LÉVY PROCESSES WITH VALUES IN LOCALLY CONVEX SUSLIN SPACES

FLORIAN BAUMGARTNER

Abstract. We provide a Lévy-Itô decomposition of sample paths of Lévy processes with values in complete locally convex Suslin spaces. This class of state spaces contains the well investigated examples of separable Banach spaces, as well as Fréchet or distribution spaces among many others. Sufficient conditions for the existence of a pathwise compensated Poisson integral handling infinite activity of the Lévy process are given.

1. Introduction

The present paper is concerned with Lévy processes with infinite-dimensional state spaces beyond Banach spaces. We assume the state space to be a complete locally convex Suslin space. A fundamental result in the analysis of Lévy processes is the decomposition of sample paths into independent diffusion and jump components. As a main result of this paper we obtain this so-called Lévy-Itô decomposition:

Theorem (Lévy-Itô-decomposition). Let $X$ be a Lévy process in a locally convex Suslin space $E$ with characteristics $(\gamma, Q, \nu, K)$ and let the Lévy measure $\nu$ be locally reducible with reducing set $K$. Then there exist an $E$-valued Wiener process $(W_t)_{t \in T}$ with covariance operator $Q$, an independently scattered Poisson random measure $N$ on $T \times E$ with compensator $\lambda \otimes \nu$ and a set $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ such that for all $\omega \in \Omega_0$ one has

$$X_t(\omega) = \gamma t + W_t(\omega) + \int_{[0,t] \times K} x \, d\tilde{N}(s, x)(\omega) + \int_{[0,t] \times K^c} x \, dN(s, x)(\omega)$$

for all $t \in T$. Furthermore, all the summands in (1.1) are independent and the convergence of the first integral in the sense of (7.8) is a.s. uniform in $t$ on bounded intervals in $E_K$ and $E$.

Additionally, if the compact sets of $E$ admit a fundamental system of compact separable Banach disks, then there exists a separable compactly embedded Banach space $F \subseteq E$ such that the first three summands in the Lévy-Itô decomposition take values in $F$ a.s.

A compact separable Banach disk $K$ is a compact set such that its linear hull is a separable Banach space with respect to the closed unit ball $K$. The crucial notion of local reducibility and the precise definition of all terms in the theorem will be given and investigated in this paper.

This theorem extends and unifies almost all known results of Lévy processes with values in topological vector spaces. Lévy processes with values in $\mathbb{R}^d$ [32, 3] and in Banach spaces [1, 2, 31, 42, 13] have been well-established but beyond the Banach space case little is known.

2010 Mathematics Subject Classification: Primary: 60B11. Secondary: 60G51 60G17 28C20. Key words and phrases: Lévy processes, Lévy-Itô decomposition, locally convex space, Suslin space, stochastic process in infinite dimensions, Poisson integral, weak metric.
Nevertheless, much can be found regarding more general stochastic processes or special cases of Lévy processes in topological vector spaces: Wiener processes with values in locally convex spaces \cite{15, 6} or Markov processes in completely regular Suslin spaces \cite{35} were considered; K. Itô presents a theory of stochastic processes with values in distribution spaces in order to solve abstract Cauchy problems, cf. \cite{21}. In different settings, SPDEs with solution processes in nuclear and duals of nuclear spaces have been investigated by various authors, cf. \cite{26, 27, 39, 24, 25, 8, 9, 28, 18}.

Returning to Lévy processes, Üstünel presented a Lévy-Itô decomposition in a class of Suslin nuclear duals of nuclear spaces \cite{40}. However, the decomposition had some shortcomings, as missing independence of components and an \(L^2\)-converging integral not allowing to claim a desirable pathwise decomposition. Recently, C. Fonseca Mora was able to both enlarge the class of state spaces by new methods, cf. \cite{16} and prove a satisfactory Lévy-Itô decomposition for duals of reflexive nuclear spaces.

In this paper, a different approach based on the works of Dettweiler and Tortrat on infinitely divisible Radon measures on locally convex spaces, cf. \cite{11, 38}, allows to drop any nuclearity assumptions on the state space \(E\). This extends and unifies the results of Üstünel (in the nuclear setting) and Dettweiler \cite{13} (for Banach spaces).

For example, this approach allows to treat a Lévy process in the locally convex direct sum \(L^p([0,1]) \oplus D(\mathbb{R}^n)\) which is obviously neither a Banach nor a nuclear space but a locally convex Suslin. Also, projecting on the components does not work in general as they might not be independent (as in the finite-dimensional case). So, even in this simple example, a unified approach is necessary.

The main technique is the reduction of the small jumps to a compactly embedded separable Banach subspace \(E_K\) of \(E\) – if possible, i.e., if the Lévy measure \(\nu\) is locally reducible, cf. Definition \ref{def:local_reducibility}. This will guarantee a.s. uniform convergence of a Poisson integral representing the small jumps of a Lévy process:

\textbf{Theorem.} Let \(\nu\) be locally reducible with reducing set \(K\). For \(t \in T\), the quantity

\begin{equation}
J_t := \int_{(0,t] \times K} x \, d\tilde{N}(s, x) := \sum_{n=1}^{\infty} J([0,t] \times C_n)
\end{equation}

is a series of independent random variables in \(E_K = \bigcup_{n \in \mathbb{N}} nK\) and converges almost surely in \(E_K\) and \(E\). The convergence is uniform in \(t\) on bounded intervals of \(T\). Finally, \((J_t)_{t \in T}\) is a càdlàg Lévy process in \(E\) with characteristics \((0, 0, \nu|_K, K)\).

In Section 4 we investigate this reduction technique and characterise locally reducible Lévy measures (Theorem 4.9). It should be pointed out that this result is even a little better than in \cite[Proof of Theorem 2.1]{13}, as the series is converging a.s. uniformly without any subsequence arguments.

On the other hand, a zero-one law for generalised Poisson exponentials by Janssen \cite{22} will allow us to impose a simple and natural condition to the state space obtaining the following useful result:

\textbf{Theorem.} If the compact sets of \(E\) admit a fundamental system of compact separable Banach disks, then every Lévy measure on \(E\) is locally reducible.

This property is simpler to check and satisfied e.g. by separable Fréchet and all common distribution spaces.

Another peculiarity happens taking into account the possibly uncountable neighbourhood bases of \(E\). Measurability problems in connection with limits can be overcome introducing a weak metric on \(E\) and exploiting the Suslin property of \(E\) which is essential in this approach, cf. Section 5.
The paper is organised as follows. After some preliminaries on spaces and measures, vector valued Lévy processes will be defined in Section 3. Section 4 is dedicated to the reduction of the small jumps to the Banach subspace, Section 5 deals with càdlàg functions with Suslin space values, Section 6 with Wiener processes. The main results are Theorem 7.5 in Section 7 and the Lévy-Itô decomposition, Theorem 8.1, in Section 8.

2. Preliminaries

2.1. Spaces. Throughout this paper, we will make the following assumption on the state space $E$ (unless explicitly stated differently):

(S1) $E$ is a real locally convex space,
(S2) $E$ complete, and
(S3) $E$ is a Suslin space.

A complete locally convex space has the property that every Cauchy net has a limit. A Suslin space is a Hausdorff (topological) space which is a surjective continuous image of a polish space. We denote by $E'$ the topological dual of $E$ and $B(E)$ is the Borel-$\sigma$-algebra of $E$. The cylindrical $\sigma$-algebra $\mathcal{E}(E)$ is generated by the elements of $E'$. The above assumptions guarantee the following frequently used properties: $E$ is separable, Hausdorff and completely regular (i.e. a point and a closed set can be separated by a continuous function), $\mathcal{E}(E) = B(E)$ and $(E, B(E))$ is a measurable vector space, i.e., addition and scalar multiplication are measurable. Furthermore, there exists a sequence of elements in $E'$ separating the points of $E$ and therefore one has a continuous metric on $E$.

The following essential result is due to L. Schwartz:

Proposition 2.1 (Corollary 2, p. 101 of [34]). Let $\tau_1$ and $\tau_2$ two comparable Suslin topologies on a set $F$. Then, the respective Borel-$\sigma$-algebras coincide.

2.2. Measures. The set of (nonnegative) Borel measures on $E$ is denoted by $\mathcal{M}(E)$, its subset of bounded, resp. probability measures by $\mathcal{M}^b(E)$, $\mathcal{M}^1(E)$, respectively. A measure $\mu \in \mathcal{M}^b(E)$ is called Radon, if for $B \in B(E)$ and $\varepsilon > 0$ there exists a compact set $K \subseteq B$ such that $\mu(B \setminus K) < \varepsilon$. Every finite Borel measure on any Suslin space is Radon [34, Thm. 10, p. 122]. The Fourier transform of a measure $\mu \in \mathcal{M}^b(E)$ is given by

$$\hat{\mu} : E' \to \mathbb{C} \quad \hat{\mu}(a) := \int_E e^{i(x,a)} \, d\mu(x).$$

Any two finite measures with equal Fourier transform coincide, cf. [41, Theorem 2.2, p. 200]. For $\mu, \nu \in \mathcal{M}(E)$ the convolution is defined by

$$\mu * \nu(B) = (\mu \otimes \nu)_\alpha(B) = \mu \otimes \nu(\alpha^{-1}(B)),$$

where $\alpha : E \times E \to E$, $(x,y) \mapsto x + y$, cf. [6, Appendix, p. 373f].

For $A \in B(E)$, the restriction of a measure $\mu \in \mathcal{M}(E)$ to $A$ is defined by $\mu|_A(B) := \mu(B \cap A)$, $B \in B(E)$. Note that $\mu|_A \in \mathcal{M}(E)$. If $\mu|_A$ is only considered on $B(A)$, we denote it by $\mu|_A \in \mathcal{M}(A)$.

2.3. Infinitely divisible measures. A measure $\mu \in \mathcal{M}^1(E)$ is called Poissonian provided that there exists $\nu \in \mathcal{M}^b(E)$ such that its Fourier transform satisfies

$$\hat{\mu}(a) = e^{\nu(E)} e^{i(x,a)} \, d\nu(x).$$

For a measure $\nu \in \mathcal{M}^b(E)$ its Poisson exponential can be defined by

$$e(\nu) := e^{-\nu(E)} \sum_{n=0}^{\infty} \frac{\nu^n}{n!},$$  

(2.1)
with a setwise converging series. In this case, \( e(\nu) \) is indeed a Poissonian measure with associated measure \( \nu \), cf. [11, p. 288]. A measure \( \nu \in \mathcal{M}^1(E) \) is called Gaussian if for any \( a \in E' \) the measure \( \varphi \circ a^{-1} \) is Gaussian. A measure \( \mu \in \mathcal{M}^1(E) \) is called infinitely divisible if for every \( n \in \mathbb{N} \) there exists a measure \( \mu_n \in \mathcal{M}^1(E) \) such that \( \mu = \mu_n \). Gaussian and Poisson measures are infinitely divisible. The set \( \mathcal{I}(E) \) of infinitely divisible measures on \( E \) is closed in \( \mathcal{M}^1(E) \) in the weak topology, cf. [11, Korollar 1.10].

A set \( M \subseteq \mathcal{M}^b(E) \) of finite measures on \( E \) is uniformly tight if \( \sup_{\mu \in M} \mu(E) < \infty \) and if for all \( \varepsilon > 0 \) there exists a compact set \( K \subseteq E \) such that \( \mu(K^c) < \varepsilon \) for all \( \mu \in M \). The set \( M \) is called shift tight, if for every \( \mu \in M \) there exists an \( x_\mu \in E \) such that the family \( (\mu * \delta_{x_\mu})_{\mu \in M} \) is uniformly tight.

The following definition is from [11, 38].

**Definition 2.2.** A measure \( \nu \in \mathcal{M}(E) \) is called Lévy measure if it satisfies

1. \( \nu(\{0\}) = 0 \);
2. there exists an upwards directed set of finite measures \( (\nu_i)_{i \in I} \) with \( \nu_i \leq \nu \) and \( \sup_i \nu_i = \nu \) (setwise) and such that the family of Poisson measures \( \{e(\nu_i)\}_{i \in I} \) is shift tight.

If \( E \) is a separable Banach space, this definition coincides with the following characterisation of a Lévy measure: \( \nu(\{0\}) = 0 \), \( \nu(B_{\delta}^c) < \infty \) for all \( \delta > 0 \) and where \( B_{\delta} \) is the ball with radius \( \delta \), and for each positive sequence \( \delta_n \searrow 0 \), the set \( \{e(\nu_b), n \in \mathbb{N}\} \) is shift tight, cf. [19, Theorem 3.4.9]. The above definition is thus indeed an extension of well-known concepts.

There exists a Lévy-Khintchine-decomposition of infinitely divisible measures:

**Theorem 2.3 (Dettweiler, [11, Satz 2.5]).** For \( \mu \in \mathcal{I}(E) \) there exist \( \gamma \in E \), a linear symmetric and positive definite operator \( Q : E' \to E \), an absolutely convex and compact set \( K \subseteq E \) and a Lévy measure \( \nu \) such that the characteristic function of \( \mu \) has the form

\[
\hat{\mu}(a) = \exp \left( i\langle \gamma, a \rangle - \frac{1}{2} \langle Qa, a \rangle + \int_E (e^{i\langle x, a \rangle} - 1 - i\langle x, a \rangle \mathbb{1}_K(x))d\nu(x) \right)
\]

for every \( a \in E' \). \( \nu \) and \( Q \) are uniquely determined by \( \mu \), \( \gamma \) is unique after the choice of \( K \).

Conversely, every measure \( \mu \in \mathcal{M}^1(E) \) with a Fourier transform of this type is infinitely divisible.

One says that \( \mu \) has characteristics \( (\gamma, Q, \nu, K) \) if \( \mu \) admits the above decomposition. The covariance operator is symmetric in the sense that \( \langle Qa, b \rangle = \langle Qb, a \rangle \) for \( a, b \in E' \).

**Lemma 2.4 (Dettweiler [11], Lemma 1.5/Proof of Satz 2.5).** Let \( \mu \in \mathcal{I}(E) \). Then there exists a unique Lévy measure \( \nu \) associated to \( \mu \) and some absolutely convex and compact set \( K \) such that \( \nu(K^c) < \infty \).

**Definition 2.5.** Let \( \nu \) be a Lévy measure and \( x_\mu \in E \) be chosen in a way such that the family \( (e(\mu) * \delta_{x_\mu})_{\mu \leq \nu} \) is uniformly tight. The generalised Poisson exponential \( \tilde{e}(\nu) \) is an accumulation point of this family.

The generalised Poisson exponential is unique up to translations [11, p. 288]. We will sometimes use \( \tilde{e}(\nu) \) in relations like

\[
\tilde{e}(\nu) = \tilde{e}(\nu|_K) * e(\nu|_{K^c})
\]

meaning that choosing certain representatives of the generalised Poisson exponential, the equality holds up to a convolution with a Dirac measure on one side.

By Prokhorov’s theorem, cf. [41, Theorem 1.3.6], the mentioned family is weakly relatively compact which justifies the definition as an accumulation point. We will
use the fact that for a Lévy measure, its generalised Poisson exponential $\overline{c}(\nu)$ is infinitely divisible. This follows from $\mathcal{I}(E)$ being closed in $\mathcal{M}^1(E)$.

