Identities of symmetry for generalized Bernoulli polynomials

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Abstract. In this paper, we derive eight basic identities of symmetry in three variables related to generalized Bernoulli polynomials and generalized power sums. All of these are new, since there have been results only about identities of symmetry in two variables. The derivations of identities are based on the \( p \)-adic integral expression of the generating function for the generalized Bernoulli polynomials and the quotient of \( p \)-adic integrals that can be expressed as the exponential generating function for the generalized power sums.

Key words: generalized Bernoulli polynomial, generalized power sums, \( p \)-adic integral, identities of symmetry.

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1. Introduction and preliminaries

Let \( p \) be a fixed prime. Throughout this paper, \( \mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}_p \) will respectively denote the ring of \( p \)-adic integers, the field of \( p \)-adic rational numbers and the completion of the algebraic closure of \( \mathbb{Q}_p \). Let \( d \) be a fixed positive integer. Then we let

\[
X = X_d = \lim_{\mathbb{N}} \mathbb{Z}/dp^N\mathbb{Z},
\]

and let \( \pi : X \to \mathbb{Z}_p \) be the map given by the inverse limit of the natural maps

\[
\mathbb{Z}/dp^N\mathbb{Z} \to \mathbb{Z}/p^N\mathbb{Z}.
\]

If \( g \) is a function on \( \mathbb{Z}_p \), we will use the same notation to denote the function \( g \circ \pi \). Let \( \chi : (\mathbb{Z}/d\mathbb{Z})^* \to \overline{\mathbb{Q}}^* \) be a (primitive) Dirichlet character of conductor \( d \). Then it will be pulled back to \( X \) via the natural map \( X \to \mathbb{Z}/d\mathbb{Z} \). Here we fix, once and for all, an imbedding \( \overline{\mathbb{Q}} \to \mathbb{C}_p \), so that \( \chi \) is regarded as a map of \( X \) to \( \mathbb{C}_p \) (cf. [4]).

For a uniformly differentiable function \( f : X \to \mathbb{C}_p \), the \( p \)-adic integral of \( f \) is defined (cf. [3]) by

\[
\int_X f(z) d\mu(z) = \lim_{N \to \infty} \frac{1}{dp^N} \sum_{j=0}^{dp^N-1} f(j).
\]

Then it is easy to see that

\[
\int_X f(z + 1) d\mu(z) = \int_X f(z) d\mu(z) + f'(0).
\]
More generally, we deduce from (1.1) that, for any positive integer \( n \),

\[
\int_X f(z + n)\,d\mu(z) = \int_X f(z)\,d\mu(z) + \sum_{a=0}^{n-1} f'(a).
\]

Let \( |p| \) be the normalized absolute value of \( \mathbb{C}_p \) such that \( |p|_p = \frac{1}{p} \), and let

\[
E = \{ t \in \mathbb{C}_p \mid |t|_p < p^{-\frac{1}{p}} \}.
\]

Then, for each fixed \( t \in E \), the function \( e^{zt} \) is analytic on \( \mathbb{Z}_p \) and hence considered as a function on \( X \), and, by applying (1.2) to \( f \) with \( f(z) = \chi(z) e^{zt} \), we get the p-adic integral expression of the generating function for the generalized Bernoulli numbers \( B_{n,\chi} \) attached to \( \chi \):

\[
\int_X \chi(z) e^{zt}\,d\mu(z) = \frac{t}{e^{zt} - 1} \sum_{a=0}^{d-1} \chi(a) e^{at} = \sum_{n=0}^{\infty} B_{n,\chi}(x) \frac{t^n}{n!} (t \in E).
\]

Also, from (1.1) we have:

\[
\int_X e^{zt}\,d\mu(z) = \frac{t}{e^{zt} - 1}.
\]

Let \( S_k(n, \chi) \) denote the \( k \)th generalized power sum of the first \( n + 1 \) nonnegative integers attached to \( \chi \), namely

\[
S_k(n, \chi) = \sum_{a=0}^{n} \chi(a) a^k = \chi(0) 0^k + \chi(1) 1^k + \cdots + \chi(n) n^k.
\]

From (1.4), (1.6), and (1.7), one easily derives the following identities: for \( w \in \mathbb{Z}_{>0} \),

\[
\frac{wd \int_X \chi(x) e^{zt}\,d\mu(x)}{\int_X e^{waxt}\,d\mu(y)} = \frac{e^{wadt} - 1}{e^{zt} - 1} \sum_{a=0}^{d-1} \chi(a) e^{at}
\]

\[
= \sum_{a=0}^{\infty} \chi(a) e^{at}
\]

\[
= \sum_{k=0}^{\infty} S_k(wd - 1, \chi) \frac{t^k}{k!} (t \in E).
\]

In what follows, we will always assume that the p-adic integrals of the various (twisted) exponential functions on \( X \) are defined for \( t \in E \) (cf. (1.3)), and therefore it will not be mentioned.

