Bifurcation analysis of two-DOF nonlinear vibration isolation system with internal resonance

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Abstract. The equations of motion of a two-degree-of-freedom nonlinear vibration isolation system were formulated where the nonlinear restoring force was approximated as a polynomial. The averaging method was applied to obtain the bifurcation equations for the two cases: 1) quadratic nonlinear stiffness with primary resonance and 1:2 internal resonance; 2) quadratic, cubic nonlinear stiffness with primary resonance and 2:1 internal resonance. By means of singularity theory, the bifurcation behaviours of the amplitude with respect to a parameter (which is related to the amplitude of the external force) were studied. The high-codimensional universal unfoldings were given and the transition sets in the parameter planes and the bifurcation diagrams were plotted.

1. Introduction
Two-stage vibration isolation system (VIS), which has been developed during the past 20 years, is widely used in practice, especially for power machinery on ships. The mathematical model of the two-stage VIS is usually simplified to a two-degree-of-freedom (DOF) mass-spring system [1-2]. Another example of the two-DOF mass-spring system is the single-stage VIS equipped on an elastic base, which is familiar in ships or multi-story factory buildings [1]. Under this condition, the elastic base is reduced to a rigid mass supported by a spring and a damper to reveal the primary characteristics of the VIS. The linear dynamics of the two-DOF system has been extensively studied [1-2]. However, the world around us is inherently nonlinear and real mechanical systems always possess some types of nonlinearity. The abundant and complex dynamics of multi-DOF nonlinear system has drawn increasing attentions in recent years [3-7], but little literatures concerned the nonlinear VIS and its bifurcation behaviors directly. Although some results may be useful, there is still a need to focus on the dynamics of the two-DOF nonlinear VIS.

The aim of this paper is to analyze the static bifurcation of the two-DOF nonlinear VIS using averaging method and singularity theory for the two cases: 1) quadratic nonlinear stiffness in primary resonance and 1:2 internal resonance; 2) quadratic, cubic nonlinear stiffness in primary resonance and 2:1 internal resonance. Firstly, we formulate the equations of motion of the two-DOF nonlinear VIS with expressing the nonlinearity as a polynomial in Section 2. The general form of the equations of motion is transformed to a standard form in Section 3. In Section 4, the averaging method is applied to

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obtain the bifurcation equations, and the universal unfoldings are established using singularity theory. The transition sets and bifurcation diagrams are also presented in Section 4.

2. Equations of motion
Consider a two-DOF nonlinear vibration isolation system, as shown in Fig.1. \( M_1 \) is the isolated equipment, and \( M_2 \) denotes the intermediate mass (for two-stage VIS) or the base (for the simplified elastic base). \( M_1 \) is supported by a vibration isolator combining a linear damper and a nonlinear spring, and \( M_2 \) is connected with a fixed plane using a linear damper and a linear spring. When the origins of the coordinates are set at the places where the springs are not compressed, as shown in figure 1(a), the equations of the two-DOF nonlinear spring-mass system can be formulated as following:

\[
\begin{align*}
M_1\ddot{X}_1 + C_1(\dot{X}_1 - \dot{X}_2) + F(X_1, X_2) = P\cos\Omega t - M_1g, \\
M_2\ddot{X}_2 + C_2\dot{X}_2 + K_2X_2 = C_1(\dot{X}_1 - \dot{X}_2) + F(X_1, X_2) - M_2g. 
\end{align*}
\]

\[(1)\]

where: \( C_1 \) is the damping coefficient of the nonlinear vibration isolator; \( F(X_1, X_2) \) is the nonlinear restoring force, and its expression will be discussed in the next paragraph. \( C_2 \) is the damping coefficient of the damper between \( M_2 \) and the fixed plane; \( K_2 \) is the stiffness coefficient of the linear spring between \( M_2 \) and the fixed plane; \( P \) and \( \Omega \) are the amplitude and frequency of the harmonic excitation, respectively.