**Corollary 2.6.** For a measure $\nu \in \mathcal{M}(E)$ the following assertions are equivalent:

1. $\nu$ is a Lévy measure on $E$.
2. There exists a measure $\varepsilon \in \mathcal{M}^1(E)$ such that its Fourier transform equals
   $$\int_E e^{i\langle x,a \rangle} \, d\varepsilon(x) = \exp\left(\int_E \left( e^{i\langle x,a \rangle} - 1 - i\langle x,a \rangle \right) \, d\nu(x) \right), \quad a \in E'$$
   for some absolutely convex and compact set $K \subseteq E$.

**Proof.** ($2 \implies 1$): If $\hat{\varepsilon}$ has the above form, $\varepsilon \in \mathcal{I}(E)$ and $\nu$ is the unique Lévy measure corresponding to $\varepsilon$ by Theorem 2.3.

($1 \implies 2$): If $\nu$ is a Lévy measure, it follows from the proof of the Lévy-Khintchine decomposition that there exists a generalised Poisson exponential $\varepsilon$ with the given Fourier transform (end of point 1 in the proof of Satz 2.5, [11], where this measure is called $\nu_0 \ast e(F_0)$).

3. **LÉVY PROCESSES IN LOCALLY CONVEX SPACES**

An $E$-valued random vector $X$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a measurable map $X : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$. An $E$-valued stochastic process $(X_t)_{t \in T}$ is a collection of $E$-valued random vectors over the parameter space $T = [0, t_{\text{max}}]$ with $t_{\text{max}} > 0$ or $T = [0, \infty)$. If there is no risk of confusion, we will omit the emphasis that a process is $E$-valued and call it simply a stochastic process.

A family $(\mu_t)_{t \in T}$ of probability measures on $(E, \mathcal{B}(E))$ is called a convolution semigroup, if $\mu_{t+s} = \mu_t \ast \mu_s$ for all $s, t, s + t \in T$ and $\mu_0 = \delta_0$. It is weakly continuous, if $\mu_t$ converges to $\delta_0$ for $t \searrow 0$ in the weak topology of measures.

**Definition 3.1 (Lévy processes).** An $E$-valued stochastic process $(X_t)_{t \in T}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with distributions $\mu_t := \mathbb{P}_X_t$, is a Lévy process, if

1. $X_t - X_s$ is independent of $\mathcal{F}_s := \sigma(X_r : r \leq s)$ for any $0 \leq s < t$;
2. the distributions of $X_{t+s} - X_t$ and $X_s$ are equal for all $t$;
3. $X_0 = 0$ a.s.; and
4. the family $(\mu_t)_{t \in T}$ is a weakly continuous convolution semigroup.

If $E = \mathbb{R}^d$, the above definition yields the notion of a Lévy process in law, cf. [32, Definition 1.6], where (L4) is substituted by the equivalent property (cf. [3, Proposition 1.4.1]) of stochastic continuity.

**Proposition 3.2.** Let $\mu \in \mathcal{I}(E)$.

1. The $n$-th root $\mu_{1/n} \in \mathcal{M}^1(E)$ is unique.
2. Let $a : E' \to \mathbb{C}$ be the characteristic exponent of $\mu$ in Theorem 2.3. Then, $\hat{\mu}(a) = e^{a\eta(t)}, a \in E', t \in T$.
3. There is a unique convolution semigroup $\mu_t := \mu^{*t}$ embedded into $\mu$ such that $\mu_1 = \mu$.

**Proof.** (1): On a locally convex Suslin space, $\mathcal{E}(E) = \mathcal{B}(E)$, therefore $\mu$ is uniquely determined on the generating $\pi$-system of cylindrical sets of the form $C = \{a_1^{-1}, \ldots, a_d^{-1}\}(B)$ with $B \in \mathcal{B}(\mathbb{R}^d)$ and $a_1, \ldots, a_d \in E'$. So, let $\varrho_1$ and $\varrho_2$ be two different $n$-th roots of $\mu$. Then, there exist $a_1, \ldots, a_d \in E'$ such that $\varrho_1 \circ (a_1, \ldots, a_d)^{-1} \neq \varrho_2 \circ (a_1, \ldots, a_d)^{-1}$. These are two different $n$-th roots of $\mu \circ (a_1, \ldots, a_d)^{-1} \in \mathcal{I}(\mathbb{R}^d)$, a contradiction.

(2) and (3): From (1) it follows that $\mu_q := \mu^{*q}$ is unique and definable for all rational $q$ by setting $\mu_q := \mu_{1/q}^{*q}$. For $a \in E'$, the measure $\mu_a^{*q} := \mu^{*q} \circ a^{-1}$ is infinitely divisible, and from the one-dimensional case it follows that its Fourier
transform is $\varphi_\eta^a(u) = e^{\eta u} = e^{\eta(a)}$, where $\eta: E' \to \mathbb{C}$ is the characteristic exponent of $\mu$, i.e., $\mu(a) = e^{\eta(a)}$. This follows from $\nu_\omega$ being a Lévy measure on $\mathbb{R}$ (cf. [19, Theorem 3.4.9]).

For $q \not\in \tau$, the Fourier transforms $\varphi_\eta^a(u) \to \varphi_\eta^a(u)$ for all $u \in \mathbb{R}$, which yields (2). In other words, $\widehat{\mu^{*q}}(a) \to \widehat{\mu^{*q}}(a)$ for all $a \in E'$. The family $\{\mu^{*q}\}_{q \not\in \tau}$ is uniformly tight by [36, Satz 6.4] and therefore weakly relatively compact by Prokhorov’s theorem cf. [41, Theorem I.3.6]. This is sufficient to apply [41, Theorem IV.3.1] and obtain weak convergence of the net $\mu^{*q}$, $q \in \mathbb{Q}$, $q \geq t$ for $q \not\in \tau$. □

The previous proposition allows to keep the characteristics of a Lévy process over time. The following existence theorem is proved in [4] and relies on the results of [35].

**Proposition 3.3.** Let $(\mu_t)_{t \in T}$ be a weakly continuous convolution semigroup on $E$. Then, there exists a Lévy process $(X_t)_{t \in T}$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $E$ and such that $\mathbb{P}_{X_t} = \mu_t$ for all $t \in T$. Furthermore, one can choose a probability space such that the set of càdlàg paths has probability one.

From now on, we assume that an $E$-valued Lévy process $X = (X_t)_{t \in T}$ is always given on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that almost all paths of $X$ are càdlàg.

Let $\Omega_0 := \{\omega \in \Omega: t \mapsto X_t(\omega) \text{ is càdlàg}\}$. For $\omega \in \Omega_0$ we define

$$X_{t^-}(\omega) := \lim_{s \uparrow t} X_s(\omega) \quad \text{and} \quad \Delta X_t(\omega) := X_t(\omega) - X_{t^-}(\omega).$$

For $\omega \in \Omega_0$, one sets $\Delta X_t(\omega) := 0$ for all $t \in T$.

4. Separable Banach subspaces

In this section, suitable conditions on the Lévy measure, allowing a reduction of the small jumps part to a Banach subspace, will be investigated.

For the functional analytic background of the following cf. [23]. A disk in a locally convex space is a bounded and absolutely convex set. We define the sets $B_0(E, \tau)$ resp. $K_0(E, \tau)$ of closed and compact disks, respectively. If there is no risk of confusion we omit the dependency on $E$ or its topology. For $B \in B_0$ (and therefore $K_0$) the linear hull

$$E_B := \bigcup_{n \in \mathbb{N}} n \cdot B$$

is a Banach space with respect to the gauge function $\|x\|_B := \inf\{\rho > 0: x \in \rho \cdot B\}$, $x \in E_B$. By boundedness of $B$ in $E$, the canonical injection $i: E_B \hookrightarrow E$ is continuous (for $B \in K_0$ even a compact mapping).

4.1. Measures on different underlying topologies. We give some lemmas on certain types of measures with respect to different topologies.

**Lemma 4.1.** Let $\tau$ denote the given topology on $E$ and let $\tau'$ be another comparable locally convex Suslin topology on $E$. $\tau$ and $\tau'$ need not be complete. A measure $\varrho$ is a Gaussian measure with respect to $(E, \tau)$ if and only if it is a Gaussian measure with respect to $\tau'$.

**Proof.** Let $\varrho$ be Gaussian on $\mathcal{B}(E_{\tau}) = \mathcal{E}(E, E'_{\tau})$. By [6, Proposition 2.2.10], this is equivalent to the statement that for the map $\psi: X \times X \to X$, defined by $(x, y) \mapsto x \sin \varphi + y \cos \varphi$ it holds that $(\varrho \otimes \varrho)(\psi^{-1}(B)) = \varrho(B)$ for all $B \in \mathcal{B}(E_{\tau}) = \mathcal{B}(E_{\tau'}) = \mathcal{E}(E, E'_{\tau'})$. This means, $\varrho$ is Gaussian on $E_{\tau'}$. □
Lemma 4.2. Let \( \tau \) denote the given topology on \( E \) and let \( \tau' \) be another comparable locally convex Suslin topology on \( E \). \( \tau \) and \( \tau' \) need not be complete. A measure \( \mu \in \mathcal{M}^1(E) \) is a Poisson measure with respect to \( \tau \) if and only if it is a Poisson measure with respect to \( \tau' \).

Proof. As recalled above, the Borel-\( \sigma \)-algebras of both topological spaces coincide and one obtains the result by considering the setwise converging series (2.1) which only depends on the Borel structure. \( \square \)

Lemma 4.3. Let \( \tau \) denote the given topology on \( E \) and let \( \tau' \) be another comparable locally convex Suslin topology on \( E \). A measure \( \nu \) is a Lévy measure with respect to \( (E, \tau) \) if and only if it is a Lévy measure with respect to \( \tau' \).

Proof. Without loss of generality assume \( \tau' \supseteq \tau \). If \( \nu \) is a Lévy measure on \( (E, \tau') \), it follows from shift tightness of the defining family of the Poisson measures in \( (E, \tau') \) and thus \( (E, \tau) \) that it is a Lévy measure on \( (E, \tau) \).

Conversely, if \( \nu \in \mathcal{M}(E) \) is a Lévy measure with respect to \( \tau \), then there exists an infinitely divisible measure \( \mu := \tilde{e}(\nu) \in \mathcal{M}^1(E) \). Now we consider \( \tau' \) as underlying topology. Note that infinite divisibility depends only on the Borel structure. Because \( \mu \) is infinitely divisible, the Lévy-Khintchine decomposition (w.r.t. \( \tau' \)) (Theorem 2.3) provides a unique Lévy measure \( \nu' \). It follows from the converse implication that \( \nu' = \nu \). \( \square \)

4.2. Restriction to a subspace. We begin with some lemmas, introduce the notion of local reducibility and a fundamental system of compact separable Banach disks and present the main result of this section, Theorem 4.9.

Lemma 4.4. (1) Let \( K \) be a Borel-measurable disk. The subspace \( E_K \) is a Suslin space with the induced topology.

(2) Let \( K \) be a closed disk such that \( (E_K, \| \cdot \|_K) \) is a separable Banach space. Then,

\[
\mathcal{E}(E_K) = \mathcal{B}(E_K) = \mathcal{B}(E) \cap E_K = \mathcal{E}(E) \cap E_K
\]

where \( \mathcal{B}(E_K) \) is induced by the norm topology in \( E_K \) and \( \mathcal{E}(E_K) := \mathcal{E}(E_K, E_K') \), where \( E_K' \) denotes the linear and continuous functionals on \( E_K \) with respect to the norm topology.

Proof. (1) follows from \( E_K \) being Borel and every Borel subset of a Suslin space is Suslin [34, Thm. 3, p. 96].

(2): \( E_K \) is a Suslin space due to its separability, thus \( \mathcal{B}(E_K) = \mathcal{E}(E_K) \), the same is true for the induced topology on \( E_K \) which is Suslin by (1). Finally, two comparable Suslin topologies have the same Borel sets by Proposition 2.1. \( \square \)

Lemma 4.5. Let \( M \subseteq \mathcal{M}^1(E) \) be a family of measures with the following properties:

(1) There is a measurable linear subspace \( L \subseteq E \) such that \( \mu(L) = 1 \) for all \( \mu \in M \).

(2) \( M \) is shift tight, i.e. for \( \varepsilon > 0 \) there is a compact set \( K \subseteq E \) such that for all \( \mu \in M \) there exist \( x_\mu \in E \) with

\[
\mu \ast \delta_{x_\mu}(K) > 1 - \varepsilon.
\]

Then, there exist \( x_\mu^0 \in L \) such that

\[
\mu \ast \delta_{x_\mu^0}(2K^\circ) > \mu \ast \delta_{x_\mu^0}(2(K^\circ \cap L)) > 1 - \varepsilon,
\]

where \( K^\circ \) denotes the bipolar of \( K \) (the the closed absolutely convex hull of \( K \)).
We give sufficient conditions and examples for spaces with a fundamental system.

**Proof.** Without loss of generality we can assume that $K$ is absolutely convex as the absolutely convex hull of a compact set in a complete locally convex space is compact, cf. [23, Proposition 6.7.2]. We have that $\mu(L) = 1$ for every $\mu \in M$ and therefore, $\mu * \delta_x(L - x) = 1$. We define

$$K_\mu := (L - x) \cap K, \quad \text{and} \quad \overline{K_\mu} := -(L + x) \cap K$$

and obtain $\mu * \delta_{x}(K_\mu) = \mu * \delta_{x}(K) > 1 - \varepsilon$. We show that there exists a $\pi \in \overline{K_\mu}$ with $x = \pi \in L$. This follows from $1 - \varepsilon < \mu * \delta_{x}(K_\mu) = \mu((x + K_\mu) \cap L)$. So, for every $\mu$, one can find such an element, we call it $\overline{\pi}_\mu \in \overline{K_\mu}$ such that $x_\mu := x - \overline{\pi}_\mu \in L$. The following properties hold for every $\mu \in M$:

1. $\mu * \delta_x - \pi_\mu(K_\mu + \overline{\pi}_\mu) > 1 - \varepsilon$.
2. $K_\mu + \overline{\pi}_\mu \subseteq 2K = K + K$.
3. $K_\mu + \overline{\pi}_\mu \subseteq L$, because $\overline{\pi}_\mu + K_\mu \subseteq \overline{\pi}_\mu - x + L = L$ by definition of $K_\mu$.

In particular, $\mu * \delta_\mu(2K \cap L) \mu * \delta_x(K_\mu + \overline{\pi}_\mu) > 1 - \varepsilon$ for all $\mu \in M$. \qed

**Lemma 4.6.** Let $\mu \in \mathcal{I}(E)$ and $L$ a linear subspace of $E$ with $\mu(L) = 1$. Then, $\mu_{1/n}(L) = 1$. In particular, $\mu_{1/n}$ is infinitely divisible on $L$.

**Proof.** Set $\varrho = \mu_{1/n}$ and assume a set $C \in B(E)$ with $C \cap L = \emptyset$ and $\varrho(C) > 0$. Then, $\varrho^j(C + \ldots + C) \geq \varrho(C)^j > 0$ for every $j \in \mathbb{N}$. We set $C_1 := C$. If $\varrho^j((C_{j-1} + C) \cap L) = 0$ define $C_j := (C_{j-1} + C) \setminus L$ and carry on by induction. If $j = n$ we obtain a contradiction.

