Markov approximations of Gibbs measures for long-range interactions on 1D lattices

Cesar Maldonado\textsuperscript{1} and Raúl Salgado-García\textsuperscript{2}

\textsuperscript{1} Instituto de Física, Universidad Autónoma de San Luis Potosí, Avenida Manuel Nava 6, Zona Universitaria, 78290, San Luis Potosí S.L.P., Mexico
\textsuperscript{2} Facultad de Ciencias, Universidad Autónoma del Estado de Morelos, Avenida Universidad 1001, Colonia Chamilpa, 62209, Cuernavaca Morelos, Mexico
E-mail: coma@dec1.ifisica.uaslp.mx and raulsg@uaem.mx

Received 25 June 2013
Accepted 2 August 2013
Published 27 August 2013

Online at stacks.iop.org/JSTAT/2013/P08012
doi:10.1088/1742-5468/2013/08/P08012

Abstract. We study one-dimensional lattice systems with pair-wise interactions of infinite range. We show projective convergence of Markov measures to the unique equilibrium state. For this purpose we impose a slightly stronger condition than summability of variations on the regularity of the interaction. With our condition we are able to explicitly obtain stretched exponential bounds for the rate of mixing of the equilibrium state. Finally we show convergence for the entropy of the Markov measures to that of the equilibrium state via the convergence of their topological pressure (free energy).

Keywords: rigorous results in statistical mechanics
1. **Introduction**

Two concepts that are of fundamental importance in the rigorous study of statistical mechanics are the Gibbs measures and the equilibrium states. Many important results concerning the latter within the thermodynamical formalism were first established in lattice systems and then generalized to much more abstract settings such as topological dynamical systems. The concept of equilibrium state as a translationally invariant measure was introduced within the study of the statistical mechanics of $d$-dimensional lattice systems mainly motivated by the occurrence of the phenomenon known as *spontaneous symmetry breakdown* [9, 13]. In the late 1960s Bowen and Ruelle [2, 14] independently introduced the *variational principle* to characterize equilibrium states. On the other hand, for lattice systems, Gibbs measures associated with sufficient regular interactions$^3$ are known to be equivalent to equilibrium states [13, 8]. Its characterization from the modern point of view is due to Bowen [3]. It is worth mentioning that much of the theory of the hyperbolic dynamical systems has been mainly motivated on the statistical physics of lattices systems. Nowadays, from the rigorous point of view, there are still open problems related to this systems in dimensions greater than one and to the phase transition phenomena.

$^3$ In the thermodynamical formalism are called potentials.
In this paper we are concerned with the problem of the uniqueness of Gibbs measures in a one-dimensional lattice with long-range pair interactions. On this class of systems several conditions for the existence and uniqueness of Gibbs measures have been established. For instance, Ruelle gave sufficient conditions on the regularity of the interaction to prove the uniqueness of the equilibrium state. Bowen extended such results to general dynamical systems defined by expansive homeomorphisms on compact metric spaces. He proved also that the unique equilibrium state is partially mixing. Years later, but in the same context, Walters \[17\] proved the Ruelle convergence theorem for $g$-measures using the operator techniques proposed in \[14\].

Under the Bowen–Walters condition, uniqueness of the equilibrium state has been proved following several approaches \[2, 9, 14, 17\]. The classical argument, which can be found in \[9\] for instance, goes as follows. First one defines an atomic measure on a periodic sequence, playing the role of a finite-volume measure in the canonical ensemble. Then one shows that as the period goes to infinity (i.e., in the thermodynamic limit) there are equilibrium states. In this way the existence part is proven. To prove uniqueness it is shown that equilibrium states are Gibbs measures under the Bowen–Walters condition. Then, by the ergodic decomposition theorem uniqueness follows since any two Gibbs measures are absolutely continuous with respect to each other in dimension one. Different conditions have been proposed to prove uniqueness since then \[15\].

More recently the authors of \[5\] proposed a technique of approximation of Gibbs measures for Hölder-continuous interactions on sofic subshifts by means of Markov approximations on sofic subshifts. The strategy is based on an explicit construction of a sequence of periodic Gibbs measures for Hölder-continuous interactions on subshifts of finite type ‘enveloping’ the sofic subshift. Then it is shown that the sequence of periodic measures has a unique accumulation point in the weak-$*$ topology (via the Perron–Frobenius theorem) when both the period goes to infinity and the subshift of finite type tends to the sofic subshift. Such an accumulation point is actually the unique Gibbs measure associated with the Hölder-continuous interaction on the sofic subshift.

With that technique the authors of \[5\] are able to obtain upper bounds for the rate of mixing of the limiting measure, as well as to prove convergence of the entropy.

Here we use the same technique of \[5, 6\] in order to prove similar results but in the context of one-dimensional lattice systems with long-range interactions. Within this context it is possible to enhance the rate of mixing of the limiting measure, that is, the bounds for the speed of decay of correlations of square integrable observables.

We ought to mention that simulations of any model of interacting particles in a lattice need finite-range interactions in order to be performed, that is, Markov approximations. Thus, from the physical point of view, it is interesting to study these Markov approximations. One wants to have approximations that reflect the properties of the limiting measure (state), one is concerned with the way in which those approximations converge. Using the projective convergence (defined below) it is possible to keep that control under the properties of the limiting measure in the approximations.

