A finitely presented group with two non-homeomorphic asymptotic cones

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Abstract

We give an example of a finitely presented group $G$ with two non-$\pi_1$-equivalent asymptotic cones.

1 Introduction

Asymptotic cones of groups were introduced by Gromov [6] to prove that a group of polynomial growth is virtually nilpotent. In [6], the concept was generalized to arbitrary finitely generated groups. By definition, an asymptotic cone of a group depends on the choice of an ultrafilter and a choice of an increasing sequence of numbers $d_n \to \infty$. Nevertheless, in many cases all asymptotic cones of a group turn out to be homeomorphic. In particular, this is the case for hyperbolic groups [2], for nilpotent groups [6], [14], etc. In [8] (see Question 2.B1 (c)), Gromov asked whether there exists a finitely generated (finitely presented) group with two non-homeomorphic asymptotic cones.

S. Thomas and B. Velicovic [16] gave an example of a finitely generated group $H$ with two non-homeomorphic asymptotic cones. C. Drutu and M. Sapir [5] gave an example of a finitely generated group with continuum pairwise non-homeomorphic (and even non-$\pi_1$-equivalent) asymptotic cones. On the other hand, by L. Kramer, S. Shelah, K. Tent and S. Thomas [10], if the Continuum Hypothesis is true then continuum is the maximal number of non-isometric asymptotic cones a finitely generated group can have. If the Continuum Hypothesis is not true, they give an example of a finitely presented group with $2^{2^{\aleph_0}}$ pairwise non-homeomorphic asymptotic cones (if the Continuum Hypothesis is true, their group has unique asymptotic cone).

It is essential in the proofs in [16] and [5] that the groups in both papers are limits of hyperbolic groups but non-hyperbolic themselves (all asymptotic cones of non-elementary hyperbolic groups are isometric [5], [2]). Therefore these groups are not finitely presented. Moreover, some of the asymptotic cones of these groups are $\mathbb{R}$-trees, so they cannot be finitely presented because it has been proven by M. Kapovich and B. Kramer [9] that if an asymptotic cone of a finitely presented group is an $\mathbb{R}$-tree then the group is hyperbolic and so all its asymptotic cones are isometric.

The question of whether for a finitely presented group asymptotic cones can be non-homeomorphic (non-isometric) independently of the Continuum Hypothesis was open. The goal of this note is to give a positive answer to this question.

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We use Dehn functions of groups. In [11], the first author constructed (using some ideas from [12]) a finitely presented group $G$ whose Dehn function $f(n)$ satisfies the following two properties:

(P1) there are sequences of positive numbers $d_i \to \infty$ and $\lambda_i \to \infty$ such that $f(n) \leq cn^2$ for arbitrary integer $n \in \bigcup_{i=1}^{\infty} \left[ \frac{n}{\lambda_i}, \lambda_i d_i \right]$ and some constant $c$,

(P2) there is a positive constant $c'$ and an increasing sequence of numbers $n_i \to \infty$ such that $f(n_i)/n_i^2 \to \infty$ but for every $i$, and for every integer $n$ with $n \leq c' n_i$, we have $f(n) \leq cn_i^2$.

**Theorem 1.1.** Let $G$ be a finitely presented group satisfying (P1) and (P2). Then $G$ has two asymptotic cones, one of which is simply connected and another one is not.

It is known [13] that if the Dehn function of a group is quadratic then all its asymptotic cones are simply connected. We slightly modify Papasoglu’s argument and show that property (P1) implies that one of the asymptotic cones of $G$ is simply connected.

On the other hand, by a result of Gromov [8] (see also [14]), if the Dehn function of a group is not bounded by a polynomial then the group has a non-simply connected asymptotic cone. The Dehn function of $G$ is bounded by $n^3$ (in fact, by $n^2 \log n / \log \log n$) but Property (P2) allows us to apply essentially Gromov’s argument and show that $G$ has a non-simply connected asymptotic cone.

