One-loop beta functions in topologically massive gravity

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Abstract
We calculate the running of the three coupling constants in cosmological, topologically massive 3D gravity. We find that \(\nu\), the dimensionless coefficient of the Chern–Simons term, has a vanishing beta function. The flow of the cosmological constant and Newton’s constant depends on \(\nu\), and for any positive \(\nu\) there exist both a trivial and a nontrivial fixed point.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Einstein gravity in three dimensions with or without cosmological constant and supplemented with a gravitational Chern–Simons (CS) term has been a subject of many investigations since long [1]. This theory is of considerable interest, as the presence of the cosmological term makes it possible to have a black hole solution, while the CS term is responsible for the presence of a single propagating massive graviton. For the latter reason, this model is referred to as the topologically massive gravity (TMG).

The properties of this theory have been the subject of intense scrutiny. For a generic value of the CS coupling, either black hole states (if \(G < 0\)) or graviton states (if \(G > 0\)) will have negative mass. In [2] it was observed that in the case \(G > 0\), if one chooses a critical value of the CS coupling, called the chiral point, the negative mass graviton mode can be confined to propagate only on the boundary, provided that suitable boundary conditions are imposed. The prospects of AdS/CFT duality are enhanced in this framework. There have also been alternative views in which the bulk graviton is maintained but the negative energy black hole solution is viewed as being possibly irrelevant [3].

TMG appears to be renormalizable [4–6]. However, not much is known about its renormalization group (RG). For example, does the chiral point have any special properties?
Another motivation for addressing this issue comes from the asymptotic safety approach to quantum gravity [7, 8], which requires the existence of a fixed point (FP) with finitely many UV attractive directions. Such a nontrivial FP has been found in the Einstein–Hilbert truncation of gravity in any dimension $d > 2$ [9–11], but in $d = 3$ this theory does not have propagating degrees of freedom. A nontrivial FP in TMG could provide an example of a three-dimensional asymptotically safe theory of gravity with physical degrees of freedom.

Our purpose in this paper is to calculate the one-loop beta functions for TMG, to determine the RG flow of the couplings and to discuss their FPs. We find that many subtleties and technically nontrivial issues arise due to the presence of the Lorentz CS. For example, the time honored heat kernel expansion method is not well suited in this case, essentially due to the fact that the eigenvalues of the cubic in the derivative wave operator present in the theory are a cubic polynomial in the lowest weight labels, as opposed to being quadratic. There is the delicate matter of how to choose the cutoff scheme to do these computations as well. Our results are best discussed after spelling out all these issues, and are summarized in the section 7. As we shall see, one of the key results we will obtain is that the dimensionless coefficient, $\nu$, of the CS term has a vanishing beta function. Moreover, we find that the flow of the cosmological constant and Newton’s constant (both made dimensionless by multiplying with a suitable power of the cutoff) depends on the values of $\nu$, and for any positive $\nu$ there exist both a trivial and a nontrivial FP. This is related to the structure of divergences of this theory. Various aspects of these results are discussed in section 7.

This paper is organized as follows. In section 2, we review briefly the Wilsonian approach to the study of RG equations. In section 3, we determine the gauge fixed inverse propagator of TMG on a maximally symmetric background. In section 4, we define the IR cutoff of each spin component separately and we give the harmonic expansion of the fields. In section 5, we evaluate the functional traces that arise in the computation of the one-loop beta functions, using the Euler–Maclaurin formula. The resulting beta functions and their FPs are given in section 6. In section 7, we first summarize our results, and then discuss various aspects of them in a number of subsections. Four appendices are provided, of which the first two contain useful lemmas and the details of the Euclideanization procedure and harmonic expansions on $S^3$. In appendix C, we address in detail the role of the tadpoles in an off-shell beta function computation, and in appendix D we give a heat kernel evaluation of the beta functions of pure gravity with cosmological constant, which provide a useful check of the techniques used in section 5.

2. Wilsonian method for computing the RGE

There is no single approach to the study of RG equations. Here, we shall adopt a Wilsonian method. The central lesson of Wilson’s analysis of QFT is that the ‘effective’ (as in ‘effective field theory’) action describing physical phenomena at a momentum scale $k$ can be thought of as the result of having integrated out all fluctuations of the field with momenta larger than $k$. Since $k$ can be regarded as the lower limit of some functional integration, we will usually refer to it as the infrared cutoff. The dependence of the ‘effective’ action on $k$ is the Wilsonian RG flow.

There are several ways of implementing this idea in practice, resulting in several forms of the RG equation [13]. In the specific implementation that we shall use [14], instead of introducing a sharp cutoff in the functional integral, we suppress the contribution of the field

4 At this general level of discussion, it is not necessary to specify the physical meaning of $k$: for each application of the theory one will have to identify the physically relevant variable acting as $k$. In scattering experiments $k$ is usually identified with some external momentum. See [12] for a discussion of this choice in concrete applications to gravity.
modes with momenta lower than $k$. This is obtained by modifying the low momentum end of the propagator, and leaving all the interactions unaffected. We describe first the general idea, and comment on its application to gravity later. We start from a bare action $S[\phi]$ for some fields $\phi$, and we add to it a suppression term $\Delta S_k[\phi]$ that is quadratic in the field. In flat space this term can be written simply in momentum space. In order to have a procedure that works in an arbitrary curved spacetime we choose a suitable differential operator $O$ whose eigenfunctions $\phi_n$, defined by $O\phi_n = \lambda_n \phi_n$, can be taken as a basis in the functional space we integrate over:

$$\phi(x) = \sum_n \phi_n \phi_n(x), \quad (2.1)$$

where $\phi_n$ are generalized ‘Fourier components’ of the field. (We will use a notation that is suitable for an operator with a discrete spectrum.) Then, the additional term can be written in either of the following forms:

$$\Delta S_k[\phi] = \frac{1}{2} \int dx \phi(x) R_k(O) \phi(x) = \frac{1}{2} \sum_n \phi_n^2 R_k(\lambda_n). \quad (2.2)$$

The kernel $R_k(O)$ will also be called ‘the cutoff’. It is arbitrary, except for the general requirements that $R_k(z)$ should be a monotonically decreasing function in both $z$ and $k$, that $R_k(z) \to 0$ for $z \gg k$ and $R_k(z) \neq 0$ for $z \ll k$. These conditions are enough to guarantee that the contribution to the functional integral of field modes $\phi_n$ corresponding to eigenvalues $\lambda_n \ll k^2$ is suppressed, while the contribution of field modes corresponding to eigenvalues $\lambda_n \gg k^2$ is unaffected. We will further require that $R_k(z) \to k^2$ for $k \to 0$. We define a $k$-dependent generating functional of the connected Green functions by

$$\exp \left\{ -W_k[J] \right\} = \int D\phi \exp \left\{ -S[\phi] - \Delta S_k[\phi] - \int dx J \phi \right\} \quad (2.3)$$

and a modified $k$-dependent Legendre transform

$$\Gamma_k[\phi] = W_k[J] - \int dx J \phi - \Delta S_k[\phi], \quad (2.4)$$

where $\Delta S_k[\phi]$ has been subtracted. The functional $\Gamma_k$ is sometimes called the ‘effective average action’, because it is closely related to the effective action for fields that have been averaged over volumes of order $k^{-d}$ (d being the dimension of spacetime). The ‘classical fields’ $\delta W_k/\delta J$ are denoted again by $\phi$ for notational simplicity. In the limit $k \to 0$ this functional tends to the usual effective action $\Gamma[\phi]$, the generating functional of one-particle irreducible Green functions. It is similar in spirit to the Wilsonian effective action, but differs from it in the details of the implementation.

The average effective action $\Gamma_k[\phi]$, used at tree level, should give an accurate description of processes occurring at momentum scales of order $k$. In the spirit of effective field theories, one assumes that $\Gamma_k$ exists and admits a derivative expansion of the form

$$\Gamma_k(\phi, g_i) = \sum_{n=0}^{\infty} \sum_i g_i^{(n)}(k) O_i^{(n)}(\phi), \quad (2.5)$$

where $g_i^{(n)}(k)$ are coupling constants and $O_i^{(n)}$ are all possible operators constructed with the field $\phi$ and $n$ derivatives, which are compatible with the symmetries of the theory. The index $i$ is used here to label different operators with the same number of derivatives. At one loop, $\Gamma_k$ is given by

$$\Gamma_k^{(1)} = S + \frac{1}{2} \text{Tr} \log(S^{(2)} + R_k) \quad (2.6)$$
where $S^{(2)}$ denotes the second variation of the bare action. The only dependence on $k$ is contained in the cutoff term $R_k$ and therefore

$$k \frac{d \Gamma_k^{(1)}}{dk} = \frac{1}{2} \text{Tr} \left( S^{(2)} + R_k \right)^{-1} k \frac{d R_k}{dk}. \quad (2.7)$$

Note that due to the properties of $R_k(z)$, the factor $k \frac{d R_k}{dz}$ goes to zero for $z > k^2$. Thus, even though $R_k(z)$ is introduced in the functional integral as an infrared cutoff, in the evaluation of the beta functions it acts effectively as UV cutoff. One can extract from this functional equation the one-loop beta functions, by making an ansatz for $\Gamma_k^{(1)}$ of the form (2.5), possibly containing a finite number of terms, inserting it on the left-hand side of (2.7), expanding the right-hand side and comparing term by term.

When applied to scalar, fermion and gauge field theories, this method reproduces the well-known one-loop beta functions. In this paper we shall apply this technique to TMG. In general, in applying Wilsonian ideas to gravity one has to confront the fact that the definition of a cutoff generally makes use of a metric and since the metric is now to be treated as a dynamical field it is less clear what one means by cutoff. We will follow [15] and use the background field method, thereby effectively replacing the dynamical metric $g_{\mu\nu}$ by a spin-two field $h_{\mu\nu}$ propagating in a fixed but unspecified background $\bar{g}_{\mu\nu}$. In addition to appearing in the gauge fixing, the background metric can be used to unambiguously distinguish what is meant by long and short distances, and hence low and high energies. In practice, it allows us to write a cutoff term that is purely quadratic in the quantum field, as required in the preceding discussion. The effective average action is then a functional of two fields $\Gamma_k(\bar{g}_{\mu\nu}, h_{\mu\nu})$; in practice, aside from the gauge fixing and cutoff terms, which are quadratic in $h_{\mu\nu}$, we will only need the restriction of the functional $\Gamma_k$ to the space where $\langle h_{\mu\nu} \rangle = 0$. This is sufficient to extract beta functions for the couplings we are interested in.

3. The quadratic gauge fixed action for TMG

TMG is described by the action

$$S = Z \int d^3x \sqrt{-g} \left( R - 2\Lambda + \frac{1}{2\mu} e^{\lambda_{\mu\nu}} \Gamma_{\lambda\sigma} \left( \partial_{\mu} \Gamma_{\nu\sigma} + \frac{2}{3} \Gamma_{\mu\tau} \Gamma_{\nu\rho} \Gamma_{\sigma\tau} \right) \right), \quad (3.1)$$

where $Z = \frac{1}{16\pi G}$. At this point we need not specify the signs of the couplings $G$, $\Lambda$ and $\mu$. It will be useful to define the dimensionless combinations

$$\nu = \mu G; \quad \tau = \Lambda G^2; \quad \phi = \mu / \sqrt{|\Lambda|}, \quad (3.2)$$

Note in particular that $\frac{1}{32\pi G}$ is the coefficient of the CS term. We will expand around background metric $\bar{g}_{\mu\nu}$ as $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$. Next, we add to the action the gauge fixing term

$$S_{GF} = -\frac{Z}{2\alpha} \int d^3x \sqrt{-g} \chi_{\mu} \tilde{g}^{\mu\nu} \chi_{\nu}, \quad (3.3)$$

where

$$\chi_{\nu} = \partial_{\nu} h^{\mu\nu} - \frac{\beta + 1}{4} \partial_{\nu} h. \quad (3.4)$$

Note that $\beta = 1$ corresponds to the familiar de Donder gauge. The ghost action corresponding to the gauge (3.4) is given by

$$S_{gh} = -\int d^3x \sqrt{-g} \bar{C}_{\mu} \left( \tilde{g}^{\mu}_{\nu} \Box + \frac{1 - \beta}{2} \nabla_{\mu} \nabla^{\nu} + R_{\mu\nu}^{\rho\sigma} \right) C_{\nu}. \quad (3.5)$$

Note that $\beta = 1$ corresponds to the familiar de Donder gauge. The ghost action corresponding to the gauge (3.4) is given by

$$S_{gh} = -\int d^3x \sqrt{-g} \bar{C}_{\mu} \left( \tilde{g}^{\mu}_{\nu} \Box + \frac{1 - \beta}{2} \nabla_{\mu} \nabla^{\nu} + R_{\mu\nu}^{\rho\sigma} \right) C_{\nu}. \quad (3.5)$$