If there is a $k \in \{1, \ldots, n\}$ such that $\varrho^k((C_{k-1} + C) \cap L) > 0$ we have that also $\varrho^k(L) > 0$.

Now, let $n = kl + m$ for $m < k$ and some integer $l$. By construction, $C_m \cap L = \emptyset$ and $\varrho^m(C_m) = 0$ holds, in particular, $(C_m + L) \cap L = \emptyset$. We obtain

$$\mu(C_m + L) = \varrho^m * (\varrho^k)^{\ast l}(C_m + L) \geq \varrho^m(C_m)(\varrho^k(L))^l > 0,$$

a contradiction. \qed

**Definition 4.7.**

1. We denote by

$$B_0^c := B_0^c(E) := \{ B \in B_0^c(E) : E_B \text{ separable with respect to } \| \cdot \|_B \},$$

and $K_0^c := K_0^c(E) := B_0^c(E) \cap K_0(E)$. The elements of $B_0^c$ ($K_0^c$) are called (compact) separable Banach disks.

2. If for every $K \in K_0(E)$ there exists $B \in B_0^c(E)$ (resp. $K_0^c$) such that $K \subseteq B$, the system $B_0^c(E)$ (resp. $K_0^c$) is said to be fundamental (for $K_0(E)$). In this case, $E$ is said to possess a fundamental family of separable Banach disks (resp. compact separable Banach disks).

3. A Lévy measure $\nu \in \mathcal{M}(E)$ is called locally reducible if there exists a compact set $K \in K_0^c$ with $\nu(K^c) < \infty$ such that there exists a generalised Poisson exponential of $\nu_K$ with $\nu_K(E_K) = 1$. In this case, $K$ is called $\nu$-reducing. An infinitely divisible distribution is locally reducible if its corresponding Lévy measure has this property.

In Appendix A we give sufficient conditions and examples for spaces with a fundamental system $B_0^c(E)$. Concerning local reducibility, we immediately get:

**Lemma 4.8.** If $\nu$ is locally reducible and $K \in K_0^c$ is $\nu$-reducing, then every $H \in K_0^c$ with $H \supseteq K$ is $\nu$-reducing.

**Theorem 4.9.** Let $\mu \in \mathcal{I}(E)$ and $\nu \in \mathcal{M}(E)$ its corresponding Lévy measure. The following assertions are equivalent.

1. $\mu$ is locally reducible.
2. $\varrho(\nu_K)(E_K) = 1$ for some $K \in K_0^c$ with $\nu(K^c) < \infty$, i.e. $\nu$ is locally reducible.
(3) The function
\[ \varphi(a) = \exp \left( \int_{E_K} e^{i(x,a)} - 1 - i\langle x, a \rangle 1_K(x) \, d\nu(x) \right), \quad a \in E' \]
is a Fourier transform of some probability measure on \( B(E_K) \) for some \( K \in K^0_0(E) \) with \( \nu(K^c) < \infty \).

(4) The restriction of \( \nu|_K \) to the subspace \( (E_K, \| \cdot \|_K) \) is a Lévy measure for some \( K \in K^0_0(E) \) with \( \nu(K^c) < \infty \).

(5) The restriction of \( \nu|_K \) to the subspace \( (E_K, \tau) \) is a Lévy measure for some \( K \in K^0_0(E) \) with \( \nu(K^c) < \infty \).

(6) For some \( K \in K^0_0(E) \) with \( \nu(K^c) < \infty \) the following holds: There are \( x_\varrho \in E \) such that for every \( \varepsilon > 0 \) there exists an \( n \in \mathbb{N} \) such that for all \( \varrho \leq \nu|_K \) one has
\[ e(\varrho) * \delta_{x_\varrho}(n \cdot K) > 1 - \varepsilon. \]

Furthermore, if there is \( K \in K^0_0(E) \) such that one of the assertions (2)–(6) holds, one can take the same set also in all other assertions.

Proof. (1) \( \Leftrightarrow \) (2) holds by definition, and (4) \( \Leftrightarrow \) (5) follows by taking a suitable compact set \( K \) by Lemma 4.3.

(3) \( \Rightarrow \) (4): In a Banach space, \( \nu|_{E_K} \) is a Lévy measure if and only if \( \varphi \) is the characteristic functional of a some probability measure, cf. [19, Theorem 3.4.9], where \( \varphi \) is evaluated in all \( a \in E_K' \). The set \( E'|_{E_K} \) separates the points of \( E_K \) and therefore, \( \varphi \) uniquely determines a measure on \( B(E_K) = \mathcal{E}(E_K, E'|_{E_K}) \). This implies that \( \nu|_{E_K} \) and hence \( \nu|_{E_K} \) is a Lévy measure on \( E_K \), assertion (4).

(4) \( \Rightarrow \) (3): Again using [19, Theorem 3.4.9], it suffices to note \( E'|_{E_K} \subseteq E_K' \) and \( \nu(K^c) < \infty \).

(2) \( \Rightarrow \) (3): By assumption, \( \tilde{c}(\nu|_K)(E \setminus E_K) = 0 \) and \( \tilde{c}(\nu|_K) \|_{E_K} \) have the same Fourier transform if we use the set of continuous functionals \( E' \) generating \( B(E) \) and \( B(E_K) \). One obtains
\[ \tilde{c}(\nu|_K) \|_{E_K}(a) = \exp \left( \int_{E_K} e^{i(x,a)} - 1 - i\langle x, a \rangle 1_K(x) \, d\nu(x) \right), \quad a \in E'|_{E_K} \]
which yields (3) if one takes the convolution of this measure with the Poisson measure of \( (\nu|_{E_K \setminus E_K}) \|_{E_K} \in M^0(E_K) \).

(4) \( \Rightarrow \) (6): If \( \nu' := (\nu|_K) \|_{E_K} \) is a Lévy measure on \( E_K \), there exist \( x_\varrho \in E_K \subseteq E \) such that the family \( (e(\varrho) * \delta_{x_\varrho})_{\varrho \leq \nu'} \) is uniformly tight, i.e., for every \( \varepsilon \) there exists a \( \| \cdot \|_K \)-norm compact set \( H_\varepsilon \) with \( e(\varrho) * \delta_{x_\varrho}(H_\varepsilon) > 1 - \varepsilon \) for all \( \varrho \leq \nu' \). As every compact set is bounded, there exists an \( n(\varepsilon) \in \mathbb{N} \) with \( H_\varepsilon \subseteq n(\varepsilon) \cdot K \) which implies assertion (6).

(6) \( \Rightarrow \) (2): If \( \nu \) is finite, \( e(\nu)(E \setminus E_K) = 0 \) because \( \nu^n(E \setminus n \cdot K) = 0 \), due to \( n \cdot K \subseteq E_K \). So let \( \nu \) be a Lévy measure which is not finite. The measure \( \nu|_K \) is a Lévy measure on \( E \). By assumption, there exist \( x_\varrho \in E \) such that the family of shifted Poisson measures satisfies the \( n \cdot K \)-tightness condition in (6). By Lemma 4.5, the shifts \( x_\varrho \) can be taken in \( E_K \) without loss of generality. As \( \tilde{c}(\nu|_K) \) is an accumulation point of the family \( (e(\varrho) * \delta_{x_\varrho})_{\varrho \leq \nu'} \), for all \( \varepsilon > 0 \), all \( f \in C_b(E) \) and all \( \varrho_0 \) there exists a measure \( \varrho \geq \varrho_0 \) such that
\[ |\tilde{c}(\nu|_K)(f) - e(\varrho) * \delta_{x_\varrho}(f)| < \varepsilon. \]

Assuming that there exists a Borel set \( B \subseteq E \setminus E_K \) with positive measure \( \tilde{c}(\nu|_K) \), we find also compact set \( C \subseteq B \) with positive measure. There exists a continuous function \( g \colon E \to [0, 1] \) such that \( g = 0 \) on the closed set \( n \cdot K \) and \( g = 1 \) on the compact set \( C \) due to complete regularity of \( E \). We have \( \tilde{c}(\nu|_K)(g) \geq \tilde{c}(\nu|_K)(C) > 0 \). Furthermore, we see that \( e(\varrho) * \delta_{x_\varrho}(E_K \setminus n \cdot K) \leq \varepsilon \) for \( n \) large enough which can
be chosen independently of $g$ and $x_0$ by assumption. But for all $\varepsilon > 0$ one finds a $g$ and $n \in \mathbb{N}$ such that
\[
|\tilde{c}(\nu|_K)(g)| = |\tilde{c}(\nu|_K)(g) - e(g) \ast \delta_{x_0}(g) + e(g) \ast \delta_{x_0}(g)| \\
\leq |\tilde{c}(\nu|_K)(g) - e(g) \ast \delta_{x_0}(g)| + |e(g) \ast \delta_{x_0}(g)| \leq \varepsilon + \varepsilon
\]
for $g$ constructed as above. This is a contradiction to the claim that the generalised Poisson measure of $E \setminus E_K$ was positive and the proof is complete. \hfill \Box

The null extension of $\mu \in \mathcal{M}(E_K)$ to $E$ is defined by $\mu^0(B) := \mu(B \cap E_K)$ for $B \in \mathcal{B}(E)$. By the previous lemmas we can identify Gaussian, Poissonian and infinitely divisible measures on $E_K$ and $E$ by restriction resp. extension provided that $E \setminus E_K$ has measure zero and $K \in \mathcal{B}_0^2(E)$. Furthermore, we can identify Lévy measures on $E_K$ and $E$ if there is a $\nu$-reducing set $K \in \mathcal{K}_0^2$ by Theorems 4.9.

A locally reducible Lévy measure has a simpler characterisation than in Corollary 2.6:

**Proposition 4.10.** $K \in \mathcal{K}_0^2(E)$ is $\nu$-reducing for a Lévy measure $\nu \in \mathcal{M}(E)$ if and only if the following is satisfied:

(i) $\nu(\{0\}) = 0$.
(ii) $\nu|_{(\alpha K)^c} \in \mathcal{M}^b(E)$ for some (all) $\alpha > 0$, and
(iii) the family $\{e(\nu|_{K \setminus nK})\} \in E_K : n \in \mathbb{N}\}$ is shift tight (w.r.t. $E_K$) for every (some) positive null sequence $(\delta_n)_n$.

**Proof.** $\Longrightarrow$: If $K$ is $\nu$-reducing, $(\nu|_K)\|_{E_K}$ is a Lévy measure on the separable Banach space $E_K$ by Theorem 4.9. By [19, Proposition 3.4.9] and Prokhorov’s theorem, we have that $(\nu|_K)\|_{E_K}$ satisfies (i) $(\nu|_K)\|_{E_K}(\{0\}) = 0$, (ii) $(\nu|_K)\|_{E_K}(\delta K) < \infty$ and (iii) for all (some) null sequences $\delta := (\delta_n)_n$ the set $M(\delta, K)$ is shift tight w.r.t. $E_K$. The assertion follows from taking null extensions to the original space and $\nu(K^c) < \infty$.

$\Longleftarrow$: Let $\nu$ satisfy (i)-(iii), then $(\nu|_K)\|_{E_K}$ is a Lévy measure on $E_K$ by [19, Proposition 3.4.9]. In particular, $M(\delta, K) := \{e(\nu|_{K \setminus nK})\} \in E_K$ and, a fortiori w.r.t. $(E_K, \tau)$ thus $E$. We note that $M' := M(\delta, K)^0 \ast (\nu|_K)$ (elementwise convolution) is shift tight in $E$: If for $\varepsilon \in (0, 1)$ the set $K_{\varepsilon/2} \subseteq K_0(E_K) \subseteq K_0(E)$ satisfies $\mu^0(E \setminus K_{\varepsilon/2}) \leq 2\mu(E \setminus K_{\varepsilon/2}) < \varepsilon/2$ for all $\mu \in M$, and $H_{\varepsilon/2} \subseteq K_0(E)$ satisfies $e(\nu|_{K_{\varepsilon/2}})(H_{\varepsilon/2}) < \varepsilon/2$, then
\[
\mu \ast (\nu|_{K_{\varepsilon}})((K_{\varepsilon/2} + H_{\varepsilon/2})^c) < 1 - \mu(K_{\varepsilon/2})e(\nu|_{K_{\varepsilon}})(H_{\varepsilon/2}) < \varepsilon.
\]
Furthermore, $\nu = \sup_{\mu \in M, \mu \in \mathcal{M}(E)}$ which proves that $\nu$ is a Lévy measure on $E$. The first remark gives local reducibility of $\nu$ to $E_K$ with reducing set $K$. \hfill \Box

### 4.3. Zero-One Laws and Reducibility
We show that in spaces with a fundamental system of separable Banach disks, every Lévy measure is locally reducible. For convenience, we quote a key result of Janssen.

**Theorem 4.11 (Janssen, [22, Theorem 9]).** Let $\nu \in \mathcal{M}(E)$ be a Lévy measure and $\mu := \tilde{c}(\nu)$ its generalised Poisson exponential. If $H$ is a measurable linear subspace of $E$ with $\nu(H^c) = 0$ and $x \in E$, then $\mu(x + H) \subseteq \{0, 1\}$.

The following simple fact can be straightforwardly checked.

**Lemma 4.12.** If $K_1, K_2 \in \mathcal{K}_0^2(E)$ then $K_1 + K_2 \in \mathcal{K}_0^2(E)$.

**Proposition 4.13.** If $E$ admits a fundamental system of compact separable Banach disks one has:

1. For every generalised Poisson measure $\mu$ with Lévy measure $\nu$ satisfying $\nu(K_0^c) = 0$, $K_0 \in \mathcal{K}_0^2(E)$, there exists $K \in \mathcal{K}_0^2(E)$ such that $\mu(E_K) = 1$.
2. Every infinitely divisible measure $\mu \in \mathcal{I}(E)$ is locally reducible.
Proof. (1) Let $\mu = \overline{e}(\nu)$. As $E$ has a fundamental system of compact separable Banach disks and $\mu$ is tight, we have that there exists $K_1 \in K_0^c(E)$ with $\mu(E_{K_1}) \geq \mu(K_1) > 0$. Furthermore, there exists a set $K_2 \in K_0^c(E)$ with $K_0 \subseteq K_2$ such that $\nu(K_2) = 0$. We set $K := K_1 + K_2 \in K_0^c(E)$ by Lemma 4.12. Then, we have that $\nu(E_K) \leq \nu(K_2) = 0$ and $\mu(E_K) \geq \mu(K_1) > 0$. Theorem 4.11 implies that $\mu(E_K) = 1$. In other words, $\mu$ is locally reducible.

(2) Let $\mu$ be infinitely divisible with Lévy measure $\nu$. According to (1), there exists $K_0 \in K_0(E)$ such that $\nu|_{K_0^c}$ is a finite measure, and a set $K \supseteq K_0$ with $K \in K_0^c(E)$ such that $\overline{e}(\nu|_{K_0})(E_K) = 1$. Noting that

$$\overline{e}(\nu|_{K_0})(E_K) = e(\nu|_{K_0})(E_K + E_K) \geq e(\nu|_{K_0})(E_K) \overline{e}(\nu|_{K_0})(E_K) = 1$$

we obtain the assertion.

\[ \square \]

**Theorem 4.14.** If $E$ admits a fundamental system of separable Banach disks, every infinitely divisible measure $\mu \in \mathcal{I}(E)$ is locally reducible.

