The limiting spectral distribution in terms of spectral density
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Abstract

For a large class of symmetric random matrices with correlated entries, selected from stationary random fields of centered and square integrable variables, we show that the limiting distribution of eigenvalue counting measure always exists and we describe it via an equation satisfied by its Stieltjes transform. No rate of convergence to zero of correlations is imposed, therefore the process is allowed to have long memory. In particular, if the symmetrized matrices are constructed from stationary Gaussian random fields which have spectral density, the result of this paper gives a complete solution to the limiting eigenvalue distribution. More generally, for matrices whose entries are functions of independent identically distributed random variables the result also holds.

1 Introduction

This paper is a contribution to the limiting spectral distribution for symmetric matrices with correlated entries. Among the first results in this direction are papers by Khorunzhy and Pastur [11], Boutet de Monvel and Khorunzhy [5], who treated Gaussian random fields with absolutely summable covariances. Khorunzhy [12] considered matrices with correlated entries imposing rates of convergence on some mixing coefficients, without assuming the variables are Gaussian. On the other hand, there is interest in studying linear filters of independent random variables as entries of a matrix. Anderson and Zeitouni [1] considered symmetric matrices with entries that are linear processes of finite range of independent random variables. In all the papers mentioned above the correlation between variables are diminishing with time at certain polynomial rates. Such a dependence is considered of weak type, since distant variables have weak interactions.
Very recently, results in several papers indicate that the weak dependence is not needed for the existence of the limiting spectral distribution, and much milder regularity conditions, or ergodic type, are responsible for this line of problems. More precisely, Chakrabarty et al. [7] considered symmetrized random matrices selected from a large class of stationary Gaussian random fields and argued that the limiting spectral density always exists.

On the other hand, in two recent papers Banna et al. [4], and Merlevède and Peligrad [14] showed that this type of general result is not restricted to Gaussian fields. In [4] was studied symmetric random matrices whose entries are functions of zero mean square integrable independent and identically distributed (i.i.d.) real-valued random variables. Such kind of processes provide a general framework for stationary and ergodic random fields. For this case, by using the substitution method, in [4], the study of the empirical eigenvalue distribution was reduced to the same problem for a matrix with Gaussian entries and the same covariance structure, without any other additional assumption. This universality result was combined with a known result for the Gaussian case in [11] to extend it to stationary random fields. However, because the result for the Gaussian case given in [11] applies only to some weakly dependent random fields, the characterization of the limit in [11] is restricted to the case when covariances are absolutely summable.

In this paper we obtain a characterization of the limiting empirical spectral distribution for symmetric matrices with entries selected from a stationary Gaussian field under the sole condition that its spectral density exists. This result, combined with the universality result in Banna et al. [4] shows that for random matrices with entries functions of i.i.d., the limiting empirical spectral distribution exists and is characterized via an equation satisfied by its Stieltjes transform which involves the spectral density of the field. Applications of this result to linear and nonlinear filters of a stationary random field are pointed out.

A general characterization of the limiting spectral distribution in terms of the spectral density is also expected to hold for the covariance (Gram) matrices whose entries are selected from stationary random fields. This fact is suggested by results in papers by Boutet de Monvel et al. [6], Hachem et al. [10], Bai and Zhou [9], Yao [19], Banna et al. [4], Merlevède and Peligrad [14], among many others. The study of covariance matrices is beyond the scope of this paper.

Our paper is organized as follows. In Section 2, we give definitions and state the main results. Section 3 contains the proofs. Section 4 is dedicated to examples.

2 Results

Here are some notations used throughout the paper. For a matrix $A$, we denote by $\text{Tr}(A)$ its trace. We shall use the notation $\|X\|_r$ for the $L_r$-norm ($r \geq 1$) of a real valued random variable $X$, namely...
\[ \|X\|_r = \mathbb{E}(|X|^r). \] For a set \( B \) we denote by \( B' \) its complement. For the convergence in distribution we use the notation \( \Rightarrow \). The Lebesgue measure on \( \mathbb{R} \) will be denoted by \( \lambda \). The set of complex numbers with positive imaginary part is denoted by \( \mathbb{C}^+ \).

For any square matrix \( A_n \) of order \( n \) with real eigenvalues \( \lambda_1 \leq \cdots \leq \lambda_n \), its empirical spectral measure and its empirical spectral distribution function are respectively defined by

\[ \nu_{A_n} = \frac{1}{n} \sum_{k=1}^{n} \delta_{\lambda_k} \quad \text{and} \quad F_{A_n}^n(x) = \frac{1}{n} \sum_{k=1}^{n} 1\{\lambda_k \leq x\}. \]

The Stieltjes transform of \( F_{A_n}^n \) is given by

\[ S_{A_n}(z) = \int \frac{1}{x-z} dF_{A_n}^n(x) = \frac{1}{n} \text{Tr}(A_n - zI_n)^{-1}, \]

where \( z \in \mathbb{C}^+ \), and \( I_n \) is the identity matrix of order \( n \). It is well-known that the Stieltjes transform determines the measure.

The Lévy distance between two distribution functions \( F \) and \( G \) is defined by

\[ L(F, G) = \inf \{ \varepsilon > 0 : F(x - \varepsilon) - \varepsilon \leq G(x) \leq F(x + \varepsilon) + \varepsilon \}. \]

We mention that a sequence of distribution functions \( F_n(x) \) converges to a distribution function \( F \) at all continuity points \( x \) of \( F \) if and only if \( L(F_n, G) \to 0 \).

Let \((X_{k,\ell})_{(k,\ell)\in\mathbb{Z}^2}\) be an array of real-valued random variables, and consider its associated symmetric random matrix \( X_n \) of order \( n \) defined by

\[ (X_n)_{\ell,j} = X_{\ell,j} \quad \text{if} \quad 1 \leq j \leq \ell \leq n \quad \text{and} \quad (X_n)_{\ell,j} = X_{j,\ell} \quad \text{if} \quad 1 \leq \ell < j \leq n. \] \tag{1}

Then, define the symmetric matrix of order \( n \) by

\[ \mathcal{X}_n := \frac{1}{n^{1/2}} X_n. \] \tag{2}

The aim of this paper is to study the limiting empirical spectral distribution function of the symmetric matrix \( \mathcal{X}_n \) defined by (2) when the random field \((X_{k,\ell})_{(k,\ell)\in\mathbb{Z}^2}\) is strictly stationary given by the following dependence structure: for any \((k,\ell)\in\mathbb{Z}^2\),

\[ X_{k,\ell} = g(\xi_{k-u,\ell-v} : (u, v) \in \mathbb{Z}^2), \] \tag{3}

where \((\xi_{i,j})_{(i,j)\in\mathbb{Z}^2}\) is an array of i.i.d. real-valued random variables given on a common probability space \((\Omega, \mathcal{K}, \mathbb{P})\), and \( g \) is a measurable function from \( \mathbb{R}^{2^2} \) to \( \mathbb{R} \) such that \( \mathbb{E}(X_{0,0}) = 0 \) and \( \|X_{0,0}\|_2 < \infty \).

