ON COMPETITION MODELS UNDER ALLEE EFFECT: ASYMPTOTIC BEHAVIOR AND TRAVELING WAVES

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ABSTRACT. In this article, we study a reaction-diffusion model on infinite spatial domain for two competing biological species ($u$ and $v$). Under one-side Allee effect on $u$-species, the model demonstrates complexity on its coexistence and $u$-dominance steady states. The conditions for persistence, permanence and competitive exclusion of the species are obtained through analysis on asymptotic behavior of the solutions and stability of the steady states, including the attraction regions and convergent rates depending on the biological parameters. When the Allee effect constant $K$ is large relative to other biological parameters, the asymptotic stability of the $v$-dominance state $(0, 1)$ indicates the competitive exclusion of the $u$-species. Applying upper-lower solution method, we further prove that for a family of wave speeds with specific minimum wave speed determined by several biological parameters (including the magnitude of the $u$-dominance states), there exist traveling wave solutions flowing from the $u$-dominance states to the $v$-dominance state. The asymptotic rates of the traveling waves at $\xi \to \pm \infty$ are also explicitly calculated. Finally, numerical simulations are presented to illustrate the theoretical results and population dynamics of coexistence or dominance-shifting.

1. Introduction. Reaction-diffusion models for growth of single or multiple species with Logistic or Lokta-Volterra equations have been extensively studied. In those models, the average growth rate of each species has linear dependence on densities of all species. The interspecies interactions (predation or competition) in various models are analyzed for coexistence and competitive exclusion, asymptotic stability, as well as traveling wave solutions and asymptotic rates [1, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 24, 25, 28, 31, 32, 33, 34].

The Allee effect refers to a decrease in per-capita growth rate at low population densities, which arises from a number of sources such as difficulties in finding mates, social dysfunction and inbreeding depression. For single species population, the logistic model assumes that per-capita growth rate declines monotonically with density. But for populations subject to an Allee effect, per-capita growth rate shows a humped curve increasing (from negative to positive) at low density, up to a maximum at intermediate density and then declining. The following single-species ODE model was proposed after incorporating an Allee-like effect into the logistic

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many biological parameters affect the persistence or competitive exclusion of the
constructed upper and lower solutions. It can be seen, in Theorem 2.3 and 2.4, that
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In Model (1.2), the quantities u(x, t) and v(x, t) are population densities of the
two competing species at t > 0 and x ∈ R, where α and β are the respective

tion-diffusion format, is given [24, 29] as following:
\[
\begin{aligned}
\frac{\partial u}{\partial t} - D_1 \frac{\partial^2 u}{\partial x^2} &= u \left[ b_1 \left( 1 - \frac{u + \alpha v}{R_1} \right) \left( \frac{u}{u + K} \right) - d_1 \right], (x,t) \in \mathbb{R} \times \mathbb{R}^+ \\
\frac{\partial v}{\partial t} - D_2 \frac{\partial^2 v}{\partial x^2} &= v \left[ b_2 \left( 1 - \frac{v + \beta u}{R_2} \right) - d_2 \right].
\end{aligned}
\]

(1.2)

In Model (1.2), the quantities u(x, t) and v(x, t) are population densities of the
two competing species at t > 0 and x ∈ R, where α and β are the respective
competition coefficients. u/(u + K) is the term for the Allee effect on the u-species.
The bigger K is, the stronger Allee effect will be. For i = 1, 2, D_i is the diffusion
rate. In absence of interspecific competition and Allee effects, b_i > 0 is the per-
capita birth rate without any interference from other individuals, and d_i > 0 is the
death rate of the species. R_i is the density at which the real per-capita birth rate
is 0 such that the carrying capacity for each Logistic equation is R_i(b_i - d_i)/b_i.

For convenience of discussions in later sections, we scale the density functions and
ecological parameters as follows:
\[
\bar{u} = u/R_1, \bar{\alpha} = \alpha/R_1, \bar{v} = v/R_2(b_2 - d_2), \bar{\beta} = \beta/R_2(b_2 - d_2),
\bar{b} = b_1, \bar{d} = d_1, \bar{K} = K/R_1, \bar{r} = b_2 - d_2.
\]

After the above scaling and dropping all the bar signs, the reaction-diffusion
model is now represented as
\[
\begin{aligned}
\frac{\partial u}{\partial t} - D_1 \frac{\partial^2 u}{\partial x^2} &= u \left[ b \left( 1 - u - \alpha v \right) \left( \frac{u}{u + K} \right) - d \right], (x,t) \in \mathbb{R} \times \mathbb{R}^+ \\
\frac{\partial v}{\partial t} - D_2 \frac{\partial^2 v}{\partial x^2} &= rv \left( 1 - v - \beta u \right).
\end{aligned}
\]

(1.3)

Based on the ecological nature of the competition model, the following assump-
tions are made throughout this paper:

\textbf{H1}: D_1, D_2 > 0, \alpha, \beta > 0, b > d > 0, r > 0, and K > 0.

This system demonstrates complexity in its steady states. There are 6 possible
nonnegative constant steady states: the extinction state (0, 0), the v-dominance
state (0, 1); two u-dominance states (u_*, 0) and two coexistence states (u^*, v^*).

In section 2, under various conditions on the biological parameters, we will first
analyze the asymptotic behavior of the density function (u(x, t), v(x, t)) related to
persistence, permanence and competitive extinction of the species through suitably
constructed upper and lower solutions. It can be seen, in Theorem 2.3 and 2.4, that
many biological parameters affect the persistence or competitive exclusion of the u-
species under Allee effect, and it will be persistent given relatively small Allee effect
constant K along with sizable initial density function u_0. We also obtain a condition
on sufficiently large Allee effect constant K that makes the v-dominance state (0, 1)
asymptotically stable with specifically given attraction region and exponential rates.
For all initial density functions falling into the attraction region given by Theorem 2.5, the solution of (1.3) converges to \((0, 1)\) uniformly on \(\mathbb{R}\) with the rate \(e^{-pt}\) (where \(p\) depends on several parameters including \(K\)) as \(t \to \infty\).

In Section 3 we study the existence of traveling wave solutions for competitive exclusion of \(u\)-species under the Allee effect. With the assumptions of persistence of the \(v\)-species and presence of the \(u\)-dominance states, we obtain balanced conditions on all ecological parameters (including the size of \(u_s\)) such that the dominance shifting from \(u\)-species to \(v\)-species is in the traveling wave format and evolves with both time and space. Using the theory of upper-lower solutions and monotone iteration, we prove in Theorem 3.5 that for each wave speed \(c \geq 2\overline{D}\sqrt{r(1 - \beta u_s)/D}\) (with \(\overline{D} = \max\{D_1, D_2\}\) and \(\underline{D} = \min\{D_1, D_2\}\)), there exists a positive and monotone traveling wave solution \((u(x + ct), v(x + ct))\) of (1.3) with \((u(-\infty), v(-\infty)) = (u_s, 0)\) and \((u(\infty), v(\infty)) = (0, 1)\). Section 4 is devoted to deriving the explicit expressions on asymptotic rates of the traveling wave solution \((u(\xi), v(\xi))\) at \(\pm \infty\), where the shifting of dominance ends with competitive exclusion of \(u\)-species because of the Allee effect.

