Azumaya structure on D-branes
and deformations and resolutions of a conifold revisited:
Klebanov-Strassler-Witten vs. Polchinski-Grothendieck

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Abstract

In this sequel to [L-Y1], [L-S-Y], and [L-Y2] (respectively arXiv:0709.1515 [math.AG], arXiv:0809.2121 [math.AG], and arXiv:0901.0342 [math.AG]), we study a D-brane probe on a conifold from the viewpoint of the Azumaya structure on D-branes and toric geometry. The details of how deformations and resolutions of the standard toric conifold $Y$ can be obtained via morphisms from Azumaya points are given. This should be compared with the quantum-field-theoretic/D-brane picture of deformations and resolutions of a conifold via a D-brane probe sitting at the conifold singularity in the work of Klebanov and Witten [K-W] (arXiv:hep-th/9807080) and Klebanov and Strasser [K-S] (arXiv:hep-th/0007191). A comparison with resolutions via noncommutative desingularizations is given in the end.

Key words: D-brane, Azumaya structure, Polchinski-Grothendieck Ansatz, Azumaya point, conifold.

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In memory of a young string theorist Ti-Ming Chiang, whose path I crossed accidentally and so briefly.†

†From C.-H.L. During the years I was attending Prof. Candelas’s group meetings, I learned more about Calabi-Yau manifolds and mirror symmetry and got very fascinated by the works from Brian Greene’s group. Because of this, I felt particularly lucky knowing later that I was going to meet one of his students, Ti-Ming, - a young string theorist with a PhD from Cornell at his very early 20’s - and perhaps to cooperate with him. Unfortunately that anticipated cooperation never happened. Ti-Ming had become unwell just before I resettled. Except the visits to him at the hospital and some chats when he showed up in the office, I didn’t really get the opportunity to interact with him intellectually. Further afterwards I was informed of Ti-Ming’s passing away. Like a shooting star he reveals his shining so briefly and then disappears. The current work is the last piece of Part 1 of the D-brane project. It is grouped with the earlier D(1), D(2), D(3) under the hidden collective title: “Azumaya structure on D-branes and its tests”. Here we address in particular a conifold from the viewpoint of a D-brane probe with an Azumaya structure. This is a theme Ti-Ming may have felt interested in as well, should he still work on string theory, since conifolds have play a role in understanding the duality web of Calabi-Yau threefolds - a theme Ti-Ming once worked on - and D-brane resolution of singularities is a theme Brian Greene’s group once pursued vigorously. We thus dedicate this work to the memory of Ti-Ming.
0. Introduction and outline.

Conifolds, i.e. Calabi-Yau threefolds with ordinary double-points, have been playing special roles at various stages of string theory. In this sequel to [L-Y1], [L-L-S-Y], and [L-Y2], we study a D-brane probe on a conifold from the viewpoint of Azumaya structure on D-branes and toric geometry. This should be compared with the quantum-field-theoretic/D-brany picture of deformations and resolutions of a conifold in the work of Klebanov and Strasser [K-S] and Klebanov and Witten [K-W].

Effective-space-time-filling D3-brane at a conifold singularity.

In [K-W], Klebanov and Witten studied the $d = 4$, $N = 1$ superconformal field theory (SCFT) on the D3-brane world-volume $X \simeq \mathbb{R}^4$ topologically that is embedded in the product space-time $M^{3+1} \times Y$ as $M^{3+1} \times \{0\}$ and its supergravity dual - a compactification of $d = 10$, type-IIB supergravity theory on $AdS^5 \times (S^3 \times S^2)$ - along the line of the AdS/CFT correspondence of Maldacena [Ma]. Here $M^{3+1}$ is the $d = 3 + 1$ Minkowski space-time, $Y$ is the conifold singularity on $Y$, and $AdS^5$ is the $d = 4 + 1$ anti-de Sitter space-time.

In the simplest case when there is a single D3-brane sitting at the conifold point of $Y$, the classical moduli space of the supersymmetric vacua of the associated $U(1)$ super-Yang-Mills theory coupled with matter on the D3-brane world-volume comes from the $D$-term of the vector multiplet and the coefficient $\zeta \in \mathbb{R}$ of the Fayet-Iliopoulos term in the Lagrangian. By varying $\zeta$, one realizes the two small resolutions, $Y_+$ and $Y_-$, of $Y$ as the classical moduli space $Y_\zeta$ of the above $d = 4$ SCFT. A flop $X_+ \rightarrow Y_-$ happens when $Y_\zeta$ crosses over $\zeta = 0$.

To describe the physics for $N$-many parallel D3-branes sitting at the conifold singularity, Klebanov and Witten proposed to enlarge the gauge group for the super-Yang-Mills theory on the common world-volume of the stacked D3-brane to $U(N) \times U(N)$ (rather than the naive $U(N)$) and introduce a superpotential $W$ for the chiral multiplets. The classical moduli space of the theory comes from a system with equations of the type above (i.e. D-term equations) and equations from the superpotential term $W$ (i.e. F-term equations). In particular, the $N$-fold symmetric product $Sym^N Y$ of $Y$ can be realized as the classical moduli space of the $d = 4$ SCFT on the D3-brane world-volume with $\zeta = 0$.

In [K-S], Klebanov-Strassler studied further $d = 4$, $N = 1$ supersymmetric quantum field theory (SQFT) on the D3-brane world-volume that arises from a D3-brane configuration with both $N$-many above full/free D3-branes and $M$-many new fractional/trapped D3-branes sitting.

1 Readers are referred to, for example, [C-diO] (1989); [Stro], [G-M-S], [C-G-G-K] (1995); [G-V] (1998); [Be], [C-F-I-K-V] (2001) and references therein to get a glimpse of conifolds in string theory around the decade 1990s.

2 There will be a few standard physicists’ conventional notations in this highlight of the relevant part of [K-W] and [K-S]: $N$ that counts the number of supersymmetries (susy) via the multiple number of minimal susy numbers in each space-time dimension vs. $N$ that appears in the gauge group $U(N)$ or $SU(N)$ vs. $N$ that counts the multiplicity of stacked D-branes.

3 In string-theorist’s terminology, the D3-brane is “sitting at the conifold singularity”. We will also adopt this phrasing for convenience. Note that in such a setting, the internal part is a D0-brane on the conifold $Y$. The latter is what we will study in this work.

4 $\zeta$ is part of the parameters to give local coordinates of the Wilson’s theory-space in the problem; cf. [L-Y2: Introduction] for brief words. See also [W-B] for the standard SUSY jargon.

5 See also [Wi] and [D-M] for details of such a construction.

6 See [G-K] and references therein for the detail of such fractional D-branes.
at the conifold singularity $0$ of $Y$. For infrared physics, the theory now has the gauge group $SU(N + M) \times SU(N)$. It follows from the work of Affleck, Dine, and Seiberg [A-D-S7] that an additional term to the previous superpotential $W$ is now dynamically generated. This deforms the classical moduli space of SUSY vacua of the $d = 4$ SQFT on the D3-brane world-volume. In the simplest case when $N = M = 1$, this enforces a deformation of the classical moduli space from a conifold to a deformed conifold $Y''$ ($\simeq T^*S^3$ topologically). Cf. Figure 0-1.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure01.png}
\caption{A D-brane configuration without fractional branes, a D-brane configuration with a fractional brane, the moduli space of its supersymmetric vacua.}
\end{figure}

While giving only a highlight of key points in [K-W] and [K-S] that are most relevant to us, we should remark that, in addition to further quantum-field-theoretical issues on the gauge theory side, there is also a gravity side of the story that was studied in [K-W] and [K-S].

**Azumaya structure on D-branes and its tests.**

In D(1) [L-Y1], D(2) [L-L-S-Y], D(3) [L-Y2] and the current work D(4), we illuminate the Azumaya geometry as a key feature of the geometry on D-brane world-volumes in the algebro-geometric category. These four together center around the very remark of Polchinski:

\begin{quote}
\((\text{[Po: vol. 1, Sec. 8.7, p. 272]})\) \textit{“For the collective coordinate $X^\mu$, however, the meaning is mysterious: the collective coordinates for the embedding of $n$ D-branes in space-time are now enlarged to $n \times n$ matrices. This ‘noncommutative geometry’ has proven to play a key role in the dynamics of D-branes, and there are conjectures that it is an important hint about the nature of space-time.”,}
\end{quote}

\footnote{See also [Arg: Chapter 3] and [Te: Chapter 9].}

\footnote{See [A-G-M-O-O] and [Stra] for a review with more emphasis on respectively the gravity and the gauge theory side in the correspondence; e.g. [G-K], [K-N] for developments between [K-W] and [K-S]; and e.g. [D-K-S] for a more recent study.}
which was taken as a guiding question as to what a D-brane is in this project, cf. [L-Y1: Sec. 2.2]. D(2), D(3), and the current D(4) are meant to give more explanations of the highlight [L-Y1: Sec. 4.5]. In this consecutive series of four, we learned that:

**Lesson 0.1 [Azumaya structure on D-branes].** *This “enhancement to \( n \times n \) matrices” Polchinski alluded to says even more fundamentally the nature of D-branes themselves, i.e. the Azumaya structure thereupon. This structure gives them the power to detect the nature of space-time. We also learned that Azumaya structures on D-branes and morphisms therefrom can be used to reproduce/explain several stringy/brany phenomena of stringy or quantum-field-theoretical origin that are very surprising/mysterious at a first mathematical glance.*

This is a basic test to ourselves to believe that Azumaya structures play a special role in understanding/describing D-branes in string theory. Having said this, we should however mention that D-brane remains a very complicated object and the Azumaya structure addressed here is only a part of it. Further issues are investigated in separate works.

**Convention.** Standard notations, terminology, operations, facts in (1) physics aspects of strings and D-branes; (2) algebraic geometry; (3) toric geometry can be found respectively in (1) [Po], [Jo]; (2) [Ha]; (3) [Fu].

