Curvature tensor under the Ricci flow

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Abstract
Consider the unnormalized Ricci flow \((g_{ij})_t = -2R_{ij}\) for \(t \in [0, T)\), where \(T < \infty\). Richard Hamilton showed that if the curvature operator is uniformly bounded under the flow for all times \(t \in [0, T)\) then the solution can be extended beyond \(T\). We prove that if the Ricci curvature is uniformly bounded under the flow for all times \(t \in [0, T)\), then the curvature tensor has to be uniformly bounded as well.

1 Introduction
The Ricci flow equation is the evolution equation \((g_{ij})_t = -2R_{ij}\) introduced by Richard Hamilton in his seminal paper [3]. Short time existence for solutions to the Ricci flow on a compact manifold was first shown in [3], using the Nash-Moser theorem. Shortly after that De Turck ([1]) showed the same thing by modifying the flow by a reparametrization using a fixed background metric to break the symmetry that comes from the fact that the Ricci tensor is invariant under the whole diffeomorphism group of a manifold.

The Ricci flow equation is a weakly parabolic equation and many nice regularity theorems have already been proved. A very nice corollary of the regularity of the Ricci flow is a result on the maximal existence time for a solution, proved by R. Hamilton in [5].

**Theorem 1 (Hamilton).** For any smooth initial metric on a compact manifold there exists a maximal time \(T\) on which there is a unique smooth solution to the Ricci flow for \(0 \leq t < T\). Either \(T = \infty\) or the curvature is unbounded as \(t \to T\).

The main theorem in this paper is the following

**Theorem 2.** Let \((g_{ij})_t = -2R_{ij}\), for \(t \in [0, T)\) be a Ricci flow on a compact manifold \(M\) with \(T < \infty\) and with uniformly bounded Ricci curvatures along the flow. Then the curvature tensor stays uniformly bounded along the flow.

The proof of the theorem uses Perelman’s noncollapsing theorem for the unnormalized Ricci flow that has been proved in [7].

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2 Background

$M$ will always denote a compact manifold, and $(g_{ij})_t = -2R_{ij}$ is the unnormalized Ricci flow on $M$. Perelman introduced the following functional in [7]:

$$W(g, f, \tau) = (4\pi \tau)^{-\frac{n}{2}} \int_M e^{-f}[\tau(|\nabla f|^2 + R) + f - n]dV_g.$$

Perelman has showed that $W$ is increasing along the flow. A very nice application of the monotonicity formula for $W$ is Perelman’s noncollapsing theorem for the unnormalized Ricci flow.

**Definition 3.** Let $g_{ij}(t)$ be a smooth solution to the Ricci flow $(g_{ij})_t = -2R_{ij}(t)$ on $[0, T)$. We say that $g_{ij}(t)$ is locally collapsing at $T$, if there is a sequence of times $t_k \to T$ and a sequence of metric balls $B_k = B(p_k, r_k)$ at times $t_k$, such that $\frac{r_k^2}{t_k}$ is bounded, $|Rm|(g_{ij}(t_k)) \leq \frac{1}{r_k^2}$ in $B_k$ and $\frac{1}{r_k^n}\text{Vol}(B_k) \to 0$.

**Theorem 4.** If $M$ is closed and $T < \infty$, then $g_{ij}(t)$ is not locally collapsing at $T$.

The corollary of the theorem above states

**Corollary 5.** Let $g_{ij}(t), t \in [0, T)$ be a solution to the Ricci flow on a closed manifold $M$, where $T < \infty$. Assume that for some sequences $t_k \to T$, $p_k \in M$ and some constant $C$ we have $Q_k = |Rm|(x, t) \leq CQ_k$, whenever $t < t_k$. Then a subsequence of scalings of $g_{ij}(t_k)$ at $p_k$ with factors $Q_k$ converges to a complete ancient solution to the Ricci flow, which is $\kappa$-noncollapsed on all scales for some $\kappa > 0$.

Before we start proving our theorem, we will mention some estimates for ball sizes under evolution by the Ricci flow derived by Glickenstein in [2], since we will use them in the proof of our theorem [2]. His estimates are improvements of Hamilton’s work on distance comparison at different times under Ricci flow that can be found in [3].

