Generalized Bose-Einstein phase transition in large-$m$ component spin glasses

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(Dated: October 29, 2018)

It is proposed to understand finite dimensional spin glasses using a $1/m$ expansion, where $m$ is the number of spin components. It is shown that this approach predicts a replica symmetric state in finite dimensions. The point about which the expansion is made, the infinite-$m$ limit, has been studied in the mean-field limit in detail and has a very unusual phase transition, rather similar to a Bose-Einstein phase transition but with $N^{2/5}$ macroscopically occupied low-lying states.

PACS numbers: 75.50.Lk, 05.50.+q

After almost three decades of research, the nature of the low temperature phase of finite dimensional spin glasses is not understood. The usual approach is to start from the exactly soluble Sherrington-Kirkpatrick (SK) mean field model and expand about it towards finite dimensions. This method resulted in the monumental mean field model and expand about it towards finite dimensions. This method resulted in the monumental SK limit, down to 6 dimensions, below which all known calculational tools break down and not much is known analytically. Yet there exists a mathematical proof that this picture cannot hold in any finite dimension. This proof is unfortunately non-constructive, so it does not give any insight into the nature of the spin glass phase and it cannot decide between alternative scenarios such as the droplet or the chaotic pairs picture. Neither does it show where the replica field theory goes wrong. We therefore propose an alternative method to investigate the finite dimensional spin glass phase in order to bypass conventional replica field theory and to obtain a theory which does not contradict the exact mathematical results. To this end we shall expand about the infinite component limit of the $m$-component spin glass in a power series in $1/m$. As we will see, an expansion of this type preserves the replica symmetry found in the $m = \infty$ model and will therefore give a picture of the spin glass phase which is quite different to that of the usual replica field theory. As part of our programme it is necessary to analyse the $m = \infty$-component SK spin glass phase.

The low temperature phase of this model is a generalisation of a Bose-Einstein condensation in the sense that the spins condense into a $n_0$-dimensional subspace of the $m$-dimensional space they can occupy, where $n_0$ is a number of order $N^{2/5}$. We study the spin glass model defined by the Hamiltonian

$$\mathcal{H} = -\frac{1}{2} \sum_{ij} J_{ij} \mathbf{s}_i \cdot \mathbf{s}_j - \sum_i h_i \mathbf{s}_i,$$

where the $N$ spins $\mathbf{s}_i$ are vectors with $m$ components and we use the normalization $\mathbf{s}_i^2 = m$. The vectors $h_i$ are $m$-component gaussian random fields with correlator

$$\overline{h_i h_j} = h^2 \delta_{ij} \delta_{ab}$$

and field strength $h$.

In this paper our numerical work is on the mean-field limit where the off-diagonal $J_{ij}$ are independent gaussian random variables with variance $1/N$, and $J_{ii} = 0$. Some of our results, however, also apply for $J_{ij}$ corresponding to a finite dimensional system. In all cases the partition function at a temperature $T = 1/\beta$ can be written as

$$Z = \int_{-\infty}^{\infty} \left( \prod_{i=1}^{\infty} ds_i^a \right) \left( \prod_{i} \delta(s_i^2 - m) \right) e^{-\beta \mathcal{H}}$$

$$= \int_{-\infty}^{\infty} \left( \prod_{i=1}^{\infty} ds_i^a \right) \int_{-\infty}^{\infty} \left( \prod_{i} \frac{\beta dH_i}{4\pi} \right)$$

$$\times \exp \left[ \frac{\beta}{2} \sum_i H_i (m - s_i^2) + \frac{\beta}{2} \sum_{ij} J_{ij} s_i s_j + \beta \sum_i h_i s_i \right],$$

where we have introduced an integral representation for the $\delta$ functions. The integrals over the spin components $s_i^a$ can be done, and making use of Eq. (2) this results in

$$Z = \int_{-\infty}^{i \infty} \left( \prod_{i} \frac{dH_i \beta}{4\pi} \right)$$

$$\times \exp \left[ \frac{\beta m}{2} \left( \sum_i (H_i + h^2 \chi_{ii}) + \frac{1}{\beta} \ln \det(\chi/\beta) \right) \right],$$

where the matrix $\chi$ is defined by

$$\chi_{ij} = (A^{-1})_{ij} \quad \text{with}$$

$$A_{ij} = H_i \delta_{ij} - J_{ij}. \quad (7)$$

For large $m$, the integral in Eq. (5) can be evaluated by a steepest descent calculation in the $H_i$, giving rise to the conditions

$$\beta = \chi_{ii} + \beta h^2 (\chi^2)_{ii} \quad i = 1, \ldots, N \quad (8)$$
which determine the $H_i$. Eq. (8) can be rewritten using the eigenvector decomposition of $A$, the inverse of $\chi$, as

$$\beta = \sum_n \frac{(a^n_i)^2}{\lambda_n} \left(1 + \frac{\beta h^2}{\lambda_n}\right),$$

(9)

where $\lambda_n$ are the eigenvalues and $a^n_i$ are the orthonormal eigenvectors of $A$.

