THE LIE ALGEBRA OF CYCLIC COINVARIANTS OF A SYMPLECTIC SPACE

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ABSTRACT. We exhibit a natural Lie algebra structure on the graded space of cyclic coinvariants of a symplectic vector space.

1. INTRODUCTION

Let $V$ be a vector space endowed with a symplectic form. It is well known that the adjoint representation of $\text{Sp}(V)$ is equivalent to the symmetric square of $V$, i.e., to the coinvariants of $\mathbb{Z}/2\mathbb{Z}$ acting by cyclic permutation on $V \otimes V$. This allows us to endow $(V^\otimes 2)_{\mathbb{Z}/2\mathbb{Z}}$ with the structure of a (symplectic) Lie algebra. In this note, we show that this is just the degree 0 part of a larger object: the graded vector space

$$\bigoplus_{n=0}^{\infty} (V^\otimes n^2)_{\mathbb{Z}/(n+2)\mathbb{Z}}$$

has a natural structure of Lie algebra determined by the symplectic form on $V$. In the special case that $\dim V = 2$, for every associative algebra $A$, this algebra acts infinitesimally on fibers of the commutator map $[,] : A^2 \rightarrow A$.

This paper developed from our interest in the fibers of the commutator map on the Lie algebra $\mathfrak{sl}_n$. After we wrote it, B. Tsygan called our attention to a paper of M. Kontsevich which arrived at similar conclusions, coming from a rather different (and more sophisticated) point of view. Kontsevich considered the tensor algebra of a symplectic space $V$ over $\mathbb{Q}$ and looked at the Lie algebra of derivations preserving its symplectic form. He asserted that this algebra is naturally isomorphic to

$$\bigoplus_{n=0}^{\infty} (V^\otimes n^2)_{\mathbb{Z}/(n+2)\mathbb{Z}}.$$ 

Of course in characteristic zero, there is a natural isomorphism between invariants and coinvariants for any finite group, which is not the case in characteristic $p$. By contrast, we give a characteristic free formula for the Lie bracket which is naturally defined on the coinvariant space. In addition, we indicate a dihedral theory which may be useful in analyzing the commutator fibers for $\mathfrak{so}_n$ and $\mathfrak{sp}_{2n}$.

2. THE LIE ALGEBRA

Let $K$ be a field and $V$ a vector space over $K$ equipped with an alternating bilinear form $\langle , \rangle$. For any $l$, $\mathbb{Z}/l\mathbb{Z}$ acts on $V^\otimes l$ in the obvious way and we form the graded $K$-vector space

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\[ L(V) = \bigoplus_{l=2}^{\infty} (V^\otimes l)_{\mathbb{Z}/l\mathbb{Z}}. \]

where \((V^\otimes l)_{\mathbb{Z}/l\mathbb{Z}}\) has degree \(l-2\). We now define a Lie bracket on \(L(V)\). To do this, let us first introduce some additional notation. Fix a basis \(B\) of \(V\). Given integers \(i, j\) such that \(1 \leq i, j \leq l\), we set

\[
D_{i,j}(\alpha_1 \otimes \cdots \otimes \alpha_l) = \begin{cases} 
\alpha_{i+1} \otimes \cdots \otimes \alpha_{j-1} & \text{if } i+1 \leq j-1, \\
0 & \text{if } i = j-1, \\
\alpha_{i+1} \otimes \cdots \otimes \alpha_l \otimes \alpha_1 \otimes \cdots \otimes \alpha_{j-1} & \text{if } i \geq j,
\end{cases}
\]

where \(\alpha_i \in B\). Now if \(\alpha = \alpha_1 \otimes \cdots \otimes \alpha_l\) and \(\beta = \beta_1 \otimes \cdots \otimes \beta_m\) are tensor products of basis elements, we define

\[
[\alpha, \beta] = \sum_{i=1}^{l} \sum_{j=1}^{m} \langle \alpha_i, \beta_j \rangle D_{i,i}(\alpha) \otimes D_{j,j}(\beta).
\]