A representation as in (3) includes as special cases, linear as well as many widely used nonlinear random fields models.

We are interested to establish the weak convergence, on a set of probability one, of \( \nu_{\mathcal{X}_n} \) to a nonrandom probability measure. This means that

\[ \mathbb{P}(L(F_{\mathcal{X}_n}^n(\omega), F) \to 0) = 1. \] \tag{4}
In the sequel we shall denote this convergence as $F^X_n(\omega) \Rightarrow F$ a.s.

In this paper, for the model defined by (2), we shall study the limit of the type (4) and specify the limiting distribution $F(t)$ by giving an equation satisfied by its Stieltjes transform.

Relevant to our result is the notion of spectral density for a weakly stationary field. In the context of weakly stationary random fields it is known that, according to Herglotz representation, there exists a unique measure on $[0, 2\pi]^2 = [0, 2\pi] \times [0, 2\pi]$, such that
\[
\gamma_{k,\ell} = \text{cov}(X_{0,0}, X_{k,\ell}) = \int_{[0,2\pi]^2} e^{i(ku+\ell v)} F(du, dv), \quad \text{for all } k, \ell \in \mathbb{Z}.
\]

If $F$ is absolutely continuous with respect to the Lebesgue measure $\lambda \times \lambda$ on $[0, 2\pi]^2$ then, the Radon-Nikodym derivative $f$ of $F$ with respect to the Lebesgue measure satisfies
\[
\gamma_{k,\ell} = \int_{[0,2\pi]^2} e^{i(ku+\ell v)} f(u, v) du dv, \quad \text{for all } k, \ell \in \mathbb{Z}.
\]
The function $f(u, v)$ is called spectral density.

It should be noted that, by a recent result in Lifshitz and Peligrad [13] for random fields defined by (3), the spectral density exists. It is convenient to scale $f(u, v)$ and we define
\[
b(x, y) = (2\pi)^2 f(2\pi x, 2\pi y). \tag{5}
\]

One of the main results of this paper is the following theorem, which points out the relationship between the limiting spectral distribution and the spectral density.

**Theorem 1** Let $(X_{k,\ell})_{(k,\ell) \in \mathbb{Z}^2}$ be a real-valued random field given by (3) with a spectral density $f(x, y)$. Define its scaling $b(x, y)$ by (5) and assume that $\gamma_{k,\ell} = \gamma_{\ell,k}$ for all $(k, \ell) \in \mathbb{Z}^2$. Then, the convergence (4) holds, namely $F^X_n(\omega) \Rightarrow F$ a.s., where $F$ is a nonrandom limiting distribution function whose Stieltjes transform $S(z)$ is uniquely defined by the relations: for every $z \in \mathbb{C}^+$
\[
S(z) = \int_0^1 g(x, z) dx, \tag{6}
\]
where, for any $x \in [0, 1]$, $g(x, z)$ is analytic in $z \in \mathbb{C}^+$.

There is $J \subset [0, 1]$, with $\lambda(J) = 1$ such that, for any $x \in J$ and $z \in \mathbb{C}^+$, $g(x, z)$ satisfies the equation
\[
g(x, z) = - \left( z + \int_0^1 g(y, z) b(x, y) dy \right)^{-1}. \tag{7}
\]
Moreover, for any $x \in [0, 1]$ and $z \in \mathbb{C}^+$
\[
\text{Im} g(x, z) > 0, \quad |g(x, z)| \leq (\text{Im} z)^{-1}. \tag{8}
\]

This theorem is related to Theorem 3 in [4]. The main difference is that Theorem 3 in [4] is obtained under the condition that the covariances are absolutely summable. This summability condition implies
that the spectral density is continuous and bounded, case known under the name of short memory. By removing this condition, our Theorem 1 can be applied to any symmetric random field defined by (3), therefore the memory is not restricted to short memory. The other difference is that (7) holds only on a set of Lebesgue 1 which does not depend on $z$. But keep in mind that the function given by (7) is integrated to give (6) so $S(z)$ is well determined. Also, this set $J$ can be used to obtain a version of the spectral density such that equation (7) holds for all $x \in [0,1]$.

By using a closely related approach we used to prove Theorem 1, we can easily study another symmetrized model based on the random field defined by (3). Instead of $X_n$ defined by (1) we can consider the symmetrized model

$$(X'_n)_{k,j} = X_{k,j} + X_{j,k} \text{ if } 1 \leq j, k \leq n \text{ and } X'_n := \frac{1}{n^{1/2}}X'_n. \tag{9}$$

For this model, we shall formulate the following result:

**Theorem 2** Let $(X_{k,\ell})_{(k,\ell)\in\mathbb{Z}^2}$ be a real-valued random field given by (3) with spectral density $f(x, y)$. Then, the convergence (7) holds for $F^{X_n} (\omega)$ to a nonrandom distribution function $F$, whose Stieltjes transform, $S(z), z \in \mathbb{C}^+$, is uniquely defined by the relations (6), (7), (8), with

$$b(x, y) = (2\pi)^2(f(2\pi x, 2\pi y) + f(2\pi y, 2\pi x)).$$

There are certainly connections between the models given by random matrices (2) and (9), as argued in Lemma 19 in [4]. However, Theorem 2 does not follow directly from Theorem 1 since the random field $(X_{k,j} + X_{j,k})_{k,j}$ is no longer stationary. Their proofs are similar.

It is worth mentioning that a Gaussian random field which has spectral density is a function of i.i.d., so both Theorem 1 and Theorem 2 apply to this situation.

