Unconstrained Hamiltonian Formulation of $SU(2)$ Gluodynamics

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$SU(2)$ Yang-Mills field theory is considered in the framework of the generalized Hamiltonian approach and the equivalent unconstrained system is obtained using the method of Hamiltonian reduction. A canonical transformation to a set of adapted coordinates is performed in terms of which the Abelianization of the Gauss law constraints reduces to an algebraic operation and the pure gauge degrees of freedom drop out from the Hamiltonian after projection onto the constraint shell. For the remaining gauge invariant fields two representations are introduced where the three fields which transform as scalars under spatial rotations are separated from the three rotational fields. An effective low energy nonlinear sigma model type Lagrangian is derived which out of the six physical fields involves only one of the three scalar fields and two rotational fields summarized in a unit vector. Its possible relation to the effective Lagrangian proposed recently by Faddeev and Niemi is discussed. Finally the unconstrained analog of the well-known nonnormalizable groundstate wave functional which solves the Schrödinger equation with zero energy is given and analysed in the strong coupling limit.

I. INTRODUCTION

One of the main issues in the Hamiltonian formulation of Yang-Mills theories is to find the projection from the phase space of canonical variables constrained by the non-Abelian Gauss law to the “smaller” phase space of unconstrained gauge invariant coordinates only. Dealing with the problem of the elimination of the pure gauge degrees of freedom two approaches exist, the perturbative and the nonperturbative one, with complementary features. The conventional perturbative gauge fixing method works successfully for the description of high energy phenomena, but fails in applications in the infrared region. The correct nonperturbative reduction of gauge theories $^1$- $^5$, on the other hand, leads to representations for the unconstrained Yang-Mills systems, which are valid also in the low energy region, but unfortunately up to now have been rather complicated for practical calculations. The guideline of these investigations is the search for a representation of the gauge invariant variables which is suitable for the description the infrared limit of Yang-Mills theory. To get such a representation for the unconstrained system we are following the Dirac generalized Hamiltonian formalism $^6$- $^9$ using the method of Hamiltonian reduction (see $^10$- $^12$ and references therein) instead of the conventional gauge fixing approach $^13$. In previous work $^13$ it was demonstrated that for the case of the mechanics of spatially constant $SU(2)$ Dirac-Yang-Mills fields an unconstrained Hamiltonian can be derived which has a simple practical form. The elimination of the gauge degrees of freedom has been achieved by performing a canonical transformation to a new adapted coordinates, in terms of which both the Abelianization of the Gauss law constraints and the projection onto the constraint shell is simplified. The obtained unconstrained system then describes the dynamics of a symmetric second rank tensor under spatial rotations. The main-axis-transformation of this symmetric tensor allowed us to separate the gauge invariant variables into scalars under ordinary space rotations and into “rotational” degrees of freedom. In this final form the physical Hamiltonian can be quantized without operator ordering ambiguities.

In this work we shall generalize our approach from non-Abelian Dirac-Yang-Mills mechanics $^13$ to field theory. We shall give a Hamiltonian formulation of classical $SU(2)$ Yang-Mills field theory entirely in terms of gauge invariant variables, and separate these into scalars under ordinary space rotations and into “rotational” degrees of freedom. It will be shown that this naturally leads to their identification as fields with “nonrelativistic spin-2 and spin-0”.

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Furthermore the separation into scalar and rotational degrees of freedom will turn out to be very well suited for the study of the infrared limit of unconstrained Yang-Mills theory. We shall obtain an effective low energy theory involving only two of the three rotational fields and one of the tree scalar fields, and shall discuss its possible relation to the effective soliton Lagrangian proposed recently in [24]. Finally we shall analyse the well-known nonnormalizable solution of the functional Schrödinger equation with zero energy in the framework of the unconstrained formulation of the SU(2) Yang-Mills theory.

The outline of the article is as follows. In Section II we present the Hamiltonian reduction of SU(2) Yang-Mills field theory. We perform the canonical transformation to a new set of adapted coordinates, Abelianize the Gauss law constraints, and achieve the unconstrained description for SU(2) Yang-Mills theory. In section III two representations for the physical field in terms scalars and rotational degrees are described. Section IV is devoted to the study of the infrared limit of unconstrained gluodynamics. In Section V the well-known nonnormalizable solution of the functional Schrödinger equation with zero energy is analysed in our unconstrained formulation of the SU(2) Yang-Mills theory. Finally, in Section VI, we give our Conclusions. In an Appendix we list several formulae for nonrelativistic spin-0, spin-1 and spin-2 used in the text.

II. REDUCTION OF GAUGE DEGREES OF FREEDOM

The degenerate character of the conventional Yang-Mills action for SU(2) gauge fields \( A_\mu^a(x) \)

\[
S[A] := -\frac{1}{4} \int d^4x \, F_{\mu\nu}^a F^{a\mu\nu} , \quad F_{\mu\nu}^a := \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\epsilon^{abc} A^b_\mu A^c_\nu , \tag{2.1}
\]

requires the use of the generalized Hamiltonian approach \([14]\). From the definition of the canonical momenta \( P_\alpha := \partial L/\partial (\partial_0 A^a_\alpha) \), \( E_{ai} := \partial L/\partial (\partial_0 A_{ai}) \) it follows, that the phase space spanned by the variables \((A_\mu^a, P^a)\), \((A_{ai}, E_{ai})\) is restricted by the three primary constraints \( P^a(x) = 0 \). According to the Dirac procedure in this case the evolution of the system is governed by the total Hamiltonian containing three arbitrary functions \( \lambda_\alpha(x) \)

\[
H_T := \int d^4x \left[ \frac{1}{2} (E_{ai}^2 + B_{ai}^2(A)) - A_0^a (\partial_\mu E_{ai} + g\epsilon_{abc} A_{bi} E_{ci}) + \lambda_\alpha(x) P^a(x) \right] , \tag{2.2}
\]

where \( B_{ai}(A) := \epsilon_{ijk} (\partial_j A_{ak} + \frac{1}{2} g\epsilon_{abc} A_{bj} A_{ck}) \) is the non-Abelian magnetic field. From the conservation of the primary constraints \( P^a = 0 \) in time one obtains the non-Abelian Gauss law constraints

\[
\Phi_\alpha := \partial_\mu E_{ai} + g\epsilon_{abc} A_{ci} E_{bi} = 0 . \tag{2.3}
\]

Although the total Hamiltonian \( (2.2) \) depends on the arbitrary functions \( \lambda_\alpha(x) \) it is possible to extract the dynamical variables which have uniquely predictable dynamics. Furthermore they can be chosen to be free of any constraints. Such an extracted system with predictable dynamics without constraints is called unconstrained.

The non-Abelian character of the secondary constraints,

\[
\{ \Phi_\alpha(x), \Phi_\beta(y) \} = g\epsilon_{abc} \Phi_c(x) \delta(x - y) , \tag{2.4}
\]

is the main obstacle for the corresponding projection to the unconstrained phase space. For Abelian constraints \( \Psi_\alpha (\{\Psi_\alpha, \Psi_\beta\} = 0) \) the projection to the reduced phase space can be simply achieved in the following two steps. One performs a canonical transformation to new variables such that part of the new momenta \( P_\alpha \) coincide with the constraints \( \Psi_\alpha \). After the projection onto the constraint shell, i.e. putting in all expressions \( P_\alpha = 0 \), the coordinates canonically conjugate to the \( P_\alpha \) drop out from the physical quantities. The remaining canonical pairs are then gauge invariant and form the basis for the unconstrained system. For the case of non-Abelian constraints \( (2.4) \) it is clearly impossible to obtain such a canonical basis via canonical transformations alone. The way to avoid this difficulty is to replace the set of non-Abelian constraints \( (2.4) \) by a new set of Abelian constraints which describe the constraint surface in phase space. This Abelianization procedure reduces the problem to the Abelian case. There are several methods of Abelianization of constraints (see e.g. \([20], [21]\) and references therein). In general the procedure can be reduced to the solution of linear differential equations with partial derivatives \( [20] \).

