THE QUANTUMNESS OF A SET OF QUANTUM STATES

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We propose to quantify how “quantum” a set of quantum states is. The quantumness of a set is the worst-case difficulty of transmitting the states through a classical communication channel. Potential applications of this measure arise in quantum cryptography, where one might like to use an alphabet of states most sensitive to quantum eavesdropping, and in lab demonstrations of quantum teleportation, where it is necessary to check that entanglement has indeed been used.

1 Introduction

How quantum can a single quantum state be? Does this question make sense? One gets the impression it does with even a small perusal of the quantum-optics literature, where coherent states are often called “classical” states of light. Despite the nomenclature, we suggest there is no robust notion of the classicality of a single quantum state. Consider any two coherent states $|\alpha\rangle$ and $|\beta\rangle$. The inner product of these states is nonzero. Thus, if a single mode is prepared secretly in one of these states, there is no automatic device that can amplify the signal reliably into a two-mode state $|\gamma\rangle|\gamma\rangle$, where $\gamma = \alpha, \beta$ depending upon the input. Nonorthogonal states cannot be cloned, and this holds whether such states are called “classical” or not.

A notion of the quantumness of states can thus only be attached to a set of states. The members of a set of states can be more or less quantum with respect to each other, but there is no good sense in which each one alone is intrinsically quantum or not. A set of two nonorthogonal states $|\psi_0\rangle$ and $|\psi_1\rangle$, with $x = |\langle\psi_0|\psi_1\rangle|$, provides a good example. To set the stage, let us work within the metaphor of no-cloning. There, we might take the degree of “clonability” as measure of quantumness of the two states.

A cloning attempt is a unitary operation $U$ that gives $|\psi_i\rangle|0\rangle \rightarrow |\Psi_i\rangle$, $i = 0, 1$, where $|\Psi_i\rangle$ is a state whose partial trace over either subsystem gives identical density operators. An optimal cloning attempt is one that maximizes the fidelity between the output and the wished-for target state $|\psi_i\rangle|\psi_i\rangle \equiv |\psi_i\rangle|\psi_i\rangle$, i.e., maximizes $F_{\text{try}} = \frac{1}{2} |\langle\Psi_0|\psi_0,\psi_0\rangle|^2 + \frac{1}{2} |\langle\Psi_1|\psi_1,\psi_1\rangle|^2$. It can be shown that

$$F_{\text{clone}} = \max_U F_{\text{try}} = \frac{1}{2} \left(1 + x^3 + (1 - x^2)\sqrt{1 + x^2}\right).$$

(1)
Viewing Eq. (1) as a measure of the quantumness of two states—i.e., the smaller $F_{\text{clone}}$, the more quantum the set of states—one finds that two states are the most quantum with respect to each other when $x = 1/\sqrt{3}$.

Eq. (1), though we will not ultimately adopt it, exhibits some of the main features a measure of quantumness ought to have. In particular, two states are the most classical with respect to each other when they are either orthogonal or identical. Moreover, the set is most quantum when the states are somewhere in between, in this case when they are 54.7° apart. This point draws the most important contrast between the notions of quantumness and distinguishability.\footnote{\label{footnote1}As an example, in communication theory it is important to understand the best probability with which a signal can be guessed correctly after a quantum measurement has been performed on its carrier. For the case at hand, the measure of optimal distinguishability is then given by $P_s = \frac{1}{2} \left( 1 + \sqrt{1 - x^2} \right)$. This quantity is monotone in the parameter $x$. No measure of quantumness should have this character. Instead, quantumness should capture how difficult it is to make a copy of the quantum state after some of the information about its identity has been deposited in another system.}

The optimal cloning fidelity in Eq. (1) is not completely satisfactory for our purposes, though. One reason is that the idea of cloning does not lead uniquely to Eq. (1). Under minor modifications of the clonability criterion, the particular $x$ for which two states are the most quantum with respect to each other changes drastically. For instance, by another measure in Ref.\footnote{\label{footnote2}In communication theory, it is important to understand the best probability with which a signal can be guessed correctly after a quantum measurement has been performed on its carrier. For the case at hand, the measure of optimal distinguishability is then given by $P_s = \frac{1}{2} \left( 1 + \sqrt{1 - x^2} \right)$. This quantity is monotone in the parameter $x$. No measure of quantumness should have this character. Instead, quantumness should capture how difficult it is to make a copy of the quantum state after some of the information about its identity has been deposited in another system.}

