Quantum mechanics as a solutions to the classical self-force problem

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Abstract
It is argued that, contrary to conventional wisdom, no trustworthy universal self-force/radiative corrections to the Lorentz force equation, can be derived from the basic tenets of classical electrodynamics. This concords with the apparent randomness observed in quantum mechanical scattering experiments and with the absence of any experimental support for such universality. In a recent paper [11], the statistical effect of radiative corrections to the motion of charged bodies has been derived from the basic tenets and does take a universal form, described by quantum mechanical wave equations—again conforming with experiment. As that derivation assumes nothing about the size, mass or composition of the body, it is conjectured that quantum mechanics is the appropriate framework for dealing also with radiative corrections to the motion of macroscopic bodies.

1 Introduction

Classical electrodynamics of point charges (CE) is neither a valid, an invalid or an approximate theory. It is a non-theory. The problem is that the electromagnetic (EM) potential is non differentiable exactly where one needs such differentials—on the world lines of particles—and the electromagnetic energy density is non integrable there.

The above pathologies notwithstanding, CE proves to be an immensely practical tool by employing a variety of ad hoc ‘cheats’, applicable in limited domains. The validity domain of each ‘cheating method’—and cheating is absolutely necessary for an ill defined mathematical apparatus to produce definite results—is defined solely by the experimental success of the method. It is not a sub-domain of the global non-theory in which the latter becomes well defined, nor is it the domain of an approximate theory; CE has no predictions—hence no approximations either—in any domain.

Nevertheless, the results of all those cheating methods can collectively be derived from a concise set of assumptions, dubbed henceforth the basic tenets of CE, which are: Maxwell’s equations, plus local conservation of a symmetric energy-momentum (e-m) tensor, the EM part of which is the canonical tensor. That is, for all practical purposes, CE is some well defined realization of the basic tenets by means of small charged bodies.

Acknowleding the pivotal role played by the basic tenets, leads to two approaches to the self-force problem: Explicit and implicit. In the former, explicit equations governing the ‘matter part’ of the e-m tensor are postulated (the counterpart of Maxwell’s equations, from which the EM part is constructed). We shall review some intuitive such attempts, showing through their failure to satisfy the basic tenets, the non triviality of implementing the explicit approach. This, however, does not imply that the task is necessarily all that difficult. One
only needs to covariantly define some phenomenological, short range attractive force which
counters the Coulomb repulsion, thereby allowing for the formation of a stable charged body
(nonetheless, the author knows of no such example, which is clearly much more challenging
than in the similar, gravitational case, where gravity naturally supplies the ‘glue’ holding
the body together).

An explicit phenomenological construction, by definition involving statements about the
properties of charged matter, goes beyond what is normally considered to be the domain of
CE, and can be challenged in a variety of ways. For example, to avoid using the ill defined
concept of a point charge, a continuous charge distribution must be used which, in turn,
represents a local average of elementary, physical charges. Can a phenomenological con-
struction be considered consistent without proving its compatibility with the fundamental,
extPLICIT association with the constituents of the body? Or perhaps, the latter
belongs to the realm of quantum physics? But then, what is the compatibility criterion given
the shaky conceptual foundations of QM? Such questions are usually dismissed as being “ex-
cessively philosophical” in the case of macroscopic bodies but we argue to the contrary. The
effect of radiative corrections to Lorentz trajectories, predicated by explicit methods, is so
minute for macroscopic bodies, that it has never been directly observed in any experiment.
Since countless assumptions are involved in any phenomenological description, both explicit
and implicit, their consistency—mutual and with nature—cannot yet be trusted; It is one of
those cases in which the previously mentioned “cheating methods” is pending experimental
confirmation. And as for microscopic bodies, from electrons to large ions and molecules, the
predictions of all explicit models are plain wrong; Local self-force corrections to scattering
cross sections, for example, do not reproduce the experimental result predicted by QM.

It is rather obvious that no mortal can tailor an explicit construction to the huge body
of knowledge regarding the nature of matter. If one exists, it must emerge from basic
principles. One such construction, dubbed extended charge dynamics (ECD) [4], which is
greatly elaborated upon in the current paper, is based on the principle of scale covariance,
a symmetry of CE which we consider to be just as important as its Poincaré covariance.
Although not (yet) proven to be the unique such construction, it is extremely difficult to
merge scale covariance with the basic tenets. It turns out, then, that in order to merely
satisfy the basic tenets in a scale covariant way, charged (classical) particles must probably
be much richer objects than previously expected, and the universe containing them—a much
more bizarre place. So rich and bizarre, that quantum mechanics (QM) becomes just a
natural statistical description of such a classical ontology [11]. There are also indications
[12] that, ECD alone, could be the ontology underlying all forms of known matter and that
visible matter, thus represented, suffices in explaining the outstanding observations currently
requiring for this task the contrived notions of dark-matter, dark-energy and inflation [12].
Most importantly in the context of the self-force problem, ECD conforms with the apparent
indeterminism and non locality observed in scattering experiments. Small deviations from
classical cross sections (even with big charged molecules), described by QM, are, according
to ECD, just small, radiative corrections to the Lorentz force equation. It then becomes
clear why these corrections disappear for massive bodies and why they don’t for uncharged
particles; Those could still have a radiating dipole moment.

The implicit approach to the self-force problem, aims at extracting directly from the basic tenets, irrespective of their realization, some universal behaviour on the part of accelerating charged bodies. A simple known accomplishment of this approach is that, the Lorentz force equation must be satisfied by any, sufficiently massive body, in an external potential which is slowly varying on the scale set by the body’s size. However, the implicit approach yields radiative corrections to the Lorentz force equation, having a universality domain which already depends on the details of the realization. The fact that those corrections have never been demonstrated in an experiment, indicates one of the following: Either additional assumptions entering the analysis, beside the basic tenets, are wrong, or else that the universality domain of radiative corrections has hitherto not been reached in an experiment. In the latter case, their physical relevance should be seriously questioned in light of the huge experimental body of knowledge currently available.

The failure of the implicit approach in describing universal radiative corrections to the Lorentz force equation, should be contrasted with the result of [11]. Starting with the basic tenets (not necessarily in their ECD realization), a statistical description is derived for an ensemble of bodies, each satisfying them. QM wave equations then emerge as the simplest such description. ECD, then, explains the apparent randomness in QM experiments; The universality of the statistical description appears as a direct consequence of the basic tenets.

The analysis in [11] is only approximate, valid for sufficiently small accelerations, but this domain is contained in the validity domain of all proposed solutions to the self-force problem, whether implicit or explicit. QED radiative corrections, in this regard, only slightly modify a result which, according to ECD, already incorporates radiative corrections in an essential way.

2 Manifestly scale covariant classical electrodynamics

The following is a brief review of classical electrodynamics of interacting point charges. It is equivalent to the presentation appearing in any standard book on the matter, but contains a few novel twists.

Classical electrodynamics of $N$ interacting charges in Minkowski’s space $M$ is given by the set of world-lines $k_{\gamma_k} \equiv k_{\gamma_k}(s): \mathbb{R} \mapsto M$, $k = 1 \ldots N$, parametrized by the Lorentz scalar $s$, and by an EM potential $A$ for which the following action is extremal

$$I[\{\gamma\}, A] = \int d^4x \left\{ \frac{1}{4} F^2 + \sum_{k=1}^{N} \int ds \left( \frac{1}{2} k_{\dot{\gamma_k}}^2 + qA \cdot k_{\dot{\gamma_k}} \right) \delta^{(4)}(x - k_{\gamma_k}) \right\}. \quad (1)$$

Above, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the antisymmetric Faraday tensor, $q$ some coupling constant, and $F^2 \equiv F_{\mu\nu} F^{\mu\nu}$.

1Expressing the ensemble density, $\rho_{\text{ens}}$, appearing in [11] as $\rho_{\text{ens}} = \phi^* \phi$, Schrödinger’s equation appears as the simplest equation for $\phi$, consistent with the equations derived $\rho_{\text{ens}}$. 

3
Variation of (1) with respect to any $\gamma$ yields the Lorentz force equation, governing the motion of a charge in a fixed EM field

$$\ddot{\gamma}^\mu = q F^\mu_{\nu} \dot{\gamma}^\nu. \quad (2)$$

Multiplying both sides of (2) by $\dot{\gamma}_\mu$ and using the antisymmetry of $F$, we get that $\frac{d}{ds} \dot{\gamma}^2 = 0$, hence $\dot{\gamma}^2$ is conserved by the $s$-evolution. This is a direct consequence of the $s$-independence of the Lorentz force, and can also be expressed as the conservation of a ‘mass-squared current’

$$b(x) = \int_{-\infty}^{\infty} ds \delta^{(4)}(x - \gamma_s) \dot{\gamma}^2_s \dot{\gamma}_s. \quad (3)$$

Defining $m = \sqrt{\dot{\gamma}^2} \equiv \frac{dr}{ds}$ with $\tau = \int^s \sqrt{(d\gamma)^2}$ the proper-time, equation (2) takes the familiar form

$$m \ddot{x}^\mu = q F^\mu_{\nu} \dot{x}^\nu, \quad (4)$$

with $x(\tau) = \gamma(s(\tau))$ above standing for the same world-line parametrized by proper-time. We see that the (conserved) effective mass $m$ emerges as a constant of motion associated with a particular solution rather than entering the equations as a fixed parameter. Equation (2), however, is more general than (4), and supports solutions conserving a negative $\dot{\gamma}^2$ (tachyons — irrespective of their questionable reality) as well as a vanishing $\dot{\gamma}^2$.

The second ingredient of classical electrodynamics, obtained by variation of (1) with respect to $A$, is Maxwell’s inhomogeneous equations, prescribing an EM potential given the world-lines of all charges

$$\partial_{\nu} F^{\nu\mu} \equiv \partial^2 A^\mu - \partial^\mu (\partial \cdot A) = \sum_{k=1}^N k_j^\mu, \quad (5)$$

with

$$k_j(x) = q \int_{-\infty}^{\infty} ds \delta^{(4)}(x - k_s \gamma_s) \dot{\gamma}_s. \quad (6)$$

the electric current associated with charge $k$. Applying $\partial^\mu$ to (5) and using the antisymmetry of $F^{\nu\mu}$, implies the conservation of the combined current on the r.h.s. of the equation. This can be explicitly verified for the individual currents $k_j$,

$$\partial_{\mu} j^\mu = q \int_{-\infty}^{\infty} ds \partial_{\mu} \delta^{(4)}(x - \gamma_s) \dot{\gamma}^\mu_s = -q \int_{-\infty}^{\infty} ds \partial_{\gamma} \delta^{(4)}(x - \gamma_s) = 0. \quad (7)$$

The current on the r.h.s. of (5) obviously defines $F$ only up to a solution to the homogeneous Maxwell’s equation $\partial_{\nu} F^{\nu\mu} = 0$.

### 2.1 Scale covariance

The above unorthodox formulation of classical electrodynamics highlights its scale covariance, a much ignored symmetry of CE which, nevertheless, is just as appealing a symmetry
as translational covariance (Poincaré covariance in general). Any privileged scale appearing in the description of nature, just like any privileged position, should better be an attribute of a specific solution and not of the equations themselves which ought to support all properly scaled versions of a solution. As there seems to be some confusion regarding scale covariance, we try to clarify its exact meaning next.

The Poincaré group plays a fundamental role in any theory, whether covariant or not. In particular, this means that

**a.** If some coordinate system is suitable for describing the theory then so is any other system related to the first by a Poincaré transformation.

**b.** Under the above change in coordinate systems, the parameters of the theory must transform under some representation of the Poincaré group. Poincaré covariant theories are those distinguished theories containing only Poincaré invariant parameters.

**c.** The physical content of the theory is identified with invariants of the Poincaré group, viz., attributes transforming under its trivial representation which are therefore independent of the coordinate system.

Elevating the one-parameter group of scale transformations to the status of the Poincaré group amounts to extending the latter with a dilation operation, \( x \mapsto \lambda x \) for any \( \lambda > 0 \). By **b** above, we should also assign a scaling dimension, \( D_\Omega \), to each object, \( \Omega \), dictating the latter’s transformation under scaling of space-time, \( \Omega \mapsto \lambda^{-D_\Omega} \Omega \), and by **c**, only dimensionless quantities have physical meanings (The custom of attaching ‘dimensional units’ to measurable quantities, such as a kilo or a meter, guarantees that in addition to the scale dependent measurement, another scale dependent gauge is specified, yielding a scale independent ratio). Note, however, that the assignment of scaling dimensions to objects of a theory is not unique unless the theory is scale covariant, viz., contains parameters of scaling dimension zero only (even in this latter case one can distinguish between theories leaving an action invariant thereby facilitating the derivation of a conserved current associated with scaling symmetry, and those theories only preserving the equations. CE falls into the first category).

