Intrinsic Time in Geometrodynamics of Closed Manifolds

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The global time in Geometrodynamics is defined in a covariant under diffeomorphisms form. An arbitrary static background metric is taken in the tangent space. The global intrinsic time is identified with the logarithm of the mean value of the square root of the ratio of the metric determinants. The procedures of the Hamiltonian reduction and deparametrization of dynamical systems are implemented. The explored Hamiltonian system appeared to be non-conservative. The Hamiltonian equations of motion of gravitational field in the global time are written. Relations between different time intervals (coordinate, intrinsic, proper) are obtained.

Keywords: Geometrodynamics, many-fingered intrinsic time, global time, deparametrization, Hamiltonian reduction

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1. Introduction

Geometrodynamics is a theory describing space and time in its inner essence and the spatial metric carries information about the intrinsic time. “Time has a double face, as the Time-god, Janus, has in the ancient representations. One quantity has two interpretations – chronometrical, and dynamical”\textsuperscript{1}. The Hamiltonian dynamics of gravitational field is commonly formulated by construction in redundant variables in the extended functional phase space as a consequence of covariant description of Einstein’s theory. A time parameter should be conjugated to the Hamiltonian constraint. The Hamiltonian formulation of the theory makes it possible to reveal the physical meaning of the geometrical variables. The problem of the global time definition is a basic one in General Relativity\textsuperscript{2,3}. It was demonstrated that to construct a quantum theory one needs to use the so-called bubble-time derivatives instead of the ordinary ones\textsuperscript{4,5}, that complicates the physical problem. The quantum Wheeler–DeWitt (or Schrödinger) equation for wave functions is then constructed in functional derivatives. The conservative energy is well defined in asymptotically flat spaces because of the existence of the preferable rectangular coordinate frame with a coordinate time\textsuperscript{6}. The quantities are defined by boundary integrals\textsuperscript{7}.

The Hamiltonian formalism was developed with the improved Hamiltonian\textsuperscript{7} or through modification the Poisson brackets\textsuperscript{8}. In the case of closed manifolds, the constant mean curvature slicing is preferable. And in this case one can’t introduce an outer observer with his watch. The global time can then be found from the Hamiltonian constraint in homogeneous models, see, e.g., refs.\textsuperscript{2,9,10}. The conformal Dirac’s mapping\textsuperscript{11} allows to extract a local intrinsic time. The Cauchy problem was solved in conformal variables successfully\textsuperscript{12,13}. Misner introduced the intrinsic time as a logarithm of hypersurface volume\textsuperscript{10} in a homogeneous mixmaster universe.

In this paper we describe in detail a self-consistent procedure that allows to introduce an intrinsic time in a closed manifold and discuss the corresponding implications in the Hamiltonian reduction.
The notation and preliminaries are given in Sect. 2. Introduction of a many-fingered intrinsic time is discussed in Sect. 3. Sect. 4 presents the main result of the paper which consists in the definition of the global intrinsic time. Using the defined global time, the Hamiltonian reduction and deparametrization are performed in Sect. 5. Residue dynamical variables in a perturbed Friedmann universe are discussed in Sect. 6. Hamiltonian equations of motion are derived in Sect. 7. Relations between the intrinsic and coordinate time intervals are given in Sect. 8. Sect. 9 contains discussion and conclusions. Some detail of calculations are put in the Appendix.

2. ADM variational functional. Notations

The spacetime $\mathcal{M} = \mathbb{R}^1 \times \Sigma_t$ with the metric tensor field

$$g := g_{\mu\nu}(t, x) dx^\mu \otimes dx^\nu$$

can be foliated into a family of space-like (hyper)surfaces $\Sigma_t$, labeled by the time coordinate $t$ with just three spatial coordinates on each slice $(x^1, x^2, x^3)$. Then, the evolution of space in time can be described in a natural way. The components of the metric tensor in the Arnowitt–Deser–Misner (ADM) form are

$$(g_{\mu\nu}) = \left( -N^2 + N_i N^i \frac{N_j}{\gamma_{ij}} \right).$$

(1)

Here and below Latin indices run as $i, j = 1, 2, 3$. The scalar field $N$ and the 3-vector field $\mathbf{N}$ extend the coordinate system out of $\Sigma_t$. The first quadratic form

$$\gamma := \gamma_{ik}(t, x) dx^i \otimes dx^k$$

(2)

defines the induced metric on every slice $\Sigma_t$.

The components of the extrinsic curvature tensor $K_{ij}$ of every slice are constructed out of the second quadratic form of the hypersurface and can be defined as

$$K_{ij} := -\frac{1}{2} \ell_n \gamma_{ij},$$

(3)

where $\ell_n$ denotes the Lie derivative along the time-like unit normal to the slice vector $\mathbf{n}$. The components of the extrinsic curvature tensor can be found by using the Lie derivative along the vector field $\mathbf{N}$:

$$K_{ij} = \frac{1}{2N} \left( \ell_N \gamma_{ij} - \frac{d\gamma_{ij}}{dt} \right) = \frac{1}{2N} \left( \nabla_i N_j + \nabla_j N_i - \frac{d\gamma_{ij}}{dt} \right),$$

(4)

where $\nabla_k$ is the Levi–Civita connection associated with metric $\gamma_{ij}$: $\nabla_k \gamma_{ij} = 0$. The phase space $\Gamma$ is coordinatized by the 3-metric $\gamma_{ij}$ and its conjugate momentum density of weight 1 is defined by the components

$$\pi^{ij} := -\sqrt{\gamma}(K^{ij} - K \delta^{ij}).$$

(5)

Here we introduced the notation

$$K^{ij} := \gamma^{ik} \gamma^{jl} K_{kl}, \quad K := \gamma^{ij} K_{ij}, \quad \gamma := \det(\gamma_{ij}), \quad \gamma_{ij} \gamma^{jk} = \delta^k_i.$$  

(6)

By contracting the tensors on both sides in (4), one obtains

$$\sqrt{\gamma}K = -\ell_n \sqrt{\gamma}.$$  

The extrinsic curvature tensor $K_{ij}$ provides a measure of the bending of the hypersurface $\Sigma_t$ with respect to the external space. The tensor is proportional to the time derivative of $\gamma_{ij}$. So it is naturally connected with physical quantities which are the momentum densities $\pi^{ij}$ of the metric $\gamma_{ij}$.
To construct a functional phase space $\Gamma$ in the Banach space, one should first define functionals. The Poisson bracket is a bilinear operation on two arbitrary functionals $F[\gamma_{ij}, \pi^{ij}], G[\gamma_{ij}, \pi^{ij}]$ over the hypersurface $\Sigma_t$

$$\{F, G\} := \int_{\Sigma_t} d^3x \left( \frac{\delta F}{\delta \gamma_{ij}(t, x)} \frac{\delta G}{\delta \pi^{ij}(t, x)} - \frac{\delta G}{\delta \gamma_{ij}(t, x)} \frac{\delta F}{\delta \pi^{ij}(t, x)} \right).$$

