Euler class groups and motivic stable cohomotopy

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Abstract

We study motivic cohomotopy sets in algebraic geometry using the Morel-Voevodsky $\mathbb{A}^1$-homotopy category: these sets are defined in terms of maps from a smooth scheme to a motivic sphere. Following Borsuk, we show that in the presence of suitable dimension hypotheses on the source, our motivic cohomotopy sets can be equipped with abelian group structures. We then explore links between these motivic cohomotopy groups, Euler class groups à la Nori–Bhatwadekar–Sridharan and Chow-Witt groups.

Using these links, we show that, at least for $k$ an infinite field having characteristic unequal to 2, the Euler class group of codimension $d$ cycles on a smooth affine $k$-variety of dimension $d$ coincides with the codimension $d$ Chow-Witt group. As a byproduct, we describe the Chow group of zero cycles on a smooth affine $k$-scheme as the quotient of the free abelian group on zero cycles by the subgroup generated by reduced complete intersection ideals; this answers a question of S. Bhatwadekar and R. Sridharan.

Contents

1 Motivic stable cohomotopy 5
   1.1 Abelian group structures on mapping sets ............................................... 5
   1.2 Motivic cohomotopy sets made concrete ................................................. 11

2 Group structures and naive $\mathbb{A}^1$-homotopy classes 14
   2.1 Naive homotopies of maps to quadrics .................................................. 14
   2.2 A geometric description of the composition on cohomotopy groups ............. 16

3 Segre classes and Euler class groups 22
   3.1 Euler class groups and the Segre class homomorphism .............................. 22
   3.2 Euler class, Chow-Witt and Chow groups ............................................... 26

Introduction

Suppose $k$ is a field, and $X$ is a smooth affine $k$-scheme. The goal of this paper is to establish concrete connections between “obstruction groups” attached to $X$ in the sense of M. Nori–S. Bhatwadekar–R. Sridharan (e.g., Euler class or weak Euler groups) and “motivic groups” attached to $X$ (e.g., Chow or Chow-Witt groups). In brief, we will show that (i) obstruction groups can be thought of as “cohomotopy groups” and (ii) using an algebro-geometric analog of the classical

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comparison between cohomotopy and cohomology groups, identify obstruction groups with motivic
groups in a number of cases.

We begin by explaining a concrete consequence of our techniques. Let $Z_0(X)$ be the group
of zero cycles on $X$, $CI_0(X) \subset Z_0(X)$ be the subgroup generated by zero-dimensional reduced
complete intersections in $X$, and set $E_0(X) := Z_0(X)/CI_0(X)$ (see Definition 3.2.2). Cycles in
the subgroup $CI_0(X)$ are known to be rationally equivalent to zero, and there is an induced
homomorphism $E_0(X) \to CH_0(X)$ [BS99, Lemma 2.5].

**Theorem 1** (See Theorem 3.2.4). If $k$ is an infinite (perfect) field having characteristic unequal to 2,
then for any smooth affine $k$-scheme $X = \text{Spec } R$ of dimension $d \geq 2$ the map $E_0(X) \to CH_0(X)$
is an isomorphism.

**Remark 2.** The idea that $CH_0(X)$ should be related to complete intersection subvarieties goes
back to the work of M. Pavaman Murthy and R. Swan in the 1970s [MS76, Theorem 2] (see also
[Wei84] and the references therein). The question of whether the homomorphism in Theorem 1 is an
isomorphism was posed explicitly by S. Bhatwadekar–R. Sridharan [BS99, Remark 3.13] (see also
the survey of Murthy [Mur99, Question 5.3]). That question was already known to have a positive
answer if: (i) $k$ is algebraically closed [Mur99, Theorem 5.2], (ii) if $k$ is the field of real numbers
[BS99, Theorem 5.5], or (iii) if $\dim X \leq 2$ (unpublished work of Bhatwadekar).

As will be clear, our approach differs completely from those studied previously. To explain our
techniques, we begin by recalling some classical homotopy theory. Borsuk showed [Bor36] that, if
$M$ is a manifold of dimension $d \leq 2n - 2$, the set of homotopy classes of maps $[M, S^n]$ admits a
(functional) abelian group structure; this set is called the $n$-th cohomotopy group of $M$. The
Hopf classification theorem [Hop33] states that if $\dim M = n$, then the group $[M, S^n]$ coincides
with the cohomology group $H^n(M, \mathbb{Z})$; the isomorphism is induced by a (dual) Hurewicz map
$[M, S^n] \to H^n(M, \mathbb{Z})$ [Spa49].

More generally Borsuk showed that if $X$ was any $(n - 1)$-connected space, then the set $[M, X]$ admits an abelian group structure, functorially in $X$. The (dual) Hurewicz map is then induced by
the map $S^n \to K(\mathbb{Z}, n)$ appearing in the first stage of the Postnikov tower of $S^n$. Hopf’s result
can then be deduced by an obstruction theory argument. Via the Freudenthal suspension theorem,
one may view Borsuk’s group structure as a “stable” phenomenon in the sense of stable homotopy
theory and thus view the cohomotopy groups as part of a cohomology theory.

The idea of algebro-geometric cohomotopy groups goes back (at least) to van der Kallen’s group
law on orbit sets of unimodular rows [vdK83]. We work in the setting of the Morel-Voevodsky $\mathbb{A}^1$-
homotopy category [MV99] in order to have access to basic homotopic constructions. Let $Q_{2n}$ be
the even-dimensional smooth affine quadric in $\mathbb{A}^{2n+1}$ given by the equation $\sum_{i=1}^{n} x_i y_i = z(z+1)$.
We observed in [ADF14, Theorem 2.2.5] that $Q_{2n}$ is a sphere from the standpoint of the Morel-
Voevodsky $\mathbb{A}^1$-homotopy theory [MV99]. As a consequence, if $X$ is a smooth scheme, by analogy
with the ideas of Borsuk, we will call the set $[X, Q_{2n}]_{\mathbb{A}^1}$ of morphisms in the $\mathbb{A}^1$-homotopy category
a *motivic cohomotopy set* (see Definition 1.2.1).

Our goals are (i) to equip $[X, Q_{2n}]$ with a functorial abelian group structure, (ii) to describe
$[X, Q_{2n}]_{\mathbb{A}^1}$ in terms of generators and relations, and (iii) to study analogs of the Hurewicz homomorphism and Hopf classification theorem. Write $CH^n(X)$ for the Chow-Witt groups defined by
J. Barge–F. Morel [BM00] and studied in detail in [Fas08]. If $X$ has dimension $d \leq 2n - 2$, write
Theorem 3 (See Theorems 1.2.7, 3.1.9 and 3.2.1). Suppose $k$ is an infinite perfect field having characteristic unequal to 2. Suppose $X$ is a smooth affine $k$-scheme of dimension $d$. For any integer $n \geq 2$, if $d \leq 2n - 2$, the following statements hold:

1. the set $[X, Q_{2n}]_{A^1}$ has a functorial abelian group structure;
2. there is a functorial “Segre class” homomorphism $E^n(X) \to [X, Q_{2n}]_{A^1}$, which is an isomorphism;
3. there is a functorial “Hurewicz” homomorphism $[X, Q_{2n}]_{A^1} \to \overline{CH}^n(X)$, which is an isomorphism if $d \leq n$;

In particular, if $X$ is a smooth affine $k$-scheme of dimension $d$, then there is a functorial isomorphism $E^d(X) \sim \overline{CH}^d(X)$.

Remark 4. The final statement of Theorem 3 answers in the affirmative an old question regarding the comparison between Euler class groups and Chow-Witt groups (see [BM00], Remarque 2.4) and also [Mor12, Remark 1.33(2)]). Moreover, our construction clarifies the functorial and cohomological properties (e.g., pullbacks, Mayer–Vietoris type sequences, products) of Euler class groups developed in [MY10] and [MY12]; see Remark 3.1.10 for more details. See Remark 1.1.5 for more detailed discussion of assumptions on the base.

Section 1 is devoted to equipping the set $[X, Q_{2n}]_{A^1}$ with a functorial abelian group structure and studying its properties in detail. The existence of a functorial abelian group structure on this set of homotopy classes of maps is largely formal—at least given $A^1$-analogs of classical connectivity results. The necessary connectivity results follow from F. Morel’s unstable $A^1$-connectivity theorem and his $A^1$-analog of Freudenthal’s suspension classical suspension theorem [Mor12, §6].

We give two equivalent constructions of the abelian group structure; the first, using the aforementioned suspension theorem, is useful for analyzing various functorial properties of the group structure, and the second, modeled on Borsuk’s original construction, is useful for giving a geometric interpretation of the composition. In particular, we observe that the abelian group structure on $[X, Q_{2n}]_{A^1}$ arises via an identification with stable cohomotopy groups. Since stable cohomotopy is a ring cohomology theory, we obtain Mayer–Vietoris-type exact sequences and product structures on these groups.

Results of Morel imply that the first non-trivial stage of the Postnikov tower for $Q_{2n}$ corresponds to a morphism $Q_{2n} \to K(K_{n}^{MW}, n)$ where $K_{n}^{MW}$ is Morel’s unramified Milnor-Witt $K$-theory sheaf [Mor12, Chapter 3] and $K(K_{n}^{MW}, n)$ is an Eilenberg–Mac Lane space representing sheaf cohomology. Since Chow-Witt groups are, essentially by definition, identified with $H^n(X, K_{n}^{MW})$, the functoriality of our construction yields the Hurewicz homomorphism in the statement and techniques of obstruction theory (as studied, e.g., in [AF14b] or [AF15]) imply the isomorphism statement.

In Section 2 we provide a concrete description of the abelian group structure on $[X, Q_{2n}]_{A^1}$; this uses two tools. First, we appeal to the affine representability results of [AHW15]; these results allow us to identify the abstract set $[X, Q_{2n}]_{A^1}$ in terms of “naive” $A^1$-homotopy classes of
mappings of \( k \)-schemes \( X \to Q_{2n} \). Then, we use the relationship between naive \( \mathbb{A}^1 \)-homotopy classes of morphisms with target \( Q_{2n} \) and complete intersection ideals studied in \([\text{Fas}15]\). In short, morphisms \( f : \text{Spec} R \to Q_{2n} \) determine (non-uniquely) pairs \((I, \omega_I)\), where \( I \subset R \) is an ideal, and \( \omega_I : R/I^{2n} \to I/I^2 \) is a surjection; the naive \( \mathbb{A}^1 \)-homotopy class of \( f \) essentially only depends on \( I \). Combining these ideas, we give an explicit “ideal-theoretic” description of the product in Theorem 2.2.9.

Finally, Section 3 studies applications of the above ideas. Using our concrete description of the product in cohomotopy, the existence of the “Segre class” homomorphism is straightforward and described in Section 3 after some recollections on Euler class groups (we slightly recast the definition so it is more natural in our context). To prove injectivity of the Segre class homomorphism depends on \([\text{Fas}15, \text{Theorem 3.2.7}]\), from which the second author deduced Murthy’s complete intersection conjecture. Surjectivity in the case of interest follows from a moving lemma. Finally, Theorem 1 is deduced from the comparison of Euler class groups and Chow-Witt groups by appeal to results of \([\text{DAZ15}] \) or \([\text{vdK15}]\).

**Notation/Preliminaries**

In this paper, the word ring will mean always mean commutative unital ring. Let \( R \) be a ring and let \( a = (a_1, \ldots, a_m) \in R^m \). We write \( \langle a \rangle \subset R \) for the ideal generated by \( a_1, \ldots, a_m \). If \( I \) is a finitely generated ideal, we write \( ht(I) \) for the height of \( I \).

Let \( Q_{2n-1} \) be the smooth quadric in \( \mathbb{A}^2 \) defined by the equation \( \sum_{i=1}^n x_i y_i = 1 \). Let \( Q_{2n} \) be the smooth quadric in \( \mathbb{A}^2 \) defined by the equation \( \sum_{i=1}^n x_i y_i = z(1 - z) \). This quadric is isomorphic to the quadric of the same name considered in \([\text{ADF14}]\) (change variables \( x_i \mapsto -x_i \), and \( z \mapsto -z \) for the isomorphism). Consider the scheme \( Q_{2n} \) as pointed by the class of \( x_1 = \cdots = x_n = y_1 = \cdots = y_n = 0, z = 1 \).

Fix a base field \( k \) and write \( \text{Sm}_k \) for the subcategory of schemes over \( \text{Spec} k \) that are separated, smooth and have finite type over \( \text{Spec} k \). We consider the categories \( \text{Spck} \) and \( \text{Spck}_\ast \) of simplicial and pointed simplicial presheaves on \( \text{Sm}_k \); objects of these categories will be called (pointed) \( k \)-spaces; if \( k \) is clear from context, we will simply call objects of this category (pointed) spaces.

We equip the category of (pointed) simplicial presheaves on \( \text{Sm}_k \) with its usual injective Nisnevich local model structure \([\text{Jar15, \S 5.1}]\); the associated homotopy category, denoted \( \mathcal{H}^\text{Nis}(k) \) (\( \mathcal{H}^\ast(k) \)), will be referred to as the (pointed) simplicial homotopy category. If \((\mathcal{X}, x)\) and \((\mathcal{Y}, y)\) are pointed spaces, we set \( \text{Hom}_{\mathcal{H}^\text{Nis}(k)}(\mathcal{X}, \mathcal{Y}) \) and \([\mathcal{X}, x], (\mathcal{Y}, y)]_{\mathcal{H}^\ast(k)} := \text{Hom}_{\mathcal{H}^\ast(k)}(\mathcal{X}, \mathcal{Y}) \). An element of \([\mathcal{X}, \mathcal{Y}]_{\mathcal{H}^\ast(k)} \) will also be called a free simplicial homotopy class of maps.