We shall compare Theorem 2 to Theorem 2 in Khorunzhy and Pastur [11] (see also in Theorem 17.2.1. in [15]) concerning Gaussian random fields. Indeed, if we define

$$B(j, k) = \text{cov}(X_{0,0}, X_{j,k}) + \text{cov}(X_{0,0}, X_{k,j}),$$

then, a straightforward computation shows that

$$\text{cov}(X_{j,k} + X_{k,j}, X_{u,v} + X_{v,u}) = B(j - u, k - v) + B(j - v, k - u).$$

Therefore, condition (17.2.3) of Theorem 17.2.1. in [15] is satisfied. Note that $B(j, k) = B(k, j) = B(-j, -k)$. As a matter of fact, as noticed in [4], by a careful analysis of the proof, the condition $B(j, k) = B(j, -k)$ included in (17.2.3) in [15] can be omitted in the stationary case we consider. Therefore, for the model treated in Theorem 2, we have the covariance structure required by Theorem 2 in Khorunzhy and Pastur [11]. We can see from these comments that, in the context of the symmetrized...
model ([20]), our Theorem 2 extends Theorem 2 by Khorunzhy and Pastur [11] (given also in Theorem 17.2.1 in [15]) in two directions. The result of Khorunzhy and Pastur [11] concerning Gaussian random fields is given under the condition that the covariances are absolutely summable, implying that the spectral density exists and is continuous and bounded. We removed this condition in Theorem 2 and for the Gaussian case we can assume only that the spectral density exists in order to obtain the characterization of the limit. Our result is also true for random fields which are not necessarily Gaussian, but are functions of i.i.d.

**Remark 3** If the spectral density of \((X_{k,\ell})_{(k,\ell)\in\mathbb{Z}^2}\) has the structure \(f(x, y) = u(x)u(y)\) for some real valued positive function \(u\), then the conclusion of Theorem 2 can be given in the following simplified form: the convergence (4) holds where \(F\) is a nonrandom distribution function whose Stieltjes transform \(S(z)\), \(z \in \mathbb{C}^+\) is given by the relation

\[
S(z) = -\frac{1}{z}(1 + v^2),
\]

(10)

where \(b(x) = u(2\pi x)\) and \(v(z)\) is solution to the equation

\[
v(z) = -\int_0^1 \frac{b(y)dy}{z + b(y)v(z)}, \quad z \in \mathbb{C}^+.
\]

(11)

with \(v(z)\) analytic, \(\text{Im } v(z) > 0\), and \(|v(z)| \leq (\text{Im } z)^{-1}||X_{0,0}||_2\).

In this form, we can see that one can obtain explicit polynomial equations for \(S(z)\) when \(u(x)\) is a positive step function. In particular, if \((X_{k,\ell})_{(k,\ell)\in\mathbb{Z}^2}\) is an array of i.i.d. random variables with zero mean and variance \(\sigma^2\), then \(u\) is constant and \(S(z)\) given in Remark 3 satisfies the equation

\[
\sigma^2 S^2 + S + z^{-1} = 0.
\]

Specifying the square root of a complex number as the one with positive imaginary part, the solution with positive imaginary part is \(S = -(z - \sqrt{z^2 - 4\sigma^2})(2\sigma^2)^{-1}\) which is the well-known Stieltjes transform of the semicircle law obtained by Wiener [18] (see Lemma 2.11. in [2]).

In the last section, we are going to provide two examples of random fields, one linear and one nonlinear, where our results apply. It is remarkable that for both these examples the only condition required is that the random fields are well defined in \(L_2\).

### 3 Proofs

Before proving the results we shall give some results and facts which will be used in their proofs.

#### 3.1 Preliminary considerations

1. **Continuity results for Stieltjes transform of limiting spectral distribution.**
We start by proving a continuity result related to the equation satisfied by the Fourier transform in the theorems. As mentioned by Khorunzhy and Pastur [11], it should be noted that these equations appear for the first time in Wegner [17], in the context of studying \( n \)-component generalization of discrete Schrödinger equation with a random \( n \times n \) matrix. More precisely, equations (17.2.28) and (17.2.29) in [15] are comparable to (3.19) and (3.14) respectively in [17]. Therefore, next proposition has interest in itself.

**Proposition 4** Assume that \( b_m(x, y) \) is a sequence of real positive and bounded functions, which is convergent in \( L_1([0, 1]^2, \mathbb{B}^2, \lambda^2) \) to \( b(x, y) \). For any \( z \in \mathbb{C}^+ \) define

\[
S_m(z) = \int_0^1 h_m(x, z)dx, \tag{12}
\]

where \( h_m(x, z) \) is a solution to the equation

\[
h_m(x, z) = - \left( z + \int_0^1 h_m(y, z)b_m(x, y)dy \right)^{-1}, \tag{13}
\]

with \( h_m(x, z) \) analytic in \( z \in \mathbb{C}^+ \) and for any \( x \in [0, 1], \)

\[
\text{Im} h_m(x, z) > 0, \quad |h_m(x, z)| \leq (\text{Im} z)^{-1}. \tag{14}
\]

Then, for any \( z \in \mathbb{C}^+ \) we have \( S_m(z) \to S(z) \), where \( S(z) \) is a Stieltjes transform of a probability measure uniquely determined by the equations (6), (7) and (8 ).

**Proof.** It is convenient to represent relation (13) in an equivalent form, by introducing the following transformation. For all \( z \in \mathbb{C}^+ \) and \( x \in [0, 1] \) set

\[
h_m(x, z) = -(z + \pi_m(x, z))^{-1}, \tag{15}
\]

which is well defined since \( \text{Im} h_m(x, z) > 0 \). Equation (13) becomes

\[
\pi_m(x, z) = - \int_0^1 \frac{b_m(x, s)}{z + \pi_m(s, z)}ds.
\]

Also, because

\[
\pi_m(x, z) = \int_0^1 b_m(x, s)h_m(s, z)ds
\]

and by the fact that \( \text{Im} h_m(x, z) > 0 \) and \( b_m \) is positive, it follows that for all \( x \in [0, 1] \) and \( z \in \mathbb{C}^+ \)

\[
\text{Im} \pi_m(x, z) \geq 0. \tag{16}
\]

Finally, set \( S_m(z) = \int_0^1 h_m(x, z)dx \).

