Stabilized vortex solitons in layered Kerr media

Gaspar D. Montesinos and Víctor M. Pérez-García
Departamento de Matemáticas, Escuela Técnica Superior de Ingenieros Industriales, Universidad de Castilla-La Mancha, 13071 Ciudad Real, Spain

Humberto Michinel and José R. Salgueiro
Área de Optica, Facultade de Ciencias de Ourense, Universidade de Vigo, As Lagoas s/n, Ourense, ES-32005 Spain.
(Dated: March 30, 2022)

In this letter we demonstrate the possibility of stabilizing beams with angular momentum propagating in Kerr media. Large propagation distances without filamentation can be achieved in layered media with alternating focusing and defocusing nonlinearities. Stronger stabilization can be obtained with the addition of an incoherent beam.

PACS numbers: 42.65.Tg, 05.45.Yv, 03.75.Lm

Introduction. Vortices have been a source of fascination since the works of Empedocles, Aristotle and Descartes, who tried to explain the formation of the Earth, its gravity and the dynamics of the solar system as due to primordial cosmic vortices. Many interesting problems related to vortices are open in different fields such as: fluid mechanics, high Tc superconductivity, superfluidity, light propagation, Bose-Einstein condensation (BEC), cosmology, biosciences, or solid state physics [1, 2, 3, 4].

In wave mechanics a vortex is a screw phase dislocation, or defect [4] where the amplitude of the field vanishes. The phase around the singularity has an integer number of windings, ℓ, which plays the role of an angular momentum. For fields with non-vanishing boundary conditions, this number is a conserved quantity and governs the interactions between vortices as if they were endowed with electrostatic charges. Thus, ℓ is usually called the “topological charge” of the defect.

In Optics there has been a strong interest on the so called “vortex solitons”, i.e. robust distributions of light of vortex type in which nonlinearity could compensate diffraction leading to stationary propagation. However, in self-focusing Kerr media, a finite size beam containing a vortex always destabilizes and forms a filamentary structure [5]. This also stands for saturable self-focusing nonlinearities [6]. Vortex solitons have been studied in many other different optical systems (see e.g. the review [5]) and in most realistic cases they tend to be unstable.

In this paper we propose to use layered Kerr media, which are self-focusing in average, to obtain stable propagation of vortex solitons up to very long distances. Our ideas are also extended to the field of matter waves.

Stabilized solitons. In Ref. [5] an idea to prevent collapse was proposed based on the modulation of the Kerr coefficient of an optical material along the propagation direction. In that way, a propagating beam focuses and expands in alternating regions and becomes stabilized in average. This idea has been further explored in [10, 11, 12, 13, 14, 15].

The propagation of a paraxial monochromatic beam in a Kerr medium is modeled by equations of the type (in adimensional units)

\[ i \frac{\partial u}{\partial z} = -\frac{1}{2} \Delta u + g(z)|u|^2u, \]

where \( u(x, y, z) : \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{C} \) is the slowly varying amplitude of the beam envelope, \( \Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 \), and \( g(z) \) is a periodic function accounting for the modulation of the nonlinearity. It is well known that, if \( g \) is constant, there is a stationary radially symmetric solution of Eq. (1) (the so-called Townes soliton): \( u(r, z) = \Phi_0(r)e^{i\lambda z} \). This solution is unstable since any generic slight perturbation of the initial condition will yield either collapse or spread of the distribution. In previous works, it has been shown that the structure which arises when the nonlinearity is modulated as described above is a stabilized Townes soliton (STS) [13, 14].

