Abstract. A guaranteed upper bound is proved for the time complexity of the
list-coloring problem on graphs.

For a positive integer \( n \), the set \( \{1, \ldots, n\} \) is denoted by \([1, n]\). Throughout
this note, \( n \) will denote the order of a graph, that is the number of vertices.

A proper k-coloring (or simply, a k-coloring) of a graph \( G = (V, E) \) is a
function \( f : V \rightarrow [1, k] \) such that \( f(v) \neq f(w) \) whenever \( vw \in E \). The vertex
coloring problem, whose input is a graph \( G \) and a positive integer \( k \), consists of
deciding whether \( G \) is properly \( k \)-colorable or not. In the list-coloring problem,
every vertex \( v \) comes equipped with a list of permitted colors \( L(v) \subseteq [1, \kappa] \)
for some given positive integer \( \kappa \), and we require the proper coloring to respect these
lists, i.e., \( f(v) \in L(v) \).

We say that a constant real number \( t > 1 \) fits \( G \) if for any \( m \)-element subset
\( M \) of \( V \), the induced subgraph \( G[M] \) has at most \( t^m \) inclusion-maximal indepen-
dent sets. It is well-known that \( 1.44225 \) fits any graph because \( 1.44225^3 > 3 \); see
[3] for a proof of the fact that an \( n \)-vertex graph contains at most \( 3^{n/3} \) inclusion-
maximal independent sets. Similarly, \( 1.41422 \) fits any triangle-free graph because
\( 1.41422^2 > 2 \); see [8] for a proof of the fact than a triangle-free graph contains
at most \( 2^{n/2} \) inclusion-maximal independent sets. Furthermore it was proved
in [9] that, for any fixed natural number \( k \), graphs not containing an induced
matching with \( k + 1 \) edges have at most \( O(n^{2k}) \) inclusion-maximal independent
sets. Hence for every \( k \) and every real \( \epsilon > 0 \) there exists an \( n_0 = n_0(k, \epsilon) \) such
that \( t = 1 + \epsilon \) fits every such graph of order \( n > n_0 \).

A special case of the list-coloring problem is the precoloring extension with
\( \kappa \) colors. Here each list is either \([1, \kappa]\) or a one-element subset of it. (See
[5, 6, 7] for more details on precoloring extension.) Another special case is the
\( k \)-choosability problem where each color list has size \( k \) for a fixed \( k \). (See [4, 10]
for more details on choosability.)

The main result in the present note, Theorem 1, claims that there exists a
polynomial \( p(n) \) such that if \( t \) fits a graph \( G \) with \( n \) vertices, then it can be
decided in \( p(n) \cdot (1 + t)^n + O(\kappa n) \) time whether \( G \) is properly list-colorable or not. (Here the polynomial \( p \) is independent of \( t \), and the \( O(\kappa n) \) term is irrelevant except for reading an input with rather long vertex lists.) In case of a positive answer, we will construct a proper list-coloring needing no extra time. We will extend the methods and results of Lawler [11].

Given a graph \( G = (V, E) \) on \( n \) vertices, we fix a permutation of the nonempty subsets of \( V \) for which \( W, U \subseteq V \) and \( |W| > |U| \) imply that \( W \) is before \( U \) in the fixed permutation. For any nonempty \( W \subseteq V \) let \( \langle W \rangle \) denote the position of \( W \) in the fixed permutation. As a special case, \( \langle V \rangle = 1 \), and \( |W| = n - 1 \) implies that \( \langle W \rangle \in \{ 2, 3, \ldots, n + 1 \} \), furthermore \( |W| = 1 \) implies that \( \langle W \rangle \in \{ 2^n - n, 2^n - n + 1, \ldots, 2^n - 1 \} \). We will consider a zero-one sequence \((a_m), m = 1, 2, \ldots, 2^n - 1 \). We say that the sequence has the partial increasing property (PIP, for short) if \( W \supset U \) implies \( a_{\langle W \rangle} \leq a_{\langle U \rangle} \). In words, if \( W \) is before \( U \) in the permutation and if \( a_{\langle W \rangle} = 1 \), then \( a_{\langle U \rangle} \) holds by pip.