The paper is organized as follows. Section 2 contains the necessary definitions, generalities and notational remarks. Section 3 establishes our main results: (i) that the Markov approximations converge to the unique equilibrium state in the projective sense, (ii) an explicit upper bound for the rate of mixing of the limiting measure and (iii) the convergence of the entropies of the Markov approximations to the entropy of the
equilibrium state. Section 4 is devoted to develop explicitly the proofs of theorems and lemmas stated.

2. Setting

2.1. Generalities

We consider a one-dimensional infinite lattice. At each site of the lattice is placed a particle that interacts with the other particles via pair interactions. In order to model such pair interactions between particles let us consider a finite set $A$, called alphabet, representing the possible ‘states’ of the particle. As usual, the interaction of every pair of particles will depend on its specific states and the distance between them. Let $X = A^{\mathbb{N}_0}$ be the space of all infinite sequences made up of symbols taken from $A$, in other words, the set of all possible configurations of our system. A word of length $k$ ($k \in \mathbb{N}_0$) is the finite sequence $a_{k} := a_1 a_2 \cdots a_k$ where $a_i \in A$ for all $i = 1, \ldots, k$. We endow $A$ with the discrete topology. The space $X$ is supplied with the product topology, and thus it is compact. We consider the Borel sigma algebra generated by the cylinder sets defined by $[a_0^k] := \{x \in X : x_0^k = a_0^k\}$.

Next, the dynamics in $X$ is generated by the shift map $T$, that is, $(Tx)_i = x_{i+1}$ for any $x \in X$. Let $\mathcal{M}_T(X)$ be the space of all $T$-invariant probability measures on $X$. Consider the following distance on $\mathcal{M}_T(X)$:

$$D(\mu, \nu) := \sum_{k=0}^{\infty} 2^{-k+1} \left( \sum_{a \in A^k} |\mu[a] - \nu[a]| \right).$$

It is well known that this distance is a metric for the weak-$*$ topology and on the same topology $\mathcal{M}_T(X)$ is a compact and convex set.

Note: This definition has been proposed by Ugalde and Trejo [16].

**Notational remark:** From now on, we denote by $a = c^{\pm 1}$ the inequalities $c^{-1} \leq a \leq c$. Analogously $a = \exp(\pm b)$ stands for $\exp(-b) \leq a \leq \exp(b)$.

The following definition will be important for stating our results.

**Definition 2.1.** (Projective convergence) Let $r, n \in \mathbb{N}_0$. The sequence of measures $\{\nu_r\}$ is said to converge in the projective sense to $\nu$ if there exists a sequence $\{\varepsilon_r\}$, such that $\varepsilon_r \to 0$ as $r \to \infty$ for any cylinder $[a_0^{n-1}]$, for which the following inequalities hold:

$$\frac{\nu_r[a_0^{n-1}]}{\nu[a_0^{n-1}]} = e^{\pm n \varepsilon_r}.$$

2.2. Pair-wise interactions

We are interested in quantifying the ‘energy’ of a given configuration $x$. The local energy (centred at 0) of the configuration $x$ is a function $u : X \to \mathbb{R}$ defined as follows:

$$u(x) := \sum_{k=0}^{\infty} \psi_k(x_0, x_k),$$

where $\psi_k : A \times A \to \mathbb{R}$ is a distance-dependent pair-wise interaction. A classical example is an Ising-type model studied by Dyson, for which $A = \{1, -1\}$ and the pair-wise interaction

\[4\text{ This definition has been proposed by Ugalde and Trejo [16].}\]
is
\[\psi_k(x_0, x_k) := -\beta J(x_0 x_k)/(k^\alpha),\]
where \(\beta\) is the inverse temperature and \(J\) is a ‘coupling constant’. It is known that, for \(\alpha > 2\), the Bowen–Walters condition holds and thus there is a unique equilibrium state [11].

We will assume the following condition.

**Condition 1.** Let \(p : \mathbb{R} \to \mathbb{R}\) be a function such that \(p(r)/r \to \infty\) as \(r \to \infty\), we consider pair-wise interactions \(\psi_k\) satisfying
\[
\sum_{r=0}^{\infty} p(r) \|\psi_r\| < \infty.
\]

**Remark 2.1.** Observe that this condition is slightly stronger than the Bowen–Walters condition and the Ruelle condition [14]. We emphasize the advantage of considering this condition instead.

### 2.3. Gibbs measures and equilibrium states

The states of the systems are described by probability measures. The probability measure \(\mu \in \mathcal{M}_T(X)\) is said to be an *equilibrium state* for \(u\) if
\[
h(\mu) + \int u \, d\mu = \sup_{\nu \in \mathcal{M}_T(X)} \left\{ h(\nu) + \int u \, d\nu \right\},
\]
where \(h(\nu)\) is the measure-theoretic entropy of \(\nu\) defined as
\[
h(\nu) = -\lim_{n \to \infty} \frac{1}{n+1} \sum_{a \in A^{n+1}} \nu[a] \log(\nu[a]).
\]
This definition is the well-known variational principle [3]. For lattice systems it is known that the measures fulfilling the variational principle are also Gibbs measures which are defined as follows [13].

