Wigner time delay and related concepts
Application to transport in coherent conductors

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Abstract
The concepts of Wigner time delay and Wigner-Smith matrix allow to characterize temporal aspects of a quantum scattering process. The article reviews the statistical properties of the Wigner time delay for disordered systems; the case of disorder in 1D with a chiral symmetry is discussed and the relation with exponential functionals of the Brownian motion underlined. Another approach for the analysis of time delay statistics is the random matrix approach, from which we review few results. As a practical illustration, we briefly outline a theory of non-linear transport and AC transport developed by Büttiker and coworkers, where the concept of Wigner-Smith time delay matrix is a central piece allowing to describe screening properties in out-of-equilibrium coherent conductors.

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1. Introduction

The purpose of this article is to review several results linked to a concept which has been very influential in the work of Markus Büttiker, namely the concept of Wigner time delay, and several of its extensions – traversal time, Wigner-Smith matrix, injectance, etc. In my opinion, this topic allows one to have a good flavor of Markus Büttiker’s style as a physicist: a combination between formal developments with mathematical elegance and motivations from practical questions of fundamental condensed matter physics. Besides, the choice of this theme is also related to a more personal anecdote as it was the subject of my first scientific exchange with Markus Büttiker, which I rediscovered during the preparation of this article, finding in a box a copy of an old email, dated the 8th July 1997, addressed to me and my PhD advisor Alain Comtet. Here are few sentences extracted from this message: “I noticed yesterday your paper on the cond-mat network \cite{Texier97}. I like to react to two things: (...) In your introduction you jump from the work of Fyodorov and Sommers immediately to the work of Brouwer et al. Both of these are of course fine works. But the work of Brouwer et al. as they make clear in very strong terms takes its starting point from my work by Gopar and Mello \cite{Gopar89}, and that alone should have been a good enough reason to not simply leave it out. (...) Therefore, it is my hope that you can give this work the place it deserves. Sincerely, Markus Büttiker”. Of course, he was absolutely right and we soon after amended our paper according to his remarks. Despite this somewhat awkward beginning, Markus Büttiker hired me as a postdoctoral assistant at the University of Geneva two years later and I enjoyed very much the pleasant and stimulating atmosphere of the Physics Department. I especially appreciated the freedom which Markus gave to his postdocs and learnt a lot from our regular exchanges.

The notion of time delay was introduced in the fifties by Eisenbud and Wigner in the context of the quantum theory of scattering. It allows to capture temporal aspects of the scattering process \cite{Eisenbud56, Wigner55, Wigner55b}. The most fundamen-
tal aspects of time delays have been reviewed by Carvalho and Nussenzveig [13]. Some other articles have reviewed more specific aspects: Beenakker has considered the case of wave guides in the localised regime [13]. In Ref. [107], Kottos has reviewed the random matrix approach for ergodic systems (chaotic cavities) and the case of (non ergodic) disordered systems – diffusive or strongly localised. We have emphasized in [24] the connection with the theory of exponential functional of the Brownian motion. The purpose of the present article is to review few results on time delays from different contexts and to emphasize the diversity of the physical situations where this concept has found some applications.

The outline is the following: in Section 2 we discuss several definitions of times characterizing the scattering process. Section 3 reviews the statistical analysis of time delays for disordered systems. In Section 4 we discuss few aspects of the random matrix approach. Section 5 introduces several generalized concepts (partial DoS, injectance, etc.) which will find practical applications for the analysis of non-linear transport (Section 6) and AC transport (Section 7).

2. Wigner time delay and other characteristic times

2.1. Scattering on the half line

The most simple situation allowing to introduce the concept of time delay is the scattering problem on the half line (this also corresponds to project a rotational invariant problem in an orbital momentum channel, in higher dimensions). We consider the Schrödinger equation

\[-\psi''_\varepsilon(x) + V(x)\psi_\varepsilon(x) = \varepsilon \psi_\varepsilon(x)\]  \hspace{1cm} (1)

for \(x \in \mathbb{R}^+\) (with \(\hbar^2/(2m) = 1\)), describing the scattering of a particle by a potential defined on a finite interval [0, \(L\)] (Fig. 1). At \(x = 0\) we choose to impose a Dirichlet boundary condition, \(\psi_\varepsilon(0) = 0\). In the “asymptotic” region (\(x > L\)), the stationary scattering state of energy \(\varepsilon = k^2\) is the superposition of an incident wave \(e^{-ik(x-L)}\) and a reflected wave \(r e^{+ik(x-L)}\):

\[\psi_\varepsilon(x) = \frac{1}{\sqrt{2\pi \hbar v}} (e^{-ik(x-L)} + r e^{ik(x-L)}) . \]  \hspace{1cm} (2)

The normalisation constant, involving the group velocity \(v = (1/\hbar) d\varepsilon/dk = 2k\), corresponds to associate a measure \(d\varepsilon\) to the eigenstate. The reflection probability amplitude has unit modulus as a consequence of current conservation : \(r = e^{ik}\).

The Wigner time delay is defined as the derivative of the reflection phase

\[\tau_W(\varepsilon) = -i\hbar r^* \frac{\partial r}{\partial \varepsilon} = \hbar \frac{\partial \delta_r(\varepsilon)}{\partial \varepsilon} . \]  \hspace{1cm} (3)

It measures the time spent by a wave packet \(\varepsilon\) in the domain [0, \(L\)]. This can be easily understood by considering the time evolution of a wave packet \(\Phi(x; t) = \int_0^\infty d\varepsilon \psi_\varepsilon(x) e^{-i\varepsilon t}\), where \(\Phi(\varepsilon)\) is a narrow function centered around an energy \(\varepsilon_0\) (from now on, we will set \(h = 1\) for simplicity). We can split the wave packet into an incident part and a reflected part : \(\Phi(x; t) = \Phi_{\text{inc}}(x; t) + \Phi_{\text{ref}}(x; t)\). Neglecting the effect of dispersion, \(\Phi_{\text{inc}}(x; t) = (1/\sqrt{4\pi}) \int_0^\infty d\varepsilon e^{-1/4} \Phi(\varepsilon) e^{-i\sqrt{\varepsilon(t-x-L)-\varepsilon^2 t}}\) can be rewritten under the form \(\Phi_{\text{inc}}(x; t) \propto f(-x + L - \nu_0 t)\), where \(\nu_0\) is the group velocity at energy \(\varepsilon_0\) and \(f\) the function \(f(X) \approx \int dK \Phi(\varepsilon_0 + \nu_0 K) e^{iKX}\).

In the reflected wave packet, the expansion of the phase shift produces a shift in time and we obtain \(\Phi_{\text{ref}}(x; t) \propto f(x - L - \nu_0 t + \nu_0 \tau_W(\varepsilon_0))\) (cf. also chapter 10 of [170]). The interpretation of \(\tau_W\) as the delay in time relies on following the motion of a wave packet with sufficiently narrow dispersion in energy. As trivial illustrations we can consider the case of an impenetrable region when \(\delta_r = \pi\), leading to \(\tau_W = 0\), and a free region \((V(x) = 0)\), leading to \(\delta_r = \pi + 2kL\) and \(\tau_W = 2L/v\).

2.2. The scattering problem in one dimension and characteristic times

The scattering problem on the infinite line illustrates that, although the situation is still simple, many other characteristic times can be already introduced. We consider a plane wave sent from

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1 This choice of normalisation ensures that orthonormalisation reads \(\langle \psi_\varepsilon | \psi_\varepsilon' \rangle = \delta(\varepsilon - \varepsilon')\) and closure relation \(f \, d\varepsilon \langle \psi_\varepsilon | \psi_\varepsilon | \rangle = 1\).
The stationary scattering state $\psi_{c,L}(x)$ describes the scattering of an incident plane wave from the left, which is encoded by a reflection amplitude $r$ and a transmission amplitude $t$. 

\[ r, t \in (-\infty, +\infty) \] on a potential defined on the interval $[-L/2, L/2]$ (Fig. 2). The scattering properties can be encoded in two stationary scattering states $\psi_{c,L}(x)$ (Fig. 2) and $\psi_{c,R}(x)$, controlled by two pairs of reflection/transmission amplitudes $r$, $t$ and $r'$, $t'$ characterizing transmission from the left and from the right, respectively. A general scattering state is a linear combination

$$\Psi_c(x) = A_L \psi_{c,L}(x) + A_R \psi_{c,R}(x),$$

where $A = (A_L, A_R)$ are the amplitudes of the two incoming plane waves. They are related to the amplitudes $B = (B_L, B_R)$ of the outgoing plane waves by the $2 \times 2$ scattering matrix $B = S A$:

$$S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}. \tag{5}$$

As a consequence of current conservation, the scattering matrix is a unitary matrix $S \in U(2)$, parametrised by four independent real parameters, what is made clear by expressing $S$ in the polar representation:

$$S = e^{i\Phi_f/2} \begin{pmatrix} \sqrt{1-T} e^{i\alpha} & i\sqrt{T} e^{-i\chi} \\ i\sqrt{T} e^{i\chi} & \sqrt{1-T} e^{-i\alpha} \end{pmatrix}. \tag{6}$$

$T = |t|^2 = |t'|^2 \in [0, 1]$ denotes the transmission probability. The three independent phases are a global phase $\Phi_f = \det S$, called the “Friedel phase”, the phase $\alpha$ controlling the left/right asymmetry and a magnetic phase $\chi$ (which can be removed by a gauge transformation in the 1D case). As the scattering process is characterised by a matrix, there is much more freedom to define characteristic times.

### 2.2.1. Friedel phase and Wigner time delay

The most simple generalization of the definition \[3\] is

$$\tau_W(\epsilon) = -\frac{i}{N} \text{Tr} \left\{ S^\dagger \frac{\partial S}{\partial \epsilon} \right\} = \frac{1}{N} \frac{\partial \Phi_f}{\partial \epsilon}, \tag{7}$$

where $N$ is the number of scattering channels (here $N = 2$). Although this is an important quantity, as we will explain in the § 2.3, $\tau_W$ does not have a simple interpretation as a wave packet’s delay in time.

#### 2.2.2. Transmission and reflection group delays

A plane wave sent on the scattering region from the left is splitted into two parts to which may be associated two phases : a reflection phase $\delta_r = \Phi_f/2 + \alpha$ and a transmission phase $\delta_t = \Phi_f/2 + \chi + \pi/2$. Extending the argument exposed in § 2.3 would lead to introduce three characteristic times (only two for a symmetric potential), like the transmission group delay $\tilde{\tau}_t = \partial \delta_t / \partial \epsilon$, etc.

### 2.2.3. Partial time delays : diagonalisation and derivation

The two scattering states $\psi_{c,a}(x)$, with $a \in \{L, R\}$, describing a particle incoming from the left/right (Fig. 2), may be recombined in order to form other basis of solutions. An important case corresponds to the two partial scattering states, which behave “asymptotically” as

$$\phi_{\epsilon,a}(x) = [C_a \theta_H(-x) + D_a \theta_H(x)] \cos(\kappa|x| + \eta_a) \tag{8}$$

with $a \in \{1, 2\}$. $\theta_H(x)$ is the Heaviside function. Decomposing $\phi_{\epsilon,a}(x)$ over the left/right scattering states, as in Eq. 4, leads to 175 176

$$\det(S - e^{2i\eta_a} 1_2) = 0,$$

i.e. the two partial waves are eigenvectors of the scattering matrix with eigenvalues $\{e^{2i\eta_a}\}$. We can write :

$$S = U \begin{pmatrix} e^{2i\eta_1} & 0 \\ 0 & e^{2i\eta_2} \end{pmatrix} U^\dagger, \tag{9}$$

where $U$ is a unitary matrix and $\{\eta_a\}$ the scattering phase shifts. In particular, using the polar decomposition, one finds the expressions of the two phase shifts

$$e^{2i\eta_2} = e^{i\Phi_f/2} \left( \sqrt{1-T} \cos \alpha \pm i \sqrt{1-(1-T) \cos^2 \alpha} \right), \tag{10}$$

leading to $2(\eta_1 + \eta_2) = \Phi_f$, as it should. Two other characteristic times could be introduced, which are the two partial time delays

$$\tilde{\tau}_a = 2 \frac{\partial \eta_a}{\partial \epsilon}. \tag{11}$$

As the partial waves \[8\] may be combined in order to form wave packets, the partial time delays can be interpreted as the delay of such wave packets.

\[2\] For a symmetric potential $V(x) = V(-x)$, the two partial waves are the symmetric and antisymmetric solutions (i.e. $C_1 = D_1$ and $C_2 = -D_2$).
2.2.4. Proper time delays: derivation and diagonalisation

The determination of the partial time delays involves a diagonalisation of the scattering matrix and a derivation of its eigenvalues. The converse of these two operations leads to introduce the proper time delays \( \{ \tau_a \} \), which are eigenvalues of the Wigner-Smith time delay matrix

\[
\mathcal{Q} = -i S^\dagger \frac{\partial S}{\partial \varepsilon} = V \left( \begin{array}{cc} \tau_1 & 0 \\ 0 & \tau_2 \end{array} \right) V^\dagger,
\]

where \( V \) is a unitary matrix, which differs from \( U \) involved in \([9]\) in general (see Appendix B). Hermiticity \( \mathcal{Q} = \mathcal{Q}^\dagger \) follows from unitarity of \( S \). Proper time delays and partial time delays have an important difference: being derivatives of the \( S \)-matrix eigenvalues, the partial time delays are intrinsic properties of the scattering process, whereas the proper time delays depend on the particular choice of basis in which the \( S \) matrix is expressed (see Appendix B). Although they do not coincide in general, they satisfy the sum rule

\[
-i \frac{\partial}{\partial \varepsilon} \ln \det S = \sum_{a=1}^N \tilde{\tau}_a = \sum_{a=1}^N \tau_a = N \tau_W
\]

where \( N \) is the number of scattering channels (here \( N = 2 \)).