and extend by linearity. (Here \(\bar{\gamma}\) denotes the class of \(\gamma \in V^\otimes n\) in \((V^\otimes n)_{\mathbb{Z}/n\mathbb{Z}}\).) Note that if \(\sigma\) denotes the canonical generator of \(\mathbb{Z}/l\mathbb{Z}\) then

\[
D_{i,i}(\sigma^r(\alpha)) = \begin{cases} 
D_{r+i,r+i}(\alpha) & \text{if } i \leq l-r, \\
D_{r+i-l,r+i-l}(\alpha) & \text{if } i > l-r.
\end{cases}
\]

Thus \([, ] : (V^\otimes l)_{\mathbb{Z}/l\mathbb{Z}} \times (V^\otimes m)_{\mathbb{Z}/m\mathbb{Z}} \to (V^\otimes (l+m-2))_{\mathbb{Z}/(l+m-2)}\) is well-defined. Moreover, it does not depend on the choice of basis \(B\).

For each \(x \in B\), we also define \(D_x : L(V) \to L(V)\) by

\[
D_x(\alpha) = \sum_{i=1}^{l} \delta_{\alpha_i,x} D_{i,i}(\alpha),
\]

where \(\delta\) is the Kronecker delta and \(\alpha\) is as before. We extend \(D_x\) to a (well-defined) endomorphism of \(L(V)\).

**Theorem 1.** The above bracket makes \(L(V)\) into a Lie algebra. Moreover each \(D_x\) is a derivation.

**Proof.** It is easily verified that the bracket is bilinear and antisymmetric. We now prove that the Jacobi identity holds. Suppose \(\alpha = \alpha_1 \otimes \cdots \otimes \alpha_l, \beta = \beta_1 \otimes \cdots \otimes \beta_m\)
and $\gamma = \gamma_1 \otimes \cdots \otimes \gamma_n$; then we have

$$[\alpha, [\beta, \gamma]] + [\beta, [\gamma, \alpha]] + [\gamma, [\alpha, \beta]] =$$

$$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} \sum_{r=1}^{m} \sum_{k \neq j}^{m} \langle \beta_j, \gamma_k \rangle \langle \alpha_i, \beta_r \rangle D_{i,i}(\alpha) \otimes D_{r,j}(\beta) \otimes D_{k,k}(\gamma) \otimes D_{j,r}(\beta)$$

$$+ \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} \sum_{r=1}^{m} \sum_{k \neq j}^{m} \langle \beta_j, \gamma_k \rangle \langle \alpha_i, \gamma_r \rangle D_{i,i}(\alpha) \otimes D_{r,k}(\gamma) \otimes D_{j,j}(\beta) \otimes D_{k,r}(\gamma)$$

$$+ \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} \sum_{r=1}^{m} \sum_{k \neq j}^{m} \langle \gamma_k, \alpha_i \rangle \langle \beta_j, \gamma_r \rangle D_{j,j}(\beta) \otimes D_{r,k}(\gamma) \otimes D_{i,i}(\alpha) \otimes D_{k,r}(\gamma)$$

$$+ \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} \sum_{r=1}^{m} \sum_{k \neq j}^{m} \langle \gamma_k, \alpha_i \rangle \langle \beta_j, \gamma_r \rangle D_{j,j}(\beta) \otimes D_{r,k}(\gamma) \otimes D_{i,i}(\alpha) \otimes D_{k,r}(\gamma)$$

$$+ \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} \sum_{r=1}^{m} \sum_{k \neq j}^{m} \langle \gamma_k, \beta_r \rangle D_{k,k}(\gamma) \otimes D_{r,j}(\beta) \otimes D_{i,i}(\alpha) \otimes D_{j,r}(\beta)$$.

