Cotangent Bundle over
Hermitian Symmetric Space \( E_7/E_6 \times U(1) \)
from Projective Superspace

\[ a,b \] Masato Arai\(^1 \), \[ b,c \] Filip Blaschke\(^2 \)

\(^a\) Fukushima National College of Technology
Fukushima 970-8034, Japan

\(^b\) Institute of Experimental and Applied Physics,
Czech Technical University in Prague
128 00 Prague 2, Czech Republic

\(^c\) Institute of Physics, Silesian University in Opava
746 01 Opava, Czech Republic

Abstract

We construct an \( \mathcal{N} = 2 \) supersymmetric sigma model on the cotangent bundle over the Hermitian symmetric space \( E_7/(E_6 \times U(1)) \) in the projective superspace formalism, which is a manifest \( \mathcal{N} = 2 \) off-shell superfield formulation in four-dimensional spacetime. To obtain this model we elaborate on the results developed in arXiv:0811.0218 and present a new closed formula for the cotangent bundle action, which is valid for all Hermitian symmetric spaces. We show that the structure of the cotangent bundle action is closely related to the analytic structure of the Kähler potential with respect to a uniform rescaling of coordinates.
1 Introduction

The number of supersymmetry is intimately related to the target space geometry in the supersymmetric sigma models [1]. The target spaces in the four-dimensional $\mathcal{N} = 2$ supersymmetric sigma models are hyperkähler manifolds [2]. Therefore supersymmetric field theory is a powerful tool to construct new hyperkähler manifolds. In order to construct new $\mathcal{N} = 2$ supersymmetric nonlinear sigma models, it is convenient to use a superfield formalism which makes supersymmetry manifest. Such an $\mathcal{N} = 2$ off-shell superfield formulation is the projective superspace formalism\(^1\) [3]. The projective superspace extends superspace at each point by an additional bosonic coordinate $\zeta$ which is a projective coordinate on $\mathbb{C}P^1$; action is written using contour integral over $\zeta$, and reality conditions are imposed, using complex conjugation of $\zeta$ composed with the antipodal map \([3, 5, 6, 7]\).

\(^1\)Another $\mathcal{N} = 2$ off-shell superfield formulation exists, which is called the harmonic superspace formalism [4].
In this formalism, there have been developments in construction of $\mathcal{N} = 2$ supersymmetric sigma models on the tangent bundles of the large class of the Hermitian symmetric spaces as well as using the generalized Legendre transform [5], the cotangent bundles corresponding to the hyperkähler metrics [8, 9, 10, 11, 12, 13]. The method used in [8, 9, 10, 11] essentially rested on finding solutions to the $\mathcal{N} = 2$ projective superspace auxiliary field equations in Kähler normal coordinates at a point in the Hermitian symmetric space $\mathcal{M}$ and then extending the solutions, using cleverly chosen coset representatives. This method was originally introduced in [8, 9] and was illustrated on the example of $\mathcal{M} = \mathbb{C}P^1$. The case of $\mathcal{M} = \mathbb{C}P^n$ was worked out in [9, 10]. The classical Hermitian symmetric spaces as well as their non-compact versions were worked out in [11]. The tangent bundle over $Q^n = SO(n + 2)/(SO(n) \times U(1))$ was obtained in [10].

Although the method in [8, 9, 10, 11] is viable, it has a disadvantage when more complicated spaces involving the exceptional groups, such as $E_6/(SO(10) \times U(1))$ and $E_7/(E_6 \times U(1))$, are considered. For this reason, another way of deriving the (co)tangent bundle was elaborated in [12], which is based on the property of the Hermitian symmetric spaces and supersymmetry considerations. It allows one to obtain the closed form expression of the tangent bundle Lagrangian. Using this expression, the tangent bundle over $E_6/(SO(10) \times U(1))$ was obtained. The cotangent bundle was also obtained with the use of the Legendre transformation. The method in [12] can be used for the case of the $E_7/(E_6 \times U(1))$ as well, but the Legendre transformation to the cotangent bundle is very cumbersome. Meanwhile the closed form expression of the cotangent bundle over any Hermitian symmetric space was obtained in [13].

The purpose of the paper is to derive the tangent and cotangent bundles over the Hermitian symmetric space $E_7/(E_6 \times U(1))$. We use closed expressions given in [12] and [13] for the tangent bundle and the cotangent bundle actions over any Hermitian symmetric space, respectively. The latter one in [13] was presented in a matrix form. We first elaborate on this result in order to provide a new closed formula for cotangent bundle action, more suitable for calculations. This reformulation will indicate a relation between the Kähler potential and the cotangent bundle action for any Hermitian symmetric space. More precisely, it will turn out that pole structure of derivatives of the Kähler potential under uniform rescaling of the coordinates completely determines the cotangent bundle action. It was also pointed out in [14] that there is a relationship between the tangent bundle action and the Kähler potential. Our reformulation also shows the existence of such a relation for the cotangent bundle action.

The paper is organized as follows. In Section 2 we give the background material on the
\( \mathcal{N} = 2 \) sigma model in projective superspace. The tangent bundle construction in [12] and [13] is briefly reviewed in Section 3. We also provide the relationship of the two results. In Section 4, we explain the cotangent bundle construction in [13] and reformulate their result for application to \( E_7/(E_6 \times U(1)) \). In Section 5, we derive the tangent and cotangent bundles over \( E_7/(E_6 \times U(1)) \) with the use of the results in Sections 3 and 4. Finally, in Section 6 we discuss the relation between the Kähler potential and the cotangent bundle action for all the Hermitian symmetric spaces. In the Appendix, to illustrate the power of our approach we provide a detailed calculation of the tangent and cotangent bundle action for \( E_6/(SO(10) \times U(1)) \) and show the equivalence of our result to those in [12].

2 \( \mathcal{N} = 2 \) Sigma Models in the Projective Superspace

We start with a family of four-dimensional \( \mathcal{N} = 2 \) off-shell supersymmetric nonlinear sigma models that are described in ordinary \( \mathcal{N} = 1 \) superspace by the action\(^2\)

\[
S[\Upsilon, \tilde{\Upsilon}] = \frac{1}{2\pi i} \oint \frac{d\zeta}{\zeta} \int d^8 z \, K(\Upsilon^I(\zeta), \tilde{\Upsilon}^J(\zeta)), \quad z^M = (x^\mu, \theta_\alpha, \bar{\theta}^\dot{\alpha}),
\]

where \( \mu = 0,1,2,3 \) and \( \alpha, \dot{\alpha} = 1,2 \). The action is formulated in terms of the so-called polar multiplet [5, 6] (see also [7]), one of interesting \( \mathcal{N} = 2 \) multiplets living in the projective superspace. The polar multiplet is described by the so-called arctic superfield \( \Upsilon(\zeta) \) and antarctic superfield \( \tilde{\Upsilon}(\zeta) \) that are generated by an infinite set of ordinary \( \mathcal{N} = 1 \) superfields:

\[
\Upsilon(\zeta) = \sum_{n=0}^\infty \Upsilon_n \zeta^n = \Phi + \Sigma \zeta + O(\zeta^2), \quad \tilde{\Upsilon}(\zeta) = \sum_{n=0}^\infty \tilde{\Upsilon}_n (-\zeta)^{-n},
\]

where \( \tilde{\Upsilon} \) is the conjugation of \( \Upsilon \) under the composition of complex conjugation with the antipodal map on the Riemann sphere, \( \tilde{\zeta} \to -1/\zeta \). Here \( \Phi \) is chiral, \( \Sigma \) complex linear,

\[
\bar{D}_\tilde{\alpha} \Phi = 0, \quad \bar{D}^2 \Sigma = 0,
\]

and the remaining component superfields are unconstrained complex superfields. The above theory is obtained as a minimal \( \mathcal{N} = 2 \) extension of the general four-dimensional \( \mathcal{N} = 1 \) supersymmetric nonlinear sigma model [1]

\[
S[\Phi, \bar{\Phi}] = \int d^8 z \, K(\Phi^I, \bar{\Phi}^J),
\]