(7)

The canonical variables satisfy the fundamental relations

$$\{\gamma_{ij}(t, x), \pi^{kl}(t, x')\} = \delta_{ij}^{kl} \delta(x - x'),$$

(8)

where

$$\delta_{ij}^{kl} := \frac{1}{2} (\delta_i^k \delta_j^l + \delta_i^l \delta_j^k),$$

and $\delta(x - x')$ is the Dirac’s $\delta$-function for the volume of $\Sigma_t$ defined without recourse to the metric of space by

$$\int_{\Sigma_t} d^3x \delta(x - x') f(x') = f(x)$$

for an arbitrary scalar probe function.

The super-Hamiltonian of the gravitational field is the functional

$$H_{ADM} := \int_{\Sigma_t} d^3x \left( N(x) H_\perp(x) + N^i(x) H_i(x) \right),$$

(9)

where $N$ (lapse) and $N^i$ (shift) are Lagrange multipliers; $H_\perp$ and $H_i$ have the sense of constraints. In particular,

$$H_\perp(x) := \frac{1}{2} \pi^{ij}(x) G_{ijkl}(x) \pi_{kl}(x) - \sqrt{\gamma}(x) R[\gamma](x) + \sqrt{\gamma}(x) T_{\perp \perp}$$

is obtained from the scalar Gauss relation of the embedding hypersurfaces theory and called the Hamiltonian constraint. Here $R[\gamma]$ is the Ricci scalar of the space,

$$G_{ijkl} := (\gamma_{ik} \gamma_{jl} + \gamma_{il} \gamma_{jk} - \gamma_{ij} \gamma_{kl})$$

is the supermetric of the 6-dimensional hyperbolic Wheeler–DeWitt (WDW) superspace. The matter density

$$T_{\perp \perp} := n^\mu n^\nu T_{\mu \nu}$$

is defined in a normal reference frame. Momentum constraints

$$H^i(x) := -2 \nabla_j \pi^{ij}(x) + \sqrt{\gamma}(x) T_{\perp \perp}^i$$

are obtained from the contracted Codazzi equations of the embedding hypersurfaces theory. The components of the matter momentum density

$$(T_{\perp})_i := n^\mu T_{\mu i}$$

are defined in an observer’s normal frame reference. The constraints impose restrictions on possible data $\gamma_{ij}(x, t), \pi^{ij}(x, t)$ on a space-like hypersurface $\Sigma_t$. Momentum constraints generate 3-diffeomorphisms. To prove the statement let them be smeared with a vector field $\xi$

$$\langle \xi_i|H^i \rangle = \int_{\Sigma_t} d^3x \xi_i(x) H^i(x)$$

and take the Poisson bracket with the metric

$$\{\gamma_{ij}(x), (\xi_i|H^i)\} = \nabla_i \xi_j + \nabla_j \xi_i \equiv L \xi \gamma_{ij}.$$ 

(12)
By this rule an infinitesimal diffeomorphism acts on a metric field. The Killing equations follow from (12) as condition of the Lie derivative of the metric along $\xi$ to be zero. Constraints (10) and (11) are of the first class since they belong to a closed algebra.

Then the Hamiltonian dynamics is built of the ADM variational functional

$$S = \int_{t_i}^{t_f} dt \int d^3x \left( \pi^{ij} \frac{d\gamma_{ij}}{dt} - N \mathcal{H}_+ - N^i \mathcal{H}_i \right),$$

where the common ADM units $c = 1, 16\pi G = 1$ were used. Action (13) is obtained from the Hilbert functional after the procedure of $(3 + 1)$ foliation and the Legendre transformation are executed.

3. Many-fingered intrinsic time

The Dirac’s mapping to conformal variables has a restricted domain of applicability: they can be used only for coordinate systems with a dimensionless determinant of the metric. But often one needs to perform calculations, e.g., in a spherical coordinate system $(r, \theta, \varphi)$. Then one has to use the spherical Schwarzschild’s coordinates

$$x^1 = r^3, \quad x^2 = -\cos \theta, \quad x^3 = \varphi$$

with a unit metric determinant. To expand the region of applicability of the Dirac’s mapping, let us implement the following conformal transformation

$$\gamma_{ij} := \phi^4 \tilde{\gamma}_{ij}, \quad \phi^4 := \frac{3}{\sqrt{\gamma f}},$$

where, in addition to the determinant $\gamma$, the background static metric determinant $f$ has appeared

$$f := \det(f_{ij}).$$

Introduction of the background metric is the key point of this paper. It will allow us to consider not only asymptotically flat spaces but also general closed manifolds. In addition to the space metric (2), the background metric of the tangent space, Lie-dragged along the coordinate time evolution vector, can be introduced:

$$f := f_{ik}(x) dx^i \otimes dx^k.$$ (15)

The Minkowskian metric as the background one was used in for description of gravitational problems in asymptotically flat spacetimes. We claim that the background metric has to be chosen from the physical point of view. The mapping of the Riemannian space with metric $\gamma$ to the background space with the metric $f$ should be bijective. For this reason we suggest to consider a closed manifold. The conformal metric

$$\tilde{\gamma} := \tilde{\gamma}_{ik}(t, x) dx^i \otimes dx^k$$

is a tensor field, i.e., it is transformed according to the tensor representation of the group of diffeomorphisms. The scaling factor $(\gamma/f)$ is a scalar field, i.e., it is invariant under the diffeomorphisms. To the conformal variables

$$\tilde{\gamma}_{ij} := \frac{\gamma_{ij}}{\sqrt{\gamma/f}}, \quad \tilde{\pi}^{ij} := \sqrt{\frac{3}{f}} \left( \pi^{ij} - \frac{1}{3} \pi \gamma^{ij} \right),$$

we add the canonical pair: the local intrinsic time $D$ and the trace of momentum density $\pi$:

$$D := -\frac{2}{3} \ln \sqrt{\frac{3}{f}}, \quad \pi = 2K \sqrt{\gamma}.$$ (18)