5 We use the conventions $R_{\mu\nu}^{\rho\sigma} = \partial_{\nu} \Gamma_{\rho\sigma} - \partial_{\sigma} \Gamma_{\rho\nu} + \Gamma_{\rho\tau} \Gamma_{\nu\tau} - \Gamma_{\nu\rho} \Gamma_{\tau\nu}$.
The one-loop average effective action $\Gamma^{(1)}_k$ contains terms of the same form as $S + S_{GF} + S_{gh}$, except that all the couplings in $S$ are now to be interpreted as renormalized couplings$^6$. Their beta functions will be obtained from equation (2.7), which for this theory has the form

$$\frac{d\Gamma^{(1)}_k}{dk} = \frac{1}{2} \frac{d}{dk} \left( \frac{\delta^2(S + S_{GF})}{\delta h_{\mu\nu} \delta h_{\rho\sigma}} + R_{\mu\nu\rho\sigma} \right)^{-1} k \frac{dR_{\mu\rho\nu\sigma}}{dk} - \Tr \left( \frac{\delta^2 S_{gh}}{\delta C_{\mu} \delta C_{\nu}} + R_{\mu} \right)^{-1} k \frac{dR_{\mu\nu}}{dk}. \tag{3.6}$$

Expanding in powers of $h_{\mu\nu}$, and discarding total derivative terms, the quadratic part of the action is given by

$$S^{(2)} + S_{GF} = \frac{1}{4} \int d^3x \sqrt{g} \left[ h_{\mu\nu} \left( \frac{2}{3} \frac{R}{\Lambda} - \frac{2R}{3} + 2\Lambda \right) h_{\mu\nu} + \frac{2(1 - \alpha)}{\alpha} h_{\mu\nu} \nabla^{\mu} \nabla^{\nu} h_{\mu\nu} - \frac{1}{\Lambda} \frac{(\beta + 1)^2}{8\alpha} h \Box h + \frac{1}{6} h(R - 6\Lambda) h + \frac{1}{\Lambda} \epsilon^{\lambda\mu\nu} h_{\lambda\sigma} \left( \nabla^{\lambda} h_{\mu\sigma} - \frac{R}{3} \nabla^{\sigma} h_{\mu\lambda} - \nabla^{\mu} h_{\sigma\lambda} \right) \right]. \tag{3.7}$$

The raising and lowering of indices are understood to be with the background metric, on which we drop the bar for notational simplicity. For our purposes it will be sufficient to consider maximally symmetric backgrounds, for which

$$R_{\mu\nu\rho\sigma} = \frac{R}{6} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}), \quad R_{\mu\nu} = \frac{R}{3} g_{\mu\nu}. \tag{3.8}$$

Here $R = \pm 6/\ell^2$, with the $+$ sign for de Sitter, and the $-$ sign for anti-de Sitter space, and $\ell$ is the ‘radius’$^3$. It is important to note that, in computing the gauge fixed quadratic action and the beta functions, we do not use the equation of motion which for this class of metrics would imply the relation $\ell = 1/\sqrt{|\Lambda|}$. As we shall see later, this is necessary to compute the beta functions for $\Lambda$ and $G$ separately.

In order to achieve partial diagonalization of the inverse propagator we decompose the quantum fluctuation $h_{\mu\nu}$ into irreducible parts:

$$h_{\mu\nu} = h^T_{\mu\nu} + \nabla_{\mu} \xi_{\nu} + \nabla_{\nu} \xi_{\mu} + \nabla_{\nu} \sigma - \frac{1}{2} g_{\mu\nu} \Box \sigma + \frac{1}{4} g_{\mu\nu} \xi, \tag{3.9}$$

where $h = g^{\mu\nu} h_{\mu\nu}$, $\xi_{\mu}$ satisfies $\nabla^{\mu} \xi_{\mu} = 0$ and $h^T_{\mu\nu}$ satisfies $g^{\mu\nu} h^T_{\mu\nu} = 0$ and $\nabla^{\lambda} h^T_{\mu\lambda} = 0$. It is important to note that in passing from a path integral over the field $h_{\mu\nu}$ to one over the fields $(h^T_{\mu\nu}, \xi, \sigma, h)$, one has to take into account the appropriate Jacobian factors. These Jacobians will be exactly canceled by other Jacobians arising from suitable redefinitions of $\xi$ and $\sigma$, as we will see later (see (4.2)).

Inserting (3.9) into (3.7), and using the lemmas in appendix A, we find

$$S^{(2)} + S_{GF} = \frac{1}{4} \int d^3x \sqrt{g} \left[ h^T_{\mu\nu} \left( \left( \frac{2}{3} \frac{R}{\Lambda} + 2\Lambda \right) h^T_{\mu\nu} + \frac{1}{\mu} h^T_{\lambda\sigma} \left( \frac{R}{3} - \frac{R}{\Lambda} \right) \right) \epsilon^{\lambda\mu\nu} \nabla^{\lambda} h^T_{\mu\sigma} \right. \left. - \frac{2}{\alpha} \xi_{\mu} \left( \left( \frac{2}{3} \frac{R}{\Lambda} + 2\Lambda \right) \xi_{\mu} + \frac{1}{\alpha} \frac{R}{3} \right) \xi^{\mu} + \frac{2(4 - \alpha)}{9\alpha} \sigma \left( \frac{R}{2} + \frac{2}{4 - \alpha} \frac{R}{4 - \alpha} \right) \sigma \right] \left[ \frac{2(\alpha - 3\beta + 1)}{9\alpha} \sigma \left( \frac{R}{2} + \frac{R}{18} + \frac{\Lambda_k}{3} \right) h \right]. \tag{3.10}$$

$^6$ We allow only the couplings in $S$ to run, whereas the gauge fixing parameters will be kept fixed.
Note the factor $1/4$ instead of the traditional $1/2$. The additional, irrelevant factor of $1/2$ has been extracted in order to have an operator where the coefficient of $\Box$ acting on $h_T^{\mu\nu}$ is $1$. Note that the CS term affects only the propagation of $h_T^{\mu\nu}$. We decompose the ghost field as

$$C_\mu = V_\mu + \partial_\mu S,$$

(3.11)

where $\nabla_\mu V^\mu = 0$, and similarly for $\bar{C}^\mu$. This leads to

$$S_{\text{ghost}}^{(2)} = - \int d^3x \sqrt{-g} \left[ V^\mu \left( \Box + \frac{R}{3} \right) V_\mu - \frac{3 - \beta}{2} S \Box \left( \Box + \frac{4}{3(3 - \beta)} R \right) S \right].$$

(3.12)

Changing ghost variables to $(V_\mu, S)$ again produces a Jacobian which will be canceled by another Jacobians coming from a redefinition of $S$ (see (4.2)).

4. The cutoff

In this section we define the cutoff and we set up the calculation of the traces in (3.6) by means of harmonic expansions on $S$, viewed as Euclideanized de Sitter space. From here on we will work in the ‘diagonal’ gauge in which $\sigma - h$ couplings vanish. This fixes

$$\beta = \frac{2\alpha + 1}{3}. $$

(4.1)

Furthermore we make the field redefinitions

$$\Box + \frac{R}{3} \xi_\mu = \hat{\xi}_\mu, \quad \Box \left( \Box + \frac{R}{2} \right) \sigma = \hat{\sigma}, \quad \Box S = \hat{S},$$

(4.2)

whose Jacobian determinants cancel those coming from (3.9) and (3.11) [16, 17]. Then the action (3.10) becomes

$$S^{(2)} + S_{\text{GF}} = \frac{1}{4} Z k \int d^3x \sqrt{-g} \left[ h_T^{\mu\nu} \Delta_2 h_T^{\mu\nu} h_T^{\rho\sigma} + c_1 \xi_\mu \Delta_1 h \bar{\xi}_\nu + c_\sigma \Delta_\sigma \bar{\hat{S}} + c_h \Delta_\bar{h} \bar{h} \right],$$

(4.3)

where we have defined the operators

$$\Delta_2 h_T^{\mu\nu} = \left( \Box - \frac{2R}{3} + 2\Lambda \right) \delta_\mu^{(\rho} \delta_\nu^{\sigma)} + \frac{1}{\mu} \xi_\mu \delta_\delta \left( \Box - \frac{R}{3} \right),$$

$$\Delta_1 h_T^{\mu\nu} = \left( \Box + \frac{1 - \alpha}{3} R + 2\alpha \Lambda \right) \delta_\mu^{\nu},$$

$$\Delta_\sigma = \left( \Box + \frac{2 - \alpha}{4 - \alpha} R + \frac{6\alpha \Lambda}{4 - \alpha} \right) \delta_\sigma,$$

(4.4)

$$\Delta_\bar{h} = \left( \Box + \frac{R}{4 - \alpha} + \frac{6\Lambda}{4 - \alpha} \right) \delta_\bar{h},$$

and coefficients

$$c_1 = -\frac{2}{\alpha}, \quad c_\sigma = \frac{2(4 - \alpha)}{9\alpha}, \quad c_h = -\frac{4 - \alpha}{18}.$$

(4.5)

Similarly, the ghost action (3.12) in the diagonal gauge becomes

$$S_{\text{ghost}}^{(2)} = - \int d^3x \sqrt{-g} \left[ \bar{V}_\mu \Delta V_\mu \bar{V}_\nu + c_S \bar{\bar{\delta}} \Delta \bar{\bar{S}} \right],$$

(4.6)

where $c_S = (\alpha - 4)/3$ and

$$\Delta V_\mu = \left( \Box + \frac{R}{3} \right) \delta_\mu, \quad \Delta \bar{\bar{S}} = \Box + \frac{2}{4 - \alpha} R.$$

(4.7)
It would be tempting to choose $\alpha = 1$, which corresponds to the de Donder gauge and simplifies the inverse propagator, but we shall not do so in order to probe the $\alpha$-dependence of our results. We shall come back to this issue later. We note that the gauge $\alpha = 4$ which has been used in [2] is not singular, as can be seen from (3.10). However, in this gauge the $\sigma^2$, $h^2$, $S^2$ terms do not contain $\Box$ and the method we shall use below to compute the beta functions will turn out to be not suitable to deal with this case. Therefore, in the rest of this paper we will assume that $\alpha \neq 4$.

Now we are ready to discuss the choice of the cutoff. For each spin component we choose the cutoff to be a function of the corresponding operator given in (4.4). Then, defining the gauge fixed inverse propagator

$$O = Z \begin{pmatrix} \Delta_2 & c_1 \Delta_1 \\ c_\sigma \Delta_\sigma & c_h \Delta_h \end{pmatrix}$$  \hspace{1cm} (4.8)

we choose the cutoffs to have the following forms:

$$R_k = Z \begin{pmatrix} R_k(\Delta_2) & c_1 R_k(\Delta_1) \\ c_\sigma R_k(\Delta_\sigma) & c_h R_k(\Delta_h) \end{pmatrix}$$  \hspace{1cm} (4.9)

where $R_k(z)$ is a real profile function with the properties described in section 1. Then the denominator of the first term in (3.6) is

$$\frac{c_1}{2 \Gamma^2} \left[ \text{Tr} \left\{ W(\Delta_2) + \text{Tr} \left\{ W(\Delta_1) + \text{Tr} \left\{ W(\Delta_\sigma) + \text{Tr} \left\{ W(\Delta_h) \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \r
so $W(z) = 2 \theta(k^2 - |z|)$. Dividing the numerator and denominator by $k^2$, they are given by
\[
k \frac{dG_k}{dk} = \sum_n m_n^{T \pm} \theta(1 - \tilde{\lambda}_n^{T \pm}) + \sum_n m_n^{\xi} \theta(1 - \tilde{\lambda}_n^{\xi}) + \sum_n m_n^{\sigma} \theta(1 - \tilde{\lambda}_n^{\sigma}) + \sum_n m_n^{\mu} \theta(1 - \tilde{\lambda}_n^{\mu}) + \sum_n m_n^{V} \theta(1 - \tilde{\lambda}_n^{V}) + \sum_n m_n^{S} \theta(1 - \tilde{\lambda}_n^{S}).
\]
\[\text{(4.13)}\]

Here $\tilde{\lambda}_n^{(i)} = \lambda_n^{(i)} / k^2$ are the distinct dimensionless eigenvalues of the Euclideanized operator $\Delta_i$, and $m_n^{(i)}$ their multiplicities. Using the results of appendix B, the eigenvalues are given by
\[
\lambda_n^{T \pm} = \frac{R}{6} \left( n^2 + 2n + 2 \right) - 2 \Lambda \pm \frac{1}{\mu} \left( \frac{R}{6} \right)^{3/2} n(n + 1)(n + 2), \quad n \geq 2,
\]
\[
\lambda_n^{\xi} = \frac{R}{6} \left( n^2 + 2n - 3 + 2\alpha \right) - 2\alpha \Lambda, \quad n \geq 2,
\]
\[
\lambda_n^{\sigma} = \frac{R}{6} \left( n^2 + 2n - 6 \left( 2 - \alpha \right) \right) - \frac{6\alpha \Lambda}{4 - \alpha}, \quad n \geq 2,
\]
\[
\lambda_n^{h} = \frac{R}{6} \left( n^2 + 2n - 6 \right) - \frac{6\Lambda}{4 - \alpha}, \quad n \geq 0,
\]
\[
\lambda_n^{V} = \frac{R}{6} \left( n^2 + 2n - 3 \right), \quad n \geq 1,
\]
\[
\lambda_n^{S} = \frac{R}{6} \left( n^2 + 2n - 12 \right), \quad n \geq 1,
\]
\[\text{(4.14)}\]

and the respective multiplicities are
\[
m_n^{T \pm} = m_n^{T -} = 2(n^2 + 2n - 3),
\]
\[
m_n^{\xi} = m_n^{\xi} = 2(n^2 + 2n),
\]
\[
m_n^{\sigma} = m_n^{\sigma} = 2(n^2 + 2n + 1).
\]
\[\text{(4.15)}\]

We see that the effect of the cutoff function is simply to terminate each sum over eigenvalues at some maximal value $n_{\text{max}}$. In the next section we discuss the way in which one can evaluate these finite sums, emphasizing the subtleties that arise in the spin-two sector.

