Finsler–Lagrange Geometries and Standard Theories in Physics: New Methods in Einstein and String Gravity

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Abstract

In this article, we review the current status of Finsler–Lagrange geometry and generalizations. The goal is to aid non–experts on Finsler spaces, but physicists and geometers skilled in general relativity and particle theories, to understand the crucial importance of such geometric methods for applications in modern physics. We also would like to orient mathematicians working in generalized Finsler and Kähler geometry and geometric mechanics how they could perform their results in order to be accepted by the community of ”orthodox” physicists.

Although the bulk of former models of Finsler–Lagrange spaces where elaborated on tangent bundles, the surprising result advocated in our works is that such locally anisotropic structures can be modelled equivalently on Riemann–Cartan spaces, even as exact solutions in Einstein and/or string gravity, if nonholonomic distributions and moving frames of references are introduced into consideration.

We also propose a canonical scheme when geometrical objects on a (pseudo) Riemannian space are nonholonomically deformed into generalized Lagrange, or Finsler, configurations on the same manifold or on a corresponding tangent bundle. Such canonical transforms are defined by the coefficients of a prime metric (it can be a solution of the Einstein equations) and generate target spaces as generalized Lagrange structures, their models of almost Hermitian/ Kähler, or nonholonomic Riemann spaces with constant curvature, for some Finsler like connections. There are formulated the criteria when such constructions can be redefined equivalently in terms of the Levi Civita connection.

Finally, we consider some classes of exact solutions in string and Einstein gravity modelling Lagrange–Finsler structures with solitonic pp–waves and speculate on their physical meaning.

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1 Introduction

The main purpose of this survey is to present an introduction to Finsler–Lagrange geometry and the anholonomic frame method in general relativity and gravitation. We review and discuss possible applications in modern physics and provide alternative constructions in the language of the geometry of nonholonomic Riemannian manifolds (enabled with nonintegrable distributions and preferred frame structures). It will be emphasized the approach when Finsler like structures are modelled in general relativity and gravity theories with metric compatible connections and, in general, non-trivial torsion.

Usually, gravity and string theory physicists may remember that Finsler geometry is a quite "sophisticate" spacetime generalization when Riemannian metrics $g_{ij}(x^k)$ are extended to Finsler metrics $g_{ij}(x^k, y^l)$ depending both on local coordinates $x^k$ on a manifold $M$ and "velocities" $y^l$ on its tangent bundle $TM$. Perhaps, they will say additionally that in order to describe local anisotropies depending only on directions given by vectors $y^l$, the Finsler metrics should be defined in the form $g_{ij} \sim \frac{\partial F^2}{\partial y^i \partial y^j}$, where

\footnote{We emphasize that Finsler geometries can be alternatively modelled if $y^l$ are considered as certain nonholonomic, i. e. constrained, coordinates on a general manifold $V$, not only as "velocities" or "momenta", see further constructions in this work.}
$F(x^k, \zeta y^l) = |\zeta| F(x^k, y^l)$, for any real $\zeta \neq 0$, is a fundamental Finsler metric function. A number of authors analyzing possible locally anisotropic physical effects omit a rigorous study of nonlinear connections and do not reflect on the problem of compatibility of metric and linear connection structures. If a Riemannian geometry is completely stated by its metric, various models of Finsler spaces and generalizations are defined by three independent geometric objects (metric and linear and nonlinear connections) which in certain canonical cases are induced by a fundamental Finsler function $F(x, y)$. For models with different metric compatibility, or non-compatibility, conditions, this is a point of additional geometric and physical considerations, new terminology and mathematical conventions. Finally, a lot of physicists and mathematicians have concluded that such geometries with generic local anisotropy are characterized by various types of connections, torsions and curvatures which do not seem to have physical meaning in modern particle theories but (may be?) certain Finsler like analogs of mechanical systems and continuous media can be constructed.

There were published a few rigorous studies on perspectives of Finsler like geometries in standard theories of gravity and particle physics (see, for instance, Refs. [26, 247]) but they do not analyze any physical effects of the nonlinear connection and adapted linear connection structures and the possibility to model Finsler like spaces as exact solutions in Einstein and string gravity [228]). The results of such works, on Finsler models with violations of local Lorentz symmetry and nonmetricity fields, can be summarized in a very pessimistic form: both fundamental theoretic consequences and experimental data restrict substantially the importance for modern physics of locally anisotropic geometries elaborated on (co) tangent bundles, see Introduction to monograph [228] and article [201] and reference therein for more detailed reviews and discussions.

Why we should give a special attention to Finsler geometry and methods and apply them in modern physics? We list here a set of contr–arguments and discuss the main sources of ”anti–Finsler” skepticism which (we hope) will explain and re–move the existing unfair situation when spaces with generic local anisotropy are not considered in standard theories of physics:

1. One should be emphasized that in the bulk the criticism on locally anisotropic geometries and applications in standard physics was motivated only for special classes of models on tangent bundles, with violation of local Lorentz symmetry (even such works became very

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2 In result of such opinions, the Editors and referees of some top physical journals almost stopped to accept for publication manuscripts on Finsler gravity models. If other journals were more tolerant with such theoretical works, they were considered to be related to certain alternative classes of theories or to some mathematical physics problems with speculations on geometric models and ”nonstandard” physics, mechanics and some applications to biology, sociology or seismology etc
important in modern physics, for instance, in relation to brane gravity [51] and quantum theories [89]) and nonmetricity fields. Not all theories with generalized Finsler metrics and connections were elaborated in this form (on alternative approaches, see next points) and in many cases, like [26, 247], the analysis of physical consequences was performed not following the nonlinear connection geometric formalism and a tensor calculus adapted to nonholonomic structures which is crucial in Finsler geometry and generalizations.

2. More recently, a group of mathematicians [17, 147] developed intensively some directions on Finsler geometry and applications following the Chern's linear connection formalism proposed in 1948 (this connection is with vanishing torsion but noncompatible with the metric structure). For non–experts in geometry and physics, the works of this group, and other authors working with generalized local Lorentz symmetries, created a false opinion that Finsler geometry can be elaborated only on tangent bundles and that the Chern connection is the "best" Finsler generalization of the Levi Civita connection. A number of very important constructions with the so–called metric compatible Cartan connection, or other canonical connections, were moved on the second plan and forgotten. One should be emphasized that the geometric constructions with the well known Chern or Berwald connections can not be related to standard theories of physics because they contain nonmetricity fields. The issue of nonmetricity was studied in details in a number of works on metric–affine gravity, see review [70] and Chapter I in the collection of works [228], the last one containing a series of papers on generalized Finsler–affine spaces. Such results are not widely accepted by physicists because of absence of experimental evidences and theoretical complexity of geometric constructions. Here we note that it is a quite sophisticate task to elaborate spinor versions, supersymmetric and noncommutative generalizations of Finsler like geometries if we work with metric noncompatible connections.

3. A non–expert in special directions of differential geometry and geometric mechanics, may not know that beginning E. Cartan (1935) [42] various models of Finsler geometry were developed alternatively by using metric compatible connections which resulted in generalizations to the geometry of Lagrange and Hamilton mechanics and their higher order extensions. Such works and monographs were published by prominent schools and authors on Finsler geometry and generalizations from Romania and Japan [107, 108, 103, 104, 112, 106, 85, 86, 80, 159, 245, 102, 124, 125, 127, 128, 129, 24, 25] following approaches quite different from the geometry of sympletic mechanics and generalizations [97, 99, 101, 91]. As a matter of principle, all geometric constructions
with the Chern and/or sympletic connections can be redefined equivalently for metric compatible geometries, but the philosophy, aims, mathematical formalism and physical consequences are very different for different approaches and the particle physics researches usually are not familiar with such results.

4. It should be noted that for a number of scientists working in Western Countries there are less known the results on the geometry of nonholonomic manifolds published in a series of monographs and articles by G. Vrănteanu (1926), Z. Horak (1927) and others [241, 243, 73], see historical remarks and bibliography in Refs. [25, 228]. The importance for modern physics of such works follows from the idea and explicit proofs (in quite sophisticated component forms) that various types of locally anisotropic geometries and physical interactions can be modelled on usual Riemannian manifolds by considering nonholonomic distributions and holonomic fibrations enabled with certain classes of special connections.

5. In our works (see, for instance, reviews and monographs [178, 179, 180, 181, 183, 225, 229, 230, 201, 228, and references therein), we re-oriented the research on Finsler spaces and generalizations in some directions connected to standard models of physics and gauge, supersymmetric and noncommutative extensions of gravity. Our basic idea was that both the Riemann–Cartan and generalized Finsler–Lagrange geometries can be modelled in a unified manner by corresponding geometric structures on nonholonomic manifolds. It was emphasized, that prescribing a preferred nonholonomic frame structure (equivalently, a nonintegrable distribution with associated nonlinear connection) on a manifold, or on a vector bundle, it is possible to work equivalently both with the Levi Civita and the so–called canonical distinguished connection. We provided a number of examples when Finsler like structures and geometries can be modelled as exact solutions in Einstein and string gravity and proved that certain geometric methods are very important, for instance, in constructing new classes of exact solutions.

This review work has also pedagogical scopes. We attempt to cover key aspects and open issues of generalized Finsler–Lagrange geometry related to a consistent incorporation of nonlinear connection formalism and moving/deformation frame methods into the Einstein and string gravity and analogous models of gravity, see also Refs. [201, 228, 108, 24, 145] for general reviews, written in the same spirit as the present one but in a more comprehensive, or inversely, with more special purposes forms. While the article is essentially self–contained, the emphasis is on communicating the underlying ideas and methods and the significance of results rather than on presenting
systematic derivations and detailed proofs (these can be found in the listed literature).

The subject of Finsler geometry and applications can be approached in different ways. We choose one of which is deeply rooted in the well-established gravity physics and also has sufficient mathematical precision to ensure that a physicist familiar with standard textbooks and monographs on gravity [69, 113, 244, 158, 151] and string theory [53, 139, 148] will be able without much efforts to understand recent results and methods of the geometry of nonholonomic manifolds and generalized Finsler–Lagrange spaces. In other turn, in order to keep the article to a reasonable size, and avoid overwhelming non–experts, we have to leave out several interesting topics, results and viewpoints. We list the most important alternative directions and comment references in Appendix in order to orient experts in gravity and field theories in existing literature and researches related to applications of Finsler geometry methods in modern physics. This is meant that the work is an introduction into some subjects and new geometric methods which seem to be very important in standard physics rather then an exhaustive review of them.

We shall use the terms "standard" and "nonstandard" models in geometry and physics. In connection to Finsler geometry, we shall consider a model to be a standard one if it contains locally anisotropic structures defined by certain nonholonomic distributions and adapted frames of reference on a (pseudo) Riemannian or Riemann–Cartan space (for instance, in general relativity, Kaluza–Klein theories and low energy string gravity models). Such constructions preserve, in general, the local Lorentz symmetry and they are performed with metric compatible connections. The term "nonstandard" will be used for those approaches which are related to metric non–compatible connections and/or local Lorentz violations in Finsler spacetimes and generalizations. Sure, any standard or nonstandard model is rigorously formulated following certain purposes in modern geometry and physics, geometric mechanics, biophysics, locally anisotropic thermodynamics and stochastic and kinetic processes and classical or quantum gravity theories. Perhaps, it will be the case to distinguish the class of "almost standard" physical models with locally anisotropic interactions when certain geometric objects from a (pseudo) Riemannian or Riemann–Cartan manifolds are lifted on a (co) tangent or vector bundles and/or their supersymmetric, non–commutative, Lie algebroid, Clifford space, quantum group ... generalizations. There are possible various effects with "nonstandard" corrections, for instance, violations of the local Lorentz symmetry by quantum effects but in some classical or quantum limits such theories are constrained to correspond to certain standard ones.

This contribution is organized as follows:

In section 2, we outline an unified approach to the geometry of nonholonomic distributions on Riemann manifolds and Finsler–Lagrange spaces.
The basic concepts on nonholonomic manifolds and associated nonlinear connection structures are explained and the possibility of equivalent (non)holonomic formulations of gravity theories is analyzed.

Section 3 is devoted to nonholonomic deformations of manifolds and vector bundles. There are reviewed the basic constructions in the geometry of (generalized) Lagrange and Finsler spaces. We show how effective algebroid structures can be generated by nonholonomic transforms. A general ansatz for constructing exact solutions, with effective (algebroid) Lagrange and Finsler structures, in Einstein and string gravity, is analyzed.

In section 4, the Finsler–Lagrange geometry is formulated as a variant of almost Hermitian and/or Kähler geometry with additional Lie algebroid structure. We show how the Einstein gravity can be equivalently reformulated in terms of almost Hermitian geometry with preferred frame structure.

Section 5 is focused on explicit examples of exact solutions in Einstein and string gravity when (generalized) Finsler–Lagrange structures are modelled on (pseudo) Riemannian and Riemann–Cartan spaces. We analyze some classes of Einstein metrics which can be deformed into new exact solutions characterized additionally by Lagrange–Finsler configurations. For string gravity, there are constructed explicit examples of locally anisotropic configurations describing gravitational solitonic pp–waves and their effective Lagrange spaces. We also analyze some exact solutions for Finsler–solitonic pp–waves on Schwarzschild spaces.

Conclusions and further perspectives of Finsler geometry and new geometric methods for modern gravity theories are considered in section 6.

We provide an Appendix containing historical and bibliographical comments on (generalized) Finsler geometry and physics.

Finally, we should note that our list of references is minimalist, trying to concentrate on reviews and monographs rather than on original articles. More complete reference lists can be found in the books [228, 183, 225, 108, 112]. Various guides for learning, both for experts and beginners on geometric methods and further applications in modern physics, with references, can be found in [228, 108, 112, 24, 145].

Notational remarks:

We shall consider geometric and physical objects on different spaces. There were elaborated very sophisticate systems of denotations and terminology in various approaches to general relativity, string theory and generalized Finsler–Lagrange geometry. In this work, one follows the conventions from [228, 201]. We shall use "boldface" letters, \( \mathbf{A}, \mathbf{B}^{\alpha}, \ldots \) for geometric objects and spaces adapted to (provided with) a nonlinear connection structure. In general, small Greek indices are considered as abstract ones, which may split into horizontal (h) and vertical (v) indices, for instance \( \alpha = (i, a), \beta = (j, b), \ldots \) where with respect to a coordinate basis they run values of type...
\[i, j, ... = 1, 2, ..., n\] and \(a, b, ... = n + 1, n + 2, ... n + m,\) for \(n \geq 2\) and \(m \geq 1.\) One shall be considered primed indices, \(\alpha' = (i', a'), \beta' = (j', b'),\) working with respect to a nonholonomically transformed bases, or underlined indices, \(\underline{\alpha} = (i, \underline{a}), \underline{\beta} = (j, \underline{b}),\) in order to emphasize that coefficients of geometric objects are defined with respect to a coordinate basis. Various types of left "up" and "low" labels of geometric objects will be used, for instance, \(RC\) means that the manifold \(V\) is a Riemann–Cartan one, the Levi Civita connection will be labelled \(\bar{\nabla} = \nabla\) and the corresponding Riemannian and Ricci tensors will be written \(\bar{\mathcal{R}} = \{ \bar{R}_{\beta\gamma} \},\) and \(\bar{\text{Ric}}(\bar{\nabla}) = \{ \bar{R}_{\beta\gamma} \}.\) We shall omit labels and indices if that will not result in ambiguities. Finally, we note that we shall write with boldface letters a new term if it is introduced for the first time in the text.

2 Nonholonomic Einstein Gravity and Finsler–Lagrange Spaces

In this section we present in a unified form the Riemann–Cartan and Finsler–Lagrange geometry. The reader is supposed to be familiar with well–known geometrical approaches to gravity theories \([69, 113, 244, 158, 151]\) but may not know the basic concepts on Finsler geometry and nonholonomic manifolds. The constructions for locally anisotropic spaces will be derived by special parametrizations of the frame, metric and connection structures on usual manifolds, or vector bundle spaces, as we proved in details in Refs. \([228, 201].\)

2.1 Metric–affine, Riemann–Cartan and Einstein manifolds

Let \(V\) be a necessary smooth class manifold of dimension \(\text{dim} V = n + m,\) when \(n \geq 2\) and \(m \geq 1,\) enabled with \textbf{metric}, \(g = g_{\alpha\beta} e^\alpha \otimes e^\beta,\) and \textbf{linear connection}, \(D = \{ \Gamma^\alpha_{\beta\gamma} \},\) structures. The coefficients of \(g\) and \(D\) can be computed with respect to any local \textbf{frame}, \(e_{\alpha},\) and \textbf{co–frame}, \(e^\beta,\) bases, for which \(e_{\alpha} \cdot e^\beta = \delta^\alpha_\beta,\) where \(\cdot\) denotes the interior (scalar) product defined by \(g\) and \(\delta^\alpha_\beta\) is the Kronecker symbol. A local system of coordinates on \(V\) is denoted \(u_{\alpha} = (x^i, y^a),\) or (in brief) \(u = (x, y),\) where indices run correspondingly the values: \(i, j, k, ... = 1, 2, ..., n\) and \(a, b, c, ..., = n + 1, n + 2, ... n + m\) for any splitting \(\alpha = (i, a), \beta = (j, b), ...\) We shall also use primed, underlined, or other type indices: for instance, \(e_{\alpha'} = (e_i', e_a')\) and \(e'^\beta = (e^j', e^b'),\) for a different sets of local (co) bases, or \(\underline{e}_{\alpha} = e_{\underline{a}} = \partial_{\underline{a}} = \partial/\partial u^{\underline{a}},\) \(\underline{e}^\beta = e_i = \partial_{\underline{a}} = \partial/\partial u_{\underline{a}},\) \(\underline{e}_{\alpha} = e_{\underline{a}} = \partial_{\underline{a}} = \partial/\partial y^{\underline{a}}\) if we wont to emphasize that the coefficients of geometric objects (tensors, connections, ...) are defined with respect to a local \textbf{coordinate basis}. For simplicity, we shall omit underlining or priming of indices and symbols if that will not result in ambiguities. The Einstein’s summation rule on repeating ”up-low” indices will be applied.
Frame transforms of a local basis $e_\alpha$ and its dual basis $e^\beta$ are parametrized in the form
\[
e_\alpha = A_\alpha^\alpha'(u)e_{\alpha'} \quad \text{and} \quad e^\beta = A^\beta_\beta'(u)e^{\beta'},
\]
where the matrix $A^\beta_\beta'$ is inverse to $A_\alpha^\alpha'$. In general, local bases are nonholonomic (equivalently, anholonomic, or nonintegrable) and satisfy certain anholonomy conditions
\[
e_\alpha e_\beta - e_\beta e_\alpha = W^\gamma_{\alpha\beta}e_\gamma
\]
with nontrivial anholonomy coefficients $W^\gamma_{\alpha\beta}(u)$. We consider the holonomic frames to be defined by $W^\gamma_{\alpha\beta} = 0$, which holds, for instance, if we fix a local coordinate basis.

Let us denote the covariant derivative along a vector field $X = X^\alpha e_\alpha$ as $D_X = D$. One defines three fundamental geometric objects on manifold $V$: nonmetricity field,
\[
Q_X \equiv D_X g,
\]
torsion,
\[
T(X,Y) \equiv D_X Y - D_Y X - [X,Y],
\]
and curvature,
\[
R(X,Y)Z \equiv D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z,
\]
where the symbol "\(\equiv\)" states "by definition" and $[X,Y] \equiv XY - YX$. With respect to fixed local bases $e_\alpha$ and $e^\beta$, the coefficients $Q = \{Q_{\alpha\beta\gamma} = D_\alpha g_{\beta\gamma}\}, T = \{T^\alpha_{\beta\gamma}\}$ and $R = \{R^n_{\alpha\beta\gamma}\}$ can be computed by introducing $X \rightarrow e_\alpha, Y \rightarrow e_\beta, Z \rightarrow e_\gamma$ into respective formulas (3), (4) and (5).

In gravity theories, one uses three others important geometric objects: the Ricci tensor, $Ric(D) = \{R_{\beta\gamma} = \bar{R}^\alpha_{\beta\gamma}\}$, the scalar curvature, $R \equiv g^{\alpha\beta}R_{\alpha\beta}$ ($g^{\alpha\beta}$ being the inverse matrix to $g_{\alpha\beta}$), and the Einstein tensor, $\mathcal{E} = \{E_{\alpha\beta} = \bar{R}_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R\}$.

A manifold $\text{ma}V$ is a metric–affine space if it is provided with arbitrary two independent metric $g$ and linear connection $D$ structures and characterized by three nontrivial fundamental geometric objects $Q, T$ and $R$.

If the metricity condition, $Q = 0$, is satisfied for a given couple $g$ and $D$, such a manifold $\text{RC}V$ is called a Riemann–Cartan space with nontrivial torsion $T$ of $D$.

A Riemann space $\text{RV}$ is provided with a metric structure $g$ which defines a unique Levi Civita connection $\bar{D} = \nabla$, which is both metric compatible, $\mathcal{Q} = \nabla g = 0$, and torsionless, $\mathcal{T} = 0$. Such a space is pseudo- (semi-
Riemannian if locally the metric has any mixed signature \((\pm 1, \pm 1, \ldots, \pm 1)\). In brief, we shall call all such spaces to be Riemannian (with necessary signature) and denote the main geometric objects in the form \(\mathcal{R} = \{\mathcal{R}^\alpha_{\beta\gamma\tau}\}, \mathcal{Ric}(\mathcal{D}) = \{\mathcal{R}_{\beta\gamma}\}, \mathcal{R} \) and \(\mathcal{E} = \{\mathcal{E}_{\alpha\beta}\} \).

The **Einstein gravity theory** is constructed canonically for \(\dim R^V = 4\) and Minkowski signature, for instance, \((-1, +1, +1, +1)\). Various generalizations in modern **string and/or gauge gravity** consider Riemann, Riemann–Cartan and metric–affine spaces of higher dimensions.

The **Einstein equations** are postulated in the form

\[
\mathcal{E}(D) \triangleq \mathcal{Ric}(D) - \frac{1}{2} \mathcal{g} \mathcal{S}c(D) = \Upsilon,
\]

where the source \(\Upsilon\) contains contributions of matter fields and corrections from, for instance, string/brane theories of gravity. In a physical model, the equations (6) have to be completed with equations for the matter fields and torsion (for instance, in the **Einstein–Cartan theory** [70], one considers algebraic equations for the torsion and its source). It should be noted here that because of possible nonholonomic structures on a manifold \(V\) (we shall call such spaces to be locally anisotropic), see next section, the tensor \(\mathcal{Ric}(D)\) is not symmetric and \(D[\mathcal{E}(D)] \neq 0\). This imposes a more sophisticated form of conservation laws on spaces with generic "local anisotropy", see discussion in [228] (a similar situation arises in Lagrange mechanics [97, 99, 101, 91, 108] when nonholonomic constraints modify the definition of conservation laws).

For **general relativity**, \(\dim V = 4\) and \(D = \nabla\), the field equations can be written in the well–known component form

\[
\mathcal{E}_{\alpha\beta} = \mathcal{R}_{\beta\gamma} - \frac{1}{2} \mathcal{R} = \Upsilon_{\alpha\beta}
\]

when \(\nabla(\mathcal{E}_{\alpha\beta}) = \nabla(\Upsilon_{\alpha\beta}) = 0\). The coefficients in equations (7) are defined with respect to arbitrary nonholonomic frame [11].