The category of (pointed) simplicial presheaves can be further localized to obtain the Morel-Voevodsky \( \mathbb{A}^1 \)-homotopy category \( \mathcal{H}(k) \) (\( \mathcal{H}_\ast(k) \)); this localization is a left Bousfield localization of \( \text{Spck} \) \([\text{MV99}]\). In particular, there is an endo-functor \( L_{\mathbb{A}^1} \) of the category of (pointed) simplicial presheaves, together with a natural transformation \( \theta : \text{Id} \to L_{\mathbb{A}^1} \) such that if \( \mathcal{Y} \) is a space, then \( \mathcal{Y} \to L_{\mathbb{A}^1} \mathcal{Y} \) is a cofibration and \( \mathbb{A}^1 \)-weak equivalence and \( L_{\mathbb{A}^1} \mathcal{Y} \) is simplicially fibrant and \( \mathbb{A}^1 \)-local. We refer the reader to \([\text{AWW15, Proposition 2.2.1}]\) for a convenient summary of properties of the \( \mathbb{A}^1 \)-localization functor and note in passing that \( L_{\mathbb{A}^1} \) commutes with the formation of finite products. We set \( [\mathcal{X}, \mathcal{Y}]_{\mathbb{A}^1} := \text{Hom}_{\mathcal{H}(k)}(\mathcal{X}, \mathcal{Y}) \) and \([\mathcal{X}, x], (\mathcal{Y}, y)]_{\mathbb{A}^1} := \text{Hom}_{\mathcal{H}(k)}(\mathcal{X}, \mathcal{Y}) \). Analogous to the terminology used above, an element of \([\mathcal{X}, \mathcal{Y}]_{\mathbb{A}^1} \) will be called a free \( \mathbb{A}^1 \)-homotopy class of maps.
maps.

We begin by recalling some notation. Write $S^1$ for the simplicial circle and $S^i$ for the simplicial $i$-sphere, i.e., the $i$-fold smash product of $S^1$ with itself. More generally, set $S^{i+j} := S^i \wedge G_m^j$. If $\mathcal{Y}$ is any pointed space, then we write $\Sigma^n \mathcal{Y}$ for the smash product $S^n \wedge \mathcal{Y}$ and $\Omega^n \mathcal{Y}$ for the derived $n$-fold loop space i.e., $\text{Hom}_\ast(S^n, \mathcal{Y})$, where $(-)^f$ is a functorial fibrant replacement functor. Looping and suspension are adjoint, and the unit map of the loop-suspension adjunction yields a functorial morphism $\mathcal{Y} \to \Omega^n \Sigma^n \mathcal{Y}$ for any integer $i$.

A pointed space $(\mathcal{X}, x)$ will be called simplicially $m$-connected if its stalks are all $m$-connected simplicial sets. Similarly, we will say that $\mathcal{X}$ is $A_1$-$m$-connected, if $L_{A_1} \mathcal{X}$ is simplicially-$m$-connected. Finally, we freely use the homotopy theoretic formulation of sheaf cohomology; we refer the reader to [Jar15, Part III] for more details.

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1 Motivic stable cohomotopy

In this section, we establish an analog in the $A_1$-homotopy category of a result of Borsuk: if $U \in \text{Sm}_k$ has Krull dimension $\leq 2n - 2$ and $(\mathcal{X}, x)$ is a pointed $A_1$-$(n - 1)$-connected space, then, if $n \geq 2$, the set $[U, \mathcal{X}]_{A_1}$ of free $A_1$-homotopy classes of maps admits an abelian group structure, functorial in both inputs. Section 1.1 studies the properties of the construction in this generality. In Section 1.2 we specialize these results to the cases of interest. The main results used from this section in subsequent sections are Theorems 1.2.4 and 1.2.7.

1.1 Abelian group structures on mapping sets

In this section, we give two equivalent constructions of a functorial abelian group structure on sets of homotopy classes of maps. The second is essentially Borsuk’s classical construction, transplanted in our context; this version has the benefit that it can be made very explicit and will be used to derive concrete formulas for the composition in special cases (see Proposition 1.1.7). The first construction we present is a modernized version of Borsuk’s construction (see Proposition 1.1.4); this version has the benefit of rendering various properties of the group structure entirely formal. The equivalence of the two constructions is established in Proposition 1.1.9 and various functorial properties, including the existence of the Hurewicz homomorphism, are studied in Proposition 1.1.10.

$A_1$-cohomological dimension

We begin by introducing a useful notion of cohomological dimension. To make the definition, we need to recall two pieces of terminology from [Mor12]. Recall that a Nisnevich sheaf of groups $G$ is
called \textit{strongly }$\mathbb{A}^1$-\textit{invariant} if the simplicial classifying space $BG$ is $\mathbb{A}^1$-local. A Nisnevich sheaf of abelian groups $G$ is called \textit{strictly }$\mathbb{A}^1$-\textit{invariant} if the cohomology presheaves $U \mapsto H^i_{\text{Nis}}(U, A)$ are homotopy invariant for any $i \geq 0$. If $A$ is a strictly $\mathbb{A}^1$-invariant sheaf of groups, then one can show that the Eilenberg-Mac Lane space $K(A, i)$ is $\mathbb{A}^1$-local for every $i \geq 0$. If $k$ is infinite and perfect, Morel showed in [Mor12, Theorem 5.46] that strongly $\mathbb{A}^1$-invariant sheaves of abelian groups are already strictly $\mathbb{A}^1$-invariant.

\textbf{Definition 1.1.1.} If $X \in \text{Sm}_k$, then say that $X$ has $\mathbb{A}^1$-cohomological dimension $\leq d$ if, for any strictly $\mathbb{A}^1$-invariant sheaf $B$, $H^i_{\text{Nis}}(X, B)$ vanishes for $i > d$. Likewise, say $X$ has $\mathbb{A}^1$-cohomological dimension $d$ (write $d = cd_{\mathbb{A}^1}(X)$) if $d$ is the smallest integer such that $X$ has $\mathbb{A}^1$-cohomological dimension $\leq d$.

\textbf{Example 1.1.2.} If $X \in \text{Sm}_k$ is isomorphic in $\mathscr{H}(k)$ to a smooth scheme $Y$ that has Krull dimension $d$, then $X$ has $\mathbb{A}^1$-cohomological dimension $\leq d$. For instance, the scheme $Q_{2n-1}$ has dimension $2n-1$, but $cd_{\mathbb{A}^1}(Q_{2n-1}) \leq n$ since $Q_{2n-1}$ is $\mathbb{A}^1$-weakly equivalent to $\mathbb{A}^n \setminus 0$. By [ADF14, Theorem 2.2.5], one knows that $Q_{2n}$ is $\mathbb{A}^1$-weakly equivalent to $\Sigma^1 \mathbb{A}^n \setminus 0$. By the suspension isomorphism for cohomology, it follows that $cd_{\mathbb{A}^1}(Q_{2n}) \leq n + 1$.

\textbf{Lemma 1.1.3.} If $k$ is an infinite perfect field, and $n \geq 2$ is an integer, then $cd_{\mathbb{A}^1}(Q_{2n-1}) = n - 1$ while $cd_{\mathbb{A}^1}(Q_{2n}) = n$.

\textbf{Proof.} The first statement then follows from [AF14a, Lemma 4.5] (appeal to this result imposes the hypothesis on $k$), via the $\mathbb{A}^1$-weak equivalence $Q_{2n-1} \rightarrow \mathbb{A}^n \setminus 0$. The second statement follows from the first using the suspension isomorphism and [ADF14, Theorem 2.2.5], which implies $Q_{2n} \cong \Sigma^1 \mathbb{A}^n \setminus 0$. \hfill $\square$

\textbf{Group structures on mapping sets: a modern approach}

For any integer $i \geq 1$, the space $S^i$ has the structure of an $h$-co-group [Ark11, Definition 2.2.7]. This structure induces functorial $h$-cogroup structures on $\Sigma^i \mathscr{Y}$ for any pointed space $\mathscr{Y}$. By duality, $\Omega^i \mathscr{Y}$ then carries a functorial $h$-group structure [Ark11, Definition 2.2.1]; if $i \geq 2$ it has a functorial structure of homotopy commutative $h$-group. In particular, for any $U \in \text{Sm}_S$, the set of pointed maps $[U, \Omega^i \mathscr{Y}]$ has the structure of a group, functorially in both inputs. Now, we explain how to use the Freudenthal suspension theorem to transport this group structure to one on the set of free homotopy classes of maps $[U, \mathscr{Y}]_{\mathbb{A}^1}$, at least under suitable hypotheses on the Krull dimension of $U$ and on the connectivity of $\mathscr{Y}$.

\textbf{Proposition 1.1.4.} Fix an integer $n \geq 2$ and let $k$ be an infinite (perfect) field. If $\mathscr{X}$ is a pointed $\mathbb{A}^1-(n-1)$-connected space and $U \in \text{Sm}_k$ has $\mathbb{A}^1$-cohomological dimension $d \leq 2n - 2$, then for any integer $i \geq 1$ the map

$$[U, \mathscr{X}]_{\mathbb{A}^1} \cong [U_+, (\mathscr{X}, x)]_{\mathbb{A}^1} \rightarrow [U_+, \Omega^i \Sigma^i (\mathscr{X}, x)]_{\mathbb{A}^1}$$

induced by the pointed map $(\mathscr{X}, x) \rightarrow \Omega^i \Sigma^i (\mathscr{X}, x)$ is a bijection, functorial in both inputs. Moreover, the group structure on $[U, \mathscr{X}]_{\mathbb{A}^1}$ induced in this fashion is abelian, and functorial with respect to both inputs.
Remark 1.1.5. The assumption that \( k \) is perfect is not strictly necessary here; it appears implicitly by way of appeal to [Mor12, Theorem 5.46]. In fact, if we modify the hypothesis of the statement to simply assert that \( U \) has dimension \( d \), as opposed to \( \mathbb{A}^1 \)-cohomological dimension, then one only needs to know that homotopy sheaves are strongly \( \mathbb{A}^1 \)-invariant [Mor12, Theorem 6.1]. The latter fact implicitly appeals to [Mor12, Lemma 1.15], the published proof of which requires \( k \) is infinite. Thus, the credulous reader might believe this result holds without restriction on the base field. Henceforth, if hypotheses on the base field are placed in parentheses, it is possible to modify the proof to remove them.

Proof. Under the stated assumptions on \( k \), we may appeal to the results of Morel in [Mor12, §6]; see [AWW15, §2-3] for a more axiomatic treatment of these results. By assumption \( \mathcal{X} \) is \( \mathbb{A}^1 \)-connected, \( (n-1) \)-connected, for \( n \geq 2 \). In particular, it is \( \mathbb{A}^1 \)-connected. Under this hypothesis, one knows that the canonical map \( L_{\mathbb{A}^1} \Omega \Sigma \mathcal{X} \to \Omega L_{\mathbb{A}^1} \Sigma \mathcal{X} \) is a simplicial weak equivalence by [Mor12, Theorem 6.46]. For later use, observe also that arguing by induction, one concludes that the map \( L_{\mathbb{A}^1} \Omega^i \Sigma^j \mathcal{X} \to \Omega^i L_{\mathbb{A}^1} \Sigma^j \mathcal{X} \) is a simplicial weak equivalence.

Morel’s \( \mathbb{A}^1 \)-suspension theorem [Mor12, Theorem 6.61] states that, under the above hypotheses, the map \( \mathcal{X} \to \Omega L_{\mathbb{A}^1} \Sigma \mathcal{X} \) is an \( \mathbb{A}^1 \)-\((2n-2)\)-equivalence. Now, suppose we are given a pointed map \( f : U_+ \to \Omega L_{\mathbb{A}^1} \Sigma \mathcal{X} \). We analyze the Moore-Postnikov factorization of the map \( \mathcal{X} \to \Omega L_{\mathbb{A}^1} \Sigma \mathcal{X} \) [AF15, Theorem 6.1.1] (the discussion there is considerably simplified because \( \pi_1^{\mathbb{A}^1}(\Omega L_{\mathbb{A}^1} \Sigma \mathcal{X}) \) is, by assumption, trivial). Furthermore, the homotopy sheaves of \( \Omega L_{\mathbb{A}^1} \Sigma \mathcal{X} \) are all strictly \( \mathbb{A}^1 \)-invariant by [Mor12, Corollary 6.2]. Using the assumption on the \( \mathbb{A}^1 \)-cohomological dimension of \( U \) and appealing to the lifting procedure described on [AF15, p. 1055], one sees that \( f \) lifts uniquely up to \( \mathbb{A}^1 \)-homotopy to a map \( \tilde{f} : U_+ \to \mathcal{X} \). By [AF14a, Lemma 2.1], the connectivity hypothesis on \( \mathcal{X} \) guarantees that the map \( [U_+, \mathcal{X}]_{\mathbb{A}^1} \to [U, \mathcal{X}]_{\mathbb{A}^1} \) and the corresponding map with \( \mathcal{X} \) replaced by \( \Omega \Sigma \mathcal{X} \) are bijections, functorially in both inputs. By transport of structure, \( [U, \mathcal{X}]_{\mathbb{A}^1} \) inherits a group structure.

To see that the group structure on \( [U, \mathcal{X}]_{\mathbb{A}^1} \) described above is abelian is also straightforward. Indeed, since \( \mathcal{X} \) is \( \mathbb{A}^1 \)-\((n-1)\)-connected, \( L_{\mathbb{A}^1} \mathcal{X} \) is simplicially \((n-1)\)-connected, \( \Sigma L_{\mathbb{A}^1} \mathcal{X} \) is simplicially \( n \)-connected and thus by the unstable \( \mathbb{A}^1 \)-connectivity theorem [Mor12, Theorem 6.38] \( \Sigma L_{\mathbb{A}^1} \mathcal{X} \) is also \( \mathbb{A}^1 \)-\( n \)-connected. Since \( \mathcal{X} \to \Sigma L_{\mathbb{A}^1} \mathcal{X} \) is an \( \mathbb{A}^1 \)-weak equivalence, by [MV99, §2 Lemma 2.13], we conclude that \( \Sigma \mathcal{X} \) is \( \mathbb{A}^1 \)-\( n \)-connected. Therefore, arguing as in the previous paragraph, by the simplicial suspension theorem the map \( \Sigma \mathcal{X} \to \Omega \Sigma \Sigma \mathcal{X} \) is \( \mathbb{A}^1 \)-\( 2n \)-connected and the map \( \Omega \Sigma \Sigma \mathcal{X} \to \Omega^2 \Sigma^2 \mathcal{X} \) is \( \mathbb{A}^1 \)-\((2n-1)\)-connected. Since the map in the previous line is an \( h \)-map and \( \Omega^2 \Sigma^2 \mathcal{X} \) is homotopy commutative, for \( U \) as in the statement the induced map \( [U, \Omega \Sigma \mathcal{X}]_{\mathbb{A}^1} \to [U, \Omega^2 \Sigma^2 \mathcal{X}]_{\mathbb{A}^1} \) is an isomorphism of groups and thus the former is necessarily abelian; this also establishes the result for \( i = 2 \).