Back to the case of classical electrodynamics, we can see that the scaled variables

\[
A'(x) = \lambda^{-1}A(\lambda^{-1}x), \quad \gamma'(s) = \lambda\gamma(\lambda^{-2}s),
\]

also solve (2) and (5), without scaling of \( q \), hence CE is scale covariant. From (8) one can also read the following scaling dimensions: \([x] = [\gamma] = 1; [s] = 2; [A] = [m] = -1; [j] = -3\), and by virtue of scale covariance \([q] = 0\). Poincaré symmetry combined with (8), forms the symmetry group of CE.

The simplicity in which scale covariance emerges in classical electrodynamics is due to the representation of a charge by a mathematical point, obviously invariant under scaling of space-time. As we shall see, achieving scale covariance with extended charges is a lot more difficult, as no dimensionful parameter may be introduced into the theory from which the charge can inherit its typical scale.
2.2 The basic tenets of CE

Associated with each charge is a ‘matter’ energy-momentum (e-m) tensor,

\[ m_{\nu\mu} = \int_{-\infty}^{\infty} ds \dot{\gamma}^\nu \dot{\gamma}^\mu \delta^{(4)}(x - \gamma_s), \]  

formally satisfying

\[ \partial_\nu k_{\nu\mu} = F^{\mu\nu} j_\nu, \]  

\[ \partial_\nu m_{\nu\mu} = \int ds \dot{\gamma}^\nu \dot{\gamma}^\mu \partial_\nu \delta^{(4)}(x - \gamma_s) = -\int ds \dot{\gamma}^\mu \partial_s \delta^{(4)}(x - \gamma_s) \]

\[ = \int ds \dot{\gamma}^\mu \delta^{(4)}(x - \gamma_s) = \int ds qF^{\mu\nu} \dot{\gamma}^\nu \delta^{(4)}(x - \gamma_s) = F^{\mu\nu} j_\nu. \]

Likewise, associated with the EM potential is a unique gauge invariant and symmetric EM e-m tensor

\[ \Theta^{\nu\mu} = \frac{1}{4} g^{\nu\mu} F^2 + F^{\nu\rho} F_{\rho}^{\mu} \]  

formally satisfying Poynting’s theorem

\[ \partial_\nu \Theta^{\nu\mu} = -F^{\mu\nu} \sum_k k_{\nu}\,, \]  

where only use of (5) and the identity

\[ \partial^\mu F^{\nu\rho} + \partial^\rho F^{\mu\nu} + \partial^\nu F^{\mu\rho} = 0 \]  

has been made in establishing (12). Summing (10) over k and adding to (12) we get a symmetric conserved e-m tensor of the combined matter-radiation system,

\[ p^{\nu\mu} := \Theta^{\nu\mu} + \sum_k k_{\nu\mu} \Rightarrow \partial_\nu p^{\nu\mu} = 0, \]  

the conservation of which can also be established form the invariance of the action (1) under translations. Note that the obvious coupling between matter and radiation notwithstanding, the conserved e-m tensor in (14) splits into two pure contributions. Conservation of a generalized angular momentum tensor follows straightforwardly from the symmetry of p and (14).

Finally, for future reference, we note that associated with the scaling symmetry (8) is an interesting conserved ‘dilatation current’

\[ \xi^\nu = p^{\nu\mu} \dot{x}_\mu - \sum_{k=1}^{n} \int ds \delta^{(4)}(x - k_\gamma_s) s \dot{k}_{\gamma_s}^2 k_{\gamma_s}^\nu. \]  

\[ ^2\text{The symmetry of the e-m tensor is mandatory if it to be a } g_{\mu\nu} \rightarrow \eta_{\mu\nu} \text{ limit of its general relativistic version as there, symmetry follows from its definition. See [12]} \]
However, the conserved dilatation charge, $\int d^3x \xi^0$, depends on the choice of origin for both space-time, and the $n$ parameterizations of $k_\gamma$, and is therefore difficult to interpret.

Equation (10) and Maxwell’s equations (5), together with electric charge conservation of individual currents $k_j$, are dubbed in this paper the basic tenets of CE, and henceforth shall assume a status of axioms rather than derived relations. The rational behind such a step lies in the fact that, the infinitely detailed dynamics of point charges, or the singular EM field generated by them, are never the actual subject of observation in experiments to which CE is successfully applied, but rather the basic tenets in their integral forms. For example, the thin tracks left by charges in particle detectors, accurately described by the Lorentz force equation, are consistent with a hypothetical pair \{j, m\}, localized about a common world line, satisfying the basic tenets (10) and (7) (see appendix D). Likewise, the phenomenon of radiation resistance, whether in wires or particle accelerators, is a demonstration of Poynting’s theorem (12) and e-m conservation (14), and not of a specific, damping self-force resisting the motion of the charges.

As yet another example, consider the EM energy stored in a capacitor or a solenoid. Its subsequent conversion to mechanical energy (heat) in a resistor is a demonstration of (14). Similarly, the familiar $r^{-1}$ dependence of the Coulomb potential between two charged bodies, is hidden in an integral of the $E \cdot E$ part of $\Theta^{00}$ over the entire space, when one assumes that the self-energy of each charge is finite. Likewise, the potential energy of an infinitesimal circulating current (a magnetic dipole) in an external field, is just due to the $B \cdot B$ term.

The basic tenets are not only verified by any experiment—including QM ones—but moreover, it seems impossible for any theory not satisfying the basic tenets to be consistent with the full range of experiments associated even with CE (let alone QM), a small sample of which was described above.

2.3 The classical self-force problem

The self-force problem of CE refers to the fact that the EM potential, $A$, generated by (5) is non differentiable everywhere on the world line $\bar{\gamma} \equiv \bigcup_s \gamma_s$, traced by $\gamma$, rendering ill defined the Lorentz force—the r.h.s. of (2)—as well as the r.h.s. of the constitutive relation (10) (even in the distributional sense). A reminder of this appears in the form of non integrable singularities on the $\gamma$’s of the EM energy density $\Theta^{00}$, making the energy of a system of particles likewise ill defined.

Fixing the self-force problem amounts to turning a non-theory into a (mathematically well defined) theory and there is no obvious ‘right way’ of doing so. The simplest way, which often leads to good agreement with experiment, is to eliminate the self generated field from $F$ when computing the Lorenz force acting on a particle. For this to be possible one needs to be able to uniquely define the contribution of each charge to the total field $F$, and the prevailing method is to take the retarded Lienard-Wiechert potential of the charge

$$A_{\text{ret}}(x) = q \int ds \delta \left[ (x - \gamma_s)^2 \right] \gamma_s \theta \left( x^0 - \gamma^0_s \right),$$

as that field. The r.h.s. of (2) is rendered well defined this way, but the basic tenets no
longer hold true even in a formal way, their validity follows from the existence of an action, (1), not discriminating between the contributions of different charges to \( F \).

In his celebrated work on the self force problem, [1], Dirac attempts to salvage the basic tenets by retaining the self-retarded potential, writing it as

\[
A_{\text{ret}} = \frac{1}{2} (A_{\text{ret}} + A_{\text{adv}}) + \frac{1}{2} (A_{\text{ret}} - A_{\text{adv}}),
\]

with the advanced Lienard-Wiechert potential

\[
A_{\text{adv}}(x) = q \int ds \delta \left[ \left( x - \gamma s \right)^2 \right] \dot{\gamma}_s \theta \left( x^0 + \gamma^0 s \right),
\]

and, de facto, ignoring the ill-defined Lorentz force derived from the first term in (17). The well defined force derived from the second term modifies the Lorentz force equation into the third order Abraham-Lorentz-Dirac (ALD) equation

\[
\ddot{\gamma} = qF\dot{\gamma} + q^2 \frac{2}{3} (\gamma^2 - \gamma^2 \dot{\gamma}),
\]

which is not a formal Euler-Lagrange equation of the action (1) nor of any known alternative action. Consequently, in the general case of a set of point charges interacting according to Dirac, it is not even known if any expression (let alone (14)) exists which can be interpreted as e-m conservation (see appendix D for a detailed account of Dirac’s equation and related implicit solutions to the self-force problem).

In another classic work [9][10], Wheeler and Feynman gave a surprising new look at Dirac’s electrodynamics. Elaborating the formalism of action-at-a-distance electrodynamics, they found a locally conserved and integrable e-m tensor for a set of point charges interacting through their half advanced plus half retarded Lienard-Wiechert potentials, without self interaction. Under certain assumptions, a subset of charges surrounded by sufficiently many other charges, behaves in accordance with Dirac’s theory. Nevertheless, the form of that integrable EM tensor is radically different from (11), admitting both negative values for its energy density component as well as nonzero values at places where the EM field due to all charges vanishes (implying, among else, gravitational curvature in a generally covariant extension). In fact, the very notion of localization of EM e-m is absent from Wheeler and Feynman’s theory, making it impossible to apply e-m conservation to isolated subsystems—probably the most well tested prediction of CE. Their proposal, therefore, can hardly be claimed to be consistent with the full range of experiments to which CE is successfully applied. Instead, it is some well defined theory of interacting point charges sharing with CE a common symmetry group and admitting an integrable and conserved e-m tensor—but it is not CE.

2.3.1 Extended currents

Insisting on retaining both the form (11) of the canonical EM tensor and a point charge, inevitably leads to a non-integrable energy density and consequently to violation of the basic
tenets. In a second, explicit class of attempts to solve the self-force problem, one therefore substitutes for the distributions (6) and (9) regular currents, both localized about \( \bar{\gamma} \). The regularity of the electric current implies a smooth potential on \( \bar{\gamma} \), rendering the Lorentz force (2) well defined and the canonical EM tensor—integrable. Various proposals can be found in the literature, all utilizing a ‘rigid construction’ in the sense that the extended currents are uniquely determined by \( \gamma \). This is not only the simplest way to eliminate the singularity of \( A \) on \( \bar{\gamma} \) but also the only one allowing to retain the Lorentz force equation (2). Below, we shall employ a rigid construction which is equivalent to the one employed in [5] but via a different method which will serve us later.

The idea is to substitute for \( \delta^{(4)} \) in (6) a finite approximation of a delta function, respecting the symmetries of the theory. In Euclidean four dimensional space this is straightforward: \( \delta^{(4)}(x) \mapsto a^{-4}h(x/a) \) for any normalized spherically symmetric \( h \) and some small \( a \). In Minkowski’s space this is more tricky due to the non-compactness of Lorentz invariant manifolds \( x^2 = \text{const} \), so first we note that the current

\[
\int ds \frac{1}{\epsilon} h \left[ \frac{(x - \gamma s)^2}{\epsilon} \right] \dot{\gamma}_s, \tag{19}
\]

is conserved and significantly differs from the \( \epsilon \)-independent current

\[
\int ds \delta \left[ (x - \gamma s)^2 \right] \dot{\gamma}_s, \tag{20}
\]

only up to a distance from \( \gamma_s \) on the order of \( \sqrt{\epsilon} \) (in the rest frame of \( \gamma_s \)). Taking the derivative of (19) with respect to \( \epsilon \) we therefore get a conserved current

\[
j(x) = q \frac{\partial}{\partial \epsilon} \int ds \frac{1}{\epsilon} h \left[ \frac{(x - \gamma s)^2}{\epsilon} \right] \dot{\gamma}_s, \tag{21}
\]

which is significant only inside a ball of radius \( \sim \sqrt{\epsilon} \) in the rest frame of \( \gamma \), reducing to the line current (6) in the limit \( \epsilon \to 0 \) for a properly normalized \( h \). Pushing the derivative into the integral, the regular function

\[
\frac{\partial}{\partial \epsilon} \frac{1}{\epsilon} h \left( \frac{x^2}{\epsilon} \right), \tag{22}
\]

appears (up to a normalization constant) as a finite approximation to the invariant \( \delta^{(4)}(x) \) entering (6). This can indeed be directly verified. Note, however, that even for a compactly supported \( h \), (22) is non vanishing in some neighborhood of the light-cone \( x^2 = 0 \) for an arbitrarily large (light like) \( x \).\(^{3}\) Consequently, the current (21) is never compactly supported and can be shown to have an (integrable) algebraically decaying ‘halo’. We see that the obvious way of covariantly generalizing Lorentz’s construction of a finite-size electron, leads to weakly localized currents.