At last, in Section 5, we give several numerical examples to illustrate the theoretical results obtained in Sections 2-4. The first example gives the permanence effect in the ecological system shown in Theorem 2.4, with sufficiently large initial density for \(u\)-species and relatively small Allee effect constant \(K\). The second example demonstrates the asymptotic stability of the \(v\)-dominance state \((0, 1)\) shown in Theorem 2.5 under larger Allee effect constant \(K\), and the uniform convergence of the solution \((u(x, t), v(x, t))\) with initial density functions in the attraction area. The third example shows the case of competitive exclusion for \(u\)-species in the format of traveling wave solutions. With the presence of two \(u\)-dominance states, each satisfying one of the two criteria given by Theorem 3.5, we display the simulation of the traveling waves for the competition system (1.3) satisfying limiting behavior at \(\pm \infty\): \((u(-\infty), v(-\infty)) = (u_s, 0)\) and \((u(\infty), v(\infty)) = (0, 1)\).

2. Persistence, permanence and competitive exclusion. In this section we study the impacts of resource competition and Allee effect on the asymptotic behavior of the density functions which indicates the ultimate outcomes (persistence, permanence or competitive exclusion) in the ecological system. We now consider the following reaction-diffusion system

$$
\begin{align*}
\frac{\partial u}{\partial t} - D_1 \frac{\partial^2 u}{\partial x^2} &= u \left[ b (1 - u - \alpha v) \left( \frac{u}{u + K} \right) - d \right], (x, t) \in \mathbb{R} \times \mathbb{R}^+ \\
\frac{\partial v}{\partial t} - D_2 \frac{\partial^2 v}{\partial x^2} &= rv \left( 1 - v - \beta u \right).
\end{align*}
$$

(2.1)

with initial conditions

$$
u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \mathbb{R}
$$

(2.2)

where \(u_0\) and \(v_0\) are nonnegative bounded smooth functions on \(\mathbb{R}\).

Let

$$f_1(u, v) = u \left[ b (1 - u - \alpha v) \left( \frac{u}{u + K} \right) - d \right]$$

and

$$f_2(u, v) = rv \left( 1 - v - \beta u \right),$$

with

$$f_1(u, v) = u \left[ b (1 - u - \alpha v) \left( \frac{u}{u + K} \right) - d \right]$$

and

$$f_2(u, v) = rv \left( 1 - v - \beta u \right),$$

with
then

\[
\frac{\partial f_1}{\partial v} = -\frac{\alpha bu^2}{(u + K)}, \quad \frac{\partial f_2}{\partial u} = -r\beta v.
\]

Model (2.1)-(2.2) is quasi-monotone non-increasing \[20\] for all \(u \geq 0\) and \(v \geq 0\). For the existence and asymptotic behavior of the time-dependent solution \((u(x, t), v(x, t))\) in \(\mathbb{R} \times \mathbb{R}^+\), we apply the method of upper-lower solutions based on the existence-comparison argument given in \[16, 20\]. Specifically, for the Cauchy problem (2.1)-(2.2), we refer to Definition 2.1 on page 296 of \[20\].

**Definition 2.1.** A pair of nonnegative bounded smooth functions \((\hat{u}(x, t), \hat{v}(x, t))\) and \((\tilde{u}(x, t), \tilde{v}(x, t))\) are called coupled upper and lower solutions of the Cauchy problem (2.1)-(2.2) on \(\mathbb{R} \times \mathbb{R}^+\) if \((\hat{u}, \hat{v}) \geq (\tilde{u}, \tilde{v})\) satisfy the growth condition

\[
|\hat{u}(x, t)|, |\hat{v}(x, t)|, |\tilde{u}(x, t)|, |\tilde{v}(x, t)| \leq C_0
\]

for some \(C_0 > 0\), and further satisfy the following differential and initial inequalities:

\[
\begin{align*}
\hat{u}_t - D_1 \hat{u}_{xx} &\geq \hat{u} \left( b(1 - \hat{u} - \alpha \hat{v}) \left( \frac{\hat{u}}{\hat{u} + K} \right) - d \right), \\
\tilde{v}_t - D_2 \tilde{v}_{xx} &\geq \tilde{v} \left( 1 - \tilde{v} - \beta \hat{u} \right), \\
\hat{u}_t - D_1 \hat{u}_{xx} &\leq \hat{u} \left( b(1 - \hat{u} - \alpha \hat{v}) \left( \frac{\hat{u}}{\hat{u} + K} \right) - d \right), \\
\tilde{v}_t - D_2 \tilde{v}_{xx} &\leq \tilde{v} \left( 1 - \tilde{v} - \beta \hat{u} \right), \quad \text{in } \mathbb{R} \times \mathbb{R}^+ \\
\hat{u}(x, 0) &\geq u_0(x) \geq \hat{u}(x, 0), \tilde{v}(x, 0) \geq v_0(x) \geq \tilde{v}(x, 0) \quad \text{in } \mathbb{R}.
\end{align*}
\]

It is known from \[16, 20\] that if there exist coupled upper and lower solutions \((\hat{u}, \hat{v})\) and \((\tilde{u}, \tilde{v})\) on \(\mathbb{R} \times \mathbb{R}^+\), then the Cauchy problem (2.1)-(2.2) has a unique solution \((u(x, t), v(x, t))\) with \((\hat{u}, \hat{v}) \geq (u, v) \geq (\tilde{u}, \tilde{v})\) on \(\mathbb{R} \times \mathbb{R}^+\). One can verify that \((M_1, M_2)\) and \((0, 0)\), with \(M_1 = \max\{(b - d)/b, \|u_0\|_\infty\}\) and \(M_2 = \max\{1, \|v_0\|_\infty\}\), are a pair of coupled upper-lower solutions. As a direct application of the comparison argument and Definition 2.1, the following theorem states the existence and uniqueness of a bounded solution for (2.1)-(2.2).

**Theorem 2.2.** For constants \(M_1\) and \(M_2\) with \(M_1 = \max\{(b - d)/b, \|u_0\|_\infty\}\) and \(M_2 = \max\{1, \|v_0\|_\infty\}\), the system (2.1)-(2.2) has a unique solution \((u(x, t), v(x, t))\) with \((0, 0) \leq (u(x, t), v(x, t)) \leq (M_1, M_2)\) on \(\mathbb{R} \times \mathbb{R}^+\).

We now state a result on conditions for persistence of the \(u\)-species under Allee effect, with relatively small \(K\) and sufficiently large initial density function \(u_0\).

**Theorem 2.3.** Assume that \(d < b(1 - \alpha)\). If \(K \leq [b(1 - \alpha) - d]^2 / Abd\), then the \(u\)-species in Model (2.1)-(2.2) is persistent as long as

\[
\frac{b(1 - \alpha) - d}{2b} \leq u_0(x) \leq \frac{b - d}{b} \quad \text{and} \quad 0 \leq v_0(x) \leq 1
\]

for all \(x \in \mathbb{R}\).