\(^2\)The study of such models in this context was initiated in [8, 9, 15]. They correspond to a subclass of the general hypermultiplet theories in projective superspace [5, 6].
where $K$ is the Kähler potential of a Kähler manifold $\mathcal{M}$.

The extended supersymmetric sigma model (2.1) inherits all the geometric features of its $\mathcal{N} = 1$ predecessor (2.4), such as the Kähler invariance and invariance under the holomorphic reparametrization of the Kähler manifold. The latter property implies that the variables $(\Phi^I, \Sigma^J)$ parametrize the tangent bundle $T\mathcal{M}$ of the Kähler manifold $\mathcal{M}$ [15].

To describe the theory in terms of the physical superfields $\Phi$ and $\Sigma$ only, all the auxiliary superfields have to be eliminated with the aid of the corresponding algebraic equations of motion

$$
\int \frac{d\zeta}{\partial \zeta} \zeta^n \frac{\partial K(\Upsilon, \bar{\Upsilon})}{\partial \Upsilon^I} = \int \frac{d\zeta}{\partial \zeta} \zeta^{-n} \frac{\partial K(\Upsilon, \bar{\Upsilon})}{\partial \bar{\Upsilon}^\bar{I}} = 0, \quad n \geq 2. \tag{2.5}
$$

Let $\Upsilon_*(\zeta) \equiv \Upsilon_*(\zeta; \Phi, \bar{\Phi}, \Sigma, \bar{\Sigma})$ denote a unique solution subject to the initial conditions

$$
\Upsilon_*(0) = \Phi, \quad \dot{\Upsilon}_*(0) = \Sigma. \tag{2.6}
$$

For a general Kähler manifold $\mathcal{M}$, the auxiliary superfields $\Upsilon_2, \Upsilon_3, \ldots$, as well as their conjugates, can be eliminated only perturbatively. Their elimination can be carried out, using the ansatz [16]

$$
\Upsilon_n^I = \sum_{p=0}^{\infty} G_j^{I_1 \ldots J_{n+p} L_1 \ldots L_p}(\Phi, \bar{\Phi}) \Sigma^{I_1} \ldots \Sigma^{I_{n+p}} \bar{\Sigma}^{L_1} \ldots \bar{\Sigma}^{L_p}, \quad n \geq 2. \tag{2.7}
$$

Upon elimination of the auxiliary superfields, the action (2.1) should take the form [8, 9]

$$
S_{tb}[\Phi, \Sigma] = \int d^8 z \left\{ K(\Phi, \bar{\Phi}) + \mathcal{L}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) \right\}, \tag{2.8}
$$

where

$$
\mathcal{L}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) = \sum_{n=1}^{\infty} \mathcal{L}^{(n)}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) \tag{2.9}
$$

$$
= \sum_{n=1}^{\infty} \mathcal{L}_{I_1 \ldots I_n J_1 \ldots J_n}(\Phi, \bar{\Phi}) \Sigma^{I_1} \ldots \Sigma^{I_n} \bar{\Sigma}^{J_1} \ldots \bar{\Sigma}^{J_n}.
$$

The first term in the expansion (2.9) is given as $\mathcal{L}_{IJ} = -g_{IJ}(\Phi, \bar{\Phi})$ and the tensors $\mathcal{L}_{I_1 \ldots I_n J_1 \ldots J_n}$ for $n > 1$ are functions of the Riemann curvature $R_{IJKL}(\Phi, \bar{\Phi})$ and its covariant derivatives. Eq. (2.8) is written by the base manifold coordinate $\Phi$ and the tangent space coordinate $\Sigma$. Therefore, this is the tangent bundle action.
The complex linear tangent variables $\Sigma$'s in (2.8) can be dualized into chiral one-forms in accordance with the generalized Legendre transform [5]. To construct a dual formulation, we consider the first-order action

$$
S = \int d^8z \left\{ K(\Phi, \bar{\Phi}) + \mathcal{L}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) + \Psi_I \Sigma^I + \bar{\Psi}_{\bar{I}} \bar{\Sigma}^{\bar{I}} \right\},
$$

(2.10)

where the tangent vector $\Sigma^I$ is now a complex unconstrained superfield, while the one-form $\Psi$ is chiral, $\bar{D}_a \Psi_I = 0$. Integrating out $\Sigma$'s and their conjugate should give the cotangent-bundle action

$$
S_{\text{ctb}}[\Phi, \Psi] = \int d^8z \left\{ K(\Phi, \bar{\Phi}) + \mathcal{H}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) \right\},
$$

(2.11)

where

$$
\mathcal{H}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) = \sum_{n=1}^{\infty} \mathcal{H}^{(n)}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi})
$$

$$
= \sum_{n=1}^{\infty} \mathcal{H}^{I_1 \cdots I_n, \bar{J}_1 \cdots \bar{J}_n}(\Phi, \bar{\Phi}) \Psi_{I_1} \cdots \Psi_{I_n} \bar{\Psi}_{\bar{J}_1} \cdots \bar{\Psi}_{\bar{J}_n},
$$

(2.12)

with $\mathcal{H}^{IJ}(\Phi, \bar{\Phi}) = g^{IJ}(\Phi, \bar{\Phi})$. The variables $(\Phi^I, \Psi_J)$ parametrize the cotangent bundle $T^*M$ of the Kähler manifold $M$. The action (2.11) is written by chiral superfields only, so the action gives the hyperkähler potential.

Above, we have given a brief sketch of a derivation of the tangent and cotangent bundle actions (2.8) and (2.11). Detailed calculations of $\mathcal{L}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma})$ and $\mathcal{H}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi})$ for the case when $M$ is a Hermitian symmetric space are given in [8, 9, 10, 11, 12, 13]. For our purpose the most useful formulation is to use the method as presented in [13]. Below, we briefly review the result in [13] and, in particular, reformulate the result of the cotangent bundle to be better suited for our purpose.