### 2.2 Nonholonomic manifolds and adapted frame structures

A **nonholonomic manifold** \((M, D)\) is a manifold \(M\) of necessary smooth class enabled with a nonholonomic distribution \(D\), see details in Refs. [25, 228]. Let us consider a \((n + m)\)–dimensional manifold \(\mathbf{V}\), with \(n \geq 2\) and \(m \geq 1\) (for a number of physical applications, it will be considered to model a physical or geometric space). In a particular case, \(\mathbf{V} = TM\), with \(n = m\) (i.e. a tangent bundle), or \(\mathbf{V} = \mathbf{E} = (E, M)\), \(\dim M = n\), is a vector bundle on \(M\), with total space \(E\) (we shall use such spaces for traditional definitions of Finsler and Lagrange spaces [107, 108, 102, 24, 145, 17, 147]). In a

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3 mathematicians usually use the term semi–Riemannian but physicists are more familiar with pseudo–Riemannian; we shall apply both terms on convenience.
general case, a manifold $V$ is provided with a local fibred structure into conventional "horizontal" and "vertical" directions defined by a nonholonomic (nonintegrable) distribution with associated nonlinear connection (equivalently, nonholonomic frame) structure. Such nonholonomic manifolds will be used for modelling locally anisotropic structures in Einstein gravity and generalizations \cite{181, 229, 230, 201, 228}.

### 2.2.1 Nonlinear connections and $N$–adapted frames

We denote by $\pi^\top : TV \to TM$ the differential of a map $\pi : V \to V$ defined by fiber preserving morphisms of the tangent bundles $TV$ and $TM$. The kernel of $\pi^\top$ is just the vertical subspace $vV$ with a related inclusion mapping $i : vV \to TV$.

A **nonlinear connection** ($N$–connection) $N$ on a manifold $V$ is defined by the splitting on the left of an exact sequence

$$0 \to vV \xrightarrow{i} TV \to TV/vV \to 0,$$

i.e. by a morphism of submanifolds $N : TV \to vV$ such that $N \circ i$ is the unity in $vV$.

Locally, a $N$–connection is defined by its coefficients $N_i^a(u)$,

$$N = N_i^a(u)dx^i \otimes \frac{\partial}{\partial y^a}. \quad (8)$$

In an equivalent form, we can say that any $N$–connection is defined by a **Whitney sum** of conventional horizontal (h) space, $(hV)$, and vertical (v) space, $(vV)$,

$$TV = hV \oplus vV. \quad (9)$$

The sum (9) states on $TV$ a nonholonomic (equivalently, anholonomic, or nonintegrable) distribution of h- and v–space. The well known class of linear connections consists on a particular subclass with the coefficients being linear on $y^a$, i.e.

$$N_i^a(u) = \Gamma_{ij}^a(x)y^j. \quad (10)$$

The geometric objects on $V$ can be defined in a form adapted to a $N$–connection structure, following decompositions which are invariant under parallel transports preserving the splitting (9). In this case, we call them to be distinguished (by the $N$–connection structure), i.e. **d–objects**. For instance, a vector field $X \in TV$ is expressed

$$X = (hX, vX), \text{ or } X = X^a e_a = X^i e_i + X^a e_a,$$

where $hX = X^i e_i$ and $vX = X^a e_a$ state, respectively, the adapted to the $N$–connection structure horizontal (h) and vertical (v) components of the
In brief, \( X \) is called a distinguished vector, \( d \)-vector. In a similar fashion, the geometric objects on \( V \like tensors, spinors, connections, ... \) are called respectively \( d \)-tensors, \( d \)-spinors, \( d \)-connections if they are adapted to the \( N \)-connection splitting (9).

The \( N \)-connection curvature is defined as the Neijenhuis tensor

\[
\Omega(X, Y) \doteq v[X, vY] + v[X, Y] - v[vX, Y] - v[X, vY].
\] (11)

In local form, we have for (11)

\[
\Omega = \frac{1}{2} \Omega^a_{ij} d^i \wedge d^j \otimes \partial_a,
\]

with coefficients

\[
\Omega^a_{ij} = \frac{\partial N_i^a}{\partial x^j} - \frac{\partial N_j^a}{\partial x^i} + N^b_i \frac{\partial N^a_j}{\partial y^b} - N^b_j \frac{\partial N^a_i}{\partial y^b}.
\] (12)

Any \( N \)-connection \( N \) may be characterized by an associated frame (vielbein) structure \( e_\nu = (e_i, e_a) \), where

\[
e_i = \frac{\partial}{\partial x^i} - N_i^a(u) \frac{\partial}{\partial y^a} \quad \text{and} \quad e_a = \frac{\partial}{\partial y^a},
\] (13)

and the dual frame (coframe) structure \( e^\mu = (e^i, e^a) \), where

\[
e^i = dx^i \quad \text{and} \quad e^a = dy^a + N_i^a(u) dx^i,
\] (14)

see formulas (11). These vielbeins are called respectively \( N \)-adapted frames and coframes. In order to preserve a relation with the previous denotations \( [201, 228] \), we emphasize that \( e_\nu = (e_i, e_a) \) and \( e^\mu = (e^i, e^a) \) are correspondingly the former "\( N \)-elongated" partial derivatives \( \delta_\nu = \partial/\partial u^\nu = (\delta_i, \delta_a) \) and \( N \)-elongated differentials \( \delta^\mu = \delta u^\mu = (d^i, \delta^a) \). This emphasizes that the operators (13) and (14) define certain "\( N \)-elongated" partial derivatives and differentials which are more convenient for tensor and integral calculations on such nonholonomic manifolds. The vielbeins (14) satisfy the nonholonomy relations

\[
[e_\alpha, e_\beta] = e_\alpha e_\beta - e_\beta e_\alpha = W^\gamma_{\alpha \beta} e_\gamma
\] (15)

with (antisymmetric) nontrivial anholonomy coefficients \( W^b_{ia} = \partial_a N_i^b \) and \( W^a_{ji} = \Omega^a_{ij} \) defining a proper parametrization (for a \( n + m \) splitting by a \( N \)-connection \( N^a_{ii} \)) of (12).

\( ^{\text{4}}\)We shall use always "boldface" symbols if it would be necessary to emphasize that certain spaces and/or geometrical objects are provided/adapted to a \( N \)-connection structure, or with the coefficients computed with respect to \( N \)-adapted frames.
2.2.2 N–anholonomic manifolds and d–metrics

For simplicity, we shall work with a particular class of nonholonomic manifolds: A manifold \( V \) is **N–anholonomic** if its tangent space \( T V \) is enabled with a N–connection structure.\(^5\)

A distinguished metric (in brief, **d–metric**) on a N–anholonomic manifold \( V \) is a usual second rank metric tensor \( g \) which with respect to a N–adapted basis \(^1\) can be written in the form

\[
g = g_{ij}(x, y) \, e^i \otimes e^j + h_{ab}(x, y) \, e^a \otimes e^b \tag{16}\]

defining a N–adapted decomposition \( g = h g \oplus N v g = [h g, v g] \).

A metric structure \( \tilde{g} \) on a N–anholonomic manifold \( V \) is a symmetric covariant second rank tensor field which is not degenerated and of constant signature in any point \( u \in V \). Any metric on \( V \), with respect to a local coordinate basis \( du^\alpha = (dx^i, dy^a) \), can be parametrized in the form

\[
\tilde{g} = g_{\alpha\beta}(u) \, du^\alpha \otimes du^\beta \tag{17}\]

where

\[
g_{\alpha\beta} = \begin{bmatrix}
g_{ij} + N_i^a N_j^b h_{ab} & N_j^e h_{ae} \\
N_i^e h_{be} & h_{ab}
\end{bmatrix}. \tag{18}\]

Such a metric is generic off–diagonal, i.e. it can not be diagonalized by coordinate transforms if \( N_i^a(u) \) are any general functions.

In general, a metric structure is not adapted to a N–connection structure, but we can transform it into a d–metric

\[
g = h g(h X, h Y) + v g(v X, v Y) \tag{19}\]

adapted to a N–connection structure defined by coefficients \( N_i^a \). We introduce denotations \( h \tilde{g}(h X, h Y) = h g(h X, h Y) \) and \( v \tilde{g}(v X, v Y) = v g(v X, v Y) \) and try to find a N–connection when

\[
\tilde{g}(h X, v Y) = 0 \tag{20}\]

for any d–vectors \( X, Y \). In local form, for \( h X \to e_i \) and \( v Y \to e_a \), the equation (20) is an algebraic equation for the N–connection coefficients \( N_i^a \),

\[
\tilde{g}(e_i, e_a) = 0, \text{ equivalently, } g_{ia} - N_i^b h_{ab} = 0, \tag{21}\]

\(^5\)In a similar manner, we can consider different types of (super) spaces and low energy string limits \([180, 183, 216, 170]\), Riemann or Riemann–Cartan manifolds \([228]\), noncommutative bundles, or superbundles and gauge models \([189, 217, 52, 218, 201]\), Clifford–Dirac spinor bundles and algebroids \([202, 178, 181, 223, 230, 229]\), Lagrange–Fedosov manifolds \([57]\)... provided with nonholonomic (super) distributions \([0]\) and preferred systems of reference (supervielbeins).
where $g_{ia} \doteq g(\partial/\partial x^i, \partial/\partial y^a)$, which allows us to define in a unique form the coefficients $N^b_i = h^{ab} g_{ia}$ where $h^{ab}$ is inverse to $h_{ab}$. We can write the metric $\hat{g}$ with ansatz (18) in equivalent form, as a d–metric (16) adapted to a N–connection structure, if we define $g_{ij} \doteq g(e_i, e_j)$ and $h_{ab} \doteq g(e_a, e_b)$ and consider the vielbeins $e_α$ and $e^α$ to be respectively of type (13) and (14).

A metric $\hat{g}$ (17) can be equivalently transformed into a d–metric (16) by performing a frame (vielbein) transform

$$e_α = e_α^α \partial_α \quad \text{and} \quad e^β = e^β_β du^β,$$

with coefficients

$$e_α^α(u) = \begin{bmatrix} e_{i}^i(u) & N^b_i(u)e^a_b(u) \\ 0 & e_{i}^a(u) \end{bmatrix},$$

$$e^β_β(u) = \begin{bmatrix} e^{i}_i(u) & -N^k_i(u)e^{k}_i(u) \\ 0 & e^a_k(u) \end{bmatrix},$$

being linear on $N^a_i$.

It should be noted here that parametrizations of metrics of type (18) have been introduced in Kaluza–Klein gravity [130] for the case of linear connections (10) and compactified extra dimensions $y^a$. For the five (or higher) dimensions, the coefficients $\Gamma_a^b(x)$ were considered as Abelian or non–Abelian gauge fields. In our approach, the coefficients $N^b_i(x,y)$ are general ones, not obligatory linearized and/or compactified on $y^a$. For some models of Finsler gravity, the values $N^a_i$ were treated as certain generalized nonlinear gauge fields (see Appendix to Ref. [107]), or as certain objects defining (semi) spray configurations in generalized Finsler and Lagrange gravity [107, 108, 5].

The N–connection coefficients can be associated to certain off–diagonal metric coefficients $N^b_i$ in (18) when a $(n+m)$–splitting is prescribed for a manifold $V$ (such a manifold may be a Riemannian or an Einstein space). We can also say that such a splitting and corresponding coefficients $N^a_i$ induce preferred (in general, nonholonomic) frame and/or coframe structures, respectively (13) and/or (14). In general, this does not violate the frame and coordinate diffeomorphisms invariance because any formulas can be written in any system of references, or coordinates. Nevertheless, there is a class of frame transforms (22) with coefficients (23) and (24) preserving the prescribed nonintegrable $(n+m)$–splitting (9). This is like on a Schwarzschild space when we prefer the spherical symmetry but all formulas can be written in arbitrary coordinates (frames), for instance, in Cartesian coordinates.

Formal $(n+m)$–splitting exist naturally on vector/ tangent bundles when $x^i$ label the base space coordinates and $y^a$ label the fiber coordinates. If such splitting are defined by nonintegrable distributions, we also get N–connection structures. In order to give to the N–connections a gauge like
interpretation, we can say that they broke nonholonomically the spacetime symmetry and define certain type of nonlinear gauge fields. In this work we shall not consider gauge like models of locally anisotropic gravity (see [189, 217, 52, 218, 201, 228]).

On N–anholonomic manifolds, we can say that the coordinates \( x^i \) are holonomic and the coordinates \( y^a \) are nonholonomic (on N–anholonomic vector bundles, such coordinates are called respectively to be the horizontal and vertical ones). We conclude that a N–anholonomic manifold \( V \) provided with a metric structure \( \tilde{g} \) (equivalently, with a d–metric (16)) is a usual manifold (in particular, a pseudo–Riemannian one) with a prescribed nonholonomic \( n + m \) splitting into conventional “horizontal” and “vertical” subspaces (9) induced by the “off–diagonal” terms \( N^b_i(u) \) and the corresponding preferred nonholonomic frame structure (15).

2.2.3 d–torsions and d–curvatures

From the general class of linear connections which can be defined on a manifold \( V \), and any its N–anholonomic versions \( V \), we distinguish those which are adapted to a N–connection structure \( \mathcal{N} \).

A distinguished connection (d–connection) \( D \) on a N–anholonomic manifold \( V \) is a linear connection conserving under parallelism the Whitney sum (9). For any d–vector \( X \), there is a decomposition of \( D \) into h– and v–covariant derivatives,

\[
D_X = hD_X + vD_X = DhX + DvX = hD_X + vD_X.
\]

The symbol “\( ] \)” in (25) denotes the interior product defined by a metric (17) (equivalently, by a d–metric (16)). The N–adapted components \( \Gamma^\gamma_{\alpha\beta} \) of a d–connection \( D \) are defined by the equations

\[
D_\alpha e_\beta = \Gamma^\gamma_{\alpha\beta} e_\gamma, \quad \Gamma^\gamma_{\alpha\beta} (u) = (D_\alpha e_\beta)|e_\gamma.
\]

The N–adapted splitting into h– and v–covariant derivatives is stated by

\[
hD = \{ D_k = (L^i_{jk}, L^a_{bk}) \}, \quad vD = \{ D_c = (C^i_{jc}, C^a_{bc}) \},
\]

where

\[
L^i_{jk} = (D_k e_j)|e^i, \quad L^a_{bk} = (D_k e_b)|e^a, \quad C^i_{jc} = (D_c e_j)|e^i, \quad C^a_{bc} = (D_c e_b)|e^a.
\]

The components \( \Gamma^\gamma_{\alpha\beta} = (L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc}) \) completely define a d–connection \( D \) on a N–anholonomic manifold \( V \). We shall write conventionally that \( D = (hD, vD) \), or \( D_\alpha = (D_i, D_a) \), with \( hD = (L^i_{jk}, L^a_{bk}) \) and \( vD = (C^i_{jc}, C^a_{bc}) \), see (26).

The torsion and curvature of a d–connection \( D = (hD, vD) \), d–torsions and d–curvatures, are defined similarly to formulas (4) and
with further h– and v–decompositions. The simplest way to perform computations with d–connections is to use N–adapted differential forms like
\[ \Gamma^\alpha_{\beta \gamma} e^\gamma \] (27)
with the coefficients defined with respect to (13) and (14). For instance, torsion can be computed in the form
\[ T^i = \partial_i N^a_j - \partial_j N^a_i - L^a_i b_j, T^a = C^a b_c - C^a c_b. \] (29)

By a straightforward d–form calculus, we can compute the N–adapted components
\[ R^i_{j k} = e^i L^j_h j_k - e^j L^i_h k - L^m_{h j} L^i_m j - L^m_{h k} L^i_m j - C^i_{h a} \Omega^a_{k j}, \]
\[ R^a_{b j} = e^a L^b_m j k + L^c_{b j} L^a_c k - L^c_{b k} L^a_c j - C^a_{b c} \Omega^c_{k j}, \]
\[ R^i_{j k a} = e^a L^i_j k - D_k C^i_{j a} + C^i_{j b} T^b_{j k a}, \]
\[ R^c_{b k a} = e^a L^c_{b k} - D_k C^c_{b a} + C^c_{b a} T^c_{b k a}, \]
\[ R^i_{j b c} = e^i C^j_{b c} - e^b C^j_{b c} + C^h_{j b} C^i_{h c} - C^h_{j c} C^i_{h b}, \]
\[ R^a_{b c d} = e^a C^b_{c d} - e^c C^a_{b d} + C^b_{c b} C^a_{c d} - C^b_{c d} C^a_{c b}. \] (31)

Contracting respectively the components of (31), one proves that the Ricci tensor
\[ R^i_{\alpha \beta} = \frac{1}{2} g^{i \gamma} R^\gamma_{\alpha \beta \gamma} \] is characterized by h- v–components, i.e. d–tensors,
\[ R_{i j} = \frac{1}{2} R^k_{i j k}, R_{i a} = \frac{1}{2} R^b_{i k a}, R_{i a} = \frac{1}{2} R^b_{a i b}, R_{a b} = \frac{1}{2} R^c_{a b c}. \] (32)

It should be noted that this tensor is not symmetric for arbitrary d–connections D, i.e. \( R_{\alpha \beta} \neq R_{\beta \alpha} \).

The scalar curvature of a d–connection is
\[ ^s R = g^{i \gamma} R^\gamma_{i j} + h^{a b} R_{a b}, \] (33)
defined by a sum the h– and v–components of (32) and d–metric (16).

The Einstein d–tensor is defined and computed similarly to (7), but for d–connections,

\[ E_{\alpha \beta} = R_{\alpha \beta} - \frac{1}{2} g_{\alpha \beta} \, ^s R \] (34)

This d–tensor defines an alternative to \( E_{\alpha \beta} \) (nonholonomic) Einstein configuration if its d–connection is defined in a unique form for an off–diagonal metric (13).
2.2.4 Some classes of distinguished or non–adapted linear connections

From the class of arbitrary d–connections $\mathbf{D}$ on $\mathbf{V}$, one distinguishes those which are metric compatible (metrical d–connections) satisfying the condition

$$Dg = 0$$

including all $h$– and $v$–projections

$$D_j g_{kl} = 0, D_a g_{kl} = 0, D_j h_{ab} = 0, D_a h_{bc} = 0.$$ 

Different approaches to Finsler–Lagrange geometry modelled on $\mathbf{T}\mathbf{M}$ (or on the dual tangent bundle $\mathbf{T}^*\mathbf{M}$, in the case of Cartan–Hamilton geometry) were elaborated for different d–metric structures which are metric compatible \[42, 107, 108, 104, 112, 106, 180, 183, 225\] or not metric compatible \[17\].

For any d–metric $g = [hg, vg]$ on a N–anholonomic manifold $\mathbf{V}$, there is a unique metric canonical d–connection $\tilde{\mathbf{D}}$ satisfying the conditions $\tilde{\mathbf{D}}g = 0$ and with vanishing $h(hh)$–torsion, $v(vv)$–torsion, i. e. $h\tilde{T}(hX, hY) = 0$ and $v\tilde{T}(vX, vY) = 0$. By straightforward calculations, we can verify that

$$\tilde{\Gamma}_{\alpha\beta}^\gamma = \left(\tilde{\Gamma}_{jk}^i, \tilde{T}_a^{jk}, \hat{C}_i^{jc}, \hat{C}_bc\right),$$

where

$$\tilde{\Gamma}_{jk}^i = \frac{1}{2}g^{ir}(e_k g_{jr} + e_j g_{kr} - e_r g_{jk}),$$

$$\tilde{T}_a^{jk} = e_b (N^a_k) + \frac{1}{2}h^{ac} (e_k h_{bc} - h_{dc} e_b N^d_k - h_{db} e_c N^d_k),$$

$$\hat{C}_i^{jc} = \frac{1}{2}h^{ad} (e_c h_{bd} + e_d h_{cd} - e_d h_{bc}),$$

result in $\tilde{T}_{jk}^i = 0$ and $\tilde{T}_a^{bc} = 0$ but $\tilde{T}_{ja}^i, \tilde{T}_{ji}^a$ and $\tilde{T}_a^{bi}$ are not zero, see formulas \[29\] written for this canonical d–connection.

For any metric structure $g$ on a manifold $\mathbf{V}$, there is a unique metric compatible and torsionless Levi Civita connection $\nabla = \{\Gamma^\alpha_{\beta\gamma}\}$ for which $\mathcal{T} = 0$ and $\nabla g = 0$. This is not a d–connection because it does not preserve under parallelism the N–connection splitting \[9\] (it is not adapted to the N–connection structure). Let us parametrize its coefficients in the form

$$\Gamma^\alpha_{\beta\gamma} = \left(L_{jk}^i, L_a^{jk}, L_i^{jk}, L^a_{bk}, C^i_{jb}, C^a_{jb}, C^{i}_{bc}, C^{a}_{bc}\right),$$

where

$$\nabla_{e_k} (e_j) = L_{jk}^i e_i + L_a^{jk} e_a, \nabla_{e_k} (e_b) = L_{bk}^i e_i + L_a^{bk} e_a,$$

$$\nabla_{e_b} (e_j) = C^i_{jb} e_i + C^a_{jb} e_a, \nabla_{e_c} (e_b) = C^i_{bc} e_i + C^a_{bc} e_a.$$
A straightforward calculus shows that the coefficients of the Levi–Civita connection can be expressed in the form

\[ L^i_{jk} = \frac{1}{2} \Omega^i_{jk} h_{ab} - \frac{1}{2} \Omega^a_{jk}, \]

\[ L^i_{bk} = \frac{1}{2} \Omega^i_{bk} h_{ab} g^{ji} - \frac{1}{2} (\delta^i_j \delta^h_k - g_{jk} g^{ih}) C^i_{hb}, \]

\[ L^a_{bk} = \frac{1}{2} (\delta^a_d \delta^b_d + h_{cd} h^{ab}) [L^c_{bk} - e_b(N^c_k)], \]

\[ C^i_{kb} = C^i_{kb} + \frac{1}{2} \Omega^a_{jk} h_{cb} g^{ji} + \frac{1}{2} (\delta^i_j \delta^h_k - g_{jk} g^{ih}) C^i_{hb}, \]

\[ C^a_{jb} = -\frac{1}{2} (\delta^a_d \delta^b_d + h_{cd} h^{ab}) [L^c_{dj} - e_d(N^c_j)], \]

\[ C^a_{bc} = C^a_{bc}, \]

\[ \Gamma^\gamma_{\alpha\beta} = \tilde{\Gamma}^\gamma_{\alpha\beta} + \gamma^\gamma_{\alpha\beta} \]

where the explicit components of \textbf{distortion tensor} \( \gamma^\gamma_{\alpha\beta} \) can be defined by comparing the formulas (37) and (38):

\[ \gamma^i_{jk} = 0, \text{ and } \gamma^i_{jk} = -C^i_{jk} h_{ab} - \frac{1}{2} \Omega^a_{jk}, \]

\[ \gamma^i_{bk} = \frac{1}{2} \Omega^i_{bk} h_{ab} g^{ji} - \frac{1}{2} (\delta^i_j \delta^h_k - g_{jk} g^{ih}) C^i_{hb}, \]

\[ \gamma^a_{bk} = \frac{1}{2} (\delta^a_d \delta^b_d + h_{cd} h^{ab}) [L^c_{bk} - e_b(N^c_k)], \]

\[ \gamma^i_{kb} = \frac{1}{2} \Omega^i_{bk} h_{ab} g^{ji} + \frac{1}{2} (\delta^i_j \delta^h_k - g_{jk} g^{ih}) C^i_{hb}, \]

\[ \gamma^a_{jb} = -\frac{1}{2} (\delta^a_d \delta^b_d + h_{cd} h^{ab}) [L^c_{dj} - e_d(N^c_j)], \]

\[ \gamma^a_{bc} = 0, \]

It should be emphasized that all components of \( \Gamma^\gamma_{\alpha\beta}, \tilde{\Gamma}^\gamma_{\alpha\beta} \), and \( \gamma^\gamma_{\alpha\beta} \) are uniquely defined by the coefficients of d–metric \( \Omega^a_{jk} \) and N–connection \( N^c_k \), or equivalently by the coefficients of the corresponding generic off–diagonal metric \( \Omega \).