2.2.5. Other characteristic times

The time delay interpretation of the various characteristic times deduced from scattering phases may lead to paradoxical conclusions in the presence of tunneling barriers, like superluminal propagation. For this reason “clock approaches” have been proposed in order to provide more satisfactory answers to the question “how much time needs a wave packet to travel in a given region?”. We refer to the review articles \([99, 116, 30, 115, 42, 63, 77]\) and Büttiker’s contributions \([44, 45]\).

2.3. Krein-Friedel relation and Virial expansion

An important aspect behind the notion of time delay is its link with the spectral properties of open systems with continuous spectra, which can be understood as follows. Let us come back to the simple situation of scattering on the semi-infinite line for simplicity (Fig. 1). The scattering states \( \phi \) satisfying the Dirichlet boundary condition \( \psi_0(0) = 0 \) may be used as a basis in order to consider the spectral problem on the finite interval \([0, L]\), which involves a second boundary condition at \( x = L \). We choose a Neumann boundary condition \( \psi'_0(L) = 0 \) for instance. The eigenenergies \( \{ \varepsilon_n \} \) are solutions of the quantisation equation \( \delta_r(\varepsilon_n) = 2n\pi r \) for \( n \) integer. When two successive energies are sufficiently close, we can expand the relation \( \delta_r(\varepsilon_{n+1}) - \delta_r(\varepsilon_n) = 2\pi r \delta_n \) as \( \delta_r(\varepsilon_n) \delta_n \approx 2\pi r \), where \( \delta_n = \varepsilon_{n+1} - \varepsilon_n \) is the level spacing. We deduce the expression of the density of states \( \nu(\varepsilon) \approx 1/\delta_n \approx \tau_W(\varepsilon)/(2\pi \hbar) \). A more precise connection between spectral and scattering properties was obtained by Friedel \([77]\) and Smith \([166]\), who established the relation

\[
\int_0^L dx \left| \psi_r(x) \right|^2 = \frac{1}{2\pi} \left( 2W(\varepsilon) + \frac{\sin \delta_r(\varepsilon)}{2\varepsilon} \right)
\]

(see also \([179]\)). The left hand side of the equation is simply related to the local density of states (DoS) \( \nu(\varepsilon) \).

Thus, Eq. (14) shows that the Wigner time delay provides a measure of the DoS of the interval \([0, L]\), and may be written in the more general form

\[
\nu(\varepsilon) = \int_0^L dx \nu(x; \varepsilon) = \frac{1}{2\pi} \text{Tr} \left\{ S^\dagger \frac{\partial S}{\partial \varepsilon} + S - S^\dagger \right\},
\]

so that it also applies to more general situations such as the one of Fig. \([2]\) or even complex structures like metric graphs \([175, 178, 180]\). Such equalities are known as Krein-Friedel relations \([76, 109, 77, 34]\) (or Birman-Krein formula \([22]\)). \footnote{The choice of the normalisation for the scattering state \( \psi \) is important.} They have a long history and first appeared in the virial expansion of the equation of state of real gases \( p = \nu T [1 + B_2(T) n + B_3(T) n^2 + \cdots] \), where \( p \) is the pressure, \( \nu \) the density and \( T \) the temperature. As shown by Beth and Uhlenbeck \([188, 20]\), the second virial coefficient can be written as \( B_2(T) = \ldots \). 

\footnotetext{4}{The choice of the normalisation for the scattering state \( \phi \) is important.}
\[-2^{-5/2} \Lambda_T^3 (\pm 1 + 16 Z_{\text{int}}) \text{ for bosons (+) and fermions (−), where } \Lambda_T = \sqrt{2 \pi \hbar^2 / (mT)} \text{ is the thermal length and } m \text{ the mass of the particles. The first contribution to } B_2 \text{ encodes the quantum correlations arising from the symmetrisation postulate, whereas the second term describes the correlations due to the interaction, } Z_{\text{int}} \text{ being the partition function of the two body problem (in the relative coordinates)}\]

\[Z_{\text{int}} = \sum_\ell (2\ell + 1) \int_0^\infty \frac{d\varepsilon}{2\pi \hbar} \tau_\ell(\varepsilon) e^{-\varepsilon/T}, \quad (17)\]

where summation runs over orbital momentum. \(\tau_\ell = 2 \partial \eta_\ell / \partial \varepsilon\) is the time delay related to the scattering phase shift \(\eta_\ell\) characterizing the scattering in the channel of orbital momentum \(\ell\) (cf. § 77 of [114]) ; in the presence of bound states, the partition function receives additional contributions. This approach was later made more systematic by Dashen, Ma and Bernstein [62]. Note that a similar analysis of real gases based on time delay can be developed within a purely classical frame, as explained by Ma [123].

2.4. Few remarks

Characterisation of the spectral properties of open systems has important applications as it allows to express several physical observables in terms of scattering properties at the heart of the Landauer-Büttiker description of quantum transport. For example, (16) allows to measure the charge inside a conductor in terms of its scattering matrix, an idea which turns out to be central in many developments of Büttiker involving screening, which are reviewed below. Several remarks :

- The Krein-Friedel relation was studied in metric graphs in [173, 178]. In such systems with non trivial topology, the system might possess so-called “bound states in the continuum” (BIC), i.e. a discrete spectrum superimposed onto the continuum spectrum (this occurs when symmetries allow some bound states to remain uncoupled to the continuum) [192, 171]. In this case, the Krein-Friedel relation only characterises the continuous part of the spectrum [178, 150] (see also [175, 178] for illustrations).

- A related problem concerns the role of the transmission phase. As it is clear from the polar representation [9], the Friedel phase \(\Phi_f\), which enters the relation (16) as \(\det S = e^{i\Phi_f}\), is related to the transmission phase \(\delta_t = (\Phi_f + \pi)/2\) in 1D, as the magnetic field can always be removed by a gauge transformation on the infinite line. Thus the DoS can be as well related to \(\delta_t\) in 1D [111]. If the system has a complex topology, the transmission may vanish which causes transmission phase jumps. As a consequence of these phase jumps, the Friedel phase and the transmission phase differ ; the density of states is then only related to the Friedel phase, as discussed by Taniguchi and Büttiker [173]. This question has been examined numerically in quantum dots by Lévy Yeyati and Büttiker [119].

- In Ref. [108], the Krein-Friedel relation was considered in the quasi-one-dimensional situation (wave guide) which is relevant in order to describe the electric contacts of mesoscopic structures.

3. Disordered systems (localised regime)

The review [54] has underlined the close relationship between Wigner time delay for 1D disordered systems and exponential functionals of the Brownian motion, which have been widely studied in the mathematical literature [195] (see also [56] for a brief review). Let us first recall few properties which will be useful for the following.

3.1. Exponential functionals of the Brownian motion

We introduce the random variable

\[Z_X^{(\mu)} = \int_0^X \frac{dx e^{-2(\mu x + B(x))}}{x}, \quad (18)\]

where \(B(x)\) is a normalised Brownian motion starting from \(B(0) = 0\) (Wiener process). The distribu-
tion of (18) was found in Refs. [136, 155] (see also [56] :)

\[ \psi_X(\mu, \nu) = 2 e^{-1/(2Z)} \sum_{0 \leq n, \mu \leq \nu} (-1)^n (\mu - 2n) \times e^{-2X(n-\mu-n)} (2Z)^n L_n^{\mu-2n} \left( \frac{1}{2Z} \right) \]

\[ + \frac{e^{-1/(4Z)}}{2\pi^2 (2Z)^{1+\mu}/2} \int_0^\infty ds \sinh(\pi s) \left| \Gamma \left( \frac{is - \mu}{2} \right) \right|^2 \times W_i(1+\mu)/2, s/2 \left( \frac{1}{2Z} \right)^{e^{-(X/2)(s^2 + s^2)}} , \quad (19) \]

where \( L_n^{\mu}(x) \) is a Laguerre polynomial and \( W_{\mu, \nu}(z) \) the Whittaker function [97]. Eq. (19) shows that the random variable admits the limit law

\[ \psi_\infty(\mu) = \frac{1}{2\pi^2 (\mu) (2\pi)^{1+\mu}} e^{-1/(2Z)} \quad \text{for} \quad \mu > 0. \quad (20) \]

For finite \( X \), the functional is characterised by exponential moments

\[ \langle (X_X(\mu))^{n} \rangle \approx 2^{-n} \frac{\Gamma(n - \mu)}{\Gamma(2n - \mu)} \cdot e^{n(n - \mu)X} \quad (21) \]

The precise expression of the moments can be found in Ref. [133] for \( \mu = 0 \) and Ref. [136] in the general case. \[ \square \]

3.2. Disorder on the half line

The interest for time delays in disordered systems has started with the work of Sulem and coworkers [79] on stochastic resonances (see also [147]). The idea that localised states could produce sharp resonances was later developed by Azbel [12]. The first statistical analysis of the Wigner time delay in a one-dimensional situation was provided by Jayannavar, Vijayagovindan and Kumar [104]. These authors have studied the time delay distribution \( P_\tau(x) \) for the Schrödinger equation (4) on the half line (Fig. 1) when \( V(x) \) is a Gaussian white noise. They have identified the existence of a limit law with power law tail \( P_\infty(x) \sim \tau^{-2} \) for an infinitely long disordered region, \( L \to \infty \), although their expression of \( P_\infty(x) \) was partly incorrect (with a non vanishing distribution as \( \tau \to 0 \)). Similar conclusions were obtained in Ref. [105] when the potential \( V(x) \) is the integral of a white noise, i.e. a Brownian motion. As it was recognised later [155], the incorrect non vanishing \( P_\infty(0) \) of Refs. [104, 105] was due to some inappropriate averaging over the fast phase variable in the weak disorder regime. Short after, Heinrichs identified that the exponential moments, \( \langle \tau^n \rangle \sim \exp[2n(n-1)L/\xi] \) where \( \xi \) is the localisation length, are characteristic of a log-normal tail \( P_\tau(\tau) \) when the length \( L \) is finite [100]. Inspired by the work of Faris and Tsay [72], we established in Ref. [57] the connection between the Wigner time delay and exponential functionals of the Brownian motion. We analysed two different disordered models exhibiting qualitatively different spectral and localisation properties, namely the Schrödinger Hamiltonian \( H = -\partial^2_x + V(x) \) and the supersymmetric Hamiltonian \( H = -\partial^2_x + m(x)^2 + m'(x) \) [i.e. the square of the Dirac Hamiltonian (29)], where the potential \( V(x) \) and the mass \( m(x) \) are Gaussian white noises. This analysis suggested the universality of the statistical properties, which was later demonstrated in Ref. [179]. This can be understood from the integral representation (14) : in the weak disorder regime we may write \( \tau_W \sim 2\pi \int_0^L dx \psi^2(x) \). Using that the logarithm of the envelope of the wave function is a Brownian motion with drift, we have related the time delay to the functional (18) [179, 54]. \[ \tau_W \approx 2\pi \left( \frac{L}{\xi} \right)^{\frac{1}{2}}, \quad (22) \]

where \( \xi \) is the localisation length [7] and \( \tau_\xi = \xi/v = \xi/(2k) \) is the time needed for a ballistic motion on the localisation length \( \xi \). The value of the drift \( \mu = 1 \) originates from the equality

\[ \langle \ln |\psi(x)| \rangle \approx \text{Var} \langle \ln |\psi(x)| \rangle \quad \text{for} \quad x \gg \xi, \quad (23) \]

valid in the weak disorder regime [10]. Eq. (23) is known as "single parameter scaling" [7, 53], as the full distribution of \( \ln |\psi(x)| \) (or of the conductance) is characterised by a unique length scale, the localisation length defined by \( \lim_{x \to \infty} (1/x) \ln |\psi(x)| \). This question has been re-discussed more recently for 1D disordered systems [64, 65, 103, 156]. The universality of the Wigner

\[ ^7 \text{Note however that Eq. 4.4 of Ref. [136] for } \mu = 0 \text{ contains a misprint: } C_{2n}^k \text{ should be replaced by } C_{2n}^{k-n}. \]

\[ ^8 \text{An equality in law } (\text{law}) \text{ relates two quantities with the same statistical properties. For example, the well-known scaling properties of the Brownian motion may be conveniently written } B(x) \approx \text{law} \chi^{1/2} B(x), \text{ where } B(x) \text{ is a Brownian motion.} \]

\[ ^9 \text{For the model (1) with } (V(x)V(x') = \sigma \delta(x - x'), \text{ the localisation length is } \xi^2 \approx 8\xi/\sigma \text{ for weak disorder } \sigma^{2/3} \ll \epsilon. \]
The time delay statistical properties are thus understood as a direct consequence of the universality of the localisation properties in 1D in the weak disorder regime. In other terms, the Wigner time delay in the universal (weak disorder) regime, leading to the same conclusions as for the Schrödinger equation, Eqs. (22, 24), provided that the localisation length is modified, \( \tau_L = \xi \simeq 2/g \) for \( \varepsilon \gg g \) (in the Dirac equation, the velocity is equal to unity). The energy \( \varepsilon \) can be viewed as a parameter which tunes the chiral symmetry breaking: at the symmetry point \( \varepsilon = 0 \), while chiral symmetry holds, the Wigner time delay presents different properties as we now discuss.

