QUANTIZED SL(2) REPRESENTATIONS OF KNOT GROUPS

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ABSTRACT. For a braided Hopf algebra $A$ with braided commutativity, we introduce the space of $A$ representations of a knot $K$ as a generalization of the $G$ representation space of $K$ defined for a group $G$. By rebuilding the $G$ representation space from the viewpoint of Hopf algebras, it is extended to any braided Hopf algebra with braided commutativity. Applying this theory to $\text{BSL}(2)$ which is the braided quantum $\text{SL}(2)$ introduced by S. Majid, we get the space of $\text{BSL}(2)$ representations. It is a non-commutative algebraic scheme which provides quantized $\text{SL}(2)$ representations of $K$.

1. INTRODUCTION

The discovery of the Jones polynomial brought us a new method to study knots and links, but its relation to the geometric properties of the knot complement was unclear at that moment. After Witten’s interpretation in terms of $\text{SU}(2)$ Chern-Simons theory, R. Kashaev [12] observed a precise relation between quantum invariants and the hyperbolic volume of the knot complement. This was reinterpreted as a relation between the colored Jones invariant and the hyperbolic volume by H. Murakami and the first author in [17]. Moreover, it was observed by Q. Chen and T. Yang in [8] that such relation also holds for the Witten-Reshetikhin-Turaev invariant of closed 3-manifolds. These relations between quantum invariants and hyperbolic volumes are not rigorously proved yet in general and are known as the volume conjecture. In some sense, the volume conjecture means that the colored Jones invariants represent a quantization of the hyperbolic volume. Viewing the hyperbolic structure as a particular flat $\text{SL}(2,\mathbb{C})$ connection, the above was given an interpretation in terms of topological quantum field theory with gauge group $\text{SL}(2,\mathbb{C})$, [9].

Once we got a relation like the volume conjecture, it is natural to think about quantization of other geometric properties. Let $K$ be a knot in $S^3$ and $\Gamma_K$ be the fundamental group of the complement of $K$, which is isomorphic to a quotient of the hyperbolic space $\mathbb{H}^3/\Gamma$ by a discrete subgroup group $\Gamma$ in $\text{SL}(2,\mathbb{C})$, where $\Gamma$ is isomorphic to $\Gamma_K$. So, if there is a good quantization of the Lie group $\text{SL}(2,\mathbb{C})$ and its discrete subgroup $\Gamma$, then the geometric structure of the complement of $k$ can be quantized. As the first step of this purpose, we construct a quantum deformation of the $\text{SL}(2,\mathbb{C})$ representation space of $K$ by using the braided $\text{SL}(2,\mathbb{C})$ introduced by S. Majid [14].
We start by briefly recalling the construction of the space of representations of the knot group $\Gamma_K$ into $\text{SL}(2)$ that we aim to generalize/quantize in this work. The space of representations is described by an ideal in a tensor power of the coordinate algebra $\mathbb{C}[\text{SL}(2)]$. The coordinate algebra is generated by the four matrix entries and any presentation of $\Gamma_K$ allows us to express the relations as polynomial equations in these matrix entries.

This construction works for any finitely presented group and any affine algebraic group and is independent of the chosen presentation, see [6, Proposition 8.2]. However, it is not clear how to generalize this ideal to an ideal in a non-commutative deformation (i.e. quantizing) because one would need some way to order the variables that no longer commute. In the case of knots we are able to generalize the ideal $I_b$ describing the representation space of the knot $K$ by presenting it as the closure of a braid $b$ and using the Wirtinger presentation where all relations are given by conjugation.

To prepare our generalization to the non-commutative world we construct the ideal $I_b$ using the commutative Hopf algebra structure of the coordinate ring $\mathbb{C}[\text{SL}(2)]:$

\[\Delta : \mathbb{C}[\text{SL}(2)] \to \mathbb{C}[\text{SL}(2)] \otimes \mathbb{C}[\text{SL}(2)] \quad \text{with} \quad \Delta(f)(a_1 \otimes a_2) = f(a_1 a_2) \quad \text{(coproduct)},\]

\[S : \mathbb{C}[\text{SL}(2)] \to \mathbb{C}[\text{SL}(2)] \quad \text{with} \quad S(f)(a) = f(a^{-1}) \quad \text{(antipode)},\]

\[\varepsilon : \mathbb{C}[\text{SL}(2)] \to \mathbb{C} \quad \text{with} \quad \varepsilon(f) = f(E) \quad (E : \text{the identity matrix of } \text{SL}(2, \mathbb{C})) \quad \text{(counit)}.]\]

If the braid $b$ is a product of the standard generators $\sigma_1, \ldots, \sigma_n$ the ideal $I_b$ is generated by

\[\psi_b - \text{id} : \mathbb{C}[\text{SL}(2)]^\otimes n \to \mathbb{C}[\text{SL}(2)]^\otimes n \quad (i = 1, 2, \cdots, n)\]

where $\psi_b$ is given by

\[\psi_b(\alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_n) = \left(\psi_{\sigma_{i_1}} \circ \psi_{\sigma_{i_2}} \circ \cdots \circ \psi_{\sigma_{i_n}} \right) (\alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_n),\]

and $\psi_{\sigma_{\pm 1}}$ is given by

(1) \[\psi_{\sigma_{i}}(\alpha_1 \otimes \cdots \otimes \alpha_i \otimes \alpha_{i+1} \otimes \cdots \otimes \alpha_n) = (\alpha_1 \otimes \cdots \otimes \alpha_{i-1}^{(2)} \otimes \alpha_i S(\alpha_{i+1}^{(1)}) \alpha_{i+1}^{(3)} \otimes \cdots \otimes \alpha_n),\]

\[\psi_{\sigma_{i}}^{-1}(\alpha_1 \otimes \cdots \otimes \alpha_i \otimes \alpha_{i+1} \otimes \cdots \otimes \alpha_n) = (\alpha_1 \otimes \cdots \otimes \alpha_i^{-1} S(\alpha_{i+1}^{(3)}) \alpha_{i+1} \otimes \alpha_i^{(2)} \otimes \cdots \otimes \alpha_n).\]

