Non-linear Liouville and Shrödinger equations in phase-space

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Abstract

Unitary representations of the Galilei group are studied in phase space, in order to describe classical and quantum systems. Conditions to write in general form the generator of time translation and Lagrangians in phase space are then established. In the classical case, Galilean invariance provides conditions for writing the Liouville operator and Lagrangian for non-linear systems. We analyze, as an example, a generalized kinetic equation where the collision term is local and non-linear. The quantum counter-part of such unitary representations are developed by using the Moyal (or star) product. Then a non-linear Schrödinger equation in phase space is derived and analyzed. In this case, an association with the Wigner formalism is established, which provides a physical interpretation for the formalism.
1 Introduction

Phase space (Γ) is the natural manifold for formulation of the kinetic theory; and as such, is the basic starting point for exploring symmetry. For classical particles and fields, the Poisson bracket, a symplectic two-forms in Γ, is mapped in the Lie product of Lie algebras \([1, 2]\), giving rise to representations of kinematical groups. In particular, unitary representations constructed from a Hilbert space in Γ are of interest for their practical appeal: taking advantage of the notion of linear space to develop, for instance, perturbative techniques.

For the Galilei group, describing non-relativistic systems, unitary representations for classical statistical systems were proposed by Schönberg \([4, 5, 6]\), who introduced the notion of Fock-space in Γ and the study of unitary symplectic representations. Numerous other developments then followed, to consider the Brownian motion, stochastic processes, classical kinetic theory and generalizations of these ideas to the quantum domain \([7]-[22]\).

The phase space for quantum systems is introduced by the Wigner function \([23]-[27]\). In such an approach each operator, \(A\), defined in the usual Hilbert space, \(\mathcal{H}\), is associated with a function, \(a_W(q, p)\), in Γ \([24]-[29]\). Then there is a mapping \(\Omega_W: A \rightarrow a_W(q, p)\), such that, the associative algebra of operators defined in \(\mathcal{H}\) turns out to be an associative algebra in Γ, given by \(\Omega_W: AB \rightarrow a_W * b_W\), where the star (or Moyal)-product, *, is given as

\[
a_W * b_W = a_W(q, p) \exp \left[ \frac{i\hbar}{2} \left( \frac{\partial}{\partial q} \frac{\partial}{\partial p} - \frac{\partial}{\partial p} \frac{\partial}{\partial q} \right) \right] b_W(q, p),
\]

This provides a non-commutative algebraic structure in the phase space, that has been explored in different ways \([24]-[51]\). The study of unitary representations of Lie groups in phase space for quantum systems has been achieved \([30]-[32]\), by using the Weyl operators, \(\hat{a} = a_W *\), that are introduced as a mapping on functions \(b_W\), such that \(\hat{a}(b_W) = a_W * b_W\). This symplectic representation provides a way to consider a perturbative approach for Wigner functions based on symmetry groups. One example is the \(\lambda\phi^4\) field theory in phase-space, giving rise to a relativistic kinetic equation with a local Boltzmann-like collision term. It is important to emphasize that, although associated with the Wigner formalism, the symplectic representations have a Hamiltonian, and not a Liouville, operator as generator of time translations.

From a conceptual standpoint, formulations of physics in phase space are such that the generator of time translation, as the classical and quantum
Liouvillian operators, is usually defined by using the Hamiltonian. In both cases, it is necessary to know the Hamiltonian first and then it is possible to proceed to the Liouvillian formulation. This has been recognized as a hindrance to exploring a variety of phenomena in kinetic theory and stochastic problems, involving non-linear elements and irreversibility [13, 16, 21, 48, 51].

By using unitary symplectic representations, we show here that such a path to find the generator of time translation in phase space is not necessary. With the Galilean symmetries applied to time evolution of physical states, we find algebraic relations, which the generator of time translation must satisfy. Then it is possible to infer the form of the classical Liouville operator without previous knowledge of the Hamiltonian. The Liouville operator is interpreted independently of the Hamiltonian form, having a life of its own. In a similar way, the Hamiltonian in phase space describing quantum systems is constructed. In addition, by using the Hilbert space defined in $\Gamma$ and the Galilei symmetry, we analyze the Lagrangian formalism. This procedure opens numerous possibilities to introduce interactions and non-linear effects in the kinetic theory.

We explore these possibilities by studying a non-linear Schrödinger (or a Gross-Pitaevskii-like) equation in phase space [52, 53]. The association of this formalism with the Wigner function is then discussed. In the case of classical systems, we analyze a Liouville-like equation with a non-linear source term. These non-linear equations are solved perturbatively, showing a systematic procedure to use the group theory analysis to improve and to explore the kinetic theory.

The paper is organized as follows. In Section 2, we briefly review symplectic manifolds in order to define unitary representations in phase space. In Section 3, we consider classical systems and the non-linear Liouville equation. In Section 4, we study quantum representations using the Moyal product. In Section 5, the non-linear Schrödinger equation in phase space is studied. Finally, in Section 6, some concluding remarks are presented.