Let us show that \( \pi_m(x, z) \) is pointwise convergent for all \( x \in [0, 1] \) and \( z \in \mathbb{C}^+ \). Obviously

\[
\pi_n(x, z) - \pi_m(x, z) = \int_0^1 \left[ \frac{b_m(x, s)}{z + \pi_m(s, z)} - \frac{b_n(x, s)}{z + \pi_n(s, z)} \right]ds = \int_0^1 \left[ \frac{b_m(x, s) - b_n(x, s)}{z + \pi_m(s, z)} + b_n(x, s) \frac{\pi_n(s, z) - \pi_m(s, z)}{(z + \pi_m(s, z))(z + \pi_n(s, z))} \right]ds.
\]
Now, by (15) and (14) it follows that

\[ |z + \pi_m(x, z)| = |h_m^{-1}(x, z)| \geq \text{Im } z. \tag{17} \]

Also, since \( b_m \)'s are bounded it follows from (13) that we can find positive constants \( K_m \), such that

\[ |\pi_m(x, z)| \leq \int_0^1 \frac{b_m(x, s)}{|z + \pi_m(s, z)|} ds \leq \frac{1}{\text{Im } z} K_m. \]

So, for \( z \in \mathbb{C}^+ \), it follows by the above considerations that

\[
\sup_{x \in [0,1]} |\pi_n(x, z) - \pi_m(x, z)| \leq \frac{1}{\text{Im } z} \int_{[0,1]^2} |b_m(x, s) - b_n(x, s)| ds + \frac{1}{(\text{Im } z)^2} \sup_{x \in [0,1]} |\pi_n(x, z) - \pi_m(x, z)| \int_{[0,1]^2} b_n(x, s) ds < \infty. \tag{18}
\]

By integrating with \( x \) on \( [0, 1] \) we obtain

\[
\sup_{x \in [0,1]} |\pi_n(x, z) - \pi_m(x, z)| \leq \frac{1}{\text{Im } z} \int_{[0,1]} |b_m(x, s) - b_n(x, s)| ds dx + \frac{1}{(\text{Im } z)^2} \sup_{x \in [0,1]} |\pi_n(x, z) - \pi_m(x, z)| \int_{[0,1]} b_n(x, s) ds dx. \]

Now, since \( b_n \to b \) in \( L_1[0,1] \) we have

\[
\lim_{n \to \infty} \int_{[0,1]^2} b_n(x, s) ds dx = \int_{[0,1]^2} b(x, s) ds dx = B. \tag{19}
\]

Define the domain \( D \) by

\[ D = \{ z \in \mathbb{C}^+ : B < \text{Im } z \}. \tag{20} \]

For \( z \in D \), by (18), (19) and simple algebra, we have that

\[
\lim_{m \to \infty} \sup_{x \in [0,1]} |\pi_n(x, z) - \pi_m(x, z)| \leq \frac{\text{Im } z}{(\text{Im } z)^2 - B} \lim_{m \to \infty} \int_{[0,1]^2} |b_m(x, s) - b_n(x, s)| ds dx.
\]

As a consequence, for \( z \in D \) we obtain that

\[
\lim_{m \to \infty} |\pi_n(x, z) - \pi_m(x, z)| = 0 \text{ uniformly in } x,
\]

and by the Lebesgue dominated convergence theorem we also have

\[
\lim_{m \to \infty} \int_0^1 |\pi_n(x, z) - \pi_m(x, z)| dx = 0.
\]

It follows that for \( z \) in \( D \) and \( x \in [0,1] \), we have both \( \pi_m(x, z) \) is convergent pointwise and in \( L_1[0,1] \) to a function we shall denote by

\[ \lim_{m \to \infty} \pi_m(x, z) =: \pi(x, z). \]
By (15) we know that $h_m(x, z) = -(z + \pi(x, z))^{-1}$, and by (14), $|h_m(x, z)| \leq 1 / \text{Im}(z)$. Therefore for $z$ in $D$

$$\lim_{m \to \infty} h_m(x, z) = -(z + \pi(x, z))^{-1} =: h(x, z),$$

(21)

pointwise and in $L_1[0, 1]$. Since for all $x \in [0, 1]$, $(h_m(x, z))_{m \geq 1}$ are analytic and uniformly bounded functions on all compacts of $\mathbb{C}^+$, by applying Lemma 3 in [8], the limit $h(x, z)$ is also analytic on $D$ for all $x$.

We shall remove now the restriction about $z \in D$ and we shall argue that actually, for all $x$, $h_m(x, z)$ is convergent on $\mathbb{C}^+$ to an analytic function $g(x, z)$ which coincides with $h(x, z)$ on $D$. Since $h_m(x, z)$ is a sequence of analytic functions on $\mathbb{C}^+$, uniformly bounded on compacts of $\mathbb{C}^+$, by a classical result (see Lemma 3 in [8] and references therein) every subsequence of $h_m(x, z)$ has a subsequence that converges uniformly on compact subsets of $\mathbb{C}^+$ to an analytic function on $\mathbb{C}^+$. Let us consider now a convergent subsequence $h_m'(x, z) \to g_1(x, z)$ with $g_1(x, z)$ analytic on $\mathbb{C}^+$. Clearly on $D$ we have $g_1(x, z) = h(x, z)$. Now if we have another convergent subsequence $h_m''(x, z) \to g_2(x, z)$ with $g_2(x, z)$ analytic on $\mathbb{C}^+$, we have $g_2(x, z) = h(x, z)$ on $D$. Therefore, both $g_1(x, z)$ and $g_2(x, z)$ are analytic on $\mathbb{C}^+$ and $g_1(x, z) - g_2(x, z) = 0$ on $D$. It is a well-known fact that two analytic functions on a connected domain, which coincide on a subset of the domain with an accumulation point, coincide everywhere on the domain. Therefore we have that $g_1(x, z) = g_2(x, z)$ on $\mathbb{C}^+$. As a consequence, for each $x$ fixed, $h_m(x, z) \to g(x, z)$ for all $z \in \mathbb{C}^+$ where $g(x, z)$ is analytic on $\mathbb{C}^+$ and coincides with $h(x, z)$ on $D$.

We shall pass now to the limit in the equations (12), (13) and (14). Since $h_m(x, z)$ are bounded, by passing to the limit in (12), we obtain

$$S(z) = \int_0^1 g(x, z)dx$$

for all $z \in \mathbb{C}^+$, so (6) holds.

Also, by passing to the limit in (14), we immediately obtain

$$|g(x, z)| \leq \text{Im} z^{-1}. \quad (22)$$

By passing to the limit in equation (13) we have

$$g(x, z) = -\left(z + \lim_{m \to \infty} \int_0^1 h_m(y, z)b_m(x, y)dy\right)^{-1}. \quad (23)$$

Then, since $\text{Im} h_m(x, z) > 0$ and $b_m(x, y)$ is positive, we deduce that $\text{Im} g(x, z) > 0$ for all $x \in [0, 1]$ and $z \in \mathbb{C}^+$. It follows that (8) holds.