\(^{3}\)The ‘pickup’ property of (22) is achieved by means of its rapid oscillation across the light cone, i.e., near large light-like \( x \), (22) takes both positive and negative values.
Choosing the retarded solution of Maxwell’s equations with the current (21) as source, one arrives at the following expansion for the self-force correction to the Lorentz force at $\gamma_s$, when $s$ is chosen as the proper time:

$$-\frac{q^2C}{\sqrt{\epsilon}} \ddot{\gamma} + f_{\text{ALD}} + q^2O(\sqrt{\epsilon}).$$

(23)

Above, $C$ is some positive constant depending only on $h$, the $O(\sqrt{\epsilon})$ term also depends on the local form of $\gamma$, and $f_{\text{ALD}}$ stands for the standard Abraham-Lorentz-Dirac self-force, $q^2\frac{2}{3}(\ddot{\gamma} - \dot{\gamma}^2\dot{\gamma})$.

There are three major difficulties with the above extended current approach to the self-force problem. First, it introduces an arbitrary function—an infinite set of parameters—into single-parameter CE. Second, the dimensionful parameter $\epsilon$ spoils the scale-covariance of CE. Finally, the basic tenets are still not satisfied, the problem being with (10). To show this, we plug a most general covariant ansatz

$$m^\mu^\nu(x) = M \frac{\partial}{\partial \epsilon} \int \frac{1}{\epsilon} h_1 \left[ \frac{(x - \gamma_\tau)^2}{\epsilon} \right] \dot{\gamma}^\mu \dot{\gamma}^\nu + g^\mu^\nu \frac{1}{\epsilon} h_2 \left[ \frac{(x - \gamma_\tau)^2}{\epsilon} \right] \dot{\gamma}^2 d\tau,$$

(24)

for some suitably normalized functions $h_1$, $h_2$, and notice that the value of the l.h.s. of (10) at any $x$ depends only on the value of the external field on $\dot{\gamma}$, whereas the r.h.s. depends also on its the local value at $x$. No choice of $h$’s can lead to point-wise equality in (10), the only way to do so is to take the limit $\epsilon \to 0$, restricting the support of both sides of (10) to $\dot{\gamma}$. The basic tenets are then, indeed, satisfied but only because the dynamics of the bodies trivialize to uniform motion due to their infinite mass (the first term in (23) which renormalizes the mass $M$).

A rather contrived domain in which the above extended body approach may still be valid is obtained by rescaling the charge and mass, $q \mapsto \sqrt{\epsilon}q$, $M \mapsto \sqrt{\epsilon}M$. Adding (23) with the scaled charge to a Lorenz force equation with scaled parameters, and dividing by $\sqrt{\epsilon}$, the zero order effect is a finite renormalization of the mass, $M \mapsto M + q^2C$, in the original Lorenz force equation. The ‘universality’ of the ALD self-force in this case, is equivalent to the statement that $\sqrt{\epsilon}f_{\text{ALD}}$ is the leading correction to the Lorentz force in an $\epsilon$ expansion (note the $\sqrt{\epsilon}$ multiplying the standard ALD force!).

An implicit approach, leading to the same result appears in [7] (see also appendix D for a lighter version with the same conclusion). Higher order corrections, however, cannot be computed using the implicit approach without introducing extra assumptions, beyond the commitment to retarded solutions. The $\epsilon \to 0$ limit is void of any physical meaning, as both the charge and the mass, though retaining a fixed ratio in the limit, vanish. Consequently, the dynamics of a group of such interacting charges must trivialize to uniform motion—as in our previous $\epsilon \to 0$ limit. If a physically realistic $\epsilon$ is used instead of the limit, and the effects

\[4\] There are, of course, other possibilities, such as $m^\mu^\nu = aj^\mu j^\nu + bg^\mu^\nu j^2$ for some constants $a$ and $b$. The static case of this ansatz was analyzed in [8] and was found to necessitate the same Coulomb singularity on $\dot{\gamma}$ as in the point charge case, in order for (10) to be satisfied.
of the ALD term (e.g., via the reduced order ALD equation; see [7]) are still undetectable in an experiment, this could only mean that, either the assumption of using only retarded solutions (which is at odd with ECD) is inconsistent with nature’s realization of the basic tenets, or that higher order corrections are not negligible, meaning that whatever extra assumptions were used to bound the higher order corrections, are invalid. The fact that no experiment has ever found such universal deviations from Lorentz trajectories, renders the physical relevance of the implicit approach moot.

Summarizing, CE of point charges cannot satisfy the basic tenets, if only because of the singularity in the canonical tensor, while CE of rigid extended charges, though eliminating the former problem, further spoils scale covariance and introduces infinitely many new parameters. The implicit approach, while rigorous, has no specific validity domain and is therefore experimentally moot. To these approaches to the self-force problem one may add nonlinear electrodynamics, notably the Born-Infeld version, in which the singularity in a (modified) canonical EM tensor is rendered integrable at the cost of modifying Maxwell’s equations and making them nonlinear. The potential $A$ is non-differentiable at the source hence the paths of charges are still ill-defined in those theories which further suffer from broken scale covariance, and manifestly violate the basic tenets in their experimentally established form. Finally, solitary solutions of non linear PDE’s, as possible solutions to the self-force problem, are ruled out below.

3 Extended Charge Dynamics

Our starting point in the construction of currents satisfying the basic tenets is the electric current (21) and expression (24) for the e-m tensor $m$. We saw above that the ‘rigidity’ of the covariant integrands in both currents leads to violation of the basic tenets, while their nonsingular nature further spoils scale covariance. To fix both problems we substitute for them more ‘vibrant’ integrands which do depend on the local field $F$, and whose characteristic scale surfaces naturally without introducing extra dimensionfull parameters. To this end, let us look at the proper-time Schrödinger equation (also known as a five dimensional Schrödinger equation, or Stueckelberg’s equation),

$$\left[i\hbar\partial_s - \mathcal{H}(x)\right]\phi(x, s) = 0, \quad \mathcal{H} = -\frac{1}{2}D^2 + V,$$

(25)

with

$$D_\mu = \bar{\hbar}\partial_\mu - iqA_\mu$$

(26)

the gauge covariant derivative, $A$ and $V$ some vector and scalar potentials respectively, $\bar{\hbar}$ a real dimensionless ‘quantum parameter’, not to be confused with $\hbar$, and $q$ some EM coupling constant. It can be shown by standard means that solutions of (25) satisfy a continuity equation

$$\partial_s \rho = \partial \cdot J, \quad \text{with} \quad J = q \text{Im } \phi^* D \phi, \quad \rho = q |\phi|^2,$$

(27)
and four relations

\[ q^{-1} \partial_s J^\mu = F^{\mu\nu} J_\nu + q^{-1} \partial^\mu V \rho - \partial_{\nu} M^{\mu\nu}, \]

with \( M^{\mu\nu} = g^{\mu\nu} \left( \frac{i\hbar}{2} (\phi^* \partial_s \phi - \partial_s \phi^* \phi) - \frac{1}{2} (D^\lambda \phi)^* D_{\lambda} \phi \right) + \frac{1}{2} (D^\nu \phi (D^\mu \phi)^* + c.c.) \).

The common implications of the non relativistic counterparts of (27) and (28) are probability conservation and Ehrenfest’s theorem, and readily carry to the relativistic case. Multiplying (27) by \( q^{-1} x \) and integrating its r.h.s. by parts over four-space, we get (for a normalized \( \phi \), but this is immaterial to the result we wish to establish) the four momentum of a wave packet. Integrating (28) over four-space, and substituting the above momentum for the l.h.s., localized wave-packets can then be shown to trace classical paths when the EM field varies slowly over their extent and the scalar potential \( V \) vanishes.

Yet, another implication of (27) and (28) which has no direct nonrelativistic counterpart is obtained by integrating the two equations over \( s \) rather than space-time. The \( s \)-independent current

\[ j(x) = \int_{-\infty}^{\infty} ds J(x, s) \]

is conserved and, for a vanishing \( V \), the constitutive relation (10) is satisfied by \( j \) and

\[ m(x) = \int_{-\infty}^{\infty} ds M(x, s). \]

Associating a unique \( \phi \) with each particle and taking the sum of the corresponding currents, \( j \), as the source of Maxwell’s equations (5), the basic tenets are fully satisfied, and the full symmetry group—scale covariance in particular—is retained.

The above realization of the basic tenets, nevertheless, is apparently inconsistent with the condition of localized \( j \) and \( m \). The dispersion inherent in the Schrödinger evolution (25) implies that a localized wave-packet gradually spreads even in a potential free space-time. In collisions with an external potential the situation is even worse, and may result in a rapid loss of localization. This means that the wave-packet could maintain its localization under the \( s \)-evolution (25) only if somehow the EM potential generated by its associated current \( j \), creates a binding trap, but the prospects of such a solution are dim as the self generated Coulomb potential is repulsive rather than attractive. It is further unlikely that such a self-trapping solution, even if it exists in some otherwise potential free region of space-time, would retain its localization following violent (realistic) interactions with EM potential generated by other charges. Finally, it can be shown that equation (25) and its associated currents admit a much more natural interpretation in terms of an ensemble of particles, making the single particle interpretation seem rather contrived.

It appears inevitable that for (25) to be useful in the realization of the basic tenets by means of localized currents, an additional localization mechanism for the wave packet must be introduced into the formalism. In [4], this mechanism takes the form of a (point) ‘delta function potential’, \( V = \delta^{(4)}(x - \gamma_s) \), moving along some \( \gamma \) in Minkowski’s space, which is
plugged into the Hamiltonian in (25), preventing the wave function from spreading by the binding action of the potential. Note that essentially any other choice of binding potential would lead to violation of scale covariance.

3.1 The central ECD system

In order for the previous results established for the case $V \equiv 0$, viz., Ehrenfest’s theorem and the basic tenets, to still hold true for $V = \delta^{(4)}(x - \gamma s)$, we need to have $\partial \delta^{(4)}(x - \gamma s) \rho = 0$. As both Ehrenfest’s theorem and the basic tenets in their integral form (the only form directly related to physical observables) involve an integral over four-space, our proviso is equivalent to $\partial \rho (\gamma s, s) = 0$. Associated with each particle, then, is a pair $\{k \phi, k \gamma\}$, $k = 1 \ldots N$, performing a tightly coordinated ‘dance’: $\gamma$ points to $\phi$ where to focus, but simultaneously follows the extremum of its modulus squared. In mathematical terms this dance takes the form of two coupled equations dubbed the central ECD system. The first is an integral version of (25) containing a delta function potential (see [4] for a formal derivation) and reads (omitting the particle index on $\phi$ and $\gamma$)

$$\phi(x, s) = -2\pi^2 \hbar^2 \epsilon i \int_{-\infty}^{s-\epsilon} ds' G(x, \gamma s'; s - s') \phi(\gamma s', s')$$

$$+ 2\pi^2 \hbar^2 \epsilon i \int_{s+\epsilon}^{\infty} ds' G(x, \gamma s'; s - s') \phi(\gamma s', s')$$

$$\equiv -2\pi^2 \hbar^2 \epsilon i \int_{-\infty}^{\infty} ds' G(x, \gamma s'; s - s') \phi(\gamma s', s') \mathcal{U}(\epsilon; s - s'),$$

with $\mathcal{U}(\epsilon; \sigma) = \theta(\sigma - \epsilon) - \theta(-\sigma - \epsilon)$,

and the second equation is naturally

$$\partial_x |\phi(x, s)|^2 \bigg|_{x=\gamma s} \equiv \partial_x |\phi(\gamma s, s)|^2 = 0. \quad (32)$$

Above, $G(x, x'; s)$ is the propagator of a proper-time Schrödinger equation, viz., solution of (25) satisfying the initial condition (in the distributional sense),

$$G(x, x'; s) \underset{s \to 0}{\longrightarrow} \delta^{(4)}(x - x'). \quad (33)$$

The extra parameter, $\epsilon$, of dimension 2, which is needed for the construction of the scale-invariant delta function potential, is ultimately taken to zero and is discussed below.

A delta function potential, inserted into a differential equation, cannot possibly go as smoothly as presented hitherto. Indeed, the central ECD system, first derived in [4], is just a background intuition for turning that formal potential into a well defined mathematical object and it turns out that the best way to do so is to look at the basic tenets. As $\phi(x, s)$ solving Schrödinger’s equation (25) in the presence of a point potential $\delta^{(4)}(x - \gamma s)$, is formally a solution of the free Schrödinger equation (25) for any $(x, s) \neq (\gamma s, s)$, it appears that the basic tenets would be respected by $j$ (29), and $m$ (30), for an arbitrary $\gamma$ and $x \notin \bar{\gamma}$. The
only way, therefore, for the basic tenets to be sensitive to the choice of $\gamma$ (the path taken by the ‘center’ of the particle!) is if the exclusion of $\bar{\gamma}$ from their domain somehow affects their validity. And indeed, as shown in the appendix, a ‘wrong’ choice of $\gamma$ leads to ‘leakage’ of mechanical e-m associated with $m$, and to electric charge associated with $j$, to ‘world-sinks’ on $\bar{\gamma}$, rendering the (local) basic tenets useless by preventing their conversion into integral conservation laws via Stoke’s theorem. The derivation of the ‘fine tuned’ central ECD system, shown in the appendix, is guided by this no-leakage criteria.

