Critical Parameter of the Frog Model on Homogeneous Trees with Geometric Lifetime

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Abstract
We consider the frog model with geometric lifetime (parameter $1 - p$) on homogeneous trees of dimension $d$. In 2002, Alves et al. (Electron J Probab 7:21, 2002) proved that there exists a critical lifetime parameter $p_c \in (0, 1)$ above which infinitely many frogs are activated with positive probability, and they gave lower and upper bounds for $p_c$. Since then, the literature on this model focussed on refinements of the upper bound. In the present paper, we improve the bounds for $p_c$ on both sides. We also provide a discussion comparing the bounds of the literature and their proofs. Our proofs are based on coupling.

Keywords Frog models · Renewal theory · Multi-types branching processes · Critical parameter

Mathematics Subject Classification Primary: 60K35 · 05C81 · Secondary: 60K05

1 Introduction

Frog models are simple models for the propagation of information (or rumor, or disease) through a graph: active (informed/ill) particles perform independent random walks on the graph, activating (informing/infecting) frogs of the visited vertices. This class of models seems to have been first introduced by Telcs and Wormald [11] under the name of egg model: particles making independent simple symmetric random walks on $\mathbb{Z}^d$ transform eggs of the visited vertices into active particles. In our terminology, particles are frogs and eggs (inactive particles) are inactive frogs.
The literature on frog models has grown very fast over the last two decades, with variations depending on the lifetime of the frogs, the underlying graph structure on which the frogs wander around, the type of random walk the frogs perform, the initial number of frogs per vertex, etc... In the present paper, we stand in the line of Alves et al. [1] who considered the case where frogs have geometric lifetime and make simple symmetric random walks, and we focus specifically on homogeneous infinite trees.

More precisely, consider the homogeneous tree $T_d$, that is, the rooted tree in which each vertex has $d+1$ neighbors. One frog is placed on each vertex and all but the root one start inactive. Active frogs perform simple symmetric random walks on $T_d$ for a geometric number of steps (parameter $(1-p)$, $p \in [0, 1]$), activating the inactive frogs of the visited vertices. (We use the definition of the geometric distribution taking values in $\{0, 1, 2, \ldots\}$.) After its geometric number of steps, the active frog “dies”: it remains inactive forever. The process survives if infinitely many frogs are activated.

For any $p \in [0, 1]$ we denote the law of the process by $P_p$. Naturally, if $p = 0$ then the frog of the root dies and $P_p(\text{survival}) = 0$. On the other hand, if $p = 1$, frogs never die and $P_p(\text{survival}) = 1$. Moreover, it is clear that $P_p(\text{survival})$ is non-decreasing in $p$, thus we can define the critical parameter for the model on $T_d$ as

$$p_c = p_c(d) := \inf\{p \in [0, 1] : P_p(\text{survival}) > 0\}.$$ 

In the present paper, we are interested in obtaining tight bounds for $p_c$, as functions of $d$. Let us quickly enumerate the results from the literature concerning this issue. In their seminal paper in 2002, Alves et al. [1] proved that

$$\frac{d + 1}{2d + 1} \leq p_c \leq \frac{d + 1}{2d - 2}. \quad (1)$$

Three years later, Lebensztayn et al. [9] obtained the following improvement of the upper bound

$$p_c \leq \frac{d + 1}{2d}.$$ 

Recently, a new improvement of the upper bound was obtained by Gallo and Rodriguez [5] as a consequence of the study of a percolation model on oriented trees:

$$p_c \leq \frac{(d + 1)\left[(7d - 1) - \sqrt{(7d - 1)^2 - 14}\right]}{d(7d - 1)^2 - 7d + 2 - d(7d - 1)\sqrt{(7d - 1)^2 - 14}}$$

and one year later, Lebensztayn and Utria [10] devoted their work to a further improvement of the upper bound, but the expression they obtained is very complicated (see Definition 2.1 and Theorem 2.2 therein).

Notice that the above improvements all refer to the upper bound of $p_c$. The lower bound in display (1), obtained by Alves et al. [1], was not improved so far.

In this paper, we prove the following result.

**Theorem 1** For any $d \geq 2$,

$$\frac{2(d + 1)}{\sqrt{4d^2 + 4d - 3} + 2d + 1} \leq p_c(d) \leq \frac{(d + 1)(4d + \frac{1}{4d^2} - 2)(8d - 5)}{d(8d - 5)^2 + \left(4d + \frac{1}{4d^2} - 2\right)^2}. \quad (2)$$
Thus, since the definition of the model by Alves it et al. [1], two papers have been devoted exclusively to improving the upper bound (Lebensztayn et al. [9] and Lebensztayn and Utria [10]) and Gallo and Rodriguez [5] obtained an improvement as a consequence of a more general result. Here we get better bounds on both sides: we provide the first improvement of the lower bound since [1], and get a tighter and simpler upper bound than Lebensztayn and Utria [10].

By (1), we know that $p_c(d)$ converges to $1/2$ when $d$ diverges, and for large $d$’s it gives

$$
\frac{1}{2} + \frac{1}{4d} - \frac{1}{8d^2} + O \left( \frac{1}{d^3} \right) \leq p_c(d) \leq \frac{1}{2} + \frac{1}{d} + \frac{1}{d^2} + O \left( \frac{1}{d^3} \right).
$$

Theorem 1 gives, after some algebra,

$$
\frac{1}{2} + \frac{1}{4d} + O \left( \frac{1}{d^4} \right) \leq p_c(d) \leq \frac{1}{2} + \frac{7}{16d} - \frac{5}{128d^2} + O \left( \frac{1}{d^3} \right).
$$

We emphasize also that the asymptotics given by the righthand side of (3) cannot be improved using our method of proof. That point is discussed in Sect. 4.4.