**Definition 2.2** (Gibbs measures). Let \((X, T)\) be the full shift and \(u : X \to \mathbb{R}\) be a continuous interaction. The measure \(\mu\) associated with \(u\) is a Gibbs measure if there exist constants \(C > 1\) and \(P(u) > 0\) such that for all \(x \in X\)
\[
\frac{\mu[x_0^{n-1}]}{\exp(U_n(y) + nP(u))} = C^\pm,
\]
for all \(y \in [x_0^{n-1}]\), and here \(U_n(y) := \sum_{k=0}^{n-1} u(T^k y)\). The constant \(P(u)\) is the topological pressure, which in the thermodynamical formalism is defined as
\[
P(u) = h(\mu) + \int u \, d\mu,
\]
where \(\mu\) is the measure that fulfills the variational principle.

At this point we should stress that the definitions given above are the same as those used in statistical mechanics up to sign conventions and constants. More precisely, a word \(a = a_0 a_1 \ldots a_n\) stands for a configuration of \(n + 1\) spins. The energy associated with such configuration times the inverse temperature \(\beta\) is then given by the function \(-U_n\), or in other words \(-\beta E_n = U_n\). Along the same line, the thermodynamic entropy \(S\) equals \(nh(\mu)\) and minus the Helmholtz free energy \(A\) times \(\beta\) is the topological pressure \(P(u)\),
i.e., $-\beta A =nP(u)$. Thus, the Gibbs measure defined above says that the probability of observing a given spin configuration $a$ is given by $\mu[a]$ or, equivalently
\[\mu[a] \propto \exp (-\beta \mathcal{E}_n - \beta \mathcal{A}),\]
which is the well-known form in statistical mechanics of the Gibbs measure. Furthermore, the average of $u$ with respect to $\mu$ times $n$ is actually the thermodynamic energy $\mathcal{E}$ times $\beta$,
\[-n \int u \, d\mu = \beta \mathcal{E}.
\]
In this way, the variational principle states that the equilibrium state is the maximum over all invariant probability measures of minus the reduced Helmholtz free energy $A$ times $\beta$, i.e.,
\[
\sup_{\nu \in \mathcal{M}_T(X)} \left\{ h(\nu) + \int u \, d\nu \right\} = \sup_{\nu \in \mathcal{M}_T(X)} \{S/n - \beta \mathcal{E}/n\},
\]
which clearly is equivalent to saying that the measure that minimizes the Helmholtz free energy $A = -\beta^{-1}S + \mathcal{E}$ is the equilibrium state.

2.4. Markov approximations to Gibbs measures

In this work we use the same technique as the one implemented in [5] to characterize the Gibbs measures associated with infinite-range interactions. The strategy goes as follows. We will use the fact that every Gibbs measure can be obtained as the limit of a sequence of periodic measures, defined as follows.

**Definition 2.3 (Periodic measures).** Let $p \in \mathbb{N}_0$ be a positive integer and denote by $\text{Per}_p(X)$ the set of all periodic sequences $x \in X$ of period $p$. For every $n \in \mathbb{N}_0$ and $p > n$,
\[
\mathcal{P}(p)[a_0^{n-1}] := \sum_{y \in \text{Per}_p(X) \cap [a_0^{n-1}]} \exp(U_n(y)) = \sum_{y \in \text{Per}_p(X) \cap [a_0^{n-1}]} \exp(U_n(y)).
\]

In order to explicitly build up the sequence of measures having as a limit the Gibbs measure for the infinite-range interaction $u : X \to \mathbb{R}$, an intermediate step is necessary. We need to construct a Gibbs measure for interactions of range $r$. The latter will be taken as ‘approximations’ to the infinite-range interaction we are interested in. The $r$-range approximation $u_r : X \to \mathbb{R}$ to the interaction $u$ is defined as
\[
u(a_k, x_{k+1}, \ldots, x_{r-1}) = u_r(x) \text{ for every } x = x_0 x_1 x_2 \cdots \in X.\]
Then we define the transition matrix $M_r : \mathcal{A}^r \times \mathcal{A}^r \to \mathbb{R}^+$ associated with $u_r$, by
\[
M_r(a, b) := \exp (u_r(ab_{r-1})) \text{ if } a_{r-1}^r = b_{r-2}^r,
\]
otherwise.

\[\text{doi:10.1088/1742-5468/2013/08/P08012} 6\]
Markov approximations of Gibbs measures

In the last expression $ab_{r-1}$ means the concatenation of the word $a$ and the last letter of the word $b$. We will denote by $L_r : A \to \mathbb{R}$ and $R_r : A \to \mathbb{R}$ the corresponding left and right eigenvectors of $M_r$. Then the measure defined by

$$\mu_r[a_0^{n-1}] := L_r(a_r^{n-1}) \prod_{j=0}^{n-r} \frac{M_r(a_j^{j+r-1}, a_{j+1}^{j+r})}{\rho_j^{n-r+1}} R_r(a_0^{n-1})$$

for each $n \in \mathbb{N}_0$ such that $n \geq r$, is a shift-invariant probability measure. Moreover, $\mu_r$ is the unique Gibbs measure associated with the interaction $u_r$ [6].