[1], [2], [4], [7] and [8] are some of the previous works on identities of symmetry involving Bernoulli polynomials and power sums. For the brief history, one is referred to those papers.
In this paper, we will produce 8 basic identities of symmetry in three variables \(w_1, w_2, w_3\) related to generalized Bernoulli polynomials and generalized power sums (cf. (4.8)-(4.10), (4.12), (4.14)-(4.17)). All of these seem to be new, since there have been results only about identities of symmetry in two variables in the literature ([5]). The following is stated as Theorem 2 and an example of the full six symmetries in \(w_1, w_2, w_3\).

\[
\sum_{k+l+m=n} \binom{n}{k, l, m} B_{k, \chi}(w_1 y_1) B_{l, \chi}(w_2 y_2) S_m(w_3 d - 1, \chi) w_1^{l+m} w_2^{k+m} w_3^{l-i-1} \\
= \sum_{k+l+m=n} \binom{n}{k, l, m} B_{k, \chi}(w_1 y_1) B_{l, \chi}(w_3 y_2) S_m(w_3 d - 1, \chi) w_1^{l+m} w_3^{k+m} w_2^{l-i-1} \\
= \sum_{k+l+m=n} \binom{n}{k, l, m} B_{k, \chi}(w_2 y_1) B_{l, \chi}(w_1 y_2) S_m(w_3 d - 1, \chi) w_2^{l+m} w_1^{k+m} w_3^{l-i-1} \\
= \sum_{k+l+m=n} \binom{n}{k, l, m} B_{k, \chi}(w_2 y_1) B_{l, \chi}(w_3 y_2) S_m(w_3 d - 1, \chi) w_2^{l+m} w_3^{k+m} w_1^{l-i-1} \\
= \sum_{k+l+m=n} \binom{n}{k, l, m} B_{k, \chi}(w_3 y_1) B_{l, \chi}(w_2 y_2) S_m(w_3 d - 1, \chi) w_3^{l+m} w_2^{k+m} w_1^{l-i-1} \\
= \sum_{k+l+m=n} \binom{n}{k, l, m} B_{k, \chi}(w_3 y_1) B_{l, \chi}(w_1 y_2) S_m(w_3 d - 1, \chi) w_3^{l+m} w_1^{k+m} w_2^{l-i-1}.
\]

The derivations of identities are based on the \(p\)-adic integral expression of the generating function for the generalized Bernoulli polynomials in (1.5) and the quotient of integrals in (1.8)-(1.10) that can be expressed as the exponential generating function for the generalized power sums. These abundance of symmetries would not be unearthed if such \(p\)-adic integral representations had not been available. We indebted this idea to the paper [5].

2. SEVERAL TYPES OF QUOTIENTS OF \(p\)-ADIC INTEGRALS

Here we will introduce several types of quotients of \(p\)-adic integrals on \(X\) or \(X^3\) from which some interesting identities follow owing to the built-in symmetries in \(w_1, w_2, w_3\). In the following, \(w_1, w_2, w_3\) are positive integers and all of the explicit expressions of integrals in (2.2), (2.4), (2.6), and (2.8) are obtained from the identities in (1.4) and (1.6). To ease notations, from now on we suppress \(\mu\) and denote, for example, \(d\mu(x)\) simply by \(dx\).

(a) Type \(\Lambda^i_{23}\) (for \(i = 0, 1, 2, 3\))

