On models of a nondeterministic computation

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Abstract

In this paper we consider a nondeterministic computation by deterministic multi-head 2-way automata having a read-only access to an auxiliary memory. The memory contains additional data (a guess) and computation is successful iff it is successful for some memory content.

Also we consider the case of restricted guesses in which a guess should satisfy some constraint.

We show that the standard complexity classes such as $L$, $NL$, $P$, $NP$, PSPACE can be characterized in terms of these models of nondeterministic computation. These characterizations differ from the well-known ones by absence of alternation.

Keywords: automaton, nondeterminism, language, complexity class.

The standard way to define a nondeterministic computation by an automaton or a Turing machine is to change a transition function by a transition relation. In a nondeterministic state of a computational device a computation branches into several computation paths.

There is another way to introduce a nondeterminism. Suppose that a computational device has an additional data (a guess or a certificate or a proof of correctness) and performs a deterministic computation operating with an input data and a guess data.

Sometimes these variants of introducing nondeterminism lead to equivalent computational models. The class $NP$, for example, can be defined in both ways using Turing machines.

If we restrict computational power of a computational device these variants may differ drastically. The aim of this paper is to investigate models of nondeterminism based on the second variant for multi-head 2-way automata.

It is well-known that computation abilities of multi-head 2-way automata are equivalent to Turing machines with a logarithmically bounded memory. So, they recognize languages from the class $L$.

Nondeterministic (in the sense of transition relation) multi-head 2-way automata recognize languages from the class $NL$. One can rewrite a definition of a nondeterministic automaton using the second way of introducing nondeterminism. Let’s imagine that a guess data are written on an auxiliary tape, which is 1-way read-only. It is easy to see that using an 1-way guess tape leads to an equivalent definition of a nondeterministic automaton.

In this paper we consider a more general model of an auxiliary read-only memory (see definitions in Section 1). Guess data are stored in cells of a memory and at each moment of time an automaton has an access to the exactly one memory cell. Possible transitions between memory cells form a directed graph (the memory graph). An automaton can choose between finite number of variants only. So, the natural condition on the memory graph is a finite fan-out in each vertex (i.e. a memory cell).

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1 O. H. Ibarra [10] attributed this result to A. Cobham and coauthors referring to an unpublished manuscript.
The most natural variant of the auxiliary memory is a 2-way tape. The corresponding computational model appears to be very close to nonerasing nondeterministic stack automata (NENSA) [8, 10]. Similarly to multi-head NENSA, the automata with 2-way read only guess tape recognize all languages from the class PSPACE.

It is possible to define in our settings a deterministic computation as a specific case of a nondeterministic one. The deterministic automata with 2-way guess tape are similar to nonerasing deterministic stack automata (NEDSA) and also recognize the languages from the class PSPACE.

We focus our attention on a more restricted memory model, so-called 1.5-way tape. It was used in research of quantum automata [1]. For classic automata 1.5-way tape means an 1-way tape with an additional possibility to return into the first cell.

The nondeterministic automata with 1.5-way tape also recognize the class PSPACE (Theorem 2 below). But deterministic automata with this memory type recognize the class P only (Theorem 1). These results show that the 1.5-way guess tape is potentially more suitable to characterize various complexity classes.

Also we introduce a nondeterministic computation with a restricted guess. An example of restricted guess is a sparse guess. Sparseness of a guess means that a guess tape contains the only one (or finitely many) non-empty symbol and the rest symbols stored on the tape are empty. Using this model of a nondeterministic computation gives the class NP.

An interesting feature of all these results is a formal absence of resource bounds in characterizations of resource-bounded classes such as P, NP and so on. It should be noted that there is a primary result of this sort: many heads are equivalent to logarithmic space. The rest of results are based on this fact.

The main technical tool in study of the 1.5-way tape is calculations modulo polynomially bounded integer. These calculations can be performed on logarithmic space. To compute a length of a part of the guess tape we use the simple algorithm: go along the part and increase a counter modulo $p$. The latter operation can be done on logarithmic space. The length can be restored from these data due to the Chinese remainder theorem.

There are many results on characterizations of complexity classes in terms of some sort of automata. The classes L, NL, P, PSPACE have the well-known characterizations by deterministic, nondeterministic, alternating and synchronized alternating 2-way automata [4, 6, 9]. There are also characterizations of NP, the polynomial hierarchy and some other complexity classes in terms of alternating auxiliary stack automata [7].

Our results differ from these characterization because the models considered in this paper do not use alternation.

It is worth to mention a paper [3], which contains the characterizations of P, NP and PSPACE in terms of nondeterminism and so close to our results. The difference is in the nature of nondeterminism introduced. In [3] nondeterministic colorings of $n$-dimensional words are considered. Contrary, our main results concern the case of 1-dimensional guess memory.

The rest of paper is organized in the following way. In Section 1 we introduce our basic computational model: multi-head 2-way automata with a nondeterministic auxiliary memory. Section 2 contains results about the 1-way, the 1.5-way and the 2-way guess tapes. In Section 3 we introduce a model of a restricted guess formally and give characterizations of NP in terms of this model. Section 4 contains some additional remarks on possible variants of defining nondeterministic computation.

### 1 Automata with an auxiliary memory

In this section we provide definitions for a model of nondeterministic computation by automata with an auxiliary read-only memory. The definitions fix an informal idea explained in the introductory section. They follow the standard way of definition for computational models.
Definition 1. A memory model is a directed graph \((M, E)\), the initial cell \(m_0 \in M\) and a marking map \(g : E \rightarrow G\) from the edges of the graph to some finite set \(G\). The marking map satisfies the following conditions:

- \(g(u, v) \neq g(u, w)\) for \(v \neq w\) (different edges outgoing from the same vertex have different marks);
- for each \(u \in M\) and \(a \in G\) there is an edge \((u, v) \in E\) such that \(g(u, v) = a\).