For \( i \geq 2 \), one proceeds inductively and shows that the map \( [U, \Omega^i \Sigma^i \mathcal{X}]_{\mathbb{A}^1} \to [U, \Omega^i \Sigma^i \mathcal{X}]_{\mathbb{A}^1} \) is always a bijection. \(\square\)

We now show that \( [U, \mathcal{X}]_{\mathbb{A}^1} \) can be identified with maps in a stable homotopy category. To formalize this, we work in the \( S^1 \)-stable \( \mathbb{A}^1 \)-homotopy category (see, for example, [Mor05, Definition 4.1.1]). If \( \mathcal{Y} \) is a pointed space, we write \( \Sigma^\infty \mathcal{Y} \) for the \( S^1 \)-suspension spectrum attached to \( \mathcal{Y} \). If \( E \) and \( E' \) are \( S^1 \)-spectra, we will abuse notation and write \( [E, E']_{\mathbb{A}^1} := \text{Hom}_{S^1_{\mathbb{A}^1}}(E, E') \). This
result says that $[U, \mathcal{X}]_{\mathbb{A}^1}$ can, under suitable hypotheses, be identified in terms of mappings in the $S^1$-stable homotopy category.

**Proposition 1.1.6.** Fix an integer $n \geq 2$, and suppose $\mathcal{X}$ is a pointed $\mathbb{A}^1$-$(n - 1)$-connected space. If $U \in Sm_S$ has $\mathbb{A}^1$-cohomological dimension $d \leq 2n - 2$, then the map

$$[U, \mathcal{X}]_{\mathbb{A}^1} \to [\Sigma^\infty U_+, \Sigma^\infty \mathcal{X}]_{\mathbb{A}^1}$$

is a bijection, functorial in both inputs.

**Proof.** This result is essentially a compactness argument, and we give a detailed outline of the proof; related results are established in [DI05, §9]. It will be helpful to use a slightly different model for the unstable and $S^1$-stable $\mathbb{A}^1$-homotopy categories, namely the motivic model structure of [DR03, Theorem 2.12]. Recall that a simplicial presheaf $\mathcal{F}$ is fibrant with respect to this model structure if for every smooth scheme $X$ the following conditions hold: (i) $\mathcal{F}(X)$ is a Kan complex, (ii) the projection $X \times \mathbb{A}^1 \to X$ induces a weak equivalence of simplicial sets $\mathcal{F}(X) \to \mathcal{F}(X \times \mathbb{A}^1)$, (iii) $\mathcal{F}$ satisfies Nisnevich excision, i.e., $\mathcal{F}$ takes Nisnevich distinguished squares to homotopy pullback squares and $\mathcal{F}(\emptyset)$ is contractible. The category $Spc_{k,\bullet}$ is a simplicial model category with the usual notion of simplicial mapping space $Map(\mathcal{F}, \mathcal{Y})$ [MV99, p. 47].

Likewise, write $Spt_k$ for the category of $S^1$-spectra of motivic spaces (this is constructed just as in [DR03, §2.2] but uses the simplicial circle instead of the Thom space of the trivial bundle; the fact that the threefold permutation on $S^1 \wedge S^1 \wedge S^1$ acts as the identity follows from the corresponding fact for simplicial sets). A spectrum $E$ is fibrant if and only if it is levelwise fibrant and an $\Omega$-spectrum. There are then standard simplicial Quillen adjunctions

$$\Sigma^\infty : Spc_{k,\bullet} \rightleftarrows Spt_k : \Omega^\infty$$

Now, let $(E_n)_{n \geq 0}$ be a level-wise fibrant replacement of $\Sigma^\infty \mathcal{X}$, i.e., $E_n$ is a fibrant replacement of $\Sigma^n \mathcal{X}$ and let $E$ be a fibrant replacement of $\Sigma^\infty \mathcal{X}$. By [DR03, Corollary 2.16], we conclude that filtered colimits in $Spc_{k,\bullet}$ preserve fibrant objects. From that observation, it follows that there is a simplicial weak equivalence of the form $\Omega^\infty E \cong \colim_n \Omega^n E_n$.

Now, if $\mathcal{F}$ is a pointed fibrant space, then there is an identification of unpointed mapping spaces $Map(U, \mathcal{F}) = \mathcal{F}(U)$. Thus, we conclude that there is an identification of pointed mapping spaces $Map(U_+, \mathcal{F}) = \mathcal{F}(U)$ as well. Since $U_+$ is $\omega$-compact in $Spc_{k,\bullet}$ we conclude that $Map(U_+, \colim_n \Omega^n E_n) \cong \colim_n Map(U_+, \Omega^n E_n)$. Combining this fact with the evident adjunctions, we see that there are a sequence of homotopy equivalences of Kan complexes of the form

$$Map(\Sigma^\infty U_+, \Sigma^\infty \mathcal{X}) \cong Map(\Sigma^\infty U_+, E)$$

$$\cong Map(U_+, \Omega^\infty E)$$

$$\cong Map(U_+, \colim_n \Omega^n E_n)$$

$$\cong \colim_n Map(U_+, \Omega^n E_n)$$

$$\cong \colim_n Map(\Sigma^n U_+, E_n)$$

Since $\pi_0$ preserves filtered colimits of simplicial sets, we conclude that

$$\colim_n [\Sigma^n U_+, \Sigma^\infty \mathcal{X}]_{\mathbb{A}^1} \cong [\Sigma^\infty U_+, \Sigma^\infty \mathcal{X}]_{\mathbb{A}^1}.$$
Now, by adjunction we conclude that there are functorial identifications \([U_+, \Omega^n \Sigma^n X]_{\mathbb{A}^1} \cong [\Sigma^n U_+, \Sigma^n X]_{\mathbb{A}^1}\) and using Proposition 1.1.4 we conclude that the transition maps in \(\operatorname{colim}_n [\Sigma^n U_+, \Sigma^n X]_{\mathbb{A}^1}\) are all bijections; this is precisely what we wanted to prove.

**Borsuk’s original construction**

In this section, we give the analog of Borsuk’s classical construction of the composition on cohomotopy. If \(\mathcal{Y}\) is a pointed space, we use the following notation: \(\Delta : \mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}\) is the diagonal map, \(\nabla : \mathcal{Y} \vee \mathcal{Y} \to \mathcal{Y}\) is the fold map, and \(\mathcal{Y} \vee \mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}\) is the canonical cofibration.

If \(\mathcal{X}\) is an \((n-1)\)-connected space, \(U \in \text{Sm}_k\), and \(f, g : U \to \mathcal{X}\) are morphisms, then we can contemplate the following diagram:

\[
\begin{array}{ccc}
\mathcal{X} \vee \mathcal{X} & \xrightarrow{\nabla} & \mathcal{X} \\
\downarrow & & \\
U & \xrightarrow{\Delta} & U \times U \xrightarrow{f \times g} \mathcal{X} \times \mathcal{X}.
\end{array}
\]

If we can lift the composite \((f \times g) \circ \Delta\) to a morphism \((f, g) : U \to \mathcal{X} \vee \mathcal{X}\), then composing with the fold map, we obtain a map \(\nabla(f, g)\) that we can think of as the product of \(f\) and \(g\). In general, there is an obstruction to producing such a lift, and even if a lift exists it needs not be unique. However, with suitable dimension restrictions imposed on \(U\), lifts exist and will be unique.

**Proposition 1.1.7.** Assume \(k\) is an infinite perfect field and \(n \geq 2\) is an integer. If \(\mathcal{X}\) is a pointed \(\mathbb{A}^1\)-(\(n-1\))-connected space and if \(U \in \text{Sm}_k\) has \(\mathbb{A}^1\)-cohomological dimension \(d \leq 2n - 2\), then there is a bijection

\([U, \mathcal{X} \vee \mathcal{X}]_{\mathbb{A}^1} \to [U, \mathcal{X} \times \mathcal{X}]_{\mathbb{A}^1}\)

functorial in both \(U\) and \(\mathcal{X}\).

**Proof.** It follows from [AWW15, Corollary 3.3.11] that the map \(\mathcal{X} \vee \mathcal{X} \to \mathcal{X} \times \mathcal{X}\) is an \(\mathbb{A}^1\)-(2\(n-2\))-equivalence and the argument is a straightforward obstruction theory argument completely parallel to that in Proposition 1.1.4.

**Definition 1.1.8.** Assume \(k\) is an infinite perfect field, \(n \geq 2\) is an integer, that \(\mathcal{X}\) is a pointed \(\mathbb{A}^1\)-(\(n-1\))-connected space and if \(U \in \text{Sm}_k\) has \(\mathbb{A}^1\)-cohomological dimension \(d \leq 2n - 2\). Given two elements \(f, g \in [U, \mathcal{X}]_{\mathbb{A}^1}\), we set \(\tau(f, g) := \nabla(f, g)\) where \((f, g) : U \to \mathcal{X} \vee \mathcal{X}\) is the unique lift of \((f \times g) \circ \Delta : U \to \mathcal{X} \times \mathcal{X}\) guaranteed to exist by Proposition 1.1.7.

**Comparing the composition operations**

The next result shows that when the composition operations of Propositions 1.1.4 and 1.1.7 are both defined, they coincide.

**Proposition 1.1.9.** Assume \(k\) is an infinite perfect field and fix an integer \(n \geq 2\). If \(\mathcal{X}\) is an \(\mathbb{A}^1\)-(\(n-1\))-connected space and \(U \in \text{Sm}_k\) has \(\mathbb{A}^1\)-cohomological dimension \(d \leq 2n - 2\), then given \(f, g \in [U, \mathcal{X}]_{\mathbb{A}^1}\), the class \(\tau(f, g)\) coincides with the product of \(f, g\) from Proposition 1.1.4.
Proof. If \( Y \) is any pointed \( h \)-space with multiplication \( m \), then the unit condition can be phrased as the existence of a homotopy commutative diagram of the form

\[
\begin{array}{ccc}
Y \vee Y & \xrightarrow{\nabla} & Y \\
\downarrow & & \downarrow \\
Y \times Y & \xrightarrow{m} & Y.
\end{array}
\]

We now apply this observation with \( Y = X \) and \( Y = \Omega \Sigma X \) where \( X \) is \( A^1 \)-\((n - 1)\)-connected and contemplate some obvious diagrams.

First, there is a commutative diagram of the form

\[
\begin{array}{ccc}
X \vee X & \xrightarrow{\nabla} & \Omega \Sigma X \vee \Omega \Sigma X \\
\downarrow & & \downarrow \\
X \times X & \xrightarrow{\Omega \Sigma X \times \Omega \Sigma X} & \Omega \Sigma X \times \Omega \Sigma X.
\end{array}
\]

Thus, if \( h \in [U, X \times X]_{A^1} \) lifts to \( \tilde{h} \in [U, X \vee X]_{A^1} \), then the induced class in \([U, \Omega \Sigma X \times X]_{A^1}\) lifts to \([U, \Omega \Sigma X \vee X]_{A^1}\) as well.

Now, the commutativity of Diagram 1.1 together with functoriality of the fold map yields a commutative diagram of the form:

\[
\begin{array}{ccc}
X \vee X & \xrightarrow{\nabla} & X \\
\downarrow & & \downarrow \\
\Omega \Sigma X \vee \Omega \Sigma X & \xrightarrow{\Omega \Sigma X \times \Omega \Sigma X} & \Omega \Sigma X.
\end{array}
\]

Given maps \( f, g \in [U, X]_{A^1} \), since the map \([U, X]_{A^1} \to [U, \Omega \Sigma X]_{A^1}\) is a bijection, the commutativity of the above diagram together with the lifting observation of the previous paragraph show that \( \tau(f, g) = m(f, g) \).

Further functoriality properties

If \( X \) is an \( A^1 \)-\((n - 1)\)-connected space, setting \( \pi = \pi^{A^1}_n(X) \), the first non-trivial layer of the \( A^1 \)-Postnikov tower provides an \( A^1 \)-homotopy class of maps \( X \to K(\pi, n) \). The space \( K(\pi, n) \) is also an \( A^1 \)-\((n - 1)\)-connected space. If \( U \in Sm_k \) has Krull dimension \( \leq 2n - 2 \), then the set \([U, K(\pi, n)]_{A^1}\) a priori admits two abelian group structures: one coming from Proposition 1.1.4 (i.e., induced by the \( h \)-group structure on \( \Omega \Sigma K(\pi, n) \)) and one from the \( h \)-space structure of Eilenberg-Mac Lane spaces. The stabilization map \( K(\pi, n) \to \Omega \Sigma K(\pi, n) \) is a map of \( h \)-groups, and thus in the range of dimensions under consideration, these two \( h \)-space structures coincide. The map \( X \to K(\pi, n) \) yields a homomorphism

\[
[U, X]_{A^1} \rightarrow H^n(U, \pi)
\]
arising from the functoriality clause of Proposition 1.1.4; this homomorphism is often referred to as a Hurewicz homomorphism. The map \( \mathcal{X} \to K(\pi, n) \) corresponds to a canonically defined cohomology class \( \alpha_\mathcal{X} \in H^n(\mathcal{X}, \pi) \) (in fact, the coskeletal description of the Postnikov tower [DK84, §1.2(vi)] provides a canonical representing cocycle) that we will refer to as a fundamental class.

**Proposition 1.1.10.** Assume \( k \) is an infinite perfect field. Suppose \( X \) is a pointed \( A^1 \)-\((n - 1)\)-connected space, \( \pi := \pi_{A^1}^n(X) \), and \( U \in \text{Sm}_k \) is a smooth scheme of \( A^1 \)-cohomological dimension \( d \leq 2n - 2 \).

1. The Hurewicz homomorphism

\[
[U, \mathcal{X}]_{A^1} \to H^n(U, \pi).
\]

is surjective if \( d \leq n + 1 \) and an isomorphism if \( d \leq n \).

2. If \( f \in [U, \mathcal{X}]_{A^1} \) is an \( A^1 \)-homotopy class of maps, and \( \alpha_\mathcal{X} \in H^n(\mathcal{X}, \pi) \) is the fundamental class, then the image of \( f \) under the Hurewicz homomorphism is \( f^* \alpha_\mathcal{X} \), where \( f^* : H^n(\mathcal{X}, \pi) \to H^n(U, \pi) \) is the pullback by \( f \).