- *Noncommutative algebraic geometry* is a very technical topic. For the current work, [Art] of Artin, [K-R] of Kontsevich and Rosenberg, and [leB1] of Le Bruyn are particularly relevant. See [L-Y1: References] for more references.

**Outline.**

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1 D-branes in an affine noncommutative space.

We recall definitions and notions in [L-Y1] that are needed for the current work. Readers are referred to ibidem for more details and references. See also [L-L-S-Y] and [L-Y2] for further explanations and examples.

Affine noncommutative spaces and their morphisms.

An affine noncommutative space over \( \mathbb{C} \) is meant to be a “space” \( \text{Space} \, R \) that is associated to an associative unital \( \mathbb{C} \)-algebra \( R \). In general, it can be tricky to truly realize \( \text{Space} \, R \) as a set of points with a topology in a natural/functorial way. However, “geometric” notions can still be pursued - despite not knowing what \( \text{Space} \, R \) really is - via imposing the fundamental geometry/algebra ansatz:

\[
\text{geometry} = \text{algebra}
\]

The correspondence \( R \leftrightarrow \text{Space} \, R \) gives a contravariant equivalence between the category \( \text{Alg}_\mathbb{C} \) of associative unital \( \mathbb{C} \)-algebras and the category \( \text{AffineSpace}_\mathbb{C} \) of “affine noncommutative spaces” over \( \mathbb{C} \).

Example 1.2. [noncommutative affine space]. ([K-R: Sec. 2: Example (E1)].) The noncommutative affine \( n \)-space \( \text{NA}_n := \text{Space}(\mathbb{C}(\xi_1, \ldots, \xi_n)) \) over \( \mathbb{C} \) is smooth. Here \( \mathbb{C}(\xi_1, \ldots, \xi_n) \) is the associative unital \( \mathbb{C} \)-algebra freely generated by the elements in the set \( \{ \xi_1, \ldots, \xi_n \} \).

Example 1.3. [Azumaya-type noncommutative space]. ([C-Q: Sec. 5 and Proposition 6.2], [K-R: Sec. 1.2, Examples (E2) and (C4)].) Let \( M_r(R) \) be the \( \mathbb{C} \)-algebra of \( r \times r \)-matrices with entries in a commutative regular \( \mathbb{C} \)-algebra \( R \). Then the Azumaya-type noncommutative space \( \text{Space} \, M_r(R) \) is smooth (over \( \mathbb{C} \)). Furthermore, it is also smooth over \( \text{Spec} \, R \).

As a consequence of the Geometry/Algebra Ansatz, a morphism \( \varphi : X = \text{Space} \, R \to Y = \text{Space} \, S \) is defined contravariantly to be a \( \mathbb{C} \)-algebra homomorphism \( \varphi^\sharp : S \to R \). The image, denoted \( \text{Im} \varphi \) or \( \varphi(X) \), of \( X \) under \( \varphi \) is defined to be \( \text{Space}(S/\text{Ker} \varphi^\sharp) \). The latter is canonically included in \( Y \) via the morphism \( i : \varphi(X) \hookrightarrow Y \) defined by the \( \mathbb{C} \)-algebra quotient-homomorphism \( i^\sharp : S \to S/\text{Ker} \varphi^\sharp \). This extends what is done in Grothendieck’s theory of (commutative) schemes. The benefit of thinking a morphism between affine noncommutative spaces this way is actually two folds:
(1) As a functor of point: The space $X = \text{Space } R$ defines a functor

$$h_X : \text{AffineSpace}_\mathbb{C} \rightarrow \text{Set}^\circ$$

$$Y \mapsto \text{Mor}(Y, X);$$

i.e. a functor

$$h_R : \text{Alg}_\mathbb{C} \rightarrow \text{Set}$$

$$S \mapsto \text{Hom}(R, S).$$

Here $\text{Set}$ is the category of sets, $\text{Set}^\circ$ its opposite category, and $\text{Hom}(R, S)$ is the set of $\mathbb{C}$-algebra-homomorphisms.

(2) As a probe: $X = \text{Space } R$ defines another functor

$$g_X : \text{AffineSpace}_\mathbb{C} \rightarrow \text{Set}^\circ$$

$$Y \mapsto \text{Mor}(X, Y);$$

i.e. a functor

$$g_R : \text{Alg}_\mathbb{C} \rightarrow \text{Set}^\circ$$

$$S \mapsto \text{Hom}(S, R).$$

Aspect (1) is by now standard in algebraic geometry. It allows one to define the various local geometric properties of a “space” via algebra-homomorphisms; for example, Definition 1.1. It suggests one to think of $X$ as a sheaf over $\text{AffineSpace}_\mathbb{C}$. Thus, after the notion of coverings and gluings is selected, it allows one to extend the notion of a noncommutative space to that of a “noncommutative stack”. Aspect (2) is especially akin to our thought on D-branes. It says, in particular, that the geometry of $X = \text{Space } R$ can be revealed through an $\mathbb{C}$-subalgebra of $R$.

Example 1.4. [Azumaya point]. Consider the Azumaya point of rank $r : \text{Space } M_r(\mathbb{C})$. Its only two-sided prime ideal is $(0)$, the zero ideal. Thus, naively, one would expect $\text{Space } M_r(\mathbb{C})$ to behave like a point with an Artin $\mathbb{C}$-algebra as its function ring. However, for example, from the $\mathbb{C}$-algebra monomorphism $\times^r \mathbb{C} \hookrightarrow M_r(\mathbb{C})$ with image the diagonal matrices in $M_r(\mathbb{C})$, one sees that $\text{Space } M_r(\mathbb{C})$ - which is topologically a one-point set if one adopts its interpretation as $\text{Spec } M_r(\mathbb{C})$ - can dominate $\Pi_r \text{Spec } \mathbb{C}$ - which is topologically a disjoint union of $r$-many points -. Furthermore, consider, for example, the morphism $\varphi : \text{Space } M_r(\mathbb{C}) \rightarrow \mathbb{A}^1 = \text{Spec } \mathbb{C}[z]$ defined by $\varphi^\sharp : \mathbb{C}[z] \rightarrow M_r(\mathbb{C})$ with $\varphi^\sharp(z) = m$ that is diagonalizable with $r$ distinct eigenvalues $\lambda_1, \ldots, \lambda_r$. Then $\text{Im } \varphi$ is a collection of $r$-many $\mathbb{C}$-points on $\mathbb{A}^1$, located at $z = \lambda_1, \cdots, \lambda_r$ respectively. In other words, the Azumaya noncommutativity cloud $M_r(\mathbb{C})$ over the seemingly one-point space $\text{Space } M_r(\mathbb{C})$ can really “split and condense” to a collection of concrete geometric points! Cf. Figure 1-1. See [L-Y1: Sec. 4.1] for more examples. Such phenomenon generalizes to Azumaya schemes; in particular, see [L-L-S-Y] for the case of Azumaya curves.

Definition 1.5. [surrogate associated to morphism]. Given $X = \text{Space } R$, let $R' \hookrightarrow R$ be a $\mathbb{C}$-subalgebra of $R$. Then, the space $X' := \text{Space } R'$ is called a surrogate of $X$. By definition, there is a built-in dominant morphism $X \rightarrow X'$, defined by the inclusion $R' \hookrightarrow R$. Given a morphism $\varphi : \text{Space } R \rightarrow \text{Space } S$ defined by $\varphi^\sharp : S \rightarrow R$, then $\text{Space } R_\varphi$, where $R_\varphi$ is the image $\varphi^\sharp(S)$ of $S$ in $R$, is called the surrogate of $X$ associated to $\varphi$.

As Example 1.4 illustrates, commutative surrogates may be used to manifest/reveal the hidden geometry of a noncommutative space.
Despite that $\text{Space} \, M_r(\mathbb{C})$ may look only one-point-like, under morphisms the Azumaya “noncommutative cloud” $M_r(\mathbb{C})$ over $\text{Space} \, M_r(\mathbb{C})$ can “split and condense” to various schemes with a rich geometry. The latter schemes can even have more than one component. The Higgsing/un-Higgsing behavior of the Chan-Paton module of D0-branes on $Y$ occurs due to the fact that when a morphism $\varphi : \text{Space} \, M_r(\mathbb{C}) \to Y$ deforms, the corresponding push-forward $\varphi_\ast \mathcal{C}^r$ of the fundamental module $\mathcal{C}^r$ on $\text{Space} \, M_r(\mathbb{C})$ can also change/deform. These features generalize to morphisms from Azumaya schemes to $Y$. Here, a module over a scheme is indicated by a dotted arrow $\cdots$.
Definition 1.6. [push-forward of module]. Given a morphism \( \varphi : X = Space R \rightarrow Y = Space S \), defined by \( \varphi^\sharp : S \rightarrow R \), and a (left) \( R \)-module \( M \), the push-forward of \( M \) from \( X \) to \( Y \) under \( \varphi \), in notation \( \varphi^* M \) or \( S^M \) when \( \varphi \) is understood, is defined to be \( M \) as a (left) \( S \)-module via \( \varphi^\sharp \). Since \( \text{Ker} \varphi^\sharp \cdot M = 0 \), we say that the \( S \)-module \( \varphi^* M \) on \( Y \) is supported on \( \varphi(X) \subset Y \).

In particular, any \( R \)-module \( M \) on \( X = Space R \) has a push-forward on any surrogate of \( X \).

D-branes in an affine noncommutative space à la Polchinski-Grothendieck Ansatz.

A \( D \)-brane is geometrically a locus in space-time that serves as the boundary condition for open strings. Through this, open strings dictate also the fields and their dynamics on D-branes. In particular, when a collection of D-branes are stacked together, the fields on the D-brane that govern the deformation of the brane are enhanced to matrix-valued, cf. Polchinski in [Po: vol. I, Sec. 8.7]. This open-string-induced phenomenon on D-branes, when re-read from Grothendieck’s contravariant equivalence between the category of geometries and the category of algebras, says that D-brane world-volume carries an Azumaya-type noncommutative structure. I.e.