**Proof.** We construct a pair of upper-lower solutions

\[
(\hat{u}, \hat{v}) = \left( \frac{b - d}{b}, 1 \right) \quad \text{and} \quad (\tilde{u}, \tilde{v}) = (\mu, 0),
\]

where \(\mu\) is a positive constant which will be determined. It can be seen that \(\hat{u}, \hat{v}\) and \(\tilde{u}\) automatically satisfy the differential inequalities in (2.3). For \(\hat{u} = \mu\), we need
to verify that with \( \tilde{v} = 1 \),
\[
\mu \left[ b \left( 1 - \mu - \alpha \tilde{v} \right) \left( \frac{\mu}{\mu + K} \right) - d \right] \geq 0.
\]
This inequality is equivalent to the following
\[
(1 - \alpha - \mu) \left( \frac{\mu}{\mu + K} \right) \geq \frac{d}{b}, \quad \text{and} \quad \mu > 0,
\]
or
\[
\mu^2 - \frac{b(1 - \alpha) - d}{b} \mu + \frac{dK}{b} \leq 0 \quad \text{for some} \quad \mu > 0.
\] (2.7)
The vertex of this parabola is at \( 0 < \mu_0 = \frac{b(1 - \alpha)}{2b} < 1 - \frac{d}{b} = \tilde{u} \), with
\[
\mu_0^2 - \frac{b(1 - \alpha) - d}{b} \mu_0 + \frac{dK}{b} = \frac{dK}{b} - \mu_0^2.
\]
Therefore, if \( K \leq \frac{b_0 \mu_0^2}{d} \), then the positive constant \( \mu_0 \) satisfy the required order and differential inequality for \( \hat{u} \) in (2.3). By the Existence-Comparison argument [16, 20], the \( u \)-density function satisfies
\[
u(x, t) \geq \mu_0 \quad \text{as long as} \quad u_0(x) \geq \mu_0 \quad \text{for all} \quad x \in \mathbb{R}.
\]
In the case of \( K = 0 \), it is well-known that the following condition \( H2 \):
- \( H2-a: \ d < b(1 - \alpha) \) and \( H2-b: \ 0 < \beta(b - d) < b \)
will ensure the permanence effect in the Lotka-Volterra resource competition model.

The next theorem gives conditions for permanence in the competition model (2.1)-(2.2) under Allee effect.

**Theorem 2.4.** Assume that \( H2 \) holds. If \( K \leq \left[ b(1 - \alpha) - d \right]^2 / 4bd \) and the initial functions satisfy
\[
\frac{b(1 - \alpha) - d}{2b} \leq u_0(x) \leq \frac{b - d}{b} \quad \text{and} \quad \epsilon \leq v_0(x) \leq 1,
\] (2.8)
with some \( \epsilon > 0 \) for all \( x \in \mathbb{R} \), then the density function \( (u(x, t), v(x, t)) \) in (2.1)-(2.2) satisfies
\[
\liminf_{t \to +\infty} u(x, t) \geq \frac{b(1 - \alpha) - d}{2b} > 0, \quad \liminf_{t \to +\infty} v(x, t) \geq \frac{b - \beta(b - d)}{b} > 0
\] (2.9)
for all \( x \in \mathbb{R} \).

**Proof.** From condition \( H2-b \), let
\[
\delta = r\left[ b - \beta(b - d) \right] > 0.
\] (2.10)
Given any \( \epsilon > 0 \), there exists a \( L > 0 \) such that \( \delta / (r + L) < \epsilon \). we construct a pair of upper-lower solutions
\[
(\tilde{u}, \tilde{v}) = \left( \frac{b - d}{b}, 1 \right) \quad \text{and} \quad (\hat{u}, \hat{v}) = \left( \frac{b(1 - \alpha) - d}{2b}, m(t) \right),
\] (2.11)
where the positive function \( 0 < m(t) < 1 \) will be determined. The differential inequalities in (2.3) are satisfied by the given \( \tilde{v} \) and \( \hat{u} \), based on the proof of the previous theorem. For \( \tilde{u} \), we can verify that
\[
\hat{u} \left[ b \left( 1 - \tilde{u} - \alpha \tilde{v} \right) \left( \frac{\tilde{u}}{\tilde{u} + K} \right) - d \right] \leq \hat{u} \left[ b(1 - \tilde{u}) - d_1 \right] = 0,
\]
therefore the corresponding inequality for \( \hat{u} \) in (2.3) also holds.
Finally, for $\hat{v}$, the following relation needs to hold:
\[ m'(t) \leq rm(t) \left[ 1 - m(t) - \beta \hat{u} \right] = rm(t) \left[ 1 - \frac{\beta(b - d)}{b} - m(t) \right], \]

or equivalently,
\[ m'(t) - \delta m(t) \leq -rm^2(t). \tag{2.12} \]

This inequality leads to the choice of
\[ m(t) = \frac{\delta}{r + Le^{-\delta t}} \tag{2.13} \]

with $m(0) = \delta/(r + L) < \epsilon$. From the existence-comparison argument, the solution for (2.1)-(2.2) satisfies
\[ (\hat{u}, \hat{v}) \leq (u(x,t), v(x,t)) \leq (\tilde{u}, \tilde{v}) \]

with $\hat{u} = b(1 - \alpha) - d/2b$ and $\hat{v} = m(t) = \delta/(r + Le^{-\delta t})$ which leads to
\[ \liminf_{t \to +\infty} u(x,t) \geq \frac{b(1 - \alpha) - d}{2b} \]

and
\[ \liminf_{t \to +\infty} v(x,t) \geq \lim_{t \to +\infty} m(t) = \frac{\delta - \beta(b - d)}{b} \]

for all $x \in \mathbb{R}$. \hfill \square

After scaling of the reaction-diffusion system in Section 1, all possible point-wise nonnegative steady states for (2.1) are given here: the extinction state $(0, 0)$ and the $v$-dominance state $(0, 1)$; under the assumption that

H3-a: $b \geq d + 2\sqrt{bdK}$,

there are one or two $u$-dominance states $(u_s, 0)$ with

\[ u_s = \frac{b - d \pm \sqrt{(b - d)^2 - 4bdK}}{2b}; \]

under the assumption that

H3-b: $b(1 - \alpha) \geq d + 2\sqrt{(1 - \alpha\beta)bdK}$,

there are one or two coexistence states $(u^*, v^*)$ with

\[ u^* = \frac{b(1 - \alpha) - d \pm \sqrt{[b(1 - \alpha) - d]^2 - 4(1 - \alpha\beta)bdK}}{2b(1 - \alpha\beta)}, \quad v^* = 1 - \beta u^*. \]

There is no coexistence state if $b(1 - \alpha) < d + 2\sqrt{(1 - \alpha\beta)bdK}$.

It is seen from above that while the Lotka-Volterra competition model is permanent, sufficiently significant Allee effect (large constant $K$) eliminates the presence of $u$-dominance and coexistence states. The following theorem gives the asymptotic stability of the $v$-dominance state $(0, 1)$, which indicates the competitive exclusion of $u$-species under Allee effect. Specifically, Theorem 2.5 below gives stability conditions for $(0, 1)$ in relation to the Allee effect constant $K$ and other ecological parameters. For all $(u_0, v_0)$ in a given rectangular region, the time-dependent solution of (2.1)-(2.2) converges to the equilibrium $(0, 1)$ uniformly with the rate $e^{-pt}$ for $t \to \infty$ where $p$ is a positive number depending on many ecological parameters.