A. Canonical transformation & Abelianization of the Gauss law constraints

The problem of Abelianization is considerably simplified when studied in terms of coordinates adapted to the action of the gauge group. The knowledge of the SU(2) gauge transformations

\[ \begin{array}{ll}
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which leaves the Yang-Mills action [2,4] invariant, directly prompts us with the choice of adapted coordinates by using the following point transformation to the new set of Lagrangian coordinates \( q_j \) \( (j = 1, 2, 3) \) and the six elements \( Q_{ik} = Q_{ki} \) \( (i, k = 1, 2, 3) \) of the positive definite symmetric \( 3 \times 3 \) matrix \( Q \)

\[
A_{ai} (q, Q) := O_{ai} (q) Q_{ki} - \frac{1}{2g} \epsilon_{abc} \left( O(q) \partial_i O^T (q) \right)_{bc},
\]

where \( O(q) \) is an orthogonal \( 3 \times 3 \) matrix parametrised by the \( q_i \). In the following we shall show that in terms of these variables the non-Abelian Gauss law constraints \( \text{(2.3)} \) only depend on the \( q_i \) and their conjugated momenta \( p_i \) and after Abelianization become \( p_i = 0 \). The unconstrained variables \( Q_{ik} \) and their conjugate \( P_{ik} \) are gauge invariant, i.e. commute with the Gauss law, and represent the basic variables for all observable quantities [3]. The transformation \( \text{(2.6)} \) induces a point canonical transformation linear in the new canonical momenta \( P_{ik} \) and \( p_i \). Using the corresponding generating functional depending on the old momenta and the new coordinates,

\[
F_3 [E; q, Q] := \int d^3 z \ E_{ai} (z) A_{ai} (q(z), Q(z)),
\]

one can obtain the transformation to new canonical momenta \( p_i \) and \( P_{ik} \)

\[
p_j (x) := \frac{\delta F_3}{\delta q_j (x)} = - \frac{1}{g} \Omega_{jr} \left( D_i (Q) D^T E \right)_{ri},
\]

\[
P_{ik} (x) := \frac{\delta F_3}{\delta Q_{ik} (x)} = - \frac{1}{2} \left( E^T O + O^T E \right)_{ik}.
\]

Here

\[
\Omega_{ji} (q) := - \frac{i}{2} \text{Tr} \left( O^T (q) \frac{\partial O (q)}{\partial q_j} J_i \right),
\]

with the \( 3 \times 3 \) matrix generators of \( SO(3) \), \((J_i)_{mn} := i \epsilon_{mn} \), and the corresponding covariant derivative \( D_i (Q) \) in the adjoint representation

\[
(D_i (Q))_{mn} := \delta_{mn} \partial_i - ig \left( J^k \right)_{mn} Q_{ki}.
\]

A straightforward calculation based on the linear relations \( \text{(2.8)} \) and \( \text{(2.9)} \) between the old and the new momenta leads to the following expression for the field strengths \( E_{ai} \) in terms of the new canonical variables

\[
E_{ai} = O_{ai} (q) \left[ P_{ki} + \epsilon_{kis} * D^{-1} (Q) \left[ (\Omega^{-1})_i - S_i \right] \right].
\]

Here \( * D^{-1} \) is the inverse of the matrix operator

\[
* D_{ik} (Q) := -i \left( J^m D_m (Q) \right)_{ik},
\]

and

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1. In the strong coupling limit the representation [2,4] reduces to the so-called polar representation for arbitrary quadratic matrices for which the decomposition can be proven to be well-defined and unique (see for example [26]). In the general case we have the additional second term which takes into account the inhomogeneity of the gauge transformation and \( D^T \) has to be regarded as a set of partial differential equations for the \( q_i \) variables. The uniqueness and regularity of the suggested transformation \( \text{(2.6)} \) depends on the boundary conditions imposed. In the present work the uniqueness and regularity of the change of coordinates is assumed as a reasonable conjecture without search for the appropriate boundary conditions.

2. The freedom to use other canonical variables in the unconstrained phase space corresponds to another fixation of the six variables \( Q \) in the representation \( \text{(2.6)} \). This observation clarifies the connection with the conventional gauge fixing method. We shall discuss this point in forthcoming publications (see also ref. [5]).
\[ S_k(x) := \epsilon_{klm} (PQ)_{lm} - \frac{1}{g} \partial_l P_{kl}. \] (2.14)

Using the representations (2.6) and (2.12) one can easily convince oneself that the variables \( Q \) and \( P \) make no contribution to the Gauss law constraints (2.3)
\[ \Phi_a = O_{as}(q) \Omega^{-1}_{sj}(q) p_j = 0. \] (2.15)
Here and in (2.12) we assume that the matrix \( \Omega \) is invertible. The equivalent set of Abelian constraints is
\[ p_a = 0. \] (2.16)
They are Abelian due to the canonical structure of the new variables.

### B. The Hamiltonian in terms of unconstrained fields

After having rewritten the model in terms of the new canonical coordinates and after the Abelianization of the Gauss law constraints, the construction of the unconstrained Hamiltonian system is straightforward. In all expressions we can simply put \( p_a = 0 \). In particular, the Hamiltonian in terms of the unconstrained canonical variables \( Q \) and \( P \) can be represented by the sum of three terms
\[
H[Q, P] = \frac{1}{2} \int d^3x \left[ \text{Tr}(P)^2 + \text{Tr}(B^2(Q)) + \frac{1}{2} \vec{E}^2(Q, P) \right].
\] (2.17)
The first term is the conventional quadratic “kinetic” part and the second the “magnetic potential” term which is the trace of the square of the non-Abelian magnetic field
\[ B_{sk} := \epsilon_{klm} \left( \partial_l Q_{sm} + \frac{g}{2} \epsilon_{sbc} Q_{bl} Q_{cm} \right). \] (2.18)
It is interesting that after the elimination of the pure gauge degrees of freedom the magnetic field strength tensor is the commutator of the covariant derivatives \( F_{ij} = [D_i(Q), D_j(Q)] \).

The third, nonlocal term in the Hamiltonian (2.17) is the square of the antisymmetric part of the electric field (2.12),\[ E_s := (1/2)\epsilon_{sij} E_{ij}, \] after projection onto the constraint surface. It is given as the solution of the system of differential equations\[ ^* D_{ls}(Q) E_s = g S_l, \] (2.19)
with the derivative \(^* D_{ls}(Q)\) defined in (2.13). Note that the vector \( S_l(x) \), defined in (2.14), coincides up to divergence terms with the spin density part of the Noetherian angular momentum, \( S_l(x) := \epsilon_{ijk} A^k \partial_x E_{sk} \), after transformation to the new variables and projection onto the constraint shell. The solution \( \vec{E} \) of the differential equation (2.19) can be expanded in a \( 1/g \) series. The zeroth order term is
\[ E_s^{(0)} = \gamma_{sk}^{-1} \epsilon_{klm} (PQ)_{lm}, \] (2.21)
with \( \gamma_{ik} := Q_{ik} - \delta_{ik} \text{Tr}(Q) \), and the first order term is determined as
\[ E_s^{(1)} := \frac{1}{g} \gamma_{sk}^{-1} \left[ \left( \text{rot} \vec{E}^{(0)} \right)_l - \partial_l P_{kl} \right]. \] (2.22)

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3 We remark that for the solution of this equation we need to impose boundary conditions only on the physical variables \( Q \), in contrast to Eq. (2.6) for which boundary conditions only for the unphysical variables \( q_i \) are needed.

4 Note that the presence of this divergence term destroys the \( so(3) \) algebra of densities due to the presence of Schwinger terms
\[ \{ S_l(x), S_j(y) \} = \epsilon_{ijk} S_k(x) \delta(x - y) + \epsilon_{ijk} P_{sk}(x) \delta(x - y), \] (2.20)
but maintains the value of spin and its algebra if one neglects the surface terms.
from the zeroth order term. The higher terms are then obtained by the simple recurrence relations

$$E_s^{(n+1)} := \frac{1}{g} s^{-1}_s (\text{rot} \ E_s^{(n)})_s .$$  \hfill (2.23)

One easily recognizes in these expressions the conventional definition of the covariant curl operation in terms of the covariant derivative

$$\text{curl} S(e_i, e_j) := \langle \nabla e_i S, e_j \rangle - \langle \nabla e_j S, e_i \rangle ,$$

calculated in the basis $e_i := (\gamma^{1/2})_{ij} \partial_j$ and $\gamma_{ij} := \langle e_i, e_j \rangle$ with the corresponding connection $\nabla e_i e_j = \Gamma^l_{ij} e_l$, e.g.

$$E^{(1)}_{ij} = \text{curl} S(e_i, e_j) .$$  \hfill (2.24)

III. THE UNCONSTRAINED HAMILTONIAN IN TERMS OF SCALAR AND ROTATIONAL DEGREES OF FREEDOM

In the previous section we have obtained the unconstrained Hamiltonian system in terms of physical fields represented by a positive definite symmetric matrix $Q$. The initial gauge fields $A_i$ transformed as vectors under spatial rotations. We now would like to study the transformation properties of the corresponding reduced matrix field $Q$.