Here we use Sweedler notation, i.e. $\alpha^{(1)} \otimes \alpha^{(2)} \otimes \alpha^{(3)}$ means $\Delta(\Delta(\alpha)) = \sum_j \alpha_j^{(1)} \otimes \alpha_j^{(2)} \otimes \alpha_j^{(3)}$.

Of course each generator just acts by conjugation as in the Wirtinger presentation. A diagrammatic interpretation of (1) is given in Figure 1. This construction of $I_b$ works not only for $\mathbb{C}[\text{SL}(2)]$ but also for any commutative Hopf algebra. Our main result is that it also works for braided commutative (braided) Hopf algebras.

A braided Hopf algebra $A$ is a generalization of a Hopf algebra where the braiding is used instead of the usual flip sending $x \otimes y$ to $y \otimes x$ as in Figure 2. Braided commutativity is a generalization of the commutativity property of usual Hopf algebras, which is given in Definition 3.
To generalize the above construction of the ideal $I_b$ to get a space of $A$ representations, we modify the relation at the crossing as in Figure 3. With this modification, we define a two-sided ideal $I_b$ and get the space of $A$ representation of $K$ as a quotient of $A^\otimes n$ divided by $I_b$ as before. Our main result is that this quotient algebra is proved to be an invariant of $K$, see Theorem 2.

The typical example of braided Hopf algebras is BSL(2), which is the braided one-parameter deformation of the coordinate ring of $\text{SL}(2, \mathbb{C})$. By applying the above construction, we get the space of BSL(2) representation which is a quantization of the $\text{SL}(2, \mathbb{C})$ representation space of $K$. This algebra gives a non-commutative algebraic scheme which should be the coordinate ring of the ‘space’ of quantized $\text{SL}(2, \mathbb{C})$ representations in the sense of non-commutative algebraic geometry.

This paper is organized as follows. In Section 2, we introduce the braided Hopf algebra with a focus on the braided commutative case. In Section 3, we construct representation of the braid group $B_n$ in $\text{End}(A^\otimes n)$ for any braided Hopf algebra $A$. Here we use the
braided version of the Wirtinger presentation given in Figure 3. In Section 4, we define the space of \( A \) representations of a knot \( K \) for any braided Hopf algebra \( A \) with braided commutativity. Let \( b \) be a braid in \( B_n \), whose closure \( \hat{b} \) is isotopic to \( K \), and construct the space of \( A \) representations of \( b \) as a quotient of \( A^\otimes n \) by using the construction in Section 3. Then it is shown that this space only depends on the isotopy type of \( \hat{b} \). In Section 5, we apply the above construction to the trefoil knot and the Hopf link. In Section 6, we recall the definition and properties of the braided \( \text{SL}(2) \), which is a typical example of the braided Hopf algebra. At the final section, some remarks and comments are given, including the definition of quantum deformation of \( \text{SL}(2) \) character variety, which is the co-adjoint invariant subspace of the space of \( B\text{SL}(2) \) representations.

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2. Braided Hopf Algebra and Braided Commutativity

2.1. Braided Hopf algebra. A braided Hopf algebra is a version of a Hopf algebra having an extra operation called braiding. It may also be viewed as a Hopf object in a braided monoidal category, see for example, [16]. Through the process of transmutation any quasi-triangular Hopf algebra gives rise to a braided Hopf algebra.

Definition 1. An algebra \( A \) over a field \( k \) is called a \textit{braided Hopf algebra} if it is equipped with following linear maps described by the diagrams in Figure 4 satisfying the relations given in Figure 5.

- **multiplication**: \( \mu : A \otimes A \to A \),
- **comultiplication**: \( \Delta : A \to A \otimes A \),
- **unit**: \( 1 : k \to A \),
- **counit**: \( \varepsilon : A \to k \),

\[ 
\begin{align*}
\text{multiplication} & : \mu & & \text{comultiplication} & : \Delta \\
\text{braiding} & : \Psi & & \text{inverse braiding} & : \Psi^{-1} \\
\text{antipode} & : S & & S^{-1} \text{ unit} & : 1 \text{ counit} & : \varepsilon
\end{align*}
\]

**Figure 4.** The operations of the braided Hopf algebra \( A \).
antipode: $S: A \to A$,
braiding: $\Psi: A \otimes A \to A \otimes A$.

Figure 5. The relations of a braided Hopf algebra, read from top to bottom.

**Definition 2.** A diagram expressing a linear mapping from $A^\otimes m$ to $A^\otimes n$ consists of the components corresponding to the operations of the Hopf algebra structure of $A$ given in Figure 4 is called a braided Hopf diagram. Let $\text{BHD}(m, n)$ denote the set of braided Hopf diagrams expressing linear homomorphisms from $A^\otimes m$ to $A^\otimes n$.