\[I(\Lambda^i_{23})\]
(2.1) \[
\frac{d^i}{dx^i} \chi(x_1) \chi(x_2) \chi(x_3) e^{(w_2 w_3 x_1 + w_1 w_2 x_2 + w_1 w_3 x_3 + (\sum_{j=1}^{d-1} y_j) t) x_1} dx_2 dx_3
\]
(2.2) \[
\frac{(w_1 w_2 w_3)^{2-i} i^{i-1} e^{w_1 w_2 w_3 \left(\sum_{j=1}^{d-1} y_j\right) t} (e^{dw_1 w_2 w_3 t} - 1)^i}{(e^{dw_2 w_3 t} - 1)(e^{dw_1 w_3 t} - 1)(e^{dw_1 w_2 t} - 1)}
\times \left(\sum_{a=0}^{d-1} \chi(a) e^{a w_2 w_3 t}\right) \left(\sum_{a=0}^{d-1} \chi(a) e^{a w_1 w_3 t}\right) \left(\sum_{a=0}^{d-1} \chi(a) e^{a w_1 w_2 t}\right)
\]
(b) Type \( \Lambda_{13} \) (for \( i = 0, 1, 2, 3 \))
(2.3) \[
I(\Lambda_{13}) = \frac{d^i}{dx^i} \chi(x_1) \chi(x_2) \chi(x_3) e^{(w_1 x_1 + w_2 x_2 + w_3 x_3 + w_1 w_2 w_3 \left(\sum_{j=1}^{d-1} y_j\right) t) x_1} dx_2 dx_3
\]
(2.4) \[
\frac{(w_1 w_2 w_3)^{1-i} i^{-1} e^{w_1 w_2 w_3 \left(\sum_{j=1}^{d-1} y_j\right) t} (e^{dw_1 w_2 w_3 t} - 1)^i}{(e^{dw_1 w_3 t} - 1)(e^{dw_2 w_3 t} - 1)}
\times \left(\sum_{a=0}^{d-1} \chi(a) e^{a w_1 w_3 t}\right) \left(\sum_{a=0}^{d-1} \chi(a) e^{a w_1 w_2 t}\right) \left(\sum_{a=0}^{d-1} \chi(a) e^{a w_1 w_2 t}\right)
\]
(c-0) Type \( \Lambda_{12}^0 \)
(2.5) \[
I(\Lambda_{12}^0) = \frac{(w_1 w_2 w_3)^{3} e^{(w_1 w_3 + w_2 w_3 + w_1) y t}}{(e^{dw_1 w_3 t} - 1)(e^{dw_2 w_3 t} - 1)}
\times \left(\sum_{a=0}^{d-1} \chi(a) e^{a w_1 t}\right) \left(\sum_{a=0}^{d-1} \chi(a) e^{a w_2 t}\right) \left(\sum_{a=0}^{d-1} \chi(a) e^{a w_3 t}\right)
\]
(c-1) Type \( \Lambda_{12}^1 \)
(2.7) \[
I(\Lambda_{12}^1) = \frac{d^3}{dx^3} \chi(x_1) \chi(x_2) \chi(x_3) e^{(w_1 x_1 + w_2 x_2 + w_3 x_3) t} dx_1 dx_2 dx_3
\]
(2.8) \[
\frac{(e^{dw_2 w_3 t} - 1)(e^{dw_1 w_3 t} - 1)(e^{dw_1 w_2 t} - 1)}{w_1 w_2 w_3 (e^{dw_1 w_3 t} - 1)(e^{dw_2 w_3 t} - 1)(e^{dw_3 t} - 1)}
\times \left(\sum_{a=0}^{d-1} \chi(a) e^{a w_1 t}\right) \left(\sum_{a=0}^{d-1} \chi(a) e^{a w_2 t}\right) \left(\sum_{a=0}^{d-1} \chi(a) e^{a w_3 t}\right)
All of the above \( p \)-adic integrals of various types are invariant under all permutations of \( w_1, w_2, w_3 \), as one can see either from \( p \)-adic integral representations in \((2.1), (2.3), (2.5), \) and \((2.7)\) or from their explicit evaluations in \((2.2), (2.4), (2.6), \) and \((2.8)\). 

3. Identities for generalized Bernoulli polynomials 

All of the following results can be easily obtained from \((1.3)\) and \((1.8)-(1.10)\). First, let’s consider Type \( \Lambda_{23}^i \), for each \( i = 0, 1, 2, 3 \).

\[(a-0)\]

\[
I(\Lambda_{23}^0) = \int_X \chi(x_1)e^{w_2w_3(x_1-w_1y_1)t}dx_1 \int_X \chi(x_2)e^{w_1w_3(x_2+w_2y_2)t}dx_2 
\times \int_X \chi(x_3)e^{w_1w_2(x_3+w_3y_3)t}dx_3 
= \left( \sum_{k=0}^{\infty} \frac{B_{k,\chi}(w_1y_1)}{k!} \right) \left( \sum_{l=0}^{\infty} \frac{B_{l,\chi}(w_2y_2)}{l!} \right) \left( \sum_{m=0}^{\infty} \frac{B_{m,\chi}(w_3y_3)}{m!} \right) \frac{t^n}{n!},
\]

where the inner sum is over all nonnegative integers \( k, l, m \), with \( k + l + m = n \), and

\[
\left( \begin{array}{c} n \\ k, l, m \end{array} \right) = \frac{n!}{k! l! m!}.
\]

\[(a-1)\]