Returning to the extra parameter, $\epsilon$, appearing in (31), in the appendix we show that the $\epsilon \to 0$ limit of a family of solutions to the $\epsilon$-dependent central ECD system, indeed exists, but not without the usual toll paid for manipulating formal mathematical objects. For it turns out, that solutions of the central ECD system develop a distribution on the light cone of $\gamma_s$ in the limit $\epsilon \to 0$. As both $J$ and $M$—the integrands of $j$ (29), and $m$ (30), respectively—are bilinears in $\phi$ and its adjoint, a meaningless product of two distributions is formed as a result of taking the $\epsilon \to 0$ limit of $\phi$ and only then plugging it into $J$ and $M$. A similar product of distributions is the source of much of the troubles in QFT and is overcome by two steps: covariant regularization of the distributions, followed by ‘renormalization’, viz., making sense of possible infinities arising from the removal of the regulator. Likewise, in ECD a covariant regulator, $\epsilon$, is built into the formalism, and the counterpart of the renormalization step takes the form of a simple covariant prescription

$$j \mapsto \lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon} \epsilon^{-1} j, \quad x \notin \bar{\gamma}$$

$$m \mapsto \lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon} \epsilon^{-1} m, \quad x \notin \bar{\gamma},$$

namely, plug a finite-$\epsilon$ $\phi$ into $j$ and $m$, apply a certain ‘infinity removal’ operation: $\partial_\epsilon \epsilon^{-1}$ (see appendix A for details), taking the $\epsilon \to 0$ limit only as a final step. This trick yields a locally conserved smooth electric current which does not leak to a world-sink on $\bar{\gamma}$ and is integrable. Likewise, the e-m tensor of a system of interacting particles, $\Theta + \sum_k k m$, is locally conserved, non leaking and integrable. It follows that Stoke’s theorem can be freely applied to the local basic tenets, as if their domain included all of space-time.

**Spin.** A lot has been said about non-integer spin being one of the hallmarks of QM, but while the above procedure of realizing the basic tenets involves scalar $k \phi$’s, a similar method exists in which each $k \phi$ transforms under an arbitrary representation of the Lorentz group. The spin of a particle is a nonphysical ‘label’ of the particular method used to construct $j$ and $m$, both transforming under integer representations of the Lorentz group, whose internal such currents depend on that method. An example of spin-$\frac{1}{2}$ ECD is discussed in appendix E.

### 3.2 The nature of particles in ECD

The simplest possible problem in ECD is that of single a stationary particle in an otherwise void universe. That is, the very existence of a particle is due to a nontrivial localized solution,
viz. $A \neq 0$ up to a gauge transformation, for the coupled ECD-Maxwell system. Using a small-$\hbar$ approximation of the propagator, we show in appendix B that such solutions must indeed be particle-like, represented by integrable currents which are localized about their center $\gamma$, and this conclusion is not an artifact of the small-$\hbar$ analysis but rather a direct consequence of equation (31).

In a naive approach, finding a particle solution in ECD amounts to guessing a potential $A$, then solving the central ECD system (31),(32) for a pair, $\{\phi, \gamma\}$, from which the electric current (34) is computed, and ‘hoping’ that this current, along with the initial guess, $A$, indeed solves Maxwell’s equation (5).

By the scale covariance of ECD, to each such isolated solution there corresponds an infinite family of scaled versions, sharing the same electric charge and spin but differing on their self energy which has dimension $-1$. It is hypothesized that different elementary particles are just scaled versions of each other, hence their common charge. A possible explanation for the observed ‘spontaneous scaling symmetry breaking’, viz., the absence of an observed continuum of masses, and its dramatic implications to cosmology, appear in [12].

An elementary particle solution, or any other solution for that matter, must come with an ‘antiparticle’ solution to the ECD equations. This is a consequence of the symmetry of ECD under a ‘CPT’ transformation:

$$
A(x) \mapsto -A(-x), \quad \gamma(s) \mapsto -\gamma(s) \quad \phi(x, s) \mapsto \phi(-x, s)
$$

$$
\Rightarrow j(x) \mapsto -j(-x), \quad m(x) \mapsto m(-x).
$$

3.2.1 Comparison with solitons

The possibility of representing elementary particles by solitary solutions of nonlinear PDE’s has been extensively studied in the past. Coupled Maxwell–Dirac [6] or Maxwell–Klein-Gordon [3] systems (some also add a nonlinearity to either the KG or Dirac equations, whose purpose is not entirely clear as the original system is highly nonlinear to begin with) can even be shown to possess spherically symmetric localized solutions which satisfy the basic tenets, although the mass term spoils scale-covariance. Classically speaking, a self-trapping ‘charged dust cloud’ is obviously impossible due to the repulsive Coulomb self-force (one cannot add a non-electromagnetic force countering this repulsion without violating the basic tenets and further breaching scale-covariance) and it turns out that this intuition also applies to such coupled systems. To counter the Coulomb self-repulsion, one must therefore add a point-charge with an opposite sign at the center of the soliton (or, equivalently, impose boundary conditions at the origin, forcing the radial electric field there to diverges as $r^{-2}$). This means that the electric field at the origin behaves as baldly as in the point charge case, rendering the self-energy ill-defined. Moreover, the monopole of that solution can be shown to vanish so charged particles cannot be represented by such solutions anyhow (see [6] and references therein). A similar singularity is also found in [8], where the ansatz $m^{\mu \nu} = aj^\mu j^\nu + bg^{\mu \nu} j^2$ is plugged into (10)(5) and solved for a spherically symmetric $j$. It should be noted in this regard that ECD charged particles likewise have a divergent electric field on $\bar{\gamma}$, but this field
leads to an integrable self-energy. Moreover, the dynamics of ECD particles is based on the condition that no electric charge nor e-m leakage occurs at those singularities—an analysis which is missing altogether from [6] and related work.

An apparent way out of this dead end is to add gravity, leading to a coupled Einstein–Maxwell–Dirac system, satisfying the generally covariant basic tenets. Given the relative weakness of gravitational attraction compared with electrostatic repulsion, it is rather surprising that such localized, non-singular solutions actually exist [2]. Nevertheless, the stability of those (static) solutions is only demonstrated for a limited class of—small, by definition—perturbations. More generally, the very ambition of modelling particles—objects maintaining their localization and identity notwithstanding strong interaction with the environment—by means of fields, extending throughout space-time, without introducing a focusing/stabilising center, such as \( \gamma \) in ECD, seems like an entirely hopeless program—a point we have already made with regard to possible Maxwell–Stueckelberg solitons, discussed at the end of section 3.

### 3.3 The necessity for advanced solutions of Maxwell’s equations

In a universe in which no particles imply no EM field, a solution of Maxwell’s equations is uniquely determined by the conserved current, \( j \), due to all particles. The most general such dependence which is both Lorentz and gauge covariant takes the form

\[
A^\mu(x) = \int d^4x' \left[ \alpha_{\text{ret}}(x')K_{\text{ret}}{}^{\mu\nu}(x-x') + \alpha_{\text{adv}}(x')K_{\text{adv}}{}^{\mu\nu}(x-x') \right] j_\nu(x') ,
\]

for some (Lorentz invariant) space-time dependent functionals, \( \alpha \)'s, of the current \( j \), constrained by \( \alpha_{\text{ret}} + \alpha_{\text{adv}} \equiv 1 \), where \( K_{\text{ret}}^{\mu\nu} \) are the advanced and retarded Green’s function of (5), defined by

\[
(g_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu) K_{\text{ret}}{}^{\nu\lambda}(x) = g_{\mu\lambda} \delta^{(4)}(x) ,
\]

\[
K_{\text{ret}}(x) = 0 \quad \text{for} \quad x^0 \leq 0 .
\]

In ill defined CE of section 2, \( \alpha_{\text{adv}} \equiv 0 \) is taken as a definition. Modulo the self force problem, the fact that CE admits a formulation in terms of a Cauchy initial value problem (IVP) means that indeed, solutions of CE may be found containing only retarded fields. This proviso, however, is incompatible with the ECD equations, and \( \alpha_{\text{adv}} \) would generally differ from zero and vary across space-time. In particular, the fact that the ECD current also depends on \( A \), both explicitly through the gauge covariant derivative \( D \), and implicitly via \( \phi \)'s dependence on \( A \), means that the solution of even a single radiating ECD particle must include advanced components as these cannot be eliminated by the addition of a solution of the homogeneous Maxwell’s equations, as in CE. More generally, the values of both \( \alpha \)'s must be read from a global ECD solution, involving both fields and currents, rather than being imposed on it.

\footnote{More accurately, (38) and (39) do not uniquely define \( K \) but the remaining freedom can be shown to translate via (37) to a gauge transformation \( A \rightarrow A + \partial \Lambda \), consistent with the gauge covariance of ECD.}
That advanced solutions of Maxwell’s equations are on equal footing with retarded ones is outraging from the perspective of the (almost) consensual paradigm which accepts only retarded solutions as physically meaningful. One can think of two major reasons for this outrage. The first is the parallelism which which is often drawn with ‘contrived’ advanced solutions of other physical wave equations (e.g. surface waves in a pond converging on a point and ejecting a pebble). This parallelism, however, is a blatant repetition of the historical mistake which led to the invention of the aether. The formal mathematical similarity between the d’Alembertian—the only linear, Lorentz invariant second-order differential operator—and other (suitably scaled) wave operators, is no more than a misfortunate coincidence. Has this coincidence had some real substance to it, then application of the Lorentz transformation to the wave equation describing the propagation of sound, for example, would have yielded a meaningful result. It is quit remarkable that over a century after the existence of the aether was refuted, and the geometrization revolution of Minkowski in mind, terms such as ‘wave’ and ‘propagation’ are still as widely used in the context of electromagnetic phenomena as in the nineteenth century.

The second, stronger case for rejecting advanced solutions is observational. While at the microscopic scale we challenge this assertion in [11] and [12], indeed, no macroscopic object is observed anywhere spontaneously increasing its energy content by the convergence of advanced radiation on it. Nevertheless, this stronger argument does not imply that no advanced fields are involved even in such macroscopic processes, but rather that our universe has a macroscopic radiation arrow-of-time, discussed next. When, for example, a LED converts the electrostatic energy, previously stored in a capacitor, into light which, in turn, recharges a second capacitor via a (perfectly efficient...) photoelectric cell, we can only deduce what is the imbalance between the advanced and retarded Poynting fluxes integrated across a surface containing either systems (and that the two integrals are equal in magnitude but with opposite signs).

There remains the question of why there is a macroscopic arrow-of-time, and why the observed direction rather than the reversed. One legitimate answer relies on the anthropic principle: Without the observed arrow-of-time (its direction included) which could very well be just a peculiarity of a specific ECD solution, we wouldn’t exist to raise the question. Alternatively, it may turn out that when global cosmological considerations are included in the analysis, only the observed arrow-of-time becomes possible. But there is another possibility, more specific to ECD: The coupled ECD-Maxwell system is not covariant with respect to time-reversal; CPT symmetry (36) is the closest one gets to the notion of ‘running the movie backward’. The observed direction of the arrow-of-time is therefore intimately linked with the manifest imbalance between particles and their antiparticles.

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Appendices
A The ‘fine tuned’ central ECD system

As all ECD currents are computed in the limit $\epsilon \to 0$, the central ECD system (31) and (32) is given below an operational definition for small $\epsilon$ only. To this end, we would need the small-$s$ form of the propagator $G$. Plugging the ansatz

$$G(x, x', s) = G_t e^{i\Phi(x, x', s)/\hbar}$$

(40)

into (25), with

$$G_t(x, x'; s) = \frac{i}{(2\pi\hbar)^2} \frac{e^{\frac{i(x-x')^2}{2s}}}{s^2} \text{sign}(s),$$

(41)

the free propagator computed for $A \equiv 0$, and expanding $\Phi$ (not necessarily real) in powers of $s$, $\Phi(x, x', s) = \Phi_0(x, x') + \Phi_1(x, x')s + \ldots$, higher orders of $\Phi_k$ can recursively be computed with $\Phi_0$ alone incorporating the initial condition (33) in the form $\Phi_0(x', x') = 0$ (note the manifest gauge covariance of this scheme to any order $k$). For our purpose, $\Phi_0$ is enough. A simple calculation gives the gauge covariant phase

$$\Phi_0(x, x') = q \int_{x'}^x d\xi \cdot A(\xi),$$

(42)

where the integral is taken along the straight path connecting $x'$ with $x$.