2.2. **Adjoint coaction.** A $k$-vector space $M$ is called a right $A$-comodule if there is a linear map

$$\Delta: M \to M \otimes A$$
satisfying the coassociativity
\[(\Delta \otimes id)(\Delta) = (id \otimes \Delta)(\Delta)\].

Then \(A\) itself is a right \(A\)-comodule with the following adjoint coaction \(\text{ad} : A \to A \otimes A\).
\[
\text{ad}(x) = (id \otimes \mu)(\Psi \otimes id)(S \otimes \Delta)\Delta(x),
\]

where \(\mu : A \otimes A \to A\) is the multiplication of \(A\), i.e. \(\mu(x \otimes y) = xy\).

**Proposition 1.** The adjoint coaction satisfies the following relations.

\[(2) \quad (id \otimes id \otimes \mu)(id \otimes \Psi \otimes id)(ad \otimes ad)\Delta(x) = (\Delta \otimes id)\text{ad}(x),
\]

\[(3) \quad (ad \otimes id)\text{ad} = (id \otimes \Delta)\text{ad}.
\]

\[(4) \quad (\varepsilon \otimes id)\text{ad} = 1 \circ \varepsilon \quad (id \otimes \varepsilon)\text{ad} = id
\]

**Proof.** The relations (2) and (3) are proved by the graphical computation in Figure 6. The relations (4) come from the properties of the unit 1 and the antipode \(\varepsilon\). 

\[
\text{(2)} : \qquad \text{(3)} : \qquad \text{(4)} :
\]

**Figure 6.** Graphical proof of Proposition 1.
2.3. Braided commutativity. We introduce the notion of the braided commutativity, which implies the compatibility of the adjoint coaction with respect to the product $\mu$, the braiding $\Psi$, and the antipode $S$.

**Definition 3.** The braided Hopf algebra $A$ is **braided commutative** if it satisfies

\[(id \otimes \mu)(\Psi \otimes id)(id \otimes ad)\Psi = (id \otimes \mu)(ad \otimes id)\]

In the remainder of this section we assume $A$ is braided commutative.

**Proposition 2.** The adjoint coaction commutes with the multiplication, i.e.

\[(ad \circ \mu) = (\mu \otimes \mu)(id \otimes \Psi \otimes id)(ad \otimes ad)\]

*Proof.* The relation (6) is proved by the graphical computation in Figure 7. At the second to last equality, we use the braided commutativity. \(\square\)

**Proposition 3.** The adjoint coaction commutes with the braiding $\Psi$ as follows.

\[(id \otimes id \otimes \mu)(id \otimes \Psi \otimes id)(ad \otimes ad) \Psi = (\Psi \otimes id)(id \otimes id \otimes \mu)(id \otimes \Psi \otimes id)(ad \otimes ad)\]

*Proof.* This relation comes from the braided commutativity as explained in Figure 8. \(\square\)

**Proposition 4.** The adjoint coaction commutes with the antipode $S$, i.e.

\[ad \circ S = (S \otimes id) \circ ad\]
Figure 8. The adjoint coaction is commutative with the braiding.

Proof. This relation comes from the braided commutativity as explained in Figure 9. The braided commutativity is used at the second equality. In the fourth equality we used the antipode axiom and in the final equation the axiom relating $S$ and multiplication as used.

Figure 9. The adjoint ad commutes with $S$.

3. Braid representation

In this section, we construct a homomorphism from the braid group $B_n$ on $n$ strings to $\text{End}(A^\otimes n)$ by using the braided Hopf algebra structure of $A$. In this section, we do not require the braided commutativity of $A$.

3.1. Representation of generators. The braid group $B_n$ is defined by the following generators and relations.

$$B_n = \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i \quad (|i-j| \geq 2) \rangle.$$

We define a braided Hopf diagram corresponding to the braid generators by generalizing the definition of $R_{ad}$ in [7], which is based on [15]. These are braided version of the Wirtinger presentation for the fundamental group of a knot complement.

$$R = \quad R^{-1} = \quad R$$

For $\sigma_i \in B_n$, let

$$\rho(\sigma_i) = id^{\otimes (i-1)} \otimes R \otimes id^{\otimes (n-i-1)} \in A^{\otimes n}.$$
3.2. **Adjoint coaction at crossing.** We define an adjoint coaction
\[ \text{Ad} : A^\otimes m \to A^\otimes m \otimes A \]
as in Figure 10.

![Figure 10. Adjoint coaction Ad : A^\otimes n \to A^\otimes n \otimes A.](image)

**Proposition 5.** The adjoint coaction \( \text{Ad} \) commutes with \( R \), i.e.
\[ \text{Ad} \circ R = (R \otimes \text{id}) \circ \text{Ad} \quad \text{for } b \in B_n. \]

**Proof.** This is proved by the graphical computation in Figure 11.

![Figure 11. Commutativity of R and Ad.](image)

3.3. **Representation of braid groups.** Now we construct a representation of braid groups in \( \text{End}(A^\otimes n) \).

**Theorem 1.** The map \( \rho \) defined for generators of \( B_n \) in (9) extends to an algebra homomorphism from the group algebra \( \mathbb{C}B_n \) to \( \text{End}(A^\otimes n) \).

