central regions of the overall potential well in relatively large galaxies, we approximate their mass profile as \( M_\text{ej}(r) \approx F_\text{ej} M_*(r) \) with \( F_\text{ej} \) a constant. The value of \( F_\text{ej} \) comes from the same single-burst population-synthesis models that we used in §3.1 to calculate stellar mass-to-light ratios. Namely, for a Kroupa (2001) stellar IMF and stellar ages greater than several Gyr, Maraston (2005) gives the ratio of current-to-total mass in stars (and remnants) as \( \approx 0.58 \). Thus, in our notation, \( (1 + F_\text{ej}) \approx 1/0.58 \). The value of \( F_\text{ej} \) is robust to any changes in the star formation history, with a \( < 2\% \) increase for extended star formation.

With dimensionless radii, stellar densities and one-dimensional velocity dispersions defined as

\[
\bar{r} \equiv \frac{r}{R_e}; \quad \bar{\rho}_* \equiv \frac{\rho_*}{M_*,/R_e^2}; \quad \bar{\sigma}_*^2 \equiv \frac{\sigma_*^2}{GM_*,/R_e}.
\]

the isotropic and spherical Jeans equation is

\[
\frac{d}{d\bar{r}} \left[ \bar{\rho}_* \bar{\sigma}_*^2(\bar{r}) \right] = -\frac{\bar{\rho}_*}{\bar{r}^2} M_*(\bar{r}) \left( 1 + F_\text{ej} \right) + \frac{1}{f_s(\bar{r})}.
\]

(35)

The profiles \( \bar{\rho}_*(\bar{r}) \) and \( M_*(\bar{r})/M_{*,\text{tot}} \) are given by equations (2) and (3) in §2.1 for a Hernquist model, while \( (1 + F_\text{ej}) = 1/0.58 \) as just mentioned. The function \( f_s(\bar{r}) \equiv M_*(\bar{r})/M_{*,\text{tot}} \) is known in full for any specific value of \( M_{*,\text{tot}} \) (and choice of dark-matter density profile) as discussed in §3.4. Subject to the boundary condition that \( \bar{\rho}_* \bar{\sigma}_*^2 \to 0 \) as \( \bar{r} \to \infty \), equation (35) can therefore be solved for the dimensionless \( \bar{\sigma}_*^2/(GM_*,/R_e) \) as a function of \( r/R_e \) in a galaxy with any given total stellar mass.

The aperture velocity dispersion over a circular disc on the plane of the sky comes from projecting \( \sigma_\text{p}(r) \) along the line of sight and then taking a luminosity-weighted average. Defining the dimensionless projected radius \( \bar{R} \equiv R/R_e \), the stellar surface-density profile is first obtained as

\[
\Sigma_\text{p}(\bar{R}) \equiv \frac{\Sigma_*(R)}{M_{*,\text{tot}}/R_e^2} = 2 \int_{\bar{R}}^{\infty} \bar{\rho}_*(\bar{r}) \frac{\bar{r}}{(\bar{r}^2 - \bar{R}^2)^{1/2}} \, d\bar{r}; \quad (36)
\]

then the projected stellar velocity-dispersion profile is

\[
\bar{\sigma}_\text{p}^2(\bar{R}) \equiv \frac{2}{\Sigma_\text{p}(\bar{R})} \int_{\bar{R}}^{\infty} \bar{\rho}_*(\bar{r}) \bar{\sigma}_*^2(\bar{r}) \frac{\bar{r}}{(\bar{r}^2 - \bar{R}^2)^{1/2}} \, d\bar{r}; \quad (37)
\]

and the aperture dispersion within projected radius \( R_{\text{ap}} \) is

\[
\frac{\sigma_{\text{ap}}^2(R_{\text{ap}})}{GM_*,/R_e} = \left[ \int_0^{R_{\text{ap}}/R_e} \bar{\sigma}_\text{p}^2(\bar{R}) \Sigma_\text{p}(\bar{R}) \bar{R} \, d\bar{R} \right]^{-1} \times \left[ \int_0^{R_{\text{ap}}/R_e} \Sigma_\text{p}(\bar{R}) \bar{R} \, d\bar{R} \right]^{-1}.
\]

(38)

The right-hand side of this is determined entirely by \( M_{*,\text{tot}} \) once a halo model has been chosen and a value of \( R_{\text{ap}} \) specified. Setting \( R_{\text{ap}} = R_e \) yields the model \( \sigma_{\text{ap}}(R_e) \) that corresponds to the measured velocity dispersions in the McConnell & Ma (2013) compilation of SMBH \( M-\sigma \) data.

Panel (h) of Figure 4 shows the calculated \( \sigma_{\text{ap}}(R_e) \) versus \( M_{*,\text{tot}} \) for each of the four different dark-matter halo models. The points are data for the ATLAS3D galaxies, taken again from Cappellari et al. (2011, 2013a,b), the ACS/UVES galaxies included in the plot of \( R_e \) versus \( M_{*,\text{tot}} \) (do not have published velocity dispersions). All of the cuspy halos give curves that run through the middle of the \( \sigma_{\text{ap}}(R_e) \) data, while the cored Burkert (1995) halo predicts velocity dispersions that are higher for a given \( M_{*,\text{tot}} \). A Burkert halo has relatively more of its mass at larger radii than the cuspy halos do. The unprojected \( \sigma_\text{p}(r) \) is substantially higher around and beyond \( r \sim R_e \) as a result, which inflates the line-of-sight dispersion even inside \( R_e \) and boosts the aperture dispersion noticeably.

The dimensionless aperture dispersion inside \( R_e \) for a self-consistent Hernquist sphere of stars only, with no ejecta or dark matter \( F_\text{ej} = 0 \) and \( 1/f_s(\bar{r}) \equiv 0 \), is \( \sigma_{\text{ap}}(R_e)/(GM_*,/R_e)^{1/2} \approx 0.389 \). Based on this, the dispersion with ejecta and dark matter included can be usefully approximated by the function

\[
\frac{\sigma_{\text{ap}}(R_e)}{GM_*,/R_e} \approx 0.389 \sqrt{(1 + F_\text{ej}) + \frac{0.86}{f_s(\bar{r})}}.
\]

© 2015 RAS, MNRAS 000, 1–20
where the term under the square-root represents the ratio of an “effective” total mass to the total stellar mass. This formula reproduces the values from our full calculations with relative error < 2.5% for any $f_s(R_e) > 0.1$ in any of an NFW, Hernquist or Dehnen & McLaughlin halo.

We have also calculated $\sigma_{ap}(R_e)/(GM_{*, tot}/R_e)^{1/2}$ for self-gravitating Sérsic (1968) $R^{1/n}$ spheres without any dark matter. For indices $n \leq 5$—which apply to giant ellipticals and dwarfs with masses down to $M_{*,tot} \sim 10^8-10^9 M_\odot$—we find $0.36 \lesssim \sigma_{ap}(R_e) \lesssim 0.43$, as compared to $\sigma_{ap}(R_e) \approx 0.389$ for the Hernquist model. Thus, over most of the mass range in Figure 4, the model curves for $\sigma_{ap}(R_e)$ are vulnerable at only the $\lesssim 10\%$ level to bias (a slight tilt) resulting from our use of a Hernquist profile to describe all of the stellar distributions. Very massive ellipticals with $M_{*,tot} \gtrsim 2-3 \times 10^{11} M_\odot$ are generally fitted by Sérsic indices $n \approx 5-7$, for which $\sigma_{ap}(R_e)/(GM_{*, tot}/R_e)^{1/2} \approx 0.43-0.49$ rather than 0.389. However, a small compensation in our parametrisation of $R_e$ versus $M_{*,tot}$ at high masses then suffices to yield essentially the same $\sigma_{ap}(R_e)$ as the curve in Figure 4(h).