## 2 Symplectic manifolds and Hilbert space

Consider an analytical manifold $\mathbb{M}$ where each point is specified by Euclidean coordinates $q^i$, with $i = 1, 2, 3$. The coordinates of each point in the cotangent-bundle $\Gamma = T^*\mathbb{M}$ is denoted by $(q^i, p^i)$. The space $\Gamma$ is equipped
with a symplectic structure by the 2-form
\[ \omega = \sum_{i=1}^{3} dq^i \wedge dp^i \] (2)

Let us define the following,
\[ \Lambda = \sum_{i=1}^{3} \frac{\partial}{\partial q^i} \frac{\partial}{\partial p^i} - \frac{\partial}{\partial p^i} \frac{\partial}{\partial q^i}, \] (3)
such that for \( C^\infty \) functions, \( f(q,p) \) and \( g(q,p) \), we have
\[ \omega(f \Lambda, g \Lambda) = f \Lambda g = \{f,g\}, \] (4)
where \( \{f,g\} = \sum_{i=1}^{3} (\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p^i} - \frac{\partial f}{\partial p^i} \frac{\partial g}{\partial q^i}) \) is the Poisson bracket. We identify vector fields in \( \Gamma \) by
\[ f \Lambda = X_f = \sum_{i=1}^{3} (\frac{\partial f}{\partial q^i} \frac{\partial}{\partial p^i} - \frac{\partial f}{\partial p^i} \frac{\partial}{\partial q^i}), \] (5)
where \( f = f(q^i,p^i) \in C^\infty(\Gamma) \). The space \( \Gamma \), endowed with this symplectic structure, is called the phase space.

In order to construct a Hilbert space over \( \Gamma \), let \( \mu \) be an invariant measure on the cotangent bundle. If \( \varphi \) is a mapping: \( \Gamma \to \mathbb{R} \) which is measurable. Then we define the integral of \( \varphi \) with respect to \( \mu \) as
\[ \int_{\Omega} \varphi(\mathbf{z})d\mu(\mathbf{z}), \] (6)
where \( \mathbf{z} \in \Gamma \). Let \( \mathcal{H}(\Gamma) \) be a linear subspace of the space of \( \mu \)-measurable functions \( \psi : \Gamma \to \mathbb{C} \) which are square integrable, such that
\[ \int_{\Gamma} | \psi(\mathbf{z}) |^2 d\mu(\mathbf{z}) < \infty. \] (7)

We equip \( \mathcal{H}(\Gamma) \) with an inner product, \( \langle \cdot | \cdot \rangle \) by
\[ \langle \psi_1 | \psi_2 \rangle = \int_{\Gamma} \psi_1(q,p)^\dagger \psi_2(q,p) d\mu(q,p), \] (8)
where we take \( z = (q^i, p^j) = (q, p) \), and \( \psi(q, p) \in C^\infty(\Gamma) \) is such that
\[
\int d^3p d^3q \psi^\dagger(q, p) \psi(q, p) < \infty.
\]
(9)
Then \( \mathcal{H}(\Gamma) \) is a Hilbert space.

In this case, we have \( \psi(q, p) = \langle q, p|\psi \rangle \), with
\[
\int d^3p d^3q \langle q, p|\psi \rangle\langle q, p| = 1,
\]
(10)
such that, the kets \( |q, p\rangle \) are defined from the set of operators \( \tilde{Q} \) and \( \tilde{P} \), such that
\[
\tilde{Q}|q, p\rangle = q|q, p\rangle, \quad \tilde{P}|q, p\rangle = p|q, p\rangle,
\]
satisfying the commutation condition \( [\tilde{Q}, \tilde{P}] = 0 \). The state of a system is described by functions \( \phi(q, p) \), with the condition
\[
\langle \psi|\phi \rangle = \int d^3p d^3q \psi^\dagger(q, p) \phi(q, p) < \infty
\]
(11)
This Hilbert space, \( \mathcal{H}(\Gamma) \), is taken as the representation space to provide a general scheme to study unitary representations of the Galilei group. A unitary transformation in \( \mathcal{H}(\Gamma) \) is the mapping \( U: \mathcal{H}(\Gamma) \to \mathcal{H}(\Gamma) \) such that \( \langle \psi_1|\psi_2 \rangle \) is invariant.

Observe that a general associative product in \( \mathcal{H}(\Gamma) \) is introduced as a mapping \( e^{ia\Lambda} = *: \Gamma \times \Gamma \to \Gamma \), called the Moyal (or star) product, as given in Eq. (1), i.e.
\[
f * g = f(q, p) e^{ia\Lambda} g(q, p),
\]
(12)
where \( f \) and \( g \) are functions in phase-space and \( \partial_z = \partial/\partial z \) (\( z = p, q \)). The constant \( a \) is used at this point to fix units, without any special meaning. The usual associative product is obtained by taking \( a = 0 \). In addition, to each function, say \( f(q, p) \), we introduce operators in the form \( \hat{f} = f(q, p)\ast \).

Such an operator will be used as the generator of unitary transformations.

In the following sections these two types of representations are analyzed explicitly. We take into account the Lie algebra for the Galilei group, \( \mathfrak{g} \), given by
\[
\begin{align*}
[\hat{L}_i, \hat{L}_j] &= i\epsilon_{ijk} \hat{L}_k, & [\hat{K}_i, \hat{P}_j] &= i a_0 \delta_{ij}, \\
[\hat{L}_i, \hat{K}_j] &= i\epsilon_{ijk} \hat{K}_k, & [\hat{K}_i, \hat{H}] &= i\hat{P}_j, \\
[\hat{L}_i, \hat{P}_j] &= i\epsilon_{ijk} \hat{P}_k, & [\hat{P}_i, \hat{P}_j] &= 0, \\
[\hat{L}_i, \hat{H}] &= 0, & [\hat{P}_i, \hat{H}] &= 0,
\end{align*}
\]
(13)
where $\hat{P}, \hat{K}, \hat{L}$ and $\hat{H}$ are the generators of translations, boost, rotations and time translations, respectively. The constant $a_0$ is the central extension of the group. Defining two operators $Q$ and $P$ that are transformed by the boost according to

$$\exp(-iv \cdot \hat{K}) Q_j \exp(iv \cdot \hat{K}) = Q_j + v_j t$$  \hspace{1cm} (14)

$$\exp(-iv \cdot \hat{K}) P_j \exp(iv \cdot \hat{K}) = P_j + mv_j,$$  \hspace{1cm} (15)

then the physical content of the algebra become obvious. The operators $Q$ and $P$ are interpreted as position and momentum, respectively. These are basic relations used to derive physical conditions to study representations. In the following sections we consider classical and quantum representations.

