TWISTED CYCLES AND TWISTED PERIOD RELATIONS
FOR LAURICELLA’S HYPERGEOMETRIC FUNCTION $F_C$

YOSHIKI GOTO

Abstract. We study Lauricella’s hypergeometric function $F_C$ by using twisted (co)homology groups. We construct twisted cycles with respect to an integral representation of Euler type of $F_C$. These cycles correspond to $2^m$ linearly independent solutions to the system of differential equations annihilating $F_C$. Using intersection forms of twisted (co)homology groups, we obtain twisted period relations which give quadratic relations for Lauricella’s $F_C$.

1. Introduction

Lauricella’s hypergeometric series $F_C$ of $m$-variables $x_1,\ldots,x_m$ with complex parameters $a, b, c_1,\ldots,c_m$ is defined by

$$F_C(a, b, c; x) = \sum_{n_1,\ldots,n_m=0}^{\infty} \frac{(a, n_1 + \cdots + n_m)(b, n_1 + \cdots + n_m)}{(c_1, n_1)\cdots(c_m, n_1)! \cdots n_m!} x_1^{n_1} \cdots x_m^{n_m},$$

where $x = (x_1,\ldots,x_m)$, $c = (c_1,\ldots,c_m)$, $c_1,\ldots,c_m \notin \{0, -1, -2, \ldots\}$ and $(c_1, n_1) = \Gamma(c_1 + n_1)/\Gamma(c_1)$. This series converges in the domain

$$D_C := \{(x_1,\ldots,x_m) \in \mathbb{C}^m \mid \sum_{k=1}^{m} \sqrt{|x_k|} < 1\},$$

and admits the integral representation (3). The system $E_C(a, b, c)$ of differential equations annihilating $F_C(a, b, c; x)$ is a holonomic system of rank $2^m$ with the singular locus $S$ given in (1). There is a fundamental system of solutions to $E_C(a, b, c)$ in a simply connected domain in $D_C - S$, which is given in terms of Lauricella’s hypergeometric series $F_C$ with different parameters, see (2) for their expressions.

In the case of $m = 2$, the series $F_C(a, b, c; x)$ and the system $E_C(a, b, c)$ are called Appell’s hypergeometric series $F_4(a, b, c; x)$ and system $E_4(a, b, c)$ of differential equations, which are studied in [6] by twisted (co)homology groups concerning with the integral representation (3) for $m = 2$. We construct twisted cycles corresponding to four solutions to $E_4(a, b, c)$ expressed by the series $F_4$, and evaluate their intersection numbers. We also evaluate the intersection matrix for a basis of the twisted cohomology group. By using these results, we determine the monodromy representation of $E_4(a, b, c)$ and give twisted period relations, which are quadratic relations between two fundamental systems of $E_4$ with different parameters.

In this paper, we construct $2^m$ twisted cycles which represent elements of the $m$-th twisted homology group concerning with the integral representation (3). They imply integral representations of the solutions (2) expressed by the series $F_C$. We evaluate the intersection numbers of these $2^m$ twisted cycles. We also evaluate the intersection numbers of some elements of the twisted cohomology group. These

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results imply twisted period relations for two fundamental systems of \( E_C \) with different parameters, refer to Theorem 6.1 for their explicit forms.

In the study of twisted homology groups, twisted cycles given by bounded chambers are useful. However there are few bounded chambers in our case. By avoiding this difficulty, we succeed in constructing \( 2^m \) twisted cycles. We explain our idea in the construction. Our twisted homology group is defined by the multi-valued function

\[
  u(t) = \prod_{k=1}^{m} t_k^{1-c_k + b} \cdot v^{\sum_{k=1}^{m} c_k - a - m + 1} \cdot w^{-b},
\]

where \( v = 1 - \sum_{k=1}^{m} t_k, \quad w = \prod_{k=1}^{m} t_k \cdot (1 - \sum_{k=1}^{m} x_k/t_k) \),

for fixed small positive real numbers \( x_1, \ldots, x_m \). Let \( \{i_1, \ldots, i_r\} \) be a subset of \( \{1, \ldots, m\} \) of cardinality \( r \) and let \( \{j_1, \ldots, j_m-r\} \) be its complement. We embed the direct product of an \( r \)-simplex \( \tau_{i_1 \cdots i_r} \) and an \( (m-r) \)-simplex \( \tau_{j_1 \cdots j_m-r} \) into the bounded chamber

\[
  \{(t_1, \ldots, t_m) \in \mathbb{R}^m \mid t_1, \ldots, t_m, \quad v, \quad w > 0\}.
\]

We consider an \( m \)-dimensional twisted chain given by the image of \( \tau_{i_1 \cdots i_r} \times \tau_{j_1 \cdots j_m-r} \) with loading a branch of \( u(t) \) on it. As is in Section 3.2.4 of [1], we can eliminate its boundary by regarding the image of \( \tau_{i_1 \cdots i_r} \) as the simplex

\[
  \{(s_{i_1}, \ldots, s_{i_r}) \in \mathbb{R}^r \mid s_{i_1}, \ldots, s_{i_r}, 1 - \sum_{p=1}^{r} s_{i_p} \geq \varepsilon\},
\]

and that of \( \tau_{j_1 \cdots j_{m-r}} \) as the simplex

\[
  \{(t_{j_1}, \ldots, t_{j_{m-r}}) \in \mathbb{R}^{m-r} \mid t_{j_1}, \ldots, t_{j_{m-r}}, 1 - \sum_{q=1}^{m-r} t_{j_q} \geq \varepsilon\},
\]

where \( s_{i_p} = x_{i_p}/t_{i_p} \) for \( p = 1, \ldots, r \) and \( \varepsilon \) is a certain positive real number. In this elimination of the boundary, we must regulate the difference of branches of \( u(t) \) by a different way from the usual regularization. We realize this elimination by using the twisted homology group defined by another multi-valued function, see Section 3 for details. Our first main theorem states that this twisted cycle corresponds to the solution to \( E_C(a, b, c) \) with the power function \( \prod_{p=1}^{r} x_{i_p}^{1-c_{i_p}} \). Our construction of twisted cycles is also useful in the study on the system of differential equations annihilating Lauricella’s hypergeometric series \( F_A \). Refer to [2], for the twisted cycles corresponding to the solutions expressed by the series \( F_A \) and twisted period relations for \( F_A \).

By our first main theorem and the proof of Lemma 4.1 in [6], it turns out that the intersection matrix becomes diagonal. Moreover, our construction and results in [3] enable us to evaluate the diagonal entries of the intersection matrix. If the intersection matrix for bases of twisted homology groups is evaluated, then the intersection numbers of some elements of twisted cohomology groups imply twisted period relations, which are originally identities among the integrals given by the pairings of elements of twisted homology and cohomology groups. Our first main theorem transforms these identities into quadratic relations among hypergeometric series \( F_C \)’s. Our second main theorem states these formulas in Section 6.

As is in [4], the irreducibility condition of the system \( E_C(a, b, c) \) is known to be

\[
  a - \sum_{p=1}^{r} c_{i_p}, \quad b - \sum_{p=1}^{r} c_{i_p} \not\in \mathbb{Z}
\]
for any subset \( \{i_1, \ldots, i_r\} \) of \( \{1, \ldots, m\} \). Since our interest is in the property of solutions to \( E_C(a, b, c) \) expressed in terms of the hypergeometric series \( F_C \), we assume throughout this paper that the parameters \( a, b \) and \( c = (c_1, \ldots, c_m) \) satisfy the condition above and \( c_1, \ldots, c_m \not\in \mathbb{Z} \).