At zero temperature, Eqs. (8) and (9) are no longer well-defined. In the ground state all spins are aligned parallel to their local field, i.e.

$$H_i s_i = \sum_j J_{ij} s_j.$$  

(10)

For the large-$m$ limit there exists a unique stable solution of these equations for the $H_i$, as opposed to the case of the finite $m$ spin glass which has an exponentially large number of stable solutions [3]. This fact allows us to solve Eqs. (10) numerically without difficulty. We note that it is not immediately obvious that the $H_i$ in Eq. (10) are in any way related to the $H_i$, buried in the matrix $\chi$ in the finite temperature problem, Eq. (8). It has been shown in [3], however, that the $H_i$ in Eq. (10) are equal to the limit of the $H_i$ in Eq. (8) as $\beta \to \infty$. This observation allows us to cover the complete temperature range including $T = 0$ within one framework.

Given the $H_i$ determined by Eq. (8) and disregarding any irrelevant prefactors, the partition function is then

$$Z = \exp \left[\frac{\beta m}{2} \left(\sum_i (H_i + h^2 \chi_i) + \frac{1}{\beta} \ln \det(\chi/\beta)\right)\right] \times \exp \left(-\frac{1}{2} \ln \det(B/\beta^2) - \frac{N}{2} \ln m\right),$$

(11)

where we have included the first order fluctuation corrections around the steepest descent values for $H_i$ which involve the matrix $B$ defined by

$$B_{ij} = \frac{1}{2} \chi^2_{ij} + \beta h^2 \chi_{ij} (\chi^2)_{ij}.$$  

(12)

This expansion is valid for any spatial dimension $d$ including $\infty$ since we have so far made no assumptions about the matrix $J_{ij}$.

It has been shown by de Almeida et al. [5] that the $m = \infty$ SK spin glass is replica symmetric and in the thermodynamic limit has the same free energy as the spherical spin glass model [4] (but despite having the same free energy, the physics of the low temperature phase is in fact very different from the spherical model). All the $H_i$ are equal in the thermodynamic limit, and the distribution of eigenvalues of the matrix $A$ follows the Wigner semicircle law. The phase transition (at $h = 0$) then follows from the fact that Eq. (12) only has a solution for $0 \leq \beta \leq \beta_c = 1$, which is

$$H_i = \beta + 1/\beta.$$  

(13)

At the critical temperature $T_c = 1/\beta_c$, the smallest eigenvalue becomes zero.

If the field is non-zero, however, there is no phase transition since the term involving $1/\lambda_n^2$ allows for a solution at any temperature. This can be demonstrated by evaluating Eq. (10) under the assumption that again all $H_i$ are equal to $H$ in the thermodynamic limit and thus the Wigner semicircle law holds, which then yields after some algebra

$$\beta^2 h^2 = (\beta^2 + 1 + \beta^2 h^2 - \beta H)(H^2 - 4).$$  

(14)

This equation has a positive physically relevant solution $H$ for any $\beta$, as long as $h \neq 0$.

The scenario in finite dimensions has been described in [5]. This work was an extension of the Bose glass theory of Hertz, Fleishman, and Anderson [8]. For temperatures $T > T_c$, the eigenfunctions corresponding to the smallest eigenvalues of the matrix $A$ are localized but become extended at criticality. The $H_i$ vary from site to site. However, Eq. (8) still applies. The second term on the right hand side of Eq. (8) involves the replicon susceptibility $\chi^2$. Assuming there is a phase transition in non-zero field, this is the quantity which diverges on the critical line [5]. This, however, makes it impossible for Eq. (8) to be satisfied, thereby ruling out the existence of a phase transition in a field, i.e. an Almeida-Thouless line [10], by contradiction. As this line marks the onset of replica symmetry breaking, we deduce from its absence that the large $m$ limit and the straightforward expansion in $1/m$ about it, produces a theory which is replica symmetric. The absence of such a line is a prediction of the droplet theory of spin glasses. However, usually with $1/m$ expansion methods one does not adopt the direct expansion approach in powers of $1/m$ but instead uses (ad-hoc) self-consistent approximations, such as could be generated by making the sum of the terms in $m$ and of $O(1)$ in the exponents in Eq. (10) stationary with respect to the $H_i$, and it is possible that with such an approach replica symmetry breaking might emerge.