3. Results

Our main results are the following.

**Theorem 3.1.** Consider the infinite-range interaction $u$ given by (1) and let $u_r$ be its finite-range approximation. Let $\mu_r$ be the Markov measure associated with $u_r$. If $u$ satisfies condition 1, then $\mu_r$ converges to $\mu$ in the projective sense. As a consequence $\mu_r$ converges to $\mu$ in the weak-$\ast$ topology.

The technique used in this paper plus the condition assumed on the interaction allows us to obtain a stretched exponential rate of mixing for the limiting measure. In our context this result enhances the rate of mixing obtained in [5] which is of order $\sqrt{s}$. Our result is the following.

**Theorem 3.2.** Assume that the infinite-range interaction $u$ satisfies condition 1, then there exist constants $C > 1$, $c > 0$ and $\xi \in (0, 1)$ such that for any $a, b \in A^r$ there exists $s^* = s(a, b)$ for which

$$\left| \frac{\mu([a] \cap T^{-s}[b])}{\mu[a] \mu[b]} - 1 \right| \leq Ce^{-cs^\xi},$$

for all $s \geq s^*$.

Finally we also are able to obtain a result for the convergence of the entropies of the approximating Markov measures to the entropy of the equilibrium state.

**Theorem 3.3.** Let $u : X \to \mathbb{R}$ be a long-range interaction as defined above fulfilling condition 1. Then, there are constants $C > 0$ and $\epsilon > 0$ such that

$$|h(\mu) - h(\mu_r)| \leq Cr^{-\epsilon}.$$  

3.1. Example of applications

It is well known that the mixing property is equivalent to decay of the correlation of square integrable observables (see for instance [4, Chapter 4.3.]). So, the latter result gives improved bounds for the classical problem of a lattice of spins with polynomially decaying pair interactions, i.e. $\psi_k = -\beta J x_0 x_k k^{-\alpha}$ with $\alpha > 2$—in which, according to the operator theory [1, Chapter 1.4.], the two-site correlation function will decay polynomially with the distance between spins. In contrast, the projective convergence gives estimates for the correlation function as a stretched exponential with the distance between spins. The latter is in agreement with recent numerical simulations in which the observed decay of correlation is exponential (see for example the estimated correlation functions reported in [7]).

doi:10.1088/1742-5468/2013/08/P08012
3.2. Concluding remarks

Our main contribution to the study of the Gibbs measures on one-dimensional lattices with long-range pair interactions (with a finite state space) is the improvement of previous results within the framework of thermodynamical formalism. Particularly, we have proved projective convergence of a sequence of finite-range Markov approximations to the unique Gibbs measure (Theorem 3.1) corresponding to a given interaction. We have also proved that the latter is mixing with a stretched exponential rate (Theorem 3.2). Finally, we have proved that the entropies of the finite-range approximations also converges, a result which is a direct consequence of the projective convergence (Theorem 3.3). For the particular case of polynomially decaying pair interactions, i.e. \( \psi_k = -\beta J x_0 x_k^{-\alpha} \) (with \( \beta \) as the inverse temperature and \( J \) as a ‘coupling’ constant), it is known that for \( \alpha > 2 \) there is a unique ergodic measure for all the values of \( \beta \) [10]. Since this potential complies with condition 1 for \( \alpha > 2 \), our analysis concludes that for all \( \beta \geq 0 \) the Markov approximations converge in the projective sense. Moreover, the above theorems let us obtain explicit formulae for the mixing rate and for the convergence rate of the entropy. On the other hand, for the case \( \alpha \leq 2 \) it is known that for some \( \beta_c \), for all the values of \( \beta < \beta_c \), the Gibbs measure \( \mu_\beta \) (associated with the interaction \( J_u \)) is the convex combination of two ergodic measures. In such a case the weak-* limit does exist and has two ergodic components. However, the Markov approximations cannot converge projectively by a theorem about the projective limit of ergodic measures due to [16]. Indeed, according to [16] the loss of projective convergence may be a signature of non-uniqueness of ergodic measures. They prove that if a sequence of ergodic measures converge in the projective sense then the limit is ergodic. In our work we have proved that, for the case of one-dimensional lattices with, to some extent, arbitrary pair interactions, the Markov approximations converges projectively to the unique Gibbs measure when the interaction fulfills condition 1. If condition 1 does not hold we have the possibility that the Gibbs measure might be decomposed into two or more ergodic measures. As mentioned above, in this case such a limit measure cannot be reached by a sequence of Markov approximations because they are ergodic (pure states) and the ‘limit measure’ is not. It would be interesting to see if for the case \( \beta > \beta_c \) and \( 1 < \alpha \leq 2 \) (for which the Gibbs measure is still in a pure state [12]) we can prove projective convergence of Markov approximations, a fact that could give us a criterion to detect phase transitions.

