COMPACTON EQUATIONS AND INTEGRABILITY: THE ROSENAU-HYMAN AND COOPER-SHEPARD-SODANO EQUATIONS

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Abstract. We study integrability –in the sense of admitting recursion operators– of two nonlinear equations which are known to possess compacton solutions: the $K(m,n)$ equation introduced by Rosenau and Hyman
\[ D_t(u) + D_x(u^m) + D_x^3(u^n) = 0 , \]
and the CSS equation introduced by Cooper, Shepard, and Sodano,
\[ D_t(u) + u^{l-2}D_x(u) + \alpha p D_x(u^{p-1}u_x^2) + 2\alpha p D_x^2(u^p u_x) = 0 . \]
We obtain a full classification of integrable $K(m,n)$ and CSS equations; we present their recursion operators, and we prove that all of them are related (via nonlocal transformations) to the Korteweg-de Vries equation. As an application, we construct isochronous hierarchies of equations associated to the integrable cases of CSS.

1. Introduction. We begin by quoting Rosenau [30]: "We define a compact wave as a robust solitary wave with compact support beyond which it vanishes identically. We then define a compacton as a compact wave that preserves its shape after interacting with other compactons". Rosenau and Hyman found examples of compactons while studying generalizations of the Korteweg-de Vries equation for which the dispersion term is nonlinear. Their model equation is the so-called $K(m,n)$ equation
\[ u_t + (u^m)_x + (u^n)_{xxx} = 0 , \] (1)
and an example of a compacton bearing equation within the family (1) is $K(2,2)$. In this case, the function $u(x,t) = (4c/3) \cos^2((x-ct)/4)$ for $|x-ct| \leq 2\pi$ and $u(x,t) =

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0 otherwise, is a compacton solution. Further works on compactons are \cite{22,23} and the comprehensive review \cite{29}.

It turns out that solutions to equations within the $K(m, n)$ can exhibit very complex behaviors; we refer the reader to \cite{2–4}, and to the papers \cite{23,29,37} authored by Rosenau and his coworkers, for general discussions. Here, we just mention one example: in \cite{31} the authors present four local conservation laws of $K(2, 2)$, and credit P.J. Olver with the observation that no further local conservation laws seem to exist\footnote{This observation has been proven rigorously by Vodová in 2013, see \cite{36}.}. This (non)existence of conservation laws has an important analytic implication, see \cite{37}: initially nonnegative, smooth and compactly supported solutions to $K(m, n)$ lose their smoothness within a finite time.

We wonder if this complex behavior has to do with (lack of) integrability. In this work we present a detailed study of the integrability properties of $K(m, n)$. We find that, module a rather general space of allowable transformations, the only integrable equations belonging to the $K(m, n)$ family are the KdV and modified KdV equations, and that integrable equations within the $K(m, n)$ family cannot have compacton solutions. In particular, we recover the observation in \cite{19,36} that $K(2, 2)$ is not integrable.

In order to obtain this result we classify all integrable $K(m, n)$ equations using the theory of formal symmetries (to be summarized in Section 2). The power of this approach has been amply demonstrated by the classification results for evolution equations and systems of equations due to researchers such as Shabat, Fokas, Svinolupov, Sokolov, Mikhailov and others (see \cite{16–18,20,25,35}), and also by the important papers \cite{32,33} on the classification of integrable scalar evolution equations satisfying a homogeneity condition.

Since our search for integrable compacton bearing equations within the $K(m, n)$ class does not yield examples, we also investigate a related family, the Cooper-Shepard-Sodano (CSS) family of equations

$$u_t + u^{l-2} u_x - \alpha p D_x (u^{p-1} u_x^2) + 2\alpha D_x^2 (u^p u_x) = 0, \quad \alpha \neq 0,$$ \hspace{1cm} (2)

introduced in \cite{13}. We quote from this paper: “These equations have the same terms as the equations considered by Rosenau and Hyman, but the relative weights of the terms are quite different leading to the possibility that the integrability properties might be different“. The authors of \cite{13} then proceed to show that their family of equations indeed admits compacton-bearing equations. One such equation is (2) with $l = 3, p = 2$. This family of equations is further studied in \cite{15,21}.

Encouraging properties of (2) are the facts that it admits a Hamiltonian formulation, and that it possesses three physically interesting conservation laws: area, mass and energy. Regrettfully, we prove herein that they are not integrable in general. Using formal symmetries once more, we obtain six integrable equations within the (2) family. None of them can support compacton solutions.

Since the existence of compacton solutions is a rather extraordinary occurrence in the nonlinear world, we believe that our results are not only important by themselves, but also because they seem to express certain rigidity in our present algebraic/geometric/analytic approach to integrability. In other words, $K(2, 2)$ say, must be “special”, and so far we have not been able to uncover the deeper source of its special character.

Our paper is organized as follows. We review the theory of formal symmetries and integrability in Section 2 after, essentially, \cite{25}, and in Section 3 we use this
theory to classify integrable $K(m, n)$ equations. We note that a previous classification has appeared in [19]. One integrable case was missing therein and we single it out here. Fortunately, the missing case does not alter the conclusion in [19] that the only $K(m, n)$ integrable case are (essentially, module a class of allowable transformations specified in Section 3) the KdV and mKdV equations. The present classification also differs from the one appearing in [19] in that here we explain in detail how to connect our integrable cases to KdV (or, to the linear equation) and because in Section 5 we exhibit explicit recursion operators for all our integrable $K(m, n)$ equations. In Section 4 we study integrability of the Cooper-Shepard-Sodano family (2) and again we are able to explain how to connect its integrable cases to KdV (or, to the linear equation), and to exhibit recursion operators. Finally in Section 6 we present an application of our results: we construct integrable isochronous equations, after [9, 11, 12], starting from the equations in our classification of integrable CSS equations, explain how to obtain their point symmetries, and present their corresponding recursion operators.

2. Formal symmetries and integrability. The formal symmetry approach to integrability [25, 26] begins with the observation that standard (systems of) partial differential equations which are integrable (for instance, in the sense of Calogero, see [8]) usually admit an infinite set of (generalized) symmetries of arbitrarily large differential order. A.B. Shabat and his collaborators, see for instance [25, 26], realized that it is possible to weaken the notion of a (generalized) symmetry to the notion of a formal symmetry—to be defined precisely below—and that this new concept provides a computationally efficient tool for defining integrability and classifying integrable equations.

Here and henceforth we use standard notation from the geometric theory of differential equations as presented in [27], see also [25]. A general system of partial differential equations can be written in the form

$$\Delta_a(x^i, u^\alpha, u^\alpha_{x^i}, \ldots) = 0$$

(3)

where $x^i$ and $u^\alpha$ are respectively the independent and dependent variables, and sub-indexed variables $u^\alpha_{x^i}$ denote the corresponding partial derivatives. The operator $D_i$ (or $D_{x^i}$) denotes the total derivative with respect to $x^i$. A total derivative acts on a differential function $F = F(x^i, u^\alpha, u^\alpha_{x^i}, \ldots)$ through the chain rule, e.g. $D_i u^\alpha = u^\alpha_{x^i}$, and $D_x(u^m) = mu^{m-1}u_x$. Composition of total derivatives is denoted using a multi-index, e.g. $D_{ij} = D_i \circ D_j$, and $u_{ij}$ denotes $D_{ij}u$.

A differential function $G = (G^\alpha)$ is a symmetry of system (3) if [25, 27]

$$\Delta_a(G) = 0$$

(4)

whenever $u^\alpha(x^i)$ is a solution to $\Delta_a = 0$, where $\Delta_s$ is the linearization operator of $\Delta_a$, that is,

$$\Delta_s = \left( \sum_L \frac{\partial \Delta_a}{\partial u^\alpha_L} D_L \right)$$

(in the sum $L$ denotes any multi-index, including the null one, i.e terms $\frac{\partial \Delta_a}{\partial u^\alpha_L}$). If the system consists of just one scalar evolution equation,

$$\Delta = u_t - F,$$

(5)

then equation (4) becomes $D_t G = F_s(G)$ or, equivalently,

$$D_t G = D_x F,$$

(6)
where
\[ D_t = \sum_{\#K \geq 0} D_K(G) \frac{\partial}{\partial u_K} \]
(#K denotes the order of the multi-index K).

