Eventual regularization of the slightly supercritical fractional Burgers equation

Chi Hin Chan, Magdalena Czubak and Luis Silvestre

November 26, 2009

Abstract

We prove that a weak solution of a slightly supercritical fractional Burgers equation becomes Hölder continuous for large time.

1 Introduction

We consider the fractional Burgers equation

$$\theta_t + \theta \cdot \theta_x + (-\Delta)^s \theta = 0.$$  \hfill (1.1)

It is well known that solutions $\theta$ of the subcritical ($s > 1/2$) and critical ($s = 1/2$) Burgers equation are smooth \cite{9}, \cite{7}, \cite{4}.

There are parallel results for the quasi-geostrophic equation. In the subcritical case, the solutions are smooth \cite{5}. In the critical case the solutions are also smooth, which was proved independently by Kiselev, Nazarov and Volberg \cite{8} and Caffarelli and Vasseur \cite{3} using different methods. The proof by Kiselev, Nazarov and Volberg is based on their previous work on the Burgers equation and consists of showing that certain modulus of continuity (that is essentially Lipschitz for nearby points) is preserved by the flow. The proof by Caffarelli and Vasseur is more involved and consists in proving a Hölder continuity result using classical ideas of De Giorgi.

The two different methods were also used in the context of the critical Burgers equation. The method of modulus of continuity was used in \cite{9} to show smoothness of solutions in the periodic setting. On the other hand, the parabolic De Giorgi method developed in \cite{3} was used in \cite{4} to show smoothness of solutions in the non-periodic setting.

For the case of the supercritical quasi-geostrophic equation, it was shown that the solutions are smooth for large time if $s = 1/2 - \varepsilon$ for a small $\varepsilon$ \cite{13} extending the methods of Caffarelli and Vasseur. More precisely the idea is to use the extra room in the improvement of oscillation lemma to compensate for the bad scaling.

In this article, we prove that the solutions of a slightly supercritical fractional Burger’s equation become regular for large time. It is a similar result to the one shown in \cite{13} for the quasi-geostrophic equation.

It is important to point out that in \cite{9,11,7} it was shown that singularities indeed occur for any $s < 1/2$. What we show here is that they disappear after a certain amount of time. Even though singularities may (and sometimes do) appear during an interval of time $[0, T]$, for $t > T$ they do not occur any more. The amount of time $T$ that we need to wait depends on the initial data and the value of $s$. For any given initial data, $T \to 0$ as $s \to 1/2$. The essential idea of the proof is to combine the ideas from \cite{3} and \cite{13}. On the other hand, we can present a completely self contained proof which has been simplified considerably.
The idea in the proofs in this paper is still to make the improvement of oscillation in parabolic cylinders compete with the deterioration of the equation due to scaling. The improvement of oscillation lemma is the lemma which allows us to show Hölder continuity when we iterate it at different scales (as in the classical methods of De Giorgi). We present a simple and completely self contained proof of this crucial lemma in this paper (section 4). An alternative approach could be to redo the proof in [4] adapted to general powers of the Laplacian using the extension in [2]. We find a few advantages in the choice of presenting this new proof of the oscillation lemma in this article. One is that it makes the paper self contained. It also provides a proof that does not use the extension argument and thus it could be generalized to other integral operators instead of the fractional Laplacian. The new proof is essentially a parabolic adaptation of the ideas in [12]. This proof uses strongly that the equation is non-local. This idea is also used in [11] to obtain a Hölder estimate for critical advection diffusion equations for bounded flows that are not necessarily divergence free.

We now state the main result.

**Theorem 1.1.** There exists a universal constant $\alpha \in (0, \frac{1}{2})$ such that if $\theta$ is a solution of (1.1) in $\mathbb{R} \times [0, +\infty]$ with $1 - \frac{\alpha}{2} < s \leq 1$ and initial data $\theta_0 \in L^2$, then there exists $T^* > 0$ such that when $t > T^*$, $\theta(t)$ is $C^\alpha(T^* \text{ depending only on } \|\theta_0\|_{L^2})$.

**Remark 1.2.** We note that we believe this could be extended to data in any $L^p$, $1 \leq p < \infty$, but for simplicity we do not pursue this here.

**Notation:**

$Q_r = [-r, r] \times [-r^{2s}, 0]$.

$\text{osc}_{Q_r} \theta = \sup_{Q_r} \theta - \inf_{Q_r} \theta$.

## 2 Preliminaries

### 2.1 The notion of a solution and vanishing viscosity approximation

By a **solution** of (1.1) we mean a weak solution (a solution in the sense of distributions) that can be obtained through the vanishing viscosity method. In other words it is a limit as $\varepsilon_1 \to 0$ of solutions satisfying

$$
\theta_t + \theta \cdot \nabla_x + (\nabla^s \theta - \varepsilon_1 \Delta \theta = 0,
\theta(\cdot, 0) = \theta_0 \in L^2(\mathbb{R}),
$$

(2.1)

where $\theta_0$ is an initial data for (1.1).

For every $\varepsilon_1 > 0$ and $\theta_0 \in L^2$, the equation (2.1) has a solution $\theta$ which is $C^\infty$ for all $t > 0$. We list the properties of such solution in the next elementary lemma.

**Lemma 2.1.** For every $\varepsilon_1 > 0$ and $\theta_0 \in L^2$, the equation (2.1) is well posed and its solution $\theta$ satisfies

1. $\theta(\cdot, t) \in C^\infty$ for every $t > 0$.

2. **Energy equality:**

$$
\|\theta(\cdot, t)\|_{L^2(\mathbb{R})}^2 + \int_0^t \|\theta(\cdot, t)\|_{H^s(\mathbb{R})}^2 + \varepsilon_1 \|\theta(\cdot, t)\|_{H^1(\mathbb{R})}^2 \, dt = \|\theta_0\|_{L^2(\mathbb{R})}^2.
$$

where $H^s$ stands for the homogeneous Sobolev space.
3. For every $t > 0$, $\theta(x, t) \to 0$ as $x \to \pm \infty$.