**Theorem 2.5.** Assume that condition $H2$ holds. Let
\[ K_0 = \frac{(b - d)[b(1 - \alpha) - d]}{bd}. \tag{2.14} \]
If $K > K_0$, then the $v$-dominance state $(0, 1)$ is asymptotically stable. For
\[
p = \min \left\{ r \left[ 1 - \frac{\beta(b-d)}{b} \right], \ d - \frac{b(1 - \alpha)}{1 + bK/(b-d)} \right\}, \ q = r \left[ 1 - \frac{\beta(b-d)}{b} \right], \quad (2.15)
\]
the solution $(u,v)$ for (2.1)-(2.2) satisfies
\[
(0, 1 - m(t)) \leq (u(x,t), v(x,t)) \leq \left( \frac{(b-d)m(t)}{b}, 1 + m(t) \right)
\]
on $[0, \infty) \times \mathbb{R}$ as long as the above inequalities hold for $t = 0$, where
\[
m(t) = \left[ \frac{q}{p} + \left( \frac{1}{m(0)} - \frac{q}{p} \right) e^{pt} \right]^{-1}, \quad \text{with } 0 < m(0) < \frac{p}{q} \leq 1. \quad (2.17)
\]

Proof. We will show that $(\hat{u}, \hat{v}) = ((b-d)m(t)/b, 1 + m(t))$ and $(\tilde{u}, \tilde{v}) = (0, 1 - m(t))$
are a pair of coupled upper-lower solutions as defined in (2.3)-(2.4). One can easily
see that $\hat{u} = 0$ satisfies the required inequalities in (2.3)-(2.4).

We now start with $\tilde{v} = 1 + m(t)$. It can be seen from (2.3) that $m(t)$ needs to satisfy the differential inequality
\[
m'(t) \geq r \left( 1 + m(t) \right) [1 - (1 + m(t))] = -rm(t) [1 + m(t)]. \quad (2.18)
\]
So we need to have
\[
m'(t) + rm(t) \geq -rm^2(t). \quad (2.19)
\]
We can also look at $\hat{v} = 1 - m(t)$. Again from (2.3), $m(t)$ needs to satisfy the differential inequality
\[
-m'(t) \leq r \left( 1 - m(t) \right) \left[ 1 - (1 - m(t)) - \frac{\beta(b-d)}{b} m(t) \right]
\]
\[
=r \left( 1 - \frac{\beta(b-d)}{b} \right) m(t) [1 - m(t)]. \quad (2.20)
\]
Therefore with $0 < m(t) < 1$, the differential inequalities for both $\hat{v}$ and $\tilde{v}$ will hold if
\[
m'(t) + r \left[ 1 - \frac{\beta(b-d)}{b} \right] m(t) \geq r \left[ 1 - \frac{\beta(b-d)}{b} \right] m^2(t). \quad (2.21)
\]

Finally, we check on $\hat{u} = (b-d)m(t)/b$. From (2.3), the following differential inequality needs to be satisfied
\[
m'(t) \geq m(t) \left[ b(1 - \alpha) - (b-d)m(t)/b + am(t) \right] \left( \frac{(b-d)m(t)/b}{(b-d)m(t)/b + K} \right) - d
\]
\[
= m \left[ b(1 - \alpha) \left( \frac{(b-d)m(t)/b}{(b-d)m(t)/b + K} \right) - d - \frac{b(1 - \alpha)m^2(t)}{m(t) + bK/(b-d)} \right]. \quad (2.22)
\]
It suffices to ensure that $d - \frac{b(1 - \alpha)}{1 + bK/(b-d)} > 0$, or equivalently $K > K_0 = \frac{b - d(b(1 - \alpha) - d)}{kd}$,
and to find some function $0 < m(t) < 1$ with
\[
m'(t) + \left[ d - \frac{b(1 - \alpha)}{1 + bK/(b-d)} \right] m(t) \geq 0. \quad (2.23)
\]
Combining the two differential inequalities (2.21) and (2.23) that need to be satisfied, for $p$ and $q$ given in (2.15) we just need to look for a function $m(t)$ as the solution of the differential equation
\[
m'(t) + p m(t) = q m^2(t), \quad \text{with } 0 < m(0) < \frac{p}{q} \leq 1, \quad (2.24)
\]
which leads to the function $m(t)$ given in (2.17). From the construction of upper-lower solutions and the existence-comparison argument, the solution $(u(x,t), v(x,t))$ satisfies (2.16) as long as the initial inequalities in (2.4)

$$(0, 1 - m(0)) \leq (u_0(x), v_0(x)) \leq \left(\frac{(b - d)m(0)}{b}, 1 + m(0)\right)$$

hold for all $x \in \mathbb{R}$. The theorem is now proven. 

Theorem 2.5 demonstrates that under balanced ecological parameters and with sufficiently large Allee effect constant $K$, for all the initial density functions $(u_0, v_0)$ in the rectangular area $\left(0, \frac{(b-d)p}{bq}\right) \times \left(1 - \frac{\sqrt{2}}{2}, 1 + \frac{\sqrt{2}}{2}\right)$, the solution $(u(x,t), v(x,t))$ of the system (2.1)-(2.2) converges to the $v$-dominance state $(0, 1)$ uniformly on $\mathbb{R}$ as $t \to \infty$ with the rate $e^{-\gamma t}$. It is seen from (2.15) that the exponential rate $p$ depends on all of the ecological parameters in the reaction functions: $r$, $\alpha$, $\beta$, $b$, $d$, and $K$.

**Corollary 2.6.** Assume that condition $H2$ holds, and $d > \delta$ for $\delta$ given as (2.10). Let

$$K_* = \frac{(b - d)[b(1 - \alpha) - d + \delta]}{b(d - \delta)}.$$  

(2.25)

If $K \geq K_*$, then the $v$-dominance state $(0, 1)$ is globally asymptotically stable.

**Proof.** When $d > \delta$ and $K \geq K_*$, we have

$$r \left[1 - \frac{\beta(b - d)}{b}\right] \leq d - \frac{b(1 - \alpha)}{1 + bK/(b - d)}$$

which makes $p = q$ in (2.15). This leads to the function $m(t)$ in (2.17) with any $0 < m(0) < 1$. Hence for all the initial density functions

$$(0, 0) < (u_0, v_0) < \left(\frac{b - d}{b}, 2\right),$$

the solution $(u(x,t), v(x,t))$ of the system (2.1)-(2.2) converges to the $v$-dominance state $(0, 1)$ uniformly on $\mathbb{R}$ as $t \to \infty$. 

3. **Competitive exclusion with traveling waves.** In this section, we look into the traveling wave solutions $(u(x,t), v(x,t)) = (u(x + ct), v(x + ct)) = (u(\xi), v(\xi))$ flowing from the $u$-dominance states $(u_*, 0)$ to the $v$-dominance state $(0, 1)$, which indicates the competitive exclusion of $u$-species under resource competition and Allee effect. From the discussions on multiple steady states in Section 2, under condition $H3$-a: $b \geq d + 2\sqrt{bdK}$, there exist one or two $u$-dominance states $(u_*, 0)$ with

$$0 < u_* = \frac{b - d \pm \sqrt{(b - d)^2 - 4bdK}}{2b} < \frac{b - d}{b}.$$ 

The traveling wave solution of (2.1) connecting $(u_*, 0)$ and $(0, 1)$ has the form $(u(x,t), v(x,t)) = (u(x + ct), v(x + ct)) = (u(\xi), v(\xi))$ with $\xi \in \mathbb{R}$ and $c > 0$, and satisfies the following equations with limiting conditions:

$$\begin{cases} D_1u'' - cu' + u \left[b(1 - u - \alpha v)\frac{u}{u + K} - d\right] = 0 \\ D_2v'' - cv' + v[1 - \beta u - v] = 0 \end{cases}$$