Following [25, 26], we apply a second linearization to formula (6). We obtain, using some formulae appearing in [25],
\[ (D_t G)_s = (D_t F)_s \iff D_t(G_s) + G_s \circ F_s = D_t(F_s) + F_s \circ G_s, \] (7)
in which, if \( G_s = \sum_L a_L D_L \), then \( D_t(G_s) = \sum_L D_t(a_L) D_L \), and the last equality holding on solutions to (5). The expression \( D_t(F_s) \) is defined analogously. We interpret our symmetry condition (7) using commutators:
\[ D_t(G_s) - [F_s, G_s] = D_t(F_s). \] (8)

Let us consider the degree of the operators appearing in (8). The degree of \( F_s \) as a differential operator —let us denote it by \( \text{deg}(F_s) \)— is the differential order of \( F \), i.e. the order of the differential equation (5) and thus, it is fixed. The degree of a differential operator —let us denote it by \( \text{deg}(F) \) — is the differential order of \( F \) in which, if \( G_s = \sum_L a_L D_L \), then \( D_t(G_s) = \sum_L D_t(a_L) D_L \), and the last equality holding on solutions to (5). The expression \( D_t(F_s) \) is defined analogously. We interpret our symmetry condition (7) using commutators:
\[ D_t(G_s) - [F_s, G_s] = D_t(F_s). \] (8)

Following [25, 26], and partially motivated by the theory of recursion operators, see [27], we define formal symmetries using the left-hand side of Equation (8):

**Definition 2.1.** Let
\[ u_t = F \] (9)
be an evolution equation with \( F \) a function of two independent variables \( x, t \), one dependent variable \( u \) and a finite number of derivatives of \( u \) with respect to \( x \). A formal symmetry of rank \( k \) of (9) is a formal pseudo-differential operator
\[ \Lambda = l_0 D^r + l_{-1} D^{r-1} + \cdots + l_0 + l_{-1} D^{-1} + l_{-2} D^{-2} + \cdots, \quad D = D_x \] (10)
with \( l_i \) being functions of \( t, x, u \) and finite numbers of \( x \)-derivatives of \( u \), that satisfies the equation
\[ D_t(\Lambda) = [F_s, \Lambda] \] (11)
everywhere \( u \) is a solution to \( u_t = F \), up to a pseudo-differential operator of degree \( r + \text{deg}(F_s) - k \). A formal symmetry of infinite rank is a pseudo-differential operator (10) such that (11) holds identically whenever \( u \) is a solution to \( u_t = F \).

Note that if \( G \) is a symmetry of order \( p \) of \( u_t = F \), then (8) implies that \( G_s \) is a formal symmetry of rank \( p \). We also remark that a formal symmetry of infinite rank is a formal recursion operator, see [27]. Thus, it generates, in principle, an infinite number of generalized symmetries of the equation at hand. For example, see [14], it can be proven that application of a quasilocal recursion operator (in the sense of [14, Section 1]) to a given symmetry yields a (generalized) symmetry, and so such an operator could indeed generate an infinite chain of (generalized) symmetries.

The main technical point behind Definition 2.1 is that the space of solutions of equation (11) is much richer and structured than that of equation (8) or even (4).

\[ ^2 \text{This equation must be understood as valid under the usual rules of the algebra of pseudo-differential series, see [25, 27].} \]
For example, powers and roots of formal symmetries are also formal symmetries. In fact, this observation was one of the original motivations for the use of formal pseudo-differential operators in Definition 2.1, because the $r$th root of a differential operator (10) is usually a pseudo-differential operator.

Now we explain why Definition 2.1 restricts the function $F$. A theorem due to M. Adler, see [1], states that the residue (the coefficient of $D^{-1}$) of a commutator of formal pseudo-differential operators is always a total derivative. If we apply this result to different powers $\Lambda^{i/r}$ of a generic formal symmetry (10) of rank $k$ inserted into (11), we obtain

$$D_t(\text{residue}(\Lambda^{i/r})) = \text{residue} D_t(\Lambda^{i/r}) = \text{residue}[F, \Lambda^{i/r}] = D\sigma_i$$

for some differential functions $\sigma_i$, i.e. a sequence of conservation laws

$$D_t\rho_i \doteq D_x\sigma_i, \quad i = -1, 1, \ldots, \quad (12)$$

which are, together with the special case $D_t\rho_0 = D_t(l/r - 1) = D_x\sigma_0$, the so called canonical conservation laws. The symbol $\doteq$ means that equations (12) must hold on solutions of (5), i.e. all derivatives with respect to $t$ must be substituted using the equation and its differential consequences.

As observed in [25], the canonical densities $\rho_i$ and conserved fluxes $\sigma_i$ are differential functions which can be recursively written in terms of the right hand side $F$ of equation (9) and its derivatives. The fact that the left-hand side of (12) must be a total derivative with respect to $x$ for all $i = -1, 0, 1, \ldots$, produces obstructions that are necessary conditions for the existence of (generalized/formal) symmetries $G$, i.e. for integrability.

For example, for evolution equations of third order,

$$u_t = F(x, u, u_x, u_{xx}, u_{xxx}), \quad (13)$$

the first canonical density is $\rho_{-1} = \left(\partial F/\partial u_{xxx}\right)^{-1/3}$, see [25]. Therefore, a first integrability condition is requiring $D_t\rho_{-1}$ to be the total derivative of a local function $\sigma_{-1}$. The second canonical density imposes further differential restrictions on $F$, and so forth. Usually, after a small number of steps our family of equations either fails to satisfy the integrability conditions, or the right hand side $F$ becomes so specific that we are able to produce a formal symmetry of infinite rank and therefore, in principle, a sequence of generalized symmetries of $u_t = F$. If $u_t = F$ represents a family of equations, this procedure allows us to find all integrable cases in the family. We are thus led to the following precise definition of integrability, after [25,27]:

**Definition 2.2.** A system of evolution equations is integrable if and only if it possesses a formal symmetry of infinite rank.

Let us write down the first five canonical densities for a third order equation (13) following [25]. We will use them in the next section to study the integrability of the Rosenau-Hyman and Cooper-Shepard-Sodano equations (1) and (2):

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3The foregoing discussion implies that instead of a general formal symmetry $\Lambda$ of degree $r$, we can consider its $r$th root $\Lambda^{1/r}$ of degree 1 without loss of generality, see [25] for details.
Proposition 1. Let \( u_t = F(x,u,u_x,u_{xx},u_{xxx}) \) be an arbitrary third order evolution equation. The first five canonical conserved densities can be written explicitly as
\[
\rho_{-1} = \left( \frac{\partial F}{\partial u_{xxx}} \right)^{-1/3}, \\
\rho_0 = \rho_{-1}^3 \frac{\partial F}{\partial u_{xx}}, \\
\rho_1 = D_x \left( 2\rho_{-1}^2 u_x + \rho_{-1}^2 \frac{\partial F}{\partial u_{xx}} \right) + \rho_{-1}^3 (D_x \rho_{-1})^2 + \frac{1}{3} \rho_{-1}^2 \left( \frac{\partial F}{\partial u_{xx}} \right)^2 \\
+ \rho_{-1} (D_x \rho_{-1}) \frac{\partial F}{\partial u_{xx}} - \rho_{-1} \rho_{-1} \frac{\partial F}{\partial u_x} + \rho_{-1} \sigma_{-1} \\
\rho_2 = -\frac{1}{3} (D_x^2 \rho_{-1}) \frac{\partial F}{\partial u_{xx}} - (D_x \rho_{-1}) \frac{\partial F}{\partial u_{xx}} + \rho_{-1} \frac{\partial F}{\partial u} + \rho_{-1}^2 (D_x \rho_{-1})^2 \frac{\partial F}{\partial u_{xx}} \\
- \frac{1}{3} \rho_{-1}^4 \frac{\partial F}{\partial u_x} + \frac{1}{3} \rho_{-1}^3 (D_x \rho_{-1}) \left( \frac{\partial F}{\partial u_{xx}} \right)^2 \\
+ \frac{2}{3} \rho_{-1}^7 \left( \frac{\partial F}{\partial u_{xx}} \right)^3 + \frac{1}{3} \rho_{-1} \sigma_0, \\
\rho_3 = \rho_{-1} \sigma_1 - \rho_1 \sigma_{-1}.
\]