3 Symplectic classical mechanics

Let us consider unitary representations in phase space describing a classical system. This is achieved by using the vector field in phase space given in Eq. (5), i.e. we introduce unitary operators with the definition $\hat{f} = i X_f$. We consider the following set of operators:

$$\hat{P}_i = i X_{p_i} = -i \frac{\partial}{\partial q_i},$$

$$\hat{K}_i = i X_{K_i} = im \frac{\partial}{\partial p_i} - it \frac{\partial}{\partial q_i},$$

$$\hat{J}_i = i X_{J_i} = \hat{L}_i + \hat{S}_i,$$

$$\hat{H} = i \frac{\partial}{\partial \hat{t}},$$

where

$$\hat{L}_i = i X_{L_i} = i \varepsilon_{ijk} \left( q_k \frac{\partial}{\partial q_j} + p_k \frac{\partial}{\partial p_j} \right),$$

with $L_i = \varepsilon_{ijk} q_j p_k$. The boost operator is constructed with the function $K_i = mq_i - tp_i$. The operators $\hat{S}_i$ are the spin operators (a representation of $SO(3)$ such that $\hat{S}$ commutes with every operator defined on the phase space. We take here $\hat{S}_i = 0$. This set of operators fulfills the relations given in Eq. (13) with $a_0 = 0$. 


Let us define the linear operators $P_i$ and $Q_i$ by

$$P_i|p_i, q_i⟩ = p_i|p_i, q_i⟩, \quad Q_i|p_i, q_i⟩ = q_i|p_i, q_i⟩,$$

such that $⟨q, p|θ⟩ = θ(p, q)$ is a vector in the phase space representation, which is a vector in $H(Γ)$. Notice that $[P_i, Q_j] = 0$. Let us evaluate the physical consequences of this representation.

The operators $P$ and $Q$ are interpreted as the momentum and position operators, since they satisfy the Galilei boost conditions, namely

$$⟨θ| exp (−iv^j K_j) Q_i exp (iv^j K_j)|φ⟩ = ⟨θ|Q_i|φ⟩ + vt⟨θ|φ⟩,$$

and

$$⟨θ| exp (−iv^j K_j) P_i exp (iv^j K_j)|φ⟩ = ⟨θ|P_i|φ⟩ + mv⟨θ|φ⟩.$$

where $|θ⟩$ and $|φ⟩$ $(∈ H)$ are arbitrary states of the system, and $[P_i, Q_j] = −imδ_{ij}$. Then, $L$ is the angular momentum, and $H$ is the Hamiltonian.

Since $[P_i, K_j] = 0$, then $a_0 = 0$ in Eq. (13). However, by introducing the c-number operator $K = mQ − tP$, we have $[P_i, K_j] = [P_i, K_j] = −imδ_{ij}$. Taking this relation, together with Eqs. (16) and (17), we find that the constant $m$ is mass. These relations among $K, P$ and $Q$ are similar to those used in quantum mechanics, but $Q$ and $P$ commute with one another, since they describe the position and momentum of a classical system.

The expectation value of a dynamical variable $A$ in a state $|θ⟩$ is defined by

$$⟨A⟩ = ⟨θ|A|θ⟩.$$

(18)

On the other hand, the temporal evolution of $A$ is given by

$$⟨θ_0| exp (it\hat{H})A exp((−it\hat{H})|θ_0⟩ = ⟨θ_0|t(A)|θ_0⟩,$$

where $\hat{H}$ is called the Liouvillian. Therefore, we have defined a Heisenberg picture for the temporal evolution of the dynamical variables, and from Eq. (19) we obtain

$$i∂_t A = [A, \hat{H}].$$

(20)

In the Schrödinger picture, the evolution of the state is given by

$$i∂_t|θ(t)⟩ = \hat{H}|θ(t)⟩,$$

(21)
Using the orthonormality of the states $|q, p\rangle$, 
\[ \langle q, p|q', p'\rangle = \delta(q - q')\delta(p - p') \text{ and } \int |q, p\rangle\langle q, p|dqdp = 1, \]
we write 
\[ i\partial_t \theta(q, p; t) = \int \langle q, p|\hat{H}|q', p'\rangle \langle q', p'|\theta(t)\rangle dq'dp', \quad (22) \]
where $|\theta(t)\rangle$ is in $H(\Gamma)$ and $\theta(q, p; t) = \langle q, p|\theta(t)\rangle$. Assuming 
\[ \langle q, p|\hat{H}|q', p'\rangle = \delta(q - q')\delta(p - p')\langle q, p|\hat{H}|q, p\rangle, \]
we have 
\[ i\partial_t \theta(q, p; t) = \mathcal{L}(q, p)\theta(q, p; t), \quad (23) \]
where $\mathcal{L}(q, p) = \langle q, p|\hat{H}|q, p\rangle$ is the classical Liouville operator, and the connection with the Liouville equation is to be derived. A simple solution is to consider $\mathcal{L}(q, p) = iX_H = i\{H,\cdot\}$, where $H = p^2/2m$ \[14\]. A central point here is to find a general form for $\mathcal{L}(q, p)$. We solve this problem by using the symmetry properties of the Galilei group. To proceed further, let us discuss some additional aspects about this symplectic representation for classical physics, as it was first proposed by Schöenberg [4, 5, 6]. A set of rules for physical interpretation rules has to be established. For an $n$-particle system described in the phase space, these rules are the following.