- Polchinski-Grothendieck Ansatz: D-brane has a geometry that is generically locally associated to algebras of the form \( M_r(R_0) \), where \( R_0 \) is an \( \mathbb{R} \)-algebra.

See [L-Y1: Sec. 2.2] for detailed explanations.

For this work, we will be restricting ourselves to affine situations in noncommutative algebraic geometry with \( R_0 \) a commutative Noetherian \( \mathbb{C} \)-algebra. Thus:

Definition 1.7. [affine D-brane in affine target]. A \( D \)-brane (or \( D \)-brane world-volume) in an affine noncommutative space \( Y = Space S \) is a triple that consists of

- a \( \mathbb{C} \)-algebra \( R \) that is isomorphic to \( M_r(R_0) \) for an \( R_0 \),
- a (left) generically simple \( R \)-module \( M \), which has rank \( r \) as an \( R_0 \)-module,
- a morphism \( \varphi : Space R \rightarrow Y \), defined by a \( \mathbb{C} \)-algebra-homomorphism \( \varphi^\sharp : S \rightarrow R \).

We will write \( \varphi : (Space R, M) \rightarrow Y \) for simplicity of notations. \( \varphi(X) = \text{Im} \varphi \) is called the image-brane on \( Y \). \( M \) is called the fundamental module on \( Space R \) and the push-forward \( \varphi_* M \) is called the Chan-Paton module on the image-brane \( \varphi(X) \).

Definition/Example 1.8. [D0-brane as morphism from Azumaya point with fundamental module]. A \( D0 \)-brane of length \( r \) on an affine noncommutative space \( Y = Space S \) is given by a morphism \( \varphi : (Space \text{End}(V), V) \rightarrow Y \), where \( V \simeq \mathbb{C}^r \). In other words, a D0-brane on \( Y \) is given by

- a finite-dimensional \( \mathbb{C} \)-vector space \( V \) and a \( \mathbb{C} \)-algebra-homomorphism: \( \varphi^\sharp : S \rightarrow \text{End}(V) \).

This is how one would think of a D-brane to begin with. Later development of string theory enlarges this picture considerably. See [L-Y1: References] to get a glimpse.
This is precisely a realization of a finite-dimensional $\mathbb{C}$-vector space $V$ as an $S$-module.\footnote{Thus, a D0-brane on $\text{Space } S$ is precisely an $S$-module that is of finite dimension as a $\mathbb{C}$-vector space. Such a direct realization of a D-brane as a module on a target-space is a special feature for D0-branes. For high dimensional D-branes, such modules on the target-space give only a subclass of D-branes that describe solitonic branes in space-time.} A morphism from $\varphi_1 : (\text{Space } \text{End}(V_1), V_1) \rightarrow Y$ to $\varphi_2 : (\text{Space } \text{End}(V_2), V_2) \rightarrow Y$ is a $\mathbb{C}$-vector-space isomorphism $h : V_2 \sim V_1$ such that the following diagram commutes

\[
\begin{array}{ccc}
\text{End}(V_1) & \xrightarrow{\varphi_1} & S \\
\downarrow h & & \downarrow \ & \\
\text{End}(V_2) & \xrightarrow{\varphi_2} & S
\end{array}
\]

Here, the $h$-induced isomorphism $\text{End}(V_2) \xrightarrow{\sim} \text{End}(V_1)$ is also denoted by $h$. In other words, a morphism between $\varphi_1$ and $\varphi_2$ is an isomorphism of the corresponding $V_1$ and $V_2$ as $S$-modules.

It follows from the above definition/example that the moduli stack $\mathcal{M}_{D0}^Y(Y)$ of D0-branes of length $r$ on $Y = \text{Space } S$ has an atlas given by the representation scheme $\text{Rep}(S, M_r(\mathbb{C}))$ that parameterizes all $\mathbb{C}$-algebra-homomorphisms $S \rightarrow M_r(\mathbb{C})$. The latter commutative scheme serves also as the moduli space of morphisms $\text{Space } M_r(\mathbb{C}) \rightarrow Y$ with $M_r(\mathbb{C})$ treated as fixed. From [K-R] and [leB1], one expects that noncommutative geometric structures/properties of $Y = \text{Space } S$ are reflected in properties/structures of the discrete family of commutative schemes $\text{Rep}(S, M_r(\mathbb{C}))$, $r \in \mathbb{Z}_{\geq 0}$. This anticipation from noncommutative algebraic geometry rings hand in hand with the stringy philosophy to use D-branes as a probe to the nature of space-time!

2 Deformations of a conifold via an Azumaya probe.

Using a toric setup for a conifold that is meant to match Klebanov-Witten [K-W], we discuss how an Azumaya probe “sees” deformations of the conifold in a way that resembles Klebanov-Strassler [K-S].

A toric setup for the standard local conifold.

The standard local conifold $Y = \text{Spec } (\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1z_2 - z_3z_4))$ can be given an affine toric variety description as follows. Let $N = \bigoplus_{i=1}^4 \mathbb{Z}e_i$ be the rank 4 lattice and $\Delta$ be the fan in $N$ that consists of the single non-strongly convex polyhedral cone $\sigma = \bigoplus_{i=1}^6 \mathbb{R}_{\geq 0} v_i$ in $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$, where

\[
v_1 = e_1, \quad v_2 = e_2, \quad v_3 = e_3, \quad v_4 = -e_1 + e_2 + e_3,
\]

\[
v_5 = e_1 - e_2 - e_3 + e_4, \quad v_6 = -v_5 = -e_1 + e_2 + e_3 - e_4.
\]

Let $M = \text{Hom}(N, \mathbb{Z})$ be the dual lattice of $N$, with the dual basis $\{e_1^*, e_2^*, e_3^*, e_4^*\}$. Then, the dual cone $\sigma^\vee$ of $\sigma$ is given by $\text{Span}_{\mathbb{R}_{\geq 0}} \{e_1^* + e_2^*, e_3^* + e_4^*, e_1^* + e_3^*, e_2^* + e_4^*\} \subseteq \mathbb{R}_{\geq 0}$. This determines a commutative semigroup

\[
S_\sigma = \sigma^\vee \cap M = \text{Span}_{\mathbb{Z}_{\geq 0}} \{e_1^* + e_2^*, e_3^* + e_4^*, e_1^* + e_3^*, e_2^* + e_4^*\}
\]
with generators \( e_1^* + e_2^* + e_3^* + e_4^* + e_1^* + e_2^* + e_3 + e_4 \). The corresponding group-algebra

\[
\mathbb{C}[S_\sigma] = \mathbb{C}[\xi_1 \xi_2, \xi_3 \xi_4, \xi_1 \xi_3, \xi_2 \xi_4] \subset \mathbb{C}[\xi_1, \xi_2, \xi_3, \xi_4],
\]

where \( \xi_i = \exp(e_i^*) \), \( i = 1, 2, 3, 4 \), defines then the conifold

\[
Y = U_\sigma = \text{Spec} \left( \mathbb{C}[S_\sigma] \right) = \text{Spec} \left( \mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4) \right),
\]

where

\[
z_1 = \xi_1 \xi_2, \quad z_2 = \xi_3 \xi_4, \quad z_3 = \xi_1 \xi_3, \quad z_4 = \xi_2 \xi_4.
\]

Note that built into this construction is the morphism

\[
A^4_{[\xi_1, \xi_2, \xi_3, \xi_4]} := \text{Spec} \left( \mathbb{C}[\xi_1, \xi_2, \xi_3, \xi_4] \right) \rightarrow Y \hookrightarrow A^4_{[z_1, z_2, z_3, z_4]} := \text{Spec} \left( \mathbb{C}[z_1, z_2, z_3, z_4] \right),
\]

where the first morphism is surjective.

An Azumaya probe to a noncommutative space and its commutative descent.

Guided by [K-W] and [K-S], where \( \xi_i \)'s here play the role of scalar component of chiral superfields involved in bidem, consider the noncommutative space

\[
\Xi := \text{Space} (R_\Xi)
\]

\[
:= \text{Space} \left( \mathbb{C}(\xi_1, \xi_2, \xi_3, \xi_4) \left/ \left( [\xi_1 \xi_3, \xi_2 \xi_4], [\xi_1 \xi_2, \xi_1 \xi_4], [\xi_2 \xi_4, \xi_1 \xi_4], [\xi_2 \xi_4, \xi_2 \xi_3], [\xi_1 \xi_4, \xi_2 \xi_3] \right) \right. \right),
\]

where \( \mathbb{C}(\xi_1, \xi_2, \xi_3, \xi_4) \) is the associative unital \( \mathbb{C} \)-algebra generated by \{\( \xi_1, \xi_2, \xi_3, \xi_4 \), \( \cdots \)\} in the denominator is the two-sided ideal generated by \( \cdots \), and \( [\bullet, \bullet'] \) is the commutator. Here, \( \text{Space} (\bullet) \) is the would-be space associated to the ring \( \bullet \). We do not need its detail as all we need are morphisms between spaces which can be contravariantly expressed as ring-homomorphisms. By construction, the scheme-morphism \( A^4_{[\xi_1, \xi_2, \xi_3, \xi_4]} \rightarrow A^4_{[z_1, z_2, z_3, z_4]} \), whose image is \( Y \), extends to a morphism

\[
\pi^\Xi : \Xi \rightarrow A^4_{[z_1, z_2, z_3, z_4]},
\]

whose image is now the whole \( A^4_{[z_1, z_2, z_3, z_4]} \). The underlying ring-homomorphism is given by

\[
\pi^{\Xi, #} : \mathbb{C}[z_1, z_2, z_3, z_4] \longrightarrow R_\Xi
\]

\[
\begin{align*}
z_1 &\mapsto \xi_1 \xi_3, \\
z_2 &\mapsto \xi_2 \xi_4, \\
z_3 &\mapsto \xi_1 \xi_4, \\
z_4 &\mapsto \xi_2 \xi_3.
\end{align*}
\]

Consider a D0-brane moving on the conifold \( Y \) via the chiral superfields. In terms of Polchinski-Grothendieck Ansatz, this is realized by the descent of morphisms \( \tilde{\varphi} : \text{Spec} M_1(\mathbb{C}) = \text{Spec} \mathbb{C} \rightarrow \Xi \) to \( \varphi : \text{Space} M_1(\mathbb{C}) = \text{Spec} \mathbb{C} \rightarrow Y \) by the specification of ring-homomorphisms

\[
\tilde{\varphi} : \xi_1 \mapsto a_1; \quad \xi_2 \mapsto a_2; \quad \xi_3 \mapsto b_1; \quad \xi_4 \mapsto b_2.
\]

The corresponding

\[
\varphi : z_1 \mapsto a_1 b_1; \quad z_2 \mapsto a_2 b_2; \quad z_3 \mapsto a_1 b_2; \quad z_4 \mapsto a_2 b_1
\]

gives a morphism \( \varphi : \text{Spec} \mathbb{C} \rightarrow Y \), i.e. a \( \mathbb{C} \)-point on the conifold \( Y \).
Deformations of the conifold via an Azumaya probe: descent of noncommutative superficially-infinitesimal deformations.