There is yet another motivation behind the search for tighter bounds, which we now explain. Fontes et al. [3] proved that the critical parameter of the frog model, in general graphs, is not always a monotonic function of the graph. For homogeneous trees, the question of whether or not $p_c(d)$ is monotonically decreasing in $d$ remains open. Denote by $l(d)$ (resp. $r(d)$) the lower (resp. upper) bound of (2). What we can show (using the software Mathematica) is that $l(d) > r(ad)$ for any $a \geq 1.75$. This implies that $p_c(d) > p_c(ad)$ for those values of $a$, meaning for instance that $p_c(2) > p_c(4) > p_c(7) > p_c(13)$. A proof of monotonicity using this argument needs the bounds to be strengthened, and we believe the present paper also provides new insights in this direction.

Further related literature will be provided in Sect. 4. We will also take the opportunity to discuss further the monotonicity issue, and clarify the relationship between the upper bounds of the literature and their respective proofs. As we will explain there, no significant improvement on the upper bound can be achieved using our method of proof, meaning that a different approach has to be developed to prove monotonicity by comparisons of lower and upper bounds.

The paper is organized as follows: we prove the lower bound in Sect. 2, we prove the upper bound in Sect. 3, and we conclude with a discussion in Sect. 4.

## 2 Proof of the Lower Bound

The idea of the proof is to slightly modify the dynamics of the frog model so that it can be easily coupled with a two-type branching process (TTBP) in such a way that the latterdominates the former. This TTBP is defined in Subsection 2.1. In Subsection 2.2, we give the alternative construction of the frog model, and although it is not a branching process, we call it the frog model branching process (FMBP) by abuse of terminology. We prove the lower bound of Theorem 1 in Subsection 2.3, by constructing the coupling between the TTBP and the FMBP.
2.1 A Two-Type Branching Process

Consider a two-type branching process in which, at each time step, exactly one particle is chosen, dies, and gives birth to a random number of individuals of each type, independently of everything else. For \( a = 1, 2 \), we denote by \( p_a(i, j) \) the probability that a particle of type \( a \) generates \( i \) particles of type 1 and \( j \) particles of type 2:

\[
p_1(0, 0) = 1 - p, \quad p_1(1, 0) = 0, \quad p_1(2, 0) = \frac{pd}{d + 1}, \quad p_1(0, 1) = \frac{p}{d + 1}, \quad \text{(4)}
\]

\[
p_2(0, 0) = 1 - p, \quad p_2(1, 0) = \frac{p}{d + 1}, \quad p_2(2, 0) = \frac{p(d - 1)}{d + 1}, \quad p_2(0, 1) = \frac{p}{d + 1}. \quad \text{(5)}
\]

It is well-known that a multi-types Galton-Watson process has probability zero to survive if, and only if, the largest eigenvalue of the first moment matrix is smaller or equal to 1 (see [2] for instance). Simple calculations show that this matrix is

\[
M = \begin{pmatrix}
\frac{2dp}{d+1} & \frac{p}{d+1} \\
\frac{p}{d+1} & \frac{p}{d+1}
\end{pmatrix}
\]

and its largest eigenvalue is

\[
\rho_M = \frac{\sqrt{4d^2 + 4d - 3} + 2d + 1}{2(d + 1)} p.
\]

In other words, if

\[
p \leq \frac{2(d + 1)}{\sqrt{4d^2 + 4d - 3} + 2d + 1},
\]

then the TTBP defined above will generate finitely many individuals with probability 1.

2.2 Modification of the Dynamics of the Frog Model

We define the following modification in the dynamics of the frog model, which does not alter its survival probability but simplifies the construction of a coupling with the TTBP.

(1) We consider the frog model in a way that frogs move one at a time, and at each time, the frog that is chosen to make the step is arbitrary. That is, the frog of the origin makes a move (with probability \( p \)), activating the sleeping frog of the visited site. Then, we choose any frog among the activated ones, to make a move (a move that is made with probability \( p \) too) while the others remain frozen (not moving) and active, and can be chosen to make a move at a future time step. This procedure slows down the process in the sense that it propagates more slowly on the tree, but since the random walks of each activated frog are independent, this does not change anything in terms of survival.

(2) For ease of comparison with a two-type branching process, we will interpret “moves” differently. At each step, instead of saying that the chosen frog moves to a neighboring site, we will think that it dies, and with probability \( p \), gives birth to frogs at one neighboring site: one frog if the chosen neighboring site has been already visited, and two frogs otherwise.

(3) For any \( t \geq 0 \), we denote by \( T_t \subset T_d \) the set of visited sites at time \( t \). Notice that \( T_t \subseteq T_{t+1}, t \geq 0 \).
(4) We consider the model starting from the random time \( K = \inf \{ t \geq 0 : |T_t| = d + 3 \} \) where \( | \cdot | \) denotes the cardinality of the set. This ensures that at least one frog within a distance of 2 from the root has been activated (see the next item). Notice that \( K = \infty \) has a positive probability to occur.