The scattering problem on the half line for the Dirac equation is settled as follows: we consider the Dirac equation with a random mass (see §3.1). The importance of Azbel resonances was emphasized as it allowed to recover the moments by some heuristic argument earlier by Heinrichs [100] and providing the pre-exponential factor (a more precise expression of the moments was given in Ref. [179]). A lattice model was studied in Ref. [145], additional remarks were given in Ref. [179]. Note that the log-normal distribution is characterised by moments of the form \( \exp[\alpha n^2] \); the fact that the moments increase with a lower rate with \( n \), as \( \sim \exp[2n(n-1)L/\xi] \), reflects the existence of a limit law with tail \( \tau^{-2} \) (see §3.1). The importance of Azbel resonances was emphasized as it allowed to recover the moments by some heuristic argument [179]. A lattice model was studied in Ref. [135], leading to the same conclusions.

### 3.3. Disorder in 1D with a chiral symmetry

The localisation properties of disordered systems are mainly controlled by their dimensionality and symmetries. A first classification of symmetries was given by Wigner and Dyson in the context of random matrix theory, depending on the presence of time reversal symmetry and spin rotational symmetry, leading to the orthogonal (\( \beta = 1 \)), unitary (\( \beta = 2 \)) and symplectic classes (\( \beta = 4 \)) [129].

These three symmetry classes were later completed by others, identified according to the presence or not of two other types of discrete symmetries which might occur in condensed matter physics: the chiral (or sublattice) symmetry and the particle-hole symmetry. These has led to the classification in terms of ten symmetry classes [197, 6, 71].

A one-dimensional disordered model with a chiral symmetry which has attracted a lot of attention is the Dirac equation with a random mass (see reviews [26, 58, 181]).

\[
\mathcal{H}_D \Psi(x) = \varepsilon \Psi(x)
\]

with

\[
\mathcal{H}_D = i \sigma_2 \partial_x + \sigma_1 m(x),
\]

where \( \sigma_i \) are the Pauli matrices. A chiral symmetry is the anticommutation of the Hamiltonian with a unitary operator, here \( \sigma_2 \mathcal{H}_D \sigma_1 = -\mathcal{H}_D \).
The matching with (30) shows that the phase shift is given by \( \delta_\varepsilon = 2 \theta(L) \). We choose the initial condition \( \theta(0) = 0 \) or \( \pi/2 \), which corresponds to an infinite mass \( m(x) = +\infty \) or \( m(x) = -\infty \) for \( x < 0 \), respectively (these boundary conditions confine the particle on \( \mathbb{R}_+ \); they are the only ones which do not break the chiral symmetry \([181]\)). Following \([57]\) we introduce the variable \( Z(x) = 2 \partial \theta(x)/\partial \varepsilon \) which obeys

\[
\frac{\partial Z(x)}{\partial x} = 2 + 2m(x) Z(x) \cos [2\theta(x)]
\]

with initial condition \( Z(0) = 0 \) and provides the value of the Wigner time delay \( \tau_W = Z(L) \).

The analysis of the symmetry point \( \varepsilon = 0 \) is easy: the phase remains locked at 0 or \( \pi/2 \) (corresponding to \( m(x) = \pm \infty \) for \( x < 0 \)), therefore

\[
\tau_W = 2 \int_0^L dx \, e^{x^2} \int_x^L dx' \, m(x') \quad \text{at } \varepsilon = 0 \ .
\]

For the sake of concreteness, we now consider the situation where the mass is a Gaussian white noise \( [m(x)(x')]_c = g \delta(x - x') \) with mean value \( \langle m(x) \rangle = \mu g \). The representation \((33)\) makes clear the identity

\[
\tau_W = \frac{2}{g} P^{(\varepsilon=0)}(2\mu g) \quad \text{at } \varepsilon = 0 \, ,
\]

i.e. the distribution is \([12]\)

\[
P_L(\tau) = \frac{g}{2} \Gamma\left(\frac{\varepsilon}{2}\right) \left(\frac{g\tau}{2}\right)^{\frac{\varepsilon}{2}} \quad ,
\]

where the dimensionless function was defined above, Eq. [19].

3.3.1. The critical case \( \langle m(x) \rangle = 0 \)

The distribution for \( \mu = 0 \) was obtained in \([170]\) (although the distribution has been already determined in another context \([136]\), what was used in \([174]\)). It is characterised by a log-normal tail \( P_L(\tau) \sim \exp \left[ -1/(8gL) \ln^2(\tau) \right] \) and exponential moments

\[
\langle \tau^n \rangle \simeq \frac{(n - 1)!}{(2n - 1)!} g^{n-2} e^{2n^2gL} \quad \text{for } n \geq 1 \ .
\]

Although the distribution has no limit law, the large \( L \) limit presents the power law behaviour \([136]\)

\[
P_L(\tau) \sim \frac{1}{\sqrt{2\pi gL\tau}} e^{-1/(g\tau)} \quad (37)
\]
cut off by the log-normal tail for \( \tau \to \infty \). The absence of a limit law may be associated with the delocalisation of the model for \( \varepsilon = 0 \), as all time scales extracted from the distribution increase with \( L \) (see \([26, 58, 181]\) for reviews).

3.3.2. The case \( \langle m(x) \rangle = \mu g \neq 0 \): time delay as a probe for zero mode

We now analyse the case \( \langle m(x) \rangle = \mu g \neq 0 \) which was not considered in the literature. Although the spectral properties of the model are invariant under the change of the sign of the mass, this is not the case for the scattering properties: neither for the phase distribution \([52]\) nor for the Wigner time delay distribution, as we demonstrate here. The distribution \( P_L(\tau) \) has an interesting property: the existence of a limit law is correlated with the choice of the boundary condition at \( x = 0 \) and the sign of the average mass \( \langle m(x) \rangle = \mu \). If we choose the boundary condition corresponding to \( m(x) = \pm \infty \) for \( x < 0 \), we obtain the limit law only for \( \mp \mu > 0 \)

\[
P_\infty(\tau) = \frac{g}{\Gamma(\mu)(g\tau)^{\frac{\mu}{2}} e^{-1/(g\tau)} \ .
\]

Conversely, for \( \pm \mu > 0 \), there is no limit law as \( L \to \infty \). We can correlate the existence of the limit law with the presence of a chiral zero mode located at the boundary \( x = 0 \), what occurs when the sign of the mass changes at the boundary:

| \( m(x) \) | \( \langle m(x) \rangle = \mu g \) | zero mode | limit law \( P_\infty(\tau) \) |
|---|---|---|---|
| for \( x < 0 \) | for \( x > 0 \) | | |
| \( -\infty \) | \( > 0 \) | yes | yes |
| \( -\infty \) | \( < 0 \) | no | no |
| \( +\infty \) | \( > 0 \) | no | no |
| \( +\infty \) | \( < 0 \) | yes | yes |

(see \([26, 58, 181]\) for reviews). The existence of zero modes in the multichannel Dirac equation with random mass was recently studied in Ref. \([52]\), where their topological nature was underlined).

3.4. Related questions and final remarks

- An alternative derivation of the stationary distribution \([24]\) was proposed in Ref. \([174]\), by using the relation between the time delay and the reflection coefficient in the presence of a
constant imaginary component in the potential (wave amplification), i.e. making use of analytic properties of the scattering matrix (see also [174, 50]).

- A discrete Anderson model with Cauchy disorder (Lloyd model [124]) was studied in Ref. [63], for which the relation between the two cumulants of $|\psi_k(x)|$ presents an extra factor 2 (i.e. drift $\mu = 1/2$), compared with the usual form for the single parameter scaling, Eq. (23). Some numerical investigations have shown that the distribution of the Wigner time delay is however still characterised by a power law tail $\tau^{-2}$ (and not $\tau^{-3/2}$). It would thus be interesting to clarify this observation in connection with the discussion of the section.

- Correlations in energy $\langle \tau_W(\varepsilon)\tau_W(\varepsilon') \rangle$ were analysed by Titov and Fyodorov [185].

- Time delay and resonance width are closely related (see the review [107]). The resonance width distribution was studied in [113, 96].

- We have focused the discussion on the Wigner time delay characterizing the reflection problem on the half line. The time delay describing the transmitted wave through a disordered medium was analysed in [25].

3.5. Beyond 1D

3.5.1. Multichannel disordered wires

The extension to the quasi-1D situation (a wave guide with $N$ channels) has been considered by several authors (e.g. see [135, 190]). Using the relation between the time delays and the reflection probabilities in an absorbing or amplifying medium (i.e. the analyticity of the $\mathcal{S}$-matrix), Beenakker and Brouwer obtained the distribution of the proper time delays describing the scattering on a semi-infinite disordered wave guide [13, 17]:

$$\mathcal{P}(\gamma_1, \ldots, \gamma_N) \propto \prod_{i<j} |\gamma_i - \gamma_j|^\beta \prod_k e^{-\beta \gamma_k/2}, \quad (39)$$

where $\gamma_k = \tau_s/\tau_k$ is the inverse of the proper time and $\tau_s$ a characteristic time related to the disorder strength. The result relies on the isotropy assumption (among the channels), i.e. describes the weakly disordered (quasi-1D diffusive) regime; in this case the localisation length increases with the number of channels as [13] $\xi^{(\text{quasi 1D})} = \xi^{(1D)} [1 + \beta(N - 1)/2]$. The distribution (39) is a particular instance of the Laguerre ensemble of random matrix theory (RMT). For one channel $N = 1$, one recovers the limit law (25) by setting $\beta \tau_s/2 = \tau_L$.

Starting from (39), the distribution of the Wigner time delay $\tau_W = (1/N) \sum_i \gamma_i^{-1}$ was obtained in Ref. [63]:

$$\mathcal{P}_N^{(\beta)}(\tau) \sim \frac{C_\beta}{\tau^2} \times \exp \left\{ -\frac{27\beta}{64\pi^2} + \left( 1 - \frac{\beta}{2} \right) \frac{9(2 - \sqrt{3})}{4\pi} \right\} \quad (40)$$

where $\beta$ is the Dyson index for orthogonal ($\beta = 1$) or unitary ($\beta = 2$) symmetry classes. $C_\beta$ is a normalisation constant. The distribution presents the same power law tail as in the strictly one-dimensional case $\mathcal{P}_N^{(3)}(\tau) \sim 1/\tau^2$. This can be understood from the fact that the physics at large time (i.e. large scale) is expected to be dominated by a single channel (the less localised one).

Denoting by $\nu_L(\varepsilon_F)$ the DoS of the multichannel disordered wire of length $L$, [10] can be interpreted as the limit law for $\tau_W \simeq 2\pi \nu_L(\varepsilon_F)/N$ in the $L \to \infty$ limit. The fact that all moments of $\tau_W$ are infinite in this case is thus simply related to the divergence of the DoS when $L \to \infty$, as in the strictly 1D situation.

3.5.2. Higher dimensions

- Some measurements of time delays were performed with electromagnetic waves in Ref. [163, 59, 50].

- The case of higher dimensions was considered by Ossipov and Fyodorov [114] who analysed a two dimensional situation in the diffusive regime. The localised regime was studied by numerical simulations in Ref. [194].

- A study of time delay at a critical point like the metal-insulator transition was performed in Ref. [108, 80, 144] (see the review [107]).

4. Random matrix approach for quantum dots

In systems with ergodic properties, like a chaotic cavity with narrow contacts (Figs. 5 and 6), or a
weakly disordered cavity in the ergodic regime, it is natural to make a maximum entropy assumption \[132\] \cite{73,63,130,131} leading to postulate that the scattering matrix is uniformly distributed over the unitary group (for perfect couplings at the contacts). Although this approach has led to many successes in the description of several properties of coherent conductors (conductance, shot noise,...) \[13\] \cite{131}, it does not provide any information about the energy dependence of the scattering matrix, which is probed by the Wigner-Smith matrix \[12\]. Two approaches were proposed to develop a random matrix description for the energy dependence of the scattering matrix:

- the “Hamiltonian Approach” (HA) of chaotic scattering pushed forward by Fyodorov, Savin, Sommers and coworkers,
- the “Alternative Stochastic Approach” (ASA), pioneered by the work of Brouwer and Büttiker \[28\] and mostly developed by Brouwer, Beenakker and coworkers.

One of the first result within RMT was the calculation of the two point correlation function of the time delay \[117\], Eq. (63). The analysis the distributions was considered soon after by Fyodorov and Sommers \[83,84\] who derived the marginal law for time delays \[117\], Eq. (63). The analysis the distribution of the two point correlation function of the scattering matrix is uniformly distributed over the SN matrix description for the energy dependence of the cavity (of more correctly the “mean resonance spacing” as we deal with a scattering problem).

An important advance was the work of Beenakker, Brouwer and Frahm who were able to obtain the joint distribution of the proper time delays within ASA in Ref. \[29,30\]. They established the relationship with the Laguerre ensemble of random matrices: the inverse of the Wigner-Smith matrix is a Wishart matrix \(Q^{-1} = X^T X\) where \(X\) is a \(N \times (2N-1+2/\beta)\) matrix with identical and independent Gaussian elements (this interpretation only holds for \(\beta = 1, 2\)). The joint distribution of the inverse of the proper times, \(\tau_i = \tau_{1i}/\tau_i\) being given by \[29,30\]

\[
P(\tau_1, \ldots, \tau_N) \propto \prod_{i<j} |\tau_i - \tau_j|^\beta \prod_k \frac{N!}{2k} e^{-\gamma_k/2}.
\]

(42)

For \(N = 1\), the Gamma-law \(P(\tau_1) \propto |\tau_1|^{\gamma_1/2} e^{-\gamma_1/2}\) corresponds to Gopar, Mello and Büttiker’s result, Eq. (41).