**Lemma 6.** Suppose $(M, g(t))$ are solutions to the Ricci flow for $t \in [0, T)$ such that the Ricci curvatures are uniformly bounded by some constant $C$. Then for all $\delta > 0$ there exists an $\eta > 0$ such that if $|t - t_0| < \eta$ then

$$|d_{g(t)}(q, q') - d_{g(t_0)}(q, q')| \leq \delta^2d_{g(t_0)}(q, q'),$$

for all $q, q' \in M$.

One can choose $\delta = e^{2C|t-t_0|} - 1$ and $\eta = \frac{(\delta+1)}{2C}$.

**Lemma 7 (Glickenstein).** If $g(t)$ is a solution to the Ricci flow such that the Ricci curvature is bounded by constant 1 then for all $\rho > 0$

$$B_{g(t_0)}(0, \rho(t)\rho) \subset B_{g(0)}(0, \rho),$$
where $r(t) = \frac{1}{1+(e^{2t}-1)^{\frac{3}{2}}}$.  

The proof of lemma 7 uses lemma 4 and triangle inequality for distances.

3 Blow up argument

In this section we will prove theorem (2).

Proof. Assume that under the assumptions of our theorem the curvature blows up at a finite time $T$. That means there exist sequences $t_i \to T$ and $p_i \in M$, such that

$$Q_i = |\text{Rm}(p_i, t_i)| \geq C^{-1} \max_{M \times [0, t_i]} |\text{Rm}(x, t)|,$$

and $Q_i \to \infty$ as $i \to \infty$. By corollary the scaled metrics $g_i(t) = Q_i g(t_i + \frac{t}{Q_i})$ converge to a complete ancient solution to the Ricci flow, $\kappa$-noncollapsed on all scales for some $\kappa > 0$. We have that $\{(M, g_i(t_i), p_i)\} \to (N, \bar{g}(t), p)$ in a pointed Gromov-Hausdorff metric, for all $t \in (-\infty, a)$ where $a > 0$. Since the Ricci curvatures of our original flow are uniformly bounded, we have that $|R(t)| \leq C$, where $R(t)$ is a scalar curvature of $g(t)$. Since $R_i(t) = \frac{R(t)}{Q_i}$ for all $t \in [-t_i Q_i, (T-t_i) Q_i]$, we get that $R_i(t) \to 0$ as $i \to \infty$, i.e. our ancient solution $\bar{g}(t)$ has zero scalar curvature for every $t \in (-\infty, a)$, where $a > 0$.

The evolution equation for a scalar curvature $\bar{R}$ is:

$$\frac{d}{dt} \bar{R} = \Delta \bar{R} + 2|\text{Ric}|^2.$$

Since $\bar{R}(t) \equiv 0$ for all $t \in (-\infty, a)$ the evolution equation for $\bar{R}$ implies that $\bar{\text{Ric}}(t) = 0$, for all $t$, i.e. our solution $\bar{g}$ is stationary. Therefore it can be extended for all times $t \in (-\infty, +\infty)$ to an eternal solution, where $\bar{g}(t) = \bar{\bar{g}}$.

Take any $r > 0$. Then:

$$\frac{\text{Vol} B(p, r)}{r^n} = \lim_{i \to \infty} \frac{\text{Vol}_i B_i(p_i, r)}{r^n},$$

where the volume and the ball $B(p, r)$ on the LHS of (1) are considered in metric $\bar{g}$, while on the RHS of (1) we consider metric $g_i(0) = Q_i g(t_i)$. Furthermore,

$$\frac{\text{Vol} B(p, r)}{r^n} = \lim_{i \to \infty} \frac{\text{Vol}_{g(t_i)} B_{g(t_i)}(p_i, r Q_i^{-\frac{1}{2}})}{(r Q_i^{-\frac{1}{2}})^n}.$$

Since the Ricci curvatures of our original metrics $g(t)$ are uniformly bounded, from the evolution equation for Vol we get that $\text{Vol}_{g(t)} M \leq \bar{C}$ for all $t \in [0, T)$, for some...
constant $\bar{C}$ that does not depend on $t$. Take any $\epsilon > 0$. Choose $i_0$ such that for all $i \geq i_0$ we have the following estimates:

$$e^{-\frac{C}{n}}|t_i - t_{i_0}| > (1 - \frac{\epsilon}{2}), \tag{2}$$

$$\left(\frac{1}{1 + (e^{2C(t_i - t_{i_0})} - 1)^2}\right)^n > 1 - \frac{\epsilon}{2}, \tag{3}$$

and

$$\frac{\text{Vol}(p, r)}{r^n} \geq \frac{\text{Vol}_{g(t_i)}B_{g(t_i)}(p_i, r\sqrt{Q_i}^{-\frac{2}{n}})}{(r\sqrt{Q_i}^{-\frac{2}{n}})^n} - \frac{\epsilon}{2}, \tag{4}$$

hold for all $i \geq i_0$, where $C$ is a constant in the statement of our theorem. We can choose such $i_0$ because of Lemma 4 and the fact that $t_i \to T$ as $i \to \infty$.