Focusing first on (31), we see that, for fixed $\gamma$ and $G$, it is in fact an equation for a function $f^R(s) \equiv \phi(\gamma_s, s)$. Indeed, plugging an ansatz for $f^R$ into the r.h.s. of (31), one can compute $\phi(x, s) \forall s, x$, and in particular for $x = \gamma_s$, which we call $f^L(s)$. The linear map $f^R \mapsto f^L$ (which, using $G(x, x'; s) = G^s(x, x'; -s)$, can be shown to be formally self-adjoint) must therefore send $f^R$ to itself, for (31) to have a solution. Now, the universal, viz. $A$-independent, $i/(2\pi\hbar s)^2$ divergence of $G(y, y, s)$ for $s \to 0$ and any $y$, implies $f^R \mapsto f^R + O(\epsilon)$, so the nontrivial content of (31) is in this $O(\epsilon)$ term, which we write as $\epsilon f^r$. In [4], $\lim_{\epsilon \to 0} f^r = 0$ was implied as the content of (31). While this may turn out to be true for some specific solutions (a freely moving particle, for example), equation (31) should take a more relaxed form

$$\text{Im} \left( \lim_{\epsilon \to 0} f^r \right) f^R = 0,$$

(43)

where, as usual, ‘Im’ is the imaginary part of the entire product to its right.

Moving next to the second ECD equation, (32), conveniently rewritten as

$$\text{Re} \, \hbar \partial_x \phi(\gamma_s, s) \phi^*(\gamma_s, s) = 0,$$

(44)

a similar isolation of the nontrivial content exists. For further use, however, we first want to isolate the contribution of the small $s$ divergence of $G$ to $\phi(x, s)$, for a general $x$ other than $\gamma_s$. Substituting (40) into (31), and expanding the integrand around $s$ to first order in $s' - s$: $\gamma_{s'} \sim \gamma_s + \gamma_s(s' - s)$, $\Phi_0(x, \gamma_{s'}) \sim \Phi_0(x, \gamma_s)$, $\phi(\gamma_{s'}, s') \sim f^R(s)$, leads to a gauge covariant definition of the singular part of $\phi$

$$\phi^s(x, s) = f^R(s) e^{i\left(\Phi_0(x, \gamma_s) + \gamma_s \xi\right)/\hbar} \text{sinc} \left( \frac{\xi^2}{2\hbar \epsilon} \right)$$

(45)

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with $\xi \equiv x - \gamma_s$. Consequently, the residual (or regular) wave-function is defined via the gauge covariant equation
\[ \epsilon \phi^r(x, s) = \phi(x, s) - \tilde{\phi}^\ast(x, s). \] (46)

Note the implicit $\epsilon$-dependence of all the $\phi$’s in (46) which are omitted for economical reasons.

Using $\partial_x \Phi_0(x, \gamma_s) = q A(\gamma_s)$, we have
\[ \phi^\ast(\gamma_s, s) = f^R(s), \quad \tilde{\hbar} \partial_x \phi^\ast(\gamma_s, s) = i [\gamma_s + A(\gamma_s)] f^R(s), \] (47)

and (44) is automatically satisfied up to an $O(\epsilon)$, gauge invariant term
\[ \epsilon \Re \tilde{\hbar} \partial_x [\phi^\ast(\gamma_s, s) \phi^\ast(\gamma_s, s)^\ast] = \epsilon \Re D \phi^r(\gamma_s, s) \phi^\ast(\gamma_s, s)^\ast, \] (48)

where the above equality follows from (47), $\phi^\ast(\gamma_s, s) = f^r(s)$ and (43). The fine-tuned definition of (32) is therefore
\[ \lim_{\epsilon \to 0} \Re D \phi^r(\gamma_s, s) \phi^\ast(\gamma_s, s)^\ast = 0. \] (49)

Using the above definitions, (43) can also be written as
\[ \lim_{\epsilon \to 0} \Im \phi^r(\gamma_s, s) \phi^\ast(\gamma_s, s)^\ast = 0. \] (50)

More insight into this fine tuned central ECD system is given in the sequel. For the time being, let us just note that it is invariant under the original symmetry group of ECD. In particular, the system is invariant under
\[ \phi^\ast \mapsto C \phi^\ast, \quad \phi^r \mapsto C \phi^r, \quad C \in \mathbb{C}, \] (51)

under a gauge transformation
\[ A \mapsto A + \partial \Lambda, \quad G(x, x', s) \mapsto G e^{i[q\Lambda(x)-q\Lambda(x')]/\hbar}, \quad \phi^\ast \mapsto \phi^\ast e^{i q \Lambda / \hbar}, \quad \phi^r \mapsto \phi^r e^{i q \Lambda / \hbar}, \] (52)

and under scaling of space-time
\[ A(x) \mapsto \lambda^{-1} A(\lambda^{-1} x), \quad \epsilon \mapsto \lambda^2 \epsilon, \quad \gamma(s) \mapsto \lambda \gamma (\lambda^{-2} s), \]
\[ \phi^\ast(x, s) \mapsto \phi^\ast (\lambda^{-1} x, \lambda^{-2} s), \quad \phi^r(x, s) \mapsto \lambda^{-2} \phi^r (\lambda^{-1} x, \lambda^{-2} s), \] (53)

directly following from the transformation of the propagator under scaling
\[ A(x) \mapsto \lambda^{-1} A(\lambda^{-1} x) \Rightarrow G(x, x'; s) \mapsto \lambda^{-4} G (\lambda^{-1} x, \lambda^{-1} x'; \lambda^{-2} s). \]

Regarding this last symmetry, two points should be noted. First, for a finite $\epsilon$ it relates between solutions of different theories, indexed by different values of $\epsilon$. It is only because $\epsilon$ is ultimately eliminated from all results, via an $\epsilon \to 0$ limit, that scaling can be considered a symmetry of ECD. The second point concerns the scaling dimension of $\phi^\ast$ (and $\phi^r$). By the symmetry (51), this dimension can be an arbitrary number $D (D-2$ respectively). However, to comply with scale covariance $j$ must have dimension $-3$, hence $D = 0$. 

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A.1 ECD currents

All ECD currents have the common form

\[ j = \partial_\epsilon \epsilon^{-1} \int ds \, B[\phi, \phi^*], \]  

where \( B \) is some bilinear in \( \phi \) and \( \phi^* \). Using the decomposition (46) we write

\[ B[\phi, \phi^*] = \sum_{a, b \in \{r, s\}} O_a \phi^a O_b \phi^{*b}, \]  

for some local operators \( O \)'s (containing an \( \epsilon \) multiplier in the case of \( r \)). There are therefore three types of contributions: \( \{a, b\} = \{s, s\}, \{r, r\}, \) and \( \{r, s\}, \{s, r\} \) taken as one. Let us examine each for the typical case of the electric current \( j_\text{ss} \)—(34).

The \( \{s, s\} \) term reads

\[ j^{ss}(x) = \partial_\epsilon \epsilon^{-1} \frac{q}{\hbar} \int ds \left( \dot{\gamma}_s - qA(x) + \partial_s \Phi(x, \gamma_s) \right) |f^R(s)|^2 \text{sinc}^2 \left( \frac{(x - \gamma_s)^2}{2\hbar\epsilon} \right). \]  

By the same arguments as in section 2.3.1, \( j^{ss}(x) \) can be shown to reduce to the line current

\[ \alpha \int ds \ |f^R(s)|^2 \delta^{(4)}(x - \gamma_s) \dot{\gamma}_s, \]  

for some real constant \( \alpha \), which is not necessarily conserved as \( |f^R(s)|^2 \) may be \( s \)-dependent, and is discarded of in ECD. Likewise, the \( \{s, s\} \) contribution of all ECD currents is a distribution supported on \( \bar{\gamma} \) albeit generally containing more complex distributions, involving also derivatives of line distributions.

Moving to the \( \{r, r\} \) term, this piece gives a nonsingular contribution which is well localized around \( \gamma \) in a region referred to as the core. The localization mechanism of the core is explained within the semiclassical approximation in appendix B.

Finally, the most singular part of the \( \{r, s\} \) term gives a regular, well localized piece, coming from the \( r \) part, multiplying an \( x \)-derivative of a \( \delta(\xi^2) \) coming from the \( s \) part (in combination with the \( \epsilon^{-1} \)). This piece generates, at worst, an integrable \( r^{-2} \) singularity on \( \gamma \) to which no charge leaks by virtue of (31) (see appendix C). We say ‘at worst’ because condition (50) could counter the the \( r^{-2} \) divergence coming from the delta-function, depending on the form of \( \phi^s(x, s) \) around \( \gamma_s \). Indeed, in a semiclassical analysis the divergence is moderated to \( r^{-1} \) only.

An \( r^{-2} \) divergence in the \( j^0 \) component implies an \( r^{-1} \) divergence of the electric field on \( \bar{\gamma} \) which, in turn, leads to an integrable \( r^{-2} \) divergence of \( \Theta^{00} \)—the electrostatic energy density (as oppose to the non-integrable \( r^{-4} \) divergence in the case of a point charge.). Similar arguments show that \( m^{00} \) has an integrable singularity at worst.
B A semiclassical analysis of ECD

In this section we analyze the consistency of the central ECD system using a small \( \hbar \) approximation of the propagator known as the semiclassical propagator,

\[
G_{sc}(x, x'; s) = \frac{i \text{sign}(s)}{(2\pi \hbar)^2} \mathcal{F}(x, x'; s) e^{iI(x, x'; s)/\hbar}.
\]

Above,

\[
I = \int_0^s d\sigma \frac{1}{2} \dot{\beta}_\sigma^2 + qA(\beta_\sigma) \cdot \dot{\beta}_\sigma,
\]

is the action of the classical path \( \beta \) such that \( \beta_0 = x' \) and \( \beta_s = x \), and \( \mathcal{F} \) — the so-called Van-Vleck determinant — is the gauge-invariant classical quantity, given by the determinant

\[
\mathcal{F}(x, x'; s) = \left| \partial_{x_\mu} \partial_{x'_\nu} I(x, x'; s) \right|^{1/2}.
\]

The semiclassical propagator becomes exact for small \( s \), so the singular-regular decomposition (46) of \( \phi \) is consistent with the approximation, the latter affecting only the accuracy of \( \phi^r \).

Let us next show that to leading order in \( \hbar \) and some fixed potential \( A \), the fine-tuned central ECD system is solved by any classical \( \gamma \), and by a corresponding ansatz of the form

\[
f^R(s') = Ce^{iI(\gamma_s, \gamma_0, s')/\hbar},
\]

with \( C \in \mathbb{C} \) an arbitrary constant.

Substituting in (31), \( G \rightarrow G_{sc} \), \( x' \rightarrow \gamma_{s'} \) and \( x \rightarrow \gamma_s \), we first note that \( \gamma \) is the classical path in \( A \), connecting \( \gamma_{s'} \) with \( \gamma_s \). Using

\[
I(\gamma_s, \gamma_{s'}, s - s') + I(\gamma_{s'}, \gamma_0, s') = I(\gamma_s, \gamma_0, s)
\]

we get

\[
\phi(\gamma_s, s) = \frac{eC}{2} e^{iI(\gamma_s, \gamma_0, s)/\hbar} \int_{-\infty}^\infty ds' \mathcal{F}(\gamma_s, \gamma_{s'}; s - s') \text{sign}(s-s')\mathcal{U}(\epsilon; s-s')
\]

\[
\Rightarrow \phi^f(\gamma_s, s) = \frac{C}{2} e^{iI(\gamma_s, \gamma_0, s)/\hbar} \left[ R(s, \epsilon) - \frac{2}{\epsilon} \right] = \frac{1}{2} f^R(s) \left[ R(s, \epsilon) - \frac{2}{\epsilon} \right],
\]

with

\[
R(s, \epsilon) = \int_{-\infty}^\infty ds' \mathcal{F}(\gamma_s, \gamma_{s'}; s - s') \text{sign}(s-s')\mathcal{U}(\epsilon; s-s').
\]

We shall assume that for a given \( s \), there exists a unique path connecting \( x' \) with \( x \). The existence of a plurality of classical paths is inconsequential to our analysis.

The approximation involved in the computation of the semiclassical propagator amounts to ignoring a ‘quantum potential’ term in the dynamics of a classical particle originating from \( x' \). This potential reads \( \hbar^2 \Box R/2R \), with \( R \) the modulus of the exact propagator. Granted that the latter’s form is (40) for small \( s \), the modulus of \( G \) is independent of \( x \) and the quantum potential vanishes.
some real functional of the EM field and its first derivative (its local neighborhood in an exact analysis) on \( \bar{\gamma} \), such that \( \lim_{\epsilon \to 0} [R(s, \epsilon) - 2/\epsilon] \) is finite, implying that (50) is satisfied.