Partially stabilized vortex solitons. However, Townes solitons are not the only stationary solutions of Eq. (1) for constant \( g \). There also exist vortex-type solutions of the form \( u(r, z) = \Phi_0(r)e^{i\theta(r)}e^{i\lambda z} \) which are unstable as well. Therefore, one could naively expect that the same stabilization mechanism proposed in Ref. [5] could be applied to achieve stabilization of these solutions, i.e. to induce alternative expanding and squeezing cycles of the vortex by a periodic modulation of the nonlinear coefficient. To study further this possibility we have computed numerically the profiles of vortex solutions of Eq. (1) corresponding to a constant value of \( g_0 = -24.15 \) (the critical one for vortex solutions), by using a standard shooting method. We have then studied the evolution of this stationary solution numerically using a special pseudospectral method described in Ref. [14] in different situations. Obviously if we compute the evolution of this vortex solution for constant \( g > g_0 \) we would have expansion while for \( g = g_0 < g_v \) it would collapse. In Fig. (a,b,c) we plot the collapse of a vortex with supercritical value of \( g_0 = -8\pi \), while in Fig. (d,e,f) we
present some snapshots of the evolution of the vortex after the addition of a stabilizing term to the nonlinearity 
\[ g(z) = g_0 + g_1 \cos \Omega z \] 
for \( g_0 = -8\pi, g_1 = 20\pi, \Omega = 40 \). We can see how the periodic modulation of the nonlinearity retards the filamentation of the vortex, which now propagates for a longer distance before breaking into several stabilized solitons. The splitting length and number of outgoing stabilized solitons depend on the specific parameters of the nonlinearity. For instance choosing a different set of parameters \( g_0 = -27, g_1 = 20\pi, \Omega = 40 \) increases the stability of the vortex and decreases the number of filaments. Since each emerging beam is close to an STS the excess energy is eliminated in the form of radiation which is removed by the absorbing boundary conditions of our numerical scheme. In this paper we have chosen a smooth form for the modulation \( g(z) \) but similar results are obtained when \( g(z) \) is taking as piecewise constant.

We have made an extensive search in the parameter space for modulations of the form \( g(z) = g_0 + g_1 \cos \Omega z \) and we have not been able to find any parameter combination allowing indefinite stabilization of the vortex soliton. Concerning finite-dimensional reduced models for the evolution of the effective width of the solutions, such as those successfully used for nodeless beams in Refs. [10, 11, 12, 13], we must stress that as pointed out in Ref. [10] these formulations do not reflect correctly the dynamics and instabilities of vortex solutions.

Vector systems.- From the previous analysis it seems that a vortex can only be partially stabilized in the framework of Eq. (1), i.e. in scalar systems. Recent works point out the important fact that the incoherent interaction of two components could provide, in saturable media, an effective waveguide for the vortex, leading to a more stable behavior [16]. Following this idea we consider now a vector two-component system with Kerr interactions of the form:

\[ i \frac{\partial u_1}{\partial z} = -\frac{1}{2} \Delta u_1 + g(z) (a_{11} |u_1|^2 + a_{12} |u_2|^2) u_1, \]
\[ i \frac{\partial u_2}{\partial z} = -\frac{1}{2} \Delta u_2 + g(z) (a_{21} |u_1|^2 + a_{22} |u_2|^2) u_2, \]

where \( a_{jk} \in \mathbb{R} \) are the nonlinear coupling coefficients and \( g(z) \) accounts for the modulation of the nonlinearity. We will denote \( I_j = \int_{\mathbb{R}^2} |u_j|^2 dx dy \). Although this system is conservative, in the numerical simulations to be shown later we incorporate absorbing boundary conditions in order to get rid of the radiation. Therefore, in practice, there will be a decrease of \( I_j \) during the propagation.

Eqs. (2) are an extension of the Manakov system [17] to two transverse dimensions. Among other situations these equations model the propagation of two circularly polarized beams with opposite polarizations leading to specific factors \( a_{11} = a_{22} = 1, a_{12} = a_{21} = 2 \). In the context of BEC, these equations (with an additional trapping term) describe the dynamics of multicomponent quasi-two dimensional condensates, \( u_j \) being the wavefunctions for the atomic species involved. The formation of vector solitons composed of appropriate fractions of Townes states has been studied in Ref. [18].

Our idea is to choose \( g(z) \) to get the Townes soliton \( u_1 \) stabilized. As the coupling terms in Eq. (2) would provide an effective waveguide for \( u_2 \), it seems reasonable that, when \( I_2 \ll I_1 \), the guiding effect will dominate over self-interaction and the vortex could become stabilized.