Note that the group $G$ from [11] is an $S$-machine in the terminology of the second author, so it is a multiple HNN extension of a free group with finitely generated associated subgroups (see [15], [12]). In particular, $G$ has cohomological dimension 2.

## 2 Proof

Recall the definition of an asymptotic cone. A non-principal ultrafilter $\omega$ is a finitely additive measure defined on all subsets $S$ of $\mathbb{N}$, such that $\omega[S] \in \{0, 1\}$ and $\omega[S] = 0$ if $S$ is a finite subset. For a bounded function $f : \mathbb{N} \to \mathbb{R}$ the limit $\lim_{\omega} f(i)$ with respect to $\omega$ is the unique real number $a$ such that $\omega(\{i \in \mathbb{N} : |f(i) - a| < \epsilon\}) = 1$ for every $\epsilon > 0$.

Let $(X, \text{dist})$ be a metric space. Fix an arbitrary $x_0 \in X$, and a sequence of scaling constants $d_i \to \infty$. Consider the set of sequences $g : \mathbb{N} \to X$ such that $\text{dist}(f(i), x_0) \leq cd_i$ for some constant $c = c(f)$. Two sequences of this set $\mathcal{F}$ are said to be equivalent if $\lim_{\omega} \frac{\text{dist}(f(i), g(i))}{d_i} = 0$. The asymptotic cone $\text{Con}^{\omega}(X, (d_i))$ is the quotient space $\mathcal{F}/ \sim$ where the distance between equivalence classes $[f]$ and $[g]$ is equal to $\lim_{\omega} \frac{\text{dist}(f(i), g(i))}{d_i}$. The asymptotic cone is a complete space; it is a geodesic metric space if $X$ is a geodesic metric space ($8$; $13$). If $[f]$ is an element of $\text{Con}^{\omega}(X, (d_i))$ then we say that $f(i)$ converges to $[f]$. Note that $\text{Con}^{\omega}(X, (d_n))$ does not depend on the choice of $x_0$.

An asymptotic cone of a group $G$ with a word metric is isometric to the asymptotic cone of its Cayley graph (considered as the 1-skeleton of the Cayley complex). The asymptotic cones of the same group relative to two finite generating sets are bi-Lipschitz equivalent. Note that in [8], [13] and other papers, a more restrictive definition of asymptotic cone was used: it was always assumed that $(d_i) = (i)$. It was observed in [16], however, that if, say, all $d_i$’s are different integers then $\text{Con}^{\omega}(G, (d_i))$ is isometric to a cone $\text{Con}^{\omega'}(G, (i))$ for some $\omega'$. Since in all the asymptotic cones considered in this paper, all $d_i$’s are different integers, they are isometric to restricted asymptotic cones.

As in [13] p. 792], we define an $n$-gone $P$ in a geodesic metric space $(X, \text{dist})$ as a map from the set of vertices of the standard regular $n$-gon $S_n$ in the plane into $X$. If $X$ is a Cayley graph of a group, we shall always assume that elements of $P$ are vertices of the graph, i.e. they belong
Lemma 2.1. Let \((X, \text{dist})\) be a geodesic metric space, and \(P\) a polygon in \(\text{Con}^\omega(X, (d_i))\) with vertices \(P_1, \ldots, P_n\). Assume that \(P\) satisfied the following Loop Division Condition:

\[
\text{LDC}(k): \text{There exists a sequence of polygons } Q^i = (P^i_1, \ldots, P^i_n) \text{ in } X \text{ such that } P_j^i \text{ converges to } P_j \text{ for } j = 1, \ldots, n, \text{ and every } Q^i \text{ can be partitioned into } k \text{ pieces whose perimeters are less than or equal to } \frac{1}{2} \text{perimeter}(Q^i).
\]

Then the polygon \(P\) can be partitioned in \(\text{Con}^\omega(X, (d_i))\) into \(k\) pieces whose perimeters do not exceed \(\frac{1}{2}\) of the perimeter of \(P\).