We now consider what happens if we add a D0-brane to the conifold point of $Y$. This D0-brane together with the D0-brane probe is the image of a morphism from the Azumaya point $Space M_2(\mathbb{C})$ to $Y$. Thus we should consider morphisms $\tilde{\varphi} : Space M_2(\mathbb{C}) \to \Xi$ of noncommutative spaces and their descent $\varphi$ on related commutative spaces.

**Definition 2.1. [superficially infinitesimal deformation].** Given finitely-presented associative unital rings, $R = \langle r_1, \ldots, r_m \rangle/\sim$ and $S$, and a ring-homomorphism $h : R \to S$. A superficially infinitesimal deformation of $h$ with respect to the generators $\{r_1, \ldots, r_m\}$ of $R$ is a ring-homomorphism $h_\varepsilon : R \to S$ such that $h_\varepsilon(r_i) = h(r_i) + \varepsilon_i$ with $\varepsilon_i^2 = 0$, for $i = 1, \ldots, m$.

**Remark 2.2. [commutative $S$].** Note that when $S$ is commutative, a superficially infinitesimal deformation of $h_\varepsilon : R \to S$ is an infinitesimal deformation of $h$ in the sense that $h_\varepsilon(r) = h(r) + \varepsilon_r$ with $(\varepsilon_r)^2 = 0$, for all $r \in R$. This is no longer true for general noncommutative $S$.

To begin, consider the diagram of morphisms of spaces

\[
\begin{array}{ccc}
Space M_2(\mathbb{C}) & \xrightarrow{\varphi} & Y \\
\downarrow & & \downarrow \\
Space M_2(\mathbb{C}) & \xrightarrow{\tilde{\varphi}} & A^4 \\
\end{array}
\]

given by ring-homomorphisms

\[
\begin{array}{ccc}
M_2(\mathbb{C}) & \xleftarrow{\varphi^\sharp} & R_{\Xi} \\
\downarrow & & \uparrow \\
M_2(\mathbb{C}) & \xleftarrow{\tilde{\varphi}^\sharp} & C[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4) \\
\end{array}
\]

with

\[
\begin{align*}
A_1; & \quad A_2; \quad B_1; \quad B_2 \\
& \xleftarrow{\varphi^\sharp} \xi_1; \xi_2; \xi_3; \xi_4 \\
& \xleftarrow{\tilde{\varphi}^\sharp} \xi_1 \xi_3; \xi_2 \xi_4; \xi_1 \xi_4; \xi_2 \xi_3 \\
A_1 B_1; & \quad A_2 B_2; \quad A_1 B_2; \quad A_2 B_1 \\
& \xleftarrow{\varphi^\sharp} z_1; \quad z_2; \quad z_3; \quad z_4 \\
& \xleftarrow{\tilde{\varphi}^\sharp} z_1; \quad z_2; \quad z_3; \quad z_4 \\
\end{align*}
\]

where

\[
A_1 = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} a_2 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} b_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} b_2 & 0 \\ 0 & 0 \end{bmatrix}.
\]

The image D-brane $\varphi(Space M_2(\mathbb{C}))$ is supported on a subscheme $Z$ of $Y$ associated to the ideal

\[
Ker\varphi = \begin{cases}
(z_1, z_2, z_3, z_4) \cap (z_1 - a_1 b_1, z_2 - a_2 b_2, z_3 - a_1 b_2, z_4 - a_2 b_1) \\
\text{if the tuple } (a_1 b_1, a_2 b_2, a_1 b_2, a_2 b_1) \neq (0, 0, 0, 0), \\
(z_1, z_2, z_3, z_4) \text{ if the tuple } (a_1 b_1, a_2 b_2, a_1 b_2, a_2 b_1) = (0, 0, 0, 0).
\end{cases}
\]

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The former corresponds to two simple non-coincident D0-branes, each with Chan-Paton module $\mathbb{C}$, on the conifold $Y$ with one of them sitting at the conifold point $0$ and the other sitting at the $\mathbb{C}$-point with the coordinate tuple $(a_1 b_1, a_2 b_2, a_1 b_2, a_2 b_1)$ while the latter corresponds to coincident D0-branes at $0$ with the Chan-Paton module enhanced to $\mathbb{C}^2$ at $0$. In both situations, the support $Z$ of the D-brane is reduced. This is the transverse-to-the-effective-space-time part of the D3-brane setting in [K-W] and [K-S].

Consider now a superficially infinitesimal deformation of $\bar{\varphi}$ given by:

$\text{Space } M_2(\mathbb{C}) \quad \xrightarrow{\bar{\varphi}(\delta_1, \delta_2, \eta_1, \eta_2)} \quad \Xi = \text{Space } R_{\Xi}$

$M_2(\mathbb{C}) \quad \xleftarrow{\varphi^*(\delta_1, \delta_2, \eta_1, \eta_2)} \quad R_{\Xi}$

$A_1; A_2; B_1; B_2 \quad \xleftarrow{} \quad \xi_1; \xi_2; \xi_3; \xi_4$

where

$A_1 = \begin{bmatrix} a_1 & \delta_1 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} a_2 & \delta_2 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} b_1 & 0 \\ \eta_1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} b_2 & 0 \\ \eta_2 & 0 \end{bmatrix}.$

Should $\text{Space } M_2(\mathbb{C})$ be a commutative space, this would give only an infinitesimal deformation of $\varphi$. However, $\text{Space } M_2(\mathbb{C})$ is not a commutative space and, hence, the naive anticipation above could fail. Indeed, the descent $\varphi(\delta_1, \delta_2, \eta_1, \eta_2)$ of $\bar{\varphi}(\delta_1, \delta_2, \eta_1, \eta_2)$ is given by

$\text{Space } M_2(\mathbb{C}) \quad \xrightarrow{\varphi(\delta_1, \delta_2, \eta_1, \eta_2)} \quad \mathbb{A}^4$

$M_2(\mathbb{C}) \quad \xleftarrow{\varphi^*(\delta_1, \delta_2, \eta_1, \eta_2)} \quad \mathbb{C}[z_1, z_2, z_3, z_4]$

$A_1 B_1; A_2 B_2; A_1 B_2; A_2 B_1 \quad \xleftarrow{} \quad z_1; z_2; z_3; z_4,$

i.e.

$\begin{bmatrix} a_1 b_1 + \delta_1 \eta_1 & 0 \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} a_2 b_2 + \delta_2 \eta_2 & 0 \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} a_1 b_2 + \delta_1 \eta_2 & 0 \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} a_2 b_1 + \delta_2 \eta_1 & 0 \\ 0 & 0 \end{bmatrix}$

$\rightarrow \quad z_1; z_2; z_3; z_4.$

The image $Z := \varphi(\delta_1, \delta_2, \eta_1, \eta_2) \ (\text{Space } M_2(\mathbb{C}))$ of the Azumaya point $\text{Space } M_2(\mathbb{C})$ under $\varphi(\delta_1, \delta_2, \eta_1, \eta_2)$ remains a 0-dimensional reduced scheme, consisting of either two $\mathbb{C}$-points - with one of them at $0$ - or $0$ alone. However,

$z_1 z_2 - z_3 z_4 = \begin{vmatrix} z_1 & z_3 \\ z_4 & z_2 \end{vmatrix} = \begin{vmatrix} a_1 & \delta_1 \\ a_2 & \delta_2 \end{vmatrix} \begin{vmatrix} b_1 & b_2 \\ \eta_1 & \eta_2 \end{vmatrix}$

vanishes if and only if either $\begin{vmatrix} a_1 & \delta_1 \\ a_2 & \delta_2 \end{vmatrix}$ or $\begin{vmatrix} b_1 & b_2 \\ \eta_1 & \eta_2 \end{vmatrix}$ is 0. In other words, while the image $\varphi(\delta_1, \delta_2, \eta_1, \eta_2) \ (\text{Space } M_2(\mathbb{C}))$ still contains the conifold-point $0$ in $Y$, as a whole it may longer lie completely even in any infinitesimal neighborhood of the conifold $Y$ in $\mathbb{A}^4$. I.e.:

**Lemma 2.3. [deformation from descent of superficially infinitesimal deformation].**

The descent $\varphi(\delta_1, \delta_2, \eta_1, \eta_2)$ of a superficially infinitesimal deformation of $\bar{\varphi}$ can truly deform $\varphi$. Thus, an appropriate choice of a subspace of the space of morphisms $\bar{\varphi}(\bullet): \text{Space } M_2(\mathbb{C}) \rightarrow \Xi$ can descend to give a space of morphisms $\varphi(\bullet): \text{Space } M_2(\mathbb{C}) \rightarrow \mathbb{A}^4$ that is parameterized by a deformed conifold $Y'$. 