(3.1)

$$\begin{pmatrix} u \\ v \end{pmatrix}(-\infty) = \begin{pmatrix} u_* \\ 0 \end{pmatrix}, \quad \begin{pmatrix} u \\ v \end{pmatrix}(+\infty) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$
Using a transformation $U = u_s - u$ and $V = v$, the above system is then changed into a quasi-monotone non-decreasing system which is more convenient for our analysis:

\[
\begin{align*}
\begin{cases}
D_1 U_{\xi\xi} - c U_{\xi} + (u_s - U) \left[ d - \frac{b(1 - u_s + U - \alpha V)(u_s - U)}{K + u_s - U} \right] = 0, \\
D_2 V_{\xi\xi} - c V_{\xi} + r V (1 - \beta u_s + \beta U - V) = 0,
\end{cases}
\end{align*}
\tag{3.2}
\]

It is easy to verify that system (3.2) is quasi-monotone nondecreasing in the region $G = \{(U, V) | 0 \leq U \leq (b - d)/b, 0 \leq V \leq 1\}$. We next define the upper- and lower-solution pairs for the traveling wave system (3.2) based on the monotone iteration methods established in early literature [4, 10, 23, 24, 26].

**Definition 3.1.** A $C^2(\mathbb{R}) \times C^2(\mathbb{R})$ function $(\bar{U}(\xi), \bar{V}(\xi))$ is an upper solution of (3.2) if it satisfies the following differential and limit inequalities

\[
\begin{align*}
\begin{cases}
D_1 \bar{U}_{\xi\xi} - c \bar{U}_{\xi} + (u_s - \bar{U}) \left[ d - \frac{b(1 - u_s + \bar{U} - \alpha \bar{V})(u_s - \bar{U})}{K + u_s - \bar{U}} \right] \leq 0, \\
D_2 \bar{V}_{\xi\xi} - c \bar{V}_{\xi} + r \bar{V} (1 - \beta u_s + \beta \bar{U} - \bar{V}) \leq 0,
\end{cases}
\end{align*}
\tag{3.3}
\]

A lower solution $(\tilde{U}(\xi), \tilde{V}(\xi))$ of (3.2) is defined in a similar way by reversing all the inequalities in (3.3).

We will construct a pair of upper- and lower-solutions for the traveling wave system (3.2) by applying a well-known result [25] on the solution of the following KPP equation (where $a > 0$) with limiting conditions:

\[
\begin{align*}
\begin{cases}
Y_{\xi\xi} - c Y_{\xi} + a Y(1 - Y) = 0, \\
Y(-\infty) = 0, Y(+\infty) = 1.
\end{cases}
\end{align*}
\tag{3.4}
\]

**Lemma 3.2.** Corresponding to every $c \geq 2\sqrt{a}$, there is a unique solution (up to a translation of the origin) $Y$ of (3.4). Such solution satisfies $Y'(\xi) > 0$ for $\xi \in \mathbb{R}$ and has the following asymptotic behaviors at infinities.

For the wave solution $Y(\xi)$ with non-critical speed $c > 2\sqrt{a}$, we have

\[
\begin{align*}
\begin{cases}
Y(\xi) = Ae^{c - \sqrt{c^2 - 4a}\xi} + o(e^{c - \sqrt{c^2 - 4a}\xi}), \text{ as } \xi \to -\infty; \\
Y(\xi) = 1 - Be^{-\sqrt{c^2 + 4a}\xi} + o(e^{-\sqrt{c^2 + 4a}\xi}), \text{ as } \xi \to +\infty.
\end{cases}
\end{align*}
\tag{3.5}
\]

where $A, B > 0$ are constants.

For the wave solution $Y(\xi)$ with critical speed $c = 2\sqrt{a}$ we have

\[
\begin{align*}
\begin{cases}
Y(\xi) = (A_c + D_c \xi)e^{\sqrt{c}\xi} + o(e^{\sqrt{c}\xi}), \text{ as } \xi \to -\infty; \\
Y(\xi) = 1 - B_c e^{-(\sqrt{2} - 1)\sqrt{c}\xi} + o(e^{-(\sqrt{2} - 1)\sqrt{c}\xi}), \text{ as } \xi \to +\infty.
\end{cases}
\end{align*}
\tag{3.6}
\]

where $A_c \in \mathbb{R}$, $D_c < 0$, $B_c > 0$ are constants.
It is seen from previous section that the condition \( \textbf{H2-b}: \beta(b - d) < b \), ensures the persistence of the \( v \)-species in (2.1)-(2.2). Since both \( u_s < (b - d)/b < 1 \), then \( 1 - \beta u_s > 0 \) as long as \( \textbf{H2-b} \) holds. Let

\[
D = \max\{D_1, D_2\}, \quad \text{and} \quad \bar{D} = \min\{D_1, D_2\}.
\]

For all \( c \geq 2D \sqrt{r(1 - \beta u_s)/\bar{D}} \), let \( Y \) be the solution of

\[
\begin{align*}
Y_{\xi\xi} - \frac{c}{D} Y_{\xi} + \frac{r(1 - \beta u_s)}{D} Y(1 - Y) &= 0, \\
Y(\pm\infty) &= 0, \quad Y(+\infty) = 1.
\end{align*}
\]

Lemma 3.3. Assume that conditions

\( \textbf{H2-b}: 0 < \beta(b - d) < b \) and \( \textbf{H3-a}: b \geq d + 2\sqrt{bdK} \) hold.

If a \( u \)-dominance state \((u_s, 0)\) with \( u_s = \frac{b - d + \sqrt{(b - d)^2 - 4bdK}}{2b} \) satisfies

\[
\alpha \leq u_s \leq r - \frac{d}{\beta r}
\]

or

\[
u_s \leq \alpha \quad \text{and} \quad d + \frac{bu_s(\alpha - u_s)}{K + u_s} \leq r(1 - \beta u_s),
\]

then for each \( c \geq 2D \sqrt{r(1 - \beta u_s)/\bar{D}} \),

\[
\left( \begin{array}{c} \hat{V}(\xi) \\ \hat{V}^{\prime}(\xi) \end{array} \right) = \left( \begin{array}{c} u_s Y(\xi) \\ Y(\xi) \end{array} \right) \quad \xi \in R
\]

is an upper solution of system (3.2), where \( Y \) is the solution of the traveling wave problem (3.8).