For systems possessing some rigid symmetry it is well known to be very useful for practical calculations to pass to a coordinate basis such that a subset of variables is invariant under the action of the symmetry group. In this section we shall therefore carry out the explicit separation of the rotational degrees of freedom, which vary under rotations, from the scalars.

A. Transformation properties of the unconstrained fields under space rotations

In order to search for a parametrization of the unconstrained variables in Yang-Mills theory adapted to the action of the group of spatial rotations we shall study the corresponding transformation properties of the field $Q$. The total Noetherian angular momentum vector for $SU(2)$ gluodynamics is

$$I_i = \epsilon_{ijk} \int d^3 x \left( E_{aj} A_k^a + x_k E_{al} \frac{\partial A_{il}}{\partial x^j} \right) .$$  \hfill (3.1)

After elimination of the gauge degrees of freedom it reduces to

$$I_i = \int d^3 x \ \epsilon_{ijk} \left( (P Q)_{jk} + x_k \text{Tr} (P \partial_j Q) \right) ,$$  \hfill (3.2)

where surface terms have been neglected.

Under infinitesimal rotations in 3-dimensional space, $\delta x_i = \omega_{ij} x_j$, generated by (3.2), the physical field $Q$ transforms as

$$\delta Q_{ij} = \epsilon_{s mn} \omega_{mn} \{Q_{ij}, I_s \} = \omega_{mn} (S^{mn} Q)_{ij} + \text{orbital part transf.}$$  \hfill (3.3)

with the matrices

$$(S)^{mn}_{(kl)} := (\delta^m_i \delta^n_j - \delta^m_j \delta^i_n) + (m \leftrightarrow n) ,$$  \hfill (3.4)

which describes the $SO(3)$ rotations of a 3-dimensional second rank tensor field

$$Q'_{ik} = R_{il}(\omega) R_{km}(\omega) Q_{lm} .$$  \hfill (3.5)

It is well known that any symmetric second rank tensor can be decomposed into its irreducible components, one spin-0 and the five components of a spin-2 field by extraction of its trace. On the other hand it can be diagonalized via a main axis transformation, which corresponds to a separation of diagonal fields, which are invariant under rotations, from the rotational degrees of freedom. In the following paragraphs we shall investigate both representations and their relation to each other.
B. The unconstrained Hamiltonian in terms of spin-2 and spin-0 fields

As shown in the preceding paragraph the six independent elements of the matrix field $Q$ can be represented as a mixture of fields with nonrelativistic spin-2 and spin-0. In order to put the theory into a more transparent form explicitly showing its rotational invariance, it is useful to perform a canonical transformation to the corresponding spin-2 and spin-0 fields as new variables. To achieve this let us decompose the symmetric matrix $Q$ into the irreducible representations of $S0(3)$ group

$$Q_{ij}(x) = \frac{1}{\sqrt{2}} Y_A(x) \ T^A_{ij} + \frac{1}{\sqrt{3}} \Phi(x) \ I_{ij} ,$$

(3.6)

with the field $\Phi$ proportional to the trace of $Q$ as spin-0 field and the 5-dimensional spin-2 vector $Y(x)$ with components $Y_A$ labeled by its value of spin along the $z$-axis, $A = \pm 2, \pm 1, 0$. $\mathbb{I}$ is the $3 \times 3$ unit matrix and the five traceless $3 \times 3$ basis matrices $T^A$ are listed in the Appendix.

The momenta $P_A(x)$ and $P_\Phi(x)$ canonical conjugate to the fields $Y_A(x)$ and $\Phi(x)$ are the components of the corresponding expansion for the $P$ variable

$$P_{ij}(x) = \frac{1}{\sqrt{2}} P_A(x) \ T^A_{ij} + \frac{1}{\sqrt{3}} P_\Phi(x) \ I_{ij} .$$

(3.7)

For the magnetic field $B$ we obtain the expansion

$$B_{ij}(x) = \frac{1}{\sqrt{2}} H_A(x) \ T^A_{ij} + \frac{1}{\sqrt{2}} \ h_\alpha(x) \ J^\alpha_{ij} + \frac{1}{\sqrt{3}} \ b(x) \ I_{ij} ,$$

(3.8)

with the components

$$H_A := \frac{1}{2} c^{(2)}_{A\beta B} \partial_\beta Y^B + \frac{g}{\sqrt{3}} \left( \frac{1}{\sqrt{2}} \ast Y_A - \Phi Y_A \right) ,$$

(3.9)

$$h_\alpha := \frac{1}{2} d^{(1)}_{\alpha B_\gamma} \partial_\gamma Y^B + \frac{2}{\sqrt{3}} \partial_\alpha \Phi ,$$

(3.10)

$$b := \frac{g}{\sqrt{3}} (\frac{1}{2} Y^2 - \Phi^2) ,$$

(3.11)

in terms of the structure constants $c^{(2)}_{A\beta B}$ and $d^{(1)}_{\alpha B_\gamma}$ of the algebra of the spin-1 matrices $J^\alpha$ and the spin-2 matrices $T^A$, listed in the Appendix, and another five-dimensional vector

$$\ast Y_C := d^{(2)}_{CAB} Y^A Y^B ,$$

(3.12)

with constants $d^{(2)}_{ABC}$ given explicitly in the Appendix. Finally we obtain the reduced Hamiltonian in terms of spin-2 and spin-0 field components

$$H[\mathbf{P}, \mathbf{Y}, P_\phi, \Phi] := \frac{1}{2} \int d^3x \ \left( \mathbf{P}^2(x) + \mathbf{E}^2(x) + P_\phi^2(x) + \mathbf{H}^2(x) + \mathbf{\tilde{h}}^2(x) + b^2(x) \right) ,$$

(3.13)

with expressions \(3.3\) for the magnetic field components and the antisymmetric part $\mathbf{\tilde{E}}$ of the electric field given by \(2.21\) - \(2.23\), expressing $Q$ and $P$ in terms of $\mathbf{Y}$, $\Phi$ and $\mathbf{P}$, $P_\phi$ via \(3.6\) and \(3.7\). In order to discuss the transformation properties of the spin-2 fields $\mathbf{Y}$ under spatial rotations we rewrite the angular momentum vector \(3.2\) in terms of the fields $\mathbf{Y}$, $\mathbf{P}$ and $\Phi$, $P_\phi$.

$$I_i = S_i + \epsilon_{ijk} \int d^3x \ x_j (P_\phi \partial_k \Phi + P_A \partial_k Y^A ) ,$$

(3.14)

Everywhere in the article 3-dimensional vectors are topped by an arrow and their Cartesian and spherical components are labeled by small Latin and Greek letters respectively, while the 5-dimensional spin-2 vectors are written in boldface and their “spherical” components labeled by capital Greek letters. For the lowering and raising of the indices of 5-dimensional vectors the metric tensor $\eta_{AB} = (-1)^A \delta_{A,-B}$ is used.
with the spin part

\[ S_i = i(J_i)_{AB} Y^A e^B. \]  

(3.15)

Here the three $5 \times 5$ matrices $J^k$ are the elements of the $so(3)$ algebra. They are shown explicitly in the Appendix. The $J_i$ generate the transformation of the 5-dimensional vector $Y$ under infinitesimal rotations in 3-dimensional space

\[ \delta \omega^A = \omega_k \{ Y^A, S_k \} = -i(J^k)_{AB} Y^B. \]  

(3.16)

For finite spatial rotations $R(\omega)$ we therefore have

\[ Y'_A = D_{AB}(\omega) Y_B, \]  

(3.17)

with the well-known 5-dimensional spin-2 $D$-functions [28] related to the $3 \times 3$ orthogonal matrix $R(\omega)$ via

\[ D_{AB}(\omega) = \text{Tr}(R(\omega)^T M_A R(\omega)^T M_B). \]  

(3.18)

The transformation rule (3.17) is in accordance with (3.5).

Note that for a complete investigation of the transformation properties of the reduced matrix field $Q$ under the whole Poincaré group one should also include the Lorentz transformations. But we shall limit ourselves here to the isolation of the scalars under spatial rotations and can treat $Q$ in terms of “nonrelativistic spin-0 and spin-2 fields” in accordance with the conclusions obtained in the work [3]. The study of the nonlinear representation of the whole Poincaré group in terms of the unconstrained variables will be the subject of further investigation.