**Proof.** We first show that \( \rho(\sigma_i) \rho(\sigma_i^{-1}) = \rho(\sigma_i^{-1}) \rho(\sigma_i) = 1 \). To show these, we prove
\[ RR^{-1} = R^{-1} R = \text{id} \otimes \text{id} \]
by the graphical computation in Figure 12 and Figure 13. The braid relation \( \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \) comes from
\[ (R \otimes \text{id})(\text{id} \otimes R)(R \otimes \text{id}) = (\text{id} \otimes R)(R \otimes \text{id})(\text{id} \otimes R), \]
which is shown by the graphical computation in Figure 14. We also have \( \sigma_i \sigma_j = \sigma_j \sigma_i \) for \( j - i \geq 2 \) since
\[ R_i R_j = \text{id}^\otimes(i-1) \otimes R \otimes \text{id}^\otimes(n-i-j-2) \otimes R \otimes \text{id}^\otimes(n-i-1) = R_j R_i. \]
where $R_i = id^\otimes(i-1) \otimes R \otimes id^\otimes(n-i-1)$. Hence the relations of $B_n$ are all satisfied. □

3.4. **Adjoint coaction on braids.** From the commutativity of $Ad$ and $\rho(\sigma_i)$ given in Proposition 5, we get the following.
Proposition 6. The adjoint coaction $\text{Ad}$ commutes with $\rho(b)$ for $b \in B_n$, i.e.

$$\text{Ad} \circ \rho(b) = (\rho(b) \otimes \text{id}) \circ \text{Ad}.$$ 

4. Space of braided Hopf algebra representations of a knot

Let $A$ be a braided Hopf algebra with braided commutativity. For a knot $K$, we construct the space of $A$ representations of $K$ as a quotient algebra of $A^\otimes n$ by the two-sided ideal $I$ determined by a braid whose closure is $K$. The number $n$ and the ideal $I$ depend on the choice of the braid, but it is shown that the resulting quotient spaces are isomorphic if the closures of the braids are the same knot $K$.

4.1. Knots as braid closures. Let $K$ be a knot in $S^3$, then it is known that there is a braid $b \in B_n$ for certain $n \in \mathbb{N}$ such that $K$ is isotopic to the closure of $b$. The closure of $b$ is obtained by connecting the top points and bottom points of $b$ as in Figure 15, and is denoted by $\hat{b}$.

4.2. Space of $A$ representations. For $b \in B_n$, let $d(b)$ be the corresponding braided Hopf diagram. Then $d(b)$ represents $\rho(b) \in \text{End}(A^\otimes n)$. Let us assign $x_1, x_2, \cdots, x_n \in A$ to the top points of $d(b)$. Then the corresponding element in $A^\otimes n$ at the bottom of $d(b)$ is $\rho(b)(x_1 \otimes x_2 \otimes \cdots \otimes x_n)$. Let $I_{d(b)}$ be the left ideal generated by $\rho(b)(x_1 \otimes x_2 \otimes \cdots \otimes x_n) - x_1 \otimes x_2 \otimes \cdots \otimes x_n$ for all $x_1, x_2, \cdots, x_n \in A$.

Lemma 1. For $b \in B_n$, the left ideal $I_{d(b)}$ is an $A$-comodule with respect to the adjoint coaction $\text{Ad}$ of $A$.

Proof. For $\mathbf{x}, \mathbf{y} \in A^\otimes n$, we have

$$\text{Ad} \circ (d(b) - id^\otimes n)(\mathbf{x}) = \left( (d(b) - id^\otimes n) \otimes \text{id} \right) \circ \text{Ad}(\mathbf{x}) \in I_{d(b)} \otimes A$$

by Proposition 6. Hence $\text{Ad}(I_{d(b)}) \subset I_{d(b)} \otimes A$. \qed
Lemma 2. For \( x \in A^{\otimes n} \), we have
\[
  d(b) \mu(x \otimes y) = \mu((d(b) x) \otimes (d(b) y)).
\]

Proof. It is enough to show that
\[
  R \mu(x \otimes y) = \mu(R \otimes R)(x \otimes y)
\]
for the product \( \mu : A^{\otimes 2} \otimes A^{\otimes 2} \to A^{\otimes 2} \) and \( x = x_1 \otimes x_2, y = y_1 \otimes y_2 \in A^{\otimes 2} \), which is proved graphically in Figure 16. Proposition 2 is used at the first equality and the braided commutativity is used at the second equality. □

![Figure 16](image_url)

Figure 16. \( R \) is distributive over the multiplication.

Proposition 7. The left ideal \( I_{d(b)} \) is a two-sided ideal of \( A^{\otimes n} \).

Proof. To show that \( I_{d(b)} \) is a right ideal, we prove that \( \mu((d(b) x) \otimes y) - \mu(x \otimes y) \in I_{d(b)} \) for any \( x, y \in A^{\otimes n} \). Since \( d(b^{-1}) x - x \in I_{d(b)} \) and \( I_{d(b)} \) is a left ideal, \( \mu(x \otimes (d(b^{-1}) y)) - \mu(x \otimes y) \) is also contained in \( I_{d(b)} \). From Lemma 2, we have
\[
  \mu(x \otimes (d(b^{-1}) y)) - \mu(x \otimes y) = d(b^{-1}) \mu((d(b) x) \otimes y) - \mu(x \otimes y) \in I_{d(b)}.
\]
On the other hand,
\[
  d(b^{-1}) \mu((d(b) x) \otimes y) - \mu((d(b) x) \otimes y) \in I_{d(b)}.
\]
Therefore
\[
  \mu((d(b) x) \otimes y) - \mu(x \otimes y)
  = - \left( d(b^{-1}) \mu((d(b) x) \otimes y) - \mu((d(b) x) \otimes y) \right) + \left( d(b^{-1}) \mu((d(b) x) \otimes y) - \mu(x \otimes y) \right)
  \in I_{d(b)}
\]
and so \( I_{d(b)} \) is a right ideal. □

