Conditions for a pure toroidal dipole source

Adrià Canós Valero

ITMO University, Kronversky Prospect 49, 197101, St Petersbourg Russia
adria.canos@optomech.ifmo.ru

Abstract. Recently, the physical significance of dynamic toroidal multipoles in the context of electrodynamics has been put under discussion. Indeed, the latter can be shown to arise simply from a Taylor series of the exact source (Cartesian) multipole moments. The split into elementary and toroidal parts was demonstrated to lead to an unphysical result were forbidden components of the momentum transform of the current could radiate into free space. In this contribution, we elaborate the conditions that a current distribution must necessarily satisfy to be considered a 'pure' toroidal dipole source. We demonstrate for the first time that symmetry prevents such current distribution to radiate as an elementary electric dipole moment, without leading to an unphysical result. Thus, while both elementary electric dipole and toroidal dipoles are indistinguishable outside the source, they display topologically distinct characteristics within the smallest spherical surface enclosing the source itself and have different physical origin. Based on our results, a pure 'toroidal' source can be designed. We believe the outcome of our investigations will help clarify further the formal meaning of the toroidal multipoles.

INTRODUCTION

In modern physics, multipole expansions are commonly used tools to understand a broad range of phenomena in optics, nuclear physics and condensed matter [1,2]. It is nowadays well-known that the so-called ‘Cartesian’ or multipole expansions of the source, which arise in a Taylor series of the vector potential [3,4], require the introduction of dynamic toroidal moments [4]. The most well-known constituent of the toroidal family is the toroidal dipole, first identified [5] and observed as a static current distribution. It was later shown to naturally give rise to a different form of long-range order in ferroics violating both space and time inversion symmetries [6], and predicted to have a strong impact in the electronic properties of several molecular structures [2]. Nowadays, dynamic toroidal moments in electrodynamics are a subject of renewed scientific and technological interest in the growing field of all-dielectric nanophotonics [7–12] due to their role in the formation of the nonradiating sources known as anapoles [13].

Their physical relevance, as well as their formal meaning in electrodynamics, is still however a matter of on-going discussion [14,15]. Corbaton et.al. rigorously showed in [15] how the toroidal moments appear as high order terms of a Taylor series of the exact source multipoles. Taking as a case example the electric and electric toroidal dipole, the split into elementary and toroidal terms leads to an unphysical result if considered separately, i.e. the toroidal dipole cannot, in fact, be regarded differently from the elementary electric dipole moment.

In this work, we show that such a restriction can in fact be circumvented under a few special conditions, which effectively allow a ‘pure’ naturally broadband electric toroidal dipole to arise in the absence of the elementary electric dipole. With our analysis, we demonstrate that, while indistinguishable for an observer outside the source, the physical
origin of the elementary electric dipole and the toroidal dipole are different and must be distinguished when investigating the effects taking place within the volume of the smallest sphere surrounding the source.

**GENERAL THEORETICAL CONSIDERATIONS**

We start with the dipolar vector of electric parity in the exact form [15]:

\[
a_i^\alpha = -\frac{1}{i\omega} \left\{ \int d^3r J_\omega j_0(kr) + \frac{k^2}{2} \int d^3r \left[ 3(r \cdot J_\omega) r - r^2 J_\omega \right] j_2(kr) \right\}.
\]  

(1)

The Bessel functions within the integral act as ‘filters’ for the momentum components of the current, such that only those with total momentum \(|\mathbf{p}| = \omega / c\) contribute to radiation. A Taylor series of Eq.(1) leads to the well-known expressions of the elementary dipole \(p^\alpha\) and toroidal dipole \(t^\alpha\) moments:

\[
a_i^\alpha = p_i^\alpha + t_i^\alpha \approx -\frac{1}{i\omega} \left\{ \int d^3r J_\omega j_0(kr) + \frac{k^2}{2} \int d^3r \left[ (r \cdot J_\omega) r - 2r^2 J_\omega \right] j_2(kr) \right\}.
\]  

(2)

In what follows, it will be convenient to recall the expression for the momentum transform of a localized current distribution:

\[
J_\omega(\mathbf{p}) = \frac{1}{(2\pi)^3} \int d^3r J_\omega \exp(-i\mathbf{p} \cdot \mathbf{r}).
\]  

(3)

Comparing Eqs.(2)-(3) one immediately notices that \(p_1^\alpha\) corresponds to \(J_\omega(\mathbf{p} = 0)\), which cannot by definition radiate into free space. Therefore, the toroidal terms in the Taylor series of Eq. (1) are linked to the first order term and must necessarily contain the \(J_\omega(\mathbf{p} = 0)\) that cancels the first one when summed together. Altogether, from the previous authors in Ref. [15] concluded that, since each term in the expansion contains nonradiative terms, the elementary dipole and the toroidal dipole moment couple identically to electromagnetic radiation, and, as such, there is no need for separating their contributions outside the source.

Here, we focus however on the way that the elementary (electric) and toroidal terms are excited within the source itself. Within the smallest spherical volume enclosing the source, there is no restriction as to which components \(J_\omega(\mathbf{p})\) can contribute to the electromagnetic fields, and therefore the previous arguments do not apply.

