Research Article

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A systolic inequality with remainder in the real projective plane

Abstract: The first paper in systolic geometry was published by Loewner’s student P. M. Pu over half a century ago. Pu proved an inequality relating the systole and the area of an arbitrary metric in the real projective plane. We prove a stronger version of Pu’s systolic inequality with a remainder term.

Keywords: systole, geometric inequality, Riemannian submersion, Cauchy-Schwarz theorem, probabilistic variance

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1 Introduction

Loewner’s systolic inequality for the torus and Pu’s inequality [1] for the real projective plane were historically the first results in systolic geometry. Great stimulus was provided in 1983 by Gromov’s paper [2] and later by his book [3].

Our goal is to prove a strengthened version with a remainder term of Pu’s systolic inequality $\text{sys}(g) \leq \frac{\pi}{2} \text{area}(g)$ (for an arbitrary metric $g$ on $\mathbb{R}P^2$), analogous to Bonnesen’s inequality $L^2 - 4\pi A \geq \pi^2(R - r)^2$, where $L$ is the length of a Jordan curve in the plane, $A$ is the area of the region bounded by the curve, $R$ is the circumradius and $r$ is the inradius.

Note that both the original proof in Pu ([1], 1952) and the one given by Berger ([4], 1965, pp. 299–305) proceed by averaging the metric and showing that the averaging process decreases the area and increases the systole. Such an approach involves a five-dimensional integration (instead of a three-dimensional one given here) and makes it harder to obtain an explicit expression for a remainder term. Analogous results for the torus were obtained in ref. [5] with generalizations in ref. [6–17].

2 The results

We define a closed three-dimensional manifold $M \subseteq \mathbb{R}^3 \times \mathbb{R}^3$ by setting

$$M = \{(v, w) \in \mathbb{R}^3 \times \mathbb{R}^3 : v \cdot v = 1, w \cdot w = 1, v \cdot w = 0\},$$

where $v \cdot w$ is the scalar product on $\mathbb{R}^3$. We have a diffeomorphism $M \to \text{SO}(3, \mathbb{R}), (v, w) \mapsto (v, w, v \times w)$, where $v \times w$ is the vector product on $\mathbb{R}^3$. Given a point $(v, w) \in M$, the tangent space $T_{(v, w)}M$ can be identified by differentiating the three defining equations of $M$ along a path through $(v, w)$. Thus,
\[ T_{(v,w)} M = \{(X, Y) \in \mathbb{R}^3 \times \mathbb{R}^3 : X \cdot v = 0, Y \cdot w = 0, X \cdot w + Y \cdot v = 0\}. \]

We define a Riemannian metric \( g_M \) on \( M \) as follows. Given a point \((v, w) \in M\), let \( n = v \times w \) and declare the basis \((0, n), (n, 0), (w, -v)\) of \( T_{(v,w)} M \) to be orthonormal. This metric is a modification of the metric restricted to \( M \) from \( \mathbb{R}^3 \times \mathbb{R}^3 = \mathbb{R}^6 \). Namely, with respect to the Euclidean metric on \( \mathbb{R}^6 \) the above three vectors are orthogonal and the first two have length 1. However, the third vector has Euclidean length \( \sqrt{2} \), whereas we have defined its length to be 1. Thus, if \( A \subseteq T_{(v,w)} M \) denotes the span of \((0,n)\) and \((n,0)\), and \( B \subseteq T_{(v,w)} M \) is spanned by \((w,-v)\), then the metric \( g_M \) on \( M \) is obtained from the Euclidean metric \( g \) on \( \mathbb{R}^6 \) (viewed as a quadratic form) as follows:

\[
\frac{1}{2} g \big|_A + \frac{1}{2} g \big|_B. \tag{1}
\]

Each of the natural projections \( p, q : M \to S^2 \) given by \( p(v, w) = v \) and \( q(v, w) = w \) exhibits \( M \) as a circle bundle over \( S^2 \).

**Lemma 2.1.** The maps \( p \) and \( q \) on \((M, g_M)\) are Riemannian submersions, over the unit sphere \( S^2 \subseteq \mathbb{R}^3 \).

