Different solutions of the diffusion equation and its applications

Khaled S. M. Essa1*, Soad M. Etman1, Maha S. El-Otaify1, M. Embaby1, Ahmed M. Mosallem1 and Ahmed S. Shalaby2

Abstract

In this report, we solved the advection–diffusion equation under pollutants deposition on the ground surface, taking wind speed and vertical diffusion depend on the vertical height. Also, we estimated a simple diffusion model from point source in an urban atmosphere and the conservative material with downwind was evaluated. Then, we calculated the extreme ground-level concentration as a function of stack height and plume rise in two cases. Comparison between the proposed models and the emission from the Egyptian Atomic Research Reactor at Inshas had been done. Lastly, we discussed the results in this report.

Keywords: Advection–diffusion equation, Pollutants deposition, Extreme ground-level concentration, Egyptian atomic research reactor, Urban atmosphere, The conservative material, Continuous point source, Maximum concentration value, The logarithmic law of wind velocity, Contaminants in the planetary boundary layer

1 Background

The Gaussian plume equation was semi-analytical solution, assuming that wind velocity and eddy diffusivities were constant. The non-Gaussian models agree well with the observed data by Hinrichsen [17]. Wortmann et al. [35] and Essa et al. [10] calculated a new technique to get the concentration of contaminants in the planetary boundary layer.

Early solutions used only two dimensions [26, 28, 34]. The Green’s function method was studied by Stakgold [30]. Yeh [37] calculated advection–diffusion equation in three dimensions under boundary conditions. Also, these equations have been studied in turbulent models for unbounded domain by [18, 38]. Also, [33, 32] utilized dispersion modeling. All techniques have been restricted to a single isolated point source located at the origin. The details of several solutions were investigated by Carslaw and Jaeger [7], Sutton [31], Yaglom [36], Pasquill [24], Berlyand [4] and Lin and Hildemann [21]. Also, the diffusion from a point source in an urban atmosphere was estimated by Essa and El-Otaify [9]. Heines and Peters [16], and Bennett [3] investigated the physical effect of model for the dry deposition of pollutants to a rough surface. Recently, Essa et al. [11, 12] estimated the solution of the advection–diffusion equation in two dimensions with variable vertical eddy diffusivity and wind speed using Hankel transform. Also, Essa et al. [11, 12] found the solution of advection–diffusion equation in three dimensions using Hankel transform.

In the first part: Pasquill and Smith [22, 23] introduced the advection–diffusion equation in steady state for a continuous point source as follows:

\[ u(z) = \frac{\partial C}{\partial x} = \frac{\partial}{\partial z} \left[ K(z) \frac{\partial C}{\partial z} \right] \tag{1} \]

where \( u(z) \) and \( K(z) \) are the mean horizontal wind velocity and vertical eddy diffusivity as a function of vertical height. Assuming that \( u(z) = 0 \) at \( z = 0 \), as defined by [29] as follows:

*Correspondence: mohamedksm56@yahoo.com
1 Mathematics and Theoretical Physics Department, NRC, Egyptian Atomic Energy Authority, Cairo, Egypt
Full list of author information is available at the end of the article
\[
\frac{du}{dz} = \frac{u_*}{kz} \varphi_m(z/L)
\]

where \( \varphi_m(z/L) \) is the non-dimensional wind gradient, \( L \) is the Monin–Obukhov length, \( u_* \) is the friction velocity, and \( k \) is the von-Karman's constant equals 0.4. One can get by integrating Eq. (2) w.r.t \( z \) the following velocity profiles [9]:

\[
u(z) = \frac{u_*}{k} \ln \left( \frac{z + z_0}{z_0} \right), \text{ neutral conditions} \quad (3)
\]

\[
u(z) = \frac{u_*}{k} \left[ \ln \left( \frac{z + z_0}{z_0} \right) + \frac{5.2z}{L} \right], \text{ stable conditions} \quad (4)
\]

\[
u(z) \left[ \ln \left( \frac{1 + \frac{16(z + z_0)}{L}}{1 + \frac{16(z + z_0)}{L}} \right)^{1/4} + 1 \right] + 2 \tan \left[ 1 + \frac{16(z + z_0)}{L} \right]^{1/4} \text{ unstable conditions} \quad (5)
\]