Initially, the sequence \((a_m), m = 1, 2, \ldots, 2^n - 1 \) will be set as the characteristic sequence of the independent sets in \( G \). In other words, \( a_{\langle W \rangle} = 1 \) holds if and only if the induced subgraph \( G[W] \) is edgeless. Later some zero values in the sequence will be changed to one; however, the pip will be managed.

Clearly, list-colorability can be reformulated as follows: The vertex set \( V \) can be partitioned into independent set color classes \( V_1, \ldots, V_\kappa \) such that for each \( j \in [1, \kappa] \) and for each \( v \in V_j \) the relation \( j \in L(v) \) holds. Here we allow the empty set to occur among \( V_1, \ldots, V_\kappa \) any number of times.

For each \( j \in [1, \kappa] \) and for each \( v \in V \) we define \( L^j(v) \) as \( L(v) \cap [1, j] \). Clearly \( L^1(v) \subseteq \ldots \subseteq L^\kappa(v) = L(v) \).

For each \( j \in [1, \kappa] \) we will consider the \( j \)th version of the list-coloring problem, denoted by \( \mathcal{LC}^j \) for all induced subgraphs \( G[W] \) by considering the lists \( L^j(v) \). Note that \( \mathcal{LC}^1 \) is trivial since \( G[W] \) is list-colorable (with respect to the lists \( L^j(v), w \in W \)) if and only if, on the one hand, \( W \) is an independent set in \( G \), and on the other hand, \( 1 \in L(w) \) for all \( w \in W \). Starting from the list \( \mathcal{I} \) of all inclusion-maximal independent sets of \( G = (V, E) \), the sequence \((a_m), m = 1, 2, \ldots, 2^n - 1 \) can be computed as follows:

1. Let all \( a_m \) get the value 0.
2. For each \( I \in \mathcal{I} \), consider \( I^{(1)} = \{ v \in I : 1 \in L(v) \} \).
and set $a_{(W)} = 1$.

3. Manage the above mentioned PIP of the sequence $(a_m)$.

For this purpose we go forward in the sequence and for any $a_{(W)} = 1$ and for any $w \in W$ we set $a_{(W-w)} = 1$. (Here $W - w$ simply denotes $W \setminus \{w\}$.)

**Lemma 1.** After performing the above three steps, the final values in the sequence $(a_{(W)})$ give the true answer for the list-coloring problem for each $G[W]$ with respect to the lists $L^1(v)$. This means that $a_{(W)} = 1$ holds if and only if $G[W]$ is list-colorable with respect to the lists $L^1(v)$, $v \in W$.

**Proof.** For any inclusion-maximal vertex set $I$, in step 2 the sequence element $a_{(I)}$ gets value 1, and in step 3, for any other independent set $W$, the final value of $a_{(W)}$ will also be 1.

Now for any fixed $j = 2, 3, ..., \kappa$, we will recursively solve the list-colorability problem $\mathcal{LC}^j$ for all $G[W]$ by using the lists consisting of the sets

$I^{(j)} = \{v \in I : j \in L(v)\}$

obtained from all inclusion-maximal independent sets $I$ in $G$. For a fixed $j \in \{2, 3, ..., \kappa\}$, we assume that the problem $\mathcal{LC}^{j-1}$ is completely solved, and the result is properly recorded in the sequence $(a_m)$, $m = 1, 2, ..., 2^n - 1$, that is for any $G[W]$ this induced subgraph is list-colorable with respect to the color lists $L^{j-1}(v)$, $v \in W$, if and only if $a_{(W)} = 1$.

**Lemma 2.** The graph $G = (V, E)$ is list-colorable with respect to the color lists $L^j(v)$, $v \in V$, if and only if there is at least one independent set $I^{(j)}$ for which either $I^{(j)} = V$ or $a_{(V \setminus I^{(j)})} = 1$ holds as a result of the $\mathcal{LC}^{j-1}$ problem.