### 3 The Tangent Bundle

In what follows, we consider the case when $M$ is a Hermitian symmetric space:

$$
\nabla_L R_{I_1 \bar{J}_1 I_2 \bar{J}_2} = \bar{\nabla}_L R_{I_1 \bar{J}_1 I_2 \bar{J}_2} = 0.
$$

(3.1)

Then, the algebraic equations of motion (2.5) are known to be equivalent to the holomorphic geodesic equation (with complex evolution parameter) [8, 9]

$$
\frac{d^2 \Upsilon^I_*(\zeta)}{d\zeta^2} + \Gamma^I_{JK} \left( \Upsilon^J_*(\zeta), \bar{\Phi} \right) \frac{d \Upsilon^J_*(\zeta)}{d\zeta} \frac{d \Upsilon^K_*(\zeta)}{d\zeta} = 0,
$$

(3.2)
under the same initial conditions (2.6). Here $\Gamma^I_{JK}(\Phi, \Phi)$ are the Christoffel symbols for the Kähler metric $g_{I \bar{J}}(\Phi, \bar{\Phi}) = \partial_I \partial_{\bar{J}} K(\Phi, \bar{\Phi})$. In particular, we have

$$\Upsilon^I_2 = -\frac{1}{2} \Gamma^I_{JK}(\Phi, \bar{\Phi}) \Sigma^J \Sigma^K. \quad (3.3)$$

According to the principles of projective superspace [5, 6], the action (2.1) is invariant under the $\mathcal{N} = 2$ supersymmetry transformations

$$\delta \Upsilon^I(\zeta) = i \left( \varepsilon^{\alpha}_I \bar{Q}_\alpha + \varepsilon^{\dot{\alpha}}_I \dot{Q}^{\dot{\alpha}} \right) \Upsilon^I(\zeta), \quad (3.4)$$

when $\Upsilon^I(\zeta)$ is viewed as a $\mathcal{N} = 2$ superfield. Here $i = 1, 2$ is the $SU(2)_R$ index. However, since the action is given in $\mathcal{N} = 1$ superspace, it is only the $\mathcal{N} = 1$ supersymmetry which is manifestly realized. The second hidden supersymmetry can be shown to act on the physical superfields $\Phi$ and $\Sigma$ as follows (see, e.g. [7])

$$\begin{align*}
\delta \Phi^I &= \bar{\varepsilon}^I_\alpha D^I_\alpha, \\
\delta \Sigma^I &= -\varepsilon^\alpha D^I_\alpha \Phi^J - \frac{1}{2} \bar{\varepsilon}^I_\alpha D^I_\alpha \left\{ \Gamma^I_{JK}(\Phi, \bar{\Phi}) \Sigma^J \Sigma^K \right\}.
\end{align*} \quad (3.5)$$

Upon elimination of the auxiliary superfields, the action (2.8), which is associated with the Hermitian symmetric space $\mathcal{M}$, is invariant under

$$\begin{align*}
\delta \Phi^I &= \bar{\varepsilon}^I_\alpha D^I_\alpha, \\
\delta \Sigma^I &= -\varepsilon^\alpha D^I_\alpha \Phi^J - \frac{1}{2} \bar{\varepsilon}^I_\alpha D^I_\alpha \left\{ \Gamma^I_{JK}(\Phi, \bar{\Phi}) \Sigma^J \Sigma^K \right\}.
\end{align*} \quad (3.6)$$

The action (2.8) varies under (3.6) as follows

$$\delta S_{tb}[\Phi, \Sigma] = \int d^8 z \left\{ \frac{\partial L}{\partial \Phi^I} - \frac{\partial L}{\partial \Sigma^L} \Gamma^L_{IJ} \Sigma^J \right\} \bar{\varepsilon}^I_\alpha D^I_\alpha + \int d^8 z \left\{ \frac{1}{2} R_{KIJL} \frac{\partial L}{\partial \Sigma^K} \Sigma^L + \frac{\partial L}{\partial \Sigma^I} \Sigma^I \right\} \bar{\varepsilon}^I_\alpha D^I_\alpha + c.c. \quad (3.7)$$

Here the variation in the first line vanishes, since the curvature is covariantly constant. Thus, we obtain the first-order differential equation

$$\frac{1}{2} R_{KIJL} \frac{\partial L}{\partial \Sigma^K} \Sigma^L + \frac{\partial L}{\partial \Sigma^I} \Sigma^I + g_{I J} \Sigma^I = 0. \quad (3.8)$$

The solution to the equation (3.8) was first presented in [12] in the form

$$L = -\frac{e^{R_{\Sigma, \bar{\Sigma}}} - 1}{R_{\Sigma, \bar{\Sigma}}} |\Sigma|^2, \quad |\Sigma|^2 := g_{I J} \Sigma^I \Sigma^J. \quad (3.9)$$

Then an alternative form of the solution was obtained in [13]. In the rest of this section we show that the two results are equivalent. The solution in [13] is given by

$$L(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) = -\frac{1}{2} \Sigma^T g \ln \left( 1 + \frac{R_{\Sigma, \bar{\Sigma}}}{R_{\Sigma, \bar{\Sigma}}} \right) \Sigma, \quad (3.10)$$
where
\[
\Sigma = \left( \begin{array}{c} \Sigma^I \\ \Sigma^J \end{array} \right), \quad g = \left( \begin{array}{cc} 0 & g_{IJ} \\ g_{IJ} & 0 \end{array} \right),
\]
(3.11)
\[
R_{\Sigma,\bar{\Sigma}} = \left( \begin{array}{cc} 0 & (R_{\Sigma})^I_J \\ (R_{\Sigma})^I_J & 0 \end{array} \right) = \left( \begin{array}{cc} 0 & \frac{1}{2} R_{I}^J K L K L^J \\ \frac{1}{2} R_{I}^J K L K L^J & 0 \end{array} \right).
\]
(3.12)

We shall rewrite (3.10), using the following differential operators
\[
R_{\Sigma,\bar{\Sigma}} = - (R_{\Sigma})^I_J \frac{\partial}{\partial \Sigma^I}, \quad \bar{R}_{\Sigma,\bar{\Sigma}} = -(R_{\Sigma})^I_J \Sigma^J \frac{\partial}{\partial \bar{\Sigma}^I}.
\]
(3.13)

These operators were originally introduced in [12]. They satisfy the relation
\[
R_{\Sigma,\bar{\Sigma}} \bar{L}^{(n)} = \bar{R}_{\Sigma,\bar{\Sigma}} \Sigma^J \frac{\partial}{\partial \Sigma^I} F^{(n)}(n),
\]
(3.14)

Performing the Taylor expansion for (3.10), we have
\[
L = \sum_{n=1}^{\infty} c_n F^{(n)}(n), \quad c_n = \frac{(-1)^n}{n},
\]
(3.15)

where the functions \( F^{(n)}(n) \) given by
\[
F^{(2k+2)} = \Sigma^I g_{IJ} \left( (R_{\Sigma} R_{\Sigma})^J K L K L \right)^{k+1} \Sigma^J, \quad k = 0, 1, 2 \ldots
\]
(3.16)
\[
F^{(2k+1)} = \Sigma^I g_{IJ} \left( (R_{\Sigma} R_{\Sigma})^J K L K L \right)^{k} \Sigma^J, \quad k = 0, 1, 2 \ldots
\]
(3.17)
satisfy relations
\[
\Sigma^I \frac{\partial}{\partial \Sigma^J} F^{(n)}(n) = \Sigma^I \frac{\partial}{\partial \Sigma^J} F^{(n)}(n) = n F^{(n)}(n),
\]
(3.18)
\[
\frac{\partial}{\partial \Sigma^J} F^{(2k+2)} = (2k+2) g_{IJ} \left( (R_{\Sigma} R_{\Sigma})^J K L K L \right)^{k+1} \Sigma^J, \quad k = 0, 1, 2 \ldots
\]
(3.19)
\[
\frac{\partial}{\partial \Sigma^J} F^{(2k+1)} = (2k+1) g_{IJ} \left( (R_{\Sigma} R_{\Sigma})^J K L K L \right)^{k} \Sigma^J.
\]
(3.20)