In other words, the map \(g\) restricted to the set of edges outgoing from a vertex is a bijection.

For any finite alphabet \(\Delta\) a memory content \(\mu\) is a map \(\mu : M \rightarrow \Delta\).

Definition 2. An \(h\)-head automaton \(A\) with an auxiliary memory of model \(M\) is characterized by

- a finite state set \(Q\),
- a finite input alphabet \(\Sigma \cup \{\triangleleft, \triangleright\}\),
- a finite memory alphabet \(\Delta\),
- a transition function \(\delta\), which maps a \((h + 2)\)-tuple (the current state, symbols of the input word under the heads, the symbol in the current memory cell) to a \((h + 2)\)-tuple (a new state, a motion command for each head, a command of changing memory cell),
- the initial state \(q_0 \in Q\),
- the set of accepting states \(Q_a \subset Q\).

Heads can move along the input words by one position per step. So, a motion command for a head is an element from the set \((-1, 0, +1)\). A command of changing memory cell is just an element of the marking set \(G\) or an empty command (do not change the cell).

An automaton \(A\) operates on an input word \(w \in \Sigma^*\) in natural way. We assume that the input word is extended by markers \(\{\triangleleft, \triangleright\}\) of the beginning and the end of the word. The automaton starts from the initial state \(q_0\), the initial position of each head is the leftmost symbol of the input word, the initial memory cell is \(m_0\). The automaton applies the transition function on each step of operation to modify its state, head positions and a memory cell. For a fixed content of the auxiliary memory it generates a sequence of configurations. The automaton stops iff it reaches an accepting state.

Definition 3. The automaton \(A\) accepts the input word \(w\) iff for some memory content \(\mu\) it stops an operation.

The automaton recognizes the language \(L\) iff for any \(w \in L\) it accepts \(w\) and for any \(w \notin L\) it do not accept \(w\).

We denote by \(M\)-NFA the class of languages recognized by automata with an auxiliary memory of model \(M\). We denote by \(M\)-NFA\((h)\) the subclass of languages recognized by automata with \(h\) heads.

1.1 Determinization

As a specific case of a nondeterministic memory one can regard deterministic automata equipped with a WORM-memory (write once, read many). Such an automaton should fill a new memory cell by a symbol when it enter the cell the first time. In further operation it can not change the cell. Let’s introduce a formal definition suitable for our purposes.

Definition 4. A WORM-memory automaton on memory model \(M\) is characterized by
• a finite state set $Q$,
• a finite input alphabet $\Sigma \cup \{\triangleleft, \triangleright\}$,
• a finite memory alphabet $\Delta \cup \{\text{void}\}$,
• a transition function $\delta$, which maps a $(h+2)$-tuple (the current state, symbols of the input word under the heads, the symbol in the current memory cell) to a $(h+2)$-tuple (a new state, a motion command for each head, a command of changing memory cell),
• the initial state $q_0 \in Q$,
• the set of accepting states $Q_a \subset Q$,
• the set of writing states $Q_w \subset Q$,
• a filling memory function $\varphi: Q_f \to \Delta$.

At the start of operation all memory cells are void. A WORM-memory automaton operates in the same way as a nondeterministic $M$-automaton except the moments of entering a writing state. In that moment the filling function is applied to the current state of the automaton. If the current memory cell is visited at first time then the value of the filling function is assigned to the cell and the automaton continues operation by application of the transition function. An attempt to change the content of a cell visited before causes the error as well as an attempt to apply the transition function being at a void cell. In the case of an error the automaton stops the operation and do not accept the input word.

So, during a successful operation the automaton enters a new memory cell in a writing state. Also note that if the automaton writes the non-void symbol $d$ to the cell containing the symbol $d$ then no error occurs. We call this property ‘a freedom of writing the same’.

We denote by $M$-DFA the class of languages recognized by deterministic automata with an auxiliary WORM-memory of model $M$.

**Lemma 1.** $M$-DFA $\subseteq M$-NFA.

**Proof.** Let $A$ be a WORM-$M$ automaton recognizing the language $L$ and $Q$ is the state set of $A$. The state set of a nondeterministic $M$-automaton $A'$ recognizing the language $L$ is $Q \cup \{r\}$, where $r$ is an additional rejecting state. The transition function of $A'$ coincide with the transition function of $A$ except writing states and the rejecting state. In a writing state $q \in Q_w$ the automaton compares the content $d$ of the current memory cell with $\varphi(q)$. If $d = \varphi(q)$ then the value of transition function is the same as for the automaton $A$. Otherwise, the transition leads to the rejecting state. In the rejecting state the automaton do nothing and the rejecting state is absorbing.

An operation of the WORM-$M$ automaton $A$ on an input word $w$ gives a partial memory content $\eta: M \to \Delta$ for memory cells visited during the operation. We denote by $T(A, w)$ the set of memory contents extending $\eta$. In other words, each memory content $\mu \in T(A, w)$ has in each cell visited by $A$ during the operation on the word $w$ the symbol written by $A$.

Let $w \in L$. The automaton $A'$ accepts the word $w$ on any memory content from the set $T(A, w)$. Indeed, it operates exactly in the same way as $A$ on this memory content.

Let $w \notin L$. Let’s consider the cases of memory content for the nondeterministic automaton $A'$.

1. $A'$ is operating on $\mu \in T(A, w)$. In this case its operation is also the same as for $A$. Here we use the property of freedom of writing the same. So, $A'$ do not accept.
2. $A'$ is operating on $\mu' \notin T(A, w)$. In this case $\eta(m) \neq \mu'(m)$ for some memory cell $m$ visited by the automaton $A$. Following the operation of the $A$ choose the first such memory cell $m_1$. Before entering $m_1$ operation $A$ and $A'$ is the same. When entering $m_1$
the automaton $A'$ reads a symbol $d \neq \phi(q)$, where $q$ is the current state due to the choice of $m_1$. It means that $A'$ goes to the rejecting state and do not accept the word $w$ on the memory content $\mu'$.