(5) At each time step \( t \), frogs that are at the tip of \( T_t \) (meaning they have \( d \) unvisited neighboring sites) are classified as Type 1, and the other frogs are classified as Type 2. Note that once we consider times \( t \geq K \), at least one neighboring site of a Type 2 frog is surrounded by at least two already visited sites. This means that any activated frog in the system can be classified as either type 1 or type 2.

Note that, if a frog, when born, is of type 1, depending on the position of its vertex \( v \) in the evolving set \( T_t \), \( t \geq 1 \), it may transform into type 2 (notice the difference in the preceding sentence between “transform” and “give birth to”). However, the reverse cannot occur, a frog that is born type 2 cannot, over time, transform into a type 1 frog.

Fix \( t \geq K \) and \( T_t = T \). We denote by \( p_v(i, j | T) \) the offspring distribution of the frog located at \( v \) inside \( T \) which has been chosen to make a move, where \( i \) and \( j \) are respectively the numbers of offspring of types 1 and 2. For any \( v \), the distribution \( p_v \) lives on \( \{ (0, 0), (1, 0), (0, 1), (2, 0) \} \). The location of \( v \) inside \( T_t = T \) specifies the type 1 or 2 of the frog:

- Suppose it has type 1. Then, independently of \( v \) and \( T \)
  \[
  p_v(0, 0| T) = 1 - p, \quad p_v(1, 0| T) = 0, \quad p_v(2, 0| T) = \frac{pd}{d+1}, \quad p_v(0, 1| T) = \frac{p}{d+1}.
  \]
- Suppose it has type 2. Then, there exist two integers (which depend on the location of \( v \) in \( T \) ) \( a \), \( b \), with \( a \geq 1 \) and \( a + b \geq 2 \), such that
  \[
  p_v(0, 0| T) = 1 - p, \quad p_v(1, 0| T) = \frac{pb}{d+1}, \quad p_v(2, 0| T) = p - p \frac{a+b}{d+1}, \quad p_v(0, 1| T) = \frac{pa}{d+1}.
  \]

2.3 Proof of the Lower Bound

We are now ready to prove our lower bound.

**Proof of the Lower Bound of Theorem 1** Let \( M_i^t, i = 1, 2 \) (resp. \( N_i^t, i = 1, 2 \)) count the number of active particles of type \( i \) in the system at time \( t \) in the FMBP (resp. in the TTBP).

Since we are only interested in bounding the critical parameter of the frog model, instead to start from \( T_0 = \{ o \} \) we can start the frog model from any of the configurations satisfying \( |T_0| = d + 3 \). So let us start from any vector \( (M_0^1, M_0^2) \) having positive probability to be produced in the FM with \( |T_0| = d + 3 \), and put \((N_0^1, N_0^2) = (M_0^1, M_0^2)\). If we can couple these processes in such a way that, for any \( t \geq 0 \), \( N_t^1 \geq M_t^1 \) and \( N_t^1 + N_t^2 \geq M_t^1 + M_t^2 \), then, in particular, the total number of particles in the TTBP is at least as large as the total number of activated frogs in the FMBP, at each time step. Together with what we said in Subsection 2.1, if

\[
p \leq \frac{2(d+1)}{\sqrt{4d^2 + 4d - 3} + 2d + 1},
\]
then the FMBP would not survive with probability 1, which would conclude the proof of the theorem.

So it only remains to prove that we can couple these processes in such a way that, for any \( t \geq 0 \), \( N^I_t \geq M^I_t \) and \( N^1_t + N^2_t \geq M^1_t + M^2_t \). The inequalities are satisfied at \( t = 0 \) by definition. We assume that the inequalities are satisfied at time \( t \), and we now want to prove that they are still satisfied at time \( t + 1 \). To couple the processes at \( t + 1 \), we need to couple the probability distributions \( p_1 \), \( p_2 \) and \( p_1(\cdot | T) \). We do this using a random variable \( U_{t+1} \) uniformly distributed in \([0, 1]\) (independent of everything) and several partitions of \([0, 1]\). To construct \( p_1 \), \( p_2 \) we use the following partitions

\[
\mathcal{P}_1 = \{ I^{0,1}_1, I^{1,0}_1, I^{0,0}_1 \}
\]

\[
\mathcal{P}_2 = \{ I^{0,1}_2, I^{1,0}_2, I^{2,0}_2, I^{0,0}_2 \}
\]

where

\[
I^{0,1}_1 = \left[ 0, \frac{p}{d+1} \right], \quad I^{1,0}_1 = \left[ \frac{p}{d+1}, p \right], \quad I^{0,0}_1 = [p, 1]
\]

and

\[
I^{0,1}_2 = \left[ 0, \frac{p}{d+1} \right], \quad I^{1,0}_2 = \left[ \frac{p}{d+1}, \frac{2p}{d+1} \right], \quad I^{2,0}_2 = \left[ \frac{2p}{d+1}, p \right], \quad I^{0,0}_2 = [p, 1].
\]

We refer to Fig. 1 for a pictorial representation of these partitions.