Remark 1. The condition of existence of a formal symmetry of infinite rank is strictly weaker than the condition of existence of an infinite number of generalized symmetries. Indeed, the 2 component system
\[
\begin{align*}
  u_t &= u_{xxx} + v^2 \\
  v_t &= v_{xxx},
\end{align*}
\]
considered by Bakirov in [5], possesses exactly one generalized symmetry, as proved by Beukers, Sanders and Wang [6]. On the other hand, it does possess a recursion operator [7].

Now, a very important remark, see [25] and also [27], is that the use of transformations between equations is a very convenient way to proceed when seeking classifications. In this paper we deal with integrable equations of the type
\[
u_t = f(u)u_{xxx} + g(u,u_x,u_{xx}) ,
\]
see [13, 31]. The general strategy we use to classify these equations consists in performing a sequence of convenient point transformations and differential substitutions that preserve integrability, and then apply and compute the integrability conditions associated to (14)–(18). Our general procedure is as follows:

First, if \( f'(u) \neq 0 \) the point transformation \( u \rightarrow f(u)^{-1/3} \) converts\(^4\) Equation (19) into another one of the form
\[
u_t = D_x \left[ \frac{u_{xx}}{u^3} + f_1(u,u_x) \right] + f_2(u,u_x,u_{xx}) ,
\]
where \( f_2(u,u_x,u_{xx}) \) is not a total \( x \)-derivative. This form is very convenient because it follows from (14) that the integrability condition \( D_t \rho_{-1} = D_x \sigma_{-1} \) is equivalent to requiring that \( f_2 = 0 \). This condition greatly restricts the form of the equation. Once \( f_2 = 0 \), the equation admits a potentiation \( u \rightarrow u_x \) that brings it into the

\(^4\)The notation \( u \rightarrow f(u) \) denotes a change of variables or substitution \( u = f(v) \) producing an equation in \( v \) which is then rewritten in terms of the letter \( u \) again.
form \( u_t = u_{xxx}/u_x^3 + f_1(u_x, u_{xx}) \). A subsequent hodograph\(^5\) transformation \( x \to u \), \( u \to x \) simplifies it to one of the form

\[ u_t = u_{xxx} + h(u_x, u_{xx}). \]

This equation can be “antipotentiated” \( (u_x \to u) \) to get

\[ u_t = u_{xxx} + D_x h(u_x, u_{xx}). \]

Our integrability conditions imply that the integrable cases of this equation are all of the form

\[ u_t = u_{xxx} + h_2(u)x^2 + (a + bu)u_x + h_0(u). \quad (21) \]

If \( h_2(u) \neq 0 \) a further point transformation \( \int \exp \left[ \frac{2}{3} \int h_2(u) \, du \right] \, du \to u \) transforms this equation into another one of the form

\[ u_t = u_{xxx} + (a + bu)u_x + f_1(u)x^3 + f_2(u)x^2 + f_3(u)u_x. \quad (22) \]

We will show that all the integrable cases of the RH and CSS equations (1) and (2) can be written in the form (22), as linear equations, KdV or mKdV, or the Calogero-Degasperis-Fokas (CDF) equation (see below: the CDF is Miura-transformable to KdV). Thus, if we “pullback” the recursion operator of KdV (or, the linear equation) by the foregoing transformations, we can construct recursion operators of the original equations, and therefore we obtain an explicit proof of integrability in terms of Definition 2.2. In actual fact, we seldom perform this pullback operation explicitly. Once we know that a given equation is integrable, it is usually straightforward to compute its recursion operator from first principles, as in [28].

Now we carry out this plan.

3. Integrability of the Rosenau-Hyman equation. We consider the compacton equation of Rosenau and Hyman (see [31])

\[ D_t(u) + D_x(u_m) + D_x^2(u^n) = 0, \quad n \neq 0. \quad (23) \]

The case \( n = 1 \) i.e.

\[ u_t + mu^{m-1}u_x + u_{xxx} = 0, \]

is well-known, see [25, Section 4.1]: the only integrable cases are \( m = 0, 1, 2, 3 \), i.e. the linear equation, the KdV and the modified KdV equations. We write them as (using the point transformation \( x \to -x \))

\[ u_t = u_{xxx} + \alpha u_x + \beta, \quad \alpha, \beta \in \mathbb{C}, \quad (24) \]

\[ u_t = u_{xxx} + 2 uu_x, \quad (25) \]

\[ u_t = u_{xxx} + 3u^2u_x. \quad (26) \]

If \( n \neq 1 \), the point transformation \( x \to -x, t \to t/n, u \to u^{3/(1-n)} \) changes (23) into equations of the form (20), namely:

\[ u_t = D_x \left[ \frac{u_{xx}}{u^3} - \frac{m(n-1)}{n(3m-n-2)} \frac{u^{2-3m+n}}{u^{3m+1}} - \frac{9n}{2(n-1)} \frac{u_x^2}{u^4} + \frac{(n+2)(2n+1)}{(n-1)^2u^5} \right] \quad (27) \]

if \( 3m - n - 2 \neq 0 \), and

\[ u_t = D_x \left[ \frac{u_{xx}}{u^3} - \frac{9n}{2(n-1)} \frac{u_x^2}{u^4} + \frac{(n+2)}{3n} \log u \right] + \frac{(n+2)(2n+1)}{(n-1)^2u^5} \quad (28) \]

if \( 3m - n - 2 = 0 \).

\(^5\)See [25] for a review of the relevant theory of transformations and differential substitutions of pde’s.
The first integrability condition \( D_t \rho_{-1} = D_x \sigma_{-1} \) or, equivalently, \( u_t \in \text{Im} \, D_x \) (i.e. \( D_t \rho_{-1} \) is a total \( x \)-derivative) implies that either \( n = -2 \) or \( n = -1/2 \) in both cases, because the last term in (27) and (28) must be zero. The composition of a potentiation, a hodograph and an antipotentiation yields the following equations of the form (21):

\[
\begin{align*}
    u_t &= u_{xxx} + D_x \left[ \frac{3n + 2}{2} u_x^2 + \frac{m(n - 1)}{n(3m - n - 2)} u^{3(m-1)} \right], \\
    u_t &= u_{xxx} + D_x \left[ \frac{3}{2} \left( \frac{n + 2}{n - 1} \right) \frac{u_x^2}{u} + \frac{n + 2}{3n} u \log u \right].
\end{align*}
\]

If \( n = -2 \), Equation (29) becomes

\[
    u_t = u_{xxx} - \frac{m - 1}{2} u^{-m} u_x
\]

whose integrable cases are again \( m = 1, 0, -1, -2 \), corresponding to linear equations, KdV and mKdV. On the other hand, if \( n = -2 \), Equation (30) becomes a linear equation included in case (24).

