COLOR CONFINEMENT IN LATTICE LANDAU GAUGE

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The problem of color confinement in lattice Landau gauge QCD is briefly reviewed in the light of Kugo-Ojima criterion and Gribov-Zwanziger theory. The infrared properties of the Landau gauge QCD is studied by the measurement of the gluon propagator, ghost propagator, Kugo-Ojima parameter and the running coupling via $\beta = 6.0, 16.4, 32.4, 64.4$ and $\beta = 6.4, 32.4, 48.4$ lattice simulation. The data are analyzed in the principle of minimum sensitivity (PMS) in the $\tilde{\text{MOM}}$ scheme. We observe the running coupling $\alpha_s(q)$ has a maximum about 1 and decreases near $q = 0$. The Kugo-Ojima parameter, which is expected to be 1 was about 0.8.

1. Introduction

In 1978 Gribov proposed that the integration range of the gauge field should be restricted to the region where its Faddeev-Popov determinants are positive and that the restriction will cancel the infrared singularity of the gluon propagator and results in the singularity of the ghost propagator, which makes the linear potential between colored sources, and the color confinement will be realized\(^1\). Almost at the same time, Kugo and Ojima proposed color confinement criterion based on the BRST(Becchi-Rouet-Stora-Tyutin) symmetry without Gribov’s problem taken into account\(^2\).

Kugo-Ojima two-point function is defined in the lattice simulation as

$$
(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}) u^{ab} (p^2) = \frac{1}{V} \sum_{x,y} e^{-ip(x-y)} \left\langle \text{tr}(\lambda^a \dagger D_\mu - \frac{1}{\partial D}[A_\nu, \lambda^b])_{xy} \right\rangle
$$

(1)

where $u^{ab}(0) = \delta^{ab} u(0)$ and $1 + u(0) = Z_1/Z_3 = 1/\tilde{Z}_3$. $Z_3$ and $\tilde{Z}_3$ are gluon- and ghost- wave function renormalization factor, respectively and $Z_1$ is the gluon vertex renormalization factor. Kugo expressed the ghost propagator...
in the infrared asymptotic region as\(^3\) \(G(q^2) \sim q^2(1 + O(q^2))\).

In 1994, Zwanziger analyzed various regions of the Landau gauge in connection with Gribov’s problem, and developed the formulation of lattice Landau gauge. From L-periodic link variables, \(U_{x,\mu} \in SU(n)\) \((n = 3)\), we define gauge field \(A_{\mu}(U)\), and then there are two possible options of \(A_{\mu}(U)\), \(U\)-linear (Zwanziger’s) type \(A_{x,\mu} = (U_{x,\mu} - U_{x,\mu}^{1})/2\) \([\text{traceless part}]\), and log \(U\) (ours)\(^7\) type \(A_{x,\mu} = e^{A_{x,\mu}}\). Here brief review is given below in terms of common language to both options of the definition. In each definition, \(\delta A_{\mu}\) under infinitesimal gauge transformation \(g = e^\epsilon\) is given as \(\delta A_{\mu} = D_{\mu}(U)\epsilon\) with the covariant derivative defined as \(D_{\mu}(U)\phi = S(U_{\mu})\partial_{\mu}\phi + [A_{\mu}, \phi]\), where operations \(\partial_{\mu}\) and \(\epsilon\mu\) on the scalar \(\phi\) to give vectors are defined as \((\partial_{\mu}\phi)_{x,\mu} = \phi(x + \mu) - \phi(x)\) and \((\epsilon\mu)_{x,\mu} = (\phi(x + \mu) + \phi(x))/2\), respectively, and the operation \(S(U_{\mu})\) on a vector \(B_{\mu}\) is defined for each option as \(S(U_{\mu})B_{\mu} = (1/2) \{(U_{x,\mu} + U_{x,\mu}^{1})/2, B_{x,\mu}\}\) \([\text{traceless part}]\) in \(U\)-linear definition, \(S(U_{\mu})B_{\mu} = S(A_{x,\mu})B_{x,\mu} = \{(A_{x,\mu}2)/\text{th}(A_{x,\mu}/2)\}B_{x,\mu}\) in log \(U\) definition with \(A_{x,\mu} = adj_{A_{x,\mu}} = [A_{x,\mu}, \cdot]\).