Here we write \( I(\Lambda_{23}^1) \) in two different ways:

\[(1)\]

\[
I(\Lambda_{23}^1) = \frac{1}{w_3} \int_X \chi(x_1)e^{w_2w_3(x_1-w_1y_1)t}dx_1 \int_X \chi(x_2)e^{w_1w_3(x_2+w_2y_2)t}dx_2 
\times \int_X \chi(x_3)e^{w_1w_2(x_3+w_3y_3)t}dx_3 
= \frac{1}{w_3} \left( \sum_{k=0}^{\infty} B_{k,\chi}(w_1y_1) \frac{(w_2w_3t)^k}{k!} \right) \left( \sum_{l=0}^{\infty} B_{l,\chi}(w_2y_2) \frac{t^l}{l!} \right) 
\times \left( \sum_{m=0}^{\infty} \frac{S_m(w_3d-1, \chi) t^m}{m!} \right) 
\]

\[(3.3)\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{k+l+m=n} \left( \begin{array}{c} n \\ k, l, m \end{array} \right) B_{k,\chi}(w_1y_1) B_{l,\chi}(w_2y_2) S_m(w_3d-1, \chi) \frac{t^n}{n!} \right).
\]

(2) Invoking \((1.4), (2.2)\) can also be written as
Invoking (1.9), (3.5) can also be written as
(3.4)
\[
I(\Lambda_{23}) = \frac{1}{w_3} \sum_{a=0}^{w_3-1} \chi(a) \int_X \chi(x_1) e^{w_3 x_3} (1 + w_3 y_1)^t \, dx_1 \int_X \chi(x_2) e^{w_1 w_2} (x_2 + w_2 y_2 + \frac{w_2}{w_3})^t \, dx_2
\]
\[
= \frac{1}{w_3} \sum_{a=0}^{w_3-1} \chi(a) \left( \sum_{k=0}^{w_3-1} B_{k,\chi}(w_1 y_1) \left( \frac{w_2 w_3 t}{k!} \right)^k \right) \left( \sum_{l=0}^{w_1 w_2} B_l, \chi \left( \frac{w_2 w_3 t}{l!} \right) \right) \int_X e^{w_1 w_2} (x_2 + w_2 y_2 + \frac{w_2}{w_3})^t \, dx_2
\]
\[
= \sum_{n=0}^{w_3-1} \sum_{k=0}^{w_3-1} \binom{n}{k} B_{k,\chi}(w_1 y_1) \sum_{a=0}^{w_3-1} \chi(a) B_{n-k,\chi}(w_2 y_2 + \frac{w_2}{w_3}) x_2^{n-k} \frac{t^n}{n!}
\]
(a-2) Here we write \( I(\Lambda_{23}) \) in three different ways:
(1)
\[
(3.5)
I(\Lambda_{23}) = \frac{1}{w_2 w_3} \int_X \chi(x_1) e^{w_2 w_3 x_3} (x_3 + w_2 y_1)^t \, dx_1 \int_X \chi(x_2) e^{w_1 w_2} (x_2 + w_2 y_2 + \frac{w_2}{w_3})^t \, dx_2
\]
\[
= \frac{1}{w_2 w_3} \left( \sum_{k=0}^{w_3-1} B_{k,\chi}(w_1 y_1) \left( \frac{w_2 w_3 t}{k!} \right)^k \right) \left( \sum_{l=0}^{w_1 w_2} S_l(w_2 d - 1, \chi) \left( \frac{w_1 w_2 t}{l!} \right)^l \right)
\]
\[
\times \left( \sum_{m=0}^{w_3-1} S_m(w_3 d - 1, \chi) \frac{(w_1 w_2 t)^m}{m!} \right)
\]
(3.6)
\[
= \sum_{n=0}^{w_3-1} \sum_{k+l+m=n} \binom{n}{k, l, m} B_{k,\chi}(w_1 y_1) S_l(w_2 d - 1, \chi)
\]
\[
\times S_m(w_3 d - 1, \chi) x_2^{n-k} \frac{t^n}{n!} w_2^{k+l-1} w_3^{k-l} \frac{t^n}{n!}
\]
(2) Invoking (1.9), (3.5) can also be written as
\[
(3.7)
I(\Lambda_{23}) = \frac{1}{w_2 w_3} \int_X \chi(x_1) e^{w_2 w_3 x_3} (x_3 + w_2 y_1 + \frac{w_2}{w_3})^t \, dx_1 \int_X \chi(x_2) e^{w_1 w_2} (x_2 + w_2 y_2 + \frac{w_2}{w_3})^t \, dx_2
\]
\[
= \frac{1}{w_2 w_3} \left( \sum_{k=0}^{w_3-1} B_{k,\chi}(w_1 y_1 + \frac{w_1}{w_2}) \left( \frac{w_2 w_3 t}{k!} \right)^k \right) \left( \sum_{l=0}^{w_1 w_2} S_l(w_2 d - 1, \chi) \left( \frac{w_1 w_2 t}{l!} \right)^l \right)
\]
\[
= \sum_{n=0}^{w_3-1} \sum_{k=0}^{w_3-1} \binom{n}{k} \chi(a) B_{k,\chi}(w_1 y_1 + \frac{w_1}{w_2}) S_n-k(w_3 d - 1, \chi) x_2^{n-k} \frac{t^n}{n!} w_2^{k-1} \frac{t^n}{n!}
\]
(3) Invoking (1.9) once again, (3.7) can be written as
identities for Type \( \Lambda \)