We shall show now that equation (7) is satisfied for all $x$ in a set of Lebesgue measure 1 which does not depend on $z$. With this goal in mind we shall start from relation (23). Since it gives that for any $x \in [0, 1]$ and $z \in \mathbb{C}^+$ the limit $\lim_{m \to \infty} \int_0^1 h_m(y, z)b_m(x, y)dy$ exists, in order to verify (7) it is enough to find a subsequence $(n')$ such that for all $z \in \mathbb{C}^+$ and all $x \in J \subset [0, 1]$ with $\lambda(J) = 1$, 

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\[
\lim_{m' \to \infty} \int_0^1 h_{m'}(y, z) b_{m'}(x, y) dy = \int_0^1 g(y, z) b(x, y) dy. \tag{24}
\]

We have
\[
\int_0^1 |h_m(y, z) b_m(x, y) - g(y, z) b(x, y)| dy \leq \int_0^1 |h_m(y, z) - g(y, z)| b(x, y) dy +
\]
\[
\int_0^1 |h_m(y, z)| |b_m(x, y) - b(x, y)| dy =: I_m(x, z) + II_m(x, z).
\]

Note that for all \(x \in [0, 1]\) and \(z \in \mathbb{C}^+\)

\[
|h_m(y, z) - g(y, z)| b(x, y) \leq \frac{2}{\text{Im } z} b(x, y)
\]

and for \(\lambda\)-almost all \(x \in [0, 1]\), \(b(x, y)\) is integrable. Therefore, by the Lebesgue dominated convergence theorem, \(I_m(x, z) \to 0\) for \(x\) in a subset of \([0, 1]\) of Lebesgue measure 1 and any \(z \in \mathbb{C}^+\).

By (14),

\[
II_m(x, z) \leq \frac{1}{\text{Im } z} \int_0^1 |b_m(x, y) - b(x, y)| dy.
\]

Since \(b_m(x, y) \to b(x, y)\) in \(L_1[0, 1]^2\), by Fubini Theorem, \(\int_0^1 |b_m(x, y) - b(x, y)| dy\) converges to 0 in \(L_1[0, 1]\); therefore there is a subsequence \((m')\) which does not depend on \(z\) such that \(\int_0^1 |b_{m'}(x, y) - b(x, y)| dy \to 0\) for \(\lambda\)-almost all \(x \in [0, 1]\). These considerations show that (24) holds and therefore equation (7) is satisfied for all \(z \in \mathbb{C}^+\) and \(x \in J \subset [0, 1]\) with \(\lambda(J) = 1\).

We shall verify now that \(S\) is a Stieltjes transform of a probability measure. Note that since \(S\) is a pointwise limit in \(\mathbb{C}^+\) of Stieltjes transforms, according to Theorem 1 in Geronimo and Hill [8], we have to show that

\[
\lim_{u \to \infty} iuS(iu) = -1. \tag{25}
\]

A simple computation shows that, by the definition of \(g(x, iu)\), we have for \(x \in J\)

\[
iu g(x, iu) = - \left(1 + \frac{1}{iu} \int_0^1 g(y, iu) b(x, y) dy\right)^{-1}.
\]

Note that, by (22),

\[
\left|\frac{1}{iu} g(x, iu)\right| \leq \frac{1}{u^2} \to 0 \text{ as } u \to \infty.
\]

Therefore we can conclude that for all \(x \in J\)

\[
iu g(x, iu) \to -1.
\]

Now, again by (22) we have \(|iu g(x, iu)| \leq u/u = 1\). We can apply next the Lebesgue dominated convergence theorem and obtain

\[
iu S(iu) = \int_0^1 iyh(x, iu) dx \to -1.
\]
So, (25) follows, showing that $S$ is indeed a Stieltjes transform of a probability measure with distribution $F$.

It is easy to see that $S(z)$ is uniquely determined by the relations (6)-(8). It is convenient to work with the equivalent form of equation (7), namely

$$h(x, z) = -\left(\frac{x}{x + \pi(x, z)}\right) - 1.$$  

If we have two functions $\pi_1(x, z)$ and $\pi_2(x, z)$, both analytic in $z$, we shall write equation (26) for $\pi_1(x, z)$ and $\pi_2(x, z)$, and by similar manipulations done at the beginning of the proof for $\pi_m(x, z)$ and $\pi_n(x, z)$, we get

$$(1 - B(\text{Im } z)^2) \sup_{x \in J} |\pi_1(x, z) - \pi_2(x, z)| = 0 \quad \text{for } x \in J.$$  

So, for all $x \in J$, $\pi_1(x, z) = \pi_2(x, z)$ and therefore $h_1(x, z) = -(z + \pi_1(x, z))^{-1} = -(z + \pi_2(x, z))^{-1} = h_2(x, z)$. The uniqueness follows after we integrate $h_1(x, z)$ and $h_2(x, z)$ with respect to $x$. The proof of Proposition 4 is now complete.

2. Facts about universality results for limiting spectral distribution

Proposition A below, proved in [4], shows a universality scheme for the random matrix $X_n$ when each $X_{k,\ell}$ is a function of i.i.d. random variables defined by (3). Next, we shall introduce next a Gaussian random field with the same covariance structure.

Let $(G_{k,\ell})_{(k,\ell) \in \mathbb{Z}^2}$ be a real-valued centered Gaussian random field, with covariance function given by

$$\mathbb{E}(G_{k,\ell}G_{u,v}) = \mathbb{E}(X_{k,\ell}X_{u,v}) \text{ for any } (k, \ell) \text{ and } (u, v) \text{ in } \mathbb{Z}^2.$$  

(27)

Let $G_n$ be the symmetric random matrix defined by

$$(G_n)_{\ell,j} = G_{\ell,j} \text{ if } 1 \leq j \leq \ell \leq n \text{ and } (G_n)_{\ell,j} = G_{j,\ell} \text{ if } 1 \leq \ell < j \leq n.$$  

(28)

Denote

$$G_n = \frac{1}{n^{1/2}}G_n.$$  

(29)

The following is Proposition 1 in [4] which shows that the study of the empirical distribution function of a class of processes which are functions of i.i.d. random variables can be reduced to the study of a matrix with Gaussian entries.

**Proposition A.** (Banna-Merlevede-Peligrad) Define $(X_{\ell,j})$ by (3), the centered Gaussian random field $(G_{k,\ell})$ satisfying (27), and the symmetric matrices $X_n$ and $G_n$ by (2) and (29) respectively. Then, for any $z \in C^+$,

$$\lim_{n \to \infty} \left|S_{X_n}(z) - \mathbb{E}\left(S_{G_n}(z)\right)\right| = 0 \text{ almost surely.}$$
The following corollary stated in [4], is a direct consequence of Proposition A together with Theorem B.9 in Bai-Silverstein [2]:

**Corollary B.** Assume that \( X_n \) and \( G_n \) are as in Proposition A. Furthermore, assume that there exists a nonrandom distribution function \( F \) such that

\[
E(F^{G_n}(t)) \to F(t) \quad \text{for all continuity points } t \in \mathbb{R} \text{ of } F.
\]

Then (30) holds.