Proof. We consider the operator that maps $\theta$ to the solution of

$$\hat{\theta}_t + (-\Delta)^s \hat{\theta} - \varepsilon_1 \Delta \hat{\theta} = -\theta \partial_x \theta.$$  

Then we see that the map $A : \theta \mapsto \hat{\theta}$ is a contraction in the norm

$$|||\theta||| = \sup_{[0,T]} \|\theta(\cdot, t)\|_{L^2} + t^{1/2} \|\partial_x \theta(\cdot, t)\|_{L^2}$$  

To see that we note

$$|||e^{-t((\cdot)^s - \varepsilon_1 \Delta)}(\cdot)\theta_0||| \leq C ||\theta_0||_{L^2}.$$  

(This is an elementary computation using Fourier transform). Given $\theta_1$ and $\theta_2$ such that $|||\theta_i||| \leq R$ for $i = 1, 2$, we estimate $|||A\theta_1 - A\theta_2|||$ using Duhamel formula. On one hand we have

$$||A\theta_1(\cdot, t) - A\theta_2(\cdot, t)||_{L^2} \leq C \int_0^t \|\theta_1(\cdot, r)\partial_x \theta_1(\cdot, r) - \theta_2(\cdot, r)\partial_x \theta_2(\cdot, r)\|_{L^2} \, dr$$

$$\leq C \int_0^t \|\theta_1 - \theta_2\|_{L^\infty} \|\partial_x \theta_1\|_{L^2} + \|\theta_2\|_{L^\infty} \|\partial_x \theta_1 - \partial_x \theta_2\|_{L^2} \, dr$$

Using the interpolation inequality: $||f||_{L^\infty} \leq ||f||_{L^2}^{1/2} ||f'||_{L^2}^{1/2}$,

$$\leq CR ||\theta_1 - \theta_2|| \int_0^t (t - r)^{-1/4} \, dr \leq CR ||\theta_1 - \theta_2|| t^{3/4}.$$  

On the other hand, we also estimate

$$t^{1/2} \|\partial_x A\theta_1(\cdot, t) - \partial_x A\theta_2(\cdot, t)\|_{L^2} \leq C t^{1/2} \int_0^t (t - r)^{-1/2} \|\theta_1(\cdot, r)\partial_x \theta_1(\cdot, r) - \theta_2(\cdot, r)\partial_x \theta_2(\cdot, r)\|_{L^2} \, dr$$

$$\leq CR t^{1/2} ||\theta_1 - \theta_2|| \int_0^t (t - r)^{-3/4} \, dr \leq CR ||\theta_1 - \theta_2|| t^{3/4}$$

Thus, if we choose $T$ small enough (depending on $R$), $A$ will be a contraction in the ball of radius $R$ with respect to the norm $|||\cdot|||$. Therefore, the equation (2.1) has a unique solution locally in time for which the norm $|||\cdot|||$ is bounded. A standard bootstrap argument proves that moreover $|||\partial_x^k \theta|||_{L^2} \leq C t^{-k/2}$ for all $k \geq 0$. This proves 1. and 3. for short time.

The energy equality 2. follows immediately by multiplying equation (2.1) by $\theta$ and integrating by parts. Since the $L^2$ norm of the solution is non increasing, the solution can be continued forever, thus 1. and 3. hold for all time. \qed

If we let $\varepsilon_1 \to 0$, the energy estimate allows us to obtain a subsequence of solutions of the approximated problem that converges weakly in $L^\infty(L^2) \cap L^2(\dot{H}^s)$ to a weak solution for which the energy inequality holds. In a later section, we will also prove a bound of the $L^\infty$ norm of $\theta(\cdot, t)$ for $t > 0$, that is also independent of $\varepsilon_1$, thus we can also find a subsequence that converges weak-* in $L^\infty((t, +\infty) \times \mathbb{R})$ for every $t > 0$.  


2.2 A word about scaling

There is a one-parameter group of scalings that keeps the equation invariant. It is given by \( \theta_r = r^{2s-1} \theta(rx, r^{2s}t) \). If \( \theta \) solves \((1.1)\), then so does \( \theta_r \). In the critical case \( s = 1/2 \), the scaling of the equation keeps the \( L^\infty \) norm fixed. This case is critical because the scaling coincides with the a priori estimate given by the maximum principle.

We can consider a one parameter scaling that preserves Hölder spaces. The function \( \theta_r = r^{-\alpha} \theta(rx, r^{2s}t) \) has the same \( C^\alpha \) semi-norm as \( \theta \). If we want to prove that \( \theta \in C^\alpha \), we will have to deal with this type of scaling, but in this case the equation is not conserved. Instead, if \( \theta \) satisfies \((1.1)\), \( \theta_r \) satisfies

\[
\partial_t \theta_r + r^{2s-1+\alpha} \theta_r \cdot \partial_x \theta_r + (-\Delta)^s \theta_r = 0.
\]

We have an extra factor in front of the nonlinear term. Note that if \( \alpha > 1 - 2s \) (only slightly supercritical) and \( r < 1 \) (zoom in), this factor is smaller than one.

In the case of the equation with the extra term \( \varepsilon_1 \Delta \theta \), the viscosity will have a larger effect in smaller scales. Indeed, if \( \theta \) satisfies \((2.1)\), \( \theta_r \) satisfies

\[
\partial_t \theta_r + r^{2s-1+\alpha} \theta_r \cdot \partial_x \theta_r + (-\Delta)^s \theta_r + r^{2s-2} \varepsilon_1 \Delta \theta_r = 0.
\]

3 \( L^\infty \) Decay

First, as an immediate consequence of the energy equality in Lemma 2.1 we have the following lemma.

**Lemma 3.1.** If \( \theta \) is a solution of \((1.1)\), then

\[
\|\theta(t)\|_{L^2(\mathbb{R})} \leq \|\theta_0\|_{L^2(\mathbb{R})}.
\]

Nonincreasing properties of \( L^p \) norms as above for general \( 1 < p \leq \infty \) for the quasi-geostrophic equations were showed in \([10],[6]\). Now we have a theorem about the decay of the \( L^\infty \) norm. See also \([9],[3],[4],[13]\).

**Theorem 3.2.** If \( \theta \) is a solution of \((1.1)\), then

\[
\sup_{x \in \mathbb{R}} |\theta(x,t)| \leq C(s)t^{-\frac{s+2}{2s}} \|\theta_0\|_{L^2(\mathbb{R})}, \quad (3.1)
\]

where \( C(s) = \frac{2s}{C_1} \frac{2}{\sqrt{1+4s}} \), and \( C_1 \) is the constant appearing the integral formulation of the fractional Laplacian below.