According to Weierstrass Approximation Theorem [8], which states: Let \( f \in C([a,b],\mathbb{R}) \), and then there is a sequence of polynomials \( p_n(x) \) that converges uniformly to \( f(x) \) on \([a, b]\), the nonlinear restoring force can be expressed, without loss of generality, as the polynomial \( F(\xi) = K\xi + Q\xi^2 + U\xi^3 + \cdots \), in which \( \xi \) presents the amount of compression of the nonlinear spring and equals \( X_1 - X_2 \) in the two-DOF VIS model as shown in figure 1. The order of the polynomial depends on the stiffness characteristics of the nonlinear spring, and usually is selected as two and/or three. This paper will focus on the bifurcations of the two-DOF nonlinear VIS in two situations: quadratic nonlinear stiffness with 1:2 internal resonance and quadratic, cubic nonlinear stiffness with 2:1 internal resonance.

Given that the nonlinear restoring force is expressed as a cubic and quadratic polynomial \( F(\xi) = K\xi + Q\xi^2 + U\xi^3 \), the equations of motion of the two-DOF nonlinear VIS are rewritten as:

\[
\begin{align*}
M_1\ddot{X}_1 + C_1(\dot{X}_1 - \dot{X}_2) + K(X_1 - X_2) + Q(X_1 - X_2)^2 + U(X_1 - X_2)^3 = P\cos\Omega t - M_1g, \\
M_2\ddot{X}_2 + C_2\dot{X}_2 + K_2X_2 = C_1(\dot{X}_1 - \dot{X}_2) + K(X_1 - X_2) + Q(X_1 - X_2)^2 + U(X_1 - X_2)^3 - M_2g. 
\end{align*}
\]

\[(2)\]
Note that the origins are not the equilibrium points of this system, which is inconvenient for further analyses, and hence the coordinate transformation should be carried out. As shown in figure 1(b), the origins of the new coordinates are located at the equilibrium places of these springs, i.e. the places where these springs are compressed with stillness. The relations between the old and new coordinates are: 11 12 2 2 ,  X = − = − .

In the equilibrium state, the gravitation terms can be eliminated from the right-hand side of the equations of motion by using the following relations:

\[ KH - QH^2 + UH^3 = M_1g, \quad K_2h_2 = M_2g + M_1g, \]

where \( H = h_1 - h_2 \). Substituting the coordinate transformation 11 12 2 2 , X = − = − and (3) into (2), and setting 11 12 3 4 , x = , 12 3 4 , x = , 23 4 , x = , 24 4 , x = , the equations of motion can be expressed in the first order form. Here, we assume the nonlinear stiffness coefficients, damping coefficients and the amplitude of exciting force are small quantities and introduce an additional parameter \( \varepsilon \) into the equations to present this kind of smallness. Then (2) can be rewritten in the general form:

\[ \frac{dx}{dt} = Ax + \varepsilon N(x, f), \] (4)

where \( x = [x_1, x_2, x_3, x_4]^T \),

\[ A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k_1 & 0 & k_1 & 0 \\ 0 & 0 & 0 & 1 \\ wk_1 & 0 & -k_2 - wk_1 & 0 \end{bmatrix}, \quad N(x, f) = \begin{bmatrix} 0 \\ -\delta_1(x_2 - x_4) - q(x_1 - x_3)^2 - u(x_1 - x_3)^3 + f \cos \Omega t \\ 0 \\ -\delta_2 x_4 + w\delta_1 (x_2 - x_4) + wq (x_1 - x_3)^2 + wu(x_1 - x_3)^3 \end{bmatrix}, \]

\[ k_1 = \frac{(K_1 - 2QH + 3UH^2)}{M_1}, \quad \delta_1 = \frac{C_1}{M_1}, \quad q = \frac{Q - 3UH}{M_1}, \quad u = \frac{U}{M_1}, \quad f = \frac{P}{M_1}, \quad k_2 = \frac{K_2}{M_2}, \quad \delta_2 = \frac{C_2}{M_2}, \quad w = \frac{M_1}{M_2}. \]

If only the quadratic term of the nonlinear restoring force is considered, letting the cubic stiffness coefficient \( U \) be zero, (4) is also the equations of motion in this situation.