Using these identities, one can easily prove
\[
F^{(n+1)} = \frac{(-R_{\Sigma,\bar{\Sigma}})^n}{n!} |\Sigma|^2.
\]
(3.21)

Putting this into (3.15), we find out that
\[
L = - \sum_{n=1}^{\infty} \frac{(R_{\Sigma,\bar{\Sigma}})^{n-1}}{n!} |\Sigma|^2 = - \int_0^1 dt e^{t R_{\Sigma,\bar{\Sigma}}} |\Sigma|^2 = \frac{e^{R_{\Sigma,\bar{\Sigma}}} - 1}{R_{\Sigma,\bar{\Sigma}}} |\Sigma|^2.
\]
(3.22)

This form coincides with (3.9).
4 The Cotangent Bundle

We start with (2.10) to construct a dual formulation [12]. The action can be shown to be invariant under the following supersymmetry transformations

\[\delta \Phi^I = \frac{1}{2} \bar{D}^2 \{ \bar{\varepsilon} \theta \Sigma^I \}, \]

\[\delta \Sigma^I = -\varepsilon^\alpha D_\alpha \Phi^I - \frac{1}{2} \bar{\varepsilon} \bar{D}^\dot{\alpha} \{ \Gamma^I_{JK}(\Phi, \bar{\Phi}) \Sigma^J \Sigma^K \} - \frac{1}{2} \bar{\varepsilon} \theta \Gamma^I_{JK}(\Phi, \bar{\Phi}) \Sigma^J \bar{D}^2 \Sigma^K,\]

\[\delta \Psi_I = -\frac{1}{2} \bar{D}^2 \{ \bar{\varepsilon} \theta K_I(\Phi, \bar{\Phi}) \} + \frac{1}{2} \bar{D}^2 \{ \bar{\varepsilon} \theta \Gamma^K_{IJ}(\Phi, \bar{\Phi}) \Sigma^J \} \Psi_K. \quad (4.23)\]

These transformations induce the supersymmetry transformations under which the action (2.11) is invariant

\[\delta \Phi^I = \frac{1}{2} \bar{D}^2 \{ \bar{\varepsilon} \theta \Sigma^I(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) \}, \]

\[\delta \Psi_I = -\frac{1}{2} \bar{D}^2 \{ \bar{\varepsilon} \theta K_I(\Phi, \bar{\Phi}) \} + \frac{1}{2} \bar{D}^2 \{ \bar{\varepsilon} \theta \Gamma^K_{IJ}(\Phi, \bar{\Phi}) \Sigma^J(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) \} \Psi_K, \quad (4.24)\]

with

\[\Sigma^I(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) = \frac{\partial}{\partial \Psi_I} \mathcal{H}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) \equiv \mathcal{H}^I. \quad (4.25)\]

The requirement of invariance under such transformations can be shown to be equivalent to the following nonlinear equation on \( \mathcal{H} \) [12]:

\[\mathcal{H}^I g_{IJ} - \frac{1}{2} \mathcal{H}^K \mathcal{H}^L R_{KIJL} \Psi_I = \Psi_J. \quad (4.26)\]

This equation also results directly from (3.8), using the definition of the \( \Psi \)'s, or, if one wants, as a consequence of the superspace Legendre transform. Detailed discussion of the above is given in [12].

The closed form expression of \( \mathcal{H} \) for any Hermitian symmetric space was obtained as a solution of (4.26), which is given by [13]

\[\mathcal{H}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) = \frac{1}{2} \Psi^T g^{-1} \mathcal{F}(-R_{\Psi, \bar{\Psi}}) \Psi, \quad (4.27)\]

where

\[\Psi = \begin{pmatrix} \Psi_I \\ \bar{\Psi}_I \end{pmatrix}, \quad g^{-1} = \begin{pmatrix} 0 & g^{IJ} \\ g^{IJ} & 0 \end{pmatrix}, \quad (4.28)\]

\[R_{\Psi, \bar{\Psi}} = \begin{pmatrix} 0 & (R_{\Psi})_I^J \\ (R_{\bar{\Psi}})_I^J & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} R_{IJ}^{KL} \Psi_I \Psi_L \Psi_K \Psi_L \\ \frac{1}{2} R_{IJ}^{KL} \Psi_K \Psi_L \end{pmatrix}, \quad (4.29)\]
and
\[ F(x) = \frac{1}{x} \left[ \sqrt{1 + 4x} - 1 - \ln \left( \frac{1 + \sqrt{1 + 4x}}{2} \right) \right]. \] (4.30)

We rewrite (4.27) to a convenient form for our purpose. In a similar way to the tangent bundle case, we introduce the following differential operators
\[ R_{\Phi, \bar{\Phi}} = -(R_{\Phi})^I_J \frac{\partial}{\partial \Psi^I}, \quad \bar{R}_{\Phi, \bar{\Psi}} = -(R_{\bar{\Phi}})^I_J \frac{\partial}{\partial \bar{\Psi}^I}. \] (4.31)

They satisfy
\[ R_{\Phi, \bar{\Phi}} \mathcal{H}^{(n)} = \bar{R}_{\Phi, \bar{\Psi}} \mathcal{H}^{(n)}. \] (4.32)

Performing the Taylor expansion for (4.27), we have
\[ \mathcal{H} = \sum_{n=1}^{\infty} c_n G^{(n)}, \quad c_n = \frac{(-1)^{n-1} F^{(n-1)}(0)}{(n-1)!}, \] (4.33)

where the terms \( G^{(n)} \) are given by
\[ G^{(2k+2)} = \Psi_I g^{IJ} \left( (R_{\Phi} R_{\Phi})^k \right)_J^K (R_{\bar{\Phi}})_{KL}, \quad k = 0, 1, 2 \ldots \] (4.34)
\[ G^{(2k+1)} = \bar{\Psi}_I g^{IJ} \left( (R_{\Phi} R_{\Phi})^k \right)_J^K \bar{\Psi}_K, \quad k = 0, 1, 2 \ldots \] (4.35)

and satisfy the following relations
\[ \Psi_I \frac{\partial}{\partial \Psi^I} G^{(n)} = \bar{\Psi}_I \frac{\partial}{\partial \bar{\Psi}^I} G^{(n)} = n G^{(n)}, \] (4.36)
\[ \frac{\partial}{\partial \Psi^I} G^{(2k+2)} = (2k+2) g^{IJ} \left( (R_{\Phi} R_{\Phi})^k \right)_J^K (R_{\bar{\Phi}})_{KL}, \] (4.37)
\[ \frac{\partial}{\partial \bar{\Psi}^I} G^{(2k+1)} = (2k+1) g^{IJ} \left( (R_{\Phi} R_{\Phi})^k \right)_J^K \bar{\Psi}_K. \] (4.38)