3. Facts about stationary Gaussian fields with spectral density

Now we mention several facts about stationary Gaussian random fields with spectral density \( f(x, y) \). These facts are also used in [7] and, for the case of Gaussian sequences, explained in Ch. 6, Section 6.6. in Varadhan [16].

A centered Gaussian field \((G_{k,\ell})\) has a spectral density \( f(x, y) \) if and only if \((G_{k,\ell})\) is distributed as

\[
G_{k,\ell} = \sum_{(u,v) \in \mathbb{Z}^2} a_{u,v} \xi_{k-u,\ell-v},
\]

with \((\xi_{k,\ell})\) a Gaussian field of i.i.d. random variables centered and square integrable with variance 1 and

\[
\sum_{(u,v) \in \mathbb{Z}^2} a_{u,v}^2 < \infty.
\]

It is known that

\[
f^{1/2}(x, y) = \frac{1}{2\pi} \left| \sum_{(u,v) \in \mathbb{Z}^2} a_{u,v} e^{-i(xu+yv)} \right|.
\]

Denote

\[
G^{m}_{k,\ell} = \sum_{-m \leq u,v \leq m} a_{u,v} \xi_{k-u,\ell-v}.
\]

Let \( f_m \) be the spectral density of \( G^{m}_{k,\ell} \). Then

\[
f^{1/2}_m(x, y) = \frac{1}{2\pi} \left| \sum_{-m \leq u,v \leq m} a_{u,v} e^{-i(xu+yv)} \right|.
\]

Let us show that \( f^{1/2}_m(x, y) \) converges in \( L^2([0,2\pi]^2) \) to \( f^{1/2}(x, y) \). Indeed, simple estimates show that

\[
\int_{[0,2\pi]^2} |f^{1/2}_m(x, y) - f^{1/2}(x, y)|^2 \, dx \, dy \\
\leq \frac{1}{(2\pi)^2} \int_{[0,2\pi]^2} \left| \sum_{-m \leq u,v \leq m} a_{u,v} e^{-i(xu+yv)} - \sum_{(u,v) \in \mathbb{Z}^2} a_{u,v} e^{-i(xu+yv)} \right|^2 \, dx \, dy \\
= \frac{1}{(2\pi)^2} \sum_{(-m \leq u,v \leq m)} a_{u,v}^2 \to 0 \quad \text{as } m \to \infty.
\]
This convergence implies that $f_m(x, y)$ converges to $f(x, y)$ in $L_1([0, 2\pi]^2)$, since, by the Cauchy-Schwarz inequality,

$$\left( \int_{[0, 2\pi]^2} |f_m(x, y) - f(x, y)| \, dx \, dy \right)^2 \leq \int_{[0, 2\pi]^2} |f_m^{1/2}(x, y) - f^{1/2}(x, y)|^2 \, dx \, dy \int_{[0, 2\pi]^2} (f_m^{1/2}(x, y) + f^{1/2}(x, y))^2 \, dx \, dy$$

and

$$\lim_{m \to \infty} \mathbb{E}(G^m_{0,0})^2 = \sum_{u,v} a^2_{u,v} = \mathbb{E}(G^2_{0,0}).$$

### 3.2 Proof of Theorem 1

The proof has several steps. The idea of proof is that, according to Corollary B, it is enough to show that (30) holds for a Gaussian random field with the same spectral density. Then, we shall approximate, as above, the spectral density $f(x, y)$ of $(G_{k,\ell})$ by the spectral density $f_m(x, y)$ of $(G^m_{k,\ell})$ defined by (32). Since for $m$ fixed the sequence of matrices associated to $(G^m_{k,\ell})$ satisfies the conditions of Theorem 3 in [4], we know how to describe the Stieltjes transform of the nonrandom limiting empirical spectral distribution $F_m$ of $F^m_n(\omega)$. Then, by using Proposition 4 we show that $F_m \Rightarrow F$ as $m \to \infty$, where $F$ is a distribution function of a probability measure, and we shall specify the equations satisfied by $S$, the Stieltjes transform of $F$. Finally we show that $F^G_n(\omega) \Rightarrow F$ a.s.

#### Step 1. We analyze an associated finite range dependent random field and a sequence of random matrices.

Let us construct the random field $(G^m_{k,\ell})$ as in (32) and consider the random matrix

$$(G^m_n)_{\ell,j} = G^m_{\ell,j} \text{ if } 1 \leq j \leq \ell \leq n \text{ and } (G^m_n)_{\ell,j} = G^m_{j,\ell} \text{ if } 1 \leq \ell < j \leq n.$$  

Note that, by definition,

$$\text{cov}(G^m_{0,0}, G^m_{k,\ell}) = 0 \text{ for } k^2 + \ell^2 > 8m^2,$$

and therefore, denoting by $M_m = \{(k, \ell) \in \mathbb{Z}^2, k^2 + \ell^2 \leq 8m^2\}$,

$$\sum_{(k,\ell) \in \mathbb{Z}^2} |\text{cov}(G^m_{0,0}, G^m_{k,\ell})| = \sum_{(k,\ell) \in M_m} |\text{cov}(G^m_{0,0}, G^m_{k,\ell})| < \infty.$$  

By Theorem 3 in [4] we conclude that the convergence $F^G_n(\omega) \Rightarrow F_m$ a.s. holds for $F^m_n(\omega)$, where $F_m$ is a nonrandom distribution function whose Stieltjes transform $S_m(z), z \in \mathbb{C}^+$ is uniquely determined by the relations: (12)-(14), where

$$b_m(x, y) = \sum_{(j,k) \in M_n} \text{cov}(G^m_{0,0}, G^m_{j,k}) e^{-2\pi i(xj+yk)} = (2\pi)^2 f_m(2\pi x, 2\pi y).$$  

13
The last equality above follows, because \( \text{(34)} \) implies that \( f_m(x, y) \) is bounded and, by the inversion Fourier formula, the following representation for the spectral density holds:

\[
    f_m(x, y) = \frac{1}{(2\pi)^2} \sum_{(j,k) \in M_n} \text{cov}(G_{0,0}^m, G_{j,k}^m) e^{-i(xj + yk)}.
\]

**Step 2.** Here we show that \( F_m \Rightarrow F \) and the Stieltjes transform of \( F \) satisfies the equations of Theorem 1

Recall that \( f_m(x, y) \) converges to \( f(x, y) \) in \( \mathbb{L}_1([0,1]^2) \), as shown in \( \text{(35)} \). Taking into account definition \( \text{(5)} \), note that

\[
    B = \int_{[0,1]^2} b(x,s) dxds = \int_{[0,2\pi]^2} f(x,s) dxds = ||X_{0,0}||^2.
\]

With the notation \( b_m(x, y) \) from \( \text{(35)} \), note that, after a change of variables,

\[
    \int_{[0,1]^2} |b_m(x, y) - b(x, y)| dydx = \int_{[0,2\pi]^2} |f_m(x, y) - f(x, y)| dydx
\]

and so \( b_m(x, y) \) converges in \( \mathbb{L}_1([0,1]^2) \) to \( b(x, y) \).