### 3.6 Discussion

#### 3.6.1 Dwarf galaxies

There are more physical considerations than the validity of a Hernquist profile for the stellar distribution, which affect how well our models might be able to describe galaxies with stellar masses less than a few $\times 10^9 M_\odot$.

In order to calculate velocity dispersions in §3.5, we assumed that stellar ejecta are retained at the bottom of any galaxy’s potential well. However, supernova-driven winds will have expelled the ejecta from many dwarf ellipticals to far beyond the stellar distributions. In this case, $F_{ej} = 0$ in equations (35) and (39) is more appropriate than $(1 + F_{ej}) = 1/0.58$. This lowers the expected $\sigma_{ap}(R_e)$ by $\approx 30\%$ at a given $M_{*,tot}$ for a given halo density profile.

On the other hand, the same galactic winds may cause changes in the central structures of the dark-matter halos of dwarfs, from initially steep density cusps to shallower profiles perhaps closer to the Burkert (1995) model (e.g., Burkert & Silk 1997; Pontzen & Governato 2012); while subsequent tidal stripping could have led to further modifications at large radii in the halos. Substantial, systematic alterations to the dark-matter density profiles may impact the values we infer for $V_{ap, pk}$, $f_s(R_e)$ and $\sigma_{ap}(R_e)$ from a given $M_{*,tot}$, $R_e$ and $M_{dark}$. And in any case, the relationship connecting $M_{*,tot}$ to $M_{dark}$ in equation (20), from Moster et al. (2010), may itself be in error if extrapolated to halo masses much below $M_{dark} \lesssim 10^{11} M_\odot$ (see Behroozi et al. 2013).

All in all, while the model curves in Figure 4 can be viewed as broadly indicative of the situation for dwarf galaxies, they should also be seen as provisional in that regime. More comprehensive modelling is required to be confident of how these kinds of average trends extrapolate to stellar masses much less than several $\times 10^9 M_\odot$ (or, roughly, $\sigma_{ap}(R_e) \lesssim 60-70$ km s$^{-1}$).

#### 3.6.2 Intracluster baryons

As already discussed in §3.2, we can safely ignore any small differences that intracluster baryons (whether gas or stars) might make to the virial radii and masses we calculate for halos centred on the most massive galaxies. Equation (39) in §3.5 now provides a way to assess the effects of intracluster baryons on the stellar velocity dispersions in the central galaxies of groups and clusters.

If additional baryonic mass is distributed spatially like the dark matter, then it can be accounted for in the Jeans equation (35), and hence in equation (39), by decreasing $f_s(r) \equiv M_s(r)/M_d(r)$ by a constant factor. This factor will be largest if the global baryon fraction in a halo is equal to the cosmic average value but only a trace amount is actually contained in the central galaxy itself. Thus, an “effective” $f_s(r)$ in the Jeans equation might be lower than the Moster et al. (2010) value by a factor of $(1 - \Omega_b,0/\Omega_m,0)^{-1}$ at most, which is $\approx 1.18$ for a 2013 Planck cosmology. This could plausibly be the situation in halos with $M_d(\nu_{vir}) \sim 10^{12} M_\odot$ (which have $M_{*,tot} \sim 10^{12} M_\odot$ for the central galaxy), but the total baryon fraction decreases systematically with decreasing (sub-)-halo mass (e.g., Gonzalez et al. 2013; Zhang et al. 2011; McGaugh et al. 2010). In galaxy-sized halos, it is generally consistent with the mass of stars, remnants and stellar ejecta in the galaxy proper, which we have already accounted for fully.

The maximum effect on $\sigma_{ap}(R_e)$ in the central galaxy can be estimated by comparing the value of equation (39) with $(1 + F_{ej}) = 1/0.58$ and $f_s(R_e) = 0.5$—the lowest value in any of our models at $M_{*,tot} = 10^{12} M_\odot$ or $M_{dark} \sim 10^{12} M_\odot$ in Figure 4—to the value using $f_s(R_e) = 0.5/1.18$ instead. The result is an increase of $< 5\%$ in the velocity dispersion. This is of the same order as the maximum effect on our values for the halo virial radii. We have chosen to ignore intracluster baryons altogether rather than introduce detailed additional modelling just to make adjustments that are at most so small.

#### 3.6.3 Comparisons to individual systems

In Appendix, we make some checks on the average scalings represented in Figure 4, by comparing various numbers extracted from them to relevant data in the literature for the Milky Way, M87 and M49 (the central galaxies of Virgo sub-clusters A and B) and NGC 4889 (the brightest galaxy in the Coma Cluster). The stellar masses and velocity dispersions of these systems span the range covered by the local early-type galaxies used to define empirical black hole $M-\sigma$ relations. It is notable in particular that, starting with just the galaxies' total stellar masses, the scalings imply detailed properties of the cluster-sized dark-matter halos around each of M87, M49 and NGC 4889, which are in reasonably good agreement with literature values.

### 4  THE BLACK HOLE $M-\sigma$ RELATION

The scalings in §3 give typical virial masses and peak circular speeds for dark-matter halos, along with stellar velocity dispersions inside an effective radius, as one-to-one functions of galaxy stellar mass at $z = 0$. Therefore, they can be recast to give $M_{dark}(\nu_{vir})$ and $V_{ap, pk}$ directly as functions of the observable $\sigma_{ap}(R_e)$. If a theory ties $M_{BH}$ to the properties of halos at some time in the past, then in order to predict the dependence of $M_{BH}$ on $\sigma_{ap}(R_e)$ or any other
of gas-rich mergers. We denote this redshift by

$$z_{\text{qso}} = 3$$

and SMBH accretion-rate densities in the Universe: namely, $z_{\text{qso}} = 3$ in most cases (e.g., Richards et al. 2006; Hopkins et al. 2007a; Deltrevchio et al. 2014; also Di Matteo et al. 2008; Sijacki et al. 2007, 2015).

In this Section, we apply our calculations from §2.6 to find typical values of $M_{\text{d,vir}}(z_{\text{qso}})$ and $V_{d,\text{pk}}(z_{\text{qso}})$ for the most massive progenitors of halos, and hence estimate an expected $M_{\text{BH}}$ in their central galaxies, as functions of the stellar $\sigma_{\text{ap}}(R_e)$ at $z = 0$. This involves an assumption that the most massive progenitor halo at $z_{\text{qso}} > 0$ is the one that ultimately defines the centre of the larger potential well at $z = 0$. This is statistically accurate but not always true in every individual case—see, for example, the discussion in van den Bosch et al. (2014) of the distinction between “most massive” and “most contributing” progenitors. Glossing over this subtlety could lead to a small amount of scatter in the SMBH $M-\sigma$ data relative to our final curves.

The model $M_{\text{BH}}-\sigma_{\text{ap}}(R_e)$ relations we obtain do not include any growth of the SMBH itself at redshifts $z < z_{\text{qso}}$, which can occur by coalescences in gas-poor galaxy mergers at the centre of a halo. However, this is distinct from the growth of the halo as a whole; many sub-halos can be accreted at low redshift that do not sink to the bottom of the potential well and thus do not grow the central SMBH. We discuss this further in §4.2.