(i) The states of an $n$-particle system are vectors in Hilbert state $H(\Gamma)$. Each vector is given by a wave function $\theta_n = \theta(z_1, \cdots, z_n)$ with $(z_i := (q_i, p_i))$, such that, the probability density in the classical phase space is written as $f_n := f(z_1, \cdots, z_n) = |\theta_n|^2$. The state $\theta_n$ satisfies Eq. (23), that is written in the notation of $n$-particle systems, explicitly, as 
\[ \frac{\partial \theta_n}{\partial t} = \{H, \theta_n\}_n = -i\mathcal{L}_n \theta_n \quad (24) \]
where, 
\[ \{H, \theta_n\}_n = \sum_{i=1}^{3n} \left( \frac{\partial H_n}{\partial q_i} \frac{\partial \theta_n}{\partial p_i} - \frac{\partial H_n}{\partial p_i} \frac{\partial \theta_n}{\partial q_i} \right) \quad (25) \]
and $H_n$ is the Hamiltonian of a classical $n$-particle system.
(ii) To each physical quantity $a(z_1, \ldots, z_n) \equiv a(q, p)$ in phase space, two hermitian operators on the space $\mathcal{H}(\Gamma)$ are associated; i.e. a diagonal operator $A$ and a differential operator $\hat{A} = i \sum_{i=1}^{3n} \left( \frac{\partial a}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial a}{\partial q_i} \frac{\partial}{\partial p_i} \right) \equiv i\{a, \cdot\}$. The operators $A$ are the usual physical observables, whereas operators of type $\hat{A}$ are called dynamical generators of symmetries. The possible values of a physical quantity represented by $A$ are its eigenvalues. It follows that the average value of the quantity $A$ in the state $\theta_n$ is $\langle A \rangle = \int dz\theta_n^* A \theta_n = \int dz |\theta_n|^2 a(q, p)$; i.e, the classical result.

(iii) In the symplectic representation, we introduce three pictures for the state vector $|\theta(t)\rangle$ as well as for operators of the theory. They are, the classical Schrödinger picture, the classical Heisenberg picture and the classical interaction picture [4, 14]. We have used above the classical Heisenberg and Schrödinger pictures, in the analysis of the symplectic representation.

At this point it is instructive to compare this representation for classical systems with the usual formulation of quantum physics. In quantum mechanics, symmetry transformations are represented by unitary operators acting on the Hilbert space [54]. The Hamiltonian operator, for example, represents the infinitesimal generator of time translations whereas momentum generates spatial translations, and so on. This can be formulated in terms of representations of the Galilei group in the Hilbert space. In the classical symplectic representation, on the other hand, symmetry operations are mapping on $\mathcal{H}(\Gamma)$, and one example is the Liouville operator $L_n$, that is the generator of time translations. This conclusion is reached by using for instance the Hamiltonian $H_n$. Using Galilean invariance, however, we have to show the general algebraic conditions for the representation of time translation generator, $L_n$, irrespective of any Hamiltonian formulation previously assumed.

Let $|\theta_n\rangle \in \mathcal{H}(\Gamma)$ be an arbitrary $n$-particle state prepared by an observer $\mathcal{O}$ at the instant $t_0$, and let $|\theta_n; v\rangle$ be the state having the same properties at time $t_0$ insofar as an observer $\mathcal{O}'$, who is moving with velocity $v$ relative to $\mathcal{O}$, is concerned. Let us assume that at $t = 0$ the two coordinate frames coincide. The expectation value of the operators $Q_i := (Q_{ix}, Q_{iy}, Q_{iz})$, $P_i := (P_{ix}, P_{iy}, P_{iz})$ and $\hat{F}$ (an arbitrary physical quantity in phase space) in these
states is then related by the Galilei transformations,
\[
\langle \theta_n; \mathbf{v} | \mathbf{Q}_i | \theta_n; \mathbf{v} \rangle = \langle \theta_n | \mathbf{Q}_i | \theta_n \rangle + \mathbf{v} t_0 \tag{26}
\]
\[
\langle \theta_n; \mathbf{v} | \mathbf{P}_i | \theta_n; \mathbf{v} \rangle = \langle \theta_n | \mathbf{P}_i | \theta_n \rangle + m_i \mathbf{v} \tag{27}
\]
\[
\langle \theta_n; \mathbf{v} | \hat{\mathbf{F}}_i | \theta_n; \mathbf{v} \rangle = \langle \theta_n | \hat{\mathbf{F}}_i | \theta_n \rangle + \Delta_F \mathbf{v}, \tag{28}
\]
with \( \Delta_F \) being a quantity to be determined for each \( \hat{\mathbf{F}} \). Considering an infinitesimal Galilei transformation, we replace \( \mathbf{v} \) by \( \delta \mathbf{v} \) and define the infinitesimal unitary operator
\[
\tilde{\Gamma}(t_0, \delta \mathbf{v}) = 1 - i \delta \mathbf{v} \cdot \hat{\mathbf{K}}, \quad \hat{\mathbf{K}} = \hat{\mathbf{K}}^\dagger, \tag{29}
\]
by requiring that \( \tilde{\Gamma} | \theta_n \rangle = | \theta_n; \delta \mathbf{v} \rangle \). We assume that \( \hat{\mathbf{K}} = \mathbf{K}(\mathbf{q}, \mathbf{p}, \frac{\partial}{\partial \mathbf{q}}, \frac{\partial}{\partial \mathbf{p}}) \), in Eq. (29).

It is to be noted that for \( \hat{\mathbf{F}} = iX_f = i\{f, \cdot\} \) and \( \hat{\mathbf{A}} = iX_a = i\{a, \cdot\} \), we have:
\[
[A, \hat{\mathbf{F}}] = i\{a(q, p), f(q, p)\}, \tag{30}
\]
\[
[i\{f, \cdot\}, i\{a, \cdot\}] = -\{\cdot, \{a, f\}\} \tag{31}
\]
and
\[
[A, \mathbf{F}] = 0, \tag{32}
\]
where \( A \) and \( \mathbf{F} \) are diagonal operators in \( \mathcal{H}(\Gamma) \).