We turn now to a more detailed study of the phase transition mechanism at zero field in the SK limit. First we examine the ground state properties in zero field. Eq. (10) can be written as

$$\sum_j (H_i \delta_{ij} - J_{ij}) s_j^2 = 0,$$

(15)

where $\alpha = 1, \ldots, m$ labels the spin components, showing that the matrix $A_{ij} = H_i \delta_{ij} - J_{ij}$ has at least one eigenvalue which is exactly 0. On the other hand, it was shown in [3] that there is an upper limit of the number of null eigenvalues $n_0$, which is $n_0 < \sqrt{2N}$.

The number of null eigenvalues will turn out to be the key quantity of the low temperature phase of the large-$m$ spin glass. Here, we can already see some implications. The fact that there are $n_0$ null eigenvalues reduces the
effective number of spin components from $m$ to $n_0$. To see this, consider the matrix formed by the entries of the spins, $s_i^a$. Since the rows of this matrix (regarding $a$ as the row index) correspond to null eigenvectors of $A$ and there are only $n_0$ linearly independent ones, the row rank of this matrix is at most $n_0$. But since the row rank and the column rank of any matrix are equal, the column rank is also at most $n_0$. Therefore the $N$ columns, being the spins, can only span a $n_0$-dimensional subspace of the $m$ dimensional space they live in. Making use of the global rotation invariance of the spins, it is therefore sufficient to set $m = n_0$, or at least $m = \sqrt{2N} > n_0$ when $n_0$ is yet unknown. Obviously, this accelerates numerical simulations, allowing one to go to relatively large system sizes, as we will see in the following.

We have solved Eqs. (10) numerically for the spins $s_i$ by straightforward iteration, setting $H_i = |\sum_j J_{ij}s_j|/\sqrt{m}$ at each step, until the average angular deviation of the spins from their local field directions was smaller than some prescribed accuracy (we used $10^{-8}$). From the ground state configuration found in this way we determined the matrix $A$ and analysed its eigenvalues. We found that the smallest non-null eigenvalues are always at least 5 orders of magnitude larger than the null eigenvalues (which we find numerically to be of the order $10^{-8}$ or smaller). The average number of null eigenvalues as a function of the number of spins is plotted in Fig. 1. In

$$\langle n_0 \rangle = \sum_{n=0}^{\infty} n \rho(n)$$

the range of system sizes accessible to us, it behaves as $n_0 \sim N^{2/5}$. This agrees with the prediction from [3]. The calculation leading to this result was, however, based on an approximation of the density of states of the matrix $A$ which, at finite values of $N$, had a gap in the density of states and a square root singularity at the edge of the gap. Numerically we found a very different behavior, see Fig. 2. Instead of a gap there is an enhancement in the density of states at low eigenvalues (rather similar to the behavior produced in the three-dimensional XY spin-glass model by finite size effects [11]).

A new argument is therefore needed to explain the observed value of the exponent $2/5$. From [1, 2] it can be deduced that the ground state energy per spin of the $m$-component spin glass, divided by $m$, goes as $-1 + 1/4m + O(1/m^2)$. Since $n_0$ is equal to the effective number of spin components, the system would at first sight be able to attain its lowest energy state by choosing $n_0$ as large as possible, i.e. equal to its upper bound. However, this calculation of the energy was done taking the thermodynamic limit first, such that $m$ (or $n_0$) is always much less than any power of $N$. When $n_0$ is comparable to some power of $N$, there are additional energy costs, whose magnitude can be estimated by the following argument. Starting from all $H_i$ being equal and then tuning them in such a way that there are $n_0$ null eigenvalues will result in a downward shift in the eigenvalue spectrum of $A$ which is of order $(n_0/N)^{2/3}$ (the lowest $n_0$ eigenvalues in a Wigner semicircle reach this far from the band edge, and the shift is expected to be of the same order). The ground state is reached when the energy arising from these two competing terms $1/4n_0 + \text{const.}(n_0/N)^{2/3}$ is minimized, which is the case when $n_0 \sim N^{2/5}$.

It is striking that in the ground state the individual spins condense into a $n_0$-dimensional subspace of their original $m$-dimensional space. This behaviour is a generalization of the conventional Bose-Einstein condensation where the constituents condense into a single (one-dimensional) state, which is also what happens in the spherical spin glass model [5].