The Landau gauge \(\partial A = 0\) can be characterized\(^6\) such that \(\delta F_{U}(g) = 0\) for any \(\delta g\), in use of the optimizing functions \(F_{U}(g)\); \(F_{U}(g) = \sum_{x,\mu} \text{tr} \left\{ 2 - (U_{x,\mu}^g + U_{x,\mu}^{g,1}) \right\}\) in \(U\)-linear definition, and \(F_{U}(g) = \sum_{x,\mu} \text{tr} \left( A_{x,\mu}^g A_{x,\mu}^{g,1} \right) = \{A_{x,\mu}^g A_{x,\mu}^{g}\}\) in log \(U\) definition. The variation \(\Delta F_{U}(g)\), under infinitesimal gauge transformation \(g^{-1}\delta g = \epsilon\), reads as \(\Delta F_{U}(g) = -2\langle \partial A\rangle \phi + \langle \epsilon \rangle - \partial D(A\epsilon)\rangle\phi + \cdots\). The fundamental modular region is specified by the global minimum along the gauge orbits, i.e., \(\Lambda_{L} = \{U | A = A(U), F_{U}(1) = \min_{U} F_{U}(g)\}\), \(\Lambda_{L} \subset \Omega_{L}\), where \(\Omega_{L}\) is called as Gribov region (local minima), and \(\Omega_{L} = \{U \mid -\partial D(U) \geq 0, \partial A = 0\}\). So far, all field variables are supposed to be \(L\)-periodic. Zwanziger further defined the core region \(\Xi_{L}\) as a set of the global minimum points of \(F_{U}(g)\) in the extended gauge transformation \(g = e^{\omega e^{\theta x}}\) where \(U\) and \(\omega\) are \(L\)-periodic, and constant \(\theta_{\mu}\)’s belong to an arbitrary Cartan subalgebra, i.e., \([\theta_{\mu}, \theta_{\nu}] = 0\): \(\Xi_{L} = \{U \mid F_{U}(1) = \min_{\mu = e^{\omega e^{\theta x}} F_{U}(g)} \subset \Lambda_{L}\}\). Putting \(g = e^{\omega e^{\theta x}}\) with arbitrary \(\omega\) and \(\theta_{\mu}\) of order \(\epsilon\), one has the variation of \(F_{U}(g)\) up to \(O(\epsilon^2)\) for \(U \in \Xi_{L}\) as \(\Delta F_{U}(g) = 2\langle A_{\mu} \theta_{\mu} + \omega \epsilon \rangle - \partial D(\omega \epsilon)\rangle\theta_{\mu} = \text{traceless part} \theta_{\mu} \geq 0\), where \(\omega = \omega - (-\partial D)^{-1} \partial \theta_{\mu}\). The non-negative sum of the third and fourth terms is expressed by \(-\langle \theta_{\mu} H_{\mu\nu} \theta_{\nu} \rangle\) with the horizon tensor \(H_{\mu\nu} = -D_{\mu}(-D)^{-1} D_{\nu} - \delta_{\mu\nu} S(U_{\mu})\). Taking the trace of the operator \(H_{\mu\nu}\) with respect to the normalized constant colored vectors \(\eta_{\mu}^{\epsilon, a} = \delta_{\mu \nu} \lambda_{a}^{\nu}\).
with $\text{tr}\lambda^{a\dagger}\lambda^b = \delta_{ab}$, one defines the **horizon function** $H(U)$ as $H(U) = \sum_{\nu,a}\langle \eta_{\mu,a}^{\dagger}|H_{\mu\nu}|\eta_{\nu,a}^{\mu}\rangle = \sum_{\mu,a}\langle \eta_{\mu,a}^{\dagger}|D_\mu(-\partial D)^{-1}D_\mu|\eta_{\mu,a}^{\mu}\rangle - (n^2-1)E(U) \equiv h(U)V$ where $(n^2-1)E(U) = \sum_{x,\mu}\text{tr}(\lambda^{a\dagger}S(U_{x,\mu})\lambda^a)$. Thus one has for $U \in \Xi_L$ that $\overline{A}_\mu = V^{-1}\sum_x A_{x,\mu} = 0$ and $H(U) \leq 0$, where $V = L^4$. Zwanziger hypothesized that the dynamics on $\Xi_L$ tends to that on $\Lambda_L$ in the infinite volume limit, and derived the **horizon condition**, statistical average $\langle h(U) \rangle = 0$, in the infinite volume limit. Taking the Fourier transform of the tensor propagator of the color point source, $\langle -D_\mu(-\partial D)^{-1}D_\nu \rangle_{x,a,y,b}$, one has $G_{\mu\nu}(p)\delta^{ab} = (c/d)(p_\mu p_\nu/p^2)\delta^{ab} - \{\delta_{\mu\nu} - (p_\mu p_\nu/p^2)\}u^{ab}(p^2)$, where $c = \langle E(U) \rangle/V$ and dimension $d = 4$. Putting Kugo-Ojima parameter as $u(0) = -c$ and comparing $\lim_{p_\mu \to +0} G_{\mu\nu}$ with $\langle h(U) \rangle = 0$ one finds that the horizon condition reduces to