\[ I(\Lambda^2_{23}) = \frac{1}{w_2 w_3} \sum_{a=0}^{w_2 d-1} \chi(a) \sum_{b=0}^{w_3 d-1} \chi(b) \int_X \chi(x_1) e^{w_2 w_3 (x_1 + w_1 y_1 + \frac{w_1}{w_2} a + \frac{w_1}{w_3} b)} dx_1 \]

\[ I(\Lambda^2_{23}) = \frac{1}{w_2 w_3} \sum_{a=0}^{w_2 d-1} \chi(a) \sum_{b=0}^{w_3 d-1} \chi(b) \left( \sum_{n=0}^{\infty} B_{n, \chi}(w_1 y_1 + \frac{w_1}{w_2} a + \frac{w_1}{w_3} b) \frac{(w_2 w_3 t)^n}{n!} \right) \]

\[ \left( w_2 w_3 \right)^{n-1} \sum_{a=0}^{w_2 d-1} \sum_{b=0}^{w_3 d-1} \chi(ab) B_{n, \chi}(w_1 y_1 + \frac{w_1}{w_2} a + \frac{w_1}{w_3} b) \frac{t^n}{n!}. \]

(a-3)

\[ I(\Lambda^3_{23}) = \frac{1}{w_1 w_2 w_3} \sum_{a=0}^{w_1 d-1} \chi(a) \sum_{b=0}^{w_2 d-1} \chi(b) \sum_{c=0}^{w_3 d-1} \chi(c) \int_X \chi(x_1) e^{w_1 w_2 w_3 x_1^4} dx_1 \]

\[ I(\Lambda^3_{23}) = \frac{1}{w_1 w_2 w_3} \sum_{a=0}^{w_1 d-1} \sum_{b=0}^{w_2 d-1} \sum_{c=0}^{w_3 d-1} \chi(ab) B_{n, \chi}(w_1 y_1 + \frac{w_1}{w_2} a + \frac{w_1}{w_3} b) \frac{t^n}{n!}. \]

\[ \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) S_k(w_1 d - 1, \chi) \frac{(w_2 w_3 t)^k}{k!} \left( \sum_{l=0}^{\infty} S_l(w_2 d - 1, \chi) \frac{(w_1 w_3 t)^l}{l!} \right) \times \left( \sum_{m=0}^{\infty} S_m(w_3 d - 1, \chi) \frac{(w_1 w_2 t)^m}{m!} \right) \times w_1^{l+m-1} w_2^{k+m-1} w_3^{k+l-1} \frac{t^n}{n!}. \]

(b) For Type \( \Lambda^1_{13} \) (\( i = 0, 1, 2, 3 \)), we may consider the analogous things to the ones in (a-0), (a-1), (a-2), and (a-3). However, these do not lead us to new identities. Indeed, if we substitute \( w_2 w_3, w_1 w_3, w_1 w_2 \) respectively for \( w_1, w_2, w_3 \) in (2.1), this amounts to replacing \( t \) by \( w_1 w_2 w_3 t \) in (2.3). So, upon replacing \( w_1, w_2, w_3 \) respectively by \( w_2 w_3, w_1 w_3, w_1 w_2 \) and dividing by \( (w_1 w_2 w_3)^n \), in each of the expressions of (3.1), (3.3), (3.4), (3.6), (3.8), (3.9), we will get the corresponding symmetric identities for Type \( \Lambda^1_{13} \) (\( i = 0, 1, 2, 3 \)).