By using the above identities, one can show that
\[ G^{(n+1)} = \frac{(-R_{\Phi, \bar{\Psi}})^n}{n!} |\Psi|^2, \quad |\Psi|^2 := g^{IJ} \Psi_I \bar{\Psi}_J. \] (4.39)

Putting together (4.33) and (4.39), we find
\[ \mathcal{H} = \sum_{n=0}^{\infty} \frac{\mathcal{F}^{(n)}(0) R^n_{\Phi, \bar{\Psi}}}{n!} \frac{\Psi^n}{n!} |\Psi|^2. \] (4.40)

Now we make use of the formula
\[ \frac{x^n}{n!} = \oint_C \frac{d\xi}{2\pi i} e^{\xi x}, \] (4.41)
where, at this point, contour $C$ can be any closed loop containing the origin with a positive (counterclockwise) orientation. Using this identity, we obtain

$$
\mathcal{H} = \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} \oint_{C} \frac{d\xi}{2\pi i} \frac{e^{\xi R_{\Phi,\bar{\Psi}}}}{\xi^{n+1}} |\Psi|^2, \tag{4.42}
$$

where now the contour $C$ must be chosen such that it lies in the region where the factor $e^{\xi R_{\Phi,\bar{\Psi}}}|\Psi|^2$ is analytic. With this in mind, we can rewrite eq. (4.42) as

$$
\mathcal{H} = \oint_{C} \frac{d\xi}{2\pi i} \frac{F^{(1/\xi)}}{\xi} e^{\xi R_{\Phi,\bar{\Psi}}}|\Psi|^2. \tag{4.43}
$$

The function $F(1/\xi)/\xi$ gives a branch cut between $-4$ and $0$. Therefore, the contour has to be chosen so that it does not cross the branch cut, for the $\mathcal{H}$ is real and regular. Combining both requirements, we see that the contour $C$ in addition to avoiding the branch cut must be also bounded by the poles of the factor $e^{\xi R_{\Phi,\bar{\Psi}}}|\Psi|^2$ (see fig. 1). In the subsequent section, we derive the tangent bundle as well as the cotangent bundle over $E_7/[E_6 \times U(1)]$ by using (3.22) and (4.43).

5 The Hermitian Symmetric Space $E_7/(E_6 \times U(1))$

The Kähler potential for the Hermitian symmetric space $E_7/(E_6 \times U(1))$ was computed in [17, 18]. Here we will use the Kähler potential derived in [18]. In order to be consistent with the notation adopted in [18], we will use small Latin letters to label indices. Upper
indices will be used for the base-space ($\Phi^I \to \Phi^i$) and tangent coordinates ($\Sigma^I \to \Sigma^i$),
while lower indices are reserved for one-forms ($\Psi^I \to \Psi^i$).

Locally, the Hermitian symmetric space $E_7/(E_6 \times U(1))$ can be described by complex variables $\Phi^i$ transforming in the $27$ representation of the $E_6$ and their conjugates.

$$\Phi^i, \quad \bar{\Phi}^i := (\Phi^i)^* , \quad i = 1, \ldots, 27 .$$
(5.1)

The Kähler potential is

$$K(\Phi, \bar{\Phi}) = \ln \left( 1 + \Phi^i \bar{\Phi}_i + \frac{1}{4} | \Gamma_{ijk} \Phi^j \phi^k |^2 + \frac{1}{36} | \Gamma_{ijk} \Phi^i \phi^j \phi^k |^2 \right) .$$
(5.2)

Here we have used the notation $| \Gamma_{ijk} \Phi^j \phi^k |^2 = (\Gamma_{ijk} \Phi^j \phi^k)(\Gamma^{ilm} \bar{\Phi}_l \bar{\Phi}_m)$ and $| \Gamma_{ijk} \Phi^i \phi^j \phi^k |^2 = (\Gamma_{ijk} \Phi^i \phi^j \phi^k)(\Gamma^{lmn} \bar{\Phi}_l \bar{\Phi}_m \bar{\Phi}_n)$ where $\Gamma_{ijk}$ is a rank-3 symmetric tensor covariant under the $E_6$. Its complex conjugate is denoted by $\Gamma^{ijk}$.

Products of these tensors satisfy the identity [19]

$$\Gamma_{ijk} \Gamma^{ijl} = 10 \delta^l_k .$$
(5.3)

And

$$\Gamma_{ijk} \left( \Gamma^{ilm} \Gamma^{jnp} + \Gamma^{ipl} \Gamma^{jmn} + \Gamma^{imn} \Gamma^{jnp} \right) = \delta^j_k \Gamma^{mnp} + \delta^p_k \Gamma^{mnj} + \delta^m_k \Gamma^{njp} + \delta^j_m \Gamma^{ijn} ,$$
(5.4)

which is called the Springer relation [20].

Let us calculate the tangent bundle Lagrangian by using (3.22). In our notation, the first-order differential operator defined in (3.13) is given by

$$R_{\Sigma, \bar{\Sigma}} = -\frac{1}{2} \Sigma^i \Sigma^j \Sigma^k R_{i \ k \ l}(g^{-1})_m^i \frac{\partial}{\partial \Sigma^m} ,$$
(5.5)

where $(g^{-1})^{ij} = (g^{ij})^{-1}$ is the inverse metric of $g^{ij}$, that is $g^{k}(g^{-1})^{ij} = \delta^j_i$. Since we consider a symmetric space, it is actually sufficient to carry out the calculations at a particular point, say at $\Phi = 0$. The Riemann tensor at $\Phi = 0$ is given as

$$R_{i \ k \ l} \bigg|_{\Phi = 0} = \partial_i \partial_k g^{jl} - (g^{-1})^m_n \partial_n g^{ij} \partial^m g^{kl} \bigg|_{\Phi = 0} = -\delta^j_i \delta^l_k + \Gamma_{mik} \Gamma^{mlj} - \delta^j_i \delta^l_k .$$
(5.6)

Substituting this into (5.5), we have

$$R_{\Sigma, \bar{\Sigma}} = xD - \frac{1}{2} y \partial_x - \frac{1}{3} z \partial_y , \quad D := x \partial_x + y \partial_y + z \partial_z ,$$
(5.7)
where we have used (5.4) and invariant quantities under the $E_6$ action

\[ x := \Sigma^i \bar{\Sigma}_i, \]
\[ y := (\Gamma_{ijk} \Sigma^j \Sigma^k)(\Gamma^{ilm} \bar{\Sigma}_i \bar{\Sigma}_m), \]
\[ z := (\Gamma_{ijk} \Sigma^j \Sigma^k)(\Gamma^{lnm} \bar{\Sigma}_l \bar{\Sigma}_m \bar{\Sigma}_n). \]

Using the Baker-Campbell-Hausdorff expansion formula, one can show that

\[ e^{tR_{\Sigma, \bar{\Sigma}}} = e^{txD} e^{-\frac{t^2}{4} y \partial_x} e^{-\frac{t^3}{3} z \partial_y} e^{\frac{t^2}{4} yD} e^{-\frac{t^3}{3} zD}. \]  

By a straightforward application of each exponential we obtain

\[ e^{tR_{\Sigma, \bar{\Sigma}}} = -\partial_t \ln \Omega(t; x, y, z), \quad \Omega(t; x, y, z) := 1 - tx + \frac{t^2}{4} y - \frac{t^3}{36} z. \]