*Such results were originally considered by R. Miron and M. Anastasiei for vector bundles provided with N–connection and metric structures, see Ref. [108]. Similar proofs hold true for any nonholonomic manifold provided with a prescribed N–connection structure [228].
2.3 On equivalent (non)holonomic formulations of gravity theories

A N–anholonomic Riemann–Cartan manifold \( RC^\nu \) is defined by a d–metric \( g \) and a metric d–connection \( D \) structures. We can say that a space \( \hat{R}^\nu \) is a canonical N–anholonomic Riemann manifold if its d–connection structure is canonical, i.e. \( D = \hat{D} \). The d–metric structure \( g \) on \( RC^\nu \) is of type \( (16) \) and satisfies the metricity conditions \( (35) \). With respect to a local coordinate basis, the metric \( g \) is parametrized by a generic off–diagonal metric ansatz \( (18) \). For a particular case, we can treat the torsion \( \hat{T} \) as a nonholonomic frame effect induced by a nonintegrable N–splitting. We conclude that a manifold \( \hat{R}^\nu \) is enabled with a nontrivial torsion \( (29) \) (uniquely defined by the coefficients of N–connection \( (8) \), and d–metric \( (16) \) and canonical d–connection \( (36) \) structures). Nevertheless, such manifolds can be described alternatively, equivalently, as a usual (holonomic) Riemann manifold with the usual Levi Civita for the metric \( (17) \) with coefficients \( (18) \). We do not distinguish the existing nonholonomic structure for such geometric constructions.

Having prescribed a nonholonomic \( n + m \) splitting on a manifold \( V \), we can define two canonical linear connections \( \nabla \) and \( \hat{\nabla} \). Correspondingly, these connections are characterized by two curvature tensors, \( R_{\beta\gamma\delta}^\alpha(\nabla) \) (computed by introducing \( \Gamma_{\beta\gamma}^\alpha \) into \( (27) \) and \( (30) \)) and \( R_{\beta\gamma\delta}^\alpha(\hat{D}) \) (with the N–adapted coefficients computed following formulas \( (31) \)). Contracting indices, we can commute the Ricci tensor \( Ric(\nabla) \) and the Ricci d–tensor \( Ric(\hat{D}) \) following formulas \( (32) \), correspondingly written for \( \nabla \) and \( \hat{\nabla} \). Finally, using the inverse d–tensor \( g^{\alpha\beta} \) for both cases, we compute the corresponding scalar curvatures \( s_Ric(\nabla) \) and \( s_Ric(\hat{D}) \), see formulas \( (33) \) by contracting, respectively, with the Ricci tensor and Ricci d–tensor.

The standard formulation of the Einstein gravity is for the connection \( \nabla \), when the field equations are written in the form \( (7) \). But it can be equivalently reformulated by using the canonical d–connection, or other connections uniquely defined by the metric structure. If a metric \( (18) \) \( g_{\alpha\beta} \) is a solution of the Einstein equations \( E_{\alpha\beta} = \Upsilon_{\alpha\beta} \), having prescribed a \( (n + m) \)–decomposition, we can define algebraically the coefficients of a N–connection, \( N_1^\alpha \), N–adapted frames \( e_\alpha \) \( (13) \) and \( e_\beta \) \( (14) \), and d–metric \( g_{\alpha\beta} = [g_{ij}, h_{ab}] \) \( (16) \). The next steps are to compute \( \hat{\Gamma}_{\alpha\beta}^\gamma \), following formulas \( (36) \), and then using \( (31), (32) \) and \( (33) \) for \( \hat{D} \), to define \( \hat{E}_{\alpha\beta} \) \( (34) \). The Einstein equations with matter sources, written in equivalent form by using the canonical d–connection, are

\[
\hat{E}_{\alpha\beta} = \Upsilon_{\alpha\beta} + Z \Upsilon_{\alpha\beta},
\]

where the effective source \( Z \Upsilon_{\alpha\beta} \) is just the deformation tensor of the Einstein tensor computed by introducing deformation \( (38) \) into the left part of \( (7) \); all decompositions being performed with respect to the N–adapted
co–frame (14), when $E_{\alpha\beta} = \hat{E}_{\alpha\beta} = \hat{\mathbf{z}} \gamma_{\alpha\beta}$. For certain matter field/ string gravity configurations, the solutions of (40) also solve the equations (7). Nevertheless, because of generic nonlinear character of gravity and gravity–matter field interactions and functions defining nonholonomic distributions, one could be certain special conditions when even vacuum configurations contain a different physical information if to compare with usual holonomic ones. We analyze some examples:

In our works [201, 228, 202], we investigated a series of exact solutions defining N–anholonomic Einstein spaces related to generic off–diagonal solutions in general relativity by such nonholonomic constraints when $\text{Ric}(\hat{\mathbf{D}}) = \text{Ric}(\nabla)$, even $\hat{\mathbf{D}} \neq \nabla \hat{\mathbf{D}}$. In this case, for instance, the solutions of the Einstein equations with cosmological constant $\lambda$,

$$\hat{\text{R}}_{\alpha\beta} = \lambda g_{\alpha\beta}$$

(41)
can be transformed into metrics for usual Einstein spaces with Levi Civita connection $\nabla$. The idea is that for certain general metric ansatz, see section 3.3, the equations (41) can be integrated in general form just for the connection $\hat{\mathbf{D}}$ but not for $\nabla$. The nontrivial torsion components

$$\hat{T}_{ja} = -\hat{T}_{aj} = \hat{C}_{ja}, \quad \hat{T}_{ai} = \hat{\Omega}^a_{ji}, \quad \hat{T}_{bi} = -\hat{T}_{ib} = \frac{\partial N^a_i}{\partial y^b} - \hat{L}^a_{bi},$$

(42)

see (29), for some configurations, may be associated with an absolute antisymmetric $H$–fields in string gravity [53, 139], but nonholonomically transformed to N–adapted bases, see details in [201, 228].

For more restricted configurations, we can search solutions with metric ansatz defining Einstein foliated spaces, when

$$\hat{\Omega}^c_{jk} = 0, \quad \hat{L}^c_{bk} = e_b(N^c_k), \quad \hat{C}^a_{jb} = 0,$$

(43)

and the $d$–torsion components (42) vanish, but the N–adapted frame structure has, in general, nontrivial anholonomy coefficients, see (15). One present a special interest a less constrained configurations with $\hat{T}^c_{jk} = \hat{\Omega}^c_{jk} \neq 0$ when $\text{Ric}(\hat{\mathbf{D}}) = \text{Ric}(\nabla)$ and $\hat{T}^i_{ja} = \hat{T}^a_{bi} = 0$, for certain general ansatz $\hat{T}^i_{ja} = 0$ and $\hat{T}^a_{bi} = 0$, but $\hat{\text{R}}^\alpha_{\beta\gamma\delta} \neq \text{R}^\alpha_{\beta\gamma\delta}$. In such cases, we constrain the integral varieties of equations (41) in such a manner that we generate integrable or nonintegrable distributions on a usual Einstein space defined by $\nabla$. This is possible because if the conditions (43) are satisfied,}{7}

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7One should be emphasized here that different type of connections on N–anholonomic manifolds have different coordinate and frame transform properties. It is possible, for instance, to get equalities of coefficients for some systems of coordinates even the connections are very different. The transformation laws of tensors and d–tensors are also different if some objects are adapted and other are not adapted to a prescribed N–connection structure.
the deformation tensor \( Z^\gamma_{\alpha\beta} = 0 \), see (39). For \( \lambda = 0 \), if \( n + m = 4 \), for corresponding signature, we get foliated vacuum configurations in general relativity.

For N–anholonomic manifolds \( V^{n+n} \) of odd dimensions, when \( m = n \), and if \( g_{ij} = h_{ij} \) (we identify correspondingly, the h- and v–indices), we can consider a canonical d–connection \( \tilde{D} = (h\tilde{D}, v\tilde{D}) \) with the nontrivial coefficients with respect to \( e_\nu \) and \( e_\mu \) parametrized respectively \( \tilde{\Gamma}^\alpha_{\beta\gamma} = (\tilde{L}^i_{\ jk}, \tilde{C}^a_{\ bc}) \) for

\[
\tilde{L}^i_{\ jk} = \frac{1}{2} g^{ih}(e_k g_{jh} + e_j g_{kh} - e_h g_{jk}), \\
\tilde{C}^a_{\ bc} = \frac{1}{2} g^{ae}(e_b g_{ec} + e_c g_{eb} - e_e g_{bc}),
\]

defining the generalized Christoffel symbols. Such nonholonomic configurations can be used for modelling generalized Finsler–Lagrange, and particular cases, defined in Refs. [107, 108] for \( V^{n+n} = TM \), see below section 3.1.

There are only three classes of d–curvatures for the d–connection (44),

\[
\tilde{R}^i_{\ hjk} = e_k \tilde{L}^i_{\ hj} - e_j \tilde{L}^i_{\ hk} + \tilde{L}^m_{\ hj} \tilde{L}^i_{\ mk} - \tilde{L}^m_{\ hk} \tilde{L}^i_{\ mj} - \tilde{C}^i_{\ ha} \Omega^a_{\ kj}, \\
\tilde{P}^i_{\ jka} = e_a \tilde{L}^i_{\ jk} - \tilde{D}_k \tilde{C}^i_{\ ja}, \\
\tilde{S}^a_{\ bed} = e_d \tilde{C}^a_{\ bc} - e_e \tilde{C}^a_{\ bd} + \tilde{C}^e_{\ bc} \tilde{C}^a_{\ ed} - \tilde{C}^e_{\ bd} \tilde{C}^a_{\ ec},
\]

where all indices \( a, b, ..., i, j, ... \) run the same values and, for instance, \( C^a_{\ bc} \rightarrow C^i_{\ jk}, ... \). Such locally anisotropic configurations are not integrable if \( \Omega^a_{\ kj} \neq 0 \), even the d–torsion components \( \tilde{T}^i_{\ jk} = 0 \) and \( \tilde{T}^a_{\ bc} = 0 \). We note that for geometric models on \( V^{n+n} \), or on \( TM \), with \( g_{ij} = h_{ij} \), one writes, in brief, \( \tilde{\Gamma}^\alpha_{\beta\gamma} = (\tilde{L}^i_{\ jk}, \tilde{C}^a_{\ bc}) \), or, for more general d–connections, \( \Gamma^\alpha_{\beta\gamma} = (L^i_{\ jk}, C^a_{\ bc}) \), see below section 3.1, on Lagrange and Finsler spaces.

## 3 Nonholonomic Deformations of Manifolds and Vector Bundles

This section will deal mostly with nonholonomic distributions on manifolds and vector/ tangent bundles and their nonholonomic deformations modelling, on Riemann and Riemann–Cartan manifolds, different types of generalized Finsler–Lagrange geometries.

### 3.1 Finsler–Lagrange spaces and generalizations

The notion of Lagrange space was introduced by J. Kern [87] and elaborated in details by R. Miron’s school, see Refs. [107, 108, 103, 104, 112, 106], as
a natural extension of Finsler geometry [42, 145, 102, 24] (see also Refs. [180, 183, 216, 201, 228], on Lagrange–Finsler super/noncommutative geometry). Originally, such geometries were constructed on tangent bundles, but they also can be modelled on N–anholonomic manifolds, for instance, as models for certain gravitational interactions with prescribed nonholonomic constraints deformed symmetries.

3.1.1 Lagrange spaces

A differentiable Lagrangian \( L(x, y) \), i.e. a fundamental Lagrange function, is defined by a map \( L : (x, y) \in TM \rightarrow L(x, y) \in \mathbb{R} \) of class \( C^\infty \) on \( \widetilde{TM} = TM \setminus \{0\} \) and continuous on the null section \( 0 : M \rightarrow TM \) of \( \pi \). A regular Lagrangian has non-degenerate Hessian

\[
L_{g_{ij}}(x, y) = \frac{1}{2} \frac{\partial^2 L(x, y)}{\partial y^i \partial y^j},
\]

when \( rank |g_{ij}| = n \) and \( L^g_{ij} \) is the inverse matrix. A Lagrange space is a pair \( L^n = [M, L(x, y)] \) with \( L_{g_{ij}} \) being of fixed signature over \( V = \widetilde{TM} \).

One holds the results: The Euler–Lagrange equations

\[
\frac{d}{d\tau} \left( \frac{\partial L}{\partial y^i} \right) - \frac{\partial L}{\partial x^i} = 0
\]

where \( y^i = \frac{dx^i}{d\tau} \) for \( x^i(\tau) \) depending on parameter \( \tau \), are equivalent to the “nonlinear” geodesic equations

\[
\frac{d^2 x^a}{d\tau^2} + 2G^a_i(x, y) \frac{dx^b}{d\tau} = 0
\]

defining paths of a canonical semispray

\[
S = y^i \frac{\partial}{\partial x^i} - 2G^a_i(x, y) \frac{\partial}{\partial y^a}
\]

where

\[
2G^a_i(x, y) = \frac{1}{2} L_{g^{ij}} \left( \frac{\partial^2 L}{\partial y^i \partial x^k} y^k - \frac{\partial L}{\partial x^i} \right).
\]

There exists on \( V \simeq \widetilde{TM} \) a canonical N–connection

\[
L_{N^a_j} = \frac{\partial G^a_i(x, y)}{\partial y^j}
\]

defined by the fundamental Lagrange function \( L(x, y) \), which prescribes nonholonomic frame structures of type \([13]\) and \([14]\), \( L e^\nu = (L e_i, e_a) \) and \( L e^\mu = (e^i, L e^a) \). One defines the canonical metric structure

\[
L g = L_{g_{ij}}(x, y) \, e^i \otimes e^j + L_{g_{ij}}(x, y) \, L e^i \otimes L e^j
\]
constructed as a Sasaki type lift from \( M \) for \( L_{gij}(x,y) \), see details in [248, 107, 108].

There is a unique canonical d–connection \( \hat{\mathcal{D}} = (h, L\hat{D}, v) \) with the coefficients \( L\hat{\Gamma}^\alpha_{\beta\gamma} = (L\hat{\Gamma}_{ijk}^a, L\hat{\mathcal{C}}_{bc}^a) \) computed by formulas (44) for the d–metric (48) with respect to \( L^e \) and \( L\hat{e}^{\mu} \). All such geometric objects, including the corresponding to \( L\hat{\Gamma}^\alpha_{\beta\gamma} \), \( Lg \) and \( LN^a \) d–curvatures \( L\hat{R}^\alpha_{\beta\gamma\delta} = (L\hat{R}_{ijk}^i, L\hat{P}_{i jka}, L\hat{S}_{abc}^a) \), see (45), are completely defined by a Lagrange fundamental function \( L(x,y) \) for a nondegenerate \( Lg_{ij} \).

Let us consider how a Lagrange mechanics can be modelled on nonholonomic Riemann, usual Riemann, or Riemann–Cartan manifolds. We take a manifold \( V^{n+n}, \dim V = n+n \), and consider a metric structure (18) for a particular case when the values \( g_{ij} \) and \( N^a_i \) are respectively of type (46) and (47) for a function \( L(x,y) \) on \( V \), with nondegenerate \( Lg_{ij} \). The preferred frame structure on \( V \) is defined by introducing \( LN^a_i \) in the class of vierbein transforms (22) with coefficients (23) and (24). All data can be redefined for a d-metric (16) but generated by \( Lg \) in a form equivalent to (48), with that difference that the first geometric object is defined on a N–anholonomic manifold but the second one is considered on a TM.

The next step of modelling is to decide what kind of linear connection we chose. There are two canonical, equivalent, possibilities. If we take (36), on \( V^{n+n} \), we model a Riemann–Cartan manifold with induced torsion (42), in this case, completely defined by \( L \) and respective \( Lg_{ij} \) and \( LN^a_i \). We can simplify the constructions for a normal canonical d–connection (44) and generate a nonholonomic Riemann manifold with nonintegrable structure \( L\Omega^a_{k} \). Finally, we note that all constructions can be re–defined for the Levi Civita connection if we consider \( L\Gamma^\gamma_{\alpha\beta} = L\hat{\Gamma}^\gamma_{\alpha\beta} + L\hat{Z}^\gamma_{\alpha\beta} \) of type (37), where the values are computed following formulas (38) and (39) (also completely defined by \( L \) and respective \( Lg_{ij} \) and \( LN^a_i \)). Such constructions are not adapted to the N–connection structure: we work with arbitrary frame and coordinate transforms and hidden Lagrange structure which appear in explicit form only with respect to certain preferred, N–adapted, frames of reference.

We conclude that any regular Lagrange mechanics can be geometrized as a nonholonomic Riemann manifold \( L\hat{V} \) equipped with the canonical N–connection \( LN^a_j \) (47). This geometrization was performed in such a way that the N–connection is induced canonically by the semispray configurations subjected the condition that the generalized nonlinear geodesic equations are equivalent to the Euler–Lagrange equations for \( L \). Such mechanical models and semispray configurations can be used for a study of certain classes of nonholonomic effective analogous of gravitational interactions. The approach can be extended for more general classes of effective metrics, then those parametrized by (48), see next sections. After Kern and Miron and Anastasiei works, it was elaborated the so–called "analogous gravity" ap-
proach \[15\] with similar ideas modelling related to continuous mechanics, condensed media.... It should be noted here, that the constructions for higher order generalized Lagrange and Hamilton spaces \[103, 104, 112, 106\] provided a comprehensive geometric formalism for analogous models in gravity, geometric mechanics, continuous media, nonhomogeneous optics etc etc.

### 3.1.2 Finsler spaces

Following the ideas of the Romanian school on Finsler–Lagrange geometry and generalizations, any Finsler space defined by a **fundamental Finsler function** \( F(x, y) \), being homogeneous of type \( F(x, \lambda y) = |\lambda| F(x, y) \), for nonzero \( \lambda \in \mathbb{R} \), may be considered as a particular case of Lagrange space when \( L = F^2 \) (on different rigorous mathematical definitions of Finsler spaces, see \[145, 102, 107, 108, 24, 17\]; in our approach with applications to physics, we shall not constrain ourself with special signatures, smooth class conditions and special types of connections). Historically, the bulk of mathematicians worked in an inverse direction, by generalizing the constructions from the Cartan’s approach to Finsler geometry in order to include into consideration regular Lagrange mechanical systems, or to define Finsler geometries with another type of nonlinear and linear connection structures. The Finsler geometry, in terms of the normal canonical \( d\)-connection (44), derived for respective \( F g_{ij} \) and \( F N_{ja} \), can be modelled as for the case of Lagrange spaces considered in the previous section: we have to change formally all labels \( L \rightarrow F \) and take into consideration possible conditions of homogeneity (or \( TM \), see the monographs \[107, 108\]).

For generalized Finsler spaces, a \( N\)-connection can be stated by a general set of coefficients \( N_{ja} \) subjected to certain nonholonomy conditions. Of course, working with homogeneous functions on a manifold \( V^{n+n} \), we can model a Finsler geometry both on holonomic and nonholonomic Riemannian manifolds, or on certain types of Riemann–Cartan manifolds enabled with preferred frame structures \( F e_{\nu} = (F e_i, e_a) \) and \( F e_{\mu} = (e^i, F e^a) \). Bellow, in the section 3.3, we shall discuss how certain type Finsler configurations can be derived as exact solutions in Einstein gravity. Such constructions allow us to argue that Finsler geometry is also very important in standard physics and that it was a big confusion to treat it only as a “sophisticated” generalization of Riemann geometry, on tangent bundles, with not much perspectives for modern physics.

In a number of works (see monographs \[107, 108, 17\]), it is emphasized that the first example of Finsler metric was considered in the famous inauguration thesis of B. Riemann [144], long time before P. Finsler [58]. Perhaps, this is a reason, for many authors, to use the term Riemann–Finsler geometry. Nevertheless, we would like to emphasize that a Finsler space is not
completely defined only by a metric structure of type

\[ F_{g_{ij}} = \frac{1}{2} \frac{\partial^2 F}{\partial y^i \partial y^j} \]  

(49)

originally considered on the vertical fibers of a tangent bundle. There are
necessary additional conventions about metrics on a total Finsler space,
N–connections and linear connections. This is the source for different ap-
proaches, definitions, constructions and ambiguities related to Finsler spaces
and applications. Roughly speaking, different famous mathematicians, and
their schools, elaborated their versions of Finsler geometries following some
special purposes in geometry, mechanics and physics.

The first complete model of Finsler geometry exists due to E. Cartan
[42] who in the 20-30th years of previous century elaborated the concepts of
vector bundles, Riemann–Cartan spaces with torsion, moving frames, de-
veloped the theory of spinors, Pfaff forms ... and (in coordinate form) operated
with nonlinear connection coefficients. The Cartan’s constructions were per-
formed with metric compatible linear connections which is very important
for applications to standard models in physics.

Latter, there were proposed different models of Finsler spaces with metric
not compatible linear connections. The most notable connections were those
by L. Berwald, S. -S. Chern (re–discovered by H. Rund), H. Shimada and
others (see details, discussions and bibliography in monographs [107, 108,
[17, 145]). For d–connections of type (44), there are distinguished three cases
of metric compatibility (compare with h- and v-projections of formula (35)):
A Finsler connection \( F^{D}_\alpha = ( F^{D}_k, F^{D}_a ) \) is called h–metric if
it is called v–metric if \( F^{D}_a F_{g_{ij}} = 0 \) and it is metrical if both conditions are
satisfied.

Here, we note four of the most important Finsler d–connections having
their special geometric and (possible) physical merits:

1. The \textbf{canonical Finsler connection} \( F^{\tilde{D}} \) is defined by formulas (44),
but for \( F_{g_{ij}} \), i.e. as \( F^{\tilde{\Gamma}}_{\alpha \beta \gamma} = \left( F^{\tilde{L}}_{i j k}, F^{\tilde{C}}_{a b c} \right) \). This d–connection is
metrical. For a special class of N–connections \( C N^a_j(x^k, y^h) = y^k C L^i_{kj} \),
we get the famous \textbf{Cartan connection} for Finsler spaces,
\( C^{\Gamma}_{\alpha \beta \gamma} = \left( C L^i_{jk}, C^{C}_{a b c} \right) \), with

\[ C L^i_{jk} = \frac{1}{2} F_{gh} \left( C e_k F_{g_{jh}} + C e_j F_{g_{kh}} - C e_h F_{g_{jk}} \right), \]  

(50)

\[ C^{C}_{a b c} = \frac{1}{2} F_{ae} \left( e_b F_{g_{ec}} + e_c F_{g_{eb}} - e_e F_{g_{bc}} \right), \]

where

\[ C e_k = \frac{\partial}{\partial x^k} - C N^a_j \frac{\partial}{\partial y^a} \] \text{and} \[ e_b = \frac{\partial}{\partial y^b}, \]
which can be defined in a unique axiomatic form [102]. Such canonical and Cartan–Finsler connections, being metric compatible, for nonholonomic geometric models with local anisotropy on Riemann or Riemann–Cartan manifolds, are more suitable with the paradigm of modern standard physics.

2. The Berwald connection $B^D_{\Gamma}$ was introduced in the form $B^D_{\Gamma}^{\alpha}_{\beta\gamma} = \left( \frac{\partial C_N^b}{\partial y^a}, 0 \right)$ [27]. This d–connection is defined completely by the N–connection structure but it is not metric compatible, both not h–metric and not v–metric.