**Proof.** For the first statement, consider the \( A^1 \)-fiber sequence \( X \langle n \rangle \to X \to K(\pi, n) \). The existence of a homomorphism as in the statement follows immediately from the functoriality assertion in Proposition 1.1.4 together with the identification \( H^n(U, \pi) \cong [U_+, K(\pi, n)]_{A^1} \). That the group structure coming from Proposition 1.1.4 coincides with the usual group structure on \( [U_+, K(\pi, n)]_{A^1} \) follows from the discussion before the statement. The only thing that needs to be checked is surjectivity. To this end, observe because \( \mathcal{X} \langle n \rangle \) is \( A^1 \)-\( n \)-connected, by the Blakers-Massey theorem [AF13] the cone \( \mathcal{C} \) of the map \( \mathcal{X} \to K(\pi, n) \) is at least \( A^1 \)-\((n + 1)\)-connected. The result then follows from the long exact sequence obtained by applying \([U_+, -]\) to the cofiber sequence \( \mathcal{X} \to K(\pi, n) \to \mathcal{C} \).

The second statement follows immediately from the definition of the Hurewicz map and the fundamental class. \( \square \)

### 1.2 Motivic cohomotopy sets made concrete

We now specialize the results of the previous section to the cases of interest in this paper, namely the quadrics \( Q_{2n-1} \) and \( Q_{2n} \). To this end, recall that there are isomorphisms in \( H_*^{\text{mot}}(k) \) of the form \( Q_{2n-1} \cong S^{2n-1,n} \) and \( Q_{2n} \cong S^{2n,n} \) (the latter using [ADF14, Theorem 2.2.5]). Thus, these quadrics are “motivic spheres.” In this context, we define motivic cohomotopy sets, observe in Theorem 1.2.4 that these group structures have concrete algebro-geometric interpretations, and then study the Hurewicz homomorphism in great detail (Theorem 1.2.7).

#### Motivic cohomotopy groups

The following definition is a motivic analog of the classical notion of cohomotopy (we have chosen indexing to agree with the indexing in motivic cohomology); by the observations just made, these cohomotopy sets can be identified as maps into suitable quadrics in certain cases. We refer the reader to the preliminaries for our conventions regarding spheres.
Definition 1.2.1 (Motivic cohomotopy). Assume $i, j$ are positive integers and $U \in \text{Sm}_S$. The unstable motivic cohomotopy sets of $U$ are defined by the formula:

$$\pi_{i,j}^U(U) := [U, S^{i,j}]_{\mathbb{A}^1}.$$ 

The stable motivic cohomotopy groups are defined by the formula:

$$\pi_{s,i,j}^U(U) := [\Sigma^\infty U, \Sigma^\infty S^{i,j}]_{\mathbb{A}^1}.$$ 

Theorem 1.2.2. Assume $k$ is an infinite perfect field. The following statements are true.

1. If $n \geq 2$, $i - j \geq n$, and $U \in \text{Sm}_k$ has $\mathbb{A}^1$-cohomological dimension $d \leq 2n - 2$, then the evident map $\pi_{i,j}^U(U) \rightarrow \pi_{s,i,j}^U(U)$ is a bijection.

2. If $j : W \rightarrow X$ is an open immersion and $\varphi : V \rightarrow X$ is an étale morphism such that the pair $(j, \varphi)$ give rises to a Nisnevich distinguished square, i.e., if $\varphi^{-1}(X \setminus W)_{\text{red}} \rightarrow (X \setminus W)_{\text{red}}$ is an isomorphism, then there is a Mayer-Vietoris long exact sequence of the form

$$\cdots \rightarrow \pi_{s,i,j}^U(X) \rightarrow \pi_{s,i,j}^U(W) \oplus \pi_{s,i,j}^U(V) \rightarrow \pi_{s,i,j}^U(W \times_X V) \rightarrow \pi_{s,i,j+1}^U(X) \rightarrow \cdots$$

Proof. Since $k$ is an infinite perfect field, and $i - j \geq n$, Morel’s unstable $\mathbb{A}^1$-connectivity theorem [Mor12, Theorem 6.38] implies that the space $S^{i,j}$ is at least $(n-1)$-connected. Bearing this in mind, the first statement is then an immediate consequence of Proposition 1.1.7.

The second statement follows from the existence of a Mayer-Vietoris distinguished triangle:

$$\Sigma^\infty(V \times_X U)_+ \rightarrow \Sigma^\infty(V \coprod U)_+ \rightarrow \Sigma^\infty X_+.$$ 

\[ \square \]

Remark 1.2.3. There are also evident external product map for motivic cohomotopy sets. Indeed, there is an $\mathbb{A}^1$-weak equivalence $Q_{2m} \wedge Q_{2n} \cong Q_{2(n+m)}$. Therefore, given morphisms $f : X \rightarrow Q_{2m}$ and $g : X \rightarrow Q_{2n}$, we can form the composite morphism

$$X_+ \xrightarrow{\delta} X_+ \wedge X_+ \xrightarrow{f \wedge g} Q_{2m} \wedge Q_{2n} \xrightarrow{\sim} Q_{2(n+m)}.$$ 

This composite defines a functorial morphism $\pi^{2m,m}(X) \times \pi^{2n,n}(X) \rightarrow \pi^{2(n+m),m+n}(X)$. The resulting composite depends on the chosen weak-equivalence $Q_{2m} \wedge Q_{2n} \cong Q_{2(n+m)}$ only up to isomorphism, and a straightforward computation of the $\mathbb{A}^1$-homotopy class of the map switching factors [AWW15, Proof of Theorem 4.4.1] shows that the resulting product is $(-\varepsilon)$-graded commutative.

Naive homotopy classes

If $R$ is a ring, write as usual $\Delta_R^\bullet$ for the cosimplicial affine space over $R$, i.e.,

$$\Delta_R^n := \text{Spec } R[t_0, \ldots, t_n]/(\sum_i t_i = 1).$$

If $Y$ is a pointed smooth scheme, then $\text{Sing}_{\mathbb{A}^1}Y$ is defined by the assignment $R \mapsto \text{Sing}_{\mathbb{A}^1}Y(R) = Y(\Delta_R^\bullet)$ [MV99, p. 87]. There is an associated functor from the category of rings to the category of
pointed sets sending a ring $R$ to the set of connected components $\pi_0(\text{Sing}^{A^1}_Y(R))$ pointed by the image of the base point. The following result connects the set of naive homotopy classes of maps to “true” $A^1$-homotopy classes of maps as studied in the previous section; these results allow us to interpret the cohomotopy groups above in “concrete” terms.

**Theorem 1.2.4** ([AHW15, Theorems 4.2.1 and 4.2.2.]). Assume $k$ is a field, $X = \text{Spec} R$ is a smooth affine $k$-scheme and $n \geq 0$ is an integer; if $n$ is even and $\geq 6$ assume further that $k$ is infinite and has characteristic different from 2. The map

$$\pi_0(\text{Sing}^{A^1}_YQ_n(R)) \rightarrow [X, Q_n]_{A^1}$$

is a bijection, functorially in $X$.

**Remark 1.2.5.** Below, we will routinely use the following consequence of this result: every element of $f \in [X, Q_n]_{A^1}$ can be represented by an actual morphism of schemes $f : X \rightarrow Q_n$. In particular, building off this result and Proposition 1.1.9, we will give a geometric description of the abelian group structure on $[X, Q_n]_{A^1}$ from Proposition 1.1.4.

### The Hurewicz map made concrete

If the base field $k$ is infinite and perfect, the identifications of quadrics as spheres combined with [Mor12, Corollary 6.43] imply that, $\pi_{n-1}^{A^1}(Q_{2n-1}) \cong K^{MW}_n$ and $\pi_{n}^{A^1}(Q_{2n}) \cong K^{MW}_n$. There are identifications $H^{n-1}(Q_{2n-1}, K^{MW}_n) \cong K^{MW}_0(k)$ and $H^n(Q_{2n}, K^{MW}_n) \cong K^{MW}_0(k)$ by [AF14, Lemma 4.5] and [AF13, Lemma 3.5.8].

Proposition 1.1.10(2) describes the Hurewicz homomorphism in terms of a fundamental class; in the cases above, we obtain a sequence of classes $\alpha_n := \alpha_{Q_n}$: if $n = 2m - 1$, these lie in $H^{m-1}(Q_{2m-1}, K^{MW}_m)$, while if $n = 2m$, these lie in $H^m(Q_{2m}, K^{MW}_m)$. In each case, the groups housing the fundamental class are free $K^{MW}_0(k)$-modules of rank 1 and via the identifications of the previous paragraph, the fundamental class $\alpha_n$ can be viewed as a generator of this module. We now give a more geometric description of this generator.

The sheaves $K^{MW}_n$ restricted to the small Nisnevich site admit an explicit flasque resolution [Mor12, §5.1 and Corollaries 5.43-5.44], which allows one to identify their Zariski and Nisnevich cohomology; we assume the reader is familiar with this setup. The group $H^0_{\text{Zar}}(X, K^{MW}_n)$ coincides with the Chow-Witt group $\CH^0(X)$ as defined, for instance in [Fas08, Définition 10.2.14]. The closed subscheme $Z_n \subset Q_{2n}$ defined by $x_1 = \cdots = x_n = z = 0$ has trivial normal bundle (e.g., since it is isomorphic to $\mathbb{A}^n$) and $Q_{2n} \setminus Z_n$ is $\mathbb{A}^1$-contractible, e.g., by [ADF14, Theorem 3.1.1]. Thus, the localization sequence in Chow-Witt groups [Fas08, Corollaire 10.4.11] (combined with $\mathbb{A}^1$-representability of Chow-Witt groups) implies that $\CH^n(Q_{2n}) \cong K^{MW}_0(k)$ is generated by the the image of $\CH^0(Z_n)$.

The generic point of $Z_n$ is an element $z_n \in Q^{(n)}_{2n}$. The maximal ideal of $\mathcal{O}_{Q_{2n}, z_n}$ is generated by the classes of $x_1, \ldots, x_n$ and we can consider $x_1 \wedge \cdots \wedge x_n \in \wedge^n(m_{z_n}/m^2_{z_n})$. Finally, we obtain an element $\alpha \in \langle 1 \rangle \otimes x_1 \wedge \cdots \wedge x_n$ in $K^{MW}_0(k(z_n), \wedge^n(m_{z_n}/m^2_{z_n}))$; the following result shows that this element provides an explicit generator for $\CH^n(Q_{2n})$.

**Lemma 1.2.6** ([ADF14, Lemma 4.2.6]). The element $\alpha_n \in K^{MW}_0(k(z_n), \wedge^n(m_{z_n}/m^2_{z_n}))$ is a cycle and its class generates $\CH^n(Q_{2n})$ as a $K^{MW}_0(k)$-module.
Now, combining Propositions 1.1.4, 1.1.7 and 1.1.10, if \( U \in \text{Sm}_k \) is an infinite perfect field, then the Hurewicz homomorphism can be viewed as a morphism

\[
\pi_0(\text{Sing}^{A^1}Q_{2n})(U) \longrightarrow H^n_{\text{Nis}}(U, K^n_{\text{MW}}) \cong \widehat{CH}^n(U)
\]

In particular, it assigns to a map \( f : U \to Q_{2n} \) an element of \( \widehat{CH}^n(U) \). The next result then follows immediately by combining all of the observations we have made so far with the conclusion of Lemma 1.2.6.

**Theorem 1.2.7.** Fix an integer \( n \geq 2 \). If \( k \) is an infinite field perfect having characteristic unequal to 2 and \( U \) is a smooth affine \( k \)-scheme of \( A^1 \)-cohomological dimension \( d \leq 2n - 2 \), then the binary operation \( \tau \) of Definition 1.1.8 equips \( \pi^{2n,n}(X) \) with a functorial structure of abelian group. The Hurewicz homomorphism

\[
\pi^{2n,n}(U) = [U, Q_{2n}]_{A^1} \longrightarrow \widehat{CH}^n(U),
\]

has the following properties:

1. it is surjective if \( d \leq n + 1 \) and an isomorphism if \( d \leq n \);
2. it sends a map \( f : U \to Q_{2n} \) representing a class in \( \pi^{2n,n}(X) \) to \( f^*\alpha_n \) where the class \( f^*\alpha_n \) differs from the the image of the canonical generator of \( \widehat{CH}^n(Q_{2n}) \) by a constant unit \( \lambda \in K^n_{\text{MW}}(k) \).

**Proof.** It only remains to check the final statement. Suppose \( f : U \to Q_{2n} \) is a morphism of schemes. One can consider the pullback \( f^* : \widehat{CH}^n(Q_{2n}) \to \widehat{CH}^n(U) \). It suffices to observe that if then the Hurewicz image of \( [f] \in \pi_0(\text{Sing}^{A^1}Q_{2n})(U) \to \widehat{CH}^n(U) \) is of the form \( \lambda \cdot f^*(\alpha_n) \) for \( \alpha_n \) the fundamental class and some constant unit \( \lambda \in K^n_{\text{MW}}(k) \) (this follows, for example, by considering the universal case where \( U = Q_{2n} \) and \( f \) is the identity map; this makes sense after Lemma 1.1.3). \( \square \)

## 2 Group structures and naive \( A^1 \)-homotopy classes

This section is the algebraic and geometric heart of the paper. If we use the identification \( \pi_0(\text{Sing}^{A^1}Q_{2n})(U) \cong [U, Q_{2n}]_{A^1} \) for \( U \) a smooth \( k \)-scheme from Theorem 1.2.4, then if \( U \) has dimension \( \leq 2n - 2 \), \( \pi_0(\text{Sing}^{A^1}Q_{2n})(U) \) inherits a functorial abelian group structure from Proposition 1.1.4. However, elements of \( \pi_0(\text{Sing}^{A^1}Q_{2n})(U) \) are represented by actual morphisms \( U \to Q_{2n} \). In Section 2.1, we review some results of [Fas15] and some preliminary moving lemmas that allow us to identity maps \( U \to Q_{2n} \) in “ideal-theoretic” terms. In Section 2.2 we use these ideas to give a completely algebraic description of the composition operation; this is achieved in Theorem 2.2.9.