Formulae (17) and (18) define the scaled Dirac’s mapping

$$(\gamma_{ij}, \pi^{ij}) \mapsto (D, \pi; \tilde{\gamma}_{ij}, \tilde{\pi}^{ij}).$$ (19)
The symplectic potential takes the form
\[ \omega^1 := \pi^{ij} d\gamma_{ij} = \tilde{\pi}^{ij} d\tilde{\gamma}_{ij} - \pi dD. \] (20)

The Lie algebra of new variables in the extended phase space \( \Gamma_D \) are the same as in 7 because of the static nature of the background metric
\[
\{ D(t, x), \pi(t, x') \} = -\delta(x - x'),
\]
(21)
\[
\{ \tilde{\pi}^{ij}(t, x), \tilde{\pi}^{kl}(t, x') \} = \tilde{\delta}^{ij}_{kl} \delta(x - x'),
\]
(22)
\[
\{ \tilde{\pi}^{ij}(t, x), \tilde{\pi}^{kl}(t, x') \} = \frac{1}{3} (\tilde{\gamma}^{kl} \tilde{\pi}^{ij} - \tilde{\pi}^{ij} \tilde{\gamma}^{kl}) \delta(x - x'),
\]
(23)
where
\[
\tilde{\delta}^{ij}_{kl} := \delta^{i}_k \delta^{j}_l + \delta^{i}_l \delta^{j}_k - \frac{1}{3} \tilde{\gamma}^{kl} \tilde{\gamma}_{ij}
\]
is the conformal Kronecker symbol.

The Hamiltonian constraint in the new variables yields the Lichnerowicz–York differential equation
\[
\left( \hat{\Delta} - \frac{1}{8} \hat{R} \right) \phi + \frac{1}{8} \hat{\pi}_{ij} \hat{\pi}^{ij} \phi^{-7} - \frac{1}{12} K^2 \phi^5 + \frac{1}{8} \hat{T}_{\perp \perp} \phi^5 = 0.
\]
(24)
Here \( \hat{\nabla}_k \) is the conformal connection associated with the conformal metric \( \hat{\gamma}_{ij} \); quantity \( \hat{R} \) is the conformal Ricci scalar related to the standard Ricci scalar \( R \):
\[
R = \frac{1}{\phi^4} \hat{R} - \frac{8}{\phi^5} \hat{\Delta} \phi.
\]
(25)

The matter density is transformed according to
\[
\tilde{T}_{\perp \perp} := \phi^8 T_{\perp \perp},
\]
(26)
where \( T_{\perp \perp} \) is a component of the energy-momentum tensor along the future pointing normal to \( \Sigma_t \).

4. Intrinsic global time

The momentum density \( \pi \) enters into the conformal Hamiltonian constraint quadratically [23] as usual for relativistic theories. So, with the plus sign, it can be expressed from the Hamiltonian constraint [10]. Taking into account the role of the Hamiltonian \( H(x) \), the integral over the hypersurface of \( \pi(x) \) should give us the canonically conjugated global time \( T \). Let us extract the zero mode out of the scalar field that is the square root of ratio of the determinants of metric tensors \( \sqrt{\gamma/\tilde{f}}(x) \)
\[
\sqrt{\gamma}(x) = < \sqrt{\gamma/\tilde{f}}(x) >, \]
(27)
where the mean value of it over the hypersurface \( \Sigma_t \) is
\[
< \sqrt{\gamma/\tilde{f}} > := \frac{\int_{\Sigma_t} d^3y \sqrt{\gamma}(y) \sqrt{\gamma/\tilde{f}}(y)}{\int_{\Sigma_t} d^3y \sqrt{\gamma}(y)}.
\]
(28)
According to the construction [28], the second term in (27) is the residue of the \( \sqrt{\gamma/\tilde{f}}(x) \) with a zero mean value over a hypersurface \( \Sigma_t \)
\[
\int_{\Sigma_t} d^3y \sqrt{\gamma}(y) \sqrt{\gamma/\tilde{f}}(y) = 0.
\]
(29)
Now we can define the global time as discussed in our preprint [19]
\[
T(t) := -\frac{2}{3} \ln < \sqrt{\gamma/\tilde{f}} >
\]
(30)
as the logarithmic function of the mean value over a hypersurface $\Sigma_t$ at every instant $t$. The commutator of the global time $T(t)$ with the integral characteristics $P(t)$ of the field $\pi(x)$

$$P(t) := \int_{\Sigma_t} d^3x \pi^{ij}(x)\gamma_{ij}(x)$$

(31)

is

$$\{T, P\} = -1.$$  

(32)

So, they form a global canonical pair.

Then, we extract the zero mode of the field $\pi(x)$, that is a scalar density

$$\pi(x) = \sqrt{\gamma}(x) < \pi > + \bar{\pi}(x),$$

(33)

where the mean value of $\pi(x)$ over a hypersurface $\Sigma_t$ is

$$< \pi > := \frac{\int_{\Sigma_t} d^3y \pi(y)}{\int_{\Sigma_t} d^3\sqrt{\gamma(y)}}.$$  

(34)

Two thirds of the average of $\pi$ over $\Sigma_t$ is the global York time $^{20}$ and the conjugated variable (the Hamiltonian) is the volume of the hypersurface in $^{21}$. Our approach is different because the essence of these geometric characteristics is treated in the opposite way. Note that in our approach the integral characteristics $P(t)$ of the field $\pi(x)$ is the canonical partner of the global time. In general, canonical momenta can’t be defined within a hypersurface. They should refer to the motion in time of the original $\Sigma_t$. While the intrinsic time is the variable constructed entirely out of the metric of the hypersurface. In this way, we establish the roles of the Hamiltonian and the global time in accord with the general principles.