C. Separation of scalar and rotational degrees of freedom

In this paragraph we would like to introduce a parametrization of the 5-dimensional $Y$ field in terms of three Euler angles and two variables which are invariant under spatial rotations. The transformation property (3.17) prompts us with the parametrization

\[ Y_A(x) = D_{AB}(\chi(x)) M^B(x), \]  

(3.19)

in terms of the three Euler angles $\chi_i = (\phi, \theta, \psi)$ and some 5-vector $M$. The special choice

\[ M(\rho, \alpha) = \rho \left(-\frac{1}{\sqrt{2}} \sin \alpha, 0, \cos \alpha, 0, -\frac{1}{\sqrt{2}} \sin \alpha\right) \]  

(3.20)

corresponds to the main-axis-transformation of the original symmetric $3 \times 3$ matrix field $Q(x)$,

\[ Q(x) = R^T(\chi(x)) Q_{\text{diag}}(\phi_1(x), \phi_2(x), \phi_3(x)) R(\chi(x)), \]  

(3.21)

with the $D(\chi)$ related to $R(\chi)$ via (3.18) and the rotational invariant variables $\Phi, \rho, \alpha$ related to the diagonal elements $\phi_i$ via [29]

\[ \phi_1 := \frac{1}{\sqrt{3}} \Phi + \sqrt{\frac{2}{3}} \rho \cos(\alpha + \frac{2\pi}{3}), \]

\[ \phi_2 := \frac{1}{\sqrt{3}} \Phi + \sqrt{\frac{2}{3}} \rho \cos(\alpha + \frac{4\pi}{3}), \]

\[ \phi_3 := \frac{1}{\sqrt{3}} \Phi + \sqrt{\frac{2}{3}} \rho \cos \alpha. \]  

(3.22)

\[ ^6 \text{Similar variables have been used as density and deformation variables in the collective model of Bohr in Nuclear Physics [29] and as a parametrization for the square of the eigenvalues of the rotational invariant part of the gauge field by [30] in the representation proposed in [3].} \]

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As it was mentioned in the first part of the paper, the matrix $Q$ is symmetric positive definite. The variables $\phi_i$ are therefore positive

\[ \phi_i \geq 0, \quad i = 1, 2, 3, \]  

and the domain of definition for the variables $\alpha$ and $\rho$ can correspondingly be taken as

\[ 0 \leq \rho \leq \sqrt{2}\Phi, \quad \alpha \leq \frac{\pi}{3}. \]  

The main-axis-transformation of the symmetric second rank tensor field $Q$ therefore induces a parametrization of the five spin-2 fields $Y^A$ in terms of the three rotational degrees of freedom, the Euler angles $\chi_i = (\psi, \theta, \phi)$, which describe the orientation of the “intrinsic frame”, and the two invariants $\rho$ and $\alpha$ represented by the 5-vector $\mathbf{M}$. As the three scalars under spatial rotations we can hence use either $\rho$, $\alpha$, and the spin-0 field $\Phi$, or the three fields $\phi_i (i = 1, 2, 3)$.

In the following we shall use the main-axis-representation (3.21). The momenta $\pi_i$ and $p_{\chi_i}$, canonical conjugate to the diagonal elements $\phi_i$ and the Euler angles $\chi_i$, can easily be found using the generating function

\[ F_3 [\phi_i, \chi_i; P] := \int d^3x \ Tr (QP) = \int d^3x \ Tr \ (R^T(\chi)Q_{\text{diag}}(\phi)R(\chi)P) \]  

as

\[ \pi_i (x) = \frac{\partial F_3}{\partial \phi_i (x)} = \text{Tr} \left( P R^T \vec{\pi}_i R \right), \]

\[ p_{\chi_i} (x) = \frac{\partial F_3}{\partial \chi_i (x)} = \text{Tr} \left( \frac{\partial R^T}{\partial \chi_i} R [P Q - Q P] \right). \]  

Here $\vec{\pi}_i$ are the diagonal matrices with the elements $(\vec{\pi}_i)_{lm} = \delta_{li} \delta_{mi}$. Together with the off-diagonal matrices $(\alpha_i)_{lm} = |\epsilon_{ilm}|$ they form an orthogonal basis for symmetric matrices, shown explicitly in the Appendix. The original physical momenta $P_{ik}$ can then be expressed in terms of the new canonical variables as

\[ P(x) = R^T(x) \left( \sum_{s=1}^3 \pi_s (x) \vec{\pi}_s + \frac{1}{\sqrt{2}} \sum_{s=1}^3 \mathcal{P}_s (x) \alpha_s \right) R(x), \]  

with

\[ \mathcal{P}_i (x) := \frac{\xi_i (x)}{\phi_j (x) - \phi_k (x)} \quad (\text{cyclic permutation } i \neq j \neq k), \]  

where the $\xi_i$ are given in terms of Euler angles $\chi_i = (\psi, \theta, \phi)$ as

\[ \xi_k (x) := \mathcal{M}(\theta, \psi)_{kl} p_{\chi_l}, \]  

\[ \mathcal{M}(\theta, \psi) := \begin{pmatrix} \sin \psi / \sin \theta & \cos \psi & -\sin \psi \cot \theta \\ -\cos \psi / \sin \theta & \sin \psi & \cos \psi \cot \theta \\ 0 & 0 & 1 \end{pmatrix}. \]  

Note that the $\xi_i$ are the $SO(3)$ left-invariant Killing vectors satisfying the algebra

\[ \{ \xi_i (x), \xi_j (y) \} = -\epsilon_{ijk} \xi_k (x) \delta (x - y). \]  

The antisymmetric part $\vec{E}$ of the electric field appearing in the unconstrained Hamiltonian (2.17) is given by the following expansion in a $1/g$ series, analogous to (2.21) - (2.23),

\[ E_i = R^T_{2s} \sum_{n=0}^{\infty} \xi^{(n)}_s, \]  

with the zeroth order term

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E_i := - \frac{\xi_i}{\phi_j + \phi_k} \quad (cycl. \text{ permut. } i \neq j \neq k), \quad (3.32)

the first order term given from \( E^{(0)} \) via

\[ E_i^{(1)} := \frac{1}{g} \frac{1}{\phi_j + \phi_k} \left[ (\nabla_{X_j} \hat{E}^{(0)})_k - (\nabla_{X_k} \hat{E}^{(0)})_j \right] - \Xi_i, \quad (3.33) \]

with cyclic permutations of \( i \neq j \neq k \), and the higher order terms of the expansion determined via the recurrence relations

\[ E_i^{(n+1)} := \frac{1}{g} \frac{1}{\phi_j + \phi_k} \left[ (\nabla_{X_j} \hat{E}^{(n)})_k - (\nabla_{X_k} \hat{E}^{(n)})_j \right]. \quad (3.34) \]

Here the components of the covariant derivatives \( \nabla_{X_k} \) in the direction of the vector field \( X_i(x) := R_{ik} \partial_k \),

\[ (\nabla_{X_i} \hat{E})_b := X_i \hat{E}_b + \Gamma^d_{ib} \hat{E}_d, \quad (3.35) \]

are determined by the connection depending only on the Euler angles

\[ \Gamma^b_{ia} := (R_X R^T)_{ab}, \quad (3.36) \]

It is easy to check that the connection \( \Gamma^b_{ia} \) can be written in the form

\[ \Gamma^b_{ia} = i(J^*)_{ab}(M^{-1})_{sk} X_i \chi_k, \quad (3.37) \]

using the matrix \( M \) given in terms of the Euler angles \( \chi_i = (\psi, \theta, \phi) \) in (3.30), which expresses the dual nature of the Killing vectors \( \xi_i \) in (3.30) and the Maurer-Cartan one-forms \( \omega^j \) defined by

\[ RdR^T =: \omega^j J^i, \quad \omega^j = (M^{-1})^i_k d\chi_k. \quad (3.38) \]

The source terms \( \Xi_k \) in (3.33), finally, are given as

\[ \Xi_1 = \Gamma_{12}^1 (\pi_1 - \pi_2) + \frac{1}{2} X_1 \pi_1 - \Gamma_{23}^1 P_2 - \Gamma_{32}^1 P_3 - 2 \Gamma_{12}^1 P_3 + X_2 P_3 + (2 \leftrightarrow 3), \quad (3.39) \]

and its cyclic permutations \( \Xi_2 \) and \( \Xi_3 \).