**RESULTS AND DISCUSSION**

First, we focus on a simple way to obtain a pure ‘toroidal’ source, i.e. with \(p_1^\alpha = 0\). A current distribution validating this condition would have \(J_\omega(\mathbf{p} = 0) = 0\), therefore allowing the toroidal part to radiate by itself. This idea was discussed in Ref. [15] and discarded as trivial, since in the far field pure toroidal radiation cannot be distinguished from that of the elementary dipole. However, a current distribution validating such condition has a very well-defined topology, very different from that of an elementary electric dipole. Such a current must, in fact, not only be divergence-less \(\nabla \cdot J_\omega = 0\) and poloidal \(\mathcal{L} \cdot J_\omega = 0\), where \(\mathcal{L}\) is the orbital angular momentum operator), but also it must not establish a surface charge density \(\sigma_{e_\omega}\) at the boundary of the source volume. To satisfy these requirements, we propose a current distribution of the form:

\[
J_\omega(p, \theta) = \frac{K\mathcal{H}(\Delta \rho)}{R_0 + p\cos\theta} e_\theta.
\]  

(4)
FIGURE 1. Scheme depicting a cross-section of the considered toroidal current distribution. The toroidal coordinate system utilized for the parametrization of Eq. (4) is also illustrated.

This current can be rigorously obtained from imposing the conditions above in a toroidal coordinate system as depicted in Fig.1, where $K$ is a constant and $H(\Delta \rho)$ is a step function in the coordinate $\rho$. Eq.(4) effectively corresponds to a current flowing along the poloidal axis $e_\theta$ of a torus. We remark that this result is independent on the extension of the torus. Indeed, even if the dimensions of the source are larger than the wavelength, an elementary electric dipole will never occur in this system. In addition, since the current is poloidal, the system can also never support magnetic moments; they are proportional to $L \cdot J_{\theta \rho}$, and therefore zero.

We now obtain analytical results for the expected behavior of the toroidal part. Indeed, integrating Eq. (2) with Eq. (4) yields $p_{10}^\omega = 0$, while toroidal moments remain non-zero. The toroidal dipole takes the form:

$$t_{1}\omega = \frac{k^2 \pi^2}{3} K p_0 R_0 e_z.$$  \hspace{1cm} (5)

CONCLUSION

We have discussed the possibility to excite the dynamic toroidal dipole in the absence of the elementary electric dipole and elaborate a sufficient set of conditions that a current distribution must satisfy in order to support it. Based on the latter, we propose a current easily imprintable on the surface of a metallic body that cannot, by definition, excite an elementary electric dipole. Outside the smallest spherical surface enclosing the source, the system cannot be distinguished from that of an elementary electric dipole. Inside, however, the topology of the fields and associated electromagnetic quantities differ significantly.

ACKNOWLEDGEMENTS

The author acknowledges the support of grant Num. 20-52-00031 of the Russian Foundation for Basic Research (RFBR). This project is inscribed within the Grant of the President of the Russian Federation.

REFERENCES

[1] Zimmermann A S, Meier D and Fiebig M 2014 Ferroic nature of magnetic toroidal order Nat. Commun. 5 1–6
[2] Ceulemans A, Chibotaru L F and Fowler P W 1998 Molecular anapole moments Phys. Rev. Lett. 80 1861–4
[3] Dubovik V M and Tugushev V V. 1990 Toroid moments in electrodynamics and solid-state physics Phys. Rep. 187 145–202
[4] Gurvitz E A, Ladutenko K S, Dergachev P A, Evlyukhin A B, Miroshnichenko A E and Shalin A S 2019 The High-Order Toroidal Moments and Anapole States in All-Dielectric Photonics Laser Photon. Rev. 13
1800266

[5] Zel’Dovich 1957 Electromagnetic interaction with parity violation Sov. J. Exp. Theor. Phys. 6 1184–6

[6] Hayami S, Kusunose H and Motome Y 2014 Toroidal order in metals without local inversion symmetry Phys. Rev. B - Condens. Matter Mater. Phys. 90 1–12

[7] Baryshnikova K, Filonov D, Simovski C, Evlyukhin A, Kadochkin A, Nenasheva E, Ginzburg P and Shalin A S 2018 Giant magnetoelectric field separation via anapole-type states in high-index dielectric structures Phys. Rev. B 98 1–9

[8] Kostina N, Ivinskaya A, Sukhov S, Bogdanov A, Toftul I, Nieto-Vesperinas M, Ginzburg P, Petrov M and Shalin A 2017 Optical binding via surface plasmon polariton interference Phys. Rev. B 99 125416

[9] Terekhov P D, Shamkhi H K, Gurvitz E A, Baryshnikova K V., Evlyukhin A B, Shalin A S and Karabchevsky A 2019 Broadband forward scattering from dielectric cubic nanoantenna in lossless media Opt. Express 27 10924

[10] Kozlov V, Filonov D, Shalin A S, Steinberg B Z and Ginzburg P 2016 Asymmetric backscattering from the hybrid magneto-electric meta particle Appl. Phys. Lett. 109 203503

[11] Terekhov P D, Baryshnikova K V., Greenberg Y, Fu Y H, Evlyukhin A B, Shalin A S and Karabchevsky A 2019 Enhanced absorption in all-dielectric metasurfaces due to magnetic dipole excitation Sci. Rep. 9 3438

[12] Shalin A S and Moiseev S G 2009 Optical properties of nanostructured layers on the surface of an underlying medium Opt. Spectrosc. 106 916–25

[13] Valero A C, Gurvitz E A, Benimetskiy F A, Pidgayko D A, Samusev A, Evlyukhin A B, Redka D, Tribelsky M I, Rahmani M, Kamali K Z, Pavlov A A, Miroshnichenko A E and Shalin A S 2020 Theory, observation and ultrafast response of novel hybrid anapole states

[14] Papasimakis N, Fedotov V A, Savinov V, Raybould T A and Zheludev N I 2016 Electromagnetic toroidal excitations in matter and free space Nat. Mater. 15 263–71

[15] Fernandez-Corbaton I, Nanz S and Rockstuhl C 2017 On the dynamic toroidal multipoles from localized electric current distributions Sci. Rep. 7