**Proof.** Let \( T > 0 \) and suppose \( \theta \) is a solution of \((2.1)\). Define

\[
F(x,t) = t^\frac{1}{p} \theta(x,t),
\]

for some \( p \) to be chosen later. By Lemma 2.1 there must exist a point \((x_0,t_0)\) such that

\[
\sup_{x \times [0,T]} F(x,t) = F(x_0,t_0) < \infty.
\]

Observe that \( F \) satisfies the following equation

\[
F_t - \epsilon \Delta F + (-\Delta)^s F = \frac{1}{pt} F - \frac{1}{t^\alpha} F \cdot F_x. \quad (3.2)
\]
At \( (x_0, t_0) \) we have
\[
F_t \geq 0, \quad F_x = 0, \quad -\Delta F \geq 0.
\]
Then by (3.2)
\[
(-\Delta)^s F(x_0, t_0) \leq \frac{1}{p|t_0|} F(x_0, t_0). \tag{3.3}
\]
Using \( F(x_0, t_0) - F(y, t_0) \geq 0 \) for all \( y \in \mathbb{R} \), we compute a lower bound for \((-\Delta)^s F(x_0, t_0)\) as follows
\[
(-\Delta)^s F(x_0, t_0) = C_s \int_{\mathbb{R}} \frac{F(x_0, t_0) - F(y, t_0)}{|x-y|^{1+2s}} \, dy
\geq C_s \int_{|x-y|>R} \frac{F(x_0, t_0) - F(y, t_0)}{|x-y|^{1+2s}} \, dy, \quad \text{for any } R > 0
= \frac{C_s}{sR^{2s}} F(x_0, t_0) - C_s \int_{|x-y|>R} \frac{F(y, t_0)}{|x-y|^{1+2s}} \, dy. \tag{3.4}
\]
Next by Cauchy Schwarz
\[
\int_{|x-y|>R} \frac{F(y, t_0)}{|x-y|^{1+2s}} \, dy \leq \frac{\tilde{C}_s}{R^{1/2+2s}} \|\theta(0)\|_{L^2(\mathbb{R})}
= \frac{\tilde{C}_s t_0^{\frac{1}{p}}}{R^{1/2+2s}} \|\theta(0)\|_{L^2(\mathbb{R})}, \tag{3.5}
\]
where the last inequality follows from Lemma 3.1 and \( \tilde{C}_s = \left( \frac{s}{1+2s} \right)^{\frac{1}{p}} \). Combine (3.3)-(3.5) to obtain
\[
\frac{1}{p|t_0|} F(x_0, t_0) \geq C_s \left( \frac{1}{sR^{2s}} F(x_0, t_0) - \frac{\tilde{C}_s t_0^{\frac{1}{p}}}{R^{1/2+2s}} \|\theta(0)\|_{L^2(\mathbb{R})} \right),
\]
or equivalently
\[
\left( \frac{C_s}{sR^{2s}} - \frac{1}{p|t_0|} \right) F(x_0, t_0) \leq \tilde{C}_s C_s t_0^{\frac{1}{p}} \|\theta(0)\|_{L^2(\mathbb{R})}.
\]
Let \( p = 4s \), and choose \( R \) so that \( \frac{C_s}{sR^{2s}} = \frac{1}{2|t_0|} \). Rearranging we have
\[
F(x_0, t_0) \leq C(s) \|\theta(0)\|_{L^2(\mathbb{R})},
\]
with \( C(s) \) as in the statement of the theorem. Finally, from the definition of \( F \)
\[
\sup_{\mathbb{R} \times [0,T]} t^{\frac{1}{2s}} \theta(x, t) \leq C(s) \|\theta(0)\|_{L^2(\mathbb{R})},
\]
or
\[
\sup_{\mathbb{R} \times [0,T]} \theta(x, t) \leq t^{-\frac{1}{2s}} C(s) \|\theta(0)\|_{L^2(\mathbb{R})},
\]
and since the estimate is independent of \( \epsilon_1 \) and \( T \) is arbitrary, the theorem follows (note this gives an upper bound for \( \theta \). To obtain a lower bound we can redo the proof with \( F \) defined by \(-t^{\frac{1}{2s}} \theta(x, t)\).

\textbf{Remark 3.3.} Note that an estimate like (3.3) could be obtained using any \( L^p \) norm instead of \( L^2 \). We chose to use \( L^2 \) because it is the norm that is easiest to show that it stays bounded (using the energy inequality).
4 The oscillation lemma

Lemma 4.1. Let $M_0 \geq 2$ and $s \in [\frac{1}{2}, \frac{3}{2}]$. Assume $\theta \leq 1$ in $\mathbb{R} \times [-\frac{2}{M_0}, 0]$ and $\theta$ is a subsolution of

\[ \theta_t + M\theta \cdot \theta_x + (-\Delta)^s \theta - \varepsilon_1 \Delta \theta \leq \varepsilon_0, \]

in the set $[-5, 5] \times [-\frac{2}{M_0}, 0]$ where $|M| \leq M_0$ and $0 < \varepsilon_1 \leq 10^{3/2}$. Assume also that

\[ |\{\theta \leq 0\} \cap ([-1, 1] \times [-\frac{2}{M_0}, -\frac{1}{M_0}])| \geq \mu. \]

Then, if $\varepsilon_0$ is small enough (depending only on $\mu$ and $M_0$) there is a $\lambda > 0$ (depending only on $\mu$ and $M_0$) such that $\theta \leq 1 - \lambda$ in $[-1, 1] \times [-\frac{1}{M_0}, 0]$.

We will apply the lemma above only to the case when $M$ is constant in $Q_1$. This is not necessary to prove the lemma as it will be apparent in the proof. We are not aware of any possible application of the lemma with variable $M$ (even discontinuous).