In the large \(N\) limit, the marginal law for proper time delays takes the form

\[
w_N^\beta(\tau) \sim N \rho(N \tau)
\]

where \[30\] \cite{182} \[19\]

\[
\rho(y) = \frac{1}{2\pi y^2} \sqrt{(x_+ - y)(y - x_-)}
\]

(44)

with \(x_\pm = (\sqrt{2} \pm 1)^2\). The marginal law is not strictly zero out of the interval \([x_-/N, x_+/N]\) but presents finite \(N\) corrections with asymptotics \[19\]

\[
w_N^\beta(\tau) \sim \tau^{-2-3\beta N/2} e^{-\beta/(2\tau)}
\]

\[
\sim \tau^{-2-\beta N/2}.
\]

(45)

(46)

\[14\] Note that a similar route was followed in Ref. \[163\]; however the result of the reference is partly incorrect as explained in \[74\].

\[15\] The distribution is related to the Marčenko-Pastur law by \(\rho(y) = (1/y^2) \rho_{MP}(1/y)\), where \(\rho_{MP}(x) = (1/(2\pi x)) \sqrt{x(x - \sigma)(x + \sigma)}\) is the distribution of the rescaled eigenvalues \(\gamma = N x\) for \(N \times N\) matrices in the Laguerre ensemble, Eq. (12).

\[16\] In the general case, the Laguerre ensemble describes \(N \times N\) random Hermitian matrices \(\Gamma\) with positive eigenvalues distributed according to \(P(\Gamma) \propto \det(\Gamma)^{a-1}\exp\left[-\beta/2 \text{Tr}\{\Gamma^2\}\right]\) (Eq. (42) corresponds to \(a = 1\) and \[39\] to \(a = 0\)). The Marčenko-Pastur law \(w_\beta(y) \sim (1/N) \rho_{MP}(\gamma/N)\) for the density of eigenvalues has support \([N x_-, N x_+]\) where \(x_\pm = (\sqrt{2\pm \sigma} \pm 1)^2\). Finite \(N\) corrections to the Marčenko-Pastur law for \(\gamma \in [0, N x_-] \cup [N x_+, \infty]\) were obtained by Forrester in Ref. \[75\]. Asymptotic expansions of Forrester’s result are \(w_N(\gamma) \sim \gamma^{\alpha N/2}/2\) for \(\gamma \to 0\) and \(w_N(\gamma) \sim \gamma^{(n+2)/2}(N/N^{-1/2})^{\alpha/2}\) for \(\gamma \to \infty\).
We can deduce the moments:

\[ \langle \tau_i \rangle = \frac{1}{N} \quad \text{and} \quad \text{Var}(\tau_i) \approx \frac{1}{N^2} \quad \forall \beta. \]  

(47)

The approach of Ref. [110] provides the precise expression of the variance [161]:

\[ \text{Var}(\tau_i) = \frac{N[\beta(N-1)+2]+2}{N^2(N+1)(\beta N-2)}. \]  

(48)

The marginal law for partial time delays \( \tilde{\tau}_a \) is given in [2], a method providing a systematic determination of the cumulants was proposed by Mezzadri and Simm in Ref. [135]. The marginal distribution was shown to present the tail \( \tilde{\tau}_N(\beta)(\tau) \sim g^{-1/2} \tau^{-3/2} \) in the intermediate range \( g^{-1} \ll \tau \ll g \), whereas the far tail is not affected \( \tilde{\tau}_N(\beta)(\tau) \sim \tau^{-2-\beta N/2} \) [82].

**Wigner time delay \( \tau_W \).** Despite the joint distribution for the proper times was exactly known [29], the statistical properties of their sum, the Wigner time delay \( \tau_W = (1/N) \sum_i \tau_i = (1/N) \sum_a \tilde{\tau}_a \), remained unknown for a while. The distribution for \( N = 2 \) channels was deduced in Ref. [160]:

\[ \mathcal{P}_2^{\beta}(\tau) = \frac{\beta^{\beta+2} \Gamma(3(\beta+1)/2)}{\Gamma(\beta+1) \Gamma(3\beta+2)} \times \tau^{-3(\beta+1)} U\left(\frac{\beta+1}{2}, 2(\beta+1); \beta/\tau\right) e^{-\beta/\tau}, \]  

(51)

where \( U(a, b; z) \) is a Kummer function (confluent hypergeometric function) [2]. A method providing a systematic determination of the cumulants was proposed by Mezzadri and Simm in Ref. [135]. The authors gave explicitly the first four cumulants

\[ \langle \tau_W \rangle = \frac{\tau_W}{N} \]  

(52)

\[ \langle \tau_W^2 \rangle_c = \frac{4 \tau_W^2}{N^2(N+1)(N\beta-2)} \]  

(53)

\[ \langle \tau_W^3 \rangle_c = \frac{96 \tau_W^3}{N^3(N+2)(N+1)(N\beta-2)(N\beta-4)} \]  

(54)

\[ \langle \tau_W^4 \rangle_c = \begin{cases} 
\frac{96(53N^3-68N-156)\tau_W^4}{N^4(N+3)(N+2)(N+1)(N\beta-4)(N\beta-6)} & \text{for } \beta = 1 \\
\frac{12(53N^3+77)\tau_W^4}{N^4(N+3)(N+2)(N+1)(N\beta-2)(N\beta-3)} & \text{for } \beta = 2 \\
\frac{12(53N^3+34N-39)\tau_W^4}{N^4(N+3)(N+2)(N+1)(2N-4)(N\beta-3)} & \text{for } \beta = 4 
\end{cases} \]  

(55)

Remarks:

- In the unitary case, a general formula for the marginal law for partial time delays \( \tilde{\tau}_N(\beta)(\tau) \) for \( N \) arbitrary (fixed) couplings between lead and dot was also obtained in Ref. [84], which reduces to (49) when all channels are perfectly coupled.

- The crossover between orthogonal (\( \beta = 1 \)) and unitary (\( \beta = 2 \)) classes was studied in Ref. [82] with the effect of non perfect coupling: in the weak coupling limit, when \( g = 2/T - 1 \gg 1 \) where \( T \) is the transmission probability, the marginal distribution was shown to present the tail \( \tilde{\tau}_N(\beta)(\tau) \sim g^{-1/2} \tau^{-3/2} \) in the intermediate range \( g^{-1} \ll \tau \ll g \), whereas the far tail is not affected \( \tilde{\tau}_N(\beta)(\tau) \sim \tau^{-2-\beta N/2} \) [82]. [84].

\[ \mathcal{P}_2^{\beta}(\tau) = \frac{\beta^{\beta+2} \Gamma(3(\beta+1)/2)}{\Gamma(\beta+1) \Gamma(3\beta+2)} \times \tau^{-3(\beta+1)} U\left(\frac{\beta+1}{2}, 2(\beta+1); \beta/\tau\right) e^{-\beta/\tau}, \]  

(51)

Figure 3: Marginal law for the proper time delay.
The analysis of the variance is instructive: we can write
\[
\text{Var}(N \tau W) = \text{Var}\left( \sum_i \tau_i \right) \sim \text{Var}(\tau_i)
\]
\[
= \text{Var}\left( \sum_a \tilde{\tau}_a \right) \approx 2N \text{Var}(\tilde{\tau}_a).
\]  

Writing \( \text{Var}(N \tau W) = N \text{Var}(\tau_i) + N(N - 1) \text{Cov}(\tau_i, \tau_j) \) and using \(48\), provides the covariance \( \text{Cov}(\tau_i, \tau_j) = -1/[N^2(N + 1)] \), showing that the proper times are anti-correlated. We have
\[
\frac{\text{Cov}(\tau_i, \tau_j)}{\sqrt{\text{Var}(\tau_i) \text{Var}(\tau_j)}} = -\frac{\beta N - 2}{N[\beta(N - 1) + 2]} + 2 \approx \frac{1}{N}. 
\]

A similar argument gives the covariance for partial times, \( \text{Cov}(\tilde{\tau}_a, \tilde{\tau}_b) = 2/[N^2(N + 1)(N\beta - 2)] \), resulting in
\[
\frac{\text{Cov}(\tilde{\tau}_a, \tilde{\tau}_b)}{\sqrt{\text{Var}(\tilde{\tau}_a) \text{Var}(\tilde{\tau}_b)}} = \frac{1}{N + 1} \approx 1. 
\]

Note that the joint probability distribution for two partial time delays was obtained in Ref. 160 (Eq. 23).

A systematic expression of the moments \( \langle \tau^q \rangle \) of the Wigner time delay for \( \beta = 2 \) was given more recently by Novaes 141. We see that the variance diverges for \( N\beta \leq 2 \), and the third cumulant for \( N\beta \leq 4 \). Writing the denominator of the fourth cumulant as \( N^4(N + 3)(N + 2)(N + 1)^2(N - 2/\beta)^2(N - 4/\beta)(N - 6/\beta) \), we see that \( \langle \tau^4 \rangle_c \) diverges for \( N\beta \leq 6 \). These observations suggest the power law tail \( P_N^{(\beta)}(\tau) \sim \tau^{-1-2\beta/N^2} \), conjectured in Ref. 84 for \( \beta = 2 \) on the basis of some heuristic argument involving resonances [i.e. identifying the tail of \( P^{(\beta)}_N(\tau) \) with the one of \( \tilde{w}^{(\beta)}_N(\tau) \), Eq. 49].

Using an Edgeworth expansion, Mezzadri and Simm concluded that the distribution weakly converges towards a Gaussian form as \( N \) grows. The full distribution of the Wigner time delay for \( N \gg 1 \) was however shown to present a richer structure in Ref. 182 where the large deviations were studied in detail, leading to the behaviours
\[
P^{(\beta)}_N(\tau) \sim \tau^{-3\beta N^2/4} e^{-\beta N/(2\tau)} \quad \text{for} \quad \tau \to 0 
\]
\[
\sim e^{-\left(\beta N^2/8\right)(N\tau - 1)^2} \quad \text{for} \quad \tau \sim 1/N 
\]
\[
\sim (N\tau - 1)^{-2-\beta N/2} \quad \text{for} \quad N\tau - 1 \gg \sqrt{(2/N) \ln N} 
\]
sketched on Fig. 4. In particular, the transition between the sharp Gaussian peak and the power law tail was shown to be related to a phase transition in underlying the Coulomb gas 182.

Figure 4: Wigner time delay distribution in the limit \( N \gg 1 \): sketch of the rescaled distribution \( P_N(s) = \tau_N \tilde{P}^{(\beta)}_N(\tau = s\tau_N) \), where \( \tau_N = h/(N\Delta) \) is the dwell time. The small curves are sketches of the optimal distributions of eigenvalues of \( Q^{-1} \) with the constraint that \( \text{Tr}\{Q\} \) is fixed. The transition between the sharp Gaussian peak and the power law tail is associated with a phase transition in the density of eigenvalues. Figure from 182.

Few remarks:

- A systematic analysis of the “moments” \( \langle \text{Tr}\{Q^n\} \rangle \) within RMT was carried out by several authors: 133, 134, 60, 141.

- The determination of the variance 115 and the covariance 116 in Ref. 91 (and also 191) has involved the correlation between the two linear statistics \( \text{Tr}\{Q\} \) and \( \text{Tr}\{Q^2\} \). A systematic analysis of the covariance \( \text{Cov}\{ \text{Tr}\{Q^n\}, \text{Tr}\{Q^{n'}\} \} \) was carried out by Cunden recently 60, based on the recent work 59 where a general formula for the covariance of two linear statistics was derived with the Coulomb gas method. More recently, the authors were also able to further analyse the correlations \( \langle \text{Tr}\{Q^{n}\}\text{Tr}\{Q^{n'}\}\cdots \rangle \) in Ref. 61.

### 4.1.2. Energy correlations

The knowledge of the correlation function \( \langle \tau W(\varepsilon)\tau W(\varepsilon') \rangle = \langle \tau W(\varepsilon)\tau W(\varepsilon') \rangle - \langle \tau W(\varepsilon) \rangle \langle \tau W(\varepsilon') \rangle \) is of importance and has practical applications: for example, this information was used by Polianiski and Böttiker in order to study the effect of thermal fluctuations on the non-linear conductance of a quantum dot 151. The correlation function was obtained by Lehmann et al. 117 by a random matrix analysis in the orthogonal case \( (\beta = 1) \), within
the HA of random matrices (see also [81]):
\[
\frac{\langle \tau_W(\varepsilon) \rangle}{\tau_W^2} \approx 1 - \frac{1}{2N^2 (1 + \omega \tau_d)^2},
\]
where \( \omega = \varepsilon - \varepsilon' \).

We have introduced
\[
\tau_d = \frac{2\pi}{N\Delta} = \frac{\tau_h}{N},
\]
the dwell time for an electron in the cavity (Eq. 63), valid for strongly overlapping resonances, \( 1/\tau_d \gg \Delta \), i.e., \( N \gg 1 \). The result (63) was later generalised [82] in order to include parametric correlations and describe the crossover between orthogonal and unitary cases.