5. Evaluation of the functional traces

The most familiar way of evaluating the asymptotic expansion of functional traces is the method of the heat kernel. However, in the presence of the $\mu$–dependent terms in the trace over the spin-two fields, the heat kernel coefficients are not known. Since on the other hand for a maximally symmetric background the eigenvalues are known explicitly, we will directly evaluate the expansion of the traces using the Euler–Maclaurin formula. In the present context, where the function to be traced and all its derivatives vanish at $+\infty$, it states that
\[
\sum_{n=0}^{\infty} F(n) = \int_{n_0}^{\infty} dx \ F(x) + \frac{1}{2} F(n_0) - \frac{B_2}{2!} F'(n_0) - \frac{B_4}{4!} F''(n_0) + R
\]
\[\text{(5.1)}\]

where $B_n$ are the Bernoulli numbers and $R$ is a remainder. For the evaluation of the beta functions of interest, the terms with three or more derivatives of $F$ give no contribution, so it is enough to include the term containing $B_2 = 1/6$ and the remainder can be neglected.

The evaluation of the sums can be done using algebraic manipulation software. There are some subtleties that need to be emphasized. For each type of field, the effect of the step
function is that the integral extends up to a finite value of $n$. Thus, in any spin sector in (4.13) we have to evaluate schematically

$$\int_{n_0}^{\infty} dx \ m(x) \theta(1 - \tilde{\lambda}(x)) = \int_{n_0}^{n_{\max}} dx \ m(x),$$

(5.2)

where $n_{\max}$ is a (real and positive) root of the equation\(^7\)

$$\tilde{\lambda}(x) - 1 = 0.$$  

(5.3)

When the eigenvalues are quadratic in $n$, equation (5.3) has two roots. Since the cutoff must be positive, it is always clear which one to choose. For $\alpha < 4$, which is the case for the gauges we use, the trace $\tilde{h}$ has a negative definite kinetic term. This is the well-known problem of the indefiniteness of the gravitational action. In the evaluation of the sums (4.13) this sign has no effect. One can choose $n_{\max}$ as for the other components and this does not lead to any pathology in the flow equation (see [15] for a discussion).

Things are more complicated in the presence of the CS term. The eigenvalues of the TT field contain a term which is cubic in $n$, and this term occurs with opposite signs in the two chirality sectors. If we tried to define directly the one-loop effective action, this would lead to convergence problems similar to those discussed in [19]. For us, convergence should never be a problem because we are only interested in the chirality sectors. If we tried to define directly the one-loop effective action, this would lead to pathology in the flow equation (see [15] for a discussion).

For the $\tilde{\lambda}^\pm$ equation (5.3) is cubic, one has to choose one among its three roots

$$r = s + t; \quad r + e^{i\pi/3} s - e^{-i\pi/3} t; \quad r + e^{-i\pi/3} s - e^{i\pi/3} t,$$

(5.4)

where $r, s, t$ depend on $\tilde{R} = R/k^2$, $\tilde{\Lambda} = \Lambda/k^2$ and $\tilde{\mu} = \mu/k$, respectively. Superficially the first root seems to be real and the others seem to be complex, so one could try to define the cutoff using the first root. However, the expressions $s$ and $t$ actually involve cubic roots and there are regions in parameter space where these roots develop imaginary parts. In fact, there is no root which is real for all parameter values.

One can understand this better by considering the plot of the functions $\tilde{\lambda}^T\pm(x)$ for $x > 0$, see figures 1–3. Consider first the case when $\tilde{\mu} > 0$. The function $\tilde{\lambda}^T\pm(x)$ is monotonically increasing, and for this function equation (5.3) has a single real root. Thus there is no ambiguity for the positive chirality modes. In the case of the negative chirality modes, however, for small $x$ the function $\tilde{\lambda}^-T(x)$ initially grows with $x$, until the cubic term prevails; hereafter it decreases. When $\tilde{R}$ and $\tilde{\Lambda}$ can be neglected, which is the situation we are interested in, the maximum is equal to $4\tilde{\mu}^2/27$. Thus if $\tilde{\mu} > \sqrt{27}/4$, equation (5.3) has two positive roots. It is clear that the smaller of the two has to be chosen, namely the one where the function is growing. For this reason we will call this an ‘ascending root cutoff”.

On the other hand if $0 < \tilde{\mu} < \sqrt{27}/4$, equation (5.3) has no positive root and the ascending root does not exist. However, we observe that the equation $\tilde{\lambda}^T^- = -1$ has a real positive root for any $\tilde{\mu}$. As mentioned in section 3, in the evaluation of the beta functions it is not important whether the modes have positive or negative eigenvalue. Therefore we can use this descending root to define the cutoff. We call this a ‘descending root cutoff”. When $\tilde{\mu} < 0$, the discussion can be repeated interchanging the roles of $\tilde{\lambda}^T+$ and $\tilde{\lambda}^T-$.

Since the descending solution of $\tilde{\lambda}^T- = -1$ always exists, while the ascending solution of $\tilde{\lambda}^T- = 1$ only exists if $|\tilde{\mu}| > \sqrt{27}/4$, one wonders why not use always the former. The reason\(^7\) in general $n_{\max}$ is not an integer. One can think that the sums can be evaluated by terminating them at the largest integer $n \leq n_{\max}$. However, it is easy to check that his procedure applied, for example, to the evaluation of the known heat kernel coefficients will not give the right answer.

\(^7\) In general $n_{\max}$ is not an integer. One can think that the sums can be evaluated by terminating them at the largest integer $n \leq n_{\max}$. However, it is easy to check that his procedure applied, for example, to the evaluation of the known heat kernel coefficients will not give the right answer.
Figure 1. The eigenvalues $\tilde{\lambda}^+_n$ (solid curve) and $\tilde{\lambda}^-_n$ (dashed curve) as functions of $n$, for $\tilde{R} = \tilde{\Lambda} = 0.01$. Left panel: small $\tilde{\mu}$ regime (here $\tilde{\mu} = 0.3$). Right panel: large $\tilde{\mu}$ regime (here $\tilde{\mu} = 3$).

Figure 2. The real (left panel) and imaginary (right panel) parts of the roots of the equation $\tilde{\lambda}^+_n = 1$, for $\tilde{R} = \tilde{\Lambda} = 0.01$, as functions of $\tilde{\mu}$. The first root (thick dotted line) always has a positive real part and is complex for $-\sqrt{27/4} < \tilde{\mu} < 0$; the second root (dashed line) always has a negative real part and is complex for $0 < \tilde{\mu} < \sqrt{27/4}$; the third root (solid line) is complex for $-\sqrt{27/4} < \tilde{\mu} < \sqrt{27/4}$. The solutions of the equation $\tilde{\lambda}^-_n = 1$ are obtained by the reflection $\tilde{\mu} \rightarrow -\tilde{\mu}$.

Figure 3. The real (left panel) and imaginary (right panel) parts of the roots of the equation $\tilde{\lambda}^-_n = -1$, for $\tilde{R} = \tilde{\Lambda} = 0.01$, as functions of $\tilde{\mu}$. The first root (thick dotted) always has a positive real part and is complex for $\tilde{\mu} < 0$; the second root (dashed) always has a negative real part and is complex for $\tilde{\mu} > 0$; the third root (solid line) is always complex. The solutions of the equation $\tilde{\lambda}^+_n = -1$ are obtained by the reflection $\tilde{\mu} \rightarrow -\tilde{\mu}$.
is that the information one can get from these different cutoffs is complementary. For example if we restrict ourselves to $\tilde{\mu} > 0$, when $\tilde{\mu}$ becomes very large, the descending solution of $\tilde{\lambda} T^\nu_T - n = -1$ is much larger than the ascending solution of $\tilde{\lambda} T^\nu_T + n = 1$. In fact the sum over negative chirality modes is divergent in the limit $\tilde{\mu} \to \infty$. The beta functions computed in this way will have fictitious singularities in this limit and it will not be possible to flow smoothly to the case when the CS term is absent. As we will see later, this also implies that the beta functions have fictitious singularities near $\tilde{G} = 0$, so it will not be possible to study the Gaussian FP in this cutoff scheme. On the other hand, the ascending solution has an imaginary part when $|\tilde{\mu}| < \sqrt{27/4}$ and therefore cannot be used to study the small $\tilde{\mu}$ region. The conclusion is that in order to properly understand the behavior of the theory over the whole range of values of $\tilde{\mu}$ one has to use both schemes: the ‘descending’ cutoff for small $\tilde{\mu}$ and the ‘ascending’ cutoff for large $\tilde{\mu}$.

6. The beta functions

Applying the Euler–Maclaurin formula to each of the sums in (4.13) and extracting from each of the integrals the appropriate powers of $R, t h e r h s o f (4.13)$ can be written as

$$\partial_t \Gamma_k = \sum \left[ C_0 R^{-3/2} + C_2 R^{-1/2} + C_{3/2} + \frac{1}{2} F(n_0) - \frac{B_2}{2!} F'(n_0) \right]$$

(6.1)

where the sum is over $h^T, \xi, \sigma, h, V, S$.

The contributions of each spin component to the $c_{3/2}$ term are independent of the cutoff scheme. They are listed in the following table:

| $n_0$ | $C_{3/2}$ | $F(n_0)$ | $F'(n_0)$ |
|-------|-----------|----------|------------|
| $h^T_{\mu\nu}$ | 2 | 12 | 20 | 24 |
| $\xi^\mu$ | 2 | -24 | 32 | 24 |
| $\sigma$ | 2 | -18 | 18 | 12 |
| $h$ | 0 | -$\frac{7}{2}$ | 2 | 4 |
| $C^\mu$ | 1 | -$\frac{8}{7}$ | 12 | 16 |
| $\tilde{C}$ | 1 | -$\frac{16}{7}$ | 8 | 8 |

In the second column, the lower end of integration in (5.1) is shown. Summing all the contributions we conclude that the coefficient of the $R$-independent term is exactly zero. More precisely, the sum $C_{3/2} + \frac{1}{2} F(n_0) - \frac{B_2}{2!} F'(n_0)$ is zero separately for the trace-free part of $h_{\mu\nu}$, (the sum of the first three lines), for the trace part $h$ and for the ghosts (the sum of the last two lines). These are the components of the fields that can be defined by purely algebraic conditions. The cancellation does not occur for components defined by differential constraints, such as the transversality condition. To some extent, the overall cancellation of the $R$-independent terms is expected. In the simpler setting of pure gravity without the CS term, or any matter field coupled to gravity, the sums can be evaluated using the heat kernel expansion. On a manifold without boundary the trace of the heat kernel contains only integer powers of $R$, and in a three-dimensional manifold the volume prefactor is proportional to $R^{-3/2}$, so the expansion of $\partial_t \Gamma_k$ contains only odd powers of $R$, and there is no $R$-independent term. So the CS term will not be induced, if one starts without it. In appendix D we show that the sums done with the Euler–Maclaurin method explained above exactly reproduce the heat kernel results for 3D gravity without the CS term at one loop. The fact that all the contributions listed in
the above table exactly cancel when properly summed is therefore a nontrivial check of our calculation.

To obtain the rest of the beta functional \( \partial_t \Gamma_k \) there remain to sum the contributions of type \( C_0 \) and \( C_2 \) in (6.1); the final result has the following structure:

\[
\partial_t \Gamma_k = \frac{V(S^3)}{16\pi} [k^3 A(\Lambda, \bar{\mu}) + k B(\Lambda, \bar{\mu}) R + O(R^2)],
\]

(6.2)

where we have inserted powers of \( k \) such that the \( A \)- and \( B \)-coefficients are dimensionless. The volume of \( S^3 \) with radius \( \ell \) is \( V(S^3) = 2\pi^2 \ell^3 \) where \( \ell = \sqrt{\frac{k}{R}} \).