The unconstrained Hamiltonian therefore takes the form

\[ H = \frac{1}{2} \int d^3 x \left( \sum_{i=1}^3 \pi_i^2 + \frac{1}{2} \sum_{cycl.} \frac{\xi_i^2}{(\phi_j - \phi_k)^2} + \frac{1}{2} \hat{E}^2 + V \right), \quad (3.40) \]

where the potential term \( V \)

\[ V[\phi, \chi] = \sum_{i=1}^3 V_i[\phi, \chi], \quad (3.41) \]

is the sum of

\[ V_1[\phi, \chi] = (\Gamma_{12}^1 (\phi_2 - \phi_1) - X_2 \phi_1)^2 + (\Gamma_{13}^1 (\phi_3 - \phi_1) - X_3 \phi_1)^2 + (\Gamma_{23}^1 \phi_3 + \Gamma_{32}^1 \phi_2 - g \phi_2 \phi_3)^2, \quad (3.42) \]

and its cyclic permutations. We see that through the main-axis-transformation of the symmetric second rank tensor field \( Q \) the rotational degrees of freedom, the Euler angles \( \chi \) and their canonical conjugate momenta \( p_\chi \), have been isolated from the scalars under spatial rotations and appear in the unconstrained Hamiltonian only via the three Killing vector fields \( \xi_k \), the connections \( \Gamma \), and the derivative vectors \( X_k \).
IV. THE INFRARED LIMIT OF UNCONSTRAINED SU(2) GLUODYNAMICS

A. The strong coupling limit of the theory

From the expression (3.40) for the unconstrained Hamiltonian one can analyse the classical system in the strong coupling limit up to order $O(1/g)$. Using the leading order (3.33) of the $\mathcal{E}$ we obtain the Hamiltonian

$$ H_S = \frac{1}{2} \int d^3x \left( \sum_{i=1}^{3} \pi_i^2 + \sum_{\text{cycl.}} \xi_i^2 \frac{\phi_j^2 + \phi_k^2}{(\phi_j^2 - \phi_k^2)^2} + V(\phi, \chi) \right). $$

(4.1)

For spatially constant fields the integrand of this expression reduces to the Hamiltonian of SU(2) Yang-Mills mechanics considered in previous work [23]. For the further investigation of the low energy properties of SU(2) field theory a thorough understanding of the properties of the leading order $g^2$ term in (3.41), containing no derivatives,

$$ V_{\text{hom}}[\phi] = g^2[\phi_1^2 + \phi_2^2 + \phi_3^2] , $$

(4.2)

is crucial. The stationary points of the potential term (4.2) are

$$ \phi_1 = \phi_2 = 0, \quad \phi_3 \text{ arbitrary} , $$

(4.3)

and its cyclic permutations. Analysing the second order derivatives of the potential at the stationary points one can conclude that they form a continuous line of degenerate absolute minima at zero energy. In other words the potential has a “valley” of zero energy minima along the line $\phi_1 = \phi_2 = 0$. They are the unconstrained analogs of the toron solutions [35] representing constant Abelian field configurations with vanishing magnetic field in the strong coupling limit. The special point $\phi_1 = \phi_2 = \phi_3 = 0$ corresponds to the ordinary perturbative minimum.

In terms of the variables $\rho$, $\Phi$ and $\alpha$ the homogeneous potential (4.2) reads

$$ V_{\text{hom}} := \frac{g^2}{3} \left( \Phi^4 + \frac{3}{4} \rho^4 - \sqrt{2} \Phi \rho^3 \cos 3\alpha \right) , $$

(4.4)

showing that the $\alpha$ parametrizes the strength of the coupling between the spin-0 and spin-2 fields. The valley of minima is given by $\rho = \sqrt{2} \Phi$, $\alpha = 0$, $\Phi$ arbitrary, and the perturbative vacuum by $\rho = \Phi = \alpha = 0$.

For the investigation of configurations of higher energy it is necessary to include the part of the kinetic term in (1.3) containing the angular momentum variables $\xi_i$. Since the singular points of this term just correspond to the absolute minima of the potential there will be a competition between an attractive and a repulsive force. At the balance point we shall have a local minimum corresponding to a classical configuration with higher energy.

B. Nonlinear sigma model type effective action as the infrared limit of the unconstrained system

We would like to find in this paragraph the effective classical field theory to which the unconstrained theory reduces in the limit of infinite coupling constant $g$, if we assume that the classical system spontaneously chooses one of the classical zero energy minima of the leading order $g^2$ part (4.2) of the potential. As discussed in the proceeding section these classical minima include apart from the perturbative vacuum, where all fields vanish, also field configurations with one scalar field attaining arbitrary values. Let us therefore put without loss of generality (explicitly breaking the cyclic symmetry)

$$ \phi_1 = \phi_2 = 0, \quad \phi_3 \text{ arbitrary} , $$

(4.5)

such that the potential (4.2) vanishes. In this case the part of the potential (3.41) containing derivatives takes the form

$$ V_{\text{inhom}} = \phi_3 (x)^2 \left[ (\Gamma_{13}^2)(x) \right]^2 + (\Gamma_{23}^2(x))^2 + (\Gamma_{33}^2(x))^2 + (\Gamma_{11}^3(x))^2 + (\Gamma_{21}^3(x))^2 + (\Gamma_{31}^3(x))^2 \right] + \left[ X_1 \phi_3^2 + (X_2 \phi_3)^2 \right] + 2 \phi_3(x) \left[ \Gamma_{31}^3(x) X_1 \phi_3 + \Gamma_{32}^3(x) X_2 \phi_3 \right] . $$

(4.6)

Introducing the unit vector

$$ n_i(\phi, \theta) := R_{5i}(\phi, \theta) , $$

(4.7)
pointing along the 3-axis of the “intrinsic frame”, one can write
\[ V_{\text{inhom}} = \phi_3(x)^2 (\partial_t \vec{n})^2 + (\partial_t \phi_3)^2 - (n_i \partial_i \phi_3)^2 - (n_i \partial_i n_j) \partial_j (\phi_3^2) \].

(4.8)

Concerning the contribution from the nonlocal term in this phase, we obtain for the leading part of the electric fields
\[ E_1^{(0)} = -\xi_1/\phi_3 \], \[ E_2^{(0)} = -\xi_2/\phi_3 \].

(4.9)

Since the third component \( E_3^{(0)} \) and \( P_3 \) are singular in the limit \( \phi_1, \phi_2 \to 0 \), it is necessary to have \( \xi_3 \to 0 \). The assumption of a definite value of \( \xi_3 \) is in accordance with the fact that the potential is symmetric around the 3-axis for small \( \phi_1 \) and \( \phi_2 \), such that the intrinsic angular momentum \( \xi_3 \) is conserved in the neighbourhood of this configuration. Hence we obtain the following effective Hamiltonian up to order \( O(1/g) \)
\[ H_{\text{eff}} = \frac{1}{2} \int d^3x \left[ \frac{1}{\phi_3^2} \xi_1^2 + \frac{1}{\phi_3^2} \xi_2^2 + (\partial_t \phi_3)^2 + \phi_3^2 (\partial_t n)^2 - (n_i \partial_i \phi_3)^2 - (n_i \partial_i n_j) \partial_j (\phi_3^2) \right] \].

(4.10)

After the inverse Lagrangian transformation we obtain the corresponding nonlinear sigma model type effective Lagrangian for the unit vector \( \vec{n}(t, \vec{x}) \) coupled to the scalar field \( \phi_3(t, \vec{x}) \)
\[ L_{\text{eff}}[\phi_3, \vec{n}] = \frac{1}{2} \int d^4x \left[ (\partial_t \phi_3^2)^2 + \phi_3^2 (\partial_t n)^2 + (n_i \partial_i \phi_3)^2 + n_i (\partial_i n_j) \partial_j (\phi_3^2) \right] \].

(4.11)

In the limit of infinite coupling the unconstrained field theory in terms of six physical fields equivalent to the original \( SU(2) \) Yang-Mills theory in terms of the gauge fields \( A_\mu \) reduces therefore to an effective classical field theory involving only one of the three scalar fields and two of the three rotational fields summarized in the unit vector \( \vec{n} \). Note that this nonlinear sigma model type Lagrangian admits singular hedgehog configurations of the unit vector field \( \vec{n} \). Due to the absence of a scale at the classical level, however, these are unstable. Consider for example the case of one static monopole placed at the origin,
\[ n_i := x_i/r \], \[ \phi_3 = \phi_3(r) \], \[ r := \sqrt{x_1^2 + x_2^2 + x_3^2} \].