3. The Chern connection $Ch^D_{\Gamma}$ was considered as a minimal Finsler extension of the Levi Civita connection, $Ch^D_{\Gamma}^{\alpha}_{\beta\gamma} = \left( C^i L^j_{ik}, 0 \right)$, with $C^i L^j_{ik}$ defined as in (50), preserving the torsionless condition, being h–metric but not v–metric. It is an interesting case of nonholonomic geometries when torsion is completely transformed into nonmetricity which for physicists presented a substantial interest in connection to the Weyl nonmetricity introduced as a method of preserving conformal symmetry of certain scalar field constructions in general relativity, see discussion in [70]. Nevertheless, it should be noted that the constructions with the Chern connection, in general, are not metric compatible and can not be applied in direct form to standard models of physics.

4. There is also the Hashiguchi connection $H^D_{\Gamma}$ was defined by the coefficients of Finsler type d–metric and N–connection structure (equivalently, by the coefficients of corresponding generic off–diagonal metric of type (18)) following well defined geometric conditions. From such d–connections, we can always ‘extract’ the Levi Civita connection, using formulas of type (37), (38) and (39), and work in ‘non–adapted’ (to N–connection) form. From geometric point of view, we can work with all types of Finsler connections and elaborate equivalent approaches even different connections have different merits in some directions of physics. For instance, in [107, 108], there are considered the Kawaguchi metrization procedure and the Miron’s method of computing all metric compatible Finsler connections starting with a canonical one. It was analyzed also the problem of transforming one Finsler connection into different ones on tangent bundles and the formalism of mutual transforms of connections was reconsidered for nonholonomic manifolds, see details in [228].

Different models of Finsler spaces can be elaborated in explicit form for different types of d–metrics, N–connections and d–connections. For in-
stance, for a Finsler Hessian \( \{49\} \) defining a particular case of \( d \)-metrics \( \{48\}, \) or \( \{16\} \), denoted \( F^g \), for any type of connection (for instance, canonical \( d \)-connection, Cartan–Finsler, Berwald, Chern, Hashigushi etc), we can compute the curvatures by using formulas \( \{45\} \) when "hat" labels are changed into the corresponding ones "\( C, B, Ch, H\)...". This way, we model Finsler geometries on tangent bundles, like it is considered in the bulk of monographs \( \{42, 102, 145, 107, 108, 24, 17\} \), or on nonholonomic manifolds \( \{241, 243, 73, 25, 228\} \).

With the aim to develop new applications in standard models of physics, let say in classical general relativity, when Finsler like structures are modelled on a (pseudo) Riemannian manifold (we shall consider explicit examples in the next sections), it is positively sure that the canonical Finsler and Cartan connections, and their variants of canonical \( d \)-connection on vector bundles and nonholonomic manifolds, should be preferred for constructing new classes of Einstein spaces and defining certain low energy limits to locally anisotropic string gravity models. Here we note that it is a very difficult problem to define Finsler–Clifford spaces with Finsler spinors, noncommutative generalizations to supersymmetric/ noncommutative Finsler geometry if we work with nonmetric \( d \)-connections, see discussions in \( \{228, 225, 183\} \).

We cite a proof \( \{17\} \) that any Lagrange fundamental function \( L \) can be modelled as a singular case for a certain class of Finsler geometries of extra dimension (perhaps, the authors were oriented to prove from a mathematical point of view that it is not necessary to develop Finsler geometry as a new theory for Lagrange spaces, or their dual constructions for the Hamilton spaces). This idea, together with the method of Kawaguchi–Miron transforms of connections, can be related to the H. Poincare philosophical concepts about conventionality of the geometric space and field interaction theories \( \{137, 138\} \). According to the Poincare’s geometry–physics dualism, the procedure of choosing a geometric arena for a physical theory is a question of convenience for researches to classify scientific data in an economical way, but not an action to be verified in physical experiments: as a matter of principle, any physical theory can be equivalently described on various types of geometric spaces by using more or less "simple" geometric objects and transforms.

Nevertheless, the modern physics paradigm is based on the ideas of objective reality of physical laws and their experimental and theoretical verifications, at least in indirect form. The concept of Lagrangian is a very important geometrical and physical one and we shall distinguish the cases when we model a Lagrange or a Finsler geometry. A physical or mechanical model with a Lagrangian is not only a "singular" case for a Finsler geometry but reflects a proper set of concepts, fundamental physical laws and symmetries describing real physical effects. We use the terms Finsler and Lagrange spaces in order to emphasize that they are different both from geometric and physical points of view. Certain geometric concepts and methods (like
the N–connection geometry and nonholonomic frame transforms ... are very important for both types of geometries, modelled on tangent bundles or on nonholonomic manifolds. This will be noted when we use the term Finsler–Lagrange geometry (structures, configurations, spaces).

One should be emphasized that the author of this review should not be considered as a physicist who does not accept nonmetric geometric constructions in modern physics. For instance, the Part I in monograph [228] is devoted to a deep study of the problem when generalized Finsler–Lagrange structures can be modelled on metric-affine spaces, even as exact solutions in gravity with nonmetricity [70], and, inversely, the Lagrange-affine and Finsler-affine spaces are classified by nonholonomic structures on metric-affine spaces. It is a question of convention on the type of physical theories one models by geometric methods. The standard theories of physics are formulated for metric compatible geometries, but further developments in quantum gravity may request certain type of nonmetric Finsler like geometries, or more general constructions. This is a topic for further investigations.

3.1.3 Generalized Lagrange spaces

There are various application in optics of nonhomogeneous media and gravity (see, for instance, Refs. [108, 228, 57, 201]) considering metrics of type $g_{ij} \sim e^{\lambda(x,y)} L g_{ij}(x, y)$ which can not be derived directly from a mechanical Lagrangian. The ideas and methods to work with arbitrary symmetric and nondegenerated tensor fields $g_{ij}(x, y)$ were concluded in geometric and physical models for generalized Lagrange spaces, denoted $GL^n = (M, g_{ij}(x, y))$, on $\tilde{T}M$, see [107, 108], where $g_{ij}(x, y)$ is called the fundamental tensor field. Of course, the geometric constructions will be equivalent if we shall work on N–anholonomic manifolds $V^{n+n}$ with nonholonomic coordinates $y$. If we prescribe an arbitrary N–connection $N^a_i(x, y)$ and consider that a metric $g_{ij}$ defines both the h– and v–components of a d–metric (16), we can introduce the canonical d–connection (44) and compute the components of d–curvature (45), define Ricci and Einstein tensors, elaborate generalized Lagrange models of gravity.

If we work with a general fundamental tensor field $g_{ij}$ which can not be transformed into $Lg_{ij}$, we can consider an effective Lagrange function

$$\mathcal{L}(x, y) \doteq g_{ab}(x, y) y^a y^b$$

and use

$$\mathcal{L}_{g_{ab}} \doteq \frac{1}{2} \frac{\partial^2 \mathcal{L}}{\partial y^a \partial y^b} \quad (51)$$

\footnote{In [107, 108], it is called the absolute energy of a $GL^n$–space, but for further applications in modern gravity the term ”energy” may result in certain type ambiguities}
as a Lagrange Hessian (48). A space $GL^n = (M, g_{ij}(x, y))$ is said to be with a weakly regular metric if $L^n = \left[ M, L = \sqrt{L} \right]$ is a Lagrange space. For such spaces, we can define a canonical nonlinear connection structure

\[ \mathcal{L} N^a_j(x, y) \triangleq \frac{\partial \mathcal{L} G^a}{\partial y^j}, \]  

for

\[ \mathcal{L} G^a = \frac{1}{4} \mathcal{L} g^{ab} \left( y^k \frac{\partial \mathcal{L}}{\partial y^b \partial x^k} - \frac{\partial \mathcal{L}}{\partial x^a} \right) = \frac{1}{4} \mathcal{L} g^{ab} \left( \frac{\partial g_{bc}}{\partial y^d} + \frac{\partial g_{bd}}{\partial y^c} - \frac{\partial g_{cd}}{\partial y^b} \right) y^c y^d, \]

which allows us to write $\mathcal{L} N^a_j$ is terms of the fundamental tensor field $g_{ij}(x, y)$. The geometry of such generalized Lagrange spaces is completely similar to that of usual Lagrange one, with that difference that we start not with a Lagrangian but with a fundamental tensor field.

In our papers [204, 3], we proposed to see also nonholonomic transforms of a metric $\mathcal{L} g_{a'b'}(x, y)$

\[ g_{ab}(x, y) = e_a^a(x, y)e_b^b(x, y) \mathcal{L} g_{a'b'}(x, y) \]

where

\[ \mathcal{L} g_{a'b'} = \frac{1}{2} \left( e_a^a e_b^b \mathcal{L} + e_b^b e_a^a \mathcal{L} \right) = {^0} g_{a'b'}, \]

for $e_a^a = {^a_a}^a(x, y) \frac{\partial}{\partial x^a}$, where $^{0} g_{a'b'}$ are constant coefficients (or in a more general case, they should result in a constant matrix for the d–curvatures (31) of a canonical d–connection (36)). Such constructions allowed to derive proper solitonic hierarchies and bi–Hamilton structures for any (pseudo) Riemannian or generalized Finsler–Lagrange metric. The point was to work not with the Levi Civita connection (for which the solitonic equations became very cumbersome) but with a correspondingly defined canonical d–connection allowing to apply well defined methods from the geometry of nonlinear connections. Having encoded the "gravity and geometric mechanics" information into solitonic hierarchies and convenient d–connections, the constructions were shown to hold true if they are "inverted" to those with usual Levi Civita connections.

### 3.2 Effective Finsler–Lagrange (algebroid) structures

All valuable physical solutions in general gravity and generalizations are characterized by corresponding symmetries of spacetime metrics, see reviews of such constructions in Refs. [90, 28] and, for nonholonomic solutions, [228]. For instance, a special importance is given to spacetimes with spherical, cylindrical, or toroidal symmetries, and, in general, to gravitational
distributions characterized by Killing symmetries, or by metrics conformal to the flat Minkowski metric. In this section, we show that, as a matter of principle, any (pseudo) Riemannian manifold can be deformed by nonholonomic trasforms into some classes of N–anholonomic manifolds and/or gravitational Lie algebroid configurations [202].

We note that applying any general frame or coordinate transforms (1), when the metric transforms are of type $g_{\alpha \beta} = A^\alpha_\alpha(u) A^\beta_\beta(u) g_{\alpha' \beta'}$, a Lagrange, or Finsler, structure, characterized by a ”prime” d–metrics of type (48), with coefficients (46), or (49), became "hidden" into some general formulas for metrics and linear connections. The bulk of solutions in Einstein gravity and generalizations can be associated to certain models of generalized Lagrange (or Finsler) spaces not in explicit form but via some special types of nonholonomic deformations (transforms) of the frame/ N–connection, metric and linear connection structures.

The aim of this section is to examine some important examples of nonholonomic transforms preserving the N–connection splitting (9).

### 3.2.1 N–adapted nonholonomic transforms

We use the term "transforms/ transformations" of geometric objects if we work with usual transforms of the local frames. The spacetime geometry and geometrical objects are not changed under such transforms. Fixing a frame structure (holonomic or nonholonomic one), we can consider "deformations" of geometric objects induced by deformations of the metric, or linear connection structure.

The spacetime geometry and geometric objects are changed under holo- nomic of nonholonomic deformations. For instance, fixing a co-frame $e^\alpha' = du^\alpha$, we define conformal maps of metrics as local re–scaling of metric coefficients,

$$ g = g_{\alpha' \beta'} e^{\alpha'} \otimes e^{\beta'} \rightarrow \omega g = \omega^2 g_{\alpha \beta} e^\alpha \otimes e^\beta, $$

i.e. $\omega g_{\alpha \beta} = \omega^2(u)g_{\alpha \beta}(u)$, when $e^\alpha = \delta^\alpha_\alpha' e^{\alpha'}$ which mean that we deform the metric structure and the spacetime geometry is changed under such maps (we get another types of connections and curvatures). In an alternative form, we can say that a conformal transform of metric, $\omega g$, is generated from $g$ by an active frame transform $e^\alpha' \rightarrow e^\alpha = \omega^\alpha_\alpha' e^{\alpha'}$. Such (active) frame transforms preserve the spacetime geometry. Similar properties exist for more general classes of transforms (deformations) on N–anholonomic manifolds.

In the simplest case, for a fixed trivial N–connection structure in (23), when $N^a_i = 0$ and $e^a_i(u) = \omega(u) \delta^a_i$ and $e^2_i(u) = \omega(u) \delta^2_i$, we model confor- mal transforms $\omega g_{\alpha \beta} = \omega^2(u) g_{\alpha \beta}'$, where $g_{\alpha \beta}'$ is a flat metric on a manifold $V^{n+m}$. For nontrivial N–connection structures, it is convenient to distinguish four general classes of nonholonomic frame transforms (deformations):

31
General frame transforms on N–anholonomic manifolds:

Any N–anholonomic structure can be induced by a series of chains of one, two, three,..., k,... frame transforms (1):

\[
\begin{align*}
e^\alpha &= e^\alpha(u)e^\alpha, \\
^2e^\alpha &= ^2e^\alpha(u)e^\alpha = A^\alpha u)A^\alpha(u)e^\alpha, \\
^3e^\alpha &= ^3e^\alpha(u)e^\alpha = A^\alpha u)A^\alpha(u)A^\alpha(u)e^\alpha, \\
&\vdots \\
k e^\alpha &= ke^\alpha(u)e^\alpha = A^\alpha u)\ldots A^\alpha u)A^\alpha(u)e^\alpha.
\end{align*}
\]

where the left–up index label the number of transforms in a chain and we can chose a coordinate base \(e^\alpha = \partial^\alpha\). The \((n+m)\times(n+m)\) dimensional matrices \(A^\alpha u), A^\alpha u), A^\alpha u)\) parametrize arbitrary frame transforms but subjected to the condition that their product \(k e^\alpha\) results is a triangle matrix of type (23), which induces at the final step of transforms, with fixed \((n+m)\)–splitting, a N–connection structure (13).

Having generated a N–anholonomic frame structure \(e^\alpha\), applying superpositions of nonholonomic transforms, we get, in general, hidden N–anholonomic frame structures of type

\[
e^\alpha = k A^\alpha(u)e^\alpha = A^\alpha(u)\ldots A^\alpha(u)A^\alpha(u)e^\alpha.
\]

We conclude that if a \((n+m)\)–splitting is prescribed by a N–connection \(N^\alpha_i\) on a nonholonomic manifold \(V\), we always can model this structure by certain chains of nonholonomic frames even the elements of the chains may not result in explicit forms of N–anholonomic frames. If the N–anholonomic structure of \(V^n+n\) is, for instance, of Lagrange type with canonical N–connection (13), by chains of transforms (55), we can hide the Lagrange structure (and, inversely, we can extract the Lagrange structure by chains of frame transforms (54) from certain special Riemannian or Riemann–Cartan configutations). In a more general context, we can work with generalized Lagrange structures \(\sharp \gamma_{ab}(u)\) and \(\sharp N^\alpha_j(u)\), see respectively (51) and (52), and (53), hidden or embedded in explicit form into a nonholonomic Riemannian space. The aim of such transforms is to relate a (pseudo) Riemannian metric structure (it can be a solution of the gravitational field equations) to certain generalized Lagrange, or Finsler, geometries which allows to apply new geometric methods and define additional symmetries and conservation laws, for instance, associated to bi–Hamilton structures and solitonic hierarchies.

**N–adapted frame transforms:**

For the class of general nonholonomic transforms considered above, we can can associate subclasses of matrices of type (23) for any element of the
chains,

\[ e_\alpha = e_\alpha \frac{\partial (u)}{\partial e_\alpha} \tag{56} \]

\[ 2e_\alpha = 2e_\alpha (u) \frac{\partial e_\alpha}{\partial \alpha} = e_\alpha \frac{\partial (u)}{\partial e_\alpha} e_\alpha \]

\[ 3e_\alpha = 3e_\alpha (u) \frac{\partial e_\alpha}{\partial \alpha} = e_\alpha \frac{\partial (u)}{\partial e_\alpha} e_\alpha \frac{\partial (u)}{\partial e_\alpha} e_\alpha \]

\[ \ldots \]

\[ k e_\alpha = k e_\alpha (u) \frac{\partial e_\alpha}{\partial \alpha} = e_\alpha \frac{\partial (u)}{\partial e_\alpha} \ldots \frac{\partial (u)}{\partial e_\alpha} \frac{\partial (u)}{\partial e_\alpha} e_\alpha, \]

\[ \ldots \]

which, for every step, transform an N–adapted base into another one, in general, with different N–connection coefficients, but with the same \( n \) and \( m \) for the \((n + m)\)–splitting. The reason to introduce such transforms is to relate a d–metric and N–connection structure to a special type ansatz for which the Einstein equations became integrable in general form, see section 3.3. There are possible chains of N–anholonomic frame transforms when some Lagrange spaces are nonholonomically related to generalized Lagrange spaces and then to some classes of exact solutions of the Einstein equations. The superpositions of nonholonomic frame transforms may be defined to depend on some classes of parameters, see details in ref. [209]. All types of nonholonomic deformations may be considered to change the signature of metrics if such constructions are necessary.

**N–connection transforms with fixed h– and v–metrics:**

For some purposes, for instance, in constructing exact solutions, or defining analogous models of gravity, it is useful to work with more special classes of nonholonomic deformations. The transforms of vielbeins (23) of type

\[ e_\alpha = \begin{bmatrix} e_i^i (u) & N_i^b (u) e_b^a (u) & \eta e_\alpha \\ 0 & 0 & e_\alpha (u) \end{bmatrix} \rightarrow \eta e_\alpha = \begin{bmatrix} e_i^i (u) & N_i^b (u) e_b^a (u) & \eta e_\alpha \\ 0 & 0 & e_\alpha (u) \end{bmatrix}, \]

preserve the h– and v–components of a d–metric (16). \( \eta g_{ij} = g_{ij} = e_i^i e_j^j g_{\alpha \alpha} \) and \( \eta g_{ab} = g_{ab} = e_a^a e_b^b g_{\alpha \alpha} \) for some prescribed values \( g_{\alpha \alpha} \) and \( g_{\alpha \alpha} \)

transform the N–connection,

\[ N_j^i \rightarrow \eta N_j^i = \eta_i^b N_i^b, \tag{57} \]

where we do not consider summation on repeating both up/low indices. Usually, it is supposed that such deformations of the N–connection structure preserve the \( n \)– and \( m \)–dimensions of splitting (9).

**N–anholonomic transforms of h– and v–metrics:**

We can fix a N–connection \( N_i^b (u) \) and consider frame transforms of type

\[ e_\alpha = \begin{bmatrix} e_i^i & N_i^b e_b^a \\ 0 & e_\alpha \end{bmatrix} \rightarrow \eta e_\alpha = \begin{bmatrix} e_i^i & N_i^b e_b^a \\ 0 & e_\alpha \end{bmatrix}, \]

33
which result in deformations of the h– and v–metrics, respectively,

\[
g_{ij}(u) = e_i^\xi(u) e_j^\lambda(u) g_{ij}^\lambda, \quad N g_{ij}(u) = \eta_i^\xi(u) \eta_j^\lambda(u) e_i^\xi(u) e_j^\lambda(u) g_{ij}^\lambda
\]

and

\[
g_{ab}(u) = e_a^\lambda(u) e_b^\lambda(u) g_{ab}, \quad N g_{ab}(u) = \eta_a^\lambda(u) \eta_b^\lambda(u) e_a^\lambda(u) e_b^\lambda(u) g_{ab},
\]

where we do not consider summation on indices for \(\eta_i^\xi e_i^\lambda\) but the Einstein summation rule is applied, for instance to ’up–low’ repeating indices like on underlined ones in \(e^\xi_a e^\lambda_b g_{ab}\).

For \(n + m = 2 + 2\), for simplicity, we can work only with diagonalized matrices for the h– and v–components of d–metrics of type (16), when \(g_{ij} = \text{diag}[g_1(u), g_2(u)]\) and \(h_{ab} = \text{diag}[h_3(u), h_4(u)]\) (such diagonalizations can be performed by coordinate transforms for matrices of dimension \(3 \times 3\) and \(2 \times 2\)). We can write in effectively diagonalized forms the deformations of h– and v–metrics, respectively, (58) and (59),

\[
g_{\alpha} = [g_i(u), h_a(u)] \rightarrow \eta g_{\alpha} = [\eta g_i = \eta_i(u) g_i(u), \eta h_i = \eta_a(u) h_a(u)],
\]

where \(\eta_a(u) = [\eta_i(u), \eta_a(u)]\) are called polarization functions, see exact solutions with such polarization functions in Ref. [228, 201]. The reason to introduce such polarizations was that for small polarizations \(\eta \sim 1 + \varepsilon_a(u)\), where \(|\varepsilon_a| \ll 1\), it was possible to generate small nonholonomic deformations of solutions of the Einstein equations, belonging in general to a class of exact solutions, but for small deformations possessing, for instance, black ellipsoid properties.

### 3.2.2 Lie algebroids and N–connections

A **Lie algebroid** \(A \doteq (E, [\cdot, \cdot], \rho)\) is defined as a vector bundle \(E = (E, \pi, M)\), with a surjective projection \(\pi : E \rightarrow M, \dim M = n \) and \(\dim E = n + m\), provided with **algebroid structure** \(([\cdot, \cdot], \rho)\), where \([\cdot, \cdot]\) is a Lie bracket on the \(C^\infty(M)\)–module of sections of \(E\), denoted \(\text{Sec}(E)\), and the ’anchor’ \(\rho\) is defined as a bundle map \(\rho : E \rightarrow M\) such that

\[
[X, fY] = f[X, Y] + \rho(X)(f)Y
\]

for \(X, Y \in \text{Sec}(E)\) and \(f \in C^\infty(M)\), see [98, 100] for general results and some applications of Lie algebroid geometry. In local form, the Lie algebroid structure is defined by its **structure functions** \(\rho_a^i(x)\) and \(C_{ab}^d(x)\) on \(M\), determined by the relations

\[
\rho(e_a) = \rho_a^i(x)e_i, \quad \quad (60)
\]
\[
[e_a, e_b] = C_{ab}^d(x)e_d \quad \quad (61)
\]
and subjected to the structure equations

\[ \rho^i_a \frac{\partial \rho^j_b}{\partial x^i} - \rho^j_b \frac{\partial \rho^i_a}{\partial x^j} = \rho^i_d C^d_{ab} \quad \text{and} \quad \sum_{cyclic(a,b,c)} \left( \rho^d_a \frac{\partial C^d_{bc}}{\partial x^j} + C^d_{af} C^f_{bc} \right) = 0, \quad (62) \]

for any local basis \( e_a = (e_i, e_b) \) on \( E \).

We extended the Lie algebroid constructions for nonholonomic manifolds and vector bundles provided with \( N \)-connection structure \( N = \{ N^a_i (x, y) \} \), respectively, on \( V \) and \( E \), and introduced the concept of \( N \)-anholonomic manifold \( N_A \equiv (V, [\cdot, \cdot], \hat{\rho}) \), see details and references in [202, 210] (we note that in this work we use \( \hat{\rho} \) instead of \( \hat{\rho} \)).

The Lie algebroid and \( N \)-connection structures prescribe a subclass of local both \( N \)-adapted and \([\cdot, \cdot]\)-adapted frames

\[ \hat{e}_a = (e_i = \frac{\partial}{\partial x^i} - \hat{N}^a_i \hat{e}_a, \hat{e}_b) \quad (63) \]

and dual coframes

\[ \hat{e}^a = (e^i, \hat{e}^b = \hat{e}^b + \hat{N}^b_i dx^i), \quad (64) \]

for some

\[ \hat{e}_b = \hat{e}^b \partial_{\hat{e}_b} \quad (65) \]

when \( \hat{e}_c, \hat{e}^b = \delta^b_c \). The \( N \)-connection coefficients may be redefined as

\[ N = N^a_i (u) dx^i \otimes \frac{\partial}{\partial y^a} = \tilde{N}^b_i (u) e^i \otimes \hat{e}_b, \]

where \( \tilde{N}^b_i = \hat{e}^b_a N^a_i \) and there are underlined the indices defining the coefficients with respect to a local coordinate basis.