Let $\nu$ be the Lévy measure of $\mu$ and $K \in K_0(E)$ with $\nu(K) < \infty$. Without loss of generality assume that $\mu = \overline{e}(\nu)$. As $K_1 + K_2 \in K_0(E)$ for $K_1, K_2 \in K_0$ by continuity of addition, we can assume that $\overline{e}(\nu|_{K_0})(E_K) > 0$ as in the proof of Proposition 4.13, (2). Let now $B \in \mathcal{B}_0^c(E)$ with $B \supseteq K$ which exists by assumption. We obtain that $\mu_B := \overline{e}(\nu|_B)(E_B) > 0$ and $\nu(B^c) < \infty$. In particular, $\mu_B(E_B) = 1$ and $\mu_B(E_B) \in \mathcal{I}(E_B)$ by Theorem 4.11 and Lemma 4.6 with Lévy measure $\nu_B = \nu|_B$ by uniqueness of the Lévy measure. As in the separable Banach space $E_B$ there is a fundamental family of $\| \cdot \|_B$-compact separable Banach disks $K_0^c(E_B)$, we obtain that $\mu_B$ is locally reducible on $E_B$, i.e. there exists $H \in K_0^c(E_B)$ with $\overline{e}(\nu_B|_H)(E_B) = 1$ and $\nu_B(E_B \setminus H) < \infty$. The assertion follows by $H \in K_0^c(E_B) \subseteq K_0^c(E)$ and noting that $\mu = \mu_B * e(\nu|_B^c)$.

The theorem above states in particular that this property depends only on the duality $(E, E^\prime)$.

**Example 4.15.** The following complete locally convex Suslin spaces have a fundamental system of compact separable Banach disks: All separable Fréchet and Banach spaces, furthermore, the well-known spaces of test functions and distributions $\mathcal{D}, \mathcal{D}^\prime, E, E^\prime, O_M, O^\prime_M, O_C, O^\prime_C, S, S^\prime$, cf. [34], pp. 115-177 and 233] and Appendix A. A separable Banach or Fréchet space with the weak topology satisfies the condition of Theorem 4.14 as the property of sets being bounded only depends on duality. Furthermore, finite direct sums and closed subspaces of spaces with a fundamental system of (compact) separable Banach disks share the same property, cf. Appendix A.

Open questions. If $E$ has a fundamental system of separable Banach disks, then all $\mu \in \mathcal{I}(E)$ are locally reducible. However, the converse implication is not known.

We assume for the rest of the paper that $\mathbb{P}_{X_1} = \mu_1$ is locally reducible.

5. **Cádlág functions in Suslin spaces**

Before we begin our investigations on random measures we present some results for càdlág functions with values in locally convex Suslin spaces. The results are similar to those in [5, Chapter 3]. But due to the possibly uncountable neighbourhood bases, they are not standard.

We denote by $\mathcal{D}(T; E)$ the space of càdlág functions $\xi: T \rightarrow E$. This space will always be endowed with the $\sigma$-algebra $\mathcal{F}_T$ of cylinder sets on $\mathcal{D}(T; E)$ which are generated by the coordinate functions $x_t: \mathcal{D}(T; E) \rightarrow E$ defined by $x_t(\xi) := \xi(t)$, $t \in T$. For $t \in T$ the left limit mapping $x_{t^-}: \mathcal{D}(T; E) \rightarrow E$ is defined by $x_{t^-}(\xi) := \lim_{s \searrow t} x_s(\xi)$ and the jump function by $\Delta \xi_t := x_t(\xi) - x_{t^-}(\xi)$, $t \in T$. 

Given a dense subset \( T_0 \subseteq T \), càdlàg functions are defined as follows: \( \xi : T_0 \to E \) is càdlàg if and only if for all increasing or decreasing Cauchy sequences \((t_n)_{n \in \mathbb{N}}\) in \( T_0 \) the limits of \( \xi(t_n) \) exist and if \( t_n \searrow t \in T_0 \) the limit equals \( \xi(t) \). As above, \( \mathcal{D}(T_0; E) \) denotes the set of such càdlàg functions. \( \xi \in \mathcal{D}(T_0; E) \) is said to have a jump in \( s \in T \), if the limits
\[
y_s := \lim_{r \uparrow s} \xi(r) \quad \text{and} \quad y_s := \lim_{r \downarrow s} \xi(r)
\]
are different. \( \Delta \xi(s) := y_s - y_{s-} \) is the jump size in \( s \) and \( s \mapsto \Delta \xi(s) \) is the jump function corresponding to \( \xi \).

**Lemma 5.1.** Let \( T_0 \subseteq [0, t_{\text{max}}] \) be dense. For a càdlàg function \( \xi \in \mathcal{D}(T_0; E) \), a continuous seminorm \( p \) and \( \varepsilon > 0 \) there exist finitely many points \( 0 = t_0 < t_1 < \ldots < t_n = t_{\text{max}} \) such that
\[
(5.1) \quad \sup \{ p(\xi(t) - \xi(s)) : s, t \in [t_{i-1}, t_i) \cap T_0 \} < \varepsilon, \quad i = 1, \ldots, n.
\]
Furthermore, for any weaker metric \( d \) on \( E \), there are finitely many points with
\[
\sup \{ d(\xi(t), \xi(s)) : s, t \in [t_{i-1}, t_i) \cap T_0 \} < \varepsilon, \quad i = 1, \ldots, n.
\]

**Proof.** Let \( \pi : E \to E/p^{-1}(\{0\}) \) be the canonical projection associated to \( p \) which is continuous. Furthermore, \( \pi \circ \xi \in \mathcal{D}(T; E/p^{-1}(\{0\})) \) and all the limits exist by completeness of \( E \). The target space is normed, but not necessarily complete. The expression in (5.1) remains the same if the càdlàg function is projected onto the quotient space. Exactly as in [5, Chapter 3, Lemma 1, p. 110], where completeness is not needed, one obtains the first assertion.

If \( d \) is a continuous metric on \( E \), \( \text{id} : (E, \tau) \to (E, d) \) is continuous and by the same reasoning as above the assertion follows. \( \square \)

**Lemma 5.2.** A càdlàg function with values in a complete locally convex Suslin space has at most countably many jumps.

**Proof.** Let \( (E, \tau) \) be the Suslin space with the original topology and \( (E, d) \) a metric space, where \( d \) is a weaker metric. Let \( \text{id} : (E, \tau) \to (E, d) \) be the continuous identity map. If \( \xi \in \mathcal{D}(T_0; E) \) has a jump in \( t_0 \in T \) then \( \text{id} \circ \xi \) has a jump in \( t_0 \). Thus, \( \text{id} \circ \xi \) has at least as many jumps as \( \xi \) and is càdlàg as well by continuity of the identity map. Furthermore, if \( \xi \) is continuous in \( t_0 \), this carries over to \( \text{id} \circ \xi \), thus the jumps of \( \xi \) and \( \text{id} \circ \xi \) are the same. But for càdlàg functions with values in metric spaces it is well-known that there are at most countably many jumps [14, Lemma 4.5.1]. \( \square \)

Given \( \xi \in \mathcal{D}(T; E) \) one can number the jumps of \( \xi \) in the following way: Choose the metric \( d \) of the proof above and measure the jumps as above by \( d(\xi_t, \xi_{t-}) \). Then, by Lemma 5.1, there are only finitely many jumps on bounded intervals if \( d(\xi_t, \xi_{t-}) \) is larger than 1. Therefore, one can denote these jump times by \( t_{1,1}(\xi), t_{1,2}(\xi), \ldots \) (setting \( t_{1,k+1}(\xi) = t_{1,k+2}(\xi) = \ldots := \infty \) if there are \( k \) jumps larger than 1) and obtain thus a numbering. If \( d(\xi_{t_i}, \xi_{t_{i-}}) \in (\frac{1}{n+1}, \frac{1}{n}] \) for some \( n \geq 1 \), we can do the same procedure using \( t_{n,1}(\xi), t_{n,2}(\xi), \ldots \) and so on.

**Lemma 5.3.** Given \( n \) and \( j \), the map \( t_{n,j} : \mathcal{D}(T; E) \to T \cup \{\infty\} \) is \( \mathcal{F}_T \)-measurable.

The proof is exactly the same as in [32, Proof of Lemma 20.9] replacing the modulus by \( d \) which is a continuous metric on \( E \).

**Lemma 5.4.** For \( t \in T, t > 0 \), the left limit mapping \( x_{t-} : \mathcal{D}(T; E) \to E \) is \( \mathcal{F}_T - \mathcal{B}(E) \)-measurable.
Proof. Let $d$ be a weaker metric on $E$. We note that $B(E,\tau) = B(E,d)$ and the Borel sets are generated by open $d$-balls $B_d(\varepsilon) := \{y \in E : d(x,y) < \varepsilon\}$, $x \in E$ due to $(E,d)$ being a separable metric space and therefore second countable. Furthermore, the $\sigma$-algebra $\mathcal{F}_D$ on $D(T; (E,\tau))$ only depends on $B(E)$ and not the precise topology generating it. We define $x_{t-}^d(\xi) := d - \lim_{s \uparrow t} x_s(\xi)$ for $\xi \in D(T; (E,\tau))$ and show that $x_{t-}^d(\xi) = x_{t-}(\xi)$. But this follows from $(E,d)$ being Hausdorff and the continuity of the injection $(E,\tau) \hookrightarrow (E,d)$ and therefore

$$x_{t-}(\xi) = \lim_{s \uparrow t} x_s(\xi) = d - \lim_{s \uparrow t} x_s(\xi) = x_{t-}^d(\xi) \quad \text{for } \xi \in D(T; (E,\tau)).$$

We test measurability on a generator of $B(E)$ and obtain

$$(x_{t-})^{-1}(B_x(\varepsilon)) = (x_{t-}^d)^{-1}(B_x(\varepsilon)),$$

$$= \{\xi \in D(T; (E,\tau)) : \exists n \in \mathbb{N} \forall q \in \mathbb{Q} \cap (0,1/n): x_{t-q}(\xi) \in B_x(\varepsilon)\} \in \mathcal{F}_D,$$

which proves the lemma. □

**Lemma 5.5.** The maps $x : D(T; E) \times T \to E$ defined by $(\xi,t) \mapsto x_t(\xi)$ and $\pi : D(T; E) \times T \to E$ defined by $(\xi,t) \mapsto x_{t-}(\xi)$ are $\mathcal{F}_D \otimes \mathcal{B}(T) - B(E)$-measurable.

**Proof.** For all $t \in T$ the map $x_t : D(T; E) \to E$ is $\mathcal{F}_D - B(E)$-measurable. Using right-continuity of $x$ in the $d$-topology we obtain that the $\mathcal{F}_D \otimes \mathcal{B}(T)$-measurable maps

$$x_s^{(n)}(\xi) := \sum_{k=0}^{2^n-1} x_{\max\{k+1/2^n, t\}}(\xi) \mathbb{I}_{(\frac{k}{2^n}, \frac{k+1}{2^n}]}(s), \quad \text{if } s \in T = [0, t_{\max}]$$

or

$$x_s^{(n)}(\xi) := \sum_{k=0}^{4^n-1} x_{\max\{k+1/2^n, t\}}(\xi) \mathbb{I}_{(\frac{k}{4^n}, \frac{k+1}{4^n}]}(s), \quad \text{if } s \in T = [0, \infty),$$

respectively, converge to $x_s(\xi)$ in the $\tau$- and the $d$-topology as $n$ tends to infinity. The latter convergence yields the desired measurability of the joint map.

For $\pi$ the proof is exactly the same taking the left endpoints $x_{t_{\max}k2^{-n}}$ resp. $x_{k2^{-n}}$ in the approximating sums. □

6. **Wiener processes in locally convex spaces**

We consider Gaussian random variables with values in a locally convex space $E$. Probably the most comprehensive monograph concerning Gaussian measures on locally convex topological vector spaces is [6], the following concepts are taken mainly from chapters 2 and 3. See also e.g. [5, 29]. A random variable $X : \Omega \to E$ is called Gaussian, if for all $a \in E'$ the real valued random variables $(X,a)$ are Gaussian. Let $\varrho$ be a Gaussian measure. A mapping $q : E \to \mathbb{R}_+$ is a $\varrho$-measurable seminorm if there exists a $B(E)$-measurable linear subspace $E_0 \subseteq E$ with $\varrho(E_0) = 1$ and such that the restriction $q|_{E_0}$ is a seminorm on $E_0$. Obviously, $\| \cdot \|_K$ is a $\varrho$-measurable seminorm if $\varrho(E_K) = 1$.

**Lemma 6.1 (Zero-one law for Gaussian measures, [6, Theorem 2.5.5]).** Let $E_0 \subseteq E$ be a $B(E)$-measurable affine subspace and $\varrho$ a Gaussian measure. Then, $\varrho(E_0) \in \{0,1\}$.

For later use, we formulate a proposition about Gaussian measures which can be reduced to a separable Banach subspace.

**Proposition 6.2.** Let $\varrho$ be a centered Gaussian measure on a locally convex Suslin space $E$. If there exists a set $K \in K_0^*(E)$ of positive measure, one has

1. $\varrho(E_K) = 1$,
2. $\varrho$ is a Gaussian measure on $E_K$ and
there exists $\alpha > 0$ such that $\int_{E_K} e^{\alpha \|x\|} dx < \infty$. 

In particular, if $E$ has a fundamental system of compact separable Banach disks, every Gaussian measure on $E$ has a Banach support.

The assertions follow from the above cited zero-one law, Lemma 4.1 and by Fernique’s theorem for measurable seminorms, [6, Theorem 2.8.5]. The last assertion guarantees the existence of all moments of Gaussian random variables, whenever there exists a compact set of positive measure. But this follows from $E$ being a Suslin, thus a Radon space.

Let $w_t$ be a real-valued standard Brownian motion, $\sigma \geq 0$ and $\gamma \in \mathbb{R}$. We consider every real-valued process of the form $y_t = \sigma w_t + \gamma t$ a one-dimensional Brownian motion.

**Definition 6.3.** A continuous Lévy process with values in $E$ is called **Wiener process**.

We provide the following theorem characterising Wiener processes in $E$.

**Theorem 6.4.** For a stochastic process $(W_t)_t$ with values in $E$ the following assertions are equivalent:

1. $(W_t)_t$ is a Wiener process.
2. $(W_t)_t$ is a continuous process with independent increments such that $\langle W_t, a_t \rangle_t$ is a (possibly degenerate) one-dimensional Brownian motion for all $a \in E'$.
3. $(W_t)_t$ is a càdlàg Lévy process with Gaussian distributed increments.
4. $(W_t)_t$ is a càdlàg Gaussian process determined by the mean $\gamma \in E$ (i.e. $\gamma$ satisfies $\mathbb{E}(W_t, a) = \langle \gamma, a \rangle$ for all $a \in E'$) and a symmetric, positive semidefinite operator $Q \in \mathcal{L}(E', E)$, where $E'$ is equipped with the Mackey topology $\tau_\mu(E', E)$, such that

$\mathbb{E}(W_t - \gamma t, a)(W_s - \gamma s, b) = \langle Qa, b \rangle \min\{s, t\}$

for all $s, t \in T$ and $a, b \in E'$.