For the coupling, if a type \( k = 1, 2 \) particle is chosen at time \( t + 1 \) from the TTBP, its offspring is chosen with \( U_{t+1} \), using partition \( \mathcal{P}_k \) in the following way: we put \((i, j)\) if and only if \( U_{t+1} \in I^{i,j}_k \). This gives the correct offspring distribution since

\[
P(U_{t+1} \in I^{i,j}_1) = p_1(i, j) \quad \text{and} \quad P(U_{t+1} \in I^{i,j}_2) = p_2(i, j).
\]

For moving frogs of type 1 in the FMBP, no matter what is the pair \((v, T)\), we can use partition \( \mathcal{P}_1 \) similarly (and with the same uniform \( U_{t+1} \)) since for such frogs \( p_1(\cdot | T) = p_1 \).

According to (7), for moving frogs of type 2 we define the partition (recall that \( a \geq 1 \) and \( a + b \geq 2 \))

\[
\mathcal{P}_{(a,b)} = \{ I^{0,1}_{(a,b)}, I^{1,0}_{(a,b)}, I^{2,0}_{(a,b)}, I^{0,0}_{(a,b)} \}
\]

where

\[
I^{0,1}_{(a,b)} = \left[ 0, \frac{ap}{d+1} \right], \quad I^{1,0}_{(a,b)} = \left[ \frac{ap}{d+1}, \frac{(a+b)p}{d+1} \right], \quad I^{2,0}_{(a,b)} = \left[ \frac{(a+b)p}{d+1}, p \right], \quad I^{0,0}_{(a,b)} = [p, 1].
\]
Observe that, for any $a$, $b$ and $i$, $j$, we have

$$P(U_{t+1} \in I_{(a,b)}^{i,j}) = p_v(i, j|T).$$

The coupling is thus performed by updating both processes at time $t+1$ using the same uniform random variable $U_{t+1}$. We can now establish the recursion from $t$ to $t+1$, recalling that the recursion hypothesis is $N_t^1 \geq M_t^1$ and $N_t^1 + N_t^2 \geq M_t^1 + M_t^2$:

1. If the chosen frog for the next FMBP move is of type 1, then a TTBP type 1 particle is chosen to give birth to its offspring. Since $N_t^1 \geq M_t^1$, this choice is always possible. In this case, according to our coupling, the two chosen particles (one in each process) give birth to the same offspring, thus maintaining the inequalities.

2. Suppose now that the chosen frog is of type 2, then, either a type 2 particle of the branching process is chosen to give birth to its offspring, or, if there is none, a type 1 particle is chosen.

   According to the location $v$ in $T$ of the chosen frog of type 2, a pair $(a, b)$ of integers is associated, as explained before. For any fixed pair $(a, b)$, we have the following possibilities:

   (a) The particle of the TTBP is also a type 2 particle: then
   
   - if $U_{t+1} \leq \frac{p}{d+1}$, then one particle of type 2 is created in each process;
   - if $U_{t+1} \geq p$, then the chosen particle dies in each process;
   - if $U_{t+1} \in \left[\frac{p}{d+1}, \frac{dp}{d+1}\right]$ (and therefore $a \geq 2$ since otherwise this interval half open to the right is empty) the branching process produces two type 1 particles while the frog model produces one type 2 particle;
   - if $U_{t+1} \in I_{(a,b)}^{1,0}$ then either one (if $a = 1$) or two (if $a \geq 2$) type 1 particles are created in the branching process, while the frog model creates one type 1 particle;
   - if $U_{t+1} \in I_{(a,b)}^{2,0}$ then two type 1 particles are created in both processes.

   Therefore, for this case, no matter where the uniform random variable $U_{t+1}$ falls, the inequalities of the recursion hypothesis are maintained at time $t+1$.

   (b) There is no more type 2 particles in the TTBP, and we have to choose a type 1 particle.

   In this case, the reasoning is the same as above, observing moreover that, necessarily, $N_t^1 \geq M_t^1 + 1$.

   Thus, in any case, we have $N_{t+1}^1 \geq M_{t+1}^1$ and $N_{t+1}^1 + N_{t+1}^2 \geq M_{t+1}^1 + M_{t+1}^2$, establishing the recursion.

The proof of the lower bound of Theorem 1 is concluded.

3 Proof of the Upper Bound

We will proceed in three steps. In the first step of the proof, we explain the comparison of the frog model with its oriented version. This part of the proof is common to [5, 9, 10]. In the second step, we state Theorem 2, a very nice result that identifies the critical parameter of the oriented version with the root of a power series. Notice that, although this result was already proved in [5] (see Theorem 1 therein), we will partially prove it to give support
to our discussion in Sect. 4. Based on this theorem, the two last steps are dedicated to obtaining bounds for the critical parameter of the oriented version, and these steps only rely on calculus. This is done by first finding finite degree polynomials (of degree 6 and degree 5) which approximate the power series from below and from above, and next, finding root approximations for these polynomials.

Let us now give some notation. We denote by \( d(v, v') \) the distance between any two vertices of \( v, v' \) of \( T_d \), defined as the number of edges of the unique path connecting them. We write \( v \leq v' \) if \( v \) is one of the vertices on the path between the root \( o \) and \( v' \), and we naturally write \( v < v' \) if \( v \leq v' \) and \( v \neq v' \).

3.1 Step One: Oriented Version of the Frog Model

Consider the following modification of the frog model, that we call orientated frog model: when a frog, initially at vertex \( v \in T_d \) is activated and makes its random walk, it only activates the frogs of vertices \( v' \) such that \( v' > v \).