The second term in (33) is the residue of the $\pi(x)$ with zero mean value over a hypersurface $\Sigma_t$

$$\int_{\Sigma_t} d^3x \bar{\pi}(x) = 0.$$  

(35)

Thus, the mapping of the phase space $\Gamma_D$ onto the phase space $\bar{\Gamma}$ after extraction of the global variables $T$ and $H$ is executed:

$$(D, \pi; \bar{\gamma}_{ij}, \bar{\pi}^{ij}) \mapsto (T, P, \bar{\pi}, \sqrt{\gamma/f}; \bar{\gamma}_{ij}, \bar{\pi}^{ij}).$$

New variables are expressed via the old ones by the following formulae:

$$T(t) = -\frac{2}{3} \ln \left[ \frac{\int_{\Sigma_t} d^3y \sqrt{T(y)} \exp[-3D(y)]}{\int_{\Sigma_t} d^3y \sqrt{T(y)} \exp[-(3/2)D(y)]} \right],$$

$$P(t) = \int_{\Sigma_t} d^3y \pi(y),$$

$$\sqrt{\gamma/f}(x) = \exp[-(3/2)D(x)] - \left[ \frac{\int_{\Sigma_t} d^3y \sqrt{T(y)} \exp[-3D(y)]}{\int_{\Sigma_t} d^3y \sqrt{T(y)} \exp[-(3/2)D(y)]} \right],$$

$$\bar{\pi}(x) = \pi(x) - \sqrt{f(x)} \exp[-(3/2)D(x)] \frac{\int_{\Sigma_t} d^3y \pi(y)}{\int_{\Sigma_t} d^3y \sqrt{T(y)} \exp[-(3/2)D(y)]};$$
The extrinsic curvature scalar is expressed through the dynamical field characteristics

\[ \{T, P, \pi, \sqrt{\gamma/f} \} \] appear to be nonlinear, they read

\[ \{T, P\} = -1, \quad \{\sqrt{\gamma/f}(x), P\} = \frac{3}{2} \sqrt{\gamma/f}(x), \quad \{\pi(x), P\} = 0, \]

\[ \{\pi(x), T\} = \frac{2}{V_t} \sqrt{f(x)} \sqrt{\gamma/f}(x) \left(1 + \sqrt{\gamma/f}(x) \exp[(3/2)T]\right), \]

\[ \{\pi(x), T\} = \frac{3}{V_t} \sqrt{f(x)} \left(\sqrt{\gamma/f}(x) \exp[-(3/2)T]\right) \]

The relation between the time intervals depends on the global time

\[ \phi \]

Functions of \( \phi \) are expressed via the global time and the residues, e.g., function \( \phi^6(x) \) is a sum of two parts:

\[ \phi^6(x) = \phi_t^6(x) = \sqrt{\frac{\gamma}{f}(x)} + \sqrt{\frac{\gamma}{f}(T)} = \exp\left(-\frac{3}{2}T\right) + \sqrt{\frac{\gamma}{f}(x)}. \]
The Hamiltonian density (43) can be rewritten in the explicit form

\[ H(x) = 4\sqrt{3} \left[ \frac{\tilde{\pi}_{ij} \tilde{\pi}^{ij}}{8 \left( \exp\left[-\left(\frac{3}{2}\right)T\right] + \sqrt{\gamma/f} \right)^{5/6}} \times \left( \tilde{\Delta} - \frac{1}{8} \tilde{R} \right) \left( \exp\left[-\left(\frac{3}{2}\right)T\right] + \sqrt{\gamma/f} \right)^{1/6} + \frac{1}{8} \tilde{T}_{\perp \perp} \right]^{1/2}. \]  

(45)

The residue field \( \sqrt{\gamma/f} \) does not admit to the Hamiltonian equations of motion. From the ADM reduced action (41) one obtains the equation to the field

\[ \frac{\delta H}{\delta \sqrt{\gamma/f}} = 0. \]  

(46)

6. Residue terms in a perturbed Friedmann universe

Let us consider spatial metric perturbations in a Friedmann model and elucidate the role of the residue term in (27). For this purpose we use the harmonic analysis of linear geometric perturbations using irreducible representations of the isometry group of a constant curvature space. The eigenfunctions of the Laplace–Beltrami operator form a basis of unitary representations of the group \( \tilde{\gamma} \)

\[ Y_{nlm}(\chi, \theta, \varphi) = 2l! \sqrt{\frac{2(n + 1)(n - l)!}{\pi(n + l + 1)!}} \sin^l \chi C_{n-l}^{l+1}(\cos \chi)Y_{lm}(\theta, \varphi). \]  

(47)

Here, \( C_{n-l}^{l+1}(\cos \chi) \) are the Gegenbauer polynomials \( \tilde{\gamma} \), and \( Y_{lm}(\theta, \varphi) \) are spherical harmonics. Indices run over the following values

\[ n = 0, 1, 2, \ldots; \quad l = 0, 1, \ldots, n; \quad m = -l, -l + 1, \ldots, l. \]

The first eigenfunctions are

\[ Y_{000} = \frac{1}{\sqrt{2\pi}}, \quad Y_{100} = \frac{\sqrt{2}}{\pi} \cos \chi, \quad Y_{110} = \frac{\sqrt{2}}{\pi} \sin \chi \cos \theta, \quad Y_{11, \pm 1} = \pm \frac{1}{\pi} \sin \chi \sin \theta e^{\pm i\varphi}. \]  

(48)

The orthogonality condition and normalization of the basis functions (47) are well known

\[ \int_0^\pi d\chi \sin^2 \chi \int_0^{2\pi} d\theta \sin \theta \int_0^{2\pi} d\varphi Y^*_{n'l'm'}(\chi, \theta, \varphi) Y_{nlm}(\chi, \theta, \varphi) = \delta_{nn'}\delta_{ll'}\delta_{mm'}. \]  

(49)

Omitting the quantum indexes of the basis functions \( \tilde{\gamma} \), the equation for eigenvalues can be represented in the following symbolic form

\[ ((\hat{\Delta}^{(1)}) + k^2)Y^{(s)}(x) = 0, \]  

(50)

where \( -k^2 \) is an eigenvalue of the Laplace–Beltrami operator \( \hat{\Delta}^{(1)} \) on a 3-sphere of unit radius. The connection \( \hat{\nabla} \) is associated with this metric \( \hat{f}_{ij} \). For a positive curvature space we have \( k^2 = n(n + 2) \). So, we get the expressions for the residue terms in the countable basis \( Y_{nlm}(\chi, \theta, \varphi) \)

\[ \sqrt{\gamma}(x) = \sum_{n=1}^{\infty} \sum_{l=0}^{n} \sum_{m=-l}^{l} \gamma_{nlm} Y_{nlm}(\chi, \theta, \varphi), \]

\[ \pi(x) = \sum_{n=1}^{\infty} \sum_{l=0}^{n} \sum_{m=-l}^{l} \pi_{nlm} Y_{nlm}(\chi, \theta, \varphi) \]

with the expansion coefficients \( \gamma_{nlm} \) and \( \pi_{nlm} \).
7. Hamiltonian equations of motion