Equation (6.1) is an expansion in \( R \), whose coefficients are functions of \( \Lambda, \mu \) and \( k^2 \). For reasons that will become apparent below, we shall restrict ourselves to the parameter region where \( \Lambda \) and \( R \) are of the same order. Therefore we shall also expand the coefficients in (6.1) in powers of \( \Lambda \), namely in \( C_0 \) we keep at most terms linear in \( \Lambda \) while in \( C_2 \) we only keep the \( \Lambda \)-independent terms. We will give explicit expressions for \( A \) and \( B \) later.

Evaluating the (Euclidean version of the) renormalized TMG action (3.1) on the \( S^3 \) background, it can be written in the form

\[
\Gamma_k = \frac{V(S^3)}{16\pi} \left( \frac{2\Lambda}{16\pi G} - \frac{1}{12\sqrt{6\pi G \mu}} R^{3/2} + O(R^2) \right),
\]

(6.3)

where we have used that the integral of the CS term on \( S^3 \) is given by \( \int \text{tr}(\omega d\omega + \frac{3}{2} \omega^3) = 32\pi^2 \). The couplings \( \Lambda, G, \mu \) are now renormalized couplings evaluated at scale \( k \). Rescaling the coupling constants as

\[
G = \tilde{G} k^{-1}, \quad \Lambda = \tilde{\Lambda} k^2, \quad \mu = \tilde{\mu} k,
\]

(6.4)

so as to make them dimensionless, and comparing the \( t \)-derivative of (6.3) with (6.2), we obtain

\[
\frac{1}{8\pi \tilde{G}} \left( \partial_t \tilde{\Lambda} - \frac{\partial_t \tilde{G}}{\tilde{G}} \tilde{\Lambda} \right) = -\frac{3\tilde{\Lambda}}{8\pi \tilde{G}} + \frac{\Lambda}{16\pi},
\]

(6.5)

\[
\frac{\partial_t \tilde{G}}{16\pi \tilde{G}^2} = \frac{1}{16\pi \tilde{G}} + \frac{B}{16\pi},
\]

(6.6)

\[
\frac{1}{12\sqrt{6\pi \tilde{\mu} \tilde{G}}} \left( \frac{\partial_t \tilde{G}}{\tilde{G}} + \frac{\partial_t \tilde{\mu}}{\tilde{\mu}} \right) = 0.
\]

(6.7)

The last equation results from the fact that the terms of order \( R^{3/2} \) in (6.2) cancel, and it implies that the dimensionless combination \( v = G \mu = \tilde{G} \tilde{\mu} \) does not run. From the other two equations one obtains the one-loop beta functions of \( \tilde{G} \) and \( \tilde{\Lambda} \):

\[
\partial_t \tilde{G} = \tilde{G} + B(\tilde{\mu}) \tilde{G}^2,
\]

(6.8)

\[
\partial_t \tilde{\Lambda} = -2\tilde{\Lambda} + \frac{1}{2} \tilde{G}(A(\tilde{\mu}, \tilde{\Lambda}) + 2B(\tilde{\mu}) \tilde{\Lambda}).
\]

These equations have exactly the same form as in pure gravity with cosmological constant, except that the coefficients \( A \) and \( B \) are now \( \tilde{\mu} \)-dependent.

It may be useful to note that the beta functions are related to the structure of divergences of the theory. Inserting (6.4) into (6.3) we see that for \( k \to \infty \) the first two terms in the action would be cubically and linearly divergent, respectively. A nonzero constant on the rhs of (6.7) would correspond to a logarithmic divergence in the third term of (6.3), but such a divergence is absent, as we have seen. We also note that when dimensionful couplings are
present, the definition of FP requires that the rescaled couplings (6.4) tend to constants. This implies that the original, dimensionful couplings run according to their canonical dimension, and this makes the FP action scale invariant.

In order to solve for the RG flow, we use that \( \nu \) does not run and substitute \( \tilde{\mu} = \nu / \tilde{G} \). This yields two ordinary first-order differential equations for \( \tilde{G}(t) \) and \( \tilde{\Lambda}(t) \), depending on the fixed constant value of the external parameter \( \nu \). Rather surprisingly, despite the very different functional form, the resulting flow is numerically quite similar to that of pure gravity with cosmological constant. Unlike in pure gravity, however, here we cannot give the solution in closed form. We will now describe the results for different cutoff schemes.

7. Results

7.1. Ascending root cutoff

For \( |\tilde{\mu}| > \sqrt{27/4} \) we define the cut on the spin-two modes as the smallest root of the equation \( \tilde{\lambda}^T = 1 \). In the case of positive \( \tilde{\mu} \), the cutoff on \( \tilde{\lambda}^T + \) corresponds to the thick dotted line in figure 2, while the cutoff on \( \tilde{\lambda}^T - \) is obtained by reflecting the solid line. (Note that the two roots are different, so the number of modes of positive and negative chirality is different.) Calculating the sums yields beta functions that are real and well defined for \( 0 < \tilde{G} < \sqrt{4/27} \nu \).

For negative \( \tilde{\mu} \) the roles of \( \tilde{\lambda}^T + \) and \( \tilde{\lambda}^T - \) are interchanged. Calculating the sums yields beta functions that are real and well defined for \( -\sqrt{4/27} \nu < \tilde{G} < 0 \). The two calculations match smoothly along the line \( \tilde{G} = 0 \), so one can put them together to obtain a RG flow on the whole region \( \tilde{G}^2 < 4 \nu / 27 \).

With this prescription we calculate the coefficients \( A \) and \( B \). These arise as complicated functions involving cubic roots, but after some manipulations can be reduced to the following relatively simple form:

\[
A(\tilde{\Lambda}, \tilde{\mu}) = -\frac{16}{3\pi} + \frac{9(2\sqrt{3} \cos 2\theta - \sqrt{3} \cos 3\theta + 8(\cos \theta)^3 \sin \theta)}{\pi (\cos 3\theta)^3} \tilde{\Lambda} + \frac{8(3 + 9\alpha - 2\alpha^2)}{\pi (4 - \alpha)} \Lambda + \frac{48(\cos \theta - \sqrt{3} \sin \theta)}{\pi \sin 6\theta} \tilde{\Lambda},
\]

\[
B(\tilde{\mu}) = -\frac{4(11 + 9\alpha - 2\alpha^2)}{3\pi (4 - \alpha)} = -\frac{2(\sqrt{3} \sin \theta - \cos \theta) + 22(\sqrt{3} \sin 5\theta + \cos 5\theta)}{3\pi \sin 6\theta},
\]

where we have introduced the angle

\[
\theta = \frac{1}{3} \arctan \sqrt{\frac{4\tilde{\mu}^2}{27} - 1}.
\]

The beta functions admit a Taylor expansion around \( \tilde{\Lambda} = \tilde{G} = 0 \):

\[
\partial_t \tilde{G} = \tilde{G} - \frac{4(47 - 2\alpha^2)\tilde{G}^2}{3\pi (4 - \alpha)} - \frac{95\tilde{G}^4}{6\pi v^2} - \frac{2233\tilde{G}^6}{32\pi v^4} + O(\tilde{G}^7),
\]

\[
\partial_t \tilde{\Lambda} = -2\tilde{\Lambda} - \frac{4(14 - 27\alpha + 4\alpha^2)\tilde{G}\tilde{\Lambda}}{3\pi (4 - \alpha)} + \frac{\tilde{G}^3 (42 + 115\tilde{\Lambda})}{6\pi v^2} + \frac{11\tilde{G}^5 (78 + 343\tilde{\Lambda})}{32\pi v^4} + O(\tilde{G}^6).
\]

We observe, as a nontrivial check, that in the limit \( \tilde{\mu} \to \infty \) the beta functions agree with the result for gravity without the CS term, which are calculated in appendix D using a different method.
Figure 4. The flow in the $\tilde{\Lambda}$-$\tilde{G}$ plane for $\alpha = 0, \nu = 5$, in the ascending root cutoff scheme. Right panel: enlargement of the region around the origin, showing the Gaussian FP. The beta functions become singular at $|\tilde{G}| = 1.9245$, outside the domain of the picture, but this singularity is an artifact of the scheme.

The flow is shown, in the case $\nu = 5$, in figure 4. We note the main features, which hold for any value of $\nu > 0$. There is a FP, called the Gaussian FP, in the origin, which is seen to be attractive in the $\tilde{\Lambda}$ direction and repulsive in the $\tilde{G}$ direction. To make this statement more quantitative, one has to study the linearized equation

$$k \frac{d(\tilde{g}_i - \tilde{g}_{i*})}{dk} = M_{ij}(\tilde{g}_j - \tilde{g}_{j*}).$$

(7.4)

where

$$M_{ij} = \left. \frac{\partial \beta_i}{\partial \tilde{g}_j} \right|_{*}, \quad \beta_i = k \frac{d\tilde{g}_i}{dk}.$$  

(7.5)

Here $\tilde{g}_i$ are the couplings and $\tilde{g}_{i*}$ their values at the FP. At the Gaussian FP $\tilde{g}_{i*} = 0$ one finds

$$M_{ij} = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}. $$

(7.6)

The eigenvalues of this matrix (‘scaling exponents’) correspond to the canonical mass dimensions of $\tilde{\Lambda}$ and $\tilde{G}$, as expected. An important fact (which had been previously observed in the absence of the CS term in [9]) is that the eigenvectors coincide with the coordinate axes. This is not the case in higher dimensional gravity, where the eigenvector with eigenvalue equal to the canonical dimension of $\tilde{G}$ has a nonvanishing component along $\tilde{\Lambda}$.

In addition to the Gaussian FP there is a nontrivial FP at $\tilde{\Lambda}_* = 0.000490.471$ and $\tilde{G}_* = 0.200.016$. It is seen to be UV attractive in both directions. The eigenvalues of the corresponding stability matrix are found to be equal to $-2.294.01$ and $-1.005.15$.

The position of the FP and the eigenvalues of the stability matrix are given, for other values of $\nu$, in figure 5. Note that $\tilde{\Lambda}$ is always positive but very small. This is due to the absence of a term of order $\tilde{G}^2$ in the expansion of $\partial_t \tilde{\Lambda}$ in (7.3). Actually if one plots the contours of the beta functions for $\tilde{\Lambda}$, one may see this as an effect of the deformation of the flow due to the presence of the boundary at $\tilde{G} = \nu/\sqrt{4/27}$ (which occurs at $\tilde{G} = 1.924$ in figure 4). This can probably be regarded as a scheme artifact.
Finally we observe that in the limit $\tilde{\mu} \to \infty$ the beta functions reduce to

$$\partial_t \tilde{G} = \tilde{G} - \frac{4}{3\pi} \frac{47 - 2\alpha^2}{4 - \alpha} \tilde{G}^2,$$

$$\partial_t \tilde{\Lambda} = -2\tilde{\Lambda} - \frac{4}{3\pi} \frac{14 - 27\alpha + 4\alpha^2}{4 - \alpha} \tilde{\Lambda} \tilde{G}$$

which agree with the result for pure gravity without the CS term.

### 7.2. Descending root cutoff

In this scheme the cut on the positive and negative chirality spin-two modes is defined by two different equations, as follows:

| $\tilde{\mu}$ | $\tilde{\mu}$ |
|---------------|---------------|
| $> 0$         | $< 0$         |
| $\tilde{\lambda}^+ = 1$ | $\tilde{\lambda}^- = -1$ |
| $\tilde{\lambda}^- = -1$ | $\tilde{\lambda}^+ = 1$ |

Using the criteria described in the previous section, in the case of positive $\tilde{\mu}$, the cutoff on $\tilde{\lambda}^+$ corresponds once again to the thick dotted line in figure 2, while the cutoff on $\tilde{\lambda}^-$ corresponds to the thick dotted line in figure 3. For negative $\tilde{\mu}$ the roles of $\tilde{\lambda}^+$ and $\tilde{\lambda}^-$ are interchanged. Again the two roots are different, so the number of modes of positive and negative chirality is different. In fact in this case the behavior is drastically different, since the cut on the descending mode grows linearly with $\tilde{\mu}$. This will give a singularity in the beta functions for $\tilde{\mu} \to \infty$, and therefore, when we make the substitution $\tilde{\mu} = v/G$, for $G \to 0$. Thus this scheme is really useful only for sufficiently small $\tilde{\mu}$. 

---

**Figure 5.** Position of the FP (left panel) and eigenvalues of the stability matrix (right panel) for the nontrivial FP with $\alpha = 0$, $1 < \nu < 40$, in the ascending root cutoff scheme. Note that for this range of $\nu$ the singularity is always above the FP. In the left panel, $\nu$ grows from right to left. Note that $\tilde{\mu}/\Lambda^* > 0$ in this scheme. See section 7.1 for a discussion. For large $\nu$, $G_*$ tends to 0.2005 and the eigenvalues tend to $-1$ and $-2.298$. 