It is obvious that, if this oriented model survives, then the original model survives as well. So if we find an upper bound for the critical parameter \( \hat{p}_c \) of the oriented model, it will also be an upper bound for the critical parameter of the original model.

Our objective in the remaining of the proof will be to find tight upper and lower bounds for \( \hat{p}_c \). Notice that even though the lower bound on \( \hat{p}_c \) is not necessary to get our upper bound for \( p_c \), we will partially provide it to show that the upper bound of \( \hat{p}_c \) is already very accurate as an estimate of \( \hat{p}_c \) (see the discussion in Sect. 4).

3.2 Step Two: \( \hat{p}_c \) as the Root of a Power Series

For any vertices \( v', v \), we denote by \( \{ v \rightarrow v' \} \) the event that the frog at \( v \), if it were active, would visit \( v' \) during his random walk. Lemma 2.1 of Lebensztayn et al. [9] states that for any \( v, v' \) such that \( d(v, v') = n \geq 1 \),

\[
\mathbb{P}_p(v \rightarrow v') = r^n
\]

where

\[
r = r(p, d) := \frac{d + 1 - \sqrt{(d + 1)^2 - 4dp^2}}{2dp},
\]

a fact that will have importance later on. Notice in particular that, writing \( r_c = r(\hat{p}_c, d) \), we can now focus on \( r_c \) directly since \( r \) is a continuous bijection on \([0, 1]\).

The objective of the present step is to prove the following very nice theorem which gives the critical parameter as the root of a power series. It is proved, indirectly, in [5], so we include its proof here for the sake of completeness.

**Theorem 2** \( \sum_{k \geq 1} d^k r(\hat{p}_c, d)^k \prod_{i=1}^{k-1} (1 - r(\hat{p}_c, d)^i) = 1 \).

**Proof** We denote by \( \{ v \rightarrow v' \} \) the event that either \( \{ v \rightarrow v' \} \), or there exist \( k \geq 1 \) and a sequence of vertices \( v_1, \ldots, v_k \) such that \( v := v_0 < v_1 < \ldots < v_k < v_{k+1} := v' \) such that \( \bigcap_{i=0}^k \{ v_i \rightarrow v_{i+1} \} \). In words, \( \{ v \rightarrow v' \} \) means that \( v \) has started a chain of activation of frogs which in particular activates \( v' \). By symmetry, we can use \( u_n = u_n(r, d) := \mathbb{P}_p(o \rightarrow v) \) for any \( v \) such that \( d(o, v) = n \).
On the one hand, we have

$$\Pr_p \left( \bigcup_{v : d(o, v) = n} \{ o \xrightarrow{c} v \} \right) \leq \Pr_p \text{(survival of the oriented model)}. $$

Thus

$$\Pr_p \left( \bigcup_{v : d(o, v) = n} \{ o \xrightarrow{c} v \} \right) \leq dn \Pr_p (o \xrightarrow{c} v) = : d^n u_n. $$

In other words,

$$d^n u_n \to 0 \Rightarrow \Pr_p \text{(survival of the oriented model)} = 0. \quad (10)$$

On the other hand, the expected number of vertices at distance $n$ of the root which have been visited by frogs is, also by symmetry

$$\sum_{v : d(o, v) = n} \Pr_p (o \xrightarrow{c} v) = d^n u_n. $$

Fix some $N \geq 1$ and consider the following process, obtained from the oriented model via the following recursive procedure:

- At each vertex of level $N$ which has been visited (at time $N$, since the process is oriented), we keep only one activated frog.
- Each frog activated at level $iN$, $i \geq 1$ is started, and at each vertex of level $(i + 1)N$ which has been visited, we keep only one activated frog.

Observe first that this modified process is dominated by the oriented process since at each step $i \geq 1$ of the recursive procedure we only keep 1 activated frog at the visited vertices at distance $iN$ of the root. Moreover, the frog we keep at $iN$, given it has reached this level, can be substituted by a new active frog, because of the loss of memory property of the geometric distribution (8). It follows that the number of visited vertices at level $iN$, $i \geq 1$ of this modified process has the same distribution as the number of individuals at the $i$th generation in a Galton-Watson tree with expected offspring $d^N u_N$. If $d^N u_N > 1$, the Galton-Watson has a positive probability to survive, and thus so does the oriented model. In other words

there exist $N \geq 1 : d^N u_N > 1 \Rightarrow \Pr_p \text{(survival of the oriented model)} > 0. \quad (11)$

To continue, we need to study $u_n = \Pr_p (o \xrightarrow{c} v)$ and its limiting properties, and it is precisely in that study that [5] differs from [9, 10] (see Sect. 4 for a discussion). Indeed, Gallo and Rodriguez [5] used a result of Gallo et al. [4] implying that $u_n$ is the probability that an undelayed renewal sequence, with inter-renewal distribution $f_k = r^k \prod_{i=1}^{k-1} (1 - r^i)$, $k \geq 1$, has a renewal at time $n$. This allowed them to conclude, using renewal theory (see Sect. 4 in [5]) that

$$u_\infty (r, d) := \lim u_n^{1/n} \text{ exists, is continuous in } r, \text{ and it satisfies the equality}$$

$$\sum_{k \geq 1} r_k u_\infty^{k-1} \prod_{i=1}^{k-1} \left( 1 - r^i \right) = 1. \quad (12)$$