3. **Standard form of the equations of motion**

Standard form of the equations of motion refers to the equations where the vector fields are proportional to the small parameter \( \varepsilon \). In order to transform the general form (4) of the equations of motion into the standard form, the coordinate transformation \( x = \sum_{j=1}^{2} a_i \varphi_{ij}(\theta_j), (s = 1, 2, 3, 4) \) should be introduced. \( \varphi_{ij} \) (s=1..4, j=1,2) are the elementary solutions of the derived system \( dx/dt = Ax \) of (4). Let \( \varphi_{ij} \) (s=1..4, j=1,2) be the elementary solutions of the conjugate equation of the equation \( dx/dt = Ax \). In view of the orthogonality between \( \varphi_{ij} \) and \( \varphi_{ij} \), the general form of the equations of motion can be transformed to the standard form [9]:

\[ \frac{da_1}{dr} = -\varepsilon \sin(\theta_1) N_1 k_1 + wk_1 N_2 - \lambda_1^2 N_4, \quad a_1 \frac{d\theta_1}{dr} = a_1 \lambda_1 + \frac{\varepsilon \cos(\theta_1) (N_4 k_1 + wk_1 N_2 - \lambda_1^2 N_4)}{\Delta_1 k_1 w \lambda_1}, \]

\[ \frac{da_2}{dr} = -\varepsilon \sin(\theta_2) N_1 k_1 + wk_1 N_2 - \lambda_2^2 N_4, \quad a_2 \frac{d\theta_2}{dr} = a_2 \lambda_2 + \frac{\varepsilon \cos(\theta_2) (N_4 k_1 + wk_1 N_2 - \lambda_2^2 N_4)}{\Delta_2 k_1 w \lambda_2}, \] (5)

where \( \lambda_{1,2} \) are the natural frequencies of the derived system and the expressions are

3
\[
\lambda_{1,2}^2 = \frac{1}{2} \left[ w k_1 + k_2 \pm \left( w^2 k_1^2 + 2 w k_1 k_2 + k_2^2 - 2 k_2 k_1 + k_1^2 \right)^{1/2} \right],
\]

\[
\Delta_{1,2} = \frac{-w k_1^2 \lambda_{1,2}^2 + 2 \lambda_{1,2}^3 k_1 - \lambda_{1,2}^4 k_1^2}{-w \lambda_{1,2}^2 k_1^2}.
\]

The 1:2 and 2:1 internal resonance mentioned above means the ratios \( \lambda_1 : \lambda_2 = 1:2 \) and \( \lambda_1 : \lambda_2 = 2:1 \) should be satisfied. At the same time, the situation that the frequency \( \Omega \) of the exciting force approaches to the first natural frequency \( \lambda_1 \), i.e. the primary resonance, will be considered, which makes the bifurcation analysis become a challenging task.

4. Bifurcation analysis

Bifurcation analysis is a mathematical technique that enables determination of the stability of a system with respect to a parameter. Bifurcation diagrams describe the dependence of a state variable on a continuous change in a chosen system parameter, termed a bifurcation parameter. It is well known that singularity theory [10, 11] plays an important role in the static bifurcation analysis. In the singularity theory approach, the universal unfolding of a bifurcation equation can reveal all possible bifurcation behavior when the original system is subjected to a small perturbation. In this section, we will obtain the single-variable bifurcation equation through the averaging method and establish the universal unfolding for the bifurcation equation using singularity theory.

In order to apply the averaging method, the KB transformation is introduced:

\[
a_i(t) = \gamma_i(t) + eU_i(t, y, \rho), \quad \theta_i(t) = \lambda_{i0} + \rho_i(t) + e\gamma_i(t, y, \rho), (i = 1, 2),
\]

(6)

The derivatives of the new parameters should satisfy the conditions:

\[
\frac{dy_i(t)}{dt} = e\gamma_i(t) + e^2 \gamma_i''(t, y, \rho), \quad \frac{d\rho_i(t)}{dt} = \sigma_i + e\gamma_i(t) + e^2 \gamma_i''(t, y, \rho), (i = 1, 2),
\]

(7)

where \( \sigma_i \) and \( \sigma_2 \) are tune parameters, and \( \sigma_1 = \lambda_1 - \lambda_{10}, \sigma_2 = \lambda_2 - \lambda_{20} \).