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This realizes a deformed conifold as a moduli space of morphisms from an Azumaya point and is the reason why the Azumaya probe can see a deformation of the conifold $Y$ from the viewpoint of Polchinski-Grothendieck Ansatz. Figure 2-1.

**Remark 2.4.** [generalization] This phenomenon can be generalized beyond a conifold. In particular, recall that an $A_n$-singularity on a complex surface is also a toric singularity. Similar mechanism/discussion can be applied to deform a transverse $A_n$-singularity via morphisms from an Azumaya probe.
Deformations of the conifold via an Azumaya probe: details.

We now give an explicit construction that realizes Lemma 2.3. For convenience, we will take $Space M_2(\mathbb{C})$ as fixed, and is equipped with the defining fundamental (left) $M_2(\mathbb{C})$-module $\mathbb{C}^2$. Then, the space $Mor^a (\text{Space } M_2(\mathbb{C}), \Xi)$ of admissible morphisms of the form $\tilde{\varphi}(\bullet)$ in the previous theme is naturally realized as a subscheme $\text{Rep}^a(R_\Xi, M_2(\mathbb{C}))$ of the representation scheme $\text{Rep}(R_\Xi, M_2(\mathbb{C}))$ that parameterizes elements in $\text{Mor}_{C, \text{Alg}} (R_\Xi, M_2(\mathbb{C}))$. From the previous discussion,

$$\text{Rep}^a(R_\Xi, M_2(\mathbb{C})) = \text{Spec } \mathbb{C}[a_1, a_2, \delta_1, \delta_2, b_1, b_2, \eta_1, \eta_2]$$

$$=: \mathbb{A}^8_{[a_1, a_2, \delta_1, \delta_2, b_1, b_2, \eta_1, \eta_2]} = \mathbb{A}^4_{[a_1, a_2, \delta_1, \delta_2]} \times_C \mathbb{A}^4_{[b_1, b_2, \eta_1, \eta_2]}.$$

Consider also the space $Mor^a (\text{Space } M_2(\mathbb{C}), \mathbb{A}^4)$ of morphisms from Azumaya point to $\mathbb{A}^4_{[z_1, z_2, z_3, z_4]}$ with the associated $\mathbb{C}$-algebra-homomorphism of the form

$$z_1 \mapsto \begin{bmatrix} c_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad z_2 \mapsto \begin{bmatrix} c_2 & 0 \\ 0 & 0 \end{bmatrix}, \quad z_3 \mapsto \begin{bmatrix} c_3 & 0 \\ 0 & 0 \end{bmatrix}, \quad z_4 \mapsto \begin{bmatrix} c_4 & 0 \\ 0 & 0 \end{bmatrix}.$$

Denote the associated representation scheme by

$$\text{Rep}^a (\mathbb{C}[z_1, z_2, z_3, z_4], M_2(\mathbb{C})),$$

which is $\text{Spec } \mathbb{C}[c_1, c_2, c_3, c_4] =: \mathbb{A}^4_{[c_1, c_2, c_3, c_4]}$.

The $\mathbb{C}$-algebra homomorphism $\pi^{\Xi, 8} : \mathbb{C}[z_1, z_2, z_3, z_4] \rightarrow R_\Xi$ induces a morphism of representation schemes

$$\pi_{\text{Rep}} : \mathbb{A}^8_{[a_1, a_2, \delta_1, \delta_2, b_1, b_2, \eta_1, \eta_2]} \longrightarrow \mathbb{A}^4_{[c_1, c_2, c_3, c_4]}$$

with $\pi_{\text{Rep}}^\sharp$ given in a matrix form by

$$\pi_{\text{Rep}}^\sharp : \begin{bmatrix} c_1 & c_3 \\ c_4 & c_2 \end{bmatrix} \mapsto \begin{bmatrix} a_1 & \delta_1 \\ a_2 & \delta_2 \end{bmatrix} \cdot \begin{bmatrix} b_1 & b_2 \\ \eta_1 & \eta_2 \end{bmatrix}.$$

Lemma 2.5. [enough superficially infinitesimally deformed morphisms].

$$\pi_{\text{Rep}} : \mathbb{A}^8_{[a_1, a_2, \delta_1, \delta_2, b_1, b_2, \eta_1, \eta_2]} \longrightarrow \mathbb{A}^4_{[c_1, c_2, c_3, c_4]}$$

is surjective.

There are three homeomorphism classes of fibers of $\pi_{\text{Rep}}$ over a closed point of $\mathbb{A}^4_{[c_1, c_2, c_3, c_4]}$, depending on the rank of $\begin{bmatrix} c_1 & c_3 \\ c_4 & c_2 \end{bmatrix}$.

Lemma 2.6. [topological type of fibers of $\pi_{\text{Rep}}$]. Let $C^3_{[a_1, a_2, c_1, c_2]}$ be the subvariety of $\mathbb{A}^4_{[c_1, c_2, c_3, c_4]}$ associated to the ideal $(c_1 c_2 - c_3 c_4)$. Similarly, for $C^3_{[a_1, a_2, \delta_1, \delta_2]}$ and $C^3_{[b_1, b_2, \eta_1, \eta_2]}$.

Then:

(0) Over $\mathbf{0}$, the fiber is given by $\mathbb{A}^4_{[a_1, a_2, \delta_1, \delta_2]} \cup \mathbb{A}^4_{[b_1, b_2, \eta_1, \eta_2]} \cup \Pi^5$, where $\Pi^5$ is a 5-dimensional irreducible affine scheme meeting $\mathbb{A}^4_{[a_1, a_2, \delta_1, \delta_2]} \cup \mathbb{A}^4_{[b_1, b_2, \eta_1, \eta_2]}$ along $C^3_{[a_1, a_2, \delta_1, \delta_2]} \cup C^3_{[b_1, b_2, \eta_1, \eta_2]}$.

11If $Space M_2(\mathbb{C})$ is not fixed, then one studies Artin stacks that parameterizes morphisms in question from $Space M_2(\mathbb{C})$ to $Space R_\Xi$, the conifold $Y$, and $\mathbb{A}^4_{[z_1, z_2, z_3, z_4]}$ respectively. The discussion given here is then on an atlas of the stack in question.
(1) Over a closed point of $C^3_{[c_1,c_2,c_3,c_4]} - \{0\}$, the fiber is the union $\Pi^4_1 \cup \Pi^4_2$ of two irreducible 4-dimensional affine scheme meeting at a deformed conifold.

(2) Over a closed point of $\mathbb{A}^4_{[c_1,c_2,c_3,c_4]} - C^3_{[c_1,c_2,c_3,c_4]}$, the fiber is isomorphic to $\mathbb{A}^4_{[a_1,a_2,\delta_1,\delta_2]} - C^3_{[a_1,a_2,\delta_1,\delta_2]} = \mathbb{A}^4_{[b_1,b_2,\eta_1,\eta_2]} - C^3_{[b_1,b_2,\eta_1,\eta_2]}$.

The lemma follows from a straightforward computation. Note that the fundamental group as an analytic space is given by

$$\pi_1(\mathbb{A}^4_{[c_1,c_2,c_3,c_4]} - C^3_{[c_1,c_2,c_3,c_4]}) \simeq \pi_1(\mathbb{A}^4_{[a_1,a_2,\delta_1,\delta_2]} - C^3_{[a_1,a_2,\delta_1,\delta_2]}) \simeq \pi_1(\mathbb{A}^4_{[b_1,b_2,\eta_1,\eta_2]} - C^3_{[b_1,b_2,\eta_1,\eta_2]}) \simeq \mathbb{Z}$$

and that the smooth bundle-morphism

$$\pi_{\text{Rep}} : \mathbb{A}^8_{[a_1,a_2,\delta_1,\delta_2,b_1,b_2,\eta_1,\eta_2]} - \pi^{-1}_{\text{Rep}}(C^3_{[c_1,c_2,c_3,c_4]}) \longrightarrow \mathbb{A}^4_{[c_1,c_2,c_3,c_4]} - C^3_{[c_1,c_2,c_3,c_4]}$$

exhibits a monodromy behavior which resembles that of a Dehn twist.

The map $\pi_{\text{Rep}} : \mathbb{A}^8_{[a_1,a_2,\delta_1,\delta_2,b_1,b_2,\eta_1,\eta_2]} \rightarrow \mathbb{A}^4_{[c_1,c_2,c_3,c_4]}$ admits sections, i.e. morphism $s : \mathbb{A}^4_{[c_1,c_2,c_3,c_4]} \rightarrow \mathbb{A}^8_{[a_1,a_2,\delta_1,\delta_2,b_1,b_2,\eta_1,\eta_2]}$ such that $\pi_{\text{Rep}} \circ s = \text{the identity map on } \mathbb{A}^4_{[c_1,c_2,c_3,c_4]}$.

Example 2.7. [section of $\pi_{\text{Rep}}$]. Let $t \in GL_2(\mathbb{C})$, then a simple family of sections of $\pi_{\text{Rep}}$

$$s_t : \mathbb{A}^4_{[c_1,c_2,c_3,c_4]} \longrightarrow \mathbb{A}^8_{[a_1,a_2,\delta_1,\delta_2,b_1,b_2,\eta_1,\eta_2]}$$

is given compactly in a matrix expression by (with $t$ also in its defining 2 × 2-matrix form)

$$s^*_t : \begin{pmatrix} a_1 & \delta_1 \\ a_2 & \delta_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ \eta_1 \end{pmatrix} \longrightarrow \begin{pmatrix} c_1 & c_3 \\ c_4 & c_2 \end{pmatrix} \cdot t^{-1}, t.$$

Through any section $s : \mathbb{A}^4_{[c_1,c_2,c_3,c_4]} \rightarrow \mathbb{A}^8_{[a_1,a_2,\delta_1,\delta_2,b_1,b_2,\eta_1,\eta_2]}$, one can realize $Y' \Pi \{0\}$, where $Y'$ is a deformation of the conifold $Y$ in $\mathbb{A}^4 = \mathbb{A}^4_{[c_1,c_2,c_3,c_4]}$ and 0 is the singular point on $Y$, as the descent of a family of superficially infinitesimal deformations of morphisms from Azumaya point to the noncommutative space $\Xi$. In string theory words,

- deformations of a conifold via a D-brane probe are realized by turning on D-branes at the singularity appropriately; the conifold is deformed and becomes smooth while leaving the trapped D-branes at the singularity behind.