The vertical diffusion coefficient was taken from Hanna et al. [13, 14] as follows:

\[
K(z) = K_o + k u_* z \quad \text{ neutral conditions} \quad (6)
\]

\[
K(z) = K_o + k u_* \left( 1 + \frac{5.2z}{L} \right)^{-1} \quad \text{ stable conditions} \quad (7)
\]

\[
K(z) = K_o + k u_* \left( 1 + \frac{16z}{L} \right)^{1/4} \quad \text{ unstable conditions} \quad (8)
\]

where \( K_o \) is a constant value. Equation (1) is solved using the boundary conditions as follows:

1. The mixing layer was assumed not allowing fluid to pass through to the pollutants
   \[
   C(x, z) = 0, \quad \text{at} \quad z = h
   \]
   \[
   \frac{\partial C(x, z)}{\partial z} = 0, \quad \text{at} \quad z = h \quad (9a)
   \]

2. Assuming that there is a deposition at the ground surface as follows:
   \[
   K(z) \frac{\partial C(x, z)}{\partial z} = v_d C(x, z) \quad \text{at} \quad z = 0
   \]
   \[
   \text{where} \ v_d \text{ is the deposition velocity.}
   \]

The solution of Eq. (1) is solved theoretically under different stabilities [10] as follows:

\[
C(x, z) = F(x) \left( 1 - \frac{z}{h} \right)^2 \quad (11)
\]

Integrating Eq. (1) from zero to \( h \) with respect to \( z \), and applying Eqs. (9b) and (10), one gets:

\[
\frac{d}{dx} \int_0^h u(z) C(x, z) \, dz = -v_d C(x, 0) \quad (12)
\]

2 For neutral conditions

Substituting \( C(x, z) \) and \( u(z) \) by Eqs. (11) and (3) in Eq. (12), one obtains:

\[
\frac{u_*}{k} \frac{dF(x)}{dx} \int_0^h \left( 1 - \frac{z}{h} \right)^2 \ln \left( \frac{z + z_0}{z_0} \right) \, dz = -v_d F(x) \quad (13)
\]

Equation (13) becomes:

\[
\frac{u_* N}{k} \frac{dF(x)}{dx} = -v_d F(x) \quad (14)
\]

where

\[
N = \int_0^h \left( 1 - \frac{z}{h} \right)^2 \ln \left( \frac{z + z_0}{z_0} \right) \, dz
\]

\[
= \frac{z_0^3}{3h^2} \left( 1 + \frac{h}{z_0} \right)^3 \ln \left( 1 + \frac{h}{z_0} \right) - \frac{z_0^2}{18h} \left[ 6 + \frac{15}{z_0} + \frac{11h^2}{z_0} \right] \quad (15)
\]

Equation (14) becomes:

\[
F(x) = F_0 \exp \left( -\frac{v_d k}{u_* N} x \right) \quad (16)
\]

where \( F_0 \) is a constant. Then, the solution in the neutral case was obtained as follows:
where \( x_{dn} \) is the decay distance of pollutant in neutral condition. Equation (17) becomes:

\[
C(x, z) = F_0 \exp\left(- \frac{vd k}{u_e N} x \right) \left(1 - \frac{z}{h}\right)^2
\]  (17)

Taking

\[
x_{dn} = \frac{u_e N}{vd k}
\]  (18)

Now, \( F_0 \) can be determined by using the formula:

\[
Q = \int_{x=0}^{x_{dn}} h \int_{z=0}^{h} u(z) C(x, z) dx dz
\]  (20)

where \( Q \) is the emission rate which after substitution of Eqs. (3) and (19), one gets:

\[
Q = \frac{u_e F_0}{k} \int_{x=0}^{x_{dn}} \exp\left(- \frac{x}{x_{dn}}\right) dx \int_{z=0}^{h} \left(1 - \frac{z}{h}\right)^2 \ln\left(\frac{z + z_0}{z_0}\right) dz
\]

\[
= \frac{u_e F_0 N x_{dn}}{k} \left(1 - \frac{1}{e}\right)
\]