Finally we observe that in the limit $\tilde{\mu} \to \infty$ the beta functions reduce to}

$$\partial_t \tilde{G} = \tilde{G} - \frac{4}{3\pi} \frac{47 - 2\alpha^2}{4 - \alpha} \tilde{G}^2,$$

$$\partial_t \tilde{\Lambda} = -2\tilde{\Lambda} - \frac{4}{3\pi} \frac{14 - 27\alpha + 4\alpha^2}{4 - \alpha} \tilde{\Lambda} \tilde{G}$$

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Finally we observe that in the limit $\tilde{\mu} \to \infty$ the beta functions reduce to
Figure 6. The flow in the $\tilde{\Lambda}$–$\tilde{G}$ plane for $\alpha = 0$, $\nu = 0.1$, in the descending root cutoff scheme. Right panel: enlargement of the region around the origin, showing that there is no Gaussian FP. The beta functions diverge on the $\Lambda$ axis.

With this prescription we find

$$A(\tilde{\mu}, \tilde{\Lambda}) = -\frac{16}{3\pi} + \frac{\sqrt{3}}{\pi} \left[ \left( \frac{2 \cosh 2\eta - 1}{\cosh 3\eta} \right)^3 + \left( \frac{2 \cosh 2\psi + 1}{\cosh 3\psi} \right)^3 \right]$$

$$+ \frac{8(3 + 11\alpha - 2\alpha^2)}{\pi(4 - \alpha)} \frac{1}{\tilde{\Lambda}} + \frac{8\sqrt{3}}{\pi} \left[ \frac{1}{2 \cosh \eta + \cosh 3\eta} + \frac{1}{2 \sinh \psi - \sinh 3\psi} \right]$$

$$B(\tilde{\mu}) = -\frac{4(11 + 9\alpha - 2\alpha^2)}{3\pi(4 - \alpha)} - 20\sqrt{3} \frac{\cosh 2\eta + \cosh 2\psi}{3\pi}$$

$$+ \frac{2}{3\sqrt{3}\pi} \left( \frac{8 \cosh 2\eta - 1}{\cosh 3\eta + 2 \cosh \eta} + \frac{8 \cosh 2\psi + 1}{\sinh 3\psi - 2 \sinh \psi} \right)$$

where we have defined

$$\eta = \frac{1}{3} \arctanh \sqrt{1 - \frac{4\tilde{\mu}^2}{27}}$$

$$\psi = \frac{1}{3} \arccoth \sqrt{1 + \frac{4\tilde{\mu}^2}{27}}.$$  \hspace{1cm} (7.8)$$

A representative flow is shown in figure 6 for $\nu = 0.1$. Superficially this may look similar to figure 4, but there are some important differences. First and foremost, there is no Gaussian FP, as is clear from the enlargement of the area around $\tilde{\Lambda} = \tilde{G} = 0$. The FP is wiped out by the singularity at $\tilde{G} = 0$, where the beta function of $\tilde{\Lambda}$ blows up. Note that the flow lines run in opposite directions on the two sides of this line. All the flow lines which, for positive $\tilde{G}$, seem to come in from plus infinity actually originate from minus infinity, and follow closely the $\tilde{G}$ axis before bending over. Similarly for $\tilde{G} < 0$ all the flow lines that seem to come in from minus infinity actually come from plus infinity and run close to the $\tilde{G} = 0$ line. There is thus an infinite accumulation of flow lines along the $\tilde{\Lambda}$ axis. Another difference with figure 4 is that the flow lines near the $\tilde{G}$ axis are tilted in the opposite direction, and by a much larger amount.

For $\nu = 0.1$ the nontrivial FP occurs at $\tilde{\Lambda}_* = -0.0565337$, $\tilde{G}_* = 0.30036$ and the eigenvalues of the stability matrix are $-2.76746$, $-0.781453$. The position of the FP and the eigenvalues, for $10^{-6} < \nu < 0.5$, are shown in figure 7. The FP occurs at positive $\tilde{\Lambda}$ for
Figure 7. Position of the FP (left panel) and eigenvalues of the stability matrix (right panel) for the nontrivial FP with $\alpha = 0$, $10^{-6} < \nu < 0.5$, in the descending root cutoff scheme. In the left panel, $\nu$ decreases from right to left. The cosmological constant changes sign for $\nu = 0.18$. The rightmost point ($\nu = 0.5$) has $\tilde{\mu} \approx 3 > \sqrt{27/4}$ and therefore is in the region where the scheme becomes unreliable.

For $\nu > 0.18$ and negative $\tilde{\Lambda}$ for $\nu < 0.18$. Given that this cutoff scheme is more dependable for small $\tilde{\mu}$ (and hence, at fixed $\tilde{G}$, for small $\nu$), it is again possible that the positive values of $\tilde{\Lambda}$ are a scheme artifact.

Just as the beta functions in the ascending root scheme tend to a very simple form in the limit $\tilde{\mu} \to \infty$, the beta functions in the descending root scheme tend to a very simple form in the limit $\tilde{\mu} \to 0$:

$$\partial_t \tilde{G} = \tilde{G} - \frac{4}{3\pi} \frac{11 + 9\alpha - 2\alpha^2}{4 - \alpha} \tilde{G}^2,$$

$$\partial_t \tilde{\Lambda} = -2\tilde{\Lambda} - \frac{8}{3\pi} \left( 1 + \frac{1}{4} - \frac{12\alpha + 2\alpha^2}{4 - \alpha} \right) \tilde{G}. \tag{7.10}$$

These beta functions have a FP at $\tilde{\Lambda}_* = -1/3$, $\tilde{G}_* = \frac{3\pi(4-\alpha)}{4(11+9\alpha-2\alpha^2)}$, which for $\alpha = 0$ is $\tilde{G}_* \approx 0.8568$. The eigenvalues of the stability matrix are $-2.182$ and $-1$ corresponding to the directions $(1, 0)$ and $(-0.5499, 0.8352)$.

### 7.3. Spectrally balanced cutoff

In the preceding sections we have described two calculations of the beta functions for TMG. Both schemes are ‘spectrally unbalanced’ in the sense that the summations over the spin-two fields in (4.13) contained a different number of positive and negative chirality modes. Consideration of the behavior of the roots in figures 1–3 shows that the ascending scheme becomes balanced for large $\tilde{\mu}$ while the descending scheme becomes balanced for small $\tilde{\mu}$. These are the regimes where these schemes are most reliable. One thus wonders whether one can work in a scheme where the sums always contain equal numbers of positive and negative chirality modes, by construction.

Such a scheme can be constructed by tweaking the cutoff profile. First, we must allow different profiles for different spin components. The spin-one and spin-zero components will be treated as before. For the spin-two components we allow the more general form $R_k(z) = (q(z)k^2 - z)\theta(q(z)k^2 - z)$ where $q(z)$ is a dimensionless function. For any choice of the function $q(z)$ we have $\partial_t R_k(z) = 2k^2\theta(q(z)k^2 - z)$ and for $z < k^2$, $P_k(z) = k^2$. 


so $W(z) = 2\theta(q(z) - z/k^2)$. With this cutoff choice the traces are still finite sums of the multiplicity, up to a maximal value $n_{\text{max}}$ which is determined as a root of the equation

$$\tilde{\lambda}(x) = q(\tilde{\lambda}(x)).$$

(7.11)

Thus the sums are still calculable by the same technique used previously. A choice of $n_{\text{max}}$ implicitly defines a choice of the function $q$. The sums on the spin-two components in (4.13) are now replaced by

$$\sum_{n=2}^{\infty} m_n^i \theta(q^i(\tilde{\lambda}_n^i) - \tilde{\lambda}_n^i) = 2 \sum_{n=2}^{n_{\text{max}}} m_n^i.$$

(7.12)

If $\tilde{\mu} > 0$, we choose $n_{\text{max}}$ to be the unique positive root of the equation $\tilde{\lambda}_n^{\ast} = 1$; then we cut both sums at this value. In this case the function $q$ is $q^i = 1$ for the positive chirality modes, while $q^-$ is determined by the condition $q^- (\tilde{\lambda}_n^{\ast} (x)) = \tilde{\lambda}_n^{\ast} (x) / \tilde{\lambda}_n^{\ast} (x)$. Likewise, if $\tilde{\mu} < 0$, we cut at the root of $\tilde{\lambda}_n^{\ast} = 1$.

With this prescription we get the following results:

$$A(\tilde{\mu}, \tilde{\lambda}) = \frac{16}{3\pi} + \frac{2\sqrt{3}}{\pi(cosh \eta)^5} + \frac{8(3 + 11\alpha - 2\alpha^2)}{\pi(4 - \alpha)} \tilde{\lambda} + \frac{16\sqrt{3}}{\pi(cosh 3\eta + 2cosh \eta)} \tilde{\lambda},$$

$$B(\tilde{\mu}) = -\frac{4(11 + 9\alpha - 2\alpha^2)}{3\pi(4 - \alpha)} - \frac{8\sqrt{3}}{9\pi} \left( \frac{8 + 11 \cosh 2\eta}{\cosh 3\eta + 2 \cosh \eta} \right),$$

(7.13)

where $\eta$ is given by (7.9). Thus, we observe that the apparent poles at $a = 0$, which correspond to $\eta = 0$, are actually absent as can be readily deduced from (7.13), and the coefficients $A, B$ are real. Note also that $\eta = i\theta$ where $\theta$ is the angle that we have encountered in section 7.1, see (7.2). Furthermore $\cosh 3\eta = 3\sqrt{3}/|\tilde{\mu}|$, and the results depend on the absolute value of $\tilde{\mu}$.

In the limit $\tilde{\mu} \to \infty$ we get

$$A \to \frac{8(11 + 9\alpha - 2\alpha^2)}{\pi(4 - \alpha)} \tilde{\lambda}, \quad B \to -\frac{4(47 - 2\alpha^2)}{3(4 - \alpha)}.$$

(7.14)

This limit corresponds to neglecting the CS term, and it is reassuring that we find agreement with the result of the heat kernel calculation in appendix D.

The Taylor expansion of the beta functions in powers of $\tilde{G}$ around the Gaussian FP is

$$\partial_v \tilde{G} = \tilde{G} - \frac{4(47 - 2\alpha^2)\tilde{G}^2}{3\pi(4 - \alpha)} + \frac{28\tilde{G}^3}{3\pi v} - \frac{95\tilde{G}^4}{6\pi v^2} + O(\tilde{G}^5),$$

$$\partial_v \tilde{\lambda} = -\frac{2\tilde{\lambda}}{3\pi(4 - \alpha)} - \frac{4(14 + 27\alpha + 4\alpha^2)\tilde{G}\tilde{\lambda}}{3\pi v} - \frac{4\tilde{G}^2(3 + 5\tilde{\lambda})}{3\pi v} + \frac{4\tilde{G}^3(42 + 115\tilde{\lambda})}{6\pi v^2} + O(\tilde{G}^4).$$

(7.15)

The $v$-independent terms coincide with the result for pure gravity, as described in appendix D.

In the limit $\tilde{\mu} \to 0$ the beta function of $\tilde{\lambda}$ has the same expression as in (7.10), but the beta function of $\tilde{G}$ becomes singular. Still, the position of the FP seems to approximate closely the one that was found in the descending root cutoff. For $v = 10^{-6}$ we find $\tilde{\lambda}_* = -0.315$, $\tilde{G}_* = 0.830$. This should not come as a surprise, since in this limit the two positive roots of the equations $\tilde{\lambda}_n^{\ast} = 1$ and $\tilde{\lambda}_n^{\ast} = -1$ become equal, so the descending root cutoff becomes spectrally balanced.

8. Discussion

8.1. Summary

We begin by summarizing the main results. We have calculated the beta functions of $\Lambda$, $G$ and $\mu$ and we have studied the corresponding RG flows. It is generally the case that
the beta functions of dimensionful couplings (by which we mean also the beta functions of the corresponding dimensionless ratios that we denoted by a tilde) are scheme dependent even to leading order. This is in contrast to the more familiar case of the dimensionless couplings, whose beta functions are scheme independent to leading order. It is therefore not too surprising that the three schemes we have considered give different pictures. Out of these different pictures one can however extract some common features that are presumably scheme independent. We summarize them here.

The first main result is that the dimensionless combination \( \nu = \mu G \), which is the coefficient of the CS term, does not run. We comment further on this in the subsection on topological properties. Due to this fact, the remaining beta functions describe a flow in the \( \hat{\Lambda} - \hat{G} \) plane, whose properties depend on the value of the fixed constant \( \nu \). This two-dimensional flow is governed by two FPs: a Gaussian FP in the origin and a non-Gaussian FP at positive \( \hat{G} \). The Gaussian FP is UV attractive in the \( \hat{\Lambda} \) direction and UV repulsive in the \( \hat{G} \) direction, with scaling exponents equal to the canonical dimensions of \( \hat{\Lambda} \) and \( \hat{G} \). This FP is not seen in the descending root cutoff scheme, but this is clearly an artifact of the unphysical singularity of the flow for large \( \mu \) (hence small \( \hat{G} \)). For the same reason one should not trust the properties of the Gaussian FP that reappears in that scheme in the limit \( \nu \to 0 \).