(c-0) \[ I(\Lambda^0_{12}) \]

\[ I(\Lambda^0_{12}) = \int_X \chi(x_1) e^{w_1 (x_1 + w_2 y)} dt \int_X \chi(x_2) e^{w_2 (x_2 + w_3 y)} dt \int_X \chi(x_3) e^{w_3 (x_3 + w_1 y)} dx_3 \]

\[ I(\Lambda^0_{12}) = \sum_{n=0}^{\infty} B_{k, \chi}(w_2 y) (w_1 t)^k \sum_{l=0}^{\infty} B_{l, \chi}(w_3 y) (w_2 t)^l \sum_{m=0}^{\infty} B_{m, \chi}(w_1 y) (w_3 t)^m \]
\[ (3.11) \]

\[
= \sum_{n=0}^{\infty} \left( \sum_{k+l+m=n} \binom{n}{k, l, m} B_{k, \chi}(w_2 y) B_{l, \chi}(w_3 y) B_{m, \chi}(w_1 y) w_1^k w_2^l w_3^m \right) \frac{t^n}{n!}.
\]

(c-1)

\[
I(\Lambda_{12}) = \frac{1}{w_1 w_2 w_3} \frac{dw_1 \int_X \chi(x_1) e^{w_1 x_1 t} dx_1}{\int_X e^{w_1 x_1 t} dx_1} \times \frac{dw_2 \int_X \chi(x_2) e^{w_2 x_2 t} dx_2}{\int_X e^{w_2 x_2 t} dx_2} \times \frac{dw_3 \int_X \chi(x_3) e^{w_3 x_3 t} dx_3}{\int_X e^{w_3 x_3 t} dx_3} = \frac{1}{w_1 w_2 w_3} \left( \sum_{k=0}^{\infty} S_k(w_2 d - 1, \chi) \left( \frac{(w_1 t)^k}{k!} \right) \left( \sum_{l=0}^{\infty} S_l(w_3 d - 1, \chi) \left( \frac{(w_2 t)^l}{l!} \right) \left( \sum_{m=0}^{\infty} S_m(w_1 d - 1, \chi) \left( \frac{(w_3 t)^m}{m!} \right) \right) \right) \right.
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{k+l+m=n} \binom{n}{k, l, m} S_k(w_2 d - 1, \chi) S_l(w_3 d - 1, \chi) S_m(w_1 d - 1, \chi) \right. \times \frac{w_1^{k-1} w_2^{l-1} w_3^{m-1}}{n!} \frac{t^n}{n!}.
\]

4. Main theorems

As we noted earlier in the last paragraph of Section 2, the various types of quotients of $p$-adic integrals are invariant under any permutation of $w_1, w_2, w_3$. So the corresponding expressions in Section 3 are also invariant under any permutation of $w_1, w_2, w_3$. Thus our results about identities of symmetry will be immediate consequences of this observation.

However, not all permutations of an expression in Section 3 yield distinct ones. In fact, as these expressions are obtained by permuting $w_1, w_2, w_3$ in a single one labeled by them, they can be viewed as a group in a natural manner and hence it is isomorphic to a quotient of $S_3$. In particular, the number of possible distinct expressions are 1, 2, 3 or 6. (a-0), (a-1(1)), (a-1(2)), and (a-2(2)) give the full six identities of symmetry, (a-2(1)) and (a-2(3)) yield three identities of symmetry, and (c-0) and (c-1) give two identities of symmetry, while the expression in (a-3) yields no identities of symmetry.

Here we will just consider the cases of Theorems 4 and 8, leaving the others as easy exercises for the reader. As for the case of Theorem 4, in addition to (4.11)-(4.13), we get the following three ones:
But, by interchanging \( l \) and \( m \), we see that (4.4), (4.5), and (4.6) are equal to (1.11), (1.12), and (1.13).

As to Theorem 8, in addition to (4.17) and (4.18), we have:

\[
\sum_{k+l+m=n} \binom{n}{k,l,m} B_{k,\chi}(w_1 y_1) S_{l}(w_3 d - 1, \chi) S_{m}(w_2 d - 1, \chi) w_1^{l+m} w_3^{k+m-1} w_2^{k+l-1},
\]

(4.2)

\[
\sum_{k+l+m=n} \binom{n}{k,l,m} B_{k,\chi}(w_1 y_1) S_{l}(w_1 d - 1, \chi) S_{m}(w_3 d - 1, \chi) w_2^{l+m} w_1^{k+m-1} w_3^{k+l-1},
\]

(4.3)

\[
\sum_{k+l+m=n} \binom{n}{k,l,m} B_{k,\chi}(w_3 y_1) S_{l}(w_1 d - 1, \chi) S_{m}(w_1 d - 1, \chi) w_3^{l+m} w_2^{k+m-1} w_1^{k+l-1}.
\]