## 2 Complexity classes recognized by automata with auxiliary tape memory

### 2.1 1-way tape

Let $W_1$ be an infinite 1-way tape (Fig. 1). The class $W_1$-NFA is just the class NL. Indeed, a $W_1$-automaton can read a symbol from the guess tape once. This symbol can be used to make a nondeterministic choice in a transition relation for the case of the standard definition of nondeterministic automaton.

Note also, that $W_1$-DFA = L because we can simply ignore the symbols written to the 1-way tape.

![Figure 1: 1-way tape $W_1$](image1)

![Figure 2: 2-way tape $W_2$](image2)

### 2.2 2-way tape

Let $W_2$ be an infinite 2-way tape (Fig. 2). For graphs of fan-out $> 1$ we should also indicate the marking of edges. In the case of $W_2$ the marking is natural: mark '+' is placed on the edges going from a vertex $n$ to the vertex $n + 1$, mark '-' is placed on the edges going into the opposite direction.

It was mentioned above that $W_2$-NFA = PSPACE because $W_2$-automata is almost the same as nonerasing nondeterministic stack automata and NENSA recognize the class PSPACE [10].

The only difference between NENSA and $W_2$-automata is an ability of NENSA to make arbitrary nondeterministic transitions while an $W_2$-automaton should follow data read from the guess tape. It means that $W_2$-automata are weaker than NENSA, so $W_2$-NFA $\subseteq$ PSPACE. The reverse inclusion is valid even for deterministic $W_2$-automata. Indeed, a deterministic $W_2$-automaton is able to write a computational history of a Turing machine computation on a polynomially bounded space. For this purpose the automaton should move on distances polynomially bounded by the input size. But many heads are equivalent to logarithmic space and it is easy to count polynomially many times using logarithmic memory.

Thus, $W_2$-NFA $\subseteq$ PSPACE $\subseteq$ $W_2$-DFA $\subseteq$ $W_2$-NFA (the last inclusion is due to Lemma 1).

### 2.3 1.5-way tape

The memory model $W_{1.5}$ is pictured on the Fig. 3. Edges going to the right are marked by '+' and edges going to the initial vertex are marked by '−'.

**Theorem 1.** $W_{1.5}$-DFA = P.

We start from two simple observations.

**Lemma 2.** Let $A$ be a $W_{1.5}$-automaton and $\#Q$ be the number of its states. Then any accepting computation of $A$ includes no more than $\#Q$ moves to the initial cell.
Proof. After each return move the automaton $A$ scans the same tape content and its behavior is deterministic. So, if $A$ starts the scan process from the same state twice it loops and never reach an accepting state.

Thus, the number of return moves is no more than the number of the states. \hfill \qed

Proposition 1. Let $A$ be a WORM-$W_{1.5}$ automaton, $k$ be the number of heads, $n$ be the length of the input word $w$ and $\#Q$ is the number of the states of $A$. If $A$ accepts $w$ then between two subsequent return moves the automaton visits no more than $n^k \#Q$ new cells.

Proof. There are no more than $n^k \#Q$ surface configurations of $A$. Surface configurations are tuples (state, positions of heads). If the automaton pass through more than $n^k \#Q$ new cells, some surface configuration occurs twice. It means that the automaton loops and moves to the right infinitely. \hfill \qed

Now we are ready to prove Theorem 1.

Proof of Theorem 1. The inclusion $P \subseteq W_{1.5}$-DFA follows from the fact that a WORM-$W_{1.5}$ automaton is able to simulate a WORM-$W_2$ automaton on a polynomially bounded part of the memory tape. For this purpose one can use a polynomially bounded counter keeping the index of the current position on the guess tape. When the $W_2$-automaton goes to the left, the simulating $W_{1.5}$-automaton returns to the initial position along the ‘−’ marked edge and makes the required number of steps to the right according to the value of the position counter.

In this way WORM-$W_{1.5}$ automaton can write down a computational history of a deterministic Turing machine computation polynomially bounded in time.

Now we prove the reverse inclusion. Let $L \in W_{1.5}$-DFA, $A$ be a WORM-$W_{1.5}$ automaton recognizing $L$, $Q$ be the state set of $A$, $k$ be the number of heads, $n$ be the length of the input word $w$.

It follows from Lemma 2 and Proposition 1 that an accepting computation of $A$ uses no more than $n^k (\#Q)^2$ cells. So, the automaton works on polynomially bounded auxiliary read only tape. It means that the total number of steps in an accepting computation is also polynomially bounded. It does not exceed $n^k \cdot n^k (\#Q)^2$.

Polynomially bounded in space and time computation of $W_{1.5}$-automaton can be simulated in polynomial time. \hfill \qed

Theorem 1 shows that deterministic $W_{1.5}$-automata are much weaker than deterministic $W_2$-automata. As for nondeterministic automata, 1.5-way tape provides the same computational power as 2-way tape.

Theorem 2. $W_{1.5}$-NFA = PSPACE.

Proof. The statement is obvious in one direction: $W_{1.5}$-NFA $\subseteq W_2$-NFA = PSPACE.

To prove the reverse inclusion we show that a $W_{1.5}$-automaton is able to check correctness of a computational history for a Turing machine computation on a polynomially-bounded space.

Without loss of generality we assume that the machine uses the binary alphabet \{0, 1\}. Recall that a computational history is a sequence of a Turing machine configurations. A configuration is a word of form $\ell q a r$, where $\ell$ is the tape content to the left of the head position, $q$ is the state of the machine, $a$ is a currently read symbol, and $r$ is the tape content to the right of the head position.
It is convenient to fix a length of a configuration. That is possible because we simulate a space bounded computation. For a computation on a space \( s \) it is sufficient to deal with configurations of length \( 2s \).