**Proof.** For the projection \( p \), given \((v, w) \in M\), the vector \((0, n)\) as defined above is tangent to the fiber \( p^{-1}(v) \). The other two vectors, \((n, 0)\) and \((w, -v)\), are thus an orthonormal basis for the subspace of \( T_{(v,w)} M \) normal to the fiber and are mapped by \( dp \) to the orthonormal basis \( n, w \) of \( T_v S^2 \).

The projection \( p \) maps the fiber \( q^{-1}(w) \) onto a great circle of \( S^2 \). This map preserves length since the unit vector \((n,0)\), tangent to the fiber \( q^{-1}(w) \) at \((v, w)\), is mapped by \( dp \) to the unit vector \( n \in T_v S^2 \). The same comments apply when the roles of \( p \) and \( q \) are reversed.

In the following proposition, integration takes place, respectively, over great circles \( C \subseteq S^2 \), over the fibers in \( M \), over \( S^2 \), and over \( M \). The integration is always with respect to the volume element of the given Riemannian metric. Since \( p \) and \( q \) are Riemannian submersions by Lemma 2.1, we can use Fubini’s theorem to integrate over \( M \) by integrating first over the fibers of either \( p \) or \( q \), and then over \( S^2 \); cf. [18, Lemma 4]. By the remarks above, if \( C = p(q^{-1}(w)) \) and \( f : S^2 \to \mathbb{R} \), then \( \int_{q^{-1}(w)} f \circ p = \int_C f \).

**Proposition 2.2.** Given a continuous function \( f : S^2 \to \mathbb{R}^+ \), we define \( m \in \mathbb{R} \) by setting

\[
m = \min \left\{ \int_C f : C \subseteq S^2 \text{ a great circle} \right\}.
\]

Then,

\[
\frac{m^2}{\pi} \leq \frac{1}{4\pi} \left( \int_{S^2} f \right)^2 \leq \int_{S^2} f^2,
\]

where equality in the second inequality occurs if and only if \( f \) is constant.

**Proof.** Using the fact that \( M \) is the total space of a pair of Riemannian submersions, we obtain

\[
\int_{S^2} f = \int_{S^2} \left( \frac{1}{2\pi} \int_{p^{-1}(v)} f \circ p \right) = \frac{1}{2\pi} \int_{M} f \circ p = \frac{1}{2\pi} \int_{S^2} \left( \int_{q^{-1}(w)} f \circ p \right) \geq \frac{1}{2\pi} \int_{S^2} m = 2m,
\]

proving the first inequality. By the Cauchy-Schwarz inequality, we have
proving the second inequality. Here, equality occurs if and only if \( f \) and 1 are linearly dependent, i.e., if and only if \( f \) is constant.

We define the quantity \( V_f \) by setting 

\[
V_f = \int_{S^2} f^2 - \frac{1}{4\pi} \left( \int_{S^2} f \right)^2.
\]

Then, Proposition 2.2 can be restated as follows.

**Corollary 2.3.** Let \( f : S^2 \to \mathbb{R}^+ \) be continuous. Then,

\[
\int_{S^2} f^2 - \frac{m^2}{\pi} \geq V_f \geq 0,
\]

and \( V_f = 0 \) if and only if \( f \) is constant.

**Proof.** The proof is obtained from Proposition 2.2 by noting that \( a \leq b \leq c \) if and only if \( c - a \geq c - b \geq 0 \).

We can assign a probabilistic meaning to \( V_f \) as follows. Divide the area measure on \( S^2 \) by \( 4\pi \), thus turning it into a probability measure \( \mu \). A function \( f : S^2 \to \mathbb{R}^+ \) is then thought of as a random variable with expectation \( E_{\mu}(f) = \frac{1}{4\pi} \int_{S^2} f \). Its variance is thus given by

\[
\text{Var}_{\mu}(f) = E_{\mu}(f^2) - (E_{\mu}(f))^2 = \frac{1}{4\pi} \int_{S^2} f^2 - \left( \frac{1}{4\pi} \int_{S^2} f \right)^2 = \frac{1}{4\pi} V_f.
\]

The variance of a random variable \( f \) is non-negative, and it vanishes if and only if \( f \) is constant. This reproves the corresponding properties of \( V_f \) established above via the Cauchy-Schwarz inequality.

Now let \( g_0 \) be the metric of constant Gaussian curvature \( K = 1 \) on \( \mathbb{RP}^2 \). The double covering \( \rho : S^2 \to (\mathbb{RP}^2, g_0) \) is a local isometry. Each projective line \( C \subseteq \mathbb{RP}^2 \) is the image under \( \rho \) of a great circle of \( S^2 \).