(4.12)

Minimizing its total energy \( E \)
\[ E[\phi_3] = 4\pi \int dr \phi_3^2(r) \]

(4.13)

with respect to \( \phi_3(r) \) we find the classical solution \( \phi_3(r) \equiv 0 \). There is no scale in the classical theory. Only in a quantum investigation a mass scale such as a nonvanishing value for the condensate \( <0|\phi_3^2|0> \) may appear, which might be related to the string tension of flux tubes directed along the unit-vector field \( \vec{n}(t, \vec{x}) \). The singular hedgehog configurations of such string-like directed flux tubes might then be associated with the glueballs. The pure quantum object \( <0|\phi_3^2|0> \) might be realized as a squeezed gluon condensate \[31\]. Note that for the case of a spatially constant condensate,
\[ <0|\phi_3^2|0> := 2m^2 = \text{const.} \].

(4.14)

the quantum effective action corresponding to \[4.11\] should reduce to the lowest order term of the effective soliton Lagrangian discussed very recently by Faddeev and Niemi \[24\]
\[ L_{\text{eff}}[\vec{n}] = m^2 \int d^3x (\partial_t n)^2 \].

(4.15)

As discussed in \[24\], for the stability of these knots furthermore a higher order Skyrmion-like term in the derivative expansion of the unit-vector field \( \vec{n}(t, \vec{x}) \) is necessary. To obtain it from the corresponding higher order terms in the strong coupling expansion of the unconstrained Hamiltonian \[4.40\] is under present investigation.

First steps towards a quantum treatment of the unconstrained formulation obtained in the preceeding paragraphs will be undertaken in the next section.

V. QUANTUM GROUND STATE WAVE FUNCTIONAL AND THE CLASSICAL CONFIGURATION OF LOWEST ENERGY

In this section we shall discuss several interesting relations between the quantum ground state wave functional for the unconstrained Hamiltonian system and the corresponding classical state configurations with zero energy.
A. Exact ground state solution of the Schrödinger equation

For the original constrained system of $SU(2)$ gluodynamics in terms of the gauge fields $A_i^a(x)$ with the Hamiltonian

$$\mathcal{H}(A) := \frac{1}{2} \int d^3x \left( -\left( \frac{\delta}{\delta A_i^a(x)} \right)^2 + B^2(x) \right)$$

(5.1)

and the Gauss law operators

$$\mathcal{G}_a(x) := \left( \partial_i \delta^a_i - g\epsilon^{abc} A^c_i(x) \right) \frac{\delta}{\delta A_i^a(x)}$$

(5.2)

in the Schrödinger functional formalism, a physical state has to satisfy both the functional Schrödinger equation and the Gauss law constraints

$$\mathcal{H}\Psi[A] = E\Psi[A],$$

$$\mathcal{G}_a(x)\Psi[A] = 0.$$  

(5.3, 5.4)

Remarkably, an exact solution for the ground state wave functional $\Psi[A]$ can be given

$$\Psi[A] = \exp\left(-8\pi^2 W[A]\right)$$

(5.5)

in terms of the so called “winding number functional” $W[A]$ defined as the integral over 3-space

$$W[A] := \int d^3x \ K_0(x)$$

(5.6)

of the zero component of the Chern-Simons secondary characteristic class vector

$$K^\mu(A) := -\frac{1}{16\pi^2} \epsilon^{\mu\nu\sigma\kappa} \text{Tr} \left( F_{\nu\sigma} A_\kappa - \frac{2}{3} g A_\nu A_\sigma A_\kappa \right).$$

(5.8)

Since $W[A]$ obeys the functional differential equation

$$\frac{\delta}{\delta A_i^a(x)} W[A] = B_i^a(x).$$

(5.9)

the wave functional (5.5) satisfies the above Schrödinger equation. However this exact solution for the functional Schrödinger equation with the zero energy is known to be nonnormalizable and hence does not seem to have a physical meaning.

In the following we shall now analyse, how such an exact solution arises in our unconstrained formalism.

B. Exact ground state solution for the unconstrained Hamiltonian

Quantizing the variables $Q$ and $P$ of the unconstrained Hamiltonian (2.17) analogously to the $A_i^a$ above we have

$$q[A] := \int d^4x \partial_\mu K^\mu(A),$$

(5.7)

and the corresponding Pontryagin density $\text{Tr} \left( * F^{\mu\nu} F_{\nu\mu} \right)$, are gauge invariant quantities, the Chern-Simons vector $K^\mu$ is not gauge invariant.

Note that whereas the topological invariant, the Pontryagin index

$$q[A] := \int d^4x \partial_\mu K^\mu(A),$$

and the corresponding Pontryagin density $\text{Tr} \left( * F^{\mu\nu} F_{\nu\mu} \right)$, are gauge invariant quantities, the Chern-Simons vector $K^\mu$ is not gauge invariant.

Note that due to the positive definiteness of the elements of the matrix field $Q$ we have to solve the Schrödinger equation in a restricted domain of functional space. Special boundary conditions have to be imposed on the wavefunctional such that all operators are well-defined (e.g. Hermicity of the Hamiltonian). A discussion on this subject in gauge theories can be found in the review.

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\[ H = \frac{1}{2} \int d^3x \left( -\left( \frac{\delta}{\delta Q_{ij}(x)} \right)^2 + B^2(x) + \frac{1}{2} E^2(Q, \frac{\delta}{\delta Q}) \right), \quad (5.10) \]

and hence the functional Schrödinger equation
\[ H \Psi[Q] = E \Psi[Q]. \quad (5.11) \]

The Gauss law has already been implemented by the reduction to the physical variables.

A corresponding exact zero energy solution can indeed be found for our reduced Schrödinger equation (5.11). For this we note the following two important properties of the potential terms present in the Schrödinger equation (5.11).

Firstly, the reduced magnetic field \( B_{ij}(Q) \) can be written as the functional derivative of the functional \( W[Q] \)
\[ W[Q] := \frac{1}{32\pi^2} \int d^3x \left[ \text{Tr}(BQ) - \frac{1}{12} g \left( (\text{Tr}(Q^3) + \text{Tr}^3(Q) - 2\text{Tr}(Q)\text{Tr}(Q^2)) \right) \right], \quad (5.12) \]

as
\[ \frac{\delta}{\delta Q_{ij}(x)} W[Q] = B_{ij}(x). \quad (5.13) \]

Furthermore, the nonlocal term in the Schrödinger equation (5.11) annihilates \( W[Q] \)
\[ \vec{E}^2(Q, \frac{\delta}{\delta Q_{ij}(x)} W[Q]) = 0. \quad (5.14) \]

The last equation can easily be found to hold if one takes into account that the magnetic field \( B_i = * F_0^i \) satisfies the Bianchi identity
\[ D_i * F_0^i = 0. \]

Thus the corresponding ground state wave functional solution for the unconstrained Hamiltonian is
\[ \Psi[Q] = \exp \left( -8\pi^2 W[Q] \right). \quad (5.15) \]

In order to investigate the relation of our \( W[Q] \) to the above winding number functional \( W[A] \) we write the zero component of the the Chern-Simons secondary characteristic class vector \( K^\mu \), given in (5.8), in terms of the new variables \( Q \) and \( q_i \)
\[ K_0(Q, q) = K_0(Q) - \frac{1}{24\pi^2} \epsilon^{ijk} \left[ 2 g^3 \text{Tr} (\Omega_i \Omega_j \Omega_k) - \partial_i \text{Tr} (Q_j \Omega_k) \right]. \quad (5.16) \]

The first term
\[ K_0(Q) := -\frac{1}{16\pi^2} \epsilon^{ijk} \text{Tr} \left( F_{ij} Q_k - \frac{2}{3} g Q_i Q_j Q_k \right) \quad (5.17) \]

is a functional only of the physical \( Q \) of a form similar to that of the original Chern-Simons secondary characteristic class vector. Here we have introduced the \( SU(2) \) matrices \( Q_i := Q_i \tau_i \), with the Pauli matrices \( \tau_i \), and
\[ \Omega_i(q) := \frac{1}{g} U^{-1}(q) \partial_i U(q) = \frac{1}{g} \Omega_{is}(q) \tau^s \left( \frac{\partial q_i}{\partial x_i} \right), \quad (5.18) \]

with the \( SU(2) \) matrices \( U(q) \) related to the \( 3 \times 3 \) orthogonal matrix \( O(q) \) via \( O_{ab}(q) = \frac{1}{2} Tr(O(q)\tau_a O^T(q)\tau_b) \) and the \( 3 \times 3 \) matrix \( \Omega_{ij} \) defined in (2.10).