Limit of small \( u_2 \).- Let us first consider the case of constant \( g \) and \( I_2 \ll I_1 \) so that Eqs. (2) become:

\[ i \frac{\partial u_1}{\partial z} \simeq -\frac{1}{2} \Delta u_1 + g a_{11} |u_1|^2 u_1, \]
\[ i \frac{\partial u_2}{\partial z} \simeq \left( \frac{1}{2} \Delta + g a_{21} |u_1|^2 \right) u_2. \]

Taking \( u_1(r, z) = \Phi_0(r)e^{i\lambda_0 z} \), then Eq. (3b) is a linear two-dimensional Schrödinger problem in which the role of the potential is played by \( |\Phi_0|^2 \). Following Ref. [18] we can bound the number of \( k \)-wave bound states in this

FIG. 1: [Color online] Evolution of initial data of vortex type obtained as a stationary solution of Eq. (1) for \( g_0 = -24.15 \) under different parameter variations. (a-c) Evolution for constant \( g = -8\pi \). (d-f) Evolution with modulated nonlinearity \( g(z) = -8\pi + 20\pi \cos 40z \). (g) Isosurface plot of \( u(x, y, z) \) of the same simulation as in (d-f) showing the development of the instability. (h-j) Evolution with modulated nonlinearity \( g(z) = -27 + 20\pi \cos 40z \).

\[ \Phi_0(r) = \frac{1}{\sqrt{2\pi}} \left( e^{i\lambda_0 z} + e^{-i\lambda_0 z} \right), \Phi_1(r) = \frac{1}{\sqrt{2\pi}} \left( e^{i\lambda_0 z} - e^{-i\lambda_0 z} \right), \Phi_2(r) = \frac{1}{\sqrt{2\pi}} \left( e^{i\lambda_0 z} + e^{-i\lambda_0 z} \right). \]
potential, by $N_\ell < (a_{12} g / \ell \int_{\mathbb{R}^+} r |u_1|^2 dr)$. Vortex-type solutions with smallest topological charge are those with $\ell = 1$. Thus, taking into account that for a Townes soliton $g I_1 = 0.931$ we get $N_{\ell=1} < 0.931 a_{12}$. Therefore, we can expect the existence of a unique $\ell = 1$ stationary vortex solution of Eq. \eqref{eq:3.1}, in the case $a_{21} = 2$. This will be denoted hereafter as $u_2 = V(r)e^{i\theta}e^{i\lambda_0 z}$. Let us notice that for $\ell \geq 2$ we get always $N_{\ell\geq2} < 1$ thus ruling out the possibility of obtaining higher order vortices. We have numerically found the profile of this vortex solution $V(r)$ by using a standard shooting method.

Choosing an appropriate modulation for $g$ allows us to stabilize $u_1$. For Eq. \eqref{eq:3.1}, the potential –the stabilized Townes soliton– oscillates with a fast frequency of the order of $\Omega$ and another slower one of dynamical origin \cite{13}. In this linear case we can apply to $u_2$ the quantum mechanical theory of fast perturbations \cite{13} to account for the effect of the fast modulation on the vortex. The main result of this theory is that the vortex will remain unaffected by the fast perturbation in the potential provided the modulation period $T$ satisfies $\Delta u_2 \tilde{H} \ll 1/T$. In our case $\tilde{H} = \frac{1}{T} \int_0^T \left[ -\frac{1}{2} \Delta + g(t) \right] u_2 \bar{u}_2 |u_1|^2 dt$ and this inequality imposes $T \ll 1.12$ which requires $\Omega \gg 5.6$. The STS oscillation of lower frequency will induce the same modulation in the vortex. To verify our previous considerations we have simulated Eqs. \eqref{eq:2} with initial conditions $u_1(r, 0) = \Phi_0(r), u_2(r, 0) = 0$ with $\alpha = 0.1$ and $g(z) = -2\pi + 8\pi \cos 40z$. Full stabilization of the vortex is observed, its oscillations following the pattern predicted above, i.e. there is only a residual fast oscillation in the vortex component and its slow oscillation follows that of the stabilized Townes soliton in $u_1$.