Moving next to the second refined ECD equation, (49), and pushing \( \partial \) into the integral in (31),

\[
\hbar \partial \phi(\gamma_s, s) = \frac{\epsilon C}{2} e^{i H(\gamma_s, \gamma_0, s)/\hbar} \int_{-\infty}^{\infty} ds' \left[ i \partial_x I(x, \gamma_{s'}; s - s') \bigg|_{x=\gamma_s} \mathcal{F}(\gamma_s, \gamma_{s'}; s - s') \right. \\
+ \left. \hbar \partial_x \mathcal{F}(x, \gamma_{s'}; s - s') \bigg|_{x=\gamma_s} \right] \text{sign}(s - s') \mathcal{U}(\epsilon; s - s') .
\]

(65)

The \( \hbar \partial F \) term in (65) can be neglected for small \( \hbar \). Using a relativistic variant of the Hamilton-Jacobi theory (see appendix B in [4]), we can write

\[
\partial_x I(\gamma_s, \gamma_{s'}, s - s') = p(s) \equiv \dot{\gamma}_s + qA(\gamma_s)
\]

which is independent of \( s' \). Together with (63) we therefore get

\[
\hbar \partial \phi(\gamma_s, s) = ip(s) \phi(\gamma_s, s) \quad \Rightarrow \quad \hbar \partial \phi^\dagger(\gamma_s, s) = ip(s) \phi^\dagger(\gamma_s, s)
\]

\[
\Rightarrow \lim_{\epsilon \to 0} \text{Re} \ D \phi^\dagger(\gamma_s, s) f^{R*}(s) = -\dot{\gamma}_s \lim_{\epsilon \to 0} \text{Im} \ \phi^\dagger(\gamma_s, s) f^{R*}(s) ,
\]

(67)

which vanishes by (50), hence (49) is satisfied.

### B.1 ECD currents in the semiclassical approximation

For \( x \) other than \( \gamma_s \), applying the semiclassical approximation to (31) gives

\[
\phi(x, s) = \frac{\epsilon C}{2} \int_{-\infty}^{\infty} ds' \mathcal{F}(x, \gamma_{s'}; s - s') e^{i (U(x, \gamma_{s'}, s - s') + U(\gamma_{s'}, \gamma_0, s'))/\hbar} \text{sign}(s - s') \mathcal{U}(\epsilon; s - s') .
\]

The phase of the integrand is independent of \( s' \) only for \( x = \gamma_s \), as manifested in (62). Otherwise, the family of paths connecting \( \gamma_{s'} \) with \( x \), and that connecting \( \gamma_{s'} \) with \( \gamma_0 \), traverse different parts of the potential and do not even lie on the same mass-shell. The phase is therefore a rapidly oscillating function of \( s' \) for small \( \hbar \) and/or \( x \) lying far from \( \gamma_s \), rendering \( \phi(x, s) \) arbitrarily localized around \( \gamma_s \) in the limit \( \hbar \to 0 \). Combined with (66) and a suitably chosen \( C \), the ECD electric current (34) reduces to the CE electric current (6) in that limit.

In the terminology of section A.1, using the accuracy of the semiclassical propagator for small \( s \), it can readily be shown that the generic integrable singularity of the \( \{ r, s \} \) contribution to the ECD electric current (34) survives the semiclassical approximation. This singularity in \( j^0 \), moderated to \( r^{-1} \) by virtue of (66), implies a discontinuous EM field at \( \bar{\gamma} \) (a non differentiable \( A \) there) which means that the fully coupled Maxwell-ECD system cannot be consistently solved in the semiclassical approximation as for \( x \) and \( x' \) both lying on \( \bar{\gamma} \), the semiclassical propagator (58) is ill defined when self fields are taken into account (Note
that this sensitivity to a discontinuity in the EM field is just an artifact of the semiclassical approximation and does not carry to an exact analysis.

In summary, the semiclassical analysis of (scalar) ECD has lead us back to the two well defined but mutually incompatible ingredients of CE: the Lorentz force equation and the line current associated with a point charge. We see once more that a solution to the classical self force problem requires an essentially “quantum” ($\hbar \neq 0$) treatment.

C The basic tenets

To prove the conservation of the ECD electric current (34), we first need the following lemma, whose proof is obtained by direct computation.

**Lemma.** Let $f(x, s)$ and $g(x, s)$ be any (not necessarily square integrable) two solutions of the homogeneous Schrödinger equation (25), then

$$\frac{\partial}{\partial s}(fg^*) = \partial_\mu \left[ \frac{i}{2}(D^\mu fg^* - (D^\mu g)^* f) \right].$$

This lemma is just a differential manifestation of unitarity of the Schrödinger evolution—hence the divergence.

Turning now to equation (31),

$$\phi(x, s) = -2\pi^2 \hbar^2 \epsilon i \int_{-\infty}^{\infty} ds' G(x, \gamma_{s'}; s - s') f^R(s') U(\epsilon; s - s'),$$

and its complex conjugate,

$$\phi^*(x, s) = 2\pi^2 \hbar^2 \epsilon i \int_{-\infty}^{\infty} ds'' G^*(x, \gamma_{s''}; s - s'') f^R(s'') U(\epsilon; s - s''),$$

we get by direct differentiation

$$q \frac{\partial}{\partial s} \left[ -2\pi^2 \hbar^2 \epsilon i \int_{-\infty}^{\infty} ds' f^R(s') \ 2\pi^2 \hbar^2 \epsilon i \int_{-\infty}^{\infty} ds'' f^R(s'') U(\epsilon; s - s') G(x, \gamma_{s''}; s - s'') \right]$$

$$= -2q\pi^2 \hbar^2 \epsilon i \int_{-\infty}^{\infty} ds' f^R(s') \ 2\pi^2 \hbar^2 \epsilon i \int_{-\infty}^{\infty} ds'' f^R(s'')$$

$$\partial_s \left[ G(x, \gamma_{s'}; s - s') G^*(x, \gamma_{s''}; s - s'') \right] U(\epsilon; s - s') U(\epsilon; s - s'')$$

$$+ \left[ \partial_s U(\epsilon; s - s') U(\epsilon; s - s'') + U(\epsilon; s - s') \partial_s U(\epsilon; s - s'') \right] G(x, \gamma_{s'}; s - s') G^*(x, \gamma_{s''}; s - s'').$$
Focusing on the first term on the r.h.s. of (71), we note that, as \( G \) is a homogeneous solution of Schrödinger’s equation, we can apply our lemma to that term, which therefore reads

\[
-2q\pi^2\hbar^2\epsilon i \int_{-\infty}^{\infty} ds' f^R(s') - 2q\pi^2\hbar^2\epsilon i \int_{-\infty}^{\infty} ds'' f^{R*}(s'')
\]

\[
\partial_\mu \left[ \frac{i}{2} \left( D^\mu G(x, \gamma_{s''}; s - s')G^*(x, \gamma_{s''}; s - s'') - (D^\mu G(x, \gamma_{s''}; s - s''))^*G(x, \gamma_{s'}; s - s') \right) \right]
\]

\[
U(\epsilon; s - s')U(\epsilon; s - s'').
\]

Integrating (71) with respect to \( s \), the left-hand side vanishes (we can safely assume it goes to zero for all \( x, s', s'' \) as \(|s| \to \infty\)), and the derivative \( \partial_\mu \) can be pulled out of the triple integral in the first term. The reader can verify that this triple integral, after application of \( \lim_{\epsilon \to 0} \partial_\epsilon \epsilon^{-1} \), is just \( \partial_\mu j^\mu \), with \( j \) given by (34) and \( \phi, \phi^* \) are explicated using (69), (70) respectively. The ECD electric current is therefore conserved, provided the \( s \) integral over the second term in (71), after application of \( \lim_{\epsilon \to 0} \partial_\epsilon \epsilon^{-1} \) to it, vanishes in the distributional sense.

Let us then show that this is indeed the case. Integrating the second term with respect to \( s \), and using \( \partial_\epsilon U(\epsilon; s - s') = \delta(s - s' - \epsilon) + \delta(s - s' + \epsilon) \), that term reads

\[
-2q\pi^2\hbar^2\epsilon i \int_{-\infty}^{\infty} ds' f^R(s') - 2q\pi^2\hbar^2\epsilon i \int_{-\infty}^{\infty} ds'' f^{R*}(s'')
\]

\[
U(\epsilon; s' - s - s'')G(x, \gamma_{s''}; -\epsilon)G^*(x, \gamma_{s''}; s' - \epsilon - s'')
\]

\[
+U(\epsilon; s' + \epsilon - s'')G(x, \gamma_{s''}; +\epsilon)G^*(x, \gamma_{s''}; s' + \epsilon - s'')
\]

\[
+U(\epsilon; s'' - \epsilon - s')G(x, \gamma_{s'}; s'' - \epsilon - s')G^*(x, \gamma_{s'}; -\epsilon)
\]

\[
+U(\epsilon; s'' + \epsilon - s')G(x, \gamma_{s'}; s'' + \epsilon - s')G^*(x, \gamma_{s'}; +\epsilon).
\]

Using (69) and (70), this becomes

\[
\text{Re} - 4q\pi^2\hbar^2\epsilon i \int_{-\infty}^{\infty} ds' f^R(s') \left[ \phi^*(x, s' - \epsilon)G(x, \gamma_{s''}; -\epsilon) + \phi^*(x, s' + \epsilon)G(x, \gamma_{s''}; +\epsilon) \right].
\]

Writing \( \phi = \phi^\delta + \epsilon \phi^\gamma \) above, and using the short-\( s \) propagator (40) plus the explicit form, (45), of \( \phi^\delta \), one can show that application of \( \lim_{\epsilon \to 0} \partial_\epsilon \epsilon^{-1} \) to (74) results in a distribution supported on \( \gamma \)—a ‘line sink’—which is composed of two pieces: one coming from \( \phi^\delta \) and one—from \( \phi^\gamma \). The \( s \) piece is just the (not necessarily vanishing) divergence of the line current (57) and is therefore of no concern to us. The second piece reads

\[
\lim_{\epsilon \to 0} -8q\pi^2\hbar^2 \int_{-\infty}^{\infty} ds \text{ Re } i f^R(s) \phi^{\gamma^*}(\gamma_s, s) \delta^{(4)}(x - \gamma_s) =
\]

\[
\lim_{\epsilon \to 0} 8q\pi^2\hbar^2 \int_{-\infty}^{\infty} ds \text{ Im } f^R(s) \phi^{\gamma^*}(\gamma_s, s) \delta^{(4)}(x - \gamma_s)
\]

(75)
and represents a ‘line sink in Minkowski’s space’ associated with the singularity of \( j \) on \( \bar{\gamma} \).

By virtue of (50), no leakage of charge occurs at those sinks, as one can establish the time-independence of the charge by integrating \( \partial \cdot j = 0 \) over a volume in Minkowski’s space, and apply Stoke’s theorem, to get a conserved quantity. A more explicit way of demonstrating the conservation of charge, avoiding the use of distributions, is shown next.

### C.1 Line sinks in Minkowski’s space

To gain a more explicit geometrical insight into the meaning of a ‘line sink in Minkowski’s space’, consider a small space-like three-tube, \( T \), surrounding \( \bar{\gamma} \), the construction of which proceeds as follows. Let \( \beta(\tau) = \gamma(s(\tau)) \) be the world line \( \bar{\gamma} \), parametrized by proper time \( \tau = \int s \sqrt{(d\gamma)^2} \), and let \( x \mapsto \tau_r \) be the retarded light-cone map defined by the relations

\[
\eta^2 \equiv (x - \beta_{\tau_r})^2 = 0, \quad \text{and} \quad \eta^0 > 0.
\]  

Let the ‘retarded radius’ of \( x \) be

\[
r = \eta \cdot \dot{\beta}_{\tau_r}.
\]  

Taking the derivative of (76), treating \( \tau_r \) as an implicit function of \( x \), and solving for \( \partial \tau_r \), we get

\[
\partial \tau_r = \frac{\eta}{r} \Rightarrow \partial r = 1 + \beta_{\tau_r} \cdot \frac{\eta}{r}.
\]  

The (retarded) three-tube of radius \( \rho \) is defined as the time-like three surface

\[
T_\rho = \{ x \in M : r(x) = \rho \}.
\]  

It can be shown in a standard way that the directed surface element normal to \( x \in T_\rho \) is

\[
d^\mu T_\rho = \partial ^\mu r |_{r=\rho} \rho^2 d\tau d\Omega,
\]  

where \( d\Omega \) is the surface element on the two-sphere.