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For these reasons, we adopt a metaphor more akin to eavesdropping in quantum cryptography. For any set of pure states $S = \{ \Pi_i = |\psi_i\rangle\langle\psi_i| \}$, let us act as if there is a source emitting systems with states drawn according to a probability distribution $\pi_i$. (This distribution is an artifice; ultimately it will be discarded after an optimization.) The systems are then passed to an eavesdropper who is required to measure them one by one and thereafter fully discard the originals. To make sure the latter process is enforced, we might imagine that the eavesdropper really takes the form of two people, perhaps Eve and Yves, separated by a classical channel. Eve may perform any quantum measurement imaginable, but then Yves will have access to nothing beyond the classical information obtained to attempt to reproduce the original state.

The question is, how intact can the states remain in this process? To gauge the intactness, we take the average fidelity between the initial and final states. Operationally this corresponds to Yves handing his newly prepared quantum system back to the source. The preparer checks to see whether the
system has kept its identity; the probability it passes the test is the fidelity.

Considering the best measurement and resynthesis strategies Eve and Yves can perform gets us most of the way toward a notion of quantumness. The final ingredient is to imagine that the source makes this task as hard as possible. Conceptually, we do this by adjusting the probabilities \( \pi_i \) so that the maximum average fidelity is as small as it can be. The resulting fidelity is what we take to be the quantumness of \( S \). The intuition behind this definition is simple. It captures in a clear-cut way how difficult the eavesdropper’s task can be made for reconstructing the set of states. And it does this disregarding the more subtle task of quantifying how much information Eve learns about the state’s identity in the process. In a way, it captures the raw sensitivity to eavesdropping that can be imparted to the states in \( S \).

The problem promoted here has its roots in the “state estimation” scenario studied in great detail in the recent literature.\(^7\) The main differences are that we have relaxed the condition that the states in \( S \) be associated with a uniform distribution on Hilbert space, and we have added an extra optimization over the probability distribution \( \pi_i \). Moreover, the traditional use of “state estimation” has been for purposes of defining a notion of distinguishability for quantum states.\(^5\),\(^8\) As explained above, this is exactly what we are not trying to get at with a notion of quantumness.\(^9\)

### 2 Building an Expression for Quantumness

Imagine \( S \) equipped with a probability distribution \( \pi_i \). Such a set of states along with a set of assigned probabilities, we call an ensemble \( \mathcal{P} \). (There are no restrictions on the number of elements in \( S \), nor on \( d \), the dimension of Hilbert space for which \( |\psi_i\rangle \in \mathcal{H}_d \).) Eve performs a single quantum measurement, i.e., some POVM \( \mathcal{E} = \{E_b\} \), on the signal.

Yves makes use of the information Eve obtains—some explicit index \( b \)—by preparing his system in a quantum state \( \sigma_b \). Since the preparation is based solely on classical information, there need be no restrictions on the mapping \( \mathcal{M} : b \to \sigma_b \). For instance, it need not be completely positive, etc. Moreover, the \( \sigma_b \) may be mixed states. This corresponds to the possibility of a randomized output strategy on the part of Yves. The conjunction of \( \mathcal{E} \) and \( \mathcal{M} \) constitutes a protocol for the eavesdropping pair.