4. Proofs

In order to give the proof of Theorem 3.1, let us first introduce the following definition. Let \( n, r \in \mathbb{N}_0 \) be fixed and \( n \geq r \). Consider \( A^n \subseteq A^n \). For every \( p > n + r \) the \( r \)-Markov approximation \( P^{(p)}_r \) of the periodic measure (2) is given by

\[
P^{(p)}_r[a_0^{n-1}] := \frac{\sum_{y \in \text{Per}_p(X) \cap [a_0^{n-1}]} e^{U_n,r(y)}}{\sum_{y \in \text{Per}_p(X)} e^{U_n,r(y)}},
\]

where \( U_n,r(y) := \sum_{k=0}^{n-1} \nu_r(T^k y) \) is the \( r \)-range approximation of \( U_n \). These measures are also called elementary Gibbs measures. We have the following lemma.

**Lemma 4.1.** Let \( \mu_r \) be the \( r \)-range Markov approximation of the measure \( \mu \). And let \( P^{(p)}_r \) be the \( r \)-range approximation of the periodic measure \( P^{(p)} \). Then there exist constants...
$C_r > 0$ and $\eta < 1$ for all $a^n \in A^n$, such that

$$\mu_r[a^n] = \mathcal{P}(a^n)[\mu_r[a^n]] \cdot \exp \left( \pm 12rC_r e^{2C_r} \cdot \eta^{(p-n)/r-2} \right).$$

Let us assume for the moment this lemma—the proof will be given after the proof of Theorem 3.1. The explicit expressions for the constants $C_r$ and $\eta$ are given below.

### 4.1. Proof of Theorem 3.1

Consider two consecutive Markov approximations. We prove that they accumulate towards a limit measure as the range of the approximation diverges. The strategy is to use the periodic approximation of the Markov measure as follows. Using Lemma 4.1 for $(r+1)$-range interactions, one obtains that

$$\mu_{r+1}[a^n] = \mathcal{P}(a^n)[\mu_{r+1}[a^n]] \cdot \exp \left( \pm (r+1)C_{r+1} e^{2C_{r+1}} \cdot \eta^{(p-q)/(r+1)-2} \right),$$

where $\eta := 1 - e^{2\sum_{i=1}^{\infty} \lambda_i}$ and $q := \max\{n, r+1\}$. For the sake of clarity, let us drop the indices on $a^n$. We compare two consecutive Markov measures,

$$\frac{\mu_r[a]}{\mu_{r+1}[a]} = \frac{\mathcal{P}(a)[\mu_r[a]]}{\mathcal{P}(a)[\mu_{r+1}[a]]} \times \frac{\exp \left( \pm 12rC_r e^{2C_r} \cdot \eta^{(p-n)/r-2} \right)}{\exp \left( \pm 12(r+1)C_{r+1} e^{2C_{r+1}} \cdot \eta^{(p-q)/(r+1)-2} \right)},$$

In order to have a bound depending only on the interaction we compare two consecutive periodic measures. Since $|U_{r+1}(y) - U_{r+1}(y)| = |\sum_{p=0}^{\infty} \psi_{r+1}(T^k y_0, T^k y_1)| \leq (p+1)\|\psi_{r+1}\|$ we obtain that

$$\frac{\mathcal{P}_r[a]}{\mathcal{P}_{r+1}[a]} = e^{\pm 2(p+1)\|\psi_{r+1}\|}.$$

Finally, putting together these previous bounds yields

$$\frac{\mu_r[a]}{\mu_{r+1}[a]} = \exp \left( \pm (2(p+1)\|\psi_{r+1}\|) \cdot \exp \left( \pm 24(r+1)C_{r+1} e^{2C_{r+1}} \cdot \eta^{(p-q)/(r+1)-2} \right) \right)$$

$$= \exp \left( \pm (2(p+1)\|\psi_{r+1}\| + 24(r+1)C_{\infty} e^{2C_{\infty}} \cdot \eta^{(p-n-r-1)/(r+1)-2}) \right),$$

where $C_{\infty} = \sum_{k=1}^{\infty} k\|\psi_k\|$. Let $p := p(s)$, then for any word $a \in A^n$ and any $r' > r \in \mathbb{N}_0$, one has that

$$\frac{\mu_r[a]}{\mu_{r'}[a]} = \exp \left( \pm \left( 2 \sum_{s=r}^{r'} p(s)\|\psi_s\| + 24R \sum_{s=r}^{r'} s\eta^{(p(s)/s)} \right) \right),$$

where $R := C_{\infty} e^{2C_{\infty}}$. Since $\eta < 1$ and assuming condition 1, the right-hand side of the previous expression tends to zero as $r$ diverges. Thus, the limit $\mu^*[a] := \lim_{r \to \infty} \mu_r[a]$, exists for any word $a \in A^n$, proving the weak-$*$ convergence. Moreover, one has that

$$\frac{\mu_r[a]}{\mu^*[a]} = \exp \left( \pm \left( 2 \sum_{s \geq r} p(s)\|\psi_s\| + 24R \sum_{s \geq r} s\eta^{(p(s)/s)} \right) \right), \quad (5)$$

doi:10.1088/1742-5468/2013/08/P08012
for every $r \in \mathbb{N}_0$ and any word $a \in A^n$. So one also has that $\mu_r[a]/\mu^*[a] = e^{\pm n \varepsilon_r}$, where $n$ is the length of the word $a$, and $\varepsilon_r$ is given by

$$\varepsilon_r := 2 \sum_{s \geq r} p(s)\|\psi_s\| + 24R \sum_{s \geq r} s \eta^{p(s)/s},$$

which proves the projective convergence of the Markov approximations $\mu_r$ to $\mu^*$. The limiting measure $\mu^*$ turns out to be absolutely continuous to $\mu$ since we use the elementary Gibbs measures to approximate the Markov ones. The proof of this statement is found in [6, section 6.3.3.]. That concludes the proof.