**Proof.** If $I^{(j)} = V$, then $G$ is clearly list-colorable since $f(v) = j$ suffices for all $v \in V$. If there is an $I^{(j)} \neq V$ for which $a_{(V \setminus I^{(j)})} = 1$ holds, then any list-coloring $f$ of $G[V \setminus I^{(j)}]$ with respect to the lists $L^{j-1}(v)$, $v \in V \setminus I^{(j)}$, is clearly extendible to obtain a list-coloring of $G$ by defining $f(v) = j$ for all $v \in I^{(j)}$. On the other hand, in any list-coloring $f$ of $G$ with respect to the lists $L^j(v)$, $v \in V$, the vertices $v$ with $f(v) = j$, must form an independent set $J$, and this $J$ must be contained in at least one $I^{(j)}$, and now the restriction of $f$ onto $V \setminus I^{(j)}$ gives us a list-coloring of $G[V \setminus I^{(j)}]$ with respect to the lists $L^{j-1}(v)$, $v \in V \setminus I^{(j)}$.

Now Lemma 2 shows how we can solve the $\mathcal{LC}^j$ problem for $G$. Similarly we can solve the $\mathcal{LC}^j$ problem for any induced subgraph $G[W]$. Here we can use the
same sequence \((a_m)\), \(m = 1, 2, ..., 2^n - 1\). For each \(W\) we scan the sets \(I^{(j)}\) for all inclusion-maximal independent sets \(I\) in \(G[W]\), and update \(a_{(W)}\) from 0 to 1 if \(a_{(W \setminus I^{(j)})} = 1\) holds in the sequence after round \(j - 1\). In this procedure PIP ensures that the elements \(a_{(W)}\) may be updated only when they are not needed anymore for updates of any other elements.

If \(t\) fits \(G\), then the total number of steps is not larger than \(p(n)\) times the following number, where \(p(n)\) is a polynomial independent of \(G\) and independent of \(t\), too:
\[
\sum_{W \neq \emptyset} \left( \frac{n}{|W|} \right)^{t |W|}
\]
By the binomial theorem, the above sum is \((1 + t)^n - 1\). In summary, we obtain the following theorem.

**Theorem 1.** There exists a polynomial \(p\) for which if \(t\) fits \(G\), an \(n\)-vertex graph, then all the problems \(LC^j\), \(j = 1, 2, ..., \kappa\) can be solved in \(p(n) \cdot (1 + t)^n + O(\kappa n)\) time.

**Proof.** While reading the input, we put aside all vertices whose lists have length at least \(n\). Once the rest of the graph is properly list-colored, the vertices with long lists can be colored in an obvious greedy way. For this reason we can assume from now on that \(\kappa < n\) holds.

It is proved in [2] that if a graph on \(n\) vertices has \(N\) inclusion-maximal independent sets, then those \(N\) sets can be listed in \(O(n^3N)\) time. We apply this method \(\kappa\) times for all induced subgraphs \(G[W]\). The input graph \(G\) is list colorable if and only if \(a_{(V)} = 1\) holds after round \(\kappa\). Thus there exists a suitable polynomial \(p(n) = O(n^4)\).

If, in case of list-colorability, we want to output a proper list-coloring \(f : V \to [1, \kappa]\), we simply have to memorize an independent set \(I^{(\kappa)}\) for which \(G[V \setminus I^{(\kappa)}]\) is list-colorable with respect to the lists \(L^{\kappa-1}(v)\), \(v \in V \setminus I^{(\kappa)}\), and for any \(w \in I^{(\kappa)}\) we can set \(f(w) = \kappa\). Then we decrease the value of \(\kappa\) by one, and we repeat the procedure for the decreased graph instead of the old one, and for the new \(LC^\kappa\) problem instead of the old \(LC^\kappa\) problem. And so on.
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