Proof. We can first verify that

\[
\hat{V}_{\xi\xi} - \frac{c}{D_2} \hat{V}_{\xi} + \frac{r\hat{V}}{D_2} \left( 1 - \beta u_s - \hat{V} + \beta\hat{u} \right)
\]

\[
\leq Y_{\xi\xi} - \frac{c}{D} Y_{\xi} + \frac{rY}{D} (1 - \beta u_s) (1 - Y) = 0.
\]

Next, we show that \( \hat{U} = u_s Y \) satisfies the differential inequality given in (3.3). Since (noting the fact that \( Y_{\xi} > 0 \))

\[
\hat{U}_{\xi\xi} - \frac{c}{D_1} \hat{U}_{\xi} + \frac{(u_s - \hat{U})}{D_1} \left[ d - b \left( 1 - u_s + \hat{U} - \alpha\hat{V} \right) \left( \frac{u_s - \hat{U}}{K + u_s - \hat{U}} \right) \right]
\]

\[
\leq u_s \left( Y_{\xi\xi} - \frac{c}{D} Y_{\xi} \right) + \frac{u_s(1 - Y)}{D_1} \left[ d - b \left( 1 - u_s(1 - Y) - \alpha Y \right) \left( \frac{u_s(1 - Y)}{K + u_s(1 - Y)} \right) \right],
\]

then by the relation that \( d - bu_s(1 - u_s)/(k + u_s) = 0 \), from equation (3.8) and condition (3.9) or (3.10), we can verify that

\[
\hat{U}_{\xi\xi} - \frac{c}{D} \hat{U}_{\xi} + \frac{(u_s - \hat{U})}{D_1} \left[ d - b \left( 1 - u_s + \hat{U} - \alpha\hat{V} \right) \left( \frac{u_s - \hat{U}}{K + u_s - \hat{U}} \right) \right]
\]

\[
\leq u_s Y(1 - Y) \left[ \frac{r(1 - \beta u_s)}{D} + \frac{bu_s}{D_1[1 - K + u_s(1 - Y)]} \left( \frac{K(1 - u_s)}{K + u_s} + \alpha u_s(1 - Y) \right) \right]
\]

\[
\leq u_s Y(1 - Y) \left[ d - r(1 - \beta u_s) + \frac{bu_s(\alpha - u_s)}{K + u_s} \right] \leq 0.
\]

\[\Box\]
We now construct a lower-solution for the traveling wave system (3.2). For all $c \geq 2D\sqrt{r/(1-\beta u_s)}/D$, let $Z$ be the solution of
\[
\begin{cases}
Z_{\xi \xi} - \frac{c}{D} Z_{\xi} + \frac{r}{D} Z(1-\beta u_s - Z) = 0, \\
Z(-\infty) = 0, Z(+\infty) = 1 - \beta u_s.
\end{cases}
\tag{3.13}
\]

**Lemma 3.4.** Assume that conditions

**H2-b:** $0 < \beta(b-d) < b$ and **H3-a:** $b \geq d + 2\sqrt{bdK}$ hold.

Given a $u$-dominance state $(u_s, 0)$ with $u_s = \frac{b-d+\sqrt{(b-d)^2-4bdK}}{2b}$, for each $c \geq 2D\sqrt{r(1-\beta u_s)}/D$ and sufficiently small $0 < l < u_s$,
\[
\begin{pmatrix}
\dot{U}(\xi) \\
\dot{V}(\xi)
\end{pmatrix}
= \begin{pmatrix}
IZ(\xi) \\
Z(\xi)
\end{pmatrix},
\quad \xi \in R
\tag{3.14}
\]

is a lower solution of system (3.2), where $Z$ is the solution of the traveling wave problem (3.13).

**Proof.** We will show that $(\hat{U}, \hat{V}) = (lZ, Z)$ is a lower solution for some $0 < l < u_s$. First, since $Z_{\xi} > 0$,
\[
\hat{V}_{\xi \xi} - \frac{c}{D^2} \hat{V}_{\xi} + \frac{r}{D} \hat{V} \left(1 - \beta u_s + \beta \hat{U} - \hat{V}\right)
\geq Z_{\xi \xi} - \frac{c}{D} Z_{\xi} + \frac{r}{D} Z \left(1 - \beta u_s + \beta lZ - Z\right) \geq 0.
\]
Next, we show that by choosing $l$ small enough with $0 < l < u_s$, the inequality for $\hat{U}$ required in Definition 3.1 will also be satisfied.
\[
\begin{align*}
\hat{U}_{\xi \xi} - \frac{c}{D_1} \hat{U}_\xi + \frac{(u_s - \hat{U})}{D_1} \left[d-b(1-u_s+\hat{U}-\alpha \hat{V}) \frac{u_s-\hat{U}}{K+u_s-\hat{U}}\right]
\geq &\left[Z_{\xi \xi} - \frac{c}{D} Z_{\xi} + \frac{(u_s-lZ)}{D} \left[d-b(1-u_s+lZ-\alpha Z) \frac{u_s-lZ}{K+u_s-lZ}\right]\right] \\
\geq &-\frac{r l Z}{D} (1-\beta u_s - Z) + \frac{bZ(u_s-lZ)}{D} (\alpha - l) \frac{u_s-lZ}{K+u_s-lZ} \\
+ &\frac{(u_s-lZ)}{D} \left[d-b(1-u_s) \frac{u_s-lZ}{K+u_s-lZ}\right].
\end{align*}
\]

From the relation $d - b(1 - u_s) \frac{u_s}{K+u_s} = 0$, we know that for any $0 < l < u_s$, $d - b(1 - u_s) \frac{u_s-lZ}{K+u_s-lZ} \geq 0$. Also, the continuous function
\[
F(l) = \frac{b(u_s-l)^2(\alpha - l)}{K+u_s-l} - lr(1-\beta u_s)
\]
gives $F(0) = \frac{ab(u_s)^2}{K+u_s} > 0$. Therefore we can choose $l$ to be small enough such that
\[
\begin{align*}
\hat{U}_{\xi \xi} - \frac{c}{D_1} \hat{U}_\xi + \frac{(u_s - \hat{U})}{D_1} \left[d-b(1-u_s+\hat{U}-\alpha \hat{V}) \frac{u_s-\hat{U}}{K+u_s-\hat{U}}\right]
\geq &\left[Z_{\xi \xi} - \frac{c}{D} Z_{\xi} + \frac{(u_s-lZ)}{D} \left[d-b(1-u_s+lZ-\alpha Z) \frac{u_s-lZ}{K+u_s-lZ}\right]\right] \\
\geq &-\frac{r l Z}{D} (1-\beta u_s - Z) + \frac{bZ(u_s-lZ)}{D} (\alpha - l) \frac{u_s-lZ}{K+u_s-lZ} \\
+ &\frac{(u_s-lZ)}{D} \left[d-b(1-u_s) \frac{u_s-lZ}{K+u_s-lZ}\right].
\end{align*}
\]

It is seen that the constructed $(\hat{U}, \hat{V})$ and $(\hat{U}, \hat{V})$ are a pair of smooth and ordered upper- and lower-solutions for the traveling wave system (3.2), where $(\hat{U}, \hat{V})$
also satisfies the limiting conditions in (3.2) at ±∞. Starting from this upper-
lower solution pair and following the monotone iteration scheme given in [4, 27],
one can generate a non-increasing sequence \((\{U(n), V(n)\})\) and a non-decreasing se-
quence \((\{\tilde{U}(n), \tilde{V}(n)\})\) which preserves the ordered property, monotonicity in \(\xi\)
and limit values at ±∞. The sequence \((\{U(n), V(n)\})\) then converges to a positive and
smooth solution \((U(\xi), V(\xi))\) for the traveling wave system 3.2 which is increasing
in \(\xi\). By the existence theorems given in [4, 10, 23, 24, 27], we can obtain the exis-
tence and monotonicity of a positive, smooth traveling wave solution flowing from
\(u\)-dominance state \((u_s, 0)\) to \(v\)-dominance state \((0, 1)\), indicating the competitive
exclusion of \(u\)-species.

**Theorem 3.5.** Assume that conditions

\(H2-b: 0 < \beta(b-d) < b\) and \(H3-a: b \geq d + 2\sqrt{bdK}\) hold.