There is then a nontrivial FP which always occurs for positive \( \hat{G} \). Whether this FP has positive or negative cosmological constant seems to be somewhat unclear. The ascending root cutoff, which is reliable for large \( \nu \), and the spectrally balanced cutoff both indicate that for large \( \nu \) the FP tends toward the FP that is known to occur in the absence of the CS term, which has \( \hat{\Lambda}_c = 0 \). The descending root cutoff, which is reliable for small \( \nu \), and the spectrally balanced cutoff both indicate that for small \( \nu \) the FP tends toward the values \( \hat{\Lambda}_c \approx -0.33 \), \( \hat{G}_c \approx 0.85 \). We conclude that for small \( \nu \) the cosmological constant is certainly negative. The fact that for \( \nu > 0.18 \) in the descending cutoff scheme \( \hat{\Lambda}_c \) becomes positive is probably an unphysical effect of the singularity occurring at \( \hat{G} = 0 \), as figure 7 suggests. Whether for intermediate values of \( \nu \) \( \hat{\Lambda}_c \) could be positive is an issue that we leave for future more refined analyses. On balance, we think that figure 8 probably gives the most reliable picture.

The final universal feature of the flow is that the nontrivial FP is UV attractive in both directions, and the existence of a trajectory that connects the nontrivial FP in the UV to the Gaussian FP in the IR. The scaling exponents of the nontrivial FP depend on \( \nu \) but both for
very large and very small $\nu$ they tend to similar values which are close to $-1$ and $-2.3$ (exactly $-1$, for large $\nu$).

8.2. Small $\tilde{G}$ expansion

We have confined ourselves to the one-loop approximation, which is of the lowest order in an expansion in $\tilde{G}$. From this point of view one could be surprised by the fact that the beta functions seem to contain arbitrarily high powers of $\tilde{G}$ (see equations (7.3) and (7.15)). But note that the original beta functions of $\Lambda$, $G$ and $\tilde{\mu}$ are indeed at most quadratic in $\tilde{G}$: it is only when we use the constancy of $\nu$ to solve for $\tilde{\mu}$ as a function of $\tilde{G}$ that the beta functions become nonpolynomial in $\tilde{G}$. Since the beta functions will anyway receive higher order corrections, one may want to truncate the expansion in (7.15) to terms at most quadratic in $\tilde{G}$.

In the case of the ascending root cutoff there are no terms of order $\tilde{G}^2$ beyond those that are already present in the absence of the CS term, so to leading order in that scheme the FP is the same as in the absence of the CS term. In the case of the descending root cutoff one cannot even pose the question, since the beta functions are singular in the origin. In the case of the spectrally balanced cutoff one obtains

$$
\partial_t \tilde{G} = \frac{4(47 - 2\alpha^2)\tilde{G}^2}{3\pi (4 - \alpha)},
$$

$$
\partial_t \Lambda = -2\Lambda - \frac{4(14 - 27\alpha + 4\alpha^2)\tilde{G}\Lambda}{3\pi (4 - \alpha)} - \frac{4\tilde{G}^2(3 + 5\Lambda)}{3\pi \nu}.
$$

In this case the beta function for $\tilde{G}$ is the same as for pure gravity without the CS term, but the beta function of $\Lambda$ receives a $\nu$-dependent correction. In this approximation the location of the FP is at

$$
\tilde{G}_* = \frac{3\pi}{47}, \quad \Lambda_* = -\frac{3\pi}{5\nu + 423\nu},
$$

and the eigenvalues of the linearized flow equation are $-1$ and $-\frac{108}{47} - \frac{108\nu}{229\nu}$.

In any case we believe that the vanishing of the beta function of $\nu$ will also be true when higher order corrections are taken into account. This provides some justification for keeping also the higher order terms in the beta functions.

8.3. The chiral point

One of the motivations for this work was to determine whether the chiral point defined by the condition $\mu = \pm \sqrt{\Lambda}$ has any special properties under RG. We can recast this issue in a slightly more general way as follows. With the dimensionful couplings $\Lambda$, $G$ and $\mu$ one can construct the dimensionless combinations (3.2). The more general question is whether any of these combinations is invariant under the flow. We have seen that this is the case for $\nu$. The conditions $\phi = \text{const}$, $\tau = \text{const}$ determine two-dimensional subspaces in the space of couplings. The intersection of these subspaces with the $\Lambda$–$\tilde{G}$ plane defines congruences of curves. The lines of constant $\tau$ are hyperbolas with the equation

$$
\tilde{G}^2 = \frac{\tau}{\Lambda},
$$

Since $\nu$ is RG invariant, using the relation

$$
\phi = \frac{\nu}{\sqrt{\tau}},
$$

we also see that $\phi$ is constant on these hyperbolas. It is easy to see that the flow lines will generally not preserve $\phi$: they will intersect the lines of constant $\phi$. If we choose the initial
conditions for the flow to lie on one hyperbola, then the flow will move away from it. In this sense, the condition \( \phi = 1 \) is not a RG invariant.

One can still ask whether there exist some FP that satisfy the chirality condition. For each value of \( \nu \), the corresponding FP in the \( \Lambda - \tilde{G} \) plane also determines a FP value for \( \tilde{\mu} \); with these FP values, one can compute the dimensionless combinations \( \nu \), \( \tau \) and \( \phi \). With the spectrally balanced cutoff, there exists a unique value \( \nu_{\text{crit}} \approx 0.0956016 \), for which the FP lies exactly on the corresponding hyperbola. For this choice the chirality condition is UV attractive, but even in this case the flow does not preserve the hyperbola.

The only point in the \( \Lambda - \tilde{G} \) plane where \( \phi \) becomes approximately RG invariant is near the Gaussian FP, where the dimensionful \( \Lambda \), \( G \) and \( \mu \) become constant. In terms of the dimensionless \( \tilde{\Lambda} \) and \( \tilde{G} \) this means that in their beta functions one only keeps the first term on the rhs of (7.7). Then the flow lines approximate the hyperbolas of constant \( \tau \) (see the right panel in figure 4). The approximation becomes better when \( \tau \) is small. If we want to have \( \phi = 1 \) approximately constant, this can be achieved for \( \nu = \sqrt{\tau} \to 0 \).

8.4. On-shell properties and the sign of \( \Lambda \)

Given that the effective action should be gauge independent on-shell, i.e. for \( R = 6 \Lambda \), also its \( t \)-derivative should have this property. From equation (6.2), this implies that \( A + 6B \tilde{\Lambda} \) has to be \( \alpha \)-independent. Indeed one finds, for all cutoffs, that

\[
A + 6B \tilde{\Lambda} = -\frac{16}{3\pi} (1 + 3\tilde{\Lambda}) + \mu \text{-dependent terms}
\] (8.5)

so the \( \alpha \)-dependence cancels. Another way to see this is to observe that on-shell the Einstein-Hilbert action is proportional to \( 1/\sqrt{\tau} = 1/(G\sqrt{\Lambda}) \), up to terms of order \( \sqrt{\Lambda} \). Thus the beta function for this particular dimensionless combination of couplings should also be gauge independent. Indeed, using (6.8) one finds that this beta function is proportional to \( A + 6B \tilde{\Lambda} \), and therefore \( \alpha \)-independent. In view of (8.4), the same is true for the beta function of \( \phi = \mu/\sqrt{\Lambda} \). Note however that the beta functions of \( \tau \) and \( \phi \) are not functions of \( \tau \) and/or \( \phi \) alone but depend on \( \tilde{\Lambda} \) and \( \tilde{G} \) separately. Therefore, to calculate their flow one needs the beta functions of \( \tilde{\Lambda} \) and \( \tilde{G} \), which have to be computed with off-shell backgrounds.

It is perhaps appropriate to add some further comments on the significance of the on-shell condition. One may worry that off-shell backgrounds will give rise to corrections proportional to the equations of motion. We show in appendix C that this is not the case. One may also worry that since the sums (4.13) are performed on a three sphere, which has positive curvature, the resulting beta functions are only correct in the domain \( \Lambda > 0 \). In particular, could this not affect the existence of the FP, which actually occurs (in the descending and balanced schemes) for negative \( \Lambda \)? Would one not have to perform the calculation on an AdS background? It would indeed be desirable to perform such a calculation, but since the Euclidean continuation of AdS spacetime is a noncompact hyperboloid, the harmonic analysis would be much trickier. Fortunately there are good reasons to believe that in the UV limit such a calculation would yield exactly the same beta functions that we have derived here. The reason is that any positive definite metric looks the same at sufficiently short distance scales, so the beta functions would be the same as long as \( k^2 \gg R \). Furthermore we observe that for large \( \nu \) the actual values of the cosmological constant, in units of \( k \), can be made arbitrarily small and negative, while the background can be chosen to have \( R \) arbitrarily small and positive, so there exist FPs which are as close to the mass shell as one wishes.
8.5. Topological issues

The vanishing of the beta function of the coefficient of the CS term calls for a topological interpretation. The analogous phenomenon in gauge theories has been investigated in [29, 30]. It was found that in spite of the finiteness of the theory, the coefficient of the CS term receives a finite renormalization proportional to the Casimir of the gauge group in the adjoint representation. This however is an infrared effect [31]; in the context of the flow of the effective average action, which we used here, one can see that the coefficient of the CS term is constant for all finite $k$ except for a finite jump in the limit $k \to 0$ [32]. Since our beta functions are only valid in the case $k \gg R$, we do not see such effect and we cannot exclude that such a finite renormalization happens.

One can interpret the vanishing beta function of $\nu$ as the consequence of some quantization condition. This issue has been discussed in [33] in the special case of flat space with asymptotically flat boundary conditions, and no quantization was found to be necessary. However, there will be topologies for which quantization is necessary. The vanishing of the beta function of $\nu$ is consistent with the expectation that our beta functions should hold for all topologies.

We worked here on a manifold without boundary. In the presence of a boundary, the CS term would give rise to a gravitational anomaly in the boundary [34]. Thus the nonrenormalization of the coefficient of the CS term implies the nonrenormalization of the coefficient of the anomaly in a certain two-dimensional conformal field theory. This is consistent with the results of [35], where the running of the central charges in such a theory was calculated using holographic methods.

8.6. Asymptotic safety and higher derivative terms

In four-dimensional gravity, a nontrivial FP has been found in the leading order of the $2 + \epsilon$ expansion [20] and of a $1/N$ expansion [21], in perturbation theory [22, 23] and in a variety of truncations of an exact RG equation [24–26] also in the presence of matter [27]. The nontrivial FP we have found for each value of $\nu$ in TMG is similar to the FP that is found in the Einstein–Hilbert truncation of gravity also in higher dimensions, except for the fact that the cosmological constant turns out to be negative. There are however two features that make our result particularly interesting.

The first is that for large $\tilde{\mu}$ the FP value of $\tilde{G}$ is securely within the perturbative domain. There is therefore every reason to believe that the one-loop calculation we have performed correctly captures the main features of the Wilsonian flow of this theory. The analogous result in pure three-dimensional Einstein Hilbert gravity is physically less appealing because in the absence of the CS term the theory does not have propagating degrees of freedom.

The second has to do with the consistency of the truncation. In principle the effective average action will also contain higher derivative terms, which in our computation have been neglected. However, in three dimensions the Riemann tensor is expressed entirely as a function of the Ricci tensor, and consequently it is possible to use field redefinitions to eliminate higher derivative terms by means of field redefinitions, order by order in perturbation theory [28]. In three dimensions the higher derivative operators are technically redundant and therefore the truncation considered here is consistent. At least in perturbation theory, it is not necessary to consider an infinite set of operators. However, the redefinitions used to eliminate the higher derivative terms change the cosmological constant and Newton’s constant. Therefore, the beta functions of $\tilde{\Lambda}$ and $\tilde{G}$ will receive corrections which have not been considered here.
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Appendix A. Useful lemmas

In implementing various cutoff schemes, it is useful to work out separately the decomposition of various terms occurring in the quadratic action $S^{(2)} + S_{GF}$ given in (3.7). The decomposition of the terms that come from the Einstein–Hilbert action with a cosmological term is as follows:

$$
\int d^3x \sqrt{-g} h_{\mu\nu} \Box h^{\mu\nu} = \int d^3x \sqrt{-g} \left[ h_{\mu\nu}^{T} \Box h_{\mu\nu}^{T} - 2\xi_{\mu} \left( \Box + \frac{R}{3} \right) \left( \Box + \frac{2R}{3} \right) \xi_{\mu} \right] + \frac{2}{3} \sigma \Box \left( \Box + \frac{R}{2} \right) \left( \Box + \frac{R}{2} \sigma + \frac{1}{3} h \Box h \right).
$$

$$
\int d^3x \sqrt{-g} h_{\mu\nu} \nabla^\mu \nabla^\nu h^{\mu\nu} = \int d^3x \sqrt{-g} \left[ -\xi_{\mu} \left( \Box + \frac{R}{3} \right)^2 \xi_{\mu} + \frac{4}{9} \sigma \Box \left( \Box + \frac{R}{2} \right)^2 \sigma + \frac{2}{3} h \Box \left( \Box + \frac{R}{2} \right) \sigma + \frac{1}{3} h^2 \right].
$$