### 2.1 Naive homotopies of maps to quadrics

If \( R \) is a ring, then by definition an element of \( Q_{2n}(R) \) corresponds to a sequence of elements \((x_1, \ldots, x_n, y_1, \ldots, y_n, z) \in R^{2n+1} \) satisfying the equation defining \( Q_{2n} \); we will write \((x, y, z)\) for such a triple with implicit understanding that \( x = (x_1, \ldots, x_n) \), and \( y = (y_1, \ldots, y_n) \). Given \( v \in Q_{2n}(R) \), we can consider the ideal \( I(v) := \langle x_1, \ldots, x_n, z \rangle \subset R \). We write \( \overline{v} \) for the image of \( x \) under the map \( I \to I/I^2 \).
Lemma 2.1.1 ([Fas15, Lemma 2.0.1] or [MK77, Lemma p. 533]). Given an ideal \( I \) and a sequence \((a_1,\ldots,a_n)\) of elements in \( I \) such that \( I/I^2 = \langle \overline{a_1},\ldots,\overline{a_n} \rangle \), there exist an element \( s \in I \) and \( b_1,\ldots,b_n \in R \) such that \( I = \langle a_1,\ldots,a_n, s \rangle \) and \((a, b, s)\) is an element of \( Q_{2n} \).

Using this observation, the naive \( A^1 \)-homotopy class of an element of \( v \in Q_{2n}(R) \) is essentially determined by \( I(v) \); we now explain the precise sense in which this is true.

Lemma 2.1.2 ([Fas15, Lemma 2.0.5]). Suppose \( R \) is a commutative ring and \( I \subset R \) is a finitely generated ideal. If \((a, b, s)\) and \((a', b', s')\) determine elements of \( Q_{2n}(R) \) satisfying the following properties:

i) \( a_1,\ldots,a_n, a'_1,\ldots,a'_n \in I; \)

ii) \( I/I^2 = \langle \overline{a_1},\ldots,\overline{a_n}, \overline{a'_1},\ldots,\overline{a'_n} \rangle; \)

iii) \( I/I^2 = \langle \overline{a_1},\ldots,\overline{a_n} \rangle; \)

then the naive \( A^1 \)-homotopy classes \([a, b, s]\) and \([a', b', s']\) coincide in \( \pi_0(\Sing A^1 Q_{2n})(R) \).

Moving maps with target \( Q_{2n} \)

In order to describe the abelian group structure on \( \pi_0(\Sing A^1 Q_{2n})(R) \) geometrically, it will be helpful to be able to move maps to \( Q_{2n} \) into “general position”. We now describe a procedure to do this, inspired by [MY10, Lemma 4.3] (or [BS00, Corollary 2.14] when \( n = \dim(R) \)). To fix notation, let \( v = (a_1,\ldots,a_n, b_1,\ldots,b_n, s) \in Q_{2n}(R) \) and \( I(v) = \langle a_1,\ldots,a_n, s \rangle \). The ideal \( I(v) \) need not have height \( n \), however, Lemma 2.1.3 will demonstrate that the sequence \( v \) is \( A^1 \)-homotopic to one for which the associated ideal \textit{does} have height \( n \).

Lemma 2.1.3. Suppose \( R \) is a Noetherian ring and \( a = (a_1,\ldots,a_n) \), \( b = (b_1,\ldots,b_n) \) in \( R^n \) and \( s \in R \) are such that \( ab^t = s(1-s) \). If \( J_1,\ldots,J_r \subset R \) are ideals such that \( \dim(R/J_i) \leq n-1 \) for \( i = 1,\ldots,r \), then, there exists a sequence \( \mu = (\mu_1,\ldots,\mu_n) \in R^n \) and an ideal

\[
N := \langle a + \mu(1-s)^2, s + \mu b^t(1-s) \rangle,
\]

such that the following statements hold:

1. the sequence \( (a + \mu(1-s)^2, (1-b^t\mu), s + \mu b^t(1-s)) \) yields an element of \( Q_{2n}(R) \);
2. in \( \pi_0(\Sing A^1 Q_{2n}(R)) \) the equality \( (a, b, s) = (a + \mu(1-s)^2, (1-b^t\mu), s + \mu b^t(1-s)) \) holds;
3. \( \ht(N) \geq n; \) and
4. \( J_i + N = R \) for \( i = 1,\ldots,r \).

Proof. We appeal to the results of Eisenbud-Evans [EE73], as generalized by Plumstead [Plu83, p. 1420]. To this end, let \( A \subset \Spec R \) be the set of prime ideals \( p \subset R \) such that \((1-s) \notin p \) and \( \ht(p) \leq n-1 \). Let moreover \( B_i := V((J_i)_{(1-s)}) \subset \Spec R_{(1-s)} \subset \Spec R \) for \( i = 1,\ldots,r \) and \( B = \cup_i B_i \). Since \( R \) is Noetherian, the restriction of the usual dimension function on \( \Spec(R_{(1-s)}) \) to \( A \) is a generalized dimension function \( d : A \to \mathbb{N} \) in the sense of [Plu83, Definition p. 1419] (cf. [Plu83, Example 1]). Likewise, let \( d_p : B_i \to \mathbb{N} \) be the usual dimension function on \( V((J_i)_{(1-s)}) \).

As in [Plu83, Example 2], we obtain a generalized dimension function \( \delta : A \cup B \to \mathbb{N} \) such that \( \delta(p) \leq n-1 \) for any \( p \in A \cup B \).
Now, we apply the Eisenbud-Evans theorems to the finitely generated free $R$-module $R^n$ with the generalized dimension function $\delta$ on $A \cup B$. Then, if $(a, (1 - s)^2)$ is unimodular on $A \cup B$, we conclude that there exists a sequence $\mu = (\mu_1, \ldots, \mu_n) \in R^n$ such that the row $(a_1 + \mu_1(1 - s)^2, \ldots, a_n + \mu_n(1 - s)^2)$ is unimodular on $A \cup B$ (a priori the Eisenbud-Evans results are formulated in terms of basic elements, but [EE73, Lemma 1] and the subsequent remark guarantee that the result is unimodular).

To establish points (1) and (2), observe that if we set $A = a + T(1 - s)^2 \mu \in R[T]^n$, then

\[ Ab^t = ab^t + T(1 - s)^2 \mu b^t = s(1 - s) + T(1 - s)^2 \mu b^t = (1 - s) - (1 - s)^2 (1 - T \mu b^t). \]

Multiplying both sides by $(1 - T \mu b^t)$, we obtain the equality

\[ Ab^t (1 - T \mu b^t) = (1 - s)(1 - T \mu b^t) - (1 - s)^2 (1 - T \mu b^t)^2. \]

Setting $B = (1 - T \mu b^t)b$, one deduces that $(A, B, (1 - s)(1 - T \mu b^t)) \in Q_{2n}(R[T])$. It follows that $(A, B, 1 - (1 - s)(1 - T \mu b^t)) = (A, B, s + T \mu b^t(1 - s)) \in Q_{2n}(R[T])$.

To establish points (3) and (4), observe that

\[ \langle a + \mu(1 - s)^2 \rangle = \langle a + \mu(1 - s)^2, (1 - s)(1 - T \mu b^t) \rangle \cap \langle a + \mu(1 - s)^2, s + \mu b^t(1 - s) \rangle = \langle a + \mu(1 - s)^2, (1 - s)(1 - \mu b^t) \rangle \cap N. \]

If $p$ is a prime ideal such that $N \subset p$, then it follows that $(1 - s)(1 - \mu b^t) \notin p$ and therefore that $(1 - s) \notin p$. Moreover, $\langle a + \mu(1 - s)^2 \rangle \subset N \subset p$. As $(a_1 + \mu_1(1 - s)^2, \ldots, a_n + \mu_n(1 - s)^2)$ is unimodular on $A \cup B$, it follows that $p \notin A \cup B$. Consequently, $p \notin \cup_i V(J_i)$ and condition (4) follows. Similarly, $p \notin A$ and condition (3) is also satisfied.

2.2 A geometric description of the composition on cohomotopy groups

In order to describe the sum on $[U, Q_{2n}]_{\mathbb{A}^1}$ explicitly, it is convenient to use the description of the product given in Definition 1.1.18; recall that this composition coincides with that described in Proposition 1.1.4 by appealing to Proposition 1.1.9. More precisely, given $f, g : U \to Q_{2n}$, we consider the following diagram

\[
\begin{array}{ccc}
Q_{2n} \vee Q_{2n} & \xrightarrow{\nabla} & Q_{2n} \\
\downarrow & & \\
U \xrightarrow{\Delta} U \times U & \xrightarrow{f \times g} & Q_{2n} \times Q_{2n}
\end{array}
\]

We begin by providing concrete models for the map from the wedge sum to the product and the fold map in terms of maps of suitable varieties.

The sum-to-product and fold maps

Begin by recalling that by [ADF14, Theorem 3.1.1] the quadric $Q_{2n}$ has an open subscheme $X_{2n}$ that is $\mathbb{A}^1$-contractible. The subscheme $X_{2n}$ can be defined as the complement of the closed subscheme $Z_n := \{x_1 = \cdots = x_n = z = 0\}$.
Lemma 2.2.1. The sum-to-product map \( Q_{2n} \vee Q_{2n} \to Q_{2n} \times Q_{2n} \) factors as

\[
Q_{2n} \vee Q_{2n} \longrightarrow (Q_{2n} \times Q_{2n}) \setminus (Z_n \times Z_n) \longrightarrow Q_{2n} \times Q_{2n};
\]

the first morphism is an \( \mathbb{A}^1 \)-weak equivalence, and the second morphism is the inclusion morphism.

Proof. Since \( X_{2n} \) is \( \mathbb{A}^1 \)-contractible, the inclusion of the base-point in \( X_{2n} \) yields a commutative diagram of the form

\[
\begin{array}{ccc}
Q_{2n} & \overset{*}{\longrightarrow} & Q_{2n} \\
\downarrow & & \downarrow \\
Q_{2n} \times X_{2n} & \overset{}{\longrightarrow} & X_{2n} \times X_{2n} & \overset{}{\longrightarrow} & X_{2n} \times Q_{2n},
\end{array}
\]

where all vertical morphisms are \( \mathbb{A}^1 \)-weak equivalences. In particular, the evident map of homotopy colimits is also an \( \mathbb{A}^1 \)-weak equivalence.

Since all the horizontal morphisms in this diagram are cofibrations, the homotopy colimit of each row coincides with the actual colimit. The colimit of the top row is, by definition, the wedge sum. On the other hand, the colimit of the bottom row is simply the union in \( Q_{2n} \times Q_{2n} \) of \( Q_{2n} \times X_{2n} \) and \( X_{2n} \times Q_{2n} \), i.e., it is the open subscheme of the product \( Q_{2n} \times Q_{2n} \) whose closed complement is \( Z_n \times Z_n \).

Remark 2.2.2. Provided \( n > 0, (Q_{2n} \times Q_{2n}) \setminus (Z_n \times Z_n) \) is a strictly quasi-affine smooth \( k \)-scheme (i.e., quasi-affine and not affine). Therefore, every morphism \( (Q_{2n} \times Q_{2n}) \setminus (Z_n \times Z_n) \to Q_{2n} \) extends uniquely to a morphism \( Q_{2n} \times Q_{2n} \to Q_{2n} \). Tracing through the definitions, an extension of \( \nabla \) to a \( Q_{2n} \times Q_{2n} \to Q_{2n} \) yields an \( h \)-space structure on \( Q_{2n} \). However, if \( k \) has characteristic 0, using complex realization one sees that existence of such an \( h \)-space structure contradicts the Hopf invariant 1 theorem (see the introduction to [Ada60]). Thus, one expects that no such extension of \( \nabla \) exists in general.

There is a geometric model of the fold map, which we obtain in two steps. First, by the Jouanolou trick there exists an affine vector bundle torsor \( \phi : Q_{2n} \tilde{\times} Q_{2n} \to (Q_{2n} \times Q_{2n}) \setminus (Z_n \times Z_n) \) [Wei89, Definition 4.2]. Second, since \( Q_{2n} \tilde{\times} Q_{2n} \) is a smooth affine \( k \)-scheme, the class of the fold map in \( \nabla \in [Q_{2n} \tilde{\times} Q_{2n}, Q_{2n}]_{\mathbb{A}^1} \) is, by means of Theorem 1.2.4, represented by a unique up to naive \( \mathbb{A}^1 \)-weak equivalence morphism of smooth \( k \)-schemes \( \nabla : Q_{2n} \tilde{\times} Q_{2n} \to Q_{2n} \). We begin by providing an explicit Jouanolou device, that will aid our computations below.

Example 2.2.3. Suppose \( X \) is any regular affine scheme, and \( Z \subset X \) is a closed subscheme equipped with a choice \( f_1, \ldots, f_n \) of generators of the ideal of \( Z \). There is an induced morphism \( f : X \to \mathbb{A}^n \) such that \( f^{-1}(0) = Z \). Pulling back the morphism \( Q_{2n-1} \to \mathbb{A}^n \setminus 0 \) (which one can check is a torsor under a trivial vector bundle of rank \( n - 1 \)) along \( f \) one obtains a torsor under a vector bundle \( X \setminus Z \to X \setminus Z \). More explicitly, the scheme \( X \setminus Z \) is the closed subscheme of \( X \times \mathbb{A}^n \) defined by the equation \( \sum_i f_i z_i = 1 \) and the map to \( X \setminus Z \) is induced by projection onto the first factor.

One appeals to this construction to obtain a Jouanolou device for \( (Q_{2n} \times Q_{2n}) \setminus (Z_n \times Z_n) \). Indeed, setting \( x = (x_1, \ldots, x_n) \) and similarly for \( x', y, y', u \) and \( v \), a model for the Jouanolou
device has coordinate ring:

\[ k[x, y, z, x', y', z', u, u_{n+1}, v, v_{n+1}]/\langle xy^t = z(1 - z), \]
\[ x'(y')^t = z'(1 - z'), \]
\[ ux^t + v(x')^t + u_{n+1}z + v_{n+1}z' = 1 \]

and, in these coordinates, the projection morphism is induced by projection onto \( x, y, z, x', y', z' \).