Then, the concentration of pollutants in neutral condition becomes:

\[
C(x, z) = \frac{Q}{0.63 v_d x_{dn}^2} \exp\left(- \frac{x}{x_{dn}}\right) \left(1 - \frac{z}{h}\right)^2
\]  (21)

Similarly, \( C(x, z) \) in stable condition takes form [10]:

\[
C(x, z) = \frac{Q}{0.63 v_d x_{du}^2} \exp\left(- \frac{x}{x_{du}}\right) \left(1 - \frac{z}{h}\right)^2
\]  (22)

\[
where X_{ds} is given in the stable case as follows:
\[
x_{ds} = \frac{u_e M}{v_d k}
\]  (23)

where \( M = N - \frac{5.2 h^2}{12L} \).

Also, \( C(x, z) \) in unstable condition is given [10] as follows:

\[
C(x, z) = \frac{Q}{0.63 v_d x_{du}^2} \exp\left(- \frac{x}{x_{du}}\right) \left(1 - \frac{z}{h}\right)^2
\]  (24)

where \( X_{du} \) is given in unstable case as follows:

\[
x_{du} = \frac{u_e D}{v_d k}
\]  (25)

where \( D \) is taken from [10].

In the second part: We assumed two-dimensional structure with homogeneity in the lateral coordinate, where \( \bar{u}(z) \) is described in Fig. 1. The ground surface is treated with no deposition of matter exist.

The conservation of mass can be taken from [9]:
\[ Q = \int_0^H \pi(z) C(z) dz \]  

(26)

where

\[ \pi(z) \] is the average wind velocity, \( C(z) \) is the concentration of pollutants, and \( H \) is the effective stack height of the plume.

### 3 The Effective stack Height (H)

We estimated the height of the plume as follows:

\[ \Delta h = 3(w_0/u)D_1 \]  

(27)

where \( w_0 \) is the exit velocity of the pollutants (\( m/s \)) and \( D_1 \) is the inside stack diameter (\( m \)). Then, \( H \) equals:

\[ H = h_s + \Delta h = h_s + 3(w_0/u)D_1 \]  

(28)

The concentration profile will be assumed in form [9]:

\[ C_0 = \frac{kHQ}{\nu_s} \left\{ \frac{1}{4H} \left[ 2(H + z_0)(2H + aH - \alpha z_0) \ln \frac{H + z_0}{z_0} - H(4H + aH - 2\alpha z_0) \right] + \frac{5.2H^2}{L} \left( \frac{1}{2} \right) \alpha \right\} \]  

(34)

### 5 Stable case

After integration, \( C_0 \) takes the form:

\[ Q = \int_0^H \left[ \frac{\nu_s}{k} \ln \left( \frac{z + z_0}{z_0} \right) + \frac{5.2z}{L} \right] C_0(1 + \alpha(z/H)) dz \]  

(33)

### 6 Unstable case

\[ C/C_0 = 1 + \alpha_1(z/H) + \alpha_2(z/H)^2 + ... \]  

(29)

where \( C_0 \) is the concentration value at the edge of the plume. \( C(z) \) is the concentration at vertical height; \( \alpha_1 \) and \( \alpha_2 \), etc., are constants.

The fitting in Fig. 2 is obtained by taking the first two terms as follows:

\[ C/C_0 = 1 + \alpha_1(z/H) \]  

(30)

If \( \alpha_1 \) is a \( r \) percentage of the concentration as follows:

\[ \alpha_1 = 0.01 r - 1 \]  

(30a)

if \( r = 0 \), then

\[ C/C_0 = 1 - (z/H) \]  

(30b)

After substituting, Eq. (26) becomes using different stabilities as follows:

### 4 Neutral case

\[ D = \int_0^t C dt \]  

(36)

The time dosage integral of the concentration “D” is defined as:

\[ Q = \int_0^z \int_0^t uCdZdt = \text{constant} \]  

(37)

