RADIATION REACTION IN CURVED SPACE-TIME: LOCAL METHOD.*

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Although consensus seems to exist about the validity of equations accounting for radiation reaction in curved space-time, their previous derivations were criticized recently as not fully satisfactory: some ambiguities were noticed in the procedure of integration of the field momentum over the tube surrounding the world-line. To avoid these problems we suggest a purely local derivation dealing with the field quantities defined only on the world-line. We consider point particle interacting with scalar, vector (electromagnetic) and linearized gravitational fields in the (generally non-vacuum) curved space-time. To properly renormalize the self-action in the gravitational case, we use a manifestly reparameterization-invariant formulation of the theory. Scalar and vector divergences are shown to cancel for a certain ratio of the corresponding charges. We also report on a modest progress in extending the results for the gravitational radiation reaction to the case of non-vacuum background.

1. Introduction

Study of the radiation reaction problem in classical electrodynamics initiated by Lorentz and Abraham by the end of the 19-th century ¹, remained an area of active research during the whole 20-th century. Although with the development of quantum electrodynamics this problem became somewhat academic, it still attracts attention in connection with new applications and new ideas in fundamental theory. Current understanding of the radiation reaction has emerged in the classical works of Dirac ², Ivanenko and Sokolov ³, Rohrlich ⁴, Teitelboim ⁵ and some other (for a recent discussion see ⁶).

The Lorentz-Dirac equation was covariantly generalized to curved space-time with arbitrary metric by DeWitt and Brehme in 1960. In their paper

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an elegant technique of covariant expansion of two-point tensor quantities was introduced which later became a basis of the perturbative quantum field theory calculations in curved space-time. It was used for calculation of the field momentum within a small tube surrounding the world-line of a point charge which resulted in the charge equation of motion with the radiation damping term. The main difference with the flat space case was the presence of the tail term depending on the entire history of a charge. Its presence signals violation of the Huygens’ principle in curved space: a sharp pulse of light does not in general remain sharp, but gradually develops a “tail”. The equation (as extended by Hobbs to non-vacuum metrics) reads

\[ m \ddot{z}^\alpha = e F_{\alpha \beta}^{\text{in}} \dot{z}^\beta + \frac{2}{3} e^2 (\dot{z}^2 - \ddot{z}^2) \]

where \( F_{\alpha \beta}^{\text{in}} \) is the incoming electromagnetic field \( R_{\alpha \beta} \) is the Ricci tensor and \( f_{\alpha \beta \gamma} \) is some two-point function taken on the world-line. This results was later generalized to other massless fields — scalar and linearized gravitational. The gravitational radiation reaction was discussed long ago by C. DeWitt-Morette and Jing (9), and more recently reconsidered in detail by Mino, Sasaki and Tanaka (10) for vacuum background metrics. In an extension of DeWitt-Brehme method was used consisting in integration of the gravitational field momentum flux across the small world-tube surrounding the particle world-line. The equation of motion obtained in (10) can be presented as

\[ \ddot{z}^\mu = -\frac{1}{2} (g^{\mu \nu} + \dot{z}^\mu \dot{z}^\nu) (2 h^{\text{tail}}_{\nu \lambda \rho} - h^{\text{tail}}_{\lambda \rho}) \dot{z}^\lambda \dot{z}^\rho. \]

where (contrary to the case of (1)) the motion is supposed to be geodesic, so the local higher-derivative terms vanishes within the linear approximation, and the reaction force is given entirely by the tail term

\[ h^{\text{tail}}_{\mu \nu \lambda} = 4mG \int_{-\infty}^{\tau} \left( G_{\mu \nu \tau \sigma ; \lambda} - \frac{1}{2} g_{\mu \nu} G_{\rho \tau \sigma ; \lambda} \right) z(\tau), z(\tau') \dot{z}(\tau') \dot{z}(\tau') d\tau'. \]

Although there is consensus about the validity of these equations, their previous derivation is somewhat problematic. In fact, as was discussed recently 11, all calculations involving integration of the field momentum located the small tube surrounding the particle world-line contain some yet
unsolved problems. One such problem consists in computing the contributions of "caps" at the ends of the chosen tube segment: contributions of "caps" were rather conjectured than rigorously calculated. Another problem constitutes the singular integral over the internal boundary of the tube which was simply discarded in these calculations with no clear justification. In addition, the usual mass-renormalization is not directly applicable in the gravitational case since, due to the equivalence principle, the mass does not enter at all into the geodesic equations (in the mass parameter was actually reintroduced by hand).