Suppose given a Jouanolou device \( \phi : \widetilde{Q_{2n}} \rightarrow (Q_{2n} \times Q_{2n}) \setminus (Z_n \times Z_n) \). Note that the maps \( Q_{2n} \rightarrow Q_{2n} \times Q_{2n} \) obtained by inclusion of the base-point in one component factor through closed immersions \( i_l, i_r : Q_{2n} \rightarrow (Q_{2n} \times Q_{2n}) \setminus (Z_n \times Z_n) \). Since the pullback of a torsor under a vector bundle is a torsor under a vector bundle, and since torsors under vector bundles on affine schemes are simply vector bundles, we conclude that the pullback of \( \phi \) along either \( i_l \) or \( i_r \) is a vector bundle of rank \( n - 1 \) on \( Q_{2n} \). The zero section of this vector bundle then yields morphisms \( Q_{2n} \rightarrow Q_{2n} \setminus Q_{2n} \) that factor \( i_l \) and \( i_r \). By universality, there is an induced \( \mathbb{A}^1 \)-weak equivalence \( Q_{2n} \setminus Q_{2n} \rightarrow Q_{2n} \setminus Q_{2n} \) that factors the \( \mathbb{A}^1 \)-weak equivalence \( Q_{2n} \setminus Q_{2n} \rightarrow (Q_{2n} \times Q_{2n}) \setminus (Z_n \times Z_n) \) of Lemma 2.2.5. With this in mind, we now give a geometric construction of the fold map.

**Construction 2.2.4.** Using the model of the Jouanolou device given in Example 2.2.3 we can uniquely specify the \( \mathbb{A}^1 \)-homotopy class of the fold map as follows. Consider the ideals \( I := \langle x, z \rangle \) and \( I' = \langle x', z' \rangle \) and set \( J := II' \). The third defining equation implies that the ideals \( I \) and \( I' \) are comaximal and therefore \( II' = I \cap I' \).

Set

\[ c_i := x_i(1d_i x + u_{n+1}z) + x_i(1d_i x' + v_{n+1}z'). \]

In that case, one checks that \( J/J^2 = (\bar{c}_1, \ldots, \bar{c}_n) \). Note that, by construction, \( c \equiv x \mod I \) and \( c \equiv x' \mod I' \). Therefore, \( I \equiv \langle c \rangle \equiv I^2 \) and \( I' \equiv \langle c \rangle + I'^2 \). By appeal to [Fas15, Lemma 2.0.1] we conclude that:

i) there exist elements \( w \in I \) and \( w' \in I' \) such that \( I = \langle c, w \rangle \) and \( I' = \langle c, w' \rangle \), and
ii) there exist \( n \)-tuples of regular functions \( d, d' \) on the explicit Jouanolou device such that \( w(1 - w) = cd \) and \( w'(1 - w') = cd' \).

Then, one can check \( J = \langle c, w, w' \rangle \) and the equation \( w w'(1 - w w') = c \delta^t \) is satisfied with \( \delta = (c(d')^t + (u')^2d + u^2d') \). The sequence \( (c, \delta, w, w') \) corresponds to a morphism \( \nabla \) from the Jouanolou device to \( Q_{2n} \).

The next result follows immediately from Construction 2.2.4 by observing that the restriction of the morphism described above along either closed immersion \( i_l \) or \( i_r \) is the identity map \( Q_{2n} \rightarrow Q_{2n} \).

**Lemma 2.2.5.** The map \( \nabla : \widetilde{Q_{2n}} \rightarrow Q_{2n} \) described in Construction 2.2.4 is a model for the fold map.

**A geometric lift**

Suppose that \( U \) is a smooth affine \( k \)-scheme of dimension \( d \leq 2n - 2 \) and \( f, g : U \rightarrow Q_{2n} \). We consider the map \( (f \times g) \circ \Delta : U \rightarrow Q_{2n} \times Q_{2n} \). While \( (f \times g) \circ \Delta \) does not necessarily factor
through \((Q_{2n} \times Q_{2n}) \setminus (Z_n \times Z_n)\), we now show that, up to replacing \(f\) and \(g\) by naively \(\mathbb{A}^1\)-homotopic maps, such a lift does exist. In fact, we will establish slightly more: we will show that we can choose \(f'\) and \(g'\) such that \((f')^{-1}(Z_n)\) and \((g')^{-1}(Z_n)\) are disjoint, i.e., the corresponding ideals in \(U\) are comaximal.

**Lemma 2.2.6.** Suppose \(U\) is a smooth affine \(k\)-scheme of dimension \(d \leq 2n - 2\) and \(f,g : U \to Q_{2n}\). There exist \(f',g' : U \to Q_{2n}\) such that

1. \([f] = [f']\) and \([g] = [g']\) in \(\pi_0(\text{Sing}^{\mathbb{A}^1}Q_{2n}(R))\),
2. \((f' \times g') \circ \Delta : U \to (Q_{2n} \times Q_{2n}) \setminus (Z_n \times Z_n)\), and
3. the unique lift of \((f \times g) \circ \Delta\) to \([U, Q_{2n} \vee Q_{2n}]\) guaranteed to exist by Proposition 1.1.7 is represented by the class of \((f' \times g') \circ \Delta\) under the \(\mathbb{A}^1\)-weak equivalence of Lemma 2.2.1.

**Proof.** Suppose \(U = \text{Spec } R\) for some \(k\)-algebra \(R\). Morphisms \(f,g : U \to Q_{2n}\) correspond to sequences \((a,b,s)\), \(a = (a_1,\ldots,a_n)\), \(b = (b_1,\ldots,b_n)\) and \((c,d,t)\), \(c = (c_1,\ldots,c_n)\), \(d = (d_1,\ldots,d_n)\) of elements of \(R\). By Lemma 2.1.2, the \(\mathbb{A}^1\)-homotopy class \([f]\) only depends on the ideal \(f^{-1}(Z_n) =: I_1 = (a,s)\); likewise, \([g]\) only depends on \(g^{-1}(Z_n) =: I_2 = (c,t)\).

First, by appeal to Lemma 2.1.3 (with all \(J_i\) equal to the unit ideal) we can assume without loss of generality that \(I_2\) has height \(\geq n\). Now, by replacing \(f\) by a naively \(\mathbb{A}^1\)-homotopic map if necessary, we can assume that \(f^{-1}(Z_n)\) and \(g^{-1}(Z_n)\) are disjoint subschemes, i.e. that \(f \times g\) misses \(Z_n \times Z_n\) in \(Q_{2n} \times Q_{2n}\), which is precisely what we wanted to show. To this end, set \(J_1 = I_2\) (and take \(J_1\) to be the unit ideal otherwise). Since the height of \(J_1\) is assumed \(\geq n\), we see that \(\dim R/J_1 \leq 2n - 2 - n = n - 2\). The existence of the required homotopy is then guaranteed by Lemma 2.1.3 (with \(J_i\) the unit ideal if \(i \neq 1\)).

We now give a geometric construction that we will compare to the composition operation in \(\pi_0(\text{Sing}^{\mathbb{A}^1}Q_{2n}(R))\) defined previously.

**Construction 2.2.7.** Suppose \(X = \text{Spec } R\) is a smooth affine \(k\)-scheme of dimension \(d \leq 2n - 2\) and \(f,g : X \to Q_{2n}\). Assume that \(f\) and \(g\) correspond with sequences \(v = (a,b,s)\) and \(v' = (a',b',s')\) \(\in Q_{2n}(R)\) such that \(I(v) = (a,s)\) and \(I(v') = (a',s')\) are comaximal. Since \(I(v)\) and \(I(v')\) are comaximal, we conclude that \(I(v)I(v') = I(v) \cap I(v')\) and we set:

\[ J := I(v)I(v') = I(v) \cap I(v'). \]

As \(J/J^2 \cong I(v)/I(v)^2 \times I(v')/I(v')^2\), we see that there exist elements \(c_1,\ldots,c_n \in J\) such that \(c = (c_1,\ldots,c_n)\) satisfies \(c \equiv a \pmod{I(v)}\), \(c \equiv a' \pmod{I(v')}\) and \(J = \langle c \rangle \pmod{J^2}\). In particular, the following hold:

\[ I(v) = \langle c \rangle + I(v)^2, \text{ and} \]
\[ I(v') = \langle c \rangle + I(v')^2. \]

By appeal to [Fas15, Lemma 2.0.1], we conclude the following:

i) there exist elements \(u \in I(v)\) and \(u' \in I(v')\) such that \(I(v) = \langle c, u \rangle\) and \(I(v') = \langle c, u' \rangle\),

ii) there exist elements \(d, d' \in R^n\) such that the equations \(u(1-u) = cd^t\) and \(u'(1-u') = c(d')^t\) are satisfied.
Using these relations, we can write \( J = \langle c, uu' \rangle \) and the equation \( uu'(1 - uu') = cx^t \), with \( x = (c(d')^t)d + (u')^2d + u^2d' \) is satisfied. The \((2n+1)\)-uple \((c, x, uu')\) yields a morphism \( h : X \to Q_{2n} \).

**Lemma 2.2.8.** Suppose \( k \) is an infinite field having characteristic different from \( 2 \). Assume \( n \geq 2 \) is an integer. Suppose \( U = \text{Spec} \ R \) is a smooth affine \( k \)-scheme of dimension \( d \leq 2n - 2 \) and we are given two morphisms \( f, g : U \to Q_{2n} \). Replacing \( f \) and \( g \) by naively \( A^1 \)-homotopic maps \( f', g' \) with \((f')^{-1}(Z_n)\) and \((g')^{-1}(Z_n)\) disjoint and sending the resulting pair \((f', g')\) to the output \( h \) of Construction 2.2.7 passes to a well-defined function

\[
\tau : \pi_0(\text{Sing}^A Q_{2n}(R)) \times \pi_0(\text{Sing}^A Q_{2n}(R)) \to \pi_0(\text{Sing}^A Q_{2n}(R)),
\]

\[
([f], [g]) \mapsto [h].
\]

**Proof.** By appeal to Lemma 2.2.6, we may always suppose that \((f')^{-1}(Z_n)\) and \((g')^{-1}(Z_n)\) are disjoint. We now trace through Construction 2.2.7 to see how the output depends on the chosen representatives.

Following the notation of Construction 2.2.7, the element \( h \) depends on the choices of elements \( c, x \in \mathbb{R}^n \) and \( u, u' \in \mathbb{R}^n \). Pick \( c' \in \mathbb{R}^n \) such that \( c' \equiv a'( \mod I(V)) \), \( c' \equiv a'( \mod I(v')) \) and \( J = \langle c' \rangle ( \mod J^2) \). In that case, \( I(v) = \langle c' \rangle + I(v)^2 \) and \( I(v') = \langle c' \rangle + I(v')^2 \). From this we conclude that \( c_i - c'_i \in I(v)^2 \) and also in \( I(v')^2 \). Now, as in Construction 2.2.7, we build elements \( \mu \in I(v) \) and \( \mu' \in I(v') \) such that \( I(v) = \langle c', \mu \rangle \) and \( I(v') = \langle c', \mu' \rangle \) and elements \( \delta, \delta' \in \mathbb{R}^n \) such that \( \mu(1 - \mu) = c'\delta \) and \( \mu'(1 - \mu') = c'\delta' \). Then, \( J = \langle c', \mu\mu' \rangle \) and by setting \( x' = c'(\delta')^t \delta + (\mu')^2\delta + \mu^2\delta' \), the equation \( \mu\mu'(1 - \mu') = c'(x')^t \) is satisfied. By appeal to Lemma 2.1.1, one concludes that \((c, x, uu')\) and \((c', x', \mu\mu')\) yield the same class in \( \pi_0(\text{Sing}^A Q_{2n}(R)) \), irrespective of the choice of \( \mu, \mu' \) and \( x' \). \( \square \)

**Theorem 2.2.9.** Assume \( k \) is an infinite (perfect) field having characteristic unequal to \( 2 \) and \( U = \text{Spec} \ R \) is a smooth affine \( k \)-scheme. The composition \([f], [g] \to \tau([f], [g])\) on \( \pi_0(\text{Sing}^A Q_{2n}(R)) \) given in Lemma 2.2.8 coincides with the operation from Proposition 1.1.4.

**Proof.** By Proposition 1.1.9, it suffices to prove that the composition \( \tau \) is the same as that described in Definition 1.1.8. Consider the following diagram:

\[
\begin{array}{ccc}
Q_{2n} \vee Q_{2n} & \xrightarrow{\nabla} & Q_{2n} \\
\Phi & \downarrow & \\
(\mathbb{Q}_n \times \mathbb{Q}_n) \setminus (\mathbb{Z}_n \times \mathbb{Z}_n) & \leftarrow & \mathbb{Q}_n \times \mathbb{Q}_n \\
\mathbb{U} & \xrightarrow{\Delta} & \mathbb{U} \times \mathbb{U} & \xrightarrow{f \times g} & \mathbb{Q}_n \times \mathbb{Q}_n
\end{array}
\]

the morphisms \( \Phi \) and \( \nabla \) in this diagram are described in Lemmas 2.2.1 and 2.2.5 (and the intervening discussion).

We now describe the map labelled \( \exists \). Suppose \( f \) and \( g \) are such that \( f^{-1}(Z_n) \) and \( g^{-1}(Z_n) \) are disjoint, which we can assume by Lemma 2.2.6. In that case, we know that \((f \times g) \circ \Delta \) lifts, up to naive \( A^1 \)-homotopy, to a morphism \( h : U \to (\mathbb{Q}_n \times \mathbb{Q}_n) \setminus (\mathbb{Z}_n \times \mathbb{Z}_n) \).
2.2 A geometric description of the composition on cohomotopy groups

Next, note that the map \( \text{Sing}^A_1 \mathbb{Q}_{2n} \to \text{Sing}^A_1(Q_{2n} \times Q_{2n}) \setminus (Z_n \times Z_n) \) induced by \( \phi \) is a Zariski fiber space with affine space fibers by construction; therefore by appeal to [AHW15, Lemma 4.2.4] we conclude that it is a weak equivalence when evaluated on any smooth affine \( k \)-scheme. Thus, for any smooth affine \( k \)-scheme \( U \) the induced map \( \pi_0(\text{Sing}^A_1 \mathbb{Q}_{2n}) \to \pi_0(\text{Sing}^A_1(Q_{2n} \times Q_{2n}) \setminus (Z_n \times Z_n)(U)) \) is a bijection, i.e., any morphism of \( k \)-schemes \( h : U \to (Q_{2n} \times Q_{2n}) \setminus (Z_n \times Z_n) \) lifts uniquely up to naive \( A_1 \)-homotopy to a morphism of \( k \)-schemes \( \tilde{h} : U \to Q_{2n} \).