If a \(u\)-dominance steady state \((u_s, 0)\) satisfies (3.9) or (3.10), then for all \(c \geq
2\tilde{D}\sqrt{r(1 - \beta u_s)}/\tilde{D}\) there exists a smooth traveling wave solution \((u(x + ct), v(x +
ct))\) for the competition model (3.1) under Allee effect connecting \((u_s, 0)\) to the \(v\-
dominance state \((0, 1)\), with \(u(\xi)\) decreasing and \(v(\xi)\) increasing on \(\mathbb{R}\).

**Remark 3.6.** As seen in the above theorem, whenever one or both of the \(u\)-dominance
states satisfy the condition (3.9) or (3.10), there exist traveling wave solutions flow-
ing from \((u_s, 0)\) to \((0, 1)\) which indicate the competitive exclusion of the \(u\)-species.
The size of the steady state also affects the minimal wave speed of the traveling
wave solutions.

4. **Asymptotic rates of the traveling wave solution.** In this section we study
the asymptotic rates of the traveling wave solutions at ±∞. For simplicity of cal-
culations, we let the diffusion rates for both species to be equal: \(D_1 = D_2 = 1\). For
the case of unequal diffusion rates, the asymptotic rates can be estimated through
similar methods. Recalling the known results in Lemma 3.2 on the asymptotic rates
of the solution for the K.P.P equation (3.4), we see the following asymptotic rates
for the increasing and smooth function \(Y(\xi)\) given by system (3.8):

For non-critical speed \(c > 2\sqrt{r(1 - \beta u_s)}\),

\[
\begin{align*}
Y(\xi) &= Ae^{-\sqrt{r(1 - \beta u_s)} \xi} + o(e^{-\sqrt{r(1 - \beta u_s)} \xi}) \quad \text{as} \ \xi \to -\infty; \\
Y(\xi) &= 1 - Be^{-\sqrt{r(1 - \beta u_s)} \xi} + o(e^{-\sqrt{r(1 - \beta u_s)} \xi}) \quad \text{as} \ \xi \to +\infty.
\end{align*}
\]

where \(A, B > 0\) are constants.

For the wave with critical speed \(c = 2\sqrt{r(1 - \beta u_s)}\), we have

\[
\begin{align*}
Y(\xi) &= (A_c + D_c \xi)e^{\sqrt{r(1 - \beta u_s)} \xi} \quad \text{as} \ \xi \to -\infty; \\
Y(\xi) &= 1 - Bce^{(c^2 - 1)\sqrt{r(1 - \beta u_s)} \xi} + o(e^{-\sqrt{r(1 - \beta u_s)} \xi}) \quad \text{as} \ \xi \to +\infty.
\end{align*}
\]

where \(A_c \in \mathbb{R}, D_c < 0, B_c > 0\) are constants.

Since \(\tilde{D} = D\), then by linearizing (3.13) we see that the increasing and smooth
function \(Z(\xi)\) has the same asymptotic rate as \(Y(\xi)\) when \(\xi \to -\infty\). From the
ordered relation

\[
\begin{pmatrix}
\dot{L}(\xi) \\
\dot{Z}(\xi)
\end{pmatrix} = \begin{pmatrix}
\dot{U}(\xi) \\
\dot{V}(\xi)
\end{pmatrix} \leq \begin{pmatrix}
U(\xi) \\
V(\xi)
\end{pmatrix} \leq \begin{pmatrix}
\tilde{U}(\xi) \\
\tilde{V}(\xi)
\end{pmatrix} = \begin{pmatrix}
u_s Y(\xi) \\
Y(\xi)
\end{pmatrix} \quad \xi \in \mathbb{R},
\]
and the fact that the constructed upper and lower solutions \((\tilde{U}, \tilde{V})\) and \((\hat{U}, \hat{V})\) both converge to \((0, 0)\) as \(\xi \to -\infty\) with the same asymptotic rate, we can conclude on the asymptotic rate of \(u(\xi) = r - U(\xi), v(\xi) = V(\xi)\) at \(-\infty\) later stated in Theorem 4.1.

Next, the asymptotic rate of \(u(\xi)\) and \(v(\xi)\) at \(+\infty\) is obtained by linear decomposition. In order to find out the asymptotic rate of the traveling wave solutions \((U(\xi), V(\xi))\) at \(+\infty\), we look into \((\mu(\xi), \nu(\xi)) = (U'(\xi), V'(\xi))\) which satisfies

\[
\begin{align*}
\mu_{\xi\xi} - c\mu_{\xi} + \frac{\partial G_1}{\partial U} \mu + \frac{\partial G_1}{\partial V} \nu = 0, \quad &
u_{\xi\xi} - c\nu_{\xi} + \frac{\partial G_2}{\partial U} \mu + \frac{\partial G_2}{\partial V} \nu = 0,
\end{align*}
\]

where

\[
\begin{align*}
G_1 &= (u_s - U) \left[ d - b(1 - u_s + U - v) \frac{u_s - U}{u_s - U + K} - d \right],
G_2 &= rV \left[ 1 - \beta u_s + \beta U - V \right].
\end{align*}
\]

Looking into \((\mu(\xi), \nu(\xi)) = (U'(\xi), V'(\xi))\) at \(\xi = +\infty\), the limit system is calculated as

\[
\begin{pmatrix}
\mu^+(\xi) \\
\nu^+(\xi)
\end{pmatrix} =
\begin{pmatrix}
1 & 0 \\
\beta r & 0 \\
1 & -d
\end{pmatrix}
\begin{pmatrix}
C_1 e^{\frac{c\xi - \sqrt{c^2 - 4d}}{2}} \\
C_2 e^{\frac{c\xi + \sqrt{c^2 - 4d}}{2}}
\end{pmatrix}.
\]

Integrating for \((U(\xi), V(\xi))\) with observance of its monotonicity and limit at \(+\infty\),

\[
\begin{pmatrix}
U(\xi) \\
V(\xi)
\end{pmatrix} =
\begin{pmatrix}
C_1 e^{\frac{c\xi - \sqrt{c^2 - 4d}}{2} - B_1 \left( \frac{1}{\beta r} - \frac{1}{1 - d} \right) e^{\lambda_1 \xi} - B_2 \left( \frac{1}{\beta r} - \frac{1}{1 - d} \right) e^{\lambda_2 \xi} + o(e^{\xi})
\end{pmatrix}
\]

as \(\xi \to +\infty\), where \(\lambda_1 = \frac{c - \sqrt{c^2 - 4d}}{2}, \lambda_2 = \frac{c + \sqrt{c^2 - 4d}}{2}, \bar{\lambda} = \max\{\lambda_1, \lambda_2\}\).

Finally, using the transformation \(u(\xi) = u_s - U(\xi)\) and \(v(\xi) = V(\xi)\), we have

**Theorem 4.1.** Assume that all conditions in Theorem 3.5 hold, and \((u_s, 0)\) is a 1-dominance steady state satisfying (3.9) or (3.10). If \(D_1 = D_2 = 1\), then for all \(c \geq 2\sqrt{r(1 - \beta u_s)}\), there exists a traveling wave solution \((u(x + ct), v(x + ct))\) for (2.1) where \(u(\xi)\) is decreasing and \(v(\xi)\) is increasing on \(\mathbb{R}\).