Let \( A_b \) be the quotient algebra given by the two-sided ideal \( I_{d(b)} \), i.e.
\[
  A_b = A^{\otimes n} / I_{d(b)}.
\]

Then we have the following.
Theorem 2. If the closures of two braids $b_1 \in B_{n_1}$ and $b_2 \in B_{n_2}$ are isotopic, then $A_{b_1}$ and $A_{b_2}$ are isomorphic algebras. Moreover, $A_{b_1}$ and $A_{b_2}$ are isomorphic $A$-comodules. In other words, $A_b$ is an invariant of the knot (or link) $\hat{b}$.

Definition 4. The quotient algebra $A_b = A^{\otimes n}/I_{d(b)}$ is called the space of $A$ representations of the closure $\hat{b}$.

Since $I_{d(b)}$ is a two-sided ideal, $A_b$ is a non-commutative algebra which defines a non-commutative algebraic variety.

4.3. Markov moves. It is known that the closures of two braids $b_1 \in B_{n_1}$ and $b_2 \in B_{n_2}$ are isotopic in $S^3$ if and only if there is a sequence of the following two types of moves connecting $b_1$ to $b_2$. These moves are called the Markov moves and such $b_1$ and $b_2$ are called Markov equivalent.

**First Markov move (MI):** $bb' \leftrightarrow b'b$ for $b, b' \in B_n$.

**Second Markov move (MII):** $b \in B_n \leftrightarrow \sigma_n^{\pm 1} b \in B_{n+1}$.

![Figure 17. The Markov moves.](image)

4.4. Equivalent pair. For $b \in B_n$, we present $I_{d(b)}$ by $d(b) \sim d(e)$ where $e$ is the unit element of $B_n$. Similarly, for two braided Hopf diagrams $d_1, d_2 \in \text{BHD}(m, n)$, $d_1 \sim d_2$ present a two-sided ideal $I_{d_1,d_2}$ in $A^{\otimes m}$ which is spanned by $d_1(x_1 \otimes \cdots \otimes x_m) - d_2(x_1 \otimes \cdots \otimes x_m)$ for $x_1, \ldots, x_m \in A$. Such $d_1$ and $d_2$ are called the equivalent pair of diagrams corresponding to the two-sided ideal $I_{d_1,d_2}$ and the quotient algebra $A^{\otimes m}/I_{d_1,d_2}$.

Lemma 3. Let $d_1 \sim d_2$ be an equivalent pair of BHD($m, n$) and let $d'_1 \sim d'_2$ be the equivalent pair where $d'_1$ and $d'_2$ are obtained from $d_1$ and $d_2$ respectively by one of the following operations (1), (2), (3), (3S), (4L), (4LS), (4R), (4RS) illustrated in Figure 18. Then the corresponding ideals $I_{d_1,d_2}$ and $I_{d'_1,d'_2}$ are equal.

(1) Add $S$ or $S^{-1}$ to the same position of $d_1$ and $d_2$ at the top.
(2) Apply a braiding to the same position of $d_1$ and $d_2$ at the top.
(3) Add an arc connecting the adjacent strings to the same position of $d_1$ and $d_2$ at the top.
(3S) Add an arc with $S$ connecting the adjacent arcs to the same position of $d_1$ and $d_2$ at the top.
(4L) Add an arc connecting the leftmost top arc and some bottom arc.
(4LS) Add an arc with $S$ connecting the leftmost top arc and some bottom arc.
(4R) Add an arc connecting the rightmost top arc and some bottom arc.
(4RS) Add an arc with $S$ connecting the rightmost top arc and some bottom arc.

Figure 18. Operations applied simultaneously to $d_1, d_2$ to make $d'_1, d'_2$ in Lemma 2.

Proof. The operations (1), (2), (3), (3S) are invertible operations at the top and they do not change the image of $d_1 - d_2$. Hence we have $\text{Im}(d_1 - d_2) = \text{Im}(d'_1 - d'_2)$ and so $I_{d_1,d_2} = I_{d'_1,d'_2}$.

For (4L), (4LS), (4R), (4RS), we see that $I_{d'_1,d'_2} \subset I_{d_1,d_2}$ since $I_{d_1,d_2}$ is a two-sided ideal. But (4L) and (4LS), (4R) and (4RS) are mutually inverse respectively and so we have $I_{d_1,d_2} \subset I_{d'_1,d'_2}$. Hence we get $I_{d'_1,d'_2} = I_{d_1,d_2}$. □
4.5. Invariance under MII move. We show that the quotient algebra keeps its structure by MII move.

**Proposition 8.** For \( b \in B_n \), the algebras \( A^{\otimes n}/I_{d(b)} \) and \( A^{\otimes (n+1)}/I_{d(b\sigma_n)} \) are isomorphic. They are also isomorphic as \( A \)-comodules.