**Proposition 2.4.** Given a function \( f : \mathbb{RP}^2 \to \mathbb{R}^+ \), we define \( \overline{m} \in \mathbb{R} \) by setting

\[
\overline{m} = \min \left\{ \int_C f : C \subseteq \mathbb{RP}^2 \text{ a projective line} \right\}.
\]

Then,

\[
\frac{2\overline{m}^2}{\pi} \leq \frac{1}{2\pi} \left( \int_{\mathbb{RP}^2} f \right)^2 \leq \int_{\mathbb{RP}^2} f^2,
\]

where equality in the second inequality occurs if and only if \( f \) is constant.

**Proof.** We apply Proposition 2.2 to the composition \( f \circ \rho \). Note that we have \( \int_{\rho^{-1}(C)} f \circ \rho = 2 \int_C f \) and

\[
\int_{S^2} f \circ \rho = 2 \int_{\mathbb{RP}^2} f.
\]

The condition for \( f \) to be constant holds since \( f \) is constant if and only if \( f \circ \rho \) is constant.

For \( \mathbb{RP}^2 \) we define 

\[
V_f = \int_{\mathbb{RP}^2} f^2 - \frac{1}{2\pi} \left( \int_{\mathbb{RP}^2} f \right)^2 = \frac{1}{2} V_{f \circ \rho}.
\]

We obtain the following restatement of Proposition 2.4.
Corollary 2.5. Let \( f : \mathbb{RP}^2 \to \mathbb{R} \) be a continuous function. Then,
\[
\int_{\mathbb{RP}^2} f^2 - \frac{2\pi}{\pi} \geq \nabla f \geq 0,
\]
where \( \nabla f = 0 \) if and only if \( f \) is constant.

Relative to the probability measure induced by \( \frac{1}{2\pi}g_0 \) on \( \mathbb{RP}^2 \), we have \( E(f) = \frac{1}{2\pi} \int_{\mathbb{RP}^2} f^2 \), and therefore \( \text{Var}(f) = \frac{1}{2\pi} \nabla f \), providing a probabilistic meaning for the quantity \( \nabla f \), as before.

By the uniformization theorem, every metric \( g \) on \( \mathbb{RP}^2 \) is of the form \( g = f^2 g_0 \), where \( g_0 \) is of constant Gaussian curvature +1, and the function \( f : \mathbb{RP}^2 \to \mathbb{R} \) is continuous. The area of \( g \) is \( \int_{\mathbb{RP}^2} f^2 \), and the \( g \)-length of a projective line \( C \) is \( \int_C f \). Let \( L \) be the shortest length of a noncontractible loop. Then, \( L \leq \bar{m} \) where \( \bar{m} \) is defined in Proposition 2.4, since a projective line in \( \mathbb{RP}^2 \) is a noncontractible loop. Then, Corollary 2.5 implies \( \text{area}(\mathbb{RP}^2, g) - \frac{2L^2}{\pi} \geq \nabla f \geq 0 \). If \( \text{area}(\mathbb{RP}^2, g) = \frac{2L^2}{\pi} \), then \( \nabla f = 0 \), which implies that \( f \) is constant, by Corollary 2.5. Conversely, if \( f \) is a constant \( c \), then the only geodesics are the projective lines, and therefore, \( L = c \pi \). Hence, \( \frac{2L^2}{\pi} = 2\pi c^2 = \text{area}(\mathbb{RP}^2) \). We have thus completed the proof of the following result strengthening Pu’s inequality.

Theorem 2.6. Let \( g \) be a Riemannian metric on \( \mathbb{RP}^2 \). Let \( L \) be the shortest length of a noncontractible loop in \( (\mathbb{RP}^2, g) \). Let \( f : \mathbb{RP}^2 \to \mathbb{R} \) be such that \( g = f^2 g_0 \), where \( g_0 \) is of constant Gaussian curvature +1. Then,
\[
\text{area}(g) - \frac{2L^2}{\pi} \geq 2\pi \text{Var}(f),
\]
where the variance is with respect to the probability measure induced by \( \frac{1}{2\pi}g_0 \). Furthermore, equality \( \text{area}(g) = \frac{2L^2}{\pi} \) holds if and only if \( f \) is constant.

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