The momentum constraints generate spatial diffeomorphisms. Below, by studying the dynamics, we choose the semi-geodesic slicing, ignoring the action of the generators of diffeomorphisms \([11]\). The energy of the universe in our situation is not conserved, it exponentially increases in time \(T\). The Hamiltonian flow is governed by the Hamiltonian \([12]\) with the Poisson brackets \([22]\) and \([23]\) (see details in Appendix):

\[
\frac{d}{dT} \tilde{\gamma}_{ij}(x) = \int_{\Sigma_t} d^3x' \left\{ \tilde{\gamma}_{ij}(x), \tilde{\pi}^{kl}(x') \right\} \frac{\delta}{\delta \tilde{\pi}^{kl}(x')} H,
\]

\[
\frac{d}{dT} \tilde{\pi}^{ij}(x) = \int_{\Sigma_t} d^3x' \left\{ \tilde{\pi}^{ij}(x), \tilde{\pi}^{kl}(x') \right\} \frac{\delta}{\delta \tilde{\pi}^{kl}(x')} H + \int_{\Sigma_t} d^3x' \left\{ \tilde{\pi}^{ij}(x), \tilde{\gamma}_{kl}(x') \right\} \frac{\delta}{\delta \tilde{\gamma}_{kl}(x')} H.
\]  

(51) (52)

The functional derivative with respect to the momentum density reads

\[
\frac{\delta}{\delta \tilde{\pi}^{kl}(x')} H[\phi, \tilde{\pi}^{ij}, \tilde{\gamma}_{ij}] = \frac{6\gamma(x')}{\phi^{12}(x')} H[\phi, \tilde{\pi}^{ij}, \tilde{\gamma}_{ij}; x'] \tilde{\pi}^{kl}(x').
\]

(53)

The derivative of the conformal metric with respect to the global time \([51]\) after application of \([22]\) and \([53]\) becomes

\[
\frac{d}{dT} \tilde{\gamma}_{ij}(x) = \frac{12\gamma(x)\tilde{\pi}_{ij}(x)}{\phi^{12}(x) H[\phi, \tilde{\pi}^{ij}, \tilde{\gamma}_{ij}; x]}.
\]

(54)

Thus, the relation between the derivative of the generalized coordinates \(\tilde{\gamma}_{ij}\) with respect to the global time \(T\) with the conjugate generalized momenta \(\tilde{\pi}^{ij}\) is obtained.

The first term in \([54]\) is calculated easily

\[
\int_{\Sigma_t} d^3x' \left\{ \tilde{\pi}^{ij}(x), \tilde{\pi}^{kl}(x') \right\} \frac{\delta}{\delta \tilde{\pi}^{kl}(x')} H = -\frac{2\gamma(x)\tilde{\gamma}^{ij}(x)\tilde{\pi}^{kl}(x)\tilde{\pi}_{kl}(x)}{\phi^{12}(x) H[\phi, \tilde{\pi}^{ij}, \tilde{\gamma}_{ij}; x]}.
\]

(55)

One can write the functional derivative of the Hamiltonian density \(H\) with respect to the conformal metric components \(\tilde{\gamma}_{kl}\) as

\[
\frac{\delta}{\delta \tilde{\gamma}_{kl}(x)} H[\phi; \tilde{\gamma}_{ij}, \tilde{\pi}^{ij}] = \int_{\Sigma_t} d^3y \left( \frac{3\gamma(y)}{\phi^{4}(y) H(y)} \frac{\delta}{\delta \tilde{\gamma}_{kl}(x)} R[\tilde{\gamma}_{ij}, y] \right)
\]

\[
+ \frac{24\gamma(y)}{\phi^{5}(y) H(y)} \frac{\delta}{\delta \tilde{\gamma}_{kl}(x)} \tilde{\Delta} \phi(y) \right).
\]

The functional derivative of the Ricci scalar with respect to the metric coefficients reads

\[
\frac{\delta}{\delta \tilde{\gamma}_{kl}(x)} R[\tilde{\gamma}_{ij}, y] = \left( -\tilde{R}^{kl}[\tilde{\gamma}_{ij}; y] + \tilde{\gamma}^{kl}(y) \tilde{\Delta} y - \tilde{\nabla}^k \tilde{\nabla}^l \right) \delta(x - y).
\]

By summing up these four terms we obtain the functional derivative

\[
\frac{\delta}{\delta \tilde{\gamma}_{kl}(x)} H[\phi, \tilde{\gamma}_{ij}, \tilde{\pi}^{ij}] = \frac{3\gamma(x)}{\phi^{4}(x) H(x)} \tilde{R}^{kl} + 3 \left( \tilde{\nabla}^k \tilde{\nabla}^l - \tilde{\gamma}^{kl}(x) \tilde{\Delta} x \right) \frac{\gamma(x)}{\phi^{5}(x) H(x)}
\]

\[+ 12(\tilde{\gamma}^{km} \tilde{\gamma}^{ln} - \tilde{\gamma}^{kl} \tilde{\gamma}^{mn}) \tilde{\nabla}^m \frac{\gamma(x)}{\phi^{5}(x) H(x)} \tilde{\nabla}^n \phi(x).
\]

(56)

Finally, the derivative of the conformal momentum density with respect to the global time \([52]\) with application of the commutation relations between conformal phase variables and taking into
A.B. Arbuzov and A.E. Pavlov account (55) and (56), becomes
\[
\frac{d}{dT} \tilde{\pi}_{ij}(x) = -\frac{6\gamma(x)}{\phi^2(x)\mathcal{H}(x)} \left( \tilde{R}^{ij} - \frac{1}{6} \tilde{\nabla}^j \tilde{R} \right) - \frac{2\gamma(x)}{\phi^2(x)\mathcal{H}(x)} \tilde{\pi}^{ij}(x) \tilde{\pi}^{kl} \tilde{\pi}_{kl}
\]
\[
-3 \left( \tilde{\nabla}^i \tilde{\nabla}^j + \tilde{\nabla}^i \tilde{\nabla}^k \tilde{\nabla}_k \right) \left[ \frac{\gamma(x)}{\phi^2(x)\mathcal{H}(x)} \right]
\]
\[
-24 \left( \tilde{\pi}^{ik} \tilde{\pi}^{jl} + \tilde{\pi}^{jk} \tilde{\pi}^{il} - \frac{2}{3} \tilde{\pi}^{ij} \tilde{\pi}^{kl} \right) \tilde{\nabla}_i \phi \tilde{\nabla}_k \left[ \frac{\gamma(x)}{\phi^2(x)\mathcal{H}(x)} \right].
\]
(57)