Any Lie algebroid structure can be adapted to a prescribed \( N \)-connection and resulting frame structures (63) and (64). This can be done following the procedure: Let us re–define the coefficients of the anchor and structure functions with respect to the \( e_a \) and \( e^a \), when

\[ \rho^i_b (x) \rightarrow \tilde{\rho}^i_b (x, y) = e_i^j (x, y) e^b_j (x, y) \rho^j_b (x), \]
\[ C^d_{ab} (x) \rightarrow C^d_{ab} (x, y) = e^f_i (x, y) e^d_j (x, y) e^b_j (x, y) C^d_{ab} (x), \]

where the transform \( e \)-matrices are linear on coefficients \( N^a_i \) as can be obtained from (23). In terms of \( N \)-adapted anchor \( \tilde{\rho}^i_b (x, y) \) and structure functions \( C^d_{ab} (x, y) \) (which depend also on variables \( y^a \)), the structure equations of the Lie algebroids (60), (61) and (62) transform respectively into

\[ \tilde{\rho} (\hat{e}_b) = \tilde{\rho}^i_b (x, y) e_i, \quad (66) \]
\[ [\hat{e}_d, \hat{e}_b] = C^d_{ab} (x, y) \hat{e}_f. \quad (67) \]

35
and
\[ \sum_{\text{cyclic}(a,b,e)} \left( \tilde{\rho}_a^i e_j(C_{bc}^f) + C_{ag}^f C_{be}^g - C_{b'e'}^f \tilde{\rho}_a^{f' e'} Q_{f' e' \text{bej}}^{f'} \right) = 0, \]
for \( Q_{f' e' \text{bej}}^{f'} = e_{f'}^b e_{e'}^e e_{f}^f \) computed for the values \( e_{f'}^b \) and \( e_{f}^f \) taken from (23) and (24).

Using \( N \)-anholonomic Lie algebroid structures, we can apply certain methods of Finsler and Lagrange geometry to spacetimes provided with arbitrary d–metrics (16) when \( g^{ij} \neq h^{ab} \), see details and examples of exact solutions in Refs. \[210\]. In the simplest case, for a \( N \)-anholonomic manifold \( V^{n+n} \), we can work with a trivial anchor structure \( \tilde{\rho}_a^i = 0 \), when the conditions (66) and (68) are satisfied for certain nonholonomic frame configurations, but with the bracket \([\cdot, \cdot]\) induced by decomposition \( h^{ab}(u) = \tilde{e}_{a'}^a(u) \tilde{e}_{b'}^b(u) g^{a'b'}(u) \)
with
\[ \tilde{g} = g_{ij}(x,y) \left( dx^i \otimes dx^j + \tilde{e}_{a}^i \tilde{e}_{b}^j dy^a \otimes dy^b \right) = g_{ij}(x,y) \left( dx^i \otimes dx^j + \tilde{e}^i \otimes \tilde{e}^j \right) \]
where \( \tilde{e}^i = \tilde{e}_{a}^i dy^a \) states the coefficients \( C_{ag}^f(x,y) \) for (67) \[10\]. We can say that if \( g_{ij} \neq h_{ab} \), on \( V^{n+n} \), or \( TM \), we work similarly with the generalized Lagrange spaces but with modified prescribed frame structures (63) and (64) when nonholonomy coefficients are nontrivial both for the \( h \)-part and \( v \)-part. In a particular case, we can consider that \( g_{ij}(x,y) \) from (69) is of type \( L g_{ij}(x,y) \), see (16) and (18) (in certain more general or particular cases we can take (53), (51) or (49)), which models a Lagrange N–algebroid structure (respectively, generalized Lagrange, or Finsler, N–algebroid structure).

To work on generalized Lagrange N–algebroids with d–metrics of type (69) is convenient if we apply some methods from almost Hermitian/ Kähler geometry, see section 4. The Einstein equations for the d–metric (69) and canonical d–connection defined with respect to N–adapted bases (63) and (64) are equivalent to equations (40) for the canonical d–connection (36), redefined with respect to N–algebroid bases.

### 3.3 An ansatz for constructing exact solutions

We consider a four dimensional (4D) manifold \( V \) of necessary smooth class and conventional splitting of dimensions \( \text{dim} V = n + m \) for \( n = 2 \) and \( m = 2 \).

\[10\] see Refs. \[202, 210\] for constructions with nontrivial \( \tilde{\rho} \)-algebroid models with nontrivial anchor maps are useful in extra dimension gravity, for instance, with nonholonomic splitting of dimensions of type \( n + m \geq 5 \), when \( n \geq 2 \) and \( m \geq n \); for simplicity, we omit such constructions in this work.
The local coordinates are labelled in the form $u^a = (x^1, y^a) = (x^1, x^2, y^3 = v, y^4)$, for $i = 1, 2$ and $a, b, ..., = 3, 4$.

The ansatz of type (10) is parametrized in the form

$$
g = g_1(x^i) dx^1 \otimes dx^1 + g_2(x^i) dx^2 \otimes dx^2 + h_3(x^k, v) \delta v \otimes \delta v + h_4(x^k, v) \delta y \otimes \delta y, \tag{70}$$

$$\delta v = dv + w_i(x^k, v) dx^i, \delta y = dy + n_i(x^k, v) dx^i$$

with the coefficients defined by some necessary smooth class functions of type

$$g_{1,2} = g_{1,2}(x^1, x^2), h_{3,4} = h_{3,4}(x^3, v), w_i = w_i(x^k, v), n_i = n_i(x^k, v).$$

The off–diagonal terms of this metric, written with respect to the coordinate dual frame $dx^a = (dx^i, dy^a)$, can be redefined to state a N–connection structure $N = [N_3^i = w_i(x^k, v), N_4^i = n_i(x^k, v)]$ with a N–elongated co–frame (14) parametrized as

$$e^1 = dx^1, e^2 = dx^2, e^3 = \delta v = dv + w_i dx^i, e^4 = \delta y = dy + n_i dx^i. \tag{71}$$

This vielbein is dual to the local basis

$$e_i = \frac{\partial}{\partial x^i} - w_i(x^k, v) \frac{\partial}{\partial v} - n_i(x^k, v) \frac{\partial}{\partial y^5}, e_3 = \frac{\partial}{\partial v}, e_4 = \frac{\partial}{\partial y^5}, \tag{72}$$

which is a particular case of the N–adapted frame (13). The metric (70) does not depend on variable $y^4$, i.e. it possesses a Killing vector $e_4 = \partial/\partial y^4$, and distinguish the dependence on the so–called ”anisotropic” variable $y^3 = v$.

Computing the components of the Ricci and Einstein tensors for the metric (70) and canonical d–connection (see details on tensors components’ calculus in Refs. [210, 228]), one proves that the Einstein equations (10) for a diagonal with respect to (71) and (72) source,

$$\Upsilon^\alpha_\beta + Z \Upsilon^\beta_\beta = [\Upsilon_1 = \Upsilon_2(x^i, v), \Upsilon_2^2 = \Upsilon_2(x^i, v), \Upsilon_3 = \Upsilon_4(x^i), \Upsilon_4^4 = \Upsilon_4(x^i)] \tag{73}$$

transform into this system of partial differential equations:

$$\hat{R}_1^1 = \hat{R}_3^3 = \frac{1}{2g_1g_2} \left[ g_1 g_2^* + (g_2^*)^2 - g_2^* + \frac{g'_1 g'_2}{2g_2} + \frac{(g'_1)^2}{2g_1} - g''_1 \right] = -\Upsilon_4(x^i), \tag{74}$$

$$\hat{S}_3^3 = \hat{S}_4^4 = \frac{1}{2h_3 h_4} \left[ h_4^* \left( \ln \sqrt{|h_3 h_4|} \right)^* - h_4^{**} \right] = -\Upsilon_2(x^i, v), \tag{75}$$

$$\hat{R}_{3i} = -w_i(\beta - \alpha_i) = 0, \tag{76}$$

$$\hat{R}_{4i} = -\frac{h_3}{2h_4} [n_i^{**} + \gamma n_i^*] = 0, \tag{77}$$

37
where, for \( h_{3,4}^* \neq 0 \),

\[
\alpha_i = h_i^* \partial_i \phi, \quad \beta = h_4^* \phi^*, \quad \gamma = \frac{3h_4^*}{2h_4} - \frac{h_3^*}{h_3}, \quad (78)
\]

\[
\phi = \ln \left| \frac{h_3^*}{\sqrt{|h_3 h_4^*|}} \right|, \quad (79)
\]

when the necessary partial derivatives are written in the form \( a^* = \partial a / \partial x^1 \), \( a' = \partial a / \partial x^2 \), \( a^* = \partial a / \partial v \). In the vacuum case, we must consider \( \Upsilon_{2,4} = 0 \). We note that we use a source of type \( 73 \) in order to show that the anholonomic frame method can be applied also for non–vacuum solutions, for instance, when \( \Upsilon_2 = \lambda_2 = \text{const} \) and \( \Upsilon_4 = \lambda_4 = \text{const} \), defining locally anisotropic configurations generated by an anisotropic cosmological constant, which in its turn, can be induced by certain ansatz for the so–called \( H \)–field (absolutely antisymmetric third rank tensor field) in string theory \( 201, 228, 210 \). Here we note that the off–diagonal gravitational interactions can model locally anisotropic configurations even if \( \lambda_2 = \lambda_4 \), or both values vanish.

In string gravity, the nontrivial torsion components and source \( \kappa \Upsilon_{\alpha \beta} \) can be related to certain effective interactions with the strength (torsion) \( H_{\mu \nu \rho} = e_\mu B_{\nu \rho} + e_\rho B_{\nu \mu} + e_\nu B_{\rho \mu} \) of an antisymmetric field \( B_{\mu \rho} \), when

\[
R_{\mu \nu} = -\frac{1}{4} H_{\mu}^{\ \nu \rho} H_{\nu \lambda \rho} \quad (80)
\]

and

\[
D_\lambda H_{\lambda \mu \nu} = 0, \quad (81)
\]

see details on string gravity, for instance, in Refs. \( 53, 139 \). The conditions \( 80 \) and \( 81 \) are satisfied by the ansatz

\[
H_{\mu \nu \rho} = \tilde{Z}_{\mu \nu \rho} + \tilde{H}_{\mu \nu \rho} = \lambda[H] \sqrt{|g_{\alpha \beta}|} \varepsilon_{\nu \lambda \rho} \quad (82)
\]

where \( \varepsilon_{\nu \lambda \rho} \) is completely antisymmetric and the distorsion (from the Levi–Civita connection) and

\[
\tilde{Z}_{\mu \alpha \beta} e^\mu = e_\beta | T_\alpha - e_\alpha | T_\beta + \frac{1}{2} (e_\alpha | e_\beta | T_\gamma) e^\gamma
\]

is defined by the torsion tensor \( 28 \). Our \( H \)–field ansatz is different from those already used in string gravity when \( \tilde{H}_{\mu \nu \rho} = \lambda[H] \sqrt{|g_{\alpha \beta}|} \varepsilon_{\nu \lambda \rho} \). In our approach, we define \( H_{\mu \nu \rho} \) and \( \tilde{Z}_{\mu \nu \rho} \) from the respective ansatz for the \( H \)–field and nonholonomically deformed metric, compute the torsion tensor for the canonical distinguished connection and, finally, define the ‘deformed’ \( H \)–field as \( \tilde{H}_{\mu \nu \rho} = \lambda[H] \sqrt{|g_{\alpha \beta}|} \varepsilon_{\nu \lambda \rho} - \tilde{Z}_{\mu \nu \rho} \).
Summarizing the results for an ansatz (70) with arbitrary signatures 
$\epsilon_{\alpha} = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ (where $\epsilon_{\alpha} = \pm 1$) and $h_3^* \neq 0$ and $h_4^* \neq 0$, one proves that any off–diagonal metric 

$$
\circ g = e^{\psi(x^i)} \left[ \epsilon_1 \, dx^1 \otimes dx^1 + \epsilon_2 \, dx^2 \otimes dx^2 \right] 
+ \epsilon_3 h_0^2(x^i) \left[ f^* (x^i, v) \right]^2 \varsigma (x^i, v) \mid \delta v \otimes \delta v 
+ \epsilon_4 \left[ f (x^i, v) - f_0(x^i) \right]^2 \delta y^4 \otimes \delta y^4, 
$$

$$
\delta v = dv + w_k (x^i, v) \, dx^k, \quad \delta y^4 = dy^4 + n_k (x^i, v) \, dx^k, 
$$

(83)
declare variable \ \psi(x^i) \ \ is \ \ a \ \ solution \ \ of \ \ the \ \ 2D \ \ equation

$$
\epsilon_1 \psi^{**} + \epsilon_2 \psi'' = \Upsilon_4,
$$

for a given source $\Upsilon_4 (x^i)$, 

$$
\varsigma (x^i, v) = \varsigma_0 (x^i) - \frac{\epsilon_3}{8} h_0^2(x^i) \int \Upsilon_2(x^k, v) f^* (x^i, v) \left[ f (x^i, v) - f_0(x^i) \right] \, dv,
$$

and the N–connection coefficients $N_i^3 = w_i(x^k, v)$ and $N_i^4 = n_i(x^k, v)$ are computed following the formulas

$$
w_i = - \frac{\partial_i \varsigma (x^i, v)}{\varsigma^* (x^i, v)} 
$$

(84)

and

$$
n_k = 1 n_k (x^i) + 2 n_k (x^i) \int \frac{[f^* (x^i, v)]^2}{[f (x^i, v) - f_0(x^i)]^3} \varsigma (x^i, v) \, dv, 
$$

(85)
declare variable \ \ define \ \ an \ \ exact \ \ solution \ \ of \ \ the \ \ system \ \ of \ \ Einstein \ \ equations \ \ (74)–(77). \ \ It \ \ should \ \ be \ \ emphasized \ \ that \ \ such \ \ solutions \ \ depend \ \ on \ \ arbitrary \ \ nontrivial \ \ functions \ \ f (x^i, v) \ \ (with \ \ f^* \neq 0), \ \ f_0(x^i), \ \ h_0^2(x^i), \ \ \varsigma_0 (x^i), \ \ 1 n_k (x^i) \ \ and \ \ 2 n_k (x^i), \ \ and \ \ sources \ \ \Upsilon_2(x^k, v), \ \ \Upsilon_4 (x^i). \ \ Such \ \ values \ \ for \ \ the \ \ corresponding \ \ signatures \ \ \epsilon_{\alpha} = \pm 1 \ \ have \ \ to \ \ be \ \ defined \ \ by \ \ certain \ \ boundary \ \ conditions \ \ and \ \ physical \ \ considerations. \ \ These \ \ classes \ \ of \ \ solutions \ \ depending \ \ on \ \ integration \ \ functions \ \ are \ \ more \ \ general \ \ than \ \ those \ \ for \ \ diagonal \ \ ansatz \ \ depending, \ \ for \ \ instance, \ \ on \ \ one \ \ radial \ \ variable \ \ like \ \ in \ \ the \ \ case \ \ of \ \ the \ \ Schwarzschild \ \ solution \ \ (when \ \ the \ \ Einstein \ \ equations \ \ are \ \ reduced \ \ to \ \ an \ \ effective \ \ nonlinear \ \ ordinary \ \ differential \ \ equation, \ \ ODE). \ \ In \ \ the \ \ case \ \ of \ \ ODE, \ \ the \ \ integral \ \ varieties \ \ depend \ \ on \ \ integration \ \ constants \ \ which \ \ can \ \ be \ \ defined \ \ from \ \ certain \ \ boundary/ \ \ asymptotic \ \ and \ \ symmetry \ \ conditions, \ \ for \ \ instance, \ \ from \ \ the \ \ constraint \ \ that \ \ far \ \ away \ \ from \ \ the \ \ horizon \ \ the \ \ Schwarzschild \ \ metric \ \ contains \ \ corrections \ \ from \ \ the \ \ Newton \ \ potential. \ \ Because \ \ the \ \ ansatz (70) \ \ results \ \ in \ \ a \ \ system \ \ of \ \ nonlinear \ \ partial \ \ differential \ \ equations \ \ (74)–(77), \ \ the \ \ solutions \ \ depend \ \ not \ \ only \ \ on \ \ integration \ \ constants, \ \ but \ \ on \ \ very \ \ general \ \ classes \ \ of \ \ integration \ \ functions.
The ansatz of type (70) with \( h^*_3 = 0 \) but \( h^*_4 \neq 0 \) (or, inversely, \( h^*_3 \neq 0 \) but \( h^*_4 = 0 \)) consist more special cases and request a bit different method of constructing exact solutions. Nevertheless, such type solutions are also generic off–diagonal and they may be of substantial interest (the length of paper does not allow to include an analysis of such particular cases).

A very general class of exact solutions of the Einstein equations with nontrivial sources (73), in general relativity, is defined by the ansatz

\[
\mathfrak{g} = e^{\psi(x^i)} \left[ \epsilon_1 \, dx^1 \otimes dx^1 + \epsilon_2 \, dx^2 \otimes dx^2 \right] + h_3 (x^i, v) \, \delta v \otimes \delta v + h_4 (x^i, v) \, \delta y^4 \otimes \delta y^4,
\]

\[
\delta v = dv + w_1 (x^i, v) \, dx^1 + w_2 (x^i, v) \, dx^2,
\]

\[
\delta y^4 = dy^4 + n_1 (x^i) \, dx^1 + n_2 (x^i) \, dx^2,
\]

with the coefficients restricted to satisfy the conditions

\[
\epsilon_1 \psi'' + \epsilon_2 \psi'' = \Upsilon_4,
\]

\[
h^*_3 \phi/h^*_4 = \Upsilon_2,
\]

\[
w_1' - w_2^* + w_2 w_1^* - w_1 w_2^* = 0,
\]

\[
n_1' - n_2^* = 0,
\]

for \( w_i = \partial_i \phi/\phi^* \), see (79), for given sources \( \Upsilon_4(x^k) \) and \( \Upsilon_2(x^k, v) \). We note that the second equation in (87) relates two functions \( h_3 \) and \( h_4 \) and the third and forth equations satisfy the conditions (43).

Even the ansatz (70) depends on three coordinates \( (x^k, v) \), it allows us to construct more general classes of solutions for \( d \)-metrics, depending on four coordinates: such solutions can be related by chains of nonholonomic transforms (54), (56), or (58) and (59). New classes of generic off–diagonal solutions will describe nonholonomic Einstein spaces related to string gravity, if one of the chain metric is of type (83), or in Einstein gravity, if one of the chain metric is of type (86). The geometries of such spacetimes are modelled equivalently by corresponding classes of N–anholonomic algebroids provided with metric structures of type (69).

### 4 Einstein Gravity and Lagrange–Kähler Spaces

We show how nonholonomic Riemannian spaces and generalized Lagrange algebroids can be transformed into almost Hermitian manifolds enabled with nonintegrable almost complex structures.

#### 4.1 Generalized Lagrange–Hermitian algebroids

Let \( \tilde{e}_a = (e_i, \tilde{e}_b) \) (63) and \( \tilde{e}^a = (e^i, \tilde{e}^b) \) (64) be, respectively, some N–adapted frames and co–frames on \( V^{n+n} \). In a particular case, \( V^{n+n} = TM \).
We introduce a linear operator \( \tilde{F} \) acting on the vectors on \( V^{n+n} \) following formulas
\[
\tilde{F}(e_i) = -\tilde{e}_i \quad \text{and} \quad \tilde{F}(\tilde{e}_i) = e_i,
\]
where the superposition \( \tilde{F} \circ \tilde{F} = -I \), for \( I \) being the unity matrix. On \( TM \), for vertically integrable distributions, when \( C^{ij}_{ab} = 0 \) for \( 67 \), we get an \textbf{almost Hermitian model} of generalized Lagrange space \([107, 108]\). The operator \( \tilde{F} \) reduces to a \underline{complex structure} if and only if both the \( h- \) and \( v- \)distributions are integrable.

The metric \( g \) \([59]\) induces a 2–form associated to \( \tilde{F} \) following formulas
\[
\tilde{\theta}(X, Y) = \tilde{g}(\tilde{F}X, Y)
\]
for any \( d- \)vectors \( X \) and \( Y \). In local form, we have
\[
\tilde{\theta} = g_{ij}(x, y)\tilde{e}^i \wedge dx^j.
\]
For \( v- \)integrable distributions, we shall write \( F \) and \( \theta \) in order to emphasize that we model generalized Lagrange structures.

We compute
\[
d\tilde{\theta} = \frac{1}{6} \sum_{(ijk)} g_{is} \tilde{\Omega}^s_{jk} dx^i \wedge dx^j \wedge dx^k
\]
\[
+ \frac{1}{2} (g_{ij} \tilde{g}_{k} + g_{ik} \tilde{g}_{j}) \tilde{e}^i \wedge dx^j \wedge dx^k + \frac{1}{2} (\tilde{e}^i \tilde{g}_{ij} - \tilde{e}^i \tilde{g}_{kj}) \tilde{e}^k \wedge \tilde{e}^j \wedge dx^i,
\]
where
\[
g_{ij} \tilde{g}_{k} = e_k g_{ij} - \tilde{B}_{ik}^s g_{sj} - \tilde{\tilde{B}}_{jk}^s g_{is},
\]
for \( \tilde{B}_{ik}^s = \tilde{e}_i \tilde{N}_k^s \) and \( \tilde{N}_i^b = \tilde{e}_b \tilde{N}_i \).

An \textbf{almost Hermitian model} of a generalized \( N \)-algebroid structure is defined by a triple \( \tilde{H}^2n = (V^{n+n}, \tilde{g}, \tilde{F}) \). A space \( \tilde{H}^2n \) is almost Kähler if and only if \( d\tilde{\theta} = 0 \), i.e.
\[
\tilde{\Omega}_{jk} = 0, g_{ij} \tilde{g}_{k} = g_{ik} \tilde{g}_{j}, \tilde{e}_k g_{ij} = \tilde{e}_i g_{kj}.
\]

A generalized Lagrange algebroid is not reducible to a Lagrange one if \( \tilde{e}_k g_{ij} \neq \tilde{e}_i g_{kj} \), i.e. the almost symplectic structure \( \tilde{\theta} \) \([88]\) is not integrable.

One considers also \( h- \) (\( v- \)) Hermitian spaces \( \tilde{H}^{2n} \) if the \( h- \) (\( v- \)) distributions are integrable. For instance, the almost Hermitian model of a Finsler (or Lagrange) space is an almost Kähler space \([102]\) (or \([126]\)).