Together with (10) and (11), the existence of $u_\infty (r, d)$ gives us

$$u_\infty > 1/d \Rightarrow \Pr_p \text{(survival of the oriented model)} > 0$$

$$u_\infty < 1/d \Rightarrow \Pr_p \text{(survival of the oriented model)} = 0.$$
The continuity of \( u_\infty(r, d) \) allows us to conclude that \( r_c \) satisfies
\[
u_\infty(r_c, d) = \frac{1}{d}. \tag{13}\]

Putting (13) together with (12) concludes the proof of the theorem. \( \square \)

3.3 Third Step: Bounds for \( \hat{r}_c \) as Zeros of Polynomials

Using (9) and Theorem 2, the way to proceed now is to find \( r = r(d) \) and \( \bar{r} = \bar{r}(d) \) such that
\[
\underline{r} \leq r_c \leq \bar{r} \tag{14}\]

which yields
\[
\frac{(d + 1)r}{1 + dr^2} \leq \hat{r}_c \leq \frac{(d + 1)\bar{r}}{1 + d\bar{r}^2}. \tag{15}\]

The first characterization that we will give of \( \underline{r} \) and \( \bar{r} \) is as zeros of polynomials.

**Proposition 3** Let \( \underline{r} \) (resp. \( \bar{r} \)), denote the unique root of the polynomial \( L(r) := -d^3r^6 + d^3r^5 + d^2r^3 - 2dr + 1 \) (resp. of the polynomial \( U(r) := d^3r^5 + d^2r^3 - 2dr + 1 \)) in \( r \in (0, 1/d) \). Then
\[
\underline{r} \leq r_c \leq \bar{r}. \tag{16}\]

**Proof** We have the inequalities for any \( k \geq 3 \)
\[
1 - r - r^2 < (1 - r - r^2 + r^k) \leq \prod_{i=1}^{k-1} (1 - r^i) \leq (1 - r)(1 - r^2) \tag{16}\]
where the lower bound between parenthesis follows by recursion and the two others are trivial. Inequalities (16) imply that
\[
f_{\text{inf}}(r) \leq \sum_{k \geq 1} d^kr^k \prod_{i=1}^{k-1} (1 - r^i) \leq f_{\text{sup}}(r) \]
with
\[
f_{\text{inf}}(r) = dr + d^2(1 - r)r^2 + (1 - r - r^2) \left( -\frac{d^3r^3}{dr - 1} \right) \]
and
\[
f_{\text{sup}}(r) = dr + d^2(1 - r)r^2 + (1 - r)(1 - r^2) \left( -\frac{d^3r^3}{dr - 1} \right). \]

With Theorem 2 we conclude that
\[
f_{\text{inf}}(r_c) \leq 1 \leq f_{\text{sup}}(r_c). \]

With some further algebra, we notice that
\[
f_{\text{inf}}(r_c) \leq 1 \iff d^3r_c^5 + d^2r_c^3 - 2dr_c + 1 \geq 0 \]
\[
f_{\text{sup}}(r_c) \geq 1 \iff -d^3r_c^6 + d^3r_c^5 + d^2r_c^3 - 2dr + 1 \leq 0. \]

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But since both polynomials are decreasing on \((0, 1/d)\), then we conclude that the root \(r_L\) of 
\(L(r) = -d^3 r^6 + d^3 r^5 + d^2 r^3 - 2dr + 1\) and the root \(r_U\) of 
\(U(r) = d^3 r^5 + d^2 r^3 - 2dr + 1\) will satisfy 
\[ r_L \leq r_c \leq r_U \]
as stated by the lemma.

### 3.4 Fourth (Last) Step: Explicit Bounds for \(\hat{p}_c\) as Functions of \(d\)

In principle, we could simply seek for the exact expression of the zeros stated in Proposition 3. However, this leads to very complicated expressions. So we will do one further step to get approximations of \(r_L\) and \(r_U\).

**Lemma 4** For \(d \geq 2\),

\[ r(d) \geq r_L := \frac{5 - 8d - 16d^2 + 64d^3}{12d - 20d^2 - 48d^3 + 128d^4} \]

and

\[ \bar{r}(d) \leq r_U := \frac{2 - \frac{1}{14d^2} - 4d}{5d - 8d^2} \]

Putting Display (15), Proposition 3 and Lemma 4 together, we get explicit lower and upper bounds for \(\hat{p}_c\) as a function of \(d\). In conjunction with Step 1, this in particular proves the upper bound given in Theorem 1.

### 4 Discussion on the Bounds and their Proofs

#### 4.1 Further Literature on Related Models

Under the name “frog model”, a vast literature has been developed in recent years. Here we highlight two other papers focusing on trees. The first one [6], considers the speed of the spread and the final proportion of activated frogs, on finite trees and with frogs having a.s. finite lifetime (not necessarily starting with one sleeping frog per site). Although the situation and problem are slightly different than ours here, the paper also gives a nice account of the recent literature for general frog models. The second one, much more related to ours, is [8], which proves the transience and recurrence of the frog model on infinite trees when the frogs have an infinite lifetime (and starting with one sleeping frog per site). In that paper, the authors use a similar argument as the one used here to get the upper bound, designing a multi-type branching process that dominates the frog model of interest. We will come back to this in Sect. 4.3 when we will discuss lower bounds.

#### 4.2 The Upper Bounds and their Proofs

Our objective here is to compare the literature’s upper bounds (and their proofs). Specifically, we have to compare [5, 9, 10], and the present work.