Time delay correlations can also be determined within the stochastic approach: the model described in Refs. [5, 148, 31] and used by Polianski and Büttiker [149, 150, 151] provides some information about the energy and magnetic field dependence of the \( S \)-matrix correlator, which reads [151]
\[
\langle S_{ab}(\varepsilon, B)S^*_{cd}(\varepsilon', B') \rangle = \delta_{ac}\delta_{bd} \rho_{\varepsilon-\varepsilon'} + \delta_{ad}\delta_{bc} \rho_{\varepsilon-\varepsilon'}. \tag{65}
\]
\( \rho_{\omega} \) and \( \mathcal{C}_{\omega} \) are the (zero-dimensional) analogues of the Diffuson and the Cooperon appearing in the diagrammatic approach for weakly disordered metals [4, 18] given by [19]
\[
\rho_{\omega} = \frac{1}{N\tau_d 1/\tau_{\mathcal{G}_{\omega}}, \varepsilon - \omega},
\]
The two characteristic times
\[
\frac{1}{\tau_{\mathcal{G}_{\omega}, \varepsilon'}} = \frac{1}{\tau_d} + \frac{1}{\tau(2\pi B')/2}, \tag{67}
\]
combines a contribution describing the escape rate \( 1/\tau_d \) from the cavity and dephasing due to the magnetic field. The magnetic dephasing rate is \( 1/\tau_B = (v_F l/\text{Surf}) (\Phi/\phi_0)^2 \) where \( v_F \) is the Fermi velocity, \( \Phi = \text{Surf} B \) the magnetic flux through the cavity and \( \phi_0 = h/e \) the quantum flux. \( l \) is the size of the cavity in the ballistic case or the elastic mean free path in the weakly disordered (diffusive) case.

As a simple application of the formula we get
\[
\langle \tau_W(\varepsilon) \rangle \approx \tau_d (N^2 \rho_0^2 + N'\rho_0^2), \tag{68}
\]
where we have used \(-i\partial_\varepsilon \rho_{\varepsilon-\varepsilon'} = \tau_h \rho_{\varepsilon-\varepsilon'}^2 \). We deduce \( \langle \tau_W(\varepsilon) \rangle / \tau_d \approx 1 + O(1/N) \), whose leading order term coincides with [52].

The analysis of the correlations requires additional information: simply applying Wick’s theorem with the correlator (63), we get
\[
\langle \tau_W(\varepsilon, B)\tau_W(\varepsilon', B') \rangle \approx \frac{1}{\tau_h} \left( |\mathcal{D}_{\varepsilon-\varepsilon'}|^4 + |\mathcal{C}_{\varepsilon-\varepsilon'}|^4 \right), \tag{69}
\]
which leads, at zero magnetic field, to the incomplete expression
\[
\langle \tau_W(\varepsilon, 0)\tau_W(\varepsilon', 0) \rangle_{\text{Gaussian}} / \langle \tau_W \rangle^2 \approx (2/N^2) \left[ 1 + (\omega \tau_d)^{-2} \right]. \tag{70}
\]
This calculation, known as the “diagonal approximation” in the context of semiclassical methods, disagrees with [63]: not only the energy dependence differs but also the value at \( \omega = 0 \) is half the correct result [compare with the variance (53)]. This emphasizes the importance of non Gaussian contributions to the correlator, which are taken into account in the more precise expression of the four-point correlation function \( \langle S_{ab}S_{cd}'S_{de}'S_{el}' \rangle \) given in Appendix B of Ref. [81] (see also [27]).

This question has been much discussed in semiclassical approach: this discrepancy was underlined by Lewenkopf and Vallejos [120]. Kuijpers and Sieber [112] identified the nature of the contributions (“trajectory quadruplets”) to be added to the diagonal approximation in order to recover the random matrix result (68) (so that they have reconciled the two semiclassical approaches for a Wigner time delay analysis: the periodic orbit expansion based on the relation with the density of states [57], and the scattering approach involving trajectories entering and leaving the system). The expression of the correlator, determined earlier within a semiclassical approach by Vallejos et al. [189] (for \( B = B' \)), is the sum of two contributions which can be identified as Diffuson and Cooperon contributions. Reintroducing the effect of the field mismatch, we get the structure obtained within the HA of random
matrices in Ref. [82, 84]:

$$\langle \tau_W(\varepsilon, B)\tau_W(\varepsilon^\prime, B^\prime) \rangle \sim \frac{2}{\tau_H^2} \left\{ \frac{1}{1/\tau^2_\varepsilon - \omega^2} + \frac{1/\tau^2_\varepsilon - \omega^2}{1/\tau^2_\varepsilon + \omega^2} \right\},$$

(68)

where we recall that $\tau_H = 2\pi/\Delta$. This expression now agrees with (65) for $\tau_g = \tau_\varepsilon = \tau_d = \tau_H/N$.

**Few remarks:**

- The time delay may be represented in terms of $\tau_W(\varepsilon) = (1/N) \sum_\alpha \Gamma_\alpha/[(\varepsilon - \varepsilon_\alpha)^2 + \Gamma^2_\alpha/4]$, where the sum runs over the resonances [117, 84]. This establishes a connection between Wigner time delay fluctuations and Ericson fluctuations of the cross-section for strongly overlapping resonances $1/\tau_d \gg 1/\tau_H \sim \Delta$ (i.e. $N \gg 1$) [189].

- More recently, an improved semiclassical approach for the analysis of the Wigner time delay statistics was developed by Kuipers, Savin and Sieber [110], who carried out a diagrammatic calculation of the moments of the Wigner time delay. A semiclassical derivation of the moments of the time delay was also proposed by Novaes [142].

### 4.2. Other symmetry classes

As we have already mentioned, the three Wigner-Dyson symmetry classes (denoted AI, A and AII in the Altland-Zirnbauer classification) were completed by three chiral classes (chiral orthogonal BDI, chiral unitary AIII and chiral symplectic CII) and four Bogoliubov-de Gennes classes (C, CI, D, DIII) [197, 6, 71]. The new symmetry classes have attracted a lot of attention during the last years in relation with topological insulators [98, 153] and topological superconductors [15, 16] (see also [158] where we have neglected the term $\text{Tr} \{ S^1 \} /\varepsilon$ in Eq. (16) (this is justified in the metallic regime). Such relation allows to characterize the amount of charge injected in an open coherent conductor, a crucial tool which has been used by Büttiker in order to describe screening properties. The out-of-equilibrium situation where the conductor is connected to several contacts (terminals) with different chemical potentials however requires to identify the contributions of each terminal to the DoS, which has led Büttiker to introduce several generalisations of (70) as the partial DoS, the injectance and the emittance. These concepts have been used to develop a theory of non-linear transport [58, 51] and AC transport [38, 147, 49, 152, 52] in coherent conductors (see the reviews [193, 93, 111, 10]).

#### 5.1. Partial DoS, injectance and emittance

Let us consider a multi-terminal structure, whose scattering properties are characterised by a basis of
stationary scattering states $\psi_{\nu,\alpha}(x)$ where the index $\alpha$ labels the terminals. We assume that the terminals support each a single conducting channel (i.e. contact wires are effectively one-dimensional), what simplify the analysis (the generalisation to contacts with several channels is straightforward). Furthermore, this would allow to illustrate the discussion by explicit formulae by considering the case of metric graphs for which explicit construction of the scattering matrix is possible.

Büttiker introduced the concept of "partial density of states" [38, 109]

$$\nu_{\alpha\beta}(\epsilon) \approx \frac{1}{4\pi} \left( S_{\alpha\beta}^* \frac{\partial S_{\alpha\beta}}{\partial \epsilon} - \frac{\partial S_{\alpha\beta}^*}{\partial \epsilon} S_{\alpha\beta} \right)$$

measuring the contribution to the DoS [70] of particles incoming from terminal $\beta$ and outgoing at contact $\alpha$. Another quantity which appears in Büttiker’s work is the "injectance", obtained by summation of (70) over the first index

$$\nu_{\alpha\beta}(\epsilon) \approx \frac{1}{4\pi} \left( S_{\alpha\beta}^* \frac{\partial S_{\alpha\beta}}{\partial \epsilon} - \frac{\partial S_{\alpha\beta}^*}{\partial \epsilon} S_{\alpha\beta} \right)$$

It provides the contribution to the DoS of the scattering states incoming from terminal $\alpha$. Summation over the second index leads to the "emittance"

$$\nu_{\alpha\beta}(\epsilon) \approx \frac{1}{2\pi} \left( \frac{\partial S}{\partial \epsilon} S^\dagger \right)_{\alpha\alpha}$$

which corresponds to the contribution of particles outgoing at terminal $\beta$. Injectance and emittance are related by magnetic field reversal,

$$\nu_{\alpha}(\epsilon; B) = \nu_{\alpha}(\epsilon; -B)$$

which follows from $S(-B) = S(B)^\dagger$. Obviously

$$\nu(\epsilon) = \sum_{\alpha} \nu_{\alpha}(\epsilon) = \sum_{\alpha} \nu_{\alpha\beta}(\epsilon) = \sum_{\alpha,\beta} \nu_{\alpha\beta}(\epsilon).$$

Injectance and emittance allow for a preselection and a postselection, respectively, when identifying the contribution to the density inside the conductor from particles passing through it [88].

Few remarks:

- All formulae can be straightforwardly generalised to the case of multichannel contacts by adding some terms over channels.

- These concepts have been further generalised in order to deal with local properties, leading to the concept of partial local DoS, injectivity and emissivity [38, 88]. This can be understood from the relation

$$-\frac{1}{2\pi} \left( S^\dagger \frac{\delta S}{\delta \mathcal{U}(x)} \right)_{\alpha\beta} \approx \psi_{\nu,\alpha}(x) \psi_{\nu,\beta}(x)$$

where $\mathcal{U}(x)$ is the potential (see Appendix of Ref. [40] or Refs. [173, 178, 180]). This leads to introduce the injectivity

$$\nu_{\alpha}(x; \epsilon) = -\frac{1}{2\pi} \left( S^\dagger \frac{\delta S}{\delta \mathcal{U}(x)} \right)_{\alpha\alpha} \approx |\psi_{\nu,\alpha}(x)|^2$$

which measures the contribution of the scattering state to the local DoS $\nu(\epsilon; x) = \sum_{\alpha} \nu_{\alpha}(x; \epsilon)$. Similarly, Büttiker introduced the concepts of partial local DoS and emissivity [38, 88].

- The Fisher and Lee relation [74]

$$S_{\alpha\beta}(\epsilon) = -\delta_{\alpha\beta} + i/\sqrt{\nu_{\alpha\beta}} G^{\text{R}}(\epsilon, \alpha; \beta; \epsilon),$$

where $\nu_{\alpha\beta}$ is the group velocity in terminal $\alpha$ and $G^{\text{R}}$ the retarded Green’s function is measured at the two terminals, makes the functional derivation transparent : $\delta S_{\alpha\beta}/\delta \mathcal{U}(x) = i/\sqrt{\nu_{\alpha\beta}} G^{\text{R}}(\epsilon, \alpha; \beta; \epsilon)$. Cf. also the discussion in [178].

- An illustration: the knowledge of the injectivities allows to express the density of electrons in the conductor out-of-equilibrium

$$n(x) = \sum_{\alpha} \int d\epsilon f(\epsilon - eV_\alpha) \nu_{\alpha}(x; \epsilon),$$

where $f(\epsilon)$ is the Fermi function and $V_\alpha$ the electric potential at contact $\alpha$.

- The introduction of the injectivities allows to avoid the approximation made in (70) by neglecting the term $\text{Tr} \left\{ S - S^\dagger \right\} / \epsilon$ of Eq. (16), and provides a better definition for the partial DoS, injectance and emittance. Introducing a uniform potential $\mathcal{U}(x) = U$ inside the conductor (but not in the contacts), we can write

$$-\frac{1}{2\pi} \left( S^\dagger \frac{\delta S}{\delta U} \right)_{\alpha\beta} \approx \int_{\text{QD}} dx \psi_{\nu,\alpha}^*(x) \psi_{\nu,\beta}(x),$$

where $\int_{\text{QD}} \cdots$ denotes integration inside the conductor [the boundaries of integration are
given by the place where the scattering states are matched with plane waves in order to define scattering amplitudes, like in Eq. (2). In other terms \[ \mathbf{S} \frac{\partial \mathbf{S}}{\partial U} = \mathbf{S} \frac{\partial \mathbf{S}}{\partial \varepsilon} + \mathbf{S} - \mathbf{S} \frac{1}{4\varepsilon}. \] (80)

Therefore, instead of (72) which involves a high energy approximation, a more rigorous definition of the injection should be

\[ \nu_{\alpha}(\varepsilon) = -\frac{1}{2\pi} \left( \mathbf{S} \frac{\partial \mathbf{S}}{\partial U} \right)_{\alpha}. \] (81)

The remark also holds for partial DoS and emittances, Eqs. (71) [28]. The DoS takes the form \[ \nu(\varepsilon) = -(2\pi)^{-1} \partial U/\det \mathbf{S}. \]

5.2. A generalisation of the Feynman-Hellmann theorem for a continuous spectrum

Büttiker’s idea of relating the spectral properties to the scattering matrix is not limited to the DoS and the local DoS and can be generalised. This may be viewed as an extension of the famous Feynman-Hellmann theorem which applies to bounded problems with discrete energy spectra. Let us first recall this well-known theorem: consider a Hamiltonian \( \mathbf{H} \) characterised by the discrete spectrum \( \{ \varepsilon_n, |\psi_n\rangle \} \). The determination of the diagonal matrix elements of an observable \( X = -\partial \mathbf{H}/\partial f \), where \( f \) is a conjugate force, does not require the knowledge of the eigenvectors but only of the eigenvalues: \( \langle \psi_n|X|\psi_n\rangle = -\partial \varepsilon_n/\partial f \). In Ref. [180], we proposed that an extension of this theorem to the case of Hamiltonians with continuous spectra, characterised by scattering stationary states \( \{|\psi_{\varepsilon,\alpha}\rangle\} \), is

\[ \langle \psi_{\varepsilon,\alpha}|X|\psi_{\varepsilon,\beta}\rangle = \frac{1}{2\pi} \left( \mathbf{S} \frac{\partial \mathbf{S}(\varepsilon)}{\partial f} \right)_{\alpha\beta}. \] (82)

This relation was proved in [178, 180] for various observables in metric graphs [28] for example, choosing the local density \( X \rightarrow \rho(x) = |x\rangle\langle x| \) with \( f \rightarrow -\partial \mathbf{U}(x) \), we recover (76). In [180], we considered also the case of the current density \( I_a \) in a wire \( (a) \) of the graph, which involves a derivation with respect to the magnetic flux \( \phi_a \) along the wire \( \langle \psi_{\varepsilon,\alpha}|I_a|\psi_{\varepsilon,\beta}\rangle = (2\pi)^{-1} \left( \mathbf{S} \frac{\partial \mathbf{S}}{\partial \phi_a} \right)_{\alpha\beta}. \) A

\[ \text{trace over indices leads to the well-known formula obtained by Akkermans, Auerbach, Avron and Shapiro [3]: } \langle I \rangle_\varepsilon = (2i\pi)^{-1} \partial \varepsilon \ln \det \mathbf{S}. \] Note that similar considerations were used in order to analyse current fluctuations at equilibrium in Ref. [172] (a formula for current correlations out-of-equilibrium was obtained in [180]).