**Fully nonlinear regime.** To account for the case of finite $u_2$ we have studied numerically the solutions of Eqs. \eqref{eq:2} taking as initial data $u_1(r, 0) = \Phi_0(r), u_2(r, 0) = \alpha V(r)e^{i\theta}$ and $g(z) = -2\pi + 8\pi \cos 40z$. We have repeated most simulations starting with the full stationary solutions of Eqs. \eqref{eq:2} and found similar results.

For small values of $\alpha$ (e.g. $\alpha = 0.1$ as in Fig. \ref{fig:2}) the vortex is fully stabilized up to the maximum propagation distances studied. However, a continuous loss of energy is observed during propagation. We think that this power damping is due to radiation emitted by the vortex and it is related to the continuous background oscillations of the stabilizing Townes beam. We have tested these numerical simulations both by the spectral numerical scheme of Ref. \cite{14}, for which radiation is eliminated by means of an absorbing potential and by a finite-difference Crank-Nicholson type scheme with transparent boundary conditions with the same results.

**FIG. 2:** [Color online] Solutions of Eqs. \eqref{eq:2} for initial data $u_1(r, 0) = \Phi_0(r), u_2 = \alpha V(r)e^{i\theta}$ with $\alpha = 0.1$ in a grid of 810 x 810 points on the spatial region [-40,40] showing stable but dissipative due to the effect of radiation propagation of the vortex in the full range $z \in [0, 500]$. (a) Iso surface plot of $u_2(x, y, z)$ spanning all the propagation range. (b) Evolution of the norms $I_1^{1/2}(z), I_2^{1/2}(z)$ and of the amplitudes $A_1(z) = \max_{(x, y)} |u_1|, A_2(z) = \max_{(x, y)} |u_2|$ of both components.

**FIG. 3:** [Color online] Same as Fig. \ref{fig:2} for $\alpha = 0.32$. (a)-(e) Pseudocolor plots of $u_2(x, y, z)$ for $z = 0, 200, 295, 305, 450$. (f) Iso surface plot of $u_2(x, y, z)$ spanning all the propagation range. (g) Details of the region in which the vortex spirals out of $u_2$ for $z \in [280, 319]$. (h) Evolution of the norms of both components $I_1^{1/2}(z), I_2^{1/2}(z)$ showing the readjustment of the norms after the vortex is ejected and a stabilized vector soliton is formed. (i) Amplitudes $A_1(z) = \max_{(x, y)} |u_1|$ and $A_2(z) = \max_{(x, y)} |u_2|$ of both components.
For larger values of \( \alpha \) (e.g. \( \alpha = 0.32 \) as in Fig. 3), the vortex destabilizes at long propagation distances due to the effect of nonlinear interactions between the guiding Townes soliton and the vortex. We can see that, although the perturbation is not small, the vortex propagates for very long distances of about 300 propagation units (compare this with the results shown in Fig. 1). In the region of stable propagation the vortex amplitude slowly decays due to the emission of radiation until a stabilized vector soliton is formed. In this process, both components emit radiation to readjust their norms to satisfy the relation \( I_1 + I_2 = I_{\text{Townes}} \). From Fig. 4(i) it is clear that the oscillations of the vortex amplitude basically contain only the slower frequency and that our previous arguments apply here. The addition of noise to the initial data of 1% in amplitude triggers the instability faster but even in that case the vortex propagates for more than 125 dimensional units before the instability sets in.

Finally let us briefly comment that for \( \alpha = O(1) \) there is a different branch of bound states of Eqs. (2) in the fully nonlinear regime, corresponding to thin Townes solitons localized near the bottom of the vortex solutions. This soliton act as a pinning potential for the vortex but it is not enough to stabilize vortex-type solutions.