4.1. Quadratic stiffness with 1:2 internal resonance

As mentioned above, if only the quadratic stiffness is considered, the cubic stiffness coefficient \( U \) in (4) should be set as zero. Given there exists the resonance relation \( \Omega : \lambda_{10} : \lambda_{20} = 1:1:2 \), that is to say considering not only the internal resonance but also the primary resonance, substituting (6) and (7) into (5), and collecting terms of the first order in \( e \), the following average equations are obtained:

\[
\frac{dy_1(t)}{dt} = e \left( -\frac{1}{2} \frac{f \sin(\gamma_1)}{\Delta_1 \lambda_1} + \frac{1}{2} \frac{\lambda_1 \lambda_2 q y_1 y_2 \sin(\gamma_1)}{k_1^2 \Delta_1} + \frac{1}{2} A_1 y_1 \right),
\]

\[
\frac{dy_2(t)}{dt} = e \left( -\frac{1}{4} \frac{\lambda_2 \lambda_2 q y_2^2 \sin(\gamma_2)}{k_1^2 \Delta_2} + \frac{1}{2} A_2 y_2 \right),
\]

\[
\frac{d\rho_1(t)}{dt} = \sigma_1 + e \left( -\frac{1}{2} \frac{\lambda_1 \lambda_2 q y_2 \cos(\gamma_2)}{k_1^2 \Delta_1} + \frac{1}{2} f \cos(\gamma_1) \right),
\]

\[
\frac{d\rho_2(t)}{dt} = \sigma_2 + e \left( -\frac{1}{2} \frac{\lambda_2 \lambda_2 q y_2^2 \cos(\gamma_2)}{k_1^2 \Delta_2} \right).
\]

where \( A_1 = (-\delta k_1^2 \lambda_{10} - 2\delta k_2^2 \lambda_{10}^2 - \lambda_{10}^4 k_1^2 + \lambda_{10}^2 \delta k_1) \), \( \gamma_1 = \rho_1 \), \( \gamma_2 = 2\rho_2 - \rho_1 \). Let \( dy_1(t)/dt \), \( dy_2(t)/dt \), \( d\gamma_1(t)/dt \) and \( d\gamma_2(t)/dt \) be zero, then the steady-state solution is obtained:
where:
\[
\mu = \frac{f^2 k_1^2 \Delta_2^2 \left( A_3 e^2 + 16 \sigma_2^2 \right)}{A_4 A_5 A_6 A_7}, \quad \alpha_1 = \frac{1}{4} \Delta_2^2 \Delta_3^2 k_1^2 \left( e^2 A_3^2 + 16 \sigma_2^2 \right) \left( e^2 A_4^2 + 16 \sigma_2^2 \right),
\]
\[
\alpha_2 = \frac{\Delta_4 \Delta_5 k_1^2 \left( -16 \sigma_2 \sigma_3 + e^2 A_3 A_4 \right)}{A_6^2 A_7^2 q^2 e^2}.
\]

It is obvious that the bifurcation equation (8) takes on the $Z_2$ symmetry, i.e. $G(-y_2, \mu) = -G(y_2, \mu)$. Selecting $g(y_1, \mu) = y_1^2 - \mu y_1$ as the germ, the following relation is obtained: $g(y_1, \mu) = r(v, \mu) y_1$, where $r(v, \mu) = v^3 - \mu$, $g(y_1, \mu) \in \epsilon_{v, \mu}(Z_2)$, $v = y_1^2$. For analyzing the bifurcation behaviors of the equation (8), singularity theory provides the following proposition.

**Proposition 1.** Let the germ be $g = (\rho v^4 + \delta \mu)x$, where $v = x^2$, and $\rho, \delta$ are sign functions: $\rho = \pm 1$, $\delta = \pm 1$. Then the $Z_2$-codimension of the germ $g$ is $k-1$, and the universal unfolding of $g$ is given by
\[
(\rho v^4 + \delta \mu + a_1 v + \cdots + a_{k-1} v^{k-1})x.
\]

The proof of this proposition had been discussed in detail in literature [10]. Obviously, the bifurcation equation (8) is the universal unfolding of the normal form $g(y_1, \mu) = y_1^2 - \mu y_1$, and the $Z_2$-codimension is 2, i.e. there exist two unfolding parameters, which all are the combinations of the stiffness, damping coefficients and the tune parameters, as expressed above.