5.3. Statistical analysis in quantum dots

In the case of chaotic quantum dots with several contacts with perfect couplings (Fig. 3), Brouwer and Büttiker [28] obtained the first two moments of the partial DoS (71) within the ASA of random matrices [28]

\[ \langle \nu_{\mu\nu} \rangle = \frac{1}{\Delta} \left[ N_\mu N_\nu \left( \frac{2}{\beta} - 1 \right) \left( \frac{N_\mu N_\nu}{\Delta^2} - \frac{N_\mu \delta_{\mu\nu}}{\Delta} \right) \right] \] (83)

where \( N_\mu \) is the number of open channels at contact \( \mu \) and \( N = \sum_\mu N_\mu \). We deduce \( \langle \nu_{\alpha\alpha} \rangle = \langle \nu \rangle = (N_\alpha/\Delta^{-1}) \), as expected in an ergodic device.

In the unitary case \( (\beta = 2) \), the covariances are [28]

\[ \text{Cov} \langle \nu_{\rho\sigma}, \nu_{\mu\nu} \rangle = \frac{1}{(\Delta^2)} \left[ N_\mu N_\nu \left( \delta_{\mu\rho} N_\sigma + \delta_{\nu\sigma} N_\rho \right) \right] \] (84)

The correlator for the orthogonal case \( (\beta = 1) \) is obtained by adding similar terms obtained by permutation \( \rho \leftrightarrow \sigma \) in order to fulfill the symmetry \( \nu_{\rho\sigma} = \nu_{\sigma\rho} \) (for \( \mathbf{B} = 0 \)); the symplectic case receives additionally a factor \( 1/4 \). Eq. (84) shows that mesoscopic fluctuations are of order \( \delta \nu_{\alpha\beta} \sim (\nu_{\alpha\beta})/N \sim 1/(\Delta^2) \).

The correlations of injectances and emittances are also of interest. We deduce (for \( \beta = 1, 2 \))

\[ \text{Cov} \langle \nu_{\mu}, \nu_{\nu} \rangle = \text{Cov} \langle \nu_{\mu}, \nu_{\nu} \rangle \] (85)

\[ = \frac{1}{(\Delta^2)^2} \left[ N_\mu N_\nu + \delta_{\mu\nu} N_\mu N_\nu + \left( \frac{2}{\beta} - 1 \right) \frac{2N_\mu N_\nu}{\Delta^2} \right] \] (86)

\[ \text{Note that the expression (82) does not account for the contributions of BICs if present, as the DoS (180).} \]

\[ \text{Brouwer and Büttiker considered the dimensionless AC conductances } G_{\alpha\beta}(\omega) = G_{\alpha\beta}(0) - i\omega E_{\alpha\beta} + O(\omega^2), \text{ which, in the unscreened limit, can be related to the partial DoS (71): } \lim_{\Delta \to \infty} E_{\alpha\beta} = E_{\alpha\beta} = 2\pi f \partial \varepsilon \langle I \rangle_{\varepsilon} \] (87)
and

\[
\text{Cov}(\tau_\mu, \nu_\nu) = \frac{1}{(N\Delta)^2} \times \left[ \frac{2N_\mu N_\nu}{N^2} + \left( \frac{2}{\beta} - 1 \right) \left( \frac{N_\mu N_\nu}{N^2} + \delta_\mu \frac{N_\nu}{N} \right) \right].
\]

(86)

The interchange of the two contributions between (85) and (86) is analogous to the exchange between Diffuson and Cooperon in weakly disordered metals [183]. These expressions allow us to quantify the difference between injectance and emittance

\[
(\langle \tau_\alpha - \nu_\alpha \rangle)^2 = \left( 2 - \frac{2}{\beta} \right) \frac{2}{(N\Delta)^2} \frac{N_\alpha}{N} \left( 1 - \frac{N_\alpha}{N} \right).
\]

(87)

The difference vanishes in the orthogonal case (\beta = 1), as a consequence of the symmetry [74]. This quantity has found some application in [159].

6. Non-linear transport in coherent conductors

This section gives a brief presentation of the theory of non-linear transport in coherent conductors proposed by Büttiker and Christen [38, 51, 43], illustrating the interest of the quantities introduced in the previous section. We consider the case of conductors of mesoscopic dimensions such that they can be considered in the ergodic regime (like quantum dots), which slightly simplifies the presentation, although the theory was developed in a more general context in Ref. [38, 51].

![Figure 5: A conductor of mesoscopic dimension connected to M = 3 macroscopic reservoirs and capacitively coupled to a gate. When the potential at contact 0 is raised, current is injected in the conductor from this contact (blue arrows).](image)

6.1. Non-linear conductances

We consider a multiterminal mesoscopic structure (Fig. 5). For small voltages \( V_\alpha \to 0 \), current in terminal 0 can be written under the form of an expansion

\[
I_\alpha = \frac{2e^2}{h} \sum_\beta g_{\alpha\beta} V_\beta + \frac{2e^2}{h} \sum_\beta g_{\alpha\beta\gamma} V_\beta V_\gamma + \cdots
\]

(88)

where 2\( e \) is the spin degeneracy, \( g_{\alpha\beta} \) is the dimensionless Landauer-Büttiker linear conductance [37]

\[
g_{\alpha\beta} = \int d\varepsilon \left( -\frac{\partial f}{\partial \varepsilon} \right) g_{\alpha\beta}(\varepsilon).
\]

(89)

\( f(\varepsilon) \) is the Fermi distribution and \( g_{\alpha\beta}(\varepsilon) = N_\alpha \delta_{\alpha,\beta} - \text{Tr}\{S_{\alpha,\beta}(\varepsilon)S_{\alpha,\beta}(\varepsilon)\} \) the zero temperature dimensionless conductance, \( N_\alpha \) being the number of conducting channels in contact \( \alpha \).

The Landauer-Büttiker formula \( I_\alpha = (2e^2/h) \sum_\beta \int d\varepsilon f(\varepsilon - eV_\beta) g_{\alpha\beta}(\varepsilon) \) already produces a contribution to the non-linear conductance

\[
g_{\alpha\beta\gamma} = \frac{1}{2} \int d\varepsilon \left( -\frac{\partial f}{\partial \varepsilon} \right) g'_{\alpha\beta\gamma}(\varepsilon).
\]

(90)

Büttiker emphasized the importance of screening effect which produces a second contribution that we now determine. The first step is to analyse the electrostatic inside the conductor.

6.2. Characteristic potentials

Let us recall the main ingredients involved in the description of screening in good metals [199], which Büttiker adapted to deal with the non-equilibrium situation. These analysis is based on three ingredients. (i) When charges are introduced in the conductor, the total density of carriers \( \delta n_{\text{in}}(x) = \delta n_{\text{ext}}(x) + \delta n_{\text{int}}(x) \) is the sum of the density \( \delta n_{\text{ext}}(x) \) of charges injected in the conductor and the density \( \delta n_{\text{int}}(x) \) induced by screening.

The concept of injectance introduced above allows us to write the number of injected charges when the conductor is out-of-equilibrium. An expansion of \( \delta n_{\text{ext}}(x) = n(x) - n_{\text{equil}}(x) \) where \( n(x) \) is given by [78] and \( n_{\text{equil}}(x) \) is the equilibrium density, gives

\[
\delta n_{\text{ext}} = \int_{Q\text{D}} dx \delta n_{\text{ext}}(x) \simeq \sum_{\alpha=1}^M eV_\alpha \varphi_\alpha(\varepsilon_F),
\]

(91)

where the summation runs over the \( M \) contacts. We assume zero temperature for simplicity; the finite temperature formula involves an additional convolution with a Fermi function \( \int d\varepsilon \left( -\partial f/\partial \varepsilon \right) \cdots.\)
(ii) The induced density is related to the potential energy $\mathcal{W}(x)$ by linear response theory
\begin{equation}
\delta n_{\text{ind}}(x) = -\int dz' \Pi(x, x') \mathcal{W}(x') \tag{92}
\end{equation}
where $\Pi(x, x')$ is the (static) density-density correlation function of the non interacting electron gas (Lindhard function [33]). Assuming ergodic properties, the potential energy can be considered uniform in the conductor $\mathcal{W}(x) \simeq U$. Thomas-Fermi approximation relates the response to the DoS:
\begin{equation}
\delta n_{\text{ind}} = \int_{QD} dx \delta n_{\text{ind}}(x) \simeq -\nu(\varepsilon_F) U . \tag{93}
\end{equation}
(iii) The potential energy and the density are related by the Poisson equation $\Delta \mathcal{W}(x) = -4\pi e^2 \delta n_{\text{tot}}(x)$ which encodes the Coulomb interaction in the conductor. The fact that $\Pi$ involves the potential $\mathcal{W}(x)$ and not the electrostatic potential related to the external density $\delta n_{\text{ext}}(x)$ makes the approach self-consistent. For a uniform potential, the Poisson equation is replaced by
\begin{equation}
C(U - eV_0) = e^2(\delta n_{\text{ext}} + \delta n_{\text{ind}}) \tag{94}
\end{equation}
where $C$ is the capacitance of the conductor and $V_0$ the electrostatic potential of the gate, labelled with index $0$ (Fig. 5). Excess charge inside the conductor occurs when the system is driven out-of-equilibrium by the external potentials. At linear order, it is convenient to decompose the potential over contributions of the different external voltages
\begin{equation}
U \simeq \sum_{\alpha=0}^{M} u_{\alpha} eV_{\alpha} \tag{95}
\end{equation}
where the “characteristic potential” $u_{\alpha}$ measures the response of the potential $U$ to a shift of the external voltage $V_{\alpha}$ [38, 51] (one should not forget the response of the gate voltage controlled by $u_0$). Injecting (91, 93, 95) in (94) provides the characteristic potentials
\begin{equation}
u(u_{\alpha}) = \frac{C_\mu}{C} \frac{\nu(\varepsilon_F)}{\nu(\varepsilon_F)} \tag{96}
\end{equation}
for $\alpha \in \{1, \ldots, M\}$ and
\begin{equation}
u(u_0) = 1 - \frac{C_\mu}{C} = \frac{C_\mu}{C_q} \tag{97}
\end{equation}
We have introduced the “mesoscopic capacitance”
\begin{equation}
\frac{1}{C_\mu} = \frac{1}{C} + \frac{1}{C_q} , \tag{98}
\end{equation}
combining the “geometric capacitance” $C$ and the “quantum capacitance”
\begin{equation}
C_q = e^2 \nu(\varepsilon_F) . \tag{99}
\end{equation}
We can check the sum rule
\begin{equation}
\sum_{\alpha=0}^{M} u_{\alpha} = 1 . \tag{100}
\end{equation}

The ratio $\gamma_{\text{int}} = C_\mu/C$ in Eq. (96) may be viewed as an interaction constant as it measures the efficiency of screening ($\gamma_{\text{int}} \simeq 1$ for efficient screening in a good metal and $\gamma_{\text{int}} \ll 1$ for weak screening). The response of the gate $u_0 = 1 - \gamma_{\text{int}}$ is thus important when screening is weak, however the presence of the gate (or the surrounding medium) can be forgotten in good metals.