4.1 Halo masses and peak circular speeds at $z > 0$

The top panel of Figure 5 shows the scaling of halo virial mass at $z = 0$ versus stellar velocity dispersion $\sigma_{\text{ap}}(R_e)$ in the central galaxy at $z = 0$, obtained directly from the results of §3 (combining panels (d) and (h) of Figure 4). The next panel down shows $M_{d,\text{vir}}$ for the most massive progenitor of a halo at redshift $z_{\text{qso}} = 3$ (obtained from $M_{d,\text{vir}}(0)$ as described in §2.6; see Figure 3) against $\sigma_{\text{ap}}(R_e)$ in the central galaxy at $z = 0$.

The blue curves in Figure 5 correspond to Dehnen & McLaughlin (2005) models for the halo density profiles; the red curves, to Hernquist (1990) models. These bracket the

$$M_{\text{BH}} \simeq 1.14 \times 10^8 M_\odot \left( \frac{f_0}{0.2} \right) \left( \frac{V_{d,\text{pk}}}{200 \text{ km s}^{-1}} \right)^4.$$  

(1)

As discussed in §1.1, this equation is limited by simplifying assumptions: for example, about the nature of quasar-mode SMBH feedback (taken to be purely momentum-conserving) and the distribution of gas in protogalaxies (taken to be virialised, with ongoing cosmic infall ignored). Within these limitations it has the advantage of generality, being applicable to dark matter halos with any density profile.

In equation (1), $V_{d,\text{pk}}$ measures the potential well of a protogalaxy that just fails to contain the quasar-mode feedback of an SMBH with mass $M_{\text{BH}}$. It thus refers to conditions at a redshift marking the end of rapid SMBH growth by accretion at Eddington or supercritical rates in a series of gas-rich mergers. We denote this redshift by $z_{\text{qso}}$. It will be different for different systems, but we expect the general range to coincide with the epoch of peak quasar number and SMBH accretion-rate densities in the Universe: namely, $z_{\text{qso}} \sim 2–4$ in most cases (e.g., Richards et al. 2006; Hopkins et al. 2007a; Deltrevchio et al. 2014; also Di Matteo et al. 2008; Sijacki et al. 2007, 2015).

The blue curves in Figure 5 correspond to Dehnen & McLaughlin (2005) halos; red curves have Hernquist (1990) halos. Next two panels: Peak dark-matter circular speed (in km s$^{-1}$) at $z = 0$ and at $z_{\text{qso}} = 3$, versus $\sigma_{\text{ap}}(R_e)$ at $z = 0$. Blue curves are for galaxy models with Dehnen & McLaughlin (2005) halos; red curves have Hernquist (1990) halos. Next two panels: Peak dark-matter circular speed (in km s$^{-1}$) at $z = 0$ and at $z_{\text{qso}} = 3$, versus $\sigma_{\text{ap}}(R_e)$ at $z = 0$. Blue and red curves correspond again to Dehnen & McLaughlin and Hernquist halo density profiles. The straight, dashed line shows $V_{d,\text{pk}} = \sqrt{2} \sigma_{\text{ap}}(R_e)$. Bottom panel: SMBH mass (in $M_\odot$) calculated from equation (1) with $f_0 = 0.18$ using the dark-matter $V_{d,\text{pk}}$ at $z = 0$ (dot-dash blue and red curves) and at $z_{\text{qso}} = 3$ (solid blue and red curves), all plotted against $\sigma_{\text{ap}}(R_e)$ at $z = 0$. The dashed straight line is equation (1) with $V_{d,\text{pk}} = \sqrt{2} \sigma_{\text{ap}}(R_e)$. Data points are for the 53 ellipticals and lenticulars in McConnell & Ma (2013).
scalings obtained using NFW halo profiles, while (as discussed in §3), the cored halo profiles of Burkert (1995) are not appropriate in the galaxy mass range plotted here. Velocity dispersions $\sigma_{ap}(R_e) \geq 70$ km s$^{-1}$ at $z = 0$ correspond to stellar masses $M_{*\tot} \geq 8 \cdot 10^9 M_\odot$ at $z = 0$.

The next panel in the Figure shows the peak dark-matter circular speed at $z = 0$ versus stellar velocity dispersion at $z = 0$, again from §3 [combining panels (f) and (a) of Figure 4]. Just below this is the scaling of $V_{d, pk}$ in the most massive progenitor at $z_{qso} = 3$ [obtained from $V_{d, pk}(0)$ and $M_{d, vir}(0)$ as in §2.6 and Figure 3] versus $\sigma_{ap}(R_e)$ in the central galaxy at $z = 0$. The straight, dashed (black) line in these panels traces out $V_{d, pk} = \sqrt{2} \sigma_{ap}(R_e)$. This is clearly a poor substitute for the actual relationship between the two velocities at $z = 0$ in galaxies with $\sigma_{ap}(R_e) \gtrsim 200$ km s$^{-1}$ (or $M_{*\tot} \gtrsim 3 \times 10^{11} M_\odot$). It does come closer in this mass range to correctly estimating the dependence of $V_{d, pk}$ at $z_{qso} = 3$ on $\sigma_{ap}(R_e)$ at $z = 0$; but this appears to be entirely coincidental, and the situation is reversed for $\sigma_{ap}(R_e) \lesssim 200$ km s$^{-1}$.

At a given value for $\sigma_{ap}(R_e)$, the downwards “corrections” to $M_{d, vir}$ and $V_{d, pk}$, from their values at $z = 0$ to the progenitors at $z_{qso} = 3$, are systematically larger for larger systems. This is a restatement of the flattening towards higher masses in the dependence of $M_{d, vir}(z)$ on $M_{d, vir}(0)$, which we showed in Figure 3 and discussed there. Again, it is fundamentally because in a (Λ)CDM cosmology, more massive halos were assembled and virialised more recently.

A given range of halo mass or circular speed at $z = 0$ thus corresponds to a narrower range at any $z_{qso} > 0$, and the contrast is greater for higher masses. In Figure 5, this works to make the slopes of $M_{d, vir}$ and $V_{d, pk}$ versus $z = 0$ velocity dispersions significantly shallower for the halo progenitors at $z_{qso} = 3$ than for the halos themselves at $z = 0$.

The equations from §2.6 that underpin these results are approximations to the mass accretion histories of simulated halos in van den Bosch et al. (2014). These simulations extend up to halo masses $M_{d, vir}(0) \lesssim 10^{15} M_\odot$, corresponding to stellar $\sigma_{ap}(R_e) \lesssim 350–400$ km s$^{-1}$ at $z = 0$. Beyond this, our analysis is not only approximate but an extrapolation. Thus, for example, the peaks around $\sigma_{ap}(R_e) \approx 400$ km s$^{-1}$ in the panels of Figure 5 for $M_{d, vir}$ and $V_{d, pk}$ at $z_{qso} = 3$ may not be accurate. What is secure is the simple fact of the relative flatness in these curves for high stellar velocity dispersions. The same effect must appear to a greater or lesser degree for any other $z_{qso} > 0$, and it directly impacts any prediction for an observable SMBH $M-\sigma$ relation at $z = 0$ from a model like our equation (1) or similar.

### 4.2 $M_{BH}$ versus $\sigma_{ap}(R_e)$

The bottom panel of Figure 5 shows SMBH mass versus $\sigma_{ap}(R_e)$ at $z = 0$. The data points are for the E and S0 galaxies in the compilation of McConnell & Ma (2013). (Their data for the bulges of late-type galaxies can be seen in Figure 1. We do not show them here because our calculations for $\sigma_{ap}(R_e)$ versus $M_{*\tot}$ do not allow for discs.) The dashed straight line (black), which we show purely for reference, is equation (1) evaluated with a protogalactic gas fraction of $f_0 \equiv \Omega_{b, 0}/(\Omega_{m, 0} – \Omega_{b, 0}) = 0.18$ (for the 2013 Planck cosmology) and the simplistic substitution $V_{d, pk} \equiv \sqrt{2} \sigma_{ap}(R_e)$. The other curves (blue and red for Dehnen & McLaughlin and Hernquist halo density profiles) also come from equation (1) with $f_0 = 0.18$, but with $V_{d, pk}$ depending on $\sigma_{ap}(R_e)$ as shown in the other panels of Figure 5.