Each step of computation changes the configuration of the machine. We will describe this change using arithmetic encoding of binary words \([11, 12]\). Namely, a word \( w \in \{0, 1\}^* \) is encoded by a positive integer \( c(w) \) written in binary as \( 1w \).

We will encode a configuration \( \ell_qar \) by a 4-tuple \((c(\ell), q, a, c(\ell)R)\), where \( c(\ell)R \) denote the word \( r \) is the reversal of the word \( r \).

Changes of these data during a computation step are represented in the following table:

| Left move | \( c(\ell) \) | \( q \) | \( a \) | \( c(\ell)R \) |
|-----------|----------------|--------|--------|----------------|
| \( [c(\ell)/2] \) | \( q\) | \( a \) | \( a + 2c(\ell)R \) |

| Right move | \( c(\ell) \) | \( q \) | \( a \) | \( c(\ell)R \) |
|------------|----------------|--------|--------|----------------|
| \( a + 2c(\ell) \) | \( q \) | \( c(\ell)R \) mod 2 | \( [c(\ell)/2] \) |

It is clear from the table that correctness of a computational history in the arithmetic encoding is equivalent to very simple arithmetic relations between neighbor pairs of configurations in the history. Depending on the pair \( q, a \) and parities of \( c(\ell), c(\ell)R \) each relation has a form

\[
y = 2x, \ y = 2x + 1, \ x = 2y + 1, \ x = 2y, \tag{1}
\]

where \( x \) is the old value and \( y \) is the new value of \( c(\ell) \) or \( c(\ell)R \).

Recall that we consider a computational history of a computation on a polynomially bounded space. So, \( c(\ell) = 2^{\text{poly}(n)}, c(\ell)R = 2^{\text{poly}(n)} \), where \( n \) is the input length. Thus, the relations (1) can be verified by calculations modulo \( 1, 2, \ldots, m = \text{poly}(n) \). This fact follows from the Chinese remainder theorem and the prime number theorem [2].

Now we are ready to describe a \( W_{1.5} \)-automaton verifying a computational history on the input word \( u \) using a space \( s \). The automaton expects a guess in form

\[
u(\ell_0)q_0a_0u(r_0)\#u(\ell_1)q_1a_1u(r_1)\#\ldots\#u(\ell_i)q_ia_iu(r_i)\#\#,
\]

where \( \ell_0 = 0^*, a_0r_0 = w0^{-[w]}, \ell_i, q_i, a_i, r_i \) are components of the \( i \)th configuration in the computational history, \( q_0 \) is a final state of the simulated Turing machine. The function \( u(\ell) \) is the unary encoding of the number \( c(\ell) \), i.e. \( u(\ell) = 1^{c(\ell)} \), where * is the special symbol.

The automaton makes \( m = \text{poly}(s) \) stages of computation. On the \( p \)th stage it verifies relations modulo \( p \). It should verify the correctness of the the first block of the guess and the relations \( (1) \).

The correctness of the first block on the input word \( w = w_1w_2\ldots w_n \) means that \( c(\ell_0) = 2^{s+1}, a_0 = w_1 \) and \( c(r_0) = 2^{s+1} + w_nw_{n-1}\ldots w_2 \). Note that the right hand sides of these equalities can be computed modulo \( p \) on a logarithmic memory without using the guess tape. After that the automaton computes residues modulo \( p \) for the lengths of \( u(\ell_0) \) and \( u(r_0) \) in natural way: go along a word and count modulo \( p \).

The relations (1) are verified in the same manner: the automaton keeps in its logarithmic memory residues modulo \( p \) of lengths \( u(\ell_i), u(r_i) \) as well as \( q_i, a_i \) and compares them to the data of \( (i + 1) \)th block computing residues modulo \( p \) in natural way.

If all checks are passed successfully for each residue and the state \( q_\ell \) is a final state of the Turing machine then the automaton accepts the word \( w \). Otherwise, it rejects (say, moves to the right infinitely).

It is clear from the construction that if the simulated Turing machine accepts the word \( w \) then the automaton also accepts it. Now suppose that the automaton accepts a word \( w \).
It means that there is a guess in form (2) such that the automaton accepts \( w \) on this guess. Because all relations (1) are true and the first block corresponds to the input we conclude that taking lengths \( u(\ell_i), u(r_i) \mod \text{lcm}(1, 2, \ldots, m) \) give us the valid accepting computational history on the input \( w \).

**Remark 1.** A simulation in Theorem 2 fails for a nondeterministic Turing machine computation. In the deterministic case there is the unique valid computational history for the computation on the input word \( w \). In the nondeterministic case there are several computation paths. So, correctness modulo small integers do no imply the total correctness.

### 2.4 A perversed 1.5-way guess tape

Let \( \omega \) be an infinite binary word. A modification \( W_{\omega}^{\omega} \) of the 1.5-way guess tape differs from \( W_{1.5}^{\omega} \) in the edge marking. The mark '+' is assigned to the edges outgoing to the right from the vertex \( n \) iff \( \omega_n = 1 \). The rest of edges are marked by '-'.

The marked graph \( W_{\omega}^{\omega} \) bears an information about the word \( \omega \). There are continually many infinite binary words. So, it is natural to expect non-decidable languages in some classes \( W_{\omega}^{\omega} \)-NFA. We present an example in the next theorem.

**Theorem 3.** Let \( L \) be a tally language (all its words are \( 1^n \)). Denote by \( \omega_L \) the infinite word such that \( \omega_{2n-1} = 1 \) and \( \omega_{2n} = 1 \) iff \( 1^n \in L \). Then \( L \in W_{1.5}^{\omega} \)-NFA.

**Proof.** A \( W_{1.5}^{\omega} \)-automaton recognizing \( L \) expects a guess in a special form: each vertex contains an information about the direction of the edge marked by '+' and the initial vertex has a special root label.