Cf. Figure 2-1.

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\[^{12}\text{It is very instructive to think of the fibration } \pi_{\text{Rep}} : \mathbb{A}^8_{[a_1,a_2,\delta_1,\delta_2,b_1,b_2,\eta_1,\eta_2]} \rightarrow \mathbb{A}^4_{[c_1,c_2,c_3,c_4]} \text{ as defining a one-matrix-parameter family of "matrix nodal curves" in the sense of noncommutative geometry.}\]
3 Resolutions of a conifold via an Azumaya probe.

In this section, we consider resolutions of the conifold $Y = \text{Spec} (\mathbb{C}[z_1, z_2, z_3]/(z_1 z_2 - z_3 z_4))$ from the viewpoint of an Azumaya probe. Recall the following diagram of resolutions of $Y$ from blow-ups of $Y$:

\[
\begin{array}{ccc}
Y & \xrightarrow{\pi} & Y \\
Y_+ & \xleftarrow{\pi} & Y_-
\end{array}
\]

where

- $\pi : \tilde{Y} = \text{Bl}_I Y = \text{Proj}(\bigoplus_{i=0}^{\infty} I^i) \to Y$ with $I = (z_1, z_2, z_3, z_4)$,
- $\pi_+ : Y_+ = \text{Bl}_{I_+} Y = \text{Proj}(\bigoplus_{i=0}^{\infty} I_+^i) \to Y$ with $I_+ = (z_1, z_3)$, and
- $\pi_- : Y_- = \text{Bl}_{I_-} Y = \text{Proj}(\bigoplus_{i=0}^{\infty} I_-^i) \to Y$ with $I_- = (z_1, z_4)$

are blow-ups of $Y$ along the specified subschemes $V(\bullet)$ associated respectively to the ideals $I, I_+$, and $I_-$ of $\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4)$ as given. Here, we set $I_+^0 = \mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4)$. Let $0 = V(z_1, z_2, z_3, z_4)$ be the singular point of $Y$. Then the exceptional locus in each case is given respectively by $\pi^{-1}(0) \simeq \mathbb{P}^1 \times \mathbb{P}^1$, $\pi_+^{-1}(0) \simeq \mathbb{P}^1$, and $\pi_-^{-1}(0) \simeq \mathbb{P}^1$; $Y_+$ and $Y_-$ as schemes/Y are related by a flop; and the restriction of birational morphisms $f_\pm : \tilde{Y} \to Y_\pm$ to $\pi^{-1}(0)$ corresponds to the projections of $\mathbb{P}^1 \times \mathbb{P}^1$ to each of its two factors.

D-brane probe resolutions of a conifold via the Azumaya structure.

An atlas for the stack of morphisms from $\text{Space} \ M_2(\mathbb{C})$ to $Y$ is given by the representation scheme $\text{Rep}(\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4), M_2(\mathbb{C}))$ with the $\text{PGL}_2(\mathbb{C})$-action induced from the $\text{GL}_2(\mathbb{C})$-action on the fundamental module $\mathbb{C}^2$. For convenience, we will also call this a $\text{GL}_2(\mathbb{C})$-action on $\text{Rep}(\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4), M_2(\mathbb{C}))$. Let

\[ W = \text{Rep}^{\text{singleton}} (\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4), M_2(\mathbb{C})) \]

be the subscheme of $\text{Rep}(\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4), M_2(\mathbb{C}))$ that parameterizes D0-branes $\varphi : (\text{Spec} \mathbb{C}, M_2(\mathbb{C}), \mathbb{C}^2) \to Y$ with $(\text{Im} \varphi)_{\text{red}}$ a single $\mathbb{C}$-point on $Y$. Explicitly, $W$ is the image scheme of

\[ \text{GL}_2(\mathbb{C}) \times W_{\text{ut}} \rightarrow \text{Rep}(\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4), M_2(\mathbb{C})) \]

where

\[ W_{\text{ut}} = \left\{ \rho : \mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4) \rightarrow M_2(\mathbb{C}) \mid \rho(z_i) \text{ is of the form } \begin{bmatrix} a_i & \varepsilon_i \\ 0 & a_i \end{bmatrix} \right\} \]

\[ \subset \text{Rep}(\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4), M_2(\mathbb{C})) \]

and the morphism $\rightarrow$ is from the restriction of the $\text{GL}_2(\mathbb{C})$-group on $\text{Rep}(\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4), M_2(\mathbb{C}))$. Using this notation, as a scheme,

\[ W_{\text{ut}} = \text{Spec} (\mathbb{C}[a_1, a_2, a_3, a_4, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4]/(a_1 a_2 - a_3 a_4, a_2 \varepsilon_2 - a_4 \varepsilon_3 - a_3 \varepsilon_4)) \]

\[ \subset \text{Spec} (\mathbb{C}[a_1, a_2, a_3, a_4, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4]) =: \mathbb{A}^8_{[a_1, a_2, a_3, a_4, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4]} \]
Imposing the trivial $GL_2(\mathbb{C})$-action on $Y$, then by construction, there is a natural $GL_2(\mathbb{C})$-equivariant morphism

$$\pi^W : W \to Y$$

defined by $\pi^W_i(z_i) = \frac{1}{2} Tr \rho(z_i) = a_i$ in the above notation. This is the morphism that sends a $\varphi : (Spec \mathbb{C}, M_2(\mathbb{C}), \mathbb{C}^3) \to Y$ under study to $(Im \varphi)_{red} \in Y$.

**Lemma 3.1. [Azumaya probe to conifold singularity].** There exists $GL_2(\mathbb{C})$-invariant subschemes $Y', Y'_+$, and $Y'_-$ of $W$ such that their geometric quotient $\tilde{Y}'/GL_2(\mathbb{C})$, $Y'_+/GL_2(\mathbb{C})$, $Y'_-/GL_2(\mathbb{C})$ under the $GL_2(\mathbb{C})$-action exist and are isomorphic to $\tilde{Y}$, $Y_+$, and $Y_-$ respectively. Furthermore, under these isomorphisms, the restriction of $\pi^W : W \to Y$ to $\tilde{Y}'$, $Y'_+$, and $Y'_-$ descends to morphisms from the quotient spaces $\tilde{Y}'/GL_2(\mathbb{C})$, $Y'_+/GL_2(\mathbb{C})$, $Y'_-/GL_2(\mathbb{C})$ to $Y$ that realize the resolution diagram

\[
\begin{array}{ccc}
\tilde{Y}' & \xrightarrow{f_+} & Y' \\
\downarrow \pi & & \downarrow \pi \\
Y' & \xrightarrow{f_-} & Y
\end{array}
\]

of $Y$ at the beginning of this section.

It is in the sense of the above lemma we say that

- an Azumaya point of rank $\geq 2$ and hence a D-brane probe of multiplicity $\geq 2$ can “see” all the three different resolutions of the conifold singularity.

It should also be noted that Lemma 3.1 is a special case of a more general statement that reflects the fact that the stack of morphisms from Azumaya points to a (general, possibly singular, Noetherian) scheme $Y$ is a generalization of the notion of jet-schemes of $Y$. Cf. [L-Y2: Figure 0-1, caption].

**An explicit construction of $\tilde{Y}'$, $Y'_+$, and $Y'_-$.**

An explicit construction of $\tilde{Y}'$, $Y'_+$, and $Y'_-$, and hence the proof of Lemma 3.1, follows from a lifting-to-$W$ of an affine atlas of $Proj(\oplus_{i=0}^{\infty} I^i(\pm))$.

To construct $\tilde{Y}'$, recall that $I = (z_1, z_2, z_3, z_4)$. An affine atlas of $\tilde{Y}$ is given by the collection

$$U(z_i) = Spec(\oplus_{j=0}^{\infty} I^j)[z_i^{-1}]_0 \simeq \begin{cases} 
Spec(\mathbb{C}[z_1, u_2, u_3, u_4]/(u_2 - u_3 u_4)) & \simeq A^3_{[z_1, u_2, u_3, u_4]} \text{ for } i = 1; \\
Spec(\mathbb{C}[u_1, z_2, u_3, u_4]/(u_1 - u_3 u_4)) & \simeq A^3_{[z_2, u_3, u_4]} \text{ for } i = 2; \\
Spec(\mathbb{C}[u_1, u_2, z_3, u_4]/(u_1 u_2 - u_4)) & \simeq A^3_{[u_1, u_2, u_3]} \text{ for } i = 3; \\
Spec(\mathbb{C}[u_1, u_2, u_3, z_4]/(u_1 u_2 - u_3)) & \simeq A^3_{[u_1, u_2, z_3]} \text{ for } i = 4. 
\end{cases}$$

Here, $z_i \in I$ has grade 1 and $(\oplus_{j=0}^{\infty} I^j)[z_i^{-1}]_0$ is the grade-0 component of the graded algebra $(\oplus_{j=0}^{\infty} I^j)[z_i^{-1}]$. Each $U(z_i)$ is equipped with a built-in morphism $\pi^{(i)} : U(z_i) \to Y$ in such a way that, when all four are put together, they glue to give the resolution $\pi : \tilde{Y} \to Y$.  