Now note that

$$D_{i,i}(\alpha) \otimes D_{r,j}(\beta) \otimes D_{k,k}(\gamma) \otimes D_{j,r}(\beta) = D_{k,k}(\gamma) \otimes D_{j,r}(\beta) \otimes D_{i,i}(\alpha) \otimes D_{r,j}(\beta)$$

in $(V \otimes (l+m+n-4)) / (l+m+n-4) \mathbb{Z}$. Therefore, interchanging the sums over $j$ and $r$ in the last term, we see that it is exactly the negative of the first term (since $\langle \alpha_i, \beta_j \rangle \langle \gamma_k, \beta_r \rangle = -\langle \beta_r, \gamma_k \rangle \langle \alpha_i, \beta_j \rangle$). Similarly, the second term cancels the third and the fourth term cancels the fifth. The Jacobi identity now follows by linearity.

Finally, we have

$$D_x([\alpha, \beta]) = \sum_{i=1}^{l} \sum_{j=1}^{m} \langle \alpha_i, \beta_j \rangle D_x(D_{i,i}(\alpha) \otimes D_{j,j}(\beta))$$

$$= \sum_{i=1}^{l} \sum_{j=1}^{m} \langle \alpha_i, \beta_j \rangle \sum_{k=1}^{m} \delta_{\alpha_k,x} D_{k,k}(\gamma) \otimes D_{i,i}(\alpha) \otimes D_{j,j}(\beta)$$

$$+ \sum_{i=1}^{l} \sum_{j=1}^{m} \langle \alpha_i, \beta_j \rangle \sum_{k=1}^{m} \delta_{\beta_k,x} D_{k,k}(\gamma) \otimes D_{i,i}(\alpha) \otimes D_{j,j}(\beta)$$

$$= [D_x(\alpha), \beta] + [\alpha, D_x(\beta)]$$

If $\phi : V \to W$ is a linear transformation such that $\langle \phi(\alpha), \phi(\beta) \rangle_W = \langle \alpha, \beta \rangle_V$ for all $\alpha, \beta \in V$ then we have an obvious Lie algebra homomorphism $L(\phi) : L(V) \to L(W)$ induced by $\alpha_1 \otimes \cdots \otimes \alpha_l \mapsto \phi(\alpha_1) \otimes \cdots \otimes \phi(\alpha_l)$. 

$\square$
The case \( \dim V = 2 \) is of particular interest. Let \( \{x, y\} \) be a basis with \( \langle x, y \rangle = 1 \). We identify \( V^\otimes l \) with the noncommutative homogeneous polynomials of degree \( l \) in \( x \) and \( y \).

Suppose that \( A \) is an associative \( K \)-algebra considered as a Lie algebra via the usual bracket \([X, Y] = XY - YX\). Let \( N_l \) denote the trace with respect to the \( \mathbb{Z}/l\mathbb{Z}\)-action on \( V^\otimes l \); then the \( N_l \) induce a well-defined map \( N : L(V) \to \bigoplus_{l \geq 2}(V^\otimes l)/\mathbb{Z}/l\mathbb{Z} \). Let \( K[\varepsilon] \) denote the ring of dual numbers. If \( \alpha \in V^\otimes l \) represents a class in \( L(V) \), write \( N(\alpha) = xp_\alpha - yq_\alpha \) and consider the vector field

\[
F_\alpha(X, Y) = (X - \varepsilon q_\alpha(X, Y), Y - \varepsilon p_\alpha(X, Y))
\]
on \( A \times A \). Note that \( N(\alpha) \), \( p_\alpha \), \( q_\alpha \), and \( F_\alpha \) depend only on \( \bar{\alpha} \).

**Theorem 2.** The map \( \bar{\alpha} \mapsto F_\alpha \) is a homomorphism of Lie algebras from \( L(V) \) to the algebra of vector fields on \( A \times A \) tangent to the fibers of the commutator map.