2. Differential equations and integral representations

In this section, we collect some facts about Lauricella’s \( F_C \) and the system \( E_C \) of hypergeometric differential equations annihilating it.

**Notation 2.1.** Throughout this paper, the letter \( k \) always stands for an index running from 1 to \( m \). If no confusion is possible, \( \sum_{k=1}^{m} \) and \( \prod_{k=1}^{m} \) are often simply denoted by \( \sum \) (or \( \sum_k \)) and \( \prod \) (or \( \prod_k \)), respectively. For example, under this convention \( F_C(a, b, c; x) \) is expressed as

\[
F_C(a, b, c; x) = \sum_{n_1, \ldots, n_m = 0}^{\infty} \frac{(a, \sum n_k)(b, \sum n_k)}{\prod n_k!} \prod x_k^n.
\]

Let \( \partial_k \) (\( k = 1, \ldots, m \)) be the partial differential operator with respect to \( x_k \). Lauricella’s \( F_C(a, b, c; x) \) satisfies hypergeometric differential equations

\[
\left[ x_k(1-x_k)\partial_k^2 - x_k \sum_{1 \leq i \leq m, i \neq k} x_i \partial_i \partial_k - \sum_{1 \leq i \neq j \leq m} x_i x_j \partial_i \partial_j \
+ (c_k - (a + b + 1)x_k)\partial_k - (a + b + 1) \sum_{1 \leq i \leq m, i \neq k} x_i \partial_i - ab \right] f(x) = 0,
\]

for \( k = 1, \ldots, m \). The system generated by them is called Lauricella’s system \( E_C(a, b, c) \) of hypergeometric differential equations.

**Proposition 2.2** ([7], [9]). The system \( E_C(a, b, c) \) is a holonomic system of rank \( 2^m \) with the singular locus

\[
S := \left( \prod_{k=1}^{m} x_k \cdot \prod_{\varepsilon_1, \ldots, \varepsilon_m = \pm 1} (1 + \sum_{k=1}^{m} \varepsilon_k \sqrt{x_k}) = 0 \right) \subset \mathbb{C}^m.
\]

If \( c_1, \ldots, c_m \not\in \mathbb{Z} \), then the vector space of solutions to \( E_C(a, b, c) \) in a simply connected domain in \( D_C - S \) is spanned by the following \( 2^m \) elements:

\[
(2) \quad f_{i_1 \cdots i_r} := \prod_{p=1}^{r} x_i^{1-c_{i_p}} \cdot F_C \left( a + r - \sum_{p=1}^{r} c_{i_p}, b + r - \sum_{p=1}^{r} c_{i_p}, c^{1-i_r}; x \right).
\]

Here \( r \) runs from 0 to \( m \), indices \( i_1, \ldots, i_r \) satisfy \( 1 \leq i_1 < \cdots < i_r \leq m \), and the row vector \( c^{1-i_r} \) is defined by

\[
c^{1-i_r} := e + 2 \sum_{p=1}^{r} (1 - c_{i_p}) e_{i_p},
\]

where \( e_i \) is the \( i \)-th unit row vector of \( \mathbb{C}^m \).

For the above \( i_1, \ldots, i_r \), we take \( j_1, \ldots, j_{m-r} \) so that \( 1 \leq j_1 < \cdots < j_{m-r} \leq m \) and \( \{i_1, \ldots, i_r, j_1, \ldots, j_{m-r}\} = \{1, \ldots, m\} \). It is easy to see that the \( i_p \)-th entry of \( c^{1-i_r} \) is \( 2 - c_{i_p} \) \( (1 \leq p \leq r) \) and the \( j_q \)-th entry is \( c_{i_q} \) \( (1 \leq q \leq m-r) \).

We denote the multi-index “ \( i_1 \cdots i_r \) ” by a letter \( I \) expressing the set \( \{i_1, \ldots, i_r\} \). Note that the solution \( (2) \) for \( r = 0 \) is \( f(= f_0) = F_C(a, b, c; x) \).
Proposition 2.3 (Integral representation of Euler type, Example 3.1 in [1]). For sufficiently small positive real numbers \( x_1, \ldots, x_n \), if \( c_1, \ldots, c_m, a - \sum c_k \notin \mathbb{Z} \), then \( F_C(a, b; c; x) \) admits the following integral representation:

\[
F_C(a, b; c; x) = \frac{\Gamma(1 - a)}{\prod \Gamma(1 - c_k) \cdot \Gamma(\sum c_k - a - m - 1)} \cdot \int_\Delta \prod t_k^{c_k} \cdot (1 - \sum t_k)^{\sum c_k - a - m} \cdot (1 - \sum \frac{x_k}{t_k})^{-b} dt_1 \wedge \cdots \wedge dt_m,
\]

where \( \Delta \) is the twisted cycle made by an \( m \)-simplex in Sections 3.2 and 3.3 of [1].

In fact, this cycle is one of twisted cycles constructed in Section 4.

3. Twisted homology groups

We review twisted homology groups and the intersection form between twisted homology groups in general situations, by referring to Chapter 2 of [1] and Chapters IV, VIII of [8].

For polynomials \( P_j(t) = P_j(t_1, \ldots, t_m) (1 \leq j \leq n) \), we set \( D_j := \{ t \mid P_j(t) = 0 \} \subset \mathbb{C}^m \) and \( M := \mathbb{C}^m - (D_1 \cup \cdots \cup D_n) \). We consider a multi-valued function \( u(t) \) on \( M \) defined as

\[
u(t) := \prod_{j=1}^n P_j(t)^{\lambda_j}, \quad \lambda_j \in \mathbb{C} - \mathbb{Z} \quad (1 \leq j \leq n).
\]

Let \( \sigma \) be a \( k \)-simplex in \( M \), we define a loaded \( k \)-simplex \( \sigma \odot u \) by \( \sigma \) loading a branch of \( u \) on it. We denote the \( C \)-vector space of finite sums of loaded \( k \)-simplexes by \( C_k(M; u) \), called the \( k \)-th twisted chain group. An element of \( C_k(M; u) \) is called a twisted \( k \)-chain. For a loaded \( k \)-simplex \( \sigma \odot u \) and a smooth \( k \)-form \( \varphi \) on \( M \), the integral \( \int_{\sigma \odot u} u \cdot \varphi \) is defined by

\[
\int_{\sigma \odot u} u \cdot \varphi := \int_{\sigma} [\text{the fixed branch of } u \text{ on } \sigma] \cdot \varphi.
\]

By the linear extension of this, we define the integral on a twisted \( k \)-chain.

We define the boundary operator \( \partial^u : C_k(M; u) \rightarrow C_{k-1}(M; u) \) by

\[
\partial^u(\sigma \odot u) := \partial(\sigma) \odot u|_{\partial(\sigma)},
\]

where \( \partial \) is the usual boundary operator and \( u|_{\partial(\sigma)} \) is the restriction of \( u \) to \( \partial(\sigma) \). It is easy to see that \( \partial^u \circ \partial^u = 0 \). Thus we have a complex

\[
C_*(M; u) : \cdots \xrightarrow{\partial^u} C_k(M; u) \xrightarrow{\partial^s} C_{k-1}(M; u) \xrightarrow{\partial^s} \cdots,
\]

and its \( k \)-th homology group \( H_k(C_*(M; u)) \). It is called the \( k \)-th twisted homology group. An element of ker \( \partial^u \) is called a twisted cycle.