(4.4)

However, (4.4) and (4.5) are equal to (4.17), as we can see by applying the permutations \( k \to l, l \to m, m \to k \) for (4.4) and \( k \to m, l \to k, m \to l \) for (4.5). Similarly, we see that (4.6) and (4.7) are equal to (4.18), by applying permutations \( k \to l, l \to m, m \to k \) for (4.6) and \( k \to m, l \to k, m \to l \) for (4.7).
Theorem 4.1. Let $w_1, w_2, w_3$ be any positive integers. Then we have:

$$
\sum_{k+l+m=n} \binom{n}{k, l, m} B_{k, \chi}(w_1 y_1) B_{l, \chi}(w_2 y_2) B_{m, \chi}(w_3 y_3) w_1^{l+m} w_2^{-k-m} w_3^{k+l} = \sum_{k+l+m=n} \binom{n}{k, l, m} B_{k, \chi}(w_1 y_1) B_{l, \chi}(w_2 y_2) B_{m, \chi}(w_3 y_3) w_1^{l+m} w_2^{-k-m} w_3^{k+l} = \sum_{k+l+m=n} \binom{n}{k, l, m} B_{k, \chi}(w_2 y_1) B_{l, \chi}(w_1 y_2) B_{m, \chi}(w_3 y_3) w_2^{l+m} w_1^{-k-m} w_3^{k+l} = \sum_{k+l+m=n} \binom{n}{k, l, m} B_{k, \chi}(w_3 y_1) B_{l, \chi}(w_1 y_2) B_{m, \chi}(w_2 y_3) w_3^{l+m} w_1^{-k-m} w_3^{k+l} = \sum_{k+l+m=n} \binom{n}{k, l, m} B_{k, \chi}(w_3 y_1) B_{l, \chi}(w_2 y_2) B_{m, \chi}(w_1 y_3) w_3^{l+m} w_2^{-k-m} w_1^{k+l}
$$

(4.8)

Theorem 4.2. Let $w_1, w_2, w_3$ be any positive integers. Then we have:

$$
\sum_{k+l+m=n} \binom{n}{k, l, m} B_{k, \chi}(w_1 y_1) B_{l, \chi}(w_2 y_2) B_{m, \chi}(w_3 y_3) w_2^{l+m} w_1^{-k-m} w_3^{k+l-1} = \sum_{k+l+m=n} \binom{n}{k, l, m} B_{k, \chi}(w_1 y_1) B_{l, \chi}(w_3 y_2) B_{m, \chi}(w_3 y_3) w_2^{l+m} w_3^{-k-m} w_1^{k+l-1} = \sum_{k+l+m=n} \binom{n}{k, l, m} B_{k, \chi}(w_2 y_1) B_{l, \chi}(w_1 y_2) B_{m, \chi}(w_3 y_3) w_2^{l+m} w_1^{-k-m} w_3^{k+l-1} = \sum_{k+l+m=n} \binom{n}{k, l, m} B_{k, \chi}(w_3 y_1) B_{l, \chi}(w_1 y_2) B_{m, \chi}(w_2 y_3) w_3^{l+m} w_1^{-k-m} w_2^{k+l-1} = \sum_{k+l+m=n} \binom{n}{k, l, m} B_{k, \chi}(w_3 y_1) B_{l, \chi}(w_2 y_2) B_{m, \chi}(w_1 y_3) w_3^{l+m} w_2^{-k-m} w_1^{k+l-1}
$$

(4.9)

Theorem 4.3. Let $w_1, w_2, w_3$ be any positive integers. Then we have:

$$
\sum_{k=0}^{w_1-1} \binom{n}{k} B_{k, \chi}(w_3 y_1) \sum_{a=0}^{w_1 d-1} \chi(a) B_{n-k, \chi}(w_2 y_2 + \frac{w_2}{w_1} a) w_3^{n-k} w_2^k = \sum_{k=0}^{w_1-1} \binom{n}{k} B_{k, \chi}(w_2 y_1) \sum_{a=0}^{w_1 d-1} \chi(a) B_{n-k, \chi}(w_3 y_2 + \frac{w_3}{w_1} a) w_2^{n-k} w_3^k
$$
Theorem 4.6. \( w_2^{-1} \sum_{k=0}^{n} \binom{n}{k} B_{k, \chi}(w_3 y_1) \sum_{a=0}^{w_2 d-1} \chi(a) B_{n-k, \chi}(w_1 y_2 + \frac{w_1}{w_2} a) w_3^{n-k} w_1^k \)