Substituting from Eq. (36) in Eq. (37), one gets:

\[ Q = \int_0^z uDZ = \text{constant} \]  

(38)
The second research reactor in E.A.E.A., (ETRR-2) has stack height \( H_1 \) equals 27 m which is a point source of \((^{131}I)\), \( (H) \) equals 31.29 m, \( (Q) \) equals 35 Bq, the wind velocity \( (u) \) equals 2.8 m/s, and the lapse rate \((\Delta T/\Delta Z)\) \((\circ C/100 \text{ m})\) is 0.36, which is a stable case. Using Eq. (34), one gets \( (C_o) \) equals 1.208 Bq/m³. Then, the concentration at ground level becomes:

\[
C(\text{ground}) = 1.208 \left(1 - \frac{H_1}{H}\right) = 0.17 \text{ Bq/m}^3
\] (39)

The observed concentration at \( x=300 \text{ m} \) and \( H=31.29 \text{ m} \) was 0.16 Bq/m³. The corrected source strength becomes:

\[
Q(\text{corrected}) = \frac{(0.16)(35)}{0.17} = 37.06 \text{ Bq}
\]

Then, the corrected \( C(\text{ground}) = 0.18 \text{ Bq/m}^3 \).

**In the third part:** The maximum ground-level concentration with two cases was estimated [9]. The concentration at the ground reaches a maximum and thereafter decreases as given by Hans et al.[15] with the formula:

\[
C(x, y) = \left[Q/\sigma_x \sigma_y \sigma_z u\right] \exp \left[-\frac{H^2}{2\sigma_y^2} \exp \left[-\frac{y^2}{2\sigma_y^2}\right]\right] \exp \left[-\frac{H^2}{2\sigma_z^2} \exp \left[-\frac{z^2}{2\sigma_z^2}\right]\right] (40)
\]

\( \sigma_1 \) (i = y, z): are dispersion parameters of concentration in the lateral and vertical directions (m).

The extreme concentration at ground occurs along the plume centerline \((y = 0)\) at ground \((z = 0)\) [25].

\( \sigma_y \) and \( \sigma_z \) can be estimated as follows:

\[
\sigma_y = a x^b, \quad \sigma_z = c x^d
\] (41)

where \( a, b, c, d \) values are taken from Curtiss [8].

The ground-level concentration along centerline was estimated from Eq. (40) taking \( \sigma_y \) and \( \sigma_z \) from Eq. (41) one gets:

\[
\chi(x, 0, 0; H) = \frac{Q}{\pi \alpha u x^{d+b}} \exp \left[-\frac{H^2(x)}{2\sigma_y^2}\right] \exp \left[-\frac{y^2}{2\sigma_y^2}\right] \exp \left[-\frac{H^2}{2\sigma_z^2}\right] \exp \left[-\frac{z^2}{2\sigma_z^2}\right] (42)
\]

The maximum concentration value can be obtained after solving the following equation:

\[
xH \frac{dH}{dx} - H^2 d + c^2 x^{2d} (b + d) = 0
\] (43)

The maximum concentration and effective height were obtained as follows:

\[
\chi^*(x, 0, 0, H) = \left[Q / (\pi \sigma_y \sigma_z u_h)\right] \exp \left[-\frac{1}{2\sigma_z^2} \left[\frac{c^2 x^{2d} (b + d)}{0.5(\frac{1.6F^{1/3}}{u_h}) x^{2/3} + h_s d}\right]\right] (52)
\]

### 7 First: the effective height as a function of x

**A—Unstable or neutral conditions:**

The buoyancy flux parameter, \( F_b \), was written [5] as follows:

\[
F_b = g v_s^2 (T_s - T_a) / T_i (44)
\]

For \((h_2 < 305 \text{ m})\), \( g \) is the acceleration \((\text{ms}^{-2})\), \( v_s \) is the exit vertical speed \((\text{ms}^{-1})\), \( T_s \) is the gas exit temperature \((\text{K})\), and \( T_a \) is the ambient temperature \((\text{K})\) at \( h_2 \) [27].