The automaton should be able to verify the correctness of the guess. The algorithm of guess verification for the first \( 2n \) vertices checks the root label in the initial vertex and after that it makes \( 2n \) moves 'to the right' according to the instructions of the guess, then it makes 'the return move' also following the instructions of the guess. If the root label appears on the last step only then the automaton adopts the guess. Otherwise, it reject the guess as well as the input.

It is easy to see that the algorithm indeed adopts the guesses of the form described above because any wrong instruction leads the automaton to the initial cell.

After verification step the automaton can move along the guess tape following the instructions of the guess. It accepts the input word \( 1^n \) iff \( 2n \)th instruction do not lead to the initial cell.

### 3 The restricted guess case

One can put a restriction on the form of a guess. In the proofs above we already use this technique. In this Section we consider the notion of nondeterminism that arises in the restricted guess settings.

**Definition 5.** Let \( T \subseteq \Delta^M \) be a subset of possible memory contents. We say that an automaton \( A \) accepts a word \( w \) with a \( T \)-restricted guess iff it accepts \( w \) working on some memory content \( \mu \) from the set \( T \).

We denote by \( M(T) \)-NFA the corresponding class of languages recognizable by \( M \)-automata with a \( T \)-restricted guess.

Of course, in general \( M(T) \)-NFA \( \not\subseteq M \)-NFA. For example, let \( T \) is the set of all valid computational histories of a Turing machines. Then \( W_2(T) \)-NFA contains all recursively enumerable languages. Indeed, a \( W_2 \)-automaton can verify the correctness of the first block of the history and the correctness of all local changes of the machine state and symbols around it. It is sufficient by the definition of the restriction.
To guarantee the inclusion $M(T)$-NFA $\subseteq M$-NFA it is sufficient to construct an automaton $V$ that checks compatibility of memory content $\eta$ in visited cells with the set $T$. Compatibility means that $\eta$ can be extended to some $\tau \in T$. Below we apply this idea in specific cases.

We are interested in restrictions that describe subclasses of $M$-NFA. As an example of this kind of restriction we introduce sparse guesses.

**Sparse guess.** Suppose that $\Delta = \{0\} \cup \Delta'$. A $k$-sparse guess contains no more than $k$ symbols from the $\Delta'$.

We denote by $U_k$ the set of $k$-sparse guesses.

Below we consider sparse guesses for tape memories.

### 3.1 Sparse guesses for 1.5-way tape

An informal idea of guess verification described above gives us in the case of the 1.5-way guess tape the following lemma.

**Lemma 3.** $W_{1.5}(U_k)$-NFA $\subseteq W_{1.5}$-NFA for any $k$.

**Proof.** Let $L$ be a language recognized by a $W_{1.5}$-automaton $A$ with a $U_k$-guess. An automaton $A'$ recognizing $L$ with the unrestricted guess runs in parallel the automaton $A$ and a special verifying automaton $V$. The automaton $V$ has rejecting states which are absorbing. If $V$ is in a rejecting state then $A'$ rejects. Otherwise, it accepts if $A$ accepts.

The automaton $V$ do not move itself. It looks at memory cells passed in motion of the automaton $A$ and change its state. Informally, it keeps an information about the number of non-zero symbols to the left of the current position. So, the states of the $V$ are the set $\{0, 1, \ldots, k + 1\}$. The state $k + 1$ is rejecting and thus is absorbing.

At the start and after each return move the state $V$ is set to 0 (except the case of state $k + 1$). After reading a non-zero symbol and passing to the right $V$ changes the state $i$ by $i + 1$ provided $i \leq k$.

If $w \in L$ then the automaton $A$ accepts it on a guess $\tau \in U_k$. The automaton $A'$ is also accepts $w$ on a guess $\tau$ because the state $k + 1$ of the automaton $V$ can not be reached.

If $w \notin L$ then no $U_k$-guess can enforce the automaton $A$ to accept $w$. The same holds for $A'$ and $U_k$-guesses. Suppose now that $A'$ accepts on a guess $\tau \notin U_k$. By construction $A'$ do not visit more than $k$ different cells filled by non-zero symbols (otherwise, the automaton $V$ rejects). Let $\eta$ be the memory content of cells visited by $\tau$ during the accepting computation. Then $\eta$ can be extended to some memory content $\tau' \in U_k$. The automaton $A$ works on the $\mu'$ in the same way as $A'$. In particular, it accepts on this guess. So, $w \in L$ and we come to a contradiction. Thus, $A'$ rejects on any guess.

The following inclusions are proved along the same lines.

**Lemma 4.** $W_{1.5}(U_1)$-NFA $\subseteq W_{1.5}(U_k)$-NFA.

**Proof.** Let $L$ be a language recognized by a $W_{1.5}$-automaton $A$ with a $U_1$-guess. Now we construct for $k \geq 2$ an automaton $A'$ that recognizes $L$ with $U_k$-guess. The automaton $A'$ runs in parallel $A$ and a verifying automaton $V$ counting the number of non-zero symbols read. The construction of $V$ is the same as in the proof of the previous lemma. But now the state 2 is rejecting for $V$.

If $w \in L$ then the automaton $A$ accepts it on a guess $\tau$. The automaton $A'$ is also accepts $w$ on a guess $\tau'$ such that it coincides with $\tau$ in cells visited by $A$.

If $w \notin L$ then no $U_k$-guess can enforce the automaton $A$ to accept $w$. The same holds for $A'$: $A'$ works in the same way as $A$ until reading the second non-zero symbol in which case the $A'$ rejects.

Thus, $L \in W_{1.5}(U_k)$-NFA. 

\[ \square \]
Now we give a characterization of the classes $W_{1.5}(U_k)$-NFA.

**Theorem 4.** $W_{1.5}(U_k)$-NFA = NP for $k \geq 1$.

The proof of Theorem 4 is split naturally into two parts.