By virtue of Eqs. (26), (27) and (28), and the definition given in Eq. (29), we get
\[
t_0 \delta \mathbf{v} = -i[\mathbf{Q}_i, \delta \mathbf{v} \cdot \hat{\mathbf{K}}], \tag{33}
\]
\[
m_i \delta \mathbf{v} = -i[\mathbf{P}_i, \delta \mathbf{v} \cdot \hat{\mathbf{K}}], \tag{34}
\]
\[
\Delta_F \delta \mathbf{v} = -i[\hat{\mathbf{F}}, \delta \mathbf{v} \cdot \hat{\mathbf{K}}]. \tag{35}
\]
Non-trivial solutions of these equations are found by taking \( \hat{\mathbf{K}} = iX_k = i\{k, \cdot\} \), where \( k = k(q, p) \) is a vector-function in phase space. Hence,
\[
t_0 \delta \mathbf{v} = -i[\mathbf{Q}_i, i\{k, \cdot\}] \cdot \delta \mathbf{v} = \frac{\partial}{\partial \mathbf{p}_i} \delta \mathbf{v} \cdot \mathbf{k}, \tag{36}
\]
\[
m_i \delta \mathbf{v} = -i[\mathbf{P}_i, i\{k, \cdot\}] \cdot \delta \mathbf{v} = -\frac{\partial}{\partial \mathbf{q}_i} \delta \mathbf{v} \cdot \mathbf{k}, \tag{37}
\]
\[ \Delta_{\mathbf{Q}} \delta \mathbf{v} = -i \delta \mathbf{v} \cdot \{ , \frac{\partial \mathbf{k}}{\partial \mathbf{p}_i} \}, \quad (38) \]
\[ \Delta_{\mathbf{P}} \delta \mathbf{v} = i \delta \mathbf{v} \cdot \{ , \frac{\partial \mathbf{k}}{\partial \mathbf{q}_i} \}, \quad (39) \]

where we have used for \( \tilde{F} \) the operators \( \tilde{\mathbf{Q}}_i \) and \( \tilde{\mathbf{P}}_i \).

As \( \delta \mathbf{v} \) is arbitrary, these equations reduce to:

\[ t_0 = \frac{\partial k_x}{\partial p_{i_x}}, \]
\[ m_i = \frac{\partial k_x}{\partial q_{i_x}}, \]
\[ i \Delta_x = \frac{\partial^2 k_x}{\partial p_{i_x}^2} \frac{\partial}{\partial q_{i_x}} - \frac{\partial^2 k_x}{\partial q_{i_x} \partial p_{i_x} \partial p_{i_x}}, \]
\[ -i \Delta_{p_x} = \frac{\partial^2 k_x}{\partial p_{i_x} \partial q_{i_x} \partial q_{i_x}} - \frac{\partial^2 k_x}{\partial q_{i_x}^2 \partial p_{i_x}}. \]

A solution for these equations is derived considering that the Poisson brackets of coordinates \( q_i \) and momenta \( p_i \) are invariant under Galilei transformations, i.e. \( \Delta_x = \Delta_{p_x} = 0 \), such that,

\[ \mathbf{k}_{t_0} = t_0 \mathbf{p} - M \mathbf{R}, \quad \mathbf{R} = \sum_i m_i \mathbf{q}_i, \quad M = \sum_i m_i, \quad \mathbf{p} = \sum_i \mathbf{p}_i \quad (40) \]

Let \( U(t_1, t_0) = e^{-i \mathcal{L}_n (t_1 - t_0)} \) be the time-evolution operator on \( \mathcal{H}(\Gamma) \), such that

\[ U(t_1, t_0) \tilde{\Gamma}(t_0, \mathbf{v}) = \tilde{\Gamma}(t_1, \mathbf{v}) U(t_1, t_0). \quad (41) \]

This equation imposes a condition on the operator \( \mathcal{L}_n \). In order to obtain this condition it suffices to replace \( \mathbf{v} \) by the infinitesimal \( \delta \mathbf{v} \). Therefore, with the aid of Eq. (29), we obtain

\[ U(t_1, t_0)(1 + \delta \mathbf{v} \cdot \{ \mathbf{k}_{t_0}, \cdot \}) = (1 + \delta \mathbf{v} \cdot \{ \mathbf{k}_{t_1}, \cdot \}) U(t_1, t_0). \quad (42) \]

Using Eq. (40), this equation reads

\[ -t_0 U(t_1, t_0) \frac{\partial}{\partial \mathbf{q}_i} = U(t_1, t_0) m_i \frac{\partial}{\partial \mathbf{p}_i} \]
\[ -t_1 \frac{\partial}{\partial \mathbf{q}_i} U(t_1, t_0) - m_i \frac{\partial}{\partial \mathbf{p}_i} U(t_1, t_0). \quad (43) \]
Since the system is invariant under space translations, the operator $\partial/\partial q_i$ commutes with $U(t_1, t_0)$. Moreover, with $\delta t \simeq 0$, from Eq. (43), we find

$$[m_i \frac{\partial}{\partial p_i}, 1 - i L_n \delta t] = -\delta t(1 - i L_n \delta t) \frac{\partial}{\partial q_i}, \quad (44)$$

or

$$i[m_i \frac{\partial}{\partial p_i}, L_n] \delta t = \delta t \frac{\partial}{\partial q_i}. \quad (45)$$

With definitions of $M, R$ and $p$, we obtain

$$i \sum_i [m_i \frac{\partial}{\partial p_i}, L_n] = \sum_i \frac{\partial}{\partial q_i}, \quad (46)$$

This is a basic result, showing the algebraic condition that has to be fulfilled by the Liouville operator.