1. For the wave speed \(c > 2\sqrt{r(1 - \beta u_s)}\), we have

\[
\begin{pmatrix}
u(\xi) \\
v(\xi)
\end{pmatrix} =
\begin{pmatrix}
u(0) \\
u(0)
\end{pmatrix} + \begin{pmatrix}
-A_1 & A_2 \\
A_2 & -A_1
\end{pmatrix} e^{\lambda \xi} + o(e^{\lambda \xi})
\]

as \(\xi \to -\infty\), with \(\lambda = \frac{c - \sqrt{c^2 - 4r(1 - \beta u_s)}}{2}\) and \(A_1, A_2 > 0\);

2. For the critical wave speed \(c = 2\sqrt{r(1 - \beta u_s)}\), we have

\[
\begin{pmatrix}
u(\xi) \\
v(\xi)
\end{pmatrix} =
\begin{pmatrix}
u(0) \\
v(0)
\end{pmatrix} + \begin{pmatrix}
(A_{c(1)}^{(11)} + A_{c(12)}^{(12)}) & \lambda_1 \\
(A_{c(21)}^{(21)} + A_{c(22)}^{(22)}) & \lambda_2
\end{pmatrix} e^{\lambda \xi} + o(e^{\lambda \xi})
\]

as \(\xi \to -\infty\), where \(\lambda = \sqrt{r(1 - \beta u_s)}, A_{c(12)}^{(12)} > 0, A_{c(22)}^{(22)} < 0, A_{c(11)}^{(11)}, A_{c(21)}^{(21)} \in \mathbb{R}\).
3. For the wave speed \( c \geq 2\sqrt{r(1-\beta u)} \), we have

\[
\begin{pmatrix}
  u(\xi) \\
  v(\xi)
\end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + B_1 \begin{pmatrix} 1 \\ -\beta r \end{pmatrix} e^{\lambda_1 \xi} - B_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{\lambda_2 \xi} + o(e^{\lambda \xi})
\]

(4.8)

as \( \xi \to +\infty \), where \( \lambda_1 = \frac{-\sqrt{c^2+4d}}{2} \), \( \lambda_2 = \frac{-\sqrt{\beta}}{2} \), \( \lambda = \max\{\lambda_1, \lambda_2\} \), and some \( B_1, B_2 > 0 \).

5. Numerical simulations of the model. In this last section, we work on numerical simulations of the density functions for two competing species with one under Allee effect on a rectangular domain \([-L, L] \times [0, T]\), and the traveling wave solutions. After we discretize the differential equation system (2.1)-(2.2) into finite-difference systems, numerical solutions can be obtained through the monotone iterative scheme developed in several earlier articles (see for example, [19, 21, 22]).

Example 5.1. Figure 1. Permanence effect in the competition model under Allee effect.

As seen in Theorem 2.4 with condition \( \textbf{H2} \) and \( K \leq [b(1-\alpha) - d]^2/4bd \), both the competing species will persist as long as and the initial functions satisfy

\[
\frac{b(1-\alpha) - d}{2b} \leq u_0(x) \leq \frac{b-d}{b}
\]

and

\[
\epsilon \leq v_0(x) \leq 1
\]

for some \( \epsilon > 0 \). Furthermore, the density function \((u(t, x), v(t, x))\) in (2.1)-(2.2) satisfies

\[
\liminf_{t \to +\infty} u(x, t) \geq \frac{b(1-\alpha) - d}{2b} > 0
\]

and

\[
\liminf_{t \to +\infty} v(x, t) \geq \frac{b-\beta(b-d)}{b} > 0
\]

for all \( x \in \mathbb{R} \). We choose the following set of biological parameters: \( D_1 = 0.2, D_2 = 0.3, K = 0.15, b = 0.7, d = 0.2, r = 0.7, \alpha = 0.3, \beta = 0.9 \), which makes all required conditions in Theorem 2.4 hold. The initial function

\[
(u_0, v_0) = (0.45 + 0.24 \cos(\pi x), 0.07 + 0.05 \cos(\pi x))
\]

ensures the persistence of both species with \( \liminf_{t \to +\infty} u(x, t) \geq \frac{b-\beta(b-d)}{b} = 5/14 \).

Example 5.2. Figure 2. Asymptotic stability of the \( v \)-dominance state \((0, 1)\).

As seen in Theorem 2.5, with conditions \( \textbf{H2} \) and \( K > K_0 = (b-d)[b(1-\alpha)-d]/bd \), the \( v \)-dominance state \((0, 1)\) has a rectangular attraction region \((0, \frac{b-d}{bd}) \times (1 - \frac{p}{q}, 1+\frac{q}{p})\) where \( p \) and \( q \) are defined in (2.15). We choose the following set of biological parameters: \( D_1 = 0.2, D_2 = 0.3, K = 0.9, b = 0.5, d = 0.4, r = 0.52, \alpha = 0.11, \beta = 0.6 \), which makes \( K_0 = 0.225, p = 0.3191, q = 0.4576, p/q = 0.6973 \). The initial function

\[
(u_0, v_0) = (0.1 + 0.035 \cos(\pi x), 0.5 + 0.19 \cos(\pi x))
\]

ensures that it falls into the attraction region, and the solution \((u(x, t), v(x, t))\) of (2.1)-(2.2) converges to \((0, 1)\) uniformly under exponential rate \( e^{-\rho t} \).
Figure 1. Permanence in the competition model under Allee effect.

Figure 2. Asymptotic stability of the steady state \((0, 1)\), \(v\) species dominance.

Example 5.3. Figure 3-4. Traveling wave front flowing from the \(u\)-dominance state \((u_s, 0)\) to the \(v\)-dominance state \((0, 1)\).

When conditions H2-b and H3-a hold, and the \(u\)-dominance state \((u_s, 0)\) satisfies (3.9) or (3.10), Theorem 3.5 indicates that for any \(c \geq 2D \sqrt{r(1 - \beta u_s)}/D\), the system (2.1) has a traveling wave solution \((u(x + ct), v(x + ct))\) flowing from \((u_s, 0)\) to the \(v\)-dominance state \((0, 1)\). For \(D_1 = D_2 = 1\), the asymptotic rates of \((u(\xi), v(\xi))\) at \(\xi = \pm \infty\) are also given in Theorem 4.1. We choose the following set of biological parameters: \(D_1 = 1.0\), \(D_2 = 1.0\), \(K = 0.3125\), \(b = 0.8\), \(d = 0.2\), \(r = 0.5\), \(\alpha = 0.5\), \(\beta = 0.4\), which makes conditions H2-b and H3-a hold. There exist two \(u\)-dominance states: \((u_s^{(1)}, 0) = (0.625, 0)\) that satisfies condition (3.9) and \((u_s^{(2)}, 0) = (0.125, 0)\) that satisfies condition (3.10). Using this set of biological parameters and fixing the initial function \((u_0, v_0)\) as a small perturbation of \((u_s^{(1)}, 0)\) or \((u_s^{(2)}, 0)\), we demonstrate the traveling wave front for competitive exclusion of \(u\)-species in Figure 3 and Figure 4.
Figure 3. Traveling wave front connecting \((u_s^{(1)},0)\) to \((0,1)\), competitive exclusion of \(u\)-species.

Figure 4. Traveling wave front connecting \((u_s^{(2)},0)\) to \((0,1)\), competitive exclusion of \(u\)-species.
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