By considering \( u^{-1} = 1/u \) instead of \( u \), we have \( H_k(C_*(M; u^{-1})) \). There is the intersection pairing \( I_h \) between \( H_m(C_*(M; u)) \) and \( H_m(C_*(M; u^{-1})) \) (in fact, the intersection pairing is defined between \( H_k(C_*(M; u)) \) and \( H_{2m-k}(C_*(M; u^{-1})) \), however we do not consider the cases \( k \neq m \)). Let \( \Delta \) and \( \Delta' \) be elements of \( H_m(C_*(M; u)) \) and \( H_m(C_*(M; u^{-1})) \) given by twisted cycles \( \sum_i \alpha_i \cdot \sigma_i \odot u_i \) and \( \sum_j \alpha'_j \cdot \sigma'_j \odot u_j^{-1} \) respectively, where \( u_i \) (resp. \( u_j^{-1} \)) is a branch of \( u \) (resp. \( u^{-1} \)) on \( \sigma_i \) (resp. \( \sigma'_j \)). Then their intersection number is defined by

\[
I_h(\Delta, \Delta') := \sum_{i,j} \sum_{s \in \sigma_i \cap \sigma'_j} \alpha_i \alpha'_j \cdot (\sigma_i \cdot \sigma'_j)_s \cdot \frac{u_i(s)}{u_j(s)}.
\]
where \((\sigma_i, \sigma'_j)_s\) is the topological intersection number of \(m\)-simplexes \(\sigma_i\) and \(\sigma'_j\) at \(s\).

In this paper, we mainly consider
\[
M := \mathbb{C}^m - \left( H_1 \cup \cdots \cup H_m \cup H \cup D \right),
\]
where
\[
H_k := (t_k = 0) \ (1 \leq k \leq m), \ H := (v = 0), \ D := (w = 0), \ v := 1 - \sum t_k, \ w := \prod t_k \cdot (1 - \sum \tfrac{x_k}{t_k})).
\]
Note that \(w\) is a polynomial in \(t_1, \ldots, t_m\). We consider the twisted homology group on \(M\) with respect to the multi-valued function
\[
\begin{align*}
u := \prod t_k^{1-c_a+b} \cdot \prod c_a - a - m + 1 \cdot w^{-b} \\
= \prod t_k^{1-c_k} \cdot (1 - \sum t_k) \cdot \prod c_k - a - m + 1 \cdot \left( 1 - \sum \tfrac{x_k}{t_k} \right)^{-b}
\end{align*}
\]
(the second equality holds under the coordination of branches). Proposition 2.3 means that the integral
\[
\int_B u \varphi, \ \varphi := \frac{dt_1 \wedge \cdots \wedge dt_m}{\prod t_k \cdot (1 - \sum t_k)}
\]
represents \(F_{\mathcal{C}}(a, b, c; x)\) modulo Gamma factors.

4. Twisted cycles corresponding to local solutions \(f_{i_1 \ldots i_r}\)

In this section, we construct \(2^m\) twisted cycles in \(M\) corresponding to the solutions \(2\) to \(E_{\mathcal{C}}(a, b, c)\).

Let \(0 \leq r \leq m\) and subsets \(\{i_1, \ldots, i_r\}\) and \(\{j_1, \ldots, j_{m-r}\}\) of \(\{1, \ldots, m\}\) satisfy \(i_1 < \cdots < i_r, \ j_1 < \cdots < j_{m-r}\) and \(\{i_1, \ldots, i_r, j_1, \ldots, j_{m-r}\} = \{1, \ldots, m\}\).

**Notation 4.1.** From now on, the letter \(p\) (resp. \(q\)) is always stands for an index running from 1 to \(r\) (resp. from 1 to \(m - r\)). We use the abbreviations \(\sum, \prod\) for the indices \(p, q\) as are mentioned in Notation 2.1.

We set
\[
M_{i_1 \ldots i_r} := \mathbb{C}^m - \left( \bigcup_k (s_k = 0) \cup (v_{i_1 \ldots i_r} = 0) \cup (w_{i_1 \ldots i_r} = 0) \right),
\]
where
\[
v_{i_1 \ldots i_r} := \prod_p s_{i_p} \cdot (1 - \sum_{i_p} \tfrac{x_{i_p}}{s_{i_p}} - \sum q s_{j_q}), \ w_{i_1 \ldots i_r} := \prod q s_{j_q} \cdot (1 - \sum p s_{i_p} - \sum q \tfrac{x_{j_q}}{s_{j_q}})
\]
are polynomials in \(s_1, \ldots, s_m\). Let \(u_{i_1 \ldots i_r}\) and \(\varphi_{i_1 \ldots i_r}\) be a multi-valued function and an \(m\)-form on \(M_{i_1 \ldots i_r}\), defined as
\[
u_{i_1 \ldots i_r} := \prod_k s_{C_k} \cdot v_{i_1 \ldots i_r}^{-A} \cdot w_{i_1 \ldots i_r}^{B}, \ \varphi_{i_1 \ldots i_r} := \frac{ds_1 \wedge \cdots \wedge ds_m}{s_{1} \cdots s_{m} v_{i_1 \ldots i_r}}
\]
where
\[
A := \sum c_k - a - m + 1, \ B := -b, \ C_{i_p} := c_{i_p} - 1 - A, \ C_{j_q} := 1 - c_{j_q} - B.
\]
We construct a twisted cycle \(\tilde{\Delta}_{i_1 \ldots i_r}\) in \(M_{i_1 \ldots i_r}\) with respect to \(u_{i_1 \ldots i_r}\). Note that if \(\{i_1, \ldots, i_r\} = \emptyset\), then these settings coincide with those in the end of Section 3.
We choose positive real numbers \( \varepsilon_1, \ldots, \varepsilon_m \) and \( \varepsilon \) so that \( \varepsilon < 1 - \sum \varepsilon_k \). And let \( x_1, \ldots, x_m \) be small positive real numbers satisfying
\[
x_k < \frac{\varepsilon_k}{m} \varepsilon
\]
(for example, if
\[
\varepsilon_k = \varepsilon = \frac{1}{3m}, \quad 0 < x_k < \frac{1}{9m^3}
\]
these conditions hold). Thus the closed subset
\[
\sigma_{\varepsilon_1, \ldots, \varepsilon_m} := \left\{ (s_1, \ldots, s_m) \in \mathbb{R}^m \mid s_{ip} \geq \varepsilon_{ip}, \quad 1 - \sum s_{ip} \geq \varepsilon, \quad 1 - \sum s_{iq} \geq \varepsilon \right\}
\]
is nonempty, since we have \( \left( \varepsilon_1 + \frac{\varepsilon}{2m^2}, \ldots, \varepsilon_m + \frac{\varepsilon}{2m^2} \right) \) in \( \sigma_{\varepsilon_1, \ldots, \varepsilon_m} \), where \( \delta := 1 - \sum \varepsilon_k - \varepsilon > 0 \). Further, \( \sigma_{\varepsilon_1, \ldots, \varepsilon_m} \) is contained in the bounded domain
\[
\left\{ (s_1, \ldots, s_m) \in \mathbb{R}^m \mid s_k > 0, \quad 1 - \sum \frac{s_{ip}}{s_{ip}} - \sum s_{iq} > 0, \quad 1 - \sum s_{ip} - \frac{m - r}{m} \varepsilon > 0 \right\} \subset (0, 1)^m,
\]
and is a direct product of an \( r \)-simplex and an \( (m-r) \)-simplex. Indeed, \( (s_1, \ldots, s_m) \in \sigma_{\varepsilon_1, \ldots, \varepsilon_m} \) satisfies
\[
1 - \sum \frac{x_{ip}}{s_{ip}} - \sum s_{iq} > 1 - \frac{r}{m} \varepsilon - \sum s_{iq} > 1 - \sum s_{iq} - \varepsilon \geq 0,
\]
\[
1 - \sum s_{ip} - \sum \frac{x_{ip}}{s_{ip}} > 1 - \sum s_{ip} - \frac{m - r}{m} \varepsilon > 0 - \sum s_{ip} - \varepsilon \geq 0.
\]