**Proof.** We first prove that the quotient algebras \( A^{\otimes n}/I_{d(b)} \) and \( A^{\otimes (n+1)}/I_{d(b\sigma_n)} \) are isomorphic. The two-sided ideal \( I_{b\sigma_n} \) is given by the equivalent diagrams \( d(b\sigma_n) \sim d(e) \) as in the left hand side of Figure 19. By adding a string with \( S \) from the right as the operation (4RS) in the previous lemma, we get new equivalent diagram as the right hand side of Figure 19. These equivalent diagrams are deformed as in Figure 20 and the last equivalent diagrams mean that the ideal \( I_{d(b\sigma_n)} \) is generated by

**Figure 19.** Apply (4RS) to \( d(b\sigma_n) \) and \( d(e) \).

**Figure 20.** Deform the equivalent diagrams \( d(b\sigma_n) \sim d(e) \) modified by (4RS). Symbols under the arrows show the operations in Lemma 3. At the first equality sign the relation between \( \mu \) and \( \Delta \) is used.
\[ d(b)(x_1 \otimes \cdots \otimes x_{n-1} \otimes \mu (\Psi_{n,n+1}(x_n, x_{n+1})) \otimes 1 - x_1 \otimes \cdots \otimes x_n \otimes x_{n+1} \]

for any \( x_1, \cdots, x_{n+1} \in A \). Let \( p_n \) be the surjection from \( A^{\otimes (n+1)} \) to \( A^{\otimes n} \) defined by

\[ p_n(x_1 \otimes \cdots \otimes x_n \otimes x_{n+1}) = x_1 \otimes \cdots \otimes x_{n-1} \otimes \mu (\Psi_{n,n+1}(x_n, x_{n+1})). \]

Then \( I_{d(\sigma_{n\alpha})} \) is generated by \((d(b) \circ p_n)(x) \otimes 1 - x\) for \( x \in A^{\otimes (n+1)} \) and \( p_n(I_{d(\sigma_{n\alpha})}) = I_{d(b)} \).

For \( x \in \text{Ker} p_n \),

\[ (d(b) \circ p_n)(x) \otimes 1 - x = -x, \]

and so \( x \in I_{d(\sigma_{n\alpha})} \). This means that \( \text{Ker} p_n \subset I_{d(\sigma_{n\alpha})} \), which implies \( p_n^{-1}(I_{d(b)}) = I_{d(\sigma_{n\alpha})} \) since \( p_n(I_{d(\sigma_{n\alpha})}) = I_{d(b)} \). Therefore \( p_n \) gives an isomorphism \( A^{\otimes (n+1)}/I_{d(\sigma_{n\alpha})} \cong A^{\otimes n}/I_{d(b)} \).

The multiplication \( \mu \) and the braiding \( \Psi \) commute with the adjoint \( \text{Ad} \) so we have \( \text{Ad} \circ p_n = (p_n \otimes \text{id}) \circ \text{Ad} \). This means that \( p_n \) is a comodule map with respect to the coaction \( \text{Ad} \), and the isomorphism \( A^{\otimes (n+1)}/I_{d(\sigma_{n\alpha})} \cong A^{\otimes n}/I_{d(b)} \) is also an isomorphism as a \( A \)-comodule.

The proof for \( A^{\otimes (n+1)}/I_{d(b)} \cong A^{\otimes n}/I_{d(\sigma_{n\alpha}^{-1})} \) is similar to the previous one except the sequence of deformations of the diagram, which is given in Figure 21.

\[ \text{Figure 21. Deform the diagram } d(b\sigma_{n\alpha}^{-1}). \]

### 4.6. Proof of Theorem 2

Let \( b_1, b_2 \) are braids in \( B_{n_1}, B_{n_2} \) respectively. We show that the quotient algebra \( A^{\otimes n_1}/I_{d(b_1)} \) and \( A^{\otimes n_2}/I_{d(b_2)} \) are isomorphic if there is a sequence of Markov moves which sends \( b_1 \) to \( b_2 \).

First, consider the MI move. Let \( b_1, b_2 \in B_n \) such that \( b_1 = b \cdot b' \) and \( b_2 = b' \cdot b \). Then there is an algebra isomorphism \( f : A^{\otimes n}/I_{d(b_1)} \to A^{\otimes n}/I_{d(b_2)} \) given by \( f(x) = d(b')(x) \).

The map \( f \) is well-defined because \( f(d(bb')(x) - x) = d(b'b)(y) - y \) with \( y = d(b')(x) \) and the inverse is given by \( f^{-1}(x) = (d(b')^{-1})(x) \).

Next, consider the MII move. For this case, Proposition 8 shows that \( A^{\otimes n}/I_{d(b)} \) and \( A^{\otimes (n+1)}/I_{d(\sigma_{n\alpha}^{-1})} \) are isomorphic algebras.
Moreover, $A^\otimes n/I_{d(b)}$ and $A^\otimes (n+1)/I_{d(b^{n+1})}$ are isomorphic as $A$-comodules since all the braided Hopf diagrams in Figures 19, 20, 21 commute with $\text{Ad}$. The same is true for the isomorphism $f$ dealing with the first Markov move. □

5. Examples

Let $A$ be a braided Hopf algebra. We construct the space of $A$ representations for the Hopf link and the trefoil knot.

5.1. Hopf link. The Hopf link is the closure of $\sigma^2_1$ in $B_2$. So we can get the space of $A$ representations of the Hopf link by the sequence of deformations of the equivalent pair $\sigma^2_1 \sim e$ given in Figure 22. From the resulting diagrams, the space of $A$ representations of the Hopf link $A_{\text{Hopf}}$ is given by

$$A_{\text{Hopf}} = A \otimes A/I,$$

where $I$ is the two-sided ideal generated by

$$\left( (\Psi - \text{id}) \circ \Delta \right)(x)$$

for all $x \in A$.