$$\left\langle \frac{h(U)}{n^2-1} \right\rangle = \left(\frac{e}{d}\right) + (d-1)c - e = (d-1) \left(\frac{c - \frac{e}{d}}{d} \right) \equiv (d-1)h = 0. \quad (2)$$

Kugo-Ojima’s and Zwanziger’s arguments emerge to be consistent with each other provided the lattice covariant derivative meets with the continuum one $c/d = 1$. Accordingly, Zwanziger derived independently the same characteristic singular behavior of the ghost propagator as Kugo’s, both perturbatively in the sense that the diagrammatic expansion was used. Recent Dyson-Schwinger analyses and numerical simulations are to be noticed in this respect as well. It should be pointed out here that our numerical simulation does not pursue the selective core region dynamics, but rather checks the Zwanziger hypothesis in our standard method.

### 2. The gluon propagator

The gluon propagator and its dressing function are defined numerically as

$$D_{\mu\nu}(q) = \frac{V^{-1}}{n^2-1} \sum_{x,y} e^{-i\eta(x-y)\text{tr}A_\mu(x)A_\nu(y)} = (\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2})DA(q^2) \quad (3)$$

and $Z(q^2) = q^2DA(q^2)$, respectively. The infrared behavior of $Z(q^2)$ is parametrized as $(q^{-2})^{\alpha_D}$, and we observed that $\alpha_D < 0$ for $L \geq 24$.

As a kinematical structure of the Gribov region, Zwanziger gave a bound on the constant mode $\overline{A}_\mu = V^{-1}\sum_x A_{x,\mu}$ for $A \in \Omega_L$, as $|\overline{A}_\mu| \leq \frac{2}{\alpha}\tan\left(\frac{\pi}{L}\right) \equiv \sigma(L)$ with a lattice size $L$, $V = L^4$, and $\alpha = \sqrt{3/2}$ in $SU(3)$ and $\alpha = \sqrt{2}$ in $SU(2)$, respectively. This fact can be derived irrespective of the gauge field options. The Zwanziger’s bound was checked to hold in case $\beta = 6.4$, $L = 48$, as $|\overline{A}_\mu| \leq 0.005$ vs. $\sigma(L) = 0.10$.
for each sample. The propagator at zero momentum has an extra factor $V$, as $D_A(0) \sim V A^2$. Although $\sqrt{V} \langle A_\mu \rangle = 0.23 \pm 4.2$, the propagator has the value $3D_A(0) = \frac{3}{4} V \langle \sum_{\mu,a} A_\mu a^2 \rangle \approx 150 \pm 30$ (averaged over samples). It implies that the gluon propagator is infrared finite. Since there is a strong cancellation between samples, the sample average is $\frac{3}{4} V \sum_\mu \langle A_\mu \rangle^2 = 0.16 \pm 31$.