The orientation of \( \sigma_{\varepsilon_1, \ldots, \varepsilon_m} \) is induced from the natural embedding \( \mathbb{R}^m \subset \mathbb{C}^m \). We construct a twisted cycle from \( \sigma_{\varepsilon_1, \ldots, \varepsilon_m} \). We may assume that \( \varepsilon_k = \varepsilon \) (the above example satisfies this condition), and denote them by \( \varepsilon \). Set \( L_1 := (s_1 = 0, \ldots, s_m = 0) \), \( L_{m+1} := (1 - \sum s_{ip} = 0) \), \( L_{m+2} := (1 - \sum s_{iq} = 0) \), and let \( U(\mathbb{C}^m) \) be the bounded chamber surrounded by \( L_1, \ldots, L_{m+1}, L_{m+2} \), then \( \sigma_{\varepsilon_1, \ldots, \varepsilon_m} \) is contained in \( U \). Note that we do not consider the hyperplane \( L_{m+1} \) (resp. \( L_{m+2} \)) when \( r = 0 \) (resp. \( r = m \)). For \( J \subset \{ 1, \ldots, m+2 \} \), we consider \( L_J := \cap_{j \in J} L_J, U_J := \overline{U} \cap L_J \) and \( T_J := \varepsilon \)-neighborhood of \( U_J \). Then we have
\[
\sigma_{\varepsilon_1, \ldots, \varepsilon_m} = U - \bigcup_J T_J.
\]

Using these neighborhoods \( T_J \), we can construct a twisted cycle \( \tilde{\Delta}_{\varepsilon_1, \ldots, \varepsilon_m} \) in the same manner as Section 3.2.4 of \( \mathbb{H} \) (notations \( L \) and \( U \) correspond to \( H \) and \( \Delta \) in \( \mathbb{H} \), respectively). Note that we have to consider contribution of branches of \( s_{ip}^A \cdot v_{i_1, \ldots, i_r} \), when we deal with the circle associated to \( L_{ip} \) \( (p = 1, \ldots, r) \). Indeed, for fixed positive real numbers \( s_k \) \( (k \neq i_p) \), \( s_{ip} \) satisfying \( 1 - \sum \frac{s_{ip}}{s_{ip}} = \sum s_{iq} = 0 \) belongs to \( \mathbb{R} \) and we have
\[
s_{ip} = \frac{x_{ip}}{1 - \sum s_{iq}} < \frac{m \varepsilon}{\varepsilon - (r-1) \frac{m}{m}} = \varepsilon, \quad \frac{m \varepsilon}{m - r - 1} < \varepsilon.
\]
Thus the exponent about this contribution is
\[
C_{ip} + A = c_{ip} - 1.
\]
The exponents about the contributions of the circles associated to \( L_{jq} \), \( L_{m+1} \), \( L_{m+2} \) are also evaluated as
\[
C_{jq} + B = 1 - c_{jq}, \quad B = -b, \quad A = \sum c_k - a - m + 1,
\]
respectively. We briefly explain the expression of \( \tilde{\Delta}_{\varepsilon_1, \ldots, \varepsilon_m} \). For \( j = 1, \ldots, m + 2 \), let \( l_j \) be the \( (m+1) \)-face of \( \sigma_{\varepsilon_1, \ldots, \varepsilon_m} \), given by \( \sigma_{\varepsilon_1, \ldots, \varepsilon_m} \cap T_j \), and let \( S_j \) be a positively
oriented circle with radius $\varepsilon$ in the orthogonal complement of $L_j$ starting from the projection of $l_j$ to this space and surrounding $L_j$. Then $\Delta_{i_1 \cdots i_r}$ is written as

$$\sigma_{i_1 \cdots i_r} + \sum_{\emptyset \neq J \subseteq \{1, \ldots, m+2\}} \prod_{j \in J} \frac{1}{d_j} \cdot \left( \left( \bigcap_{j \in J} l_j \right) \times \prod_{j \in J} S_j \right),$$

where

$$d_{i_r} := \gamma_{i_r} - 1, \quad d_{j_2} := \gamma_{j_2}^{-1} - 1, \quad d_{m+1} := \beta^{-1} - 1, \quad d_{m+2} := \alpha^{-1} \prod \gamma_k - 1,$$

and $\alpha := e^{2\pi \sqrt{-1} a}$, $\beta := e^{2\pi \sqrt{-1} b}$, $\gamma_k := e^{2\pi \sqrt{-1} c_k}$. Note that we define an appropriate orientation for each $(\bigcap_{j \in J} l_j) \times \prod_{j \in J} S_j$, see Section 3.2.4 of [1] for details.

**Example 4.2.** We give explicit forms of $\Delta$ and $\Delta_1$, for $m = 2$.

**(i)** In the case of $I = \emptyset$ ($r = 0$), we have

$$\Delta = \sigma + \frac{S_1 \times l_1}{1 - \gamma_1} \times \frac{S_2 \times l_2}{1 - \gamma_2} \times \frac{S_4 \times l_4}{1 - \beta^{-1}\gamma_1\gamma_2} + \frac{S_1 \times S_2}{(1 - \gamma_1)(1 - \gamma_2)} \times \frac{S_2 \times S_4}{(1 - \gamma_2^{-1})(1 - \alpha^{-1}\gamma_1\gamma_2)} \times \frac{S_4 \times S_1}{(1 - \alpha^{-1}\gamma_1\gamma_2)(1 - \gamma_1^{-1})},$$

where the 1-chains $l_j$ satisfy $\partial \sigma = l_1 + l_2 + l_4$ (see Figure 1), and the orientation of each direct product is induced from those of its components. Note that the face $l_3$ does not appear in this case.

![Figure 1. $\Delta$ for $m = 2$.](image_url)

**(ii)** In the case of $I = \{1\}$, we have

$$\Delta_1 = \sigma_1 + \frac{S_1 \times l_1}{1 - \gamma_1} \times \frac{S_2 \times l_2}{1 - \gamma_2} \times \frac{S_4 \times l_4}{1 - \beta^{-1}\gamma_1\gamma_2} + \frac{S_1 \times S_2}{(1 - \gamma_1)(1 - \gamma_2)} \times \frac{S_2 \times S_4}{(1 - \gamma_2^{-1})(1 - \alpha^{-1}\gamma_1\gamma_2)} \times \frac{S_4 \times S_1}{(1 - \beta^{-1})(1 - \alpha^{-1}\gamma_1\gamma_2)} \times \frac{S_3 \times S_4}{(1 - \alpha^{-1}\gamma_1\gamma_2)(1 - \gamma_1)},$$

where the 1-chains $l_j$ satisfy $\partial \sigma = \sum_{j=1}^4 l_j$ (see Figure 2), and the orientation of each direct product is induced from those of its components.
We consider the following integrals:

\[ F_{i_1 \cdots i_r} := \int_{\tilde{\Delta}_{i_1 \cdots i_r}} u_{i_1 \cdots i_r} \varphi_{i_1 \cdots i_r} \]

\[ = \int_{\tilde{\Delta}_{i_1 \cdots i_r}} \prod_{p=1}^r s_{i_p}^{c_{i_p} - 2} \prod_{q=1}^r s_{j_q}^{c_{j_q}} \cdot \left( 1 - \sum_{p=1}^r \frac{x_{i_p}}{s_{i_p}} - \sum_{q=1}^r \frac{x_{j_q}}{s_{j_q}} \right)^{\sum c_k - a - m} \]

\[ \cdot \left( 1 - \sum_{p=1}^r s_{i_p} - \sum_{q=1}^r s_{j_q} \right)^{-b} ds_1 \wedge \cdots \wedge ds_m. \]