To show that the behavior we observed at zero temperature is in fact the generic behaviour in the low temperature phase, we solved Eqs. (5) numerically in zero
field. Since there is no phase transition for a finite number of spins, we were able to solve Eqs. 8 numerically over a very large temperature range, both in the high and low temperature regions. We used a standard Newton-Raphson iteration scheme for this purpose. This method converges very quickly for high temperatures \( \beta < 1 \) and fails to converge for low temperatures unless the starting configuration for the iteration is already sufficiently close to the solution. In order to ensure this, we employed the exact differential equation

\[
\frac{dH_i}{d\beta} = -\frac{1}{2} \sum_j (B^{-1})_{ij},
\]

with \( B \) as defined in Eq. 12 (with \( h = 0 \)). This equation, which can easily be derived from Eq. 5, was used to project from a solution found at \( \beta \) to a good initial configuration at \( \beta + \Delta \beta \). Using this method we were able to track the solution of Eq. 5 over the temperature range \( \beta = 0.1 \ldots 100 \).

We have found it useful to split Eq. 9 at \( h = 0 \) into two parts corresponding to the eigenvalues that are going to zero as \( \beta \to \infty \) and the rest,

\[
\beta = \sum_{\lambda_n \text{ going to 0}} \frac{(a_n^\alpha)^2}{\lambda_n} + \sum_{\lambda_n \text{ staying finite}} \frac{(a_n^\alpha)^2}{\lambda_n}.
\]

In a finite system at finite temperature, all eigenvalues are naturally non-zero, such that this equation is well-defined. However, for \( \beta > \beta_c \) the eigenvalues which become exactly 0 at \( T = 0 \) decrease with system size as \( N^{-b} \) at finite temperature, where \( b \) is an unknown positive exponent, and so are equal to 0 in the thermodynamic limit throughout the whole low temperature phase. In this situation, the second sum in Eq. 17 is equal to the diagonal elements of the Moore-Penrose inverse (see, e.g., [12]) of \( A \) which corresponds to the physical susceptibility \( \chi_{ij} = \partial(s_i^\alpha)/\partial h_i^\alpha = \beta(\langle s_i^\alpha s_j^\alpha \rangle - \langle s_i^\alpha \rangle \langle s_j^\alpha \rangle) \). For \( \beta > \beta_c \), Eq. 17 can therefore be written as

\[
\beta = \beta(\langle s_i^\alpha \rangle^2 + \bar{\chi}_{ii}),
\]

while for \( \beta < \beta_c \), \( \bar{\chi}_{ii} = \chi_{ii} = \beta \). In Fig. 3 we have plotted \( \bar{\chi} \), defined as \( \bar{\chi}_{ii} \) averaged over sites, and its variance \( \text{Var}(\bar{\chi}_{ii}) \) as a function of \( \beta \). The plot shows that above the transition temperature \( \bar{\chi} \) is equal to \( \beta \), whereas below the transition temperature \( \bar{\chi} \) remains essentially frozen at a value of \( \beta_c \). The site-to-site variance of \( \bar{\chi} \) decreases with system size, roughly following a power law \( \text{Var}(\bar{\chi}_{ii}) \sim N^{-1/3} \) (data not shown here). Eq. 15 then implies that the Edwards-Anderson order parameter \( q = \langle s_i^\alpha \rangle^2 \) is equal to \( 1 - \beta_c/\beta = 1 - T/T_c \), i.e. the frozen components of the spin are those associated with the null eigenvalues of the matrix \( A \).

We believe that the approach to spin glasses via a \( 1/m \) expansion method, while it is not simple to carry out, seems to be the only calculational technique which can avoid the problems associated with the unwanted replica symmetry breaking of the loop expansion. The point about which the expansion takes place, viz the large-\( m \) limit, is also interesting in its own right as the unusual phase transition mechanism found in the SK limit would be expected to carry over to finite dimensions. The work by Viana [13], who found that in the large-\( m \) limit the upper critical dimension and the lower critical dimension are both equal to 8, indicates that below the upper critical dimension there is no Edwards-Anderson order but instead perhaps chiral order as suggested by Kawamura [14, 15] (but contested by [16]). Our approach might in the future be developed into a tool to study this controversy from a new perspective.

We thank M. B. Hastings for valuable discussions. TA acknowledges financial support by the German Academic Exchange Service (DAAD) and the European Community’s Human Potential Programme under contract HPRN-CT-2002-00307, DYGLAGEMEM.

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