Let \( \Sigma_1 \) and \( \Sigma_2 \) be two space-like surfaces, intersecting \( T_\rho \) and \( T_R \). Applying Stoke’s theorem to the interior of the three surface composed of \( T_\rho \), \( T_R \), \( \Sigma_1 \) and \( \Sigma_2 \), and using \( \partial \cdot j = 0 \) there, we get

\[
\int_{\Sigma_2} d\Sigma_2 \cdot j + \int_{\Sigma_1} d\Sigma_1 \cdot j = -\int_{T_\rho} dT_\rho \cdot j - \int_{T_R} dT_R \cdot j.
\]  

Realistically assuming that the second term on the r.h.s. of (80) vanishes for \( R \to \infty \), we get that the ‘leakage’ of the charge, \( \int_{\Sigma_2} d\Sigma_2 \cdot j - \int_{\Sigma_1} d\Sigma_1 \cdot j \), equals to \( -\lim_{\rho \to 0} \int_{T_\rho} dT_\rho \cdot j \).

As \( dT_\rho = O(\rho^2) \), the leakage only involves the piece of \( j \) diverging as \( r^{-2} \). This piece, reads

\[
2q\hbar^2 \int ds \text{ Im } \phi^*(x,s) f^R(s) \partial \frac{1}{2 \hbar \epsilon} \text{ sinc } \left( \frac{\xi^2}{2 \hbar \epsilon} \right) \xrightarrow{\epsilon \to 0} 2q\hbar^2 \pi \int ds \text{ Im } \phi^*(x,s) f^R(s) \partial \delta (\xi^2)
\]

\[
\sim 2q\hbar^2 \pi \partial \int ds \text{ Im } \phi^*(\gamma_s,s) f^R(s) \delta (\xi^2) = q\hbar^2 \pi \sum_{s=s_s, s_f} \text{ Im } \phi^*(\gamma_s,s) f^R(s) \partial \frac{1}{|\xi \cdot \gamma_s|},
\]  

25
where \( s_r = s(\tau_r) \), and \( \gamma_{sa} \) is the corresponding advanced point on \( \bar{\gamma} \), defined by
\[
\xi^2 \equiv (x - \gamma_{sa})^2 = 0, \quad \xi^0 < 0.
\]
Focusing first on the contribution of \( s_r \), and using a technique similar to that leading to (78), we get
\[
\frac{\partial}{\partial \xi} \frac{1}{\xi \cdot \dot{\gamma}_{sr}} = -\dot{\gamma}_{sr} \frac{(\dot{\gamma}_{sr}^2 + \dot{\gamma}_{sr} \cdot \xi)}{(\xi \cdot \dot{\gamma}_{sr})^2} \xi \sim -\frac{\dot{\beta}_{\tau r} m r^2 + \eta_{mr}}{m r^3},
\]
(81)
where \( m = d\tau/ds \) needs not be constant. In the limit \( \rho \rightarrow 0 \), using \( \partial \frac{1}{\xi \dot{\gamma}_{sr}} \cdot \partial r \bigg|_{r=\rho} \rightarrow m^{-1} \), the contribution of \( s_r \) to the flux across \( T_\rho \) is most easily computed
\[
\int_{T_\rho} dT_\rho \cdot j = q\hat{h}^2 \pi \int d\Omega \int d\tau m^{-1} \text{Im} \phi^*(\beta_{\tau r}, \tau_r) f^R(\tau_r)
\]
\[
= 4q\hat{h}^2 \pi \int ds_r \text{Im} \phi^*(\beta_{sr}, s_r) f^R(s_r).
\]
(82)
The contribution of \( s_a \) to the flux of \( j \) is more easily computed across a different, (advanced) \( T_\rho \), and gives the same result in the limit \( \rho \rightarrow 0 \). The fact that \( \rho \) can be taken arbitrarily small, in conjunction with the conservation of \( j(x) \) for \( x \notin \bar{\gamma} \), implies that the flux of \( j \) across any three-tube, \( T_\rho = \partial C \), with \( C \) a three-cylinder containing \( \bar{\gamma} \), equals twice the value in (82), when \( C \) is shrunk to \( \bar{\gamma} \). Changing the dummy variable \( s_r \mapsto s \) in (82), the formal content of (75) receives a clear meaning using Stoke’s theorem
\[
\int_C d^4x \partial \cdot j = 8q\hat{h}^2 \pi \int ds \text{Im} \phi^*(\beta_{sa}, s) f^R(s) \int_C d^4x \delta^{(4)}(x - \gamma_s) = \int_T dT \cdot j,
\]
which vanishes by virtue of (50).

### C.2 Energy-momentum conservation

The conservation of the ECD energy momentum tensor can be established by the same technique used in the previous section. To explore yet another technique, as well as to illustrate the role played by symmetries of ECD in the context of conservation laws, we cautiously apply Noether’s theorem to the following functional
\[
L \left[ \{ k_\phi \}^n_1, \mathcal{A} \right] = \sum_{k=1}^n L_m \left[ k_\phi, \mathcal{A} \right] - \int_M d^4x \frac{1}{4} F^{\mu\nu} F_{\mu\nu}
\]
(83)
with
\[
L_m [\phi, \mathcal{A}] = -\int_{-\infty}^{\infty} ds \int_M d^4x \frac{i\hat{h}}{2} (\phi^* \partial_s \phi - \partial_s \phi^* \phi) - \frac{1}{2} (D^\lambda \phi)^* D_\lambda \phi.
\]
(84)
Above, \( \mathcal{A} \) is a vector-potential which appears in the covariant derivative, \( D, \mathcal{F} \) the associated Faraday tensor, and \( \phi \) a complex scalar function of space-time and of \( s \).

Next, let us look at a solution to the coupled ECD-Maxwell system of \( n \) interacting particles. As an \( \epsilon \) limit is involved in that solution, we assume that in addition to a vector
potential, \( A \), we also also have a family of \( n \) pairs, \( \{ k \phi_e, k \gamma \} \), such that \( \lim_{\epsilon \to 0} \epsilon^{-1} k \phi_e \) exists in the distributional sense (the \( \epsilon^{-1} \) is needed to render the limit non trivial).

Plugging \( A \) and the form (69) of \( \phi_e \) into (83), we compute the first variation of (83) around them:  

\[
\delta L = \int_{-\infty}^{\infty} ds \int_M d^4 x - \left( \sum_k q \Im \phi^*_k D^\mu \phi_k - \partial_\nu F^{\nu \mu} \right) \delta A^\mu - \sum_k 2 \Re (i \partial_s - \mathcal{H}) \phi_k \delta \phi^*_k \tag{85}
\]

where \( \mathcal{H} \) is defined in (25)—without a scalar potential. Note that (obvious) particle labels, \( k \), have been dropped for economical reasons.

Choosing \( \delta \phi_e = \partial \phi_e \cdot a \), and \( \delta A^\mu = \partial_\nu A^\mu a^\nu \), corresponding to an infinitesimal shift, \( x \mapsto x + a(x) \), of the coordinates, we get after some integrations by part

\[
\delta L = \int_M d^4 x \partial_\nu D^\nu a^\mu \quad \text{by eq. (85)} - \left( \sum_k q \Im \phi^*_k D^\mu \phi_k - \partial_\nu F^{\nu \mu} \right) \partial_\nu A^\mu a^\nu - \sum_k \delta A^\mu a^\nu \tag{86}
\]

\[
\int_{-\infty}^{\infty} ds \int_M d^4 x \Re 4\pi \hbar^2 \epsilon \left[ G(x, \gamma_{s+\epsilon}; +\epsilon) f^R(s - \epsilon) + G(x, \gamma_{s+\epsilon}; -\epsilon) f^R(s + \epsilon) \right] \partial^\mu \phi^*(x, s) a_\mu ,
\]

with \( p \) the Noether (canonical) e-m tensor associated with the action (83), computed for \( \phi_e \) and \( A \), and the second line follows from

\[
(i \partial_s - \mathcal{H}) \phi = 2\pi \hbar^2 \epsilon \left[ G(x, \gamma_{s-\epsilon}; +\epsilon) f^R(s - \epsilon) + G(x, \gamma_{s+\epsilon}; -\epsilon) f^R(s + \epsilon) \right],
\tag{87}
\]

which, in turn, directly follows from (69).

Applying now \( \lim_{\epsilon \to 0} \partial \epsilon^{-1} \) to (86), the first term on the r.h.s. vanishes by virtue of Maxwell’s equations and the definition, (34), of the electric current, while the terms that follow can be analyzed using the same technique used in the computation of (74). This gives

\[
8\pi^2 \hbar^2 \int_{-\infty}^{\infty} ds \int_M d^4 x \Re f^R(s) \delta^{(4)}(x - \gamma_s) \partial \phi^*(\gamma_s, s) \cdot a(x, s) =
\]

\[
8\pi^2 \hbar^2 \int_{-\infty}^{\infty} ds \Re f^R(s) \partial \phi^*(\gamma_s, s) \cdot a(\gamma_s, s)
\tag{88}
\]

which vanishes by virtue of (49) for any \( a \). The arbitrariness of \( a \) implies the vanishing of \( \partial_\nu (\lim_{\epsilon \to 0} \partial \epsilon^{-1} p^{\mu \nu}) \) in the distributional sense. Just like the electric current \( j \), the e-m tensor \( p \) can easily be shown to be a smooth function of \( x \), implying point-wise conservation. Equation (49) in the central ECD system, by which (88) vanishes, appears therefore as the condition that no energy or momentum leak into a world-sink on \( \gamma \).

The Noether e-m tensor \( p \) is not symmetric nor gauge invariant. This is an artefact of a non generally covariant treatment and the standard way of dealing with it (other than by

\[\text{We restrict the space of variations to those which go sufficiently fast to zero at infinity to justify integrations by part.}\]
setting a \( g_{\mu\nu} = \eta_{\mu\nu} \) in the generally covariant version) is to add to \( p \) a conserved chargeless piece \( \partial_\lambda (F^\mu_\lambda A_\mu) \) which together, again using Maxwell’s equations, turn \( p \) into \( \Theta + \sum_k h_k \), with \( \Theta \) the canonical tensor (11) and \( h_k \) the ‘mechanical’ e-m tensor (35).

C.3 Charges leaking into world sinks

Both methods used above, can be applied to prove the conservation of the mass-squared current — the counterpart of (3)

\[
 b(x) = \lim_{\epsilon \to 0} \partial_\epsilon^{-1} \int ds B(x, s) \equiv \lim_{\epsilon \to 0} \partial_\epsilon^{-1} \int ds \text{Re} \ h \partial_s \phi^* D\phi, \quad \text{for } x \notin \bar{\gamma}. \tag{89}
\]

In the first method, used to establish the conservation of \( j \), the counterpart of (68) is \( \partial_s (g^* \mathcal{H} f) = \partial \cdot (\text{Re} \ h \partial_s g^* Df) \), corresponding to the invariance of the Hamiltonian (in the Heisenberg picture) under the Schrödinger evolution. In the variational approach, the conservation follows from the (formal) invariance of (83) \( \phi(x, s) \mapsto \phi(x, s + s_0) \). However, the leakage to the sink on \( \bar{\gamma} \), between \( \gamma_{s_1} \) and \( \gamma_{s_2} \), is given by

\[
 8\pi^2 \hbar^3 \int_{s_1}^{s_2} ds \text{Re} \ \partial_s \phi^* (\gamma_{s_1}, s) f^{R}(s), \tag{90}
\]

is not guaranteed to vanish. Note that this leakage (whether positive or negative) is a ‘highly quantum’ phenomenon — proportional to \( \hbar^2 \) (the term \( \partial_s \phi^* \) generally diverges as \( \hbar^{-1} \)).

Similarly, associated with the formal invariance of (83) under

\[
 A(x) \mapsto \lambda^{-1} A(\lambda^{-1} x), \quad \phi(x, s) \mapsto \lambda^{-2} \phi (\lambda^{-1} x, \lambda^{-2} s),
\]

is a locally conserved dilatation current, the counterpart of the classical current (15),

\[
 \xi^\mu = p^{\mu\nu} x_\nu - \lim_{\epsilon \to 0} \partial_\epsilon^{-1} \sum_k 2 \int_{-\infty}^{\infty} ds \ s^k B, \quad \text{with } B \text{ defined in (89).} \tag{91}
\]

The leakage to the sinks on \( \bar{\gamma} \) is due to the second term, involving the mass-squared of the particles. A leakage of mass, therefore, also modifies the scale-charge of a solution.