Let $r_i = \frac{1}{1 + (e^{2C(t_i - t_{i_0})} - 1)^2}$. Lemma 7 tells us how the size of balls change with the Ricci flow. Since $t_i \geq t_{i_0}$, we have that $B_{g(t_{i_0})}(p_i, r\sqrt{Q_i}^{-\frac{2}{n}}) \subset B_{g(t_i)}(p_i, r\sqrt{Q_i}^{-\frac{2}{n}})$. From this observation and estimate 4 we have:

$$\frac{\text{Vol}(p, r)}{r^n} \geq \frac{\text{Vol}_{g(t_{i_0})}B_{g(t_{i_0})}(p_i, r\sqrt{Q_i}^{-\frac{2}{n}})}{(r\sqrt{Q_i}^{-\frac{2}{n}})^n} - \frac{\epsilon}{2}.$$  

Denote by $\bar{r}_i = r\sqrt{Q_i}^{-\frac{2}{n}}$. From the evolution equation for Vol, we get

$$\text{Vol}_{t_i} = \text{Vol}_{t_{i_0}} e^{\int_{t_{i_0}}^{t_i} R\;du}. \tag{5}$$

Integrating equation 5 over a ball $B_{g(t_{i_0})}(p_i, \bar{r}_i)$ gives

$$\text{Vol}_{g(t_{i_0})}B_{g(t_{i_0})}(p_i, \bar{r}_i) = \int_{B_{g(t_{i_0})}} e^{\int_{t_{i_0}}^{t_i} R\;du} dV_{g(t_{i_0})} \geq \text{Vol}_{g(t_{i_0})}B_{g(t_{i_0})}(p_i, \bar{r}_i)e^{-\int_{t_i}^{t_{i_0}} \frac{R}{n}} \geq (1 - \frac{\epsilon}{2})\text{Vol}_{g(t_{i_0})}B_{g(t_{i_0})}(p_i, \bar{r}_i), \tag{6}$$

where we have used estimate 2 and the fact that $|R|^2 \leq |\text{Ric}|^2 \leq \frac{\epsilon^2}{n}$. This gives the following estimate

$$\frac{\text{Vol}(p, r)}{r^n} \geq (1 - \frac{\epsilon}{2})\frac{\text{Vol}_{g(t_{i_0})}B_{g(t_{i_0})}(p_i, \bar{r}_i)}{(r\sqrt{Q_i}^{-\frac{2}{n}})^n} - \frac{\epsilon}{2}. \tag{7}$$

We have a fixed metric $g(t_{i_0})$ on the RHS of the inequality 7. Our manifold $\bar{M}$ is compact and therefore we have a uniform asymptotic volume expansion:

$$\text{Vol}_{t_{i_0}}B_{t_{i_0}}(p_i, \bar{r}_i) = \bar{r}_i^n \omega_n (1 - \frac{R(g(t_{i_0}))r^2}{6(n + 2)Q_i(1 + (e^{2C(t_i - t_{i_0})} - 1)^\frac{n}{2} + o(r_i^2)), \tag{8}$$
where $\omega_n$ is a volume of a unit euclidean ball.

Combining the estimate \[7\] with the asymptotic expansion \[8\] and letting $i \to \infty$ we get:

$$\frac{\text{Vol}B(p, r)}{r^n} \geq r^n \omega_n \left(1 - \frac{\epsilon}{2}\right) - \epsilon.$$ 

By estimate \[8\] we have that $r^n_i > 1 - \frac{\epsilon}{2}$ for all $i \geq i_0$ and therefore

$$\frac{\text{Vol}B(p, r)}{r^n} \geq \omega_n \left(1 - \frac{\epsilon}{2}\right)^2 - \epsilon.$$ 

Since $\epsilon > 0$ was arbitrary, we have that $\frac{\text{Vol}B(p, r)}{r^n} \geq \omega_n$.