Now we apply Proposition 4 and deduce that \( S_m(z) \to S(z) \) on \( \mathbb{C}^+ \) and \( S(z) \) is a non-random Stieltjes transform of a probability measure, uniquely determined by the equations \( \text{(6)}, \text{(7)} \) and \( \text{(8)} \). Furthermore, since \( S_m(z) \to S(z) \) on \( \mathbb{C}^+ \), there is a probability measure with distribution function \( F \), such that \( F_m \Rightarrow F \) (see for instance Theorem B.9 in Bai-Silverstein \( \text{[2]} \)).

**Step 3.** Now we show that actually \( F^G_n(\omega) \Rightarrow F \text{ a.s.} \)

This is equivalent to proving that for \( z \in \mathbb{C}^+ \) we have \( S_{\mathbb{C}^n}(z) \to S(z) \) almost surely. Since \( G_{i,j} \) are functions of \( i.i.d. \), by Proposition A, it is enough to show that for all \( z \in \mathbb{C}^+ \), \( \mathbb{E}(S_{\mathbb{C}^n}(z)) \to S(z) \).

By using the triangle inequality,

\[
    |\mathbb{E}(S_{\mathbb{C}^n}(z)) - S(z)| \leq |\mathbb{E}(S_{\mathbb{C}^n}(z)) - \mathbb{E}(S_{\mathbb{C}^m}(z))| + |\mathbb{E}(S_{\mathbb{C}^m}(z)) - S_m(z)| + |S_m(z) - S(z)|.
\]

By the Step 1 of the proof, \( F^G_n(\omega) \Rightarrow F \text{ a.s.} \). Therefore, for all \( z \in \mathbb{C}^+ \), we have \( S_{\mathbb{C}^m}(z) \to S_m(z) \) a.s. But since the Stieltjes transforms are bounded, we also have \( \mathbb{E}(S_{\mathbb{C}^m}(z)) \to S_m \). By the Step 2 of the proof, we know that \( S_m(z) \to S(z) \). In order to conclude that \( \mathbb{E}(S_{\mathbb{C}^n}(z)) \to S(z) \), it suffices to show only that

\[
    \lim_{m \to \infty} \limsup_n |\mathbb{E}(S_{\mathbb{C}^n}(z)) - \mathbb{E}(S_{\mathbb{C}^m}(z))| = 0 \text{ for all } z \in \mathbb{C}^+.
\]

(36)

By Lemma 2.1 in Götze et al. \( \text{[9]} \) we have

\[
    |S_{\mathbb{C}^n}(z) - S_{\mathbb{C}^m}(z)|^2 \leq \frac{1}{|\text{Im } z|^2 n} \text{Tr}(G_n(\omega) - G_n^m(\omega))^2.
\]
Clearly,
\[ \text{ETr}(G_n(\omega) - G_n^m(\omega))^2 \leq \frac{2}{n} \sum_{1 \leq j, \ell \leq n} \mathbb{E}(G_{\ell,j} - G_{\ell,j}^m)^2 = \]
\[ \leq \frac{2}{n} \sum_{1 \leq j, \ell \leq n} \mathbb{E}\left( \sum_{-m \leq u,v \leq m} a_{u,v} E_{u-j,v-\ell} \right)^2 \leq 2n \left( \sum_{-m \leq u,v \leq m} a_{u,v}^2 \right). \]

So, for all \( n \) and \( m \) we have
\[ \mathbb{E}|S_{G_n}(z) - S_{G_m}(z)|^2 \leq \frac{2}{n} \sum_{-m \leq u,v \leq m} a_{u,v}^2 \]
and therefore (36) follows by (31). The proof of Theorem 1 is complete.

**Proof of Theorem 2**

Its proof is a variation of the proof of Theorem 1. We shall mention only the differences. As before, we associate to \( (X_{k,\ell})_{(k,\ell) \in \mathbb{Z}^2} \) a real-valued centered Gaussian random field \( (G_{k,\ell})_{(k,\ell) \in \mathbb{Z}^2} \) satisfying (27). Let \( G_n' \) and \( G_n'' \) be the symmetric random matrix defined by
\[ (G_n')_{k,j} = G_{k,j} + G_{j,k} \quad \text{if } 1 \leq j, k \leq n; \quad G_n'' = \frac{1}{n^{1/2}} G_n'. \quad (37) \]

Then, we notice that Proposition A and Corollary B (i.e. Theorem 1 and Corollary 2 in [4]) also hold for \( X_n' \), defined by (9) and \( G_n' \), defined by (37), replacing \( X_n \) and \( G_n \). To see this, we have to follow the proof in [4] with the following change. We introduce the following random field: for any integers \( k \geq j \) and any positive integer \( \ell \) define
\[ Y_{k,j}^\ell = \mathbb{E}(X_{k,j}|F_{k,j}^\ell) + \mathbb{E}(X_{j,k}|F_{j,k}^\ell), \]
where \( F_{k,j}^\ell = \sigma(\xi_{u,v}; |u-k| \leq \ell, |v-j| \leq \ell) \). Note that the Euclidian distance \( d \) between the two points: \( (k, j) \) with \( k \geq j \) and \( (p, q) \) with \( p \geq q \), is the same as the distance between the sets \( \{(k, j) \cup (j, k)\} \) and \( \{(p, q) \cup (q, p)\} \). It follows that \( Y_{k,j}^\ell \) and \( Y_{p,q}^\ell \) are independent when \( d^2((k, j), (p, q)); k \geq j, p \geq q > 8\ell^2 \).
By using this remark, all the arguments in the proof of Proposition A (i.e. Theorem 1 in [4]) work unchanged.