Here we review Zwanziger’s strong argument for the vanishing of zero momentum connected Green functions in the infinite volume limit. Let a partition function be $Z(J) = \int dA \rho(A) e^{J \cdot A}$, and for constant $J_x = h$, it reads that $Z(h) = \int dA \rho(A) e^{h \cdot \tilde{A}(0)} = \int dA \rho(A) e^{V h \cdot \tilde{A}}$, where $\tilde{A}_\mu(0) = \delta^d \sum_x A_\mu(x)$. Let $W(h) = \log Z(h)$, and then the connected Green function $V G^{(n)}_c(0,\ldots,0) = \langle (\tilde{A}(0))^{n} \rangle_c$ is given as $G^{(n)}_c(0,\ldots,0) = \frac{1}{V} \frac{\partial^n}{\partial h^n} W(h)$. Note from positivity of $\rho(A)$ that $e^{-V h \sigma(L)} \leq Z(h) \leq e^{V h \sigma(L)}$. Putting $w(h) = W(h)/V$, we obtain $|w(h)| \leq h \sigma(L) \rightarrow 0$ as $L \rightarrow \infty$. Hence it proves that $G^{(n)}_c(0,\ldots,0) \rightarrow 0$ in the infinite volume limit. However, it should be noted that this infinite volume limit is only proved directly for connected Green functions.

In Fig1, we plot the gluon dressing function of 24$^4$, 32$^4$, and 48$^4$ lattices. The infrared properties of the gluon dressing function is studied in the principle of minimum sensitivity(PMS) in $\tilde{MOM}$ scheme. The parameter $y = 0.02227$ is fixed at $q = 1.97 GeV$ and $q^2 D_A(q^2) = Z(q^2,y)\big|_{y=0.02227}$ is compared with the lattice data.
3. The ghost propagator

In use of color source $|\lambda^a x\rangle$ normalized as $\text{tr} \langle \lambda^a x | \lambda^b x_0 \rangle = \delta^{ab} \delta_{x,x_0}$, the ghost propagator is given by the Fourier transform of

$$D_G^{ab}(x,y) = \langle \text{tr} \langle \lambda^a x | \{-\partial D(U)\}^{-1} | \lambda^b y \rangle \rangle \quad (4)$$

where the outmost $\langle \rangle$ denotes an average over samples $U$. The ghost dressing function is defined as $G_{ab}^{\rho^2} = q^2 D_{G}^{ab}(q^2)$, and in the infrared region, it is parametrized as $(p^{-2})^{\alpha c}$. Although the ghost dressing function is monotonic in the case of $L \leq 32$, the lowest momentum point of $L = 48$ is suppressed.

We compare in Fig.2 the result of the PMS method in $\tilde{\text{MOM}}$ scheme\textsuperscript{14} $D_G(q^2) = -Z_3(q^2,y)/q^2|_{y=0.02142}$ and the lattice data of $\beta = 6.0, L = 24, 32$ and $\beta = 6.4, L = 48$.

4. The Kugo-Ojima parameter

We directly measured the Kugo-Ojima parameter on $16^4, 24^4, 32^4$ and $48^4$ lattices. Results are summarized in Table1.

| $\beta$ | $L$ | $c_1$ | $e_1/d$ | $h_1$ | $c_2$ | $e_2/d$ | $h_2$ |
|---------|-----|-------|---------|-------|-------|---------|-------|
| 6.0     | 16  | 0.576(79) | 0.860(1) | -0.28 | 0.628(94) | 0.943(1) | -0.32 |
| 6.0     | 24  | 0.695(63) | 0.861(1) | -0.17 | 0.774(76) | 0.944(1) | -0.17 |
| 6.0     | 32  | 0.706(39) | 0.862(1) | -0.15 | 0.777(46) | 0.944(1) | -0.16 |
| 6.4     | 32  | 0.650(39) | 0.883(1) | -0.23 | 0.700(42) | 0.953(1) | -0.25 |
| 6.4     | 48  | 0.720(49) | 0.982(1) | -0.26 |

In Fig3, the value $c$ is plotted as a function of $\log Z_3(1.97 GeV)$ for $\beta = 6.0, L = 16, 24$ and $U$-linear and $\log \hat{U}$ definitions of the gauge field, and for $\beta = 6.4, L = 32, 48$ and $\log \hat{U}$ definition. The crossing point of the extrapolation of the points corresponding to the $\log \hat{U}$ and $U$-linear definitions of the gauge field suggests $c$ in the continuum limit $c = 1 - Z_1/Z_3$ is consistent to 1.