It is worth noting that, taking the Liouville operator to be in the form $L_n = i \{H_n, \cdot \}$, we obtain,

$$-\sum_i (m_i \frac{\partial}{\partial p_i} \{H_n, \cdot \}) - \{H_n, \cdot \} \sum_i m_i \frac{\partial}{\partial p_i} = \sum_i \frac{\partial}{\partial q_i},$$

and hence

$$-\frac{\partial^2 H_n}{\partial p_i \partial q_i} \frac{\partial}{\partial p_i} + \frac{\partial^2 H_n}{\partial p_i^2} \frac{\partial}{\partial q_i} = \frac{1}{m_i} \frac{\partial}{\partial q_i}$$

or

$$\frac{\partial^2 H_n}{\partial p_i \partial q_i} = 0, \quad \frac{\partial^2 H_n}{\partial p_i^2} = \frac{1}{m_i},$$

whose solution is

$$H_n = \sum_{i=1}^n \frac{p_i^2}{2m_i} + V(q_1, \cdots, q_n),$$

where $V(q_1, \cdots, q_n) \equiv V(q)$ is an arbitrary function. Therefore, the condition specified by Eq. (45) gives, in particular, the standard expression for $L_n$. However, Eq. (45) is general and is satisfied, in principle, for systems where $H$ is not defined.

Using the Galilean invariance, a general equation of motion is derived by writing the Lagrangian associated with Eq. (23) in the form

$$\mathcal{L} = \theta^i \left( i \partial_t + \frac{p_i}{m} \partial_q + iF(q) \partial_p \right) \theta + g(\theta \theta^i), \quad (47)$$
where $F(q) = -\partial_q V(q)$ and $g$ is an arbitrary functional of the wave functions. This Lagrangian gives rise to Eq. (23) with $g = 0$. Let us consider, as an example, $(1+1)$-dimensions with $F(q) = 0$ and $g(\theta\theta^\dagger) = -\frac{\lambda}{4}(\theta\theta^\dagger)^2$ such that Eq. (17) leads to

\[
\left( i \partial_t + i \frac{p}{m} \partial_q \right) \theta = \lambda (\theta\theta^\dagger) \theta.
\]

This equation describes a flow of particles in phase space, without external field and with a local non-linear collision term. Writing $\theta(q,p;t) = \phi(q,p) \exp(-i\nu t)$, we have

\[
\left( \nu + i \frac{p}{m} \partial_q \right) \phi(q,p) = \lambda \phi(q,p)^3.
\]

The zero-order ($\lambda = 0$) solution is $\phi_0(q,p) = Ae^{i\nu mq/p}$. And a solution, up to first order in $\lambda$, is $\phi(q,p) \simeq \phi_0(q,p) + \lambda \phi_1(q,p)$, that reads

\[
\phi(q,p) = \phi_0(q,p) + \frac{2A\lambda}{\nu(3i + 1)} e^{i3\nu mq/p}.
\]

The distribution function in phase space is $f(q,p) = \theta^\dagger = \phi^\dagger(q,p)\phi(q,p)$. As an illustration, is is important to take the average of an observable as the momentum, giving rise to the momentum flow. Following the previous prescription we have,

\[
\langle P \rangle = \int dqdp \theta^\dagger(q,p;t) \hat{P} \theta(q,p;t) = \int dqdp f(q,p;t),
\]

that is consistent with the usual result.

### 4 Symplectic quantum mechanics

In this section we consider representations using the star-product. For simplicity we treat a one-particle system. The generalization for an $n$ particle system is obtained by following a procedure similar to the classical case. The representation space is still $\mathcal{H}(\Gamma)$ but equipped with the star-product. The Galilei-Lie algebra in phase space is constructed by using the operators given by $f^*$, according to Eq. (12). We proceed by selecting the following set of functions in $\Gamma$: $p_i, q_i, \ell_i = \epsilon_{ijk} q_j p_k$, $k_i = mq_i - tp_i$ (sum over repeated indices
is assumed). This set is a hint to look for Weyl operators fulfilling the Galilei Lie algebra, Eq. (13); we have,

\( \hat{\mathcal{P}} = p^* = p - \frac{i\hbar}{2} \partial_q, \)  
\( \hat{\mathcal{Q}} = q^* = q + \frac{i\hbar}{2} \partial_p, \)  
\( \hat{\mathcal{K}} = k^* = mq^* - tp^* = m\hat{Q} - t\hat{P}, \)  
\( \hat{\mathcal{L}}_i = \epsilon_{ijk} \hat{Q}_j \hat{P}_k \)  
\( \hat{H} = i\hbar \frac{\partial}{\partial t}. \)

The physical content of this representation is derived, first, by observing that \( \hat{\mathcal{Q}} \) and \( \hat{\mathcal{P}} \) are transformed by the boost according to

\[ \exp \left( -iv \cdot \frac{\hat{\mathcal{K}}}{\hbar} \right) \hat{\mathcal{Q}}_j \exp \left( iv \frac{\hat{\mathcal{K}}}{\hbar} \right) = \hat{\mathcal{Q}}_j + v_j t, \]  
\[ \exp \left( -iv \cdot \frac{\hat{\mathcal{K}}}{\hbar} \right) \hat{\mathcal{P}}_j \exp \left( iv \cdot \frac{\hat{\mathcal{K}}}{\hbar} \right) = \hat{\mathcal{P}}_j + mv_j. \]

Furthermore

\[ \left[ \hat{\mathcal{Q}}_j, \hat{\mathcal{P}}_l \right] = i\hbar \delta_{jl}. \]