Several new derivations were suggested during past few years. One is based on the redefinition of the Green’s functions of massless fields in curved space-time proposed by Detweiler and Whiting. But this scheme involves an axiomatic assumption about the nature of the singular term and so it does not help to solve the above problems. Another scheme was suggested by Quinn and Wald under the name of an "axiomatic approach to radiation reaction". This scheme makes use of some intuitive "comparison axioms", which do not follow from the first principles either. A recent attempt by Sanchez and Poisson to overcome the above difficulties in fact is tight to a particular model of an extended particle (a "dumbbell" model). Thus, though the results obtained within several different approaches agree in the final form of non-divergent terms, their consistent derivation from the first principles is still lacking. Also, the derivation of the gravitation radiation reaction force for non-vacuum background metrics remains an open problem.

Here we briefly report on the new derivation (more detailed version will be published elsewhere) of the radiation reaction in curved space-time which has an advantage to deal with the fields only on the world-line and not to involve the volume integrals over the world-tube at all. This provides a more economic calculation and at the same time removes objections raised in. The problem of divergences is relocated to the definition of the delta function with the support lying on the boundary of the integration domain, but this is exactly the same problem which is encountered in the flat space case where the approved prescription amounts to the point-splitting procedure. Thus the local method is free from ambiguities which were met in the previous curved space calculations and can be considered as an adequate solution of the radiation reaction problem in curved space-time from the first principles. In general, our final results are conformal with those derived previously; in addition we partly remove the restriction by vacuum metrics in the gravitational case. Our signature is ($-+++$).
2. Non-geodesic motion

Here we consider the non-geodesic motion of point particle along the affinely parametrized world-line \( x^\mu = z^\mu(\tau) \) interacting with the scalar \( \phi \) and vector \( A^\mu \) fields. The total action is the sum \( S = S_p + S_f \), where the world-line part reads

\[
S_p = -m_0 \int (1 + q\phi)\sqrt{-\dot{z}^2}d\tau + e \int A^\mu \dot{z}_\mu d\tau,
\]

\( (m_0 \) being the bare mass, \( q, e \) – the scalar and electric charges) while the volume part is

\[
S_f = -\frac{1}{4\pi} \int \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{4} F^2 \right) \sqrt{-g} d^4x.
\]

Our local approach to radiation reaction simply consists in the substitution of the proper fields into the particle equation of motion (for brevity we omit the external force)

\[
m_0(1 + q\phi) \ddot{z}^\mu = eF^\mu_\nu \dot{z}^\nu - m_0 q \Pi^\mu_\nu \phi_{,\nu}
\]

where the velocity-transverse projector in the gauge \( \dot{z}^2 = -1 \) reads

\[
\Pi^\mu_\nu = g^\mu_\nu + \dot{z}_\mu \dot{z}_\nu.
\]

The equations for the scalar field and the 4-potential \((F = dA)\) read

\[
\Box \phi = 4\pi \rho
\]

\[
\Box A_\mu - R^\nu_\mu A_\nu = -4\pi j^\mu
\]

where the covariant D’Alembert operator is understood \( \Box = D_\mu D^\mu \), and the source terms are standard. The Green’s functions are defined in the usual way starting with the Hadamard solution (in the scalar case):

\[
G^{(1)}(x, z) = \frac{1}{(2\pi)^2} \left[ \frac{\Delta^{1/2}}{\sigma} + v \ln \sigma + w \right],
\]

where \( \sigma(x, z) \) is the Synge’s world function, \( \Delta \) is van Vleck determinant, and \( v, w \) satisfy the system

\[
\Box v = 0,
\]

\[
2v + (2v_\mu - v\Delta^{-1}\Delta_{,\mu})\sigma_{,\mu} + \Box \sigma + \Delta^{1/2} + \sigma \Box w = 0.
\]

In the vector case

\[
G^{(1)}_{\mu\alpha} = \frac{1}{(2\pi)^2} \left( \frac{u_{\mu\alpha}}{\sigma} + v_{\mu\alpha} \ln \sigma + w_{\mu\alpha} \right),
\]