![Figure 22. A sequence of deformations for the equivalent pair $\sigma^2_1 \sim e$.](image)

5.2. Trefoil knot. Now we assume that $A$ is braided commutative. The trefoil knot is the closure of $\sigma^3_1$ in $B_2$. So we can get the space of $A$ representations of the trefoil knot by the sequence of deformations of the equivalent pair $\sigma^3_1 \sim e$ given in Figure 23. From the resulting diagrams, the space of BSL(2) representations of the trefoil $A_{\text{trefoil}}$ is given by

$$A_{\text{trefoil}} = A \otimes A/I,$$

where $I$ is the two-sided ideal generated by

$$\left( (\text{id} \otimes \mu) \circ (\Psi \otimes \text{id}) - (\mu \otimes \text{id}) \circ (\text{id} \otimes \Psi) \right) \circ (\text{id} \otimes \Delta) \circ \Delta(x)$$
for all $x \in A$. 

\[
\begin{array}{c}
\text{Figure 23. A sequence of deformations for the equivalent pair } \sigma_1^3 \sim e. \\
\end{array}
\]

6. Braided SL(2)

In this section, we review the braided Hopf algebra BSL(2) introduced by S. Majid in \cite{Majid}, which is a quantization of the coordinate ring of SL(2). We will see that it is in fact braided commutative so that the constructions of the previous sections apply to this case. For a knot $K$ in $S^3$, we construct a quantum version of the space of SL(2) representations of $\pi_1(S^3 \setminus K)$ as a quotient algebra of a tensor product of BSL(2).

6.1. Definition of the braided SL(2) Throughout this paper, let $t$ be an indeterminate. As a $\mathbb{C}(t)$-algebra, BSL(2) is generated by 4 elements $a$, $b$, $c$, and $d$ corresponding to the matrix elements of the $2 \times 2$ matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and satisfying the following relations.

\[
\begin{align*}
ba &= ta b, & ca &= t^{-1} a c, & da &= ad, \\
db &= bd + (1 - t^{-1}) a b, & cd &= dc + (1 - t^{-1}) c a, \\
bc &= cb + (1 - t^{-1}) a (d - a), & ad - t cb &= 1.
\end{align*}
\]

(14)

It follows directly from the relations and proposition I.8.17 of \cite{Majid} that BSL(2) is Noetherian.

Remark 1. Among the relations of BSL(2), only the determinant relation $ad - t cb = 1$ is non-homogeneous. So if we introduce a degree 2 indeterminate $\delta$ and replace the determinant relation by $ad - t cb = \delta$, then every relation of BSL(2) is homogeneous.
and the BSL(2)-comodule BSL(2)_b = BSL(2)^{\otimes n}/I_{d(b)} can be regarded as a projective non-commutative algebraic scheme.

The Hopf algebra structure of BSL(2) is given as follows.

\[ \Delta(a) = a \otimes a + b \otimes c, \quad \Delta(b) = a \otimes b + b \otimes d, \]
\[ \Delta(c) = c \otimes a + d \otimes c, \quad \Delta(d) = c \otimes b + d \otimes d, \]
\[ S(a) = (1 - t)a + td \quad S(b) = -tb, \quad S(c) = -tc, \quad S(d) = a, \]
\[ \varepsilon(a) = 1, \quad \varepsilon(b) = 0, \quad \varepsilon(c) = 0, \quad \varepsilon(d) = 1, \]
\[ \Psi(x \otimes 1) = 1 \otimes x, \quad \Psi(1 \otimes x) = x \otimes 1, \]
\[ \Psi(a \otimes c) = c \otimes a + (1 - t)b \otimes c, \quad \Psi(a \otimes b) = b \otimes a, \]
\[ \Psi(b \otimes a) = a \otimes b + (1 - t)(d - a) \otimes (d - a), \quad \Psi(b \otimes b) = t b \otimes b, \]
\[ \Psi(b \otimes c) = t^{-1}c \otimes b + (1 + t)(1 - t^{-1})b \otimes c - (1 - t^{-1})(d - a) \otimes (d - a), \]
\[ \Psi(b \otimes d) = d \otimes b + (1 - t^{-1})b \otimes (d - a), \]
\[ \Psi(c \otimes a) = a \otimes c, \quad \Psi(c \otimes b) = t^{-1}b \otimes c, \]
\[ \Psi(c \otimes c) = t c \otimes c, \quad \Psi(c \otimes d) = d \otimes c, \]
\[ \Psi(d \otimes a) = a \otimes d + (1 - t^{-1})b \otimes c, \quad \Psi(d \otimes b) = b \otimes d, \]
\[ \Psi(d \otimes c) = c \otimes d + (1 - t^{-1})(d - a) \otimes c, \quad \Psi(d \otimes d) = d \otimes d - t^{-1}(1 - t^{-1})b \otimes c. \]

6.2. Braided commutativity of braided SL(2). We give a simple and self-contained proof that the braided Hopf algebra BSL(2) has the braided commutativity property. Recall that this means for all \( x, y \in BSL(2) \) we have \((id \otimes \mu)(\Psi \otimes id)(id \otimes ad)\Psi(x \otimes y) = (id \otimes \mu)(ad \otimes id)(x \otimes y)\). The result also follows from the general theory of transmutation [15], theorem 4.1.