Furthermore, we can write down an “explicit” formula for the map \( \tilde{h} \), at least up to naive \( A_1 \)-homotopy. Let \( \varphi : k[x, y, z]/(xy = z(1 - z)) \to k[U] \) and \( \psi : k[x, y, z]/(xy = z(1 - z)) \to k[U] \) be the maps corresponding to \( f \) and \( g \). We write down a ring map from the coordinate ring displayed in Example 2.2.3 to \( k[U] \). The images of the variables \( (x, y, z, x', y', z') \) are determined by \( \varphi \) and \( \psi \). By assumption, we know that the ideals \( I := \langle \varphi(x), \varphi(z) \rangle \) and \( I' := \langle \psi(x'), \psi(z') \rangle \) are comaximal. Therefore we can find \( n + 1 \)-tuples \( (a, a_{n+1}) \) (where, as above \( a = (a_1, \ldots, a_n) \)) and \( (b, b_{n+1}) \) such that

\[
a \varphi(x) + a_{n+1} \varphi(z) + b \psi(x') + b_{n+1} \psi(z') = 1.
\]

Then, sending \( u \mapsto a \) and \( v \mapsto b \), we obtain the required morphism.

Thus, to establish the result, it suffices to show that \( [\tilde{h}] \) as in Lemma 2.2.8 is a model for the composite of this lift and the geometric \( \nabla \). Given the constructions above, the result follows by comparison with the explicit formula for \( \nabla \) given in Construction 2.2.4 combined with Lemma 2.2.5.

Remark 2.2.10. The formula in Theorem 2.2.9 is motivated by van der Kallen’s group structure on orbit sets of unimodular rows [vdK83]. In fact, it is possible to use the formula in Construction 2.2.7 to establish directly that the composition so-defined is unital, associative and commutative. Following this course likely produces a composition in greater generality than we have established here (e.g., presumably one could make a statement, suitably modified, that holds for \( X \) the spectrum of an arbitrary commutative Noetherian ring). Nevertheless, we have not pursued this approach for two reasons. From a theoretical point of view, we felt the homotopy theoretic techniques in Section 1 give a nice “explanation” for the formulas regarding composition and yield strong functoriality properties without significant additional work. From the standpoint of applications, in Section 3 our appeal to the main results of [Fas15] will force us to assume (for totally different reasons) that we are working with smooth affine \( k \)-algebras with \( k \) infinite and having characteristic unequal to 2.

Remark 2.2.11. Assume \( k \) is an infinite perfect field. The variety \( Q_{2n-1} \) is \( A_1 \)-weakly equivalent to \( A^n \setminus 0 \) and therefore \( A_1-(n - 2) \)-connected. As a consequence, if \( X = \text{Spec} R \) is a smooth \( k \)-scheme of dimension \( d \leq 2n - 4 \), we conclude that \([X, Q_{2n-1}]_{A_1} \) inherits a group structure via Proposition 1.1.4. On the other hand, by [AHW15, Theorem 4.2.1], we know that \( \pi_0(\text{Sing}^A_1 Q_{2n-1})(R) \to [X, Q_{2n-1}]_{A_1} \) is a bijection for any smooth affine \( k \)-scheme \( X \). On the other hand, by [Fas11, Theorem 2.1] we know that \( \pi_0(\text{Sing}^A_1 Q_{2n-1})(R) \cong Um_n(R)/E_n(R) \) and therefore the latter is equipped with an abelian group structure. On the other hand, van der Kallen showed \( Um_n(R)/E_n(R) \) admits an abelian group structure [vdK89, Theorem 5.3]. Analogous to the results of Section 2.2, it seems reasonable to expect that these two group structures coincide.
3 Segre classes and Euler class groups

In this section, we finally connect with other classical results. Section 3.1 is devoted to construction of the homomorphism from Euler class groups to motivic cohomotopy; the main result is Theorem 3.1.9. In Section 3.2 we establish some applications of this comparison result: we compare Euler class groups, weak Euler class groups, Chow-Witt groups and Chow groups in certain situations.

3.1 Euler class groups and the Segre class homomorphism

We begin by recalling some key results of [Fas15] related to Segre classes. Then, we record the definition of Euler class groups à la Bhatwadekar–Sridharan, slightly adapted for our purposes in Definition 3.1.3. After recording a helpful moving lemma, we construct the Segre class homomorphism and then establish its main properties. The main result of this section is Theorem 3.1.9, which establishes a portion of Theorem 3 in the introduction.

Universal Segre classes

As before, suppose $R$ is a commutative ring and $f : \text{Spec} R \to Q_{2n}$ is a morphism given by a sequence of elements $(x,y,z) \in R$. If $I$ is the ideal $(x_1,\ldots,x_n,z)$, then the quotient $I/I^2$ is generated by $\{x_1,\ldots,x_n\}$ and there is a surjective homomorphism $\omega_I : (R/I)^n \to I/I^2$; such a homomorphism is sometimes called a local $n$-orientation [Man15, Definition 2.3].

Definition 3.1.1. If $R$ is a commutative ring and $n \in \mathbb{N}$, then write $\text{Ob}_n(R)$ for the set consisting of pairs $(I,\omega_I)$ where $I$ a finitely generated ideal in $R$ and $\omega_I : (R/I)^n \to I/I^2$ a surjective homomorphism.

Given $(I,\omega_I) \in \text{Ob}_n(R)$, by Lemma 2.1.1 there exist elements $a_1,\ldots,a_n, s \in I$ and $b_1,\ldots,b_n \in R$ such that $I = (a_1,\ldots,a_n,s)$ and $(a_1,\ldots,a_n,b_1,\ldots,b_n,s) \in Q_{2n}(R)$. We write $s(I,\omega_I)$ for any such $(2n+1)$-tuple; evidently, $s(I,\omega_I)$ depends on a number of choices, but it is shown in [Fas15] that the class of $s(I,\omega_I) \in \pi_0(\text{Sing}^b Q_{2n}(R))$ is well-defined; more precisely, the following result holds.

Theorem-Definition 3.1.1 ([Fas15, Theorem 2.0.7]). If $R$ is a Noetherian commutative ring and $(I,\omega_I) \in \text{Ob}_n(R)$, then the class of $s(I,\omega_I)$ in the pointed set $\pi_0(\text{Sing}^b Q_{2n}(R))$ (pointed by the image $v_0$ of the base-point of $Q_{2n}$) is well-defined, independent of the choices made to describe $s(I,\omega_I)$. The class $s(I,\omega_I)$ will be called the universal Segre class of $(I,\omega_I)$.

Remark 3.1.2. The terminology Segre class is inspired by the work of Murthy [Mur94, §5]. See the introduction to [Fas15] for more discussion of this point of view and the connection with the notions of [Ful98, §4.2].

Euler class groups

Let $R$ be a commutative Noetherian ring with $\dim(R) = d$. Euler class groups of $R$ were defined for height $d$ ideals in [BS00, p. 197] and more generally in [BS02, p. 146]. We slightly recast the
3.1 Euler class groups and the Segre class homomorphism

definition here and to do so, we recall some notation. Define \( Ob_n(R) \subset Ob_n(R) \) to be the subset of the obstruction set defined in Definition 3.1.1 consisting of pairs \((I, \omega_I)\) with \(I\) an ideal of height \(n\). Given a pair \((I, \omega_I)\), if \(I\) is generated by \(a_1, \ldots, a_n\), we say that \(\omega_I\) is induced by \(\overline{a}_1, \ldots, \overline{a}_n\) if \(\omega_I\) is the morphism sending the standard basis vector \(e_i\) in the free \(R/I\)-module \(R/I^n\) to the element \(\overline{a}_i\). For any commutative ring \(A\), let \(E_n(A) \subset GL_n(A)\) be the subgroup consisting of elementary (shearing) matrices. Note that there is a left action of \(E_n(R/I)\) on \(Ob_n(R)\) or \(Ob_n'(R)\) given as follows: for an element \(\sigma \in E_n(R/I)\), \(\sigma \cdot (I, \omega) = (I, \omega \circ \sigma^{-1})\).

**Definition 3.1.3.** The Euler class group \(E_n(R)\) is the quotient of the free abelian group \(\mathbb{Z}[Ob'_n(R)]\) by the ideal generated by the following relations:

1. If \(I = \langle a_1, \ldots, a_n \rangle\) and \(\omega_I : (R/I)^n \to I/I^2\) is induced by \(\overline{a}_1, \ldots, \overline{a}_n\), then \((I, \omega_I) = 0\).
2. If \(\sigma \in E_n(R/I)\), then \(\sigma \cdot (I, \omega_I) = (I, \omega_I)\).
3. If \(I = JK\), where \(J, K\) are height \(n\) ideals with \(K + J = R\), then a surjection \(\omega_J : (R/I)^n \to I/I^2\) induces surjections \(\omega_K : (R/K)^n \to K/K^2\) and \(\omega_J : (R/J)^n \to J/J^2\) and the relation is \((I, \omega_I) = (K, \omega_K) + (J, \omega_J)\).

**Remark 3.1.4.** Definition 3.1.3 is equivalent to that given in [BS02, p. 147] even though it looks (slightly) formally different. More precisely, Bhatwadekar and Sridharan consider the free abelian group on equivalence class of pairs \((I, \omega)\) with \(\text{Spec } R/I\) connected; conditions (2) and (3) are imposed precisely to compare with that situation.

Note also that Definition 3.1.3 always makes sense, but it is mainly of interest when \(d \leq 2n - 3\) because in that context it is closely related with the problem of when the ideal \(I\) can be generated by \(n\) elements ([BS02, Theorem 4.2] or [MY12, Theorem 2.4]).

**The Segre class homomorphism**

We begin with a result that is an immediate consequence of Theorems 1.2.4 and 3.1.1.

**Lemma 3.1.5.** Assume \(k\) is an infinite (perfect) field having characteristic unequal to 2, and suppose \(X = \text{Spec } R\) be a smooth affine \(k\)-scheme of dimension \(d \leq 2n - 2\). The assignment \((I, \omega_I) \mapsto s(I, \omega_I)\) passes to a homomorphism

\[
s : \mathbb{Z}[Ob'_n(R)] \longrightarrow [X, Q_{2n}]_{\mathbb{A}^1},
\]

where \([X, Q_{2n}]_{\mathbb{A}^1}\) has the abelian group structure of Proposition 1.1.7.

The next result shows that the homomorphism \(s\) from Lemma 3.1.5, which we will refer to as the Segre class homomorphism, factors through the Euler class group of Definition 3.1.3 under suitable hypotheses.

**Proposition 3.1.6.** Fix an integer \(n \geq 2\) and suppose \(d\) is an integer \(\leq 2n - 2\). Assume \(k\) is an infinite (perfect) field having characteristic unequal to 2 and suppose \(X = \text{Spec } R\) is a smooth affine \(k\)-scheme having \(\mathbb{A}^1\)-cohomological dimension \(d \leq 2n - 2\). The Segre class homomorphism factors through a homomorphism:

\[
s : E^n(R) \longrightarrow [X, Q_{2n}]_{\mathbb{A}^1}.
\]
Proof. We prove that the relations in Definition 3.1.3 are satisfied in \([X, Q_{2n}]\).

**Step 1.** Relation 1 holds, i.e., if \(I = (a_1, \ldots, a_n)\) is an ideal of height \(n\) and \(\omega_I : (R/I)^n \to I/I^2\) is given by \(e_i \mapsto \alpha_i\), then \(s(I, \omega_I) = 0\). Indeed, the Segre class of \((I, \omega_I)\) is given, for instance, by \(v := (a_1, \ldots, a_n, 0, \ldots, 0) \in Q_{2n}(R)\). Now, \((a_1T, \ldots, a_nT, 0, \ldots, 0) \in Q_{2n}(R[T])\) provides an explicit homotopy between \(v\) and \(v_0 = (0, \ldots, 0)\).

**Step 2.** Relation 2 holds, i.e., if \((I, \omega_I)\) is a generator of the Euler class group and \(\sigma \in E_n(R/I)\), then \(s(I, \omega_I) = s(\sigma \cdot (I, \omega_I))\). By definition, any element \(\sigma \in E_n(R/I)\) can be factored as a product of elementary matrices. Therefore, it suffices to establish the result for \(\sigma = 1 + e_{ij}(\lambda)\), \(i \neq j\) with \(\lambda \in R\) an arbitrary element.

Choose any representative \((a, b, s)\) of \((I, \omega_I)\); such a representative exists by [Fas15, Lemma 2.0.1]. In that case, set \(\sigma = 1 + e_{ij}(\lambda)\) and observe that we can choose the representative \((\sigma - a, \sigma^{-1}b, s)\) for \(\sigma \cdot (I, \omega_I)\). Thus, \(\sigma(t) := 1 + e_{ij}(\lambda t)\) determines an explicit homotopy between \((a, b, s)\) and \((\sigma - a, \sigma^{-1}b, s)\) (cf. Relation (3) in [Fas15, §3.2]).

**Step 3.** Relation 3 holds, i.e., suppose \(I, J, K\) are ideals of height \(n\) such that \(J\) and \(K\) are comaximal, \(I = JK\), \(\omega_I : (R/I)^n \to I/I^2\) is a surjection, and \(\omega_J\) and \(\omega_K\) are the surjections induced by \(\omega_I\); we will show that \(s(I, \omega_I) = s(J, \omega_J) + s(K, \omega_K)\). To this end, recall Construction 2.2.7: if \(J\) and \(K\) correspond to elements \(v, v' \in Q_{2n}(R)\), we observed there how to construct an element of \(Q_{2n}(R)\) corresponding with \(I\). With that in mind, relation 3 follows immediately from Theorem 2.2.9.

**The Segre class homomorphism is an isomorphism**

We now aim to prove that the Segre class homomorphism is an isomorphism. To this end, we establish some preliminary “moving” results, which will be useful in the course of establishing injectivity. We begin with the following lemma (which is a simplified version of [MY12, Lemma 2.2]).