Proof. Let $m : [-\frac{2}{M_0}, 0] \to \mathbb{R}$ be the solution of the following ODE:

\[ m(-\frac{2}{M_0}) = 0, \]

\[ m'(t) = c_0|\{x \in [-1, 1] : \theta(x, t) \leq 0\}| - C_1m(t). \tag{4.1} \]

The above ODE can be solved explicitly and $m(t)$ has the formula

\[ m(t) = \int_{-\frac{2}{M_0}}^t c_0|\{x : \theta(x, s) \leq 0\}| \cap B_1|e^{-C_1(t-s)} \, ds. \]

We will show that if $c_0$ is small and $C_1$ is large, then $\theta \leq 1 - m(t) + \varepsilon_0$ in $[-1, 1] \times [-\frac{1}{M_0}, 0]$. This naturally implies the result of the lemma since for $t \in [-\frac{1}{M_0}, 0]$,

\[ m(t) \geq c_0e^{\frac{-2c_1}{M_0}}|\{\theta \leq 0\} \cap [-1, 1] \times [-\frac{2}{M_0}, -\frac{1}{M_0}]| \geq c_0e^{\frac{-2c_1}{M_0}} \mu. \]

So we can set $\lambda = c_0e^{\frac{-2c_1}{M_0}} \mu/2$ for $\varepsilon_0$ small.

Let $\beta : \mathbb{R} \to \mathbb{R}$ be a fixed smooth nonincreasing function such that $\beta(x) = 1$ if $x \leq 1$ and $\beta(x) = 0$ if $x \geq 2$. Moreover, we can take $\beta$ with only one inflection point between 0 and 2, so that if $\beta \leq \beta_0$ then $\beta'' \geq 0$.

Let $b(x, t) = \beta(|x| + M_0t) = \beta(|x| - M_0|t|)$. As a function of $x$, $b(x, t)$ looks like a bump function for every fixed $t$. By construction $b_{xx} \geq 0$ if $b \leq \beta_0$. Moreover, at those points where $b = 0$ (precisely where $|x| \geq 2 - M_0t = 2 + M_0|t|$), $(-\Delta)^sb < 0$. Since $b$ is smooth, $(-\Delta)^sb$ is continuous and it remains negative for $b$ small enough. Thus, there is some constant $\beta_1$ such that $b_{xx} \geq 0$ and $(-\Delta)^sb \leq 0$ if $b \leq \beta_1$.

Assume that $\theta(x, t) > 1 - m(t) + \varepsilon_0(1 + t)$ for some point $(x, t) \in [-1, 1] \times [-\frac{1}{M_0}, 0]$. We will arrive to a contradiction by looking at the maximum of the function

\[ w(x, t) = \theta(x, t) + m(t)b(x, t) - \varepsilon_0(1 + t). \]

We are assuming that there is one point in $[-1, 1] \times [-\frac{1}{M_0}, 0]$ where $w(x, t) > 1$. Let $(x_0, t_0)$ be the point that realizes the maximum of $w$:

\[ w(x_0, t_0) = \max_{\mathbb{R} \times [-\frac{1}{M_0}, 0]} w(x, t). \]
(Note \((x_0, t_0)\) exists by the definition of \(w\) and Lemma \([24]\)) Since \(w(x_0, t_0) > 1\), by using the fact that \(\theta(x_0, t_0) \leq 1\), we deduce \(m(t_0)b(x_0, t_0) > \varepsilon_0(1 + t_0) > 0\), which further implies \(m(t_0) > 0\) (this tells us that \(t_0 > -\frac{2}{M_0}\)) and \(b(x_0, t_0) > 0\), so \(|x_0| < 2 + M_0|t_0| \leq 4\).

Since the function \(w\) realizes a maximum at \((x_0, t_0)\), we have the following elementary inequalities:

\[
w(x_0, t_0) > 1
\]
\[
w_t(x_0, t_0) \geq 0
\]
\[
w_x(x_0, t_0) = 0
\]
\[
\Delta w(x_0, t_0) \leq 0
\]
\[
(-\Delta)^s w(x_0, t_0) \geq 0
\]

The last inequality can be turned into a more useful estimate by recalling the integral formula of \((-\Delta)^s w\) and looking at the set of points where \(\theta \leq 0\).

\[
(-\Delta)^s w(x_0, t_0) = C_s \int_{\mathbb{R}} \frac{w(x_0, t_0) - w(y, t_0)}{|x_0 - y|^{1+2s}} \, dy
\]

(Note the integrand is nonnegative)

\[
\geq C_s \int_{\{y \in [-1,1] : \theta(y, t_0) \leq 0\}} (w(x_0, t_0) - w(y, t_0)) 5^{-1-2s} \, dy
\]
\[
\geq C_s (1 - m(t_0)) 5^{-1-2s} \{y \in [-1,1] : \theta(y, t_0) \leq 0\}
\]
\[
\geq C_s \frac{25}{25} (1 - m(t_0)) \{y \in [-1,1] : \theta(y, t_0) \leq 0\},
\]

where the last inequality is valid since \(5^{1+2s} \leq 25\) for \(\frac{1}{4} \leq s \leq \frac{1}{2}\). We choose the constant \(c_0\) in order to make sure that \(m(t)\) stays below \(1/4\) (simply by choosing \(c_0 < 1/8\)), and we choose \(c_0 \leq \frac{3}{4} C_s\), so that

\[
(-\Delta)^s w(x_0, t_0) \geq c_0 \{y \in [-1,1] : \theta(y, t) \leq 0\}.
\]

(4.2)

Note that the constant \(C_s\) in the integral form of the fractional Laplacian stays bounded and away from zero as long as \(s\) stays away from 0 and 1. We can consider \(C_s\) bounded above and below independently of \(s\) as long as \(s\) stays in a range away from 0 and 1, like for example \(s \in [1/4, 1/2]\).

Now we recall that \(w = \theta + mb - \varepsilon_0(1 + t)\) and we rewrite the inequalities in terms of \(\theta\).

\[
1 \geq \theta(x_0, t_0) \geq 1 - m(t_0)b(x_0, t_0) \geq \frac{3}{4}
\]
\[
\theta_t(x_0, t_0) \geq -m'(t_0)b(t_0, x_0) + m(t_0)M_0|b_x(x_0, t_0)| + \varepsilon_0
\]
\[
\theta_x(x_0, t_0) = -m(t_0)b_x(x_0, t_0)
\]
\[
\Delta \theta(x_0, t_0) \leq -m(t_0)\Delta b(x_0, t_0)
\]
\[
(-\Delta)^s \theta(x_0, t_0) \geq -m(t_0)(-\Delta)^s b(x_0, t_0) + c_0 \{y \in [-1,1] : \theta(y, t_0) \leq 0\}
\]

We consider two cases and obtain a contradiction in both. Either \(b(x_0, t_0) > \beta_1\) or \(b(x_0, t_0) \leq \beta_1\).