Proof. This assertion is proved in [13]. (See the proof of the Proposition formulated on page 793; that proof works without changes, although the formulation of Lemma 2.1 slightly differs from the formulation of the cited Proposition.)

Lemma 2.2. Let \(X\) be a complete geodesic metric space such that for some integer \(k\), every polygon \(P\) of the asymptotic cone \(\text{Con}^\omega(X, (d_i))\) satisfies LDC\((k)\). Then \(X\) is simply connected.

Proof. Again, it suffices to repeat the proof of the Proposition formulated on the bottom of page 793 of [13] (though our formulation differs from that in [13], and one should refer to Lemma 2.1 now).

The following version of Papasoglu’s lemma is now formulated for arbitrary planar triangular map, i.e. for a map whose faces are of (combinatorial) perimeter at most 3.

Lemma 2.3. Let \(\Delta\) be a triangular map whose perimeter \(n\) is at least 200. Assume that the area of \(\Delta\) does not exceed \(Mn^2\). Then there is \(k\) depending on \(M\) only, such that \(\Delta = \Gamma_1 \cup \cdots \cup \Gamma_k\) where \(\Gamma_i, (i = 1, \ldots, k)\) are submaps of \(\Delta\), and \(\Gamma_i \cap \Gamma_j (0 \leq i < j \leq k)\) is empty or a vertex, or a simple path, and perimeter \(|\partial \Gamma_i|\) is at most \(n/2\) for all \(i = 1, \ldots, k\).

Proof. The proof of the Theorem formulated in [13], page 799, does not use the labels of diagram edges, and so it also works for maps. Although it is assumed in [13], that \(\text{area}(\Delta) \leq M|\partial \Delta|^2\) for all minimal van Kampen diagrams over a triangular group presentation, the proof uses this quadratic isoperimetric inequality only for one diagram \(\Delta\). The assertion of Lemma 2.3 is therefore correct.

Let \(Q\) be a polygon in the Cayley graph of \(G\). Connect the vertices of each side of \(Q\) by a geodesic, then the product of labels of these geodesics viewed as a cyclic word is called a label of \(Q\) (a label depends on the choices of the geodesics, of course).

Lemma 2.4. Let \(f\) be the Dehn function of a finite group presentation \(G = \langle A \mid R \rangle\) satisfying (P1). Then the asymptotic cone \(\text{Con}^\omega(G, (d_i))\) is simply connected for arbitrary non-principal ultrafilter \(\omega\).
Proof. Let \( P = (P_1, \ldots, P_m) \) be a polygon in \( \text{Con}^\omega(X, (d_i)) \) with pairwise distinct vertices \( P_1, \ldots, P_m \). Consider a sequence of polygons \( Q^i = (Q^i_1, \ldots, Q^i_m) \) in the Cayley graph \( \Gamma(G,A) \) of the group \( G \), such that \( Q^i_j \) converges to \( P_j \) for every \( j \). Since \( \text{dist}(P_j, P_j') > 0 \) in the cone for \( j \neq j' \), there are constants \( \alpha \) and \( \beta \) independent of \( i \) and \( j \) such that \( l_i = \text{perimeter}(Q^i_i) \in [ad_i, \beta d_i] \) for almost all \( i \)-s (with respect to \( \omega \)). Property (P1) implies that

\[
\omega(I) = 1 \text{ for } I = \{ i \mid l_i \in \left[ \frac{d_i}{\lambda_i}, \lambda_i d_i \right] \}
\] (2.1)