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Consider the lifting \( \{ \pi^{(i)} : U^{(z_i)} \to W \mid i = 1, 2, 3, 4 \} \) of the atlas \( \{ \pi^{(i)} : U^{(z_i)} \to Y \mid i = 1, 2, 3, 4 \} \) of \( \tilde{Y} \) that is given by the lifting \( \{ \pi^{(i)'} : U^{(z_i)} \to W_{ul} \subset W \mid i = 1, 2, 3, 4 \} \) defined by

\[
\begin{align*}
\pi^{(1)'} : & \quad a_1, a_2, a_3, a_4, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \quad \mapsto 
\quad z_1, z_1 u_2, z_1 u_3, z_1 u_4, 1, u_2, u_3, u_4 & \text{ respectively,} \\
\pi^{(2)'} : & \quad a_1, a_2, a_3, a_4, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \quad \mapsto 
\quad z_2 u_1, z_2, z_2 u_3, z_2 u_4, u_1, 1, u_3, u_4 & \text{ respectively,} \\
\pi^{(3)'} : & \quad a_1, a_2, a_3, a_4, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \quad \mapsto 
\quad z_3 u_1, z_3 u_2, z_3, z_3 u_4, u_1, u_2, 1, u_4 & \text{ respectively,} \\
\pi^{(4)'} : & \quad a_1, a_2, a_3, a_4, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \quad \mapsto 
\quad z_4 u_1, z_4 u_2, z_4 u_3, z_4, u_1, u_2, u_3, 1 & \text{ respectively.}
\end{align*}
\]

\( \pi^{(i)'} \), \( i = 1, 2, 3, 4 \), are now embeddings into \( W \) with the property that for any geometric point \( p \in U^{(z_i)} \times \tilde{Y} \), \( \pi^{(i)'}(p) \) and \( \pi^{(i)}(p) \) lies in the same \( GL_2(\mathbb{C}) \)-orbit in \( W \). In other words, up to the pointwise \( GL_2(\mathbb{C}) \)-action, they are gluable. Let \( \tilde{Y}' \) be the image scheme of the morphism

\[
GL_2(\mathbb{C}) \times (U^{(z_1)} \amalg U^{(z_2)} \amalg U^{(z_3)} \amalg U^{(z_4)}) \to W
\]

via \( \pi^{(1)'} \amalg \pi^{(2)'} \amalg \pi^{(3)'} \amalg \pi^{(4)'} \) and the \( GL_2(\mathbb{C}) \)-action on \( W \). Then it follows that the geometric quotient \( \tilde{Y}'/GL_2(\mathbb{C}) \) exists and is equipped with a built-in isomorphism \( \tilde{Y}'/GL_2(\mathbb{C}) \xrightarrow{\sim} \tilde{Y} \), as schemes over \( Y \), through the defining embeddings \( U^{(z_i)} \to \tilde{Y} \), \( i = 1, 2, 3, 4 \).

For \( Y'_+ \), recall that \( I_+ = (z_1, z_3) \). An affine atlas of \( Y_+ \) is given by the collection

\[
U^{(z_i)}_+ = Spec ((\bigoplus_{j=0}^\infty I^j)[z_i^{-1}]) \simeq \left\{ \begin{array}{ll}
\text{Spec} (\mathbb{C}[z_1, z_2, u_3, z_4]/(z_2 - z_4 u_3)) \simeq \mathbb{A}^3_{[z_1, u_3, z_4]} & \text{ for } i = 1 ; \\
\text{Spec} (\mathbb{C}[u_1, z_2, z_4]/(z_2 u_1 - z_4)) \simeq \mathbb{A}^3_{[u_1, z_2, z_4]} & \text{ for } i = 3 .
\end{array} \right.
\]

Each \( U^{(z_i)}_+ \) is equipped with a built-in morphism \( \pi^{(i)}_+ : U^{(z_i)}_+ \to Y \) in such a way that, when both are put together, they glue to give the resolution \( \pi_+ : Y_+ \to Y \).

Consider the lifting \( \{ \pi^{(i)'}_+ : U^{(z_i)}_+ \to W \mid i = 1, 3 \} \) of the atlas \( \{ \pi^{(i)}_+ : U^{(z_i)}_+ \to Y \mid i = 1, 3 \} \) of \( Y_+ \) that is given by the lifting \( \{ \pi^{(i)'}_+ : U^{(z_i)}_+ \to W_{ul} \subset W \mid i = 1, 3 \} \) defined by

\[
\begin{align*}
\pi^{(1)'}_+ : & \quad a_1, a_2, a_3, a_4, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \quad \mapsto 
\quad z_1, z_4 u_3, z_1 u_3, z_4, 1, 0, u_3, 0 & \text{ respectively,} \\
\pi^{(3)'}_+ : & \quad a_1, a_2, a_3, a_4, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \quad \mapsto 
\quad z_3 u_1, z_2, z_3, z_2 u_1, u_1, 0, 1, 0 & \text{ respectively.}
\end{align*}
\]

The pair, \( \pi^{(1)'}_+ \) and \( \pi^{(3)'}_+ \), are now embeddings into \( W \) that, as in the case of \( \tilde{Y} \), are gluable up to the pointwise \( GL_2(\mathbb{C}) \)-action. Same construction as in the case of \( \tilde{Y} \) gives then a \( GL_2(\mathbb{C}) \)-invariant subscheme \( Y'_+ \) of \( W \) whose geometric quotient \( Y'_+ /GL_2(\mathbb{C}) \) exists and is equipped with a built-in isomorphism \( Y'_+/GL_2(\mathbb{C}) \xrightarrow{\sim} Y_+ \) as schemes over \( Y \).

For \( Y'_- \), recall that \( I_- = (z_1, z_4) \). The construction is identical to that in the case of \( Y_+ \) after relabelling. An affine atlas of \( Y_- \) is given by the collection

\[
U^{(z_i)}_- = Spec ((\bigoplus_{j=0}^\infty I^j)[z_i^{-1}]) \simeq \left\{ \begin{array}{ll}
\text{Spec} (\mathbb{C}[z_1, z_2, z_3, u_4]/(z_2 - z_3 u_4)) \simeq \mathbb{A}^3_{[z_1, z_3, u_4]} & \text{ for } i = 1 ; \\
\text{Spec} (\mathbb{C}[u_1, z_2, z_3, z_4]/(z_2 u_1 - z_4)) \simeq \mathbb{A}^3_{[u_1, z_2, z_4]} & \text{ for } i = 4 .
\end{array} \right.
\]

Each \( U^{(z_i)}_- \) is equipped with a built-in morphism \( \pi^{(i)}_- : U^{(z_i)}_- \to Y \) in such a way that, when both are put together, they glue to give the resolution \( \pi_- : Y_- \to Y \).

Consider the lifting \( \{ \pi^{(i)'}_- : U^{(z_i)}_- \to W \mid i = 1, 4 \} \) of the atlas \( \{ \pi^{(i)}_- : U^{(z_i)}_- \to Y \mid i = 1, 4 \} \) of \( Y_- \) that is given by the lifting \( \{ \pi^{(i)'}_- : U^{(z_i)}_- \to W_{ul} \subset W \mid i = 1, 4 \} \) defined by
\[\pi^{(1),+} : a_1, a_2, a_3, a_4, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \rightarrow z_1, z_3u_4, z_3, z_1u_4, 1, 0, 0, u_4 \text{ respectively,} \]
\[\pi^{(1),-} : a_1, a_2, a_3, a_4, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \rightarrow z_4u_1, z_2, z_2u_1, z_4, u_1, 0, 1 \text{ respectively.} \]

The pair, \(\pi^{(1),+}\) and \(\pi^{(1),-}\), are now embeddings into \(W\) that are glueable up to the pointwise \(GL_2(\mathbb{C})\)-action. Same construction as in the case of \(\bar{Y}\) gives then a \(GL_2(\mathbb{C})\)-invariant subscheme \(Y'_w\) of \(W\) whose geometric quotient \(Y'_w/GL_2(\mathbb{C})\) exists and is equipped with a built-in isomorphism \(Y'_w/GL_2(\mathbb{C}) \rightarrow Y_-\) as schemes over \(Y\).

This concludes the explicit construction.

**Remark 3.2. [lifting to jet-scheme].** Note that there is a one-to-one correspondence between \(GL_2(\mathbb{C})\)-orbits in \(W\) and isomorphism classes of 0-dimensional torsion sheaves of length 2 on the conifold \(Y\) (i.e. the push-forward Chan-Paton sheaves on \(Y\) under associated morphisms from the Azumaya point \(\text{Space } M_2(\mathbb{C})\) with the fundamental module \(\mathbb{C}^2\)) with connected support. Under this correspondence, the various special liftings-to-\(W\) in the construction above:

\[ (\pi^{(1),+}, \pi^{(2),+}, \pi^{(3),+}, \pi^{(4),+}), (\pi^{(1),-}, \pi^{(2),-}, \pi^{(3),-}, \pi^{(4),-}), \]
and the glueing property, up to the pointwise \(GL_2(\mathbb{C})\)-action, in each tuple follow from the underlying lifting property to the related jet-schemes, which is the total space of the tangent sheaf \(\mathcal{T}_Y\) of \(Y\) in our case.

**A comparison with resolutions via noncommutative desingularizations.**

Consider the conifold algebra defined by\(^{13}\)

\[\Lambda_c := \frac{\mathbb{C}\langle \xi_1, \xi_2, \xi_3 \rangle}{(\xi_1^2\xi_2 - \xi_2\xi_1^2, \xi_1\xi_3^2 - \xi_3\xi_1^2, \xi_1\xi_3 + \xi_3\xi_1, \xi_2\xi_3 + \xi_3\xi_2, \xi_3^2 - 1)},\]

where the numerator is the associative unital \(\mathbb{C}\)-algebra generated by \(\{\xi_1, \xi_2, \xi_3\}\) and the denominator is the two-sided ideal generated by the elements of \(\mathbb{C}\langle \xi_1, \xi_2, \xi_3 \rangle\) as indicated.