Let us start with the latter. If \(b(x_0, t_0) \leq \beta_1\), then \(\Delta b(x_0, t_0) \geq 0\) and \((\Delta)^s b(x_0, t_0) \leq 0\), then

\[
\Delta \theta(x_0, t_0) \leq -m(t_0)\Delta b(x_0, t_0) \leq 0
\]
\[
(-\Delta)^s \theta(x_0, t_0) \geq c_0 \{y \in [-1,1] : \theta(y, t_0) \leq 0\}
\]

Therefore

\[
\varepsilon_0 \geq \theta_t + M\theta_x + (-\Delta)^s \theta - \varepsilon_1 \Delta \theta \geq \varepsilon_0 - m'(t_0)b(x_0) + c_0 \{y \in [-1,1] : \theta(y, t_0) \leq 0\},
\]

7
where in the last inequality, we have implicitly used the fact that
\[ m(t_0) (M_0 |b_x(x_0, t_0)| - M \theta(x_0, t_0) b_x(x_0, t_0)) \geq 0, \]
since \( 1 \geq \theta(x_0, t_0) \geq \frac{3}{4} \) and \( |M| \leq M_0 \).

So we obtain
\[ -m'(t_0)b(x_0) + c_0 |\{ y \in [-1,1] : \theta(y, t_0) \leq 0 \}| \leq 0, \]
buts this is a contradiction with (4.1) for any \( C_1 \geq 0 \).

Let us now analyze the case \( b(x_0, t_0) > \beta_1 \). Since \( b \) is a smooth, compactly supported function, there is some constant \( C \) (depending on \( M_0 \)), such that \( |\Delta b| \leq C \) and \( |(-\Delta)^{\alpha} b| \leq C \). Then we have the bounds
\[ \Delta \theta(x_0, t_0) \leq -m(t_0) \Delta b(x_0, t_0) \leq C m(t_0) \]
\[ (-\Delta)^{\alpha} \theta(x_0, t_0) \geq c_0 |\{ y \in [-1,1] : \theta(y, t_0) \leq 0 \}| - C m(t_0) \]

Therefore
\[ \varepsilon_0 \geq \theta_t + M \theta \cdot \theta_x + (-\Delta)^{\alpha} \theta - \varepsilon_1 \Delta \theta \geq \varepsilon_0 - m'(t_0)b(x_0, t_0) - C m(t_0) + c_0 |\{ y \in [-1,1] : \theta(y, t_0) \leq 0 \}| \]
and we have
\[ -m'(t_0)b(x_0, t_0) - C m(t_0) + c_0 |\{ y \in [-1,1] : \theta(y, t_0) \leq 0 \}| \leq 0. \]

We replace the value of \( m'(t_0) \) in the above inequality using (4.1) and obtain
\[ (C_1 b(x_0, t_0) - C) m(t_0) + c_0 (1 - b(x_0, t_0)) |\{ y \in [-1,1] : \theta(y, t) \leq 0 \}| \leq 0. \]

Recalling that \( b(x_0, t_0) \geq \beta_1 \), we arrive at a contradiction if \( C_1 \) is chosen large enough.

\begin{lemma}
Let \( s \in \left[ \frac{1}{4}, \frac{3}{4} \right] \), and let \( \theta \) be a solution of
\[ \theta_s + M \theta \cdot \theta_x + (-\Delta)^{\alpha} \theta - \varepsilon_1 \Delta \theta \leq 0, \quad (4.3) \]
where \( |M| \leq 1 \) and \( \varepsilon_1 \leq 1 \). Assume that \( |\theta| \leq 1 \) in \( Q_1 \) and \( |\theta(x)| \leq |500x|^{2s} \) for \( |x| > 1 \). Then if \( \alpha \) is small enough, there is a \( \lambda > 0 \) (which does not depend on \( \varepsilon_1 \)) such that \( \text{osc}_{Q_{1/400}} \theta \leq 2 - \lambda \).
\end{lemma}

There is no deep reason for the choice of the number 500 in the above lemma. But the smaller the cube is, say \( Q_{\frac{1}{400}} \), on which the improved oscillation occurs, we need a number, say 500, which is greater than 400 in order to make inequality (4.2) hold. In principle, 500 can be replaced by any number greater than 400.

\begin{proof}
We want to apply Lemma 4.2 to \( \theta \). We check if we have the required hypothesis. We set \( M_0 = 2 \cdot 10^{1/2} \). (The reason for this choice will become clear shortly.) Next, \( \theta \) will be either nonnegative or nonpositive in half of the points in \([-10, 10] \times [-\frac{2}{M_0}, \frac{1}{M_0}] \) (in measure). Let us assume \( \{(x, t) \in [-1, 1] \times [-\frac{2}{M_0}, \frac{1}{M_0}] : \theta(x, t) \leq 0 \} \geq \mu = \frac{1}{M_0} \). (Otherwise, we would continue the proof with \( -\theta \) instead of \( \theta \) and \( -M \).) Next, the hypothesis that we are missing is that \( \theta \) may be larger than 1 outside \( Q_1 \). Thus we define
\[ \overline{\theta} = \min(\theta, 1). \]

We show below \( \overline{\theta} \) satisfies
\[ \overline{\theta}_s + M \overline{\theta} \cdot \overline{\theta}_x + (-\Delta)^{\alpha} \overline{\theta} - \varepsilon_1 \Delta \overline{\theta} \leq \varepsilon_0. \quad (4.4) \]