Plugging (5.12) into (3.22), we get the tangent bundle action at $\Phi = 0$.

\[ L = \ln \left( 1 - x + \frac{1}{4} y - \frac{1}{36} z \right). \]

This result can be extended to an arbitrary point $\Phi$ of the base manifold by replacing

\[ x \to g^i_j \Sigma^i \bar{\Sigma}_j, \]
\[ \frac{1}{4} y \to \frac{1}{4}(g^i_j \Sigma^i \bar{\Sigma}_j)^2 + \frac{1}{4} R^j_i \Sigma^i \Sigma^j \Sigma^k \bar{\Sigma}_l, \]
\[ -\frac{1}{36} z \to -\frac{1}{6}(g^i_j \Sigma^i \bar{\Sigma}_j)^3 - \frac{1}{4}(g^i_j \Sigma^i \bar{\Sigma}_j)(R^m_i \Sigma^k \bar{\Sigma}_m \Sigma^l \bar{\Sigma}_n) \]
\[ - \frac{1}{12} |g^i_j R^k_i \Sigma^m \Sigma^l \bar{\Sigma}_m|^2. \]

Let us turn to the cotangent bundle action. Again we restrict the calculations to the origin of the base manifold $\Phi = 0$. Defining $E_6$ invariant quantities in terms of the cotangent vector

\[ \tilde{x} := \Psi^i \bar{\Psi}_i, \]
\[ \tilde{y} := (\Gamma^{ijk} \Psi^j \Psi^k)(\Gamma_{ilm} \bar{\Psi}_l \bar{\Psi}_m \bar{\Psi}_n), \]
\[ \tilde{z} := (\Gamma^{ijk} \Psi^j \Psi^k)(\Gamma_{lnm} \bar{\Psi}_l \bar{\Psi}_m \bar{\Psi}_n), \]

we obtain the differential operator (4.31) of the form

\[ R_{\Psi, \bar{\Psi}} = \tilde{x} \tilde{D} - \frac{1}{2} \tilde{y} \partial_{\tilde{x}} - \frac{1}{3} \tilde{z} \partial_{\tilde{y}}, \quad \tilde{D} := \tilde{x} \partial_{\tilde{x}} + \tilde{y} \partial_{\tilde{y}} + \tilde{z} \partial_{\tilde{z}}. \]

Repeating the same calculation as below (5.11), we find

\[ e^{\xi R_{\Psi, \bar{\Psi}}} \tilde{x} = -\partial_\xi \ln \Omega(\xi; \tilde{x}, \tilde{y}, \tilde{z}), \]
where the function $\Omega$ is given in (5.12). Eq. (4.43) with the above leads to

\[
\mathcal{H} = -\oint_C \frac{d\zeta}{2\pi i} \frac{\mathcal{F}(1/\zeta)}{\zeta} \left( -\bar{x} + \frac{\zeta \bar{y}}{2} - \frac{\zeta^2 \bar{z}}{12} \right) .
\]  

(5.22)

Note that the factor in (4.42) $e^{\xi R_{\Psi, \Phi} |\Psi|^2}$ produces poles which are given by the roots of the cubic equation:

\[
1 - \xi x + \frac{\xi^2 y}{4} - \frac{\xi^3 z}{36} = \frac{z}{36} (\xi_1 - \xi)(\xi_2 - \xi)(\xi_3 - \xi) = 0 .
\]  

(5.23)

By construction, these poles are not inside the contour which only encircles the branch cut between the origin $\xi = 0$ and $\xi = -4$. Since the function $\mathcal{F}(1/\xi)/\xi$ has no singularity at infinity in the $\xi$-plane, the contour can be equivalently respected as encircling the poles $\xi_1, \xi_2$ and $\xi_3$ in the opposite direction. Thus, we can apply the Residue theorem, picking additional minus sign from the orientation of contour (which kills another minus given by eq. (5.21)) to obtain

\[
\mathcal{H} = \frac{\mathcal{F}(1/\xi_1)}{\xi_1} + \frac{\mathcal{F}(1/\xi_2)}{\xi_2} + \frac{\mathcal{F}(1/\xi_3)}{\xi_3}.
\]  

(5.24)

The explicit form of $\xi_1, \xi_2$ and $\xi_3$ are given by

\[
\xi_1 = \frac{3\bar{y}}{\bar{z}} + \frac{2^{1/3}(-81\bar{y}^2 + 108\bar{x} \bar{z})}{3\bar{z}(-1458\bar{y}^3 + 2916\bar{x}\bar{y}\bar{z} - 972\bar{z}^2 + A)^{1/3}},
\]

\[
-\frac{2^{1/3}(-1458\bar{y}^3 + 2916\bar{x}\bar{y}\bar{z} - 972\bar{z}^2 + A)^{1/3}}{3 \cdot 2^{1/3} \bar{z}},
\]  

\[
\xi_2 = \frac{3\bar{y}}{\bar{z}} - \frac{(1 + i\sqrt{3})(-81\bar{y}^2 + 108\bar{x} \bar{z})}{3 \cdot 2^{2/3}\bar{z}(-1458\bar{y}^3 + 2916\bar{x}\bar{y}\bar{z} - 972\bar{z}^2 + A^{1/3})},
\]

\[
+ \frac{1 - i\sqrt{3}}{6 \cdot 2^{1/3} \bar{z}}(-1458\bar{y}^3 + 2916\bar{x}\bar{y}\bar{z} - 972\bar{z}^2 + A^{1/3}),
\]  

\[
\xi_3 = \frac{3\bar{y}}{\bar{z}} - \frac{(1 - i\sqrt{3})(-81\bar{y}^2 + 108\bar{x} \bar{z})}{3 \cdot 2^{2/3}\bar{z}(-1458\bar{y}^3 + 2916\bar{x}\bar{y}\bar{z} - 972\bar{z}^2 + A^{1/3})},
\]

\[
+ \frac{1 + i\sqrt{3}}{6 \cdot 2^{1/3} \bar{z}}(-1458\bar{y}^3 + 2916\bar{x}\bar{y}\bar{z} - 972\bar{z}^2 + A^{1/3}),
\]  

(5.25-27)

where $A = \sqrt{4(-81\bar{y}^2 + 108\bar{x} \bar{z})^2 + (-1458\bar{y}^3 + 2916\bar{x}\bar{y}\bar{z} - 972\bar{z}^2)^2}$. The result at an arbitrary point of $\Phi$ can be obtained by the following replacements

\[
\bar{x} \rightarrow (g^{-1})_i^j \Psi_j \bar{\Psi}^i,
\]  

\[
\frac{1}{4} \bar{y} \rightarrow \frac{1}{2} ((g^{-1})_i^j \Psi_j \bar{\Psi}^i)^2 + \frac{1}{4} \bar{R}^{-1}_i^k \bar{\Psi}^i \Psi_j \bar{\Psi}^k \Psi_j,
\]  

\[
-\frac{1}{36} \bar{z} \rightarrow -\frac{1}{6} ((g^{-1})_i^j \Psi_j \bar{\Psi}^i)^3 - \frac{1}{4} ((g^{-1})_i^j \Psi_j \bar{\Psi}^i)(\bar{R}^{-1}_i^k \Psi_j \bar{\Psi}^k \Psi_j \bar{\Psi}^m \Psi_m)
\]

\[
- \frac{1}{12} ((g^{-1})_i^j \bar{R}^{-1}_j^k \Psi_k \bar{\Psi}^i \Psi_m)^2 ,
\]  

(5.28-30)
where $\tilde{R}_{ij}^k (g^{-1})_i^m (g^{-1})_j^n (g^{-1})_k^p R_m^n q$.