An almost Hermitian connection \( \tilde{D} \) is of Lagrange type if it preserves by parallelism the vertical distribution and is compatible with the almost Hermitian structure \( (\tilde{g}, \tilde{F}) \), i.e. \( \tilde{D}_X \tilde{g} = 0 \) and \( \tilde{D}_X \tilde{F} = 0 \) for any \( d- \)vector \( X \). Considering the canonical metrical \( d- \)connections for \( \tilde{g} \), we construct a canonical almost Hermitian connection \( \tilde{D} \), for \( \tilde{H}^{2n} \), with the coefficients \( \tilde{F}^{a}_{b \gamma} = \left( \tilde{L}^{a}_{jk}, \tilde{C}^{a}_{bc} \right) \), when
\[
\tilde{L}^a_{jk} = \frac{1}{2} g^{ij} (e_k g_{jh} + e_j g_{kh} - e_h g_{jk}),
\]
\[
\tilde{C}^{a}_{bc} = \tilde{e}^a_{\beta} \tilde{e}^b_{\gamma} e^c_{\alpha} \tilde{C}^{\alpha}_{be} + \tilde{e}^a_{\alpha} \partial_{\beta} \tilde{e}^b_{\gamma},
\]
where $\tilde{e}_b = e_b \partial_b$ (65) and $	ilde{C}^a_{bc} = \frac{1}{2} g^{ae} (e_b g_{ec} + e_c g_{eb} - e_e g_{bc})$ computed as in (44) but with respect to the “underlined” basis, when $g^{ae}$ with respect to $e_i \otimes e_j$ has the same values as $g^{ij}$ with respect to $e_e \otimes e_e$.

The curvature of (90) is

$$\tilde{R}_{hjk} = e_k \tilde{L}_{ij} - e_j \tilde{L}_{hk} \tilde{L}_{mj} - \tilde{L}_{hj} \tilde{L}_{mk} - \tilde{C}^i_{ha} \tilde{\Omega}^{ik}_{kj},$$

$$\tilde{P}_{jka} = \tilde{e}_a \tilde{L}_{jk} - \tilde{D}_k \tilde{C}^{ia}_{ja}, \text{ for } \tilde{S}_{bcd} = \tilde{e}_a \tilde{\Omega}^{ia}_{bc} \tilde{e}_b \tilde{\Omega}^{ja}_{cd} \tilde{e}_c \tilde{\Omega}^{ka}_{de} \tilde{S}^{ab}_{cd}, \quad (91)$$

for $\tilde{S}_{bcd} = e_d \tilde{C}^a_{bc} - e_e \tilde{C}^a_{bd} + \tilde{C}^e_{be} \tilde{C}^a_{ed} - \tilde{C}^e_{bd} \tilde{C}^a_{ec}$ computed as in (45) but with respect to the “underlined” base.

### 4.2 Almost Hermitian connections and general relativity

We prove that the Einstein gravity on a (pseudo) Riemannian manifold $V^{n+n}$ can be equivalently redefined as an almost Hermitian model for N–anholonomic Lie algebroids if a nonintegrable N–connection splitting is prescribed. The Einstein theory can be also modified by considering certain canonical lifts on tangent bundles. The first class of Finsler–Lagrange like models [215] preserves the local Lorentz symmetry and can be applied for constructing exact solutions in Einstein gravity or for developing some approaches to quantum gravity following methods of geometric/deformation quantization. The second class of such models [214] can be considered for some extensions to canonical quantum theories of gravity which can be elaborated in a renormalizable form, but, in general, result in violation of local Lorentz symmetry by such quantum effects.

#### 4.2.1 Nonholonomic deformations in Einstein gravity

Let us consider a metric $g_{\alpha \beta}$ (18), which for a $(n+n)$–splitting by a set of prescribed coefficients $N^a_i(x, y)$ can be represented as a d–metric $g$ (15), or $\tilde{g}$ (69), in N–algebroid form. Respectively, we can write the Einstein equations in the form (7), or, equivalently, in the form (40) with the source $Z \Upsilon_{\alpha \beta}$ defined by the off–diagonal metric coefficients of $g_{\alpha \beta}$, depending linearly on $N^a_i$, and generating the distorsion tensor $Z \Upsilon_{\alpha \beta}$ (39).

Computing the Ricci and Einstein d–tensors by contracting the indices in (91), we conclude that the Einstein equations written in terms of the almost Hermitian d–connection can be also parametrized in the form (40). Such geometric structures are nonholonomic: working respectively with $g$, $\tilde{g}$ or $\tilde{g}$, we elaborate equivalent geometric and physical models on $V^{n+n}$, $\tilde{V}^{n+n}$, or $N \mathcal{A} \tilde{=} (V^{n+n}, [\cdot, \cdot], \tilde{\rho})$ and $H^{2n}(V^{n+n}, \tilde{g}, \tilde{F})$. Even for vacuum configurations, when $\Upsilon_{\alpha \beta} = 0$, in the almost Hermitian model of the Einstein gravity, we have an effective source $Z \Upsilon_{\alpha \beta}$ induced by the coefficients of generic off–diagonal metric. Nevertheless, there are possible integrable configurations, when the conditions (43) are satisfied. In this case, $Z \Upsilon_{\alpha \beta} = 0$, and we
can construct effective Hermitian configurations defining vacuum Einstein foliations.

One should be noted that the geometry of nonholonomic $2+2$ splitting in general relativity, with nonholonomic frames and $d$-connections, or almost Hermitian connections, is very different from the geometry of the well known $3+1$ splitting ADM formalism, see [113], when only the Levi Civita connection is used. Following the anholonomic frame method, we work with different classes of connections and frames when some new symmetries and invariants are distinguished and the field equations became exactly integrable for some general metric ansatz. Constraining or redefining the integral varieties and geometric objects, we can generate, for instance, exact solutions in Einstein gravity and compute quantum corrections to such solutions.

4.2.2 Conformal lifts of Einstein structures to tangent bundles

Let us consider a pseudo–Riemannian manifold $M$ enabled with a metric $\bar{g}_{ij}(x)$ as a solution of the Einstein equations. We define a procedure lifting $\bar{g}_{ij}(x)$ conformally on $TM$ and inducing a generalized Lagrange structure and a corresponding almost Hermitian geometry. Let us introduce

$$\varpi L(x,y) \doteq \varpi^2(x,y)g_{ab}(x)y^ay^b$$

and use

$$\varpi g_{ab} \doteq \frac{1}{2} \frac{\partial^2 \varpi L}{\partial y^a \partial y^b}$$  \hspace{1cm} (92)$$

as a Lagrange Hessian for (48). A space $GL^n = (M, \varpi g_{ij}(x,y))$ possess a weakly regular conformally deformed metric if $L^n = \left[ M, L = \sqrt{\varpi L} \right]$ is a Lagrange space. We can construct a canonical $N$–connection $\varpi N^a_i$ following formulas (52), using $\varpi L$ instead of $L$ and $\varpi g_{ab}$ instead of $L_{g_{ab}}$ (51), and define a $d$–metric on $TM$,

$$\varpi g = \varpi g_{ij}(x,y) \, dx^i \otimes dx^j + \varpi g_{ij}(x,y) \, \varpi e^i \otimes \varpi e^j,$$  \hspace{1cm} (93)$$

where $\varpi e^i = dy^i + \varpi N^j_i dx^j$. The canonical $d$–connection and corresponding curvatures are constructed as in generalized Lagrange geometry but using $\varpi g$.

It is possible to reformulate the model with $d$–metric (93) for almost Hermitian spaces as we considered in section 4.1. For the $d$–metric (93), the model is elaborated for tangent bundles with holonomic vertical frame structure. The linear operator $F$ defining the almost complex structure acts on $TM$ following formulas

$$F(\varpi e_i) = -\partial_i \quad \text{and} \quad F(\partial_i) = \varpi e_i,$$
when $F \circ F = -I$, for $I$ being the unity matrix. The operator $F$ reduces to a complex structure if and only if the $h$–distribution is integrable.

The metric $\varpi \mathbf{g} (\mathbf{g})$ induces a 2–form associated to $F$ following formulas $\varpi \theta (\mathbf{X}, \mathbf{Y}) \triangleq \varpi \mathbf{g} (F \mathbf{X}, \mathbf{Y})$ for any $d$–vectors $\mathbf{X}$ and $\mathbf{Y}$. In local form, we have

$$\varpi \theta = \varpi g_{ij} (x,y) dy^i \wedge dx^j,$$

similarly to (88). The canonical $d$–connection $\varpi \hat{\mathbf{D}}$, with $N$–adapted coefficients $\varpi \Gamma_\alpha^{\beta\gamma} = \left( \varpi \hat{\mathcal{L}}_j^{\beta} + \varpi \hat{\mathcal{C}}_{bc} \right)$, and corresponding $d$–curvature can be computed respectively by the formulas (90) and (91) where $\hat{\varepsilon}_b^b = \delta_b^b$ and $\varpi g_{ij}$ are used instead of $g_{ij}$.

The model of almost Hermitian gravity $H^{2n}(\mathbf{T}M, \varpi \mathbf{g}, F)$ can be applied in order to construct different extensions of general relativity to geometric quantum models on tangent bundle [214]. Such models will result positively in violation of local Lorentz symmetry, because the geometric objects depend on fiber variables $y^a$. The quasi–classical corrections can be obtained in the approximation $\varpi \sim 1$. We omit in this work consideration of quantum models, but note that Finsler methods and almost Kähler geometry seem to be very useful for such generalizations of Einstein gravity.

5 Finsler–Lagrange Metrics in Einstein & String Gravity

We consider certain general conditions when Lagrange and Finsler structures can be modelled as exact solutions in string and Einstein gravity. Then, we analyze two explicit examples of exact solutions of the Einstein equations modelling generalized Lagrange–Finsler geometries and nonholonomic deformations of physically valuable equations in Einstein gravity to such locally anisotropic configurations.

5.1 Einstein spaces modelling generalized Finsler structures

In this section, we outline the calculation leading from generalized Lagrange and Finsler structures to exact solutions in gravity.

Let us consider a $d$–metric of type (69), which is also nonholonomically deformed in the $h$–part, when

$$\varepsilon \mathbf{g} = \varepsilon g_{i'j'} (x^{k'}, y^{l'}) \left( e'^i \otimes e'^j + \varepsilon e'^i \otimes \varepsilon e'^j \right),$$

where $\varepsilon g_{i'j'}$ can be any metric defined by nonholonomic transforms (53) or a $v$–metric $\varepsilon g_{ij}$ (51), $L g_{ij}$ (46), or $F g_{ij}$ (49). The co–frame $h$– and $v$–bases are

$$e'^i = e'^i (x,y) \ dx^i,$$

$$\varepsilon e'^i = \varepsilon e'^i (x,y) \ dy^a = \varepsilon g^a \left( dy^a + N_i^a dx^i \right) = e'^a + N_i^a e'^i,$$
for \( e^{\alpha'} = \tilde{e}^{\alpha'}_a dy^a \) and \( \tilde{N}^{a}_{\alpha'} = \tilde{\bar{\varepsilon}}^{\alpha'}_a \tilde{N}^{a}_{\alpha'} \), when there are considered nonholonomic transforms of type (58), (59) and (57),

\[
\varepsilon g_{\alpha'\nu'} = e^{\alpha'}_i e^{\nu'}_i \cdot g_{ij}, \quad h_{ab} = \varepsilon g_{a'b'} \tilde{\varepsilon}^{\alpha'}_a \tilde{\varepsilon}^{\beta'}_b, \quad \varepsilon N_i^a = \eta_i^a(x,y) \varepsilon N_i^a, \tag{95}
\]

where we do not consider summation on indices for "polarization" functions \( \eta_i^a \) and \( \varepsilon N_i^a \) is a canonical connection corresponding to \( \varepsilon g_{\alpha'\nu'} \).

The \( d\)-metric (94) is equivalently transformed into the \( d\)-metric

\[
\begin{align*}
\tilde{\mathbf{g}} & = g_{ij}(x)dx^i \otimes dx^j + h_{ab}(x,y)dy^a \otimes dy^b, \\
\delta y^a & = dy^a + \varepsilon N_i^a(x,y)dx^i,
\end{align*}
\]

where the coefficients \( g_{ij}(x) \), \( h_{ab}(x,y) \) and \( \varepsilon N_i^a(x,y) \) are constrained to be defined by a class of exact solutions (83), in string gravity, or (86), in Einstein gravity. If it is possible to get the limit \( \eta_i^a \to 1 \), we can say that an exact solution (96) models exactly a respective (generalized) Lagrange, or Finsler, configuration. We argue that we define a nonholonomic deformation of a Finsler (Lagrange) space given by data \( \varepsilon g_{\alpha'\nu'} \) and \( \varepsilon N_i^a \) as a class of exact solutions of the Einstein equations given by data \( g_{ij} \), \( h_{ab} \) and \( N_i^a \), for any \( \eta_i^a(x,y) \neq 1 \). Such constructions are possible, if certain nontrivial values of \( \varepsilon g_{\alpha'\nu'} \) and \( \varepsilon N_i^a \) can be algebraically defined from relations (95) for any defined sets of coefficients of the \( d\)-metric (94) and (96).

Expressing a solution in the form (94), we can define the corresponding almost Hermitian 1-form

\[
\tilde{\mathbf{f}}(e_{\nu'}) = -\tilde{e}_{\nu'} \quad \text{and} \quad \tilde{\mathbf{f}}(\tilde{e}_{\nu'}) = e_{\nu'},
\]

when \( e_{\nu'} = e_{\nu'} \left( \frac{\partial}{\partial x^a} - \varepsilon N_i^a \frac{\partial}{\partial y^a} \right) = e_{\nu'} - \varepsilon N_i^a \tilde{e}_{\nu'} \). This is convenient for further applications to certain models of quantum gravity and geometry. For explicit constructions of the solutions, it is more convenient to work with parametrizations of type (96).

Finally, in this section, we note that the general properties of integral varieties of such classes of solutions are discussed in Refs. 209, 228.

### 5.2 Deformation of Einstein exact solutions into Lagrange–Finsler metrics

Let us consider a metric ansatz \( g_{a\beta} \) with quadratic metric interval

\[
ds^2 = g_1(x^1, x^2) (dx^1)^2 + g_2(x^1, x^2) (dx^2)^2 + h_3(x^1, x^2, v) [dv + w_1(x^1, x^2, v) dx^1 + w_2(x^1, x^2, v) dx^2]^2 + h_4(x^1, x^2, v) [dy^4 + n_1(x^1, x^2, v) dx^1 + n_2(x^1, x^2, v) dx^2]^2 \tag{97}
\]
defining an exact solution of the Einstein equations (7), for the Levi–Civita connection, when the source $\Upsilon_{\alpha\beta}$ is zero or defined by a cosmological constant. We parametrize the coordinates in the form $u^\alpha = (x^1, x^2, y^3 = v, y^4)$ and the N–connection coefficients as $N^3 = w_i$ and $N^4 = n_i$.

We nonholonomically deform the coefficients of the primary d–metric (97), similarly to (95), when the target quadratic interval

$$ds^2_{\eta} = g_i (dx^i)^2 + h_a (dy^a + N^a_i dx^i)^2$$

similarly to ansatz (70), and defines a solution of type (83) (with N–connection coefficients (84) and (85)), for the canonical d–connection, or a solution of type (86) with the coefficients subjected to solve the conditions (87).

The class of target metrics (98) and (99) defining the result of a nonholonomic deformation of the primary data $[g_i, h_a, N^b_i]$ to a Finsler–Lagrange configuration $[\varepsilon g_{ij}, \varepsilon N^b_j]$ are parametrized by values $\varepsilon_{i}^i, \varepsilon_{a}^a$ and $\eta_i^a$. These values can be expressed in terms of some generation and integration functions and the coefficients of the primary and Finsler like d–metrics and N–connections in such a manner when a primary class of exact solutions is transformed into a "more general" class of exact solutions. In a particular case, we can search for solutions when the target metrics transform into primary metrics under certain infinitesimal limits of the nonholonomic deforms.

In general form, the solutions of equations (41) transformed into the system of partial differential equations (74)–(77), for the d–metrics (98), equivalently (99), are given by corresponding sets of frame, N–connections and d–metric coefficients which state for the h–part
and for the v–part

\[ e_{1}' = \sqrt{|\eta_1|} \sqrt{|g_1|} \times \sqrt{|\varepsilon g_{1'1'} \varepsilon g_{2'2'} \left[ (\varepsilon g_{1'1'})^2 \varepsilon g_{2'2'} + (\varepsilon g_{1'2'})^3 \right]^{-1}|, \]

\[ e_{2}' = \sqrt{|\eta_2|} \sqrt{|g_1|} / \sqrt{|\varepsilon g_{1'1'} \varepsilon g_{2'2'} \left[ (\varepsilon g_{1'1'})^2 \varepsilon g_{2'2'} + (\varepsilon g_{1'2'})^3 \right]|, \]

\[ e_{1}' = -\sqrt{|\eta_1|} \sqrt{|g_2|} \times g_{1'2'} / \sqrt{|\varepsilon g_{1'1'} \left[ (\varepsilon g_{1'1'}) \varepsilon g_{2'2'} - (\varepsilon g_{1'2'})^2 \right]|, \]

\[ e_{2}' = \sqrt{|\eta_2|} \sqrt{|g_2|} \times \sqrt{|\varepsilon g_{1'1'} / \left[ (\varepsilon g_{1'1'}) \varepsilon g_{2'2'} - (\varepsilon g_{1'2'})^2 \right]|, \]

(100)

and for the v–part

\[ e_{3}' = \sqrt{|\eta_3|} \sqrt{|h_3|} \times \sqrt{|\varepsilon g_{1'1'} \varepsilon g_{2'2'} \left[ (\varepsilon g_{1'1'})^2 \varepsilon g_{2'2'} + (\varepsilon g_{1'2'})^3 \right]^{-1}|, \]

\[ e_{4}' = \sqrt{|\eta_4|} \sqrt{|h_4|} / \sqrt{|\varepsilon g_{1'1'} \varepsilon g_{2'2'} \left[ (\varepsilon g_{1'1'})^2 \varepsilon g_{2'2'} + (\varepsilon g_{1'2'})^3 \right]|, \]

\[ e_{3}' = -\sqrt{|\eta_3|} \sqrt{|h_4|} \times g_{1'2'} / \sqrt{|\varepsilon g_{1'1'} \left[ (\varepsilon g_{1'1'}) \varepsilon g_{2'2'} - (\varepsilon g_{1'2'})^2 \right]|, \]

\[ e_{4}' = \sqrt{|\eta_4|} \sqrt{|h_4|} \times \sqrt{|\varepsilon g_{1'1'} / \left[ (\varepsilon g_{1'1'}) \varepsilon g_{2'2'} - (\varepsilon g_{1'2'})^2 \right]|, \]

(101)

where h–polarizations \( \eta_j \) are defined from \( g_j = \eta_j \cdot g_j(x^i) = \varepsilon_j \varepsilon^\psi(x^i) \), with signatures \( \varepsilon_i = \pm 1 \), for \( \psi(x^i) \) being a solution of the 2D equation

\[ \varepsilon_1 \psi^{**} + \varepsilon_2 \psi^{''} = \lambda, \]

(102)

for a given source \( \Upsilon_4 (x^i) = \lambda \), and the v–polarizations \( \eta_a \) defined from the data \( h_a = \eta_1, h_2 \), for

\[ h_3 = \varepsilon_3 h_0^2(x^i) \left[ f^*(x^i, v) \right]^2 | \lambda_\zeta (x^i, v)|, \]

\[ h_4 = \varepsilon_4 \left[ f(x^i, v) - f_0(x^i) \right]^2, \]

(103)

where

\[ \lambda_\zeta (x^i, v) = \zeta_0 (x^i) - \frac{\varepsilon_3}{8} \lambda h_0^2(x^i) \int f^*(x^i, v) \left[ f(x^i, v) - f_0(x^i) \right] dv, \]

for \( \Upsilon_2(x^k, v) = \lambda \). The polarizations \( \eta_i^a \) of N–connection coefficients

\[ N_{xi}^3 = w_i = w_{\eta_i}(x^k, v) \varepsilon w_i(x^k, v), \]

\[ N_{xi}^4 = n_i = n_{\eta_i}(x^k, v) \varepsilon n_i(x^k, v) \]

are computed from the respective formulas

\[ w_{\eta_i} \varepsilon w_i = -\frac{\partial_i \lambda_\zeta (x^k, v)}{\lambda_\zeta^* (x^k, v)} \]

(104)
\[ n_{k} \varepsilon n_{k} = 1 n_{k} (x^i) + 2 n_{k} (x^i) \int \frac{[f^* (x^i, v)]^2}{[f (x^i, v) - f_0 (x^i)]^3} \lambda \varsigma (x^i, v) \, dv. \quad (105) \]

We generate a class of exact solutions for Einstein spaces with \( \Upsilon_2 = \Upsilon_4 = \lambda \) if the integral varieties defined by \( g_j, h_a, w_i \) and \( n_i \) are subjected to constraints (87).

### 5.3 Solitonic pp–waves and their effective Lagrange spaces

Let us consider a d–metric of type (97),

\[ \delta s^2_{[pw]} = -dx^2 - dy^2 - 2\kappa (x, y, v) \, dv^2 + dp^2 / 8\kappa (x, y, v), \quad (106) \]

where the local coordinates are \( x^1 = x, \, x^2 = y, \, y^3 = v, \, y^4 = p \), and the nontrivial metric coefficients are parametrized

\[ g_1 = -1, \quad g_2 = -1, \quad h_3 = -2\kappa (x, y, v), \quad h_4 = 1 / 8 \kappa (x, y, v). \]

This is vacuum solution of the Einstein equation defining pp–waves [135]: for any \( \kappa (x, y, v) \) solving

\[ \kappa_{xx} + \kappa_{yy} = 0, \]

with \( v = z + t \) and \( p = z - t \), where \( (x, y, z) \) are usual Cartesian coordinates and \( t \) is the time like coordinate. Two explicit examples of such solutions are given by

\[ \kappa = (x^2 - y^2) \sin v, \]

defining a plane monochromatic wave, or by

\[ \kappa = \frac{xy}{(x^2 + y^2)^2} \exp \left[ \frac{v_0^2 - v^2}{v_0^2} \right], \quad \text{for } |v| < v_0; \]
\[ = 0, \quad \text{for } |v| \geq v_0, \]

defining a wave packet travelling with unit velocity in the negative \( z \) direction.

We nonholonomically deform the vacuum solution (106) to a d–metric of type (99)

\[ ds^2_{0} = -e^{\psi (x, y)} \left[ (dx)^2 + (dy)^2 \right] \]
\[ -\eta_3 (x, y, v) \cdot 2\kappa (x, y, v) \left[ dv + w \eta_1 (x, y, v, p) \varepsilon w_i (x, y, v, p) dx^i \right]^2 \]
\[ +\eta_4 (x, y, v) \cdot \frac{1}{8\kappa (x, y, v)} \left[ dy^4 + n \eta_1 (x, y, v, p) \varepsilon n_i (x, y, v, p) dx^i \right]^2, \]

where the polarization functions \( \eta_1 = \eta_2 = e^{\psi (x, y)}, \eta_3, \eta_4 \) and \( n \eta_i (x, y, v) \) have to be defined as solutions in the form (102), (103).
and (105) for a string gravity ansatz (82), \( \lambda = \lambda_H^2/2 \), and a prescribed (in this section) analogous mechanical system with

\[
\lambda = \frac{H}{2}.
\]

A class of 3D solitonic configurations can be defined by taking a polarization function \( \eta_4(x, y, v) = \eta(x, y, v) \) as a solution of solitonic equation \(^{11}\)

\[
\eta^{**} + \epsilon(\eta' + 6\eta \eta^* + \eta^{***})* = 0, \quad \epsilon = \pm 1,
\]

and \( \eta_1 = \eta_2 = e^{\psi(x,y)} \) as a solution of (102) written as

\[
\psi^{**} + \psi'' = \frac{\lambda^2_H}{2}.
\]

Introducing the above stated data for the ansatz (107) into the equation (75), we get two equations relating \( h_3 = \eta_3, h_4 = \eta_4 \),

\[
\eta_4 = 8\kappa(x, y, v) \left[ h_4^0(x, y) + \frac{1}{2\lambda_H^2} e^{2\eta(x,y,v)} \right],
\]

and

\[
|\eta_3(x, y, v)| = \frac{e^{-2\eta(x,y,v)}}{2\kappa^2(x, y, v)} \left[ \sqrt{\eta_4^*(x, y, v)} \right]^2,
\]

where \( h_4^0(x, y) \) is an integration function.