Using this new version of Proposition A and Corollary B, as in the proof of Theorem 1 we reduce the problem to the study of Gaussian random matrices selected from stationary Gaussian random fields with the same spectral density.

We define \( G_{i,j}^m \) by \( (32) \) and
\[ (G_n^m)_{i,j} = G_{i,j}^m + G_{j,i}^m \quad \text{if } 1 \leq j, i \leq n; \quad G_n^m = \frac{1}{n^{1/2}} G_n^m. \]

A straightforward computation shows that
\[ \text{cov}(G_{j,k}^m + G_{k,j}^m, G_{u,v}^m + G_{v,u}^m) = B_m(j-u, k-v) + B_m(j-v, k-u), \]
where
\[ B_m(j, k) = \text{cov}(G_{0,0}^m, G_{j,k}^m) + \text{cov}(G_{0,0}^m, G_{k,j}^m). \]

Therefore condition (17.2.3) of Theorem 17.2.1. in [15] is satisfied. By Theorem 17.2.1. in [15] we conclude that the convergence \(F_{S_n} \Rightarrow F_m'\) a.s. holds, where \(F_m'\) is a nonrandom distribution function whose Stieltjes transform \(S_m(z), z \in \mathbb{C}^+\) is uniquely defined for \(z \in \mathbb{C}^+\) by the relations (12)-(14) with
\[ b_m(x, y) = (2\pi)^2(f_m(2\pi x, 2\pi y) + f_m(2\pi y, 2\pi x)). \]

From here on, the proof of Theorem 2 is identical to the proof of Theorem 1.

**Proof of Remark 3.**

By the conditions of this remark, note that \(b(x, y) = t(x)t(y)\), with \(t(x) = (2\pi u)(2\pi x)\), which together with (7) gives
\[ g(x, z) = -((z + t(x)v(z)))^{-1}, \tag{38} \]
where
\[ v(z) = \int_0^1 g(y, z)t(y)dy. \]

By multiplying (38) by \(t(x)\) and integrating with \(x\) we get the equation (11).

Now from equation (6) we have
\[ S(z) = -\int_0^1 \frac{dx}{z + t(x)v(z)}. \]

Multiplying equation (11) by \(v(z)\) and adding it to \(zS(z)\) we obtain \(v^2(z) + zS(z) = -1\), leading to (10). Next, by (8), we notice that \(v(z)\) is analytic, has strictly positive imaginary part and
\[ |v(z)| \leq \frac{1}{\text{Im} z} \int_0^1 t(y)dy = \frac{1}{\text{Im} z} \int_0^{2\pi} u(x)dx = \frac{1}{\text{Im} z} \|X_{0,0}\|_2. \]

### 4 Examples

#### 4.1 Linear processes

Let \((a_{k,\ell})_{(k,\ell) \in \mathbb{Z}^2}\) be a double indexed sequence of real numbers such that
\[ \sum_{(k,\ell) \in \mathbb{Z}^2} a_{k,\ell}^2 < \infty, \tag{39} \]
and let \((X_{u,v})_{(u,v) \in \mathbb{Z}^2}\) be the linear random field defined by: for any \((u, v) \in \mathbb{Z}^2\),
\[ X_{u,v} = \sum_{(k,\ell) \in \mathbb{Z}^2} a_{k,\ell}\xi_{k+u,\ell+v}, \tag{40} \]
where the variables \((\xi_{k,\ell})\) are i.i.d. centered and square integrable. Note that \(X_{u,v}\) is well defined in \(L_2\) if and only if (39) holds. The spectral density is defined by

\[
f(x, y) = \frac{1}{(2\pi)^2} \left| \sum_{(u,v) \in \mathbb{Z}^2} a_{u,v} e^{-i(xu + vy)} \right|^2.
\]

We can apply our results to the random matrices associated to the linear random field and obtain the following corollary, describing the limiting spectral distribution.

**Corollary 5** Assume that condition (39) is satisfied and \((X_{u,v})_{(u,v)\in \mathbb{Z}^2}\) is defined by (40). Then \(X_{u,v}\)'s are well defined in \(L_2\) and the conclusion of Theorem 2 holds. If \(a_{k,\ell} = a_{\ell,k}\) for any \((k, \ell)\in \mathbb{Z}^2\), then the conclusion of Theorem 1 holds. If \(a_{k,\ell} = a_k a_{\ell}\) for some real numbers \(a_k\) with \(\sum_{k\in \mathbb{Z}} a_k^2 < \infty\), then Remark 3 applies.

### 4.2 Volterra-type processes

Other classes of stationary random fields having the representation (3) are Volterra-type processes which play an important role in the nonlinear system theory. For any \(k = (k_1, k_2) \in \mathbb{Z}^2\), define the Volterra-type expansion as follows:

\[
X_k = \sum_{u,v \in \mathbb{Z}^2} b_{u,v} \xi_{k-u} \xi_{k-v},
\]

where \(b_{u,v}\) are real numbers satisfying

\[
b_{u,v} = 0 \text{ if } u = v, \quad \sum_{u,v \in \mathbb{Z}^2} b_{u,v}^2 < \infty,
\]

and \((\xi_k)_{k \in \mathbb{Z}^2}\) is an i.i.d. random field of centered and square integrable variables. Under the above conditions, the random field \(X_k\) exists, is stationary, centered and square integrable. By (13), the field has spectral density since it is a function of i.i.d. The covariance structure is given by: for any \(u = (u_1, u_2)\) and \(v = (v_1, v_2)\) in \(\mathbb{Z}^2\),

\[
\gamma_k = \|\xi_0\|^4_2 \sum_{u,v \in \mathbb{Z}^2} b_{u,v}(b_{u+k,v+k} + b_{v+k,u+k}) \text{ for any } k \in \mathbb{Z}^2.
\]

Therefore the following corollary holds:

**Corollary 6** Assume that condition (42) is satisfied and \((X_k)_{k \in \mathbb{Z}^2}\) is defined by (41). Then the conclusion of Theorem 3 holds for the model given by (4).

If we impose additional symmetry conditions to the coefficients \(b_{u,v}\) defining the Volterra random field (11), we can derive the limiting spectral distribution of its associated symmetric matrix \(X_n\) defined by (2).
If for any \( \mathbf{u} = (u_1, u_2) \) and \( \mathbf{v} = (v_1, v_2) \) in \( \mathbb{Z}^2 \), \( b_{\mathbf{u}, \mathbf{v}} = b_{u_1, v_1} b_{u_2, v_2} \), the conclusion of Theorem 1 holds.

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