**Proof.**

(1) $\implies$ (2): If $W$ is a continuous Lévy process, the same holds for $(W, a)$ and it is well-known that a continuous Lévy process in $\mathbb{R}$ is a Brownian motion with drift of the form $\sigma_a w_t + \gamma_a t$ with $\sigma_a \geq 0$ and $\gamma_a \in \mathbb{R}$ and $w = (w_t)_t$ is a real-valued standard Brownian motion.

(2) $\implies$ (3): If $W$ is continuous, it is càdlàg. For the stationary increments property we note that the distribution of $W_t - W_s$ equals the distribution of $W_{t-s}$ on the $\pi$-system of cylindrical sets, thus on $\mathcal{E}(E) = \mathcal{B}(E)$. Furthermore, for all $a \in E'$ we have that $\mathbb{P}_{(W_t, a)}$ is real-valued Gaussian which means that $W_t$ is Gaussian distributed for all $t$. Gaussian distributed increments follow from stationarity.

(3) $\implies$ (4) If $W_t$ obeys a Gaussian law, there exists an element $\gamma_t \in E$ and a continuous operator $Q_t : E' \to E$ (where $E'$ is equipped with the Mackey topology) such that $\langle \gamma_t, a \rangle = \mathbb{E}(W_t, a)$ and $\langle Q_t a, b \rangle = \mathbb{E}(W_t - \gamma t, a)(W_t - \gamma t, b)$ for all $a, b \in E'$ by cf. [6, Lemma 3.2.1].

For simplicity, assume for the following that $\gamma = 0$. The independent increments property yields

$\mathbb{E}(W_t, a)(W_s, b) = \mathbb{E}(W_t - W_s, a)(W_s, b) + \mathbb{E}(W_s, a)(W_t, b) = \langle Q_s a, b \rangle$

and one obtains the equality

$\langle Q_t a, b \rangle = n(Q_{t/n} a, b)$

for every $n \in \mathbb{N}$ and $t \in T$ by writing $W_t$ as a telescoping sum over an equidistant time net and using independent and stationary increments. Finally, one obtains
$Q_t = tQ_1 = tQ$ by \( W_t \) being càdlàg and using

\[
\langle Q_t, a \rangle = \lim_{s \to t} \mathbb{E}(W_s, a) = \lim_{s \to t} s(Q, a) = t(Q, a)
\]

and polarisation. Similarly one obtains $\gamma_t = t\gamma_1 = t\gamma$.

(4) $\implies$ (3): It suffices to check independent and stationary increments of the given process, but this follows immediately by the covariance structure of the occurring Gaussian vectors.

(3) $\implies$ (1): In the sequel, let

\[
S_Q := \begin{cases} \{S \cap Q \} \cup \{ \max S \} & \text{max } S \text{ exists,} \\ S \cap Q & \text{else} \end{cases}
\]

for some interval $S \subseteq [0, \infty)$. Let $\varrho_t := \mathbb{P}_{W_t}$ and $H \subseteq E$ be an absolutely convex compact set with $\varrho_1(H) > 0$. Without loss of generality we assume that $\mathbb{E}W_1 = 0$ for all $t$, i.e. all Gaussian distributions are centered. Then, the zero-one law for Gaussian measures, cf. [6, Theorem 2.5.5.], implies that $\varrho_1(E_H) = 1$, where $E_H$ is the linear hull of $H$. Furthermore, for every $m \in \mathbb{N}$, one has

\[
\varrho_m(E_H) = \varrho_1^{*m}(E_H) = \varrho_1^{*m}(E_H + \ldots + E_H) \geq \varrho(E_H)^m \geq 1
\]

which implies together with Lemma 6.5 below that $\varrho_1(E_H) = 1$ for all $t \in T$. The compact sets $n \cdot H \subseteq E$ are metrizable, [6, Propositions A.1.7 and A.3.16] and we denote by $d_n$ a metric inducing the same topology on $n \cdot H$.

We take $T = [0, t_0], t_0 \geq 0$. If the claim holds on all bounded intervals, it is true on $\mathbb{R}$. First, let us prove that $\mathbb{P}(W_t \in E_H, t \in T) = 1$. As $\mathbb{P}(W_t(E_H) = \varrho_t(E_H) = 1$ for every $t \in T$, one has $\mathbb{P}(W_t \in E_H : t \in T_Q) = 1$. For an $\omega$ in this set of full measure we consider the trajectory $t \mapsto W_t(\omega)$. We follow [6, Proof of Proposition 7.2.3] and show that there exists an $n(\omega) \in \mathbb{N}$ such that $\{W_t(\omega) : t \in T_Q\} \subseteq n(\omega) \cdot H$.

To this end, we define the (possibly infinity-valued) process $\eta_k := ||W_t||_H$. Let $d$ be a metric on $E$ inducing a weaker topology. Defining the continuous function $\text{id} : (E, \tau) \to (E, d)$, the set $Q := \text{id}(H)$ is compact and absolutely convex in $(E, d)$.

The measurable seminorm $|| \cdot ||_Q$ on $(E, d)$ satisfies $|| \cdot ||_Q = \sup_{a \in H} a$, with $a \in (E, d)' \subseteq E'$, cf. [6, Problem A.3.27]. Furthermore, $||x||_H = ||\text{id}(x)||_Q$ for all $x \in E$. As in [6, Proof of Proposition 7.2.3] one deduces from $a_t(W_t)$ being a real-valued martingale that $\eta$ is a submartingale.

Due to Doob’s inequality and Fernique’s theorem (cf. [6, Theorem 2.8.5]) one has

\[
\mathbb{E} \sup_{t \in T_Q} \eta_t^2 \leq 2\mathbb{E} \eta_0^2 = 2 \int_E ||x||_H^2 d\varrho_0(x) < \infty
\]

which implies for $\omega \in \Omega_0$, a set of measure one, that $\sup_{t \in T_Q} ||W_t(\omega)||_H < \infty$. But this yields the existence of an $n(\omega) \in \mathbb{N}$ with $W_t(\omega) \in n(\omega) \cdot H$ for all $t \in T_Q$. The limit

\[
W_t(\omega) = \lim_{s \to t} W_s(\omega)
\]

is an element of $n(\omega) \cdot H$ by completeness. It follows that $\{W_t(\omega) : t \in T\} \subseteq n(\omega) \cdot H \subseteq E_H$ for $\omega$ in a set of measure one.

The second part of the proof follows an idea of [17, Proposition A.1]. In [15, Proposition 5] a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ and a process $(X_t)_{t \in T'}$ are constructed such that the latter has the same finite-dimensional distributions as $W$ and continuous trajectories in $E$. As above, it is argued that they are almost surely in $E_H$. 

LÉVY PROCESSES WITH VALUES IN LOCALLY CONVEX SUSLIN SPACES 15
Lemma 6.5. Let \((\varrho_t)_{0 \leq t \leq 1}\) be a semigroup of Gaussian measures on \(E\). Then,

1. \(\varrho_t(H) \geq \varrho_1(H)\) for every measurable absolutely convex set \(H\) and \(t \in [0,1]\).
2. If \(H\) is bounded and \(\varrho_1(H) > 0\), \(\varrho_t(E_H) = 1\) for all \(t\).

Proof. By the semigroup property we have \(\varrho_t \ast \varrho_{1-t} = \varrho_1\). Therefore,

\[
\varrho_1(H) = \varrho_t \ast \varrho_{1-t}(H) = \int_E \int_E 1_H(x+y) \, d\varrho_t(x) \, d\varrho_{1-t}(y) = \int_E \varrho_t(H-y) \, d\varrho_{1-t}(y) \leq \int_E \varrho_t(H) \, d\varrho_{1-t}(y) = \varrho_t(H),
\]

where \(H \in \mathcal{K}_0(E)\) with \(\varrho_1(H) > 0\). Let

\[
\Omega'_0 \coloneqq \{ \omega \in \Omega' : X_t(\omega) : t \in [0,t_0]Q \subseteq E_H \} \quad \text{and}
\Omega'_n \coloneqq \{ \omega \in \Omega' : X_t(\omega) : t \in [0,t_0]Q \subseteq n \cdot H \}, \quad n = 1,2,\ldots
\]

The latter sets form an ascending chain \(\Omega'_n \subseteq \Omega'_{n+1}\) and their union equals \(\Omega'_0\). By hypothesis, \(t \mapsto X_t(\omega)\) is continuous for \(\omega \in \Omega'_0\) and it is uniformly continuous as a mapping from \([0,t_0]Q\) to \(n(\omega) \cdot H\). In particular, its restriction to \([0,t_0]Q\) is uniformly continuous as well. The set

\[
O'_n := \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{s,t \in [0,t_0]Q} \left\{ \omega \in \Omega'_m : d_n(X_t(\omega),X_s(\omega)) \leq \frac{1}{k} \right\} \subseteq \Omega'_n
\]

equals \(\Omega'_n\) and thus \(\mathbb{P}(O'_n) = \mathbb{P}(\Omega'_n)\). This property depends only on the distribution of the process. Setting

\[
\Omega_n := \{ \omega \in \Omega' : X_t(\omega) : t \in [0,t_0]Q \subseteq n \cdot H \}, \quad n = 1,2,\ldots
\]

one obtains that

\[
O_n := \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{s,t \in [0,t_0]Q} \left\{ \omega \in \Omega_n : d_n(W_t(\omega),W_s(\omega)) \leq \frac{1}{k} \right\} \subseteq \Omega_n
\]

has the same measure, \(\mathbb{P}(O_n) = \mathbb{P}(O'_n)\) and the sets are increasing. Let \(O := \bigcup_{n=1}^{\infty} O_n\) the set of uniformly continuous paths from \([0,t_0]Q\) to \(E_H\). Then,

\[
\mathbb{P}(O) = \lim_{n \to \infty} \mathbb{P}(O) = \lim_{n \to \infty} \mathbb{P}(O'_n) = \mathbb{P}(\Omega'_0) = 1,
\]

therefore, \(W(\omega)\) is uniformly continuous on \([0,t_0]Q\) for almost all \(\omega \in \Omega\). One can define a unique continuous extension of the uniformly continuous function \(W(\omega) : [0,t_0]Q \to n(\omega) \cdot H\) to the closure \([0,t_0]\) of its domain by setting

\[
\widetilde{W}_t(\omega) := \begin{cases} W_t(\omega), & t \in Q \cap [0,t_0], \text{ and } \omega \in O_0, \\ \lim_{s \to t} W_s(\omega), & t \in Q^c \cap [0,t_0], \text{ and } \omega \in O_0, \\ 0, & \text{else.} \end{cases}
\]

But as

\[
W_{t-}(\omega) = \lim_{s \to t} W_s(\omega) = \lim_{s \to t} W_s(\omega) = \widetilde{W}_t(\omega) = \lim_{s \to t} W_s(\omega) = W_t(\omega)
\]

for \(\omega \in O_0\) and for every \(t\) by the càdlàg property of \(W\), one obtains that \(W\) has already a.s. continuous sample paths. \(\square\)
because for a Gaussian measure $\mu$, $a \in E$ and every absolutely convex set $A$ one has $\mu(A + a) \leq \mu(A)$ by [6, Theorem 2.8.10]. The second assertion immediately follows by the zero-one law for linear subspaces, cf. [6, Theorem 2.5.5]. \hfill \Box

7. Jump Processes and Random Measures

Let $X$ be a Lévy process in $E$ and $\nu \in \mathcal{M}(E)$ the Lévy measure of $\mu_1 = \mathbb{P}_X$, and locally reducible. Let $K \in K^*_0$ be a $\nu$-reducing set. In particular, the properties of Proposition 4.10 are valid. The main results of this section will be Theorem 7.5 and Proposition 7.11.

7.1. A constructed process. We begin with some general facts about Poisson random measures, which can be found in [32, Chapters 19 and 20]. Given a $\sigma$-finite measure space $(\Theta, \mathcal{B}, \rho)$ one can define a Poisson random measure $\{N'(B), B \in \mathcal{B}\}$ on some probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ with intensity measure $\rho$. For now, we choose this measure space to be the finite measure space $(T \times E, \mathcal{B}(T \times E), \lambda \otimes \nu|_{K'})$. Then, for $\omega \in \Omega'_0 \in \mathcal{F}'$ with $\mathbb{P}'(\Omega'_0) = 1$, the measure $N'(\cdot, \omega)$ is supported on a finite number of points of mass 1 on all bounded intervals, and $N'(\{s\} \times E, \omega)$ has values in $\{0, 1\}$ for all $s \in T$, cf. [32, Lemma 20.1].

**Definition 7.1.** Let $B \in \mathcal{B}(E)$. The Poisson integral with respect to $N'$ is defined by

\[
Y_t(B)(\omega) := \int_{(0,t] \times B} x \, dN'(s, x)(\omega), \quad \omega \in \Omega'_0
\]

and $Y_t(\omega) := 0$ for $\omega \in \Omega'_0 \setminus \Omega'_0$. If $B = E$, we write $Y_t := Y_t(E)$.

**Proposition 7.2.**

1. For $t \in T, B \in \mathcal{B}(E)$, the random variable $Y_t(B)$ defined in (7.1) is finite.
2. $Y_t(B)$ is a càdlàg Lévy process in $E$ and $\mathbb{P}_{Y_t(B)}$ has distribution $e^{t (\nu|_{K' \cap B})}$. In particular, its characteristics are $(0, 0, \nu|_{K' \cap B}, K)$.
3. $N'(B)(\omega) = \#\{s : (s, \Delta Y_s(\omega)) \in B \setminus \{0\}\}$

**Proof.** (1) follows from the property that $N'(\cdot, \omega)$ is a.s. supported on a finite number of points.

(2) This essentially follows from [32, Proposition 19.5]. For convenience, we adapt the situation to ours: The functional $a \in E'$ is measurable. We consider the real-valued random variable

\[
Y_t^a(B)(\omega) := \langle Y_t(B)(\omega), a \rangle = \int_{(0,t] \times B} \langle x, a \rangle \, dN'(s, x)(\omega),
\]

which has Fourier transform

\[
\mathbb{E} e^{izY_t^a(B)} = \exp \left( t \int_E \left( e^{iz(x,a)} - 1 \right) \, d\nu|_{K' \cap B}(x) \right)
\]

\[
= \exp \left( \int_{(0,t] \times E} \left( e^{iz(x,a)}1_B(x) - 1 \right) \, d\lambda \otimes \nu|_{K' \cap B}(s,x) \right), \quad z \in \mathbb{R}.
\]

Since this holds for arbitrary $a \in E'$, the $E$-valued random variable $Y_t(B)$ has distribution $e^{t (\nu|_{K' \cap B})}$ on $E$. Independent increments follow from $N'$ being independently scattered, stationary increments from the structure of the intensity measure. The map $t \mapsto e^{t (\nu|_{K' \cap B})} = \mathbb{P}_{Y_t(B)}$ is a convolution semigroup. Weak
continuity of the semigroup of distributions of \( Y_t \) follows from

\[
\left| \left[ e^{(t\nu|K^c)} - \delta_0 \right] (f) \right| \leq \left| e^{-t\nu|K^c}(E) \sum_{n=1}^{\infty} \frac{t^n\nu|K^c(E)^n M^n}{n!} \right|
\]

\[
e^{-t\nu|K^c}(E) \left[ e^{t\nu|K^c(E)M} - 1 \right] \to 0 \quad \text{for } t \searrow 0,
\]

where \( f \in C^b(E) \) and \( \text{im}(f) \subseteq [-M, M] \). Weak continuity of \( P_{y_t(B)} \) is shown analogously substituting \( K^c \) by \( K^c \cap B \) in the previous calculation.