**Lemma 3.3. [center of \(\Lambda_c\)].** ([leB-S: Lemma 5.4.]) The \(\mathbb{C}\)-algebra monomorphism

\[\tau^g : \mathbb{C}[z_1, z_2, z_3, z_4]/(z_1z_2 - z_3z_4) \rightarrow \Lambda_c \]
\[z_1 \mapsto \xi_1^g\]
\[z_2 \mapsto \xi_2^g\]
\[z_3 \mapsto \frac{1}{g}(\xi_1\xi_2 + \xi_2\xi_1) + \frac{1}{g}(\xi_1\xi_2 - \xi_2\xi_1)\xi_3\]
\[z_4 \mapsto \frac{1}{g}(\xi_1\xi_2 + \xi_2\xi_1) - \frac{1}{g}(\xi_1\xi_2 - \xi_2\xi_1)\xi_3\]

realizes \(\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1z_2 - z_3z_4)\) as the center of \(\Lambda_c\).

**Proposition 3.4. [representation variety of \(\Lambda_c\)].** ([leB-S: Proposition 5.7.]) The representation variety \(\text{Rep}(\Lambda_c, M_2(\mathbb{C}))\) is a smooth affine variety with three disjoint irreducible components. Two of these components are a point. The third \(\text{Rep}^0(\Lambda_c, M_2(\mathbb{C}))\) has dimension 6.

\(^{13}\)The highlight here follows [leB-S] with some change of notations for consistency and mild rephrasings to link ibidem directly with us.
This implies\(^{14}\) that \(\Lambda_c\) is a smooth order over \(\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1z_2 - z_3z_4)\) and, if one defines \(\text{Spec} \Lambda_c\) to be the set of two-sided prime ideals of \(\Lambda_c\) with the Zariski topology, then the natural morphism
\[
\text{Spec} \Lambda_c \to \text{Spec}(\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1z_2 - z_3z_4))
\]
by intersecting a two-sided prime ideal of \(\Lambda_c\) with the center of \(\Lambda_c\) gives a smooth noncommutative desingularization of \(Y\). ([leB-S: Proposition 5.7,])

Up to the conjugation by an element in \(GL_2(\mathbb{C})\), a \(\mathbb{C}\)-algebra homomorphism \(\rho : \Lambda_c \to M_2(\mathbb{C})\) can be put into one of the following three forms: (In (1) and (2) below, 0 and \(Id\) are respectively the zero matrix and the identity matrix in \(M_2(\mathbb{C})\).)

\begin{enumerate}
\item \(\rho(\xi_1) = 0, \rho(\xi_2) = 0, \rho(\xi_3) = Id;\)
\item \(\rho(\xi_1) = 0, \rho(\xi_2) = 0, \rho(\xi_3) = -Id;\)
\item \(\rho(\xi_1) = \begin{bmatrix} 0 & a_1 \\ b_1 & 0 \end{bmatrix}, \rho(\xi_2) = \begin{bmatrix} 0 & a_2 \\ b_2 & 0 \end{bmatrix}, \rho(\xi_3) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.\)
\end{enumerate}

Form (1) and Form (2) correspond to the two point-components in \(\text{Rep}(\Lambda_c, M_2(\mathbb{C}))\) and Form (3) corresponds to elements in \(\text{Rep}^0(\Lambda_c, M_2(\mathbb{C}))\). On the subvariety \(A^4_{[a_1, b_1, a_2, b_2]}\) of \(\text{Rep}^0(\Lambda_c, M_2(\mathbb{C}))\) that parameterizes \(\rho\) of the form (3), the \(GL_2(\mathbb{C})\)-action on \(\text{Rep}^0(\Lambda_c, M_2(\mathbb{C}))\) reduces to the \(\mathbb{C}^* \times \mathbb{C}^*\)-action
\[
(a_1, b_1, a_2, b_2) \xrightarrow{(t_1, t_2)} (t_1 t_2^{-1} a_1, t_1^{-1} t_2 b_1, t_1 t_2^{-1} a_2, t_1^{-1} t_2 b_2),
\]
where \((t_1, t_2) \in \mathbb{C}^* \times \mathbb{C}^*\). The pair \((\rho(\xi_1), \rho(\xi_2))\) in Form (3) realizes this \(A^4_{[a_1, b_1, a_2, b_2]}\) as the representation variety of the quiver
\[
\begin{array}{ccc}
\bullet & \overset{a_1}{\longrightarrow} & \bullet \\
\vline & \vline & \vline \\
a_2 & \nearrow & b_1 \\
\vline & \vline & \vline \\
\bullet & \overset{a_2}{\longrightarrow} & \bullet
\end{array}
\]

Impose the trivial \(GL_2(\mathbb{C})\)-action on \(Y\), then note that there is a natural \(GL_2(\mathbb{C})\)-equivariant morphism from \(\text{Rep}(\Lambda_c, M_2(\mathbb{C}))\) to \(Y\), as the composition
\[
\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1z_2 - z_3z_4) \xrightarrow{\tau^t} \Lambda_c \xrightarrow{\rho} M_2(\mathbb{C})
\]
has the form
\[
z_i \mapsto 0, \quad i = 1, 2, 3, 4,
\]
for \(\rho\) conjugate to Form (1) or Form (2);
\[
z_1 \mapsto a_1 b_1 Id, \quad z_2 \mapsto a_2 b_2 Id, \quad z_3 \mapsto a_1 b_2 Id, \quad z_4 \mapsto a_2 b_1 Id
\]
for \(\rho\) conjugate to Form (3)\(^{15}\). One can now follow the setting of [Ki] to define the stable structures for the \(GL_2(\mathbb{C})\)-action on \(\text{Rep}^0(\Lambda_c, M_2(\mathbb{C}))\). There are two different choices, \(\theta_+\) and

\(^{14}\) Readers are referred to [leB1] for a general study of the several notions involved in this paragraph. We do not need their details here.

\(^{15}\) Note that when restricted to \(A^4_{[a_1, b_1, a_2, b_2]} \subset \text{Rep}^0(\Lambda_c, M_2(\mathbb{C}))\), this is the morphism \(A^4_{[\xi_1, \xi_2, \xi_3, \xi_4]} \to Y\) in Sec. 2 after the substitution: \(a_1\) (here) \(\to \xi_1\) (there), \(a_2 \to \xi_2, b_1 \to \xi_3, b_2 \to \xi_4\).
θ−, of such structures in the current case. The corresponding stable locus on the quiver variety \( A^4_{[a_1,b_1,a_2,b_2]} \) is given respectively by

\[
A^4_{[a_1,b_1,a_2,b_2]} + V(b_1,b_2) \quad \text{and} \quad A^4_{[a_1,b_1,a_2,b_2]} - V(a_1,a_2),
\]

where \( V(a_1,a_2) \) (resp. \( V(b_1,b_2) \)) is the subvariety of \( A^4_{[a_1,b_1,a_2,b_2]} \) associated to the ideal \((a_1,a_2)\) (resp. \((b_1,b_2)\)). The corresponding GIT quotients recover

\[
\begin{align*}
\text{Rep}^0(\Lambda_c, M_2(\mathbb{C}))/\!/^{\theta+} GL_2(\mathbb{C}) & \quad Y \\
\pi^{\theta+} & \quad \pi^{\theta-} \\
Y & \quad Y + \quad Y -
\end{align*}
\]

at the beginning of the section. See [leB-S], [leB2] for the mathematical detail and [Be], [B-L], [K-W] for the SQFT/stringy origin.

From the viewpoint of the Polchinski-Grothendieck Ansatz, both the Azumaya-type noncommutative structure on D-branes and a noncommutative structure over \( Y \) described by \( Space \Lambda_c \) come into play in the above setting. As indicated by the explicit expression for \( \rho \circ \tau^\sharp \) above, any morphism \( \tilde{\varphi} : Space M_2(\mathbb{C}) \to Space \Lambda_c \) has the property:

- The composition

\[
Space M_2(\mathbb{C}) \xrightarrow{\tilde{\varphi}} Space \Lambda_c \xrightarrow{\tau} Y
\]

is a morphism \( \varphi := \tilde{\varphi} \circ \tau \) from the Azumaya point \( pt^{\Lambda_c} = Space M_2(\mathbb{C}) \) to \( Y \) with the associated surrogate \( pt_\varphi \simeq Spec \mathbb{C} \).

Thus, the new ingredient of target-space noncommutativity comes into play as another key role toward resolutions of \( Y \) in the above setting while the generalized-jet-resolution-of-singularity picture in our earlier discussion disappears.

Remark 3.5. [world-volume noncommutativity vs. target-space(-time) noncommutativity]. Such a “trading” between a noncommutativity target and morphisms from Azumaya schemes to a commutative target suggests a partial duality between D-brane world-volume noncommutativity and target space(-time) noncommutativity.

Figure 3-1.
Figure 3-1. Trading of morphisms from \( \text{Space} M_2(\mathbb{C}) \) directly to the conifold \( Y \) with those to the noncommutative space \( \text{Space} \Lambda_c \) over \( Y \). Note that for generic \( \rho \in \text{Rep}(\Lambda_c, M_2(\mathbb{C})) \) such that \( \rho \circ \tau^\sharp = 0 \), \( \rho(\Lambda_c) \) is similar to the \( \mathbb{C} \)-subalgebra \( U \) of upper triangular matrices in \( M_2(\mathbb{C}) \). The noncommutative point \( \text{Space} U \) is also smooth, with \( \text{Spec} U \) consisting of two \( \mathbb{C} \)-points connected by a directed nilpotent bond. It is thus represented by a quiver \( \bullet \rightarrow \bullet \) in the figure. Furthermore, let \( \tilde{\varphi} : \text{Space} M_2(\mathbb{C}) \rightarrow \text{Space} \Lambda_c \) be the corresponding morphism. Then \( \tilde{\varphi} \) determines also a flag in the Chan-Paton module \( \tilde{\varphi}_* \mathbb{C}^2 \) on the image D0-brane \( \text{Im} \tilde{\varphi} \). On the other hand, over a generic \( p \neq 0 \) on \( Y \), the generic image of a \( \varphi' \) that maps to \( p \) after the composition with \( \tau \) will be simply \( \text{Space} M_2(\mathbb{C}) \).
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