The broken blue and red curves come from those for $V_{d, pk}$ at $z = 0$ versus $\sigma_{ap}(R_e)$ at $z = 0$ in the middle panel of Figure 5. These are the predictions of equation (1) for the critical SMBH masses required to clear halos filled with virialised gas in an 18% mass ratio, via quasar-mode feedback now. It is no surprise that such predictions overshoot the $M-\sigma$ data for normal early-type galaxies, quite substantially for $\sigma_{ap}(R_e) \gtrsim 200$ km s$^{-1}$.

The solid blue and red curves of $M_{BH}$ versus $\sigma_{ap}(R_e)$, which run through the data, are based on the curves of $V_{d, pk}$ at $z_{qso} = 3$ versus $\sigma_{ap}(R_e)$ at $z = 0$ in the fourth panel of Figure 5. These are predictions for the $M-\sigma$ relation in quiescent galaxies at $z = 0$, if it came from an $M_{BH} \propto V_{d, pk}^\gamma$ relationship established by quasar-mode feedback and blowout from gaseous protogalaxies at $z_{qso} = 3$ (with negligible subsequent SMBH growth via coalescence in mergers).

Figure 6 gives an expanded view of $M_{BH}$ versus $\sigma_{ap}(R_e)$. Now, the solid (blue) curves show SMBH masses obtained from equation (1) after using our scalings to relate stellar velocity dispersion at $z = 0$ to the typical $V_{d, pk}$ in progenitor halos at a wider range of possible $z_{qso} = 0, 1, 2, 3$ and 4. All of these curves assume a Dehnen & McLaughlin density profile for the dark matter; the results for NFW or Hernquist profiles are very similar. The dashed, straight (black) reference line is again equation (1) with $V_{d, pk} \equiv \sqrt{2} \sigma_{ap}(R_e)$.

Most of the $M-\sigma$ data at $z = 0$ lie between model curves in which an $M_{BH}-V_{d, pk}$ relation emerged from the clearing of protogalaxies by quasar-mode feedback at redshifts $2 \lesssim z_{qso} \lesssim 4$. The correspondence of this range with the epoche of peak quasar activity and SMBH accretion rate in both observations and cosmological simulations is encouraging. Equation (1) represents a highly simplified, broad-brush picture of just a few processes at a critical stage of galaxy and black hole formation; but the fundamental connection it makes between protogalactic dark-matter halos and SMBH masses appears to be along the right lines.

The upward bends around $\sigma_{ap}(R_e) \approx 140$ km s$^{-1}$ in all of the $M_{BH}-\sigma_{ap}(R_e)$ predictions in Figure 6 trace back to the peak at $M_{*\tot} \approx 3.4 \times 10^{10} M_\odot$ [at $z = 0$; see Figure 4(b)] in $f_{\nu, vir}$, the global stellar-to-dark matter mass fraction. Thus, a linear relation $\log(M_{BH}) \sim 4 \log(V_{d, pk})$ is strongly distorted by a non-linear “conversion” from halo circular speeds and virial masses to stellar masses and velocity dispersions.

The curves with $2 \lesssim z_{qso} \lesssim 4$ in Figure 6 have average slopes $\Delta \log M_{BH}/\Delta \log \sigma_{ap}(R_e) \approx 1.5-2$ for galaxies with $50 \lesssim \sigma_{ap}(R_e) \lesssim 100$ km s$^{-1}$, but $\Delta \log M_{BH}/\Delta \log \sigma_{ap}(R_e) \approx 5-7$ in the range $200 \lesssim \sigma_{ap}(R_e) \lesssim 300$ km s$^{-1}$. However, this curvature is easily accommodated by the data. It is reminiscent of the ad hoc, log-quadric fits to local $M-\sigma$ samples by Wyithe (2006a,b) (see also Gültekin et al. 2009; McConnell & Ma 2013).

Equally important is the flattening of the model $M_{BH} - \sigma_{ap}(R_e)$ relations away from the $z_{qso} = 0$ curve, which occurs at high $\sigma_{ap}(R_e) \gtrsim 300$ km s$^{-1}$ and is more pronounced for larger $z_{qso}$. This is just the behaviour seen in Figures 3 and 5 above: the masses $M_{d, vir}(z)$ and circular speeds $V_{d, pk}(z)$ of the most massive progenitors of halos (which directly determine $M_{BH}$ here) have flatter dependences at higher $z$ on the final mass $M_{d, vir}(0)$ [related to $\sigma_{ap}(R_e)$ at $z = 0$ by the

© 2015 RAS, MNRAS 000, 1–20
Figure 6. SMBH mass versus stellar velocity dispersion measured inside $R_e$ at $z = 0$. Data points represent the 53 galaxies flagged as early types in McConnell & Ma (2013). The solid, blue curves are our models for $M_{\text{BH}}$ versus $\sigma_{\text{ap}}(R_e)$ at $z = 0$ if a relation $M_{\text{BH}} \propto V_{\text{d, pk}}^3$ was established by accretion-driven feedback, according to equation (1), at redshift $z_{\text{qso}} = 0, 1, 2, 3$ or 4. The curves all assume a Dehnen & McLaughlin (2005) model for the dark-matter halo density profile, and a spatially constant gas-to-dark matter mass ratio $f_0 = 0.18$ in the progenogalaxies. They do not include any SMBH growth between $0 < z < z_{\text{qso}}$; see text for discussion. For reference only, the dashed black line shows equation (1) with $V_{\text{d, pk}} \equiv \sqrt{2}\sigma_{\text{ap}}(R_e)$.

4.2.1 Dry mergers at low redshift

It is also in the highest-$\sigma_{\text{ap}}$ regime that gas-poor galaxy mergers at $z < z_{\text{qso}}$ may have increased $M_{\text{BH}}$ the most from any value determined by quasar-mode feedback at $z_{\text{qso}}$.

Volonteri & Ciotti (2013) perform cosmological simulations of black hole growth in the central galaxies of halos with masses at $z = 0$ of $10^{13} M_\odot \lesssim M_{\text{d, vir}}(0) \lesssim 10^{15} M_\odot$. They track contributions from gas accretion and from SMBH coalescences in gas-poor mergers separately. The results they show for six example halos with $M_{\text{d, vir}}(0) = 10^{15} M_\odot$ have the central SMBH growth by accretion essentially finished in all cases at a redshift $z \approx 2$–3. We would associate this here with $z_{\text{qso}}$. Coalescences in dry mergers then drive the growth for $z < z_{\text{qso}}$, and especially at $z \lesssim 1$. Ultimately the SMBH masses are increased by a wide range of factors, $f_{\text{co}} \equiv M_{\text{BH}}(0)/M_{\text{BH}}(z_{\text{qso}}) \simeq 1$–30. For a larger sample of $10^{15}$-$M_\odot$ halos, Volonteri & Ciotti report an average $\langle f_{\text{co}} \rangle \approx 11 \pm 10$.