**Proof.** As \( \sigma(N_l(\alpha)) = N_l(\alpha) \), \( (xp_\alpha - yq_\alpha) = (p_\alpha x - q_\alpha y) \), so

\[
[X - \varepsilon q_\alpha(X, Y), Y - \varepsilon p_\alpha(X, Y)] \equiv [X, Y] \quad \text{(mod } \varepsilon\text{)}.
\]

Thus, \( F_\alpha \) is always tangent to the fibers of the commutator map. To see that \( \bar{\alpha} \mapsto F_\alpha \) is a Lie algebra homomorphism, it suffices to check the case that \( \alpha = \alpha_1 \otimes \cdots \otimes \alpha_l \) and \( \beta = \beta_1 \otimes \cdots \otimes \beta_m \) are tensor products of basis vectors. Now,

\[
p_\alpha = \sum_{i=1}^l \delta_{\alpha_i, x} D_{i,i}(\alpha), \quad q_\alpha = -\sum_{i=1}^l \delta_{\alpha_i, y} D_{i,i}(\alpha).
\]

(Note that \( \overline{p_\alpha} = D_x(\alpha) \) and \( \overline{q_\alpha} = D_y(\alpha) \).) Regarding \( F_\alpha \) and \( F_\beta \) as sections \( A \times A \to A[[\varepsilon]] \times A[[\varepsilon]] \) of the evaluation at zero map,

\[
F_\alpha(F_\beta(X, Y)) = F_\alpha \left( X + \varepsilon \sum_{j=1}^m \delta_{\beta_j, y} D_{j,j}(\beta)(X, Y), Y - \varepsilon \sum_{j=1}^m \delta_{\beta_j, x} D_{j,j}(\beta)(X, Y) \right) =
\]

\[
\left( X + \varepsilon \sum_{j=1}^m \delta_{\beta_j, y} D_{j,j}(\beta)(X, Y) \right)
\]

\[
+ \varepsilon \sum_{i=1}^l \delta_{\alpha_i, y} D_{i,i}(\alpha) \left( X + \varepsilon \sum_{j=1}^m \delta_{\beta_j, y} D_{j,j}(\beta)(X, Y), Y - \varepsilon \sum_{j=1}^m \delta_{\beta_j, x} D_{j,j}(\beta)(X, Y) \right),
\]

\[
Y - \varepsilon \sum_{j=1}^m \delta_{\beta_j, x} D_{j,j}(\beta)(X, Y)
\]

\[
- \varepsilon \sum_{i=1}^l \delta_{\alpha_i, x} D_{i,i}(\alpha) \left( X + \varepsilon \sum_{j=1}^m \delta_{\beta_j, y} D_{j,j}(\beta)(X, Y), Y - \varepsilon \sum_{j=1}^m \delta_{\beta_j, x} D_{j,j}(\beta)(X, Y) \right) \right).
\]

If \( \gamma = \gamma_1 \otimes \cdots \otimes \gamma_n \) is a tensor monomial regarded as a noncommutative homogeneous polynomial, then

\[
\gamma(X + \varepsilon X_1, Y + \varepsilon Y_1) \equiv \gamma(X, Y) + \varepsilon \sum_{k=1}^n D_{\alpha,k}(\gamma)(X, Y)(\delta_{\gamma_{k,x}, x} X_1 + \delta_{\gamma_{k,y}, y} Y_1) D_{k,1}(\gamma)(X, Y) \quad \text{(mod } \varepsilon^2\text{)}.
\]

Therefore,
Extending \( F_\alpha(F_\beta(X,Y)) - F_\beta(F_\alpha(X,Y)) \)

\[
\equiv \varepsilon^2 \left( \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{l} \delta_{\alpha_i,y}(\delta_{\alpha_k,x} \delta_{\beta_j,y} - \delta_{\alpha_k,y} \delta_{\beta_j,x}) \left( D_{i,k}(\alpha) D_{j,i}(\beta) D_{k,i}(\alpha) \right)(X,Y),
\right.