D The Lorentz force from the basic tenets

The derivation of the Lorentz force equation (2) given below, applies to a general, sufficiently isolated pair \( j, m \), satisfying the basic tenets and co-localized—in any sensibly meaning of the word—around a world-line \( \bar{\gamma} = \bigcup_s \gamma_s \).

Let \( \Sigma(s) \) be a foliation of \( M \), viz., a one-parameter family of non intersecting space-like surfaces, each orthogonally intersecting the world line \( \bar{\gamma} \) at \( \gamma_s \), \( C \) a four-cylinder containing \( \bar{\gamma} \), and \( p^\mu(\tau) \) the corresponding four-momenta

\[
 p^{\mu}(s) = \int_{\Sigma(s) \cap C} d\Sigma_\nu m^{\nu\mu}, \tag{92}
\]
where $d\Sigma$ is the Lorentz covariant directed surface element, orthogonal to $\Sigma(s)$. Let also $C(s, \delta) \in C$ be the volume enclosed between $\Sigma(s - \delta/2)$ and $\Sigma(s + \delta/2)$, and $T(s, \delta)$ its time-like boundary (see figure 1). Integrating (10) over $C(s, \delta)$, applying Stoke’s theorem to the l.h.s., and dividing by $\delta$ we get

$$\frac{p^\mu(s + \delta/2) - p^\mu(s - \delta/2)}{\delta} + \delta^{-1} \int_T dT_\nu m^{\nu\mu} = \delta^{-1} \int_{C(s, \delta)} d^4 x F^{\mu\nu} j_\nu,$$  

(93)

with $dT$ the outward pointing directed surface element, orthogonal to $T$. Assuming that $m$ is sufficiently localized about $\bar{\gamma}$, the second, surface term, on the l.h.s. of (93) can be ignored.

Both sides of (93) depend on the details of the foliation $\{\Sigma_s\}$, and may rapidly fluctuate if the particle experiences internal vibrations. Both, nevertheless, are well defined—unlike in the point-charge case.

To translate (93) into an equation for $\bar{\gamma}$—roughly speaking the center of the current—we first ‘low-pass’ (93), viz., convolve it with a normalized kernel, $w(s)$, to remove possible fluctuations which are due to internal vibrations in the particle. It is easy to see then that for a sufficiently wide $w$, the r.h.s. of (93) becomes independent of the details of the foliation hence also the l.h.s. of (93) (remember that we ignore the second term on the l.h.s.). Next, we make the reasonable assumptions that the low-passed $p$ is locally (s-wise) proportional to the low-passed $\dot{\gamma}$, with an $s$-independent proportionality constant $M_m$. This latter assumption is nothing but the condition that the same particle is being investigated at different $s$’s, namely, that the average mechanical momentum of the particle can be deduced from its...
average velocity. We further assume that the low-passed momentum is slowly changing on time scales on the order of $\delta$. Under these assumptions, using the same notation for the low-passed $\gamma$, (93) becomes

$$M_m \ddot{\gamma}^\mu = \delta^{-1} \int \! d^4 x \, \bar{w}(s, x) F^{\mu\nu}(x) j_\nu(x) \equiv \langle F^{\mu\nu} j_\nu \rangle_{\gamma_s},$$

(94)

with $\bar{w}(s, x)$ defined by $x \in \Sigma_{s'} \Rightarrow \bar{w}(s, x) = w(s - s')$, and $M_m = \sqrt{p^2}$ is the 'mechanical mass'.

For a sufficiently isolated particle, expression (37) for $A$ provides a convenient decomposition of $F$ in (94) into a self field, $F_{\text{sel}}$ generated by the isolated particle, and an external field $F_{\text{ext}}$ generated by the rest of the particles. For a slowly varying $F_{\text{ext}}$ on the scale set by $w$ the r.h.s. of (94) can be written

$$Q F_{\text{ext}}^{\mu\nu}(\gamma_s) \dot{\gamma}_\nu + \langle F_{\text{sel}}^{\mu\nu} j_\nu \rangle_{\gamma_s},$$

(95)

with $Q = \int_{\Sigma_{s'}} \! d\Sigma \cdot j$ the $s$-independent electric charge.

The self-force term in (95) is dealt with by noting that $j$ generates $F_{\text{sel}}$, hence we can locally apply Poynting theorem (12) to them, calling $\Theta_{\text{sel}}$ the associated canonical tensor. Further noting that (12) is formally equivalent to (10) used above with $m \mapsto -\Theta_{\text{sel}}$, we can hope that, insofar as as the particle does not (significantly) radiate, the following would also be true

$$\langle F_{\text{sel}}^{\mu\nu} j_\nu \rangle_{\gamma_s} = -M_{\text{EM}} \ddot{\gamma},$$

(96)

with $M_{\text{EM}} = \sqrt{p_{\text{EM}}^2}$, where

$$p_{\text{EM}}^{\mu} = \int_{\Sigma(s) \cap C} \! d\Sigma_\nu \Theta_{\text{sel}}^{\nu\mu},$$

(97)

is the EM contribution to a particle’s e-m. This is indeed the case as long as only the Coulomb part (with its integrable $r^{-4}$ tail) significantly contributes to (97), which is true for either a sufficiently small acceleration or particle size. Combining (94), (95) and (96), we get the Lorentz force equation for a particle with an effective mass equal to $M_m + M_{\text{EM}}$ when $s$ is chosen as proper time $\tau$, with $d\tau = \sqrt{d\gamma^2}$.

The non integrable $r^{-4}$ divergence of the Coulomb energy density associated with a point charge, implies that the limit of a finitely charged particle shrinking to a point can only be the trivial, infinite mass particle, therefore experiencing no acceleration.

The above analysis demonstrates that when the radiation field of a particle can be neglected, the Lorentz force equation is reproduced on scales larger than the extent of the particle. This explains the partial success of simply ignoring the self force as a solution to the self force problem. In some cases, in contrast, the self force dominates the dynamics, leading to such a colossal failure of this approximation that physicist mistakenly reasoned that CE must be abandoned altogether.

To incorporate radiation corrections to a particle’s bulk motion, we need to better approximate the integral

$$\delta^{-1} \int_{\partial C(s, \delta)} \! dC_\nu \Theta_{\text{sel}}^{\nu\mu},$$

(98)
with dC standing for either dT or dΣ, beyond the Coulomb approximation. To this end, one must commit to a particular mixture of advanced and retarded solutions (which is incompatible with ECD’s mathematical structure; recall section 3.3). Adhering to the convention of using only retarded solutions, the integral (98) can be evaluated using a multipole expansion of the EM field generated by the particle. For a sufficiently small particle, the dipole term is responsible for the leading contribution away from $\vec{\gamma}$. In addition, the Coulomb contribution to the self-energy gets modified by a term whose leading order is proportional to $\gamma_s$, coming from the vicinity of $\vec{\gamma}$. The combined result is the ALD self-force $Q_2^2 2^3 \left( \cdots \right) \gamma^2 \dot{\gamma}$, which is orthogonal to the momentum. The result from section 2.3.1, of an explicit extended body model, is therefore reproduced, and has exactly the same experimentally problematic validity domain.

A more symmetric treatment of ‘matter’ and the EM field is provided by the conservation of the total e-m, $p$ in (14). Applying Stoke’s theorem to $\partial p = 0$, and using the same construction as in figure 1, we get

$$p^\mu(s + \delta'/2) - p^\mu(s - \delta'/2) = - \int_T dT_\nu p^{\nu \mu}, \quad (99)$$

with

$$p^\mu(s) = \int_{\Sigma(s) \cap C} d\Sigma_\nu p^{\nu \mu}, \quad (100)$$

the total four-momentum content of $\Sigma(s) \cap C$, including the EM part coming from $\Theta$—the full canonical tensor this time. If we assume, as previously, that the flux of $p$ across $T$ is purely of EM origin, we arrive at the conclusion that, for a sufficiently isolated particle (or a bound aggregate of particles), the change in momentum can be read from the flux of the Poynting vector across a time-like surface surrounding it. Note that no approximation whatsoever is involved this time.

### E Spin-$\frac{1}{2}$ ECD

In a spin-$\frac{1}{2}$ version of ECD, the following modifications are made. The wave-function $\phi$ is a bispinor ($\mathbb{C}^4$-valued), transforming in a Lorentz transformation according to

$$\rho(e^\omega) \phi \equiv e^{-i/4 \sigma_{\mu\nu} \omega^{\mu\nu}} \phi, \quad \text{for } e^\omega \in SO(3, 1), \quad (101)$$

where $\sigma_{\mu\nu} = \frac{i}{2} \left[ \gamma_\mu, \gamma_\nu \right]$, with $\gamma_\mu$ Dirac matrices (not to be confused with $\gamma$ the trajectory).

The propagator is now a complex, $4 \times 4$ matrix, transforming under the adjoint representation, satisfying

$$i\hbar \partial_s G(x, x', s) = -\frac{1}{2} \hbar^2 G(x, x', s), \quad \hat{a} \equiv \gamma^\mu a_\mu, \quad (102)$$

with the initial condition (33) at $s \to 0$ reading $\delta^{(4)}(x - x') \delta_\alpha_\beta$, where $\delta_\alpha_\beta$ is the identity operator in spinor-space.
The transition to spin-$\frac{1}{2}$ ECD is rendered easy by the observation that all expressions in scalar ECD are sums of bilinears of the form $a^* b$, which can be seen as a Lorentz invariant scalar product in $\mathbb{C}^1$. Defining an inner product in spinor space (instead of $\mathbb{C}^1$)

$$(a, b) \equiv a^\dagger \gamma^0 b,$$  

with $\gamma^0$ the Dirac matrix diag$(1, 1, -1, -1)$ (again, not to be confused with $\gamma$ the trajectory) and substituting $a^* b \mapsto (a, b)$ in all bilinears, all the results of scalar ECD are retained. The Lorentz invariance of (103) follows from the Hermiticity of $\sigma^{\mu \nu}$ with respect to that inner product, viz. $(\sigma^{\mu \nu})^\dagger = \gamma^0 \sigma^{\mu \nu} \gamma^0$, and from $(\gamma^0)^2 = 1$.

Let us illustrate this procedure for important cases. By a direct calculation of the short-s propagator of (102), as in section A, the spin can be shown to affect the $O(s)$ terms in the expansion of $\Phi$, leading to an equally simple $\phi_s$, the counterpart of (45), from which the regular part of all ECD currents can be obtained. The action, (84), from which all conservation laws can be derived, is modified to

$L_m[\phi, A] = \int_\mathbb{M} d^4x \int ds \left[ (\phi, \partial_s \phi) - (\partial_s \phi, \phi) \right] - \frac{1}{2} \left( \mathcal{D} \phi, \mathcal{D} \phi \right), \quad (104)$

while the counterpart of the electric current, (34), derived from $\phi$, is now a sum of an ‘orbital current’ and a ‘spin current’

$$j^\mu(x) \equiv j^{\text{orb} \mu} + j^{\text{spn} \mu} = \lim_{\epsilon \to 0} \partial_\epsilon \epsilon^{-1} \int ds \, q \text{Im} \left( \phi, D^\mu \phi \right) - \overline{\hbar} \partial_\nu \left( \phi, \sigma^{\nu \mu} \phi \right), \quad \text{for } x \notin \bar{\gamma}. \quad (105)$$

Each of the terms composing $j$ is individually conserved and gauge invariant. The conservation of the sum is follows from the $U(1)$ invariance of (104), while conservation of the spin current follows directly from the antisymmetry of $\sigma$. This current has an interesting property that its monopole vanishes identically. Calculating in an arbitrary frame, using the antisymmetry of $\sigma$, and assuming $j^{\text{spn} \nu}(x) \to 0$ for $|x| \to \infty$

$$\int d^3x \, j^{\text{spn} 0} = \lim_{\epsilon \to 0} \partial_\epsilon \epsilon^{-1} \int d^3x \int ds \partial_0 (\phi, \sigma^{00} \phi) - \partial_0 (\phi, \sigma^{00} \phi) = 0 - 0 = 0. \quad (106)$$

As $\sum_k k^j$ generates $A$, we clearly have

$$\partial_\nu \Theta^{\nu \mu} + \sum_k F^\mu_\nu \left( k^j \text{orb} \nu + k^j \text{spn} \nu \right) = 0, \quad \text{for } x \notin \bar{\gamma}. \quad (107)$$

Repeating the procedure from section C.2 with the modified action, (104), we get a conserved e-m tensor, $p$. However, the simple symmetrization trick used for the scalar case doesn’t work in the current case and a symmetric, gauge invariant $p$, is more easily derived via a flat-space limit of a the fully generally covariant $p$, guaranteed to be both. This is a straightforward exercise in GR involving spinor fields.

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