6.3. Non-linear conductances

We can now determine the non-linear conductances as follows. The redefinition of the electrostatic potential inside the conductor leads to a modification of its scattering properties which depends on the potential, what contributes to the non-linear response : $g_{\alpha\beta}(\varepsilon_F) \rightarrow g_{\alpha\beta}(\varepsilon_F - U) \simeq g_{\alpha\beta}(\varepsilon_F) - g_{\alpha\beta}(\varepsilon_F) \sum_\gamma u_\gamma eV_\gamma$. After symmetrisation with respect to indices $\beta \leftrightarrow \gamma$ and reintroducing the Fermi function, we get the expression of the non-linear conductance [31, 49]
\begin{equation}
g_{\alpha\beta\gamma} = \frac{1}{2} \int d\varepsilon \left( -\frac{\partial f}{\partial \varepsilon} \right) \times \left[ g_{\alpha\beta}(\varepsilon) \delta_{\beta\gamma} - g_{\alpha\beta}(\varepsilon) u_\gamma - g_{\alpha\gamma}(\varepsilon) u_\beta \right] , \tag{101}
\end{equation}
where the first term comes from the noninteracting theory, Eq. (96). For consistency, the $T = 0$ expression (96) should be replaced by
\begin{equation}
u(u_{\alpha}) = \int d\varepsilon \left( -\partial_\varepsilon f \right) \nu(\varepsilon) . \tag{102}
\end{equation}
The second term of the denominator is the finite temperature quantum capacitance $C_q = e^2 \int d\varepsilon \left( -\partial_\varepsilon f \right) \nu(\varepsilon)$.

Few remarks :
- As emphasized above, all quantities involved in Eqs. (101, 102), $g_{\alpha\beta}(\varepsilon)$, $\nu(\varepsilon)$ and $\nu(\varepsilon)$, are expressed in terms of the $S$-matrix. The derivative of the conductance $g'_{\alpha\beta}(\varepsilon)$ can also be related to the sensitivities introduced by Gasparian, Christen and Büttiker [88] :
\begin{equation}
\eta_{\alpha\beta}(\varepsilon) \simeq \frac{1}{4\pi} \left( S_{\alpha\beta}^* \frac{\partial S_{\alpha\beta}}{\partial \omega} + \frac{\partial S_{\alpha\beta}^*}{\partial \omega} S_{\alpha\beta} \right) . \tag{103}
\end{equation}
A statistical analysis of the non-linear conductance in chaotic quantum dots requires the statistical properties of the conductance’s derivative provided in Ref. [32] and those of the characteristic potentials, i.e. of the injectivities, obtained in Ref. [28], see Eqs. (83, 84).

6.4. Recent developments

- As mentioned, the Büttiker’s scattering formalism can be extended to discuss conductors which are not in the ergodic regime [38, 51, 43]. The equivalence between this formalism and the non equilibrium Green’s function approach for non-linear transport was established by Hernández and Lewenkopf [102] (see also [183]).

- An interesting aspect was identified by Sánchez and Büttiker [159] and Spivak and Zyuzin [169]: the interacting part of the non-linear conductance \( g_{\alpha\beta\gamma}^{\text{int}} = -(1/2)[g^{\prime}_{\alpha\beta}(\varepsilon_F) u_{\gamma} + g^{\prime}_{\alpha\gamma}(\varepsilon_F) u_{\beta}] \) is not constrained to any specific symmetry under reversal of the magnetic field, contrary to the non-interacting part \( g_{\alpha\beta\gamma}^{0} = (1/2)g^{\prime}_{\alpha\beta}(\varepsilon_F)\delta_{\beta\gamma} \).

- This is due to the asymmetry of the characteristic potentials \( u_{\alpha} \) (i.e. of the injectance \( \nu_{\alpha} \)). The analysis of the asymmetry of the non-linear conductance under magnetic field reversal \( \frac{g_{111}(B) - g_{111}(-B)}{2} = -(C_{\mu}/C) g^{\prime}_{11}(\varepsilon_F) (\tau_1 - \tau_1)/(2\nu) \) was thus proposed as a new way for probing electronic interactions in mesoscopic structures (the parameter \( \tau_{\text{int}} = C_{\mu}/C \) introduced above was measured in Ref. [9] for a mesoscopic ring). Spivak and Zyuzin borrowed arguments from diagrammatic techniques in order to analyse the low magnetic field regime [169]. Sánchez and Büttiker [159] proposed a random matrix approach in the unitary case (strong magnetic field), which was later improved by Büttiker and Polianski who extended the statistical analysis in order to describe the crossover from orthogonal to unitary cases and include thermal effects [149, 150] (for more details, cf. the excellent review article [151]). Note also the study for a ring made of strictly 1D wires [101].

- Several experimental groups have analysed the non-linear transport in mesoscopic structures and specifically the asymmetry in magnetic field [118, 126, 138, 9, 8].

- The case of diffusive wires is considered in Ref. [183].

7. AC transport

The search for fast control and manipulation of charge in coherent conductor has stimulated many developments (see the reviews [86, 23]). Büttiker, Prêtre and Thomas have proposed a theory of time-dependent response in coherent conductors [47, 48, 49] based on the scattering approach and a Hartree-Fock treatment (with Thomas-Fermi approximation) of electronic interactions. In order to emphasize few ideas, we will restrict ourselves here to the case of the “quantum RC circuit” which was studied experimentally in the integer quantum Hall regime during the PhD of Gabelli [85, 173, 87] (Fig. 6). The RC circuit was later studied from the perspective of an emitter of electronic wave packets in the PhD of Fève [73] and as such is a building block for “electron optics experiment” (see the review [23] and article [103] of this special issue). These latter developments have mostly considered the RC circuit in the integer quantum Hall regime, such that the current is carried by a single edge state along the boundary of the system, when one conducting channel is open at the constriction (quantum point contact, QPC). Below, we do not consider the integer quantum Hall regime but we rather discuss the situation where the RC circuit of Fig. 6 is submitted to a weak magnetic field, such that the electron dynamics inside the cavity is chaotic. This justifies a random matrix approach. In this case, the opening of the constriction controls the number of channels \( N \).

Figure 6: The quantum RC circuit: a coherent conductor patterned in a 2DEG (blue) is closed by a quantum point contact (QPC) controlled by side gates. A capacitive coupling to a top gate allow to analyse its AC response. Figure from Ref. [97].
7.1. The AC response

By using arguments similar to the ones described in Section 6, Büttiker and coworkers have shown that the Coulomb interaction can be taken into account by adding the impedance of the non-interacting electrons, \(1/G_0(\omega)\), and the classical impedance of the capacitance \(Z(\omega) \equiv 1/G(\omega) = 1/G_0(\omega) + 1/(\omega \varepsilon C)\). The admittance is \([47]\)

\[
G_0(\omega) = \frac{2e^2}{h} \int d\varepsilon \text{ Tr} \left\{ 1 - S^\dagger(\varepsilon)S(\varepsilon + \omega) \right\} \times \frac{f(\varepsilon) - f(\varepsilon + \omega)}{\omega},
\]

(104)

where \(2s\) denotes the spin degeneracy. One can write the low frequency expansion of the impedance under the form given by the elementary laws of electrokinetics

\[
Z(\omega) = \frac{1}{-i\omega C_\mu} + R_q + O(\omega),
\]

(105)

where the “mesoscopic capacitance” \(C_\mu\) and the “charge relaxation resistance” \(R_q\) carry some information about the dynamics of electrons in a quantum coherent regime, as they control the relaxation time \(\tau_{RC} = R_q C_\mu\). The presence of the mesoscopic capacitance \([38]\) in the AC response comes from the \(\omega \to 0\) expansion of \(G_0(\omega)\), whose first terms involve the Wigner-Smith matrix \([12]\). However, this latter controls both the quantum capacitance \([59]\)

\[
C_q = \frac{2s e^2}{h} \text{ Tr} \{Q\}
\]

(106)

and the charge relaxation resistance

\[
R_q = \frac{h}{2s} \times \frac{\text{ Tr} \left\{ Q^2 \right\}}{\left( \text{ Tr} \{Q\} \right)^2}.
\]

(107)

In these equations, the Wigner-Smith matrix is taken at Fermi energy \(\varepsilon_F\).

7.2. The charge relaxation resistance

In this section we forget about spin degeneracy factor \(2s\) and base the discussion on the expression

\[
R_q = \frac{h}{2e^2} \frac{\text{ Tr} \left\{ Q^2 \right\}}{\left( \text{ Tr} \{Q\} \right)^2} = \frac{h}{2e^2} \frac{\sum r_\alpha^2}{\left( \sum r_\alpha \right)^2}.
\]

(108)

A first important observation is that the resistance belongs to the interval:

\[
\frac{h}{2Ne^2} \leq R_q \leq \frac{h}{2e^2}.
\]

(109)

In particular the \(N = 1\) channel case leads to the universal value \(R_q = h/(2e^2)\) \([45]\) (this is a property of a coherent conductor, cf. remarks closing the section).

Case \(N = 2\) —. Assuming a random matrix description of the quantum dot, we can deduce the distribution of \(R_q\) starting from \([12]\). The calculation only involves a double integral \([34]\):

\[
p_\beta(r_q) = A_\beta (2r_q - 1)^{(\beta-1)/2} (1 - r_q)^\beta
\]

(110)

where \(r_q = (N/e^2/h) R_q\) is the dimensionless charge relaxation resistance per channel : \(A_\beta = 2^{\beta+1}/B(\beta + 1, (\beta + 1)/2)\) is a normalisation constant. We deduce the mean value

\[
\langle R_q \rangle = \frac{h}{3e^2}
\]

(111)

which is surprisingly independent on the symmetry class.

Remark : The distribution of \(R_q\) for \(N = 2\) channels was analysed by Pedersen, van Langen and Büttiker in \([140]\), however these authors introduced a slightly different averaging procedure : following \([32]\) the authors weighted the joint distribution of proper times by the DoS, corresponding to a “canonical” averaging. They deduced \(p_\beta^{(\text{PvLB})}(r_q) = B_\beta (2r_q - 1)^{(\beta-1)/2} (1 - r_q)^{\beta-1}\) where the normalisation is \(B_\beta = 2^{\beta+1}/B(\beta + 1, (\beta + 1)/2)\). As a result the mean resistance depends on the symmetry index \(\langle R_q \rangle^{(\text{PvLB})} = (3/8) h/e^2\) for \(\beta = 1\) and \(\langle R_q \rangle^{(\text{PvLB})} = (5/14) h/e^2\) for \(\beta = 2\).

Mean value —. Büttiker, Prêtre and Thomas \([48]\) already pointed out that the charge relaxation resistance should scale with the number of channels as \(R_q \sim 1/N\), which corresponds to the addition of resistances in parallel, although the precise prefactor was not given. The fact that the one channel case (spin polarised) leads to the universal half quantum of resistance \(R_q = h/(2e^2)\) has produced a certain confusion for the large \(N\) case, between \(R_q = h/(2Ne^2)\) (incorrect) and \(R_q = h/(Ne^2)\) (correct).

The first precise calculation of the mean value \(\langle R_q \rangle\) in the \(N \gg 1\) limit is contained in the study of the complex admittance \(G(\omega)\) by Brouwer and Büttiker \([28]\) within the ASA of random matrices : the low frequency expansion of \(\langle G(\omega) \rangle\) (Eq. 12 of
Ref. [28] leads to
\[
\langle R_q \rangle = \frac{h}{Ne^2} \left[ 1 + \left( \frac{2}{\beta} - 1 \right) \frac{1}{N} + O(N^{-2}) \right], \tag{112}
\]
(without spin degeneracy $2_s$). The second term can be interpreted as a weak localisation correction. A first review paper by Büttiker [39] incorrectly quoted this result by introducing a spurious factor of 1/2 (see Eq. 17 of Ref. [39]). In a second review [40], Büttiker gave the correct result [Eq. 9 of this reference]
\[
\langle R_q \rangle \approx \frac{h}{e^2} \frac{1}{N \tau_T} \tag{113}
\]
for a non perfect contact with $NT \gg 1$ (without spin degeneracy), where $\tau_T$ is the transmission probability through the contact. Büttiker referred to a private communication with Carlo Beenakker and some unpublished work with myself [117], that I describe in Appendix A. This value coincides with the DC resistance of the constricton, what was expected in the large $N$ limit.

We can summarize this discussion in the following table giving the dimensionless charge relaxation resistance per channel $r_q = (Ne^2/h)R_q$:

| $N$  | $r_q$  |
|------|--------|
| 1    | 1/2    |
| 2    | 2/3    |
| $N \gg 1$ | $1 + (2/\beta - 1)N^{-1}$ |

Fluctuations. As the quantum capacitance is directly proportional to the Wigner time delay, its statistical properties were determined in [182] for a contact with many channels. More recently, we have studied with Grabisch [111] the statistical properties of the charge relaxation resistance, which involves the analysis of the ratio of two “linear statistics” $\sum \rho^2 / (\sum \rho^2)$. We have obtained the mean values $\langle C_q^2 \rangle \simeq e^2/\Delta$ and $\langle R_q \rangle \simeq h/(Ne^2)$ (without spin degeneracy), in correspondence with the DC resistance of the QPC, as it should. The variance of the capacitance is given by [117, 28, 182, 135, 60]
\[
\frac{\langle \delta C_q^2 \rangle}{\langle C_q^2 \rangle} \simeq \frac{4}{\beta N^2} \tag{114}
\]
(this coincides with the leading order term of cumulant [53] for $N \gg 1$). The variance of the resistance
\[
\frac{\langle \delta R_q^2 \rangle}{\langle R_q^2 \rangle} \simeq \frac{8}{\beta N^2} \tag{115}
\]
and the correlations with the quantum capacitance
\[
\frac{\langle \delta C_q \delta R_q \rangle}{\sqrt{\langle \delta C_q^2 \rangle \langle \delta R_q^2 \rangle}} = +1/\sqrt{2} \tag{116}
\]
were found in Ref. [111], where we have also analysed the large deviations.