### 4.1.1. Proof of Lemma 4.1.

Here we use that every Markov measure can be seen as a limit of measures with support on the periodic points as was proved in [6, Appendix 6.3.1.]. From the calculations in that paper, one has explicit bounds for the Markov measure $\mu_r$ in terms of the finite-range periodic approximation $P_r(p)$ as follows:

$$\mu_r[a] = P_r(p)[a] \exp\left(\pm \frac{rD_0}{1 - \eta}(P - n)/r - 2\right),$$

where $\eta := \eta(M_r^r)$, is the Birkhoff coefficient of the matrix $M_r^r$ and $D_0$ is a constant bounded by

$$D_0 \leq 2 \max_{b,c,c' \in A^r} \log \left(\frac{M_r^r(c,b)M_r^{r+1}(c',b)}{M_r^{r+1}(c,b)M_r^r(c',b)}\right) = \frac{e^{U_{r,r}(cb)}e^{U_{r+1,r}(c'b)}}{e^{U_{r+1,r}(cb)}e^{U_{r,r}(c'b)}}.$$

Using that

$$|U_{r,r}(bc') - U_{r+1,r}(bc')| \leq 3 \sum_{k=1}^r k\|\psi_k\|,$$

for $b \in A^r$ and $c' \in A$, one gets

$$D_0 \leq 2C_r,$$

where $C_r := \sum_{k=1}^r k\|\psi_k\|$. And so, one has

$$\mu_r[a] = P_r(p)[a] \cdot \exp\left(\pm 12rC_r e^{2C_r} \times \eta^{(P - n)/r - 2}\right),$$

which proves the lemma for $\eta = 1 - e^{-2\sum_{k=1}^\infty k\|\psi_k\|}$ and $C_r = \sum_{k=1}^r k\|\psi_k\|$. □

### 4.2. Proof of Theorem 3.2

Here we mimic the proof of Theorem 3.3 in [5]. As before we make use of the periodic approximation of the limiting measure through the following lemma.

**Lemma 4.2.** Let $P_r^{(p)}$ be the $r$-range approximation of the periodic measure $P^{(p)}$. Let $s, r \in \mathbb{N}_0$. For any $a,b \in A^r$ one has that

$$P_r^{(p)}([a] \cap T^{-s}[b]) = P_r^{(p)}[a]P_r^{(p)}[b] \times \exp\left(\pm 4 \frac{rD}{1 - \eta} \eta^{s/r}\right),$$

for a constant $D > 0$ and where $\eta$ is the same constant as in Lemma 4.1.

doi:10.1088/1742-5468/2013/08/P08012
From inequality (5) applied to the limiting measure one can easily obtain that

$$|\mu([a] \cap T^{-s}[b]) - \mu[a] \mu[b]| \leq |\mathcal{P}_r^{(p)}([a] \cap T^{-s}[b]) - \mathcal{P}_r^{(p)}[a] \mathcal{P}_r^{(p)}[b]|$$

$$\times \exp \left( 2 \left[ 2 \sum_{k \geq r} p(k) \| \psi_k \| + 24 R \sum_{k \geq r} k \eta p(k)/k \right] \right).$$

Using the inequality (6) from our previous lemma yields

$$\left| \frac{\mu([a] \cap T^{-s}[b])}{\mu[a] \mu[b]} - 1 \right| \leq \exp \left( 4 r \frac{D}{1 - \eta} \eta^{1/s(r)} \right) - 1$$

$$\times \exp \left( 2 \left[ 2 \sum_{k \geq r} p(k) \| \psi_k \| + 24 R \sum_{k \geq r} k \eta p(k)/k \right] \right).$$

By the hypothesis made on the interaction, the second factor in the right-hand side of the last inequality can be bounded by above by a constant $C > 1$ for every $r \geq 1$. The larger $r$ we take, the smaller the sufficient upper bound $C$ becomes.