\end{proof}
where, in the last inequality, we have used the assumption that \( \frac{1}{2} \leq s \leq \frac{1}{2} \). Notice \( \omega(\alpha) \to 0 \) as \( \alpha \to 0 \). So we can choose \( \alpha > 0 \) such that \( \omega(\alpha) < \varepsilon_0 \). Hence \( \bar{\theta} \) satisfies (4.3) over \( Q_{1/2} \) as claimed. However, in order to apply Lemma 4.1 we need to rescale so that we can have that the inequality holds on \([-5, 5] \times \left[-\frac{2}{M_0}, 0\right]\). Since we also need to preserve the condition \( \bar{\theta} \leq 1 \) after rescaling, we choose to work with the function \( \bar{\theta}(x, t) = \bar{\theta}\left(\frac{x}{10^s}, \frac{t}{10^{2s}}\right) \). Observe that \( \bar{\theta}' \) satisfies the following differential inequality over \( Q_5 \):

\[
\bar{\theta}'_t + 10^{-2s}M\bar{\theta}' \cdot \bar{\theta}''_x + (\Delta)^s\bar{\theta}' - 10^{-2s}\varepsilon_1\Delta\bar{\theta} \leq \frac{\varepsilon_0}{10^{2s}} \leq \varepsilon_0.
\]  

(4.5)

Observe that with \( M_0 = 2 \cdot 10^\frac{1}{2} \), \([-5, 5] \times \left[-\frac{2}{M_0}, 0\right] \subset Q_5 \), and \( 10^{2-2s}M \leq M_0 \). Also \( 10^{2-2s}\varepsilon_1 \leq 10^{3/2} \), and since by construction \( \bar{\theta}' \leq 1 \in \mathbb{R} \times \left[-\frac{2}{M_0}, 0\right] \), we now finally can apply Lemma 4.1 and obtain that \( \bar{\theta}' \leq 1 - \lambda \) over \([-1, 1] \times \left[-\frac{1}{M_0}, 0\right] \), where \( \lambda \) depends only on \( M_0 = 2 \cdot 10^\frac{1}{2} \). However, since we would like to have an improved oscillation on a parabolic cube, we note that \( Q_{1/40} = \left[-\frac{1}{10}, \frac{1}{10}\right] \times \left[-\frac{1}{M_{40}}, 0\right] \subset [-1, 1] \times \left[-\frac{1}{M_0}, 0\right] \), for \( \frac{1}{10} \leq s \leq \frac{1}{2} \). So we have \( \bar{\theta}' \leq 1 - \lambda \) over \( Q_{1/40} \). Hence by rescaling \( \theta = \bar{\theta} \leq 1 - \lambda \) in \( Q_{1/400} \). This completes the proof.

\[\square\]

5 Proof of the main result

To simplify the exposition of the proof of theorem 5.2, we first state and establish the following technical but elementary lemma.

Lemma 5.1. For any \( \rho \in (0, \frac{1}{100}) \), there exists some \( \alpha_1 \in (0, \frac{1}{2}) \), depending only on \( \rho \), such that for any \( 0 < \alpha < \alpha_1 \), the following holds:

\[
1 < \frac{1}{400\rho} - \frac{1}{\rho}(1 - \rho^\alpha),
\]  

(5.1)

\[
\rho^{-\alpha}(2 - \rho^\alpha) < 5002\alpha\left\{\frac{1}{400\rho} - \frac{1}{\rho}(1 - \rho^\alpha)\right\}^{2\alpha},
\]  

(5.2)

\[
\rho^{-\alpha}(5002\alpha + 1 - \rho^\alpha) < 5002\alpha\left\{\frac{1}{\rho} - \frac{1}{\rho}(1 - \rho^\alpha)\right\}^{2\alpha}.
\]  

(5.3)

Proof. (5.1) is immediate by the assumptions on \( \rho \). So is (5.2) after we observe that it is equivalent to

\[
\rho^{-\frac{1}{2}}(2 - \rho^\alpha)^{\frac{1}{\rho}} < 500\left(\frac{1}{400\rho} - \frac{1}{\rho}(1 - \rho^\alpha)\right).
\]

Since \( \lim_{\alpha \to 0} \rho^{-\frac{1}{2}}(2 - \rho^\alpha)^{\frac{1}{\rho}} = \frac{1}{\rho} < \frac{500}{400\rho} = \lim_{\alpha \to 0} 500\left(\frac{1}{400\rho} - \frac{1}{\rho}(1 - \rho^\alpha)\right) \), by continuity, the above inequality holds for sufficiently small \( \alpha > 0 \).
We rearrange (5.3), and note that it follows from showing that
\[ f(\alpha) = \rho^\alpha (500^2\alpha + 1 - \rho^\alpha) - 500^2\alpha \rho^{2\alpha}, \]
has a local maximum at 0. This is indeed true, since \( f(0) = f'(0) = 0, \) and
\[ f''(0) = \ln \rho (4 \ln 500 - 4 - 2 \ln \rho) < 0, \]
for any fixed \( \rho \in (0, \frac{1}{400}). \)

**Theorem 5.2.** Let \( \theta \) be a solution of (5.1) with \( |\theta| \leq 1 \) in \( \mathbb{R} \times [-1,0] \). There is a small \( \alpha \in (0, \frac{1}{2}) \) such that if \( \frac{\lambda}{2} < s < \frac{1}{2} \), then \( \theta \) satisfies
\[ |\theta(y,0) - \theta(x,0)| \leq C|x-y|^\alpha \]
for some constant \( C \) (independent of \( \varepsilon_1 \)) and for all points such that \( |x-y| > \varepsilon_1^{1-2s} \).

**Proof.** Fix \( \rho \in (0, \frac{1}{400}) \). Let \( \alpha_0, \) and \( \alpha_1 \) be as in Lemma 4.2 and Lemma 5.1 respectively. Take \( \alpha = \min\{\frac{\alpha_0}{2}, \frac{\alpha_1}{2}\} \) (\( \alpha \) depends only on \( \rho \)). Next let \( \lambda \) be as in Lemma 4.2. Then if necessary, we can make \( \lambda \) or \( \alpha \) smaller, so that \( 2 - \lambda = 2\rho^\alpha \). Finally, set \( \frac{\lambda}{2} < s < \frac{1}{2} \).

We define the sequence \( \theta_k \) recursively for all nonnegative integers \( k \) such that \( \rho^{2-2s}k \geq \varepsilon_1 \). We will do it so that every \( \theta_k \) satisfies
\[
\partial_t \theta_k + M_k \theta_k \partial_x \theta_k + (-\Delta)^s \theta_k - \rho^{2s-2k} \varepsilon_1 \Delta \theta_k = 0 \quad \text{in } Q_1 \text{ with } M_k \leq 1, \quad (5.4)
\]
\[ |\theta_k(x,t)| \leq 1 \quad \text{for } (x,t) \in Q_1, \quad (5.5) \]
\[ |\theta_k(x,t)| \leq 500^2 \alpha |x|^{2\alpha} \quad \text{for } |x| \geq 1 \text{ and } t \in [-1,0], \quad (5.6) \]
For all \( k \), we will have \( \theta_k(x,0) = \rho^{-\alpha k} \theta(\rho^k x,0) \). So (5.5) implies immediately the result of this theorem.