8. Relations between time intervals

To obtain the relation between an intrinsic time interval and a coordinate time interval one should find the derivative of a hypersurface volume with respect to the coordinate time
\[
\frac{dV_t}{dt} = \int_{\Sigma_t} d^3x \frac{\partial}{\partial t} \sqrt{\gamma} = \int_{\Sigma_t} d^3x \left( -NK + \nabla_i N^i \right) \sqrt{\gamma}.
\]
(58)
The above equality follows from the definition (4)
\[
\nabla \cdot N - NK = \frac{1}{2} \frac{\partial}{\partial t} \gamma_{ij} = \frac{1}{\sqrt{\gamma}} \frac{d}{dt} \sqrt{\gamma}.
\]
The second integrand is zero according to the Gauss–Ostrogradsky theorem
\[
\int_{\Sigma_t} d^3x \nabla_i N^i \sqrt{\gamma} = \int_{\Sigma_t} d^3x \frac{\partial}{\partial x^i} \left( N^i \sqrt{\gamma} \right) = 0.
\]
Hence, the speed of the volume change depends on the lapse function only
\[
\frac{dV_t}{dt} = -\int_{\Sigma_t} d^3x \sqrt{\gamma} NK.
\]
(59)
The maximal slicing corresponds to the vanishing of the scalar \( K = 0 \). Analogously, the relation between the intrinsic time interval \( dD \) (18) and the coordinate time interval \( dt \) is obtained
\[
\frac{dD}{dt} = \frac{2}{3} \left( NK - \nabla_i N^i \right).
\]
(60)
It is composed of normal and tangent components.

In this way, we obtain the relation between the intrinsic global time interval \( dT \) (30) and the coordinate time interval \( dt \)
\[
\frac{dT}{dt} = \frac{4}{3} \frac{\sqrt{\gamma/f} (NK - \nabla_i N^i)}{\sqrt{\gamma/f}} - \frac{2}{3} < NK >.
\]
(61)
The conformal Friedmann equation also can be written if the conformal time interval is introduced \( d\eta = dt/a \). The lapse function \( N \) and the shift vector \( N \) are contained in (61). According to the thin sandwich conjecture \( 24 \), they can be found from the theory. In the case of semi-geodesic slicing \( N = 0 \), one obtains the simple relation between the time intervals \( d\tau = N dt \), demonstrating the notion of the lapse of the proper watch.

The application of the perfect cosmological principle to the Friedman interval yields the ratio of universe radius \( a(t) \) to the present day one \( a_0 \) from the standard cosmological conception
\[
\left( \frac{a(t)}{a_0} \right)^2 = e^{-T}.
\]
Here, for the background space a sphere of the present day radius \( a_0 \) is taken. The global time is related to the observable redshift \( 25 \) in the standard way:
\[
z(t) = \frac{a_0 - a(t)}{a(t)}.
\]
9. Discussion and conclusions

We demonstrated that in Geometrodynamics of closed manifolds, it is possible to generalize the Misner approach and introduce the global time. After application of the Hamiltonian reduction procedure, we got differential evolution equations for conformal metric components and conformal momentum densities. They do not contain Lagrange multipliers contrary to the ones obtained in 6. In opposite to the case of asymptotically flat spacetimes, for closed manifolds we got a non-conservative Hamiltonian systems.

It was assumed \(26, 27, 28, 29\) that the extrinsic time is the most useful variable in dealing with Einstein solutions on spatially compact surfaces. Our approach is different since these geometric characteristics possess in our case quite an opposite essence. In general, we claim that canonical momenta are not defined within the hypersurface. They refer to the motion in time of the original \(\Sigma_t\). And the intrinsic time is the variable constructed entirely out of the metric of the hypersurface. So, the roles of the Hamiltonian and global time are interchanged with respect to the ones adopted in refs. \(26, 27, 28, 29\).

As discussed in Sect. 4, our treatment of the Hamiltonian and the global time is in accord with the general principles.

The deviation from the mean value of the global time behaves as a classical scalar field. It deserves an additional attention to be physically interpreted. Note that it emerged without any modification of the Einstein’s theory. The Wheeler’s thin sandwich conjecture in General Relativity known to be valid even for higher dimensional theories of gravity under positive lapse and some restriction \(30\). Thus, the lapse function can be found from the Hamiltonian constraint. It was conjectured \(31\) that the deviations from the mean value of the global time can play the role of static gravitational potentials.

Earlier to construct a scalar, the Minkowskian metric as a background one was used for asymptotically flat spaces \(18\). The intrinsic time interval \(\delta D\) as a scalar was implemented in the symplectic 1-form \(32, 33\). For splitting one degree of freedom, the average of the trace of the momentum density was used as the York time in the shape dynamics \(21\). The key difference of our study is the consideration of closed manifolds without the asymptotically flat space condition. Our choice is motivated by cosmological applications where models with closed manifolds are of great interest.

For interpretation of the latest data of the Hubble diagram, the global time as the scale factor of the Friedmann model was successfully implemented in refs. \(34, 35, 36\). The choice of conformal variables allows to suggest a new interpretation of the redshift of distant stellar objects. Both the changing volume of the Universe in standard cosmology and the changing of masses of elementary particles in conformal cosmology \(37\) can serve as the measure of time. We have shown that the cosmological observable redshift effect can be directly related to the global time of the Universe. In general, we claim that the intrinsic time can be considered as an evolution parameter of cosmological evolution.

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Appendix

To obtain functional derivatives we need some useful results for variations with respect to metric in Banach functional space

\[
\begin{align*}
\delta \gamma &= \gamma^{ij} \delta \gamma_{ij} = -\gamma_{ij} \delta \gamma^{ij}, \\
\delta \sqrt{\gamma} &= \frac{1}{2} \sqrt{\gamma} \gamma^{ij} \delta \gamma_{ij}, \\
\delta \gamma^{ij} &= -\gamma^{ik} \gamma^{jl} \delta \gamma_{kl}.
\end{align*}
\]

(62)