\( (u_{\mu\alpha} \) and \( v_{\mu\alpha} \) being the vector Green’s functions).
where $u_{\mu\alpha}(x, z)$, $v_{\mu\alpha}(x, z)$ and $w_{\mu\alpha}(x, z)$ are bi-vectors, satisfying a similar system. In particular, $u_{\mu\alpha} = \bar{g}_{\mu\alpha}\Delta_{1/2}$, where $\bar{g}_{\mu\alpha}$ is the bi-vector of parallel transport. Notation is that of DeWitt and Brehme: indices $\alpha, \beta, \ldots$ are associated with the “emission” point $z$, while $\mu, \nu, \ldots$ — with the “observation” point $x$. When both points are taken on the world-line, we use the first set to denote $z^{\alpha}(\tau')$ associated with the integration variable $\tau'$, and the second set to denote $z^{\mu}(\tau)$, where $\tau$ is the proper time in the resulting equation. In terms of these quantities the retarded solutions of the field equation read

$$\phi_{\text{ret}}(x) = m_0 q \int_{-\infty}^{\tau_{\text{ret}}(x)} [-u\delta(\sigma) + v\theta(-\sigma)] d\tau', \quad (11)$$

$$F_{\mu\nu}^{\text{ret}}(x) = -2e \int_{-\infty}^{\tau_{\text{ret}}(x)} \left[ u[\mu\alpha, \nu]\delta'(\sigma) + (u[\mu\alpha, \nu] + v[\mu\alpha, \sigma_{\nu}])\delta(\sigma) + v[\mu\alpha, \nu] \right] \dot{z}^\alpha d\tau', \quad (12)$$

where an anti-symmetrization over the indices $\mu$ and $\nu$ is used.

When substituted into the equations of motion, two-point functions become localized on the world-line, so we are led to use the expansions of the Synge’s function $\sigma(z(\tau), z(\tau'))$(in terms containing delta-function and its derivative) as follows

$$\sigma(z(\tau), z(\tau')) = \sum_{k=0}^{\infty} \frac{1}{k!} D_k \sigma(\tau, \tau)(\tau - \tau')^k \quad (13)$$

In what follows we will denote by dots the quantities

$$\dot{\sigma} = \sigma_\alpha \dot{z}^\alpha, \quad \ddot{\sigma} = \sigma_{\alpha\beta} \ddot{z}^\alpha \dot{z}^\beta + \sigma_\alpha \dot{z}^\alpha, \quad \text{etc.} \quad (14)$$

Denoting the difference as $s = \tau - \tau'$, we get for $\sigma$ an expansion similar to that in the flat space

$$\sigma(s) = -\frac{s^2}{2} - \ddot{z}^2(\tau) \frac{s^2}{24} + \mathcal{O}(s^4), \quad (15)$$

but with dots corresponding to covariant derivatives along the world-line. Similarly, for the derivative of $\sigma$ with respect to $z^\mu(\tau)$ one finds

$$\sigma^\mu(s) = s \left( \dot{z}^\mu - \ddot{z}^\mu \frac{s^2}{2} + \dot{z}^\mu \frac{s^2}{6} \right) + \mathcal{O}(s^4). \quad (16)$$

This quantity is a vector at the point $z(\tau)$ and a scalar at the point $z(\tau')$ where the index $\mu$ now corresponds to the point $z(\tau)$: $\sigma^\mu = \partial \sigma(z, z')/\partial z_\mu$. 
An expansion for the delta-function reads
\[ \delta(-\sigma) = \delta(s^2/2) + s^4 \frac{\dot{z}^2(\tau)}{24} \delta'(s^2/2) + ... \] (17)
and an analogous expansion is easily obtained for its derivative. Since the most singular term is \( u\delta'(\sigma)\sigma \nu \) the maximal expansion order to be retained is \( s^3 \). This also means that we have to retain for the delta-function only the leading term: \( \delta(\sigma) = \delta(s^2/2) + O(s^4) \). The delta-function of the squared argument can be regularized in exactly the same way as in the flat space as
\[ \theta(s)\delta(s^2/2) \rightarrow \theta(s)\delta([s^2 - \varepsilon^2]/2) = \delta(s - \varepsilon)/\varepsilon, \]
where the positive regularization parameter \( \varepsilon \rightarrow 0 \) of the dimension of length is introduced.

We have to perform the expansions up to the third order in \( s \) in all terms containing delta-function and its derivative (local terms). For the function \( u \) we will have Ricci-terms involved into expansions, up to third order one finds:
\[ u = 1 + \frac{1}{12} R_{\sigma\tau} \dot{z}^\tau \dot{z}^\sigma s^2, \quad u_{\nu} = \frac{1}{6} R_{\nu\tau} \dot{z}^\tau s, \] (18)
and similarly in the vector case
\[ u_{\nu\alpha} \sigma^{\mu} \dot{z}^\nu \dot{z}^\alpha = -s \dot{z}^\mu + \frac{s^2}{2} \ddot{z}^\mu - s^3 \left( \frac{1}{6} \dot{z}^\mu + \frac{1}{12} R_{\lambda\nu} \dot{z}^\lambda \dot{z}^\nu + \frac{1}{2} \dddot{z}^\mu \right). \] (19)
The functions of \( v \)-type in the tail term can not be found in a closed form.