In singularity theory, two bifurcation diagrams are defined to be similar if the number, order, and orientation of the steady-state solutions change in an identical way as the bifurcation parameter is varied. Different bifurcation diagrams are divided by some boundaries over the unfolding parameter space, and the boundaries usually consist of one of three types of set: bifurcation point set (BS), hysteresis set (HS), and double limit point set (DLS). The union of these three kinds of set is called transition set, on which the construction of the bifurcation diagram will change qualitatively when a tiny perturbation is introduced into the system.

According to the definition of the transition set, the transition set of the universal unfolding (8) is obtained. Bifurcation point set: $B_0 = \Phi$, $B_1 = \Phi$; Hysteresis set: $H_0 = \{\alpha_1 = 0\}$, $H_1 = \{\alpha_1 = \alpha_2^2/3, \alpha_2 \leq 0\}$; Double limit point set: $DL = \{\alpha_1 = \alpha_2^2/4, \alpha_2 \leq 0\}$. So the transition set is $\Sigma = B_0 \cup B_1 \cup H_0 \cup H_1 \cup DL$.

![Figure 2. Transition set and bifurcation diagrams of the bifurcation equation (8)](image)

The transition set and the bifurcation diagrams in different regions are shown in figure 2. From figure 2, we can obtain the complex bifurcation of the nonlinear VIS with quadratic stiffness coefficient and 1:2 internal and primary resonance. When the unfolding parameters are in the region 2,
the change of the amplitude of vibration is the smoothest and the phenomena of jump never happen. It is necessary to remark that as for the bifurcation problem in engineering, the bifurcation parameter or the unfolding parameters may be constrained by some conditions. For example, the bifurcation parameter $\mu$ and unfolding parameter $\alpha_1$ in (8) should be non-negative. Therefore, in order to obtain some results applicable in engineering, the analyses should be carried out with considering the transition set and bifurcation diagrams as shown in figure 2, as well as the constrained conditions. It is difficult to transform the transition set from the unfolding parameter space into the space of the physical parameters, such as $\sigma_1$, $\sigma_2$, so a practical way is to calculate the unfolding parameters based on the real values of the physical parameters and then judge which region the unfolding parameters fall into. Eventually, using the bifurcation diagrams in the unfolding parameter space, the bifurcation behaviors under real values of the physical parameters can be obtained.

4.2. Quadratic, cubic stiffness with 2:1 internal resonance

If the nonlinear restoring force is expressed as the polynomial with quadratic and cubic stiffness, (4) and (5) are the general and standard forms of the equations of motion with $U \neq 0$. Consider the primary and internal resonance: $\Omega : \lambda_0 = 2 : 2 : 1$, then the following averaging equation is obtained:

$$\begin{align*}
\frac{d\gamma_1(t)}{dt} &= \epsilon \left( \frac{f \sin(\gamma_1)}{2\Delta_1,\lambda_1} + \frac{\lambda_2^2 \gamma_1^2 \sin(\gamma_2)}{4k_1^2,\Delta_1} + \frac{A_1,\gamma_1}{2k_1^2,\Delta_1,\omega_1} \right), \\
\frac{d\gamma_2(t)}{dt} &= \epsilon \left( \frac{\lambda_2^2 \gamma_1^2 \sin(\gamma_2)}{2k_1^2,\Delta_2} + \frac{A_2,\gamma_2}{2k_1^2,\Delta_2,\omega_2} \right), \\
\frac{d\rho_1(t)}{dt} &= \sigma_1 + \epsilon \left( -\frac{\lambda_1^2 \lambda_2^4 \gamma_1^2 \cos(\gamma_2)}{4k_1^2,\Delta_1,\gamma_1} - f \cos(\gamma_1) \frac{3\mu_1,\gamma_1^2}{2k_1^2,\Delta_1,\omega_1} + 6u_1,\gamma_1^2 \lambda_1^2 \gamma_2^2 \right), \\
\frac{d\rho_2(t)}{dt} &= \sigma_2 + \epsilon \left( -\frac{\lambda_1^2 \lambda_2^2 \gamma_1^2 \cos(\gamma_2)}{2k_1^2,\Delta_2,\gamma_1} - 3u_1,\gamma_1^2 \lambda_1^2 \lambda_2^2 \gamma_1^2 \right),
\end{align*}$$