On the other hand Bishop-Gromov volume comparison principle applied to a Ricci flat, complete manifold $(N, \bar{g})$ gives

$$\frac{\text{Vol}B(p, r)}{r^n} \leq \frac{\text{Vol}B(p, \delta)}{\delta^n},$$

for all $\delta \leq r$. When $\delta \to 0$, the RHS of the previous inequality tend to $\omega_n$. Therefore for every $r > 0$ we would have that $\frac{\text{Vol}B(p, r)}{r^n} = w_n$. Ricci flat, complete manifold with this property has to be the Euclidean space and therefore $\text{Rm} \equiv 0$ on $N$. This contradicts the fact that $\text{Rm}(p, 0) = 1$.

Therefore, a curvature of an unnormalized flow can not blow up in a finite time when the Ricci curvatures are uniformly bounded for all times for which a solution of a Ricci flow exists.

We have immediately the following simple corollary of theorem \[2\]

**Corollary 8.** Let $g(t)$ be a solution to $(g_{ij})_t = -2R_{ij}$ with $|\text{Ric}| \leq C$ uniformly for all times when the solution exist. Then the solution exists for all times $t \in [0, \infty)$.

**Proof.** This is a simple consequence of theorems \[2\] and \[1\].

**Corollary 9.** Let $(g_{ij})_t = -2R_{ij} + \frac{1}{2\tau}g_{ij}$ be a flow on a closed manifold $M$, which solution exists for $t \in [0, T)$, where $T < \infty$. If $|\text{Ric}| \leq C$ for all $t \in [0, T)$, then the solution can be extended past time $T$.

**Proof.** Let $\bar{g}(s) = g(t(s))c(s)$, where $c(s) = 1 - \frac{s}{\tau}$ and $t(s) = -\tau \ln(1 - \frac{s}{\tau})$. Then $|\text{Ric}|_{\bar{g}} \leq \frac{C}{c(s)} \leq Cc(s)^{\frac{2}{2}}$ for all $s \in [0, \tau(1 - e^{-\frac{T}{\tau}})]$. We can apply theorem \[2\] to the unnormalized flow $\bar{g}(s)$ to get that it can be extended past time $\tau(1 - e^{-\frac{T}{\tau}})$. If we go back to flow $g(t)$, it means that $g(t)$ can be extended past time $T$.

Since Perelman’s noncollapsing theorem plays an important role in a study of the Ricci flow, we would like to say few words about it in the case of normalized flow.
Claim 10. Let \((g_{ij})_t = -2R_{ij} + \frac{1}{t}g_{ij}\) for \(t \in [0, \infty)\) on a closed manifold \(M\). Then \(g_{ij}(t)\) is not locally collapsing at \(\infty\).

Before we start with the proof of the claim, we will write down the definition for collapsing at the \(\infty\).

Definition 11. Let \(g_{ij}(t)\) be a solution to \((g_{ij})_t = -2R_{ij} + \frac{1}{t}g_{ij}\). We say that \(g_{ij}(t)\) is locally collapsing at \(\infty\), if there is a sequence of times \(t_i \to \infty\) and a sequence of metric balls \(B_k = B(p_k, r_k)\) at times \(t_k\), such that \(\frac{r_k^2}{t_k}\) is bounded, \(|\text{Rm}(g_{ij}(t_k))| \leq r_k^{-2}\) and \(r_k^{-n}\text{Vol}(B_k) \to 0\).

Notice that in the proof of 10 we need just that \(|\text{Ric}|(g_{ij}(t_k))| \leq r_k^{-2}\).

Proof of claim 10. Assume that there exists a sequence of collapsing balls \(B_k = B(p_k, r_k)\) at times \(t_k \to \infty\) where \(\frac{r_k^2}{t_k}\) is bounded. Let \(\tilde{g}(s) = c(s)g(t(s))\), where \(c(s) = 1 - \frac{s}{\tau}\) and \(t(s) = -\tau \ln(1 - \frac{s}{\tau})\). This choice of \(c(s)\) and \(t(s)\) gives us that \(\tilde{g}(t)\) satisfies \((\tilde{g}_{ij})_t = -2R_{ij}\), for \(s \in [0, \tau]\). Perelman’s noncollapsing theorem 11 gives us that \(\tilde{g}(s)\) is not locally collapsing at \(\tau\). Let \(s_k\) be a sequence such that \(t(s_k) = t_k\). By our assumption there exists a sequence \(\epsilon_k \to 0\) as \(k \to \infty\) such that