From §3, at $z = 0$ the central galaxies in halos with $M_{\text{d, vir}}(0) = 10^{15} M_\odot$ typically have $M_{\star, \text{tot}} \simeq 10^{12} M_\odot$ and $\sigma_{\text{ap}}(R_e) \approx 350$–400 km s$^{-1}$ (depending on the assumed dark-matter density profile). The rightmost and highest data point in Figure 6 sits near this region; it represents NGC 4889 in the Coma Cluster, with $\sigma_{\text{ap}}(R_e) = 347 \pm 17$ km s$^{-1}$ (McConnell et al. 2012). This may well be a system where low-redshift merging grew $M_{\text{BH}}$ substantially above a feedback-limited value at $z_{\text{qso}} = 2$–3.

At lower halo and galaxy masses, there is generally much less SMBH growth through late mergers. For the central galaxies of halos with $2 \times 10^{13} M_\odot \lesssim M_{\text{d, vir}}(0) \lesssim 10^{14} M_\odot$ (corresponding to $M_{\star, \text{tot}} \simeq 2$–4 $\times 10^{11} M_\odot$ and $\sigma_{\text{ap}}(R_e) \approx 220$–275 km s$^{-1}$ at $z = 0$), Volonteri & Ciotti give averages of $\langle f_{\text{co}} \rangle \approx 2 \pm 1$. For a set of $10^{13}$-$M_\odot$ halos (corresponding to $M_{\star, \text{tot}} \simeq 1.4 \times 10^{11} M_\odot$ and $\sigma_{\text{ap}}(R_e) \approx 200$ km s$^{-1}$), they find $\langle f_{\text{co}} \rangle = 1.8 \pm 1.8$, suggestive of a small systematic effect with a few strong outliers.$^2$

Thus, we can expect dry mergers to scatter data at the top end of the $M$--$\sigma$ relation significantly upwards from curves like those in Figure 6. This would mask any flattening of the curves at $\sigma_{\text{ap}}(R_e) \gtrsim 300$ km s$^{-1}$ and could appear as a much steeper, even near-vertical mean relation there (the so-called “saturation” discussed by, e.g., Kormendy & Ho 2013 and McConnell & Ma 2013). Among systems with more moderate velocity dispersions at $z = 0$, dry merging can still introduce some scatter, but not as much. The net shift

$^2$ Volonteri & Ciotti do not show explicitly for any of their halos with $M_{\text{d, vir}}(0) < 10^{15} M_\odot$ that accretion-driven growth of the central-galaxy SMBH is negligible after $z_{\text{qso}} \approx 2$–3. However, other simulations imply this is generally the case (and, indeed, suggest larger $z_{\text{qso}}$ in some instances); see, e.g., Sijacki et al. (2007) and Di Matteo et al. (2008).
up from curves for \( M_{\text{BH}} \) limited by feedback at \( z_{\text{qso}} \approx 2-3 \) could plausibly amount to a factor of \( \approx 2-3 \) in the regime \( 200 \lesssim \sigma_{\text{ap}}(R_e) \lesssim 300 \text{ km s}^{-1} \), and probably less for lower \( \sigma_{\text{ap}}(R_e) \lesssim 150-200 \text{ km s}^{-1} \). This should largely preserve the overall shape of such curves.

### 4.2.2 Discussion

Incorporating the generally modest systematic effects of low-redshift mergers in the models shown in Figure 6 would primarily move the curves upwards on the plot. [Mergers at all redshifts are already included in how \( V_{\text{d,pk}} \) in a progenitor halo at \( z_{\text{qso}} > 0 \) is connected to \( \sigma_{\text{ap}}(R_e) \) in the central galaxy at \( z = 0 \); only the value of \( M_{\text{BH}} \) needs to be adjusted.] However, a few factors could lower the starting \( M_{\text{BH}}-V_{\text{d,pk}} \) relation predicted by equation (1) at any given \( z_{\text{qso}} \).

First, if the baryon-to-dark mass fraction in a protogalaxy at \( z_{\text{qso}} \) were less than \( f_0 = 0.18 \)—the cosmic average, assumed for all of the curves in Figure 6—then the critical \( M_{\text{BH}} \) for blow-out would be decreased proportionately. Second, equation (1) ignores any prior work done by a growing SMBH to push the protogalactic gas outwards before the point of final blow-out, and thus it overestimates the mass required to clear the halo completely at \( z_{\text{qso}} \). Related to this, lower SMBH masses may suffice to quench quasar-mode accretion by clearing gas from the inner regions to “far enough” away from a central SMBH, without expelling it fully past the virial radius.

Cosmological simulations are required to evaluate the balance between these effects pushing the model \( M_{\text{BH}}-\sigma_{\text{ap}}(R_e) \) curves downwards in Figure 6, and the competing effects of late, dry mergers pulling upwards. But at this level, the more fundamental simplifications underlying equation (1)—among others, the idea that quasar-mode feedback is always momentum-driven—need to be improved first.

Likewise, low-redshift mergers are just one possible source of intrinsic scatter in the empirical \( M-\sigma \) relation at \( z = 0 \). Another is different values in different systems for the precise redshift at which the main phase of accretion-driven SMBH growth was ended by quasar-mode feedback. Even if there were a single \( z_{\text{qso}} \), there must be real scatter in the data around any trend line such as those in Figure 6, because of the scatter around the constituent scalings from §2 and §3 for halos, halo evolution and central galaxies. It is important, but beyond the scope of this paper, to understand the physical content of the observed \( M-\sigma \) scatter in detail. Part of the challenge is to know the “correct” trend for \( M_{\text{BH}} \) versus \( \sigma_{\text{ap}}(R_e) \) at \( z = 0 \), around which scatter should be calculated. In the context of feedback models, this again requires improving on equation (1) for the prediction of \( M_{\text{BH}} \) values at \( z_{\text{qso}} > 0 \).

### 5 SUMMARY

We have examined how a simple relationship between SMBH masses \( M_{\text{BH}} \) and the circular speeds \( V_{\text{d,pk}} \) in protogalactic dark-matter halos, established by quasar-mode feedback at redshift \( z_{\text{qso}} > 0 \), is reflected in a correlation between \( M_{\text{BH}} \) and the stellar velocity dispersions \( \sigma_{\text{ap}}(R_e) \) in early-type galaxies at \( z = 0 \). Straightforward but non-trivial approximations for halo growth and scalings between halos and their central galaxies transform a power-law \( M_{\text{BH}}-V_{\text{d,pk}} \) relation at \( z_{\text{qso}} \) into a decidedly non-power-law \( M_{\text{BH}}-\sigma_{\text{ap}}(R_e) \) relation at \( z = 0 \). This relation nevertheless compares well to current data, for assumed values of \( z_{\text{qso}} \approx 2-4 \).

We worked with two-component models for spherical galaxies. Because the stellar properties most relevant to us are those at (or averaged inside) an effective radius, it sufficed to assume Hernquist (1990) density profiles for the stars inside any galaxy. Because dark-matter halos are key to determining SMBH mass in the feedback scenario we focused on, we allowed for any of four different halo density profiles: those of Navarro et al. (1996, 1997), Hernquist (1990), Dehnen & McLaughlin (2005) and Burkert (1995).