Van Kampen’s Lemma provides us with a minimal diagram \( \Delta_i \) over the presentation \( G = \langle A \mid R \rangle \) such that the boundary label of \( \Delta_i \) is a label of the polygon \( Q^i \) in the Cayley graph \( \Gamma(G,A), i \in I \). By Lemma 2.2 and formula (2.1), there is a constant \( k = k(c) \) such that \( \Delta_i \) can be partitioned into subdiagrams \( \Gamma^i_1, \ldots, \Gamma^i_k \) with perimeters at most \( l_i/2 \). Then the polygon \( Q^i \) can be accordingly partitioned into discs \( D^i_1, \ldots, D^i_k \) in the Cayley graph, and the perimeters of these discs do not exceed \( l_i/2 \). Hence, in the cone, every polygon \( P \) satisfies LDC(\( k \)), and, by Lemma 2.2, the cone \( \text{Con}^\omega(G, (d_i)) \) is simply connected.

Lemma 2.5. Let \( f \) be the Dehn function of a finite group presentation \( G = \langle A \mid R \rangle \) satisfying property (P2). Then, for arbitrary non-principal ultrafilter \( \omega \), the asymptotic cone \( \text{Con}^\omega(G,(n_i)) \) is not simply connected.

Proof. Property (P2) implies existence of a positive constant \( c' < 1 \) such that \( f(n_i)/f(c'n_i) \to \infty \).

Assume there is a number \( k \) such that for every \( i \), an arbitrary polygon \( Q \) of the Cayley graph \( \Gamma(G,A) \) of length \( l \), \( c'n_i \leq l \leq n_i \), can be partitioned into at most \( k \) pieces of perimeter \( l/2 \). It follows that every loop of length \( n_i \) can be partitioned into at most \( K = k^{1-\log_2 c'} \) pieces of perimeter at most \( c'n_i \).

Let \( \Delta_i \) be a van Kampen diagram with perimeter \( n_i \) and area \( f(n_i) \) that has minimal area among all diagrams with the same boundary label. For each \( i \), consider the loop \( Q^i \) in the Cayley graph \( \Gamma(G,\text{dist}) \) of \( G \), whose label is equal to the boundary label of \( \Delta_i \)'s. It follows from our assumption that for every \( i \), there is a partition of \( Q^i \) into at most \( K \) pieces of perimeter \( \leq c'n_i \). The smallest area van Kampen diagram having the same label as \( Q^i \) has area at most \( f(c'n_i) \). Therefore the area of the minimal diagram \( \Delta \) cannot exceed \( K f(c'n_i) \). Hence \( f(n_i)/f(c'n_i) \leq K \) for every \( i \). This contradicts property (P2).

Therefore our assumption was false, and there is no such number \( k \). Also there is a constant \( c_0 \) such that the radius of \( Q^i \) (i.e. \( \max(\text{dist}(x,y) \mid x,y \in Q^i) \)) is at least \( c_0 n_i \). Otherwise we could easily partition \( Q^i \) into a bounded number of loops with length \( \leq c'n_i \) (which can be ruled out as in the previous paragraph). Consider the \( \omega \)-limit of the sequence of (finite) sets \( Q^i \), i.e. the set of all elements \( \{x_i\} \) where \( x_i \in Q^i \). It is easy to see (cf., for example, 1) that the \( \omega \)-limit of \( Q^i \) is a loop \( P \) in \( \text{Con}^\omega(G,(n_i)) \) of length at least \( c_0 \). Indeed, one can parametrise each loop \( Q^i \) by its arc length by a function \( x_i : [0,1] \to (\Gamma,\text{dist}/n_i) \), then \( P \) has parametrization \( x : [0,1] \to \text{Con}^\omega(G,(n_i)) \) where \( x(t) = \{x_i(t)\} \) for each \( t \in [0,1] \).

The loop \( P \) has no finite partition into pieces \( P_1, P_2, \ldots, P_k \) whose perimeters do not exceed half of the perimeter of \( P \). Therefore the loop \( P \) is not contractible. (For more details justifying the last two phrases, see 3, 13 or the proof of Theorem 4.4 in 2.)

Proof of Theorem 1.1 The theorem follows from lemmas 2.2 and 2.5.

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