\[
\text{Vol}_{\tilde{g}}B_{\tilde{g}}(p_k, r_k) \leq \epsilon_k r_k^n,
\]

at time \(t_k\). This we can write as

\[
\text{Vol}_{\tilde{g}}B_{\tilde{g}}(p_k, \sqrt{c(s_k)}r_k) \leq \epsilon_k (r_k \sqrt{c(s_k)})^n,
\]

at time \(s_k\). \(\frac{c(s_k)r_k^2}{s_k^2}\) is bounded (for \(\frac{c(s_k)r_k^2}{s_k^2} = (\frac{\tau \ln(1 - \frac{s_k}{\tau})}{\tau^2}) (\frac{r_k^2}{t_k})\), where the first bracket is bounded since \(s_k \to 1\) and \(\ln(1 - s_k)(1 - s_k) \to 0\) as \(k \to \infty\) and the second bracket is bounded by assumption).

This means a sequence of balls \(B(p_k, \sqrt{c(s_k)}r_k)\) at times \(s_k\) is a sequence of collapsing balls at \(\tau\) for flow \(\tilde{g}_{ij}(s)\), which contradicts theorem 11.

Lemma 12. Let \(g_{ij}(t)\), \(t \in [0, \infty)\) be a solution to the Ricci flow \((g_{ij})_t = -2R_{ij} + \frac{1}{t}g_{ij}\) on a closed manifold \(M\), with uniformly bounded Ricci curvatures. Assume that for some sequences \(t_k \to \infty\), \(p_k \in M\) and some constant \(C\) we have \(Q_k = |\text{Rm}|(p_k, t_k) \to \infty\) and \(|\text{Rm}|(x, t) \leq CQ_k\) when \(t \leq t_k\). Then a subsequence of scalings of \(g_{ij}(t_k)\) at \(p_k\) with factors \(Q_k\) converges to a complete, eternal solution to the Ricci flow that is noncollapsed on all scales for some \(\kappa > 0\).

Proof. This lemma is a simple corollary of claim 10. The proof is the same as a proof of a corollary of Perelman’s noncollapsing theorem in section 4 of paper 7.
4 The volume form under the Ricci flow in dimension three

In this section we would like to mention one nice application of the proof of theorem 2 to 3 dimensional manifolds, observed by Richard Hamilton.

**Theorem 13.** Fix $t_0 \in [0, T)$. Let $(g_{ij})_t = -2R_{ij}$ be the Ricci flow on a 3-dimensional compact manifold, for $t \in [0, T)$, where $T < \infty$. If $g(t)$ is singular at time $T$, then $\lim_{t \to T} \hat{\text{Vol}}_t = 0$, where $\hat{\text{Vol}}_t = \min_{x \in M} \text{Vol}_t$.

**Proof.** We will prove theorem 13 by contradiction. Assume that there exist a sequence of times $t_i \to T$ and $\delta > 0$ so that $\hat{\text{Vol}}_{t_i} > \delta$ for all $i$. That would imply $\frac{\text{Vol}_{t_i}}{\text{Vol}_{t_0}} > \delta$ for all $i$ and all $x \in M$.

**Claim 14.** There exists $C = C(g(0), T)$ so that $\frac{\text{Vol}_{t_i}}{\text{Vol}_{t_0}} \leq C$ for all $t \in [0, T)$ and all $x \in M$.

**Proof.** The evolution equation for $\ln \text{Vol}_t$ is

$$\frac{d}{dt} \ln \text{Vol}_t = -R. \tag{9}$$

The evolution equation for the scalar curvature $R(t)$

$$\frac{d}{dt} R = \Delta R + 2|\text{Ric}|^2,$$

implies by a straightforward maximum principle argument that at any time $t \in [0, T)$

$$R(t) \geq \frac{1}{(\min R(0))^{-1} - 2t/3} \geq -\frac{3}{2(t + C)},$$

for some constant $C$. If we integrate the equation (9) over $s \in [t_0, t]$ for any $t \in [t_0, T)$ we will get

$$\ln \text{Vol}_t - \ln \text{Vol}_{t_0} = -\int_{t_0}^{t} R(s) ds \leq \int_{0}^{T} \frac{3}{2(s + C)} ds \leq C_1.$$

Now we easily get the statement of the claim.