Proposition 4.3.

\[ F_{i_1 \cdots i_r} = \prod_p \Gamma(c_{i_p} - 1) \cdot \prod_q \Gamma(1 - c_{j_q}) \cdot \frac{\Gamma(\sum c_k - a - m + 1) \Gamma(1 - b)}{\Gamma(\sum c_k - a - r + 1) \Gamma(\sum c_k - b - r + 1)} \]

\[ \cdot F_C \left( a + r - \sum_{p=1}^r c_{i_p}, b + r - \sum_{p=1}^r c_{i_p}, e^{i_1 \cdots i_r} ; x \right). \]

Proof. We compare the power series expansions of the both sides. Note that the coefficient of \( x_1^{n_1} \cdots x_m^{n_m} \) in the series expression of \( F_C(a + r - \sum_{p=1}^r c_{i_p}, b + r - \sum_{p=1}^r c_{i_p}, e^{i_1 \cdots i_r} ; x) \) is

\[ A_{n_1 \cdots n_m} := \frac{\Gamma(a + r - \sum_{p=1}^r c_{i_p} + \sum_k n_k)}{\Gamma(a + r - \sum_{p=1}^r c_{i_p})} \cdot \frac{\Gamma(b + r - \sum_{p=1}^r c_{i_p} + \sum_k n_k)}{\Gamma(b + r - \sum_{p=1}^r c_{i_p})} \]

\[ \cdot \prod_p \Gamma(2 - c_{i_p}) / \Gamma(2 - c_{i_p} + n_{i_p}) \cdot \prod_q \Gamma(c_{j_q}) / \Gamma(c_{j_q} + n_{j_q}) \cdot \prod_k 1 / n_k! \]

On the other hand, we have

\[ \left( 1 - \sum_p \frac{x_{i_p}}{s_{i_p}} - \sum_{q=1}^r \frac{x_{j_q}}{s_{j_q}} \right)^{\sum c_k - a - m} \]

\[ = \sum_{n_{i_1} \cdots n_{i_r}} \frac{\Gamma(a - \sum_k c_k + m + \sum_{p=1}^r n_{i_p})}{\Gamma(a - \sum_k c_k + m)} \cdot \prod_p s_{i_p}^{-n_{i_p}} \cdot (1 - \sum_{q=1}^r \frac{x_{j_q}}{s_{j_q}})^{\sum c_k - a - m - \sum_{q=1}^r n_{j_q}} \cdot \prod_p s_{i_p}^{n_{i_p}} \]
and

\[
(1 - \sum_{p} s_{ip} - \sum_{q} x_{jq})^{-b} = \sum_{n_{j_{1}}, \ldots, n_{j_{m-r}}} \frac{\Gamma(b + \sum_{q} n_{jq})}{\Gamma(b) \cdot \prod_{q} n_{jq}} \prod_{q} s_{jq}^{-n_{jq}} \cdot (1 - \sum_{p} s_{ip})^{-b - \sum_{q} n_{jq}} \cdot \prod_{q} x_{jq}^{n_{jq}}.
\]

When \( r = 0 \) (resp. \( r = m \)), we do not need the first (resp. second) expansion. The convergences of these power series expansions are verified as follows. We explain only the first one. By the construction of \( \bar{\Delta}_{i_{1} \cdots i_{r}} \), we have

\[
x_{k} < \frac{\varepsilon_{k}}{m} \varepsilon, \quad |s_{ip}| \geq \varepsilon_{ip}, \quad 1 - \sum_{q} s_{jq} \geq \varepsilon.
\]

Thus the uniform convergence on \( \bar{\Delta}_{i_{1} \cdots i_{r}} \) follows from

\[
\left| \frac{1}{1 - \sum_{q} s_{jq}} \cdot \sum_{p} \frac{x_{ip}}{s_{ip}} - \frac{1}{1 - \sum_{q} s_{jq}} \cdot \sum_{p} \frac{|x_{ip}|}{s_{ip}} \right| < \varepsilon \cdot \sum_{p} \frac{|s_{ip}|}{m} \varepsilon = \frac{r}{m} \leq 1.
\]

Since \( \bar{\Delta}_{i_{1} \cdots i_{r}} \) is constructed as a finite sum of loaded (compact) simplexes, we can exchange the sum and the integral in the expression of \( F_{i_{1} \cdots i_{r}} \). Then the coefficient of \( x_{1}^{r_{1}} \cdots x_{m}^{r_{m}} \) in the series expansion of \( F_{i_{1} \cdots i_{r}} \) is

\[
(4) \quad B_{n_{1} \cdots n_{m}} := \frac{\Gamma(a - \sum c_{k} + m + \sum n_{jq})}{\Gamma(a - \sum c_{k} + m)} \cdot \frac{\Gamma(b + \sum n_{iq})}{\Gamma(b)} \cdot \prod_{k} \frac{1}{n_{k}!} \cdot \int_{\bar{\Delta}_{i_{1} \cdots i_{r}}} \prod_{p} s_{ip}^{-c_{ip} - n_{ip} - 1} \cdot (1 - \sum_{p} s_{ip})^{-b - \sum n_{jq}} \prod_{q} s_{jq}^{-c_{jq} - n_{jq} - 1} \cdot (1 - \sum_{q} s_{jq}) \sum c_{k} - a - m - \sum n_{ip} ds_{1} \wedge \cdots \wedge ds_{m}.
\]

By the construction, the twisted cycle \( \bar{\Delta}_{i_{1} \cdots i_{r}} \) of this integral is identified with the usual regularization of the domain

\[
\left\{ (s_{1}, \ldots, s_{m}) \in \mathbb{R}^{m} \mid s_{ip} > 0, \quad 1 - \sum_{q} s_{jq} > 0, \quad 1 - \sum_{q} s_{jq} > 0 \right\}
\]

for the multi-valued function

\[
\prod_{p} s_{ip}^{-c_{ip} - n_{ip} - 1} \cdot (1 - \sum_{p} s_{ip})^{-b - \sum n_{jq} + 1} \cdot \prod_{q} s_{jq}^{-c_{jq} - n_{jq} + 1} \cdot (1 - \sum_{q} s_{jq}) \sum c_{k} - a - m - \sum n_{ip} + 1
\]

on \( C^{m} - \left( \bigcup_{k} \{ s_{k} = 0 \} \cup (1 - \sum s_{ip} = 0) \cup (1 - \sum s_{jq} = 0) \right) \). Hence the integral in (4) is equal to

\[
\frac{\prod \Gamma(c_{ip} - n_{ip} - 1) \cdot \Gamma(-b - \sum c_{jq} + 1)}{\Gamma(-b + \sum c_{ip} - \sum n_{k} - r + 1)} \cdot \frac{\prod \Gamma(-c_{jq} + n_{jq} + 1) \cdot \Gamma(\sum c_{k} - a - m - \sum n_{ip} + 1)}{\Gamma(\sum c_{ip} - a - \sum n_{k} - r + 1)}.
\]

Using the formula

\[
(5) \quad \Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin(\pi z)}.
\]
we thus have
\[ \frac{B_{n_1 \ldots n_m}}{A_{n_1 \ldots n_m}} = \prod_p \Gamma(c_p - 1) \cdot \prod_q \Gamma(1 - c_q) \]
\[ \cdot \frac{\Gamma(\sum c_k = a - m + 1)\Gamma(1 - b)}{\Gamma(\sum c_k = a - r + 1)\Gamma(\sum c_k = b - r + 1)}. \]
which implies the proposition. \(\square\)

We define a bijection \(\iota_{i_1 \ldots i_r}: M_{i_1 \ldots i_r} \to M\) by
\[ \iota_{i_1 \ldots i_r}(s_1, \ldots, s_m) := (t_1, \ldots, t_m); \ t_p = \frac{x_{i_p}}{s_{i_p}}, \ t_q = s_q. \]
For example, \(\iota(= \iota_0)\) is the identity map on \(M = M_0\), and \(\iota_{1 \ldots m}\) defines an involution on \(M = M_{1 \ldots m}\). We state our first main theorem.