If \( n = -1/2 \), Equation (29) can be written in the form (22), that is,

\[
    u_t = u_{xxx} - \frac{1}{2} u_x^2 - \frac{4m(m - 1)}{2m - 1} e^{(1-2m)u} u_x,
\]

after a point transformation \( u \to e^u \). This family of equations satisfies the first two integrability conditions. The third integrability condition is \( D_t \rho_1 \in \text{Im} \, D_x \), and the canonical conserved density \( \rho_1 \) satisfies

\[
    D_t(\rho_1) \sim -2(m - 1)m(2m - 3)(2m + 1)e^{-2mu}u_x^3,
\]

in which the symbol \( \sim \) denotes equality except for the addition of a total \( x \)-derivative. Thus, integrability can be achieved only in the cases \( m = -1/2, 0, 1, 3/2 \) which are all subcases of the Calogero-Degasperis-Fokas (CDF) equation [10,16]

\[
    u_t = u_{xxx} - \frac{1}{2} u_x^2 + (\alpha e^{2u} + \beta e^{-2u} + \gamma) u_x.
\]

Finally, when \( n = -1/2 \) Equation (30) becomes

\[
    u_t = u_{xxx} - 3u_x u_{xx}/u + \frac{3}{2} u_x^3 - (1 + \log u) u_x
\]

and for it

\[
    D_t(\rho_1) \sim -2u_x^3/u^3
\]

so this case is not integrable.

**Remark 2.** We note that the CDF equation (31) can be related to the KdV equation through the Miura transformation

\[
    \frac{3}{2} u_{xx} - \frac{3}{4} u_x^2 - \sqrt{6} \beta e^{-u} u_x - \frac{1}{2} \alpha e^{2u} - \frac{1}{2} \beta e^{-2u} - \frac{\gamma}{2} \to u.
\]

We summarize the integrable cases of the Rosenau-Hyman family (23) in the following theorem. We make the point transformation \( x \to -x, t \to t/n \) and write

\[
    u_t = \frac{1}{n} D_x(u^n) + \frac{1}{n} D_x^2(u^n), \quad n \neq 0
\]

instead of (23). This transformation is invertible and does not affect integrability.

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\[
\text{[6]} \text{We note that the case } n = -1/2, m = 3/2 \text{ was missing in [19].}
\]
Theorem 3.1. The integrable cases of the Rosenau-Hyman family (32) are

1. \( n = 1, m = 0, 1, 2, 3 \), corresponding to Equations (24), (25) and (26), namely,
   \[
   \begin{align*}
   u_t &= u_{xxx} + \alpha u_x + \beta, \quad \alpha, \beta \in \mathbb{C}, \quad (33) \\
   u_t &= u_{xxx} + 2u u_x, \quad (34) \\
   u_t &= u_{xxx} + 3u^2 u_x. \quad (35)
   \end{align*}
   \]

2. \( n = -2, m = -2, -1, 0, 1 \), corresponding to Equations
   \[
   \begin{align*}
   u_t &= D_x \left[ D_x \left( \frac{u_x}{u^3} \right) - \frac{1}{2u^2} \right], \quad (36) \\
   u_t &= D_x \left[ D_x \left( \frac{u_x}{u^{3/2}} \right) - \frac{1}{2u} \right], \quad (37) \\
   u_t &= -\frac{1}{2} D_x^2 \left[ u^{-2} \right], \quad (38) \\
   u_t &= D_x \left[ D_x \left( \frac{u_x}{u^{3/2}} \right) - \frac{1}{2u} \right] \quad (39)
   \end{align*}
   \]
   respectively.

3. \( n = -\frac{1}{2}, m = \frac{3}{2}, 1, 0, -\frac{1}{2} \), corresponding to Equations
   \[
   \begin{align*}
   u_t &= D_x \left[ D_x \left( \frac{u_x}{u^{3/2}} \right) - 2u^{3/2} \right], \quad (40) \\
   u_t &= D_x \left[ D_x \left( \frac{u_x}{u^{3/2}} \right) - 2u \right], \quad (41) \\
   u_t &= D_x^2 \left[ u^{3/2} \right], \quad (42) \\
   u_t &= D_x \left[ D_x \left( \frac{u_x}{u^{3/2}} \right) - 2 \frac{1}{u^{1/2}} \right], \quad (43)
   \end{align*}
   \]
   respectively.

All these equations are related to the linear equation or to the KdV equation through differential substitutions.

Remark 3. It is clear that the equations appearing above cannot admit solutions with compact support, let alone compactons. As explained in Section 1, this theorem extends and enriches the discussion on \( K(m, n) \) appearing in [19]. We mention that J. Vodová classified conservation laws of the \( K(m, m) \) equations and observed that \( K(-2, -2), K(-1/2, -1/2) \) are integrable; integrability of \( K(-2, -2), K(-1/2, -1/2), \) and \( K(-1, -2) \), is also observed in the later review [29].

4. Integrability of Cooper-Shepard-Sodano. In this section we study equations of the form
   \[
   u_t + u^{l-2} u_x - \alpha p D_x (u^{p-1} u_x^2) + 2\alpha D_x^2 (u^p u_x) = 0, \quad \alpha \neq 0. \quad (44)
   \]
   We consider the case \( p = 0 \) first. Equation (44) becomes \( u_t + u^{l-2} u_x + 2\alpha u_x = 0 \), which is integrable if and only if \( l = 2, 3, 4 \), i.e. in the linear, KdV and mKdV case, as observed in [25]. Let us now consider \( p \neq 0 \). We use the same strategy as with the Rosenau-Hyman case: first we apply the change \( t \to -t, u \to (2\alpha u^3)^{-1/p} \) to obtain
   \[
   u_t = D_x \left[ \frac{u_{xxx}}{u^3} - \frac{3(2p + 3) u_x^2}{2p u^3} - \frac{p(2\alpha)^{2/l}}{3l - p - 6} \frac{u^{6 - 3l + p}}{u^p} \right] + \frac{(p + 3)(p + 6) u_x^3}{2p^2 u^5}. \quad (45)
   \]
if $3l - p - 6 \neq 0$, and
\[
ul = D_x \left[ \frac{uxx}{u^3} - \frac{3(2p + 3)}{2p} \frac{u_x^2}{u^4} + \frac{1}{\sqrt{2}a} \log u \right] + \frac{(p + 3)(p + 6)u_x^3}{2p^2u^5} \tag{46}
\]
if $3l - p - 6 = 0$.

Thus, the first integrability condition of Proposition 1 fixes the values $p = -3$ or $p = -6$. If $p = -3$, a combination of potentiation, hodograph, antipotentiation and exponential point transformation $u \to e^u$, with a scaling to absorb the constant $\alpha$, change (45) and (46) into the equations
\[
ul = u_{xxx} - \frac{u_x^2}{2} + \frac{(l - 2)e^{u - lu_x}}{l - 1} \quad \text{and} \quad ul = u_{xxx} - \frac{u_x^2}{2} + u u_x
\]
respectively. On the other hand, if $p = -6$, the same combination of transformations, using $u \to u^2$ instead of the exponential, changes (45) and (46) into
\[
ul = u_{xxx} + \frac{(l - 2)}{l} \frac{u_x}{u^2} \quad \text{and} \quad ul = u_{xxx} + \log(u)u_x
\]
respectively.

Let us assume that $3l - p - 6 \neq 0$. The integrability condition in $\rho_1$ implies that if $p = -3$ we must have $l = -1, 2, 3$, and if $p = -6$, then $l = -2, -1, 2$ and we recover again the linear, KdV, mKdV and CDF equations. When $3l - p - 6 = 0$, we have the CSS equations
\[
ul = \frac{a}{u^6}u_{xxx} - 12\frac{a}{u^7}u_x u_{xx} + 21\frac{a}{u^8}u_x^3 + \frac{u_x}{u^2}
\]
($p = -6$ and $l = 0$) and
\[
ul = \frac{a}{u^3}u_{xxx} - 6\frac{a}{u^4}u_x u_{xx} + 6\frac{a}{u^5}u_x^3 + \frac{u_x}{u},
\]
($p = -3$ and $l = 1$), in which $a = 2\alpha$. The integrability condition for $\rho_2$ implies that both equations are not integrable.