Therefore, the operators \( \hat{\mathcal{Q}} \) and \( \hat{\mathcal{P}} \) correspond to the physical observables of position and momentum, respectively, with Eqs. (54) and (55) describing, consistently, the way \( \hat{\mathcal{Q}} \) and \( \hat{\mathcal{P}} \) transform under the Galilei boost. The Heisenberg commutation relation is given by Eq. (56) and \( m \) is the mass. As a consequence, and for consistency, the generators \( \hat{\mathcal{L}}_i \) and \( \hat{H} \) are interpreted as the angular momentum and the Hamiltonian operators, respectively. It is important is to determine a general form for \( H \), that is accomplished by using the Galilei group. The time evolution of an observable \( \hat{A} \) is specified by

\[ \exp \left( it\hat{H}/\hbar \right) \hat{A}(0) \exp \left( -it\hat{H}/\hbar \right) = \hat{A}(t), \]

which results in

\[ i\hbar \frac{\partial}{\partial t} \hat{A}(t) = [\hat{A}(t), \hat{H}]. \]
For a homogeneous system, the commutation relations \([\hat{K}, \hat{H}] = i\hat{P}\), leads to
\[
[mq + i \frac{\partial}{\partial p}, H(q, p)\ast] = ip + \frac{1}{2} \frac{\partial}{\partial q},
\]
where
\[
H(q, p)\ast = H(q + i \frac{\partial}{2 \partial p}, p - i \frac{\partial}{2 \partial q}).
\]
A solution is \(H(q, p) = \frac{p^{2}}{2m} + V(q)\). This result provides a general functional for the Hamiltonian
\[
H(q, p)\ast = \frac{p^{2}}{2m} + V(q)\ast
\]
\[
= \frac{p^{2}}{2m} - \frac{\hbar^{2}}{8m} \frac{\partial^{2}}{\partial q^{2}} - \frac{i \hbar p}{2} \frac{\partial}{\partial q}
\]
\[+
\tilde{V}(q + \frac{i \hbar}{2} \frac{\partial}{\partial p}).\]

It is worthy of noting that this expression for the Hamiltonian cannot be derived by using the Casimir invariant of the Galilei-Lie Algebra \(I = \hat{H} - \hat{P}^{2}/2m\). In the next section we set forth a set of rules for a complete physical interpretation of the theory in terms of the notion of states.

5 Non-linear Schrödinger equation in phase space

Let us introduce a frame in the Hilbert space for the representation analyzed in the previous section. We define the operators
\[
\bar{Q} = \hat{Q} - \frac{\hbar}{2} \frac{\partial}{\partial p} \quad \text{and} \quad \bar{P} = \hat{P} + \frac{\hbar}{2} \frac{\partial}{\partial q}
\]
transform as
\[
\exp \left( -\frac{i}{\hbar} vK \right) \exp \left( \frac{i}{\hbar} vK \right) = 2\bar{Q} + vt \tag{59}
\]
and
\[
\exp \left( -\frac{i}{\hbar} vK \right) \exp \left( \frac{i}{\hbar} vK \right) = 2\bar{P} + mv. \tag{60}
\]
As for the observables $\hat{P}$ and $\hat{Q}$ in Eqs. (14) and (15), we find that $\hat{Q}$ and $\hat{P}$ also transform as position and momentum. However, since $[\hat{Q}, \hat{P}] = 0$, $\hat{Q}$ and $\hat{P}$ cannot be interpreted as observables, although they can be used to construct a frame in the Hilbert space with the content of the phase space. Then we introduce $|q,p\rangle$ such that

$$\hat{Q}|q,p\rangle = q|q,p\rangle \quad \text{and} \quad \hat{P}|q,p\rangle = p|q,p\rangle,$$

with

$$\langle q,p|q',p'\rangle = \delta(q-q')\delta(p-p'),$$

and

$$\int dqdp|q,p\rangle \langle q,p| = 1.$$ (63)

Then we have,

$$\psi(q,p,t) = \langle q,p|\psi,t\rangle.$$ (64)

Here $\psi(q,p,t)$ is a wave function but not with the content of the usual quantum mechanical state, for $q$ and $p$ are the eigenvalues of the operators $\hat{Q}$ and $\hat{P}$ which are ancillary variables and not observables.

From Eq. (63), we have

$$\langle \psi|\phi\rangle = \langle \psi|\left(\int dqdp|q,p\rangle \langle q,p|\right)\rangle = \int d\hat{q}d\hat{p}\psi^\dagger(q,p)\phi(q,p).$$ (65)

Using the definition of the star-product, we also have

$$\langle \psi|\phi\rangle = \int dqdp\psi^\dagger(q,p) * \phi(q,p).$$

The average of a physical observable $\hat{A}(q,p) = a(q,p;\hat{t})\star$, in the state $\psi(q,p)$ is given by

$$\langle \hat{A}\rangle = \int dqdp\psi^\dagger(q,p)\hat{A}(q,p) \psi(q,p)$$

$$= \int dqdp\psi^\dagger(q,p)[a(q,p) \star \psi(q,p)]$$

$$= \int dqdp\ a(q,p)[\psi(q,p) \star \psi^\dagger(q,p)]$$

(66)

The quantity $\langle \hat{A}\rangle$ will be real if the spectrum of $\hat{A}$ is real.
The equation of motion is determined by the Lie algebra, resulting in the Heisenberg-equation in phase space

\[ i\partial_t \hat{A}(q, p; t) = [\hat{A}(q, p; t), \hat{H}(\hat{q}, \hat{p})]. \]