As already mentioned, to obtain their upper bounds, all the above-cited works used Step 1 in their proofs: they consider the oriented version of the frog model. The main difference is in
Step 2. Lebensztayn et al. [9] used the fact that, finding a solution, in $p$, for $d^k u_k(p, d) = 1$ inside the interval $(0, 1/d)$, yields an upper bound for $p_c$. This is the content of Theorem 3.1 therein, and it is a fact that can also be concluded from (11) above. The problem is that this yields a bound that depends on $k$, and as they observe, it is not obvious whether this sequence of upper bounds is decreasing, so they cannot make the limiting procedure at this step. Instead, they consider a sequence $v_k \leq u_k$ and work on the asymptotics of the sequence of solutions of $d^k v_k$ as a function of $r$, or, equivalently, of $p$ (see Lemmas 4.1, 4.2 and 4.3 therein). It is interesting how this approach differs from ours. Formally, what Lebensztayn et al. [9] obtained is that (see Display (4.2) therein)

$$u_k = r^k \prod_{i=1}^{k-1} (1 - r^i) + \sum_{j=1}^{k-1} r^{k-j} u_j \prod_{l=1}^{k-j-1} (1 - r^l). \quad (17)$$

At this point what they decided to do, instead of studying the asymptotic behavior in $k$, is to take

$$v_k = r^k \prod_{i=1}^{k-1} (1 - r) + \sum_{j=1}^{k-1} r^{k-j} u_j \prod_{l=1}^{k-j-1} (1 - r) = r^k (1 - r)^{k-1} + \sum_{i=1}^{k-1} r^{k-j} u_j (1 - r)^{k-j-1}$$

which clearly satisfies $v_k \leq u_k$. Observe that this amounts simply to substitute $r^j$ by $r$ into the products of (17). They can then work asymptotically with the solutions of $d^k v_k = 1$ and obtain their bound $r_c \leq 1 - \sqrt{(d – 1)/d}$, which is actually the solution of $dr(2 – r) = 1$ for $r \in (0, 1/d)$. It was also remarked by Lebensztayn et al. [9] that substituting the $r^j$ by $r^2$ for any $k \geq 2$ would naturally yield tighter, yet more complicated bounds for $r_c$. Indeed, they state that $r_c$ would be the root $\tilde{r}_U$ of $\bar{U}(r) := dr^4 - d(d + 1)r^3 + 2dr - 1$ in $(0, 1/d)$. It is precisely what Lebensztayn and Utria [10] used, yielding yet another refinement of the bound, although very complicated (see Definition 2.1 and Theorem 2.2 therein).

On the contrary, what we do (and what was done by Gallo and Rodriguez [5]) in Step 2 is that we directly work asymptotically on (17) using renewal theory, and this yields Theorem 2. In particular, notice that the bounds in [9, 10] are direct consequences of Theorem 2 as well: indeed, substituting, in the products, $r^j$ and $r^l$ by $r$, yields the polynomial $dr(2 – r) = 1$ used in [9], and substituting by $r^j$ and $r^l$ by $r^2$ (for $i, l \geq 2$) yields the polynomial $\bar{U}(r) := dr^4 - d(d + 1)r^3 + 2dr - 1$ used in [10].

Our upper bound (2) is simpler than the one of Lebensztayn and Utria [10]. To see that it is also tighter, it is enough to notice (we did this using Mathematica) that $\bar{U}(r_U) < 0$ and that $\bar{U}$ is monotonically increasing on $(0, 1/d)$ (see Lemma 4 for the definition of $r_U$).

### 4.3 The Lower Bounds and their Proofs

The original lower bound of Alves et al. [1] was based on a simple coupling with a one-type branching process in which each particle could have 0 offspring with probability $1 – p$, 1 offspring with probability $p/(d + 1)$ (for frogs coming back) and 2 offsprings with probability $pd/(d + 1)$. In other words, their coupling took into account the fact that except for the frog initially at the root, an active frog necessarily has at least one adjacent vertex which has already been visited. What we noticed in the present paper is that after a certain number of steps (almost-surely finite), any activated frog which is not at the tip of the visited cluster has at least two visited neighboring sites. To take this into account, we needed to consider a two-type branching process.
The frog model can be seen as an “infinite types” branching process, and it is naturally possible to improve further our reasoning. The idea would be to compare the frog model with branching processes having more and more types. For instance, a simple modification of the multi-type branching process used by Hoffman et al. [8] in the proof of their Proposition 19 could be used as well in our setting. This amounts essentially to adding types corresponding to keeping track of two simultaneously activated particles. Doing so, we are able, for instance, to prove that $p_c(2) < p_c(3)$, but it is impossible to get explicit expression holding for any $d$ based on so many types. More generally, we tried such a method up to a certain level and obtained slightly tighter lower bounds with very involved expressions, and we preferred to keep it simple at the cost of precision.