The critical \( x^* \) is given by:

\[
x^* = 2.16 F_b^{2/5} h_s^{3/5}
\] (45)

For \( x \leq x^* \), we have:

\[
\Delta h(x) = \text{const.} F_b^{1/3} (u)^{-1} x^{2/3}
\] (46)

Taking the constant equals 1.6 [6], one gets:

\[
H = h_s + 1.6 F_b^{1/3} (u)^{-1} x^{2/3}
\] (47)

This equation is used if \( T_s > T_a \) (2/3 law) [27]. \( u_h \) can be calculated at 10 m as follows:

\[
u_h = u_{10} (H/10m)^p
\] (48)

where the parameter \( p \) is given from [20].

Substituting from (47) into (43), one gets:

\[
H^* = \frac{-c^2 x^{2d} (b + d)}{2/3(\frac{1.6F^{1/3}}{u_h}) x^{2/3} - d(\frac{1.6F^{1/3}}{u_h}) x^{2/3} - h_s d}
\] (49)

where \( H^* \) is the maximum value.

In unstable condition, taking \( d = 1.17 \) [8] \( H^* \) becomes:

\[
H^* = \frac{c^2 x^{2d} (b + d)}{0.5(\frac{1.6F^{1/3}}{u_h}) x^{2/3} + h_s d}
\] (50)

Then,

\[
H^* = \left[\frac{c^2 x^{2d} (b + d)}{0.5(\frac{1.6F^{1/3}}{u_h}) x^{2/3} + h_s d}\right]^{1/2}
\] (51)

The maximum value for \( x^* \) has the form:

\[
\chi^*(x, 0, 0, H) = \left[Q / (\pi \sigma_y \sigma_z u_h)\right] \exp \left[-\frac{1}{2\sigma_z^2} \left[\frac{c^2 x^{2d} (b + d)}{0.5(\frac{1.6F^{1/3}}{u_h}) x^{2/3} + h_s d}\right]\right] (52)
\]
In neutral condition, taking \( d = 0.95 \) [8], \( \chi^* \) becomes:

\[
\chi^*(x, 0, 0, H) = \left[ \frac{Q}{\pi \sigma_y \sigma_z u_b} \right] \exp \left( -\frac{1}{2\sigma_x^2} \right) \left[ \frac{c^2 \chi^{2d}(b + d)}{0.28 \left( \frac{1.65^{1/3}}{u_b} \right) x^{2/3} + h_s d} \right]^2
\]

In stable stability, there are two methods for maximum concentration at \( z = 0 \) as follows:

(First)—In stable, where \( d = 0.67 \) [8], the maximum value for \( \chi^* \) is:

\[
\chi^*(x, 0, 0, H) = \left[ \frac{Q}{\pi \sigma_y \sigma_z u_b} \right] \exp \left( -\frac{c^2 \chi^{2d}(b + d)^2}{2d^2 h_s^2} \right)
\]

(Second)—In stable (E and F), the stability parameter is written as follows:

\[
s = \frac{g}{T_a} \left( \frac{\Delta h}{\Delta Z} \right)
\]

where \( \left( \frac{\Delta h}{\Delta Z} \right) = 0.02 \) K/m for E and \( \left( \frac{\Delta h}{\Delta Z} \right) = 0.035 \) K/m for F ([27]).

\( \Delta h \) is written as follows:

\[
\Delta h = 2.6 \left( \frac{F_b}{u_b^8} \right)^{1/3}
\]

\( H^* \) becomes:

\[
H^* = \left( \frac{c^2 \chi^{2d}(b + d)}{d} \right)^{0.5}
\]

Also, \( \chi^* \) has the form:

\[
\chi^*(x, 0, 0, H) = \left[ \frac{Q}{\pi \sigma_y \sigma_z u_b} \right] \exp \left( -\frac{(b + d)}{2d} \right)
\]

8 Second case: \( H \) is a constant of downwind distance:

If \( H \) is constant, then \( \frac{nH}{\sigma_x} = 0 \), assuming \( H_c = h_s + 3w_o D/u \), where \( H_c \) is constant. \( H^* \) becomes:

\[
H^* = \left[ \frac{c^2 \chi^{2d}(b + d)}{d} \right]^{0.5}
\]