(3) If \( s \in T \) such that \( N'(\{s\} \times E, \omega) = 0 \), then \( \Delta Y_s(\omega) = 0 \). If there is an \( x \in B \setminus \{0\} \) such that \( N'(\{(s, x)\}, \omega) = 1 \), then \( \Delta Y_s(\omega) = x \). \( \square \)

For \( A \in \mathcal{B}(T \times E) \) satisfying \( A \subseteq T \times (K \setminus \varepsilon K) \) for some \( \varepsilon \in (0, 1) \) define the Bochner integral

\[
(7.2) \quad \int_A x \, d(\lambda \otimes \nu)(t, x).
\]

Indeed, \( (\lambda \otimes \nu)|A \) is finite and concentrated on \( (a \text{ subset of}) T \times K \). The map \( \psi : T \times K \to E_K \), \( (t, x) \mapsto x \), satisfies the conditions of Bochner integrability: It takes values in a Banach space \( E_K \subseteq E \) and is Bochner measurable (which is equivalent to being measurable in separable Banach spaces); for Bochner integrals in locally convex spaces cf. \cite[Definition 15, p. 75]{37}. Therefore, \( (7.2) \) can be considered as a Bochner integral in \( E_K \) or in \( E \).

The measure \( \lambda \otimes \nu \) is \( \sigma \)-finite on \( \mathcal{B}(T \times E) \) using the partition \( E = \bigcup_n C_n \) with \( C_0 = K^c \) and \( C_n := \frac{1}{n}K \setminus \frac{1}{n+1}K \) for \( n = 1, 2, \ldots \). Indeed, \( \nu(C_n) < \infty \) by Proposition 4.10 (ii). By \( \sigma \)-finiteness of \( \lambda \otimes \nu \), one can construct a Poisson random measure \( N' \) on a probability space \( (\Omega', \mathcal{F}', \mathbb{P}') \) with intensity measure \( \lambda \otimes \nu \), cf. \cite[Proposition 19.4]{32}. Without loss of generality, \( N' \) from above is just the restriction of this new Poisson random measure to \( T \times K^c \).

Let \( B \in \mathcal{B}(E), B \subseteq C_n \) and \( t \in T \). Analogously as in \( (7.1) \) the Lévy processes

\[
Y_{\lambda}(B)(\omega) := \int_{[0, t] \times B} x \, dN'(s, x)(\omega)
\]

are finite for almost all \( \omega \in \Omega' \), contained in \( E_K \) for \( n \neq 0 \) and Poisson distributed on \( E \) and \( E_K \) by Lemma 4.2.

**Definition 7.3.** Let \( B \in \mathcal{B}(E) \) with \( B \subseteq C_n \) and \( t \in T \). The *compensated Poisson integral* is defined by

\[
J'(\{0, t\} \times B) := \int_{[0, t] \times B} x \, dN'(t, x) := \int_{[0, t] \times B} x \, dN'(t, x) - \int_{[0, t] \times B} x \, d(\lambda \otimes \nu)(t, x).
\]

**Definition 7.4.** Let \( Z^n : T \times \Omega \to E, n = 1, 2, \ldots \) be càdlàg stochastic processes. They are said to *converge almost surely uniformly on bounded intervals of \( T \)*, if there exists a set \( \Omega_0 \) of measure one, such that for all \( \omega \in \Omega_0 \), for all seminorms \( p \) generating the topology \( \tau \) on \( E \) and all bounded intervals \( T_0 \subseteq T \) one has that

\[
\sup_{t \in T_0} p(Z^n_t(\omega) - Z^m_t(\omega)) < \varepsilon
\]

for every \( \varepsilon > 0 \) and \( n, m \in \mathbb{N} \) large enough.

**Theorem 7.5 (Compensated Poisson integral).** With the notation from above, for \( t \in T \), the quantity

\[
(7.3) \quad J'_t := \int_{[0, t] \times K} x \, d\widetilde{N}'(s, x) := \sum_{n=1}^{\infty} J'(\{0, t\} \times C_n)
\]
is a series of independent random variables in \( E_K \) and converges almost surely in \( E_K \) and \( E \). The convergence is uniform in \( t \) on bounded intervals of \( T \). Finally, \((J'_t)_{t \in T}\) is a càdlàg Lévy process in \( E \) with characteristics \((0, 0, \nu|_K, K)\).

**Remark 7.6.** We understand the process \( J'_t(\omega) \) in the following sense: \( J'_t(\omega) \) takes the value of the right-hand side for all \( t \in T \), if \( \omega \in \Omega'_T \) such that the series converges uniformly on bounded intervals of \( T \), and \( J'_t := 0 \) for \( \omega \in \Omega'_T^c \).

**Proof.** Define \( x_n := -\int_{C_n} x \, d\nu(x) \in E_K \subseteq E \). The summands \( J'(]0,1] \times C_n) \) have distributions \( e(\nu|_{C_n}) \ast \delta_{x_n} \) on \( E \), or, by restriction \( e(\nu|_{C_n}) \parallel E_K \ast \delta_{x_n} \in \mathcal{M}^1(E_K) \). Their sums converge weakly to \( \tilde{e}(\nu|_K) \parallel E_K \in \mathcal{M}^1(E_K) \), cf. proof of [19, Theorem 3.4.9]. By the same arguments as in [13, Proof of Theorem 2.1], convergence in distribution of the partial sums of the independent random variables yields a.s. convergence in \( E_K \) (cf. [19, Theorem 3.1.6]). Similarly, it converges for every \( t \in T \). \((J'_t)_{t \in T}\) is a Lévy process in \( E_K \) with characteristics \((0, 0, (\nu|_K) \parallel E_K, K)\). Indeed, the independent and stationary increments follow from the definition of \( N' \). The semigroup of distributions equals \( t \mapsto \tilde{e}(t(\nu|_K) \parallel E_K) \) and is weakly continuous, cf. [19, Theorem 2.3.9].

By Lemma 4.4 we have that all random elements on \( E_K \) are also measurable with respect to \( \mathcal{B}(E_K) \). As \( K \) is bounded and closed, the injection \( i: E_K \hookrightarrow E \) is continuous and thus the series converges a.s. in \( E \). In order to establish uniform convergence on bounded intervals of \( T \), we fix \( t \in T \), \( t > 0 \) and again use the injection \( i \). Continuity implies \( p(i(x)) \leq c_p \parallel x \parallel_K \) for all \( x \in E_K \) and all seminorms \( p \) generating \( \tau \) and suitable constants \( c_p > 0 \). Therefore,

\[
\sup_{s \in [0,t]} p \left( i \left( J'_s - \sum_{n=1}^{N} J'(]0,s] \times C_n) \right) \right) \leq c_p \sup_{s \in [0,t]} \left\| J'_s - \sum_{n=1}^{N} J'(]0,s] \times C_n) \right\| \parallel E_K \parallel_K
\]

due to the fact that \( J' \) is a Lévy process in \( E_K \).

Applying [32, Lemma 20.4] (which can be proven in exactly the same way for Banach spaces) yields a.s. uniform convergence on bounded intervals of the sequence of processes in \( E_K \), a càdlàg limiting process \( J' \) in \( E_K \) as \( D([0, t]; E_K) \) is closed under uniform convergence. The inequality yields uniform convergence in \( E \) and càdlàg functions in \( E \) as well injecting the \( D([0, t]; E_K) \) functions into \( D([0, t]; E) \) by virtue of \( \xi \mapsto i \circ \xi \). It remains to state that \( J' \) is a Lévy process in \( E \). Continuity of \( i \) implies weak continuity of the semigroup \( t \mapsto \tilde{e}(t(\nu|_K), \) the null extension of the family of distributions on \( E_K \) and the proof is complete. \( \square \)

As mentioned above, the generalised Poisson exponential is unique up to a convolution. From now on we will use a certain representative given by the construction above in order to avoid ambiguities: We define \( \tilde{e}(\nu|_K) \) as the weak limit of \( \tilde{e}(\nu|_{C_n}) \ast \delta_{x_n} \) with \( x_n \) from the previous proof.

Given an infinitely divisible distribution \( \varrho \) with characteristics \((\gamma, Q, 0, K)\) on a locally convex Suslin space, where \( Q: E' \to E \) is a covariance operator associated to a Gaussian measure \( \varrho \), one can define a Wiener process \( W'_t \) on a probability space \((\Omega'_t, \mathcal{F}_t, P')\) such that \( W'_t \sim \varrho^{t*} \), cf. [15, Proposition 5]. The stochastic process \( (W'_t)_{t} \) is continuous, has values in \( E \) and

\[
\mathbb{E}(W'_t - \gamma, t) \langle W'_s - \gamma, s \rangle = \min\{s, t\}\langle Qa, b \rangle.
\]

For details, we confer to Theorem 6.4.

**Proposition 7.7.** Let \( X'_t = J'_t + L'_t + W'_t \), \( t \in T \), be defined on a complete probability space \((\Omega', \mathcal{F}', \mathbb{P}')\) where \( L'_t := Y_t(K') \) is defined as in (7.1), and let the summands be constructed in a way such that they are independent. Then,
(1) \((X_t^i)_{t \in T}\) is a càdlàg Lévy process and
\(X_t^i\) has characteristics \((\gamma, Q, \nu, K)\).

(2) \(X_t^i\) has characteristics \((\gamma, Q, \nu, K)\).

(3) The random measure \(N'\) satisfies \(N'(B) = \#\{s : (s, \Delta X^i_s) \in B \setminus \{0\}\}\) for \(B \in B(T \times E)\).

**Proof.** (1) is clear. (2) One obtains the characteristics by convolution of the ingredients and noting that \(\tilde{\nu}(\nu_1) * \nu_2 = \tilde{\nu}(\nu_1 + \nu_2)\) (as can be seen using the Fourier transform) for a Lévy measure \(\nu_1\) and \(\nu_2 \in \mathcal{M}_0(E)\). In our situation, \(\nu_1 = \nu|_K\), \(\nu_2 = \nu|_{\bar{K}^c}\).

(3) This property holds for each \(\varepsilon > 0\) if \(B \cap (T \times \varepsilon K) = \emptyset\). Observing
\[
B \setminus \{0\} = \bigcup_{n=0}^{\infty} (T \times C_n) \cap B
\]
with \(C_n, n \in \mathbb{N}^*\), as defined above (and \(C_0 := E \setminus K\)), one has
\[
N'(B) = \sum_{n=0}^{\infty} N'((T \times C_n) \cap B), \quad \text{a.s.}
\]
and the assertion follows. \(\square\)

### 7.2. Definitions on the original space.
This section is following the ideas of Sato in [32, Section 20] for the proof of the Lévy-Itô decomposition of finite dimensional Lévy processes. Extra considerations for measurability issues have been carried out in lemmas 5.4 and 5.5 above.

**Definition 7.8.** Let \(B \in B(T \times E)\) and \(X = (X_t)_{t \in T}\) the (original) Lévy process given on \((\Omega, \mathcal{F}, \mathbb{P})\). Let \(\xi \in \mathcal{D}(T; E)\) be a càdlàg function and \(\Delta x_t(\xi) := x_t(\xi) - x_{t-}(\xi)\). For \(\omega \in \Omega\) resp. \(\xi \in \mathcal{D}(T; E)\) define
\[
N(B, \omega) := \#\{s : (s, \Delta X_s(\omega)) \in B \setminus \{0\}\} \quad \text{resp.}
\]
\[
n(B, \xi) := \#\{s : (s, \Delta x_t(\xi)) \in B \setminus \{0\}\}
\]
and for all sets \(B\) with \(\lambda \otimes \nu(B) < \infty\) set
\[
(7.4) \quad \tilde{N}(B, \omega) := N(B, \omega) - (\lambda \otimes \nu)(B)
\]
\[
\tilde{n}(B, \xi) := n(B, \xi) - (\lambda \otimes \nu)(B).
\]

\(N\) is called the Poisson random measure associated to \(X\) and \(\tilde{N}\) the compensated Poisson random measure corresponding to \(X\).

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be the original probability space and \((\Omega', \mathcal{F}', \mathbb{P}')\) the probability space of the constructed process of the previous section. Both processes \(X\) and \(X'\) consist of càdlàg paths only by construction. We define the following maps to the space of càdlàg functions:
\[
\psi : \Omega \rightarrow \mathcal{D}(T; E), \quad \psi(\omega) := X.(\omega),
\]
\[
\psi' : \Omega' \rightarrow \mathcal{D}(T; E), \quad \psi'(\omega) := X'.(\omega).
\]

One obtains \(\mathbb{P} \circ \psi^{-1} = \mathbb{P}' \circ (\psi')^{-1}\), where this measure, call it \(\mathbb{P}'^D\), is defined on \(\mathcal{F}'_D\). This is due to equality of distributions of \(X^i_t\) and \(X_t\) on \(E\) and therefore of \(X'\) and \(X\) on \(\mathcal{D}(T; E)\). The following lemma shows that \(\lambda \otimes \nu\) is the compensator of \(N\) and \(n\).

**Lemma 7.9.** Let \(B \in B(T \times E)\) with \(\lambda \otimes \nu(B) < \infty\). Then,
\[
\mathbb{P}_N(B) = \mathbb{P}_n'(B) = \mathbb{P}^D_n(B).
\]
In particular, \(N\) and \(n\) are a Poisson random measures with intensity measure \(\lambda \otimes \nu\).
Proof. We follow the proof of Sato, cf. \cite[p. 132]{32}. We have that \(n(B, \psi(\omega)) = N(B, \omega)\) and \(n(B, \psi'(\omega)) = N'(B, \omega)\) and furthermore they are equal in law, provided that they are \(F_D\)-measurable. But this follows from lemmas 5.3 and 5.4 and the fact that \(x_{t_n,j}(\xi)\) and \(x_{t_n,j}(-\xi)\) are \(F_D\)-measurable due to Lemma 5.5, thus
\[
G(m, j) := \{\xi \in D(T; E) : t_{m,j} < \infty \text{ and } \Delta x_{t_{m,j}}(\xi) \in B\} \in F_D.
\]
Noting that \(n(B, \xi)\) can be written as the \(m, j\)-series over indicator functions of \(G(m, j)\) yields the assertion.
\[\square\]

For a Borel set \(A \in \mathcal{B}(E)\) let \(\Omega_0\) be the intersection of all subsets \(\Omega^n_0\) of \(\Omega\) of full measure such that \(N([0, m] \times A, \omega) < \infty, m \in \mathbb{N}\), if \(T = [0, \infty)\). In the case \(T = [0, \max]\), the set \(\Omega_0\) consists of all \(\omega \in \Omega\) such that \(N([0, \max] \times A, \omega) < \infty\).