The scaling relations we developed are trend lines connecting average stellar properties at \( z = 0 \) [total masses \( M_{\text{tot}} \), effective radii \( R_e \), aperture velocity dispersions \( \sigma_{\text{ap}}(R_e) \) and dark-matter mass fractions] to the typical virial masses \( M_{\text{d,vir}} \) and peak circular speeds \( V_{\text{d,pk}} \) of dark-matter halos at \( z = 0 \) and their most massive progenitors up to \( z \lesssim 4-5 \). These scalings are constrained by theoretical work in the literature on the global structures, baryon contents and redshift-evolution of dark-matter halos (§2) and by data in the literature for local elliptical galaxies (§3). They are robust for normal early-type systems with stellar masses greater than several \( 10^9 M_{\odot} \) at \( z = 0 \), corresponding to velocity dispersions \( \sigma_{\text{ap}}(R_e) \gtrsim 60-70 \text{ km s}^{-1} \), but are largely untested against lower-mass dwarf galaxies (see §3.6).

We applied the scalings to show in §4 how a relationship of the form \( M_{\text{BH}} \propto V_{\text{d,pk}}^\gamma \) at a range of redshifts \( z_{\text{qso}} > 0 \) (equation [1]; McQuillan & McLaughlin 2012) appears as a much more complicated \( M_{\text{BH}}-\sigma_{\text{ap}}(R_e) \) relation at \( z = 0 \). The specific form for an initial \( M_{\text{BH}}-V_{\text{d,pk}} \) relation comes from a simplified theoretical analysis of momentum-conserving SMBH feedback in isolated and virialised gaseous protogalaxies with non-isothermal dark-matter halos. Some of the simplifying assumptions involved thus need to be relaxed and improved in future work. Meanwhile, the highly “non-linear” observable \( M_{\text{BH}}-\sigma_{\text{ap}}(R_e) \) relation we infer from it does describe the data for local early types if the redshift of quasar-mode blow-out was \( z_{\text{qso}} \approx 2-4 \). This range is reassuringly similar to the epoch of peak quasar density and SMBH accretion rate in the Universe.

This lends support to the notion that the empirical \( M-\sigma \) relation fundamentally reflects some close connection due to accretion feedback between SMBH masses in galactic nuclei and the dark matter in their host (proto)galaxies. It also demonstrates that the true, physical relationship between \( M_{\text{BH}} \) and stellar velocity dispersion at \( z = 0 \) is not necessarily a pure power law. The shape in our analysis has an upwards bend around \( \sigma_{\text{ap}}(R_e) \approx 140 \text{ km s}^{-1} \) (Figure 6), corresponding to stellar masses \( M_{\text{tot}} \approx 3-4 \times 10^{10} M_{\odot} \) and halo masses \( M_{\text{d,vir}}(0) \approx 10^{12} M_{\odot} \) at \( z = 0 \). This bend comes from a sharp maximum at these masses in the global stellar-to-dark matter fractions, \( M_{\text{tot}}/M_{\text{d,vir}}(0) \) (e.g., Moster et al. 2010). Consequently, there is a sharp upturn in the dependence of halo circular speeds \( V_{\text{d,pk}} \) on the stellar \( \sigma_{\text{ap}}(R_e) \) (see Figures 4 and 5).

Our models also show a flattening of \( M_{\text{BH}} \) versus \( \sigma_{\text{ap}}(R_e) \) at \( z = 0 \) for velocity dispersions above \( 300 \text{ km s}^{-1} \) or so, for any blow-out redshift \( z_{\text{qso}} > 0 \) but more so for higher \( z_{\text{qso}} \) (Figure 6). This is due to the way that dark-matter halo masses grow and circular speeds increase...
through hierarchical merging in a ΛCDM cosmology after $M_{\text{BH}}$ is set by feedback and the halo properties at $z_{\text{qso}}$ (see Figure 3). However, the values we calculate for $M_{\text{BH}}$ include only the growth by accretion up to $z = z_{\text{qso}}$; further growth through SMBH–SMBH coalescences in gas-poor mergers at lower redshifts is neglected. (The effects of such mergers on halo masses and circular speeds, and stellar velocity dispersions at $z = 0$, are accounted for.)

As discussed in §4.2, simulations by Volonteri & Ciotti (2013) suggest that low-redshift merging has a significant effect on the SMBH masses in systems with large $\sigma_{\text{ap}}(R_e) \gtrsim 300-350$ km s$^{-1}$ at $z = 0$. There, dry mergers can scatter $M_{\text{BH}}$ values strongly upwards from the values at $z_{\text{qso}}$, essentially erasing the flattening that might otherwise be observed at $z = 0$ and “saturating” the empirical $M-\sigma$ relation. In galaxies with lower $\sigma_{\text{ap}}(R_e) \lesssim 300$ km s$^{-1}$, where most current data fall, such scatter up from feedback-limited SMBH masses will be much more modest in general. The expected $M_{\text{BH}}-\sigma_{\text{ap}}(R_e)$ relations at $z = 0$ should then have the same basic shape as when late mergers are ignored.

Although we have focussed on the observed $M-\sigma$ relation, other SMBH–bulge correlations exist that may be just as strong intrinsically. These include the $M_{\text{BH}}-M_{\text{bulge}}$ correlation and multivariate, “fundamental-plane” relationships between $M_{\text{BH}}$ and non-trivial combinations of $M_{\text{* tot}}$, $R_e$ and $\sigma_{\text{ap}}(R_e)$. They should also reflect any underlying SMBH–dark matter connection at some $z_{\text{qso}} > 0$, and the techniques of this paper can be applied to look at them as well. However, this will best be done with close attention also paid to the inevitable scatter around all of the scalings we have adopted for both stellar and dark-matter halo properties. It remains to be understood how the numerous individual sources of scatter combine to produce SMBH correlations with apparently so little net scatter at $z = 0$.

More sophisticated predictions of critical SMBH masses for quasar-mode blow-out in terms of protogalactic dark-matter halo properties are required. The simple relation $M_{\text{BH}} \propto V_{\text{vir}}^{2/3}$ that we have used makes very specific assumptions about the mechanism (e.g., momentum-driven) and the setting (spherical protogalaxies with no stars, initially virialised gas, smooth outflows) of the feedback that establishes it. We mentioned in §1.1 and §4.2 several ways to improve on these assumptions. Our work in this paper is readily adaptable to help test any refinements.

ACKNOWLEDGEMENTS

ACL has been supported by an STFC studentship.

REFERENCES

Behroozi P.S., Wechsler R.H., Conroy C., 2013, ApJ, 770, 57
Blakeslee J.P., et al., 2009, ApJ, 694, 556
Bryan G.L., Norman M.L., 1998, ApJ, 495, 80
Bruzual G., Charlot S., 2003, MNRAS, 344, 1000
Bullock J.S., Kolatt T.S., Sigad Y., Somerville R.S., Kravtsov A.V., Klypin A.A., Primack J.R., Dekel A., 2001, MNRAS, 321, 559
Burkert A., 1995, ApJ, 447, L25