**Theorem 4.4.** We define a twisted cycle \(\Delta_{i_1 \ldots i_r}\) in \(M\) by
\[ \Delta_{i_1 \ldots i_r} := (-1)^r (\iota_{i_1 \ldots i_r})* (\Delta_{i_1 \ldots i_r}). \]
Then we have
\[ \int_{\Delta_{i_1 \ldots i_r}} \prod_k t_k^{c_k} \cdot (1 - \sum t_k)\sum c_k = a - m \cdot \left(1 - \sum \frac{x_k}{t_k}\right)^{-b} dt_1 \wedge \cdots \wedge dt_m \]
\[ = \int_{\Delta_{i_1 \ldots i_r}} w^\varphi = \prod_p \frac{1 - c_{i_p}}{c_{i_p}} \cdot F_{i_1 \ldots i_r}, \]
and hence this integral corresponds to the local solution \(f_{i_1 \ldots i_r}\) to \(E_C(a, b, c)\) given in Proposition 2.3.

**Proof.** Pull-back the integral of the left hand side by \(\iota_{i_1 \ldots i_r}\). Indeed, we have
\[ t_p = \frac{x_{i_p}}{s_{i_p}}, \ dt_{i_p} = -\frac{x_{i_p}}{s_{i_p}} ds_{i_p} \]
by the definition of \(\iota_{i_1 \ldots i_r}\). Note that the sign \((-1)^r\) arising from the pull-back of \(dt_1 \wedge \cdots \wedge dt_m\) is canceled by that in \((6)\). The second claim is followed from the first equality and Proposition 4.3. \(\square\)

**Remark 4.5.**
(i) The sign \((-1)^r\) in \((6)\) implies that the orientation of the direct product \(\iota_{i_1 \ldots i_r}(\sigma_{i_1 \ldots i_r})\) of two simplexes in \(\Delta_{i_1 \ldots i_r}\) is coincide with that of \(\mathbb{R}^m\) induced from the natural embedding \(\mathbb{R}^m \subset \mathbb{C}^m\).
(ii) The twisted cycle \(\Delta\) (for \(r = 0\)) equals to that mentioned in Proposition 2.3.

The replacement \(u \mapsto u^{-1} = 1/u\) and the construction same as \(\Delta_{i_1 \ldots i_r}\) give the twisted cycle \(\Delta_{i_1 \ldots i_r}^\vee\), which represents an element in \(H_m(C_\ast(M, u^{-1}))\). We obtain the intersection numbers of the twisted cycles \(\{\Delta_{i_1 \ldots i_r}\}\) and \(\{\Delta_{i_1 \ldots i_r}^\vee\}\).

**Theorem 4.6.**
(i) For \(I, J \subset \{1, \ldots, m\}\) such that \(I \neq J\), we have \(I_h(\Delta, \Delta) = 0\).
(ii) The self-intersection number of \(\Delta_{i_1 \ldots i_r}\) is
\[ I_h(\Delta_{i_1 \ldots i_r}, \Delta_{i_1 \ldots i_r}) = (-1)^r \cdot \frac{\prod_q \gamma_{i_q} \cdot (\alpha - \prod_p \gamma_{i_p}) (\beta - \prod_p \gamma_{i_p})}{\prod_k (\gamma_k - 1) \cdot (\alpha - \prod_p \gamma_{i_p}) (\beta - 1)}. \]
Proof. (i) Since $\Delta_{i_1 \cdots i_r}$’s represent local solutions \([2]\) to $E_C(a, b, c)$ by Theorem 4.3, this claim is followed from similar arguments to the proof of Lemma 4.1 in \([6]\).

(ii) By using $i_1 \cdots i_r$, the self-intersection number of $\Delta_{i_1 \cdots i_r}$ is equal to that of $\Delta_{1 \cdots 1}$ with respect to the multi-valued function $u_{i_1 \cdots i_r}$. To calculate this, we apply results of M. Kita and M. Yoshida (see \([8]\)). Since we construct the twisted cycle $\Delta_{i_1 \cdots i_r}$ from the direct product $\sigma_{i_1 \cdots i_r}$ of two simplexes, the self-intersection number of $\Delta_{i_1 \cdots i_r}$ is obtained as the product of those of simplexes. Thus we have

$$I_k(\Delta_{i_1 \cdots i_r}, \Delta_{1 \cdots 1}) = \frac{1 - \prod_p \gamma_{i_p} \cdot \beta^{-1}}{\prod_p (1 - \gamma_{i_p}) \cdot (1 - \beta^{-1})} \cdot \frac{1 - \prod_q \gamma_{i_q} \cdot \alpha^{-1}}{\prod_q (1 - \gamma_{i_q}) \cdot (1 - \alpha^{-1})}.$$  

\[ \square \]

5. Intersection Numbers of Twisted Cohomology Groups

In this section, we review twisted cohomology groups and the intersection form between twisted cohomology groups in our situation, and evaluate some self-intersection numbers of twisted cocycles.

Recall that

$$M = \mathbb{C}^m - \left( \bigcup_k \{ t_k = 0 \} \cup \{ u = 0 \} \cup \{ w = 0 \} \right),$$

$$u = \prod_k t_k^{c_k + b} \cdot \prod \sum c_{k} - a + m + 1 \cdot \prod \gamma_{k} - b.$$  

We consider the logarithmic 1-form

$$\omega := d \log u = \frac{du}{u}.$$  

We denote the $\mathbb{C}$-vector space of smooth $k$-forms on $M$ by $\mathcal{E}^k(M)$. We define the covariant differential operator $\nabla_{\omega} : \mathcal{E}^k(M) \to \mathcal{E}^{k+1}(M)$ by

$$\nabla_{\omega}(\psi) := d\psi + \omega \wedge \psi, \quad \psi \in \mathcal{E}^k(M).$$

Because of $\nabla_{\omega} \circ \nabla_{\omega} = 0$, we have a complex

$$\mathcal{E}^*(M) : \cdots \xrightarrow{\nabla_{\omega}} \mathcal{E}^k(M) \xrightarrow{\nabla_{\omega}} \mathcal{E}^{k+1}(M) \xrightarrow{\nabla_{\omega}} \cdots,$$

and its $k$-th cohomology group $H^k(M, \nabla_{\omega})$. It is called the $k$-th twisted de Rham cohomology group. An element of ker $\nabla_{\omega}$ is called a twisted cocycle. By replacing $\nabla_{\omega}$ with the $\mathbb{C}$-vector space $\mathcal{E}^k(M)$ of smooth $k$-forms on $M$ with compact support, we obtain the twisted de Rham cohomology group $H^k_{\text{tw}}(M, \nabla_{\omega})$ with compact support. By \([2]\), we have $H^k(M, \nabla_{\omega}) = 0$ for all $k \neq m$. Further, by Lemma 2.9 in \([1]\), there is a canonical isomorphism

$$j : H^m(M, \nabla_{\omega}) \to H^m_{\text{tw}}(M, \nabla_{\omega}).$$

By considering $u^{-1} = 1/u$ instead of $u$, we have the covariant differential operator $\nabla_{-\omega}$ and the twisted de Rham cohomology group $H^k(M, \nabla_{-\omega})$. The intersection form $I_c$ between $H^m(M, \nabla_{\omega})$ and $H^m(M, \nabla_{-\omega})$ is defined by

$$I_c(\psi, \psi') := \int_M j(\psi) \wedge \psi', \quad \psi \in H^m(M, \nabla_{\omega}), \quad \psi' \in H^m(M, \nabla_{-\omega}),$$

which converges because of the compactness of the support of $j(\psi)$.