The maximum value for \( \chi^* \) has the form:

\[
\chi^* (x, 0, 0; H) = \frac{Q}{\pi a c u_h x^{b+d}} \exp \left( -\frac{(b + d)}{2d} \right)
\]

9 First case study

The decay distance and the concentration of pollutant at Inshas, Nuclear Research Center, Egyptian Atomic Energy Authority, were calculating. The stability was found in stable condition by using these data. The value of \( L \) was taken from Gifford work [13]. One uses Eq. (4) to calculate the values of \( u* \) in terms of the wind velocity.
u(z) at 10 m height; hence, the values of the mixing height could be estimated from \([2]\) for stable condition:

\[
h = 0.4 \left( \frac{u_s}{|f|} \right)^{1/2}, \text{ for } h/L > 0
\]  

where \(f\) is the Coriolis parameter. \(v_d\) equals 0.04 m/s for Iodine \([19]\). Hence, the decay distance of iodine has been estimated using Eq. (23). The friction velocity and the mixing height with downwind distance are illustrated in Fig. 3. One gets the change of the friction velocity and mixing height from 0.25 to 0.3 m/s and 180 to 190 m, respectively \([10]\).

The values of mixing height proportional with the decay distance are shown in Fig. 4; the normalized concentration (C/Q) of iodine at heights \(z = 0, 10, 50\) m as a function of downwind distance has been evaluated by Eq. (22). One finds that the maximum normalized ground-level concentration equals \(18 \times 10^{-6}\) at 100 m and then decreases to reach near minimum value at 5000 m as shown in Fig. 5.

**10 Second case study**

\(C_o/Q\) is applied on the first research reactor at Inshas, Nuclear Research Center, Egyptian Atomic Energy Authority. \(h_s\) equals 43 m, \(D\) equals 1 m, \(w_o\) equals 4 m/s, and the total ventilation equals 39,965 m\(^3\)/hr. Also, one takes \(\alpha = -1\). Figure 6 shows that a straight-line fit well to these data in three conditions between \(C_o/Q\) and \(H\).

**11 Third case study**

One gets the maximum concentration on the second research reactor at Inshas, Nuclear Research Center, Egyptian Atomic Energy Authority. \(h_s\) equals 27 m, \(D\) equals 1 m, \(w_o\) equals 4 m/s, and the total ventilation equals 39,965 m\(^3\)/hr. Also, one takes \(\alpha = -1\). Figure 6 shows that a straight-line fit well to these data in three conditions between \(C_o/Q\) and \(H\).
equals 1 m, \( w_o \) equals 0.3 m/s, and the total ventilation equals 3400 m\(^3\)/hr \([1]\). One finds that the maximum concentrations are \( 6.2 \times 10^{-4}, 4.8 \times 10^{-4} \) and \( 2 \times 10^{-4} \) Bq/m\(^3\) at 40 m in unstable, neutral and stable conditions, respectively; when plume rise depends on downwind distance as shown in Fig. 7, these values are less than the maximum system activity (MSA).

One finds that the effective height is quadratic with downwind distance in unstable case and the correlation coefficient (R) equals 0.99, \( (x_{\text{max}}) \) equals 1000 m, and \( (H_{\text{max}}) \) reaches 978 m. In neutral case, the curve is still quadratic and smaller than that in the unstable case, \( R=1 \). Also, in the stable case, the relation is linear, \( R=0.984 \), \( x_{\text{max}} \) reaches to maximum value, and \( H_{\text{max}} \) will reach 59 m as shown in Fig. 8.