For \(\omega \in \Omega_0\) define
\[
X_t(A)(\omega) := \sum_{0 \leq s \leq t} \Delta X_s(\omega) \mathbb{1}_{A \setminus \{0\}}(\Delta X_s(\omega))
\]
and if \(\omega \in \Omega^n_0\) we set the trajectory \(X_t(A)(\omega)\) to zero. Furthermore, carrying out the same construction with \(n\) instead of \(N\), one can define
\[
x_t(A)(\xi) := \sum_{0 \leq s \leq t} \Delta x_s(\xi) \mathbb{1}_{A \setminus \{0\}}(\Delta x_s(\xi)),
\]
for \(\xi \in D_0\) of full measure \(\mathbb{P}^D\) and also \(X'_t(A)(\omega)\) on a set \(\Omega^n_0' \in \mathcal{F}'\) with \(\mathbb{P}'(\Omega^n_0') = 1\)
setting the whole trajectories to zero on the complements \(D^n_0\) and \(\Omega^n_0'\).
Then, for all \(\omega \in \Omega\) and \(\omega' \in \Omega'\), it holds that
\[
X_t(A)(\omega) = x_t(A)(\psi(\omega)) \quad \text{and} \quad X'_t(A)(\omega') = x_t(A)(\psi'(\omega')).
\]
Letting
\[
(7.5) \quad L_t := X_t(K^c) \quad \text{and} \quad l_t := x_t(K^c)
\]
we have
\[
(7.6) \quad L_t(\omega) = l_t(\psi(\omega)) \quad \text{and} \quad L'_t(\omega) = l_t(\psi'(\omega)) \quad \text{for all} \ \omega \in \Omega, \ \omega' \in \Omega'.
\]

\textbf{Lemma 7.10.} Let \(L_t\) be defined as in (7.6) and the random measures \(N\) and \(n\) be defined as in Definition 7.8. Then, for almost all \(\omega \in \Omega\) and \(\xi \in \mathcal{D}(T; E)\) one has
\[
(7.7) \quad L_t(\omega) = \int_{[0, t] \times K^c} x \, dN(s, x)(\omega) \quad \text{and} \quad l_t(\xi) = \int_{[0, t] \times K^c} x \, dn(s, x)(\xi)
\]
and \(L_t\) and \(l_t\) are càdlàg Lévy processes with characteristics \((0, 0, \nu|_{K^c}, K)\).

\textbf{Proof.} From \(\lambda \otimes \nu([0, t] \times K^c) < \infty\) we deduce \(N([0, t] \times K^c, \omega) < \infty\) for \(\omega \in \Omega^1_1\) for some \(\Omega^1_1 \in \mathcal{F}\) with \(\mathbb{P}(\Omega_1) = 1\) by Lemma 7.9. For such \(\omega \in \Omega^1_1\), there are \(s_1(\omega), \ldots, s_m(\omega) \in [0, t]\) with \(N(\{s_k(\omega)\} \times K^c, \omega) = 1\) and for all other \(s \in [0, t]\) one has that \(N(\{s\} \times K^c, \omega) = 0\). Therefore one obtains
\[
\int_{[0, t] \times K^c} x \, dN(s, x)(\omega) = \sum_{k=1}^{m(\omega)} \Delta X_{s_k}(\omega) = \sum_{s \in [0, t]} \Delta X_s(\omega) \mathbb{1}_{K^c}(\Delta X_s(\omega)).
\]
If
\[
\omega \in \Omega_1 := \bigcap_{m=1}^{\infty} \Omega^m_1,
\]
the above expression exists for every \(t \in T = [0, \infty)\). If \(T = [0, \max]\), \(\Omega_1 := \Omega^m_1\).
By (7.7) the same follows for \(l_t\) taking \(\xi \in \mathcal{D}_1\) with \(\mathbb{P}^D\)-measure one.

The distributions of \(L_t\), \(L'_t\) and \(L'_t\) are the same by (7.7). So the weakly continuous semigroup of distributions of \(L'\) carries over to \(L\) and \(l\). Therefore, the finite-dimensional distributions on the path space of \((L_t)_{t \in T}, (l_t)_{t \in T}\) and \((L'_t)_{t \in T}\)
coincide which implies that the processes \( L \) and \( l \) have independent and stationary increments. Furthermore, \( L_t \) is càdlàg by construction. \( \Box \)

Also the compensated integral with respect to \( N \) resp. \( n \) can be constructed analogously as above by setting
\[
J_j := \int_{[0,t] \times C_m} x \, dN(s,x) - \int_{[0,t] \times C_m} x \, d(\lambda \otimes \nu)(s,x) \quad \text{resp.}
\]
\[
j_j := \int_{[0,t] \times C_m} x \, dn(s,x) - \int_{[0,t] \times C_m} x \, d(\lambda \otimes \nu)(s,x)
\]
for \( m \in \mathbb{N} \). Again,
\[
J(\omega) = j(\psi(\omega)) \quad \text{and}
\]
\[
J'(\omega') = j'(\psi'(\omega'))
\]
for all \( \omega \in \Omega \) resp. \( \omega' \in \Omega' \).

**Proposition 7.11.** Defining
\[
J_t := \int_{[0,t] \times K} x \, d\tilde{N}(s,x) := \sum_{m=1}^{\infty} J(\omega) \quad \text{resp.}
\]
\[
j_t := \int_{[0,t] \times K} x \, d\tilde{n}(s,x) := \sum_{m=1}^{\infty} j(\omega)
\]
one has the following:

1. The series (7.8) resp. (7.9) converge almost surely in \( E_K \) (thus in \( E \)) uniformly in \( t \) on bounded intervals.
2. \( J_t(\omega) = j_t(\psi(\omega)) \) and \( J'_t(\omega') = j_t'(\psi'(\omega')) \) for almost all \( \omega \in \Omega \) resp. \( \omega \in \Omega' \).
3. \( J_t \) and \( j_t \) are càdlàg Lévy processes with characteristics \((0,0,\nu|_K,K)\).

**Proof.** Analogously to Lemma 7.10, one obtains that the processes
\[
J^m := \sum_{k=1}^{m} J(\omega) \quad \text{and}
\]
\[
j^m := \sum_{k=1}^{m} j(\omega)
\]
are equal in distribution, where it has been proved above that the distributions of \((J^m_t, m = 1, 2, \ldots)\) converge to an infinitely divisible measure with characteristics \((0,0,\nu|_K,K)\) for \( m \to \infty \) and \( t \in T \).

1. Let \( \Omega' \) be the set of full \( \mathbb{P}' \)-measure such that the series (7.3) for \( J'_t \) converges in Theorem 7.5. It equals the intersection over all sets \( \Omega'_2(N) \) of \( \mathbb{P}' \)-measure one \((N = 1, 2, \ldots)\) where
\[
\Omega'_2(N) := \left\{ \omega \in \Omega' : \lim_{m \to \infty} \sup_{t,k \geq m} \sup_{t \in [0,N]} \| (J'_t)^k(\omega) - (J'_t)^k(\omega) \|_K = 0 \right\}.
\]
Let \( \mathcal{D}' \) resp. \( \Omega \) be the intersection of sets \( \mathcal{D}'(N) \) resp. \( \Omega(N) \) from above with \( \Omega' \) replaced by \( \mathcal{D}(T;E) \) resp. \( \Omega \) and \( J' \) by \( j \) and \( J \), respectively. Then,
\[
1 = \mathbb{P}'(\Omega'_2) = \mathbb{P}(\mathcal{D}'(\mathcal{D}) = \mathbb{P}(\Omega)
\]
as it only depends on the distribution of the processes. This yields a.s. uniform convergence on bounded intervals of \( j_t \) and \( J_t \) in \( E_K \) and consequently in \( E \).

2. For all \( \omega \in \Omega'_2 \) and all \( n \in \mathbb{N} \) we have \((J'_t)^n(\omega) = j^n_t(\psi(\omega))\). As the left-hand side and therefore the right-hand side converges uniformly we obtain that \( \psi(\omega) \in \mathcal{D}' \) and \( j^n_t(\psi(\omega)) \to j_t(\psi(\omega)) \) uniformly in \( t \) on bounded intervals by definition of \( \mathcal{D}' \). Then, \( J'_t(\omega) = j_t(\psi(\omega)) \) for \( \omega \in \Omega'_2 \). The same arguments hold for \( J \) on \( \Omega \).

3. \( J \) is càdlàg by uniform convergence and \( J \) and \( j \) are Lévy processes by equality of their finite dimensional distributions with those of \( J' \). \( \Box \)
Finally, we set $Y_t(\omega) := X_t(\omega) - L_t(\omega) - J_t(\omega)$ and $y_t(\xi) := x_t(\xi) - l_t(\xi) - j_t(\xi)$ and $Y_t'(\omega) := X_t'(\omega) - L_t'(\omega) - J_t'(\omega) = W_t'(\omega) + \gamma t$. From the equality of finite-dimensional distributions of $(J, L, Y)$, $(j, l, y)$ and $(J', L', Y')$ we obtain:

**Proposition 7.12.** The three processes $(J_t, L_t, Y_t)_{t \in T}$ are independent.

8. Lévy-Itô-decomposition

**Theorem 8.1** (Lévy-Itô-decomposition). Let $X$ be an $E$-valued Lévy process with characteristics $(\gamma, Q, \nu, K)$ and $\nu$ locally reducible with reducing set $K$. Then there exist an $E$-valued Wiener process $(W_t)_{t \in T}$ with covariance operator $Q$, an independently scattered Poisson random measure $N$ on $T \times E$ with compensator $\lambda \otimes \nu$ and a set $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ such that for all $\omega \in \Omega_0$ one has

$$X_t(\omega) = \gamma t + W_t(\omega) + \int_{[0,t] \times K} x \, d\tilde{N}(s, x)(\omega) + \int_{[0,t] \times K'} x \, dN(s, x)(\omega)$$

for all $t \in T$. Furthermore, all the summands in (8.1) are independent and the convergence of the first integral in the sense of (7.8) is a.s. uniform in $t$ on bounded intervals in $E_K$ and $E$.

**Proof.** As the distribution $\mu_1$ of $X_1$ is locally reducible with reducing set $K \in K_0^s(E)$ it holds that $\nu(K^c) < \infty$ and $\nu|_K$ is a Lévy measure on $E_K$.

Let $L_t$ and $J_t$ be defined as in (7.6) and (7.8), respectively. By means of the mappings $\psi$ and $\psi'$ from Section 7.2, the processes $(X_t, L_t, J_t)_{t \in T}$ and $(X'_t, L'_t, J'_t)_{t \in T}$ have the same finite-dimensional distributions. Therefore, the process $Z = (Z_t)_{t \in T}$ defined by $Z_t := X_t - L_t$ has the same distribution as $X'_t - L'_t = J'_t + W'_t + \gamma t$ and is therefore a Lévy process. The process $(Z_t)_{t \in T}$ has jumps of a size in $K$ and its characteristics are $(\gamma, Q, 0, K)$, which is obtained by the characteristics of $X'_t - L'_t$.

A.s. uniform convergence of the series in (7.8) and the fact that for every jump of $Z_t$ there exists an $n \in \mathbb{N}$ such that the jump exceeds $n^{-1} \cdot K$ imply that for almost all $\omega \in \Omega$ the trajectory $Y_t(\omega) := Z_t(\omega) - J_t(\omega)$ is continuous as all jumps are erased. The distribution of $J_t$ equals $\mathbb{E}(tv|_K)$ on $E$ by Theorem 7.5. Furthermore, $Y_t$ has characteristics $(\gamma, Q, 0, K)$, namely the same as those of $X'_t - L'_t - J'_t$.

In addition, the continuous process $Y_t$ has stationary and independent increments, thus it is a continuous Lévy process, a Wiener process with drift in $E$ by Theorem 6.4. The element $\gamma \in E$ satisfies $\mathbb{E}(\gamma, a) = \langle \gamma, a \rangle$ for all $a \in E'$. This is nothing else than $\gamma \in EY_1$ in the Pettis sense. Setting $W_t := Y_t - \gamma t$ we obtain a centered Wiener process $W_t$.

Independence of the summands follows from Proposition 7.12. \[\square\]

We can even strengthen the result and let the process $X^0_t := \gamma t + W_t + J_t$ live in a Banach space $E_K$. In order to obtain this, we use Lemma 4.12.

**Corollary 8.2.** Let $E$ have a fundamental system of separable Banach disks. Then, there exists $K \in K_0^s$ such that $X_t = X^0_t + L_t$ and additionally,

1. $X^0_t$ has values in $E_K$ for all $t \in T$ a.s.
2. $X^0_t$ is Bochner integrable and square integrable in $E_K$ for every $t \in T$.

**Proof.** Define $K := K_1 + K_2 + K_3$, where $K_1 \in K_0^s$ is $\nu$-reducing which exists due to Theorem 4.14, the set $K_2 \in K_0^s$ has positive (centered) Gaussian measure and the whole trajectory of $W$ stays in $E_{K_2}$ by the same arguments as in the proof of Theorem 6.4. Indeed, $K_2$ exists, as there must be $K' \in K_0$ of positive measure, therefore $B \in B_0^s$ with $B \supseteq K'$ as $B_0^s$ is fundamental. The separable Banach space $E_B$ has full Gaussian measure. As $K_0^s(E_B)$ is fundamental in $E_B$,
there exists $K_2 \in K_0^0(E_B) \subseteq K_0^0(E)$ of positive Gaussian measure, therefore $E_{K_2}$ has full Gaussian measure.

The set $K_3$ is the absolutely convex hull of $\gamma \in E$ and $E_{K_1} \cong \mathbb{R}$ is clearly separable. Lemma 4.12 yields that $K \subseteq K_0^0$. Note that $K$ is $\nu$-reducing by Lemma 4.8, $K$ has positive centered Gaussian measure (and therefore measure one by [6, Theorem 2.5.5]) and $\gamma \in K$. Therefore, $X_0^\gamma$ has values in $E_K$ a.s. and can be actually considered as a process in the Banach space $E_K$ by analogous arguments as above. (Square) integrability follows from Proposition 6.2 and [30, Corollary 3.4].

**Proposition 8.3.** Let $E$ be a space with a fundamental system of separable Banach disks and $(X_t)_{t \in T}$ a Lévy process with values in $E$ and characteristics $(\gamma, Q, \nu, K)$. The following assertions are equivalent:

1. There exists a $t_0 \in T \setminus \{0\}$ such that $X_{t_0}$ takes values a.s. in a separable Banach space $E_1$ with closed unit ball compact in $E$.
2. There exists a $t_0 \in T \setminus \{0\}$ such that $\mathbb{P}_{X_{t_0}}$ has a Banach support $E_1$ with closed unit ball compact in $E$.
3. For all $t \in T$ one has $X_t \in E_1$ a.s. for a separable Banach space $E_1$ with closed unit ball compact in $E$.
4. For all $t \in T$ the distribution $\mathbb{P}_{X_t}$ has a Banach support $E_1$ with closed unit ball compact in $E$.

5. $\nu$ has a Banach support $E_2$ with closed unit ball compact in $E$.

Given (5), one can choose $E_1 = E_K + E_2 + E_3 + \mathbb{R} \gamma$, where $E_3$ is a suitable Banach support of the Gaussian part of $X$ and $K$ is $\nu$-reducing.

**Proof.** The equivalence of (1) and (2) resp. (3) and (4) and the implication (4) $\Rightarrow$ (1) are obvious and (2) $\Rightarrow$ (4) follows from Lemma 4.6.