\[
\left. \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{l} \delta_{\alpha_i,x}(-\delta_{\alpha_k,x} \delta_{\beta_j,y} + \delta_{\alpha_k,y} \delta_{\beta_j,x}) \left( D_{i,k}(\alpha) D_{j,i}(\beta) D_{k,i}(\alpha) \right)(X,Y) \right) \right)
\]

\[
- \varepsilon^2 \left( \sum_{j=1}^{l} \sum_{i=1}^{m} \sum_{k=1}^{l} \delta_{\beta_j,y}(\delta_{\beta_k,x} \delta_{\alpha_i,y} - \delta_{\beta_k,y} \delta_{\alpha_i,x}) \left( D_{j,k}(\beta) D_{i,j}(\alpha) D_{k,j}(\beta) \right)(X,Y),
\right.

\[
\left. \sum_{j=1}^{l} \sum_{i=1}^{m} \sum_{k=1}^{l} \delta_{\beta_j,x}(-\delta_{\beta_k,x} \delta_{\alpha_i,y} + \delta_{\beta_k,y} \delta_{\alpha_i,x}) \left( D_{j,k}(\beta) D_{i,j}(\alpha) D_{k,j}(\beta) \right)(X,Y) \right) \right)
\]

\[
\equiv \varepsilon^2 \left( \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{l} \delta_{\alpha_i,y}(\alpha_k, \beta_j) \left( D_{i,k}(\alpha) D_{j,j}(\beta) D_{k,i}(\alpha) \right)(X,Y),
\right.

\[
\left. - \sum_{j=1}^{l} \sum_{i=1}^{m} \sum_{k=1}^{l} \delta_{\beta_j,y}(\beta_k, \alpha_i) \left( D_{j,k}(\beta) D_{i,j}(\alpha) D_{k,j}(\beta) \right)(X,Y),
\right.
\]

\[
\left. \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{l} \delta_{\alpha_i,x}(\beta_j, \alpha_k) \left( D_{i,k}(\alpha) D_{j,j}(\beta) D_{k,i}(\alpha) \right)(X,Y),
\right.
\]

\[
\left. - \sum_{j=1}^{l} \sum_{i=1}^{m} \sum_{k=1}^{l} \delta_{\beta_j,x}(\alpha_i, \beta_k) \left( D_{j,k}(\beta) D_{i,i}(\alpha) D_{k,j}(\beta) \right)(X,Y) \right) \right)
\]

\[
\equiv \varepsilon^2 \left( -q_{(\alpha, \beta)}(X,Y) \right) \mod \varepsilon^3.
\]

\[\square\]

Each \( V^\otimes l \) has a natural involution \( \iota \) induced by

\[
\iota(\alpha_1 \otimes \cdots \otimes \alpha_l) = (-1)^l \alpha_1 \otimes \cdots \otimes \alpha_1.
\]

Extending \( \iota \) to all of \( L(V) \), we get a Lie algebra involution; to see this, observe that

\[
\iota(D_{i,l}(\alpha) \otimes D_{j,j}(\beta)) = (-1)^{l+m} D_{m-j+1,m-j+1}(\iota(\beta)) \otimes D_{l-i+1,l-i+1}(\iota(\alpha))
\]

\[
= (-1)^{l+m} D_{l-i+1,l-i+1}(\iota(\alpha)) \otimes D_{m-j+1,m-j+1}(\iota(\beta)).
\]
and since $(\iota(\alpha))_{t-i+1} = \alpha_i$, $(\iota(\beta))_{m-j+1} = \beta_j$ and $(-1)^{t+m-2} = (-1)^{t+m}$, it follows that $[[\iota(\alpha), \iota(\beta)]] = \iota[[\alpha, \beta]]$. We will also need to consider the unsigned involution $I$ induced by $I(\alpha_1 \otimes \cdots \otimes \alpha_l) = \alpha_1 \otimes \cdots \otimes \alpha_l$.