Therefore, the Schrödinger picture is derived, from the average of \( \hat{A} \), that is given by

\[ \langle \hat{A} \rangle = \int dq dp \psi^\dagger(t) \hat{A}(0) \psi(t) \]

\[ = \int dq dp \psi^\dagger(t) a(0) \ast \psi(t), \quad (67) \]

where \( \psi(t) = e^{-i\hat{H}t} \psi(t) \). Then we obtain the Shrödinger equation in phase space

\[ i\hbar \partial_t \psi(q, p; t) = \hat{H}(q, p) \psi(q, p; t), \]

\[ = \left( \frac{p^2}{2m} - \frac{\hbar^2}{8m} \frac{\partial^2}{\partial q^2} - \frac{i\hbar}{2m} \frac{\partial}{\partial q} \right) \psi(q, p; t) \]

\[ + \hat{V} \left( q + i\hbar \frac{\partial}{2p} \right) \psi(q, p; t). \quad (68) \]

A fundamental physical result in this formalism is the connection of \( \psi(q, p; t) \) with the Wigner function, \( f_W(q, p) \), that is given by

\[ f_W(q, p) = \psi(q, p) \ast \psi^\dagger(q, p), \quad (70) \]

fulfilling the Liouville-von Neumann equation \[30\]. Using the star-product, the probability density in the configuration space is defined by

\[ \rho(q) = \int dp \psi(q, p) \ast \psi^\dagger(q, p) = \int dp \psi(q, p) \psi^\dagger(q, p), \quad (71) \]

while in momentum space it is

\[ \rho(p) = \int dq \psi(q, p) \ast \psi^\dagger(q, p) = \int dq \psi(q, p) \psi^\dagger(q, p). \quad (72) \]

The wave function, \( \psi(q, p) \), is then interpreted as a quasi-probability amplitude describing the state of the system.
The Galilean invariant Lagrangian density for bosons with a non-linear self interaction is
\[ L = \frac{i\hbar}{2}(\psi^\dagger \partial_t \psi - \psi \partial_t \psi^\dagger) + \frac{i\hbar}{4m}p(\psi^\dagger \partial_q \psi - \psi \partial_q \psi^\dagger) \]
\[ - \frac{p^2}{2m} \psi \psi^\dagger + \hat{V}(q)(\psi \psi^\dagger) - \frac{\hbar^2}{8m} \partial_q \psi \partial_q \psi^\dagger + (\psi \psi^\dagger)^2. \]

Then the Euler-Lagrange equation is
\[ i\hbar \partial_t \psi(t) = \left( \frac{p^2}{2m} - \frac{\hbar^2}{8m} \frac{\partial^2}{\partial q^2} - \frac{i\hbar p}{2m} \frac{\partial}{\partial q} \right) \psi(t) \]
\[ + \hat{V} \left( q + \frac{i\hbar}{2} \frac{\partial}{\partial p} \right) \psi(t) + \lambda (\psi^\dagger \psi) \psi. \]

This describes an extension of the Gross-Pitaevskii equation to the phase space.

Let us consider, as an example, \( \hat{V} = 0 \), and \( \lambda \ll 1 \). Then a linear approximation can be used, i.e. \( \psi(q,p;t) = \psi_0(q,p;t) + \lambda \psi_1(q,p;t) \), where \( \psi_0(q,p;t) \) is the solution of the linear equation,
\[ i\hbar \partial_t \psi_0(t) = \left( \frac{p^2}{2m} - \frac{\hbar^2}{8m} \frac{\partial^2}{\partial q^2} - \frac{i\hbar p}{2m} \frac{\partial}{\partial q} \right) \psi_0(t). \]

For simplicity we analyze the (1+1)-dimensional case. A particular solution for \( \psi_0(t) \) is
\[ \psi_0(q,p;t) = \phi_0(q,p) e^{-iEt/\hbar}, \]
where \( \phi_0(q,p) = A e^{k_\pm q} \) with
\[ k_\pm = \frac{p}{\hbar} 4i[1 \pm \frac{1}{p} (2mE)^{1/2}]. \]

In addition
\[ \psi_1(q,p;t) = \phi_1(q,p) e^{-iEt}, \]
where \( \phi_1(q,p) = B e^{3k_\pm q} \), with
\[ B = \frac{8mA^3}{(\hbar k_\pm)^2 - i8\hbar k_\pm - 8mE}. \]
The Wigner function, up to first order in $\lambda$, is given by
\[
f_w(q, p; t) = \psi_0(q, p; t) \ast \psi_0^\dagger(q, p; t) + \lambda \psi_1(q, p; t) \ast \psi_1^\dagger(q, p; t) + \lambda \psi_0(q, p; t) \ast \psi_1^\dagger(q, p; t).
\]
The star product has to be explicitly developed, providing a non-trivial result for the Wigner function.

The average of the momentum, as an example, is given by
\[
\langle \hat{P} \rangle = \int dq dp \psi^\dagger(q, p; t) \hat{P} \psi(q, p; t) = \int dp dq pf_w(q, p; t),
\]
where we have used Eqs. (66) and (70). Physically, this result is consistent with the Wigner formalism and describes a quantum flow of bosons in phase space, with the collision term of the kinetic equation being local and non-linear.

6 Conclusion

In this paper we have studied symplectic (unitary) representations of the Galilei group for classical and quantum systems, developing two aspects. First, we have found the general conditions that the generator of time translation in phase space has to satisfy, such that its explicit form is derive from general elements of symmetry. Second, we derive non-linear equations in phase space, associated with the kinetic theory.

For the classical systems, the generator of time translation is the Liouville differential operator, and from the Lagrangian formalism, a classical Liouville equation is derived with a local and non-linear collision term. For quantum systems, the time generator is a Hamiltonian written in phase space, and the analysis of the Lagrangian leads to a non-linear Schrödinger equation in phase space. These classical and quantum equations are solved perturbatively, as an example, to emphasize the usefulness of such representations in non-relativistic kinetic theory. At the same time, these results open doors for further developments, such as the analysis of quantum dynamical systems in phase space.

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