\(\bullet\) Let us prove the statement \(52\). Calculate the functional derivative of the ratio of two functionals
Integrating over the hypersurface $\Sigma_t$, i.e.,

$$\delta \frac{\gamma_{ij}(x)}{\delta \gamma_{ij}(x)} = \frac{1}{V_i} \left( \frac{\gamma(x) \gamma^{ij}(x)}{\sqrt{f(x)}} V_i - \frac{1}{2} \sqrt{\gamma(x)} \gamma^{ij}(x) \int d^3 y \sqrt{\gamma(y)} \sqrt{\frac{\gamma}{f(y)}} \right)$$

$$= \frac{1}{V_i} \sqrt{\gamma(x)} \gamma^{ij}(x) \left( \sqrt{\frac{\gamma}{f(x)}} - \frac{1}{2} < \sqrt{\frac{\gamma}{f}} > \right), \tag{63}$$

where $V_i$ is the volume of the hypersurface $\Sigma_t$: $V_i = \int_{\Sigma_t} d^3 y \sqrt{\gamma(y)}$. The functional derivative of $P$ is the metric tensor

$$\frac{\delta P}{\delta \pi^{ij}(x)} = \gamma_{ij}(x). \tag{64}$$

Then we calculate the Poisson bracket of the functionals $\{x, y\}$ with use of the derivatives $\{x, y\}$ and $\{x, y\}$

$$\{< \sqrt{\frac{\gamma}{f}} | \psi >, P[\gamma_{ij}, \pi^{ij}] \} = \frac{3}{V_i} \int_{\Sigma_i} d^3 y \sqrt{\gamma(y)} \sqrt{\frac{\gamma}{f(y)}} - \frac{3}{2} < \sqrt{\frac{\gamma}{f}} > = \frac{3}{2} < \sqrt{\frac{\gamma}{f}} >.$$

Rewrite the bracket obtained

$$- \frac{2}{3} \log \begin{Bmatrix} \frac{\sqrt{\gamma}}{f} \psi >, P[\gamma_{ij}, \pi^{ij}] \end{Bmatrix} = \{ T, P \} = -1. \tag{65}$$

Thus we prove the canonical commutation relation $\{x, y\}$ and get the global time $\{x, y\}$. The global time $\{x, y\}$ is a scalar. If one consider diffeomorphisms

$$\delta \gamma_{ij} = \mathbf{N} \gamma_{ij} \equiv \nabla_i N_j + \nabla_j N_i,$$

then, applying it to $\{x, y\}$, one obtains

$$= \frac{1}{V_i} \sqrt{\gamma(x)} \gamma^{ij}(x) \left( \nabla_i N_j + \nabla_j N_i \right) \left( \sqrt{\frac{\gamma}{f}}(x) - \frac{1}{2} < \sqrt{\frac{\gamma}{f}} > \right).$$

Integrating over the hypersurface $\Sigma_t$ one gets the integral of the divergent term

$$\int_{\Sigma_t} d^3 x \delta < \sqrt{\frac{\gamma}{f}} > = \frac{2}{V_i} \int_{\Sigma_t} d^3 x \frac{\partial}{\partial x^i} \left( \gamma^{ij}(x) N_j \left( \sqrt{\frac{\gamma}{f}(x)} - \frac{1}{2} < \sqrt{\frac{\gamma}{f}} > \right) \right) = 0.$$

So, the global time is a scalar, i.e., it is conserved under spatial diffeomorphisms.

Let us calculate the bracket

$$\{ \pi(x), T \} = \{ \pi(x), T \} - \sqrt{\gamma(x)} \{ < \pi >, T \}.$$

We calculate the following functional derivatives

$$\frac{\delta}{\delta \pi^{ij}(x')} \pi(x) = \frac{\delta}{\delta \pi^{ij}(x')} \left[ \gamma_{ij}(x) \pi^{ij}(x) \right] = \gamma_{ij}(x) \delta(x - x'), \tag{66}$$

$$\frac{\delta}{\delta \pi^{ij}(x')} \pi(x) = \gamma_{ij}(x) \delta(x - x') - \frac{1}{V_i} \sqrt{\gamma(x)} \gamma_{ij}(x'). \tag{67}$$

$$\frac{\delta}{\delta \gamma_{ij}(x')} T = - \frac{2}{3} \frac{1}{\sqrt{\gamma/f}} \frac{1}{V_i} \sqrt{\gamma(x')} \gamma^{ij}(x') \left( \sqrt{\frac{\gamma}{f}(x')} - \frac{1}{2} < \sqrt{\frac{\gamma}{f}} > \right). \tag{68}$$

Multiplying $\{x, y\}$ and $\{x, y\}$ with sign minus we get the bracket

$$\{\pi(x), T\} = \frac{2}{V_i} \sqrt{\gamma/f(x)} \sqrt{f(x)} \left( 1 + \sqrt{\gamma/f(x)} \exp[(3/2)T] \right). \tag{69}$$
By subtracting (72) from (70), we obtain

\begin{equation}
\{ \sqrt{\gamma/f}(x), P \} = \{ \sqrt{\gamma/f}(x), \int d^3y \pi(y) \} - \{ \sqrt{\gamma/f}(x) > , \int d^3y \pi(y) \}.
\end{equation}

We calculate functional derivatives

\begin{align*}
\frac{\delta}{\delta \gamma_{ij}(x')} \sqrt{\gamma/f}(x) &= \frac{1}{2} \sqrt{\gamma/f}(x) \gamma^{ij}(x) \delta(x - x'), \\
\frac{\delta}{\delta \pi^{ij}(x')} \int d^3y \pi^{ij}(y) \gamma_{ij}(y) &= \gamma_{ij}(x').
\end{align*}

Then

\begin{equation}
\{ \sqrt{\gamma/f}(x), P \} = \frac{1}{2} \int d^3x' \sqrt{\gamma/f}(x) \gamma^{ij}(x) \gamma_{ij}(x) \delta(x - x') = \frac{3}{2} \sqrt{\gamma/f}(x).
\end{equation}

(70)

\begin{equation}
\{ \sqrt{\gamma/f} > , P \} = \frac{1}{V_t} \int d^3x' \sqrt{\gamma/f}(x) \gamma^{ij}(x') \gamma_{ij}(x') \left( \sqrt{\gamma/f}(x') - \frac{1}{2} < \sqrt{\gamma/f} > \right)
= \frac{3}{V_t} \int d^3x' \sqrt{\gamma(x') \gamma/f}(x') \sqrt{\gamma/f}(x') - \frac{3}{2V_t} \int d^3x' \sqrt{\gamma(x')} < \sqrt{\gamma/f} >
= \frac{3}{2} < \sqrt{\gamma/f} >.
\end{equation}

(71)

(72)

By subtracting (72) from (70), we obtain

\begin{equation}
\{ \sqrt{\gamma/f}(x), P \} = \frac{3}{2} \sqrt{\gamma/f}(x) - \frac{3}{2} < \sqrt{\gamma/f} > = \frac{3}{2} \sqrt{\gamma/f}(x).
\end{equation}

(73)