(4.10)

\[= w_2^{-1} \sum_{k=0}^{n} \binom{n}{k} B_{k, \chi}(w_1 y_1) \sum_{a=0}^{w_2 d-1} \chi(a) B_{n-k, \chi}(w_3 y_2 + \frac{w_3}{w_2} a) w_1^{n-k} w_3^k \]

(4.11)

\[= w_3^{-1} \sum_{k=0}^{n} \binom{n}{k} B_{k, \chi}(w_2 y_1) \sum_{a=0}^{w_3 d-1} \chi(a) B_{n-k, \chi}(w_1 y_2 + \frac{w_1}{w_3} a) w_2^{n-k} w_1^k \]

(4.12)

\[= w_3^{-1} \sum_{k=0}^{n} \binom{n}{k} B_{k, \chi}(w_3 y_1) \sum_{a=0}^{w_3 d-1} \chi(a) B_{n-k, \chi}(w_2 y_2 + \frac{w_2}{w_3} a) w_3^{n-k} w_2^k \]

(4.13)

Theorem 4.4. Let \( w_1, w_2, w_3 \) be any positive integers. Then we have the following three symmetries in \( w_1, w_2, w_3 \):

(4.14)

\[= w_3^{-1} \sum_{k=0}^{n} \binom{n}{k} B_{k, \chi}(w_3 y_1) \sum_{a=0}^{w_3 d-1} \chi(a) B_{n-k, \chi}(w_1 y_2 + \frac{w_1}{w_3} a) w_2^{n-k} w_1^k \]

(4.15)

Theorem 4.5. Let \( w_1, w_2, w_3 \) be any positive integers. Then we have:

(4.16)

\[= w_3^{-1} \sum_{k=0}^{n} \binom{n}{k} B_{k, \chi}(w_3 y_1) \sum_{a=0}^{w_3 d-1} \chi(a) B_{n-k, \chi}(w_2 y_2 + \frac{w_2}{w_3} a) w_1^{n-k} w_2^k \]

(4.17)

Theorem 4.6. Let \( w_1, w_2, w_3 \) be any positive integers. Then we have the following three symmetries in \( w_1, w_2, w_3 \):
\[(w_1 w_2)^{n-1} \sum_{a=0}^{w_1 d-1} \sum_{b=0}^{w_2 d-1} \chi(ab) B_{n,\chi}(w_3 y_1 + \frac{w_3}{w_1} a + \frac{w_3}{w_2} b)\]

(4.15) \[= (w_2 w_3)^{n-1} \sum_{a=0}^{w_2 d-1} \sum_{b=0}^{w_3 d-1} \chi(ab) B_{n,\chi}(w_1 y_1 + \frac{w_1}{w_2} a + \frac{w_1}{w_3} b)\]

(4.16) \[= (w_3 w_1)^{n-1} \sum_{a=0}^{w_3 d-1} \sum_{b=0}^{w_1 d-1} \chi(ab) B_{n,\chi}(w_2 y_1 + \frac{w_2}{w_3} a + \frac{w_2}{w_1} b).\]

**Theorem 4.7.** Let \(w_1, w_2, w_3\) be any positive integers. Then we have the following two symmetries in \(w_1, w_2, w_3:\)

\[
\sum_{k+l+m=n} \binom{n}{k, l, m} B_{k,\chi}(w_1 y) B_{l,\chi}(w_2 y) B_{m,\chi}(w_3 y) w_3^{k} w_1^{l} w_2^{m}.
\]

(4.17) \[
= \sum_{k+l+m=n} \binom{n}{k, l, m} B_{k,\chi}(w_1 y) B_{l,\chi}(w_3 y) B_{m,\chi}(w_2 y) w_2^{k} w_1^{l} w_3^{m}.
\]

**Theorem 4.8.** Let \(w_1, w_2, w_3\) be any positive integers. Then we have the following two symmetries in \(w_1, w_2, w_3:\)

\[
\sum_{k+l+m=n} \binom{n}{k, l, m} S_k(w_1 d - 1, \chi) S_l(w_2 d - 1, \chi) S_m(w_3 d - 1, \chi) w_3^{k-1} w_1^{l-1} w_2^{m-1}.
\]

(4.18) \[
= \sum_{k+l+m=n} \binom{n}{k, l, m} S_k(w_1 d - 1, \chi) S_l(w_3 d - 1, \chi) S_m(w_2 d - 1, \chi) w_2^{k-1} w_1^{l-1} w_3^{m-1}.
\]

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