Denote by $P_+(V)$ and $P_-(V)$, respectively, the +1 and −1 eigenspaces for $\iota$. Then of course $P_+(V)$ is a Lie subalgebra of $L(V)$ and $[P_-(V), P_+(V)] \subseteq P_+(V)$.

Similarly, we let $P_{1,+}$ and $P_{1,-}$ denote the +1 and −1 eigenspaces for $\iota$ in $V^{\otimes l}$. To simplify the notation, we will assume that $V$ is finite-dimensional, so that $B = \{x_1, \ldots, x_r\}$.

**Lemma.** Suppose $\alpha \in (V^{\otimes l})^{\mathbb{Z}/2\mathbb{Z}}$ and write $\alpha = \sum x_i \otimes p_i$, where the $p_i \in V^{\otimes l-1}$. Then $\alpha \in P_{1,+}$ (resp., $P_{1,-}$) if and only if $p_1, \ldots, p_r \in P_{1,-}$ (resp., $P_{1,+}$).

**Proof.** This is clear, as $\iota(\alpha) = -\sum \iota(p_i) \otimes x_i$ and $\sigma(\alpha) = \sum p_i \otimes x_i$.

As an application of these definitions, we turn to the case of a Lie algebra $\mathfrak{g}$ defined over $K$ and identify $V^{\otimes l}$ with noncommutative polynomials of degree $l$ in $x_1, \ldots, x_r$. Our first observation is that if $\mathfrak{g}$ is either $\mathfrak{so}_n$ or $\mathfrak{sp}_n$ and $p = p(x_1, \ldots, x_r)$ is in $\bigoplus_l P_{l,-}$ then for all $X_1, \ldots, X_r \in \mathfrak{g}$, $p(X_1, \ldots, X_r) \in \mathfrak{g}$. Indeed write $p = \sum p_i$, where each $p_i$ is a monomial. Then if $\mathfrak{g} = \mathfrak{so}_n$,

$$p(X_1, \ldots, X_r)^l = \sum p_i(X_1, \ldots, X_r)^l = \sum I(p_i)(X_1^l, \ldots, X_r^l)$$

$$= \sum (-1)^{\deg p_i} I(p_i)(X_1, \ldots, X_r) = \iota(p)(X_1, \ldots, X_r)$$

$$= -p(X_1, \ldots, X_r).$$

If $\mathfrak{g} = \mathfrak{sp}_n$ let $J$ denote the matrix of the nondegenerate alternating form which defines $\mathfrak{g}$. Then

$$Jp(X_1, \ldots, X_r) = \sum Jp_i(X_1, \ldots, X_r) = \sum (-1)^{\deg p_i} p_i(X_1^l, \ldots, X_r^l)J$$

$$= (\iota(p)(X_1, \ldots, X_r))^l J = -p(X_1, \ldots, X_r)^l J.$$

With $\dim V = 2$ and $\mathfrak{g}$ of the above type (in particular, $\mathfrak{g} \subset \mathfrak{gl}_n$) we consider the above map $\tilde{\alpha} \mapsto F_\alpha$ from $L(V)$ to the algebra of vector fields on $\mathfrak{gl}_n \times \mathfrak{gl}_n$.

**Proposition.** The image of $P_+(V)$ under this map consists of vector fields on $\mathfrak{g} \times \mathfrak{g}$ tangent to the fibers of the commutator map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$.

**Proof.** If $\alpha \in V^{\otimes l}$ then $\iota(\sigma(\alpha)) = \sigma^{-1}(\iota(\alpha))$, which implies that $N(P_+(V)) \subset \bigoplus_l P_{l,+}$. Moreover if $\alpha \in P_+(V)$ and $N(\alpha) = xp_\alpha - yq_\alpha$ then $p_\alpha, q_\alpha \in \bigoplus_l P_{l,-}$ by the Lemma. Now we are done, as $p_\alpha(X,Y), q_\alpha(X,Y) \in \mathfrak{g}$ whenever $X, Y \in \mathfrak{g}$ by the preceding discussion.

**References**

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