By the Poincaré duality (see Lemma 2.8 in \([1]\)), we have

\begin{align*}
\dim H_k(C_\bullet(M, u)) &= 0 \quad (k \neq m), \\
\dim H_m(C_\bullet(M, u)) &= \dim H^m(M, \nabla_{\omega}).
\end{align*}
Proposition 5.1. Let $x_1, \ldots, x_m$ be generic.

(i) We have $\dim H_m(C^*_u(M, u)) = 2^m$.

(ii) The twisted cycles $\{\Delta_I\}_I$ form a basis of $H_m(C^*_u(M, u))$.

(iii) The integrations of $u\varphi$ on twisted cycles give an isomorphism between $H_m(C^*_u(M, u))$ and the space of local solutions to $E_C(a, b, c)$.

Proof. We prove (i). By (7) and Theorem 2.2 in [1], we have
\[
\dim H_m(C^*_u(M, u)) = (-1)^m \chi(M),
\]
where $\chi(M)$ is the Euler characteristic of $M$. It is sufficient to show that $\chi(M) = (-1)^m \cdot 2^m$. Let $h \in \mathbb{C}[T_0, \ldots, T_m]$ be a homogeneous polynomial defined by
\[
h(T) := \prod_{i=0}^m T_i \cdot \left( \sum_{i=0}^m \prod_{j \neq i} T_j \right),
\]
and let $D(h) = \{T \in \mathbb{P}^m \mid h(T) \neq 0\}$. Then we have $\chi(M) = \chi(D(h) - L)$ for some generic hyperplane $L \in \mathbb{P}^m$. We consider the gradient map
\[
\text{grad}(h) : D(h) \to \mathbb{P}^m; \quad [T_0 : \cdots : T_m] \mapsto \left[ \frac{\partial h}{\partial T_0}(T) : \cdots : \frac{\partial h}{\partial T_m}(T) \right].
\]
It is easy to see that the degree of $\text{grad}(h)$ is equal to $2^m$. By Theorem 1 in [1], we obtain $\chi(D(h) - L) = (-1)^m \deg(\text{grad}(h)) = (-1)^m \cdot 2^m$, which shows (i). (The author thanks to J. Kaneko for pointing out this fact.)

The claim (ii) follows from (i), since the determinant of the intersection matrix $(I_h(\Delta_I, \Delta_J))$ is not zero by Theorem 1.6.

We show (iii). Let $\text{Sol}$ be the space of local solutions to $E_C(a, b, c)$. By Theorem 4.4, integrals of $u\varphi$ on linear combinations of $\Delta_i$'s are in $\text{Sol}$. Then (ii) implies that the linear map
\[
\Phi : H_m(C^*_u(M, u)) \to \text{Sol}; \quad C \mapsto \int_C u\varphi
\]
is defined. Proposition 2.2 and Theorem 4.4 imply that $\Phi$ is surjective. Therefore $\Phi$ is isomorphic because of $\dim H_m(C^*_u(M, u)) = \dim \text{Sol} = 2^m$. $\square$

We evaluate some intersection numbers of the twisted cocycles
\[
\varphi = \frac{dt_1 \wedge \cdots \wedge dt_m}{\prod t_k \cdot (1 - \sum t_k)},
\]
\[
\varphi' := \frac{dt_1 \wedge \cdots \wedge dt_m}{vw} = \frac{dt_1 \wedge \cdots \wedge dt_m}{\prod t_k \cdot (1 - \sum t_k) \cdot (1 - \sum \frac{1}{t_k})}.
\]

Theorem 5.2. (i) The self-intersection number of $\varphi$ is
\[
I_c(\varphi, \varphi) = (2\pi \sqrt{-1})^m \left( \sum c_k - a - m + 1 + \frac{1}{b + m - \sum c_k} \right) \cdot \sum_{i \in \{1, \ldots, m\}} \prod_{r=1}^{m-1} \frac{1}{b + r - \sum c_{i(r)}},
\]
where $\{I^{(r)}\}$ runs sequences of subsets of $\{1, \ldots, m\}$, which satisfy
\[
\{1, \ldots, m\} \supseteq I^{(m-1)} \supseteq \cdots \supseteq I^{(2)} \supseteq I^{(1)} \neq \emptyset,
\]
and we write $I^{(r)} = \{i_1^{(r)}, \ldots, i_{r}^{(r)}\}$.

(ii) We have $I_c(\varphi, \varphi') = 0$. 

Proof. (i) The hypersurfaces $H_1, \ldots, H_m, H, D$ do not form a normal crossing divisor, because of $H_i \cap H_j \subset D$ for $i \neq j$. Thus we blow up $\mathbb{C}^m$ along some intersections of hyperplanes so that the pole divisor of $\varphi$ is normally crossing. Firstly we consider the blow up at the origin ($= H_1 \cap \cdots \cap H_m$). Secondly we blow up this along $H_1 \cap \cdots \cap H_k \cap \cdots \cap H_m$ ($k = 1, \ldots, m$). Repeat the blowing up process, lastly we blow up along $H_i \cap H_j$ ($1 \leq i < j \leq m$). For $i_1 < \cdots < i_r$, $r \geq 2$, the exceptional divisor $E_{i_1, \ldots, i_r}$ arising from the blow up along $H_{i_1} \cap \cdots \cap H_{i_r}$ has the exponent

$$\sum_{p=1}^{r} (1 - c_p + b) -(r -1)b = b + \sum_{p=1}^{r} c_p.$$  

By expressing $\varphi$ in each coordinates system, results in \[10\] give the self-intersection number of $\varphi$.

(ii) By the definition, the pole divisor of $\varphi'$ does not contain the exceptional divisors. Hence the pole divisors of $\varphi$ and $\varphi'$ do not have $m$ or more common factors, which implies $I_c(\varphi, \varphi') = 0$. □

Remark 5.3. Precisely speaking, to evaluate intersection numbers of twisted cocycles, we should blow up a compactification of $\mathbb{C}^m$ ($\mathbb{P}^m$ or $(\mathbb{P}^1)^m$) so that the pole divisor of $\omega = \text{d} \log u$ is normally crossing. Though the blowing up process in the above proof is not enough, it is shown that the exceptional divisors arising from the other blowing up processes do not appear as the components of the pole divisor of $\varphi$. Thus the proof is completed.

Remark 5.4. It seems difficult to evaluate the self-intersection number $I_c(\varphi', \varphi')$. For $m = 2$ (i.e., Appell’s $F_4$), this number ($= I_c(\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ in \[6\]) is written by the parameters and the factor of the defining equation of the singular locus (see \[6\]). Furthermore, the author does not find $m$-forms which form a basis of $H^m(M, \nabla \omega)$.