7.3. Final remarks

- The multiterminal version of the general formulae reviewed in the section was provided in Refs. [18, 40].
- The finite temperature formulae are obtained by introducing convolutions with Fermi functions $\int d\epsilon (-\partial f/\partial \epsilon) : \text{tr} \{ Q(\epsilon) \}$ with the correlator [68], the model leads to the following expression
\[
\frac{\langle \delta C_q \delta C_q' \rangle}{\langle C_q^2 \rangle^2} \simeq \int d\omega \frac{\delta_T(\omega)}{\omega^2} \tag{117}
\]
\[
\times \frac{1}{\tau_n^2} \frac{\partial^2}{\partial \omega^2} \ln \left[ (1 + (\omega \tau_n^2)^2) (1 + (\omega \tau_n \epsilon)^2) \right]
\]
where $\delta_T(\omega)$ is a (normalised) thermal function of width $T$ which arises from the convolution of the two Fermi function’s derivatives. For $T\tau_n \gg 1$ this leads to the decay $\langle \delta C_q (B^2) \rangle / \langle C_q \rangle^2 \simeq \left[ \kappa / (\beta N^2) \right] (T\tau_n)^{-2}$ where $\kappa = -\int_0^\infty dx F'(x)/x \simeq 0.365$ (the function $F(x)$ is defined in the footnote). The dependence in the Dyson index $\beta$ can be replaced by the precise magnetic field dependence straightforwardly. Note that in the metallic limit $(e^2/C \gg \Delta)$, i.e. $C < C_q$, the fluctuation of the mesoscopic capacitance is $\delta C_{\mu} \simeq (C/\langle C_q \rangle)^2 \delta C_q$.\(^{22}\)

\(^{22}\) Although the geometries are different, we can compare this expansion with those of the DC resistance of a coherent quantum dot closed by two constrictions with $N_1$ and $N_2$ channels [13] : $\langle R_{q_{\text{class}}} \rangle \sim \frac{1}{N_1 + 1/N_2}$ where $R_{\text{class}} = (h/e^2)(1/N_1 + 1/N_2)$ and $N = N_1 + N_2$.

\(^{23}\) $\delta_T(\omega) = F(\omega/(2T))/(2T)$ with $F(x) = (x \coth x - 1)/\sinh^2 x$.\(^{23}\)
• A more precise discussion of the capacitive response of quantum dots has been provided in Ref. [5], beyond mean field and also analysing the limit of a weakly transmitting constriction in the presence of Coulomb blockade.

• Nigg and Böttiker have considered in Ref. [139] the case of a single spin polarised channel. They showed that strong enough dephasing in the cavity can induce the transition from the universal value $R_q = \hbar/(2e^2)$ to the DC resistance $R_{dc} = \hbar/(e^2T)$ of the QPC, where $T$ is the transmission probability through the QPC.

• The universal charge relaxation resistance $R_q = \hbar/(2e^2)$ for a single spin non-degenerate channel [48, 49, 40] has stimulated several works. This value was later obtained within more accurate treatments of electronic interactions: within a mean-field Hartree-Fock analysis [140, 157] improving the original Thomas-Fermi treatment, and beyond mean field [97, 137], within Matveev formalism [128] of QPC (see also [70]). These treatments have accounted for possible Coulomb blockade effects.

The treatment of interaction in a multichannel contact was extended more recently in [66].

8. Conclusion

Despite the huge diversity of Böttiker’s work, this article has tried to show that a possible Ariane’s thread is furnished by the concept of time delay and related quantities. Böttiker’s interest in this matter goes back to his early fundamental study of times in quantum mechanics; this question has led to many possible definitions, from scattering phase shifts (Wigner time delay, transmission group delays,...), clock approaches, etc. When randomness is introduced, these quantities are characterised by non-trivial distributions. I have reviewed in Section 3 the statistical analysis for the most simple of these fundamental times in the 1D situation in the presence of disorder. In particular I have emphasized the role of symmetry by considering the effect of a chiral symmetry. In this case, a new observation was the relation between the presence of a chiral zero mode and the existence of a limit law for the Wigner time delay when the size of the disordered region goes to infinity.

One of Böttiker’s major motivations for Wigner-Smith time delay matrix’ analysis comes from its central role in the scattering approach for non-linear (Section 6) and AC (Section 7) coherent electronic transport. This has led to introduce several new concepts (Section 5) which have been analysed within the frame of random matrix theory: Böttiker, with Gopar and Mello, obtained one of the first important result in the Wigner time delay statistical analysis [90], motivated by the statistical analysis of the mesoscopic capacitance of a chaotic quantum dot. The first moments of the AC response were obtained with Brouwer [28] within the random matrix theory and the non-linear response with Sánchez and Polianski [159, 149]. In the present article I have provided a review of the known results on time delay statistics within the random matrix approach: joint distribution of proper times, moments of Wigner time delay and its correlation function and statistical properties of the partial DoS.

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Appendix A. Calculation of $\langle R_q \rangle$ for $N \gg 1$ – Mapping to the study of absorption in the cavity

This section reproduces some unpublished contribution due to myself and Böttiker [177] (quoted in Ref. [40] but remained unpublished so far).

Our aim is to estimate the charge relaxation resistance for a one contact conductor in the good contact limit $NT \gg 1$, where $N$ is the number of conducting channels and $T$ the transmission probability through the QPC. Our starting point is the expression [108] (without spin degeneracy) [13]. As we mentioned in the text, the trace in the denominator is a measure of the density of states (DoS) of the cavity $\nu \sim \text{Tr} \{Q\}/\hbar$, therefore $\langle \text{Tr} \{Q\} \rangle = 2\pi/\Delta$ where $\Delta$ is the mean level spacing and the main question left is to estimate
The relation between the dynamical problem (real $\omega$) and the static problem with absorption (complex $\omega$) was used in another context in Refs. [106, 154, 14, 17, 80].

Tr $\{Q^2\}$. We remark that it appears in the expansion of the dimensionless AC conductance \[ G_0(\omega) = N - \text{Tr} \left\{ S^\dagger (\varepsilon - \frac{i}{2} \omega) S (\varepsilon + \frac{i}{2} \omega) \right\} \]
\[ = -i\omega \text{Tr} \{Q\} + \frac{\omega^2}{2} \text{Tr} \{Q^2\} + \cdots \quad (A.1) \]

Büttiker suggested to use analytic properties of the scattering matrix, $\omega \to i\Gamma$, i.e., to map the problem of AC transport onto the study of DC transport for a cavity in the presence of absorption, whose dimensionless conductance reads

\[ \tilde{G} = G_0(i\Gamma) = N - \text{Tr} \left\{ S^\dagger (\varepsilon - \frac{i}{2} \Gamma) S (\varepsilon + \frac{i}{2} \Gamma) \right\} \]
\[ = \Gamma \text{Tr} \{Q\} - \frac{\Gamma^2}{2} \text{Tr} \{Q^2\} + \cdots. \quad (A.2) \]

This corresponds to a situation where absorption takes place uniformly in space with absorbing rate $\Gamma$. In the ergodic regime where RMT holds, this also describes an absorbing contact (Fig. A.7, 24).

Figure A.7: A conductor closed by a constriction with $N$ channels (quantum point contact, QPC) and with an absorbing probe.

The idea is then to estimate $\tilde{G}$ by noticing that a conductor with one contact and an absorbing probe is equivalent to a two terminal conductor (Fig. A.8). In this latter case, the conductance is dominated by the resistance of the constrictions, hence $1/\tilde{G} = 1/(NT) + 1/(N_a T_a)$, where $N_a$ and $T_a$ characterise the second contact. We deduce

\[ N - \tilde{G} = N \frac{N T + N_a T_a (1 - T)}{N T + N_a T_a}. \quad (A.3) \]

Expansion (A.2) in powers of $\Gamma$ and expansion of (A.3) in powers of $N_a T_a$ can be identified by setting $N_a T_a = c (\Gamma / \Delta)$, where $c$ is a dimensionless constant. Hence

\[ \langle \text{Tr} \left\{ S^\dagger (\varepsilon - \frac{i}{2} \Gamma) S (\varepsilon + \frac{i}{2} \Gamma) \right\} \rangle = N \frac{NT + (1 - T) c \frac{\Gamma}{\Delta}}{NT + c \frac{\Gamma}{\Delta}} \]
\[ = N - \frac{\Gamma}{\Delta} + \frac{1}{NT} \left( \frac{\Gamma}{\Delta} \right)^2 + \mathcal{O}(\Delta^3). \quad (A.4) \]

Comparison with (A.2) and the fact that $\langle \text{Tr} \{Q\} \rangle = 2\pi / \Delta$ shows that $c = 2\pi$. We deduce

\[ \langle \text{Tr} \{Q^2\} \rangle = \sum_i \tau_i^2 \simeq \frac{8\pi^2}{NT\Delta^2}, \quad (A.5) \]

leading to (13).

Appendix B. Partial versus proper times

Let us make few remarks on the relation between partial time delays and proper time delays. We consider a scattering situation with $N$ channels. The $S$-matrix may be represented under the form

\[ S = U e^{i\Theta} U^\dagger \quad (B.1) \]

where $\Theta = \text{diag}(2\eta_1, \cdots, 2\eta_N)$ gathers the $N$ phase shifts. We deduce the following representation for the Wigner-Smith matrix:

\[ Q = U (\partial_x \Theta) U^\dagger + i \left[ S^\dagger (U \partial_x U^\dagger) S - U \partial_x U^\dagger \right] \quad (B.2) \]

The first term involves the partial time delays : $\partial_x \Theta = \text{diag}(\tau_1, \cdots, \tau_N)$, which are intrinsic characteristics of the scattering matrix (independent on the basis). The second term controlled by $U$, i.e., by the choice of basis, is the origin of the difference between partial and proper times.

We now say a little bit more on the unitary matrix $U$ in the $N = 2$ channel case. Going back to the notations introduced in § 2.2, we can rewrite the relation between partial waves (8) (eigenstates of the $S$-matrix) and the left/right scattering states in the matricial form

\[ \begin{pmatrix} \phi_{e,1}(x) \\ \phi_{e,2}(x) \end{pmatrix} = \begin{pmatrix} A_{L,1} & A_{R,1} \\ A_{L,2} & A_{R,2} \end{pmatrix} \begin{pmatrix} \psi_{e,L}(x) \\ \psi_{e,R}(x) \end{pmatrix} \quad (B.3) \]
which involves the transpose of the matrix which diagonalises the $S$-matrix

$$
U(\varepsilon) = \begin{pmatrix} A_{L,1} & A_{L,2} \\ A_{R,1} & A_{R,2} \end{pmatrix} \quad (B.4)
$$

therefore, when eigenstates are properly normalised (associated to a measure $d\varepsilon$), we can write $\langle \phi_{\varepsilon,a} | \psi_{\varepsilon,\,b} \rangle = U_{ba}(\varepsilon) \delta(\varepsilon - \varepsilon')$ where $a \in \{1, 2\}$ and $b \in \{L, R\}$.

**Appendix B.1. Symmetric case**

We first consider the symmetric case $r = r'$ and $t = t'$ [in 1D this corresponds to a symmetric potential $V(-x) = V(x)$]. The left and right scattering states are simply related by $\psi_{x, R}(x) = \psi_{x, L}(-x)$ and the two partial waves are the symmetric/antisymmetric eigenstates $\phi_{\varepsilon,a}(-x) = (-1)^{a+1}\phi_{\varepsilon,a}(x)$, hence the unitary matrix is independent on the energy in this case

$$
U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (B.5)
$$

and the partial and proper times coincide

$$
\tau_1 = \tilde{\tau}_1 \quad \text{and} \quad \tau_2 = \tilde{\tau}_2. \quad (B.6)
$$

**Figure B.9:** Proper time delays (continuous blue line) and partial time delays (dashed red line) as a function of $k$ in a particular example of 1D Hamiltonian with potential $V(x) = \lambda_1 \delta(x) + \lambda_2 \delta(x-a)$ with $\lambda_1 = 2$ and $\lambda_2 = 4$ ($a = 1$).

**Appendix B.2. Asymmetric case**

In general the matrix $U$ carries some energy dependence. In order to illustrate the difference between the two sets of characteristic times, we consider a specific 1D example: we compute these times for the Hamiltonian $H = -\partial_x^2 + \lambda_1 \delta(x) + \lambda_2 \delta(x-a)$. We consider the basis of left and right scattering states in which the $S$-matrix takes the form $[5]$. The three coefficients $r$, $t$ and $r'$ can be computed easily thanks to a transfer matrix approach. Diagonalisation of the $S$-matrix provides the phase shifts and hence the partial time delays $\tilde{\tau}_1$ and $\tilde{\tau}_2$. We also compute the Wigner-Smith matrix in the basis of left/right scattering states (cf. §2.2) and diagonalise it, we get the proper time delays $\tau_1$ and $\tau_2$. We check that for the symmetric potential ($\lambda_1 = \lambda_2$) the two sets exactly coincide, Eq. (B.6). In the asymmetric case ($\lambda_1 \neq \lambda_2$) the four times are plotted in Fig. B.9 as a function of $\sqrt{\varepsilon}$. We see that proper and partial times are very close (they get closer as the energy grows). At the crossing points where the two partial times are equal we can however observe that partial times and proper times differ. This follows from the fact that the second term of Eq. (B.2) can be understood as a perturbation which produces anticrossing of the eigenvalues of $Q$.

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