Summarizing, we have the following theorem.

**Theorem 4.1.** The integrable equations of family (44) are

1. $p = 0$, $l = 2, 3, 4$, corresponding to the linear equation, KdV equation, mKdV equation;
2. $p = -6$, $l = -2, -1, 2$, corresponding to the Equations
   \[
   ul = \frac{a}{u^6}u_{xxx} - 12\frac{a}{u^7}u_x u_{xx} + 21\frac{a}{u^8}u_x^3 + \frac{u_x}{u^2}, \tag{47}
   ul = \frac{a}{u^3}u_{xxx} - 12\frac{a}{u^4}u_x u_{xx} + 21\frac{a}{u^5}u_x^3 + \frac{u_x}{u^2}, \tag{48}
   ul = \frac{a}{u^3}u_{xxx} - 12\frac{a}{u^4}u_x u_{xx} + 21\frac{a}{u^5}u_x^3 + u_x, \tag{49}
   \]
   respectively.
3. $p = -3$, $l = -1, 2, 3$, corresponding to the Equations
   \[
   ul = \frac{a}{u^3}u_{xxx} - 6\frac{a}{u^4}u_x u_{xx} + 6\frac{a}{u^5}u_x^3 + \frac{u_x}{u^2}, \tag{50}
   ul = \frac{a}{u^4}u_{xxx} - 6\frac{a}{u^5}u_x u_{xx} + 6\frac{a}{u^6}u_x^3 + u_x, \tag{51}
   ul = \frac{a}{u^4}u_{xxx} - 6\frac{a}{u^5}u_x u_{xx} + 6\frac{a}{u^6}u_x^3 + u_x, \tag{52}
   \]
   respectively.
All these equations are related to the linear equation or to the KdV equation through differential substitutions.

As in Section 3, this theorem implies that no integrable CSS equation admits solutions with compact support.

5. Integrability and recursion operators. In this section we construct explicit recursion operators for the equations appearing in the above theorems using the work [28]. First of all, we note that these equations are all in [25]. We have (when we write (4.x.xx) we are referring to the corresponding equation in [25]):

1. Equation (36) is a special case of (4.1.27), namely,
\[ u_t = D_x \left( \frac{u_{xx}}{u^3} - 3 \frac{u_x^2}{u^4} + \frac{1}{2u^2} \right) = -9 \frac{u_x u_{xx}}{u^5} + 12 \frac{u_x^3}{u^5} - \frac{u_x}{u^3} + \frac{u_{xxx}}{u^3}. \]  

2. Equation (37) is equivalent to a subcase of (4.1.25)
\[ u_t = D_x \left( \frac{u_{xx}}{u^3} - 3 \frac{u_x^2}{u^4} + \frac{3}{2u} + cu \right) = -9 \frac{u_x u_{xx}}{u^4} + 12 \frac{u_x^3}{u^5} + \frac{3u_x}{2u^2} + cu_x + \frac{u_{xxx}}{u^3}. \]  

3. Equations (38) and (39) are equivalent to subcases of (4.1.34)
\[ u_t = D_x \left( \frac{u_{xx}}{u^3} - 3 \frac{u_x^2}{u^4} + c_1 \frac{u_x}{u^2} + c_2 u \right) \]
\[ = -9 \frac{u_x u_{xx}}{u^5} + 12 \frac{u_x^3}{u^5} - 2c_1 \frac{u_x}{u^3} + c_2 u_x + c_1 \frac{u_{xx}}{u^2} + \frac{u_{xxx}}{u^3}. \]  

4. Equations (40)-(43) are all equivalent to subcases of (4.1.30)
\[ u_t = D_x \left[ \frac{u_{xx}}{u^3} - \frac{3}{2} \frac{u_x^2}{u^4} - \frac{3}{2} \frac{\lambda u_x^2}{u^6} \right] + c_1 \frac{(\lambda u + 1)^3}{u^2} + c_2 \frac{u^2}{\lambda u + 1} + c_3 u \]
with \( \lambda = 0 \), that is,
\[ u_t = -6 \frac{u_x u_{xx}}{u^4} + 6 \frac{u_x^3}{u^5} - 2c_1 \frac{u_x}{u^3} + 2c_2 u_x u_x + c_3 u_x + \frac{u_{xxx}}{u^3}, \]  

after applying the point transformation \( u \to u^2 \).

Now, Equations (53), (54) and (55) are special cases (for the values of the constant numbers \( c_1, c_2 \) relevant to us) of Equation (81) in [28], namely,
\[ u_t = \frac{u_{xx}}{u^3} - 9 \frac{u_x u_{xx}}{u^4} + 12 \frac{u_x^3}{u^5} + \frac{2}{3} \lambda_1 \frac{u_x}{u^3} + \frac{1}{2} \lambda_2 \frac{u_x}{u^2} - cu_x, \]  
while Equation (56) is Equation (85) in [28], see below. The recursion operator for Equation (57) is as follows:
\[ R[u] = u^{-2} D_x^2 - 5u^{-3} u_x D_x - 4u^{-3} u_{xx} + 12u^{-4} u_x^2 + \frac{2\lambda_1}{3} u^{-3} + \frac{2\lambda_2}{3} u^{-2} \]
\[ - u_x D_x^{-1} \circ 1 + u_x D_x^{-1} \circ \left( \frac{\lambda_2}{6} u^{-2} - c \right). \]

Repeatedly applying the operator \( R[u] \) on the t-translation symmetry \( u_t \), yields a corresponding symmetry-integrable hierarchy of order \( 2m + 3 \), namely
\[ u_t = R^m[u] \left( \frac{u_{xx}}{u^3} - 9 \frac{u_x u_{xx}}{u^4} + 12 \frac{u_x^3}{u^5} + \frac{2}{3} \lambda_1 \frac{u_x}{u^3} + \frac{1}{2} \lambda_2 \frac{u_x}{u^2} - cu_x \right), \]
\[ m = 0, 1, 2, \ldots, \]
and we note that for the \( x \)-translation symmetry we obtain
\[
R[u](u_x) = 0.
\]

Let us now consider the integrable cases of the CSS equations. We see that Equations (47), (48) and (49) are special cases of Equation (90) in [28], namely
\[
u_t = \frac{a u_{xxx}}{u^6} - 12 \frac{a u_x u_{xx}}{u^7} + 21 \frac{a u_x^3}{u^8} + \beta_1 u_x + \beta_2 \frac{u_x}{u^4} + \beta_3 u_x,
\]
where \( a, \beta_1, \beta_2 \) and \( \beta_3 \) are arbitrary constants and \( a \neq 0 \). This equation admits the following recursion operator:
\[
R_1[u] = \frac{1}{u} D_x^2 + \frac{6 u_x}{u^5} D_x - \frac{6 u_{xxx}}{u^5} + \frac{2 u_x^2}{u^6} + \frac{2 \beta_2}{u} + \frac{1}{a} \frac{1}{u} + \beta_1 \frac{1}{3a u^2} - \frac{2}{a} u_1 D_x^{-1} \circ u
\]
\[+ u_x D_x^{-1} \circ \left( -\frac{2 u_{xxx}}{u^6} + \frac{6 u_x^2}{u^7} + \beta_2 \frac{1}{a} \frac{1}{u^4} + \frac{2 \beta_1}{3a u^3} + \frac{2 \beta_3}{a} \right).
\]

Acting \( R_1[u] \) on the \( t \)-translation symmetry \( u_t \), we obtain a corresponding symmetry-integrable hierarchy of order \( 2m + 3 \), namely
\[
u_t = R_1^m[u] \left( \frac{a u_{xxx}}{u^6} - 12 \frac{a u_x u_{xx}}{u^7} + 21 \frac{a u_x^3}{u^8} + \beta_1 u_x + \beta_2 \frac{u_x}{u^4} + \beta_3 u_x \right),
\]
\[m = 0, 1, 2, \ldots ,
\]
and we note that for the \( x \)-translation symmetry we obtain
\[
R_1[u](u_x) = 0.
\]