Supposing the source emits the state \( \Pi_i \), and Eve obtains the outcome \( b \) for her measurement, then the fidelity Yves achieves is \( F_{b,i} = \langle \psi_i | \sigma_b | \psi_i \rangle \). However there is no predictability of Eve’s measurement outcome beyond what quantum mechanics allows. Similarly, the most we can say about the actual state the source produces is through the probability distribution \( \pi_i \). Therefore, the average fidelity for the protocol is

\[
F_P(\mathcal{E}, \mathcal{M}) = \sum_{b,i} \pi_i \text{tr}(\Pi_i E_b) \text{tr}(\Pi_i \sigma_b) .
\]

(2)
A convenient intermediate quantity comes from optimizing Yves’ strategy alone. For a given \( \mathcal{E} \), we define the achievable fidelity with respect to the measurement to be

\[
F_P(\mathcal{E}) = \max_{\mathcal{M}} F_P(\mathcal{E}, \mathcal{M}).
\]

In analogy to the quantity known as accessible information in the theory of quantum channel capacities, let us define the accessible fidelity of the ensemble \( \mathcal{P} \) to be

\[
F_P = \max_{\mathcal{E}} F_P(\mathcal{E}).
\]

Finally, the quantumness of the set \( \mathcal{S} \) is defined by

\[
Q(\mathcal{S}) = \min_{\{\pi_i\}} F_P.
\]

3 The Accessible Fidelity

It is easy to derive an exact expression for the achievable fidelity \( F_P(\mathcal{E}) \) for any given \( \mathcal{E} \). First note that \( F_P(\mathcal{E}, \mathcal{M}) \) is linear in the \( \sigma_b \). Thus, in any decomposition of \( \sigma_b \) into a mixture of pure states, we might as well delete \( \sigma_b \) and replace it with the most advantageous element in the decomposition. Therefore, it never hurts to take the \( \sigma_b \) to be pure states, \( \sigma_b = |\phi_b\rangle\langle\phi_b| \).

Rewriting \( F_P(\mathcal{E}, \mathcal{M}) \) under this assumption, we obtain

\[
F_P(\mathcal{E}, \mathcal{M}) = \sum_b \langle\phi_b|M_b|\phi_b\rangle,
\]

where \( M_b = \sum_i \pi_i \Pi_i E_b \Pi_i = \sum_i \pi_i \text{tr}(\Pi_i E_b \Pi_i) \). Now the pure states \( |\phi_b\rangle \) are arbitrary. Thus we can optimize each term in Eq. (5) separately. This is done by remembering that the largest eigenvalue \( \lambda_1(A) \) of any Hermitian operator \( A \) can be characterized by

\[
\lambda_1(A) = \max_{\alpha} \langle\alpha|A|\alpha\rangle.
\]

Therefore,

\[
F_P(\mathcal{E}) = \sum_b \lambda_1 \left( \sum_i \pi_i \text{tr}(\Pi_i E_b \Pi_i) \right).
\]

Unfortunately, this is where the easy part of the development ends. No explicit expression for the accessible fidelity exists in general. The most one can hope is to understand some of its general properties, a few explicit examples, and perhaps some useful bounds.

In this regard, the first question to ask is how much can said about the measurements achieving equality in Eq. (3). Are we even sure that optimal measurements exist? The answer is yes, and the reason is essentially the same as for the existence of an optimal measurement for the accessible information. Because \( \lambda_1(A+B) \leq \lambda_1(A) + \lambda_1(B) \), \( F(\mathcal{E}) \) is a convex function over the the set of POVMs. Since the set of POVMs is a compact set and \( \lambda_1(A) \) is a continuous function, it follows that \( F_P(\mathcal{E}) \) achieves its supremum on an extreme point of the set. Furthermore, by the reasoning of Refs. \[11\] \[12\], we know that for any extreme point \( \mathcal{E} \), all the nonvanishing operators \( E_b \) within it must be linearly independent. Thus, we can restrict the maximization in
Eq. 3 to POVMs with no more than $d^2$ outcomes. Finally, because of the subadditivity of $\lambda_1(A)$, these $d^2$ operators can be chosen to be rank-one.