The decomposition of the terms arising in the gravitational CS terms gives

$$
\int d^3x h_{\sigma\rho} \epsilon^{\lambda\mu\nu} \nabla_\mu h_{\nu}^\sigma = \int d^3x \left[ h_{\sigma\rho}^{T} \epsilon^{\lambda\mu\nu} \nabla_\mu h_{\nu}^\sigma - \xi_{\lambda} \epsilon^{\lambda\mu\nu} \nabla_\mu \left( \Box + \frac{R}{3} \right) \left( \Box + \frac{2R}{3} \right) \xi_{\nu} \right].
$$

$$
\int d^3x h_{\sigma\rho} \epsilon^{\lambda\mu\nu} \nabla_\mu h_{\nu}^\sigma = \int d^3x \left[ h_{\sigma\rho}^{T} \epsilon^{\lambda\mu\nu} \nabla_\mu h_{\nu}^\sigma - \xi_{\lambda} \epsilon^{\lambda\mu\nu} \nabla_\mu \left( \Box + \frac{R}{3} \right) \xi_{\nu} \right].
$$

$$
\int d^3x h_{\sigma\rho} \epsilon^{\lambda\mu\nu} \nabla_\mu \nabla^\rho h_{\nu}^\sigma = \int d^3x \left[ -\xi_{\lambda} \epsilon^{\lambda\mu\nu} \nabla_\mu \left( \Box + \frac{R}{3} \right)^2 \xi_{\nu} \right].
$$

Appendix B. Euclidean continuation and harmonic expansions on $S^3$

The issue of defining the Euclidean continuation for a theory of gravity is quite subtle and we will not try to address it here in generality. In general one would have to think of the real (three-dimensional) Minkowskian spacetime as being a section of a three-dimensional complex manifold, and find another real section where the metric is positive definite, and where the action is real and possibly bounded from below. The Wick rotation procedure which is employed in perturbative path integral formulation of non-gravitational field theories runs into severe problems in gravity in which the notion of time is affected by diffeomorphisms. Nonetheless, here we will adapt the analog of the standard field theoretic Wick rotation,
since we are interested in one-loop beta functions, which are calculable from the gauge
fixed quadratic part of the action when expanded about maximally symmetric backgrounds
classified in (3,8). The nature of the Wick rotation still depends on the choice of
background metric. For our purposes, it is convenient to work with the dS3 metric, which can be represented as
\[ ds^2 = -d\rho^2 + \cosh^2 \rho (d\theta^2 + \sin^2 \theta d\phi^2). \] (B.1)
Upon Wick rotation \( \rho \to -i\rho \), this turns into a metric on \( S^3 \) given by
\[ ds^2 = d\rho^2 + \cos^2 \rho (d\theta^2 + \sin^2 \theta d\phi^2). \] (B.2)
Thus in the path integral \( e^{iS[g]} \) becomes \( e^{-S_E[g_E]} \), where
\[ S(g) = i \int d\rho d\theta d\phi \sqrt{-R} \mathcal{L}[\Psi] \to - \int d\rho d\theta d\phi \sqrt{-\mathcal{L}[\Psi]} = -S_E[g_E]. \] (B.3)
With this procedure the Euclidean action differs from (3,1) just by an overall sign.
To perform harmonic analysis on \( S^3 \), we decompose all the fields that carry irreducible
representations of the stability group \( SO(3) \subset SO(4) \), where \( SO(4) \) is the isometry group of
\( S^3 \). These fields admit harmonic expansion in terms of all the representation functions of
\( SO(4) \) that contain the given \( SO(3) \) representation (see, for example, [36]). Thus, the divergence-free
and trace-free symmetric tensor \( h_{\mu\nu}^T \), the divergence-free vector \( \xi \) and a generic scalar \( \phi \) are expanded on \( S^3 \) as\(^8\)
\[
\begin{align*}
  h_{\mu\nu}^T(x) &= \sum_{n=2}^{\infty} \left( h_{q(n,2)}^{(n,2)} Y_{\mu\nu,q}^{(n,2)}(x) + h_{q(n,1)}^{(n,-2)} Y_{\mu\nu,q}^{(n,-2)}(x) \right), \\
  \xi_\mu(x) &= \sum_{n=1}^{\infty} \left( \xi_{q(n,1)}^{(n,1)} Y_{\mu,q}^{(n,1)}(x) + \xi_{q(n,1)}^{(n,-1)} Y_{\mu,q}^{(n,-1)}(x) \right), \\
  \phi(x) &= \sum_{n=0}^{\infty} \phi_{q(n,0)}^{(n,0)} Y_{0,q}^{(n,0)}(x),
\end{align*}
\] (B.4)
where \( q \) labels a unitary irreducible representation (UIR) of \( SO(4) \), the functions \( Y_{\mu\nu,q} \), \( Y_{\mu,q} \)
and \( Y_{0,q} \) are respectively the \( \mu\nu \)th, \( \mu \)th and 0th row, and \( q \)th column of the Wigner functions for
\( SO(4) \) UIRs is labeled by the highest weight \( (n, s) \), with \( n \geq |s| \). In expanding a field in spin-\( j \)
representation of \( SO(3) \) only the \( SO(4) \) UIRs that contain the given \( SO(3) \) representation occur,
and hence the restriction \( n \geq j \geq |s| \). Moreover, the \( SO(3) \) representation is contained
only once. The expansion coefficients \( h_{q(n,\pm2)}^{(n,\pm2)} \) are constants, and \( x \) labels the coordinates
on \( S^3 \). The Wigner functions occurring in these expansions have the same symmetry, trace
and divergence properties as the corresponding fields. Note the presence of the zero modes
in the form of the single constant mode \( Y_{0,0}^{(0,0)} \), the six Killing vectors \( Y_{\mu}^{(1,\pm1)} \) in the
adjoint representation of \( SO(4) \) and the four conformal Killing vectors \( \partial_\mu Y_{\nu}^{(10)} \). These obey
\[
\begin{align*}
  \partial_\mu Y_{0,0}^{(0,0)} &= 0, & \nabla_{[\mu} \nabla_{\nu]} Y_{\rho}^{(10)} &= 0, & \nabla_{[\mu} Y_{\nu]}^{(1,\pm1)} &= 0,
\end{align*}
\] (B.5)
where the notation \([\mu, \nu]\) means the symmetric and traceless part. The Killing vectors and the
conformal Killing vectors together generate the conformal group \( SO(4, 1) \) that acts on \( S^3 \).

\(^8\) We have absorbed the normalization factor \( \sqrt{\frac{2n+1}{4\pi}} \) occurring in the expansion of a spin-\( j \) field into the definition
of the expansion coefficients [36], where \( d(n,s) \) is the dimension of an \( SO(4) \) UIR with highest weight \( (n, s) \).
Next, we make use of the fact that the group theoretical considerations yield the following equations:

\[ -\Box Y^{(n,\pm 2)}_{\mu,\nu,q}(x) = \frac{R}{6} [(n + 1)^2 - 3] Y^{(n,\pm 2)}_{\mu,\nu,q}(x), \quad (B.6) \]

\[ -\Box Y^{(n,\pm 1)}_{\mu,q}(x) = \frac{R}{6} [(n + 1)^2 - 2] Y^{(n,\pm 1)}_{\mu,q}(x), \quad (B.7) \]

\[ \nabla [\mu Y^{(n,\pm 1)}_{\nu,q}(x)] = \pm \frac{1}{2} \sqrt{\frac{R}{6}} (n + 1) \bar{\varepsilon}_{\mu \nu} Y^{(n,\pm 1)}_{\nu,q}(x), \quad (B.8) \]

\[ -\Box Y^{(n,0)}_{0,q}(x) = \frac{R}{6} [(n + 1)^2 - 1] Y^{(n,0)}_{0,q}(x), \quad (B.9) \]

\[ \nabla [\mu Y^{(n,\pm 2)}_{\nu,q}(x)] = \pm i \sqrt{\frac{R}{6}} (n + 1) \bar{\varepsilon}_{\mu \nu} Y^{(n,\pm 2)}_{\nu,q}(x) \quad (B.10) \]

with multiplicities as given in (4.15). In deriving these formulae, we use the well-known formulae for the Casimir eigenvalues of the second-order Casimir operator \( C_2(n, s) \) for an \( SO(4) \) representation labeled by \( (n, s) \), and its dimension \( d(n,s) \) is given by

\[ C_2(n, s) = n(n + 2) + s^2, \quad d(n,s) = (n - 1)^2 - s^2. \quad (B.11) \]

Furthermore, we use the formula

\[ -\Box Y^{(n,s)}_{j,q}(x) = \frac{R}{6} [C_2(n, s) - C_2(j)] Y^{(n,s)}_{j,q}(x), \quad (B.12) \]

where \( Y^{(n,s)}_{j,q}(x) \) is the Wigner function for the \( SO(4) \) UIR in the \( (n, s) \) representation, viewed as a matrix, restricted in its row to the lowest spin-\( j \) representation it contains, and \( C_2(j) = j(j + 1) \) is the eigenvalue of the second-order Casimir operator for \( SO(3) \) in spin-\( j \) representation.

**Appendix C. The off shell one-loop beta functional**

In the evaluation of the one-loop effective action using the background field method one usually assumes that the background is a solution of the classical equations of motion. On the other hand in the derivation of the beta functions for gravity we have to evaluate \( \partial_t / \Gamma_1 \) for off-shell backgrounds. This is necessary in order to disentangle the beta functions of \( G \) and \( \Lambda \).

One may worry that for off-shell backgrounds the simple equation (2.7) is no longer correct.

We show here that this is not the case.

In order to do this, we review the derivation of that equation within the background field method. We consider a classical action \( S(\varphi) \) and expand \( \varphi = \bar{\varphi} + \eta \) where the background \( \bar{\varphi} \) is arbitrary. We Taylor expand

\[ S(\varphi) = S(\bar{\varphi}) + \int \frac{\delta S}{\delta \varphi} \frac{\delta S}{\delta \bar{\varphi}} \eta + \frac{1}{2} \int \eta \Delta_\varphi(\varphi) \eta + \cdots \]

(C.1)

where \( \Delta_\varphi = \frac{\delta^2 S}{\delta \varphi \delta \varphi} \big|_{\bar{\varphi}} \). The \( k \)-dependent generating functional analogous to (2.3) can now be written as

\[ e^{-W_k(\bar{\varphi},j)} = \int D\eta \exp \left\{ -S(\bar{\varphi} + \eta) - \Delta S_\varphi(\eta) - \int j \eta \right\}, \quad (C.2) \]

such that

\[ \varphi \equiv \langle \eta \rangle = \frac{\delta W_k}{\delta j}. \quad (C.3) \]
The background-dependent effective average action is the $k$-dependent modified Legendre transform
\[ \Gamma_k[\tilde{\phi}, \varphi] = W_k[j] - \int j \varphi - \Delta S_k[\varphi], \tag{C.4} \]
where $j$ has to be expressed as a function of $\varphi$, solving (C.3). From here one finds that
\[ \frac{\delta \Gamma_k}{\delta \varphi} = -j - R_k \varphi; \quad \frac{\delta^2 \Gamma_k}{\delta \varphi \delta \varphi} = -\left( \frac{\delta^2 W_k}{\delta j \delta j} \right)^{-1} - R_k. \tag{C.5} \]

Let us evaluate these functionals at one loop. We have
\[ W_k^{(1)}[\tilde{\phi}, j] = S(\tilde{\phi}) + \frac{1}{2} \text{Tr} \log(\Delta_\tilde{\phi} + R_k(\Delta_\tilde{\phi})) - \frac{1}{2} \int \int (\Delta_\tilde{\phi} + R_k(\Delta_\tilde{\phi}))^{-1} j. \tag{C.6} \]
From here solving equation (C.3) we get
\[ j = - (\Delta_\tilde{\phi} + R_k(\Delta_\tilde{\phi})) \varphi. \tag{C.7} \]

Therefore the one-loop average effective action $\Gamma_k^{(1)}$ is given by
\[ \Gamma_k^{(1)}(\tilde{\phi}, \varphi) = S(\tilde{\phi}) + \frac{1}{2} \text{Tr} \log(\Delta_\tilde{\phi} + R_k(\Delta_\tilde{\phi})) + \frac{1}{2} \int \int \varphi \Delta_\tilde{\phi} \varphi. \tag{C.8} \]

The main point to observe here is that the terms of the form $\int \varphi R_k \varphi$ have exactly canceled. Therefore, the only dependence on $k$ is in the trace term, and the beta functional (2.7) is unaffected. In a gauge theory, the presence of the gauge fixing terms does not change this conclusion.

Appendix D. Gravity without the CS term

Here we derive the beta functions of $\tilde{\Lambda}$ and $\tilde{G}$ using heat kernel techniques. This has been described elsewhere in a variety of gauges and cutoff schemes, but not using the cutoff scheme described in section 3. For the sake of comparison with the beta functions given in section 5 we give here this calculation.