**Proposition 9.** The braided Hopf algebra BSL(2) is braided commutative.

**Proof.** By the linearity, it is enough to prove for the case that \( x \) and \( y \) are both monomials. The length of a monomial \( x \) is the length of the word \( w \) explaining \( x \) consists of the generators \( a, b, c, d \). Now we prove the proposition by inductions on the lengths of \( x \) and \( y \).

The case where \( x, y \in \{a,b,c,d\} \) can be proven by direct computations. For example, if \( x = y = a \), we have

\[ (id \otimes \mu)(\Psi \otimes id)(id \otimes ad)\Psi(a \otimes a) = (id \otimes \mu)(ad \otimes id)(a \otimes a) = a \otimes a^2d + t^{-1}b \otimes acd - t^2c \otimes a^2b - td \otimes acb. \]

Another example is for \( x = b \) and \( y = c \).

\[ (id \otimes \mu)(\Psi \otimes id)(id \otimes ad)\Psi(b \otimes c) = (id \otimes \mu)(ad \otimes id)(b \otimes c) = \]
\[
a \otimes \left( - (1 - t^{-1}) a^2 d + (1 - t^{-1}) a d^2 + c b d \right) + b \otimes \left( - (t^{-1} - t^{-3}) a c d + (1 - t^{-1}) c^2 b + c d^2 \right) +
\]
\[
c \otimes \left( (t^2 - t + t^{-1} - t^{-2}) a^2 b - (t - t^{-1}) a b d - c b^2 \right) + d \otimes \left( (1 - t^{-1}) a^2 d - (1 - t^{-1}) a d^2 - c b d \right).
\]

The other cases can be checked similarly.

Next, we assume that (5) holds for \( x \) with \( \text{length}(x) \leq m - 1 \) and \( y \in \{a, b, c, d\} \). For \( x \) with \( \text{length}(x) = m \), let \( x = x_1 x_2 \) with \( \text{length}(x_1) = m - 1 \) and \( \text{length}(x_2) = 1 \), (5) is proved by the following graphical computation in Figure 24. The induction hypothesis is used at the 2nd, 3rd and 6th equalities.

![Figure 24. Step 2 of the proof of the braided commutativity.](image_url)

Now we assume that (5) holds for \( x, y \) with \( \text{length}(x) = m \) and \( \text{length}(y) = n - 1 \). For \( y \) with \( \text{length}(y) = n \), let \( y = y_1 y_2 \) with \( \text{length}(y_1) = 1 \) and \( \text{length}(y_2) = n - 1 \), (5) is proved by the graphical computation in Figure 25. Hence the proposition is proved for all the cases.

![Figure 25. Step 3 of the proof of the braided commutativity.](image_url)
6.3. **Quantized SL(2) representations.** Let $K$ be a knot in $S^3$ and let $b$ be a braid in $B_n$ whose closure $\hat{b}$ is isotopic to $K$. Since BSL(2) is an example of a braided commutative braided Hopf algebra, we may take $A = \operatorname{BSL}(2)$ in Definition 4. The resulting space of BSL(2) representations of $K$ is $\operatorname{BSL}(2)^{\otimes n}/I_{d(b)}$. We also call it the space of quantized SL(2) representations of $K$. Since BSL(2) is Noetherian, the ideal $I_{d(b)}$ is finitely generated.

The BSL(2)-comodule $\operatorname{BSL}(2)^{\otimes n}/I_{d(b)}$ depends on the choice of $b$, but is unique up to comodule isomorphisms by Theorem 2. By putting $A = \operatorname{BSL}(2)$, the examples in Section 5 give the spaces of BSL(2) representations for the Hopf link and the trefoil knot.

7. **Final remarks**

For a braided commutative braided Hopf algebra $A$ we constructed a space of $A$ representations $A_b$ of knot $K$. Here $K$ is the closure of braid $b$. It may be valuable to consider the adjoint coaction of $A$ on $A_b$ given by Figure 10. Let $A_b^A$ be the Ad invariant subspace, i.e.

$$A_b^A = \{x \in A^{\otimes n}/I_{d(b)} \mid \operatorname{Ad}(x) = x \otimes 1\}$$

**Definition 5.** We call $A_b^A$ the **quantum $A$ character variety** of $K$. If $A = \operatorname{BSL}(2)$, we also call it the **quantum SL(2) character variety**.

Such a definition of a quantum analogue of character variety is also given by Habiro [11]. It would also be interesting to compare our quantization to the quantization based on ideal triangulations given in [12] and also with the quantization procedure of [11].

In particular in the case $A = \operatorname{BSL}(2)$ we constructed quantum version of the space of SL(2) representations of a knot group. Another way to construct a quantum version of the knot group is to quantize the crossings of the paths in the fundamental group. In the fundamental group, two elements corresponding to paths which are different only in a small neighbourhood of a point where they are as in Figure 26 are considered to be the same element. However, there is a way to distinguish such paths by applying the skein theory coming from the Jones polynomial. Such a skein theory is considered by many researchers, and it should have some relation to BSL(2), for example through the bigons in [13].

![Two types of crossings.](image)

**Figure 26.** Two types of crossings.

For surface groups, skein theory gives a nice quantization of the fundamental groups as shown in [2], [3], [4]. The knot group of the Hopf link can be identified with a surface
group of the 2-torus $T^2 = S^1 \times S^1$, so it may be interesting to compare the quantization of the surface groups of $T^2$ and the space of BSL(2) representations of the Hopf link.

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