Next, let us consider $r = r(s)$ such that $r(s) = o(s)$. Then there exists a constant $0 < \xi < 1$ such that for every pair $a, b \in A^r$ there exists a $s^* \in \mathbb{N}_0$ for which

$$\left| \frac{\mu([a] \cap T^{-s}[b])}{\mu[a] \mu[b]} - 1 \right| \leq C \xi \eta^s,$$

for every $s \geq s^*$. Now, by introducing an adequate positive constant $c$ we change the base and it is possible to rewrite the previous inequality into the form

$$\left| \frac{\mu([a] \cap T^{-s}[b])}{\mu[a] \mu[b]} - 1 \right| \leq C e^{-c s},$$

again for every $s \geq s^*$. And that finishes the proof of the theorem. \hfill \Box

4.2.1. Proof of Lemma 4.2. First, let $r \in \mathbb{N}$, $p > 2r$ and let $M_r$ be the transition matrix given by (4). Consider any $a \in A^r$. We have

$$\mathcal{P}_r^{(p)}[a] = \frac{\sum_{x \in \text{Perp}(X) \cap [a]} e^{U_{r,p}(x)}}{\sum_{x \in \text{Perp}(X)} e^{U_{r,p}(x)}} = \frac{e^\dagger_a M_r e_a}{\sum_{x \in A^r} e^\dagger_x M_r e_x},$$

where $e_x$ is a vector with an entry equal to 1 at position $x$ and 0 elsewhere. For any $s \in \mathbb{N}_0$, it is easy to see that

$$\mathcal{P}_r^{(p)}([a] \cap T^{-s}[b]) = \sum_{w \in A^s} \mathcal{P}_r^{(p)}[awb].$$

Next, choose an arbitrary but fixed $w \in A^s$. We are able to write $\mathcal{P}_r^{(p)}[awb]$ in terms of the transition matrix $M_r$, as follows:

$$\mathcal{P}_r^{(p)}[awb] = \frac{e^\dagger_a M_r^{p-a} e_{awb} e^\dagger_{awb} M_r^s e_a}{\sum_a \sum_{awb} e^\dagger_a M_r^{p-a} e_{awb} e^\dagger_{awb} M_r^s e_a}.$$
Here we use the enhanced Perron–Frobenius theorem [6, Corollary 2.16.], yielding
\[
P_r^{(p)}[awb] = \frac{e_a^\dagger(L^1e_{wb})\rho^{p-s}R\cdot e_a^\dagger(L^1e_a)\rho^p R}{\sum_a \sum_{wb} e_a^\dagger(L^1e_{wb})\rho^{p-s}R\cdot e_a^\dagger(L^1e_a)\rho^p R} 
\times \exp \left( \pm 2r\delta(e_{wb}, Fe_{wb}) \eta^{[p-s]/r} \right) 
\times \exp \left( \pm 2r\delta(e_a, Fe_a) \eta^{[s]/r} \right),
\]
where \( F \) is the contraction defined on the simplex and \( \delta \) is a distance defined on the simplex—both are explicitly defined in [5, Appendix 1.] and [6, Appendix 6.1.]. Since \( L^1e_x = L(x) \) and \( e_a^\dagger R = R(x) \) one has that
\[
P_r^{(p)}[awb] = \frac{L(wb)R(a)\cdot L(a)R(wb)}{\sum_a \sum_{wb} L(wb)R(a)\cdot L(a)R(wb)} 
\times \exp \left( \pm 2r\delta(e_{wb}, Fe_{wb}) \eta^{[p-s]/r} \right) 
\times \exp \left( \pm 2r\delta(e_a, Fe_a) \eta^{[s]/r} \right)
\]
= \( P_r^{(p)}(a)P_r^{(p)}(wb) \)
\times \exp \left( \pm 2r\delta(e_{wb}, Fe_{wb}) \eta^{[p-s]/r} \right) 
\times \exp \left( \pm 2r\delta(e_a, Fe_a) \eta^{[s]/r} \right)
\]
where in the last inequality we have used the fact that \( P \) is a probability measure. Finally, one obtains that
\[
P_r^{(p)}([a] \cap T^{-s}[b]) = \sum_{w \in A^s} P_r^{(p)}(a)P_r^{(p)}(wb) 
\times \exp \left( \pm 2r\delta(e_{wb}, Fe_{wb}) \eta^{[p-s]/r} \right) 
\times \exp \left( \pm 2r\delta(e_a, Fe_a) \eta^{[s]/r} \right)
\]
= \( P_r^{(p)}(a)P_r^{(p)}(b) \times \exp \left( \pm 4rD \frac{1}{1-\eta} \eta^{[s]/r} \right) \)
for some constant \( D > 0 \). Recall that in our particular case the Birkhoff coefficient \( \eta \) is giving by \( \eta = 1 - e^{-2\sum_{k=1}^n k\|\psi_k\|} \).

4.3. Proof of Theorem 3.3

First notice that given \( r > 0 \) and given \( a \in A^k \) with \( k > 0 \) we have that for every \( x, y \in [a] \)
\[
|u(x) - u(y)| = \sum_{n=r}^{\infty} \psi_n(x_0, x_n)
\]
then we have that, for every \( k > 0 \) the sum difference \( |U_k(x) - U_{k,r}(y)| \) can be bounded by
\[
|U_k(x) - U_{k,r}(y)| = \sum_{n=r}^{\infty} \sum_{j=0}^{k-1} \psi_n(x_j, x_{j+n}) \leq k \sum_{n=r}^{\infty} \|\psi_n\|
\]
Since \( \psi \) has summable variation it is clear that the last sum is finite and goes to zero as \( r \to \infty \). Moreover, for polynomially decaying interaction, such can be bounded by
\[
\sum_{n=r}^{\infty} \|\psi_n\| \leq C'r^{-\varepsilon}
\]
doi:10.1088/1742-5468/2013/08/P08012
Markov approximations of Gibbs measures