We have to construct the sequence \( \theta_k \). We start with \( \theta_0 = \theta \) and \( M_0 = 1 \) which clearly satisfy the assumptions. Now we define the following ones recursively. Let us assume that we have constructed up to \( \theta_k \) and let us construct \( \theta_{k+1} \).

Given the assumptions (5.4), (5.5) and (5.6), we can apply Lemma 4.2 as long as \( \varepsilon_1 < \rho^{2-2s}k \) and obtain that \( \text{osc}_{Q_1/400} \theta_k \leq 2 - \lambda = 2\rho^\alpha \). If \( \varepsilon_1 \geq \rho^{2-2s}k \), we stop the iteration, i.e., we iterate only until the viscosity term becomes large.

Since \( \text{osc}_{Q_1/400} \theta_k \leq 2 - \lambda \), there is a number \( d \in [-\lambda/2, \lambda/2] \) such that
\[ -1 + \lambda/2 \leq \theta_k - d \leq 1 - \lambda/2, \quad \forall (x,t) \in Q_{1/400}, \quad (5.7) \]
Now we define \( \theta_{k+1} \) as follows,
\[ \theta_{k+1}(x,t) = \rho^{-\alpha k} \theta_k(\rho(x + L_k), \rho^{2s} t - \rho L_k), \]
where \( L_k = \rho^{2s-1} M_k dt \). The function \( \theta_{k+1} \) satisfies the equation
\[ \partial_t \theta_{k+1} + \rho^{\alpha+2s-1} M_k \theta_{k+1} \partial_x \theta_{k+1} + (-\Delta)^s \theta_{k+1} - \rho^{2s-2(k+1)} \varepsilon_1 \Delta \theta_k = 0 \]
so we define \( M_{k+1} = \rho^\alpha + 2s-1 M_k \). Due to the fact that \( \alpha+2s-1 > 0 \) for our choice of \( s \in (\frac{1-\alpha}{2}, \frac{1}{2}) \), we have \( M_{k+1} \leq M_k \). Hence, we know that \( \theta_{k+1} \) satisfies (5.3).

Now, since the graph of \( 500^{2\alpha} |x|^{2\alpha} \) is symmetric about the y-axis, without loss of generality, suppose \( d < 0 \), so \( L_k > 0 \).

To establish (5.5) for \( \theta_{k+1} \), we first note that by (5.7) we have
\[ -1 + \lambda/2 \leq \theta_k(\rho(x + L_k), \rho^{2s} t - \rho L_k) - d \leq 1 - \lambda/2, \quad \forall x \in [-\frac{1}{400 \rho} - L_k, \frac{1}{400 \rho} - L_k], t \in [0,1]. \quad (5.8) \]
Next we show that the absolute value of the transport term $L_t = \rho^{2s-1} M_k dt$ is small enough, so that $[-1, 1] \subset [-\frac{1}{400\rho} - L_t, \frac{1}{400\rho} - L_t]$. Indeed, since $M_k dt \leq \frac{\rho}{\lambda} = (1 - \rho^\alpha)$ we have

$$\frac{1}{400\rho} - \rho^{2s-1} M_k dt \geq \frac{1}{400\rho} - \rho^{2s-1}(1 - \rho^\alpha) \geq \frac{1}{400\rho} - \rho(1 - \rho^\alpha) > 1,$$

which holds by (5.1). We conclude $[-1, 1] \subset [-\frac{1}{400\rho} - L_t, \frac{1}{400\rho} - L_t]$. Thus by (5.8) for all $(x, t) \in Q_1$

$$|\theta_{k+1}(x, t)| \leq \rho^{-\alpha} |\theta_k(\rho(x + L_t), \rho^{2s}t) - d| \leq \frac{1}{1 - \lambda/2}(1 - \lambda/2) = 1,$$

so (5.3) holds as needed.

Now we introduce

$$\psi(x) = \begin{cases} 
1 & \text{if } |x| < 1, \\
500^{2a} |x|^{2a} & \text{if } |x| \geq 1.
\end{cases}$$

By the inductive hypothesis

$$|\theta_k(x, t)| \leq \psi(x), \quad t \in [-1, 0].$$

Then observe that by definition of $\theta_{k+1}$, in order to establish (5.6) for $\theta_{k+1}$, it is enough to show

$$(\rho^{-\alpha} \psi(\rho(x + L_t)) + \rho^{-\alpha} |d|) \chi_{\{x + L_t \mid 1 \leq x \leq \frac{1}{400\rho}\}} \leq \psi(x). \quad (5.9)$$

First we note that

$$(\rho^{-\alpha} \psi(\rho(x + L_t)) + \rho^{-\alpha} |d|) \chi_{\{x + L_t \mid \frac{1}{400\rho} \leq x \leq \frac{1}{400\rho}\}} \leq \phi_1(x) + \phi_2(x),$$

where $\phi_1(x) = \rho^{-\alpha} (2 - \rho^\alpha) \chi_{\{\frac{1}{400\rho} \leq x \leq \frac{1}{400\rho}\}}$ and $\phi_2(x) = \rho^{-\alpha} \psi(\rho(x + L_t)) + \rho^{-\alpha} (1 - \rho^\alpha) \chi_{\{x + L_t \mid x \geq \frac{1}{2}\}}$.

So (5.9) will follow if we can show that $\phi_1 < \psi$ and $\phi_2 < \psi$.