**Lemma 3.1.7.** Let \(R\) be a Noetherian ring of dimension \(d\), \(I \subset R\) an ideal of height \(n\) and \(\omega_I : R^n \to I/I^2\) a surjective homomorphism. For \(I_1, \ldots, I_r\) arbitrary ideals of \(R\), there exists an ideal \(K \subset R\) and a homomorphism \(f : R^n \to I \cap K\) having the following properties:

1. the ideals \(I^2\) and \(K\) are comaximal;
2. the composite \(R^n \xrightarrow{f} I \cap K \subset R \to I/I^2\) is equal to \(\omega_I\);
3. the ideal \(K\) has height \(\geq n\);
4. for any integer \(1 \leq i \leq r\), the inequality \(\text{ht}((I_i + K)/I_i) \geq n\) holds.

**Corollary 3.1.8.** Let \(R\) be a Noetherian ring of dimension \(d \leq 2n - 1\) and let \(\alpha \in E^n(R)\). There exists then an ideal \(I\) of height \(n\) and a surjective homomorphism \(\omega_I : (R/I)^n \to I/I^2\) such that \(\alpha = (I, \omega_I) \in E^n(R)\).

**Proof.** As the Euler class group is a quotient of the free abelian group on \(\text{Ob}^n(R)\) by some relations, we can write

\[
\alpha = \sum_{i=1}^{m} (I_i, \omega_{i}) - \sum_{j=1}^{m} (J_j, \omega_{j}).
\]
Using Lemma 3.1.7, we see that there exists an ideal $K$ of height $\geq n$ which is comaximal with $I_i^2$. Moreover, we have $\text{ht}(\langle K + I_i \rangle / I_i) \geq n$ for any $i = 2, \ldots, r$ and $\text{ht}(\langle K + J_j \rangle / J_j) \geq n$ for any $j = 1, \ldots, m$. As the ideals $I_2, \ldots, I_n, J_1, \ldots, J_m$ are of height $n$ and $R$ is of dimension $2n - 1$, we obtain that $K$ is pairwise comaximal with all of them. Moreover, the surjective homomorphism $f : R^n \to I \cap K$ yields a surjective homomorphism $\omega_K : R^n \to K/K^2$ and we have $(I, \omega_I) + (K, \omega_K) = 0$ in $E^n(R)$. It follows that

$$\alpha = \sum_{i=2}^{n} (I_i, \omega_i) - \sum_{j=1}^{m} (J_j, \omega_j) = (K, \omega_K).$$

Arguing inductively, we see finally that we can suppose that the ideals in the expression

$$\alpha = \sum_{i=1}^{n} (I_i, \omega_i) - \sum_{j=1}^{m} (J_j, \omega_j),$$

are pairwise comaximal. Now, the relation (3) in Definition 3.1.3 shows that $\alpha = (I, \omega_I) - (J, \omega_J)$ for comaximal ideals $I, J$ of height $n$. Using again Lemma 3.1.7 as above, we see that there exists an ideal $K$ of height $n$, comaximal with $I$ and a surjective homomorphism $\omega_K : R^n \to K/K^2$ such that $\alpha = (I, \omega_I) + (K, \omega_K)$. We conclude using again the relation (3) in Definition 3.1.3.

**Theorem 3.1.9.** Fix an integer $n \geq 2$, and assume $d \leq 2n - 2$. If $k$ is an infinite (perfect) field, having characteristic unequal to 2, and $X = \text{Spec} R$ is a $d$-dimensional smooth affine $k$-algebra, then the Segre class homomorphism

$$s : E^n(R) \longrightarrow [X, Q_{2n}]_{\mathbb{A}^1}$$

is an isomorphism.

**Proof.** In order to demonstrate injectivity, suppose that $\alpha \in E^n(R)$ has $s(\alpha) = 0$. By Corollary 3.1.8, we have $\alpha = (I, \omega_I)$ in $E^n(R)$ and it suffices to show that $(I, \omega_I) = 0$ provided $s(I, \omega_I) = 0$. By [Fas15, Theorem 3.2.7], we know that if $s(I, \omega_I) = (a, b, s)$ is trivial then $I = \langle a_1 + s^2, \ldots, a_n + s^2 \rangle$ and it follows that $\omega_I$ lifts to a surjection $R^n \to I$. It follows that $\alpha = (I, \omega_I) = 0$ by the relation (1) in Definition 3.1.3.

To establish surjectivity, fix $v = (a, b, s) \in Q_{2n}(R)$. By Lemma 2.1.3 we can find $v' = (a', b', s')$ such that (i) $I := \langle a', s' \rangle$ has height $\geq n$ and (ii) the class of $v'$ in $\pi_0(\text{Sing}_{\mathbb{A}^1} Q_{2n}(R))$ coincides with that of $v$. Since $a'(b')^t = s'(1 - s')$, we conclude that the localization $I_{1-s'} = \langle a' \rangle$. If $I_{1-s'}$ is a proper ideal, it follows from Krull’s Hauptidealsatz that $\text{ht}(I_{1-s'}) \leq n$ and therefore that $\text{ht}(I) \leq n$ as well. Thus, in that case, we conclude that $\text{ht}(I) = n$. Defining $\omega_I : (R/I)^n \to I/I^2$ by $e_i \mapsto \omega_i$, it follows that $s(I, \omega_I) = (a', b', s')$. On the other hand, if $I_{1-s'} = R_{1-s'}$, then $I = R$ and it follows that $(a', b', s') = (a, b, s) = 0$ in $\pi_0(\text{Sing}_{\mathbb{A}^1} Q_{2n}(R))$.

**Remark 3.1.10.** Theorem 3.1.9 immediately implies that Euler class groups satisfy a number of functorial properties. For example, $\mathbb{A}^1$-homotopy invariance of $E^n(X)$ is immediate (cf. [Das03, Proposition 5.7]). Similarly, if $f : X \to Y$ is any arbitrary morphism of smooth affine $k$-varieties (of dimension $d \leq 2n - 2$), the map $f^* : [Y, Q_{2n}]_{\mathbb{A}^1} \to [X, Q_{2n}]$ induced by composition yields a
pull-back morphism for Euler class groups. By appeal to Theorem 1.2.2 one obtains Mayer-Vietoris sequences. More precisely, if $X$ is a smooth affine scheme of dimension $d \leq 2n - 2$ and we have $j : U \to X$ an open immersion of an affine $k$-scheme $U$ and $\varphi : V \to X$ an étale morphism from an affine $k$-scheme $V$ such that the induced map $(V \setminus U \times X V)_{\text{red}} \to (X \setminus U)_{\text{red}}$ is an isomorphism, then there is an exact sequence of the form

$$[X, Q_{2n}]_{\mathbb{A}^1} \to [U, Q_{2n}]_{\mathbb{A}^1} \times [V, Q_{2n}]_{\mathbb{A}^1} \to U \times_X V, Q_{2n}]_{\mathbb{A}^1}. $$

Furthermore, if $d \leq 2n - 4$, there is a map from $[U \times_X V, Q_{2n-1}]$ to $[X, Q_{2n}]$ which extends the exact sequence further to the left. Finally, Remark 1.2.3 equips Euler class groups with a product operation as well. It would be interesting to compare the excision and product operations studied in [MY12] and [MY10].

**Remark 3.1.11.** As a further consequence Theorem 3.1.9, given a ring $R$ satisfying the hypotheses, it is possible to attach to a pair $(I, \omega_I)$ consisting of an ideal $I \subset R$ and a surjection $\omega_I : (R/I)^n \to I/I^2$ an element in the Euler class group that detects if $\omega_I$ lifts to a surjection $R^n \to I$. In particular, one can extend (and partially generalize) the work of M. Das and R. Sridharan [DS10].

### 3.2 Euler class, Chow-Witt and Chow groups

Finally, we put everything together: if $X$ is a smooth $k$-scheme of dimension $d$, we compare Euler class groups and Chow-Witt groups in top codimension. The explicit form of the Hurewicz homomorphism from Theorem 1.2.7 plays a role in the comparison with previous constructions. In Theorem 3.2.1 we show that Euler class groups can be identified with Chow-Witt groups in certain cases. Finally, Theorem 3.2.4 establishes the connection between weak Euler class groups and Chow groups by appeal to an exact sequence studied in [DAZ15] and [vdK15].

#### Euler class groups vs. Chow-Witt groups

In view of Theorems 1.2.7 and 3.1.9, there is a homomorphism $E^n(R) \to \widetilde{CH}^n(X)$ which we make more explicit when $n = d := \dim(X)$. In that case, the Euler class group is generated by pairs $(m, \omega_m)$ where $m \subset R$ is a maximal ideal and $\omega_m : (R/m)^d \to m/m^2$ is a surjection (and indeed an isomorphism). Let $m_1, \ldots, m_n$ be elements of $m$ such that $\omega_m(e_i) = \overline{m_i}$. In that case, the description of the Hurewicz homomorphism from Theorem 1.2.7 shows that the Hurewicz image of $(s(m, \omega_m))$ is given (up to the unit $\lambda \in K_0^{MW}(k)$) by the class of the cycle $\langle 1 \rangle \otimes \overline{m_1} \wedge \ldots \wedge \overline{m_d}$. In other words, the composite homomorphism

$$E^d(R) \xrightarrow{s} [X, Q_{2d}] \longrightarrow \widetilde{CH}^d(X)$$

coincides (up to the unit $\lambda \in K_0^{MW}(k)$ with the homomorphism defined in [Fas08, Proposition 17.2.8]. Using these observations, we establish the following result, which establishes another part of Theorem 3 from the introduction.

**Theorem 3.2.1.** Suppose $k$ is a field that is infinite, perfect and has characteristic unequal to 2. If $X = \text{Spec } R$ is a smooth affine $k$-scheme of dimension $d \geq 2$, then composite homomorphism

$$E^d(R) \to \widetilde{CH}^d(X).$$
described above is an isomorphism.

**Proof.** If \( d = 2 \), the result holds by [Fas08, Corollaire 17.4.2] and the discussion just preceding the statement. Therefore, assume \( d \geq 3 \). It suffices to show that the composite

\[
E^d(R) \xrightarrow{s} [X, Q_{2d}]_{\mathbb{A}^1} \xrightarrow{\sim} \overline{CH}^d(X)
\]

is an isomorphism. In view of Theorem 1.2.7, this follows from Theorem 3.1.9. \( \square \)

**Weak euler class groups and Chow groups**

Let \( R \) be a smooth affine algebra of dimension \( d \) over a field \( k \). Recall the notion of weak Euler class groups \( E^d_0(R) \) (cf. [BS99, Definition 2.2]; these groups were mentioned in the introduction).

**Definition 3.2.2.** If \( X = \text{Spec} \ R \) is a smooth affine \( k \)-scheme of dimension \( d \), the weak Euler class group \( E^d_0(X) \) is the quotient of the free abelian group on the set of maximal ideals \( m \subset R \) subject to the relation \( \sum_i m_i = 0 \) if \( I = \bigcap_i m_i \) is a reduced complete intersection ideal.

Associating with a maximal ideal \( m \) its class in \( CH^d(X) \) yields a well-defined surjective homomorphism \( s' : E^d_0(R) \rightarrow CH^d(X) \) [BS99, Lemma 2.5]. By [Fas08, Proposition 17.2.10], there is a commutative diagram

\[
\begin{array}{ccc}
E^d(R) & \xrightarrow{s} & E^d_0(R) \\
\downarrow s' & & \downarrow s' \\
\overline{CH}^d(X) & \xrightarrow{\sim} & CH^d(X)
\end{array}
\]

where the horizontal maps are surjective.

Recall that \( Um_{d+1}(R) \) is the set of unimodular rows of length \( d + 1 \) in \( R \), i.e., sequences \( (a_1, \ldots, a_{d+1}) \) that admit a right inverse. [DAZ15, Theorem 3.8] shows that there is an exact sequence of the form:

\[
Um_{d+1}(R)/E_{d+1}(R) \rightarrow E^d(R) \rightarrow E^d_0(R) \rightarrow 0
\]

where the map \( Um_{d+1}(R)/E_{d+1}(R) \rightarrow E^d(R) \) is defined as follows. One first defines the map \( \phi \) on “special” unimodular rows: suppose \( (a_1, \ldots, a_{d+1}) \) is a unimodular row such that the ideal \( I := (a_1, \ldots, a_d) \) has height \( d \). In this case, observe that \( a_{d+1} \) is a unit modulo \( I \) and define \( \omega_I : (R/I)^d \rightarrow I/I^2 \) by sending \( e_i \mapsto \pi_i \). Then set

\[
\phi(a_1, \ldots, a_{d+1}) := (I, a_{d+1} \omega_I).
\]

One can then show that (i) any unimodular row can be moved by multiplication by elementary matrices to a “special” unimodular row and (ii) the function \( \phi \) so extended is constant on orbits of \( E_{d+1}(R) \) (for an alternative proof, see [vdK15, Theorem 3.4]).

**Remark 3.2.3.** The morphism \( \phi \) described above coincides with that studied in [BS02, §5] using Euler classes if \( d \) is even, but differs if \( d \) is odd.
In [Fas11, Proposition 3.1, Theorem 4.9], a homomorphism \( \phi' : Um_{d+1}(R)/E_{d+1}(R) \to \widetilde{CH}^d(X) \) is defined; moreover, it is established that this homomorphism sits in an exact sequence of the form:

\[
Um_{d+1}(R)/E_{d+1}(R) \xrightarrow{\phi'} \widetilde{CH}^d(X) \longrightarrow CH^d(X) \longrightarrow 0.
\]

One may check that \( s\phi = \phi' \) and therefore obtain a commutative diagram with exact rows of the form:

\[
egin{array}{ccc}
Um_{d+1}(R)/E_{d+1}(R) & \xrightarrow{\phi} & E^d(R) \\
\downarrow s & & \downarrow s' \\
Um_{d+1}(R)/E_{d+1}(R) & \xrightarrow{\phi'} & \widetilde{CH}^d(X) \\
\end{array}
\longrightarrow CH^d(X) \longrightarrow 0.
\]

Then, appealing to Theorem 3.2.1 and the five lemma, we obtain the following result, which establishes Theorem 1 from the introduction.

**Theorem 3.2.4.** Suppose \( k \) is a field that is infinite, perfect and has characteristic unequal to 2. If \( X = \text{Spec} R \) is a smooth affine \( k \)-scheme of dimension \( d \geq 2 \), then the homomorphism \( s' : E^d_0(R) \to CH^d(X) \) is an isomorphism.

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