Burkert A., Silk J., 1997, ApJ, 488, L55
Cappellari M., et al., 2011, MNRAS, 413, 813
Cappellari M., et al., 2013a, MNRAS, 432, 1709
Cappellari M., et al., 2013b, MNRAS, 432, 1862
Chen C-W., et al., 2010, ApJ, 191, 1
Costa T., Sijacki D., Haehnelt M.G., 2014, MNRAS, 444, 2355
Côté P., McLaughlin D.E., Cohen J.G., Blakeslee, J.P., 2003, ApJ, 591, 850
Côté P., McLaughlin D.E., Hanes D.A., Bridges T.J., Geisler D., Merritt D., Hesser J.E., Harris G.L.H., Lee, M.G., 2001, ApJ, 559, 828
Côté P., et al., 2004, ApJS, 153, 223
Dehnen W., McLaughlin D. E., 2005, MNRAS, 362, 1057
Dehnen W., McLaughlin D. E., Sachaniia J. 2006, MNRAS, 369, 1688
DelleVecchio I. et al., 2014, MNRAS, 439, 2736
Di Matteo T., Collberg J., Springel V., Hernquist L., Sijacki D. 2008, ApJ, 676, 33
Dubinski J., Carlberg R. G., 1991, ApJ, 378, 496
Dutton A., Maccio A., 2014, MNRAS, 441, 3359
Einasto J., 1965, Trudy Inst. Astrof. Alma-Ata, 57, 87
Fabian A.C., 1999, MNRAS, 308, L39
Ferrarese L., Ford H., 2005, SSRv, 116, 523
Ferrarese L., Merritt D., 2000, ApJ, 539, L9
Freeman K.C., 1985, IAU Symp. 106, 113F
Gebhardt K. et al., 2000, ApJ, 539, L13
Gioiini S. et al., 2009, ApJ, 703, 982
Gonzalez A.H., Sivanandam S., Zabludoff A.I., Zaritsky D., 2013, ApJ, 778, 14
Graham A., Colless M., 1997, MNRAS, 287, 221
Graham A. W., Merrill D., More B., Diemand J., Terzic B., 2006, AJ, 132, 2711
Gultekin K. et al., 2009, ApJ 698, 198
Hopkins P.F., Richards G.T., Hernquist L., 2007a, ApJ, 654, 731
Hopkins P.F., Cox T.J., Robertson B., Krause E., 2007b, ApJ, 669, 45
Hopkins P.F., Hernquist L., Cox T.J., Robertson B., Krause E., 2007c, ApJ, 669, 67
Hernquist L., 1990, ApJ, 356, 359
Hébrard N., Rix H.-W., 2004, ApJ, 604, L89
King A., Pounds K. A., 2003, MNRAS, 345, 657K
King A., 2003, ApJ, 596, L27
King A., 2005, ApJ, 635, L121
King A., 2010, MNRAS, 402, 1516
Kormendy J., Ho L.C., 2013, ARAA, 51, 511
Kroupa P., 2001, MNRAS, 322, 231
Lokas E.L., Mamon, G.A., 2003, MNRAS, 343, 401
Magorrian J. et al., 1998, AJ, 115, 2285
Maraston C., 1998, MNRAS, 300, 872
Maraston C., 2005, MNRAS, 362, 799
Marconi A., Hunt L.K., 2003, ApJ, 589, L21
McConnell N.J., Ma C.-P., 2013, ApJ, 764, 184
McConnell N.J., Ma C.-P., Gebhardt K., Wright S.A., Murphy J.D., Lauer T.R., Graham J.R., Richstone D.O., 2011, Nature, 480, 215
McConnell N.J., Ma C.-P., Murphy J.D., Gebhardt K., Lauer T.R., Graham J.R., Wright S.A., Richstone D.O., 2012, ApJ, 756, 179
McGaugh S.S., Schombert J.M., de Blok W.J.G., Zagursky M.J., 2010, ApJ, 708, L14
APPENDIX A: MODEL CHECKS AT $z = 0$

Here we collect some properties from the literature for a few galaxies and halos spanning the range of mass and stellar velocity dispersion covered by local galaxy samples used to define empirical SMBH $M-\sigma$ relations. We then extract numerical values from the $z = 0$ scalings in §3 (Figure 4) to compare with the measurements.

A1 Stellar and halo properties from the literature

Table A1 lists observed stellar properties of the Milky Way, M87 (at the centre of Virgo subcluster A), M49 (at the centre of Virgo B) and NGC 4889 (in the Coma Cluster). Properties of the dark matter halos are also given, from dynamical modelling in the literature. Our analysis is clearly not meant to describe disc galaxies, but we have included the Milky Way as a useful check on the implications for $\sim L^*$ galaxies in general.

A1.1 The Milky Way

In the first row of Table A1, the total stellar mass, the radius $r_{200}$ of mean overdensity $\Delta \equiv 200$ and the dark-matter mass $M_{\Delta,200}$ inside this are all taken from McMillan (2011). Combining his best-fitting NFW concentration, $r_{200}/r_{-2} \approx 9.55$, with his values of $M_{\Delta,200}$ and $r_{200}$ plus $r_{pk}/r_{-2} = 2.16258$ for an NFW halo, yields $r_{pk} \approx 52$ kpc and $V_{\text{ap},pk} \approx 185$ km s$^{-1}$. These are consistent with separate modelling of the Milky Way by Dehnen et al. (2006).

The second row of Table A1 contains the total stellar mass of the Milky Way bulge only, according to McMillan (2011). He does not record the effective radius of the bulge or the aperture dispersion inside it, so we take $R_e \approx 2.7$ kpc from Freeman (1985) and $\sigma_{\text{ap}}(R_e) \approx 103$ km s$^{-1}$ from McConnell & Ma (2013).

A1.2 M87 and M49

For M87 and M49, Table A1 quotes total stellar masses based on three different sources: the ATLAS$^{3D}$ survey (Cappellari et al. 2011), the ACSVCS (Chen et al. 2010) and McConnell & Ma (2013). The original authors give total luminosities, to which we have applied mass-to-light ratios from Maraston (2005) models for a Kroupa (2001) IMF for $M_{*,\text{tot}}/L_K \approx 0.88 M_\odot L_\odot^{-1}$ for the ATLAS$^{3D}$ luminosity, $M_{*,\text{tot}}/L_\odot \approx 1.7 M_\odot L_\odot^{-1}$ for the ACSVCS value and $M_{*,\text{tot}}/L_V \approx 3.15 M_\odot L_\odot^{-1}$ for McConnell & Ma (2013). Both galaxies have $R_e$ values in the ATLAS$^{3D}$ survey and the ACSVCS, and velocity dispersions in ATLAS and McConnell & Ma (2013).

McLaughlin (1999) and Côté et al. (2001) fitted the kinematics of stars and globular clusters in M87, plus the kinematics of Virgo-cluster galaxies and the total mass profile derived from intracluster X-ray gas, with a two-component mass model comprising the stars (plus remnants and stellar ejecta) in the body of M87 and an NFW dark-matter halo with $r_{200} \approx 1.55$ Mpc and $M_{\Delta,200} \approx 4.2 \times 10^{14} M_\odot$. This clearly identifies the dark matter in and around M87 with the halo of the entire Virgo A subcluster. McLaughlin and Côté et al. have an NFW concentration of $r_{200}/r_{-2} = 2.8 \pm 0.7$ for the M87/Virgo A halo, so (with $r_{pk}/r_{-2} = 2.16258$ again) $r_{pk} \approx 1.2$ Mpc and $V_{\text{ap},pk} \approx 1100$ km s$^{-1}$.

For M49/Virgo B, Côté et al. (2003) similarly use a two-component mass model consisting of the galaxy’s stars plus a single NFW dark-matter halo, to fit the stellar and globular cluster kinematics on $\lesssim 50$-kpc scales and the X-ray mass profile out to $\sim$Mpc radii. The Côté et al. analysis implies $r_{200} \approx 950$ kpc with $M_{\Delta,200} \approx 9.4 \times 10^{13} M_\odot$, and $r_{200}/r_{-2} \approx 4.8$. The dark-matter circular speed therefore peaks at $r_{pk} \approx 425$ kpc, where $V_{\text{ap},pk} \approx 710$ km s$^{-1}$.