**Applications to matter waves.** The previous results have also implications in the field of matter waves because of the close analogy of Eqs. (2) with the equations of evolution of a multi-component Bose-Einstein condensate in the mean field approximation. The analysis of vortices in dilute-gas BECs has been a very hot topic in the last years, specially after their experimental generation with different setups. In multicomponent BEC systems the interaction coefficients \( a_{ij} \) are proportional to the respective scattering lengths and the effective two-dimensionality can be achieved by confining the condensate tightly along one specific direction. The condition \( N_1 < 0.931a_{12} \) has relevance since it imposes restrictions to the atomic species which can be used to trap a vortex. For instance the cross-interaction coefficient in multicomponent condensates made of different hyperfine levels of \(^{87}\text{Rb} \) does not satisfy this condition. However, bosonic K-Rb mixtures such as the one described in Refs. 21, 22 could be used because of the large scattering length of the collisions K-Rb. In this scenario the results presented here could be extended to tightly confined BEC systems. Our predictions would imply the existence of self-supported vortex solitons which could be generated using Feschbach resonance management techniques.

In conclusion, we have shown stabilization of vortex solitons for long propagation distances in stratified Kerr media by control of the nonlinear coefficient and stronger stabilization by the use of two combined beams. Our results can also be used to stabilize vortices in certain types of multicomponent Bose-Einstein condensates.

This work has been partially supported by Ministerio de Ciencia y Tecnologia (Spain) under grants BFM2000-0521, BFM2003-02832, TIC-2000-1105-C03-01 and by the Junta de Comunidades de Castilla-La Mancha under grant PAC-02-002. G. D. M. acknowledges support from grant AP2001-0535 from MECD.

[1] H.J. Lugt, *Vortex Flow in Nature and Technology* (Krieger, Malabar, FL, 1995).
[2] L.M. Pismen, *Vortices in Nonlinear Fields* (Clarendon, Oxford, UK, 1999).
[3] F. Sols, Physica C 369, 125 (2001).
[4] M. R. Matthews et al., Phys. Rev. Lett. 83, 2498 (1999); K.W. Madison et al., Phys. Rev. Lett. 84, 806 (2000); F. Chevy, K.W. Madison, J. Dalibard, Phys. Rev. Lett. 85, 2223 (2000); J.R. Abo–Shaer et al., Science 292, 476 (2001).
[5] J. F. Nye and M. V. Berry, Proc. R. Soc. London A 336, 165-190 (1974).
[6] V. I. Kruglov and V. M. Volkov, Phys. Lett. A 111, 401-404 (1985).
[7] W. J. Firth and D. V. Skryabin, Phys. Rev. Lett. 79, 2450 (1997).
[8] Y. Kivshar, E. A. Ostrovskaya, Optics & Photonics News, April, 26-30 (2001).
[9] L. Berge et al., Opt. Lett. 25, 1037 (2000).
[10] I.Towers and B.A.Malomed, J. Opt. Soc. Am. B 19, 537 (2002).
[11] H. Saito and M. Ueda, Phys. Rev. Lett. 90, 040403 (2003).
[12] F. Abdullaev et al., Phys. Rev. A 67, 013605 (2003).
[13] G. D. Montesinos, V. M. Pérez-García and P. Torres, Physica D (Amsterdam) 191, 193 (2004).
[14] G. D. Montesinos, V. M. Pérez-García, Math. Comp. Simul. (to appear, 2004).
[15] G. D. Montesinos, V. M. Pérez-García, H. Michinel, Phys. Rev. Lett. 92 133901 (2004).
[16] Z. H. Musslimani et al., Phys. Rev. Lett. 84, 1164 (2000); J. J. García-Ripoll, et al., Phys. Rev. Lett. 85, 82 (2000); J. Yang, D. E. Pelinovsky, Phys. Rev. E 67, 016608 (2003); J. R. Salgueiro, Y. Kivshar, Ukr. Jour. Phys. 49 (2004) 327.
[17] S. V. Manakov, Sov. Phys. JETP 38, 248 (1974).
[18] K. Chadan et al., Jour. Math. Phys. 44, 406 (2003).
[19] A. Galindo, *Quantum Mechanics II*, Springer, Berlin (1991).
[20] G. Modugno et al., Science 294, 1320 (2001).
[21] A. Simoni et al., Phys. Rev. Lett. 90, 163202 (2003).