**Lemma 5.** NP ⊆ $W_{1.5}(U_1)$-NFA.

**Proof.** Let $L$ be an NP-language. It means that there is a (deterministic) Turing machine $M$ and a polynomial $p$ such that for any $w \in L$ there is a certificate $y$ of polynomial size in the length of $w$ ($|y| = p(|w|)$) such that $M$ accepts the input pair $w, y$ and for any $w \not\in L$ there is no such certificate.

A history of computation of $M$ on the input pair $(w, y)$ can be verified by a multi-head 2-way automaton $V$ with the indexed access to the history. It means that $V$ is equipped by a logarithmically small query tape which is read/write. The automaton $V$ has a special query state. Entering this state $V$ sends a query to the storage containing a string and receives in answer the value of the $i$th symbol of the string, where $i$ is written in binary on the query tape.

It is easy to see that using polynomially small counters the automaton can verify a computational history of polynomial size.

Now we are going to simulate the indexed access by a $U_1$-guess. In other words, we construct a $W_{1.5}$-automaton $I$ such that for any sequence $b_1, \ldots, b_m$, where $m = \text{poly}(n)$ and $0 \leq b_i < b = O(1)$, there is a $U_1$-guess $\xi$ such that the automaton $I$ can restore $b_i$ operating on the guess $\xi$.

At first we note that using a space $s$ one can compute the $k$th prime number $p_k$ for $1 \leq k \leq 2^{s/C}$, where $C$ is the absolute constant. Indeed, the check of primality of an integer $n$ written in binary on the space $\log n$ can be done by use of $O(\log n)$ additional memory (containing auxiliary counters). Thus, using one more counter to keep the number of the last prime found one can compute $p_k$ on the space $s \leq C \log p_k$. From the prime number theorem [2] we conclude that $p_k \sim k \ln k$, hence, $\log p_k \sim \log k + C_1 \log \log k$ and for sufficiently large $k$ the computation can be done on space $s \sim (C + 1) \log k$.

The automaton $I$ works in the following way. To compute a value of $b_i$ it computes $p_i$ on its own logarithmic memory. Then it starts a motion along the guess tape and counts modulo $p_i$. When it reaches the non-zero symbol it returns the current residue modulo $p_i$ as the value of $b_i$ if $b_i < b$. Otherwise, it rejects.

The Chinese remainder theorem implies that for any sequence $b_i$ there is an integer $N$ such that $N \equiv b_i \pmod{p_i}$ for all $0 \leq i \leq m$. So, $I$ returns correct values of $b_i$ on the guess $0^{N-1}10\ldots$.

The $W_{1.5}$-automaton $R$ with a $U_1$-guess recognizing the language $L$ is combined from the automata $V$ and $I$. It substitute calls of $I$ instead of queries of $V$.

By construction, if $w \in L$ then $R$ accepts it. Let $w \not\in L$. Consider an operation of $R$ on the input $w$. Possible results of operation $I$ form a sequence $(b_i')$ and the $V$ part of the automaton $R$ verifies it as a valid computational history. Thus, the automaton $R$ rejects because there are no accepting computation.

**Lemma 6.** $W_{1.5}(U_k)$-NFA ⊆ NP for any $k$.

**Proof.** We should construct a nondeterministic polynomial time algorithm to verify that a $W_{1.5}$-automaton $A$ accepts an input word $w$ on some guess $\tau \in U_k$.

From $A$ and $w$ we construct in deterministic polynomial time an auxiliary automaton $B$. The states of $B$ are surface configurations of $A$, i.e. $(h+1)$-tuples (a state of $A$, head positions). So the number of states of $B$ is polynomially bounded. The automaton $B$ moves along the 1.5-way guess tape in the same way as the automaton $A$ do on the input $w$ except steps that do not change a memory cell. Following along the transitions of the automaton $A$ one can determine the next ‘moving’ step in polynomial time. The automaton $B$ jumps
to this step immediately. Accepting states of $B$ are surface configurations such that $A$ is in an accepting state.

Hence the problem is reduced to verification that there is a $U_k$-guess such that $B$ accepts on this guess. For this purpose we need the following claim.

**Claim 1.** If $B$ accepts on some $U_k$-guess then it accepts on a $U_k$-guess of exponential length.

Consider an operation of $B$ on the guess $0^{y_0} s_0 0^{y_1} s_1 \ldots 0^{y_{k-1}} s_{k-1} 0^{y_k}$, $\ldots$ Let $N$ be $\text{lcm}(1, \ldots, \#Q(B))$, where $\#Q(B)$ is the number of the states of $B$. Let’s prove an intermediate claim.

**Claim 2.** The operation of $B$ on the guess $0^{y_0} s_0 0^{y_1} s_1 \ldots 0^{y_{k-1}} s_{k-1} 0^{y_k}$ gives the same result as the operation of $B$ on the guess $0^{y_0} s_0 0^{x_1} s_1 \ldots 0^{x_{k-1}} s_{k-1} 0^{x_k}$ provided $y_i \equiv x_i \pmod{N}$ for $x_i > \#Q(B)$ and $y_i = x_i$ for $x_i \leq \#Q(B)$.

Indeed, a sequence of states of $B$ working on a part of the tape filled by zeroes is obtained by iterations of a map $\alpha_0 : Q(B) \to Q(B)$. After $\leq \#Q(B)$ iterations the sequence $\alpha^n(q)$ became periodic. The period depends on $q$ but in any case it is a divisor of $N$. Claim 2 is proved.

Now the Claim 1 follows from the bound $N < 2^{\#Q(B)^2}$. (Actually, the bound is more tight.)

Note that the parameters $x_i$ of an exponentially bounded guess can be written in binary nondeterministically in polynomial time.

To complete a proof we construct a (deterministic) polynomial time algorithm verifying that $B$ accepts on the guess with parameters $x_i$.