A1.3 NGC 4889

NGC 4889 is the brightest galaxy in Coma and not far from the nominal central galaxy, NGC 4874. According to McConnell & Ma (2013), NGC 4889 has $L_V \approx 3.0 \times 10^{11} L_\odot$
and hence (for $M_*/L_V \approx 3.15 M_\odot L_\odot^{-1}$ from the Maraston 2005 population-synthesis models) $M_{*,\text{tot}} \approx 9.5 \times 10^{11} M_\odot$. It is at the uppermost end of the range of stellar masses plotted for our relations in Figure 4 (but it does not appear on those plots since it is not in the ATLAS3D survey), and it hosts one of the largest supermassive black holes yet measured: $M_{BH} = (2.1 \pm 1.6) \times 10^{10} M_\odot$ (McConnell et al. 2011, 2012). The effective radius $R_e = 27$ kpc and velocity dispersion $\sigma_{ap}(R_e) = 347$ km s$^{-1}$ in Table A1 are from McConnell & Ma (2013) and McConnell et al. (2011, 2012).

The global dark matter properties of the Coma Cluster are taken from dynamical modelling by Lokas & Mamon (2003). They give values for $r_{vir}$ and $M_{d,\text{vir}}$, rather than $r_{200}$ and $M_{d,200}$ like the other galaxies in Table A1, and a best-fitting NFW concentration of $c_{\text{vir}} = 9.4$. Together these imply $r_{pk} \approx 670$ kpc and $V_{d, pk} \approx 1585$ km s$^{-1}$.

### A2 Comparison to models

Taking the total stellar mass $M_{*,\text{tot}}$ as a starting point for each of the systems in Table A1, we now find their other stellar and halo properties from the scaling relations developed in §3. Table A2 shows the results for $R_e$, $\sigma_{ap}(R_e)$, $V_{d, pk}$, $r_{pk}$, $M_{d,200}$ or (for NGC 4889/Coma) $M_{d,\text{vir}}$, and $r_{200}$ or (for NGC 4889/Coma) $r_{\text{vir}}$.

#### A2.1 $L^*$ galaxies: $\sigma_{ap}(R_e) \sim 100-150$ km s$^{-1}$

For $M_{*,\text{tot}} \approx 6.4 \times 10^{10} M_\odot$ (the total Milky Way mass), our scalings give the stellar effective radius as $R_e \approx 3$ kpc and the velocity dispersion as $\sigma_{ap}(R_e) \approx 160$ km s$^{-1}$. This dispersion is rather higher than the value typically used to put the Milky Way on the black hole $M-\sigma$ relation: for example, McConnell & Ma (2013) take $\sigma_{ap}(R_e) = 103$ km s$^{-1}$ for the Galaxy. However, this value is meant to represent the bulge only. For the bulge mass of $M_{*,\text{tot}} \approx 9 \times 10^{10} M_\odot$, our relations give $R_e \approx 1.4$ kpc and $\sigma_{ap}(R_e) \approx 90$ km s$^{-1}$.

For the total Galactic stellar mass of $6.4 \times 10^{10} M_\odot$ and assuming an NFW halo, the scalings lead to a peak circular speed of $V_{d, pk} \approx 200$ km s$^{-1}$, occurring at $r_{pk} \approx 75$ kpc. Using equations (5), (24) and (23) to go from the virial radius implied by $M_{*,\text{tot}}$ to the radius of mean overdensity $\Delta = 200$, we find $M_{d,200} \approx 2 \times 10^{12} M_\odot$ and $r_{200} \approx 270$ kpc. For the mass of the bulge alone, $M_{*,\text{tot}} \approx 9 \times 10^{9} M_\odot$, we obtain $V_{d, pk} \approx 120$ km s$^{-1}$, $r_{pk} \approx 35$ kpc, $M_{d,200} \approx 3.6 \times 10^{11} M_\odot$ and $r_{200} \approx 150$ kpc.

#### A2.2 M87 and M49: $\sigma_{ap}(R_e) \sim 250$ km s$^{-1}$

For each of these galaxies, we take the mean of $M_{*,\text{tot}}$ from the three different values in Table A1. Thus, $M_{*,\text{tot}} = 3.3 \times 10^{11} M_\odot$ for M87, and $M_{*,\text{tot}} = 4.2 \times 10^{11} M_\odot$ for M49. Our parametrisation of $R_e$ versus $M_{*,\text{tot}}$ in §3.1 then gives the values recorded in Table A2, which broadly agree with the measurements of $R_e$. The model values in Table A2 for $\sigma_{ap}(R_e)$, $V_{d, pk}$, $r_{pk}$, $M_{d,200}$ and $r_{200}$ assume an NFW halo around each galaxy (as the analyses from the literature do). The predicted velocity dispersions compare well to the measurements for M87 and M49 in the ATLAS3D survey but...
not quite as well to the values recorded by McConnell & Ma
(2013), which are 20% higher.

The value of r\textsubscript{200} for M87/Virgo A in Table A1, from
McLaughlin (1999), is \(\simeq 80\%\) bigger than the one in Ta-
ble A2, implied by our models here. McLaughlin’s \(M_{d,200}\)
is consequently larger by about a factor of \(1.8^3 \simeq 6\). Sim-
ilarly, the circular-speed curve of the halo in McLaughlin
(1999) peaks at \(r_{pk} \sim 1.2\) Mpc (with a very large uncer-
tainty) rather than \(r_{pk} \simeq 330\) kpc as expected here, and it
has \(V_{d, pk} \simeq 1100\) km s\(^{-1}\) rather than \(V_{d, pk} \simeq 600\) km s\(^{-1}\).

These discrepancies for M87/Virgo A may simply re-
fect the inevitable scatter in the properties of individual
systems around the typical values given by our trend lines.
For M49/Virgo B, all of the halo properties in Table A2 ob-
tained from our scalings are remarkably close to the values
in Table A1 from Côté et al. (2003).

A2.3 NGC 4889: \(\sigma_{ap}(R_e) \sim 350\) km s\(^{-1}\)

For \(M_{*, tot} = 9.5 \times 10^{11}\) M\(_{\odot}\), our scalings give \(R_e = 15.2\) kpc and
(assuming an NFW halo) \(\sigma_{ap}(R_e) \simeq 345\) km s\(^{-1}\). The
velocity dispersion agrees with the value in McConnell et al.
(2011, 2012), although the effective radius is smaller than
their adopted 27 kpc. Further, we find \(r_{vir} \simeq 2.45\) Mpc and
\(M_{d, vir} \simeq 8.0 \times 10^{14}\) M\(_{\odot}\), which compare well to the values
in Table A1 determined by Lokas & Mamon (2003). (This is
even though NGC 4889 is not precisely at the centre of the
Coma Cluster).

Assuming an NFW halo density profile, our models im-
ply \(r_{pk} \simeq 925\) km s\(^{-1}\) and \(V_{d, pk} \simeq 1285\) km s\(^{-1}\) for the peak
of the dark-matter circular speed in NGC 4889/Coma—
different by \(\sim 30\%\) from the Lokas & Mamon numbers. Com-
paring to the peak radii and speeds above for M87/Virgo A
and M49/Virgo B emphasises the clear visual impression
given by Figure 4: In large galaxies \(V_{d, pk}\), along with \(M_{d, vir}\),
is a much more sensitive function of galaxy stellar mass than
the stellar \(\sigma_{ap}(R_e)\) is. (This follows directly from the steep
decline at high masses in the cosmological connection be-
tween \(M_{*, tot}\) and \(M_{d, vir}\) adopted from Moster et al. 2010.)
It therefore seems natural to expect much more scatter and
many more apparent “outliers” in \(M_{BH}\) among very mas-
sive galaxies, if SMBH masses are connected fundamentally
to the global properties of dark-matter halos rather than to
stellar velocity dispersions directly.