\bullet \text{ Let us calculate the bracket}

\begin{equation}
\{ \bar{\pi}(x), P \} = \{ \pi(x), \int d^3y \pi(y) \} - \{ \sqrt{\gamma}(x) < \pi > , P \}.
\end{equation}

The first term is zero, calculate the second one. We need the following functional derivatives

\begin{align*}
\frac{\delta}{\delta \gamma_{ij}(x')} (\sqrt{\gamma}(x) < \pi >)
= \frac{1}{2} \sqrt{\gamma}(x) \gamma^{ij}(x) \delta(x - x') < \pi > + \sqrt{\gamma}(x) \frac{\delta}{\delta \gamma_{ij}(x') < \pi >}, \\
\frac{\delta}{\delta \gamma_{ij}(x')} < \pi > &= \frac{1}{V_t} \pi^{ij}(x') - \frac{1}{2V_t} \sqrt{\gamma(x')} \gamma^{ij}(x') < \pi >,
\end{align*}

\begin{equation}
\frac{\delta}{\delta \pi^{ij}(x')} P = \gamma_{ij}(x'), \quad \frac{\delta}{\gamma_{ij}(x')} P = \pi^{ij}(x'),
\end{equation}

\begin{equation}
\frac{\delta}{\delta \pi^{ij}(x')} (\sqrt{\gamma}(x) < \pi >) = \frac{\sqrt{\gamma(x)}}{V_t} \gamma_{ij}(x').
\end{equation}

(74)

(75)

(76)

(77)

Collecting all Eqs. (74)–(77), we obtain

\begin{equation}
\{ \bar{\pi}(x), P \} = \frac{3}{2} \sqrt{\gamma}(x) < \pi > + \sqrt{\gamma}(x) < \pi > \\
- \frac{3}{2} \sqrt{\gamma}(x) < \pi > - \sqrt{\gamma}(x) < \pi > = 0.
\end{equation}

(78)

\bullet \text{ Let us calculate the bracket}

\begin{equation}
\{ \bar{\pi}(x), \sqrt{\gamma/f}(y) \} = \{ \bar{\pi}(x), \sqrt{\gamma/f}(y) \} - \{ \bar{\pi}(x), < \sqrt{\gamma/f} > \}.
\end{equation}
We need the following functional derivatives
\[\frac{\delta}{\delta \pi^{ij}(x')} \bar{\pi}(x) = \gamma_{ij}(x) \delta(x - x') - \frac{1}{V_t} \sqrt{\gamma}(x) \gamma_{ij}(x'),\]
\[\frac{\delta}{\delta \gamma^{ij}(x')} \sqrt{\frac{\gamma}{f}(y)} = \frac{1}{2} \sqrt{\frac{\gamma}{f}(y)} \gamma^{ij}(y) \delta(y - x') - \frac{1}{V_t} \sqrt{\gamma(x')} \gamma^{ij}(x') \left( \sqrt{\frac{\gamma}{f}(x')} - \frac{1}{2} \sqrt{\frac{\gamma}{f}} > \right).\] (79)

Multiplying (79) to (9) with sign minus, we get the bracket
\[\{ \bar{\pi}(x), \sqrt{\gamma/f}(y) \} = \frac{3}{2} \sqrt{\frac{\gamma}{f}(y)} \delta(x - y) + 3 V_t \sqrt{\gamma(x)} \left( \sqrt{\frac{\gamma}{f}(x)} + \frac{1}{2} \sqrt{\frac{\gamma}{f}} - \sqrt{\frac{\gamma}{f}} > \right).\] (80)

- Variations for components of metric connections
  \[\delta \Gamma^k_{ij} = \frac{1}{2} \gamma^{kl} (\nabla_i (\delta \gamma_{lj}) + \nabla_j (\delta \gamma_{li}) - \nabla_l (\delta \gamma_{ij})),\]
  \[\delta \Gamma^k_{ik} = \frac{1}{2} \gamma^{kl} \nabla_i (\delta \gamma_{kl});\] (81)

variations of components of Riemannian tensor
\[\delta R^l_{jkl} = \nabla_l (\delta \Gamma^l_{jk}) - \nabla_k (\delta \Gamma^l_{jl});\]

variations of components of Ricci tensor known as Palatini identity
\[\delta R_{ij} = \nabla_j (\delta \Gamma^k_{ik}) - \nabla_k (\delta \Gamma^k_{ij});\] (82)

a variation of the Ricci scalar
\[\delta R = \delta (R_{ij} \gamma^{ij}) = -R^{ij}(\delta \gamma_{ij}) + \gamma^{ij} \delta R_{ij}.\] (83)

Using formulae (81) and (82), one yields
\[\delta R_{ij} = \frac{1}{2} \nabla_j \nabla_i \left( \gamma^{kl} \delta \gamma_{kl} \right) + \frac{1}{2} \nabla_i \nabla^l (\delta \gamma_{lj}) - \frac{1}{2} \nabla_k \nabla_j \left( \gamma^{kl} \delta \gamma_{li} \right) - \frac{1}{2} \nabla_k \nabla_i \left( \gamma^{kl} \delta \gamma_{lj} \right).\]

Contracting the tensor, we get
\[\gamma^{ij} \delta R_{ij} = \gamma^{ij} \Delta (\delta \gamma_{ij}) - \nabla^i \nabla^j (\delta \gamma_{ij}),\]

where
\[\Delta := \gamma^{ij} \nabla_i \nabla_j \equiv \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^i} \left( \sqrt{\gamma} \gamma^{ij} \frac{\partial}{\partial x^j} \right)\]
is the invariant Laplace operator.

Finally, the expression for the Ricci scalar variation (83) that we need:
\[\delta R = -R^{ij}(\delta \gamma_{ij}) + \gamma^{ij} \Delta (\delta \gamma_{ij}) - \nabla^i \nabla^j (\delta \gamma_{ij}).\] (84)

- Let us prove a useful formula. Here and below, using the Gauss–Ostrogradsky theorem, we get
For Hermitian operator

Taking a functional derivative of the expression by metric coefficients, we prove the next useful formula

Then, we simplify the following integral

rig out of divergent terms:

Thus we proved Hermiticity of the Laplace operator. We need also the relation

Taking a functional derivative of the expression by metric coefficients, we prove the next useful formula for Hermitian operator

Then, we simplify the following integral

Now we can find a functional derivative of it

Here, assuming $F$ and $\phi$ to be scalar fields, we replaced the partial derivatives with covariant ones, and introduced the Beltrami invariant operator

$$\Delta_1(F, \phi) =: \gamma^{kl} \nabla_k F \nabla_l \phi.$$
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