Collecting all the contributions, one obtains the following expressions for the scalar and electromagnetic reaction forces:
\[ f_{sc} = m^2 q^2 \left[ \frac{\dot{z}^\mu}{2\varepsilon} + \Pi^{\mu\nu} \left( \frac{1}{3} \ddot{z}^\nu + \frac{1}{6} R_{\nu\tau} \dot{z}^\tau - \int_{-\infty}^{\tau} v_{\nu} d\tau' \right) - \dot{z}^\mu \int_{-\infty}^{\tau} v d\tau' \right], \] (20)
\[ f_{em} = e^2 \left[ -\frac{\dddot{z}^\mu}{2\varepsilon} + \Pi^{\mu\nu} \left( \frac{2}{3} \ddot{z}^\nu + \frac{1}{3} R_{\nu\alpha} \dot{z}^\alpha \right) + 2 \dot{z}^\nu(\tau) \int_{-\infty}^{\tau} v^{[\mu}_{\alpha\beta]} \dot{z}^\alpha(\tau') d\tau' \right]. \] (21)

Divergent terms can be absorbed by the renormalization of mass.
\[ m = m_0 + \frac{1}{2\varepsilon} (m_0^2 q^2 - e^2) \] (22)
Note different signs of the divergent terms for scalar and vector self-forces, so for a "BPS" particle with the ratio of charges $\frac{m|q|}{|e|}$ the model is free of divergencies. Finite contributions coincide with the previous results obtained via (somewhat questionable) world-tube derivations. Thus, in view of the criticisms in 11, our derivation may be considered as confirmation of these results. Technically, the local calculation is substantially simpler than the world-tube one.

3. Neutral particle at geodesic motion

In the case of gravitational radiation reaction we have the only one parameter — particle mass, which actually does not enter into the geodesic equation. Thus it is not possible to use the above renormalization scheme. This problem can be remedied using a manifestly reparametrization invariant treatment. This is done by introducing the einbein $e(\tau)$ on the world line acting as a Lagrange multiplier:

$$S[z^\mu, e] = -\frac{1}{2} \int \left[e(\tau) g_{\mu\nu} \dot{z}^\mu \dot{z}^\nu + \frac{m^2}{e(\tau)}\right] d\tau, \quad (23)$$

Varying (23) with respect to $z^\mu(\lambda)$ and $e(\tau)$ gives

$$\frac{D}{d\tau}(e \dot{z}^\mu) = 0, \quad e = \frac{m}{\sqrt{-g_{\mu\nu} \dot{z}^\mu \dot{z}^\nu}}, \quad (24)$$

and we obtain the geodesic equation in a manifestly reparametrization invariant form

$$\frac{D}{d\tau}\left(\frac{\dot{z}^\lambda}{\sqrt{-g_{\mu\nu} \dot{z}^\mu \dot{z}^\nu}}\right) = 0. \quad (25)$$

We split the total metric into background and perturbation due to point particle $g_{\mu\nu} \rightarrow \hat{g}_{\mu\nu} = g_{\mu\nu} + \kappa h_{\mu\nu}$. Assuming now that the particle motion with no account for radiation reaction is geodesic on the background metric, the perturbed equation in the leading order in $\kappa$ will read

$$\ddot{z}^\mu = \frac{\kappa}{2} \left(g^{\mu\nu} - \frac{\dot{z}^\mu \dot{z}^\nu}{\dot{z}^2}\right) \left(h_{\lambda\rho;\nu} - 2h_{\nu;\lambda;\rho}\right) \dot{z}^\lambda \dot{z}^\rho, \quad (26)$$

where contractions are with the background metric.

The derivation of the field equations for the metric perturbation in the general case of non-vacuum background is non-trivial 14. It is expected that the particle stress-tensor $T_{\mu \nu}^{(P)}$ has to be divergence-free with respect to the background metric, this allows one to derive the equation for the metric
perturbation in a harmonic gauge, which leads to a manageable Green’s function. But expanding the Bianchi identities for the full metric one finds that this is only possible if the Einstein tensor $G_{\mu\nu}$ and metric perturbation $h_{\mu\nu}$ satisfies an additional equation

$$G^{\lambda}_{\mu} h_{\lambda\cdot\lambda} - G_{\lambda\rho} h^{\lambda\rho:\mu} = O(\ddot{z}^\mu).$$

(27)