for some $\epsilon > 0$. The last constant depends on how the pair-wise interaction $\psi_k$ decays with the ‘distance’ $k$. From the above we can easily see that

$$\exp \{ U_k(a^*) \} = \exp \{ U_{k,r}(a^*) \} \exp (\pm kC'r^{-\epsilon}).$$

This result lets us bound the difference of topological pressures as follows:

$$0 \leq |P(u, X) - P(u_r, X)| \leq \frac{1}{k} \log \left( \frac{\sum_{a \in \text{Per}_k} \exp \{ U_k(a^*) \}}{\sum_{b \in \text{Per}_k} \exp \{ U_{k,r}(b^*) \}} \right) \leq C'r^{-\epsilon},$$

where $a^*$ ($b^*$ respectively) is some point $X$ belonging to $[a]$ ($[b]$ respectively).

Now, since both $\mu_r$ and $\mu$ satisfy the variational principle [3] we have that

$$P(u, X) = \int_X u \, d\mu + h(\mu) \quad \text{and} \quad P(u_r, X) = \int_X u_r \, d\mu_r + h(\mu_r),$$

which allows us to write

$$|h(\mu) - h(\mu_r)| \leq |P(u, X) - P(u_r, X)| + \left| \int_X u \, d\mu - \int_X u_r \, d\mu_r \right|. \quad (7)$$

Now, by (5) we know that

$$\mu = \mu_r \exp(\pm \varepsilon_r),$$

which allows us to write

$$\left| \int_X u \, d\mu - \int_X u_r \, d\mu_r \right| \leq (1 - \exp(-\varepsilon_r)) \|u\|.$$

It is clear that the right-hand side of the above equation vanishes exponentially, which allows us to conclude that the left-hand side of inequality (7) is bounded by $C'r^{-\epsilon}$ choosing an appropriate constant $C > 0$,

$$|h(\mu) - h(\mu_r)| \leq C'r^{-\epsilon},$$

which proves the theorem. $\Box$

Acknowledgments

We thank Edgardo Ugalde for introducing us to this interesting topic of research, for his suggestions, corrections and interesting discussions. CM was supported by a postdoctoral fellowship PROMEP, UASLP-CA-188 and thanks the warm hospitality at CiC, Cuernavaca. RS-G thanks IF-UASLP for its hospitality, and acknowledges the financial support of CONACyT through grant no. CB-2012-01-183358.

References

[1] Baladi V, 2000 Positive Transfer Operators and Decay of Correlations (Advanced Series in Nonlinear Dynamics) vol 16 (River Edge, NJ: World Scientific)

[2] Bowen R, Some systems with unique equilibrium states, 1974/75 Math. Syst. Theory 8 193

[3] Bowen R, 2008 Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms (Lecture Notes in Mathematics) revised edition, vol 470, ed J-R Chazottes (Berlin: Springer)

[4] Brin M and Stuck G, 2002 Introduction to Dynamical Systems (Cambridge: Cambridge University Press)

[5] Chazottes J-R, Ramirez L and Ugalde E, Finite type approximations of Gibbs measures on sofic subshifts, 2005 Nonlinearity 18 445

doi:10.1088/1742-5468/2013/08/P08012
Markov approximations of Gibbs measures

[6] Chazottes J-R and Ugalde E, On the preservation of Gibbsianness under symbol amalgamation, 2011. Entropy of Hidden Markov Processes and Connections to Dynamical Systems (London Math. Soc. Lecture Note Ser, vol 385) (Cambridge: Cambridge University Press) pp 72–97

[7] Cinti F, Portmann O, Pescia D and Vindigni A, One-dimensional Ising ferromagnet frustrated by long-range interactions at finite temperatures, 2009 Phys. Rev. B 79 214434

[8] Dobrushin R L, Gibbsian random fields for lattice systems with pairwise interactions, 1968 Funct. Anal. Appl. 2 292

[9] Dobrushin R L, The problem of uniqueness of a Gibbsian random field and the problem of phase transitions, 1968 Funct. Anal. Appl. 2 302

[10] Dyson F J, Existence of a phase-transition in a one-dimensional Ising ferromagnet, 1969 Commun. Math. Phys. 12 91

[11] Dyson F J, Non-existence of spontaneous magnetization in a one-dimensional Ising ferromagnet, 1969 Commun. Math. Phys. 12 212

[12] Fröhlich J and Spencer T, The phase transition in the one-dimensional Ising model with 1/r^2 interaction energy, 1982 Commun. Math. Phys. 84 87

[13] Lanford O E III and Ruelle D, Observables at infinity and states with short range correlations in statistical mechanics, 1969 Commun. Math. Phys. 13 194

[14] Ruelle D, Statistical mechanics of a one-dimensional lattice gas, 1968 Commun. Math. Phys. 9 267

[15] Stenflo ¨O, Uniqueness in g-measures, 2003 Nonlinearity 16 403

[16] Trejo L and Ugalde E, 2013 in preparation

[17] Walters P, Ruelle’s operator theorem and g-measures, 1975 Trans. Am. Math. Soc. 214 375