One might wish for a further refinement in what can be said of optimal measurements for accessible information. For instance, that the number of measurement outcomes not exceed the number of inputs in analogy to the case of quantum hypothesis testing\footnote{This intuition is captured by asking rhetorically, what can Eve possibly do better than make her best guess and pass that information on to Yves? If such is the case, though, a proof remains to be seen. Indeed, because of the nonlinearity in Eq. (3), there may be counterevidence from the case of accessible information.\footnote{13}}. Indeed, because of the nonlinearity in Eq. (3), there may be counterevidence from the case of accessible information.\footnote{13}

This brings up the question of how to draw a comparison between the success probability in hypothesis testing and the achievable fidelity for any given measurement. The usual way of posing the hypothesis testing problem is to assume a one-to-one correspondence between inputs $\Pi_i$ and POVM elements $E_i$, each element signifying the guess one should make. In that way of writing the problem, the average success probability $P_s$ takes the form $P_s = \sum_i \pi_i \text{tr}(\Pi_i E_i)$. Here, however, we cannot make such a restriction on the number of outcomes. So, we must pose the hypothesis testing problem in a more general way.

Suppose Eve performs a measurement $\mathcal{E}$ and observes outcome $b$ to occur. This information will cause her to update here probabilities for the various inputs according to Bayes’ rule:

$$p(i|b) = \frac{p(b, i)}{p(b)} = \frac{\pi_i \text{tr}(\Pi_i E_b)}{\text{tr}(\rho E_b)}, \quad (7)$$

where $p(b) = \text{tr}(\rho E_b)$ and $\rho = \sum_i \pi_i \Pi_i$. Maximum likelihood dictates that Eve’s success probability will be optimal if she chooses the value $i$ for which $p(i|b)$ is maximum. Thus her average success probability will be

$$P_s(\mathcal{E}) = \sum_b p(b) \max_i \{p(i|b)\} = \sum_b \max_i \{\pi_i \text{tr}(\Pi_i E_b)\}. \quad (8)$$

Eq. 3 compares to $F_p(\mathcal{E})$ through a simple inequality. To see this, note that $F_p(\mathcal{E}) = \sum_b p(b) \lambda_1(\rho_b)$, where $\rho_b = \sum_i p(i|b) \Pi_i$. Suppose $i = j$ maximizes $p(i|b)$. Then $\lambda_1(\rho_b) = \max_\phi \langle \phi | \rho_b | \phi \rangle \geq \langle \psi_j | \rho_b | \psi_j \rangle = \sum_i p(i|b) \langle \psi_j | \psi_i \rangle^2 \geq p(j|b) + \sum_{i \neq j} p(i|b) \langle \psi_j | \psi_i \rangle^2 \geq \max_i \{p(i|b)\}$. Therefore, $P_s(\mathcal{E}) \leq F_p(\mathcal{E})$. This inequality is not tight, however. For instance, for $\mathcal{P}$ describing a uniform distribution of states on a qubit, $P_s \to 0$, while $F_p = 2/3$.\footnote{13}

Tighter bounds, both upper and lower, on Eq. 3 would be useful. An obvious lower bound comes directly from the convexity of the $\lambda_1(A)$ function. Note in particular that $\rho = \sum_b p(b) \rho_b$. Therefore $F_p(\mathcal{E}) \geq \lambda_1(\rho)$. This inequality is generally tighter than the previous one in that it never falls below $1/d$; moreover, there is a measurement $\mathcal{E}$ that achieves it.

A more interesting lower bound comes about by considering the behavior of $F_p(\mathcal{E})$ with respect to the “square-root measurement.”\footnote{This POVM is...} This POVM is
constructed from the ensemble decomposition of $\rho$ by multiplying it from the left and right by $\rho^{-1/2}$. Inserting this measurement into $F_P(\mathcal{E})$, we obtain

$$F_P \geq F_{P_{GM}} = \sum_i \lambda_i \left( \sum_j \pi_i \pi_j \Pi_j \rho^{-1/2} \Pi_i \rho^{-1/2} \Pi_j \right).$$

(9)

4 Conclusion

Much more can be said about accessible fidelity and quantumness, but lack of space prevents us from saying it here. We end instead with a question. One can define the quantumness of a Hilbert space by

$$Q_d = \inf_S Q(S),$$

(10)

where the infimum is taken over all sets of states living on $\mathcal{H}_d$. One has to wonder whether this quantity might not indicate a deep defining property for the quantum system itself—its ultimate “sensitivity to the touch.”

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