(2) $\Rightarrow$ (5): If (2) holds, $\mathbb{P}_{X_{t_0}}$ is infinitely divisible on the Banach space $E_1$ and thus, there exists a Lévy measure $\nu'$ on $E_1$. Injecting $E_1$ into $E$, finding the null extensions of $\mathbb{P}_{X_{t_0}}$ and $\nu'$ and using Theorem 4.9 and uniqueness of the Lévy measure of an infinitely divisible distribution, one obtains $\nu^0 = \nu$ and therefore, $\nu$ has Banach support $E_1$. One can choose $E_2 = E_1$ and obtain assertion (5).

(5) $\Rightarrow$ (2): Let $K \subseteq K_0^0$ be $\nu$-reducing. Then, $\bar{c}(\nu) = \bar{c}(\nu|_{E_1}) * e(\nu|_{E_2})$ has Banach support $E_K + E_2$. Let $K_2 \subseteq K_0^0$ be the closed unit ball of $E_2$. The Gaussian part $\varrho$ has Banach support $E_3 = E_H$ for some $H \subseteq K_0^0$ and $\delta_H$ has Banach support $\mathbb{R} \gamma$. Put $K' := K + K_2 + H + [-1, 1] \cdot \gamma$ which is in $K_0^0$ by Lemma 4.12. Then, $\mu := \bar{c}(\nu) * \varrho * \delta \ast \delta$, has Banach support $E_1 := E_{K'} = E_H + E_2 + E_3 + \mathbb{R} \gamma$ and as $\mu_{E_1}$ is infinitely divisible on $E_1$, there exists a root $(\mu_{E_1})^{s_{E_1}} = (\mu^{s_{E_1}})_{E_1}$ by Lemma 4.6. Therefore, $\mathbb{P}_{X_{t_0}} = \mu_{t_0} = \mu^{s_{E_1}}$ also has Banach support $E_1$ which is the assertion.

**Appendix A. Some Functional Analysis**

In this appendix we investigate conditions for $K_0^0(E)$ or $B_0^0(E)$ being fundamental in $K_0(E)$.

**Proposition A.1.** In Fréchet spaces $K_0^0(E)$ is fundamental in $K_0(E)$.

**Proof.** The proof is essentially given in [6, Theorem 3.6.5] but presented here for convenience: If $K \subseteq K_0$ there exists $A \subseteq K_0$ such that $K$ is compact in $E_A$, cf. [20, Lemma p.18]. Again, in $E_A$ there exists $C \in K_0(E_A) \subseteq K_0(E)$ with $K$ compact in $E_C$ and $C$ is compact in $E_A$. A factorisation lemma of Davis-Fiegel-Johnson-Pelčiński [10, Corollary 1] for weakly compact operators provides a Banach space $Y$ which is reflexive and continuously embedded into $E_A$ and $C$ is bounded in $Y$. This continuity of $E_C \rightarrow Y$ yields that $K$ is compact in $Y$. Taking the closure $L$ in $Y$ of the linear hull of $K$ yields a reflexive subspace of $Y$ which is separable as
it is the closure of the image of the compact mapping $J: E_K \to Y$, where $J$ is the natural embedding). By blowing up by a suitable factor, the closed unit ball $K_0$ of $L$ can be chosen such that $K$ is contained in $K_0$ by continuity. Finally, $K_0$ is compact in $E$ as reflexivity implies that $K_0$ is weakly compact in $L$ and therefore weakly compact in $E$ by continuity of the natural embedding. But this implies that $K_0$ is weakly closed and by convexity and precompactness in $E$ we have $K_0 \in K_0(E)$. □

The following example establishes the connection of our approach to the work of Üstünel, cf. [40], who considered Lévy processes with values in strong duals of nuclear spaces which are nuclear and Suslin. In these spaces $K_0^*$ is fundamental in $K_0$.

**Example A.2.** Let $E'$ be a nuclear Suslin space and a strong dual of a separable barreled nuclear space $E$. Then, if $K$ is a compact set in $E'$, there exists an absolutely convex compact set $S \supset K$ such that $E_S$ is a separable Hilbert space. Thus, $K_0^*(E')$ is fundamental in $K_0(E')$.

**Proof.** In a nuclear space there exists a neighbourhood base $U$ such that for all $U \in U$ the completion of the space $E/p_{U^{-1}}(\{0\})$ is a separable Hilbert space $E(U)$. Its dual space can be identified with $E'_U^* = \{U^*: U \in U\}$ which is a fundamental system of closed bounded sets in $E'$ because $E$ is barreled, cf. [33, 5.2, p.141]. As $E'$ is nuclear, all bounded sets are precompact, cf. [33, p. 101, Corollary 2] and $K'$ consists of compact sets only. Choosing a compact set $K$ in $E'$ one finds an $S \subseteq K'$ with $S = U^*, U \in U$, such that $K \subseteq S$. Consequently, $K \subseteq E_S = (E(U))'$ which is a separable Hilbert space. □

**Remark A.3.** Üstünel claims in his proof of Theorem III.1 [40] that in his setting for a given $K \in K_0^*(E')$ one can always choose $S \subseteq K'$ such that $K \subseteq S$. This means $K'$ is a fundamental system of bounded sets, which is only the case if $E$ is barreled. But this assumption is missing in the mentioned paper.

One easily verifies the following stability properties:

**Lemma A.4.**

1. If $E_1, \ldots, E_n$ have fundamental systems of Banach disks, so does the locally convex direct sum $E_1 \oplus \ldots \oplus E_n$.

2. If $E$ has a fundamental system of Banach disks, so does every closed subspace $F$.

**A.1. A sufficient condition for $K_0^*(E)$ being fundamental.** The construction in Proposition A.1 is only known for Fréchet spaces. If for every compact disk $K$ in $E$ one can find a larger compact disk $B$ with compact embeddings $E_K \hookrightarrow E_B \hookrightarrow E$ one obtains a similar result.

**Proposition A.5.** Let $K \in K_0(E)$. If there exists a compact disk $B \subseteq E$ containing $K$ and such that the canonical injection $J: E_K \to E_B$ is compact, then there is also a compact disk $K_0 \in K_0^*(E)$ containing $K$.

**Proof.** First we note that $K$ is compact in $E_B$ as it is precompact by definition and closed by virtue of continuity of $E_B \hookrightarrow E$. The Banach space $E_B$ allows to find the desired compact set $K_0 \in K_0^*(E_B) \subseteq K_0^*(E)$ by Proposition A.1. □

In the literature, e.g. [20, 23], the notion of a co-Schwartz space is well-established. Let $\mathcal{S}$ be a system of bounded absolutely convex sets in $E$. A locally convex Hausdorff space $E$ is an $\mathcal{S}$-co-Schwartz space if for every $B \in \mathcal{S}$ there exists $C \supseteq B$, $C \in \mathcal{S}$ such that the natural embedding of the normed spaces $J_{BC}: E_B \to E_C$ admits a compact extension (to the completions). If $\mathcal{S}$ is the space of closed disks, $E$ is called a co-Schwartz space. In the following, we choose $\mathcal{S} = K_0(E)$. By Proposition A.5 we obtain:
Corollary A.6. In $K_0$-co-Schwartz spaces the family $K_0^s(E)$ is fundamental.

Corollary A.7. Let $K \in K_0(E)$. If there exists a bounded closed disk $B \subseteq E$ containing $K$ and such that the canonical injection $J: E_K \to E_B$ is compact, then there is also a compact disk $K_0 \in K_0^s(E)$ containing $K$.

Proof. In view of Proposition A.1, $K$ is compact in some Banach space $E_B$, $B \in \mathcal{B}_0(E)$ and there is a compact disk $K_0 \in K_0^s(E_B) \subseteq K_0^s(E)$ with $K_0 \supseteq K$.

The property of $B$ being a Banach disk only depends on duality. In fact, the factorisation theorem in Proposition A.1 tells that the property of $E$ being a $K_0$-co-Schwartz space only depends on duality as the compact operator $\iota: E_K \to E_B$ can be factored by two consecutive compact operators $E_K \hookrightarrow E_{K_0} \hookrightarrow E_B$ and $K$ is compact in $E_{K_0}$. This yields

Corollary A.8. Let $(E, \tau)$ be a locally convex space and $E' = (E, \tau)'$. If on $E$ there is a $K_0$-co-Schwartz locally convex topology $\tau'$ which is compatible with duality $(E, E')$, then $(E, \tau)$ is $K_0$-co-Schwartz. In this case, the family $K_0^s(E)$ is fundamental. Furthermore, then one can always choose compact sets from $K_0^s(E, \mu(E, E'))$, i.e., separable compact Banach disks in the Mackey topology.

Corollary A.9. Fréchet spaces are $K_0$-co-Schwartz.

Proposition A.10. Co-Schwartz spaces are $K_0$-co-Schwartz.

Proof. If $E$ is a co-Schwartz space, it is quasi-complete, cf. [20, Chapter 1, Theorem (4d)]. Let $K \in K_0(E)$. It suffices to show that there exists $B \in K_0(E)$ such that the canonical embedding $J_{KB}$ is compact. For $K$ one finds a larger (not necessarily compact) disk $C$ such that the extension of the canonical embedding $J_{KC}$ is compact. Without loss of generality $C$ is closed (e.g. take the closure of a suitable disk), so we assume that $C$ be closed, thus complete by quasi-completeness of $E$. Its linear hull $E_C$ is a Banach space. In particular, it is $K_0$-co-Schwartz by Corollary A.9. Therefore, one finds a compact disk $B \supseteq K$ in $E_C$ (and therefore in $E$) and the assertion follows.

Remark A.11. Interestingly, although we need our assumptions for different purposes, Dettweiler posed essentially the same two conditions in [12, Section 3]: In $E$ there should exist a fundamental system $\mathcal{K}_H^s$ of $\mathcal{K}_H$ of compact Hilbert disks ($E_K$ is a separable Hilbert space for all $K \in \mathcal{K}_H^s$). A second condition requests that for every $K \in \mathcal{K}_H$ there is an $L \in \mathcal{K}_H^s$, $K \subseteq L$, such that $\iota: E_K \to E_L$ is compact, i.e., it is a $K_0^s$-co-Schwartz space.

Acknowledgements. I want to express my deepest gratitude to my PhD supervisor Stefan Geiss for all the fruitful discussions and for his careful reading and valuable contributions to this work.

REFERENCES

[1] S. Albeverio and B. Rüdiger. Stochastic Integrals and Lévy-Itô decomposition on separable Banach spaces. In 2nd MaPhySto Lévy Conference, 2002.
[2] D. Applebaum. Lévy Processes and Stochastic Integrals in Banach Spaces. Probab. Math. Statist., 27:75–88, 2007.
[3] D. Applebaum. Lévy Processes and Stochastic Calculus. Cambridge University Press, Cambridge, 2nd edition, 2009.
[4] F. Baumgartner. Stochastic Analysis for Lévy Processes. PhD thesis, Universität Innsbruck, 2015.
[5] P. Billingsley. Convergence of probability measures. John Wiley & Sons, New York, 2 edition, 1999.
[6] V. I. Bogachev. Gaussian measures. Number 62 in Mathematical Surveys and monographs. American Mathematical Society, Rhode Island, 1991.
[7] V. I. Bogachev. Gaussian measures on linear spaces. *J. Math. Sci.*, 79(2):933–1034, 1996.
[8] T. Bojdecki and L. G. Gorostiza. Langevin equations for $S'$-valued Gaussian processes and fluctuation limits of infinite particle systems. *Probab. Theory Relat. Fields*, 73(2):227–244, 1986.
[9] T. Bojdecki and J. Jakubowski. The Girsanov theorem and weak solutions of stochastic differential equations in the dual of a nuclear space. *Stochastic Anal. Appl.*, 9(4):401–428, 1991.
[10] W. J. Davis, T. Figiel, W. B. Johnson, and A. Pelczynski. Factoring weakly compact operators. *J. Functional Analysis*, 17:311–327, 1974.
[11] E. Dettweiler. Grenzwertsätze für Wahrscheinlichkeitsmaße auf Badrikianschen Räumen. *Probability Theory and Related Fields*, 34:285–311, 1976.
[12] E. Dettweiler. Stabile Maße auf Badrikianschen Räumen. *Math. Z.*, 146(2):149–166, 1976.
[13] E. Dettweiler. Banach space valued processes with independent increments and stochastic integration. In *Probability in Banach spaces IV (Oberwolfach 1982)*, volume 990 of *Lectures Notes in Mathematics*, pages 54–83, 1983.
[14] S. N. Ethier and T. G. Kurtz. *Markov processes*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York, 1986. Characterization and convergence.
[15] H. Heyer. *Structural Aspects in the Theory of Probability*. World Scientific, Singapore, 2nd edition, 2010.
[16] K. Itô. *Foundations of stochastic differential equations in infinite-dimensional spaces*, volume 47 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1984.
[17] A. Janssen. Zero-one laws for infinitely divisible probability measures on groups. *Z. Wahrsch. Verw. Gebiete*, 60(1):119–138, 1982.
[18] H. Jarchow. *Locally convex spaces*. Teubner, Stuttgart, 1981. Introductory course on nuclear and conuclear spaces in the light of the duality “topology-bornology”, Notas de Matemática [Mathematical Notes], 79.
[19] K. Itô. *Foundations of stochastic differential equations in infinite-dimensional spaces*, volume 47 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1984.
[20] R. Kumar. Current fluctuations for independent random walks in multiple dimensions. *J. Theoret. Probab.*, 24(4):1170–1195, 2011.
[32] K. Sato. Lévy Processes and Infinitely Divisible Distributions. Cambridge University Press, Cambridge, 1999.

[33] H. H. Schaefer and M. P. Wolff. Topological vector spaces, volume 3 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, 1999.

[34] L. Schwartz. Radon measures on arbitrary topological spaces and cylindrical measures. Published for the Tata Institute of Fundamental Research, Bombay by Oxford University Press, London, 1973. Tata Institute of Fundamental Research Studies in Mathematics, No. 6.

[35] L. Schwartz. Processus de Markov et désintégrations régulières. Ann. Inst. Fourier (Grenoble), 27(3):xi, 211–277, 1977.

[36] E. Siebert. Einbettung unendlich teilbarer Wahrscheinlichkeitsmasse auf topologischen Gruppen. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 28:227–247, 1973/74.

[37] G. E. F. Thomas. Integration of Functions With Values in Locally Convex Suslin Spaces. Transactions of the American Mathematical Society, 212:pp. 61–81, 75.

[38] A. Tortrat. Sur la structure des lois indéfiniment divisibles (classe $t(X)$) dans les espaces vectoriels $X$ sur le corps réel). Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 11:311–326, 1969.

[39] A. S. Üstünel. Stochastic Integration on Nuclear Spaces. Annales de l’Institut Henri Poincaré, 18(2):165–200, 1982.

[40] A. S. Üstünel. Additive Processes on Nuclear Spaces. Ann. Probab., 12(3):858–868, 1984.

[41] N. N. Vakhania, V. I. Tarieladze, and S. A. Chobanyan. Probability distributions on Banach spaces, volume 14 of Mathematics and its Applications (Soviet Series). D. Reidel Publishing Co., Dordrecht, 1987. Translated from the Russian and with a preface by Wojbor A. Woyczynski.

[42] G. Ziglio and B. Rüdiger. Itô formula for stochastic integrals w.r.t. compensated Poisson random measures on separable Banach spaces. Stochastics, 78(3):377–410, 2006.