On the other hand, Equations (50), (51) and (52) are special cases of Equation (85) in [28], namely
\[
u_t = \frac{a u_{xxx}}{u^3} - 6 \frac{a u_x u_{xx}}{u^4} + 6 \frac{a u_x^3}{u^5} + \beta_1 \frac{u_x}{u^4} + \beta_2 u u_x + \beta_3 u_x,
\]
where \( a, \beta_1, \beta_2 \) and \( \beta_3 \) are arbitrary constants and \( a \neq 0 \). This equation admits the following recursion operator:
\[
R_2[u] = \frac{1}{u^2} D_x^2 - \frac{3 u_x}{u^3} D_x - \frac{3 u_{xxx}}{u^4} + \frac{6 u_x^2}{u^5} + \beta_2 \frac{u_x}{u^4} + \beta_1 \frac{1}{a} \frac{1}{u^2}
\]
\[- \frac{1}{a} u_1 D_x^{-1} \circ 1 + \frac{4 \beta_2}{3a} u_x D_x^{-1} \circ u + \frac{\beta_3}{a} u_x D_x^{-1} \circ 1.
\]

Acting \( R_2[u] \) on the \( t \)-translation symmetry \( u_t \frac{\partial}{\partial u} \), we obtain a symmetry-integrable hierarchy of order \( 2m + 3 \), namely
\[
u_t = R_2^m[u] \left( \frac{a u_{xxx}}{u^3} - 6 \frac{a u_x u_{xx}}{u^4} + 6 \frac{a u_x^3}{u^5} + \beta_1 \frac{u_x}{u^4} + \beta_2 u u_x + \beta_3 u_x \right),
\]
\[m = 0, 1, 2, \ldots ,
\]
and we note that for the \( x \)-translation symmetry we obtain
\[
R_2[u](u_x) = 0.
\]
6. The isochronous equations derived from (58) and (61). In this section we construct new integrable evolution equations starting from what we will call the Cooper-Shepard-Sodano model equations (58) and (61). Our new equations are isochronous in the sense of Calogero, see [9, Chapter 7] and [11, 12, 24]: they are autonomous evolution PDEs which depend on a positive parameter $\omega$ and possess many solutions which are time-periodic with period $T = 2\pi/\omega$. For completeness, we also explain how to obtain the Lie point symmetries of our equations and present their recursion operators.

6.1. Isochronous equations. Following [9,11,12,24], we introduce a new dependent variable $v(r,s)$, where $r$ and $s$ are new independent variables, as follows:

$$u(x,t) = e^{-i\lambda \omega s} v(r,s)$$  \hspace{1cm} (64a)

$$x = re^{i\mu \omega s}$$  \hspace{1cm} (64b)

$$t = \frac{1}{i\omega} (e^{i\omega s} - 1).$$  \hspace{1cm} (64c)

The prolongations are

$$u_t = e^{-i(\lambda + 1)\omega s} [v_s - i\lambda \omega v - i\mu \omega r v_r]$$

$$u_{nx} = e^{-i(\lambda + n\mu)e_s} v_{nr}, \hspace{1cm} n = 1, 2, 3, \ldots,$$

where

$$u_{nx} = \frac{\partial^n u}{\partial x^n}, \hspace{1cm} v_{nr} = \frac{\partial^n v}{\partial r^n}.$$  

With the change of variables (64a) – (64c), equation (58) takes the form

$$v_s - i\lambda \omega v - i\mu \omega r v_r = e^{i(6\lambda - 3\mu + 1)\omega s} \left( \frac{av_{rrr}}{v^6} - \frac{12av_r v_{rr}}{v^7} + \frac{21av^3}{v^8} \right)$$

$$+ \beta_1 e^{i(4\lambda - \mu + 1)\omega s} \left( \frac{v_{r}}{v^4} \right) + \beta_2 e^{i(3\lambda - \mu + 1)\omega s} \left( \frac{v_{r}}{v^3} \right) + \beta_3 e^{i(-\mu + 1)\omega s} v_r.$$

This equation can become autonomous for $\alpha \neq 0$ only if

$$\mu = 2\lambda + \frac{1}{3},$$  

so that (66) then takes the form

$$v_s - i\lambda \omega v - i \left( 2\lambda + \frac{1}{3} \right) \omega r v_r = \frac{av_{rrr}}{v^6} - \frac{12av_r v_{rr}}{v^7} + \frac{21av^3}{v^8}$$

$$+ \beta_1 e^{i(2\lambda + \frac{2}{3})\omega s} \left( \frac{v_{r}}{v^4} \right) + \beta_2 e^{i(\lambda + \frac{2}{3})\omega s} \left( \frac{v_{r}}{v^3} \right) + \beta_3 e^{i(2\lambda + \frac{2}{3})\omega s} v_r.$$  \hspace{1cm} (67)

Clearly (67), and therefore (66), becomes autonomous in the following three cases:

**Case 1.1.** $\lambda = -\frac{1}{3}$ and $\mu = -\frac{1}{3}$ with $\beta_2 = \beta_3 = 0$. Then (66) becomes

$$v_s + \frac{i}{3} \omega v + i \frac{1}{3} \omega r v_r = \frac{av_{rrr}}{v^6} - \frac{12av_r v_{rr}}{v^7} + \frac{21av^3}{v^8} + \beta_1 v_r.$$  \hspace{1cm} (68)

**Case 1.2.** $\lambda = -\frac{2}{3}$ and $\mu = -1$ with $\beta_1 = \beta_3 = 0$. Then (66) becomes

$$v_s + \frac{i}{3} \omega v + i \omega r v_r = \frac{av_{rrr}}{v^6} - \frac{12av_r v_{rr}}{v^7} + \frac{21av^3}{v^8} + \beta_2 v_r.$$  \hspace{1cm} (69)
Equation (58) also admits the following point symmetries:

\[ v_s - i \frac{1}{3} \omega v - i \mu \omega v_r = \frac{a v_{rrr}}{v^3} - \frac{12 a v_r v_{rr}}{v^4} + \frac{21 a v_r^3}{v^5} + \beta_3 v_r. \]  

(70)

Let us now consider (61). With the change of variables (64a) – (64c), Equation (61) takes the form

\[ v_s - i \lambda \omega v - i \mu \omega v_r = e^{i(3\lambda - 3\mu + 1)\omega s} \left( \frac{a v_{rrr}}{v^3} - 6 a v_r v_{rr} + 6 a v_r^3 \right) \]

\[ + \beta_1 e^{i(3\lambda - \mu + 1)\omega s} \left( \frac{v_r}{v^3} \right) + \beta_2 e^{i(-\lambda - \mu + 1)\omega s} v v_r + \beta_3 e^{i(-\mu + 1)\omega s} v_r. \]  

(71)

This equation can become autonomous for \( a \neq 0 \) only if

\[ \mu = \lambda + \frac{1}{3}, \]

so that (71) then takes the form

\[ v_s - i \lambda \omega v - i \left( \lambda + \frac{1}{3} \right) \omega v_r = \frac{a v_{rrr}}{v^3} - \frac{6 a v_r v_{rr}}{v^4} + \frac{6 a v_r^3}{v^5} \]

\[ + \beta_1 e^{i(2\lambda + \frac{2}{3})\omega s} \left( \frac{v_r}{v^3} \right) + \beta_2 e^{i(-2\lambda + \frac{2}{3})\omega s} v v_r + \beta_3 e^{i(-\lambda + \frac{2}{3})\omega s} v_r. \]  

(72)

Clearly (72), and therefore (71), becomes autonomous in the following three cases:

**Case 1.3.** \( \lambda = \frac{1}{3} \) and \( \mu = 1 \) with \( \beta_1 = \beta_2 = 0 \). Then (71) becomes

\[ v_s + \frac{1}{3} \omega v = \frac{a v_{rrr}}{v^3} - \frac{6 a v_r v_{rr}}{v^4} + \frac{6 a v_r^3}{v^5} + \beta_1 v_r. \]  

(73)

**Case 2.1.** \( \lambda = -\frac{1}{3} \) and \( \mu = 0 \) with \( \beta_2 = \beta_3 = 0 \). Then (71) becomes

\[ v_s - i \frac{1}{3} \omega v - i \lambda \omega v_r = \frac{a v_{rrr}}{v^3} - \frac{6 a v_r v_{rr}}{v^4} + \frac{6 a v_r^3}{v^5} + \beta_1 v_r. \]  

(74)

**Case 2.2.** \( \lambda = \frac{1}{3} \) and \( \mu = \frac{2}{3} \) with \( \beta_1 = \beta_3 = 0 \). Then (71) becomes

\[ v_s - \frac{2}{3} \omega v - \frac{2}{3} \omega v_r = \frac{a v_{rrr}}{v^3} - \frac{6 a v_r v_{rr}}{v^4} + \frac{6 a v_r^3}{v^5} + \beta_2 v v_r. \]  

(75)

Summarizing, the constructed isochronous equations are (68)–(70) and (73)–(75).

### 6.2. Symmetries

Let us now list the Lie point symmetries of equation (58): i.e.

\[ u_t = \frac{a u_{xx}}{u^6} - 12 \frac{a u_x u_{xx}}{u^7} + 21 \frac{a v_r^3}{u^8} + \beta_1 \frac{u_x}{u^4} + \beta_2 \frac{u_x}{u^4} + \beta_3 u_x. \]

Besides the obvious \( x \)-translation, \( Z_x = \frac{\partial}{\partial x} \), and \( t \)-translation symmetry, \( Z_t = \frac{\partial}{\partial t} \), Equation (58) also admits the following point symmetries:

a) For \( \beta_1 = \beta_2 = \beta_3 = 0 \):

\[ Z_1 = x \frac{\partial}{\partial x} + 3t \frac{\partial}{\partial t}, \quad Z_2 = u \frac{\partial}{\partial u} + 6t \frac{\partial}{\partial t}. \]

b) For \( \beta_1 = \beta_2 = 0 \) and \( \beta_3 \neq 0 \):

\[ Z_1 = (x - 2t \beta_3) \frac{\partial}{\partial x} + 3t \frac{\partial}{\partial t}, \quad Z_2 = u \frac{\partial}{\partial u} + 6t \frac{\partial}{\partial t} - 6t \beta_3 \frac{\partial}{\partial x}. \]
c) For $\beta_1 = \beta_3 = 0$ and $\beta_2 \neq 0$:

$$Z_1 = -\frac{2}{3} u \frac{\partial}{\partial u} + x \frac{\partial}{\partial x} - t \frac{\partial}{\partial t}.$$ 

d) For $\beta_2 = \beta_3 = 0$ and $\beta_1 \neq 0$:

$$Z_1 = -u \frac{\partial}{\partial u} + x \frac{\partial}{\partial x} - 3t \frac{\partial}{\partial t}.$$  

Let us now list the Lie point symmetries of equation (61), that is

$$u_t = \frac{au_{xxx}}{u^3} - 6 \frac{au_x u_{xx}}{u^4} + 6 \frac{a u_x^3}{u^5} + \beta_1 \frac{u_x}{u^3} + \beta_2 uu_x + \beta_3 u_x.$$  

Besides the obvious $x$-translation symmetry and $t$-translation symmetry, (61) also admits the following point symmetries:

a) For $\beta_1 = \beta_2 = \beta_3 = 0$:

$$Z_1 = x \frac{\partial}{\partial x} + 3t \frac{\partial}{\partial t}, \quad Z_2 = u \frac{\partial}{\partial u} + 3t \frac{\partial}{\partial t}, \quad Z_3 = xu \frac{\partial}{\partial u} - \frac{1}{2}x^2 \frac{\partial}{\partial x}.$$  

b) For $\beta_1 = \beta_2 = 0$ and $\beta_3 \neq 0$:

$$Z_1 = (x - 2t \beta_3) \frac{\partial}{\partial x} + 3t \frac{\partial}{\partial t}, \quad Z_2 = u \frac{\partial}{\partial u} + 3t \frac{\partial}{\partial t} - 3t \beta_2 \frac{\partial}{\partial x},$$

$$Z_3 = u(x + \beta_3 t) \frac{\partial}{\partial u} - \frac{1}{2}(x + \beta_3 t)^2 \frac{\partial}{\partial x}.$$  

c) For $\beta_1 = \beta_3 = 0$ and $\beta_2 \neq 0$:

$$Z_1 = u \frac{\partial}{\partial u} - 2x \frac{\partial}{\partial x} - 3t \frac{\partial}{\partial t}.$$  

d) For $\beta_2 = \beta_3 = 0$ and $\beta_1 \neq 0$:

$$Z_1 = u \frac{\partial}{\partial u} + 3t \frac{\partial}{\partial t}.$$  

$$Z_2 = u \sin \left( a^{-1/2} \beta_{1/2}^2 x \right) \frac{\partial}{\partial u} + a^{1/2} \beta_1^{-1/2} \cos \left( a^{-1/2} \beta_{1/2}^2 x \right) \frac{\partial}{\partial x}.$$  

$$Z_3 = u \cos \left( a^{-1/2} \beta_{1/2}^2 x \right) \frac{\partial}{\partial u} - a^{1/2} \beta_1^{-1/2} \sin \left( a^{-1/2} \beta_{1/2}^2 x \right) \frac{\partial}{\partial x}.$$  

We can now obviously map the symmetries of (58) and (61) with the (64a) – (64c) to symmetries of the isochronous equations (66) and (71). For example, in vertical form the $x$-translation symmetry

$$u_x \frac{\partial}{\partial u},$$

then takes the form

$$e^{-i\mu \omega s} v_r \frac{\partial}{\partial v},$$

for (66) and (71), whereas the $t$-translation symmetry

$$u_t \frac{\partial}{\partial u},$$

becomes the symmetry

$$e^{-i\omega s} \left( v_s - i\lambda \omega v - i\mu \omega rv_r \right) \frac{\partial}{\partial v}$$

for (66) and (71).
6.3. The isochronous hierarchies for (58) and (61). For equation (58) we have the hierarchy (60), namely

\[ u_t = R_1^m[u] \left( \frac{au_{xxx}}{u^6} - 12\frac{au_xu_{xx}}{u^7} + 21\frac{au_x^3}{u^8} + \beta_1 \frac{u_x}{u^4} + \beta_2 \frac{u_x^3}{u^5} + \beta_3 u_x, \right) \]

\[ m = 0, 1, 2, \ldots, \]

where \( R_1[u] \) is given by (59). Corresponding to the above Case 1.1, Case 1.2 and Case 1.3, the isochronous hierarchies are the following:

**Case 1.1.** We consider the hierarchy (60) with \( \beta_2 = \beta_3 = 0 \). This leads to the following isochronous hierarchy

\[ v_s + i \left( \frac{1}{2m+3} \right) \omega v + i \left( \frac{1}{2m+3} \right) \omega r v_r = R_{11}^m[v] \left( \frac{av_{rrr}}{v^6} - \frac{12av_r v_{rr}}{v^7} + \frac{21av_r^3}{v^8} + \beta_1 \frac{v_r}{v^4} \right), \quad m = 0, 1, 2, \ldots, \]

where

\[ R_{11}[v] = \frac{1}{v^4} D_r^2 - \frac{6v_r}{v^5} D_r - \frac{6v_{rr}}{v^5} + \frac{22v_r^2}{v^6} + \frac{4\beta_1}{3a v^2} \]

\[ - \frac{2}{a} \left( \frac{av_{rrr}}{v^6} - \frac{12av_r v_{rr}}{v^7} + \frac{21av_r^3}{v^8} + \beta_1 \frac{v_r}{v^4} \right) D_r^{-1} \circ v \]

\[ + v_r D_r^{-1} \circ \left( \frac{-2v_{rr}}{v^6} + \frac{6v_r^2}{v^7} + \frac{2\beta_1}{3a v^3} \right). \]

The first member of the hierarchy (76) for \( m = 0 \) is the equation (68).