We find that the maximum concentrations are \( 2.5 \times 10^{-4}, 2.1 \times 10^{-4} \) and \( 2 \times 10^{-4} \) Bq/m\(^3\) in neutral, unstable and stable conditions, respectively, at 40 m when \( H \) is a constant as shown in Fig. 9. Also, the relation between "H" and "x" is linear in unstable condition, \( H_{\text{max}} = 163 \) m when \( x_{\text{max}} = 1000 \) m and \( R=0.997 \). In neutral case, the relation is quadratic and \( R=1 \) and \( H_{\text{max}} = 83 \) m at \( x_{\text{max}} = 1000 \) m. Also, in stable case the relation is still quadratic and \( R=0.998 \), \( H_{\text{max}} = 58.5 \) m at \( x_{\text{max}} = 1000 \) m as shown in Fig. 10. We must take \( H \) that depends on \( x \) because one gets the values of "H" and the concentration near the true. This is the best result to reach it.
12 Conclusions
The concentration released from an elevated point source in the presence of both deposition and elevated mixed layer is estimated. The decay distance in the downwind distance is calculated. The maximum normalized concentration equals $18 \times 10^{-6}$ (s/m$^3$) at 100 m and then decreases to reach minimum value at 5000 m at the ground level.

The logarithmic law of wind velocity in different conditions is used to get the plume rise, effective stack height and the normalized concentration at the axis of the plume from the reactor release through different stability classes. We find that the ground-level concentration of iodine ($^{131}$I) agrees with the observed concentrations.

The values of extreme concentration are $6.2 \times 10^{-4}$, $4.8 \times 10^{-4}$ and $2 \times 10^{-4}$ Bq/m$^3$ at 40 m in unstable, neutral and stable conditions, respectively; when the plume rise depends on downwind distance and then decreases to values less than MSA, after that the values are close to zero at 400 m from the stack. In stable case, the maximum values of the ground-level concentration of air...
pollutants were similar in the two cases. One obtains a good model when the plume rise depends on downwind distance and it is the best result.

Abbreviations
PBL: Planetary boundary layer is (PBL); \( X_{dn} \): Decay distance of pollutant in neutral condition; \( X_{ds} \): Decay distance of pollutant in the stable condition; \( X_{du} \): Decay distance of pollutant in the unstable condition; \( D \): Time dosage integral of the concentration; \( H \): Effective stack height of the plume; \( C_0 \): Concentration value at the edge of the plume, MSA: Maximum system activity.

Acknowledgements
Authors should obtain permission to acknowledge a chief editor and reviewers in this journal and all members in my department.

Authors’ contributions
KE, SE, MS, ME, AM and AS contributed in all sections. All authors read and approved the final manuscript.

Funding
All sources of funding for the research reported should be paid by Beni-Suef University, Egypt.

Availability of data and materials
Not applicable in this section.

Declarations
Ethics approval and consent to participate
Not applicable.

Consent for publication
My manuscript does not contain any individual person’s data in any form (including individual details, images or videos).

Competing interests
The authors declare that they have no competing interests in this section.

References
1. AEA (2003) 0767–5325–3IBLI‑001–10: ETRR‑2, Safety Analysis Report. AEA, Egypt
2. Arya SP (1999) Air pollution meteorology and dispersion. OxfordUniversity Press, New York, p P98
3. Bennett M (1988) Atmos Environ 22(12):2701–2705
4. Briggs GA (1971) Some recent analysis of plume rise observations. In: Proceedings of the 2nd international clean air congress, England. Academic Press, New York
5. Briggs GA (1969) Plume rise. USACE Critical Review Series, TID– 25075 National Technical Information Service, Springfield, Virginia
6. Briggs JA (1971) Some recent analysis of plume rise observations. In: Proceedings of the 2nd international clean air congress, England. Academic Press, New York
7. Carslaw HS, Jaeger JC (1959) Conduction of heat in solids, 2nd edn. Oxford University Press, London, p 510
8. Curtiss PS, Rabl A (1996) Impacts of air pollution: general relationships and site dependence. Atmos Environ 30(19):3331–3347
9. Essa KSM, El-Otrafy MS (2006) Diffusion from a point source in an urban atmosphere. Meteorol Atmos Phys 92:95–101
10. Essa KSM, Soad M, Etman ME (2007) New analytical solution of the dispersion equation. Atmos Res 85:337–343
11. Essa KSM, Shalaby AS, Ibrahim MAE, Mosallem AM (2020) Analytical solutions of the advection–diffusion equation with variable vertical eddy diffusivity and wind speed using hankel transform. Pure Appl Geophys. https://doi.org/10.1007/s00024-020-02496-y
12. Essa KSM, Mosallem AM, Shalaby AS (2020) Evaluation of analytical solution of advection diffusion equation in three dimensions using Hankel transform. Under publication in Izvestiya Atmospheric and Oceanic Physics