By Lemma 2 there are no more than $\#Q(B)$ return moves during an accepting operation of $B$. So, the algorithm can call a procedure $F$ that by a state $q$ determine the behavior of $B$ starting from the initial cell: either it reaches an accepting state or it makes the return move to the state $q'$.

This procedure can be constructed easily using calls of the simpler procedure $F_0$ answering the same question concerning a behavior of the automaton on the part of tape filled by zeroes. More exactly, an input of the procedure is an integer $x$ written in binary and a state $q \in Q(B)$. The procedure $F_0$ should output the result of operation in one of three following forms:

(a) $B$ reaches an accepting state working on the part $0^x$ of the tape without return moves;

(b) $B$ reaches a return state and goes to the initial cell in the state $q'$;

(c) $B$ passes the part $0^x$ and leaves it in the state $q'$.

To answer these questions the procedure $F_0$ represents the map $\alpha_0$ in a Boolean matrix form and applies fast algorithm of matrix exponentiation.

Let $B'$ be a modified automaton such that all accepting and return states of $B$ are changed by absorbing states. Let $\alpha'$ be a Boolean matrix of $\alpha_0$ for the automaton $B'$:

$$(\alpha')_{q'q''} = 1 \text{ iff } \alpha_0(q') = q''.$$

The Boolean matrix multiplication is defined similarly to the usual matrix multiplication but addition and multiplication are changed by disjunction and conjunction respectively.

The Boolean multiplication is associative due to distributive law for disjunction and conjunction. So, a Boolean power $(\alpha')^n$ can be computed in time $\text{poly}(\log n)$ in usual way: by writing binary representation of $n$ and using subsequent squaring. Let $q'$ be an accepting or return state. Then it can be easily verified by a straightforward induction that

- $((\alpha')^n)_{qq'} = 0$ if $q'$ is not reached during the operation of $B'$ on the string $0^n$,

- $((\alpha')^n)_{qq'} = 1$ if $q'$ is reached during the operation of $B'$ on the string $0^n$, where $m \leq n$. 


Computing Boolean powers of $\alpha'$ helps to choose between the above variants (a)–(c).
Indeed, if $((\alpha')^n)_{qq'} = 0$ for each accepting or return state then we have the variant (c).
The state $q'$ in question is in this case the only state such that $((\alpha')^n)_{qq'} = 1$.
Otherwise, some accepting or return state is reached within the region $0^x$. To determine the state we apply a binary search to find out the smallest $n$ such that $((\alpha')^n)_{qq'} = 1$ for some accepting or return state $q'$. Looking at the state $q'$ we can easily distinguish the variants (a) and (b) and compute the data required in each case.

Proof of Theorem
From Lemmata 4, 5, 6 we conclude that $NP \subseteq W_1(U_1)-NFA \subseteq W_1(U_k)-NFA \subseteq NP$.

Remark 2. In similar way it is possible to determine the result of operation of a $W_1(U_1)$-automaton on a guess containing polynomially many non-zero symbols.

3.2 Sparse guesses for 2-way tape

Lemma 7. $W_2(U_k)$-NFA $\subseteq W_2$-NFA for any $k$.

Sketch of proof. The idea is the same as for Lemma 4. We use a combined automaton that runs in parallel the recognizing and the verifying automata. The latter should be modified to include the moves to the left. The modification is straightforward.

The class $W_2(U_1)$-NFA is rather weak. The reason is the absence of the root label in the initial cell. Using a non-zero symbol as the root label we obtain a subclass of $W_2(U_1)$-NFA that coincides with the class $Aux2DC$ of languages recognized by deterministic 2-way counter automata with a logarithmic auxiliary memory. The inclusion $Aux2DC \subset P$ follows from the Cook theorem [5]. The Cook theorem claims that $Aux2PDA = AuxN2PDA = P$,

where $Aux2PDA$ is the class of languages recognized by deterministic 2-way pushdown automata with a logarithmic auxiliary memory and $AuxN2PDA$ is the class of languages recognized by nondeterministic 2-way pushdown automata with a logarithmic auxiliary memory.

To upperbound the class $W_2(U_1)$-NFA we state a rather obvious proposition.

Proposition 2. A trajectory of motion of a 2-way automaton $B$ along the tape filled by zeroes either became periodic with the period width bounded by $\#Q(B)$, where $\#Q(B)$ is the number of the states of $B$, or is an infinite repetition of right shifts by a distance $s$ along periodically repeated route. Here $s \leq Q(B)$.

Proof. After $t \leq Q(B)$ steps a sequence of states became periodic. From this moment of time one of variants listed in the proposition became true.

Theorem 5. $W_2(U_1)$-NFA $\subseteq Aux2NC \subset P$, where $Aux2NC$ is the class of languages recognized by nondeterministic 2-way counter automata with a logarithmic auxiliary memory.

Proof. Let $L \in W_2(U_1)$-NFA is recognized by a $W_2$-automaton $A$ with an $U_1$-guess.
Proposition 2 implies that if $A$ accepts the input word $w$ on some $U_1$-guess then it accepts the word $w$ on a guess such that a non-zero symbol is placed at polynomially bounded distance from the initial cell. (Look at the behavior of the automaton after visiting the non-zero symbol the first time.)
The auxiliary counter automaton $B$ guesses nondeterministically the distance between the initial cell and the cell containing the non-zero symbol and keeps it in its logarithmic
auxiliary memory. After that \( B \) simulates an operation of \( A \). The counter helps to simulate a behavior of \( A \) when \( A \) is to the right of the non-zero symbol. For the rest moments of time \( B \) simulates the behavior of \( A \) using the auxiliary memory. It keeps a polynomially bounded counter indicating the position of \( A \) on the guess tape to the left of the non-zero symbol.

The second inclusion in theorem follows from the Cook theorem mentioned above. \( \square \)

Theorem 4 implies that \( W_2(U_2)\)-NFA \( \supseteq \) NP because two non-zero symbols can be used to mark the initial cell and provide a \( U_1 \)-guess for a \( W_{1.5} \)-automaton. The latter can be simulated by a \( W_2 \)-automaton working on a guess of this kind.