To show $\phi_1 < \psi$, we observe that, by (5.1), we have

$$\phi_1(\frac{1}{400\rho} - L_t) = \rho^{-\alpha} (2 - \rho^\alpha) < \psi(\frac{1}{400\rho} - \frac{1}{\rho} (1 - \rho^\alpha)) \leq \psi(\frac{1}{400\rho} - \rho^{2s-1}(1 - \rho^\alpha)) \leq \psi(\frac{1}{400\rho} - L_t).$$

Since $\phi_1$ is constant over $[-\frac{1}{400\rho} - L_t, \frac{1}{400\rho} - L_t]$, and $\psi(x)$ is strictly increasing for $x \geq \frac{1}{400\rho} - L_t$, it follows that $\phi_1(\frac{1}{400\rho} - L_t) < \psi(\frac{1}{400\rho} - L_t)$ implies $\phi_1 \chi_{\{\frac{1}{400\rho} - L_t, \frac{1}{400\rho} - L_t\}} < \psi$. On the other hand, it is quite obvious that we must have $\phi_1 \chi_{\{\frac{1}{400\rho} - L_t, \frac{1}{400\rho} - L_t\}} < \psi$. Hence we deduce that $\phi_1 < \psi$.

To prove $\phi_2 < \psi$, we just need to observe that by (5.3)

$$\phi_2(\frac{1}{\rho} - L_t) = \rho^{-\alpha} \{500^{2a} + 1 - \rho^\alpha\} < \psi(\frac{1}{\rho} - \frac{1}{\rho} (1 - \rho^\alpha)) \leq \psi(\frac{1}{\rho} - \rho^{2s-1}(1 - \rho^\alpha)) \leq \psi(\frac{1}{\rho} - L_t).$$

Now, for any point $x \in \begin{cases} \frac{1}{\rho} - L_t, +\infty \end{cases}$ the derivative of $\phi_2$ at $x$ is strictly less than the derivative of $\psi$ at $x$. Because of this, $\phi_2(\frac{1}{\rho} - L_t) < \psi(\frac{1}{\rho} - L_t)$ at once implies that $\phi_2 \chi_{\{\frac{1}{\rho} - L_t, +\infty\}} < \psi$. On the other hand, we also have $\phi_2 \chi_{\{-\infty, -\frac{1}{\rho} - L_t\}} < \psi$. Hence we conclude that $\phi_2 < \psi$, and this completes the proof.

\[\Box\]

**Corollary 5.3.** Let $\theta$ be a solution of (1.1) with $|\theta| \leq 1$ in $\mathbb{R} \times [-1, 1]$. There is a small $\alpha \in (0, \frac{1}{2})$ such that if $\frac{1}{2\alpha} < s < \frac{1}{2}$ then $\theta(t, t) \in C^\alpha$ for all $t \geq 0$. 

Proof. For every $\varepsilon_1$, we have a solution $\theta^{\varepsilon_1}$ of (2.1) for which we can apply Theorem 5.2 in any interval of time $[-1 + t, t]$. Since neither constant $\alpha$ or $C$ depend on $\varepsilon_1$, then for any $h \in \mathbb{R}$,

$$\theta^{\varepsilon_1}(x + h, t) - \theta^{\varepsilon_1}(x, t) \leq C|h|^\alpha$$

for all $\varepsilon_1$ small enough (depending on $|h|$). This estimate passes to the limit as $\varepsilon_1 \to 0$ since $\theta^{\varepsilon_1}(., 0) \to \theta(., 0)$ weak-* in $L^\infty$. Moreover, it will hold for all $h$ at the limit, which finishes the proof.

Now the proof of the main result follows immediately.

Proof of Theorem 1.1. For any initial data $\theta_0 \in L^2$, by Theorem 3.2 $\|\theta(\cdot, t)\|_{L^\infty(\mathbb{R})}$ decays. So all we have to do is wait until it is less than one, and we can apply Corollary 5.3.

Remark 5.4. The only part of the paper where we use that the solution is in $L^2$ is in the proof of the decay of the $L^\infty$ norm (Theorem 3.2). For the rest of the paper, all we use is that the $L^\infty$ norm of $\theta$ will eventually become smaller than one so that we can apply Corollary 5.3. Of course there is nothing special about the number one, and a similar estimate can be obtained just by assuming that $\|\theta\|_{L^\infty} \leq C$. However, the value of $\alpha$ would depend on this $C$.

Acknowledgment

Luis Silvestre was partially supported by NSF grant DMS-0901995 and the Alfred P. Sloan foundation.

References

[1] Nathaël Alibaud, Jérôme Droniou, and Julien Vovelle. Occurrence and non-appearance of shocks in fractal Burgers equations. *J. Hyperbolic Differ. Equ.*, 4(3):479–499, 2007.

[2] L. Caffarelli and L. Silvestre. An extension problem related to the fractional Laplacian. *Communications in partial differential equations*, 32(8):1245–1260, 2007.

[3] L. Caffarelli and A. Vasseur. Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation. *Arxiv preprint math/0608447*, 2006.

[4] C.H. Chan and M. Czubak. Regularity of solutions for the critical $N$-dimensional Burgers’ equation. *Arxiv preprint arXiv:0810.3055*, 2008.

[5] Peter Constantin and Jiahong Wu. Behavior of solutions of 2D quasi-geostrophic equations. *SIAM J. Math. Anal.*, 30(5):937–948 (electronic), 1999.

[6] Antonio Córdoba and Diego Córdoba. A maximum principle applied to quasi-geostrophic equations. *Comm. Math. Phys.*, 249(3):511–528, 2004.

[7] Hongjie Dong, Dapeng Du, and Dong Li. Finite time singularities and global well-posedness for fractal Burgers equations. *Indiana Univ. Math. J.*, 58(2):807–821, 2009.

[8] A. Kiselev, F. Nazarov, and A. Volberg. Global well-posedness for the critical 2D dissipative quasi-geostrophic equation. *Inventiones mathematicae*, 167(3):445–453, 2007.

[9] Alexander Kiselev, Fedor Nazarov, and Roman Shterenberg. Blow up and regularity for fractal Burgers equation. *Dyn. Partial Differ. Equ.*, 5(3):211–240, 2008.
[10] Serge Resnick. Dynamical problems in nonlinear advective partial differential equations. *Ph.D. Thesis, University of Chicago*, 1995.

[11] L. Silvestre. On the differentiability of the solution to the hamilton-jacobi equation with critical fractional diffusion. *Preprint*.

[12] L. Silvestre. Holder estimates for solutions of integro-differential equations like the fractional Laplace. *Indiana University Mathematics Journal*, 55(3):1155–1174, 2006.

[13] L. Silvestre. Eventual regularization for the slightly supercritical quasi-geostrophic equation. *Arxiv preprint arXiv:0812.4901*, 2008.