We observe that the space integral over the first term coincides with the above functional \( W[Q] \) of (5.12)
\[ \int d^3x K_0(Q) = W[Q]. \quad (5.19) \]

Using the usual boundary condition [1]

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[9] Note that for the behaviour of the unphysical variables \( q_i \) we have no information. For example the requirement of the finiteness of the action usually used to fix the behaviour of the physical fields does not apply for the unphysical field \( q_i \).
we see that the space integral over the second term is proportional to the natural number $n$ representing the winding of the mapping of compactified three space into $SU(2)$

$$\frac{g^3}{24\pi^2} \int d^3x \, e^{ijk} \text{Tr}(\Omega_i \Omega_j \Omega_k) = n.$$  \hspace{1cm} (5.21)

Assuming here the vanishing of the physical field $Q$ at spatial infinity there is no contribution from the third term. Hence we obtain the relation

$$\Psi[A] = \exp\left[-\frac{8\pi^2}{g^2} n\right] \Psi[Q]$$  \hspace{1cm} (5.22)

between the groundstate wave functional (5.15) of the extended quantization scheme and the reduced (5.15). We find that the winding number of the original gauge field $A$ only appears as an unphysical normalization prefactor originating from the second term in (5.16), which depends only on the unphysical $q$. Furthermore, we note that the power $\frac{8\pi^2}{g^2} n$ is the classical Euclidean action of $SU(2)$ Yang-Mills theory of self-dual fields [36] with winding number $n$.

The physical part of the wavefunction, $\Psi[Q]$, on the other hand however, has the same unpleasant property as (5.5) that it is nonnormalizable. In order to shed some light on the reason for its nonnormalizability, it is useful to limit to the homogeneous case and to analyse the properties of $\Psi[Q]$ in the neighbourhood of the classical minima of the potential.

C. Analysis of the exact groundstate wave functional in the strong coupling limit

In the strong coupling limit the groundstate wave functional (5.15) reduces to the very simple form

$$\Psi[\phi_1, \phi_2, \phi_3] = \exp\left[-g\phi_1 \phi_2 \phi_3\right].$$  \hspace{1cm} (5.23)

This wave functional is obviously nonnormalizable. In difference to the Abelian case, however, where an analogous nonnormalizable exact zero energy solution exists [10] the exponent of (5.23) is free of any sign ambiguities due to the positivity of the symmetric matrix $Q$ in the polar representation (2.6), and hence the positivity of the diagonal fields $\phi_i$, see (3.23). In order to investigate the reason for the nonnormalizability of (5.23) we analyze it near the classical zero-energy minima, that is, without loss of generality, in the neighbourhood of the line $\phi_1 = \phi_2 = 0$ of minima of the classical potential (4.2). It is useful to pass from the variables $\phi_1$ and $\phi_2$ transverse to the valley to the new variables $\phi_\perp$ and $\gamma$ via

$$\phi_1 = \phi_\perp \cos \gamma \quad \phi_2 = \phi_\perp \sin \gamma \quad \left(\phi_\perp \geq 0, \quad 0 \leq \gamma \leq \frac{\pi}{2}\right).$$  \hspace{1cm} (5.24)

The classical potential then reads

$$V(\phi_3, \phi_\perp, \gamma) = g^2 \left(\phi_3^2 \phi_\perp^2 + \frac{1}{4} \phi_\perp^4 \sin^2(2\gamma)\right),$$  \hspace{1cm} (5.25)

and the groundstate wave function (5.23) becomes

$$\Phi[\phi_3, \phi_\perp, \gamma] = \exp\left[-\frac{g^2}{2} \phi_3^2 \phi_\perp^2 \sin(2\gamma)\right].$$  \hspace{1cm} (5.26)

We see that close to the bottom of the valley, for small $\phi_\perp$, the potential is that of a harmonic oscillator and the wave functional correspondingly a Gaussian with a maximum at the classical minimum line $\phi_\perp = 0$. The height of the maximum is constant along the valley. The nonnormalizability of the groundstate wave function in the infrared region is therefore due to the outflow of the wave function with constant values along the valley to arbitrarily large values of the field $\phi_3$. This may result in the formation of condensates with macroscopically large fluctuations of the field amplitude. To establish the connection between this phenomenon and the model of the squeezed gluon condensate [31] will be an interesting task for further investigation.

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10 In (Abelian) electrodynamics the unconstrained form of the corresponding exact zero-energy groundstate wave functional is an exponential of $\int dk a_1(k) a_2^*(k)$, where $a_1, a_2$ are the (momentum space) polarization modes of $A$. This false ground state is nonnormalizable due to the sign indefiniteness of the exponent. We thank the Referee for pointing this out to us.
VI. CONCLUDING REMARKS

Following the Dirac formalism for constrained Hamiltonian systems we have formulated several representations for the classical SU(2) Yang-Mills gauge theory entirely in terms of unconstrained gauge invariant local fields. All transformations which have been used, canonical transformations and the Abelianization of the constraints, maintain the canonical structures of the generalized Hamiltonian dynamics. We identify the unconstrained field with a symmetric positive definite second rank tensor field under spatial rotations. Its decomposition into irreducible representations under spatial rotations leads to the introduction of two fields, a five-dimensional vector field $Y(x)$ and a scalar field $\Phi(x)$. Their dynamics is governed by an explicitly rotational invariant non-local Hamiltonian. It is different from the local Hamiltonian obtained by Goldstone and Jackiw [1] as well as by Izergin et.al. [3]. They used the so-called electric field representation with vanishing antisymmetric part of the electric field. A representation for the Hamiltonian with a nonlocal interaction of the unconstrained variables similar to ours has been derived in the work of Y. Simonov [8] based on another separation of scalar and rotational degrees of freedom. Our separation of the unconstrained fields into scalars under spatial rotations and into rotational degrees of freedom, however, leads to a simpler form of the Hamiltonian, which in particular is free of operator ordering ambiguities in the strong coupling limit. Our unconstrained representation of the Hamiltonian furthermore allows us to derive an effective low energy Lagrangian for the rotational degrees of freedom coupled to one of the scalar fields suggested by the form of the classical potential in the strong coupling limit. The dynamics of the rotational variables in this limit is summarized by the unit vector describing the orientation of the intrinsic frame. Due to the absence of a scale in the classical theory the singular hedgehog configurations of the unit vector field is found to be unstable classically. In order to obtain a nonvanishing value for the vacuum expectation value for one of the three scalar field operators, which would set a scale, a quantum treatment at least to one loop order is necessary and is under present investigation. For the case of a spatially constant scalar quantum condensate we expect to obtain the first term of a derivative expansion proposed recently by Faddeev and Niemi [24]. As shown in their work such a soliton Lagrangian allows for stable massive knotlike configurations which might be related to glueballs. For the stability of the knots higher order terms in the derivative expansion, such as the Skyrme type fourth order term in [24], are necessary. Their derivation in the framework of the unconstrained theory, proposed in this paper, is under investigation. First steps towards a complete quantum description have been done in this paper. In particular, we have investigated the famous groundstate wave functional, which solves the Schrödinger equation with zero energy eigenvalue. In conclusion we would like to emphasize that our investigation of low energy aspects of non-Abelian gauge theories directly in terms of the physical unconstrained fields offers an alternative to the variational calculations based on the gauge projection method [23,28].

The reason for trying to construct the physical variables entirely in internal terms without the use of any gauge fixing is the aspiration to maintain all local and global properties of the initial gauge theory. Several questions in connection with the global aspects of the reduction procedure are arising at this point. In the paper we describe how to project SU(2) Yang-Mills theory onto the constraint shell defined by the Gauss law. It is well known that the exponentiation of infinitesimal transformations generated by the Gauss law operator can lead only to homotopically trivial gauge transformations, continuously deformable to unity. However, the initial classical action is invariant under all gauge transformations, including the homotopically nontrivial ones. What trace does the existence of large gauge transformations leave on the unconstrained system? First steps towards a clarification of these important issues have been undertaken in this paper, a more complete analysis is under present investigation.