(10)

where $A_{1,2}$ is the same with that in Section 4.1; $\gamma_1 = \rho_1$, $\gamma_2 = \rho_1 - 2\rho_2$. Letting $d\gamma_1(t)/dt$, $d\gamma_2(t)/dt$, $d\rho_1(t)/dt$ and $d\rho_2(t)/dt$ be zero, and setting $x^2 = \lambda_1^2 q^2 \lambda_1^2 k_1^2 w^2 y_1^2 - A_1^2 k_1^2$, the steady-state solution is obtained:

$$G(x, \mu, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = x^6 - \mu + \alpha_1 x^4 + \alpha_2 x^3 + \alpha_3 x^2 + \alpha_4 x = 0,$$

(11)

where $\mu$ is the bifurcation parameter which is relate to the amplitude of exciting force, and $\alpha_1, \ldots, \alpha_4$ are relate to the stiffness coefficients, damping coefficients, tune parameters and so on. The expressions of these parameters are omitted here due to the limitation of the length of this paper. In singularity theory, the following proposition can be used to analyze the bifurcation behaviors of (11).

**Proposition 2.** Let the germ be $g = \rho x^6 + \delta \mu$, where $\rho$, $\delta$ are sign function: $\rho = \pm 1$, $\delta = \pm 1$, then the codimension of the germ $g$ is $k = 2$, and the universal unfolding of $g$ is given by

$$\epsilon x^6 + \delta \mu + a_1 x + \cdots + a_{k-2} x^{k-2}.$$

(12)

The proof of this proposition also had been discussed in some literatures [10,11]. According to the proposition 2, the bifurcation equation (11) is the universal unfolding of the normal form $x^6 - \mu$, and the codimension is 4 with unfolding parameters $\alpha_1, \ldots, \alpha_4$, which all are the combinations of the physical parameters of the system.

According to the definition of the transition set, the transition set of the universal unfolding (11) is obtained. However, from the fact that the universal unfolding (11) is a high codimensional problem with codimension 4, it is very difficult to discuss the bifurcation behavior of (11) in full detail.
Therefore, it is necessary to analyze all forms of two parameter unfoldings contained in (11). In this paper, only the transition set and bifurcation diagrams in the $\alpha_1-\alpha_2$ plane are presented.

Setting $\alpha_1 = 0$, $\alpha_4 = 0$, we obtain $B = \Phi$, $H = \{\alpha_1 = 0\}$, $DL = \{(\alpha_1/4)^{1} = (\alpha_2/3)^{1}\}$. The transition set in the $\alpha_1-\alpha_2$ plane is $\Sigma_{\alpha_1-\alpha_2} = B \cup H \cup DL$, as shown in figure 3. From figure 3, we can find the bifurcation of the nonlinear VIS with quadratic, cubic nonlinear stiffness and 2:1 internal resonance is also considerable complex. As the same with the situation discussed in section 4.1, the bifurcation and unfolding parameters are constrained by some conditions, so for the application in engineering, these constrained conditions should be considered carefully.

![Diagram](image)

**Figure 3.** Transition set and bifurcation diagrams of the bifurcation equation (11)

## 5. Conclusions

In this work we first, presented the equations of motion of the two-DOF nonlinear VIS where the nonlinear restoring force was approximated as a polynomial. The averaging method was applied to obtain the single-variable bifurcation equation. The bifurcation behaviors of the nonlinear VIS for two cases, quadratic nonlinear stiffness with the primary resonance and 1:2 internal resonance, and quadratic, cubic nonlinear stiffness with the primary resonance and 2:1 internal resonance, were considered. According to singularity theory, the universal unfoldings of $Z_2$-codimension 2 and codimension 4 were obtained respectively, and the transition sets and bifurcation diagrams were plotted in unfolding parameters. The results of this work may be used to analyze the bifurcation problem in engineering with considering the constrained conditions of the physical parameters.

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