Using Proposition 2 one can prove the reverse statement. The proof is similar to the proof of Lemma 6. An arbitrary guess is replaced by an exponentially bounded guess. After that one can develop an algorithm computing the result of operation on an exponentially bounded guess represented by parameters \( x_i \) as in the proof of Lemma 6. So, we came to the theorem

**Theorem 6.** \( W_2(U_k)\)-NFA = NP for \( k \geq 2 \).

## 4 Some other memory models and variants of nondeterminism

In this final section we briefly outline several interesting variants of memory models and possible extensions of definitions.

### 4.1 Monoid memory

Let \( G \) be a monoid generated by a set \( G' = \{ g_1, \ldots, g_n \} \). Then the memory of type \((G, G')\) is defined by the Cayley graph of the monoid \( M \): the vertex set is \( G \), an edge marked \( g_k \) goes from a vertex \( x \) to the vertex \( xg_k \).

1-way and 2-way tapes are examples of monoid memory. It follows immediately from definitions that \( W_1\)-NFA = \((\mathbb{N}, \{+1\})\)-NFA. Also it is easy to see that \( W_2\)-NFA = \((\mathbb{Z}, \{+1, -1\})\)-NFA. For the inclusion \( W_2\)-NFA \( \supseteq \) \((\mathbb{Z}, \{+1, -1\})\)-NFA one should apply a useful trick converting a tape infinite in both directions to a tape infinite to one direction. For the reverse inclusion it is useful to use a root labeling. Walking around \( \mathbb{Z} \), an automaton is able to check that there is the only one vertex labeled as the root in the region visited.

There is a weak upper bound for an arbitrary monoid memory.

**Theorem 7.** Let \( M \) be a monoid. If the word problem for \( M \) is decidable then \( M\)-NFA \( \subseteq \) R.e, where R.e is the class of recursively enumerable languages.

Let make some general remarks before explaining the proof of the theorem.

From an \( M \)-automaton \( A \) one can construct in polynomial time an automaton \( B \) with polynomially many states that walks on \( M \) in the same way as \( A \) do. The construction is in fact described in the proof of Lemma 6.

**Definition 6.** An \( M \)-walking automaton \( B \) is called halting iff it reaches an accepting state on some memory content.

Let \( S_M \) be a language consisting of descriptions of halting \( M \)-automata.

An upper bound on the class \( M \)-NFA follows from the fact.

**Proposition 3.** Any language \( L \in M\)-NFA is polynomially reducible to \( S_M \).
The proof of this proposition repeats the argument from the proof of Lemma 6.

The halting problem for an $M$-automata $B$ is in fact a problem of conditional reachability in the state graph of $B$. Correctness conditions stem from the fact that if the automaton comes to the same cell of memory it should follow the same guess symbol stored in the cell. In other words, a route $q_0, q_1, q_2, \ldots$ along the state graph of $B$ induces a route $m_0, m_1, \ldots$ along the memory graph. The route is correct if all transitions in moments corresponding to the same cell $m$ go along the edges with the same mark $d \in \Delta$.

Let put this more formally. For any route $q_0, q_1, \ldots$ along the state graph the corresponding route $m_0, m_1, \ldots$ along the memory graph introduces an equivalence relation between positions in the route: $i \sim_M j$ iff $m_i = m_j$. On the other hand, the route $q_0, q_1, \ldots$ determines the word $\tau \in (G \times \Delta)^*$ generated by routes from the start state to some accepting state. By definition the language $L(B)$ is regular.

The halting words are in the language $L(B)$. A word is halting iff $d_i = d_j$ for all $i, j$ such that $i \sim_M j$.

**Sketch of proof of Theorem 8**

Let prove that $S_M$ is recursively enumerable.

Since the word problem for $M$ is decidable there is an algorithm computing for each route from $(G \times \Delta)^*$ the equivalence relation $\sim_M$.

To enumerate halting automata the enumeration algorithm starts an enumeration of all pairs $(B, \xi)$, $\xi \in L(B)$. For each pair the algorithm computes the relation $\sim_M$ and checks the correctness conditions. If the conditions hold then the algorithm outputs $B$.

Application of Proposition 3 completes the proof.

For many monoids and groups the bound of Theorem 7 is exact.

### 4.2 $\mathbb{Z}^2$ memory

The generators of $\mathbb{Z}^2$ are chosen naturally: $(\pm 1, 0)$ and $(0, \pm 1)$.

**Theorem 8.** $\mathbb{Z}^2$-NFA = R.e.

**Sketch of proof.** The word problem for $\mathbb{Z}^2$ is decidable. So, by Theorem 7 $\mathbb{Z}^2$-NFA $\subseteq$ R.e.

On the other hand, a $\mathbb{Z}^2$-automaton is able to verify the correctness of computational history of an arbitrary Turing machine computation. The automaton expects a guess containing subsequent Turing machine configurations in subsequent rows of $\mathbb{Z}^2$. Correctness of computational history in this form is a conjunction of local conditions that can be verified by the automaton walking on $\mathbb{Z}^2$.

**Corollary 1.** Let $G$ be a group with decidable word problem and $\mathbb{Z}^2 < G$. Then $G$-NFA = R.e.

### 4.3 Multi-head access to the guess data

Our definitions permit a local access to the guess data. Typically, a relaxation of this property leads to the class R.e of recursively enumerable languages.

For example, if we allow two heads on 1-way tape we already get the class R.e. Indeed, one can verify an arbitrary computational history using two 1-way heads.

Note that even for a sparse encoding two heads on the 2-way guess tape are too much and we get R.e. Indeed, two parts of an arbitrary length can be used to simulate an automaton with two counters. But such an automaton is able to make an universal computation.
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