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APPENDIX A: NOTATIONS AND SOME FORMULAS

1. Spin 1 matrices and eigenvectors

For generators of spin-1 obeying the algebra \([J_i, J_j] = i \epsilon_{ijk} J_k\) we use the following matrix realizations

\[
J^1 = i \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{pmatrix}, \quad J^2 = i \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix}, \quad J^3 = i \begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

Furthermore the representation of rotations \(R(\chi)\) in terms of Euler angles \(\chi = (\theta, \psi, \phi)\) is used

\[
R(\psi, \theta, \phi) = e^{-i\psi J_3} e^{-i\theta J_1} e^{-i\phi J_3}.
\] (A1)

The eigenfunctions of \(J^2\) and \(J_3\) are

\[
\vec{e}_{+1} = \frac{1}{\sqrt{2}} \begin{pmatrix}
-1 \\
0 \\
0
\end{pmatrix}, \quad \vec{e}_0 = \begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}, \quad \vec{e}_{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix},
\]

which are orthogonal with respect to the metric \(\eta_{\alpha\beta} := (-1)^\alpha \delta_{\alpha,-\beta}\)

\[
(\vec{e}_\alpha \cdot \vec{e}_\beta) = \eta_{\alpha\beta}
\] (A2)

and satisfy the completeness condition

\[
\delta^{ij} = e_i^\alpha e_j^\beta \eta^{\alpha\beta}.
\] (A3)

2. Spin-0, spin-1 and spin-2 tensors basis

To obtain a matrix representation for spin-0, spin-1 and spin-2 basis matrices we use the Clebsh-Gordon decomposition for the direct product of spin-1 eigenvectors \(e_i^\alpha\) into the irreducible components \(3 \otimes 3 = 0 \oplus 1 \oplus 2\). To distinguish the matrices corresponding to the different spins we use boldface notation for spin 2.

For spin-0 they read explicitly

\[
I_0 := \frac{1}{\sqrt{3}} (\vec{e}_0 \otimes \vec{e}_0 - \vec{e}_1 \otimes \vec{e}_{-1} - \vec{e}_{-1} \otimes \vec{e}_1) = \frac{1}{\sqrt{3}} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

for spin-1

\[
J^+ := (\vec{e}_0 \otimes \vec{e}_{+1} - \vec{e}_{+1} \otimes \vec{e}_0) = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & i \\
-1 & -i & 0
\end{pmatrix},
\]

\[
J^- := (\vec{e}_{-1} \otimes \vec{e}_0 - \vec{e}_0 \otimes \vec{e}_{-1}) = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & -i \\
-1 & i & 0
\end{pmatrix},
\]

\[
J^0 := (\vec{e}_{-1} \otimes \vec{e}_1 - \vec{e}_1 \otimes \vec{e}_{-1}) = \begin{pmatrix}
0 & -i & 0 \\
i & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

for spin-2
\[
T_{+2} = \sqrt{2} (\vec{e}_{+1} \otimes \vec{e}_{+1}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_{-2} = \sqrt{2} (\vec{e}_{-1} \otimes \vec{e}_{-1}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i & 0 \\ -i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
T_{+1} := (\vec{e}_{+1} \otimes \vec{e}_0 + \vec{e}_0 \otimes \vec{e}_{+1}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -i \\ -1 & -i & 0 \end{pmatrix},
\]

\[
T_{-1} := (\vec{e}_{-1} \otimes \vec{e}_0 + \vec{e}_0 \otimes \vec{e}_{-1}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ 1 & -i & 0 \end{pmatrix},
\]

\[
T_0 := \frac{1}{\sqrt{3}} (\vec{e}_{+1} \otimes \vec{e}_{-1} + 2\vec{e}_0 \otimes \vec{e}_0 + \vec{e}_{-1} \otimes \vec{e}_{+1}) = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.
\]

They obey the following orthonormality relations:

\[\text{Tr} (T_A T_B) = 2\eta_{AB}, \quad \text{Tr} (T_A J_\alpha) = 0, \quad \text{Tr} (J_\alpha J_\beta) = 2\eta_{\alpha\beta}, \quad (A4)\]

the completeness condition

\[\frac{1}{10} \sum_A (T_A)_{il} (T_A)_{km} + (I_0)_{il} (I_0)_{km} = \frac{1}{4} (\delta_{im}\delta_{lk} + \delta_{il}\delta_{mk}) \quad (A5)\]

and the following commutation and anticommutation relation

\[[T_A, T_B]_+ = \frac{4}{\sqrt{3}} \eta_{AB} I_0 + \frac{2}{\sqrt{3}} d^{(2)}_{ABC} T_C, \quad (A6)\]

\[[T_A, T_B]_- = c^{(2)}_{ABC} J_\gamma, \quad (A7)\]

\[[J_\alpha, J_\beta]_+ = \frac{4}{\sqrt{3}} \eta_{\alpha\beta} I_0 + d^{(1)}_{\alpha\beta C} T_C, \quad (A8)\]

\[[J_\alpha, J_\beta]_- = c^{(1)}_{\alpha\beta\gamma} J_\gamma, \quad (A9)\]

\[[J_\alpha, T_B]_+ = d^{(1)}_{\alpha\beta B} J_\gamma, \quad (A10)\]

\[[J_\alpha, T_B]_- = c^{(2)}_{BDA} T^D. \quad (A11)\]

The coefficients \(c^{(1)}_{\alpha\beta\gamma}\) are totally antisymmetric with \(c^{(1)}_{\alpha\beta\gamma} = 1\). The coefficients \(d^{(1)}_{\alpha\beta C}, d^{(2)}_{ABC}\) and \(c^{(2)}_{ABC}\) are given in Tables 1 to 3.
The corresponding Cartesian components satisfying the $SO(3)$ algebra

\[ d_{\alpha\beta\gamma}^{(1)} = (T_A)^{\alpha\beta} c_\alpha^\alpha c_\beta^\beta. \]  

(A12)

3. Generators for the D-functions

The following five-dimensional spin matrices has been used in the main text

\[
\begin{align*}
(J^+)_A^B & = \begin{pmatrix}
0 & \sqrt{2} & 0 & 0 & 0 \\
0 & 0 & -\sqrt{3} & 0 & 0 \\
0 & 0 & 0 & -\sqrt{3} & 0 \\
0 & 0 & 0 & 0 & \sqrt{2} \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \\
(J^-)_A^B & = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & -\sqrt{2} & 0 & 0 & 0 \\
0 & 0 & \sqrt{3} & 0 & 0 \\
0 & 0 & 0 & \sqrt{3} & 0 \\
0 & 0 & 0 & 0 & -\sqrt{2}
\end{pmatrix}, \\
(J^0)_A^B & = \begin{pmatrix}
2 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -2
\end{pmatrix}.
\end{align*}
\]

The corresponding Cartesian components \((J^i)_A^B := \eta^{ij} e_i^j (J_j)_A^B\) are

\[
(J^i)_A^B = i
\begin{pmatrix}
0 & -1 & 0 & 0 & 0 \\
1 & 0 & \sqrt{3/2} & 0 & 0 \\
0 & -\sqrt{3/2} & 0 & \sqrt{3/2} & 0 \\
0 & 0 & \sqrt{3/2} & 0 & -1 \\
0 & 0 & 0 & -1 & 0
\end{pmatrix}, \quad (J^2)_A^B = i
\begin{pmatrix}
0 & -1 & 0 & 0 & 0 \\
1 & 0 & \sqrt{3/2} & 0 & 0 \\
0 & -\sqrt{3/2} & 0 & \sqrt{3/2} & 0 \\
0 & 0 & \sqrt{3/2} & 0 & -1 \\
0 & 0 & 0 & -1 & 0
\end{pmatrix}, \quad (J^3)_A^B = (J^0)_A^B.
\]

satisfying the $SO(3)$ algebra

\[ [J_a, J_b] = i\epsilon_{abc} J_c. \]  

(A13)

Note that

\[ c^{(2)}_{\alpha\beta\gamma} = i(J^e)_{\alpha\beta\gamma}. \]  

(A14)

We use the D-functions as representation of rotations in 3-space defined in terms of Euler angles \(\chi = (\theta, \psi, \phi)\)

\[ D(\psi, \theta, \phi) = e^{-i\psi J_3} e^{-i\theta J_1} e^{-i\phi J_3}. \]  

(A15)

They can be obtained from the corresponding 3-dimensional representation \([8]\) via the formula

\[ D(\chi)_{AB} = \frac{1}{2} \text{Tr} (R(\chi) T_A R^T(\chi) T_B). \]  

(A16)

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4. Basis for symmetric matrices

We use the orthogonal basis $\alpha_A = (\bar{\alpha}_i, \alpha^i)$ for symmetric matrices. They read explicitly

\[
\bar{\alpha}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{\alpha}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{\alpha}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \\
\alpha^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \alpha^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \alpha^3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

They obey the following orthonormality relations:

\[
\text{tr} (\bar{\alpha}_i \bar{\alpha}_j) = \delta_{ij}, \quad \text{tr} (\alpha^i \alpha_j) = 2\delta_{ij}, \quad \text{tr} (\bar{\alpha}_i \alpha_j) = 0. \tag{A17}
\]
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