### 4.4 Concluding Remark

Tighter bounds on $\hat{p}_c$ can be obtained from Theorem 2, this is a matter of root approximation for the power series of Theorem 2. However, as pointed out by Table 1, such improvements would be almost insignificant compared to the distance to the lower bounds of $p_c$ since we are limited by the lower bound on $\hat{p}_c$, already very close to the upper bound. This is also made clear by Lemma 4 which yield, for large $d$’s,

$$r_c(d) = \frac{1}{2d} + \frac{1}{16d^2} + \frac{5}{128d^3} + O\left(\frac{1}{d^4}\right)$$

and therefore, together with Display (15),

$$\hat{p}_c(d) = \frac{1}{2} + \frac{7}{16d} - \frac{5}{128d^2} + O\left(\frac{1}{d^3}\right).$$

For this reason, the Introduction mentioned that the upper bound cannot be further improved based on the comparison with the oriented model. Future works should either improve the lower bound or find another approach to get upper bounds for $p_c$. 

---

**Table 1** Lower bound (LB) and upper bound (UB) of $p_c$ from Theorem 1 and lower bound on $\hat{p}_c$ from Lemma 4

| $d$ | LB on $p_c$ | LB on $\hat{p}_c$ | UB on $\hat{p}_c$ and $p_c$ |
|-----|-------------|-----------------|-----------------------------|
| 2   | 0.6261364   | 0.7103674       | 0.7137989                   |
| 3   | 0.5835921   | 0.6419859       | 0.6428580                   |
| 4   | 0.5625890   | 0.6071563       | 0.6074957                   |
| 5   | 0.5500385   | 0.5860557       | 0.5862210                   |
| 6   | 0.5416859   | 0.5719015       | 0.5719940                   |
| 7   | 0.5357250   | 0.5617475       | 0.5618043                   |
| 8   | 0.5312564   | 0.5541074       | 0.5541448                   |
| 9   | 0.5277818   | 0.5481503       | 0.5481761                   |
| 10  | 0.5250027   | 0.5433751       | 0.5433937                   |
| 20  | 0.5125001   | 0.5217793       | 0.5217815                   |
| 50  | 0.5050000   | 0.5087345       | 0.5087346                   |
| 100 | 0.5025000   | 0.5043711       | 0.5043711                   |
5 Appendix: Root Approximations

As promised, we now prove Lemma 4.

Proof For the lower bound, we use the Newton-Raphson method (see for instance [7, Sect. 10.11]) for root approximation of \( L(r) = -r^3 + d^3 + d^2 r^3 - 2 dr + 1 \). The second derivative of \( L(r) \) is positive for all \( r \in (0, 1/d) \), so \( L(r) \) is convex on this interval. Thus, the approximation calculated from the Newton-Raphson method will be smaller than the root \( r \).

Recall that the Newton-Raphson iterative method is started from some value \( t_0 \) and for any \( n \geq 1 \)

\[
t_n = t_{n-1} - \frac{L(t_{n-1})}{L'(t_{n-1})}.
\]

Starting with \( t_0 = 0 \) and iterating two times, we get after some algebraic manipulations

\[
t_2 = \frac{5 - 8d - 16d^2 + 64d^3}{12d - 20d^2 - 48d^3 + 128d^4}.
\]

This concludes the proof of the lower bound.

For the upper bound, the proof of the Newton-Raphson method is much longer so we will proceed simply proving that \( r_U \geq \bar{r} \), which is faster. We proceed in two steps: we first show that the polynomial \( U \) is monotonically decreasing in \( r \) around the root of interest, and secondly, that \( U(r_U) < 0 \) for any \( d \geq 2 \).

For the first step, notice that

\[
U'(r) < 0 \iff 5d^2 r^4 + 3dr^2 - 2 < 0,
\]

This is equivalent to \( r < \sqrt{1/(5d)} \). Thus it only remains to show that \( r_U < \sqrt{1/(5d)} \). To see this, we come back to the proof of Proposition 3 and notice that

\[
(1 - r)^{k-1} \leq \prod_{i=1}^{k-1} (1 - r^i)
\]

leads to an intermediary polynomial \( f_{\inf}(r) \leq dr(2 - r) \leq \sum_{k \geq 1} d^k r^k \prod_{i=1}^{k-1} (1 - r^i) \). Thus \( dr_U(2 - r_U) \leq 1 \), meaning that indeed \( r_U \leq 1 - \sqrt{(d-1)/d} \) which is strictly smaller than \( \sqrt{1/(5d)} \) for any \( d \geq 2 \).

To prove the second step, we used the software Mathematica to write \( U(r_U(d)) < 0 \) as

\[
-\frac{1}{537824 d^{12} (8d - 5)^5} \bar{U}(d) < 0
\]

where

\[
\bar{U}(d) = 211441664 d^{14} - 801511424 d^{13} + 988904672 d^{12} - 496642048 d^{11} + 72342816 d^{10} + 14993216 d^9 - 2579360 d^8 - 918064 d^7 + 203840 d^6 + 26460 d^5 - 7840 d^4 - 280 d^3 + 140 d^2 - 1.
\]

We are done if we prove that \( \bar{U}(d) > 0 \) for any \( d \geq 2 \). Using Cauchy bounds, we know that the largest root of this polynomial is smaller or equal to (for \( i = 0, \ldots, 14 \) we write \( a_i \) for the coefficient of order \( i \))

\[
m := 1 + \max \left\{ \frac{a_{13}}{a_{14}}, \ldots, \frac{a_0}{a_{14}} \right\} = 1 + \frac{988904672}{211441664} < 6.
\]

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The proof is concluded noticing that for \( d = 2, \ldots, 6 \), \( \bar{U}(d) > 0 \).

\[ \square \]

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**Declarations**

**Conflict of interest** The authors declare no conflict of interest.

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