6. Twisted period relations

By Theorems \[16\] and \[52\] we state our second main theorem.

Theorem 6.1 (Twisted period relations). We have

$$I_c(\varphi, \varphi) = \sum_I \frac{1}{I_I(\Delta_{i_1, \ldots, i_r}, \Delta_{i_1, \ldots, i_r}')} \cdot g_{i_1, \ldots, i_r},$$

$$I_c(\varphi, \varphi') = \sum_I \frac{1}{I_I(\Delta_{i_1, \ldots, i_r}, \Delta_{i_1, \ldots, i_r}')} \cdot g_{i_1, \ldots, i_r} \cdot h_{i_1, \ldots, i_r}'.$$

where

$$g_{i_1, \ldots, i_r} = \int_{\Delta_{i_1, \ldots, i_r}} u \varphi, \quad g_{i_1, \ldots, i_r}' = \int_{\Delta_{i_1, \ldots, i_r}'} u^{-1} \varphi, \quad h_{i_1, \ldots, i_r}'' = \int_{\Delta_{i_1, \ldots, i_r}'} u^{-1} \varphi'.$$

Further, under the notations

$$a_{i_1, \ldots, i_r} := a - \sum c_p + r, \quad b_{i_1, \ldots, i_r} := b - \sum c_p + r, \quad c_{i_1, \ldots, i_r} := (2, \ldots, 2) - c_{i_1, \ldots, i_r},$$
the equalities (3) and (4) are reduced to

\( \sum_{l} (-1)^{l-1} \frac{a_{i_1 \cdots i_r}}{b_{i_1 \cdots i_r}} \cdot \gamma (a_{i_1 \cdots i_r}, b_{i_1 \cdots i_r}, c_{i_1 \cdots i_r}; x) \cdot \gamma (2-a_{i_1 \cdots i_r}, b_{i_1 \cdots i_r}, c_{i_1 \cdots i_r}; x) \)

\( \frac{\Gamma(\sum c_k - a - m + 1)\Gamma(1 - b)}{\Gamma(\sum c_k - a + r + 1)\Gamma(\sum c_k - b + r + 1)} \cdot \prod_{p=1}^{m-r} x_{i_p}^{-a_{i_p}} \cdot \gamma (a - r + \sum c_{i_p}, b + r - \sum c_{i_p}, c_{i_1 \cdots i_r}; x) \).

(10)

On the other hand, we can express the equalities (8) and (9) are reduced to

\( \sum_{l} (-1)^{l-1} \frac{a_{i_1 \cdots i_r}}{b_{i_1 \cdots i_r}} \cdot \gamma (a_{i_1 \cdots i_r}, b_{i_1 \cdots i_r}, c_{i_1 \cdots i_r}; x) \cdot \gamma (2-a_{i_1 \cdots i_r}, b_{i_1 \cdots i_r}, c_{i_1 \cdots i_r}; x) \)

\( = 0, \)

respectively.

**Proof.** Because of the compatibility of intersection forms and pairings obtained by integrations (see [3]), we obtain the equalities (3) and (4). We show that (3) is reduced to (10). By Proposition 4.3 and Theorem 4.4, we have

\[ g_{i_1 \cdots i_r} = \prod_{p=1}^{r} \Gamma(c_{i_p} - 1) \cdot \prod_{q=1}^{m-r} \Gamma(1 - c_{i_q}) \cdot \frac{\Gamma(\sum c_k - a - m + 1)\Gamma(1 - b)}{\Gamma(\sum c_k - a + r + 1)\Gamma(\sum c_k - b + r + 1)} \cdot \prod_{p=1}^{r} x_{i_p}^{-a_{i_p}} \cdot \gamma (a - r + \sum c_{i_p}, b + r - \sum c_{i_p}, c_{i_1 \cdots i_r}; x). \]

On the other hand, we can express \( g_{i_1 \cdots i_r} \) like this by the replacement

\( (a, b, c) \mapsto (2 - a, -b, (2, \ldots, 2) - c), \)

since \( u^{-1} \varphi \) is written as

\[ u^{-1} \varphi = \prod_{p=1}^{r} x_{i_p}^{c_{i_p} - 2} \cdot (1 - \sum_{k} t_k - \sum c_k + a + m - 2 \cdot \left( 1 - \sum \frac{x_k}{t_k} \right)) dt_1 \wedge \cdots \wedge dt_m. \]

Thus we obtain

\[ g_{i_1 \cdots i_r} = \prod_{p=1}^{r} \Gamma(1 - c_{i_p}) \cdot \prod_{q=1}^{m-r} \Gamma(c_{i_q} - 1) \]

\[ \frac{\Gamma(- \sum c_k + a + m - 1)\Gamma(1 + b)}{\Gamma(- \sum c_k + a + r + 1)\Gamma(- \sum c_k + b + r + 1)} \cdot \prod_{p=1}^{r} x_{i_p}^{-a_{i_p}} \cdot \gamma (2-a-r+ \sum c_{i_p}, b-r+ \sum c_{i_p}, (2, \ldots, 2) - c_{i_1 \cdots i_r}; x). \]

By the formula (5), we have

\[ \prod \Gamma(c_{i_p} - 1)\Gamma(1 - c_{i_p}) \cdot \frac{\Gamma(\sum c_k - a - m + 1)\Gamma(- \sum c_k + a + m - 1)}{\Gamma(\sum c_k - a + r + 1)\Gamma(- \sum c_k + a + r + 1)} \cdot \Gamma(1 - b) \Gamma(1 + b) \]

\[ \frac{\Gamma(\sum c_k - b + r + 1)\Gamma(- \sum c_k + b + r + 1)}{\Gamma(\sum c_k - b - r + 1)\Gamma(- \sum c_k + b + r + 1)} \]

\[ = (2\pi \sqrt{-1})^m \cdot \prod (1 - c_{i_p}) \cdot (a + m - 1 - \sum c_k) \]

\[ \cdot (-1)^r \cdot \frac{a + r - 1 - \sum c_{i_p}}{b + r - \sum c_{i_p}} \cdot \gamma (\Delta_{i_1 \cdots i_r}, \Delta_{i_1 \cdots i_r}). \]
Hence, under the notations \( a_{i_1 \ldots i_r}, b_{i_1 \ldots i_r} \) and \( \tilde{c}_{i_1 \ldots i_r} \), the equality (5) is reduced to

\[
\left( \frac{1}{1 - a_{i_1 \ldots i_r}} + \frac{1}{b_{i_1 \ldots i_r}} \right) \cdot \sum_{\{i^{(r)}\}} \prod_{r=1}^{m-1} \frac{1}{b_{i^{(r)}}} = \prod_{k}(1 - c_k) \cdot (a_{i_1 \ldots i_r} - 1) \cdot \sum_{I} (-1)^{a_{i_1 \ldots i_r} - 1} F_C(a_{i_1 \ldots i_r}, b_{i_1 \ldots i_r}, c_{i_1 \ldots i_r}; x) \cdot F_C(2 - a_{i_1 \ldots i_r}, -b_{i_1 \ldots i_r}, \tilde{c}_{i_1 \ldots i_r}; x).
\]

By multiplying \( (1 - a_{i_1 \ldots i_r}) \cdot \prod_{k}(1 - c_k) \), we obtain (10). The similar calculation shows that (9) is reduced to (11). □

Note that (10) and (11) are generalizations of some equalities in Corollary 6.1 of [6].

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Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan

E-mail address: y-goto@math.sci.hokudai.ac.jp