□

From the equation (9) we get

$$\ln \text{Vol}_{t_0} - \ln \text{Vol}_{t_i} = -\int_{t_0}^{t_i} R(s) ds.$$

Since $\ln \delta < \ln \frac{\text{Vol}_{t_i}}{\text{Vol}_{t_0}} \leq \ln C$, where $C$ is a constant from the claim above, we get
\[ | \int_{t_0}^{T} R(s) ds | \leq C, \]

where \( C \) may be some other constant depending on \( g(t_0) \) and \( T \), but nonetheless we will denote all such constants by the same symbol.

In dimension 3 we have a pinching estimate, i.e. there exists constants \( C_1 \) and \( C_2 \) so that

\[ |\text{Ric}| \leq C_1(R + C_2), \]

for all times \( t \in [0, T) \). This gives that \( \int_{t_0}^{T} |\text{Ric}| ds \leq C_1 \int_{t_0}^{T} (R(s) + C_2) ds \leq C \) and the estimate does not depend on \( x \in M \).

If \( g(t) \) has a singularity at time \( T \), by theorem 2 we know that there exist a sequence of times \( s_i \to T \), sequence of points \( x_i \in M \) and a constant \( C \) so that \( Q_i = |\text{Ric}|(x_i, s_i) \geq C^{-1} \max_{M \times [0, s_i]} |\text{Ric}|(x, t) \) and \( Q_i \to \infty \). In dimension 3 the curvature tensor is controled by the Ricci curvature. Let \( g_i(s) = Q_i g(s_i + sQ_i^{-1}) \). By Perelman’s noncollapsing theorem and Hamilton’s compactness theorem there exists a subsequence \( g_i \) so that \((M, g_i(s), x_i) \to (N, \bar{g}(s), x)\), where \((N, \bar{g}(s))\) is a complete, non-flat, ancient solution to the Ricci flow. When the Ricci curvature blows up, that happens with the scalar curvature as well because of the pinching estimate in dimension 3. Because of the pinching estimate, when the scalar curvature \( R \) is big, any negative curvature is small compared to the positive ones. That is why after rescaling by a factor \( Q_i \) and after taking a limit when \( i \to \infty \) we get that our ancient, complete, non-flat solution \((N, \bar{g}(s))\) has bounded, non-negative sectional curvatures. This allows us to apply the relative volume comparison theorem to \((N, \bar{g}(s))\) to conclude that for every \( r > 0 \)

\[ \frac{\text{Vol}_{\bar{g}(0)} B(x, r)}{r^n} \leq w_n, \]

where \( w_n \) is the volume of a unit euclidean ball. Fix \( \epsilon > 0 \). We can show that

\[ \frac{\text{Vol}_{\bar{g}(0)} B(x, r)}{r^n} \geq w_n - \epsilon, \]

in the same way as in theorem 2, since we used only integral bound on \( |\text{Ric}| \) in order to control changes in volumes and distances for \( t \) close to \( T \). Since this estimate holds for every \( \epsilon > 0 \), we can conclude that \((N, \bar{g}(0))\) would have to be the euclidean space, which is not possible since \( |\text{Ric}|(x, 0) = 1 \).

\[ \square \]

References

[1] D. DeTurck: Deforming metrics in the direction of their Ricci tensors, J. Differential Geom. 18 (1983), no.1, 157–162.
[2] D. Glickenstein: Precompactness of solutions to the Ricci flow in the absence of injectivity radius estimates; preprint arXiv:math.DG/0211191v2

[3] R. Hamilton: Three-manifolds with positive Ricci curvature, Journal of Differential Geometry 17 (1982) 225–306

[4] R. Hamilton: A compactness property for solutions of the Ricci flow, Amer.J.Math. 117 (1995) 545–572

[5] The formation of singularities in the Ricci flow, Surveys in Differential Geometry, vol. 2, International Press, Cambridge, MA (1995) 7–136

[6] R. Hamilton: Non-singular solutions of the Ricci flow on 3 manifolds, Communications in analysis and geometry vol. 7 (1999) 695–729

[7] G. Perelman: The entropy formula for the Ricci flow and its geometric applications, preprint