In the absence of the CS term, and in diagonal gauge, all the operators appearing in the functional traces (4.12) are minimal Laplace-type operators of the form $\Delta = -\nabla^2 + E$ where $E$ is a linear map acting on the spacetime and internal indices of the fields. In our applications it will have the form $E = (q_1 R + q_2 \Delta) \mathbf{1} +$ where $\mathbf{1}$ is the identity in the space of the fields and $q_1, q_2$ are real numbers.

The trace of a function $W$ of the operator $\Delta$ can be written as
\[ \text{Tr} W(\Delta) = \sum_i W(\lambda_i) = \int_0^\infty ds \text{Tr} K(s) \tilde{W}(s) \tag{D.1} \]
where $\lambda_i$ are the eigenvalues of $\Delta$, $\tilde{W}$ is the Laplace anti-transform of $W(s)$ and Tr$K(s) = \sum_i e^{-s \lambda_i}$ is the trace of the heat kernel of $\Delta$. The UV behavior of the theory is governed by the lower end of the integration over $s$. We use the asymptotic expansion for $s \to 0$:
\[ \text{Tr} (e^{-s \Delta}) = \frac{1}{(4\pi)^{d/2}} \left[ B_0(\Delta) s^{-\frac{d}{2}} + B_2(\Delta) s^{-\frac{d+2}{2}} + \cdots + B_{d+2}(\Delta) s + \cdots \right] \tag{D.2} \]
where $B_n = \int d^d x \sqrt{g} \text{tr} b_n$ and $b_n$ are linear combinations of curvature tensors and their covariant derivatives containing $n$ derivatives of the metric. Then (D.1) becomes
\[ \text{Tr} W(\Delta) = \frac{1}{(4\pi)^{d/2}} \left[ Q_0(\Delta) B_0(\Delta) + Q_1(\Delta) B_2(\Delta) + \cdots + Q_0(\Delta) B_{d+2}(\Delta) + \cdots \right], \tag{D.3} \]
where
\[ Q_n(W) = \int_0^\infty ds s^{-n} \tilde{W}(s) = \frac{1}{\Gamma(n)} \int_0^\infty dz z^{n-1} W(z). \quad (D.4) \]

The last equality only holds for \( n > 0 \).

For the function \( W = \frac{\delta R}{\ell_p^2} \) of interest in the evaluation of (4.12) the integrals are
\[ Q_{3/2}(W) = \frac{8}{3\sqrt{\pi}} k^3; \quad Q_{1/2}(W) = \frac{4}{\sqrt{\pi}} k. \quad (D.5) \]

The only remaining ingredient that is needed for the evaluation of (6.2) is the coefficients \( B_0 \) and \( B_2 \) of the operators \( \Delta_1 \) given in (4.4). These can be evaluated using standard methods (see for example appendix B in [11]) and are listed in the table.

| \( h \)  | \( T \) | \( \xi \) | \( \sigma \) | \( h \) | \( V \) | \( S \) |
|---------|-------|------|----|----|----|----|
| \( \text{tr}b_0 \) | 2 | 2 | \frac{11-9\alpha-2\alpha^2}{1}+4\alpha\Lambda | 1 | 1 | \frac{16-\alpha^2}{6(4-\alpha)} | \frac{6\alpha}{4-\alpha} | \frac{16-\alpha^2}{6(4-\alpha)} |
| \( \text{tr}b_2 \) | \(-3\Lambda + 4\Lambda \Lambda \) | \frac{11-9\alpha-2\alpha^2}{1}+4\alpha\Lambda | \frac{16-\alpha^2}{6(4-\alpha)} | \frac{6\alpha}{4-\alpha} | \frac{16-\alpha^2}{6(4-\alpha)} |

The result is
\[ \partial_t \Gamma = \int d^3x \sqrt{g} \frac{6(11-9\alpha-2\alpha^2)\Lambda - (47-2\alpha^2)R}{12\pi^2(4-\alpha)} k, \quad (D.6) \]
from which we read off
\[ A = \frac{8(11+9\alpha-2\alpha^2)\Lambda}{\pi(4-\alpha)}; \quad B = -\frac{4(47-2\alpha^2)}{3\pi(4-\alpha)}. \quad (D.7) \]

These results agree, for \( \alpha = 1 \), with those reported in section IV D of [11]. As discussed there, the values of \( A \) and \( B \) do depend on the cutoff scheme. This is normal for the beta functions of dimensionful couplings. It is reassuring that this scheme dependence does not change the picture qualitatively. Furthermore, there are certain aspects of the flow that are scheme independent. The first one is the beta function of the dimensionless coupling \( V \), which is zero independently on the scheme. Another one is the beta function of the dimensionless combination \( \tau \); it is equal to
\[ \partial_t (\Lambda G^2) = -\frac{36\Lambda G^3}{\pi}, \quad (D.8) \]
in both cases. This accords with the gauge independence of the same quantity, as discussed in section 6. Higher terms of the expansion in \( \Lambda \) will probably not be likewise universal.

We conclude by giving the properties of the nontrivial FP in this scheme. It occurs at \( \Lambda = 0; G = 0.2005 \); its scaling exponents are \(-1\) and \(-2.298\), with eigenvectors aligned with the coordinate axes.

References
[1] Deser S, Jackiw R and Templeton S 1982 Topologically massive gauge theories Ann. Phys. 140 372
Deser S, Jackiw R and Templeton S 1988 Topologically massive gauge theories Ann. Phys. 185 409–49 (erratum)
[2] Li W, Song W and Strominger A 2008 Chiral gravity in three dimensions J. High Energy Phys. JHEP04(2008)082 (arXiv:0801.4566 [hep-th])
[3] Carlip S, Deser S, Waldron A and Wise D K 2009 Cosmological topologically massive gravitons and photons Class. Quantum Grav. 26 075008 (arXiv:0803.3998 [hep-th])
[4] Deser S and Yang Z 1990 Is topologically massive gravity renormalizable? Class. Quantum Grav. 7 1603
[5] Keszthelyi B and Klappe G 1992 Renormalizability of D = 3 topologically massive gravity Phys. Lett. B 281 33
[6] Oda I 2009 Renormalizability of topologically massive gravity arXiv:0905.1536 [hep-th]
[7] Weinberg S 1979 General Relativity: An Einstein centenary survey ed S W Hawking and W Israel (Cambridge: Cambridge University Press) pp 790–831
[8] Niedermayer M and Reuter M 2006 The asymptotic safety scenario in quantum gravity Living Rev. Rel. 9 5
Niedermayer M 2007 The asymptotic safety scenario in quantum gravity: an introduction Class. Quantum Grav. 24 R171 (arXiv:gr-qc/0610018)
Percacci R 2009 Asymptotic Safety Approaches to Quantum Gravity: Towards a New Understanding of Space, Time and Matter ed D Oriti (Cambridge: Cambridge University Press) (arXiv:0709.3851 [hep-th])
Litim D F 2008 Fixed points of quantum gravity and the renormalisation group PoS(QG-Ph)/024 (arXiv:0810.3675 [hep-th])
[9] Lauscher O and Reuter M 2002 Phys. Rev. D 65 025013 (arXiv:hep-th/0108040)
[10] Fischer P and Litim D F 2006 Fixed points of quantum gravity in extra dimensions Phys. Lett. B 638 497 (arXiv:hep-th/0602203)
[11] Codello A, Percacci R and Rahmede C 2009 Investigating the ultraviolet properties of gravity with a Wilsonian renormalization group equation Ann. Phys. 324 414 (arXiv:0805.2909 [hep-th])
[12] Hewett J and Rizzo T 2007 Collider signals of gravitational fixed points J. High Energy Phys. JHEP12(2007)009 (arXiv:0707.3182 [hep-ph])
[13] Wegner F J and Houghton A 1973 Renormalization group equation for critical phenomena Phys. Rev. A 8 401–12
Polchinski J 1984 Renormalization and effective Lagrangians Nucl. Phys. B 231 269–95
Morris T R 1994 Derivative expansion of the exact renormalization group Phys. Lett. B 329 241 (arXiv:hep-ph/9403340)
[14] Wetterich C 1993 Exact evolution equation for the effective potential Phys. Lett. B 301 90
[15] Reuter M 1998 Nonperturbative evolution equation for quantum gravity Phys. Rev. D 57 971 (arXiv:hep-th/9605030)
[16] Fradkin E S and Tseytlin A A 1982 Off-shell one loop divergences in gauged O(N) supergravities Phys. Lett. B 117 303
[17] Dou D and Percacci R 1998 The running gravitational couplings Class. Quantum Grav. 15 3449 (arXiv:hep-th/9707239)
[18] Litim D F 2001 Optimised renormalisation group flows Phys. Rev. D 64 105007 (arXiv:hep-th/0103195)
[19] Witten E 2010 Analytic continuation Of Chern–Simons theory arXiv:1001.2933 [hep-th]
[20] Hewett J and Rizzo T 2007 Collider signals of gravitational fixed points J. High Energy Phys. JHEP12(2007)009 (arXiv:0707.3182 [hep-ph])
[21] Witten E 2010 Analytic continuation Of Chern–Simons theory arXiv:1001.2933 [hep-th]
[22] Hewett J and Rizzo T 2007 Collider signals of gravitational fixed points J. High Energy Phys. JHEP12(2007)009 (arXiv:0707.3182 [hep-ph])
[23] Niedermayer M R 2009 Gravitational fixed points from perturbation theory Phys. Rev. Lett. 103 101303
[24] Lauscher O and Reuter M 2002 Phys. Rev. D 65 025013 (arXiv:hep-th/0108040)
[25] Lauscher O and Reuter M 2002 Is quantum Einstein gravity nonperturbatively renormalizable? Class. Quantum Grav. 19 483 (arXiv:hep-th/0110021)
[26] Lauscher O and Reuter M 2002 Towards nonperturbative renormalizability of quantum Einstein gravity Int. J. Mod. Phys. A 17 993 (arXiv:hep-th/0112089)
Reuter M and Saueressig F 2002 Renormalization group flow of quantum gravity in the Einstein–Hilbert truncation Phys. Rev. D 65 065016 (arXiv:hep-th/0110054)
Lauscher O and Reuter M 2002 Flow equation of quantum Einstein gravity in a higher-derivative truncation Phys. Rev. D 66 025026 (arXiv:hep-th/0205062)
[27] Codello A, Percacci R and Rahmede C 2008 Ultraviolet properties of f(R)-gravity Int. J. Mod. Phys. A 23 143 (arXiv:0705.1769 [hep-th])
Machado P F and Saueressig F 2008 On the renormalization group flow of f(R)-gravity Phys. Rev. D 77 124045 (arXiv:0712.0445 [hep-th])
[28] Benedetti D, Machado P F and Saueressig F 2009 Asymptotic safety in higher-derivative gravity Mod. Phys. Lett. A 24 2233 (arXiv:0901.2984 [hep-th])
Benedetti D, Machado P F and Saueressig F 2010 Taming perturbative divergences in asymptotically safe gravity Nucl. Phys. B 824 168 (arXiv:0902.4630 [hep-th])
[29] Percacci R and Perini D 2003 Constraints on matter from asymptotic safety Phys. Rev. D 67 081503 (arXiv:hep-th/0207013)
Percacci R and Perini D 2003 Asymptotic safety of gravity coupled to matter Phys. Rev. D 68 044018 (arXiv:hep-th/0304222)
Narain G and Percacci R 2009 Renormalization group flow in scalar-tensor theories: I arXiv:0911.0386 [hep-th]
Narain G and Rahmede C 2009 Renormalization group flow in scalar-tensor theories: II arXiv:0911.0394 [hep-th]

[28] Gupta R K and Sen A 2008 Consistent truncation to three dimensional (super)-gravity J. High Energy Phys. JHEP03(2008)015 (arXiv:0710.4177 [hep-th])

[29] Pisarski R D and Rao S 1985 Topologically massive chromodynamics in the perturbative regime Phys. Rev. D 32 2081

[30] Witten E 1989 Quantum field theory and the Jones polynomial Commun. Math. Phys. 121 351

[31] Shifman M A 1991 Four-dimension aspect of the perturbative renormalization in three-dimensional Chern–Simons theory Nucl. Phys. B 352 87

[32] M and Reuter M 1996 Effective average action of Chern–Simons field theory Phys. Rev. D 53 4430 (arXiv:hep-th/9511128)

[33] Percacci R 1987 On the topological mass in three dimensional gravity Ann. Phys. 177 27

[34] Kraus P and Larsen F 2006 Holographic gravitational anomalies J. High Energy Phys. JHEP01(2006)022 (arXiv:hep-th/0508218)

[35] Hotta K, Hysuktakte Y, Kubota T, Nishinaka T and Tanida H 2009 Left-right asymmetric holographic RG flow with gravitational Chern–Simons term Phys. Lett. B 680 279–85 (arXiv:0906.1255 [hep-th])

[36] Salam A and Strathdee J A 1982 On Kaluza–Klein theory Ann. Phys. 141 316