**Case 1.2.** We consider the hierarchy (60) with \( \beta_1 = \beta_3 = 0 \). This leads to the following isochronous hierarchy

\[ v_s + i \left( \frac{2}{2m+3} \right) \omega v + i \left( \frac{3}{2m+3} \right) \omega r v_r = R_{12}^m[v] \left( \frac{av_{rrr}}{v^6} - \frac{12av_r v_{rr}}{v^7} + \frac{21av_r^3}{v^8} + \beta_1 \frac{v_r}{v^4} \right), \quad m = 0, 1, 2, \ldots, \]

where

\[ R_{12}[v] = \frac{1}{v^4} D_r^2 - \frac{6v_r}{v^5} D_r - \frac{6v_{rr}}{v^5} + \frac{22v_r^2}{v^6} + \frac{2\beta_2}{a v} \]

\[ - \frac{2}{a} \left( \frac{av_{rrr}}{v^6} - \frac{12av_r v_{rr}}{v^7} + \frac{21av_r^3}{v^8} + \beta_2 \frac{v_r}{v^4} \right) D_r^{-1} \circ v \]

\[ + v_r D_r^{-1} \circ \left( \frac{-2v_{rr}}{v^6} + \frac{6v_r^2}{v^7} + \frac{\beta_2}{a v^2} \right). \]

The first member of the hierarchy (77) for \( m = 0 \) is the equation (69).

**Case 1.3.** We consider the hierarchy (60) with \( \beta_1 = \beta_2 = 0 \). This leads to the following isochronous hierarchy

\[ v_s + i \lambda \omega v + i \left( 2\lambda + \frac{1}{2m+3} \right) \omega r v_r = R_{13}^m[v] \left( \frac{av_{rrr}}{v^6} - \frac{12av_r v_{rr}}{v^7} + \frac{21av_r^3}{v^8} + \beta_3 v_r \right), \quad m = 1, 2, \ldots \]
where \( \lambda \) is arbitrary and

\[
R_{13}[v] = \frac{1}{v^4} D_r^2 - \frac{6v_r}{v^5} D_r - \frac{6v_{rr}}{v^3} + \frac{22v_r^2}{v^6} - \frac{2}{a} \left( \frac{av_{rrr}}{v^6} - \frac{12av_r v_{rr}}{v^7} + \frac{21av_r^3}{v^8} + \beta_3 v_r \right) D^{-1}_r \circ v
\]

+ \( v_r D^{-1}_r \circ \left( -\frac{2v_{rr}}{v^6} + \frac{6v_r^2}{v^9} + \frac{2\beta_3}{a} v_r \right) \).

Note that equation (70) does not correspond to \( m = 0 \) in (78). The reason is rather obvious: since

\[
\text{R}_{13}^m[v] (\beta_3 v_r) = 0
\]

for all \( m = 1, 2, \ldots \), the \( \beta_3 \) term disappears in (78) and there remains only one constraint on \( \lambda \) and \( \mu \) to assure that the hierarchy does not depend explicitly on \( s \), namely

\[
\mu - 2\lambda - \frac{1}{2m + 3} = 0.
\]

For the equation (61) we have the hierarchy (63), namely

\[
u_t = R^m_2[u] \left( \frac{a v_{xxx}}{v^3} - \frac{6 a u_{xx} u_{xx}}{u^4} + \frac{6 a u_x^3}{u^5} + \left( \frac{1}{a} \frac{v_r}{v^4} + \beta_1 v_r \right) \right),
\]

where \( R^m_2[u] \) is given by (62). Corresponding to the above Case 2.1, Case 2.2 and Case 2.3, the isochronous hierarchies are the following:

**Case 2.1.** We consider the hierarchy (63) with \( \beta_1 = \beta_3 = 0 \). This leads to the following isochronous hierarchy

\[
v_s + i \left( \frac{1}{2m + 3} \right) \omega v = R^m_{21}[v] \left( \frac{a v_{rrr}}{v^3} - \frac{6 a v_{r r} v_{rr}}{v^4} + \frac{6 a v_r^3}{v^6} + \beta_1 v_r \right)
\]

\[
\quad m = 0, 1, 2, \ldots,
\]

where

\[
R_{21}[v] = \frac{1}{v^2} D_r^2 - \frac{3v_r}{v^3} D_r - \frac{3v_{rr}}{v^3} + \frac{v_r^2}{v^4} + \frac{\beta_1}{a} \frac{1}{v^2}
\]

\[
- \frac{1}{a} \left( \frac{a v_{rrr}}{v^3} - \frac{6 a v_{r r} v_{rr}}{v^4} + \frac{6 a v_r^3}{v^6} + \beta_1 v_r \right) D^{-1}_r \circ 1.
\]

**Case 2.2.** We consider the hierarchy (63) with \( \beta_1 = \beta_3 = 0 \). This leads to the following isochronous hierarchy

\[
v_s - i \left( \frac{1}{2m + 3} \right) \omega v - i \left( \frac{2}{2m + 3} \right) \omega rv_r =
\]

\[
= R^m_{22}[v] \left( \frac{a v_{rrr}}{v^3} - \frac{6 a v_{r r} v_{rr}}{v^4} + \frac{6 a v_r^3}{v^6} + \beta_2 v v_r \right)
\]

\[
\quad m = 0, 1, 2, \ldots,
\]
where
\[
R_{22}^2[v] = \frac{1}{v^3} \frac{D_r^2}{v^3} - \frac{3v_r}{v^3} D_r - \frac{3v_r}{v^3} v^2 + 6 \frac{v_r^2}{v^4} + \frac{\beta_2}{3a} v^2
- \frac{1}{a} \left( \frac{av_{rrr}}{v^3} - \frac{6av_r v_{rr}}{v^4} + \frac{6av^3}{v^5} + \frac{\beta_2 v_r}{v^3} \right) D_r^{-1} \circ 1 + 4 \frac{\beta_2}{3a} v_r D_r^{-1} \circ v. \tag{81}
\]

**Case 2.3.** We consider the hierarchy (63) with $\beta_1 = \beta_2 = 0$. This leads to the following isochronous hierarchy
\[
v_s - i\lambda \omega v - i \left( \lambda + \frac{1}{2m+3} \right) \omega rv_r
= R_{23}^m[v] \left( \frac{av_{rrr}}{v^3} - \frac{6av_r v_{rr}}{v^4} + \frac{6av^3}{v^5} + \beta_3 v_r \right) \tag{82}
\]
where
\[
R_{23}^m[v] = \frac{1}{v^3} \frac{D_r^2}{v^3} - \frac{3v_r}{v^3} D_r - \frac{3v_r}{v^3} v^2 + 6 \frac{v_r^2}{v^4}
- \frac{1}{a} \left( \frac{av_{rrr}}{v^3} - \frac{6av_r v_{rr}}{v^4} + \frac{6av^3}{v^5} + \beta_3 v_r \right) D_r^{-1} \circ 1 + \frac{\beta_3}{a} v_r D_r^{-1} \circ 1.
\]

Note that equation (75) does not correspond to $m = 0$ in (82) for the same reason as in Case 1.3. That is, since
\[
R_{23}^m[v] (\beta_3 v_r) = 0
\]
for all $m = 1, 2, \ldots$, the $\beta_3$ term disappears in (82) and there remains only one constraint on $\lambda$ and $\mu$ to assure that the hierarchy does not depend explicitly on $s$, namely
\[
\mu - 2\lambda - \frac{1}{2m+3} = 0.
\]

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