Otherwise, the “naive” particle stress tensor on a given background is not enough, and construction of a reliable source term for metric perturbation becomes a complicated problem. Clearly, there is no general solution to the constraint (27), but some particular cases can be found. One can notice that this equation is identically satisfied for Einstein metrics $R_{\mu\nu} = -\Lambda g_{\mu\nu}$. Another case is conformally-flat metrics with some special conformal factor (details will be given elsewhere). Provided the Eq. (27) holds, one can derive (using the results of Sciama et al. 14) the following equation for the trace-reversed perturbation $\psi_{\mu\nu} = h_{\mu\nu} - g_{\mu\nu} h_{\lambda\lambda}/2$:

$$\psi_{\mu\nu};\xi + 2R_{\xi \rho}^{\mu \nu} \psi^{\xi \rho} - 2\psi^{(\mu} R_{\nu)\sigma} + \psi_{\mu\nu} R - g_{\mu\nu} R_{\alpha\beta} \psi^{\alpha\beta} = -2\kappa^2 T_{\mu\nu}^{(P)}.$$

(28)

The corresponding Hadamard’s function is similar to (10) with four-index bi-tensors $u_{\lambda\rho\alpha\beta}$, $v_{\lambda\rho\alpha\beta}$. The retarded solutions is constructed in an analogous way and substituted into the equation of motion (26) leading to an integral involving bi-tensor quantities depending on two points on the world-line $\tau, \tau’$. As before, one can distinguish four different contributions: terms proportional to the delta function, the derivative of the delta function, the derivative of the Heaviside function, and the tail term. Performing series expansions in the first three cases up to the third order in $\tau - \tau’ = s$ one finds a local contribution to the self-force. To facilitate expansions of bi-tensors one has first to reduce them to scalars at the point of expansion, e.g. quantities like $u_{\nu\lambda\alpha\beta} z^\alpha(\tau’) z^\beta(\tau’)$ behave as a scalar with respect to the point $z(\tau’) and tensors with respect to $z(\tau)$. Then expansion in $s$ around $\tau$ is straightforward.

Contrary to the non-geodesic case, now we have to drop all (covariant) derivatives with respect to the proper time of the second and higher order, with the only exception of divergent term. Actually, the divergent term is proportional to the acceleration, so formally it vanishes for the geodesic motion. But still it may be argued that its infinite value demands some renormalization to be made, and we have to find an appropriate quantity to be renormalized. Since the mass does not enter the equation, we invoke the einbein $e(\tau)$, and insert as the corresponding background quantity
Recall, that Ricci term is not arbitrary here, but has to be proportional to the metric for consistency. So actually the only local term here is proportional to the four-velocity \( \dot{z}_\nu \) and vanishes by virtue of the projector. The tail term is

\[
F_{\text{tail}}^\mu = \frac{\kappa^2 e_0}{4} \Pi^{\mu \nu} \int_{-\infty}^\tau \left[ 4v_{\nu \lambda \alpha \beta} - (g_{\nu \lambda} v_\sigma^{\alpha \beta \rho} + v_{\lambda \rho \alpha \beta}) - g_{\lambda \rho} v^{\sigma \tau \alpha \beta} - 1/2 \right] \dot{z}^\alpha \dot{z}^\beta \dot{z}^\lambda \dot{z}^\rho d\tau.
\]

Renormalization of the einbein is performed as

\[
\left( \frac{1}{e_0} - \frac{7 \kappa^2}{2e} \right) \ddot{e}_0^\mu = e^\mu.
\]

Finally the choice of the affine parameter \( z^2 = -1 \), equivalent to setting \( e = m \), leads to the final form

\[
\ddot{z} = m \kappa^2 \Pi^{\mu \nu} \left[ - \int_{-\infty}^\tau \left[ (2v_{\nu \lambda \alpha \beta} - g_{\nu \lambda} v_\sigma^{\alpha \beta \rho} + v_{\lambda \rho \alpha \beta}) - \right] \dot{z}^\alpha \dot{z}^\beta \dot{z}^\lambda \dot{z}^\rho \right].
\]

This coincides with the result obtained in \(^{10,13}\). We have thus shown that this equation remains valid for a class on non-vacuum metrics, in particular, for Einstein spaces.

### 4. Discussion

We have presented a local calculation of radiation reaction in both geodesic and non-geodesic cases which do not involve any ambiguous integrals outside the world-line and thus is free from the corresponding problems. It is based on the expansion of bi-tensor quantities only on the world line and technically is much simpler than the DeWitt-Brehme type approach. We have also generalized the gravitational radiation reaction to some non-vacuum metrics, however, satisfying an additional condition. The general non-vacuum case can not be treated within the test body/external field approximation because the global Bianchi identity demands to take into account the perturbation of the background matter as well.
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