5. The QCD running coupling

Using the renormalized gauge field $A_{\tau\mu}^a(x) = Z_3^{-1/2} A_{\mu}^a(x)$ and the ghost field $c_b^h(x) = \tilde{Z}_3^{-1/2} c^b(x)$, one can show that the vertex renormalization fac-
Figure 3. The Kugo-Ojima parameter $c$ as the function of $\log Z_3 (1.97 \, \text{GeV})$. 
$\beta = 6.0, 16^4, 24^4$ in $U$-linear (diamond) in $\log U$ (triangle) and $\beta = 6.4, 32^4, 48^4$ in $\log U$ version (star).

Figure 4. The running coupling $\alpha_s (q)$ of $\beta = 6.0, 24^4$ (box), $32^4$ (triangle), $\beta = 6.4, 32^4$ (diamond) and $48^4$ (star) as a function of momentum $q (\text{GeV})$ and the result of the PMS method in $\tilde{MOM}$ scheme.

The tor of the ghost field $\tilde{Z}_1 = Z_\gamma Z_3^{1/2} \tilde{Z}_3 = 1$, implies $g^2 Z_3 \tilde{Z}_3^2$ is renormalization group invariant.

We estimated the running coupling from the two-point function (gluon- and ghost-dressing functions)

$$
\alpha_s (p^2) = \left\{ \frac{g^2}{(4\pi)} \right\} Z(p^2) G(p^2)^2 \simeq (p a)^{-(\alpha_D + 2\alpha_G)}.
$$

(5)

Dyson-Schwinger approach predicts $\alpha_G = 0.5953, \alpha_D = -1.1906$ and $\alpha_s (0) = 2.972$, while the $\tilde{MOM}$ scheme approach of Orsay group and the PMS method predict that $\alpha_s (0)$ is small and close to 0.

Table 2. The exponent of gluon dressing function near the zero momentum $\alpha_D$, near the $p a = 1 \alpha_D' = \alpha_G$, the exponent of ghost dressing function near the zero momentum $\alpha_G$ and $\alpha_D + 2\alpha_G$ in log $U$ type simulation.

| $\beta$ | $L$ | $\alpha_D$ | $\alpha_D'$ | $\alpha_G$ | $\alpha_D + 2\alpha_G$ |
|---------|-----|------------|-------------|------------|------------------------|
| 6.0     | 32  | -0.375     | 0.392       | 0.174      | -0.03 (10)             |
| 6.4     | 48  | -0.273     | 0.288       | 0.109      | -0.06 (10)             |

ALPHA collaboration derived scheme independent running coupling in the Schrödinger functional method. Our data of $\alpha_s$ for $q \geq 1 \, \text{GeV}$ is consistent with those of Schrödinger functional.

6. Discussion and Conclusion

In the lattice simulation of $\tilde{MOM}$ scheme, the running coupling decreases in the infrared region and $\Lambda_{\text{MS}} = 295 \pm 20 \, \text{MeV}$ was reported. Orsay group
claims that the non-perturbative gluon condensate reduces the $\Lambda_{\overline{MS}}$ in the $\overline{MOM}$ scheme from 295MeV to 238MeV and becomes consistent with that in the Schrödinger functional method$^{11}$.

In the PMS$^{13}$ one assumes that there is an infrared fixed point and one bridges the gap between ultraviolet asymptotic free and the infrared non-perturbative regions. In our analysis of lattice data we considered optimum coupling constant $y$ at $q = 1.97 GeV$ that corresponds to $1/a$ of $\beta = 6.0$ and observed that the PMS in $\overline{MOM}$ scheme fits the lattice data, $q > 0.5 GeV$ in the ghost propagator, and $q > 1 GeV$ in the gluon propagator and the running coupling.

In the infrared region, we also analyzed the data via the contour-improved perturbation series which is expressed by the Lambert W function$^{15}$. We find that the result is close to that of the static quark potential derived by using the perturbative gluon condensate dynamics $\tilde{\alpha}_V$ shown in the analysis of hypothetical $\tau$ lepton decay coupling constant$^{16}$. Results will be presented elsewhere.

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