6 Generalization

We have constructed the tangent and cotangent bundles over $E_7/(E_6 \times U(1))$ by using the projective superspace formalism. Our results have been obtained and based on the results in [12] and [13], which give the closed form expressions of the tangent and cotangent bundles. We have rewritten the expression of the cotangent bundle to a form convenient for our purpose. The new formula for the cotangent bundle action (4.43) is useful, for it drastically simplifies calculations and allows us to obtain the cotangent bundle action.

Our reformulation suggests an alternative expression for the cotangent bundle over any compact Hermitian symmetric space. In order to explain it, let us look at the equation (5.22). We can rewrite it as follows

$$H = -\oint_C \frac{d\xi}{2\pi i} \frac{F(\xi)}{\xi} \partial_\xi \ln \Omega(\xi; \tilde{x}, \tilde{y}, \tilde{z}).$$

(6.31)

It is easy to see that $\Omega$ is related to the Kähler potential $K(\Phi, \bar{\Phi})$ by uniform rescaling of coordinates

$$\ln \Omega(\xi; \tilde{x}, \tilde{y}, \tilde{z}) = K(\Phi \to -\sqrt{\xi} \bar{\Psi}, \bar{\Phi} \to \sqrt{\xi} \Psi) := K_\xi(\Psi, \bar{\Psi}).$$

(6.32)

Thus, we see that the cotangent bundle action for any Hermitian symmetric space is given by

$$H = -\oint_C \frac{d\xi}{2\pi i} \frac{F(\xi)}{\xi} \partial_\xi K_\xi(\Psi, \bar{\Psi}),$$

(6.33)

where, as in the $E_7/(E_6 \times U(1))$ case, the contour $C$ must encircle the branch cut of $F(1/\xi)/\xi$ and it must be bounded by the poles of $\partial_\xi K_\xi$, in the same manner as depicted in fig. 1. Since for all Hermitian symmetric spaces the $\Omega$ is a finite-order polynomial in $\xi$, we may write

$$\Omega \sim \Pi_i (\xi - \xi_i(\Psi, \bar{\Psi})),$$

(6.34)

where $\xi_i$ are the solutions to characteristic equation $\Omega = 0$. Since we are interested in (the derivative of) the logarithm of the above quantity, the constant of proportionality is actually not important. Thus, we are led to

$$\partial_\xi K_\xi = \sum \frac{1}{\xi - \xi_i}.$$  

(6.35)
Substituting this back into (6.33) and using the Residue theorem, we obtain

\[ \mathcal{H} = -\sum_i \oint_C \frac{d\xi}{2\pi i} \frac{\mathcal{F}(1/\xi)}{\xi - \xi_i} = \sum_i \frac{\mathcal{F}(1/\xi_i)}{\xi_i}, \]

where minus sign is absorbed because the contour \( C \) encircles poles in the clockwise direction. We checked that the results obtained by this formula for all Hermitian symmetric spaces perfectly coincide with the previous results in [8, 9, 11, 12]. In the Appendix, as an illustration, we apply the formula (6.36) to \( E_6/[SO(10) \times U(1)] \) case and show that the result coincides with the one in [11].

7 Conclusion

In this paper we have presented closed formulas for tangent and cotangent bundle action over Hermitian symmetric space \( E_7/(E_6 \times U(1)) \) for the first time, starting from the result in [13]. This particular case helped us to establish the correspondence between a cotangent bundle action and poles of derivatives of the Kähler potential under uniform rescaling of coordinates. Based on this correspondence, we have presented a general formula for a cotangent bundle action for any Hermitian symmetric space in Eq. (6.36). In the former methods in [8, 9, 11, 12], the Legendre transform is used to obtain a cotangent bundle action. This procedure, although in principle applicable to the \( E_7/(E_6 \times U(1)) \) case, is not straightforward and one often must find a suitable ansatz to solve involved algebraic equations. Our simple formula for the cotangent bundle action obtained from the result in [13] avoids such a complication and clearly demonstrates the reduction of work in constructing a cotangent bundle action.

Acknowledgements:
M.A. thanks Sergei M. Kuzenko for useful discussions and comments at the early stage of this work. F.B. thanks Petr Blaschke for his comments about the contour integral in the cotangent bundle formulation and Veronika Portešová for her aid in correcting the manuscript. The work of the authors is supported in part by the Research Program MSM6840770029, by the project of International Cooperation ATLAS-CERN of the Ministry of Education, Youth and Sports of the Czech Republic, and by the Japan Society for the Promotion of Science (JSPS) and Academy of Sciences of the Czech Republic (ASCR) under the Japan - Czech Republic Research Cooperative Program.
A Cotangent Bundle over $E_6/\text{SO}(10) \times U(1)$

Kähler potential or $E_6/\text{SO}(10) \times U(1)$ is given as

$$K(\Phi, \bar{\Phi}) = \ln \left( 1 + \Phi^a \bar{\Phi}^a + \frac{1}{8} (\bar{\Psi}^a (\sigma_A)_{\alpha\beta}(\sigma_A)_{\gamma\delta}) (\Psi^\dagger_{\gamma}\sigma^\dagger_A)^{\alpha\gamma}\Psi^\dagger_{\delta} \right),$$  \hspace{1cm} (A.37)

where $(\sigma_A)_{\alpha\beta} = (\sigma_A)_{\beta\alpha}$ are the $16 \times 16$ sigma-matrices which generate, along with their Hermitian-conjugates $(\sigma_A)_{\alpha\beta}^{\dagger}$, the ten-dimensional Dirac matrices in the Weyl representation. The sigma-matrices obey the anti-commutation relations

$$(\sigma_A \sigma_B^{\dagger} + \sigma_B \sigma_A^{\dagger})_{\alpha\beta} = 2\delta_{AB}\delta_{\alpha\beta}.$$  \hspace{1cm} (A.38)

As usual, we limit our discussion only to the base point $\Phi = \bar{\Phi} = 0$, where the metric and Riemann tensor are given as

$$g^\beta_{\alpha} \rightarrow \delta^\beta_{\alpha}, \hspace{2cm} (A.39)$$

$$R^\alpha_{\beta \gamma \delta} \rightarrow -\delta^\alpha_{\beta} \delta^\gamma_{\delta} + \frac{1}{2} (\sigma_A)_{\beta\delta}(\sigma_A^{\dagger})_{\alpha\gamma}. \hspace{2cm} (A.40)$$