Having defined the coefficients \( h_a \), we can solve the equations (76) and (77) expressing the coefficients (78) and (79) through \( \eta_3 \) and \( \eta_4 \) defined by pp– and solitonic waves as in (112) and (111). The corresponding solutions are

\[
w_1 = \eta_1^L w_1 = (\phi^*)^{-1} \partial_x \phi, \quad w_2 = \eta_1^L w_1 = (\phi^*)^{-1} \partial_y \phi,
\]

for \( \phi^* = \partial \phi/\partial v \), see formulas (79) and

\[
n_i(x, y, v) = n_i^0(x, y) + n_i^1(x, y) \int |\eta_3(x, y, v)|^{3/2} (x, y, v) \, dv,
\]

where \( n_i^0(x, y) \) and \( n_i^1(x, y) \) are integration functions.

The values \( e^{\psi(x,y)}, \eta_3 \) (112), \( \eta_4 \) (111), \( w_1 \) (113) and \( n_i \) (114) for the ansatz (107) completely define a nonlinear superpositions of solitonic and pp–waves as an exact solution of the Einstein equations in string gravity if

\(^{11}\) as a matter of principle we can consider that \( \eta \) is a solution of any 3D solitonic, or other, nonlinear wave equation.
there are prescribed some initial values for the nonlinear waves under consideration. In general, such solutions depend on some classes of generation and integration functions.

It is possible to give a regular Lagrange analogous interpretation of an explicit exact solution (107) if we prescribe a regular Lagrangian \( \varepsilon = L(x,y,v,p) \), with Hessian \( L_{g'j'} = \frac{1}{2} \frac{\partial^2 L}{\partial y' \partial y'} \), for \( x' = (x,y) \) and \( y' = (v,p) \).

Introducing the values \( L_{g'j'} \), \( \eta_1 = \eta_2 = e^\psi \), \( \eta_3, \eta_4 \) and \( \eta_5 \), \( \eta_6 \), all defined above, into (100) and (101), we compute the vierbein coefficients \( e_i ' \) and \( \tilde{e}_a ' \) which allows us to redefine equivalently the quadratic element in the form (98) as for a Lagrange N–algebroid for which the N–connection coefficients \( L_N^a \) (47) are nonholonomically deformed to \( N_i^a \) (108). With respect to such nonholonomic frames of references, an observer "swimming in a string gravitational ocean of interacting solitonic and pp–waves" will see his world as an analogous mechanical model defined by a regular Lagrangian \( L \).

5.4 Finsler–solitonic pp–waves in Schwarzschild spaces

We consider a primary quadratic element

\[
\delta s_0^2 = -d\xi^2 - r^2(\xi) \, d\vartheta^2 - r^2(\xi) \sin^2 \vartheta \, d\varphi^2 + \varpi^2(\xi) \, dt^2,
\]

(115)

where the local coordinates and nontrivial metric coefficients are parametrized in the form

\[
\begin{align*}
x^1 &= \xi, x^2 = \vartheta, x^3 = \varphi, x^4 = t, \\
g_{1} &= -1, g_{2} = -r^2(\xi), h_{3} = -r^2(\xi) \sin^2 \vartheta, h_{4} = \varpi^2(\xi),
\end{align*}
\]

(116)

for

\[
\xi = \int dr \left| 1 - \frac{2\mu}{r} \right|^{1/2}
\]

and \( \varpi^2(r) = 1 - \frac{2\mu}{r} \).

For \( \mu \) being a point mass, the element (115) defines the Schwarzschild solution written in spacetime spherical coordinates \( (r, \vartheta, \varphi, t) \).

Our aim, is to find a nonholonomic deformation of metric (115) to a class of new vacuum solutions modelled by certain types of Finsler geometries.

The target stationary metrics are parametrized in the form similar to (99), see also (86),

\[
d_{s_0}^2 = -e^{\psi(\xi,\vartheta)} \left[ (d\xi)^2 + r^2(\xi)(d\vartheta)^2 \right] \\
- \eta_3(\xi,\vartheta,\varphi) \cdot r^2(\xi) \sin^2 \vartheta \left[ d\varphi + \eta_i(\xi,\vartheta,\varphi,t) \right] F w_i(\xi,\vartheta,\varphi,t) dx^i]^2 \\
+ \eta_4(\xi,\vartheta,\varphi) \cdot \varpi^2(\xi) \left[ dt + \eta_i(\xi,\vartheta,\varphi,t) F n_i(\xi,\vartheta,\varphi,t) dx^i \right]^2.
\]

(117)

The polarization functions \( \eta_1 = \eta_2 = e^{\psi(\xi,\vartheta)} \), \( \eta_3(\xi,\vartheta,\varphi) \), \( \eta_4(\xi,\vartheta,\varphi) \) and \( \eta_i(\xi,\vartheta,\varphi) \) have to be defined as solutions of (87) for \( Y_2 = Y_4 = 0 \) and a
prescribed (in this section) locally anisotropic, on \( \varphi \), geometry with

\[
N^a_i = \{ w_i(\xi, \vartheta, \varphi) = w_i n_i(\xi, \vartheta, \varphi) = n_i(\xi, \vartheta, \varphi) \},
\]

for \( \varepsilon = F^2(\xi, \vartheta, \varphi, t) \) considered as a fundamental Finsler function for a Finsler geometry modelled on a N–anholonomic manifold with holonomic coordinates \((r, \vartheta)\) and nonholonomic coordinates \((\varphi, t)\). We note that even the values \( w_i n_i \) can depend on time like variable \( t \), such dependencies must result in N–connection coefficients of type \( N^a_i(\xi, \vartheta, \varphi) \).

Putting together the coefficients solving the Einstein equations (75)–(77) and (87), the class of vacuum solutions in general relativity related to (117) can be parametrized in the form

\[
\begin{align*}
 ds^2 & = -e^{\psi(\xi, \vartheta)} \left[ (d\xi)^2 + r^2(\xi)(dt)^2 \right] \\
 & \quad -h_0^i \left[ b^*(\xi, \vartheta, \varphi) \right]^2 \left[ d\varphi + w_1(\xi, \vartheta)d\xi + w_2(\xi, \vartheta)d\vartheta \right]^2 \\
 & \quad + \left[ b(\xi, \vartheta, \varphi) - b_0(\xi, \vartheta) \right]^2 \left[ dt + n_1(\xi, \vartheta)d\xi + n_2(\xi, \vartheta)d\vartheta \right]^2,
\end{align*}
\]

where \( h_0 = \text{const} \) and the coefficients are constrained to solve the equations

\[
\begin{align*}
 \psi^{\bullet\bullet} + \psi'^2 & = 0, \\
 w_1' - w_2'' + w_2 w_1'' - w_1 w_2'' & = 0, \\
 n_1' - n_2'' & = 0,
\end{align*}
\]

for instance, for \( w_1 = (b^*)^{-1}(b + b_0) \), \( w_2 = (b^*)^{-1}(b + b_0) \), \( n_2^2 = \partial n_2 / \partial \xi \) and \( n_1 = \partial n_1 / \partial \vartheta \). The polarization functions relating (118) to (117), are computed in the form

\[
\begin{align*}
 \eta_1 & = \eta_2 = e^{\psi(\xi, \vartheta)}, \quad \eta_3 = \left[ h_0 b^*/r(\xi) \sin \vartheta \right]^2, \quad \eta_4 = [(b - b_0) / \varphi]^2, \quad (120) \\
 w_\eta_i & = w_i(\xi, \vartheta) / F w_i(\xi, \vartheta, \varphi, t), \quad ^a\eta_i = n_i(\xi, \vartheta) / F n_i(\xi, \vartheta, \varphi, t).
\end{align*}
\]

The next step is to chose a Finsler geometry which will model (118), equivalently (117), as a Finsler like d-metric (58). For a fundamental Finsler function \( F = F(\xi, \vartheta, \varphi, t) \), when \( x^\xi = (\xi, \vartheta) \) are h–coordinates and \( y^{\varphi'} = (\varphi, t) \) are v–coordinates, we compute \( F g_{\xi'\xi'} = (1/2) \partial^2 F / \partial y^\varphi \partial y^\varphi \) following formulas (49) and parametrize the Cartan N–connection as \( C N^a_i = \{ F w_i, F n_i \} \). Introducing the values (116), \( F g_{\varphi'\varphi'} \) and polarization functions (120) into (100) and (101), we compute the vierbein coefficients \( e^i_i \) and \( e^a_a \) which allows us to redefine equivalently the quadratic element in the form (58), in this case, for a Finsler N–algebroid for which the N–connection coefficients \( C N^a_i \) are nonholonomically deformed to \( N^a_i \) satisfying the last two conditions in (119). With respect to such nonholonomic frames of references, an observer "swimming in a locally anisotropic gravitational ocean" will see the nonholonomically deformed Schwarzschild geometry as an analogous Finsler model defined by a fundamental Finsler function \( F \).
6 Outlook and Conclusions

In this review article, we gave a self-contained account of the core developments on generalized Finsler–Lagrange geometries and their modelling on (pseudo) Riemannian and Riemann–Cartan manifolds provided with preferred nonholonomic frame structure. We have shown how the Einstein gravity and certain string models of gravity with torsion can be equivalently reformulated in the language of generalized Finsler and almost Hermitian/Kähler geometries. It was also argued that former criticism and conclusions on experimental constraints and theoretical difficulties of Finsler like gravity theories were grounded only for certain classes of theories with metric noncompatible connections on tangent bundles and/or resulting in violation of local Lorentz symmetry. We emphasized that there were omitted the results when for some well defined classes of nonholonomic transforms of geometric structures we can model geometric structures with local anisotropy, of Finsler–Lagrange type, and generalizations, on (pseudo) Riemann spaces and Einstein manifolds.

Our idea was to consider not only some convenient coordinate and frame transforms, which simplify the procedure of constructing exact solutions, but also to define alternatively new classes of connections which can be employed to generate new solutions in gravity. We proved that the solutions for the so-called canonical distinguished connections can be equivalently re-defined for the Levi Civita connection and/or constrained to define integral varieties of solutions in general relativity.

The main conclusion of this work is that we can avoid all existing experimental restrictions and theoretical difficulties of Finsler physical models if we work with metric compatible Finsler like structures on nonholonomic (Riemann, or Riemann–Cartan) manifolds but not on tangent bundles. In such cases, all nonholonomic constructions modelled as exact solutions of the Einstein and matter field equations (with various string, quantum field ... corrections) are compatible with the standard paradigm in modern physics.

In other turn, we emphasize that in quantum gravity, statistical and thermodynamical models with local anisotropy, gauge theories with constraints and broken symmetry and in geometric mechanics, nonholonomic configurations on (co) tangent bundles, of Finsler type and generalizations, metric compatible or with nonmetricity, seem to be also very important.

Various directions in generalized Finsler geometry and applications has matured enough so that some tenth of monographs have been written, including some recent and updated: we cite here [102, 107, 108, 103, 104, 112, 106, 24, 228, 183, 225, 17, 147, 4, 8, 13, 14, 16, 29]. These monographs approach and present the subjects from different perspectives depending, of course, on the authors own taste, historical period and interests both in geometry and physics. The monograph [228] summarizes and develops the results oriented to application of Finsler methods to standard theories.
of gravity (on nonholonomic manifolds, not only on tangent bundles) and their noncommutative generalizations; it was also provided a critical analysis of the constructions with nonmetricity and violations of local Lorentz symmetry.

Finally, we suggest the reader to see a brief outline and comments on main directions related to nonstandard applications in physics of Finsler geometries and generalizations in Appendix.

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A Historical and Bibliographical Comments

One can be found by 3500 titles on key word “Finsler” in MathSciNet and almost 150 titles in arxiv.org. It is not possible to review in an article all known mathematical constructions and applications in various directions in science related to Finsler geometry and generalizations. We shall sketch only some very important lines of developments of such researches and comment only a small part of results on nonstandard theories which, in our subjective opinion, seem to have importance and certain perspectives to be redefined for standard theories of gravity, mechanics and field interactions.

A.1 Moving frames, N–connections and nonholonomic (super) manifolds

There are well known textbooks and monographs [69, 113, 244, 158, 151] where (pseudo–) Riemann geometry and general relativity are formulated in arbitrary frame bases. The approach originates from the E. Cartan moving frame method [41, 43] being developed both in abstract and coordinate forms in modern gravity, see also supersymmetric generalizations related to string gravity [148, 53, 139].

The global definition of nonlinear connection (N–connection) is due to W. Barthel (1963) [19]. In coefficient form, the N–connections can be found in the E. Cartan’s book on Finsler geometry [42] (1935) and in A. Kawaguchi (1937,1952) works [85, 86]. The concept of N–connection is also known as the Ehresmann connection [55] (1955), see also Grifone’s works [66]. The N–connection geometry was developed in a series of works of the Romanian school of Finsler, Lagrange and Hamilton geometries and higher order generalizations [107, 108, 103, 104, 112, 106, 24] (beginning the end of 50th of previous century).
Then, the constructions with N–connections were generalized for supersymmetric fiber variables [24] (1989,1990) following the DeWitt approach to supermanifolds [54]. A definition of nonholonomic supermanifolds was not possible in those works because it was not not yet elaborated the concept of spinors for generalized Finsler spaces, see details in section A.6.2. Having accepted any global or local constructions for supermanifolds and superbundles, for a well defined nonholonomic spinor structure, it was possible to introduce N–connections by a corresponding class of super–distributions and/or preferred systems of superfields (i.e. super–vielbeinds). That allowed us to consider nonholonomic generalizations of the geometry of supermanifolds, superstrings and supergravity [180, 183, 216]. Recently, the geometry of semi–spray and N–connection structures on supermanifolds was developed in Refs. [54, 143]. The geometry of nonholonomic supermanifolds has a number of perspectives in such supergravity and superstring models when non–compactified configurations and constraints on the superfield dynamics are introduced into consideration.

The geometry on N–connections was extended and applied in gauge and Einstein gravity with anholonomic/ noncommutative variables [218, 217, 52, 201], for Clifford/ spinor bundles and algebroids provided with N–connection structure [173, 178, 181, 230, 225, 202] and on Fedosov–Lagrange manifolds [57]. Here, we emphasize that the concept of N–connection has to be not confused with the linear connections in gauge models with nonlinear realizations of some gauge groups related to non–Abelian gauge potentials. Such constructions are completely different.

One should be noted that the idea of nonholonomic manifolds (as a geometric background for geometric mechanics and generalized geometries) exists in rigorous mathematical form due to the works of G. Vrăncăeanu and Z. Horak [241, 242, 243, 73] (1926, 1931,1957,1927), see further developments in [119, 105] and a modern approach and references in [25] (2006). For different classes of connections on nonholonomic manifolds, there were computed the torsion and Riemannian tensors and investigated the geometric properties of such spaces and considered certain applications in geometric mechanics.

In Ref. [95], see also references therein, the author argues that he was able to define the Riemann tensor for a general nonholonomic manifold. His homological considerations and analysis of former works on nonholonomic (super) spaces was based on reviews [96, 238, 239] on supermanifolds, nonholonomic manifolds and mechanics. We note here that the Riemannian tensors and Einstein equations were defined and computed rigorously in various approaches to nonholonomic manifolds, Finsler and Lagrange spaces and superbundles provided with nonlinear connections much before mentioned publications and reviews by former Soviet mathematicians (see, for instance, [243, 27, 142, 145, 107, 108, 24, 133, 180]).

The approaches developed by the Romanian school on Finsler geome-
try and generalizations have a number of connections to G. Vrânceanu's results (he published a four volume Course on Differential Geometry, in Romanian, and some of them were translated in French, by 1957). In the G. Vrânceanu, R. Miron, M. Anastasiei, A. Bejancu and other authors on the geometry of nonholonomic manifolds and generalized Finsler–Lagrange spaces there are not considered any methods of constructing exact solutions in gravity theories (they were elaborated during the last decade, see below section A.5). Nevertheless, the mentioned authors (and a number of their co-authors) elaborated various applications in geometric mechanics, Finsler generalizations of gravity, electro–gravitational fields, gauge models and locally anisotropic supersymmetric variables. A brief summary of R. Miron school’s results on Finsler, Lagrange, Hamilton and higher order generalizations and further perspectives in gravity and field theories is given in Ref. [200].

A.2 Finsler and Lagrange algebroid structures

Lie algebroids were introduced as a generalization of the concepts of Lie algebra and integrable distribution, see details in Ref. [100]. An extension of the theory of Lagrangians and Euler–Lagrange equations on Lie algebroids was considered by A. Weinstein [246], see also Refs. [98, 49].

In our approach, we tried to model Lie algebroid structures as exact solutions in gravity [193, 194, 195]. Such solutions were defined by generic off–diagonal metrics and nonholonomic frames of references. They were constructed following the anholonomic frame method, outlined in Ref. [228] (see also the references from section A.5) when nonlinear connections are defined by splitting the gravitational degrees of freedom into holonomic and anholonomic ones.

If in the usual approaches to Lie algebroids the geometric constructions are related to Lie algebra generalizations and sections of vector bundles, in order to define algebroid structures as solutions of the Einstein equations, in general, it is necessary to work on nonholonomic manifolds modelling certain types of Lie algebroid or Clifford–Lie algebroid structures. Some explicit examples of such solutions were considered in Ref. [194] and the geometry of Clifford–Finsler algebroids and nonholonomic Einstein–Dirac structures was elaborated in Ref. [202]. In a general context, the theory of nonholonomic algebroids in relation to nonholonomic manifolds, Finsler geometry and Lagrange–Hamilton spaces was elaborated in Ref. [210]. Here we note that some topics on Lie algebroids and Finsler and Lagrange geometry, without connections to modern gravity but with certain orientation to application in mechanics, are considered alternatively in Refs. [70, 71, 2], see also references therein.
A.3 Higher order extensions of Lagrange and Hamilton spaces

The geometric approach to Lagrange and Hamilton mechanics elaborated as a generalization of Finsler geometry was developed in a direction to include higher order mechanics [109, 110], see Ref. [91] as a summary of alternative directions. The nonlinear connection formalism was developed for higher order (co) tangent bundles which resulted in a series of monographs on higher order Lagrange–Finsler and Hamilton–Cartan spaces [103, 104, 112, 106], see also a recent work [38] and a brief review [200]. Such higher order geometric mechanical constructions are naturally adapted to corresponding (semi)spray configurations, from which canonical nonlinear and distinguished connections can be derived. Sure, this geometric formalism would be very important for developing analogous models of gravity.

In parallel to the mentioned works on geometric mechanics, there were elaborated certain new directions related to "higher order anisotropic" configurations in high energy physics, gravity and string theory. The idea was to consider higher order "shells" of extra dimensions which are not completely compactified like in the Kaluza Klein theory and to take into account certain possible correlations between spacetime dimensions and extra dimensions (the higher dimension interactions being modelled by effective higher order nonlinear connections). Such configurations can be derived as exact solutions of the Einstein equations in extra dimension gravity, or in certain low energy limits, but locally anisotropic, in string theory. It was necessary to elaborate a corresponding (super) geometric formalism for the higher order gauge theories, including gauge and Einstein gravity, see [182], higher order super spaces [180] and Clifford bundles and spinor theory [181, 230], the bulk of results being summarized in monographs [183, 225].

A.4 Almost Kähler and nonholonomic structures

The constructions transforming nonholonomic Riemannian spaces and generalized Lagrange algebroids into almost symplectic structures, presented in section[1] originate from a series of works on almost Kähler models for Finsler [102] and Lagrange spaces and generalizations [124, 125, 126, 127, 128, 129]. We note that V. Oproiu and co-authors performed the bulk of their lifts on (co) tangent bundles working with linearized N–connections $N^a_i(u) = \Gamma^a_{bj}(x)y^b(10)$ but a number of formulas hold true for more general nonholonomic structures with arbitrary $N^a_i(u)$.

In order to model gravitational interactions by analogous Finsler–Lagrange algebroid structures, we have to consider arbitrary nonholonomic h– and v–frames and N–connections. In such cases, the effective geometric models are almost Hermitian ones with nonzero 2–form $\bar{d}\theta$ [89]. This is not a problem for the anholonomic frame method of constructing exact solutions.
but result in more sophisticate nonholonomic relations if we try to develop the approach, for instance, for geometric quantization of Lagrange–Fedosov spaces, see [57] and references therein.

Finally, we note that from physical point of view there are two different directions of elaborating analogous almost Hermitian / Kähler models of gravitational interactions. If we work with nonholonomic deformations of geometric objects on the same class of manifolds, for instance, on a semi–Riemannian one, we can preserve, in general, the local Lorentz symmetry. Any lifts on tangent bundles and additional nonholonomic transforms, positively result in violation of the local Lorentz symmetry and modification of various types of possible gauge symmetries and conservation laws. Both classes of such models present a substantial interest in modern physics but only the first one can be related to the so–called standard theories.

A.5 Finsler methods and exact solutions

We applied the methods of Finsler–Lagrange geometry in order to elaborate the so–called anholonomic frame method of constructing exact solutions in Einstein, gauge, string and brane gravity and various locally anisotropic/ noncommutative generalizations [188, 229, 220, 221, 222, 223, 217, 52, 201, 228]. The idea was to define such N–connection structures associated to nonholonomic frame transforms, when the gravitational field equations are transformed into general systems of partial differential equations which can be integrated in general form. It was possible to construct, following geometric methods, new classes of generic off–diagonal metrics, nonholonomic frames and linear connections depending on 2–4 variables in 3–5 dimensional gravity. It became obvious that the N–connection formalism can be applied also on Riemann, Riemann–Cartan and metric–affine spaces and that an unified geometric approach, with nonholonomic distributions, can be elaborated in order to model generalized Finsler–Lagrange spaces both on nonholonomic manifolds or on any (super, noncommutative, spinor...) bundle provided with N–connection splitting. The main results are summarized in monograph [228] (2005) and the basic constructions and some ansatz and examples are considered in section 5.

Here, we briefly outline some additional results giving a number of examples of exact solutions with local anisotropy in standard and nonstandard models of gravity:

The paper [184] was the first one containing the idea how the N–connection formalism can be applied for generating generic off–diagonal solutions in gravity. The work [188] presented a number of examples with anholonomic soliton–dilaton and black hole solutions in general relativity and extensions to string or Finsler generalized theories. Two papers [229, 220] contain a research on nonholonomic deformations of Taub NUT spinning metrics and locally anisotropic solitons and Dirac spinor waves in such spaces.

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A series of works [221, 222, 223] is based on a collaboration of authors on constructing exact solutions defining locally anisotropic defining various types of locally anisotropic (ellipsoidal, toroidal, warped ...) wormhole / flux tubes and black holes moving in nontrivial solitonic backgrounds in four and five dimensional gravity. The anholonomic frame method was shown to be the most general one allowing to generate exact solutions following geometric methods. There were elaborated a number of examples which together with the the details on analytic computations were summarized in Parts I and II of monograph [228], see also review [192].

There were constructed solutions defining static black ellipsoids [190, 191] which seem to be stable in Einstein gravity, with mater fields and/or geometric distorsions, and various noncommutative and metric-affine generalizations of gravity, see [201] and Parts I and III in monograph [228]. Solutions with generic local anisotropy, of Finsler type and generalied ones, were constructed in gauge gravity [217, 52], for various black hole and cosmological configurations, see Part II in [228], and by modelling explicit examplas of (disk, black hole, solitonic and spinor waves ...) of solitonic spacetimes with Lie algebroid symmetry [193, 194, 195]. The bulk of recent results on parametric solutions and solitonic hierarchies are summarized in [209, 204, 3]. The anholonomic frame method provides also a unique general geometric scheme for constructing exact solutions of Ricci flow equations, see section A.6.5.