Fig. 10 The variation of (H) for Cs‑137 via (x) when H is a constant
13. Hanna SR (1982) Review of atmospheric diffusion models for regulatory applications. WMO Tech. Note No. 177, WMO, Geneva, 37
14. Hanna Steven R, Briggs GA, Hosker RP, Jr (1982) Handbook on atmospheric diffusion of energy. (DoEl / TIC-11223 (DE 82002045))
15. Hans AP, Jhon AD (1984) Atmospheric turbulence, models and methods for engineering applications. Willy, New York
16. Heines TS, Peters LK (1974) The effect of ground level absorption on the dispersion of pollutants in the atmosphere. Atmos Environ 8:1143–1153
17. Hinrichsen K (1986) Comparison of four analytical dispersion models for near-surface releases above a grass surface. Atmos Environ 20:29–40
18. Huang CH (1979) A theory of dispersion in turbulent shear flow. Atmos Environ 13:141–144
19. IAEA (1982) Generic models and parameters for assessing the environmental transfer of radio nuclides from routine release.
20. Irwin JS (1979) A theoretical variation of the wind profile power law exponent as a function of surface roughness and stability. Atmos Environ 13:191–194
21. Lin JS, Hildemann LM (1997) A generalized mathematical scheme to analytically solve the atmospheric diffusion equation with dry deposition. Atmos Environ 31:59–71
22. Pasquill F, Smith FB (1983) Atmospheric diffusion, 3rd edn. D. Van Norstrand, London
23. Pasquill F, Smith FB (1983) Atmospheric diffusion. Wiley, New York
24. Pasquill F (1974) Atmospheric diffusion. 2nd ed., Wiley, New York
25. Richard EF, Shultis KJ (1993) Radiological assessment, sources and exposures. PTR Prentice-Hall, Englewood Cliffs
26. Rounds W (1955) Solution of the two-dimensional diffusion equation. Trans. Am. Geophys. 36:395–405
27. Karl BS Jr, Dey PR (2000) Atmospheric dispersion modeling compliance guide. McGraw Hill Companies, New York
28. Smith FB (1957) The diffusion of smoke from a continuous elevated point-source into a turbulent atmosphere. J Fluid Mech 2:49–76
29. Smith FB (1990) College on atmospheric boundary layer physics “air pollution modelling for environmental impact assessment” international center for theoretical physics (4–15 June). SMR/462:1
30. Stakgold I (1968) Boundary value problems of mathematical physics. Vol. 1. MacMillan, New York
31. Sutton OG (1953) Micrometeorology. McGraw-Hill, New York
32. Tirabassi T (1989) Analytical air pollution advection and diffusion models. Water Air Soil Pollut 47:19–24
33. Tirabassi T, Tagliazucca M, Zannetti P (1986) KAPPA-G, a non-Gaussian plume dispersion model description and evaluation against tracer measurements. JAPCA 36:592–596
34. Walters TS (1957) Diffusion from an infinite line source lying perpendicular to the mean wind velocity of a turbulent flow. Quart J Mech Appl Math 5:214–219
35. Wortmann S, Turbulence VM, Moreira DM, Buske D (2005) A new approach to simulate the pollutant dispersion in the PBL. Atmos. Env. 39:2171–2178
36. Yaglom AM (1972) Turbulent diffusion in the surface layer of the atmosphere. Izv Atmos Ocean Phys 8(6):333–340
37. Yeh GT (1975) Green’s function of a diffusion equation. Geophys Res Lett 2:293–296
38. Yeh GT, Huang CH (1975) Three-dimensional air pollution modeling in the lower atmosphere. Boundary Layer Met 9:381–390

Publisher’s Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.