In this limit, the operator $R_{\Sigma, \bar{\Sigma}}$ (3.13) can be written in the form

$$R_{\Sigma, \bar{\Sigma}} \rightarrow \Sigma^a_{\alpha} \bar{\Sigma}^a_{\alpha} \frac{\partial}{\partial \Sigma^a_{\alpha}} - \frac{1}{4} (\Sigma^a_{\alpha} (\sigma_A)_{\alpha\beta} \Sigma^a_{\beta}) (\sigma_A^{\dagger})_{\gamma\delta} \Sigma^\gamma \frac{\partial}{\partial \Sigma^\delta}. \hspace{2cm} (A.41)$$

As before, we use the coordinates given by two algebraically independent invariants

$$x := \Sigma^a_{\alpha} \bar{\Sigma}^a_{\alpha}, \hspace{2cm} y := (\Sigma^a_{\alpha} (\sigma_A)_{\alpha\beta} \Sigma^a_{\beta}) (\Sigma^\gamma (\sigma_A)_{\gamma\delta} \bar{\Sigma}^\delta), \hspace{2cm} (A.42)$$

so that the operator $R_{\Sigma, \bar{\Sigma}}$ becomes

$$R_{\Sigma, \bar{\Sigma}} = xD - \frac{1}{4} y \partial_x, \hspace{2cm} D := x \partial_x + y \partial_y. \hspace{2cm} (A.43)$$

Using the Baker–Campbell–Hausdorf’s formula, one finds that

$$e^{tR_{\Sigma, \bar{\Sigma}}} = e^{txD} e^{-\frac{ty}{4} \partial_x} e^{-\frac{t^2y}{8} D}, \hspace{2cm} (A.44)$$

$$e^{tR_{\Sigma, \bar{\Sigma}}x} = -\partial_t \ln \left( 1 - tx + t^2y \right). \hspace{2cm} (A.45)$$

Plugging this into the formula (3.22), we obtain the tangent bundle action

$$\mathcal{L} \rightarrow \ln \left( 1 - x + \frac{y}{8} \right), \hspace{2cm} (A.46)$$
which is in a complete agreement with the result given in [12].

The cotangent bundle action of $E_6/(SO(10) \times U(1))$ can be computed by a direct use of (6.36) after finding the roots of

$$1 - \xi x + \frac{\xi^2 y}{8} = 0.$$  \hspace{1cm} (A.47)

Those are

$$\xi_1 = \frac{4x}{y} - \frac{2\sqrt{4x^2 - 2y}}{y}, \quad \xi_2 = \frac{4x}{y} + \frac{2\sqrt{4x^2 - 2y}}{y}.$$  \hspace{1cm} (A.48)

Thus

$$\mathcal{H} = \frac{\mathcal{F}(1/\xi_1)}{\xi_1} + \frac{\mathcal{F}(1/\xi_2)}{\xi_2}.$$  \hspace{1cm} (A.49)

The result given in [12] at the base point $\Phi = \bar{\Phi} = 0$ is formulated as

$$\mathcal{H} = -\ln\left(\Lambda + \sqrt{\Lambda + x}\right) + \Lambda + \sqrt{\Lambda + x} - \frac{1}{2} \frac{y}{\Lambda + \sqrt{\Lambda + x}},$$  \hspace{1cm} (A.50)

where

$$\Lambda := \frac{1}{2} + \sqrt{\frac{1}{4} + x + \frac{y}{2}}.$$  \hspace{1cm} (A.51)

It can be easily proved that our results are in complete agreement with this one up to the irrelevant constant factor $\ln 2 - 2$.

References

[1] B. Zumino, “Supersymmetry and Kähler manifolds,” Phys. Lett. B 87 (1979) 203.

[2] L. Alvarez-Gaumé and D. Z. Freedman, “Geometrical structure and ultraviolet finiteness in the supersymmetric sigma model,” Commun. Math. Phys. 80 (1981) 443.

[3] A. Karlhede, U. Lindström and M. Roček, “Self-interacting tensor multiplets in N=2 superspace,” Phys. Lett. B 147 (1984) 297.

[4] A. Galperin, E. Ivanov, S. Kalitsyn, V. Ogievetsky and E. Sokatchev, “Unconstrained N=2 Matter, Yang-Mills and Supergravity Theories in Harmonic Superspace,” Class. Quant. Grav. 1 (1984) 469; A. S. Galperin, E. A. Ivanov, V. I. Ogievetsky and E. S. Sokatchev, “Harmonic superspace,” Cambridge, UK: Univ. Pr. (2001) 306 p.

[5] U. Lindström and M. Roček, “New hyperkähler metrics and new supermultiplets,” Commun. Math. Phys. 115 (1988) 21.
[6] U. Lindström and M. Roček, “N=2 super Yang-Mills theory in projective superspace,” Commun. Math. Phys. 128 (1990) 191.

[7] F. Gonzalez-Rey, U. Lindström, M. Roček, S. Wiles and R. von Unge, “Feynman rules in N = 2 projective superspace. I: Massless hypermultiplets,” Nucl. Phys. B 516 (1998) 426 [hep-th/9710250].

[8] S. J. Gates Jr. and S. M. Kuzenko, “The CNM-hypermultiplet nexus,” Nucl. Phys. B 543 (1999) 122 [hep-th/9810137].

[9] S. J. Gates Jr. and S. M. Kuzenko, “4D N = 2 supersymmetric off-shell sigma models on the cotangent bundles of Kähler manifolds,” Fortsch. Phys. 48 (2000) 115 [hep-th/9903013].

[10] M. Arai and M. Nitta, “Hyper-Kähler sigma models on (co)tangent bundles with SO(n) isometry,” Nucl. Phys. B 745, 208 (2006) [hep-th/0602277].

[11] M. Arai, S. M. Kuzenko and U. Lindström, “Hyperkähler sigma models on cotangent bundles of Hermitian symmetric spaces using projective superspace,” JHEP 0702 (2007) 100 [hep-th/0612174].

[12] M. Arai, S. M. Kuzenko and U. Lindstrom, “Polar supermultiplets, Hermitian symmetric spaces and hyperkahler metrics,” JHEP 0712 (2007) 008 [arXiv:0709.2633 [hep-th]].

[13] S. M. Kuzenko and J. Novak, “Chiral formulation for hyperkahler sigma-models on cotangent bundles of symmetric spaces,” JHEP 0812 (2008) 072 [arXiv:0811.0218 [hep-th]].

[14] S. M. Kuzenko, “On superconformal projective hypermultiplets,” JHEP 0712 (2007) 010 [arXiv:0710.1479 [hep-th]].

[15] S. M. Kuzenko, “Projective superspace as a double punctured harmonic superspace,” Int. J. Mod. Phys. A 14 (1999) 1737 [hep-th/9806147].

[16] S. M. Kuzenko and W. D. Linch, “On five-dimensional superspaces,” JHEP 0602, 038 (2006) [hep-th/0507176].

[17] F. Delduc and G. Valent, “Classical and quantum structure of the compact Kählerian sigma models,” Nucl. Phys. B 253, 494 (1985).

[18] K. Higashijima and M. Nitta, “Supersymmetric nonlinear sigma models as gauge theories,” Prog. Theor. Phys. 103 (2000) 635 [hep-th/9911139].

[19] T. W. Kephart and M. T. Vaughn, “Tensor Methods For The Exceptional Group E6,” Annals Phys. 145 (1983) 162.

[20] T. A. Springer, Proc. Kon. Ned. Akad. Wet. A65 (1962) 259.