A.6 On standard and nonstandard models of Finsler geometry and physics

It is worth mentioned certain important directions additionally to those mentioned in the points 1–5 from Introduction section and the previous sections of the Attachment. Due to limits of space we have to leave out a number of interesting developments: we discuss here briefly the contributions of some authors and outline a few open issues, see also Introduction to [228].

A.6.1 Finsler structures in gauge gravity and noncommutative gravity

The first original ideas to consider gauge group transforms and additional gauge fields on Finsler spaces and generalizations belong to Y. Takano [159, 160], S. Ikeda [80] and G. Asanov and co–authors [13, 16, 14, 12]. Here one should be noted that the monograph [107] was the first one on Finsler and Lagrange spaces written rigorously in the language of the geometry of vector and tangent bundles and related geometric structures (fiber bundles and linear connections provide the standard geometric formalism for the theory of gauge fields). Various types of nonlinear connections (N–connections) on nonholonomic spaces (considered to define a special class
of nonlinear gauge fields) can be found in a number of constructions in Finsler geometry, see [12, 85, 86]. The monograph [24] contains a study of Yang–Mills theories in Finsler spaces. In the same line of research, elaborating gauge and field theories on bundle spaces, can be considered papers [149, 156]. Such models can not be related to standard approaches in modern physics. Nevertheless, they provide a number of geometric ideas which can be applied for noncompactified higher dimension gravity models.

A series of works [175, 177, 182, 218] is devoted to gauge models of matter and gravity fields on generalized Lagrange and Finsler spaces in relation to the Poincare, affine and de Sitter structure groups with actions distinguished by the N–connection structure. The idea was to elaborate gauge models which would embed Finsler like gravities into the class of former theories on gauge gravity [165, 140, 141, 161]. Such constructions, with nonholonomic frames and spinor–gauge transforms, were related to the results on twistor–gauge gravity [166, 167, 168, 169, 170, 171], and further supersymmetric generalizations were summarized in the monograph [183]. This class of locally anisotropic gauge models was elaborated following the frame bundle and generalized connection formalism. They have strong connections to standard gauge and gravity models in physics: for instance the nonholonomic structures, metrics and connections can be defined for (pseudo) Riemannian spaces and fiber bundles on such spaces.

Gauge models of higher order anisotropic gravity and nearly autoparallel maps and their connections to Einstein and gauge models were analyzed in Refs. [217, 52]. The approach was developed for noncommutative versions of the Einstein and gauge gravity [189, 216, 197, 198, 199] and provided with explicit examples of exact solutions for nonholonomic noncommutative structures in gravity.

A.6.2 Clifford structures, gerbes, and Lagrange–Finsler spinors

The first who considered spinor variables in Finsler geometry was Y. Takano [160]. Perhaps, we can cite here the work [1] on modelling spinors on Hilbert manifolds, even explicit constructions related to Finsler spaces are not given there (the author latter had fundamental contributions in Finsler–Lagrange geometry and modelling such geometries on Hilbert spaces). There were published a series of papers on geometries and physical models with metrics depending on spinor variables, see [120, 121, 122, 123, 16, 157, 152, 156]. It should be emphasized here that all mentioned works do not contain a definition of spinor for Finsler like spaces and generalizations. In the bulk, they provide certain constructions when the existence of the bundle of two–spinors (i.e. two dimensional spinor spaces) is supposed to exist on a Finsler like manifold for which the metric and connections are considered to depend on two spinor variables. Such models belong to ”nonstandard” approaches to Finsler geometry and generalizations.
A rigorous definition of spinors for Finsler spaces should contain a fundamental relation between the Clifford structure and the metric structure (like in Einstein gravity, one has to consider an anticommutator of ”gamma” matrices, generating the corresponding Clifford algebra, and the metric quadratic form, all defined with respect to a local orthonormalized frame basis). Such constructions for Finsler–Lagrange spaces were elaborated in Refs. [173, 178] in the language of Clifford bundles provided with N–connection structure. We note that the condition of metric compatibility of Finsler–Lagrange connection plays a fundamental role in definition of spinors on spaces provided with N–connection structure. For instance, it is not possible to define directly the concept spinors in Finsler geometries with metric noncompatible connections like [27, 17]. Without fermions / spinors, it is not possible to elaborate viable physical models. It was necessary to construct the spinor geometry for Finsler spaces and generalizations. The direction was analyzed in details, with a number of examples and exact solutions in gravity models and generalizations to higher order Finsler, Lagrange and Hamilton spaces, in Refs. [181, 183, 230]. Such constructions belong to the class of standard Clifford–Finsler structures. Both standard and nonstandard approaches are analyzed in details in monograph [225].

The standard constructions with Finsler–Lagrange spinors have further developments for noncommutative Finsler geometry, see [199] and the Part III in [228], for explicit solutions see [201] and for Clifford–Lagrange algebroid structures see [202, 194]. One could be topological restrictions in definition of spinor structures on general Finsler or Lagrange spaces: certain generalized constructions for such cases were performed following the gerbe formalism [196, 219].

A.6.3 Stochastic processes and locally anisotropic kinetics and thermodynamics

The famous P. Finsler’s thesis [58] (1918) was written under supervision of Prof. Caratheodory who had classical contributions in thermodynamics. A theory of kinetics and thermodynamics based on distribution functions on Finsler spaces was elaborated in monograph [240].

Both the Riemann and Finsler geometry were applied to various problems in information thermodynamics [81, 82, 83] and geometric thermodynamics [115, 84, 116, 142]. The approaches were summarized in review [146]. There were proposed certain applications of geometric thermodynamics with

12As a matter of principle, we can define Finsler–spinors using the metric compatible Cartan connections and then to deform the geometric constructions to those for the Chern connection, by using corresponding deformations tensors for connections. So, following certain more sophisticate geometric constructions, we can define nonholonomic Clifford bundles provided with metric noncompatible linear connection structure. But there are not physical arguments for such nonmetric structures.
A number of works were published on locally anisotropic diffusion in two series of works elaborated in parallel by two collaborations of authors. The first one is related to publications \[9, 10, 11, 6, 7, 5, 8, 4\] and the second one to \[172, 174, 176, 186\] and Chapter 10 in monograph \[183\]. The second approach was developed in relation to the theory of kinetic processes and locally anisotropic thermodynamics \[187, 186\] (the works propose certain applications in modern astrophysics and cosmological models, see also such locally anisotropic cosmological solutions in Refs. \[217, 154\]). Chapter 5 in \[183\] contains the results on diffusion theory on locally anisotropic superspaces.

Generalized Riemann–Finsler structures in thermodynamics and kinetic and stochastic processes can be described in terms of both metric compatible and noncompatible connections. For such constructions, there were not yet elaborated criteria on standard and nonstandard models. The results cited in this section can be elaborated following covariant calculus with different generalized Finsler connections. Priorities should be given to certain more ”simple” of theoretical models with possible experimental verification and applications, for instance, in modern thermodynamics, astrophysics and cosmology.

### A.6.4 Nonholonomic curve flows and bi–Hamiltonian structures

A new direction of applications of the nonlinear connection formalism was elaborated in Refs. \[204, 3\]. It was proposed to consider such nonholonomic distributions and related frames on (semi) Riemannian and generalized Finsler–Lagrange spaces when the curvature \[31\] of the canonical d–connection \[36\], with respect to certain N–adapted bases, is parametrized by constant matrix coefficients. The geometric information for such spaces can be encoded into bi–Hamilton structures and solitonic hierarchies characterized by corresponding invariants and conservation laws. The approach was generalized for nonholonomic Ricci flows \[208\].

### A.6.5 Nonholonomic Ricci flows and Lagrange–Finsler spaces

Recently, it was elaborated a new direction in Riemann–Finsler geometry, gravity with nonholonomic distributions and geometric mechanics, the theory of nonholonomic Ricci flows and generalizations which is positively related to standard theories of physics and can be generalized similarly for evolution models on tangent and vector bundles, i.e. for nonstandard theories. The idea was not just to extend the R. Hamilton \[67, 68\] and Grisha Perelman \[132, 133, 134\] results, on Ricci flows of Riemannian metrics (see comprehensive reviews of results in Refs. \[40, 39, 88, 114\]), to more sophisticated classes of geometries, like Finsler and Lagrange geometry. A series...
of works [205, 207, 208, 211, 212] was written as a research of the Ricci flows when (semi) Riemannian metrics are subjected to certain classes of nonholonomic constraints. We proved that under well defined conditions, for instance, a Finsler like metric can evolve into a Riemannian one, and inversely. In general, the Ricci flows of such nonholonomic manifolds contain nontrivial torsion structures but such flows can be alternatively, and equivalently, described by flows of metrics and Levi Civita connections. Nonholonomic distributions on Riemannian manifolds result in some geometric multi–connection structures with, in general, nonzero torsion coefficients induced by some off–diagonal metric coefficients transformed into the N–connection coefficients. This is typical for metric compatible Finsler like connections; as a matter of principle, the constructions can be generalized for nonsymmetric metrics or Finsler connections with nonmetricity.

The theory of nonholonomic Ricci flows and Perelman’s functionals adapted to the N–connection structure allowed to formulate a new statistical interpretation of Finsler–Lagrange spaces and related nonholonomic and/or mechanical systems, see Ref. [206].

It should be noted that the anholonomic frame method works effectively not only for generating new classes of exact solutions in gravity, as we discussed in section A.5, but also for constructing exact solutions for Ricci flow evolution of valuable physical metrics in gravity (like solitonic and pp-waves, Taub NUT spaces, nonholonomically deformed Schwarzschild metrics...), see Refs. [203, 231, 232]. Perhaps, this is still the unique method which allows us to construct exact solutions of Ricci flow equations in general form, by using geometric methods and ideas from Finsler geometry.

### A.6.6 Quantum gravity and Finsler methods

There are a few ”standard” works related to nonholonomic Lagrange–Fedosov spaces [57] and geometric quantization of the Einstein gravity transformed equivalently as an almost Hermitian / Kähler model [213, 214, 215], see discussion in section A.4. This direction is under elaboration for gravitational gauge models and certain nonholonomic generalizations of gravity on manifolds and tangent bundles. Some results on quantum models connected to Finsler geometry and gravity belonging to the class of nonstandard theories will be analyzed in the next section.

### A.6.7 On some nonstandard but important contributions to Finsler geometry and physics

We shall cite and briefly comment here some series of works concerning nonstandard Finsler geometric and physical models for which a number of important results can be re–defined on nonholonomic manifolds and may
present a substantial interest in the so-called standard theories, or are related to certain recent developments in modern physics.

**Gauge transforms on tangent bundle and Kaluza–Klein theory:**
Additionally to the discussion on Finsler geometry and generalized gauge theories in section A.6.1, we refer to a series of works by R. G. Beil [20, 21, 22, 23]. The author considered a Kaluza–Klein like theory following the idea that the nature of spacetime is Finslerian and the extra dimension is time-like, i.e. not compactified. The geometry of moving frame transports on Finsler spacetimes was related to local Poincare and Lorentz transforms. The corresponding gauge transforms on tangent bundle resulted in a Kaluza–Klein type theory, but not in a Yang–Mills one, which was connected to a new type of quantum field theory. It should be noted that A. Bejancu also elaborated gauge models in tangent bundle summarized in his monograph [24]. Such gauge Finsler–Kaluza–Klein theories can be considered in the usual gauge gravity or string gravity models with nonholonomic distributions if the constructions are performed for nonholonomic manifolds, see for instance, [228, 183, 201].

**Generalized/ broken local Lorentz symmetries:**
We mention two directions on Finsler spacetime field theories with generalized Lorentz and gauge symmetries. The first one, recently, is connected to the so-called Finsleroid structures [15], with anisotropic kinematics and Finsler like generalizations of the local Minkowski metric. A number of former investigations on locally anisotropic gauge models, jet models, and Finsler like corrections to the Einstein gravity can found in [12, 13, 14, 16] and references therein.

Group transforms defining a generalized local Lorentz invariance on a Finsler like spacetime, instead a local Minkowski space are considered in [30], see also models with generalized Lorentz invariants and Dirac equation [31], and the so-called relativistic theory of gravity generalized to Finsler like spaces [29]. This class of theories by definition belong to the class of nonstandard ones. It presents certain interest for some approaches in modern physics related to violation of local Lorentz invariance and violation of principle of equivalence.

**Classical & quantum theories with local Lorentz invariance:**
An "almost standard" approach with Finslerian fields and applications of Finsler geometry methods is developed by H. E. Brandt. In work [32], he proposed a Cartan like Finsler theory with Kaluza–Klein generalizations on tangent bundle with Kähler geometrization starting with Christoffel symbols on the base and considering local Lorentz transforms on fibers. On
total space, the theory possesses a nontrivial almost complex structure and nontrivial torsion.

Further constructions are with maximal acceleration invariant quantum fields, formulated in terms of the differential geometric structure of the spacetime tangent bundle \[33\]. It was proposed a physically based Planck scale effective regularization with a spectral cutoff at the Planck mass. There were also considered Finslerian fields, strings and p–branes on tangent bundle with local Lorentz invariance and maximal invariance, quantum fields. There are Kaluza–Klein like fields but in general not compactified.

It was also attempted to elaborate a quantum field theory with Finslerian quantum fields (scalar fields on tangent bundle/Finsler spacetime) and microcausality with corresponding commutators of scalar fields \[35\]. The general idea was to work with Lorentz–invariant quantum fields and maximal–acceleration invariants (for such quantum fields) in the spacetime tangent bundle by using a Plank scale, causal domains and microcausal constructions \[36, 37\].

We also cite a series of works published recently in a Russian journal "Hypercomplex Numbers in Physics and Geometry", see \[64, 65\] and references therein, with the aim to construct of pseudo–Riemannian geometry on the basis of Berwald–Moore geometry, by using certain classes of relativistic invariant Finsler geometry generating functions.

This class of theories is related to standard models of gravity and strings but proposes new Finsler alternatives for quantum theories.

**Finslerian teleparallel and Kähler–Clifford structures:**
In this section, we shall comment on a series of works by J. Vargas and D. Torr \[233, 234, 235, 236, 237\]. A very important idea for applications to standard theories of physics (even, in general, the authors work with locally anisotropic geometric models on tangent bundle) is that a Riemann structure can be reconfigured on the Finsler bundle without loss of information but with increased structural richness. This allows us to consider canonical connections of Finsler metrics and Finslerian connections on Riemannian metrics \[233\]. Some constructions are similar, but inverse, to our constructions \[228\] when Finsler geometries and generalizations are modelled on (pseudo) Riemannian spaces.

Perhaps a source for such investigations can be found in a rigorous study by E. Cartan and A. Einstein (1929) when a theory was elaborated in which the electromagnetic field constitutes the time–like 2-form part of the torsion of Finslerian teleparallel connections on pseudo–Riemannian metrics \[235\]. The research by Vargas and Torr were performed following a comparative analysis of fundamental geometric constructions elaborated by Riemann, Cartan, Weyl, Klein, Clifford and Kähler and their explicit realization in Finsler geometry and generalizations.
For B. Riemann, at a time when the theory of continuous groups had not yet been founded, the fundamental geometric notion is that of distance (see Riemann’s famous inaugural dissertation ‘On the hypothesis…’, 1854, [144]). Then, a geometric approach of a completely different nature has developed between 1867 and 1914 by F. Klein, see an analysis oriented to Finsler geometry in [234]. For Klein, the fundamental geometric notion is contained in the axiom of geometric equality, interpreted in the light of the notion of group. We note here that the concept of geometric equality vary from geometry to geometry and is contained in the axioms of each geometry.

Weyl geometry is considered as the first type of Yang–Mills theories, ahead of the times. It is directly related to the geometry of base manifolds endowed with metric–compatible affine connection of the particular type that Cartan called metric connections. One thus has to leave open the possibility that some Yang–Mills theories may eventually become part of classical differential geometry of some more general type, with some more general (for instance, Finsler like) structure.

Various Finsler geometry models were elaborated following different relations between metrics and connections (connections are in general nonlinear but the authors tried to introduce certain effective linear ones). The Vargas–Torr idea is to derive Finsler spaces from certain spacetime structure moving the constructions on tangent bundles. This can be considered as an "almost standard" modelling of physical interactions on spaces with more rich (than Riemann geometry) structures. They formulated corresponding Kähler and Clifford calculus for Finsler like connections, analyzed Clifford structure of Kaluza–Klein spaces and tried to generate "metrics without metric tensors". Such constructions present a substantial interest also for standard models of physics because we can model them by nonholonomic distributions on (semi) Riemannian manifolds and Riemann–Cartan spaces.

In [236], one works with Finsler bundles using the language of differential forms but with nonmetricity, like Chern, see [17], which is not compatible with the standard models of physics. The authors develop the concept of affine Finsler connection involving bundles. Such Finsler connections are defined even in the absence of a metric function and/or metric. For the corresponding tangent bases and special bases (frames) and affine Finsler connection, it is elaborated a teleparallel and Kähler calculus. They compare their results with other approaches to teleparallelism and Kaluza–Klein reformulation of Finslerian teleparallelism. It is also considered the Kähler–Dirac equation for such Finsler spaces.

It should be emphasized that the Kähler equations for forms are equivalent for the Dirac equations only on flat spaces. For general curved spaces, this result is not true. In general, it is not clear how to formulate the Dirac equation for metric noncompatible connections if there are not used certain auxiliary constructions for metric compatible ones. The original contributions by Vargas–Torr is that they elaborated a Kähler calculus with
Clifford–valued multiforms for certain classes of Finsler connections and found an analogous (Kähler equation) to the Dirac equation for such cotangent spaces provided with Finsler like structure. Nevertheless, this is not a theory of spinors for Finsler–Lagrange spaces which was elaborated in Refs. [173, 178], see discussion in section A.6.2.

For applications of Finsler geometry methods to standard models of physics, the most important is the idea that the classical and quantum geometric structure of the spacetime and field/string dynamics may be richer than it presently appears to be on Riemann, or Riemann–Cartan spaces. If in [237] it is suggested to consider Clifford algebras for nonsymmetric quadratic forms and generalized Finsler spaces, we consider that by nonholonomic distributions on (semi) Riemann geometries we can also model very rich geometric structures but preserving compatibility with the present days paradigm of standard physics.

**Clifford manifolds and Finsler spaces:**

A model with maximal speed of light and maximal–acceleration relativity principle in the spacetime tangent bundle and in the phase spaces (cotangent bundle) was elaborated in Ref. [47] following the approach on relativity in C–spaces (Clifford manifolds), see [48] and references to the cited papers. The idea of maximal–acceleration is similar to that from [36, 37] but it is developed on C–spaces.

The constructions on C–spaces are related to Finsler geometry by considering dependencies on speed and accelerations following a ”new program” in physical theories [43, 46]. We also cite the work [45] on W geometry from Fedosov’s deformation quantization. It should be noted that C. Castro’s works conventionally belong to the class of nonstandard models of physics because they use constructions for the tangent and cotangent bundles, or jets, even such spaces are modelled by C–space structures. In other turn, they can be re–defined for nonholonomic manifolds by using nonholonomic Clifford bundles [173, 178, 183, 225] and Fedosov–Lagrange manifolds [57]. Such results can be positively related to standard models of physics.

Finally, in this section, we would mention the Hull’s formulation [78, 79] of $W_\infty$ gravity as a gauge theory of the group of sympletic diffeomorphisms of the cotangent bundle of two–dimensional surface considering a generalized (Finsler like) scalar line element

$$ds = (g_{\mu_1...\mu_n}dx^{\mu_1} ... dx^{\mu_n})^{1/n}.$$  

If for this element one considers a canonical d–connection and extracts the Levi–Civita connection, we generate a nonholonomic effective Riemannian space with more rich structure. In this case, we can establish further relations to noncommutative geometry and M(atrix) theory [201, 228]. We conclude that such models can be elaborated both in standard and nonstandard fashions.
Deterministic quantum Finsler models:

A series of works on Finsler geometry and applications [59, 60, 61, 62, 63] has the aim to solve certain problems of quantum mechanics with dissipation and loss of information. The constructions are based on 't Hooft’s proposal [71, 72] to use deterministic models in order to describe physical systems at the Planck scale through a Hilbert space formulation of these models. R. Gallego uses Finsler geometry with Chern connection in order to elaborate such deterministic quantum models and relate dissipation to average of Chern connection, with nonmetricity, in order to get the Levi Civita connection [60] and define a corresponding Hamiltonian following a model of dissipative dynamics.

It was elaborated a corresponding formalism when translation of results is considered from Riemannian to Finsler spaces and proposed the notions of complete Finsler manifold with the ”average” to a Riemannian manifold [61]. The dynamics at the quantum Planck scale with loss of information is constructed supposing that a Finsler structure on tangent bundle $TM$ evolves to a Riemannian structure also in $TM$ [62]. The canonical quantization is considered on the dual tangent - tangent bundle $T^*TM$ which is supposed to be the arena for deterministic Finslerian models and dynamical systems with corresponding Poisson structures. Finally, a nonstandard approach to quantum gravity with maximal acceleration is considered in [63].

It is not obligatory to use in researches on deterministic quantum model only the Chern connection [17] (similarly, and in a more simple form one can be elaborated quantum Cartan–Finsler models with further developments on Finsler, Lagrange and Hamilton geometry with metric compatible connections [107, 108, 24, 183, 180]). We note that having a canonical d–connection, we can always define exactly the Levi–Civita connection, because both such linear connections are uniquely defined by a generic off–diagonal metric tensor of type (18). It is also not obligatory to average the Chern connection with nonmetricity in order to get a metric compatible Levi Civita connection: we can do this by subtracting from the Cartan, Chern, or other canonical connection the corresponding deformation tensor (this topic is discussed in details in Refs. [205, 207]). The experimental data does not constrain us to use at Planck scale nonmetric connections (even there are quantum uncertainty relations, there are not proofs that they are related strongly to nonmetricity). A deterministic quantum dynamics can be modelled alternatively with metric compatible Finsler like connections. Working with nonholonomic manifolds and metric compatible connections, such constructions will belong to the class of standard models, contrary to those performed on tangent spaces and for nonmetric connections.

Other directions:
There were also proposed a number of other possible applications of Finsler geometry in order to solve important geometric and physical problems. In this final section, we mention a few of them.

The works by P. Stavrinos and co-authors \[150, 153\] elaborate certain Finsler like generalizations (nonstandard ones) in modern cosmology.

The idea to consider the Fermat principle on Finsler spaces was approached by V. Perlick \[136\] following a variational principle with a corresponding Lagrangian and Lagrangian, Euler–Lagrange equations. The topic of nonlinear connections and distinguished connections is not discussed in his work but it appears from the corresponding semi–spray dynamics if the geometric constructions are adapted to the nonlinear connection structure.

Possible hidden connections between general relativity and Finsler geometry are considered in Ref. \[131\]. An approach with homogeneous and symmetric Finsler spaces, with Chern connection, as a generalization of similar constructions for homogeneous/ symmetric Riemannian spaces, is developed by authors \[92, 93, 94\]. It is not clear if such results hold true for nonholonomic Riemannian spaces and Finsler models defined by metric compatible Finsler connections (for instance, with the Cartan connection). Generalized Lagrange–Weyl structures and compatible connections are analyzed in Ref. \[50\].

Geometric methods of Finsler–Lagrange geometry have been applied for a study of systems of partial differential equations, multi–time Lagrange spaces and dynamical systems and jet geometry \[162, 163, 164, 117, 118\].

Finally, we note that the geometry of induced structures on submanifolds and almost product Riemannian and/or Finsler manifolds \[74, 75\] has certain applications in geometric quantization \[57\].

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